Abstract. For \( u \in H^p \), \( 0 < p \leq 2 \), with Haar expansion \( u = \sum x_I h_I \) we constructively determine the Pietsch measure of the 2-summing multiplication operator \( \mathcal{M}_u : \ell^\infty \to H^p \), \( (\varphi_I) \mapsto \sum \varphi_I x_I h_I \).

Our method yields a constructive proof of Pisier’s decomposition of \( u \in H^p \)

\[
|u| = |x|^{1-\theta} |y|^{\theta} \quad \text{and} \quad \|x\|_{X_0}^{1-\theta} \|y\|_{H^2}^{\theta} \leq C\|u\|_{H^p},
\]

where \( X_0 \) is Pisier’s extrapolation lattice associated to \( H^p \) and \( H^2 \). Our construction of the Pietsch measure for the multiplication operator \( \mathcal{M}_u \) involves the Haar coefficients of \( u \) and its atomic decomposition.

1. Introduction

The spaces of this paper are dyadic Hardy spaces \( H^p \) with the Haar system \( \{h_I\}_{I \in D} \) (indexed by the dyadic intervals \( D \)) as their unconditional basis. For \( u = \sum_{I \in D} x_I h_I \) we have

\[
\|u\|_{H^p} = \left( \int_0^1 \left( \sum_{I \in D} x_I^2 1_I(t) \right)^{\frac{p}{2}} dt \right)^{\frac{1}{p}}.
\]

The operators of this paper are multipliers acting on the Haar system: For \( u \in H^p \) with Haar expansion \( u = \sum_{I \in D} x_I h_I \) the multiplier \( \mathcal{M}_u \) is defined by

\[
\mathcal{M}_u : \ell^\infty(D) \to H^p \quad \text{and} \quad \mathcal{M}_u(\varphi) = \sum_{I \in D} \varphi_I x_I h_I.
\]

Clearly \( \mathcal{M}_u \) is a bounded operator,

\[
(1.1) \quad \|\mathcal{M}_u \varphi\|_{H^p} \leq \|u\|_{H^p} \sup |\varphi_I|,
\]

since \( \{h_I\} \) is an unconditional basis of \( H^p \). Moreover, for \( 0 < p \leq 2 \), \( H^p \) is of cotype 2. Therefore, by the work of Dubinsky, Pelczyński, Rosenthal [DPR72] \( \mathcal{M}_u \) is 2-summing. Hence, by the work of Pietsch [Pie67] \( \mathcal{M}_u \) has a Pietsch measure. It is easy to show (Step 2 and 3 in the proof of Theorem 2.4) that the measure has
the following form. There exists \( \omega = (\omega_I)_{I \in D} \) with \( \omega_I \geq 0 \) and \( \sum \omega_I \leq 1 \) such that we have a significant strengthening of the basic estimate (1.1):

\[
\|M_u \varphi\|_{H^p} \leq C \|u\|_{H^p} \left( \sum_{I \in D} |\varphi_I|^2 \omega_I \right)^{\frac{1}{q}}.
\]

The existence of the weight \( \omega = (\omega_I)_{I \in D} \) is guaranteed by abstract theory (Pietsch measure, Hahn-Banach theorems).

The main result. In (Theorem 3.1) we give explicit formulas for the weights \( \omega = (\omega_I) \), using as input the Haar coefficients \( (x_I)_{I \in D} \) of \( u \in H^p \). We obtain several extensions and variants of the formulas referred to above. These include vector-valued dyadic Hardy spaces and Triebel-Lizorkin spaces.

Multiplier operators, as described above, arise with interpolation and extrapolation of Banach lattices. Here we refer to Pisier's proof in [Pis79a] of the equation

\[
X = (X_0)^{1-\theta} (L^2)^{\theta},
\]

where \( X \) is a \( q \)-concave and \( q' \)-convex Banach lattice, \( \theta = \frac{2}{q} \) and \( X_0 \) is Pisier's extrapolation lattice for \( X \) and \( L^2 \) (see Section 4). The equation (1.3) asserts that for \( u \in X \) there is \( y \in L^2(\Omega, \Sigma, \mu) \) so that

\[
\left( |u| |y|^{-\theta} \right)^{\frac{1}{\theta}} \in X_0.
\]

In order to obtain \( y \in L^2(\Omega, \Sigma, \mu) \) the proof in [Pis79a] sets up the following multiplication operator

\[
M_u : L^{\infty}(\Omega, \Sigma, \mu) \to X
\]

by putting

\[
M_u(\varphi)(t) = u(t) \varphi(t), \quad t \in \Omega.
\]

Exploiting the work of Maurey [Mau74a], Rosenthal [Ros76] and Pietsch [Pie67], Pisier shows in [Pis79a] that there exists a density \( \omega \in L^1(\Omega, \Sigma, \mu) \) so that

\[
\|M_u \varphi\|_X \leq C \|u\|_X \left( \int_{\Omega} |\varphi(t)|^q \omega(t) d\mu(t) \right)^{\frac{1}{q}}.
\]

To obtain (1.3) and (1.4) Pisier [Pis79a] puts finally

\[
y(t) = \omega(t)^{\frac{1}{2}}.
\]

The application. We constructively determine for the special Banach lattice \( X = H^p \) the density \( \omega \in \ell^1(D) \) and therefore have a constructive proof of

\[
H^p = (X_0)^{1-\theta} (f_q^{\theta}),
\]

where \( X_0 \) is Pisier’s extrapolation lattice for \( H^p \) and \( H^2 \), see Theorem 4.1.

The organization of this paper is the following: Section 2 contains the preliminaries. Section 3 contains the main result of this paper (Theorem 3.1). We constructively determine Pietsch measures for absolutely summing multipliers into \( H^p \) spaces. We prove extensions to Triebel-Lizorkin spaces and vector-valued \( H^p_X \) spaces. In Section 4 we apply our approach to determine constructively the Calderón product

\[
f_q^\theta = (X_0)^{1-\theta} (f_q^{\theta}),
\]

where \( X_0 \) is Pisier’s extrapolation lattice for the Triebel-Lizorkin spaces \( f_q^\theta \) and \( f_q^\theta \).
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2. Preliminaries

2.1. Banach space preliminaries.

Kahane’s inequality and Kahane’s contraction principle. See [Kah85]. Let \( \{r_n\}_{n \in \mathbb{N}} \) denote the independent Rademacher system. For any \( 0 < p < q < \infty \) there is a constant \( C_{p,q} \) such that for any Banach space \( X \) and for any sequence \( (x_n)_{n=1}^N \) in \( X \) we have

\[
\left( \int_0^1 \left\| \sum_{n=1}^N r_n(t)x_n \right\|^q_X dt \right)^{\frac{1}{q}} \leq C_{p,q} \left( \int_0^1 \left\| \sum_{n=1}^N r_n(t)x_n \right\|^p_X dt \right)^{\frac{1}{p}}.
\]

Let \( X \) be a Banach space and \((x_n)_{n=1}^N\) a sequence in \( X \). Then for all sequences \((a_n)_{n=1}^N\) of real numbers and for all \( 1 \leq p < \infty \) one has

\[
\left( \int_0^1 \left\| \sum_{n=1}^N r_n(t)a_n x_n \right\|^p_X dt \right)^{\frac{1}{p}} \leq \sup_{1 \leq n \leq N} |a_n| \left( \int_0^1 \left\| \sum_{n=1}^N r_n(t)x_n \right\|^p_X dt \right)^{\frac{1}{p}}.
\]

Cotype of a Banach space. See e.g. [LT91]. Let \( 2 \leq q \leq \infty \). Let again \( \{r_n\}_{n \in \mathbb{N}} \) denote the independent Rademacher system. A Banach space \( X \) is called of cotype \( q \) if there is a constant \( C \) such that for all finite sequences \((x_n)\) in \( X \)

\[
\left( \sum_{n=1}^N \|x_n\|^q \right)^{\frac{1}{q}} \leq C \left( \int_0^1 \left\| \sum_{n=1}^N r_n(t)x_n \right\|^2_X dt \right)^{\frac{1}{2}}.
\]

We denote by \( C_q(X) \) the smallest possible constant \( C \). A Banach space \( X \) is of nontrivial cotype if it is of cotype \( q < \infty \).

Every \( L^p \)-space is of cotype \( \max(p,2) \). If \( X \) is a Banach space of cotype \( q \), then \( L^p_X \) is of cotype \( \max(r,q) \), cf. [LT91].

\( p \)-summing operators. See e.g. [Pie67]. Let \( X, Y \) be Banach spaces and let \( 1 \leq p \leq \infty \). An operator \( T \in L(X,Y) \) is called \( p \)-summing if there is a constant \( K \) so that for every choice of an integer \( n \) and vectors \( \{x_i\}_{i=1}^n \) in \( X \), we have

\[
\left( \sum_{i=1}^n \|Tx_i\|^p \right)^{\frac{1}{p}} \leq K \sup_{\|x\| \leq 1} \left( \sum_{i=1}^n |x^*(x_i)|^p \right)^{\frac{1}{p}}.
\]

The smallest possible constant \( K \) is denoted by \( \pi_p(T) \). The class of all \( p \)-summing operators in \( L(X,Y) \) is denoted by \( \Pi_p(X,Y) \). \( \pi_p \) defines a norm on \( \Pi_p(X,Y) \) with \( \|T\| \leq \pi_p(T) \) for all \( T \in \Pi_p(X,Y) \).

Maurey’s theorem. We next recall Maurey’s theorem on \( p \)-summing operators from a \( C(K) \)-space into a Banach space of nontrivial cotype. We refer to [Mau74b] and the exposition of Maurey’s theorem in [DJT95].

Theorem 2.1 ([Mau74b]). For all \( 2 \leq r < \infty \) and \( r < s < \infty \) there exists a positive constant \( K_{s,r} \) such that for every Banach space \( Y \) of cotype \( r \) with cotype-\( r \) constant \( C_r(Y) \) and for every compact Hausdorff space \( K \) we have

\[
L(C(K),Y) = \Pi_s(C(K),Y)
\]
and for every \( T \in L(C(K), Y) \)

\[
\pi_s(T) \leq K_{s,r}C_r(Y)\|T\|.
\]

The case when the target space \( Y \) is of cotype 2 allows the following strengthening of the conclusion. This is the context of the theorem in [DPR72].

**Theorem 2.2** ([DPR72]). There exists a positive constant \( B \) such that for every Banach space \( Y \) of cotype 2 with cotype-2 constant \( C_2(Y) \) and for every compact Hausdorff space \( K \) we have

\[
L(C(K), Y) = \Pi_2(C(K), Y)
\]

and for every \( T \in L(C(K), Y) \)

\[
\pi_2(T) \leq BC_2(Y)^2\|T\|.
\]

**Banach lattices.** For general reference on Banach lattices we refer to [LT79] and on quasi Banach lattices to [Kal84], see also [Pis79a].

Let \((\Omega, \Sigma, \mu)\) be a measure space and \( L^q(\Omega, \Sigma, \mu) \) the space of all measurable functions with real values. Let \( X \) be a (quasi) Banach space, whose elements form a subspace of \( L^q(\Omega, \Sigma, \mu) \). We call \( X \) a (quasi) Banach lattice over the measure space \((\Omega, \Sigma, \mu)\), if for all \( f \in X \) and for all \( g \in L^q(\Omega, \Sigma, \mu) \) with \( |g| \leq |f| \) holds that \( g \in X \) and \( \|g\|_X \leq \|f\|_X \).

**q-convexity and q-concavity of Banach lattices.** We refer to [LT79] and for quasi Banach lattices to [CT86]. Let \( 0 < q \leq \infty \). A (quasi) Banach lattice \( X \) is called \( q \)-convex, if there exists a constant \( M > 0 \) such that

\[
\left( \sum_{i=1}^n |x_i|^q \right)^{1/q} \leq M \left( \sum_{i=1}^n \|x_i\|^q \right)^{1/q},
\]

for every choice of vectors \( \{x_i\}_{i=1}^n \) in \( X \). The smallest possible constant \( M \) is denoted by \( M^{(q)}(X) \). A (quasi) Banach lattice \( X \) is called \( q \)-concave, if there exists a constant \( M > 0 \) such that

\[
\left( \sum_{i=1}^n \|x_i\|^q \right)^{1/q} \leq M \left( \sum_{i=1}^n |x_i|^q \right)^{1/q},
\]

for every choice of vectors \( \{x_i\}_{i=1}^n \) in \( X \). The smallest possible constant \( M \) is denoted by \( M_{(q)}(X) \).

Every \( q \)-concave Banach lattice with \( q \geq 2 \) is of cotype \( q \), cf. [LT79, DJT95].

**Multiplication operators.** Here we collect crucial information on multiplication operators on \( C(K) \) spaces with values in a \( q \)-concave Banach lattice.

Let \( K \) be a compact Hausdorff space and \( \mu \) a regular Borel measure on \( K \). Let \( 1 \leq q < \infty \) and \( X \) a \( q \)-concave Banach lattice over the measure space \((K, \mu)\). Each \( x \in X \) induces a bounded multiplication operator

\[
\mathcal{M}_x : C(K) \to X, \varphi \mapsto \varphi \cdot x, \quad \|\mathcal{M}_x\| = \|x\|_X.
\]

Pisier [Pis79a] asserts that the multiplication operator is \( q \)-summing with \( \pi_q(\mathcal{M}_x) = M_{(q)}(X)\|x\|_X \), where \( M_{(q)}(X) \) is the \( q \)-concave constant of \( X \). Explicitly this means
that for $\varphi_1, \ldots, \varphi_n \in C(K)$

\[
\left( \sum_{i=1}^{n} \|M_x \varphi_i\|^q_X \right)^{\frac{1}{q}} \leq M_{(q)}(X) \|x\|_X \sup_{\varphi^* \in C(K)^*} \left( \sum_{i=1}^{n} |\varphi^*(\varphi_i)|^q \right)^{\frac{1}{q}}.
\]

(2.7)

2.2. Dyadic Hardy spaces.

Dyadic intervals. An interval $I \subseteq [0,1]$ is called a dyadic interval, if there exists $n \in \mathbb{N}_0$ and $1 \leq k \leq 2^n$ such that

\[
I = \left[ \frac{k - 1}{2^n}, \frac{k}{2^n} \right].
\]

Let $D = \{ I \subseteq [0,1] : I \text{ is dyadic interval} \}$ and $D_n = \{ I \in D : |I| \geq 2^{-n} \}$. The set of dyadic intervals $D$ is "nested" in the following sense: if $I, J \in D$ are not disjoint, then either $I \subseteq J$ or $J \subseteq I$.

The spaces $\ell^1(D)$ and $\ell^\infty(D)$. The space $\ell^1(D)$ is the space of all summable sequences $s = (s_I)_{I \in D}$ indexed by the dyadic intervals, i.e.

\[
\sum_{I \in D} |s_I| < \infty,
\]

equipped with the norm

\[
\|s\|_1 = \sum_{I \in D} |s_I|.
\]

The space $\ell^\infty(D)$ is the space of all bounded sequences $s = (s_I)_{I \in D}$ indexed by the dyadic intervals equipped with the norm

\[
\|s\|_\infty = \sup_{I \in D} |s_I|.
\]

Carleson constant. See [Mul05]. Let $E \subseteq D$ be a non-empty collection of dyadic intervals. Then the Carleson constant of $E$ is given by

\[
[E] = \sup_{I \in E} \frac{1}{|I|} \sum_{J \in E, J \subseteq I} |J|.
\]

(2.8)

Blocks of dyadic intervals. Let $C$ be a collection of dyadic intervals. We say that $C(I) \subseteq C$ is a block of dyadic intervals in $C$ if the following conditions hold:

1. The collection $C(I)$ has a unique maximal interval, namely the interval $I$.
2. If $J \in C(I)$ and $K \in C$, then $J \subseteq K \subseteq I$ implies $K \in C(I)$.

The Haar system. We define the $L^\infty$- normalised Haar system $\{h_I\}_{I \in D}$ indexed by dyadic intervals $I$ as follows:

\[
h_I = \begin{cases} 
1 & \text{on the left half of } I, \\
-1 & \text{on the right half of } I, \\
0 & \text{otherwise.}
\end{cases}
\]
Dyadic Hardy Spaces $H^p$, $0 < p \leq 2$. See [Mü05; GMP05]. Let $(x_I)_{I \in \mathcal{D}}$ be a real sequence. We define $f = (x_I)_{I \in \mathcal{D}}$ to be the real vector indexed by the dyadic intervals. We define the square function of $f$ as follows

\begin{align}
S(f)(t) &= \left( \sum_{I \in \mathcal{D}} x_I^2 1_I(t) \right)^{\frac{1}{2}},
\end{align}

for $t \in [0, 1]$. The space $H^p$, $0 < p \leq 2$, consists of vectors $f = (x_I)_{I \in \mathcal{D}}$ for which

\begin{align}
\|f\|_{H^p} &= \|S(f)\|_{L^p([0,1])} < \infty.
\end{align}

For $1 \leq p \leq 2$, (2.10) defines a norm and $H^p$ is a Banach space. For $0 < p < 1$, (2.10) defines a quasi norm and the resulting Hardy spaces $H^p$ are quasi Banach spaces, cf. [Woj97]. The lattice structure on the $H^p$ spaces is induced by the natural lattice structure on sequence spaces (cf. [LT79]) and therefore they are (quasi) Banach lattices over the dyadic intervals $\mathcal{D}$ equipped with the counting measure. They are $2$-concave with $2$-concavity constant $M(2)(H^p) = 1$: let $x^1, \ldots, x^n \in H^p$, by Minkowski’s inequality for $\frac{2}{p} \geq 1$ we have

\begin{align}
\left( \sum_{i=1}^{n} \|x^i\|_{H^p}^2 \right)^{\frac{1}{2}} &= \left( \sum_{i=1}^{n} \left( \int_{0}^{1} \left( \sum_{I \in \mathcal{D}} |x^i_I|^2 1_I(t) \right)^{\frac{p}{2}} dt \right)^{\frac{2}{p}} \right)^{\frac{1}{2}} \\
&\leq \left( \int_{0}^{1} \left( \sum_{i=1}^{n} \sum_{I \in \mathcal{D}} |x^i_I|^2 1_I(t) \right)^{\frac{2}{p}} dt \right)^{\frac{1}{p}} \\
&= \left\| \left( \sum_{i=1}^{n} |x^i|^2 \right)^{\frac{1}{2}} \right\|_{H^p}.
\end{align}

Analogous we get that $H^p$, $0 < p \leq 2$, is $p$-convex with $p$-convexity constant $M(p)(H^p) = 1$.

For convenience we identify $f = (x_I)_{I \in \mathcal{D}} \in H^p$ with its formal Haar series

\[ f = \sum_{I \in \mathcal{D}} x_I h_I. \]

The Haar support of $f$ is defined as the following collection of dyadic intervals:

\[ \{ J \in \mathcal{D} : x_J \neq 0 \}. \]

Discrete Triebel-Lizorkin spaces $f^q_p$, $0 < p \leq q < \infty$. For general information on Triebel-Lizorkin spaces we refer to [FJ90]. Let $(x_I)_{I \in \mathcal{D}}$ be a real sequence. We define $f = (x_I)_{I \in \mathcal{D}}$ to be the real vector indexed by the dyadic intervals. We define the $q$-variation of $f$ as follows

\begin{align}
S_q(f)(t) &= \left( \sum_{I \in \mathcal{D}} |x_I|^q 1_I(t) \right)^{\frac{1}{q}},
\end{align}

\footnote{The Triebel-Lizorkin spaces $f^q_p$ are special cases of the discrete Triebel-Lizorkin spaces $f_{p, \alpha, q}$, defined in [FJ90] page 47, for the value $\alpha = -\frac{1}{2}$.}
for $t \in [0,1]$. The spaces $f_p^q$, $0 < p \leq q < \infty$, consist of vectors $f = (x_I)_{I \in D}$ for which
\begin{equation}
\|f\|_{f_p^q} = \|S_q(f)\|_{L^p([0,1])} < \infty.
\end{equation}
For $1 \leq p \leq q < \infty$ \eqref{2.13} defines a norm, otherwise it defines only a quasi norm. Therefore, the Triebel-Lizorkin spaces $f_p^q$, $0 < p \leq q < \infty$ are (quasi) Banach spaces.

As in the case of Hardy spaces the lattice structure on the Triebel-Lizorkin spaces is induced by the natural lattice structure on sequence spaces and therefore they are (quasi) Banach lattices over the dyadic intervals equipped with the counting measure.

The Triebel-Lizorkin spaces $f_p^q$ are the $\frac{q}{2}$-convexification of the Hardy space $H^{\frac{2q}{p}}$, where $\frac{2q}{p} \in (0,2]$, meaning that $f_p^q$ can be identified with the space of all sequences $u = (x_I)_{I \in D}$ such that $|u|^{\frac{q}{2}} \in H^{\frac{2q}{p}}$ endowed with the norm $\||u|^{\frac{q}{2}}\|_{H^{\frac{2q}{p}}}$, cf. [LT79, CT86]. The absolute value is defined by $|u|^{\frac{q}{2}} = \left(\|x_I\|^{\frac{q}{2}}\right)_{I \in D}$ and it can be identified with its formal Haar series $|u|^{\frac{q}{2}} = \sum_{I \in D} |x_I|^{\frac{q}{2}} h_I$.

Recall that the spaces $H^p$, $0 < p \leq 2$, are 2-concave and $p$-convex (quasi) Banach lattices. Since $\frac{2q}{p} \in (0,2]$ the theory of convexification ([LT79, CT86]) yields that the Triebel-Lizorkin spaces are $q$-concave and $p$-convex Banach lattices. The $q$-concavity and $p$-convexity constants $M(q)(f_p^q)$ and $M(p)(f_p^q)$ are equal to one.

### 2.3. Vector-valued dyadic Hardy Spaces $H^p_X$, $0 < p \leq 2$.

Let $X$ be a Banach space and $(x_I)_{I \in D}$ be a sequence in $X$. We define $f = (x_I)_{I \in D}$ to be the $X$-valued vector indexed by the dyadic intervals. We define the square function of $f$ as follows:
\begin{equation}
\mathbb{S}(f)(t) = \lim_{n \to \infty} \left( \int_0^1 \left\| \sum_{I \in D_n} r_I(s) x_I h_I(t) \right\|_X^2 \, ds \right)^{\frac{1}{2}}, \quad t \in [0,1].
\end{equation}
where $\{r_I\}_{I \in D}$ is an enumeration of the independent Rademacher system. Let $0 < p \leq 2$. We say $f \in H^p_X$, if
\begin{equation}
\|f\|_{H^p_X} = \|\mathbb{S}(f)\|_{L^p([0,1])} < \infty.
\end{equation}

We identify $f = (x_I)_{I \in D}$ with its formal Haar series
\[ f = \sum_{I \in D} x_I h_I. \]

The Haar support of $f$ is defined as the following collection of dyadic intervals: $\{J \in D : x_J \neq 0\}$.

The following theorem states the decomposition of an element in $H^p_X$ into absolutely summing elements with disjoint Haar support and bounded square function. The decomposition is done by a stopping time argument that may be regarded as a constructive algorithm. The decomposition originates in the work of S. Janson and P.W. Jones [JJ82] and is described, for example, in [Müller05].
Theorem 2.3 (Atomic decomposition). For all $0 < p \leq 2$ there exist constants $a_p$, $A_p$ such that for every $u \in H^p_X$ with Haar expansion

$$u = \sum_{J \in \mathcal{D}} x_J h_J, \quad x_J \in X$$

there exists an index set $\mathcal{N} \subseteq \mathbb{N}$ and a sequence $(G_i)_{i \in \mathcal{N}}$ of blocks of dyadic intervals such that for

$$u_i = \sum_{J \in G_i} x_J h_J, \quad i \in \mathcal{N}$$

the following holds:

i) $(G_i)_{i \in \mathcal{N}}$ is a disjoint partition of $\mathcal{D}$.

ii) $I_i := \bigcup_{J \in G_i} J$ is a dyadic interval and $\mathcal{E} := \{ I_i : i \in \mathcal{N} \}$ satisfies $|\mathcal{E}| \leq 4$.

iii) (2.16) $a_p \|u\|^p_{H^p_X} \leq \sum_{i \in \mathcal{N}} \|u_i\|^p_{H^p_X} \leq \sum_{i \in \mathcal{N}} |I_i| \|\mathcal{S}(u_i)\|_\infty^p \leq A_p \|u\|_{H^p_X}$.

The family $(u_i, G_i, I_i)_{i \in \mathcal{N}}$ is called the atomic decomposition of $u \in H^p_X$.

Remark. If we set $X = \mathbb{R}$ in the above theorem, we get the atomic decomposition of $u \in H^p$. Note that in this case $a_p = 1$ for all $0 < p \leq 2$. The scalar valued decomposition procedure, particularly the inequalities (2.16) in the case $X = \mathbb{R}$, can be found in [Müller05]. The right-hand side inequality in (2.16) transfers directly from the scalar valued case (cf. [Müller05]) to the vector valued case. For the left-hand side estimate in (2.16) we have to consider two cases. In the case $0 < p \leq 1$ use the well known triangle inequality

$$\|f + g\|_{H^p_X} \leq \|f\|_{H^p_X} + \|g\|_{H^p_X}.$$

In the case $1 < p \leq 2$ there are some differences between the scalar and the vector valued case. In the scalar valued case, the left-hand side inequality follows immediately from the disjoint decomposition of $\mathcal{D}$ into blocks of dyadic intervals. In the vector valued case one can adapt the proof of [GM08 Lemma 3.3]. We include the Appendix (Section 5) in order to give the proof in detail.

However, note that the left-hand side inequality of (2.16) depends only on the fact that $(G_i)_{i \in \mathcal{N}}$ is a sequence of disjoint blocks of dyadic intervals and that the Carleson constant of $\mathcal{E}$ is finite. Therefore, for $\varphi = (\varphi_I)_{I \in \mathcal{D}} \in \ell^\infty(\mathcal{D})$ we have

$$ \left\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \right\|^p_{H^p_X} \leq \frac{1}{a_p} \sum_{i \in \mathcal{N}} \left\| \sum_{J \in G_i} \varphi_I x_J h_J \right\|^p_{H^p_X}. $$

(2.17)

Remark. Applying the atomic decomposition procedure to the function $|u|^p = \sum_{I \in \mathcal{D}} |x_I|^p h_I \in H^p_X$, yields the atomic decomposition of $u = (x_I)_{I \in \mathcal{D}} \in f^p$, denoted by $(u_i, I_i, G_i)$, where $u_i = (x_I)_{I \in G_i}$ and $I_i, G_i$ are as in Theorem 2.3.

2.4. Maurey’s Factorization Theorem. The following theorem follows directly from Maurey’s theorem (Theorem 2.1) and from Pietsch’s factorization theorem ([Wo91]).

Theorem 2.4. Let $X$ be a Banach space of cotype $r$, $2 \leq r < \infty$. For all $s > r$ there exists a constant $K_{s,r} > 0$ such that the following holds. For all $f \in H^s_X$ with
Haar expansion

\[ f = \sum_{I \in \mathcal{D}} f_I h_I, \quad f_I \in X, \]

there exists a \( \mu \in \ell^1(\mathcal{D}) \) (not depending on \( s \)), with

\[ \mu_I \geq 0, \quad \forall I \in \mathcal{D} \]

and

\[ \sum_{I \in \mathcal{D}} \mu_I = 1 \]

such that for each \( \varphi \in \ell^\infty(\mathcal{D}) \)

\[ \| \varphi \cdot f \|_{H^2_X} \leq K_{s,r} C_r(X) \| f \|_{H^2_X} \left( \sum_{I \in \mathcal{D}} |\varphi_I|^s \mu_I \right)^{\frac{1}{s}}, \]

where \( C_r(X) \) is the cotype-\( r \) constant of \( X \) and

\[ \varphi \cdot f = \sum_{I \in \mathcal{D}} \varphi_I f_I h_I. \]

Proof. We obtain from Kahane’s inequality that \( H^2_X \) is of cotype \( r \). Let \( f \in H^2_X \).

We consider the multiplication operator \( \mathcal{M}_f : \ell^\infty(\mathcal{D}) \to H^2_X, \varphi \mapsto \varphi \cdot f. \)

From Kahane’s contraction principle we get \( S(\varphi \cdot f)(t) \leq \sup_{I \in \mathcal{D}} |\varphi_I| S(f)(t) \). Hence, \( \| \mathcal{M}_f \| \leq \| f \|_{H^2_X} \).

**Step 1:** We assume that \( f \) has finite Haar support \( \mathcal{D}' \subset \mathcal{D} \). Then, applying Maurey’s theorem (Theorem 2.1) to the multiplication operator \( \mathcal{M}_f \in L(\ell^\infty(\mathcal{D}'), H^2_X) \) we obtain that \( \mathcal{M}_f \) is \( s \)-summing for all \( s > r \) and

\[ \pi_s(\mathcal{M}_f) \leq K_{s,r} C_r(X) \| \mathcal{M}_f \| \leq K_{s,r} C_r(X) \| f \|_{H^2_X}. \]

Since \( \mathcal{D}' \) is finite, we can apply Pietsch’s factorization theorem [Woj91, Theorem III.8.5] and get the following. There exists a \( \mu \in \ell^1(\mathcal{D}) \) with \( \mu_I \geq 0 \) for all \( I \in \mathcal{D} \) and \( \sum_{I \in \mathcal{D}} \mu_I = 1 \) such that for each \( \varphi \in \ell^\infty(\mathcal{D}) \)

\[ \| \mathcal{M}_f(\varphi) \|_{H^2_X} \leq \pi_s(\mathcal{M}_f) \left( \sum_{I \in \mathcal{D}} |\varphi_I|^s \mu_I \right)^{\frac{1}{s}}. \]

**Step 2:** Let \( f \in H^2_X \) with its formal Haar series \( f = \sum_{I \in \mathcal{D}} x_I h_I \), where the partial sums are listed according to the lexicographic order in \( \mathcal{D} \). The conditional expectation \( \mathbb{E}_N \) with respect to the \( \sigma \)-algebra generated by the set \( \{ I \in \mathcal{D} : |I| = 2^{-N} \} \) is given by

\[ \mathbb{E}_N(f) = \sum_{I \in \mathcal{D}_{N-1}} x_I h_I. \]

Then we have

\[ f = \mathbb{E}_N(f) + f - \mathbb{E}_N(f) \]

and for each \( \varepsilon > 0 \) there exists \( N(\varepsilon) \in \mathbb{N} \) such that for all \( N \geq N(\varepsilon) \)

\[ \| \mathbb{E}_N(f) - f \|_{H^2_X} \leq \varepsilon. \]

To confirm equation (2.21) we exploit the finite cotype of \( X \) and invoke Kwapien’s theorem ([DJT95, p.259]) and Hoffmann-Jørgensen’s theorem ([DJT95, Theorem...].
where $r$ such that we have that (2.25)

\[
\phi \in \mathcal{D}.
\]

From the construction of the sequence $(f I)_{I \in \mathcal{D}}$ and define a sequence of independent, symmetric random variables with values in $L^2_X([0,1], dt)$. Let $I \in \mathcal{D}$ and

\[
R_I : [0,1] \to L^2_X, \quad s \mapsto \left( t \mapsto x_I h_I(t)r_I(s) \right),
\]

where $r_I$ is an enumeration of the Rademacher system. Since $f = \sum_{I \in \mathcal{D}} x_I h_I \in H^2_X$ we have that

\[
\sup_{N \in \mathbb{N}} \left( \int_0^1 \int_0^1 \left\| \sum_{I \in \mathcal{D}_N} x_I h_I(t)r_I(s) \right\|^2_X dt ds \right)^{1/2} < \infty.
\]

Therefore, the partial sums of random variables $\sum_{I \in \mathcal{D}_N} R_I$ are bounded in $L^2_Z([0,1], ds)$, where $Z = L^2_X([0,1], dt)$. $X$ has finite cotype, hence $L^2_X$ has finite cotype and Kwapień’s theorem yields that there exists a limit of the partial sums given by

\[
R(s) := \lim_{N \to \infty} \sum_{I \in \mathcal{D}_N} R_I(s), \quad \text{a.e.}
\]

Hoffmann-Jørgensen’s theorem asserts that the partial sums $\sum_{I \in \mathcal{D}_N} R_I$ converge in $L^2_Z([0,1], ds)$, i.e.

\[
\lim_{N \to \infty} \left\| R - \sum_{I \in \mathcal{D}_N} R_I \right\|_{L^2_Z} = 0,
\]

and hence (2.21) holds. Note that we identified $f = (x_I)_{I \in \mathcal{D}}$ with the convergent series $R = \sum_{I \in \mathcal{D}} x_I h_I \otimes r_I \in L^2_Z$, where $Z = L^2_X$.

Iterating (2.20) and (2.21) there exists a monotonically increasing sequence $(N_i)_{i \geq 0}$ of natural numbers and a sequence $(f_i)_{i \geq 0}$ in $H^2_X$ given by

\[
f_0 = \mathbb{E}N_0(f), \quad f_i = \mathbb{E}N_i(f) - \mathbb{E}N_{i-1}(f), \quad i \geq 1
\]

such that

\[
\|f_i\|_{H^2_X} \leq 4^{-i} \|f\|_{H^2_X}.
\]

From the construction of the sequence $(f_i)_{i \geq 0}$ we get that each $f_i$ has finite Haar support $D_i \subset \mathcal{D}$.

**Step 3:** We apply Step 1 to the sequence $(f_i)_{i \geq 0}$. There exists a sequence $(\mu_i)_{i \geq 0}$, $\mu_i \in \ell^1(D_i)$ with $\mu_i \geq 0$ for all $I \in D_i$ and $\sum_{I \in D_i} \mu_i = 1$ such that for each $\varphi \in \ell^\infty(D)$

\[
\|\mathcal{M}_{f_i}(\varphi)\|_{H^2_X} \leq \pi_s(\mathcal{M}_{f_i}) \left( \sum_{I \in D_i} |\varphi_I|^{\mu_i} \right)^{1/2}.
\]

Step 2 yields $f = \sum_{i=0}^{\infty} f_i$. Therefore,

\[
\|\mathcal{M}_{f}(\varphi)\|_{H^2_X} \leq \sum_{i=0}^{\infty} \|\mathcal{M}_{f_i}(\varphi)\|_{H^2_X} \leq \sum_{i=0}^{\infty} \pi_s(\mathcal{M}_{f_i}) \left( \sum_{I \in D_i} |\varphi_I|^{\mu_i} \right)^{1/2}.
\]
Using (2.19) and (2.24) yields
\[
\|M(f)(\varphi)\|_{H^2_X} \leq K_{s,r} C_r \sum_{i=0}^{\infty} \|f_i\|_{H^2_X} \left(\sum_{I \in D_i} |\varphi_I|^s \mu_I^s\right)^{\frac{1}{s}}.
\]

Let \(\frac{1}{s} + \frac{1}{s'} = 1\). Hölder’s inequality yields
\[
\|M(f)(\varphi)\|_{H^2_X} \leq K_{s,r} C_r \|f\|_{H^2_X} \left(\sum_{i=0}^{\infty} 2^{-is'} \sum_{I \in D_i} |\varphi_I|^s \mu_I^s\right)^{\frac{1}{s}}
\]
\[
= K_{s,r} C_r \|f\|_{H^2_X} \left(\sum_{I \in D} |\varphi_I|^s \sum_{i=0}^{\infty} 2^{-i} 1_{D_i}(I) \mu_I^s\right)^{\frac{1}{s}}
\]
\[
= K_{s,r} C_r \|f\|_{H^2_X} \left(\sum_{I \in D} |\varphi_I|^s \nu_I\right)^{\frac{1}{s}},
\]
where \(\nu_I = \sum 2^{-i} 1_{D_i}(I) \mu_I^s\) satisfies \(\nu_I \geq 0\) for all \(I \in D\) and
\[
\sum_{I \in D} \nu_I = \sum_{i=0}^{\infty} 2^{-i} \sum_{I \in D_i} \mu_I^s = \sum_{i=0}^{\infty} 2^{-i} = 2.
\]

Remark. The proof above has a direct extension to \(H^p_X, 1 \leq p < \infty\), where again the measure \(\mu \in \ell^1(D)\) has its origin in the abstract version of Pietsch’s theorem.

Remark. If \(X\) is of cotype 2 then we know from Theorem (2.2) that the statement of Theorem (2.3) is valid for \(s \geq 2\). Especially it is valid for \(s = 2\).

3. The main Theorem

We investigate multiplication operators acting on the Haar system in the Hardy spaces \(H^p, 0 < p \leq 2\). We fix \(u \in H^p\) with Haar expansion \(u = \sum x_I h_I\) and define \(M_u : \ell^\infty(D) \to H^p\) by
\[
M_u(\varphi) = \sum_{I \in D} \varphi_I x_I h_I.
\]

We frequently use the ”lattice convention”
\[
\varphi \cdot u = M_u(\varphi)
\]
to emphasize that \(\varphi \in \ell^\infty(D)\) is acting as a multiplier on \(u \in H^p\). Since the Haar basis is 1-unconditional in \(H^p\) we have
\[
\|M_u : \ell^\infty(D) \to H^p\| \leq \|u\|_{H^p}.
\]

Note that \(H^p, 0 < p \leq 2\), is 2-concave as a sequence space. Hence, our multiplication operator is 2-summing, see (2.7), and has a Pietsch measure. Precisely, by
Step 2 and 3 in the proof of Theorem 2.4 this measure is given as follows. There exists a nonnegative sequence \( \omega_I, I \in \mathcal{D} \) so that \( \sum \omega_I \leq 1 \) and for all \( \varphi \in \ell^\infty(\mathcal{D}) \)

\[
\|M_u(\varphi)\|_{H^p} \leq \|u\|_{H^p} \left( \sum_{I \in \mathcal{D}} |\varphi_I|^2 \omega_I \right)^{\frac{1}{2}}.
\]

Since \( M_u \) is determined by \( u \in H^p \), also the Pietsch measure \( (\omega_I)_{I \in \mathcal{D}} \) is given by \( u \in H^p \). Note however that the existence of Pietsch measures comes from an application of the Hahn-Banach theorem and therefore Pietsch measures are not given constructively.

### 3.1. Construction of Pietsch measures

The construction of the Pietsch measure is the contribution of our paper: for multipliers ranging in the Hardy spaces \( H^p \), we are able to find explicit formulae for the Pietsch measure \( (\omega_I)_{I \in \mathcal{D}} \). The input for our construction is the atomic decomposition of \( u \in H^p \), Theorem 2.3. The output is the equation (3.5) determining \( \omega_I \) explicitly.

Recall that Theorem 2.3 asserts that \( u = \sum x_I h_I \in H^p, 0 < p \leq 2 \) admits an atomic decomposition, that is a triple \( (u_i, G_i, I_i)_{i \in \mathbb{N}} \) so that

\[
\|u\|_{H^p} \leq \sum_{i \in \mathbb{N}} \|u_i\|_{H^p} \leq \sum_{i \in \mathbb{N}} |I_i| \|S(u_i)\|_{\ell^\infty} \leq A_p \|u\|_{H^p}.
\]

**Theorem 3.1.** Let \( 0 < p \leq 2 \). Let \( u \in H^p \) with Haar expansion

\[
u = \sum_{I \in \mathcal{D}} x_I h_I
\]

and atomic decomposition \( (u_i, G_i, I_i)_{i \in \mathbb{N}} \). Then the sequence \( (\omega_I)_{I \in \mathcal{D}} \), defined by

\[
\omega_I = \frac{1}{A_p} \frac{|I_i|^{1-\frac{p}{2}} |x_I|^2 |I_i|^{1-\frac{p}{2}}}{\|u_{I_i}\|_{H^p}^{2-p} \|u\|_{H^p}^{p}}, \quad I \in G_i,
\]

satisfies

\[
\sum_{I \in \mathcal{D}} \omega_I \leq 1
\]

and there exists a constant \( C_p > 0 \) such that for each \( \varphi \in \ell^\infty(\mathcal{D}) \)

\[
\|\varphi \cdot u\|_{H^p} \leq C_p \|u\|_{H^p} \left( \sum_{I \in \mathcal{D}} |\varphi_I|^2 \omega_I \right)^{\frac{1}{2}}.
\]

**Proof.** From \( \|u\|_p^{p} \leq \sum_{i \in \mathbb{N}} \|u_i\|_p^{p} \)

we get the estimate

\[
\|u\|_{H^p} \leq \sum_{i \in \mathbb{N}} \|u_i\|_{H^p}^{p} |I_i|^{1-\frac{p}{2}}.
\]
We get from
\[ \sum_{i \in \mathcal{N}} |I_i| \|S(u_i)\|_\infty^p \leq A_p \|u\|_{H^p}^p \]
that
\[ (3.9) \quad \sum_{i \in \mathcal{N}} \|u_i\|_2^p |I_i|^{1 - \frac{p}{q}} \leq A_p \|u\|_{H^p}^p. \]

This follows from
\[ \|u_i\|_2^p |I_i|^{1 - \frac{p}{q}} \leq \|Su_i\|_\infty^p |I_i|. \]

By (3.8) and the remark following Theorem 2.3 we get for \( \varphi \in \ell^\infty(\mathcal{D}) \)
\[ \left\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \right\|_{H^p}^p = \left\| \sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \varphi_I x_I h_I \right\|_{H^p}^p \]
\[ \leq \sum_{i \in \mathcal{N}} \left\| \sum_{I \in \mathcal{G}_i} \varphi_I x_I h_I \right\|_{2}^p \|I_i|^{1 - \frac{q}{2}} \]
\[ = \sum_{i \in \mathcal{N}} \left\| \sum_{I \in \mathcal{G}_i} \varphi_I \frac{x_I}{u_i} \right\|_{2}^p \|u_i\|_2^p |I_i|^{1 - \frac{q}{2}}. \]

With
\[ \left\| \sum_{I \in \mathcal{G}_i} \varphi_I \frac{x_I}{u_i} \right\|_{2}^p = \left( \sum_{i \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{u_i^2} |I_i| \right)^{\frac{p}{2}} \]
we get
\[ \left\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \right\|_{H^p}^p \leq \sum_{i \in \mathcal{N}} \left( \sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{u_i^2} |I_i| \right)^{\frac{p}{2}} \|u_i\|_2^p |I_i|^{1 - \frac{q}{2}}. \]

Applying Hölder’s inequality with \( \frac{p}{2} + 1 - \frac{q}{2} = 1 \)
\[ \sum_{i \in \mathcal{N}} \left( \sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{u_i^2} |I_i| \right)^{\frac{p}{2}} \|u_i\|_2^p |I_i|^{1 - \frac{q}{2}} \left( \sum_{i \in \mathcal{N}} \|u_i\|_2^p |I_i|^{1 - \frac{q}{2}} \right)^{\frac{1}{2}}. \]

we get
\[ \left\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \right\|_{H^p}^p \leq \left( \sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{u_i^2} |I_i| \right)^{\frac{p}{2}} \left( \sum_{i \in \mathcal{N}} \|u_i\|_2^p |I_i|^{1 - \frac{q}{2}} \right)^{\frac{1}{2}}. \]

Applying (3.9) to the second term on the right-hand side we obtain the estimate
\[ \left\| \sum_{I \in \mathcal{D}} \varphi_I x_I h_I \right\|_{H^p}^p \leq A_p^{1 - \frac{q}{2}} \|u\|_{H^p}^{p(1 - \frac{q}{2})} \left( \sum_{i \in \mathcal{N}} \sum_{I \in \mathcal{G}_i} \varphi_I^2 \frac{x_I^2}{u_i^2} |I_i| \right)^{\frac{p}{2}} \left( \sum_{i \in \mathcal{N}} \|u_i\|_2^p |I_i|^{1 - \frac{q}{2}} \right)^{\frac{1}{2}}. \]

Recall
\[ (3.11) \quad \|u_i\|_2^2 = \sum_{I \in \mathcal{G}_i} x_I^2 |I|. \]

By (3.9) and (3.11) we obtain for the sequence \( (\omega_I)_{I \in \mathcal{D}} \), defined by
\[ \omega_I = \frac{|I_i|^{1 - \frac{q}{2}} |I| x_I^2}{A_p \|u\|_{H^p}^p \|u_i\|_2^{2 - p}}, \quad I \in \mathcal{G}_i, \]
the following estimate
\[
\sum_{I \in D} \omega_I = \frac{1}{A_p \|u\|_{H_p}^p} \sum_{I \in \mathcal{N}} \sum_{i \in \mathcal{V}_i} |I_i|^{1-\frac{q}{p}} |x_i|^q \leq \frac{1}{A_p \|u\|_{H_p}^p} \sum_{I \in \mathcal{N}} |I_i|^{1-\frac{q}{p}} \|u_i\|_2^q \\
\leq 1.
\]

3.2. Extension to Triebel-Lizorkin spaces. We can extend the construction to the Triebel-Lizorkin spaces \( f_p^q \), \( 0 < p \leq q < \infty \). Recall that \( f_p^q \) is the \( \frac{q}{p} \)-convexification of \( H_{\frac{q}{p}} \), where \( \frac{q}{p} \in (0, 2] \), with
\[
\|u\|_{f_p^q} = \left\| |u|^q \right\|_{H_{\frac{q}{p}}}.
\]

Therefore, we have for \( \varphi \in \ell^\infty(D) \)
\[
\|\varphi \cdot u\|_{f_p^q} = \left\| |\varphi|^q \cdot |u|^q \right\|_{H_{\frac{q}{p}}}.
\]

Therefore, we get from (3.4), (3.12) and (3.13) the following statement:

Let \( 0 < p \leq q < \infty \). For all \( u \in f_p^q \) with \( u = (x_I)_{I \in D} \) there exists a non negative sequence \( \omega_I, I \in D \) so that \( \sum \omega_I \leq 1 \) and for all \( \varphi \in \ell^\infty \)
\[
\|\varphi \cdot u\|_{f_p^q} \leq \|u\|_{f_p^q} \left( \sum_{I \in D} |\varphi|^q \omega_I \right)^{\frac{1}{q}}.
\]

where
\[ \varphi \cdot u = (\varphi_I x_I)_{I \in D}. \]

We are now able to give an explicit formula for \( (\omega_I)_{I \in D} \) using again (3.12) and (3.13). Let \( (u_i, G_i, I_i)_{i \in \mathcal{N}} \) be the atomic decomposition of \( u = (x_I)_{I \in D} \in f_p^q \). Then \( (|u_i|^q, G_i, I_i)_{i \in \mathcal{N}} \) is the atomic decomposition of \( |u|^q \in H_{\frac{q}{p}} \). Theorem 3.1 asserts that there exists a constant \( C_{2q} \) such that
\[
\left\| |\varphi|^q \cdot |u|^q \right\|_{H_{\frac{q}{p}}} \leq C_{2q} \left( \sum_{I \in D} |\varphi|^q I_i \right)^{\frac{1}{q}} \|u\|_{f_p^q},
\]
where
\[ \omega_I = \frac{1}{A_p \|u_i\|_{f_p^q}^q \|u\|_{f_p^q}} \frac{|I_i|^{1-\frac{q}{p}} |x_i|^q |I|}{\|u_i\|_{f_p^q}^{2-\frac{q}{p}} \|u\|_{f_p^q}^{\frac{q}{p}}}, \quad I \in G_i. \]

Summarizing we get the following statement for Triebel-Lizorkin spaces as corollary of Theorem 3.1.

**Corollary 3.2.** Let \( 0 < p \leq q < \infty \). Let \( u = (x_I)_{I \in D} \in f_p^q \) with atomic decomposition \( (u_i, G_i, I_i)_{i \in \mathcal{N}} \). Then the sequence \( (\omega_I)_{I \in D} \), defined by
\[
\omega_I = \frac{1}{A_p \|u_i\|_{f_p^q}^q \|u\|_{f_p^q}} \frac{|I_i|^{1-\frac{q}{p}} |x_i|^q |I|}{\|u_i\|_{f_p^q}^{2-\frac{q}{p}} \|u\|_{f_p^q}^{\frac{q}{p}}}, \quad I \in G_i,
\]
satisfies
and there exists a constant \(C_{p,q} > 0\) such that for each \(\varphi \in \ell^\infty(D)\)

\[
\|\varphi \cdot u\|_{f^p} \leq C_{p,q} \|u\|_{f^p} \left( \sum_{I \in D} |\varphi_I|^q \omega_I \right)^{\frac{1}{q}}.
\]

### 3.3. Extension to vector-valued Hardy spaces.

Let \(X\) be a Banach space. Fix a sequence \((x_I)_{I \in D}\) in \(X\), then for \(u = \sum_{I \in D} x_I h_I \in H^p_X\) we define the multiplication operator

\[
\mathcal{M}_u : \ell^\infty(D) \to \overline{\text{span}} \{x_I h_I : I \in D\} \subseteq H^p_X,
\]

by

\[
\mathcal{M}_u(\varphi) = \sum_{I \in D} \varphi_I x_I h_I.
\]

By Kahane’s contraction principle (2.2), for fixed \((x_I)_{I \in D}\), the sequence \((x_I h_I)_{I \in D}\) is an unconditional basic sequence in \(H^p_X\), \(0 < p \leq 2\). This remark links the present work on vector-valued \(H^p_X\) spaces with the lattices of the previous sections.

As an application of Theorem 2.4 and the atomic decomposition in \(H^p_X\) we obtain the following statement. The atomic decomposition works as extrapolation tool, transferring the Pietsch measure of multiplication operators into \(H^2_X\) to Pietsch measures for multiplication operators into \(H^p_X\), \(0 < p < 2\).

Here the result is only partially constructive. The formulae for the Pietsch measures for atoms are obtained by applying Theorem 2.4 invoking the abstract version of Pietsch’s theorem.

**Theorem 3.3.** Let \(X\) be a Banach space of cotype \(r\), \(2 \leq r < \infty\). Let \(0 < p \leq 2\) and let \(u \in H^p_X\) with Haar expansion

\[
u = \sum_{I \in D} x_I h_I
\]

and atomic decomposition \((u_i, G_i, I_i)_{i \in N}\) (see Theorem 2.3). Then for each \(i \in N\) there exists a \(\mu^{(i)} \in \ell^1(G_i)\) with

\[
\mu^{(i)}_I \geq 0, \forall I \in G_i
\]

and

\[
\sum_{I \in G_i} \mu^{(i)}_I = 1
\]

so that the following holds: The sequence \((\omega_I)_{I \in D}\), defined by

\[
\omega_I = \frac{\|u_I\|_{H^p_X} |I_i|^{\frac{1}{p}}}{A_p \|u\|_{H^p_X}} \mu^{(i)}_I, \quad I \in G_i,
\]

satisfies

\[
\sum_{I \in D} \omega_I \leq 1
\]
and for all $s > r$ there exists a constant $K_{s,r,p} > 0$ such that for each $\varphi \in \ell^\infty(D)$

$$
\|\varphi \cdot u\|_{H_X^p} \leq K_{s,r,p} C_r(X) \|u\|_{H_X^p} \left( \sum_{I \in D} |\varphi_I|^s \omega_I \right)^{\frac{p}{s}}.
$$

(3.21)

We point out that the exponent $s$ in (3.21) is determined by the cotype of $X$ alone. In particular it is not depending on $0 < p \leq 2$.

**Proof.** Recall the inequalities in (2.16) from the atomic decomposition in Theorem 2.3. We have with

$$
\|u\|_{H_X^p}^p \leq \frac{1}{a_p} \sum_{i \in N} \|u_i\|_{H_X^p}^p
$$

the following estimate

$$
\|u\|_{H_X^p}^p \leq \frac{1}{a_p} \sum_{i \in N} \|u_i\|_{H_X^p}^p |I_i|^{1 - \frac{p}{s}}.
$$

(3.22)

We also get from

$$
\sum_{i \in N} |I_i| \|S(u_i)\|_{\infty}^p \leq A_p \|u\|_{H_X^p}^p
$$

that

$$
\sum_{i \in N} \|u_i\|_{H_X^p}^p |I_i|^{1 - \frac{p}{s}} \leq A_p \|u\|_{H_X^p}^p.
$$

(3.23)

This follows from

$$
\|u_i\|_{H_X^p}^p |I_i|^{1 - \frac{p}{s}} \leq \|S u_i\|_{\infty}^p |I_i|.
$$

By (3.22) and the remark following Theorem 2.3 we get for $\varphi \in \ell^\infty(D)$

$$
\left\| \sum_{I \in D} \varphi_I x_I h_I \right\|_{H_X^p}^p \leq \left\| \sum_{i \in N} \sum_{I \in G_i} \varphi_I x_I h_I \right\|_{H_X^p}^p |I_i|^{1 - \frac{p}{s}}
$$

$$
\leq \frac{1}{a_p} \sum_{i \in N} \left\| \sum_{I \in G_i} \varphi_I x_I h_I \right\|_{H_X^p}^p |I_i|^{1 - \frac{p}{s}}
$$

$$
= \frac{1}{a_p} \sum_{i \in N} \left\| \sum_{I \in G_i} \varphi_I \|u_i\|_{H_X^p}^p |I_i|^{1 - \frac{p}{s}}.
$$

(3.24)

We apply Theorem 2.4 with the specification $f = u_i$, recall that $u_i = \sum_{I \in G_i} x_I h_I$. Therefore, we obtain the following statement: For all $s > r$ there exists a constant $K_{s,r}$ such that

$$
\left\| \sum_{I \in G_i} \varphi_I \|u_i\|_{H_X^p}^p h_I \right\|_{H_X^p}^p \leq K_{s,r} C_r(X)^p \left( \sum_{I \in G_i} |\varphi_I|^s \mu_I^{(i)} \right)^{\frac{p}{s}},
$$

where $\mu_i \in \ell^1(G_i)$ with $\mu_I^{(i)} \geq 0$ for all $I \in D$ and $\sum_{I \in G_i} \mu_I^{(i)} = 1$. Summing up we get from Maurey’s theorem (in particular Theorem 2.4) the following statement

$$
\left\| \sum_{I \in D} \varphi_I x_I h_I \right\|_{H_X^p}^p \leq \frac{K_{s,r} C_r(X)^p}{a_p} \sum_{i \in N} \left( \sum_{I \in G_i} |\varphi_I|^s \mu_I^{(i)} \right)^{\frac{p}{s}} \|u_i\|_{H_X^p}^p |I_i|^{1 - \frac{p}{s}}.
$$

(3.25)
We rewrite the sum on the right-hand side in an appropriate way and apply Hölder’s inequality with \( \frac{p}{q} + 1 - \frac{q}{p} = 1 \):

\[
\sum_{i \in N} \left( \sum_{I \in G_i} |\varphi_I|^{s} \mu_I^{(i)} \right)^{\frac{q}{p}} \|u_i\|_{H_p^X}^{p} |I_i|^{1 - \frac{p}{q}}
\]

(3.26)\[
= \sum_{i \in N} \left( \sum_{I \in G_i} |\varphi_I|^{s} \|u_i\|_{H_p^X} |I_i|^{1 - \frac{p}{q}} \mu_I^{(i)} \right)^{\frac{q}{p}} \left( \|u_i\|_{H_p^X} |I_i|^{1 - \frac{p}{q}} \right)^{1 - \frac{p}{q}}
\]

\[
\leq \left( \sum_{i \in N} \sum_{I \in G_i} |\varphi_I|^{s} \|u_i\|_{H_p^X} |I_i|^{1 - \frac{p}{q}} \mu_I^{(i)} \right)^{\frac{q}{p}} \|u\|_{H_p^X} \left( \sum_{i \in N} \sum_{I \in G_i} |\varphi_I|^{s} \|u_i\|_{H_p^X} |I_i|^{1 - \frac{p}{q}} \mu_I^{(i)} \right)^{1 - \frac{q}{p}}.
\]

By (3.23) we get an estimate for the second term on the right-hand side and therefore

\[
\sum_{i \in N} \left( \sum_{I \in G_i} |\varphi_I|^{s} \mu_I^{(i)} \right)^{\frac{q}{p}} \|u_i\|_{H_p^X} |I_i|^{1 - \frac{p}{q}}
\]

(3.27)\[
\leq \left( \sum_{i \in N} \sum_{I \in G_i} |\varphi_I|^{s} \|u_i\|_{H_p^X} |I_i|^{1 - \frac{p}{q}} \mu_I^{(i)} \right)^{\frac{q}{p}} A_p^{1 - \frac{q}{p}} \|u\|_{H_p^X}^{p(1 - \frac{q}{p})}
\]

\[= A_p \|u\|_{H_p^X} \left( \sum_{i \in N} \sum_{I \in G_i} |\varphi_I|^{s} \|u_i\|_{H_p^X} |I_i|^{1 - \frac{p}{q}} \mu_I^{(i)} \right)^{1 - \frac{q}{p}}.
\]

Combining (3.25) and (3.27) yields

\[
\left\| \sum_{I \in D} \varphi_I x_I h_I \right\|_{H_p^X} \leq K_{s,r,p} C_r(X) \|u\|_{H_p^X} \left( \sum_{i \in N} \sum_{I \in G_i} |\varphi_I|^{s} \|u_i\|_{H_p^X} |I_i|^{1 - \frac{p}{q}} \mu_I^{(i)} \right)^{\frac{q}{p}},
\]

where \( K_{s,r,p} \) is dependent on the constant \( K_{s,r} \) from Theorem 2.4 and the constants \( a_p, A_p \) from the atomic decomposition (Theorem 2.3). Now we set

\[
(3.28) \quad \omega_I = \frac{\|u_i\|_{H_p^X} |I_i|^{1 - \frac{p}{q}}}{A_p \|u\|_{H_p^X} \mu_I^{(i)}}, \quad I \in G_i
\]

and obtain

\[
\left\| \sum_{I \in D} \varphi_I x_I h_I \right\|_{H_p^X} \leq C_r(X) K_{s,r,p} \|u\|_{H_p^X} \left( \sum_{I \in D} |\varphi_I|^{s} \omega_I \right)^{\frac{q}{p}}.
\]

The sequence \((\omega_I)_{I \in D}\), defined by (3.28), satisfies

\[
\sum_{I \in D} \omega_I = \frac{1}{A_p \|u\|_{H_p^X}} \sum_{i \in N} \|u_i\|_{H_p^X} |I_i|^{1 - \frac{p}{q}} \sum_{I \in G_i} \mu_I^{(i)}.
\]

Recall that \( \mu_i \) is a probability measure on \( G_i \), thus we get from (3.23)

\[
\sum_{I \in D} \omega_I = \frac{1}{A_p \|x\|_{H_p^X}} \sum_{i \in N} \|u_i\|_{H_p^X} |I_i|^{1 - \frac{p}{q}} \leq 1.
\]

□
Remark. If \( X \) is of cotype 2 then the statement of Theorem 3.3 is valid for all \( s \geq 2 \) including \( s = 2 \), cf. Theorem 2.2 and the remark following Theorem 2.4.

4. Application to Pisier’s extrapolation lattices

Fix \( 0 < \theta < 1 \) and \( 1 < q < \infty \). Define \( p, r \) as follows
\[
\frac{1}{p} = 1 - \theta + \frac{\theta}{q} \quad \text{and} \quad \frac{1}{r} = \frac{\theta}{q},
\]
Let \( X \) be a lattice over a measure space \((\Omega, \Sigma, \mu)\) which we assume \( r \)-concave and \( p \)-convex (with constants one). Let \( X_1 = L^q(\Omega, \Sigma, \mu) \). The lattice \( X_0 \subseteq L^q(\Omega, \Sigma, \mu) \) introduced by Pisier in \cite{Pis79a} and \cite{Pis79b} is defined by putting \( x \in X_0 \) iff
\[
\|x\|_{X_0} = \sup \left\{ \|x^{1-\theta}|y|\|^\theta \|y\|_{X_1} : \|y\|_{X_1} \leq 1 \right\} < \infty.
\]
Pisier’s theorem, \cite{Pis79a}, \cite{Pis79b}, asserts that \( X_0 \) is a Banach lattice and
\[
X = (X_0)^{1-\theta}(X_1)^\theta.
\]
The lattice \((X_0)^{1-\theta}(X_1)^\theta\) is called Calderon product of the Banach lattices \( X_0, X_1 \) and is defined as follows (cf. \cite{Cal64}). The lattice \((X_0)^{1-\theta}(X_1)^\theta\) is the space of those functions \( u \in L^q(\Omega, \Sigma, \mu) \) such that \( |u| = |x|^{1-\theta}|y|\) with \( x \in X_0 \) and \( y \in X_1 \) equipped with the norm
\[
\|u\|_{(X_0)^{1-\theta}(X_1)^\theta} = \inf \{ \|x\|_{X_0}^{1-\theta} \|y\|_{X_1}^\theta : |u| = |x|^{1-\theta}|y|\}.
\]
Specifically, (4.2) states that given \( u \in X \) there is \( y \in X_1 \) and \( x \in X_0 \) so that
\[
|u| = |x|^{1-\theta}|y|\quad \text{and} \quad \|x\|_{X_0}^{1-\theta} \|y\|_{X_1}^\theta \leq C \|u\|_X.
\]
The proof in \cite{Pis79a} obtains \( y \in X_1 \) (and hence \( x \in X_0 \)) by a Hahn-Banach argument, as explained in the third paragraph of the introduction. For specific examples of lattices it may however be possible to obtain \( y \in X_1 \) constructively.

These considerations were the stimulus for our work on multiplication operators into Hardy spaces \( H^p \) (real-valued and vector-valued) and into Triebel-Lizorkin spaces. Recall, the Triebel-Lizorkin spaces \( f^p_q \), \( 1 < p < q < \infty \), are \( p \)-convex and \( q \)-concave Banach lattices over the dyadic intervals equipped with the counting measure. Indeed, \( f^p_q \) is \( r \)-concave for all \( r \geq q \). M.Frazier and B.Jawerth showed in \cite{FJ90} that \( f^p_q \simeq (f^1_1)^{1-\theta}(f^q_q)^\theta \), where \( 0 < \theta < 1 \) and \( \frac{1}{p} = 1 - \theta + \frac{\theta}{q} \). Moreover, Pisier’s theorem (\cite{Pis79a} Theorem 2.10, Remark 2.13)) asserts that
\[
f^p_q \simeq (X_0)^{1-\theta}(f^q_q)^\theta,
\]
where \( X_0 \) is the lattice of all elements \( f = (x_t)_{t \in \mathcal{D}} \) for which
\[
\|f\|_{X_0} = \sup \left\{ \left\| \|x_t|^{1-\theta}|y_t|\|_{f^q_q} \right\|_{f^p_q} : y = (y_t)_{t \in \mathcal{D}} \in f^q_q, \|y\|_{f^q_q} \leq 1 \right\} < \infty.
\]
As we pointed out, the factorization of \( u \in f^p_q \) as
\[
|u| = |x|^{1-\theta}|y|, \quad \|x\|_{X_0}^{1-\theta} \|y\|_{f^q_q} \leq C_{p,q} \|u\|_{f^p_q}
\]
uses a Pietsch measure for the multiplication operator
\[
\mathcal{M}_u : \ell^{\infty}(\mathcal{D}) \rightarrow f^p_q, \quad \varphi \mapsto (\varphi \cdot u).
\]
Using our explicit formulas for the Pietsch measure of (4.8), we obtain the decomposition (4.7) constructively without invoking Hahn Banach theorems.
Theorem 4.1. Let $1 < p \leq q < \infty$ and $\theta = \frac{q-1}{q-1/p}$. Let $X_0$ be the Banach lattice defined by (4.7). Let $u = (u_I)_{I \in \mathcal{D}} \in f_q^p$ and let $\omega \in \ell^1(\mathcal{D})$ be the weight defined by (4.10). Then $y \in f_q^p$ and $x \in X_0$ defined by
\[ y = \left( \frac{\omega_I}{|I|} \right)^{\frac{1}{q}}_{I \in \mathcal{D}} \quad \text{and} \quad x = \left( |u_I||y_I|^{-\theta} \right)^{\frac{1}{1-\theta}}_{I \in \mathcal{D}} 
\] satisfy
\begin{align*}
(4.9) \quad |u| &= |x|^{1-\theta} |y|^\theta 
\text{and} \quad ||x||_{X_0}^{1-\theta} ||y||_p^\theta \leq C_{p,q} ||u||_{f_q^p}.
\end{align*}

Proof. We have from Corollary 3.2 that
\[ ||y||_{f_q^p}^q = \sum_{I \in \mathcal{D}} \omega_I \leq 1. \]
Let $z = (z_I)_{I \in \mathcal{D}} \in f_q^p$ with $\|z\| \leq 1$. Since $\frac{p(q-1)}{p-1} \geq q$ we have from Hölder’s inequality
\begin{align*}
(4.10) \quad \left( \sum_{I \in \mathcal{D}} |y_I|^{-\theta} |z_I|^\theta \omega_I \right)^{\frac{1}{1}} &\leq \left( \sum_{I \in \mathcal{D}} |y_I|^{-\frac{p(q-1)}{q-1}} |z_I|^\frac{p(q-1)}{q-1} \omega_I \right)^{\frac{q-1}{q}}.
\end{align*}
Using the definition of $y$ and $\theta$ we get
\begin{align*}
(4.11) \quad \sum_{I \in \mathcal{D}} |y_I|^{-\frac{p(q-1)}{q-1}} |z_I|^\frac{p(q-1)}{q-1} \omega_I &= \sum_{I \in \mathcal{D}} |\omega_I|^{-\frac{p(q-1)}{q-1}} |I|^{\frac{p(q-1)}{q-1}} |\omega_I|^{\frac{p(q-1)}{q-1}}
\quad = \sum_{I \in \mathcal{D}} |z_I|^q |I| = \|z\|^q \leq 1.
\end{align*}
Therefore, combining (4.10) and (4.11)
\begin{align*}
(4.12) \quad \left( \sum_{I \in \mathcal{D}} |y_I|^{-\theta} |z_I|^\theta \omega_I \right)^{\frac{1}{1}} &\leq 1.
\end{align*}
We can apply Corollary 3.2 to the sequence $\varphi_I = |y_I|^{-\theta} |z_I|^\theta$ and get with (4.12)
\begin{align*}
(4.13) \quad \left\| |u||y|^\theta |z|^\theta \right\|_{f_q^p} \leq C_{p,q} ||u||_{f_q^p} \left( \sum_{I \in \mathcal{D}} |y_I|^{-\theta} |z_I|^\theta \omega_I \right)^{\frac{1}{q}}
\quad \leq C_{p,q} ||u||_{f_q^p}.
\end{align*}
Recall that $x = (x_I)_{I \in \mathcal{D}}$ with $x_I = \left( |u_I||y_I|^{-\theta} \right)^{-\frac{1}{1-\theta}}$. Then invoking (4.1), the defining equation for the norm in $X_0$, the estimate (4.13) translates into
\[ ||x||_{X_0}^{1-\theta} \leq C_{p,q} ||u||_{f_q^p}. \]
As $||y||_{f_q^p} \leq 1$ we have
\[ ||x||_{X_0}^{1-\theta} ||y||_p^\theta \leq C_{p,q} ||u||_{f_q^p}. \]
Since for $I \in \mathcal{D}$
\[ |y_I|^\theta |x_I|^{1-\theta} = \left( \frac{\omega_I}{|I|} \right)^{\frac{1}{q}} |u_I| \left( \frac{\omega_I}{|I|} \right)^{-\frac{1}{q}} = |x_I|, \]
we have $|u| = |y|^\theta |x|^{1-\theta}$. \qed
Remark. The uniqueness theorem of Cwickel and Nilsson ([CNS03]) gives the identification of the Banach lattice $X_0$ defined by (4.6) as $f^p_{1q}$. Let $\frac{1}{p} = 1 - \theta + \frac{\theta}{q}$ and $f = (x_I)_{I \in D} \in f^p_{1q}$, then there exists a constant $c$ such that
\begin{equation}
(4.14) \quad c\|f\|_{f^p_{1q}} \leq \sup \left\{ \left\| \frac{1}{1-\theta} |y_I|^\theta \right\|_{f^p_{1q}} : y = (y_I)_{I \in D} \in f^p_{1q}, \|y\|_{f^q_{1q}} \leq 1 \right\} \leq \|f\|_{f^p_{1q}}.
\end{equation}

Our Theorem 4.1 complements the constructive proofs for (4.14) given by [GMP05] and [Bow13]. The common denominator of Theorem 4.1, [GMP05] and [Bow13] is the use of atomic decomposition as starting point for the proof.

5. Appendix

Proof of the left-hand side of inequality (2.16). The left-hand side inequality of (2.16) was stated without proof in [Mül12]. Since we use this inequality repeatedly in this paper we provide the proof here. It uses the ideas of [GM08, Lemma 3.3], who in turn exploit the ideas of [JO74, Theorem 1] and [Joh76, p.336]. For the following definitions and statements we refer to [Mül05] and [GM08].

Let $E \subseteq D$ be a non-empty collection of dyadic intervals. We denote by $E^*$ the set covered by $E$, i.e. $E^* = \bigcup_{I \in E} I$. In the following we define consecutive generations of $E$. We define $G_0(E)$ to be the maximal dyadic intervals of $E$, where maximal refers to inclusion. Note that the maximal intervals of a collection $E$ are pairwise disjoint intervals and that $G_0(E)$ covers the same set as $E$. Suppose that we have already defined the generations $G_0(E), \ldots, G_{n-1}(E)$, then we define $G_n(E) = G_0(E \setminus (G_0(E) \cup \cdots \cup G_{n-1}(E)))$.

Given $I \in D$, let $I \cap E = \{J \in E : J \subseteq I\}$ and put $G_0(I, E) = G_0(I \cap E)$, for $\ell \in \mathbb{N}$.

We fix pairwise disjoint blocks of dyadic intervals $\{C(I) : I \in E\}$ so that (5.1)-(5.3) hold:
\begin{align}
(5.1) \quad [E] = \sup_{I \in E} \frac{1}{|I|} \sum_{J \in E, J \subseteq I} |J| < \infty, \\
(5.2) \quad C(I)^* = I, \\
(5.3) \quad \text{The sigma algebra generated by } \{h_J : J \in C(I)\} \text{ is purely atomic.}
\end{align}

We denote by $B(I)$ the set of atoms.

For every $I \in E$ we have
\begin{equation}
|G_0^I(I, E)| \leq 4 \cdot 2^{-\frac{3\ell}{2^{\ell+1}}} |I|.
\end{equation}

Therefore, by the properties above we get for every $I \in E$ and every $B \in B(I)$
\begin{equation}
|B \cap G_0^I(I, E)| \leq 4 \cdot 2^{-\frac{3\ell}{2^{\ell+1}}} |B|.
\end{equation}

Let $x_J \in X$, $J \in D$ and put for $I \in E$
\begin{equation}
u_I = \sum_{J \in C(I)} x_J h_J.
\end{equation}

Note that by property (5.3) $\nu_I$ is constant on every atom $B \in B(I)$. 

Let $1 \leq p < \infty$. Then for $u = \sum_{I \in \mathcal{E}} u_I$ we claim

$$(5.6) \quad \|u\|_{H^p_X}^p \leq c_p \sum_{I \in \mathcal{E}} \|u_I\|_{H^p_X}^p. $$

In order to prove inequality (5.6) we define for each $I \in \mathcal{E}$ the set

$$A_I = I \setminus \bigcup_{J \in \mathcal{G}_I(I, \mathcal{E})} J. $$

Note that by construction $\{A_I : I \in \mathcal{E}\}$ is a collection of pairwise disjoint and measurable sets such that $\bigcup_{I \in \mathcal{E}} A_I = \mathcal{E}^*$, where $\mathcal{E}^*$ is the set covered by $\mathcal{E}$, cf. [Müll12 Proposition 1]. Therefore,

$$(5.7) \quad \left\| \sum_{I \in \mathcal{E}} u_I \right\|_{H^p_X}^p = \left( \int_0^1 \left\| \int_{A_K} S^p \left( \sum_{I \in \mathcal{E}} u_I \right)(t) dt \right\|_p^p \right)^\frac{1}{p}. $$

By the definition of $A_I$ we get

$$(5.8) \quad \left( \sum_{K \in \mathcal{E}} \int_{A_K} S^p \left( \sum_{I \in \mathcal{E}} u_I \right)(t) dt \right)^\frac{1}{p} = \left( \sum_{K \in \mathcal{E}} \int_{A_K} S^p \left( \sum_{I \in \mathcal{E}} u_I \right)(t) dt \right)^\frac{1}{p}. $$

We know that $\mathcal{G}_0(K, \mathcal{E}) = K$. There exists a shortest dyadic interval $\mathcal{G}_{-1}(K, \mathcal{E}) \in \mathcal{E}$ such that $K$ is strictly contained in $\mathcal{G}_{-1}(K, \mathcal{E})$. Then there exists a shortest dyadic interval $\mathcal{G}_{-2}(K, \mathcal{E}) \in \mathcal{E}$ such that $\mathcal{G}_{-1}(K, \mathcal{E})$ is strictly contained in $\mathcal{G}_{-2}(K, \mathcal{E})$. We continue this pattern $n(K)$ steps until $\mathcal{G}_{-n(K)}(K, \mathcal{E})$ is a maximal interval in $\mathcal{E}$ and therefore not contained in any interval in $\mathcal{E}$. We have

$$K = \mathcal{G}_0(K, \mathcal{E}) \subset \mathcal{G}_{-1}(K, \mathcal{E}) \subset \mathcal{G}_{-2}(K, \mathcal{E}) \subset \cdots \subset \mathcal{G}_{-n(K)}(K, \mathcal{E}). $$

Thus

$$(5.9) \quad \sum_{I \in \mathcal{E}, I \geq K} u_I = \sum_{\ell = 0}^{n(K)} u_{\mathcal{G}_{-\ell}(K, \mathcal{E})}. $$

Summarizing the equations (5.7), (5.8) and (5.9) we have

$$(5.10) \quad \left\| \sum_{I \in \mathcal{E}} u_I \right\|_{H^p_X}^p = \left( \sum_{K \in \mathcal{E}} \int_{A_K} \left( \sum_{\ell = 0}^{\infty} 1_{[0, n(K)]}(\ell) u_{\mathcal{G}_{-\ell}(K, \mathcal{E})} \right)(t) dt \right)^\frac{1}{p}. $$

Put $C_{\ell, K} = C(\mathcal{G}_{-\ell}(K, \mathcal{E}))$, then $u_{\mathcal{G}_{-\ell}(K, \mathcal{E})} = \sum_{J \in C_{\ell, K}} x_j h_J$ and we have

$$(5.11) \quad \left\| \sum_{I \in \mathcal{E}} u_I \right\|_{H^p_X}^p = \left( \sum_{K \in \mathcal{E}} \int_{A_K} \left( \int_0^1 \left\| \sum_{\ell = 0}^{\infty} 1_{[0, n(K)]}(\ell) \sum_{J} x_j h_J(t) r_J(s) \right\|_X^2 ds \right)^\frac{1}{2} dt \right)^\frac{1}{p}. $$

We define the sequence $(a_\ell)_{\ell = 0}^\infty$ of elements in $L^p_{\mathcal{E} \times \mathcal{E}}(A_K \times \mathcal{E}, dt dz)$, where $dz$ is the counting measure on $\mathcal{E}$, as follows: $a_\ell(t, K) = 1_{[0, n(K)]}(\ell) \sum_j x_j h_J(t) r_J$. By the
triangle inequality we get

\[(5.12) \quad \left( \sum_{K \in \mathcal{E}} \int_{A_K} \left\| \sum_{\ell=0}^{\infty} a_{\ell}(t,K) \right\|_{L^2_X}^p dt \right)^{\frac{1}{p}} \leq \sum_{K \in \mathcal{E}} \left( \sum_{\ell=0}^{\infty} \left( \int_{A_K} \left\| a_{\ell}(t,K) \right\|_{L^2_X}^p dt \right) \right) \]
We get the following estimate for the right-hand side in (5.18)

\[
\sum_{B \in \mathcal{B}(I)} S^p(u_I)(B) \leq 4 \cdot 2^{-\frac{2^p}{2^p}} \sum_{K \subseteq B} \mathcal{S}^p(u_I)(B)|B|
\]

\[
= 4 \cdot 2^{-\frac{2^p}{2^p}} \int_0^1 S^p(u_I)(t)dt
\]

\[
= 4 \cdot 2^{-\frac{2^p}{2^p}} \|u_I\|_{H^p_X}^p.
\]

Combining inequalities (5.15), (5.16), (5.17), (5.18) and (5.19) we obtain

\[
\left\| \sum_{I \in \mathcal{E}} u_I \right\|_{H^p_X}^p \leq \left( 1 + 4 \cdot 2^{\frac{p}{p-1}} \sum_{l=1}^{\infty} 2^{-\frac{2^p}{2^p(l+1)}} \left( \sum_{I \in \mathcal{E}} \|u_I\|_{H^p_X}^p \right)^{\frac{1}{p}} \right)^p.
\]

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