Weak solutions to the Navier–Stokes equations

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ABSTRACT: We prove the equivalence of being a Leray-Hopf weak solution of the Navier-Stokes equations in $\mathbb{R}^m$, $m \geq 3$, to satisfying a well known integral equation and use this equation to derive some regularity properties of these weak solutions.

1 Introduction

The purpose of this work is to prove that the property of being a Leray-Hopf weak solution, $u(t)$, of the Navier-Stokes equations in $\mathbb{R}^m$, $m \geq 3$, is equivalent to $u(t)$ satisfying the well-known perturbation equation (see Theorem 3.1)

$$u(t) = e^{t\Delta}u_0 - \mathbb{P}\int_0^t e^{(t-s)\Delta}(u(s) \cdot \nabla u(s))ds; \text{ a.e. } t \geq 0$$

(1.1)

including $t = 0$ and to prove some properties of these solutions (see Section 4). The equivalence is known with some definition (probably weaker than that of Leray and Hopf $[1, 5]$) of weak solution (see $[9]$ and for an earlier proof in $\mathbb{R}^3$ $[8]$). We use the definition of weak solution for which Leray $[1]$ and Hopf $[5]$ proved the existence of global in time solutions given an arbitrary divergence free initial velocity $u_0 \in L^2$. In (1.1) $\mathbb{P}$ is the Leray projector onto divergence free fields and $\Delta$ is the Laplacian. We call (1.1) the perturbation equation because in some sense it treats the Navier-Stokes equations as a perturbation of the heat equation (which in many ways it is not). We make the following definition:

Definition $u(t)$ is a weak solution of the Navier-Stokes equations in $\mathbb{R}^m$ if for each $T > 0$, $u \in L^\infty([0, T]; L^2(\mathbb{R}^m)) \cap L^2([0, T]; H^1(\mathbb{R}^m))$, $\nabla_x \cdot u(t, x) = 0$ (distributional derivatives) and

$$\int_0^t [-(u(s), \partial_x \phi(s)) + (\nabla u(s), \nabla \phi(s)) + (u(s) \cdot \nabla u(s), \phi(s))]ds =$$

$$(u_0, \phi(0)) - (u(t), \phi(t)); \text{ a.e. } t \geq 0, \text{ including } t = 0.$$
We do not incorporate the energy inequality (see [1, 5]) in our definition of weak solution.

Here $(\cdot, \cdot)$ is the usual inner product in $L^2$ where we are abbreviating $(u, v) = \sum_{i=1}^{m}(u_i, v_i)$ and $(\nabla u, \nabla v) = \sum_{i,j}(\partial_x u_j, \partial_x v_j)$ when $u$ and $v$ are vector fields. Sometimes we use the notation $(f, g)$ when possibly neither $f$ nor $g$ is in $L^2$ but the product $fg$ is in $L^1$. The test functions, $\phi$, are in $C_0^\infty([0, T] \times \mathbb{R}^m)$ and satisfy $\nabla \cdot \phi(s, x) = 0$. The function $u_0 \in L^2$ is the initial fluid velocity. Setting $t = 2$ in (1.2) we find $u(0) = u_0$. The space $H^1$ is the $L^2$ Sobolev space of functions $f$ with $||\nabla f||_2^2 + ||f||_2^2 < \infty$. Weak solutions (with the definition above) are known to exist globally, i.e. for all $t \geq 0$, given an arbitrary divergence free initial velocity $u_0 \in L^2(\mathbb{R}^m)$, see [1, 3]. Weak solutions are apriori only defined for almost every $t$, but as is shown in [2] for $m = 3$ we can choose $u(t)$ for each $t \in [0, \infty)$ so that $u$ is weakly continuous in $L^2$. For the reader’s convenience we give a proof for $m \geq 3$ in Section 2.

In Section 2 we extend the set of test functions to \{ $\phi \in S(\mathbb{R}^{m+1})$ : supp$\phi$ $\subset K \times \mathbb{R}^m$, $\nabla \cdot \phi(s, x) = 0$ \}, where $K$ is a compact subset of $(-\infty, T)$ and $S(\mathbb{R}^{m+1})$ is the Schwartz space of $C^\infty(\mathbb{R}^{m+1})$ functions of rapid decrease. Then after another small extension, in Section 3 we derive (1.1) and show that the validity of this equation is equivalent to the Leray-Hopf definition of weak solution given above. In Section 4 we use the perturbation equation to show that $v(t) = u(t) - e^{\Delta u_0}$ is in $L^1(\mathbb{R}^m)$ and has nearly one distributional space derivative in $L^1$ (in a sense to be made precise). In addition $v(t)$ is $L^1$ - Hölder continuous in time of order $\alpha$ for any $\alpha < 1/2$ and $L^1$ - Hölder continuous in space of order $r$ for any $r < 1$. It would be very interesting to know if this regularity can be improved. The implications of these results for $L^p$, $1 < p < 2$ are given.

## 2 Extending the space of test functions

We use Kato’s [3] method of approximating a divergence free field, $f$, by one of compact support. Let $z_R(x) = \tilde{z}(R^{-1} \log(|x|^2 + 1))$ where $\tilde{z} \in C^\infty(\mathbb{R})$ with $\tilde{z}(t) = 1$ for $t \leq 1$ and $\tilde{z}(t) = 0$ for $t \geq 2$. For $k = 0$ and 1, define

$$Q_k(f)(x) = \int_0^1 t^{n-1-k} f(tx) dt,$$

Kato’s approximation of $f$ is

$$f_R(x)_j = z_R(x)f_j(x) + \sum_{i=1}^{m} \partial_i z_R(x)(x \wedge Q_1(f)(x))_{ij}.$$

Here $(x \wedge g(x))_{ij} = x_ig_j(x) - x_jg_i(x)$. A calculation which we omit gives

$$\nabla \cdot f_R = z_R \nabla \cdot f + (x \cdot \nabla z_R)Q_0(\nabla \cdot f).$$

Thus if $f$ is divergence free, $f_R$ has compact support and is divergence free. We estimate $||f_R - f||_p$ and $||\nabla(f_R - f)||_p$. First we have ($p'$ is the dual index to $p$)
\[ \frac{1}{L^2} \leq \int |\xi|^2 |\hat{f}(\xi)|^2 d\xi = \sum_{|\alpha| \leq k} \int |\partial^\alpha f(x)|^2 dx \] for all time we can ensure that \( \chi_r(\xi) = e^{-r/|\xi|^2}, \) \( r > 0, \) defined to be 0 when \( \xi = 0. \) If the vector function \( f \in S(\mathbb{R}^m) \) then the function \( f_r \) with Fourier transform given by \( \chi_r(\xi) \hat{f}(\xi) \) has the property that \( \mathbb{P} f_r \) is also in \( S(\mathbb{R}^m) \) since \( \hat{f}_r \) and all its derivatives are zero at the origin so that no singularity arises from applying \( \mathbb{P}. \) With \( H^k(\mathbb{R}^m) \) the \( L^2 \) Sobolev space with norm \( ||f||_{H^k}^2 = \sum_{|\alpha| \leq k} \int |\partial^\alpha f(x)|^2 dx \) we find

\[
\lim_{r \to 0} ||\mathbb{P}(f - f_r)||_{H^k(\mathbb{R}^m)} = 0.
\]

Before replacing the test functions in \( \ref{test_functions} \) with a larger class we follow the ideas in \( \ref{test_functions} \) to show that by making a particular definition of \( u(t) \) for all time we can ensure that \( u \) is \( L^2 \) weakly continuous. First choosing \( \alpha(s) \in C_0^\infty(\mathbb{R}) \) with \( \alpha(s) = 1 \) for \( s \in [t_1, t_2] \) and letting \( \phi(s, x) = \alpha(s)\phi(x) \) with \( \phi \in C_0^\infty(\mathbb{R}^m) \) with \( \nabla \cdot \phi = 0 \) we subtract \( \ref{test_functions} \) with \( t = t_1 \) from \( \ref{test_functions} \) with \( t = t_2 \) to obtain

\[
\int_{t_1}^{t_2} \left( (\nabla u(s), \nabla \phi) + (u(s) \cdot \nabla u(s), \phi) \right) ds = (u(t_1), \phi) - (u(t_2), \phi)
\]

for a.e. \( t_1 \) and \( t_2 \) but including \( t_1 = 0. \) With \( m' \) the dual index to \( m \)

\[
||u(s) \cdot \nabla u(s)||_{m'} \leq c ||\nabla u(s)||_2^2.
\]

This follows from

\[
||u \cdot \nabla u||_{m'} \leq c' ||u||_{(2^{-1} - m^{-1})^{-1}} ||\nabla u||_2 \leq c ||\nabla u||_2^2,
\]

where the first inequality is just Hölder’s inequality and the last is a Sobolev inequality.

It is convenient to replace \( \int_{t_1}^{t_2} (u(s) \cdot \nabla u(s), \phi) ds \) with \( \int_{t_1}^{t_2} (u(s) \cdot \nabla u(s), \phi) ds \) and then use the estimates above to replace the class of test functions in \( \ref{test_functions} \) by \( \phi \in S(\mathbb{R}^m). \) That is this is
possible follows easily. We now would like to show that we can replace the class $S(\mathbb{R}^m)$ with $B = H^1(\mathbb{R}^m) \cap L^m(\mathbb{R}^m)$, in other words that $S(\mathbb{R}^m)$ is dense in $B$. Let $\chi, \psi \in C_0^\infty(\mathbb{R}^m)$ with $\chi(x) \in [0, 1]$, $\chi(x) = 1$ for $|x| \leq 1$, and $\psi \geq 0$ with $\int \psi dx = 1$. Define $\chi_\delta(x) = \chi(\delta x)$, $\psi_\epsilon(x) = \epsilon^{-m} \psi(x/\epsilon)$. Then if $\phi \in B$, $\phi_{t,\delta} = \chi_\delta(\psi_\epsilon) \in S$ and converges to $\phi$ in both $H^1$ and $L^m$. Then we have

$$u(t) - u_0 = \int_0^t g(s) ds, a.e.$$

(2.3)

where $g(s) \in B^*$ for almost every $s$. We have

$$\langle g(s), \phi \rangle = - (\nabla u(s), \nabla \phi) - (\mathbb{P}(u(s) \cdot \nabla u(s)), \phi)$$

so that $||\langle g(s), \phi \rangle \leq c||\nabla u(s)||_2^2||\phi||_m + ||\nabla u(s)||_2^2||\phi||_{H^1}$. Thus $||g(s)||_{B^*}$ is integrable over $[0,T]$ for any $T$. We choose $u(t)$ so that $u(t) - u_0 \in C([0,T],B^*)$, $T > 0$. We would like to show that for our choice of $u(t)$, $||u(t)||_2 \leq ||u||_{L^2([0,T];L^2)} =: c_0$ for all $t \in [0,T]$. Choose $t_n \to t$ with $||u(t_n)||_2 \leq c_0$. Then there is some subsequence of $\{t_n\}$ for which $u(t_n)$ converges weakly to some $v \in L^2$. We relabel the subsequence so that $u(t_n)$ converges weakly to $v$ in $L^2$. Suppose $f \in S(\mathbb{R}^m)(\subset B)$. Then $(u(t_n), f) \to (v, f)$ and since $u(t) - u_0 \in C([0,T];B^*)$, $(u(t_n), f) \to (u(t), f)$. Thus $u(t) = v$ and since $c_0 \geq \lim sup ||u(t_n)||_2 \geq ||v||_2$ we have $||u(t)||_2 \leq c_0$. Using the convergence of $u(t_n) - u_0$ to $u(t) - u_0$ in $B^*$ for any sequence $t_n \to t$ we have $(u(t_n), f) \to (u(t), f)$ for $f \in S$. Since $||u(t_n) - u(t)||_2 \leq 2c_0$ this follows for all $f \in L^2$. Thus we have for our choice of $u(t)$,

**Lemma 2.1** $u(\cdot)$ is $L^2$ - weakly continuous.

We now show how to replace the test functions in $(1.2)$ with a larger class. We first see how to replace those test functions in $C_0^\infty(0, T) \times \mathbb{R}^m$ and satisfying $\nabla_x \cdot \phi_x(s, x) = 0$ with functions $\phi \in \mathcal{T}_1$. Consider each term in $(1.2)$ where $0 < t < T$: With $\phi \in \mathcal{T}_1$, as $R \to \infty$,

$$| \int_0^t (u(s), \partial_s(\phi_R(s, \cdot)) - \phi(s, \cdot))ds | \leq \int_0^t ||u(s)||_2(||1 - z_R)\partial_s\phi(s)||_2 + c'R^{-1}||\partial_x\phi(s)||_2)ds \to 0.$$

Similarly

$$| \int_0^t (\nabla u(s), \nabla(\phi_R(s, \cdot) - \phi(s, \cdot)))ds |$$

$$\leq \int_0^t ||\nabla u(s)||_2(||(1 - z_R)\nabla \phi(s)||_2 + c''R^{-1}||\phi(s)||_2 + ||\nabla \phi(s)||_2)) \to 0$$

where we use the Schwarz inequality and the fact that $\int_0^t ||\nabla u(s)||_2^2ds < \infty$. For the $u \cdot \nabla u$ term we have

$$| \int_0^t (u(s) \cdot \nabla u(s), \nabla(\phi_R(s, \cdot) - \phi(s, \cdot)))ds | \leq \int_0^t ||u(s) \cdot \nabla u(s)||_{m'} ||\nabla(\phi_R(s, \cdot) - \phi(s, \cdot))||_m ds.$$
As we showed above

\[ ||\nabla (\phi_R(s,\cdot) - \phi(s,\cdot))||_p \to 0 \]

for all \( p \in [2, \infty] \) and this is uniform for \( s \in [0,t] \) so this term \( \to 0 \). The remaining two terms in (1.2) also converge to what they are supposed to and thus we can use \( \phi \in \mathcal{T}_1 \) in (1.2). Basically the same ideas allows us to replace test functions in \( \mathcal{T}_1 \) with test functions in \( \mathcal{T} \). We need to show that the terms in (1.2) with \( \phi(s) \) replaced with \( \mathbb{P}(\phi_r(s,\cdot) - \phi(s,\cdot)) \) tend to 0. Here we use that \( \mathbb{P} \) is bounded on \( L^p(\mathbb{R}^m) \) for \( 1 < p < \infty \). We omit the details. Using the \( L^2 \) - weak continuity of \( u(t) \) it is easy to see that (1.2) is true for all \( t \).

3 The perturbation equation

In this section we prove the following theorem.

**Theorem 3.1** Suppose \( u \) is a weak solution (see (1.2) of the Navier-Stokes equations so that \( u \in L^\infty([0,T];L^2(\mathbb{R}^m)) \cap L^2([0,T];H^1(\mathbb{R}^m)), \nabla \cdot u(t,x) = 0 \) for all \( T > 0 \) and with \( u(0) = u_0 \), \( 0 \leq t \leq T \),

\[
\int_0^t \left[ -(u(s), \partial_s \phi(s)) + (\nabla u(s), \nabla \phi(s)) + (u(s) \cdot \nabla u(s), \phi(s)) \right] ds = (u_0, \phi(0)) - (u(t), \phi(t)), \text{ a.e. } t > 0
\]

for all \( \phi \in C_0^\infty([0,T] \times \mathbb{R}^m) \) satisfying \( \nabla \cdot \phi(s,x) = 0 \). Then \( u \) satisfies the perturbation equation

\[
u(t) = e^{t\Delta} u_0 - \mathbb{P} \int_0^t e^{(t-s)\Delta} (u(s) \cdot \nabla u(s)) ds \tag{3.2}
\]

for a.e. \( t \geq 0 \), including \( t = 0 \). Conversely if for all \( T > 0 \), \( u \in L^\infty([0,T];L^2(\mathbb{R}^m)) \cap L^2([0,T];H^1(\mathbb{R}^m)), \nabla \cdot u(t,x) = 0 \) and the perturbation equation (3.2) holds for a.e. \( t \geq 0 \) including \( t = 0 \), then \( u \) is a weak solution of the Navier-Stokes equations.

**Proof** Fix \( t \) with \( 0 < t < T, \epsilon \in (0,t) \), and \( \alpha_\epsilon \in C_0^\infty((-\infty,T)) \) with \( \alpha_\epsilon(s) = 1 \) for \( s \in [0,t-\epsilon] \) and \( \alpha_\epsilon(s) = 0 \) for \( s \geq t \). In addition we take \( 0 \geq \alpha'_\epsilon(s) \) for \( s \in (t-\epsilon,t) \). We define a test function \( \phi_\epsilon(s,\cdot) = \mathbb{P}e^{(t-s)\Delta} \alpha_\epsilon(s)f(\cdot) \) where \( f \in \mathcal{S}(\mathbb{R}^m) \). Notice that \( \phi_\epsilon \in \mathcal{T} \). We consider

\[
\int_0^t (u(s), \partial_s \phi_\epsilon(s)) ds = \int_{t-\epsilon}^t (u(s), e^{(t-s)\Delta} f) \alpha'_\epsilon(s) ds - \int_0^t (u(s), e^{(t-s)\Delta} f) \alpha_\epsilon(s) ds.
\]

From (1.2) it follows that

\[
- \int_{t-\epsilon}^t (u(s), e^{(t-s)\Delta} f) \alpha'_\epsilon(s) ds + \int_0^t (u(s) \cdot \nabla u(s), \mathbb{P}e^{(t-s)\Delta} f) \alpha_\epsilon(s) ds = (u_0, e^{t\Delta} f).
\]

The first term does not change if we choose \( u(s) \) for all \( s \) such that \( u(s) \) is weakly continuous (see Lemma 2.1). Then since \( u(s) \) converges weakly to \( u(t) \) as \( s \to t \) and \( e^{(t-s)\Delta} f \) converges strongly to \( f \) as \( s \uparrow t \), \( (u(s), e^{(t-s)\Delta} f) \) converges to \( (u(t), f) \) as \( s \uparrow t \). Since \( \int_{t-\epsilon}^t \alpha'_\epsilon(s) ds = 0 \),

\[
\int_0^t (u(s), e^{(t-s)\Delta} f) \alpha_\epsilon(s) ds = (u_0, e^{t\Delta} f),
\]

\[
\int_0^t (u(s), e^{(t-s)\Delta} f) \alpha_\epsilon(s) ds = (u_0, e^{t\Delta} f).
\]
Integrating on the interval \([0, t]\),\( \alpha'_\epsilon(s)ds = 1\), the first term above converges to \((u(t), f)\) as \(\epsilon \to 0\). In the second term we use the Lebesgue dominated convergence theorem, \(|\alpha_\epsilon(s)| \leq 1\) in \([0, t]\) and the fact that the integrand is bounded by \(c||u(s) \cdot \nabla u(s)||_m||P e^{(t-s)\Delta} f||_m \leq c ||\nabla u(s)||_2^2||f||_m\) which is independent of \(\epsilon\) and integrable. Thus \(\alpha_\epsilon\) can be replaced by 1 in the limit.

It follows that for the weakly continuous version of \(u\), for every \(t > 0\),

\[
(u(t), f) + \left( \int_0^t (Pe^{(t-s)\Delta} u(s) \cdot \nabla u(s), f) \right) = (e^{t\Delta} u_0, f)
\]

for all \(f \in \mathcal{S}({\mathbb{R}}^m)\) and thus we have (1.1) for all \(t \geq 0\) for the weakly continuous version and for almost every \(t \geq 0\) for any version.

Conversely suppose \(u \in L^\infty([0, T]; L^2({\mathbb{R}}^m)) \cap L^2([0, T]; H^1({\mathbb{R}}^m)), \nabla_x \cdot u(t, x) = 0\) and (1.1) holds for almost all \(t > 0\) and for \(t = 0\). Choose a test function \(\phi \in C_0^\infty((-\infty, T) \times {\mathbb{R}}^m)\) with \(\nabla_x \cdot \phi(t, x) = 0\) and for \(0 < t < T\) consider

\[
(u(t), \partial_t \phi(t)) = (e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} (u(s) \cdot \nabla u(s))ds, \partial_t \phi(t)) =
\]

\[
(u_0, -e^{t\Delta} \Delta \phi(t) + \partial_t (e^{t\Delta} \phi(t))) + \int_0^t (u(s) \cdot \nabla u(s), e^{(t-s)\Delta} \Delta \phi(t))ds - \int_0^t (u(s) \cdot \nabla u(s), \partial_t (e^{(t-s)\Delta} \phi(t)))ds =
\]

\[
-(u(t), \Delta \phi(t)) + (u_0, \partial_t (e^{t\Delta} \phi(t))) - \int_0^t (u(s) \cdot \nabla u(s), \partial_t (e^{(t-s)\Delta} \phi(t)))ds.
\]

Integrating on the interval \([0, t]\) we have

\[
\int_0^t (u(s), \partial_s \phi(s))ds =
\]

\[
\int_0^t (\nabla u(s), \nabla \phi(s))ds + \int_0^t \partial_s (u_0, e^{s\Delta} \phi(s))ds - \int_0^t \int_0^s (u(\tau) \cdot \nabla u(\tau), \partial_s (e^{(s-\tau)\Delta} \phi(\tau)))d\tau ds =
\]

\[
\int_0^t (\nabla u(s), \nabla \phi(s))ds + (u_0, e^{t\Delta} \phi(t)) - \int_0^t (e^{(t-s)\Delta} (u(\tau) \cdot \nabla u(\tau)), \phi(t))d\tau - (u_0, \phi(0))
\]

\[
+ \int_0^t (u(\tau) \cdot \nabla u(\tau), \phi(\tau))d\tau =
\]

\[
\int_0^t (\nabla u(s), \nabla \phi(s))ds + (u(t), \phi(t)) - (u_0, \phi(0)) + \int_0^t (u(s) \cdot \nabla u(s), \phi(s))ds.
\]

From (1.1), the last equality holds a.e. This is (1.2).

\[\square\]
4 Consequences of the perturbation equation: \(L^1\) regularity

In this section we show that if \(u(t)\) is a weak solution to the Navier-Stokes equations, then \(v(t) = u(t) - e^{t\Delta}u_0\) is in \(L^1(\mathbb{R}^m)\) with some mild smoothness properties in the \(x\) variable; more explicitly in some sense almost one distributional derivative in \(L^1\). In addition, in the \(L^1\) norm, we prove Hölder continuity in the \(t\) variable of degree \(\alpha\) for any \(\alpha < 1/2\). We consider the consequences of these results for \(L^p, 1 < p < 2\).

We will work with the spaces \(L^p_r(\mathbb{R}^m) \subset L^p(\mathbb{R}^m)\) defined as \((1 - \Delta)^{-r/2} L^p(\mathbb{R}^m)\) with norm \(\|f\|_{L^p_r} = \|(I - \Delta)^{r/2} f\|_p\) (see \([6]\) and \([7]\) for a discussion of these spaces). The next Proposition shows that \(v(t) \in L^r(\mathbb{R}^m)\) for \(0 \leq r < 1\). And see an extension in Corollary 4.3.

**Theorem 4.1** Suppose \(u(t)\) is a weak solution to the Navier-Stokes equations in dimension \(m \geq 3\). Then \(v(t) = u(t) - e^{t\Delta}u_0 \in C([0, \infty); L^1_r(\mathbb{R}^m))\) for \(0 \leq r < 1\) with norms bounded uniformly for \(t\) and \(r\) in compact subsets of \([0, \infty)\) and \([0, 1)\) respectively.

**Proof** We have \((I - \Delta)^{r/2} v(t) = -\mathbb{P} \int^t_0 (I - \Delta)^{r/2} e^{(t-s)\Delta} u(s) \cdot \nabla u(s) ds\). The fact that \(\mathbb{P}\) is not a bounded operator on \(L^1\) complicates the proof. Since \(\mathbb{P} = I - (I - \mathbb{P})\) we begin by estimating \((I - \mathbb{P})(I - \Delta)^{r/2} K_t(x)\) where \(K_t\) is the integral kernel for the operator \(e^{t\Delta}\) in \(\mathbb{R}^m\), \(K_t(x-y) = (4\pi t)^{-m/2} e^{-|x-y|^2/4t}\). We have \(\mathcal{F}((I - \mathbb{P})f)(\xi) = \sum_j \xi^j |\xi_j|^{-2} \mathcal{F} f_j(\xi)\) where \(\mathcal{F}\) is the Fourier transform. Thus working in \(x\)-space with \(y = x/\sqrt{t}\) we have

\[
\partial_{x_i} \partial_{x_j} (I - \Delta)^{r/2} K_t(x) = -(2\pi)^{-m} t^{-m/2} \left( t \sum_j \frac{\partial^2}{\partial x_j \partial x_{j'}} (I - \Delta)^{r/2} K_t(x) \right) \cdot \eta_j \cdot \partial_j \int (t + |\eta|^2)^{r/2} |\eta|^{-2} e^{-|\eta|^2} \frac{i^{\eta_j}}{\sqrt{t}} d\eta.
\]

A bit more calculation gives

\[
(I - \mathbb{P})_{i,j} (I - \Delta)^{r/2} K_t(x) = ct^{-(m+r)/2} (\omega_i \omega_j g_{r,t}(|y|)) + |y|^{-1} \left( \delta_{i,j} - \omega_i \omega_j \right) g_{r,t}(|y|),
\]

where \(\omega_i = y_i/|y|\) and

\[
g_{r,t}(w) = \int_0^\infty \int_{-1}^1 (t + s^2)^{r/2} s^{m-2} e^{-s^2} e^{isw\lambda} (1 - \lambda^2)^{(m-3)/2} d\lambda ds.
\]

Here \(c\) is an \(m\) dependent constant.

It is shown in the Appendix that the \(C^\infty\) function \(g_{r,t}\) and its derivatives obey the following estimates

\[
|g^{(n)}_{r,t}(w)| \leq C_n (1 + |w|)^{-(m+n-1)},
\]

for all \(r \geq 0\), uniformly for \(t\) in compact sets of \([0, \infty)\).

We need to estimate the following \(L^1\) norm:

\[
\int_0^t \int_0^t \int_0^t |\partial_{x_i} \partial_{x_j} (I - \Delta)^{r/2} K_s(x-z) f_j(t-s, z) dz dz ds| dx
\]
where \( f_j(s, z) = u(s, z) \cdot \nabla u_j(s, z) \). For a reason that will be apparent only at the end of the proof we decouple the \( j \) indices and replace \( f_j \) by \( f_a \) where now \( i, j, a \) can be all different.

We first look at the term involving the derivative \( g_{r,t} \) in (4.1):

\[
\int \int_{0}^{t} s^{-(m+r)/2} \int \omega_j \omega_j g_{r,s}'(|y|) f_a(t - s, z) dz ds |dx|
\]

where \( y = (x - z)/\sqrt{s} \) and \( \omega_i = y_i/|y| \). We write

\[
f_a(s, z) = \sum_k (\partial/\partial z_k)(u_k(s, z)u_a(s, z))
\]

and integrate by parts in the \( z \) integral. We use \((\partial/\partial z_k)|y| = O(s^{-1/2}), (\partial/\partial z_k)\omega_i = O(|y|^{-1}s^{-1/2})\) to get

\[
|(\partial/\partial z_k)(\omega_i \omega_j g_{r,t}'(|y|))| \leq cs^{-1/2}|y|^{-1}(1 + |y|)^{-m}
\]

which gives

\[
\int_{0}^{t} \int_{s}^{r} s^{-(m+r)/2} \int |(\partial/\partial z_k)[\omega_i \omega_j g_{r,s}'(|y|)]|dx)|u_k(t - s, z)u_a(t - s, z)|dz ds \leq (4.3)
\]

For the term involving \(|y|^{-1} g_{r,s} \) in (4.1) we again integrate by parts and use

\[
|(\partial/\partial z_k)(|y|^{-1} g_{r,s}(|y|))| \leq cs^{-1/2}(|y|^{-2}g_{r,s}(|y|) + |y|^{-1}g_{r,s}'(|y|)) \leq C s^{-1/2}|y|^{-2}(1 + |y|)^{-m+1}.
\]

This term is then estimated in the same way as the previous term involving \( g_{r,t} \), which results in

\[
\| (I - \mathbb{P}) \int_{0}^{t} (I - \Delta)^{r/2} e^{(t-s)\Delta} u(s) \cdot \nabla u(s) ds \|_{L^1} \leq C_{r,t}
\]

where \( C_{r,t} \) is bounded uniformly for \( r \) and \( t \) in compacts of \([0, 1]\) and \([0, \infty)\) respectively. To show that

\[
\| \mathbb{P} \int_{0}^{t} (I - \Delta)^{r/2} e^{(t-s)\Delta} u(s) \cdot \nabla u(s) ds \|_{L^1}
\]

is bounded we note \( \mathbb{P} = I - (I - \mathbb{P}) \) so we just need to deal with the identity term. But \( \sum_j \partial x_j \partial x_j \Delta^{-1} = I \) so that the bound proved for

\[
(I - \mathbb{P})_{i,j} \int_{0}^{t} (I - \Delta)^{r/2} e^{(t-s)\Delta} u(s) \cdot \nabla u_a(s) ds
\]

immediately gives the desired estimate.
It remains to prove the continuity in $t$. With $\delta > 0$ we have

$$\|\mathbb{P} \int_0^{t+\delta} (I - \Delta)^{r/2} e^{(t+\delta-s)\Delta} u(s) \cdot \nabla u(s) ds - \mathbb{P} \int_0^t (I - \Delta)^{r/2} e^{(t-s)\Delta} u(s) \cdot \nabla u(s) ds\|_1 \leq \|\mathbb{P} \int_t^{t+\delta} (I - \Delta)^{r/2} e^{(t+\delta-s)\Delta} u(s) \cdot \nabla u(s) ds\|_1 + \|\mathbb{P} \int_t \int_0^{t+\delta} (I - \Delta)^{r/2} e^{(t+s)\Delta} u(s) \cdot \nabla u(s) ds\|_1.$$  

The first term is just

$$\|/(e^{\delta\Delta} - I)(I - \Delta)^{r/2} v(t)\|_1$$

which tends to zero as $\delta \downarrow 0$ since $\{e^{t\Delta}\}$ is a $C_0$ semi-group on $L^1$. For the second term we refer to (4.3). The estimate we need is the last term where $t$ is replaced by $t + \delta$ and 0 is replaced by $t$. We obtain

$$\|\mathbb{P} \int_t^{t+\delta} (I - \Delta)^{r/2} e^{(t+\delta-s)\Delta} u(s) \cdot \nabla u(s) ds\|_1 \leq c \int_t^{t+\delta} (t + \delta - s)^{-1+1/2} ds\|u\|^2_{L^\infty([0,T],L^2)} =$$

$$= 2c(1 - r)^{-1} \delta^{1-r/2}\|u\|^2_{L^\infty([0,T],L^2)}$$

where $T$ can be taken any number larger than say $t + 1$ (where $\delta < 1$).

When we consider $v(t - \delta) - v(t)$ with $\delta > 0$ the proof needs the following lemma (which we will also make use of in the corollary which follows):

**Lemma 4.2** Suppose $\epsilon \in (0,1]$. Then

$$\|/(I - e^{h\Delta})(I - \Delta)^{-\epsilon}\|_{L^1 \rightarrow L^1} \leq Ch^\epsilon$$

**Proof** We write

$$(I - e^{h\Delta})(I - \Delta)^{-\epsilon} = \Gamma(\epsilon)^{-1}(I - e^{h\Delta}) \int_0^\infty t^{-1} e^{-t} e^{t\Delta} dt =$$

$$\Gamma(\epsilon)^{-1}\left(\int_0^\infty t^{-1} e^{-t} e^{t\Delta} dt - \int_0^\infty t^{-1} e^{-t} e^{(t+h)\Delta} dt\right) =$$

$$\Gamma(\epsilon)^{-1}\left(\int_h^\infty (t^{-1} e^{-t} - (t-h)^{-1} e^{-(t-h)}) e^{t\Delta} dt + \Gamma(\epsilon)^{-1}\left(\int_0^h t^{-1} e^{-t} e^{t\Delta} dt\right)\right).$$

Since $\|e^{t\Delta}\|_{L^1 \rightarrow L^1} = 1$,

$$\|/(I - e^{h\Delta})(I - \Delta)^{-\epsilon}\|_{L^1 \rightarrow L^1} \leq \Gamma(\epsilon)^{-1}\left(\int_0^h t^{-1} e^{-t} dt + \int_0^\infty ((t-h)^{-1} e^{-(t-h)} - t^{-1} e^{-t}) dt\right).$$

Here we have used the positivity of the last integrand. Thus we have

$$\|/(I - e^{h\Delta})(I - \Delta)^{-\epsilon}\|_{L^1 \rightarrow L^1} \leq \Gamma(\epsilon)^{-1} 2h^\epsilon (\epsilon \Gamma(\epsilon))^{-1} = 2h^\epsilon \Gamma(1 + \epsilon)^{-1}. $$
Proof (continued)

\[ ||P\int_0^{t-\delta} (I-\Delta)^{r/2} e^{(t-\delta-s)\Delta} u(s) \cdot \nabla u(s) ds - P\int_0^t (I-\Delta)^{r/2} e^{(t-s)\Delta} u(s) \cdot \nabla u(s) ds||_1 \leq \]

\[ ||P\int_0^{t-\delta} (I-\Delta)^{r/2} (e^{(t-s)\Delta} - e^{(t-\delta-s)\Delta}) u(s) \cdot \nabla u(s) ds||_1 + ||P\int_0^t (I-\Delta)^{r/2} e^{(t-s)\Delta} u(s) \cdot \nabla u(s) ds||_1. \]

The first term above is

\[ ||(I-e^{\delta \Delta})(I-\Delta)^{-\epsilon/2}P\int_0^{t-\delta} (I-\Delta)^{(r+\epsilon)/2} e^{(t-\delta-s)\Delta} u(s) \cdot \nabla u(s) ds||_1. \]

Since \( r < 1 \) we can choose \( \epsilon > 0 \) so that \( r+\epsilon < 1 \). Then using the lemma and the boundedness of the \( L^1 \) norm of

\[ P\int_0^{t-\delta} (I-\Delta)^{(r+\epsilon)/2} e^{(t-\delta-s)\Delta} u(s) \cdot \nabla u(s) ds \]

which we have shown above for \( r+\epsilon < 1 \) (uniformly for \( 0 \leq t-\delta \leq T \), \( T \) fixed but arbitrary) we have

\[ ||P\int_0^{t-\delta} (I-\Delta)^{r/2} (e^{(t-s)\Delta} - e^{(t-\delta-s)\Delta}) u(s) \cdot \nabla u(s) ds||_1 \leq C\delta^{r/2}. \]

The remaining term is handled the same way as with \( t + \delta \) so that we have

\[ ||P\int_{t-\delta}^t (I-\Delta)^{r/2} e^{(t-s)\Delta} u(s) \cdot \nabla u(s) ds||_1 \leq C\int_{t-\delta}^t s^{-(r+1)/2} ds ||u||^2_{L^\infty([0,T],L^2)} \leq 2(1-r)^{-1}\delta^{(1-r)/2} ||u||^2_{L^\infty([0,T],L^2)}. \]

The following corollary is actually a corollary of the proof of Theorem 4.1:

**Corollary 4.3** \( v(\cdot) \) is Hölder continuous in the norm of \( L^1_r(R^m) \) for \( 0 < r < 1 \).

\[ ||v(t+h) - v(t)||_{L^1_r} \leq c_\alpha |h|^\alpha \quad (4.4) \]

if \( 0 < \alpha < (1-r)/2, t, t+h \geq 0, |h| \leq 1. c_\alpha \) is bounded uniformly in \( t \) for \( t \) in any compact interval of \([0, \infty)\).

**Proof** See the last two inequalities above which apply in the cases \( h = \delta \) and \( h = -\delta, \delta > 0. \)
Theorem: If for $r > 0$ we define $h_{r,t}$ by the equation
\[
(I - \Delta)^{r/2}K_t(x) = (2\pi)^{-m-1} t^{-m/2} e^{i\eta x} dy = \int (t + |\eta|^2)^{r/2} e^{-|\eta|^2} e^{iy\eta} dy = Ct^{-(m+r)/2} h_{r,t}(|y|),
\]
with $C$ defined so that
\[
h_{r,t}(w) := \int_0^\infty \int_{\eta}^1 e^{is\lambda} (t + s^2)^{r/2} s^{m-1} e^{-s^2} (1 - \lambda^2)^{(m-3)/2} d\lambda ds,
\]
it can be shown that
\[
|h_{r,\epsilon}(w)| \leq C(1 + |w|)^{-(m+\epsilon)}
\]
for $\epsilon \in [0, r)$. Here $C$ is uniformly bounded for $t$ in compacts of $[0, \infty)$. Thus for $r > 0$ the integration by parts in the above proof is not necessary to show $|| (I - \Delta)^{r/2} \int_0^t e^{(t-s)\Delta} u(s) \cdot \nabla u(s) ds ||_1 < \infty$. In fact we have
\[
|| (I - \Delta)^{r/2} K_t ||_1 \leq c t^{r/2} \int \nabla h_{r,t}(|y|) dy \leq c't^{r/2}.
\]

Thus
\[
|| (I - \Delta)^{r/2} \int_0^t e^{(t-s)\Delta} u(s) \cdot \nabla u(s) ds ||_1 \leq c' \int_0^t (t-s)^{-r/2} || u(s) ||_2 || \nabla u(s) ||_2 ds \leq c' || u ||_{L^\infty([0, T]; L^2)} (T^{1-r}/(1-r))^{1/2} \left( \int_0^T || \nabla u(s) ||^2 ds \right)^{1/2}
\]
where $T \geq t$.

The following proposition is a simple consequence of Theorem 4.1, Corollary 4.3 and interpolation:

**Proposition 4.4** $v(t) \in L^p_s(\mathbb{R}^m)$ and $v(\cdot)$ is Hölder continuous in the $L^p_s(\mathbb{R}^m)$ norm whenever $1 \leq p < m/(m - 1 + s)$. Explicitly given $s \geq 0$ and $p$ so that $1 \leq p < m/(m - 1 + s)$ then
\[
|| v(t + h) - v(t) ||_{L^p} \leq C \epsilon |h|^\epsilon
\]
for $0 < \epsilon < (m/2)(p^{-1} - p_0^{-1})$, $p_0 = m/(m - 1 + s)$. If $1 \leq p < 2$, $v(t) \in L^p$ and $v(\cdot)$ is Hölder continuous in the $L^p$ norm:
\[
|| v(t + h) - v(t) ||_p \leq C' \epsilon |h|^\epsilon
\]
for $0 < \epsilon < p^{-1} - 2^{-1}$. Note that if $s = 0$, $p^{-1} - 2^{-1} - (m/2)(p^{-1} - p_0^{-1}) = ((m/2) - 1)p^{-1} \geq 0$. 

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4 CONSEQUENCES OF THE PERTURBATION EQUATION: $L^1$ REGULARITY

**Proof** From the representation $(1 - \Delta)^{-w/2} f(x) = \int k_w(x - y) f(y) dy$ with

$$k_w(x) = \Gamma(w/2)^{-1} \int_0^\infty e^{-t(w-2)/2} K_t(x) dt$$

for $w > 0$, it is easy to derive the bound $k_w(x) \leq C_w e^{-|x|^2/2} |x|^{-(m-w)}$ for $0 < w < 1$. Given $p$ and $s \geq 0$ with $1 \leq p < p_0$, we choose $0 < 1 - r < m(p-1 - p_0^{-1})$ or what is the same $p < m/(m - r + s)$. We have

$$||(I - \Delta)^{s/2}(v(t + h) - v(t))||_p = ||(I - \Delta)^{(r-s)/2}(I - \Delta)^{r/2}(v(t + h) - v(t))||_p.$$  

Using Minkowski’s inequality we have

$$||(I - \Delta)^{s/2}(v(t + h) - v(t))||_p \leq ||k_{r-s}||_p||(I - \Delta)^{r/2}(v(t + h) - v(t))||_1.$$  

Clearly $||k_{r-s}||_p < \infty$ if $p(m - (r-s)) < m$ which combined with Corollary 4.3 gives the first result. The fact that $v(t) \in L^p_t$ follows from a similar interpolation.

To prove the second inequality, for $1 \leq p < 2$ we interpolate to obtain with $\theta = 2(p^{-1} - 2^{-1})$

$$||v(t+h) - v(t)||_p \leq ||v(t+h) - v(t)||_1^{1-\theta}||v(t+h) - v(t)||_1^{\theta} \leq ||v(t+h) - v(t)||_1^{\theta}||v(t+h) - v(t)||_1^{1-\theta}$$

for $T$ large enough. The desired Hölder continuity follows from Corollary 4.3. Basically the same interpolation gives $v(t) \in L^p$.

\[\blacksquare\]

The somewhat abstract condition of belonging to the space $L^p_T(\mathbb{R}^m)$ has a more down to earth consequence. We prove the following Hölder condition: For $|h| \leq 1$

**Proposition 4.5**

$$||v(t)(\cdot + h) - v(t)(\cdot)||_p \leq c(r)|h|^r||v(t)||_{L^p_T}.$$  

where $1 \leq p < m/(m - 1 + r)$ and $0 \leq r < 1$.

**Proof** This is a result which has nothing to do with the Navier-Stokes equation and is probably well-known. We give a simple proof a result which holds for $1 \leq p < \infty$ and $0 < r$. Define $(U(h)w)(x) = w(x + h)$. We will estimate the norm of $(U(h) - \bar{I})(-\Delta + \bar{I})^{-r/2}$ as a self-map of $L^p(\mathbb{R}^m)$. We consider the integral kernel of this operator

$$(U(h) - \bar{I})(-\Delta + \bar{I})^{-r/2}(x, y) = \Gamma(r/2)^{-1} \int_0^\infty e^{-t^{r/2}} K_t(x + h - y) - K_t(x - y) dt / t.$$  

We will use Minkowski’s inequality in integral form so we calculate with $y = \frac{x}{2\sqrt{t}}$ and $h = (\lambda, 0, ..., 0)$ with $\lambda \geq 0$

$$\int |K_t(x+h) - K_t(x)| dx = (4\pi)^{-m/2} \int \int e^{-|x+h|^2/4t} - e^{-|x|^2/4t} dx = \pi^{-m/2} \int e^{-|y| + \frac{h}{\lambda^2}} - e^{-|y|^2} |dy| =$$

\[\int \int e^{-|x+h|^2/4t} - e^{-|x|^2/4t} |dx| = \pi^{-m/2} \int e^{-|y| + \frac{h}{\lambda^2}} - e^{-|y|^2} |dy| = \pi^{-m/2} \int e^{-|y|^2} |dy| = \pi^{-m/2} \pi^{m/2} = \pi^0 = 1.$$
\[ \pi^{-1/2} \int_{-\infty}^{\infty} |e^{-(s+w/2)^2} - e^{-(s-w/2)^2}|ds. \]

where \( w = \frac{\lambda}{2\sqrt{t}} \). Thus

\[ \int |K_t(x+h) - K_t(x)|dx = 2\pi^{-1/2} \int_0^{\infty} \int_0^1 \left( d/dt \right) [e^{-(s-tw/2)^2} - e^{-(s+tw/2)^2}]dt ds = \]
\[ w\pi^{-1/2} \int_0^1 \int_0^{\infty} [(s-tw/2)e^{-(s-tw/2)^2} + (s+tw/2)e^{-(s+tw/2)^2}]ds dt. \]

We have \( \int_0^{\infty} (s-tw/2)e^{-(s-tw/2)^2}ds = \int_{-\infty}^{\infty} u e^{-u^2}du \leq \int_0^{\infty} u e^{-u^2}du = 1/2 \), while similarly \( \int_0^{\infty} (s+tw/2)e^{-(s+tw/2)^2}ds \leq 1/2 \). Since we also have \( \int |K_t(x+h) - K_t(x)|dx \leq 2 \) we obtain
\[ \int |K_t(x+h) - K_t(x)|dx \leq cw(1+w)^{-1} \]

Finally
\[ \|(U(h) - I)(-\Delta + I)^{-r/2}\|_{L^p \rightarrow L^p} \leq c \int_0^{\infty} e^{-t} t^{r/2} \frac{2\sqrt{t}}{1 + \frac{\lambda}{2\sqrt{t}}} dt \]

This integral is easy to estimate. We find
\[ \|(U(h) - I)(-\Delta + I)^{-r/2}\|_{L^p \rightarrow L^p} \leq c(r) |h|^r; r < 1 \]
\[ \leq c|h| \log |h|^{-1}; r = 1 \]
\[ \leq c(r) |h|; r > 1 \]

5 Acknowledgement

Thanks go to my colleague Zoran Grujic for pointing out that a version of Theorem 3.1 is proved in [9] and in [8].

6 Appendix

In this appendix we derive bounds on \( (I - P)_{i,j}(I - \Delta)^{r/2}K_t(x) \) where \( K_t \) is the integral kernel for the operator \( e^{t\Delta} \) in \( \mathbb{R}^m \). Explicitly, \( K_t(x-y) = (4\pi t)^{-m/2}e^{-|x-y|^2/4t} \). We have
\[ (I - P)_{i,j}(I - \Delta)^{r/2}K_t(x) = ct^{-(m+r)/2}(\omega_i \omega_j g_{r,t}(|y|)) + |y|^{-1}(\delta_{i,j} - \omega_i \omega_j)g_{r,t}(|y|), \]
where \( y = x/\sqrt{t} \), \( \omega_i = y_i/|y| \) and
\[ g_{r,t}(w) = \int_0^{\infty} \int_{-1}^{1} \left( t + s^2 \right)^{r/2} s^{m-2} e^{-s^2} e^{isw\lambda}(1 - \lambda^2)^{(m-3)/2}d\lambda ds. \] (6.1)

Here \( c \) is an \( m \) dependent constant.
Proposition 6.1 \( g_{r,t} \in C^\infty(\mathbb{R}) \) and
\[
|g_{r,t}^{(n)}(w)| \leq C_n(1 + |w|)^{-(m+n-1)} \tag{6.2}
\]
for any \( r \geq 0 \) uniformly for \( t \in [0, T] \), for any \( T > 0 \).

**Proof** We easily derive \((d/dw)^n e^{isw\lambda} = w^{-n} \sum_{j=1}^{n} n_j (sD)^j e^{isw\lambda}\) where the \( n_j \) are integers and \( D = d/ds \). Integrating by parts in the \( s \) integral we obtain
\[
g_{r,t}^{(n)}(w) = w^{-n} \int_{-1}^{1} \int_{0}^{\infty} e^{isw\lambda} h(s) \lambda (1 - \lambda^2)^{(m-3)/2} ds d\lambda
\]
where
\[
h(s) = \sum_{j=1}^{n} n_j (-Ds)^j [(t + s^2)^{r/2}s^{m-2}e^{-s^2}]
\]
and \( D \) is \( d/ds \). In the following we use \(|D^{l}(t + s^2)^{r/2}| \leq C_l(t + s^2)^{(r-l)/2} \) with \( C_l = C_l(t) \) bounded for \( t \) in compacts of \([0, \infty)\).

We integrate by parts \( 2l \) times using \([-w^2\lambda s)^{-1} (\partial^2 / \partial s \partial \lambda) e^{isw\lambda} = e^{isw\lambda} \) and \([- (d/d\lambda) \lambda^{-1}] (1 - \lambda^2)^{k/2} = k\lambda(1 - \lambda^2)^{(k-2)/2} \). We obtain
\[
g_{r,t}^{(n)}(w) = w^{-2l-n}c_{l,m} \int_{0}^{\infty} \int_{-1}^{1} e^{isw\lambda} (Ds^{-1})^l h(s) \lambda (1 - \lambda^2)^{-l+(m-3)/2} d\lambda ds. \tag{6.3}
\]

If \( m \) is odd we take \( l = (m-3)/2 \) and obtain
\[
g_{r,t}^{(n)}(w) = w^{-(m+n-3)}c_{m} \int_{0}^{\infty} \int_{-1}^{1} e^{isw\lambda} (Ds^{-1})^{(m-3)/2} h(s) \lambda d\lambda ds.
\]

Integrating by parts once more we have
\[
g_{r,t}^{(n)}(w) = 2iw^{-(m+n-1)}c_{m} \int_{0}^{\infty} \frac{\sin(ws)}{s} D(Ds^{-1})^{(m-3)/2} h(s) ds,
\]
and integrate by parts one final time to obtain
\[
g_{r,t}^{(n)}(w) = -2iw^{-(m+n-1)}c_{m} \int_{0}^{\infty} (\int_{0}^{\infty} \frac{\sin(\tau)}{\tau} d\tau) D^2(Ds^{-1})^{(m-3)/2} h(s) ds \tag{6.4}
\]
If \( r = 0 \) then \( D^2(Ds^{-1})^{(m-3)/2} h(s) = \) (polynomial) \( e^{-s^2} \).

If \( r > 0 \) and \( t = 0 \) terms of the form \( s^{-1+r}e^{-s^2} \) arise which look singular for small \( r \) at \( s = 0 \). This is because we are not keeping track of coefficients. We do not lose any information if we keep \( r \) away from 0 since bounds on \( ||(I - \Delta)^{r/2} f|| \) for small \( r \geq 0 \) follow from those for larger \( r \). It is not hard to see from (6.4) that \( g_{r,t}^{(n)}(w) = O(w^{-(m+n-1)}) \) uniformly for \( t \) in compact subsets of \([0, \infty)\). To see the situation more explicitly it might help to note that \( (Ds^{-1})^l = \sum_{j=1}^{l} m_j D^j s^{-2l-j} \) for certain integers \( m_j \).
Now we turn to even $m$. Using (6.3) with $l = (m - 2)/2$ we find
\[ g_{r,t}(w) = w^{-(m+n-2)}c'_m \int_0^\infty \int_{-1}^1 e^{iws\lambda}(Ds^{-1})(m-2)/2 h(s)\lambda(1 - \lambda^2)^{-1/2} d\lambda ds. \]

Another integration by parts in the $s$ integral gives
\[ g_{r,t}(w) = w^{-(m+n-1)}ic'_m \int_0^\infty \int_{-1}^1 e^{iws\lambda}D(Ds^{-1})(m-2)/2 h(s)(1 - \lambda^2)^{-1/2} d\lambda ds + \]
\[ +w^{-(m+n-1)}ic'_m [(Ds^{-1})(m-2)/2 h(s)]_{s=0} \int_{-1}^1 (1 - \lambda^2)^{-1/2} d\lambda \]
so that $g_{r,t}(w) = O(w^{-(m+n-1)})$. From (6.1) it is clear that $g_{r,t}^{(n)}$ is bounded for all $r \geq 0$ uniformly for $t$ in compacts of $[0, \infty)$.

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