New static black holes solutions in Einstein and Einstein-Gauss-Bonnet gravity with topology SO(n) × SO(n)

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We obtain new static black hole solutions in Einstein and Einstein-Gauss-Bonnet gravity with product two-spheres topology, \( \text{SO}(n) \times \text{SO}(n) \), in higher dimensions. There is an unusual new feature of Gauss-Bonnet black hole that in order to prevent occurrence of non-central naked singularity, a positive \( \Lambda > 0 \), and a prescription of range of values for black hole mass in terms of \( \Lambda \) are required. For the Einstein-Gauss-Bonnet black hole a limited window of negative values for \( \Lambda \) is also permitted. This topology encompasses black string and brane as well as a generalized Nariai metric. We also give new solutions with product two-spheres of constant curvatures.

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I. INTRODUCTION

The study of spaces with some rotational symmetries in general relativity has been strongly motivated by the property that it provides a rich spectrum of different phases of black objects with distinct topologies for horizons, see for a review \([1]\). The simplest realization of it is provided by the usual 4-dimensional Schwarzschild black hole with an extra dimension of radius \( L \) added. If the Schwarzschild radius is much smaller than radius of extra dimension, it would resemble to 5-dimensional Schwarzschild black hole with horizon topology \( S^3 \). On the other hand if the black hole radius is bigger than extra dimension radius, it would describe a black string with horizon topology \( S^2 \times S^1 \). This means there does occur a local change in horizon topology as radius of hole increases. It turns out that this change is brought about \([2]\), see also \([3]\), through a cone over \( S^2 \times S^2 \) (for brevity, we shall term it two-spheres topology) by seeking a Ricci flat metric for the cone. This construction could as well be looked upon as solid angle deficit for each \( S^2 \). Note that angle deficit describes a cosmic string for which the Riemann curvature vanishes while solid angle deficit for which the Riemann curvature is non-zero describes a global monopole \([4]\). The interesting question that arises is whether contribution of solid angle deficit of one sphere exactly cancels out that of the other giving rise to Ricci flat spacetime. This is precisely what happens and that is why we are able to put in a static black hole in this setting. The two-spheres topology harbours a static black hole.

In this paper, we would like to study the more general case of \( S^{d_1} \times S^{d_2} \) for \( \Lambda \)-vacuum solutions of Einstein, Gauss-Bonnet (GB) and Einstein-Gauss-Bonnet (E-GB) equation. We show that a \((d = d_1 + d_2 + 2)\)-dimensional spacetime harbours a static black hole with two-spheres topology \( S^{d_1} \times S^{d_2} \) for Einstein and \( S^{d_0} \times S^{d_0} \) with \( d_1 = d_2 = d_0 \) for GB and E-GB gravity. These are new black hole solutions that we wish to report in this paper. One of the new features of these GB and E-GB black holes is that there can occur a non-central naked singularity which could however be avoided by prescribing a range for black hole mass in terms of a given \( \Lambda \). It is noteworthy that presence of positive \( \Lambda \) is therefore necessary for existence of these black holes for GB gravity. That is, \( \Lambda \) plays a very critical role in this setting as is the case for stability of pure Lovelock black hole where it renders otherwise unstable black hole stable \([11]\).

The paper is organized as follows: we begin with the general metric ansatz and set up the framework in the next section followed by new vacuum solutions representing black hole in Einstein, GB and E-GB gravity and also study of their thermodynamical parameters. We end up with a discussion. We also devote some appendices to give solutions of Einstein, GB and E-GB with spheres of constant curvature.

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II. EINSTEIN BLACK HOLE

We begin with the general spherically symmetric metric with two-spheres topology \( R^2 \times S^{d_1} \times S^{d_2} \) which is written as follows:

\[
ds^2 = -A(r) \, dt^2 + B(r) \, dr^2 + C(r) \, dS^2_{(d_1)} + D(r) \, dS^2_{(d_2)},
\]

in \( d = d_1 + d_2 + 2 \) dimensions. We use the notation of indices \((0, 1)\) for \((t, r)\), \((a, b, ...)\) for the angular coordinates of the first sphere \( S^{d_1} \) and \((a', b', ...)\) for those of the second sphere \( S^{d_2} \). Keeping the four functions \( A(r), B(r), C(r), D(r) \) as the unknown variables is a consistent truncation ansatz, which means that the direct substitution of \((1)\) into the Lagrangian will give the same equations of motion (EOM) (from the truncated Lagrangian) as if directly substituted into the EOM of the original Lagrangian (see details concerning consistent truncations in \([12, 13]\)).

Under this generic ansatz, the only nonvanishing components of the Riemann tensor are:

\[
\begin{align*}
R_{01}^{01} &= \frac{A(r)A'(r)B'(r) + B(r)[A'(r)^2 - 2A(r)A''(r)]}{4A(r)^2B(r)^2} =: L(0, 1), \\
R_{0a}^{0a} &= -\frac{A'(r)C'(r)}{4A(r)B(r)C(r)} =: L(0, a), \\
R_{1a}^{1a} &= \frac{C(r)B'(r)C'(r) - 2B(r)C(r)C''(r) + B(r)C'^2(r)}{4B(r)^2C(r)^2} =: L(1, a), \\
R_{ab}^{ab} &= -\frac{C'(r)^2}{4B(r)C(r)^2} =: L(a, b), \quad (a \neq b) \\
R_{0a'}^{0a'} &= -\frac{A'(r)D'(r)}{4A(r)B(r)D(r)} =: L(0, a'), \\
R_{1a'}^{1a'} &= \frac{D(r)B'(r)D'(r) - 2B(r)D(r)D''(r) + B(r)D'^2(r)}{4B(r)^2D(r)^2} =: L(1, a'), \\
R_{a'b'}^{a'b'} &= \frac{1}{D(r)} - \frac{D'(r)^2}{4B(r)D(r)^2} =: L(a', b'), \quad (a' \neq b'), \\
R_{aa'}^{aa'} &= -\frac{C'(r)D'(r)}{4B(r)C(r)D(r)} =: L(a, a').
\end{align*}
\]

The Einstein Lagrangian \( \sqrt{-g}(R - 2\Lambda) \), for the metric \([1]\) reads as follows:

\[
L_{EH} = \sqrt{-g} \left[ 2L(0, 1) + 2d_1 \left( L(0, a) + L(1, a) \right) + 2d_2 \left( L(0, a') + L(1, a') \right) \\
+ d_1(d_1 - 1)L(a, b) + d_2(d_2 - 1)L(a', b') + 2d_1d_2L(a, a') - 2\Lambda \right].
\]

where the density factor \( \sqrt{-g} \) is (up to the volume of the spheres, here irrelevant)

\[
\sqrt{-g} \rightarrow \sqrt{A(r)B(r)C(r)} \frac{d_1}{k_1} D(r) \frac{d_2}{k_2}
\]

It is well known that null energy condition as well as the fact that the radial photon experiences no acceleration \([5]\) require \( B(r) = \frac{1}{A(r)} \), and we set \( C(r) = \frac{r^2}{k_1}, D(r) = \frac{r^2}{k_2} \) where \( k_1, k_2 \) are constants.

The metric thus takes the form

\[
ds^2 = -A(r) \, dt^2 + \frac{1}{A(r)} \, dr^2 + \frac{r^2}{k_1} \, dS^2_{(d_1)} + \frac{r^2}{k_2} \, dS^2_{(d_2)},
\]

The constants, \( k_i \) are fixed as \( k_i = \frac{d-3}{d-1} \) by solving the EOM for \((3)\) for \( A(r) = 1 \), which obtains the results already given in \([2]\). It turns out that EOM for the truncated Lagrangian \([3]\) ultimately reduces to a single first order differential equation that is given by

\[
\frac{d}{dr} \left( r^{d-3} (1 - A(r)) - \frac{2\Lambda}{(d-1)(d-2)} \right) = 0.
\]
This readily solves to give the solution

\[ A = 1 - \frac{2\Lambda}{(d-1)(d-2)} r^2 - \frac{M}{r^{d-3}} \]  

(6)

and thus we have the static black hole metric as

\[ ds^2 = -\left(1 - \frac{2\Lambda}{(d-1)(d-2)} r^2 - \frac{M}{r^{d-3}}\right) dt^2 + \left(1 - \frac{2\Lambda}{(d-1)(d-2)} r^2 - \frac{M}{r^{d-3}}\right) dr^2 + \frac{1}{d-3}(d-1) dS^2_{(d_1)} + \frac{1}{d-3}(d-2) dS^2_{(d_2)}, \]  

(7)

with \( d_1 > 1, d_2 > 1 \). Notice that constant coefficient before \( dS^2 \) indicates a solid angle deficit which depends upon dimension of sphere. The metric (7) describes a black hole with horizon topology \( S^{d_1} \times S^{d_2} \). Note that for \( \Lambda = M = 0 \) spacetime is not Minkowski because of solid angle deficits which produce non-zero Riemann curvature as could be seen from the Kretschmann scalar, \( K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) which reads as

\[ K = \frac{d_1 d_2 (d-4)(d-3)}{2(d_1-1)(d_2-1)} \frac{1}{r^4} + \left(1 + \frac{2}{d-1} - \frac{2}{d-2}\right) \Lambda^2 + \frac{1}{4}(d-3)(d-2)(d-1) \frac{M^2}{r^{2d-17}}. \]  

(8)

Clearly it is non-zero when \( \Lambda = M = 0 \) and spacetime has singularity at \( r = 0 \). However it has weaker divergence (\( 1/r^4 \)) as compared to black hole \( (1/r^{2(d-1)}) \). When \( \Lambda \) and \( M \) vanish, the solution coincides with the one proposed in [2], see also [3], as the mediator solution in some topology changing transitions in the space of higher dimensional black hole solutions. Note that the black hole potential, \( M/r^{d-3} \), remains unaltered by two-spheres topology (i.e. \( SO(d_1) \times SO(d_2) \) symmetry of the metric). It only rescales \( \Lambda \), else it makes no difference at all. Thus for Einstein gravity, the black hole solution is neutral to product topology; i.e. it doesn’t matter whether it is \( S^{d_1} \times S^{d_2} \) or simply \( S^{d_1+d_2} \).

1. **Black string and brane**

For \( d_1 = 1 \), it turns out that solution cannot accommodate \( \Lambda \) because 1-sphere (circle) has no intrinsic curvature. Instead we have the well known solution

\[ ds^2 = -\left(1 - \frac{M}{r^{d-3}}\right) dt^2 + \left(1 - \frac{M}{r^{d-3}}\right) dr^2 + dz^2 + r^2 dS^2_{(d-3)}, \quad (z\text{ periodic}). \]  

(9)

This is the uniform black string solution in which a flat direction is added to a Schwarzschild black hole [5].

On the other hand if we take one of spheres to be of constant curvature and the other without solid angle deficit, then it would give a black brane with the metric,

\[ ds^2 = -A(r) dt^2 + \frac{1}{A(r)} dr^2 + r^2 dS^2_{(d_1)} \pm \frac{1}{k} d\Sigma^2_{(d_2)}, \]  

(10)

where

\[ A(r) = 1 - \frac{(d_2 - 1)}{d_1 + 1} k r^2 - \frac{M}{r^{d_1-1}}, \quad \Lambda = \frac{1}{2}(d_2 - 1)(d_1 + d_2) k. \]  

(11)

Here \( \Sigma \) is a space of constant curvature, sphere \((k > 0)\) or hyperboloid \((k < 0)\) or flat \((k = 0)\). That is, a constant curvature space is added to a Schwarzschild-dS/AdS black hole in \( d = d_1 + 2 \) dimension and hence it may be taken as a uniform black brane [6].
2. Generalized Nariai metric

For $M = 0$, we have product of two spaces of constant curvature, $R^{d_1+2} \times S^{d_2}$, it is a generalized Nariai solution \cite{7, 8} of $R_{ab} = \Lambda g_{ab}$. Let us consider product of two constant curvature spaces, $R^2 \times S^2$. If the curvatures are equal, it is Nariai solution of $R_{ab} = \Lambda g_{ab}$, on the other hand if they are equal and opposite in sign, then it is an Einstein-Maxwell solution \cite{9, 10} describing gravitational field of uniform electric field. Contrary to general behaviour of such spacetimes, the former is conformally non-flat while the latter is conformally flat. Note that both product spaces are of the same dimension, while in our case, they are of different dimensions, $R^{d_1+2} \times S^{d_2}$, and that is why we call it generalized Nariai metric. For $d_2 = d_1 = 2$, the generalized Nariai metric would take the form,

$$ds^2 = -\left(1 - \frac{\Lambda}{6} r^2\right) dt^2 + \frac{1}{1 - \frac{\Lambda}{6} r^2} dr^2 + r^2 dS_2^2 + \frac{2}{\Lambda} d\Sigma_2^2.$$  \hspace{1cm} (12)

Thus the two-spheres ansatz we have considered encompasses black objects like hole, string and brane as well as generalized Nariai solution.

III. GB BLACK HOLE

We can use the same method above to write down the reduced -consistently truncated- GB Lagrangian in term of the variables $A(r), B(r), C(r), D(r)$ given in \cite{1} and with the use of equations \cite{2}. So we start by considering the GB Lagrangian (with cosmological constant)

$$L_{GB} = \sqrt{-g}\left(-2\Lambda + R^2 - 4R_{\mu}^\nu R_{\nu}^\mu + R_{\mu\nu}^{\rho\sigma} R_{\rho\sigma}^{\mu\nu}\right)$$  \hspace{1cm} (13)

and truncate it by implementing the ansatz \cite{1} into it, analogously to \cite{3}. Since it is not particulary illuminating, the resulting truncated Lagrangian is given in detail in Appendix B.

We begin with the metric (4),

$$ds^2 = -A(r) dt^2 + \frac{1}{A(r)} dr^2 + \frac{r^2}{k_1} dS_{(d_1)}^2 + \frac{r^2}{k_2} dS_{(d_2)}^2,$$  \hspace{1cm} (14)

with $d_1 > 1, d_2 > 1$. It turns out that we are able to find analytic solutions when the two-spheres have the same dimension $d_1 = d_2 =: d_0$. This means $d = d_1 + d_2 + 2 = 2(d_0 + 1)$ and $k_1 = k_2 = k = \frac{2d_0 - 1}{d_0 - 1}$. Let’s define

$$\Psi(r) := 1 - A(r),$$  \hspace{1cm} (15)

then EOM (from the truncated Lagrangian (13)) again becomes a single first order differential equation,

$$\frac{d}{dr}\left(r^{2d_0 - 3}(\psi^2 + \frac{d_0}{(d_0 - 1)^2(2d_0 - 3)}) - \frac{\Lambda r^{2d_0 + 1}}{2(2d_0 + 1)(2d_0 - 1)(d_0 - 1)d_0}\right) = 0.$$  \hspace{1cm} (16)

It integrates to give

$$A(r) = 1 \pm \sqrt{-\frac{d_0}{(d_0 - 1)^2(2d_0 - 3)} + \frac{r^4\Lambda}{2(2d_0 + 1)(2d_0 - 1)(d_0 - 1)d_0} + \frac{M}{r^{2d_0 - 3}}} ,$$  \hspace{1cm} (17)

where $M$ is an integration constant proportional to mass of the configuration. It represents a black hole in an asymptotically dS spacetime. This is a new black hole solution with two-spheres topology in GB gravity. The sign $\pm$ is chosen by requiring the solution to asymptotically go over to Schwarzschild-dS spacetime in $d = 2(d_0 + 1)$ dimension. Thus we choose negative sign to describe a black hole in this setting.

It is obvious that the solution cannot admit $M = \Lambda = 0$ limit and clearly reality of the metric as well as existence of horizons would prescribe a bound on mass of the black hole in relation to $\Lambda$. That is what we next consider.

A. Reality and physical bounds for (17)

Since in absence of black hole ($M = 0$), $\Lambda$ must be positive, and hence we shall take both $M$ and $\Lambda$ to be always non-negative. For concreteness let’s set $d_0 = 2$ which means we are considering 6-dimensional black hole solution.
Further for simplicity we define $\tilde{\Lambda} = \frac{\Lambda}{15}$, we write the solution (17) for $d_0 = 2$ as

$$A(r) = 1 - \sqrt{-2 + \frac{1}{4} \tilde{\Lambda} r^4 + \frac{M}{r}},$$

(18)

where $\tilde{\Lambda}$ and $M$ are taken to be non-negative.

Clearly for reality of the solution the discriminant should be $\geq 0$ as well as $A \geq 0$ for the existence of black hole horizon. Both these conditions should hold good simultaneously which means

$$2 \leq h(r) \leq 3$$

(19)

where $h(r) := \frac{1}{4} \tilde{\Lambda} r^4 + \frac{M}{r}$. The lower bound guarantees non-negativity of the discriminant while the upper ensures existence of horizons bounding a regular region of spacetime. The function $h(r)$ has a single minimum at

$$r_0 = \left(\frac{M}{\tilde{\Lambda}}\right)^{\frac{1}{2}}$$

(20)

and

$$h(r_0) = \frac{5}{4} \tilde{\Lambda}^{\frac{1}{2}} M^{\frac{1}{2}}.$$  

(21)

As will be discussed below, for physical viability of black hole we must have $2 \leq h(r_0) \leq 3$ which implies

$$2 \leq \frac{5}{4} \tilde{\Lambda}^{\frac{1}{2}} M^{\frac{1}{2}} \leq 3,$$

(22)

or, equivalently,

$$\left(\frac{8}{5}\right)^5 \leq \tilde{\Lambda} M^4 \leq \left(\frac{12}{5}\right)^5.$$  

(23)

The lower bound is given by the discriminant being non-negative while the upper by existence of horizons. The horizons are given by $h(r) = 3$ which is a fifth degree equation and can have two positive roots giving two horizons, lower ($r_-$) and upper ($r_+$), respectively for black hole and cosmological, dS-like, as shown in Fig. 1. There is an unusual feature of this class of black holes that there occurs a curvature singularity at vanishing of the discriminant, $h(r) = 2$. As a matter of fact the Ricci scalar for GB black hole (18) is given by

$$R = \frac{70 M^2 + 6 M r \left(15 \tilde{\Lambda} r^4 - 56\right) + r^2 \left(3 \tilde{\Lambda} r^4 - 8\right) \left(5 \tilde{\Lambda} r^4 - 48\right)}{r^4 \left(\frac{4M}{r} + \tilde{\Lambda} r^4 - 8\right)^{3/2}}.$$

which clearly diverges for $h(r_1) = 2$ unless numerator also vanishes at $r_1$. The numerator and denominator both vanish simultaneously only for $\tilde{\Lambda} M^4 = \left(\frac{8}{5}\right)^5$ at $r_1 = \frac{5 M}{8} = \left(\frac{8}{5 \tilde{\Lambda}}\right)^{\frac{1}{2}}$ making $R$ finite. This marks the limiting minimum for
black hole mass at which $r_1$ becomes the minimum $r_0$ as shown in Fig. 2. The remarkable property of this class of black holes is therefore existence of extremal value for mass which is a minimum.

This is in addition to the central singularity at $r = 0$. For a black hole, the latter is always covered by a horizon, now the question arises how to manage the former. One of the possibilities is to cover it with a horizon but there is no physical source that could produce a horizon to cover it. There are only $M$ and $\Lambda$ which could only produce the familiar horizons, the former black hole horizon covering the central singularity and the latter giving cosmological horizon. The only option then left for an acceptable black hole spacetime is therefore not to let it occur. The above bounds on mass for a given $\Lambda$ precisely do that as demonstrated in Figs refregion2, 2. This requires both $\Lambda$, $M$ to be non-zero. It can be easily seen that either of them being zero makes non-central ($h(r) = 2$) singularity naked.

Not only that it remains naked for $\Lambda < 0, M > 0$ and hence $\Lambda$ must always be positive. This is a new property of this class of black holes. In this setting, black hole can thus exist only in asymptotically de Sitter spacetime. That is, presence of positive $\Lambda$ is critical for black hole existence. Very recently a similar result has also been obtained for stability of pure Lovelock black holes [11] where $\Lambda$ makes otherwise unstable black hole stable.

IV. E-GB BLACK HOLE

Now we consider the E-GB Lagrangian

$$L_{E-GB} = \sqrt{-g} \left( -2\Lambda + \alpha_1 R + \alpha_2 (R^2 - 4R_{\mu \nu} R_{\nu}^{\mu} + R_{\mu \nu} \sigma \tau R_{\sigma \tau}^{\mu \nu}) \right),$$

(24)

with separate parameters $\alpha_1$, $\alpha_2$, so we can recover the GB and EH cases as limits with either parameter vanishing. The consistent truncation of (24) under (2) is given by (3) and (38) (Appendix B).

Again we consider the specific metric,

$$ds^2 = -A(r) dt^2 + \frac{1}{A(r)} dr^2 + \frac{d_0 - 1}{d - 3} r^2 \left( dS_{(d_0)}^2 + dS_{(d_0)}^2 \right),$$

(25)

Now the EOM for the variable $A(r)$ takes the form

$$\frac{d}{dr} \left( \alpha_1 r^{2d_0 - 1} \left( \frac{1}{2(2d_0^2 - 3d_0 + 1)} \right) \Psi + \alpha_2 r^{2d_0 - 3} \left( \Psi^2 + \frac{d_0}{(d_0 - 1)^2(2d_0 - 3)} \right) - \frac{\Lambda r^{2d_0 + 1}}{2(2d_0 + 1)(2d_0 - 1)(d_0 - 1)d_0} \right) = 0,$$

(26)
where $\Psi(r)$ has been defined in [15]. This integrates to give the solution

$$A(r) = 1 + \frac{\alpha_1 r^2}{4(2d_0 - 1)(d_0 - 1)\alpha_2} - \left( -\frac{d_0}{(d_0 - 1)^2(2d_0 - 3)} + \frac{\alpha_1^2 r^4}{4^2(2d_0 - 1)^2(d_0 - 1)^2\alpha_2^2} + \frac{\alpha_2^2 r^4}{2(2d_0 + 1)(2d_0 - 1)(d_0 - 1)\alpha_2^2} + \frac{M}{\alpha_2 r^{2d_0 - 3}} \right)^{\frac{1}{2}},$$

(27)

where we have chosen negative sign before the radical for the same reason as for GB case. In the limit $\alpha_2 \to 0$ we recover

$$\lim_{\alpha_2 \to 0} A(r) = 1 - \frac{\Lambda}{(2d_0^2 + d_0)\alpha_1} r^2 - \frac{2M((2d_0 - 3)d_0 + 1)}{\alpha_1 r^{2d_0 - 1}},$$

(28)

the corresponding Schwarzschild-dS solution [1] for $d_0 = \frac{d_2}{2} - 1$, $\alpha_1 = 1$ with an appropriate redefinition of the mass parameter, $M$.

Let us now set $d_0 = 2$ and then

$$A(r) = 1 + \frac{\alpha_1}{12\alpha_2} r^2 - \sqrt{-\frac{2}{\alpha_1} + \left( \frac{\alpha_1^2}{(12\alpha_2)^2} + \frac{\Lambda}{60\alpha_2} \right) r^4 + \frac{M}{\alpha_2 r}}.$$  

(29)

It is interesting to compare this solution with the solution with one-sphere topology,

$$ds^2 = -A(r) dt^2 + \frac{1}{A(r)} dr^2 + r^2 dS^2_{(d-2)},$$

(30)

with

$$A(r)_{\text{one-sphere}} = 1 + \frac{\alpha_1}{12\alpha_2} r^2 - \sqrt{-\frac{2}{\alpha_1} + \left( \frac{\alpha_1^2}{(12\alpha_2)^2} + \frac{\Lambda}{60\alpha_2} \right) r^4 + \frac{M}{\alpha_2 r}}.$$  

It is indeed the same as the above without $-2$ under the radical. Note that the former is not asymptotically flat for $\Lambda = 0$ while the latter is asymptotically flat, Minkowski. The other difference of course is that in the former metric, each sphere has a solid angle deficit which cancel out each other to give a $\Lambda$-vacuum spacetime. This is true more generally for $d = 2d_0 + 2$ where the former will have $\frac{d_0}{(d_0 - 1)r^{2d_0 - 3}}$ under the radical while the latter would be free of it.

A. Physical bounds for (27)

Let us rewrite (27) as

$$A(r) = 1 + Dr^2 - \sqrt{f(r)} ,$$

$$f(r) = -C + \frac{1}{4} E^{2d_0 + 1} r^4 + \frac{1}{2d_0 - 3} B^{2d_0 + 1} \frac{1}{r^{2d_0 - 3}} ,$$

(31)

with

$$E^{2d_0 + 1} = \frac{\alpha_1^2}{4(2d_0 - 1)^2(d_0 - 1)^2\alpha_2^2} + \frac{\Lambda}{2(2d_0 + 1)(2d_0 - 1)(d_0 - 1)\alpha_2^2}$$

$$B^{2d_0 + 1} = \frac{d_0}{(d_0 - 1)^2(2d_0 - 3)}$$

$$C = \frac{M}{\alpha_2}$$

$$D = \frac{\alpha_1}{4(2d_0 - 1)(d_0 - 1)\alpha_2}$$

(32)
Note that since their exponent is odd, the signs of $E$ and $B$ are those of their respective right hand side in the definition.

Let us consider the case $\alpha_1 > 0$, $\alpha_2 > 0$ (the same sign for the $EH$ and $GB$ coefficients is required for the theory to be ghost free\(^{[14]}\)) as well as $\Lambda > 0$ and $M > 0$, which imply $E > 0$, $B > 0$. Note from \(^{[32]}\) that there is a window for negative $\Lambda$ and still keeping $E > 0$.

Following on the same lines as before (see section (III A)), we obtain the bounds as follows:

$$C \leq \frac{2d_0 + 1}{4(2d_0 - 3)} E^{2d_0 - 3} B^4 \leq C + (1 + D \frac{B}{E})^2$$

(33)

where $C$, given in \(^{[32]}\), is a positive quantity determined by spacetime dimension. Since $E, B, D$ are positive, it is clear, for given $\alpha_1, \alpha_2$, there exists a range of values for $E$ and $B$ (i.e. for $\Lambda$ and $M$) fulfilling the bounds \(^{[33]}\). Thus as before non-central curvature singularity at the vanishing of the radical could be avoided by suitable prescription on black hole mass for given $\Lambda$.

\section{V. THERMODYNAMICS OF BLACK HOLES}

We will use the notation of Eq \(^{[32]}\) and of course we assume that the conditions given in Eq \(^{[33]}\) hold true, which guarantee existence of horizons. Let’s denote black hole horizon by $r_h$. Of course $r_h$ and $M$ could be traded for each other, just by requiring the function $M(r_h)$ to keep $A(r_h) = 0$ while varying $r_h$. The entropy is calculated from the First Law of thermodynamics by following the standard procedure.

We write the identity

$$A(r, M(r)) = 0$$

as an implicit equation for the function $M(r)$ introduced above, therefore we have the identity

$$A'(r, M(r)) + \frac{\partial A}{\partial M} |_{A=0} M'(r) = 0,$$

where $A'(r, M(r))$ denotes derivative relative to first argument $r$. Employing the Euclidean method (with periodic time to eliminate a conical singularity), we identify $A'(r, M(r))$ as the Hawking temperature $T(r) = \frac{A(r, M(r))}{4\pi}$. Thus we have

$$\frac{M'(r)}{T} = -4\pi \frac{1}{\frac{\partial A}{\partial M}} |_{A=0} = 8\pi \alpha_2 (1 + D r^2) r^{2d_0 - 3},$$

and we can compute the entropy by integrating the First Law

$$S = \int \frac{dM}{T} = \int \frac{M'(r)}{T(r)} dr = 8\pi \alpha_2 \int (1 + D r^2) r^{2d_0 - 3} dr = 8\pi \alpha_2 r^{2d_0} (\frac{1}{(2d_0 - 2) r^2} + \frac{D}{2d_0}).$$

(34)

where we have assumed that the entropy vanishes when the horizon shrinks to zero.

Of course the parameter $M$ used in our derivation is identified with the mass except for an overall factor that will depend on the dimension of the spacetime and linearly on area of two-spheres at unit radius. With this factor installed we get

$$S \simeq A \times \left( \frac{\alpha_2}{(2d_0 - 2) r^2} + \frac{\alpha_2 D}{2d_0} \right) = A \times \left( \frac{\alpha_2}{(2d_0 - 2) r^2} + \frac{\alpha_1}{8d_0 (2d_0 - 1) (d_0 - 1)} \right) = A \times \left( \bar{\alpha}_2 r^2 + \bar{\alpha}_1 \right),$$

(35)

where $A$ denotes horizon area and with $\bar{\alpha}_1 = \alpha_1/(8d_0 (2d_0 - 1) (d_0 - 1))$ and $\bar{\alpha}_2 = \alpha_2/(2d_0 - 2)$.

The temperature, in terms of $r_h$, is

$$T = \frac{1}{2(1 + D r^2) r_h} \left( (1 + D r^2) (2d_0 - 3) + (2d_0 + 1) D r^2_h + (2d_0 - 3) C - \frac{2d_0 + 1}{4} E^{2d_0 + 1} r^4_h \right).$$
For instance, for the pure GB case ($\alpha_2 = 1, \alpha_1 = 0 \Rightarrow D = 0$) and $d_0 = 2$ ($d = 6$), we obtain

$$T = \frac{1}{2\pi} \left( -\frac{3}{r_h} + \frac{5}{4} \frac{M}{r^2} \right), \quad S \simeq \frac{A}{r^2_h} \simeq r^2_h \simeq A^{\frac{2}{3}}. $$

It is worth noting that these parameters bear the same universal relation to $r_h$ as established in [15] for pure Lovelock one-sphere topology in the critical dimension $d = 2N + 2$, here for the $N = 2$ case. In particular, for pure GB black hole, $T = \frac{1}{2\pi} \left( -\frac{1}{r_h} + \frac{5}{4} \frac{M}{r^2} \right)$ and $S = 4\pi r^2_h$. Thus thermodynamics parameters do not distinguish between one or two-sphere topology, save for numerical factors.

VI. DISCUSSION

It is well-known that vacuum equation for Einstein as well as for general Lovelock gravity in spherical symmetry ultimately reduces to a single first order equation which is an exact differential[16–21] and hence can be integrated trivially. As a matter of fact, the equation then turns purely algebraic for one-sphere topology with $SO(d - 2)$ symmetry. Interestingly it turns out that this feature is carried through even for two-spheres topology with $SO(d_0) \times SO(d_0)$ symmetry where $d = 2(d_0 + 1)$. In particular the equations, (5), (16) and (26), refer respectively to solved to Einstein, GB and E-GB gravity which yield static black hole solutions. This result obviously raises the question as to whether this feature is also carried over to Lovelock gravity in general. The answer is in affirmative and it would be taken up separately in a forthcoming paper[22].

For black string, there occurs local topology change change as horizon radius increases from that of a black hole $S^{d_0}$ to that of black string $S^{d_0-1} \times S^1$. This change is negotiated through[2] a Ricci flat space over a double cone formed by two spheres with solid angle deficit. The main result of the paper is that we have been able to put on a black hole with this two-spheres topology. For Einstein black hole, the topology is $S^{d_1} \times S^{d_2}$ while for GB and E-GB it is $S^{d_0} \times S^{d_0}$. It should be noted that all these are new black hole solutions. The two-spheres setting also encompasses black string and brane as well as generalized Nariai metric.

For GB and E-GB cases, what $SO(n) \times SO(n)$ symmetry entails is the occurrence of an additional non-central curvature singularity which could be let not to occur by suitable prescription on black hole mass for a given $\Lambda > 0$. The two extremal limits for mass are defined by non-occurrence of non-central naked singularity (intersection with lower line in Fig. 2) and existence of horizons (intersection with upper line in Fig. 1). The range for mass is given in Eqs (23) and (33) which ensures absence of naked singularity for GB and E-GB black hole in dS spacetime. Also non-central singularity cannot be avoided when $M = 0$ or $\Lambda \leq 0$. Thus $\Lambda$ plays a very critical role for existence of this class of black holes and it prescribes a range of values for mass. This reminds one of the recently obtained result in which $\Lambda$ makes otherwise unstable pure Lovelock black hole stable by similarly prescribing a range of values for mass[11].

Further it turns out that black hole thermodynamics does not however distinguish between two-spheres and one-sphere topology as expressions for temperature and entropy of black hole remain essentially the same. For pure Lovelock black hole with spherical symmetry thermodynamics is universal; i.e. temperature and entropy bear the same relation to horizon radius in all odd ($d = 2N + 1$) and even ($d = 2N + 2$) dimensions where $N$ is the degree of Lovelock Lagrangian[15]. It is interesting that this universality continues to hold true even for two-spheres topology black holes in GB and E-GB gravity.

Whether these new black hole solutions find any application and relevance in string theory inspired or otherwise understanding of gravitational dynamics in higher dimensions is however an open question.

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Appendix A. EH solutions with spheres of constant curvature

The following spacetimes are solutions of the eom of $L_{EH}$,

$$ds^2 = -(1 - kr^2) dt^2 + \frac{1}{1 - kr^2} dr^2 + \frac{d_1 - 1}{k} dS^2_{(d_1)} + \frac{d_2 - 1}{k} dS^2_{(d_2)},$$

(36)

with $d_1 > 1, d_2 > 1$, arbitrary $k > 0$, and $\Lambda = \frac{1}{2}(d_1 + d_2)k > 0$. The topology of these spacetimes is $dS^2 \times S^{d_1} \times S^{d_2}$ (in our notation $dS^2$ is 2-dimensional de Sitter) and they are generalizations of the Nariai solutions.

Note that the curvature of each sphere, $\frac{1}{d_i - 1}$, is different according to its dimensionality. This is the way for this solution with two-spheres to be sustained by a cosmological constant.

Actually we can find solutions with hyperboloids ($k < 0$) instead of spheres. In such case the spacetimes are

$$ds^2 = -(1 + |k|r^2) dt^2 + \frac{1}{1 + |k|r^2} dr^2 + \frac{d_1 - 1}{|k|} dH^2_{(d_1)} + \frac{d_2 - 1}{|k|} dH^2_{(d_2)},$$

(37)

Now we have $\Lambda = \frac{1}{2}(d_1 + d_2)k < 0$. The topology of these spacetimes is $AdS^2 \times H^{d_1} \times H^{d_2}$.

The case $k \to 0$ is Minkowski spacetime (the spheres or hyperboloids acquire infinite radius and become flat), though the ansatz (37) is no longer convenient to describe such a limit.

Appendix B. The truncated GB Lagrangian

After some combinatorics, the GB Lagrangian, reduced under (2), becomes

$$L_{GB} = \sqrt{-g} \left( 8 d_1 d_2 L(0,1) L(a, a') + 4 (d_1 - 1) d_1 L(0, 1) L(a, b) + 8 (d_1 - 1) d_1 d_2 L(a, a') \left( L(0, a) + L(1, a) \right) \right) + 4 \left( (d_1 - 1) d_1 L(a', b') \left( L(0, a) + L(1, a) \right) + 8 d_1 d_2 L(0, a) L(1, a') + 4 (d_2 - 1) (d_1 - 1) d_1 L(a, b) \left( L(0, a) + L(1, a) \right) \right) + 8 (d_1 - 1) d_1 L(0, a) L(1, a') + 8 d_1 d_2 L(0, a') L(1, a) + 8 (d_2 - 1) d_2 L(a, a') \left( L(0, a') + L(1, a') \right) \right) + 4 (d_1 - 1) d_1 d_2 L(0, a) L(a', a') + 4 (d_1 - 1) (d_2 - 1) d_2 L(a, a')^2 + 4 (d_2 - 1) (d_1 - 1) d_1 d_2 L(a, a') L(a, b) + 4 (d_1 - 1) d_1 L(a, a') L(a', b') + 2 (d_1 - 1) d_1 (d_2 - 1) d_2 L(a, b) L(a', b') + (d_1 - 3) (d_1 - 2) (d_1 - 1) d_1 L(a, b)^2 + 4 (d_2 - 1) d_2 L(0, 1) L(a', b') + 4 (d_2 - 1) (d_2 - 1) d_2 L(a', b') \left( L(0, a') + L(1, a') \right) + 8 (d_2 - 1) d_2 L(0, a') L(1, a') \right) + (d_2 - 3) (d_2 - 2) (d_2 - 1) d_2 L(a', b')^2. \right)$$

(38)

Appendix C. GB solutions with spheres of constant curvature

With the ansatz $(d_1 > 1, d_2 > 1)$,

$$ds^2 = -A(r) dt^2 + \frac{1}{A(r)} dr^2 + \frac{1}{k_1} dS^2_{(d_1)} + \frac{1}{k_2} dS^2_{(d_2)},$$

(39)

we get solutions with $k_1$ and $k_2$ constrained to satisfy the third degree polynomial equation,

$$(d_1 - 1)(d_2 - 1) k_1^2 k_2 \left( (d_1 - 2)^2 (d_1 + 3) - 2((d_1 - 3)d_1 + 3)d_2 \right) + (d_1 - 1)(d_2 - 1) k_1 k_2^2 \left( 2d_1 ((d_2 - 3)d_2 + 3) - (d_2 - 2)^2 (d_2 + 3) \right) + (3 - d_1)(d_1 - 2)(d_1 - 1)^2 d_1 k_1^3 + (d_2 - 3)(d_2 - 2)(d_2 - 1)^2 d_2 k_2^3 = 0,$$

(40)

with $\Lambda$ (here $\Lambda$ includes a dimensional factor originated by the dimensionality of the GB Lagrangian) determined by

$$\Lambda = \frac{1}{8} \left( 2(d_1 - 1)d_1(d_2 - 1)d_2 k_1 k_2 + (d_1 - 3)(d_1 - 2)(d_1 - 1)d_1 k_1^2 + (d_2 - 3)(d_2 - 2)(d_2 - 1)d_2 k_2^2 \right),$$

(41)
and (after trivial redefinitions of the coordinates $r, t$, with translations and dilatations) with $A(r)$ given by

$$A(r) = 1 - \left(\frac{(d_1 - 1)k_1((d_1 - 3)(d_1 - 2)k_1 + (d_2 - 1)d_2k_2)}{(d_1 - 2)(d_1 - 1)k_1 + (d_2 - 1)d_2k_2}\right)r^2. \tag{42}$$

**Case of spheres of the same dimension.**

The above expressions simplify notably when $d_1 = d_2 =: d_0$. Then we obtain $k_1 = k_2 =: k > 0$ and

$$\Lambda = \frac{1}{2}(d_0 - 1)d_0((d_0 - 3)d_0 + 3)k^2, \quad A(r) = 1 - \frac{((d_0 - 3)d_0 + 3)}{d_0 - 1}kr^2. \tag{43}$$

**Appendix D. E-GB Solutions with spheres of constant curvature**

With the ansatz

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + \frac{1}{k_1}ds^2_{(d_1)} + \frac{1}{k_2}ds^2_{(d_2)}, \tag{44}$$

we can find solutions with $k_1, k_2$, related by a third degree polynomial equation as in Appendix C. Instead of giving the details of the general case, we focus on the simplest case $d_1 = d_2 =: d_0$, which is solved by $k_1 = k_2 =: k$,

$$\Lambda = \frac{1}{2}(d_0 - 1)d_0k(2\alpha_1 + 4((d_0 - 3)d_0 + 3)k\alpha_2) \tag{45}$$

and

$$A(r) = 1 - \frac{(d_0 - 1)(\alpha_1 + 4((d_0 - 3)d_0 + 3)k\alpha_2)}{\alpha_1 + 4(d_0 - 1)^2k\alpha_2}kr^2, \tag{46}$$

which for $\alpha_2 \to 0$ yields the solutions given in Appendix A whereas for $\alpha_1 \to 0$ gives the solutions in Appendix C.
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