Representations of the vertex operator algebra $V_{A_4}^{L_2}$

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Abstract
The rationality and $C_2$-cofiniteness of the orbifold vertex operator algebra $V_{A_4}^{L_2}$ are established and all the irreducible modules are constructed and classified. This is part of classification of rational vertex operator algebras with $c = 1$.

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1 Introduction
Motivated by the classification of rational vertex operator algebras with $c = 1$, we investigate the vertex operator algebra $V_{A_4}^{L_2}$ where $L_2$ is the root lattice of type $A_1$ and $A_4$ is the alternating group which is a subgroup of the automorphism group of lattice vertex operator algebra $V_{L_2}$. The $C_2$-cofiniteness and rationality of $V_{A_4}^{L_2}$ are obtained, and the irreducible modules are constructed and classified.

Classification of rational vertex operator algebras with $c = 1$ goes back to [G] and [K] in the literature of physics at character level under the assumption that each irreducible character is a modular function over a congruence subgroup and the sum of the square norm of irreducible characters is invariant under the modular group. According to [K], the character of a rational vertex operator algebra with $c = 1$ must be the character of one of the following vertex operator algebras: (a) lattice vertex operator algebras $V_{L}$ associated with positive definite even lattices $L$ of rank one, (b) orbifold vertex operator algebras $V_{L}^+$ under the automorphism of $V_{L}$ induced from the $-1$ isometry of $L$, (c) $V_{Z_2}^G$ where $(\alpha, \alpha) = 2$ and $G$ is a finite subgroup of $SO(3)$ isomorphic to one of $\{A_4, S_4, A_5\}$. As it is pointed out in [DJ1] that this list is not correct if the effective central charge $\tilde{c}$ [DM2] is not equal to $c$. The vertex operator algebra $V_{L}$ for any positive definite even lattice $L$ has been characterized by using $c$, the effective central charge $\tilde{c}$ and the rank of the weigh one subspace as a Lie algebra [DM2]. The orbifold vertex operator algebras $V_{L}^+$ for rank one lattices $L$ have also been characterized in [DJ1]-[DJ3] and [ZD]. But the vertex operator algebra $V_{Z_2}^G$ has not been understood well as $G$ is not a cyclic group. Although $V_{Z_2}^G$ is in the above list of rational vertex operator algebras, the rationality of $V_{Z_2}^G$ was unknown. The present paper deals with the case $G = A_4$.

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The main idea is to realize $V_{Z,\alpha}^G$ as $(V_{Z,\beta})^{(\sigma)}$ where $(\beta, \beta) = 8$ and $\sigma$ is an automorphism of $sl(2, \mathbb{C})$ of order 3. The vertex operator algebra $V_{Z,\beta}^+$ is well understood (see [DN1]-[DN3], [A1]-[A2]). Also it is easier to deal with the cyclic group $\langle \sigma \rangle$ than nonabelian group $A_4$. One key step is to give an explicit expression of the generator $u^{(9)}$ of weight 9. Another key step is to prove the $C_2$-cofiniteness of $V_{A_4}^L$. We achieve this by using the fusion rules of the Virasoro vertex operator algebra $L(1, 0)$ and technical calculations. The rationality follows from the $C_2$-cofiniteness [M2]. For the classification of irreducible modules, we follow the standard procedure. We first construct the irreducible $\sigma^2$-twisted $V_{Z,\beta}^+$-modules, and then give the irreducible $(V_{Z,\beta}^+)^{(\sigma)}$-submodules. According to [M1], these irreducible modules should give a complete list of irreducible $(V_{Z,\beta}^+)^{(\sigma)}$-modules.

It is expected that the ideas and techniques developed in this paper will work for $V_{S_4}^L$ as well. The case $G = A_5$ might be more complicated. Once the rationality of $V_{L_2}^G$ is established for all $G$, the classification of rational vertex operator algebras with $c = 1$ is equivalent to the following conjecture: If $V$ is a simple, rational vertex operator algebra of CFT type such that $\dim V < 3$ then $V$ is isomorphic to $V_{L_2}^G$ for $G = A_4, S_4, A_5$.

The paper is organized as follows. We recall various notions of twisted modules from [DLM1] in Section 2. We also briefly discuss lattice vertex operator algebras $V_L$ [FLM] and $V_{L_2}^+$ including the classification of irreducible modules and rationality [DN1]-[DN3], [A2], [AD], [DJL]. In Section 3, we identify the vertex operator algebra $V_{L_2}^{A_4}$ with $(V_{Z,\beta}^+)^{(\sigma)}$ and discuss several special vectors (which play important roles in later sections) in both $V_{Z,\beta}^+$ and $(V_{Z,\beta}^+)^{(\sigma)}$. The rationality and $C_2$-cofiniteness of $(V_{Z,\beta}^+)^{(\sigma)}$ are established in Section 4. The classification of the irreducible $(V_{Z,\beta}^+)^{(\sigma)}$-modules is achieved in Section 5.

2 Preliminaries

We first recall weak twisted-modules and twisted-modules for vertex operator algebras from [DLM2]. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra [B], [FLM] and $g$ an automorphism of $V$ of finite order $T$. Denote the decomposition of $V$ into eigenspaces with respect to the action of $g$ as

$$V = \bigoplus_{r \in \mathbb{Z}/T \mathbb{Z}} V^r$$

where $V^r = \{ v \in V | gv = e^{-2\pi iv/T} v \}$.

**Definition 2.1.** A weak $g$-twisted $V$-module $M$ is a vector space equipped with a linear map

$$V \to (\text{End } M) \{ z \}$$

$$v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Q}} v_n z^{-n-1} \quad (v_n \in \text{End } M)$$
which satisfies the following for all $0 \leq r \leq T - 1$, $u \in V^r$, $v \in V$, $w \in M$,
\[
Y_M(u, z) = \sum_{n \in \mathbb{Z}_+} u_n z^{-n-1} \quad (2.2)
\]
\[
u_l w = 0 \quad \text{for} \quad l \gg 0 \quad (2.3)
\]
\[
Y_M(1, z) = 1 \quad (2.4)
\]
\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1) Y_M(v, z_2) = z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y_M(v, z_2) Y_M(u, z_1)
\]
\[
= z_1^{-1} \left( \frac{z_2 + z_0}{z_1} \right)^{r/T} \delta \left( \frac{z_2 + z_0}{z_1} \right) Y_M(Y(u, z_0)v, z_2). \quad (2.5)
\]

It is known that (see [DLM2], etc) the twisted-Jacobi identity is equivalent to the following two identities.

\[
[u_{m+i}^{\tau}, v_{n+i}^{\tau}] = \sum_{i=0}^{\infty} \binom{m+r}{i} (u_i v)_{m+n+i}^{\tau},
\]
\[
\sum_{i \geq 0} \binom{s}{i} (u_{m+i}^{\tau})_{n+i}^{\tau} = \sum_{i \geq 0} (-1)^i \binom{m}{i} (u_{m+i}^{\tau})_{n+i}^{\tau} - (-1)^{m} v_{m+n+i}^{\tau},
\]

where $p, n \in \mathbb{Z}$, $u \in V^s$, $v \in V^t$.

**Definition 2.2.** An admissible $g$-twisted $V$-module $M = \bigoplus_{n \in \mathbb{Z}_+} M(n)$ is a $1/T\mathbb{Z}_+$-graded weak $g$-twisted module such that $u_m M(n) \subset M(w \mu - m - 1 + n)$ for $u \in V$ and $m, n \in 1/T\mathbb{Z}$.

**Definition 2.3.** A $g$-twisted $V$-module $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ is a $\mathbb{C}$-graded weak $g$-twisted $V$-module with $M_\lambda = \{ u \in M | L(0) u = \lambda u \}$ such that $M_\lambda$ is finite dimensional and for fixed $\lambda \in \mathbb{C}$, $M_{\lambda + n/T} = 0$ for sufficiently small integer $n$.

We now review the vertex operator algebras $M(1)^+, V_L^+$ and related results from [A1], [A2], [AD], [ADL], [DN1], [DN2], [DN3], [DJL], [FLM].

Let $L = \mathbb{Z}\alpha$ be a positive definite even lattice of rank one. That is, $(\alpha, \alpha) = 2k$ for some positive integer $k$. Set $\mathfrak{h} = \mathbb{C} \otimes \mathbb{C} L$ and extend $(\cdot, \cdot)$ to a $\mathbb{C}$-bilinear form on $\mathfrak{h}$. Let $\hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C} K$ be the affine Lie algebra associated to the abelian Lie algebra $\mathfrak{h}$ so that

\[
[\alpha(m), \alpha(n)] = 2km\delta_{m+n,0}K \quad \text{and} \quad [K, \hat{\mathfrak{h}}] = 0
\]

for any $m, n \in \mathbb{Z}$, where $\alpha(m) = \alpha \otimes t^m$. Then $\hat{\mathfrak{h}}^{\geq 0} = \mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C} K$ is a commutative subalgebra. For any $\lambda \in \mathfrak{h}$, we define a one-dimensional $\hat{\mathfrak{h}}^{\geq 0}$-module $\mathbb{C} e^\lambda$ such that $\alpha(m) \cdot e^\lambda = (\lambda, \alpha) \delta_{m,0} e^\lambda$ and $K \cdot e^\lambda = e^\lambda$ for $m \geq 0$. We denote by

\[
M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}^{\geq 0})} \mathbb{C} e^\lambda \cong S(t^{-1}\mathbb{C}[t^{-1}]) \quad \text{(linearly)}
\]
the \( \mathfrak{h} \)-module induced from \( \mathfrak{h} \geq 0 \)-module \( \mathbb{C} e^\lambda \). Set

\[ M(1) = M(1, 0). \]

Then there exists a linear map \( Y : M(1) \rightarrow \text{End} M(1)[[z, z^{-1}]] \) such that \( (M(1), Y, 1, \omega) \) carries a simple vertex operator algebra structure and \( M(1, \lambda) \) becomes an irreducible \( M(1) \)-module for any \( \lambda \in \mathfrak{h} \) (see [FLM]). The vacuum vector and the Virasoro element are given by \( 1 = e^0 \) and \( \omega = \frac{1}{2\pi} \alpha(-1)^2\mathbf{1} \), respectively.

Let \( \mathbb{C}[L] \) be the group algebra of \( L \) with a basis \( e^\beta \) for \( \beta \in L \). The lattice vertex operator algebra associated to \( L \) is given by

\[ V_L = M(1) \otimes \mathbb{C}[L]. \]

The dual lattice \( L^o \) of \( L \) is

\[ L^o = \{ \lambda \in \mathfrak{h} \mid (\alpha, \lambda) \in \mathbb{Z} \} = \frac{1}{2k} L. \]

Then \( L^o = \bigcup_{i=-k+1}^k (L + \lambda_i) \) is the coset decomposition with \( \lambda_i = \frac{i}{2k} \alpha. \) In particular, \( \lambda_0 = 0. \) Set \( \mathbb{C}[L + \lambda_i] = \bigoplus_{\beta \in L} \mathbb{C} e^{\beta + \lambda_i}. \) Then each \( \mathbb{C}[L + \lambda_i] \) is an \( L \)-submodule in an obvious way. Set \( V_{L+\lambda_i} = M(1) \otimes \mathbb{C}[L + \lambda_i]. \) Then \( V_L \) is a rational vertex operator algebra and \( V_{L+\lambda_i} \) for \( i = -k+1, \ldots, k \) are the irreducible modules for \( V_L \) (see [B], [FLM], [D2]).

Define a linear isomorphism \( \theta : V_{L+\lambda_i} \rightarrow V_{L-\lambda_i} \) for \( i \in \{-k+1, \ldots, k\} \) by

\[ \theta(\alpha(-n_1)\alpha(-n_2) \cdots \alpha(-n_s) \otimes e^{\beta+\lambda_i}) = (-1)^{k} \alpha(-n_1)\alpha(-n_2) \cdots \alpha(-n_s) \otimes e^{-\beta-\lambda_i} \]

where \( n_j > 0 \) and \( \beta \in L. \) Then \( \theta \) defines a linear isomorphism from \( V_{L^o} = M(1) \otimes \mathbb{C}[L^o] \) to itself such that

\[ \theta(Y(u, z)v) = Y(\theta u, z)\theta v \]

for \( u \in V_L \) and \( v \in V_{L^o}. \) In particular, \( \theta \) is an automorphism of \( V_L \) which induces an automorphism of \( M(1). \)

For any \( \theta \)-stable subspace \( U \) of \( V_{L^o} \), let \( U^\pm \) be the \( \pm 1 \)-eigenspace of \( U \) for \( \theta. \) Then \( V_L^+ \) is a simple vertex operator algebra.

Also recall the \( \theta \)-twisted Heisenberg algebra \( \mathfrak{h}[-1] \) and its irreducible module \( M(1)(\theta) \) from [FLM]. Let \( \chi_s \) be a character of \( L/2L \) such that \( \chi_s(\alpha) = (-1)^s \) for \( s = 0, 1 \) and \( T_{\chi_s} = \mathbb{C} \) the irreducible \( L/2L \)-module with character \( \chi_s. \) It is well known that \( V_L^{T_{\chi_s}} = M(1)(\theta) \otimes T_{\chi_s} \) is an irreducible \( \theta \)-twisted \( V_L \)-module (see [FLM], [D2]). We define actions of \( \theta \) on \( M(1)(\theta) \) and \( V_L^{T_{\chi_s}} \) by

\[ \theta(\alpha(-n_1)\alpha(-n_2) \cdots \alpha(-n_p)) = (-1)^p \alpha(-n_1)\alpha(-n_2) \cdots \alpha(-n_p) \]

\[ \theta(\alpha(-n_1)\alpha(-n_2) \cdots \alpha(-n_p) \otimes t) = (-1)^p \alpha(-n_1)\alpha(-n_2) \cdots \alpha(-n_p) \otimes t \]

for \( n_j \in \frac{1}{2} + \mathbb{Z}_+ \) and \( t \in T_{\chi_s}. \) We denote the \( \pm 1 \)-eigenspaces of \( M(1)(\theta) \) and \( V_L^{T_{\chi_s}} \) under \( \theta \) by \( M(1)(\theta)^\pm \) and \( (V_L^{T_{\chi_s}})^\pm \) respectively. We have the following results:
Theorem 2.4. Any irreducible module for the vertex operator algebra \( M(1)^+ \) is isomorphic to one of the following modules:

\[ M(1)^+, M(1)^-, M(1, \lambda) \cong M(1, -\lambda) \quad (0 \neq \lambda \in \mathfrak{h}), M(1)(\theta)^+, M(1)(\theta)^-. \]

Theorem 2.5. Any irreducible \( V_L^+ \)-module is isomorphic to one of the following modules:

\[ V_L^\pm, V_{\lambda + L}^\pm(i \neq k), V_{\lambda_k + L}^\pm, (V_L^{TX})^\pm. \]

Theorem 2.6. \( V_L^+ \) is rational.

We remark that the classification of irreducible modules for arbitrary \( M(1)^+ \) and \( V_L^+ \) are obtained in [DN1]-[DN3] and [AD]. The rationality of \( V_L^+ \) is established in [A2] for rank one lattice and [DJL] in general.

We next turn our attention to the fusion rules of vertex operator algebras. Let \( V \) be a vertex operator algebra, and \( W^i \ (i = 1, 2, 3) \) be ordinary \( V \)-modules. We denote by \( I_V \left( \begin{array}{c} W^3 \\ W^1 W^2 \end{array} \right) \) the vector space of all intertwining operators of type \( \left( \begin{array}{c} W^3 \\ W^1 W^2 \end{array} \right) \). For a \( V \)-module \( W \), let \( W' \) denote the graded dual of \( W \). Then \( W' \) is also a \( V \)-module [FHL]. It is well known that fusion rules have the following symmetry (see [FHL]).

Proposition 2.7. Let \( W^i \ (i = 1, 2, 3) \) be \( V \)-modules. Then

\[ \dim I_V \left( \begin{array}{c} W^3 \\ W^1 W^2 \end{array} \right) = \dim I_V \left( \begin{array}{c} W^3 \\ W^2 W^1 \end{array} \right), \quad \dim I_V \left( \begin{array}{c} W^3 \\ W^1 W^2 \end{array} \right) = \dim I_V \left( \begin{array}{c} (W^2)' \\ W^1 (W^3)' \end{array} \right). \]

Recall that \( L(c, h) \) is the irreducible highest weight module for the Virasoro algebra with central charge \( c \) and highest weight \( h \) for \( c, h \in \mathbb{C} \). It is well known that \( L(c, 0) \) is a vertex operator algebra. The following two results were obtained in [M] and [DJ1].

Theorem 2.8. (1) We have

\[ \dim I_{L(1,0)} \left( \begin{array}{c} L(1, k^2) \\ L(1, m^2) L(1, n^2) \end{array} \right) = 1, \quad k \in \mathbb{Z}_+, \ |n - m| \leq k \leq n + m, \]

\[ \dim I_{L(1,0)} \left( \begin{array}{c} L(1, k^2) \\ L(1, m^2) L(1, n^2) \end{array} \right) = 0, \quad k \in \mathbb{Z}_+, \ k < |n - m| \text{ or } k > n + m, \]

where \( n, m \in \mathbb{Z}_+ \).

(2) For \( n \in \mathbb{Z}_+ \) such that \( n \neq p^2 \), for all \( p \in \mathbb{Z}_+ \), we have

\[ \dim I_{L(1,0)} \left( \begin{array}{c} L(1, n) \\ L(1, m^2) L(1, n) \end{array} \right) = 1, \]

\[ \dim I_{L(1,0)} \left( \begin{array}{c} L(1, k) \\ L(1, m^2) L(1, n) \end{array} \right) = 0, \]

for \( k \in \mathbb{Z}_+ \) such that \( k \neq n \).
3 The vertex operator subalgebra $V_{L_2}^{A_4}$

Let $L_2 = \mathbb{Z}\alpha$ be the rank one positive-definite even lattice such that $(\alpha, \alpha) = 2$ and $V_{L_2}$ the associated simple rational vertex operator algebra. Then $(V_{L_2})_1 \cong sl(2, \mathbb{C})$ and $(V_{L_2})_1$ has an orthonormal basis:

$$x^1 = \frac{1}{\sqrt{2}}\alpha(-1)1, \quad x^2 = \frac{1}{\sqrt{2}}(e^\alpha + e^{-\alpha}), \quad x^3 = \frac{i}{\sqrt{2}}(e^\alpha - e^{-\alpha}).$$

Let $\tau_i \in Aut(V_{L_2})$, $i = 1, 2, 3$ be such that

$$\tau_1(x^1, x^2, x^3) = (x^1, x^2, x^3) \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\tau_2(x^1, x^2, x^3) = (x^1, x^2, x^3) \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\tau_3(x^1, x^2, x^3) = (x^1, x^2, x^3) \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$}

Let $\sigma \in Aut(V_{L_2})$ be such that

$$\sigma(x^1, x^2, x^3) = (x^1, x^2, x^3) \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$}

Then $\sigma$ and $\tau_i$, $i = 1, 2, 3$ generate a finite subgroup of $Aut(V_{L_2})$ isomorphic to the alternating group $A_4$. We simply denote this subgroup by $A_4$. It is easy to check that the subgroup $K$ generated by $\tau_i$, $i = 1, 2, 3$ is a normal subgroup of $A_4$ of order 4. Let

$$J = h(-1)^41 - 2h(-3)h(-1)1 + \frac{3}{2}h(-2)^21, \quad E = e^\beta + e^{-\beta}$$

where $h = \frac{1}{\sqrt{2}}\alpha$, $\beta = 2\alpha$. The following lemma comes from [DG].

**Lemma 3.1.** $V_{L_2}^K \cong V_{Z\beta}^+$, and $V_{Z\beta}^+$ is generated by $J$ and $E$. Moreover, $(V_{L_2}^K)_4$ is four dimensional with a basis $L(-2)^21, L(-4)1, J, E$.

By Lemma 3.1 we have $V_{L_2}^{A_4} = (V_{Z\beta}^+)^{\langle \sigma \rangle}$. A direct calculation yields that

**Lemma 3.2.** We have

$$\sigma(J) = -\frac{1}{2}J + \frac{9}{2}E, \quad \sigma(E) = -\frac{1}{6}J - \frac{1}{2}E.$$
Let
\[ X^1 = J - \sqrt{2}iE, \quad X^2 = J + \sqrt{2}iE. \]  
(3.1)
Then it is easy to check that
\[ \sigma(X^1) = -1 + \frac{\sqrt{3}i}{2}X^1, \quad \sigma(X^2) = -1 - \frac{\sqrt{3}i}{2}X^2. \]  
(3.2)
It follows that \((V_{\mathbb{Z},\beta}^+)\) of \((V_{\mathbb{Z},\beta}^+)\) is an irreducible \((V_{\mathbb{Z},\beta}^+)\)-module generated by \(X_i\) with lowest weight 4, \(i = 1, 2\).

For convenience, we call highest weight vectors for the Virasoro algebra primary vectors. Note from [DG] that \((V_{\mathbb{Z},\beta}^+)\) contains two linearly independent primary vectors \(J\) and \(E\) of weight 4 and one linearly independent primary vector of weight 9. It is straightforward to compute that
\[ J_3J = -72L(-4)1 + 336L(-2)21 - 60J, \quad E_3E = -\frac{8}{3}L(-4)1 + \frac{112}{9}L(-2)21 + \frac{20}{9}J \]  
(cf. [DJ3]). By Theorem 2.8 and Lemma 3.1, we have for any \(n \in \mathbb{Z}\),
\[ X^1_nX^2 \in (V_{\mathbb{Z},\beta}^+) \]

The following lemma follows from Theorem 3 in [DM1] and (3.2).

**Lemma 3.3.** We have decomposition
\[ V_{\mathbb{Z},\beta}^+ = (V_{\mathbb{Z},\beta}^+)^0 \oplus (V_{\mathbb{Z},\beta}^+)^1 \oplus (V_{\mathbb{Z},\beta}^+)^2, \]
where \((V_{\mathbb{Z},\beta}^+)^0 = (V_{\mathbb{Z},\beta}^+)^0\) is a simple vertex operator algebra and \((V_{\mathbb{Z},\beta}^+)^i\) is the irreducible \((V_{\mathbb{Z},\beta}^+)^0\)-module generated by \(X^i\) with lowest weight 4, \(i = 1, 2\).

Set
\[ u^0 = -\frac{8}{3}L(-4)1 + \frac{112}{9}L(-2)21 \]  
(3.3)
\[ u^1 = -\frac{16}{9}L(-5)1 + \frac{112}{9}L(-3)L(-2)1 \]  
(3.4)
\[ u^2 = (\frac{-1856}{135}L(-6) - \frac{2384}{135}L(-4)L(-2) + \frac{1316}{135}L(-3)^2 + \frac{1088}{135}L(-2)^2)1 \]  
(3.5)
\[ u^3 = (\frac{-464}{45}L(-7) - \frac{928}{45}L(-5)L(-2) + \frac{40}{9}L(-4)L(-3) + \frac{544}{45}L(-3)L(-2)^2)1 \]  
(3.6)
\[ v^2 = \left( \frac{28}{75} L(-2) + \frac{23}{300} L(-1)^2 \right) J, \quad (3.7) \]
\[ v^3 = \left( \frac{14}{75} L(-3) + \frac{14}{75} L(-2)L(-1) - \frac{1}{300} L(-1)^3 \right) J. \quad (3.8) \]
\[ v^4 = \left( \frac{28}{75} L(-2) + \frac{23}{300} L(-1)^2 \right) E, \quad (3.9) \]
\[ v^5 = \left( \frac{14}{75} L(-3) + \frac{14}{75} L(-2)L(-1) - \frac{1}{300} L(-1)^3 \right) E. \quad (3.10) \]

By Lemma 2.5 of [DJ3], we have the following lemma.

**Lemma 3.4.** Let \( E \) and \( J \) be as before. Then
\[ E_3E = u_0^0 + \frac{20}{9} J, \quad J_3J = 27u_0^0 - 60J, \quad J_3E = 60E, \]
\[ E_2E = u_1^1 + \frac{10}{9} L(-1)J, \quad J_2J = 27u_1^1 - 30L(-1)J, \quad J_2E = 30L(-1)E, \]
\[ E_1E = u_2^2 + \frac{20}{9} v^2, \quad J_1J = 27u_2^2 - 60v^2, \quad J_1E = 60v^4, \]
\[ E_0E = u_3^3 + \frac{20}{9} v^3, \quad J_0J = 27u_3^3 - 60v^3, \quad J_0E = 60v^5. \]

Using Lemma 3.4, one can check directly that
\[ (J_{-2}E - E_{-2}J)_{8}J = -10800E, \quad (J_{-2}E - E_{-2}J)_{8}E = 400J. \quad (3.11) \]

As a result, we have
\[ (J_{-2}E - E_{-2}J)_{8}X_{1} = -400\sqrt{27i}X_{1}, \quad (J_{-2}E - E_{-2}J)_{8}X_{2} = 400\sqrt{27i}X_{2}. \quad (3.12) \]

By (3.12), we immediately know that \( J_{-2}E - E_{-2}J \) is a non-zero primary vector of weight 9. Recall from [DG] that \( V_{_{2\beta}}^+ \) has one primary vector of weight 9 up to a constant. A direct calculation yields that

**Lemma 3.5.** The vector
\[ u^{(9)} = -\frac{1}{\sqrt{2}}(15h(-4)h(-1) + 10h(-3)h(-2) + 10h(-2)h(-1)^3) \otimes (e^{\beta} + e^{-\beta}) \]
\[ + (6h(-5) + 10h(-3)h(-1)^2 + \frac{15}{2}h(-2)^2h(-1) + h(-1)^5) \otimes (e^{\beta} - e^{-\beta}) \]

is a non-zero primary vector of weight 9 and \( u^{(9)} \in C(J_{-2}E - E_{-2}J). \)

Note from [L1] that there is a non-degenerate symmetric invariant bilinear form \((\cdot, \cdot)\) on \( V_{_{2\beta}}^+ \). The next lemma gives a relation between \( u^{(9)} \) and \( J_{-2}E - E_{-2}J \).
Lemma 3.6. We have
\[ J_{-2}E - E_{-2}J = -2\sqrt{2}u^{(9)}, \]
\[ (u^{(9)}, u^{(9)}) = 5400. \]

Proof: By Lemma 3.5 and (3.11), we have
\[ (V_{\beta}^+)_4 = Ch(-3)h(-1)1 \oplus Ch(-2)^21 \oplus Ch(-1)^41 \oplus CE. \]

Let \( W_4 \) be the subspace of \((V_{\beta}^+)_4\) linearly spanned by \( E, h(-3)h(-1)1 \) and \( h(-2)^21 \). Then
\[ h(-1)^41 \equiv J (\text{mod } W_4). \]
Furthermore, we have
\[ (h(-4)h(-1)E)_8E \equiv \sum_{i=0}^{\infty}(-1)^{i+1} \left( \begin{array}{c} 4 \\ i \end{array} \right) (h(-1)E)_4^{-i}h(i)E (\text{mod } W_4) \]
\[ \equiv -\sqrt{8}(h(-1)E)_4F (\text{mod } W_4) \]
\[ \equiv -\sqrt{8} \sum_{i=0}^{\infty}(-1)^i \left( \begin{array}{c} 1 \\ i \end{array} \right) (h(-1-i)E_{4+i} + E_{3-i}h(i))F (\text{mod } W_4) \]
\[ \equiv -\sqrt{8}h(-1)E_4F - 8E_3E (\text{mod } W_4). \]

Similarly,
\[ (h(-3)h(-2)E)_8E \equiv -8E_3E (\text{mod } W_4), \]
\[ (h(-2)h(-1)^3E)_8E \equiv -\sqrt{8}h(-1)^3E_3F - 24h(-1)^2E_5E - 24\sqrt{8}h(-1)E_4F - 64E_3E (\text{mod } W_4), \]
\[ (h(-5)F)_8E \equiv \sqrt{8}F_3F (\text{mod } W_4), \]
\[ (h(-3)h(-1)^2F)_8E \equiv \sqrt{8}h(-1)^2F_5F + 16h(-1)F_4E + 8\sqrt{8}F_3F (\text{mod } W_4). \]
\[ (h(-2)^2h(-1)F)_8E \equiv 8h(-1)F_4E + 8\sqrt{8}F_3F (\text{mod } W_4), \]
\[ (h(-1)^5F)_8E \equiv 5\sqrt{8}h(-1)^4F_7F + 80h(-1)^3F_6E + 80\sqrt{8}h(-1)^2F_5F + 320h(-1)F_4E + 64\sqrt{8}F_3F (\text{mod } W_4). \]

It is then easy to check that
\[ u_8^{(9)}E = -100\sqrt{2}h(-1)^41 (\text{mod } W_4). \]

This implies that
\[ u_8^{(9)}E = -100\sqrt{2}J. \]

Then by (3.11),
\[ J_{-2}E - E_{-2}J = -2\sqrt{2}u^{(9)}. \]
Note that
\[ (J_{-2}E - E_{-2}J, J_{-2}E) = (J_8(J_{-2}E - E_{-2}J), E) = -((J_{-2}E - E_{-2}J)_8J, E) \]
and
\[ (J_{-2}E - E_{-2}J, E_{-2}J) = (E_8(J_{-2}E - E_{-2}J), J) = -(J_{-2}E - E_{-2}J)_8E, J). \]
Since
\[ (E, E) = 2, \ (J, J) = 54, \]
(see [DJ2]) it follows from (3.11) that
\[ (J_{-2}E - E_{-2}J, J_{-2}E - E_{-2}J) = 43200 \] (3.13)
and
\[ (u^{(9)}, u^{(9)}) = 5400. \]
The proof is complete. \(\square\)

4 \textbf{C}_2\text{-cofiniteness and rationality of} \ V_{L_2}^{A_4}

The \textit{C}_2\text{-cofiniteness and rationality of} \ V_{L_2}^{A_4} \text{ is established in this section. The proof involves some very hard computations.}

By Lemma 3.2 we have
\[ J_{-9}J + 27E_{-9}E \in (V_{Z,\beta}^+)^{(\sigma)}. \]
Then it is clear that
\[ J_{-9}J + 27E_{-9}E = x^0 + X^{(16)} + 27(e^{2\beta} + e^{-2\beta}), \] (4.1)
where \(x^0 \in L(1,0),\) and \(X^{(16)}\) is a non-zero primary element of weight 16 in \(M(1)^+.\)
Denote
\[ u^{(16)} = X^{(16)} + 27(e^{2\beta} + e^{-2\beta}). \] (4.2)
Then \(u^{(16)} \in (V_{Z,\beta}^+)^{(\sigma)}\) is a non-zero primary vector of weight 16.

\textbf{Lemma 4.1.} We have the following:
\[ u_1^{(9)}u^{(9)} - 58800u^{(16)} \in L(1,0). \]

\textbf{Proof:} Denote \(E^2 = e^{2\beta} + e^{-2\beta}.\) By Theorem 2.8 and the skew-symmetry, we may assume that
\[ u_1^{(9)}u^{(9)} = v + cu^{(16)}, \]
for some \( v \in L(1,0) \) and \( c \in \mathbb{C} \). To determine \( c \) we just need to consider \((u_1^{(9)} u^{(9)}, E^2)\) by (4.2). Recall that

\[
u^{(9)} = -\frac{1}{\sqrt{2}}(15h(-4)h(-1) + 10h(-3)h(-2) + 10h(-2)h(-1)^3) \otimes E
\]

\[
+ (6h(-5) + 10h(-3)h(-1)^2 + \frac{15}{2} h(-2)^2 h(-1) + h(-1)^5) \otimes F,
\]

where \( F = e^\beta - e^{-\beta} \). To calculate \(((h(-4)h(-1) \otimes E)_1 (h(-4)h(-1) \otimes E), E^2)\), we only need to consider the coefficient of the monomial \(E^2\) in \((h(-4)h(-1) \otimes E)_1 (h(-4)h(-1) \otimes E)\). Then direct calculation yields that

\[
((h(-4)h(-1) \otimes E)_1 (h(-4)h(-1) \otimes E), E^2) = (972 E^2, E^2).
\]

Calculations for other monomials are similar. For example,

\[
((h(-3)h(-2) \otimes E)_1 (h(-2)h(-1)^3 \otimes E), E^2) = (304 E^2, E^2).
\]

Then one can check that

\[
(u_1^{(9)} u^{(9)}, E^2) = (1587600 E^2, E^2).
\]

It follows that \( c = 58800 \).

**Lemma 4.2.** The following hold: (1) \((V_{z,\beta}^+)^{(\sigma)}\) is generated by \(u^{(9)}\).

(2) \((V_{z,\beta}^+)^{<\sigma>}\) is linearly spanned by

\[
L(-m_s) \cdots L(-m_1) v_n^{(9)} u^{(9)}, \quad L(-m_s) \cdots L(-m_1) w_{-k}^{p} \cdots w_{-k_1}^{p} w,
\]

where \( w, w^1, \ldots, w^p \in \{u^{(9)}, u^{(16)}\}, \quad k_p \geq \cdots \geq k_1 \geq 2, \quad n \in \mathbb{Z}, \quad m_s \geq \cdots \geq m_1 \geq 1, \quad s, p \geq 0.\)

**Proof:** By Lemma 3.6 \( \omega \) can be generated by \(u^{(9)}\). It follows from [DGR] that \((V_{z,\beta}^+)^{(\sigma)}\) is generated by \(u^{(9)}\) and \(u^{(16)}\). Then (1) follows from Lemma 4.1.

By (3.2) in [A1] and (3.3) in [A3], we have

\[
M(1, 2\sqrt{2}m) = \bigoplus_{p=0}^{\infty} L(1, (2m + p)^2), \quad (4.3)
\]

\[
V_{z,\beta}^+ = M(1)^+ \bigoplus_{m=1}^{\infty} M(1, 2\sqrt{2}m)) = M(1)^+ \bigoplus_{m=1}^{\infty} \bigoplus_{p=0}^{\infty} L(1, (2m + p)^2). \quad (4.4)
\]

By (4.4) the subspace \(U^1\) linearly spanned by primary elements of weight 16 in \(V_{z,\beta}^+\) is three dimensional. Obviously \(U^1\) is invariant under \(\sigma\). Note that \(e^{2\beta} + e^{-2\beta} \in U^1\). Consider the \(M(1)^+\)-submodule \(W\) of \(V_{z,\beta}^+\) generated by \(e^{2\beta} + e^{-2\beta}\). If \(e^{2\beta} + e^{-2\beta} \in (V_{z,\beta}^+)^{<\sigma>}\), then by the fusion rule of \(M(1)^+\) (also see [DN2]), \(J \in W \cdot W = \langle u_n v | u, v \in W, n \in \mathbb{Z} \rangle\). So
Consider the $M$-W for some is a non-zero primary element $w$ of $(\mathbf{1})$. A proof similar to that of Lemma 4.3 in \cite{DN2} gives (2) with the help of (1), (4.5) and elements in $U$ of $(\mathbf{9})$, elements in $V$ of $(\mathbf{9})$. Let $U$ and $U$ be the $L(1,0)$-submodules of $(V^+_\mathbb{Z})_{\sigma}$ generated by $u$ and $u$, respectively. Then by Part (1) and the skew-symmetry, any element of weight 25 in $(V^+_\mathbb{Z})_{\sigma}$ is a linear combination of elements in $L(1,0) \oplus U \oplus U$ and $U \cdot U = \langle u, v | u \in U, v \in U \rangle$. By Lemma 3.5 and (4.2), elements in $U$ and $U$ of $(\mathbf{9})$ have the forms: $u \otimes (e^\beta + e^{-\beta}) + v \otimes (e^\beta - e^{-\beta})$, where $u \in M(1)^+$ and $v \in M(1)^-$. So we know that $w \notin (V^+_\mathbb{Z})_{\sigma}$. This proves that $\sigma|_{U^2}$ has eigenvalues not equal to 1. Since $\sigma = 1$ and dim$_\mathbb{C} U^2 = 2$, it follows that $\sigma|_{U^2}$ has two eigenvalues $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$. So we immediately have

$$(V^+_\mathbb{Z})_{\sigma} = L(1,0) \oplus L(1,9) \oplus L(1,16) \oplus (\sum_{n \geq 5} a_n L(1,n^2)).$$

A proof similar to that of Lemma 4.3 in \cite{DN2} gives (2) with the help of (1), (4.5) and Theorem 2.8.

**Lemma 4.3.** We have

$$u_{-3} u = s^1 + \frac{162770}{99} L(-4) u + \frac{5204015}{1584} L(-3) L(-1) u + \frac{1154225}{792} L(-2) L(-1)^2 u + \frac{354895}{3168} L(-1)^4 u,$$

$$u_{-4} u = s^2 - \frac{653871670}{6306 \cdot 27} L(-6) u + \frac{3303230375}{2018016 \cdot 27} L(-5) L(-1) u + \frac{489993820}{346772585} L(-4) L(-2) u + \frac{69658220}{9009 \cdot 27} L(-3)^2 u + \frac{42042 \cdot 27}{16816 \cdot 27} L(-4) L(-1)^2 u + \frac{3338006885}{1055175305} L(-3) L(-2) L(-1) u + \frac{19408720}{7007 \cdot 27} L(-2)^3 u + \frac{14067649205}{4036032 \cdot 27} L(-3) L(-1)^3 u + \frac{1055175305}{252252 \cdot 27} L(-2)^2 L(-1)^2 u + \frac{1185150565}{8072064 \cdot 27} L(-2) L(-1)^4 u + \frac{119070745}{119070745} L(-1)^6 u,$$

where $s^1, s^2 \in L(1,0)$.
Proof: By Theorem 2.8 and the skew-symmetry, we may assume that

\[ u^{(9)}u^{(9)} = s^1 + y^1, \quad u^{(9)}u^{(9)} = s^2 + y^2, \]

where \( s^1, s^2 \in L(1,0), \ y^1, y^2 \in U^{(16)} \cong L(1,16) \) which is an \( L(1,0) \)-submodule of \( (V_{Z^3}^+)^{(\sigma)} \) generated by \( u^{(16)} \). Then we may assume that

\[
y^1 = a_1 L(-4)u^{(16)} + a_2 L(-3)L(-1)u^{(16)} + a_3 L(-2)^2u^{(16)}
+ a_4 L(-2)L(-1)^2u^{(16)} + a_5 L(-1)^4u^{(16)}
= \sum_{i=1}^{5} a_i w^i.
\]

To determine \( a_i, 1 \leq i \leq 5 \), we consider \( (u^{(9)}u^{(9)}, w^i), (w^i, w^j), i, j = 1, 2, \ldots, 5 \). Then by Lemma 4.1 and direct calculation, we have

\[
\begin{bmatrix}
133 & 224 & 387 & 576 & 1920 \\
224 & 3328 & 480 & 10560 & 49920 \\
387 & 480 & 17673/2 & 13152 & 57600 \\
576 & 10560 & 13152 & 162336 & 1267200 \\
1920 & 49920 & 57600 & 1267200 & 30159360
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5
\end{bmatrix}
= 58800
\begin{bmatrix}
43 \\
560 \\
675 \\
7344 \\
93024
\end{bmatrix}.
\]

We get that

\[
a_1 = \frac{162770}{99}, \quad a_2 = \frac{5204015}{1584},
\]

\[
a_3 = \frac{14760}{11}, \quad a_4 = \frac{1154225}{792}, \quad a_5 = \frac{354895}{3168}.
\]

The first formula follows. The proof for the second one is similar. We omit it. \( \square \)

Let \( v \) be any element in \( V_{Z^3}^+ \) of weight \( m \leq 22 \). Then \( v \) is a linear combination of an element in \( V^{(4)} \oplus V^{(16)} \) and elements in \( M(1)^+ \) having the form \( h(-n_t) \cdots h(-n_1)1 \) such that \( n_t \geq \cdots \geq n_1 \geq 1 \) and \( \sum_{t=1}^{l} n_i = m \), where \( V^{(4)} \) and \( V^{(16)} \) are \( M(1)^+ \)-submodules of \( V_{Z^3}^+ \) generated by \( E \) and \( E^2 \) respectively. We denote by \( c(v) \) the coefficient of the monomial \( h(-1)^m1 \) in the linear combination. Then we have the following lemma.

Lemma 4.4.

\[
c(u^{(9)}u^{(9)}) = -\frac{447232}{19 \cdot 17 \cdot 11 \cdot 7^2 \cdot 5^2 \cdot 3},
\]

\[
c(u^{(9)}u^{(9)}) = -\frac{328099328}{19 \cdot 17 \cdot 13 \cdot 11^2 \cdot 7^3 \cdot 5^2 \cdot 3^6}.
\]
Further, by Lemma 4.5 and Lemma 4.3, we have

\[ c(u_{-k}^{(9)}u^{(9)}) = -2700c(E_{-k-10}E) - 13500c(h(-1)^2E_{-k-8}E') - 18000\sqrt{2}c(h(-1)^3E_{-k-7}E) - 31500c(h(-1)^4E_{-k-6}E) - 15300\sqrt{2}c(h(-1)^5E_{-k-5}E) - 9060c(h(-1)^6E_{-k-4}E) - 1620\sqrt{2}c(h(-1)^7E_{-k-3}E) - 345c(h(-1)^8E_{-k-2}E) - 20\sqrt{2}c(h(-1)^9E_{-k-1}E) - c(h(-1)^{10}E_{-k}E) \]

Note that for \( m, n \in 2\mathbb{Z}, m, n \leq 7 \),

\[ c(E_mE) = \frac{2 \cdot (\sqrt{8})^{-m}}{(7 - m)!}, \quad c(E_nF) = \frac{2 \cdot (\sqrt{8})^{-n}}{(7 - n)!}. \]

Let \( k = -3, k = -5 \) respectively, we then get the lemma.  

As defined in [Z], a vertex operator algebra \( V \) is called \( C_2 \)-cofinite, if \( V/C_2(V) \) is finite-dimensional, where \( C_2(V) = \text{span}_C\{u_{-2}v|u, v \in V\} \). The following lemma comes from [Z].

Lemma 4.5. (1) \( L(-1)u \in C_2(V) \) for \( u \in V \);

(2) \( u_{-k}v \in C_2(V) \), for \( u, v \in V \) and \( k \geq 2 \);

(3) \( u_{-1}v \in C_2(V) \), for \( u \in V, v \in C_2(V) \).

We are now in a position to state the main result of this section.

Theorem 4.6. \( (V_{Z,\delta}^+(\sigma)) \) is \( C_2 \)-cofinite and rational.

Proof: Let \( s^1, s^2 \in L(1, 0) \) be the same as in Lemma 4.3. Then \( s^1 \) and \( s^2 \) are linear combinations of linearly independent elements having the forms \( L(-m_1) \cdots L(-m_s) \) and \( L(-n_1) \cdots L(-n_t) \) respectively such that \( m_1 \geq \cdots \geq m_s \geq 2, n_1 \geq \cdots \geq n_t \geq 2 \) and \( \sum_{i=1}^{s} m_i = 20, \sum_{i=1}^{t} n_i = 22 \). Assume the coefficients of \( L(-2)^{10}1 \) and \( L(-2)^{11}1 \) in the two linear combinations are \( a_1 \) and \( a_2 \) respectively. Then by Lemma 4.5,

\[ s^1 - a_1L(-2)^{10}1, \quad s^2 - a_2L(-2)^{11}1 \in C_2((V_{Z,\delta}^+(\sigma)). \]

Further, by Lemma 4.3 and Lemma 4.3, we have

\[ s^1 + \frac{14760}{11}L(-2)^2u^{(16)}, \quad s^2 + \frac{19408720}{7007 \cdot 27}L(-2)^3u^{(16)} \in C_2((V_{Z,\delta}^+(\sigma)). \]

So

\[ a_1L(-2)^{10}1 + \frac{14760}{11}L(-2)^2u^{(16)}, \quad a_2L(-2)^{11}1 + \frac{19408720}{7007 \cdot 27}L(-2)^3u^{(16)} \in C_2((V_{Z,\delta}^+(\sigma)). \]

Thus by Lemma 4.5

\[ a_1L(-2)^{11}1 + \frac{14760}{11}L(-2)^3u^{(16)}, \quad a_2L(-2)^{11}1 + \frac{19408720}{7007 \cdot 27}L(-2)^3u^{(16)} \in C_2((V_{Z,\delta}^+(\sigma)). \]

(4.6)
On the other hand, note from the definition of $L(-2)^1$ that $c(L(-2)^k 1) = 2^k$. This implies that

\[
c(u_{-3}^{(9)}u^{(9)}) = \frac{1}{2^{10}}a_1 + \frac{1}{4} \cdot \frac{14760}{11} c(X^{(16)}),
\]

\[
c(u_{-5}^{(9)}u^{(9)}) = \frac{1}{2^{11}}a_2 + \frac{1}{8} \cdot \frac{19408720}{7007 \cdot 27} c(X^{(16)}).
\]

So by Lemma 4.4,

\[
\frac{1}{2^{10}}a_1 + \frac{1}{4} \cdot \frac{114760}{11} c(X^{(16)}) = -\frac{447232}{19 \cdot 17 \cdot 11 \cdot 7^2 \cdot 5^2 \cdot 3},
\]

\[
\frac{1}{2^{11}}a_2 + \frac{1}{8} \cdot \frac{19408720}{7007 \cdot 27} c(X^{(16)}) = -\frac{32809328}{19 \cdot 17 \cdot 13 \cdot 11^2 \cdot 7^3 \cdot 5^2 \cdot 3^6}.
\]

(4.7)

(4.8)

If

\[
a_1/a_2 = \frac{14760}{11} / \frac{19408720}{7007 \cdot 27},
\]

then by (4.7) and (4.8), we have

\[
-\frac{447232}{2 \cdot 19 \cdot 17 \cdot 11 \cdot 7^2 \cdot 5^2 \cdot 3} - \frac{32809328}{19 \cdot 17 \cdot 13 \cdot 11^2 \cdot 7^3 \cdot 5^2 \cdot 3^6} = \frac{14760}{19 \cdot 17 \cdot 11 \cdot 7^2 \cdot 5^2 \cdot 3}.
\]

But

\[
-\frac{447232}{32809328} - \frac{32809328}{19 \cdot 17 \cdot 13 \cdot 11^2 \cdot 7^3 \cdot 5^2 \cdot 3^6} = \frac{32688117}{2563276} \neq \frac{6346431}{485218} = \frac{14760}{19408720}.
\]

This means that

\[
a_1/a_2 \neq \frac{14760}{11} / \frac{19408720}{7007 \cdot 27}.
\]

By (4.6), we have

\[
L(-2)^{11} 1, L(-2)^3 u^{(16)} \in C_2((V_{2\beta}^+)^{(\sigma)}).
\]

Then it follows from Lemma 4.2 that $(V_{2\beta}^+)^{(\sigma)}$ is $C_2$-cofinite. Since $V_{2\beta}^+$ is rational and $(V_{2\beta}^+)^{(\sigma)}$ is self-dual, it follows that $(V_{2\beta}^+)^{(\sigma)}$ satisfies the Hypothesis I in [M2]. Then by Corollary 7 in [M2], $(V_{2\beta}^+)^{(\sigma)}$ is rational.

\[
\square
\]

5 Classification and construction of irreducible modules of $(V_{2\beta}^+)^{(\sigma)}$

We will first construct all the irreducible $\sigma^i$-twisted modules of $V_{2\beta}^+$, $i = 1, 2$. We have the following lemma.

**Lemma 5.1.** There are at most two inequivalent irreducible $\sigma$-twisted modules of $V_{2\beta}^+$.
Proof: Let $(W, Y)$ be an irreducible $V_{z, \beta}^+$-module. Define a linear map

$$Y^\sigma : V_{z, \beta}^+ \to (\text{End} W)[[z, z^{-1}]]$$

by

$$Y^\sigma(u, z)w = Y(\sigma^{-1}(u), z)w$$

where $u \in V_{z, \beta}^+$, $w \in W$. Recall from [DLM1] that $(W, Y^\sigma)$ is still an irreducible module of $V_{z, \beta}^+$, which we denote by $W^\sigma$. As in [DLM1], if $W \cong W^\sigma$, we say $W$ is stable under $\sigma$. Recall from [DN2] that all the irreducible modules of $V_{z, \beta}^+$ are

$$V_{z, \beta}^\pm, V_{z, \beta + \frac{\alpha}{2}}^\pm (1 \leq r \leq 3), V_{z, \beta}^{T_1, \pm}, V_{z, \beta}^{T_2, \pm}$$

with the following tables

| $V_{z, \beta}^+$ | $V_{z, \beta}^-$ | $V_{z, \beta + \frac{1}{2}}^+$ | $V_{z, \beta + \frac{3}{2}}$ | $V_{z, \beta + \frac{1}{2}}^-$ | $V_{z, \beta + \frac{3}{2}}^-$ |
|----------------|----------------|------------------|------------------|------------------|------------------|
| $\omega$      | 0              | $\frac{1}{16}$  | $\frac{1}{4}$   | $\frac{1}{16}$  | 1                |
| $E$           | 0              | 0                | 0               | 0               | 1                |
| $J$           | 0              | $-\frac{3}{64}$ | 0               | $\frac{1}{64}$  | 3                |

| $V_{z, \beta}^{T_1, +}$ | $V_{z, \beta}^{T_1, -}$ | $V_{z, \beta}^{T_2, +}$ | $V_{z, \beta}^{T_2, -}$ |
|------------------------|------------------------|------------------------|------------------------|
| $\omega$              | $1/16$                  | $9/16$                 | $1/16$                 |
| $E$                   | $1/128$                  | $-15/128$              | $-1/128$               |
| $J$                   | $3/128$                  | $-45/128$              | $3/128$                |

It is easy to check that

$$V_{z, \beta}^+ \cong (V_{z, \beta})^\sigma, (V_{z, \beta + \frac{\alpha}{4}})^\sigma \cong V_{z, \beta + \frac{\alpha}{4}}^+,$$

$$(V_{z, \beta}^-)^\sigma \cong V_{z, \beta + \frac{\alpha}{4}}^-, (V_{z, \beta + \frac{\alpha}{4}}^-)^\sigma \cong V_{z, \beta + \frac{\alpha}{4}}^+,$$

$$(V_{z, \beta + \frac{\alpha}{4}})^\sigma \cong V_{z, \beta + \frac{\alpha}{4}}^+, (V_{z, \beta + \frac{\alpha}{4}}^+)^\sigma \cong V_{z, \beta + \frac{\alpha}{4}}^-,$$

Then the lemma follows from [A2], [Y] and Theorem 10.2 in [DLM1].

Next we will prove that there are exactly two inequivalent irreducible $\sigma$-twisted $V_{z, \beta}^-$ modules. We first construct irreducible $\sigma$-twisted $V_{L_2}$-modules. Let $x^i, i = 1, 2, 3$ be defined as in Section 3. Set

$$h' = \frac{1}{3\sqrt{6}}(x^1 + x^2 - x^3),$$

16
$$y^1 = \frac{1}{\sqrt{3}}(x^1 + \frac{-1 + \sqrt{3}i}{2}x^2 + \frac{1 + \sqrt{3}i}{2}x^3),$$
$$y^2 = \frac{1}{\sqrt{3}}(x^1 + \frac{-1 - \sqrt{3}i}{2}x^2 + \frac{1 - \sqrt{3}i}{2}x^3).$$

Then

$$L(n)h' = \delta_n, h'(n)h' = \frac{1}{18} \delta_{n,1} 1, \quad n \in \mathbb{Z},$$

$$h'(0)y^1 = \frac{1}{3}y^1, \quad h'(0)y^2 = -\frac{1}{3}y^2, \quad y^1(0)y^2 = 6h'.$$

It follows that $h'(0)$ acts semisimply on $V_{L_2}$ with rational eigenvalues. So $e^{2\pi ih'(0)}$ is an automorphism of $V_{L_2}$ (see [L2], [DG], etc.). Since

$$e^{2\pi ih'(0)}h' = h', \quad e^{2\pi ih'(0)}y^1 = \frac{-1 + \sqrt{3}i}{2}y^1, \quad e^{2\pi ih'(0)}y^2 = \frac{-1 - \sqrt{3}i}{2}y^2,$$

it is easy to see that

$$e^{2\pi ih'(0)} = \sigma.$$

Let

$$\Delta(h', z) = z^{h'(0)} \exp\left(\sum_{k=1}^{\infty} \frac{h'(k)}{-k} (-z)^{-k}\right),$$

and

$$W^1 = V_{L_2}, \quad W^2 = V_{L_2 + \frac{\alpha}{2}}.$$

Then $W^1$ and $W^2$ are all the irreducible $V_{L_2}$-modules and

$$W^1(0) = \mathbb{C}1, \quad W^2(0) = \mathbb{C}e^{\frac{\alpha}{2}} \bigoplus \mathbb{C}e^{-\frac{\alpha}{2}}.$$

Let

$$w^1 = e^{\frac{\alpha}{2}} + \frac{(\sqrt{3} - 1)(1 + i)}{2} e^{-\frac{\alpha}{2}},$$

$$w^2 = \frac{1}{\sqrt{2}} [(\sqrt{3} - 1)e^{\frac{\alpha}{2}} - (1 + i)e^{-\frac{\alpha}{2}}].$$

Then $W^2 = \mathbb{C}w^1 \bigoplus \mathbb{C}w^2$ and

$$h'(0)w^1 = \frac{1}{6} w^1, \quad h'(0)w^2 = -\frac{1}{6} w^2,$$

$$y^1(0)w^1 = 0, \quad y^1(0)w^2 = w^1, \quad y^2(0)w^1 = w^2.$$

From [L2], we have the following lemma.

**Lemma 5.2.** $(W^{i,T}, Y_\sigma(\cdot, z)) = (W^i, Y(\Delta(h', z)\cdot, z))$ are irreducible $\sigma$-twisted modules of $V_{L_2}, i = 1, 2.$
Direct calculation yields that
\[
\Delta(h', z)L(-2)1 = L(-2)1 + z^{-1}h'(-1)1 + \frac{1}{36}z^{-2}1,
\]
(5.1)
\[
Y_\sigma(h', z) = Y(h' + \frac{1}{18}z^{-1}, z),
\]
(5.2)
\[
Y_\sigma(y^1, z) = z^{\frac{1}{3}}Y(y^1, z),
\]
(5.3)
\[
Y_\sigma(y^2, z) = z^{-\frac{1}{3}}Y(y^2, z).
\]
(5.4)
To distinguish the components of \(Y(u, z)\) from those of \(Y_\sigma(u, z)\) we consider the following expansions
\[
Y_\sigma(u, z) = \sum_{n \in \mathbb{Z}^+} u_n z^{-n-1}, \quad Y(u, z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1},
\]
where \(u \in V_{L_2}\) such that \(\sigma(u) = e^{-\frac{2\pi i}{3} u}\). By (5.2)-(5.4) and direct calculation, we have the following lemma.

**Lemma 5.3.** Write \(W_{i,T_1} = \bigoplus_{n \in \frac{1}{3} \mathbb{Z}^+} W_{i,T_1}(n)\) as admissible \(\sigma\)-twisted module. Then
\[
W_{1,T_1}(0) = \mathbb{C}1, \quad W_{1,T_1}\left(\frac{1}{3}\right) = \mathbb{C}y_1^{\frac{1}{3}}1 = 0,
\]
\[
W_{1,T_1}\left(\frac{2}{3}\right) = \mathbb{C}y_2^{\frac{2}{3}}1 = \mathbb{C}y_2, \quad W_{1,T_1}\left(\frac{4}{3}\right) = \mathbb{C}y_2^{\frac{4}{3}}1 = \mathbb{C}y_2^1,
\]
\[
W_{2,T_1}(0) = \mathbb{C}w^2, \quad W_{2,T_1}\left(\frac{1}{3}\right) = \mathbb{C}y_2^{\frac{1}{3}}w^2 = \mathbb{C}w^1,
\]
\[
W_{2,T_1}\left(\frac{2}{3}\right) = \mathbb{C}y_2^{\frac{2}{3}}w^2 = 0, \quad W_{2,T_1}\left(\frac{5}{3}\right) = \mathbb{C}y_2^{\frac{5}{3}}w^2 = \mathbb{C}y_2^2(-2)w^2,
\]
\[
L(0)|_{W_{1,T_1}(0)} = \frac{1}{36}id, \quad L(0)|_{W_{2,T_1}(0)} = \frac{1}{9}id.
\]
We have the following result.

**Theorem 5.4.** \(W_{1,T_1}\) and \(W_{2,T_1}\) are the only two irreducible \(\sigma\)-twisted modules of \(V_{Z,\beta}^+\).

**Proof:** By Lemma 5.3 \(W_{1,T_1}\) and \(W_{2,T_1}\) are inequivalent \(\sigma\)-twisted modules of \(V_{Z,\beta}^+\). Note that \(W_{1,T_1}\) and \(W_{2,T_1}\) have irreducible quotients which are \(\sigma\)-twisted modules of \(V_{Z,\beta}^+\) with lowest weights \(\frac{1}{36}\) and \(\frac{1}{9}\), respectively. If \(W_{i,T_1}\) is not irreducible for some \(i\), then the lowest weight \(\lambda\) of the maximal proper submodule is different from \(\frac{1}{36}\) and \(\frac{1}{9}\). By [Y], \(V_{Z,\beta}^+\) is \(C_2\)-cofinite. It follows from [DLM1] that \(V_{Z,\beta}^+\) has an irreducible \(\sigma\)-twisted module with lowest weight \(\lambda\). This means that there are at least three inequivalent irreducible \(\sigma\)-twisted modules of \(V_{Z,\beta}^+\), which contradicts Lemma 5.1. So \(W_{1,T_1}\) and \(W_{2,T_1}\) are irreducible inequivalent \(\sigma\)-twisted \(V_{Z,\beta}^+\)-modules. Then the theorem follows from Lemma 5.1. \(\Box\)
Note that 
\[ \sigma^2 = \sigma^{-1} = e^{2\pi i (-h'(0))}, \]
and 
\[ e^{2\pi i (-h'(0))}(-h') = -h', \quad e^{2\pi i (-h'(0))} y^1 = \frac{-1 - \sqrt{3}i}{2} y^1, \quad e^{2\pi i (-h'(0))} y^2 = \frac{-1 + \sqrt{3}i}{2} y^2. \]
So we similarly have

**Lemma 5.5.** \((W_{i,T_2}, Y_{\sigma^{-1}}(\cdot, z)) = (W_i, Y(\Delta(-h', z), z))\) are irreducible \(\sigma^{-1}\)-twisted modules of \(V_{L_2}, i = 1, 2\).

It is easy to see that \(\Delta(-h', z)L(-2)1 = L(-2)1 - z^{-1}h'(-1)1 + \frac{1}{36} z^{-2}1, \)
\[ \tag{5.5} \]
\(Y_{\sigma^{-1}}(-h', z) = Y(-h' + \frac{1}{18} z^{-1}, z), \)
\[ \tag{5.6} \]
\(Y_{\sigma^{-1}}(y^1, z) = z^{-\frac{4}{3}} Y(y^1, z), \)
\[ \tag{5.7} \]
\(Y_{\sigma^{-1}}(y^2, z) = z^{\frac{4}{3}} Y(y^2, z). \)
\[ \tag{5.8} \]

By (5.5)-(5.8), we have
\[ W^{1,T_2}(0) = \mathbb{C}1, \quad W^{1,T_2}(\frac{1}{3}) = \mathbb{C}y_{\frac{2}{3}} 1 = 0, \]
\[ W^{1,T_2}(\frac{2}{3}) = \mathbb{C}y_{\frac{2}{3}} 1 = \mathbb{C}y^1, \quad W^{1,T_2}(\frac{4}{3}) = \mathbb{C}y_{\frac{2}{3}} 1 = \mathbb{C}y^2, \]
\[ W^{2,T_2}(0) = \mathbb{C}w^1, \quad W^{2,T_2}(\frac{1}{3}) = \mathbb{C}y_{\frac{2}{3}} w^1 = \mathbb{C}w^2, \]
\[ W^{2,T_2}(\frac{2}{3}) = \mathbb{C}y_{\frac{2}{3}} w^1 = 0, \quad W^{2,T_2}(\frac{5}{3}) = \mathbb{C}y_{\frac{2}{3}} w^1 = \mathbb{C}y^1(-2) w^1, \]
\[ L(0)|_{W^{1,x_2(0)} = \frac{1}{36} id, \quad L(0)|_{W^{2,x_2(0)} = \frac{1}{9} id.} \]

Similar to Theorem 5.4, we have

**Theorem 5.6.** \(W^{1,T_2}\) and \(W^{2,T_2}\) are the only two irreducible \(\sigma^2\)-twisted modules of \(V_{Z,5}^+.\)

We finally classify all the irreducible modules of \(V_{L_2}^{A_4}\). Recall that \((V_{Z,5}^+)^{(\sigma)} = V_{L_2}^{A_4}\). We prove, in particular, that any irreducible \((V_{Z,5}^+)^{(\sigma)}\)-module is contained in some irreducible \(\sigma^i\)-twisted \(V_{Z,5}^+\)-module, \(i = 0, 1, 2\).

Let \(X^1\) and \(X^2\) be defined as in (3.1). By Lemma 3.3, \(X^i\) generates an irreducible \((V_{Z,5}^+)^{(\sigma)}\)-module with lowest weight 4, denoted by \((V_{Z,5}^+)^{i}, i = 1, 2\).
Note that $W^{i,T_1}$, $W^{i,T_2}$, $i = 1, 2$ can also be regarded as $(V^+_{Z\beta})^{(\sigma)}$-modules. Set

$$
\begin{align*}
& w^{1,T_1,1} = 1 \in W^{1,T_1}(0), \quad w^{1,T_1,2} = y^2 \in W^{1,T_1}(2/3), \quad w^{1,T_1,3} = y^1 \in W^{1,T_1}(4/3), \\
& w^{2,T_1,1} = w^2 \in W^{2,T_1}(0), \quad w^{2,T_1,2} = w^1 \in W^{2,T_1}(1/3), \quad w^{2,T_1,3} = y^2(-2)w^2 \in W^{2,T_1}(5/3), \\
& w^{1,T_2,1} = 1 \in W^{1,T_2}(0), \quad w^{1,T_2,2} = y^1 \in W^{1,T_2}(2/3), \quad w^{1,T_2,3} = y^2 \in W^{1,T_2}(4/3), \\
& w^{2,T_2,1} = w^1 \in W^{2,T_2}(0), \quad w^{2,T_2,2} = w^2 \in W^{2,T_2}(1/3), \quad w^{2,T_2,3} = y^1(-2)w^1 \in W^{2,T_2}(5/3).
\end{align*}
$$

Then we have the following lemma.

**Lemma 5.7.** Let $W^{i,T,j}$ be the $(V^+_{Z\beta})^{(\sigma)}$-module generated by $w^{i,T,j}$, where $i, j = 1, 2$, $k = 1, 2, 3$. Then $W^{i,T,j}$, $i, j = 1, 2, k = 1, 2, 3$ are irreducible $(V^+_{Z\beta})^{(\sigma)}$-modules such that

$$
\begin{align*}
& L(0)w^{1,T_1,1} = \frac{1}{36} w^{1,T_1,1}, \quad L(0)w^{1,T_2,1} = \frac{1}{36} w^{1,T_2,1}, \\
& L(0)w^{1,T_1,2} = \frac{25}{36} w^{1,T_1,2}, \quad L(0)w^{1,T_2,2} = \frac{25}{36} w^{1,T_2,2}, \\
& L(0)w^{1,T_1,3} = \frac{49}{36} w^{1,T_1,3}, \quad L(0)w^{1,T_2,3} = \frac{49}{36} w^{1,T_2,3}, \\
& L(0)w^{2,T_1,1} = \frac{1}{9} w^{2,T_1,1}, \quad L(0)w^{2,T_2,1} = \frac{1}{9} w^{2,T_2,1}, \\
& L(0)w^{2,T_1,2} = \frac{4}{9} w^{2,T_1,2}, \quad L(0)w^{2,T_2,2} = \frac{4}{9} w^{2,T_2,2}, \\
& L(0)w^{2,T_1,3} = \frac{16}{9} w^{2,T_1,3}, \quad L(0)w^{2,T_2,3} = \frac{16}{9} w^{2,T_2,3}.
\end{align*}
$$

**Proof:** The lemma follows from a general result: Let $U$ be a vertex operator algebra with an automorphism $g$ of order $T$. Let $M = \sum_{n \in Z_+} M(n)$ be an irreducible $g$-twisted admissible $U$-module. Then $M^i = \bigoplus_{n \in Z_+} M(n)$ is an irreducible $V^g$-module for $i = 0, ..., T-1$ (cf. [DMI]).

We have the following lemma from [DMI].

**Lemma 5.8.** As an $(V^+_{Z\beta})^{(\sigma)}$-module,

$$
V_{Z+4\beta} = (V^+_{Z\beta})^0 \oplus (V^+_{Z\beta})^{1} \oplus (V^+_{Z\beta})^{2}
$$

such that $(V^+_{Z\beta})^0$, $(V^+_{Z\beta})^{1}$ and $(V^+_{Z\beta})^{2}$ are irreducible $(V^+_{Z\beta})^{(\sigma)}$-modules generated by $e^{eta/4} + e^{-\beta/4}$, $h(-2) \otimes (e^{eta/4} + e^{-\beta/4}) - \sqrt{2} h(-1)^2 \otimes (e^{eta/4} + e^{-\beta/4}) + a(e^{eta/4} + e^{-\beta/4})$ and $h(-2) \otimes (e^{eta/4} + e^{-\beta/4}) - \sqrt{2} h(-1)^2 \otimes (e^{eta/4} + e^{-\beta/4}) - a(e^{eta/4} + e^{-\beta/4})$ for some $0 \neq a \in \mathbb{C}$ with weights $\frac{1}{4}$, $\frac{9}{4}$ and $\frac{9}{4}$ respectively.
We are now in a position to state the main result of this section. Recall that \((V^+_{\mathbb{Z}_\beta})^0 = (V^+_{\mathbb{Z}_\beta})^{(\sigma)}\).

**Theorem 5.9.** There are exactly 21 irreducible modules of \((V^+_{\mathbb{Z}_\beta})^{(\sigma)}\). We give them by the following tables 1-4.

| \(\omega\) | \((V^+_{\mathbb{Z}_\beta})^0\) | \((V^+_{\mathbb{Z}_\beta})^1\) | \((V^+_{\mathbb{Z}_\beta})^2\) | \(V^+_{\mathbb{Z}_\beta+\frac{1}{2}\beta}\) | \(V^+_{\mathbb{Z}_\beta+\frac{3}{2}\beta}\) |
|---|---|---|---|---|---|
| \(\frac{1}{3}\) | \(\frac{1}{4}\) | \(\frac{1}{4}\) | \(\frac{1}{4}\) | \(\frac{1}{4}\) |

| \(\omega\) | \(W_{1,1}\) | \(W_{1,2}\) | \(W_{1,3}\) | \(W_{2,1}\) | \(W_{2,2}\) | \(W_{2,3}\) |
|---|---|---|---|---|---|---|
| \(\frac{1}{3}\) | \(\frac{1}{3}\) | \(\frac{1}{3}\) | \(\frac{1}{3}\) | \(\frac{1}{3}\) | \(\frac{1}{3}\) |

| \(\omega\) | \((V^+_{\mathbb{Z}_\beta+\frac{1}{2}\beta})^0\) | \((V^+_{\mathbb{Z}_\beta+\frac{3}{2}\beta})^1\) | \((V^+_{\mathbb{Z}_\beta+\frac{3}{2}\beta})^2\) |
|---|---|---|---|
| \(\frac{1}{3}\) | \(\frac{1}{3}\) | \(\frac{1}{3}\) |

**Proof:** It follows from the proof of Lemma \[5.1\] and Theorem 6.1 in [DM1] that \(V^-_{\mathbb{Z}_\beta}\), \(V^+_{\mathbb{Z}_\beta+\frac{1}{2}\beta}\) and \(V^+_{\mathbb{Z}_\beta+\frac{3}{2}\beta}\) are irreducible \((V^+_{\mathbb{Z}_\beta})^{(\sigma)}\)-modules and as \((V^+_{\mathbb{Z}_\beta})^{(\sigma)}\)-modules,

\[
V^-_{\mathbb{Z}_\beta} \cong V^-_{\mathbb{Z}_\beta+\frac{1}{2}\beta} \cong V^+_{\mathbb{Z}_\beta+\frac{1}{2}\beta} \cong V^+_{\mathbb{Z}_\beta+\frac{3}{2}\beta} \cong V^+_{\mathbb{Z}_\beta+\frac{3}{2}\beta} \cong V^+_{\mathbb{Z}_\beta}. \ 
\]

Then the theorem follows from Lemma \[3.3\], Lemma \[5.8\], Theorems \[5.4\] and \[5.6\], Theorem \[4.6\] and Theorem A in [M1].

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