Conformally Covariant Bi-differential Operators for Differential Forms

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Abstract: The classical Rankin–Cohen brackets are bi-differential operators from $C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$. They are covariant for the (diagonal) action of $SL(2, \mathbb{R})$ through principal series representations. We construct generalizations of these operators, replacing $\mathbb{R}$ by $\mathbb{R}^n$, the group $SL(2, \mathbb{R})$ by the group $SO_0(1, n + 1)$ viewed as the conformal group of $\mathbb{R}^n$, and functions by differential forms.

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1. Introduction

The \textit{Rankin–Cohen brackets} are the most famous examples of conformally covariant bi-differential operators. For a presentation of these operators from our point of view based on harmonic analysis of the group $SL(2, \mathbb{R})$, we refer the reader to the introduction of [2]. In [15] Ovsienko and Redou introduced their analogs for conformal analysis on $\mathbb{R}^n$. Later, Somberg [16], Kobayashi and Pevzner [13] investigated these conformally covariant bi-differential operators on $\mathbb{R}^n$ using the F-method.

A new construction of these covariant bi-differential operators was proposed by Beckmann and the second present author in [1], where (although implicitly) the source
operator method was introduced. In our situation, the source operator is a differential operator on \( \mathbb{R}^n \times \mathbb{R}^n \), covariant for the diagonal action of the conformal group \( \text{SO}_0(1, n+1) \). The covariant bi-differential operators are obtained by composing the source operator with the restriction map from \( \mathbb{R}^n \times \mathbb{R}^n \) to the diagonal. This technique has shown to be very efficient to produce new examples of covariant differential operators in many different contexts. In [2], we constructed covariant bi-differential operators in the context of simple real Jordan algebras. The article [3] contains an alternative construction of the covariant differential operators introduced by Juhl [9] in the context of the restriction of \( \mathbb{R}^n \) to \( \mathbb{R}^{n-1} \). In the same geometric context, Fischmann, Ørsted and Somberg [5] recently obtained a new construction of the covariant differential operators for differential forms, previously obtained by Kobayashi, Kubo and Pevzner in [14] and by Fischmann, Juhl and Somberg in [6]. Finally, it is worthwhile mentioning that a more general notion of symmetry breaking operators (not necessarily differential) has been studied by Kobayashi and his collaborators (see, e.g., [11–13]).

In the present paper, we construct bi-differential operators acting on spaces of differential forms which are covariant for the conformal group of \( \mathbb{R}^n \); more precisely for the group \( G = \text{SO}_0(1, n+1) \). To build these bi-differential operators, we use again the source operator method. The source operators are constructed as a composition of the multiplication operator by the function \( \|x - y\|^2 \) (using its transformation rule under the action of the conformal group) and classical Knapp–Stein intertwining operators. These intertwining operators on differential forms, previously obtained by Kobayashi, Kubo and Pevzner in [14] and by Fischmann, Juhl and Somberg in [6]. Finally, it is worthwhile mentioning that a more general notion of symmetry breaking operators (not necessarily differential) has been studied by Kobayashi and his collaborators (see, e.g., [11–13]).

The construction relies ultimately on two main identities stated in Theorem 3.2 and Theorem 3.3 (see also Theorem 3.4), the second one being the Euclidean Fourier transform of the first one. As they involve purely Euclidean harmonic analysis they are presented in Sect. 3, independently of the conformal context. In Sect. 4 we give some background on the conformal group of \( \mathbb{R}^n \) needed to describe the noncompact model for the principal series representations of \( \text{SO}_0(1, n+1) \) in Sect. 5. The conformal properties of the source operator are given in Sect. 6, where harmonic analysis of the group \( \text{SO}_0(1, n+1) \) plays a crucial role.

The corresponding covariant bi-differential operators are constructed in Sect. 7. The lack of a manageable decomposition of the tensor product \( \sigma_k \otimes \sigma_\ell \), \( 0 \leq k, \ell \leq n \), where \( \sigma_k \) denotes the representation of \( \text{SO}(n) \) on the space \( \Lambda^k \) of complex-valued alternating \( k \)-forms on \( \mathbb{R}^n \), prevents us from giving explicit formulas for the corresponding bi-differential operators, but this can be done at least for the Cartan factor \( \Lambda^{k+\ell} \) with \( 0 \leq k + \ell \leq n \), appearing in the decomposition of \( \Lambda^k \otimes \Lambda^\ell \), (see (7.1)).

### 2. Background on Differential Forms

Let \( \langle \cdot, \cdot \rangle \) be the standard Euclidean scalar product in \( \mathbb{R}^n \) and \( \| \cdot \| \) the corresponding norm. Let \( (e_1, e_2, \ldots, e_n) \) be the standard orthonormal basis of \( \mathbb{R}^n \) and let \( (e^*_1, e^*_2, \ldots, e^*_n) \) be its dual basis.

For \( 0 \leq k \leq n \), we denote by \( \Lambda^k = \Lambda^k(\mathbb{R}^n) \otimes \mathbb{C} \) the vector space of complex-valued alternating multilinear \( k \)-forms on \( \mathbb{R}^n \). A basis of the space \( \Lambda^k \) is given by

\[
\{ e^*_i := e^*_{i_1} \wedge \cdots \wedge e^*_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n \}.
\]

If \( \omega \in \Lambda^k \) and \( \eta \in \Lambda^\ell \), then \( \omega \wedge \eta \in \Lambda^{k+\ell} \). Furthermore,

\[
\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega.
\]
The exterior algebra $\Lambda := \bigoplus_{k=0}^{\infty} \Lambda^k = \bigoplus_{k=0}^{n} \Lambda^k$ is an associative algebra, graded with respect to the degree $k$.

The interior product of a $k$-form $\omega$ with a vector $x \in \mathbb{R}^n$ is the $(k - 1)$-form defined by

$$i_x \omega(x_1, \ldots, x_{k-1}) = \omega(x, x_1, \ldots, x_{k-1}).$$

Moreover,

$$i_{e_j} (e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*) = \begin{cases} 0 & \text{if } j \neq \text{any } i_r \\ (-1)^{r-1} e_{i_1}^* \wedge \cdots \wedge \hat{e}_{i_r}^* \wedge \cdots \wedge e_{i_k}^* & \text{if } j = i_r, \end{cases}$$

\hspace{1cm} (2.2)

where the “cap” over $e_{i_r}^*$ means that it is deleted from the exterior product. One may check that

$$i_x i_y + i_y i_x = 0.$$ \hspace{1cm} (2.3)

Given $x \in \mathbb{R}^n$, the exterior product of a $k$-form $\omega$ with the linear form $x^*$ is the $(k + 1)$-form defined by

$$i_x \omega = x^* \wedge \omega.$$ From the associativity of the wedge product and (2.1), it follows

$$i_x i_y + i_y i_x = 0.$$ \hspace{1cm} (2.4)

There is the following useful anticommutation relation

$$i_x i_y + i_y i_x = \langle x, y \rangle \text{Id}_\Lambda, \quad x, y \in \mathbb{R}^n.$$ \hspace{1cm} (2.5)

This is a fairly straightforward consequence of (2.2). Further, by (2.5), (2.4) and (2.3) we have

$$i_x i_y i_x = \langle x, y \rangle i_x, \quad i_y i_x i_y = \langle x, y \rangle i_y.$$ We denote by $i_j$ the interior product with the basis vector $e_j$ and by $e_j$ the exterior products with $e_j^*$. The following lemma is needed for later use.

**Lemma 2.1.** On $\Lambda^k$ we have

$$\sum_{j=1}^{n} e_j i_j = k \text{Id}_{\Lambda^k}, \quad \sum_{j=1}^{n} i_j e_j = (n - k) \text{Id}_{\Lambda^k}.$$  

**Proof.** Let $I = \{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$. In view of (2.2) and the fact that $e_j^* \wedge e_j^* = 0$ for every $j$, clearly we have

$$\left( \sum_{j \in I} e_j i_j \right) e_j^* = k e_j^*, \quad \left( \sum_{j \notin I} e_j i_j \right) e_j^* = 0.$$  

Similarly,

$$\left( \sum_{j \notin I} e_j i_j \right) e_I^* = 0, \quad \left( \sum_{j \notin I} i_j e_j \right) e_I^* = (n - k) e_I^*.$$  

The lemma is now a matter of putting pieces together. \hspace{1cm} $\square$
From now on, we will identify the dual of $\mathbb{R}^n$ with $\mathbb{R}^n$. For $0 \leq k \leq n$, let

$$\mathcal{E}^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n, \Lambda^k) \simeq C^\infty(\mathbb{R}^n) \otimes \Lambda^k(\mathbb{R}^n)$$

be the space of smooth complex-valued differential forms of degree $k$ on $\mathbb{R}^n$. An element of $\mathcal{E}^k(\mathbb{R}^n)$ can be uniquely represented as

$$\omega(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega_{i_1, \ldots, i_k}(x) e_{i_1}, \ldots, e_{i_k}, \quad (2.6)$$

where the coefficients $\omega_I$ are complex-valued smooth functions on $\mathbb{R}^n$. In particular, $\mathcal{E}^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$.

The direct sum

$$\mathcal{E}(\mathbb{R}^n) := \bigoplus_{k=0}^n \mathcal{E}^k(\mathbb{R}^n)$$

is the linear space of all smooth differential forms.

From the properties of the interior product $\iota_x$ and the exterior product $\varepsilon_x$ on $\Lambda^k$ which we discussed above, one gets analogue properties on $\mathcal{E}^k(\mathbb{R}^n)$.

We now define the exterior differential $d : \mathcal{E}^k(\mathbb{R}^n) \rightarrow \mathcal{E}^{k+1}(\mathbb{R}^n)$ by

$$d = \sum_{j=1}^n \varepsilon_j \partial_j,$$

and the co-differential $\delta : \mathcal{E}^{k+1}(\mathbb{R}^n) \rightarrow \mathcal{E}^k(\mathbb{R}^n)$ by

$$\delta = -\sum_{j=1}^n \iota_j \partial_j,$$

where $\partial_j$ is the directional derivative in the direction of the basis vector $e_j$. We set $\delta = 0$ on $\mathcal{E}^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$. In the light of (2.3), (2.4) and (2.5), direct computations show $d \circ d = \delta \circ \delta = 0$ and

$$d \iota_j + \iota_j d = \partial_j, \quad \delta \varepsilon_j + \varepsilon_j \delta = -\partial_j, \quad 0 \leq j \leq n. \quad (2.7)$$

We close this section by introducing the Hodge Laplacian on differential forms, defined by

$$\Delta := -(d \delta + \delta d) = \sum_{j=1}^n \partial_j^2. \quad (2.8)$$

Henceforth we will denote the Hodge Laplacian by $Q(\frac{\partial}{\partial x})$ where

$$Q(x) := x_1^2 + \cdots + x_n^2. \quad (2.9)$$
3. The Main Identities for Riesz Distributions on Differential Forms

Let $s$ be a complex parameter and consider, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the formula

$$
\langle \mathcal{R}_s, \varphi \rangle = \pi^{-\frac{n}{2}} 2^{-n-s} \frac{\Gamma \left( -\frac{s}{2} \right)}{\Gamma \left( \frac{n+s}{2} \right)} \int_{\mathbb{R}^n} \varphi(x) \|x\|^s dx.
$$

(3.1)

For $-n < \text{Re} \, s < 0$, this formula defines a tempered distribution depending holomorphically on $s$. The normalization factor is chosen for convenience, as we shall see below (see (3.2)). By standard argument (see [7]) it can be analytically continued to $\mathbb{C}$, yielding a meromorphic family of tempered distributions, called the Riesz distributions.

We follow the following convention for the Fourier transform on Schwartz functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$:

$$
\mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{i(x,\xi)} dx,
$$

which extends to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. It is known (see [7] or [17, p. 38]) that the image of $\mathcal{R}_s$ by the Fourier transform is

$$
\mathcal{F}(\mathcal{R}_s)(\xi) = \|\xi\|^{-n-s}.
$$

(3.2)

The classical Riesz distributions offer a good motivation for defining Riesz distributions for differential forms on $\mathbb{R}^n$ with coefficients in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

For $0 \leq k \leq n$, let $\mathcal{SE}^k(\mathbb{R}^n)$ (resp. $\mathcal{S}'E^k(\mathbb{R}^n)$) be the space of differential $k$-forms represented as in (2.6) with coefficients $\omega_{i_1,\ldots,i_k}$ in $\mathcal{S}(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$). We pin down that we may extend the Fourier transform $\mathcal{F}$ on the space $\mathcal{SE}^k(\mathbb{R}^n)$ by acting on the coefficients $\omega_{i_1,\ldots,i_k}$ of the form (2.6). For $0 \leq k \leq n$ and a complex parameter $s$, let $\mathcal{R}_s^k$ be the Riesz distribution on differential forms defined by

$$
\langle \mathcal{R}_s^k, \omega \rangle = \pi^{-\frac{n}{2}} 2^{-s-n+1} \frac{\Gamma \left( -\frac{s}{2} + 1 \right)}{\Gamma \left( \frac{s+n}{2} \right)} \int_{\mathbb{R}^n} \|x\|^{s-2}(\tau_x \xi_x + \xi_x \tau_x)\omega(x) dx,
$$

(3.3)

with $\omega \in \mathcal{SE}^k(\mathbb{R}^n)$. For $-n < \text{Re} \, s < 0$, this formula defines a tempered distribution depending holomorphically on $s$. By [4, §3.2], $\mathcal{R}_s^k$ can be analytically continued to $\mathbb{C}$, giving a meromorphic family of tempered distributions. When $k = 0$, the identity $\tau_x \xi_x + \xi_x \tau_x = \|x\|^2 \text{Id}_\mathcal{E}$ implies immediately that $\mathcal{R}_s^0$ is nothing but the classical Riesz distribution (3.1), up to $-s$.

In [4, Theorem 3.2] the authors proved that the Fourier transform of $\mathcal{R}_s^k$ is given by

$$
\mathcal{F}(\mathcal{R}_s^k)(\xi) = \|\xi\|^{-s-n-2}(-(s+2k)\tau_\xi \xi_\xi + (s + 2n - 2k)\xi_\xi \tau_\xi)
$$

(3.4)

for $s$ not a pole. Due to the facts $\tau_\xi = 0$ and $\xi_\xi = \|\xi\|^2 \text{Id}_\mathcal{E}$ on $\mathcal{SE}^0(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$, it follows that (3.4) for $k = 0$ coincides with (3.2), up to $-s$.

In order to simplify notation, it is convenient to let

$$
\mathcal{Z}_s^k := \mathcal{F}(\mathcal{R}_s^k)
$$

(3.5)

i.e.

$$
\mathcal{Z}_s^k(x) = \|x\|^{s-2}\left((s+n-2k)\tau_x \xi_x - (s-n+2k)\xi_x \tau_x\right).
$$
We should point out that in all arguments below we first prove the desired result when the complex parameter $s$ is so that everything makes sense, and then we extend it meromorphically to the complex plane $\mathbb{C}$.

We shall need the following crucial result (which might be of some interest in its own right):

**Theorem 3.1.** The distribution $\mathcal{Z}^k_s$ satisfies the following properties:

\[
\mathcal{Z}^k_s(x) = \mathcal{Z}^k_{s-2}(x) (a_{k,s} t_x \varepsilon_x + b_{k,s} \varepsilon_x t_x),
\]

\[
\partial_{x_j} \mathcal{Z}^k_s(x) = \mathcal{Z}^k_{s-2}(x) (s x_j \operatorname{Id} + c_{k,s} t_x \varepsilon_j + d_{k,s} \varepsilon_x t_j),
\]

\[
Q (\partial_{x}) \mathcal{Z}^k_s(x) = s(s + n) \mathcal{Z}^k_{s-2}(x),
\]

with

\[
a_{k,s} = \frac{s + n - 2k}{s + n - 2k - 2}, \quad b_{k,s} = \frac{s - n + 2k}{s - n + 2k - 2},
\]

and

\[
c_{k,s} = \frac{2s}{s + n - 2k - 2}, \quad d_{k,s} = \frac{2s}{s - n + 2k - 2}.
\]

Above $Q (\partial_{x})$ denotes the Hodge Laplacian (see (2.8)).

**Proof.** (1) On the one hand, we may rewrite $\mathcal{Z}^k_s(x)$ as

\[
\mathcal{Z}^k_s(x) = \|x\|^2 \mathcal{Z}^k_{s-2}(x) - 2s(s + n)\|x\|^{s-2}(t_x \varepsilon_x - \varepsilon_x t_x).
\]

On the other hand, using the fact $t_x \varepsilon_x + \varepsilon_x t_x = \|x\|^2 \operatorname{Id}_\varepsilon$, we may rewrite the term $\|x\|^2(t_x \varepsilon_x - \varepsilon_x t_x)$ as follows:

\[
\|x\|^2(t_x \varepsilon_x - \varepsilon_x t_x) = ((s + n - 2k - 2)t_x \varepsilon_x - (s + n + 2k - 2)\varepsilon_x t_x)
\]

\[
\left( a_{k,s}' t_x \varepsilon_x + b_{k,s}' \varepsilon_x t_x \right),
\]

where

\[
a_{k,s}' = \frac{1}{s + n - 2k - 2}, \quad b_{k,s}' = \frac{1}{s - n + 2k - 2}.
\]

Thus,

\[
\mathcal{Z}^k_s(x) = \|x\|^2 \mathcal{Z}^k_{s-2}(x) + 2 \mathcal{Z}^k_{s-2}(x) (a_{k,s}' t_x \varepsilon_x + b_{k,s}' \varepsilon_x t_x).
\]

Using again the identity $t_x \varepsilon_x + \varepsilon_x t_x = \|x\|^2 \operatorname{Id}_\varepsilon$ to deduce the first statement.

(2) First we have

\[
\partial_{x_j} (t_x \varepsilon_x) = t_j \varepsilon_x + t_x \varepsilon_j = x_j \operatorname{Id}_\varepsilon - \varepsilon_x t_j + t_x \varepsilon_j,
\]

\[
\partial_{x_j} (\varepsilon_x t_x) = \varepsilon_j t_x + \varepsilon_x t_j = x_j \operatorname{Id}_\varepsilon + \varepsilon_x t_j - t_x \varepsilon_j,
\]
where, for abbreviation, $t_j$ (resp. $e_j$) denotes $t_{e_j}$ (resp. $e_{e_j}$). Above we have used the identity (2.5). Then
\[
\partial_{x_j} Z^k_s(x) = (s - 2)x_j \|x\|^{s-4} \left( (s + n - 2k)t_x e_x - (s - n + 2k)e_x t_x \right)
+ \|x\|^{s-2} \left( (s + n - 2k)(x_j \text{Id}_e - e_x t_j - t_x e_j) \right)
- (s - n + 2k)(x_j \text{Id}_e + e_x t_j - t_x e_j)
\]
\[
= (s - 2)x_j \|x\|^{s-4} \left( ((s + n - 2k)t_x e_x - (s - n + 2k)e_x t_x) \right)
+ \|x\|^{s-2} \left( 2(s - 2)(t_x e_j - e_x t_j) \right)
\]
\[
= (s - 2)x_j Z^k_{s-2}(x) + 2s \|x\|^{s-2}(t_x e_j - e_x t_j).
\]

Now using again the trick (3.11) we obtain
\[
\partial_{x_j} Z^k_s(x) = sx_j Z^k_{s-2}(x) + 2s Z^k_{s-2}(x)(a'_{k,s} t_x e_j + b'_{k,s} e_x t_j),
\]
and (3.7) follows.

(3) The definition of $Z^k_s(x)$ and the intertwining property $Q(\partial_x) \circ F = -F \circ \| \cdot \|_2^2$ imply
\[
Q(\partial_x) Z^k_s(x) = -F(\| \cdot \|_2^2 R^k_{s-n})(x)
= s(s + n)F(R^k_{s-n+2})(x) = s(s + n) Z^k_{s-2}(x).
\]

For $0 \leq k, \ell \leq n$, we define the space $E^{k,\ell}(\mathbb{R}^n \times \mathbb{R}^n)$ as the space of smooth functions on $\mathbb{R}^n \times \mathbb{R}^n$ with values in $\Lambda^k \otimes \Lambda^\ell$. More generally, denote by $D E^{k,\ell}(\mathbb{R}^n \times \mathbb{R}^n)$ (resp. $S E^{k,\ell}(\mathbb{R}^n \times \mathbb{R}^n)$, $S' E^{k,\ell}(\mathbb{R}^n \times \mathbb{R}^n)$) the space of differential forms of bidegree $(k, \ell)$ on $\mathbb{R}^n \times \mathbb{R}^n$ with coefficients in $D(\mathbb{R}^n \times \mathbb{R}^n)$ (resp. $S(\mathbb{R}^n \times \mathbb{R}^n)$, $S'(\mathbb{R}^n \times \mathbb{R}^n)$).

**Theorem 3.2.** For every $\omega \in SE^{k,\ell}(\mathbb{R}^n \times \mathbb{R}^n)$, the following formula holds true
\[
Q(\partial_x - \partial_y) \left( Z^k_s(x) \otimes Z^\ell_t(y) \omega(x, y) \right) = Z^k_{s-2}(x) \otimes Z^\ell_{t-2}(y) D^{k,\ell}_{s,t}(x, y),
\]
where $D^{k,\ell}_{s,t}$ is the differential operator on $(k, \ell)$-differential forms given by
\[
D^{k,\ell}_{s,t} = \left( a_{k,s} t_x e_x + b_{k,s} e_x t_x \right) \otimes \left( a_{\ell,t} t_y e_y + b_{\ell,t} e_y t_y \right) \circ Q(\partial_x - \partial_y)
+ 2 \sum_{j=1}^n (sx_j \text{Id}_e + c_{k,s} t_x e_j + d_{k,s} e_x t_j) \otimes \left( a_{\ell,t} t_y e_y + b_{\ell,t} e_y t_y \right) \circ (\partial_{x_j} - \partial_{y_j})
- 2 \sum_{j=1}^n (a_{k,s} t_x e_x + b_{k,s} e_x t_x) \otimes (ty_j \text{Id}_e + c_{\ell,t} t_y e_j + d_{\ell,t} e_y t_j) \circ (\partial_{x_j} - \partial_{y_j})
- 2 \sum_{j=1}^n (sx_j \text{Id}_e + c_{k,s} t_x e_j + d_{k,s} e_x t_j) \otimes (ty_j \text{Id}_e + c_{\ell,t} t_y e_j + d_{\ell,t} e_y t_j)
+ s(s + n) \text{Id}_e \otimes (a_{\ell,t} t_y e_y + b_{\ell,t} e_y t_y) + t(t + n)(a_{k,s} t_x e_x + b_{k,s} e_x t_x) \otimes \text{Id}_e.$
The coefficients \( a_{k,s}, \ b_{k,s}, \ c_{k,s} \) and \( d_{k,s} \) are given by (3.9) and (3.10) (and similarly when the subscripts \( k, s \) are replaced by \( \ell, t \)).

**Proof.** A routine calculation gives

\[
\begin{align*}
Q (\partial_x - \partial_y) \left( Z^k_s(x) \otimes Z^\ell_t(y) \omega(x, y) \right) &= \left( Q (\partial_x - \partial_y) Z^k_s(x) \otimes Z^\ell_t(y) \right) \omega(x, y) \\
+ Z^k_s(x) \otimes Z^\ell_t(y) Q (\partial_x - \partial_y) \omega(x, y) &+ \tilde{Q} (\partial_x, \partial_y) \left( Z^k_s(x) \otimes Z^\ell_t(y) \omega(x, y) \right),
\end{align*}
\]

where

\[
\begin{align*}
\tilde{Q} (\partial_x, \partial_y) \left( Z^k_s(x) \otimes Z^\ell_t(y) \omega(x, y) \right) \\
= 2 \sum_{j=1}^n \left( \partial_{x_j} Z^k_s(x) \otimes Z^\ell_t(y) \partial_{y_j} \omega(x, y) + Z^k_s(x) \otimes \partial_{y_j} Z^\ell_t(y) \partial_{x_j} \omega(x, y) \right) \\
- 2 \sum_{j=1}^n \left( Z^k_s(x) \otimes \partial_{y_j} Z^\ell_t(y) \partial_{x_j} \omega(x, y) - \partial_{x_j} Z^k_s(x) \otimes Z^\ell_t(y) \partial_{y_j} \omega(x, y) \right).
\end{align*}
\]

Firstly, in view of the identities (3.7) and (3.8), we have

\[
\begin{align*}
Q (\partial_x - \partial_y) Z^k_s(x) \otimes Z^\ell_t(y) \omega(x, y) \\
&= Q (\partial_x) Z^k_s(x) \otimes Z^\ell_t(y) \omega(x, y) + Z^k_s(x) \otimes Q (\partial_y) Z^\ell_t(y) \omega(x, y) \\
&- 2 \sum_{j=1}^n \partial_{x_j} Z^k_s(x) \otimes \partial_{y_j} Z^\ell_t(y) \omega(x, y) \\
&= s(s + n) Z^k_{s-2}(x) \otimes Z^\ell_t(y) \omega(x, y) + t(t + n) Z^k_s(x) \otimes Z^\ell_{t-2}(y) \omega(x, y) \\
&- 2 Z^k_s(x) \otimes Z^\ell_{t-2}(y) \\
&\left\{ \sum_{j=1}^n (s x_j \text{Id}_{C^\ell} + c_{k,s} t_x e_j + d_{k,s} e_x t_j) \otimes (t y_j \text{Id}_{C^\ell} + c_{\ell,t} t_y e_j + d_{\ell,t} e_y t_j) \right\} \omega(x, y),
\end{align*}
\]

thus,

\[
\begin{align*}
Q (\partial_x - \partial_y) Z^k_s(x) \otimes Z^\ell_t(y) \omega(x, y) &= Z^k_{s-2}(x) \otimes Z^\ell_{t-2}(y) \\
\left\{ s(s + n) \text{Id}_{C^\ell} \otimes (a_{\ell,t} t_y e_y + b_{\ell,t} e_y t_y) + t(t + n) (a_{k,s} t_x e_x + b_{k,s} e_x t_x) \otimes \text{Id}_{C^\ell} \\
- 2 \sum_{j=1}^n (s x_j \text{Id}_{C^\ell} + c_{k,s} t_x e_j + d_{k,s} e_x t_j) \otimes (t y_j \text{Id}_{C^\ell} + c_{\ell,t} t_y e_j + d_{\ell,t} e_y t_j) \right\} \omega(x, y).\end{align*}
\]

Secondly, the identity (3.6) gives

\[
Z^k_s(x) \otimes Z^\ell_t(y) Q (\partial_x - \partial_y) \omega(x, y) \\
= Z^k_{s-2}(x) \otimes Z^\ell_{t-2}(y) \left\{ (a_{k,s} t_x e_x + b_{k,s} e_x t_x) \otimes (a_{\ell,t} t_y e_y + b_{\ell,t} e_y t_y) \right\} Q (\partial_x - \partial_y) \omega(x, y).
\]
Finally, using again (3.6) and (3.7) we obtain
\[
\tilde{Q} \left( \partial_x, \partial_y \right) \left( \mathcal{Z}^k_s(x) \otimes \mathcal{Z}^\ell_t(y) \omega(x, y) \right) = 2 \mathcal{Z}^k_{s-2}(x) \otimes \mathcal{Z}^\ell_{t-2}(y) \Big\{ \sum_{j=1}^n (sx_j \, \text{Id}_{E^k} + c_{k,s} t_x e_j + d_{k,s} e_x t_j) \otimes (a_{\ell,t} t_y e_y + b_{\ell,t} e_y t_y) \, \partial_{x_j} \omega(x, y) \\
+ \sum_{j=1}^n (a_{k,s} t_x e_x + b_{k,s} e_x t_x) \otimes (t y_j \, \text{Id}_{E^\ell} + c_{\ell,s} t_y e_j + d_{\ell,s} e_y t_j) \, \partial_{y_j} \omega(x, y) \\
- \sum_{j=1}^n (a_{k,s} t_x e_x + b_{k,s} e_x t_x) \otimes (t y_j \, \text{Id}_{E^\ell} + c_{\ell,s} t_y e_j + d_{\ell,s} e_y t_j) \, \partial_{x_j} \omega(x, y) \\
- \sum_{j=1}^n (s x_j \, \text{Id}_{E^k} + c_{k,s} t_x e_j + d_{k,s} e_x t_j) \otimes (a_{\ell,t} t_y e_y + b_{\ell,t} e_y t_y) \, \partial_{y_j} \omega(x, y) \Big\},
\]
therefore,
\[
\tilde{Q} \left( \partial_x, \partial_y \right) \left( \mathcal{Z}^k_s(x) \otimes \mathcal{Z}^\ell_t(y) \omega(x, y) \right) = 2 \mathcal{Z}^k_{s-2}(x) \otimes \mathcal{Z}^\ell_{t-2}(y) \Big\{ \sum_{j=1}^n (sx_j \, \text{Id}_{E^k} + c_{k,s} t_x e_j + d_{k,s} e_x t_j) \otimes (a_{\ell,t} t_y e_y + b_{\ell,t} e_y t_y) \, \partial_{x_j} \omega(x, y) \\
+ \sum_{j=1}^n (a_{k,s} t_x e_x + b_{k,s} e_x t_x) \otimes (t y_j \, \text{Id}_{E^\ell} + c_{\ell,s} t_y e_j + d_{\ell,s} e_y t_j) \, \partial_{y_j} \omega(x, y) \\
- \sum_{j=1}^n (a_{k,s} t_x e_x + b_{k,s} e_x t_x) \otimes (t y_j \, \text{Id}_{E^\ell} + c_{\ell,s} t_y e_j + d_{\ell,s} e_y t_j) \, \partial_{x_j} \omega(x, y) \\
- \sum_{j=1}^n (s x_j \, \text{Id}_{E^k} + c_{k,s} t_x e_j + d_{k,s} e_x t_j) \otimes (a_{\ell,t} t_y e_y + b_{\ell,t} e_y t_y) \, \partial_{y_j} \omega(x, y) \Big\}.
\]
This finishes the proof of the theorem. \(\Box\)

For \(s \in \mathbb{C}\), let
\[
J^k_s \omega(x) := \int_{\mathbb{R}^n} \mathcal{R}^k_s(x - y) \omega(y) \, dy, \quad \omega \in \mathcal{S} \mathcal{E}^k(\mathbb{R}^n). \tag{3.12}
\]
We may see \(J^k_s\) as a convolution operator with the distribution \(\mathcal{R}^k_s\),
\[
J^k_s \omega = \mathcal{R}^k_s \ast \omega.
\]
So (3.12) defines a meromorphic family of operators from \(\mathcal{S} \mathcal{E}^k(\mathbb{R}^n)\) to \(\mathcal{S}' \mathcal{E}^k(\mathbb{R}^n)\).

Recall the following formulas for the Fourier transform:
\[
\mathcal{F} \left( \partial_{y_j} \omega \right)(x) = -\sqrt{-1} x_j \mathcal{F}(\omega)(x), \quad \mathcal{F}(y_j \omega)(x) = -\sqrt{-1} \partial_{x_j} \mathcal{F}(\omega)(x) \tag{3.13}
\]
\[
\mathcal{F}(d \omega)(x) = -\sqrt{-1} e_x \mathcal{F}(\omega)(x), \quad \mathcal{F}(e_y \omega)(x) = -\sqrt{-1} d \mathcal{F}(\omega)(x) \tag{3.14}
\]
\[
\mathcal{F}(\delta \omega)(x) = \sqrt{-1} t_x \mathcal{F}(\omega)(x), \quad \mathcal{F}(t_y \omega)(x) = \sqrt{-1} \delta \mathcal{F}(\omega)(x). \tag{3.15}
\]
For $s \in \mathbb{C}$, let

$$
\alpha_{k,s} = (s + n - 2k)(s - n + 2k - 2) \quad \beta_{k,s} = (s - n + 2k)(s + n - 2k - 2) \quad (3.16)
$$

$$
\gamma_{k,s} = 2s(s - n + 2k - 2) \quad \delta_{k,s} = 2s(s + n - 2k - 2) \quad (3.17)
$$

and

$$
\kappa_{k,s} = (s - n + 2k - 2)(s + n - 2k - 2). \quad (3.18)
$$

**Theorem 3.3.** The following identity holds true

$$
-k_{s,t} \|x - y\|^2 (J_{s-n}^k \otimes J_{t-n}^\ell) = (J_{s-n+2}^k \otimes J_{t-n+2}^\ell) \circ E_{s,t}^{k,\ell},
$$

where $E_{s,t}^{k,\ell}$ is the differential operator with polynomial coefficients in $x, y$ (and also in $s, t$) defined on $(k, \ell)$-differential forms by

$$
E_{s,t}^{k,\ell} = - (\alpha_{k,s} \delta d + \beta_{k,s} d \delta) \otimes (\alpha_{t,\ell} \delta d + \beta_{t,\ell} d \delta) \circ \|x - y\|^2
$$

$$
-2 \sum_{j=1}^{n} (sk_{k,s} \partial x_j - \gamma_{k,s} \delta e_j + \delta_{k,s} d t_j) \otimes (\alpha_{t,\ell} \delta d + \beta_{t,\ell} d \delta) \circ (x_j - y_j)
$$

$$
+2 \sum_{j=1}^{n} (\alpha_{k,s} \delta d + \beta_{k,s} d \delta) \otimes (tk_{k,s} \partial y_j - \gamma_{t,\ell} \delta e_j + \delta_{t,\ell} d t_j) \circ (x_j - y_j)
$$

$$
+2 \sum_{j=1}^{n} (sk_{k,s} \partial x_j - \gamma_{k,s} \delta e_j + \delta_{k,s} d t_j) \otimes (tk_{k,s} \partial y_j - \gamma_{t,\ell} \delta e_j + \delta_{t,\ell} d t_j)
$$

$$
+ s(s + n) \kappa_{k,s} \text{Id}_{C^k} \otimes (\alpha_{t,\ell} \delta d + \beta_{t,\ell} d \delta) + t(t + n) \kappa_{t,\ell} (\alpha_{k,s} \delta d + \beta_{k,s} d \delta) \otimes \text{Id}_{C^\ell}.
$$

(3.19)

This is merely the Fourier transform version of Theorem 3.2, using (3.5) and formulas (3.13), (3.14) and (3.15). We omit details.

Next, we need to rewrite $E_{s,t}^{k,\ell}$ in its normal form (i.e. multiplications after differentiations). Before stating the result, let us introduce for $0 \leq k \leq n, 1 \leq j \leq n$ the following differential operators:

$$
\Box_{k,s} := \alpha_{k,s} \delta d + \beta_{k,s} d \delta,
$$

$$
= (s + n - 2k)(s - n + 2k - 2)\delta d + (s - n + 2k)(s + n - 2k - 2)d \delta
$$

$$
\nabla_{k,s,j} := (2\alpha_{k,s} \partial x_j - 4(n - 2k) e_j \delta + 4(n - 2k) d t_j) - (s\alpha_{k,s} \partial x_j + \gamma_{k,s} e_j \delta + \delta_{k,s} d t_j)
$$

$$
= (2 - s) [(s + n - 2k)(s - n + 2k - 2)\partial x_j
$$

$$
+ 2(s + n - 2k) e_j \delta + 2(s + n - 2k) d t_j].
$$

where the coefficients $\alpha_{k,s}, \beta_{k,s}, \gamma_{k,s}$ and $\delta_{k,s}$ are given by (3.9) and (3.10). Similarly, we introduce the operators $\Box_{t,\ell}$ and $\nabla_{t,\ell,j}$ with respect to the $y$-variable.
Theorem 3.4. The operator $E_{s,t}^{k,\ell}$ in Theorem 3.3 can be rewritten in the following normal form:

$$
E_{s,t}^{k,\ell} = -\|x - y\|^2 \Box_{k,s} \otimes \Box_{\ell,t} + 2 \sum_{j=1}^n (x_j - y_j) \left\{ \nabla_{k,s,j} \otimes \Box_{\ell,t} - \nabla_{k,s} \otimes \nabla_{\ell,t,j} \right\} + 2 \sum_{j=1}^n \nabla_{k,s,j} \otimes \nabla_{\ell,t,j} \\
+ (t - 2)(t - n - 2)(t - n + 2\ell)(t + n - 2\ell) \Box_{k,s} \otimes \text{Id}_{E_{\ell, t}} \\
+ (s - 2)(s - n - 2)(s - n + 2k)(s + n - 2k) \text{Id}_{E_k} \otimes \Box_{\ell, t}.
$$

The proof is straightforward, but long and tedious. We first need some elementary formulas.

Lemma 3.5. Let $\omega$ be a k-form. Then, for fixed $y \in \mathbb{R}^n$,

$$
\delta d (x_j - y_j) \omega = (x_j - y_j) \delta d \omega - 2 \partial x_j \omega - \epsilon_j \delta \omega + d \iota_j \omega
$$

$$
d \delta (x_j - y_j) \omega = (x_j - y_j) d \delta \omega + \epsilon_j \delta \omega - d \iota_j \omega
$$

$$
\delta d \|x - y\|^2 \omega = \|x - y\|^2 \delta d \omega + 2 \sum_{j=1}^n (x_j - y_j)(-2 \partial x_j - \epsilon_j \delta + d \iota_j) \omega - 2(n - k) \omega
$$

$$
d \delta \|x - y\|^2 \omega = \|x - y\|^2 d \delta \omega + 2 \sum_{j=1}^n (x_j - y_j)(\epsilon_j \delta - d \iota_j) \omega - 2k \omega.
$$

Derivations are taken with respect to the x-variable.

Proof. This is a direct consequence of the identities

$$
\begin{cases}
  d (x_j - y_j) \omega = \epsilon_j \omega + (x_j - y_j) d \omega \\
  \delta (x_j - y_j) \omega = - \iota_j \omega + (x_j - y_j) \delta \omega,
\end{cases}
$$

and

$$
\begin{cases}
  d \|x - y\|^2 \omega = \|x - y\|^2 d \omega + 2 \sum_{j=1}^n (x_j - y_j) \epsilon_j \omega \\
  \delta \|x - y\|^2 \omega = \|x - y\|^2 \delta \omega - 2 \sum_{j=1}^n (x_j - y_j) \iota_j \omega.
\end{cases}
$$

\[\square\]

We may now start the proof of Theorem 3.4. In the light of Lemma 3.5 (and its version with respect to y) together with the anti-commutator laws in (2.7) we arrive at
the expressions below for the first three terms of the operator \( E_{s,t}^{k,\ell} \) in Theorem 3.3. The first term of the operator \( E_{s,t}^{k,\ell} \) in (3.19) can be rewritten as:

\[
-\|x - y\|^2 \left( \alpha_{k,s} \delta d + \beta_{k,s} d \delta \right) \otimes \left( \alpha_{\ell,t} \delta d + \beta_{\ell,t} d \delta \right)
+ 4 \sum_{j=1}^{n} (x_j - y_j) \left( -2(n - 2k)(e_j \delta - dt_j) + \alpha_{k,s} \partial x_j \right) \otimes \left( \alpha_{\ell,t} \delta d + \beta_{\ell,t} d \delta \right)
- 4 \sum_{j=1}^{n} (x_j - y_j) \left( \alpha_{k,s} \delta d + \beta_{k,s} d \delta \right) \otimes \left( -2(n - 2\ell)(e_j \delta - dt_j) + \alpha_{\ell,t} \partial y_j \right)
+ 8 \sum_{j=1}^{n} \left( -2(n - 2k)(e_j \delta - dt_j) + \alpha_{k,s} \partial x_j \right) \otimes \left( -2(n - 2\ell)(e_j \delta - dt_j) + \alpha_{\ell,t} \partial y_j \right)
+ 2(\ell - \ell) \alpha_{\ell,t} + \ell \beta_{\ell,t} \left( \alpha_{k,s} \delta d + \beta_{k,s} d \delta \right) \otimes \text{Id}_E \ell
+ 2(\ell - k) \alpha_{k,s} + k \beta_{k,s} \text{Id}_E \ell \otimes \left( \alpha_{\ell,t} \delta d + \beta_{\ell,t} d \delta \right).
\]

The second term of the operator \( E_{s,t}^{k,\ell} \) in (3.19) can be rewritten as:

\[
-2 \sum_{j=1}^{n} (x_j - y_j) \left( s \alpha_{k,s} \partial x_j + \gamma_{k,s} e_j \delta + k_{s} \delta d t_j \right) \otimes \left( \alpha_{\ell,t} \delta d + \beta_{\ell,t} d \delta \right)
- 4 \sum_{j=1}^{n} \left( s \alpha_{k,s} \partial x_j + \gamma_{k,s} e_j \delta + k_{s} \delta d t_j \right) \otimes \left( -2(n - 2\ell)(e_j \delta - dt_j) + \alpha_{\ell,t} \partial y_j \right)
- 2 \left( nsk_{k,s} + (n - k) \gamma_{k,s} + k \delta_{k,s} \right) \text{Id}_E \ell \otimes \left( \alpha_{\ell,t} \delta d + \beta_{\ell,t} d \delta \right).
\]

Finally, the third term of the operator \( E_{s,t}^{k,\ell} \) in (3.19) can be rewritten as:

\[
2 \sum_{j=1}^{n} (x_j - y_j) \left( \alpha_{k,s} \delta d + \beta_{k,s} d \delta \right) \otimes \left( t \alpha_{\ell,t} \partial y_j + \gamma_{\ell,t} e_j \delta + \delta_{\ell,t} d t_j \right)
- 4 \sum_{j=1}^{n} \left( -2(n - 2k)(e_j \delta - dt_j) + \alpha_{k,s} \partial x_j \right) \otimes \left( t \alpha_{\ell,t} \partial y_j + \gamma_{\ell,t} e_j \delta + \delta_{\ell,t} d t_j \right)
- 2 \left( nt \kappa_{\ell,t} + (n - \ell) \gamma_{\ell,t} + \ell \delta_{\ell,t} \right) \left( \alpha_{k,s} \delta d + \beta_{k,s} d \delta \right) \otimes \text{Id}_E \ell.
\]

It is worthwhile noting that by the anti-commutator laws in (2.7) we may rewrite the fourth term of the operator \( E_{s,t}^{k,\ell} \) in (3.19) as:

\[
2 \sum_{j=1}^{n} \left( s \alpha_{k,s} \partial x_j + \gamma_{k,s} e_j \delta + k_{s} \delta d t_j \right) \otimes \left( t \alpha_{\ell,t} \partial y_j + \gamma_{\ell,t} e_j \delta + \delta_{\ell,t} d t_j \right).
\]

It remains to sum up all terms to finish the proof of Theorem 3.4.

We close this section by writing the operator \( E_{s,t}^{0,0} \) in the particular case \( k = \ell = 0 \). Here the operator \( E_{s,t}^{0,0} \) will act on the space \( S \mathcal{E}^{0,0}(\mathbb{R}^n \times \mathbb{R}^n) = S(\mathbb{R}^n \times \mathbb{R}^n) \). Since \( \delta = 0 \) and \( t_j = 0 \) on scalar functions, the operators \( \square_{0,s} \) and \( \nabla_{0,s,j} \) reduce to

\[
\square_{0,s} = \alpha_{0,s} \delta d = -\alpha_{0,s} Q(\partial s),
\]

\[
\nabla_{0,s,j} = \gamma_{0,s} e_j \delta + \delta_{0,s} d t_j.
\]
(see (2.8)), and
\[ \nabla_{0,s,j} = (2 - s) \alpha_{0,s} \partial_{x_j}. \]

Similar identities hold with respect to \( y \). Hence
\[ E_{s,t}^{0,0} = \alpha_{0,s} \alpha_{0,t} \left\{ -\|x - y\|^2 Q(\partial_x) \otimes Q(\partial_y) - 2 \sum_{j=1}^{n} (t - 2)(x_j - y_j) Q(\partial_x) \otimes \partial_{y_j} \right. \]
\[ + 2 \sum_{j=1}^{n} (s - 2)(x_j - y_j) \partial_{x_j} \otimes Q(\partial_y) - (t - 2)(t - n) Q(\partial_x) \otimes \text{Id}_E \]
\[ + 2 \sum_{j=1}^{n} (s - 2)(t - 2) \partial_{x_j} \otimes \partial_{y_j} - (s - 2)(s - n) \text{Id}_E \otimes Q(\partial_y) \left\} \right. \]
\[ = (s + n)(s - n - 2)(t + n)(t - n - 2) \]
\[ \times \left\{ -\|x - y\|^2 Q(\partial_x) \otimes Q(\partial_y) - 2 \sum_{j=1}^{n} (t - 2)(x_j - y_j) Q(\partial_x) \otimes \partial_{y_j} \right. \]
\[ + 2 \sum_{j=1}^{n} (s - 2)(x_j - y_j) \partial_{x_j} \otimes Q(\partial_y) - (t - 2)(t - n) Q(\partial_x) \otimes \text{Id}_E \]
\[ + 2 \sum_{j=1}^{n} (s - 2)(t - 2) \partial_{x_j} \otimes \partial_{y_j} - (s - 2)(s - n) \text{Id}_E \otimes Q(\partial_y) \left\}. \right. \]

Up to the normalization constant \((s + n)(s - n - 2)(t + n)(t - n - 2)\) and the change of variables \( s \) by \( 2s \) and \( t \) by \( 2t \), the operator \( E_{s,t}^{0,0} \) coincides with the differential operator obtained in [2, Proposition 10.3] to build covariant bi-differential operators under the (diagonal) action of the Lie group \( O(n + 1, 1) \).

4. Background on the Conformal Group of \( \mathbb{R}^n \)

Let \( \mathbb{R}^{1,n+1} \) be the \( n + 2 \)-dimensional real vector space equipped with the Lorentzian quadratic form
\[ [x, x] = x_0^2 - (x_1^2 + \cdots + x_{n+1}^2), \quad x = (x_0, x_1, \ldots, x_{n+1}). \]

Let \( \Xi \) be the isotropic cone defined by
\[ \Xi = \{ x \in \mathbb{R}^{1,n+1} \setminus \{0\} : [x, x] = 0 \}. \]

For \( x \in \mathbb{R}^{1,n+1} \setminus \{0\} \), denote by \([x] = \mathbb{R}^*x\) the ray through \( x \) and consider the space of isotropic rays, i.e. the quotient space \( \Xi / \mathbb{R}^* \).

The subspace \( \{ x \in \mathbb{R}^{1,n+1} : x_0 = 0 \} \) will be identified with \( \mathbb{R}^{n+1} \) under the isomorphism
\[ \mathbb{R}^{n+1} \ni x' \longmapsto (0, x') \in \mathbb{R}^{1,n+1}. \]

Denote by \( S^n \) the unit sphere of \( \mathbb{R}^{n+1} \). The map
\[ S^n \ni x' \longmapsto \mathbb{R}^*(1, x') \]
yields an isomorphism of $S^n$ with $\mathbb{E}/\mathbb{R}^*$; the inverse isomorphism being described by

$$\mathbb{E}/\mathbb{R}^* \ni \mathbb{R}^* x \mapsto \mathbb{R}^* x \cap \{ x_0 = 1 \}.$$ 

Let $G = \text{SO}_0(1, n + 1)$ be the connected component of the identity in the group of isometries for the Lorentzian form on $\mathbb{R}^{1,n+1}$. Then $G$ acts on $\mathbb{E}$ and commutes with the action of $\mathbb{R}^*$ on $\mathbb{R}^{1,n+1}$, so that $G$ acts on $\mathbb{E}/\mathbb{R}^*$ and yielding an action of $G$ on $S^n$.

Let $G = \text{SO}_0(1, n + 1)$ be the connected component of the identity in the group of isometries for the Lorentzian form on $\mathbb{R}^{1,n+1}$. Then $G$ acts on $\mathbb{E}$ and commutes with the action of $\mathbb{R}^*$ on $\mathbb{R}^{1,n+1}$, so that $G$ acts on $\mathbb{E}/\mathbb{R}^*$ and yielding an action of $G$ on $S^n$.

Let us give more details on the action of $G$ on the unit sphere $S^n$. For $x' = (x'_1, \ldots, x'_{n+1}) \in S^n$ and $g \in G$, observe that $(g(1, x'))_0 > 0$ and define $g(x') \in S^n$ by

$$(1, g(x')) = (g(1, x'))^{-1}_0 g(1, x').$$

For $g \in G$ and $x' \in S^n$, set

$$c(g, x') = (g(1, x'))^{-1}_0.$$(4.1)

Clearly $c(g, x')$ is a smooth and strictly positive function on $G \times S^n$. Moreover, the function $c$ satisfies the cocycle property

$$c(g_1 g_2, x') = c(g_1, g_2(x')) c(g_2, x'), \quad g_1, g_2 \in G, \ x' \in S^n.$$

This action turns out to be conformal on $S^n$, i.e. for any $g \in G$, $x' \in S^n$ and arbitrary $\xi \in T_x S^n$, the differential $Dg(x')$ satisfies

$$\|Dg(x')\xi\| = c(g, x') \|\xi\|,$$

and the term $c(g, x')$ is called the conformal factor of $g$ at $x'$.

Let $e_{n+1} = (0, 0, \ldots, 0, 1)$, and let

$$\kappa : \mathbb{R}^n \longrightarrow S^n \setminus \{-e_{n+1}\}$$

defined by

$$(x_1, \ldots, x_n) \longmapsto \left( \frac{2x_1}{1 + |x|^2}, \ldots, \frac{2x_n}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2} \right)$$

be the inverse map of the stereographic projection. The action of $G$ on $S^n$ can be transferred to a rational action (not everywhere defined) on $\mathbb{R}^n$, for which we still use the notation $G \times \mathbb{R}^n \ni (g, x) \longmapsto g(x) \in \mathbb{R}^n$.

The map $\kappa$ is conformal and hence, the rational action of $G$ on $\mathbb{R}^n$ transferred from its action on $S^n$ is conformal. For $g \in G$ defined at $x \in \mathbb{R}^n$, denote by $\Omega(g, x)$ the corresponding conformal factor. Below, among other things, we will find an expression for $\Omega(g, x)$. 


Choose $e_{n+1} = (0, 0, \ldots, 0, 1)$ as origin on the sphere $S^n$, and let $[(1, e_{n+1})]$ be the corresponding isotropic ray. The stabilizer of $[(1, e_{n+1})]$ is a parabolic subgroup $P$ of $G$, which has the Langlands decomposition $P = MAN$ with

$$M = \left\{ \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} : m \in SO(n) \right\} ,$$

$$A = \left\{ a_t := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\} ,$$

$$N = \left\{ n_x := \begin{pmatrix} 1 + \frac{1}{2} \|x\|^2 & x^t & -\frac{1}{2} \|x\|^2 \\ x & \text{id}_n & -x \\ \frac{1}{2} \|x\|^2 & x^t & 1 - \frac{1}{2} \|x\|^2 \end{pmatrix} : x \in \mathbb{R}^n \right\} .$$

Denote by $\overline{N}$ the opposite nilpotent subgroup,

$$\overline{N} = \left\{ \overline{n}_x := \begin{pmatrix} 1 + \frac{1}{2} \|x\|^2 & x^t & \frac{1}{2} \|x\|^2 \\ x & \text{id}_n & x \\ -\frac{1}{2} \|x\|^2 & -x^t & 1 - \frac{1}{2} \|x\|^2 \end{pmatrix} : x \in \mathbb{R}^n \right\} .$$

The origin $e_{n+1}$ on the sphere $S^n$ corresponds to the point $0 = (0, 0, \ldots, 0)$ in $\mathbb{R}^n$, and hence the parabolic subgroup $P$ is the stabilizer of $0$. The group $M \simeq SO(n)$ acts on $\mathbb{R}^n$ by its natural action and $A$ acts on $\mathbb{R}^n$ by

$$a_t(x) = e^{-t}x , \quad a_t \in A .$$

The group $\overline{N}$ acts on $\mathbb{R}^n$ by translations,

$$\overline{n}_y(x) = x + y , \quad \overline{n}_y \in \overline{N} .$$

The explicit action of $N$ on $\mathbb{R}^n$ (which is rational) will not be needed, but it is easily verified that

$$Dn_y(0) = \text{Id}_n , \quad n_y \in N .$$

Up to a closed subset of null Haar measure, the group $G$ is equal to $\overline{N} \times M \times A \times N$. The corresponding decomposition of $g \in G$ is

$$g = \overline{n}(g)m(g)a_{t(g)}n(g) .$$
An elementary computation gives

\[ Dg(0) = \text{Ad}m(g)(0)|_\pi \circ \text{Ad}a_t(g)|_\pi = e^{-t(g)} m(g), \quad (4.2) \]

where \( \pi = \text{Lie}(\mathbb{N}) \).

Now let \( x \in \mathbb{R}^n \) and let \( g \in G \) be defined at \( x \). Since \( g(x) = g\pi_x(0) \), it follows from (4.2) that

\[ Dg(x) = D(g\pi_x)(0) = e^{-t(g\pi_x)} m(g\pi_x). \quad (4.3) \]

As a consequence, we obtain

\[ \Omega(g, x) = e^{-t(g\pi_x)}, \quad g \in G, \ x \in \mathbb{R}^n. \quad (4.4) \]

We close this paragraph by the following standard result.

**Lemma 4.1.** Let \( x, y \in \mathbb{R}^n \) and let \( g \in G \) be defined at \( x \) and \( y \). Then

\[ \|g(x) - g(y)\|^2 = \Omega(g, x) \|x - y\|^2 \Omega(g, y). \quad (4.5) \]

5. The Principal Series Representations of \( SO_0(1, n + 1) \) on the Space of Differential Forms

Let \( \mathcal{E}^k(S^n) \) be the space of differential \( k \)-forms on the unit sphere \( S^n \) (\( 0 \leq k \leq n \)). For \( \lambda \in \mathbb{C} \), let \( \rho^k_\lambda \) be the representation of \( G = SO_0(1, n + 1) \) on \( \mathcal{E}^k(S^n) \) given by

\[ \rho^k_\lambda(g)\omega(x') = c(g^{-1}, x')^\lambda \left( L^*_{g^{-1}} \right) (x'), \quad g \in G, \ \omega \in \mathcal{E}^k(S^n), \]

where \( L_g \) is the diffeomorphism \( x' \mapsto g(x') \) on \( S^n \) and \( L^*_{g^{-1}} \) is the induced action on differential forms. Here \( c(g^{-1}, x') \) is the conformal factor given by (4.1).

Below we will describe the noncompact model for this series of representations, obtained from the present model through the stereographic projection.

Denote by \( \mathcal{D}\mathcal{E}^k(\mathbb{R}^n) \) the space of \( k \)-forms represented as in (2.6) with the complex-valued coefficients \( \omega_{i_1, \ldots, i_k} \) in \( \mathcal{D}(\mathbb{R}^n) \).

Now, for \( g \in G \) and \( \omega \in \mathcal{E}^k(\mathbb{R}^n) \) let

\[ \pi^k_\lambda(g)\omega(x) = \Omega(g^{-1}, x)^\lambda L^*_{g^{-1}} \omega(x), \quad (5.1) \]

where \( \Omega(g^{-1}, x) \) is given by (4.4). This formula defines formally a representation of \( G \). As it stands, the representation is not globally defined. In what follows, it will be enough to observe that for a relatively compact open subset \( U \) of \( \mathbb{R}^n \), there exists a small neighborhood \( V \) of the neutral element in \( G \) (depending on \( U \)) such that for any \( g \in V \), \( g^{-1} \) is defined on \( U \). Hence, for any smooth differential \( k \)-form \( \omega \) with \( \text{Supp}(\omega) \subset U \), the object \( \pi^k_\lambda(g)\omega \) is well defined, it belongs to \( \mathcal{E}^k(\mathbb{R}^n) \) and has a compact support. This allows us to define the corresponding infinitesimal representation \( d\pi^k_\lambda \) of the Lie algebra \( \mathfrak{g} = \text{Lie}(G) \) by

\[ d\pi^k_\lambda(X)\omega := \frac{d}{dt} \pi^k_\lambda(\exp(tX))\omega \bigg|_{t=0}, \quad X \in \mathfrak{g}. \]
The expression is well defined when \( \omega \in \mathcal{D}\mathcal{E}^k(\mathbb{R}^n) \). The operator \( d\pi^k_\lambda(X) \) is a first order differential operator with polynomial coefficients and hence can be extended to \( \mathcal{S}\mathcal{E}^k(\mathbb{R}^n) \).

The representations \( \pi^k_\lambda \) can also be viewed as principal series representations. Indeed, rewrite (5.1) as

\[
\pi^k_\lambda(g)\omega(x) = \Omega(g^{-1}, x)^{\lambda}\omega(g^{-1}(x)) \circ Dg^{-1}(x),
\]

where \( \omega(g^{-1}(x)) \circ Dg^{-1}(x) \) is the \( k \)-form given by

\[
\omega(g^{-1}(x)) \circ Dg^{-1}(x)(v_1, \ldots, v_k) = \omega(g^{-1}(x))(Dg^{-1}(x)v_1, \ldots, Dg^{-1}(x)v_k).
\]

Now using (4.3) and (4.4) we get

\[
\pi^k_\lambda(g)\omega(x) = e^{-(\lambda+k)t(g^{-1} \pi_x)}\sigma_k(m(g^{-1} \pi_x))^{-1} \omega(g^{-1}(x)),
\]

(5.2)

where \( \sigma_k \) is the representation of \( M = \text{SO}(n) \) on \( \Lambda^k = \Lambda^k(\mathbb{R}^n) \otimes \mathbb{C} \). The presentation (5.2) is just the noncompact realization of a principal series representation (cf [10]). This yields the identification

\[
\pi^k_\lambda \simeq \text{Ind}_G^G(\sigma_k \otimes \chi_{\lambda+k} \otimes 1),
\]

(5.3)

where, for \( \lambda \in \mathbb{C} \), we denote by \( \chi_{\lambda} \) the character of \( A \) given by \( \chi_{\lambda}(a_t) = e^{\lambda t} \).

We pin down that the representation \( \sigma_k \) is an irreducible representation of \( \text{SO}(n) \), except for the case where \( n \) is even and \( k = \frac{n}{2} \) (see [6,8]), but for our purpose, this makes no difference.

The Knapp–Stein intertwining operators play a crucial role in semi-simple harmonic analysis. In the present situation they are given as follows (see [4]),

\[
I^k_\lambda\omega(x) = \int_{\mathbb{R}^n} R^k_{-2n+2\lambda}(x-y) \omega(y)dy,
\]

\( \omega \in \mathcal{S}\mathcal{E}^k(\mathbb{R}^n) \)

where \( R^k_{-2n+2\lambda} \) is the tempered distribution defined by (3.3). In the notations of the previous section, \( I^k_\lambda \) is nothing but the convolution operator \( J^k_\lambda \), defined in (3.12), with \( s = -2n+2\lambda \). The operators \( I^k_\lambda \), defined first for \( \frac{n}{2} < \text{Re} \lambda < n \) so that the integral converges for \( \omega \in \mathcal{S}\mathcal{E}^k(\mathbb{R}^n) \), can be analytically continued to the complex \( \lambda \)-plane as a meromorphic family of convolution operators by tempered distributions, thus mapping \( \mathcal{S}\mathcal{E}^k(\mathbb{R}^n) \) into \( \mathcal{S}'\mathcal{E}^k(\mathbb{R}^n) \). The following (a priori formal) relation holds for any \( g \in G \):

\[
I^k_\lambda \circ \pi^k_\lambda(g) = \pi^k_{\lambda-n}(g) \circ I^k_\lambda.
\]

(5.4)

The relation is first proved when \( \frac{n}{2} < \text{Re} \lambda < n \), and shown (using the covariance property (4.5) of \( \|x-y\|^2 \)) to be valid for forms in \( \mathcal{D}\mathcal{E}^k(\mathbb{R}^n) \) and \( g \) in an appropriate small neighborhood of the neutral element of \( G \). The corresponding infinitesimal form of the intertwining relation (5.4) is

\[
I^k_\lambda \circ d\pi^k_\lambda(X) = d\pi^k_{\lambda-n}(X) \circ I^k_\lambda.
\]

(5.5)

for \( X \in \mathfrak{g} \) and valid on \( \mathcal{S}\mathcal{E}^k(\mathbb{R}^n) \). By analytic continuation, it is then extended meromorphically in \( \lambda \).

The following property will be required later on.
Proposition 5.1. For generic \( \lambda \), the operator \( I_k^\lambda \) is injective on the space \( \mathcal{SE}^k(\mathbb{R}^n) \).

Proof. Since \( I_k^\lambda \) is the convolution product with the tempered distribution \( \mathcal{R}_k^{2n+2\lambda} \), then saying that \( I_k^\lambda \) is injective is equivalent to prove that (for generic \( \lambda \)) the multiplication operator by the Fourier transform \( \mathcal{F}(\mathcal{R}_k^{2n+2\lambda}) \) is injective on \( \mathcal{SE}^k(\mathbb{R}^n) \). Recall from (3.4) that, generically in \( \lambda \), we have

\[
\mathcal{F}(\mathcal{R}_k^{2n+2\lambda})(x) = 2\|x\|^{n-2\lambda-2}((n-k-\lambda)t_x e_x + (\lambda - k) e_x t_x).
\]

Recall also from Sect. 2 that

\[
\begin{align*}
(t_x e_x)^2 &= \|x\|^2 t_x e_x, \\
(e_x t_x)^2 &= \|x\|^2 e_x t_x, \\
(t_x e_x)(e_x t_x) &= 0, \\
(t_x e_x)^2 + (e_x t_x)^2 &= \|x\|^4 \text{Id}.
\end{align*}
\]

If we assume, in addition, that \( n - k - \lambda \neq 0 \) and \( \lambda - k \neq 0 \), then the identities (5.6) imply that for \( x \neq 0 \) the operator

\[
(n - k - \lambda)t_x e_x + (\lambda - k) e_x t_x
\]

is invertible. Let \( \omega \in \mathcal{SE}^k(\mathbb{R}^n) \) and \( \lambda \) as above so that \( \mathcal{F}(\mathcal{R}_k^{2n+2\lambda})(x)\omega(x) = 0 \). As \( \mathcal{F}(\mathcal{R}_k^{2n+2\lambda})(x) \) is invertible for \( x \neq 0 \), it follows that \( \omega(x) = 0 \) for \( x \neq 0 \), and therefore \( \omega \equiv 0 \) on \( \mathbb{R}^n \). □

6. The Covariance Property of the Source Operator

We can now start to give the conformal interpretation of Theorem 3.3. On the one hand, we saw in the previous section that the convolution operators \( J_k^s \) are related to the Knapp–Stein intertwining operators. So it remains to understand the conformal property of the multiplication by \( \|x - y\|^2 \).

The group \( G = \text{SO}_0(1, n+1) \) acts rationally on the space \( \mathbb{R}^n \times \mathbb{R}^n \) by the diagonal extension of its action on \( \mathbb{R}^n \), and hence on \( \mathcal{E}^{k,\ell}(\mathbb{R}^n \times \mathbb{R}^n) \), giving a realization of the tensor product representation \( \pi_\lambda^k \otimes \pi_\mu^\ell \). More explicitly

\[
\pi_\lambda^k \otimes \pi_\mu^\ell(g)(\omega(x, y)) = \Omega(g^{-1}, x)^k \Omega(g^{-1}, y)^\ell L^*_g \omega(x, y).
\]

Define the multiplication operator \( M : \mathcal{E}^{k,\ell}(\mathbb{R}^n \times \mathbb{R}^n) \rightarrow \mathcal{E}^{k,\ell}(\mathbb{R}^n \times \mathbb{R}^n) \) by

\[
M(\omega)(x, y) = \|x - y\|^2 \omega(x, y).
\]

The covariance property (4.5) of \( \|x - y\|^2 \) immediately implies the following result.

Proposition 6.1. The operator \( M \) satisfies

\[
M \circ (\pi^k_\lambda \otimes \pi^\ell_\mu)(g) = (\pi^k_{\lambda-1} \otimes \pi^\ell_{\mu-1})(g) \circ M.
\]
Here again the relation is valid when applied to differential forms in $D\mathcal{E}^{k,\ell}(\mathbb{R}^n \times \mathbb{R}^n)$ and $g$ in a small enough neighborhood of the neutral element of $G$. The rigorous infinitesimal version reads as follows: For every $X \in \mathfrak{g}$, we have

$$M \circ d\left(\pi^k_\mathcal{E} \otimes \pi^\ell_\mathcal{E}\right)(X) = d\left(\pi^{k-1}_\mathcal{E} \otimes \pi^{\ell-1}_\mathcal{E}\right)(X) \circ M,$$

where, by definition,

$$d\left(\pi^k_\mathcal{E} \otimes \pi^\ell_\mathcal{E}\right)(X) := d\pi^k_\mathcal{E}(X) \otimes \text{id} + \text{id} \otimes d\pi^\ell_\mathcal{E}(X).$$

Recall from above that the Knapp–Stein intertwining operator $I_k^\mathcal{E}$ is nothing but the convolution operator $J^\mathcal{E}_s$ with $s = -2n + 2\lambda$. For convenience let

$$F_{k,\mu}^{\ell,i} := -E_{n-2\lambda,n-2\mu}^{\ell,i},$$

where $E_{n,\mu}^{\ell,i}$ is the differential operator (3.20). Explicitly

$$F_{k,\mu}^{\ell,i} = 16\left\{\|x - y\|^2 \tilde{\Box}_{k,\lambda} \otimes \tilde{\Box}_{\ell,\mu} \right.$$

$$= -2 \sum_{j=1}^n (x_j - y_j) \left((2\lambda - n + 2)\tilde{\nabla}_{k,\lambda,j} \otimes \tilde{\nabla}_{\ell,\mu,j} - (2\mu - n + 2)\tilde{\Box}_{k,\lambda} \otimes \tilde{\nabla}_{\ell,\mu,j}\right)$$

$$- 2(2\lambda - n + 2)(2\mu - n + 2) \sum_{j=1}^n \tilde{\nabla}_{k,\lambda,j} \otimes \tilde{\nabla}_{\ell,\mu,j}$$

$$- 2(2\mu - n + 2)(\mu + 1)(\mu - \ell)(\mu - n + \ell)\tilde{\Box}_{k,\lambda} \otimes \text{Id}_{\mathcal{E}^\ell}$$

$$- 2(2\lambda - n + 2)(\lambda + 1)(\lambda - k)(\lambda - n + k)\text{Id}_{\mathcal{E}^k} \otimes \tilde{\Box}_{\ell,\mu}\left\},
$$

where

$$\tilde{\Box}_{k,\lambda} = (\lambda - n + k)(\lambda - k + 1) \delta d + (\lambda - n + k + 1)(\lambda - k) d \delta$$

$$\tilde{\nabla}_{k,\lambda,j} = (\lambda - n + k)(\lambda - k + 1)\partial_{x_j} - (\lambda - k)\epsilon_j \delta - (\lambda - n + k) \delta t_j,$$

and similarly for $\tilde{\Box}_{\ell,\mu}$ and $\tilde{\nabla}_{\ell,\mu,j}$. We will call $F_{k,\mu}^{\ell,i}$ the source operator.

Theorem 3.3 can now be reformulated as follows.

**Theorem 6.2.** The differential operator $F_{k,\mu}^{\ell,i}$ acts on $\mathcal{E}^{k,\ell}(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies

$$\kappa_{k,\mu}^{\ell,i} M \circ (I_k^\mathcal{E} \otimes I_\mu^\mathcal{E}) = (I_{k+1}^\mathcal{E} \otimes I_{\mu+1}^\mathcal{E}) \circ F_{k,\mu}^{\ell,i},$$

where $\kappa_{k,\mu}^{\ell,i} = 16(\lambda - k + 1)(\lambda - n + k + 1)(\mu - \ell + 1)(\mu - n + \ell + 1)$.

Moreover, we have the following covariance property of the source operator $F_{k,\mu}^{\ell,i}$.

**Theorem 6.3.** For all $\lambda, \mu \in \mathbb{C}$ and for any $X \in \mathfrak{g}$, we have

$$F_{k,\mu}^{\ell,i} \circ d\left(\pi^k_\mathcal{E} \otimes \pi^\ell_\mathcal{E}\right)(X) = d\left(\pi^{k+1}_{\mathcal{E}} \otimes \pi^{\ell+1}_{\mathcal{E}}\right)(X) \circ F_{k,\mu}^{\ell,i}.$$
Proof. In the light of Theorem 6.2, Proposition 6.1 and the identity (5.5), we have
\[(I_{\lambda+1}^k \otimes I_{\mu+1}^\ell) \circ F_{\lambda,\mu}^{k,\ell} \circ d(\pi^k_{\lambda} \otimes \pi^\ell_{\mu})(X)\]
\[= \kappa_{\lambda,\mu}^{k,\ell} M \circ (I_{\lambda}^k \otimes I_{\mu}^\ell) \circ d(\pi^k_{\lambda} \otimes \pi^\ell_{\mu})(X)\]
\[= \kappa_{\lambda,\mu}^{k,\ell} d\left(\pi^k_{n-\lambda} \otimes \pi^\ell_{n-\mu}\right)(X) \circ (I_{\lambda}^k \otimes I_{\mu}^\ell)\]
\[= \kappa_{\lambda,\mu}^{k,\ell} d\left(\pi^k_{n-\lambda-1} \otimes \pi^\ell_{n-\mu-1}\right)(X) \circ M \circ (I_{\lambda}^k \otimes I_{\mu}^\ell)\]
\[= d\left(\pi^k_{n-\lambda-1} \otimes \pi^\ell_{n-\mu-1}\right)(X) \circ (I_{\lambda+1}^k \otimes I_{\mu+1}^\ell) \circ F_{\lambda,\mu}^{k,\ell}\]

Now, use Proposition 5.1 to finish the proof. □

As G is connected, the infinitesimal covariance property of the operator $F_{\lambda,\mu}^{k,\ell}$ implies first its covariance under the group G, that is

$F_{\lambda,\mu}^{k,\ell} \circ (\pi^k_{\lambda} \otimes \pi^\ell_{\mu})(g) = \left(\pi^k_{\lambda+1} \otimes \pi^\ell_{\mu+1}\right)(g) \circ F_{\lambda,\mu}^{k,\ell}$

for $g \in \mathcal{D}^k,\ell(\mathbb{R}^n \times \mathbb{R}^n)$ and $g \in G$ is such that $g^{-1}$ is defined on a neighborhood of the support of $\omega$.

Remark 6.4. Let us mention that improving on our results, it is possible to construct a differential operator $F_{\lambda,\mu}^{k,\ell}$ on $S^n \times S^n$, which admits $F_{\lambda,\mu}^{k,\ell}$ as its local expression on $S^n \setminus \{(0,0,\ldots,0,-1)\} \times S^n \setminus \{(0,0,\ldots,0,1)\} \simeq \mathbb{R}^n \times \mathbb{R}^n$ and which is covariant for $G$ with respect to $(\rho_{\lambda}^k \otimes \rho_{\mu}^\ell, \rho_{\lambda+1}^k \otimes \rho_{\mu+1}^\ell)$. We skip the proof as this corresponds to general standard results. See for instance Section 8.2 in [2] or Fact 3.3 in [14].

It is possible to compose the source operators, to yield more covariant differential operators. Indeed, for arbitrary integer $m \geq 1$, we set

$F_{\lambda,\mu;m}^{k,\ell} := F_{\lambda+m,\mu+m-1}^{k,\ell} \circ \cdots \circ F_{\lambda+1,\mu+1}^{k,\ell} \circ F_{\lambda,\mu}^{k,\ell}$.

Then $F_{\lambda,\mu;m}^{k,\ell}$ intertwines the representations $\pi^k_{\lambda} \otimes \pi^\ell_{\mu}$ and $\pi^k_{\lambda+m} \otimes \pi^\ell_{\mu+m}$.

The following statement gives another insight into these operators.

Proposition 6.5. For any integer $m \geq 1$, we have

$\kappa_{\lambda,\mu;m}^{k,\ell} M^m \circ (I_{\lambda}^k \otimes I_{\mu}^\ell) = (I_{\lambda+m}^k \otimes I_{\mu+m}^\ell) \circ F_{\lambda,\mu;m}^{k,\ell}$,

where $\kappa_{\lambda,\mu;m}^{k,\ell} = \kappa_{\lambda+1,\mu+1}^{k,\ell} \cdots \kappa_{\lambda+m,\mu+m-1}^{k,\ell}$ and $M^m = M \circ \cdots \circ M$, $m$-times.

Proof. For $m = 1$, this is Theorem 6.2. Assume $m \geq 2$. By induction on $m$ we have

$\kappa_{\lambda,\mu;m}^{k,\ell} M^m \circ (I_{\lambda}^k \otimes I_{\mu}^\ell) = \kappa_{\lambda,\mu;m}^{k,\ell} M \circ M^{m-1} \circ (I_{\lambda}^k \otimes I_{\mu}^\ell) = \kappa_{\lambda+1,\mu+1}^{k,\ell} \circ M \circ (I_{\lambda+m}^k \otimes I_{\mu+m}^\ell) \circ F_{\lambda,\mu;m}^{k,\ell} = (I_{\lambda+m}^k \otimes I_{\mu+m}^\ell) \circ F_{\lambda+m,\mu+m-1}^{k,\ell} \circ F_{\lambda,\mu;m}^{k,\ell}$.
7. Conformally Covariant Bi-differential Operators on Differential Forms

From the source operators $F_{k,\ell}^{\lambda,\mu}$, one can construct covariant bi-differential operators under the action of $G = \text{SO}_0(1, n + 1)$. First, introduce the restriction map

$$\text{res} : \mathcal{E}^{k,\ell} (\mathbb{R}^n \times \mathbb{R}^n) \longrightarrow C^\infty (\mathbb{R}^n, \Lambda^k \otimes \Lambda^\ell)$$

defined by

$$(\text{res} \omega)(x) = \omega(x, x),$$

where $C^\infty (\mathbb{R}^n, \Lambda^k \otimes \Lambda^\ell)$ denotes the space of complex-valued smooth functions on $\mathbb{R}^n$ with values in $\Lambda^k \otimes \Lambda^\ell$. Let $G$ act on $\mathcal{E}^{k,\ell} (\mathbb{R}^n \times \mathbb{R}^n)$ by $\pi^k_\lambda \otimes \pi^\ell_\mu$. Using the realization of $\pi^k_\lambda$ and $\pi^\ell_\mu$ as principal series representations (see (5.3)), the following result is immediate.

**Proposition 7.1.** For any $(\lambda, \mu)$ the map $\text{res}$ intertwines the representations $\pi^k_\lambda \otimes \pi^\ell_\mu$ and $\text{Ind}_G^G (\sigma^\lambda_k \otimes \sigma^\ell_\mu \otimes \chi^k_\lambda + \chi^\ell_\mu + 1)$. 

As a representation of $M = \text{SO}(n)$, the representation $\sigma^k_\lambda \otimes \sigma^\ell_\mu$ is in general not irreducible. Let $\Gamma$ be a minimal invariant subspace of $\Lambda^k \otimes \Lambda^\ell$ under the action of $\text{SO}(n)$. Let $\sigma_\Gamma$ be the corresponding irreducible representation of $\text{SO}(n)$ on $\Gamma$ and let $p_\Gamma$ be the orthogonal projection on $\Gamma$. Define the map $\text{res}_\Gamma$ by

$$\text{res}_\Gamma = p_\Gamma \circ \text{res}.$$

We can refine the previous proposition as follows.

**Proposition 7.2.** For any $(\lambda, \mu)$ the map $\text{res}_\Gamma$ intertwines the representations $\pi^k_\lambda \otimes \pi^\ell_\mu$ and $\text{Ind}_G^G (\sigma_\Gamma \otimes \chi^k_\lambda + \chi^\ell_\mu + 1)$. 

Now define the bi-differential operators

$$B_{k,\ell;\lambda,\mu;m}^{\lambda,\mu} : \mathcal{E}^{k,\ell} (\mathbb{R}^n \times \mathbb{R}^n) \longrightarrow C^\infty (\mathbb{R}^n, \Gamma)$$

by

$$B_{k,\ell;\lambda,\mu;m}^{\lambda,\mu} := \text{res}_\Gamma \circ F_{k,\ell}^{\lambda,\mu},$$

where $C^\infty (\mathbb{R}^n, \Gamma)$ denotes the space of smooth functions on $\mathbb{R}^n$ with values in $\Gamma \subset \Lambda^k \otimes \Lambda^\ell$.

**Theorem 7.3.** The operator $B_{k,\ell;\lambda,\mu;m}^{\lambda,\mu}$ is a bi-differential operator covariant with respect to $\pi^k_\lambda \otimes \pi^\ell_\mu$ and $\text{Ind}_G^G (\sigma_\Gamma \otimes \chi^k_\lambda + \chi^\ell_\mu + 2m \otimes 1)$.

In some cases, it is possible to give an explicit expression for these covariant bi-differential operators. For instance, assume that $0 \leq k + \ell \leq n$, then the representation $\Lambda^{k+\ell}$ appears in the decomposition of the tensor product $\Lambda^k \otimes \Lambda^\ell$ with multiplicity one and the projection (up to a normalization factor) is given by

$$p_{\Lambda^{k+\ell}} (\omega \otimes \eta) = \omega \wedge \eta.$$
For $m = 1$, the bi-differential operator $B_{k,\ell;1}^{k,\ell;1}$ is given by
\begin{align*}
B_{k,\ell;1}^{k,\ell;1}(\omega \otimes \eta)(x) &= -32 \left\{ (2\mu - n + 2)(\mu + 1)(\mu - \ell)(\mu - n + \ell) \tilde{\square}_{k,\ell} \omega(x) \wedge \eta(x) \\
&\quad + (2\lambda - n + 2)(2\mu - n + 2) \sum_{j=1}^{n} \tilde{\nabla}_{k,\ell,j} \omega(x) \wedge \tilde{\nabla}_{\ell,j} \eta(x) \\
&\quad + (2\lambda - n + 2)(\lambda + 1)(\lambda - k)(\lambda - n + k) \omega(x) \wedge \tilde{\square}_{\ell,\mu} \eta(x) \right\}.
\end{align*}
(7.1)

If in addition $k = \ell = 0$, i.e. $\omega, \eta \in C^\infty(\mathbb{R}^n)$, then
\begin{align*}
B_{0,0;1}^{0,0;1}(\omega \otimes \eta)(x) &= -64(\lambda + 1)(\lambda - n)(\mu + 1)(\mu - n) \left\{ \mu(\mu - n^2 + 1) \left( Q(\partial_x) \omega \right)(x) \eta(x) \\
&\quad + 2(\lambda - n^2 + 1)(\mu - n^2 + 1) \sum_{j=1}^{n} \left( \partial_{x,j} \omega \right)(x) \left( \partial_{x,j} \eta \right)(x) \\
&\quad + \lambda(\lambda - n^2 + 1) \omega(x) \left( Q(\partial_x) \eta \right)(x) \right\},
\end{align*}
where $Q$ is the quadratic form (2.9). Hence we recover the multidimensional Rankin–Cohen operators in [2, Section 10].

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