Examples of Calabi-Yau threefolds with small Hodge numbers

A O Shishanin

1 Bauman Moscow State Technical University, Moscow, Russia;
2 Lomonosov Moscow State University, Moscow, Russia

E-mail: shishandr@rambler.ru

Abstract. We observe some suitable examples of Calabi-Yau threefolds for heterotic superstring compactifications. It is reasonable to seek CY threefolds with Euler characteristic equals \( \pm 6 \) because of generation’s number. Hosotani mechanism for violations of the gauge group by the Wilson loops requires such CY space has a non-trivial fundamental group. These spaces can be obtained by factoring the complete intersection Calabi-Yau spaces by the free action of some discrete group. Also we shortly discuss cases when discrete groups act with fixed point sets.

1. Introduction

Calabi-Yau (CY) spaces and structures associated with them are one of the most important objects of study in modern mathematical physics. CY spaces appear naturally in areas such as two-dimensional supersymmetric field theories, conformal field theory, topological strings, and mirror symmetry [1]. In string theory [2], [3], [4], [5] CY spaces arise as objects for obtaining reasonable compactifications.

Candelas et al [6] had studied the compactification of \( E_8 \times E_8 \) heterotic string on \( M \times X \). Here \( M \) is 4d Minkowski space and \( X \) is some six-dimensional manifold. They wondered when such a compactification leads to 4d \( N = 1 \) supersymmetric field theory. They demonstrated [6] that supersymmetry requires the existence of a covariant constant spinor on \( X \). The presence of such spinors gives the \( SU(3) \) holonomy group on \( X \). For any n-dimensional Riemann manifold holonomy group is subgroup of \( SO(2n) \). If this manifold is Kähler then holonomy group is subgroup of \( U(n) \). For Calabi-Yau manifolds maximal holonomy group reduces to \( SU(n) \).

The embedding of the spin connection in the heterotic string leads to the violation of the gauge group from \( E_8 \times E_8 \) to \( E_6 \times E_6 \). Adjoint representation of \( E_8 \) with dimension 248 is reducible to the group \( SU(3) \times E_6 \) as

\[
248 = (8,1) + (1,178) + (3,27) + (\bar{3}, \bar{27}).
\]

The generation number of elementary particles equals difference between the number for generation of particles (3,27) and the number for anti-generation of particles (\( \bar{3}, \bar{27} \)). These numbers (3,27) and (\( \bar{3}, \bar{27} \)) equal \( h^{2,1} \) and \( h^{1,1} \) respectively. Thus it turns out that the number for generations of elementary particles is half of Euler characteristic of \( \chi \) \( |h^{2,1} - h^{1,1}| = |\chi|/2 \). Then quest of CY threefolds with \( \chi = \pm 6 \) represents significant interest because of the generation number is expected to be no more than 3. The simplest and most important approach to break \( E_8 \) gauge group is Hosotani mechanism [7], [8]. The Hosotani mechanism uses Wilsonian loops. It requires the fundamental group \( \pi(X) \) of CY threefold \( X \) to be non-trivial.
2. General information about Calabi-Yau spaces

There are many definitions of Calabi-Yau spaces. It is mathematically correct to treat of Calabi-Yau spaces as Kähler manifolds with a trivial canonical class $K = 0$. This means that the holomorphic form of the highest degree $\Omega$ has no poles and zeros anywhere. Physically Calabi-Yau spaces are Ricci-flat (zero first Chern class) Kähler-Einstein manifolds.

The low-dimensional examples of Calabi-Yau spaces are torus and famous $K3$-surface. There are many examples of $K3$-surface a quartic in $\mathbb{C}P^3$, the intersection of a quadric and a cubic in $\mathbb{C}P^4$, the intersection of three quadrics in $\mathbb{C}P^5$, a Kummer surface and so on. One important example of CY manifolds is elliptic fibrations. $K3$-surface can be obtain, for instance, as elliptic fibration on weighted projective space

$$ y^2 = x^3 + f(z)x + g(z). $$

If $f(z)$ is polynomial degree 8, $g(z)$ is polynomial degree 12, $x$ has weight 4 and $y$ has weight 6 then this equation defines smooth $K3$-surface. Also we can obtain some $K3$-fibration Calabi-Yau threefold if polynomials $f(z,w)$ and $g(z,w)$ have coordinates $z$ and $w$ on $K3$-surface.

Let recall that Hodge numbers are dimensions of Dolbeault cohomology groups

$$ h^{p,q} = \text{dim} H^{p,q}. $$

For complex manifolds, the numbers $h^{p,q}$ consist some table called the Hodge diamond. The Hodge diamond for CY threefold is defined as

\[ \begin{array}{cccccc}
  &  &  &  &  & 1 \\
  &  &  &  & 0 & \\
  &  &  & 1 & h^{1,1} & 1 \\
  &  &  & 0 & h^{1,1} & 0 \\
  &  &  & 0 & 0 & \\
  &  &  & 1 & & \\
\end{array} \]

Hence the Euler characteristic $\chi$ equals $2(h^{1,1} - h^{2,1})$. Numbers $h^{1,1}$ are called Kähler moduli and $h^{2,1}$ are complex moduli. Mirror manifold for CY threefold with Hodge numbers $(h^{1,1}, h^{2,1})$ has Hodge numbers $(h^{2,1}, h^{1,1})$.

The most known example of CY threefold is quintic in $\mathbb{C}P^4$. Let $(x_0, x_1, x_2, x_3, x_4)$ homogeneous coordinates in $\mathbb{C}P^4$ then the quintic equation has form

$$ x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0 $$

with $h^{1,1} = 1, h^{2,1} = 101$ and $\chi = -200$.

There are few examples of complete intersections CY (CICY) threefolds in $\mathbb{C}P^N$ for some $N$. $\mathbb{C}P_N[d_1, …, d_k]_X$ denotes complete intersection of $k$ homogeneous polynomials with $d_i$ degrees in $\mathbb{C}P^N$. Then the total Chern class is given by adjunction formula

$$ c = \frac{(1 + J)^{N+1}}{\prod_{i=1}^{k}(1 + d_i J)} $$

(1)

where $J$ is some 2-form obtained by normalizing the Kähler form. For CY threefolds, the condition zero first Chern class $c_1 = 0$ gives expression

$$ \sum_{i=1}^{k} d_i = k + 1. $$

There are five nonlinear examples of such manifolds the quintic $\mathbb{C}P_4[5]_{-200}$ and $\mathbb{C}P_5[2,4]_{-176}$, $\mathbb{C}P_3[3,3]_{-144}$, $\mathbb{C}P_6[2,2,3]_{-144}$, $\mathbb{C}P_7[2,2,2,2]_{-128}$.

Hübsch notation [9] for CICY threefold is defined $N$ polynomials in $\mathbb{C}P^{i_1} \times \mathbb{C}P^{i_2} \times \cdots \times \mathbb{C}P^{i_n}$
Each column of this table corresponds to a polynomial in $\mathbb{C}P^1 \times \mathbb{C}P^2 \times \cdots \times \mathbb{C}P^n$.

The easiest way to lower the Hodge numbers is by factoring the manifold $X_0$ by some discrete group $\Gamma$.

If $\Gamma$ has free action on $X$ (without fixed points) then the Euler characteristic of the quotient manifold $X = X_0 / \Gamma$ equals

$$
\chi(X_0) = \frac{\chi(X_0)}{|\Gamma|},
$$

where $|\Gamma|$ is the order of $\Gamma$. This equality is true for spaces of odd dimension, as in our case. It can be obtained using the Atiyah-Bott fixed point formula. It is necessary to check the transformation of the form of the highest degree $\Omega$ [10]. Number $h^{0,1}$ is usually easy to find. Then the number $h^{1,1}$ of $X$ is given by

$$
h^{1,1} = h^{2,1} + \frac{\chi(X_0)}{2|\Gamma|}.
$$

For example, the quintic is factorized by $\mathbb{Z}_5 \times \mathbb{Z}_5$ and has Euler characteristic $-8$.

Such way the first example of CY threefold with Euler characteristics $|\chi| = 6$ was constructed by S.T. Yau in 1985 [11]. This CY threefold with $\chi = -6$ is called Tian-Yau space. Let us denote quotient complete intersection Calabi-Yau manifold as QCICY.

3. Tian-Yau space and its twins

There exist complete intersection $K_0$ in $\mathbb{C}P^3 \times \mathbb{C}P^3$ with table

$$
p^2 \begin{bmatrix} 1 & 3 & 0 \end{bmatrix}^{14,23}, \quad p^3 \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}^{18}.
$$

Let us denote $(x_0, x_1, x_2, x_3)$ four homogeneous coordinates of the first $\mathbb{C}P^3$ and for the second $\mathbb{C}P^3$ homogeneous coordinates $(y_0, y_1, y_2, y_3)$. $K_0$ is given by polynomials

$$
x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0,
$$

$$
y_0^3 + y_1^3 + y_2^3 + y_3^3 = 0,
$$

$$
x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 = 0.
$$

There is a group $\mathbb{Z}_3$ with free action. This action is given by $g \in \mathbb{Z}_3$

$$
g: (x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3) \rightarrow (x_0, \alpha^2 x_1, \alpha x_2, \alpha x_3, y_0, \alpha y_1, \alpha^2 y_2, \alpha^2 y_3),
$$

where $\alpha^3 = 1, \alpha \neq 1$. The Tian-Yau space $K_1 = K_0 / \mathbb{Z}_3$ has Hodge numbers $(6,9)$ and Hodge diamond

$$
\begin{array}{cccc}
1 & & & \\
0 & 0 & & \\
0 & 6 & 0 & \\
1 & 9 & 9 & 1 \\
0 & 6 & 0 & \\
0 & 0 & & \\
1 & & & \\
\end{array}
$$

An explanation of obtaining Hodge numbers for Tian-Yau space can be found in [12]. The fundamental group of Tian-Yau space is $\mathbb{Z}_3$.

Schimmrigk had found another example of factorized CICY with same Hodge numbers and fundamental group [13]. Consider complete intersection in $\mathbb{C}P^2 \times \mathbb{C}P^3$ with table

$$
p^2 \begin{bmatrix} 3 & 0 \end{bmatrix}^{8,35}, \quad p^3 \begin{bmatrix} 1 & 3 \end{bmatrix}^{54}.
$$
This CICY can be factorized by $A \times B$ group. Both group are isomorphic $\mathbb{Z}_3$. Action of group A is specified by element $g_1$ of order 3

$$g_1: \mathbb{C}P^2 \times \mathbb{C}P^3 \rightarrow \mathbb{C}P^2 \times \mathbb{C}P^3,$$

$$g_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

This action is free. The group B acts only in $\mathbb{C}P^2$

$$g_2: (x_0, x_1, x_2, y_0, y_1, y_2, y_3) \rightarrow (x_0, \alpha x_1, \alpha^2 x_2, y_0, y_1, y_2, y_3).$$

Here there are three fixed point sets which define three invariant tori

\begin{align*}
&(1,0,0) \times \{y_0^2 + y_1^2 + y_2^3 = 0\}, \\
&(0,1,0) \times \{y_0^3 + y_1^3 + y_2^3 = 0\}, \\
&(0,0,1) \times \{y_0^3 + y_1^3 + y_2^3 = 0\}.
\end{align*}

It is possible to resolve singularities without leaving class of CY threefolds [11].

There is other sample of CY threefold with same Hodge diamond (2). Let us consider bicubic $L$

$$L = p^2 \mathbb{Z}_{128} \mathbb{Z}_{162}^2.$$ 

$L$ is factorized by three groups with fixed point acting groups $G_1, G_2$ and free acting group $G_3$. Resolved $L/G_1 \times G_2 \times G_3$ has Hodge diamond and fundamental group as Tian-Yau and Schimmrigk spaces.

4. **More modern examples of QCICY**

Many examples of “three generation” manifolds with $\chi = \pm 6$ was constructed in [15]. Two more samples were built by Braun, Candelas and Davies [16]. CICY is denoted in [16] as $Y_{8,44}$. This manifold is defined by following Hübisch table

\[
\begin{pmatrix}
  p^2 & 1 & 1 & 1 & 0 & 0 \\
p^2 & 0 & 0 & 1 & 1 & 1 \\
p^2 & 1 & 1 & 1 & 0 & 0 \\
p^2 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}_{-72}^{8,44}.
\]

Two groups of order 12 act freely on $Y_{8,44}$. One group is abelian $\mathbb{Z}_{12}$. And other group is non-Abelian dicyclic group $DiC_3$ (semidirect product $\mathbb{Z}_3$ and $\mathbb{Z}_4$). The quotient manifolds for both these groups have Hodge numbers $(h^{1,1}, h^{2,1}) = (1, 4)$. Fundamental groups these manifolds are $\mathbb{Z}_{12}$ and $DiC_3$ respectively.

A huge number of examples of CY threefolds with small Hodge numbers are given in the table article [17]. Many examples with a non-trivial fundamental group provide quotients for complete intersections of four quadrics $\mathbb{C}P_7[2,2,2,2]_{-128}$.

This conifold transition shifts the Hodge numbers as follows

$$\delta(h^{1,1}, h^{2,1}) = (1, -1).$$

The conifold transition does not change the fundamental group. However, there is a hyperconifold transition [21], [22] that can change the fundamental group.

In this text we have given some examples CY threefolds that are interesting for compactifications of heterotic string theory. Mirror symmetry [1] is an actual way to study Calabi-Yau spaces. Euler characteristic for mirror manifold is same. Therefore, from the modern point of view, it is important to explore mirror manifolds with small Hodge numbers.

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