The Equations of Magnetoquasigeostrophy

O. M. Umurhan

1 School of Physics and Astronomy, Queen Mary University of London, London E1 4NS, U.K.
2 School of Natural Sciences, University of California Merced, Merced, CA 95343, USA

ABSTRACT

Context. The dynamics contained in magnetized layers of exoplanet atmospheres are important to understand in order to characterize what observational signatures they may provide for future observations. It is important to develop a framework to begin studying and learning the physical processes possible under those conditions and what, if any, features contained in them may be observed in future observation missions.

Aims. The aims of this study is to formally derive, from scaling arguments, a manageable reduced set of equations for analysis, i.e. a magnetic formulation of the equations of quasigeostrophy appropriate for a multi-layer atmosphere. We check these derived equations for consistency with respect to similar equations currently used in the literature like the magnetized shallow water equations and their precursors. The main goal is to provide a simpler theoretical platform to explore the dynamics possible within confined magnetized layers of exoplanet atmospheres.

Methods. We primarily use scaling arguments to derive the reduced equations of “magnetoquasigeostrophy” which assumes dynamics to take place in an atmospheric layer which is vertically thin compared to its horizontal scales. Furthermore, the derivation exploits the fact that the Rossby Numbers of the emergent flows are small and that the Cowling Number is also near 1, the latter of which measures the relative content the energy per unit volume contained in the magnetic field to the corresponding kinetic energy of the flow.

Results. The magnetized incarnation of the quasigeostrophic equations for a two-layer system are fully rederived from scaling arguments. The resulting equation set retains features existing in standard shallow-water magnetohydrodynamic equations but are absent in more classical derivations of the quasi-geostrophic limit, namely, the non-divergence of the in-plane components of the magnetic field. We liken this non-divergence of the in-plane magnetic fields as indicative of a quantity whose behaviour mimics a two-dimensional “pseudo”-magnetic monopole source. We also find, using the same scaling argument procedures, appropriate limits of the fundamental parameters of the system which yield reduced equations describing the flow dynamics primarily characterized by magnetostrophic balance.

Conclusions. The standard scaling arguments employed here show how traditional magnetized quasigeostrophic equations connect to their magnetized shallow water forms. The equations derived are amenable to analysis using well-known techniques.

Key words. Hydrodynamics, Magnetohydrodynamics, Exoplanets

1. Introduction

It is now generally accepted that hot extrasolar giant planets (“hot-Jupiters”) are commonplace in the Galaxy. Many orbit their parent stars in so close that recent calculations by Koskinen et al. (2010) indicate that the upper atmospheric layers of these hot-Jupiters are sufficiently irradiated by the parent star’s UV radiation field (and therefore sufficiently ionized) to treat the gas as a plasma. It is important therefore to understand what sorts of observational indications these upper layers may show to future telescope missions. Model atmosphere calculations show that these upper layers are likely to be stably stratified (like the Earth’s stratosphere) and, therefore, it is justifiable to use global circulation models to describe and explore their flow dynamics (Cooper & Showman 2002, Cho et al. 2003, Menou & Rauscher 2009, to name only a few). Preliminary examinations of the response of magnetized shallow-water models of exoplanets reported by Cho (2008) indicate that under not-too unreasonable conditions, the atmospheres of irradiated exoplanets could settle into a collection of stationary, planetary scale, magnetized vortices with zero total circulation. If this turns out to be a robust feature for hot irradiated Jupiters (for instance) then there will be observational consequences which may be detectable by the next generation of space missions.

The aim of this work is to begin to formulate a systematic approach toward understanding the dynamics of stably stratified flows on exoplanets under the influence of magnetohydrodynamic effects which are usually absent in standard meteorological modeling. The simplest place to begin exploring these kind of dynamics is in a quasi-geostrophic (“QG” for short) framework which is a setting representing dynamics (i) occurring on synoptic scales (horizontal scale dynamics which are much smaller than the planetary radius), (ii) described by order 1 Burger numbers and (iii) characterized by small Rossby number flows. The last of these is a measure of the ratio of the typical rotation time of the planet to the circulation time of a synoptic scale vortical structure. The Burger number is also a ratio between the planet’s rotation...
time to the gravity wave propagation time across the synoptic scale. If such an atmosphere is also magnetic, then another important number governing the quality of the dynamics will be the Cowling number which measures the relative importance of the Lorentz force. The Cowling Number here will be understood as the ratio of the energy contained in the magnetic field to the kinetic energy in vortical motions.

When the Cowling number becomes an order 1 quantity QG dynamics will be substantially modified. Examination of magnetohydrodynamic modified quasigeostrophic flow goes back to the original series of studies in the thesis work of Gilman (Gilman 1967a-c) motivated by the problem of solar differential rotation. The model equations derived in Gilman’s original studies are the magnetohydrodynamic analog of the classical quasigeostrophic equations on a \(\beta\)-plane. In these equations the magnetic field is strictly two-dimensional and horizontal, where it is assumed that the magnetic energy content contained in this layer is outweighed by the energy contained in the hydrostatic configuration. As a result, the in-plane magnetic field does not figure into the lowest order geostrophic balance. Furthermore, the field lines, although they behave two-dimensionally, can generate vertical field lines through the action of vertical layer undulations. The variation scales of these vertical field fluctuations are, however, small compared to the horizontal divergences (at next order) and, thus, though vertical field lines may be generated they play no dynamical role in the equations of motion at lowest order.

These equations also support Rossby waves and Alfvén waves, often referred to by their hybridized forms as ‘hydromagnetic-planetary’ waves (Acheson & Hide 1973). Unlike the situation in classical QG, however, the magnetohydrodynamic version of classical shallow water equations (hereafter SWMHD) was developed in Gilman (2000) in order to address the problem of the solar tachocline (Spiegel & Zahn, 1992). The SWMHD are model equations confined to a thin-shell of magnetized gas which is stably (or marginally) stratified. In the familiar non-magnetized setting, the shallow water equations retain the action of gravity waves while the more simplified QG models do not. Gravity waves are absent in the latter because the Burger numbers are order 1 in QG, and since the timescales for dynamics in QG are much longer than the local planetary rotation time (because the Rossby number is small), it means that gravity waves are filtered out. In this sense, one may regard the shallow water equations as containing “more” physics over the QG set. The same is true of the SWMHD model in that it contains the hydromagnetic planetary waves found in the MGQ set and it further supports magnetically modified gravity waves whose properties and structure has received recent scrutiny (Schechter et al. 2001, Zaqarashvili et al. 2007, Heng & Spitkovsky 2009).

Besides the absence of magnetically modified gravity waves, the SWMHD equations and the original MGQ models also diverge from one another in content because the horizontal divergence of the magnetic fields are not zero in SWMHD as they are in the original classical MGQ equations. This means that, for example, the dynamical evolution of the horizontal fields cannot be formulated in terms of the evolution of a scalar ‘flux-function’ as is usually done for horizontal-divergence free fields. It is shown in this study that one can view this non-divergence of the horizontal fields as corresponding to the presence of a globally conserved, two-dimensionally distributed “pseudo”-magnetic monopole charge, \(q^{(m)}(x, y, t)\). Therefore, in this sense, one may characterize the solutions contained within the various model approximations according to whether or not \(q^{(m)}\) is zero. For example, the MGQ developed in the original papers of Gilman from 1967-8 have \(q^{(m)} = 0\) while the SWMHD models have \(q^{(m)} \neq 0\) in general.

What is achieved in this work: Since classical QG offers a sound conceptual platform to focus on and understand the dynamics of (mainly) small Rossby number vortical flows, it would be logical to start studying magnetized exoplanet atmospheres from a similar vantage point. It makes sense to begin this exploration from a MGQ framework and, once having uncovered some of the dynamics that this set of equations contain, verifying that some facet of these dynamics are contained in the SWMHD equations. However, some ambiguity persists as to the fate of the so-called “pseudo”-magnetic monopole distribution in the reduced MGQ system as it is identically zero in Gilman’s original studies but clearly present in SWMHD. Does the MGQ framework contain some features of this “pseudo”-magnetic monopole field \(q^{(m)}\)? The answer is yes and it is one of our goals here to develop an intermediate framework to study magneto-vortical dynamics free of gravity waves but not necessarily free of the \(q^{(m)}\) field. In this work the MGQ equations for arbitrary number of vertical layers of constant density is re-derived using familiar scale analysis techniques as, for instance, found in Vallis (2006) (see also Pedlosky 1987). We display the derived equations for a two-layer model. We show how the various aforementioned features between equation sets relate to one another.

The procedure implemented to derive the MGQ equations are sufficiently general also to establish the scalings which lead to magnetostrophic balance (Acheson & Hide, 1973). Magnetostrophy is a limiting form for small Rossby numbers in which dominant dynamical balance exists between Coriolis, pressure and Lorentz forces. The magnetostrophic approximation has been extensively explored to study the Earth’s geodynamo and associated turbulence (e.g. Fearn 1997, Moffat 2008). In our analysis and overall setting we find that the magnetostrophic balance at low Rossby numbers is achieved when (i) the Cowling number is sufficiently large in proportion to the inverse Rossby number and (ii) the Burger
number is small in proportion to the Rossby number. After establishing this fundamental magnetostrophic balance we proceed to derive a set of self-contained equations describing dynamics which are analogous to the MGQ set. We only derive the resulting equations here and shall return to a more thorough discussion of them in a follow-up study.

In Section 2 the equations and assumptions are presented. In Section 3 the equations of motion are expanded around a given latitude, scaled and analyzed. In this section we formally define the non-dimensional parameters of the system and scale the equations of motion. Most importantly, we highlight the fundamental relationships that must be met between the various parameters in order to attain the aforementioned MGQ and magnetostrophic balancing. The actual expansion procedure completing the derivation of the MGQ equations is detailed in Section 4 while the same is done for the magnetostrophic set in Section 5. Because the main focus of this study is about the MGQ equations, in Section 6 they are summarized followed by a brief discussion demonstrating how a MGQ model reduction does exist which retains the \( q^{(m)} \) field and how, most importantly, the classical MGQ system is recovered when \( q^{(m)} \) is set to zero.

2. Equations and assumptions: an overview

The general equations of motion of an ideal, incompressible MHD fluid in a frame rotating with rotation vector \( \Omega \) are

\[
\frac{dU}{dt} + 2\Omega \cdot \text{curl}U = 1\frac{\text{grad} P - g\hat{r}}{\rho} + \frac{1}{\rho} \text{curl} B, \tag{1}
\]

\[
\text{div} \cdot U = 0, \tag{2}
\]

\[
\frac{dB}{dt} + \text{curl}(U \cdot \text{curl} B) = 0, \tag{3}
\]

\[
\text{div} \cdot B = 0, \tag{4}
\]

where “div”, “grad” and “curl” are the corresponding three dimensional operations of divergence, gradient and curl respectively. These equations will be considered in the next section in a Cartesianized representation around a point where “div”, “grad” and “curl” are the corresponding three dimensional operations of divergence, gradient and curl respectively.

1. The planetary \( \beta \)-plane, nominally centred on a latitudinal zone which is significantly away from the planet’s equatorial zone.

2. The dynamics of small Rossby number disturbances (i.e. \( \text{Ro} \ll 1 \)) are of interest so that we may develop the equations as a power series in this quantity.

3. As is standard practice in atmosphere modeling, we break the atmosphere up into several layers. We assume that the background density fields in each vertical layer is constant so that each layer is effectively incompressible as implied by \( \Omega \). The layers, however, may have differing densities each given by \( \rho_i \) where \( i \) denotes the layer under consideration.

4. We shall allow for either none, some or all of the layers to be electrically conducting so that MHD will be a good description of the dynamics in those layers that are magnetically active. This is why the induction equations for ideal MHD are included.

In the majority of the analysis performed in this study the dynamics are treated in two-layers. However, for the sake of generality of the procedures we implement, let us consider the possibility of an arbitrary number of constant-density layers with, as yet, unassigned conductivity. Let us suppose that the heights delineating the transition from one constant density layer to another is given by \( H_1 < H_2 \) respectively where \( Z = 0 \) represents the nominal “bottom” of the atmosphere (see Figure 1). Then the prescription for the current \( J \) will be represented in this way by

\[
J = \varphi \cdot \text{curl}B; \quad \varphi = \begin{cases} 
\varphi_N, & H_{N-1} < Z \leq H_N \\
\vdots & \vdots \\
\varphi_i, & H_{i-1} < Z \leq H_i \\
\vdots & \vdots \\
\varphi_1, & 0 \leq Z \leq H_1
\end{cases} \tag{5}
\]

The functions \( \varphi_i \) will take on either the values 0 or 1 depending upon the layer \( i \).

Sections 3-4 detail the derivation of the MGQ equations with explicit presentation of a two-layer model. The derivation follows the standard procedure outlined in Vallis (2006, § 5.3, pgs. 207-215) while exploiting some of the scaling arguments invoked by Gilman (1967a). We introduce here some of the relevant scalings and corresponding non-dimensional parameters appearing in the system analyzed. The planetary rotation as viewed from the latitude in question is scaled by \( \Omega \). If the typical meridional/zonal velocity scales are given by \( U \) then we may define the Rossby number

\[
\text{Ro} \equiv \frac{U}{f_0L},
\]

\(^2\) If the planetary rotation is given by \( \Omega \) and if one is at latitude \( \lambda \) then the rotation normal to that latitude is \( \Omega_0 = \Omega \sin \lambda \)
where \( f_0 = 2\Omega_0 \) and \( R \) is the planetary radius and \( \mathcal{L} \) represents the horizontal “synoptic” scale of the dynamics. Dynamic flow timescales are assumed to be \( 1/Ro \) times longer than the Coriolis timescale \( 1/f_0 \) - this is the basis of the quasigeostrophic analysis. The magnetic field strength is scaled by \( B \). This then leads to a natural non-dimensional quantity

\[
C \equiv \frac{B^2}{4\pi \tilde{\rho} U^2},
\]

also known as the Cowling number. The scale density \( \tilde{\rho} \) is equated to the density of the lower atmosphere layer. \( C \) is related to the inverse square of the usual \( \beta \) parameter frequently referred to in plasma physics studies. The Cowling number can be understood as a measure of the energy contained in the magnetic field versus that contained in kinetic motions. It can also be understood as the square of the magnetic Mach number, i.e. \( C = U_A^2 / U^2 \), since \( U_A^2 \equiv B^2 / 4\pi \tilde{\rho} \) is the Alfvén speed squared.

The atmosphere also has the following vertical length scales that are important: the overall vertical extent of the atmosphere \( \mathcal{H} \), and the vertical length scale of atmospheric fluctuations \( \tilde{\h} \). Therefore, the zonal and meridional dynamical lengths \((X, Y)\) are scaled by \( \mathcal{L} \) while the vertical scales \((Z)\) are characterized by \( \mathcal{H} \). Given that the flow speeds are \( O(U) \) it follows that their typical characteristic timescales \((T)\) are \( Ro^{-1} \) times longer than \( 1/f_0 \), consistent with the assertion made above. These are all formally reintroduced in the discussion appearing in the next section. Finally, the Burger number, \( Bu \), measuring the relative importance of gravity, is given by the relationship

\[
Bu^2 = \frac{g\mathcal{H}}{4\Omega_0^2 \mathcal{L}^2}.
\]

### 3. Scalings and analysis

The local Cartesianization of the equations of motion around a latitude \( \lambda_0 \) is given by,

\[
\frac{dU}{dT} + 2\Omega \hat{z} \times U = \frac{1}{\rho} \nabla \Pi - g\hat{z} + \frac{1}{4\pi \rho} (B \cdot \nabla)B, \tag{6}
\]

\[
\nabla \cdot U = 0, \tag{7}
\]

\[
\frac{dB}{dT} = (B \cdot \nabla)U, \tag{8}
\]

\[
\nabla \cdot B = 0, \tag{9}
\]

where the total pressure \( \Pi \) is given by

\[
\Pi = P + \frac{1}{8\pi} B^2. \tag{10}
\]

The coordinate \( Z \) coincides with the effective gravity direction. The component of gravity \( g \) will be taken to be constant. In this construction \( X, Y \) denote the zonal and meridional coordinates respectively.

Before analyzing these equations on the Cartesian plane, we explicitly nondimensionalize all quantities appearing in the equations of motion according to the dimensional quantities stated above. Henceforth, all lower case Latin symbols (some with hats over them) denote the corresponding non-dimensional quantity. The primary parameter that will be used for the following expansions will be the Rossby number, \( Ro \). In terrestrial and planetary studies the Rossby number is typically substantially less than one, i.e. \( Ro \ll 1 \). \( \mathcal{U} \) is the typical velocity scale of the horizontal planetary scale atmospheric motions, that is to say,

\[
U \to \mathcal{U} u, \quad V \to \mathcal{U} v, \tag{11}
\]

where \( u, v \) are the non-dimensionalized order one representations of the longitudinal and latitudinal velocities. Since \( \mathcal{U} \) measures an effective overturning speed of the synoptic scale vortex structures in the atmosphere, the corresponding time scale associated with it scales as \( Ro/\Omega_0 \). Thus we scale the horizontal lengths and time according to:

\[
X \to \mathcal{L} x, \quad Y \to \mathcal{L} y, \quad T \to t/(Ro\Omega_0)
\]

where \( x, y, t \) are now understood to be non-dimensionalized quantities. The vertical scale is measured by \( \mathcal{H} \) and this will be considered small compared to \( \mathcal{L} \). In problems in which the atmosphere is broken up into sublayers \( \mathcal{H} \) represents the full vertical extent of the whole atmosphere under consideration. In order to recover the quasi-geostrophic ordering the smallness of the ratio \( \mathcal{H}/\mathcal{L} \) needs only be \( \ll 1 \). Correspondingly, the vertical velocity will also be assumed to be small in proportion to the ratio \( \mathcal{H}/\mathcal{L} \). Thus we scale these two quantities accordingly as

\[
Z \to \mathcal{H} z, \quad W \to \left( \frac{\mathcal{H}}{\mathcal{L}} \right) \mathcal{U} w
\]
where \( z \) and \( w \) are order 1 non-dimensional quantities. The variation of the Coriolis parameter (which is the usual function of latitude on a planet) is written as

\[
\Omega = \Omega_0 \left( 1 + \frac{L}{R} \beta y + \cdots \right),
\]

where \( \Omega_0 \) is the evaluation of the projected planetary rotation at latitude \( \lambda_0 \) in which \( yL/R \equiv \lambda - \lambda_0 \). \( \Omega_0 = \dot{\Omega} \sin \lambda_0 \) and \( \beta = \cos \lambda_0 \), where \( \dot{\Omega} \) is the planetary rotation rate. The ratio \( L/R \) will be formally shown to be \( \mathcal{O}(Ro) \) in Section 4 in order to retain the planetary-\( \beta \) effect. The magnetic field strength will be measured by \( B \). Thus, in keeping with the formalism developed by Gilman (2000), the horizontal fields are scaled by this,

\[
B_x \rightarrow B b_x, \quad B_y \rightarrow B b_y,
\]

while, because of the small aspect ratios under consideration, \textit{we shall only consider} vertical fields which are smaller than the horizontal components in proportion to the ratio \( \mathcal{H}/L \),

\[
B_z \rightarrow \frac{\mathcal{H}}{L} b_z.
\]

The vertical extent of the atmosphere is subdivided into sublevels with vertical scale \( \mathcal{H}_i \) where \( i \) labels the level number. For our considerations we assume that all \( \Delta \mathcal{H}_i = \mathcal{H}_i - \mathcal{H}_{i-1} \), representing the thicknesses of each layer, are an order 1 fraction of the overall vertical scale \( \mathcal{H} \). We let \( \bar{h}_i \) represent perturbations about the mean level height \( \mathcal{H}_{i0} \), i.e.,

\[
\mathcal{H}_i = \mathcal{H}_{i0} + \bar{h}_i,
\]

If we assume that the deviations scale by a characteristic length \( \bar{h} \), then we may introduce another parameter \( \delta \) which measures the deviation against the vertical scale of the atmosphere

\[
\delta \equiv \frac{\bar{h}}{\mathcal{H}}.
\]

In typical quasigeostrophic scalings this factor \( \delta \) is assumed small, usually \( \mathcal{O}(Ro) \) for \( Ro \) small.

### 3.1. Analysis of the vertical momentum equation

We seek to develop reductions of the equations of motion which are dynamically in hydrostatic equilibrium. This means assuming the pressure scaling to be

\[
\Pi \rightarrow g \mathcal{H} \bar{\rho} \bar{\Pi},
\]

where \( \bar{\rho} \) is the density scale, typically of the lowest atmosphere layer. Examining the vertical component of the momentum equation with the scalings thus far presented reveals

\[
Ro^2 \left( \frac{\mathcal{H}}{L} \right)^2 \left( \partial_t + u \cdot \nabla + \omega \partial_z \right) w = Bu^2 \bar{\rho} \left( \partial_t \bar{\Pi} + \frac{\rho}{\bar{\rho}} \right) + Ro^2 C \bar{\rho} \left( \frac{\mathcal{H}}{L} \right)^2 (b \cdot \nabla + b_z \partial_z)b_z.
\]

Where the Burger number, \( Bu \), and the Cowling number, \( C \), are as defined at the end of Section 2. To insure that hydrostatic balance is always dominant we shall assume \( Bu \) always dominates the other terms appearing in (13). This means to say specifically that unless

\[
Bu^2 \gg Ro^2 \left( \frac{\mathcal{H}}{L} \right)^2, \quad \text{and} \quad Bu^2 \gg Ro^2 \left( \frac{\mathcal{H}}{L} \right)^2 C,
\]

the assumption of hydrostatic balance will fail. For planetary atmospheres this criterion is generally met since \( \mathcal{H}/L \ll 1 \). The new piece here is the second disparity involving the Cowling number. We return to this shortly below. As of this point we have said nothing about the specific orderings required of the both the Burger and Cowling numbers. In general for this study we shall assume that \( Ro \) and the ratio \( \mathcal{H}/L \) are sufficiently smaller than 1. It follows that the lowest order balance of the vertical momentum equation (13) is hydrostatic,

\[
\partial_t \bar{\Pi} + \frac{\rho}{\bar{\rho}} = 0.
\]

Henceforth we shall treat \( \rho \) as being non-dimensionalized by \( \bar{\rho} \) which means, in practice, that \( \rho/\bar{\rho} \) will be replaced everywhere by \( \rho \). When multiple layers are handled we shall characterize the density as being constant in each layer with value \( \rho_i \). Furthermore, to be consistent with what was stated earlier, the density of the lowest layer will always be unity, thus \( \rho_1 = 1 \). The procedure performed here essentially mimics the strategy employed in Vallis (2006). We allow the top layer to move about freely and assume the pressure is zero on the top surface. The solution for the pressure
for an N-layered atmosphere (with a flat bottom located at \(z = 0\)) expressed in terms of the non-dimensionalized lid coordinates \(H_{i}\) is given by

\[
\tilde{\Pi} = \int_{H_{N}}^{z} \frac{\rho}{\rho_{i}} dz = \begin{cases}
\rho_{N}(H_{N} - z), & H_{N} > z \geq H_{N-1} \\
\sum_{k=N}^{N} \rho_{k}(H_{k} - H_{k-1}) + \rho_{N-1}(H_{N-1} - z), & H_{N-1} > z \geq H_{N-2} \\
\vdots & \vdots \\
\sum_{k=i+1}^{N} \rho_{k}(H_{k} - H_{k-1}) + \rho_{i}(H_{i} - z), & H_{i} > z \geq H_{i-1} \\
\vdots & \vdots \\
\sum_{k=2}^{N} \rho_{k}(H_{k} - H_{k-1}) + \rho_{1}(H_{1} - z), & H_{1} \geq z \geq 0.
\end{cases}
\]  

(16)

Note that in the above formulation the summation procedure begins with layer \(i = N - 1\) and that it is explicitly absent in the top, \(i = N\), layer. As was motivated earlier, the levels \(H_{i}\) are further written in the form

\[H_{i} = H_{i0} + \delta h_{i},\]

where the set \(\{H_{i0}\}\) are constants. Henceforth the procedure outlined in the remainder of this study will focus only on a two-layer system with the understanding that it straightforwardly generalizes to a multi-layer configuration. Given the functional form postulated above we find that horizontal gradients (\(\nabla \equiv \hat{x}\partial_{x} + \hat{y}\partial_{y}\)) of the pressure are given by

\[\nabla \tilde{\Pi} = \delta \nabla \Pi = \delta \nabla \left\{ \begin{array}{l}
\hat{\Pi}_{2} = \rho_{2}h_{2}, \quad H_{2} > z \geq H_{i} \\
\hat{\Pi}_{1} = \rho_{2}h_{2} + (\rho_{1} - \rho_{2})h_{1}, \quad H_{1} \geq z \geq 0.
\end{array} \right.\]

(18)

### 3.2. Analysis of the horizontal momentum equations

Now we are prepared to move onto the horizontal momentum equations. With the scalings proposed above the equations become

\[
\begin{align*}
\text{Ro}^{2} \left( \partial_{t} + u \cdot \nabla + w \partial_{z} \right) u + 2\text{Ro} \left( 1 + \frac{\mathcal{L}}{R} \beta y \right) \hat{z} \times u & = \\
-\delta Bu^{2} \frac{1}{\rho} \nabla \hat{\Pi} + \text{Ro}^{2} \frac{C}{\rho} (b \cdot \nabla + b_{z} \partial_{z}) b + \mathcal{O} \left( \text{Ro}^{2} \frac{\mathcal{L}}{R} \right),
\end{align*}
\]

(19)

in which \(u = u_{x} \hat{x} + \nu \hat{y}\) and \(b = b_{x} \hat{x} + b_{y} \hat{y}\). We note that the curvature terms neglected at this stage come in at an ordering of \(\mathcal{O} \left( \text{Ro}^{2} \right) \cdot \mathcal{O} \left( \mathcal{L}/R \right)\), and since \(\mathcal{L}/R\) will be assumed to be \(\ll 1\), the curvature terms will be neglected throughout. It is at this stage that we explore various balances:

#### 3.2.1. Quasigeostrophic/magnetoquasigeostrophic scalings

We assume that the deviations from the mean height scale to the Rossby number, i.e. \(\delta = \mathcal{O} \left( \text{Ro} \right)\). Although it would be enough to require that \(\mathcal{L}/R \ll 1\), in order to include the planetary \(\beta\) we require that \(\mathcal{L}/R \sim \mathcal{O} \left( \text{Ro} \right)\). In Section 4 we detail the full procedure leading to the quasigeostrophic reduction, however, we show here the leading order balances to illustrate the main flavour of the results. Expanding the pressure and velocities within each layer \(i\) in powers of \(\text{Ro}\) according to, \(\hat{\Pi}_{i} = \hat{\Pi}_{i}^{(0)} + \text{Ro} \hat{\Pi}_{i}^{(1)} + \cdots\), \(u_{i} = u_{i}^{(0)} + \text{Ro} u_{i}^{(1)} + \cdots\) and \(b_{i} = b_{i}^{(0)} + \text{Ro} b_{i}^{(1)} + \cdots\). then it follows that the leading order terms of \(\hat{\Pi}_{i}^{(0)}\), which are \(\mathcal{O} \left( \text{Ro} \right)\), reduce to the fundamental statement of geostrophic balance, that is to say,

\[2\hat{z} \times u_{i}^{(0)} = -\frac{\delta Bu^{2}}{\rho_{i}} \nabla \hat{\Pi}_{i}^{(0)}.\]

(20)

Because we consider the density in each layer to be constant, taking the curl of the above equation immediately reveals that the horizontal divergence of the lowest order velocity field is identically zero

\[\partial_{x} u_{i}^{(0)} + \partial_{y} v_{i}^{(0)} = 0.\]

Therefore it follows from this that the lowest order vertical velocity \(w_{i}^{(0)}\) must zero. Requiring \(\delta\) to be less than order 1 actually follows from the horizontal divergence free condition of the lowest order horizontal velocity field. This consistency is formally shown in Section 4.

In order to understand how magnetic effects enter the resulting equations we carry out the expansion to order \(\text{Ro}^{2}\) to find,

\[\left( \partial_{t} + u_{i}^{(0)} \cdot \nabla \right) u_{i}^{(0)} + 2\hat{z} \times u_{i}^{(1)} + \beta y \hat{z} \times u_{i}^{(0)} = -\frac{\delta Bu^{2}}{\rho_{i}} \nabla \hat{\Pi}_{i}^{(1)} + \frac{C}{\rho_{i}} \left( b_{i}^{(0)} \cdot \nabla \right) b_{i}^{(0)}.\]

\(^{3}\) To avoid any ambiguity: the general form of the disturbance pressure would be given by the general formulae \(\hat{\Pi}_{i}^{(0)} = \rho_{i}h_{i} + \sum_{k=i+1}^{N} \rho_{k}(h_{k} - h_{k-1})\) with \(\hat{\Pi}_{N}^{(0)} = \rho_{n}h_{n}.\)
The scalings assumed at the beginning of this section result in the incompressibility equation in each layer rewritten as

\[ \nabla \cdot \mathbf{u} = 0. \]

The induction equations are also similarly written as

\[ (\partial_t + \mathbf{u} \cdot \nabla) \mathbf{b} = (\mathbf{b} \cdot \nabla + \mathbf{b}_z \partial_z) \mathbf{u}, \]

### Table 1. Summary of scalings and lowest order balances for hydrostatic disturbances.

All balances shown assume \( \mathcal{R}_o \ll 1 \). The quoted extreme values in describing magnetostrophic balance is the same as saying the Acheson and modified Burger numbers (denoted in the text by \( A \) and \( \tilde{B}_u \) respectively) are \( \mathcal{O}(1) \).

| Balance Name | \( \mathcal{O}(\tilde{B}_u) \) | \( \mathcal{O}(\delta) \) | \( \mathcal{O}(\mathcal{C}) \) | \( \mathcal{O}(\frac{1}{\mathcal{C}}) \) | Lowest Order Balance |
|--------------|-----------------|-----------------|-----------------|-----------------|------------------|
| QG           | 1               | \( \mathcal{R}_o \) | \( \ll 1 \)      | \( \mathcal{R}_o \) | Coriolis, Pressure Gradient |
| MQG          | 1               | \( \mathcal{R}_o \) | 1               | \( \mathcal{R}_o \) | Coriolis, Pressure Gradient |
| Magnetostrophy | \( \mathcal{R}_o^{1/2} \) | 1 | \( \mathcal{R}_o^{-1} \) | \( \ll 1 \) | Coriolis, Pressure Gradient, Lorentz Force |

The above is a restate ment of Eq. (57) where in arriving at this equation it is argued that the horizontal components of the magnetic field are independent of \( z \) at lowest order. All but the last term on the RHS of the above expression constitute the next order corrections which ultimately lead to the classical equations of quasigeostrophy. In particular if \( \mathcal{C} \ll 1 \) then this limiting form identically leads to classical QG. We see that the first non-trivial in clusion of magnetic effects comes in when \( \mathcal{C} \) is an order 1 quantity. In this event, the resulting equations will be called magnetoquasigeostrophy (MGQ). The completed derivation of the latter is found in Section 4.

#### 3.2.2. Magnetostrophic balance

When magnetic effects are relatively strong there exists another more general lowest order balance between Coriolis, horizontal pressure gradients and the horizontal components of the Lorentz force. This magnetostrophic balance has been explored in the context of the Earth’s geodynamo (Acheson & Hide 1973, Fearn 1997, Moffatt 2008, to name just a few). Inspection of (19) shows that in order to bring the Lorentz term in on the same order as the Coriolis term requires \( \mathcal{C} = \mathcal{O}(\mathcal{R}_o^{-1}) \). Thus, we define a new number

\[ A \equiv \frac{B^2}{8\pi R_d L}. \]

which we shall, henceforth, call the Acheson Number. For this three-way balance to occur the Acheson Number must be an order 1 quantity. Similar inspection of (19) shows that in order for the horizontal pressure gradient to enter the mix we must have the combination \( \delta \tilde{B}_u^2 = \mathcal{O}(\mathcal{R}_o) \). Inspection also shows that if the resulting scaling arguments are consistent with the resulting equations \( \delta \) must be \( \mathcal{O}(1) \) (see below). Thus we define a modified Burger number \( \tilde{B}_u \) such that \( \tilde{B}_u = \mathcal{R}_o^{1/2} \tilde{B}_u \), where

\[ \left( \tilde{B}_u \right)^2 \equiv \frac{g H}{2 \mathcal{R}_o \mathcal{L} L}. \]

With the assumption that (i) all quantities may be expanded in powers of \( \mathcal{R}_o \) (as we did in the previous section) and (ii) the lowest order horizontal velocity and magnetic field components are \( z \)-independent in a layer, then we find that the magnetostrophic balances in the horizontal directions are

\[ 2z \times \mathbf{u}_i^{(0)} = -\tilde{B}_u \frac{1}{\rho_i} \nabla \Pi_i^{(0)} + \frac{A}{\rho_i} (\mathbf{b}_i^{(0)} \cdot \nabla) \mathbf{b}_i^{(0)}. \]

The horizontal divergence of the lowest order velocity is not zero in general. In fact,

\[ \partial_x u_i^{(0)} + \partial_y v_i^{(0)} = \frac{A}{2 \rho_i} \left( \partial_x b_i^{(0)} + \partial_y b_i^{(0)} + b_i^{(0)} \cdot \nabla \right) J_i^{(0)} \neq 0, \]

in which \( J_i^{(0)} \) is the lowest order vertical current in the layer, \( J_i^{(0)} \equiv \partial_x b_i^{(0)} - \partial_y b_i^{(0)} \). Unlike what we encounter in the MGQ scalings, it follows that the lowest order vertical velocity cannot be zero at lowest order. Since \( \delta \) measures the vertical variations of each layer height and since the latter is related to the vertical velocity, in order for the magnetostrophic scalings to be consistent and to have the lowest order horizontal divergences to be non-zero it must be that \( \delta = \mathcal{O}(1) \), which is to say vertical motions are driven by vertical currents due to the Lorentz force at lowest order. The continued analysis of the magnetostrophic equations and development of a self-contained equation set is presented in Section 5.

In Table 1 we summarize the relative scalings of the relevant quantities that produce the reduced equations discussed to now (i.e., QG, MQG and magnetostrophy).

#### 3.3. Analysis of the incompressibility and induction equations

The scalings assumed at the beginning of this section result in the incompressibility equation in each layer rewritten as

\[ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \partial_z \mathbf{w} = 0. \]

The induction equations are also similarly written as

\[ (\partial_t + \mathbf{u} \cdot \nabla + \mathbf{w} \partial_z) \mathbf{b} = (\mathbf{b} \cdot \nabla + \mathbf{b}_z \partial_z) \mathbf{u}, \]
with the vertical field component \( b_z \), relating to the other field components through the divergence free condition

\[
\partial_x b_x + \partial_y b_y + \partial_z b_z = 0.
\]

(25)

4. Detailed derivation of the equations of magnetoquasigeostrophy

We formally develop the scaling analysis and equation reduction that we began in Section 3.2.1. Quantities in each layer \( i \) are expanded in the following way,

\[
\begin{align*}
\mathbf{u}_i &= \mathbf{u}^{(0)}_i + \text{Ro} \mathbf{u}^{(1)}_i + \cdots \\
b_i &= b^{(0)}_i + \text{Ro} b^{(1)}_i + \cdots \\
\hat{\Pi}_i &= \Pi^{(0)}_i + \text{Ro} \Pi^{(1)}_i + \cdots ,
\end{align*}
\]

(26)

where the superscript denotes which order of the Ro expansion the term represents. As we stated before, insertion of these expansions into (19) results in (20) to lowest order. Since we are considering a two-layer problem we have by layer that

\[
\begin{align*}
u^{(0)}_1 &= -\partial_y \frac{B u^2}{2 \rho_1} \Pi^{(0)}_1, & v^{(0)}_1 &= \partial_z \frac{B u^2}{2 \rho_1} \Pi^{(0)}_1, \\
u^{(0)}_2 &= -\partial_y \frac{B u^2}{2 \rho_2} \Pi^{(0)}_2, & v^{(0)}_2 &= \partial_z \frac{B u^2}{2 \rho_2} \Pi^{(0)}_2,
\end{align*}
\]

(27)

and

\[
\begin{align*}
u^{(0)}_1 &= -\partial_y \frac{B u^2}{2 \rho_1} \Pi^{(0)}_1, & v^{(0)}_1 &= \partial_z \frac{B u^2}{2 \rho_1} \Pi^{(0)}_1, \\
u^{(0)}_2 &= -\partial_y \frac{B u^2}{2 \rho_2} \Pi^{(0)}_2, & v^{(0)}_2 &= \partial_z \frac{B u^2}{2 \rho_2} \Pi^{(0)}_2,
\end{align*}
\]

(28)

It will prove to be more convenient to write the pressure fields \( \hat{\Pi}_i \) in terms of a streamfunction \( \psi_i \) such that

\[
\begin{align*}
\psi^{(0)}_1 &= \frac{B u^2}{2 \rho_1} \Pi^{(0)}_1 = \frac{B u^2}{2} \left[ \frac{\rho_2}{\rho_1} h^{(0)}_1 + \left( 1 - \frac{\rho_2}{\rho_1} \right) h^{(0)}_1 \right] , \\
\psi^{(0)}_2 &= \frac{B u^2}{2 \rho_2} \Pi^{(0)}_2 = \frac{B u^2}{2} h^{(0)}_2 ,
\end{align*}
\]

(29)

and

\[
\begin{align*}
\psi^{(0)}_1 &= \frac{B u^2}{2 \rho_1} \Pi^{(0)}_1 = \frac{B u^2}{2} \left[ \frac{\rho_2}{\rho_1} h^{(0)}_1 + \left( 1 - \frac{\rho_2}{\rho_1} \right) h^{(0)}_1 \right] , \\
\psi^{(0)}_2 &= \frac{B u^2}{2 \rho_2} \Pi^{(0)}_2 = \frac{B u^2}{2} h^{(0)}_2 ,
\end{align*}
\]

(30)

where the height deviations have been similarly expanded in powers of Ro, i.e.

\[
\begin{align*}
\hat{h}_i &= h^{(0)}_i + \text{Ro} h^{(1)}_i + \cdots .
\end{align*}
\]

This rewriting expresses the relationships more transparently

\[
\begin{align*}
u^{(0)}_1 &= -\partial_y \psi^{(0)}_1, & v^{(0)}_1 &= \partial_z \psi^{(0)}_1, \\
u^{(0)}_2 &= -\partial_y \psi^{(0)}_2, & v^{(0)}_2 &= \partial_z \psi^{(0)}_2.
\end{align*}
\]

(31)

Evidently the fields \( \mathbf{u}^{(0)}_i \) are 2-dimensional incompressible flow independent of the vertical coordinate \( z \). This means from the incompressibility equation (23)

\[
\partial_x u^{(0)}_i + \partial_y v^{(0)}_i + \partial_z w^{(0)}_i = 0,
\]

we get

\[
\partial_z w^{(0)}_i = 0.
\]

Thus the leading order vertical velocity term is zero, motivating its expansion to be

\[
w_i = \text{Ro} w^{(1)}_i + \cdots
\]

(32)

Note that this is necessarily consistent with the assumption made at the outset that \( \delta = \mathcal{O}(\text{Ro}) \). In other words this says that if the vertical velocities are small then so must be the vertical variations of height.

We turn to the leading order equation for the horizontal components of the magnetic field and we find

\[
\left( \partial_t + \mathbf{u}^{(0)}_i \cdot \nabla \right) b^{(0)}_i = \left( b^{(0)}_i \cdot \nabla + b^{(0)}_z \partial_z \right) \mathbf{u}^{(0)}_i.
\]

However, since the horizontal velocity components are independent of \( z \) the above equation reduces to

\[
\left( \partial_t + \mathbf{u}^{(0)}_i \cdot \nabla \right) b^{(0)}_i = b^{(0)}_i \cdot \nabla \mathbf{u}^{(0)}_i,
\]

(34)

if the totality of all the layers are not moving.
which implies that $b_i^{(0)}$ is independent of $z$ as well. The leading order expansion of the source free condition
\begin{equation}
\partial_t b_{x i}^{(0)} + \partial_y b_{y i}^{(0)} = -\partial_z h_i^{(0)}
\end{equation}
relates the leading order vertical field $h_i^{(0)}$ to $b_i^{(0)}$. We note that it means $h_i^{(0)}$ is at most a linear function of the coordinate $z$ within each layer. Furthermore, and quite unlike the geostrophic balance situation we encountered before for the leading order horizontal velocities, the horizontal magnetic fields are not strictly two-dimensional, i.e. $b_i^{(0)}$ is not identically zero (Gilman, 2000). It follows from this that there are modes of dynamical activity which will allow the divergence of the leading order horizontal magnetic field resulting in a certain amount of “breathing” into the vertical field component.

At order $\text{Ro}$ of the incompressibility condition we find
\begin{equation}
\partial_x u_i^{(1)} + \partial_y v_i^{(1)} = -\partial_z w_i^{(1)}.
\end{equation}

The remaining order $\text{Ro}$ terms of the horizontal momentum balance equation
\begin{equation}
(\partial_t + u_i^{(0)} \cdot \nabla) u_i^{(0)} + 2z \times u_i^{(1)} + \beta y \cdot \nabla u_i^{(0)} = -B u^2 \frac{1}{\rho_i} \nabla h_i^{(1)} + C_i (b_i^{(0)} \cdot \nabla) b_i^{(0)},
\end{equation}
where we have explicitly used the fact that $b_i^{(0)}$ is independent of $z$ in writing (37). Because an examination of the next order expansion of the vertical momentum balance equation reveals that $h_i^{(1)}$ is independent of the vertical coordinate, it follows that the next order horizontal velocity corrections $u_i^{(1)}$ are also independent of $z$. We may therefore take the curl of the above equation by operating the $y$-momentum component by $\partial_x$ and subtracting from it the $\partial_y$ operation upon the $x$-momentum equation to get
\begin{equation}
(\partial_t + u_i^{(0)} \cdot \nabla) (\partial_x u_i^{(0)} - \partial_y b_i^{(0)}) + 2\Omega_i^{(1)} = C_i (b_i^{(0)} \cdot \nabla) (\partial_x b_i^{(0)} - \partial_y b_i^{(0)}) + C_i (\partial_x b_i^{(0)} + \partial_y b_i^{(0)}) (\partial_x b_i^{(0)} - \partial_y b_i^{(0)}).
\end{equation}

with $C_i \equiv C_i / \rho_i$ and where, for notational convenience, we have defined
\begin{equation}
\Omega_i^{(1)} = - (\partial_x u_i^{(1)} + \partial_y v_i^{(1)}).
\end{equation}

In order to proceed we must determine how $\Omega_i^{(1)}$ relate to $h_i^{(0)}$. To do this we begin by determining the equation for each of the level heights $h_i$. The previous scaling analysis leads to the non-dimensionalized version for them which, at leading order, reveals that
\begin{equation}
(\partial_t + u_i^{(0)} \cdot \nabla) h_i^{(0)} = w_i^{(1)}(z = H_i)
\end{equation}
where care must be taken since one may evaluate the motion of the $j^{th}$ surface using the velocity fields from layer $i$. Specifically we explicitly write out the solution to the vertical velocities for each layer. In the bottom layer we have
\begin{equation}
w_1^{(1)}(z = H_1) = H_{10} \Omega_1^{(1)},
\end{equation}
in which we have implicitly forced the vertical velocity to be zero at the bottom $z = 0$. To leading order it follows that
\begin{equation}w_1^{(1)}(z = H_1) = H_{10} \Omega_1^{(1)}.
\end{equation}
When viewed from the lower layer then we have the motion of the lower layer’s interface to be given by
\begin{equation}
(\partial_t + u_1^{(0)} \cdot \nabla) h_1^{(0)} = H_{10} \Omega_1^{(1)}.
\end{equation}

For the upper layer we have
\begin{equation}
w_2^{(1)} = w_2^{(1)}(z = 0) + \Omega_2^{(1)} z,
\end{equation}where $w_2^{(1)}$ is a constant velocity to be determined below. Now it is evident that the motion of the bottom boundary when viewed from the upper layer is
\begin{equation}
(\partial_t + u_2^{(0)} \cdot \nabla) h_1^{(0)} = w_2^{(1)} + H_{10} \Omega_2^{(1)}.
\end{equation}while the motion of the upper boundary when viewed from the upper layer is
\begin{equation}
(\partial_t + u_2^{(0)} \cdot \nabla) h_2^{(0)} = w_2^{(1)} + H_{20} \Omega_2^{(1)}.
\end{equation}
Subtracting these two expresses the evolution of the width of the upper layer in terms of \( \Omega^{(1)} \), that is to say,

\[
\left( \partial_t + \mathbf{u}^{(0)} \cdot \nabla \right) \left( h^{(0)}_1 - h^{(0)}_2 \right) = (H_20 - H_10)\Omega^{(1)},
\]

(47)

which, we note, is purely in terms of the velocity fields \( \mathbf{u}^{(0)} \). We are now in a position to rewrite (43) layer by layer by replacing all instances of \( \Omega^{(1)} \) with its appropriate expressions in terms of \( h^{(0)}_i \) given by (43) and (47). To this end, we begin with the bottom layer

\[
\left( \partial_t + \mathbf{u}^{(0)} \cdot \nabla \right) \left[ \partial_x b^{(0)}_1 - \partial_y \mathbf{u}^{(0)}_1 - \frac{2h^{(0)}_1}{H_10} \right] = C \left( \mathbf{b}^{(0)}_1 \cdot \nabla + \partial_x b^{(0)}_y + \partial_y b^{(0)}_y \right) \left( \partial_x b^{(0)}_y - \partial_y b^{(0)}_y \right),
\]

(48)

while for the top layer we have

\[
\left( \partial_t + \mathbf{u}^{(0)} \cdot \nabla \right) \left[ \partial_x b^{(0)}_2 - \partial_y \mathbf{u}^{(0)}_2 - \frac{2h^{(0)}_2}{H_20 - H_10} \right] =
C \left( \frac{\rho_1}{\rho_2} \right) \left( \mathbf{b}^{(0)}_2 \cdot \nabla + \partial_x b^{(0)}_2 + \partial_y b^{(0)}_2 \right) \left( \partial_x b^{(0)}_y - \partial_y b^{(0)}_y \right).
\]

(49)

In order to simplify the expressions appearing above we rewrite the heights \( h^{(0)}_i \) in terms of the streamfunctions

\[
h^{(0)}_1 = \frac{2}{\text{Bu}^2} \left( \frac{\psi^{(0)}_1 - \frac{\rho_2}{\rho_1} \psi^{(0)}_2}{1 - \frac{\rho_2}{\rho_1}} \right), \quad h^{(0)}_2 = \frac{2}{\text{Bu}^2} \psi^{(0)}_2.
\]

(50)

We may now rewrite the equations in more transparent form

\[
\left( \partial_t + \mathbf{u}^{(0)} \cdot \nabla \right) Q^{(0)}_1 + \frac{\beta}{\rho} v^{(0)}_1 = C \left( \mathbf{b}^{(0)}_1 \cdot \nabla + q^{(0)}_{m1} \right) J^{(0)}_1,
\]

(51)

\[
\left( \partial_t + \mathbf{u}^{(0)} \cdot \nabla \right) Q^{(0)}_2 + \frac{\beta}{\rho} v^{(0)}_2 = C \left( \frac{\rho_1}{\rho_2} \right) \left( \mathbf{b}^{(0)}_2 \cdot \nabla + q^{(0)}_{m2} \right) J^{(0)}_2
\]

(52)

where the potential vorticity in each layer is defined by

\[
Q^{(0)}_1 = \nabla^2 \psi^{(0)}_1 - \frac{1}{L^{(0)}_1} \left( \psi^{(0)}_1 - \frac{\rho_2}{\rho_1} \psi^{(0)}_2 \right),
\]

(53)

\[
Q^{(0)}_2 = \nabla^2 \psi^{(0)}_2 - \frac{1}{L^{(0)}_2} \left( \psi^{(0)}_2 - \psi^{(0)}_1 \right),
\]

(54)

in which the \( L^{(0)}_1 \) and \( L^{(0)}_2 \) denote the Rossby radius of deformation for layers 1 and 2 respectively,

\[
\frac{1}{L^{(0)}_1} \equiv \frac{4}{\text{Bu}^2 H_10 \left( 1 - \frac{\rho_2}{\rho_1} \right)}, \quad \frac{1}{L^{(0)}_2} \equiv \frac{4}{\text{Bu}^2 (H_20 - H_10) \left( 1 - \frac{\rho_2}{\rho_1} \right)}.
\]

(55)

The form presented above diverges slightly from the form presented in Vallis wherein the upper layer is constrained by an upper boundary while here we allow for the upper layer to undulate freely. The difference then is that the ratio \( \rho_2/\rho_1 \) appearing on the RHS of (53), defining \( Q^{(0)}_1 \), would be replaced by 1. We have also defined the currents \( J^{(0)}_i \) appropriate to layer \( i \)

\[
J^{(0)}_1 \equiv \partial_x b^{(0)}_y - \partial_y b^{(0)}_y, \quad J^{(0)}_2 \equiv \partial_x b^{(0)}_y + \partial_y b^{(0)}_y,
\]

(56)

and the corresponding layer densities \( q^{(0)}_{m1} \),

\[
q^{(0)}_{m1} \equiv \partial_x b^{(0)}_y + \partial_y b^{(0)}_y, \quad q^{(0)}_{m2} \equiv \partial_x b^{(0)}_y + \partial_y b^{(0)}_y.
\]

(57)

All the terms in these equations, as appearing in the summary Section 6, will be expressed with their individual superscripts “(0)” removed. Additionally, the expression for Bu is rewritten in terms the definition of the Rossby Deformation Radii.
5. Further development of the magnetostrophic limit and the derivation of a closed set of equations

The leading order magnetostrophic balance \((21)\) leads to a diagnostic specification of the horizontal velocities which we explicitly write here

\[
2e_i^{(0)} = - \left( \frac{B_i^2}{\rho_i} \frac{\partial_t \Pi_i^{(0)}}{N} + \frac{A}{\rho_i} b_i^{(0)} \cdot \nabla b_i^{(0)} \right),
\]

\[
2u_i^{(0)} = - \left( \frac{B_i^2}{\rho_i} \frac{\partial_t \Pi_i^{(0)}}{N} + \frac{A}{\rho_i} b_i^{(0)} \cdot \nabla b_i^{(0)} \right).
\]

(58)

We shall assume henceforth that the horizontal components of the velocity and magnetic fields are z-independent within a given layer. Thus, the leading order induction equation is the same as \((54)\) which we rewrite here for convenience,

\[
\left( \partial_t + u_i^{(0)} \cdot \nabla \right) b_i^{(0)} = b_i^{(0)} \cdot \nabla u_i^{(0)}.
\]

The horizontal velocity components are expressed in terms of \(b_i^{(0)}\) and the perturbation height fluctuations \(h_i\) are contained in the perturbation pressure fields \(\hat{\Pi}_i^{(0)}\). Because the horizontal divergence of the lowest order velocity field was shown to be non-zero and subsequently assumed to be independent of the vertical coordinate \(z\), we can write the leading order vertical velocity to be

\[
w_i^{(0)} = w_{i0}^{(0)} + z\Omega_{i}^{(0)},
\]

where, as before, \(w_{i0}^{(0)}\) is a z-independent undetermined vertical velocity. Because of the relationships set up between the velocities according to the leading order continuity equation

\[
\partial_x u_i^{(0)} + \partial_y v_i^{(0)} + \partial_z w_i^{(0)} = 0,
\]

together with the help of \((22)\), we can express the quantity \(\Omega_i^{(0)}\) in terms of the horizontal magnetic field quantities as

\[
\Omega_i^{(0)} = -\frac{A}{2\rho_i} \left( \partial_x b_{i1}^{(0)} + \partial_y b_{i2}^{(0)} + b_i^{(0)} \cdot \nabla \right) J_i^{(0)}.
\]

(60)

We develop an equation for the height fluctuations in the same manner as was done in the development of the MGQ equations. Because we have argued the fluctuations are on the same scale as the mean height of any given layer (i.e. because \(\delta = O(1)\)), instead of expressing the height \(H_i\) as the sum \(H_{i0} + \delta h_i\) we shall simply stick with the expression \(H_i\) since it now makes no difference as both quantities are the same order of magnitude. Thus the vertical motion of layer height with position \(z = H_i\) as viewed from the layer with index \(i\) is to leading order

\[
\left( \partial_t + u_i^{(0)} \cdot \nabla \right) H_i = w_{i0}^{(0)} + H_i\Omega_i^{(0)}.
\]

Similarly the motion of the layer height with position \(z = H_{i-1}\) as viewed from the same layer with index \(i\) is to leading order

\[
\left( \partial_t + u_i^{(0)} \cdot \nabla \right) H_{i-1} = w_{i0}^{(0)} + H_{i-1}\Omega_i^{(0)}.
\]

Subtracting these two equations and reordering the results reveals

\[
\left( \partial_t + u_i^{(0)} \cdot \nabla \right) \ln (H_i - H_{i-1}) = \Omega_i^{(0)} = -\frac{A}{2\rho_i} \left( \partial_x b_{i1}^{(0)} + \partial_y b_{i2}^{(0)} + b_i^{(0)} \cdot \nabla \right) J_i^{(0)}.
\]

(61)

As presented, this magnetostrophic limit involves evolving equations \((22)\) and \((61)\) with the horizontal flow quantities determined by the magnetostrophic balances explicitly given in \((58)\) and \((59)\), where the perturbation pressure fields \(\hat{\Pi}_i^{(0)}\), given in \((18)\), are re-expressed in terms of \(H_i\). We remind the reader that \((18)\) explicitly shows the pressure field quantities for a two-layer system. In general, for layers \(i < N\) with \(h_i\) expressed in terms of \(H_i\), we would have,

\[
\hat{\Pi}_i^{(0)} = \rho_i H_i + \sum_{k=i+1}^{N} \rho_k (H_k - H_{k-1}),
\]

(62)

and for layer \(N\) we have \(\hat{\Pi}_N^{(0)} = \rho_N H_N\).

To illustrate how these equations appear we write them out for a single layer atmosphere where we drop the superscripts,

\[
\left( \partial_t + u_1 \cdot \nabla \right) b_1 = b_1 \cdot \nabla u_1,
\]

\[
\left( \partial_t + u_1 \cdot \nabla \right) \ln H_1 = - (A/2) \left( \partial_x b_{11} + \partial_y b_{12} + b_1 \cdot \nabla \right) J_1,
\]

(63)

(64)
in which the velocities and currents are diagnostically given by
\begin{align}
2v_1 &= -\tilde{B}u \partial_x H_1 - Ab_1 \cdot \nabla b_x, \\
2u_1 &= \tilde{B}u \partial_y H_1 + Ab_1 \cdot \nabla b_y, \\
J_1 &= \partial_y b_y - \partial_x b_x.
\end{align}

Note that these equations are governed by the two parameters, the modified Burger number \( \tilde{B}u \) and the Acheson number \( A \).

### 6. Summary and brief discussion of the equations of magnetoquasigeostrophy

The result of the lengthy procedure detailed in Section 4 are summarized for two layers. The equations for the potential vorticity in each layer are given explicitly,
\begin{align}
(\partial_t + \mathbf{u}_1 \cdot \nabla) Q_1 + \beta v_1 &= \varphi_1 C \left( q_1^{(m)} + \mathbf{b}_1 \cdot \nabla \right) J_1, \\
(\partial_t + \mathbf{u}_2 \cdot \nabla) Q_2 + \beta v_2 &= \varphi_2 C \left( \frac{\rho_1}{\rho_2} q_2^{(m)} + \mathbf{b}_2 \cdot \nabla \right) J_2,
\end{align}
in which the horizontal velocities in vector form are given by \( \mathbf{u}_i = u_i \hat{x} + v_i \hat{y} \). The gradient operator \( \nabla \) is now understood to be two-dimensional so that, for instance, \( \mathbf{u}_i \cdot \nabla = u_i \partial_x + v_i \partial_y \). The potential vorticities for each layer, \( Q_i \), are given by
\begin{align}
Q_1 &= \nabla^2 \psi_1 - \frac{1}{L_{10}^2} \left( \psi_1 - \frac{\rho_2}{\rho_1} \psi_2 \right), \\
Q_2 &= \nabla^2 \psi_2 - \frac{1}{L_{21}^2} \left( \psi_2 - \psi_1 \right),
\end{align}
where the streamfunctions \( \psi_i \) relate to the velocity in each layer according to
\begin{align}
u_i &= -\partial_b \psi_i, \\
v_i &= \partial_x \psi_i,
\end{align}
and where the 2 dimensional Laplacian operator is given by \( \nabla^2 \rightarrow \partial_x^2 + \partial_y^2 \). Note that because the leading order horizontal velocities are geostrophic it follows that their horizontal divergences are zero, i.e.
\[ \nabla \cdot \mathbf{u}_i = 0. \]

The (non-dimensionalized) Rossby radii of deformation for each layer are given by
\begin{align}
L_{10}^2 &= \frac{g \mathcal{H}_{10}}{4 \Omega_0^2 L^2} \left( 1 - \frac{\rho_2}{\rho_1} \right), \\
L_{21}^2 &= \frac{g \mathcal{H}(H_{20} - H_{10})}{4 \Omega_0^2 L^2} \left( 1 - \frac{\rho_2}{\rho_1} \right),
\end{align}
where \( \rho_1 \) and \( \rho_2 \) are the densities of each corresponding layer. The dimensional value of each level height \( \mathcal{H}_i \) are given respectively according to their non-dimensional fractional measures \( H_{i0} \). In other words if \( H_{i0} \) represents the vertical coordinate where the transition from density \( \rho_i \) to \( \rho_{i+1} \) occurs in steady state then the following may be defined \( H_{10} \equiv \mathcal{H}_{10}/\mathcal{H}, \quad H_{20} \equiv \mathcal{H}_{20}/\mathcal{H} \). Since only two layers are considered here the value of \( H_{20} = 1 \) as \( \mathcal{H}_{20} = \mathcal{H} \) by construction. It follows that \( H_{10} \) is some number less than 1. The horizontal magnetic field in each layer is denoted in vector form with \( \mathbf{b}_i = b_{xi} \hat{x} + b_{yi} \hat{y} \) and the equations for their evolution are given by
\begin{align}
(\partial_t + \mathbf{u}_1 \cdot \nabla) \mathbf{b}_1 &= (\mathbf{b}_1 \cdot \nabla) \mathbf{u}_1, \\
(\partial_t + \mathbf{u}_2 \cdot \nabla) \mathbf{b}_2 &= (\mathbf{b}_2 \cdot \nabla) \mathbf{u}_2.
\end{align}

As in the study considered by Gilman (2000), and unlike what is encountered for the horizontal velocity components, the horizontal magnetic field components are not automatically divergence-free. Of course, the source free condition is met in three dimensions so that a non-zero divergence of the horizontal magnetic field components will result in the generation of a linearly varying (with respect to the vertical coordinate) vertical magnetic field \( b_{zi} \), as expressed in relationship (35). It so happens that for these orderings the vertical field (if there is one dynamically, see below) effects the dynamics diagnostically through the pseudo-source term \( q_i^{(m)} \) defined by
\begin{align}
q_i^{(m)} &= \nabla \cdot \mathbf{b}_i = \partial_x b_{xi} + \partial_y b_{yi},
\end{align}
and its influence can be seen in the Lorentz terms in the equations for the potential vorticity (68-69). The source term \( q_i^{(m)} \) is the pseudo-magnetic monopole distribution referred to in the Introduction. Finally the \( z \)-directed currents for each layer are \( J_i \) and defined by
\begin{align}
J_i &= \partial_x b_{yi} - \partial_y b_{xi}.
\end{align}
An equation for the source terms \( q_i^{(m)} \) may be developed by operating each equation (74) and (75) by the divergence operator and making use of the fact \( \nabla \cdot \mathbf{u} = 0 \). This procedure results in,

\[
\left( \partial_t + \mathbf{u}_i \cdot \nabla \right) q_i^{(m)} = 0.
\]

(78)

This equation says something important: that if \( q_i^{(m)}(x, y) = 0 \) initially then it remains zero for subsequent times. When the case, the consequence of this is that the horizontal magnetic field components obey a divergence-free condition and the evolution of the magnetic field is strictly two-dimensional derivable from a flux function. Note also that \( q_i^{(m)}(x, y) = 0 \) solutions of the MGQ equation set form a self-contained subspace of all solutions of the MGQ equations. When examination is focused wholly on this subclass, i.e. when

\[
q_1^{(m)} = q_2^{(m)} = 0,
\]

(79)

it automatically follows that \( \nabla \cdot \mathbf{b}_i = 0 \). Consequently, the flux functions relating to the magnetic field components are,

\[
b_{x1} = -\partial_y \phi_1, \quad b_{y1} = \partial_x \phi_1, \quad b_{x2} = -\partial_y \phi_2, \quad b_{y2} = \partial_x \phi_2.
\]

(80)

Instead of solving the vector equations (74) and (75), the following simpler scalar equations must solved,

\[
\begin{align*}
\left( \partial_t + \mathbf{u}_1 \cdot \nabla \right) \phi_1 &= 0, \\
\left( \partial_t + \mathbf{u}_2 \cdot \nabla \right) \phi_2 &= 0.
\end{align*}
\]

(81)

(82)

In terms of flux functions, then, the currents in each layer are given by

\[
J_1 = \nabla^2 \phi_1, \quad J_2 = \nabla^2 \phi_2.
\]

(83)

Thus the evolution equations given by (88,89), with \( q_i^{(m)} \) set to zero, and (81,82), together with all of the supporting ancillary definitions form the basis of the equations appropriate for this subclass. These are the same equations analyzed by Gilman in 1967-8.

In an upcoming series of publications, we study the linear and nonlinear response of these equations which are amenable to relatively unencumbered theoretical analyses using well-known techniques used in both the meteorological and fluid dynamics literature for studies of the QG equations. To this end, we shall use as a guide and expand upon the work laid out by Gilman and collaborators. Our initial investigations will focus upon analyzing the physical stability of the compact magnetic vortices observed to emerge in the simulations of Cho (2008) and to hopefully, as a result, understand the nature of those structures and answer why they may be manifesting in those simulations with the observed robustness reported.

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