MULTIPLICITIES IN THE TRACE COCHARACTER SEQUENCE
OF TWO 4 × 4 MATRICES

VESSELIN DRENSKY AND GEORGI K. GENOV

Abstract. We find explicitly the generating functions of the multiplicities in
the pure and mixed trace cocharacter sequences of two 4×4 matrices over a field
of characteristic 0. We determine the asymptotic behavior of the multiplicities
and show that they behave as polynomials of 14th degree.

Introduction

Let us fix an arbitrary field \( F \) of characteristic 0 and two integers
\( n, d \geq 2 \). Consider the \( d \) generic \( n \times n \) matrices \( X_1, \ldots, X_d \). We denote by \( C \) the pure trace
algebra generated by the traces of all products \( \text{tr}(X_{i_1} \cdots X_{i_k}) \), and by \( T \) the mixed
trace algebra generated by \( X_1, \ldots, X_d \) and \( C \), regarding the elements of \( C \) as scalar
matrices. The algebra \( C \) coincides with the algebra of invariants of the general
linear group \( GL_n(F) \) acting by simultaneous conjugation on \( d \) matrices of size \( n \).
The algebra \( T \) is the algebra of matrix concominants under a suitable action of
\( GL_n(F) \). See e.g. the books \[12\], \[11\], or \[5\] as a background on
\( C \) and \( T \) and their
numerous applications.

The algebras \( C \) and \( T \) are graded by multidegree. The Hilbert (or Poincaré)
series of \( C \) is
\[
H(C) = H(C, t_1, \ldots, t_d) = \sum \dim C^{(k)} t_1^{k_1} \cdots t_d^{k_d},
\]
where \( C^{(k)} \) is the homogeneous component of multidegree \( k = (k_1, \ldots, k_d) \). In the
same way one defines the Hilbert series of \( T \). These series are symmetric functions
and decompose as infinite linear combinations of Schur functions \( S_{\lambda}(t_1, \ldots, t_d) \),
\[
H(C) = \sum m_{\lambda}(C) S_{\lambda}(t_1, \ldots, t_d),
\]
where \( \lambda = (\lambda_1, \ldots, \lambda_d) \) is a partition in not more than \( d \) parts, and the \( m_{\lambda}(C) \)s are
nonnegative integers which are 0 if \( d > n^2 \) and \( \lambda_{n^2+1} > 0 \). A similar expression
holds for the Hilbert series of \( T \). The multiplicities \( m_{\lambda}(C) \) and \( m_{\lambda}(T) \) have important
combinatorial properties and ring theoretical meanings. In particular, they are
equal to the multiplicities of the irreducible \( S_\lambda \)-characters in the sequences of pure
and mixed trace cocharacters, respectively, and give estimates for the multiplicities
in the “ordinary” cocharacters of the polynomial identities of the \( n \times n \) matrix
algebra.

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Research.
The multiplicities of \( m_\lambda(C) \) and \( m_\lambda(T) \) are explicitly known in very few cases. For \( n = 2 \) and any \( d \), Formanek \( [9] \) showed that

\[
m_\lambda(T) = (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1),
\]

if \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \), and \( m_\lambda(T) = 0 \) if \( \lambda_5 \neq 0 \). He also found a more complicated expression for the multiplicities in the Hilbert series of \( C \). With some additional work, based on representation theory of symmetric and general linear groups, these results can be also derived from the paper by Procesi \( [14] \). The Hilbert series of \( C \) and \( T \) can be expressed as multiple integrals, evaluated by Teranishi \( [15, 16] \) for \( C \) and \( n = 3, 4 \) and \( d = 2 \). Van den Bergh \( [17] \) suggested graph theoretical methods for calculation of \( H(C) \) and \( H(T) \). Berele and Stembridge \( [3] \) calculated the Hilbert series of \( C \) and \( T \) for \( n = 3 \), \( d \leq 3 \) and for \( n = 4 \), \( d = 2 \), correcting also some typographical errors in the expression of \( H(C) \) for \( n = 4 \) and \( d = 2 \) in \( [16] \). Using the Hilbert series of \( C \), \( n = 3 \), \( d = 2 \), Berele \( [1] \) found an asymptotic expression of \( m_{(\lambda_1, \lambda_2)}(C) \). The explicit form of the generating function of \( m_{(\lambda_1, \lambda_2)}(C) \) was found by the authors of the present paper \( [6] \) correcting also a technical error in \( [1] \). Later they \( [7] \) suggested a method to find the coefficients of the Schur functions in the expansion of a class of rational symmetric functions in two variables. Jointly with Valenti \( [8] \) they have used the expression of the Hilbert series \( H(T) \) found by Berele and Stembridge \( [3] \) and have calculated explicitly the multiplicities \( m_{(\lambda_1, \lambda_2)}(T) \) for \( n = 3 \) and \( d = 2 \).

The purpose of the present paper is to calculate, for \( n = 4 \) and \( d = 2 \), the generating functions of \( m_{(\lambda_1, \lambda_2)}(C) \) and \( m_{(\lambda_1, \lambda_2)}(T) \). In principle, this allows to give explicit expressions for the multiplicities. Since the formulas are quite complicated, we prefer to give the asymptotics only, as in \( [1] \). It turns out that the multiplicities \( m_{(\lambda_1, \lambda_2)}(C) \) and \( m_{(\lambda_1, \lambda_2)}(T) \) behave as polynomials of degree 14 in \( \lambda_1 \) and \( \lambda_2 \). As in \( [3] \), our approach is to apply the methods of \( [7] \) to the explicit form of the Hilbert series of \( C \) and \( T \) found in \( [3] \). As in \( [8] \), results of Formanek \( [9, 10] \) imply that the values of the multiplicities \( m_\lambda(M_4(F)) \) for \( \lambda = (\lambda_1, \ldots, \lambda_{16}) \) coincide with these of \( m_{(\lambda_1 - \lambda_1, \lambda_2 - \lambda_1)}(T) \) when \( \lambda_3 = \cdots = \lambda_{16} = 2 \).

Carbonara, Carini and Remmel \( [4] \) determined, for any \( n \), the behaviour of \( m_\lambda(C) \), when \( \lambda_2, \ldots, \lambda_n \) is fixed and \( \lambda_1 \) sufficiently large. Our results give more detail, for \( n = 4 \), in this direction. A general result of Berele \( [2] \) describes the multiplicities of Schur functions for the class of rational symmetric functions with denominators which are products of binomials \( 1 - t_1^{k_1} \cdots t_d^{k_d} \), in any number of variables. It covers the Hilbert series of relatively free algebras and pure and mixed trace algebras. Again, our results give some detail in the partial case which we consider in this paper.

1. Preliminaries

We refer to the book by Macdonald \( [13] \) for a background on theory of symmetric functions. We shortly summarize the facts we need for our exposition. We consider the algebra \( \text{Sym}[[x, y]] \) of formal power series which are symmetric functions in two variables over \( \mathbb{C} \). Every element \( f(x, y) \in \text{Sym}[[x, y]] \) can be expressed in a unique way as an infinite linear combination

\[
f(x, y) = \sum_\lambda m(\lambda)S_\lambda(x, y),
\]
where \( S_\lambda(x, y) \) is the Schur function related with the partition \( \lambda = (\lambda_1, \lambda_2) \), and \( m(\lambda) \in \mathbb{C} \) is the multiplicity of \( S_\lambda(x, y) \) in the decomposition of \( f(x, y) \). The Schur function \( S_\lambda(x, y) \) has the form

\[
S_\lambda(x, y) = (xy)^{\lambda_2} \left( x^p + x^{p-1}y + \cdots + xy^{p-1} + y^p \right) = \frac{(xy)^{\lambda_2} \left( x^{p+1} - y^{p+1} \right)}{x - y},
\]

where we have denoted \( p = \lambda_1 - \lambda_2 \). In \([7]\) we introduced the multiplicity series of \( f(x, y) \)

\[
M(f)(t, u) = \sum_{\lambda_1 \geq \lambda_2 \geq 0} m(\lambda_1, \lambda_2) t^{\lambda_1} u^{\lambda_2} \in \mathbb{C}[t, u].
\]

Introducing a new variable \( v = tu \), it accepts the more convenient form

\[
M'(f)(t, v) = M(f)(t, u) = \sum_{\lambda_1 \geq \lambda_2 \geq 0} m(\lambda_1, \lambda_2) t^{\lambda_1} v^{\lambda_2} \in \mathbb{C}[t, v].
\]

The relation between symmetric functions and their multiplicity series is given, see (11) in \([6]\), by

\[
f(x, y) = \frac{x M'(f)(x, y) - y M'(f)(y, xy)}{x - y}. \tag{1}
\]

The following theorem from \([7]\) describes the symmetric functions with rational multiplicity series and gives hints how to calculate the multiplicity series for such functions.

**Theorem 1.** The multiplicity series of \( f(x, y) \in \text{Sym}[[x, y]] \) is rational if and only if \( f(x, y) \) has the form

\[
f(x, y) = \frac{p(x, xy) + p(y, xy)}{q(x, xy)q(y, xy)},
\]

where \( p(x, z), q(x, z) \) are polynomials in \( x \) with coefficients which are rational functions in \( z \). Then

\[
M_1(f; t, v) = \frac{h(t, v)}{q(t, v)},
\]

where \( h(t, v) \in \mathbb{C}(v)[t] \) and \( \deg h \leq \max(\deg q, \deg g - 2) \).

The polynomial \( h(t, v) \) can be found from (1) by the method of unknown coefficients. The calculations can be simplified if we know the decomposition of \( q(x, z) \). We need the following easy lemma which is a slight generalization of Lemma 14 in \([6]\).

**Lemma 2.** Let \( K \) be any field, let \( \xi \) be an arbitrary element from \( K \) and let \( f(w), g(w) \in K[w] \) be two polynomials such that \( f(1/\xi), g(1/\xi) \neq 0 \). Then, in the decomposition as a sum of elementary fractions,

\[
\frac{f(w)}{(1 - \xi w)^k g(w)} = \frac{\alpha_k}{(1 - \xi w)^k} + \frac{\alpha_{k-1}}{(1 - \xi w)^{k-1}} + \cdots + \frac{\alpha_1}{1 - \xi w} + \frac{b(w)}{g(w)} + c(w), \tag{2}
\]

where \( \alpha_1, \ldots, \alpha_k \in K \) and \( b(w), c(w) \in K[w] \), the coefficient \( \alpha_k \) has the form

\[
\alpha_k = \frac{f(1/\xi)}{g(1/\xi)}.
\]
2. Main Results

In the sequel we fix $n = 4$ and $d = 2$ and denote, respectively, by $C$ and $T$ the pure and mixed trace algebras generated by two generic $4 \times 4$ matrices $X$ and $Y$. We replace the variables $t_1, t_2$ with $x, y$ and denote $e_2 = xy$. The Hilbert series of $C$ and $T$ found by Berele and Stembridge are

$$h_C = \frac{P_C(x, y)}{Q_C(x, y)}, \quad h_T = \frac{P_T(x, y)}{Q_T(x, y)}$$

Theorem 3. The multiplicity series $m_C$ of the Hilbert series $H(C, x, y)$ of the pure trace algebra of two generic $4 \times 4$ matrices is

$$m_C = \frac{\alpha_4}{(1-t)^4} + \frac{\alpha_3}{(1-t)^3} + \frac{\alpha_2}{(1-t)^2} + \frac{\alpha_1}{1-t} + \frac{\beta_2}{(1+t)^2} + \frac{\beta_1}{1+t} + \frac{\gamma_0}{1+t + t^2} + \frac{\delta_0}{1+t + t^2} + \frac{\delta_1}{1+t + t^2} + \frac{\epsilon_2}{1-vt} + \frac{\epsilon_1}{1-vt} + \frac{\varphi_0 + \varphi_1 t}{1-vt^2},$$

where

$$\alpha_4 = \frac{(1-v^2)(1-v^2)^2 + 4v^3 + 4v^4 - v^5 + v^6}{24(1-v)^2(1+v)^5(1+v+v^2)^2(1+v^2)};$$

$$\alpha_3 = \frac{(1+v+v^2)(1-v^2)^2}{24(1-v)+1(1+v)^5(1+v+v^2)^4(1+v^2)^2};$$

$$\alpha_3' = 3 - 4v - 2v^2 + 4v^3 - 3v^4 - 20v^5 - 12v^6 - 12v^7 + 7v^8 + 12v^9 + 4v^{10} - 4v^{11} + 5v^{12};$$

$$\alpha_2 = \frac{288(1-v)^2(1+v)^2(1+v+v^2)^2(1+v^2)}{288(1-v)^2(1+v)^2(1+v+v^2)^2(1+v^2)};$$

$$\alpha_2' = 59 - 97v - 26v^2 + 223v^3 + 675v^4 + 840v^5 + 2501v^6 + 4049v^7 + 6799v^8 + 7754v^9 + 6367v^{10} + 3473v^{11} + 2189v^{12} + 768v^{13} + 747v^{14} + 271v^{15} - 26v^{16} - 97v^{17} + 107v^{18};$$

$$\alpha_1 = \frac{144(1-v)^2(1+v)^2(1+v+v^2)^2(1+v^2)}{144(1-v)^2(1+v)^2(1+v+v^2)^2(1+v^2)^2};$$

$$\alpha_1' = 34 - 86v - 62v^2 + 106v^3 + 459v^4 - 624v^5 - 1887v^6 - 6630v^7 - 12804v^8 - 24712v^9 - 40531v^{10} - 57622v^{11} - 62642v^{12} - 57622v^{13} - 40531v^{14} - 24712v^{15} - 12804v^{16} - 6630v^{17} - 1887v^{18} - 624v^{19} + 459v^{20} + 106v^{21} - 62v^{22} - 86v^{23} + 34v^{24};$$

$$\beta_2 = \frac{1 + v^4}{32(1-v)^2(1+v)^2(1+v^2)^2}.
we know that we want to say a couple of words about how we have calculated \( \phi \). The coefficients \( \phi \) are obtained using both \( v \) and \( t \). Applying Theorem 1 \( C \) gives also that \( \deg \), we apply Lemma 2 for \( \phi \).

Proof. Direct calculations, which we have performed using Maple, show that, replacing the expression (4) of \( \phi \) instead of \( \phi' \) in the right hand side of (1) we obtain \( s \) from (3). This completes the proof, because of the injectivity of \( \phi \). We want to say a couple of words how we have calculated \( mC \). Applying Theorem 1 we know that \( mC \) is of the form

\[
mC(t, v) = \frac{r(t, v)}{q(t, v)},
\]

where \( r(t, v) \in C(v)[t] \) and \( q(t, v) = (1 - t)^4(1 + t)^2(1 + t + t^2)(1 + t^2)(1 - vt)^2(1 - vt^2). \]

Theorem 1 gives also that \( \deg r \leq \deg q = 2 = 12 \). Decomposing \( mC \) as a sum of elementary fractions with coefficients from \( C(v) \) and denominators which are powers of \( 1 - t, 1 + t, 1 + t + t^2, 1 + t^2, 1 - vt, \) and \( 1 - vt^2 \), we obtain that \( mC \) is as in (4). In order to calculate the coefficients \( \alpha_i, \beta_i, \gamma_i, \delta_i, \varepsilon_i, \varphi_{\pm}, \) we replace \( hC \) and \( mC \) in the left and right hand sides of (1), respectively. Replacing \( y \) with \( c_2/x \), we apply Lemma 2 for \( K = C(e_2) \) and \( \xi \) a zero of the denominator \( q(x, e_2) \). In this way we obtain the coefficients of the denominators of highest degree of \( 1 - t, 1 + t, \) and \( 1 - vt \). Replacing the obtained coefficients in (4), we calculate step by step all the coefficients \( \alpha_i, \beta_i, \varepsilon_i \). The coefficients \( \gamma_i, \delta_i, \varphi_{\pm}, i = 0, 1, \) are obtained using both the zeros of \( 1 + t + t^2, 1 + t^2 \) and \( 1 - vt^2 \). □

The proof of the following theorem is similar.
Theorem 4. The multiplicity series \( m_T \) of the Hilbert series \( H(T,x,y) \) of the mixed trace algebra of two generic \( 4 \times 4 \) matrices is

\[
m_T = \frac{\alpha_4}{(1-t)^4} + \frac{\alpha_3}{(1-t)^3} + \frac{\alpha_2}{(1-t)^2} + \frac{\alpha_1}{1-t} + \frac{\beta}{1+t} + \frac{\gamma_0 + \gamma_1 t}{1+t+t^2} + \frac{\varepsilon_2}{(1-v)t^2} + \frac{\varepsilon_1}{1-vt} + \frac{\varphi_0 + \varphi_1 t}{1-vt^2},
\]

where

\[
\begin{align*}
\alpha_4 &= \frac{1 - v + 3 v^2 - v^3 + v^4}{6(1-v)^{12}(1+v)(1+v^2)^2}, \\
\alpha_3 &= \frac{3 - 8 v + 4 v^2 - 9 v^3 - 8 v^4 - 5 v^5 + 12 v^6 - 8 v^7 + 7 v^8}{12(1-v)^{13}(1+v)(1+v^2)^3}, \\
\alpha_2 &= \frac{\alpha'_2}{72(1-v)^{14}(1+v)(1+v^2)^4}, \\
\alpha'_2 &= \frac{-17 - 55 v + 124 v^2 + 304 v^3 + 540 v^4 + 777 v^5 + 1332 v^6 + 687 v^7 + 468 v^8 + 280 v^9 + 124 v^{10} - 73 v^{11} + 47 v^{12}}{144(1-v)^{15}(1+v)^6(1+v^2)^5}, \\
\beta &= \frac{1 - v + v^2 - v^3 + v^4}{16(1-v)^{5}(1+v)^6(1+v^2)^2(1-v+v^2)}, \\
\gamma_0 &= \frac{1}{9(1-v)^4(1+v)(1+v^2)^5(1-v+v^2)}, \\
\gamma_1 &= \frac{1}{9(1-v)^3(1+v+v^2)^5(1-v+v^2)}, \\
\varepsilon_2 &= \frac{1 - v + v^2 - v^3 + v^4}{(1-v)^{14}(1+v)^5(1+v^2)^2(1+v^2)(1+v^2+v^3+v^4)}, \\
\varepsilon_1 &= \frac{-1 - v^8}{(1-v)^{15}(1+v)^6(1+v+v^2)^3(1+v^2)^2(1+v^2+v^3+v^4)^2(1-v+v^2)^2}, \\
\varepsilon'_1 &= (3 + 3 v + 4 v^2 + 4 v^3 + 4 v^4 + 3 v^5 + 3 v^6)(3 + v + 5 v^2 + 3 v^3 + 5 v^4 + v^5 + 3 v^6), \\
\varphi_0 &= \frac{2 v^4 \varphi_0}{(1-v)^{15}(1+v)(1+v+v^2)^3(1+v+v^2+v^3+v^4)^2}, \\
\varphi_0' &= \frac{2 v^4 \varphi_0'}{(1-v)^{15}(1+v+v^2+v^3+v^4)^2}, \\
\varphi_1 &= \frac{1 + v + 12 v^2 + 18 v^3 + 34 v^4 + 37 v^5 + 42 v^6 + 37 v^7 + 34 v^8 + 18 v^9 + 12 v^{10} + 5 v^{11} + v^{12}}{v^4 \varphi_1'}, \\
\varphi'_1 &= 1 + 5 v + 12 v^2 + 18 v^3 + 34 v^4 + 37 v^5 + 42 v^6 + 37 v^7 + 34 v^8 + 18 v^9 + 12 v^{10} + 5 v^{11} + v^{12}.
\end{align*}
\]

Theorems 3 and 4 give the explicit (but very complicated) form of the multiplicities \( m_\lambda(C) \) and \( m_\lambda(T) \) for any \( \lambda = (\lambda_1, \lambda_2) \). We prefer to present the results in more compressed, but more convenient form.
Proposition 5. The multiplicity series \( m_C \) and \( m_T \) are linear combinations of fractions
\[
\frac{v^a t^b}{\pi^k(v) \rho(t, v)},
\]
where \( 0 \leq a < \deg_v \pi(v) \), \( 0 \leq b < \deg_t \rho(t, v) \), and \( \pi(v) \) and \( \rho(t, v) \) are the following polynomials
\[
\pi(v) = 1 \pm v, 1 \pm v + v^2, 1 + v + v^2 + v^3 + v^4,
\]
\[
\rho(t, v) = 1 \pm t, 1 + t + t^2, 1 + t^2, 1 - vt, 1 - vt^2.
\]
The degrees of the denominators satisfy the inequality \( k + l \leq 16 \). The linear combinations \( M_C \) and \( M_T \) of fractions with denominators of degree 16 of \( m_C \) and \( m_T \), respectively, are
\[
M_C = \frac{1}{(1 - v)^{12}} \left( \frac{1}{2^{8}3^{2}(1-t)^{4}} - \frac{1}{2^{8}3^{3}(1-v) (1-t)^{5}} \right).
\]
\[
+ \frac{127}{2^{10}3^{4}(1-v)^{2}(1-t)^{2}} \frac{1}{2^{8}3^{5}(1-v)^{3}(1-t)}
\]
\[
- \frac{7}{2^{10}3^{2}5(1-v)^{2}(1-vt)^{2}} \frac{2^{4}(1+t)}{2^{8}3^{2}(1-v)^{3}(1-vt) + 3^{5}5^{2}(1-v)^{3}(1-vt^2)}
\]
\[
M_T = 16 M_C.
\]

Proof. Decomposing the rational functions \( \alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i, \phi_i \) from Theorem 3 as linear combinations of elementary fractions, we obtain that \( m_C \) is a linear combination of fractions of the form (5). We also see that the maximum degree \( k + l \) is equal to 16 and this maximum is reached for the fractions
\[
\frac{1}{(1 - v)^{16-l}(1-t)^{l}}, \quad l = 1, 2, 3, 4,
\]
\[
\frac{1}{(1 - v)^{16-l}(1-vt)^{l}}, \quad l = 1, 2, \quad \frac{t^b}{(1-v)^{15}(1-vt^2)}, \quad b = 0, 1.
\]
The explicit coefficients are found using Lemma 2. The calculations for \( m_T \) are similar, applying Theorem 4. \( \square \)

The following theorem is the main result of our paper. It is in the spirit of the description of the multiplicities of the pure trace algebra of two \( 3 \times 3 \) matrices given by Berele [4].

Theorem 6. Let \( \lambda = (\lambda_1, \lambda_2) \) be any partition.

(i) The multiplicities \( m_\lambda(C) \) of the pure trace cocharacter of \( 4 \times 4 \) matrices satisfy the condition
\[
m_\lambda(C) = \begin{cases} 
m_1 + \mathcal{O}((\lambda_1 + \lambda_2)^{13}), & \text{if } \lambda_1 > 3 \lambda_2, \\
m_1 + m_2 + \mathcal{O}((\lambda_1 + \lambda_2)^{13}), & \text{if } 3 \lambda_2 \geq \lambda_1 > 2 \lambda_2, \\
m_1 + m_2 + m_3 + \mathcal{O}((\lambda_1 + \lambda_2)^{13}), & \text{if } 2 \lambda_2 \geq \lambda_1,
\end{cases}
\]
where
\[
m_1 = \frac{(\lambda_1 - \lambda_2)^3 \lambda_2^2}{2^{8}3^{2}11!3!} + \frac{(\lambda_1 - \lambda_2)^2 \lambda_1^2}{2^{8}3^{3}12!2!} + \frac{127(\lambda_1 - \lambda_2) \lambda_2^3}{2^{10}3^{4}13!} - \frac{305 \lambda_2^4}{2^{9}3^{5}14!},
\]
\[
m_2 = \frac{(3 \lambda_2 - \lambda_1)^{14}}{2^{10}3^{5}5^{2}14!},
\]
Comparing the coefficients of the expansion of \( p, q, r, s, w \) are constants. In this way, the coefficients \( b_1 \) satisfy

\[
\text{Proof.} \quad 8 \, \text{VESSELIN DRENSKY AND GEORGI GENOV}
\]

where \( |a_q| \) is bounded by a polynomial of degree \( k - 1 \) in \( q \). A similar fact holds for \( 1/\rho(t, v) \) when \( p = 1 + t, 1 + t + t^2, 1 + t^2 \). Finally, the coefficients of the expansion

\[
\frac{1}{(1 - vt)^2} = \sum_{p \geq 0} (p + 1)(vt)^p
\]

are linear functions in \( p \) and those of

\[
\frac{1}{1 - vt} = \sum_{p \geq 0} (vt)^p, \quad \frac{1}{1 - vt^2} = \sum_{p \geq 0} (vt^2)^p
\]

are constants. In this way, the coefficients \( b_{pq} \) of the expansion

\[
\frac{1}{\pi^k(v)}\rho^l(t, v) = \sum_{p, q \geq 0} b_{pq} p^l v^q
\]

are bounded by polynomials of degree \( k + l - 2 \) in \( p, q \) and satisfy \( b_{pq} = \mathcal{O}((p + q)^{k+l-2}) \). Since \( k + l \leq 16 \), the contribution of maximum degree 14 to the multiplicities \( m_{\lambda}(C) \) comes from the expansion of \( M_C \). Using (6) we obtain that \( M_C \) is equal to

\[
\sum_{p, q \geq 0} \left( q^{11} \left( \frac{p^3}{283^311!13!} - \frac{p^2 q}{283^312!2!} + \frac{127 pq^2}{2103^413!} - \frac{305 q^3}{293^314!} \right) + \mathcal{O}((p + q)^{13}) \right) p^l v^q
\]

\[
+ 2^4(1 + t) \sum_{r, s \geq 0} \left( \frac{w^{13} \left( \frac{p}{2103^25^213!} + \frac{7 w}{293^25^214!} \right) + \mathcal{O}((p + w)^{13}) \right) (tv)^p v^w.
\]

Comparing the coefficients of the expansion of \( M_C \) with those of \( M_C \) we obtain that

\[
m_{\lambda}(C) = q^{11} \left( \frac{p^3}{283^311!13!} - \frac{p^2 q}{283^312!2!} + \frac{127 pq^2}{2103^413!} - \frac{305 q^3}{293^314!} \right)
\]

\[
+ 2^4 s^{14} \left( \frac{p}{2103^25^213!} + \frac{7 w}{293^25^214!} \right) + \mathcal{O}((\lambda_1 + \lambda_2)^{13})
\]

where \( p, q, r, s, w \geq 0 \) are such that

\[
\lambda_1 - \lambda_2 = p, \quad \lambda_2 = q = r = s = p + w,
\]

\[
\lambda_1 - \lambda_2 = 2s \text{ or } \lambda_1 - \lambda_2 = 2s + 1.
\]
This gives the conditions

\[ s = \frac{1}{2} (\lambda_1 - \lambda_2), \quad r = \frac{1}{2} (3\lambda_2 - \lambda_1), \]

if \( \lambda_1 - \lambda_2 \) is even,

\[ s = \frac{1}{2} (\lambda_1 - \lambda_2 - 1), \quad r = \frac{1}{2} (3\lambda_2 - \lambda_1 + 1), \]

if \( \lambda_1 - \lambda_2 \) is odd, and \( w = 2\lambda_2 - \lambda_1 \). The expression depending on \( p, q \) gives the contribution \( m_1 \) for any \( \lambda_1 \geq \lambda_2 \). The part with \( r, s \geq 0 \) gives \( m_2 \) when \( 3\lambda_2 \geq \lambda_1 \). Finally, \( p, w \geq 0 \) gives \( m_3 \) when \( 2\lambda_2 \geq \lambda_1 \).

\[ \square \]

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Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria

E-mail address: drensky@math.bas.bg, gguenov@hotmail.com