Universal Gaussian Quantization With Side-Information Using Polar Lattices

Shubham Jha, Graduate Student Member, IEEE

Abstract—We consider universal quantization with side information for Gaussian observations, where the side information is a noisy version of the sender’s observation with noise variance unknown to the sender. In this paper, we propose a universally rate optimal and practical quantization scheme for all values of unknown noise variance. Our scheme uses Polar lattices from prior work, and proceeds based on a structural decomposition of the underlying auxiliaries so that even when recovery fails in a round, the parties agree on a common “reference point” that is closer than the previous one. We also present the finite blocklength analysis showing an sub-exponential convergence for distortion and exponential convergence for rate. The overall complexity of our scheme is $O(N^2 \log^2 N)$ for any target distortion and fixed rate larger than the rate-distortion bound.

Index Terms—Universal quantization, Wyner-Ziv coding, polar codes, polar lattices, side information.

I. INTRODUCTION

DISTRIBUTED quantization with side information at the decoder is a classic multiterminal information theory problem, studied first in the seminal work of [26]. We consider the Gaussian setting (cf. [19], [22], [27]) where the encoder and the decoder observe correlated Gaussian random variables (rvs) $X^N$ and $Y^N$, respectively. We are interested in the universal version of this problem where the variance of $X_i$ is known, but the variance of noise between $X_i$ and $Y_j$ is not known at the encoder. In this setting, we seek universally rate optimal and practical schemes.

This is indeed a well-studied problem. Perhaps the best understood variant is where the encoder observation statistics are unknown but the channel to the side information is known; see, for instance, [8], [12], [18], [20], [25] for results in this setting. Fewer results are available when the statistics of channel from $X$ to $Y$ are unknown, which is our setting of interest; the theory for general sources was studied in [24] and practical schemes using LDPC codes was considered in [5]. However, there is still no provably universally rate-optimal, practical code for this problem.

Drawing on results on interactive schemes for universal Slepian-Wolf problem from [3], [4], [23], we propose a practical universal Wyner-Ziv code between the encoder and the decoder. Unlike the preliminary version [9] of this work, only encoder is allowed to send messages in the current setting. Our scheme uses Polar lattices from [15], [17], where they were used for the Gaussian Wyner-Ziv problem with known channel statistics. In our scheme, the encoder communicates its messages in multiple rounds. In each round, encoder considers a new guess from the set of noise variances and use a code designed for that guess. We assume that the set of noise variances used by encoder is predefined and shared upfront with the decoder. That way, the decoder exactly knows the round to form the desired estimate. The total message bits communicated to the decoder till this round constitute the overall rate used by our scheme.

As a solution, it is well-known that the encoder needs to sample from an appropriate auxiliary rv $X'$ that forms the following Markov chain structure: $X' \rightarrow X \rightarrow Y$. However, the joint distribution of $(X', X, Y)$ depends on the channel statistics between $(X, Y)$-pair, unavailable to the encoder. To that end, our scheme uses more than one auxiliary and exploits a long resulting Markov chain structure. By using a structural observation for the underlying auxiliary rvs, our scheme ensures that even when the guess fails, the encoder and the decoder agree on a closer “reference point” which can be subtracted from both $X^N$ and $Y^N$. One of the key property of Polar codes used for the Wyner-Ziv problem (known channel statistics) in [17] is that the polarization operation preserves degradedness [29]. Further, this property has been instrumental in designing a rateless coding scheme [13] based on Polar codes. Our scheme too utilizes this property in order to ensure the optimal rate-distortion tradeoff in every round. Our presentation below focuses on describing the scheme and presenting the underlying theoretical guarantees that lead to it, which are technical. Sometimes, we use $P_1$ and $P_2$ to represent encoder and decoder, respectively.

Notation: Random variables and vectors are denoted in capital letters without and with bold fonts, respectively. Their realizations are expressed as their small letter counterparts. $X \sim P$ implies $X$ is distributed as $P$. $\var{X}$ denotes the variance of $X$, and $\cov{X}$ denotes the covariance of a joint rv $X$. $\var{P}{Q}$ and $\KL{P}{Q}$ denote the total variational distance and Kullback-Leibler (KL) divergence between two distributions $P$ and $Q$, respectively. $x$ is the shorthand for the vector $(x_1, \ldots, x_n)$, and $x(i:j)$ for subvector $(x_i, x_{i+1}, \ldots, x_j)$. $x_{ij}$ denotes the sequence.
of vectors $\mathbf{x}_i, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n$. $\mathbf{M}^T$ denotes the transpose of a matrix $\mathbf{M}$. $\mathbb{R}$ and $\mathbb{Z}$ represent the sets of real numbers and integers, respectively. $O$ represents the standard “Big O” notation, and we write $f(x) = O(g(x))$ as $x \to \infty$ if $\limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty$.

II. PRELIMINARIES

A. The Discrete Gaussian Distribution

An $n$-dimensional lattice $\Lambda$ is given by the set $\Lambda = \{ \lambda = \mathbf{u} \cdot \mathbf{C} : \mathbf{u} \in \mathbb{Z}^n \}$, where $\mathbf{C} = [c_1, c_2, \ldots, c_n]^T$ is a full rank $(n \times n)$-generator matrix. For a vector $\mathbf{x} \in \mathbb{R}^n$, we define the nearest-neighbor quantizer $Q$ associated with $\Lambda$ as $Q(\mathbf{x}) = \arg\min_{\lambda \in \Lambda} \| \lambda - x \|$, where ties are resolved arbitrarily. We also define the modulo lattice operation for $\mathbf{x}$ as $\mathbf{x} \mod \Lambda := \mathbf{x} - Q(\mathbf{x})$. The joint probability density function (pdf) of an $n$-dimensional Gaussian random vector $\mathbf{x}$, with mean $\mu \in \mathbb{R}^n$ and variance $\sigma^2$ for each independent coordinate, is given by

$$f_{\mu, \sigma}(\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp \left( -\frac{\| \mathbf{x} - \mu \|^2}{2\sigma^2} \right), \, \mathbf{x} \in \mathbb{R}^n.$$  

Given a lattice $\Lambda$, the discrete Gaussian distribution over $\Lambda \subset \mathbb{R}^n$ centered at $\mu$ is defined as

$$D_{\Lambda, \sigma, \mu}(\lambda) = \frac{f_{\mu, \sigma}(\lambda)}{\sum_{\lambda' \in \Lambda} f_{\mu, \sigma}(\lambda')} = \epsilon_\Lambda(\sigma) \frac{f_{\mu, \sigma}(\lambda)}{\sum_{\lambda' \in \Lambda} f_{\mu, \sigma}(\lambda')}, \, \lambda \in \Lambda.$$  

Namely, it is a probability mass function (pmf) over points of $\Lambda$ with mass of point $\lambda \in \Lambda$ proportional to the Gaussian density $f_{\mu, \sigma}(\lambda)$ at that point. Also, define the flatness factor $\epsilon_\Lambda(\sigma)$ [14]

$$\epsilon_\Lambda(\sigma) := \max_{x \in \mathcal{R}(\Lambda)} |V(\Lambda) \sum_{\lambda' \in \Lambda} f_{\mu, \sigma}(\lambda') - 1|,$$

where $\mathcal{R}(\Lambda)$ denotes the fundamental region of $\Lambda$ and $V(\Lambda)$ denotes its volume. That $\epsilon_\Lambda(\sigma)$ is a decreasing function of $\sigma$, and the normalizing factor in (1) is bounded as

$$1 - \epsilon_\Lambda(\sigma) \leq \frac{1}{V(\Lambda)} \sum_{\lambda' \in \Lambda} f_{\mu, \sigma}(\lambda') \leq 1 + \epsilon_\Lambda(\sigma) \frac{1}{V(\Lambda)}.$$  

The following result brings in the importance of flatness factor and shows that the distance between the output distributions for an additive Gaussian channel, when the input is Gaussian and discrete lattice Gaussian, can be controlled using the associated flatness factor of the lattice. In other words, lattices with small flatness factors can very well approximate the output distribution for any discrete lattice Gaussian input.

Lemma 1 [14, [21]: Consider multivariate rvs $\mathbf{X}, \mathbf{Y}, \tilde{\mathbf{X}}, \tilde{\mathbf{Y}}$ such that

$$\mathbf{Y} = \mathbf{X} + \mathbf{Z} \text{ and } \tilde{\mathbf{Y}} = \tilde{\mathbf{X}} + \mathbf{Z},$$

where $\mathbf{X} \sim \mathcal{N}(0, \sigma_X^2 \mathbf{I}_n)$, $\mathbf{Z} \sim \mathcal{N}(0, \sigma_Z^2 \mathbf{I}_n)$ and $\tilde{\mathbf{X}} \sim D_{\Lambda, \sigma_X, \mathbf{0}}$. Then, the total variational distance between distributions $P_Y$ and $P_{\tilde{Y}}$ is bounded as

$$d_{TV}(P_Y, P_{\tilde{Y}}) \leq 2\epsilon,$$

where $\epsilon = \epsilon_\Lambda(\frac{\sigma_X}{\sqrt{\sigma_X^2 + \sigma_Z^2}})$ denotes the flatness factor of noise variance scaled by Minimum Mean Square Error (MMSE) coefficient $\sigma_x/\sqrt{\sigma_x^2 + \sigma_z^2}$.

B. Polar Codes

It will be convenient to recall a general definition of Bhattacharyya parameter for our discussion on Polar codes.

Definition 1: For a channel $P_{Y|X}$ with a binary input $X$ and (possibly continuous) output $Y$, the Bhattacharyya parameter $Z(X \mid Y)$ is given by

$$Z(X \mid Y) = \int \frac{P_Y(y)\sqrt{P_{X|Y}(0|y)P_{X|Y}(1|y)}}{dy}. $$

The following proposition relates the parameter $Z(X \mid Y)$ with the conditional entropy $H(X \mid Y)$ of the rv $X$ given the rv $Y$.

Proposition 1 [2, Proposition 2]: For rvs $X$ and $Y$ with $X \in \{0, 1\}$, we have

$$Z(X \mid Y)^2 \leq H(X \mid Y) \leq Z(X \mid Y).$$  

In a Polar code, the input $X^N$ to $N = 2^k$ copies of a binary input channel $P_{Y|X}$ is transformed using the generator matrix $G_N = G^m \otimes$, where $G = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\otimes$ denotes the Kronecker product. The transformed bits $U^N = X^N G_N^{-1}$ under the binary field $\mathbb{F}_2 = \{0, 1\}$ operations are treated as new inputs, which we try to decode using channels $W_i^{(N)}$ from $U_i$ to $(Y_i, U_i^{-1})$. The seminal result of [2] states that the Bhattacharyya parameters of the channels $W_i^{(N)}$ tend to 0 or 1 as $N$ tends to infinity, and the fraction of indices $i$ for which it tends to 0 is exactly the symmetric-capacity of the channel. That is, the channels are “polarized” into perfect and useless channels. The indices of the bits with small (close to 0) Bhattacharyya parameters constitute the set of information bits and those with large (close to 1) ones constitute the set of frozen bits. The bits indexed in the information set can be determined almost error-free, provided that all the bits $U_i$s indexed in the frozen set are shared in advance. In this paper, we will be using Polar codes for degraded channels, which we define next.

Definition 2: Consider two channels $W_1 : \mathcal{X} \to \mathcal{Y}_1$ and $W_2 : \mathcal{X} \to \mathcal{Y}_2$. The channel $W_1$ is (stochastically) degraded with respect to $W_2$, denoted $W_2 \succeq W_1$, if there exists a channel $V : \mathcal{Y}_2 \to \mathcal{Y}_1$ such that $W_1(y_1|x) = \int W_2(y_2|x)V(y_1|y_2)dy_2$.

We use the fact that the information set for $W_2$ such that $W_2 \succeq W_1$ contains the information set for $W_1$ (cf. [11, Lemma 1.8]).

C. Polar Lattices

For a pair of lattices $(\Lambda, \Lambda')$ satisfying $\Lambda' \subseteq \Lambda$, $\Lambda'$ is said to be nested within $\Lambda$. $\Lambda/\Lambda'$ denotes the partition of $\Lambda$ into $m = \frac{V(\Lambda)}{V(\Lambda')} \frac{\Lambda}{\Lambda'}$ cosets of $\Lambda'$ in $\Lambda$. We call this as “binary partition” if $m = 2$. Consider a binary partition chain $\Lambda_0/\Lambda_1/\ldots/\Lambda_{\ell}$. For each partition $\Lambda_i/\Lambda_{i+1}$, a code $C_i$ selects a sequence of representatives $a_i \in \{0, 1\}$ for the cosets of $\Lambda_{i+1}$. Construction D [6] requires a set of linear binary codes $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_{\ell}$.

For our problem, we use the Polar lattices [15] which construct capacity achieving Polar codes on each level (based on Construction D) and are known to exhibit a natural nested
structure across levels. It has been shown in [17] that Polar lattices have the potential for Gaussian Wyner-Ziv problem, where the solution consists of two nested Polar lattices – one is AWGN-good and the other is Gaussian rate-distortion bound achieving. This is in accordance with results in [28], where authors have shown that the Wyner-Ziv problem can be solved by nested quantization-good and AWGN-good lattices.

We refer the interested readers to [17], [28] for more details solved by nested quantization-good and AWGN-good lattices. where authors have shown that the Wyner-Ziv problem can be on the goodness properties of such lattices.

III. PROBLEM FORMULATION

We consider the Gaussian rate-distortion problem where the observations are independent copies of jointly Gaussian rvs \((X, Y)\) given by

\[
X = Y + Z,
\]

where \(Y\) and \(Z\) are independent Gaussian rvs with zero means and variances \(\sigma_Y^2\) and \(\sigma_Z^2\), respectively. For our setting, it is more convenient to fix the variance \(\sigma_Y^2\) of \(X\) and express \(\sigma_Y^2\) as \(\sigma_X^2 - \sigma_Z^2\).

Specifically, let \(\{(X_i, Y_i)\}_{i=1}^N\) be \(N\) independent and identically distributed (i.i.d.) copies of Gaussian rvs with joint pdf \(P_{XY} = \mathcal{N}((0 0)^T, K)\), where for \(\sigma_Y^2 > \sigma_X^2\) \(K\) is the covariance matrix given by

\[
\begin{bmatrix}
\sigma_X^2 & -\sigma_X^2 \\
-\sigma_X^2 & \sigma_Z^2 - \sigma_X^2
\end{bmatrix}.
\]

For brevity, we use the abbreviation \((X, Y) := (X_i, Y_i)_{i=1}^N\). Throughout this paper, for an estimate \(\hat{x}\) of any vector \(x\), we fix the distortion measure \(D(x, \hat{x})\) to be the squared Euclidean distance given by

\[
D(x, \hat{x}) := \|x - \hat{x}\|^2.
\]

While there can be various possible applications, our formulation is guided by the following application. Suppose parties \(P_1\) and \(P_2\) have access to two correlated files \(F_1\) and \(F_2\), respectively, such that the “amount” of correlation between files is known only to \(P_2\). \(P_1\) observes \(F_1\) and prepares a compressed version comprising multiple small fragments. On the other hand, \(P_2\) observes \(F_2\) and uses its knowledge of correlation between \(F_1\) and \(F_2\) to download as few number of compressed segments as needed to recover \(F_1\) to a prescribed distortion using these compressed fragments. The challenge here is that \(P_1\) is not aware of the correlation between files, which makes it difficult to compress \(F_1\) appropriately. When \(P_1\) knows the correlation, it can compress \(F_1\) using standard Wyner-Ziv codes. However, in the absence of this knowledge, we need a universal coding scheme. We capture the requirement above formally as follows:

Parties \(P_1\) and \(P_2\) observe the sequences \(X\) and \(Y\), respectively, generated according to \(P_{XY}\), and the goal for \(P_2\) is to estimate \(P_1\’s\) observation \(X\) within a fixed target distortion \(\Delta\). We assume that number of samples observed is large enough such that parties can infer the marginal moments up to an acceptable accuracy. Further, to model the nescience of correlation and \(Y\’s\) uncertainty at \(P_1\), and thereby, to capture the universal behaviour, we make the following assumption.

Assumption 1: The variance \(\sigma_Y^2\) is known to both \(P_1\) and \(P_2\), but \(\sigma_X^2\) is known only to \(P_2\). Further, \(\sigma_X^2\) lies in a closed positive interval \(I \subseteq \mathbb{R}_+\).

We consider schemes where \(P_1\) encodes \(X\) using a finite sequence of \(r\) increasing rates \(R_1, \ldots, R_r\), representing \(r\) different fragments of encoded data. \(P_2\) downloads the first \(k\) segments of total rate \(R_1 + \cdots + R_k\), where \(k\) is decided by \(P_2\) using its knowledge of the correlation \(\sigma_Y^2\).

More formally, we consider \(r\)-round Wyner-Ziv (WZ) codes consisting of encoders and decoders \((e_i, d_i), i \in [1, \ldots, r]\). Each \(e_i\) is an encoder of rate \(R_i\) whose output \(C_i\), given by \(C_i = e_i(X)\), is an \(NR_i\) length bit-string and each \(d_i\) is a decoder that uses \(C_1, \ldots, C_i\) along with the side-information \(Y\) to form an estimate of \(X\). \(P_2\) forms the estimate \(\hat{X}\) in any round \(k\) by applying the decoder \(d_k\) given by \(\hat{X} = d_k(C_1, \ldots, C_k, Y)\). Note that \(d_k\) may use previous decoder outputs till round \(k - 1\).

Recall that when \(\sigma_Y^2\) is known at both encoder and decoder, for \(N\) sufficiently large, the minimum rate \(R^*\) required to attain distortion \(\Delta\) is roughly \(\frac{1}{2} \log \frac{\sigma_Y^2}{\Delta}\) [17], [26]. Our goal is to design codes attaining this rate universally (that is, even when \(P_1\) doesn’t know \(\sigma_Y^2\)) for all values \(\sigma_Y^2\), which are known to be lying in a fixed interval \(I\). Specifically, the universal \(r\)-round WZ code definition below requires that for all (unknown) noise variances \(\sigma_Y^2\), there exists a integer \(k \leq r\) such that \(d_k\) can recover an estimate of \(X\) from \((C_1, \ldots, C_k, Y)\) and the total rate used \(R_1 + \cdots + R_k\) is roughly \(\frac{1}{2} \log \frac{\sigma_Y^2}{\Delta}\) (the optimal rate for known \(\sigma_Y^2\)).

Definition 3 (Universal WZ Codes): For \(\delta, \epsilon > 0\), a fixed \(\Delta > 0\) and a closed interval \(I \subseteq \mathbb{R}_+\), an \(r\)-round WZ code is \((\epsilon, \delta)\)-universal at distortion level \(\Delta\) for \(I\) if for every \(\sigma_Y^2 \in I\), there exists a \(k \leq r\) such that

\[
\sum_{j=1}^k R_j \leq \frac{1}{2} \log \frac{\sigma_Y^2}{\Delta} + \epsilon, \quad \mathbb{E}\|X - \hat{X}\|^2 \leq N\Delta + \delta.
\]

We emphasize that we don’t consider the related problem of identifying the appropriate \(k\) using \(X\) and \(Y\). A particular method for this, which requires \(P_2\) to form an estimate \(\hat{X}\) and compare it with \(X\) using its hash, was considered in an earlier version of this paper [9]. In the current setting, \(P_2\) decides the right \(k\) in the beginning itself based on the knowledge \(\sigma_Y^2\). We describe this later in next section.

It is important to note that for \(\sigma_Y^2 \leq \Delta\), the estimate \(\hat{X} = Y\) constitutes an acceptable estimate, and therefore, we are interested in the case when \(\sigma_Y^2 \geq \Delta\). Accordingly, we assume that \(\omega \geq \Delta\), for all \(\omega \in I\).

A remark on terminology: For consistency with the earlier conference version of the paper, which was addressing a slightly different interactive variant of the problem, we will use the phrase “\(k\)-round code” to represent the code corresponding to encoders \(e_1, \ldots, e_k\) and decoder \(d_k\).

IV. PROPOSED SCHEME FOR UNIVERSAL QUANTIZATION

In this section, we propose the strategies for \(P_1\) and \(P_2\) achieving the rate-distortion bound universally.

A. A Review of the Basic Polar Code Based Scheme

We will review first the classic scheme from [26], which forms the basis of many practical schemes. In that setting, \(\sigma_Y^2\) is assumed to be known at the encoder \(P_1\) too. The
scheme uses an auxiliary rv $X'$, which minimizes the conditional mutual-information $I(X \wedge X' \mid Y)$ and is independent of $Y$ given $X$. For the Gaussian case which is of interest to us, this auxiliary takes a simple form given by (cf. [19])

$$X' = X + T,$$  

(5)

where $T$ is a Gaussian rv with mean zero and variance $\tilde{\Delta} = \frac{\sigma^2}{\sigma^2 - \Delta}$, independent of $(X, Y)$. Denote by $P_{X'XY}$ the joint distribution of rvs due to (4) and (5). Given the pair of rvs $(X', Y)$, one can form the MMSE estimate $\hat{X}_{MMSE}$ of $X$ given $X' \wedge Y$ by

$$\hat{X}_{MMSE} \leftarrow \mathbb{E}_{P_{X'XY}}[X \mid X', Y],$$

for which the MSE is $\Delta$. Then, a naive solution for $A_1$ is to generate $N$ independent samples $X'^N$ from this auxiliary such that $(X'_i, X_i, Y_i)_{i=1}^N$ are i.i.d. and send them to $P_2$, who in turn, uses $X'^N$ and $Y^N$ to construct $\hat{X}_{MMSE}$. However, this will require too much communication. To alleviate that, one can use shared randomness to simulate these samples $X'^N$ at $P_1$ and send them to $P_2$ using much less communication. This is an interpretation of the classic Wyner-Ziv scheme; the scheme in [15], too, can be interpreted in this manner.

To facilitate the simulation mentioned above, it is more appropriate to consider an alternative form of the Markov model in (5). Specifically, we consider the following generative model:

$$X = \frac{\sigma^2}{\sigma^2 + \tilde{\Delta}}X' + T' \quad \text{and} \quad \tilde{Y} = X + Z',$$  

(6)

where $T'$ and $Z'$ are independent Gaussians with zero means and variances $\frac{\sigma^2}{\sigma^2 + \tilde{\Delta}}$ and $\frac{\tilde{\Delta}}{\sigma^2 - \tilde{\Delta}}$, respectively. Let $A := \frac{\sigma^2}{\sigma^2 + \tilde{\Delta}}X'$, and denote by $Q_{XY}$ the joint distribution of rvs from (6). Then, the above generative model satisfies $P_{X'XY} = Q_{XY}$, and the pair $(A, \tilde{Y})$ still allows us to form an estimate of $X$ which is as accurate as that can be formed using $(X', Y)$. In particular, the corresponding MMSE estimate $\mathbb{E}_{Q_{XY}}[X \mid X', Y] = \hat{X}_{MMSE}$ and the MSE value is $\Delta$. Also, the mutual information $I(A \wedge X \mid \tilde{Y})$ equals $\frac{1}{2} \log \frac{\sigma^2}{\tilde{\Delta}}$, the optimal rate for getting distortion $\Delta$.

However, the following problem still remains. We need to quantize the samples before communicating. Towards that, several quantization methods have been proposed using structured codes of which the most recent, [15], is using Polar codes from [2]. The idea is to use a lattice Gaussian rv from (1) instead of a continuous one. In particular, a discrete Gaussian rv $\tilde{A}$ is considered over a one-dimensional lattice $\Lambda = N^{-1/2} \mathbb{Z}$ instead of the Gaussian rv $A$, and (6) is modified as

$$\tilde{X} = \tilde{A} + T' \quad \text{and} \quad \tilde{Y} = \tilde{X} + Z'.$$

Note that the choice for $\Lambda$ is not arbitrary. It was shown in [17] that for $c = O(N^{-1/2})$, the “flatness factor” associated with the lattice $c \cdot \mathbb{Z}$ is negligible, which further ensures that the induced distribution $P_{\tilde{X} \tilde{Y}}$ is close to $P_{XY}$ in total variational distance (cf. Lemma 1). For the sake of completeness, we present this in Proposition 3. Further, to ease our presentation, we take the orderwise constant in the value of $c$ to be unity. Note that while rv $\tilde{A}$ takes values only in the lattice $N^{-1/2} \mathbb{Z}$, rvs $T'$ and $Z'$ take values in $\mathbb{R}$.

Further, due to this closeness in joint pdfs, the samples $X^N$ to be quantized can be approximated as $N$ independent copies $\tilde{X}^N$ of $\tilde{X}$. This is tantamount to viewing $X'^N$ as being generated by first generating $\tilde{A}^N$ and then adding Gaussian noise to it. From here on, we will simply view our observations as coming from this new modified distribution.

The rest of the scheme proceeds as before, and the parties use structured codes to simulate $\tilde{A}^N$. However, this new auxiliary is still an infinite-precision number. The last component of lattice construction in [15] is the observation that we need not recover $A$s completely, and it suffices to agree on the $\ell$ least significant bits with $\ell = O(\log N)$. It is useful to note that this choice of $\ell$ together with the lattice $N^{-1/2} \mathbb{Z}$ has been crucial in establishing a sub-exponential convergence to the optimal rate-distortion bound $R^*(\Delta)$.

Finally, the scheme uses Polar codes to simulate and share the $\ell$ least significant bits of each coordinate of $\tilde{A}^N$ at $P_2$. Specifically, $P_1$ uses Polar codes as a covering code to recover the information bits of $\tilde{A}^N$ at each level $1 \leq j \leq \ell$, and $P_2$ uses it as a packing code for the channel from $\tilde{A}$ to $\tilde{Y}$; the common frozen bits are sampled from shared randomness.

B. The Universal Scheme

Coming to our universal case, since $\sigma^2$ is not known to $P_1$, it cannot fix the distribution of $A$ upfront. Instead, we consider $r$ distinct auxiliaries for our scheme motivated by the infinite divisibility property of Gaussians. Each of these auxiliaries corresponds to a different possible value of the unknown $\sigma^2$. Without loss of generality, let $I$ (c.f. Definition 3) takes the form $I := [\sigma_0^2, \sigma_2^2]$ for some $\sigma_0, \sigma_2 > 0$. Further, consider a finite grid of points to cover the entire continuum $I$. In particular, let $\sigma_0^2 \leq \cdots \leq \sigma_k^2$ be an increasing $r$-tuple partitioning $I$ into $r$ sub-intervals. We assume that the tuple $(\sigma_0^2, \ldots, \sigma_k^2)$ is known to both $P_1$ and $P_2$. For $1 \leq k \leq r$, the value $\sigma_k^2$ corresponds to the possibility that $\sigma^2 \in [\sigma_{k-1}, \sigma_k]$.

Denote by $A^{(1)}, \ldots, A^{(r)}$ the optimal auxiliaries corresponding to noise variances $\sigma_0^2, \ldots, \sigma_k^2$, respectively. These auxiliaries can be viewed as forming a Markov chain depicted in Fig. 1. This Markov chain which couples all these auxiliaries is instrumental in the design of our scheme. Specifically, we observe that the auxiliary $A^{(k)}$ used in round $k$ can be decomposed

$$A^{(k)} = \sum_{j=0}^{k-1} B_j,$$

(7)

where rvs $B_j$ are specified by following sequence of distributions: $B_0 \sim N(0, \sigma_1^2 \Lambda_1), B_i \sim N(0, \alpha_i \Lambda_i - \alpha_{i+1} \Lambda_{i+1})$, $1 \leq i \leq r - 1$, and $B_r \sim N(0, \alpha_r \Delta_r)$, with $\Delta_i = \frac{\sigma_i^2}{\sigma_i^2 - \Delta}$. Then, $A^{(k)} \sim N(0, \sigma_k^2 \Lambda_k)$.
\[
\frac{\sigma_k^2}{\sigma_{k+1}^2}, \quad \forall i \in [r]. \quad \text{That implies, for } 1 \leq k \leq r, A(k) \text{ is a Gaussian rv with mean zero and variance } \alpha_k \sigma_k^2. \quad \text{This decomposition is the key step in designing our rate-optimal strategy.}
\]

In round \(k\), we subtract the previously recovered “parts” \(B_0, \ldots, B_{k-2}\) and treat the residue as the new observation, i.e., the pair \((X, \bar{Y})\) is replaced by \((X - A(k-1), \bar{Y} - A(k-1))\). The main idea driving our scheme is that even when \(\sigma_k^2 \geq \sigma_{k+1}^2\), both \(P_1\) and \(P_2\) will end up recovering \(B_{k-1}\) and thereby \(A(k)\), which is an optimal auxiliary for the noise variance \(\sigma_k^2\). In round \(k+1\), this can be subtracted from both \(X\) and \(\bar{Y}\) by \(P_1\) and \(P_2\), respectively.

Heuristically, when the parties begin, they only agree on the origin 0 as the “reference point”. But in each round they agree on a (on average) closer reference point, which they subtract from both their observations. Since, \(P_2\) knows the value of \(\sigma_k^2\), it exactly knows the number of rounds \(k\) needed for its estimation task. The new observation pairs for round \(k\), denoted as \((X(k), Y(k))\), is obtained by subtracting \(A(k-1)\) from the pair \((X, \bar{Y})\). We illustrate this distribution in Fig. 2.

Note the Markov chain for round \(k\):
\[
X(k) = B_{k-1} + T_k \quad \text{and} \quad Y(k) = X(k) + Z',
\]
where \(T_k = \sum_{i=k}^{r} B_i\) has distribution \(N(0, \alpha_k \Delta_k)\).

For consistency, we take \((X(1), Y(1)) = (X, \bar{Y})\). Further, as described earlier, we simulate samples from the lattice Gaussian distribution \(B_{k-1} \sim D_{N-1/2,Z\text{var} (B_{k-1})} O\) instead of \(B_{k-1}\) and modify (8) as
\[
\tilde{X}(k) = \tilde{B}_{k-1} + T_k \quad \text{and} \quad \tilde{Y}(k) = \tilde{X}(k) + Z'.
\]

Henceforth, we consider \(N\) i.i.d. samples from the distribution in (9), i.e.,
\[
\tilde{X}(k) = \tilde{B}_{k-1} + T_k, \quad \text{and} \quad \tilde{Y}(k) = \tilde{X}(k) + Z'.
\]

For \(1 \leq l \leq \ell\), let \(U(l)i \equiv H_i G_N^{-1}\) where each coordinate \(H_i(j)\) of \(H_i\) corresponds to the \(j\)-th least significant bit in the binary representation of scaled \(j\)-th coordinate \(N^{1/2}B_{k-1}(j) \in \mathbb{Z}_2\). Recall that \(G_N\) is the \(N \times N\) generator matrix for Polar codes. It has been noted in [15], [17] that \(\forall i \in \{1, \ldots, \ell\}\), the channel between \(H_i\) and \(\tilde{X}(k)\) conditioned on the event \(\{H_{1,1} = h_{1,1}\} = h_{1,1}\) may not be symmetric in general. For \(P_1\), the index set \([N]\) is partitioned into the information set \(I_{1,1}^{(k)}\) and the frozen set \(F_{1,1}^{(k)}\) defined as follows: \(F_{1,1}^{(k)}\) is the set of
\[
\begin{align*}
\text{Algorithm 1} & \quad P_1\text{'s Strategy in Round } k \\
\text{Require:} & \quad \sigma_k^2, \Delta, \Lambda = N^{-1/2}Z, \ell, \{dF_{1,l}^{(k)}\}_{l \in [\ell]}, x^{(1)} = x, a^{(k-1)}\text{'s shared randomness (13)} \\
\text{Initialize:} & \quad x^{(k)} \leftarrow x - a^{(k-1)}, u_{1,l}^{(k)} \leftarrow \{0\}_N \\
1: & \quad \text{for } l \in [\ell] \text{ do} \\
2: & \quad \text{for } j \in [N] \text{ do} \\
3: & \quad \text{if } j \in T_{1,l}^{(k)} \text{ then} \\
4: & \quad \quad \text{Compute } p_{(k)}^{(j)} \text{ as in (12)} \\
5: & \quad \quad \text{Set } u_{1,l}^{(k)}(j) \leftarrow \text{Bernoulli}(1/1 + p_{(k)}^{(j)}) \\
6: & \quad \quad \text{else} \quad \text{Use (13) to determine } u_{1,l}^{(k)}(j) \\
7: & \quad \quad \text{Send } u_{1,l}^{(k)}(dF_{1,l}^{(k)}) \text{ to } P_2 \\
8: & \quad \quad \quad \quad \quad \quad \quad \quad \text{set difference of frozen-bit sets} \\
9: & \quad \quad \quad \quad \quad \quad \quad \quad \text{Lattice point} \\
10: & \quad \quad \quad \quad \quad \quad \quad \quad \text{Auxiliary update} \\
\end{align*}
\]

indices \(j \in [N]\) satisfying for any \(\beta \in (0, 1/2),
\[
Z(U_l^{(k)}(j) | U_l^{(k)}(1 : j - 1), U_l^{(k)}(1 : l - 1), \tilde{X}(k)) \geq 1 - 2^{-N^o}, \quad \text{or}
\]
\[
Z(U_l^{(k)}(j) | U_l^{(k)}(1 : j - 1), U_l^{(k)}(1 : l - 1), \tilde{X}(k)) \leq 2^{-N^o}. \quad (10)
\]

Similarly, the index set \([N]\) is partitioned into \(I_{2,l}^{(k)}\) and \(F_{2,l}^{(k)}\).

These definitions of information and frozen sets are from [7] where Polar codes for asymmetric channels were analysed. It has a slightly different form in comparison to the original definition in [2]. Note that we have defined the frozen set for \(P_1\) and the information set for \(P_2\). The reason for this distinction is that we use Polar codes to construct a covering (source) code for \(P_1\) and a packing (channel) code for \(P_2\); see [7]. Since the channel between each \(H_i\), \(1 \leq l \leq \ell\), and \(Y_k\) is perfectly known to the decoder, it constructs its frozen set \(F_{2,l}^{(k)}\) in advance for all the levels and shares them with \(P_1\).

With that, we specify the overall protocol to be used in round \(k\) by \(P_1\) and \(P_2\) in Algorithms 1 and 2, respectively. \(P_2\) uses MAP rule for realizing its information bits with posterior \(p_{2,l}^{(k)}(j_u)\) given by
\[
p_{2,l}^{(k)}(j_u) = P(U_l^{(k)}(j_u) | U_l^{(k)}(1 : j - 1), U_{1,l - 1}^{(k)}, X(k)) \left( u \mid u_{1,l - 1}^{(k)}(1 : j - 1), \tilde{X}(k) \right),
\]
where as \(P_1\) uses randomized MAP rule with quantity \(p_{1,l}^{(k)}(j)\) defined as ratio
\[
\begin{align*}
P_{1,l}^{(k)}(j_u) & \left( U_l^{(k)}(1 : j - 1), U_{1,l - 1}^{(k)}, X(k) \right) \left( 0 | u_{1,l - 1}^{(k)}(1 : j - 1), u_{1,l - 1}^{(k)}(1 : j - 1), u_{1,l - 1}^{(k)}(1 : j - 1), \tilde{X}(k) \right) \\
P_{1,l}^{(k)}(j_u) & \left( U_l^{(k)}(1 : j - 1), U_{1,l - 1}^{(k)}, X(k) \right) \left( 1 | u_{1,l - 1}^{(k)}(1 : j - 1), u_{1,l - 1}^{(k)}(1 : j - 1), u_{1,l - 1}^{(k)}(1 : j - 1), \tilde{X}(k) \right)
\end{align*}
\]
Algorithm 2 $P_2$’s Strategy in Round $k$

Require: $\sigma^2_i, \Delta, \Lambda = N^{-1/2}Z_i, \ell', \{dF_{\ell'}^{(k)}\}_{l \in \ell'}, \tilde{y} = \frac{\sigma^2}{\sigma^2 - \sigma^2_2} \cdot y$, $\hat{a}^{(k-1)}$, shared randomness (13)

Communication received: $\{u_i^{(k)}(dF_{\ell'}^{(k)})\}_{l \in \ell'}$

Initialize: $y^{(k)} \leftarrow \tilde{y} - \hat{a}^{(k-1)}, \hat{u}_{1:2} \leftarrow [0]_{N \times \ell}$

1: for $l \in \ell'$ do
2:     for $j \in [N]$ do
3:         if $j \in \mathcal{Z}_2^{(k)}$ then
4:             Compute $p_{2, l}^{(k)}(j_a)$ as in (11)
5:             $u_i^{(k)}(j) \leftarrow \arg\max_{j_a} p_{2, l}^{(k)}(j_a) \quad \triangleright \text{MAP rule}$
6:         else if $j \in dF_{\ell'}^{(k)}$ then
7:             $u_i^{(k)}(j) \leftarrow u_i^{(k)}(j)$ \quad \triangleright \text{Received bits}
8:         else Use (13) to determine $u_i^{(k)}(j)$
9:     $\hat{b}_{k-1} \leftarrow N^{-1/2} \left(u_i^{(k)}(1) + \ldots + 2^{l-1} u_i^{(k)}(2^l - 1)\right) \mod 2^l \mathbb{Z} \quad \triangleright \text{Lattice point}$
10: $\hat{a}^{(k)} \leftarrow \hat{a}^{(k-1)} + \hat{b}_{k-1} \quad \triangleright \text{Auxiliary update}$
11: $\hat{z}^{(k)} \leftarrow \hat{a}^{(k)} + \frac{\sigma^2_2 \hat{\Delta} (\sigma^2 - \sigma^2_2)}{\sigma^2 \sigma^2_2 (\sigma^2 - \sigma^2_2) + \sigma^2_1 \sigma^2_2} (\tilde{y} - \hat{a}^{(k)}) \quad \triangleright \text{Reconstruction}$

for the same. The paper [10] discusses several advantages of using a randomized MAP rule while encoding.

Also, due to a (stochastically) degraded channel structure between the channels from $H_l$ to $\tilde{X}_k$ and $H_l$ to $\tilde{Y}$ conditioned on $rvs H_{1:l-1}$, we notice that $F_{\ell'}^{(k)} \subseteq F_{\ell'}^{(k)} \quad \forall l \geq 1$. Therefore, $P_1$ and $P_2$ use randomized mapping to realize the bits in all the layers $\{dF_{\ell'}^{(k)}\}_{l \geq 1}$, while remaining bits in $\{dF_{\ell'}^{(k)} \setminus F_{\ell'}^{(k)}\}_{l \geq 1}$ are communicated as $C_l$. This randomized mapping can be shared in advance and realized using pseudo-random numbers generated as follows (see [7]): For all $l \geq 1, j \in [N],$

$$u_i^{(k)}(j) \sim \text{Ber}\left(P \left[ u_i^{(k)}(j) \mid u_i^{(k)}(j-1) \right] \right) \left(1 \mid u_i^{(k)}(1 : j - 1), u_i^{(k)}(j) \right)$$

Once $P_2$ decodes all its frozen bits located in $F_{\ell'}^{(k)}$ using communicated bits $C_l$ and the shared randomness, the remaining bits in $\mathcal{Z}_2^{(k)}$ are recovered with high probability using MAP rule (See Line 5, Algorithm 2). This is essentially due to the capacity-achieving property of Polar codes. The final step in Algorithm 2 is to add all the previously recovered lattice points (auxiliaries) and form the MMSE estimate.

V. ANALYSIS

Our analysis can be understood as follows. First considering the Gaussian distribution in (8), and then moving, in steps, to the discrete Gaussian distributions $\{\tilde{A}^{(i)} : i \in [r]\}$, and finally to the distribution simulated using Polar codes, retaining only the $\ell$ least significant bits. Following [17], we show that all these distributions are close to each other and mean-squared error guarantees for one translates to that for the other. Indeed, following a result from [14], we notice that for the discrete Gaussian auxiliary $\tilde{A}^{(i)}$ taking $N$ to be sufficiently large renders $P_{X,Y}$ close to $P_{X,Y}$. Further, a covering bound for Polar codes for asymmetric channels from [7] ensures that the samples simulated using Polar codes in our algorithm are close in distribution to those obtained by sampling the lattice based distribution in (9).

We note that the rate used in the scheme of [17] is close to $I(\tilde{A} \wedge \tilde{X} \mid \tilde{Y})$, which is shown to be close to $I(A \wedge X \mid Y)$ using similar approximations as those above. The key observation we make is that we can decompose this rate into those corresponding to different grid points in $[\sigma^2_1, \ldots, \sigma^2_2]$ for $\sigma^2_2$, whereby even when $\sigma_2 \leq \sigma_1$ in round $j$, the parties will agree on the optimal auxiliary corresponding to the grid point. This auxiliary can then be subtracted from both $X$ and $Y$, resulting in small variances for both. Since the mean-square distance between the resulting input pairs remains the same even after this subtraction, we end-up having another instance of the same Wyner-Ziv set up. Formally, we observe the following for continuous rvs.

Lemma 2: Consider the Markov chain in (8). For every $1 \leq k \leq r$ and the optimal auxiliary $A_k$ for $\sigma^2_2$ (7), the mutual information quantity $I(A_k \wedge X \mid Y)$ equals to

$$\sum_{j=1}^{k} I(B_{j-1} \wedge (X - A_{j-1}) \mid (\tilde{Y} - A_{j-1})), \quad (14)$$

with $A^{(0)} = 0$. Moreover, the MMSE estimate of $X$ given $\tilde{Y}$ and $A^{(k)}$ is given by

$$\hat{X}^{(k)} = A^{(k)} + \gamma (\tilde{Y} - A^{(k)}), \quad (14)$$

where $\gamma := \frac{\sigma^2_2 \hat{\Delta} (\sigma^2 - \sigma^2_2)}{\sigma^2_2 \sigma^2_2 (\sigma^2 - \sigma^2_2) + \sigma^2_1 \sigma^2_2}$, under which, the distortion achieved is

$$\sigma^2_2 \sigma^2_2 (\sigma^2 - \sigma^2_2)$$.

In practice, we form the estimate of $X$ by replacing $A^{(k)}$ with its decoded proxy; see Line 12, Algorithm 2. The final form of the estimate suggests that we can simply subtract $A^{(k)}$, once recovered, from $Y$ and $X$. This provides a clear justification for our algorithm for Gaussian rvs. The main technical step is to retain these claims when we move to discrete lattice Gaussian distribution, which we do in the manner outlined above. The following proposition characterizes the rate-loss incurred when one uses discrete lattice Gaussian auxiliaries instead of continuous.

Proposition 2 [14, Th. 2]: Consider the Markov chains in (8) and (9) with $\Lambda = N^{-1/2}Z_i$. Denote by $\tilde{\sigma}_{l-1}$ the variance of $B_{l-1}$. Let $\tilde{\sigma}_{l-1} := \tilde{\sigma}_{l-1} \sqrt{\text{Var}(\tilde{X}_l + Z_l)}$ and $\varepsilon(\tilde{\sigma}_{l-1})$ be the associated flatness factor. Then, if $\varepsilon = \varepsilon(\tilde{\sigma}_{l-1}) < 0.5$ and $\frac{\pi \varepsilon_l}{1 - \varepsilon_l} \leq \varepsilon$, we have for all $k \in [r]$ that

$$I(B_{k-1} \wedge \tilde{Y}^{(k)}) > I(B_{k-1} \wedge Y^{(k)}) - 5 \varepsilon. \quad (15)$$
A similar lower bound also holds in the source coding counterpart (cf. [17, Th. 1]) for $I(\hat{B}_{k-1} \wedge \hat{X}^{(k)})$ with an equivalent variance $\sigma_{cov}^2 = \theta_{k-1} \frac{\text{var}(Y)}{\text{var}(X)^2}$ instead of $\sigma_{\text{pack}}^2$, i.e.,

$$I(\hat{B}_{k-1} \wedge \hat{X}^{(k)}) \geq I(\hat{B}_{k-1} \wedge X^{(k)}) - 5\beta. \quad (16)$$

As evident from (15) and (16), the mutual-information losses incurred due to using discrete lattice auxiliaries is almost $5\beta$. As a result, the gap to the optimal rate depends on flatness factors for the equivalent noise variances associated with $\Lambda$. Motivated by this, our aim is to choose a lattice $\Lambda$ such that the associated flatness factor becomes negligible. Also, as described above, we consider working with only $\ell = O(\log N)$ least significant bits. We capture all these together formally in the following proposition.

**Proposition 3** ([17, Proposition 1]): Let $\eta = O(N^{-1/2})$ and $X$ be the discrete Gaussian r.v over the lattice $\eta Z$ distributed as $D_{\eta Z, \sigma_{\eta}^2}$ as defined in (1). Consider an additive Gaussian noise channel with mean zero and variance $\sigma_{\eta}^2$ with input $X$ and output $Y$. For $\bar{\sigma} \triangleq \frac{\sigma_{\eta}^2}{\sqrt{\sigma_{\eta}^2 + \sigma_{\eta}^0}}$, the flatness factor $\varepsilon_{\Lambda}(\bar{\sigma}) = O(\varepsilon_{\Lambda})$.

Moreover, let $X_1X_2 \ldots$ be the binary sequence equivalent to the scaled lattice point $X/\eta$. Then, there exists an $\ell = O(\log N)$ such that $\sum_{i=1}^{\ell} I(Y \wedge X_i | X_{i-1}) = O(\varepsilon_{\Lambda})$.

Proposition 3 says that choosing the lattice to be $\eta Z$ ensures an exponentially small flatness factor and that, considering only the first $\ell$ least significant bits incur a capacity loss that decays exponentially in $N$. Recall that in order to sample from the discrete auxiliary distributions, our scheme uses Polar codes. Let $P_{U^{(i)}_{1:k} \hat{X}^{(i)}(i)}$ denotes the joint distribution obtained without Polar coding, i.e., not utilizing the polarization phenomenon and shared randomness at all, while simply applying randomized MAP rule (cf. Line 5, Algorithm 1) using $\hat{X}$ to sample the first $\ell = O(\log N)$ least significant bits for each coordinate $i \in [N]$ in all $k$ rounds. On the other hand, let $\hat{Q}_{U^{(i)}_{1:k} \hat{X}^{(i)}(i)}$ be the joint distribution of simulated $\text{rvs}$ obtained using the Polar coding in Algorithms 1 and 2. In the next lemma, we show that the simulated distribution is close to the joint distribution without Polar coding in Kullback-Leibler (KL) divergence.

**Lemma 3:** For $1 \leq i \leq \ell$ and $1 \leq k \leq r$, the KL-divergence

$$D_{\text{KL}}(P_{U^{(i)}_{1:k} \hat{X}^{(i)}(i)} || \hat{Q}_{U^{(i)}_{1:k} \hat{X}^{(i)}(i)}) = O(kN^{2^{-\beta}}),$$

where $\beta \in (0, 1/2)$ is a constant.

Using the additivity of KL-divergence and the inherent Markov chain structure $U^{(i)}_{1:k} \rightarrow \hat{X}^{(i)} \rightarrow \hat{Y}^{(i)}$ given $U^{(i)}_{1:k-1}$, $i \in [\ell]$, we can obtain the following corollary.

**Corollary 1:** For every $k \in [\ell]$, $i \in [\ell]$, we have

$$D_{\text{KL}}(P_{U^{(i)}_{1:k} \hat{X}^{(i)} || \hat{Q}_{U^{(i)}_{1:k} \hat{X}^{(i)}}}) = O(k\beta N^{2^{-\beta}}),$$

and

$$D_{\text{KL}}(P_{U^{(i)}_{1:k} \hat{X}^{(i)} || \hat{Q}_{U^{(i)}_{1:k} \hat{X}^{(i)}}}) = O(k\beta N^{2^{-\beta}}),$$

where $\beta$ is the same as in Lemma 3.

Next, we capture the overall performance of the proposed r-round WZ scheme. Recall that the unknown noise variance $\sigma_{\eta}^2 \in [\sigma_{\eta 0}^2, \sigma_{\eta}^2]$, and the proposed scheme works only in finitely many rounds to cover this interval. Thus, to clearly present our ideas, we first consider the case when $\mathbb{P}_2$ have a partial knowledge about the noise variance being one of the grid points, namely that $\sigma_{\eta}^2 \in \{\sigma_{\eta 0}^2, \sigma_{\eta}^2\}$, and the set $\{\sigma_{\eta 0}^2, \sigma_{\eta}^2\}$ of possible values of the noise variance $\sigma_{\eta}^2$ is known apriori. For this case, the rate-distortion bound achieved by the r-round WZ scheme is given below.

**Theorem 1:** Suppose that $\sigma_{\eta} = \sigma_k$ for some $k \in \{1, 2, \ldots, r\}$. Then, using the scheme of Algorithm 1 and Algorithm 2 with the decoder for $\sigma_k$, the reconstructed vector $\hat{X}_k$ satisfies the following distortion bound

$$E\|X - \hat{X}^{(k)}\| \leq N\Delta + O((kN \log N)^{3/2}2^{-N^{\beta'}}) + O((kN)^2e^{-N}),$$

where $\beta'$ is a constant in $(0, 1/2)$. Further, the total rate of communication used is roughly $1/2 \log \frac{\sigma_{\eta}^2}{\Delta} + O(k(\log N)^2e^{-N}).$

We now state our result for a general case when $\sigma_{\eta}^2$ need not belong to the set $\{\sigma_{\eta 0}^2, \sigma_{\eta}^2\}$. We remark that for this case, too, we use only finitely many rounds of communication from $\mathbb{P}_1$. In effect, our scheme uses a $\sigma_{\eta}^2 \in \{\sigma_{\eta 0}^2, \sigma_{\eta}^2\}$ that is close to $\sigma_{\eta}^2$; the grid $\{\sigma_{\eta 0}^2, \sigma_{\eta}^2\}$ must be chosen to minimize the loss due to using $\sigma_{\eta}^2$ instead of $\sigma_{\eta 0}^2$. The following result is characterization of the universal performance of the overall algorithm.

**Theorem 2:** For every $\epsilon, \delta > 0$, there exists a sufficiently large $N$ such that the scheme in Algorithm 1 and Algorithm 2 with $\ell = O(\log N)$ yields an $(\epsilon, \delta)$-universal WZ code at distortion level $\Delta$ for $[\sigma_{\eta 0}^2, \sigma_{\eta}^2]$. Thus, the proposed scheme is our desired universal rate-optimal scheme.

VI. PROOFS

A. Proof for Lemma 2

Without loss of generality, let $A^{(0)} = 0$ be a constant random variable. From the Markov chain in Fig. 1, we have that $I(A^{(j-1)} \wedge X | X, \hat{Y}) = 0$, for $1 \leq j \leq k - 1$, which further implies $I(A^{(1)}, \ldots, A^{(k)} \wedge X | \hat{Y}) = I(A^{(k)} \wedge X | \hat{Y})$ due to the chain rule of mutual information. Using the chain rule again, we also have $I(A^{(1)}, \ldots, A^{(k)} \wedge X | \hat{Y})$

$$= \sum_{j=1}^{k} I(A^{(j)} \wedge X | A^{(j-1)}, \hat{Y})$$

$$= \sum_{j=1}^{k} I(A^{(j)} \wedge X | \hat{Y} - A^{(j-1)}, A^{(j-1)})$$

$$= \sum_{j=1}^{k} H(A^{(j)} | \hat{Y} - A^{(j-1)}, A^{(j-1)}) - H(A^{(j)} | \hat{Y} - A^{(j-1)}, X - A^{(j-1)}, A^{(j-1)})$$

$$= \sum_{j=1}^{k} H(A^{(j)} - A^{(j-1)} | \hat{Y} - A^{(j-1)}) - H(A^{(j)} - A^{(j-1)} | \hat{Y} - A^{(j-1)}, X - A^{(j-1)})$$

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
where the second identity follows from independence of
\( \tilde{Y} - A^{(j-1)} \) and \( A^{(j-1)} \), the third identity follows from the independence of \( X - A^{(j-1)} \) and \( A^{(j-1)} \), and the fourth identity follows from the fact that subtracting a constant to a rv does not change its entropy, and that the difference \( A^{(j)} - A^{(j-1)} \) is a Gaussian noise independent of \( A^{(j-1)} \). In addition, the MMSE estimate of \( X \) given the rvs \( A^{(k)} \) and \( \tilde{Y} \) is the conditional expectation given by

\[
\hat{X}_k = \mathbb{E}[X | A^{(k)}, \tilde{Y}]
\]

\[
= \frac{\sigma^2 \Delta (\sigma^2 - \sigma_i^2)}{\sigma_k^2 \Delta (\sigma_k^2 - \sigma_i^2) + \sigma_i^2 \sigma_k^2 \sigma_i^2} \cdot \tilde{Y}
\]

\[
+ \left(1 - \frac{\sigma_i^2 \sigma_k^2 \Delta (\sigma_k^2 - \sigma_i^2)}{\sigma_k^2 \Delta (\sigma_k^2 - \sigma_i^2) + \sigma_i^2 \sigma_k^2 \sigma_i^2}\right) \cdot A^{(k)}.
\]

Further, from the Markov chain in Fig. 1, \( \tilde{Y} - A^{(k)} \) is independent of \( A^{(k)} \) with the distribution \( N(0, \alpha_k \Delta_k + \frac{\sigma_i^2 \sigma_k^2 \Delta (\sigma_k^2 - \sigma_i^2)}{\sigma_k^2 \Delta (\sigma_k^2 - \sigma_i^2) + \sigma_i^2 \sigma_k^2 \sigma_i^2}) \). It is then easy to see that \( \text{var}(\hat{X}_k) = \sigma_k^2 - \frac{\sigma_i^2 \sigma_k^2 \Delta (\sigma_k^2 - \sigma_i^2)}{\sigma_k^2 \Delta (\sigma_k^2 - \sigma_i^2) + \sigma_i^2 \sigma_k^2 \sigma_i^2} \) and that the distortion achieved \( \mathbb{E}[D(X, \hat{X}_k)] = \frac{\sigma_i^2 \sigma_k^2 \Delta (\sigma_k^2 - \sigma_i^2)}{\sigma_k^2 \Delta (\sigma_k^2 - \sigma_i^2) + \sigma_i^2 \sigma_k^2 \sigma_i^2} \).

B. Proof for Lemma 3

First, we consider the case when the previous \( k-1 \) auxiliaries are recovered perfectly, by which, \( \tilde{X} \) is perfectly available to be used. Using chain rule of KL-divergence, we have

\[
\mathbb{D}_{KL}(P_{U^{(k)} | U^{(0)}_{1:j-1}} \cdot \tilde{X}^{(k)} || Q_{U^{(k)} | U^{(0)}_{1:j-1}} \cdot \tilde{X}^{(k)})
\]

\[
= \sum_{j \in \mathcal{F}^{(k)}_{1:j-1}} \mathbb{D}_{KL}(P_{U^{(k)} | U^{(0)}_{1:j-1}} \cdot U^{(k)}_{1:j-1} \cdot \tilde{X}^{(k)})
\]

\[
= \sum_{j \in \mathcal{F}^{(k)}_{1:j-1}} H(U^{(k)} | U^{(0)}_{1:j-1} U^{(k)}_{1:j-1})
- H(U^{(k)} | U^{(k)}_{1:j-1} U^{(k)}_{1:j-1})
\]

\[
= \sum_{j \in \mathcal{F}^{(k)}_{1:j-1}} Z(U^{(k)} | U^{(k)}_{1:j-1} U^{(k)}_{1:j-1})^2
\]

\[
\leq 2N^2 - N^2\beta.
\]

For the other case when we may have error in recovering previous round auxiliaries, the KL divergence

\[
\mathbb{D}_{KL}(P_{U^{(k)} | U^{(0)}_{1:j-1}} \cdot \tilde{X}^{(k)} || Q_{U^{(k)} | U^{(0)}_{1:j-1}} \cdot \tilde{X}^{(k)}) \leq N \text{ almost surely.}
\]

However, the expectation of decoding error probability vanishes as \( O(k \cdot 2^{-2N\beta}) \) for some \( 0 \leq \beta' \leq 0.5 \). The proof steps are similar to that of [7, Th. 3] and [16, Th. 5]. We skip the details for brevity. Combining both these cases, the proof is completed.

C. Proof for Theorem 1

1) Calculation of Total Rate Used: The rate contribution comes from Line 7 of Algorithm 1 where the difference of the frozen sets are being communicated. Using polarization theorem of [15], we have for any round \( 1 \leq j \leq r \),

\[
\sum_{i=1}^{\infty} \frac{|F_{j,i}^0| - |F_{j,i}^1|}{N} \leq I(\tilde{B}_{j-1} \cdot X^{(j)} | \tilde{Y}^{(j)}) + \epsilon.
\]

This implies that for an arbitrary \( \epsilon > 0 \), there exists a sufficiently large \( N \) such that

\[
\sum_{i=1}^{\ell} \frac{|F_{j,i}^0| - |F_{j,i}^1|}{N} \leq I(\tilde{B}_{j-1} \cdot X^{(j)} | \tilde{Y}^{(j)}) + \mathcal{O}(e^{-N}) + \epsilon.
\]

(17)

Denote by \( C_{\ell,j} \) the bits communicated up to \( \ell \) levels in any round \( j \) is given by \( C_{\ell,j} = \sum_{i=1}^{\ell} \frac{|F_{j,i}^0| - |F_{j,i}^1|}{N} \). Further, the total rate of communication over \( k \) rounds is \( \sum_{j=1}^{k} C_{\ell,j} \). Using (17), we bound it as

\[
\sum_{j=1}^{k} C_{\ell,j} \leq \sum_{j=1}^{k} I(\tilde{B}_{j-1} \cdot X^{(j)} | \tilde{Y}^{(j)}) + \mathcal{O}(ke^{-N}) + k\epsilon
\]

\[
= \sum_{j=1}^{k} I(\tilde{B}_{j-1} \cdot X^{(j)} | \tilde{Y}^{(j)}) + \mathcal{O}(ke^{-N}) + k\epsilon
\]

\[
\leq \sum_{j=1}^{k} I(\tilde{B}_{j-1} \cdot X^{(j)} | \tilde{Y}^{(j)}) + \mathcal{O}(ke^{-N}) + \epsilon'
\]

\[
= \sum_{j=1}^{k} I(\tilde{B}_{j-1} \cdot X^{(j)} | \tilde{Y}^{(j)}) + \mathcal{O}(ke^{-N}) + \epsilon'
\]

\[
= I(A^{(k)} \cdot X | \tilde{Y}) + \mathcal{O}(ke^{-N}) + \epsilon',
\]

where the first equality is due to the underlying Markov structure, the second inequality uses Proposition 2 and choosing \( \epsilon = \epsilon'/k \), and the last equality is due to Lemma 2.
2) Distortion Achieved by the r-Round WZ Scheme: We calculate the distortion achieved for \( \tilde{X}, \tilde{Y} \) under the scenarios: without and with Polar coding. Throughout, we denote the joint distributions induced by \( P \) for the former case and by \( Q \) the latter case.

**Distortion without Polar coding:** For this case, \( P_1 \) can simply apply the randomized MAP rule for every coordinate \( i \in N \) using \( \tilde{X} \) and sample the first \( \ell = O(\log N) \) least significant bits in all rounds. All these encoded bits at \( P_1 \) can then be perfectly communicated to \( P_2 \), which further employs an MMSE estimate to reconstruct \( \tilde{X} \) within the desired distortion \( \Delta \).

Let the reconstructed source and the recovered bit vectors under the distribution \( P \) be \( \tilde{X}_P \) and \( \{\tilde{U}_{i,p}\}_{1,1 \leq i \leq k} \), respectively. Denote by \( \tilde{A}_{P,\infty}^{(k)} \) the auxiliary lattice point when recovered at all levels without any modulo operation, i.e., \( \tilde{A}_{P,\infty}^{(k)} := N^{-1/2} \sum_{t=1}^{k} (\tilde{U}_{1,p}^{(l)} + 2 \cdot \tilde{U}_{2,p}^{(l)} + \ldots + 2^{\ell-1} - \tilde{U}_{\ell,p}^{(l)} + \ldots) G_N \). However, we consider only \( \ell \) levels for this case too (cf. Line 9, Algorithm 2) to observe the corresponding auxiliary lattice point over \( N^{-1/2} \mathbb{Z} \) given by \( \tilde{A}_{P,\infty}^{(k)} := N^{-1/2} \sum_{t=1}^{k} (\tilde{U}_{1,p}^{(l)} + 2 \cdot \tilde{U}_{2,p}^{(l)} + \ldots + 2^{\ell-1} - \tilde{U}_{\ell,p}^{(l)} ) G_N \) mod \( 2^\ell \mathbb{Z} \). Note that the modulo operation \( \mod 2^\ell \mathbb{Z} \) above maps inputs to the lattice points in interval \([-2^{\ell-1}, 2^{\ell-1})\). Further, using Lemma 2, the final estimate used by \( P_2 \) is \( \tilde{X}_P = \tilde{A}_{P,\infty}^{(k)} + \gamma (\tilde{Y} - \tilde{A}_{P,\infty}^{(k)}) \), where \( \gamma \) is same as in (14). Define \( \theta := k N^{-1/2} 2^{\ell-1} \) for the ease of notation. Under the joint distribution \( P \), we then calculate the distortion as follows: \( \mathbb{E}_P[\|\tilde{X} - \tilde{X}_P\|^2] = N \mathbb{E}_P \left[ \tilde{X} - \tilde{X}_P \right]^2 \cdot \mathbb{I}_{\tilde{A}_{P,\infty}^{(k)} \geq \theta} \] + \( \mathbb{E}_P \left[ \tilde{X} - \tilde{X}_P \right]^2 \cdot \mathbb{I}_{\tilde{A}_{P,\infty}^{(k)} < \theta} \). Using the fact that \( \tilde{A}_{P,\infty}^{(k)} = \tilde{A}_{\gamma P,\infty}^{(k)} \) whenever \( \tilde{A}_{P,\infty}^{(k)} \leq \theta \), the first term in the right-hand side (RHS) simplifies to

\[
\mathbb{E}_P \left[ \tilde{X} - \tilde{X}_P \right]^2 \cdot \mathbb{I}_{\tilde{A}_{P,\infty}^{(k)} \geq \theta} = \mathbb{E}_P \left[ \left( T_k - \gamma \left( T_k + Z \right) \right)^2 \cdot \mathbb{I}_{\tilde{A}_{P,\infty}^{(k)} \geq \theta} \right] \leq N \frac{\sigma_x^2 \sigma_y^2 \Delta}{\sigma_z^2} = N \Delta, \tag{18}
\]

where the only inequality uses the fact that probability is at most 1 and the last line uses the assumption in the statement \( \sigma_x = \sigma_y \). The second term can be bounded as

\[
\mathbb{E}_P \left[ \tilde{X} - \tilde{X}_P \right]^2 \cdot \mathbb{I}_{\tilde{A}_{P,\infty}^{(k)} < \theta} \leq 2 \mathbb{E}_P \left[ \tilde{X} - \tilde{A}_{P,\infty}^{(k)} \right]^2 \cdot \mathbb{I}_{\tilde{A}_{P,\infty}^{(k)} \geq \theta} + 2 \mathbb{E}_P \left[ \tilde{Y} - \tilde{A}_{P,\infty}^{(k)} \right]^2 \cdot \mathbb{I}_{\tilde{A}_{P,\infty}^{(k)} < \theta}, \tag{20}
\]

where the last line uses the inequality: \( (a - b)^2 \leq 2(a^2 + b^2) \). With regard to RHS in (20), the first term can be further broken as

\[
\mathbb{E}_P \left[ \left( \tilde{X} - \tilde{A}_{P,\infty}^{(k)} \right)^2 \cdot \mathbb{I}_{\tilde{A}_{P,\infty}^{(k)} \geq \theta} \right] \leq \mathbb{E}_P \left[ \tilde{X}^2 \cdot \mathbb{I}_{\tilde{A}_{P,\infty}^{(k)} \geq \theta} \right] + \mathbb{E}_P \left[ \left( \tilde{A}_{P,\infty}^{(k)} \right)^2 \cdot \mathbb{I}_{\tilde{A}_{P,\infty}^{(k)} \geq \theta} \right]\]

where the first inequality uses the Chernoff bounds for the first component and integration by parts for the second component, and solving the integral for the third, and the last inequality is due the fact that for \( |a| \geq \theta \), the maximum value of RHS occurs at \( a = \theta \), which is a quadratic function in \( \theta \). Choosing sufficiently large values for \( \ell \), the function can be further bounded by \( c \theta^2 \) for some universal constant \( c > 0 \). The last inequality in (21) is due to the sub-exponential decay of discrete Gaussian distribution described below:

\[
\sum_{|a| \geq \theta} \mathbb{P}_{A_{P,\infty}}(a) = \frac{2}{\sum_{\lambda \in N^{-1/2} \mathbb{Z}_e} e^{-\frac{a^2}{2\sigma_z^2}}} \sum_{\lambda \in N^{-1/2} \mathbb{Z}_e} e^{-\frac{a^2}{2\sigma_z^2}} \frac{2}{\sum_{\lambda \in N^{-1/2} \mathbb{Z}_e} e^{-\frac{a^2}{2\sigma_z^2}}} \]

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
where the numerator 2 in the first equality is due to the symmetry of the underlying lattice around 0, the first inequality uses the fact that $e^{-2x^2/N} \leq 1$, for all $x$, the second inequality is due to the lower bound on normalization constant in denominator from (2), and the last line is due to $\epsilon_{N^{-1}/2}(\sqrt{ak_x}) = O(e^{-N})$ (see Proposition 3) and choosing sufficiently large $N$.

Similarly, the second term in the RHS of (20) can also be bounded as $O(k^2 Ne^{-N^2})$. Combining this bound and (21), we get

$$N\mathbb{E}_P \left[ \left( X - \hat{X}_P \right)^2 \right] = O\left( k^2 N^2 e^{-N^2} \right).$$

and we have the distortion under the joint distribution $P$ obtained without Polar coding

$$\mathbb{E}_P \| X - \hat{X}_P \|^2 = N\Delta + O\left( k^2 N^2 e^{-N^2} \right). \quad (23)$$

However, we must note that this case requires too many bits of communication and thus, quantization is necessary.

**Distortion under Polar coding:** Towards that, $P_1, P_2$ rely on the Polar coding technique, which uses shared randomness (cf. Algorithm 1 and 2) and exhibits much smaller communication. Let the reconstructed source and the recovered bit vectors under this distribution by $\tilde{X}^{(k)}_Q$, and $\tilde{U}^{(k)}_{Q, \ell}$, respectively. Denote by $\tilde{A}^{(k)}_{Q, \ell}$ the auxiliary lattice point over $N^{-1/2}\mathbb{Z}$ when only $\ell$ levels are recovered is given by $\tilde{A}^{(k)}_{Q, \ell} := N^{-1/2}, \sum_{k=1}^{k} (C^{(k)}_Q + 2, \tilde{U}^{(k)}_{Q, \ell} + \cdots + 2^{k-1}.C^{(k)}_{Q, \ell} \mod 2^k \mathbb{Z}).$ Note that the mod $2^k \mathbb{Z}$ operation always maps inputs to the lattice points in interval $[-2^{k-1}, 2^{k-1})$. Further, the final estimate used by $P_2$ is

$$\tilde{X}^{(k)}_Q = \tilde{A}^{(k)}_{Q, \ell} + \gamma \left( \tilde{Y} - \tilde{A}^{(k)}_{Q, \ell} \right),$$

where $\gamma$ is defined as earlier (14). Using the Minkowski’s inequality, we have

$$\sqrt{\mathbb{E}_Q \| X - \tilde{X}^{(k)}_Q \|^2} \leq \sqrt{\mathbb{E}_P \| X - \tilde{X}^{(k)}_P \|^2} + \sqrt{\mathbb{E}_\mu \| \tilde{X}^{(k)}_Q - \tilde{X}^{(k)}_P \|^2}, \quad (24)$$

for every coupling $\mu$ between $P$ and $Q$. While we already have bound for the first term using (23), the second term is bounded as follows. First note that we have

$$\mathbb{E}_\mu \| \tilde{X}^{(k)}_Q - \tilde{X}^{(k)}_P \|^2 = (1 - \gamma)^2 \mathbb{E}_\mu \| \tilde{A}^{(k)}_{Q, \ell} - \tilde{A}^{(k)}_{P, \ell} \|^2.$$
For such coupling, i.e., when $\mu = \mu^*$ in (26), we can use Pinsker’s inequality followed by the fact that for any sequence of rvs $X_1, \ldots, X_n \sim P, Y_1, \ldots, Y_n \sim Q : D_{\text{KL}}(P_{X_1 \ldots X_n} \| Q_{Y_1 \ldots Y_n}) \geq D_{\text{KL}}(P_{X_i} \| Q_{Y_i}), \forall i \in [n]$ to show

$$\sqrt{\mathbb{E}_\mu \left[ \| \hat{X}^{(k)} - \Theta^{(k)} \|^2 \cdot 1_{|\tilde{\gamma}_{\mu(x)}| \| \hat{X}^{(k)} \| \leq \theta} \right]} \leq \frac{2^{\ell-1}}{\sqrt{N}} \sum_{i=1}^{N} \mathbb{E}_\mu \left[ \left( \sum_{j=1}^{N} \left( D_{\text{KL}}(P_{X_j \cdot Y_j} \| Q_{X_j \cdot Y_j}) \right) \right)^{\frac{1}{2}} \right]$$

$$= \frac{2^{\ell-1}}{\sqrt{N}} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E}_\mu \left[ \left( D_{\text{KL}}(P_{X_j \cdot Y_j} \| Q_{X_j \cdot Y_j}) \right)^{\frac{1}{2}} \right]$$

$$+ O\left((kN \log N)^{3/2 - 2\theta/4}\right).$$

where the first identity is due to one-to-one Polar transform, and the last line uses Lemma 3 and $\beta' = \beta/4$. For the second term in (25), we have

$$\sqrt{\mathbb{E}_\mu \left[ \| \hat{X}^{(k)} - \Theta^{(k)} \|^2 \cdot 1_{|\tilde{\gamma}_{\mu(x)}| \| \hat{X}^{(k)} \| \geq \theta} \right]} \leq \sqrt{\mathbb{E}_\mu \left[ 2 \left( \| \hat{X}^{(k)} - \Theta^{(k)} \|^2 + \| \hat{X}^{(k)} \|^2 \right) \cdot 1_{|\tilde{\gamma}_{\mu(x)}| \| \hat{X}^{(k)} \| \geq \theta} \right]}$$

$$\leq \sqrt{\mathbb{E}_\mu \left[ 4\theta^2 \cdot 1_{|\tilde{\gamma}_{\mu(x)}| \| \hat{X}^{(k)} \| \geq \theta} \right]}$$

$$= O\left((kN e^{-N^2})\right).$$

where the first inequality uses $\|x - y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$, the second inequality uses the fact that both $\| \hat{X}^{(k)} \|^2$ and $\| \Theta^{(k)} \|^2$ are less than $\theta^2$ almost surely due to modulo mapping, and the last line follows by applying union bound to (22). Plugging all the obtained upper bounds in (24), we get

$$\mathbb{E}_Q \| \hat{X} - \hat{X}^{(k)} \|^2 \leq N\Delta + O\left((kN \log N)^{3/2 - 2\theta/4}\right)$$

$$+ O\left((kN e^{-N^2})\right).$$

(27)

Recall that all the calculations are done w.r.t input pair $(X, Y)$ resulting from discrete Gaussian auxiliaries, instead of actual input pair $(X, Y)$. We now show that as the joint distributions of these pairs are close in total variational distance, the resulting distortion gap is also close. For $\theta$ as defined earlier, we have

$$\mathbb{E}_Q \| X - \hat{X}^{(k)} \|^2$$

$$= \int \int \int \| x - y \|^2 Q_\mu x, y (a, x, y) dy dx da$$

$$= \int \int \int \| x - y \|^2 Q_\mu x, y (a, x, y) dy dx da$$

$$+ \int \int \int \| x - y \|^2 Q_\nu x, y (a, x, y) dy dx da$$

$$\leq \int \int \int \| x - y \|^2 Q_\mu x, y (a, x, y) dy dx da$$

$$+ \int \int \int \| x - y \|^2 Q_\nu x, y (a, x, y) dy dx da$$

$$= \int \int \int \| x - y \|^2 Q_\mu x, y (a, x, y) dy dx da$$

$$+ \int \int \int \| x - y \|^2 Q_\nu x, y (a, x, y) dy dx da$$

(28)

where bounding the second term in (28), note that each $\hat{x}(i) \leq \max\{a(i), y(i)\} \leq \theta$ a.s., which implies $\| x - y \|^2 \leq 4N\theta^2$. Further, the Polar encoding operation applied to $(X, Y)$ is same as that to modified rvs $(\tilde{X}, \tilde{Y})$, i.e., $Q_\mu x, y (a, x, y) = Q_\mu x, y (a, x, y)$. Thus, the first term is bounded by $8N\theta^2 \mathcal{D}_{TV}(P_X, P_\lambda X)$, which further evaluates to $O(N\theta^2 e^{-N})$ using Lemma 1 and Proposition 3. The first term can be bounded by $\mathbb{E}_Q \| \hat{X} - \hat{X}^{(k)} \|^2$ in (27).

Further, for the third term note that whenever there exists an $i$ such that $|y(i)| \geq \theta$, we have $\hat{x}(i) \leq \max\{a(i), y(i)\} \leq |y(i)|$ implies $\| x - y \|^2 \leq 2(\|x\|^2 + \|y\|^2)$. As a result, we have

$$\int \int \int \| x - y \|^2 Q_\mu x, y (a, x, y) dy dx da$$

$$\leq \int \int \int \| x \|^2 + \| y \|^2 f_{XY}(x, y) dy dx.$$

(29)

Using Chernoff’s bound and $\int_t^\infty y^2 dy \leq \frac{1}{\sqrt{2\pi}} te^{-\frac{t^2}{2}} + e^{-\frac{t^2}{2}}$, we have that

$$\int \int \int \| x \|^2 f_{XY}(y) dy$$

$$= \int \int \int \| y \|^2 f_{XY}(y) dy$$

$$+ \int \int \int \| y \|^2 f_{XY}(y) dy$$

$$\leq 4\sigma_1^2 \sigma_2^2 e^{-\frac{\|x\|^2}{2\sigma_1^2}} + 4\sigma_1^2 \sigma_2^2 e^{-\frac{\|y\|^2}{2\sigma_2^2}} + 8N\sigma_1^2 \sigma_2^2 e^{-\frac{\|x\|^2}{2\sigma_1^2}} e^{-\frac{\|y\|^2}{2\sigma_2^2}}$$

$$= O\left(C_1^{-N^2}\right),$$

for a constant $C_1 > 1$. Using Chernoff’s bound again, we have the second term in (29) as $O(e^{-\frac{\|x\|^2}{2\sigma_1^2}})$.

For the only remaining case when there exists an $i$ such that $|x(i)| \geq \theta$ and $y(i) \leq \theta$, we have $\| x - y \|^2 \leq 4\|x\|^2$. Using similar arguments as above, we can show that for this case, too, the integral is bounded as $O(C_2^{-N^2})$ for some constant $C_2 > 0$.

D. Proof for Theorem 2

Recall that $\sigma_i^2$ lies in the continuum $I = [\sigma_0^2, \sigma_2^2]$, but the number of rounds are finitely many. Further, we choose the finite grid points in geometric sequence: $\sigma_i = \sigma_{i-1}2^{1/N}, 1 \leq i \leq N$.
i ≤ r. This gives k = O(N). Since, p2 knows exactly the channel variance σ2, it performs the final decoding almost surely in the round k ∈ [r] iff σ2 ∈ [σk−1, σk). Consider the case when σ2 = σk−1 + ε, ε > 0, and r-round WZ uses σk as the guess. For this case, the total rate of communication as given by Theorem 1 is 1 2 log σ2 Δ + O(ke−N), which is more than the optimal rate 1 2 log σ2 Δ. Therefore, the extra rate used by the proposed universal scheme is

\[ \sum_{j \in [k]} R_j - \frac{1}{2} \log \frac{\sigma^2_k}{\Delta} \leq \sum_{j \in [k]} R_j - \frac{1}{2} \log \frac{\sigma^2_{k-1}}{\Delta} \]

\[ = \frac{1}{2} \log \frac{\sigma^2_k}{\sigma^2_{k-1}} + O(ke^{-N}) \]

\[ \leq \max_{i} \frac{1}{2} \log \frac{\sigma^2_{i}}{\sigma^2_{i-1}} + O(ke^{-N}) \]

\[ = O(1/N), \]

where the first inequality uses σ2 ≥ σk−1 and the last line holds for the choice of grid σ2 = σi−1 + 2i/N.

The distortion calculation goes same as in proof of Theorem 1 earlier, except that (18) is always bounded as

\[ \mathbb{E}_{p_X} \left[ \left( \tilde{X} - \hat{X}_P \right)^2 \right] \leq \frac{\sigma^2_k \Delta}{\sigma^2_k - \sigma^2_0} \]

because σ2 < σk, which makes the multiplicative factor less than 1. Note that the distortion bounds hold for k = O(N) in this case.

ACKNOWLEDGMENT

The author would like to thank Himanshu Tyagi for helpful discussions in formulating the problem and developing the proof ideas. He is also grateful to Ling Liu for the discussion on Polar lattices, which helped to improve the result in Theorem 1. The author also thanks anonymous reviewers for their insightful comments and suggestions, which helped to improve the quality of this manuscript.

REFERENCES

[1] D. Aldous, “Random walks on finite groups and rapidly mixing Markov chains,” in Séminaire de Probabilités XVII 1981/82. Berlin, Germany: Springer, 1983, pp. 243–297. [Online]. Available: https://link.springer.com/chapter/10.1007/BFb0068329 citeas

[2] E. Arikan, “Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels,” IEEE Trans. Inf. Theory, vol. 55, no. 7, pp. 3051–3073, Jul. 2009.

[3] S. S. Banerjee and H. Tyagi, “RT-polar: An HARQ scheme with universally competitive rates,” in Proc. IEEE Inf. Theory Workshop (ITW), 2018, pp. 1–5.

[4] S. S. Banerjee and H. Tyagi, “Practical universal data exchange using polar codes,” in Proc. IEEE Inf. Theory Workshop (ITW), 2019, pp. 1–5.

[5] E. Dupraz, A. Roumy, and M. Kieffer, “Source coding with side information at the decoder and uncertain knowledge of the correlation,” IEEE Trans. Commun., vol. 62, no. 1, pp. 269–279, Jan. 2014.

[6] G. D. Forney, M. D. Trott, and S.-Y. Chung, “Sphere-bound-achieving coset codes and multilevel coset codes,” IEEE Trans. Inf. Theory, vol. 46, no. 3, pp. 820–850, May 2000.

[7] I. Honda and H. Yamamoto, “Polar coding without alphabet extension for asymmetric models,” IEEE Trans. Inf. Theory, vol. 59, no. 12, pp. 7829–7838, Dec. 2013.

[8] S. Jalali, S. Verdu, and T. Weissman, “A universal scheme for Wyner–Ziv coding of discrete sources,” IEEE Trans. Inf. Theory, vol. 56, no. 4, pp. 1737–1750, Apr. 2010.

[9] S. K. Jha and H. Tyagi, “Universal interactive Gaussian quantization with side information,” in Proc. IEEE Inf. Theory Workshop (ITW), Riva del Garda, Italy, 2020, pp. 1–5.

[10] S. B. Korada and R. L. Urbanke, “Polar codes are optimal for lossy source coding,” IEEE Trans. Inf. Theory, vol. 56, no. 4, pp. 1751–1768, Apr. 2010.

[11] S. B. Korada, “Polar codes for channel and source coding,” Ph.D. dissertation, Dept. Comput. Sci. Commun. Inf., EPFL, Lausanne, Switzerland, 2009.

[12] S. Kuzuoka, A. Kimura, and T. Uematsu, “Universal source coding for multiple decoders with side information,” in Proc. IEEE Int. Symp. Inf. Theory, 2010, pp. 1–5.

[13] B. Li, D. Tse, K. Chen, and H. Shen, “Capacity-achieving rateless polar codes,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), 2016, pp. 46–50.

[14] C. Ling and J.-C. Belfiore, “Achieving AWGN channel capacity with lattice Gaussian coding,” IEEE Trans. Inf. Theory, vol. 60, no. 10, pp. 5918–5929, Oct. 2014.

[15] L. Liu, Y. Yan, C. Ling, and X. Wu, “Construction of capacity-achieving lattice codes: Polar lattices,” IEEE Trans. Commun., vol. 67, no. 2, pp. 915–928, Feb. 2019.

[16] L. Liu, J. Shi, and C. Ling, “Polar lattices for lossy compression,” 2015, arXiv:1501.05683.

[17] L. Liu, J. Shi, and C. Ling, “Polar lattices for lossy compression,” IEEE Trans. Inf. Theory, vol. 67, no. 9, pp. 6140–6163, Sep. 2021.

[18] N. Merhav and J. Ziv, “On the Wyner–Ziv problem for individual sequences,” IEEE Trans. Inf. Theory, vol. 52, no. 3, pp. 867–873, Mar. 2006.

[19] Y. Oohama, “Gaussian multiterminal source coding,” IEEE Trans. Inf. Theory, vol. 43, no. 6, pp. 1912–1923, Nov. 1997.

[20] A. Reani and N. Merhav, “Efficient on-line schemes for encoding individual sequences with side information at the decoder,” IEEE Trans. Inf. Theory, vol. 57, no. 10, pp. 6860–6876, Oct. 2011.

[21] O. Regev, “On lattices, learning with errors, random linear codes, and cryptography,” J. ACM, vol. 56, no. 6, p. 34, Sep. 2009.

[22] Y. Steinberg and N. Merhav, “On successive refinement for the Wyner–Ziv problem,” IEEE Trans. Inf. Theory, vol. 50, no. 8, pp. 1636–1654, Aug. 2004.

[23] H. Tyagi, P. Viswanath, and S. Watanabe, “Interactive communication for data exchange,” IEEE Trans. Inf. Theory, vol. 64, no. 1, pp. 26–37, Jan. 2018.

[24] S. Watanabe and S. Kuzuoka, “Universal Wyner–Ziv coding for distortion constrained general side information,” IEEE Trans. Inf. Theory, vol. 60, no. 12, pp. 7568–7583, Dec. 2014.

[25] T. Weissman, E. Ordentlich, G. Seroussi, S. Verdu, and M. J. Weinberger, “Universal discrete denoising: Known channel,” IEEE Trans. Inf. Theory, vol. 51, no. 1, pp. 5–28, Jan. 2005.

[26] A. Wyner and J. Ziv, “The rate-distortion function for source coding with side information at the decoder,” IEEE Trans. Inf. Theory, vol. IT-22, no. 1, pp. 1–10, Jan. 1976.

[27] M. Ye and A. Barg, “Polar codes for distributed hierarchical source coding,” Adv. Math. Commun., vol. 9, no. 1, pp. 87–103, 2015.

[28] R. Zamir, S. Shamai, and U. Erez, “Nested linear/lattice codes for structured multiterminal binning,” IEEE Trans. Inf. Theory, vol. 48, no. 6, pp. 1250–1276, Jun. 2002.

[29] E. Şaşoğlu and L. Wang, “Universal polarization,” IEEE Trans. Inf. Theory, vol. 62, no. 6, pp. 2937–2946, Jun. 2016.

Shubham Jha (Graduate Student Member, IEEE) received the B.Tech. degree in electronics and communication engineering from the Indian Institute of Information Technology Guwahati, India, in 2018. He is currently pursuing the Ph.D. degree with the Robert Bosch Center for Cyber-Physical Systems, Indian Institute of Science, Bengaluru, India. His research interests lie at the intersection of information theory and distributed quantization and optimization. He is also interested in designing communication-efficient and privacy-coupled federated learning algorithms for wireless systems.