Regularization of the spectral singularity in $\mathcal{PT}$-symmetric systems by all order nonlinearities: nonreciprocity and optical isolation

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Spectral singularities are ubiquitous with $\mathcal{PT}$-symmetry leading to infinite transmission and reflection coefficients. Such infinities imply the divergence of the fields in the medium thereby breaking the very assumption of the linearity of the medium used to obtain such singularities. We identify saturable nonlinearity retaining contributions from all orders of the field to limit the infinite growth and regularize the spectral singularity. We present explicit numerical results to demonstrate regularization. The all order nonlinear $\mathcal{PT}$-symmetric device is shown to exhibit very effective isolation or optical diode action, since transmission through such a system is nonreciprocal. In contrast, a linear system or a system with Kerr nonlinearity is known to have only reciprocal transmission. Further we demonstrate optical bistability for such a system with high contrast.

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Complex $\mathcal{PT}$-symmetric potentials [1] possessing real spectra have drawn considerable attention in recent times [2–7]. Optics has played a very fertile role to offer both theoretical and experimental testbeds for the unique properties of $\mathcal{PT}$-symmetric systems [2, 8–10]. The advantages and versatility of optics stem from the one to one correspondence between the Schrödinger equation and the Helmholtz equation, and the ability of the dielectric function to mimic the complex potential in certain cases. Though most of the studies are devoted to laterally coupled systems (mostly waveguides and directional couplers) [2, 8, 11], there have been studies on purely one dimensional systems with adjoining loss and gain sections [12–16]. Diverse effects like power oscillations, loss-induced increase in transmission, non reciprocity in reflection, invisibility etc have been reported [2, 8, 10]. It has been shown that the plasmonic realization of $\mathcal{PT}$-symmetric devices hold a great deal of promise for future technologies [17]. In view of the broad span of the underlying physics concepts and their appeal for fundamental and applied research a host of diverse systems has been studied to understand the consequences of spontaneously broken $\mathcal{PT}$-symmetry. These include Talbot imaging, coupled waveguides and periodic structures, Bloch oscillations, chaos, coupled lumped networks, $\mathcal{PT}$-CPA lasers and many other linear and nonlinear systems [10, 11, 18–22].

Perhaps the most striking feature of the $\mathcal{PT}$-symmetric systems, is their ability to possess spectral singularities which has been investigated in detail [13, 15, 16, 23]. These studies assume that the underlying equations are linear. At the spectral singularities the reflection and the transmission coefficients tend to infinity. There have been nonlinear extensions of the theory involving a dispersive Kerr type nonlinearity in a distributed feedback system [24] reporting bistability and non reciprocity in reflection. Soliton-type solutions and other nonlinear modes have been reported in several other studies [25, 26]. Two very recent studies investigate the robustness of the singularity in presence of Kerr type nonlinearity or in the framework of a nonlinear Schrödinger equation [27, 28]. It is shown that the presence of a dispersive nonlinearity does not break the parity-reflection symmetry of the spectral singularities, which are now power dependent [27]. In this letter we show that a different nonlinear mechanism is needed to limit the infinite growth of the reflection and transmission coefficients. In particular, with an example of a waveguide with equal finite segments of gain and loss, we show that the incorporation of a saturation mechanism for both gain and loss can lead to finite scattering amplitudes. Indeed, a Kerr type nonlinearity can move around the location of the singularity via an intensity-induced modification of the optical path. In view of the divergence of the field for a linear system a nonlinearity having contributions from all orders of the field is needed for the description of the realistic system. Thus, a saturable nonlinearity can lead to an intricate local field distribution affecting the intensity dependent real and imaginary parts of the dielectric function, eventually breaking the $\mathcal{PT}$-symmetry. Note that saturable nonlinearities are extremely important in optics and their role has been adequately discussed in the classic text by Allen and Eberly [29]. However, to the best of our knowledge, saturation type nonlinearities have not been addressed in the context of $\mathcal{PT}$-symmetric systems. Moving from dispersive to saturation type nonlinearities increases the complexity of the problem to the extent that analytical treatment is no longer possible and one has to revert to numerical simulation. Note that exact or approximate solutions are known for dispersive nonlinearity [30], while such are missing for saturable active/passive media.
FIG. 1: Schematic view of the $\mathcal{PT}$-symmetric waveguide with equal segments of gain and loss medium.

The proposed system has several definite advantages. Most importantly, such a device can act like a near-perfect isolator allowing only one way traffic [31]. Similar optical diode action can find many applications in chip-level optical circuitry. Indeed in the nonlinear regime it allows light to pass through only in one direction implying non reciprocity in transmission. Note that the linear counterpart can never have nonreciprocity in transmission, though reflection can be nonreciprocal [32, 33]. Even in a Kerr nonlinear system such nonreciprocity in transmission is absent [24]. Secondly, as expected from a nonlinear system, there can be multivalued response with high contrast between the ‘off’ and ‘on’ states. Needless to mention that these features are quite attractive for pure optical switching and logic operations.

$\mathcal{PT}$-symmetric system with all order nonlinearity: Consider the quasi-one dimensional system shown in Fig. 1, consisting of alternating equal segments of loss and gain. For a general power-dependent nonlinear response the Helmholtz equation for such a system can be written as:

$$\left[ \frac{\partial^2}{\partial x^2} + k^2 \epsilon \left( |\Psi|^2 \right) \right] \Psi = 0, \quad (1)$$

where $k = \omega / c$ is the vacuum wave vector and $\epsilon$ is the power dependent dielectric function. We further model the loss (gain) medium to be a collection of two-level atoms without (with) interaction. The dielectric function can then be expressed as [34]:

$$\epsilon \left( |\Psi|^2 \right) = 1 + \chi_0 \eta \left( |\Psi|^2 \right), \quad (2)$$

with

$$\eta \left( |\Psi|^2 \right) = \begin{cases} \frac{\delta + 1}{\delta + 2} |\Psi|^2 & 0 < x < L \\ \frac{\delta - 1}{\delta + 2} |\Psi|^2 & -L < x < 0 \end{cases}, \quad (3)$$

where $\delta = (\omega - \omega_0) / \Gamma$ is the detuning normalized to the total decay rate $\Gamma = \gamma + \Gamma$ with $2\gamma$ and $\Gamma$ giving the Einstein $A$ coefficient and the collisional line width, respectively. $\chi_0$ is the imaginary part of the linear susceptibility at resonance. Since $\Gamma / \omega_0$ is typically $\sim 10^{-4}$, one has $\omega = (1 + 10^{-4} \delta) \omega_0$. In Eq. (3) we introduce a binary switch parameter $a$ with values 1 and 0 to indicate the presence and absence of nonlinearity. It is clear from a close inspection of Eq. (3), that an increase in power affects both the real and imaginary parts of $\eta$. And, thus, the propagation aspects determined by Eq. (1).

An indepth nonlinear study requires a complete understanding of the linear properties of the system. Properties of the linear counterpart is now well understood [13, 21]. We recall some of the essential features for self consistency. For illumination from both sides by plane waves with amplitude $a$ and $d$, leading to scattered amplitudes $b$ and $c$, one can relate them through a $2 \times 2$ matrix $\mathcal{M}$ as:

$$\begin{bmatrix} c \\ d \end{bmatrix} = \mathcal{M} \begin{bmatrix} a \\ b \end{bmatrix}. \quad (4)$$

Particular cases of left-incident and right-incident scattering solutions can be written as [13]

$$\Psi^l(x) = \begin{cases} c_l (e^{ikx} + r_l e^{-ikx}) & x < -L \\ c_l t_l e^{ikx} & x > L \end{cases}, \quad (5)$$

and

$$\Psi^r(x) = \begin{cases} c_r t_r e^{-ikx} & x < -L \\ c_r e^{ikx} + r_r e^{ikx} & x > L \end{cases}, \quad (6)$$

where $l_i = |c_l|^2$, $r_i = |c_r|^2$ are the incident light intensities, $r_l$ and $t_l$ ($r_r$ and $t_r$) are the corresponding amplitude reflection coefficient and amplitude transmission coefficient. We can then get transmission and reflection coefficients $T_i = |t_i|^2$, $R_i = |r_i|^2$ ($R_r = |r_r|^2$).

In terms of the elements of $\mathcal{M}$, for the linear system, the amplitude transmission and reflection coefficients are given by matrix elements of $\mathcal{M}$ [13, 21]:

$$t_l = \frac{\text{det} \mathcal{M}}{M_{22}}, \quad t_r = \frac{1}{M_{22}}, \quad r_l = -\frac{M_{21}}{M_{22}}, \quad r_r = \frac{M_{12}}{M_{22}}, \quad (7)$$

and we have $\text{det} \mathcal{M} = M_{22} M_{11} - M_{12} M_{21} = 1$ [35], which gives $t_l = t_r$ [32, 33]. For a general proof directly based on Helmholtz equation, see equation 9 in [32], which holds for any linear medium. For $\mathcal{PT}$ symmetric systems, one has additional relations $M_{12} (\omega) = M_{12}^* (\omega^*)$. Based on these properties, for a real $\omega$, if $|t_l| = |t_r| = 1$, we have $M_{22} M_{11} = |M_{22}|^2 = 1$. In order to satisfy $\text{det} \mathcal{M} (\omega) = 1$, we need $M_{12} M_{21} = 0$. Thus at least one of the amplitude reflection coefficients $r_l$ and $r_r$ must be zero as will be confirmed by Fig. 2 below.

For calculations we picked two singularities, namely, $n = 0$ and $n = 10$ of the Table I in Ref. [13]. Most of the results will refer to the most discussed in literature case of $n = 0$ as proof of principle, while the case of $n = 10$ will be discussed in the context of practical realization and viability of a near perfect isolator in the optical domain. $n = 0$ singularity occurs at $\chi_0 = 1.82765566$.
and from transition frequency $\omega_0$, we may get the half-length of scattering area $L_0$ for divergence by the condition $k_0L_0 = 1.06ق68255$, where $k_0 = \frac{2\pi}{\lambda}$ is the wave vector corresponding to the transition frequency. The results for linear response (the various transmission and reflection coefficients) are shown in Fig. 2. The occurrence of the singularity is marked by letter $D$, while the critical points are labeled by $A$, $B$. The one marked by $D$ reflects the situation where all the reflection and transmission amplitudes are singular, and it has been discussed in great detail [13]. The critical points in $R_l$ and $R_r$ occur at $A$ and $B$, respectively, for left and right incidence. One can also demarcate parameters regimes of overall gain and loss, looking at the total scattering $S = T + R$ as a function of $\delta$. The gain (loss) domains is characterized by $S > 1$ ($S < 1$). For example, for the critical point in $R_l$ at $A$, we notice that a crossover at $\omega = \omega_A$ takes place from loss to gain behavior. Analogous behavior is observed at the critical point $B$ for right incident waves. Note that such cross-over behavior and both linear and nonlinear response in terms of the criticality could be very different for a different resonance, since they refer to different system sizes as compared to the wavelength. This will be further highlighted in the context of the nonlinear response.

**Regularization of spectral singularities:** Having discussed the linear properties, we ask the question how nonlinearity (see Eq. (3)) affects the spectral singularities and critical points. We concentrate mostly on the singularity at $D$, where all the coefficients diverge in the linear system (see Fig. 2). In the context of the zeroes of $R$ (the critical points), we show how the domain boundaries between gain and loss get affected by the nonlinear response. We employ a variation of the finite difference method for numerical integration of Eq. (1), with nonlinearity (3) [36]. For the power levels used in the calculations, convergence was easily achieved.

Our central result concerns the ability of the saturation type nonlinearity to limit the infinite growth, which eventually can lead to optical isolation and diode action. The results for the reflection and transmission coefficients for both left and right incidence for different power levels are shown in Fig. 3. Indeed, the singularity gets destroyed by the saturation mechanism, as would be the case in a realistic system. In fact, the peak amplitude gets less and less with increasing power level. Moreover, there is a frequency shift of the peak for higher powers with a broadening of the response.

**Nonreciprocity, optical isolation and bistability:** We now show how the gain/loss domain boundary given by $S = 1$ gets distorted by increasing power levels. The color density plot in Fig. 4 shows the dependence of the transmission and reflection coefficients as functions of $\delta$ and the incident intensity $I$. The domain boundaries for left and right incidence characterized by $R_l = 0$ and $R_r = 0$ are shown by blue lines in (a) and (c), respectively. The case $T = 1$ are depicted by the black heavy lines in (b) and (d) for left and right incidence. It is clear
that the effects of nonlinearity is not the same for left and right incidence even in case of transmission. Recall that transmission is known to be reciprocal in linear systems, and, as has been shown recently, even in nonlinear systems with Kerr type nonlinearity \[24, 32, 33\]. In our case nonlinearity can lift the parity degeneracy, and make transmission nonreciprocal. In fact, we show below, that this nonreciprocity can be amplified to the extent that this nonreciprocity can be amplified to the extent that system allows predominantly only one way transmission (optical isolation). It allows light to pass through for left incidence while it is blocked for light coming from the other side leading to the optical diode action. Another interesting feature that need be noted from the black line in Fig. 4(b) is that the same power level corresponds to three distinct values of \(\delta\) (say at \(I = 2.2\)). This is a typical signature of a nonlinear system and normally gets manifested in bistable or multi-stable response. In what follow we show both diode action and bistability in our system. To this goal we choose a system with \(L = 3L_0\), which is detuned from the spectral singularity which occurs at \(k_0L_0 = 1.06468255\). We show that bistable response of transmission coefficient for left incidence \(T_l\) as a function of the modulus of the incident amplitude. One can easily see the high contrast of the lower and upper branches. The part of the curve with negative slope is unstable as in standard hysteretic response.

The results discussed above clearly demonstrate the remarkable potentials of \(PT\)-symmetric systems with saturable nonlinearity. However, in the optical domain \(k_0L_0 = 1.06468255\) (for \(n = 0\)) corresponds to a system of size \(100 \text{ nm} - 1 \text{ \mu m}\), and it may be really challenging to create gain and loss regions. In order to overcome such difficulties, one can choose a larger system, say for \(n = 10\) with \(k_0L_0 = 32.8243878\) and \(\chi_0 = 0.17167639\). With atomic vapors such a set of parameters should be achievable. As mentioned earlier, the linear and nonlinear response could be very different for different order singularities with different \(n\) values. The results for \(n = 10\) are shown in Figs. 5(c) and 5(d). A comparison of these with Figs. 5(a) and 5(b) for \(n = 0\) reveals that one now has a completely contrasting result, since cases of left and right incidence exchange their roles. Now one has bistable response for \(T_r\), while one had multi valued response for left incidence for \(n = 0\). Even for nominal intensities below a certain threshold, one has near perfect optical isolation (Fig. 5(c)). One also has the familiar bistable response

\[\text{FIG. 4: The color density images (in log scale) of the intensity reflection and transmission coefficients as functions of normalized detuning } \delta \text{ and incident intensity } I: \ \text{(a) log}_{10} R_l, \ \text{(b) log}_{10} T_l, \ \text{(c) log}_{10} R_r, \ \text{and (d) log}_{10} T_r.} \]

\[\text{The zeros of the reflection coefficients } R_l \text{ and } R_r \text{ are shown by blue lines. For comparison, they are reproduced as thin lines in the right panels. The contour } T_l = 1 \text{ } (T_r = 1) \text{ is shown as heavy (black) lines in panel (b) (panel (d)).} \]

\[\text{FIG. 5: Optical isolation and bistability in the nonlinear } \mathcal{PT} \text{ symmetric system when } L \text{ is no more the resonant length } L_0. \ \text{(a-b) } L = 3L_0 \text{ for } n = 0, \ \text{(c-d) } L = 2.2L_0 \text{ for } n = 10. \ \text{The parameters for } n = 0 \text{ (} n = 10 \text{) were taken as } k_0L_0 = 1.06468255, \chi_0 = 1.82765566 \text{ (} k_0L_0 = 32.8243878, \chi_0 = 0.17167639 \text{) as in table I of [13].} \]

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for larger power levels (Fig. 5(d)). Clearly such effects open up new application possibilities of $\mathcal{PT}$-symmetric systems.

In conclusion, we have studied a $\mathcal{PT}$-symmetric waveguide with saturable nonlinearity. We show that the saturable nonlinearity can regularize the spectral singularities and limit the infinite growth of the scattering amplitudes. Besides, the nonlinear response can exhibit near-perfect optical isolation, showing near-total transmission for left incidence, while it is close to zero for right incidence. We stress that nonreciprocity in transmission is a property of our all order nonlinear $\mathcal{PT}$-symmetric system and is absent in linear transmission or even in transmission through Kerr-nonlinear systems. We highlight the application potentials of this optical diode action with near-perfect isolation and bistability in logic and memory devices.