MODULUS OF CONTINUITY ESTIMATES FOR FULLY NONLINEAR PARABOLIC EQUATIONS

XIAOLONG LI

Abstract. We prove that the moduli of continuity of viscosity solutions to fully nonlinear parabolic partial differential equations are viscosity subsolutions of suitable parabolic equations of one space variable. As applications, we obtain sharp Lipschitz bounds and gradient estimates for fully nonlinear parabolic equations with bounded initial data, via comparison with one-dimensional solutions. This work extends multiple results of Andrews and Clutterbuck for quasilinear equations to fully nonlinear equations.

1. Introduction

Given a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, any function $f$ satisfying

$$|u(x) - u(y)| \leq 2f \left( \frac{|x - y|}{2} \right)$$

for all $x, y \in \Omega$ is called a modulus of continuity of $u$. The (optimal) modulus of continuity $\omega$ of $u$ is defined by

$$\omega(s) = \sup \left\{ \frac{u(x) - u(y)}{2} : x, y \in \Omega, |x - y| = 2s \right\}.$$

The definitions here are consistent with \[1\]\[2\]\[3\]\[4\], but differ from the usual ones by the factors of 2, which are included for convenience and nice statements of results. For instance, if $\varphi_0(s)$ is an odd function defined on a symmetric interval and it is positive and concave for positive $s$, then its modulus of continuity is exactly $\omega(s) = \varphi_0(s)$ for all $s \geq 0$. If we then evolve $\varphi_0$ as initial data by a quasilinear parabolic equation of one space variable of the form $\varphi_t = \alpha(\varphi')\varphi''$, then it is easy to see that the solution $\varphi(\cdot, t)$ remains odd, concave and positive for positive $s$. Therefore the modulus of continuity $\omega(s, t)$ of the solution $\varphi(s, t)$ is exactly $\omega(s, t) = \varphi(s, t)$.

Inspired by the above example, Andrews and Clutterbuck \[3\] first observed that the modulus of continuity of a regular ($C^2$ in space variable and $C^1$ in time) periodic solution of the quasilinear parabolic equation $\varphi_t = \alpha(\varphi')\varphi''$ is a subsolution of the same equation. Their proof is inspired by an argument of doubling variables used by Kružkov \[15\] for linear parabolic equations of one space variable. In \[4\], they managed to generalize this to higher dimensions, showing that for a wide class of quasilinear parabolic equations including the anisotropic mean curvature flows, the modulus of continuity of a periodic regular solution is a subsolution of an associated one-dimensional equation. Moreover, the results are sharp in the sense that initial data close to a square-wave function of one of the variables will give

2010 Mathematics Subject Classification. Primary: 35K55; Secondary: 35J60, 35D40.

Key words and phrases. Modulus of continuity estimates, fully nonlinear equations, viscosity solutions, gradient estimates.
equality in the limit of lattices with large period. Equations with Dirichlet or Neumann boundary conditions were also treated in [3] with convexity assumptions on the domain. As applications, they obtained time-interior gradient estimates (more precisely, estimates on the gradient for positive times which do not depend on the initial gradient, but only on the oscillation of the initial data) for solutions of quasilinear parabolic equations with gradient-dependent coefficients, under the weakest possible assumptions on the coefficients. Such estimates have not yet been accomplished using direct estimates on the gradient except in some special cases [10]. Later on, the modulus of continuity estimates were extended to quasilinear isotropic equations on Riemannian manifolds in [6][7][20] and to viscosity solutions in [10][17]. At last, we would like to mention that the modulus of continuity and its variants have found remarkable applications in proving sharp lower bounds for lower eigenvalues of the Laplacian in [4][6][12][13][20][22], the p-Laplacian in [2] and the weighted p-Laplacian in [18][19]. We refer the reader to the nice surveys by Professor Andrews [1][2], where these ideas were further explained, various applications are discussed, and connections to other problems in geometric analysis are made.

The purpose of the present paper is to extend the above-mentioned modulus of continuity estimates for linear and quasilinear parabolic equations to fully nonlinear parabolic equations. We will show that the moduli of continuity of viscosity solutions to fully nonlinear parabolic equations are viscosity subsolutions of suitable parabolic equations of one space variable (see Theorems 1.1, 5.2 and 7.1 for precise statements). In contrast to the quasilinear case, the one-dimensional equations are determined by a structure condition (see (SC) below) that we introduce on the fully nonlinear operator, rather than canonically associated. The structure condition specifies all the properties that need to be satisfied by the one-dimensional operators, and we then provide numerous examples (see Propositions 1.1, 4.1 and 8.1) to illustrate how to choose a natural one-dimensional operator for the given fully nonlinear operator.

Consider parabolic partial differential equations of the form

\[ u_t + F(t, x, u, Du, D^2u) = 0, \]

on a domain \( \Omega \subset \mathbb{R}^n \), where \( F : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R}, S(n) \) is the set of symmetric \( n \times n \) matrices, \( u_t \) is the time derivative of \( u \), \( Du \) is the gradient of \( u \), and \( D^2u \) is the Hessian of \( u \). Throughout the paper we assume that \( F \) is degenerate elliptic, i.e.,

\[ F(t, x, r, p, X) \leq F(t, x, r, p, Y) \text{ whenever } Y \leq X. \]

We also assume \( F \) is a continuous function of its arguments, so the basic theory of viscosity solutions in [11] applies.

A fundamental difficulty in proving the modulus of continuity estimates for fully nonlinear equations is that it is not clear how to identify the one-dimensional operators. In the quasilinear isotropic case considered in [3] and [4], the one-dimensional operator is obtained by plugging into a function that depends only on one of the variables. For example, the associated one-dimensional operator of the Laplacian \( (F = -\text{tr}(X)) \) is the one-dimensional Laplacian \( (f(\varphi) = -\varphi'') \). However, if the operator has lower order terms depending on \( x \) (say \( F = -\text{tr}(X) + h(x) \) for some function \( h(x) \)) or its coefficients depending on \( x \) (say \( F = -\text{tr}(AX) \) for \( A = (a_{ij}(x)) \)), one certainly cannot identify its one-dimensional operator by plugging a solution of one space variable, not to mention for more general fully nonlinear operators. To overcome this difficulty, we introduce a structure condition on \( F \) as follows.
Given an operator $F(t, x, r, p, X)$, let $f(t, s, \varphi, \varphi', \varphi'')$ be a one-dimensional operator (also degenerate elliptic and continuous of its arguments) such that

$$
\begin{cases}
F(t, y, r, \frac{x-y}{|x-y|}, Y) - F(t, x, v, \frac{x-y}{|x-y|}, X) \leq -2f(t, s, \varphi, \varphi', \varphi'') \\
\text{for all } x, y \in \Omega \text{ with } |x - y| = 2s > 0, v, r \in \mathbb{R} \text{ with } v - r = 2\varphi > 0, \\
\text{and } X, Y \in S(n) \text{ satisfying } \left( \begin{array}{cc} X & 0 \\ 0 & -Y \end{array} \right) \leq D^2_{x,y} \left(2\varphi \left(\frac{|x-y|}{2}, t\right)\right),
\end{cases}
$$

where all derivatives of $\varphi$ are evaluated at $s = \frac{|x-y|}{2}$ and $D^2_{x,y}$ means taking the Hessian with respect to all spatial variables.

The above structure condition is inspired by the structure condition in [11, (3.13) and (3.14)], under which the comparison principle is proved for a large class of operators. Also, it will be clear in the proofs that such a structure condition is exactly what we need. Indeed, the one-dimensional operator $f$ is in some sense a modulus of continuity of $F$, as it measures the change of $F$ when its arguments change under the constrains specified above. At last, such one-dimensional operators exist for very general $F$, and we will discuss how to choose $f$ for a large class of $F$ in Proposition 1.1 after stating our main result below.

**Theorem 1.1.** Suppose $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ is a spatially periodic viscosity solution of

$$u_t + F(t, x, u, Du, D^2u) = 0 \quad (1.1)$$

Let $f(t, s, \varphi, \varphi', \varphi'')$ be a one-dimensional operator satisfying the structure condition (SC). Then the modulus of continuity

$$\omega(s, t) := \sup \left\{ \frac{u(x, t) - u(y, t)}{2} : |x - y| = 2s \right\}
$$

of $u$ is a viscosity subsolution of the one-dimensional equation

$$\omega_t + f(t, s, \omega, \omega', \omega'') = 0 \quad (1.2)$$
on $(0, \infty) \times (0, T)$.

The same conclusion as in Theorem 1.1 holds if $u$ solves (1.1) on a bounded convex domain $\Omega \subset \mathbb{R}^n$ with smooth boundary and satisfies the Neumann boundary condition. This will be proved in Section 5. The Dirichlet boundary condition can also be handled with more restrictions on both $F$ and $\omega$.

The structure condition (SC) specifies all the requirements for the one-dimensional operator, but does not provide any choice of it directly. So we provide in the next proposition some natural choices of the one-dimensional operators according to the given $F$. More examples will be given in Section 4.

**Proposition 1.1.** The following pairs of operators $F$ and $f$ satisfy the structure condition (SC).

1. $F$ is the linear elliptic operator given by

$$F(x, r, p, X) = -\text{tr}(X) + (W(x), p) - Vr - h(x),$$

where $W$ is a bounded continuous vector field, $h$ is a bounded continuous function, and $V \in \mathbb{R}$.

$$f(\varphi, \varphi', \varphi'') = -\varphi'' - K|\varphi'| - V\varphi - \omega_h,$$

where $K = \sup_x |W(x)|$ and $\omega_h$ is a modulus of continuity of $h$. 
(2) $F$ is the quasilinear isotropic operator given by

$$F(p, X) = -\text{tr} \left[ \left( \alpha(|p|) \frac{p \otimes p}{|p|^2} + \beta(|p|) \left( I - \frac{p \otimes p}{|p|^2} \right) \right) X \right],$$

where $\alpha$ and $\beta$ are nonnegative functions, $I$ denotes the identity matrix and $p \otimes q$ denotes the matrix whose $(i, j)$ entry is $pq_j$.

$$f(\varphi, \varphi'') = -\alpha(|\varphi'|)\varphi''.$$  

(3) $F(X)$ is uniformly elliptic with ellipticity constants $0 < \lambda \leq \Lambda < \infty$, i.e.,

$$-\Lambda \text{tr}(Z) \leq F(X + Z) - F(X) \leq -\lambda \text{tr}(Z) \text{ for } Z \geq 0.$$  

$$f(\varphi'') = -\lambda \varphi''.$$  

(4) $F(t, r, p, X)$ (independent of $x$) is proper, i.e.,

$$F(t, r, p, X) \leq F(t, v, p, Y) \text{ whenever } r \leq v, Y \leq X.$$  

$$f \equiv 0.$$  

(5) $F(t, x, r, p, X)$ satisfies that for all $x, y \in \Omega$, $t \in [0, T]$, $v, r \in \mathbb{R}$ and $X, Y, Z \in S(n)$,

$$|F(t, y, r, p, X) - F(t, x, r, p, X)| \leq L|x - y|,$$

$$F(t, x, r, p, X) - F(t, v, p, X) \leq K(v - r) \text{ for } v \geq r,$$

$$F(t, x, r, p, X + Z) - F(t, x, r, p, X) \leq -\lambda(|p|, t) \text{tr}(Z) \text{ for } Z \geq 0,$$

where $L, K$ are positive constants and $\lambda(s, t)$ is a nonnegative function.

$$f(s, \varphi, \varphi', \varphi'') = -\lambda(|\varphi'|, t)\varphi'' - K \varphi - Ls.$$  

Part (1) of Proposition 1.1 says that $f$ is linear if $F$ is linear. It implies exponentially gradient bounds for linear equations because the one-dimensional can be solved explicitly (choosing $\omega_h = \frac{1}{2} \sup_x |h(x)|$); see [9].

Applying Theorem 1.1 to the operators in Part (2) covers the main results obtained by Andrews and Clutterbuck in [3] and [4] for quasilinear equations. Note that quasilinear isotropic operators in Part (2) include many greatly studied elliptic operators such as the Laplacian (with $\alpha = \beta = 1$), the $p$-Laplacian (with $\alpha(s) = (p - 1)|s|^{p-2}$ and $\beta(s) = |s|^{p-2}$), and graphical mean curvature operator (with $\alpha(s) = \frac{1}{1+|s|^2}$ and $\beta = 1$).

Part (3) implies that $f$ can be chosen to be linear as long as $F$ is uniformly elliptic. In particular, if $F$ is the Pucci’s extremal operators $-\Lambda_{\alpha, \beta}(D^2u)$ with ellipticity constants $0 < \lambda \leq \Lambda < \infty$, then $f$ can be chosen to be $f(\varphi'') = -\lambda \varphi''$. As we will see in Section 6 uniformly elliptic fully nonlinear operators behave just like uniformly elliptic linear operators in terms of Lipschitz bounds and time-interior gradient estimates.

Part (4) implies that if $F$ is independent of $x$ and proper, then the parabolic equation (1.1) preserves any initial modulus of continuity for spatially periodic solutions (or with Neumann boundary condition), i.e., if $\varphi(s)$ is a modulus of continuity for $u(\cdot, 0)$, then it is also a modulus of continuity for $u(x, t)$ whenever the solution exists.

Part (5) addresses the more general situation where the ellipticity constants may depend on the norm of the gradient and and $F(t, x, v, p, X)$ is Lipschitz in $x$ and $v$. Examples of such operators include the $p$-Laplacian $\Delta_p$, $-\lambda(|Du|)\Delta u$ and $-|Du|^\gamma M_{\alpha, \beta}(D^2u)$. In these cases, the one-dimensional equations can be chosen to be quasilinear.
We conclude this section by mentioning various extensions and applications of Theorem \[1.1\] as well as discussing the organization of this paper. In Section 2, we recall the definitions of viscosity solutions for parabolic equations and state the parabolic maximum principle for semicontinuous functions, which is the key tool that we use in this paper. We then present the proofs of Theorem \[1.1\] and Proposition \[1.1\] in Sections 3 and 4, respectively. In Section 5, we prove modulus of continuity estimates when the Neumann boundary condition is imposed. Section 6 is devoted to proving Lipschitz bounds and gradient estimates for solutions to parabolic equations with bounded initial data, as applications of the modulus of continuity estimates. In Section 7, we extend the modulus of continuity estimates to fully nonlinear parabolic equations on Riemannian manifolds. In Section 8, we study the effects of curvatures on the one-dimensional operators.

2. Preliminaries on Viscosity Solutions

In this section, we collect some basics on the theory of viscosity solutions that will be needed in the sequel. The reader is encouraged to consult \[11\] for a self-contained exposition of the basic theory of viscosity solutions.

Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( O \) be an open subset of \( \Omega \times (0,T) \). The following notations are useful:

\[
\begin{align*}
\text{USC}(O) &= \{ u : O \to \mathbb{R} | \text{u is upper semicontinuous} \}, \\
\text{LSC}(O) &= \{ u : O \to \mathbb{R} | \text{u is lower semicontinuous} \}.
\end{align*}
\]

Next we introduce the notion of parabolic semijets.

**Definition 2.1.**

1. For a function \( u \in \text{USC}(O) \), the second order parabolic superjet of \( u \) at a point \((x_0,t_0)\) in \( O \) is defined by

\[
\mathcal{P}^{2,+}u(x_0,t_0) = \{ (\varphi_{t_0}(x_0), D\varphi(x_0,t_0), D^2\varphi(x_0,t_0)) : \varphi \in C^\infty(O) \text{ such that } u - \varphi \text{ attains a local maximum at } (x_0,t_0) \}.
\]

2. For \( u \in \text{LSC}(O) \), the second order parabolic subjet of \( u \) at \((x_0,t_0)\) in \( O \) is defined by

\[
\mathcal{P}^{2,-}u(x_0,t_0) = -\mathcal{P}^{2,+}(-u)(x_0,t_0).
\]

3. We also define the closures of \( \mathcal{P}^{2,+}u(x_0,t_0) \) and \( \mathcal{P}^{2,-}u(x_0,t_0) \) by

\[
\overline{\mathcal{P}}^{2,+}u(z_0) = \{ (\tau,p,X) \in \mathbb{R} \times \mathbb{R}^n \times S(n) | \text{ there is a sequence } (z_j,\tau_j,p_j,X_j) \text{ such that } (\tau_j,p_j,X_j) \in \mathcal{P}^{2,+}u(z_j) \text{ and } (z_j,u(z_j),\tau_j,p_j,X_j) \to (z_0,u(z_0),\tau,p,X) \text{ as } j \to \infty \},
\]

\[
\overline{\mathcal{P}}^{2,-}u(z_0) = -\overline{\mathcal{P}}^{2,+}(-u)(z_0).
\]

Now we can give the definition of a viscosity solution for the general equation

\[
u_t + F(t,x,u,Du,D^2u) = 0,
\]

where \( F \) is assumed to be degenerate elliptic and continuous in its arguments.

**Definition 2.2.**

1. A function \( u \in \text{USC}(O) \) is a viscosity subsolution of \( 2.1 \) in \( O \) if for all \((x,t)\) in \( O \) and \((t,p,X)\) in \( \mathcal{P}^{2,+}u(x,t) \),

\[
\tau + F(t,x,u(x,t),p,X) \leq 0.
\]
Theorem 2.3. Let \( u_i \in \text{USC}(O_i \times (0, T)) \) for \( i = 1, \ldots, k \), where \( O_i \) is a locally compact subset of \( \mathbb{R}^n \). Let \( \varphi(x_1, \ldots, x_k, t) \) be a smooth function defined on an open neighborhood of \( O_1 \times \cdots \times O_k \times (0, T) \). Suppose the function
\[
 w(x_1, \ldots, x_k, t) := u_1(x_1, t) + \cdots + u_k(x_k, t) - \varphi(x_1, \ldots, x_k, t)
\]
attains a maximum at \( (\hat{x}_1, \ldots, \hat{x}_k, \hat{t}) \) on \( O_1 \times \cdots \times O_k \times (0, T) \). Assume further that there is an \( r > 0 \) such that for every \( \eta > 0 \) there is a \( C > 0 \) such that for \( i = 1, \ldots, k \)
\[
 b_i \leq C \text{ whenever } (b_i, q_i, X_i) \in \mathcal{P}^2 u_i(t, x_i),
\]
\[
 |x_i - \hat{x}_i| + |t - \hat{t}| \leq r \text{ and } |u_i(x_i, t)| + |q_i| + \|X_i\| \leq \eta.
\]
Then for each \( \lambda > 0 \), there are \( X_i \in S(n) \) such that
\[
 (b_i, D_{x_i} \varphi(\hat{x}_1, \ldots, \hat{x}_k, \hat{t}), X_i) \in \mathcal{P}^2 u_i(t, \hat{x}_i),
\]
\[
 - \left( \frac{1}{\lambda} + \|M\| \right) I \leq \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_k \end{pmatrix} \leq M + \lambda M^2,
\]
\[
 b_1 + \cdots + b_k = \varphi_t(\hat{x}_1, \ldots, \hat{x}_k, \hat{t}),
\]
where \( M = D^2 \varphi(\hat{x}_1, \ldots, \hat{x}_k, \hat{t}) \) and \( \|M\| = \sup \{M(v, v) : \|v\| = 1\} \).

3. Modulus of Continuity Estimates: Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The proof relies on the maximum principle for semicontinuous functions.

Proof of Theorem 1.1. By definitions of viscosity solutions (see Definition 2.2), we need to show that for every smooth function \( \varphi \) touching \( \omega \) from above at \( (s_0, t_0) \in (0, \infty) \times (0, T) \), in the sense that
\[
 \begin{cases} 
 \varphi(s, t) \geq w(s, t) \text{ for all } (s, t) \text{ near } (s_0, t_0), \\
 \varphi(s_0, t_0) = \omega(s_0, t_0), 
\end{cases}
\]
it holds at the point \( (s_0, t_0) \) that
\[
 \varphi_t + f(t, s, \varphi, \varphi', \varphi'') \leq 0.
\]
By the definition of \( \omega(s, t) \), we have that
\[
 u(x, t) - u(y, t) \leq 2\omega \left( \frac{|x - y|}{2}, t \right) \leq 2\varphi \left( \frac{|x - y|}{2}, t \right)
\]
for all points \(x, y \in \mathbb{R}^n\) with \(|x - y|\) close to \(2s_0\) and \(t\) close to \(t_0\). Since \(u\) is spatially periodic, there exist \(x_0, y_0 \in \mathbb{R}^n\) with \(|x_0 - y_0| = 2s_0\) such that

\[
u(x_0, t_0) - \nu(y_0, t_0) = 2\varphi \left( \frac{|x_0 - y_0|}{2}, t_0 \right).
\]

In other words, the function

\[Z(x, y, t) := u(x, t) - u(y, t) - 2\varphi \left( \frac{|x - y|}{2}, t \right)\]

attains a local maximum zero at \((x_0, y_0, t_0)\). Now we can apply the parabolic maximum principle for semicontinuous (see Theorem 2.3) functions to conclude that for each \(\lambda > 0\), there exist \(b_1, b_2 \in \mathbb{R}\) and \(X, Y \in \mathcal{S}(n)\) such that

\[(b_1, \varphi' e, X) \in \overline{\mathcal{P}}^2_+ \nu(x_0, t_0),
-b_2, \varphi' e, Y \in \overline{\mathcal{P}}^2_- \nu(y_0, t_0),
 b_1 + b_2 = 2\varphi_t,
\left(\begin{array}{cc}
X & 0 \\
0 & -Y
\end{array}\right) \leq M + \lambda M^2,
\]

where \(e = \frac{x_0 - y_0}{|x_0 - y_0|}\), \(M = D^2_{x,y} \left(2\varphi \left( \frac{|x - y|}{2}, t \right), t \right)\), and all derivatives of \(\varphi\) are evaluated at \((s_0, t_0)\) here and in the rest of the proof.

Since \(u\) is a viscosity solution of (1.1), we have

\[b_1 + F(t_0, x_0, \nu(x_0, t_0), \varphi' e, X) \leq 0,
-b_2 + F(t_0, y_0, \nu(y_0, t_0), \varphi' e, Y) \geq 0.
\]

Therefore, we obtain by letting \(\lambda \to 0^+\) that

\[2\varphi_t = b_1 + b_2 \leq F(t_0, y_0, \nu(y_0, t_0), \varphi' e, Y) - F(t_0, x_0, \nu(x_0, t_0), \varphi' e, X) \leq -2f(t_0, s_0, \varphi, \varphi', \varphi''),
\]

where we have used (SC) in the last inequality. The proof is complete. \(\square\)

### 4. Identifying One-dimensional Operators

In this section, we first prove Proposition 1.1 and then provide more examples to illustrate how to choose \(f\) for \(F\) to make the best use of Theorem 1.1.

The following lemma will be useful.

**Lemma 4.1.** Suppose that \(X, Y \in \mathcal{S}(n)\) satisfy

\[
\left(\begin{array}{cc}
X & 0 \\
0 & -Y
\end{array}\right) \leq D^2_{x,y} \left(2\varphi \left( \frac{|x - y|}{2}, t \right), t \right).
\]

Then we have \(X \leq Y\) and

\[\text{tr}(X - Y) \leq 2\varphi'' \left( \frac{|x - y|}{2}, t \right).
\]
Proof of Lemma 4.1. Note that the Hessian of $2\varphi\left(\frac{|x-y|}{2}, t\right)$ has the form
\[
\begin{pmatrix}
P & -P \\
-P & P
\end{pmatrix},
\]
where $P = 2D^2\varphi\left(\frac{|x-y|}{2}, t\right)$. Then $X \preceq Y$ follows from the fact that the matrix
\[
\begin{pmatrix}
P & -P \\
-P & P
\end{pmatrix}
\]
annihilates vectors of the form $(x, x)$. For any matrix $C$ such that the matrix
\[
\begin{pmatrix}
I & C \\
-C & I
\end{pmatrix}
\]
is positive semidefinite, we have
\[
\text{tr}(X - Y) = \text{tr}\left(\begin{pmatrix}
P & -P \\
-P & P
\end{pmatrix}\right) \leq 2\varphi''\left(\frac{|x-y|}{2}, t\right).
\]
Choosing $C = I - 2e \otimes e$ with $e = \frac{x-y}{|x-y|}$ produces
\[
\text{tr}(\begin{pmatrix}
I & C \\
-C & I
\end{pmatrix}P) = 2P(e, e) = \varphi''\left(\frac{|x-y|}{2}, t\right).
\]
Thus we have the desired estimate.

Proof of Proposition 1.1. (1). For simplicity, we write $e = \frac{x-y}{|x-y|}$. For any $x, y \in \mathbb{R}^n$ with $|x-y| = 2s$ and $r, v \in \mathbb{R}$ with $v-r = 2\varphi$, we have
\[
F(y, r, \varphi' e, Y) = F(x, v, \varphi' e, X)
\]
\[
= -\text{tr}(Y) - (W(y), \varphi' e) - Vr - h(y) + \text{tr}(X) + (W(x), \varphi' e) + Vv + h(x)
\]
\[
= \text{tr}(X - Y) + (W(x) - W(y), \varphi' e) + V(v-r) + h(x) - h(y)
\]
\[
\leq 2\varphi'' + 2\|W\|_{L\infty}|\varphi'| + 2V\varphi + 2\omega_h(s),
\]
where we have used Lemma 4.1 and the assumption that $\omega_h$ is a modulus of continuity of $h$. Thus $f(\varphi, \varphi', \varphi'') = -\varphi'' - \|W\|_{L\infty}|\varphi'| - V\varphi - \omega_h$ satisfies [SC].

(2). Since the quasi-linear isotropic operator $F$ is invariant under rotations, we may choose an orthonormal basis $\{e_i\}_{i=1}^n$ with $e_1 = \frac{x-y}{|x-y|}$ to simplify the calculations. With respect this basis, we have
\[
D^2_{x,y} \left(2\varphi\left(\frac{|x-y|}{2}, t\right)\right) = \begin{pmatrix}
P & -P \\
-P & P
\end{pmatrix},
\]
where
\[
P = \begin{pmatrix}
\frac{2\varphi''}{|x-y|} & 0 & \cdots & 0 \\
0 & \frac{\varphi'}{|x-y|} & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & \frac{\varphi'}{|x-y|}
\end{pmatrix}
\]
Let
\[
A = \begin{pmatrix}
\alpha(|\varphi'|) & 0 & \cdots & 0 \\
0 & \beta(|\varphi'|) & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & \beta(|\varphi'|)
\end{pmatrix}
\]
and
\[
C = \begin{pmatrix}
\alpha(\|\varphi\|) & 0 & \cdots & 0 \\
0 & \beta(\|\varphi\|) & \cdots & 0 \\
0 & 0 & \ddots & \beta(\|\varphi\|)
\end{pmatrix}.
\]

It’s easy to see that \(\begin{pmatrix} A & C \\ C & A \end{pmatrix}\) is a positive semidefinite matrix. Therefore, we have
\[
F(\varphi e_1, Y) - F(\varphi e_1, X) = -\text{tr}(AY) + \text{tr}(AX)
\]
\[
= \text{tr} \begin{bmatrix} A & C \\ C & A \end{bmatrix} \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}
\]
\[
\leq \text{tr} \begin{bmatrix} A & C \\ C & A \end{bmatrix} \begin{pmatrix} P & -P \\ -P & P \end{pmatrix}
\]
\[
= 2 \text{tr} [(A - C)P]
\]
\[
= 2\alpha(\|\varphi\|) \varphi''.
\]
Thus, \(f = -\alpha(\|\varphi\|) \varphi''\) satisfies (SC).

(3). This is a special case of Part (5). By Lemma 4.1, we have
\[
F(Y) - F(X) \leq -\lambda \text{tr}(Y - X) \leq 2\lambda \varphi''.
\]
So one can take \(f(s, \varphi, \varphi', \varphi'') = -\lambda(\|\varphi\|, t) \varphi''\).

For simplicity, we write \(e = \frac{x - y}{|x - y|}\) in (4) and (5).

(4) By Lemma 4.1 we have \(X \leq Y\). In view of \(v - r = 2\varphi > 0\), properness of \(F\) implies that
\[
F(t, v, \varphi'e, X) - F(t, v, \varphi'e, X) \leq 0.
\]
It follows that \(f \equiv 0\) satisfies (SC).

(5). Using the assumptions on \(F\), we estimate that
\[
F(t, y, r, \varphi'e, Y) - F(t, x, v, \varphi'e, X)
\]
\[
\leq L|x - y| + F(t, x, r, \varphi'e, Y) - F(t, x, v, \varphi'e, X)
\]
\[
\leq L|x - y| + K(v - r) + F(t, x, v, \varphi'e, Y) - F(t, x, v, \varphi'e, X)
\]
\[
\leq L|x - y| + K(v - r) + \lambda(\|\varphi\|, t) \text{tr}(X - Y)
\]
\[
= 2Ls + 2K\varphi + 2\lambda(\|\varphi\|, t) \varphi'',
\]
where we used \(|x - y| = 2s, v - r = 2\varphi\) and Lemma 4.1 in the last inequality. Thus
\[
f(s, \varphi, \varphi', \varphi'') = -\lambda(\|\varphi\|, t) \varphi'' - K\varphi - Ls\] satisfies (SC).

\[\square\]

Next, we provide a few more examples.

**Proposition 4.1.** The following operators \(F\) and \(f\) satisfy the structure condition (SC).

(1)
\[
F(p, X) = -\text{tr} \left[ \left( I - \frac{p \otimes p}{1 + \|p\|^2} \right) X \right],
\]
\[
f = -\frac{\varphi''}{1 + (\varphi')^2}.
\]
where \( A(p,t) = a_{ij}(p,t) \) and there exists a continuous function \( \alpha(R,t) \) such that
\[
0 < \alpha(R,t) \leq R^2 \inf_{|p| = R, (v,p) \neq 0} \frac{v^T A(p,t)v}{(v \cdot p)^2},
\]
\[
f = -\alpha(\varphi',t)\varphi''.
\]

Theorem 1.1 covers [4, Theorem 2.1] with the operators in part (1) and [4, Theorem 3.1] with the operators in part (2).

**Proof of Proposition 4.1.**

(1). This is a special case of part (1) in Proposition 1.1 with \( \alpha = \frac{1}{1 + |p|^2} \) and \( \beta = 1 \).

(2). The assumption implies \( A(p,t) \geq \alpha(|p|,t)I \). So for \( Z \geq 0 \), we have
\[
F(p, X + Z) - F(p, X) = -\text{tr}(A(p,t)Z) \leq -\alpha(|p|,t) \text{tr}(Z).
\]
This becomes a special case of Part (5) of Proposition 1.1. \( \square \)

5. Estimates with Neumann Boundary Conditions

The goal of this section is to show that the same modulus of continuity estimates for spatially periodic solutions in Theorem 1.1 holds if \( u \) solves (1.1) on a bounded convex domain \( \Omega \subset \mathbb{R}^n \) with smooth boundary and satisfies the Neumann boundary condition. The difference from the periodic case is that the local maximum point may lie on \( \partial(\Omega \times \Omega) \). For regular solutions of quasilinear equations, this possibility can be ruled out easily by assuming convexity of \( \Omega \) as in [4]. However, the same argument does not work for viscosity solutions because we cannot differentiate the equation. Moreover, the Neumann boundary condition needs to be understood in a weak sense as viscosity solutions are merely continuous.

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary. We consider equations of the form
\[
\begin{align*}
\tau + F(t, x, u(x,t), p, X) &= 0, \quad \text{in } \Omega \times (0, T), \\
B(t, x, u, D^2 u) &= 0, \quad \text{on } \partial \Omega \times (0, T).
\end{align*}
\]

Here both \( F \) and \( B \) are degenerate elliptic and continuous. We recall the definition of viscosity solutions to the boundary value problem (5.1) from [11, Section 7].

**Definition 5.1.**

(1) A function \( u \in \text{USC}(\overline{\Omega} \times (0, T)) \) is a viscosity subsolution of (5.1) if
\[
\tau + F(t, x, u(x,t), p, X) \leq 0
\]
for all \( (x,t) \in \Omega \times (0, T), (\tau,p,X) \in \overline{\mathcal{P}^{2+}_{\Omega \times (0,T)}} u(x,t) \), and
\[
\min \{ \tau + F(t, x, u(x,t), p, X), B(t,x,u(x,t),p,X) \} \leq 0
\]
for all \( (x,t) \in \partial \Omega \times (0,T), (\tau,p,X) \in \overline{\mathcal{P}^{2+}_{\partial \Omega \times (0,T)}} u(x,t) \).

(2) A function \( u \in \text{LSC}(\overline{\Omega} \times (0, T)) \) is a viscosity supersolution of (5.1) if
\[
\tau + F(t, x, u(x,t), p, X) \geq 0
\]
for all \( (x,t) \in \Omega \times (0, T), (\tau,p,X) \in \overline{\mathcal{P}^{2-}_{\Omega \times (0,T)}} u(x,t) \), and
\[
\max \{ \tau + F(t, x, u(x,t), p, X), B(t,x,u(x,t),p,X) \} \geq 0
\]
for all \((x, t) \in \partial \Omega \times (0, T)\), \((r, p, X) \in \overline{\Omega}^{2-}_{\overline{\Omega}^c} \times (0, T)\) \(u(x, t)\).

(3) A viscosity solution of \((5.2)\) is a continuous function \(u\) which is both a viscosity subsolution and a viscosity supersolution of \((5.2)\). 

The main result of this section is

**Theorem 5.2.** Let \(\Omega \subset \mathbb{R}^n\) be a bounded convex domain with smooth boundary and diameter \(D\). Suppose \(u\) is a viscosity solution of

\[
\begin{cases}
    u_t + F(t, x, u, Du, D^2 u) = 0, & \text{in } \Omega \times (0, T), \\
    (Du, \nu) = 0, & \text{on } \partial \Omega \times (0, T),
\end{cases}
\]

where \(\nu\) denotes the unit outward normal vector field along \(\partial \Omega\). Let \(f(t, s, \varphi, \varphi', \varphi'')\) be a one-dimensional operator satisfying \(\mathcal{S}\). Then the modulus of continuity \(\omega\) of \(u\) is a viscosity subsolution

\[
\omega_t + f(t, s, \omega, \omega', \omega'') = 0
\]
on \((0, D/2) \times (0, T)\) whenever \(w\) is increasing.

**Proof.** As in the proof of Theorem \([14]\) we conclude that the function \(Z : \overline{\Omega} \times (0, T) \to \mathbb{R}\) defined by

\[
Z(x, y, t) := u(x, t) - u(y, t) - 2\varphi \left( \frac{|x - y|}{2}, t \right)
\]

attains a local maximum zero at \((x_0, y_0, t_0) \in \overline{\Omega} \times \overline{\Omega} \times (0, T)\) with \(|x_0 - y_0| = 2s_0\). If \((x_0, y_0) \in \Omega \times \Omega\), then the same argument as in the proof of Theorem \([14]\) would prove the theorem. So the strategy here is to perturb the equation to ensure that the maximum point always lies in the interior of \(\Omega \times \Omega\). Note that \(\omega\) is increasing implies that \(\varphi'(s_0, t_0) \geq 0\). By approximation, we may assume that \(\varphi'(s_0, t_0) > 0\).

Pick a point \(z_0 \in \Omega\) and let \(v(x) = \frac{1}{2} |x - z_0|^2\). Then \(Dv(x) = x - z_0\) and \(D^2v(x) = I\). Moreover for any \(x \in \partial \Omega\),

\[
\langle Dv(x), \nu(x) \rangle = \langle x - z_0, \nu(x) \rangle \geq d(z_0, \partial \Omega) := \delta > 0.
\]

Set

\[
u(x, t) = u(x, t) - \varepsilon v(x), \quad u^\varepsilon(x, t) = u(x, t) + \varepsilon v(x),
\]

\[
F_\varepsilon(t, x, r, p, X) = F(t, x, r + \varepsilon v, p + \varepsilon Dv, X + \varepsilon I), \quad F^\varepsilon(t, x, r, p, X) = F(t, x, r - \varepsilon v, p - \varepsilon Dv, X - \varepsilon I).
\]

Then direct calculation shows that \(u_\varepsilon\) is a viscosity subsolution of

\[
\begin{cases}
    u_\varepsilon_t + F_\varepsilon(t, x, u, Du, D^2 u) = 0, & \text{in } \Omega \times (0, T), \\
    (Du, \nu) + \varepsilon \langle Dv, \nu \rangle = 0, & \text{on } \partial \Omega \times (0, T),
\end{cases}
\]

and \(u^\varepsilon\) is a viscosity supersolution of

\[
\begin{cases}
    u^\varepsilon_t + F^\varepsilon(t, x, u, Du, D^2 u) = 0, & \text{in } \Omega \times (0, T), \\
    (Du, \nu) - \varepsilon \langle Dv, \nu \rangle = 0, & \text{on } \partial \Omega \times (0, T).
\end{cases}
\]

We consider the following approximation of the function \(Z\):

\[
Z_\varepsilon(x, y, t) = u_\varepsilon(x, t) - u^\varepsilon(y, t) - 2\varphi \left( \frac{|x - y|}{2}, t \right).
\]
Since $Z_{\epsilon}$ converges to $Z$ uniformly as $\epsilon \to 0$, we know that $Z_{\epsilon}$ has a local maximum at $(x_{\epsilon}, y_{\epsilon}, t_{\epsilon})$ with $(x_{\epsilon}, y_{\epsilon}, t_{\epsilon}) \to (x_0, y_0, t_0)$ and $2s_{\epsilon} = |x_{\epsilon} - y_{\epsilon}| \to 2s_0$ as $\epsilon \to 0$. By the parabolic maximum principle for semicontinuous functions (see Theorem 2.3), for any $\lambda > 0$, there exist $b_{1,\epsilon}, b_{2,\epsilon} \in \mathbb{R}$ and $X_{\epsilon}, Y_{\epsilon} \in S(n)$ such that

$$(b_{1,\epsilon}, \varphi'(s_{\epsilon}, t_{\epsilon})e_{\epsilon}, X_{\epsilon}) \in \mathcal{P}_{\Omega \times (0,T)}^2 u_{\epsilon}(x_{\epsilon}, t_{\epsilon}),$$

$$(-b_{2,\epsilon}, \varphi'(s_{\epsilon}, t_{\epsilon})e_{\epsilon}, Y_{\epsilon}) \in \mathcal{P}_{\Omega \times (0,T)}^2 u'(y_{\epsilon}, t_{\epsilon}),$$

$$b_{1,\epsilon} + b_{2,\epsilon} = 2\varphi(s_{\epsilon}, t_{\epsilon}),$$

$$- (\lambda^{-1} + \|M\|) I \leq \begin{pmatrix} X_{\epsilon} & 0 \\ 0 & -Y_{\epsilon} \end{pmatrix} \leq M + \lambda M^2,$$

where $e_{\epsilon} = \frac{x_{\epsilon} - y_{\epsilon}}{|x_{\epsilon} - y_{\epsilon}|}$ and $M = D_{x,y}^2 \left( 2\varphi \left( \frac{|x_{\epsilon} - y_{\epsilon}|}{2} \right), t_{\epsilon} \right)$.

Since $u$ is a viscosity solution of (5.2), we have that if $x_{\epsilon} \in \Omega$, then at $(x_{\epsilon}, t_{\epsilon})$,

$$b_{1,\epsilon} + F'(t_{\epsilon}, x_{\epsilon}, u_{\epsilon}(x_{\epsilon}, t_{\epsilon}), \varphi' e_{\epsilon}, X_{\epsilon}) \leq 0,$$

$$\min \{b_{1,\epsilon} + F(t_{\epsilon}, x_{\epsilon}, u_{\epsilon}(x_{\epsilon}, t_{\epsilon}), \varphi' e_{\epsilon}, X_{\epsilon}, \varphi'(e_{\epsilon}, \nu(x_{\epsilon}))) + \epsilon(Dv(x_{\epsilon}), \nu(x_{\epsilon}))\} \leq 0,$$

(5.5)

and if $x_{\epsilon} \in \partial \Omega$, then at $(x_{\epsilon}, t_{\epsilon})$,

$$- b_{2,\epsilon} + F'(t_{\epsilon}, y_{\epsilon}, u'(y_{\epsilon}, t_{\epsilon}), \varphi' e_{\epsilon}, Y_{\epsilon}) \geq 0,$$

$$\max \{-b_{2,\epsilon} + F(t_{\epsilon}, y_{\epsilon}, u'(y_{\epsilon}, t_{\epsilon}), \varphi' e_{\epsilon}, -Y_{\epsilon}, \varphi'(e_{\epsilon}, \nu(y_{\epsilon}))) - \epsilon(Dv(y_{\epsilon}), \nu(y_{\epsilon}))\} \geq 0.$$

Similarly, if $y_{\epsilon} \in \Omega$, then at $(y_{\epsilon}, t_{\epsilon})$,

$$- b_{2,\epsilon} + F'(t_{\epsilon}, y_{\epsilon}, u'(y_{\epsilon}, t_{\epsilon}), \varphi' e_{\epsilon}, Y_{\epsilon}) \geq 0,$$

$$\max \{-b_{2,\epsilon} + F(t_{\epsilon}, y_{\epsilon}, u'(y_{\epsilon}, t_{\epsilon}), \varphi' e_{\epsilon}, -Y_{\epsilon}, \varphi'(e_{\epsilon}, \nu(y_{\epsilon}))) - \epsilon(Dv(y_{\epsilon}), \nu(y_{\epsilon}))\} \geq 0.$$

Observe that $\varphi'(\epsilon_{\epsilon}, \nu(y_{\epsilon})) - \epsilon(Dv(y_{\epsilon}), n(y_{\epsilon})) \leq -\epsilon \delta < 0$, because $\Omega$ is convex and $\varphi' \geq 0$. Therefore, (5.6) is valid no matter $x_{\epsilon}$ lies in $\Omega$ or on $\partial \Omega$.

By passing to subsequences if necessary, we have $b_{1,\epsilon} \to b_1$, $b_{2,\epsilon} \to b_2$, $X_{\epsilon} \to X$ and $Y_{\epsilon} \to Y$ as $\epsilon \to 0$. The limits satisfy

$$b_1 + b_2 = 2\varphi(s_{0}, t_0),$$

$$- (\lambda^{-1} + \|M\|) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq M + \lambda M^2,$$

where $M = D_{x,y}^2 \left( 2\varphi \left( \frac{|x_{0} - y_{0}|}{2} \right), t_0 \right)$. Letting $\epsilon \to 0$ in (5.5) and (5.6) yields

$$b_1 + F(t_0, x_0, u(x_0, t_0), \varphi' e_0, X) \leq 0,$$

$$- b_2 + F(t_0, y_0, u(y_0, t_0), \varphi' e_0, Y) \geq 0,$$

where $e_0 = \frac{x_0 - y_0}{|x_0 - y_0|}$ and $\varphi'$ is evaluated at $(s_0, t_0)$. The rest of the proof is exactly the same as the proof of Theorem 1.1. \qed
6. Lipschitz Bounds and Gradient Estimates

An obvious application of the modulus of continuity estimates is in proving regularity of solutions. It is well known that the heat equation evolves initial data which are very singular to solutions which are smooth for any positive time. This is no longer true for more nonlinear equations even in one space dimension, particularly if the equation becomes degenerate when the gradient is large. Andrews and Clutterbuck \[3\]|4| investigated the extent to which degenerate quasilinear parabolic equations smooth out irregular initial data. In particular, they gave a necessary and sufficient condition for quasilinear parabolic equations of one space variable to smooth our irregular initial data in \[3\] and proved explicit gradient bounds for solutions of quasilinear equations such as graphical anisotropic mean curvature flows in \[4\].

In this section, we further extend their results to fully nonlinear equations, as applications of the modulus of continuity estimates derived in previous sections.

As an immediate consequence of Theorem 1.1 and 5.2, we have

**Proposition 6.1.** Suppose that $u$ is either a spatially periodic viscosity solution of (1.1) on $\mathbb{R}^n$ or a viscosity solution of (5.2) on a bounded convex domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. Let $f(t, s, \varphi, \varphi', \varphi'')$ be a one-dimensional operator satisfying \[SC\] and the comparison principle. Suppose that $\varphi(s, t)$ satisfies

1. $\varphi_t \geq f(t, s, \varphi, \varphi', \varphi'')$ (in the viscosity sense);
2. $\varphi'(s, t) \geq 0$;
3. $\varphi(0, t) \geq 0$;

for all $s$ and $t$. If $\varphi(s, 0)$ is a modulus of continuity for $u(\cdot, 0)$, i.e.,

$$|u(x, 0) - u(y, 0)| \leq 2\varphi \left(\frac{|x - y|}{2}, 0\right)$$

for all $x, y$, then $\varphi(s, t)$ is a modulus of continuity for $u(x, t)$, i.e.,

$$|u(x, t) - u(y, t)| \leq 2\varphi \left(\frac{|x - y|}{2}, t\right)$$

for all $x, y$.

Here we say the one-dimensional operator $f(t, s, \varphi, \varphi', \varphi'')$ satisfies the comparison principle if a subsolution is no bigger than a supersolution provided that this is true on the boundary and initially. This is satisfied by all examples given below.

**Proof of Proposition 6.1.** For $\varepsilon > 0$, the function $\varphi_\varepsilon = \varphi + \varepsilon e^t$ satisfies

$$(\varphi_\varepsilon)_t > f(t, s, \varphi, \varphi', \varphi''),$$

so it cannot touch the modulus of continuity $\omega$ from above by Theorem 1.1 or 5.2 \(\square\)

Proposition 6.1 provides bounds on the modulus of continuity of solutions in terms of the initial modulus of continuity and elapsed time. This then provides gradient estimates for $u$ at positive times, provided the particular solution of the one-dimensional equation has bounded gradient for positive time. Below we elaborate how to obtain such time-interior gradient estimates for fully nonlinear equations.
For convenience, we introduce Assumption (E), which will be frequently used in this section.

\[
\begin{cases}
F(t, r, p, X) \text{ is increasing in } r, \text{ and } \\
F(t, r, p, X + Z) - F(t, r, p, X) \leq -\lambda(|p|) \text{ tr}(Z) \text{ for } Z \geq 0,
\end{cases}
\]

where \(\lambda(s)\) is a nonnegative function.

The function \(\lambda(s)\) measure the ellipticity of \(F\). The time reparametrization \(t \to ct\) for \(c > 0\) changes \(F\) to \(cF\), so it suffices to consider \(\lambda(s)\) up to multiplying by a positive constant.

Let’s first consider the heat equation \(u_t = \Delta u\), which was discussed in [2, Section 2]. If the initial data \(u_0\) is bounded, say \(|u_0(x)| \leq M\), then \(\varphi_0(s) \equiv \frac{M}{2}\) is a modulus of continuity for \(u_0\). It’s easy to see that the one-dimensional equation can be chosen to be the one-dimensional heat equation \(\varphi_t = \varphi''\). With \(\varphi_0 \equiv \frac{M}{2}\) as initial data, the solution is given by

\[
\varphi(s, t) = \frac{M}{2} \text{ erf} \left( \frac{s}{2\sqrt{t}} \right),
\]

where \(\text{erf}\) stands for the error function defined by

\[
\text{erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-t^2} dt.
\]

By Proposition 6.1, we have that for all \(x\) and \(y\)

\[
|u(x, t) - u(y, t)| \leq M \text{ erf} \left( \frac{s}{4\sqrt{t}} \right).
\]

Letting \(y \to x\) then yields

\[
|Du(x, t)| \leq \frac{M}{2\sqrt{\pi t}}.
\]

for all \(x\). Moreover, the estimate is sharp, with equality holding for the error function one-dimensional solution.

It is remarkable that we can get the same conclusion for any fully nonlinear operator \(F(t, r, p, X)\) that is uniformly elliptic, independent of \(x\), and increasing in \(u\).

**Proposition 6.2.** Let \(u\) be as in Proposition 6.1. Assume further that the operator \(F(t, r, p, X)\) satisfies Assumption (E) with \(\lambda(s) \equiv 1\). If \(|u(x, 0) - u(y, 0)| \leq M\) for some \(M > 0\), then

\[
|u(x, t) - u(y, t)| \leq M \text{ erf} \left( \frac{|x - y|}{4\sqrt{t}} \right)
\]

for all \(x, y\) and \(t \in (0, T)\). In particular,

\[
|Du(x, t)| \leq \frac{M}{2\sqrt{\pi t}}
\]

whenever it exists.

**Proof.** By Part (5) of Proposition 6.1, the one-dimensional equation can be chosen to be \(\varphi_t = \varphi''\) since \(F\) satisfies (E) with \(\lambda(s) \equiv 1\). Its solution with \(\varphi(s, 0) \equiv M/2\) is given by

\[
\varphi(s, t) = \frac{M}{2} \text{ erf} \left( \frac{s}{2\sqrt{t}} \right).
\]

The desired Lipschitz estimates then follows from Proposition 6.1 and the gradient estimates follows by letting \(y \to x\). \(\square\)
The situation where the ellipticity $\lambda$ depends on $|p|$ is more complicated, even for quasilinear equations of one space variable [3]. But there are some cases where the one-dimensional equations can be solved and explicit gradient bounds can be obtained. Below we extract several examples from [4] to further illustrate how the dependence of $\lambda$ on $|p|$ affects the time-interior gradient estimates.

Let’s consider an operator $F$ satisfying (E) with $\lambda(s) = (p - 1)|s|^{p-2}$, where $1 < p < \infty$. Such $F$ could be $-(p - 1)|Du|^{p-2}\Delta u$, the $p$-Laplacian $\Delta_p u := \text{div}(|Du|^{p-2}Du)$, or $-(p - 1)|Du|^{p-2}M^+_{1,\lambda}(D^2 u)$. The solution of the one-dimensional $p$-Laplacian heat flow $\varphi_t = (p - 1)|\varphi'|^{p-2}\varphi''$ with initial data $\varphi_0 \equiv M/2$ is given by (see [4] page 359)

$$
\varphi(s, t) = \frac{M}{2} \frac{1}{2F_p(\infty)} F_p\left(\frac{s}{t^{1/p}R_p}\right),
$$

where

$$
F_p(z) = \begin{cases}
\int_0^z (1 + s^2)^{\frac{1}{p-2}} ds, & 1 < p < 2; \\
\int_0^z e^{-s^2} ds, & p = 2; \\
\int_0^z (1 - s^2)^{\frac{1}{p-2}} ds, & p > 2;
\end{cases}
$$

and

$$
R_p = \left\{ \begin{array}{ll}
\left(\frac{2-p}{2p(p-1)}\right)^{\frac{1}{p-2}} (2F_p(\infty))^{\frac{2-p}{p}}, & 1 < p < 2; \\
2, & p = 2; \\
\left(\frac{2p(p-1)}{p^2 - 2}\right)^{\frac{1}{p}} (2F_p(\infty))^{\frac{2-p}{p}}, & p > 2.
\end{array} \right.
$$

Thus we obtain with Theorem [11] and Theorem [72] that

**Proposition 6.3.** Let $u$ be as in Proposition 6.7. Assume further that the operator $F(t, r, p, X)$ satisfies Assumption (E) with $\lambda(s) \equiv (p - 1)|s|^{p-2}$, where $1 < p < \infty$. If $|u(x, 0) - u(y, 0)| \leq M$ for some $M > 0$, then

$$
|u(x, t) - u(y, t)| \leq 2\varphi\left(\frac{|x - y|}{2}, t\right)
$$

for all $x, y$ and $t \in (0, T)$, where $\varphi$ is given by (6.1). In particular,

$$
|Du(x, t)| \leq \frac{1}{2R_p F_p(\infty)} \frac{M^2}{t^p}
$$

whenever it exists.

Another example is when $F$ satisfies Assumption (E) with $\lambda(s) = \frac{1}{1+s^2}$. Such $F$ could be $\frac{1}{1+|Du|^2}\Delta u$ or $\frac{1}{1+|Du|^2}M^+_{1,\lambda}(D^2 u)$. In this case, one can use the results in [4] Theorem 2.1 to conclude that if $|u_0| \leq M$, then

$$
1 + |Du(x, t)|^2 \leq \exp\left(\frac{2M^2}{t}\right)
$$

whenever it exists. Similarly as in [4] Corollary 3.2, if $F$ satisfies Assumption (E) with $\lambda(s, t)$ satisfying that there exist positive constants $A_0$ and $P_0$ such that

$$
\lambda(s, t) \geq \frac{A_0}{s^2} \text{ for } s \geq P_0 \text{ and all } t,
$$

...
then
\[ |Du(x, t)| \leq P_0 \exp \left( 1 + \frac{M^2}{A_0 t} \right) \]
whenever it exists. Such equations include the anisotropic mean curvature flows.

Finally, under some condition on \( \lambda(|p|) \), we can use the results in [3] to get gradient bounds for any positive time if the initial data is bounded.

**Proposition 6.4.** Let \( u \) be as in Proposition 6.1. Assume further that the operator \( F(t, r, p, X) \) satisfies Assumption (E) with \( \lambda(s) \) satisfying \( \int_0^s s \lambda(s) ds \to \infty \) as \( s \to \infty \). If \( |u(x, 0) - u(y, 0)| \leq M \) for some \( M > 0 \) and \( u \) is \( C^1 \), then there exist a constant \( C \) depending only on \( M \) and \( t \) such that
\[ |Du(x, t)| \leq C \]
for all \( x, y \) and \( t \in (0, T) \).

**Proof.** By part (3) of Proposition 6.1, the one-dimensional equation can be chosen to be \( \varphi_t = \lambda(|\varphi'|)\varphi'' \). It was shown in [3], Section 5, that \( \varphi \) has its gradient bounded for all positive times by a constant depending only on \( t \) and the initial oscillation if and only if \( \lim_{s \to \infty} B(s) = \infty \), where \( B(s) = \int_0^s t \lambda(t) dt \). Thus we have the desired gradient bounds. \( \square \)

7. Extensions to Riemannian Manifolds

In this section, we extend the modulus of continuity estimates to fully nonlinear parabolic equations on Riemannian manifolds. This is mostly a straightforward matter, but requires some adaptations and modifications to overcome the non-smoothness of the Riemannian distance function. For the basic theory of viscosity solutions on manifolds, see [8][14][23].

Let \( (M^n, g) \) be an \( n \)-dimensional complete Riemannian manifold without boundary. The Riemannian distance function \( d(x, y) \) is given by
\[ d(x, y) = \inf \{ L[\gamma] : \gamma \text{ is a smooth path from } x \text{ to } y \} \]
where \( L[\gamma] \) stands for the length of \( \gamma \). The modulus of continuity \( \omega \) of a function \( u : M \to \mathbb{R} \) is defined similarly as before by
\[ \omega(s) := \sup \left\{ \frac{u(x) - u(y)}{2} : x, y \in M, d(x, y) = 2s \right\} \]

As in the Euclidean case, we consider parabolic equations of the form
\[ u_t + F(t, x, u, Du, D^2 u) = 0, \quad (7.1) \]
where \( F : [0, T] \times M \times \mathbb{R} \times TM \times \text{Sym}^2 T^* M \to \mathbb{R} \) is degenerate elliptic and continuous, \( Du \) and \( D^2 u \) denote the gradient and Hessian of \( u \) with respect to the Levi-Civita connection, and \( TM, T^* M \) and \( \text{Sym}^2 T^* M \) denote the tangent bundle, the cotangent bundle and the set of symmetric two tensors on \( M \), respectively.

We would like to introduce a structure condition that is analogous to (SC) in the Euclidean setting. One simply needs to replace \(|x - y|\) in (SC) by \( d(x, y) \), i.e.,
\[ \left\{ \begin{array}{l}
F(t, y, r, -D_y \psi, Y) - F(t, x, v, D_x \psi, X) \leq -2f(t, s, \varphi, \varphi', \varphi'') \\
\text{for all } x, y \in M \text{ with } d(x, y) = 2s > 0, v, r \in \mathbb{R} \text{ with } v - r = \psi > 0,
\end{array} \right. \quad (SCM) \]
and \( X \in \text{Sym}^2 T^* M, Y \in \text{Sym}^2 T^* M \) satisfying
\[ \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq D^2_{x, y} \psi(x, y, t), \]
where $\psi(x, y, t) = 2\varphi \left( \frac{d(x, y)}{2}, t \right)$ and all derivatives of $\psi$ are evaluated at $(x, y, t)$. Note that $d(x, y)$ is in general not a smooth function, so $\text{(SCM)}$ should be understood in the “viscosity sense”, i.e., if $d$ is not smooth at $(x, y)$, then $\text{(SCM)}$ holds with $\psi(x, y, t) = 2\varphi \left( \frac{\rho(x, y)}{2}, t \right)$ for any smooth function $\rho$ touching $d$ from above at $(x, y)$. Finally, we point out that curvatures effect the one-dimensional operator $f$ as the Hessian of the distance function depends on curvatures.

We have the modulus of continuity estimates on manifolds.

**Theorem 7.1.** Let $(M^n, g)$ be a closed Riemannian manifold of dimension $n \geq 2$ and diameter $D$. Suppose that $u : M \times [0, T) \to \mathbb{R}$ is a viscosity solution of

$$u_t + F(t, x, u, Du, D^2 u) = 0. \quad (7.2)$$

Let $f(t, s, \varphi, \varphi', \varphi'')$ be a one-dimensional operator satisfying $\text{(SCM)}$. Then the modulus of continuity $\omega$ of $u$ is a viscosity subsolution of

$$\varphi_t = f(t, s, \omega, \omega', \omega'')$$

on $(0, D/2) \times (0, T)$ whenever $\omega$ is increasing in $s$.

**Proof.** The proof is a slight modification of that of Theorem 1.1. Let $\varphi$ be a smooth function touching $w$ from above at $(s_0, t_0) \in (0, D/2) \times (0, T)$. The assumption that $\omega$ is increasing in $s$ implies $\varphi''(s_0, t_0) \geq 0$. Since $M$ is compact, there exist $x_0$ and $y_0$ in $M$ with $d(x_0, y_0) = 2s_0$ such that

$$u(x_0, t_0) - u(y_0, t_0) = 2w(s_0, t_0) = 2\varphi(s_0, t_0).$$

For $x$ close to $x_0$, $y$ close to $y_0$, and $t$ close to $t_0$, it holds that

$$u(x, t) - u(y, t) \leq 2\omega \left( \frac{d(x, y)}{2}, t \right) \leq 2\varphi \left( \frac{d(x, y)}{2}, t \right).$$

Thus the function

$$u(x, t) - u(y, t) - 2\varphi \left( \frac{d(x, y)}{2}, t \right)$$

attains a local maximum at $(x_0, y_0, t_0)$.

Now let $\rho$ be any smooth function satisfying $d(x, y) \leq \rho(x, y)$ near $(x_0, y_0)$ with equality at $(x_0, y_0)$. Since $\varphi$ is non-decreasing, the function

$$Z(x, y, t) := u(x, t) - u(y, t) - \psi(x, y, t)$$

has a local maximum zero at $(x_0, y_0, t_0)$, where $\psi(x, y, t) = 2\varphi \left( \frac{\rho(x, y)}{2}, t \right)$.

By the parabolic maximum principle for semicontinuous functions on manifolds (see [14, Section 2.2] or [17, Theorem 2.3]), we have that for each $\lambda > 0$, there exist $b_1, b_2 \in \mathbb{R}$, and
X ∈ Sym²T^*_xM, Y ∈ Sym²T^*_yM such that

\[(b_1, D_x ψ, X) ∈ P^{2^+},\]
\[(-b_2, -D_y ψ, Y) ∈ P^{2^-},\]
\[b_1 + b_2 = ψ_t = 2 \tilde{φ}_t(s_0, t_0),\]
\[
\begin{pmatrix}
X & 0 \\
0 & -Y
\end{pmatrix}
\leq M + λM^2,
\]

where \(M = D^2ψ\). Here are below, all derivatives of \(ψ\) are evaluated at \((x_0, y_0, t_0)\) and all derivatives of \(φ\) are evaluated at \((s_0, t_0)\).

Since \(u\) is a viscosity solution of (7.2), we have

\[b_1 + \mathcal{F}(t_0, x_0, u(x_0, t_0), D_x ψ, X) \leq 0,\]
\[-b_2 + \mathcal{F}(t_0, y_0, u(y_0, t_0), -D_y ψ, Y) \geq 0.\]

Therefore, we obtain by letting \(λ → 0^+\) that

\[2φ_t = b_1 + b_2 \leq \mathcal{F}(t_0, y_0, u(y_0, t_0), -D_y ψ, Y) - \mathcal{F}(t_0, x_0, u(x_0, t_0), D_x ψ, X)\]
\[\leq -2f(t_0, s_0, φ, φ', φ''),\]

where we have used the structure condition \([\text{SCM}]\) in the last inequality. This completes the proof. □

8. Effects of Curvatures on One-dimensional Operators

On curved spaces, the Hessian of the distance function depends on sectional curvatures. The well known Hessian and Laplacian comparison theorems (see for example [21]) provide sharp comparison with distance functions on spaces of constant sectional curvature \(κ ∈ \mathbb{R}\), which are spheres \((κ > 0)\), Euclidean spaces \((κ = 0)\), and hyperbolic spaces \((κ < 0)\). To control the full Hessian of the distance function, one needs bounds on the sectional curvatures. However, if we only need to estimate the Laplacian of the distance function from above, then lower bounds on Ricci curvature suffice.

We extract an example from [6] to show how the curvatures effect the one-dimensional operators.

**Proposition 8.1.** Assume the Ricci curvature of \(M\) is bounded from below by \((n - 1)κ\) for some \(κ ∈ \mathbb{R}\). Then the following pair of \(f\) and \(F\) satisfy the structure condition \([\text{SCM}]\).

\[F(p, X) = -\text{tr} \left( \alpha(|p|) \frac{p \otimes p}{|p|^2} + \beta(|p|) \left( I - \frac{p \otimes p}{|p|^2} \right) \right) X,\]

where \(α\) and \(β\) are nonnegative functions.

\[f(φ', φ'') = -α(φ')φ'' + (n - 1)β(φ')T_κ,\]

where the function \(T_κ\) is defined for \(κ ∈ \mathbb{R}\) by

\[T_κ(t) = \begin{cases} 
\sqrt{κ} \tan(\sqrt{κ}t), & κ > 0, \\
0, & κ = 0, \\
-\sqrt{-κ} \tanh(\sqrt{-κ}t), & κ < 0.
\end{cases} \tag{8.1}\]
Proof of Proposition 8.1. The proof is essentially the same as in [6]. Let \( x, y \in M \) be such \( d(x, y) = 2s > 0 \) and \( \gamma : [0, 1] \to M \) be a length-minimizing geodesic connecting \( x \) and \( y \) with \( |\gamma'| = 2s \). Choose an orthonormal frame \( \{e_i(0)\}_{i=1}^n \) for \( T_x M \) with \( e_n(0) = \gamma'(0) \) and parallel translate it along \( \gamma \) to produce orthonormal frame \( \{e_i(s)\}_{i=1}^n \) for \( T_{\gamma(s)} M \) with \( e_n(s) = \gamma'(s) \) for all \( s \in [0, 1] \).

Let \( \rho \) be a smooth function touching \( d \) from above at \( (x, y) \) and write \( \psi(x, y, t) = 2\varphi \left( \frac{d(x, y)}{2}, t \right) \). Direct calculation shows that
\[
D_x \psi(x, y, t) = \varphi'(s, t)\gamma'(0),
\]
\[
D_y \psi(x, y, t) = -\varphi'(s, t)\gamma'(1).
\]
Therefore, we have
\[
F(-D_y \psi, Y) - F(D_x \psi, X)
\]
\[
= -\alpha(\varphi')Y(e_n(1), e_n(1)) - \beta(\varphi') \sum_{i=1}^{n-1} Y(e_i(1), e_i(1))
\]
\[
+ \alpha(\varphi')X(e_n(0), e_n(0)) + \beta(\varphi') \sum_{i=1}^{n-1} X(e_i(0), e_i(0))
\]
\[
= \alpha(\varphi') \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \begin{pmatrix} (e_n(0), -e_n(1)), (e_n(0), -e_n(1)) \\ \end{pmatrix}
\]
\[
+ \beta(\varphi') \sum_{i=1}^{n-1} \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \begin{pmatrix} (e_i(0), e_i(1)), (e_i(0), e_i(1)) \\ \end{pmatrix}
\]
\[
\leq \alpha(\varphi') D^2 \psi \begin{pmatrix} (e_n(0), -e_n(1)), (e_n(0), -e_n(1)) \\ \end{pmatrix}
\]
\[
+ \beta(\varphi') \sum_{i=1}^{n-1} D^2 \psi \begin{pmatrix} (e_i(0), e_i(1)), (e_i(0), e_i(1)) \\ \end{pmatrix}.
\]

It is easy to calculate that
\[
D^2 \psi \begin{pmatrix} (e_n(0), -e_n(1)), (e_n(0), -e_n(1)) \\ \end{pmatrix}
\]
\[
= \varphi'' D\rho \otimes D\rho \begin{pmatrix} (e_n(0), -e_n(1)), (e_n(0), -e_n(1)) \\ \end{pmatrix}
\]
\[
+ \varphi' D^2 \rho \begin{pmatrix} (e_n(0), -e_n(1)), (e_n(0), -e_n(1)) \\ \end{pmatrix}
\]
\[
= 2\varphi''
\]
and
\[
\sum_{i=1}^{n-1} D^2 \psi \begin{pmatrix} (e_i(0), e_i(1)), (e_i(0), e_i(1)) \\ \end{pmatrix}
\]
\[
= \varphi'' \sum_{i=1}^{n-1} D\rho \otimes D\rho \begin{pmatrix} (e_i(0), -e_i(1)), (e_i(0), -e_i(1)) \\ \end{pmatrix}
\]
\[
+ \varphi' \sum_{i=1}^{n-1} D^2 \rho \begin{pmatrix} (e_i(0), -e_i(1)), (e_i(0), -e_i(1)) \\ \end{pmatrix}
\]
\[
\leq -2(n-1)T_{\kappa} \varphi',
\]
where we have used the Laplacian comparison theorem in the last inequality (see for example [6, page 1018] or [17, page 564]). Combining the above estimates, we obtain

\[ F(-D_y \psi, Y) - F(D_x \psi, X) \leq -2(\alpha(\varphi')\varphi'' - (n-1)T_\kappa \varphi' \beta(\varphi')) . \]

Thus \( f = -\alpha(\varphi')\varphi'' + (n-1)T_\kappa \varphi' \beta(\varphi') \) satisfies SCM.

\[ \square \]

Analogous results hold for other examples given in Proposition 1.1 and 4.1 (Part (4) of Proposition 1.1 requires nonnegative sectional curvature). As applications, we obtain the same Lipschitz bounds and gradient estimates as in Section 6 provided that \( M \) has nonnegative Ricci curvature.

References

[1] Ben Andrews. Gradient and oscillation estimates and their applications in geometric PDE. In Fifth International Congress of Chinese Mathematicians. Part 1, 2, volume 2 of AMS/IP Stud. Adv. Math., 51, pt. 1, pages 3–19. Amer. Math. Soc., Providence, RI, 2012.
[2] Ben Andrews. Moduli of continuity, isoperimetric profiles, and multi-point estimates in geometric heat equations. In Surveys in differential geometry 2014. Regularity and evolution of nonlinear equations, volume 19 of Surv. Differ. Geom., pages 1–47. Int. Press, Somerville, MA, 2015.
[3] Ben Andrews and Julie Clutterbuck. Lipschitz bounds for solutions of quasilinear parabolic equations in one space variable. J. Differential Equations, 246(11):4268–4283, 2009.
[4] Ben Andrews and Julie Clutterbuck. Time-interior gradient estimates for quasilinear parabolic equations. Indiana Univ. Math. J., 58(1):351–380, 2009.
[5] Ben Andrews and Julie Clutterbuck. Proof of the fundamental gap conjecture. J. Amer. Math. Soc., 24(3):899–916, 2011.
[6] Ben Andrews and Julie Clutterbuck. Sharp modulus of continuity for parabolic equations on manifolds and lower bounds for the first eigenvalue. Anal. PDE, 6(5):1013–1024, 2013.
[7] Ben Andrews and Lei Ni. Eigenvalue comparison on Bakry-Emery manifolds. Comm. Partial Differential Equations, 37(11):2081–2092, 2012.
[8] Daniel Azagra, Juan Ferrera, and Beatriz Sanz. Viscosity solutions to second order partial differential equations on Riemannian manifolds. J. Differential Equations, 245(2):307–336, 2008.
[9] Le Kévin Balch. Exponential bounds for gradient of solutions to linear elliptic and parabolic equations. arXiv:2006.04582, 2020.
[10] Julie Clutterbuck. Interior gradient estimates for anisotropic mean-curvature flow. Pacific J. Math., 229(1):119–136, 2007.
[11] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User’s guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.), 27(1):1–67, 1992.
[12] Xianzhe Dai, Shoo Seto, and Guofang Wei. Fundamental gap estimate for convex domains on sphere – the case n=2. Comm. Anal. Geom., to appear, arXiv:1803.01115, 2018.
[13] Chenzu He and Guofang Wei. Fundamental gap of convex domains in the spheres (with appendix B by Qi S. Zhang). Amer. J. Math., to appear, arXiv:1705.11152, 2017.
[14] Tom Ilmanen. Generalized flow of sets by mean curvature on a manifold. Indiana Univ. Math. J., 41(3):671–705, 1992.
[15] S. N. Kružkov. Nonlinear parabolic equations with two independent variables. Trudy Moskov. Mat. Obšč., 16:329–346, 1967.
[16] Xiaolong Li. Moduli of continuity for viscosity solutions. Proc. Amer. Math. Soc., 144(4):1717–1724, 2016.
[17] Xiaolong Li and Kui Wang. Moduli of continuity for viscosity solutions on manifolds. J. Geom. Anal., 27(1):557–576, 2017.
[18] Xiaolong Li and Kui Wang. Sharp lower bound for the first eigenvalue of the weighted p-laplacian. arXiv:1910.02295, 2019.
[19] Xiaolong Li and Kui Wang. Sharp lower bound for the first eigenvalue of the weighted p-laplacian II. Math. Res. Lett, to appear, arXiv:1911.04596, 2019.
[20] Lei Ni. Estimates on the modulus of expansion for vector fields solving nonlinear equations. *J. Math. Pures Appl. (9)*, 99(1):1–16, 2013.

[21] Takashi Sakai. *Riemannian geometry*, volume 149 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. Translated from the 1992 Japanese original by the author.

[22] Shoo Seto, Lili Wang, and Guofang Wei. Sharp fundamental gap estimate on convex domains of sphere. *J. Differential Geom.*, 112(2):347–389, 2019.

[23] Xuehong Zhu. Viscosity solutions to second order parabolic PDEs on Riemannian manifolds. *Acta Appl. Math.*, 115(3):279–290, 2011.

Department of Mathematics, University of California, Irvine, Irvine, CA 92697, USA

E-mail address: xiaololi@uci.edu