Abstract

We analyze M theory fivebrane in order to study the moduli space of vacua of $N = 1$ supersymmetric $Sp(N_c)$ gauge theories with $N_f$ flavors in four dimensions. We show how the $N = 2$ Higgs branch can be encoded in M theory by studying the orientifold which plays a crucial role in our work. When all the quark masses are the same, the surface of the M theory spacetime representing a nontrivial $S^1$ bundle over $\mathbb{R}^3$ develops $A_{N_f-1}$ type singularities at two points where D6 branes are located. Furthermore, by turning off the masses, two singular points on the surface collide and produce $A_{2N_f-1}$ type singularity. The sum of the multiplicities of rational curves on the resolved surface gives the dimension of $N = 2$ Higgs branch which agrees with the counting from the brane configuration picture of type IIA string theory. By rotating M theory fivebranes we get the strongly coupled dynamics of $N = 1$ theory and describe the vacuum expectation values of the meson field parameterizing Higgs branch which are in complete agreement with the field theory results. Finally, we take the limit where the mass of adjoint chiral multiplet goes to infinity and compare with field theory results. For massive case, we comment on some relations with recent work which deals with $N = 1$ duality in the context of M theory.
1 Introduction

One of the most interesting tools used to study nonperturbative dynamics of low energy supersymmetric gauge theories is to understand the D(irichlet) brane dynamics where the gauge theory is realized on the worldvolume of D brane.

This work was pioneered by Hanany and Witten [1] where the mirror symmetry of $N = 4$ gauge theory in 3 dimensions was interpreted by changing the position of the Neveu-Schwarz(NS)5 brane in spacetime. (see also [2][3]). They took a configuration of type IIB string theory which preserves 1/4 of the supersymmetry and consists of parallel NS5 branes with D3 branes suspended between them and D5 branes located between them. A new aspect of brane dynamics was the creation of D3 brane whenever a D5 brane and NS5 brane are crossing through each other. This was due to the conservation of the linking number (defined as a total magnetic charge for the gauge field coupled with the worldvolume of the both types of NS and D branes).

By T-dualizing the above configuration on one space coordinate, the passage to $N = 2$ gauge theory in 4 dimensions can be described as two parallel NS5 branes and D4 branes suspended between them in a flat space in type IIA string theory. When one change the relative orientation of the two NS5 branes [4] while keeping their common 4 spacetime dimensions intact, the $N = 2$ supersymmetry is broken to $N = 1$. The brane configuration [5, 6] preserves 1/8 of the supersymmetry and this corresponds to turning on a mass of adjoint field because the distances between D4 branes suspended between the NS5 branes relate to the vacuum expectation values (vevs). The configuration of D4 branes gives the gauge group while the D6 branes give the global flavor group. Using this configuration they described and checked a stringy derivation of Seiberg’s duality for $N = 1$ supersymmetric gauge theory with $SU(N_c)$ gauge group with $N_f$ flavors in the fundamental representation which was previously conjectured in [4]. This result was generalized to brane configurations with orientifolds which then give $N = 1$ supersymmetric theories with gauge group $SO(N_c)$ or $Sp(N_c)$ [7, 8]. In this case the NS5 branes have to pass over each other and some strong coupling phenomena have to be considered. Similar results were obtained in [9, 10, 11] where the moduli space of the supersymmetric gauge theories is geometrically encoded in the brane setup.

Another approach was initiated by Ooguri and Vafa [12] where they considered the compactification of IIA string theory on a double elliptically fibered Calabi-Yau threefold. The wrapped D6 branes around three cycles of Calabi-Yau threefold filling also a 4 dimensional spacetime. The transition between electric theory and its magnetic dual appears when a change in the moduli space of Calabi-Yau threefold occurs. Their results were generalized in the papers of [13, 14, 15, 16] to various other models which reproduce field theory results studied previously.
So far the branes in string theory were considered to be rigid without any bendings. When the branes are intersecting each other, a singularity occurs. In order to avoid that kind of singularities, a very nice simplification was obtained by reinterpreting brane configuration in string theory from the point of view of M theory as was showed by Witten in [17]. Then both the D4 branes and NS5 branes come from the fivebranes of M theory (the former is an M theory fivebrane wrapped over $S^1$ and the latter is an M theory fivebrane on $\mathbb{R}^{10} \times S^1$). That is, D4 brane’s worldvolume projects to a five manifold in $\mathbb{R}^{10}$ and NS5 brane’s worldvolume is placed at a point in $S^1$ and fills a six manifold in $\mathbb{R}^{10}$. To obtain D6 branes one has to use a multiple Taub-NUT space whose metric is complete and smooth. The $N = 2$ supersymmetry in four dimensions requires that the worldvolume of M theory fivebrane is $\mathbb{R}^{1,3} \times \Sigma$ where $\Sigma$ is uniquely identified with the curves [18] that appear in the solutions to Coulomb branch of the field theory.

The configurations involving orientifolds were considered in [19, 20]. The method of brane dynamics was used to study supersymmetric field theories in several dimensions by many authors [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. The original work [17] was suited to study the moduli space for $N = 2$ supersymmetric theories. By rotating one of the NS5 branes the $N = 2$ supersymmetry is broken to $N = 1$ [4]. In [29, 41] (see also [30, 38, 39]) this was seen from the point of the M theory interpretation, by considering the possible deformation of the curve $\Sigma$. In field theory, the supersymmetry is broken by giving a mass to the adjoint field and if this mass is finite, the $N = 1$ field theory can be compared with the previous results obtained in [42]. These papers considered the case of unitary groups.

Recently, the exact low energy description of $N = 2$ supersymmetric $SU(N_c)$ gauge theories with $N_f$ flavors in 4 dimensions in the framework of M theory fivebrane have been found in [41]. They constructed M fivebrane configuration which encodes the information of Affleck-Dine-Seiberg superpotential [12] for $N_f < N_c$. Later, this approach has been used to study the moduli space of vacua of confining phase of $N = 1$ supersymmetric gauge theories in four dimensions [33]. In terms of brane configuration of IIA string theory, this corresponds to the picture of [5] by taking multiples of NS’5 branes rather than a single NS’5 brane.

In the present paper we generalize to the case of symplectic group $Sp(N_c)$ with $N_f$ flavors. The new ingredient that is introduced is the orientifold. We find an interesting picture which differs from the one obtained for unitary group $SU(N_c)$ [41]. This is expected because for $SU(N_c)$ groups we have both baryonic and non-baryonic branches, but in the case of $Sp(N_c)$ we cannot construct any baryon, so we have only non-baryonic branch.

This paper is organized as follows. In section 2 we review the papers of [43, 44, 45] and study the moduli space of vacua of the $N = 1$ theory which is obtained from the
$N = 2$ theory by adding a mass term to the adjoint chiral multiplet. We discuss for different values of the number of flavors with respect to the number of colors. We also introduce massive matter. In section 3, we start with the setup of M theory fivebrane and discuss the Higgs branches with the resolution of singularities. In section 4, we rotate brane configuration and obtain information about the strong coupling dynamics of $N = 1$ theory. In section 5, we take the mass of adjoint field infinite and compare it with field theory results for massless or massive matter. We discuss $N = 1$ duality and compare with similar work without D6 branes obtained in [16] recently. Finally in section 6, we conclude our results and comment on the outlook in the future directions.

2 Field Theory Analysis

Let us review and summarize field theory results already known in the papers of [13, 44, 45] for future developments. We claim no originality for most of results presented in this section except that we have found the property of meson field $M_{ij}$ having only one kind matrix element which will be discussed in detail later.

2.1 $N = 2$ Theory

Let us consider $N = 2$ supersymmetric $Sp(N_c)$ gauge theory with matter in the $2N_c$ dimensional representation of $Sp(N_c)$. In terms of $N = 1$ superfields, $N = 2$ vector multiplet consists of a field strength chiral multiplet $W_{ab}$ and a scalar chiral multiplet $\Phi_{ab}$, both in the adjoint representation of the gauge group $Sp(N_c)$. The quark hypermultiplets are made of a chiral multiplet $Q_a^i$ which couples to the Yang-Mills fields where $i = 1, \cdots, 2N_f$ are flavor indices (the number of flavors has to be even) and $a = 1, \cdots, 2N_c$ are color indices. The $N = 2$ superpotential takes the form:

$$W = \sqrt{2}Q_a^i\Phi_b^{j}J_{ab}Q_c^i + \sqrt{2}m_{ij}Q_a^iJ_{ab}Q_b^j,$$

where $J_{ab}$ is the symplectic metric

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes 1_{N_c \times N_c}$$

used to raise and lower $Sp(N_c)$ color indices and $m_{ij}$ is the antisymmetric mass matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(m_f^1, \cdots, m_{N_f}^f).$$
Classically, the global symmetries are the flavor symmetry \( O(2N_f) = SO(2N_f) \times Z_2 \) in addition to \( U(1)_R \times SU(2)_R \) chiral R-symmetry. The theory is asymptotically free for \( N_f \) smaller than \( 2N_c + 2 \) and generates dynamically a strong coupling scale \( \Lambda_{N=2} \) where we denote the \( N = 2 \) theory by writing it in the subscript of \( \Lambda \). The instanton factor is proportional to \( \Lambda_{N=2}^{2N_c+2-N_f} \). Then the \( U(1)_R \) symmetry is anomalous and is broken down to a discrete \( Z_{2N_c+2-N_f} \) by instantons.

The moduli space contains the Coulomb and the Higgs branches. The Coulomb branch is parameterized by the gauge invariant order parameters

\[
 u_{2k} = \langle \text{Tr}(\phi^{2k}) \rangle, \quad \text{where } k = 1, \cdots, N_c \tag{2.4}
\]

where \( \phi \) is the scalar field in \( N = 2 \) vector multiplet. Up to a gauge transformation \( \phi \) can be diagonalized to a complex matrix, \( \langle \phi \rangle = \text{diag}(A_1, \cdots, A_{N_c}) \) where \( A_i = (a_i^0 \ 0 \ -a_i) \). At a generic point the vevs of \( \phi \) breaks the \( Sp(N_c) \) gauge symmetry to \( U(1)_{N_c} \) and the dynamics of the theory is that of an Abelian Coulomb phase. The Wilsonian effective Lagrangian in the low energy can be made of the multiplets of \( A_i \) and \( W_i \) where \( i = 1, 2, \cdots, N_c \). If \( k a_i \)'s are equal and nonzero then there exists an enhanced \( SU(k) \) gauge symmetry. When they are also zero, an enhanced \( Sp(k) \) gauge symmetry appears. On the other hand, the Higgs branches are described by gauge invariant quantities which are made from the squarks vevs and which can be written as the meson field \( M^{ij} = Q^i_a J^{ab} Q^j_b \) because we do not have any baryons.

### 2.2 Breaking \( N = 2 \) to \( N = 1 \)

We want now to break \( N = 2 \) supersymmetry down to \( N = 1 \) supersymmetry by turning on a bare mass \( \mu \) for the adjoint chiral multiplet \( \Phi \). For the moment we consider that all the squarks are massless, so the terms of \( m_{ij} \) in (2.1) will not enter into our superpotential. The superpotential is expressed as follows:

\[
 W = \sqrt{2} Q^i_a \Phi^a_b J^{bc} Q^j_c + \mu \text{ Tr}(\Phi^2). \tag{2.5}
\]

When the mass of the adjoint chiral multiplet is much smaller than \( \Lambda_{N=2} \), by turning a mass for the adjoint chiral multiplet, the structure of moduli space of vacua for \( N = 2 \) theory is changed. Most of the Coulomb branch is lifted except \( 2N_c + 2 - N_f \) points which are related to each other by the action of \( Z_{2N_c+2-N_f} \).

When the mass \( \mu \) is increased beyond \( \Lambda_{N=2} \) we can integrate out the adjoint chiral multiplet in the low-energy theory. Below the scale \( \mu \), by a one loop matching between the \( N = 1 \) and \( N = 2 \) theories we obtain the \( N = 1 \) dynamical scale, \( \Lambda_{N=1} \) to be:

\[
 \Lambda_{N=1}^{2(3N_c+3-N_f)} = \mu^{2N_c+2} \Lambda_{N=2}^{2(2N_c+2-N_f)}. \tag{2.6}
\]
If $\mu$ is much larger than $\Lambda_{N=1}$ but finite, we can integrate out the heavy field $\Phi$ and to obtain a superpotential which is quartic in the squarks and proportional to $1/\mu$. The F-term equation for $\Phi$ from \eqref{2.7} gives us to

$$Q_i^a Q_i^c + \sqrt{2}\mu J_{ab} \Phi_b^c = 0 \quad \text{(2.7)}$$

where we can read off $\Phi_b^c$

$$\Phi_b^c = \frac{1}{\sqrt{2}\mu} J^{ab} Q_a^i Q_b^i \quad \text{(2.8)}$$

or

$$\Phi^2 = -\frac{1}{2\mu^2} M^2. \quad \text{(2.9)}$$

We plug this into the superpotential equation and obtain

$$\Delta W = -\frac{1}{2\mu} \text{Tr}(M^2) \quad \text{(2.10)}$$

which is similar to the equation (2.5) of \cite{41} but without the term involving $(\text{Tr}M)^2$ because $M$ is traceless antisymmetric in our case and with a minus sign due to (2.9). Therefore, the system below the energy scale $\mu$ can be regarded as the $N=1$ SQCD with the tree level superpotential \eqref{2.10} and with the dynamical scale $\Lambda_{N=1}$ given by \eqref{2.6}. When we take the limit of $\mu \to \infty$ keeping $\Lambda_{N=1}$ fixed, the superpotential \eqref{2.10} vanishes.

Let us start with the discussion for the various values of $N_f$ as a function of $N_c$.

- $0 \leq N_f \leq N_c$

In this range of the number of flavors, as it is well known, a superpotential is dynamically generated \cite{42} by strong coupling effects. For a general value of $N_f$, the ADS superpotential is given by \cite{43}:

$$W_{\text{ADS}} = (N_c + 1 - N_f) \omega_{N_c+1-N_f} \left(2^{N_c-1-N_f} \Lambda_{N=1}^{3(N_c+1)-N_f} \text{PfM} \right)^{1/(N_c+1-N_f)} \quad \text{(2.11)}$$

where $\omega_{N_c+1-N_f}$ is an $N_c+1-N_f$ th root of unity and Pf(Pfaffian) of antisymmetric matrix $M$ has the following relation: $(\text{PfM})^2 = \det M$. For $N_f = N_c$, the gauge group is completely broken for Pf $< M >$ not zero and the ADS superpotential is generated by an instanton in the broken $Sp(N_c)$.

For large but finite values for $\mu$, the potential obtained after $N=2$ breaking into $N=1$ \eqref{2.10} can be described as a perturbation theory to the ordinary $N=1$ theory. Then the total effective superpotential is the sum of $W_{\text{ADS}}$ and $\Delta W$

$$W_{\text{eff}} = W_{\text{ADS}} + \Delta W. \quad \text{(2.12)}$$

This form for $W_{\text{eff}}$ is exact for any non-zero value of $\mu$. The argument is based on the holomorphic property. The two terms appearing in an analytic function expanded with respect to $1/\mu$ are given by $W_{\text{ADS}}$ and $\Delta W$ so a term that can be generated are of the form:

$$\mu^{-\alpha} M^\beta \Lambda_{N=1}^{3(N_c+1)-N_f} \gamma$$

(2.13)

where $\alpha, \gamma$ are non-negative integers. In order to obtain $\alpha, \beta, \gamma$ we use the fact that (2.13) is invariant under the axial flavor symmetry $U(1)_A$ and has a charge 2 under the R-symmetry $U(1)_R$ where $U(1)_R$ is the anomaly free combination of the $U(1)$ R-symmetry group. The charges for $\Lambda_{N=1}, \mu$ and $M$ are given by:

|          | $\Lambda_{N=1}^{3N_c+3-N_f}$ | $M$     | $\mu$  |
|----------|-------------------------------|---------|--------|
| $U(1)_R$ | 0                             | $\frac{2(N_f-N_c-1)}{N_f}$ | $\frac{2(N_f-2N_c-2)}{N_f}$ |
| $U(1)_A$ | $2N_f$                        | 2       | 4      |

Using these values for the charges and applying them in (2.13), the condition that the superpotential has $U(1)_A$ charge 0 gives us that $2\alpha - \beta = N_f \gamma$ and other condition that the superpotential has charge 2 under $U(1)_R$ becomes $N_f(-\alpha+\beta-1) = (N_c+1)(-2\alpha+\beta)$. The combination of these two relations leads to:

$$1 - \alpha = (N_c + 1 - N_f) \gamma.$$  

(2.14)

Because $\gamma \geq 0$ and we are considering the case of $N_f < N_c$, we have only two solutions for this equation, that is, $\alpha = 0, \gamma = 1/(N_c + 1 - N_f)$ and $\alpha = 1, \gamma = 0$, which exactly correspond to the two terms which appear in (2.13). Therefore, it turns out that the superpotential (2.13) is exact.

The moduli space of vacua is obtained by extrematizing this superpotential, and we get:

$$M^2 = -\mu \omega_{N_c+1-N_f} \left( \frac{2N_c-1}{\text{Pf}M} \Lambda_{N=1}^{3(N_c+1)-N_f} \right)^{1/N_c+1-N_f}.$$  

(2.15)

In this moment, by a similarity transformation, $M$ can be brought to different forms. In [44, 45], $M$ has been brought to a form such that $M^2 = 0$ which is the right equation for $M$ whenever we do not consider the ADS potential. But in our case, the equation (2.15) tells us that $M^2$ is not equal with 0, so we take another form for $M$ after a similarity transformation. We bring $M$ to the simplest form, i.e., with two top-right and bottom-left diagonal blocks, one being minus the other because $M$ is to be antisymmetric. Denote $m_1, \cdots, m_{N_f}$ by the elements of top-right diagonal block in $M$ and of course $-m_1, \cdots, -m_{N_f}$ by the elements of the bottom-left diagonal block. In this case we will have an equation like (2.13) for each $m_i$. Since the right hand side is the same for all the
$m_i$’s, they have to be equal. So all the diagonal entries in the top-right and bottom-left diagonal blocks of $M$ are equal. This is our new observation which will appear naturally in section 4 due to the symmetry of orientifolding and can be compared with the result of [11] for $SU(N_c)$ case where there were two cases, one with equal diagonal entries and the other with two different entries on the diagonal.

Now since all the top-right diagonal entries are equal with $m \equiv m_1 = \cdots = m_{N_f}$, we find the value for $m$ by solving (2.15):

$$m = 2^{\frac{N_c+1}{2} - N_f} \mu \Lambda_{N=2}$$

(2.16)

where we have used the renormalization group (RG) matching equation (2.6). The values of $m$ in equation (2.16) describe the moduli space of the $N = 1$ theory in the presence of a perturbation to the ADS superpotential. When $\mu \to \infty$ and $\Lambda_{N=1}$ are finite, the solution diverges. In this case $\Delta W$ is 0 and divergence of the solution coincides with the fact that there is no supersymmetric vacua in this region of the flavor.

Let us turn on quark mass terms like $\frac{1}{2}m^{ij}M_{ij}$. In this case the effective superpotential is given by

$$W_{eff} = W_{ADS} + \Delta W + \frac{1}{2} m M$$

(2.17)

where $W_{ADS}$ and $\Delta W$ are given by (2.11) and (2.10) and $m$ is an antisymmetric matrix as in (refmass) but where we take $m_f = m_1 = \cdots = m_{N_f}$. In this case the equation (2.13) is modified to contain a term $\mu m M/2$. As we consider the limit $\mu \to \infty$ by keeping $\Lambda_{N=1}$ finite, the system will be $N = 1$ SQCD with massive flavors. Only terms which are proportional to $\mu$ will resist (so the term form the LHS of (2.13) will be neglected because it does not depend on $\mu$) and thus we obtain the solution for the moduli space to be:

$$\omega_{N_c+1-N_f} \left(2^{N_c-1} \Lambda_{N=1}^{3(N_c+1)-N_f} \right)^{\frac{N_c+1-N_f}{N_c+1-N_f}} = m_f m/2$$

(2.18)

with the solution

$$m^{N_c+1} = \frac{2^{2N_c-N_f} \Lambda_{N=1}^{3(N_c+1)-N_f}}{m_f^{N_c+1-N_f}}$$

(2.19)

giving $N_c + 1$ vacua in accordance with the interpretation of the low energy physics as the pure $N = 1$ Yang-Mills theory.

Next we are now increasing the number of flavors.

- $N_f = N_c + 1$. 
Now it is obvious that the ADS superpotential vanishes and the classical moduli space of vacua is changed quantum mechanically. It is parameterized by the meson satisfying the constraint

$$\text{Pf} M = 2^{N_c-1} \Lambda_{N=1}^{2(N_c+1)}.$$  \hfill (2.20)

Again the quartic term $\Delta W$ is small for large finite $\mu$ and can be considered as a perturbation to the ordinary $N = 1$ theory. By introducing a Lagrange multiplier $X$ in order to impose the constraint (2.20), the effective superpotential will be:

$$W_{\text{eff}} = X (\text{Pf} M - 2^{N_c-1} \Lambda_{N=1}^{2(N_c+1)}) - \frac{1}{2\mu} \text{Tr}(M^2).$$  \hfill (2.21)

From the derivative with respect to $M$, we get:

$$M^2 = \mu X \text{Pf} M.$$  \hfill (2.22)

For the case of $X \neq 0$, again we can bring $M$ by a similarity transformation to the same form as before and this tells us that again all the top-right diagonal entries are the same and we obtain:

$$m_{N_c+1} = 2^{N_c-1} \Lambda_{N=1}^{2(N_c+1)}$$  \hfill (2.23)

which leads to, after using the RG equation:

$$m = 2^{N_c-1} \mu \Lambda_{N=2}$$  \hfill (2.24)

which gives the moduli space.

- $N_f = N_c + 2$

In this case, the effective potential by adding $\Delta W$ is given by:

$$W_{\text{eff}} = -\frac{\text{Pf} M}{2^{N_c-1} \Lambda_{N=1}^{2N_c+1}} - \frac{1}{2\mu} \text{Tr}(M^2).$$  \hfill (2.25)

which give us after extrematizing:

$$M^2 = -\mu \frac{\text{Pf} M}{2^{N_c-1} \Lambda_{N=1}^{2N_c+1}}.$$  \hfill (2.26)

Again $M$ can be brought to a simple form by a similarity transformation and all the diagonal entries are equal. After using the RG equation we get the moduli space given by:

$$m = 2^{N_c-1} \mu \Lambda_{N=2}.$$  \hfill (2.27)

- $N_f > N_c + 2$
The theory that we have discussed until now is the electric theory which for this range of the number of flavors has a dual description in terms of a $Sp(N_f - N_c - 2)$ gauge theory with $N_f$ flavors $q^i$ in the fundamental $(i = 1, \cdots, 2N_f)$, gauge singlets $M_{ij}$ and a superpotential

$$W = \frac{1}{4\lambda} M_{ij} q^i q^j J^{cd}.$$  \hfill (2.28)

where the scale $\lambda$ relates the scale $\Lambda_{N_f=1}$ of the electric theory and the scale $\tilde{\Lambda}_{N_f=1}$ of the magnetic theory by:

$$\Lambda_{N_f=1}^{3(N_c+1)-N_f} \tilde{\Lambda}_{N_f=1}^{3(N_f-N_c-1)-N_f} = C(-1)^{N_f-N_c-1} \lambda^{N_f}$$ \hfill (2.29)

where the constant $C$ was found in [43] to be $C = 16$. The effective superpotential is given as:

$$W_{\text{eff}} = W + W_{\text{ADS}} + \Delta W.$$ \hfill (2.30)

If the vevs for the magnetic quarks are 0, then the analysis is identical to those for the case $N_f < N_c$. If the vevs are not zero, then as in [41] we can take a limit to approach $PfM = 0$ and to use the corresponding formula in order to compare with the M theory approach. In the $SU(N_c)$ case, where baryons exist, a specific choice has been taken such that the baryons have a specific interpretation in the M theory picture.

\section{3 $N = 2$ Higgs Branch from M Theory}

In this section we study the moduli space of vacua of $N = 2$ supersymmetric QCD by analyzing M theory fivebranes. We will consider the Higgs branch in terms of geometrical picture. Let us first describe the Higgs branch in the type IIA brane configuration.

Following the paper of [3], the brane configuration contains three kind of branes: the two parallel NS5 branes extend in the direction $(x^0, x^1, x^2, x^3, x^4, x^5)$, the D4 branes are stretched between two NS5 branes and extend over $(x^0, x^1, x^2, x^3)$ and are finite in the direction of $x^6$, and the D6 branes extend in the direction of $(x^0, x^1, x^2, x^3, x^7, x^8, x^9)$. In order to study symplectic or orthogonal gauge groups, we will consider an O4 orientifold which is parallel to the D4 branes in order to keep the supersymmetry and is not of finite extent in $x^6$ direction. The D4 branes is the only brane which is not intersected by this O4 orientifold. The orientifold gives a spacetime reflection as $(x^4, x^5, x^7, x^8, x^9) \rightarrow (-x^4, -x^5, -x^7, -x^8, -x^9)$, in addition to the gauging of worldsheet parity $\Omega$. The fixed points of the spacetime symmetry define this O4 planes. Each object which does not lie at the fixed points (i.e. over the orientifold plane), must have its mirror image. Thus NS5 branes have a mirror in $(x^4, x^5)$ directions and D6 branes have a mirror in $(x^7, x^8, x^9)$ directions. Another important aspect of the orientifold is its charge, given by
the charge of $H^{(6)} = dA^{(5)}$ coming from Ramond Ramond (RR) sector, which is related to the sign of $\Omega^2$. In the natural normalization, where the D4 brane carries one unit of this charge, the charge of the O4 plane is $\pm 1$, for $\Omega^2 = \mp 1$ in the D4 brane sector.

With the above preliminary setup, let us discuss about the two different branches of the theory. The Coulomb branch can be described when all the D4 branes lie between NS5 branes where no squark has vevs. To go to the Higgs branch, the D4 branes are broken on the D6 branes and are suspended between D6 branes being allowed to move on the $(x^7, x^8, x^9)$ directions. Together with the gauge field component $A_6$ in the $x^6$ coordinate this gives two complex parameters to parameterize the location of the D4 branes. In [41], for $SU(N_c)$ case, the Coulomb branch and the Higgs branch share common directions and this comes from the fact that there are two different eigenvalues for $M$ which correspond to $r$ equal eigenvalues and $N_c - r$ equal eigenvalues. By turning on vevs for $r$ squarks, this gives rise to make the $r$ dimensional block of $M$ be nonzero. In brane language, this describes breaking $r$ D4 branes on the D6 branes and suspending the remaining $N_c - r$ D4 branes between the two NS5 branes.

In the case of $Sp(N_c)$ gauge theory, there are only two possibilities:

- All D4 branes are suspended between the two NS5 branes where no squark has vevs.
- Some of D4 branes are broken on D6 branes.

We, for simplicity, restrict ourselves to the case of all D4 branes being broken on D6 branes. See [47, 48] for more general cases.

The motion of D4 branes along D6 branes describes the Higgs branch and for each D4 brane suspended between two D6 branes there exist two massless complex scalars parameterizing the fluctuations of the D4 brane. Because of the O4 orientifold we have to take into account D4 branes stretched between two D6 branes. The s-rule [1] allows only one D4 brane (and its mirror) between a NS5 brane and a single D6 brane (and its mirror). For $N = 2$ theory because we have two NS5 branes, for both of them we have to impose the s-rule. Also, in contrast with $N = 1$ theory, there are no complex scalars which correspond to D4 branes stretched between NS5 branes and D6 branes. However remember that for $N = 1$ theory [3] there are no complex scalar corresponding to a D4 brane stretched between NS5 brane and a D6 brane. The dimension of the Higgs moduli space is obtained by counting all possible breakings of D4 branes on D6 branes as follows: the first D4 brane is broken in $N_f - 1$ sectors between the D6 branes (therefore the complex dimension is the twice of $N_f - 1$), the second D4 branes is broken

\[\text{§ There are recent papers on this issue [47, 48].}\]
in $N_f - 3$ sectors (the complex dimension is twice of $N_f - 3$) and so on. But, besides that we have to consider the antisymmetric orientifold projection which eliminates some degrees of freedom, as explained in [6]. Then the dimension of the Higgs moduli space is given by:

$$2[(2N_f - 2 - 1) + (2N_f - 6 - 1) + \cdots + (2N_f - 4N_c + 2 - 1)] = 4N_c(N_f - N_c) - 2N_c$$ (3.1)

or $4N_cN_f - 2N_c(2N_c + 1)$ where in the previous equations we have explicitly extracted 1 as a result of the antisymmetric orientifold projection. The overall factor 2 in the left hand side is due to the mirror D6 branes and the result is very similar to the field theory result except an extra multiplicative factor 2 in the right hand side, because we consider here complex dimensions. In field theory, because of the $N_f$ vevs, the gauge symmetry is completely broken and there are $4N_cN_f - 2N_c(2N_c + 1)$ massless neutral hypermultiplets for a $N = 2$ supersymmetric theory which thus exactly gives the dimension of the Higgs moduli space. Thus, the field theory results match the brane configuration results.

Let us discuss how the above brane configuration appears in M theory context in terms of a generically smooth single M fivebrane whose worldvolume is $\mathbb{R}^{1,3} \times \Sigma$ where $\Sigma$ is identified with Seiberg-Witten curves [19] that determine the solutions to Coulomb branch of the field theory. As usual, we write $v = x^4 + ix^5, s = (x^6 + ix^{10})/R, t = e^{-s}$ where $x^{10}$ is the eleventh coordinate of M theory which is compactified on a circle of radius $R$. Then the curve $\Sigma$, describing $N = 2$ $Sp(N_c)$ gauge theory with $N_f$ flavors, is given [19] by an equation in $(v, t)$ space

$$t^2 - (v^2B(v^2, u_k) + \Lambda_{N=2}^{2N_c+2-N_f} \prod_{i=1}^{N_f} m_i)t + \Lambda_{N=2}^{4N_c+4-2N_f} \prod_{i=1}^{N_f}(v^2 - m_i^2) = 0$$ (3.2)

where $B(v^2)$ is a polynomial of $v^2$ of degree $N_c$ with the coefficients depending on the moduli $u_k, v^{2N_c} + u_2v^{2N_c-2} + \cdots + u_{2N_c}$ and $m_i$ is the mass of quark.

### 3.1 Including D6 Branes

In M theory, the type IIA D6 branes are the magnetic dual of the electrically charged D0 branes, which are the Kaluza-Klein monopoles described by a Taub-NUT space. This is derived from a hyper-Kähler solution of the four-dimensional Einstein equation. But we will ignore the hyper-Kähler structure of this Taub-NUT space. Instead, we use one of the complex structures, which can be described by [13]

$$yz = \Lambda_{N=2}^{4N_c+4-2N_f} \prod_{i=1}^{N_f}(v^2 - m_i^2)$$ (3.3)

Note that this $m_i$ is nothing to do with the element of meson field $M$. Unfortunately we used same notation.
in $\mathbb{C}^3$. The D6 branes are located at $y = z = 0, v = \pm m_i$. This surface, which represents a nontrivial $S^1$ bundle over $\mathbb{R}^3$ instead of the flat four dimensional space $\mathbb{R}^3 \times S^1$ with coordinates $(x^4, x^5, x^6, x^{10})$, is the unfolding of the $A_{2n-1}$ ($n = N_f$) singularity in general. The Riemann surface $\Sigma$ is embedded as a curve in this curved surface and given by

$$y + z = v^2 B(v^2) + \Lambda_{N_c=2}^{2N_c+2-N_f} \prod_{i=1}^{N_f} m_i. \tag{3.4}$$

which reproduces to eq. (3.2) as we identify $y$ with $t$. From the symmetries existent in the type II A brane configuration, not all of them are preserved in the M-theory configuration. Our type IIA brane configuration has $U(1)_{4,5}$ and $SU(2)_{7,8,9}$ symmetries interpreted as classical $U(1)$ and $SU(2)$ R-symmetry groups of the 4 dimensional theory on the brane worldvolume. The classical brane configuration is invariant both under the rotations. One of them, only $SU(2)_{7,8,9}$ is preserved in M theory quantum mechanical configuration but $U(1)_{4,5}$ is broken. This is the same as saying that the $U(1)_R$ symmetry of the $N = 2$ supersymmetric field theory is anomalous being broken by instantons. As discussed in section 2, the instanton factor is proportional with $\Lambda_{N_c=2}^{2N_c+2-N_f}$. So we have to see what is the charge of this factor under $U(1)_{4,5}$. We see this from equations (3.3) and (3.4) by considering $v$ of charge 2. We list below the charges of coordinates and parameter in the table:

| $z$ | $y$ | $v$ | $\Lambda_{N_c=2}^{2N_c+2-N_f}$ |
|-----|-----|-----|--------------------------|
| $4N_c + 4$ | $4N_c + 4$ | $4$ | $4N_c + 4 - 2N_f$ |

In this case, the full $U(1)_{4,5}$ symmetry is restored, by assigning the instanton charge $(4N_c + 4 - 2N_f)$ to the $\Lambda$ factor.

Note that whenever some $m_i$ are the same, the smooth complex surface (3.3) develops A-type singularity. But this is misleading since the hyper-Kähler structure becomes singular only if the D6 branes have the same position in $x^6$ and not only in $v$. When D6 branes with the coincident $m_i$’s are separated in the $x^6$ direction, the singular surface (3.3) must be replaced by a smooth one which is the resolution of A-type singularity. We will briefly describe the resolution of the A-type singularity. On the resolved surface, we also describe the parity due to orientifolding.

### 3.2 Resolution of the A-type Singularity

When all bare masses are the same but not zero (say $m = m_i$), the surface (3.3) $S$ develops singularities of type $A_{n-1}$ at two points $y = z = 0, v = \pm m$. By succession of
blowing ups, we obtain a smooth complex surface $\tilde{S}$ which isomorphically maps onto the singular surface $S$ except at the inverse image of the singular points. Over each singular point, there exist $n-1$ rational curves $\mathbb{CP}^1$'s on the smooth surface $\tilde{S}$. These rational curves are called the exceptional curves. Let us denote the exceptional curves over the point $y = z = 0, v = m$ by $C_1, C_2, \ldots, C_{n-1}$ and those over the point $y = z = 0, v = -m$ by $C'_1, C'_2, \ldots, C'_{n-1}$. Here $C_i$'s (resp. $C'_i$) are arranged so that $C_i$ (resp. $C'_i$) intersects $C_j$ (resp. $C'_j$) only if $i = j \pm 1$. The symmetry due to orientifolding yields the correspondence between $C_i$ and $C'_i$.

When we turn off the bare mass, that is, $m_i = 0$ for all $i$, the singularity gets worse. Two singular points on the surface $S$ collides to create the $A_{2n-1}$ singularity. Now there are $2n-1$ exceptional curves on the resolved surface, which may be considered as a union of two previous exceptional curves $C_i$ and $C'_i$ and a new rational curve, say $E$ which connects these two exceptional curves. The orientifold provides a reflection between $C_i$ and $C'_i$ while inducing a self-automorphism on $E$. The more precise picture of the resolved surface is as follows: It is covered by $2N_f$ complex planes $U_1, U_2, \ldots, U_{2n}$ with coordinates $(y_1 = y, z_1), (y_2, z_2), \ldots, (y_n, x_n = y)$ which are mapped to the singular surface $S$ by

\[
U_i \ni (y_i, z_i) \mapsto \begin{cases} 
  y = y_i^iz_i^{i-1} \\
  z = y_i^{2N_f-i}z_i^{2N_f+1-i} \\
  v = y_iz_i
\end{cases} \quad (3.5)
\]

The planes $U_i$ are glued together by $z_iy_{i+1} = 1$ and $y_iz_i = y_{i+1}z_{i+1}$. The exceptional curve $C_i$ is defined by the locus of $y_i = 0$ in $U_i$ and $z_{i+1} = 0$ in $U_{i+1}$, the exceptional curve $E$ by $y_n = 0$ in $U_n$ and $z_{n+1} = 0$ in $U_{n+1}$ and the exceptional curve $C'_i$ by $y_{2n-i} = 0$ in $U_{2n-i}$ and $z_{2n-i+1} = 0$ in $U_{2n-i+1}$. The separation of the D6 branes in the $x^6$ direction corresponds to the infinitesimal direction on the singular surface $S$. Hence the position of the D6-brane may be interpreted as the $2N_f$ intersection points of the exceptional curves.

### 3.3 The Higgs Branch

In this section, all the bare masses are turned off. In M theory, the transition to the Higgs branch occurs when the fivebrane intersects with the D6-branes, which means that the curve $\Sigma$ given by $(3.4)$ passes through the singular point $y = z = v = 0$. As a special case, we will consider when all D4 branes are broken on D6 branes in type IIA picture. Write the right hand side of (3.4) as:

\[
v^2B(v^2) = v^2(u_2v^{2N_c-2} + \cdots + u_{2N_c}). \quad (3.6)
\]

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Then our case corresponds to $u_k = 0$ for all $k$. To describe the Higgs branch, we will study how the curve

$$y + z = v^{2N_c}$$

looks like in the resolved $A_{2N_f-1}$ surface. Here we ignored the factor $v^2$ in the right hand side of (3.6) because it is always contained in the orientifold plane $O4$ and thus does not contribute to the Higgs branch. Away from the singular point $y = z = v = 0$, we may regard the curve as embedded in the original $y - z - v$ space because there is no change in the resolved surface in this region. Near the singular point $y = z = v = 0$, we have to consider the resolved surface. On the $i$-th patch $U_i$ of the resolved surface, the equation of the curve $\Sigma$ becomes

$$y_i^j z_i^{j-1} + y_i^{2N_f-i} z_i^{2N_f+1-i} = y_i^{2N_c} z_i^{2N_c}$$

Now we may factorize this equation according to the range of $i$: For $i = 1, \ldots, 2N_c$, we have

$$y_i^j z_i^{j-1}(1 + y_i^{2N_f-2i} z_i^{2N_f+2-2i} - y_i^{2N_c-i} z_i^{2N_c+1-i}) = 0,$$

for $i = 2N_c + 1, \ldots, 2N_f - 2N_c$,

$$y_i^{2N_c} z_i^{2N_c} (y_i^{i-2N_c} z_i^{i-2N_c-1} + y_i^{2N_f-i-2N_c} z_i^{2N_f-i-2N_c+1} - 1) = 0,$$

and for $i = 2N_f - 2N_c + 1, \ldots, 2N_f$,

$$y_i^{2N_f-i} z_i^{2N_f+1-i} (y_i^{2i-2N_f} z_i^{2i-2-2N_f} + 1 - y_i^{2N_c-2N_f+i} z_i^{2N_c-2N_f+i-1}) = 0.$$  

Thus the curve consists of several components. One component, which we call $C$, is the zero of the last factor of the above equations. This extends to the one in the region away from $y = z = v = 0$ which we have already considered. The other components are the rational curves $C_1, \ldots, C_{n-1}, E, C'_1, \ldots, C'_{n-1}$ with some multiplicities. For convenience, we rename the exceptional curves $E_1, \ldots, E_{2n-1}$ so that $E_i$ is defined by $y_i = 0$ on $U_i$ and $z_{i+1} = 0$ on $U_{i+1}$. Hence we can see from the above factorization that the component $E_i$ has multiplicity $l_i = i$ for $i = 1, \ldots, 2N_c$; $l_i = 2N_c$ for $i = 2N_c + 1, \ldots, 2N_f - 2N_c$; and $l_i = 2N_f - i$ for $i = 2N_f - 2N_c + 1, \ldots, 2N_f - 1$. Note that the component $C$ intersects with $E_{2N_c}$ and $E_{2N_f-2N_c}$.

To count the dimension of the Higgs branch, recall that once the curve degenerates and $\mathbb{CP}^1$ components are generated, they can move in the $x^7, x^8, x^9$ directions [17]. This motion together with the integration of the chiral two-forms on such $\mathbb{CP}^1$'s parameterizes the Higgs branch of the four-dimensional theory. However, we have to omit the component $E_{N_f} = E$ because this corresponds to the D4 brane connecting a D6 brane and its mirror. (Recall that $E$ was created after collision of two singular points.
which were mirror to each other.) Such a D4 brane is eliminated by the antisymmetric orientifold projection. Hence we have to put $l_{N_f} = 0$. Now, after consideration of $\mathbb{Z}_2$ symmetry, the quaternionic dimension of the Higgs branch is

$$\frac{1}{2} \sum_{i=1}^{2N_f-1} l_i = \sum_{i=1}^{2N_c} i + (2N_f - 4N_c)N_c = 2N_c(N_f - N_c) - N_c,$$  

(3.12)

which is the half of the complex dimension given in (3.1). Perhaps, a more appropriate geometric setting would have been a double covering of a $A_{N_f-1}$ singular surface with the embedded Seiberg-Witten curve. We will leave this for future investigation.

4 The Rotated Configuration

As we have seen that $N = 2$ supersymmetry can be broken to $N = 1$ by inserting a mass term of the adjoint chiral multiplet in field theory approach, we analyze the corresponding configuration in M theory fivebranes. While in the context of IIA picture, this turns out to be the rotation of one of NS5 branes, in order to describe this configuration, let us introduce a complex coordinate

$$w = x^8 + ix^9.$$  

(4.1)

Of course, the fivebranes are positioned at $w = 0$ before the rotation. Notice that the D4 brane corresponds classically to an M fivebrane at $v = w = 0$ and is extended in the direction of $s$, the NS5 brane is at $s = w = 0$ and extended in $v$ and NS’5 brane is at $v = 0$, $s = s_0$ and extended in $w$. Now we rotate only the left NS5 branes and from the behavior of two asymptotic regions which correspond to the two NS5 branes with $v \to \infty$ this rotation leads to the following boundary conditions.

$$w \to \mu v \quad \text{as} \quad v \to \infty, \quad t \sim v^{2N_c+2}$$

$$w \to 0 \quad \text{as} \quad v \to \infty, \quad t \sim \Lambda_{N=2}^{2(2N_c+2-N_f)} v^{2N_f-2N_c-2}$$  

(4.2)

where the left(right) NS5 brane is related to the first(second) asymptotic boundary condition. Far from the origin of the $(v, w)$ plane which is the location of D4 branes, the location of the NS5 brane in the $s$ plane can be described by $s(NS5) = -2R(N_f - N_c - 1)\ln v$ while the NS’5 brane in the $s$ plane by $s(NS'5) = -2R(N_c + 1)\ln w$. We discuss about the R-symmetries of the rotated configuration. After rotation, $SU(2)_{7,8,9}$ is broken to $U(1)_{8,9}$. In order this to be true, because of the connection between $v$ and $w$ in (4.2), $\mu$ has to have charges under $U(1)_{4,5} \times U(1)_{8,9}$. $v$ has charge 2 under $U(1)_{4,5}$ while 0 under $U(1)_{8,9}$. $w$ has charge 0 under $U(1)_{4,5}$ and 2 under $U(1)_{8,9}$. So $\mu$ has $(-2, 2)$ charges under $U(1)_{4,5} \times U(1)_{8,9}$. From the equations (3.3), (3.4) and (4.2) we find the following values for the R-symmetry charges:
Since the rotation is only possible at points in moduli space at which all 1-cycles on the curve $\Sigma$ are degenerate [18], the curve $\Sigma$ is rational, which means that the functions $v$ and $t$ can be expressed as a rational functions of $w$ after we identify $\Sigma$ with a complex plane $w$ with some deleted points. Because of the symmetry of $v \rightarrow -v$, $w \rightarrow -w$ due to orientifolding, we can write:

$$v^2 = P(w^2), \quad t = Q(w^2). \quad (4.3)$$

Since $v$ and $t$ become infinity only if $w = 0, \infty$, these rational functions are polynomials of $w$ up to a factor of some power of $w$: $P(w^2) = w^{2a} P(w^2), Q(w^2) = w^{2b} Q(w^2)$ where $a$ and $b$ are some integers and $p(w^2)$ and $q(w^2)$ are some polynomials of $w$ with only even degree terms which we may assume non-vanishing at $w = 0$. Near one of the points at $w = \infty$, $v$ and $t$ behave as $v \sim \mu^{-1} w$ and $t \sim t^{2N_c+2}$ by (4.2). Thus the rational functions are of the form

$$P(w^2) = w^{2a} (w^{2-2a} + \cdots)/\mu^2 \quad \text{and} \quad Q(w^2) = \mu^{-2N_c-2} w^{2b} (w^{2N_c+2-2b} + \cdots). \quad (4.4)$$

Around a neighborhood $w = 0$, the Riemann surface $\Sigma$ can be parameterized by $1/v$ which goes to zero as $w \rightarrow 0$. Since $w$ and $1/v$ are two coordinates around the neighborhood $w = 0$ in the compactification of $\Sigma$ and vanish at the same point, they must be linearly related $w \sim 1/v$ in the limit $w \rightarrow 0$. The function $P(w^2)$ then takes the form

$$P(w^2) = \frac{(w^2 - w_+)(w^2 - w_-)}{\mu^2 w^2}. \quad (4.5)$$

However the equation $v^2 = P(w^2)$ implies that $P(w^2)$ must be a square. Hence we have $w_+ = w_-$ and by letting $w_0^2 = w_+$

$$P(w^2) = \frac{(w^2 - w_0^2)^2}{\mu^2 w^2} \quad (4.6)$$

which is a square of $w^2 - w_0^2/\mu w$. Since $t \sim v^{2N_f-2N_c-2}$ and $w \sim 1/v$ as $w \rightarrow 0$, we get $b = N_c + 1 - N_f$ and thus,

$$Q(w^2) = \mu^{-2N_c-2} w^{2(N_c+1-N_f)} (w^{2N_f} + \cdots). \quad (4.7)$$

For $N_f > 0$, by the equation $yz = v^{2N_f}$ defining the space-time, $t = 0$ (i.e. $y = 0$) implies $v = 0$. Therefore the zeros of the polynomial $w^{2N_f} + \cdots$ are $\pm w_0$ of $P(w^2)$. Hence we have

$$Q(w^2) = \mu^{-2N_c-2} w^{2(N_c+1-N_f)} (w^2 - w_0^2)^{N_f} \quad (4.8)$$

|   | $v$ | $w$ | $y = t$ | $z$ | $\mu$ | $\Lambda_{N=2}^{2N_c+2-N_f}$ |
|---|-----|-----|---------|-----|-------|--------------------------------|
| $U(1)_{4,5}$ | 2 | 0 | $4(N_c + 1)$ | $4(N_c + 1)$ | -2 | $4(N_c + 1) - 2N_f$ |
| $U(1)_{8,9}$ | 0 | 2 | 0 | 0 | 2 | 0 |
The value of \( w_0 \) can be determined by the fact that \( v^2 \) and \( t \) satisfy the relation

\[
t + \Lambda_{N=2}^{4N_c+4-2N_f} v^{2N_f} / t = v^2 B(v^2) + \Lambda_{N=2}^{2N_c+2-N_f} \prod_{i=1}^{N_f} m_i \tag{4.9}
\]

Then by plugging \( v^2 \) and \( t \) into the above equation we can read off \( w_0 \) from the lowest order term in power of \( w \)

\[
\mu^{-2N_c-2} w^{2(N_c+1-N_f)} (w^2 - w_0^2)^{N_f} + \Lambda_{N=2}^{4N_c+4-2N_f} v^{2N_f} \mu^{2N_c+2-N_f} w^{-2(N_c+1-N_f)} (w^2 - w_0^2)^{-N_f} = \frac{\left( w^2 - w_0^2 \right)^2}{\mu^2 w^2} B(v^2) - \Lambda_{N=2}^{2N_c+2-N_f} \prod_{i=1}^{N_f} m_i \tag{4.10}
\]

We want to calculate now \( w_0 \) from the above equation. For this, we will match the lowest order term in powers of \( w \) in this equation. Actually we will look for terms with \( w^0 \) i.e., constant terms. In the left hand side the first term will always have a power of \( w \), so does not contribute to the lowest order term. In the right hand side the last term will contain at least \( w^{2(N_c-1)} \) so again does not contribute. After using the expression for \( v \), the second term in left hand side will be

\[
\Lambda_{N=2}^{4N_c+4-2N_f} (w^2 - w_0^2)^{N_f} \mu^{2N_c+2-N_f} w^{-2N_c-2}
\]

In the right hand side, from the explicit form of \( B(v^2) \), the only term that can be independent of \( w \) is obtained when we take only the highest power of \( v \) which then will give \( v^{2N_c} \). The contribution of this in the right hand side is as follows:

\[
\frac{\left( w^2 - w_0^2 \right)^2}{\mu^{2N_c+2}} w^{-2N_c-2}
\]

We now extract the lowest order from (4.11) and (4.12) and make them equal to obtain the relation for \( w_0 \) finally:

\[
(-1)^{N_f} \Lambda_{N=2}^{4N_c+4-2N_f} w_0^{2N_f} \mu^{4N_c+4-2N_f} = w_0^{4N_c+4}
\]

This gives us the value for \( w \) to be

\[
w_0 = (-1)^{4N_c+4-2N_f} (\mu \Lambda_{N=2}).
\]

This gives us \( w_0 \) up to a \( \mathbb{Z}_{4N_c+4-2N_f} \) rotation. We have here \( 4N_c + 4 - 2N_f \) instead of \( 2N_c + 2 - N_f \) because of the symmetry \( w \leftrightarrow -w \) implied by the orientifold. The rotated curve is now completely determined.
5 \ N = 1 \ SQCD

We study the \( \mu \to \infty \) limit of our M fivebrane configuration and compare it with the known results in \( N = 1 \) supersymmetric gauge theory. We have considered the rotation of the left NS5-brane, which corresponds to the asymptotic region \( t \sim v^{2N_c+2} \) before the rotation and to \( w \to \infty, v \sim \mu^{-1}w \) and \( t \sim \mu^{-2N_c-2}w^{2N_c+2} \) after the rotation. We expect the relation \( t \sim v^{2N_c+2} \) to hold also in the \( \mu \to \infty \) because the D4 branes still end on the left NS5-brane in this limit.

In order to preserve this relation, we should rescale \( t \) by a factor \( \mu^{2N_c+2} \) and introduce a new variable

\[
\tilde{t} = \mu^{2N_c+2}t \tag{5.1}
\]

which will have the same dependence on \( v \) as \( t \) had before the rotation and this corresponds to the shift of the origin in the direction of \((x^6, x^{10})\). After putting \( y = \tilde{t} \) in (3.3), the space-time is described by

\[
\tilde{y}z = \mu^{2N_c+2} \Lambda_{N_c+4}^{4N_c+4-2N_f} \prod_{i=1}^{N_f} (v^2 - m_i^2), \tag{5.2}
\]

where \( \tilde{y} = \mu^{2N_c+2}y \). This equation describes a smooth surface in the limit \( \mu \to \infty \) provided the product

\[
\mu^{2N_c+2} \Lambda_{N_c+4}^{4N_c+4-2N_f} \tag{5.3}
\]

remains to be finite. We define this product as follows:

\[
\Lambda_{N_c=1}^{2(3N_c+3-N_f)} = \mu^{2N_c+2} \Lambda_{N_c=2}^{4N_c+4-2N_f} \tag{5.4}
\]

which is nothing but the RG matching condition of the four-dimensional field theory. Note that this spacetime and M fivebrane is under the rotation groups \( U(1)_{4,5} \) and \( U(1)_{8,9} \) in appropriate way discussed before. We want to see what are the deformations of the \( N = 2 \) Coulomb branch after the rotation and the limit \( \mu \to \infty \).

5.1 Pure \( Sp(N_c) \) Theory

Without matter, the curve (3.2) describing the \( N = 2 \) Coulomb branch is given by:

\[
t^2 - C_{N_c+1}(v^2, u_k) t + \Lambda_{N_c=2}^{4N_c+4} = 0 \tag{5.5}
\]

where \( C_{N_c+1}(v^2, u_k) = v^2 B(v^2, u_k) \). This curve is completely degenerate at \((N_c + 1)\) points on the Coulomb branch. At one these points, the curve has the following form

\[
v^2 = \Lambda_{N_c=2}^{4N_c+4} t^{-1/(N_c+1)} + t^{1/(N_c+1)}. \tag{5.6}
\]
Thus its rotation is

\[ v^2 = \mu^2 \Lambda_{N=2}^4 w^{-2} + \mu^{-2} w^2 \]  
\[ t = \mu^{-2N_c^2} w^{2N_c^2}. \]  

(5.7)

Now we rescale \( t \) as \( \tilde{t} = \mu^{2N_c^2} t \) and send \( \mu \to \infty \) by keeping \( \Lambda_{N=1} \) finite. It is easy to see that the curve becomes in this limit:

\[ v^2 = \Lambda_{N=1}^6 \tilde{t}^{-1/(N_c+1)} \]  
\[ w^2 = \tilde{t}^{1/(N_c+1)}. \]  

(5.8)

where the RG matching condition is used.

5.2 Introducing Massless Matter

For the rotated configuration we use now the expressions for \( v^2 = P(w^2) \) and \( t = Q(w^2) \) given before in terms of new variables. We again introduce the rescaled \( \tilde{t} \) which is then given by

\[ \tilde{t} = w^{2(N_c+1-N_f)}(w^2 - w_0^2)^{N_f}. \]  

(5.9)

For \( v \) and \( w \) we have the relation by remembering that the order parameters \( u_k \) are independent of \( \mu \), are powers of \( \Lambda_{N=2} \) and vanish in the \( \mu \to \infty \)

\[ \tilde{t} = \Lambda_{N=1}^{6N_c+6-2N_f} v^{2(N_f-N_c-1)}. \]  

(5.10)

When \( \mu \to \infty \), the limit for (5.9) and (4.6) is given by the behavior of \( w_0 \sim \mu \Lambda_{N=2} \). By using the relation:

\[ \mu \Lambda_{N=2} = (\Lambda_{N=1}^{3N_c+3-N_f} \mu^{N_c+1-N_f})^{1/(2N_c+2-N_f)}, \]  

(5.11)

we have three regions for \( N_f \). Let us see how the curves look like:

- \( N_f < N_c + 1 \)

(5.11) tells us that \( \mu \Lambda_{N=2} \) diverges and \( w_0 \) also diverges. Therefore the curve becomes infinite in the \( x^6 \) direction. So there is no field theory in four dimensions. This is just the same as saying that there is no supersymmetric vacua in the \( N = 1 \) theory.

- \( N_c + 1 < N_f < 2(N_c + 1) \)

In this case, from (5.11) it is easy too see that \( \mu \Lambda_{N=2} = 0 \) in the limit \( \mu \to \infty \) and (5.9), (4.6) and (5.10) transform into:

\[ \tilde{t} = w^{2(N_c+1)} \]  
\[ v w = 0 \]  
\[ v^{2(N_c+1)} \tilde{t} = \Lambda^{6(N_c+1)-2N_f} w^{2N_f}. \]  

(5.12)
As explained in [41], only the limit $\tilde{t} \neq 0, w = 0$ is allowed, so the interpretation of the previous equation is that the curve splits into two components in this limit: $C_L(\tilde{t} = w^{2(N_c+1)}, v = 0)$ and $C_R(\tilde{t} = \Lambda_{N=1}^{6N_c+6-2N_f} v^{2N_f-2N_c-2}, w = 0)$ where the component $C_L$ corresponds to the NS'5 brane which was rotated and on the other hand, $C_R$ refers to the NS5 brane and the attached D4-branes.

- $N_f = N_c + 1$

In this case, the RG matching condition tells that $\mu \Lambda_{N=2}$ is equal to $\Lambda_{N=1}^2$. The equations (5.10), (5.9) and (1.6) become:

\begin{align}
\tilde{t} &= \Lambda_{N=1}^{4(N_c+1)} \\
\tilde{t} &= (w^2 - w_0^2)^{N_c+1} \\
vw &= 0.
\end{align}

The correct interpretation of these equations is that the curve also splits into two components: $C_L(\tilde{t} = (w^2 - w_0^2)^{N_c+1}, v = 0)$ and $C_R(\tilde{t} = \Lambda_{N=1}^{4(N_c+1)}, w = 0)$.

For $SU(N_c)$ group the cases $N_f = N_c + 1$ and $N_f > N_c + 1$ differed from each other because the first different non-baryonic branch roots went to different limits and the second all non-baryonic branch roots have the same limit. That was determined by the fact that $M$, the meson matrix, had 2 different values for the diagonal entries. However in our case, for the $Sp(N_c)$ group, there is only one kind of top-right diagonal entry, so we do not see those difference appeared in $SU(N_c)$ gauge group and also do not have any baryonic branch.

### 5.3 Massive Matter

Before starting our discussion of introducing matter for our case, let us briefly examine the difference between the results of [41] and [30, 38] for the case of massive matter when we consider $SU(N_c)$ gauge group. Actually we will just compare the equation (5.28) of [41], and (4.6) and (5.1) of [38]. Notice that the second equation in (4.6) of [38] and the first one in (5.28) of [41] are the same:

$$v w = (m_f \Lambda_{N=1}^{3N_c-N_f})^{1/N_c}.$$  \hspace{1cm} (5.14)

The first equation of (4.6) in [38] is not the same as the second one of (5.28) in [41]. Rather the equation of (5.28) looks like the equation (5.1) of [38] because there we have the dependence $t - w$. Recall that the vev for $M$ are given for equal squark masses by:

$$m = m_f \Lambda_{N=1}^{3N_c-N_f}/\Lambda_{N=1}^{3N_c-N_f}.$$  \hspace{1cm} (5.15)
The relation between $\tilde{t}$ and $w$ can be rewritten as:

$$\tilde{t}w^{N_f-N_c} = (w - m)^{N_f}. \tag{5.16}$$

When we write it in terms of $t$ and $v$, this turns out:

$$\tilde{t}v^{N_c} = (-1)^{N_f} \Lambda_{N=1}^{3N_c-N_f}(v - m_f)^{N_f} \tag{5.17}$$

The relations (5.16) and (5.17) are just the equivalent of (4.6) and (5.1) in [38]. In (5.17) we have a supplementary power of $\Lambda$ as compared with [38]. This is due to the fact that in [41], $t$ has a dimension of mass by its definition, but in [38] it is dimensionless. The power of $\Lambda$ is just used to match the dimension of mass. We then find that the same curve can be written in terms of $t - w$ and $t - v$. The two descriptions correspond to the $N = 1$ duality [7], between theories with gauge groups $SU(N_c)$ and $SU(N_f - N_c)$, as we can see from the dependence of $t$ as a function of $v$ and $w$ in (5.16) and (5.17). So the electric-magnetic duality can be observed also from the set-up of [41]. The connection between [41] and [38] may become clear only after we should introduce D6 branes in the set-up of [38] which is a very interesting direction to pursue and investigate.

What happens when we consider our case? If all the quarks have equal mass $m_f$, then the curve for $\mu \to \infty$ becomes:

$$v^2 w^2 = \left( m_f^{N_f} \Lambda_{N=1}^{3(N_c+1)-N_f}\right)^{2/(N_c+1)} \tag{5.18}$$

$$\tilde{t} = w^{2(N_c+1-N_f)} \left( w^2 - \left( \frac{2^{2N_c-N_f}}{m_f^{N_c+1-N_f}} \Lambda_{N=1}^{3(N_c+1)-N_f}\right)^{2/(N_c+1)} \right)^{N_f}.$$

By the relation which connects the vev of $M$ and the masses of quarks, we obtain:

$$\tilde{t}w^{2(N_f-N_c-1)} = (w^2 - m^2)^{N_f} \tag{5.19}$$

When we write it in terms of $t$ and $v$, this gives rise to:

$$\tilde{t}v^{2N_c+2} = (-1)^{N_f} \Lambda_{N=1}^{6(N_c+1)-2N_f}(v^2 - 2^{2(N_c-N_f)}m_f^2)^{N_f} \tag{5.20}.$$
Now take the limit \( m_f \to \infty \). After we integrate out the massive flavors and use the matching of the running coupling constant
\[
\tilde{\Lambda}_{N=1}^{3(N_c+1)} = m_f^{N_f} \tilde{\Lambda}_{N=1}^{3(N_c+1)-N_f}
\]  
(5.21)
to rewrite the equation (5.18) as
\[
v^2 w^2 = \tilde{\Lambda}_{N=1}^6
\]  
(5.22)
\[
\tilde{t} = w^{2(N_c+1-N_f)} \left( w^2 - 2^{2(N_c-N_f)} \frac{\tilde{\Lambda}_{N=1}^6}{m_f^2} \right)^{N_f}.
\]

If we keep \( \tilde{\Lambda}_{N=1} \) finite while sending \( m_f \) to \( \infty \), this again reduces to the pure Yang-Mills result.

### 6 Conclusions

In the present work we considered the M theory description of the supersymmetry breaking from \( N = 2 \) to \( N = 1 \) for the case of symplectic gauge group \( Sp(N_c) \) obtaining many aspects of the strong coupling phenomena. In this case no baryon can be constructed so the only Higgs branch is the non-baryonic branch. In field theory approach, starting with a \( N = 2 \) supersymmetric gauge theory and giving mass to the adjoint chiral multiplet, the extremum of the superpotential gave us a unique solution for the expectation value for the meson matrix \( M \) in which \( \text{Tr} M = 0 \) as opposed to the \( SU(N_c) \) case where \( M \neq 0 \).

In the M theory fivebrane approach, we discussed first the unrotated configuration which corresponds to an \( N = 2 \) theory, with or without D6 branes. For the case without D6 branes, M theory fivebrane configuration is a single fivebrane with the world volume \( \mathbb{R}^{1,3} \times \Sigma \) where \( \Sigma \) is the Seiberg-Witten curve of the gauge group \( Sp(N_c) \). By introducing D6 branes, we have considered the complex structure of the corresponding Taub-NUT space which is the same one as those of the ALE space of \( A_{2n-1} \)-type and resolved the \( A_{2n-1} \) singularity.

One of the most important aspects of the \( Sp(N_c) \) gauge theory was the O4 orientifold which is parallel to D4 branes. Its antisymmetric projection which eliminates some degrees of freedom was essential in matching the dimension of the Higgs moduli space in IIA brane approach with the one of the field theory. This observation was used not only in type IIA picture when counting the number of D4 branes suspended between the D6 branes but also in the M theory picture when counting the multiplicities of the rational curves. It is known \[25, 30\] that A type singularity by imposing the \( \mathbb{Z}_2 \) symmetry,
due to the orientifolding, leads to D type singularity. We expect that our M theory fivebrane argument starting from D type singularity can go similarly and will see how the interrelation between two types singularities plays a role.

For rotated branes we used the coordinates $v = x^4 + ix^5$ and $w = x^8 + ix^9$, the position of the D4 branes in the $w$ direction being identified with the eigenvalues of the meson matrix $M$. We found at most one eigenvalue $w_0$ for the asymptotic position of the D4 branes which is consistent with the field theory result where only one eigenvalue for $M$ was found. We connected $w_0$ with the unique eigenvalue of $M$. In section 5 we have obtained the forms for the rotated curves, for pure gauge group and for massive and massless matter.

In all of our discussions as well as in many exciting works which appeared recently, many results obtained in field theory were rederived in M theory which makes M theory approach an extraordinary laboratory to derive results which were very difficult to obtain only by pure field theory methods. Now we have a clearer view over the strongly coupled phenomena of supersymmetric theories. But we still need to obtain new information like how to introduce flavors in the spinor representation and how to obtain dualities for $N = 1$ theories with gauge groups like as $SO \times SO$ or $SO \times SU$, for which M theory approach did not give yet any supplementary information compared with type IIA approach. We hope that this information will be reached in the near future.

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