Constraining and applying a generic high-density equation of state

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We discuss the “constant speed of sound” (CSS) parameterization of the equation of state of high density matter, and its application to the Field Correlator Method (FCM) model of quark matter. We show how observational constraints on the maximum mass and typical radius of neutron stars are expressed as constraints on the CSS parameters. We find that the observation of a 2M\(_{\odot}\) star already severely constrains the CSS parameters, and is particularly difficult to accommodate if the squared speed of sound in the high density phase is assumed to be around 1/3 or less.

We show that the FCM equation of state can be accurately represented by the CSS parameterization. We display the mapping between the FCM and CSS parameters, and see that FCM only allows equations of state in a restricted subspace of the CSS parameters.

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I. INTRODUCTION

There are many models of matter at density significantly above nuclear saturation density, each with their own parameters. In studying the equation of state (EoS) of matter in this regime it is therefore useful to have a general parameterization of the EoS which can be used as a generic language for relating different models to each other and for expressing experimental constraints in model-independent terms. In this work we use the previously proposed “Constant Speed of Sound” (CSS) parameterization\(^4\). We show how mass and radius observations can be expressed in terms of the CSS parameters. We then analyze a specific example, where the high-density phase is quark matter described by a model based on the Field Correlator Method (Sec.\(^4\)). We show how its parameters can be mapped on to the CSS parameter space, and how it is constrained by currently available observations of neutron stars.

The CSS parameterization is applicable to high-density equations of state for which: (a) there is a sharp interface between nuclear matter and a high-density phase which we will call “quark matter”, even when (as in Sec.\(^2\)) we do not make any assumptions about its physical nature; (b) the speed of sound in the high-density matter is pressure-independent for pressures ranging from the first-order transition pressure up to the maximum central pressure of neutron stars. One can then write the high-density EoS\(^5\) in terms of three parameters: the pressure \(p_{\text{trans}}\) of the transition, the discontinuity in energy density \(\Delta\varepsilon\) at the transition, and the speed of sound \(cQM\) in the high-density phase. For a given nuclear matter EoS \(\varepsilon_{\text{NM}}(p)\), the full EoS is then

\[
\varepsilon(p) = \begin{cases} 
\varepsilon_{\text{NM}}(p) & p < p_{\text{trans}} \\
\varepsilon_{\text{NM}}(p_{\text{trans}}) + \Delta\varepsilon + c_{QM}^{-2}(p - p_{\text{trans}}) & p > p_{\text{trans}}
\end{cases}
\]

with \(\varepsilon_{\text{trans}} = \varepsilon_{\text{NM}}(p_{\text{trans}})\). This form can be viewed as the lowest-order terms of a Taylor expansion of the high-density EoS about the transition pressure. Following Ref.\(^4\), we express the three parameters in dimensionless form, as \(p_{\text{trans}}/\varepsilon_{\text{trans}}\), \(\Delta\varepsilon/\varepsilon_{\text{trans}}\) (equal to \(\lambda - 1\) in the notation of Ref.\(^4\)) and \(c_{QM}^2\), with \(\varepsilon_{\text{trans}} = \varepsilon_{\text{NM}}(p_{\text{trans}})\).

The assumption of a sharp interface will be valid if, for example, there is a first-order phase transition between nuclear and quark matter, and the surface tension of the interface is high enough to ensure that the transition occurs at a sharp interface (Maxwell construction) not via a mixed phase (Gibbs construction); Given the uncertainties in the value of the surface tension\(^5\), this is a possible scenario. One can also formulate generic equations of state that model interfaces that are smeared out by mixing or percolation\(^8\).\(^9\).

The assumption of a density-independent speed of sound is valid for a large class of models of quark matter. For some Nambu–Jona-Lasinio models, the CSS EoS fits Eq.\(^1\)\(^5\) almost exactly\(^2\)\(^10\)\(^12\). The perturbative quark matter EoS\(^13\) has roughly density-independent \(c_{QM}^2\) with a value around 0.2 to 0.3, above the transition from nuclear matter (see Fig. 9 of Ref.\(^14\)). In the quartic polynomial parameterization\(^15\), varying the coefficient \(a_2\) between \(\pm(150\text{MeV})^2\), and the coefficient \(a_4\) between 0.6 and 1, and keeping \(p_{\text{trans}}/p_0\) above 1.5 (\(n_0 = 0.16\text{fm}^{-3}\) is the nuclear saturation density), one finds that \(c_{QM}^2\) is always between 0.3 and 0.36. It is noticeable that models based on relativistic quarks tend to have \(c_{QM}^2 \approx 1/3\), which is the value for systems with conformal symmetry, and it has been conjectured that there is a fundamental bound \(c_{QM}^2 < 1/3\)\(^16\).

In Sec.\(^2\) we show how the CSS parameterization is constrained by observables such as the maximum mass \(M_{\text{max}}\) and the radius \(R_{1.4}\) of a star of mass 1.4M\(_{\odot}\). In Secs.\(^3\) and\(^4\) we describe a specific model, based on a Brueckner-Hartree-Fock (BHF) calculation of the nuclear matter EoS and the Field Correlator Method (FCM) for the quark matter EoS. We show how the parameters of this model map on to part of the CSS parameter space, and how the observational constraints apply to the FCM model parameters. Sec.\(^4\) gives our conclusions.
II. CONSTRAINING THE CSS PARAMETERS

A. Topology of the mass-radius relation

![Phase diagram](image)

FIG. 1: Schematic phase diagram (from [1]) for hybrid star branches in the mass-radius relation of compact stars. We fix $c_{QM}^2$ and vary $\rho_{trans}/\epsilon_{trans}$ and $\Delta \epsilon/\epsilon_{trans}$. The four regions are (A) no hybrid branch (“absent”); (B) both connected and disconnected hybrid branches; (C) connected hybrid branch only; (D) disconnected hybrid branch only.

We use the term “hybrid star” to refer to stars whose central pressure is above $\rho_{trans}$ and so they contain a core of the high-density phase. The part of the mass-radius relation that arises from such stars is the “hybrid branch”. In all models of nuclear/quark matter we find the same four topologies of the mass-radius curve for compact stars: the hybrid branch may be connected to the nuclear branch (C), or disconnected (D), or both may be present (B) or neither (A). The occurrence of these as a function of the CSS parameters $\rho_{trans}/\epsilon_{trans}$ and $\Delta \epsilon/\epsilon_{trans}$ at fixed $c_{QM}^2$ is shown schematically in Fig. 1 (taken from Ref. [1]). The mass-radius curve in each region is depicted in inset plots, in which the thick green line is the hadronic branch, the thin solid red lines are stable hybrid stars, and the thin dashed red lines are unstable hybrid stars.

In the phase diagram the solid red line shows the threshold value $\Delta \epsilon_{crit}$ below which there is always a stable hybrid star branch connected to the neutron star branch. This critical value is given by

$$\frac{\Delta \epsilon_{crit}}{\epsilon_{trans}} = \frac{1}{2} + \frac{3}{2} \frac{p_{trans}}{\epsilon_{trans}}$$

and was obtained by performing an expansion in powers of the size of the core of high-density phase. Eq. (2) is an analytic result, independent of $c_{QM}^2$ and the nuclear matter EoS. The dashed and dot-dashed black lines mark the appearance-disappearance of the connected or disconnected hybrid star branch. The position of these lines depends on the value of $c_{QM}^2$ and (weakly) on the accompanying nuclear matter EoS [11].

Once a nuclear matter EoS has been chosen, any high-density EoS that is well-approximated by the CSS parameterization can be summarized by giving the values of the three CSS parameters, corresponding to a point in the phase diagram. We then know what sort of hybrid branches will be present.

B. Maximum mass of hybrid stars

| property | BHF, $Av_{18}$ + UVIX TBF | DBHF, Bonn A |
|----------|-----------------------------|--------------|
| saturation baryon density $\rho_0$ (fm$^{-3}$) | 0.16 | 0.18 |
| binding energy/baryon $E/A$ (MeV) | -15.98 | -16.15 |
| compressibility $K_0$ (MeV) | 212.4 | 230 |
| symmetry energy $S_0$ (MeV) | 31.9 | 34.4 |
| $L = 3p_0(dS/d\rho)|_{\rho_0}$ (MeV) | 52.9 | 69.4 |
| max mass of star ($M_\odot$) | 2.03 | 2.31 |
| radius of $M = 1.4M_\odot$ star (km) | 11.77 | 13.41 |

In Fig. 2 we show how mass measurements of neutron stars can be expressed as constraints on the CSS parameters. Each panel shows dependence on $p_{trans}/\epsilon_{trans}$ and $\Delta \epsilon/\epsilon_{trans}$ for fixed $c_{QM}^2$, as in Fig. 1. The region in which the transition to quark matter would occur below nuclear saturation density ($\rho_{trans} < n_0$) is excluded, because we would then have to assume that nuclei are metastable, which means we cannot assume a simple phase boundary between nuclear matter and quark matter. The contours show the maximum mass of a hybrid star as a function of the EoS parameters. The region inside the $M = 2M_\odot$ contour corresponds to EoSes for which the maximum mass is less than $2M_\odot$, so it is shaded to signify that this region of parameter space for the high-density EoS is excluded by the observation of a star with mass $2M_\odot$ [19]. For high-density EoSes with $c_{QM}^2 = 1$ (right hand plots), this region is not too large, and leaves a good range of transition pressures and energy density discontinuities that are compatible with the observation. However, for high-density matter with $c_{QM}^2 < 1/3$ (left hand plots), which is the typical value in many models (See Sec. I), the $M_{max} > 2M_\odot$ constraint eliminates a large region of the CSS parameter space [11][16]. We discuss this in more detail below.

The upper plots in Fig. 2 are for a stiffer nuclear matter EoS, Dirac-Brueckner-Hartree-Fock (DBHF) [20], and the lower plots are for a softer nuclear matter EoS, Brueckner-Hartree-Fock (BHF) [21] (see Sec. III). Properties of these nuclear matter EoSes are given in Table I. As one would expect, the stiffer EoS gives rise to heavier (and larger) stars, and therefore allows a wider range of CSS parameters to be compatible with the $2M_\odot$ measurement. In Fig. 2 the dot-dashed (red) contours are for hybrid stars on a connected branch, while the dashed (blue) contours are for disconnected branches. As discussed in Ref. [11], when crossing the near-horizontal boundary from region C to B the connected hybrid branch splits into a smaller connected
branch and a disconnected branch, so the maximum mass of the connected branch smoothly becomes the maximum mass of the disconnected branch. Therefore the red contour in the C region smoothly becomes a blue contour in the B and D regions. When crossing the near-vertical boundary from region C to B a new disconnected branch forms, so the connected branch (red dot-dashed) contour crosses this boundary smoothly.

In each panel of Fig. 2 the physically relevant allowed region is the white unshaded region. If, as predicted by many models, \( c_{QM}^2 \lesssim 1/3 \), then the existence of a \( 2M_\odot \) star constrains the other CSS parameters to two regions of parameter space (unshaded regions on the left and right sides of the two left panels of Fig. 2). In the left-hand region the transition occurs at a fairly low density \( n_{trans} \lesssim 2n_0 \). The region B, where connected and disconnected hybrid star branches can coexist, is excluded for \( c_{QM}^2 \lesssim 1/3 \), and even for larger \( c_{QM}^2 \) it is only allowed if the nuclear matter EoS is sufficiently stiff. In the right-hand allowed region the transition pressure is high, and the connected branch (red dot-dashed) contours are, except at very low \( \Delta \epsilon \), almost vertical, corresponding to EoSes that give rise to a very small connected hybrid branch which exists in a very small range of central pressures \( p_{cent} \) just above \( p_{trans} \). The maximum mass on this branch is therefore very close to the mass of the purely-hadronic matter star with \( p_{cent} = p_{trans} \). The mass of such a purely hadronic star is naturally independent of parameters that only affect the quark matter EoS, such as \( \Delta \epsilon \) and \( c_{QM}^2 \), so the contour is vertical. These hybrid stars have a tiny core of the high density phase and cover a tiny range of masses, of order \( 10^{-3} M_\odot \) or less, and so would be very rare.

C. Typical radius of hybrid stars

In Fig. 3 we show contour plots of the typical radius of a hybrid star (defined as \( R_{1.4} \), the radius of a \( 1.4M_\odot \) star) as a function of the CSS quark matter EoS parameters. The layout is
FIG. 3: Contour plots similar to Fig. 2 showing the radius at $M = 1.4\,M_\odot$ for hybrid stars as a function of the CSS parameters. The grey shaded region is excluded by the observational constraint $M_{\text{max}} > 2\,M_\odot$. The diagonally-hatched region $n_{\text{trans}} < n_0$ is excluded because nuclei would be metastable.

as in Fig. 2, each panel shows dependence on $p_{\text{trans}}/\epsilon_{\text{trans}}$ and $\Delta\epsilon/\epsilon_{\text{trans}}$ for fixed $c_{\text{QM}}^2$: the plots on the left are for $c_{\text{QM}}^2 = 1/3$ and the plots on the right are for $c_{\text{QM}}^2 = 1$; the plots on the top are for the stiffer DBHF nuclear matter EoS, while the lower plots are for the softer BHF nuclear matter EoS. As in Fig. 2, the region that is eliminated by the observation of a $2\,M_\odot$ star is shaded in grey. The solid red contours are for hybrid stars on a connected branch, while the solid blue contours are for disconnected branches. Because the contours show the radius of a $1.4\,M_\odot$ star they only fill part of the plot, the part where there are hybrid stars with that mass. The dashed (magenta) lines delimit the region where the disconnected branch contains stars of mass $1.4\,M_\odot$. The overall behavior is that, at fixed $\Delta\epsilon/\epsilon_{\text{trans}}$, the typical radius is large when the transition density is low. As the transition density rises the radius of a $1.4\,M_\odot$ star decreases at first, but then increases again. This is related to the previously noted fact [22] that when one fixes the speed of sound of quark matter and increases the bag constant (which increases $p_{\text{trans}}/\epsilon_{\text{trans}}$ and also varies $\Delta\epsilon/\epsilon_{\text{trans}}$ in a correlated way) the resultant family of mass-radius curves all pass through the same small region in the $M-R$ plane: the $M(R)$ curves “rotate” counter-clockwise around this hub (see Fig. 2 of Ref. [22]). In our case we are varying $p_{\text{trans}}/\epsilon_{\text{trans}}$ at fixed $\Delta\epsilon/\epsilon_{\text{trans}}$, so the hub itself also moves. At low transition density the hub is below $1.4\,M_\odot$, so $R_{1.4}$ decreases with $p_{\text{trans}}/\epsilon_{\text{trans}}$. At high transition density the hub is at a mass above $1.4\,M_\odot$ so $R_{1.4}$ will increase with $p_{\text{trans}}/\epsilon_{\text{trans}}$.

In Fig. 4 we show a magnified version of the radius contour plot, focusing on the region of low transition density which is allowed by the $M_{\text{max}} > 2\,M_\odot$ constraint. We see that there is a minimum radius in the allowed ($M_{\text{max}} > 2\,M_\odot$) region. For $c_{\text{QM}}^2 = 1/3$ (left hand plots), $R_{1.4} \gtrsim 12.3\,\text{km}$ for both stiff and soft nuclear EoS. It is interesting to note that the $R_{1.4} = 12\,\text{km}$ contour almost follows the $M_{\text{max}} = 2\,M_\odot$ contour. If an even heavier neutron star were observed, or if $c_{\text{QM}}^2$ turned out to be even smaller than $1/3$, the excluded (grey) area would expand, driving up the minimum radius. Conversely, if a $1.4\,M_\odot$ star were observed to have radius below about $12.3\,\text{km}$, one would have to conclude, similarly to [16], that $c_{\text{QM}}^2 > 1/3$.

For $c_{\text{QM}}^2 = 1$ (right hand plots), $R_{1.4} \gtrsim 9.5\,\text{km}$ at large values of the energy density discontinuity, and the bound rises as the discontinuity is decreased.

These findings obey the constraint noted by Lattimer [23].
(Fig. 5), that when we take into account the constraint \( M_{\text{max}} > 2M_c \), the maximally compact star for matter with \( dp/d\varepsilon = 1/3 \) has a radius of about 11 km, and for \( dp/d\varepsilon = 1 \) it has a radius of about 8.5 km.

III. THE BHF AND DBHF EOS OF NUCLEAR MATTER

We now discuss in more detail the nuclear matter equations of state that we use in this work. We adopt the Brueckner-Hartree-Fock (BHF) scheme, in which the only input needed is the realistic free nucleon-nucleon (NN) interaction \( V \) in the Brueckner-Bethe-Goldstone (BBG) equation for the reaction matrix \( G \)

\[
G[\rho; \omega] = V + \sum_{k \omega} V \frac{|k\omega|Q(k\omega)|}{\omega - e(k) - e(k)} G[\rho; \omega];
\]

where \( \rho \) is the nucleon number density, and \( \omega \) the starting energy. The propagation of intermediate baryon pairs is determined by the single-particle energy \( e(k;\rho) = \frac{k^2}{2m} + U(k;\rho) \), and the Pauli operator \( Q \). Because of the occurrence of \( G \) in the single-particle potential \( U(k;\rho) = \text{Re} \sum_{k' < k} \langle kk' | G[\rho; e(k) + e(k')]|kk' \rangle_a \), where the subscript “\( a \)” indicates antisymmetrization of the matrix element, the BBG equation (Eq. (3)) has to be solved in a self-consistent manner for several moments of the particles involved, at the considered densities.

In the non-relativistic BHF approximation the energy per nucleon is given by

\[
\frac{E}{A} = \frac{3}{5} \frac{k^2}{2m} + \frac{1}{2\rho} \sum_{k k' \leq k F} \langle kk' | G[\rho; e(k) + e(k')]|kk' \rangle_a.
\]

The nuclear EoS can be calculated with good accuracy in the Brueckner two-hole-line approximation with the continuous choice for the single-particle potential, and the results in this scheme are quite close to the calculations which include also the three hole-line contribution. The dependence on the NN interaction, also within other many-body approaches, has been systematically investigated in Ref.\[28\].

It is well known that, in order to reproduce the correct saturation point of symmetric nuclear matter, we must introduce nuclear three-body forces (TBFs). In our approach the TBF is reduced to a density-dependent two-body force by averaging over the position of the third particle, assuming that the probability of having two particles at a given distance is reduced according to the two-body correlation function.\[29\].

In this work we use the Argonne \( V_{18} \) NN potential \[31\], and the so-called Urbana model for TBFs, which consists of an attractive term due to two-pion exchange with excitation of an intermediate \( \Delta \) resonance, and a repulsive phenomenological central term.\[32\]\[33\]. Those TBFs produce a shift of about +1 MeV in energy and \(-0.01 \text{ fm}^{-3} \) in density. This adjustment is obtained by tuning the two parameters contained in the TBFs, and was performed to get an optimal saturation point\[29\][\[30\]. At present the theoretical status of microscopically derived TBFs is still quite rudimentary, however a tentative approach has been proposed using the same meson-exchange parameters as the underlying NN potential \[34\]\[35\]. In the past years, the BHF approach has been extended in order to include the hyperon degrees of freedom \[36\][\[37\], which play an important role in the study of neutron star matter. However, in this paper we will not discuss this issue.

Along with the nonrelativistic BHF EoS we consider its relativistic counterpart, the DBHF scheme \[20\] where the Bonn-A potential is used for the nucleon-nucleon interaction. In the low density region (\( \rho < 0.3 \text{ fm}^{-3} \)), the BHF (including TBF) and DBHF equations of state are very similar, whereas at higher densities the DBHF is slightly stiffer. The discrepancy between the nonrelativistic and relativistic calculation can be easily understood by recalling that the DBHF treatment is equivalent to introducing in the nonrelativistic BHF the TBF corresponding to the excitation of a nucleon-antinucleon pair, the so-called Z-diagram \[38\], which is repulsive at all densities. In the BHF treatment with Urbana TBF, both attractive and repulsive TBF are introduced and therefore a softer EoS is expected. We report in Table II the main properties of both EoSes.

IV. QUARK MATTER VIA THE FIELD CORRELATOR METHOD

A. The FCM EoS

The approach based on the Field Correlator Method (FCM) provides a natural treatment of the dynamics of confinement in terms of the Color Electric (\( D^\pm \)) and Color Magnetic (\( D^\mp \)) Gaussian correlators, the former being directly related to confinement, so that its vanishing above the critical temperature implies deconfinement \[39\]. The extension of the FCM to finite temperature \( T \) at chemical potential \( \mu_q = 0 \) gives analytical results in reasonable agreement with lattice data, giving us some confidence that it correctly describes the deconfinement phase transition \[40\][\[41\]. In order to derive an EoS of the quark-gluon matter in the range of baryon density typical of the neutron star interiors, we have to extend the FCM to non-zero chemical potential \[40\][\[41\]. In this case, the quark pressure for a single flavour is simply given by

\[
P_q/T^4 = \frac{1}{\pi^2} \left[ \phi_v \left( \frac{\mu_q - V_1/2}{T} \right) + \phi_v \left( -\frac{\mu_q + V_1/2}{T} \right) \right]
\]

where

\[
\phi_v(a) = \int_0^\infty du \left( u^4/\sqrt{u^2 + v^2} \right) \left( \exp \left[ \sqrt{u^2 + v^2} - a \right] + 1 \right)^{-1}
\]

with \( v = m_q/T \), and \( V_1 \) is the large distance static \( \eta q \) potential whose value at zero chemical potential and temperature is \( V_1(T = \mu_B = 0) = 0.8 \) to 0.9 GeV \[42\][\[43\]. The gluon contribution to the pressure is

\[
P_g/T^4 = \frac{8}{3\pi^2} \int_0^\infty d\chi \frac{1}{\exp \left( \frac{\chi + V_1}{\sqrt{\frac{m_q}{T}}} \right) - 1}
\]
FIG. 4: Magnified version of Fig. 3, showing contour plots of $R_{1.4}$, the radius at $M = 1.4M_\odot$, for hybrid stars with a low transition density. The grey shaded region is excluded by the observational constraint $M_{\text{max}} > 2M_\odot$. There is a minimum radius for hybrid stars in the allowed region.

and the total pressure is

$$P_{\text{tot}} = \sum_{j=u,d,s} P_{j}^q + P_g - \left(\frac{11 - \frac{2}{3}N_f}{32}\right) G_2 \frac{G_2}{2}$$  \hspace{1cm} (8)

where $P_{j}^q$ and $P_g$ are given in Eqs. (5) and (7), and $N_f$ is the number of flavours. The last term in Eq. (8) corresponds to the difference of the vacuum energy density in the two phases, $G_2$ being the gluon condensate whose numerical value, determined by the QCD sum rules at zero temperature and chemical potential, is known with large uncertainty, $G_2 = 0.012 \pm 0.006 \text{ GeV}^4$. At finite temperature and vanishing baryon density, a comparison with the recent available lattice calculations provides clear indications about the specific values of these two parameters, and in particular their values at the critical temperature $T_c$. Some lattice simulations suggest no dependence of $V_1$ on $\mu_B$, at least for very small $\mu_B$, while different analyses suggest a linear decreasing of $G_2$ with the baryon density $\rho_B$ [44], in nuclear matter. However, for simplicity, in the following we treat both $V_1$ and $G_2$ as numerical parameters with no dependence on $\mu_B$.

B. The FCM EoS and the CSS parameterization

The CSS parameterization will be applicable to the FCM EoS if the speed of sound in the FCM EoS depends only weakly on the density or pressure. In Fig. 5 we show that this is indeed the case. The upper panel shows the speed of sound vs. pressure in the FCM quark matter EoS for different values of the FCM parameters, displayed in the lower panel. We see that the speed of sound varies by less than 5% over the considered range of pressures along each curve, and lies in the interval $0.28 < c_{\text{QM}}^2 < 1/3$. The value of $c_{\text{QM}}^2$ shows a weak dependence on $V_1$ and extremely weak on $G_2$, which appears as an additive constant in the quark matter EoS according to Eq. (8). The transition pressure is more sensitive to the FCM parameters, increasing rapidly with $V_1$ and with $G_2$. The energy density at a given pressure increases slightly with an increase in $V_1$ or $G_2$.

To illustrate how well the CSS parameterization fits the FCM EoS, we show in the lower panel of Fig. 5 that, for the same FCM parameter choices, we can always find suitable values of the CSS parameters which fit the FCM calculation.
there would be no transition from hadronic to quark matter, as case could not give \( 2M \). Above that region, \( V_1 \) would be negative which would correspond to a repulsive quark-quark interaction, and in any case could not give \( 2M \) stars. Below that region, there would be no transition from hadronic to quark matter, as explained below.

In Fig. 5, \( V_1 \) varies from 0 up to the maximum value at which hybrid star configurations occur, which is indicated by an (orange) cross. For the BHF case that value is \( V_1 = 240 \text{ MeV}, G_2 = 0.0024 \text{ GeV}^4 \) and for the DBHF case it is \( V_1 = 255 \text{ MeV}, G_2 = 0.0019 \text{ GeV}^4 \). Along each FCM curve in Fig. 6, the parameter \( G_2 \) starts at the minimum value at which there is a phase transition from hadronic to FCM quark matter; at lower \( G_2 \) the quark and the hadronic pressures \( p(\mu) \) do not cross at any \( \mu \). On each curve one point is labelled with its value of \( G_2/(10^{-3} \text{ GeV}^4) \), and subsequent points are at intervals where \( G_2 \) increases in increments of 1 in the same units.

We observe that along each line of constant \( V_1 \), \( p_{\text{trans}}/\epsilon_{\text{trans}} \) grows with \( G_2 \). This can be explained by recalling the linear dependence of the quark pressure on \( G_2 \) in Eq. (4), so that, at fixed chemical potential, an increase of \( G_2 \) lowers the quark pressure, making quark matter less favorable, and shifting the transition point to higher chemical potential or pressure. This was already discussed in Ref. [45] for BHF nuclear matter, and is equally applicable to DBHF nuclear matter. Obviously if \( G_2 \) becomes too small, the phase transition takes place in a region of low densities where finite nuclei are present, and the homogeneous nuclear matter approach becomes invalid.

The qualitative behavior of the curves of constant \( V_1 \) can be understood in terms of the Maxwell construction between the purely hadronic phase and the quark phase. The fact that \( \Delta \epsilon/\epsilon_{\text{trans}} \) goes through a minimum (which is always at \( p_{\text{trans}}/\epsilon_{\text{trans}} \approx 0.1 \) as \( G_2 \) is increased at constant \( V_1 \) can be understood from Fig. 2 of Ref. [45], which shows pressure \( p \) as a function of baryon density \( n \) and the location of the hadron (BHF EoS) to quark (FCM EoS) transition when \( G_2 \) is varied. The hadronic EoS is strongly curved, especially at low pressure, while the FCM EoS is closer to a straight line. Consequently, the baryon density difference between the two phases at a given pressure has a minimum at densities around \( 2n_0 \), which corresponds to \( p_{\text{trans}}/\epsilon_{\text{trans}} \approx 0.1 \). As \( G_2 \) increases, the transition pressure rises, scanning through this minimum. It follows that the energy density difference also goes through a minimum, because \( \epsilon = \mu n - p \), and \( p \) and \( \mu \) are continuous at the transition, so \( \Delta \epsilon = \mu \Delta n \). The DBHF hadronic EoS is very similar to BHF at low pressure, so the curves have their minimas at the same value of \( p_{\text{trans}}/\epsilon_{\text{trans}} \) in both panels of Fig. 6.

We also see in Fig. 6 that an increase of \( V_1 \) moves the curves slightly downward and to the right. This is expected since \( V_1 \) is a measure of the attractive interparticle strength, and therefore it is inversely proportional to the pressure of the system, so the pressure decreases as \( V_1 \) is increased at fixed \( \mu \), and, as already discussed for the parameter \( G_2 \), a decrease of the quark pressure raises \( p_{\text{trans}} \). The role of \( V_1 \) and \( G_2 \) in the quark EoS discussed so far, provides in the same way a qualitative understanding of \( c_{\text{QM}}^2 \) in panel (a) in Fig. 5 although, as already noticed, the effect in Fig. 6 of the change in \( c_{\text{QM}}^2 \) is negligible.

**C. Expected properties of Mass-Radius curves**

By comparing Fig. 6 with Fig. 5 we can see that when combining FCM quark matter with BHF (soft) nuclear matter, the physically allowed range of FCM parameter values yields extremely well. This means that there exists a well-defined mapping between the FCM parameters \( V_1, G_2 \) and the CSS parameters \( (p_{\text{trans}}/\epsilon_{\text{trans}}, \Delta \epsilon/\epsilon_{\text{trans}}, c_{\text{QM}}^2) \). Note that the mapping depends on the EoS of the hadronic matter.

The mapping is displayed in Fig. 6, which shows the region of the CSS parameter space where FCM equations of state are found. As in the phase diagrams in Sec. III we show the plane whose co-ordinates are the CSS parameters \( \Delta \epsilon/\epsilon_{\text{trans}} \) and \( p_{\text{trans}}/\epsilon_{\text{trans}} \). For the hadronic EoS we use BHF (left panel) and DBHF (right panel). The lines without points represent the phase boundaries, as for the figures in Sec. III for connected and disconnected branches. Whether a given FCM EoS yields stable hybrid stars depends on which of those phase regions (see Fig. 5) it is in. The solid (green) phase boundary with a cusp at \( p_{\text{trans}}/\epsilon_{\text{trans}} \approx 0.17 \) delimits the region with a disconnected branch for \( c_{\text{QM}}^2 = 1/3 \), while the nearby dashed (green) line is for \( c_{\text{QM}}^2 = 0.28 \), so these span the range of \( c_{\text{QM}}^2 \) relevant for the FCM, as discussed in Fig. 5. It is evident that the dependence on \( c_{\text{QM}}^2 \) is tiny and negligible for practical purposes.

The thin dashed (black) line and the solid (black) line studied with circles delimit the equations of state yielded by the FCM calculation. Within that region, the lines studied with points show the CSS parameterization of the FCM quark matter EoS, where along each line we keep \( V_1 \) constant and vary \( G_2 \). Above that region, \( V_1 \) would be negative which would correspond to a repulsive quark-quark interaction, and in any case could not give \( 2M \) stars. Below that region, there would be no transition from hadronic to quark matter, as explained above.
FIG. 6: The mapping of the FCM quark matter model onto the CSS parameterization. Results are obtained using the BHF (left panel) and DBHF (right panel) nuclear matter EoS. The undecorated curves are the phase boundaries for the occurrence of connected and disconnected hybrid branches (compare Fig. 1 and 2). The thin dashed (black) line and the solid (black) line studded with circles delimit the region yielded by the FCM model. Within that region, lines decorated with symbols give CSS parameter values for FCM quark matter as \( G_2 \) is varied at constant \( V_1 \) (given in MeV). The (orange) cross denotes the EoS with the highest \( p_{\text{trans}} \), which gives the heaviest FCM hybrid star. See text for details.

EoSes that are mostly in regions C and A, where there is no disconnected hybrid branch. At the lowest transition densities the FCM EoS can achieve a large enough energy density discontinuity to yield a disconnected branch (region D).

For the DBHF (stiff) nuclear EoS there is a wider range of values of \( V_1 \) and \( G_2 \) that give disconnected branches, and some of them give simultaneous connected and disconnected branches. This difference can be understood in terms of the stiffness of the EoSes. A change from a soft hadronic EoS (BHF) to a stiff one (DBHF) produces a steeper growth of the hadronic pressure as a function of the baryon density. Referring again to Fig. 2 of Ref. [45], this pulls the DBHF \( p(n) \) curve further away from the FCM curve, giving a larger difference in baryon density at given pressure, and hence, as noted above, a larger \( \Delta \varepsilon \). This is why the curves for DBHF+FCM (right panel of Fig. 6) are shifted upwards along the \( \Delta \varepsilon /\varepsilon_{\text{trans}} \) axis compared to the BHF+FCM curves (left panel of Fig. 6).

We can calculate the maximum mass of a hybrid star containing an FCM core as a function of the FCM parameters, and then use the mapping described above to obtain the CSS parameter values for each FCM EoS, producing a contour plot of maximum mass (Fig. 7) for BHF (left panel) and DBHF (right panel) hadronic EoS. Given that the CSS parameterization is a fairly accurate representation of the FCM EoS, one would expect this to be very similar to the corresponding plot for CSS itself with \( c_{\text{QM}}^2 = 1/3 \) (Fig. 2), and this is indeed the case. The contours in Fig. 7 are restricted to the region corresponding to physically allowed FCM parameter values. As they end at the edges of that region.

The triangular shaded area at the edge of each panel shows the region of the parameter space that is accessible by the FCM and is consistent with the measurement of a 2\( M_{\odot} \), by having hybrid stars of maximum mass greater than 2\( M_{\odot} \). The (orange) cross in each panel of Fig. 6 is at the high-transition-pressure corner of that triangular area. The heaviest BHF+FCM hybrid star has a mass of 2.03\( M_{\odot} \), and the heaviest DBHF+FCM hybrid star has a mass of 2.31\( M_{\odot} \).

As noted in Sec. II B, the hybrid stars in this physically allowed and FCM-compatible region of the phase diagram lie on a very tiny connected branch, covering a very small range of central pressures and masses and radii, and would therefore occur only rarely in nature. These stars have very small quark matter cores (see Ref. 1, Figs. 5 and 6), and their mass and radius are very similar to those of the heaviest purely hadronic
FIG. 7: Contour plots, analogous to Fig. 2, showing the maximum mass of hybrid stars with FCM quark matter cores, given in terms of the corresponding CSS parameter values rather than the original FCM parameter values. As in Fig. 6, solid lines are phase boundaries (compare Figs. 1 and 2). The shaded sectors indicate the parameter regions accessible by the FCM and with \( M_{\text{max}} > 2M_\odot \). In each panel the lower border of the shaded region meets the phase boundary (red line) at the point with highest value of \( V_1 \) reported in Fig. 6 as an orange cross. Note the different scales on the x-axis for the two panels.

star, but there could be other clear signatures of the presence of the quark matter core, such as different cooling behavior.

The CSS parameterization has another region where heavy hybrid stars occur, at low transition pressure (see Fig. 2), but the FCM does not predict that the quark matter EoS could be in that region.

To characterize the radius of FCM hybrid stars we cannot construct contour plots like Fig. 3 because, as we have just seen, the FCM predicts that only hybrid stars with mass very close to the maximum mass are allowed. There are no FCM hybrid stars with mass around 1.4\( M_\odot \). Instead, in Fig. 8 we show the range of radii of stars with a given maximum mass when varying FCM parameters, for our two different hadronic EoSes. The right hand edge of each shaded region traces out the mass-radius relation for hadronic stars with the corresponding hadronic EoS. The FCM hybrid stars form very small connected branches which connects to the nuclear matter where the central pressure reaches the transition pressure (See Sec. II B) so the hybrid stars do not deviate very far from the hadronic mass-radius curve. Hence the shaded regions in Fig. 8 are narrow, especially in the observationally allowed \( (M_{\text{max}} > 2M_\odot) \) region. For BHF (soft) nuclear matter, the hadronic stars, and hence the hybrid stars, are smaller because the nuclear mantle is more compressed by the self-gravitation of the star.

V. CONCLUSIONS

We have shown how observational constraints on the maximum mass and typical radius of hybrid stars can be expressed in a reasonably generic model-independent way as constraints on the parameters of the CSS parameterization of the high-density EoS. Our analysis assumed a sharp transition from nuclear matter to a high-density phase such as quark matter, and that the speed of sound in that phase is independent of the pressure. We found that the observation of a 2\( M_\odot \) star constrains the CSS parameters significantly \([1][10]\).

If, as predicted by many models, \( \epsilon_{\text{QM}}^2 \lesssim 1/3 \), then there are two possible scenarios.
Firstly, there is a low-transition-pressure scenario, where the transition to the high density phase occurs at \( n_{\text{trans}} \lesssim 2n_0 \) (unshaded region on the left side of the two left panels of Fig. [2]).

The hybrid branch of the mass-radius relation will be connected to the nuclear branch if the energy density discontinuity at the transition is less than about 50%, otherwise it will be a separate “third family” disconnected branch. In the low-transition-pressure scenario the radius of a 1\( M_\odot \) star must be larger than about 12.3 km (Fig. [4] assuming \( n_{\text{trans}} > n_0 \); see also [23]). If a neutron star heavier than 2\( M_\odot \) were observed or if \( c_{\text{QM}}^2 \) turned out to be even smaller than 1/3, the minimum radius would become larger. Conversely, if a 1.4\( M_\odot \) star were observed to have radius below 12.3 km, this would imply that \( c_{\text{QM}}^2 > 1/3 \) (as in Ref. [15]).

Secondly, there is a high-transition-pressure scenario (white region on the right side of the left panels of Fig. [2]). This tends, in the region of near-vertical maximum-mass contours, to give a very small branch of hybrid stars with tiny quark matter cores, occurring in a narrow range of central pressures just above the transition pressure, although the branch may be larger if the nuclear EOS is stiff and energy density discontinuity at the transition is small. Their radii are determined by the hadronic matter EOS (see Fig. [3]).

If \( c_{\text{QM}}^2 \) is larger than 1/3 then a larger region of the CSS parameter space becomes allowed (right panels of Fig. [2]).

Our study is intended to motivate the use of the CSS parameterization as a framework in which the implications of observations of neutron stars for the high-density EOS can be expressed and discussed in a way that is largely model-independent.

As an application to a specific model, we performed calculations for the FCM quark matter EOS. We showed that the FCM equation of state can be accurately represented by the CSS parameterization, and we displayed the mapping between the FCM and CSS parameters. We found that FCM quark matter has a speed of sound in a narrow range around \( c_{\text{QM}}^2 = 0.3 \), and the FCM family of EOSes covers a limited region of the space of all possible EOSes (Fig. [9]). Once the observational constraint \( M_{\text{max}} > 2M_\odot \) is taken into account, the allowed region in the parameter space is drastically reduced to the shaded areas of Fig. [7]. This corresponds to the high-transition-pressure scenario, with a small connected branch of hybrid stars with tiny quark matter cores. Such stars would be hard to distinguish from hadronic stars via mass and radius measurements, but the quark matter core could be detectable via other signatures, such as cooling behavior.

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