Adaptive Gradient Descent Methods for Computing Implied Volatility *

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Abstract

In this paper, a new numerical method based on adaptive gradient descent optimizers is provided for computing the implied volatility from the Black-Scholes (B-S) option pricing model. It is shown that the new method is more accurate than the close form approximation. Compared with the Newton-Raphson method, the new method obtains a reliable rate of convergence and tends to be less sensitive to the beginning point.

Keywords: Implied volatility, Black-Scholes equation, Adaptive gradient descent methods, Newton-Raphson iterations

1. Introduction

Black and Scholes (1973)\textsuperscript{5} and Merton (1973) proposed the European call option pricing model on a stock as follows

\[ s \Phi(d_1) - ke^{-rt} \Phi(d_2) = c \]

\[ d_1 = \frac{\ln \frac{s}{k} + (r + \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \]

\[ d_2 = d_1 - \sigma \sqrt{t}, \] (1)

where \( \sigma \) is the volatility, \( s \) is the corresponding spot price, \( k \) is the strike price, \( r \) is the risk-free interest rate, \( c \) is the price of the call option, \( t \) is the time of maturity, and \( \Phi(\cdot) \) is the cumulative distribution function of standard normal distribution up to \( x \) i.e.

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\tau^2/2} d\tau. \]

All parameters except \( \sigma \) can be obtained directly from market data of the B-S equation \( \text{(1)} \). This enables a market-based estimation of a call option’s future volatility. Implied volatility can be estimated by the inverse use of \( \text{(1)} \), which infers \( \sigma \) from the market price of the call option.

Latane and Rendleman \textsuperscript{12} first proposed that implied volatility is extensively used for those that claim some predictive power over future volatility. Bandi and Perron \textsuperscript{4} suggested that implied volatility is unbiased in a long term, and is a predictor of realized volatility. Conclusively, implied volatility is an irreplaceable measure of financial risk, since the standard deviation is not a representative metric for estimating assets’ return in future.

However, the inversion of the B-S formula \( \text{(1)} \) cannot be observed directly \textsuperscript{10}. The investigations have been conducted in two directions: one is about the most common analytical approximations \textsuperscript{6–8, 13},

\*This author is partially supported by NSFC 11901393, and Natural Science Fundation of Shanghai under the grant 19ZR1436300. Code available at https://github.com/cloudy-sfu/SGD-Implied-Volatility

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Preprint submitted to Elsevier August 17, 2021
the other is numerical procedures. The Chance model and Chambers model are analytical approximations but the two methods have the same problem of ever-increasing error when the value of the stock price diverges from the strike price. However, the Corrado-Miller formula is an approximated analytical solution of the implied volatility, which is relatively accurate even when stock price deviates from the discounted strike price. Corrado-Miller formula is an approximated analytical solution of the implied volatility, which is relatively accurate even when stock price deviates from the discounted strike price. However, the Corrado-Miller formula contains a square root, so it cannot be calculated in some cases. Numerical methods are based on iterative root-finding algorithms, such as the bisection, Newton-Raphson method, and Dekker-Brent method. The bisection and Dekker-Brent method both require a pair of upper and lower bounds, and if the theoretical root exceeds the boundary, these methods cannot find the root. Meanwhile, the Newton-Raphson iterations are sensitive to the initial value of \( \sigma \).

This paper focuses on providing an accurate numerical approximation of the implied volatility. We transform the B-S formula to an equivalent optimization problem. Afterward, we use ‘Adam’ and ‘Adabelief’ adaptive gradient decent optimizers, which are originally used in fitting neural networks in data science, to find the minimum of loss function. The numerical results show this method is more accurate than the analytical approximation of Corrado-Miller approach. Meanwhile, compared with the Newton-Raphson, this method has better adaptability and is less sensitive to the beginning point.

The remainder of this paper is organized as follows. We review the Newton-Raphson and show its limitations in Section 2. In Section 3, adaptive gradient decent methods are used to get the approximation of implied volatility. In Section 4, we collect 88077 call option contracts from Shanghai Stock Exchange, use different methods to approximate their implied volatility. The results indicate that our method is accurate and robust. The conclusions are drawn in Section 5.

2. Review of Newton-Raphson method

The numerical approximation of implied volatility from B-S formula is to find the root of

\[
    g(\sigma) = s \Phi(d_1) - k e^{-rt} \Phi(d_2) - c = 0. \tag{2}
\]

According to the Newton-Raphson method, a sequence \( \sigma_0, \sigma_1, ... \) is defined as

\[
    \sigma_{i+1} = \sigma_i - \frac{g(\sigma_i)}{g'(\sigma_i)}
\]

with a beginning point \( \sigma_0 \). If the beginning point is in the neighborhood of theoretical root and \( g'(\sigma) \neq 0 \), then Newton iteration usually converge. The convergence can arrive at least quadratic if the zero has multiplicity 1.

More precisely, the following well-known theorem holds:

**Theorem 1.** Let \( g : I \to \mathbb{R} \) be a convex (or concave) differentiable function on an interval \( I \subseteq \mathbb{R} \) with at least one root. Then the sequence \( \{\sigma_n\} \) obtained from Newton iteration will converge to a root of \( g(\sigma) \), provided that \( g'(\sigma_0) \neq 0 \) and \( \sigma_1 \in I \) for the given beginning point \( \sigma_0 \in I \).

From (2), there is

\[
    g'(\sigma) = s \sqrt{\frac{t}{2\pi}} e^{-\frac{d_1^2}{2}}
\]

thus \( g'(\sigma) > 0 \) for \( \sigma > 0 \).

\[
    g''(\sigma) = s \sqrt{\frac{t}{2\pi}} e^{-\frac{d_1^2}{2}} \frac{d_1 d_2}{\sigma}
\]

thus \( g''(\sigma) = 0 \) for \( \sigma = \sqrt{\frac{2}{t} |\ln \frac{s}{k} + rt|} \).

Therefore, the function \( g(\sigma) \) has the following properties:

- For \( \sigma > 0 \), there is \( g'(\sigma) > 0 \), thus \( g \) is strictly increasing in \( \sigma > 0 \).

- \( g(\sigma) \) is convex in \([0, \lambda]\) and concave in \([\lambda, +\infty)\), where \( \lambda = \sqrt{\frac{2}{t} |\ln \frac{s}{k} + rt|} \).
We adopt two methods to implement the optimization of Algorithm 1. Let \( \eta \) be the unique solution of the equation \( h(\sigma) = \max \{0, s - k\} \). The 'Adabelief' [9] is another optimizer that tends not to pre-define schedules of adjusting the learning rate [15], preventing the objective function from fluctuate or diverge nearby the optimal value. 

The 'Adam' optimizer [3] derived from gradient descent, where adaptive moment estimation is adopted based on the gradient of symbolic variables are automatically calculated, powered by TensorFlow[1]. 

### 3. Adaptive Gradient Decent Methods

The numerical approximation (2) can be transformed into solving an optimization problem

\[
\min_{\sigma} h(\sigma) = (s\Phi(d_1) - ke^{-rt}\Phi(d_2) - c)^2.
\]

Gradient descent is an iterative method, each iteration weight \( \eta \) based on the gradient of \( h(\sigma) \):

\[
\sigma_{t+1} := \sigma_t - \eta \nabla h(\sigma_t),
\]

where \( \eta \) is the step size. If beginning point \( \sigma_0 \) is in the neighbourhood of optimum value and learning rate is relatively small, then this method achieves linear convergence [2].

The ‘Adam’ optimizer [3] derived from gradient descent, where adaptive moment estimation is adopted to solve (7). This algorithm uses the first and second moment estimators of gradient to adapt the learning rate. The first momentum of gradient is \( m_t = \beta_1 m_{t-1} + (1 - \beta_1) \nabla h(\sigma_t) \) where \( \beta_1 \) is by default equal to 0.9, \( m_0 = 0 \). We have

\[
\mathbb{E}(m_t) = \mathbb{E}(\nabla h(\sigma_t)) = \mathbb{E}((1 - \beta_1) \sum_{i=0}^{t} \beta_1^{t-i} \nabla h(\sigma_i)) = \mathbb{E}(\nabla h(\sigma_t))(1 - \beta_1) \sum_{i=0}^{t} \beta_1^{t-i} + \zeta.
\]

From (8), the bias corrected estimators for the first momentum will be \( \hat{m}_t = \frac{m_t}{1 - \beta_1^t} \).

Similarly, the second moment of gradient can be obtain \( v_t = \beta_2 v_{t-1} + (1 - \beta_2)(\nabla h(\sigma_t))^2 \), where \( \beta_2 = 0.999, v_0 = 0 \). We have

\[
\mathbb{E}(v_t) = \mathbb{E}(\nabla h(\sigma_t))^2(1 - \beta_2^t) + \zeta.
\]

Then we get bias correction for the second momentum \( \hat{v}_t = \frac{v_t}{1 - \beta_2^t} \). Adam’s update rule is

\[
\sigma_t = \sigma_{t-1} - \eta \frac{\hat{m}_t}{\sqrt{\hat{v}_t} + \epsilon},
\]

where \( \eta = 0.001 \) is learning rate and \( \epsilon \) is a small term (usually \( 10^{-7} \)) preventing division by zero. See Algorithm [1].

In this algorithm, gradient of symbolic variables are automatically calculated, powered by TensorFlow[1]. We adopt two methods to implement the optimization of \( h(\sigma) \), as for the following advantages: The ‘Adam’ optimizer tends not to pre-define schedules of adjusting the learning rate [15], preventing the objective function from fluctuate or diverge nearby the optimal value. The ‘Adabelief’ [9] is another optimizer that we use to estimate \( \sigma \). It makes an improvement based on ‘Adam’: it replaces the formula \( v_t \leftarrow \beta_2 v_{t-1} + (1 - \beta_2)\nabla h(\sigma_{t-1})^2 \) in Algorithm [1] with \( v_t \leftarrow \beta_2 v_{t-1} + (1 - \beta_2)\nabla (h(\sigma_{t-1} - m_t))^2 \).
Algorithm 1: Gradient decent method for solving implied volatility

Define:
\[
\sigma \quad \text{// Implied volatility, a symbolic variable}
\]
\[
\sigma_0 \quad \text{// The beginning point, a numerical value}
\]
\[
\eta = 0.001, \beta_1 = 0.9, \beta_2 = 0.999, \epsilon = 10^{-7} \quad \text{// Good default settings for machine learning}
\]
\[
m_0 = 0, v_0 = 0, i = 0 \quad \text{// The beginning point, a numerical value}
\]
\[
h(\sigma) \quad \text{// Objective function with parameters } \sigma
\]

while \( \sigma \) not converged do
\[
i \leftarrow i + 1
\]
\[
m_i \leftarrow \beta_1 m_{i-1} + (1 - \beta_1) \nabla h(\sigma_{i-1})
\]
\[
v_i \leftarrow \beta_2 v_{i-1} + (1 - \beta_2)(\nabla h(\sigma_{i-1}))^2
\]
\[
\hat{m}_i \leftarrow \frac{m_i}{1 - \beta_1}
\]
\[
\hat{v}_i \leftarrow \frac{v_i}{1 - \beta_2}
\]
\[
\sigma_i \leftarrow \sigma_{i-1} - \eta \frac{\hat{m}_i}{\sqrt{\hat{v}_i + \epsilon}}
\]

Return: \( \sigma_i \)

4. Experiments

This experiment collects 88077 numerical samples of call options on Shanghai Stock Exchange from 2015-02 to 2020-07. Samples meeting any of the following rules are deleted: 1) \( t = 0 \) (the solution does not exist). 2) \( g(\sigma) > 0 \) (there is no root at \( \sigma \in [0, +\infty) \) since \( g'(\sigma) > 0 \). After the pre-processing, 83427 samples remain in the data set. This data set records only original quotation of call options on Shanghai Stock Exchange, and does not include derivative indicators published by stock brokerage firms.

Suppose \( \hat{\sigma}_{j,n} (j = 1, 2, ..., N) \) is the numerical solution, \( n \) is the number of iterative steps, and \( N = 83427 \) is the number of samples. Inserting \( \hat{\sigma}_{j,n} \) into \( h(\sigma) \) we can get a numerical approximation of the price of the call option \( c_n \). We use two functions to test the efficiency of the numerical methods. One is the ratio of not converged samples, defined as

\[
NC = \frac{1}{N} \sum_{j=1}^{N} l(c_j) \quad \text{with } l(c_j) = \begin{cases} 0, & |\hat{c}_{j,n} - c_{j,n-1}| < 10^{-4} \\ 1, & \text{otherwise} \end{cases}
\]

The other is

\[
MAE = \frac{1}{N} \sum_{j=1}^{N} |\hat{c}_j - c_j| (1 - l(c_j)) \quad \text{with } N = N(1 - NC)
\]

We present the numerical results of Newton-Raphson iteration, Adam, Adabelief methods in Table 1 with different beginning points \( \sigma_0 = 0.1, 0.25, 0.4, 0.55, 0.7, 0.85, 1 \). During program execution, we consider iteration sequence \( \{\hat{\sigma}_{j,n}\} \) to be convergent when \( |\hat{c}_{j,n} - \hat{c}_{j,n-1}| < 10^{-4} \). Provided \( \sigma_0 = 0.1 \), it is clearly shown that NC is about 4 \times 10^{-4} with gradient descent methods, while NC is about 0.32 with the Newton-Raphson iteration. NC is almost close to 0 for the two gradient descent methods with other beginning points. We demonstrate that gradient descent methods have better convergence and tends to be less sensitive to the beginning point than Newton-Raphson iteration.

We give an estimation of \( \sigma \) by Corrado-Miller formula \( \int \) (analytical approximation method):

\[
\sigma \approx \sqrt{\frac{2\pi}{t}} \frac{1}{s + x} \left( c - 0.5(s - x) + \sqrt{(c - 0.5(s - x))^2 - \frac{(s - x)^2}{\pi}} \right) \quad \text{with } x = ke^{-rt}.
\]

The histogram of \( \sigma \) estimated by Corrado-Miller formula is as Figure 1. The MAE is about 3.23 \times 10^{-3} of the Corrado-Miller formula accordingly. The comparison of MAE between adaptive gradient descent methods
Table 1: Convergence and MAE of different methods and beginning points

| beginning point | method     | NC     | MAE               |
|-----------------|------------|--------|-------------------|
| 0.1             | Newton     | 0.321910 | 1.3108E-16       |
| 0.1             | GD-Adam    | 0.000407 | 3.7269E-03       |
| 0.1             | GD-Adabelief | 0.000407 | 4.3763E-03      |
| 0.25            | Newton     | 0.075097 | 1.3503E-16       |
| 0.25            | GD-Adam    | 0.000132 | 6.6421E-04       |
| 0.25            | GD-Adabelief | 0.000036 | 6.8931E-04      |
| 0.4             | Newton     | 0.027226 | 1.3291E-16       |
| 0.4             | GD-Adam    | 0.000000 | 3.2986E-04       |
| 0.4             | GD-Adabelief | 0.000012 | 2.3508E-04      |
| 0.55            | Newton     | 0.012811 | 1.3251E-16       |
| 0.55            | GD-Adam    | 0.000000 | 2.4921E-04       |
| 0.55            | GD-Adabelief | 0.000012 | 1.1105E-04      |
| 0.7             | Newton     | 0.007093 | 1.3177E-16       |
| 0.7             | GD-Adam    | 0.000000 | 2.1206E-04       |
| 0.7             | GD-Adabelief | 0.000000 | 5.9906E-05      |
| 0.85            | Newton     | 0.004402 | 1.3202E-16       |
| 0.85            | GD-Adam    | 0.000000 | 2.1036E-04       |
| 0.85            | GD-Adabelief | 0.000000 | 3.4524E-05      |
| 1               | Newton     | 0.002883 | 1.3166E-16       |
| 1               | GD-Adam    | 0.000000 | 2.5383E-04       |
| 1               | GD-Adabelief | 0.000000 | 3.1324E-05      |

and Corrado-Miller is indicated in Figure 2, where adaptive gradient descent methods has a lower error (MAE) than Corrado-Miller. During the iterative process, MAE decreases and convergent to a constant, and the learning speed slows down of the two adaptive gradient descent methods. See Figure 3a and 3b.
5. Conclusions

In this paper, we transform the approximation of implied volatility from finding the root of the B-S equation to an optimization model and use two adaptive gradient descent methods (‘Adam’ and ‘Adabelief’) to approximate the value of implied volatility. The numerical examples show that it is more accurate than Corrado-Miller, the analytical solution, in most of the cases. The dependence on the initial iterative values is small for two adaptive gradient descent methods. These methods allow financial markets to provide implied volatility for call options close to the exercise date. Moreover, with the development of innovative optimizers in deep learning, it is easy to implement them by TensorFlow and migrate them to modify the computation method further.

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