ON CERTAIN CUSP FORMS ON A DEFINITE QUATERNION

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ABSTRACT. If \( D \) is the definite quaternion over \( \mathbb{Q} \) of discriminant \( p \), we compute, for any prime \( p > 3 \), the number of infinite dimensional cusp forms on \( D^* \) which are trivial at infinity, tamely ramified at \( p \), and have given conductor \( N \) away from \( p \). We include a detail explanation of a Deuring–type correspondence between supersingular elliptic curves in characteristic \( p \) and a certain double coset arising from the adelic points of \( D^* \).

INTRODUCTION

Let \( p \) be a prime number, and \( D \) the quaternion algebra over \( \mathbb{Q} \) ramified precisely at \( p \) and infinity. Let \( \pi \) be an irreducible, unitary, automorphic representation of the multiplicative group \( D^* \) of \( D \). For a prime number \( \ell \), let \( \pi_\ell \) be the local component of \( \pi \) at \( \ell \), and let \( \pi_\infty \) be the component of \( \pi \) at the archimedean place of \( \mathbb{Q} \). The representations \( \pi_p \) and \( \pi_\infty \), of the respective local groups \((D \otimes \mathbb{Q}_p)^*\) and \((D \otimes \mathbb{R})^*\), are both finite dimensional. In this paper we will only be concerned with those \( \pi \) so that

i) \( \pi_\infty \) is the trivial, one–dimensional representation;

ii) \( \pi_p \) has a nonzero vector fixed by the maximal pro–\( p \) subgroup of \( D_p^* \).

Condition ii) can be interpreted as a tame ramification requirement for \( \pi \) at \( p \). For a given integer \( N \geq 1 \) not divisible by \( p \), let \( \mathcal{A}(p, N) \) denote the set of isomorphism classes of automorphic representations \( \pi \) of \( D^* \) of the type described above and whose prime–to–\( p \) conductor is equal to \( N \). In this paper we obtain an explicit formula for the cardinality \( \mathcal{A}(p, N) \) of \( \mathcal{A}(p, N) \), and of some distinguished subsets (cf. Thm. [13]). For simplicity \( p \) is assumed to be \( > 3 \).

Associated to any irreducible, unitary, automorphic representation \( \pi \) of \( D^* \) there is a system of Hecke eigenvalues \( \Phi_\pi = (a_\ell)_{\ell \nmid pN} \), where \( \ell \) is a prime number not dividing \( pN \), and \( N \) is an integer divisible by the prime–to–\( p \) part of the conductor of \( \pi \). When the central character \( \chi_\pi \) of \( \pi \) has finite order (always the case if \( \pi \) is trivial at infinity), then it is known that the \( a_\ell \) are algebraic integers of a certain number field \( K(\pi) \), and therefore, once a prime
of $\mathcal{Q}$ above $p$ is chosen, can be reduced mod $p$ to obtain $\Phi(\pi) = (\bar{a}_\ell)_{\ell \mid pN}$, a system of Hecke eigenvalues mod $p$.

The tameness condition imposed to the automorphic forms $\pi$ is natural when studying the reduction mod $p$ of the totality of the systems of eigenvalues arising from cusp forms on $D^*$ trivial at infinity. A theorem of Serre relates these mod $p$ systems of eigenvalues to those arising from mod $p$ modular forms. In a forthcoming work, the author will adopt Serre’s quaternionic viewpoint to estimate from above, using the formulas of this paper, the number of mod $p$ Hecke eigenforms of given conductor.

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\section{Statement of the Theorem}

\subsection{Generalities}

Let $p$ be a prime number, and $D$ the quaternion algebra over $\mathbb{Q}$ ramified precisely at $p$ and infinity. The multiplicative group $D^*$ of $D$ is an algebraic group over $\mathbb{Q}$ that can be realized as a closed subgroup of $\text{GL}_4$. For any commutative $\mathbb{Q}$-algebra $K$, the group of $K$-valued points of $D^*$ is denoted by $D^*_K$. If $K$ is also a topological algebra, then the embedding $D^*_K \subset \text{GL}_4(K)$ makes $D^*_K$ a topological group. If $\nu \in \Sigma_{\mathbb{Q}}$ is any place of $\mathbb{Q}$, and $\mathbb{Q}_\nu$ is the corresponding completion, we will write $D^*_\mathbb{Q}_\nu$ for $D^*_{\mathbb{Q}_\nu}$; when no confusion can arise we may simply write $D^*_\mathbb{Q}$ for $D^*_{\mathbb{Q}_\nu}$. A finite place $\nu \in \Sigma_{\mathbb{Q}}$ will often be denoted by its residual characteristic $\ell$.

The group $D^*_\mathbb{Q}$ sits inside the adelic group $D^*_\mathbb{A}$ as a discrete subgroup, the homogeneous space $X = D^*_\mathbb{Q} \backslash D^*_\mathbb{A}$ is locally compact and admits a measure $dx$ that is invariant under the right translation action of $D^*_\mathbb{A}$. The center $Z$ of $D^*$, viewed as a closed, algebraic subgroup, is isomorphic to the multiplicative group $\mathbb{Q}^*$, if $\psi : Z_\mathbb{A} \to \mathbb{C}^*$ is a continuous Hecke character of finite order then the space $L^2(X, \psi)$ is defined to be the space of measurable functions $\varphi : D^*_\mathbb{A} \to \mathbb{C}$ such that:

i) $\varphi(\gamma g) = \varphi(g)$, for all $\gamma \in D^*_\mathbb{Q}$, $g \in D^*_\mathbb{A}$;

ii) $\varphi(gz) = \psi(z)\varphi(g)$, for all $z \in Z_\mathbb{A}$, $g \in D^*_\mathbb{A}$;

iii) $\int_{X/Z_\mathbb{A}} |f(x)|^2\,dx < \infty$.

The space $L^2(X, \psi)$ is a Hilbert space on which $D^*_\mathbb{A}$ acts unitarily by right translation. The following fundamental theorem is proved in \cite{6}:
ON CERTAIN CUSP FORMS ON A DEFINITE QUATERNION

Theorem 1.1. The unitary representation $\rho_\psi$ of $D_\Lambda^*$ on $L^2(X, \psi)$ decomposes uniquely as the Hilbert direct sum

$$\rho_\psi = \bigoplus_{\pi \in I_\psi} m(\pi)\pi$$

of a countable collection $I_\psi$ of irreducible, pairwise non–isomorphic, subrepresentations, each appearing with finite multiplicity.

The next result is due to Jacquet–Langlands (cf. [10], Prop. 11.1.1):

Theorem 1.2. The multiplicity $m(\pi)$ of each constituent $\pi$ appearing in $\rho_\psi$ is one.

Any irreducible representation $\pi$ occurring in the above decomposition of $\rho_\psi$, for some $\psi$, will be referred to as a cusp form.

In virtue of the tensor product Theorem (cf. [6]), any cusp form $\pi$ is isomorphic to a restricted tensor product of a unique family $(\pi_\nu)_{\nu \in \Sigma_Q}$ of irreducible, unitary representations of the local groups $D_\nu^*$. For any prime number $\ell \neq p$, there is an isomorphism $D_\ell^* \simeq \text{GL}_2(Q_\ell)$, and for almost all $\ell$ the local component $\pi_\ell$ of $\pi$ defines a representation of $\text{GL}_2(Q_\ell)$ that is unramified. If $\pi$ is infinite dimensional, then it follows from the approximation theorem that for every prime $\nu$ that is unramified in $D$ the local component $\pi_\nu$ is also infinite dimensional. On the other hand, $\pi_\nu$ is finite dimensional when $\nu$ is equal to $p$ or $\infty$.

If $\pi$ is infinite dimensional and $\ell \neq p$, then the conductor $c(\pi_\ell)$ is defined (cf. [3], §2.2, or [2]), and it will be thought of as a positive integer given by the appropriate power of $\ell$, instead that of an ideal of $\mathbb{Z}$. By definition, the prime–to–$p$ conductor $N(\pi)$ of $\pi$ is

$$N(\pi) = \prod_{\ell \neq p} c(\pi_\ell),$$

it is a well defined integer since $\pi_\ell$ is unramified for almost all $\ell$.

We will be concerned only with those infinite dimensional cusp forms $\pi$ such that

i) $\pi_\infty$ is the trivial, one–dimensional representation;

ii) $\pi_p$ has a nonzero vector fixed by the maximal pro–$p$ subgroup of $D_p^*$.

In the next section we digress on the consequences of condition ii) on the nature of the unitary, local representations $\pi_p$ that may occur in the decomposition of $\pi$.

1.2. The local nature at $p$. The algebra $D_p = D \otimes Q_p$ is a quaternion, division algebra over $Q_p$, it is unique up to isomorphism. On $D_p$ a valuation function is defined and the corresponding valuation ring $O_p$ is the unique
maximal, compact subring. It consists of all the elements that are integral over \( \mathbb{Z}_p \). There is a unique maximal, two–sided ideal \( \mathfrak{m} \) of \( \mathcal{O}_p \), which is principal and generated by any uniformizer \( \varpi \in \mathfrak{m} \). The residue field \( k \) is a degree 2 extension of \( \mathbb{F}_p \), conjugation by any uniformizer \( x \to \omega x \omega^{-1} \) preserves \( \mathcal{O}_p \) and \( \mathfrak{m} \), and induces the Frobenius automorphism on \( k \). The maximal pro–\( p \) subgroup of \( \mathcal{O}_p^* \) is the left term of the exact sequence

\[
1 \to 1 + \mathfrak{m} \to \mathcal{O}_p^* \to k^* \to 1,
\]

and will be denoted by \( \mathcal{O}_p^*(1) \).

Let now \( \pi_p \) be an irreducible, unitary representation of \( D_p^* \) on a finite dimensional complex vector space \( V \). The subspace \( V' \) of elements fixed by \( \mathcal{O}_p^*(1) \) is stable under \( D_p^* \). It follows that if \( \pi_p \) satisfies ii) then, by its irreducibility, we have \( V' = V \). The group \( k^* \) acts on \( V' = V \) and there is a decomposition

\[
V = \bigoplus_{\eta \in k^*} V^\eta
\]

into isotypical components indexed by complex valued characters of \( k^* \). For every such \( \eta \), the action of a uniformizer \( \varpi \) on \( V \) induces an isomorphism of the summand \( V^\eta \) with \( V^{\eta_p} \), its Frobenius conjugate. The irreducibility of \( \pi_p \) forces the following two possibilities:

1) \( \dim V = 1 \) and \( V = V^\eta \), for a character \( \eta \) equal to its Frobenius conjugate \( \eta_p \);

2) \( \dim V = 2 \), and \( V = V^\eta \oplus V^{\eta_p} \), with \( \dim V^\eta = \dim V^{\eta_p} = 1 \), for a pair of distinct characters \( (\eta, \eta_p) \).

In the first case \( \pi_p \) is abelian and the character describing the action factors through the reduced norm map \( N_p : D_p^* \to \mathbb{Q}_p^* \) and can therefore be identified with a certain character \( \epsilon_\eta \) of \( \mathbb{Q}_p^* \) that is tamely ramified, i.e., it is trivial on the units \( \mathbb{Z}_p^*(1) \) that are congruent to 1 mod \( p \).

In the second case, if \( \varpi \in D \) is the uniformizer such that \( \varpi^2 = p \), then \( \varpi \) acts on \( V \) interchanging the two lines \( V^\eta \) and \( V^{\eta_p} \). Its square acts as multiplication by \( \chi_{\pi_p} \), where \( \chi_{\pi_p} \) is the central character of \( \pi_p \).

1.3. Statement of the Theorem. For any integer \( N \geq 1 \) not divisible by \( p \), we let \( \mathcal{A}(p, N) \) denote the set of infinite dimensional, unitary cusp forms \( \pi \) on \( D^* \) occurring in \( L^2(X, \psi) \), for some finite order character \( \psi \), satisfying i) and ii), and such that \( N(\pi) = N \).

Let \( \mathcal{A}_1(p, N) \) (resp. \( \mathcal{A}_2(p, N) \)) be the subset of \( \mathcal{A}(p, N) \) given by those cusp forms \( \pi \) for which \( \pi_p \) is one dimensional (resp. two dimensional), and denote its cardinality by \( A_1(p, N) \) (resp. \( A_2(p, N) \)). Moreover let \( \mathcal{A}_0(p, N) \) be the subset of \( \mathcal{A}_1(p, N) \) given by the cusp forms \( \pi \) so that \( \pi_p \) is trivial on \( \mathcal{O}_p^* \), i.e., in the notation used in subsection 1.2 so that the associated
character $\epsilon_q : \mathbb{Q}_p^* \to \mathbb{C}^*$ is unramified. Let $A_0(p, N)$ denote its cardinality. Theorem 1.3 below provides, for $p > 3$, closed formulas for $A_0(p, N)$, $A_1(p, N)$ and $A_2(p, N)$.

For an integer $N \geq 1$ not divisible by $p$, let $\Delta(p, N)$ be the function defined by the following table. Its value depends on $N$ and on the congruence class of $p$ modulo 12.

| $p \mod 12$ | 1     | 5     | 7     | 11    |
|-------------|-------|-------|-------|-------|
| $N = 1$     | $(p-1)^2$ | $(p-1)(p+15)$ | $(p-1)(p+11)$ | $(p-1)(p+27)$ |
| $N = 2$     | $\frac{24}{8}$ | $\frac{24}{8}$ | $\frac{24}{8}$ | $\frac{24}{8}$ |
| $N = 3$     | 0     | $\frac{2(p-1)}{3}$ | 0     | $\frac{2(p-1)}{3}$ |
| $N > 3$     | 0     | 0     | 0     | 0     |

If $f, g : \mathbb{Z}_{>0} \to \mathbb{Q}$ are functions defined over the positive integer and valued in $\mathbb{Q}$, then $(f * g)$ will denote their convolution. We say that $f$ is multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all pairs of coprime positive integers $m$ and $n$. The convolution product is associative, commutative and $(f * g)$ is multiplicative when $f$ and $g$ are. The Möbius function $\mu$ is the multiplicative function vanishing on integers that are not square free, and taking value $-1$ on every prime. For a prime $\ell$, and an integer $n \geq 1$, let $r$ be multiplicative function defined by

$$r(\ell^n) = \begin{cases} 
\ell^2 - 3 & \text{if } n = 1; \\
\ell^4 - 3\ell^2 + 3 & \text{if } n = 2; \\
\ell^{2(n-3)}(\ell^2 - 1)^3 & \text{if } n > 2;
\end{cases}$$

**Theorem 1.3.** Let $p > 3$ be a prime number. Then

$$A_0(p, N) = \frac{r(N)(p-1)}{24} + \frac{(\Delta(p, \cdot) * \mu * \mu)(N)}{p-1} - \mu(N);$$

$$A_1(p, N) = \frac{r(N)(p-1)^2}{24} + \frac{(\Delta(p, \cdot) * \mu * \mu)(N) - (p-1)\mu(N)}{p-1} - \mu(N);$$

$$A_2(p, N) = \frac{r(N)(p-1)^2}{48} - \frac{(\Delta(p, \cdot) * \mu * \mu)(N)}{2}.$$

The convolution products in the theorem are performed with respect to the second variable of $\Delta$.

**Corollary 1.4.** Let $p > 3$ be a prime number, and $N \geq 1$ an integer not divisible by $p$. The dimension of the space of cuspidal newforms of weight 2 on the group $\Gamma_1(pN)$, with trivial character locally at $p$ is equal to

$$A_0(p, N) = \frac{r(N)(p-1)}{24} + \frac{(\Delta(p, \cdot) * \mu * \mu)(N)}{p-1} - \mu(N).$$

**Proof.** From the Jacquet–Langlands global correspondence (cf. [10], or [11]) it follows that cusp forms of $A_0(p, N)$ are in bijection with cusp forms $\pi$ on...
\(GL_2\) of conductor \(pN\), with archimedean component \(\pi_\infty\) isomorphic to the principal series of lowest weight 2, and such that \(\pi_p\) is of Steinberg type at \(p\) with unramified central character.

Since any form of conductor \(pN\), and with unramified central character at \(p\) has to be of Steinberg type at \(p\), the formula follows. \(\square\)

2. Automorphic forms on \(D^*\)

We introduce here certain spaces \(S(1, N)\) of automorphic forms that intervene in the study of the cusp forms introduced in the previous section. They consist of locally constant functions on \(D^*_Q \setminus D^*_A\) and they are independent on the archimedean variable \(g_\infty \in D^*_\infty\).

Let \(R\) be a maximal order of \(D\), for a prime number \(\ell\) we set
\[
R_\ell = R \otimes \mathbb{Z}_\ell;
D_\ell = D \otimes \mathbb{Q}_\ell = R_\ell \otimes \mathbb{Q};
\]
where all tensor products are taken over \(\mathbb{Z}\). If \(\ell \neq p\) the ring \(D_\ell\) is isomorphic to the algebra \(M_2(\mathbb{Q}_\ell)\) of two–by–two matrices with entries in \(\mathbb{Q}_\ell\), and we fix an identification \(D_\ell \simeq M_2(\mathbb{Q}_\ell)\) such that \(R_\ell\) corresponds to the standard maximal \(\mathbb{Z}_\ell\)–order \(M_2(\mathbb{Z}_\ell)\).

Let \(N\) be any integer \(\geq 1\) not divisible by \(p\), using the identification \(R^*_\ell \simeq GL_2(\mathbb{Z}_\ell)\) we define a congruence subgroup \(U_\ell(N)\) of \(R^*_\ell\) as follows
\[
U_\ell(N) = \{ x \in GL_2(\mathbb{Z}_\ell) | x \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod N \}.
\]
We have that \(U_\ell = GL_2(\mathbb{Z}_\ell)\) if and only if \(\ell\) does not divide \(N\). The normalizer \(U'_\ell(N)\) of \(U_\ell(N)\) in \(R^*_\ell\) is
\[
U'_\ell(N) = \{ x \in GL_2(\mathbb{Z}_\ell) | x \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \},
\]
the quotient \(U_\ell(N)/U'_\ell(N)\) is isomorphic to \((\mathbb{Z}_\ell/N\mathbb{Z}_\ell)^*\).

The ring \(R_p\) is the unique maximal compact subring of \(D_p\), its residue field will be denoted by \(k\), and the maximal pro–\(p\) subgroup of its multiplicative group by \(R^*_p(1)\) (cf. section 1.2).

Let \(U(1, N)\) be the open subgroup of \(D^*_A\) defined by
\[
U(1, N) = \prod_{\ell \neq p} U_\ell(N) \times R^*_p(1) \times D^*_\infty,
\]
and set
\[
\Omega(1, N) = D^*_Q \setminus D^*_A / U(1, N).
\]
This double coset is finite and discrete, since the space \(D^*_Q \setminus D^*_A / D^*_\infty\) is compact.
Definition 2.1. The space $S(1, N)$ of automorphic forms on $D^*$ of level $(1, N)$ and trivial at infinity is the space of functions $f : \Omega(1, N) \to \mathbb{C}$.

Elements of $S(1, N)$ are therefore complex valued functions on $D^*$ that are invariant under translation by the discrete subgroup $D^*_Q$ to the left, and under translation by the open subgroup $U(1, N)$ to the right.

There are several operators on $S(1, N)$ that we now describe. For any prime $\ell \neq p$ the $\ell$–th Hecke operator $T_\ell$ is defined as follow. The double coset $U_\ell(N) \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} U_\ell(N)$, considered as a subset of $GL_2(\mathbb{Q}_\ell)$, is the finite union of disjoint left cosets

$$U_\ell(N) \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} U_\ell(N) = \bigsqcup_i \alpha_i U_\ell(N).$$

The number of left cosets in the above decomposition is $\ell + 1$ if $\ell \nmid pN$, and $\ell$ if $\ell \mid N$. If $f \in S(1, N)$, then $T_\ell(f)$ is defined by the formula

$$T_\ell(f)(x) = \sum_i f(x\alpha_i),$$

where $\alpha_i \in GL_2(\mathbb{Q}_\ell) \simeq D^*_\ell$ is considered an element of $D^*_\Lambda$ thank to the natural embedding $D^*_\ell \subset D^*\Lambda$. It is clear from their definition that the operators $T_\ell$ commute with each other.

In section 6 the set $\Omega(1, N)$ will be given a moduli interpretation in terms of supersingular elliptic curves. The Hecke operator $T_\ell$ corresponds to a certain averaging operator over degree $\ell$ isogenies (cf. section 6.5). It will be shown that:

Lemma 2.2. If $\ell \nmid pN$ then $T_\ell$ is a semisimple endomorphism of $S(1, N)$.

A consequence of the lemma is that $S(1, N)$ decomposes into the direct sum of common eigenspaces for the $T_\ell$, where $\ell \nmid pN$.

The normalizer of $U(1, N)$ in $D^*_\Lambda$ is

$$U'(1, N) = \prod_{\ell \neq p} U'_\ell(N) \times D^*_p \times D^*_\infty,$$

it acts naturally on $S(1, N)$ by right translation, and an action of

$$U'(1, N)/U(1, N) = (\mathbb{Z}/N\mathbb{Z})^* \times D^*_p/R^*_p(1)$$

on $S(1, N)$ can be deduced. The automorphism of $S(1, N)$ determined by $d \in (\mathbb{Z}/N\mathbb{Z})^*$ is denoted by $\langle d \rangle$ and called diamond operator.

The isomorphism class of the $k^*$ representation on $S(1, N)$ deduced from the action of $D^*_p/R^*_p(1)$ will be determined in section 4. We will see that $S(1, N)$ is close to be a multiple of the regular representation for $k^*$. 
We conclude the section describing the Hecke invariant subspace \( S(1, N)^{Eis} \) of \( S(1, N) \).

**Definition 2.3.** The subspace of \( S(1, N) \) given by those functions \( f : D_A^* \to C \) that can be factored through the reduced norm map \( \text{Nr} : D_A^* \to A^* \) is denoted by \( S(1, N)^{Eis} \).

Fix now a character \( \chi : \mathcal{F}_p^* \to C^* \) of order \( p - 1 \).

**Proposition 2.4.** The space \( S(1, N)^{Eis} \) has dimension \( (p - 1) \) and is independent of \( N \). It has a basis \( (e_1, \ldots, e_{p-1}) \) consisting of eigenvectors for all the Hecke operators \( T_\ell \), with \( \ell \neq p \), such that if \( \ell \nmid pN \) we have \( T_\ell(e_i) = (\ell + 1)\chi^{-i}(\ell)e_i \).

The meaning of the first assertion is that \( S(1, N)^{Eis} \) defines a space of locally constant functions on \( D_A^* \) which has dimension \( (p - 1) \) and that is independent on \( N \).

**Proof.** Let

\[
A^* = Q^* \times \hat{Z}^* \times R_{>0}^*
\]

be the canonical decomposition of \( A^* \) into the product of the discrete subgroup \( Q^* \) and the open subgroup \( \hat{Z}^* \times R_{>0}^* \). The image of the reduced norm map \( \text{Nr} : D_A^* \to A^* \) is the index two subgroup which, in the above decomposition of \( A^* \), corresponds to \( Q^*_0 \times \hat{Z}^* \times R_{>0}^* \). The space \( S(1, N)^{Eis} \) is identified with the space of functions

\[
f : Q_{>0}^* \times \hat{Z}^* \times R_{>0}^* \to C
\]

that are invariant under the image of \( D_Q^* \cdot U(1, N) \) with respect to the reduced norm map \( \text{Nr} \). Since

\[
\text{Nr}(D_Q^* \cdot U(1, N)) = Q_{>0}^* \times \prod_{\ell \neq p} Z_{\ell}^* \times Z_p^*(1) \times R_{>0}^*,
\]

where \( Z_p^*(1) \) is the subgroup of \( Z_p^* \) of units that are congruent to 1 modulo \( p \), the elements of \( S(1, N)^{Eis} \) are the functions on \( D_A^* \) that factor through the composition

\[
r \cdot \text{Nr} : D_A^* \to Q_{>0}^* \times \hat{Z}^* \times R_{>0}^* \to \mathcal{F}_p^*,
\]

where the first map \( \text{Nr} \) is the reduced norm, and \( r \) is the projection onto the quotient group

\[
\left( Q_{>0}^* \times \hat{Z}^* \times R_{>0}^* \right) / \left( Q_{>0}^* \times \prod_{\ell \neq p} Z_{\ell}^* \times Z_p^*(1) \times R_{>0}^* \right) = \mathcal{F}_p^*.
\]

The first part of the proposition then follows.
To complete the proof, we construct an explicit basis for $S(1,N)^Eis$ given by eigenvectors for all the Hecke operators $T_\ell$, with $\ell \neq p$. For an integer $j$ with $1 \leq j \leq p - 1$, define $e_j \in S(1,N)^Eis$ to be the composition $\chi^j \cdot r \cdot N_r$. By the linear independence of characters, we have that $(e_1, \ldots, e_{p-1})$ is a basis of $S(1,N)^Eis$. Let now $\ell \nmid pN$ be a prime number, and $x \in D_\Lambda^*$ any element. By definition of $T_\ell$, we have

$$(T_\ell e_j)(x) = \sum_i e_j(x\alpha_i),$$

where the $\alpha_i \in D_\ell^* \subset D_\Lambda^*$ are representatives for the double coset in formula (2). Observe now that $e_j(x\alpha_i) = e_j(x)e_j(\alpha_i)$, moreover $e_j(\alpha_i) = \chi^j \cdot r \cdot N_r(\alpha_i)$. For any $\alpha_i$, the idele $N_r(\alpha_i)$ is 1 at every place other than the $\ell$-adic one, and it is equal to $\ell$ at the $\ell$-adic place. It is easy to see that $r \cdot N_r(\alpha_i) = \ell^{-1} \in F_p^*$, therefore $e_j(\alpha_i) = \chi^j(\ell^{-1})$ and the proposition follows.

□

3. HECKE EIGENVALUES AND CUSP FORMS

We recall the details of the dictionary between systems of Hecke eigenvalues arising from the module $S(1,N)$ introduced in the previous section, and unitary cusp form on $D^*$ satisfying conditions i) and ii). In this correspondence, the strong multiplicity one result for $L^2(X,\psi)$ plays an important role.

**Definition 3.1.** A system of Hecke eigenvalues arising from $S(1,N)$ is a collection $\Phi = (a_\ell)_{\ell \nmid pN}$ of complex numbers such that there exists a nonzero element $f \in S(1,N)$ with $T_\ell(f) = a_\ell f$ for all primes $\ell \nmid pN$.

For a system of eigenvalues $\Phi$ we define the corresponding isotypical component of $S(1,N)$ as

$$S(1,N)^\Phi = \{f \in S(1,N) \mid T_\ell(f) = a_\ell f, \text{ for all } \ell \nmid pN\}.$$

According to lemma 2.2, there is a decomposition

$$S(1,N) = \bigoplus_{\Phi} S(1,N)^\Phi$$

of $S(1,N)$ into the direct sum of its isotypical components. Since the action on $S(1,N)$ by right translation of the normalizer $U'(1,N)$ of $U(1,N)$ commutes with that of $T_\ell$, for all primes $\ell \nmid pN$, each summand $S(1,N)^\Phi$ is a representation of $U'(1,N)/U(1,N) = (\mathbb{Z}/N\mathbb{Z})^* \times D_p^*/R_p^*(1)$.

Let now $\pi$ be an element of $A(p,N)$; that is, $\pi$ is an infinite dimensional, unitary cusp form on $D^*$ of conductor $N$, and satisfying conditions i) and ii), which occurs on a closed subspace $V(\pi)$ of $L^2(X,\psi)$, for some central
character $\psi$ whose conductor divides $N$. Associated to $\pi$ there is a system of Hecke eigenvalues $\Phi = (a_\ell)_{\ell \nmid pN}$:

**Proposition 3.2.** The space $V(\pi) \cap S(1, N)$ is non empty. For every prime $\ell \nmid pN$ the operator $T_\ell$ acts on $V(\pi) \cap S(1, N)$ via multiplication by a certain scalar $a_\ell$.

By definition, the collection $\Phi(\pi) = (a_\ell)_{\ell \nmid pN}$ is the system of eigenvalues associated to $\pi$.

**Proof.** The space $V(\pi)$ is isomorphic to a restricted tensor product $\otimes' V(\pi_\ell)$ of local representations, where we may disregard the trivial component at infinity. It follows that the space $V(\pi)_{U(1, N)}$ of $U(1, N)$–invariant vectors is the restricted tensor product

$$\otimes'_{\ell \nmid pN} V(\pi_\ell)^{U(1, N)} \otimes V(\pi_p)^{R^*_p(1)}.$$  

For any $\ell \nmid pN$ the representation $\pi_\ell$ is unramified, and the space $V(\pi_\ell)^{U(1, N)}$ is one–dimensional (cf. [3], §2.2). The Hecke operator $T_\ell$ acts on it as scalar multiplication, the proposition follows. \qed

A consequence of the strong multiplicity one result for $L^2(X, \psi)$, as we shall see below, is that in fact the space $V(\pi) \cap S(1, N)$ coincide with the full isotypical component $S(1, N)^{\Phi(\pi)}$.

Conversely, to any system of eigenvalues $\Phi$ occurring in $S(1, N)$ one can attach a cusp form $\pi(\Phi)$ on $D^*$ as follow. Let $f \in S(1, N)^{\Phi}$ be any nonzero automorphic form, and set

$$L^2(X, N) = \bigoplus_{\psi \in (\mathbb{Z}/N\mathbb{Z})^*} L^2(X, \psi).$$

Consider the smallest closed subspace $V_f$ of $L^2(X, N)$ that contains $f$ and is stable under $D^*_\Lambda$, and denote by $\pi_f$ the corresponding unitary representation of $D^*_\Lambda$. The following proposition is a standard consequence of the strong multiplicity one result for $L^2(X, N)$ (cf. [2], Thm. 2):

**Proposition 3.3.** The representation $\pi_f$ is irreducible and defines a cusp form on $D^*$. The space $V_f^{U(1, N)}$ of $U(1, N)$–invariant vectors coincide with $S(1, N)^{\Phi}$, and $V_f$ depends only on $\Phi$, and not on the choice of $0 \neq f \in S(1, N)^{\Phi}$. The action of $(\mathbb{Z}/N\mathbb{Z})^*$ on $S(1, N)^{\Phi}$ is via homotheties specified by a character $\psi_0 : (\mathbb{Z}/N\mathbb{Z})^* \rightarrow \mathbb{C}^*$. The action of $D^*_p/R^*_p(1)$ on $S(1, N)^{\Phi}$ is isomorphic to a multiple of $(\pi_f)_p$, where $(\pi_f)_p$ is the local component at $p$ of $\pi_f$, moreover $p \in D^*_p$ acts on $S(1, N)^{\Phi}$ as multiplication by $\psi_0(p)^{-1}$.
The cusp form $\pi_f$ and the Hilbert space $V_f$ will also be denoted by $\pi_\Phi$ and $V_\Phi$, respectively. The central character $\psi_\Phi$ of $\pi_\Phi$ is obtained by composing the natural map
\[ \mathbb{A}^* \longrightarrow \mathbb{Z}^* \longrightarrow (\mathbb{Z}/N\mathbb{Z})^* \]
with the character $\psi_0$ given by the Proposition 3.3. We have that $\pi_\Phi$ is an irreducible constituent of $L^2(X, \psi_\Phi)$.

The next lemma characterizes the automorphic functions $f \in S(1,N)$ so that $\pi_f$ is finite dimensional:

**Lemma 3.4.** Let $f \in S(1,N)$ be an eigenvector for all the Hecke operator $T_\ell$, with $\ell \nmid pN$. If $\pi_f$ is finite dimensional, then it is one–dimensional, and $f \in S(1,N)^{\text{Eis}}$.

**Proof.** Let $g \in D^* \subset D^*_\mathbb{A}$ be any element, write $g = g_p g^{(p)}$, where $g_p$ is the adelic element of $D^*_\mathbb{A}$ whose only non–trivial component is the $p$–th component of $g$ in $D^*_p$. Then we have that
\[ \pi_f(g_p) = \pi_f(g^{(p)})^{-1}, \]
since $\pi_f(D^*)$ is trivial. By assumption, for any $\ell \neq p$ the local representation $(\pi_f)_\ell$ is described by a character, therefore equation (4) implies that the restriction to $D^*$ of the local representation $(\pi_f)_p$ is given described by a character. Now since $(\pi_f)_p$ is trivial on an open subgroup of $D^*_p$ and is described by a character on the dense subgroup $D^*$, we have that $(\pi_f)_p$ is abelian and hence one–dimensional. It follows that $f$ factors through the reduced norm map $\text{Nr} : D^*_\mathbb{A} \rightarrow \mathbb{A}^*$.

Together with Proposition 2.4, Lemma 3.4 implies that there are precisely $(p-1)$ systems of eigenvalues $\Phi$ arising from $S(1,N)$ and for which $\pi_\Phi$ is finite dimensional. Such $\Phi$ have been explicitly described in Proposition 2.4.

Let now $\Phi = (a_\ell)_{\ell \nmid pN}$ be a system of eigenvalues arising from $S(1,N)$ such that the associated cusp form $\pi = \pi_\Phi$ is infinite dimensional. All the local components $\pi_\ell$ for $\ell \neq p$ are also infinite dimensional, and the prime to $p$ conductor of $\pi$ is denote by $N(\pi)$.

Let $t : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ be the function whose value in $n > 0$ is the number of all positive divisors of $n$.

**Lemma 3.5.** The level $N$ is divisible by $N(\pi)$, moreover
\[ \dim(S(1,N)^\Phi) = \dim(\pi_p) \cdot t(N/N(\pi)). \]

The lemma follows from a classical local result of Casselman (cf. [3] §2.2) and is the crucial ingredient that enables us to obtain a recursive formulas that the functions $A_i(p,N)$ in the main theorem have to satisfy.
4. Recurrence relations

Let \( S(1, N) = \oplus_{\eta \in k^*} S(1, N)^\eta \) be the canonical decomposition of \( S(1, N) \) into isotypical components with respect to the action of \( k^* = R^*_p/R^*_p(1) \).

Consider the decomposition

\[
S(1, N) = (\bigoplus_{\eta = \eta^p} S(1, N)^\eta) \oplus (\bigoplus_{\eta \neq \eta^p} S(1, N)^\eta),
\]

and denote by \( S_1(1, N) \) the first summand and by \( S_2(1, N) \) the second one.

We have that \( S_1(1, N) \) is the largest submodule of \( S(1, N) \) on which the action of \( D^*_p \) is abelian. Lastly, denote the isotypical component of \( S(1, N) \) with respect to the trivial character of \( k^* \) by \( S_0(1, N) \). For \( i \in \{0; 1; 2\} \) let \( u_i(p, N) \) denote the dimension of \( S_i(1, N) \).

**Proposition 4.1.** For a prime \( p \) and an integer \( N \geq 1 \) not divisible by \( p \) we have

\[
(t \ast A_0(p, .))(N) = u_0(p, N) - 1;
\]
\[
(t \ast A_1(p, .))(N) = u_1(p, N) - (p - 1);
\]
\[
(t \ast A_2(p, .))(N) = u_2(p, N)/2.
\]

Here \((t \ast A_i)\) denotes the convolution product between the multiplicative function \( t \) introduced right before Lemma 3.5 and the function \( N \to A_i(p, N) \).

**Proof.** Observe first that the subspaces \( S_0(1, N), S_1(1, N) \) and \( S_2(1, N) \), just defined are invariant under the action of the Hecke operators \( T_\ell \) for every prime \( \ell \nmid pN \). This follows from the fact that, for any system of eigenvalues \( \Phi \), the \( D^*_p \)-representation \( S(1, N)^\Phi \) is a multiple \( m(\pi_\Phi)_p \) of the local component at \( p \) of the cusp from \( \pi_\Phi \) attached to \( \Phi \) (cf. Prop. 3.3), together with the analysis of the representation of \( k^* \) that may arising from \( (\pi_\Phi)_p \) by restriction to \( R^*_p \). Again from proposition 3.3 we have that \( S_1(1, N) \) consists of the direct sum of a certain number of Hecke isotypical components \( S(1, N)^\Phi \) of \( S(1, N) \). More precisely,

\( S(1, N)^\Phi \subset S_1(1, N) \), if \( (\pi_\Phi)_p \) is one–dimensional;

\( S(1, N)^\Phi \subset S_2(1, N) \), if \( (\pi_\Phi)_p \) is two–dimensional;

furthermore, in the first case, we have that \( S(1, N)^\Phi \subset S_0(1, N) \) precisely when the character of \( Q^*_p \) describing \( (\pi_\Phi)_p \) is unramified. Another consequence of the same proposition is that

\( S(1, N)^{Eis} \subset S_1(1, N) \), and \( S(1, N)^{Eis} \cap S_0(1, N) = C \cdot e_{p-1} \),

where \( e_{p-1} \) is the constant function on \( D^*_A \) equal to 1 (cf. Prop. 2.4).
Consider the decomposition
\[
S_1(1, N) = S(1, N)^{Eis} \bigoplus_{0<d|N} \left( \bigoplus_{N(\Phi)=d} S_1(1, N)^\Phi \right),
\]
where the inner sum ranges through all the eigensystems \( \Phi \) arising from \( S(1, N) \) and giving rise to an infinite dimensional cusp form \( \pi_\Phi \) whose local component at \( p \) is one dimensional, and whose prime–to–\( p \) conductor is equal to the given positive divisor \( d \) of \( N \).

Taking dimensions on both sides of (6), and using Proposition 2.4 and Lemma 3.5, we get
\[
u_1(p, N) = (p - 1) + \sum_{0<d|N} t(N/d)A_1(p, d).
\]
In an analogous way, since \( S_0(1, N) \cap S(1, N)^{Eis} \) is one–dimensional, we see that
\[
u_0(p, N) = 1 + \sum_{0<d|N} t(N/d)A_0(p, d).
\]
Finally, working with the module \( S_2(1, N) \), we have
\[
u_2(p, N) = 2 \sum_{0<d|N} t(N/d)A_2(p, d).
\]
The proposition follows.

The relations involving the functions \( A_i(p, N) \) expressed by Proposition 4.1 can be solved for the \( A_i(p, N) \). The inverse of \( t \), with respect to the convolution product, is in fact the square \( (\mu * \mu) \) of the Möbius function. A restatement of Proposition 4.1 is then

**Proposition 4.2.** For a prime \( p \) and an integer \( N \geq 1 \) not divisible by \( p \) we have
\[
A_0(p, N) = (\mu * \mu * u_0(p, \_))(N) - \mu(N);
A_1(p, N) = (\mu * \mu * u_1(p, \_))(N) - (p - 1)\mu(N);
A_2(p, N) = (\mu * \mu * u_2(p, \_))(N)/2.
\]

Let \( \delta \) be the multiplicative function whose value at every integer \( n > 1 \) is zero, and so that \( \delta(1) = 1 \). In the restatement of Proposition 4.1 given by Proposition 4.2, we used the fact that the convolution between \( \mu \) and the constant function equal to 1 is \( \delta \), the identity for the convolution product.

To complete the proof of Theorem 1.3, we will find an explicit expression for \( u_i(p, N) \), for every prime \( p > 3 \), and every \( N \geq 1 \) not divisible by \( p \). This task will be accomplished in section 5 where the isomorphism class of the \( k^* \)–representation given by \( S(1, N) \) will be determined.
5. The functions \( u_0, u_1 \) and \( u_2 \)

The double coset \( \Omega(1, N) \) introduced in equation (1) parametrizes (in a non–canonical way) supersingular elliptic curves in characteristic \( p \) with a certain extra structure. By exploiting this correspondence, we explicitly determine, for any prime \( p > 3 \) and any integer \( N \geq 1 \) not divisible by \( p \), the isomorphism class of the linear representation of \( k^* \) given by the complex space \( S(1, N) \). This results in providing formulas for the functions \( u_i(p, N) \), thus completing the proof of Theorem 1.3.

Any supersingular elliptic curve \( E \) over \( \bar{k} \) has a canonical and functorial model \( E_0 \) over \( k \) specified by the requirement that the Frobenius endomorphism of \( E_0 \) relative to \( k \) is equal to multiplication by \( -p \) on the curve (cf. Thm. 6.1). In particular we can deduce a \( k \)–structure on the space \( t_0(E) \) of invariant 1–forms, dual to the tangent space \( t_0(E) \) of \( E \) at 0, to which we will implicitly refer when speaking of an invariant form on \( E \) defined over \( k \). The details of the next theorem are given in section 6 (cf. Thm. 6.11):

**Theorem 5.1.** There is a correspondence between \( \Omega(1, N) \) and the set of isomorphism classes of triples \( (E, \omega, x) \), where \( E \) is a supersingular elliptic curve over \( \bar{k} \), \( \omega \) is a nonzero invariant 1–form on \( E \) defined over \( k \), and \( x \in E[N](\bar{k}) \) is point of order \( N \). The permutation action of \( k^* \) on \( \Omega(1, N) \) is that induced by the twisted action

\[
\lambda.(E, \omega, x) = (E, \lambda^{p} \omega, x)
\]

by homotheties on the second entry of the triples considered.

**Remark 5.2.** The introduction of the Frobenius–twist in the permutation action of \( k^* \) as given in Theorem 5.1 results from the actual way we set up the correspondence. The correspondence can be normalized in a different way, and the twist can be removed. Observe nevertheless that the Frobenius–twisted permutation action of \( k^* \) on \( \Omega(1, N) \) is isomorphic to the untwisted action, induced by \( \lambda.(E, \omega, x) = (E, \lambda \omega, x) \). This follows from the fact that there is a permutation action of \( D^*_p/R^*_p(1) \) on \( \Omega(1, N) \) extending that of \( k^* \).

The following elementary lemma on (not necessarily supersingular) elliptic curves over a field of characteristic \( p \) will be used in the sequel. Here the assumption \( p > 3 \) plays a role.

**Lemma 5.3.** Let \( E \) be an elliptic curve over a field \( \kappa \) of characteristic \( p \), and let \( W_E \) be the group of \( \kappa \)–automorphisms of \( E \). If \( p > 3 \), then the group homomorphism \( d_0 : W_E \to \kappa^* \) describing the natural action of \( W_E \) on the tangent space \( t_0(E) \) (resp. the cotangent space \( t_0(E)^* \) ) of \( E \) at 0 is injective.

**Proof.** Let \( \sigma \in W_E \) be such that the differential \( d_0(\sigma) \) is equal to 1. Then by the linearity of the differential with respect to the addition on \( E \) we have...
$d_0(id_E - \sigma) = 0$. Therefore $id_E - \sigma$ is either zero or an inseparable isogeny, in both cases $p$ divides the degree of $id_E - \sigma$. For any pair of endomorphisms $a, b$ of $E$ over $\kappa$, we have the Cauchy–Schwarz inequality (cf. [13], III Cor. 6.3, V Lem. 1.2)

$$\deg(a + b) \leq \deg(a) + \deg(b) + 2\sqrt{\deg(a)\deg(b)}.$$ 

In particular $\deg(id_E - \sigma) \leq 1 + 1 + 2 = 4$, and its divisibility by $p > 3$ implies that $\deg(id_E - \sigma)$ is zero, and $\sigma = id_E$. \qed

**Remark 5.4.** If $E$ is a supersingular elliptic curve over $\bar{k}$, then from the lemma and from the existence of the functorial $k$–model $E_0$, it follows that the action of the group $W_E$ of $\bar{k}$–automorphisms on $t_0(E_0) \ast$ is described by an injection $W_E \cong k \ast$. 

Let $E$ be an elliptic curve over $\bar{k}$ with automorphism group $W_E$. Since the characteristic $p$ of $k$ is assumed to be $> 3$, we have that $W_E$ is cyclic of order 4, 6, or 2 according to whether $j_E$ is 1728, 0, or none of the previous values respectively (cf.[13], III §10). The next lemma indicates when the two isomorphism classes of elliptic curves with extra automorphisms are supersingular.

**Lemma 5.5.** If $p > 3$, then 1728 is a supersingular $j$–invariant if and only if $p \equiv 3 \mod 4$, whereas 0 occurs as a supersingular invariant precisely when $p \equiv 2 \mod 3$.

**Proof.** Let $E$ be an elliptic curve over $\bar{k}$ with $j_E = 1728$, and let $i \in Q$ be a primitive fourth root of unity. There exists an injective ring homomorphism $\mathbb{Z}[i] \hookrightarrow \text{End}_{\bar{k}}(E)$, sending $i$ to one of the two automorphisms of $E$ of order 4, we can deduce an embedding $Q(i) \subset \text{End}_{\bar{k}}(E) \otimes Q$. If $E$ is supersingular, then $\text{End}_{\bar{k}}(E) \otimes Q$ is a quaternion algebra over $Q$ that is ramified at $p$, therefore $Q(i) \otimes Q_p$ is a field, equivalently $Q(i)$ is not split at $p$. On the other hand if $E$ is ordinary, then $E(\bar{k})$ has a (unique) subgroup $C$ of order $p$ and we can deduce a non trivial ring homomorphism $Z[i] \to \text{End}(C) = F_p$. Which is to say that $p$ splits in $Q(i)$, since it cannot ramify for $p > 3$. Since $p$ splits in $Q(i)$ if and only if $p \equiv 1 \mod 4$ we obtain the first part of the lemma.

The second part is analogous, one has to replace the fourth root of unity $i$ by a primitive sixth root of unity $\rho \in Q$. \qed

Let now $E$ be a supersingular elliptic curve over $\bar{k}$. If $N$ is any integer $\geq 1$ with $p \nmid N$, define $\Sigma_E(1, N)$ to be the set of isomorphisms classes of pairs $(\omega, x)$, where $\omega$ is a nonzero invariant 1–form on $E$ defined over $k$, and $x$ is a $k$–valued point of $E$ of order $N$. Two such pairs $(\omega, x)$ and $(\eta, y)$ are isomorphic if there exists $u \in W_E$ such that $u.(\omega, x) := (u^*\omega, u(x))$ is equal
to \((\eta, y)\). Let moreover \(S_E(1, N)\) be the set of complex valued functions on \(\Sigma_E(1, N)\).

The group \(k^*\) permutes the elements of \(\Sigma_E(1, N)\), the action is induced by

\[
\lambda.(\omega, x) = (\lambda \omega, x).
\]

The space \(S_E(1, N)\) is therefore a linear representation of \(k^*\) via the formula

\[
\lambda.\varphi(\omega, x) = \varphi(\lambda.(\omega, x)).
\]

It follows from Theorem 5.1 that there is a decomposition

\[
S(1, N) = \bigoplus_E S_E(1, N),
\]

where the sum ranges through the isomorphism classes of supersingular elliptic curves \(E\) over \(\bar{k}\).

Consider now the set \(\Sigma_E(N)\) of isomorphisms classes of \(\bar{k}\)–valued points of \(E\) of order \(N\), by which we mean the set of \(W_E\)–orbits given by points of \(E\) of order \(N\). Let \(\alpha_1, \ldots, \alpha_r \in E[N](\bar{k})\) be a system of representatives of \(\Sigma_E(N)\), and let \(e_1, \ldots, e_r\) be the cardinalities of the subgroups of \(W_E\) given by the respective stabilizers. Notice that each \(e_i\) divides the order of \(k^*\), since \(W_E\) injects into \(k^*\) (cf. Rmk. 5.4).

If \(d\) is an integer dividing \(|k^*| = p^2 - 1\) then set

\[
I_d = \text{Ind}_{C_d}^{k^*} 1,
\]

where \(C_d\) is the subgroup of \(k^*\) of order \(d\), and 1 denotes its trivial, one–dimensional complex representation.

**Lemma 5.6.** There is an isomorphism

\[
S_E(1, N) \simeq \bigoplus_{1 \leq i \leq r} I_{e_i}.
\]

**Proof.** Using lemma 5.3 the group \(W_E\) will be identified with a subgroup of \(k^*\). Consider the permutation action of \(k^*\) on \(\Sigma_E(1, N)\). Any \(k^*\)–orbit can certainly be represented by elements of the form \((\omega, \alpha_i)\). Moreover, if \((\omega, \alpha_i)\) and \((\eta, \alpha_j)\) are in the same orbit then \(i = j\).

In order to prove the lemma it is then enough to show that the stabilizer of any class of \(\Sigma_E(1, N)\) represented by \((\omega, \alpha_i)\) is the subgroup of \(k^*\) of order \(e_i\). This follows readily from the fact that for \(\lambda \in k^*\) we have that \((\lambda \omega, \alpha_i)\) defines the same element as \((\omega, \alpha_i)\) does if and only if there is an automorphism \(u \in W_E\) such that

\[
u.(\lambda \omega, \alpha_i) = (\omega, \alpha_i).
\]

Therefore \(u \alpha_i = \alpha_i\) and \(\lambda\) has to belong to the subgroup of \(k^*\) given by the subgroup of \(W_E\) stabilizing \(\alpha_i\).

\(\square\)
If $E$ is any elliptic curve over $\bar{k}$, and $N$ any integer not divisible by $p$, we compute the integers $e_1, \ldots, e_r$. Let $\psi(N)$ be the number of elements of $(\mathbb{Z}/N\mathbb{Z})^2$ of order $N$.

**Lemma 5.7.** The sequence of integers $e_1, \ldots, e_r$ is given, up to permutation, by the following table:

| $|W_E|$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
|-------|-------|-------|-------|-------|-------|-------|
| $N = 1$ | $e_1 = 2$ |       |       |       |       |       |
| $N = 2$ | $e_i = 2$, $r = 3$ | $e_1 = 2$, $e_2 = 4$ |       |       |       |       |
| $N = 3$ | $e_i = 1$, $r = 4$ | $e_1 = 1$, $r = 2$ |       |       |       |       |
| $N > 3$ | $e_i = 1$, $r = \psi(N)/2$ | $e_i = 1$, $r = \psi(N)/4$ | $e_i = 1$, $r = \psi(N)/6$ |       |       |       |

**Proof.** If $E$ does not have extra automorphisms, then the action of $W_E$ on $E[N](\bar{k})$ is by inversion, and the first column of the table is readily established, taking in account that $\psi(2) = 3$ and that $\psi(3) = 8$.

If $|W_E| = 4$, and $\ell$ is a prime $\neq p$, then a generator $\sigma \in W_E$ acts on the $\ell$–adic Tate module $T_\ell(E)$ of $E$ via a $\mathbb{Z}_\ell$–linear, order 4 automorphism $\sigma_\ell$. Let $m = m_{\sigma_\ell} \in \text{GL}_2(\mathbb{Z}_\ell)$ be the matrix representing $\sigma_\ell$ with respect to the choice of a $\mathbb{Z}_\ell$–basis of $T_\ell(E)$. Since $m$ satisfies the polynomial $x^2 + 1$ and it cannot act a scalar (for $\text{End}_{\bar{k}}(E)$ injects into $\text{End}_{\mathbb{Z}_\ell}(T_\ell(E))$), we see that $x^2 + 1$ is its characteristic polynomial. If $\ell \neq 2$, then the roots of the reduction mod $\ell$ of $x^2 + 1$ are distinct, therefore if $N$ is not a power of 2 other than 1, the group $W_E$ acts without fixed points on the $\bar{k}$–valued points of $E$ of order $N$.

Assume then that $\ell = 2$, and let $m$ be given by

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

The mod 2 reduction $\bar{m}$ of $m$ cannot be trivial, for $\ell$ dividing $1 - m$ would prevent $m$ from having order 2, and $\bar{m}$ has order 2. It follows that there is a basis $(e_1, e_2)$ of $T_\ell(E)$ such that the matrix $m$ representing $\sigma_\ell$ has the entries $a$ and $d$ both divisible by 2. This implies that the pair $(e_1, \sigma_\ell \cdot e_1)$ is also a basis of $T_\ell(E)$, thus, up to conjugation, we can assume $m$ be given by

$$m = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

It now readily follows that $\sigma_\ell$ fixes a point of order $N = 2^n$ if and only if $n < 2$, moreover, the action of $W_E$ on points of order 2 is specified by the integers $e_1 = 4$ and $e_2 = 2$. This complete the proof of the statements in the second column of the table. The case $|W_E| = 6$ can be treated similarly and the details are omitted. □

**Remark 5.8.** If $\tilde{E}$ is one of the two elliptic curves over $\mathbb{Q}_p$ with extra automorphisms, then $\tilde{E}$ is known to have good reduction, since $j_{\tilde{E}}$ is an algebraic
The special fiber $\tilde{E}_p$ at $p$ is an elliptic curve over $\bar{k}$ with extra automorphisms, and the group $\text{Aut}(\tilde{E})$ is identified with $\text{Aut}(\tilde{E}_p)$ in a natural way (recall that $p > 3$). For $\ell \neq p$, the Tate modules $T_\ell(\tilde{E})$ and $T_\ell(\tilde{E}_p)$ are isomorphic as modules over $\text{Aut}(\tilde{E}) \simeq \text{Aut}(\tilde{E}_p)$ and one could have proved Lemma 5.7 by reducing the calculation to the complex case, where the study of a simple picture leads to the computation of the integers $e_i$ in the table.

The number of supersingular $j$–invariants characteristic $p > 3$ is given by a well–known formula due to Eichler and Deuring (cf. [7], §1):

$$p \mod 12 \begin{array}{cccc}
1 & 5 & 7 & 11
\end{array}$$

$$h \begin{array}{cccc}
(p - 1)/12 & (p + 7)/12 & (p + 5)/12 & (p + 13)/12
\end{array}$$

This formula, together with Lemmas 5.5, 5.6 and 5.7, yields:

**Proposition 5.9.** The isomorphism class of the linear representation $S(1, N)$ of $k^*$ is specified by the following table:

| $p \mod 12$ | 1 | 5 | 7 | 11 |
|-------------|---|---|---|----|
| $N = 1$     | $\frac{p-1}{12} I_2$ | $\frac{p-3}{12} I_2 + I_6$ | $\frac{p-1}{12} I_2 + I_4$ | $\frac{p-11}{12} I_2 + I_6 + I_4$ |
| $N = 2$     | $\frac{p-1}{12} I_2$ | $\frac{p-1}{12} I_2 + I_4$ | $\frac{p-3}{12} I_2 + I_4$ | $\frac{p-3}{12} I_2 + I_4$ |
| $N = 3$     | $\frac{p-1}{4} I_1$ | $\frac{p-7}{3} I_1 + I_3$ | $\frac{p-1}{2} I_1$ | $\frac{p-1}{2} I_1$ |
| $N > 3$     | $\psi(N) \frac{p-1}{24} I_1$ | $\psi(N) \frac{p-1}{24} I_1$ | $\psi(N) \frac{p-1}{24} I_1$ | $\psi(N) \frac{p-1}{24} I_1$ |

Moreover, we have:

$$u_0(p, N) = \psi(N) \frac{p-1}{24} + \Delta(p, N)$$

$$u_1(p, N) = \psi(N) \frac{(p-1)^2}{24} + \Delta(p, N)$$

$$u_2(p, N) = \psi(N) \frac{(p-1)^2 p}{24} - \Delta(p, N)$$

The second part of the proposition follows from the table in the first part, after an elementary (but a bit lengthy) calculation. The error term $\Delta(p, N)$ was defined in section 1.3.

By plugging the expressions for the functions $u_i$ obtained in Proposition 5.9 in the recurrence relation given by Proposition 4.1, we see that the proof of Theorem 1.3 is complete, once that the equality $(\mu * \mu * \psi) = r$ is verified.

### 6. An isogeny class of supersingular elliptic curves

Let $p$ be a prime number, and $k$ a finite field with $p^2$ elements. The Honda–Tate theory of abelian varieties over finite fields guarantees the existence of a $k$–isogeny class $\mathcal{C}_p$ of elliptic curves over $k$ whose objects $A$ are
those for which the equality
\[ \pi_A = -p \]
holds in the ring \( \text{End}_k(A) \) (cf. [13], Théorème 1). Here \( \pi_A : A \to A \) is the geometric Frobenius endomorphism of \( A \) relative to \( k \), i.e., \( \pi_A \) is the identity on the topological space underlying \( A \), and is given by \( s \to s^{[k]} \) on sections. Furthermore, for any such \( A \) the division algebra \( \text{End}_k(A) \otimes \mathbb{Q} \) is “the” \( \mathbb{Q} \)-quaternion ramified at \( p \) and infinity (cf. Tate, loc. cit.), and the elliptic curve \( A \) is supersingular. In fact any supersingular elliptic curve over an algebraic closure \( \overline{k} \) of \( k \) admits a canonical and functorial descent to \( C_{-p} \). It can be shown that (cf. [1], Lemma 3.21 for a brief sketch of the proof):

**Theorem 6.1.** The base extension functor \( A \to A \otimes_k \overline{k} \) induces an equivalence of categories between \( C_{-p} \) and the isogeny class of supersingular elliptic curves over \( k \).

In this section we explain how isomorphism classes of objects of \( C_{-p} \), equipped with some extra structure, are parametrized by a certain double coset arising from the adelic points of the multiplicative group of the quaternion algebra above. To describe such correspondence we make essential use of results of Tate (cf. [18], Thm. 6), describing the local structure at every prime \( \ell \) of the module \( \text{Hom}_k(A, B) \) of homomorphisms between two abelian varieties \( A, B \) over a finite field \( k \).

Since the correspondence is classical, this section will probably not appear in a more definitive version of this paper. We first learnt of this correspondence in [12], what is included here is the outcome of our effort in understanding all the details of its proof. Ghitza in [8] explains, and generalizes, a correspondence very similar to the one considered here. Our method is similar to his.

The notation adopted throughout the section is as follows. The Galois group of a fixed algebraic closure \( \bar{k} \) of \( k \) over \( k \) is denoted by \( G_k \). For an object \( A \) of \( C_{-p} \), and a prime number \( \ell \), we let \( T_\ell(A) \) denote the \( \ell \)-adic Tate module of \( A \) for \( \ell \neq p \), and the contravariant Dieudonné module attached to the \( p \)-divisible group \( A[p^\infty] \) of \( A \) for \( \ell = p \) (cf. [17], Ch. 1; [4], Ch. III). For any prime \( \ell \), we set \( V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \). If \( \ell \neq p \) then \( V_\ell(A)/T_\ell(A) \) is identified with the Galois module \( A[\ell^\infty] \) of \( \bar{k} \)-valued points of \( A \) of \( \ell \)-power torsion. If \( \psi : A \to B \) is a morphism in \( C_{-p} \), then \( \psi_\ell : T_\ell(A) \to T_\ell(B) \) denotes the corresponding morphism of Tate modules for \( \ell \neq p \), and \( \psi_p : T_p(B) \to T_p(A) \) that of Dieudonné modules. The morphism of \( \mathbb{Q}_\ell \)-vector spaces deduced from \( \psi_\ell \) is denoted by \( V_\ell(\psi) \), while \( \psi[\ell^\infty] \) denotes, for \( \ell \neq p \), the morphism from \( A[\ell^\infty] \to B[\ell^\infty] \) induced by \( \psi \). It is a basic fact that \( \psi \neq 0 \) if and only if \( \psi_\ell \) is injective for some (all) \( \ell \),
if and only if $V_\ell(\psi)$ is an isomorphism for some (all) $\ell$. If $\psi$ is an isogeny, then $\ker(\ell(\psi))$ denotes the finite subgroup of $A$ given by the $\ell$–primary part of $\ker(\psi)$. If $\ell \neq p$, then $\ker(\ell(\psi))$ will be identified with the Galois module given by $\coker(\psi_\ell)$, on the other hand $\coker(\psi_p)$ is the finite length Dieudonné module associated to $\ker_p(\psi)$.

6.1. The endomorphism ring. Let $E$ be any object of $\mathcal{C}_{-p}$, denote the ring $\text{End}_k(E)$ by $R$, and the $\mathbb{Q}$–algebra $\text{End}_k(E) \otimes \mathbb{Q}$ by $D$. If $\ell$ is any prime number, set $R_\ell = R \otimes \mathbb{Z}_\ell$ and $D_\ell = D \otimes \mathbb{Q}_\ell$. As mentioned above, $D$ is a central, division algebra over $\mathbb{Q}$ such that $D_\ell$ is isomorphic to the matrix algebra $M_2(\mathbb{Q}_\ell)$ when $\ell \neq p$, and to “the” central, division algebra over $\mathbb{Q}_\ell$ of rank four when $\ell = p$. In this section we show that $R$ is a maximal order of $D$ or, equivalently, that $R_\ell$ is a maximal $\mathbb{Z}_\ell$–order in $D_\ell$, for all $\ell$.

If $\ell$ is a prime $\neq p$, then $T_\ell(E)$ is a free $\mathbb{Z}_\ell$–module of rank two on which $G_k$ acts continuously and in a natural way. The action of the arithmetic Frobenius $\text{Frob}_k \in G_k$ on $T_\ell(E)$ is the same as that given by $\pi_{E,\ell}$, i.e., that induced by $\pi_F$ via functoriality of the Tate module. Therefore the Galois module structure of $T_\ell(E)$ is specified by the requirement that $\text{Frob}_k$ act as multiplication by $-p$. In particular we see that

$$\text{End}_{\mathbb{Z}_\ell[G_k]}(T_\ell(E)) = \text{End}_{\mathbb{Z}_\ell}(T_\ell(E)).$$

The natural map $\iota_\ell : R \to \text{End}_{\mathbb{Z}_\ell[G_k]}(T_\ell(E))$ sending $r \in R$ into the induced morphism $r_\ell$ of Tate modules extends by continuity to a map on $R_\ell$, also denote by $\iota_\ell$. A special case of a theorem of Tate (cf. [15] or [18]) yields:

**Theorem 6.2.** The map $\iota_\ell : R_\ell \to \text{End}_{\mathbb{Z}_\ell}(T_\ell(E))$ is an isomorphism, and $R_\ell$ is a maximal order of $D_\ell$.

In order to study $R_p$ it is customary to work with the contravariant Dieudonné module $T_p(E)$ attached to the $p$–divisible group $E[p^\infty]$ of $E$. If $W = W(k)$ is the ring of Witt vectors of $k$, recall that the Dieudonné ring $\mathcal{A} = \mathcal{A}_k$ over $k$ is the non–commutative, polynomial ring in two variables $W[F,V]$ subject to the relations

$$FV = VF = p;$$

$$F\lambda = \lambda^p F, \ V\lambda^\sigma = \lambda V;$$

where $\lambda \in W$ is any element, and $\lambda \to \lambda^\sigma$ is the automorphism of $W$ inducing the absolute Frobenius on the residue field $k$. Notice that in this special case where $|k| = p^2$ we have that $F$ and $V$ have the same semi–linear behavior with respect to the action on $W$, since $\sigma = \sigma^{-1}$.

The Dieudonné module $T_p(E)$ is a left $\mathcal{A}$–module that is finite and free of rank two as $W$–module. Equation (7) implies that $F^2$ acts as multiplication by $-p$, therefore the $k$–semi–linear endomorphism of $T_p(E)/pT_p(E)$ induced
by $F$ is nonzero and nilpotent. Using this one shows that there exists a $W$–basis $(e_1, e_2)$ of $T_p(E)$ such that

\begin{align*}
F(e_1) &= -pe_2; \\
F(e_2) &= e_1;
\end{align*}

moreover we must have that $V = -F$ (cf. [9] for more details). The nonzero $A$–submodules of $T_p(E)$ that are $W$–lattices of $V_p(E)$ are those given by $F^nT_p(E)$, for $n \geq 0$. In terms of the arithmetic of $E$, this implies that every finite, closed subgroup of $E$ of $p$–power rank is the kernel of a chain of successive, alternating applications of the absolute Frobenius morphisms $F_E : E \to E^{(p)}$ and $F_{E^{(p)}} : E^{(p)} \to E^{(p^2)} = E$. Notice that equation (7) says that $F_{E^{(p)}}F_E = -p$.

The ring $R$ acts on the right of $T_p(E)$ by functoriality, and determines a ring homomorphism $\iota_p : R_p^{op} \to \text{End}_A(T_p(E))$ which extends by continuity to $R_p^{op}$. There is a version at $p$ of Theorem (6.2), also due to Tate (cf. [18], Thm. 6):

**Theorem 6.3.** The map $\iota_p : R_p^{op} \to \text{End}_A(T_p(E))$ is an isomorphism.

The ring $R_p^{op}$ is thus identified with the ring of $W$–linear endomorphisms of $T_p(E)$ commuting with $F$. Using the coordinate system induced by the previous basis $(e_1, e_2)$ of $T_p(E)$, we readily compute that $R_p^{op}$ is described by matrices of the form

$$\begin{pmatrix} a^\sigma & -b \\ pb^\sigma & a \end{pmatrix},$$

where $a, b \in W$. Letting $a$ and $b$ vary in the degree two, unramified extension $L = W[1/p]$ of $Q_p$, leads to a matrix description of $D_p^{op} = R_p^{op} \otimes_{Z_p} Q_p$ whose trace (tr) and determinant (det) respectively give the reduced trace and reduced norm of $D_p^{op}$ (cf. [16], I.1). The involution $x \to \text{tr}(x) - x$ gives an isomorphism $D_p^{op} \sim D_p$, sending $R_p^{op}$ to $R_p$. Its effect on the matrix description is

$$\begin{pmatrix} a^\sigma & -b \\ pb^\sigma & a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ -pb^\sigma & a^\sigma \end{pmatrix}.$$
6.2. Isogenies in $C_{-p}$ and ideals of $\text{End}_k(E)$. In this section we prove two theorems useful to describe a correspondance between isogenies of $C_{-p}$ whose source is a fixed object $E$, and nonzero, left ideals of $R = \text{End}_k(E)$. They both follow from the results of Tate that we already encountered.

Let $\psi : E \to E_\psi$ be any isogeny in $C_{-p}$, consider the $R$–left ideal

$$I_\psi = \{ r \in R : r = r' \psi, \text{ for some } r' \in \text{Hom}_k(E_\psi, E) \}$$

consisting of all the endomorphisms of $E$ trivial on the finite subgroup $\ker(\psi)$. Pull–back by $\psi$ gives an isomorphism $\psi^* : \text{Hom}(E_\psi, E) \xrightarrow{\sim} I_\psi$ of left $R$–modules. Notice that $I_\psi$ is a $\mathbb{Z}$–lattice of $D$, since for example $\deg(\psi) \subseteq I_\psi$. If $\ell$ is any prime, the module $I_\psi \otimes \mathbb{Z}_\ell$ will be denoted by $I_{\psi,\ell}$, it is a left ideal of $R_\ell$ that is also a $\mathbb{Z}_\ell$–lattice of $D_\ell$. The isomorphism $\psi^*$ above induces an isomorphism $\text{Hom}(E_\psi, E) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} I_{\psi,\ell}$, tacitly used in what follows.

**Remark 6.4.** From the maximality of $R_\ell$, it follows that $I_{\psi,\ell}$ is principal for any prime $\ell$ (cf. [16] II.1, II.2). If $\ell \neq p$, there exists a $\mathbb{Z}_\ell$–lattice $\Lambda_0$ of $V_\ell(E)$ containing $T_\ell(E)$ such that the isomorphism $i_\ell$ (cf. Thm. [6.2]) sends $I_{\psi,\ell}$ to the left ideal $\text{End}_{\mathbb{Z}_\ell}(\Lambda_0, T_\ell(E))$. In fact we will see that $\Lambda_0$ is the pull–back of $T_\ell(E_\psi)$ via $V_\ell(\psi)$ (cf. proof of Thm. [6.5]). For $\ell = p$, there exists a $W$–lattice $M_0$ in $V_p(E)$ contained in $T_p(E)$ such that the isomorphism $i_p$ (cf. Thm. [6.3]) sends $I_{\psi,p}$ to the right ideal $\text{End}_A(T_p(E), M_0)$. From the proof of Theorem [6.5] it will follow that $M_0$ is the isomorphic image of $T_p(E_\psi)$ with respect to $V_p(\psi) : V_p(E_\psi) \to V_p(E)$. For any $\ell$, the left ideal $I_{\psi,\ell}$ is generated by any of its elements of reduced norm with minimal valuation.

Before showing that the isogeny $\psi$ can essentially be recovered from $I_\psi$ (cf. Thm. [6.5]), we make a few observations. Let $\ell$ be a prime $\neq p$, from the isogeny $\psi : E \to E_\psi$ we can deduce a commutative diagram of $G_k$–modules with exact rows

$$
\begin{array}{cccccc}
0 & \to & T_\ell(E) & \to & V_\ell(E) & \to & E[\ell^\infty] & \to & 0 \\
| & & \psi_\ell & & V_\ell(\psi) & & \psi[\ell^\infty] & | \\
0 & \to & T_\ell(E_\psi) & \to & V_\ell(E_\psi) & \to & E_\psi[\ell^\infty] & \to & 0.
\end{array}
$$

The pull–back of the Tate module $T_\ell(E_\psi)$ via the isomorphism $V_\ell(\psi)$ identifies the former with a $\mathbb{Z}_\ell$–lattice of $V_\ell(E)$ containing $T_\ell(E)$, and that will be denoted by $\Lambda_{\psi,\ell}$. Under this identification, the map $\psi_\ell$ clearly corresponds to the inclusion $T_\ell(E) \subseteq \Lambda_{\psi,\ell}$. Since the two rows of the diagram are exact, $V_\ell(\psi)$ induces an identification $\text{coker}(\psi_\ell) \simeq \ker(\psi[\ell^\infty])$, or, equivalently,

$$
\Lambda_{\psi,\ell}/T_\ell(E) \simeq \ker(\psi).
$$

(9)
The $\mathbb{Z}_\ell$-lattice $\Lambda_\psi$ depends on $\psi$ in a functorial way, in a sense that can be made precise by working with the category $\mathcal{C}_E$ that will be introduced in the next section. We observe that if $\varphi : E \to E_\varphi$ is another isogeny, and $u : E_\psi \to E_\varphi$ is a morphism, then the map $V_\ell(\varphi)^{-1}V_\ell(u)V_\ell(\psi) : V_\ell(E) \to V_\ell(E)$ sends $\Lambda_\psi$ to $\Lambda_\varphi$ and there is a commutative diagram

$$
\begin{array}{ccc}
\Lambda_\psi & \xrightarrow{V_\ell(\varphi)^{-1}V_\ell(u)V_\ell(\psi)} & \Lambda_\varphi \\
\downarrow_{\simeq} & & \downarrow_{\simeq} \\
T_\ell(E_\psi) & \xrightarrow{u_\ell} & T_\ell(E_\varphi)
\end{array}
$$

where the vertical morphisms are the natural identifications. In particular, if $u$ is an isogeny, the cokernel of the top horizontal map of the diagram is isomorphic to $\ker_\ell(u)$ in a natural way.

If $\ell = p$, the situation is formally analogous, after replacing the covariant Tate module by the contravariant Dieudonné module. The morphism $\psi_p : T_p(E_\psi) \to T_p(E)$ identifies $T_p(E_\psi)$ with an $A$-submodule $M_{\psi_p}$ of $T_p(E)$ that is a $W$-lattice of $V_p(E)$. Moreover $T_p(E)/M_{\psi_p}$ is the Dieudonné module associated to the finite group $\ker_p(\psi)$. As before, if $\varphi : E \to E_\varphi$ is an isogeny and $u : E_\psi \to E_\varphi$ is any morphism, then the map $V_p(\psi)V_p(u)V_p(\varphi)^{-1} : V_p(E) \to V_p(E)$ sends $M_{\varphi_p}$ to $M_{\psi_p}$, there is a commutative diagram

$$
\begin{array}{ccc}
M_{\psi_p} & \xrightarrow{V_p(\psi)V_p(u)V_p(\varphi)^{-1}} & M_{\varphi_p} \\
\downarrow_{\simeq} & & \downarrow_{\simeq} \\
T_p(E_\psi) & \xrightarrow{u_p} & T_p(E_\varphi)
\end{array}
$$

where the vertical morphisms are the natural identifications. Moreover the cokernel of the top horizontal map of the diagram is the Dieudonné module of $\ker_p(u)$.

**Theorem 6.5.** For any isogeny $\psi : E \to E_\psi$ and any prime number $\ell$, there exists an isogeny $r : E_\psi \to E$ whose degree is prime to $\ell$. Equivalently, there exists an isogeny $r \in I_\psi$ so that $\ker_\ell(r) = \ker_\ell(\psi)$. In particular, $\ker(\psi)$ coincides with the largest $k$-subgroup of $E$ killed by every element of $I_\psi$.

**Proof.** We begin by proving the theorem in the case $\ell \neq p$. Consider the commutative diagram

$$
\begin{array}{ccc}
\Hom(E_\psi, E) \otimes \mathbb{Z}_\ell & \longrightarrow & \Hom_{\mathbb{Z}_\ell}(T_\ell(E_\psi), T_\ell(E)) \\
\downarrow_{\simeq} & & \downarrow_{\simeq} \\
I_{\psi, \ell} & \longrightarrow & \Hom_{\mathbb{Z}_\ell}(\Lambda_{\psi, \ell}, T_\ell(E))
\end{array}
$$
24 TOMMASO GIORGIO CENTELEGHE

where the horizontal maps are induced by functoriality of the Tate module, and the vertical isomorphism on the right comes from the identification $T_\ell(E_\psi) \simeq \Lambda_{\psi_\ell}$ and is given by $u_\ell \mapsto u_\ell V(\psi)|_{T_\ell(E_\psi)}$.

Since $\text{Hom}_{\mathbb{Z}[G_k]}(T_\ell(E_\psi), T_\ell(E)) = \text{Hom}_{\mathbb{Z}_\ell}(T_\ell(E_\psi), T_\ell(E))$, the main theorem of [15] says that the top horizontal map of the previous diagram is an isomorphism, thus so is the bottom one and the isomorphism $\iota_\ell$ from Theorem 6.2 induces an identification of left ideals

\[(10) \quad I_{\psi,\ell} \simeq \text{Hom}_{\mathbb{Z}_\ell}(\Lambda_{\psi_\ell}, T_\ell(E)).\]

In particular, $I_{\psi,\ell} = R_\ell g_\ell$, where $g_\ell \in D_\ell$ is any element of $R_\ell$ mapping $\Lambda_{\psi_\ell}$ isomorphically to $T_\ell(E)$. If we pick $r \in I_\psi$ such that its image $r_\ell \in I_{\psi,\ell}$ is a generator then $\ker_\ell(r) = \ker_\ell(\psi)$ by (9), since the lattices $\Lambda_{r_\ell}$ and $\Lambda_{\psi_\ell}$ in $V_\ell(E)$ coincide.

For $\ell = p$ consider, in an analogous way, the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(E_\psi, E) \otimes \mathbb{Z}_p & \longrightarrow & \text{Hom}_A(T_p(E), T_p(E_\psi)) \\
\downarrow \simeq & & \downarrow \simeq \\
I_{\psi,p} & \longrightarrow & \text{Hom}_A(T_p(E), M_{\psi_p})
\end{array}
\]

where the horizontal maps are induced by functoriality of $T_p$, and the vertical isomorphism on the right is given by $u_p \mapsto \psi_p u_p$. Notice that the lower horizontal map is that induced by the anti–isomorphism $\iota_p : R_p \rightarrow \text{End}_A(T_p(E))$, where $\text{Hom}_A(T_p(E), M_{\psi_p})$ is regarded in a natural way as a right ideal of $\text{End}_A(T_p(E))$.

Since both horizontal maps are isomorphisms (cf. [18], Thm. 6), the principal left ideal $I_{\psi,p}$ of $R_p$ is generated by any element $r_p$ such that $\iota_p(r_p)$ defines an endomorphism of $T_p(E)$ sending $T_p(E)$ isomorphically to $M_{\psi_p}$. Choosing now $r \in I_\psi$ such that $r_p \in I_{\psi,p}$ is a generator, we see that $M_{r_p} = M_{\psi_p}$, therefore $\ker_p(r) = \ker_p(\psi)$ and the theorem follows. \square

Let now $I$ be a nonzero left ideal of $R$, consider the isogeny

\[(11) \quad \psi_I : E \longrightarrow E/H(I),\]

where

\[H(I) = \bigcap_{r \in I} \ker(r)\]

is the largest $k$–subgroup of $E$ that is killed by every element of $I$, and $\psi_I$ is the canonical isogeny. The ideal $I$ can be recovered from $\psi_I$.

**Theorem 6.6.** For any nonzero left ideal $I$ of $R$ we have $I = I_{\psi_I}$. 

Proof. Let \( J = I_{\psi I} \) be the left ideal of \( R \) given by the endomorphisms of \( E \) trivial on \( H(I) \), certainly \( I \subset J \). For any prime \( \ell \), denote by \( I_{\ell} \) and \( J_{\ell} \) the left ideals of \( R_{\ell} \) obtained from \( I \) and \( J \) after tensoring with \( \mathbb{Z}_{\ell} \). Since \( I_{\ell} \subset J_{\ell} \), it will be enough to show that \( J_{\ell} \subset I_{\ell} \) for any \( \ell \).

For \( \ell \neq p \), as explained at the beginning of the section, the Tate module \( T_{\ell}(E/H(I)) \) is identified with the \( \mathbb{Z}_{\ell} \)-lattice \( \Lambda_{\psi_{I,\ell}} \) of \( V_{\ell}(E) \) containing \( T_{\ell}(E) \) and such that the equality

\[
\Lambda_{\psi_{I,\ell}}/T_{\ell}(E) = \ker_{\ell}(\psi_{I}),
\]

holds in the \( \ell \)-divisible group \( E[\ell^{\infty}] \). Moreover, the isomorphism \( \iota_{\ell} : R_{\ell} \tilde{\rightarrow} \text{End}_{\mathbb{Z}_{\ell}}(T_{\ell}(E)) \) induces an identification \( J_{\ell} \simeq \text{Hom}_{\mathbb{Z}_{\ell}}(\Lambda_{\psi_{I,\ell}}, T_{\ell}(E)) \), as was shown in the proof of Theorem 6.5.

On the other hand, \( I_{\ell} \) is a nonzero left ideal of \( R_{\ell} = \text{End}_{\mathbb{Z}_{\ell}}(T_{\ell}(E)) \) and there exists a \( \mathbb{Z}_{\ell} \)-lattice \( \Lambda_{0} \) of \( V_{\ell}(E) \), containing \( T_{\ell}(E) \), such that \( I_{\ell} = \text{Hom}_{\mathbb{Z}_{\ell}}(\Lambda_{0}, T_{\ell}(E)) \); since \( I_{\ell} \subset J_{\ell} \), we have \( \Lambda_{0} \supset \Lambda_{\psi_{I,\ell}} \).

Now, by definition of \( \psi_{I} \) we have

\[
\ker_{\ell}(\psi_{I}) = \bigcap_{0 \neq r \in I} \ker_{\ell}(r),
\]

or, reformulating,

\[
\Lambda_{\psi_{I,\ell}} = \bigcap_{0 \neq r \in I} V_{\ell}(r)^{-1}(T_{\ell}(E)).
\]

The right hand side of (12) certainly contains \( \Lambda_{0} \), therefore \( \Lambda_{0} \subset \Lambda(\psi_{I})_{\ell} \) and \( I_{\ell} \supset J_{\ell} \).

For \( \ell = p \) the proof is similar and is omitted. As in the case \( \ell \neq p \), the key fact is that \( I_{p} \) is known a priori to be principal, thanks to the maximality of \( R_{p} \) (cf. Remark 6.4). \( \square \)

Remark 6.7. Theorem 6.6 is a special case of a more general result of Waterhouse (cf. [17], Thm. 3.15).

6.3. An equivalence of categories. We interpret the result of the previous section in a formal way, by showing the existence of an anti-equivalence of categories (cf. Thm. 6.8).

Let \( E \) be a fixed object of \( \mathcal{C}_{-p} \), and let \( \mathcal{C}_{E} \) be the category of isogenies of \( \mathcal{C}_{-p} \) with source \( E \), that is the category whose objects are isogenies in \( \mathcal{C}_{-p} \) of the form \( \psi : E \rightarrow E_{\psi} \), and whose sets of morphisms \( \text{Hom}(\psi, \varphi) \) between two objects is simply \( \text{Hom}(E_{\psi}, E_{\varphi}) \).

Let \( \psi : E \rightarrow E_{\psi} \) be any object of \( \mathcal{C}_{E} \), the natural identification \( I_{\psi} = \text{Hom}(E_{\psi}, E) = \text{Hom}(\psi, \text{id}_{E}) \) makes clear that \( I_{\psi} \) depends functorially on \( \psi \); more precisely the assignment \( \psi \rightarrow I_{\psi} \) defines a contravariant functor \( \mathcal{I} \) from \( \mathcal{C}_{-p} \) to the category \( \mathcal{P}_{R}^{(1)} \) of projective left \( R \)-modules of rank one. We
clarify that if \( u \in \text{Hom}(\psi, \varphi) \) is a morphism in \( C_E \), then \( I(u) : I_\varphi \to I_\psi \) is constructed as follows. If \( r = r'' \varphi \in I_\varphi \), consider the commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{r} & E \\
\downarrow \psi & & \downarrow \varphi \\
E_\psi & \xrightarrow{u} & E_\varphi \\
\end{array}
\]

the value of \( I(u) \) on \( r \) is the composition \( r''u\psi \in I_\psi \).

From the results of the previous section we can draw the following consequence:

**Theorem 6.8.** The functor \( I \) sending \( \psi \) to \( I_\psi \) induces an anti-equivalence of categories between \( C_E \) and \( \mathcal{P}_R^{(1)} \).

**Proof.** We show that any object of \( \mathcal{P}_R^{(1)} \) is isomorphic to one of the form \( I(\psi) \), and that \( I \) is fullyfaithful.

Any nonzero projective left \( R \)-module \( P \) of rank one is isomorphic to a finitely generated, left \( R \)-submodule of \( D \), therefore to a nonzero left ideal \( I \) of \( R \), after multiplication by a suitable integer \( N \geq 1 \). From Theorem 6.6 we see that every nonzero left ideal \( I \) of \( R \) is of the form \( I(\psi) \), for some \( \psi \) in \( C_E \).

To see that \( I \) is fullyfaithful, let \( \psi \) and \( \varphi \) be two objects of \( C_E \), and \( f : I_\varphi \to I_\psi \) any morphism of left \( R \)-modules. We have to show that there exists \( u \in \text{Hom}(\psi, \varphi) \) such that \( f = I(u) \). Since \( I_\varphi \) and \( I_\psi \) are lattices of \( D \), \( f \) extends uniquely to an endomorphism of \( D \), as left \( D \)-module, thus there is \( \lambda \in D \) such that \( f(x) = x\lambda \), for all \( x \in I_\varphi \). Let \( N \) be an integer \( \geq 1 \) such that \( N\lambda \in R \). Right multiplication by \( N\lambda \) on \( D \) clearly induces the morphism \( Nf : I_\varphi \to I_\psi \) and we will first show that there exists \( u' \in \text{Hom}(E_\psi, E_\varphi) \) such that \( I(u') = Nf \).

For \( r = r'' \varphi \in I_\varphi \) consider the commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{N\lambda} & E \\
\downarrow \psi & & \downarrow \varphi \\
E_\psi & \xrightarrow{u'} & E_\varphi \\
\end{array}
\]

The existence of the dotted arrow \( u' \) follows from the fact that \( rN\lambda \) belongs to \( I_\psi \) (in fact even to \( NI_\psi \)), and hence factors as \( r'\psi \), for some
By Theorem 6.5 we have that \( \psi \) factors as \( u' \psi \), for a certain \( u' \in \text{Hom}(E_\psi, E_\varphi) \).

If \( N \lambda = 0 \) this is clear, therefore we may assume that \( \varphi N \lambda \) is an isogeny. By Theorem \ref{thm:factorization} we have that

\[
\ker(\varphi N \lambda) = \bigcap_{r'' \in \text{Hom}(E_\psi, E_\varphi)} \ker(r'' \varphi N \lambda),
\]

but \( r'' \varphi N \lambda = r' \psi \), therefore

\[
\ker(\psi) \subset \bigcap_{r'' \in \text{Hom}(E_\psi, E_\varphi)} \ker(r'' \varphi N \lambda).
\]

The two equations readily imply \( \ker(\psi) \subset \ker(\varphi N \lambda) \), which gives the desired factorization of \( \varphi N \lambda \) as \( u' \psi \).

A further application of Theorem \ref{thm:factorization} gives that \( u' \) is divisible by \( N \) in \( \text{Hom}(E_\psi, E_\varphi) \). In fact since \( r''u' \psi \in NI_\psi \), for \( r'' \in \text{Hom}(E_\varphi, E) \), we have that \( r''u' : E_\psi \to E \) is trivial on the subgroup \( E_\psi[N] \). Since this happens for all \( r'' \in \text{Hom}(E_\varphi, E) \) we see that, by Theorem \ref{thm:factorization} \( u' \) is itself trivial on \( E_\psi[N] \), and \( u' = Nu \) for a unique \( u \in \text{Hom}(E_\psi, E_\varphi) \). Therefore \( \mathcal{I}(u) = N^{-1}\mathcal{I}(u') = f \), this completes the proof of the theorem.

6.4. Adelic description of \( \mathcal{C}_{-p} \). The functor \( \mathcal{I} \) introduced in the previous section admits an adelic description, thanks to the fact that \( I_\psi \) is locally principal, for any \( \psi \) in \( \mathcal{C}_E \) (cf. Remark \ref{rem:adelic}). Let \( \hat{R} \) denote \( R \otimes \hat{\mathbb{Z}} \) and, similarly, \( \hat{D} \) denote \( D \otimes \hat{\mathbb{Z}} \), we have \( \hat{R} = \prod R_\ell \) and \( \hat{D} = \prod D_\ell \), where the product is taken over all the primes \( \ell \).

If \( \psi \) is any object of \( \mathcal{C}_E \), and \( \ell \) is any prime, then \( I_{\psi, \ell} = R_\ell g_\ell \), for some \( g_\ell \in R_\ell \); observe that \( g_\ell \in D_\ell^* \), since \( I_{\psi, \ell} \) is a \( \mathbb{Z}_\ell \)-lattice of \( D_\ell \). The image of the adelic element \( (g_\ell) \in \hat{R} \cap \hat{D}^* \) in the coset space \( \hat{R}^* \hat{R} \cap \hat{D}^* \) depends only on \( \psi \), and not on the choices of the \( g_\ell \), and is denoted by \( a_\psi \).

Viceversa, if \( a \in \hat{R}^* \hat{R} \cap \hat{D}^* \) is represented by \( (g_\ell) \in \hat{R} \cap \hat{D}^* \), then \( R_\ell g_\ell = R_\ell \) for almost all \( \ell \), and there is a unique left ideal \( I_a \) of \( R \) such that \( I_{a, \ell} = I_a \otimes \mathbb{Z}_\ell \) is equal to \( R_\ell g_\ell \), for all \( \ell \). The isogeny corresponding to \( I_a \) will simply be denoted by \( \psi_a : E \to E_{\psi a} \), it defines an element of \( \mathcal{C}_E \).

If \( \Sigma \) is the set of isomorphism classes of objects of \( \mathcal{C}_E \), or of \( \mathcal{C}_{-p} \), a consequence of Theorem \ref{thm:adelic} is:

\textbf{Theorem 6.9.} The assignment \( \psi \to a_\psi \) induces a correspondence

\[
\tau : \Sigma \xrightarrow{\sim} \hat{R}^* \hat{R} \cap \hat{D}^* / D^*.
\]

The surjectivity of \( \tau \) follows from the fact that the inclusion \( \hat{R} \cap \hat{D}^* \subset \hat{D}^* \) induces a bijection \( \hat{R}^* \hat{R} \cap \hat{D}^* / D^* = \hat{R}^* \hat{D}^* / D^* \).
Let now $N \geq 1$ be a fixed integer not divisible by $p$. If $\psi : E \rightarrow E_\psi$ is an object of $\mathcal{C}_E$, the tangent space $t_0(E_\psi)$ of $E_\psi$ at the origin is a one–dimensional $k$–vector space, and its dual $t_0(E_\psi)^*$ is identified with the space of invariant $1$–form on $E_\psi$ (defined over $k$). Consider the set $\Sigma(1,N)$ of isomorphism classes of triples $(\psi, \omega, x)$ given by an object $\psi : E \rightarrow E_\psi$ of $\mathcal{C}_E$, a nonzero invariant $1$–form $\omega \in t_0(E_\psi)^*$, and a point $x \in E_\psi[N](\bar{k})$ of order $N$. The notion of isomorphism between two such triples is clear: $(\psi, \omega, x)$ is isomorphic to $(\varphi, \eta, y)$ if and only if there exists an isomorphism $u : E_\psi \sim E_\varphi$ such that $u^*(\eta) = \omega$ and $u(x) = y$. Notice that $k^*$ acts on $\Sigma(1,N)$ via homotheties on the second entry of the triples.

Choose now a nonzero $1$–form $\omega_0 \in t_0(E)^*$, and an element $x_0 \in E[N](\bar{k})$ of order $N$. The ring $\hat{R} = \prod R_\ell$ acts on the left of the product

$$t_0(E)^* \times \prod_{\ell \neq p} E[\ell^\infty]$$

by considering the natural action of $R_\ell$ on $E[\ell^\infty]$ for $\ell \neq p$, and on $t_0(E)^*$ for $\ell = p$. Notice that the action of $R_p$ on $t_0(E)^*$ is given by a canonical ring homomorphism $R_p \rightarrow k$, necessarily surjective because of the structure of $R_p$, that will be used to identify the residue field $R_p/m$ with $k$. The kernel of the corresponding character $R_p^* \rightarrow k^*$ is given by $R_p^*(1)$, the subgroup of units congruent to $1$ modulo the maximal, two sided ideal $m$.

Let $K(1,N)$ be the open subgroup of $\hat{R}^*$ which stabilizes the pair $(\omega_0, x_0)$. We have a decomposition

$$K(1,N) = R_p^*(1) \times \prod_{\ell \neq p} K_\ell(N),$$

for a certain collection $K_\ell(N)$ of open subgroups of $R_\ell^*$, with $\ell \neq p$. The subgroup $R_p^*$ of $\hat{R}^*$ normalizes $K(1,N)$ and left translation induces an action of $R_p^*/R_p^*(1) = k^*$ on the coset space $K(1,N) \backslash \hat{D}^*/D^*$.

**Remark 6.10.** There is a $\mathbf{Z}_\ell$–basis $(e_1, e_2)$ of $T_\ell(E)$, for $\ell \neq p$, such that the corresponding identification $R_\ell^* \simeq \text{GL}_2(\mathbf{Z}_\ell)$ sends $K_\ell(N)$ to the subgroup

$$U_\ell(N)^\gamma = \left\{ x \in \text{GL}_2(\mathbf{Z}_\ell) \mid x \equiv \begin{pmatrix} 1 & \ast \\ 0 & \ast \end{pmatrix} \pmod{N} \right\},$$

which is the image of $U_\ell(N)$ (cf. section 2) with respect to the canonical involution $\gamma$. If $\ell^s$ is the exact power of $\ell$ dividing $N$, the basis above should be chosen in such a way that the element of $T_\ell(E)/(N) = E[\ell^s](\bar{k})$ defined by $e_1 \pmod{N}$ is equal to the $\ell$–primary component of $x_0$.

**Theorem 6.11.** There is a natural bijection

$$\tau_N : \Sigma(1,N) \sim K(1,N) \backslash \hat{D}^*/D^*.$$
The permutation action of \( k^* \) on \( \Sigma(1,N) \) corresponds via \( \tau_N \) to the action induced by left translation of \( R_p^* \) on \( K(1,N) \setminus \hat{D}^* / D^* \).

**Proof.** Let \( a \in K(1,N) \setminus \hat{R} \cap \hat{D}^* \) be any element, we begin by showing how to construct a triple \((\psi_a, \omega_a, x_a)\) of the type considered out of \( a \). Let \( I_a \) be the left ideal of \( R \) defined by the adelic coset \( a \in \hat{R}^* \setminus \hat{R} \cap \hat{D}^* \) deduced from \( a \). The isogeny \( \psi_a : E \to E_{\psi_a} \) is that obtained from the \( I_a \) using the construction [11] from section 6.2.

Let \( g = (g_\ell)_{\ell} \in \hat{R} \cap \hat{D}^* \) be any element representing \( a \), i.e., such that \( a = K(1,N)g \). Recall that for \( \ell \neq p \) the module \( T_\ell(E_{\psi_a}) \) is identified with a \( \mathbb{Z}\ell\)-lattice \( \Lambda_{\psi_{a,\ell}} \) of \( V_\ell(E) \) containing \( T_\ell(E) \), moreover \( g_\ell \in D_\ell^* \) is a generator of the principal, \( R_{\ell}\)-left ideal \( \text{Hom}_{\mathbb{Z}_\ell}(\Lambda_{\psi_{a,\ell}}, T_\ell(E)) \), and defines an isomorphism of \( V_\ell(E) \) that maps \( \Lambda_{\psi_{a,\ell}} \) isomorphically to \( T_\ell(E) \). Therefore we deduce, for any \( n \geq 0 \), an identification

\[
E_\psi[\ell^n](\bar{k}) = \Lambda_{\psi_{a,\ell}} / (\ell^n) \cong T_\ell(E)/ (\ell^n) = E[\ell^n](\bar{k})
\]

depending on the choice of the generator \( g_\ell \) of \( I_{\psi,\ell} \).

If now \( \ell^e \) is the exact power of \( \ell \) dividing \( N \), then the \( \ell \)-th primary component \( x_{0,\ell} \) of \( x_0 \) is a point of order \( \ell^e \) in \( T_\ell(E)/(\ell^e) \). The inverse image \( x_{a,\ell} \in \Lambda_{\psi_{a,\ell}} / (\ell^e) \) of \( x_{0,\ell} \) by the identification (13) for \( n = e \) is a point \( E_{\psi_a}[\ell^e](\bar{k}) \) of order \( \ell^e \) that depends only on the coset \( K_\ell(N)g_\ell \), as one can easily check. The point \( x_a \in E_{\psi_a}[N](\bar{k}) \) is the one whose \( \ell \)-th primary component is \( x_{a,\ell} \), and depends only on \( a \), and not on \( g \).

To construct the invariant \( k \)-form \( \omega_a \), we use the fact that for any object \( X \) of \( \mathcal{C}_{-p} \) there is a canonical identification of \( k \)-vector spaces (cf. [5], II, Prop. 4.3)

\[
T_p(X)/FT_p(X) = t_0(X)^*.
\]

Recall that the Dieudonné module \( T_p(E_{\psi_a}) \) is identified, via \( \psi_{a,p} \), to a \( \mathcal{A} \)-submodule \( M_{\psi_{a,p}} \) of \( T_p(E) \), moreover the component \( g_p \in D_p^* \) at \( p \) of \( g \) defines a generator \( t_p(g_p) \) of \( \text{Hom}_\mathcal{A}(T_p(E), M_{\psi_{a,p}}) \) as a right ideal of \( \text{End}_\mathcal{A}(T_p(E)) \).

We have a commutative diagram of morphisms of \( \mathcal{A} \)-modules

\[
\begin{array}{ccc}
T_p(E) & \longrightarrow & M_{\psi_{a,p}} \\
\psi_{a,p} & \\ & \nearrow \\
& \sim & \\
& \sim & \\
& \sim & \\
& \sim & \\
& \sim & \\
T_p(E_{\psi_a}).
\end{array}
\]

The diagonal isomorphism to the right of the diagram is induced by the composition of \( t_p(g_p) \) corestricted to \( M_{\psi_{a,p}} \), followed by the inverse of the corestriction of \( \psi_{p,a} \) to the same lattice of \( V_p(E) \). Notice that it is not
independent on the choice of \( g_p \). From this isomorphism, using \([14]\), we can deduce an isomorphism of \( k\)-vector spaces

\[
\xi_a : \mathfrak{t}_0(E)^* \xrightarrow{\sim} \mathfrak{t}_0(E_{\psi_a})^*
\]

that depend only on the coset \( R^*_p(1)g_p \), and not on the choice of \( g_p \), as one can easily check. Setting \( \omega_a \) to be equal to \( \xi_a(\omega_{\hat{a}}) \) completes the construction of the triple \((\psi_a, \omega_a, x_a)\) attached to \( a = K(1, N)g \in K(1, N)\backslash \hat{R} \cap \hat{D}^* \).

We are left with showing that, for \( a, b \in K(1, N)\backslash \hat{R} \cap \hat{D}^* \), the triples \((\psi_a, \omega_a, x_a)\) and \((\psi_b, \omega_b, x_b)\) are isomorphic if and only if \( a = b\lambda \), for some \( \lambda \in D^* \). One way to proceed is to observe that in order for the two triples to be isomorphic certainly we must have that there exists an isomorphism \( u : E_{\psi_a} \to E_{\psi_b} \), equivalently, by Proposition \([6.9]\) if \( \hat{a}, \hat{b} \) denote the elements of \( \hat{R}^* \backslash \hat{R} \cap \hat{D}^* \) deduced from, respectively, \( a \) and \( b \), then there must be \( \lambda \in D^* \) such that \( \hat{a} = \hat{b}\lambda \). We leave to the reader the task of showing, with the help of section \([6.2]\) that the isomorphism \( u \) induces an isomorphism between \((\psi_a, \omega_a, x_a)\) and \((\psi_b, \omega_b, x_b)\) if and only if \( a = b\lambda \).

**Remark 6.12.** If \( d \) is an integer \( \geq 1 \), the natural map \( K(1, Nd) \backslash \hat{D}^*/D^* \to K(1, N) \backslash \hat{D}^*/D^* \) corresponds, under the identifications \( \tau_{Nd} \) and \( \tau_N \) to the map sending a given triple \((\psi, \omega, x)\) to \((\psi, \omega, dx)\) in \( \Sigma(1, N) \).

**Remark 6.13.** Once for each prime \( \ell \neq p \) a basis for \( T_\ell(E) \) as in Remark \([6.10]\) is chosen, we have that the canonical involution \( \gamma \) of \( D \) defines a bijection

\[
K(1, N) \backslash \hat{D}^*/D^* \xrightarrow{\sim} \Omega(1, N),
\]

where \( \Omega(1, N) \) is the double coset space introduced in section \([2]\). The effect of the main involution of \( D_p \) on the multiplicative group of the residue field is the Frobenus automorphism. This explains the occurrence of the twist for the action of \( k^* \) as given in Theorem \([5.1]\) which is just a restatement of Theorem \([6.11]\).

**6.5. Hecke operators.** For an integer \( N \geq 1 \) not divisible by \( p \), the space \( S(1, N) \) defined in section \([2]\) is identified, using Remark \([6.13]\) following Theorem \([5.1]\) with the space of complex valued functions on \( \Sigma(1, N) \). Our task is now to investigate a few basic properties of the Hecke operators on \( S(1, N) \).

**Definition 6.14.** Let \( \ell \) be a prime \( \neq p \). The \( \ell \)-th Hecke operator \( T_\ell \) on \( S(1, N) \) is defined as

\[
T_\ell(f)(\psi, \omega, x) = \sum_{\varphi} f((\psi', \omega', x')),
\]

where the sum ranges through all the degree \( \ell \) isogenies \( \varphi : E_\psi \to E_{\psi'} \) such that \( \varphi(x) \) has order \( N \) in \( E_{\psi}^/[N] \), and where remaining entries of the triple \((\psi', \omega', x')\) are defined by \( \varphi^*(\omega') = \omega, \varphi(x) = x' \).
**Proposition 6.15.** If \( \ell \nmid pN \), then the Hecke operator \( T_\ell \) acting on \( S(1,N) \) is semi–simple.

**Proof.** Consider the real inner product on \( S(1,N) \) for which the basis given by the characteristic functions of the singletons of \( \Sigma(1,N) \) is orthonormal. A standard application of the existence of the dual isogeny shows that \( T_\ell \), for \( \ell \nmid pN \) commutes with its adjoint and is therefore semi–simple (cf. [7], Prop. 2.7). □

For \( \ell \neq p \), consider the decomposition into right cosets of the double coset

\[
U_\ell(N)^t \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} U_\ell(N)^t = \bigsqcup_i U_\ell(N)^t \alpha_i.
\]

The number of left cosets in the above decomposition is \( \ell + 1 \) if \( \ell \nmid pN \), and \( \ell \) if \( \ell | N \). Using the identification

\[
\Sigma(1, N) = \left( R_p^*(1) \times \prod_{\ell \neq p} U_\ell(N)^t \right) \backslash \hat{D}^*/D^*;
\]

one can show that the operator \( T_\ell \) is described by:

**Proposition 6.16.** For every \( \ell \neq p \) we have

\[
T_\ell(f)(x) = \sum_i f(\alpha_i x).
\]

After applying the main involution, we now see that the definition of \( T_\ell \) given in section 2 corresponds to that given in this section under the bijection of Remark 6.13.

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