Conserved mass models with stickiness and chipping

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Abstract. We study a chipping model in a one-dimensional periodic lattice with continuous mass, where a fixed fraction of the mass is chipped off from a site and distributed randomly among the departure site and its neighbours; the remaining mass sticks to the site. In the asymmetric version the chipped off mass is distributed among the site and the right neighbour, whereas in the symmetric version the redistribution occurs among the two neighbours. The steady state mass distribution of the model is obtained using a perturbation method for both parallel and random sequential updates. In most cases, this perturbation theory provides a steady state distribution with reasonable accuracy.

Keywords: solvable lattice models, transport properties (theory), stationary states
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1. Introduction

Most systems in Nature are in non-equilibrium states [1], in such a way that the accompanying fluxes of mass, energy, or spin, etc are irreversible. Unlike their equilibrium counterparts where the stationary state is characterized by the Gibbs measure, these systems usually reach different and novel stationary states depending on the dynamics of the microscopic constituents. Several non-equilibrium lattice models have been proposed recently [2] to investigate the unusual steady state distributions, spatio-temporal correlations and possibility of macroscopic collective phenomena.

One of the simple non-equilibrium models is the mass transport model where each site of a lattice is associated with discrete mass (particles) following a dynamics that involves aggregation, fragmentation, adsorption or desorption [3]. Like the zero range process [4], interestingly, many of these model systems undergo a condensation phase transition as the mass density of the system is increased. The study of these models has generated considerable interest among physicists, as a wide variety of systems exhibit basic microscopic mechanisms similar to those of the simple mass transport models. These include colloidal suspensions [5], polymer gels [6], river networks [7], traffic models [8], and wealth distribution [9], etc.

A continuous version of the mass transport model was proposed recently [10] and some of its variations have also been studied [11]–[13]. Many of the mass transport models are known to evolve into a non-equilibrium steady state that has product measure. A generic criterion for a factorized steady state has been
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derived [12, 13] for mass transport models. In our effort to study models where the steady state is not factorized and only a little is known analytically, we develop a perturbation approach that provides an approximate form of the steady state distribution.

In this paper we introduce stickiness, quantified by a parameter $\lambda$, to the continuous mass transport models. At each site, $(1 - \lambda)$ fraction of the mass is chipped off (thus $\lambda$ fraction of the mass sticks to the site), which is then redistributed either asymmetrically, i.e. among the departure site and its right neighbour, or symmetrically, i.e. among the two neighbours. Accordingly, the model is referred to as the asymmetric or symmetric sticky chipping model respectively. We use a novel perturbation approach to calculate the steady state mass distributions of these models, for both parallel and random sequential updates. Although in this perturbation approach we have ignored two and three point spatial correlations, the mass distributions calculated up to second order are strikingly close to the ones obtained from Monte Carlo simulations.

The paper is organized as follows. The model and the perturbation method are outlined in section 2. In section 3 we study the asymmetric version of the model and obtain the steady state distribution. The symmetric version of the model, where the chipped off mass is distributed among both the neighbours, is discussed in section 4. Finally we discuss the main results in section 5. The perturbation results up to second order for all these models are listed in the appendix.

2. The model

The model is defined on a one-dimensional periodic lattice with sites labelled by $i = 1, 2, \ldots, L$. A continuous mass $x_i$, associated with each site $i$, evolves according to the following dynamics. At each site $i$, $(1 - \lambda)$ fraction of the mass is chipped off (thus, $\lambda$ fraction of the mass sticks to the site) and then it is distributed among the departure site $i$ and its neighbours $(i \pm 1)$. In this paper we study two different versions; in the asymmetric sticky chipping model (ASCM) the chipped off mass $(1 - \lambda)x_i$ is distributed randomly among the sites $i$ and $(i + 1)$, whereas in the symmetric sticky chipping model (SSCM) it is distributed among the neighbours $(i + 1)$ and $(i - 1)$. Both these versions are studied using parallel and random sequential update rules.

We must mention that the steady state mass distribution of the ASCM with $\lambda = 0$ has been obtained earlier by Rajesh et al [10] assuming that the steady state is factorized. This product measure assumption turns out to be exact in the case of parallel update and an excellent approximation (though not exact) in the random sequential case. We will show that in the presence of stickiness ($\lambda \neq 0$), neither the ASCM nor the SSCM can have a factorized steady state. To construct the steady state mass distribution $P(x)$, we use a novel perturbation approach. Although the spatial correlations are ignored here, the steady state distributions, calculated up to second order in $\lambda$, are found to be in excellent agreement with those obtained from Monte Carlo simulations. The general principle of this approach is described in the following subsections.
2.1. Perturbation approach I

We try to construct the steady state mass distribution \( P(x) \) perturbatively, by expressing \( P(x) \) as a power series in \( \lambda \), about \( \lambda = 0 \),

\[
P(x) = P_0(x) + \lambda P_1(x) + \lambda^2 P_2(x) + \cdots = \sum_{k=0}^{\infty} \lambda^k P_k(x),
\]

where the functions \( P_k(x) \) do not depend on \( \lambda \). Here we have omitted the argument \( \lambda \) in \( P(x) \) for notational convenience. The perturbative expansion can also be made about any other \( \lambda \) if \( P_0(x) \) can be calculated there. In the chipping models discussed here, the total mass of the system and hence the density \( \langle x \rangle = 1/L \sum x_i \) is conserved. Without any loss of generality one can fix the average mass to be unity. This imposes a condition on \( P(x) \)

\[
\int_0^\infty dx x P(x) = \langle x \rangle = 1 = \int_0^\infty dx P(x),
\]

where the last equality stands for the normalization condition. Now, for \( \lambda = 0 \), \( P(x) = P_0(x) \). Therefore, \( P_0(x) \) satisfies two conditions,

\[
\int_0^\infty dx P_0(x) = 1 \quad \text{and} \quad \int_0^\infty dx x P_0(x) = 1.
\]

Thus for any other \( k > 0 \), using equations (2) and (3) we have

\[
\int_0^\infty dx P_k(x) = 0 = \int_0^\infty dx x P_k(x).
\]

The above constraints cannot be satisfied by any real positive function (as \( x \geq 0 \)), and thus one cannot interpret \( P_k(x) \) as a probability density function.

The perturbative corrections \( P_k(x) \) can be obtained directly by knowing the moments of \( P(x) \). First let us expand the moments \( \langle x^n \rangle = \int_0^\infty dx x^n P(x) \) as a power series in \( \lambda \),

\[
\langle x^n \rangle = A^{(n)} = \sum_{k=0}^{\infty} \frac{1}{(n+1)!} \lambda^k A_k^{(n)}.
\]

Here \( A_k^{(n)} \) are constant coefficients (independent of \( \lambda \)). The usefulness of the factor \( 1/(n+1)! \), used here for notational convenience, will be clear as we discuss specific problems. \( A_k^{(n)} \) can be determined from equations (1) and (5),

\[
A_k^{(n)} = \frac{1}{(n+1)!} \int_0^\infty dx x^n P_k(x).
\]

Since the \( P_k(x) \) are constrained by equations (3) and (4) \( A_k^{(n)} \) must satisfy

\[
A_0^{(1)} = \frac{1}{2} \quad \text{and} \quad A_k^{(1)} = 0 \quad \forall k > 0.
\]

Once the \( A_k^{(n)} \) are known, one can calculate the \( P_k(x) \) as

\[
P_k(x) = \mathcal{L}^{-1} \left[ \sum_{n=0}^{\infty} (-s)^n (n+1) A_k^{(n)} \right],
\]

where \( \mathcal{L}^{-1} \) denotes the inverse Laplace transform.
2.2. Perturbation approach II

Here we assume that the mass distribution \( P(x) \) satisfies a differential or an integral equation and use the Laplace transform

\[
Q(s) = \mathcal{L}[P(x)] = \int_0^\infty dx \, e^{-sx} P(x),
\]

which usually converts it into a differential or a transcendental equation in \( Q(s) \). We proceed further by expanding \( Q(s) \) as a power series in \( \lambda \) about \( \lambda = 0 \),

\[
Q(s) = \sum_{k=0}^{\infty} \lambda^k Q_k(s),
\]

and equate the coefficients of different powers of \( \lambda \) order by order. Finally one can find the distribution

\[
P(x) = \mathcal{L}^{-1}[Q(s)] = \mathcal{L}^{-1} \left[ \sum_{k=0}^{\infty} \lambda^k Q_k(s) \right].
\]

These two approaches are equivalent as is evident from equation (8),

\[
Q_k(s) = \mathcal{L}[P_k(x)] = \sum_{n=0}^{\infty} (-s)^n (n + 1) A_k^{(n)};
\]

for any particular problem we will use whichever approach is convenient.

3. The asymmetric sticky chipping model (ASCM)

In this section we study the asymmetric version of the model where \((1 - \lambda)\) fraction of mass \( x_i \) at site \( i \) is chipped off, only \( r_i \) fraction of the chipped off mass is then transported to the right neighbour \((i + 1)\) and the rest is retained at site \( i \). In other words, from the site \( i \), \((1 - \lambda)r_i \) fraction of the mass is transported to \((i + 1)\). Here \( r_i \) is a random number uniformly distributed in the interval \((0, 1)\).

This asymmetric sticky chipping model can be mapped to the generic one-dimensional mass transport models discussed in [12], by considering that the amount of mass \( \mu \) that is transported from a site to its right neighbour is stochastic and is distributed as

\[
\phi(\mu|x) = \begin{cases} \frac{1}{(1 - \lambda)x}, & 0 < \mu < (1 - \lambda)x, \\ 0, & \text{otherwise}, \end{cases}
\]

where \( x \) is the initial mass of the departure site. It has been shown [12] that the steady state mass distribution \( P_s(x_i) \) of the generic model is factorized, i.e. \( P_s(x_i) = \prod_i f(x_i) \), only if \( \phi(\mu|x) \) has the following form:

\[
\phi(\mu|x) = \frac{v(\mu) w(x - \mu)}{f(x)},
\]

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where $v$ and $w$ are arbitrary functions and

$$
f(x) = \begin{cases} 
\int_0^x \frac{d\mu}{w(x)}v(\mu)w(x-\mu), & \text{parallel update,} \\
\frac{1}{w(x)}, & \text{random sequential update.}
\end{cases}
$$

(15)

It is evident from equations (14) and (15) that models with parallel update can have a factorized steady state only if the dynamics remains invariant when $(x-\mu)$ is transported instead of $\mu$. For the ASCM, however, $(x-\mu)$ is distributed uniformly in the range $(\lambda x, x)$ (which is different from the range of $\mu$ in equation (13)) and thus the model cannot have a factorized steady state for parallel update unless $\lambda = 0$. For random sequential update, equations (13) and (15) suggest that $f(x) = w(x) \sim x$, but this cannot be made consistent with equation (14) for any choice of $v$. Thus random sequential update cannot give a factorized steady state for any $\lambda$ (including $\lambda = 0$).

In sections 3.1 and 3.2 we show that the mass distribution $P(x)$, calculated perturbatively, matches reasonably well with the one obtained from Monte Carlo simulations.

3.1. Parallel update

First let us consider the model with parallel update, where all the sites are updated synchronously. The dynamics can be written as

$$x_i(t+1) = \lambda x_i(t) + (1-\lambda)(1-r_i)x_i(t) + (1-\lambda)r_{i-1}x_{i-1}(t)
$$

(16)

for all $i = 1, 2, \ldots, L$, where the first term on the rhs represents the mass that sticks to the site $i$, the second term corresponds to the mass that is retained at the site $i$ after $(1-\lambda)r_i x_i(t)$ is transported to $(i+1)$. The third term results from the mass that the $i$th site receives from $(i-1)$.

This model has a factorized steady state for $\lambda = 0$ as the distribution of the transferred amount $\phi(\mu|x) = 1/x$ (from equation (13)) can be written in the form suggested by equation (14) with $v(x) = 1 = w(x)$. Correspondingly $f(x) = x$, which results in $P(x) = 4xe^{-2x}$. This special case of the model has been studied earlier by Rajesh et al [10]. For any non-zero $\lambda$, however, the steady state is not factorized.

In order to find the steady state mass distribution $P(x)$ for generic $\lambda$ first we calculate the moments. In the steady state, the distribution of $x_i(t+1)$ is the same as the distribution of $x_i(t)$. Thus, one may obtain $A^{(n)} \equiv \langle x^n \rangle/(n+1)!$ from equation (16) as

$$A^{(n)} = [n\bar{\lambda} - \lambda + \lambda^{n+1} - \bar{\lambda}^{n+1}]^{-1} \sum_{k=1}^{n-1} (1 - \lambda^{k+1})\bar{\lambda}^{n-k} A^{(k)} A^{(n-k)} \forall n \geq 2,
$$

(17)

where we have used $\langle x^n x_{n-1}^m \rangle = \langle x^n \rangle \langle x_{n-1}^m \rangle$, a mean field approximation that neglects all two point spatial correlations, and $\bar{\lambda} = (1-\lambda)$. Since $A^{(n)}$ depends on all other $A^{(k)}$ with $k = 1, 2, \ldots, (n-1)$, it is usually difficult to get a general expression for $A^{(n)}$ from this recursion relation. However, it is evident from equation (17) that if $y(n)$ is a solution for $A^{(n)}$, $y(n)z^n$ is also a solution for any arbitrary $z$. The arbitrary constant $z$ must be chosen such that the average mass of the system has the desired value $\langle x \rangle = 1$ or in other words $A^{(1)} = \frac{1}{2}$.

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First let us take $\lambda = 0$. In this case equation (17) reduces to a very simple form

$$A^{(n)} = \frac{1}{n-1} \sum_{k=1}^{n-1} A^{(k)} A^{(n-k)} \quad \forall \ n \geq 2,$$  \hspace{1cm} (18)

which can be solved trivially by taking $A^{(n)} = 1$; thus a general solution is $A^{(n)} = z^n$. The boundary condition $A^{(1)} = \frac{1}{2}$ now fixes $z = \frac{1}{2}$ and thus $\langle x^n \rangle = (n + 1)!/2^n$. The corresponding steady state distribution function is then

$$P(x) = \mathcal{L}^{-1} \left[ \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} \langle x^n \rangle \right] = 4xe^{-2x}. \hspace{1cm} (19)$$

For generic $\lambda$, $\langle x^n \rangle$ can be calculated recursively using equation (17) starting from $\langle x \rangle = 1$. The first few of them are

$$\langle x^2 \rangle = \frac{3(\lambda + 1)}{4\lambda + 2},$$
$$\langle x^3 \rangle = \frac{3(\lambda^2 + 3\lambda + 2)}{2(2\lambda + 1)^2},$$
$$\langle x^4 \rangle = \frac{15(\lambda + 1)^2(\lambda^2 - \lambda + 3)}{2(2\lambda + 1)^2(2\lambda^3 - \lambda^2 + 6\lambda + 3)}$$

$$\vdots. \hspace{1cm} (20)$$

The moments $\langle x^n \rangle$ as a function of $\lambda$ become messy with increasing $n$; it becomes practically impossible to obtain a general expression for $\langle x^n \rangle$, and hence the distribution $P(x)$. It would be useful to obtain $P(x)$ perturbatively which would give all the moments correctly (within this mean field approximation) up to some $n$th order in $\lambda$.

To proceed with the perturbation approach, we first express $\langle x^n \rangle$ in a power series in $\lambda$ as in equation (5) and then equate the coefficients of different powers of $\lambda$ which gives a set of recursion relations for $A^{(n)}_k$. The recursion relation for any $n$th order perturbation in $\lambda$ can be solved using the boundary condition (7). Once the $A^{(n)}_k$ are known, we calculate $P_k(x)$ using equation (8).

In the zeroth order, equations (17) and (5) result in

$$A^{(n)}_0 = \frac{1}{n-1} \sum_{k=1}^{n-1} A^{(k)}_0 A^{(n-k)}_0 \quad \forall \ n \geq 2,$$  \hspace{1cm} (21)

which is, indeed, the same as equation (18). Thus to this order, as expected, we get

$$A^{(n)}_0 = \frac{1}{2^n}, \quad \text{and} \quad P_0(x) = 4xe^{-2x}. \hspace{1cm} (22)$$

Next we proceed to calculate the first order correction to $P(x)$. Comparing the coefficients of $\lambda$ in equation (17) we have

$$A^{(n)}_1 = \frac{1}{n-1} \sum_{k=1}^{n-1} [2A^{(k)}_0 A^{(n-k)}_1 - (n - k)A^{(k)}_0 A^{(n-k)}_0] \quad \forall \ n \geq 2,$$
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Figure 1. The steady state mass distribution $P(x)$ of the ASCM with parallel update. (a) $P(x)$ obtained from Monte Carlo simulations of a system of size $L = 1000$, for $\lambda = 0, 0.1$, and $0.2$. Panels (b) and (c) compare $P(x)$ for $\lambda = 0.1$ with the first and second order perturbation results (dashed lines) respectively.

which needs to be solved using the boundary condition $A_1^{(1)} = 0$ from equation (7). This results in

$$A_1^{(n)} = \frac{n}{2^n} \left( 1 - \gamma - \frac{\Gamma'(n+1)}{\Gamma(n+1)} \right) \quad \forall \; n \geq 0,$$

(23)

where $\Gamma(x)$ stands for usual gamma function and $\gamma$ is the Euler constant,

$$\gamma = 0.57721 \ldots$$

(24)

Finally, using equation (8) we get

$$P_1(x) = 8e^{-2x} \left[ x^2(1 - g(x)) + xg(x) - \frac{x}{4} \right],$$

(25)

where $g(x) = \ln(2x) + \gamma$.

(26)

One can proceed in a similar way to calculate higher order corrections to $P(x)$. The corrections up to second order in $\lambda$ are listed in the appendix.

To check that the perturbation results for $P(x)$ agree well with the actual steady state mass distribution we simulate the model on a one-dimensional lattice of size $L = 1000$ for $\lambda = 0, 0.1$, and $0.2$ (as shown in figure 1(a)). The distribution for $\lambda = 0.1$ (solid line) is compared with the perturbation results (dashed line) up to first and second order in figures 1(b) and (c) respectively. An excellent agreement of $P(x)$ with the simulation results indicates that the two point correlations among sites are indeed very small.

3.2. Random sequential update

In this section we study the asymmetric sticky chipping model (ASCM) with random sequential update where $(1 - \lambda)$ fraction of the mass $x_i$ from a randomly chosen site $i$ is chipped off; a part of it, $(1 - \lambda) r_i x_i$, is then transported to the right neighbour $(i + 1)$ and the rest is returned to the departure site $i$. As usual, $r_i$ is distributed uniformly in the interval $(0, 1)$. The time increment associated with each update is $\Delta t = 1/L$; in other words a unit Monte Carlo sweep (MCS) corresponds to the update of $L$ sites.

Like the parallel update, here too one can obtain all the moments,

$$\langle x^n_i \rangle = \frac{1}{2} \langle (\lambda + 1 - \lambda)(1 - r_i)^n x_i^n \rangle + \frac{1}{2} \langle (x_i + (1 - \lambda) r_{i-1} x_{i-1})^n \rangle,$$

(27)

for all $i = 1, 2, \ldots, L$. The factors $\frac{1}{2}$ here represent the fact that the probability that a randomly chosen site acts as a departure site is $\frac{1}{2}$ and the probability of acting as a
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receiving site is the same. Now, using a mean field approximation \( \langle x^m_i x^m_{i-1} \rangle = \langle x^m_i \rangle \langle x^m_{i-1} \rangle \),

\[ A^{(n)} = [n\bar{\lambda} - \lambda + \lambda^{n+1} - \bar{\lambda}^{n+1}]^{-1}\sum_{k=1}^{n-1} (n - k + 1)\bar{\lambda}^{k+1}A^{(k)}A^{(n-k)} \quad \forall \ n \geq 2, \tag{28} \]

where \( \bar{\lambda} = (1 - \lambda) \). For \( \lambda = 0 \) this recursion relation reduces to a simpler form,

\[ A^{(n)} = \frac{1}{n-1} \sum_{k=1}^{n-1} (n - k + 1)A^{(k)}A^{(n-k)} \quad \forall \ n \geq 2, \tag{29} \]

which can be solved using the boundary condition \( A^{(1)} = \frac{1}{2} \),

\[ A^{(n)} = \frac{(2n)!}{2^n n!(n+1)!} \quad \forall \ n \geq 0. \tag{30} \]

In other words we have \( \langle x^n \rangle = (2n)!/(2^n n!) \). Thus the steady state distribution for \( \lambda = 0 \) is

\[ P(x) = \mathcal{L}^{-1} \left[ \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} \langle x^n \rangle \right] = \frac{1}{\sqrt{2\pi x}} e^{-x/2}. \tag{31} \]

Note that \( P(x) \) is the same as the mass distribution of the \( \lambda = 0 \) case, obtained earlier [10] for random sequential update.

We must mention that, a closed form expression of the steady state distribution can also be obtained for a special value of stickiness \( \lambda = \frac{1}{2} \). In this case equation (28) gives

\[ A^{(n)} = \frac{1}{n-1} \sum_{k=1}^{n-1} \frac{n - k + 1}{2^k}A^{(k)}A^{(n-k)} \quad \forall \ n \geq 2. \tag{32} \]

This equation has the trivial solution \( A^{(n)} = 1 \), similarly to the case when the system evolves following a parallel update with \( \lambda = 0 \) (discussed in section 3.1). Thus, in the steady state we have the same distribution as obtained in equation (19),

\[ P(x) = 4xe^{-2x}. \tag{33} \]

In figure 2(a) we have compared this result with the steady state distribution obtained from the Monte Carlo simulation of the system with \( \lambda = \frac{1}{2} \).

For any arbitrary \( \lambda \), however, it is not easy to obtain the solution of equation (28) and we resort to the perturbation approach. As usual, first we expand \( A^{(n)} \) as a power series in \( \lambda \) as was done in equation (5) and then equate the coefficients of different powers of \( \lambda \) in equation (28). To zeroth order, i.e. by equating the terms independent of \( \lambda \), we have

\[ A^{(n)}_0 = \frac{1}{n-1} \sum_{k=1}^{n-1} (n - k + 1)A^{(k)}_0 A^{(n-k)}_0 \quad \forall \ n \geq 2. \tag{34} \]

As expected, this equation is the same as equation (29) and correspondingly

\[ A^{(n)}_0 = \frac{(2n)!}{2^n n!(n+1)!} \quad \text{and} \quad P_0(x) = \frac{1}{\sqrt{2\pi x}} e^{-x/2}. \tag{35} \]

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Figure 2. The steady state distribution $P(x)$ of the ASCM with random sequential update. (a) $P(x)$ obtained from Monte Carlo simulations of system size $L = 1000$ for $\lambda = 0, 0.1, 0.3, \text{and } 0.5$. The distribution for $\lambda = 0.5$ is compared with the analytical result equation (33) (shown as a dashed line). In (b) and (c) we have compared the distribution for $\lambda = 0.1$ with the first and second order perturbation results (dashed lines) respectively.

Now let us proceed to the first order perturbation calculations. Collecting the coefficients of $\lambda$ in equation (28) one gets

$$A_1^{(n)} = \frac{1}{n-1} \sum_{k=1}^{n-1} \left[ -(n-k+1)(k+1)A_0^{(k)}A_0^{(n-k)} + (n-k+1) \right. \left. \times (A_0^{(k)}A_1^{(n-k)} + A_1^{(k)}A_0^{(n-k)}) \right] \forall n \geq 2. \quad (36)$$

This recursion relation can be solved using the generating function

$$V_k(s) = \sum_{n=0}^{\infty} (-s)^n A_1^{(n)}. \quad (37)$$

In terms of $V_0(s)$ and $V_1(s)$, equation (36) can be written as a differential equation

$$-sV_0(s)V'_1(s) + 2sV'_0(s)V_1(s) - 2V_0(s)V_1(s) + V_1(s) + sV_0(s)V_0'(s)$$
$$+ s^2V_0''(s) - V_0(s) + 1 = 0.$$

Since $V_0(s) = (\sqrt{2s+1}-1)/s$ is known from equation (35) we can solve the above equation for $V_1(s)$ using the boundary condition $V'_1(0) = 0$ (obtained from equation (7)),

$$V_1(s) = -\frac{2s(\tilde{s} - 3) + 4(\tilde{s} - 1) + 2(s - \tilde{s} + 1) \ln(\tilde{s})}{2s\tilde{s}}. \quad (38)$$

Here $\tilde{s} = \sqrt{1+2s}$. Thus, following equation (8), we get

$$P_1(x) = \mathcal{L}^{-1}[V_1(s) + sV'_1(s)]$$
$$= \frac{1}{4\sqrt{2\pi}}e^{-x^2/2}[-x(g(x) + 2) + g(x) + 4] + \frac{1}{2}e^{-x^2/2} - 2\delta(x), \quad (39)$$

where $g(x) = \ln(2x) + \gamma$. In the above expression the term $-2\delta(x)$ is needed to ensure the condition $\int_0^\infty dx P_1(x) = 0$.

There is no particular difficulty in proceeding for higher order perturbations in $\lambda$, except that the expressions are lengthy. We have listed the $P_k(x)$ for $k = 0, 1, 2$ in the appendix.

In figure 2 we have compared these perturbation results with Monte Carlo simulations of the dynamics on a one-dimensional periodic system of size $L = 1000$. The steady state
distributions for \( \lambda = 0, 0.1, 0.3 \), and 0.5 are shown in figure 2(a). In the same figure we compare the distribution function for \( \lambda = 0.5 \) with equation (33) (dashed line). The distribution for \( \lambda = 0.1 \) (solid line) is compared with the perturbation results (dashed lines) up to first and second order respectively in figures 2(b) and (c).

4. The symmetric sticky chipping model (SSCM)

In this section we study the symmetric version of the model, namely the SSCM, where \((1-\lambda)\) fraction of the mass \( x_i \) at site \( i \) is chipped off and distributed randomly among the neighbours \((i \pm 1)\); the right neighbour receives \( \mu_R = r_i(1-\lambda)x_i \) and the left one receives the rest, \( \mu_L = (1-r_i)(1-\lambda)x_i \). Here again \( r_i \) is a random number uniformly distributed in the interval \((0,1)\).

The criterion (14) for asymmetric mass transport models to have factorized steady state [12] does not straightforwardly extend to the symmetric case. However, to have a factorized steady state for a chipping model on an arbitrary graph, it is sufficient that its chipping kernel at each site have a product form [13] similar to equation (14). In the SSCM, this condition translates to

\[
\phi_{\text{sym}}(\mu_L, \mu_R|x) = \frac{u(\mu_L)v(\mu_R)w(x - \mu_L - \mu_R)}{\int d\mu_L d\mu_R u(\mu_L)v(\mu_R)w(x - \mu_L - \mu_R)}.
\]

The chipping kernel in this model is

\[
\phi_{\text{sym}}(\mu_L, \mu_R|x) = \delta((1-\lambda)x - \mu_L - \mu_R)\phi(\mu_L|x),
\]

where \( \phi(\mu_L|x) \) is the same distribution as given by equation (13). It is evident that \( \phi_{\text{sym}}(\mu_L, \mu_R|x) \) can be cast into the form (40) when \( \lambda = 0 \), by taking the functions \( u = 1 = v \) (similarly to the ASCM) and \( w = \delta(x - \mu_L - \mu_R) \); thus the steady state is factorized. For \( \lambda \neq 0 \), however, such a product form does not exist and we proceed to calculate the steady state distribution perturbatively for both parallel and random sequential updates.

4.1. Parallel update

In this section we study the model where all the sites are updated parallelly (synchronously) using the dynamics mentioned above. Explicitly,

\[
x_i(t+1) = \lambda x_i(t) + (1-\lambda)[r_{i-1}x_{i-1}(t) + (1-r_{i+1})x_{i+1}(t)],
\]

where the first term on the rhs represents the mass that sticks to the site \( i \) during the update, the second and the third terms on the rhs correspond to the mass which the \( i \)th site receives from its neighbours \( i \pm 1 \) respectively. Thus the steady state probability that the site \( i \) has mass \( x \) is given by

\[
P(x) = \int_0^\infty dx_{i-1} \int_0^\infty dx_i \int_0^1 dr_{i-1} \int_0^1 dr_{i+1} P(x_{i-1})P(x_i)P(x_{i+1})
\]

\[
\times \delta(x - \lambda x_i - (1-\lambda)[r_{i-1}x_{i-1} + (1-r_{i+1})x_{i+1}]),
\]

where we have used a mean field approximation \( \langle x_{i-1}^mx_{i+1}^n \rangle = \langle x_{i-1}^m \rangle \langle x_{i+1}^n \rangle \). In other words, three and lower order spatial correlations are ignored.

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Here we intend to follow approach II. First let us express the above equation in terms of \( Q(s) \), which is the Laplace transform of \( P(x) \),

\[
Q(s) = Q(\lambda s)V^2((1 - \lambda)s), \tag{43}
\]

where \( V(s) \) is defined by

\[
V(s) = \int_0^1 dr Q(sr), \quad \text{or} \quad Q(s) = V(s) + sV'(s). \tag{44}
\]

In fact, equation (43) can be written in terms of the function \( V \) as

\[
V(s) + sV'(s) = [V(\lambda s) + \lambda sV'(\lambda s)]V^2((1 - \lambda)s). \tag{45}
\]

Let us begin with \( \lambda = 0 \). In this case, equation (45) takes the form

\[
sV'(s) - V(s)^2 + V(s) = 0, \tag{46}
\]

where we have used \( V(0) = 1 \), from equation (2). The solution of this differential equation, with the boundary condition \( V'(0) = -1/2 \) which corresponds to a fixed average mass \( \langle x \rangle = 1 \), gives

\[
V(s) = \frac{2}{s + 2}, \quad \text{and thus} \quad P(x) = \mathcal{L}^{-1}\{V(s) + sV'(s)\} = 4xe^{-2x}.
\]

This expression is identical to \( P(x) \) obtained in the case of the ASCM with parallel update. In the following we argue that the dynamics of the SSCM for \( \lambda = 0 \) is equivalent to that of the ASCM,

\[
\begin{align*}
x_i(t + 1) &= r_{i-1}x_{i-1}(t) + (1 - r_i)x_i(t) \quad \text{ASCM} \\
x_i(t + 1) &= r_{i-1}x_{i-1}(t) + (1 - r_i)x_{i+1}(t) \quad \text{SSCM}.
\end{align*}
\]

Since the ASCM has a product measure, the steady state remains invariant if the dynamics is changed to a mean field dynamics \( x_i \to (r_jx_j + (1 - r_k)x_k) \), where for each \( i, j \) (as well as \( k \)) is chosen randomly from the set \( \{1, 2, \ldots, L\} \) without replacement, i.e. all the sites receive exactly two fragments, \( r_j \) fraction of \( x_j \) and \( (1 - r_k) \) fraction of \( x_k \). Clearly the choice \( j = (i - 1) \) and \( k = (i + 1) \) corresponds to the SSCM. Thus, the SSCM with \( \lambda = 0 \) has a factorized steady state with mass distribution \( P(x) \) the same as the ASCM.

These arguments do not extend to the \( \lambda \neq 0 \) case, as the steady state of the corresponding asymmetric model is not factorized. We proceed with the perturbation approach. First let us expand \( V \) in the Taylor series about \( \lambda = 0 \); for any arbitrary \( a, b \),

\[
V((a + b\lambda)s) = \sum_{m=0}^{\infty} \frac{(b\lambda)^m}{m!} \frac{d^m V(s)}{ds^m} \bigg|_{s \to as}. \tag{47}
\]

Each of the \( V(s) \) and their derivatives are then expanded, similarly to equation (10), as

\[
V(s) = \sum_{k=0}^{\infty} \lambda^k V_k(s). \tag{48}
\]

Now using equations (47) and (48) in equation (45) and collecting the coefficients of different powers of \( \lambda \), order by order, one can obtain a set of differential equations in

\[1\] Note that \( V(s) \) is also a function of \( \lambda \), which we have dropped for notational convenience.
terms of $V_k(s)$, which can be solved using the boundary conditions

\begin{align}
  k = 0 : & \quad V_0(0) = 1, \quad V'_0(0) = -\frac{1}{2}, \\
  k \neq 0 : & \quad V_k(0) = 0, \quad V'_k(0) = 0.
\end{align}

(49)

These boundary conditions on $V$ are the same as equations (3) and (4), which ensure that $P(x)$ is normalized and the average mass of the system is unity. Interestingly, using equation (12) in equation (44) one gets

\begin{align}
  V_k(s) = \sum_{n=0}^{\infty} (-s)^n A_k^{(n)},
\end{align}

(50)

which implies that $V_k(s)$ is simply the generating function of $A_k^{(n)}$ used in approach I.

Following the perturbation approach, we use equations (47) and (48) in equation (45) and equate the terms which are independent of $\lambda$ and get

\begin{align}
  sV'_0(s) - V_0(s)^2 + V_0(s) = 0,
\end{align}

which is the same as equation (46), and we get

\begin{align}
  V_0(s) = \frac{2}{s + 2}, \quad \text{and} \quad P_0(x) = 4xe^{-2x}.
\end{align}

Next we move on to the first order perturbation. Collecting the coefficients of $\lambda$ in equation (45),

\begin{align}
  s(s + 2)^3V'_1(s) + (s + 2)^2(s - 2)V_1(s) + 4s^2 = 0,
\end{align}

which, along with the boundary condition $V'_1(0) = 0$ (from equation (49)), results in

\begin{align}
  V_1(s) = -\frac{4s(\ln(s + 2) - \ln(2))}{(s + 2)^2}.
\end{align}

Thus we obtain

\begin{align}
  P_1(x) = \mathcal{L}^{-1}[V_1(s) + sV'_1(s)] = 16e^{-2x} \left[ x^2(1 - g(x)) + xg(x) - \frac{1}{4} \right],
\end{align}

where $g(x) = \ln(2x) + \gamma$.

Following the same procedure one can obtain other higher order corrections to $P(x)$. In the appendix, we have listed these correction terms up to second order. As described in figure 3, the perturbation results for $P(x)$ agree well with the actual steady state mass distribution obtained from Monte Carlo simulations. In figure 3(a) we have shown the simulation results for $\lambda = 0, 0.1, \text{and} 0.2$. In the same figure we compare the result for $\lambda = 0$ with the analytically obtained result (dashed line). In figures 3(b) and (c) the distribution for $\lambda = 0.1$ (solid line) is compared with the perturbation results (dashed lines) up to first and second order respectively. Clearly, as expected, the distribution matches better with the simulation results as we go to higher order.

4.2. Random sequential update

In this section we study the symmetric sticky chipping model using random sequential update. At each step, a site is chosen randomly and from this site $i$, $(1 - \lambda)$ fraction of the mass $x_i$ is chipped off. From the chipped off mass $(1 - \lambda)x_i$, $r_i$ fraction is transported to
Figure 3. The SSCM with parallel update. (a) $P(x)$ obtained from Monte Carlo simulations of the model on a one-dimensional lattice of size $L = 1000$ for $\lambda = 0, 0.1, 0.2$. The distribution for $\lambda = 0$ is compared with the exact result $P(x) = 4xe^{-2x}$ (dashed line). Panels (b) and (c) compare $P(x)$ obtained from the first and second order perturbation (dashed lines) respectively with the same obtained from simulations, for $\lambda = 0.1$.

The right neighbour, and the rest goes to the left neighbour. With update of a single site, the time is increased by $\Delta t = 1/L$.

The moments $\langle x^n \rangle$ can be expressed as

$$
\langle x^n \rangle = \frac{1}{3} \langle (\lambda x_i)^n \rangle + \frac{1}{3} \langle (x_i + r_{i-1}(1 - \lambda)x_{i-1})^n \rangle + \frac{1}{3} \langle (x_i + (1 - r_{i+1})(1 - \lambda)x_{i+1})^n \rangle
$$

$$
= \frac{1}{3} \langle (\lambda x_i)^n \rangle + \frac{2}{3} \langle (x_i + r_{i-1}(1 - \lambda)x_{i-1})^n \rangle.
$$

The factors $\frac{1}{3}$ come from the fact that during each update, one site transports and two other sites receive the mass. Thus, the probability that a site acts as a departure site is $\frac{1}{3}$ and the probability that it acts as a receiving site is $\frac{2}{3}$.

Now, consider the case $\lambda = 0$. Then, equation (51) takes the form

$$
\langle x^n \rangle = \frac{1}{3} \delta_{n,0} + \frac{2}{3} \langle (x_i + r_{i-1}x_{i-1})^n \rangle.
$$

The additional term $\frac{1}{3} \delta_{n,0}$ appears from the fact that for $n = 0$, the first term on the rhs of equation (51) gives $\frac{1}{3}$ for any arbitrary $\lambda$ except $\lambda = 0$.

Assuming that all the two point correlations are zero in the steady state, i.e. $\langle x_i^n x_{i-1}^m \rangle = \langle x_i^n \rangle \langle x_{i-1}^m \rangle$, one can write $A(n) \equiv \langle x^n \rangle / (n + 1)!$ as

$$
A^{(n)} = \frac{2}{n - 1} \sum_{k=1}^{n-1} (n - k + 1) A^{(k)} A^{(n-k)} \quad \forall \ n \geq 2.
$$

This equation can be converted to a differential equation by using $V(s) = \sum_{n=0}^{\infty} (-s)^n A^{(n)}$, the generating function of $A^{(n)}$,

$$
V'(s) = \frac{1 - 3V(s) + 2V^2(s)}{s(3 - 2V(s))}.
$$

For the usual boundary condition $V(0) = 1$, $V'(0) = -1/2$, the solution to the above equation is

$$
V(s) = \frac{2s - 1 + \sqrt{4s + 1}}{4s},
$$

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Figure 4. Comparison of equation (53) (dashed line) with $P(x)$ obtained from Monte Carlo simulations (solid line) of the SSCM with random sequential update for $\lambda = 0$ and $L = 1000$. (a) The actual model. (b) The model with mean field (MF) dynamics.

which results in

$$P(x) = \mathcal{L}^{-1} \left[ V(s) + sV'(s) \right] = \frac{1}{4\sqrt{\pi x}} e^{-x/4} + \delta(x).$$  \hspace{1cm} (53)$$

In the above expression the term $\delta(x)$ is needed to assure the normalization of $P(x)$.

In figure 4(a) we have compared this result with the Monte Carlo simulation of the model with $\lambda = 0$. The right panel (b) shows $P(x)$ obtained from equation (53) along with the simulation results of the model with a mean field (MF) dynamics, i.e., $r_i$ and $(1-r_i)$ fraction of the chipped off mass $(1-\lambda)x_i$ are transported from site $i$ to two arbitrary sites instead of being transported to the neighbours. Clearly equation (53) is consistent with the simulation results of the MF dynamics emphasizing that equation (53) correctly describes the mean field distribution. On the other hand the same mass distribution obtained from Monte Carlo simulation of the actual model deviates substantially (figure 4(a)). This discrepancy, which originates from the mean field approximation used here (which ignores all the two point correlations), cannot be healed by adding perturbative correction terms. We do not proceed further in this case.

5. Summary and conclusion

In this paper we have studied the conserved mass transport process in the presence of stickiness, characterized by a parameter $\lambda$. The model in one dimension evolves using a parallel or random sequential update rule, where a fixed fraction (i.e. $(1-\lambda)$) of the mass from a site is chipped off and then distributed randomly among the site and its neighbours. In the asymmetric version, ASCM, the chipped off mass is distributed among the site and its right neighbour, whereas in the SSCM it is distributed among both the neighbours.

For non-zero $\lambda$ the steady state distribution of these models does not have a factorized form. We introduce a perturbation approach to obtain an approximate mass distribution function $P(x) = \sum_{k=0}^{\infty} \lambda^k P_k(x)$, and provide an explicit form of $P_k(x)$ up to second order in $\lambda$. In all cases except the SSCM with random sequential update, the perturbation results agree quite well with the distribution obtained from Monte Carlo simulation of the model, even though we have used a mean field approximation which ignores two or three point spatial correlations.

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Interestingly, the steady state distributions for the following three cases: (a) the ASCM with parallel update and \( \lambda = 0 \), (b) the ASCM with random sequential update and \( \lambda = \frac{1}{2} \), and (c) the SSCM with parallel update and \( \lambda = 0 \), are identical, \( P(x) = 4xe^{-2x} \). It turns out that product measure is exact only for models (a) and (c).

In the absence of any general formalism, it is not always possible to calculate the exact steady state distribution for non-equilibrium models. The perturbation approach we have discussed here is quite general and can be used in models with some small parameter to obtain the steady state distribution analytically within a mean field approximation that ignores only two point correlations in all cases, except for the SSCM with parallel update where both two and three point correlations are ignored.

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**Appendix**

The steady state distributions of the models studied here with parallel (p) and random sequential (rs) updates, using the perturbation approach, are summarized below.

| Model     | \( P(x) = P_0(x) + \lambda P_1(x) + \lambda^2 P_2(x) + \cdots \) |
|-----------|---------------------------------------------------------------|
| ASCM (p)  | \( P_0(x) = 4xe^{-2x} \)                                   |
|           | \( P_1(x) = 8e^{-2x}[x^2(1 - g(x)) + xg(x) - \frac{1}{2}] \) |
|           | \( P_2(x) = 8e^{-2x}[x^3(g^2(x) - 2g(x) + \frac{16\pi^2}{3}) - x^2(\frac{5}{2}g^2(x) - 2g(x)) + \frac{48 - 5\pi^2}{12} + x(\frac{g^2(x)}{3} + \frac{g(x)}{2} - \frac{\pi^2}{6} - \frac{1}{4}(g(x) + 1)] + 2\delta(x) \) |
| ASCM (rs) | \( P_0(x) = \frac{1}{\sqrt{2\pi x}} e^{-x/2} \)               |
|           | \( P_1(x) = \frac{1}{4\sqrt{2\pi x}} e^{-x/2} \{ -x[2g(x) + 2] + xg(x) + 4 \} + \frac{1}{4}e^{-x/2} - 2\delta(x) \) |
|           | \( P_2(x) = \frac{1}{32\sqrt{2\pi x}} e^{-x/2} \{ x^3(g^2(x) + 4g(x) + \frac{20 - 3\pi^2}{2}) - 4x^2(g^2(x) + 6g(x)) + \frac{43 - 3\pi^2}{6} + x(g^2(x) + 12g(x) - \frac{20 + 3\pi^2}{6} - 10) - \frac{1}{8}e^{-x/2} \{ 3g(x) + 3 \} - 2g(x) - 10 \} + \frac{2}{3}(\delta'(x) - 2\delta(x)) \) |
| SSCM (p)  | \( P_0(x) = 4xe^{-2x} \)                                   |
|           | \( P_1(x) = 16e^{-2x}\{ x^2(1 - g(x)) + xg(x) - \frac{1}{4} \} \) |
|           | \( P_2(x) = 32e^{-2x}\{ x^3(g^2(x) - 2g(x) + \frac{33 - 2\pi^2}{12}) - x^2(3g^2(x) - 2g(x)) - H(x) + \frac{21 - 2\pi^2}{6} + x(\frac{3}{2}g^2(x) + g(x) - H(x) - \frac{\pi^2}{12}) - \frac{1}{4}(g(x) + 1)] - 8\Gamma(0, 2x)(2x - 1) + 6\delta(x) \) |
| SSCM (rs) | \( P_0(x) = \frac{1}{4\sqrt{\pi x}} e^{-x/4} + \delta(x) \) |

Here, \( g(x) = \ln(2x) + \gamma \), and \( H(x) = \sum_{m=1}^{\infty} \frac{\Gamma'(m)}{\Gamma(m)} - g(x) \), \( \Gamma(0, x) \) is the incomplete gamma function and the Euler constant \( \gamma = 0.57721 \ldots \)
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