THE SEMICLASSICAL MODIFIED NONLINEAR SCHRODINGER EQUATION I: MODULATION THEORY AND SPECTRAL ANALYSIS

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Abstract. We study an integrable modification of the focusing nonlinear Schrödinger equation from the point of view of semiclassical asymptotics. In particular, (i) we establish several important consequences of the mixed-type limiting quasilinear system including the existence of maps that embed the limiting forms of both the focusing and defocusing nonlinear Schrödinger equations into the framework of a single limiting system for the modified equation, (ii) we obtain bounds for the location of discrete spectrum for the associated spectral problem that are particularly suited to the semiclassical limit and that generalize known results for the spectrum of the nonselfadjoint Zakharov-Shabat spectral problem, and (iii) we present a multiparameter family of initial data for which we solve the associated spectral problem in terms of special functions for all values of the semiclassical scaling parameter. We view our results as part of a broader project to analyze the semiclassical limit of the modified nonlinear Schrödinger equation via the noncommutative steepest descent procedure of Deift and Zhou, and we also present a self-contained development of a Riemann-Hilbert problem of inverse scattering that differs from those given in the literature and that is well-adapted to semiclassical asymptotics.

1. Introduction

This paper will be concerned with the modified nonlinear Schrödinger (MNLS) equation for a complex-valued function \( \phi(x,t) \):

\[
i \varepsilon \frac{\partial \phi}{\partial t} + \frac{\varepsilon^2}{2} \frac{\partial^2 \phi}{\partial x^2} + |\phi|^2 \phi + \frac{i\alpha \varepsilon}{\partial x} (|\phi|^2 \phi) = 0, \quad \alpha \geq 0, \quad \varepsilon > 0.
\]

(1)

The correct problem to pose for this equation when \( x \in \mathbb{R} \) is the Cauchy problem, where initial data \( \phi(x,0) \) are given at \( t = 0 \). Here \( \alpha \) and \( \varepsilon \) are real parameters which we assume throughout the paper satisfy the inequalities given in (1).

The focusing nonlinear Schrödinger (NLS) equation is a special case of (1) when \( \alpha = 0 \). The focusing NLS equation is of course very well known, arising naturally in the modeling of any nearly monochromatic, weakly nonlinear, dispersive wave propagation. In particular, it is in many circumstances an adequate model to describe the evolution of the electric field envelope of picosecond pulses in monomode optical fibers [21]. The so-called “semiclassical” or zero-dispersion limit of the focusing NLS Cauchy problem consists of taking the initial condition in the form \( \phi(x,0) = A(x) e^{iS(x)/\varepsilon} \) for fixed real functions \( A(\cdot) > 0 \) and \( S(\cdot) \) and considering the behavior of the corresponding solution \( \phi(x,t) \) of the focusing NLS equation in the limit \( \varepsilon \downarrow 0 \). This asymptotic problem is important in the design of increasingly prevalent dispersion-shifted fibers as well as being of intrinsic mathematical interest.

The utility of the focusing NLS equation as a mathematical model stems from the fact that this equation is integrable, and in principal the Cauchy problem may be solved using the scattering/inverse scattering transform [11] that is precisely adapted to this equation. However, in applications if higher-order effects become important then it is necessary to account for them in the model. To consider sub-picosecond electromagnetic pulses in monomode optical fibers, one should consider the more general evolution equation,

\[
i \varepsilon \frac{\partial \phi}{\partial t} + \frac{\varepsilon^2}{2} \frac{\partial^2 \phi}{\partial x^2} + |\phi|^2 \phi = -i \alpha' \frac{\partial}{\partial x} (|\phi|^2 \phi) + \alpha'' \frac{\partial^3}{\partial x^3} (|\phi|^2) \cdot \phi + i\alpha'' \varepsilon^3 \frac{\partial^3 \phi}{\partial x^3} ,
\]

(2)

where \( \alpha, \alpha' \), and \( \alpha'' \) are real constants and where the three additional terms on the right hand side account for nonlinear dispersion, intrapulse Raman scattering, and higher-order linear dispersion respectively [11]. In general, (2) is not integrable; however it turns out that there are special cases of (2) besides \( \alpha = \alpha' = \alpha'' = 0 \) that are integrable, although by different machinery than applies to the focusing NLS equation. In fact, the
MNLS equation (11) is such an integrable system (10) (related to the so-called “derivative” NLS equation). As the MNLS equation takes into account at least the higher-order effect of nonlinear dispersion, understanding the predictions of this equation can serve as a mathematical “stepping stone” in the understanding of the more general problem (2) as well as any weakly nonlinear, nearly monochromatic, dispersive wave propagation problem when nonlinear dispersion is not negligible.

One of the well-known features of the focusing NLS equation is the so-called modulational instability. This instability may be manifested in several different ways, and so the term “modulational instability” can mean different things in different contexts. However, the simplest context of modulational instability is the linearized perturbation theory of small disturbances of exact “plane-wave” solutions of the focusing NLS equation. Indeed, the NLS equation

\[ i\varepsilon \frac{\partial \phi}{\partial t} + \varepsilon^2 \frac{\partial^2 \phi}{\partial x^2} + \kappa|\phi|^2\phi = 0 \]  

(3)

(the focusing case is \( \kappa = 1 \) and the defocusing case is \( \kappa = -1 \)) has exact solutions (plane waves) of the form \( \phi_0(x,t) = Ae^{i(kx-\omega t)/\varepsilon} \) for real constants \( A > 0, \ k \) and \( \omega \) necessarily related by the nonlinear dispersion relation

\[ \omega = \frac{1}{2}k^2 - \kappa A^2. \]  

(4)

Consider a small perturbation of this plane-wave solution by substituting \( \phi = \phi_0 \cdot (1 + p) \) into (3), where \( p \) is a new unknown. In this way, one obtains an (equivalent) equation for the “in-phase” perturbation \( p \):

\[ i\varepsilon \frac{\partial p}{\partial t} + ik\varepsilon \frac{\partial p}{\partial x} + \frac{\varepsilon^2}{2} \frac{\partial^2 p}{\partial x^2} + \kappa A^2(p + p^* + 2|p|^2 + p^2 + |p|^2) = 0, \]  

(5)

where the asterisk denotes complex conjugation. Assuming \( p \) to be small, this equation may be linearized and written as a system for \( a = \text{Re}\{p\} \) and \( b = \text{Im}\{p\} \):

\[ \varepsilon \frac{\partial a}{\partial t} + k\varepsilon \frac{\partial a}{\partial x} + \frac{\varepsilon^2}{2} \frac{\partial^2 a}{\partial x^2} = 0 \]  

\[ \varepsilon \frac{\partial b}{\partial t} + k\varepsilon \frac{\partial b}{\partial x} - \frac{\varepsilon^2}{2} \frac{\partial^2 a}{\partial x^2} - 2\kappa A^2 a = 0. \]  

(6)

This is a linear constant-coefficient system of partial differential equations that may be solved using Fourier techniques. We look for elementary solutions of the form

\[ \begin{bmatrix} a(x,t) \\ b(x,t) \end{bmatrix} = \begin{bmatrix} \hat{a}(t) \\ \hat{b}(t) \end{bmatrix} e^{i\Delta x/\varepsilon}, \]  

(7)

where \( \Delta \) is a “relative wavenumber” (to \( k \)), which reduces (6) to a system of ordinary differential equations:

\[ \varepsilon \frac{d}{dt} \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} -ik\Delta \\ 2\kappa A^2 - \frac{1}{2}\Delta^2 \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}. \]  

(8)

There is a basis of solutions proportional to eigenvectors of the coefficient matrix on the right-hand side, and the time dependence enters through factors of the form \( e^{\sigma t/\varepsilon} \) where \( \sigma \) is the corresponding eigenvalue of the coefficient matrix. The eigenvalues are given by

\[ \sigma = -ik\Delta \pm \frac{\Delta}{2} \sqrt{4\kappa A^2 - \Delta^2}. \]  

(9)

If we are in the defocusing case (\( \kappa = -1 \)) then \( \sigma \) is always purely imaginary, and all Fourier-type solutions of (8) are oscillatory in \( t \). This behavior indicates the absence of modulational instability in the defocusing case. However, if we are in the focusing case (\( \kappa = 1 \)) then \( \sigma \) has a nonzero real part if \( \Delta^2 < 4A^2 \) and therefore there is a band of unstable relative wavenumbers about \( \Delta = 0 \) (and hence a sideband of the unperturbed wavenumber \( k \)). This is the modulational instability in its simplest form. These calculations show that in the focusing case, the only perturbations that have a chance of avoiding growth correspond to waves of length \( O(\varepsilon) \), and worse yet, the exponential growth rates for the unstable modes scale as \( 1/\varepsilon \). In this sense, the modulational instability is enhanced when \( \varepsilon \) is small.

The modulational instability may also be understood from a nonlinear perspective, via the so-called modulation equations. Without loss of generality, we write the solution \( \phi(x,t) \) of (4) in the form \( \phi(x,t) = A(x,t)e^{iS(x,t)/\varepsilon} \) where \( A(x,t) > 0 \) is a real amplitude and \( S(x,t) \) is a real phase, and then (3) implies
corresponding (real) nonlinear equations governing these two functions. If we think of $A(x, t)$ and $S(x, t)$ as functions independent of $\varepsilon$ (this assumption is of course not exactly consistent with (3)) then locally in the neighborhood of fixed $x$ and $t$, $\phi(x, t)$ resembles a plane wave solution with amplitude $A = A(x, t)$, wavenumber $k = \partial S/\partial x(x, t)$, and frequency $\omega = -\partial S/\partial t(x, t)$. This observation is the grounds for expecting a connection between the preceding perturbative calculation and what will now follow. To proceed, it is more natural to introduce $u(x, t) := \partial S/\partial x$ and $\rho(x, t) := A^2$ and obtain equations for these unknowns equivalent to (3):

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} &= 0 \\
\frac{\partial u}{\partial t} - \kappa \frac{\partial \rho}{\partial x} + u \frac{\partial u}{\partial x} &= \varepsilon^2 \left( \frac{1}{2} \frac{\partial^2 \rho}{\partial x^2} + \left( \frac{1}{2} \frac{\partial \rho}{\partial x} \right)^2 \right).
\end{align*}
\]

This time, the approximation we make (in lieu of the formal linearization step in the preceding discussion) is to drop the formally small terms in $\varepsilon$. Thus, one arrives at the modulation equations associated to (3):

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} &= 0 \\
\frac{\partial u}{\partial t} - \kappa \frac{\partial \rho}{\partial x} + u \frac{\partial u}{\partial x} &= \varepsilon^2 \left( \frac{1}{2} \frac{\partial^2 \rho}{\partial x^2} - \left( \frac{1}{2} \frac{\partial \rho}{\partial x} \right)^2 \right).
\end{align*}
\]

A system of quasilinear partial differential equations of the form (11) is called hyperbolic if the coefficient matrix of the $x$-derivatives has distinct real eigenvalues and elliptic if the eigenvalues are complex. Cauchy problems for hyperbolic systems can be solved by the method of characteristics, one implication of which is the continuous dependence of the solution on initial data for small time. The hyperbolic Cauchy problem is said to be well-posed. On the other hand, the Cauchy problem is ill-posed for elliptic systems. Worse yet, the Cauchy problem for an elliptic system can only be solved at all (by the method of Cauchy-Kovalevskaya) if the initial data $\rho(x, 0)$ and $u(x, 0)$ are analytic functions of the variable $x$; this should be regarded as a overly restrictive condition for a physically relevant mathematical model.

It is easy to check that for $u \in \mathbb{R}$ and $\rho > 0$, the system (11) is elliptic for $\kappa = 1$ and hyperbolic for $\kappa = -1$. This calculation therefore reveals the same dichotomy between the focusing and defocusing versions of the NLS equation as did linearization about a plane wave. The linearization calculation may be viewed as a “local in $x$” version of the deduction of type (hyperbolic or elliptic) of modulation equations. In general, hyperbolicity of plane-wave modulation equations is equivalent to the absence of linear instabilities of plane waves of sideband type, although it is possible for a system to have hyperbolic modulation equations while admitting instabilities of relatively short waves (33). Interestingly, analyticity of initial data as required by elliptic modulation equations plays a fundamental role in the spectral and inverse-spectral analysis of the focusing NLS equation in the semiclassical limit (27), even though the latter analysis is not based on modulation equations at all. (For an approach based directly on the Cauchy-Kovalevskaya series solution of the analytic data Cauchy problem for the elliptic modulation equations, see Gérard [19].)

One might notice that in the linearization calculation the conclusion of stability or instability was independent of the pair $(k, A)$ that characterizes the underlying plane wave. Indeed, in the focusing case, there is a band of unstable relative wavenumbers no matter what values $k$ and $A$ take. Similarly, the modulation equations (11) are elliptic or hyperbolic independently of $u \in \mathbb{R}$ and $\rho > 0$. In principle, it could have been otherwise; some waves may be stable to all sideband perturbations while others are not. For such a system, the modulation equations may be hyperbolic for some values of $u$ and $\rho$ and elliptic for others, and the system is said to admit a change of type. It will turn out that the system of modulation equations for the MNLS equation admits a change of type.

To summarize, the focusing NLS equation exhibits modulational instability, while the defocusing NLS equation is modulationally stable. Now, the MNLS equation (11) appears to be a perturbation of the focusing NLS equation (through the limit $\alpha \to 0$). One might expect on these grounds that the MNLS problem should also experience modulational instability. On the other hand, consider making the following substitution in (11):

\[
\phi(x, t) = e^{i(c \xi + c^2 \tau/2)/\varepsilon} \psi(\xi, \tau),
\]

(12)
where $\xi = x - ct$ and $\tau = t$, and $c \in \mathbb{R}$ is an arbitrary parameter. Then, by direct calculation, $\psi$ satisfies

$$i\varepsilon \frac{\partial \psi}{\partial \tau} + \frac{\varepsilon^2}{2} \frac{\partial^2 \psi}{\partial \xi^2} + (1 - \alpha \varepsilon) |\psi|^2 \psi + i \alpha \varepsilon \frac{\partial}{\partial \xi}(|\psi|^2 \psi) = 0. \quad (13)$$

If $\alpha = 0$, we recover the well-known result that the focusing NLS equation is invariant under the one-parameter group of Galilean transformations [12]. More generally, the MNLS equation is certainly not invariant under this group. In particular, one may note that by choosing $c$ so that $\alpha c > 1$, the sign of the $|\psi|^2 \psi$ term becomes negative, and thus the equation governing $\psi(\xi, \tau)$ takes the form of a perturbed defocusing NLS equation rather than a perturbation of the focusing NLS equation.

Given that when $\alpha = 0$ one expects drastically different behavior in the semiclassical limit depending on whether the sign of the cubic term is positive (strong modulational instability) or negative (modulational stability), the above calculation suggests two things: (i) the term in the MNLS equation proportional to $\alpha$ certainly does not constitute a small perturbation uniformly in the phase space of fields $\phi$ (that is, taking $\alpha \neq 0$ amounts to a singular perturbation of the NLS equation), and (ii) that part of the phase space of fields $\phi$ may evolve under the MNLS equation in such a way as to avoid all catastrophic modulational instability associated with the undifferentiated cubic term $|\phi|^2 \phi$. If this latter statement can be placed on rigorous footing, there are immediate implications for the design of optical fiber telecommunication links (for example). Indeed, one of the difficulties arising in long-distance fiber-optic systems is the so-called Gordon-Haus effect, in which pulses experience a random “jitter” in their arrival times which leads to detection errors. This effect has been shown to be a consequence of the modulational instability associated with the focusing nonlinearity (and anomalous dispersion) in the focusing NLS model. The possibility of avoiding Gordon-Haus jitter in the context of the MNLS model has already been explored [18] at the level of single soliton solutions. One of our goals is to understand this change in stability for the MNLS equation at the level of more general initial data, and in particular initial data that “contain” many solitons as is characteristic of the semiclassical scaling.

Due to its integrable structure, solving the Cauchy problem for the MNLS equation requires three steps:

**Step 1.** Calculate certain “spectral data” associated with the given initial condition for the problem, viewed here as a coefficient in a linear system of differential equations with a spectral parameter. This is a “forward transform” of the initial data.

**Step 2.** Determine how the spectral data change as the field $\phi$ evolves under the MNLS equation, and obtain the spectral data corresponding to the evolved field.

**Step 3.** Find the field $\phi(x, t)$ from the time-evolved spectral data, that is, invert the mapping from step 1. This is an “inverse transform” step that in many cases can be written as a matrix-valued Riemann-Hilbert problem from analytic function theory.

The miracle of integrability is that for an appropriate linear system of differential equations with spectral parameter, the inverse of the spectral map of step 1 that is required in step 3 actually exists, and as importantly the evolution of the spectral data in step 2 is completely explicit. For semiclassical problems, the small parameter $\varepsilon$ enters significantly in the first and third steps. Therefore, to understand the stability transition in the MNLS equation and its role in semiclassical asymptotics, we must first understand the forward transform (step 1) in the semiclassical regime. Once the the semiclassical asymptotics of the forward transform have been understood, it will remain to utilize the growing library of tools of asymptotic analysis for Riemann-Hilbert problems to complete the solution of the Cauchy problem for the MNLS equation in the limit $\varepsilon \downarrow 0$. In this paper, we will consider the spectral problem associated with the MNLS equation (see [173]) paying particular attention to results that are meaningful in the limit $\varepsilon \downarrow 0$.

This paper is organized as follows. In Section 2 we follow the procedure applicable to the NLS equation to obtain the modulation equations associated to the MNLS equation. Through the analysis of these modulation equations we will obtain a condition for modulational stability. Throughout the remainder of the paper we note the importance of this condition and the role it plays in the spectral problem. In Section 3 we establish some bounds on the discrete eigenvalues of the spectral problem associated to the MNLS equation.

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1 If $c$ is chosen so that $\alpha c = 1$ exactly, then under the Galilean boost the MNLS equation for $\phi$ reduces to the so-called derivative NLS equation for $\psi$. 
In Section 4 we use the theory of special functions to calculate the spectral data explicitly for a certain multiparameter family of initial conditions; this calculation is valid for all \( \varepsilon > 0 \) and provides an explicit avenue to the semiclassical analysis of the corresponding Cauchy problem. The appendix contains the general spectral and inverse-spectral theory associated with the MNLS equation (that is, all theoretical details of steps 1–3 above) and in particular includes a derivation of the associated Riemann-Hilbert problem that we will return to in a later publication. The theory outlined in the appendix frames all of our analysis of the MNLS equation.

**Notation.** Throughout the paper, we use boldface capital letters to refer to square matrices, with the exception of the identity matrix which we write as \( I \), the Pauli matrices

\[
\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \text{and} \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

(14)

and certain diagonal matrices denoted \( e^{a\sigma_3} \) for \( a \in \mathbb{C} \) and defined by

\[
e^{a\sigma_3} := \begin{bmatrix} e^a & 0 \\ 0 & e^{-a} \end{bmatrix}.
\]

(15)

Lowercase boldface letters refer to column vectors. We use the “dagger” notation (superscript \( \dagger \)) to indicate the transpose and elementwise complex conjugate of a matrix or vector, and we use asterisk notation (superscript \( * \)) for complex conjugation. We sometimes use prime notation for derivatives of functions of just one variable.

## 2. Modulation Equations

In this section we derive the modulation equations associated to the MNLS equation and find a condition on the local amplitude and wavenumber equivalent to the hyperbolicity of these equations. By analogy with the connection between hyperbolicity of modulation equations and stability of plane waves described in the context of the NLS equation in Section 1, this calculation gives a condition under which the MNLS equation is (locally) modulationally stable. The fact that one can have local stability without global stability is related to the fact the MNLS modulation equations admit change of type.

### 2.1. Derivation of the modulation equations. Consider the MNLS equation (1), and for some real-valued functions \( A(x,t) \) and \( S(x,t) \) write the solution \( \phi(x,t) \) as:

\[
\phi(x,t) = A(x,t)e^{iS(x,t)/\varepsilon}.
\]

(16)

Substitution into (1), canceling the phase factor \( e^{iS/\varepsilon} \), taking the imaginary part, multiplying by \( A \) and dividing by \( \varepsilon \) gives:

\[
\frac{\partial}{\partial t}A^2 + \frac{\partial}{\partial x} \left( \frac{\partial S}{\partial x} A^2 + \frac{3}{2} \alpha A^4 \right) = 0.
\]

(17)

Proceeding similarly but taking the real part instead, one arrives at:

\[
\frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} S \right)^2 + \frac{\partial}{\partial x} \left( \frac{1}{2} \left( \frac{\partial S}{\partial x} \right)^2 - A^2 + \alpha \frac{\partial S}{\partial x} A^2 \right) = \varepsilon^2 \frac{\partial}{\partial x} \left( \frac{1}{A} \frac{\partial^2 A}{\partial x^2} \right).
\]

(18)

Setting \( \rho(x,t) := A(x,t)^2 \) and \( u(x,t) := \partial S(x,t)/\partial x \), the system of modulation equations is obtained by dropping the formally small terms in \( \varepsilon \):

\[
\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ u \end{bmatrix} + \begin{bmatrix} 3\alpha \rho + u \\ \alpha \rho - 1 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \rho \\ u \end{bmatrix} = 0.
\]

(19)
2.2. Modulational stability criterion. The eigenvalues $\zeta$ of the coefficient matrix of the $x$-derivatives in \eqref{MNLS} are the roots of the equation:

$$
\zeta^2 - (4\alpha \rho + 2u) \zeta + 3\alpha u \rho + 3\alpha^2 \rho^2 + u^2 + \rho = 0. \tag{20}
$$

The discriminant of this equation is $4\rho(\alpha^2 \rho + \alpha u - 1)$. Since $\rho > 0$, the condition

$$
\alpha^2 \rho + \alpha u - 1 > 0, \tag{21}
$$

is equivalent to having real distinct eigenvalues, which makes \eqref{MNLS} a hyperbolic system. Note that for a given value of the parameter $\alpha \in \mathbb{R}$, this condition involves both dependent variables $\rho$ (square of local plane-wave amplitude) and $u$ (local wavenumber) and this dependence is characteristic of a quasilinear system that can change type. The inequality \eqref{21} will appear at every stage of our analysis of the spectral problem (see \eqref{170}) associated with the MNLS equation.

2.3. Implications of the stability criterion on Riemann-Hilbert analysis of the inverse spectral problem. While this paper is concerned mostly with the semiclassical limit of the spectral transform associated with the MNLS equation and a subsequent paper will analyze the corresponding inverse-spectral transform in this limit, it is worthwhile pointing out how we might expect the stability criterion \eqref{21} to influence the inverse-spectral part of the solution of the MNLS Cauchy problem for rapidly decaying initial data of the form $\phi(x,0) = A(x) e^{iS(x)/\varepsilon}$, $x \in \mathbb{R}$.

How do modulated plane waves emerge from the asymptotic analysis of the inverse-spectral problem? As shown in the appendix, the inverse-spectral problem for the MNLS Cauchy problem can be cast as a Riemann-Hilbert problem in which one seeks a $2 \times 2$ matrix-valued unknown depending principally upon a complex variable $k$ and parametrically upon $x$, $t$, $\alpha$, and $\varepsilon$. The unknown matrix is characterized by a normalization condition as $k \to 0$, prescribed discontinuities along the real and imaginary $k$-axes, simple poles with prescribed residues at certain other points in the complex $k$-plane, and analyticity in $k$ for all other $k \in \mathbb{C}$. Such a Riemann-Hilbert problem involving a parameter $\varepsilon$ in a singular way may be analyzed by a two-step process. First, one introduces an explicit transformation of the matrix-valued unknown designed to remove the poles at the expense of introducing new jump discontinuities. Thus one arrives at an equivalent Riemann-Hilbert problem for a piecewise-analytic matrix-valued function of $k$ whose jump discontinuities lie along a system, $\Sigma$, of contours including (generally) the real and imaginary $k$-axes. Second, one introduces a so-called $g$-function (an innovation discovered in \cite{13}), a scalar function of $k$ analytic for $k \in \mathbb{C} \setminus \Sigma$, and by multiplying the matrix-valued unknown on the right by the diagonal matrix $e^{g(k)\sigma_3/\varepsilon}$ one arrives at a second equivalent Riemann-Hilbert problem on the same system $\Sigma$ of contours but for which the jump conditions involve the boundary values of $g$. The function $g$ is then chosen to cast the jump conditions for the matrix unknown in as simple a form as possible. By the phrase “as simple a form as possible,” we really mean that $g$ is chosen so that the steepest descent method of Deift and Zhou \cite{15} applies, implying a set of conditions that one expects to uniquely determine $g$. A matrix-valued Riemann-Hilbert problem to which the steepest descent method applies is one for which the ratio of the boundary values of the unknown is a matrix taking two alternate forms in complementary systems of subintervals of $\Sigma$. The number of subintervals and their common endpoints depend on the remaining parameters of the problem, including $x$ and $t$.

The common endpoints $k = k_j(x,t)$ of these subintervals have special significance as branching points of the Lax eigenfunctions (solutions $v$ of the linear equation \eqref{170}; see the appendix) corresponding to the local asymptotic solution of the MNLS equation in a small neighborhood of $(x,t)$. If there are just four of these endpoints, then one expects the local solution to have plane-wave form $\phi(x,t) = A(x,t) e^{iS(x,t)/\varepsilon} + O(\varepsilon)$, where one may consider $\rho = A^2$ and $u = \partial S/\partial x$ to be constants. To understand the interpretation of the branching points $k_j$ in terms of $\rho$ and $u$, note that the relevant scattering problem (see \eqref{17} below) is written in terms of $\rho$ and $u$ as

$$
2\alpha \varepsilon \frac{dw}{dx} = i \begin{bmatrix} -4k^2 + 1 - \alpha u & 4\alpha \rho^{1/2} k \\ 4\alpha \rho^{1/2} k & 4k^2 - 1 + \alpha u \end{bmatrix} w, \tag{22}
$$

With $\rho$ and $u$ constant, this is a constant-coefficient linear problem with eigensolutions $w(x) = e^{E_{\pm} x/\varepsilon} w_{\pm}$, where

$$
i \begin{bmatrix} -4k^2 + 1 - \alpha u & 4\alpha \rho^{1/2} k \\ 4\alpha \rho^{1/2} k & 4k^2 - 1 + \alpha u \end{bmatrix} w_{\pm} = 2\alpha E_{\pm} w_{\pm}. \tag{23}
$$
The eigenvalues are
\[
2\alpha E_{\pm} = \pm i \sqrt{16\alpha^2 \rho k^2 + (4k^2 + \alpha u - 1)^2},
\]
and branching points of the eigenfunctions in the complex \( k \)-plane will occur where \( E_+ = E_- \) or equivalently where \( E_\pm = 0 \). There are therefore at most four branching points satisfying
\[
4k_j^2 = -2\alpha^2 \rho - \alpha u + 1 \pm 2\alpha \sqrt{(\alpha^2 \rho + \alpha u - 1)\rho}.
\]

It follows that if (i) steepest descent analysis of the inverse-spectral Riemann-Hilbert problem in the limit \( \varepsilon \downarrow 0 \) leads to a solution of the MNLS equation of the form \( \phi = A(x,t) e^{iS(x,t)/\varepsilon} + O(\varepsilon) \) where \( \rho(x,t) = A^2 \) and \( u(x,t) = \partial S/\partial x \), and (ii) the plane wave characterized by \( \rho \) and \( u \) is modulationally stable according to the hyperbolicity criterion \((\ref{hyperbolicity})\), then it is necessary that the \( g \)-function of steepest descent theory provides four endpoints \( k_j(x,t) \) all lying on the imaginary axis. On the other hand, if \( \rho \) and \( u \) characterize a modulationally unstable plane wave, then the four endpoints \( k_j(x,t) \) will need to form a quartet invariant under the reflections \( k \to k^* \) and \( k \to -k^* \).

The unstable case of course requires that the contour system, \( \Sigma \), contain components that are disjoint from the real and imaginary \( k \)-axes, and determining the correct boundary values of the function \( g \) on such a contour further requires comparing with certain spectral functions computed on the real and imaginary \( k \)-axes (see the appendix) that are analytically continued to the complex \( k \)-plane. This process of analytic continuation, that we expect to only be required in the unstable case, is the spectral analogue of analytic continuation of the functions \( \rho(x,0) \) and \( u(x,0) \) to the complex \( x \)-plane as would be necessary to solve the Cauchy initial-value problem for the elliptic modulation equations.

2.4. Other forms of the modulation equations.

2.4.1. Local conservation laws. With the dependent variable \( v := \rho u + \alpha \rho^2 \) used instead of \( u \), the system \((\ref{system})\) takes the form
\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left( v - \alpha \rho^2 + \frac{3\alpha}{2} \rho^2 \right) &= 0, \\
\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left( \frac{v^2}{\rho} + \alpha \rho u - \frac{\rho^2}{2} \right) &= 0.
\end{align*}
\]
That is, \((\ref{system})\) may be written as a system of local conservation laws for densities \( \rho \) and \( v \). Strictly speaking, the system \((\ref{system})\) is already in this form:
\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left( \rho u + \frac{3\alpha}{2} \rho^2 \right) &= 0, \\
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \alpha \rho u - \rho + \frac{1}{2} u^2 \right) &= 0;
\end{align*}
\]
however since \( u = \partial S/\partial x \), the local conservation law for \( u \) is trivial. Consequently one should seek a second conserved local density that is not a derivative with respect to \( x \) of a local expression in \( A \) and \( S \). It is well-known and easy to verify that the momentum \( v_o := \rho u \) is a nontrivial conserved local density for the modulation equations \((\ref{modulation})\) associated to the focusing NLS equation, and the nontrivial conserved local density \( v \) for \((\ref{system})\) (or equivalently, \((\ref{momentum})\)) is clearly an \( \alpha \)-perturbation of the momentum.

In the paper \((\ref{Grenier})\) the quantity \( v \) is called a noncanonical momentum. The authors of that paper give an analysis of the (equivalent, by a Galilean boost with speed \( c = \alpha^{-1} \)) derivative NLS equation in the semiclassical limit. In the spirit of the paper of Grenier \((\ref{Grenier})\), the authors use the system \((\ref{system})\) (with \( u \) replaced by \( u + \alpha^{-1} \) as-is) for the derivative NLS equation rather than the MNLS equation to model the behavior of \( \psi(x,t) \) before singularities form in the solution of the limiting system. The development in the paper \((\ref{Grenier})\) appears to assume, however, that the system of conservation laws \((\ref{system})\) is hyperbolic for arbitrary initial data, and we have seen (see \((\ref{hyperbolicity})\) which becomes simply \( \alpha^2 \rho + \alpha u > 0 \) when \( u \) is replaced by \( u + \alpha^{-1} \)) that this is not the case.
2.4.2. **Connection with focusing NLS in the semiclassical limit.** Suppose that \(\alpha^2 \rho + \alpha u - 1 < 0\) and that \(\rho > 0\), defining an open domain \(D_- (\alpha) \subset \mathbb{R}^2\) for each \(\alpha > 0\). Then consider the map \(F_- : (\rho, u) \in D_- (\alpha) \mapsto (\hat{\rho}, \hat{u}) \in \mathbb{R}^2\) defined by the formulae

\[
\hat{\rho} = -\rho \cdot (\alpha^2 \rho + \alpha u - 1), \quad \hat{u} = u + 2\alpha \rho.
\]

(28)

The mapping \(F_-\) is one-to-one and maps \(D_- (\alpha)\) onto the upper half-plane

\[
R_- = \{ (\hat{\rho}, \hat{u}) \in \mathbb{R}^2 \text{ such that } \hat{\rho} > 0 \}.
\]

(29)

The inverse \(F_-^{-1} : R_- \rightarrow D_- (\alpha)\) is given by the formulae

\[
\rho = \frac{1}{2\alpha^2} \left( \alpha \hat{u} - 1 + \sqrt{(\alpha \hat{u} - 1)^2 + 4\alpha^2 \hat{\rho}} \right), \quad u = \frac{1}{\alpha} \left( 1 - \sqrt{(\alpha \hat{u} - 1)^2 + 4\alpha^2 \hat{\rho}} \right),
\]

(30)

where the positive square root is meant. The mapping \(F_-\) is illustrated in Figure 1.

**Figure 1.** Top: the domain of \(F_-\) is shaded. Bottom: the range of \(F_-\) is shaded. Note that \(Q(\hat{u}; c, \alpha) := [(c - \alpha^{-1})^2 - (\hat{u} - \alpha^{-1})^2]/4\).

It is a direct calculation to check that when the MNLS modulation equations are restricted to functions \((\rho(x,t), u(x,t)) \in D_- (\alpha)\), they take the form

\[
\frac{\partial}{\partial t} \left[ \hat{\rho} \right] + \left[ \frac{\partial}{\partial x} \left[ \hat{u} \right] \right] + \left[ \frac{\partial}{\partial \hat{u}} \left[ \hat{\rho} \right] \right] = 0
\]

(31)

in the new variables \((\hat{\rho}, \hat{u})\). Comparing with (11) for \(\kappa = 1\) we see that on the modulationally unstable sector of the phase space the formal semiclassical limit for the MNLS equation is, when viewed in the correct variables \((\hat{\rho}, \hat{u})\), the same as that for the focusing NLS equation.

\(^2\)Note however, that the mapping \(F_-\) does not appear to establish any equivalence between the MNLS equation and the focusing NLS equation at the level of finite nonzero \(\varepsilon\). We do not know whether the \(\varepsilon\)-independent mapping \(F_-\) can be prolonged...
This identification allows us to immediately apply certain known results for the focusing NLS equation to the MNLS equation. For example, a particular solution to (31) was obtained in implicit algebraic form by Akhmanov, Sukhorukov, and Khokhlov in 1966:

$$
\dot{\rho} = (A_o^2 + t^2 \dot{\rho}^2) \text{sech}^2(x - \dot{u}t), \quad \dot{u} = -2t \dot{\rho} \tanh(x - \dot{u}t).
$$

(32)

Here $A_o > 0$ is an amplitude parameter, and of course $x$ and $t$ are the independent variables in (31). Setting $t = 0$ gives the initial conditions

$$
\rho(x, 0) = A_o^2 \text{sech}^2(x), \quad \dot{u}(x, 0) = 0,
$$

(33)

(which corresponds to a “chirp-free, return-to-zero” pulse in the optical fiber context) and for small $t$ the Implicit Function Theorem applies to (32) and allows us to solve for $\dot{\rho}(x, t)$ and $\dot{u}(x, t)$. The earliest positive $t$ for which the Implicit Function Theorem fails is $t = (2A_o)^{-1}$, and the singularity occurs for $x = 0$. For the implicit solution of Akhmanov, Sukhorukov, and Khokhlov, $\dot{\rho}(x, t)$ is an even function of $x$ and $\dot{u}(x, t)$ is an odd function of $x$ for each $t \in (0, (2A_o)^{-1})$. For any $\alpha > 0$ it is possible to compose the Akhmanov-Sukhorukov-Khokhlov solution with the transformation $F^{-1}$ and thus obtain a solution of the MNLS modulation equations that remains restricted to the modulationally unstable sector of the phase space. The transformation $F^{-1}$ breaks the even/odd symmetry of the functions $\dot{\rho}$ and $\dot{u}$. Indeed, the initial conditions (33) both become even under $F^{-1}$:

$$
\rho(x, 0) = -\frac{1}{2 \alpha^2} \left( 1 - \sqrt{1 + 4 \alpha^2 A_o^2 \text{sech}^2(x)} \right), \quad u(x, 0) = \frac{1}{\alpha} \left( 1 - \sqrt{1 + 4 \alpha^2 A_o^2 \text{sech}^2(x)} \right).
$$

(34)

These have the following asymptotics in $\alpha$:

$$
\rho(x, 0) = A_o \alpha^{-1} \text{sech}(x) + O(\alpha^2), \quad u(x, 0) = -2 \alpha A_o \text{sech}^2(x) + O(\alpha^3), \quad \text{as } \alpha \to 0 \quad (35)
$$

and

$$
\rho(x, 0) = A_o \alpha^{-1} \text{sech}(x) + O(\alpha^{-3}), \quad u(x, 0) = -2 A_o \text{sech}(x) + O(\alpha^{-2}), \quad \text{as } \alpha \to \infty, \quad (36)
$$

which shows that the MNLS pulse corresponding to the Akhmanov-Sukhorukov-Khokhlov solution has “fatter tails” when $\alpha$ is larger, but is of lower amplitude. The even symmetry of $\rho$ and $u$ at $t = 0$ is, however, irrelevant under evolution. To show the dynamics of the Akhmanov-Sukhorukov-Khokhlov solution composed with $F^{-1}$ for different $\alpha$ we have numerically inverted the implicit relations (32) and composed explicitly with $F^{-1}$, plotting the solution snapshots for six equally spaced times $t_k \in [0, (2A_o)^{-1}]$ for $A_o = 2$ and $\alpha = 1/2$ (Figure 2) and for $A_o = 2$ and $\alpha = 2$ (Figure 3). In both cases we see leftward-leaning pulses evolving toward a singularity at $t = (2A_o)^{-1} = 1/4$ and $x = 0$.

2.4.3. Connection with defocusing NLS in the semiclassical limit. Riemann invariants. Now suppose that $\alpha^2 \rho + \alpha u - 1 > 0$ and $\rho > 0$, defining an open domain $D_+(\alpha) \subset \mathbb{R}^2$ for each $\alpha > 0$, and consider the map $F_+ : (\rho, u) \in D_+(\alpha) \mapsto (\dot{\rho}, \dot{u}) \in \mathbb{R}^2$ defined by the formulae

$$
\dot{\rho} = \rho \cdot (\alpha^2 \rho + \alpha u - 1), \quad \dot{u} = u + 2 \alpha \rho.
$$

(37)

The range of the mapping $F_+$ is a strict subset of the upper half-plane:

$$
R_+ (\alpha) = \{ (\dot{\rho}, \dot{u}) \in \mathbb{R}^2 \text{ such that } 0 < \dot{\rho} \leq (\dot{u} - \alpha^{-1})/4 \}.
$$

(38)

into a gauge transformation relating the MNLS equation to the focusing NLS equation for general $\varepsilon$. However, in (22) (see also (35)) Hayashi has shown that if $\psi(\xi, \tau)$ is a solution of the derivative NLS equation

$$
i \varepsilon \frac{\partial \psi}{\partial \tau} + \frac{\varepsilon^2}{2} \frac{\partial^2 \psi}{\partial \xi^2} + i \alpha \varepsilon \frac{\partial}{\partial \xi} (|\psi|^2 \psi) = 0
$$

(which is itself equivalent to the MNLS equation via a Galilean boost with velocity $c = 1/\alpha$ as discussed in the introduction) then

$$
f := \sqrt{2 \alpha} \exp \left( \frac{2i \alpha}{\varepsilon} \int_{-\infty}^{\xi} |\psi(\xi', \tau)|^2 \, d\xi' \right) \psi \quad \text{and} \quad g := \sqrt{2 \alpha} \exp \left( \frac{2i \alpha}{\varepsilon} \int_{-\infty}^{\xi} |\psi(\xi', \tau)|^2 \, d\xi' \right) \left[ \frac{\varepsilon}{2} \frac{\partial \psi}{\partial \xi} + i \alpha |\psi|^2 \psi \right]
$$

satisfy the coupled system

$$
i \varepsilon \frac{\partial f}{\partial \tau} + \frac{\varepsilon^2}{2} \frac{\partial^2 f}{\partial \xi^2} - ig \bar{f} g^* = 0 \quad \text{and} \quad i \varepsilon \frac{\partial g}{\partial \tau} + \frac{\varepsilon^2}{2} \frac{\partial^2 g}{\partial \xi^2} + ig \bar{f} \cdot g^* = 0
$$

which reduces to the focusing NLS equation if $g = -ig$ and to the defocusing NLS equation if $g = if$. However neither of these reductions is consistent with the definition of $f$ and $g$ in terms of $\psi$. 

9
Unlike $F_-$, the map $F_+$ is not one-to-one. Each point in the interior of $R_+(\alpha)$ has exactly two preimages in $D_+(\alpha)$, one with $u < \alpha^{-1}$ and one with $u > \alpha^{-1}$. On the other hand, each point on the boundary curve $\hat{\rho} = (\hat{u} - \alpha^{-1})^2/4$ has exactly one preimage in $D_+(\alpha)$, satisfying $u = \alpha^{-1}$. See Figure 4.

Again, by direct calculation, the MNLS modulation equations for $\rho(x, t)$ and $u(x, t)$ are transformed under $F_+$ into the system

$$\frac{\partial}{\partial t} \begin{bmatrix} \hat{\rho} \\ \hat{u} \end{bmatrix} + \begin{bmatrix} \hat{u} & \hat{\rho} \\ 1 & \hat{u} \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \hat{\rho} \\ \hat{u} \end{bmatrix} = 0,$$

which upon comparison with (11) for $\kappa = -1$ we recognize as the modulation equations for the defocusing NLS equation. Once again, we are led to expect that in the semiclassical limit of $\varepsilon \downarrow 0$, the dynamics of the MNLS equation on the stable sector of its phase space should be equivalent (under the mapping $F_+$) to the dynamics of the defocusing NLS equation.

In principle, every hyperbolic quasilinear system with two dependent variables can be written in diagonal form through the introduction of new dependent variables called Riemann invariants. A suitable definition (see, for example, [26]) of Riemann invariants for the system (39) is

$$R_\pm := \frac{\hat{u}}{2} \pm \sqrt{\hat{\rho}},$$

with the same caveat as indicated in the previous footnote.
Figure 3. Snapshots of the Akhmanov-Sukhorukov-Khokhlov solution of the focusing NLS modulation equations interpreted via \( F_{-1} \) as a solution of the MNLS modulation equations for \( \alpha = 2 \). The color scheme is the same as in Figure 2.

where the positive square root is meant. Indeed, in terms of these variables, the system (39) takes diagonal form:

\[
\begin{align*}
\frac{\partial R_+}{\partial t} + \frac{1}{2} \left( 3R_+ + R_- \right) \frac{\partial R_+}{\partial x} &= 0, \\
\frac{\partial R_-}{\partial t} + \frac{1}{2} \left( R_+ + 3R_- \right) \frac{\partial R_-}{\partial x} &= 0.
\end{align*}
\] (41)

The quantities

\[ \sigma_\pm := \frac{1}{2} \left[ 3R_\pm + R_\mp \right] \] (42)

are the characteristic velocities, that is, the eigenvalues of the coefficient matrix of (39) written in terms of the Riemann invariants. The parabolae in the range \( R_+(\alpha) \) of \( F_+ \), as shown in the bottom diagram of Figure 4, are coordinate curves for the Riemann invariants; indeed, the relation \( \hat{\rho} = (\hat{u} - c)^2/4 \) corresponds to the ray \( R_+ = c/2, R_- < c/2 \) if \( u < c \) and the ray \( R_- = c/2, R_+ > c/2 \) if \( u > c \).

Diagonal form made possible by the introduction of Riemann invariants is useful for many purposes. For example, the way we discovered that the transformation \( F_+ \) transforms the MNLS modulation equations (19) on the modulationally stable sector into the defocusing NLS modulation equations (39) was by first writing (19) in diagonal form and subsequently comparing with the known diagonal form of (39). (Later we generalized this result to obtain the corresponding result that \( F_- \) links (19) on its modulationally unstable sector to (31) although as both are elliptic systems real Riemann invariants cannot exist as an intermediate step.) Another advantage of diagonal form is that one can easily see that there are nontrivial solutions for
which either $R_+$ or $R_-$ is identically constant. Such solutions are called simple waves and their dynamics are characterized by the scalar inviscid Burgers equation. Indeed, if $R_\pm = c/2$, then it is easy to show that $\sigma_\pm$ evaluated for $R_\pm \equiv c/2$ and $R_\mp$ arbitrary satisfies

$$\frac{\partial \sigma_\pm}{\partial t} + \sigma_\pm \frac{\partial \sigma_\pm}{\partial x} = 0,$$

(43)

whose general solution with arbitrary initial condition $\sigma_\pm(x,0) = f(x)$ can be written in implicit form as

$$\sigma_\pm = f(x - \sigma_\pm t).$$

(44)

As is well-known, if $f$ has critical points, then this solution only exists in the classical sense for a finite time due to the formation of shocks (derivative singularities). The number of shocks that can form is bounded above by the number of critical points of $f$ (whether or not more than one shock forms is a sensitive question of regularization that we do not address here; in particular we are ultimately interested in dispersive regularization where an early-occurring shock can “spread” into microscopic oscillations and wipe out other shocks that might occur in nonconservative regularizations). A final advantage of diagonal form that is useful in considering solutions more general than simple waves is that as long as a solution remains classical it is bounded in terms of its initial data by sharp inequalities of the form

$$\inf_{y \in \mathbb{R}} R_\pm(y,0) \leq R_\pm(x,t) \leq \sup_{y \in \mathbb{R}} R_\pm(y,0).$$

(45)

In other words, the smallest coordinate box containing the solution at a given $t \geq 0$ is in fact independent of $t$.

Perhaps with the help of Riemann invariants, the mapping $F_+$ and its double-valued inverse $F_+^{-1}$ can be used to obtain solutions of (19) from known solutions of the hyperbolic system (39), a problem that is

![Figure 4. Top: the domain of $F_+$ is shaded. Bottom: the range of $F_+$ is shaded.](image-url)
better understood than the mixed-type system [19], having received much attention in the literature (see, for example, [3]). As the mapping $F_+$ is not one-to-one, some care must be taken in using given initial data for [19] in its modulationaly stable sector to obtain corresponding initial data for [21] and then porting the solution thereof back to the domain $D_+(\alpha)$ via $F_+^{-1}$.

Consider, for example, smooth pulse-like initial data for [19] for which $\rho(x, 0) \to 0$ as $x \to \pm \infty$ and for which the modulational stability condition [21] holds for each $x \in \mathbb{R}$. If additionally $u(x, 0) > \alpha^{-1}$ holds for all $x \in \mathbb{R}$, then the image of this initial data in the $(\hat{\rho}, \hat{u})$ plane is a curve parametrized by $x \in \mathbb{R}$ that is attached to the $\hat{\rho} = 0$ axis at one or more points with $\hat{u} > \alpha^{-1}$ and that avoids the boundary curve $\hat{\rho} = (\hat{u} - \alpha^{-1})^2/4$ of the range $R_+$. As parabolic curves of the form $\hat{\rho} = (\hat{u} - c^2)/4$ for $c \geq \alpha^{-1}$ are coordinate curves of the Riemann invariants for [21], the evolution in time prior to shock formation will not alter this basic picture due to the inequalities [45]. Thus, although when applying the inverse mapping $F_+^{-1}$ pointwise in $x$ one has to make a choice between two distinct preimages, there are only two distinct pairs of continuous functions $(\rho(x, t), u(x, t))$ for each $t$ corresponding to the evolved initial data. Furthermore, exactly one of these will satisfy the boundary condition that $\rho(x, t) \to 0$ as $x \to \pm \infty$; the remaining solution of [19] will satisfy $\alpha^2 \rho(x, t) + \alpha u(x, t) - 1 \to 0$ as $x \to \pm \infty$ instead, and taking into account the initial condition this extraneous solution is not continuous in time $t$. On the other hand, if for some finite $x \in \mathbb{R}$ the initial data crosses the threshold into the subregion of the domain $D_+(\alpha)$ for which $\rho < \alpha^{-1}$, then even the image of the initial data as a curve in the $(\hat{\rho}, \hat{u})$-plane may have some unexpected properties. Indeed, a direct calculation shows that all curves in the $(\rho, u)$-plane passing through the vertical axis $u = \alpha^{-1}$ with slope $d\rho/du \neq -(2\alpha)^{-1}$ will have images in the $(\hat{\rho}, \hat{u})$-plane that are tangent to the boundary curve $\hat{\rho} = (\hat{u} - \alpha^{-1})^2/4$ while this is not the case for curves passing through with $d\rho/du = -(2\alpha)^{-1}$ (as holds for all of the coordinate parabolae). Therefore, if $d\rho/du = -(2\alpha)^{-1}$ for $u = \alpha^{-1}$ then the transformed initial data functions $\hat{\rho}(x, 0)$ and $\hat{u}(x, 0)$ will necessarily have a common critical point for some $x \in \mathbb{R}$. Moreover, for initial data crossing the $u = \alpha^{-1}$ threshold, there exist multiple preimage function pairs $(\rho(x, t), u(x, t))$ that are continuous in $x$ with the same boundary conditions as $x \to \pm \infty$. Care must be taken to select the correct preimage to have continuity for $t \geq 0$.

These facts suggest that while the modulationaly stable sector of the MNLS modulation equations [19] can be mapped to the defocusing NLS modulation equations [21] which we may consider to be a problem for which some intuition is available, new and important dynamical features of the MNLS modulation equations can be introduced simply through the change of variables $F_+$. For example, let $c > \alpha^{-1}$ and $A_0 > 0$ be fixed, and consider initial data for [19] of the form

$$\rho(x, 0) = A_0^2 \text{sech}^2(x), \quad u(x, 0) = c - 2A_0\sqrt{\alpha c - 1} \cdot \text{sech}(x).$$

(46)

The corresponding curve parametrized by $x \in \mathbb{R}$ lies along one of the coordinate parabolae in the $(\rho, u)$-plane with $u \leq c$ and $\rho \to 0$ as $x \to \pm \infty$. If $0 < A_0^2 < (c - \alpha^{-1})/(4\alpha)$, then the initial data satisfies $u > 1/\alpha$ for all $x$ and thus a single-peaked pulse is mapped to a single-peaked pulse as shown in Figure [3]. If $(c - \alpha^{-1})/(4\alpha) < A_0^2 < (c - \alpha^{-1})/\alpha$, then the initial data crosses the threshold of $u = \alpha^{-1}$ but does not yet reach the point of tangency between the coordinate parabola and the stability boundary $\alpha^2 \rho + \alpha u - 1 = 0$ (which is also mapped to $\hat{\rho} = 0$ along with $\rho = 0$). In this situation, the image in the $(\hat{\rho}, \hat{u})$-plane of the part of the initial data with $u < \alpha^{-1}$ is the same as that of part of the initial data with $u > \alpha^{-1}$, and this results in a double-peaked image pulse as shown in Figure [3]. Finally, if $A_0^2$ exceeds $(c - \alpha^{-1})/\alpha$, then the part of the coordinate parabola in the $(\hat{\rho}, \hat{u})$-plane that is retracted is the entire left-hand branch for $\hat{u} < c$, and part of the right-hand branch (along which $\hat{\rho}$ can become arbitrarily large) is also traced out in the center of the pulse. Therefore, in this situation the single-pulse initial data for the MNLS modulation equations [19] is mapped by $F_+$ to triple-pulse initial data for the defocusing NLS modulation equations [21], as shown in Figure [4]. Since the coordinate curves correspond to constant values of one or the other of the Riemann invariants, the dynamics of all of these initial conditions under the defocusing NLS modulation equations [21] will be given by simple waves. It follows that the evolution is governed by the scalar inviscid Burgers equation, and it may thus be shown that the number of shocks that can form corresponds to the number of critical points of the simple-wave initial data. Thus, simply by varying the amplitude of a single-peaked pulse-like initial condition for the MNLS modulation equations [19], one can produce dynamics that lead to either one, two, or three (dispersively regularized) shocks.

4Of course this extraneous solution may be meaningful if different boundary conditions are imposed at $x = \pm \infty$. 13
A relation similar to the equivalence between the MNLS modulation equations [19] and those for the focusing or defocusing NLS equations has been observed by Kuvshinov and Lakhin [30]. Working with the equivalent derivative NLS equation, Kuvshinov and Lakhin derive the modulation equations (Whitham equations) corresponding not to complex exponential plane waves (genus zero) but rather to periodic waves given by elliptic functions (genus one). By comparing the Riemann invariants for these Whitham equations with the corresponding well-known formulae for the NLS genus one Whitham equations they deduce complete equivalence between these two systems of modulation equations. No doubt this phenomenon extends to genera greater than one, and a deeper explanation of these coincidences surely exists at the level of the algebro-geometric description of arbitrary genus multiphase wave solutions of the NLS and derivative NLS equations. We will treat this problem in this framework in a future publication.
Figure 7. The initial data and its image under $F_+$ for $\alpha = 1$ and $c = 6$ with $A_o^2 = 80/8$.

3. Bounds on the Discrete Spectrum

In this section we establish some basic conditions on the discrete spectrum associated to the spectral problem (176) for the MNLS equation. As seen in the appendix, for each $k$ with $\text{Im}(k^2) \neq 0$ there are two one-dimensional subspaces of solutions that decay to zero as $x \to \pm \infty$ respectively. We refer to eigenvalues as values of $k$ for which these two subspaces coincide and eigenfunctions as the corresponding solutions $v$ of (176) associated to each eigenvalue that decay to zero in both directions.

Eigenvalues are exactly the zeros of the function $S_{11}(k)$ in the region $\text{Im}(k^2) < 0$ and the zeros of the function $S_{22}(k)$ in the region $\text{Im}(k^2) > 0$, where $S_{11}(k)$ and $S_{22}(k)$ are defined by (184). The integrability of the MNLS equation leads to the fact that eigenvalues do not vary in time (see (251)). Consequently, we consider the spectral problem at $t = 0$ and set $\phi$ as the initial condition for the Cauchy problem for the MNLS equation (1). Additionally, we note that due to the symmetries (230) and (231), the eigenvalues have a four-fold symmetry, where if any of $k$, $k^*$, $-k^*$ or $-k$ is an eigenvalue the others are as well. This symmetry allows us to restrict our attention to (say) the first quadrant of the complex $k$-plane. Any result established for $k$ in the first quadrant therefore immediately gives analogous results for $k$ in any other quadrant.

We will be considering the linear differential equation (176) with a potential of the form $\phi(x) = A(x)e^{iS(x)/\varepsilon}$ where $A(\cdot)$ and $S(\cdot)$ are real-valued functions such that $A > 0$ and $A$ and $S''$ decay rapidly to zero for large $|x|$. At first blush, one difficulty that presents itself is that there are rapid oscillations in the coefficients of (176) when $\varepsilon$ is small. This difficulty is, however, circumvented by the following device. If $v$ is a vector solution of (176), then $w := e^{-iS(x)/2\varepsilon\sigma^3}v$ satisfies:

$$2\alpha\varepsilon \frac{dw}{dx} = iMw,$$

(47)

where

$$M := \begin{bmatrix} -4k^2 + 1 - \alpha S'(x) & 4\alpha k A(x) \\ 4\alpha k A(x) & 4k^2 - 1 + \alpha S'(x) \end{bmatrix}. \quad (48)$$

Since we have only scaled $v$ by exponential factors with purely imaginary exponents, $w$ decays as $x \to \pm \infty$ if and only if $v$ does, that is, the discrete spectrum of (176) is the same as that of (176).

A formal approach to (177) when $\varepsilon$ is small is apply the WKB method; that is, to assume a solution $w$ of the form $w = e^{i\sigma/(2\varepsilon\sigma)}(w_0 + \varepsilon w_1 + \cdots)$. Substituting this expansion for $w$ into (177) and equating terms of the same powers in $\varepsilon$ gives a hierarchy of equations relating $\sigma$ and $\{w_k\}$. The leading order (eikonal)

5This is inexact usage, as the spectral parameter $k$ appears nonlinearly in the system (176). The concept is, however, completely analogous to the usual situation.
equation in $\varepsilon$ is:

$$\textbf{M} w_o = \frac{d\sigma}{dx} w_o,$$

and, consequently, $d\sigma/dx$ must be an eigenvalue of $\textbf{M}$. This leads us to consider the eigenvalues of $\textbf{M}$ as smooth functions of $x$.

A simple calculation shows that the eigenvalues of $\textbf{M}$ are $\pm \omega$ where

$$\omega(x; k) := \left[16\alpha^2 k^2 A(x)^2 + (4k^2 - 1 + \alpha S'(x))^2\right]^{1/2}.$$

In WKB theory, the so-called turning points are those values of $x$ for which the eigenvalues coincide: $d\sigma/dx = \pm \omega = 0$. Near such points, WKB theory breaks down as infinite derivatives appear and approximate solutions become multiply-defined. Setting $\omega(x; k) = 0$, we see that for a given $k \in \mathbb{C}$ the turning points $x \in \mathbb{R}$ are the solutions of the equation

$$4i\alpha k A(x) = \pm (4k^2 - 1 + \alpha S'(x)).$$

On the other hand, as $x \in \mathbb{R}$ varies, the equation (52) defines a parametrized curve $\Gamma$ in the four quadrants of the complex $k$-plane that we will call the turning point curve associated with the functions $A(\cdot)$ and $S'(\cdot)$. By definition, $k \in \mathbb{T}$ if and only if $\text{Im}\{k^2\} \neq 0$ and there exists at least one real turning point $x$. As (51) is quadratic in $k$, we may solve for $k$ and therefore explicitly present $\mathbb{T}$ as a parametrized curve:

$$k(x) = \frac{i}{2} \left(s_1 \alpha A(x) + s_2 \sqrt{\alpha^2 A(x)^2 + \alpha S'(x) - 1}\right),$$

where $s_1$ and $s_2$ are independent signs ($s_j = \pm 1$) yielding up to four branches of $\mathbb{T}$.

At this juncture we may make the observation that if $x$ is a point at which the hyperbolicity (modulational stability) condition (21) holds (recall that $\rho := A^2$ and $u := S'$), or in the degenerate case if $\alpha^2\rho + \alpha u - 1 = 0$, then the four $k$-values produced by the formula (52) are purely imaginary and by definition do not lie in $\mathbb{T}$. On the other hand, if $x$ is such that $\alpha^2\rho + \alpha u - 1 < 0$, then (52) gives four distinct points $k \in \mathbb{T}$:

$$\text{Im}\{k\} = s_1 \frac{\alpha}{2} A(x),$$

$$\text{Re}\{k\} = s_2 \frac{1}{2} \sqrt{1 - \alpha S'(x) - \alpha^2 A(x)^2}.$$

The equations (53) when restricted to the values of $x$ for which the strict inequality $\alpha^2\rho + \alpha u - 1 < 0$ holds ($x$-values of modulational instability) thus give a parametric representation of the turning point curve $\mathbb{T}$. In particular, if the nonstrict modulational stability condition $\alpha^2 A(x)^2 + \alpha S'(x) - 1 \geq 0$ holds for all $x \in \mathbb{R}$, then by definition the turning point curve is empty: $\mathbb{T} = \emptyset$.

Throughout this section, we will restrict our attention entirely to those values of $k \in \mathbb{C}$ with $\text{Im}\{k^2\} \neq 0$ that do not lie on the turning point curve. For such $k$, the function $\omega(x; k)$ is well-defined by (50) if we insist on continuity for all $x \in \mathbb{R}$ and impose the boundary condition that

$$\lim_{x \to \pm \infty} \omega(x; k) = 4k^2 - 1 + \alpha S'_+,$$

where $S'_+ := \lim_{x \to +\infty} S'(x)$. Suppose that $A(\cdot) > 0$ is of class $L^1(\mathbb{R})$ and is uniformly Lipschitz. In particular, this implies that $A(x)$ decays to zero as $x \to \pm \infty$. Suppose also that $S'(\cdot)$ is uniformly Lipschitz and that $S''(\cdot)$ is of class $L^1(\mathbb{R})$ (assuring the existence of limiting values of $S'(x)$ as $x \to \pm \infty$). The function $\omega(x; k)$ defined as described above then

\[\text{if } A(\cdot) > 0 \text{ is integrable on } \mathbb{R}, \text{ and if } \lim_{x \to \pm \infty} A(x) = M > 0 \text{ (we admit the possibility that } M = +\infty), \text{ then is infinite. Begin by choosing } \delta > 0 \text{ to be arbitrarily small. Let } \{x_n\} \text{ be a sequence of real numbers tending to } +\infty \text{ such that } \lim_{n \to \infty} A(x_n) = M > 0. \text{ By passing to a subsequence if necessary, we may assume that } x_{n+1} \geq x_n + \delta \text{ for all } n. \text{ We first claim that } I_n := \inf_{x_n < x < x_n + \delta} A(x) \text{ tends to zero as } x \to +\infty. \text{ Indeed, were this not the case there would exist some } \varepsilon > 0 \text{ and some } n_0 \text{ such that } I_n \geq \varepsilon \text{ holds for all } n \geq n_0. \text{ On the other hand we would then have } \int_{x_n + \delta}^{x_{n+1}} A(x) dx \geq \varepsilon \delta, \text{ for } n \geq n_0, \text{ and on the other} \]

$$\sum_{n=n_0}^{\infty} \int_{x_n}^{x_{n+\delta}} A(x) dx \leq \int_{-\infty}^{+\infty} A(x) dx < \infty,$$

which yields a contradiction. Since $A$ is continuous, there exists for each $n$ a number $y_n \in [x_n, x_n + \delta]$ such that $A(y_n) = I_n$. As $I_n$ tends to zero as $n \to \infty$, for all $n$ sufficiently large we will then have both $A(x_n) \geq 2M/3$ (or $A(x_n) \geq 2$ if $M = +\infty$).
has the following properties whenever $\text{Im}\{k^2\} \neq 0$ and $k \not\in \mathcal{T}$: $\omega(x; k)$ and its $x$-derivative $\omega'(x; k)$ are uniformly bounded, and $\omega(x; k)$ is uniformly bounded away from zero. Indeed, given the assumed properties of $A(\cdot)$ and $S'(\cdot)$ it suffices to show that $\omega(x; k)$ is bounded away from zero. But this is so since for $k \not\in \mathcal{T}$ we have $\omega(x; k) \neq 0$ for all $x \in \mathbb{R}$, while the boundary condition [54] and the limit
\[
\lim_{x \to -\infty} \omega(x; k) = \pm (4k^2 - 1 + \alpha S''),
\]
where $S'' := \lim_{x \to -\infty} S'(x)$, (55)
when combined with the condition that $\text{Im}\{k^2\} \neq 0$ shows that $\omega(x; k)$ has nonzero limiting values as $x \to \pm \infty$.

Given $\omega(x; k)$ defined as above, we set
\[
q(x; k) := \frac{2\alpha k A(x)}{\omega(x; k)} \frac{d}{dx} \log \left( \frac{A(x)}{\omega(x; k) + 4k^2 - 1 + 2 \alpha S'(x)} \right),
\]
Supposing the same conditions on $A(\cdot)$ and $S'(\cdot)$ as above, the function $q(x; k)$ is uniformly bounded for $x \in \mathbb{R}$ whenever $\text{Im}\{k^2\} \neq 0$ and $k \not\in \mathcal{T}$ and $\alpha > 0$. Indeed, writing $q(x; k)$ in the equivalent form
\[
q(x; k) = \frac{2\alpha k}{\omega(x; k)} \left( A'(x) - A(x) \frac{\omega'(x; k) + \alpha S''(x)}{\omega(x; k) + 4k^2 - 1 + 2 \alpha S'(x)} \right),
\]
our attention falls on the denominator in the second term: $D := \omega(x; k) + 4k^2 - 1 + 2 \alpha S'(x)$. This denominator cannot vanish for any $x \in \mathbb{R}$ because this would imply that $\omega(x; k) = (4k^2 - 1 + 2 \alpha S'(x))^2$ which by [54] is in contradiction with the assumptions that $\text{Im}\{k^2\} \neq 0$, $A(x) > 0$, and $\alpha > 0$. According to the boundary condition [54], $D$ also has a nonzero limit as $x \to \pm \infty$ for $\text{Im}\{k^2\} \neq 0$. A similar situation holds as $x \to -\infty$ as long as the limiting value of $\omega(x; k)$ is achieved with the “+” sign in [55]. It follows that $q(x; k)$ is bounded for large $|x|$, at least if the limit [55] holds with the “+” sign. If [55] holds with the “−” sign, then we must expand $D$ as $x \to -\infty$:
\[
D = (4k^2 - 1 + 2 \alpha S'(x)) \cdot \left( 1 - \sqrt{1 + \frac{16\alpha^2 k^2 A(x)^2}{(4k^2 - 1 + 2 \alpha S'(x))^2}} \right),
\]
whence it follows for $\text{Im}\{k^2\} \neq 0$ and $\alpha > 0$ that
\[
\frac{D'}{D} = O \left( \frac{1}{A(x)} \right), \quad \text{as} \quad x \to -\infty,
\]
so that in this limit $q(x; k)$ is again bounded for large negative $x$.

Finally, in terms of $\omega(x; k)$ and $q(x; k)$ we define, for $\text{Im}\{k^2\} \neq 0$ and $k \not\in \mathcal{T}$, the quantity
\[
L_k = \sup_{x \in \mathbb{R}} \left| \frac{d}{dx} \left( \frac{1}{\text{Im}\{\omega(x; k)\}} \right) \right| + 2 \sup_{x \in \mathbb{R}} \left| \frac{\text{Re}\{q(x; k)\}}{\text{Im}\{\omega(x; k)\}} \right|.
\]
It is possible to have $L_k = +\infty$.

3.1. The fundamental condition on eigenvalues. Our main result is the following.

**Theorem 1.** Let $A : \mathbb{R} \to \mathbb{R}_+$ be a uniformly Lipschitz function of class $L^1(\mathbb{R})$ and let $S' : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitz with $S''(\cdot)$ of class $L^1(\mathbb{R})$. Let $k$ be a fixed complex number with $\text{Im}\{k^2\} \neq 0$ and $k \not\in \mathcal{T}$. If $\phi(x) := A(x)e^{iS(x)/\varepsilon}$ is the potential in the linear differential equation [176], then the following statements hold:

(a) If $k$ is an eigenvalue, then
\[
|\text{Im}\{k\}| \leq \frac{\alpha}{2} \sup_{x \in \mathbb{R}} A(x).
\]

(b) If $L_k < (\alpha \varepsilon)^{-1}$ then $k$ is not an eigenvalue.

and $A(y_n) \leq M/3$ (or $A(y_n) \leq 1$ if $M = +\infty$) while $|x_n - y_n| \leq \delta$. Therefore, for all such $n$,
\[
\frac{|A(x_n) - A(y_n)|}{|x_n - y_n|} \geq \frac{C_M}{\delta},
\]
where $C_M$ is a finite positive number depending only on $M$. As $\delta$ was arbitrarily small, this shows that the Lipschitz constant of $A$ is arbitrarily large, and hence infinite.
Our proof of this theorem was inspired by some notes [12] we obtained from Percy Deift regarding his proof with Stephanos Venakides and Xin Zhou of the corresponding "shadow bound" estimates for eigenvalues of the nonselfadjoint Zakharov-Shabat problem. (We will explain more about their result, and show how it can be deduced from ours as a special case, at the end of this section.)

Proof. To prove part (a), suppose that $k$ is an eigenvalue and $w$ is the corresponding eigenfunction. Using (47), we calculate

$$2 \alpha \varepsilon \left( w^\dagger \sigma_3 \frac{dw}{dx} \right) = -i \left( 4k^2 - 1 + \alpha S'(x) \right) |w|^2 - 8\alpha k A(x) \text{Im}\{w_1^* w_2\}, \quad (62)$$

where $|w|^2 := |w_1|^2 + |w_2|^2$, and we adopt the notation that will be used throughout the remainder of the paper that $w_1$ and $w_2$ are the first and second components, respectively, of the vector $w$. As shown in the appendix (see (198) and (217)), the condition that $A(\cdot) \in L^1(\mathbb{R})$ guarantees that the eigenfunction $w \neq 0$ decays exponentially to the zero vector as $x \to \pm \infty$. Integration by parts then establishes the following identity:

$$\int_{-\infty}^{+\infty} w^\dagger \sigma_3 \frac{dw}{dx} dx = \int_{-\infty}^{+\infty} [w_1^* w_1 - w_2^* w_2] dx = -\left( \int_{-\infty}^{+\infty} [w_1^* w_1 - w_2^* w_2] dx \right)^*, \quad (63)$$

so that all three expressions are purely imaginary. Therefore, integration of (62) and taking the real part yields

$$0 = \text{Im}\{k\} \text{Re}\{k\} \int_{-\infty}^{+\infty} |w|^2 dx - \alpha \text{Re}\{k\} \int_{-\infty}^{+\infty} A(x) \text{Im}\{w_1^* w_2\} dx. \quad (64)$$

Since $\text{Re}\{k\} \neq 0$, (64) gives:

$$\text{Im}\{k\} = \frac{\alpha \int_{-\infty}^{+\infty} A(x) \text{Im}\{w_1^* w_2\} dx}{\int_{-\infty}^{+\infty} |w|^2 dx}. \quad (65)$$

Applying the inequality $2|\text{Im}\{w_1^* w_2\}| \leq 2|w_1||w_2| \leq |w_1|^2 + |w_2|^2 = |w|^2$ we obtain:

$$|\text{Im}\{k\}| \leq \frac{\alpha}{2} \sup_{x \in \mathbb{R}} A(x), \quad (66)$$

which proves part (a) of Theorem 1.

To prove part (b), suppose that $k$ is an eigenvalue of (70), or equivalently, of (47). We will now suppose that $L_k < (\alpha \varepsilon)^{-1}$ and derive a contradiction.

Let $T(x; k)$ be a $2 \times 2$ matrix with eigenvectors of $M$ corresponding to (distinct) eigenvalues $\omega(x; k)$ and $-\omega(x; k)$ as its first and second columns respectively. Note that $T(x; k)$ is invertible for all $x \in \mathbb{R}$ because $k$ does not lie on the turning point curve $T$. Make the gauge transformation $p = T(x; k)^{-1}w$, where by means of (47), $p$ satisfies

$$2 \alpha \varepsilon \frac{dp}{dx} = -2 \alpha \varepsilon T(x; k)^{-1} \frac{dT}{dx}(x; k)p + i\omega(x; k)\sigma_3 p. \quad (67)$$

A general eigenvector matrix is only determined up to multiplication on the right by a diagonal matrix, and we fix the gauge so that $T(x; k)^{-1} \cdot T'(x; k)$ is an off-diagonal skew-symmetric complex matrix depending on $A(x)$, $S'(x)$ and $k$. Thus, with the particular eigenvector matrix

$$T(x; k) = \begin{bmatrix} 4\alpha k A(x) & -H(x; k) \\ H(x; k) & 2\alpha k A(x) \end{bmatrix} \quad (68)$$

where $H(x; k)$ is a continuous function of $x$ satisfying

$$H(x; k)^2 = 2\omega(x; k) \left( \omega(x; k) + 4k^2 - 1 + \alpha S'(x) \right), \quad (69)$$

and $\omega(x; k)$ is defined by

$$\frac{d\omega(x; k)}{dx} = \frac{H(x; k)}{\omega(x; k)} H(x; k), \quad (71)$$

and $\alpha \varepsilon \frac{dp}{dx}$ is a real-valued matrix.

If $\text{Re}\{k\} \neq 0$, then $T(x; k)$ is a continuous matrix of matrices $T(x; k)^{-1} \cdot T'(x; k)$ is a continuous matrix. Hence, it is possible to choose a gauge transformation $p = T(x; k)^{-1}w$ such that for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$

$$\frac{dp}{dx} = \frac{d}{dx} T(x; k)^{-1} \frac{dT}{dx}(x; k)p + i\omega(x; k)\sigma_3 p.$$

Let $\text{Re}\{k\} \neq 0$, then $T(x; k)$ is a continuous matrix of matrices $T(x; k)^{-1} \cdot T'(x; k)$ is a continuous matrix. Hence, it is possible to choose a gauge transformation $p = T(x; k)^{-1}w$ such that for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$

$$2 \alpha \varepsilon \frac{dp}{dx} = -2 \alpha \varepsilon T(x; k)^{-1} \frac{dT}{dx}(x; k)p + i\omega(x; k)\sigma_3 p.$$  \quad (67)
Lemma 1. For $k \neq 0$ for all $\varepsilon > 0$.

Proof. For any complex number $x$ to find:

\[ \frac{dT}{dx}(x; k) = iq(x; k)\sigma_2, \quad (70) \]

where $q(x; k)$ is defined by (56). Moreover, $\det(T(x; k)) = 1$, and, by the same arguments as were used to deduce the boundedness of $q(x; k)$, the elements of $T(x; k)$ are uniformly bounded in $x$ (and hence so are those of $T(x; k)^{-1}$). Therefore, (67) becomes

\[ 2\alpha\varepsilon \frac{dP}{dx} = i\omega(x; k)\sigma_3P - 2i\alpha\varepsilon q(x; k)\sigma_2P. \quad (71) \]

Next we differentiate $p^\dagger\sigma_3p$ with the help of (71) and the identities $\sigma_3^2 = \mathbb{1}$, $\sigma_2\sigma_3 = i\sigma_1$, and $\sigma_3\sigma_2 = -i\sigma_1$ to find:

\[ 2\alpha\varepsilon \frac{d}{dx} (p^\dagger\sigma_3p) = -2i\text{Im}\{\omega(x; k)\} |p|^2 - 4\alpha\varepsilon \text{Re}\{q(x; k)\} p^\dagger\sigma_1p. \quad (72) \]

Since $T(x; k)^{-1}$ is uniformly bounded in $x$, $p$ decays to zero for large $|x|$. Dividing through in (72) by $\text{Im}\{\omega(x; k)\}$ and integrating gives

\[ \int_{-\infty}^{+\infty} |p|^2 \, dx = -\int_{-\infty}^{+\infty} \frac{\alpha\varepsilon}{2\text{Im}\{\omega(x; k)\}} \frac{d}{dx} (p^\dagger\sigma_3p) \, dx - \int_{-\infty}^{+\infty} \frac{2\alpha\varepsilon}{\text{Im}\{\omega(x; k)\}} p^\dagger\sigma_1p \, dx \]

where the second equality is due to integration by parts and the decay of $p$. Since

\[ |p^\dagger\sigma_3p| = |p_1|^2 - |p_2|^2 \leq |p|^2 \quad \text{and} \quad |p^\dagger\sigma_1p| = |p_2|^2 + |p_1|^2 \leq 2|p_1||p_2| \leq |p|^2, \quad (74) \]

we therefore obtain

\[ \int_{-\infty}^{+\infty} |p|^2 \, dx \leq \alpha\varepsilon L_k \int_{-\infty}^{+\infty} |p|^2 \, dx, \quad (75) \]

where $L_k$ is the constant, independent of $\varepsilon$, defined in (66). Cancellation non-zero $L^2(\mathbb{R})$-norm of $p$, we arrive at a contradiction. Therefore $k$ cannot be an eigenvalue. This establishes part (b) and completes the proof of Theorem 1. \qed

3.2. The shadow region. For $A(\cdot)$ and $S(\cdot)$ satisfying the conditions of Theorem 1 we have seen that $\omega(x; k)$ and $q(x; k)$ are uniformly bounded functions of $x \in \mathbb{R}$ whenever $k$ is a complex number with $\text{Im}\{k^2\} \neq 0$ and $k \not\in T$. It follows easily that under these conditions, for each such complex number $k$ there exists a constant $d_k > 0$, independent of $\varepsilon$, such that

\[ L_k \leq d_k \sup_{x \in \mathbb{R}} |\text{Im}\{\omega(x; k)\}|^{-2}. \quad (76) \]

It is then of interest to ask for which values of $k$ can one conclude that $\text{Im}\{\omega(x; k)\}$ is bounded away from zero for all $x \in \mathbb{R}$, because Theorem 1 implies that such a value of $k$ cannot be an eigenvalue for any value of $\varepsilon > 0$ that is sufficiently small. Note that $\text{Im}\{k^2\} \neq 0$ implies that $|\text{Im}\{\omega(x; k)\}| \to 4|\text{Im}\{k^2\}| \neq 0$ as $x \to \pm\infty$. Consequently, it is enough to show that $\text{Im}\{\omega(x; k)\} \neq 0$ for all $x \in \mathbb{R}$. We will now derive a simple geometric condition in the complex $k$-plane that implies that $\text{Im}\{\omega(x; k)\}$ is nonzero for all real $x$.

For $C \in \mathbb{R}$, let $\Sigma_C$ denote the hyperbola in the complex $k$-plane defined by the relation

\[ \Sigma_C: \{k : \text{Re}\{k^2\} + C = \text{Re}\{k\}^2 - \text{Im}\{k\}^2 + C = 0\}. \quad (77) \]

Lemma 1. Given a fixed complex value $k_o$ in the open first quadrant of the complex plane, define the value $C_o := -(\text{Re}\{k_o\}^2 - \text{Im}\{k_o\}^2)$. Let $I_o$ be the set of $x$ values that correspond to intersection points of the turning point curve $T$ and $\Sigma_{C_o}$. If $\text{Im}\{k_o\} - \alpha A(x) > 0$ for all $x \in I_o$, then $\text{Im}\{\omega(x; k_o)\} \neq 0$ for all $x \in \mathbb{R}$.

Proof. For any complex number $z$, the condition $\text{Im}\{z\} \neq 0$ can be rewritten as a condition on $z^2$:

\[ \text{Im}\{z\} \neq 0 \iff \begin{cases} \text{Im}\{z^2\} \neq 0 \\ \text{Im}\{z^2\} = 0 \text{ and } \text{Re}\{z^2\} < 0 \end{cases}. \quad (78) \]

\[ 19 \quad 19 \]
From (50) we find that \( \text{Im}\{\omega(x; k_o)\}^2 \) is given by
\[
\text{Im}\{\omega(x; k_o)\}^2 = 16 \text{Im}\{k_o\} \text{Re}\{k_o\} \left(2\alpha^2 A(x)^2 + \alpha S'(x) - 1 - 4C_o\right).
\] (79)

Since \( k_o \) is in the open first quadrant, \( \text{Re}\{k_o\} \text{Im}\{k_o\} > 0 \). Let \( J_o \) denote the set of \( x \in \mathbb{R} \) such that \( \text{Im}\{\omega(x; k_o)\}^2 = 0 \).

**Case 1:** Let \( x \in \mathbb{R} \setminus J_o \). Then by (78) we clearly have \( \text{Im}\{\omega(x; k_o)\} \neq 0 \).

**Case 2:** Let \( x \in J_o \neq \emptyset \). Consider the hyperbola \( \Sigma_{C_o} \) defined in (77). Each intersection point \( k \) of \( \Sigma_{C_o} \) with the turning point curve \( T \) corresponds to a real value of \( x \) for which \( \alpha^2 A(x)^2 + \alpha S'(x) - 1 < 0 \) according to the natural parametrization of the latter curve, given by (53). By eliminating \( \text{Re}\{k\} \) and \( \text{Im}\{k\} \) between the two equations of (53) and the definition (77) of the hyperbola \( \Sigma_{C_o} \), it is easy to verify that each real \( x \) corresponding to such an intersection point \( k \) is an element of \( J_o \). There may however be some \( x \in J_o \) that do not correspond to intersection points of \( \Sigma_{C_o} \) with the turning point curve \( T \). These are values of \( x \) for which \( k = k(x) \) defined by (92) lies on the hyperbola \( \Sigma_{C_o} \) but is a purely imaginary number because \( \alpha^2 A(x)^2 + \alpha S'(x) - 1 \geq 0 \), and hence is not in \( T \) by definition. Since \( \text{Im}\{k_o^2\} \neq 0 \) but \( \text{Im}\{\omega(x; k_o)\}^2 = 0 \), (79) shows that such points \( x \) satisfy the completely equivalent inequality \( \alpha^2 A(x)^2 - 4C_o \leq 0 \). Therefore, we define \( I_o \) to be the subset of \( J_o \) consisting of \( x \in \mathbb{R} \) such that \( 2\alpha^2 A(x)^2 + \alpha S'(x) - 1 - 4C_o = 0 \) and \( \alpha^2 A(x)^2 - 4C_o > 0 \). In other words \( I_o \) consists of all \( x \in J_o \) such that \( x \) also corresponds to a point of intersection of the turning point curve \( T \) and the hyperbola \( \Sigma_{C_o} \). It is also clear, both directly from the inequality \( \alpha^2 A(x)^2 - 4C_o > 0 \) and also from the fact that hyperbolas \( \Sigma_C \) with \( C < 0 \) contain no purely imaginary points \( k \), that \( C_o \leq 0 \) implies \( I_o = J_o \).

Let \( x \) be a fixed element of \( J_o \) and consider \( \text{Re}\{\omega(x; k)^2\} \) restricted to \( k \)-values lying on the hyperbola \( \Sigma_{C_o} \). This restriction makes \( \text{Re}\{\omega(x; k)^2\} \) a function of \( \text{Im}\{k\}^2 \):
\[
\left. \text{Re}\{\omega(x; k)^2\} \right|_{k \in \Sigma_{C_o}} = -64 \text{Im}\{k\}^4 + 64C_o \text{Im}\{k\}^2 - 4\alpha^2 A(x)^2(4C_o - \alpha^2 A(x)^2)
\]
\[
= -64 \left(\text{Im}\{k\}^2 - r_1\right) \left(\text{Im}\{k\}^2 - r_2\right),
\] (80)
where
\[
r_1 := \frac{1}{4}\alpha^2 A(x)^2, \quad \text{and} \quad r_2 := \frac{1}{4}(4C_o - \alpha^2 A(x)^2).
\] (81)

As a function of \( \text{Im}\{k\}^2 \), the right-hand side of (80) is a concave-down quadratic. Case 2 now breaks into two subcases depending on whether or not \( x \in J_o \) satisfies \( x \in I_o \).

**Subcase 2(a):** Let \( x \in J_o \setminus I_o \). Assume that \( \text{Re}\{\omega(x; k)^2\} \geq 0 \). Since obviously \( k_o \) lies on the hyperbola \( \Sigma_{C_o} \), it follows from (81) that \( \text{Im}\{k_o\}^2 \) must lie in the closed interval whose endpoints are the two roots \( r_1 \) and \( r_2 \), and in particular
\[
\text{Im}\{k_o\}^2 \leq \max(r_1, r_2).
\] (82)
Eliminating \( \text{Im}\{k_o\}^2 \) using the definition of \( C_o \), we therefore find
\[
\text{Re}\{k_o\}^2 \leq \max(r_1 - C_o, r_2 - C_o).
\] (83)

Now, \( r_2 - C_o < 0 \) by definition. The assumption that \( x \notin I_o \) however implies further that \( r_1 - C_o \leq 0 \). Therefore, \( \text{Re}\{k_o\} = 0 \) which contradicts the assumption that \( k_o \) lies in the open first quadrant. Consequently, for all \( x \in J_o \setminus I_o \), \( \text{Re}\{\omega(x; k)^2\} < 0 \) and therefore \( \text{Im}\{\omega(x; k_o)\} \neq 0 \) by (78).

**Subcase 2(b):** Let \( x \in I_o \). It is always the case that \( r_1 > 0 \) and for \( x \in I_o \) we also have \( r_2 < 0 \), by definition. Since (81) is quadratic in \( \text{Im}\{k\}^2 \) and concave down, and since \( k_o \in \Sigma_{C_o} \), the inequality \( \text{Im}\{k_o\}^2 > r_1 \) implies that \( \text{Re}\{\omega(x; k)^2\} < 0 \) for this value of \( x \). Since \( x \in I_o \subset J_o \), it follows from (78) that (for \( k_o \) in the open first quadrant) \( 2\text{Im}\{k_o\} - \alpha A(x) > 0 \) implies that \( \text{Im}\{\omega(x; k_o)\} \neq 0 \).

We have therefore shown that if \( x \in \mathbb{R} \setminus I_o \), then \( \text{Im}\{\omega(x; k_o)\} \neq 0 \), while (according to subcase 2(b) above) if \( x \in I_o \), we have \( \text{Im}\{\omega(x; k_o)\} \neq 0 \) as long as \( 2\text{Im}\{k_o\} - \alpha A(x) > 0 \), which completes the proof of the Lemma.  \( \square \)
We might remark that Lemma 1 combines with the estimate (76) and part (b) of Theorem 1 to give a weaker (ε-dependent) version of part (a) of Theorem 1 because if $2|\text{Im}\{k_o\}| > \alpha \sup_{x \in \mathbb{R}} A(x)$ then the inequality $2|\text{Im}\{k_o\}| > \alpha A(x)$ holds in particular for all $x \in I_o$.

Now define
\[
C_{\min} := \min_{x \in \mathbb{R}} \{2\alpha^2 A(x)^2 + \alpha S'(x) - 1\}, \quad C_{\max} := \max_{x \in \mathbb{R}} \{2\alpha^2 A(x)^2 + \alpha S'(x) - 1\}.
\]

**Corollary 1.** Suppose $A(\cdot)$ and $S'(\cdot)$ satisfy the conditions of Theorem 1 and let $k_o$ be a fixed complex number such that $\text{Im}\{k_o^2\} \neq 0$. If $C_o < C_{\min}$ or $C_o > C_{\max}$ then $k_o$ is not an eigenvalue for any $\varepsilon > 0$ sufficiently small.

**Proof.** By means of (79) it is clear that if $x \in J_o$, then $2\alpha^2 A(x)^2 + \alpha S'(x) - 1 = 4C_o$. Therefore, either of the inequalities $C_o < C_{\min}$ or $C_o > C_{\max}$ implies that $J_o$ is empty and thus (by Case 1 in the proof of Lemma 1) we have $\text{Im}\{\omega(x; k_o)\} \neq 0$ for all $x \in \mathbb{R}$.

Note that $\Sigma_{C_{\min}}$ and $\Sigma_{C_{\max}}$ are the osculating hyperbolas of the image of $k(x)$ defined by (62), in turn is the union of the turning point curve $T$ and (possibly) an interval $I$ of the imaginary axis. This gives Corollary 1 geometrical meaning since the condition $C_{\min} \leq C \leq C_{\max}$ is equivalent to the statement that $\Sigma_C \cap (T \cup I) \neq \emptyset$.

A refinement of Corollary 1 is an immediate consequence of Lemma 1.

**Corollary 2.** Suppose $A(\cdot)$ and $S'(\cdot)$ satisfy the conditions of Theorem 1 and let $k_o$ be a fixed complex number such that $\text{Im}\{k_o^2\} \neq 0$. If $C_{\min} \leq C_o \leq C_{\max}$ but $\Sigma_{C_o} \cap T = \emptyset$, then $k_o$ is not an eigenvalue for any $\varepsilon > 0$ sufficiently small.

**Proof.** The set $I_o$ is empty, so the proof follows from the estimate (70) and Lemma 1.

We may now formulate a more precise geometric condition to exclude the existence of eigenvalues. For each $C \in \mathbb{R}$ with $\Sigma_C \cap T \neq \emptyset$ we define a number $U_C$ as follows:
\[
U_C := \sup_{k \in \Sigma_C \cap T} \text{Im}\{k\}^2.
\]

Then set
\[
S_C := \{k \in \Sigma_C \text{ such that } \text{Im}\{k\}^2 \leq U_C\},
\]
and
\[
S := \bigcup_{C \in [C_{\min}, C_{\max}] \backslash \Sigma_C \cap T \neq \emptyset} S_C.
\]

**Corollary 3.** Assume that $A(\cdot)$ and $S'(\cdot)$ satisfy the conditions of Theorem 1. Suppose that $k_o \in C$ and $\text{Im}\{k_o^2\} \neq 0$. If also $k_o \notin S$, then $k_o$ is not an eigenvalue for any $\varepsilon > 0$ sufficiently small.

**Proof.** First note that, by Corollaries 1 and 2 it is sufficient to consider those $k_o \notin S$ for which $C_{\min} \leq C_o \leq C_{\max}$ and also $\Sigma_{C_o} \cap T \neq \emptyset$. Now, clearly $k_o$ lies on the hyperbola $\Sigma_{C_o}$. However, since $S_{C_o} \subset S$, we also have that $k_o \notin S_{C_o}$, so
\[
\text{Im}\{k_o\}^2 > U_{C_o}.
\]

By the definition of the set $I_o$ and the definition (53) of the turning point curve $T$, we see from (83) that
\[
U_{C_o} = \sup_{x \in I_o} \frac{\alpha^2 A(x)^2}{4}.
\]

Lemma 1 (and its obvious consequences in the remaining quadrants of the complex $k$-plane) then implies that $k_o$ cannot be an eigenvalue as long as $\varepsilon > 0$ is sufficiently small.

At this point we may observe an important consequence of the hyperbolicity condition (21).

**Corollary 4.** Suppose that $A(\cdot)$ and $S(\cdot)$ satisfy the conditions of Theorem 1 and also that $\alpha^2 A(x)^2 + \alpha S'(x) - 1 \geq 0$ holds for all $x \in \mathbb{R}$. The latter condition makes the modulation equations (19) (nonstrictly) hyperbolic for all $x \in \mathbb{R}$ at $t = 0$. If $k_o$ is any fixed complex number with $\text{Im}\{k_o^2\} \neq 0$, then $k_o$ is not an eigenvalue for any $\varepsilon > 0$ sufficiently small.
Proof. In this situation, the turning point curve $T$ is empty, and hence so is the set $S$, so this statement is a direct consequence of Corollary 3.

The set $S$ is the “hyperbolic shadow” of the turning point curve $T$. One can imagine light being projected along the hyperbolae $\Sigma_C$ from infinity, and the set $S$ is the shadow cast by the turning point curve in this light. Let $N$ be any open neighborhood containing the axes $\text{Im}\{k^2\} = 0$ and the shadow $S$. Then from Corollary 3 and part (a) of Theorem 1 the set $F := N \cap \{k \text{ such that } 2|\text{Im}\{k\}| \leq \alpha \sup_{x \in \mathbb{R}} A(x)\}$ contains all of the eigenvalues if $\varepsilon > 0$ is small enough. In particular, the set $N$ may be taken to be the union of $S$ with a thin “collar” surrounding the boundary of $S$ and the axes. If one wants to contain the discrete spectrum for all $0 < \varepsilon \leq \varepsilon_{\text{max}}$, then the collar must be made thicker for larger $\varepsilon_{\text{max}}$. The region $F$ is perhaps best understood visually. Figure 8 contains two heuristic plots of $F$ in relation to the turning point curve and the family of hyperbolae $\Sigma_C$.

The need for a “collar” arises from our methods of proof: we found an upper bound for $L_k$ (see (76)) and argued that to ensure that the eigenvalue exclusion inequality $L_k < (\alpha \varepsilon)^{-1}$ holds for $\varepsilon$ small enough we simply checked for finiteness of the upper bound. The bound (76) involved a constant $d_k$ that depended on $k$ in an unspecified way. With better accounting of the $k$-dependence of upper bounds for $L_k$ it is possible to obtain a sharper condition on eigenvalue exclusion that explicitly depends on $\varepsilon$; in other words, one may estimate the width of the “collar”. As a simple example of such a calculation, we present the following result.

**Corollary 5.** Assume the same conditions on $A(\cdot)$ and $S'(\cdot)$ as in Theorem 4. There exist constants $K_\alpha > 0$ and $C_\alpha > 0$ independent of $\varepsilon$, such that whenever $|k| \geq K_\alpha$ and

$$|\text{Im}\{k\}| \geq \frac{C_\alpha \varepsilon^{1/2}}{|\text{Re}\{k\}|},$$

(90)

then $k$ is not an eigenvalue.

This Corollary says that the width of the collar about the real $k$-axis scales like $\varepsilon^{1/2}$ when $|k|$ is large. Moreover, it shows that the collar becomes thinner when $|\text{Re}\{k\}|$ is larger.

**Proof.** Another upper bound for $L_k$ is clearly

$$L_k \leq \sup_{x \in \mathbb{R}} |\omega(x; k)| \cdot \sup_{x \in \mathbb{R}} \frac{1}{|\text{Im}\{\omega(x; k)\}|^2} + 2 \sup_{x \in \mathbb{R}} |q(x; k)| \cdot \sup_{x \in \mathbb{R}} \frac{1}{|\text{Im}\{\omega(x; k)\}|}.$$  

(91)

\textbf{Figure 8.} “Light” is projected along the hyperbolae and casts the shadow on the turning point curve. The shadow region is shaded green. The collar is shaded grey and the red horizontal lines correspond to the threshold in part (a) of Theorem 1.
When \( k \) is large in magnitude, we have the following asymptotics holding uniformly with respect to \( x \in \mathbb{R} \):
\[
\begin{align*}
\omega(x; k) &= 4k^2 + O(1), \\
\omega'(x; k) &= \frac{1}{2\omega(x; k)} \left( 32\alpha^2k^2A(x)A'(x) + 2(4k^2 - 1 + \alpha S'(x))\alpha S''(x) \right) = O(1), \\
q(x; k) &= O(|k|^{-1}).
\end{align*}
\] (92)

From the above it obviously follows that \( \text{Im}\{\omega(x; k)\} = 4\text{Im}\{k^2\} + O(1) \) uniformly in \( x \) for large \( k \). When \( \text{Im}\{k^2\} \) is small, this estimate is not satisfactory, so to improve it we note that \( \sqrt{1 + x + iy} = f + ig \) with
\[
\begin{align*}
f &= \sqrt{\frac{1}{2} \left( (1 + x)^2 + y^2 + (1 + x) \right)} \\
g &= \text{sgn}(y) \sqrt{\frac{1}{2} \left( (1 + x)^2 + y^2 - (1 + x) \right)}
\end{align*}
\] (93)

where \( x \) and \( y \) are real and sufficiently small, and all square roots are positive. It follows that as \( z = x + iy \) tends to zero,
\[
\sqrt{1 + z} = 1 + O(|z|) + iO(y)
\] (94)

where all order symbols denote real-valued functions. Applying this result to
\[
\omega(x; k) = 4k^2 \sqrt{1 + (2\alpha^2A(x)^2 + \alpha S'(x) - 1)\frac{1}{2k^2} + (1 - \alpha S'(x))^2} \cdot \frac{1}{16k^4},
\] (95)

and using the facts that
\[
\text{Im}\{k^{-2}\} = -\frac{\text{Im}\{k^2\}}{|k|^4} \quad \text{and} \quad \text{Im}\{k^{-4}\} = -\frac{2\text{Re}\{k^2\}\text{Im}\{k^2\}}{|k|^8},
\] (96)

we see that
\[
\text{Im}\{\omega(x; k)\} = 4\text{Im}\{k^2\} \cdot \left( 1 + O(|k|^{-2}) \right)
\] (97)

holds uniformly for \( x \in \mathbb{R} \). We therefore have shown that
\[
L_k = O \left( \frac{1}{\text{Im}\{k^2\}^2} \right) + O \left( \frac{1}{|k| \cdot \text{Im}\{k^2\}} \right) = O \left( \frac{1}{\text{Im}\{k^2\}^2} \right) + O \left( \frac{\text{Im}\{k^2\}}{|k|} \cdot \frac{1}{\text{Im}\{k^2\}^2} \right).
\] (98)

Since part (a) of Theorem 1 allows us to restrict attention to a strip in which \( \text{Im}\{k\} \) is uniformly bounded, it then follows that
\[
L_k = O \left( \frac{1}{\text{Im}\{k^2\}^2} \right)
\] (99)
as \( |k| \) tends to infinity in this strip. The proof is then complete upon recalling part (b) of Theorem 1. \( \square \)

What do shadows \( S \) look like for functions \( A(\cdot) \) and \( S(\cdot) \) that might arise in applications? Figures 9, 12 each contain plots of turning point curves \( T \) (blue), several of the hyperbola \( \Sigma_C \) for various values of \( C \) (black), the bounds on \( \text{Im}\{k\} \) from from part (a) of Theorem 1 (red), and the shadow region \( S \) (green). The figures are all for the potentials \( \phi(x) = A(x)e^{iS(x)/\varepsilon} \) with \( A(x) = \text{sech}(x) \) and \( S'(x) = \text{sech}(x) \tan h(x) \). (This example was also considered in the context of the Zakharov-Shabat spectral problem in [3] and [32].) The four figures illustrate the dependence of the shadow region \( S \) on the parameter \( \alpha \). (For clarity, these figures do not illustrate the \( \varepsilon \)-dependent “collar”.)

Figures 11 and 12 each consist of two plots of the same thing; one displayed in the \( k \)-plane and one displayed in the \( \lambda \)-plane, where \( \lambda \) and \( k \) are related by (1.79) as explained in the appendix. When \( \alpha \) is small, as in these figures, one expects features of the spectrum when viewed in the \( \lambda \)-plane to resemble known features of the spectrum for the nonselfadjoint Zakharov-Shabat spectral problem, which arises by a formal limit of \( \alpha \to 0 \). The hyperbolae \( \Sigma_C \) tend to vertical lines in this limit, and part of the turning point curve tends to a fixed curve in the \( \lambda \)-plane. Indeed, substituting for \( k \) in terms of \( \lambda \) from (1.80) in the relations...
defining the turning point curve $T$, one may select the branch of $T$ for which $\text{Re}(k) \approx 1/2$ for $\alpha$ small and pass to the limit of $\alpha \to 0$ with $\lambda$ and $x$ held fixed, to obtain the limiting curve

\begin{align}
\text{Im}\{\lambda\} &= \pm A(x), \\
\text{Re}\{\lambda\} &= -\frac{1}{2}S'(x).
\end{align}
Figure 12. The turning point curve $T$ and its hyperbolic shadow $S$ for $\alpha = 0.0894$. Left: the $k$-plane. Right: the $\lambda$-plane.

We refer to this limiting curve in the complex $\lambda$-plane as $T_0$. Deift, Venakides, and Zhou\(^7\) have established results analogous to ours in the context of the nonselfadjoint Zakharov-Shabat eigenvalue problem; in these results the curve $T_0$ plays the role of the $\alpha$-dependent curve $T$ and the family of vertical lines $\text{Re} \{\lambda\} = C$ plays the role of the family of hyperbolae $\Sigma_C$. Thus we see that the known “shadow bound” result for the Zakharov-Shabat problem can be deduced from our results as a limiting special case.

The shadow bound for the Zakharov-Shabat eigenvalue problem implies that if $S''(\cdot) \equiv 0$ (so that $S'(\cdot) \equiv C_o$ for some constant $C_o$), then the eigenvalues lie very close to the real axis or the vertical segment with endpoints $-C_o/2 \pm iA_{\text{max}}, A_{\text{max}} \coloneqq \sup_{x \in \mathbb{R}} A(x)$, in the complex $\lambda$-plane. Klaus and Shaw\(^8\) have proved that if in addition $A(\cdot)$ is a real function with a single critical point (necessarily a local maximum) then the eigenvalues of the nonselfadjoint Zakharov-Shabat spectral problem are purely imaginary numbers regardless of the value of $\varepsilon > 0$. Such a result represents a much more precise confinement of the eigenvalues than is afforded by the shadow bound with its attendant $\varepsilon$-dependent “collar”. It is an open question whether there exists such a result in the context of the MNLS spectral problem\(^{176}\). A reasonable guess might be to draw an analogy between the vertical “rays” making the shadow in the Zakharov-Shabat problem with the hyperbolic “rays” making the shadow in the MNLS spectral problem and thus suppose that if the functions $A(\cdot)$ and $S'(\cdot)$ are such that the turning point curve $T$ is contained within a single hyperbola $\Sigma_C$ for some $C < 0$:

$$2\alpha^2 A(x)^2 + \alpha S'(x) - 1 = 4C < 0, \quad \forall x \in \mathbb{R} \tag{101}$$

(this is the MNLS analogue of the condition $S'(\cdot) \equiv C_o$ in the Zakharov-Shabat case\(^1\), which makes the corresponding turning point curve $T_0$ collapse onto the imaginary axis of the $\lambda$-plane), and if an appropriate analogue of the Klaus-Shaw critical point condition is satisfied, then the eigenvalues all lie exactly for all $\varepsilon > 0$ on the hyperbola $\Sigma_C$. Noting that the Klaus-Shaw critical point condition implies that the limiting turning point curve $T_0 \subset i\mathbb{R}$ is traced out according to its natural parametrization exactly twice as $x$ increases from $-\infty$ to $+\infty$, we might further conjecture that the analogue of this condition for the MNLS problem is that $T \subset \Sigma_C$ is covered exactly twice as $x$ varies. But, from (53) we see that this occurs for $A(\cdot)$ and $S'(\cdot)$ satisfying (101) exactly when the amplitude function $A(\cdot)$ is positive and has a single critical point, necessarily a local maximum; that is, we expect that the same exact monotonicity condition will be required of $A(\cdot)$ as in the Zakharov-Shabat case to obtain exact spectral confinement to a curve, but that the relation $S'(\cdot) \equiv C_o$ must be replaced by (101).

\(^7\)This is an unpublished result, see \[6\], \[32\]

\(^8\)In fact, upon setting $C = (\alpha C_o - 1)/4$ in (101) and taking the limit $\alpha \to 0$ one obtains exactly the relation $S'(\cdot) \equiv C_o$. 

25
4. A Family of Special Initial Data

In this section, we investigate in detail the spectral problem corresponding to the MNLS equation (see (176) in the appendix) for initial data of the form \( \phi(x, 0) = A(x)e^{iS(x)/\varepsilon} \) where we assume that the amplitude and phase functions are given by

\[
A(x) = \nu \sech(x) \quad \text{and} \quad S(x) = S_0 + \int_0^x [\delta + \mu \tanh(x)] \, dx, \quad (102)
\]

where \( \nu > 0, \delta, \mu, \) and \( S_0 \) are arbitrary real parameters. Following the algorithm described in the appendix, our aim is to calculate all of the scattering data for this family. This requires solving the linear system (176), which we will do by showing that for this special initial data it is essentially a hypergeometric equation. Our analysis will be valid for all values of the data parameters \( \nu, \delta, \mu, \) and \( S_0 \) and for all values of \( \alpha \geq 0 \). More significantly, our analysis will also be valid for all \( \varepsilon > 0 \) which will allow us to carry out a complete semiclassical analysis of the inverse (Riemann-Hilbert) problem in a subsequent publication.

In connection with this latter application, the family of special initial data under consideration here is particularly interesting because it will turn out that for certain choices of parameters in the family and certain values of \( \alpha \) we can be in any of the cases where the condition (21) is satisfied for all \( x \in \mathbb{R} \), is not satisfied for any \( x \in \mathbb{R} \), or satisfied for some \( x \) and not others. Thus we have the possibility of providing a completely rigorous asymptotic explanation of the effect of the stability threshold on the semiclassical dynamics of the MNLS equation.

The idea of seeking potentials \( \phi \) for which the spectral problem (176) might be hypergeometric is an old idea for Schrödinger operators. This technique was applied to the nonselfadjoint Zakharov-Shabat operator (upon which the integrable theory of the focusing NLS equation is based) first by Satsuma and Yajima [37] who considered the case \( \phi(x, 0) = \nu \sech(x) \), and subsequently by Tovbis and Venakides [38] who generalized the technique to potentials of the general form we consider here. Our work in this section shows that potentials of Tovbis-Venakides form are hypergeometric also for the more complicated spectral problem (176) corresponding to the MNLS equation. Moreover, our results generalize those of Tovbis and Venakides in the sense that we recover their formulae by taking a suitable limit corresponding to \( \alpha \to 0 \).

4.1. Reduction to hypergeometric form. Making the substitution \( w = e^{-iS(x)/\varepsilon}v \) in the linear system (176) where \( \phi = A(x)e^{iS(x)/\varepsilon} \) has the effect of removing the oscillatory terms from the coefficient matrix. Thus, with the amplitude and phase functions \( A \) and \( S \) given by (102), we arrive at the following equation satisfied by \( w \):

\[
2\varepsilon \frac{dw}{dx} = \begin{bmatrix} 2\Omega - i\mu \tanh(x) & 4i\kappa \nu \sech(x) \\ 4i\kappa \nu \sech(x) & -2\Omega + i\mu \tanh(x) \end{bmatrix} w, \quad (103)
\]

where

\[
\Omega := \Lambda - \frac{i\delta}{2} = \frac{1}{2\varepsilon \alpha}(4k^2 + \alpha \delta - 1). \quad (104)
\]

The key observation we make, in direct analogy with [37] and [38], is that the change of independent variable \( y = \tanh(x) \) together with the substitution

\[
a := \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1 - y^2} \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \sech(x) \quad (105)
\]

reduces (103) to a linear first-order system with rational (in the new independent variable \( y \)) coefficients.

Indeed, \( a \) satisfies

\[
2\varepsilon(1 - y^2) \frac{da}{dy} = \begin{bmatrix} 2\Omega - i\mu y & 4i\kappa \nu \\ 4i\kappa \nu (1 - y^2) & -2\Omega + (i\mu - 2\varepsilon)y \end{bmatrix} a. \quad (106)
\]

This 2 × 2 system has exactly three singular points, \( y = \pm 1 \) and \( y = \infty \). Moreover, all three are regular singular points, which essentially makes (106) a hypergeometric differential equation [9]. To make this clearer, we follow some steps to convert (106) to a more standard form.

\footnotetext{Tovbis and Venakides actually assumed that \( \delta = 0 \) but elementary symmetries of the Zakharov-Shabat problem corresponding to the Galilean invariance of the focusing NLS equation imply that their results are easily transformed to completely handle the case of \( \delta \neq 0 \) as well.
The method of Frobenius applies near each of the (regular singular) points $y = \pm 1$. That is, for $y$ in a neighborhood of $y = \pm 1$ there is a solution $a(y)$ that has a convergent expansion of the form

$$a = (1 \mp y)^{\rho \pm} \sum_{n=0}^{\infty} (1 \mp y)^n a_{n\pm},$$

(107)

where $a_{n\pm}$ are vector-valued coefficients. Substituting this series into (106) and gathering the coefficients of like powers of $(1 \pm y)$ leads to a hierarchy of algebraic equations relating the coefficients $a_{n\pm}$ and the exponent $\rho \pm$. The equation arising at leading order is

$$\mp 4\varepsilon \rho \pm a_{0\pm} = \begin{bmatrix} 2\Omega + i\mu & 4ik\nu \\ 0 & -2\Omega \mp (i\mu - 2\varepsilon) \end{bmatrix} a_{0\pm}.$$  

(108)

In order for there to exist nontrivial series solutions (subsequent coefficients vanish for generic exponents if $a_{0\pm} = 0$) it is therefore necessary that $\mp 4\varepsilon \rho \pm$ agree with one of the eigenvalues of the matrix on the right-hand side. This is the \textit{indicial equation} of Frobenius theory. The exponents therefore are

$$\rho \pm = \frac{i\mu}{4\varepsilon} \mp \frac{\Omega}{2\varepsilon} \quad \text{or} \quad \rho \pm = \frac{1}{2} - \left( \frac{i\mu}{4\varepsilon} \mp \frac{\Omega}{2\varepsilon} \right).$$

(109)

The case where $k^2 \in \mathbb{R}$ will be of particular interest shortly, and for such $k$ the two exponents never differ by an integer, which means that a full basis of solutions near each of the two singular points $y = \pm 1$ may be obtained in the form of convergent Frobenius series using both exponents in each case.

The purpose of calculating the Frobenius exponents is to show that by making the substitution

$$b = (1 + y) \frac{(-2\Omega + i\mu)/(4\varepsilon)(1 - y)(2\Omega - i\mu)/(4\varepsilon)a}{(1 - y^2)} \quad \text{or} \quad b = (1 - y) \frac{(-2\Omega - i\mu)/(4\varepsilon)(1 - y)(2\Omega + i\mu)/(4\varepsilon)a}{(1 - y^2)},$$

(110)

the differential equation (106) is converted into another having the same three singular points, all regular singular, but with the property that near each of the points $y = \pm 1$ one of the Frobenius exponents has been shifted to zero. Thus the equation for $b$ equivalent to that for $a$ via (110),

$$\varepsilon (1 - y^2) \frac{db}{dy} = \begin{bmatrix} 0 & 2ik\nu \\ 2ik\nu(1 - y^2) & -2\Omega + (i\mu - \varepsilon)y \end{bmatrix} b,$$

(111)

is guaranteed to have one solution that is analytic at $y = -1$ and another that is analytic near $y = +1$. The zero that has appeared in the coefficient matrix makes it easy to eliminate one of the components and arrive at a second-order differential equation; but the process is made even easier by first making the substitution

$$c = \begin{bmatrix} 1 & 0 \\ 0 & (1 - y^2)^{-1} \end{bmatrix} b$$

(112)

leading to the the coupled system

$$\varepsilon \frac{dc_1}{dy} = 2ik\nu c_2 \quad \text{and} \quad \varepsilon (1 - y^2) \frac{dc_2}{dy} = 2ik\nu c_1 + [(i\mu + \varepsilon)y - 2\Omega] c_2.$$

(113)

At this point it is trivial to eliminate $c_2$ and obtain a second-order differential equation for $c_1$ alone. The resulting equation for $c_1$ is a standard hypergeometric equation [9] in the independent variable $z = (1 + y)/2$ with parameters

$$A = \frac{1}{\varepsilon} \left( \frac{i\mu}{2} + \sqrt{16k^2\nu^2 - \mu^2} \right),$$

$$B = \frac{1}{\varepsilon} \left( \frac{i\mu}{2} - \sqrt{16k^2\nu^2 - \mu^2} \right),$$

$$C = \frac{1}{\varepsilon} \left( \Omega + \frac{i\mu}{2} \right) + \frac{1}{2}.$$  

(114)

The standard hypergeometric function $F(A, B; C; z)$ is therefore one solution for $c_1$. 

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4.2. Euler transforms and integral representations of Jost solutions. From the point of view of the scattering theory outlined in the appendix, it is necessary to obtain appropriate bases of solutions of the equation (117), equivalent here to the hypergeometric problem (113) and use them to compute certain Wronskian determinants as functions of $k$. In principle, all of this information is available in the vast literature on special functions (of which [3] and [9] are but two sources). Rather than simply quoting the results we need, we proceed directly by deriving integral representations for the required solutions and using these to calculate the scattering data. This approach has the advantage that we will have the integral representations available for subsequent perturbative analysis of the spectral problem (176) in the semiclassical limit based on Langer transformations. See [7] for an example of this kind of analysis.

Integral representations for the Jost solutions may be obtained with the help of Euler transforms [25]. Suppose $c_1$ and $c_2$ have the integral representations:

$$
  c_1(y) = \int_\Sigma C_1(t)(t-y)^{\beta_1} \, dt \quad \text{and} \quad c_2(y) = \int_\Sigma C_2(t)(t-y)^{\beta_2} \, dt,
$$

where the contour $\Sigma$, the exponents $\beta_j \in \mathbb{C}$ and the functions $C_j(t)$ (the Euler transforms of $c_j(\cdot)$) are to be chosen so that these expressions satisfy (113). Assuming that the contour $\Sigma$ is independent of $y$ and noting that therefore

$$
  \frac{dc_1}{dy} = -\beta_1 \int_\Sigma C_1(t)(t-y)^{\beta_1-1} \, dt,
$$

the first equation of (113) is satisfied by choosing

$$
  \beta_1 := \beta_2 + 1 \quad \text{and} \quad C_1(t) := -\frac{2i k \nu}{(\beta_2 + 1) \varepsilon} C_2(t).
$$

For convenience, we write $C(t) = C_2(t)$ and $\beta = \beta_2$. With these choices, the second equation of (113) becomes

$$
  -\varepsilon \beta (1 - y^2) \int_\Sigma C(t)(t-y)^{\beta-1} \, dt = \frac{4 k^2 \nu^2}{(\beta + 1) \varepsilon} \int_\Sigma C(t)(t-y)^{\beta+1} \, dt
$$

$$
  + [(i \mu + \varepsilon) y - 2 \Omega] \int_\Sigma C(t)(t-y)^\beta \, dt. \quad (118)
$$

The aim in satisfying an equation such as this is to equate integrands rather than integrals. As a first step, we may of course move the polynomial-in-$y$ coefficients inside of the integrals; in doing this we write $y$ in the form $y = t - (t - y)$ so that we may continue to view the integrands as functions of the variables $t$ and $t - y$. Thus we obtain

$$
  \int_\Sigma \varepsilon \beta (1 - t^2) C(t)(t-y)^{\beta-1} \, dt + \int_\Sigma [(2 \varepsilon \beta + \varepsilon + i \mu) t - 2 \Omega] C(t)(t-y)^\beta \, dt
$$

$$
  = \int_\Sigma \left[ \varepsilon \beta + \varepsilon + i \mu - \frac{4 k^2 \nu^2}{(\beta + 1) \varepsilon} \right] C(t)(t-y)^{\beta+1} \, dt. \quad (119)
$$

The next step is to use integration by parts to get all of the exponents of $t - y$ to be the same. Then comparison of the integrands will lead to a linear differential equation for $C(t)$, the Euler transform of that for $c_2(y)$. Since there are three consecutive exponents, two integrations by parts will be required in general to achieve this, and we will arrive at a second-order equation for $C(t)$. This would not be advantageous, as we began (essentially) with such an equation. We want to reduce the order.

The point is that by choosing $\beta$ appropriately we may eliminate one of the three consecutive exponents in (119), a step that will lead to a first-order equation for $C(t)$. Indeed, the quantity in square brackets on the right-hand side of (119) is a constant depending on $\beta$, and it vanishes if we choose one of the two values

$$
  \beta = -1 + \frac{1}{2 \varepsilon} \left( -i \mu \pm i \sqrt{\mu^2 - 16 k^2 \nu^2} \right). \quad (120)
$$

With either of these two choices for $\beta$ (as long as $\beta \neq -1$, and we will discuss this exceptional case once we make a proper definition (151) of $\beta$ below), the right-hand side of (119) vanishes. Integrating by parts in
the first integral on the left-hand side of (119), we therefore find that
\[ \int_{\Sigma} \left[ -\varepsilon \frac{d}{dt} [(1 - t^2)C(t)] + [(2\varepsilon \beta + \varepsilon + i\mu)t - 2\Omega] C(t) \right] (t - y)^\gamma dt = 0, \] (121)
assuming the contour \( \Sigma \) is chosen so that the boundary terms vanish (we will check this later). We solve this equation by equating the integrand to zero, yielding the first-order equation for \( C(t) \):
\[ \varepsilon \frac{d}{dt} [(1 - t^2)C(t)] = [(2\varepsilon \beta + \varepsilon + i\mu)t - 2\Omega] C(t). \] (122)
The Euler transform \( C(t) \) is therefore (up to a multiplicative constant) a branch of the multi-valued function
\[ C(t) = (1 - t)^{\gamma^+}(1 + t)^{\gamma^-}, \quad \text{where} \quad \gamma_{\pm} = -\frac{1}{2\varepsilon}(2\varepsilon \beta + 3\varepsilon + i\mu + 2\Omega). \] (123)

We now suppose that \( k^2 \in \mathbb{R} \) and \( \mu^2 - 16k^2\beta^2 > 0 \), which puts \( k \) either on the imaginary axis or in the real interval \((-|\mu|/(4\nu), |\mu|/(4\nu))\). We arbitrarily make the concrete choice of the \(+\) sign in (120) (and interpret the square root there as being positive). The task at hand is to choose the contour \( \Sigma \) so that we obtain integral representations for the Jost solution \( j^{(1)}_+(x;k) \) defined by the chain of transformations \( j^{(1)}_+ = v \mapsto w \mapsto a \mapsto b \mapsto c \) and the boundary condition
\[ \lim_{x \to +\infty} j^{(1)}_+(x;k)e^{-Ax/\varepsilon} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \] (124)
We therefore attempt to represent the components of \( j^{(1)}_+(x;k) = (J_{11+}(x;k), J_{21+}(x;k))^T \) in the form
\[ J_{11+}(x;k) = -\frac{2ik\nu e^{iS_0/(2\varepsilon)}}{(\beta + 1)\varepsilon} e^{Ax/\varepsilon} \int_{\Sigma} C(t)(t - y)^{\beta^+1} dt, \]
\[ J_{21+}(x;k) = e^{-iS(x)/\varepsilon} \text{sech}(x) e^{iS_0/(2\varepsilon)} e^{Ax/\varepsilon} \int_{\Sigma} C(t)(t - y)^{\beta} dt. \] (125)
We make these formulæe concrete in two steps. First we select the branch of \( C(t) \) as follows:
\[ C(t) = C_0(t - 1)^{\gamma^+}(t + 1)^{\gamma^-} \] (126)
where \( C_0 \) is a constant to be determined, and where the principal branch of the power functions is intended; that is, \(-\pi < \arg(t \pm 1) < \pi\). Similarly we choose \(-\pi < \arg(t - y) < \pi\). Next, we select the contour \( \Sigma \) illustrated in Figure 13. Our assumptions on \( k \) imply that Re\{\( \beta \)\} = -1 and Re\{\( \gamma_{\pm} \)\} = -1/2. The integrals in (125) are therefore both convergent at \( t = \pm 1 \) for \(-1 < y < 1\). Moreover, we have
\[ (1 - t^2)C(t)(t - y)^{\beta} = O \left( (t \mp 1)^{1/2} \right), \quad \text{as} \quad t \to \pm 1, \] (127)
so the neglect of boundary terms in the integration by parts argument is indeed justified after the fact in this situation.

To choose the constant \( C_0 \) and verify the boundary condition (124) we need to calculate the behavior of the integrals appearing in (125) in the limit \( y \to 1 \), corresponding to \( x \to +\infty \). It is easy to see that both of these integrals are in fact analytic in \( y = 1 \), and a simple dominated convergence argument shows that
\[ \lim_{y \to 1} \int_{\Sigma} (t - 1)^{\gamma^+}(t + 1)^{\gamma^-}(t - y)^{\beta+j} dt = \int_{\Sigma} (t - 1)^{\gamma^+\beta+j}(t + 1)^{\gamma^-} dt, \quad j = 0, 1. \] (128)
(The factor \( (t - y)^{\beta+j} \) is uniformly bounded on \( \Sigma \) independently of \( y \) near \( y = 1 \), and the remaining factor \( (t - 1)^{\gamma^+}(t + 1)^{\gamma^-} \) is in \( L^1(\Sigma) \), so indeed dominated convergence applies.) Again recalling Re\{\( \gamma_{\pm} \)\} = -1/2 we see that these are both finite under our assumptions on \( k \). This confirms the boundary condition (124) up to the determination of the constant \( C_0 \). To get this constant, note that these limiting integrals are essentially beta integrals, which is to say that they are ratios of gamma functions. In the case \( j = 1 \) the
limiting integrand is integrable at \( t = 1 \) as well as \( t = -1 \). Thus the contour \( \Sigma \) may be collapsed to opposite sides of the branch cut giving

\[
\int_{\Sigma} (t - 1)^{\gamma + \beta + 1} (t + 1)^{\gamma - 1} dt = -2i \sin(\pi(\gamma_+ + \beta + 1)) \int_{-1}^{1} (1 - t)^{\gamma + \beta + 1} (t + 1)^{\gamma - 1} dt
\]

\[
= -2^{3+\gamma_++\gamma_-+\beta} i \sin(\pi(\gamma_+ + \beta + 1)) \int_{0}^{1} u^{\gamma - (1 - u)^{\gamma + \beta + 1}} du
\]

\[
= -2^{3+\gamma_++\gamma_-+\beta} i \sin(\pi(\gamma_+ + \beta + 1)) \frac{\Gamma(\gamma_+ + 1)\Gamma(\gamma_+ + \beta + 2)}{\Gamma(\gamma_+ + \gamma_- + \beta + 3)}
\]

\[
= 2^{3+\gamma_++\gamma_-+\beta} \pi i \frac{\Gamma(\gamma_- + 1)}{\Gamma(-1 - \gamma_+ - \beta)\Gamma(\gamma_+ + \gamma_- + \beta + 3)},
\]

where we have used the “beta-gamma” identity

\[
\int_{0}^{1} u^{A-1}(1 - u)^{B-1} du = \frac{\Gamma(A)\Gamma(B)}{\Gamma(A + B)}, \quad \text{Re}\{A\} > 0, \quad \text{Re}\{B\} > 0
\]

(130)

and Euler’s reflection formula

\[
\sin(\pi z)\Gamma(z)\Gamma(1 - z) = \pi
\]

(see [3]). According to this calculation, the boundary condition (124) requires that we choose

\[
C_0 = \frac{(\beta + 1)\varepsilon e^{-iS_0/(2\varepsilon)}}{2^{4+\gamma_++\gamma_-+\beta} \pi k \nu} \frac{\Gamma(\gamma_+ + 1)\Gamma(\gamma_+ + \beta + 2)\Gamma(-1 - \gamma_+ - \beta)}{\Gamma(\gamma_- + 1)},
\]

(132)

and with this choice \( j_+^{(1)}(x; k) \) is determined uniquely.

This completes our construction of the Jost solution \( j_+^{(1)}(x; k) \). The three remaining Jost solutions may also be found explicitly by following similar arguments, but we will be able to calculate all of the scattering data from our knowledge of \( j_+^{(1)}(x; k) \) alone. In the case of the specific class of potentials \( \phi = \phi(x; \nu, S_0, \mu, \varepsilon) \) under consideration here it is easy to confirm that for each fixed \( k \in \mathbb{C} \), whenever \( v(x;k) \) satisfies the

**Figure 13.** The contour \( \Sigma \) in the complex \( t \)-plane. The branch cuts of the integrand all lie on the wavy half-line.
differential equation (176) with \( \phi = \phi(x; \nu, S_0, \mu, \varepsilon) \), then the vector
\[
\mathbf{v}^2(x; k) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{v}(-x; k)
\]
satisfies the same differential equation for the same \( k \in \mathbb{C} \), but for a different potential in the same family, namely \( \phi^\ast = \phi(x; \nu, -S_0, -\mu, \varepsilon) \). In this way, we see that, for example,
\[
j^{(2)}_-(x; k) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} j^{(1)}_+( -x; k) \big|_{S_0 \to -S_0, \mu \to -\mu}.
\]
Since our formulae for \( j^{(1)}_+(x; k) \) are valid for all values of the parameters, we also have immediate access to integral representations for the components of \( j^{(2)}_+(x; k) \).

4.3. Continuous scattering data. We now use the integral representation for \( j^{(1)}_+(x; k) \) to obtain the entries of the scattering matrix \( S(k) \) and establish their basic properties. We also will derive formulae for the reflection coefficient \( r(k) \) and the reduced reflection coefficient \( \rho(z) \).

As described in the appendix (see (184) in particular), the scattering matrix entries \( S_{11}(k) \) and \( S_{21}(k) \) may be defined in terms of \( j^{(1)}_+(x; k) \) by the limits
\[
S_{11}(k) := \lim_{x \to -\infty} e^{-\Lambda x/\varepsilon} J_{11+}(x; k)
\]
and
\[
S_{21}(k) := \lim_{x \to -\infty} e^{\Lambda x/\varepsilon} J_{21+}(x; k).
\]
To calculate \( S_{11}(k) \) for \( k^2 \in \mathbb{R} \) with \( \mu^2 - 4k^2\nu^2 > 0 \) we must therefore evaluate the limit
\[
S_{11}(k) = -\frac{2i \kappa \nu C_0 e^{iS_0/(2\varepsilon)}}{(\beta + 1)\varepsilon} \lim_{y \to -1} \int_{\Sigma} (t - 1)^{\gamma_+}(t + 1)^{\gamma_-} (t - y)^{\beta + 1} dt.
\]
The integral here is not analytic in a neighborhood of \( y = -1 \), but since \( (t - y)^{\beta + 1} \) is uniformly bounded for \( t \) and \( y \) both near \(-1\) for the values of \( k \) under consideration, dominated convergence again applies and allows us to take the limit under the integral. Thus
\[
S_{11}(k) = -\frac{2i \kappa \nu C_0 e^{iS_0/(2\varepsilon)}}{(\beta + 1)\varepsilon} \int_{\Sigma} (t - 1)^{\gamma_+}(t + 1)^{\gamma_-} dt.
\]
Collapsing the contour to the real line and expressing the resulting beta integral in terms of gamma functions yields
\[
S_{11}(k) = \frac{2^{\gamma_+ + \gamma_- + \beta + 4} \pi \kappa \nu C_0 e^{iS_0/(2\varepsilon)} \Gamma(\gamma_- + \beta + 2)}{(\beta + 1)\varepsilon \Gamma(-\gamma_+)\Gamma(\gamma_+ + \gamma_- + \beta + 3)}
\]
\[
= \frac{\Gamma(\gamma_- + \beta + 2)\Gamma(\gamma_- + 1)}{\Gamma(-\gamma_+)\Gamma(\gamma_- + 1)}.
\]
Similarly, to calculate \( S_{21}(k) \) for \( k^2 \in \mathbb{R} \) with \( \mu^2 - 4k^2\nu^2 > 0 \) we must evaluate the limit
\[
S_{21}(k) = C_0 e^{iS_0/(2\varepsilon)} \lim_{x \to -\infty} \left[ e^{-iS(x)/\varepsilon} \text{sech}(x) e^{2\Lambda x/\varepsilon} \int_{\Sigma} (t - 1)^{\gamma_+}(t + 1)^{\gamma_-} (t - \tanh(x))^{\beta} dt \right].
\]
Here we cannot immediately pass to the limit in the integral. However, noting that the integrand is integrable at \( t = \infty \) we may first deform the contour \( \Sigma \) to the top and bottom of the branch cut for real \( t < -1 \), giving
\[
\int_{\Sigma} (t - 1)^{\gamma_+}(t + 1)^{\gamma_-} (t - y)^{\beta} dt = 2i \sin(\pi(\gamma_+ + \gamma_- + \beta)) \int_{-\infty}^{-1} (1 - t)^{\gamma_+}(1 - t - t)^{\gamma_-}(y - t)^{\beta} dt.
\]
Next, composing an appropriate \( y \)-dependent scaling with a Möbius transformation we make the substitution
\[
t = -1 - (1 + y) \frac{u}{1 - u}
\]
Now we may apply a dominated convergence argument to the resulting integral, since \( \text{Re}\{\gamma_\pm\} = -1/2 \) and \( \text{Re}\{\beta\} = -1 \) implies that

\[
\left\lvert \left[ 1 - u + \frac{1 + y}{2} u \right]^{\gamma_+} \right\rvert \leq (1 - u)^{\text{Re}\{\gamma_+\}} = (1 - u)^{-1/2}
\]

and

\[
|u^{\gamma_+ - (1 - u)^{-\gamma_+}}| = u^{\text{Re}\{\gamma_+\}}(1 - u)^{-\text{Re}\{\gamma_+ + \beta + 2\}} = u^{-1/2}.
\]

Therefore

\[
\lim_{y \to -1} \int_0^1 \left[ 1 - u + \frac{1 + y}{2} u \right]^{\gamma_+} u^{\gamma_+ - (1 - u)^{-\gamma_+}} du = \int_0^1 u^{\gamma_+ - (1 - u)^{-\gamma_+}} du = \frac{\Gamma(\gamma_+ + 1)\Gamma(-\gamma_+ - \beta - 1)}{\Gamma(-\beta)}.
\]

Since by direct calculation with \( y = \tanh(x) \) we have

\[
\lim_{x \to -\infty} \left[ e^{-iS(x)/\varepsilon} \operatorname{sech}(x)e^{2\Lambda x/\varepsilon}(1 + y)^{1 + \gamma_+} \right] = e^{-iS_0/\varepsilon} e^{2\mu y^2/(2\epsilon)} e^{-\Omega/\varepsilon^{1/2}},
\]

we find

\[
S_{21}(k) = -C_0 e^{-iS_0/(2\varepsilon)} \left( 2^{-\beta} i \sin(\pi(\gamma_+ + \gamma_- + \beta)) \right) \frac{\Gamma(\gamma_+ + 1)\Gamma(-\gamma_+ - \beta - 1)}{\Gamma(-\beta)}
\]

\[
= C_0 e^{-iS_0/(2\varepsilon)} \left( 2^{-\beta} i \sin(\pi(\gamma_+ + \gamma_- + \beta + 3)) \right) \frac{\Gamma(\gamma_+ + 1)\Gamma(-\gamma_+ - \beta - 1)}{\Gamma(-\beta)}
\]

\[
= \frac{i\varepsilon e^{-iS_0/\epsilon} 2i/\varepsilon}{2\varepsilon} \left( \beta + 1 \right) \frac{\Gamma(-\gamma_+ - \beta - 1)\Gamma(-\gamma_+ - \beta - 1)}{\Gamma(-\beta)}
\]

\[
= \frac{i\varepsilon e^{-iS_0/\epsilon} 2i/\varepsilon}{2\varepsilon} \frac{\Gamma(-\gamma_+ - \beta - 1)\Gamma(-\gamma_+ - \beta - 1)}{\Gamma(-\beta)}
\]

where we used (131) upon substituting for \( C_0 \) from (132) and the factorial identity \( \Gamma(z+1) = z\Gamma(z) \).

All of our analysis so far has assumed that \( k^2 \in \mathbb{R} \) with \( \mu^2 - 16k^2\nu^2 > 0 \). If instead we have \( k^2 \in \mathbb{R} \) with \( 16k^2\nu^2 - \mu^2 > 0 \), then we claim that all of our results remain valid if we define

\[
\beta = -1 + \frac{1}{2\varepsilon} \left( -i\mu - \sqrt{16k^2\nu^2 - \mu^2} \right), \quad 16k^2\nu^2 - \mu^2 > 0.
\]

In this case we have the inequalities \( \text{Re}\{\beta\} < -1 \) and \( \text{Re}\{\gamma_\pm\} > -1/2 \). In fact, all of the preceding arguments remain intact exactly as written with the exception of that leading from (137) through (139) and that leading from (149) through (150). We emphasize that the results of these calculations remain the same; only their proofs must be altered.

We may combine the formulae for the two sign cases for \( 16k^2\nu^2 - \mu^2 \) by introducing the function

\[
R(k) := \begin{cases} 
-i\sqrt{\mu^2 - 16k^2\nu^2}, & 16k^2\nu^2 < \mu^2 \\
\sqrt{16k^2\nu^2 - \mu^2}, & 16k^2\nu^2 > \mu^2
\end{cases}
\]

where in each situation the positive square root is meant, and by writing

\[
\beta := -1 - \frac{1}{2\varepsilon} \left( i\mu + R(k) \right), \quad \text{Im}\{k^2\} = 0.
\]

At this point, we should resolve the question of what happens if \( \beta = -1 \). From the above definitions it follows we may only have \( \beta = -1 \) for \( \text{Im}\{k^2\} = 0 \) if \( \mu \geq 0 \), and in this case we must have \( k = 0 \). When \( k = 0 \) the differential equations for \( c_1(y) \) and \( c_2(y) \) decouple and become explicitly solvable in closed-form (Euler
transforms are not required). One can check that the construction of Jost solutions for $k = 0$ agrees with what we have done for $k \neq 0$ in the sense of taking limits as $k \to 0$ along the axes. Moreover, the behavior at $k = 0$ is consistent with what is shown in the appendix (see (199), (218), (223), and (224)) to hold for more general potentials.

The function $R(k)$ has an analytic continuation from the axes $\text{Im}\{k^2\} = 0$ to the second and fourth quadrants, and we define $R(k)$ in the complex $k$-plane by introducing appropriate branch cuts as illustrated in Figure 14. Since the quantity $R(k)^2 = 16k^2\nu^2 - \mu^2$ lies in the lower half-plane when $\text{Im}\{k^2\} < 0$, it can be easily shown by taking the appropriate branch of the square root consistent with the definition of $R(k)$ for $k$ in the second quadrant, that for such $k$ we have $\text{Re}\{R(k)\} > 0$.

In terms of this function, we then have

$$S_{11}(k) = \frac{\Gamma\left(\frac{1}{2} - \frac{\Omega}{\varepsilon} - \frac{i\mu}{2\varepsilon}\right)\Gamma\left(\frac{1}{2} - \frac{\Omega}{\varepsilon} + \frac{i\mu}{2\varepsilon}\right)}{\Gamma\left(\frac{1}{2} - \frac{\Omega}{\varepsilon} - \frac{R(k)}{2\varepsilon}\right)\Gamma\left(\frac{1}{2} - \frac{\Omega}{\varepsilon} + \frac{R(k)}{2\varepsilon}\right)},$$  \hspace{1cm} (152)

and

$$S_{21}(k) = -\frac{i\varepsilon e^{-iS_0/\varepsilon} q^{i\mu/\varepsilon}}{2k\nu} \frac{\Gamma\left(\frac{1}{2} + \frac{i\mu}{\varepsilon} + \frac{R(k)}{2\varepsilon}\right)\Gamma\left(\frac{1}{2} + \frac{i\mu}{\varepsilon} - \frac{R(k)}{2\varepsilon}\right)}{\Gamma\left(\frac{1}{2} + \frac{i\mu}{\varepsilon} - \frac{R(k)}{2\varepsilon}\right)\Gamma\left(\frac{1}{2} + \frac{i\mu}{\varepsilon} + \frac{R(k)}{2\varepsilon}\right)}. \hspace{1cm} (153)$$

The definition of $R(k)$ makes the arguments of all gamma functions in (152) analytic in $k$ for $\text{Im}\{k^2\} < 0$. This same condition on $k$ then also implies that $\text{Re}\{1/2 - \Omega/\varepsilon \pm i\mu/(2\varepsilon)\} = 1/2 - 2\text{Im}\{k^2\}/(\varepsilon\alpha) > 1/2$, which verifies the expected fact that $S_{11}(k)$ extends from the axes to an analytic function of $k$ for $\text{Im}\{k^2\} < 0$. While not guaranteed by the general theory, one can nonetheless see directly from (152) that $S_{11}(k)$ is even in $R(k)$ and hence single-valued and meromorphic in the whole complex $k$-plane (there are poles if $\text{Im}\{k^2\} > 0$). As is consistent with the general theory, $S_{21}(k)$ does not have an analytic extension to any of the four quadrants; however again we can see from (153) that in this case $S_{21}(k)$ is even in $R(k)$ and hence single-valued in the complex $k$-plane. Its only singularities are simple poles lying in all four quadrants.

![Figure 14. The branch cuts of the analytic continuation of $R(k)$ from the axes are shown with wavy curves.](image-url)
Applying the symmetry relations \[230\] established in the appendix, we then obtain from \[152\] a formula for \(S_{22}(k)\):

\[
S_{22}(k) = \frac{\Gamma\left(\frac{1}{2} + \frac{\Omega}{2} + \frac{i\mu}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\Omega}{2} - \frac{i\mu}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{\Omega}{2} - \frac{R(k)^*}{2\varepsilon}\right)\Gamma\left(\frac{1}{2} + \frac{\Omega}{2} + \frac{R(k)^*}{2\varepsilon}\right)},
\]

which extends to an analytic function for \(\text{Im}\{k^2\} > 0\), and from \[153\] a formula for \(S_{12}(k)\):

\[
S_{12}(k) = -\frac{i\varepsilon e^{iS_0/\varepsilon}2^{-i\mu/\varepsilon}}{2k\nu} \frac{\Gamma\left(\frac{1}{2} - \frac{i\mu}{2} - \frac{\Omega}{2}\right)\Gamma\left(\frac{1}{2} - \frac{i\mu}{2} + \frac{\Omega}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{\Omega}{2} - \frac{R(k)^*}{2\varepsilon}\right)\Gamma\left(-\frac{i\mu}{2} + \frac{R(k)^*}{2\varepsilon}\right)} = -\frac{i\varepsilon e^{iS_0/\varepsilon}2^{-i\mu/\varepsilon}}{2k\nu} \frac{\Gamma\left(\frac{1}{2} - \frac{i\mu}{2} + \frac{\Omega}{2}\right)\Gamma\left(\frac{1}{2} - \frac{i\mu}{2} - \frac{\Omega}{2}\right)}{\Gamma\left(-\frac{i\mu}{2} + \frac{R(k)^*}{2\varepsilon}\right)\Gamma\left(-\frac{i\mu}{2} - \frac{R(k)^*}{2\varepsilon}\right)}, \quad \text{Im}\{k^2\} = 0.
\]

(The second line follows because for \(\text{Im}\{k^2\} = 0\), either \(R(k)^* = R(k)\) or \(R(k)^* = -R(k)\).) The (reflection coefficient) function \(r(k) = -S_{12}(k)/S_{22}(k)\) is therefore

\[
r(k) = \frac{i\varepsilon e^{iS_0/\varepsilon}2^{-i\mu/\varepsilon}}{2k\nu} \frac{\Gamma\left(\frac{1}{2} - \frac{i\mu}{2} - \frac{\Omega}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\Omega}{2} + \frac{R(k)^*}{2\varepsilon}\right)}{\Gamma\left(\frac{1}{2} + \frac{\Omega}{2} - \frac{R(k)^*}{2\varepsilon}\right)\Gamma\left(-\frac{i\mu}{2} + \frac{R(k)^*}{2\varepsilon}\right)}, \quad \text{Im}\{k^2\} = 0.
\]

Since \(R(0) = -i|\mu|\), it is easy to see that \(r(k)\) is an odd function of \(k\) that is regular at \(k = 0\) (in fact in this case \(r(k)\) is analytic at \(k = 0\)). Therefore the “reduced” reflection coefficient \(\rho(z)\) defined by \[274\] is a continuous function of \(z \in \mathbb{R}\) given explicitly by

\[
\rho(z) = -\frac{i\varepsilon e^{iS_0/\varepsilon}2^{-i\mu/\varepsilon}}{2z\nu} \frac{\Gamma\left(\frac{1}{2} + \frac{i\mu}{2} - \frac{\Omega}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\Omega}{2} + \frac{R}{2\varepsilon}\right)}{\Gamma\left(\frac{1}{2} + \frac{\Omega}{2} - \frac{R}{2\varepsilon}\right)\Gamma\left(-\frac{i\mu}{2} + \frac{R}{2\varepsilon}\right)}, \quad \text{Im}\{z\} = 0,
\]

where in terms of \(z = -k^2\) we have

\[
\Omega = -\frac{1}{2\alpha} (4z + 1 - \alpha\delta),
\]

and

\[
R = \begin{cases} 
-i\sqrt{\mu^2 + 16z\nu^2}, & -16z\nu^2 < \mu^2 \\
\sqrt{-16z\nu^2 - \mu^2}, & -16z\nu^2 > \mu^2.
\end{cases}
\]

Note that for \(\alpha > 0\) the reduced reflection coefficient is never (that is, for no values of the parameters \(\nu, \mu, \delta, \) and \(S_0\) of the potential, nor for any \(\varepsilon > 0\)) identically zero as a function of \(z \in \mathbb{R}\). This should be strongly contrasted with the case \(\alpha = 0\), in which one considers the same initial data in the nonselfadjoint Zakharov-Shabat problem. For that spectral problem one of the main points made by Satsuma and Yajima \[37\] is that when \(\mu = 0\) there exists a sequence of values of \(\varepsilon > 0\) tending to zero for which the reflection coefficient vanishes identically. This leads to the powerful concept of the “higher-order solitons” — exact solutions with simple initial data that can be found by a finite number of algebraic steps as the Riemann-Hilbert problem degenerates into a finite-dimensional linear algebra problem. At least in this family of special initial data there are no higher-order solitons or reflectionless potentials when \(\alpha > 0\) and therefore the Zakharov-Shabat problem must be exchanged for the more complicated problem \[176\].

4.4. **Discrete scattering data.** As discussed in the appendix, the discrete spectrum for the problem \[176\] corresponds to the zeros of \(S_{11}(k)\) with \(\text{Im}\{k^2\} < 0\) and those of \(S_{22}(k)\) with \(\text{Im}\{k^2\} > 0\). By the symmetries \[230\] and \[231\] it suffices to consider the zeros of \(S_{11}(k)\) with \(\text{Im}\{k\} > 0\) and \(\text{Re}\{k\} < 0\). From \[152\] one can see that zeros of \(S_{11}(k)\) occur where the arguments of the gamma functions in the denominator are non-positive integers. Since for \(k\) in the second quadrant we have both \(\text{Re}\{R(k)\} > 0\) and \(\text{Re}\{\Omega\} < 0\), only one of the gamma functions can contribute zeros of \(S_{11}(k)\) in this quadrant. Therefore, the zeros of \(S_{11}(k)\) with \(\text{Im}\{k\} > 0\) and \(\text{Re}\{k\} < 0\) are precisely those values of \(k = k_n\) such that

\[
\Omega + \frac{1}{2} R(k) = \left(n + \frac{1}{2}\right) \varepsilon,
\]

(160)
where $n$ is a non-negative integer. It is easy to see that $n$ coincides with $\gamma_+$. Isolating $R(k)$ and squaring, we thus find that the desired values $k$ in the second quadrant and the solutions with $\text{Im}\{k\} > 0$ and $\text{Re}\{k\} < 0$ of the equation

$$w^2 + (iL\alpha + \nu^2 \alpha^2)w + \frac{1}{16} (4\nu^2 \alpha^2 (1 - \alpha \delta) - 4\alpha \nu^2 L^2 - \alpha^2 \nu^2 \mu^2) = 0, \quad w := k^2 - \frac{1}{4}(1 - \alpha \delta), \quad (161)$$

where $L := -\varepsilon(n+1/2) < 0$. The discrete spectrum may thus be obtained by solving this quadratic equation for $w$ and keeping only those solutions for which $\text{Im}\{w\} < 0$ (corresponding to the second quadrant for $k$). This results in a finite number (possibly zero) of roots of $S_{11}(k)$ in the second quadrant of the complex $k$-plane.

If we view $L$ as a general negative real parameter rather than a discrete one, a viewpoint that is increasingly accurate in the semiclassical limit $\varepsilon \downarrow 0$, we can eliminate the parameter $L$ between the real and imaginary parts of $\text{Im}\{k\}$ and hence find a curve $C$ in the $(x, y) = (\text{Re}\{w\}, \text{Im}\{w\})$-plane that is independent of $\varepsilon$ but that necessarily contains the eigenvalues for all $\varepsilon$. The eigenvalues for this problem lie exactly $\varepsilon$-independent curves $C$.

Indeed, in terms of $x$ and $y$, the real part of (161) is

$$x^2 - y^2 - \alpha Ly + \nu^2 \alpha^2 x - \frac{\alpha^2 L^2}{4} + \frac{1}{4}\nu^2 \alpha^2 (1 - \alpha \delta) - \frac{\alpha^2 \nu^2}{16} = 0, \quad (162)$$

and the imaginary part of (161) is linear in $L$ and can be solved to yield

$$L = \frac{2x + \alpha^2 \nu^2}{\alpha} x. \quad (163)$$

Substitution of (163) into (162) eliminates $L$ and yields the relation

$$x^2Q(x) = \frac{\nu^4 \alpha^4 y^2}{4}, \quad Q(x) := x^2 + \nu^2 \alpha^2 x + \frac{1}{4}\nu^2 \alpha^2 (1 - \alpha \delta) - \frac{\alpha^2 \mu^2}{16}. \quad (164)$$

The curve $C$ defined by (164) therefore necessarily contains all eigenvalues $k$ in the second quadrant when written in terms of $w$.

However, not all of the curve $C$ is relevant, because we need to take into account the inequalities $y < 0$ (corresponding to $k$ being in the second quadrant) and $L < 0$. The latter condition can be translated easily into a condition on $x$ with the help of (163): since $y < 0$ and $L < 0$ we must have $x$ and $2x + \alpha^2 \nu^2$ being of opposite sign, which forces the conditions $-\alpha^2 \nu^2 / 2 < x < 0$. Therefore in addition to lying on the curve $C$, to correspond to an eigenvalue for sufficiently small $\varepsilon$, $w$ must also lie in the semi-infinite strip $S$ defined by

$$S := \{(x, y) \in \mathbb{R}^2 \text{ such that } y < 0 \text{ and } -\alpha^2 \nu^2 / 2 < x < 0\}. \quad (165)$$

We now claim that the curve $C$ intersects the strip $S$ if and only if $\mu^2 < 4\nu^2 (1 - \alpha \delta)$. Indeed, if this condition holds, then $Q(0) > 0$, in which case the local structure of the curve $C$ near $x = y = 0$ is a simple crossing of two real branches of opposite slopes. Therefore in this situation there is a branch of $C$ near the origin with $x < 0$ and $y < 0$, a region that is contained in the strip $S$. On the other hand, if $Q(0) < 0$, then there are two distinct real branches of $C$, one with $x > 0$ and one with $x < -\alpha^2 \nu^2$, both of which are disjoint from $S$. See Figure 15. Therefore, if $Q(0) < 0$ there are no eigenvalues for any $\varepsilon > 0$, while if $Q(0) > 0$ then there exist eigenvalues for all sufficiently small $\varepsilon > 0$.

In the context of the particular family of potentials $\phi$ under consideration here, it is interesting and useful to compare the threshold condition for creation of eigenvalues with the condition on the parameters $\nu$, $\mu$, and $\delta$ such that the stability condition in (21) is satisfied for all $x \in \mathbb{R}$. In this specific situation, (21) becomes

$$\alpha \mu \tanh(x) + \alpha \delta + \alpha^2 \nu^2 \sech^2(x) - 1 > 0. \quad (166)$$

Depending on the sign of $\mu$, the minimum of the expression on the left of the inequality in (166) occurs either as $x \to +\infty$ or as $x \to -\infty$. The minimum value is $-\alpha |\mu| - (1 - \alpha \delta)$. Consequently, if $|\mu| < -(1 - \alpha \delta) / \alpha$ then

10No superfluous solutions are introduced because we have already shown that no solutions in the second quadrant can be obtained by changing the sign of $R(k)$.

11Note however, that these curves do not turn out to be hyperbolas of the family $\Sigma_C$. This should perhaps not come as any surprise, since for the special initial data under consideration the turning point curve $T$ does not coincide with a hyperbola $\Sigma_C$ either. See the discussion at the very end of Section 3.
Figure 15. Loss of eigenvalues with the variation of $\delta$. In all plots, $\alpha = 1$, $\mu = 4$, and $\nu = 2$, which makes the threshold for intersection of $C$ and $S$ correspond to $\delta = 0$. Upper left: $\delta = -4.1$. Upper right: $\delta = -3.9$. Lower left: $\delta = -2$. Lower right: $\delta = 1$.

the condition (166) is satisfied for all $x \in \mathbb{R}$. Furthermore, if $|\mu| \geq -(1 - \alpha \delta)/\alpha$ (in particular if $1 - \alpha \delta > 0$), then the stability condition (166) fails for some $x \in \mathbb{R}$.

We may summarize the possibilities for global-in-$x$ stability and existence of discrete spectrum for sufficiently small $\varepsilon$ in the following way:

- $1 - \alpha \delta < 0$: No eigenvalues exist for any $\varepsilon > 0$.
  - If $|\mu| < -(1 - \alpha \delta)/\alpha$ then we have modulational stability for all $x \in \mathbb{R}$.
  - If $|\mu| > -(1 - \alpha \delta)/\alpha$ modulational instability occurs for some $x \in \mathbb{R}$.

- $1 - \alpha \delta > 0$: Regardless of the value of $\mu$, modulational instability occurs for some $x \in \mathbb{R}$.
  - If $|\mu| < 2|\nu|/\sqrt{1 - \alpha \delta}$ then eigenvalues exist for sufficiently small $\varepsilon > 0$.
  - If $|\mu| > 2|\nu|/\sqrt{1 - \alpha \delta}$ then eigenvalues do not exist for any $\varepsilon > 0$.

A complete calculation of the discrete scattering data must include the determination of the proportionality constants $\gamma_j$ associated with each of the eigenvalues $k_j$ in the second quadrant of the complex $k$-plane. As described in the appendix (see (227)), this constant is defined by the relation

$$j^{(1)}_+(x; k_j) = \gamma_j j^{(2)}_-(x; k_j)$$

holding as an identity for all $x \in \mathbb{R}$. Even though it is possible to write down an integral representation for $j^{(2)}_-(x; k_j)$, it is not convenient to calculate the constants $\{\gamma_j\}$ in this way (with the exception of certain problems where it is known that the reflection coefficient vanishes identically; see [27] or [8]). However, another way to proceed in special cases such as this is to examine the consequences of the scattering relation

$$j^{(1)}_+(x; k) = S_{11}(k) j^{(1)}_-(x; k) + S_{21}(k) j^{(2)}_-(x; k), \quad k^2 \in \mathbb{R}. \quad (167)$$

It is not generally the case that this equation admits analytic continuation away from the axes, but in the special case of the class of potentials $\phi$ under consideration, analytic continuation is indeed possible. By continuation along a path beginning on the real $k$-axis at a value of $k$ with $16k^2\mu^2 > \nu^2$, one can show that the (nonphysical) continuation of $j^{(1)}_-(x; k)$ into the second quadrant is generically regular at $k = k_j$. The
only poles of this continuation occur if
\[ \frac{i\mu}{2} + \Omega = -\left( m + \frac{1}{2} \right) \varepsilon, \]  
(168)

where \( m \) is a nonnegative integer. This relation also gives poles \( k_m \) of \( S_{21}(k) \) in the second quadrant lying on the hyperbola
\[ \text{Re}\{k_m\}^2 - \text{Im}\{k_m\}^2 = \frac{1}{4} (1 - \alpha \delta + \alpha \mu). \]  
(169)

We call these poles **phantom poles**. As long as the eigenvalue \( k_j \) in the second quadrant does not (accidentally) coincide with a phantom pole, we learn from (167) and the eigenvalue condition \( S_{11}(k_j) = 0 \) that \( \gamma_j = S_{21}(k_j) \). This formula is thus generically valid for the special class of potentials under consideration.

This expression for the proportionality constants has added significance when interpreted in terms of the modified proportionality constants \( c_j^0 := \gamma_j/S_{11}(k_j) \) as one then has
\[ c_j^0 = \text{Res}_{k=k_j} r(k)^*, \]  
(170)

where \( r(k) \) is the reflection coefficient \( r(k) = -S_{12}(k)/S_{22}(k) \), or equivalently in terms of the “reduced” reflection coefficient,
\[ -2c_j^0 = \text{Res}_{z=-k_j^0} \rho(z)^*. \]  
(171)

Such relations relating the pole data to residues of jump matrix data are indispensable in the analysis of Riemann-Hilbert problems of inverse scattering; see, for example [39]. The main idea is to note that the jump matrix in the jump condition (166) for the “reduced” Riemann-Hilbert problem has a factorization:
\[ \begin{bmatrix} 1 - z|\rho(z)|^2 & -z\rho(z) \\ \rho(z)^* & 1 \end{bmatrix} = \begin{bmatrix} 1 & -z\rho(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho(z)^* & 1 \end{bmatrix}, \quad z \in \mathbb{R}. \]  
(172)

The function \( \rho(z)^* \), with \( \rho(z) \) defined by (167) and meromorphically continued to the whole complex \( z \)-plane, has two types of poles in the upper half \( z \)-plane: the eigenvalues \( z_n = -k_n^2 \) with \( k_n \) satisfying (160), and the phantom poles \( z_m = -k_m^2 \) with \( k_m \) satisfying (165). If the curves supporting the eigenvalue poles and the phantom poles are disjoint, then the first and second matrix factors on the right-hand side of (172) may be deformed into the lower and upper half \( z \)-plane respectively in such a way as to remove the eigenvalue poles from the Riemann-Hilbert problem without introducing any new phantom poles. Unlike in the Zakharov-Shabat problem with hypergeometric potential studied by Tovbis and Venakides [35], there are parameter values in the MNLS case for which the curves supporting the eigenvalue poles and phantom poles intersect. Note that if \( \mu > 0 \) or \( \mu < -\alpha^2 \nu^2/2 \), the interaction of eigenvalue poles and phantom poles is prevented because the hyperbola (169) supporting the phantom poles corresponds to a vertical line in the \( w \)-plane that lies to the right or left, respectively, of the strip \( S \). Phantom poles may only interfere with the eigenvalue poles if \(-\alpha^2 \nu^2/2 \leq \mu \leq 0 \).

To illustrate these phenomena, we fix the values \( \varepsilon = 0.075, \nu = 0.6846, \delta = 0.5 \), and \( |\mu| = 0.5 \) (the discrete spectrum only depends on \( |\mu| \) while the phantom poles are sensitive to the sign of \( \mu \) as well), and allow \( \alpha \) to vary. According to the criterion established above, eigenvalues will exist for sufficiently small \( \varepsilon > 0 \) only for \( 0 \leq \alpha < (1 - \mu^2/(4\nu^2))/\delta \approx 1.733 \). In Figure 16 we show the emergence of eigenvalues as \( \alpha \) is decreased below the threshold value of \( \alpha \approx 1.733 \).

Note that when the hyperbola (169) that supports the phantom poles passes through the origin in the \( k \)-plane, that is, when \( \alpha = 1/(\delta - \mu) \), we also have \( Q(-1 - \alpha \delta)/4 = 0 \), which implies that the curve supporting the eigenvalues also passes through the origin in the \( k \)-plane. This situation leads to the interaction of the eigenvalues with the phantom poles when \( \mu = -0.5 \), as shown in Figure 17.

Further decrease of \( \alpha \) causes a new phenomenon: the separation of the curve supporting the eigenvalues from the imaginary axis. Figure 18 shows the onset of a “spectral gap” between the discrete spectrum and the continuous spectrum on the imaginary axis occurring at approximately \( \alpha = 0.927 \). The parameter \( \alpha \) is decreased further in Figure 19 where we see that the passage through the origin of the hyperbola corresponding to the left-hand side of the strip \( S \) in the \( w \)-plane (this occurs exactly when \( \alpha = -(\delta \pm \sqrt{\delta^2 + 8\nu^2})/(4\nu^2) \) leads to only a small quantitative difference in the eigenvalues and phantom poles and the curves that support them.

Figure 20 shows the final essential transition that occurs as \( \alpha \) is decreased to zero: the departure of the
Figure 16. The eigenvalues in the complex $k$-plane are shown with purple dots on the curve of the same color that supports them for all $\varepsilon > 0$. The green hyperbolae are the images of the left and right sides of the strip $S$ in the $w$-plane. The red hyperbola is the curve that supports phantom poles shown as red dots for $\mu = -0.5$ (that supporting phantom poles for $\mu = 0.5$ is not shown but is necessarily outside the region between the bounding green hyperbolae). Left: $\alpha = 1.5$. Center: $\alpha = 1.3$. Right: $\alpha = 1.1$. In this figure and all that follow in this section, $\nu = 0.6846$, $\delta = 0.5$, $|\mu| = 0.5$, and $\varepsilon = 0.075$.

Figure 17. The eigenvalues and phantom poles in the complex $k$-plane showing onset of interaction between eigenvalues and phantom poles. Left: $\alpha = 1.004$. Center: $\alpha = 1$. Right: $\alpha = 0.996$.

hyperbola supporting the phantom poles for $\mu = -0.5$ from the region containing the eigenvalue curve. This transition occurs exactly when $\alpha = -\mu/(2\nu^2)$, which is approximately $\alpha = 0.533$ for $\mu = -0.5$ and $\nu = 0.6846$.

When $\alpha$ becomes very small, one expects some correspondence between our results and those of Tovbis and Venakides [38] for the nonselfadjoint Zakharov-Shabat eigenvalue problem which is the formal limit of the MNLS spectral problem when viewed in the correct variable $\lambda$, which is defined in terms of $k$ and $\alpha$ by (179). Figure 21 shows the discrete spectrum and phantom poles as $\alpha$ tends to zero, both in the complex $k$-plane and the complex $\lambda$-plane. In this limit, the discrete spectrum all accumulates near $k = \pm 1/2$, and the transformation (179) blows up the region near $k = 1/2$ to reveal a nontrivial limiting structure. This limiting structure is in agreement with the results of Tovbis and Venakides [38] in that the limiting discrete spectrum consists of equally-spaced points on the vertical line $\text{Re}\{\lambda\} = -\delta/2$ with the phantom poles lying on a distinct vertical line: $\text{Re}\{\lambda\} = -\delta/2 + \mu/2$. 38
Figure 18. The eigenvalues and phantom poles in the complex $k$-plane showing the appearance of a spectral gap. Left: $\alpha = 0.930$. Right: $\alpha = 0.924$.

Figure 19. The eigenvalues and phantom poles in the complex $k$-plane showing a transition in one of the bounding hyperbolae. Left: $\alpha = 0.801$. Right: $\alpha = 0.798$.

Figure 20. The eigenvalues and phantom poles in the complex $k$-plane showing the phantom poles moving away from the eigenvalues as $\alpha$ is decreased. Left: $\alpha = 0.7$. Center: $\alpha = 0.5$. Right: $\alpha = 0.3$. 
Figure 21. The eigenvalues and phantom poles in the complex $k$-plane (top) and $\lambda$-plane (bottom). The red hyperbolae contain the phantom poles for $\mu = -0.5$ (shown with phantom pole locations superimposed) and for $\mu = 0.5$ (shown without superimposed phantom poles). Left: $\alpha = 0.3$. Center: $\alpha = 0.03$. Right: $\alpha = 0.003$.

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Appendix: Scattering and Inverse Scattering for the MNLS Equation

Lax pair. Zero-curvature representation. Let \( k \in \mathbb{C} \) be a parameter and define

\[
\Lambda := -\frac{2i}{\alpha} \left( k^2 - \frac{1}{4} \right).
\]

(173)

Also given a complex-valued function \( \phi \) of \((x,t) \in \mathbb{R}^2\), let

\[
L = \begin{bmatrix}
\Lambda & 2ik\phi \\
2ik\phi^* & -\Lambda
\end{bmatrix}
\]

(174)

and

\[
B = \begin{bmatrix}
i\Lambda^2 + 2ik^2|\phi|^2 & -2k(1 + \Lambda + \alpha k)|\phi|^2 - 2i\alpha k|\phi|^2 \phi^*

-2k\Lambda \phi + k\varepsilon \phi_x - 2i\alpha k|\phi|^2 \phi^* & -i\Lambda^2 - 2ik^2|\phi|^2
\end{bmatrix}.
\]

(175)

The simultaneous linear equations

\[
\varepsilon \frac{\partial v}{\partial x} = Lv
\]

(176)

and

\[
\varepsilon \frac{\partial v}{\partial t} = Bv
\]

(177)

for a two-component vector unknown \( v(x,t; k) \) are said to comprise the Lax pair for the MNLS equation (1).

For a general function \( \phi(x,t) \) the vector \( v(x,t; k) \) is overdetermined by the two equations (176) and (177).

The MNLS equation for \( \phi(x,t) \) arises as the compatibility condition for the two equations of the Lax pair.

Indeed, by cross-differentiation one sees that the zero curvature condition

\[
\varepsilon \frac{\partial L}{\partial t} - \varepsilon \frac{\partial B}{\partial x} + [L,B] = 0
\]

(178)

on the matrices \( L \) and \( B \) is required for there to exist a full basis of simultaneous solutions of (176) and (177). Furthermore, the matrix equation (178) is equivalent to the MNLS equation governing \( \phi(x,t) \) as long as \( k \neq 0 \) (the condition (178) holds trivially if \( k = 0 \)).

While one might think of the MNLS equation (1) as a perturbation of the focusing NLS equation when \( \alpha \) is small, the singular dependence on \( \alpha \) in the spectral problem (176) suggests that at the level of inverse scattering we are considering a very singular perturbation. The spectral problem (176) can be made to look more like the Zakharov-Shabat eigenvalue problem if one introduces the alternate spectral variable \( \lambda \) defined by the relation

\[
\lambda = \frac{1}{\alpha} \left( 2k - 1 \right)
\]

or, equivalently,

\[
k = \frac{1}{2}(\alpha \lambda + 1).
\]

These relations are clearly singular as \( \alpha \to 0 \). Substituting for \( k \) in (176) we may nonetheless hold \( \lambda \) fixed and formally pass to the limit of \( \alpha \to 0 \). The coefficient matrix thus becomes that of the nonselfadjoint Zakharov-Shabat problem [41]:

\[
\lim_{\alpha \to 0} \lambda \text{ fixed } L = \begin{bmatrix}
-i\lambda & i\phi \\
-i\phi^* & i\lambda
\end{bmatrix}.
\]

(180)

This relation does not help us to deal with the inverse scattering transform for the MNLS equation, but it does help us to compare some results with known results for the Zakharov-Shabat scattering problem, by looking at what happens to the spectrum of (176) for small \( \alpha \) in the \( \lambda \)-plane rather than the \( k \)-plane.

Jost solutions and the scattering matrix. Until further notice, we consider \( t \) to be fixed and suppress all explicit dependence on \( t \). Assume that \( \phi \) is a complex-valued function of \( x \) that satisfies rapidly decreasing boundary conditions as \( x \to \pm \infty \). Then as \( x \to \pm \infty \) the differential equation (176) has purely oscillatory (rather than exponentially growing and decaying) solutions if and only if \( k^2 \in \mathbb{R} \), and thus the real and imaginary axes of the complex \( k \)-plane form the analogue of continuous spectrum for the differential equation (176).

\(^{12}\)It is just an analogue because the system is not of the form \( \mathcal{L}u = ku \) for any linear operator \( \mathcal{L} \); that is, the spectral parameter \( k \) enters in an essentially nonlinear fashion.
The Jost solutions of (176) are, for $k^2 \in \mathbb{R}$, the columns of two fundamental solution matrices $J_{\pm}(x; k)$ for this equation having simple asymptotics as $x \to \pm \infty$. By definition for $k^2 \in \mathbb{R}$ the matrices $J_{\pm}(x; k)$ satisfy
\[ 2 \varepsilon \frac{\partial J_{\pm}}{\partial x} = LJ_{\pm}, \quad \lim_{x \to \pm \infty} J_{\pm}(x; k)e^{-\Lambda x / \varepsilon} = I. \tag{181} \]
Since the trace of $L$ is zero, the Wronskian of any two vector solutions of (176) is independent of $x$, so by taking determinants in the boundary conditions we see that
\[ \det(J_{\pm}(x; k)) = 1 \tag{182} \]
which shows that if the Jost solution matrices exist for a given $k$ with $k^2 \in \mathbb{R}$ then they are both fundamental solution matrices for (176). Hence they are necessarily related by a constant (independent of $x$) matrix of coefficients:
\[ J_{+}(x; k) = J_{-}(x; k)S(k). \tag{183} \]
The matrix $S(k)$ for $k^2 \in \mathbb{R}$ is called the scattering matrix for (176) corresponding to the coefficient function $\phi$. From (182) and (183) it follows that
\[ S_{11}(k) = \det(J_{+}^{(1)}(x; k), J_{+}^{(2)}(x; k)), \quad S_{12}(k) = \det(J_{+}^{(2)}(x; k), J_{-}^{(2)}(x; k)), \]
\[ S_{21}(k) = \det(J_{-}^{(1)}(x; k), J_{+}^{(1)}(x; k)), \quad S_{22}(k) = \det(J_{-}^{(1)}(x; k), J_{-}^{(2)}(x; k)), \tag{184} \]
where $J_{+}^{(1)}(x; k)$ and $J_{+}^{(2)}(x; k)$ denote the first and second columns respectively of the matrix $J_{\pm}(x; k)$.

**Neumann series representations. Analyticity properties.** The Jost solutions may be constructed from integral equations equivalent to the differential equation and boundary conditions they satisfy. We begin by introducing the matrices $Y_{\pm}(x; k)$ related to the Jost matrices $J_{\pm}(x; k)$ by
\[ Y_{\pm}(x; k) = e^{-\Lambda x / \varepsilon}J_{\pm}(x; k). \tag{185} \]
By direct calculation using the differential equation (176) satisfied by $J_{\pm}(x; k)$ we find that
\[ 2 \varepsilon \frac{\partial Y_{\pm}}{\partial x} = \begin{bmatrix} 0 & 2ik\phi(x)e^{-2\Lambda x / \varepsilon} \\ 2ik\phi(x)^*e^{2\Lambda x / \varepsilon} & 0 \end{bmatrix} Y_{\pm}. \tag{186} \]
Also, with the use of the assumption that $k^2 \in \mathbb{R}$, the boundary conditions to be satisfied by $J_{\pm}(x; k)$ are equivalent to the requirement that
\[ \lim_{x \to \pm \infty} Y_{\pm}(x; k) = I. \tag{187} \]
We therefore seek $Y_{\pm}(x; k)$ as solutions of the following integral equations:
\[ Y_{\pm}(x; k) = I + \frac{2ik}{\varepsilon} \int_{-\infty}^{\infty} \begin{bmatrix} 0 & \phi(y)e^{-2\Lambda y / \varepsilon} \\ \phi(y)^*e^{2\Lambda y / \varepsilon} & 0 \end{bmatrix} Y_{\pm}(y; k) dy. \tag{188} \]
Iterating (188) once leads to uncoupled scalar integral equations for the four matrix elements of $Y_{\pm}(x; k)$.

In particular, the diagonal elements satisfy
\[ Y_{11\pm}(x; k) = 1 + \int_{-\infty}^{x} K(x, z; k)Y_{11\pm}(z; k) dz \
Y_{22\pm}(x; k) = 1 + \int_{-\infty}^{x} K(x, z; k)^*Y_{22\pm}(z; k) dz, \tag{189} \]
where the kernel is
\[ K(x, z; k) := -\frac{4k^2}{\varepsilon^2} \phi(z)^* \int_{z}^{x} \phi(y)e^{-2\Lambda(y-z) / \varepsilon} dy. \tag{190} \]
Note that the kernel is an even function of the complex variable $k$. Assuming that $\phi \in L^1(\mathbb{R})$:
\[ \|\phi\|_1 := \int_{-\infty}^{\infty} |\phi(x)| dx < \infty, \tag{191} \]
we easily get the estimate
\[ |K(x, z; k)| \leq \frac{4|k|^2}{\varepsilon^2} \|\phi\|_1 \cdot |\phi(z)|, \quad \text{if } \text{sgn}(x - z) \cdot \text{Im}\{k^2\} \geq 0. \tag{192} \]
Seeking to approximate the diagonal matrix elements by iteration starting from the initial guess $Y_{jj}(x; k) \equiv 1$ leads to solutions of (189) in the form of Neumann series with $m$th term given by an $m$-fold integral:

$$N_{m\pm}(x; k) := \int_{\pm\infty}^{x} K(x, z_m; k) \int_{\pm\infty}^{z_m} K(z_m, z_{m-1}; k) \cdots \int_{\pm\infty}^{z_{m-1}} K(z_{m-1}, z_{m-2}; k) dz_{m-1} \cdots dz_1 \cdots dz_{m},$$

(193)

(or, in the case of $Y_{22}(x; k), N_{m\pm}(x; k^*)^*$). We also define $N_{0\pm}(x; k) \equiv 1$. It follows from (192) that

$$|N_{m\mp}(x; k)| \leq \left[ \frac{4|k|^2}{\varepsilon^2} \right] \int_{\pm\infty}^{x} |\phi(z_m)| \int_{-\infty}^{z_m} |\phi(z_{m-1})| \cdots \int_{-\infty}^{z_1} |\phi(z_1)| dz_1 \cdots dz_m$$

(194)

and likewise that

$$|N_{m\mp}(x; k)| \leq \frac{1}{m!} \left[ \frac{4|k|^2}{\varepsilon^2} \right] \int_{-\infty}^{x} |\phi(z)| \int_{-\infty}^{x} |\phi(z)| dz \right]^{m}, \quad \text{if } \text{Im}\{k^2\} \geq 0,$$

(195)

By comparison with the exponential series it follows that the Neumann series

$$Y_{11}(x; k) = \sum_{m=0}^{\infty} N_{m\pm}(x; k), \quad \pm \text{Im}\{k^2\} \leq 0,$$

(196)

$$Y_{22}(x; k) = \sum_{m=0}^{\infty} N_{m\pm}(x; k^*)^*, \quad \pm \text{Im}\{k^2\} \geq 0,$$

all converge uniformly in compact subsets of the indicated regions of the complex $k$-plane and furnish there the unique solutions of the Volterra integral equations (189). Since the kernel is analytic and even in $k$ it then follows by uniform convergence that for each fixed $x \in \mathbb{R}$ all four of these functions are even in $k$ and continuous in the indicated regions, also being analytic in the interior. Moreover we have the uniform ($L^\infty$) estimate

$$|Y_{jj}(x; k)| \leq \exp \left( \frac{4|k|^2}{\varepsilon^2} \right)$$

(197)

that is valid for $j = 1, 2$ whenever $k^2$ is in the indicated half-plane of convergence. Together with (192) and the assumption that $\phi \in L^1(\mathbb{R})$ this estimate shows that

$$\lim_{x \to \pm\infty} Y_{jj}(x; k) = 1$$

(198)

holds for each fixed $k$ in the region of convergence. Now let $x \in \mathbb{R}$ be fixed and consider how $Y_{jj}(x; k)$ behaves as $k \to 0$ from each sector of convergence in the $k$-plane. Clearly the $L^\infty(\mathbb{R})$ estimate (197) is uniform for small $k$ as is the bound (192) of the kernel $K$. Moreover, since (192) also shows that $K \to 0$ as $k \to 0$ pointwise in $x$ and $z$, a dominated convergence argument applied to the right-hand side of (189) shows that

$$\lim_{k \to 0} Y_{jj}(x; k) = 1$$

(199)

where the limit is taken from any direction within one of the sectors of convergence. Finally, we consider how $Y_{jj}(x; k)$ behaves as $k \to \infty$ within each sector of convergence in the $k$-plane. To get the desired estimate, we have to note that the apparent $k^2$ growth of the kernel $K$ should be compensated for by the exponential behavior of the integral, so we could first integrate by parts in the definition (190) to get

$$K(x, z; k) = \frac{iak^2}{\varepsilon (k^2 - 1/4)} \phi(z)^* \left[ \phi(x) e^{-2\Lambda(x-z)/\varepsilon} - \phi(z) - \int_{z}^{x} \frac{\partial \phi}{\partial y}(y) e^{-2\Lambda(y-z)/\varepsilon} dy \right].$$

(200)

The fraction outside the brackets is now uniformly bounded as $k \to \infty$, and assuming that $\partial \phi(x)/\partial x$ is in $L^1(\mathbb{R})$ (which implies that

$$\|\phi\|_\infty := \sup_{x \in \mathbb{R}} |\phi(x)|$$

(201)

is finite, so $\phi \in L^\infty(\mathbb{R})$ as well), then we may replace the estimate (192) by

$$|K(x, z; k)| \leq \frac{\alpha}{\varepsilon |k^2 - 1/4|} (2\|\phi\|_\infty + \|\phi'\|_1) \cdot |\phi(z)|, \quad \text{if } \text{sgn}(x-z) \cdot \text{Im}\{k^2\} \geq 0.$$
This bound is uniform in \( k \) large. Furthermore, it now follows from dominated convergence that
\[
\lim_{k \to \infty} K(x, z; k) = \frac{-i}{\varepsilon} \phi(z)^2 \tag{203}
\]
with the limit being taken in a strict subsector of a sector for which \( 202 \) holds. Assuming therefore that \( \phi \in L^1(\mathbb{R}) \) and that \( \phi' \in L^1(\mathbb{R}) \), we have by dominated convergence that
\[
\lim_{k \to \infty} N_{m+}(x; k) = \frac{1}{m!} \left[ \frac{-i}{\varepsilon} \int_{x}^{+\infty} |\phi(z)|^2 \, dz \right]^m, \tag{204}
\]
with the limit being taken in the appropriate subsector (note that the presumed conditions on \( \phi \) also imply that \( \phi \in L^2(\mathbb{R}) \) so the right-hand side is finite). It further follows by dominated convergence applied to the infinite Neumann series \( 196 \) that
\[
\lim_{k \to \infty} Y_{11+}(x; k) = \exp \left( \frac{i}{\varepsilon} \int_{x}^{+\infty} |\phi(z)|^2 \, dz \right), \quad \lim_{k \to \infty} Y_{11-}(x; k) = \exp \left( -\frac{i}{\varepsilon} \int_{-\infty}^{x} |\phi(z)|^2 \, dz \right), \tag{205}
\]
\[
\lim_{k \to \infty} Y_{22+}(x; k) = \exp \left( -\frac{i}{\varepsilon} \int_{x}^{+\infty} |\phi(z)|^2 \, dz \right), \quad \lim_{k \to \infty} Y_{22-}(x; k) = \exp \left( \frac{i}{\varepsilon} \int_{-\infty}^{x} |\phi(z)|^2 \, dz \right),
\]
with the limit being taken in an arbitrary strict subsector of the sector of existence in each case.

Similarly, by iterating \( 188 \) once we find that the functions
\[
\tilde{Y}_{12\pm}(x; k) := e^{2\Lambda x/\varepsilon} Y_{12\pm}(x; k) \quad \text{and} \quad \tilde{Y}_{21\pm}(x; k) := -e^{-2\Lambda x/\varepsilon} Y_{21\pm}(x; k) \tag{206}
\]
satisfy the integral equations
\[
\tilde{Y}_{12\pm}(x; k) = f_{\pm}(x; k) + \int_{x}^{\pm\infty} \tilde{K}(x, z; k) \tilde{Y}_{12\pm}(z; k) \, dz
\]
\[
\tilde{Y}_{21\pm}(x; k) = f_{\mp}(x; k^*)^* + \int_{x}^{\pm\infty} \tilde{K}(x, z; k^*)^* \tilde{Y}_{21\pm}(z; k) \, dz, \tag{207}
\]
where the modified kernel is
\[
\tilde{K}(x, z; k) := K(x, z; k)e^{-2\Lambda(z-x)/\varepsilon} = -\frac{4k^2}{\varepsilon^2} \phi(z)^* \int_{z}^{x} \phi(y)e^{-2\Lambda(y-z)/\varepsilon} \, dy, \tag{208}
\]
and where
\[
f_{\pm}(x; k) := \frac{2ik}{\varepsilon} \int_{x}^{\pm\infty} \phi(y)e^{-2\Lambda(y-z)/\varepsilon} \, dy. \tag{209}
\]
As before, the assumption that \( \phi \in L^1(\mathbb{R}) \) gives
\[
|\tilde{K}(x, z; k)| \leq \frac{4|k|^2}{\varepsilon^2} \|\phi\|_1 \cdot |\phi(z)|, \quad \text{if} \quad \text{sgn}(x - z) \cdot \text{Im}\{k^2\} \leq 0, \quad \tag{210}
\]
and we also have
\[
|f_{\pm}(x; k)| \leq \frac{2|k|}{\varepsilon} \|\phi\|_1, \quad \text{if} \quad \pm \text{Im}\{k^2\} \geq 0, \quad \tag{211}
\]
estimates that are uniformly valid for small \( k \). The further assumption that \( \phi' \in L^1(\mathbb{R}) \) gives
\[
|\tilde{K}(x, z; k)| \leq \frac{\alpha}{\varepsilon} \frac{|k|^2}{|k^2 - 1/4|} (2\|\phi\|_{\infty} + \|\phi'(1)\|_1) \cdot |\phi(z)|, \quad \text{if} \quad \text{sgn}(x - z) \cdot \text{Im}\{k^2\} \leq 0, \quad \tag{212}
\]
and
\[
|f_{\pm}(x; k)| \leq \frac{\alpha}{2} \frac{|k|}{|k^2 - 1/4|} (\|\phi\|_{\infty} + \|\phi'(1)\|_1), \quad \text{if} \quad \pm \text{Im}\{k^2\} \geq 0, \quad \tag{213}
\]
estimates that are uniformly valid for large $k$. Using (210) and (211) we see that the following Neumann series representations converge uniformly for $k$ in compact subsets of the indicated sectors:

$$\tilde{Y}_{12\pm}(x;k) = \sum_{m=0}^{\infty} \tilde{N}_{m\pm}(x;k), \quad \pm \text{Im}\{k^2\} \geq 0,$$

$$\tilde{Y}_{21\pm}(x;k) = \sum_{m=0}^{\infty} \tilde{N}_{m\pm}(x;k^*)^*, \quad \pm \text{Im}\{k^2\} \leq 0,$$  \hspace{1cm} (214)

where

$$\tilde{N}_{m\pm}(x;k) := \int_{x,\pm\infty}^{x} \tilde{K}(x,z_m;k) \int_{x,\pm\infty}^{z_m} \tilde{K}(z_m,z_{m-1};k) \cdots \int_{x,\pm\infty}^{z_2} \tilde{K}(z_2,z_1;k) f_{\pm}(z_1;k) dz_1 \cdots dz_m.$$  \hspace{1cm} (215)

The functions $\tilde{N}_{m\pm}(x;k)$ are all odd functions of $k$. Thus, $\tilde{Y}_{12\pm}(x;k)$ and $\tilde{Y}_{21\pm}(x;k)$ are odd and analytic for $\text{Im}\{k^2\} > 0$ and continuous for $\text{Im}\{k^2\} \geq 0$. Similarly, $\tilde{Y}_{12\pm}(x;k)$ and $\tilde{Y}_{21\pm}(x;k)$ are odd and analytic for $\text{Im}\{k^2\} < 0$ and continuous for $\text{Im}\{k^2\} \leq 0$. If $g(x;k)$ denotes any of these four functions in their sectors of continuity, we have the uniform estimate

$$|g(x;k)| \leq \frac{2|k|}{\varepsilon} \|\phi\|_1 \cdot \exp\left(\frac{4|k|^2}{\varepsilon} \|\phi\|^2_1\right).$$  \hspace{1cm} (216)

A dominated convergence argument applied to (207) with the help of this estimate then shows that for all $k$ in the sectors of existence,

$$\lim_{x \to \pm\infty} \tilde{Y}_{12\pm}(x;k) = 0 \quad \text{and} \quad \lim_{x \to \pm\infty} \tilde{Y}_{21\pm}(x;k) = 0,$$  \hspace{1cm} (217)

and similarly that

$$\lim_{k \to 0} \tilde{Y}_{12\pm}(x;k) = 0 \quad \text{and} \quad \lim_{k \to 0} \tilde{Y}_{21\pm}(x;k) = 0$$  \hspace{1cm} (218)

with the limits being taken from within the closed sectors of continuity. Also, using (212) and (213) we see that $kg(x;k)$ is bounded as $k \to \infty$ within the closed sector of continuity, and by dominated convergence,

$$\lim_{k \to \infty} k\tilde{N}_{m\pm}(x;k) = \frac{\alpha}{2} \phi(x) \frac{1}{m!} \left[\frac{\alpha}{\varepsilon} \int_{x,\pm\infty} \left|\phi(z)\right|^2 dz\right]^m,$$  \hspace{1cm} (219)

with the limit being taken in any strictly smaller subsector of $\pm \text{Im}\{k^2\} > 0$. It follows that

$$\lim_{k \to \infty} k\tilde{Y}_{12\pm}(x;k) = \frac{\alpha}{2} \phi(x) \exp\left(-\frac{\alpha}{\varepsilon} \int_{x,\pm\infty} \left|\phi(z)\right|^2 dz\right),$$  \hspace{1cm} (220)

$$\lim_{k \to \infty} k\tilde{Y}_{21\pm}(x;k) = \frac{\alpha}{2} \phi(x)^* \exp\left(-\frac{\alpha}{\varepsilon} \int_{x,\pm\infty} \left|\phi(z)\right|^2 dz\right),$$  \hspace{1cm} (220)

again with the limits being taken from within a strict subsector of the sector of convergence.

From this analysis we see that when $\phi$ and $\phi'$ are in $L^1(\mathbb{R})$, the Jost solutions

$$J_+^{(1)}(x;k) = \begin{bmatrix} Y_{11+(x;k)} & Y_{12+(x;k)} \end{bmatrix} e^{\Lambda x/\varepsilon} \quad \text{and} \quad J_-^{(2)}(x;k) = \begin{bmatrix} \tilde{Y}_{12-(x;k)} & \tilde{Y}_{21-(x;k)} \end{bmatrix} e^{-\Lambda x/\varepsilon}$$  \hspace{1cm} (221)

are analytic for $\text{Im}\{k^2\} < 0$ and continuous for $\text{Im}\{k^2\} \leq 0$. For fixed $k$ with $\text{Im}\{k^2\} < 0$ they decay exponentially to zero as $x \to +\infty$ and $x \to -\infty$ respectively. Similarly, the Jost solutions

$$J_+^{(1)}(x;k) = \begin{bmatrix} Y_{11-(x;k)} & Y_{12-(x;k)} \end{bmatrix} e^{\Lambda x/\varepsilon} \quad \text{and} \quad J_+^{(2)}(x;k) = \begin{bmatrix} \tilde{Y}_{12+(x;k)} & \tilde{Y}_{21+(x;k)} \end{bmatrix} e^{-\Lambda x/\varepsilon}$$  \hspace{1cm} (222)

are analytic for $\text{Im}\{k^2\} > 0$ and continuous for $\text{Im}\{k^2\} \geq 0$. For fixed $k$ with $\text{Im}\{k^2\} > 0$ they decay exponentially to zero as $x \to -\infty$ and $x \to +\infty$ respectively. From (184) it then follows that $S_{11}(k)$
is analytic for $\text{Im}\{k^2\} < 0$ and continuous for $\text{Im}\{k^2\} \leq 0$ while $S_{22}(k)$ is analytic for $\text{Im}\{k^2\} > 0$ and continuous for $\text{Im}\{k^2\} \geq 0$. From (134) we learn in addition that
\[
\lim_{k \to 0} S_{11}(k) = 1 \quad \text{and} \quad \lim_{k \to 0} S_{22}(k) = 1
\]
with the limit being taken from within a closed sector of analyticity in each case. Also
\[
\lim_{\text{Im}(k^2) = 0} S_{12}(k) = 0 \quad \text{and} \quad \lim_{\text{Im}(k^2) = 0} S_{21}(k) = 0.
\]
Furthermore, all elements of $S(k)$ are bounded for $k$ large, and we also have
\[
\lim_{k \to \infty} S_{11}(k) = \exp\left(\frac{i\alpha}{\varepsilon}\|\phi\|_2^2\right) \quad \text{and} \quad \lim_{k \to \infty} S_{22}(k) = \exp\left(-\frac{i\alpha}{\varepsilon}\|\phi\|_2^2\right)
\]
where
\[
\|\phi\|_2^2 := \int_{-\infty}^{+\infty} |\phi(x)|^2 \, dx < \infty
\]
and the limit is taken from within a strict subsector of the sector of analyticity in each case.

Let $k_j$ be a zero of $S_{11}(k)$ with $\text{Im}\{k_j^2\} \leq 0$. Since $S_{11}(k)$ is a Wronskian, it follows that the Jost solutions $j_+^{(1)}(x; k_j)$ and $j_-^{(2)}(x; k_j)$ are proportional, with a constant of proportionality that we denote by $\gamma_j$:
\[
j_+^{(1)}(x; k_j) = \gamma_j j_-^{(2)}(x; k_j), \quad \forall x \in \mathbb{R}.
\]
Moreover, if we have the strict inequality $\text{Im}\{k_j^2\} < 0$ then the left and right hand sides of (227) both represent solutions that decay exponentially to zero as $x \to \pm \infty$. In other words, these values of $k$ are precisely the analogous restricted discrete spectrum for this problem. We further assume that $\phi$ is such that:

- $S_{11}(k)$ has a finite number of zeros.
- $S_{11}(k)$ does not vanish for $\text{Im}\{k^2\} = 0$.
- All zeros of $S_{11}(k)$ are simple; that is, $S_{11}(k) = 0$ implies that $S_{11}'(k) \neq 0$.

Directly from the Neumann series representations, we observe the following symmetries of the Jost solutions:
\[
j_\pm^{(1)}(x; k^*)^* = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad j_\pm^{(2)}(x; k),
\]
an antiholomorphic symmetry, and
\[
j_\pm^{(1)}(x; -k) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad j_\pm^{(1)}(x; k) \quad \text{and} \quad j_\pm^{(2)}(x; -k) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},
\]
a holomorphic symmetry. From the definitions (134) we then have
\[
S_{22}(k) = S_{11}(k^*)^* \quad \text{and} \quad S_{21}(k) = -S_{12}(k^*)^*
\]
and
\[
S_{11}(-k) = S_{11}(k), \quad S_{22}(-k) = S_{22}(k), \quad S_{12}(-k) = -S_{12}(k), \quad \text{and} \quad S_{21}(-k) = -S_{21}(k).
\]
Consequently the discrete spectrum for this problem consists of quartets of points of the form $(k, -k, k^*, -k^*)$, distinct because (by assumption) $\text{Im}\{k^2\} \neq 0$. The proportionality relation (227) in conjunction with the symmetries of the Jost solutions then implies also that
\[
\begin{aligned}
j_+^{(1)}(x; -k_j) &= -\gamma_j j_-^{(2)}(x; k_j), \\
j_+^{(2)}(x; k_j^*) &= -\gamma_j \overline{j_-^{(1)}(x; k_j^*)}, \\
j_+^{(2)}(x; -k_j^*) &= \gamma_j^* j_-^{(1)}(x; -k_j^*).
\end{aligned}
\]

\[\text{See the previous footnote.}\]
Riemann-Hilbert problem of inverse scattering. Consider the matrix $\mathbf{M}(k; x)$ defined for $x \in \mathbb{R}$ in terms of the Jost solutions as follows:

$$
\mathbf{M}(k; x) := \begin{cases} 
\frac{e^{-Ax/x}}{S_{11}(k)} j_{+}^{(1)}(x; k), & \text{Im}\{k^2\} < 0, \\
\frac{e^{Ax/x}}{S_{22}(k)} j_{+}^{(2)}(x; k), & \text{Im}\{k^2\} > 0.
\end{cases}
$$

(233)

It follows from [15] that $\det(\mathbf{M}(k; x)) = 1$ at every point in its domain of definition (which implicitly excludes the discrete spectrum). Let the (fourfold) discrete spectrum be denoted $D$. The following other properties of $\mathbf{M}(k; x)$ are fundamental:

**Analyticity:** $\mathbf{M}(k; x)$ is analytic for $\text{Im}\{k^2\} \neq 0$ and $k \notin D$ and takes continuous boundary values on the axes $\text{Im}\{k^2\} = 0$ from each of the four sectors of its analyticity. Moreover, $\mathbf{M}(k; x)$ is uniformly bounded for large $k$.

**Jump Condition:** Let $\mathbf{M}_{\pm}(k; x)$ denote the boundary value taken from the region where $\pm \text{Im}\{k^2\} < 0$:

$$
\mathbf{M}_{\pm}(k; x) := \lim_{z \to k} \mathbf{M}(z; x), \quad \text{Im}\{k^2\} = 0.
$$

(234)

Then the boundary values are related by the formula

$$
\mathbf{M}_{+}(k; x) = \mathbf{M}_{-}(k; x) \mathbf{V}(k; x), \quad \text{Im}\{k^2\} = 0,
$$

(235)

where

$$
\mathbf{V}(k; x) = e^{Ax/x} \mathbf{V}_0(k)e^{-Ax/x},
$$

(236)

and

$$
\mathbf{V}_0(k) := \begin{bmatrix}
1 - \frac{S_{12}(k)S_{21}(k)}{S_{11}(k)S_{22}(k)} & -\frac{S_{12}(k)}{S_{22}(k)} \\
\frac{S_{11}(k)S_{22}(k)}{S_{21}(k)} & 1
\end{bmatrix}.
$$

(237)

Note that upon taking into account the symmetries (230) and (231) we see that in terms of the function

$$
r(k) := \frac{S_{12}(k)}{S_{22}(k)}
$$

(238)

we may write $\mathbf{V}_0(k)$ in the form

$$
\mathbf{V}_0(k) = \begin{bmatrix}
1 \pm |r(k)|^2 & r(k) \\
\pm r(k)^* & 1
\end{bmatrix}, \quad \pm k^2 > 0.
$$

(239)

**Singularities:** The matrix $\mathbf{M}(k; x)$ has simple poles at the points of the finite set $D$. If $k_j \in D$ with $\text{Im}\{k_j\} > 0$ and $\text{Re}\{k_j\} < 0$, then

$$
\text{Res} \mathbf{M}(k; x) = \lim_{k \to k_j} \mathbf{M}(k; x) \begin{bmatrix} 0 & 0 \\
c_j(x) & 0
\end{bmatrix}
$$

where

$$
c_j(x) := c_j^0 e^{-2Ax/x}, \quad c_j^0 := \frac{\gamma_j}{S_{11}'(k_j)}, \quad \Lambda_j := \Lambda |_{k=k_j}.
$$

(240)

Similarly,

$$
\text{Res} \mathbf{M}(k; x) = \lim_{k \to -k_j} \mathbf{M}(k; x) \begin{bmatrix} 0 & 0 \\
c_j(x) & 0
\end{bmatrix},
$$

(241)

$$
\text{Res} \mathbf{M}(k; x) = \lim_{k \to -k_j} \mathbf{M}(k; x) \begin{bmatrix} 0 & -c_j(x)^* \\
0 & 0
\end{bmatrix},
$$

(242)

and

$$
\text{Res} \mathbf{M}(k; x) = \lim_{k \to k_j} \mathbf{M}(k; x) \begin{bmatrix} 0 & -c_j(x)^* \\
0 & 0
\end{bmatrix}.
$$

(243)
The asymptotics in (205), (220), and (225) applied to the definition (233) we see that virtue of the assumption that limiting form of the matrix $B$ these factors so that (177) holds for the products. To do choose the factors, it is enough to examine the compatible. Therefore, there exists a full basis of simultaneous solutions of (176) and (177). As $J$ $x$, and controlling these derivatives in addition to the matrix elements themselves in asymptotic problems adds an unneeded layer of difficulty.

Time dependence of scattering data. Now we suppose that $\phi$ is evolving in time subject to the MNLS equation $H$, and we seek to determine the way that the scattering data corresponding to $\phi$ depend on $t$. We need to assume here that for all $t$ we have $\phi$ and $\phi_x$ in $L^1(\mathbb{R})$ so that the notion of scattering data makes sense in the way we have discussed above. We will also assume $\phi_{xx}$ in $L^1(\mathbb{R})$.

Since $\phi$ satisfies $H$, the zero-curvature condition (177) is satisfied making the Lax pair (176) and (177) compatible. Therefore, there exists a full basis of simultaneous solutions of (176) and (177). As $\phi$ now depends on $t$ as well as $x$, we now recall the explicit time dependence of the Jost matrices by writing $J_{\pm}(x,t;k)$.

Normalization: The matrix $M(k;x)$ is normalized in the sense that

$$
\lim_{k \to 0} M(k;x) = I,
$$

with the limit being taken in any direction.

(245)

Reversing the point of view by taking the function $r(k)$, the fourfold symmetric set $D$, and the numbers $c^0_l$ (which taken together form the set of scattering data associated with $\phi$) to be given, the above properties are said to constitute a Riemann-Hilbert problem for the unknown matrix $M(k;x)$. If the matrix $M(k;x)$ can be determined from the scattering data, then the function $\phi$ may also be recovered therefrom, since from the asymptotics in (205), (220), and (225) applied to the definition (233) we see that

$$
M(k;x) = \begin{bmatrix}
1 + o(1) & \frac{\alpha}{2k} \phi(x) + o(k^{-1}) \\
-\frac{\alpha}{2k} \phi(x)^* + o(k^{-1}) & 1 + o(1)
\end{bmatrix}
\begin{bmatrix}
\exp \left( -i \frac{\alpha}{\epsilon} \int_{-\infty}^{x} |\phi(z)|^2 \, dz \right) & 0 \\
0 & \exp \left( i \frac{\alpha}{\epsilon} \int_{-\infty}^{x} |\phi(z)|^2 \, dz \right)
\end{bmatrix}
$$

as $k \to \infty$ in any non-horizontal and non-vertical direction. Consequently, we have the reconstruction formula

$$
\phi(x) = \lim_{k \to \infty} \frac{2k}{\alpha} M_{12}(k;x) M_{22}(k;x).
$$

Note that this Riemann-Hilbert problem differs from others described in the literature even for this specific problem (see, for example, [17]) in that the point of normalization is $k = 0$, while the potential $\phi$ is extracted via asymptotics of the solution for $k$ large. Indeed, nowhere in the theory of matrix Riemann-Hilbert problems is it essential that the point of normalization be any specific value. On the other hand, the fact that the potential is extracted by asymptotics at $k = \infty$ is important because while $\phi$ may also be obtained from the expansion near $k = 0$, these latter formulæ involve derivatives of the elements of $M(k;x)$ with respect to $x$, and controlling these derivatives in addition to the matrix elements themselves in asymptotic problems adds an unneeded layer of difficulty.

(246)

(247)

The equations hold columnwise throughout the closed sectors of convergence in the complex $k$-plane for the corresponding Neumann series.

In particular, both columns of (248) are meaningful for $\text{Im}\{k^2\} = 0$. In this case we have the relation (183) which we rewrite here in the form

$$
(J_{+}(x,t;k)e^{i\lambda^2 t\sigma_3/\epsilon}) = (J_{-}(x,t;k)e^{i\lambda^2 t\sigma_3/\epsilon}) (e^{-i\lambda^2 t\sigma_3/\epsilon} S(k,t) e^{i\lambda^2 t\sigma_3/\epsilon}),
$$

(249)
where we have allowed time dependence in the scattering matrix \( S \). Differentiating with respect to \( t \) and applying (248) gives

\[
B \left( J_{+}(x, t; k) e^{i \Lambda t} / \varepsilon \right) = B \left( J_{-}(x, t; k) e^{i \Lambda t} / \varepsilon \right) \left( e^{-i \Lambda t} / \varepsilon S(k; t) e^{i \Lambda t} / \varepsilon \right)
\]

\[
+ \left( J_{-}(x, t; k) e^{i \Lambda t} / \varepsilon \right) \cdot \varepsilon \frac{d}{dt} \left( e^{-i \Lambda t} / \varepsilon S(k; t) e^{i \Lambda t} / \varepsilon \right) .
\]

(250)

Using (248) again and noting that \( J_{-}(x, t; k) \) and \( e^{i \Lambda t} / \varepsilon \) are invertible we see that

\[
S(k; t) = e^{i \Lambda t} / \varepsilon S(k; 0) e^{-i \Lambda t} / \varepsilon .
\]

(251)

From this relation we learn the following facts:

- \( S_{11}(k; t) \) and \( S_{22}(k; t) \) are independent of \( t \). Therefore so are the locations of the poles of \( M(k; x, t) \) in the complex \( k \)-plane, as well as the derivatives \( S_{11}'(k, j; t) \).

- The function \( r(k; t) \) satisfies

\[
r(k; t) = r(k; 0) e^{2i \Lambda t} / \varepsilon .
\]

(252)

The proportionality constants \( \gamma_{j} \) will also depend on \( t \), and to deduce the time dependence we first rewrite the defining relation (247) in the form

\[
\left( e^{i \Lambda t / \varepsilon} j_{+}^{(1)}(x, t; k) \right) = \left( e^{-i \Lambda t / \varepsilon} j_{-}^{(2)}(x, t; k) \right) \left( e^{2i \Lambda t / \varepsilon} \gamma_{j} \right).
\]

(253)

Differentiating with respect to \( t \) and applying (248) gives

\[
B \left( e^{i \Lambda t / \varepsilon} j_{+}^{(1)}(x, t; k) \right) = B \left( e^{-i \Lambda t / \varepsilon} j_{-}^{(2)}(x, t; k) \right) \left( e^{2i \Lambda t / \varepsilon} \gamma_{j} \right)
\]

\[
+ \left( e^{-i \Lambda t / \varepsilon} j_{-}^{(2)}(x, t; k) \right) \cdot \varepsilon \frac{d}{dt} \left( e^{2i \Lambda t / \varepsilon} \gamma_{j} \right) .
\]

(254)

Again using (248) and noting that \( j_{-}^{(2)}(x, t; k) \) is not the zero vector, we learn that

\[
\gamma_{j}(t) = e^{-2i \Lambda t / \varepsilon} \gamma_{j}(0).
\]

(255)

Given this time evolution of the scattering data we are led to the following algorithm for solving the Cauchy problem for the MNLS equation (1). From the initial data \( \phi(x, 0) \) given in a suitable space, calculate the scattering matrix \( S(k; 0) \). Check that the function \( S_{11}(k; 0) \) has a finite number of purely simple zeros \( k_{j} \) with \( \text{Im}\{k_{j}^{2}\} < 0 \), and calculate the constants \( \gamma_{j}^{0} \). Set \( S_{11}(k, 0) := S_{11}(k; 0) \). Set \( r(k; 0) := -S_{12}(k; 0) / S_{22}(k; 0) \) for \( \text{Im}\{k_{j}^{2}\} = 0 \), and let \( D \) denote the set of points \( \{ k \in \mathbb{C} ; S_{11}(k; 0) = 0 \text{ or } S_{22}(k; 0) = 0 \} \). The scattering data are then used to formulate a Riemann-Hilbert problem: find a 2 \times 2 matrix \( M(k; x, t) \), \( x \in \mathbb{R}, t \geq 0 \) such that the following properties are satisfied:

**Analyticity:** \( M(k; x, t) \) is analytic for \( \text{Im}\{k^{2}\} \neq 0 \) and \( k \notin D \) and takes continuous boundary values on the axes \( \text{Im}\{k^{2}\} = 0 \) from each of the four sectors of its analyticity. Moreover, \( M(k; x, t) \) is uniformly bounded for large \( k \).

**Jump Condition:** Letting \( M_{\pm}(k; x, t) \) denote the boundary value taken from the region where \( \pm \text{Im}\{k^{2}\} < 0 \) as in [234], the boundary values are related by the formula

\[
M_{+}(k; x, t) = M_{-}(k; x, t) e^{(\Lambda x + i \Lambda t) / \varepsilon} \left[ 1 \pm r(k; 0)^{2} \right] \frac{r(k; 0)}{1 \pm r(k; 0)} e^{-(\Lambda x + i \Lambda t) / \varepsilon}, \quad \pm k^{2} > 0 .
\]

(256)

**Singularities:** The matrix \( M(k; x, t) \) has simple poles at the points of the finite set \( D \). If \( k_{j} \in D \) with \( \text{Im}\{k_{j}\} > 0 \) and \( \text{Re}\{k_{j}\} < 0 \), then

\[
\text{Res}_{k = \pm k_{j}} M(k; x, t) = \lim_{k \to \pm k_{j}} M(k; x, t) \left[ \begin{array}{cc} 0 & 0 \\ c_{j}(x, t) & 0 \end{array} \right]
\]

(257)

and

\[
\text{Res}_{k = \pm k_{j}} M(k; x, t) = \lim_{k \to \pm k_{j}} M(k; x, t) \left[ \begin{array}{cc} 0 & -c_{j}(x, t)^{*} \\ 0 & 0 \end{array} \right] ,
\]

where

\[
c_{j}(x, t) := c_{j}^{0} e^{-2i(\Lambda x + i \Lambda t) / \varepsilon}, \quad \Lambda := \Lambda_{k = k_{j}}.
\]

(259)
Normalization: The matrix $M(k; x, t)$ is normalized in the sense that
\[ \lim_{k \to 0} M(k; x, t) = I, \] (260)
with the limit being taken in any direction.

At a given $x \in \mathbb{R}$ and $t \geq 0$ we say that the Cauchy problem for the MNLS equation (1) has a unique solution in the sense of inverse scattering if this Riemann-Hilbert problem has a unique solution for which the limit
\[ \phi(x, t) := \lim_{k \to \infty} \frac{2k M_{12}(k; x, t)}{\alpha M_{22}(k; x, t)} \] (261)
exists at least in the sense of an approach to $k = \infty$ that is nontangential to the axes $\text{Im}\{k^2\} = 0$. Lee [31] has generalized the analysis of Beals and Coifman [5] to study the scattering and inverse-scattering maps for the derivative NLS equation
\[ i \frac{\partial \psi}{\partial \tau} + \frac{1}{2} \nabla^2 \psi + i \alpha \nabla (|\psi|^2 \psi) = 0 \] (262)
with initial data $\psi(\xi, 0) = \psi_0(\xi)$. This problem was shown in the introduction to be related to the MNLS equation (1) by a Galilean boost with velocity $c = \alpha^{-1}$. Lee obtains global well-posedness of the Cauchy problem in the Schwartz space $\mathcal{S}(\mathbb{R})$ for a dense subset of initial data, but the theory is not entirely complete because certain “nongeneric” initial conditions must be excluded. The nongeneric initial conditions include those for which there are “spectral singularities”; that is, the spectral functions analogous to $S_{11}(k)$ and $S_{22}(k)$ have zeros on the axes $\text{Im}\{k^2\} = 0$.

The solution of the Cauchy problem in the sense of inverse scattering would be expected to agree with solutions thereof that have been shown to be unique using Fourier-based (nonlinear) methods. For example, in [23] Hayashi and Ozawa considered the Cauchy problem for the derivative NLS equation and showed that if $\psi_0$ is a Schwartz class ($\mathcal{S}(\mathbb{R})$) function, and if
\[ \|\psi_0\|_2^2 = \int_{-\infty}^{+\infty} |\psi_0(\xi)|^2 d\xi < \frac{\pi}{\alpha} \] (263)
then there exists a unique global solution $\psi(\xi, \tau)$ that is Schwartz class for all $\tau \in \mathbb{R}$ and moreover $\psi : \mathbb{R} \to \mathcal{S}$ is of class $C^\infty(\mathbb{R}; \mathcal{S})$. It has also been shown that the constraint [2030] on the $L^2(\mathbb{R})$ norm of the initial data may be dispensed with at the expense of knowing existence only for a finite time. For example, in [30], Pipolo showed that for initial data $\psi_0$ of class $H^1,\infty(\mathbb{R})$ (meaning that $\xi^k \psi_0(\xi)$, $\xi^k \psi_0'(\xi)$, $\xi^k \psi_0''(\xi)$, and $\xi^k \psi_0''''(\xi)$ are all in $L^2(\mathbb{R})$ for all integer $k \geq 0$) then there is a number $T > 0$ such that a unique solution $\psi(\xi, \tau)$ of (262) exists of class $C^\infty([-T, T] \setminus \{0\}; C^\infty(\mathbb{R}))$. There are many other results in the literature for

\[ 14 \text{It is conjectured in [23] that the inequality [2030] is sharp in the sense that there exist initial data in } \mathcal{S} \text{ violating this bound leading to solutions that blow up in } L^\infty(\mathbb{R}) \text{ in finite time. This conjecture is based upon scaling arguments that suggest that the derivative NLS equation behaves similarly to the critical collapse case for the one-dimensional semilinear Schrödinger equation:} \]
\[ \frac{\partial \psi}{\partial \tau} + \frac{1}{2} \nabla^2 \psi + |\psi|^4 \psi = 0. \]

Despite the Hayashi-Ozawa conjecture, the literature is awash with exact (global) solutions of the derivative NLS equation (262) that have finite $L^2(\mathbb{R})$ norm but that violate (263). For example, the soliton solutions of (262) given by
\[ \psi = \frac{4\text{Im}(z^2)}{\alpha} \frac{z e^{-\zeta} + z^* e^{\zeta}}{(z e^{\zeta} + z^* e^{-\zeta})^2} e^{-i\theta}, \quad \zeta := \frac{4\text{Im}(z^2)}{\alpha} \left( \zeta_0 + \frac{4}{\alpha} \text{Re}(z^2) \tau \right), \quad \theta := \frac{4}{\alpha} \text{Re}(z^2) \xi + \frac{8}{\alpha^2} \text{Re}(z^4) \tau + \theta_0, \]
where $z \in \mathbb{C}$, $\zeta_0 \in \mathbb{R}$, and $\theta_0 \in \mathbb{R}$ are arbitrary parameters, are all Schwartz class functions uniformly bounded in $L^2(\mathbb{R})$ norm for each $\tau$, but the upper bound exceeds, exactly by a factor of two, the restriction of (263).

\[ \sup_{z \in \mathbb{C}} \int_{-\infty}^{+\infty} |\psi(\xi, \tau; z)|^2 d\xi = \frac{2\pi}{\alpha}. \]
It follows also that there exist exact multisolution solutions (that is, solutions for which $r(k; 0) \equiv 0$ in the Riemann-Hilbert problem), necessarily in $\mathcal{S}$, with arbitrarily large $L^2(\mathbb{R})$ norm. To the degree that approximations by multisoliton solutions is possible, there is therefore a dense set of initial data for (262) with arbitrary $L^2(\mathbb{R})$ norm for which there exists a global Schwartz class solution. The validity of the Hayashi-Ozawa conjecture would then seem to rest upon the contribution of nonzero $r(k; 0)$ to the Riemann-Hilbert problem. Note that although Lee’s analysis [31] of the same Cauchy problem using inverse-scattering techniques avoids entirely any assumptions about the size of the $L^2(\mathbb{R})$-norm, it excludes nongeneric data leading to spectral singularities and hence is not inconsistent with the Hayashi-Ozawa conjecture.
the Cauchy problem associated to the derivative NLS equation \( \text{(202)} \) (see in addition \([10, 11, 22, 24, 34, 35]\)) all differing in detail but all granting existence of a unique solution under either a sufficiently small \( L^2(\mathbb{R}) \) norm or a finite lifetime condition.

All of these results apply equally well to the modified NLS equation \( \text{(11)} \) by a Galilean change of coordinates followed by a rescaling of \( x \) and \( t \) by \( \varepsilon \). It must be said, however, that these results are unsatisfactory from the point of view of the semiclassical \( (\varepsilon \downarrow 0) \) limit for the Cauchy problem for \( \text{(11)} \) with \( \phi(x,0) = A(x)e^{iS(x)/\varepsilon} \), because when \( A \) and \( S \) are fixed the \( L^2(\mathbb{R}) \) norm on the left-hand side of \( \text{(263)} \) becomes, after rescaling of \( x \) by \( \varepsilon \), proportional to \( 1/\varepsilon \) which appears to rule out global solutions. On the other hand, upon rescaling of \( t \) by \( \varepsilon \), the fixed lifetime of a solution guaranteed by Pipolo’s result becomes arbitrarily small in the limit \( \varepsilon \downarrow 0 \). So it seems that there are no results in the literature guaranteeing the existence of a unique solution to the Cauchy problem for the MNLS equation \( \text{(11)} \) with initial data of the form \( \phi(x,0) = A(x)e^{iS(x)/\varepsilon} \) in the vicinity of fixed \( x \) and \( t > 0 \) when \( \varepsilon \) is sufficiently small.

Therefore, to deal with semiclassical problems we are forced to take the approach of defining the solution of the Cauchy problem by the Riemann-Hilbert problem. Generally speaking, establishing the existence of solutions for the Riemann-Hilbert problem is itself a difficult problem requiring careful analysis and the correct conditions on the function \( r(k;0) \). See \([5, 12]\) for more details about the corresponding problem in a simpler situation, and \([31]\) for information relevant to the equivalent derivative NLS problem. However, when one is considering semiclassical asymptotics the problem becomes somewhat simpler because there is the possibility of the systematic construction of an explicit parametrix by means of which the Riemann-Hilbert problem is reduced to a “small-norm” problem that can be solved uniquely by iteration. This is the essence of the steepest descent method of Deift and Zhou. See \([27, 39]\) for examples of this sort of analysis for the case of the Riemann-Hilbert problem corresponding to the focusing NLS equation. We intend to carry out such a steepest descent analysis for the Riemann-Hilbert problem corresponding to the MNLS equation in a subsequent work. We may have confidence in the success of such a program also because certain convergence results for the semiclassical MNLS equation (or more correctly the derivative NLS equation) have been obtained by Desjardins, Lin, and Tso using “nonintegrable” methods \([16, 15]\). As the effective solution size and lifetime are both large in this situation, these results suggest that global solutions indeed may typically exist for large data. However, the approach of \([16]\) fails when the solutions of the associated modulation equations become nonclassical, and it is exactly at such junctures that the integrable theory typically yields its best results, explaining and giving strong approximations for the wild oscillations that are introduced by dispersive regularization.

**Even spectral symmetry.** It turns out that our Riemann-Hilbert problem for \( M(k;x,t) \) has “too much symmetry”. As we will see below, there is a relation between \( M(k;x,t) \) and \( M(-k;x,t) \) that will, in the course of semiclassical analysis by the steepest descent method, lead to Riemann surfaces with too many branch points. For example, in Section \( \text{(2.3)} \) it was shown that the solutions of the spectral problem \( \text{(173)} \) corresponding to complex exponential plane-wave solutions of the MNLS equation naturally live on a hyperelliptic curve that is two-sheeted covering of the complex \( k \)-plane with four distinct square-root type branch points. This surface has genus one, and functions on such a Riemann surface are constructed from Riemann theta functions of genus one (that is, elliptic functions). One would therefore expect that the potential \( \phi \) would itself be constructed from elliptic functions, which appears to contradict our starting point: that \( \phi \) was of the simple form \( \rho^{1/2}e^{iux} \).

In this situation, judicious use of factorization identities of Riemann theta functions may be required to see that the “true genus” is smaller than first thought. To avoid these difficulties it is desirable to formulate a new Riemann-Hilbert problem that effectively quotients out the extra symmetry. This approach was also used by Deift, Venakides, and Zhou \([14]\) in their analysis of the Korteweg-de Vries equation, and the specific reduction we use here can be inferred also from the paper of Kaup and Newell \([28]\) on the inverse-scattering transform for the derivative NLS equation.

It was pointed out earlier that the Jost functions enjoy a holomorphic symmetry; see \([22, 23]\). This symmetry, taken together with the implied relation \( \text{(231)} \) and used in the definition \( \text{(239)} \) shows that for each \( t \) the solution \( M(k;x,t) \) of the Riemann-Hilbert problem (assuming existence and uniqueness for the moment) has

\footnote{Although as pointed out in Section \( \text{(2.4.1)} \) these authors do not treat the modulationally unstable case.}
the property that
\[
M(-k; x, t) = i^{\sigma_3} M(k; x, t) i^{-\sigma_3} = \begin{bmatrix}
M_{11}(k; x, t) & -M_{12}(k; x, t) \\
M_{21}(k; x, t) & M_{22}(k; x, t)
\end{bmatrix}, \quad \text{Im}\{k^2\} \neq 0.
\] (264)

It follows that the matrix \(N(z; x, t)\) defined by
\[
N(z; x, t) := k^{\sigma_3/2} M(k; x, t) k^{-\sigma_3/2} = \begin{bmatrix}
M_{11}(k; x, t) & kM_{12}(k; x, t) \\
k^{-1}M_{21}(k; x, t) & M_{22}(k; x, t)
\end{bmatrix}
\] (265)
is well-defined as a function of \(z = -k^2\) for \(\text{Im}\{z\} \neq 0\). From the properties defining \(M(k; x, t)\) we may then deduce a Riemann-Hilbert problem solved by the first row of \(N(z; x, t)\).

It is clear that \(N(z; x, t)\) is analytic for \(\text{Im}\{z\} \neq 0\), and that the boundary values taken for \(z \in \mathbb{R}\) are continuous at least for \(z \neq 0\). To analyze the behavior near \(z = 0\), we see immediately from the normalization condition on \(M(k; x, t)\) that
\[
\lim_{z \to 0} N_{11}(z; x, t) = \lim_{z \to 0} N_{22}(z; x, t) = 1 \quad \text{and} \quad \lim_{z \to 0} N_{12}(z; x, t) = 0.
\] (266)
The statement that \(M_{21}(k; x, t) \to 0\) as \(k \to 0\) alone is not enough to determine the limiting value of \(N_{21}(k; x, t)\); however using the integral equations for \(Y_{21}(k; x, t)\) it is possible to show that
\[
\lim_{z \to 0} N_{21}(z; x, t) = \frac{2i}{\alpha} \int_{\pm \infty}^x \phi(y, t) e^{-i(y-x)/(\alpha \varepsilon)} \, dy
\] (267)
This limit cannot be used to define a normalization condition in the usual way because it has a different value in different half-planes for \(z\), and more crucially, it depends on the (unknown) potential \(\phi(x, t)\). This is the reason why we will only seek to determine the first row of \(N(z; x, t)\) as the solution of a Riemann-Hilbert problem (a similar situation occurs in [14]). Being able to find only the first row of \(N(z; x, t)\) is not an obstruction to determining the potential however, since from the reconstruction formula \([261]\) together with the antiholomorphic symmetry relations \([228]\) and \([230]\) we may equally well find \(\phi(x, t)\) from the limit
\[
\phi(x, t) = \frac{2}{\alpha} \lim_{k \to \infty} \frac{k M_{12}(k; x, t)}{M_{11}(k; x, t)^*} = \frac{2}{\alpha} \lim_{z \to \infty} \frac{N_{12}(z; x, t)}{N_{11}(z^*; x, t)^*}.
\] (268)
Next, we calculate the jump conditions satisfied by \(N(z; x, t)\) on the real \(z\)-axis. This is straightforward, and the consistency of the calculation (two values of \(k\) with \(\text{Im}\{k^2\} = 0\) correspond to each \(z \in \mathbb{R}\)) depends on the symmetry relations \([231]\) which imply that \(k^{-1}r(k; 0)\) is actually a function of \(z = -k^2\) that is bounded as \(z \to 0\). Defining the boundary values by
\[
N_{\pm}(z; x, t) := \lim_{w \to z, w \neq \pm \text{Im}(w) > 0} N(w; x, t), \quad z \in \mathbb{R},
\] (269)
and noting that in terms of \(z = -k^2\) we have
\[
\Lambda = \frac{2i}{\alpha} \left( z + \frac{1}{4} \right),
\] (270)
one finds that
\[
N_+(z; x, t) = N_-(z; x, t) e^{(\Lambda x + i\Lambda^2 t)\sigma_3/\varepsilon} \begin{bmatrix}
1 + z |r(k; 0)|^2 & k r(k; 0) \\
k^{-1} r(k; 0)^* & 1
\end{bmatrix} e^{-(\Lambda x + i\Lambda^2 t)\sigma_3/\varepsilon}, \quad z < 0.
\] (271)
Therefore, defining
\[
\rho(z) := k^{-1} r(k; 0), \quad z \in \mathbb{R},
\] (272)
(this is indeed well-defined because \(r(k; 0)\) is an odd function of \(k\)) we resolve the dichotomy of signs in \([271]\) and find simply
\[
N_+(z; x, t) = N_-(z; x, t) e^{(\Lambda x + i\Lambda^2 t)\sigma_3/\varepsilon} \begin{bmatrix}
1 - z |\rho(z)|^2 & -z \rho(z) \\
\rho(z)^* & 1
\end{bmatrix} e^{-(\Lambda x + i\Lambda^2 t)\sigma_3/\varepsilon}, \quad z \in \mathbb{R}.
\] (273)
Finally, we examine the singularities of $N(z;x,t)$. Near a pole $k = k_j$ of $M(k;x,t)$ with $\text{Im}\{k_j^2\} < 0$ we know from (257) that

$$M(k;x,t) = \begin{bmatrix} a_jc_j(k-k_j)^{-1} + O(1) & a_j + O(k-k_j) \\ b_jc_j(k-k_j)^{-1} + O(1) & b_j + O(k-k_j) \end{bmatrix}, \quad k \to k_j$$

(274)

for some scalar functions $a_j = a_j(x,t)$ and $b_j = b_j(x,t)$ (and $c_j = c_j(x,t)$ is defined by (259)). Applying the symmetry (264) we see that

$$M(k;x,t) = \begin{bmatrix} -a_jc_j(k+k_j)^{-1} + O(1) & -a_j + O(k+k_j) \\ b_jc_j(k+k_j)^{-1} + O(1) & b_j + O(k+k_j) \end{bmatrix}, \quad k \to -k_j.$$  

(275)

From these and the definition (265) it follows easily that

$$N(z;x,t) = \begin{bmatrix} a_jc_j(k-k_j)^{-1} + O(1) & a_j + O(k-k_j) \\ k_j^{-1}b_jc_j(k-k_j)^{-1} + O(1) & k_jb_j + O(k-k_j) \end{bmatrix}, \quad k \to k_j$$

(276)

and

$$N(z;x,t) = \begin{bmatrix} -a_jc_j(k+k_j)^{-1} + O(1) & -a_j + O(k+k_j) \\ -k_j^{-1}b_jc_j(k+k_j)^{-1} + O(1) & b_j + O(k+k_j) \end{bmatrix}, \quad k \to -k_j.$$  

(277)

In turn, these relations are simultaneously equivalent to

$$N(z;x,t) = \begin{bmatrix} -2k_ajc_j(z-z_j)^{-1} + O(1) & k_ja_j + O(z-z_j) \\ -2b_jc_j(z-z_j)^{-1} + O(1) & b_j + O(z-z_j) \end{bmatrix}, \quad z \to z_j := -k_j^2.$$  

(278)

Therefore, we have

$$\text{Res}_{z=z_j} N(z;x,t) = \lim_{z \to z_j} N(z;x,t) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

(279)

a relation equivalent to both relations in (257). In a similar way one derives the condition

$$\text{Res}_{z=z_j} N(z;x,t) = \lim_{z \to z_j} N(z;x,t) \begin{bmatrix} 0 & -2z_j^*c_j(x,t)^* \\ 0 & 0 \end{bmatrix},$$

(280)

a relation equivalent to both relations in (258).

We are now in a position to formulate a “less symmetric” Riemann-Hilbert problem to determine $\phi(x,t)$. Given $\rho(z)$ for $z \in \mathbb{R}$ and a set of points $z_1, \ldots, z_N$ in the upper half-plane along with a corresponding set of nonzero complex numbers $c_{1}^0, \ldots, c_N^0$, seek a two-component row vector $n^T(z;x,t) = [n_1(z;x,t), n_2(z;x,t)]$ with the following properties

**Analyticity:** The row vector $n^T(z;x,t)$ is an analytic function of $z$ for $z \in \mathbb{C}\setminus(\mathbb{R}\cup\{z_1, \ldots, z_N, z_1^*, \ldots, z_N^*\})$ that takes continuous boundary values on the real line and is uniformly bounded as $z \to \infty$.

**Jump Condition:** With the boundary values defined by

$$n^T_+(z;x,t) := \lim_{w \to z, \text{Im}(w) > 0} n^T(w;x,t), \quad z \in \mathbb{R},$$

(281)

the relation

$$n^T_-(z;x,t) = n^T_+(z;x,t)e^{(Ax+iA^2t)\sigma_3/\varepsilon} \begin{bmatrix} 1 - z|\rho(z)|^2 & -z\rho(z) \\ \rho(z)^* & 1 \end{bmatrix} e^{-(Ax+iA^2t)\sigma_3/\varepsilon}$$

(282)

holds for $z \in \mathbb{R}$.

**Singularities:** The row vector $n^T(z;x,t)$ has simple poles at the points $z_1, \ldots, z_N$ and their complex conjugates. The residues at these points satisfy

$$\text{Res}_{z=z_j} n^T(z;x,t) = \lim_{z \to z_j} n^T(z;x,t) \begin{bmatrix} 0 & 0 \\ -2c_j(x,t) & 0 \end{bmatrix},$$

(283)

and

$$\text{Res}_{z=z_j^*} n^T(z;x,t) = \lim_{z \to z_j^*} n^T(z;x,t) \begin{bmatrix} 0 & -2z_j^*c_j(x,t)^* \\ 0 & 0 \end{bmatrix},$$

(284)

for $j = 1, \ldots, N$.  

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Normalization: The row vector $n^T(z; x, t)$ satisfies
\[
\lim_{z \to 0} n^T(z; x, t) = \begin{bmatrix} 1 & 0 \end{bmatrix}
\] (285)
with the limit being taken in any direction.

From the solution of this Riemann-Hilbert problem, the corresponding solution of the Cauchy problem for
the MNLS equation (1) is then given by
\[
\phi(x, t) = \frac{2}{\alpha} \lim_{z \to \infty} \frac{n_2(z; x, t)}{n_1(z^*; x, t)^2}.
\] (286)

We wish to emphasize that this Riemann-Hilbert problem involves discontinuities only along the real $z$-
axis and poles in complex-conjugate pairs. With the extra symmetry removed, this problem is therefore
quite similar to the Riemann-Hilbert problem associated with the nonselfadjoint Zakharov-Shabat eigenvalue
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