CONSTRUCTION OF MINIMAL NON-INVERTIBLE
SKEW-PRODUCT MAPS ON 2-MANIFOLDS

JAKUB ŠOTOLA AND SERGEI TROFIMCHUK

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Abstract. Applying the Hric-Jäger blow up technique, we give an affirmative answer to the question about the existence of non-invertible minimal circle-fibered self-maps of the Klein bottle. In addition, we present a simpler construction of a non-invertible minimal self-map of two-dimensional torus.

1. INTRODUCTION

This paper deals with the minimal circle-fibered self-maps of two-dimensional manifolds. We recall that the pair \((X, f)\) consisting of a compact metric space \(X\) and its continuous endomorphism \(f : X \to X\) is called a minimal dynamical system if \(X\) does not have any non-empty compact subset \(X' \neq X\) satisfying \(f(X') \subseteq X'\). The understanding of the structure of minimal systems has a clear importance for discrete dynamics. During the last decades, much progress has been made in studying minimal subsystems of \((M, f)\) in the case when \(M\) is a low-dimensional compact connected manifold; e.g. see [1]–[14].

In particular, Auslander and Katznelson have proved [1] that the minimality of \((M, f)\) together with \(\dim M = 1\) implies that \(M = T^1\) and that \(f\) is conjugate to an irrational rotation (hence, \(f\) is a homeomorphism). If \(\dim M = 2\) then, due to the Blokh-Oversteegen-Tymchatyn Theorem [3][11], the minimal connected manifold \(M\) must be either the 2-torus \(T^2\) or the Klein bottle \(K^2\). It was also shown in [10] that, in contrast with the minimal system \((T^1, f)\), there exist minimal fiber-preserving systems \((T^2, f)\) which are not invertible. The key dynamical and topological components of the proof in [10] are, respectively, the Rees example [14] of a non-distal but point-distal torus homeomorphism and the Roberts-Steenrod characterization [15] of the monotone transformations of 2-dimensional manifolds.

Since the available constructions [5][13] of the minimal homeomorphisms of the Klein bottle are technically quite involved, the similar question about the existence of minimal non-invertible self-maps of \(K^2\) has been left open in [3][10][11]. In fact, more complicated topology of the Klein bottle (a non-trivial fiber bundle with base space \(T^1\) and fiber \(T^1\)) in comparison to the torus \(T^2 = T^1 \times T^1\) (a direct product of two circles) could potentially be an obstacle for the existence of minimal non-invertible self-maps of \(K^2\); cf. [12] Theorem C-11 and Corollary 1] and [4][6]. Nevertheless, the main result of this paper shows that

Theorem 1.1. There exists a fiber-preserving transformation \(\overline{S}\) of the Klein bottle, which is minimal and non-invertible.
Theorem 1.1 is proved in the next section of our work. The proof uses the Hric-Jäger blow-up technique proposed recently in [7]. In the cited work, the authors also sketched a new construction of a non-distal but point-distal torus homeomorphism. We develop further their construction and adopt it to a more complicated topological situation. As a by-product, even without the use of the Roberts-Steenrod theory of monotone transformations of 2-dimensional manifolds, we are able to present a relatively short and explicit construction of a fiber-preserving minimal non-invertible self-map of the 2-torus; cf. [10].

2. Proof of the main theorem

Set $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$, then $\mathbb{T}^2 = \mathbb{T}^1 \times \mathbb{T}^1$. We will fix the positive orientation of $\mathbb{T}^1$ induced by the usual order on $[0, 1)$. Each ordered pair $(x, y)$ of points in $\mathbb{T}^1$ defines two closed sub arcs of $\mathbb{T}^1$ whose endpoints are $x$ and $y$. The arc obtained by moving a point from $x$ to $y$ in the positive direction will be denoted by $[x, y] \subset \mathbb{T}^1$. Hence, $[x, y] \cup [y, x] = \mathbb{T}^1$ so that $0.5 \in [0.25, 0.75]$ and $0 \in [0.75, 0.25]$. By slightly abusing the notation, we will also write $[0, 1] = \mathbb{T}^1$, $[0, 0.5] = [1, 0.5]$, and $[0, 0] = \{0\}$. Next, consider the homeomorphism $P : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $P(x, y) = (x + 1/2, 1 - y)$. Let $\sim$ be an equivalence relation on $\mathbb{T}^2$ in which each point $(x, y)$ is identified with all its images: $P^0(x, y) = (x, y)$ and $P(x, y)$. Let $\pi : \mathbb{T}^2 \to \mathbb{K}^2$ denote the corresponding quotient map. The quotient space $\mathbb{K}^2$ is one of standard models of the Klein bottle. Notice that a transformation $Q$ of the torus induces a transformation of the Klein bottle by the quotient map $\pi$ if and only if $Q$ commutes with $P$.

The desired minimal map $\tilde{S} : \mathbb{K}^2 \to \mathbb{K}^2$ will be constructed as a factor of a minimal and non-invertible transformation $\hat{S}$ of the torus. On the other hand, the map $\tilde{S}$ will be constructed as a topological extension of the Parry minimal homeomorphism $S : \mathbb{T}^2 \to \mathbb{T}^2$ of the form $S(x, y) = (R(x), \sigma_x(y)) := (x + \alpha, y + r(x))$. Here $R(x)$ is a rotation by an irrational angle $\alpha$ and continuous function $r : \mathbb{T}^1 \to \mathbb{R}$ is such that $r(x) = -r(x + 1/2)$ (i.e. $S$ commutes with $P$). Moreover, the Fourier coefficients of $r(x)$ must satisfy several assumptions listed in [13]; in addition, we can choose them in such a way that $r(0) = r(1/2) = 0$, and $r(x) \in (0, 1/4)$ for all $x \in (0, 1/2)$. In the sequel, we will use the notation $\sigma^n_x$ for the composition $\sigma_{R^{n+1}}^{−1}(x) \circ \sigma_{R^n}^{-2}(x) \circ \cdots \circ \sigma_{R}(x) \circ \sigma_x$.

Take now some point $x_1^* \in (0.1, 0.2) \cap \mathbb{Q}$, set $x_2^* = x_1^* + 1/2$, and then choose the points $z_1^* = (x_1^*, y_1^*)$ and $z_2^* = Pz_2^* = (x_2^*, y_2^*)$ in such a way that $y_2^* \neq \sigma_{R^m}^{-m}(x_1^*)$ for each $m \in \mathbb{Z}$, $j = 1, 2$. Then the $S$-orbits of $z_1^*$, $z_2^*$ do not intersect curves $\mathbb{T}^1 \times \{0\}$ and $\{(x, -r(x)) | x \in \mathbb{T}^1\}$.

Let continuous $\psi, \phi : \mathbb{T}^1 \to \mathbb{T}^1$ have their graphs $P$-invariant and intersecting transversally at the points $z_1^*$ and $z_2^*$; see Figure 1. We will choose $\phi, \psi$ in such a way that they take zero values (recall that 1 is identified with 0) in all points except for some small open $\rho$-neighborhoods $U_j$ of $x_j^*$, $j = 1, 2$.

We define a fiber measure $\mu_x^0$ on the $\sigma$-algebra $B$ of Borelian subsets of $\mathbb{T}^1$ as

$$\mu_x^0 := \begin{cases} \delta_{y_j^*}, & \text{if } x = x_j^*, \quad j = 1, 2, \\ \frac{\lambda_{\phi(x)}(x)}{\psi(x)} \lambda_{\phi(x)}(x) \cdot \phi(x), & \text{if } \phi(x) > \psi(x), \\ \frac{\lambda_{\phi(x)}(x)}{\psi(x)} \lambda_{\phi(x)}(x) \cdot \phi(x), & \text{if } \psi(x) > \phi(x), \end{cases}$$

where $\delta_y$ denotes a probabilistic Dirac measure concentrated at $y$ (i.e. $\delta_y(y) = 1$) and $\lambda_{[a, b]}(A)$ denotes the Lebesgue measure of intersection of a measured set $A$. 

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and the arc \([a, b]\). Consider also the measures \(\mu^n_x\), \(n \in \mathbb{Z}\), and \(\mu_x\) on \(\mathcal{B}\) defined by

\[
\mu^n_x := \mu^0_{R^n(x)} \circ \sigma^n_x, \quad \mu_x := (\lambda + \sum_{n=0}^{\infty} 2^{-n-1} \mu^n_x)/2.
\]

It is clear that \(\mu_x(\mathbb{T}^1) = 1, x \in \mathbb{T}^1\). In addition, \(\mu\) inherits the symmetry properties of \(r, \phi, \psi\):

**Lemma 2.1.** For all \(x \in \mathbb{T}^1\) and \(y \in [0, 1]\), we have

\[
(2.1) \quad \mu_x[0, y] = 1 - \mu_{x+\frac{1}{2}}[0, 1 - y].
\]

**Proof.** First we prove that, for all \(x, y \in \mathbb{T}^1\), it holds that

\[
(2.2) \quad \mu^0_x[0, y] = 1 - \mu^0_{x+\frac{1}{2}}[0, 1 - y].
\]

For \(x \notin E := [U_1 \cup U_2] \setminus \{x_1^*, x_2^*\}\) (i.e. for the fibers with the usual Lebesgue or Dirac measures), this relation is immediate. Now, let \(x \in E\) be such that \(\phi(x) < \psi(x)\) and hence \(\psi(x + 1/2) = 1 - \psi(x) < \phi(x + 1/2) = 1 - \phi(x)\). Then

\[
\mu^0_{x+\frac{1}{2}}[0, 1 - y] = \begin{cases} 
0, & \text{if } 1 - y < \psi \left( x + \frac{1}{2} \right), \\
\frac{1 - y - \psi \left( x + \frac{1}{2} \right)}{\phi(x + \frac{1}{2}) - \psi \left( x + \frac{1}{2} \right)}, & \text{if } \psi \left( x + \frac{1}{2} \right) \leq 1 - y < \phi \left( x + \frac{1}{2} \right), \\
1, & \text{if } 1 - y > \phi \left( x + \frac{1}{2} \right); \\
\frac{\psi(x) - y}{\phi(x) - \psi(x)}, & \text{if } y > \psi(x), \\
1, & \text{if } \phi(x) \leq y \leq \psi(x), \\
0, & \text{if } y < \phi(x); 
\end{cases}
\]

\[
= \begin{cases} 
0, & \text{if } \phi(x) \leq y \leq \psi(x), \\
1 - \mu^0_x[0, y]. & \text{if } y < \phi(x); 
\end{cases}
\]

Similarly, \((2.2)\) holds when \(\phi(x) > \psi(x)\). Next, we claim that

\[
(2.3) \quad \mu^0_x[a, b] = 1 - \mu^0_{x+\frac{1}{2}}[1 - a, 1 - b], \text{ for } a, b \in [0, 1], \ a < b, \ [a, b] \in \mathbb{T}^1.
\]
This is trivial for the fibers over \(x_1^*\) and \(x_2^*\). For the rest of the fibers, (2.23) follows from (2.22) in view of

\[
\mu_x^0[a, b] = \begin{cases} 
\mu_x^0[0, b] - \mu_x^0[0, a], & \text{if } 0 \leq a \leq b \leq 1; \\
1 - \mu_x^0[0, a] + \mu_x^0[0, b], & \text{if } 0 \leq b < a \leq 1.
\end{cases}
\]

Finally, we will show that the superscript 0 in (2.2) can be omitted. Indeed, set \(\varrho_n(x) := r(x) + r(x + \alpha) + r(x + 2\alpha) + \ldots + r(x + (n - 1)\alpha)\). Then

\[
\mu_{x + \frac{1}{2}}[0, 1 - y] = \sum_{n \geq 0} 2^{-n-2} \mu_{\sigma_{x + \frac{1}{2}}^n}[0, 1 - y] + (1 - y)/2
\]

\[
= \sum_{n \geq 0} 2^{-n-2} \mu_{x + n\alpha + \frac{1}{2}}[0, 1 - y] + (1 - y)/2
\]

\[
= \sum_{n \geq 0} 2^{-n-2}(1 - \mu_{x + n\alpha}[0, 1 - y]) + 1/2 - y/2
\]

\[
= \sum_{n \geq 0} 2^{-n-2}1/2 - (\sum_{n \geq 0} 2^{-n-2} \mu_{\sigma_x^n}[0, 1 - y] + y/2) = 1 - \mu_x[0, y].
\]

The next two results show that \(\mu_x^0[0, y]\) and \(\mu_x[0, y]\) have discontinuities only at the points from backward orbits of \((x_1^*, y_1^*)\) and \((x_2^*, y_2^*)\).

**Lemma 2.2.** Let \(\{x_j\}, \{y_j\}, \{w_j\}\) be sequences in \(T^1\) which converge to \(x_0\) and \(y_0, w_0\), respectively. Then

\[
\limsup_{j \to \infty} \mu_{x_j}^0[y_j, w_j] \leq \mu_{x_0}^0[y_0, w_0].
\]

Moreover, if \((x_0, y_0) \neq (x_k^*, y_k^*), k = 1, 2\), then

\[
\liminf_{j \to \infty} \mu_{x_j}^0[y_j, w_j] \geq \mu_{x_0}^0[y_0, w_0].
\]

**Proof.** Since the functions \(\phi, \psi\) are continuous, this result is a straightforward consequence of the definition of \(\mu_x\). In fact, for each \(x_0 \neq x_k^*\), we have that \(\mu_{x_j}^0[y_j, w_j] \to \mu_{x_0}^0[y_0, w_0]\) as \(j \to +\infty\). \(\square\)

**Corollary 2.3.** Let \(\{x_j\}\) and \(\{y_j\}\) converge to \(x_0\) and \(y_0\), respectively. Then

\[
\mu_{x_0}^n[0, y_0] \leq \liminf_{j \to \infty} \mu_{x_j}^n[0, y_j] \leq \limsup_{j \to \infty} \mu_{x_j}^n[0, y_j] \leq \mu_{x_0}^n[0, y_0], \quad n \in \mathbb{N}.
\]

Furthermore,

\[
\mu_{x_0}[0, y_0] \leq \liminf_{j \to \infty} \mu_{x_j}[0, y_j] \leq \limsup_{j \to \infty} \mu_{x_j}[0, y_j] \leq \mu_{x_0}[0, y_0].
\]

**Proof.** Since \(\sigma, \ R\) are continuous functions and, by our assumption, \((R^n(x), \sigma_x^n(0)) \neq (x_k^*, y_k^*), k = 1, 2\), for each \(x \in T^1\), \(n \in \mathbb{N}\), we have that

\[
\mu_{x_0}^n[0, y_0] = \mu_{R^n(x_0)}^n[\sigma_{x_0}^n(0), \sigma_{x_0}^n(y_0)]
\]

\[
\leq \liminf_{j \to \infty} \mu_{R^n(x_j)}^n[\sigma_{x_j}^n(0), \sigma_{x_j}^n(y_j)] \leq \liminf_{j \to \infty} \mu_{x_j}^n[0, y_j].
\]
The proof of the second inequality in (2.4) is similar. Finally, by a direct calculation we get the following:

\[
\lim_{j \to \infty} \sup_{n \geq 0} \left( \sum_{n \geq 0} 2^{-n-2} \mu^n_{x_j} [0, y_j] + y_j/2 \right) \leq \sum_{n \geq 0} 2^{-n-2} \mu^n_{x_0} [0, y_0] + y_0/2 = \mu_{x_0} [0, y_0],
\]

\[
\lim_{j \to \infty} \inf_{n \geq 0} \left( \sum_{n \geq 0} 2^{-n-2} \mu^n_{x_j} [0, y_j] + y_j/2 \right) \geq \sum_{n \geq 0} 2^{-n-2} \mu^n_{x_0} [0, y_0] + y_0/2 = \mu_{x_0} [0, y_0].
\]

Hence, if \( \mu_{x_0} [0, y_0] \) is discontinuous at some point \((x_0, y_0)\), then \( \mu_{x_0} \{ y_0 \} > 0 \). \( \square \)

Following [7], we consider continuous fiber-preserving self-map \( T : T^2 \to T^2 \) defined by \( T(x, y) := (x, \tau_x(y)) \), where

\[
\tau_x(y) := \min \{ y' \in [0, 1] | \mu_x [0, y'] \geq y \}.
\]

The existence of this minimum can be deduced, for example, from Corollary 2.3. It is clear that \( \tau_x(0) = 0 \) and \( \tau_x(1) = 1 \) because, for all small \( \epsilon > 0 \),

\[
\mu_x [0, 1 - \epsilon] = 1 - \mu_x + \frac{1}{2} [0, \epsilon] = 1 - \sum_{n \geq 0} 2^{-n-2} \mu^n_{x+\frac{1}{2}} [0, \epsilon] - \epsilon/2 < 1.
\]

Obviously, \( \tau_x(y) \) is an increasing function of \( y \in (0, 1) \). Now, let \( \text{Orb}_R(x) = \{ R^j(x), j \in \mathbb{Z} \} \) and \( \text{Orb}^-_R(x) = \{ R^j(x), j \leq 0 \} \) denote the full and backward orbits of a point \( x \in T^1 \), respectively. As we have proved, \( \mu_x [0, y] \) is continuous at each point \((x, y)\) where \( x \not\in \mathcal{D}^- := \text{Orb}^-_R(x_1^2) \cup \text{Orb}^-_R(x_2^2) \). Therefore

\[
(2.5) \quad \mu_x [0, \tau_x(y)] = y \text{ if } x \not\in \mathcal{D}^-,
\]

which implies \( \tau_x(y_1) < \tau_x(y_2) \) for all \( 0 < y_1 < y_2 \leq 1 \) and \( x \in T^1 \setminus \mathcal{D}^- \).

**Lemma 2.4.** The map \( T : T^2 \to T^2 \) is continuous and surjective. Moreover, \( T \) commutes with \( P \) and is invertible on the set \( T^2 \setminus (\mathcal{D}^- \times T^1) \).

**Proof.** It is clear that \( T \) is continuous if and only if \( \tau_x(y) \) is a continuous function of \( x, y \). So, take some \((x_0, y_0)\) and consider sequences \( x_j \to x_0 \) and \( y_j \to y_0 \). From the definition of \( \tau \) it holds that \( \mu_x [0, \tau_x(y_j)] \geq y_j \). Suppose that \( \tau_{x_j}(y_j) \) converges to some limit point \( z \). Then Corollary 2.3 yields \( \tau_{x_0}(y_0) \leq z = \lim_{j \to \infty} \tau_{x_j}(y_j) \). Suppose for a moment that \( \tau_{x_0}(y_0) < z \) and take \( \delta > 0 \) such that \( \tau_{x_0}(y_0) + \delta < \tau_{x_j}(y_j) \) for all large \( j \). Then

\[
\mu_{x_0} [0, \tau_{x_0}(y_0) + \delta] = \sum_{n \geq 0} 2^{-n-2} \mu^n_{x_0} [0, \tau_{x_0}(y_0) + \delta] + (\tau_{x_0}(y_0) + \delta)/2 \geq y_0 + \delta/2,
\]

and therefore, due to Corollary 2.3, we have for all large \( j \) that

\[
y_j < y_0 + 0.25 \delta < \mu_{x_0} [0, \tau_{x_0}(y_0) + \delta] \leq \lim_{j \to \infty} \mu_{x_0} [0, \tau_{x_0}(y_0) + \delta].
\]

As a consequence, \( \tau_{x_j}(y_j) \leq \tau_{x_0}(y_0) + \delta \) for all large \( j \), a contradiction. Hence, \( \tau_{x_0}(y_0) = \lim_{j \to \infty} \tau_{x_j}(y_j) \) and \( \tau_x(y) \) is continuous.

Now, since \( \tau_x(y) \) depends continuously on \( x, y \) and \( \tau_x(0) = 0 \) and \( \tau_x(1) = 1 \), we obtain that \( \tau_x(T^1) = T^1 \). In addition, if \( x \not\in \mathcal{D}^- \) then \( \tau_x(y) \) is a strictly increasing function of \( y \) and therefore \( T \) is invertible on \( T^2 \setminus (\mathcal{D}^- \times T^1) \).
Next, since
\[(T \circ P)(x, y) = \left(x + \frac{1}{2}, \tau_x + \frac{1}{2}(1 - y)\right), \quad (P \circ T)(x, y) = \left(x + \frac{1}{2}, 1 - \tau_x(y)\right),\]
we find that \(T\) commutes with \(P\) if and only if
\[\tau_x + \frac{1}{2}(1 - y) = 1 - \tau_x(y)\]
for all \(x, y\).

Now, for \(x \in \mathbb{T}^1 \setminus \mathcal{D}^-\) it holds that
\[1 = \mu_x[0, \tau_x(y)] + \mu_x + \frac{1}{2}[0, 1 - \tau_x(y)] = y + \mu_x + \frac{1}{2}[0, 1 - \tau_x(y)],\]
so that
\[\mu_x + \frac{1}{2}[0, 1 - \tau_x(y)] = 1 - y = \mu_x + \frac{1}{2}[0, \tau_x + \frac{1}{2}(1 - y)].\]

Thus \(T\) commutes with \(P\) on \(\mathbb{T}^2 \setminus (\mathcal{D}^- \times \mathbb{T}^1)\). Finally, if \((\hat{x}, \hat{y}) \notin \mathbb{T}^2 \setminus (\mathcal{D}^- \times \mathbb{T}^1)\) we can find a sequence of points \((x_j, y_j) \in \mathbb{T}^2 \setminus (\mathcal{D}^- \times \mathbb{T}^1)\) such that \((x_j, y_j) \to (\hat{x}, \hat{y})\) as \(j \to +\infty\). But then
\[T \circ P(\hat{x}, \hat{y}) = \lim_{j \to +\infty} T \circ P(x_j, y_j) = \lim_{j \to +\infty} P \circ T(x_j, y_j) = P \circ T(\hat{x}, \hat{y}).\]

This completes the proof. \(\square\)

We are ready to construct the non-invertible minimal map \(\hat{S} : \mathbb{T}^2 \to \mathbb{T}^2\). Defining \(\hat{S}\) on the set \(\Lambda := \mathbb{T}^2 \setminus (\mathcal{D} \times \mathbb{T}^1)\), where \(\mathcal{D} := \text{Orb}_R(x_1^*) \cup \text{Orb}_R(x_2^*)\), by
\[
\hat{S}|_{\Lambda} := T^{-1}\Lambda \circ S|_{\Lambda} \circ T|_{\Lambda},
\]
we will extend it continuously on the whole torus.

**Lemma 2.5.** For each \(n \in \mathbb{Z}\), the map \(\theta_n(x) := \mu^r_x[-r(x), 0]\) is continuous on \(\mathbb{T}^1\).

**Proof.** We observe that \(\theta_n(0) = \theta_n(1/2) = \theta_n(1) = 0\) and thus it suffices to establish the continuity of \(\theta_n\) on the arcs \([0, 1/2]\) and \([1/2, 1]\) of \(\mathbb{T}^1\) separately. For instance, consider the arc \([0, 1/2]\) where \(-r(x) \leq 0\) so that \(\theta_n(x) = 1 - \mu^r_x[0, 1 - r(x)]\).

We recall that, by our assumptions, none of the points \((R^n(x), \sigma^2_x(1 - r(x))) = (R^n(x), \sigma^2_x(-r(x))), n \in \mathbb{Z},\) coincided with \((x_1^*, y_1^*)\) and \((x_2^*, y_2^*)\) and therefore the function \(\mu^r_x[0, y]\) is continuous at each point of the form \((x, y) = (x, 1 - r(x))\). In consequence, the map \(x \to \theta_n(x), x \in [0, 1/2]\) is continuous as a composition of two continuous applications: \(x \to (x, 1 - r(x)), x \in [0, 1/2]\), and \((x, y) \to 1 - \mu^r_x[0, y]\), \((x, y) \in \{(x, 1 - r(x)) : x \in [0, 1/2]\}\).

Next, set \(\Lambda_- := \Lambda \cap ([0, 1/2] \times \mathbb{T}^1)\) and \(\Lambda_+ := \Lambda \cap ([1/2, 1] \times \mathbb{T}^1)\).

**Lemma 2.6.** The map \(\hat{S}\) is uniformly continuous on \(\Lambda_-\).

**Proof.** It follows from (2.6) that \(\tau_x^{-1}(y) = \mu_x[0, y]\) for each pair \((x, y) \in \Lambda\). Hence,
\[
\hat{S}(x, y) = (T^{-1} \circ S \circ T)(x, y) = (R(x), \mu_{R(x)}[0, \sigma_x(\tau_x(y))]), (x, y) \in \Lambda.
\]

Since \(R : \mathbb{T}^1 \to \mathbb{T}^1\) is continuous, we need only to prove the uniform continuity of \(M(x, y) := \mu_{R(x)}[0, \sigma_x(\tau_x(y))]\) on \(\Lambda_-\). This task can be simplified if we observe that, due to the Weierstrass M-test and Lemmas 2.4 and 2.6, the function \(W(x, y) := \sum_{n \geq 0} 2^{-n - 2}\mu_x^{n+1}[-r(x), 0] + \sigma_x(\tau_x(y))/2\) is continuous on \(\mathbb{T}^2\) while
\[
M(x, y) = \sum_{n \geq 0} \frac{\mu_{R(x)}[0, \sigma_x(\tau_x(y))]}{2^{n+2}} + \sigma_x(\tau_x(y))/2 = \sum_{n \geq 0} \frac{\mu_x^{n+1}[0, \tau_x(y)]}{2^{n+2}} + W(x, y).
\]
Here we are using the relation
\[
\mu^n_{R(x)}[0, \sigma_x(\tau_x(y))] = \mu^n_{R^n(R(x))} \circ \sigma^{R^n-1}(R(x)) \circ \ldots \circ \sigma R(x) \circ \sigma x(\tau_x(y)) \\
= k^n_{R^{n+1}(x)} \circ \sigma^{R^n(x)} \circ \ldots \circ \sigma R(x) \circ \sigma x(\sigma^{-1}(0), \tau_x(y)) \\
= \mu^{n+1}_x[-r(x), 0] + \mu^{n+1}_x[0, r(x)].
\]
Recall also that \( \sigma_x(y) = y + r(x), \mu^n_{x}(0) = 0, n \in \mathbb{N}, \) and that \( 0 \in [-r(x), \tau_x(y)] \) because of non-negativity of \( r(x) \) for \( x \in [0, 1/2]. \)

In consequence, it suffices to establish the uniform continuity of the function \( A(x, y) := \sum_{n \geq 0} 2^{-n-2} \mu^{n+1}_x[0, \tau_x(y)] \) on \( \Lambda_\ast. \) In other words, we have to prove that for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for any \( x_1, x_2 \in [0, 1/2] \setminus S \) and \( y_1, y_2 \in \mathbb{T}^1 \) satisfying \( d(x_1, x_2) < \delta \) and \( d(y_1, y_2) < \delta, \) it holds that \( a := d(A(x_1, y_1), A(x_2, y_2)) < \epsilon. \) Here \( d \) denotes the metric on \( \mathbb{T}^1 \) naturally inherited from \( \mathbb{R}. \) In particular, \( d \) is shift-invariant. Observe also that we interpret \( A(x, y) \) as a point on \( \mathbb{T}^1. \) Set \( s_n := \mu^n_{x_1}[0, \tau_x(1)] - \mu^n_{x_2}[0, \tau_x(1)]. \) Since \( y_k = \mu_{x_k}[0, \tau_x(k)] \), we find that
\[
b := d(y_1, y_2) = d\left( \sum_{n \geq 0} \frac{s_n}{2^{n+2}} + (\tau_{x_1}(y_1) - \tau_{x_2}(y_2))/2, 0 \right), \quad a = d\left( \sum_{n \geq 0} \frac{s_{n+1}}{2^{n+2}}, 0 \right).
\]
So let us take arbitrary \( \epsilon > 0; \) then there exists \( N_1 \) for which
\[
\sum_{n \geq N_1} 2^{-n-2} < \frac{\epsilon}{16}, \quad \text{so that} \quad \sum_{n \geq N_1} d(2^{-n-2}s_{n+1}, 0) < \frac{\epsilon}{16}.
\]
Let \( U^n_\kappa \) denote open \( \kappa- \) neighborhood of the set \( \{ R^n(x^1), R^n(x^2) \}. \) Due to Corollary \( 2.3 \) and Lemma \( 2.4, \) the function \( \mu^n_x[0, \tau_x(y)] : (\mathbb{T}^1 \setminus \{ U^n_\kappa \}) \times \mathbb{T}^1 \rightarrow \mathbb{T}^1 \) is uniformly continuous for each \( n. \) We can choose \( \kappa \) to be small enough in order to have the closures of \( N_1 \) sets \( U^n_{2\kappa}, \) \( n = 0, 1, \ldots, N_1, \) mutually disjoint. Obviously, each function from the finite set \( \{ \tau_x(y), \mu^n_x[0, \tau_x(y)], n = 0, 1, \ldots, N_1, \} \) is uniformly continuous on \( (\mathbb{T}^1 \setminus \{ \bigcup_{n=0}^{N_1} U^n_\kappa \}) \times \mathbb{T}^1. \)

Take now \( \delta \in (0, \min\{\epsilon, 16, \kappa\}) \) small enough to assure that \( d(x_1, x_2) < \delta, \) \( d(y_1, y_2) < \delta \) imply the inequality
\[
d(\tau_{x_2}(y_2), \tau_{x_1}(y_1)) < \epsilon/8,
\]
as well as
(i) the existence of at most one integer \( n_0 \in [0, N_1] \) such that \( \{ x_1, x_2 \} \cap U^{n_0}_{2\kappa} \neq \emptyset; \)
(ii) \( d(2^{-n_0-2}s_{n_0}, 0) < \epsilon/(16(N_1 + 1)) \) once \( \{ x_1, x_2 \} \cap \bigcup_{n=0}^{N_1} U^n_\kappa = \emptyset, \) \( n = 0, 1, \ldots, N_1. \)

A key observation is that in the case (i) the distance \( d(2^{-n_0-2}s_{n_0}, 0) \) cannot be large even when \( \{ x_1, x_2 \} \cap \bigcup_{n=0}^{N_1} U^n_\kappa \neq \emptyset:
\[
d(2^{-n_0-2}s_{n_0}, 0) \leq b + d\left( \sum_{n=0, n \neq n_0}^{N_1} 2^{-n-2}s_n, 0 \right) + d\left( \sum_{n > N_1} 2^{-n-2}s_n, 0 \right) \leq \frac{\epsilon}{16} + (N_1 + 1) \frac{\epsilon}{16(N_1 + 1)} + \frac{\epsilon}{16} + \frac{\epsilon}{16} = \frac{\epsilon}{4}.
\]
Thus, estimating separately the term \(d(2^{-n_0-1}v, 0)\) (whenever \(n_0\) with properties described in (i) appears) as \(d(2^{-n_0-1}v, 0) < \epsilon/2\), we obtain

\[
a = d \left( \sum_{n=0}^{\infty} \frac{s_{n+1}}{2^{n+2}}, 0 \right) \leq \frac{\epsilon}{2} + d \left( \sum_{n<N_1, n \neq n_0} \frac{s_{n+1}}{2^{n+2}}, 0 \right) + d \left( \sum_{n \geq N_1} \frac{s_{n+1}}{2^{n+2}}, 0 \right) < \frac{\epsilon}{2} + 2N_1 \epsilon/(16(N_1 + 1)) + \epsilon/16 < \epsilon,
\]

which completes the proof. □

**Corollary 2.7.** The map \(\hat{S} \mid \Lambda\) commutes with \(P\), is uniformly continuous on \(\Lambda\), and it admits a unique continuous extension \(\hat{S}\) on \(\mathbb{T}^2\) which also commutes with \(P\).

**Proof.** Observe that all maps \(T, P, S, \hat{S} : \Lambda \to \Lambda\) are bijective and \(P(\Lambda_+) = (\Lambda_-)\). First, we will prove that \(\hat{S} | \Lambda\) commutes with \(P\). Clearly,

\[
(S \circ T \circ P)(z) = (S \circ P \circ T)(z) = (P \circ S \circ T)(z) = (P \circ T \circ \hat{S})(z) = (T \circ P \circ \hat{S})(z).
\]

But \(T\) is injective on \(\Lambda\) and therefore \((P \circ \hat{S})(z) = (\hat{S} \circ P)(z)\) for each \(z \in \Lambda\). Hence, since \(P, P^{-1}\) are linear maps, \(\hat{S} | \Lambda_+ = P^{-1} \circ \hat{S} | \Lambda_- \circ P | \Lambda_+\) is also uniformly continuous. As the maps \(\hat{S} | \Lambda_-\) and \(\hat{S} | \Lambda_+\) are uniformly continuous and \(\Lambda^-\) is dense in \([0, 1/2] \times \mathbb{T}^1\) and \(\Lambda^+\) is dense in \([1/2, 0] \times \mathbb{T}^1\), they can be uniquely continuously extended to the sets \([0, 1/2] \times \mathbb{T}^1\) or \([1/2, 0] \times \mathbb{T}^1\), respectively. Since these maps coincide on the intersection \([0, 1/2] \times \mathbb{T}^1 = ([0, 1/2] \times \mathbb{T}^1) \cap ([1/2, 0] \times \mathbb{T}^1)\), they define a continuous self-map \(\hat{S} : \mathbb{T}^2 \to \mathbb{T}^2\). Clearly, since \(\hat{S} | \Lambda\) commutes with \(P\), the set \(\Lambda\) is dense in \(\mathbb{T}^2\) and the functions \(\hat{S}, P\) are continuous on \(\mathbb{T}^2\), we obtain that \((P \circ \hat{S})(z) = (\hat{S} \circ P)(z)\) for all \(z \in \mathbb{T}^2\). Similarly, \(T \circ \hat{S} = S \circ T\) on \(\mathbb{T}^2\). □

**Lemma 2.8.** The map \(\hat{S} : \mathbb{T}^2 \to \mathbb{T}^2\) is minimal and non-invertible.

**Proof.** Let \(F \subset \mathbb{T}^2\) be a non-empty compact and \(\hat{S}\)-invariant set. But then

\[
S \circ T(F) = T \circ \hat{S}(F) \subseteq T(F).
\]

So, \(T(F)\) is also a compact \(\hat{S}\)-invariant set, which means \(T(F) = \mathbb{T}^2\) because \(S\) is minimal. But since fibers are mapped to fibers and \(T\) is bijective on \(\Lambda\), the set \(F\) must contain whole \(\Lambda\). But then \(F \supseteq \Lambda = \mathbb{T}^2\) and therefore \(\hat{S}\) is a minimal map.

Finally, we prove that \(\hat{S}\) is non-invertible. For \(\delta \in (0, 0.25)\), consider the points \(y_1 = \mu_{x_1}[0, y_1^*] > y_2 = \mu_{x_1}[0, y_1^*] - \delta\) on the circle \(\mathbb{T}^1\). For every \(y' < y_1^*\), we have that

\[
\mu_{x_1}[0, y'] = \sum_{n=0}^{\infty} \frac{\mu^0_R(x_1^n) \sigma^n_{x_1}[0, y']}{2^{n+2}} + y'/2 = \sum_{n=1}^{\infty} \frac{\mu^0_R(x_1^n) \sigma^n_{x_1}[0, y']}{2^{n+2}} + y'/2 \leq \sum_{n=1}^{\infty} \frac{\mu^0_R(x_1^n) \sigma^n_{x_1}[0, y_1^*]}{2^{n+2}} + y_1^*/2 + 0.25 - \delta = y_1^* - \delta = y_2^*.
\]

This yields immediately that \(\tau_{x_1}(y_1) = y_1^* = \tau_{x_1}(y_2)\) and therefore

\[
T \circ \hat{S}(x_1, y_1) = S(x_1^*, y_1^*) = (R(x_1^*), \sigma_{x_1}(y_1^*)) = T \circ \hat{S}(x_1^*, y_2).
\]
Since $T$ is invertible on the fiber over $\{R(x^*_1)\}$, we find that $\hat{S}(x^*_1, y_1) = \hat{S}(x^*_1, y_2)$ and therefore the non-degenerated interval $\{x^*_1\} \times [y_1 - 0.25, y_1]$ is transformed by $\hat{S}$ into a point. □

**Proof of Theorem 1.1.** So far we have obtained a non-invertible minimal self-map $\hat{S}$ of the torus. Since $\hat{S}$ commutes with $P$, it induces a transformation $\tilde{S}$ of the Klein bottle. $\tilde{S}$ is a factor of $\hat{S}$ by the fiber-preserving quotient map $\pi$, so $\tilde{S}$ is a non-invertible minimal fiber-preserving transformation $\hat{S}$ of the Klein bottle. The construction is completed. □

**Remark 2.9.** After changing the definition of $\mu_x$ (where an appropriate weighted sum of all measures $\mu^n_x$, $n \in \mathbb{Z}$, should be considered), we can similarly construct a minimal circle-fibered homeomorphism of the Klein bottle having an asymptotic pair of points. Then the Roberts-Steenrod theory [15] of monotone transformations of 2-dimensional manifolds can be applied in order to obtain a different proof of Theorem 1.1; see [10] for more detail.

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Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, 746 01, Opava, Czech Republic

*E-mail address*: Jakub.Sotola@math.slu.cz

Instituto de Matemática y Fisica, Universidad de Talca, Casilla 747, Talca, Chile

*E-mail address*: trofimch@inst-mat.ualca.cl