Establishment of a new inequality using extended Heinz type mean

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Abstract. In this paper, the Extended Heinz type function is introduced, and it is proved that the function \( F(a, b; v) \) is a mean. Further investigated the various properties of mean such as symmetricity, homogeneity, monotonicity and convexity. As an application, some inequalities have been established.

1. Introduction and preliminaries:

The well known Mathematical means in the literature [5, 16] such as Arithmetic mean \( A(a, b) = \frac{a+b}{2} \), Geometric mean \( G(a, b) = \sqrt{ab} \), Harmonic mean \( H(a, b) = \frac{2ab}{a+b} \) and Contra harmonic mean \( C(a, b) = \frac{a^2+b^2}{a+b} \) have their own importance in the field of Science and Engineering.

In [4], Bhatia introduced the Heinz mean \( F(a, b; v) = \frac{a^v b^{1-v} + a^{1-v} b^v}{2} \), for \( 0 \leq v \leq 1/2 \), \( a, b > 0 \). For different values of \( v \), Heinz mean interpolates between the arithmetic mean \( (v = 0) \) and geometric mean \( (v = 1/2) \) such that \( \sqrt{ab} = F(a, b; 1/2) \leq F(a, b; v) \leq \frac{a+b}{2} = F(a, b; 0) \), for \( 0 < v < 1/2 \).

The Heinz mean may also be defined in the same passion for positive semi definite matrices and satisfies a similar interpolation formula [3]. In [2], Audenaert proved a matrix inequality for matrix monotone functions and applied it to prove a singular value inequality for Heinz means.

In [11], Nagaraja etal., proved that \( H(a, b) + C(a, b) = 2A(a, b) \) and also studied some double inequalities involving means. In [13], Sampath Kumar etal., introduced the new family of Heinz type means and studied some basic properties such as symmetry, homogeneity, monotonicity, log convexity and concavity. Further, established the monotonic results, log convexity and concavity results and some deductions are deduced.
For any positive real number $x$, Shafer-Fink double inequality [8] for the arctangent function is given by

$$\frac{3x}{1 + 2\sqrt{1 + x^2}} < \arctan x < \frac{\pi x}{1 + 2\sqrt{1 + x^2}}$$  \hspace{1cm} (1)$$

Based on the bisection formula for the cotangent function and the related Weierstrass product [6], is one among the various proofs available to prove the double inequality (1). For any positive real number $x$, a refinement for the upper bound of the inequality (1) is obtained as

$$\arctan x < \frac{\pi x}{4 + \sqrt{2}\sqrt{1 + x^2} + x\sqrt{1 + x^2}}$$  \hspace{1cm} (2)$$

which holds for any positive real number $x$. In establishing the proof of the refinement, Sandor [15] proved a trigonometric inequality based on infinite product expansions and Riemann’s zeta function.

Ling Zhu [9] used the power series quotient monotone rule to extend some Shafer-Fink type inequalities for the inverse sine to arc hyperbolic sine.

This work motivates to develop this article.

**Definition 1.1** [16] For $a,b > 0$, a mean $M(a,b)$ is defined as the function $M(a,b) : R_+^2 \rightarrow R_+$, which has the property that

$$a \land b \leq M(a,b) \leq a \lor b$$  \hspace{1cm} (4)$$

where $a \land b = \text{Min}(a,b)$ and $a \lor b = \text{Max}(a,b)$

**Definition 1.2** [12] For $0 \leq v \leq 1$, $a,b > 0$, the extended Heinz type function is defined as

$$F^{[E]}(a,b;v) = \frac{a^{1+v}b^{1-v} + a^{1-v}b^{1+v}}{a^{\frac{1+v}{2}}b^{\frac{1-v}{2}} + a^{\frac{1-v}{2}}b^{\frac{1+v}{2}}}$$  \hspace{1cm} (5)$$

2. Results

It is to be verified that $F^{[E]}(a,b;v)$ satisfies the definition of mean and some of the standard properties.

**Lemma 2.1** For all values of $0 \leq v \leq 1$, the extended Heinz type function $F^{[E]}(a,b;v)$ for two distinct positive real values $a \& b$ is a mean.

**Proof:**

According to the definition of mean, it is necessary to verify the condition

$$\text{Min}(a,b) \leq F^{[E]}(a,b;v) \leq \text{Max}(a,b)$$

for all values of $0 \leq v \leq 1$ and $a < b$ this is achieved by considering the following two cases:

**Case (i)** consider $a - F^{[E]}(a,b;v) = a - \frac{a^{1+v}b^{1-v} + a^{1-v}b^{1+v}}{a^{\frac{1+v}{2}}b^{\frac{1-v}{2}} + a^{\frac{1-v}{2}}b^{\frac{1+v}{2}}}$ on simplification,

$$a - F^{[E]}(a,b;v) = a^{\frac{3+v}{2}}b^{\frac{1-v}{2}} \left[ 1 - \left( \frac{b}{a} \right)^{\frac{1+v}{2}} \right] + a^{\frac{3-v}{2}}b^{\frac{1+v}{2}} \left[ 1 - \left( \frac{b}{a} \right)^{\frac{1-v}{2}} \right]$$

$$\implies a - F^{[E]}(a,b;v) = a^{\frac{3+v}{2}}b^{\frac{1-v}{2}} \left[ 1 - \left( \frac{b}{a} \right)^{\frac{1+v}{2}} \right] + a^{\frac{3-v}{2}}b^{\frac{1+v}{2}} \left[ 1 - \left( \frac{b}{a} \right)^{\frac{1-v}{2}} \right]$$
Since $0 \leq v \leq 1$ and $a < b$, it gives $a - F^{[E]}(a, b; v) < 0$ and hence obviously $a < F^{[E]}(a, b; v)$. 

Case(ii) consider $F^{[E]}(a, b; v) - b = \frac{a^{1+v}b^{1-v} + a^{1-v}b^{1+v}}{a^{\frac{1}{2} - \frac{v}{2}}b^{\frac{1}{2} + \frac{v}{2}} + a^{\frac{1}{2} + \frac{v}{2}}b^{\frac{1}{2} - \frac{v}{2}}} - b$, on simplification,  

$$b - F^{[E]}(a, b; v) = ba \frac{1+v}{2} b^{1-v} \left[ \left( \frac{a}{b} \right)^{\frac{1+v}{2}} - 1 \right] + ba \frac{1-v}{2} b^{1+v} \left[ \left( \frac{a}{b} \right)^{-\frac{1-v}{2}} - 1 \right] > 0$$

Since $a < b$, it gives $b - F^{[E]}(a, b; v) > 0$ and hence $F^{[E]}(a, b; v) < b$. This completes the proof of lemma 2.1.

**Property 2.1** For $0 < v < 1$, extended Heinz type mean $F^{[E]}(a, b; v)$ interpolates between the contra harmonic mean and geometric mean.

(i) For $v = 0$, minimum value of extended Heinz type mean is given by  

$$F^{[E]}(a, b; 0) = \frac{ab + ab}{\sqrt{ab} + \sqrt{ab}} = \sqrt{ab} = G(a, b)$$

which is the lower bound of extended Heinz type mean.

(ii) For $v = 1$, maximum value of extended Heinz type mean is given by  

$$F^{[E]}(a, b; 1) = \frac{a^2 + b^2}{a + b} = C(a, b)$$

which is the upper bound of extended Heinz type mean.

This proves that $G(a, b) \leq F^{[E]}(a, b; v) \leq C(a, b)$

**Property 2.2** The extended Heinz type mean $F^{[E]}(a, b; v)$ is symmetric and homogeneous.

(i) $F^{[E]}(a, b; v) = \frac{a^{1+v}b^{1-v} + a^{1-v}b^{1+v}}{a^{\frac{1}{2} - \frac{v}{2}}b^{\frac{1}{2} + \frac{v}{2}} + a^{\frac{1}{2} + \frac{v}{2}}b^{\frac{1}{2} - \frac{v}{2}}}$, $F^{[E]}(b, a; v) = \frac{b^{1+v}a^{1-v} + b^{1-v}a^{1+v}}{b^{\frac{1}{2} - \frac{v}{2}}a^{\frac{1}{2} + \frac{v}{2}} + b^{\frac{1}{2} + \frac{v}{2}}a^{\frac{1}{2} - \frac{v}{2}}}$

Clearly, both are same and hence $F^{[E]}(a, b; v)$ is symmetric.

(ii) $F^{[E]}(at, bt; v) = \frac{(at)^{1+v}(bt)^{1-v} + (at)^{1-v}(bt)^{1+v}}{(at)^{\frac{1}{2} - \frac{v}{2}}(bt)^{\frac{1}{2} + \frac{v}{2}} + (at)^{\frac{1}{2} + \frac{v}{2}}(bt)^{\frac{1}{2} - \frac{v}{2}}}$, $tF^{[E]}(a, b; v) = \frac{t^{1+v}a^{1-v} + t^{1-v}a^{1+v}}{t^{\frac{1}{2} - \frac{v}{2}}a^{\frac{1}{2} + \frac{v}{2}} + t^{\frac{1}{2} + \frac{v}{2}}a^{\frac{1}{2} - \frac{v}{2}}}$

Clearly, both are same and hence $F^{[E]}(a, b; v)$ is homogeneous. (See [5])

**Theorem 2.1** The extended Heinz type mean $F^{[E]}(a, b; v)$ for two distinct positive real values $a \& b$ is a mean for all values of $0 \leq v \leq 1$ then

(i) $1 \leq \frac{\cosh v}{\cosh \frac{v}{2}} < \left( \frac{\sqrt{2}e^2 + 1}{\sqrt{2}e^2 + 1} \right)$

(ii) $1 \leq \frac{\cosh v}{\cosh \frac{v}{2}} \leq 1 + \frac{3v^2}{8}$

**Proof:**

Case (i) Consider $F^{[E]}(a, b; v) = \frac{a^{1+v}b^{1-v} + a^{1-v}b^{1+v}}{a^{\frac{1}{2} - \frac{v}{2}}b^{\frac{1}{2} + \frac{v}{2}} + a^{\frac{1}{2} + \frac{v}{2}}b^{\frac{1}{2} - \frac{v}{2}}}$ for $v \in [0, 1]$

put $a = e$, $b = 1$, then

$$F^{[E]}(a, b; v) = \frac{e^{1+v} + e^{-v}}{e^{\frac{1+v}{2}} + e^{\frac{1-v}{2}}} = \frac{e}{e^\frac{1+v}{2} + e^\frac{1-v}{2}} \left[ \frac{e^v + e^{-v}}{e^\frac{v}{2} + e^{-\frac{v}{2}}} \right] = e^\frac{1}{2} \left[ \frac{\cosh v}{\cosh \frac{v}{2}} \right]$$

In lemma 2.1 proved that $F^{[E]}(a, b; v)$ is mean then it satisfy $Min(a, b) \leq M(a, b) \leq Max(a, b)$ that is minimum of $F^{[E]}(a, b; v)$ is attained at $v = 0$ is equal to $\sqrt{ab}$, maximum of $F^{[E]}(a, b; v)$
is attained at \( v = 1 \) is equal to \( C(a, b) \) then \( \sqrt{ab} \leq F^{|E|}(a, b; v) \leq \frac{a^2 + b^2}{a+b} \).

For \( a = e \) and \( b = 1 \), the above inequality becomes
\[
\sqrt{e} \leq \sqrt{e} \left[ \frac{\cosh v}{\cosh \frac{v}{2}} \right] \leq \frac{e^2 + 1}{e + 1}
\]
which leads to the inequality
\[
1 \leq \frac{\cosh v}{\cosh \frac{v}{2}} \leq \frac{1}{\sqrt{e}} \left( \frac{e^2 + 1}{e + 1} \right)
\]
(6)

Since \( 1 < \sqrt{e} \) which implies that \( \frac{1}{\sqrt{e}} < 1 \), then
\[
\frac{1}{\sqrt{e}} \left( \frac{e^2 + 1}{e + 1} \right) < \left( \frac{e^2 + 1}{e + 1} \right)
\]
(7)

Combining eqs (6) and (7),
\[
1 \leq \frac{\cosh v}{\cosh \frac{v}{2}} < \left( \frac{e^2 + 1}{e + 1} \right)
\]
(8)

**Case (ii)** From Taylor’s series expansion,
\[
\frac{\cosh v}{\cosh \frac{v}{2}} = 1 + \frac{3v^2}{8} + \frac{v^4}{4} + \text{higher order degree terms}
\]
By ignoring the fourth and higher order degree terms gives
\[
\frac{\cosh v}{\cosh \frac{v}{2}} \leq 1 + \frac{3v^2}{8}
\]
(9)

From equations (8) and (9),
\[
1 \leq \frac{\cosh v}{\cosh \frac{v}{2}} \leq 1 + \frac{3v^2}{8}
\]
(10)

This completes the proof of the theorem 2.1

**Consequence**
Put \( v = ix \), to get the converse to D’Aurizio’s trigonometric inequality (see [14, 15]) in imaginary domain.
\[
1 \leq \frac{\cos x}{\cos \frac{x}{2}} \leq 1 - \frac{3x^2}{8}, \quad \text{for} \quad x \in [-i, 0].
\]
(11)

**Conclusion**
This article justifies extended Heinz type function is a symmetric, homogeneous mean and also extended Heinz type mean can be applied to get a hyperbolic inequality. As a consequence, the converse of D’Aurizio’s trigonometric inequality has been established in imaginary domain.
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