REMARKS ON EIGENVALUE PROBLEMS FOR FRACTIONAL
\( p(\cdot) \)-LAPLACIAN

ANOUAR BAHROUNI AND KY HO

Abstract. In this paper, we give some properties and remarks of the new fractional Sobolev spaces with variable exponents. We also study the eigenvalue problem involving the new fractional \( p(\cdot) \)-Laplacian.

1. Introduction

Recently, great attention has been focused on elliptic equations involving fractional operators, both for pure mathematical research and in view of concrete real-world applications. This type of operator have major applications to various nonlinear problems, including phase transitions, thin obstacle problem, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes and flame propagation, ultra-relativistic limits of quantum mechanics, multiple scattering, minimal surfaces, material science, water waves, etc. We refer to [2, 7, 13] for a comprehensive introduction to the study of nonlocal problems.

In recent years, the study of differential equations and variational problems involving variable exponent conditions has been an interesting topic. Lebesgue spaces with variable exponents appeared in the literature in 1931 in the paper by Orlicz [14]. Zhikov [18] started a new direction of investigation, which created the relationship between spaces with variable exponents and variational integrals with nonstandard growth conditions. For more details on Lebesgue and Sobolev spaces with variable exponents, we refer the reader to [5, 15].

To our best knowledge, Kaufmann et al. [11] firstly introduced some results on fractional Sobolev spaces with variable exponent \( W^{s,q(\cdot),p(\cdot)}(\Omega) \) and the fractional \( p(\cdot) \)-Laplacian. There, the authors established compact embedding theorems of these spaces into variable exponent Lebesgue spaces. As an application, they also proved an existence result for nonlocal problems involving the fractional \( p(\cdot) \)-Laplacian. In [3], Bahrouni and Rădulescu obtained some further qualitative properties of the fractional Sobolev space \( W^{s,q(\cdot),p(\cdot)}(\Omega) \) and the fractional \( p(\cdot) \)-Laplacian. After that, some studies on this kind of problems have been performed by using different approaches, see [4, 10, 20].

Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^N \). For any real \( s > 0 \) and for any functions \( q(x) \) and \( p(x,y) \), we want to define the fractional Sobolev space with variable exponent. We start by fixing \( s \in (0,1) \), \( q \in C(\overline{\Omega}, \mathbb{R}) \), and \( p \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R}) \). Throughout this paper, we assume that

\[
1 < p(x,y) = p(y,x) < \frac{N}{s} \quad \forall \ (x,y) \in \overline{\Omega} \times \overline{\Omega} \quad (P)
\]

and

\[
1 < q(x) < \frac{Np(x,x)}{N - sp(x,x)} =: p^*_s(x), \quad \forall \ x \in \overline{\Omega}. \quad (Q)
\]

2010 Mathematics Subject Classification. 35D30, 35J20, 35J60, 35P15, 35P30, 35R11, 46E35.

Key words and phrases. Fractional Sobolev spaces, variable exponents, eigenvalue problems, variational methods.
We define the fractional Sobolev space with variable exponents $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ as
\[ W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega) = \left\{ u \in L^{q(\cdot)}(\Omega) : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \, dx \, dy < \infty \right\}. \]
Let
\[ [u]_{s,p(\cdot),\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \, dx \, dy \leq 1 \right\} \]
be the corresponding variable exponent Gagliardo seminorm. In what follows, for brevity, we denote $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ by $E$ for a general $q \in C(\overline{\Omega}, \mathbb{R})$ satisfying (Q) and by $W^{s,p(\cdot,\cdot)}(\Omega)$ when $q(x) = p(x,x)$ on $\overline{\Omega}$. Also, in some places we will write $p(x)$ instead of $p(x,x)$ and in this sense, $p \in C(\overline{\Omega}, \mathbb{R})$. We equip $E$ with the norm
\[ \|u\|_E = [u]_{s,p(\cdot),\Omega} + \|u\|_{L^{q(\cdot)}(\Omega)} \]
(see Appendix for the definitions of $L^{q(\cdot)}(\Omega)$ and $\| \cdot \|_{L^{q(\cdot)}(\Omega)}$). Then, $E$ becomes a reflexive and separable Banach space. The following embedding theorem was obtained in [11] for the case $q(x) > p(x,x)$ on $\overline{\Omega}$ and then was refined in [10, 20].

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let $s \in (0,1)$. Let $p \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})$ and $q \in C(\overline{\Omega}, \mathbb{R})$ satisfy (P) and (Q) with $q(x) \geq p(x,x)$ for all $x \in \overline{\Omega}$. Let $r \in C(\overline{\Omega}, \mathbb{R})$ satisfy
\[ 1 < r(x) < p^*_s(x), \ \forall x \in \overline{\Omega}. \quad \text{(R)} \]
Then, there exists a constant $C = C(N,s,p,q,r,\Omega)$ such that
\[ \|f\|_{L^{r(\cdot)}(\Omega)} \leq C\|f\|_E, \ \forall f \in E. \]
Thus, $E$ is continuously embedded in $L^{r(\cdot)}(\Omega)$. Moreover, this embedding is compact.

Thanks to Theorem 1.1, under the assumptions (P) and (Q) with $q(x) \geq p(x,x)$ for all $x \in \overline{\Omega}$, spaces $E$ and $W^{s,p(\cdot,\cdot)}(\Omega)$ actually coincide. It is worth pointing out that for the seminorm localized on $\Omega \times \Omega$, there is no Poincaré-type inequality in general even for constant exponent case. Because of this fact, $E$ is not suitable for studying the fractional $p(\cdot)$-Laplacian problem with Dirichlet boundary data $u = 0$ in $\mathbb{R}^N \setminus \Omega$ via variational methods and hence, we need to introduce another space as our solution space. In order to do this, invoking the continuity of $p$ on $\overline{\Omega} \times \overline{\Omega}$ we extend $p$ to $\mathbb{R}^N \times \mathbb{R}^N$ by using Tietze extension theorem, such that
\[ 1 < \inf_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x,y) \leq \sup_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x,y) < \infty. \]
We now define the following space:
\[ X = \left\{ u \in W^{s,p(\cdot)}(\mathbb{R}^N) : \ u = 0 \ \text{on} \ \Omega^c \right\} \]
endowed with norm
\[ \|u\|_X = [u]_{s,p(\cdot),\mathbb{R}^N} + \|u\|_{L^{p(\cdot)}(\Omega)}, \]
where $W^{s,p(\cdot)}(\mathbb{R}^N)$ and $[u]_{s,p(\cdot),\mathbb{R}^N}$ are defined in the same ways as $W^{s,p(\cdot)}(\Omega)$ and $[u]_{s,p(\cdot),\Omega}$ with $\Omega$ replaced by $\mathbb{R}^N$. Obviously, $X$ is a closed subspace of $W^{s,p(\cdot)}(\mathbb{R}^N)$ and hence, $(X, \| \cdot \|_X)$ is a reflexive and separable Banach space.

The first aim of our paper is to present some further basic results both on the function spaces $E$ and $X$. Also, we try to improve $X$ by giving an equivalent space (see Section 2).

Our second aim is the study of the eigenvalue problem:
\[
\begin{cases}
(-\Delta)^s_p u + \alpha |u|^{p(x)-2}u + \beta |u|^{q(x)-2}u = \lambda |u|^{r(x)-2}u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where the operator \((-\Delta)^s_{p(\cdot)}\) is defined as
\[
(-\Delta)^s_{p(\cdot)} u(x) := 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{|u(x) - u(y)|^{p(x,y) - 2} (u(x) - u(y))}{|x - y|^{N + sp(x,y)}} \, dy, \quad x \in \mathbb{R}^N,
\]
where \(B(x,\varepsilon) := \{ z \in \mathbb{R}^N : |z - x| < \varepsilon \} \); \(\alpha, \beta\) are nonnegative real numbers; \(\lambda > 0\) is a real spectral parameter; and \(r \in C(\overline{\Omega}, \mathbb{R})\) satisfies (R).

In particular, we deal with the existence, nonexistence of solutions for problem (1.1). In the context of eigenvalue, problems involving variable exponent represent a starting point in analyzing more complicated equations. To our best knowledge, the first contribution in this sense is the paper by Fan et al. [8]. The authors established the existence of a sequence of eigenvalues for \(p(\cdot)\)-Laplacian \(\text{div} \left(|\nabla u|^{p(x) - 2} \nabla u\right)\) subject to Dirichlet boundary condition by using the Ljusternik-Schnirelmann theory. In [12], Mihailescu and Rădulescu studied an eigenvalue problem with non-negative weight for the Laplace operator on a bounded domain with smooth boundary in \(\mathbb{R}^N, N = 3\). They showed the existence of two positive constants \(\lambda_\alpha\) and \(\lambda_*\) with \(\lambda_\alpha \leq \lambda_*\) such that for any \(\lambda \in (0, \lambda_\alpha)\) is not an eigenvalue of the problem while any \(\lambda \in (\lambda_*, +\infty)\) is an eigenvalue of the problem. Some similar results for a class of fractional \(p(\cdot)\)-Laplacian problems involving multiple variable exponents can be found in [1, 6]. All the aforementioned results treat only the existence of at least one solution for problem (1.1) with \(\alpha = 1, \beta = 0\) or \(\alpha = 0, \beta = 1\).

This paper is organized as follows. In Section 2, we give some basic properties of fractional Sobolev spaces with variable exponents. In Sections 3, we deal with the eigenvalue problem using techniques in calculus of variations. Finally, in Appendix we give definitions and fundamental properties of the Lebesgue spaces with variable exponents.

**Notation**
\[
p^+ := \sup_{\mathbb{R}^N \times \mathbb{R}^N} p(x,y), \quad p^- := \inf_{\mathbb{R}^N \times \mathbb{R}^N} p(x,y)
\]
\[
q^+ := \sup_{x \in \Omega} q(x), \quad q^- := \inf_{x \in \Omega} q(x)
\]
\[
r^+ := \sup_{x \in \Omega} r(x), \quad r^- := \inf_{x \in \Omega} r(x)
\]

2. **Some remarks on fractional Sobolev spaces with variables exponents**

Let \(\alpha, \beta \geq 0\) with \(\alpha + \beta > 0\). Then, on \(E\) the norm \(\| \cdot \|_E\) is equivalent to the norm
\[
\|u\|_1 = \inf \left\{ \mu > 0 : \rho \left( \frac{u}{\mu} \right) \leq 1 \right\},
\]
where \(\rho : E \to \mathbb{R}\) is defined by
\[
\rho(u) = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)} \, dx \, dy + \alpha \int_{\Omega} \frac{|u|^{p(x)}}{p(x)} \, dx + \beta \int_{\Omega} \frac{|u|^{q(x)}}{q(x)} \, dx,
\]

**Lemma 2.1.** Let \(p \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})\) and \(q \in C(\overline{\Omega}, \mathbb{R})\) satisfy (P) and (Q) with \(q(x) \geq p(x, x)\) for all \(x \in \Omega\). Let \(u \in E\), then the following holds:

(i) For \(\gamma > 0\), \(\|u\|_1 = \gamma\) if and only if \(\rho(\frac{u}{\gamma}) = 1\);

(ii) \(\|u\|_1 < 1\) implies \(\|u\|_{1^+}^+ \leq \rho(u) \leq \|u\|_{1^-}^+\);

(iii) \(\|u\|_1 > 1\) implies \(\|u\|_{1^-}^- \leq \rho(u) \leq \|u\|_{1^-}^-\).

**Proof.** By invoking Proposition A.1 in Appendix, the proof can be obtained easily from the definition of norm \(\| \cdot \|_1\) and modular \(\rho\) and we omit it. \(\square\)

Next we provide some more properties on the modular \(\rho\). In what follows, \(E^*\) (resp. \(X^*\)) denotes the dual space of \(E\) (resp. \(X\)) and \(\langle \cdot, \cdot \rangle\) denote the duality pairing between \(E\) and \(E^*\) (resp. \(X\) and \(X^*\)).
Lemma 2.2. Let $p \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})$ and $q \in C(\overline{\Omega}, \mathbb{R})$ satisfy (P) and (Q) with $q(x) \geq p(x, x)$ for all $x \in \overline{\Omega}$. Then the following properties hold.

(i) The functional $\rho$ is of class $C^1$ and its Fréchet derivative $\rho' : E \to E^*$ is given by

$$
\langle \rho'(u), \varphi \rangle = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y) - 2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + sp(x,y)}} \, dx \, dy
$$

$$
+ \alpha \int_{\Omega} |u|^{p(x) - 2} u \varphi \, dx + \beta \int_{\Omega} |u|^{q(x) - 2} u \varphi \, dx, \quad \forall u, \varphi \in E.
$$

(ii) The function $\rho' : E \to E^*$ is coercive, that is, $\frac{\langle \rho'(u), u \rangle}{\|u\|_1} \to +\infty$ as $\|u\|_1 \to +\infty$.

Proof. (i) This is standard (see [3]).

(ii) By Lemma 2.1, for $\|u\|_1 > 1$, we obtain

$$
\langle \rho'(u), u \rangle \geq \rho(u) \geq \|u\|_1^{p^-}
$$

and hence, the conclusion follows. $\square$

Remark 2.3. The above results still hold true if we replace $E$ by $X$ with $\int_{\Omega \times \Omega}$ replaced by $\int_{\mathbb{R}^N \times \mathbb{R}^N}$.

As we mentioned in the introduction, when $\alpha = \beta = 0$, we need to use the space $X$ instead of $E$ to study problem (1.1) via variational methods. For this purpose, in the rest of this section we will provide further properties for the space $X$. As we discussed in the introduction, for $p \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})$ satisfying (P) we can extend $p$ the whole space $\mathbb{R}^N \times \mathbb{R}^N$ to have $p \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$ satisfying (P) with $\Omega = \mathbb{R}^N$. We introduce a new norm on $X$ as follows:

$$
\|u\|_X = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy \leq 1 \right\}.
$$

Lemma 2.4. The functional $M : X \to \mathbb{R}$ defined by

$$
M(u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy
$$

has the following properties:

(i) for $\alpha > 0$, $\|u\|_X = (>, <) \alpha$ if and only if $M\left(\frac{u}{\alpha}\right) = (>, <) 1$;

(ii) if $\|u\|_0 > 1$, then $\|u\|_0^p \leq M(u) \leq \|u\|_0^{p^+}$.

(iii) if $\|u\|_0 < 1$, then $\|u\|_0^{p^+} \leq M(u) \leq \|u\|_0^{p^-}$.

Proof. It is a direct consequence of Proposition A.1. $\square$

Now, we prove the following compact embedding type result by employing some ideas in [4].

Theorem 2.5. Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let $s \in (0, 1)$. Let $p \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})$ satisfy (P) (with $\Omega = \mathbb{R}^N$). Then, for any $r \in C(\overline{\Omega}, \mathbb{R})$ satisfying (R), there exists a constant $C > 0$ such that

$$
\|u\|_{L^r(\Omega)} \leq C \|u\|_0, \quad \forall u \in X.
$$

Moreover, the embedding $X \hookrightarrow L^r(\Omega)$ is compact.

Proof. First, we claim that there exists a constant $C_0 > 0$ such that

$$
C_0 \|u\|_{L^p(\Omega)} \leq \|u\|_0, \quad \forall u \in X,
$$

To this end, let $A = \{u \in X : \|u\|_{L^p(\Omega)} = 1\}$. Take a sequence $\{u_n\} \subset A$ such that

$$
\lim_{n \to \infty} \|u_n\|_0 = \inf_{u \in A} \|u\|_0.
$$

So, $\{u_n\}$ is bounded in $L^p(\Omega)$ and $X_0$. Hence, $\{u_n\}$ is bounded in $L^p(\Omega)$.
W^{s,p(\cdot)}(\Omega). Up to a subsequence, there exist a subsequence of \( \{u_n\} \), still denote by \( \{u_n\} \), and \( u_0 \in W^{s,p(\cdot)}(\Omega) \) such that \( u_n \rightharpoonup u_0 \) in \( W^{s,p(\cdot)}(\Omega) \). By Theorem 1.1, we get that \( u_n \rightharpoonup u_0 \) in \( L^{p(\cdot)}(\Omega) \) and \( \|u_0\|_{L^{p(\cdot)}(\Omega)} = 1 \). Now, we extend \( u_0 \) to \( \mathbb{R}^N \) by setting \( u_0 = 0 \) in \( \mathbb{R}^N \setminus \Omega \). This implies \( u_n(x) \to u_0(x) \) a.e. in \( \mathbb{R}^N \) as \( n \to \infty \). Hence, by Fatou’s Lemma, we have
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy,
\]
which joining with \( \|u_0\|_{L^{p(\cdot)}(\Omega)} = 1 \) implies that \( u_0 \in \mathcal{A} \). Set \( \lambda_0 := \inf_{u \in \mathcal{A}} \|u\|_0 \) and \( \lambda_n := \|u_n\|_0 \) \((n=1,2,\ldots)\). From the fact that \( u_n \in \mathcal{A} \) we have that \( \lambda_n > 0 \) for \( n = 0,1,2,\ldots \) and hence, by Lemma 2.4 and by Fatou’s Lemma again, we have
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy = 1.
\]
This and Lemma 2.4 yield
\[
\|u_0\|_0 \leq \lambda_0 = \inf_{u \in \mathcal{A}} \|u\|_0.
\]
Therefore, we obtain \( 0 < \|u_0\|_0 = \inf_{u \in \mathcal{A}} \|u\|_0 := C_0 \) and this proves our claim. From (2.3), it follows that
\[
\|u\|_{W^{s,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + [u]_{W^{s,p(\cdot)}(\Omega)} \leq \|u\|_{L^{p(\cdot)}(\Omega)} + \|u\|_0 \leq (1 + \frac{1}{C_0})\|u\|_0,
\]
which implies that \( X \) is continuously embedded in \( W^{s,p(\cdot)}(\Omega) \). From (2.4) and Theorem 1.1, there exists a constant \( C > 0 \) such that
\[
\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|u\|_0,
\]
Thus, (2.2) has been proved. Finally, combining the fact that \( X \hookrightarrow W^{s,p(\cdot)}(\Omega) \) and \( W^{s,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \) (applying Theorem 1.1 again) we obtain \( X \hookrightarrow L^{r(\cdot)}(\Omega) \). The proof is complete. \( \square \)

3. EIGENVALUE PROBLEM

Motivated by [8,17], in this section we are concerned with the following nonhomogeneous problem
\[
\begin{cases}
(-\Delta)^s_{p(x)} u + \alpha |u|^{p(x)-2} u + \beta |u|^{q(x)-2} u = \lambda |u|^{r(x)-2} u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where \( p \in C(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}) \) satisfy (P) (with \( \Omega = \mathbb{R}^N \)), \( q, r \in C(\overline{\Omega}, \mathbb{R}) \) satisfy (Q) and (R); \( \alpha, \beta \) are nonnegative real numbers; and \( \lambda \) is a real spectral parameter.

Note that when \( \beta = 0 \), we will regard \( q(x) = p(x) \) on \( \overline{\Omega} \) in all our statements appearing \( q \).

**Definition 3.1.** A pair \((u, \lambda) \in X \times \mathbb{R}\) is called a solution of problem (3.1) if
\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+sp(x,y)}} \, dx \, dy + \alpha \int_{\Omega} |u|^{p(x)-2} uv \, dx + \beta \int_{\Omega} |u|^{q(x)-2} uv \, dx = \lambda \int_{\Omega} |u|^{r(x)-2} uv \, dx, \quad \forall v \in X.
\]
If \((u, \lambda)\) is a solution of problem (3.1) and \( u \in X \setminus \{0\} \), as usual, we call \( \lambda \) and \( u \) an *eigenvalue* and an *eigenfunction* corresponding to \( \lambda \) for problem (3.1), respectively. A solution \((u, \lambda)\) of (3.1) with \( u \neq 0 \) is also called an *eigenpair* of problem (3.1).
We divide this section into three subsections. In the first part by employing the Ljusternik-Schnirelmann theory, we construct a sequence of eigenvalues of problem \((3.1)\). We also discuss about the positivity of the infimum of the set of eigenvalues of problem \((3.1)\) in this subsection. In the last two parts, we deal with the existence and the nonexistence of eigenvalues of problem \((3.1)\) under some additional assumptions.

In order to investigate the eigenvalues of \((3.1)\), we consider the energy functional associated with problem \((3.1)\). In particular we consider the functionals \(I, I_0, J, J_0, \Phi_\lambda : X \rightarrow \mathbb{R}\) given by

\[
I(u) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} \, dx \, dy + \alpha \int_{\Omega} \frac{|u(x)|^{p(x)}}{p(x)} \, dx + \beta \int_{\Omega} \frac{|u(x)|^{q(x)}}{q(x)} \, dx,
\]

\[
I_0(u) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy + \alpha \int_{\Omega} |u(x)|^{p(x)} \, dx + \beta \int_{\Omega} |u(x)|^{q(x)} \, dx,
\]

\[
J(u) = \int_{\Omega} \frac{|u(x)|^{r(x)}}{r(x)} \, dx, \quad J_0(u) = \int_{\Omega} |u(x)|^{r(x)} \, dx,
\]

and

\[
\Phi_\lambda(u) = I(u) - \lambda J(u). \quad (3.2)
\]

Invoking Theorem 2.5 and a standard argument, we can show that \(I, I_0, J, J_0, \Phi_\lambda \in C^1(X, \mathbb{R})\) and a critical point of \(\Phi_\lambda\) is a solution to problem \((3.1)\). In the sequel, we will make use of the following values:

\[
\gamma_1 = \inf_{u \in X \setminus \{0\}} \frac{I(u)}{J(u)} \quad \text{and} \quad \gamma_0 = \inf_{u \in X \setminus \{0\}} \frac{I_0(u)}{J_0(u)}. \quad (3.3)
\]

Clearly,

\[
\min \{p^-, q^-\} \gamma_1 \leq \gamma_0 \leq \frac{\max \{p^+, q^+\}}{r^-} \gamma_1. \quad (3.4)
\]

In what follows, unless otherwise stated on \(X\) we will make use the norm \(|\cdot|_\infty\) or \(|\cdot|_1\) given by \((2.1)\) when \(\alpha + \beta > 0\).

### 3.1. A sequence of eigenvalues.

In this subsection we construct a sequence of eigenvalues for \((3.1)\) via the Ljusternik-Schnirelmann theory. Denote

\[
\Lambda := \{\lambda : \lambda \text{ is an eigenvalue of } (3.1)\}.
\]

For \(t > 0\), define

\[
N_t := \{u \in X : I(u) = t\}.
\]

Clearly, for each \(u \in X \setminus \{0\}\), there exists a unique \(s_t = s_t(u) \in (0, \infty)\) such that \(s_t u \in N_t\). Moreover, we have

\[
s_t \to 0 \quad \text{as} \quad t \to 0^+ \quad \text{and} \quad s_t \to +\infty \quad \text{as} \quad t \to +\infty. \quad (3.5)
\]

For each \(n \in \mathbb{N}\), define

\[
K_n := \{K \subset X \setminus \{0\} : K \text{ is compact}, -K = K, \text{ and } \gamma(K) \geq n\},
\]

where \(\gamma(K)\) denote the Krasnoselskii genus of \(K\), and

\[
c_n(t) := \sup_{K \in K_n} \inf_{u \in K} J(u).
\]

Clearly, \(c_n(t)\) is well defined for all \(n \in \mathbb{N}\). Moreover, we have

\[
c_1(t) \geq c_2(t) \geq \cdots \geq c_n(t) \geq c_{n+1}(t) \geq \cdots > 0. \quad (3.6)
\]

Also, we have the following formulation

\[
c_1(t) = \sup_{u \in N_t} J(u). \quad (3.7)
\]
By the Lagrange multiplier rule, \( u \) is a critical point of \( J \) restricted to \( N_t \) if and only if \((u, \lambda)\) with
\[
\lambda = \lambda(u) := \frac{I_0(u)}{J_0(u)}
\] is a solution of (3.1) (see [19, Sections 43.9 and 44.5]). The next theorem is deduced from the Ljusternik-Schnirelmann theory (see [19, Theorem 44.A]).

**Theorem 3.2.** For each \( t > 0 \), the following assertions hold:

(i) for each \( n \in \mathbb{N} \), \( c_n(t) \) is a critical value of \( J \) restricted on \( N_t \);

(ii) \( c_n(t) \to 0^+ \) as \( n \to \infty \).

Let \( u_n \in N_t \) such that \( c_n(t) = J(u_n) \), then by (3.8), \((u_n, \lambda_n)\) is an eigenpair of (3.1) with
\[
\lambda_n = \frac{I_0(u_n)}{J_0(u_n)} \geq \min\{ p^-, q^- \} \frac{r^+}{r^+} J(u_n) = \frac{\min\{ p^-, q^- \} \ t}{c_n(t)}.
\]
Hence, the next corollary is a direct consequence of Theorem 3.2.

**Corollary 3.3.** For each \( t > 0 \), problem (3.1) admits a sequence of eigenpairs \( \{(u_n, \lambda_n)\} \) with \( u_n \in N_t \) and \( \lambda_n \to +\infty \) as \( n \to \infty \).

To have more information about the set of eigenpairs associated with \( c_n(t) \) restricted to \( N_t \), define for \( t > 0 \) and \( n \in \mathbb{N} \),
\[
K_n(t) := \{ u \in N_t : u \text{ is a critical point of } J \text{ restricted to } N_t \text{ and } J(u) = c_n(t) \}
\]
and
\[
\Lambda_n(t) := \{ \lambda(u) : u \in K_n(t) \}.
\]
By (3.8) again, we have
\[
\min\{ p^-, q^- \} \frac{t}{c_n(t)} \leq \lambda(u) = \frac{I_0(u)}{J_0(u)} \leq \max\{ p^+, q^+ \} \frac{t}{c_n(t)}, \quad \forall u \in K_n(t).
\] (3.9)

In the following, for brevity, an inequality \( \Lambda_n(t) \leq (\geq) C \) means that \( \lambda \leq (\geq) C \) for every \( \lambda \in \Lambda_n(t) \) and a limit \( \Lambda_n(t) \to a \) as \( n \to \infty \) means the limit occurs uniformly with respect to \( \lambda \in \Lambda_n(t) \). By the definitions of \( \Lambda_n(t) \) and (3.9), we easily obtain the following estimates: for each \( t > 0 \) and \( n \in \mathbb{N} \),
\[
\min\{ p^-, q^- \} \frac{t}{c_n(t)} \leq \Lambda_n(t) \leq \max\{ p^+, q^+ \} \frac{t}{c_n(t)}. \] (3.10)

From (3.10) and Theorem 3.2, we have the following.

**Theorem 3.4.** For each \( t > 0 \) and for each \( n \in \mathbb{N} \), the sets \( K_n(t) \) and \( \Lambda_n(t) \) are nonempty, \( \Lambda_n(t) \subset \Lambda \), and for any \( u \in K_n(t) \), \((u, \lambda(u))\) is a solution of (3.1). Moreover, for each \( t > 0 \), \( \Lambda_n(t) \to +\infty \) as \( n \to \infty \).

The infimum of eigenvalues

Denote
\[
\lambda_* := \inf \Lambda. \] (3.11)

By Theorem 3.4, \( \lambda_* \) is well defined and it is clear that \( \lambda_* \in [0, \infty) \). It is worth pointing out that when \( p, q, \) and \( r \) are constant functions and \( p = q = r \), we have that \( \lambda_* = \gamma_0 > 0 \) and is the first eigenvalue of (3.1). In the variable exponent case, it is not true in general. First, we have the relation of positivity of \( \gamma_0, \gamma_1 \) and \( \lambda_* \) as follows.

**Lemma 3.5.** It holds that
\[
\gamma_1 > 0 \iff \gamma_0 > 0 \iff \lambda_* > 0.
\]
Proof. By (3.4), it suffices to prove that
\[ \gamma_0 > 0 \iff \lambda_* > 0. \] (3.12)
It is clear that \( \gamma_0 \leq \lambda_* \) hence; \((\Rightarrow)\) is obvious. Now suppose \( \gamma_0 = 0 \). Then for any given \( \epsilon > 0 \), there exists \( u_\epsilon \in X \setminus \{0\} \) such that
\[ \frac{I_0(u_\epsilon)}{J_0(u_\epsilon)} < \epsilon. \] (3.13)
Let \( I(u_\epsilon) = t \). Then, from (3.6), (3.7) and (3.13) we obtain
\[ \frac{t}{c_1(t)} \leq \frac{I(u_\epsilon)}{J(u_\epsilon)} \leq \frac{\min\{p^+, q^+\} I_0(u_\epsilon)}{r^+ J_0(u_\epsilon)} < \frac{r^+}{\min\{p^-, q^-\}} \epsilon. \] (3.14)
Let \( u_t \in N_t \) such that \( J(u_t) = c_1(t) \). Then, we deduce from (3.14) that
\[ \lambda(u_t) = \frac{I_0(u_t)}{J_0(u_t)} \leq \frac{\max\{p^+, q^+\} I(u_t)}{r^- J(u_t)} = \frac{\max\{p^+, q^+\}}{r^- c_1(t)} t < \frac{r^+ \max\{p^+, q^+\}}{r^- \min\{p^-, q^-\}} \epsilon. \]
Combining this with (3.11) gives
\[ 0 \leq \lambda_* < \frac{r^+ \max\{p^+, q^+\}}{r^- \min\{p^-, q^-\}} \epsilon. \]
Since \( \epsilon > 0 \) was chosen arbitrarily, we arrive at \( \lambda_* = 0 \). This infers that \((\Leftarrow)\) also holds. That is, (3.12) holds and the proof is complete. \( \square \)

In the next two lemmas, we provide sufficient conditions to get \( \lambda_* = 0 \). We will make use of the following conditions. In these conditions, by \( h^+(V) \) (resp. \( h^-(V) \)) we mean the supremum (resp. infimum) of the function \( h \) over the set \( V \).

(A1) There exist an open subset \( U \) of \( \Omega \) such that
\[ r^+(U) < \min\{p^-(U \times \mathbb{R}^N), q^-(U)\}. \]

(A2) There exist an open subset \( \tilde{U} \) of \( \Omega \) such that
\[ r^-(\tilde{U}) > \max\{p^+(\tilde{U} \times \mathbb{R}^N), q^+(\tilde{U})\}. \]

For each \( t > 0 \), define
\[ \mu_1(t) := \frac{t}{c_1(t)} \]
and
\[ \lambda^*(t) := \inf\{\lambda(u) : u \text{ is a critical point of } J \text{ restricted to } N_t\}. \] (3.15)

Lemma 3.6. Let (A1) hold. Then, \( \mu_1(t) \to 0^+ \) and \( \lambda^*(t) \to 0^+ \) as \( t \to 0^+ \). Consequently, \( \lambda_* = 0 \).

\( \text{Proof.} \) Let \( B \) be a ball in \( \mathbb{R}^N \) such that \( \overline{B} \subset U \). Let \( \varphi \in C_c^\infty(\Omega) \) be such that \( \varphi \equiv 1 \) on \( B \) and \( \varphi \equiv 0 \) on \( \mathbb{R}^N \setminus U \). By (3.5) and the strictly increasing monotonicity of \( t \mapsto I(tu) \) on \((0, +\infty)\) for each \( t > 0 \) small enough, there exists a unique \( s_t \in (0, 1) \) such that \( s_t \varphi \in N_t \). Set \( \delta := \min\{p^-(U \times \mathbb{R}^N), q^-(U)\} - r^+(U) > 0 \). Let \( t \in (0, +\infty) \) be arbitrary and fixed. We have
\[ \mu_1(t) \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\rho'(x,y) \rho^{p(x,y)} |\varphi(x)-\varphi(y)| p(x,y) |r(x)| q(x,y)}{p(x,y) + p^+ q^+ p(x,y) q(x,y) - r(x) q(x,y)} \, dx \, dy + \alpha \int_{\Omega} \frac{\rho'(x) |\varphi(x)| p(x)}{p(x)} \, dx + \beta \int_{\Omega} \frac{s_t^{q_0}(x) |\varphi(x)| q(x)}{r(x)} \, dx \] (3.16)
We estimate each integral in the right-hand side of (3.16) as follows. We have
\[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{s_t^{p(x,y)}}{p(x,y)} |\varphi(x) - \varphi(y)|^p(x,y) \ dx \ dy \leq 2 \int_{\mathbb{R}^N} \int_{U} s_t^{p(x,y)} \frac{|\varphi(x) - \varphi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} \ dx \ dy \]
\[ \leq 2s_t^{r^+(U)+\delta} \int_{\mathbb{R}^N} \int_{U} \frac{|\varphi(x) - \varphi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}} \ dx \ dy. \]

For the second and the third integrals, we estimate
\[ \int_{\Omega} \frac{s_t^{p(x)}}{p(x)} |\varphi|^{p(x)} \ dx = \int_{U} s_t^{r^-(U)+\delta} \frac{|\varphi|^{p(x)}}{p(x)} \ dx \leq s_t^{r^+(U)+\delta} \int_{U} |\varphi|^{p(x)} \ dx, \]
\[ \int_{\Omega} \frac{s_t^{q(x)}}{q(x)} |\varphi|^{q(x)} \ dx \leq \int_{U} s_t^{r^+(U)+\delta} \frac{|\varphi|^{q(x)}}{q(x)} \ dx. \]

Finally, we have
\[ \int_{\Omega} \frac{s_t^{r(x)}}{r(x)} |\varphi(x)|^{r(x)} \ dx \geq s_t^{r^+(U)} \int_{\Omega} |\varphi(x)|^{r(x)} \ dx. \]

Utilizing the last four estimates, we deduce from (3.16) that
\[ \mu_1(t) \leq \frac{2I(\varphi)}{J(\varphi)s_t^\delta}. \]

On the other hand, by (3.9) and (3.15) we have
\[ 0 \leq \lambda^*(t) \leq \max\{p^+, q^+\} \frac{\mu_1(t)}{r^-}. \]

Combining the last two estimates and (3.5), we conclude the lemma. The proof is complete. □

**Lemma 3.7.** Let (A2) hold. Then, \( \mu_1(t) \to 0^+ \) and \( \lambda^*(t) \to 0^+ \) as \( t \to +\infty \). Consequently, \( \lambda_* = 0 \).

The proof of Lemma 3.7 is similar to that of Lemma 3.6 for which we put \( \delta := r^-(\bar{U}) - \max\{p^+ (\bar{U} \times \mathbb{R}^N), q^+ (\bar{U})\} \) and take \( t \in (0, +\infty) \) so large that \( s_t \in (1, +\infty) \). We leave the details to the reader.

Set
\[ \mu_* := \inf\{\mu_1(t) : t \in (0, \infty)\} \quad \text{and} \quad \mu^* := \sup\{\mu_1(t) : t \in (0, \infty)\}. \quad (3.17) \]

By Lemmas 3.6 and 3.7, if either (A1) or (A2) holds, then \( \mu_* = 0 \). Clearly, we always have \( \mu^* > 0 \). Moreover, if (A1) and (A2) hold, then \( \mu^* < \infty \).

**3.2. Existence results with the growth of type I.** In this subsection, we provide a precise range of small eigenvalues for problem (3.1). Using the positive constant \( \mu^* \) given by (3.17), we have the following existence result.

**Theorem 3.8.** Let (A1) hold and define \( \Phi_\lambda \) as in (3.2). Then, for any given \( \lambda \in (0, \mu^*) \), \( \Phi_\lambda \) has a nonnegative local minimizer \( u_\lambda \) such that \( \Phi_\lambda(u_\lambda) < 0 \). Consequently, for any given \( \lambda \in (0, \mu^*) \), problem (3.1) has a nontrivial nonnegative solution \( u_\lambda \) with \( \Phi_\lambda(u_\lambda) < 0 \).

**Remark 3.9.** It is worth pointing out in existing works close to our work (e.g. [1, 6, 16]) the author assumed \( q(x) \geq p(x, x) \) on \( \overline{\Omega} \) and used a sublinear type growth \( r^- < p^- < r^+ \). It is easy to see that with the additional assumption \( q(x) \geq p(x, x) \) on \( \overline{\Omega} \), the condition \( r^- < p^- \) implies the condition (A1). That is, we are dealing with a weaker condition for this type of growth.

The proof of Theorem 3.8 is similar to that of [9, Theorem 3.3] and we only sketch the proof for sake of completeness.
Proof of Theorem 3.8. Let $\lambda \in (0, \mu^*)$. By Lemma 3.6, we have $\mu_* = 0$. Since $\mu_1(t)$ is continuous with respect to $t$ on $(0, +\infty)$ (c.f. [9, Proposition 2.3]), $\mu_1((0, +\infty))$ is connected. Thus, we find $t_\lambda > 0$ such that

$$\lambda \leq \mu_1(t_\lambda) = \frac{t_\lambda}{c_1(t_\lambda)}.$$  

Set $D = \{ u \in X : I(u) \leq t_\lambda \}$. Then, $D$ is closed, bounded and convex in $X$ and $\partial D = I^{-1}(t_\lambda) = N_{t_\lambda}$. Invoking (3.7) with $t = t_\lambda$ it follows that for any $u \in \partial D$,

$$\Phi_\lambda(u) \geq t_\lambda - \lambda c_1(t_\lambda) = t_\lambda \left(1 - \frac{\lambda c_1(t_\lambda)}{t_\lambda}\right) \geq 0.$$  

Since $\Phi_\lambda : D \to \mathbb{R}$ is weakly lower semicontinuous on $D$ and $D$ is weakly compact, $\Phi_\lambda$ achieves a global minimum on $D$ at some $w_\lambda \in D$ i.e.,

$$\Phi_\lambda(w_\lambda) = \inf_{u \in D} \Phi_\lambda(u).$$  

We claim that $\Phi_\lambda(w_\lambda) < 0$. To this end, invoking Lemma 3.6 again we find $t_0 \in (0, t_\lambda)$ such that $\mu_1(t_0) < \lambda$. Let $v \in N_{t_0}$ such that $J(v) = c_1(t_0)$. This yields

$$\Phi_\lambda(v) = I(v) - \lambda J(v) = t_0 - \lambda c_1(t_0) < 0$$  

which shows that $\Phi_\lambda(w_\lambda) = \inf_{u \in D} \Phi_\lambda(u) < 0$. By letting $u_\lambda = |w_\lambda|$, we deduce that $u_\lambda$ is a local minimizer of $D$ and hence, $u_\lambda$ is a nontrivial nonnegative solution of problem (3.1). The proof is complete. \hfill \Box

3.3. Existence/Nonexistence results with the growth of type II. In this part we study the nonexistence of eigenvalue for problem (3.1) with $\alpha > 0$ and $\beta > 0$ assuming that the functions $p, q$ and $r$ satisfy the condition

$$p^+ < r^+ \leq r^+ < q^+ < \frac{Np^+}{N - sp^+}.$$  

(G)

Our main result in this subsection is given by the following theorem.

Theorem 3.10. Assume that conditions $(P)$ (with $\Omega$ replaced by $\mathbb{R}^N$) and $(G)$ are fulfilled and let $\gamma_0, \gamma_1$ be defined in (3.3). Then, $\gamma_0, \gamma_1 \in (0, \infty)$ and any $\lambda \in (\gamma_1, \infty)$ is an eigenvalue of problem (3.1) which admits a nonnegative eigenfunction and any $\lambda \in (0, \gamma_0)$ is not an eigenvalue of problem (3.1).

In the rest of this subsection, on $X$ we will make use of the equivalent norm

$$\|u\| := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}} \, dx \, dy + \int_{\Omega} \frac{|u|^{p(x)}}{\lambda} \, dx \leq 1 \right\}.$$  

We now prove Theorem 3.10 by adapting ideas used in [17]. In the rest of this section, we always that assumptions of Theorem 3.10 are fulfilled and for simplicity and clarity of our arguments, we just take $\alpha = \beta = 1$.

Lemma 3.11. It holds that

$$\gamma_1, \gamma_0 > 0.$$  

Proof. By condition $(G)$ we deduce that

$$\int_{\Omega} |u|^{q(x)} \, dx + \int_{\Omega} |u|^{p(x)} \, dx \geq \int_{\Omega} |u|^{r(x)} \, dx.$$  

Thus

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} \, dx \, dy + \int_{\Omega} |u|^{p(x)} \, dx + \int_{\Omega} |u|^{q(x)} \, dx \geq \int_{\Omega} |u|^{r(x)} \, dx.$$  

This implies that $\gamma_0 > 0$. This and (3.4) imply $\gamma_1 > 0$ and the proof is complete. \hfill \Box
Lemma 3.12. It holds that
\[
\lim_{\|u\| \to +\infty} \frac{I(u)}{J(u)} = +\infty
\]  
(3.18)
and
\[
\lim_{\|u\| \to +0} \frac{I(u)}{J(u)} = +\infty.
\]  
(3.19)

Proof. We first note that \( r^+ < q^- \) and the embedding \( X \hookrightarrow L^{r^+}(\Omega) \) imply that there is \( C_r > 1 \) such that
\[
\|u\|_{L^{r^+}(\Omega)} \leq C_r \min \{\|u\|_{L^q(\Omega)} \}, \quad \forall u \in X.
\]  
(3.20)
Using (3.20), for \( u \in X \) with \( \|u\| > 1 \) we have
\[
\frac{I(u)}{J(u)} \geq \frac{\frac{1}{p^+} \|u\|^{p^+} + \frac{1}{q^-} \min \left\{ \|u\|^{q^+}_{L^{q^+}(\Omega)}, \|u\|^{q^-}_{L^{q^+}(\Omega)} \right\}}{\frac{1}{r} \max \left\{ \|u\|^{r^+}_{L^r(\Omega)}, \|u\|^{r^-}_{L^r(\Omega)} \right\}} \geq \frac{\frac{1}{p^+} \|u\|^{p^+} + \frac{1}{q^-} \min \left\{ \|u\|^{q^+}_{L^{q^+}(\Omega)}, \|u\|^{q^-}_{L^{q^+}(\Omega)} \right\}}{\frac{C_r}{r} \max \left\{ \|u\|^{r^+}_{L^r(\Omega)}, \|u\|^{r^-}_{L^r(\Omega)} \right\}}.
\]  
(3.21)
Let \( \{u_n\} \subset X \setminus \{0\} \) be any sequence such that \( \|u_n\| \to \infty \) as \( n \to \infty \). If \( \|u_n\|_{L^q(\Omega)} \to \infty \) then, \( \frac{I(u_n)}{J(u_n)} \to \infty \) due to (3.21) and the fact that \( r^+ < q^- \). If, up to a subsequence, \( \|u_n\|_{L^q(\Omega)} \) is bounded, then we also have \( \frac{I(u_n)}{J(u_n)} \to \infty \) due to (3.21). That is, (3.18) holds.

Next, we prove (3.19). Invoking (3.20) again, for \( u \in X \) with \( 0 < \|u\| < 1 \) we have
\[
\frac{I(u)}{J(u)} \geq \frac{\frac{1}{p^+} \|u\|^{p^+}}{\frac{1}{r} \max \left\{ \|u\|^{r^+}_{L^r(\Omega)}, \|u\|^{r^-}_{L^r(\Omega)} \right\}} \geq \frac{r^- \|u\|^{p^+}}{p^+ C_r \|u\|^{r^-}}.
\]  
(3.22)
Then, (3.19) follows from (3.22) and the fact that \( p^+ < r^- \).

\[ \square \]

Lemma 3.13. The infimum \( \gamma_1 \) is achieved at some \( u \in X \setminus \{0\} \). Moreover, \( (u, \gamma_1) \) is an eigenpair of problem (3.1).

Proof. Let \( \{u_n\} \subset X \setminus \{0\} \) such that
\[
\lim_{n \to \infty} \frac{I(u_n)}{J(u_n)} = \gamma_1 > 0.
\]  
(3.23)
From this and (3.18) it follows that \( \{u_n\} \) is bounded in \( X \). Thus, up to a subsequence we have \( u_n \rightharpoonup u \) in \( X \). Since \( X \hookrightarrow L^{r^+}(\Omega) \), we easily deduce that
\[
\lim_{n \to \infty} J(u_n) = J(u).
\]  
(3.24)
On the other hand, the continuity and the convexity of \( I \) on \( X \) imply that \( I \) is weakly lower semicontinuous on \( X \). Thus, we have
\[
\liminf_{n \to \infty} I(u_n) \geq I(u).
\]  
(3.25)
We claim that \( u \neq 0 \). Indeed, suppose by contradiction that \( u = 0 \). Then, (3.24) gives \( \lim_{n \to \infty} J(u_n) = 0 \). Combining this and (3.23) we easily obtain that \( \lim_{n \to \infty} I(u_n) = 0 \) and hence, \( \lim_{n \to \infty} \|u_n\| = 0 \). From this and (3.19) jointly with (3.23), we arrive at a contradiction. That is, we have shown that \( u \neq 0 \). Thus, it follows from (3.24) and (3.25) that
\[
\lim_{n \to \infty} \frac{I(u_n)}{J(u_n)} \geq \frac{I(u)}{J(u)}.
\]
Combining this with (3.23) and the definition of $\gamma_1$ gives

$$
\gamma_1 = \frac{I(u)}{J(u)}.
$$

(3.26)

It remains to show that $(u, \gamma_1)$ is an eigenpair of problem (3.1). From the definition of $\gamma_1$ and (3.26) we deduce that for any $v \in X$,

$$
\frac{d}{dt} I(u + tv) \bigg|_{t=0} = 0.
$$

By a simple computation, the last equality and (3.26) yield

$$
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|_p(x,y) - 2(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + sp(x,y)}} \, dx \, dy + \int_{\Omega} |u|^{p(x) - 2}uv \, dx
$$

$$
+ \int_{\Omega} |u|^{q(x) - 2}uv \, dx = \gamma_1 \int_{\Omega} |u|^{p(x) - 2}uv \, dx.
$$

That is, $(u, \gamma_1)$ is an eigenpair of problem (3.1). The proof is complete. \hfill \square

**Proof of Theorem 3.10 completed.** Let $\lambda \in (\gamma_1, \infty)$. Recall that $\Phi_\lambda$ is of class $C^1(X, \mathbb{R})$ and any nontrivial critical point of $\Phi_\lambda$ is a nontrivial solution of problem (3.1), i.e., $\lambda$ is an eigenvalue of problem (3.1). By (3.18), it is clear that $\Phi_\lambda$ is coercive. Moreover, $\Phi_\lambda$ is weakly lower semicontinuous, and hence $\Phi_\lambda$ has a global minimum achieved at some $w_\lambda \in X$. Since $\lambda > \gamma_1$, we find $v_\lambda \in X \setminus \{0\}$ such that $\frac{I(v_\lambda)}{J(v_\lambda)} < \lambda$, i.e., $\Phi_\lambda(v_\lambda) < 0$. This yields $\Phi_\lambda(w_\lambda) < 0$ and hence, $w_\lambda \neq 0$. Putting $u_\lambda = |w_\lambda|$ we deduce that $\Phi_\lambda(u_\lambda) \leq \Phi_\lambda(w_\lambda)$, and hence $u_\lambda$ is also a global minimum point for $\Phi_\lambda$. Thus, $u_\lambda$ is a critical point of $\Phi_\lambda$. That is, we have shown that any $\lambda \in (\gamma_1, \infty)$ is an eigenvalue of problem (3.1) and problem (3.1) admits a nontrivial nonnegative solution.

Finally, let $\lambda \in (0, \gamma_0)$. Assuming by contradiction that there exists a $u_\lambda \in X \setminus \{0\}$ such that

$$
\langle I'(u_\lambda), v \rangle = \lambda \langle J'(u_\lambda), v \rangle, \quad \forall v \in X.
$$

Taking $v = u_\lambda$ in the above equality we get

$$
\langle I'(u_\lambda), u_\lambda \rangle = \lambda \langle J'(u_\lambda), u_\lambda \rangle,
$$

i.e.,

$$
I_0(u_\lambda) = \lambda J_0(u_\lambda).
$$

Thus,

$$
\lambda = \frac{I_0(u_\lambda)}{J_0(u_\lambda)} \geq \gamma_0,
$$

a contradiction. The proof is complete. \hfill \square

**Appendix. The Lebesgue spaces with variable exponents**

In this Appendix, we recall some necessary properties of the Lebesgue spaces with variable exponents. We refer to [8,15] and the references therein.

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$. Consider the set

$$
C_+(\overline{\Omega}) = \{ p \in C(\overline{\Omega}, \mathbb{R}) : p(x) > 1 \text{ for all } x \in \overline{\Omega} \}.
$$

For any $p \in C_+(\overline{\Omega})$, denote

$$
p^+ = \sup_{x \in \Omega} p(x) \quad \text{and} \quad p^- = \inf_{x \in \Omega} p(x)
$$
and define the \textit{variable exponent Lebesgue space} $L^{p(\cdot)}(\Omega)$ as

$$L^{p(\cdot)}(\Omega) = \left\{ u : \text{u is measurable real-valued function, } \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \right\}.$$ 

This vector space is a Banach space if it is endowed with the \textit{Luxemburg norm}, which is defined by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \mu > 0 : \int_{\Omega} \frac{|u(x)|^{p(x)}}{\mu} \, dx \leq 1 \right\}.$$ 

We point out that if $p(x) \equiv p \in [1, \infty)$ then the optimal choice in the above expression is $\mu = \|u\|_{L^{p(\cdot)}(\Omega)}$.

Let $p \in C_+([\bar{\Omega}])$ and let $L^{q(\cdot)}(\Omega)$ denote the conjugate space of $L^{p(\cdot)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$.

If $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$ then the following H"older-type inequality holds:

$$\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{q(\cdot)}(\Omega)}.$$ 

Moreover, if $p_j \in C_+([\bar{\Omega}])$ ($j = 1, 2, \ldots, k$) and

$$\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \cdots + \frac{1}{p_k(x)} = 1,$$

then for all $u_j \in L^{p_j(\cdot)}(\Omega)$ ($j = 1, \ldots, k$) we have

$$\left| \int_{\Omega} u_1 u_2 \cdots u_k \, dx \right| \leq \left( \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \right) \|u_1\|_{L^{p_1(\cdot)}(\Omega)} \|u_2\|_{L^{p_2(\cdot)}(\Omega)} \cdots \|u_k\|_{L^{p_k(\cdot)}(\Omega)}.$$ 

An important role in manipulating the generalized Lebesgue spaces is played by the \textit{modular} of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho(u) = \int_{\Omega} |u|^{p(x)} \, dx.$$ 

\textbf{Proposition A.1.} \textit{It hold that:}

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (= 1; > 1)$ $\iff$ $\rho(u) < 1 (= 1; > 1)$.

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$ $\iff$ $\|u\|_{L^{p(\cdot)}(\Omega)}^{p^-} \leq \rho(u) \leq \|u\|_{L^{p(\cdot)}(\Omega)}^{p^+}$.

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$ $\iff$ $\|u\|_{L^{p(\cdot)}(\Omega)}^{p^+} \leq \rho(u) \leq \|u\|_{L^{p(\cdot)}(\Omega)}^{p^-}$.

\textbf{Proposition A.2.} \textit{If $u, u_n \in L^{p(\cdot)}(\Omega)$ ($n \in \mathbb{N}$), then the following statements are equivalent to each other:}

(1) $\lim_{n \to \infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0$.

(2) $\lim_{n \to \infty} \rho(u_n - u) = 0$.

\textbf{References}

[1] E. Azroul, A. Benkirane, M. Shimi, Eigenvalue problems involving the fractional $p(x)$-Laplacian, \textit{Adv. Oper. Theory} \textbf{4} (2019), 539–555. 3, 9

[2] A. Bahrouni, Trudinger-Moser type inequality and existence of solution for perturbed non-local elliptic operators with exponential nonlinearity, \textit{Comm. Pure Appl. Anal.} \textbf{16} (2017), 243–252. 1

[3] A. Bahrouni, V. Rădulescu, On a new fractional Sobolev space and applications to nonlocal variational problems with variable exponent, \textit{Discrete Contin. Dyn. Syst. Ser. S.} \textbf{11} (2018), 379–389. 1, 4

[4] A. Bahrouni, Comparaison and sub-supersolution principles for the fractional $p(x)$-Laplacian, \textit{J. Math. Anal. Appl.} \textbf{458} (2018), 1363–1372. 1, 4

[5] A. Bahrouni, V. Radulescu, D. Repovs, A weighted anisotropic variant of the Caffarelli-Kohn-Nirenberg inequality and applications, \textit{Nonlinearity} \textbf{31} (2018), 1516-1534. 1
[6] N.T. Chung, Eigenvalue Problems for Fractional $p(x, y)$-Laplacian Equations with Indefinite Weight, *Taiwanese Journal Of Mathematics* **23** (2019), 1153–1173.

[7] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** (2012), 521–573.

[8] X. Fan, Q. Zhang, D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, *J. Math. Anal. Appl.* **302** (2005), 306–317.

[9] X. Fan, Remarks on eigenvalue problems involving the $p(x)$-Laplacian, *J. Math. Anal. Appl.* **352** (2009), 85–98.

[10] K. Ho, Y.-H. Kim, A-priori bounds and multiplicity of solutions for nonlinear elliptic problems involving the fractional $p(\cdot)$-Laplacian, *Nonlinear Anal.* **188** (2019), 179–201.

[11] U. Kaufmann, J. D. Rossi, R. Vidal, Fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacians, *Electron. J. Qual. Theory Differ. Equ.* **76** (2017), 1–10.

[12] M. Mihăilescu, V. Rădulescu, Eigenvalue problems with weight and variable exponent for the Laplace operator, *Anal. Appl. (Singap.)* **8** (2010), 235-246.

[13] G. Molica Bisci, V.D. Rădulescu, R. Servadei, *Variational methods for nonlocal fractional problems*, Encyclopedia of Mathematics and its Applications, 162, Cambridge University Press, Cambridge, 2016.

[14] W. Orlicz, Uber konjugierte Exponentenfolgen, *Studia Math.*, **3** (1931), 200–212.

[15] V.D. Rădulescu, D. Repovš, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*, CRC Press, Taylor & Francis Group, Boca Raton FL, 2015.

[16] M. Mihăilescu, V. Rădulescu, On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent, *Proc. Amer. Math. Soc.* **135(9)** (2007), 2929–2937.

[17] M. Mihăilescu, V. Rădulescu, Continuous spectrum for a class of nonhomogeneous differential operators, *Manuscripta Math.* **125** (2008) 157–167.

[18] V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, *Math. USSR Izv.* **29** (1987), 33–66.

[19] E. Zeidler, *Nonlinear Functional Analysis and its Applications. III*, Springer-Verlag, New York, 1985.

[20] C. Zhang, X. Zhang, Renormalized solutions for the fractional $p(x)$-Laplacian equation with $L^1$, *Nonlinear Anal.* **190** (2020), 111610.