Topologically massive gravito-electrodynamics: exact solutions

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Abstract

We construct two classes of exact solutions to the field equations of topologically massive electrodynamics coupled to topologically massive gravity in 2 + 1 dimensions. The self-dual stationary solutions of the first class are horizonless, asymptotic to the extreme BTZ black-hole metric, and regular for a suitable parameter domain. The diagonal solutions of the second class, which exist if the two Chern-Simons coupling constants exactly balance, include anisotropic cosmologies and static solutions with a pointlike horizon.

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1 Introduction

Topologically massive gravity (TMG) \[1\] is a theory of gravity in three-dimensional spacetime which includes, in addition to the otherwise dynamically trivial Einstein-Hilbert action, a gravitational Chern-Simons term. The resulting theory has non-trivial dynamics with massive excitations. A number of papers have been devoted to the construction of exact solutions to sourceless TMG \[2\]\[3\], as well as to TMG with a cosmological constant \[4\]\[5\]. On the other hand, not much is known about exact solutions to the theory with matter sources, except for the special cases of point particles with a particular value for the mass-to-spin ratio \[6\], of lightlike particle sources \[7\], and of a two-fluid source \[8\]. Of special interest is the case of TMG coupled to a spin 1 Abelian gauge (“electromagnetic”) field. In this case, it seems natural to also add to the minimal Maxwell action a Chern-Simons term for the electromagnetic field, leading to the theory of topologically massive electrodynamics (TME) \[9\]\[1\]. Exact solutions of TME coupled to Einstein gravity, generalizing solutions of Maxwell electrodynamics coupled to Einstein gravity [10–12] have been discussed in [13]. In this paper, we shall discuss classical solutions of TME coupled to TMG — topologically massive gravito-electrodynamics (TMGE), and construct two classes of exact solutions.

In the second section, we generalize to the case of the fully coupled TMGE theory with a cosmological constant the methods previously used to reduce, under the assumption of two commuting Killing vectors, the field equations of TMG \[2\]\[3\] or of TME coupled to Einstein gravity [13]. A first class of exact, self-dual stationary solutions are constructed in the third section. These horizonless solutions, which exist only for a negative or zero cosmological constant $\Lambda$, are for $\Lambda < 0$ asymptotic to the extreme BTZ black hole metric [11]. In the fourth section we investigate the existence of diagonal solutions to TMGE. It is known that there are no non-trivial static solutions (diagonal stationary solutions) either to TMG [2] or to TME coupled to Einstein gravity [13]. However one may speculate that the dynamical spins generated by the two Chern-Simons couplings could exactly balance so that spinless, static solutions would be possible. Indeed we shall find exact diagonal solutions for a particular relation between the two Chern-Simons coupling constants. These solutions are cosmologies for $\Lambda > 0$, and static solutions for $\Lambda < 0$. We summarize our results in the last section.
2 Reduction of the field equations

The action for TMGE may be written

\[ I = I_E + I_M + I_{CSG} + I_{CSE}, \]  

(2.1)

where

\[ I_E = -m \int d^3x \sqrt{|g|} (R + 2\Lambda), \]

\[ I_M = -\frac{1}{4} \int d^3x \sqrt{|g|} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}. \]  

(2.2)

are the Einstein action (with cosmological constant \( \Lambda \) and gravitational constant \( G \equiv \frac{1}{16\pi m} \)) and the Maxwell action, and

\[ I_{CSG} = -\frac{m}{2\mu_G} \int d^3x \epsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma} \left[ \partial_\mu \Gamma^\sigma_{\rho\nu} + \frac{2}{3} \Gamma^\sigma_{\mu\tau} \Gamma^\tau_{\nu\rho} \right], \]

\[ I_{CSE} = \frac{\mu_E}{2} \int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \]  

(2.3)

are the gravitational and electromagnetic Chern-Simons terms (with \( \epsilon^{\mu\nu\rho} \) the antisymmetric symbol). The Chern-Simons terms being only pseudo-invariant, the absolute signs of the Chern-Simons coupling constants \( \mu_G \) and \( \mu_E \) are unimportant (however we expect their relative sign to be of importance). In the case of sourceless TMG (\( \mu_E = 0 \)), the gravitational coupling constant \( m \) should be negative (instead of positive as in four dimensions) to avoid the appearance of ghosts \([3]\). While this argument carries over to the case of the coupled theory (\( \mu_E \neq 0 \)) treated perturbatively, we cannot rule out the possibility that the choice \( m > 0 \) might lead to a consistent non-perturbative quantum theory, so we will leave the choice of the sign of \( m \) open.

We assume that the spacetime has two commuting Killing vectors, and choose the parametrisation \([14][12]\)

\[ ds^2 = \lambda_{ab}(\rho) \, dx^a \, dx^b + \zeta^{-2}(\rho) R^{-2}(\rho) \, d\rho^2, \quad A_\mu \, dx^\mu = \psi_a(\rho) \, dx^a \]  

(2.4)

\((a, b = 0, 1)\), where \( \lambda \) is the \( 2 \times 2 \) matrix

\[ \lambda = \begin{pmatrix} T + X & Y \\ Y & T - X \end{pmatrix}, \]  

(2.5)
with \( \det \lambda = R^2 \equiv X^2 \), the Minkowski pseudo-norm of the “vector” \( X \) of components \( X^0 \equiv T, X^1 \equiv X, X^2 \equiv Y \):

\[
X^2 = X_i X_i = T^2 - X^2 - Y^2,
\]

and the scale factor \( \zeta \) allows for arbitrary reparametrizations of the variable \( \rho \). We shall discuss in this paper both the cases of solutions with \( X \) “space-like” \((R^2 < 0)\), for which the metric (2.4) is Lorentzian, \( \rho \) being the radial coordinate, or “timelike” \((R^2 > 0)\), corresponding to a Riemannian metric for \( T > 0 \), and to a cosmology for \( T < 0 \) (\( \rho \) is then the time coordinate).

The parametrization (2.4) reduces the action (2.1) to the form

\[
I = \int d^2x \int d\rho L,
\]

with the effective Lagrangian \( L \)

\[
L = \frac{1}{2} \left[ \frac{m}{\mu_G} \zeta^2 (X \cdot (\dot{X} \wedge \ddot{X})) - m \zeta \dot{X}^2 + \zeta \hat{\psi} \Sigma \cdot X \hat{\psi} - \mu E \psi \Sigma \hat{\psi} - 4 \zeta^{-1} \Lambda \right].
\]

where \( \cdot = \partial / \partial \rho \), the Minkowski scalar and wedge products are defined by \( \alpha \cdot \beta = \alpha^i \beta_i \), \( (\alpha \wedge \beta)_i = \varepsilon_{ijk} \alpha^j \beta^k \), the “Dirac” matrices \( \Sigma^i \) are

\[
\Sigma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Sigma^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \Sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.9)
\]

and \( \hat{\psi} \equiv \psi^T \Sigma^0 \) is the (real) Dirac adjoint of the “spinor” \( \psi \).

The action (2.7) is invariant under the \( \text{SL}(2,\mathbb{R}) \approx \text{SO}(2,1) \) group of transformations in the plane of the two Killing vectors \( K^0, K^1 \). This invariance leads to the conservation of the “angular momentum” vector

\[
J = L + S_G + S_E,
\]

sum of “orbital” and “spin” contributions

\[
L = -m X \wedge \dot{X},
\]

\[
S_G = -\frac{m}{2 \mu_G} \left[ 2 X \wedge (X \wedge \ddot{X}) - \dot{X} \wedge (X \wedge \ddot{X}) \right],
\]

\[
S_E = \frac{1}{2} \Pi^i \Sigma_i \psi.
\]
where $\Pi^T \equiv \partial L/\dot{\psi}$ is the moment canonically conjugate to $\psi$, and we have set the Lagrange multiplier $\zeta$ equal to 1. Variation of the Lagrangian (2.8) with respect to $\psi$ leads to the first integrals

$$\Pi^T - \frac{\mu_E}{2} \overline{\psi} = 0 \quad (2.12)$$

(we have without loss of generality chosen a gauge such that the constant right-hand side is zero). The electromagnetic spin vector then reduces to

$$S_E = \frac{\mu_E}{4} \overline{\psi} \Sigma \psi. \quad (2.13)$$

We note for future purposes that this vector is null,

$$S_E^2 = 0. \quad (2.14)$$

Varying the Lagrangian (2.8) with respect to $X$, and taking into account equations (2.12) and (2.13), we obtain the coupled dynamical equations for the vector fields $X$ and $S_E$,

$$\ddot{X} = -\frac{1}{2\mu_G} [3(\dot{X} \wedge \ddot{X}) + 2(X \wedge \dot{X})] + \frac{2\mu_E}{mR^2} S_E - \frac{4\mu_E}{mR^4} X (S_E \cdot X), \quad (2.15)$$

$$\dot{S}_E = \frac{2\mu_E}{R^2} X \wedge S_E \iff \dot{\psi} = \frac{\mu_E}{R^2} \Sigma \cdot X \psi$$

(for the choice $\zeta = 1$). Finally, the action (2.7) is also invariant under reparametrizations of $\rho$ (variations of $\zeta$), leading to the Hamiltonian constraint,

$$H \equiv -\frac{m}{2} \dot{X}^2 + \frac{m}{\mu_G} (X \cdot (\dot{X} \wedge \ddot{X})) + \frac{2\mu_E}{R^2} S_E \cdot X + 2m\Lambda = 0 \quad (2.16)$$

(where we have again taken $\zeta = 1$ after variation). Using the first equation (2.15) to eliminate $S_E \cdot X$ from (2.16), this may be rewritten in a form involving only the gravitational field $X$ and its derivatives,

$$H \equiv -\frac{m}{2} \left[ \dot{X}^2 + 2X \cdot \ddot{X} + \frac{1}{\mu_G} (X \cdot (\dddot{X} \wedge \dot{X}) - 4\Lambda) \right] = 0. \quad (2.17)$$

In the next two sections, we construct particular solutions to the coupled equations (2.13), with the integration constants constrained by (2.16).
3 Self-dual solutions

In this section we search for electromagnetically self-dual solutions \[13\] to equations (2.15), i.e. solutions such that,

\[
\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \equiv -\frac{2\mu_E}{R^2} S_E \cdot X = 0.
\]

(3.1)

It follows from equations (2.14) and (3.1) that \( X \) is spacelike, \( X^2 \equiv R^2 = -\sigma^2 \) (\( \sigma \) real), and that the vector \( S_E \wedge X \) is collinear to \( S_E \),

\[
S_E \wedge X = \pm \sigma S_E.
\]

(3.2)

This last equation reduces the second equation (2.15) to

\[
\dot{S}_E = \pm \frac{2\mu_E}{\sigma} S_E,
\]

(3.3)

which implies that the electromagnetic spin vector \( S_E \) has a constant null direction \( \beta \) (it can be checked that \( L \) and \( S_G \) are also aligned along this direction, which is that of the total angular momentum \( J \)). The vector \( X \) can be decomposed along the two orthogonal directions \( \beta \) (lightlike) and \( \alpha \) (spacelike, with the convenient normalization \( \alpha^2 = -1 \)) as

\[
X(\rho) = \alpha \sigma(\rho) + \beta M(\rho).
\]

(3.4)

The \( \alpha \) component of the first equation (2.13) gives \( \ddot{\sigma} = 0 \), which is solved by

\[
\sigma = a\rho + b.
\]

(3.5)

The Hamiltonian constraint (2.13) then fixes the value of the constant \( a \),

\[
a^2 = -4\Lambda,
\]

(3.6)

so that these self-dual solutions exist only for \( \Lambda \leq 0 \).

In the case of a negative cosmological constant \( \Lambda \equiv -l^{-2} \), equation (3.3) integrates to

\[
S_E = \mu_E k^2 \rho^{\pm i\mu E} \beta,
\]

(3.7)

where \( k \) is an integration constant proportional to the electric charge (we have chosen the origin of coordinates so that the constant \( b \) in (3.5) vanishes).
The $\beta$ component of the first equation (2.13) then reduces to the ordinary differential equation

$$2\rho \dot{M} + (3 \mp l\mu_G) \ddot{M} = \pm \frac{k^2 l^3 \mu^2_E \mu_G}{8m} \rho^{\pm l\mu_E - 2}, \quad (3.8)$$

which is solved by

$$M_\pm (\rho) = M_\infty + C \rho^{(1 \pm l\mu_G)/2} + D \rho^{\pm l\mu_E}, \quad (3.9)$$

with

$$D = \frac{k^2 l^2 \mu_E \mu_G}{2m(1 \mp l\mu_E)(1 \mp l(2\mu_E - \mu_G))} \quad (3.10)$$

(a possible linear term in (3.9) can always be taken to zero by a suitable redefinition of the constant vector $\alpha$). The parametrization $[5]$ of the vectors $\alpha$ and $\beta$,

$$\alpha = \frac{1}{2l} (1 - l^2, 1 + l^2, 0), \quad \beta = -\frac{1}{4}(1 + l^2, 1 - l^2, \mp 2l), \quad (3.11)$$

then leads to the solution for the gravitational and electromagnetic fields,

$$ds^2 = \left(\frac{2\rho}{l^2} - \frac{M_\pm (\rho)}{2}\right) dt^2 \pm lM_\pm (\rho) d\theta dt - \left(2\rho + l^2 \frac{M_\pm (\rho)}{2}\right) d\theta^2 - \frac{l^2 d\rho^2}{4\rho^2}, \quad (3.12)$$

$$A_\mu dx^\mu = k \rho^{\pm l\mu_E/2} (dt \mp l d\theta),$$

where $\theta$ varies on the unit circle. It is easily checked that this solution reduces to the self-dual solution of pure TMG $[5]$ if $k = 0$, and to the self-dual solution of TME coupled to Einstein gravity $[13]$ if $C = 0$ and $\mu_G \to \infty$. The metric (3.12) is (as in the case of the self-dual solution to the three-dimensional Einstein equations $[13]$) horizonless, and is asymptotic to the extreme BTZ metric $[11]$ with mass $M_\infty > 0$ and spin $J = \pm lM_\infty$ if

$$\pm l\mu_E < 0 \quad \text{and} \quad \pm l\mu_G < -1, \quad (3.13)$$

which we now assume (these inequalities imply that the two Chern-Simons coupling constants $\mu_E$ and $\mu_G$ have the same sign). The nature of the apparent singularity at $\rho = 0$ may be elucidated by considering the first integral of the geodesic equation $[5]$

$$\left(\frac{d\rho}{d\tau}\right)^2 + P \cdot X - \varepsilon R^2 = 0, \quad (3.14)$$

\footnote{The BTZ radial coordinate $r$ is related to our radial coordinate $\rho$ by $r^2 = 2\rho$.}
where $\tau$ is an affine parameter, $P$ a constant future lightlike vector, and $\varepsilon = +1, 0$ or $-1$. The effective potential in (3.14) is dominated near $\rho = 0$ by the term $\beta \cdot P M(\rho)$ which behaves as $-C \rho^{\pm \mu_G} / 2$ if $\mp \mu_G > 1 \mp 2\mu_E$, and as $-(1/m) \rho^{\pm \mu_E}$ if $\mp \mu_G < 1 \mp 2\mu_E$. It follows that if $C < 0$ in the first case, or $m < 0$ in the second case, all the geodesics are reflected away from the singularity, except for the spacelike geodesics ($\varepsilon = -1$) with $P \cdot X = 0$; the circle $\rho = 0$ is at infinite affine distance on these geodesics, so that the geometry and the electromagnetic field (3.12) are regular.

We now briefly consider the other case $\Lambda = 0$ ($a = 0$). Choosing vectors $\alpha$ and $\beta$ of a form similar to (3.11), with $l$ replaced by $b$, we obtain the solution

$$ds^2 = \left(1 - \frac{M_{\pm}(r)}{2}\right) dt^2 \pm b M_{\pm}(r) dt d\theta - b^2 \left(1 + \frac{M_{\pm}(r)}{2}\right) d\theta^2 - dr^2,$$

$$A_{\mu} dx^\mu = k e^{\pm \mu_E r} (dt \mp b d\theta)$$

(3.15)

where we have put $\rho \equiv br$, and the function $M_{\pm}(r)$ is given by

$$M_{\pm}(r) = Br + C e^{\pm \mu_G r} + D e^{\pm 2\mu_E r},$$

(3.16)

with

$$D = \frac{k^2 \mu_G}{2m(2\mu_E - \mu_G)} e^{\pm 2\mu_E r}.$$ 

(3.17)

The radial coordinate in (3.15) varies from $-\infty$ to $+\infty$. If $\mu_E$ and $\mu_G$ are of the same sign, the metric (3.15) is asymptotically flat at one of the points at infinity, e.g. $r \to +\infty$ for

$$\pm \mu_E < 0 \quad \text{and} \quad \pm \mu_G < 0.$$ 

(3.18)

The asymptotic metric is for $B = 0$ the cylindrical Minkowski spacetime (a conical spacetime with the extremal value $2\pi$ for the deficit angle), and for $B \neq 0$ the other extremal flat spacetime (equation (18) of [14]). The other point at infinity, $r \to -\infty$, is at infinite geodesic distance if $C < 0$ (or $C = 0$, $m > 0$) for $\mp \mu_G > \mp 2\mu_E$, or if $m > 0$ for $\mp \mu_G < \mp 2\mu_E$. The full solution (3.15) is then perfectly regular.

2In the exceptional case $C = 0$ with $\mp l \mu_G > 1 \mp 2l \mu_E$, the geometry is regular for $m > 0$. 

8
4 Diagonal solutions

Diagonal metrics, in the parametrization (2.3), are such that

\[ Y = 0. \] (4.1)

While the simplest solutions of Einstein gravity are diagonal, it is well known [2] that the highly non-linear character of the equations of sourceless TMG precludes the existence of diagonal solutions. However we shall show that, quite remarkably, the (equally non linear) equations of TMGE admit diagonal solutions in the case where the two Chern-Simons coupling constants exactly balance, \( \mu_G + \mu_E = 0 \).

For our present purpose, it is appropriate to write the dynamical equations (2.15) in component notation for the components

\[ U \equiv T + X, \quad V \equiv T - X, \quad \xi \equiv A_0, \quad \eta \equiv A_1. \] (4.2)

For \( Y = 0 \), \( U \) and \( V \) are “light-cone” coordinates, with \( R^2 = UV \). With this parametrization, the second equation (2.15) reads, for \( Y = 0 \),

\[ \dot{\xi} = \mu_E \frac{\eta}{V}, \quad \dot{\eta} = -\mu_E \frac{\xi}{U} \] (4.3)

(Maxwell-Chern-Simons equations), while the first equation (2.15) reduces to

\[ \ddot{U} = \frac{1}{m} \dot{\xi}^2, \quad \ddot{V} = \frac{1}{m} \dot{\eta}^2, \quad \ddot{Y} = -\frac{1}{4\mu_G} \left[ 3(\ddot{U}\dot{V} - \ddot{V}\dot{U}) + 2(U\dddot{V} - U\dddot{U}) \right] + \frac{1}{m} \dot{\xi}\dot{\eta} = 0 \] (4.4)

(Einstein-Chern-Simons equations; the terms in factor of \( 1/m \) are the appropriate components of the Maxwell energy-momentum tensor). The last equation (4.4) may also be rewritten, by using equations (4.3) to eliminate \( U \) and \( V \) and their first derivatives in terms of \( \xi, \eta \) and their derivatives, and the first two equations (4.4) to eliminate \( \dddot{U} \) and \( \dddot{V} \) and their first derivatives in terms of \( \dddot{\xi}, \dddot{\eta} \) and their derivatives, as the second-order differential equation

\[ \dddot{\xi}\dddot{\eta} + \dddot{\xi}\dot{\eta} + 2 \left( 3 + 2 \frac{\mu_G}{\mu_E} \right) \dddot{\xi}\dddot{\eta} = 0. \] (4.5)
We shall also use the first integrals of the equations corresponding to the conservation of the total angular momentum \((2.10)\),

\[
\begin{align*}
-\frac{m}{\mu_G} \left[ U(U\ddot{V} - \dot{U}\dot{V}) - \frac{1}{2} \dot{U}(U\dot{V} - \dot{U}\dot{V}) \right] - \mu_E \xi^2 &= 2J_u, \\
-\frac{m}{\mu_G} \left[ -V(U\ddot{V} - \dot{U}\dot{V}) + \frac{1}{2} \dot{V}(U\dot{V} - \dot{U}\dot{V}) \right] - \mu_E \eta^2 &= 2J_v, \\
-m(U\ddot{V} - \dot{U}\dot{V}) - \mu_E \xi \eta &= 2J_y.
\end{align*}
\] (4.6)

Finally, the integration constants for our system must be constrained by the Hamiltonian constraint \((2.17)\),

\[U\ddot{V} + \dot{U}\dot{V} + 4\Lambda = 0.\] (4.7)

Although the system of equations \((4.3), (4.4)\) is obviously overdetermined, as there are five equations for only four unknown functions, it is very difficult to prove that it does not (or does, for special parameter values) admit solutions. The only case we have been able to treat completely corresponds to the choice of integration constants \(J = 0\) (the right-hand sides of the three equations \((4.6)\) are zero). Then, it follows from the third equation \((4.6)\) that

\[U\ddot{V} - \dot{U}\dot{V} = -\frac{\mu_E}{m} \xi \eta.\] (4.8)

Using this equation as well as equations \((4.3)\), we are able to show that the first two equations \((4.6)\) integrate to

\[U = a \xi^2 \eta^{-2x}, \quad V = b \xi^{2x} \eta^2,\] (4.9)

where we have put \(x \equiv 1 + \mu_G/\mu_E\), and \(a, b\) are integration constants. These equations enable us to write the system \((4.3)\) as

\[\dot{\xi} = \frac{\mu_E}{b} \xi^{-2x} \eta^{-1}, \quad \dot{\eta} = -\frac{\mu_E}{a} \xi^{-1} \eta^{-2x},\] (4.10)

which enables us to reduce the crucial differential equation \((4.3)\) to the algebraic relation

\[x \left(1 + \frac{a}{b} \left(\frac{\xi}{\eta}\right)^{-2x}\right)^2 = 0.\] (4.11)
First assume \( x \neq 0 \) \((\mu_G + \mu_E \neq 0)\). In this case equation (1.11) implies that the functions \( \eta \) and \( \xi \) are proportional, so that \( \text{from equations (4.9) and (4.10)} \) \( \dot{U} \) and \( \dot{V} \) are constant, in contradiction with the first two equations (4.4). So equation (4.11) can be satisfied only if \( x = 0 \), that is if the two Chern-Simons coupling constants are related by

\[ \mu_G + \mu_E = 0. \tag{4.12} \]

Conversely we can show that the only diagonal solutions to the TMGE equations with \( \mu_G + \mu_E = 0 \) are those with \( J = 0 \). The reasoning goes as follows. When the relation (4.12) holds, equation (4.5) reduces to \((\xi \eta)\dddot{=} 0\), which is solved by

\[ \xi = c\rho + d. \tag{4.13} \]

We may then use equations (4.3) to transform the last equation (4.6) into a first-order differential equation, depending on the parameter \( J_y \), for the unknown function \( f \equiv \rho (\xi / \xi - \dot{\eta} / \eta) \). Using this equation, we are able to reduce the Hamiltonian constraint (4.7) to an algebraic relation between \( f(\rho) \) and \( \rho \), which can be satisfied only if \( J_y = 0 \); the first two equations (4.6) then give \( J_u = J_v = 0 \) as well.

We now assume \( x = 0 \). A second differentiation then decouples the system (1.11) to

\[ \frac{\ddot{\xi}}{\xi} - \frac{b}{a} \frac{\dot{\xi}}{\xi} = 0, \quad \frac{\ddot{\eta}}{\eta} - \frac{a}{b} \frac{\dot{\eta}}{\eta} = 0. \tag{4.14} \]

We must now distinguish between two possibilities. If \( b \neq a \) \((c \neq 0 \text{ in } (4.13))\), equations (4.14) integrate to

\[ \xi = \alpha \rho^p, \quad \eta = \beta \rho^{1-p}, \tag{4.15} \]

with \( p = a/(a-b) \). This leads, according to equations (4.3), to

\[ U = -\frac{\alpha \mu_E}{\beta (1-p)} \rho^{2p}, \quad V = \frac{\beta \mu_E}{\alpha p} \rho^{2-2p}. \tag{4.16} \]

The first two equations (4.4) are both satisfied provided the three constants \( \alpha, \beta \) and \( p \) are related by

\[ \alpha \beta = 2m \mu_E \frac{(2p-1)}{p (p-1)}. \tag{4.17} \]
The Hamiltonian constraint (4.7) then determines the possible values of the exponent $p$ to be

$$p_\pm = \frac{1}{2} \left[ 1 \pm \sqrt{\frac{\Lambda + \mu_E^2}{\Lambda - \mu_E^2}} \right]$$

for $\Lambda^2 > \mu_E^2$. Using the relation $p_+ + p_- = 1$, the full solution may be written, for $p = p_-,$

$$ds^2 = 4\varepsilon m \sqrt{\Lambda^2 - \mu_E^2} \left( \frac{\rho^{2p_-}}{\beta^2 p_+} (dx^0)^2 - \frac{\rho^{2p_+}}{\alpha^2 p_-} (dx^1)^2 \right) + \frac{d\rho^2}{2(\Lambda - \mu_E^2)\rho^2},$$

$$A_\mu dx^\mu = \alpha \rho^{p_-} dx^0 + \beta \rho^{p_+} dx^1,$$

where $\varepsilon = \text{sign} \Lambda$, and the two constants $\alpha$ and $\beta$ are constrained by (4.17) (the other choice $p = p_+$ leads to the same solution with the irrelevant exchange of coordinate labels $0 \leftrightarrow 1$).

Let us discuss the geometrical properties of this diagonal solution. If $\Lambda > \mu_E^2$ ($\varepsilon > 0$), the exponents $p_+$ and $p_-$ have opposite signs ($p_+ > 1$, $p_- < 0$), so that the solution (4.19) corresponds to a Riemannian spacetime for $m > 0$, or to an anisotropic cosmology (with $\rho$ as time coordinate, $x^0$ and $x^1$ as space coordinates) for $m < 0$. If $\Lambda < -\mu_E^2$ ($\varepsilon < 0$), $p_+$ and $p_-$ are both positive, so that the metric (4.19) has the Lorentzian signature, with $t = x^1$ for $m > 0$ or $t = x^0$ for $m < 0$. The static rotationally symmetric solution corresponding to the local solution (4.19) is thus

$$ds^2 = 4|m| \sqrt{\Lambda^2 - \mu_E^2} \left( \frac{\rho^{2p_+}}{\alpha^2 p_-} dt^2 - \frac{\rho^{2p_-}}{\alpha^2 p_+} d\theta^2 \right) - \frac{d\rho^2}{2(\mu_E^2 - \Lambda)\rho^2},$$

$$A_\mu dx^\mu = \alpha_+ \rho^{p+} dt + \alpha_- \rho^{p-} d\theta,$$

where we have put $\alpha = \alpha_-$, $\beta = \alpha_+$, and $\pm = \text{sign}(m)$. The study of the geodesic equation (3.14) for this case shows that the metric (4.20) is, for both signs of $m$, singular at the point $\rho = 0$. All geodesics terminate at $\rho = 0$ for $m > 0$, while for $m < 0$ (the preferred sign in TMGE) only radial geodesics terminate at $\rho = 0$. This last property suggests that the $m < 0$ static solution (4.20) is the TMGE analogue of the conical point particle solution of three-dimensional Einstein gravity. Indeed, the extreme black-hole solutions of dilaton gravity with $a > 1$ [15], which exhibit a similar property (only radial geodesics reach the pointlike horizon) have been shown
to also behave as elementary particles in other respects [16] (another “black point” metric is discussed in [17]). We also note that the \( J = 0 \) solution (4.21) reduces in the limit \( -\Lambda \equiv l^2 \rightarrow \mu^2_E, \ m \rightarrow \infty \ (G \rightarrow 0) \) to the \( L = 0 \) BTZ “vacuum” solution [11]

\[
d s^2 = \frac{2\rho}{l^2} \, dt^2 - 2\rho \, d\theta^2 - \frac{l^2 \, d\rho^2}{4\rho^2}.
\] (4.21)

For the other possibility \( b = a \ (c = 0 \) in (4.13), equations (4.14) integrate to

\[
\xi = k \, e^{\alpha \rho}, \quad \eta = h k \, e^{-\alpha \rho}.
\] (4.22)

We then find that equations (4.13)–(4.17) are satisfied for \( \Lambda = \mu^2_E \) if the constants \( h, \ k, \ \alpha \) are related by \( h = 4m\mu_E/k^2\alpha \). Redefining the coordinates \( x^0 \equiv x, \ hx^1 \equiv y, \ \alpha \rho \equiv \mu_E t \), we obtain the \( \Lambda = \mu^2_E \) diagonal solution

\[
d s^2 = \frac{k^2}{4m} \left( e^{2\mu_E t} \, dx^2 + e^{-2\mu_E t} \, dy^2 \right) + dt^2,
\]

\[
A_{\mu} \, dx^\mu = k \left( e^{\mu_E t} \, dx + e^{-\mu_E t} \, dy \right),
\] (4.23)

Corresponding to a Riemannian spacetime for \( m > 0 \), or to an anisotropic area-preserving cosmology for \( m < 0 \).

5 Conclusion

Despite the apparent complexity of the coupled field equations of topologically massive electrodynamics, we have been able to construct two families of exact solutions to the theory with two Killing vectors. The self-dual stationary solutions of Section 3 generalize similar solutions of TMG or of TME coupled to Einstein gravity. These solutions, which exist only for \( \Lambda \leq 0 \), are asymptotic to extreme BTZ metrics if the Chern-Simons coupling constants \( \mu_G \) and \( \mu_E \) have the same sign, with \( \Lambda > -\mu^2_G \), and are regular for suitable ranges of the model parameters. The diagonal solutions which we have obtained in Section 4 in the case where the two Chern-Simons coupling constants exactly balance, \( \mu_G + \mu_E = 0 \), include anisotropic cosmologies if \( \Lambda \geq \mu^2_E \) and \( G < 0 \), and static solutions with a pointlike horizon if \( \Lambda < -\mu^2_E \) and \( G < 0 \).
Although we have not been able to prove it, it seems very likely that our system does not admit static solutions for $\mu_G + \mu_E \neq 0$ (it certainly does not admit any either for $\mu_E = 0$ or for $\mu_G = 0$). So we conjecture that our solutions (4.20) are in fact the unique static rotationally symmetric solutions to TMGE. As in the case of other models [15][17], we expect that these solutions with pointlike horizons are the limit of solutions with regular horizons to the $\mu_G + \mu_E = 0$ theory. Such hypothetical black-hole solutions would necessarily be non-static.

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