Fourier coefficients of restrictions of eigenfunctions

Emmett L. Wyman$^1$, Yakun Xi$^{2,*}$ & Steve Zelditch$^3$

$^1$Department of Mathematics, University of Rochester, Rochester, NY 14627, USA; $^2$School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China; $^3$Department of Mathematics, Northwestern University, Evanston, IL 60208, USA

Email: emmett.wyman@rochester.edu, yakunxi@zju.edu.cn, s-zelditch@northwestern.edu

Received November 4, 2021; accepted August 31, 2022; published online March 22, 2023

Abstract Let $\{e_j\}$ be an orthonormal basis of Laplace eigenfunctions of a compact Riemannian manifold $(M, g)$. Let $H \subset M$ be a submanifold and $\{\psi_k\}$ be an orthonormal basis of Laplace eigenfunctions of $H$ with the induced metric. We obtain joint asymptotics for the Fourier coefficients $\langle \gamma_H e_j, \psi_k \rangle_{L^2(H)} = \int_H e_j \overline{\psi_k} \, dV_H$ of restrictions $\gamma_H e_j$ of $e_j$ to $H$. In particular, we obtain asymptotics for the sums of the norm-squares of the Fourier coefficients over the joint spectrum $\{(\mu_k, \lambda_j)\}_{j,k=0}^{\infty}$ of the (square roots of the) Laplacian $\Delta_M$ on $M$ and the Laplacian $\Delta_H$ on $H$ in a family of suitably ‘thick’ regions in $\mathbb{R}^2$. Thick regions include (1) the truncated cone $\mu_k/\lambda_j \in [a, b] \subset (0, 1)$ and $\lambda_j \leq \lambda$, and (2) the slowly thickening strip $|\mu_k - c\lambda_j| \leq w(\lambda)$ and $\lambda_j \leq \lambda$, where $w(\lambda)$ is monotonic and $1 \ll w(\lambda) \lesssim \lambda^{1/2}$. Key tools for obtaining the asymptotics include the composition calculus of Fourier integral operators and a new multidimensional Tauberian theorem.

Keywords eigenfunctions, period integrals, Kuznecov formula

MSC(2020) 35P20, 35S30, 58J40

Citation: Wyman E L, Xi Y K, Zelditch S. Fourier coefficients of restrictions of eigenfunctions. Sci China Math, 2023, 66: 1849–1878, https://doi.org/10.1007/s11425-021-2034-1

1 Introduction

1.1 Background

Let $(M, g)$ be an $n$-dimensional compact Riemannian manifold without boundary and let $\Delta_M$ denote the Laplace-Beltrami operator with respect to $g$. Let $\{e_j\}_{j=0}^\infty$ be an orthonormal basis of eigenfunctions of $-\Delta_M$ with eigenvalues $\lambda_j^2$ enumerated in the increasing order, $\Delta_M e_j = -\lambda_j^2 e_j$ and $\langle e_j, e_k \rangle = \delta_{jk}$, where the inner product is $\langle f_1, f_2 \rangle_{L^2(M)} = \int_M f_1 \overline{f_2} \, dV_g$ with $dV_g$ the volume form of $g$. Let $H \subset M$ be an embedded, closed submanifold with the dimension $d$, the Riemannian metric $g_H$ and Laplacian $\Delta_H$. Let $\{\psi_k\}_{k=0}^\infty$ be an orthonormal basis of eigenfunctions of $\Delta_H$ with $\Delta_H \psi_k = -\mu_k^2 \psi_k$. Denote by

$$\gamma_H e_j = e_j \big|_H = \sum_k \langle \gamma_H e_j, \psi_k \rangle_{L^2(H)} \psi_k$$  (1.1)
the expansion of the restriction of $e_j$ to $H$ in the basis $\psi_k$. We refer to the inner products
\[ \int e_j \overline{\psi_k} dV_H \] (1.2)
as the Fourier coefficients of $\gamma H e_j$; here, $dV_H$ denotes the volume density on $H$. The purpose of this article is to study the joint asymptotics of the Fourier coefficients (1.2) when the joint spectrum $\{(\lambda_j, \mu_k)\}_{j,k=0}^{\infty}$ falls into a family of suitably ‘thick’ regions.

Ideally, one has sharp bounds on the individual Fourier coefficients (1.2) of a subsequence $\{e_{jk}\}_{k=1}^\infty$ of eigenfunctions with $\frac{\lambda_j}{\mu_k} = c$, as $(M, g, H, c)$ vary over all the compact Riemannian manifolds, submanifolds and eigenvalue ratios. However, except in special cases (see Section 2), it is difficult to extract asymptotic information about a single Fourier coefficient for a subsequence $\{e_{jk}\}_{k=1}^\infty$ of eigenfunctions. Since
\[ \sum_k \left| \int_H e_j \overline{\psi_k} dV_H \right|^2 = \int_H |e_j|^2 dV_H, \]
giving precise information on individual Fourier coefficients is a substantial refinement on giving precise information on the $L^2(H)$-restriction of $e_j$, which itself is difficult to describe for individual eigenfunctions (see [3,15] for such estimates).

In this article, we study the asymptotics of Fourier coefficients in an average sense. We consider the weak-* limit of the measures
\[ \nu = \lim_{\lambda \to \infty} \lambda^{-n} \sum_{j,k: \lambda_j \leq \lambda} \left| \int e_j \overline{\psi_k} dV_H \right|^2 \delta_{\mu_k/\lambda_j} \] (1.3)
in the dual space of the bounded continuous functions on $\mathbb{R}$. We compute the limiting measure by obtaining asymptotics for the sum
\[ \sum_{\mu_k/\lambda_j \in [a,b]} \left| \int e_j \overline{\psi_k} dV_H \right|^2 \] (1.4)
in the style of the third author’s result in [29], where $[a,b]$ is a subinterval of $(0,1)$. We also obtain asymptotics for the ladder sum
\[ \sum_{j,k: \lambda_j \leq \lambda} \left| \int e_j \overline{\psi_k} dV_H \right|^2, \] (1.5)
where the slope $c$ is fixed and lies in the interval $(0,1)$, and the width $w(\lambda)$ of the strip is some monotonic function which slowly grows to infinity with $\lambda$. To obtain both results, we introduce a basic multidimensional Tauberian theorem—Theorem 1.5—which the first and second authors hope to refine in later work.

The motivation for studying Fourier expansions of restricted eigenfunctions originated in the setting of automorphic eigenfunctions on hyperbolic surfaces, in particular the restriction of modular forms to closed geodesics, distance circles and closed horocycles (in the finite-area cusped case). Refer to [1,2,18,20,22] for estimates and to [2,11,16,17] for a more systematic treatment of the topic. The Fourier coefficients in the negatively curved case are expected to be rather uniform in the interval $[0,1]$. This statement is unproved and does not seem to have been formulated precisely before. By comparison, the Fourier coefficients of the standard basis $Y^N_m$ of spherical harmonics of degree $N$ on the sphere $S^2$ are highly non-uniform. On a latitude circle, only the $m$-th Fourier coefficient of $Y^N_m$ is non-zero; its size depends on the relation between the ratio $\frac{N}{2}$ and the latitude (see Subsection 2.1). These observations motivate the question of how the dynamics of the geodesic flows of $(M, g)$ and of $(H, g|_H)$ determine the equidistribution properties of the restricted Fourier coefficients (1.2). When $\psi_k$ is fixed and only $e_j$ varies, the Fourier coefficients are sometimes called ‘periods’ of $e_j$ and were first studied under the name of Kuznecov sum formulae [18] in the general Riemannian context in [29]. While improvements to the remainder bounds of
that article have yet to appear, in the last 10 years there have been numerous improvements to bounds on the corresponding spectral projection operators (see, e.g., [4–6, 24, 25, 27, 28]). Fourier coefficients (1.2), or ‘periods’ in which \( \psi_k \) varies, are sometimes called ‘generalized periods’ (see [28] for some results in the case of closed hypersurfaces).

In the sequel [26], we obtain asymptotics for the refined ladder sum (1.5) when \( |\mu - c\lambda| \leq w \) for constant width \( w \). It turns out that the extremals for the individual term occur only when \( c = 1 \). The case where \( c = 1 \) and \( H \) is a totally geodesic submanifold was studied in [26]. In further work, we also plan to study the case where \( c = 1 \) and \( H \) has the non-degenerate second fundamental form, which involves Airy-type caustic effects.

### 1.2 Statements of results

In what follows, \( M \) and \( H \) will be compact, boundary less Riemannian manifolds of dimensions \( n \) and \( d \), respectively, with the isometric embedding \( H \to M \). We let \( \Delta_M \) and \( \Delta_H \) be the respective Laplace-Beltrami operators and \( e_j \) for \( j = 1, 2, \ldots \) (resp. \( \psi_k \) for \( k = 1, 2, \ldots \)) the corresponding orthonormal eigenbases with \( \Delta_M e_j = -\lambda_j^2 e_j \) and \( \Delta_H \psi_k = -\mu_k^2 \psi_k \). Our main theorem is the following asymptotics for the sum (1.4).

**Theorem 1.1.** Let \([a, b] \subset (0, 1)\). Then

\[
\sum_{\mu_k / \lambda_j \leq \lambda} \left| \int_H e_j \psi_k dV_H \right|^2 = \frac{C_{H,M}}{n} \left( \int_a^b t^{d-1} (1 - t^2)^{\frac{n-d-2}{2}} dt \right) \lambda^n + O_{[a, b]}(\lambda^{n-1}),
\]

where we have a constant \( C_{H,M} = (2\pi)^{-n} (\text{vol} S^{d-1})(\text{vol} S^{n-d-1})(\text{vol} H) \). Moreover, the remainder is uniform for \([a, b] \) contained in a compact subset of \((0, 1)\).

We acknowledge a minor abuse of notation in the theorem above, i.e., the \( e_j \) appearing in the integral denotes its pullback from \( M \) to \( H \) via the embedding \( H \to M \). We repeat this use throughout the article.

As a corollary to this theorem, we obtain a description of the empirical measure \( \nu \) in (1.3).

**Corollary 1.2.** For any bounded, continuous function \( f \) on \( \mathbb{R} \), we have

\[
\lim_{\lambda \to \infty} \lambda^{-n} \sum_{j, k, \lambda_j \leq \lambda} \left| \int e_j \psi_k dV_H \right|^2 f(\mu_k / \lambda_j) = \frac{C_{H,M}}{n} \int_0^1 f(t) t^{d-1} (1 - t^2)^{\frac{n-d-2}{2}} dt,
\]

where the constant \( C_{H,M} \) is the same as in Theorem 1.1. In other words, the limit in (1.3) indeed converges in the weak-* sense in the dual space of the bounded continuous functions, and the limit is

\[
d\nu(t) = \frac{C_{H,M}}{n} t^{d-1} (1 - t^2)^{\frac{n-d-2}{2}} \chi_{[0,1]}(t) dt,
\]

where \( \chi_{[0,1]} \) is the characteristic function of the interval \([0, 1]\).

We also obtain asymptotics for slowly thickening ladder sums as in (1.5).

**Theorem 1.3.** Fix \( c \in (0, 1) \) and let \( w(\lambda) \) be a monotone increasing function of \( \lambda \) for which \( w(\lambda) \to \infty \) and \( w(\lambda) = O(\lambda^{1/2}) \). Then

\[
\sum_{j, k, \lambda_j \leq \lambda} \left| \int e_j \psi_k dV_H \right|^2 = \frac{2C_{H,M}}{n-1} w(\lambda) c^{d-1} (1 - c^2)^{\frac{n-d-2}{2}} \lambda^{n-1} + O_{c,w}(\lambda^{n-1}).
\]

Both Theorems 1.1 and 1.3 follow as corollaries from the next two theorems. The first is an estimate on a smoothed version of the sum (1.4), which we prove in Section 4 using the Fourier integral operator (FIO) theory. The second is our basic multidimensional Tauberian theorem, which we prove in Section 5.
Theorem 1.4. Consider the measure

$$N = \sum_{j,k} \left| \int_{H} e_{j} \overline{v_{j}} dV_{H} \right|^{2} \delta_{(\mu_{j},\lambda_{j})}$$

on $\mathbb{R}^{2}$. Let $\rho$ be a Schwartz-class function on $\mathbb{R}^{2}$. Fix $[a,b] \subset (0,1)$. For $(\mu,\lambda)$ in the cone $\mu/\lambda \in [a,b]$ with $\mu, \lambda > 0$, and for $\tilde{\rho}$ supported in a compact set depending on $H$, $M$ and $[a,b]$, we have asymptotics

$$N \ast \rho(\mu,\lambda) = C_{H,M} \tilde{\rho}(0) \mu^{d-1} \lambda^{n-d-1} \left(1 - \mu^{2}/\lambda^{2}\right)^{\frac{n-d-1}{2}} + O_{\rho,[a,b]}(\lambda^{\alpha-3}).$$

The main term in the asymptotics has a nice geometric interpretation. On the bundle $T_{\mu}M$ of covectors in $M$ over points in $H$, there is a natural volume density $\omega \in |T^{*}M| \otimes |M|^{-1} \otimes |H|$, where we are using the density notation of Duistermaat and Guillemin [9]. Let $p_{H}$ and $p_{M}$ denote the principal symbols of $\sqrt{-\Delta_{H}}$ and $\sqrt{-\Delta_{M}}$, respectively, and let $i : T_{\mu}M \to T^{*}M$ be the inclusion and $\pi : T^{*}M \to T^{*}H$ be the fiberwise projection. Then given $\lambda$ and $\mu$, we have the Lagrangian density $p_{H} \circ i = \lambda$ \cap $\{p_{H} \circ \pi = \mu\}$ given by

$$\omega = \frac{n}{|d(p_{M} \circ i) \wedge d(p_{H} \circ \pi)|}.$$ 

For the sake of illustration, consider the model situation that $H = \mathbb{R}^{d}$ and $M = \mathbb{R}^{n}$, where $H \to M$ is the embedding into $\mathbb{R}^{d} \times 0^{n-d}$, and we take as symbols $p_{H}(x,\xi) = |\xi|$ and $p_{M}(z,\zeta) = |\zeta|$. Then one works out explicitly that $\omega = dx \, d\xi$ and

$$\omega = \frac{1}{\sqrt{1 - \mu^{2}/\lambda^{2}}} \, dx \, d\sigma,$$

where $\sigma$ is the restriction of the Euclidean volume to

$$\{\zeta \in \mathbb{R}^{n} : |\zeta| = \lambda \text{ and } |\pi(\zeta)| = \mu\} = (\mu S^{d-1}) \times (\sqrt{\lambda^{2} - \mu^{2}} S^{n-d-1}).$$

Pretending momentarily that $\text{vol} \, H$ is finite, we see that integrating this volume element yields

$$(\text{vol} \, H)(\text{vol} \, S^{d-1})(\text{vol} \, S^{n-d-1}) \mu^{d-1} \lambda^{n-d-1} \left(1 - \mu^{2}/\lambda^{2}\right)^{\frac{n-d-1}{2}}.$$ 

This accounts for everything but $\tilde{\rho}(0)$ and the dimensional power of $2\pi$ in the main term of Theorem 1.4. Indeed, this volume element will arise in the computation of the principal symbol of the Lagrangian distribution $\tilde{N}$ for the big singularity at the origin.

The Fourier Tauberian theorems are tools for obtaining asymptotics for a monotonically increasing function if we are given some information about (1) its derivative, and if we want finer remainders, (2) the order of the singularities of its Fourier transform away from the origin. For the background on the one-dimensional Fourier Tauberian theorems, we refer the readers to [19, 21] and the references therein.

To state our Tauberian theorem, we borrow the definition of an order function from [30]. Specifically, we say $m$ is an order function on $\mathbb{R}^{d}$ if it is positive and there exist a positive constant $C$ and a power $\nu$ for which

$$m(x) \leq C(1 + |x - y|)^{\nu} m(y) \quad \text{for all } x, y \in \mathbb{R}^{d}.$$

Theorem 1.5 (Basic multidimensional Tauberian theorem). Let $N$ be a tempered, positive Radon measure on $\mathbb{R}^{d}$ and $\rho$ be a nonnegative Schwartz-class function on $\mathbb{R}^{d}$ satisfying $\int_{\mathbb{R}^{d}} \rho(x) \, dx = 1$, and suppose $N \ast \rho(x) \leq m(x)$ for all $x \in \mathbb{R}^{d}$ for an order function $m$. Then for any Borel subset $\Omega$ of $\mathbb{R}^{d}$ with $N(\Omega) < \infty$, we have

$$\left| N(\Omega) - \int_{\Omega} N \ast \rho(x) \, dx \right| \leq C \int_{\partial \Omega} m(x) \, dx,$$

where $\partial [\Omega]$ denotes the unit thickening of the boundary of $\Omega$ and the constant $C$ does not depend on $\Omega$. 
Presently, there does not seem to be a systematic treatment of Fourier Tauberian theorems with functions of more than one parameter in the literature. The closest result the authors could find is due to de Verdière [7], where he obtained something resembling

$$N(\lambda \Omega) = \int_{\lambda \Omega} N * \rho(x)dx + O(\lambda^\nu),$$

where $\lambda \Omega$ denotes a scaling of $\Omega$ by $\lambda$ about the origin, and $\Omega$ is compact and has piecewise $C^1$-boundary among some hypotheses on $N$. Note, while this result is set in $\mathbb{R}^d$, the family of regions is necessarily a homothetic family indexed by a single real parameter. The main insight of Theorem 1.5 is the connection between the remainder and the size of the boundary of $\Omega$. This allows us to obtain asymptotics for the ladders in Theorem 1.3.

The rest of this paper is organized as follows. In Section 2, we directly verify Theorem 1.1 in the case where $\mathcal{H}$ is a coordinate plane in the flat torus. The proofs of Corollary 1.2 and Theorems 1.4 and 1.5 are contained in Sections 3, 4 and 5, respectively. In the proofs of the corollary and the Tauberian theorem, we only use elementary tools. The proof in Section 4 relies on the symbol calculus of cleanly composing FIOs as presented in [9] (see also [14] for a thorough treatment and [8] for the background).

## 2 Examples

In this section, we illustrate the definitions and results with two types of examples: (i) the standard $S^2$, and (ii) flat tori.

In particular, we illustrate the nature of the parameter $c = \frac{\mu}{\lambda}$. The sum in (1.4) is over the joint spectral points $(\mu_k, \lambda_j)$ lying in the set $[a, b]$ of the principal symbols, this set corresponds to the ‘wedge’ or ‘cone’, i.e.,

$$C_{[a,b]} := \{(x, \xi) \in T^*M \setminus 0 : \frac{|\pi_{\mathcal{H}_{\varphi_0}}(x, \xi)|}{|\xi|} = c \in [a, b]\}.$$  \hspace{1cm} (2.1)

Below, we relate (2.1) to wedges (or cones) around rays in the image of the moment map in these two examples. But for general $(M, g, \mathcal{H})$ without symmetry, the wedge (2.1) does not have such an interpretation.

### 2.1 Restrictions to curves in $S^2$

Let $S^2$ be the standard sphere. We illustrate the definitions for the standard basis $Y_{\ell}^m$ of spherical harmonics of degree $\ell$ in the case where $\mathcal{H}$ is a latitude circle (an orbit of the rotational action around the third axis).

Let $\frac{\partial}{\partial \varphi}$ generate rotations around the $x_3$ axis in $\mathbb{R}^3$, and $(\theta, \varphi)$ be the standard spherical coordinates. A latitude circle is a level set of the azimuthal coordinate $H_{\varphi_0} : \{\varphi = \varphi_0\}$ and the equator is the special case $\varphi_0 = \pi/2$. Since rotations commute with the geodesic flow, the Clairaut integral

$$p_\theta(x, \xi) = \left\langle \xi, \frac{\partial}{\partial \theta} \right\rangle = \left| \frac{\partial}{\partial \theta} \right|_{H_{\varphi_0}} \cos \angle \left( \frac{\partial}{\partial \theta}, \hat{\gamma}_{x, \xi}(0) \right), \hspace{1cm} (x, \xi) \in T^*_x S^2$$

is a constant of the motion, i.e., the components of the moment map $P := (|\xi|, p_\theta) : T^*S^2 \to \mathbb{R}^2$ Poisson commute. Let

$$u_\theta(\theta, \varphi) := \left| \frac{\partial}{\partial \theta} \right|_{H_{\varphi_0}}^{-1} \frac{\partial}{\partial \theta}$$

be the unit vector field in the direction of $\frac{\partial}{\partial \theta}$ and let $\frac{\partial}{\partial \varphi}$ be the unit vector field tangent to the meridians. Let $u_\theta^*$ and $u_\varphi^*$ be the dual unit coframe fields. The orthogonal projection $T_{H_{\varphi_0}} S^2 \to T^*_x H_{\varphi_0}$ is given by
\[ \pi_{H_{\phi_0}}(x, \xi) = \langle \xi, u_{\phi_0} \rangle u_{\phi_0}^* \]  
Thus, if we fix \( H_{\phi_0} \) and \( c \in [0, 1] \), the corresponding slice of the cone \( C_{[a, b]} \) is given by
\[ T^*_{H_{\phi_0}} := \left\{ (x, \xi) \in T^*H_{\phi_0} : \frac{|\pi_{H_{\phi_0}}(x, \xi)|}{|\xi|} = c \Leftrightarrow \frac{|p_\theta(x, \xi)|}{|\xi|} = \epsilon \left| \frac{\partial}{\partial \theta} \right|_{H_{\phi_0}} \right\}. \]  
(2.2)

In particular if \( c = 1 \), then \( (x, \xi) \in T^*_{H_{\phi_0}} \Leftrightarrow \xi = |\xi|u_{\phi_0}^* \). As (2.2) shows, the parameter \( c \) is not the usual ratio \( \frac{p_\theta(x, \xi)}{|\xi|} \) of the components of the moment map, because we choose the operator on \( H \) to be \( \sqrt{\Delta_H} \) rather than \( \frac{\partial}{\partial \theta} \).

### 2.1.1 Spectral theory

Let \( Y^m_\ell \) be the standard orthonormal basis of joint eigenfunctions of \( \Delta_{H^0} \) and the generator of rotations around the third axis. Thus, \( Y^m_\ell \) changes by the phase \( e^{im\theta} \) under a rotation of angle \( \theta \). The orthonormal eigenfunctions of \( H_{\phi_0} \) are given by \( \psi_m(\theta) = C_{\psi_0} e^{im\theta} \), where \( C_{\psi_0} = \frac{1}{\sqrt{\Delta_{H_{\phi_0}}}} \). Hence, the Fourier coefficients (1.2) are constant multiples of the Fourier coefficients relative to \( \{ e^{im\theta} \} \). It follows that the \( m \)-th Fourier coefficient of \( Y^m_\ell \) is its only non-zero Fourier coefficient along any latitude circle \( H_{\phi_0} \), and \( \int_{H_{\phi_0}} Y^m_\ell e^{-im\theta} d\theta \) is \( \| Y^m_\ell \|_{H_{\phi_0}}^2 \). This is an example where one can obtain estimates on individual Fourier coefficients of individual eigenfunctions. The situation is much more complicated on higher-dimensional spheres \( S^n \) when \( H \) is a latitude ‘sub-sphere’ \( S^d \) (see [26]).

### 2.1.2 Ladders and cones

Let \( \Lambda_a = \mathcal{P}^{-1}(a, 1) \subset S^*S^2 \) be the level set \{ \( p_\theta = a \) \}. It is a Lagrangian torus when \( a \neq \pm 1 \) and is the equatorial (phase space) geodesic when \( a = \pm 1 \). A ray or ladder in the image of the moment map \( \mathcal{P} \) is defined by \( \{ (m, E) : \frac{m}{E} = a \} \subset \mathbb{R}^2_+ \), and its inverse image under \( \mathcal{P} \) is \( \mathbb{R}_+ \Lambda_a \subset T^*S^2 \).

When \( H \) is a latitude circle, then the wedge (2.1) is a wedge around a ray in the image of the moment map, since the rays
\[ \left\{ \frac{p_\theta(x, \xi)}{|\xi|} = a \right\} \quad \text{and} \quad \left\{ \frac{|\pi_{H_{\phi_0}}(x, \xi)|}{|\xi|} = c \right\} \]
are related by the constant \( |\frac{\partial}{\partial \theta}| \).

On the quantum level, a ray corresponds to a ‘ladder’ \( \{ Y^m_\ell \}_{\ell = a} \) of eigenfunctions. The possible Weyl-Kuznecov sum formulae for latitude circles \( H = H_{\phi_0} \) thus depend on the two parameters \( (\phi_0, \frac{m}{E}) \). The first corresponds to a latitude circle, and the second to a ladder in the joint spectrum. It is better to parametrize the ladder as \( \frac{\ell}{E} = c \) as discussed above.

### 2.1.3 Caustic sequences and Gaussian beams

There are special scenarios where the ladder of eigenfunctions corresponds to the Lagrangian torus \( \{ p_\theta = a \xi \} \) = \( \Lambda_a \) in \( T^*S^2 \) and \( H_{\phi_0} \) is the caustic of this Lagrangian torus, i.e., a boundary component of its projection. In this case, the Fourier coefficients of the subsequence of \( \{ Y^m_\ell \} \) blow up at the rate \( \ell^{1/6} \). This case is outside the scope of this article because the corresponding value of \( c \) equals 1. It will be addressed in a later article.

Another extremal scenario is when \( a = \pm 1 \), i.e., the classical ray occurs on the boundary of the moment map image. The corresponding ladder of eigenfunctions consists of the Gaussian beams, \( C_0 N^2 |x_1 + ix_2|^2 \), around the equator \( \gamma \). In Fermi-normal coordinates, they have the form \( N^{(1/4)} e^{iN s} e^{-Ny^2/2} \), where \( s \) is arc-length along \( \gamma \) and \( y \) is the normal coordinate. This ladder again corresponds to \( c = 1 \) and is outside the scope of this article; general examples with \( c = 1 \) and \( H \) totally geodesic are described in [26].

### 2.1.4 \( H \) is a closed geodesic of \( S^2 \) and \( c < 1 \)

Suppose that \( 0 < c < 1 \) and \( H \) is a closed geodesic. If \( H \) is the equator, then it lies in the interior of the image of the projection of the torus \( \Lambda_a \) and the unique non-zero Fourier coefficient occurs for the spherical harmonics \( Y^m_\ell \) with \( \frac{\ell}{E} \simeq c < 1 \) uniformly bounded above.
On the other hand, one might restrict $Y_l^m$ to a meridian geodesic, in which case all the Fourier coefficients in the range $[-\ell, \ell]$ can be non-zero. This is a $c < 1$ case to which our results apply. Note that when $m = 0$, the Fourier coefficients are those of the Legendre function $P_l(\cos \varphi)$.

2.1.5 The convex surface of revolution in $\mathbb{R}^3$

All of the above remarks can be generalized to any convex surface of revolution with the equator defined as the unique rotationally invariant geodesic. There exist zonal eigenfunctions and Gaussian beams along equators of general convex surfaces of revolution, so the orders of magnitude and the eigenfunctions are of the same type. We refer to [10] for a recent study of how the restricted $L^2$ norms vary with $c$.

2.2 An example on the torus

Here, we verify Theorem 1.1 for an easy example on the torus. This is in part to check the constant in Theorem 1.4 with a direct computation. Though we are careful to track all of the dimensional constants in the computations, we find it prudent to verify the result directly.

Let $M = \mathbb{T}^n = \mathbb{R}^n/2\pi \mathbb{Z}^n$ and $H = \mathbb{T}^d$ be embedded in $M$ as the coordinate plane $T^d \times 0 \subset M$. We select the standard bases of exponential functions

$$e_j(z) = (2\pi)^{-n/2}e^{i(x,j)}$$

and

$$\psi_k(x) = (2\pi)^{-d/2}e^{i(x,k)}$$

indexed by $j \in \mathbb{Z}^n$ and $k \in \mathbb{Z}^d$, respectively. Note

$$\int_H e_j \overline{\psi}_k dV_H = (2\pi)^{-(n+d)/2} \int_{\mathbb{T}^d} e^{i(x,j'-k)} dx$$

where $j' = (j_1, \ldots, j_d)$ is the first $d$ coordinates of $j$. The sum in Theorem 1.1 is then

$$(2\pi)^{-(n-d)} \# \{ j \in \mathbb{Z}^n : |j| \leq \lambda, |j'|/|j| \in [a,b] \}.$$

Proposition 2.1. Let $M = \mathbb{T}^n$ and $H = \mathbb{T}^d \times 0$ be a coordinate plane as above. If $[a,b] \subset (0,1)$, the sum in Theorem 1.1 is

$$(2\pi)^{-(n-d)} \frac{\text{vol} S^{d-1} \text{vol} S^{n-d-1}}{n} \left( \int_a^b t^{d-1}(1-t^2)^{\frac{n+d-2}{2}} dt \right) \lambda^n + O(\lambda^{n-1})$$

by direct computations. We recall vol $H = (2\pi)^d$ and see this agrees with Theorem 1.1.

Proof. By counting cubes, we know that the sum is

$$(2\pi)^{-(n-d)} |\{ \xi \in \mathbb{R}^n : |\xi| \leq \lambda, |\xi'|/|\xi| \in [a,b] \}| + O(\lambda^{n-1}),$$

where the absolute value notation around the set denotes the Lebesgue measure in $\mathbb{R}^n$. We parametrize this set by the map

$$\Phi(r,t,\omega,\eta) = (rt\omega, r\sqrt{1-t^2}\eta),$$

where $r \in [0,\lambda], t \in [a,b], \omega \in S^{d-1}$ and $\eta \in S^{n-d-1}$. The pullback of the Euclidean metric has the form

$$g(r,t,\omega,\eta) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{r^2} & 0 & 0 \\
0 & 0 & r^2t^2 g_{S^{d-1}}(\omega) & 0 \\
0 & 0 & 0 & r^2(1-t^2) g_{S^{n-d-1}}(\eta)
\end{bmatrix},$$

and hence the pullback of the Euclidean volume density is

$$|\det g(r,t,\omega,\eta)|^{1/2} = r^{n-1} t^{d-1}(1-t^2)^{\frac{n-d-2}{2}} |\det g_{S^{d-1}}(\omega)|^{1/2} |\det g_{S^{n-d-1}}(\eta)|^{1/2}.$$
Integrating yields

\[ |\{ \xi \in \mathbb{R}^n : |\xi| \leq \lambda, |\xi'|/|\xi| \in [a, b] \}| = \frac{(\text{vol } S^{d-1})(\text{vol } S^{n-d-1})}{n} \lambda^n \int_a^b t^{d-1}(1 - t^2)^{n-d-2} dt. \]

The proposition follows.

A similar computation can be carried out to verify Theorem 1.3 on the torus, in which we sum the norm-squares of the Fourier coefficients over the thickening strip

\[ \|k - c/|\xi| \| \leq w(\lambda), \quad |j| \leq \lambda \]

with slope \( c \) and width \( w(\lambda) \).

More importantly, this example shows that we cannot obtain asymptotics for a sum over a strip of constant width \( w \), at least not without adding some more hypotheses. By the reduction before Proposition 2.1, this ladder sum is exactly

\[ (2\pi)^{-(n-d)}\# \{ j \in \mathbb{Z}^n : |j| \leq \lambda, |\xi'| - c|j| \leq w \}. \]

The set of \( \xi \in \mathbb{R}^n \) with \( |\xi'| - c|\xi| \leq w \) is asymptotic to a \( w\sqrt{1 - c^2} \)-thickening of the cone \( |\xi'| = c|\xi| \) of 'slope' \( c/\sqrt{1 - c^2} \).

In the case where \( n = 2 \) and \( d = 1 \), \( c \) may be taken so that the slope \( c/\sqrt{1 - c^2} \) is rational and \( w \) may be taken small enough to only include those lattice points in \( \mathbb{Z}^2 \) lying along the line of slope \( c/\sqrt{1 - c^2} \). Any small, non-zero change in \( w \) will not perturb the ladder sum of Theorem 1.3, yet it would be felt by the main term

\[ \frac{1}{\pi} \frac{w}{\sqrt{1 - c^2}} \lambda. \]

We conclude the remainder must be just as large as the main term. It is also possible, using a more careful computation, to locate jumps in this ladder sum of order \( \lambda \) given a change in \( w \) on the order of \( \lambda^{-1} \).

3 Proofs of Theorems 1.1 and 1.3 and Corollary 1.2

3.1 Proofs of Theorems 1.1 and 1.3

We begin by proving Theorem 1.1. We take the cone

\[ \Omega_\lambda = \{ (\mu', \lambda') \in \mathbb{R}^2 : \mu', \lambda' > 0, \lambda' \leq \lambda \text{ and } \mu'/\lambda' \in [a, b] \} \]

and realize that the sum of Theorem 1.1 is precisely \( N(\Omega_\lambda) \) with \( N \) from Theorem 1.4. Take a smooth cutoff \( \chi \in C^\infty(\mathbb{R}, [0, 1]) \) which takes the value 1 on a neighborhood of \([a, b]\) and the value 0 on a neighborhood of the complement of \((0, 1)\), and set

\[ \tilde{N} = \sum_{j, k} \left| \int_{H} e_j \overline{\psi_k} dV_H \right|^2 \chi(\mu_k/\lambda_j) \delta(\mu_k, \lambda_j). \quad (3.1) \]

Note \( N(\Omega_\lambda) = \tilde{N}(\Omega_\lambda) \), and hence we only need to show that \( \tilde{N}(\Omega_\lambda) \) satisfies the asymptotics of Theorem 1.1.

Fix a nonnegative Schwartz-class function \( \rho \) with sufficiently small Fourier support and \( \hat{\rho}(0) = 1 \). By construction,

\[ |(N - \tilde{N}) \ast \rho(\mu, \lambda)| = O_{\rho, [a, b]}(\lambda^{-\infty}) \text{ for } \mu, \lambda > 0 \text{ and } \mu/\lambda \in [a, b]. \]

This and Theorem 1.4 yield that \( \tilde{N} \ast \rho(\mu, \lambda) \) is bounded by some constant times the order function

\[ (1 + |(\mu, \lambda)|)^{n-2}. \]

Theorem 1.1 then follows from Theorem 1.5, the asymptotics in Theorem 1.4 and an elementary computation.
Next, we prove Theorem 1.3. Fix \([a, b] \subset (0, 1)\) such that \(a < c < b\) and consider the strip

\[ S_\lambda = \{ (\mu', \lambda') \in \mathbb{R}^2 : 0 < \lambda' \leq \lambda \text{ and } |\mu' - c\lambda'| \leq w(\lambda) \}. \]

Let \(W_r = \{ (\mu', \lambda') \in \mathbb{R}^2 : 0 \leq \lambda' \leq r \}\) denote the strip of width \(r\) along the first axis, and fix a constant \(C\) for which

\[ S_\lambda \setminus W_{Cw(\lambda)} \subset \{ (\mu', \lambda') : \mu'/\lambda' \in [a, b] \}. \]

We have that by the basis property of \(\{ \psi_k \}\), the local Weyl law and the hypothesis \(w(\lambda) = O(\lambda^{1/2})\),

\[ N(S_\lambda \cap W_{Cw(\lambda)}) = \sum_{j,k : \lambda_j \leq Cw(\lambda)} \left| \int_H e_j \overline{\psi}_k dV_H \right|^2 \leq \sum_{j,k : \lambda_j \leq Cw(\lambda)} \int_H |e_j|^2 dV_H = O(\lambda^{n-1}). \]

All that is left is to show that \(N(S_\lambda \setminus W_{Cw(\lambda)})\) satisfies the asymptotics of Theorem 1.3, i.e.,

\[ N(S_\lambda \setminus W_{Cw(\lambda)}) = C_{H,M} c^{d-1}(1 - c^2)^{\frac{d-2}{2}} w(\lambda)\lambda^{n-1} + O(\lambda^{n-1}), \]

where we allow the constants implicit in the big-\(O\) remainder to depend on \(c\) and \(w\). We replace \(N\) with \(\tilde{N}\) as in (3.1) and have

\[ N(S_\lambda \setminus W_{Cw(\lambda)}) = \tilde{N}(S_\lambda \setminus W_{Cw(\lambda)}) \]

by construction. Let \(\rho\) be as before so that \(\tilde{N} * \rho\) is again bounded by the same order function. Since \(w(\lambda) \leq \lambda\) for \(\lambda\) large,

\[ \int_{\partial^{(-1)}[S_\lambda \setminus W_{Cw(\lambda)}]} \rho * \tilde{N}(\mu', \lambda') d(\mu', \lambda') = O(\lambda^{n-1}). \]

By Theorem 1.5,

\[ \tilde{N}(S_\lambda \setminus W_{Cw(\lambda)}) = \int_{S_\lambda \setminus W_{Cw(\lambda)}} \rho * \tilde{N}(\mu', \lambda') d(\mu', \lambda') + O(\lambda^{n-1}) \]

\[ = \int_{Cw(\lambda)}^{\lambda} \int_{-w(\lambda)}^{w(\lambda)} \rho * \tilde{N}(c\lambda' + t, \lambda') dt d\lambda' + O(\lambda^{n-1}). \]

By Theorem 1.4, the mean value theorem and our hypotheses on the growth of \(w(\lambda)\), we have

\[ \rho * \tilde{N}(c\lambda' + t, \lambda') = \rho * \tilde{N}(c\lambda', \lambda') + O(w(\lambda)\lambda^{n-3}) \quad \text{for all } |t| \leq w(\lambda). \]

Continuing with our estimates, we have that by \(w(\lambda) = o(\lambda)\),

\[ \int_{Cw(\lambda)}^{\lambda} \int_{-w(\lambda)}^{w(\lambda)} \rho * \tilde{N}(c\lambda' + t, \lambda') dt d\lambda' = 2w(\lambda) \int_{0}^{\lambda} \rho * \tilde{N}(c\lambda', \lambda') d\lambda' + O(\lambda^{n-1} + w(\lambda)^2 \lambda^{n-2}). \]

By our hypothesis, \(O(\lambda^{n-1} + w(\lambda)^2 \lambda^{n-2}) = O(\lambda^{n-1})\), and hence the right-hand side reads

\[ 2C_{H,M} w(\lambda)c^{d-1}(1 - c^2)^{\frac{d-2}{2}} \int_{0}^{\lambda} \lambda^{n-2} d\lambda' + O(\lambda^{n-1}) \]

\[ = \frac{2C_{H,M}}{n-1} w(\lambda) c^{d-1}(1 - c^2)^{\frac{d-2}{2}} \lambda^{n-1} + O(\lambda^{n-1}) \]

by Theorem 1.4. This concludes the proof of Theorem 1.3.

### 3.2 The proof of Corollary 1.2

For each \(j\), set

\[ \nu_j = \sum_k \left| \int_H e_j \overline{\psi}_k dV_H \right|^2 \delta_{\mu_k/\lambda_j}. \]
so that we write \( \nu = \lim_{\lambda \to \infty} \lambda^{-n} \sum_{\lambda_j \leq \lambda} \nu_j \). Let \( f \) be a continuous function with support contained in \((0, 1)\). Approximating \( f \) above and below by step functions and applying Theorem 1.1 to each constant component, we obtain

\[
\lim_{\lambda \to \infty} \frac{n}{C_H,M \lambda^n} \sum_{\lambda_j \leq \lambda} \int f d\nu_j = \int_0^1 f(t) t^{d-1} (1 - t^2)^{\frac{n-d-2}{2}} dt.
\]

We now argue that this limit holds for bounded continuous functions on all of \( \mathbb{R} \).

For \( \delta > 0 \), let \( \chi_\delta \) be a continuous cutoff function which takes values in \([0, 1]\), which is supported in \((0, 1)\) and satisfies \( \chi_\delta = 1 \) on \( [\delta, 1 - \delta] \). For any bounded continuous function \( f \) on \( \mathbb{R} \), we have

\[
\frac{n}{C_H,M \lambda^n} \sum_{\lambda_j \leq \lambda} \int f \chi_\delta d\nu_j = \int_0^1 f(t) \chi_\delta(t) t^{d-1} (1 - t^2)^{\frac{n-d-2}{2}} dt + o_\delta(1).
\]

We select \( \delta(\lambda) \) decreasing to 0 as \( \lambda \to \infty \) slowly enough so that

\[
\lim_{\lambda \to \infty} \frac{n}{C_H,M \lambda^n} \sum_{\lambda_j \leq \lambda} \int f(1 - \chi_{\delta(\lambda)}) d\nu_j
\]

vanishes, after perhaps taking \( \delta(\lambda) \to 0 \) more slowly. By the triangle inequality, we have

\[
\left| \int f(1 - \chi_{\delta(\lambda)}) d\nu_j \right| \leq \|f\|_{L^\infty(\mathbb{R})} \int (1 - \chi_{[\delta(\lambda), 1 - \delta(\lambda)]}) d\nu_j,
\]

where \( \chi_{[\delta(\lambda), 1 - \delta(\lambda)]} \) denotes the characteristic function of the interval \([\delta(\lambda), 1 - \delta(\lambda)]\). The following lemma concludes the proof of Corollary 1.2.

**Lemma 3.1.** Fix \( \delta > 0 \) and let \( \chi_{[\delta, 1 - \delta]} \) denote the characteristic function of the interval \([\delta, 1 - \delta]\). Then

\[
\frac{n}{C_H,M \lambda^n} \sum_{\lambda_j \leq \lambda} \int (1 - \chi_{[\delta, 1 - \delta]}) d\nu_j \lesssim \delta^{1/2} + C_\delta \lambda^{-1},
\]

where the constant implicit in the \( \lesssim \) notation only depends on \( H \) and \( M \), and \( C_\delta \) only depends on \( H \), \( M \) and \( \delta \).

**Proof.** Since \( \psi_k \)'s form an orthonormal basis for \( L^2(H) \),

\[
\int 1 d\nu_j = \sum_k \left| \int_H e_j \overline{\psi}_k dV_H \right|^2 = \int_H |e_j|^2 dV_H.
\]

Hence by the pointwise Weyl law [14, Theorem 29.1.4],

\[
\sum_{\lambda_j \leq \lambda} \int 1 d\nu_j = (2\pi)^{-n} (\text{vol } H) (\text{vol } S^{n-1}) \lambda^n + O(\lambda^{n-1}),
\]

so

\[
\frac{n}{C_H,M \lambda^n} \sum_{\lambda_j \leq \lambda} \int 1 d\nu_j = \frac{\text{vol } S^{n-1}}{(\text{vol } S^{d-1})(\text{vol } S^{n-d-1})} + O(\lambda^{-1})
\]

\[
= \int_0^1 t^{d-1} (1 - t^2)^{\frac{n-d-2}{2}} dt + O(\lambda^{-1}),
\]
where the second line follows from a similar computation as in the proof of Proposition 2.1. Theorem 1.1 yields

\[
\frac{n}{C_{H,M}\lambda^n} \sum_{\lambda_j \leq \lambda} \int (1 - \chi_{[\delta_1 - \delta, \delta])} dv_j
\]

\[
= \int_0^\delta t^{d-1}(1 - t^2)^{\frac{n-d-2}{2}} dt + \int_{1-\delta}^1 t^{d-1}(1 - t^2)^{\frac{n-d-2}{2}} dt + O_\delta(\lambda^{-1}).
\]

The lemma follows from \(1 \leq d \leq n - 1\) and an elementary estimate.

4 The proof of Theorem 1.4

4.1 The setup

Set \(P_M = \sqrt{-\Delta_M}\) and \(P_H = \sqrt{-\Delta_H}\). These are first-order, self-adjoint, elliptic pseudo-differential operators on their respective manifolds with principal symbols

\[
p_M(z, \zeta) = \left( \sum_{i,j} g_M^i(z) \xi_i \xi_j \right)^{1/2} \quad \text{and} \quad p_H(x, \xi) = \left( \sum_{i,j} g_H^i(x) \xi_i \xi_j \right)^{1/2}.
\]

Here, \(g_M\) and \(g_H\) are the Riemannian metric tensors on \(M\) and \(H\), respectively.

Now

\[
N = \sum_{j,k} \left| \int_H e_j \bar{\psi}_k dv_H \right|^2 \delta_{(\mu_k, \lambda_j)},
\]

which is a joint spectral measure of the operators \(P_H \otimes I\) and \(I \otimes P_M\) on \(H \times M\) weighted by the norm-squared Fourier coefficients. We want to rewrite these weights using half-densities so that we can use the theory of FIOs. We change notation so that \(\psi_k\) and \(e_j\) are instead the eigendensities

\[
\psi_k = \tilde{\psi}_k |dV_H|^{1/2} \quad \text{and} \quad e_j = \tilde{e}_j |dV_M|^{1/2},
\]

where \(\tilde{\psi}_k\) and \(\tilde{e}_j\) are the corresponding eigenfunctions. Let \(i : H \to M\) be the embedding and \(\delta_i\) be the half-density distribution in \(H \times M\) for which

\[
(\delta_i, f) = \int_H \tilde{f}(x, i(x)) dv_H \quad (4.1)
\]

for smooth test half-densities \(f = \tilde{f} |dV_H |dV_M|^{1/2}\) on \(H \times M\). Then we write the Fourier coefficient as

\[
\int_H \tilde{e}_j \bar{\psi}_k dv_H = (\delta_i, \tilde{\psi}_k \otimes e_j).
\]

For technical reasons, we need to insert a pseudo-differential cutoff. Thankfully, we can do this at minimal cost.

Lemma 4.1. Let \(\chi\) be a smooth function on \(\mathbb{R}\) taking values in \([0, 1]\) for which \(\chi = 1\) on a neighborhood of \([a, b]\) and \(\supp \chi \subseteq (0, 1)\). The operator \(B\) acting on distributions over \(H \times M\) defined spectrally by \(B(\tilde{\psi}_k \otimes e_j) = \chi(\mu_k/\lambda_j)\tilde{\psi}_k \otimes e_j\)

\(^{1}\)

is a real, self-adjoint, zeroth-order pseudo-differential operator with the principal symbol

\[
\chi(p_H(x, \xi)/p_M(z, \zeta)) \quad \text{for} \quad (x, \xi, z, \zeta) \in T^*(H \times M)
\]

and essential support in

\[
\{(x, \xi, z, \zeta) : p_H(x, \xi)/p_M(z, \zeta) \in \supp \chi\}.
\]

\(^{1}\) We take the convention that \(B(\tilde{\psi}_k \otimes e_j) = 0\) whenever \(\lambda_j = 0\).
We defer the proof of the lemma until after our reduction, but not without a few words first. The operator $B$ is equal to $\chi(\frac{P_H}{PM})$. Here, $\frac{P_H}{PM}$ is not a pseudo-differential operator but $\chi(\frac{P_H}{PM})$ is a zeroth-order pseudo-differential operator due to the properties of the cutoff.

To put this statement into context, we recall the notion of polyhomogeneous symbols $S^\alpha(\Gamma)$ relative to a choice of a conic open subset $\Gamma \subset T^*M - \{0\}$ (see [14, Volume 3, p.83]). Namely, the standard symbol estimates of [14, (28.1.1)'] are only assumed to be valid in $\Gamma$. This notion is developed more systematically on manifolds in [13, p.86].

Formally, the principal symbol of the quotient operator $\frac{P_H}{PM}$ is $\frac{p_H}{PM}$. This is not a symbol on $T^*(M \times H) - \{0\}$, and indeed it is not even defined on the sub-cone $0M \times T^*H - \{0\}$. However, it is a symbol on the cone $\Gamma \subset T^*(M \times H)\backslash 0$, where $p_H(x,\zeta)/p_M(z,\eta) \in \text{supp} \chi$. Note that if $\text{supp} \chi = [a,b]$, then $p_H(x,\zeta)/p_M(z,\eta) \in \text{supp} \chi$ if and only if $ap_M(z,\eta) \leq p_H(x,\zeta) \leq bp_M(z,\eta)$. We may further assume that $p_H(x,\zeta) + p_M(z,\eta) \geq \delta > 0$ for some $\delta > 0$, since the cutoff of $\chi(\frac{P_H}{PM})$ to the complement is a smoothing operator. Then at least one of $p_H(x,\zeta), p_M(z,\eta)$ is greater than or equal to $\delta/2$, so both symbols are uniformly bounded above zero. Then the symbol $q(y,\eta,x,\xi) = \chi(p_H(x,\zeta)/p_M(z,\eta))$ is homogeneous of degree 0 and elliptic in $\Gamma$. Moreover, it is a smoothing operator on the complement of $\Gamma$.

We set

$$N_B = \sum_{j,k} \chi^2(\mu_k/\lambda_j)(\delta_{ij}, \psi_k \otimes e_j)^2 \delta(\mu_k, \lambda_j) = \sum_{j,k} |(\delta_{ij}, B(\psi_k \otimes e_j))|^2 \delta(\mu_k, \lambda_j).$$

Similarly as in the proof of Theorem 1.1,

$$|\rho * N(\mu, \lambda) - \rho * N_B(\mu, \lambda)| = O(\lambda^{-\infty}) \text{ for } \mu/\lambda \in [a,b],$$

so we may freely exchange $N$ for $N_B$ in the statement of the theorem.

Similar to (4.1), let $\delta_{ixi}$ be the half-density on $H \times H \times M \times M$ given by

$$(\delta_{ixi}, f) = \int_H \int_H \hat{f}(x,y, i(x), i(y))dV_H(x)dV_H(y)$$

for smooth test half-densities $f = \hat{f}|dV_HdV_HdV_MdV_M|^{1/2}$ on $H \times H \times M \times M$. Then we have

$$|(\delta_{ij}, B(\psi_k \otimes e_j))|^2 = \chi(\mu_k/\lambda_j)^2 |(\delta_{ij}, \psi_k \otimes e_j)^2
\begin{align*}
&= \chi(\mu_k/\lambda_j)^2(\delta_{ij} \otimes \delta_{ij}, \psi_k \otimes e_j \otimes \psi_k \otimes e_j) \\
&= (\delta_{ixi}, A(\psi_k \otimes \psi_k \otimes e_j \otimes \tau_j)) \\
&= (A\delta_{ixi}, \omega_k \otimes \psi_k \otimes e_j \otimes \tau_j),
\end{align*}$$

(4.3)

where $A$ is a pseudo-differential operator on $H^2 \times M^2$ defined spectrally by

$$A(\psi_k \otimes \psi_k \otimes e_j \otimes \tau_j) = \chi(\mu_k/\lambda_j)(\psi_k \mu_k/\lambda_j)\beta(\mu_k/\lambda_j),$$

(4.4)

where $\beta \in C^\infty(\mathbb{R}, [0,1])$ is identically 1 on a neighborhood of 1 and has support in $(1/2, 2)$. Similar to $B$, $A$ is a real, zeroth-order, self-adjoint pseudo-differential operator. The $\beta$ cutoffs are there to ensure that the symbol of $A$ is smooth near the axes. Note that the third equality in (4.3) holds since $j' = j$ and $k' = k$.

Using (4.3) and a Fourier transform, we have

$$\hat{N}_B(s,t) = \sum_{j,k} (A\delta_{ixi}, \omega_k \otimes \psi_k \otimes e_j \otimes \tau_j) e^{-it(\mu_k + \lambda_j)}
\begin{align*}
&= (A\delta_{ixi}, e^{is\mu_k} \otimes e^{-it\lambda_j}),
\end{align*}$$

interpreted in a distributional sense. We let

$$U(s, t, x, y, z, w) = e^{is\mu_k(x, y)}e^{-it\lambda_j(z, w)}$$
be the half-density distribution kernel of the tensored half-wave operators, and by an abuse of notation, we let $U$ denote the operator with the kernel above taking smooth half-densities on $H^2 \times M^2$ to half-density distributions on $\mathbb{R}^2$. Then we have

$$\hat{\mathcal{N}}_B|dsdt|^{1/2} = U \circ A \circ \delta_{1\times i}.$$  

**Proof of Lemma 4.1.** That $B$ is real and self-adjoint is clear from its definition. Furthermore, we may remove the complex conjugate over $\psi_k$ and write

$$B(\psi_k \otimes e_j) = \chi(\mu_k/\lambda_j)\psi_k \otimes e_j.$$  

This will slightly simplify the calculations to come. We must verify that it is a pseudo-differential operator with the indicated symbol and essential support. To this end, we write $B$ locally up to lower-order terms. In what follows, we write

$$b(\sigma, \tau) = \chi(\sigma/\tau),$$

where $b$ is positive-homogeneous of order 0 and smooth on $\mathbb{R}^2 \setminus 0$ since $\chi$ is smooth and has compact support in the interval $(0, \infty)$. Note that we may declare $b(0,0) = 0$ and regularize $b$ near the origin at the cost of a smooth error.

We first note that

$$e^{isP_H \otimes tP_M} = e^{isP_H} \otimes e^{itP_M},$$

since their evaluations on joint eigenfunctions $\psi_k \otimes e_j$ agree. By the Fourier inversion, we write

$$B = b(P_H, P_M) = (2\pi)^{-2} \int_{\mathbb{R}^2} \hat{b}(s,t)e^{isP_H} \otimes e^{itP_M} dsdt.$$  

Let $\rho$ be a Schwartz-class function on $\mathbb{R}^2$ such that $\rho \equiv 1$ near the origin and $\rho \equiv 0$ outside a neighborhood of the origin. Then we cut the integral into $\rho(s,t)$ and $1 - \rho(s,t)$ parts. Note that since $b$ is in class $S^0(\mathbb{R}^2)$, its Fourier transform has singular support at 0 (see the proof of [23, Theorem 4.3.1]). Hence, $\hat{b}(s,t)(1 - \rho(s,t))$ is Schwartz-class, and hence integration by parts reveals

$$(2\pi)^{-2} \int_{\mathbb{R}^2} \hat{b}(s,t)(1 - \rho(s,t))e^{isP_H} \otimes e^{itP_M}(\psi_k \otimes e_j)dsdt$$

$$= (2\pi)^{-2} \int_{\mathbb{R}^2} \hat{b}(s,t)(1 - \rho(s,t))e^{is\mu_k} \otimes e^{it\lambda_j}(\psi_k \otimes e_j)dsdt = O(|(\mu_k, \lambda_j)|^{-\infty}).$$

It now suffices to show that

$$(2\pi)^{-2} \int_{\mathbb{R}^2} \hat{b}(s,t)\rho(s,t)e^{isP_H} \otimes e^{itP_M} dsdt$$

is the desired pseudo-differential operator.

Next, as in [23], we use Hörmander’s small time parametrix for the half-wave operator to obtain

$$e^{isP_H} \otimes e^{itP_M}(x, z, y, w)$$

$$= e^{isP_H}(x, y)e^{itP_M}(z, w)$$

$$= (2\pi)^{-n-d} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+d}} e^{is\varphi_H(x, y, \xi) + it\varphi_M(z, w, \zeta) + tp_M(w, \zeta) + tp_M(z, w, \zeta)} q_H(s, x, y, \xi)q_M(t, z, w, \zeta)d\xi d\zeta$$

modulo a smooth kernel, where

$$\varphi_H(x, y, \xi) = \langle x - y, \xi \rangle + O(\|x - y\|^2|\xi|)$$

and

$$\varphi_M(z, w, \zeta) = \langle z - w, \zeta \rangle + O(\|z - w\|^2|\zeta|)$$

and $q_H$ and $q_M$ are zeroth-order symbols. Next, we examine the contribution of the integrals in $s$ and $t$ to (4.5), i.e.,

$$\int_{\mathbb{R}^2} e^{is\varphi_H(x, y, \xi) + itp_M(w, \zeta)} \hat{b}(s,t)\rho(s,t)q_H(s, x, y, \xi)q_M(t, z, w, \zeta)dsdt$$
with the following technical warning: when a derivative, for example leading term decreases with each derivative on the latter variables. We leave the details to the readers.

Again, one must repeat this argument with derivatives in the order \((x, y, z, w, \xi, \zeta)\). We use the following notation. Finally, we recall (3). We have shown that the Schwartz kernel of an operator \(B\) can be written as

\[
\rho_M^\beta \int \int \int \int e^{i(s(p_H(y, z) - \sigma) + t(p_M(w, \zeta) - \tau))} b(\sigma, \tau) \rho(s, t) q_H(s, t) dsdt d\sigma d\tau.
\]

which satisfies (2) trivially.

Next, we establish (1). We perform a change of variables and write the integral above as

\[
p_M^\beta \int \int \int \int \int e^{i(s(p_H/p_M - \sigma) + t(p_M - \tau))} b(\sigma, \tau) \rho(s, t) q_H(s, t) dsdt d\sigma d\tau.
\]

Now the phase function vanishes at its critical point \((s, t, \sigma, \tau) = (0, 0, p_H/p_M, 1)\), at which it is nondegenerate. By a routine stationary phase argument, the result is a symbol with the principal term

\[
(2\pi)^2 b(p_H/p_M, 1) \rho(0, 0) q_H(0, x, y, \xi) q_M(0, z, w, \zeta)
\]

Again, one must repeat this argument with derivatives in the order of the leading term decreases with each derivative on the latter variables. We leave the details to the readers with the following technical warning: when a derivative, for example \(\partial_y\), hits the oscillatory part, a factor such as \(s \partial_y p_H/p_M\) comes down into the amplitude. This would be troubling if not for the presence of the cutoff \(\beta\), which ensures the amplitude remains in class \(S^0\).

Finally, we move to (3). We have shown that the Schwartz kernel of \(A\) is a Fourier integral operator and compute its symbolic data.

\[
(2\pi)^{-d-n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} e^{i(q_H(x, y, \xi) + p_M(z, w, \zeta))} (2\pi)^{-2} a(x, y, z, w, \xi, \zeta) d\xi d\zeta,
\]

where \(a \in S^0\) satisfies

\[
a(x, y, z, w) = (2\pi)^2 b(p_H/p_M)
\]

modulo a lower-order term since \(q_H(0, x, y, \xi) \equiv 1\) and \(q_M(0, z, w, \zeta) \equiv 1\) (see [23, Chapter 4]). Hence, by equivalence of phase functions (see [23, Chapter 3]), \(B\) has the principal symbol \(b(p_H, p_M)\), as desired. This concludes the claim (3) and the proof of the lemma.

4.2 Symbolic data

Next, we compute the symbolic data of the Lagrangian (actually, conormal) distribution \(\delta_{ixi}\). Then we show that \(U \circ A\) is a Fourier integral operator and compute its symbolic data.

We use the following notation. \(F^m(X, \Lambda)\) denotes the space of Lagrangian distributions on \(X\) of order \(m\) associated with the conic Lagrangian \(\Lambda \subset T^* X \setminus 0\). Then \(F^m(X \times Y, C')\) is used to denote the space of Fourier integral operators from \(Y\) to \(X\), of order \(m\), associated with the canonical relation \(C \subset T^* X \setminus 0 \times T^* Y \setminus 0\). For each \(x \in H\), we let \(\pi_x : T^*_x M \to T^*_x H\) be the pullback of covectors through \(i\). We usually suppress the subscript base point in the notation, e.g., write \((x, \pi \zeta)\) instead of \((x, \pi_x \zeta)\). Finally, we recall \(q_H\) and \(q_M\) are the respective metric tensors on \(H\) and \(M\).

**Proposition 4.2.** \(\delta_{ixi} \in F^{\dim H/2}(H^2 \times M^2, \Lambda)\), where

\[
\Lambda = \{(x, \pi \zeta, y, -\pi \omega, i x, -\zeta, i y, \omega) : x, y \in H, \zeta \in T^*_x M, \omega \in T^*_y M, (\zeta, \omega) \neq 0\},
\]
and has the principal symbol (modulo a Maslov factor) equal to the transport of the half-density
\[(2\pi)^{-\text{codim} H/2} \frac{|g_H(x)|^{1/4}|g_H(y)|^{1/4}}{|g_M(ix)|^{1/4}|g_M(iy)|^{1/4}}|dx dy d\zeta d\omega|^{1/2}\]
via the implied parametrization of \(\Lambda\) by \((x, y, \zeta, \omega)\).

This proposition is an application of the following lemma to the embedding \(i \times i : H^2 \to M^2\), whose proof we defer until the end of this subsection.

**Lemma 4.3.** Let \(i : H \to M\) be a smooth embedding and let \(\Gamma_i = \{(ix, ix) : x \in H\}\) denote its graph in \(H \times M\). Then \(\delta_i\) defined by (4.1) is a conormal distribution in \(I^\text{codim} H/4(H \times M, N^*\Gamma_i \setminus 0)\), where
\[N^*\Gamma_i := \{(x, \pi \zeta, ix, -\zeta) : x \in H, \zeta \in T^*_x M\}\]
is the conormal bundle of the graph \(\Gamma_i\) of \(i\). Moreover, \(\delta_i\) has the principal symbol (modulo a Maslov factor) equal to the transport of the half-density
\[(2\pi)^{-\text{codim} H/4} \frac{|g_H(x)|^{1/4}}{|g_M(i(x))|^{1/4}}|dx d\zeta|^{1/2}\]
via the implied parametrization of \(N^*\Gamma_i\) by \((x, \zeta)\).

Next, we compute the wavefront relation of \(U\). We use \(G_M^t\) to denote the time-\(t\) homogeneous geodesic flow on \(T^* M \setminus 0\), and it is similar for \(G_M^t\) on \(H\). Recall that the half-wave kernel \(e^{-itP_M} (z, w)\) as a half-density distribution on \(\mathbb{R} \times M \times M\) is a Lagrangian distribution associated with the Lagrangian submanifold
\[\{(t, \tau, z, \omega, -\omega) : \tau + p_M(z, \omega) = 0, (z, \omega) = G_M^t(w, \omega)\},\]
or equivalently,
\[\{(t, -p_M(z, \omega), z, \zeta, G_M^t(z, -\zeta)) : t \in \mathbb{R}, (z, \omega) \in T^* M \setminus 0\}\]
with the principal symbol (modulo a Maslov factor) equal to the transport of \((2\pi)^{1/4}|dt dz d\zeta|^{1/2}\) via the implied parametrization by \(t \in \mathbb{R}\) and \((z, \omega) \in T^* M \setminus 0\) (see [14, Subsection 29.1]). We realize \(e^{-itP_M} (x, y)\) as the kernel of the Fourier integral operator \(U_M \in \mathcal{I}^{-1/4}(\mathbb{R} \times M^2, C_M^t)\) from \(M^2\) to \(\mathbb{R}\) with the canonical relation
\[C_M = \{(t, -p_M(z, \omega); z, -\zeta, G_M^t(x, \zeta)) : t \in \mathbb{R}, (z, \omega) \in T^* M \setminus 0\}.

Here, we have used that \((w, -\omega) = G_M^t(z, -\zeta)\) if and only if \((w, \omega) = G_M^t(z, \zeta)\). Similarly, \(e^{itP_M} (x, y)\) is associated with the Lagrangian manifold
\[\{(s, -p_H(x, \xi), x, \xi, G_H^s(x, -\xi)) : s \in \mathbb{R}, (x, \xi) \in T^* H \setminus 0\}
and has the principal symbol \((2\pi)^{1/4}dsdx d\xi |\xi|^{1/2}\). Again, define \(U_H \in \mathcal{I}^{-1/4}(\mathbb{R} \times H^2, C_H^t)\) as the operator with the kernel \(e^{itP_H} (x, y)\) and the canonical relation
\[C_H = \{(s, -p_H(x, \xi); x, -\xi, G_H^s(x, \xi)) : s \in \mathbb{R}, (x, \xi) \in T^* H \setminus 0\}.

By the calculus of wavefront sets, the tensored operator \(U_H \otimes U_M\) satisfies
\[\text{WF}'(U_H \otimes U_M) \subset (C_H \times C_M) \cup (C_H \times 0) \cup (0 \cup C_M).

Now, \(U\) is precisely a permutation of the variables \(U_H \otimes U_M\), and it is precisely the latter two components in the union above which prevent \(U\) from being a Fourier integral operator. Composition of \(U\) with our pseudo-differential cutoff \(A\) in (4.4) rescues us by excluding those elements
\[(s, \sigma, t, \tau; x, \xi, y, \eta, z, \zeta, w, \omega) \in \text{WF}'(U)\]
for which \(p_H(x, \xi)/p_M(z, \omega) \notin \text{supp} \chi\) or \(p_H(y, \eta)/p_M(w, \omega) \notin \text{supp} \chi\), which kills these problematic components. This cutoff also precludes any of \(\xi, \eta, \zeta\) and \(\omega\) from vanishing. We then have the following proposition.
Proposition 4.4. $U \circ A$ is a Fourier integral operator in $I^{-1/2}(\mathbb{R}^2 \times (H^2 \times M^2), C')$ with the canonical relation

$$C = \{(s, -p_H(x, \xi), t, -p_M(z, \zeta); x, -\xi, G_H^*(x, \xi), z, -\zeta, G_M^*(z, \zeta)) :$$

$$s, t \in \mathbb{R}, (x, \xi) \in T^*H \setminus 0, (z, \zeta) \in T^*M \setminus 0\}$$

and the principal symbol (modulo a Maslov factor) equal to the transport of

$$(2\pi)^{1/2}(p_H(x, \xi)/p_M(z, \zeta))^2|dxdydzd\xi|^{1/2}$$

via the implied parametrization.

This is an immediate application of the composition formula for a Fourier integral operator with a pseudo-differential operator. To simplify the resulting canonical relation, we have used that the symbols $p_H$ and $p_M$ remain constant along their respective Hamiltonian (geodesic) flows, and that both symbols are even.

Proof of Lemma 4.3. We select local coordinates $(y_1, \ldots, y_n)$ for a neighborhood of a point on $H$ for which $y_{d+1} = \cdots = y_n = 0$ parametrizes $H$. The immersion

$$(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, 0, \ldots, 0)$$

parametrizes $H$ in these local coordinates. Together, $(x, y) \in \mathbb{R}^d \times \mathbb{R}^n$ parametrize $H \times M$ in a neighborhood of a point on the graph of $i$. We let $f(x, y) = f(x, y)|g_H(x)|^{1/4}|g_M(y)|^{1/4}|dxdy|^{1/2}$ be a smooth test half-density on $H \times M$ supported in this neighborhood. By (4.1) and the Fourier inversion, we write

$$(\delta, f) = \int_{\mathbb{R}^d} \hat{f}(x, (0,0))|g_H(x)|^{1/2}dx$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(y_-(x,0),\zeta)} \hat{f}(x, y)|g_H(x)|^{1/2}dxdy\zeta$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(y_-(x,0),\zeta)} f(x, y) \frac{|g_H(x)|^{1/4}}{|g_M(y)|^{1/4}} |dxdy|^{1/2}d\zeta,$$

and hence we write

$$\delta_i(x, y) = (2\pi)^{-n+\operatorname{codim} H/4} \left( \int_{\mathbb{R}^n} e^{i(y_-(x,0),\zeta)} (2\pi)^{-\operatorname{codim} H/4} \frac{|g_H(x)|^{1/4}}{|g_M(y)|^{1/4}} |d\zeta| \right) |dxdy|^{1/2}$$

in the oscillatory form. We set $\varphi(x, y, \zeta) = (y - (x,0), \zeta)$ and note that it is a nondegenerate phase function with the critical set

$$C_\varphi = \{(x, (x,0), \zeta) : x \in \mathbb{R}^d, \zeta \in \mathbb{R}^n \setminus 0\}.$$

We let $\zeta' = (\zeta_1, \ldots, \zeta_d)$ be the first $d$ coordinates of $\zeta$. Via the map $(x, y, \zeta) \mapsto (x, y, \varphi')$, $C_\varphi$ parametrizes

$$\{(x, (x,0), -\zeta', \zeta) : x \in \mathbb{R}^d, \zeta \in \mathbb{R}^n \setminus 0\},$$

which is precisely $N^*\Gamma_i \setminus 0$ in canonical local coordinates of $T^*(H \times M)$. We conclude

$$\delta_i \in I^{\operatorname{codim} H/4}(H \times M, N^*\Gamma_i \setminus 0).$$

Finally, we compute the invariant half-density on $N^*\Gamma_i \setminus 0$. We parametrize $C_\varphi$ by $x$ and $\zeta$ as indicated above and have the Leray density

$$d_\varphi = \left| \frac{\partial(x, \theta, \varphi')}{\partial(x, y, \zeta)} \right|^{-1} |dxd\zeta|$$
\begin{align*}
&= \left| \frac{\partial(x, \zeta, y - (x, 0))}{\partial(x, y, \zeta)} \right|^{-1} |dxd\zeta| \\
&= \left| \det \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ * & I & 0 \end{bmatrix} \right|^{-1} |dxd\zeta| \\
&= |dxd\zeta|.
\end{align*}

The invariant homogeneous half-density on \( N^* \Gamma_i \setminus 0 \) is then the transport of
\[
(2\pi)^{-\text{codim } H/4} \left| \frac{g_H(x)}{g_M(y)} \right|^{1/4} |dxd\zeta|^{1/2}
\]
to \( N^* \Gamma_i \setminus 0 \) via its parametrization by \( x \) and \( \zeta \). This completes the proof.

**4.3 Cleanness of the composition**

Before computing the principal symbol of the composition \( U \circ A \circ \delta_i \times \iota \), we must first check the composition is clean.

Consider the projection \( C \rightarrow T^*(H^2 \times M^2) \) and the inclusion \( \Lambda \rightarrow T^*(H^2 \times M^2) \). The fiber product of these maps consists of a universal object \( F \) and two maps \( F \rightarrow C \) and \( F \rightarrow \Lambda \) for which the diagram
\[
\begin{array}{ccc}
F & \rightarrow & C \\
\downarrow & & \downarrow \\
\Lambda & \rightarrow & T^*(H^2 \times M^2)
\end{array}
\]
commutes. In this case, \( (4.7) \) is said to be a fiber product diagram. The composition \( C \circ \Lambda \) may be realized as the image of the composition \( F \rightarrow C \rightarrow T^*(\mathbb{R}^2) \). In our specific case, the fiber product induces relationships
\[
z = ix, \quad \xi = -\pi \zeta, \quad (y, -\pi \omega) = G^*_H(x, \xi) \quad \text{and} \quad (iy, \omega) = G^*_M(z, \zeta)
\]
amongst the elements of \( \Lambda \) and \( C \) as appear in Lemma 4.3 and Proposition 4.4, respectively, and we have
\[
F = \left\{ (s, t, x, \zeta) : s, t \in \mathbb{R}, x \in H, \zeta \in T^*_x M \setminus 0, (i \otimes I)G^*_H(x, \pi \zeta) = (I \otimes \pi)G^*_M(ix, \zeta), \frac{p_H(x, \pi \zeta)}{p_M(ix, \zeta)} \in \text{supp } \chi \right\}. \tag{4.8}
\]

Here, we have simplified the set using the change of variables \( \zeta \mapsto -\zeta \) and the identity
\[
(I \otimes -I)G^*_M(z, -\zeta) = G^*_M(z, \zeta).
\]

The projection \( F \rightarrow C \circ \Lambda \) is then
\[
(s, t, x, \zeta) \mapsto (s, -p_H(x, \pi \zeta), t, -p_M(ix, \zeta)).
\]

**Remark 4.5.** The composition \( C \circ \Lambda \) indicates that to obtain finer estimates on the joint asymptotics of Fourier coefficients, we need hypotheses which constrain the size of the set of points \( (s, t, x, \zeta) \) for which
\[
(i \otimes I)G^*_H(x, \pi \zeta) = (I \otimes \pi)G^*_M(ix, \zeta).
\]

Geometrically, such a point corresponds to a configuration of two geodesic segments, one in \( H \) and one in \( M \), which meet at their endpoints so that the velocity of the one in \( M \) coincides with the velocity of the one in \( H \) after an orthogonal projection. Such refinements are the subject of ongoing work (see [26]).
By the composition of $C$ and $\Lambda$ being clean, we mean that the fiber product diagram (4.7) is clean. In particular, the diagram (4.7) holds in the category of smooth manifolds, and for each point $p \in F$ and its corresponding images $(a; b) \in C$ and $b \in \Lambda$, the linearized diagram

$$
\begin{array}{ccc}
T_p F & \longrightarrow & T_{(a,b)} C \\
\downarrow & & \downarrow \\
T_b \Lambda & \longrightarrow & T_b T^* (H^2 \times M^2)
\end{array}
$$

is also a fiber product diagram. In this case, the excess of the diagram is

$$
e = (\dim T_p F + \dim T_b T^* (H^2 \times M^2)) - (\dim T_b \Lambda + \dim T_{(a,b)} C)
= \dim T_p F - 2
$$

and is constant on connected components of $F$.

In our situation, the fiber product diagram is clean if and only if the linearization of the relation

$$(i \otimes I)G^*_H(x, \pi \zeta) = (I \otimes \pi)G^*_M(ix, \zeta)$$

from (4.8) defines the tangent space of $F$. We introduce a little notation to make this precise. We let $i$ and $\pi$ denote, as an abuse of notation, their respective linearizations. We let $H_{pH}$ and $H_{pM}$ denote the Hamilton vector fields associated with symbols $p_H$ and $p_M$, respectively, and recall

$$
d_{ds}^*G^*_H = H_{pH} \quad \text{and} \quad d_{dt}^*G^*_M = H_{pM}.
$$

We also use the prime notation to indicate the (coefficients of the) vector associated with a variable, e.g., $(s', t', x', \zeta') \in T_{(s,t,x,\zeta)}(\mathbb{R}^2 \times T_H^* M)$. The linearization of the relation above is then written as

$$(i \otimes I)(s'H_{pH} + dG^*_H(x', \pi \zeta')) = (I \otimes \pi)(t'H_{pM} + dG^*_M(ix', \zeta')),$$

and one quickly checks that the cleanness condition is equivalent to

"If $(s', t', x', \zeta')$ satisfies (4.10), then $(s', t', x', \zeta') \in T_p F$" (4.11)

at each $p \in F$.

We now verify that the composition of $C$ and $\Lambda$ is always clean provided that the $(s, t)$ coordinates of $C$ lie in a suitably small neighborhood of the origin. We let

$$F_0 = \left\{ (0, 0, x, \zeta) : x \in H, \zeta \in T^*_x H, \frac{p_H(x, \pi \zeta)}{p_M(ix, \zeta)} \in \text{supp } \chi \right\}
$$

be the component of $F$ which lies at $(s, t) = (0, 0)$. Furthermore for an open neighborhood $O$ of the origin in $\mathbb{R}^2$, we let $C_O$ denote the intersection of $C$ with $T^*(O \times H^2 \times M^2)$.

**Lemma 4.6.** There exists an open neighborhood $O$ of the origin in $\mathbb{R}^2$ such that $F_0 = F \cap (O \times T_H^* M)$. As a consequence,

$$C_O \circ \Lambda = \{ (0, \sigma, 0, \tau) \in T^*_x \mathbb{R}^2 : \sigma, \tau < 0, \sigma/\tau \in \text{supp } \chi \}.$$

Since $C_O \circ \Lambda$ is the only part of the whole composition $C \circ \Lambda$ which contributes to the set on the right in the lemma, the symbolic data of $U \circ \Lambda \circ \delta_{x,1}$ at the origin is determined by the composition $C_O \circ \Lambda$ and the relevant calculus. Now we verify that the composition is clean.

**Proposition 4.7.** The composition $C_O \circ \Lambda$ is clean with excess $d + n - 2$.

We start by relating the Hamilton vector fields $H_{pM}$ and $H_{pH}$ along the embedded manifold $H$ in a convenient choice of coordinates. Let $(z_1, \ldots, z_n)$ be local coordinates for $M$ such that $(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d, 0, \ldots, 0)$ parametrizes a neighborhood in $H$ and the metric $g_M$ along $H$ is a block matrix

$$g_M(x, 0) = \begin{bmatrix} g_H(x) & 0 \\ 0 & h(x) \end{bmatrix}.$$
Note that $h$ is necessarily positive-definite, nonsingular and symmetric, and has entries which vary smoothly in $x$. Let $(x_1,\ldots,x_d,\xi_1,\ldots,\xi_d)$ and $(z_1,\ldots,z_n,\zeta_1,\ldots,\zeta_n)$ be local canonical coordinates for $T^*H$ and $T^*M$, respectively. Recall that the Hamiltonian field vector associated with the symbol $p_H$ is given by

$$H_{p_H} = \sum_{k=1}^d \left( \frac{\partial p_H}{\partial x_k} \frac{\partial}{\partial x_k} - \frac{\partial p_H}{\partial \xi_k} \frac{\partial}{\partial \xi_k} \right).$$

We are concerned with the coefficients $\frac{\partial p_H}{\partial \xi_k}$ of the spacial part of the vector. In particular, we want to relate these coefficients to the corresponding coefficients of $H_{p_M}$. An elementary computation yields

$$\frac{\partial p_H}{\partial \xi_k}(x,\xi) = \frac{1}{p_H(x,\xi)} \sum_{j=1}^d g^k_j(x) \xi_j$$

and

$$\frac{\partial p_M}{\partial \xi_k}(z,\zeta) = \frac{1}{p_M(z,\zeta)} \sum_{j=1}^n g^k_j(z) \zeta_j.$$

In particular, we have

$$\frac{\partial p_M}{\partial \xi_k}(ix,\zeta) = \frac{1}{p_M(ix,\zeta)} \sum_{j=1}^d g^k_j(ix) \zeta_j, \quad 1 \leq k \leq d,$$

$$\frac{\partial p_M}{\partial \xi_k}(ix,\zeta) = \frac{1}{p_M(ix,\zeta)} \sum_{j=d+1}^n h^k_j(x) \zeta_j, \quad d+1 \leq k \leq n,$$

where by yet another abuse of notation we take $h^k_j(x)$ to be the entry of the inverse matrix of $h(x)$ in the row $j - d$ and the column $k - d$. From this, we obtain the following convenient formula:

$$\frac{\partial p_M}{\partial \xi_k}(ix,\zeta) = \frac{p_H(ix,\pi \zeta)}{p_M(ix,\zeta)} \frac{\partial p_H}{\partial \xi_k}(x,\pi \zeta), \quad k \in \{1, \ldots, d\}. \quad (4.12)$$

We need (4.13) and the following elementary lemma for the proof of Lemma 4.6.

**Lemma 4.8.** There exist an open neighborhood $O$ of the origin in $\mathbb{R}^2$ and a positive constant $C$ both only depending on $H$ and $M$ such that if $(s, t) \in O$ and $(s, t, x, \zeta) \in F$ for some $x$ and $\zeta$, then

$$\left| \frac{s}{t} - 1 \right| \leq C|t|.$$

**Proof of Lemma 4.6.** Suppose that there exists a sequence of points $(s, t, x, \zeta) \in F$ for which $(s, t) \to 0$. Since $p_H/p_M$ is homogeneous of degree 0 and $S^*_H M$ is compact, we may select a subsequence for which $p_M(ix, \zeta) = 1$ and $(x, \zeta)$ converges. Let us examine the condition $(i \otimes I)G_H^1(x, \pi \zeta) = (I \otimes \pi)G_M^d(ix, \zeta)$ in $F$. In local coordinates, we write

$$\left( \frac{s}{t} \right) \frac{1}{t} ((i \otimes I)G_H^1(x, \pi \zeta) - (ix, \pi \zeta)) = \frac{1}{t} ((I \otimes \pi)G_M^d(ix, \zeta) - (ix, \pi \zeta)).$$

By taking a limit and invoking Lemma 4.8, we find for all $\epsilon > 0$ a term in the sequence for which

$$|(i \otimes I)H_{p_H}(x, \pi \zeta) - (I \otimes \pi)H_{p_M}(ix, \zeta)| < \epsilon.$$

We look to the spacial components of the Hamilton vectors and find

$$\sum_{k=1}^d \left| \frac{\partial p_H}{\partial \xi_k}(x, \pi \zeta) - \frac{\partial p_M}{\partial \xi_k}(ix, \xi) \right|^2 + \sum_{k=d+1}^n \left| \frac{\partial p_M}{\partial \xi_k}(ix, \zeta) \right|^2 < \epsilon^2.$$

In the light of (4.13), we have

$$\left| 1 - \frac{p_M(ix, \zeta)}{p_H(x, \pi \zeta)} \right|^2 \sum_{k=1}^d \left| \frac{\partial p_M}{\partial \xi_k}(ix, \xi) \right|^2 + \sum_{k=d+1}^n \left| \frac{\partial p_M}{\partial \xi_k}(ix, \zeta) \right|^2 < \epsilon^2.$$

Since $p_M(ix, \zeta) = 1$, we can use this inequality to force $p_H(x, \pi \zeta)/p_M(ix, \zeta)$ as close to 1 as we wish by taking $\epsilon$ small. This contradicts the condition that $p_H(x, \pi \zeta)/p_M(ix, \zeta) \in \text{supp} \chi$ in $F$. \qed
Proof of Proposition 4.7. By (4.11) and Lemma 4.6, it suffices to show that
\[ T_p F_0 = \{(s', t', x', \zeta') \in T_p F : (i \otimes I)(s'H_{pH}) = (I \otimes \pi)(t' H_{pM})\} \]
for each \( p = (0, 0, x, \zeta) \in F_0 \). This amounts to showing \( s' = t' = 0 \). The computation of the excess follows from \( \dim F_0 = n + d \).

By homogeneity, assume \( p_M(ix, \zeta) = 1 \). In our usual local coordinates,
\[
\begin{align*}
  s' \frac{\partial p_H}{\partial \xi_k}(x, \pi \zeta) &= t' \frac{\partial p_M}{\partial \xi_k}(ix, \zeta) \quad \text{for } k \in \{1, \ldots, d\}, \\
  0 &= t' \frac{\partial p_M}{\partial \xi_k}(ix, \zeta) \quad \text{for } k \in \{d + 1, \ldots, n\}.
\end{align*}
\]

If \( t' \neq 0 \), then by (4.12), \( \zeta_k = 0 \) for each \( k \in \{1, \ldots, d\} \). We then have \( p_M(ix, \zeta) = p_H(x, \pi \zeta) \), which is prohibited on \( F \). Hence, \( t' = 0 \) and we have
\[
  s' \frac{\partial p_H}{\partial \xi_k}(x, \pi \zeta) = 0 \quad \text{for } k \in \{1, \ldots, d\}.
\]

By (4.13),
\[
  \frac{p_H(x, \pi \zeta)}{p_M(ix, \zeta)} \frac{\partial p_H}{\partial \xi_k}(x, \pi \zeta) = \frac{\partial p_M}{\partial \xi_k}(ix, \zeta) \quad \text{for each } k \in \{1, \ldots, d\}.
\]
Hence if \( s' \neq 0 \), each \( \partial p_M/\partial \xi_k \) must vanish for \( k \in \{1, \ldots, d\} \), so again by (4.12), \( \zeta_k = 0 \) for each \( k \in \{1, \ldots, d\} \). It follows that \( p_H(x, \pi \zeta) = 0 \), which is also prohibited on \( F \). So \( s' = 0 \) and the proof is completed. \( \square \)

Proof of Lemma 4.8. For \( |s| \) and \( |t| \) smaller than the injectivity radii of \( H \) and \( M \), respectively,
\[
  \frac{s}{t} = \frac{d_H(x, y)}{d_M(ix, iy)},
\]
where \( g \) is the point on \( H \) to which both \( G_M^i(x, \pi \zeta) \) and \( G_M^i(ix, \zeta) \) project, and \( d_H \) and \( d_M \) are the Riemannian metrics on \( H \) and \( M \), respectively. Since \( \iota : H \to M \) is an isometric embedding and \( d_H(x, y) \geq d_M(ix, iy) \) always, we need only show that
\[
  d_H(x, y) - d_M(ix, iy) \leq Cd_M(ix, iy)^2
\]
wherever \( d_H(x, y) \) is bounded by a small constant. In fact in normal coordinates about \( ix \) in \( M \), \( d_M(ix, iy) = |iy| \), and \( d_H(x, y)^2 \) is a smooth function in \( y \) satisfying \( d_H(x, y)^2 = |y|^2 + O(|y|^3) \). Then we have
\[
  d_H(x, y) - d_M(ix, iy) = \frac{d_H(x, y)^2 - d_M(ix, iy)^2}{d_H(x, y) + d_M(ix, iy)} = O(|y|^2),
\]
as desired. Furthermore, the constants implicit in the big-\( O \) notation vary continuously with \( x \). The lemma follows from compactness. \( \square \)

4.4 The symbolic data of the composition

We are nearly prepared to compute the principal symbol of \( U \circ A \circ \delta_{\chi_i} \) in a neighborhood of the origin.

Proposition 4.9. The restriction of \( U \circ A \circ \delta_{\chi_i} \) to the open neighborhood \( O \) of Lemma 4.6 is a half-density distribution in \( \Gamma^{n-3/2}(\mathbb{R}^2; C_O \circ A) \), where
\[
  C \circ A_O = \{(0, \sigma, 0, \tau) \in T^*\mathbb{R}^2 : \sigma, \tau < 0 \text{ and } \sigma/\tau \in \text{supp } \chi\}
\]
with the principal symbol
\[
  (2\pi)^{-n+2} \text{vol}(S^{n-1}) \text{vol}(S^{n-d-1}) \text{vol}(H) \cdot \chi(\sigma/\tau)^2 (-\sigma)^{d-1} (-\tau)^{n-d-1} (1 - \sigma^2/\tau^2)^{-d-n/2} |d\sigma d\tau|^{1/2},
\]
wherever \( \sigma/\tau \) belongs to a given interval \( [a, b] \subset (0, 1) \), modulo multiplication by a complex unit.
The order $n - 3/2$ is computed by
\[ \text{ord}(U \circ A \circ \delta_i \chi) = \text{ord}(U \circ A) + \text{ord} \delta_i \chi + e/2. \]

Recall that $\text{ord}(U \circ A) = -1/2$ from Proposition 4.4, $\text{ord} \delta_i \chi = (n-d)/2$ from Lemma 4.3 and $e = n+d - 2$ from Proposition 4.7.

We now show how this proposition concludes the proof of Theorem 1.4. Recall
\[ U \circ A \circ \delta_i \chi = \tilde{N}_B |dsdt|^{1/2}. \]

Let $\rho$ be a Schwartz-class function on $\mathbb{R}^2$ whose Fourier support is contained in $\mathcal{O}$. We test both sides against the oscillating half-density $\tilde{\rho}(s, t) e^{-i(s \sigma + t \tau)} |dsdt|^{1/2}$ to obtain
\[ (U \circ A \circ \delta_i \chi, \tilde{\rho}(s, t) e^{-i(s \sigma + t \tau)} |dsdt|^{1/2}) = (2\pi)^2 \rho \ast \tilde{N}_B (-\sigma, -\tau). \]

Referring to Proposition 4.9 along with [14, Proposition 25.1.5] and the discussion preceding [14, Theorem 25.1.9], we obtain
\[ \rho \ast \tilde{N}_B (-\sigma, -\tau) = (2\pi)^{-n} \text{vol}(S^{d-1}) \text{vol}(S^{n-d-1}) \text{vol}(H) \]
\[ \times \tilde{\rho}(0)(-\sigma)^{d-1}(-\tau)^{n-d-1}(1 - \sigma^2/\tau^2)^{d-d/2} + O(|(\sigma, \tau)|^{n-3}) \]
(4.14)

for $\sigma/\tau \in [a, b]$. Strictly speaking, the main term is only correct up to multiplication by a complex unit. But this complex unit is 1 anyway after one takes $\rho \geq 0$ and recalls $N_B \geq 0$. Finally, the theorem follows after a change of variables $(\mu, \lambda) = (-\sigma, -\tau)$.

To compute the symbol, we first require a specialized basis for the tangent space of $T^*_H M$. In what follows, we identify our fiber product $F_0$ with
\[ F_0 = \left\{ (x, \zeta) \in T^*_H M \setminus 0 : \frac{p_H(x, \pi \zeta)}{p_M(ix, \zeta)} \in \text{supp} \chi \right\} \]
and, via the obvious inclusion, view $T_{(x_0, \zeta_0)} F_0$ as a subspace of $T_{(ix_0, \zeta_0)} T^*_H M$. For shorthand, we also use
\[ c = \frac{p_H(x_0, \pi \zeta_0)}{p_M(ix_0, \zeta_0)}. \]

**Lemma 4.10.** Fix $(x_0, \zeta_0)$ in $F_0$. Take a system of local coordinates $(z_1, \ldots, z_n)$ of $M$ about $ix_0$ and consider the symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of $T_{(ix_0, \zeta_0)} T^*_H M$ given by
\[ e_j = \frac{\partial}{\partial z_j} \text{ and } f_j = \frac{\partial}{\partial \zeta_j}, \quad j \in \{1, \ldots, n\} \]
in canonical coordinates $(z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_n)$. We may select these local coordinates such that the following hold:

1. $e_1, \ldots, e_d, f_1, \ldots, f_n$ is a basis for $T_{(x_0, \zeta_0)} F_0$.
2. $die_j = e_j$ for all $j \leq d$.
3. $dp \imath f_j = f_j$ for all $j \leq d$, and 0 if $j > d$.
4. $dp_H e_j = 0$ for all $j$.
5. $dp_H f_j = \begin{cases} 0, & j < d, \\ 1, & j = d. \end{cases}$
6. $dp_M e_j = 0$ for $j \leq d$.
7. $dp_M f_j = \begin{cases} c, & j = d, \\ \sqrt{1-c^2}, & j = n, \\ 0, & \text{otherwise}. \end{cases}$
8. $H_{p_H} = e_d$.
9. $H_{p_M} = c e_d + \sqrt{1-c^2} e_n$ plus a linear combination of $f_{d+1}, \ldots, f_n$. 

There are some minor abuses of notation in the lemma above. First, (2) is tautological since we have identified $T_{(x_0, \zeta_0)} F_0$ as a subspace of $T_{(x_0, \zeta_0)} T^* M$. Second, we are viewing $d\pi$ as an operator on $T_{(x_0, \zeta_0)} T^* M$ rather than a map to $T_{(x_0, \zeta_0)} T^* H$ and, strictly speaking, it should be written as $I \otimes d\pi$. Alternatively, we can make precise sense of the notation if we use the symplectic basis to pull everything back to the model Euclidean symplectic space $\mathbb{R}^{2n}$.

Proof of Lemma 4.10. Consider geodesic normal coordinates $(x_1, \ldots, x_d)$ about $x_0$ in $H$. Without loss of generality, we select these coordinates so that $(0, \ldots, 0, s)$ parametrizes the geodesic traced by $G^H_{H}(x_0, \pi \zeta_0)$. It follows that $(x_0, \pi \zeta_0) = (0, \ldots, 0, \rho_H(x_0, \pi \zeta_0))$ in canonical local coordinates $(x, \xi)$, and

$$\frac{\partial \rho_H}{\partial \xi_j}(x_0, \pi \zeta_0) = \begin{cases} 1, & j = d, \\ 0, & j < d. \end{cases} \quad (4.15)$$

Since $g_H^{-1}(x) = I + O(|x|^2)$ in normal coordinates, we also have

$$\frac{\partial \rho_H}{\partial x_\ell}(x_0, \pi \zeta_0) = \frac{1}{2\rho_H} \sum_{j,k=1}^d \frac{\partial g_H}{\partial x_\ell}(x_0) \xi_j \xi_k = 0 \quad \text{for } \ell \in \{1, \ldots, d\}. \quad (4.16)$$

We extend to coordinates in $M$ by first selecting a smooth orthonormal frame $v_{d+1}, \ldots, v_n$ of vectors perpendicular to $H$ and then taking coordinates $(z_1, \ldots, z_n)$ defined by the map

$$(z_1, \ldots, z_n) \mapsto \exp(z_{d+1} v_{d+1}(z') + \cdots + z_n v_n(z')),$$

where for shorthand $z' = (z_1, \ldots, z_d)$. We note

$$g_M^{-1}(z', 0) = \begin{bmatrix} g_H^{-1}(z') & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I + O(|z'|^2) & 0 \\ 0 & I \end{bmatrix}. \quad (4.17)$$

Without loss of generality, we select $v_n$ such that the $\zeta$-gradient of $p_M$ is a linear combination of $\partial/\partial z_d$ and $v_n = \partial/\partial z_n$. Since $g_M$ is the identity at the origin and $|\nabla_\zeta p_M| = 1$, it follows that

$$\frac{\partial p_M}{\partial \zeta_j} = \begin{cases} \frac{p_M}{p_M}, & j = d, \\ \sqrt{1 - p_H^2/p_M^2}, & j = n, \\ 0, & \text{otherwise} \end{cases} \quad (4.18)$$

in canonical local coordinates.

Properties (1)–(3) follow from construction. (4) follows from

$$dp_H e_j = \frac{\partial \rho_H}{\partial x_j}(x_0, \pi \zeta_0)$$

and (4.16). (5) follows from (4.15). (6) follows from the property (4) and (4.17) if $j \leq d$, and from

$$dp_M e_j = \frac{\partial p_M}{\partial z_j}(0) = \frac{1}{2p_M} \sum_{k,l=1}^n \frac{\partial g_M}{\partial z_j}(0) = \frac{1}{2p_M} \sum_{k,l=1}^d \frac{\partial g_M}{\partial x_j}(0).$$

Finally, (7) follows from (4.18). (8) and (9) follow from (4) through (7) and computations of $H_{p_M}$ and $H_{p_H}$ in local coordinates.

Now we prove Proposition 4.9. In what follows, we assume familiarity with the composition calculus in [9]. As in [9], given a vector space $V$ and a real number $\alpha$, we use $|V|^\alpha$ to denote the 1-dimensional vector space of $\alpha$-densities on $V$ (please see also [14] for an alternative exposition of the symbol calculus of FIOs, and [12] for the half-density formalism).
Proof of Proposition 4.9. Like before, fix $p = (x, \zeta) \in F_0$ and set

$$a = (0, -p_H(x, \pi \zeta), 0, -p_M(ix, \zeta)) \quad \text{and} \quad b = (x, \pi \zeta, x, -\pi \zeta, ix, -\zeta, ix, \zeta).$$

We let $\alpha : F_0 \to C \circ \Lambda$ be the projection of the fiber onto the composition and $da$ be its linearization $T_p F_0 \to T_a C \circ \Lambda$. The procedure in [9] identifies an object in $|\ker da| \otimes |T_a C \circ \Lambda|^{1/2}$ which is then integrated over the ‘excess fiber’ $\alpha^{-1}(a)$ to obtain the half-density symbol on $C \circ \Lambda$.

Following the procedure in [9], we identify the object in $|\ker da| \otimes |T_a C \circ \Lambda|^{1/2}$ with three steps.

1. Let $\tau : T_{(a,b)}H \times \tau b \Lambda \to T_b T^*(H^2 \times M^2)$ be given by $\tau((a', b), c') = b' - c'$. We use the exact sequence

$$0 \to T_p F_0 \to T_{(a,b)}H \times \tau b \Lambda \xrightarrow{\tau} T_b T^*(H^2 \times M^2) \to \ker \tau \to 0 \quad (4.19)$$

along with the symbols on $C$ and $\Lambda$ and the symplectic half-density on $T^*(H^2 \times M^2)$ to obtain a linear isomorphism $|T_p F_0|^{-1/2} \simeq |\ker \tau|^{-1/2}$, which we make explicit in Lemma 4.11 below.

2. The spaces $\ker \alpha$ and $\ker \tau$ are nondegenerately paired by the symplectic form on $T^*(H^2 \times M^2)$ (‘canonically paired’ in Duistermaat and Guillemin’s language). This induces a linear isomorphism $|\ker \tau|^{-1/2} \simeq |\ker \alpha|^{1/2}$. Both the pairing and the isomorphism are made explicit in Lemma 4.12 below.

3. The short exact sequence

$$0 \to \ker da \to T_p F_0 \xrightarrow{da} T_a (C \circ \Lambda) \to 0 \quad (4.20)$$

yields a unique element in $|\ker da|^{1/2} \otimes |T_p F_0|^{-1/2} \otimes |T_a C \circ \Lambda|^{1/2}$, which by the isomorphisms

$$|T_p F_0|^{-1/2} \simeq |\ker \tau|^{-1/2} \simeq |\ker da|^{1/2}$$

from (1) and (2), identifies a unique element in $|\ker da| \otimes |T_a C \circ \Lambda|^{1/2}$. This is made explicit in Lemma 4.13 below.

Lemma 4.11. The exact sequence (4.19) induces a linear isomorphism $|T_p F_0|^{-1/2} \simeq |\ker \tau|^{-1/2}$ which, given the $-\frac{1}{2}$-density on $T_p F_0$ which assigns the value 1 to the basis for $T_p F_0$ of Lemma 4.10, gives a $-\frac{1}{2}$-density on $\ker \tau$ which assigns $(2\pi)^{-\text{codim } H/2+1/2} \chi(c)^2(1 - c^2)^{-1/4}$ to the basis of $\ker \tau$ obtained as the image of

$$(0, 0, f_j, 0), \quad j \leq d,$$

$$(0, e_j, 0, 0), \quad j < d,$$

$$(0, 0, 0, e_j), \quad d < j < n \quad (4.21)$$

under the quotient map $T_b T^*(H^2 \times M^2) \to \ker \tau$.

The pairing of $\ker da$ and $\ker \tau$ hinted in (2) is realized as follows. By the arguments in [9], the image of $\ker da$ in $T_b T^*(H^2 \times M^2)$ via $T_p F_0 \to T_b T^*(H^2 \times M^2)$ is precisely the symplectic complement of $\im \tau$. The symplectic form then induces a well-defined nondegenerate bilinear mapping

$$\ker da \times \ker \tau \to \mathbb{R}.$$
Finally, we integrate the object identified in Lemma 4.13 over the excess fiber $\alpha^{-1}(0, \sigma, 0, \tau)$. We fix coordinates as we have above and integrate in the $\zeta$ variables first and in $x$ variables second. We note that $(x_0, \zeta) \in \alpha^{-1}(0, \sigma, 0, \tau)$ if and only if $(\zeta_1, \ldots, \zeta_d) = -\sigma$ and $(\zeta_1, \ldots, \zeta_n) = -\tau$, which may be rewritten as the Cartesian product of the sphere of radius $-\sigma$ in $\mathbb{R}^d$ and the sphere of radius $\sqrt{\tau^2 - \sigma^2}$ in $\mathbb{R}^{n-d}$.

Integrating yields an object in $|T_xH| \otimes |T_aC \circ \Lambda|^{1/2}$, which assigns the value
\[
(2\pi)^{-\text{codim}H/2 + 1/2} \text{vol}(S^{d-1}) \text{vol}(S^{n-d-1}) (-\sigma)^{d-1}(-\tau)^{n-d-1}(1 - \sigma^2/\tau^2)^{n-d-2}
\]
to the pair of bases $e_1, \ldots, e_d$ and $(b, 0), (0, h)$. Since $e_1, \ldots, e_d$ are orthonormal at $x_0$ with respect to the Riemannian inner product, integration over $H$ yields the half-density
\[
(2\pi)^{-\text{codim}H/2 + 1/2} \text{vol}(S^{d-1}) \text{vol}(S^{n-d-1}) \text{vol}(H)(-\sigma)^{d-1}(-\tau)^{n-d-1}(1 - \sigma^2/\tau^2)^{n-d-2} |d\sigma d\tau|^{1/2}.
\]

Proposition 4.9 follows from [9, Theorem 5.4] or [14, Subsection 25.2.3], which multiplies in an additional complex unit times $(2\pi)^{-\varepsilon/2} = (2\pi)^{-(n+d-2)/2}$.

All that remains is to prove Lemmas 4.11–4.13.

**Proof of Lemma 4.11.** The exact sequence (4.19) gives us an identification
\[
|\text{coker } \tau|^{-1/2} \simeq |T_pF_0|^{-1/2} \otimes |T_{(a,b)}C \times T_b\Lambda|^{1/2} \otimes |T_bT^*(H^2 \times M^2)|^{-1/2}.
\]

We begin by fixing an element in $|T_pF_0|^{-1/2}$ which takes the basis (1) of Lemma 4.10 to 1 and, through a process of forward maps, basis extensions and change of basis operations, we obtain the desired valuation of an element in $|\text{coker } \tau|^{-1/2}$ on the basis in the lemma.

In what follows, we let $g$ and $h$ constitute the standard symplectic basis of any tangent space to $T^*\mathbb{R}$. Recalling Proposition 4.4 and Lemma 4.3 and appealing to Lemma 4.10, we see that $T_{(a,b)}C$ has a basis
\[
(g, 0; 0, -e_d, 0, 0),
(0, g; 0, 0, 0, ce_d + \sqrt{1 - c^2} e_n + *),
(0, 0; e_j, e_j, 0, 0), \quad j \leq d,
(0, 0; f_j, -f_j, 0, 0), \quad j < d,
(-h, 0; f_d, -f_d, 0, 0),
(0, 0; 0, 0, e_j, e_j), \quad j \leq n,
(0, 0; 0, 0, -f_j, f_j), \quad j \neq d, n,
(0, -ch; 0, 0, -f_d, f_d),
(0, -\sqrt{1 - c^2} h; 0, 0, -f_n, f_n),
\]
where the * is a linear combination of $f_{d+1}, \ldots, f_n$, and $T_b\Lambda$ has a basis
\[
(e_j, 0, e_j, 0), \quad j \leq d,
(0, e_j, 0, e_j), \quad j \leq d,
(f_j, 0, -f_j, 0), \quad j \leq d,
(0, f_j, 0, -f_j), \quad j \leq d,
(0, 0, -f_j, 0), \quad j > d,
(0, 0, 0, -f_j), \quad j > d.
\]

Recall that the maps $T_pF_0 \to T_{(a,b)}C$ and $T_pF_0 \to T_b\Lambda$ are given by
\[
(x', \zeta') \mapsto (0, -dp_H(x', \pi\zeta'), 0, -dp_M(ix', \zeta); x', \pi\zeta', x', -\pi\zeta', ix', -\zeta', ix', \zeta'),
(x', \zeta') \mapsto (x', \pi\zeta', x', -\pi\zeta', ix', -\zeta', ix', \zeta').
\]
Appealing to Lemma 4.10, we see that the basis $e_1, \ldots, e_d, f_1, \ldots, f_n$ of $T_bF_0$ pushes forward through the map $T_bF_0 \to T_{(a,b)}C \times T_b\Lambda$ to

$$(0,0; e_j, e_j, e_j) \times (e_j, e_j, e_j), \quad j \leq d,$$
$$(0,0; f_j, -f_j, -f_j) \times (f_j, -f_j, -f_j), \quad j < d,$$
$$(0,0; 0, 0, f_j) \times (0,0, f_j), \quad j < d < n,$$
$$(0,0, f_j, f_j) \times (0,0, f_j), \quad d < j < n,$$
$$(0,0; 0, 0, f_j) \times (0,0, f_j).$$

Our current goal is to complete this to a basis for the product $T_{(a,b)}C \times T_b\Lambda$. The evaluation of the symbolic half-density on the product will determine a unique element of $|(T_{(a,b)}C \times T_b\Lambda)/\ker \tau|^{1/2}$. We add in $d + n$ elements of the form $0 \times v$, where $v$ is a basis element of $T_b\Lambda$. We also add in $2 + 2d + 2n$ elements of the form $0 \times v$, where $v$’s are the basis elements of $T_{(a,b)}C$. Specifically, we extend to a basis of the product by adding in elements

$$0 \times (e_j,0,e_j,0), \quad j \leq d,$$
$$0 \times (0,e_j,0,e_j), \quad j \leq d,$$
$$0 \times (f_j,0,-f_j,0), \quad j \leq d,$$
$$0 \times (0,-f_j,0,f_j), \quad j \leq d,$$
$$0 \times (0,0,-f_j,0), \quad d < j \leq n,$$
$$0 \times (0,0,0,f_j), \quad d < j \leq n,$$
$$g,0;0,-e_d,0,0) \times 0,$$
$$0,0;0,0,ee_d + \sqrt{1 - c^2 e_n + *}) \times 0,$$
$$0,0;0,0,e_d,e_d) \times 0, \quad j \leq n,$$
$$0,0;0,0,-f_j,f_j) \times 0, \quad j < d,$$
$$0,-ch;0,0,-f_d,f_d) \times 0.$$

A linear transformation consisting entirely of determinant $\pm 1$ row operations and perhaps some permutations takes the extended basis to the product of the bases of $T_{(a,b)}C$ and $T_b\Lambda$. Appealing to the symbolic data in Lemma 4.3 and Proposition 4.4, we see that the half-density on the product $T_{(a,b)}C \times T_b\Lambda$ assigns the value

$$(2\pi)^{-\dim \ker H/2 + 1/2} \chi(e)^2$$

to the product basis. Hence, the desired element of $|(T_{(a,b)}C \times T_b\Lambda)/\ker \tau|^{1/2}$ assigns the same value to the image of the list (4.22) by the quotient map $T_{(a,b)}C \times T_b\Lambda \to (T_{(a,b)}C \times T_b\Lambda)/\ker \tau$.

Next, we map (4.22) forward to $T_bT^*(H^2 \times M^2)$ via $\tau$. In particular, we obtain

$$(-e_j,0,-e_j,0), \quad j \leq d,$$
$$(0,-e_j,0,-e_j), \quad j \leq d,$$
$$(-f_j,0,f_j,0), \quad j \leq d,$$
$$(0,f_j,0,-f_j), \quad j \leq d,$$
$$(0,0,f_j,0), \quad d < j \leq n,$$
$$(0,0,0,-f_j), \quad d < j \leq n,$$
$$(0,-e_d,0,0),$$
$$0,0,0,ee_d + \sqrt{1 - c^2 e_n + *},$$
$$0,0,e_d,e_d, \quad j \leq n,$$
$$0,0,-f_j,f_j, \quad j < d,$$
Recall that the $*$ is a linear combination of $f_j$. We extend this by the $n + d - 2$ elements in (4.21) and, after a sequence of determinant $\pm 1$ operations and a permutation of the basis elements, we obtain the basis

\[
(e_j, 0, 0, 0), \quad j \leq d,
(0, e_j, 0, 0), \quad j \leq d,
(0, 0, e_j, 0), \quad j \leq n,
(0, 0, 0, e_j), \quad j < n,
(0, 0, 0, \sqrt{1 - c^2} e_n),
(f_j, 0, 0, 0), \quad j \leq d,
(0, f_j, 0, 0), \quad j \leq d,
(0, 0, f_j, 0), \quad j \leq n,
(0, 0, 0, f_j), \quad j < n,
\]

(4.23)

to which the symplectic $\frac{-1}{2}$-density on $T_b T^* (H^2 \times M^2)$ assigns $(1 - c^2)^{-1/4}$. Finally, the desired element in $|\ker \tau|^{-1/2}$ assigns the product of these valuations,

\[
(2\pi)^{-\text{codim} H/2 + 1/2} \chi(c)^2 (1 - c^2)^{-1/4},
\]
to the image of the basis (4.21) via the quotient $T_b T^* (H^2 \times M^2) \to \ker \tau$.

**Proof of Lemma 4.12.** The map $T_p F_0 \to T_b T^* (H^2 \times M^2)$ is

\[
(x', \zeta') \mapsto (x', d\pi \zeta', x', -d\pi \zeta', dix', -\zeta', dix', \zeta').
\]

Hence, the basis for $\ker d\alpha$ in the lemma maps forward to

\[
(e_j, e_j, e_j, e_j), \quad j \leq d,
(f_j, -f_j, -f_j, f_j), \quad j < d,
(0, 0, -f_j, f_j), \quad d < j < n.
\]

(4.23)

The restriction of the symplectic form on $T_b T^* (H^2 \times M^2)$ to $\ker d\alpha \times \ker \tau$ has the matrix

\[
\begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & -I
\end{bmatrix}
\]

with respect to the products of the bases (4.23) and (4.21). Since the absolute value of the determinant of this matrix is conveniently 1, and our $\frac{-1}{2}$-density on $\ker \tau$ assigns 1 to the basis in Lemma 4.11, the natural element in $|\ker d\alpha|^{-1/2}$ assigns 1 to the basis in the statement of the lemma.

**Proof of Lemma 4.13.** The short exact sequence (4.20) and the results of Lemmas 4.11 and 4.12 induce a natural trivialization

\[
1 \sim |\ker d\alpha|^{-1/2} \otimes |T_p F_0|^{-1/2} \otimes |T_\alpha C \circ \Lambda|^{1/2} \simeq |\ker d\alpha| \otimes |T_\alpha C \circ \Lambda|^{1/2}.
\]

Take the element in $|T_p F_0|^{-1/2}$ which assigns the value 1 to the basis in Lemma 4.10(1). Invoking both Lemmas 4.11 and 4.12, we see that the induced element in $|\ker d\alpha|^{-1/2}$ assigns the value

\[
(2\pi)^{-\text{codim} H/2 + 1/2} \chi(c)^2 (1 - c^2)^{-1/4}
\]

to the basis in Lemma 4.12. Note that the pushforward of $f_d$ and $f_n$ through $da$ is
\[ (-h, 0), \ (ch, -\sqrt{1-c^2}h), \]
and hence the object in $|\ker da| \otimes |T_a C \circ \Lambda|^{1/2}$ assigns this same value to the pair of bases
\[ e_1, \ldots, e_d, f_1, \ldots, f_{d-1}, f_{d+1}, \ldots, f_{n-1} \quad \text{and} \quad (-h, 0), (ch, -\sqrt{1-c^2}h). \]
Note that $(-h, 0), (ch, -\sqrt{1-c^2}h)$ is the image of the basis $(h, 0), (0, h)$ under a linear transformation of determinant $\sqrt{1-c^2}$. Hence, our object in $|\ker da| \otimes |T_a C \circ \Lambda|^{1/2}$ assigns the value
\[ (2\pi)^{-\mathrm{codim} H/2+1/2} \chi(c)^2 (1-c^2)^{-1/2} \]
to the pair of bases
\[ e_1, \ldots, e_d, f_1, \ldots, f_{d-1}, f_{d+1}, \ldots, f_{n-1} \quad \text{and} \quad (h, 0), (0, h). \]

This completes the proof. \qed

5 The proof of Theorem 1.5

Finally, we prove our Tauberian theorem. The idea at the core of the argument is based on the argument of de Verdière who obtained asymptotics of certain weighted counts of spectral measures in $\mathbb{R}^d$ over a homothetic family of regions with piecewise $C^1$ boundary in [7]. The main contribution here is to relate the remainder to the size of (a unit thickening of) the boundary of the region, rather than to a scaling parameter. This allows us to apply the Tauberian theorem to obtain estimates for sums over a joint spectrum in more exotic families of regions, as in Theorem 1.3.

We introduce some notation. For a subset $\Omega \subset \mathbb{R}^n$ and $0 \leq a < b$, we define
\[ \partial \Omega^{[a,b]} = \{x \in \overline{\Omega} : a \leq d(x, \Omega) \leq b\} \]
and
\[ \partial \Omega^{[-b,-a]} = \{x \in \overline{\Omega} : a \leq d(x, \Omega^c) \leq b\}, \]
and finally if $a < 0 < b$, we set $\partial \Omega^{[a,b]} = \partial \Omega^{[a,0]} \cup \partial \Omega^{[0,b]}$. As expected, $\partial \Omega^{[-1,1]}$ is the unit thickening of the boundary of $\Omega$.

To prove Theorem 1.5, we first record a couple of helpful lemmas. The first is an observation about integration of order functions over thickenings of regions in $\mathbb{R}^n$. The second uses the first and allows us to control $N(\partial \Omega^{[0,r]})$ in terms of the integral of $m$ over $\partial \Omega^{[-1,1]}$.

**Lemma 5.1.** Let $m$ be an order function on $\mathbb{R}^n$, $\Omega$ be a subset of $\mathbb{R}^n$, and $a$ and $b$ be real numbers for which $a < b$. Then there exist a constant $C$ and an exponent $\nu$ depending only on $n$ and $m$ for which
\[ \int_{\partial \Omega^{[a,b]}} m(x)dx \leq C(1 + \max(|a|, |b|))^{\nu} \int_{\partial \Omega^{[-1,1]}} m(x)dx. \]

**Lemma 5.2.** We assume all the hypotheses of Theorem 1.5 except that we do not require that $N(\Omega)$ be finite. Then for each $r \geq 1$,
\[ N(\partial \Omega^{[0,r]}) \leq C(1 + r)^{\nu} \int_{\partial \Omega^{[-1,1]}} m(x)dx \]
for some constant $C$ and exponent $\nu$ which are independent of $r$ and $\Omega$.

We first prove Theorem 1.5 and then the lemmas. Throughout the proofs, we use $a \lesssim b$ to mean that there exists a positive constant $C$ not depending on $\Omega$ for which $a \leq Cb$. We also use $a \gtrsim b$ to denote $b \gtrsim a$ and we use $a \approx b$ to mean $a \lesssim b$ and $a \gtrsim b$. 
Proof of Theorem 1.5. We write
\[
N(\Omega) - \rho \ast N(\Omega) = \int_{\mathbb{R}^n} \chi_{\Omega}(y) dN(y) - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{\Omega}(x) \rho(x - y) dN(y) dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\chi_{\Omega}(y) - \chi_{\Omega}(x)) \rho(x - y) dN(y) dx.
\]

We note
\[
\chi_{\Omega}(x) - \chi_{\Omega}(y) = \begin{cases} 
1, & x \in \Omega \text{ and } y \in \Omega^c, \\
-1, & x \in \Omega^c \text{ and } y \in \Omega, \\
0, & \text{otherwise},
\end{cases}
\]
and hence
\[
|N(\Omega) - \rho \ast N(\Omega)| \leq \int_{\Omega} \int_{\Omega^c} \rho(x - y) dN(y) dx + \int_{\Omega^c} \int_{\Omega} \rho(x - y) dN(y) dx.
\]

We claim that the first integral is bounded as
\[
\int_{\Omega} \int_{\Omega^c} \rho(x - y) dN(y) dx \lesssim \int_{\partial \Omega \setminus [1,1]} m(x) dx. \tag{5.1}
\]

As we proceed, we see that the claim holds similarly if we interchange \(\Omega\) and \(\Omega^c\), and hence the second integral satisfies the same bound. The theorem will follow.

For \(y \in \mathbb{R}^n\) and \(r \geq 0\), we let
\[
A(y, r) = \{x \in \mathbb{R}^n : r \leq |x - y| \leq r + 1\}
\]
denote the annulus centered at \(y\) with the inner radius \(r\) and the outer radius \(r + 1\). Since \(\rho\) is Schwartz-class, we write \(|\rho(x - y)| \lesssim (1 + |x - y|)^{-K}\) for \(K\) as large as we desire and \(C\) which depends on \(N\). Hence,
\[
\int_{\Omega} \int_{\Omega^c} \rho(x - y) dN(y) dx \lesssim \sum_{k=0}^{\infty} \int_{\partial \Omega^{[k+1,k+1]}} \int_{\Omega \setminus A(y,k)} |\Omega \cap A(y,k)| dN(y) dx
\]
\[
\lesssim \sum_{k=0}^{\infty} (1 + k)^{-K} \int_{\partial \Omega^{[k+1,k+1]}} (\Omega \cap A(y,k)) dN(y)
\]
\[
\lesssim \sum_{k=0}^{\infty} (1 + k)^{-K+n-1} N(\partial \Omega^{[0,k+1]})
\]
\[
\lesssim \sum_{k=0}^{\infty} (1 + k)^{-K+n+\nu-1} \int_{\partial \Omega^{[-1,1]}} m(x) dx,
\]
where the last line follows from the second lemma. (5.1) follows from taking \(K > n + \nu\) and summing the convergent series. \(\square\)

Proof of Lemma 5.1. Let \(P\) denote a maximal 1-separated subset of \(\partial \Omega^{[a,b]}\). Note
\[
m(x) \approx m(y) \quad \text{for } |x - y| \leq 1
\]
and, since the balls \(B(x,1)\) of radius 1 centered at points in \(P\) cover \(\partial \Omega^{[a,b]}\), we have
\[
\int_{\partial \Omega^{[a,b]}} m(x) dx \lesssim \sum_{x \in P} \int_{B(x,1)} m(y) dy \lesssim \sum_{x \in P} m(x).
\]

Now let \(Q\) be a maximal 1-separated subset of \(\partial \Omega^{[-1/2,1/2]}\). Note that the balls \(B(x,1/2)\) for \(X \in Q\) are disjoint and lie entirely in \(\partial \Omega^{[-1,1]}\). Hence,
\[
\int_{\partial \Omega^{[-1,1]}} m(x) dx \gtrsim \sum_{x \in Q} \int_{B(x,1/2)} m(y) dy \gtrsim \sum_{x \in Q} m(x).
\]
Hence, it suffices to show that
\[ \sum_{x \in P} m(x) \lesssim (1 + \max(|a|, |b|))^{\nu} \sum_{y \in Q} m(y). \]

For shorthand let \( r = 1 + \max(|a|, |b|) \). For each \( x \in P \), let \( y(x) \) denote some choice of \( y \in Q \) for which \( |x - y(x)| \leq r \). Such \( y(x) \) must always exist. If \( B(x, r) \cap Q \) were empty, then we would be able to place a point in \( B(x, r - 1) \cap \partial \Omega^{1 \, -2} \), which then would be 1-separated from all the other points in \( Q \).

Suppose that \( \nu' \) is the exponent in the bound on the order function \( m \), i.e., \( m(x) \lesssim (1 + |x - y|)^{\nu'} m(y) \). We have
\[
\sum_{x \in P} m(x) \lesssim r^{\nu'} \sum_{x \in P} m(y(x)) = r^{\nu'} \sum_{y \in Q} \# \{ x : y(x) = y \} m(y) \lesssim r^{\nu' + n} \sum_{y \in Q} m(y),
\]
where the last line follows from \( \# \{ x : y(x) = y \} \leq \# P \cap B(y, r) \lesssim r^n \) for each \( y \in Q \). This completes the proof of the lemma.

**Proof of Lemma 5.2.** Since \( \rho(0) > 0 \), there exist a \( \delta \in (0, 1] \) and a positive constant \( c \) for which \( \rho(x) \geq c \) for \( |x| \leq \delta \). Hence,
\[
\int_{B(x, \delta)} dN(y) \leq \frac{1}{c} \rho \ast N(x) \leq \frac{1}{c} m(x) \text{ for all } x \in \mathbb{R}^n. \tag{5.2}
\]

We let \( \tilde{N} \) denote the restriction of \( N \) to \( \partial \Omega^{[0, r]} \) and have, by Fubini’s theorem,
\[
N(\partial \Omega^{[0, r]}) = \int_{\mathbb{R}^n} d\tilde{N}(y) = \int_{\mathbb{R}^n} \int_{B(y, \delta)} \frac{1}{|B(y, \delta)|} dxd\tilde{N}(y) = \int_{\mathbb{R}^n} \int_{B(x, \delta)} \frac{1}{|B(x, \delta)|} d\tilde{N}(y) dx.
\]
Note that the inner integral is supported for \( x \) in a \( \delta \)-thickening of the support of \( \tilde{N} \), e.g., \( \partial \Omega^{[-1, 1]} \). This and \( \tilde{N} \leq N \) yield
\[
N(\partial \Omega^{[0, r]}) \leq \int_{\partial \Omega^{[-1, 1]} \setminus [B(x, \delta)]} \frac{1}{|B(x, \delta)|} \frac{1}{c |[B(0, \delta)]|} \int_{B(x, \delta)} dN(y) dx \leq \frac{1}{c |[B(0, \delta)]|} m(x) dx,
\]
where the second inequality follows from (5.2). The proof is completed by Lemma 5.1.

**Acknowledgements** This work was partially supported by National Science Foundation of USA (Grant Nos. DMS-1810747 and DMS-1502632). The second author was supported by National Natural Science Foundation of China (Grant No. 12171424). The authors are grateful to Madelyne Brown for pointing out an error in an earlier draft of this paper. The authors are also grateful to the referees for their thorough and invaluable feedback.

**References**

1. Bruggeman R W. Fourier coefficients of cusp forms. Invent Math, 1978, 45: 1–18
2. Bruggeman R W. Fourier Coefficients of Automorphic Forms. Lecture Notes in Mathematics, vol. 865. Berlin-Heidelberg: Springer-Verlag, 1981
3. Burq N, Gérard P, Tzvetkov N. Restrictions of the Laplace-Beltrami eigenfunctions to submanifolds. Duke Math J, 2007, 138: 445–486
4. Canzani Y, Galkowski J. On the growth of eigenfunction averages: Microlocalization and geometry. Duke Math J, 2019, 168: 2991–3055
5 Canzani Y, Galkowski J. Improvements for eigenfunction averages: An application of geodesic beams. J Differential Geom, 2023, in press
6 Canzani Y, Galkowski J, Toth J A. Averages of eigenfunctions over hypersurfaces. Comm Math Phys, 2018, 360: 619–637
7 de Verdière Y C. Spectre conjoint d’operateurs pseudo-differentiels qui commutent I. Le cas non integrible. Duke Math J, 1979, 46: 169–182
8 Duistermaat J J. Fourier Integral Operators. Boston: Birkhäuser, 1996
9 Duistermaat J J, Guillemin V. The spectrum of positive elliptic operators and periodic bicharacteristics. Invent Math, 1975, 29: 39–79
10 Geis M. Concentration of quantum integrable eigenfunctions on a convex surface of revolution. arXiv:2008.12482, 2020
11 Good A. Local Analysis of Selberg’s Trace Formula. Lecture Notes in Mathematics, vol. 1040. Berlin-Heidelberg-New York-Tokyo: Springer-Verlag, 1983
12 Guillemin V, Sternberg S. Semi-Classical Analysis. Boston: International Press, 2013
13 Hörmander L. Fourier integral operators I. Acta Math, 1971, 127: 79–183
14 Hörmander L. The Analysis of Linear Partial Differential Operators I–IV. New York: Springer-Verlag, 1983, 1985
15 Hu R. $L^p$ norm estimates of eigenfunctions restricted to submanifolds. Forum Math, 2009, 21: 1021–1052
16 Iwaniec H. Topics in Classical Automorphic Forms. Graduate Studies in Mathematics, vol. 17. Providence: Amer Math Soc, 1997
17 Iwaniec H. Spectral Methods of Automorphic Forms, 2nd ed. Graduate Studies in Mathematics, vol. 53. Providence: Amer Math Soc, 2002
18 Kuznecov N V. Petersson’s conjecture for cusp forms of weight zero and Linnik’s conjecture (in Russian). Sums of Kloosterman sums. Mat Sb, 1980, 111: 334–383
19 Levin L. Fourier Tauberian theorems. In: The Asymptotic Distribution of Eigenvalues of Partial Differential Operators. Translations of Mathematical Monographs, vol. 155. Providence: Amer Math Soc, 1997, 297–305
20 Rankin R A. Contributions to the theory of Ramanujan’s function $\tau(n)$ and similar arithmetical functions. III. A note on the sum function of the Fourier coefficients of integral modular forms. Math Proc Cambridge Philos Soc, 1940, 36: 150–151
21 Safarov Y. Fourier Tauberian theorems and applications. J Funct Anal, 2001, 185: 111–128
22 Selberg A. On the estimation of Fourier coefficients of modular forms. In: Proceedings of Symposia in Pure Mathematics, vol. 8. Providence: Amer Math Soc, 1965, 1–15
23 Sogge C D. Fourier Integrals in Classical Analysis, 2nd ed. Cambridge: Cambridge University Press, 2017
24 Sogge C D, Xi Y K, Zhang C. Geodesic period integrals of eigenfunctions on Riemannian surfaces and the Gauss-Bonnet theorem. Camb J Math, 2017, 5: 123–151
25 Wyman E L, Xi Y K. Improved generalized periods estimates over curves on Riemannian surfaces with nonpositive curvature. Forum Math, 2021, 33: 789–807
26 Wyman E L, Xi Y K, Zelditch S. Geodesic bi-angles and Fourier coefficients of restrictions of eigenfunctions. Pure Appl Anal, 2022, 4: 675–725
27 Xi Y K. Improved generalized periods estimates on Riemannian surfaces with nonpositive curvature. arXiv:1711.09864, 2017
28 Xi Y K. Inner product of eigenfunctions over curves and generalized periods for compact Riemannian surfaces. J Geom Anal, 2019, 29: 2674–2701
29 Zelditch S. Kuznecov sum formulae and Szegő limit formulae on manifolds. Comm Partial Differential Equations, 1992, 17: 221–260
30 Zworski M. Semiclassical Analysis. Graduate Studies in Mathematics, vol. 138. Providence: Amer Math Soc, 2012