Gap Labelling for Schrödinger Operators on Quasiperiodic Tilings

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Abstract

For a large class of tilings, including those which are obtained by the generalized dual method from regular grids, it is shown that their algebra is stably isomorphic to a crossed product with $\mathbb{Z}^d$. For $d \leq 3$ this in particular enlarges the class of tilings of which can be shown that a set of possible gap labels is completely determined by an invariant measure on the hull.

KCL-TH-94-10
**Introduction**

Quasiperiodic tilings\(^1\) serve as models for the spatial structure of quasicrystals even though they are highly idealized. In particular, tilings with non trivial orientational order are of interest and the so-called generalized dual method (GDM) is a powerful way to create such tilings among which the Penrose tilings are the most famous ones.

The gap labelling of discrete Schrödinger operators on tilings in its \(C^*\)-algebraic formulation \([1]\) is a step towards an understanding of the nature of the spectrum of such operators as well as of the influence of the geometry of the tiling on them. This is of particular interest for operators on higher dimensional non-periodic tilings for which a computation of their spectra is presently out of reach. In this formulation, where the scaled ordered \(K_0\)-group of a \(C^*\)-algebra with a normalized trace is used, the gap labelling does not refer to a specific operator but only involves this algebra. The natural candidate for this algebra is the algebra \(A_T\) of the tiling \(T\) on which the Schrödinger operator is defined. It contains all local operators involving translations and multiplications with pattern dependent functions. A local selfadjoint operator \(H\) on the tiling \(T\) is then an operator in a specific representation of \(A_T\), namely on the space of (square summable) wavefunctions \(\psi\) taking values on the tiles of \(T\)

\[
H\psi(x) = \sum_{x'} H_{x,x'}\psi(x').
\]

Here \(x, x'\) denote tiles of \(T\) and locality refers to the requirement that the matrix element \(H_{x,x'}\) depends only on a the oriented pattern class of a certain radius to which \(x, x'\) belong. The values of its integrated density of states (IDS) \(N_H(E)\) at energies \(E\) lying in a gap of its spectrum serve as labels for the gaps; they are insensitive to certain perturbations of the operator. The abstract gap labelling of Bellissard \([1]\) states that, if \(E\) lies in a gap, then

\[
N_H(E) \in \text{tr} K_0(A_T) \cap [0, 1],
\]

where \(K_0(A_T)\) is the \(K_0\)-group of \(A_T\) and \(\text{tr}\) the state on it induced by the trace \(\text{tr}\). It requires the validity of Shubin’s formula by which this trace is equated to the operator trace per volume in that representation. One motivation for using \(A_T\) is that it is expected not to yield to many values on the r.h.s. so that for “generic” \(H\) all elements of \(\text{tr} K_0(A_T)\) actually occur as labels for open gaps of \(H\). In that case, and if \(\text{tr}\) is faithful on \(A_T\), the density of the values of the IDS on gaps in \([0, 1]\) expresses the fact that the continuous part of the spectrum is a Cantor set.

The main result of this work is the extension of the equality

\[
\text{tr} K_0(A_T) = \mu(C(\Omega, \mathbb{Z})),
\]

\(\mu\) being the invariant measure on the hull of the tiling corresponding to the trace, to a large class of tilings including those which are obtained by the GDM from regular grids. In particular this fills a gap in \([2]\) for the gap labelling of the Penrose tilings where \([3]\) has only been conjectured.

\(^1\) We simply call the analogs of tilings in other dimensions than two (e.g. sequences and packings) tilings, too.
The explicit computation of such measures, which are in known cases probability measures for the frequencies of patterns occurring in the tilings, is not addressed here. Let us only mention that these may be successfully computed if the tiling allows for a substitution \[3, 4, 2\].

Apart from the application to the gap labelling the computation of the \(K\)-groups of \(A_T\) including order and scale of \(K_0\) is of interest for the characterization of the tiling: they are the topological invariants in the non commutative geometrical description of the tiling \[5\].

In the first section the construction of \(A_T\) is reviewed taking a slightly different point of view from \[2\] in the description of the groupoid. In the second, tilings that are decorations of \(\mathbb{Z}^d\) are looked at. Their algebra is a crossed product with \(\mathbb{Z}^d\). It is for a subclass of these tilings that \(\mathfrak{3}\) could be rigorously proven so far. In the third section the concept of a reduction of a tiling is introduced. It is designed to obtain a stably isomorphic algebra which therefore has the same \(K\)-groups and the same order. This is used to extend the validity of \(\mathfrak{3}\), and in section four many examples for this are given by showing that any tiling obtained by the GDM from a regular grid has a reduction which is a decoration of \(\mathbb{Z}^d\).

1 The tiling algebra \(A_T\)

As the algebra of a tiling is a groupoid-\(C^*\)-algebra groupoids play a central role. The groupoids, which are defined by the spatial structure of the tilings, are given by equivalence relations, i.e. its elements are pairs of equivalent elements of some set \(X\), multiplication, which is only defined for pairs \((x, y), (x', y')\) if \(x' = y\), is given by \((x, y)(y, z) = (x, z)\), and inversion by \((x, y) = (y, x)^{-1}\). In the terminology of \[6\] these are all principal groupoids. The topology of the groupoids in question needs in general not to coincide with the relative topology from \(X \times X\).

Sometimes the equivalence may be expressed as orbit equivalence under the (right) action \(\varphi\) of a group \(S\) acting on \(X\): \(x \sim y\) whenever \(\exists s \in S : y = \varphi(s)x\). This leads to the consideration of another kind of groupoid which is called transformation group in \[6\]. Its space is the Cartesian product of \(X\) with \(S\) (here always considered to carry the product topology) and the groupoid structure is defined by \((x, s)(x', t) = (x, st)\) provided \(x' = \varphi(s)x\) and \((x, s)^{-1} = (\varphi(s)x, s^{-1})\). We write it as \(X \times_\varphi S\). It may be viewed as the groupoid defined by orbit equivalence only if \(S\) acts freely on \(X\).

Let us review the construction of the groupoid of a tiling \(T\) from a slightly more abstract point of view as in \[4\]. A \(d\) dimensional tiling \(T\) of \(\mathbb{R}^d\) is a complete covering of \(\mathbb{R}^d\) by tiles which do not overlap (more precisely the only overlap between neighboured tiles lies in their boundary). These tiles, which for simplicity may be seen as polyhedra, may have an additional decoration, e.g. to break symmetries. However it is not the precise form of the tiles which is important but their spatial arrangement. A tiling encodes a set of allowed translations in \(\mathbb{R}^d\) as being those which lead from one tile to another. They do in general not form a group. In order to get a hold on the allowed translations,
a point in each tile of $T$ was identified in [2] with a point of $\mathbb{R}^d$, i.e. any tile got a puncture and translation from the tile with puncture $x$ to the tile with puncture $y$ became translation of $T$ in $\mathbb{R}^d$ by $y - x$ the translate being denoted by $T - (y - x)$. But it is quite fruitful to also consider the case in which only part of the tiles carry a puncture thus restricting the set of allowed translations, where it is still assumed that the tile which lies on the origin $0 \in \mathbb{R}^d$ is punctured, and its puncture is denoted by $0$. Next to the actual tiles of $T$ and its patterns, which are finite collections of tiles, we have to consider their oriented pattern classes, i.e. their equivalence classes under translations. As in [2] we require the compactness condition (there called B1) to hold, namely that there are for any given finite size only finitely many oriented pattern classes not exceeding that size.

The hull $\Omega_T$ of $T$ is given by the closure of the set of translates of $T$ by translations from the tile on $0$ to any other punctured tile of $T$. Hence in particular contains elements like $T - x$ which is $T$ translated such that the tile with puncture $x$ lies on $0$. The closure is taken with respect to a metric which assigns to two tilings a small distance if they coincide on a large $r$-ball around $0$: let $M_r(T)$ be the oriented pattern class of the smallest pattern of $T$ which covers the $r$-ball around $0 \in \mathbb{R}^d$ then $d(T, T') = \exp(-\sup\{r | M_r(T) = M_r(T')\})$ is the metric and

$$\Omega_T := \{T - x | x \in T_{pct}\}. \tag{4}$$

$\Omega_T$ contains all those tilings which look on finite patches like some patch of $T$, i.e. which are locally homomorphic to $T$. The open neighbourhoods of an element $T$ are of the form

$$U_{M_r(x)} = \{T' \in \Omega_T | M_r(T') = M_r(T)\} \tag{5}$$

and are as smaller as larger $r$ is. They are special kinds of the following type of sets: $U_{M,x} := \{T \in \Omega_T | (M, x) \subset (T, 0)\}$, where $(M, x) \subset (T, y)$ means that a pattern of the oriented pattern class $M$ occurs in $T$ such that puncture $x \in M_{pct}$ coincides with $y \in T_{pct}$. All these sets are both, open and closed, so that $\Omega_T$, which is simply denoted by $\Omega$ if no confusion arises, is a totally disconnected compact space. Denote by $M(x, y)$, $x, y \in T_{pct}$ the oriented pattern class of $T$ which covers the smallest ball that contains $x$ and $y$. To simplify the notation denote the puncture of $M(x, y)$ which corresponds to $x \in T_{pct}$ in the above identification by $x$, too.

The set $\Gamma = T_{pct} \times T_{pct}$ with discrete topology may be seen as a groupoid (in a rather simple way) defined by the equivalence relation on $T_{pct}$ by which all elements are equivalent. It ”acts” on $\Omega$ in the following sense. On

$$\Omega^\Gamma = \{(T, (x, y)) | (M(x, y), x) \subset (T, 0)\} \subset \Omega \times \Gamma \tag{6}$$

with relative topology define the map $\gamma : \Omega^\Gamma \to \Omega$

$$\gamma(T, (x, y)) = T - (y - x); \tag{7}$$

in fact $(M(x, y), x) \subset (T, 0)$ implies that there is a unique $y' \in T_{pct}$ which coincides with $y \in M(x, y)_{pct}$ in the above identification and its corresponding translation is denoted simply by $y - x$ as it would be if punctures are identified with points in $\mathbb{R}^d$. 

4
Clearly \( \gamma(\cdot, (x, y)) : U_{M(x,y),x} \to U_{M(x,y),y} \) is continuous.\(^2\) The groupoid \( \mathcal{R} \) assigned to the tiling \( \mathcal{T} \) is as a topological space the image of \( \Omega^T \) under \((\pi, \gamma)\), \( \pi \) being projection onto the first factor,

\[
\mathcal{R} := (\pi, \gamma)(\Omega^T) \subset \Omega \times \Omega
\]

with weak topology induced by \((\pi, \gamma)\) and its groupoid structure is defined by the equivalence relation \( T \sim T' \) whenever \( \exists x \in T^\text{pct} : T' = T - x \).

To make contact to \((\mathcal{I})\) let for a pattern \( M \) of \( \mathcal{T} \) and \( x, y \in M^\text{pct} \)

\[
U_{M,x,y} := \{(T, T - (y - x))|(M, x) \subset (T, 0)\}.
\]

**Lemma 1** \( \mathcal{R} \) is r-discrete and its topology is generated by the sets \( U_{M,x,y} \).

**Proof:** The topology of \( \Omega^T \) is generated by the sets \( U_{M,x} \times (x', y') \) where \((M(x', y'), x') \subset (M, x)\). There is a unique \( y \in M^\text{pct} \) which coincides with \( y' \) under the identification of \( M(x', y') \) with a subpattern of \( M \) as above. For this \( y \) we have \((\pi, \gamma)U_{M,x} \times (x', y') = U_{M,x,y}\) showing that the topology of \( \mathcal{R} \) is indeed generated as stated. Clearly \( \Omega = \cup_M U_{M,x,x} \) is open so that \( \mathcal{R} \) is r-discrete. \( \square \)

The algebra of the tiling \( \mathcal{T} \) is the reduced groupoid-C*-algebra \( C^*_\text{red}(\mathcal{R}) \). It is the closure of the *-algebra of continuous functions \( f : \mathcal{R} \to \mathbb{C} \) with compact support, multiplication and involution being given by

\[
f * g(T, T') = \sum_{T'' \sim T} f(T, T'') g(T'', T'), \tag{10}
\]

\[
f^*(T, T') = f(T', T). \tag{11}
\]

The closure is taken with respect to the norm \( \|f\|_{\text{red}} = \sup_{T \in \Omega_T} \|\pi_T(f)\| \) the supremum being taken over all representations \( \pi_T, T \in \Omega_T \), of the form

\[
(\pi_T(f)\psi)(T') := \sum_{T'' \sim T'} f(T', T'')\psi(T''), \tag{12}
\]

where \( \pi_T \) acts on (square summable) wave functions \( \psi : \{T - x|x \in T^\text{pct}\} \to \mathbb{C} \) with the usual scalar product. \( \|\pi_T(f)\| \) is the operator norm in that representation, which is bounded. \( C^*_\text{red}(\mathcal{R}) \) is called the algebra of the tiling \( \mathcal{T} \) and below denoted by \( \mathcal{A}_\mathcal{T} \). It is separable, as \( \mathcal{R} \) is second countable. Note that the operator used in \((\mathcal{I})\) belongs to such a representation, in that \( H = \pi_T(h) \) for some \( h \in \mathcal{A}_\mathcal{T} \).

Let us give a first rough characterization of \( \mathcal{A}_\mathcal{T} \) by the properties of the tiling. Remember that any \( T \in \Omega_T \) is locally homomorphic to \( \mathcal{T} \), in the sense that any pattern of \( T \) does also occur in \( \mathcal{T} \), and that \( T \) is locally isomorphic to \( \mathcal{T} \) if the converse is true as well, i.e. if any pattern of \( \mathcal{T} \) occurs in \( T \). Moreover \( \mathcal{T} \) is homogeneous if it is locally isomorphic to any \( T \in \Omega_T \) which is equivalent to \( \Omega_T = \Omega_T \) for all \( T \in \Omega_T \).

\(^2\) \( \gamma \) has properties analogous to a group action, namely \( \forall T \in U_{M(x,x'),x} \cap U_{M(x,x''),x} : \gamma(\gamma(T,(x,x')), (x', x'')) = \gamma(T,(x,x'')) \) and a similar expression holds for the inversion.
Lemma 2 \( \pi_T \) is faithful if and only if \( T \) is locally isomorphic to \( T \).

Proof: To prove faithfulness of \( \pi_T \) we may restrict to elements of \( C_c(\mathcal{R}) \). Let \( f \in C_c(\mathcal{R}) \). If \( \pi_T(f) = 0 \) then for all \( \psi \) and \( T' \sim T \): \( \sum_{T''} f(T', T'')\psi(T'') = 0 \). Hence \( f \) vanishes on \( \{(T', T'')|T', T'' \sim T\} \). If \( T \) is locally isomorphic to \( T \) then \( \{(T', T'')|T', T'' \sim T\} \) is dense in \( \mathcal{R} \) so that \( f \) has to vanish by continuity. If \( T \) is not locally isomorphic then \( T \) contains a pattern \( M \) which does not occur in \( T \). Let \( \chi_{M,x} \) be the characteristic function on \( U_{M,x} \) for some \( x \in M_{red} \), then \( \pi_T(\chi_{M,x}) = 0 \). \( \square \)

Since all the above representations are irreducible, and at least one of them, namely \( \pi_T \), is faithful, \( A_T \) cannot be semi-simple in case \( T \) is not homogeneous. The ideal structure of \( A_T \) may be investigated by studying the open invariant subsets of \( \Omega \), i.e. those open subsets which contain next to an element \( T \) all its equivalent elements. We will just look at the simplest case in which \( \Omega_T \) does not contain any proper invariant open subset which by (6) implies simplicity of \( A_T \).

Lemma 3 If \( T \) is homogeneous then \( A_T \) is simple.

Proof: Let \( U \subset \Omega_T \) be open and invariant hence \( \Omega_T \setminus U \) is invariant and closed. If \( U \) is not equal to \( \Omega_T \) let \( T \in \Omega_T \setminus U \). Then \( \Omega_T \subset \Omega_T \setminus U \) showing that under the assumption of the lemma \( U = \emptyset \). \( \square \)

Note that moreover for homogeneous \( T \) the algebra \( A_T \) is antiliminal \([7]\), namely it is neither limial itself – its primitive spectrum contains one single point whereas the representations \( \pi_T \) and \( \pi_T' \) are unitarily equivalent if and only if \( T \sim T' \) – nor can it contain a limial ideal. It follows that the spectrum of any selfadjoint \( h \in A_T \) has no discrete part, since the spectral projection onto the eigenspace of a discrete eigenvalue would have to be represented by a compact operator.

Recall that any trace \( \text{tr} \) on \( A_T \) restricted to a linear functional \( \mu \) on \( C(\Omega) \) defines a measure on \( \Omega_T \) also denoted by \( \mu \) through \( \mu(f) = \text{tr}(f) = \int f d\mu \), which is invariant in the sense that \( \mu(U_{M,x}) \) is independent of the puncture \( x \). A direct consequence is that the inclusion

\[
\mu(C(\Omega, \mathbb{Z})) \subset \text{tr}_* K_0(A_T)
\]

is always valid. Conversely any invariant normalized measure \( \mu \) on the hull \( \Omega \) defines a normalized trace through

\[
\text{tr}(f) := \int_{\Omega} P(f) d\mu
\]

where \( P : C_{red}^*(\mathcal{R}) \to C(\Omega) \) is the restriction map. \( P \) is the unique conditional expectation on \( C(\Omega) \) and is faithful \([8]\). Moreover if \( T \) is homogeneous every non trivial invariant measure has to have closed support \( \Omega \) so that \( \text{tr} \) defined by (14) is faithful. To obtain a gap labelling of a Schrödinger operator by means of the values of its IDS a trace is required which satisfies Shubin’s formula. The question under which circumstances this is the case for a given trace will not be addressed here, but see [1, 4, 2] for investigations in this directions. In fact, conclusions on the nature of the spectrum may partly be drawn without the need to connect the gap labelling with the values of the IDS, any faithful trace may be used. For instance if \( \text{tr} \) is faithful and the set of gap
labels \( \mathcal{L}_{\text{tr}}(h) = \{ \text{tr}(\chi_{h \leq E}) | E \notin \sigma(h) \} \) dense in \([0,1]\) the spectrum \( \sigma(h) \) cannot contain a proper closed interval, for, if \([a, b] \in \sigma(h)\), then by faithfulness \( \text{tr}(\chi_{h \leq E}) > \text{tr}(\chi_{h \leq a}) \) – here \( \text{tr} \) has to be extended to measurable functions over \( \sigma(h) \) – so that \([0,1]\setminus\mathcal{L}_{\text{tr}}(h) \) would contain the open interval \( (\text{tr}(\chi_{h \leq a}), \text{tr}(\chi_{h \leq b})) \). However, up to now there is no \( K \)-theoretic formulation of a condition for a Schrödinger operator \( h \) under which \( \mathcal{L}_{\text{tr}}(h) \) coincides with \( \text{tr}_*K_0(\mathcal{A}_T) \).

## 2 Decorations of \( \mathbb{Z}^d \)

**Definition 1** A tiling shall be called a decoration of \( \mathbb{Z}^d \) if there is a continuous action \( \varphi \) of \( \mathbb{Z}^d \) on its hull \( \Omega \) such that \( \mathcal{R} \) is isomorphic as a topological groupoid to the transformation group \( \Omega \times_{\varphi} \mathbb{Z}^d \).

Isomorphic groupoids lead to isomorphic groupoid-C*-algebras. To any continuous function \( f : \Omega \times \mathbb{Z}^d \to \mathbb{C} \) with compact support one may assign the function \( \hat{f} : \mathbb{Z}^d \to C(\Omega) \) through \( \hat{f}(k)(T) = f(T, k) \). Carried over from \( \Omega \) multiplication and involution then become convolution resp. involution twisted by \( \varphi \):

\[
\hat{f} \ast \hat{g}(k) = \sum_{m \in \mathbb{Z}^d} \hat{f}(m) (\hat{g}(k-m) \circ \varphi(m))
\]  

\[
\hat{f}^*(k) = \hat{f}(-k) \circ \varphi(k).
\]

In fact, the closure \( C^*_\text{red}(\Omega \times_{\varphi} \mathbb{Z}^d) \) is isomorphic to \( C(\Omega) \times_{\varphi} \mathbb{Z}^d \), the crossed product of \( C(\Omega) \) with \( \mathbb{Z}^d \) by the action \( \varphi(k)(\hat{f}(m)) = \hat{f}(m) \circ \varphi(k) \).

Decorations of \( \mathbb{Z}^d \) do not have to be periodic. Any one dimensional tiling is a decoration of \( \mathbb{Z} \) as it may be understood as a two-sided sequence of (occasionally decorated) intervals, i.e. as a map \( T : \mathbb{Z} \to \{a_1, a_2, \ldots, a_n\} \), finiteness of the alphabet \( \{a_1, a_2, \ldots, a_n\} \) representing the intervals being a consequence of the compactness condition. \( \varphi \) may be taken to be the left shift: \( \varphi(T)_i = T_{i+1} \). As well any \( d \)-fold Cartesian product of one dimensional tilings is a decoration of \( \mathbb{Z}^d \), but these are not the most general ones.

The rest of this section is a longer remark which is not essential for the sequel.

Whether a tiling is a decoration of some \( \mathbb{Z}^d \) or not seems in the examples known to the author to be easy to check by verification of the requirements of the definition. However at some point one might wonder whether not all tilings are \( \mathbb{Z} \)-decorations. In fact \( T^{\text{pct}} \) is countable, so why not identifying it with \( \mathbb{Z} \) or with \( \mathbb{Z}^d \)? The crucial point is, of course, the topology. Let us make this point a little more precise. Consider a bijection \( \tilde{\beta} : T^{\text{pct}} \to \mathbb{Z}^d \). Then \( \beta : \Gamma \to \mathbb{Z}^d \) defines a surjective homomorphism of groupoids through \( \beta(x, y) = \tilde{\beta}(y) - \tilde{\beta}(x) \), and if we had neglected the closure in \( \Omega \) \( \mathcal{R} \) would have to be replaced by \( T^{\text{pct}} \) and \( \Gamma = T^{\text{pct}} \times T^{\text{pct}} \) which through \( (\pi_1, \beta) : \Gamma \to T^{\text{pct}} \times_{\varphi} \mathbb{Z} \) is isomorphic to a transformation group, the action being \( \varphi(z)(x) = \tilde{\beta}^{-1}(\tilde{\beta}(x) - z) \). The following theorem arose from the question under which circumstances \( (\pi_1, \beta) \) extends to \( \mathcal{R} \) giving rise to an isomorphism between \( \mathcal{R} \) and \( \Omega \times_{\varphi} \mathbb{Z}^d \). Let

\[
\Gamma(x, y) := \{(x', y') | M(x, y) = M(x', y')\},
\]

\( \text{dist}(x, y) \) be the diameter of \( M(x, y) \), and \( \pi_2 \) denote projection onto the second factor.
Lemma 4 shows that it is injective. To show that it is bijective we construct a pre-image of \( \mathbb{Z}^d \).

For the proof of the theorem we need two lemmas.

Lemma 4 If \( \beta(x, y) = \beta(x', y') \) but \( (x', y') \notin \Gamma(x, y) \) then \( U_{M(x, y)} \cap U_{M(x', y'), x'} = \emptyset \).

Proof: Assume \( \beta(x, y) = \beta(x', y') \) and \( T \in U_{M(x, y)} \cap U_{M(x', y'), x'} \). Then there exists \( x, y, v \in T, v \) corresponding to the puncture \( y' \) once \( x' \) has been identified with \( 0 \) in \( T \) such that \( \beta(x, y) = \beta(x, v) \). Hence \( y = v \) and therefore \( (x', y') \in \Gamma(x, y) \).

Lemma 5 The map \( (\pi_1, \beta \circ \pi_2) \circ (\pi_1, \gamma)^{-1} \) is a homeomorphism between \( \mathcal{R} \) and \( \Omega \times \mathbb{Z}^d \).

Proof: First of all, since \( \gamma(T, (x', y')) = \gamma(T, (x, y)) \) for \( (x', y') \in \Gamma(x, y) \)

\[
(\pi_1, \gamma)^{-1}(T, T - y) = \{(T, (x', y')) | (x', y') \in \Gamma(0, y)\}
\]

which guarantees together with 2. that \( (\pi_1, \beta \circ \pi_2) \circ (\pi_1, \gamma)^{-1} \) is well defined, and Lemma 4 shows that it is injective. To show that it is bijective we construct a pre-image of \( (T, z) \). Any \( T \in \Omega \) is approximated by translates of \( T \). Choose \( x \) such that \( M_{cz}(T) = M_{cz}(T - x) \) and set \( y = \beta^{-1}(\beta(x) + z) \in T^\text{pct} \). Then \( (M(x, y), x) \subset (T - x, 0) \) and 1. implies \( (M(x, y), x) \subset (T, 0) \) so that \( (\pi_1, \beta \circ \pi_2)(T, (x, y)) = (T, z) \).

Moreover, because of

\[
(\pi_1, \beta \circ \pi_2) \circ (\pi_1, \gamma)^{-1}(U_{M(x, y)}) = U_{M, x} \times \{\beta(x, y)\}
\]

the map is open and we are left to show its continuity. Observe that by Lemma 4 and 1. the set \( S_T(z) = \{(x, y) \in \beta^{-1}(z) | (M(x, y), x) \subset (T', 0) \} \) is the same for all \( T' \in U_{M_{cz}(T)} \). Therefore

\[
(\pi_1, \beta \circ \pi_2)^{-1}(U_{M_{cz}(T)}, z) = U_{M_{cz}(T)} \times S_T(z)
\]

and, since the r.h.s. is open, its image under \( (\pi_1, \gamma) \) is open as well.

Proof of the theorem: The proof of the theorem is completed by transforming the action of \( \gamma \) into an action of \( \mathbb{Z}^d \) on \( \Omega \). For \( (T, z) \in \Omega \times \mathbb{Z}^d \) let \( (x, y) \in S_T(z) \) and define

\[
\varphi(z)(T) := \gamma(T, (x, y))
\]

which is again independent of the choice of \( (x, y) \). Continuity follows from the continuity of \( \gamma(\cdot, (x, y)) \). To show that \( \varphi(z_2) \circ \varphi(z_1) = \varphi(z_1 + z_2) \) choose a realization of \( M_{cz_1 + cz_2}(T) \) in \( T \), let’s say \( (M, x) \subset (T - y, 0) \) with \( M_{cz_1 + cz_2}(T) = M_{cz_1 + cz_2}(T - y) \). Then \( (M, x) \) contains subpatterns \( M(x, x') \) and \( M(x', x'') \) with \( (x, x') \in S_T(z_1) \) and \( (x', x'') \in S_{T - (x' - x)}(z_2) \). Hence \( \varphi(z_2) \circ \varphi(z_1)(T) = \gamma(\gamma(T, (x, x')), (x', x'')) = \gamma(T, (x, x'')) \) which is equal to \( \varphi(z_1 + z_2)(T) \) because of \( \beta(x, x'') = \beta(x, x') + \beta(x', x'') = z_1 + z_2 \). Thus \( \varphi \) is an action of \( \mathbb{Z}^d \), and by (21) equivalence with respect to \( \mathcal{R} \) may be expressed as orbit equivalence under \( \mathbb{Z}^d \). 

\[ \square \]
3 Reductions of tilings and $K_0(\mathcal{A}_\mathcal{T})$

In [2] all the tiles of a tiling $\mathcal{T}$ were considered to be punctured. This was necessary to obtain operators in $\mathcal{A}_\mathcal{T}$ which correspond to translations from a tile to its neighbours. But it turns out to be quite useful to consider as well algebras which may be obtained with fewer (or more) punctures. Restricting the allowed translations in a controlled manner, resulting in what will be called a reduction $\mathcal{T}_r$ of $\mathcal{T}$, furnishes us with an algebra $\mathcal{A}_\mathcal{T}_r$ which is for homogeneous $\mathcal{T}$ stably isomorphic to $\mathcal{A}_\mathcal{T}$. The importance of this procedure lies in the possibility that $\mathcal{T}_r$ may be a decoration of $\mathbb{Z}^d$ which brings us a step closer to the computation of $K_0(\mathcal{A}_\mathcal{T})$.

Let $\mathcal{T}_\text{pct}$ be a subset of $\mathcal{T}^{\text{pct}}$ and set $\Omega_r = \{\mathcal{T} - x | x \in \mathcal{T}_\text{pct}\}$. Let $\mathcal{T}_r$ be the tiling which has the same tiles as $\mathcal{T}$ but a different interpretation of the punctures. The punctures of $\mathcal{T}_\text{pct}\setminus\mathcal{T}_r^{\text{pct}}$ shall either be interpreted as decorations which might break symmetries or in case these symmetries are not present just be neglected whereas the punctures of $\mathcal{T}_r^{\text{pct}}$ define the hull, i.e. $\Omega_{\mathcal{T}_r} = \Omega_r$.

**Definition 2** We call $\mathcal{T}_r$ a reduction of $\mathcal{T}$ if $\Omega_r$ is an open subset of $\Omega_{\mathcal{T}_r}$.

Recall that two $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ are called stably isomorphic if $\mathcal{A} \otimes \mathcal{K}$ is isomorphic to $\mathcal{B} \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators (on an infinite dimensional separable Hilbert space).

**Lemma 6** Let $\mathcal{T}_r$ be a reduction of a homogeneous $\mathcal{T}$. Then $\mathcal{A}_{\mathcal{T}_r}$ is stably isomorphic to $\mathcal{A}_\mathcal{T}$.

**Proof:** If $\mathcal{T}_r$ is a reduction then $\chi_{\Omega_r} \in \mathcal{A}_\mathcal{T}$ and $\mathcal{A}_{\mathcal{T}_r} = \{a \in \mathcal{A}_\mathcal{T} | a\chi_{\Omega_r} = \chi_{\Omega_r}a = a\}$. Hence the latter is a hereditary subalgebra of the simple $C^*$-algebra $\mathcal{A}_\mathcal{T}$. By a theorem of [8] hereditary subalgebras of simple separable $C^*$-algebras are stably isomorphic. □

Since $K_0(\mathcal{A}_\mathcal{T})$ is obtained via Grothendieck’s construction from the monoid of projector classes of $\mathcal{A}_\mathcal{T} \otimes \mathcal{K}$, the positive elements corresponding to the elements of the monoid, it depends together with its order structure only on the stable isomorphism class of $\mathcal{A}_\mathcal{T}$. Moreover, in the above case the embedding $\iota : \mathcal{A}_{\mathcal{T}_r} \rightarrow \mathcal{A}_\mathcal{T}$ induces an isomorphism $\iota_*$ from $K_0(\mathcal{A}_{\mathcal{T}_r})$ onto $K_0(\mathcal{A}_\mathcal{T})$. In fact, for separable $C^*$-algebras, $\mathcal{A}$ being stably isomorphic to $\mathcal{B}$ is equivalent to the existence of a (strong) Morita equivalence $\mathcal{A}$-$\mathcal{B}$-bimodule which may be viewed as an element of $KK(\mathcal{A}, \mathcal{B})$ and is a special case of a $KK$-equivalence [1, 2, 3]. Any $KK$-equivalence between $\mathcal{A}$ and $\mathcal{B}$ yields an isomorphism of $KK(\mathcal{C}, \mathcal{A})$ with $KK(\mathcal{C}, \mathcal{B})$, namely by multiplying it from the right, the multiplication being the Kasparov product. Translated into $K_0$-groups, $KK(\mathcal{C}, \mathcal{A})$ being isomorphic to $K_0(\mathcal{A})$, the right multiplication of elements of $KK(\mathcal{C}, \mathcal{A}_\mathcal{T})$ with the canonical Morita equivalence $\mathcal{A}_{\mathcal{T}_r}$-$\mathcal{A}_\mathcal{T}$-bimodule, which as a linear space is $\mathcal{A}_{\mathcal{T}_r}$-$\mathcal{A}_\mathcal{T}$, presicely becomes $\iota_*$. This shall now be used to extend the equality

$$\mu(C(\Omega, \mathbb{Z})) = tr_* K_0(\mathcal{A}_\mathcal{T}), \quad (22)$$

for any normalized trace $tr$ trace on $\mathcal{A}_\mathcal{T}$ to a larger class of tilings including the Penrose and Ammann-Beenker tilings. Equality (22) is so far known for decorations of $\mathbb{Z}^d$ if
and for arbitrary \( d \) in case \( T \) is a Cartesian product of one dimensional tilings \([4]\). Its proof is based on the identification \( \mathcal{A}_T = C(\Omega) \times_{\varphi} \mathbb{Z}^d \). Since the crossed product by \( \mathbb{Z}^d \) may be understood as an iterated crossed product by \( \mathbb{Z} \), the Pimsner Voiculescu exact sequence \([13]\) may be used for the computation of the \( K \)-groups provided the sequence splits at a certain position, which has been explicitly verified in \([12]\) up to \( d = 3 \). In case the tiling decomposes into a Cartesian product, the groupoid decomposes, too, and \( \mathcal{A}_T \) becomes a tensor product of crossed products with \( \mathbb{Z} \). The Künneth formula then applies to reduce the \( K \)-groups to the one dimensional case. For higher \( d \) the \( K_0 \)-groups look more and more complicated but their image under \( \text{tr}_* \) remains as simple as \((22)\).

**Theorem 2** If a homogeneous tiling \( T \) has a reduction \( T_r \) which is a decoration of \( \mathbb{Z}^d \) with \( d \leq 3 \) or a Cartesian product of one dimensional tilings then

\[
\text{tr}_* K_0(\mathcal{A}_T) = \mu(C(\Omega, \mathbb{Z}))
\]

where \( \mu \) is the measure corresponding to the trace \( \text{tr} \) on \( \mathcal{A}_T \).

**Proof:** Denote by \( \iota : \mathcal{A}_{T_r} \to \mathcal{A}_T \) the embedding. Then \( \text{tr} \circ \iota \) is a trace on \( \mathcal{A}_{T_r} \) which is however not normalized, \( \text{tr}(\iota(1_{\mathcal{A}_{T_r}})) = \text{tr}(\chi_{\Omega_r}) = \mu(\Omega_r) \). By Lemma \([3]\), \( \mathcal{A}_T \) is stably isomorphic to \( \mathcal{A}_{T_r} \) and in particular \( \text{tr}_* K_0(\mathcal{A}_{T_r}) = K_0(\mathcal{A}_T) \). But for \( T_r \) which is a decoration of \( \mathbb{Z}^d \) \((22)\) can be used. The invariant measure on \( \Omega_r \) corresponding to \( \text{tr} \circ \iota \) being \( \mu|_{\Omega_r} \) this is

\[
\text{tr}_* \circ \iota_* K_0(\mathcal{A}_{T_r}) = \mu(C(\Omega_r, \mathbb{Z}))
\]

and therefore

\[
\text{tr}_* K_0(\mathcal{A}_T) = \mu(C(\Omega_r, \mathbb{Z})) \subset \mu(C(\Omega, \mathbb{Z})).
\]

Together with \((13)\) the statement follows. \(\square\)

\(K_0(\mathcal{A}_{T_r})\) and \(K_0(\mathcal{A}_T)\) differ only in their order units (the images of the units of the algebras in \( K_0 \)). If one identifies them as above the order unit of the former is \([\chi_{\Omega_r}]\).

### 4 Applications: tilings obtained by the GDM

Theorem \([2]\) has a wide range of applications. To show this we consider tilings whose tiles are \( d \) dimensional analogs of rhombi. We may or may not consider further decorations of the rhombi e.g. to break the symmetry. This is of importance for substitution tilings but is not relevant below where we only show the existence of a reduction which is a decoration of \( \mathbb{Z}^d \).

Given a set of \( N \) (pairwise non parallel) vectors \( \alpha_1, \ldots, \alpha_N \) which span \( \mathbb{R}^d \) and a set \( J \subset \{1, \ldots, N\} \) containing \( n \) elements, an \( n \)-facet of type \( J \) is a subset of \( \mathbb{R}^d \) which is translationally congruent to \( \{ \sum_{i \in J} c_i \alpha_i | c_i \in [0, 1] \} \). We consider \( n \leq d \) so these are \( n \) dimensional analogs of rhombi. Let \( T \) be a tiling of \( \mathbb{R}^d \) consisting of \( d \)-facets which are arranged in such a way that tiles touch, if they touch at all, at common complete

\(^3\) In \([12]\) ergodic measures have been used, but ergodicity not essential for the proof of \((22)\).
$d'$-facets, $d' < d$. The oriented pattern classes of tiles are thus the types of $d$-facets. Let all tiles of $\mathcal{T}$ carry a puncture. If $d = N$ then $\mathcal{T}$ itself is of course a decoration of $\mathbb{Z}^d$ so here we are interested in $d < N$. As before we assume that it satisfies the condition to ensure compactness of its hull.

Now consider the following reduction of $\mathcal{T}$. Fix the type $I_0$ of the tile which has puncture 0 and let $\mathcal{T}_r^{\text{sect}}$ consist of all tiles of this type. This is certainly a reduction since $\Omega_r = U_a$ with $a$ being of type $I_0$. An $i$-rope, $i \in I_0$, is an infinite line which is constructed as follows: Join in any tile which has $d-1$-facets of type $I_0\{i\}$ their middle points by a line segment. All these lines fit together to yield an infinite line which we call an $i$-rope, c.f. figure 1. It is clear that never two ropes of the same type intersect. Any type of rope shall now be given a direction so that the tiles belonging to it can be ordered. Let us assume that any $i$-rope contains in both directions infinitely many tiles of type $I_0$. Then

$$\varphi_i(T) = T - x_i$$

defines a homeomorphism of $\Omega_r$, where $x_i$ denotes the puncture of the next tile of type $I_0$ on the $i$-rope to which 0 belongs. Clearly $\varphi$ is invertible. To proof its continuity denote by $V_{i}^{(n)}$ the set of all tilings such that the next tile of type $I_0$ on rope $i$ is the $n$'th one. By the above requirement $\Omega_r = \bigcup_n V_{i}^{(n)}$ and since $\Omega_r$ is compact and the $V_{i}^{(n)}$'s are open there is an $N$ such that $\Omega_r = \bigcup_{n \leq N} V_{i}^{(n)}$. In particular there is an upper bound $b_i$ for $\text{dist}(0, x_i)$ and $\varphi_i(M_{r+2b_i}(T)) \subset M_{r+b_i}(T - x_i)$.

For given $I_0$ there are $d$ types of ropes furnishing $d$ operations of $\mathbb{Z}$ on $\Omega_r$. If

- $\varphi_i \circ \varphi_j = \varphi_j \circ \varphi_i$ for all $i, j \in I_0$

then $\varphi = \varphi_1 \circ \cdots \varphi_d$ is an operation of $\mathbb{Z}^d$ on $\Omega_r$ which is free and such that equivalence with respect to the groupoid defined by $\mathcal{T}_r$ becomes orbit equivalence, in other words $\mathcal{T}_r$ is a decoration of $\mathbb{Z}^d$ (the topologies are easily seen to coincide).

A large class of tilings of the above kind which lead to commuting actions may be constructed by the generalized dual method (GDM) which we briefly explain. A $(d$-dimensional, two-sided) linear grid (or more precisely 1-grid) is an infinite set of parallel hyperplanes. They may be ordered involving a choice of direction and two-sided refers to the requirement that any hyperplane has a lower and an upper neighbour, i.e. the hyperplanes of the grid may be given an integer number and any integer occurs. A general grid is an infinite set of non-intersecting simply connected unbounded hypersurfaces which may of course as well be ordered. A linear $N$-grid consists of $N$ linear grids and each grid (of type $i$) comes with its so-called star vector $\alpha_i$, $i \in \{1, \cdots, N\}$, which is normal to its hyperplanes, of length one, and points towards the positive direction of the order. They are all distinct and are supposed to span $\mathbb{R}^d$ ($N \geq d$). More generally for arbitrary $N$-grids it is required that any $d$ hypersurfaces of different grid type intersect at exactly one point. In this case choices for the star vectors have to be made which have to be consistent. An intersection point of $d$ hypersurfaces (necessarily of different grid type) is called regular if no other hypersurface intersects this point. It may be as well classified by a certain type, namely its type is $J$ which is the collection of types of the grids to which the intersecting hypersurfaces belong.
The actual tiling defined by such an $N$-grid is its dual in the following sense. Any intersection point (regular or not) corresponds to a tile. If the intersection point is regular and of type $J$ then the tile is a $d$-facet of type $J$ made from the star vectors. The vertices correspond to the volumes surrounded by the hypersurfaces and are given by $\sum_{i=1}^{N} k_i \alpha_i$ where $k_i$ is the number of the lower hypersurface of grid type $i$ surrounding the volume.

Now let us assume that all intersection points of the chosen type $I_0$ are regular. Then $\varphi_i$ may be carried over to an action on the $N$-grid. Denoting by $g_i(k)$ the $k$'th hypersurface of grid $i$ the homeomorphism $\varphi_i$ precisely becomes the shift from intersection point $\bigcap_{j \in I} g_j(0)$ to intersection point $\bigcap_{j \notin \{i\}} g_j(0) \cap g_i(1)$. Hence $\varphi_i \circ \varphi_j$ is the shift from $\bigcap_{k \in I} g_k(0)$ to $\bigcap_{k \notin \{i,j\}} g_k(0) \cap g_i(1) \cap g_j(1)$ which is certainly independent of the order. We thus have shown

**Theorem 3** Let $\mathcal{T}$ be a $d$ dimensional tiling which is obtained by the GDM from a regular ($d$ dimensional, two-sided, $N$-) grid. Then it has a reduction which is a decoration of $\mathbb{Z}^d$.

The introduction of decorations does not cause any problems. One may always define $\mathcal{T}_r^{\text{pct}}$ to consist of those tiles which would as undecorated tiles be of type $I_0$.

The form of the Penrose tilings in which they appear as duals of linear 5-grids \cite{14} is slightly different from the form we used in \cite{2}. But this does not matter as the triangles used there always form rhombi and choosing one of the triangle type (in its decorated version where the mirror symmetry is broken) amounts to the same as choosing a 2-facet type.

The Ammann-Beenker tilings have as well a reduction which is a decoration of $\mathbb{Z}^2$ as they are decorated variants of tilings obtained from linear 4-grids. To give an explicit example figure \cite{1} displays such a reduction where only those squares carry a puncture which have no horizontal link, i.e. $\mathcal{T}_r^{\text{pct}}$ contains precisely the squares in which two ropes intersect. For clarity this figure contains at the bottom the prototiles, i.e. the congruence classes of the tiles under all Euclidean transformations (two of them are decorated).

**Acknowledgement.** I thank Andreas Hüffmann for very fruitful conversations.
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Figure 1
bottom: prototiles of the Ammann-Beenker tilings
top: part of an Ammann-Beenker tiling with ropes