On rainbow-free colourings of uniform hypergraphs

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Abstract

We study rainbow-free colourings of k-uniform hypergraphs; that is, colourings that use k colours but with the property that no hyperedge attains all colours. We show that $p^* = (k - 1)(\ln n)/n$ is the threshold function for the existence of a rainbow-free colouring in a random k-uniform hypergraph.

1 Introduction

A k-uniform hypergraph $H$ consists of a set of vertices $V(H)$ and a collection $E(H)$ of k-element subsets of $V(H)$, called hyperedges. For a k-uniform hypergraph $H$, a map $c : V(H) \to [k]$ is called a $k$-colouring of $H$, where $[k] := \{1, \ldots, k\}$. The colouring $c$ is called rainbow-free if for every hyperedge $e = (v_1, \ldots, v_k) \in E(H)$ we have $c(e) = \{c(v_1), \ldots, c(v_k)\} \neq [k]$ and for every $i \in [k]$ there is $v \in V(H)$ with $c(v) = i$.

The $k$-rainbow-free problem is to determine whether a given k-uniform hypergraph is rainbow-free colourable with k colours.1

Contributions We initiate the study of k-rainbow-free colourings on random hypergraphs. We consider a natural generalisation of Erdős-Rényi random graphs to random (k-uniform) hypergraphs: each possible hyperedge is present with a fixed probability, independently of the other hyperedges. In Section 3, we find a threshold function for the event that a random hypergraph of the first kind is rainbow-free colourable (Theorem 7). The proof uses a second moment argument for the lowerbound and a first moment argument with an analysis of possible types of rainbow-free colourings for the upperbound.

Related work The k-rainbow-free problem is a special case of colouring mixed hypergraphs, introduced by Voloshin [11] and further extended by Král’, Kratochvíl, Proskurowski, and Voss [10]. A mixed hypergraph is a triple $(V, C, D)$ where $V$ is the vertex set and $C$ and $D$ are collections of subsets of $V$. A colouring of the vertices of a mixed hypergraph $(V, C, D)$ is called proper if each hyperedge in $C$ contains two vertices of the same colour and each hyperedge in $D$ contains two vertices of different colours. The strict $k$-colouring problem is to determine whether a given mixed hypergraph is properly colourable with exactly $k$ colours. The strict $k$-colouring

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1The $k$-rainbow-free problem is called $k$-no-rainbow-colouring in [2]. For $k = 2$, a graph is rainbow-free 2-colourable if and only if it is disconnected (cf. Remark 4).
problem restricted to $k$-uniform mixed hypergraphs with $D = \emptyset$, so-called co-hypergraphs, is precisely the $k$-rainbow-free problem. The strict $k$-colouring of co-hypergraphs was later identified, under the name of $k$-no-rainbow-colouring, in the survey by Bodirsky, Kára, and Martin [2] as an interesting case of unknown complexity of surjective constraint satisfaction problems on a three-element domain.

Constraint satisfaction problems (CSPs) are generalisations of graph homomorphisms [9]. A graph homomorphism from $G$ to $H$ is a map from the vertex set of $G$ to the vertex set of $H$ that preserves all edges (but not necessarily non-edges). For a fixed graph $H$, the $H$-colouring problems is to determine whether a given graph $G$ admits a homomorphism to $H$. For instance, taking $H = K_3$ to be the complete graph on 3 vertices, $H$-colouring is the well known 3-colouring problem. Hell and Nešetřil established that, unless $H$ contains a loop or is a bipartite graph, the $H$-colouring problem is NP-complete [8].

In an influential paper, Feder and Vardi conjectured that a similar dichotomy holds for every digraph $H$, or equivalently, for every finite relational structure (such as hypergraphs) [7]. This conjecture, known as the CSP dichotomy conjecture, was confirmed by two independent papers by Bulatov [4] and Zhuk [12], respectively. While the recent progress on the CSP dichotomy conjecture (and various CSP variants) relied heavily on the so-called algebraic approach [5], this method does not seem directly amenable to surjective CSPs, in which we require the homomorphism be surjective. A dichotomy theorem is known to hold for surjective CSPs on two-element domains by the work of Creignou and Hébrard [6]. The $k$-rainbow-free problem is equivalent to a surjective CSP on a $k$-element domain $[k]$ with a single $k$-ary relation $[k]^k - \{(x_1, \ldots, x_k) : x_1, \ldots, x_k \text{ distinct}\}$. Very recently, Zhuk has announced NP-hardness of the $k$-rainbow-free problem for $k \geq 3$ [13].

## 2 Preliminaries

If $k$ is clear from the context, we will call a $k$-colouring simply a colouring. For a colouring $c$ of a $k$-uniform hypergraph, we denote the colour classes by $C_i := c^{-1}(i)$, $i \in [k]$.

We state now same basic properties of rainbow-free colourings.

**Definition 1.** Given a $k$-uniform hypergraph $H$ and a subset of vertices $S \subseteq V(H)$, we define the *induced subhypergraph* $H_S$ as the $(k-1)$-uniform hypergraph with vertices $V(H_S) := V(H) \setminus S$ and hyperedges $E(H_S) := \{e \cap (V(H) \setminus S) : e \in E(H) \text{ and } |e \cap S| = 1\}$.

For up to $k$ disjoint sets $S_1, \ldots, S_k \subseteq V(H)$ we write $H_{S_1, \ldots, S_k} := (H_{S_1} \ldots H_{S_k})$ for the repeated induced subhypergraph.

For $k = 3$ in Definition 1, $H_S$ will be a graph. Furthermore, note that the order of the subscripts in the definition of the repeated induced subhypergraph does not matter.

This notion of induced subhypergraphs is useful because of the following proposition.

**Proposition 2.** Let $k \geq 2$ be an integer. A $k$-uniform hypergraph $H$ is rainbow-free $k$-colourable if and only if there exists a non-empty subset of vertices $\emptyset \neq S \subseteq V(H)$ such that the $(k-1)$-uniform hypergraph $H_S$ is rainbow-free $(k-1)$-colourable. In particular, this implies the existence of a colouring $c$ of $H$ with $C_k = S$.

**Proof.** First suppose that $H$ has a rainbow-free $k$-colouring $c$. Let $S$ be $C_k \neq \emptyset$ and consider $H_S$. Write $c'$ for the colouring $c$ restricted to $V(H_S) = V(H) \setminus C_k$. We will show that $c'$ is indeed a rainbow-free $(k-1)$-colouring of $H_S$. First note that $C'_1 \cup \cdots \cup C'_{k-1} = V(H_S)$, and hence every vertex of $H_S$ has a well-defined colour in $[k-1]$. Now consider a hyperedge $e' \in E(H_S)$. By definition we have $e' = V(H_S) \cap e$ for some $e \in E(H)$. We will use a proof by contradiction to show that $c'(e') \neq [k-1]$, so assume that $c'(e') = [k-1]$. This implies $[k-1] = c(e') \subseteq c(e)$. Furthermore we know that $e \cap S = e \cap C_k \neq \emptyset$, and hence $k \in c(e)$. This
implies that \( c(e) = [k-1] \cup \{k\} = [k] \), which is a contradiction. We conclude that \( c'(e) \neq [k-1] \) and hence \( C' \) is a rainbow-free colouring of \( H_S \), as required.

For the other direction assume that \( \emptyset \neq S \subseteq V(H) \) is such that \( H_S \) has a rainbow-free \((k-1)\)-colouring \( c' \). Now extend \( c' \) to a \( k \)-colouring \( c \) of \( H \) by setting \( c(v) = k \) for all \( v \in S \). Thus, we have \( C_k = S \). Let \( e \) be a hyperedge in \( H \). We wish to show that \( c(e) \neq [k] \), so that \( c \) is a rainbow-free colouring as well. If \( |e \cap S| = 0 \) we have \( k \not\in c(S) \). In the \( |e \cap S| = 1 \) case we have \( e' := e \cap (V(H) \setminus S) \in E(H_S) \). Since \( c' \) is a rainbow-free \((k-1)\)-colouring of \( H_S \) we know that \( c(e') = c'(e') \neq [k-1] \). Adding in the one vertex \( v \) of \( e \) that is in \( S = C_k \), we get \( c(e) = c(e' \cup \{v\}) \neq [k-1] \cup \{k\} = [k] \), as required. If \( |e \cap S| \geq 2 \) there are at most \( k-2 \) vertices that have a colour in \([k-1]\). Since \( k-2 < |[k-1]| \) we know that \( c(e') \) can not attain all colours in \([k-1]\). Hence, this case also implies \( c(e) \neq [k] \).

By induction, it follows that we can apply multiple steps of Proposition 2 at once.

**Corollary 3.** Let \( 2 \leq \ell < k \) be integers. A \( k \)-uniform hypergraph \( H \) is rainbow-free \( k \)-colourable if and only if there exist disjoint non-empty subsets \( S_1, \ldots, S_\ell \) of \( V(H) \) such that \( H_{S_1, \ldots, S_\ell} \) is rainbow-free \((k-\ell)\)-colourable.

**Remark 4.** We remark that Proposition 2 also applies to the corner case of \( k = 2 \). In particular, a graph \( H \) is rainbow-free 2-colourable if and only if there is a subset \( S \subseteq V(H) \) with no outgoing edges; in other words, \( H \) is disconnected.

If all possible rainbow-free hyperedges are given, not only do we know that the rainbow-free colouring is unique, but we can easily find it.

**Proposition 5.** Suppose that \( H \) is a \( k \)-uniform hypergraph with a surjective colouring \( c : V(H) \to [k] \). Furthermore assume that \( E = \{ e \in V^{(k)} | c(e) \neq [k] \} \) consists of all rainbow-free hyperedges. Write \( E := V^{(k)} \setminus E \) for the set of rainbow hyperedges with \( c(e) = [k] \). The colour classes of \( c \) are determined by \( C_{c(v)} := \{ v \} \cup \{ u \in V | \forall e \in E : \{ u, v \} \not\subseteq e \} \).

**Proof.** If \( \{u, v\} \subseteq e \) for some \( e \in E \), we have that \( c(u) \neq c(v) \), since \( e \) would be a rainbow-free hyperedge otherwise.

For the other direction assume that \( c(u) \neq c(v) \). By surjectivity of \( c \), all colour classes are non-empty and hence there exists a vertex \( x_j \) for every colour \( j \in [k] - \{c(u), c(v)\} \). Using these vertices \( x_j \) together with \( u \) and \( v \) yields a rainbow hyperedge, which is an element of \( E \). Hence there exists a rainbow hyperedge \( e \in E \) containing both \( u \) and \( v \). This implies that the condition from the statement of the proposition is both sufficient and necessary.

## 3 Random hypergraphs

The following definition of random hypergraphs is a direct generalisation of the Erdős–Rényi random graph model: every possible hyperedge is added with a given probability.

**Definition 6.** Let \( p : N \to [0, 1] \) be a given probability function. A random \( k \)-uniform hypergraph \( H_{n,p}^k \) is a \( k \)-uniform hypergraph created by the following process:

- Start with a set of vertices \( V(H_{n,p}^k) := V \) with \(|V| = n\).
- For each hyperedge \( e \in V^{(k)} \), add \( e \) to \( E(H_{n,p}^k) \) with probability \( p = p(n) \).

Let \( A \) be a hypergraph property (in our case being rainbow-free colourable). We write \( \Pr[H_{n,p}^k \models A] \) for the probability that \( H_{n,p}^k \) satisfies \( A \). A function \( r(n) \) is called a threshold function for a hypergraph property \( A \) if (i) when \( p(n) \ll r(n) \), \( \lim_{n \to \infty} \Pr[H_{n,p}^k \models A] = 0 \), (ii) when \( p(n) \gg r(n) \), \( \lim_{n \to \infty} \Pr[H_{n,p}^k \models A] = 1 \), or vice versa.

Our main result is the following theorem.
Theorem 7. The function \( p^* = (k - 1)(\ln n)/n \) is a threshold function for the event that a random \( k \)-uniform hypergraph \( H_{n,p}^k \) is rainbow-free colourable.

The two parts of the proof, one for small \( p \) and one for large \( p \), are covered by the following two lemmas. The result is well known for \( k = 2 \) [3, Theorem VII.9] and corresponds to disconnectedness (cf. Remark 4). Hence we will assume \( k \geq 3 \).

Lemma 8. For \( k \geq 3 \), the random hypergraph \( H_{n,p}^k \) is rainbow-free colourable with high probability if \( p \leq D \frac{\ln n}{n} \) for some \( D < k - 1 \).

Lemma 9. If \( p \geq D(\ln n)/n \) with \( D > k - 1 \) and \( k \geq 3 \), the random hypergraph \( H_{n,p}^k \) is not rainbow-free colourable with high probability.

In order to prove Lemma 8, we use the second moment method; i.e., use the second moment of a random variables to bound the probability that the variable is far from its mean.

Let \( X \) be a nonnegative integer-valued random variable such that \( X = \sum_{i=1}^{m} X_i \), where \( X_i \) is the indicator variable for event \( E_i \). For indices \( i, j \) write \( i \sim j \) if \( i \neq j \) and the events \( E_i \) and \( E_j \) are not independent. We set (the sum is over ordered pairs)

\[ \Delta = \sum_{i<j} \Pr[E_i \land E_j]. \]

Proposition 10 ([1, Corollary 4.3.4]). If \( \mathbf{E}[X] \to \infty \) and \( \Delta = o(\mathbf{E}[X]^2) \) then \( \Pr[X > 0] \to 1 \).

Proof of Lemma 8. Let \( H_{n,p}^k \) be a random hypergraph and let \( X \) be the number of rainbow-free colourings of \( H_{n,p}^k \) with only one colour class of size larger than one. Our goal is to show that \( X > 0 \) with high probability, and thus \( H_{n,p}^k \) is rainbow-free colourable with high probability. We will do so by invoking Proposition 10.

We first show that \( \mathbf{E}[X] \to \infty \).

Let \( c \) be a colouring of \( H_{n,p}^k \) that uses all \( k \) colours and has only one colour class of size greater than 1. We assume that \(|C_i| = 1\) for \( 1 \leq i \leq k - 1 \) and \(|C_k| = n - k + 1\). This colouring \( c \) is rainbow-free if and only if there are no hyperedges covering all \( k \) colour classes. There are \( 1 \cdot \cdots \cdot 1 \cdot (n - k + 1) = n - k + 1 \) hyperedges with this property, and hence

\[ \Pr[c \text{ is a rainbow-free colouring}] = (1 - p)^{n-k+1} = \Theta((1 - p)^n). \]

Since \( \ln(1 + x) = x + O(x^2) \) for small \( x \), we have \( 1 - p = e^{-p + O(p^2)} \) and thus

\[ \Pr[c \text{ is a rainbow-free colouring}] = \Theta\left(e^{-pn + O(n^2)}\right) = \Theta\left(e^{-D\ln n + O(D^2\ln n^2/n)}\right) = \Theta(n^{-D}). \]

The number of colourings \( c \) with one large colour class of size \( n - k + 1 \) is \( (\binom{n}{n-k+1}) = \Theta(n^{k-1}) \).

The expected number of such colourings that are rainbow-free is now given by

\[ \mathbf{E}[X] = \left(\binom{n}{n-k+1}\right)(1 - p)^{n-k+1} = \Theta(n^{k-1}n^{-D}) = \Theta(n^{k-1-D}). \]

Since \( D < k - 1 \), this implies that \( \mathbf{E}[X] \to \infty \) when \( n \to \infty \).

Enumerate all possible colourings \( c \) (up to permutations of colours) satisfying \(|C_k| = n - k + 1\) by \( c^1 \) up to \( c^\ell \). We write \( i \sim j \) if \( i \neq j \) and \(|C_k^i \cap C_k^j| = n - k \). To every colouring \( c^i \) we associate the event \( E_i \) that \( c^i \) is rainbow-free.

Consider the quantity

\[ \Delta = \sum_{i \sim j} \Pr[E_i \land E_j]. \quad (1) \]
Given rainbow-free colourings is but the hyperedge $A$ for $x \mid c$ is rainbow-free if all hyperedges of the form $e \in V \setminus n$ and an overlap of $C$. We have $|C^i_k| = |C^j_k| = n - k + 1$ and $|C^l_k| = |C^m_k| = 1$ for all $1 \leq \ell < k$.

We will prove that $\Delta = o(\text{Var}[X]^2)$ and thus finish the proof by Proposition 10. In order for Proposition 10 to be applicable, we need that (for $i \neq j$) $i \sim j$ if the events $E_i$ and $E_j$ are not independent.

By the definition of $\sim$, we have

$$\Delta = \sum_i \sum_{j \neq i, |C^i_k \cap C^j_k| = n-k} \text{Pr}[E_i \land E_j].$$

We claim that the event $E_i$ is independent from $E_j$ if $i \neq j$ and $i \sim j$. In this case, the overlap between $C^i_k$ and $C^j_k$ is at most $n - k - 1$, since an overlap of $n - k$ implies $i \sim j$ and an overlap of $n - k + 1$ implies equality. Write $A = C^i_k \setminus C^j_k$, $B = C^j_k \setminus C^i_k$, and $R = V(H_{n,p}) \setminus C^i_k \setminus C^j_k$, as illustrated in Figure 1. The colouring $\ell$ is rainbow-free if all hyperedges of the form $e_1 = B \cup R \cup \{x\}$ for $x \in C^i_k$ are not present. On the other hand, the colouring $\ell$ is rainbow-free if all hyperedges $e_2 = A \cup R \cup \{y\}$ for $y \in C^j_k$ are not present. We have that $|A| = |C^i_k| - |C^i_k \cap C^j_k| \geq (n - k + 1) - (n - k - 1) = 2$. Similarly we have $|B| \geq 2$. Since $A$ is disjoint from $B$, we now know that the hyperedges $e_1$ and $e_2$ can not be equal. Hence, the colourings $\ell$ and $\ell$ depend on different hyperedges being present, and thus these events are independent indeed.

Let $i$ and $j$ be such that $i \sim j$; i.e., $i \neq j$ and $|C^i_k \cap C^j_k| = n - k$. In this case, we have $|A| = |B| = 1$. The hyperedges that the events $E_i$ and $E_j$ depend on are of the form $A \cup R \cup \{x\}$ for $x \in C^i_k$ and $B \cup R \cup \{y\}$ for $y \in C^j_k$ respectively. We count $2 \cdot (n - k + 1)$ hyperedges in total, but the hyperedge $A \cup R \cup B$ is counted twice. Hence, the probability that $\ell$ and $\ell$ are both rainbow-free colourings is

$$\text{Pr}[E_i \land E_j] = (1 - p)^{2(n-k+1)} - e^{-p(2n-2k+1)} = \Theta(e^{-2pn}).$$

Given $\ell$ with $|C^i_k| = n - k + 1$, the number of colourings $\ell$ such that the large colour classes...
overlap in \( n - k \) positions is \((n - k + 1)(k - 1)\). Putting this back in \( \Delta \) gives

\[
\Delta = \sum_i (n - k + 1)(k - 1)(1 - p)^{2n-2k+1}
\]

\[
\leq \binom{n}{n-k+1}(n - k + 1)(k - 1)e^{-p(2n-2k+1)}
\]

\[
\leq n^{k-1} \cdot n \cdot k \cdot e^{-2D\ln n + O(\ln n/n)}
\]

\[
= O(n^k, n^{-2D}) = O(n^{k-2D}).
\]

Since \( k \geq 3 \) we have \( 0 < k - 2 \) and hence \( k - 2D < 2k - 2 - 2D = 2(k - 1 - D) \). We conclude that

\[
\Delta = O(n^{k-2D}) = o(n^{2(k-1-D)})
\]

and thus \( \Delta = o(\mathbb{E}[X]^2) \).

We will now prove the bound in the other direction, Lemma 9.

**Proof of Lemma 9.** We use the first moment method to show that the expected number of rainbow-free colourings of \( H_{n,p}^k \) goes to 0. We identify a colouring by the sequence \( (s_1, \ldots, s_k) \) where \( s_i = |C_i| \) and \( s_1 \leq \cdots \leq s_k \). We divide the set of all possible sequences into five types:

1. \( (s_i)_i = (1, \ldots, 1, n - k + 1) \). There is one such sequence.
2. \( (s_i)_i = (1, \ldots, 1, 2, n - k) \). There is one such sequence.
3. \( (s_i)_i = (1, \ldots, 1, x, n - k + 2 - x) \) with \( x \geq 3 \). This case contains \( O(n) \) sequences.
4. \( 2 \leq s_{k-2} \leq s_{k-1} \) and \( s_1 + \cdots + s_{k-1} \leq 6k \). This case contains \( O(1) \) sequences, since \( k \) is a constant.
5. \( 2 \leq s_{k-2} \leq s_{k-1} \) and \( s_1 + \cdots + s_{k-1} > 6k \). This case contains \( O(n^{k-1}) \) sequences.

In each case we will show that the expected number of rainbow-free colourings of the relevant type is \( o(1) \), from which it follows that the probability that \( H_{n,p}^k \) is rainbow-free colourable is \( o(1) \).

Before starting calculations, we introduce some notation. We write \( \Sigma = s_1 + \cdots + s_{k-1} \) so that \( s_k = n - \Sigma \geq n/k \), and we write \( \Pi = s_1 \cdots s_{k-1} \).

A colouring is rainbow-free if none of the \( s_1 \cdots s_k \) hyperedges that span all colour classes is present. This happens with probability

\[
\mathbb{Pr}[c \text{ is rainbow-free } | (s_i)_i] = (1 - p)^{s_1 \cdots s_k} \leq e^{-p s_1 \cdots s_k} \leq n^{-D/n \cdot \Pi(n-\Sigma)}.
\]

Since the number of colourings with a given sequence \( (s_i)_i \) is upper-bounded by \( n^{s_1} \cdots n^{s_{k-1}} = n^\Sigma \), the expected number of rainbow-free colourings with a given sequence \( (s_i)_i \) is bounded by

\[
\mathbb{E}[\text{number of rainbow-free colourings } | (s_i)_i] \leq n^{\Sigma - D/n \cdot \Pi(n-\Sigma)}. \tag{2}
\]

In each of the cases below we will bound the exponent of \( n \) in (2).

Write \( D = k - 1 + \delta \) for some \( \delta > 0 \).

**Case 1:** We have \( \Sigma = k - 1 \) and \( \Pi = 1 \). Putting this into (2) gives an exponent of

\[
\Sigma - D/n \cdot \Pi(n-\Sigma) = (k - 1) - D \cdot (1 - (k - 1)/n) \to -\delta.
\]

This is less than \(-\delta/2\) if \( n \) is large enough. Hence, this case is \( o(1) \).

**Case 2:** Here we have \( \Sigma = k \) and \( \Pi = 2 \). The exponent of \( n \) in (2) becomes

\[
\Sigma - D/n \cdot \Pi(n-\Sigma) = k - (k - 1 + \delta) \cdot 2 \cdot (1 - k/n) \to -k + 2 - 2\delta. \tag{3}
\]
Since this converges to a negative number, it will be less than $-1/2$ for all large enough $n$. Hence, this case is $o(1)$ as well.

**Case 3:** There are $O(n)$ sequences in this case, so each of them must give an expected value that is $o(n^{-1})$. The variables are $\Sigma = k + x - 2$ and $\Pi = x$. The exponent in (2) is a quadratic function of $x$:

\[
\Sigma - (k - 1 + \delta)/n \cdot \Pi(n - \Sigma) = k + x - 2 - (k - 1 + \delta) \cdot x \cdot (1 - (k + x - 2)/n). \tag{4}
\]

Since the leading coefficient is positive, and we want to prove an upper bound, it suffices to check the boundaries $x = 3$ and $x = n/2$. (The maximal possible value of $x$ is actually even smaller, but overestimating doesn’t hurt.) For $x = 3$ we get

\[
k + 1 - 3(k - 1 + \delta)(1 - (k + 1)/n) \to -2k + 4 - 3\delta < -1. \tag{5}
\]

Since this converges to something less than $-1$, we know that the expected value for $x = 3$ is $o(n^{-1})$ for $n$ large enough.

Since the value of (4) goes to $-\infty$ if $x = n/2$ and $n \to \infty$, the upper bound (5) on the exponent in (2) works for the $x = n/2$ case as well.

**Case 4:** We are given that $\Sigma \leq 6k$. Furthermore we have $s_{k-2} \geq 2$. The minimal value of $\Pi$ is attained if $s_1 = \cdots = s_{k-3} = 1$ and $s_{k-1} = \Sigma - (k - 3) - 2 = \Sigma - k + 1$. Thus, we have $\Pi \geq 2(\Sigma - k + 1)$. Since $\Sigma$ is a sum of $k - 1$ terms, of which the last two are at least 2, we also have $\Sigma \geq k + 1$.

\[
\Sigma - (k - 1 + \delta)/n \cdot \Pi(n - \Sigma) \leq \Sigma - (k - 1 + \delta)2(\Sigma - k + 1)(1 - \Sigma/n) \to \Sigma - (k - 1 + \delta)2(\Sigma - k + 1).
\]

The step where we take the limit is allowed because $\Sigma$ is bounded, and hence the term divided by $n$ goes to 0 indeed. We continue

\[
\Sigma - (k - 1 + \delta)2(\Sigma - k + 1) = \Sigma(1 - 2(k - 1 + \delta)) + 2(k - 1)(k - 1 + \delta) \\
\leq (k + 1)(-2k + 1 - 2\delta) + 2(k - 1)(k - 1 + \delta) \\
= -2k^2 - k + 1 - 2k\delta - 2\delta + 2k^2 + 4k + 2 + 2k\delta - 2\delta \\
= -3k + 1 + 4\delta < 0. \tag{6}
\]

As before this converges to something negative, and hence it will be $o(1)$.

**Case 5:** We are now ready for the only remaining case. Here we have $\Sigma \geq 6k$ and as before this implies $\Pi \geq 2(\Sigma - k + 1)$.

\[
\Sigma - (k - 1 + \delta)/n \cdot \Pi(n - \Sigma) \leq \Sigma - (k - 1 + \delta)/n \cdot 2(\Sigma - k + 1)(n - \Sigma).
\]

Using that $s_k = n - \Sigma \geq n/k$ and doing some rewriting gives

\[
\frac{\Sigma - (k - 1 + \delta)/n \cdot \Pi(n - \Sigma)}{n} \leq \frac{\Sigma - (k - 1 + \delta)/n \cdot 2(\Sigma - k + 1)}{k} \\
= \frac{\Sigma - (k - 1 + \delta/k)}{k} \cdot 2(\Sigma - k + 1) \\
= (1 - 2(k - 1 + \delta)/k)\Sigma + 2(1 - 1/k)(k - 1 + \delta). \\
= (-1 + 2/k - 2\delta/k)\Sigma + 2(1 - 1/k)(k - 1 + \delta).
\]

We are now at the point where we can use $\Sigma \geq 6k$. Because $-1 + 2/k - 2\delta/k < 0$ we get

\[
\Sigma - (k - 1 + \delta)/n \cdot \Pi(n - \Sigma) \leq (-1 + 2/k - 2\delta/k) \cdot 6k + 2(1 - 1/k)(k - 1 + \delta) \\
= -6k + 12 - 12\delta + 2k - 2 + 2\delta - 2 + 2/k - 2\delta/k \\
\leq -4k + 8 - 10\delta + 2/k \leq -4k + 9 - 10\delta. \tag{7}
\]

This last value is strictly less than $-k + 1$, which is just what we needed. We conclude that the total expected number of rainbow-free colourings in this case is $o(1)$ as well, and hence the random hypergraph $H_{n,p}$ is not rainbow-free colourable with high probability. \qed
Lemma 9 can be made a bit stronger with respect to the the colourings of type \((1, \ldots, 1, n - k + 1)\).

**Proposition 11.** If a random hypergraph \(H_{n,p}^k\), with \(k \geq 3\), \(p = D(\ln n)/n\), and \(D > k - 1\) is rainbow-free colourable then with high probability it has a colouring of type \((1, \ldots, 1, n - k + 1)\).

**Proof.** The proof depends heavily on the claims established in the proofs of Lemmas 8 and 9.

Let \(X_i\) be the number of rainbow-free colourings in Case \(i\) of the proof of Lemma 9. Since \(n^{1/n} = e^{(\ln n)/n} \to 1\), we know that the convergence of exponents in (2) in the proof of Lemma 9 implies that \(n\) raised to the limit of the exponent is off by at most a constant factor. Hence,

\[
\mu := E[X_1] = \Theta(n^{k-1-D}) = \Theta(n^{-\delta}),
\]

where \(D = k - 1 + \delta\). In Cases 2 to 5 of the proof of Lemma 9, Equations (3), (5), (6), and (7) imply that the expected number of rainbow-free colourings in each case is bounded by

\[
E[X_2] = O(n^{-k+2-2\delta}),
E[X_3] = O(n) \cdot O(n^{-2k+4-3\delta}) = O(n^{-2k+5-3\delta}),
E[X_4] = O(n^{-3k+1-4\delta}),
E[X_5] = O(n^{k-1}) \cdot O(n^{-4k+9-10\delta}) = O(n^{-3k+8-10\delta}).
\]

Since \(k \geq 3\), each of these terms is \(o(n^{-1-2\delta})\). Hence for \(2 \leq i \leq 5\) we have \(\Pr[X_i > 0] \leq E[X_i] = o(n^{-1-2\delta})\). To show that almost all random rainbow-free colourable hypergraphs are rainbow-free colourable with a colouring of the first type indeed, all we have to show is that \(\Pr[X_1 > 0] = \Theta(n^{-\delta})\).

As in the proof of Lemma 8 enumerate all colourings by \(c^1\) to \(c^\ell\) and suppose that \(c^i\) is a rainbow-free colouring. The probability that there is another rainbow-free colouring \(c^j\) is bounded by

\[
\sum_{j \sim 1} \Pr[c^j | c^i] + \sum_{j \not\sim i, j \not\sim i} \Pr[c^j] \leq n \cdot k \cdot e^{-p(n-k)} + n^{k-1} e^{-p(n-k+1)}
\]

\[
= O(n^{k-1} n^{-(k-1+\delta)}) = O(n^{-\delta}).
\]

Hence, the probability that the number of rainbow-free colourings is exactly 1 is at least

\[
\sum_i \Pr[c_i](1 - O(n^{-\delta})) \sim \sum_i \Pr[c_i] = \Theta(n^{k-1-D}) = \Theta(n^{-\delta}).
\]

This implies that the probability that \(H_{n,p}^k\) is rainbow-free colourable is at least \(\Theta(n^{-\delta})\). \(\square\)

Proposition 11 implies that checking colourings of the type \((1, \ldots, 1, n - k + 1)\) is sufficient to find a colouring in \(H_{n,p}^k\) with high probability if we know that the hypergraph is rainbow-free colourable.

**4 Conclusions**

We showed that a threshold function of the event that a random \(k\)-uniform hypergraph is rainbow-free colourable is \((k - 1)(\ln n)/n\). Our results do not say anything about the case when the hyperedge probability \(p\) is close to the threshold. As far as we know, the behaviour of the rainbow-free colourings of a random hypergraph in this case is open.
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