BLOW-UP SCALING AND GLOBAL BEHAVIOUR OF SOLUTIONS OF THE BI-LAPLACE EQUATION VIA PENCIL OPERATORS

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Abstract. As the main problem, the bi-Laplace equation
\[ \Delta^2 u = 0 \quad (\Delta = D_x^2 + D_y^2) \]
in a bounded domain \( \Omega \subset \mathbb{R}^2 \), with inhomogeneous Dirichlet or Navier-type conditions on the smooth boundary \( \partial \Omega \) is considered. In addition, there is a finite collection of curves
\[ \Gamma = \Gamma_1 \cup \ldots \cup \Gamma_m \subset \Omega, \]
on which we assume homogeneous Dirichlet conditions \( u = 0 \), focusing at the origin \( 0 \in \Omega \) (the analysis would be similar for any other point). This makes the above elliptic problem overdetermined. Possible types of the behaviour of solution \( u(x, y) \) at the tip 0 of such admissible multiple cracks, being a singularity point, are described, on the basis of blow-up scaling techniques and spectral theory of pencils of non self-adjoint operators. Typical types of admissible cracks are shown to be governed by nodal sets of a countable family of harmonic polynomials, which are now represented as pencil eigenfunctions, instead of their classical representation via a standard Sturm–Liouville problem. Eventually, for a fixed admissible crack formation at the origin, this allows us to describe all boundary data, which can generate such a blow-up crack structure. In particular, it is shown how the co-dimension of this data set increases with the number of asymptotically straight-line cracks focusing at 0.

1. Introduction.

1.1. Models and preliminaries. In this work we intend to ascertain the behaviour of the solutions of the bi-Laplace equation with Dirichlet boundary conditions in a bounded smooth domain \( \Omega \subset \mathbb{R}^2 \)
\[ \begin{cases} 
\Delta^2 u = 0 & \text{in } \Omega, \\
u = f(x, y) & \text{on } \Gamma, \\
u = g(x, y), \quad \frac{\partial u}{\partial n} = h(x, y) & \text{on } \partial \Omega, 
\end{cases} \quad (1.1) \]

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where $\Delta = D_x^2 + D_y^2$ is the standard Laplace operator in $\mathbb{R}^2$, $n$ stands for the unit outward normal to $\partial \Omega$, and $f$, $g$ and $h$ are given smooth functions in $\Omega$, so that $g^2(x, y) + h^2(x, y) \neq 0$. In our particular case, $\Omega$ is assumed to have, what we refer to as a multiple crack $\Gamma$ which is composed of a finite collection of $m \geq 1$ curves (to be described below)

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_m \subset \Omega$$

such that each $\Gamma_j$ passes through the origin $0 \in \Omega$.  

(1.2)

The origin is then the tip of this crack. Indeed, in the present research, we assume that, near the origin, in the lower half-plane $\{y < 0\}$ (similarly in the upper half-plane $\{y > 0\}$), all cracks asymptotically take a straight line form, i.e. as shown in Figure 1,

$$\Gamma_k : x = \alpha_k(-y)(1 + o(1)), \quad y \to 0, \quad k = 1, 2, \ldots, m, \text{ where } \alpha_1 < \alpha_2 < \ldots < \alpha_m$$

(1.3)

are given constants. Basically we are choosing cracks of the type described by (1.3) that will allow us to obtain possible types of the behaviour of solutions $u(x, y)$ of the equation (1.1). Indeed, a posteriori the analysis carried out throughout this paper will show that those types of cracks are the only admissible ones. For further extensions a different analysis must be done. Thus, the precise statement of the problem assumes that geometrical conditions such as (1.3) describe all the admissible cracks near the origin, i.e. no other straight-line cracks are considered.

Moreover, in our basic model, we assume homogeneous Dirichlet conditions on the crack for the bi-Laplacian problem (1.1)

$$u = 0 \quad \text{on} \quad \Gamma,$$

(1.4)

that makes the problem overdetermined, so that only some types of such multiple cracks (1.2), (1.3) are admissible. Note that, since we will obtain an explicit expression for the solutions of the bi-Laplace problem (1.1), a posteriori we will be able to impose any condition on the crack $\Gamma$.

Thus, our main goal is to describe all possible types of admissible multiple cracks, for which the above elliptic problem can have a solution, at least for some boundary data $g$ and $h$ in (1.1). To this end, actually, we need to describe all types of zero or nodal sets, which are admitted by oscillatory solutions $u(x, y)$ of the bi-Laplace equation.

Therefore, performing a proper rescaling and using non-self-adjoint spectral pencil operator theory (see [10, 11, 12] and Section 2 in this paper for further details about these types of operators), we are able to show special linear combinations of “harmonic polynomials” ascertaining important qualitative information about the behaviour of the solution based on the nodal set of harmonic polynomials, especially close to the tip of the crack $\Gamma$ for which we will describe all the admissible types of cracks. Specifically, we show that their nodal sets play a key role in the general multiple crack problem for various equations. In fact, the analysis is extended, as an example to future improvements, to a couple of non-linear problems, and intends to provide an alternative methodology in the analysis of similar problems with singularity points on the boundary.

Consequently, the novelty of this work consists in approaching the study of some boundary value problems, exhibiting a crack singularity, by a “blow-up” and “elliptic evolution” approach using the spectral theory of pencils of non self-adjoint operators.
In fact we base our analysis on the application of the spectral theory of pencil operators, transforming the problem appropriately and, hence, reducing it to solve a 1D spectral problem. We also believe that the analysis presented here can be extended to other problems providing a different technique to obtain important qualitative information; see for instance [3] for an application of the techniques presented here for a $p$-Laplacian problem.

Note also that, nowadays, this kind of technique is normally used for parabolic or hyperbolic equations, not elliptic (cf. [2, 8]). On the other hand, it is necessary to recall that the pioneering Kondratiev’s study in the 1960s [10, 11] of boundary regularity/asymptotics for general linear elliptic (and ultra-parabolic) equations was, for the first time, performed via an “elliptic evolution approach”, with all typical features available: suitable blow-up scaling at a boundary point, operator pencil analysis, etc.

In particular, Kondratiev applied those blow-up scalings to analyse non-smooth domains, such as domains with corner points, edges, etc, on the boundary, forming cone-type domains (boundary shape like a cone). After some variable transformations he was able to transform the equation into a pencil spectral problem permitting the analysis at those corner points.

In this work, we have just used the fundamental ideas of Kondratiev, performing a different change of variable and, hence, obtaining a different pencil of non-self-adjoint operator that allows us to ascertain the behaviour of the solutions for that multiple crack section (1.3). However, our cone is in the interior and, actually with no connection with the boundary (the tip of the crack is not on the boundary) while Kondratiev analysed problems in cone-type domains on the boundary.

Thus, as we will show below we obtained completely different functional spaces and solutions with a specific polynomial form which provide us with the behaviour at the tip of the cracks. Indeed, we will show that this internal crack problem requires polynomial eigenfunctions of different pencils of linear operators, which were not under scrutiny in Kondratiev’s works.

Those arguments were used in [5] for a case where the boundary, after blow-up scalings, is at infinity.

![Figure 1. One-crack model.](image-url)
1.2. Approach and main results. In the study of such admissible cracks, i.e., the behaviour of the solutions when \((x, y) \to (0, 0)\), it suffices to consider
\[ D = B_1 \setminus \Gamma, \quad \text{a unit ball in } \mathbb{R}^2 \text{ centered at the origin } 0 \text{ minus the crack } \Gamma. \tag{1.5} \]
For other, not pointwise blow-up estimates, we continue to consider general smooth domains \(\Omega\).

Thus, even though our main motivation to develop this work was the analysis of the bi-Laplace equation (1.1), we shall start with similar multiple crack issues for the Laplacian. Since the bi-Laplace operator is the iteration of two Laplacians, inevitably, we will need to start the analysis of the problem by using the pure single Laplacian
\[ \Delta u = 0, \quad \text{in } \Omega \text{ (}= B_1), \quad u = f (\neq 0) \text{ on } \partial \Omega, \quad u = 0 \text{ on } \Gamma, \tag{1.6} \]
to obtain those results. Note that here we have shifted \(f\) on the boundary with respect to the problem (1.1).

Additionally, we shall complete our work with the study of several other problems as well. These problems are going to be defined again under the geometrical condition (1.3) but considering various different operators, such as other semilinear related equations.

First step. Laplace equation with multiple cracks. As a by-product of our approach, we consider the problem for the Laplace equation (1.6). For this simpler problem in Section 3, we prove that all the solutions with cracks at 0 must satisfy
\[ u(x, y) = w(z, \tau) = \sum_{k \geq l} e^{-k\tau}[c_k \psi^*_{k,1}(z) + d_k \psi^*_{k-1,2}(z)], \quad \text{with } c_l^2 + d_l^2 \neq 0, \tag{1.7} \]
where
\[ z = x/(-y) \quad \text{and} \quad \tau = -\ln(-y) \quad \text{for } y < 0, \tag{1.8} \]
and, such that, they have a polynomial expression
\[ \psi^*_l(z) = \sum_{k=l, l-2, \ldots, 0} a_k z^k \quad (a_l = 1). \]
Rescaling (1.8) corresponds to a blow-up scaling near the origin, moving the singularity point at the origin into an asymptotic convergence when \(\tau\) goes to infinity. Moreover, we will keep this “blow-up scaling logic” for the rest of other similar problems to appear. Also, this scaling approach could be extended to \(y > 0\) in a similar way.

Note that we choose such an expression depending on \(z\) for convenience since, as it is shown later in Section 3, we deal with eigenfunctions of a quadratic pencil of operators and not with a standard Sturm–Liouville problem.

Indeed, after performing the rescaling (1.8) (and then the separation variables method) we transform the Laplace equation into a pencil of non-self-adjoint operator, in particular for this case, a quadratic pencil operator of the form
\[ (1 + z^2)(\psi^*)'' + 2(\lambda + 1)z(\psi^*)' + \lambda(\lambda + 1)\psi^* = 0. \]
Therefore, in the main result of Section 3, Theorem 3.2, we prove that the coefficients \( c_l, d_l \in \mathbb{R} \) in (1.7) are arbitrary constants that satisfy
\[
c_l^2 + d_l^2 \neq 0.
\]
Indeed, we observe in the first leading terms while approaching the origin, a linear combination of those two families of eigenfunctions as classic harmonic polynomials.

On the other hand, it is also proved that, if all \( \{ \alpha_k \} \) in (1.3) do not coincide with all \( m \) subsequent zeros of any nontrivial linear combination
\[
c_l \psi_{1,1}^*(z) + d_l \psi_{1,2}^*(z), \quad \text{with} \quad c_l^2 + d_l^2 \neq 0,
\]
then the multiple crack problem (1.6) cannot have a solution for any boundary Dirichlet data \( f \) on \( \partial \Omega \).

However, a solution exists if for some \( l \) the zero condition is satisfied, i.e. \( \alpha_l \) coincides with the zeros of the linear combination of harmonic polynomials (1.9), and
\[
|u(x, y)| = O(|(x, y)|^l) \quad \text{as} \quad (x, y) \to (0, 0).
\]

Moreover and obviously, restricting to \( \Gamma \) all types of admissible crack-containing expansions (1.7) (with closure in any appropriate functional space), fully describes all types of boundary data, which lead to the desired crack formation at the origin \( a \) posteriori. The previous discussion is summarised in Theorem 3.2.

Although, one can assume the function \( u(x, y) \) in (1.7) belonging to the Sobolev space \( W^{1,2}(\Omega \setminus \Gamma) = H^1(\Omega \setminus \Gamma) \), due to the expansions considered (for the Laplace problem (1.6) and also for the bi-Laplace \( W^{2,2} \)) in our analysis we actually have that the eigenfunctions are harmonic functions of Hermite-type polynomials which are complete in any appropriate \( H_\rho^1 \) or \( L_\rho^p \)-space, where the weight \( \rho \) has an exponential decay at infinite. For example
\[
\rho(z) \sim e^{-az^2} \quad \text{or} \quad e^{-a|z|}, \quad a > 0 \text{ small},
\]
would be enough; see [9] for further details about this classical analysis.

**Remark 1.1.** Furthermore, it follows from (1.9) that any admissible crack distribution governed by zeros of the polynomial (1.9), i.e. the nodal set of the polynomial eigenfunctions, represented by pencil eigenfunctions instead of the classical one from the Sturm–Liouville problem, for any \( l = 1, 2, ... \), contains a single free parameter (say, \( c_l \in \mathbb{R}, c_l \neq 0 \)).

In other words, the whole set of admissible multiple straight-line crack formations (1.3) (and basically the reason we assume such a family of cracks) comprises no more than a *countable family of one-dimensional subsets*\(^1\) (recall that this is true for arbitrary Dirichlet data \( f(x) \) on \( \partial \Omega \), which, as we mentioned above, \( a \) posteriori, can be completely described).

**Extensions to semi-linear equations.** Although not the purpose of this paper, using a couple of examples
\[
\Delta u + |u|^{p-1}u = 0 \quad \text{and} \quad \Delta u + \frac{|u|^{p-1}u}{x^2 + y^2} = 0, \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \quad \text{where} \quad p > 1,
\]
(1.10)

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\(^1\)Bearing in mind the rotational invariance of the Laplace operator (as a one-dimensional group of orthogonal transformations in \( \mathbb{R}^2 \)), the total family of *essentially distinct* admissible crack configurations becomes no more than a *countable subset*, which is described by subsequent zeros of the harmonic polynomials. At the same time, general Dirichlet data \( f(x, y) \) on \( \Omega \setminus \Gamma \) is obviously characterised as an uncountable subset.
via similar scalings and asymptotic analysis, we are able to show the types of decay patterns at the origin showing an application of the results obtained for the Laplace problem (1.6) to semi-linear equations. We just use those two equations as examples of what could happen for non-linear problems.

Indeed, for the first equation we find that the nonlinear term is negligible, i.e. it cannot affect those patterns. We will show that performing the same rescaling (1.8) as for the Laplace equation (1.6) will lead to an exponentially small perturbation in \( \tau \) of the rescaled Laplacian one. Hence, when \( \tau \) goes to infinity (this means, due to the rescaling, close to the origin for the variables \((x, y)\)) the nonlinearity does not have any effect for the decay patterns at infinity (or the origin).

However, for the second equation, for which we involve the nonlinearity in a formation of multiple zeros at the origin we show through some numerical analysis, the nodal sets of several nonlinear eigenfunctions obtained after the rescaling.

Note that we just use those two equations as examples for the asymptotic behaviour at infinity, after the rescaling. For those purposes we only need to consider that \( p > 1 \).

Finally, the bi-Laplace equation with multiple cracks. Obviously, since \( \Delta^2 = \Delta \Delta \), the solutions of the Laplace equation also solve the bi-Laplace. Therefore, some of the results obtained for the Laplacian (see Section 3 for further details) can be translated and applied to (1.1) with the same crack constraints, though, nevertheless, the latter one is more demanding. Indeed, a full description of admissible multiple crack configurations (1.3) for (1.1) is more difficult.

For the problem (1.1) we find that the solutions have an expression of the form

\[
u(x, y) = w(z, \tau)
= \sum_{(k \geq l)} e^{-k\tau} [C_k \psi^*_l(z) + D_k \psi^*_{l-1,2}(z) + E_k \psi^*_{l-2,3}(z) + F_k \psi^*_{l-3,4}(z)],
\]

(1.11)

where, again, we use the same scaling (1.8) and

\[\psi^* = \{\psi^*_{l,1}, \psi^*_{l,2}, \psi^*_{l,3}, \psi^*_{l,4}\},\]

are four harmonic polynomial eigenfunction families, with the same \( z \)-representation, which are complete in any reasonable weighted \( L^2 \) space such that

\[\psi^*_{l,1}(z) \equiv \psi^*_{l,1}(z), \quad \psi^*_{l,2}(z) \equiv \psi^*_{l-1,2}(z), \quad \psi^*_{l,3}(z) \equiv \psi^*_{l-2,3}(z), \quad \psi^*_{l,4}(z) \equiv \psi^*_{l-3,4}(z),\]

now associated with four families of negative eigenvalues

\[\lambda_{l,1} = -l, \quad l = 1, 2, 3, ..., \quad \lambda_{l,2} = -l - 1, \quad l = 0, 1, 2, 3, ..., \quad \lambda_{l,3} = -l - 2, \quad l = 0, 1, 2, 3, ..., \quad \lambda_{l,4} = -l - 3, \quad l = 0, 1, 2, 3, ...,\]

of the corresponding pencil operator

\[F^*_l \psi^* \equiv \{(\lambda^4 + 6\lambda^3 + 11\lambda^2 + 6\lambda)I + 4(\lambda^3 + 6\lambda^2 + 11\lambda)zD_z + 2(1 + 3z^2)(\lambda^2 + 5\lambda)D_z^2 + 4\lambda(1 + z^2)zD_z^3 - C^*\} \psi^* = 0.\]

Note that again we use such non-standard notations of harmonic polynomials in order to fit our operator pencil approach.

Thus, four collections of expansion coefficients \( \{C_k\}, \{D_k\}, \{E_k\}, \{F_k\} \) (which depend on boundary data on \( \partial \Omega \)) admitted all types of cracks at 0, which is obviously described via (1.11) take place so that if all \( \{\alpha_k\} \) of the multiple
cracks (1.3) do not coincide with all \( m \) subsequent zeros of any nontrivial linear combination
\[
C_1\psi_{1,1}^*(z) + D_1\psi_{1,2}^*(z) + E_1\psi_{1,3}^*(z) + F_1\psi_{1,4}^*(z), \quad \text{with} \quad C_1^2 + D_1^2 + E_1^2 + F_1^2 \neq 0,
\]
then the multiple crack problem (1.1) cannot have a solution for any boundary Dirichlet data \( g, h \) on \( \partial\Omega \) for the problem (1.1).

**Remark 1.2.** The linear combinations previously shown arise naturally from the spectral theory of the operators involved (Laplacian, bi-Laplacian). Indeed, for the Laplacian since \( u \) is harmonic it can be decomposed as a sum of homogeneous harmonic functions. In particular, for the Laplacian operators we obtain two families of eigenfunctions denoted by
\[
\{\psi_{k,1}^*\}, \quad \{\psi_{k-1,2}^*\}.
\]
The difficulty in ascertaining here the results at the tip of the cracks comes from the regularity problem we are facing here in \( \Omega = B_1 \), for the eigenvalue problem, since we have a singularity point at the origin. However, we are in the context analysed in [13] so that
\[
w(z, \tau) = e^{\lambda \tau} \psi^*(z), \quad \text{where} \quad \text{Re} \lambda < 0,
\]
and for an orthonormal basis \( \{\psi_{k,1}^*, \psi_{k-1,2}^*\} \) of harmonic polynomial eigenfunctions, we find that the solutions of the problem (1.6) are a decomposition of the form (1.7). We conclude similar arguments for the bi-Laplacian problem (1.1).

**Remark 1.3.** We would like to explain that the rescaling introduced in this paper (1.8) allows us to get solutions of the polynomial form (1.7) and (1.11) as special linear combinations of “harmonic polynomials” that show the behaviour of the problems considered here at the tip of the cracks depending on the nodal set of those “harmonic polynomials”. Indeed, our rescaling (1.8) since it is different from the one used by Kondratiev in [10, 11] produces a different pencil operator and, hence, different eigenfunctions for the corresponding pencil non-self-adjoint operators. However, the expansions (1.7) and (1.11) based on those polynomial eigenfunctions are capable of providing the behaviour of the solutions at the tip of the cracks.

**Remark 1.4.** As one of the referees of this paper pointed out for the problems under analysis in this paper (Laplace, class of perturbations of the Laplace and Bi-Laplac) solutions with a “strong zero” (\( |u(x)| = O(|x|^N) \), for any \( N > 0 \)) do not exist. Indeed, results on Unique Continuation guarantee that any solution with a strong zero is identically zero.

**Some further extensions.** Though our approach is done in two dimensions, the scaling blow-up approach applies to \( \Omega \) in \( \mathbb{R}^3 \) (or any \( \mathbb{R}^N \)), where spherical polynomials naturally occur such that their nodal sets (finite combination of nodal surfaces) of their linear combinations, as above, describe all possible local structures of cracks concentrating at the origin. However, if \( N > 2 \) the possible geometry of the crack \( \Gamma \) is far richer.

Additionally, though very difficult to achieve, one can study similar crack problems for other elliptic equations such as
\[
u_{xxxx} + u_{yyyy} = 0,
\]
where non-standard harmonic-like polynomials naturally occur, such as eigenfunctions of a quartic operator pencil, like the ones we obtained for the bi-Laplace equation (1.1).

Further extensions to quasilinear equations, but out of the scope of this paper, such as the quasilinear $p$-Laplace equation

$$\Delta_p u \equiv \nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0,$$

(1.12)
could be also carried out applying this analysis; see [3]. However, in this particular case after performing a similar blow-up scaling one arrives at a nonlinear eigenvalue problem. Since this problem is nonlinear and the ideas used for the linear case, when $p = 2$, cannot be applied directly, the corresponding nonlinear eigenfunctions of the nonlinear pencil for (1.12) should be obtained by branching from harmonic polynomials as eigenfunctions of the quadratic pencil that occurs for the Laplacian in (1.6).

The main reason is due to the fact that for the Laplace equation (1.6), as explained above, it is possible to explicitly obtain two families of negative eigenvalues associated, respectively, to two families of eigenfunctions. Indeed, from the expressions of the two families of eigenvalues and using the pencil operator theory, we can ascertain the coefficients of every eigenfunction explicitly as well. However, following the same argument for the nonlinear PDE (1.12) it is not possible, in general, to get the corresponding families of eigenvalues and, hence, the associated eigenfunctions. Therefore, the branching argument used in [3] is applied to get such information about the eigenfunctions, and hence eigenvalues of a nonlinear problem.

2. Pencils of linear operators: preliminaries. As one of the main tools we are using in this work to get to the results and in a direct connection with our blow-up evolution approach we introduce the Theory of Pencil Operators. Let us mention that pencil operator theory appeared and was crucially used in the regularity and asymptotic analysis of elliptic problems in a seminal paper by Kondratiev [11] and also for parabolic problems in [10], where spectral problems, that are nonlinear (polynomial) in the spectral parameter $\lambda$, occurred. Later on, Mark Krein and Heinz Langer [12] made a fundamental contribution to this theory analysing the spectral theory for strongly damped quadratic operator pencils. In general, a polynomial pencil operator is denoted by

$$A(\lambda) := A_0 + \lambda A_1 + \cdots + \lambda^n A_n,$$

(2.1)

where $\lambda \in \mathbb{C}$ is a spectral parameter and $A_i$, with $i = 0, 1, \cdots, n$, are linear operators acting on a Hilbert space $X$ (here we might assume for example that $X$ can be $H^1_\rho$ or $L^2_\rho$ with any reasonable weight $\rho$). Operators of the form (2.1) are sometimes called Polynomial matrix when the linear differential operators $A_i$ are matrices. A linear pencil of operators has the form

$$A(\lambda) := A - \lambda B,$$

where $A, B$ are two linear operators. In the simplest case, we have the linear pencil operator

$$A(\lambda) = A - \lambda \text{Id}, \quad \text{or} \quad A(\lambda) = \text{Id} - \lambda A,$$

which represents the usual (standard) linear spectral problems. A clear difference between those spectral linear problems and the pencil operators is essentially that,
for the simplest pencil operators, the set of eigenvalues is obtained as the roots of the characteristic equation
\[ \det A(\lambda) = 0, \]
i.e. powers of the values \( \lambda_k \), with the basis of the eigenspace as
\[ \{ \psi_k, \lambda_k \psi_k, \cdots, \lambda_k^{n-1} \psi_k \}. \]

Furthermore, the analysis of polynomial pencil operators has been under scrutiny for many years in order to study spectral problems of the form (2.1) and, as pointed out by Markus [14], arise naturally in diverse areas of mathematical physics (differential equations and boundary value problems), with applications to Elasticity, Hydrodynamics problems, among other things. In the pioneering work of M.V. Keldysh in 1951 (earlier first ever results of J.D. Tamarkin’s PhD Thesis of the 1917 should be mentioned as well; see Markus [14] for this amazing part of the history of mathematics) pencils, including multiplicity results and completeness of the set of eigenvectors, even for non-self-adjoint operators, were thoroughly analysed.

As mentioned at the beginning of this section, one of the most important contributions, and related with the analysis carried out here, was made by Krein & Langer [12] who developed further approaches for quadratic pencil operators of the form
\[ A(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0. \]
In this paper, we will use elements of this well-developed spectral theory of non-self-adjoint quadratic or of fourth order pencil polynomials, though not that profound ones, since, for linear elliptic problems, we always deal with polynomial (harmonic) eigenfunctions, which cause no problem concerning their completeness, closure, and further functional properties.

3. The Laplace equation: crack distribution via nodal sets of transformed harmonic polynomials. Since \( \Delta^2 = \Delta \Delta \), we inevitably should first consider the multiple crack problem for the Laplace equation (1.6). Obviously, these admissible crack distributions remain valid for the bi-Laplace (1.1), (1.4), but, in addition, there are other types of such “singularities” at the origin; see Section 4.1.

Recall again that, also, in representing our pencil approach (to be used later on for other linear and nonlinear elliptic problems), in dealing with the classic Laplace equation, we will re-discover several standard and well-known facts from any textbook on linear operators. However, what is more important, the general structure of such an approach will proceed in what follows.

3.1. Blow-up scaling and rescaled equation. First we show the required transformations with which we will obtain the pencil operators that eventually will provide us with the behaviour of the solutions at the tip of the crack for the Laplace problem (1.6).

Thus, assuming the crack configuration as in Figure 1, we introduce the following rescaled variables, corresponding to “blow-up” scaling near the origin 0:
\[ u(x,y) = w(z,\tau), \quad \text{with} \quad z = x/(-y) \quad \text{and} \quad \tau = -\ln(-y) \quad \text{for} \quad y < 0, \quad (3.1) \]
to get the rescaled operator
\[ \Delta_{(x,y)} u = e^{2\tau} [D^2_{\tau} + D_{\tau} + 2zD^2_{z\tau} + (1 + z^2)D^2_z + 2zD_z] w \equiv \Delta_{(z,\tau)} w. \quad (3.2) \]
Therefore, we arrive at the equation
\[ w_{\tau\tau} + w_\tau + 2zw_{z\tau} = A^*w \equiv -(1 + z^2)w_{zz} - 2zw_z. \quad (3.3) \]
Remark 3.1. Note that $A^*$ is symmetric in the standard (dual) metric of $L^2(\mathbb{R})$, 

$$A^* \equiv -D_z[(1 + z^2)D_z],$$

though we are not going to use this. Indeed, for our crack purposes, we do not need eigenfunctions of the “adjoint” pencil, since we are not going to use eigenfunction expansions of solutions of the PDE (3.3), where bi-orthogonal basis could naturally be wanted.

This blow-up analysis of (3.3) assumes a kind of “elliptic evolution” approach for elliptic problems, which is not well-posed in the Hadamard’s sense, but, in fact, can trace out the behaviour of necessary global orbits that reach and eventually decay to the singularity point $(z, \tau) = (0, +\infty)$.

By the crack condition (1.4), we look for vanishing solutions: in the mean and uniformly on compact subsets in $z$,

$$w(z, \tau) \rightarrow 0 \quad \text{as} \quad \tau \rightarrow +\infty. \quad (3.5)$$

Therefore, under the rescaling (3.1), we have converted the singularity point at 0 into an asymptotic convergence when $\tau \rightarrow \infty$.

Hence, we are forced to describe a very thin family of solutions for which we will describe their possible nodal sets to settle the multiple crack condition in (1.6). This corresponds to Kondratiev’s “evolution” approach [10, 11] of 1966, though it was there directed to different boundary point regularity (and asymptotic expansions) questions, while the current crack problem assumes studying the behaviour at an internal point $0 \in \Omega$ such as the tip of the multiple crack under consideration. We will show first that this internal crack problem requires polynomial eigenfunctions of different pencils of linear operators, which were not under scrutiny in Kondratiev’s works. Indeed, in doing so, we will “re-discover” classic harmonic polynomials, which will play a key role. Therefore, in studying such classical objects, we could omit many technical details, but, anyway, prefer to keep some of them for the sake of comparison and general logic.

3.2. Quadratic pencil and its polynomial eigenfunctions. Now, we ascertain the quadratic pencil operator associated with equation (3.3). So that, looking, as usual in linear PDE theory, for solutions of (3.3) in separate variables

$$w(z, \tau) = e^{\lambda \tau} \psi^*(z), \quad \text{where} \quad \text{Re} \lambda < 0 \quad \text{by (3.5)}, \quad (3.6)$$

yields the eigenvalue problem for a quadratic pencil of non self-adjoint operators,

$$B_\lambda \psi^* \equiv \{\lambda(\lambda + 1)I + 2azD_z - A^*\} \psi^* = 0$$

or

$$(1 + z^2)(\psi^*)'' + 2z(\psi^*)' + \lambda(\lambda + 1)\psi^* = 0. \quad (3.7)$$

Remark 3.2. The second-order operator $A^*$ is singular at the infinite points $z = \pm \infty$, so this is a singular quadratic pencil eigenvalue problem. Since the linear first-order operator in (3.7), $zD_z$, is not symmetric in $L^2$, we are not obliged to attach the whole operator to any particular functional space. Therefore, the behaviour as $z \rightarrow \infty$ is not that crucial, and any $L^2_\rho$-space setting with $\rho(z) \sim e^{-a|z|}$ (or $e^{-a|z|}$), $a > 0$ small, would be enough.

Indeed, if the solution of the problem (1.6) is smooth in certain weighted spaces $H^1_\rho$ or $L^2_\rho$, we claim that, then, the eigenfunctions $\psi^*$ of the operator (3.7) are also analytic at infinity.
Remark 3.3. The differential part in (3.7) can be reduced to a symmetric form in a weighted $L^2_{\rho_\lambda}$-metric:

$$(1 + z^2)D^2_z + 2(\lambda + 1)zD_z \equiv (1 + z^2)\frac{1}{\rho_\lambda}D_z(\rho_\lambda D_z), \quad \text{where} \quad \rho_\lambda = (1 + z^2)^{\lambda + 1}. \quad (3.8)$$

Note that this weighted metric has an essential dependence on the a priori unknown eigenvalues.

Furthermore, we cannot forget that once the rescaling (3.1) is performed, these eigenfunctions of the quadratic pencil operator (3.7) are actually harmonic polynomials (just introducing the variables (3.1)). Then, as usual in orthogonal polynomial theory, we can now state the following property for the eigenfunctions of the adjoint pencil (3.7) with respect to its family of eigenvalues that will be determined below; see [4, 6] for details about this Sturm–Liouville Theory as well as Subsection 3.6 below.

**Proposition 3.1.** The only acceptable eigenfunctions of the adjoint pencil (3.7) are finite polynomials.

Although, it is well known that these eigenfunctions of the quadratic pencil operator (3.7) are finite polynomials, it should be pointed out that this is associated with the interior elliptic regularity.

Indeed, the blow-up approach under the rescaling (3.1) just specifies local structure of multiple zeros of analytic functions at 0, and since all of them are finite (we are assuming (1.3) with a finite number of cracks) we must have finite polynomials only.

Of course, there are other formal eigenfunctions (we will present an example; see (3.16) below), but those, in the limit as $\tau \to +\infty$ in (3.6), lead to non-analytic (or even discontinuous) solutions $u(x, y)$ at 0, that are non-existent.

On the other hand, since our pencil approach, currently, is nothing more than rewriting via scaling the standard Sturm–Liouville eigenvalue problem for harmonic polynomials (see Subsection 3.6), it is quite natural to deal with nothing other than them, which, thus, should be re-built in terms of the scaling variable $z$.

Moreover, the next lemma shows the corresponding point spectrum of the pencil (3.7).

**Lemma 3.1.** The quadratic pencil operator (3.7) admits two families of eigenfunctions

$$\psi^*_l(z) \equiv \psi^*_l(z) \quad \text{and} \quad \psi^*_l(z) \equiv \psi^*_{l-2}(z), \quad \text{for any} \quad l = m, m + 1, \ldots$$

associated with two corresponding families of eigenvalues

$$\lambda^+_l = -l, \quad l = 1, 2, 3, \ldots \quad \text{and} \quad \lambda^-_l = -l - 1, \quad l = 0, 1, 2, 3, \ldots, \quad (3.9)$$

**Proof.** In order to find the corresponding point spectrum of the pencil we look for $l$th-order polynomial eigenfunctions of the form

$$\psi^*_l(z) = z^l + a_{l-2}z^{l-2} + a_{l-4}z^{l-4} + \ldots = \sum_{k=l-2, \ldots, 0} a_kz^k, \quad (a_l = 1), \quad (3.10)$$

that we already know they are harmonic polynomials. Substituting (3.10) into (3.7) and evaluating the higher order terms yields the following quadratic equation for eigenvalues:

$$O(z) : \quad \lambda^2 + (2l + 1)\lambda + l(l + 1) = 0. \quad (3.11)$$
Solving this characteristic equation yields the two families of real negative eigenvalues under the expression (3.9) associated with two families of eigenfunctions denoted by
\[ \psi_{l,1}^*(z) \equiv \psi_{l,1}^*(z) \quad \text{and} \quad \psi_{l,2}^*(z) \equiv \psi_{l-1,2}^*(z), \quad \text{for any} \quad l = m, m + 1, \ldots, (3.12) \]
for convenience.

The next result calculates those (re-structured harmonic) polynomials (3.10), (3.12) as the corresponding eigenfunctions of the pencil.

**Theorem 3.1.** The quadratic pencil (3.7) has two (admissible) discrete spectra (3.9) of real negative eigenvalues with the finite polynomial eigenfunctions given by (3.10), where the expansion coefficients satisfy a finite Kummer-type recursion corresponding to the operator in (3.7):

\[
\begin{align*}
   a_{k+2} &= -\frac{k(k-1)+2(\lambda_l^+ + 1)k + \lambda_l^+(\lambda_l^+ + 1)}{(k+2)(k+1)} a_k, \quad \text{for any} \quad k = l, l-2, \ldots, 2, \\
   a_1 &= -\frac{2(\lambda_l^+ + 1) + \lambda_l^+(\lambda_l^+ + 1)}{2(\lambda_l^+ + 1) + \lambda_l^+(\lambda_l^+ + 1)} a_0, \quad \text{and} \quad a_0 = \frac{2}{\lambda_l^+(\lambda_l^+ + 1)} a_2. \\
\end{align*}
\] (3.13)

**Proof.** It is clear by (3.9) that the quadratic pencil (3.7) has two discrete spectra of real negative eigenvalues with two families of finite polynomial eigenfunctions\(^2\) \{\psi_{l,1}^*(z)\}, \{\psi_{l,2}^*(z)\}, such that \(\psi_{l,1}^*(z) = \psi_{l,1}^*(z)\) and \(\psi_{l,2}^*(z) = \psi_{l,1,2}^*(z)\), given by (3.10) and corresponding associated with the two families of eigenvalues \(\lambda_l^+\) and \(\lambda_l^-\). Substituting \(\psi_l^* = \sum_{k=0}^{l} a_k z^k\), for any \(l \geq 0\), into (3.7) we find that, for any \(\lambda\),

\[
(1 + z^2) \sum_{k \geq 2} k(k-1)a_k z^{k-2} + 2(\lambda + 1) \sum_{k \geq 1} k a_k z^k + \lambda(\lambda + 1) \sum_{k \geq 0} a_k z^k = 0,
\]
and hence,

\[
\sum_{k \geq 2} [(k+2)(k+1)a_{k+2} + k(k-1)a_k + 2(\lambda + 1)ka_k + \lambda(\lambda + 1)a_k] z^k + [6a_3 + 2(\lambda + 1) + \lambda(\lambda + 1)]a_1 z + 2a_2 + \lambda(\lambda + 1)a_0 = 0.
\] (3.14)

Therefore, evaluating the coefficients we find that

\[
\begin{align*}
   (k+2)(k+1)a_{k+2} + k(k-1)a_k + 2(\lambda + 1)ka_k + \lambda(\lambda + 1)a_k &= 0, \\
   k &= l, l-2, \ldots, 2, \\
   6a_3 + [2(\lambda + 1) + \lambda(\lambda + 1)]a_1 &= 0, \\
   2a_2 + \lambda(\lambda + 1)a_0 &= 0,
\end{align*}
\]
and we arrive at (3.13), completing the proof. \(\square\)

**Remark 3.4.** Alternatively, we also have that

\[
a_{l-2n} = -\frac{(l-2n+2)(l-2n+1)}{(l-2n)(l-2n-1) + 2(2\lambda_l^+ + 1)(l-2n) + \lambda_l^+(\lambda_l^+ + 1)}
\]
\[
a_{l-2n+2}, \quad n = 1, 2, \ldots, \left\lfloor \frac{l}{2} \right\rfloor; \quad a_l = 1.
\]

\(^2\)Note that, within this pencil ideology, the eigenfunctions are ordered in an unusual manner, unlike the standard harmonic polynomials.
Note that even when discrete spectra coincide excluding the first eigenvalue \( \lambda_l^- \), and, more precisely,

\[
\lambda_l^- = \lambda_l^+ - 1 = \lambda_{l+1}^+ \quad l = 1, 2, 3, \ldots ,
\]

we still have two different families of eigenfunctions. For future convenience and applications for the crack problem for \( m = 1, 2, 3, \) and \( 4 \) (with \( m = l \)), we present the first four eigenvalue-eigenfunction pairs of both families of eigenfunctions for the pencil \((3.7)\), which now are ordered with respect to \( \lambda = -l, l = 0, 1, 2, \ldots \):

\[
\begin{align*}
\lambda_0 &= 0, \text{ with } \psi_0^*(z) = 1 \ (\neq 0); \\
\lambda_1 &= -1, \text{ with } \psi_{1,1}^*(z) = z, \quad \psi_{0,2}^*(z) = 1 \ (\neq 0); \\
\lambda_2 &= -2, \text{ with } \psi_{2,1}^*(z) = z^2 - 1, \quad \psi_{1,2}^*(z) = z; \\
\lambda_3 &= -3, \text{ with } \psi_{3,1}^*(z) = z^3 - 3z, \quad \psi_{2,2}^*(z) = 3z^2 - 1; \\
\lambda_4 &= -4, \text{ with } \psi_{4,1}^*(z) = z^4 - 6z^2 + 1, \quad \psi_{3,2}^*(z) = z^3 - z; \text{ etc.}
\end{align*}
\]

**Remark: about transversality.** These (harmonic) polynomials satisfy the Sturmian property (important for applications) in the sense that each polynomial \( \psi_m^*(z) \) has precisely \( m \) transversal zeros. For Hermite polynomials, this result was proved by Sturm already in 1836 [15]; see further historical comments in [7, Ch. 1].

**Remark: about analyticity.** Obviously, we exclude, in the first line of \((3.15)\), the first eigenfunction \( \psi_0^*(z) \equiv \psi_{0,1}^*(z) \equiv 1 \), since it does not vanish and has nothing to do with a multiple zero formation. However, for \( \lambda = 0 \) in \((3.7)\), there exists another obvious bounded analytic solution having a single zero:

\[
(1 + z^2)(\psi^*)'' + 2z(\psi^*)' = 0 \quad \Rightarrow \quad \tilde{\psi}^*(z) = \tan^{-1} z \to \pm \pi/2 \quad \text{as} \quad z \to \pm \infty.
\]

This \( \tilde{\psi}^*(z) \) belongs to any suitable \( L^2_{\rho} \)-space (of polynomials). However, it becomes irrelevant due to another regularity reason: passing to the limit in the corresponding expansion of \( w(x, y) \equiv w(y, \tau) \) \((3.6)\) as \( \tau \to +\infty \) \((y \to -0)\) yields the discontinuous limit sign \( x \), i.e. an impossible trace at \( y = 0 \) of any analytic solutions of the Laplace equation.

3.3. **Nonexistence result for crack problem.** Next we ascertain how the family of admissible cracks should lead to the existence of solutions for the crack problem \((1.6)\).

We have that sufficiently “ordinary” polynomials are always complete in any reasonable weighted \( L^2 \) space, to say nothing about the harmonic ones; see [9, p. 431]. Moreover, since our polynomials are not that different from standard harmonic (or Hermite) ones, this implies the completeness in such spaces. So that, sufficiently regular solutions of \((3.3)\) should admit the corresponding eigenfunction expansions over the polynomial family pair

\[
\Phi^* = \{ \psi_{1,1}^*, \psi_{-1,2}^* \},
\]

in the following sense. Bearing in mind two discrete spectra \((3.9)\), the general expansion has the form

\[
w(z, \tau) = \sum_{k \geq 0} e^{-k\tau} [c_k \psi_{k,1}^*(z) + d_k \psi_{k-1,2}^*(z)],
\]

where two collections of expansion coefficients \( \{ c_k \} \) and \( \{ d_k \} \), depending on boundary data on \( \Omega \), are presented.
We did not need to develop an “orthonormal theory” of our polynomials, which should specify the expansion coefficients in (3.17), for a given solution \( u(x, y) \) (though specifying all the coefficients declare the whole family of \( u \) with such cracks at 0). Indeed, dealing with orthonormal harmonic polynomials, we just have a standard expansion for harmonic functions, and obtain (3.17) by introducing the scaling blow-up variables (3.1).

Note that as mentioned in the introduction the linear combination (3.17) arises naturally from the spectral theory of the operator (in this case the Laplacian, later on the bi-Laplacian). Indeed, for the Laplacian \( u \) is harmonic in \( B_1 \setminus \Gamma \) and can be decomposed by homogeneous harmonic functions, here denoted by \( \psi_{k,1}^* \) and \( \psi_{k-1,2}^* \).

Even facing a difficult regularity problem in \( \Omega \setminus \Gamma \) (at the singularity boundary point) we are in the context analysed in [13], so that

\[
\psi(z, \tau) = e^{-k\tau} \psi^*(z),
\]

for an orthonormal basis \( \{\psi_{k,1}^*, \psi_{k-1,2}^*\} \) of Hermite-type polynomials eigenfunctions. Hence, we find that our solutions are decompositions of the form (3.17).

Moreover, in view of sufficient regularity of “elliptic orbits” (via standard interior elliptic regularity), such expansion is to converge not only in the mean (in \( L^2 \) with an exponentially decaying weight at infinity), but also uniformly on compact subsets. This allows us now to prove our result on nonexistence for the crack problem.

**Theorem 3.2.** Let the cracks \( \Gamma_1, \ldots, \Gamma_m \) in (1.2) be asymptotically given by \( m \) different straight lines (1.3). Then, the following hold:

(i) If all \( \{\alpha_k\} \) do not coincide with all \( m \) subsequent zeros of any non-trivial linear combination

\[
c_l \psi_{l,1}^*(z) + d_l \psi_{l-1,2}^*(z), \quad \text{with} \quad c_l^2 + d_l^2 \neq 0, \quad \text{where} \quad z = x/(-y),
\]

of two families of (re-written harmonic) polynomials \( \psi_{l,1}^*(z) \equiv \psi_{l,1}^*(z) \) and \( \psi_{l-1,2}^*(z) \equiv \psi_{l-1,2}^*(z) \) defined by (3.10), (3.13) for any \( l = m, m+1, \ldots \) and arbitrary constants \( c_l, d_l \in \mathbb{R} \), then the multiple crack problem (1.6) cannot have a solution for any boundary Dirichlet data \( f \) on \( \Omega \).

(ii) If, for some \( l \), the distribution of zeros in (i) holds and a solution \( u(x, y) \) exists, then

\[
|u(x, y)| = O(|x, y|^l) \quad \text{as} \quad (x, y) \to (0, 0).
\]

**Proof.** Condition (1.3) implies that the elliptic “evolution” problem while approaching the origin actually occurs on compact, arbitrarily large subsets for \( x/(-y) \equiv z \). Since we have converted the singularity point at \( (0, 0) \) into an asymptotic point when \( \tau \to \infty \).

Therefore, (3.17) gives all possible types of such a decay. Hence, choosing the first non-zero expansion coefficients \( c_l, d_l \) in (3.17), that satisfies \( c_l^2 + d_l^2 \neq 0 \), we obtain a sharp asymptotic behaviour of this solution

\[
w_l(y, \tau) = e^{-\tau} [c_l \psi_{l,1}^*(z) + d_l \psi_{l-1,2}^*] + O(e^{-(l+1)\tau}) \quad \text{as} \quad \tau \to +\infty.
\]

Obviously, then the straight-line cracks (1.3) correspond to zeros of the linear combination

\[
c_l \psi_{l,1}^*(z) + d_l \psi_{l-1,2}^*(z),
\]

and the full result is straightforward since by the blow-up scaling if all the \( \alpha_k \) do not coincide with zeros of the previous linear combination (3.18) (harmonic polynomials)
the crack problem does not have a solution, since
\[ z = \frac{x}{y} = \alpha_k(1 + o(1)), \ y \to 0. \]

Otherwise, if there is some \( l \) for which all the \( \alpha_k \) coincide with zeros of (3.18) we find that the crack problem (1.6) possesses a solution and (3.19) is satisfied. The proof is complete.

**Remark 3.5.** Remember that thanks to the rescaling (3.1), we have converted the singularity point at 0 into an asymptotic convergence when \( \tau \to \infty \).

**Remark 3.6.** Of course, one can “improve” such nonexistence results. For instance, if cracks have an asymptotically small “violation” of their straight line forms near the origin, which do not correspond to the exponential perturbation in (3.20) (if \( c_{l+1} \) and \( d_{l+1} \) do not vanish simultaneously; otherwise take the next non-zero term), then the crack problem is non-solvable.

Overall, we can state the following most general conclusion.

**Corollary 3.1.** For almost every straight-line crack (1.2), the crack problem (1.6) cannot have a solution for any Dirichlet data \( f \), provided that the crack behaviour at the origin is not consistent with all the eigenfunction expansions (3.17) via the above (harmonic) polynomials.

Finally, concerning the admissible boundary data for such \( l \)-cracks at the origin, these are described by all the expansions (3.17) with arbitrary expansion coefficients excluding the first ones \( c_l, d_l \), which are fixed by the multiple crack configuration (up to a common non-zero multiplier) and satisfying \( c_l^2 + d_l^2 \neq 0 \).

3.4. **Extensions to semi-linear equations: a regular perturbation.** With the idea in mind of extending the techniques performed above to non-linear problems we show a couple of examples. Especially interesting is the application of pencil operators for non-linear equations since in most cases this creates problems. See for example [3].

As a key explaining example, consider the semi-linear Laplace equation
\[ \Delta u + |u|^{p-1}u = 0, \quad p > 1. \tag{3.21} \]

One can see that performing the same rescaling (3.1) on (3.21), in view of (3.2), will lead to the following exponentially small perturbation of the rescaled Laplacian one: as \( \tau \to +\infty \),
\[ [D^2 + D_\tau + 2\tau D^2_\tau + (1 + z^2)D^2_2 + 2zD_z]w + e^{-2\tau}|w|^{p-1}w = 0. \tag{3.22} \]

Obviously, then, on any leading asymptotic pattern given by stable subspaces in (3.17), the last nonlinear term in (3.22) is negligible, so cannot affect the types of decay patterns at the origin.

3.5. **Extensions to semi-linear equations: a singular perturbation.** It is seen from the previous example that in order to involve the nonlinear term in a formation of multiple zeros at the origin, it must be singular nearby, which happens for this model:
\[ \Delta u + \frac{|u|^{p-1}u}{x^2 + y^2} = 0 \quad (p > 1). \tag{3.23} \]
Then by the same rescaling, instead of (3.22), one obtains the following operator
\[
D^2\tau + D\tau + 2zD^2z + (1 + z^2)D^2z + 2zDz \bigg| w + \frac{|w|^{p-1}w}{1 + z^2} = 0, \tag{3.24}
\]
so that the non-linear term does not have an exponentially decaying multiplier such as in (3.22).

STATIONARY PROFILES. Firstly, it is straightforward to consider bounded stationary solutions of (3.24):
\[
w(z, \tau) = f(z) \implies (1 + z^2)f'' + 2zf' + \frac{|f|^{p-1}f}{1 + z^2} = 0 \text{ in } \mathbb{R}. \tag{3.25}
\]
In order to pose necessary conditions at \( z = \infty \), consider the operator linearised at \( f = 0, z = \infty \), that yields the following roots of the characteristic equation:
\[(1 + z^2)f'' + 2zf' = 0, \quad f = zm \implies m^2 + m = 0 \implies m_1 = -1, m_2 = 0, \tag{3.26}
\]
evaluating again the higher order terms. Therefore, we first consider (3.25) with the following conditions as \( z \to \infty \),
\[f(z) = O\left(\frac{1}{z}\right); \quad f'(0) = 0 \text{ (symmetry)} \text{ or } f(0) = 0 \text{ (anti-symmetry)}. \tag{3.27}
\]
Thus, with \( m_1 = -1 \) the last condition in (3.25) corresponds to “dipole-like” profiles. Both symmetry and anti-symmetry conditions are associated with the fact that the ODE (3.25) is invariant under the reflection
\[z \mapsto -z, \quad f \mapsto -f,
\]
which allows us to extend solutions for \( z > 0 \) to \( \{z < 0\} \) in symmetric or anti-symmetric ways. Of course, stationary nonlinear eigenfunctions (3.25) correspond to usual straight-line nodal sets.

A symmetric stationary profile \( f(z) \) satisfying (3.25) is shown in Figure 2 for the cubic case \( p = 3 \). In Figure 3, we show a dipole-like profile as a solution of the ODE in (3.25), again, for the cubic nonlinearity with for \( p = 3 \).

Also, the second root \( m_2 = 0 \) in (3.26) allows us to consider stationary profiles satisfying
\[f(+\infty) = 1. \tag{3.28}\]
Figure 4 shows that such profiles exist for \( p = 3 \), for both symmetric and dipole-like (the dash-line) cases. Overall, those examples exhibit a vast variety of nonlinear eigenfunctions with different nodal sets for elliptic equations with singular nonlinear perturbations.

QUASI-STATIONARY SELF-SIMILAR SOLUTIONS. Secondly, as other “nonlinear eigenfunctions” depending on \( \tau \), we can look for an approximate self-similar solution of a standard form:
\[w(z, \tau) = \tau^{\alpha}f(\xi), \quad \text{where } \xi = \frac{z}{\tau^\beta}, \quad \text{where } \beta = \frac{\alpha(p - 1)}{2}, \tag{3.29}\]
and \( \alpha > 0 \) is an arbitrary fixed exponent. It is clear that the evolution structure (3.29) is quasi-stationary, since all three first time-dependent derivatives, after scaling, are negligible as \( \tau \to +\infty \), of the order, at least, \( \sim O\left(\frac{1}{\tau}\right) \), in comparison with the three other stationary ones. Then, the ODE for \( f \) asymptotically takes the form (cf. that in (3.25))
\[\xi^2 f'' + 2\xi f' + \frac{|f|^{p-1}f}{\xi^2} = 0, \tag{3.30}\]
Figure 2. An example of a symmetric bounded stationary self-similar solution $f(z)$ of (3.25) for $p = 3$.

Figure 3. An example of a dipole-like stationary self-similar solution $f(z)$ of (3.25) for $p = 3$.

(note that $\alpha$ does not affect this equation). One can see that (3.30) admits solutions with the same decay at infinity:

$$f(\xi) = O\left(\frac{1}{\xi}\right) \to 0 \quad \text{as} \quad \xi \to \infty.$$  \hspace{1cm} (3.31)

At $\xi = 0$, the operator is singular, so one cannot put any definite condition on it, and we just require $f$ to be bounded. Again, we are not going to study this ODE problem in any detail. In Figure 5 we just present such a self-similar profile for $p = 3$. Note that it is oscillatory as $\xi \to 0$, so that the nodal set of such an
unbounded ($\alpha > 0$) pattern consists of an infinite number of zero curves, with the following non-standard behaviour near the origin (cf. (1.3): here, there is a log-type perturbation of the crack geometry for such nonlinear patterns):

$$x_k = \xi_k (-y) |\ln(-y)|^{\beta}, \quad \text{where} \quad k = 1, 2, 3, \ldots, \quad f(\xi_k) = 0. \quad (3.32)$$

Indeed, this is a rather strange example of a multiple crack (while the solution gets unbounded at 0), but one should remember that, here, we are talking about a strongly singularly perturbed Laplace operator (3.23), for which a proper statement of the Dirichlet problem deserves certain attention.

3.6. A comment: a standard Sturm–Liouville form of the pencil. Recall the classic fact: harmonic polynomials are eigenfunctions of a standard Sturm–Liouville problem. Therefore, obviously, our pencil eigenvalue problem must admit a reduction to a similar one. It is easy to see that, e.g., this can be achieved by the transformation

$$\psi^*(z) = (1 + z^2)^{\gamma} \varphi(z),$$

with a parameter $\gamma \in \mathbb{R}$ to be determined. Then we find that the operator (3.7) can be written as

$$(1 + z^2)^{\gamma} [\lambda(\lambda + 1) \varphi + 4(\lambda + 1) \gamma z^2 (1 + z^2)^{-1} \varphi + 2(\lambda + 1) z \varphi' + 2 \gamma \varphi
+ 4z^2 \gamma (\gamma - 1)(1 + z^2)^{-1} \varphi + 4z \gamma \varphi' + (1 + z^2) \varphi''] = 0, \quad (3.33)$$

since

$$(\psi^*)' (z) = 2\gamma z(1 + z^2)^{\gamma - 1} \varphi(z) + (1 + z^2)^{\gamma} \varphi'(z), \quad \text{and}$$

$$(\psi^*)'' (z) = 2\gamma(1 + z^2)^{\gamma - 1} \varphi(z) + 4\gamma (\gamma - 1) z^2(1 + z^2)^{\gamma - 2} \varphi(z)$$

$$+ 4\gamma z(1 + z^2)^{\gamma - 1} \varphi'(z) + (1 + z^2)^{\gamma} \varphi''(z).$$
To eliminate the necessary terms in order to get a Sturm–Liouville problem, we have to cancel the term containing \( z\phi' \), i.e. to require

\[
2\lambda + 4\gamma + 2 \quad \Rightarrow \quad \gamma = -\frac{\lambda + 1}{2}.
\]

Now, rearranging terms for that specific \( \gamma \) in the equation (3.33), so that the terms with \( \phi \) are given by

\[
(1 - z^2(1 + z^2)^{-1}) (\lambda + 1)(\lambda - 1)\phi \equiv (1 + z^2)^{-1}(\lambda + 1)(\lambda - 1),
\]

we arrive at a Sturm–Liouville problem of the form

\[
A\phi = \mu\phi, \quad \text{where} \quad A = -(1 + z^2)^2 \frac{d^2}{dz^2} \quad \text{and} \quad \mu = (\lambda + 1)(\lambda - 1), \quad (3.34)
\]

in the space of functions

\[
D = L^2(\mathbb{R}, \frac{dz}{(1 + z^2)^2}).
\]

The operator \( A \) is symmetric in a weighted \( L^2 \)-space, so the eigenvalues \( \mu \) are real. Indeed, by classic Sturm–Liouville theory, we also state that there exists an eigenfunction associated with every eigenvalue \( \mu_n \) such that

\[
\mu_1 < \mu_2 < \cdots < \mu_n \to \infty.
\]

Associated with those eigenvalues we have the eigenfunctions \( \varphi_n \) which have exactly \( n - 1 \) zeros in \( \mathbb{R} \) and are the so-called \( n \)-th fundamental solution of the Sturm–Liouville problem (3.34) and form an orthogonal basis in a specific weighted \( L^2 \)-space, denoted by \( L^2_\rho \) for an appropriate weight (in this case \( \rho = (1 + z^2)^{-2} \)). Indeed, by classical spectral theory we can be assured that the first eigenvalue \( \mu_1 \) is positive and, hence all the others. Also, since the weight \( \rho \) is integrable, i.e.

\[
\int_{\mathbb{R}} \frac{dz}{(1 + z^2)^2} < \infty,
\]
by classical spectral theory we can confirm that the spectrum is formed by a discrete family of eigenvalues. Thus, our pencil eigenvalues are associated with standard $\mu$’s via the quadratic algebraic equation

$$\mu = (\lambda + 1)(\lambda - 1),$$

and the correspondence of eigenfunctions is straightforward. We do not need any further discussion, since, inevitably, once more, we are starting to re-discover classic textbook’s facts on harmonic polynomials.

4. Bi-Laplace equation and new types of admissible cracks. According to (3.2), for the bi-Laplace problem (1.1), we need to solve the iterated rescaled Laplacian:

$$\Delta(z,\tau)\Delta(z,\tau) w = 0.$$  \hspace{1cm} (4.1)

As mentioned in previous sections, the admissible crack distributions obtained for the Laplace equation will remain valid for the bi-Laplace one (1.1), (1.4), having also other types of singularities at the origin.

4.1. Regularity via Hermitian spectral theory for a pencil. We now obtain a type of pencil operator needed to tackle the problems under analysis in this paper. Also, we shall perform our analysis on the basis of a non-self-adjoint spectral pencil theory previously unknown and, probably, one of the reasons these results could not be obtained before. Indeed, eventually, we will put in charge a wider family of harmonic polynomials, which is not that surprising.

Blow-up scaling. Firstly, we perform the same “blow-up” scaling near the origin 0 for the bi-Laplace equation (3.1), which was done before for the Laplace equation (1.6).

Remember via the rescaling (1.8) we have transformed the singularity point at the origin to an asymptotic convergence when $\tau \to \infty$, as performed for the Laplace equation.

Thus, thanks to operator (3.2) we get the rescaled one,

$$\Delta(z,\tau)\Delta(z,\tau) w = e^{2\tau}[D_4^2 + 6D_3^2 + 11D_2^2 + 6D_1^2]w$$

$$+ e^{2\tau}[44zD_{z\tau}^2 + 24zD_{z\tau\tau} + 10(1 + 3z^2)D_{zz\tau\tau}\]w+$$

$$+ e^{2\tau}[4zD_{zz\tau\tau} + 2(1 + 3z^2)D_{zzz\tau\tau} + 4z(1 + z^2)D_{zzz\tau\tau}]w$$

$$+ e^{2\tau}[(1 + z^2)^2D_z^2 + 12z(1 + z^2)D_z^3 + 12(1 + 3z^2)D_z^3 + 24zD_z^3]w = 0.$$ \hspace{1cm} (4.2)

Therefore, we arrive at the equation

$$w_{\tau\tau\tau\tau} + 6w_{\tau\tau\tau} + 11w_{\tau\tau} + 6w_\tau + 44zw_{z\tau} + 24zw_{z\tau\tau} + 10(1 + 3z^2)w_{zz\tau\tau} + 4zw_{z\tau\tau\tau}$$

$$+ 2(1 + 3z^2)w_{zz\tau\tau} + 4z(1 + z^2)w_{zz\tau\tau} = C^* w,$$ \hspace{1cm} (4.3)

where the operator $C^*$ stands for

$$C^* w \equiv -(1 + z^2)^2w_{zzzz} - 12z(1 + z^2)w_{zzz} - 12(1 + 3z^2)w_{zz} - 24zw_z.$$  

Now, as for the Laplace equation we are looking for solutions such that

$$w(z,\tau) \to 0 \quad \text{as} \quad \tau \to +\infty.$$ \hspace{1cm} (4.4)
Pencil operator. Again, thanks to Kondratiev’s “evolution” approach, we will show that also for the bi-Laplace equation (1.1), with the multiple crack condition (1.2) under consideration, we need polynomial eigenfunctions of certain pencil operators.

To do so, we write the solutions of (4.3) in separate variables

\[ w(z, \tau) = e^{\lambda \tau} \psi^*(z), \quad \text{where } \Re \lambda < 0 \text{ by (3.5)}, \]

\( \lambda \) stand for the eigenvalues of the adjoint operator \( \mathbf{C}^* \), and \( \psi^* \) the corresponding eigenfunctions, arriving at an eigenvalue problem for a polynomial (quartic) pencil of non self-adjoint operators of the form

\[
\mathbf{F}_\lambda \psi^* \equiv \left\{ (\lambda^4 + 6\lambda^3 + 11\lambda^2 + 6\lambda)I + 4(\lambda^3 + 6\lambda^2 + 11\lambda)zD_z + 2(1 + 3z^2)(\lambda^2 + 5\lambda)D_z^2 + 4\lambda(1 + z^2)zD_z^3 - \mathbf{C}^* \right\} \psi^* = 0. \tag{4.5}
\]

**Remark.** The fourth-order operator \( \mathbf{C}^* \) is singular at the infinite points \( z = \pm \infty \), so this is a singular pencil eigenvalue problem. In this case, we also have that the operator is not symmetric (similarly to the Laplace equation: see Remark 3.2), since, for instance, the linear first-order operator in (4.5), \( zD_z \), is not symmetric in \( L^2 \).

One can see that introducing any weighted \( L^2 \_p \) metric does not help either. Indeed, a single weight function \( \rho(z) \) is not enough to arrange a symmetry balance. Thus, a symmetry feature is not crucial at all for a functional setting to be used, though the quality of particular functional spaces to be used remains essential for eigenvalue analysis. In particular, the analyticity properties/conditions obviously remain valid for the bi-Laplace equation, so that, for finite-order zeros at 0, harmonic polynomials must appear again.

**Polynomial eigenfunctions and families of eigenvalues.** Similarly, as we obtained for the Laplace equation in Proposition 3.1, we have that the eigenfunctions of the adjoint pencil (4.5) are finite polynomials (cf. the above analyticity demand).

Moreover, we can state the following.

**Lemma 4.1.** The pencil operator (4.5) admits four families of eigenfunctions

\[
\{ \psi^*_{1,1}(z) \}, \quad \{ \psi^*_{1,2}(z) \}, \quad \{ \psi^*_{1,3}(z) \}, \quad \{ \psi^*_{1,4}(z) \}, \tag{4.6}
\]

associated with four families of eigenvalues of the form

\[
\lambda_{l,1} = -l, \quad l = 1, 2, 3, \ldots \quad \lambda_{l,2} = -l - 1, \quad l = 0, 1, 2, 3, \ldots
\]

\[
\lambda_{l,3} = -l - 2, \quad l = 0, 1, 2, 3, \ldots \quad \text{and} \quad \lambda_{l,4} = -l - 3, \quad l = 0, 1, 2, 3, \ldots \tag{4.7}
\]

**Proof.** To find the corresponding point spectrum of the pencil we just substitute the \( l \)th-order polynomial eigenfunctions (3.10)

\[
\psi^*_l(z) = z^l + b_{l-2}z^{l-2} + b_{l-4}z^{l-4} + \ldots = \sum_{k=l,l-2,\ldots} b_k z^k, \quad (b_l = 1), \tag{4.8}
\]

into (4.5) obtaining the following equation for the eigenvalues \( \lambda \):

\[
O(z^l) : \lambda_l^4 + 2(2l + 3)\lambda_l^3 + (6l^2 + 18l + 11)\lambda_l^2 + (4l^3 + 18l^2 + 22l + 6)\lambda_l + l^4 + 6l^3 + 11l^2 + 6l = 0. \tag{4.9}
\]

Subsequently, we solve this characteristic equation ascertaining the corresponding families of eigenvalues. Thus, taking into account that the negative eigenvalues obtained for the quadratic pencil (3.7)

\[
\lambda_l^+ = -l, \quad l = 1, 2, 3, \ldots \quad \text{and} \quad \lambda_l^- = -l - 1, \quad l = 0, 1, 2, 3, \ldots,
\]
are going to be solutions of the characteristic equation (4.9), we have that (4.9) can be written by

$$(\lambda_l + l)(\lambda_l + l + 1)(\lambda_l^2 + (2l + 5)\lambda_l + l^2 + 5l + 6) = 0.$$ 

Hence, we find four families of negative eigenvalues (4.7).

Therefore, calculating the harmonic polynomials as the corresponding eigenfunctions of the pencil, we arrive at.

**Theorem 4.1.** The fourth-order pencil (4.5) has four discrete spectra (4.7) of real negative eigenvalues with the finite polynomial eigenfunctions given by (4.6), where the expansion coefficients satisfy finite Kummer-type recursion corresponding to the operator in (4.5):

$$b_{k+4} = -\frac{2\lambda_{i,i}((\lambda_{l,i}+5)+4\lambda_{l,i}+2k(k-1)+12k+12)}{(k+4)(k-3)} b_{k+2} - \frac{\lambda_{i,i}((\lambda_{l,i})^3+6(\lambda_{l,i})^2+11\lambda_{l,i}+6)+4\lambda_{l,i}[(\lambda_{l,i})^2+6\lambda_{l,i}+11]+6\lambda_{i,i}(\lambda_{l,i}+5)k(k-1)}{(k+4)(k+3)(k+2)(k+1)} b_k,$$

(4.10)

for $k \geq 4$, any $i = 1, 2, 3, 4$, and

$$\lambda_{l,i}((\lambda_{l,i})^3+6(\lambda_{l,i})^2+11\lambda_{l,i}+6)b_0 + [4((\lambda_{l,i})^2+5\lambda_{l,i})+24]b_2 + 24b_4 = 0,$$

$$[(\lambda_{l,i})^4+10(\lambda_{l,i})^3+17(\lambda_{l,i})^2+17\lambda_{l,i}+24]b_1 + 12[(\lambda_{l,i})^2+7\lambda_{l,i}+12]b_3 + 120b_5 = 0,$$

$$[(\lambda_{l,i})^4+10(\lambda_{l,i})^3+47(\lambda_{l,i})^2+110\lambda_{l,i}+120]b_2 + 24[(\lambda_{l,i})^2+9\lambda_{l,i}+20]b_4 + 360b_6 = 0,$$

$$[(\lambda_{l,i})^4+10(\lambda_{l,i})^3+71(\lambda_{l,i})^2+254\lambda_{l,i}+460]b_3 + 240[\lambda_{l,i}+5]b_5 + 840b_7 = 0.$$ 

**Proof.** Similarly to the proof of Theorem 3.1 due to the previous Lemma, via (4.7) the pencil (4.5) has four discrete spectra (4.7) of real negative eigenvalues with four families of finite (z-re-written harmonic) polynomial eigenfunctions given by (4.6), of the polynomial form (4.8), and associated with the four families of eigenvalues $\lambda^1_l$, $\lambda^2_l$, $\lambda^3_l$, and $\lambda^4_l$, such that

$$\psi_{l,1}^*(z) \equiv \psi_{l,1}^*(z), \quad \psi_{l,2}^*(z) \equiv \psi_{l-1,1}^*(z), \quad \psi_{l,3}^*(z) \equiv \psi_{l-2,3}^*(z), \quad \psi_{l,4}^*(z) \equiv \psi_{l-3,4}^*(z).$$
Remark 4.1.

Then, substituting $\psi = \sum_{k \geq 1} a_k z^k$, for any $l \geq 0$, into (4.5) we obtain that, for any $\lambda$,

$$
\lambda(\lambda^3 + 6\lambda^2 + 11\lambda + 6) \sum_{k \geq 0} b_k z^k + 4\lambda(\lambda^3 + 6\lambda^2 + 11)z \sum_{k \geq 1} kb_k z^{k-1}
$$

$$
+ 2\lambda(\lambda + 5)(1 + 3z^2) \sum_{k \geq 2} k(k - 1)b_k z^{k-2}
$$

$$
+ 4\lambda z(1 + z^2) \sum_{k \geq 3} k(k - 1)(k - 2)b_k z^{k-2}
$$

$$
+ (1 + z^2)^2 \sum_{k \geq 4} k(k - 1)(k - 2)(k - 3)b_k z^{k-4}
$$

$$
+ 12z(1 + z^2) \sum_{k \geq 3} k(k - 1)(k - 2)(k - 3)b_k z^{k-3}
$$

$$
+ 12(1 + 3z^2) \sum_{k \geq 2} k(k - 1)b_k z^{k-2} + 24z \sum_{k \geq 1} kb_k z^{k-1} = 0,
$$

and, hence, rearranging terms

$$
\sum_{k \geq 4} [\lambda(\lambda^3 + 6\lambda^2 + 11\lambda + 6) + 4\lambda(\lambda^3 + 6\lambda^2 + 11) + 6\lambda(\lambda + 5)k(k - 1)
$$

$$
+ 4\lambda k(k - 1)(k - 2) + k(k - 1)(k - 2)(k - 3) + 12k(k - 1)(k - 2) + 36k(k - 1)
$$

$$
+ 24k]b_k z^k + \sum_{k \geq 4} [2\lambda(\lambda + 5)(k + 2)(k + 1) + 4\lambda(k + 2)(k + 1)k
$$

$$
+ 2(k + 2)(k + 1)k(k - 1) + 12(k + 2)(k + 1)k + 12(k + 2)(k + 1)]b_{k+2} z^k
$$

$$
+ \sum_{k \geq 4} (k + 4)(k + 3)(k + 2)(k + 1)b_{k+4} z^k = 0.
$$

(4.11)

Also, the first four terms of the polynomial (4.8) provide us with the following equations for the first coefficients:

$$
\lambda(\lambda^3 + 6\lambda^2 + 11\lambda + 6)b_0 + [4(\lambda^3 + 5\lambda) + 24]b_2 + 24b_4 = 0,
$$

$$
(\lambda^4 + 10\lambda^3 + 17\lambda^2 + 17\lambda + 24)b_1 + 12(\lambda^2 + 7\lambda + 12)b_3 + 120b_5 = 0,
$$

$$
(\lambda^4 + 10\lambda^3 + 47\lambda^2 + 110\lambda + 120)b_2 + 24(\lambda^2 + 9\lambda + 20)b_4 + 360b_6 = 0,
$$

$$
(\lambda^4 + 10\lambda^3 + 71\lambda^2 + 254\lambda + 460)b_3 + 240(\lambda + 5)b_5 + 840b_7 = 0,
$$

proving the expression (4.10). This completes the proof. \hfill \Box

Remark 4.1. Again we can deduce that those coefficients might have the expression

$$
b_{l-2n} = -\frac{N(l, \lambda_{l,i})}{D(l, \lambda_{l,i})} b_{l-2n+2} - \frac{(l-2n+4)(l-2n+3)(l-2n+2)(l-2n+1)}{D(l, \lambda_{l,i})} b_{l-2n+4},
$$

$$
n = 1, 2, ..., \left\lceil \frac{l}{2} \right\rceil, \quad b_l = 1, \quad i = 1, 2, 3, 4,
$$
where
\[ N(l, \lambda_{i,l}) = (l-2n+2)(l-2n+1)[2(l-2n)(l-2n+11)
\quad + 12 + 2\lambda_{l,i}(\lambda_{l,i} + 5) + 4\lambda_{l,i}(l-2n)], \]

\[ D(l, \lambda_{i,l}) = 24(l-2n) + 36(l-2n)(l-2n-1) + 12(l-2n)(l-2n-1)(l-2n-2)
\quad + (l-2n)(l-2n-1)(l-2n-2)(l-2n-3)
\quad + 4\lambda_{l,i}(l-2n)(l-2n-1)(l-2n-2) + 6\lambda_{l,i}(\lambda_{l,i} + 5)(l-2n)(l-2n-1)
\quad + 4\lambda_{l,i}(l-2n)[(\lambda_{l,i})^2 + 6\lambda_{l,i} + 11] + \lambda_{l,i}[(\lambda_{l,i})^3 + 6(\lambda_{l,i})^2 + 11\lambda_{l,i} + 6]. \]

**Remark 4.2.** Note that, even though, in this case, due to the discrete spectra, we again find certain relations for the families of eigenvalues
\[ \lambda_{l,4} = \lambda_{l,1} - 3 = \lambda_{l-3,1}, \quad \lambda_{l,3} = \lambda_{l,1} - 2 = \lambda_{l-2,1} \quad \text{and} \quad \lambda_{l,2} = \lambda_{l,1} - 1 = \lambda_{l-1,1}. \]

However, we find different polynomials (four different families) depending on the considered eigenvalue. Indeed, by the analyticity, those are *harmonic* ones but represented in a different manner by using the rescaled variable \( z \).

### 4.2. Nonexistence result for the bi-Laplace crack problem.

First, we observe that our generalised polynomials \((4.8)\) are harmonic polynomials, so that these are also complete in any reasonable weighted \( L^2 \) space. Therefore, again in this situation, sufficient regular solutions of \((4.3), (4.4)\) should admit the corresponding eigenfunction expansions over the polynomial families
\[ \Phi^* = \{ \psi_{l,1}^*, \psi_{l,2}^*, \psi_{l,3}^*, \psi_{l,4}^* \}, \]

such that
\[ w(z, \tau) = \sum_{(k \geq l)} e^{-k\tau} [C_k \psi_{k-1,1}^*(z) + D_k \psi_{k-1,2}^*(z) + E_k \psi_{k-1,3}^*(z) + F_k \psi_{k-1,4}^*(z)], \]

(4.12)

where four collections of expansion coefficients \( \{C_k\}, \{D_k\}, \{E_k\} \) and \( \{F_k\} \) (which depend on boundary data on \( \Omega \)) take place and such that
\[ \psi_{l,1}^*(z) \equiv \psi_{l,1}^*(z), \quad \psi_{l,2}^*(z) \equiv \psi_{l,2}^*(z), \quad \psi_{l,3}^*(z) \equiv \psi_{l,3}^*(z), \quad \psi_{l,4}^*(z) \equiv \psi_{l,4}^*(z). \]

Thus, we state the following result:

**Theorem 4.2.** Let the cracks \( \Gamma_1, \ldots, \Gamma_m \) in (1.2) be asymptotically given by \( m \) different straight lines (1.3). Then, the following hold:

(i) If all \( \{\alpha_k\} \) do not coincide with all \( m \) subsequent zeros of any non-trivial linear combination
\[ C_l \psi_{l,1}^*(z) + D_l \psi_{l,2}^*(z) + E_l \psi_{l,3}^*(z) + F_l \psi_{l,4}^*(z), \quad \text{with} \quad C_l^2 + D_l^2 + E_l^2 + F_l^2 \neq 0, \]

(4.13)

of the finite transformed harmonic polynomials \( \psi_{l,1}^*(z), \psi_{l,2}^*(z), \psi_{l,3}^*(z), \) and \( \psi_{l,4}^*(z) \) defined by \((4.8), (4.10)\) for any \( l = m, m+1, \ldots \) and arbitrary constants \( C_l, D_l, E_l, F_l \in \mathbb{R} \), then the multiple crack problem \((1.1)\) cannot have a solution for any boundary Dirichlet data \( g, h \) on \( \Omega \).

(ii) If, for some \( l \), the distribution of zeros in (i) holds and a solution \( u(x, y) \) exists, then
\[ |u(x, y)| = O(|x|^{l}) \quad \text{as} \quad (x, y) \rightarrow (0, 0). \]

(4.14)

**Proof.** To prove Theorem 4.2 we follow a similar argument as that performed for Theorem 3.2. Indeed, we can also assure that those expansions \((4.12)\) will converge
in $L^2_\rho$, with an appropriate exponentially decaying weight at infinity and uniformly on compact subsets.

Thus, the elliptic evolution while approaching the origin actually occurs on compact, arbitrarily large subsets for $z = \frac{x}{1-y}$. Now, choosing

$$C_k^2 + D_k^2 + E_k^2 + F_k^2 \neq 0,$$

(the first non-zero expansion coefficients), we arrive at the sharp asymptotics of the solution

$$w_l(y, \tau) = C_l e^{-l\tau} [C_k^* \psi_{k,1}^* (z) + D_k \psi_{k-1,2}^* (z) + E_k \psi_{k-2,3}^* (z) + F_k \psi_{k-3,4}^* (z)] + O(e^{-(l+3)\tau}),$$

as $\tau \to +\infty$. Hence, we have that the straight-line cracks (1.3) correspond to zeros of the linear combination (4.13),

$$C_k^* \psi_{k,1}^* (z) + D_k \psi_{k-1,2}^* (z) + E_k \psi_{k-2,3}^* (z) + F_k \psi_{k-3,4}^* (z),$$

proving Theorem 4.2.

**Remark 4.3.** Concerning the positive existence counterpart of our analysis, the result is the same: multiple cracks at 0 can occur iff the boundary data is taken from the expansion (4.12). This allows us to derive the co-dimension of this linear subspace of admissible data. If the tip of the cracks is not fixed at the origin, then the unity of all data (and an appropriate closure, if necessary) should be taken in (4.12) over all tip crack points $x_0 \in \Omega$.

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