On global models for isolated rotating axisymmetric charged bodies: uniqueness of the exterior field

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Abstract
A relatively recent study by Mars and Senovilla provided us with a uniqueness result for the exterior vacuum gravitational field generated by an isolated distribution of matter in axial rotation in equilibrium in general relativity. The generalization to exterior electrovacuum gravitational fields, to include charged rotating objects, is presented here.

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1. Introduction

A proper understanding of rotating objects in equilibrium within the context of general relativity is fundamental for many astrophysical situations. Very unfortunately, finding global models for rotating objects in general relativity has proven to be extremely difficult, even for axially symmetric configurations in equilibrium. So far, there are no known complete explicit models for a self-gravitating finite body (with non-empty interior) together with its exterior except for spherical, and hence non-rotating, configurations. Worth mentioning is the existence of solutions for rotating disc shells of dust [1]. An extensive review on numerical studies of rotating relativistic stars can be found in [2].

This work focuses on the theoretical point of view of the construction of global models for finite objects by means of the matching of spacetimes: the whole configuration is composed of two regions, one region of spacetime \((\mathcal{V}^I, g^I)\) describing the interior (I) of the body and another \((\mathcal{V}^E, g^E)\) describing the exterior field (E), matched across a hypersurface \(\Sigma\). This matching hypersurface is then a common boundary of the two regions and corresponds to the limiting surface of the body at all times. The two regions can then be treated independently, taking into account that the two problems will have to satisfy compatible boundary conditions on \(\Sigma\) imposed by the matching conditions. To model the equilibrium state of the rotating configuration, the whole matched spacetime, and hence each of the interior and exterior regions, is taken to be strictly stationary. In addition, it has usually been assumed that the model is axially symmetric. To account for the isolation of the body, the exterior region is required to be asymptotically flat.
In previous works on models of isolated rotating bodies in equilibrium, only vacuum exteriors have been taken into account. Nevertheless, compact objects in astrophysics have also been considered many times as sources of electromagnetic fields (see \[3\] and \[4\] and references therein). In fact, the existence of a net charge, no matter how negligible, has drastic consequences for the global spacetime structure, in principle. The aim of the present work is to revisit the exterior problem, and generalize the uniqueness results found for the vacuum case in \[5\] to include stationary and axisymmetric electromagnetic fields without sources (electrovacuum fields). The main result deals with the uniqueness of the exterior electrovacuum field generated by an isolated distribution of charged matter in axial rotation and in equilibrium. It is worth noticing that spherically symmetric global models for stellar collapse to include charged radiating stars and voids have already been considered by several authors during the past three decades (see \[6\] and \[4\] and references therein).

The exterior stationary and axisymmetric Einstein–Maxwell electrovacuum problem consists of an elliptic PDE system, known as the Ernst equations, for a couple of complex functions \[7, 8\]: the electromagnetic potential \(\Lambda\) and the so-called Ernst potential \(\mathcal{E}\), which carry all the information about the exterior geometry and electromagnetic field. It is known that the matching of a vacuum exterior with a given interior, prescribing the identification of the interior and the exterior across \(\Sigma\) if required, determines: (i) the existence of the matching hypersurface, and if so, its form as seen from the interior, generally, (ii) the matching hypersurface as seen from the exterior and (iii) the values of \(\mathcal{E}\) up to an arbitrary additive imaginary constant, say \(i\omega\), and the normal derivative of \(\mathcal{E}\) there \([5, 9]\). These data constitute a set of boundary conditions of Cauchy type for \(\mathcal{E}\) on \(\Sigma\) that depend on the parameter \(\omega\).

In order to accommodate an electrovacuum exterior, it is necessary to, first, generalize the spacetime matching conditions for a vacuum exterior, also taking into account the matching conditions for the electromagnetic field \([10]\) (see also \[4\]). For an object without surface charges\(^1\), the electromagnetic field has to be continuous across \(\Sigma\), and hence the interior quantities determine the electromagnetic potential \(\Lambda\) up to an additive complex constant \(\lambda\). Here it will be shown how, when considering an electrovacuum exterior, the sets (i) and (ii) remain formally unchanged whereas the rest of the matching conditions constitute a generalized new set (iii) of Cauchy boundary data for \(\mathcal{E}\) on \(\Sigma\) in terms of the interior quantities and involving \(\omega\) and \(\lambda\). Summing up, given the interior, the matching conditions determine Cauchy boundary data for \(\mathcal{E}\) and \(\Lambda\) on \(\Sigma\) up to three degrees of freedom, given by \(\omega\) and \(\lambda\).

Due to the elliptic character of the exterior electrovacuum problem, the Cauchy boundary data, together with the decaying conditions at infinity, constitute an overdetermined set of boundary conditions for the exterior electrovacuum problem. Nevertheless, as shown later, the three parameters encoded in \(\omega, \lambda\) are in principle physically relevant, and therefore, the exterior problem is overdetermined but, \(a\ pri\ o\ ri\), not unique. The proof of the uniqueness of the exterior solution then has to follow two steps. In the first step, it is shown that given the values of \(\mathcal{E}\) and \(\Lambda\) on \(\Sigma\) (Dirichlet boundary data), the exterior field is unique. In the second step, it is shown that the completion to a whole set of Cauchy boundary data of a Dirichlet boundary data set that depends on \(\omega, \lambda\) determines the values of \(\omega\) and \(\lambda\), and hence the Dirichlet data, provided that the solution exists.

The paper is structured as follows. Section 2 is devoted to a brief review concerning the electrovacuum exterior problem. The dependence on the free parameters \(\omega\) and \(\lambda\) of the Cauchy boundary conditions on \(\Sigma\) is determined in section 3. For the sake of brevity in the main text, the full determination of the matching conditions is left to appendix A. Section 4

\(^1\) This assumption is not really necessary since one could simply consider a surface charge and then include that as information that has to be given with the rest of the interior configuration.
deals then with the proof of the uniqueness for the Dirichlet problem. Finally, section 5 is devoted to showing that the full set of Cauchy boundary data determine the values of $\omega$ and $\lambda$ if the solution exists. The statement of the final result is left to the conclusions section.

The units all throughout the paper are chosen so that $G = c = 1$.

2. The exterior problem

For completeness and to fix some notation, this section is devoted to a brief review of the stationary and axisymmetric electrovacuum problem. Let me refer to, e.g., [7, 8] for the details.

Given the electromagnetic field described by the 2-form $F$, it will be convenient to use the self-dual 2-form $\mathcal{F}$ defined as

$$\mathcal{F} \equiv F + i \ast F,$$

where $\ast$ stands for the Hodge dual. The self-dual 2-form $\mathcal{F}$ is then completely determined by an arbitrary non-null vector $\vec{\kappa}$ and the projection

$$f_{\vec{\kappa}}(\vec{\kappa}) = 0,$$

by

$$\mathcal{F} = \frac{1}{N_{\vec{\kappa}}} [f_{\vec{\kappa}} \wedge \kappa + i (f_{\vec{\kappa}} \wedge \kappa)],$$

where we have defined $N_{\vec{\kappa}} \equiv \vec{\kappa} \cdot \vec{\kappa} (\neq 0)$, and $\wedge$ denotes the exterior product (see, e.g., [7]).

The real and imaginary parts of the form $f_{\vec{\kappa}}$ are usually denoted by $E_{\vec{\kappa}}$ and $B_{\vec{\kappa}}$, and for unit timelike $\vec{\kappa}$, they correspond to the electric and magnetic parts of $F$ with respect to $\vec{\kappa}$, respectively, i.e. [7]

$$f_{\vec{\kappa}} = E_{\vec{\kappa}} + i B_{\vec{\kappa}}.$$ 

The Maxwell equations in terms of $\mathcal{F}$ take the form

$$d\mathcal{F} = 4\pi i * j,$$

where $j$ is the electromagnetic current source, and the energy–momentum reads

$$T_{\mu\nu} = \frac{1}{8\pi} \mathcal{F}_{\mu\alpha} \mathcal{F}_{\nu}^{\alpha},$$

(1)

where the bar denotes the complex conjugate.

Let us consider a strictly stationary\footnote{In global terms, we demand that the manifold admits an everywhere timelike Killing vector with complete orbits.} and axisymmetric [7] asymptotically flat, causal and simply connected $C^3$ spacetime with connected boundary $(\mathcal{W}_E, g^E, \Sigma^E)$, containing an electrovacuum field $\mathcal{F}$ invariant under the same group of isometries, i.e. the stationary and axisymmetric pair $(g^E, \mathcal{F})$ is a solution of the Einstein–Maxwell equations without sources ($j = 0$). The intrinsically defined axial Killing vector field \cite{11}(see also [7]) will be denoted by $\vec{\eta}$, and the axis of symmetry by $W_{E}^2$, which is non-empty by assumption. $\vec{\xi}$ will denote an everywhere timelike Killing vector field which together with $\vec{\eta}$ generates the necessarily Abelian [12](see also [7]) $G_2$ group of isometries that act on timelike surfaces $T_2$. It is also well known that the $G_2$ on $T_2$ group of isometries in a stationary and axisymmetric electrovacuum spacetime must act orthogonally transitively (see, e.g., [7]). These properties, together with the vanishing of the trace of the energy–momentum tensor for the electromagnetic field, imply the existence of a coordinate system $\{t, \phi, \rho, z\}$ with ranges within $\rho > 0$, $t, z \in \mathbb{R}$, $0 \leq \phi < 2\pi$, the so-called Weyl canonical coordinates, in which the line element for $\mathcal{W}_E/W_{E}^2$ reads locally [7]

$$ds^2 = -e^{2U} (dt + A d\phi)^2 + e^{-2U} [e^{2k}(d\rho^2 + dz^2) + \rho^2 d\phi^2].$$

(2)
where $U$, $A$, $k$ are functions of $\rho$ and $z$, the axial Killing vector is given by $\vec{\eta} = \partial_{\rho}$, the axis of symmetry $W_2^E$ is located at the limit $\rho \to 0$, and $\vec{\xi} = \partial_t$. Moreover, the coordinate $t$ can be chosen to measure proper time of an observer at infinity. In that case, the Killing vector $\vec{\xi}$ is intrinsically defined by being unity at infinity [7]. Within this setting, the coordinate freedom in (2) consists only of trivial constant shifts of $t$, $\phi$ and $z$.

In order to have our $G_2$ group defining complete orbits, the Killing vectors $\{\vec{\xi}, \vec{\eta}\}$ are required to be tangent to $\Sigma^E$. In other words, we demand that $\Sigma^E$ preserves the stationarity and axial symmetries [13]. As a consequence, $\Sigma^E$ is everywhere timelike by assumption. Furthermore, since $\mathcal{V}^E$ is assumed simply connected, connectedness of $\Sigma^E$ implies that the slices of constant $t$ of $\Sigma^E$ are homeomorphic to 2-spheres.

The existence of the Weyl canonical coordinates globally on $\mathcal{V}^E \setminus W_2^E$ had been ensured in relation to the black hole uniqueness theorems [22] (see also [15]), where $\Sigma^E$ lies on the horizon, so that $\rho|_{\Sigma^E} = 0$. In the present case, though, the boundary $\Sigma^E$ lies on the surface of the object, and therefore those results cannot be used in principle. Nevertheless, the good behaviour of the function $\rho$ can also be ensured when the boundary $\Sigma^E$ at constant $t$ is an axially symmetric surface homeomorphic to a 2-sphere, as it is the case here. Global Weyl coordinates had been used in previous works on global models of finite objects, but without proof [5, 9]. Due to its length, and so as not to overwhelm this paper, it has been preferable to leave the proof on the existence of the global Weyl coordinates in the present scenario to a separate work [16]. Taking global Weyl coordinates (2) for $(\mathcal{V}^E, g^E, \Sigma^E)$, the axially symmetric hypersurface $\Sigma^E$ is thus given parametrically by [5, 9, 16]

$$\Sigma^E : \{t = \tau, \phi = \varphi, \rho = \rho(\mu), z = z(\mu)\}. \quad (3)$$

Denoting by $\vec{\kappa}$ either of the two Killing vectors $\vec{\xi}$ or $\vec{\eta}$, the invariance of the electromagnetic field $F$ with respect to $\vec{\kappa}$, equivalent to $\mathcal{L}_\vec{\kappa} F = 0$, implies the existence of a complex scalar potential $\Lambda_{\vec{\kappa}}$ such that (see, e.g., [7, 8])

$$d\Lambda_{\vec{\kappa}} = f_{\vec{\kappa}}$$

for either $\vec{\kappa}$. Moreover, for stationary and axisymmetric electrovacuum fields, $f_{\vec{\kappa}}$ must be orthogonal to the orbits, and thus $\Lambda_{\vec{\kappa}} = \Lambda_{\vec{\kappa}}(\rho, z)$ [7, 8].

The Einstein–Maxwell equations free of sources in the simply connected manifold $\mathcal{V}^E$ imply, in turn, the existence of a real scalar $\Omega_{\vec{\kappa}}$, the so-called twist potential, such that [17] (see, e.g., [7])

$$d\Omega_{\vec{\kappa}} = w_{\vec{\kappa}} - i(\Lambda_{\vec{\kappa}} d\bar{\Lambda}_{\vec{\kappa}} - \bar{\Lambda}_{\vec{\kappa}} d\Lambda_{\vec{\kappa}}), \quad (4)$$

where $w_{\vec{\kappa}} \equiv *(\kappa \land d\kappa)$ is the twist 1-form of $\vec{\kappa}$. Recalling the definition $N_{\vec{\kappa}} \equiv (\vec{\kappa} \cdot \vec{\kappa})$, the so-called Ernst potential $E_{\vec{\kappa}}$ (with respect to $\vec{\kappa}$) reads [18, 7, 14]

$$E_{\vec{\kappa}} = -N_{\vec{\kappa}} - \Lambda_{\vec{\kappa}} \bar{\Lambda}_{\vec{\kappa}} + i\Omega_{\vec{\kappa}}.$$  

The Einstein–Maxwell equations now reduce to the elliptic system of equations for the potentials $\Lambda_{\vec{\kappa}}$ and $E_{\vec{\kappa}}$, known as the Ernst equations, plus a quadrature for the remaining function in the metric $(k(\rho, z)$ in (2)) in terms of the potentials (see appendix A for the explicit expressions). Dropping the $\vec{\kappa}$ subindices, the Ernst equations read [7, 8]

$$N\delta^{ij}\partial_i(\rho \partial_j E) + \rho \delta^{ij}\partial_i E \partial_j E + 2\Lambda \partial_j \Lambda = 0, \quad (5)$$

$$N\delta^{ij}\partial_i(\rho \partial_j \Lambda) + \rho \delta^{ij}\partial_i \Lambda = 0, \quad (6)$$

where $N = -(E + \bar{E} + 2\Lambda) / 2$, indices $i, j$ correspond to $\{\rho, z\}$, and $\delta_{ij}$ represents the $2 \times 2$ identity. The solutions of the Ernst equations for $\vec{\kappa} = \vec{\xi}$ and $\vec{\kappa} = \vec{\eta}$, that is, $(E_{\vec{\xi}}, \Lambda_{\vec{\xi}})$ and $(E_{\vec{\eta}}, \Lambda_{\vec{\eta}})$, are called conjugated to one another and are bi-uniquely related (see [8]).
Given either \((E,\Lambda_E)\) solution of the Ernst equations, all the information on the exterior electrovacuum solution is recovered. In particular, the metric for \(\left(V^E, g^E\right)\) in the form (2) is directly obtained by taking \(\kappa = \xi(= \partial_t)\): 

\[ N_\xi = -e^{2U}, \]  

(7)

and \(A\) is determined, up to a constant, by the quadrature

\[ dA = -\frac{\rho}{N_\xi^2} \star w_\xi, \]  

(8)

where \(\star\) is used to denote the particular Hodge dual on the \(\{\rho, z\}\) 2-plane, so that \(\star dz = -d\rho\) \(\star d\rho = dz\) and \(w_\xi\) is in turn obtained from taking the imaginary part of \(dE_\xi\) and using (4). The function \(k\) is then fixed, up to an additive constant by the aforementioned quadratures involving \(E_\xi\) and \(\Lambda_\xi\) [7].

The boundary conditions for the exterior problem consist of the conditions on the boundary associated with the surface of the body plus decaying conditions at infinity. Asymptotic flatness on \(\left(V^E, g^E\right)\) determines the behaviour of the Ernst potential at infinity [19] (see also [20, 8] and references therein) and, in addition, requires asymptotic conditions on the electromagnetic field [19]. The conditions at infinity read

\[ U_\xi = -M r^{-1} + O(r^{-2}), \quad \Omega_\xi = -2J r^{-3} + O(r^{-3}), \]  

(9)

\[ \Lambda_\xi = -Q r^{-1} + O(r^{-2}), \]  

(10)

where \(r = \sqrt{\rho^2 + z^2}\), for some constants \(M, J\) and \(Q\).

The next section is devoted to the boundary conditions on the hypersurface \(\Sigma\) that result from the matching conditions with a given stationary and axisymmetric interior.

3. The matching conditions: boundary conditions on \(\Sigma\)

Let us consider a given stationary and axisymmetric spacetime with boundary \(\left(V^I, g^I, \Sigma^I\right)\) containing an electromagnetic field \(F^I\), and describing the interior region matched across the stationary and axisymmetric (and thus timelike) hypersurface \(\Sigma(= \Sigma^I = \Sigma^E)\) to the stationary and axisymmetric asymptotically flat electrovacuum spacetime \(\left(V^E, g^E, \Sigma^E\right)\). Since we do not consider any superficial charge on the surface of the body, the usual junction conditions of electrodynamics [10] imply that the electromagnetic field has to be continuous across the timelike hypersurface \(\Sigma\), and thus (see, e.g., [4])

\[ F^I|\Sigma = F|\Sigma. \]

This clearly leads to a relation for the gradient of the electromagnetic potential at the exterior with respect to any vector field \(\kappa, \Lambda_\xi\), and the interior quantities, given by

\[ \Lambda_\xi|\Sigma = e_3^a F_{a\mu} \kappa^\mu|\Sigma, \quad \overline{n}(\Lambda_\xi)|\Sigma = n^a F_{a\mu} \kappa^\mu|\Sigma, \]  

(11)

where the dot denotes differentiation with respect to \(\mu, \overline{\kappa}_3\) denotes the corresponding vector field, tangent to \(\Sigma\), and \(\overline{n}\) is normal (not necessarily unit) to \(\Sigma\). It must be stressed that the absence of a surface charge is assumed here only for simplicity. One can always consider a surface charge and simply add it to the information required from the interior region. In any case, the data \(F|\Sigma\) would then be fixed by the electromagnetic matching conditions (see [4]). In short, given the interior electromagnetic field, the matching fixes

\[ d\Lambda_\xi|\Sigma. \]  

(12)
for either $\vec{\kappa}$ for the exterior problem. As a consequence, $\Lambda_\xi|_\Sigma$ are fixed up to transformations of the form
\[
\Lambda_\xi|_\Sigma \rightarrow \Lambda'_\xi|_\Sigma = \Lambda_\xi|_\Sigma + \lambda_\xi,
\]
where $\lambda_\xi$ is an arbitrary complex constant for each respective $\vec{\kappa}$.

Regarding the matching of the spacetime $(\mathcal{W}^E, g^E)$ with a given $(\mathcal{W}^k, g^k)$, it is proven in appendix A (see also [5, 9] for particular cases) that the matching conditions determine the existence and thereby the form of the matching hypersurface $\Sigma \equiv \Sigma^E = \Sigma^E$ as seen from both sides, in the general case, together with the values of the metric functions $U$ and $A$ (see (2)) and their normal derivatives on $\Sigma$. It is worth noting that the additive constants in $A$ and $k$ (see above) are then determined. Also, one still has to keep in mind that it has been assumed that the identification of the exterior and interior across $\Sigma$ has been prescribed, in order to fix two extra degrees of freedom introduced by the matching procedure, that correspond to the identification on $\Sigma$ of the Killing vector $\tilde{\partial}_t$ with a linear combination of two Killing vectors in the interior, see appendix A and [5, 13, 9]. The data on $\Sigma$ for the exterior problem introduced by the matching with a given interior are thus given by
\[
U|_\Sigma, \quad dU|_\Sigma, \\
A|_\Sigma, \quad dA|_\Sigma,
\]
which constitute Cauchy boundary data for the exterior problem.

It is now straightforward to show how the data given by (14), (15) and (13) translate into data for $\mathcal{E}$. To begin with, the data (14) and (15) imply that, given the interior, the matching conditions fix $d\Omega_\xi|_\Sigma$ up to transformations of the form
\[
d\Omega_\xi|_\Sigma \rightarrow d\Omega'_\xi|_\Sigma = d\Omega_\xi|_\Sigma - i(\lambda_\xi d\bar{\Lambda}_\xi - \bar{\lambda}_\xi d\Lambda_\xi)|_\Sigma,
\]
and thus fix $\Omega_\xi|_\Sigma$ up to
\[
\Omega_\xi|_\Sigma \rightarrow \Omega'_\xi|_\Sigma = \Omega_\xi|_\Sigma + \omega_\xi - i(\lambda_\xi \bar{\Lambda}_\xi - \bar{\lambda}_\xi \Lambda_\xi)|_\Sigma,
\]
where $\omega_\xi$ is a real arbitrary constant. Although for the sake of clarity these relations have been explicitly deduced here for $\vec{\kappa} = \vec{\xi}$, it can be shown that the matching conditions fix, in fact,
\[
\omega_\xi|_\Sigma
\]
for both $\vec{\kappa}$, and thus (17) and (18) hold for both $\vec{\kappa}$. Combining all the above, the Cauchy data on $\Sigma$ for the Ernst potential are fixed up to transformations of the form (dropping the $\vec{\kappa}$ subindices)
\[
\mathcal{E}|_\Sigma \rightarrow \mathcal{E}'|_\Sigma = \mathcal{E}|_\Sigma - \lambda \bar{\lambda} + i\omega - 2\bar{\lambda} \Lambda|_\Sigma,
\]
and
\[
d\mathcal{E}|_\Sigma \rightarrow d\mathcal{E}'|_\Sigma = d\mathcal{E}|_\Sigma - 2\bar{\lambda} d\Lambda|_\Sigma.
\]

Due to the elliptic character of the Ernst equations system, the Cauchy data induced by the matching conditions overdetermine the problem. Nevertheless, as shown above, there are still three degrees of freedom in that data, generated by $\lambda$ and $\omega$. As remarked in [5], the parameter $\omega$ is relevant, in principle, for the exterior problem because the freedom in $\Omega$ has
already been used to set its asymptotic behaviour (9). For the same reason, expression (18) makes clear that the parameter \( \lambda \) is also relevant. All this means that although the exterior problem for a given interior is an overdetermined problem, the problem is still not unique. Therefore, in order to solve the uniqueness problem, we have to address the following two questions:

(a) uniqueness of the exterior solution \((E, \Lambda)\) given Dirichlet data on \(\Sigma\), i.e. \(\{E|_\Sigma, \Lambda|_\Sigma\}\), and
(b) if a solution \((E, \Lambda)\) for given data \(\{E|_\Sigma, \Lambda|_\Sigma, dE|_\Sigma, d\Lambda|_\Sigma\}\)(\(\lambda, \omega\)), fixed up to the transformations \((20)–(21)\), exists, do \(\lambda\) and \(\omega\) get determined? In other words, do the Cauchy data determine \(\lambda\) and \(\omega\), thereby fixing the Dirichlet data, if the solution exists?

Clearly, an affirmative answer to both questions proves the uniqueness of the electrovacuum exterior field given a charged finite source.

Regarding question (a), the procedure commonly used for proving uniqueness of Dirichlet problems is based on interpreting the Ernst equations as the Euler–Lagrange equations for a certain harmonic map. The proofs rely then, in particular, on the positivity of the Dirichlet functional associated with the harmonic map corresponding to the Ernst potential. This positivity is ensured for both \(\vec{\kappa}\) in the vacuum case, and the proof presented in [5] regarding question (a) is based on choosing \(\vec{\kappa} = \vec{\xi}\). However, in the presence of Maxwell fields (see below), one needs to choose \(\vec{\kappa} = \vec{\eta}\). Adapting the proofs of electrovacuum black hole uniqueness theorems [14, 8] to the present setting, section 4 is devoted to solve the uniqueness of the present Dirichlet problem using the Mazur identity approach. The choice of the Mazur identity approach over the Bunting identity approach used in [5] has been made for convenience because the Mazur construction presented in section 4 is useful in dealing with question (b) in section 5.

The affirmative answer to question (b) amounts to showing that if a solution \((E, \Lambda)\) for given data \(\{E|_\Sigma, \Lambda|_\Sigma, dE|_\Sigma, d\Lambda|_\Sigma\}\) exists, then another solution \((E', \Lambda')\) with \(\{E'|_\Sigma, \Lambda'|_\Sigma, dE'|_\Sigma, d\Lambda'|_\Sigma\}\) given by \(d\Lambda'|_\Sigma = d\Lambda|_\Sigma, (13), (20)\) and \(21\), in terms of \(\lambda\) and \(\omega\) exists only if \(\lambda = 0 = \omega\). This is done in section 5. Since this proof does not need any preference of vector \(\vec{\kappa}\), I have preferred to present the calculations by choosing \(\vec{\kappa} = \vec{\xi}\), so that the corresponding quantities that appear have a usual and direct physical meaning.

4. Uniqueness given Dirichlet data on \(\Sigma\)

The proof follows those used in the uniqueness theorems of black holes (see [14, 8]), making use of the very rich intrinsic structure of the Ernst equations, the only difference being that the horizon is replaced by the matching hypersurface \(\Sigma\). In what follows, the \(\vec{\kappa}\) indices will be omitted for simplicity in the expressions that hold for both \(\vec{\kappa}\), whenever this does not lead to confusion.

Equations (5) and (6) can be interpreted as the Euler–Lagrange equations for the action [8]

\[
S = 4 \int \sqrt{-g} \left( \frac{1}{4N^2} |dE|^2 + 2\Lambda d\Lambda|^2 + \frac{1}{N} |d\Lambda|^2 \right) \eta,
\]

where \(|\theta|^2 = g^{ab} \theta_a \theta_b\) for any 1-form \(\theta\), and \(\eta\) is the volume element. Taking \(\Phi\) to be a Hermitian matrix in \(SU(2, 1)\) defined by [8] \((a, b : 1, 2, 3)\)

\[
\Phi_{ab} \equiv \eta_{ab} + 2 \sign(N) \bar{u}_a v_b.
\]

(22)
where \( v_a = (2\sqrt{N})^{-1}(E - 1, 2\Lambda, E + 1) \) and \( \eta_{ab} = \text{diag}(-1, 1, 1) \), one can define a \( su(2, 1) \)-valued 1-form by
\[
\mathcal{J} \equiv \Phi^{-1} \cdot d\Phi.
\]
(23)
The information carried by the solutions \( (E, \Lambda) \) has now been translated into \( \Phi \). In terms of \( \mathcal{J} \), the above action is rewritten as \([14, 21, 8]\)
\[
S = \int_{V^a} \frac{1}{2} R_{ab} \text{Tr}(\mathcal{J}^a \cdot \mathcal{J}^b) \eta,
\]
for which the variational equation reads \([14, 21, 8]\)
\[
\nabla^a \mathcal{J}_a = 0.
\]
(24)
Note that the way \( \mathcal{J} \) is defined is not unique. In fact, although the definition given above corresponds to that in \([14]\) after a trivial interchange \( \mathcal{J}_a \leftrightarrow \mathcal{J}_i \) (see also \([8]\)), later it will be more useful to refer to the construction presented in \([21]\).

Since the object \( \Phi \) is stationary and axisymmetric by definition, the 1-form \( \mathcal{J} \) depends only on \( \rho \) and \( z \), and furthermore, it is ‘tangent’ to the surfaces orthogonal to the orbits \( T_2 \), i.e. \( \mathcal{J}_a = (0, 0, \mathcal{J}_2) \). Both \( \Phi \) and \( \mathcal{J} \) can therefore be taken as defined on the Riemannian space \( \Gamma \) that corresponds to the surfaces orthogonal to the orbits in \((V^{ab}, g^{ab})\). Recalling that \( \Sigma \) must be spatially homeomorphic to a 2-sphere, the boundary of the Riemannian domain \( \Gamma \) of the solutions for \( \Phi \) is given by (see \([16]\))

(i) a connected curve \( \Upsilon_{\Sigma} \), that corresponds to the projection of \( \Sigma \) onto \( \Gamma \), with ends at the axis,

(ii) the two segments that correspond to either (connected) part of the axis, say \( \Upsilon_+ \) and \( \Upsilon_- \), where \( \rho = 0 \),

so that \( \partial \Gamma = \Upsilon_{\Sigma} \cup \Upsilon_- \cup \Upsilon_+ \) is connected. We will also use \( \Upsilon_{\infty} \) to denote infinity, at \( r^2 \equiv \rho^2 + z^2 \to \infty \), this is the ideal point of \( \Gamma \). Taking the metric \( g_{ij} \equiv |N|g^{ij}_\Sigma \) \( \otimes d\gamma^j \) on \( \Gamma \), one has \( \nabla^a \mathcal{J}_a = |N|^{-1} \nabla^i (\rho \mathcal{J}_i) \), where \( ^{(\gamma)} \nabla \) denotes the covariant derivative with respect to \( \gamma \), and thus equation (24) reads
\[
^{(\gamma)} \nabla^i (\rho \mathcal{J}_i) = 0.
\]
(25)
on \((\Gamma, \gamma)\). The Dirichlet boundary data on \( \Upsilon_{\Sigma} \), i.e. \( \Phi|_{\Upsilon_{\Sigma}} \), are obtained by construction from the data \([E|_{\Upsilon_{\Sigma}}, \Lambda|_{\Upsilon_{\Sigma}}]\) which naturally corresponds to \([E|_{\Sigma}, \Lambda|_z]\).

The first key property for the proofs of the uniqueness theorems is the positivity of \( \Phi \) (and thus, of the action), which is ensured by taking \( \bar{\kappa} = \bar{\eta} \) (since \text{sign}(N_\Sigma) = 1 \) outside the axis), see \([14, 8]\). All the quantities in the remaining of this section will be associated with \( \bar{\kappa} = \bar{\eta} \).

Let us define now \( X \equiv N_\Sigma \) (which in Weyl coordinates reads \( e^{-2\mathbb{L}} \rho^2 - e^{2\mathbb{L}} A^2 > 0 \) outside the axis), \( \bar{\gamma} \equiv \Omega_{\Sigma} \) and \( A_{\bar{\gamma}} \equiv -A e^{2\mathbb{L}} X^{-1} \). The line element (2) can be cast as
\[
d s^2 = -X^{-1} \rho^2 \, dr^2 + X (d\phi + A_{\bar{\gamma}} \, dt)^2 + X^{-1} e^{2\mathbb{L}} (d\rho^2 + dz^2).
\]
The problem with this choice is that \( X \), which appears explicitly in the action, vanishes on the axis (located at \( \rho = 0 \)), and therefore a careful analysis there is needed (see \([14]\)). It is convenient to change the coordinates \( \{\rho, z\} \) to prolate spheroidal coordinates \( \{x, y\}, x > 1, |y| < 1 \) by \( \rho^2 = v^2 (x^2 - 1)(1 - y^2), \ z = \nu xy \), where \( \nu \) is an arbitrary positive constant, so that \( d\rho^2 + dz^2 = v^2 (x^2 - y^2) \, dx^2 \) where
\[
\frac{1}{x^2 - 1} \, dx^2 + \frac{1}{1 - y^2} \, dy^2 \equiv \bar{\gamma}_{ij} \, dx^i \, dx^j.
\]

3 The usual matrix product is denoted by \((A \cdot B)_{ab} = \sum_c A_{ac} B_{cb}\) and the trace reads \( \text{Tr} A = \sum_a A_{aa} \).
Taking the metric $\tilde{\gamma} = e^{-2\Lambda} v^{-2}(x^2 - y^2)^{-2} \gamma$ on $\Gamma$, due to the conformal invariance of the equation for $\mathcal{J}$ \(25\), in $(\Gamma, \tilde{\gamma})$ one has

$$\tilde{\mathcal{V}}^i(\rho, \mathcal{J}) = 0,$$

where ‘tilded’ quantities will refer to the metric $\tilde{\gamma}$. In the $\{x, y\}$ coordinates the boundary $\partial \Gamma$ is given by $\mathcal{Y}_\pm : \{y = \pm 1\}$, while $\mathcal{Y}_\Sigma$ is a connected curve for which $x > 1$ necessarily that joins $y = 1$ with $y = -1$, and one has $\mathcal{Y}_\infty : \{x \rightarrow \infty\}$.

The core of the proof consists of considering two different sets of solutions $\Phi_{(1)}$ and $\Phi_{(2)}$, with corresponding pairs $\mathcal{J}_{(1)}$, $\mathcal{J}_{(2)}$ and field variables $X_{(1)}$, $X_{(2)}$, etc., with common Dirichlet data on $\mathcal{Y}_\Sigma$, i.e.

$$\Phi_{(1)}|_{\mathcal{Y}_\Sigma} = \Phi_{(2)}|_{\mathcal{Y}_\Sigma}. \quad (26)$$

Defining the differences $\hat{f} = f_{(1)} - f_{(2)}$, for any functional $f$ of the variables in $\Phi$, and

$$\Psi = \Phi_{(1)} \cdot \Phi_{(2)}^{-1} - 1,$$

where $1$ is the identity, so that $\Psi = 0 \iff \Phi_{(1)} = \Phi_{(2)}$, the Mazur identity follows (see \[14, 8\])

$$\tilde{\mathcal{V}}^i(\rho \mathrm{Tr} \Psi_i) = \rho \tilde{\gamma}_{ij} \mathrm{Tr} \{\mathcal{J}^i \cdot \Phi_{(2)} \cdot \mathcal{J}^j \cdot \Phi_{(1)}\} \geq 0,$$

where the positive semi-definiteness comes from the fact that $\rho \geq 0$ and $\tilde{\gamma}_{ij}$ and $\Phi$ are positive definite. Denoting by $d\mathcal{T}^i$ the vector surface element on $\partial \Gamma$ pointing from $\Gamma$ outwards, the Stokes theorem applied to the above identity leads to

$$\int_{\partial \Gamma} \rho \mathrm{Tr} \Psi_i \, d\mathcal{T}^i = \int_{\Gamma} \rho \tilde{\gamma}_{ij} \mathrm{Tr} \{\mathcal{J}^i \cdot \Phi_{(2)} \cdot \mathcal{J}^j \cdot \Phi_{(1)}\} \tilde{\eta}, \quad (27)$$

as long as $\rho \mathrm{Tr} \Psi_i \, d\mathcal{T}^i = 0$ at $\mathcal{Y}_\infty$, and the integrand on the left is continuous up to $\partial \Gamma$.

Now, defining $\tilde{f} = f_{(1)} + f_{(2)}$, one has $[14, 22, 8]$

$$\mathrm{Tr} \Psi = \frac{1}{X_{(1)} X_{(2)}} \left[ \mathcal{X}^2 + 2 \tilde{\mathcal{X}} |\mathcal{A}|^2 + |\mathcal{A}|^4 + [\mathcal{Y} + \mathrm{Im}(\mathcal{A} \tilde{\mathcal{A}})]^2 \right]. \quad (28)$$

Taking the limits of the values of $X$, $Y$ and $\Lambda$ on the axis and at infinity—with decays given by \(9\) and \(10\), but independently of the values of $M$, $J$ and $Q$—of the configuration as computed by Carter \[22\] (see also \[14, 8\]), one first obtains that $\mathrm{Tr} \Psi$ and $\mathrm{Tr} \Psi_i \, d\mathcal{T}^i|_{\mathcal{Y}_\Sigma}$ are regular ($C^1$) on the axis \[22\], so that $\rho \mathrm{Tr} \Psi \, d\mathcal{T}^i|_{\mathcal{Y}_\Sigma} = 0$, and secondly, that $\rho \mathrm{Tr} \Psi_i \, d\mathcal{T}^i|_{\mathcal{Y}_\Sigma}$ also vanishes \[22\]. As a result, \(27\) holds and the only contribution to the integral on the left-hand side can come from $\mathcal{Y}_\Sigma$. On the other hand, since the numerator of $\mathrm{Tr} \Psi$ is quadratic in the differences of the variables, $X$, etc, and $\Phi_{(2)}|_{\mathcal{Y}_\Sigma} = 0$ by assumption, one infers $d\mathcal{T} \Psi \, d\mathcal{T}^i|_{\mathcal{Y}_\Sigma} = 0$, and hence $\rho \mathrm{Tr} \Psi \, d\mathcal{T}^i|_{\mathcal{Y}_\Sigma} = 0$.

Therefore, due to the positive semi-definiteness of the integrand on the right-hand side, \(27\) implies that $\mathcal{J} = 0$ in $\Gamma$ (and thus in $\mathcal{W}^E$). Since $d\mathcal{X} = \Phi_{(1)} \cdot \mathcal{J} \cdot \Phi_{(2)}^{-1}$ by construction, $\Psi$ is constant all over $\Gamma$, and thus $\Psi = 0$ because $\Psi|_{\mathcal{Y}_\Sigma} = 0$ by assumption \(26\). This ends the proof showing that given Dirichlet data on $\mathcal{Y}_\Sigma$, and correspondingly on $\Sigma$, i.e. $\{\mathcal{E}_\Sigma|_\Sigma, \Lambda_{\Sigma}|_\Sigma\}$, the solution $(\mathcal{E}_\Gamma, \Lambda_{\Gamma})$ of the Ernst equations in the exterior region $(\mathcal{W}^E, g^E)$ is unique. The correspondence between conjugate solutions $(\mathcal{E}_\Sigma, \Lambda_{\Sigma})$ and $(\mathcal{E}_\tilde{\Sigma}, \Lambda_{\tilde{\Sigma}})$ (see \[8\]) leads to the same result for $\mathcal{Y} = \tilde{\mathcal{Y}}$. The main result found in this section can be stated as follows.

**Proposition 1.** Let $(\mathcal{W}^E, g^E, \mathcal{J}, \Sigma)$ define a simply connected stationary and axially symmetric electrovacuum spacetime with a connected boundary preserving the symmetries. If $\mathcal{Y}^E$ is asymptotically flat and the problem is given Dirichlet boundary data on $\Sigma$ for the Ernst and electromagnetic potentials, i.e. $\{\mathcal{E}|_\Sigma, \Lambda|_\Sigma\}$, then the solution $(\mathcal{E}, \Lambda)$ is unique.
5. Fixing the Dirichlet data

The purpose now is to show how the full set of Cauchy data, i.e. taking into account \( \{ d\mathcal{E}|_{\Sigma}, d\Lambda|_{\Sigma}\}(\omega, \lambda) \) on top of the family of Dirichlet data \( \{ \mathcal{E}|_{\Sigma}, \Lambda|_{\Sigma}\}(\omega, \lambda) \), fixes the values of \( \omega, \lambda \), provided that the solution exists.

The proof presented here makes use of the divergence-free fields \( \mathcal{J}_\xi \) (23) choosing \( \vec{\kappa} = \vec{\xi} \).

Decomposing \( \mathcal{J}_\xi \) on a basis of \( \mathfrak{su}(2,1) \), one can obtain eight conserved real 1-forms. For convenience, the basis used here corresponds to that presented in [21], so that the eight forms will be presented as a set of four real 1-forms and two complex 1-forms. The representation of an element \( Z \in \mathfrak{su}(2,1) = \epsilon e + \delta d + \lambda s + \pi p + \alpha h + \beta a \), where four elements of the basis of the algebra are denoted as \( e, d, s, p \) and the remaining four are encoded in \( h \) and \( a \) by ‘complexifying’ them, is chosen as [21]

\[
\mathcal{J}_\xi = \begin{pmatrix}
\frac{1}{2} \delta - \frac{1}{2} \lambda & -\sqrt{2} \alpha \\
\lambda & \frac{1}{2} \delta - \frac{1}{2} \lambda \\
\pi & \sqrt{2} \beta \\
-\frac{1}{2} \delta - \frac{1}{2} \lambda & \frac{1}{2} \lambda
\end{pmatrix}.
\]

(29)

where \( \alpha \) and \( \beta \) are complex constants and the rest are real. Nevertheless, one cannot yet revert to the decomposition of the divergence-free currents in [21] because the construction of the currents satisfying (24) is performed in [21] in a different way. In fact, whereas the vacuum case lies on \( \mathfrak{su}(1,1) \) when using (22) and (23), the construction presented in [21] leads to an isomorphic \( \mathfrak{sl}(2,\mathbb{R}) \) in the vacuum case. Denoting by \( \mathcal{J}_{BM} \) the matrix representation of the divergence-free currents as constructed in [21] for \( \vec{\kappa} = \vec{\xi} \), and defining

\[
\sigma = \begin{pmatrix}
1 & 0 & -i \\
0 & i\sqrt{2} & 0 \\
1 & 0 & i
\end{pmatrix},
\]

the isomorphism between the two constructions can be found to be given explicitly by

\[
\mathcal{J}_{BM} = \frac{1}{4} \sigma \cdot \mathcal{J}_\xi \cdot \sigma.
\]

(30)

Using (29), \( \mathcal{J}_{BM} \) is decomposed (see also [21] pp 798) in terms of a basis of four real 1-forms \( \mathcal{J}^e, \mathcal{J}^d, \mathcal{J}^s, \mathcal{J}^p \) plus two complex ones, \( \mathcal{J}^h \) and \( \mathcal{J}^a \), which correspond to the elements of the basis of the algebra \( \mathfrak{su}(2,1) \) listed above, as

\[
\mathcal{J}_{BM} = \frac{1}{2} \mathcal{J}^e e + \mathcal{J}^d d + \mathcal{J}^s s + \frac{1}{2} \mathcal{J}^p p + \frac{1}{2} i \mathcal{J}^h h + \frac{1}{2} \mathcal{J}^a a.
\]

The explicit expressions, which can be obtained from the above constructions (22) and (23) by making use of (29), (30), are given in appendix B. Using these 1-forms, one defines on \( (\Sigma, \gamma) \) a (complex) 1-form depending on a \( (\mathcal{E}, \Lambda) \) configuration plus eight real parameters as

\[
W(\mathcal{E}, \Lambda; e_e, e_d, e_s, e_p, c_{h_1}, c_{h_2}, c_{a_1}, c_{a_2})
\]

\[
= e_e \mathcal{J}^e + e_d \mathcal{J}^d + e_s \mathcal{J}^s + e_p \mathcal{J}^p + c_{h_1} \mathcal{J}^h + c_{h_2} \mathcal{J}^h + c_{a_1} \mathcal{J}^a + c_{a_2} \mathcal{J}^a,
\]

where clearly all \( \mathcal{J} = \mathcal{J}(\mathcal{E}, \Lambda) \), which by construction and (25) satisfies

\[
(\gamma) \nabla_i (\rho W_i) = 0.
\]

(31)

The only non-vanishing surface integrals at spatial infinity \( \Gamma_\infty \) of the above 1-forms are given by (see, e.g., [21])

\[
\int_{\Gamma_\infty} \rho \mathcal{J}^i d\Gamma^i = 4M, \quad \int_{\Gamma_\infty} \rho \mathcal{J}^h d\Gamma^i = \int_{\Gamma_\infty} \rho \mathcal{J}^a d\Gamma^i = 2Q.
\]
where $M(\mathcal{E}, \Lambda)$ and $Q(\mathcal{E}, \Lambda)$ relative to a given solution $(\mathcal{E}, \Lambda)$ correspond to the mass and the electric charge of the configuration, respectively, so that at spacelike infinity one has (9) and (10). As a consequence of (31), we have

$$\int_{\Sigma}\rho W_i \, d\Sigma^i = \int_{\Sigma}\rho W_i \, d\Sigma^i = 4Mc_d + 2Q(c_{h_1} + c_{a_1} + c_{h_2} + c_{a_2}).$$

Now, let us assume two exterior solutions for the same interior exist $(\mathcal{E}(1), \Lambda(1))$, $(\mathcal{E}(2), \Lambda(2))$, with corresponding $M(1), Q(1)$ and $M(2), Q(2)$ respectively, such that their Cauchy boundary data differ by $\lambda$ and $\omega$ as given by relations (12), (13), (20) and (21):

$$\Lambda(2)|_{\Sigma} = \Lambda(1)|_{\Sigma} + \lambda, \quad \mathcal{E}(2)|_{\Sigma} = \mathcal{E}(1)|_{\Sigma} - \lambda \hat{\lambda} + i\omega - 2\hat{\lambda} \Lambda|_{\Sigma},$$

$$d\Lambda(2)|_{\Sigma} = d\Lambda(1)|_{\Sigma}, \quad d\mathcal{E}(2)|_{\Sigma} = d\mathcal{E}(1)|_{\Sigma} - 2\hat{\lambda} d\Lambda|_{\Sigma}.$$

From these relations, the following equality holds on $\Sigma$, for a set of eight certain relations

$$\hat{\mathcal{E}} = \hat{\mathcal{E}}(c_{e_2}, \ldots, c_{s})|_{\Sigma}, \quad \hat{\mathcal{E}} = \hat{\mathcal{E}}(c_{e_1}, \ldots, c_{s})|_{\Sigma},$$

which are given explicitly in appendix B. The integration of (33) over $\Sigma$ using (32) leads to

$$4M(1)c_d + 2Q(1)(c_{h_1} + c_{a_1} + c_{h_2} + c_{a_2}) = 4M(2)\hat{c}_d + 2Q(2)(\hat{c}_{h_1} + \hat{c}_{a_1} + \hat{c}_{h_2} + \hat{c}_{a_2}).$$

which has to hold for arbitrary choices of the eight parameters $[c_{e_2}, \ldots, c_{s}]$. A straightforward calculation, see appendix B, leads to the fact that

$$M(1) + M(2) \neq 0 \Rightarrow \lambda = \omega = 0$$

(so that in fact $M(1) = M(2), Q(1) = Q(2)$ necessarily). It must be stressed here that the case $M(1) + M(2) = 0$ does not lead to the same result in general, see appendix B. Nevertheless, and since the mass for physically motivated solutions is positive, this result basically implies uniqueness of the Cauchy data for which the exterior field exists: the freedom in the Cauchy and Dirichlet data generated by $\lambda$ and $\omega$ is fixed. To be more precise, we have proven the following.

**Proposition 2.** Let us consider $(\mathcal{W}^E, g^E, \mathcal{F}, \Sigma)$ as in proposition 1, with a family of Dirichlet boundary data depending on three real parameters, i.e. $[\mathcal{E}|_{\Sigma}, \Lambda|_{\Sigma}](\omega, \lambda)$, so that it is determined up to transformations of the form (13) and (20). Consider now the completion to Cauchy boundary data by adding the data $[d\mathcal{E}|_{\Sigma}, d\Lambda|_{\Sigma}](\omega, \lambda)$ determined up to transformations of the form $d\mathcal{E}|_{\Sigma} = d\Lambda|_{\Sigma}$ and (21). If within the $(\omega, \lambda)$ family of Cauchy data sets there is a Cauchy data set for which a solution with $M > 0$ exists, then there is no solution with $M > 0$ for any other Cauchy data set within the family.

6. Conclusions

The combination of propositions 1 and 2 with the boundary data that a given interior infers onto the exterior problem, as shown in section 3, leads us to:

**Theorem 1.** Let $(\mathcal{W}^I, g^I)$ be a given stationary and axially symmetric region of spacetime containing an electromagnetic field, and bounded by a symmetry-preserving and simply connected hypersurface $\Sigma$, spatially homeomorphic to a 2-sphere, and containing no surface charge currents. If $(\mathcal{W}^I, g^I)$ can be matched across $\Sigma$ to an asymptotically flat (with $M > 0$) stationary and axially symmetric electrovacuum Einstein–Maxwell field $(\mathcal{W}^E, g^E, \mathcal{F})$. 
preserving the symmetry, and where the identification on \( \Sigma \) has been prescribed, then 
\((\mathcal{V}^E, g^E, \mathcal{F})\) is unique within the set of positive mass solutions.

As a first remark, the same result holds if \( \Sigma \) is allowed to contain surface electromagnetic
charge currents, as long as these currents are taken as given data. Secondly, although for
physically motivated situations the restriction of the uniqueness result to the set of positive
mass solutions is irrelevant, it may be desirable to have a complete uniqueness result. This
might be expected to follow by using existence considerations, and it is currently under study.

All in all, this final result can be stated in other words as follows: under natural assumptions
on \( \Sigma \)—preservation of the symmetries and the prescription of the identification of both sides
of \( \Sigma \) [5]—the exterior electrovacuum field generated by an isolated distribution of charged
matter in axial rotation and in equilibrium is unique, provided that it exists.

Some comments on the definition of asymptotic flatness are in order here. The asymptotic
flatness has explicitly entered the problem only through the use of the asymptotic conditions
\((9)\) and \((10)\). In principle, one could have included NUT and monopole charges in the
expressions for \( U \) and \( \Lambda \) at infinity. In fact, it can be shown that the inclusion of NUT and
monopole charges does not change the present results, provided we can use Weyl coordinates
globally on \((\mathcal{V}^E, g^E)\) (see \((2)\)). Nevertheless, the existence of global Weyl coordinates has
only been ensured when \((\mathcal{V}^E, g^E)\) admits an asymptotically flat four-end [16] (see, e.g., [23]
for definitions). Therefore, this is the definition of asymptotic flatness that has been implicitly
used throughout the paper, which implies the vanishing of the NUT and monopole charges
[19], as mentioned in section 2.

Looking at the procedure used here, several generalizations to include different fields on
the exterior region could be attempted quite straightforwardly. The starting point would be
the possibility of using the Weyl coordinates (orthogonal transitivity and a trace-free energy–
momentum tensor [7]). Then, one should consider the boundary conditions for the fields
on the matching hypersurface, to be combined with those inferred by the spacetime matching
conditions, according to proposition 3, in order to find the form of the Cauchy boundary data
for the exterior problem. Of course, the main ingredient would consist of picking up fields that
can be described by \( \sigma \)-models [21], and then using the Mazur construction for the Dirichlet
problem, and the divergence-free currents for the fixing of the Dirichlet data with the whole
set of Cauchy data.

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Appendix A. The explicit matching

This section is devoted to explicitly presenting the general matching of two stationary and
axially symmetric spacetimes (with boundary), \((\mathcal{V}^I, g^I, \Sigma^I)\) and \((\mathcal{V}^E, g^E, \Sigma^E)\), across
a matching hypersurface \( \Sigma (\equiv \Sigma^I = \Sigma^E) \) preserving the stationarity and the axial symmetry.
The latter implies that \( \Sigma \) is timelike. Furthermore, we impose on \((\mathcal{V}^E, g^E)\) the following two
requirements: (i) the \( G_2 \) on \( T_2 \) group of isometries acts orthogonally transitively (OT) and
(ii) the trace of the Einstein tensor is zero. These two conditions simply account for having
the possibility of simplifying the problem by using the Weyl coordinates (at least locally) (see [7]).

Apart from preserving the symmetries, the only implicit assumption we are making on \( \Sigma \) is that it intersects points at the axis, so that the axis intersects both \( \mathcal{V}^E \) and \( \mathcal{W}^E \), which is ensured if \( \Sigma \) is spatially homeomorphic to a 2-sphere. Conditions (i) and (ii) are satisfied when \( (\mathcal{V}^E, g^E) \) contains a stationary and axisymmetric electrovacuum field [7].

The equations for the matching with a vacuum \( (\mathcal{V}^E, g^E) \) spacetime were presented in an explicit manner in [9]. The reader is referred to that work (and also to [5, 24] for the \( (\mathcal{V}^L, g^L) \) case) for further details.

There exists a coordinate system \( \{T, \Phi, r, \zeta\} \) in which the line element for \( g^I \) reads

\[
ds_1^2 = -e^{2V}(dT + B \, d\Phi + W_2 \, d\zeta)^2 + e^{-2V}[e^{2h}(dr^2 + d\zeta^2) + \alpha^2(d\Phi + W_3 \, d\zeta)^2],
\]

where \( V, B, h, W_2, W_3, \) and \( \alpha \) are functions of \( r \) and \( \zeta \), and \( \tilde{\eta}^I = \partial_\Phi \) is the (intrinsically defined) axial Killing vector. The vector \( \tilde{\xi}^I \equiv \partial_T \) is a stationary Killing vector. By assumption, \( (\mathcal{V}^E, g^E) \) admits Weyl coordinates \( \{t, \phi, \rho, z\} \), in which its line element is given by (2), the intrinsically defined axial Killing vector is given by \( \tilde{\eta}^E = \partial_\phi \), and we take \( \tilde{\xi}^E = \partial_t \).

As mentioned in section 2, it is straightforward to show that the most general matching hypersurface preserving stationarity and axial symmetry \( \Sigma \), as seen from the \( (\mathcal{V}^E, g^E) \) side, is given in parametric form as [5, 13, 16]

\[\Sigma^E : \{\tau = t, \phi = \varphi, \rho = \rho(\mu), z = z(\mu)\},\]

where \( \{\tau, \varphi, \mu\} \) parametrizes \( \Sigma \). At this point, the matching procedure introduces two degrees of freedom, a product of the identification of \( \partial_\phi \) with any linear combination \( a(\partial_T + b\partial_\rho) \) on \( \Sigma \) [5, 13, 9]. Of course, this is of relevance if \( \partial_\rho \) is intrinsically defined, as it happens, for instance, for asymptotically flat \( (\mathcal{V}^E, g^E) \) [5]. Despite that, one can proceed with the matching avoiding the explicit appearance of \( a, b \) by absorbing them in \( (\mathcal{V}^E, g^E) \), changing coordinates in \( (\mathcal{V}^L, g^L) \) to a new set of ‘primed’ coordinates, with corresponding functions \( V', B', h', W'_2, W'_3 \) and \( \alpha' \) in (A.1), so that \( \partial_T = a(\partial_T + b\partial_\rho) \) while keeping \( \partial_\phi = \partial_\phi \) (see [9], for the explicit relations between primed and the original non-primed quantities). Nevertheless, if the interior geometry (A.1) was given beforehand, these relations involving the parameters \( a, b \) have to be taken into account. Having this in mind, and dropping the primes, the most general parametric form of \( \Sigma \) as seen from \( (\mathcal{V}^L, g^L) \) then reads

\[\Sigma^L : \{T = \tau + f_T(\mu), \Phi = \varphi + f_\phi(\mu), r = r(\mu), \zeta = \zeta(\mu)\},\]

where \( f_T(\mu) = \hat{\zeta}(BW_3 - W_2) (\mu) \) and \( f_\phi(\mu) = -\hat{\zeta} W_3(\mu) \).

Two bases for the tangent spaces to \( \Sigma \), from the \( \Sigma^L \) and \( \Sigma^E \) sides, \( \{\mathbf{e}^I_a\} \) and \( \{\mathbf{e}^E_a\} \) \( a = 1, 2, 3 \) respectively, which are to be identified eventually as \( \{\mathbf{e}_a\} \) in the final matched spacetime [26], read explicitly

\[
\begin{align*}
\mathbf{e}^I_1 &= \tilde{\xi}^I|_\Sigma, & \mathbf{e}^I_2 &= \tilde{\eta}^I|_\Sigma, & \mathbf{e}^I_3 &= r \partial_r + \hat{\zeta} \partial_\zeta|_\Sigma, \\
\mathbf{e}^E_1 &= \tilde{\xi}^E|_\Sigma, & \mathbf{e}^E_2 &= \tilde{\eta}^E|_\Sigma, & \mathbf{e}^E_3 &= \rho \partial_\rho + \hat{\zeta} \partial_\zeta|_\Sigma.
\end{align*}
\]

In principle, any choice of normal \( n \) is suitable for computing the matching conditions, but it is fixed, by convention, so that it points \( (\mathcal{V}^L, g^L, \Sigma^L) \) inwards and \( (\mathcal{V}^E, g^E, \Sigma^E) \) outwards, and by convention, that its norm is the same as that of \( \mathbf{e}_3 \). On the \( \Sigma^E \) side it reads \( n^E = e^{2(V-U)}(\hat{\zeta} \, dr + \hat{\rho} \, dz) \), while on the \( \Sigma^I \) side this is \( n^I = e^{2(V-U)}(-\hat{\zeta} \, dr + \hat{\rho} \, dz) \), where \( \epsilon \) is a sign. The equality of the first fundamental forms is equivalent to the following four equations, the so-called preliminary junction conditions,

\[
U|_\Sigma = V|_\Sigma, \quad A|_\Sigma = B|_\Sigma.
\]  
\[\text{A.2}\]
\[ \rho|_\Sigma = \alpha|_\Sigma, \quad (A.3) \]
\[ e^{2k}|_\Sigma (\rho^2 + \dot{\xi}^2) = e^{2h}|_\Sigma (r^2 + \dot{\xi}^2). \quad (A.4) \]

Since \( \Sigma \) is timelike, the remaining matching conditions simply consist of equating the second fundamental forms, \( K_{ab} = e_{a\alpha} e_{b\beta} \nabla_a n_\beta \), which are symmetric [26], as computed from \( \Sigma^E \) and \( \Sigma^1 \). Labelling by \( A, B = 1, 2 \) the components corresponding to the orbits, the set of six equations \( K_{ab}^E = K_{ab}^1 \), after using (A.2)–(A.4), can be expressed and grouped as follows:

\[ K_{AB}^E = K_{AB}^1 : \quad \Xi^E(U)|_\Sigma = \Xi^1(V)|_\Sigma, \quad \Xi^E(A)|_\Sigma = \Xi^1(B)|_\Sigma, \quad (A.5) \]
\[ \dot{\zeta} = -\Xi^1(\alpha)|_\Sigma \quad (A.6) \]
\[ K_{A3}^E = K_{A3}^1 : \quad \partial_r (BW_1 - W_2)|_\Sigma = 0, \quad \partial_r W_3|_\Sigma = 0 \quad (A.7) \]
\[ K_{33}^E = K_{33}^1 : \quad e^{2k}(\rho^2 + \dot{\xi}^2)\Xi^E(k) + e^{2h}(r^2 + \dot{\xi}^2)\Xi^1(h) = e^{2h}(r^2 + \dot{\xi}^2)\Xi^1(h) + e^{2h}(r^2 + \dot{\xi}^2)(\dot{\rho} - \dot{\xi})|_\Sigma, \quad (A.8) \]

where (A.5) has been used in (A.8).

Using all the equations, one first finds that \( \rho^2 + \dot{\xi}^2 = (\alpha^2 + \alpha^2) \), meaning that \( \alpha^2 + \alpha^2 \) cannot vanish anywhere on \( \Sigma \), and secondly, that (A.4) and (A.8) can be substituted by

\[ k|_\Sigma = h - \frac{1}{2} \log \left( \alpha^2 + \alpha^2 \right)|_\Sigma, \quad (A.9) \]
\[ \Xi^E(k)|_\Sigma = \Xi^1(h) - \epsilon \frac{\alpha e\alpha e - \alpha e\alpha e}{\alpha e + \alpha e}|_\Sigma, \quad (A.10) \]

respectively. A straightforward calculation shows that, once the rest of the equations hold, equations (A.10) and the derivative of (A.9) along \( \mu \) are equivalent to

\[ n^{\alpha\beta} e_{\alpha\beta} S_{\alpha\beta}|_\Sigma = n^{\alpha\beta} e_{\alpha\beta} S_{\alpha\beta}|_\Sigma, \quad n^{\alpha\beta} e_{\alpha\beta} S_{\alpha\beta}|_\Sigma = n^{\alpha\beta} e_{\alpha\beta} S_{\alpha\beta}|_\Sigma, \quad (A.11) \]

where \( S_{\alpha\beta} \) denotes the Einstein tensor. On the other hand, the equations in (A.7) are equivalent to [9]

\[ * (\xi^1 \wedge \eta^1 \wedge d\xi^1)|_\Sigma = 0 = * (\xi^1 \wedge \eta^1 \wedge d\eta^1)|_\Sigma \]

(see also [13]) and, in turn, equivalent to [9] (up to additive constants, but these vanish because \( \Sigma \) intersects the axis)

\[ n^{\alpha\beta} e_{\alpha\beta} S_{\alpha\beta}|_\Sigma = 0, \quad n^{\alpha\beta} e_{\alpha\beta} S_{\alpha\beta}|_\Sigma = 0. \quad (A.12) \]

Since \( n^{\alpha\beta} e_{\alpha\beta} S_{\alpha\beta} = 0 = n^{\alpha\beta} e_{\alpha\beta} S_{\alpha\beta} \) are identically satisfied due to the OT structure of the \((\Sigma^E, g^E)\) region, the two equations in (A.12) together with those in (A.11) constitute the set of the four Israel conditions.

To sum up, the whole set of matching conditions is formed by the ten equations in (A.2), (A.3), (A.5), (A.6), (A.9), (A.10) and (A.12), taking into account that up to an additive constant in \( k|_\Sigma \), equations (A.9) and (A.10) can be substituted by (A.11). If required, that constant should then be fixed by considering (A.9).

In principle, if no conditions are imposed on the matter content in \((\Sigma^E, g^E)\), equations (A.9) and (A.10), or equivalently (A.11), provide conditions on \( \Sigma \) for the function \( k \), given the interior geometry. Conversely, if one imposes conditions on the matter content in \((\Sigma^E, g^E)\), i.e. on \( S_{\alpha\beta} \), so that the function \( k \) has to satisfy equations involving the rest of the functions, equations (A.9) and (A.10) may represent conditions on the \((\Sigma^E, g^E)\) side only. In particular, this becomes clear when \((\Sigma^E, g^E)\) is vacuum because (A.11) only involve
quantities in $(\mathcal{V}^I, g^I)$, and therefore (A.9) and (A.10) consist, in fact, of conditions on $\Sigma^I$ (apart from providing the value of the additional constant in $k$).

Although not as explicitly apparent as in the vacuum case, it is straightforward to show that the same holds true when $(\mathcal{V}^E, g^E)$ is an electrovacuum solution whose electromagnetic field extends continuously across $\Sigma$ onto $(\mathcal{V}^I, g^I)$. As mentioned in section 2, $k$ is determined, up to an additive constant, by quadratures in terms of the rest of the metric functions, which satisfy the Ernst equations. Defining $\Gamma \equiv -dN + i\nu$, where this and the following quantities are referred to $\bar{k} = \tilde{k}$, the quadratures for $k$ in (2) can be cast as (see [7, 8])

$$\frac{1}{\rho} k_\rho = \frac{1}{4N^2} (\Gamma_\rho \Gamma_\rho - \Gamma_\gamma \Gamma_\gamma) + \frac{1}{N} (\Lambda_\rho \bar{A}_\rho - \Lambda_\gamma \bar{A}_\gamma),$$
$$\frac{1}{\rho} k_\gamma = \frac{1}{4N^2} (\Gamma_\gamma \Gamma_\gamma + \Gamma_\rho \Gamma_\rho) + \frac{1}{N} (\Lambda_\rho \bar{A}_\gamma + \Lambda_\gamma \bar{A}_\rho).$$

The explicit expressions in terms of $U$ and $A$ are then obtained by using (7) and (8), and a straightforward calculation on $\Sigma$ leads to

$$k|_\Sigma = \frac{\rho}{\rho^2 + z^2} (A\rho - Bz) \bigg|_{\Sigma}, \quad \tilde{n}^E(k)|_\Sigma = \frac{\rho}{\rho^2 + z^2} (A\tilde{z} + B\tilde{z}) \bigg|_{\Sigma},$$

where we have defined

$$A \equiv U^2 + \tilde{n}^E(U)^2 + \frac{e^{4U}}{4\rho^2} (\bar{A}^2 + \tilde{n}^E(A)^2) - e^{-2U}(\bar{A}\tilde{A} - \tilde{n}^E(\Lambda)\tilde{n}^E(\Lambda)) \bigg|_{\Sigma},$$
$$B \equiv 2\tilde{U}\tilde{n}^E(U) - \frac{e^{4U}}{2\rho^2} A\tilde{n}^E(A) - e^{-2U}(\tilde{n}^E(\Lambda)\tilde{A} + \Lambda\tilde{n}^E(\Lambda)) \bigg|_{\Sigma}.$$}

The continuity of the electromagnetic field across $\Sigma$ implies relations (11) for $\Lambda|_\Sigma$ and $\tilde{n}^E(\Lambda)|_\Sigma$ in terms of (I) quantities. Using these relations together with (A.2), (A.3), (A.5) and (A.6), $k|_\Sigma$ and $\tilde{n}^E(k)|_\Sigma$ can be obtained in terms of quantities of $(\mathcal{V}^I, g^I)$, and hence obtain expressions for (A.10) and the derivative of (A.9) along $\mu$ as conditions for quantities on the $(\mathcal{V}^E, g^E)$ side only. Equivalently, equations (A.11), within the set of matching conditions, provide conditions on $(\mathcal{V}^I, g^I)$ only.

On the other hand, of course, it must be stressed that conditions on the matter content in $(\mathcal{V}^E, g^E)$ only translate into conditions on the $(\mathcal{V}^I, g^I)$ geometry through the four Israel conditions (A.11) and (A.12), and thus, given an interior geometry satisfying these conditions, the rest of the conditions involve the form of the matching hypersurface at the E side, $\Sigma^E$, plus boundary data (of Cauchy type) for $U$ and $A$ on $\Sigma^E$ given the interior geometry.

All the above can be summarized as follows.

**Proposition 3.** Let $(\mathcal{V}^E, g^E)$ and $(\mathcal{V}^I, g^I)$ be two stationary and axially symmetric spacetimes which are to be matched across their respective symmetry-preserving boundaries $\Sigma^E$ and $\Sigma^I$, whose identification is denoted as $\Sigma (\equiv \Sigma^E = \Sigma^I)$. Let us also assume that $(\mathcal{V}^E, g^E)$ contains a stationary and axially symmetric electromagnetic field and satisfies the Einstein–Maxwell equations without sources. The whole set of matching conditions can be reorganized as follows.

(i) Conditions on $\Sigma^I$, given by (A.11), (A.12), which form, in principle, an overdetermined system for $(r(\mu), \zeta(\mu))$. If a solution exists the matching is possible (locally) and the hypersurface $\Sigma^I$ is determined generically.

(ii) Equations defining $\Sigma^E$, given by (A.3), (A.6).

(iii) Boundary conditions for the electrovacuum problem in $(\mathcal{V}^E, g^E)$, given by (A.2), (A.5).

If required, the additive constant in $k$ is determined by using (A.9).
Remark 3.1. The explicit expressions apply for the definitions and metric functions as introduced above in this appendix, but can be easily re-expressed in an intrinsic manner.

Appendix B. The divergence-free forms

The explicit expressions of the divergence-free 1-forms used for the decomposition of the $\mathfrak{su}(2,1)$-valued 1-form $\mathcal{J}_e$ (for either $\xi$) are given as follows, both in terms of $\mathcal{E}$ and $\Lambda$ and in terms of the sometimes more convenient $N, \Lambda, \Omega$ and $w$ (for $d\Omega$ (4)),

$$\mathcal{J}^e = -\frac{i}{2N^2}[2(\bar{\Lambda} d\Lambda - \Lambda d\bar{\Lambda}) - d(\bar{\mathcal{E}} - \mathcal{E})] = \frac{i}{N^2} \mathcal{E},$$
$$\mathcal{J}^d = \frac{1}{2N^2}[d(\mathcal{E} \bar{\mathcal{E}}) + \bar{\Lambda} \Lambda d(\mathcal{E} + \bar{\mathcal{E}}) - (\bar{\Lambda} d\Lambda - \Lambda d\bar{\Lambda})(\bar{\mathcal{E}} - \mathcal{E})]$$
$$= \frac{1}{N^2}[\Omega w + N(dN + d(\Lambda \bar{\Lambda}))],$$
$$\mathcal{J}^r = -\frac{3i}{4N^3}[\Lambda \bar{\Lambda} d(\mathcal{E} - \bar{\mathcal{E}}) - (\bar{\Lambda} d\Lambda - \Lambda d\bar{\Lambda})(\mathcal{E} + \bar{\mathcal{E}})]$$
$$= \frac{3}{2N^2}[\Lambda \bar{\Lambda} w - iN(\bar{\Lambda} d\Lambda - \Lambda d\bar{\Lambda})],$$
$$\mathcal{J}^p = \frac{1}{2N^2}[-2\mathcal{E} \bar{\mathcal{E}}(\bar{\Lambda} d\Lambda - \Lambda d\bar{\Lambda}) + (\mathcal{E} + 2\Lambda \bar{\Lambda}) \bar{\mathcal{E}} d\bar{\mathcal{E}} - (\bar{\mathcal{E}} + 2\bar{\Lambda} \Lambda) \mathcal{E} d\mathcal{E}]$$
$$= \frac{1}{N^2}[(N^2 - \Omega^2 - \Lambda^2 \bar{\Lambda}^2)w - 2N\Omega dN - 4 \text{Re}[(\Omega - i(N + \Lambda \bar{\Lambda}))N \Lambda d\Lambda]],$$
$$\mathcal{J}^b = \frac{1}{2N^2}[\Lambda d(\mathcal{E} - \bar{\mathcal{E}}) + (\mathcal{E} + \bar{\mathcal{E}}) d\Lambda + 2\bar{\Lambda}^2 d\Lambda] = \frac{1}{N^2}[i\Lambda w - N d\Lambda],$$
$$\mathcal{J}^a = \frac{1}{2N^2}[\mathcal{E} \mathcal{E} + \bar{\mathcal{E}} \bar{\mathcal{E}}] d\Lambda + 2\bar{\Lambda}^2 (\mathcal{E} d\Lambda - \Lambda d\mathcal{E}) - \Lambda d(\mathcal{E} \bar{\mathcal{E}})]$$
$$= \frac{1}{N^2}[N^2 \bar{\Lambda} \Lambda - N \Lambda dN - 2N\bar{\Lambda}^2 \Lambda d\Lambda - (\Omega + i\Lambda \bar{\Lambda})(\Lambda \bar{\Lambda} w + iN d\Lambda)].$$

The vacuum case is recovered by $\Lambda = 0$ (so that $d\Omega = w$), and thus only three 1-forms survive, namely $\mathcal{J}^r, \mathcal{J}^d$ and $\mathcal{J}^a$. Their expressions in the case $\xi = \bar{\xi}$, and after the substitution $N = -e^{\theta t}$, read $\mathcal{J}^e = e^{-\theta t} d\Omega, \mathcal{J}^d = 2 dU + e^{-\theta t} \bar{U} d\Omega$ and $\mathcal{J}^p = -4\Omega dU + (1 - e^{-\theta t} \Omega^2) d\Omega$, which were those used in constructing the divergence-free 1-form in [5].

With the above expressions, one is ready to compute relations (34) by using the differences in the Cauchy data of the two solutions in terms of $\Lambda, N, \Omega$ and $w$ (12)–(19):

$$\frac{\Lambda(2)|_{\Gamma E}}{\Lambda(1)|_{\Gamma E}} = \Lambda(2)|_{\Gamma E} + \lambda, \quad \frac{\Lambda(2)|_{\Gamma E}}{\Lambda(1)|_{\Gamma E}} = \Lambda(2)|_{\Gamma E} - \lambda,$$
$$\frac{d\Lambda(2)|_{\Gamma E}}{d\Lambda(1)|_{\Gamma E}} = \frac{d\Lambda(2)|_{\Gamma E}}{d\Lambda(1)|_{\Gamma E}} = \frac{d\Lambda(2)|_{\Gamma E}}{d\Lambda(1)|_{\Gamma E}} = \frac{d\Lambda(2)|_{\Gamma E}}{d\Lambda(1)|_{\Gamma E}},$$
$$\frac{dN(2)|_{\Gamma E}}{dN(1)|_{\Gamma E}} = \frac{dN(2)|_{\Gamma E}}{dN(1)|_{\Gamma E}}, \quad \frac{d\Omega(2)|_{\Gamma E}}{d\Omega(1)|_{\Gamma E}} = \frac{d\Omega(2)|_{\Gamma E}}{d\Omega(1)|_{\Gamma E}}, \quad \frac{dU(2)|_{\Gamma E}}{dU(1)|_{\Gamma E}} = \frac{dU(2)|_{\Gamma E}}{dU(1)|_{\Gamma E}}.$$

Indeed, (33) is recovered with

$$\hat{\mathcal{E}}_p = c_p, \quad \hat{\mathcal{E}}_{a_1} = c_{a_1} + i2c_p \lambda, \quad \hat{\mathcal{E}}_{a_2} = c_{a_2} - i2c_p \lambda,$$
$$\hat{\mathcal{E}}_d = c_d + c_{a_1} \lambda + c_{a_2} \lambda + 2c_p \omega, \quad \hat{\mathcal{E}}_s = \frac{1}{2}(c_{a_1} \lambda - c_{a_2} \lambda) - 6c_p \lambda \bar{\lambda},$$
$$\hat{\mathcal{E}}_t = c_t + (c_{a_1} \lambda + c_{a_2} \lambda) \omega - c_d (\lambda \bar{\lambda} + \omega i) + 2\lambda^2 (c_d - ic_p \lambda) + 2c_p \lambda \omega,$$
$$\hat{\mathcal{E}}_n = c_h + (c_{a_1} \lambda + c_{a_2} \lambda) \bar{\lambda} - c_d (\bar{\lambda} \lambda + \bar{\omega} i) + 2\bar{\lambda}^2 (c_d + ic_p \bar{\lambda}) + 2c_p \bar{\lambda} \omega,$$
$$\hat{\mathcal{E}}_e = c_e + i(c_h \lambda - c_h \bar{\lambda}) + \bar{\lambda} \lambda (ic_p \bar{\lambda} - ic_p \lambda - c_p \lambda \lambda + c_d) - (c_d + c_{a_1} \lambda + c_{a_2} \lambda + c_p \omega) \omega.$$
These expressions are then introduced into (35), which has to hold for arbitrary \( \{c_r, \ldots, c_s\} \).
Putting \( c_{h_1} = 1 \) and the rest zero (say, the \( c_{h_1} \)-equation; in what follows, the analogous procedure will be referred to as \( c_{a_1} \)-equations), one immediately obtains
\[
Q(1) = Q(2),
\]
whereas the equation involving \( c_s \) implies
\[
Q(2)(\bar{\lambda} - \bar{\lambda}) = 0.
\]
Using these two relations in a suitable combination of the equations involving \( c_{a_1} \) and \( c_d \), explicitly \( \lambda [\bar{\lambda} (c_d \text{-equation}) - 2(c_{a_1} \text{-equation})] \), one obtains
\[
2\lambda \bar{\lambda} (M(1) + M(2)) = Q(2) \omega \bar{\lambda} (\lambda + \bar{\lambda}),
\]
whose real part simply reads
\[
2\lambda \bar{\lambda} (M(1) + M(2)) = 0.
\]
The same procedure applied to another combination of the equations involving \( c_d \) and \( c_p \), explicitly \( \omega (c_d \text{-equation}) - 2 (c_p \text{-equation}) \), leads to
\[
2\omega (M(1) + M(2)) = -\omega Q(2) (\lambda + \bar{\lambda}).
\]
From the last two equations, if \( M(1) + M(2) \neq 0 \), then \( \lambda = \omega = 0 \), and thus the main result in section 5 follows. As expected, once \( \lambda = \omega = 0 \), it can immediately be seen that then \( M(1) = M(2) \).

It must be stressed here that the case \( M(1) + M(2) = 0 \) does not lead to the same result: by setting \( M(2) = -M(1) \neq 0 \), another calculation shows that \( \omega = 0, \lambda - \bar{\lambda} = 0 \) (so that only the real part of \( \lambda \) survives) and finally, \( \lambda = 2M(1)/Q(1) \), exhausting equation (35).

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