Diffusive and rough homogenisation in fractional noise field

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Abstract

With recently developed tools, we prove a homogenisation theorem for a random ODE with short and long-range dependent fractional noise. The effective dynamics are not necessarily diffusions, they are given by stochastic differential equations driven simultaneously by stochastic processes from both the Gaussian and the non-Gaussian self-similarity universality classes. A key lemma for this is the ‘lifted’ joint functional central and non-central limit theorem in the rough path topology.

keywords: passive tracer, fractional noise, multi-scale, functional limit theorems, rough differential equations

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1 Introduction

Fractional noise is the ‘derivative’ of a fractional Brownian motion. Its covariance at times separated by a span $s$ is $\hat{\sigma}(s) \sim 2H(2H-1)|s|^{2H-2}+2H|s|^{2H-1}\delta_s$, where $H$ is the Hurst parameter taking values in $(0,1) \setminus \{1/2\}$ and $\delta_s$ is the Dirac measure. The ‘$H = 1/2$’ case is white noise. If $H > 1/2$, $\int_0^\infty \hat{\sigma}(s) ds = \infty$ which means that the noise has non-integrable long range dependence (LRD). If $H < 1/2$, the process is negatively correlated. Just as white noise is used for modelling noise coming from a large number of independent random components, fractional noise is used for modelling Long range dependence (LRD). LRDs are observed in nature and in time series data. We study the two scale passive tracer problem, this is also called the tagged particle problem, with fractional noise.

We consider a slow/fast system in which the slow variables are given by a random ODE $\dot{x}_t = G(x_t, y_t^\varepsilon)$. This touches on two problems. The first is the passive tracer problem modelling the motion of a tagged particle in a disturbed flow, not necessarily incompressible, which allows simulation of the turbulent from the Lagrangian description. The other is the dynamical description for Brownian particles in a liquid at rest. The particle in a disturbed flow, not necessarily incompressible, which allows simulation of the turbulent from the Lagrangian description. The other is the dynamical description for Brownian particles in a liquid at rest. The slow variables evolve in their natural time scale, while the fast random environment evolves in the microscopic scale $\varepsilon$. The aim is to extract a closed effective dynamics which approximates the slow variables when $\varepsilon$ is sufficiently small. This effective dynamics will be obtained from the persistent effects coming from the fast-moving variables through adiabatic transmission. If the environment is stationary strong mixing noise with sufficiently fast rate of convergence, the homogenisation problem is synonymous with ‘diffusion creation’, and is therefore also known as diffusive homogenisation. There have been continuous explorations of the diffusive homogenisation problem, see [Gre51, Has66, Kub57, KV86, LOV00, PK74, Tay21, KLO12] and the references therein. Recently long range dependent noises are also studied in several papers in the context of homogeneous incompressible fluids, however, they inevitably fall within the central limit theorem regimes [FK00, KNR12] and the effective dynamics are either Brownian motions or fractional Brownian motions.

We will study a family of vector fields without spatial homogeneity, the resulting dynamics can take the form of a process resembles locally a fractional Brownian motion and more generally they compromise of a larger class of stochastic dynamical systems of the form

$$dx_t = \sum_{k=1}^n f_k(x_t) \circ dX_t^k + \sum_{k=n+1}^N f_k(x_t) dX_t^k, \quad x_0 = x_0, \quad (1.1)$$

where $X_t^k$ is a Wiener process for $k \leq n$ and otherwise a Gaussian or a non-Gaussian Hermite process. To our best knowledge, this presents a new effective limit class. In these equations, the symbol $\circ$ denotes the Stratonovich integral and the other integrals are in the sense of Young integrals.

The homogenisation problem we consider is:

$$\begin{align*}
\dot{x}_t^\varepsilon &= \sum_{k=1}^N \alpha_k(\varepsilon) f_k(x_t^\varepsilon) G_k(y_t^\varepsilon), \\
x_0^\varepsilon &= x_0,
\end{align*} \quad (1.2)$$

where $y_t^\varepsilon = y_t^k$ and $y_t$ are the short and long range dependent stationary fractional Ornstein-Uhlenbeck processes (fOU) with Hurst parameter $H \in (0,1) \setminus \{1/2\}$ and one time probability distribution $\mu$, the centred real valued functions $G_k \in L^p(\mu)$ transforms the noise. If $f_k$ are in $C_b^1$ and $G_k$ are bounded measurable, the solutions to the equations $\dot{x}_t^\varepsilon = \sum_{k=1}^N f_k(x_t^\varepsilon) G_k(y_t^\varepsilon)$ will be approximated by the averaged dynamics which, in this case, is the trivial ODE $\dot{x}_t = 0$, c.f. [LH19] and [LS]. A homogenisation theorem will then describe the fluctuation around this average, for this we must rescale the vector fields to arrive at a non-trivial limit. The different scales $\alpha_k(\varepsilon)$ are reflections of the non-strong mixing property of the noise, they tend to $\infty$ as
with their canonical lifts. Using rough path theory for stochastic homogenisation is a recent development, for any
value for the limit to be locally a Brownian motion. If \( m \) is smaller, the effective limit is locally a Hermite
process of rank \( m \), otherwise a Wiener process.

Our main theorem is the following. We take \( \alpha_k(\varepsilon) \) to be \( \alpha(\varepsilon, H^* (m_k)) \), the latter is defined by (1.3).

**Theorem A** Let \( H \in (0, 1) \setminus \{ \frac{1}{2} \} \), \( f_k \in C^2_0 (\mathbb{R}^d; \mathbb{R}^d) \) and \( G_k \in L^{p_k} (\mathbb{R}; \mathbb{R}, \mu) \) be real valued functions satisfying Assumption 2.10. Then the solutions of (1.2) converge weakly in \( C^\gamma \), on any finite time interval and for any \( \gamma \in \left( \frac{1}{2}, \frac{1}{2} - \frac{1}{\min_{n \leq n_p k}} \right) \), to the solution of (1.7).

The linear contraction in the Langevin equation and the exponential convergence of the solutions would lead to the belief that it mixes as fast as the Ornstein-Uhlenbeck process. But, the auto-correlation functions of the increment process, which measures how much the shifted process remembers, exhibits power law decay. For \( H > \frac{1}{2} \), the auto correlation function is not integrable. Conventional tools are not applicable here, we For the increment process, which measures how much the shifted process remembers, exhibits power law decay.

For independent identically distributed random variables, the central limit theorems (CLTs) states that
\[
\frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k \text{ converges to a Gaussian distribution. For correlated random variables, non-Gaussian distributions may appear. One of these was proved by Rosenblatt: Let } Z_n \text{ be a stationary Gaussian sequence with correlation } \rho(n) \sim n^{-d} \text{ where } d \in (0, \frac{1}{2}) \text{ and let } Y_n = (Z_n)^2 - 1 \text{ then } n^{d-1} Y_n \text{ converges to a non-Gaussian distribution. To emphasise the non-CLTs, those limit theorems with non-Gaussian limits are referred to as } \text{‘non-Central Limit Theorems’ (non-CLTs). A functional limit theorem concerns path integrals of functionals of a stochastic process } y_s. \text{ For a centred function } G, \text{ it states that } \lim_{\varepsilon \to 0} \sqrt{\varepsilon} \int_0^T G(y_s) ds \text{ converges to a Brownian motion. Non-CLTs and functional non-CLTs were extensively studied \cite{MT07, BH02, BM83, Taq75}, these were then shown to hold for a larger class of functions \cite{CNN20, NP05} with Malliavin calculus. In a nutshell, for a class of Gaussian processes and for a centred } L^2 \text{ function } G \text{ with the scaling constant depending on its Hermite rank } m, \text{ the limit of } \alpha(\varepsilon) \int_0^T G(y_s^\varepsilon) ds \text{ will be a BM if the scale is } \frac{1}{\sqrt{\varepsilon^2}} \text{ or } \frac{1}{\sqrt{|\ln(\varepsilon)|}} \text{; otherwise it is a self-similar Hermite process of degree } m \text{ with self-similar exponent } H^*(m) = \frac{m}{H^*(1) + 1}. \text{ We will use functional limit theorems for both cases.}

Let \( \alpha(\varepsilon, H^*(m)) \) be positive constants as follows, they depend on \( m, H \) and \( \varepsilon \) and tend to \( \infty \) as \( \varepsilon \to 0 \),

\[
\alpha(\varepsilon, H^*(m)) = \begin{cases} 
\frac{1}{\sqrt{\varepsilon^2}}, & \text{if } H^*(m) < \frac{1}{2}, \\
\frac{1}{\sqrt{|\ln(\varepsilon)|}}, & \text{if } H^*(m) = \frac{1}{2}, \\
\varepsilon^{H^*(m)-1}, & \text{if } H^*(m) > \frac{1}{2}.
\end{cases}
\]  

(1.3)

Observe that \( H^* \) decreases with \( m \) and \( H^*(1) = H \). If \( H \leq \frac{1}{2} \) we only see the diffusion scale. We state below our key limit theorem, the lifted joint functional limit theorem in the rough path topology, c.f. (5.2), see \( \text{[4]} \) The proof for the main theorem is finalised in \( \text{[4]} \)

**Theorem B (Lifted joint functional CLTs/ Non-CLTs)** Let \( H \in (0, 1) \setminus \{ \frac{1}{2} \} \) and fix a finite time horizon \( T \). Suppose that the \( L^2(\mu) \) functions \( G_1, \ldots, G_N \) satisfy Assumption 2.10. Let \( m_k \) denote the Hermite rank of \( G_k \). Set

\[
X_{t, \varepsilon}^k = \alpha(\varepsilon, H^*(m_k)) \int_0^t G_k(y_s^\varepsilon) ds, \quad X^\varepsilon = (X_1^{1, \varepsilon}, X_1^{2, \varepsilon}, \ldots, X_1^{N, \varepsilon}).
\]  

(1.4)
1. Then, for every \( \gamma \in (\frac{1}{2}, \frac{1}{2} - \min_{k<\infty} \frac{1}{p_k}) \), the canonical rough paths \( X^\varepsilon := (X^\varepsilon_t, X^\varepsilon_{s,t}) \) converge weakly in the rough topology \( \mathcal{C}^\gamma([0, T], \mathbb{R}^N) \) and
\[
\lim_{\varepsilon \to 0} X^\varepsilon = X := (X_t, X_{s,t} + (t-s)A)
\]

2. The precise formulation for the stochastic process \( X_t \) in the limit is given in Theorem\[2.7\]. It consists of two independent blocks: a Wiener process block and a Hermite process block. For \( 0 \leq s \leq t \leq T \), the limiting second order processes are given by \( X = (X^{i,j}) \) and \( A = (A^{i,j}) \) where
\[
X^{i,j}_{s,t} = \int_s^t (X^i_r - X^i_s) dX^j_r, \quad \begin{cases}
\text{an Itô integral,} & \text{for } i, j \leq n, \\
\text{a Young integral,} & \text{otherwise.}
\end{cases}
\]
\[
A^{i,j} = \begin{cases}
\int_0^\infty E(G_i(y_s)G_j(y_t)) ds, & \text{if } i, j \leq n, \\
0, & \text{otherwise.}
\end{cases}
\]

The Hermite processes in Theorem B are \( Z^H(m_k, m_k) \), see §2.1. They have Hölder continuous sample paths up to the order \( H^*(m_k) \). For this theorem, we use a basic functional CLT from \([GL20]\) for proving the joint convergence of the integrals and their iterated integrals in an appropriate path space, in finite dimensional distribution. For the Wiener limit part, we employ both ergodic theorems and martingale approximations. In case where the processes are not strong mixing, proving the \( L^2 \) boundedness of the martingale approximations is rather involved (this is where we had to exclude functions with Hermit rank falling into the range \( [\frac{1}{p_1}, \frac{1}{p_0}] \)). We will follow an idea in \([Hai05a, LH19]\) for fractional Brownian motions to develop a locally independent decomposition for the IOU process and use this for estimating the conditional moments. The final hurdle is the relatively compactness of the iterated integrals in the rough path topology, for which we use the diagram formula and an upper bound, from \([Taq77]\), on the number of eligible graphs of complete pairings.

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2. Previously, we proved the homogenisation theorem for \( H > \frac{1}{2} \). This was posted to the Mathematics arxiv and unpublished otherwise, see \([GL19]\). Here we can also include the \( H < \frac{1}{2} \) case. For the presentation, we did not include the basic joint functional limit theorem from \([GL19]\). Instead, an improved version is presented in \([GL20]\).

**Notation**

- \((W_t, t \in \mathbb{R})\) denotes a two-sided Wiener process.
- \(B_t\) is the fBM in the Langevin equation, \( H \) is its Hurst parameter, \( \mathcal{F}_t \) denotes its filtration.
- \( H^*(m) = m(H - 1) + 1 \).
- \( m_k \) is the Hermit rank of \( G_k \).
- Convention : \( H^*(m_k) \leq \frac{1}{2} \) for \( k \leq n \); otherwise \( H^*(m_k) > \frac{1}{2} \).
- \( C^r_k \) : bounded continuous functions with bounded continuous derivatives up to order \( r \).
- \( f \leq g \) means that there exists a constant \( c \), not depending on \( f \) or \( g \), such that \( f \leq cg \).
- \( |x|_\alpha := \sup_{x \neq 0} \frac{|x|}{|x|^{\alpha}} \) is the homogeneous Hölder semi-norm, \( 0 < \alpha < 1 \).
- For a process \( x_t \), set \( x_{s,t} := x_t - x_s \).
- We fix a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). \( L^p(\Omega) \) denotes the \( L^p \) space on \( \Omega \) and its norm is denoted by \( \| \cdot \|_{L^p} \).
- \( \mu = N(0, 1) \) is the standard Gaussian measure, \( L^p(\mu) \) denotes the corresponding \( L^p \) space.
2 Preliminaries

A fractional Brownian motion is a continuous Gaussian process with stationary increments. We take a normalised fractional Brownian motion $B_t$ so that $B_0 = 0$ and $E(B_t)^2 = 1$. Specifically, if $H$ is its Hurst parameter, then

$$E((B_t - B_s)(B_u - B_v)) = \frac{1}{2} (|t - v|^{2H} + |s - u|^{2H} - |t - u|^{2H} - |s - v|^{2H}).$$

We refer to [PT17, Sam06, CKM03] for details on fractional Brownian motions. Note that

$$E(B_t B_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) = H(2H - 1) \int_0^s \int_0^t |r_1 - r_2|^{2H-2} dr_1 dr_2,$$

and so $\frac{\partial^2}{\partial s \partial t} E(B_t B_s) = H(2H - 1)|t - s|^{2H-2}$, when $H \in (0, 1) \setminus \{\frac{1}{2}\}$. Let $X_n = B_{1+n} - B_n$ denote the increment process of a fBM. Then, the autocorrelation function of $\{X_n\}$ is not summable for $H > \frac{1}{2}$.

2.1 Hermite processes

Let $W_t$ be a one dimensional standard two-sided Brownian motion. Let $\hat{H}(m) = \frac{1}{m}(H - 1) + 1$, so $\hat{H}$ is the inverse of $H^*$.

Definition 2.1 Let $m \in \mathbb{N}$ with $\hat{H}(m) > \frac{1}{2}$. We take a standalised Hermite process of rank $m$ to be the following mean zero process:

$$Z_{t,H,m} = \int_{\mathbb{R}^m} \int_0^t \prod_{j=1}^m (s - \xi_j)^{\frac{1}{2} + \frac{H - m}{m}} ds dW(\xi_1) \ldots dW(\xi_m).$$

(2.1)

The integral over $\mathbb{R}^m$ is understood as a multiple Wiener-Itô integral (no integration along the diagonals) and the constant $K(H, m)$ is chosen so that it variance is 1 at $t = 1$. The number $H$ is its self-similarity exponent, it is also known as its Hurst parameter.

Since $\hat{H}(1) = H$, the rank 1 Hermite processes $Z_{H,1}$ are fractional BMs. Indeed (2.1) is exactly the Mandelbrot Van-Ness representation for a fBM. We emphasise this representation:

$$B_{H,t} = \int_{\mathbb{R}} \int_0^t (s - \xi)^{H-\frac{1}{2}} ds dW_\xi.$$

The Hermite processes have stationary increments, finite moments of all orders and the following covariance function:

$$E(Z_{t,H,m} Z_{s,H,m}) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

(2.2)

Therefore, using Kolmogorv’s theorem, one can show that the Hermite processes $Z_{t,H,m}$ have sample paths of Hölder regularity up to $H$. As mentioned before, they also self similar stochastic processes:

$$\lambda^H Z_{\lambda t,H,m} \sim Z_{t,H,m}.$$

The process $Z_{H,m}$ belongs to the $m^{th}$ Wiener chaos generated by $W$, in particular, two Hermite processes $Z_{H,m}$ and $Z_{H,m'}$, defined by the same Wiener process, are uncorrelated if $m \neq m'$. Further details on Hermite processes can also be found in [MT07].
Remark 2.2 We note that in some literature, e.g., [MT07], the notation for the Hermite processes are different:

\[ Z_t^{H,m} = \frac{K(H, m)}{m!} \int_{R^m} \prod_{j=1}^{m} (s - \xi_j)_{+}^{H-\frac{1}{2}} ds \, dW(\xi_1) \ldots \, dW(\xi_m). \]

These two are related by

\[ Z_t^{H^*(m), m} = \bar{Z}_t^{H, m}, \quad Z_t^{H^*(m), m} = \bar{Z}_t^{H(m), m}. \]  

(2.3)

2.2 Fractional Ornstein-Uhlenbeck processes

We gather in this section to useful facts about the stationary fractional Ornstein-Uhlenbeck process, by which we recall the following correlation decay from [CKM03],

\[ \text{the following estimates explains how to choose the appropriate scaling constants, see [GL20] for detail.} \]

Let \( \xi \) have, for \( \mu \)

\[ \text{value} \]

2.2 Fractional Ornstein-Uhlenbeck processes

We gather in this section to useful facts about the stationary fractional Ornstein-Uhlenbeck process, by which we mean\( y_t = \sigma \int_{-\infty}^{t} e^{-(t-s)} dB_s^H \) for \( B_t^H \) a two-sided fractional BM and \( \sigma \) chosen such that \( y_t \) is distributed as \( \mu = N(0, 1) \). It is the stationary solution of the Langevin equation: \( dy_t = -y_t dt + \sigma dB_t^H \) with the initial value \( y_0 = \sigma \int_{-\infty}^{0} e^{s} dB_s^H \). We take rescale the fOU process to obtain \( \bar{y}_t \), the latter is the stationery solution of

\[ dy_t = -y_t dt + \frac{\sigma}{\varepsilon^H} dB_t^H. \]  

(2.4)

Observe that \( y^\varepsilon \) and \( y^\varepsilon \) have the same distributions, furthermore, \( y_t^\varepsilon = \frac{\sigma}{\varepsilon^H} \int_{-\infty}^{t} e^{\varepsilon(t-s)} dB_s^H \). Let us denote their correlation functions by \( \varrho \) and \( \varrho^\varepsilon \) respectively:

\[ \varrho(s, t) := E(y_sy_t), \quad \varrho^\varepsilon(s, t) := E(y^\varepsilon_s y^\varepsilon_t). \]

Let \( \varrho(s) = E(y_{s}y_{s}) \) for \( s \geq 0 \) and extended to \( R \) by symmetry, so \( \varrho(s, t) = \varrho(t-s) \) and similarly for \( \varrho^\varepsilon \). We have, for \( u > 0 \) and \( H > \frac{1}{2} \),

\[ \varrho(u) = \sigma^2 H(2H - 1) \int_{-\infty}^{u} \int_{-\infty}^{0} e^{-(u-r)_{+}^2} |r|^2 H^2 H^2 dr. \]

We recall the following correlation decay from [CKM03].

Lemma 2.3 Let \( H \in (0, 1) \setminus \{ \frac{1}{2} \} \). Then, \( \varrho(s) = \sigma^2 H(2H - 1)s^{2H-2} + O(s^{2H-4}) \) as \( s \to \infty \). In particular, for any \( s \geq 0 \),

\[ |\varrho(s)| \lesssim 1 \land |s|^{2H-2}. \]  

(2.5)

By Lemma 2.3 \( \int_{0}^{\infty} \varrho(s) ds \) is finite if and only if \( H^*(m) < \frac{1}{2} \). We are not interested in \( H = \frac{1}{2} \), as the Ornstein-Uhlenbeck process admits an exponential decay of correlations and \( \varrho^m \) is integrable for any \( m \geq 1 \). The following estimates explains how to choose the appropriate scaling constants, see [GL20] for detail.

Lemma 2.4 Let \( H \in (0, 1) \setminus \{ \frac{1}{2} \} \) and fix a finite time horizon \( T \), then, for \( t \in [0, T] \) the following holds uniformly for \( \varepsilon \in (0, \frac{1}{T}] \):

\[ \left( \int_{0}^{\frac{t}{\varepsilon}} \int_{0}^{\frac{t}{\varepsilon}} |\varrho(u, r)|^m dr \, du \right)^{\frac{1}{m}} \lesssim \begin{cases} \sqrt{\frac{t \varepsilon}{\varepsilon}} \int_{0}^{\infty} \varrho(s) ds, & \text{if} \quad H^*(m) < \frac{1}{2}, \\ \sqrt{\frac{r}{\varepsilon} \ln (\frac{r}{\varepsilon})}, & \text{if} \quad H^*(m) = \frac{1}{2}, \\ \frac{t}{\varepsilon} H^*(m) - 1, & \text{if} \quad H^*(m) > \frac{1}{2}. \end{cases} \]  

(2.6)

\[ \left( \int_{0}^{\frac{t}{\varepsilon}} \int_{0}^{\frac{t}{\varepsilon}} |\varrho^\varepsilon(u, r)|^m dr \, du \right)^{\frac{1}{m}} \lesssim \begin{cases} \sqrt{\frac{t \varepsilon}{\varepsilon}} \int_{0}^{\infty} \varrho(s) ds, & \text{if} \quad H^*(m) < \frac{1}{2}, \\ \sqrt{t \varepsilon \ln (\frac{t \varepsilon}{\varepsilon})}, & \text{if} \quad H^*(m) = \frac{1}{2}, \\ \frac{t}{\varepsilon} H^*(m) - 1, & \text{if} \quad H^*(m) > \frac{1}{2}. \end{cases} \]  

(2.7)
Then, the following holds:

\[ t \int_0^t |g^\varepsilon(s)|^m ds \lesssim \frac{t(2H^*(m)/1)}{\alpha(\varepsilon, H^*(m))^2}. \tag{2.8} \]

Note, if \( H = \frac{1}{2} \), the bound is always \( \sqrt{\frac{t}{\varepsilon} \int_0^\infty g^m(s) ds} \).

### 2.3 Hermite Rank

We take the Hermite polynomials of degree \( m \) to be \( H_m(x) = (-1)^m e^{\frac{2}{x^2}} \frac{d^m}{dx^m} e^{\frac{x^2}{2}} \). Thus, \( H_0(x) = 1 \), \( H_1(x) = x \). The Hermite rank of an \( L^2(\mu) \) function with respect to a Gaussian measure is the degree of the lowest non-zero Hermite polynomial term in the Hermite polynomial expansion of \( G_k \).

**Definition 2.5** Let \( G : \mathbb{R} \rightarrow \mathbb{R} \) be an \( L^2(\mu) \) function with chaos expansion

\[ G(x) = \sum_{k=m}^\infty c_k H_k(x), \quad c_k = \frac{1}{k!} \langle G, H_k \rangle_{L^2(\mu)} \tag{2.9} \]

1. The smallest \( m \) with \( c_m \neq 0 \) is called the Hermite rank of \( G \).
2. Set \( H^*(m_k) = m_k (H - 1) + 1 \). If \( H^*(m) \leq \frac{1}{2} \) we say \( G \) has high Hermite rank (relative to \( H \)), otherwise it is said to have low Hermite rank.

### 2.4 Joint functional CLT / non-CLT

Functional limit theorems for Gaussian processes have been extensively studied. The theorem we will need is from [GL20], it is tailored for proving the lifted functional limit theorem. We first introduce the notations.

**Convention 2.6** Let \( y^\varepsilon = y_{\varepsilon}^\ast \) be the rescaled stationary fractional Ornstein-Uhlenbeck process with standard Gaussian distribution \( \mu \) and Hurst parameter \( H \in (0, 1) \setminus \{ \frac{1}{2} \} \). Each \( G_k : \mathbb{R} \rightarrow \mathbb{R} \) is a centred function in \( L^2(\mu) \) with Hermite rank \( m_k \). Let \( \alpha_k(\varepsilon) = \alpha(\varepsilon, H^*(m_k)) \). Set

\[ X^\varepsilon := (X^{1,\varepsilon}, \ldots, X^{N,\varepsilon}), \quad X^\varepsilon_k = \alpha_k(\varepsilon) \int_0^t G_k(y^\varepsilon_s) ds. \tag{2.10} \]

We further define the rough paths \( X^\varepsilon = (X^\varepsilon, X_{t,j}^{i,j,\varepsilon}) \), where

\[ X_{u,t}^{i,j,\varepsilon} := \int_u^t (X_{i,s}^{i,\varepsilon} - X_{i,u}^{i,\varepsilon}) dX_{j,s}^{j,\varepsilon} = \alpha_i(\varepsilon) \alpha_j(\varepsilon) \int_u^t \int_0^s G_i(y^\varepsilon_u) G_j(y^\varepsilon_s) dr ds. \tag{2.11} \]

The process \( X^\varepsilon = (X^\varepsilon, X_{t,j}^{i,j,\varepsilon}) \) is called the canonical lift of \( X^\varepsilon \).

Without any further assumptions on \( G_k \), \( X^\varepsilon \) can be shown to converge jointly in finite dimensional distributions. For the convergence in a Hölder topology, we assume that \( G_k \in L^{p_k}(\mu) \) for \( p_k \) sufficiently large. This means \( H^*(m_k) - \frac{1}{p_k} > 0 \) if \( G_k \) has low Hermite rank and otherwise \( \frac{1}{2} - \frac{1}{p_k} > 0 \). This condition is summarised in part (3) of Assumption \[2.10 \]

**Theorem 2.7 (Joint Functional CLT/non-CLT)** Suppose that \( G_k \) are centred and satisfies furthermore Assumption \[2.10 \](3). Write \( G_k = \sum_{i=m_k}^{\infty} c_{k,i} H_i \) and set

\[ X_{u,t}^{W,\varepsilon} = (X^{1,\varepsilon}_u, \ldots, X^{N,\varepsilon}_u), \quad X_{u,t}^{Z,\varepsilon} = (X^{n+1,\varepsilon}_u, \ldots, X^{N,\varepsilon}_u). \]

Then, the following holds:
1. There exist stochastic processes \( X^W = (X^1, \ldots, X^n) \) and \( X^Z = (X^{n+1}, \ldots, X^N) \) such that on every finite interval \([0, T]\),

\[
(X^{W, \varepsilon}, X^{Z, \varepsilon}) \longrightarrow (X^W, X^Z),
\]

weakly in \( C^\gamma([0, T], \mathbb{R}^N) \). We can take \( \gamma \) to be any number smaller than \( \frac{1}{2} - \frac{1}{\min_{k \leq n} p_k} \) if at least one component converges to a Wiener process, otherwise we can take \( \gamma < \min_{k > n} H^*(m_k) - \frac{1}{p_k} \).

2. In particular the following holds,

\[
\sup_{\varepsilon \in (0, \frac{1}{2})} \|X_{s,t}^k,\varepsilon\|_{p_k} \lesssim \left\{ \begin{array}{ll}
\sqrt{|t-s|}, & \text{if } H^*(m) \leq \frac{1}{2}, \\
|t-s| H^*(m), & \text{if } H^*(m) > \frac{1}{2}.
\end{array} \right.
\]

Furthermore, for any \( t > 0 \)

\[
\lim_{\varepsilon \to 0} \|X_{t}^{Z,\varepsilon} \to X_{t}^{Z} \|_{L^2(\Omega)} = 0.
\]

3. The limit \( X = (X^W, X^Z) \) has the following properties

(1) \( X^W \in \mathbb{R}^n \) and \( X^Z \in \mathbb{R}^{N-n} \) are independent.

(2) \( X^W = U \hat{W} \) where \( \hat{W} \) is a standard Wiener process and \( U \) is a square root of the matrix \((2A^{ij})_{i,j \leq n}\). Let \( g(r) = E(y_j y_0) \), then the entries of the matrix are given as follows:

\[
A^{i,j} = \int_0^{\infty} E(G_i(y_s)G_j(y_0)) ds = \sum_{q=0}^{\infty} c_{i,q} c_{j,q} (k!) \int_0^{\infty} g(r)^q dr
\]

In other words, \( E(X_i X_j^2) = 2(t \wedge s) A^{i,j} \) for \( i, j \leq n \).

(3) Let \( Z_{t}^{H^*(m_k),m_k} \) be the Hermite processes, represented by (2.1), and

\[
Z_{t}^{k} = \frac{m_k!}{K(H^*(m_k),m_k)} Z_{t}^{H^*(m_k),m_k}.
\]

Then,

\[
X^{Z} = (c_{n+1,m_{n+1}}, Z_{t}^{n+1}, \ldots, c_{N,m_{N}}, Z_{t}^{N}).
\]

We emphasize that the Wiener process defining the Hermite processes is the same for every \( k \), which is in addition independent of \( \hat{W} \).

### 2.5 Assumptions and Conventions

**Definition 2.8** A function \( G \in L^2(\mu), G = \sum_{i=0}^{\infty} c_i H_i \), is said to satisfy the fast chaos decay condition with parameter \( q \in \mathbb{N} \), if

\[
\sum_{i=0}^{\infty} |c_i| \sqrt{i!} (2q - 1)^{\frac{i}{2}} < \infty.
\]

For functions \( G_1, \ldots, G_N \) in \( L^2(\mu) \), we write \( m_k \) for their Hermite ranks.

**Convention 2.9** Given a collection of functions \((G_k \in L^2(\mu), k \leq N)\), we will label the high rank ones first so \( H^*(m_k) < \frac{1}{2} \) for \( k = 1, \ldots, n \), where \( n \geq 0 \) and otherwise \( H^*(m_k) > \frac{1}{2} \).

**Assumption 2.10 (CLT rough, \( \mathcal{G}^\gamma \)-assumptions)** Each \( G_k \) belongs to \( L^{p_k}(\mu) \) for some \( p_k > 2 \) and has Hermite rank \( m_k \geq 1 \). Furthermore,

(1) Each \( G_k \) satisfies the fast chaos decay condition with parameter \( q \geq 4 \).
(2) \textbf{(Integrability condition)} \(p_k\) is sufficiently large so the following holds:

\[
\min_{k \leq n} \left( \frac{1}{2} - \frac{1}{p_k} \right) + \min_{n < k \leq N} \left( H^*(m_k) - \frac{1}{p_k} \right) > 1.
\]  

(2.13)

(3) If \(G_k\) has low Hermite rank, assume \(H^*(m_k) - \frac{1}{p_k} > \frac{1}{2}\); otherwise assume \(\frac{1}{2} - \frac{1}{p_k} > \frac{1}{6}\).

(4) Either \(H^*(m_k) < 0\) or \(H^*(m_k) > \frac{1}{2}\).

\textbf{Remark 2.11}

1. If the functions \(G_k\) are polynomial functions, all assumptions stated above are automatically satisfied, except for (4).

2. The moment assumptions arise from the necessity to obtain the convergence, not just in the space of continuous functions but also in a rough path space \(\mathcal{G}\gamma\) for some \(\gamma > \frac{1}{2}\), which is naturally established by Kolmogorov type arguments, to be able to use the continuity of the solution maps in the rough path setting.

3. Let \(\eta\) denote the greatest common Hölder continuity exponent for the first \(n\) terms in \(X^\alpha\), each of these converge to a Wiener process. Let \(\tau\) denote the greatest common Hölder continuity exponent for the rest of the components of \(X^\alpha\). Then condition (2) is used for making sure \(\eta + \tau > 1\). With this, any iterated integral, in which one term converges to a Wiener and the other one to a Hermite process, can be interpreted as a Young integral.

4. In Condition (4) we have to assume \(H^*(m_k) < 0\), leaving a gap \([0, \frac{1}{2}]\). This restriction is due to Proposition\(^3,\) where we only obtain the required integrability estimates for \(H^*(m_k) < 0\).

### 3 Lifted joint functional limit theorem

If \(X^{(n)}\) and \(Y^{(n)}\) are two sequences of stochastic processes with \(X^{(n)} \to X\) and \(Y^{(n)} \to Y\) (even if the convergence is almost surely everywhere and even if \(X\) and \(Y\) are differentiable curves), we may fail to conclude that \(\int_0^t X^{(n)}_1 dY^{(n)}_s \to \int_0^t X_1 dY_s\). Take for example \(X^{(n)}_1 = \frac{1}{\sqrt{n}} \cos(nt)\) and \(Y^{(n)}_t = \frac{1}{\sqrt{n}} \sin(nt)\).

If a sequence of vector valued stochastic processes \((X^{(n)}_1, X^{(n)}_2)\) together with its canonical lift converge in the rough path topology, the limit of the iterated integrals may not be the same as the iterated integrals of the limit. We give an example for this by modifying the earlier example by pumping randomness into the \(\cos\) and \(\sin\) sequences using random variables \(\lambda(1), \lambda(2)\) taking values in \(\{1, -1\}\). Define a sequence of stochastic processes \(X^{(n)}_1\) as follows:

\[
X^{(n)}_1(t) = \begin{cases} 
\frac{1}{\sqrt{n}} \cos(nt), & \lambda(1) = 1, \\
\frac{1}{\sqrt{n}} \sin(nt), & \lambda(1) = -1,
\end{cases}
\]

and similarly \(X^{(n)}_2\). Then, \(X^{(n)}_1 \to 0\) in \(C^\alpha\) for \(\alpha < \frac{1}{2}\) and the same holds true for \(X^{(n)}_2\), however,

\[
\int_0^t X^{(n)}_1(s) dX^{(n)}_2(s) = \begin{cases} 
\frac{t}{2}, & \lambda(1) = 1, \lambda(2) = -1, \\
0, & \lambda(1) = \lambda(2), \\
-\frac{t}{2}, & \lambda(1) = -1, \lambda(2) = 1.
\end{cases}
\]

In this example, \((X^{(n)}_1, X^{(n)}_2)\) together with its canonical lift converge in the rough path topology. The limit of the iterated integrals depend on \(\lambda\). If we set \(\lambda\) so that \((\lambda(1), \lambda(2))\) is uniformly distributed, the marginals are always the same, but the joint distributions depends on the further correlation relations of the random variables \(\lambda(1)\) and \(\lambda(2)\).
In this section, we show that $X^\varepsilon = (X^\varepsilon, X^{i,j,\varepsilon})$, the canonical lift of $X^\varepsilon$, converges in the rough path topology. Specifically, we will show in (3.3) that the secondary processes $X^{i,j,\varepsilon}$, involving only $i,j \leq n$, converge jointly in finite dimensional distributions (which is more involved due to the lack of the strong mixing property). In §3.1, we prove that $\{ (X^\varepsilon, X^{i,j,\varepsilon}), \varepsilon \in (0, 1/2]\}$ is tight in the rough path topology. The tightness plus the fact that we can identify the limiting joint probability distributions with stochastic integrals $\int_0^t X_i dX_j$ shows that $(X^W, X^{i,j,\varepsilon}, i,j \leq n)$ converges in the rough path to $(X^W, X^W)$ and its lift. Furthermore we identify its remaining canonical lift parts of $(X^Z, X^Z)$ as a measurable functions of $(X^W, X^Z)$. The rest follows from Theorem 2.7.

### 3.1 Relative compactness of iterated integrals

In this section, we establish moment bounds on the iterated integrals and prove that $X^\varepsilon$ is tight in the rough path topology. Let $G_i$ and $G_j$ be two functions in $L^2(\mu)$ with Hermite ranks $m_{G_i}$ and $m_{G_j}$, respectively. Set $\alpha_i = \alpha(\varepsilon, H^*(m_{G_i}))$ and $\alpha_j(\varepsilon) = \alpha(\varepsilon, H^*(m_{G_j}))$. Recall that

$$X^{i,j,\varepsilon}_{u,t} = \alpha_i(\varepsilon)\alpha_j(\varepsilon) \int_u^t \int_u^s G_i(y^\varepsilon_s)G_j(y^\varepsilon_r)drds,$$

To obtain tightness, we assume that the coefficients $c_{n,i}$ in the Hermite expansion of $G_i$ satisfy the decay condition specified in Assumption 2.10 (1). We want to argue by Theorem 3.1 in [FH14], the rough path analogue to Kolmogorov’s theorem. Thus, we need to estimate $\|X^{i,j,\varepsilon}_{u,t}\|_{L^p(\Omega)}$, where by stationarity we may from now on assume $u = 0$.

If $G_i$ and $G_j$ are in a finite chaos of order $Q$, then

$$E\left(\left|X^{i,j,\varepsilon}_{0,t}\right|^p\right)^{1/p} = \alpha_i(\varepsilon)\alpha_j(\varepsilon)\left(\int_0^t \int_0^s \sum_{k,k'=1}^Q c_{i,k}c_{j,k''}H_k(y^\varepsilon_s)H_{k'}(y^\varepsilon_r)drds\right)^{1/p}$$

$$\leq \alpha_i(\varepsilon)\alpha_j(\varepsilon)\left(\sum_{k_1,...,k_p=m_{G_i}}^Q \sum_{k'_1,...,k'_p=m_{G_j}}^Q \prod_{l=1}^p |c_{i,k_l}c_{j,k'_l}| \int_0^t \int_0^{s_1} \cdots \int_0^{s_{p-1}} \int_0^{s_p} E\left(\prod_{l=1}^p H_{k_l}(y^\varepsilon_{r_l})H_{k'_l}(y^\varepsilon_{s_l})\right) drds\right)^{1/p}$$

$$E\left(\left|X^{i,j,\varepsilon}_{0,t}\right|^p\right)^{1/p} \leq \alpha_i(\varepsilon)\alpha_j(\varepsilon)\left(\sum_{k_1,...,k_p=m_{G_i}}^Q \sum_{k'_1,...,k'_p=m_{G_j}}^Q \prod_{l=1}^p |c_{i,k_l}c_{j,k'_l}| \int_0^t \int_0^{s_1} \cdots \int_0^{s_{p-1}} \int_0^{s_p} E\left(\prod_{l=1}^p H_{k_l}(y^\varepsilon_{r_l})H_{k'_l}(y^\varepsilon_{s_l})\right) drds\right)^{1/p}$$

This means we need to estimate the terms $E\left(\prod_{l=1}^p H_{k_l}(y^\varepsilon_{r_l})H_{k'_l}(y^\varepsilon_{s_l})\right)$. For convenience, we will re-label the indices so to write the product in the form $E\left(\prod_{l=1}^p H_{k_l}(y^\varepsilon_{r_l})\right)$. For $p = 2$, we have the identity $E(H_{m}(y^\varepsilon_{r_1})H_{n}(y^\varepsilon_{r_2})) = \delta_{m,n}(E(y^\varepsilon_{r_1}y^\varepsilon_{r_2}))^m$. For the multiple product, we use the so called diagram-formulae, see e.g. [BH02] and references therein. The diagram-formulae formula states that the expectation we are concerned with can be calculated by summing over products of covariances, similar to Isserli’s/Wick’s theorem. This can be linked to graphs. Nodes of these graphs correspond to the $y^\varepsilon_{r_l}$’s and each such node has exactly $k_l$ edges, where no edge may connect a node to itself. Each edge between $y^\varepsilon_{r_1}$ and $y^\varepsilon_{r_2}$ corresponds to a factor $E(y^\varepsilon_{r_1}y^\varepsilon_{r_2})$. The expectation we are concerned with is then given by summing over all possible graphs of such complete pairings.

For a particular graph $\Gamma$, we denote by $n(l, q)$ the number of edges connecting $l$ to $q$, so it takes values in $\{0, 1, \ldots, \min(k_l, k_q)\}$, and consider the pairings in an ordered way so that each pairing is counted only
once. We thus have \( \sum_{q=1}^{2p} n(l, q) = k_l \) and, since edges are only allowed to connect with different nodes \( n(q, q) = 0 \) for every \( q \). For any given graph this is

\[
\prod_{q=1}^{2p} \prod_{l=q+1}^{2p} \left( E(y_{s_q}^q y_{s_l}^q) \right)^n(l, q) = \prod_{q=1}^{2p} \prod_{l: |l| > q, l \in \Gamma_q} \varphi^q(s_l - s_q)^n(l, q),
\]

where \( \Gamma_q \) denotes the subgraph of nodes connected to \( q \). Thus,

\[
E \left( \prod_{l=1}^{2p} H_{k_l}(y_{s_l}^q) \right) = \sum_{\Gamma} \prod_{q=1}^{2p} \prod_{l: |l| > q, l \in \Gamma_q} \varphi^q(s_l - s_q)^n(l, q),
\]

where the sum ranges over all suitable graphs \( \Gamma \) given \((k_1, \ldots, k_{2p})\).

**Lemma 3.1**

1. Let \( \Gamma \) denote a complete pairing of 2p nodes with a suitable amount of edges \((k_1, \ldots, k_{2p})\). Define:

\[
I(\varepsilon, 2p, \Gamma) := \int_0^t \cdots \int_0^t \prod_{(s_q, s_l) \in \Gamma} \left( E(y_{s_q}^q y_{s_l}^q) \right)^n(q, l) ds_1 \cdots ds_{2p}.
\]

Then,

\[
I(\varepsilon, 2p, \Gamma) \lesssim \prod_{l=1}^{2p} \sqrt{t \int_{-t}^t |\varphi^q(s)|^{k_l} ds} \lesssim \prod_{l=1}^{2p} \frac{t^{H^*(k_l)/2}}{\alpha(\varepsilon, H^*(k_l))}. \tag{3.4}
\]

2. If \( G_i, G_j : R \rightarrow R \) are functions in finite chaos with Hermite ranks \( m_{G_i} \) and \( m_{G_j} \) respectively. Then,

\[
\|X_{0, t}^{i,j, \varepsilon}\|_{L^p(\Omega)} = \alpha_i(\varepsilon) \alpha_j(\varepsilon) \left\| \int_0^t \int_0^s G_i(y_{s_q}^q) G_j(y_{s_l}^l) dr ds \right\|_{L^p(\Omega)} \lesssim t^{H^*(m_{G_i})/2 + H^*(m_{G_j})/2}.
\]

**Proof.** For a general graph, let us start dealing with the first variable \( s_1 \). We first count forward and observe

\[
\prod_{(s_q, s_l) \in \Gamma} \left( E(y_{s_q}^q y_{s_l}^q) \right)^n(q, l) = \prod_{q=1}^{2p} \prod_{l=q+1}^{2p} \left( E(y_{s_q}^q y_{s_l}^q) \right)^n(q, l) = \prod_{q=1}^{2p} \prod_{l: |l| > q, l \in \Gamma_q} \varphi^q(s_l - s_q)^n(l, q),
\]

where \( \Gamma_q \) denotes the subgraph of nodes connected to \( q \). Using Hölder’s inequality we obtain

\[
\int_0^t \prod_{q: q > 1, q \in \Gamma_1} |\varphi^q(s_l - s_q)|^{n(1, q)} ds_1 \leq \prod_{q: q > 1, q \in \Gamma_1} \left( \int_0^t |\varphi^q(s_l - s_q)|^{k_1} ds_1 \right)^{n(1, q) / k_1} \leq \int_0^t |\varphi^q(s_l)|^{k_1} ds_1.
\]

We have used \( \sum_{q: q > 1, q \in \Gamma_1} n(1, q) = k_1 \), the number of edges at node 1. We then peel off the integrals layer by layer, and proceed with the same technique to the next integration variable. For example suppose the
remaining integrator containing \(s_2\) has the combined exponent \(\tau_2 = \sum_{q=2}^{2p} n(2, q), (\tau_1 = k_1)\). By the same procedure as for \(s_1\) we score a factor 
\[
\int_{-t}^{t} |\varphi^\varepsilon(s_2)|^{\tau_2} ds_2.
\]

By induction and putting the estimates for each integral together,
\[
\prod_{q=1}^{2p} \sum_{l>q, l \in \Gamma_q} (\varphi^\varepsilon(s_l - s_q))^{n(l, q)} ds_1 \ldots ds_{2p} \lesssim \prod_{q=1}^{2p} \int_{-t}^{t} |\varphi^\varepsilon(s)|^{\tau_q} ds.
\]

Following [BH02], we reverse the procedure in the estimation for the integral kernel. Let \(\xi_q\) denote the number of edges connected to the node \(q\) in the backward direction, so \(\xi_q = \sum_{l=1}^{q} n(l, q)\), and the same reasoning leads to the following estimate:

\[
\prod_{q=1}^{2p} \sum_{l<q, l \in \Gamma_q} (\varphi^\varepsilon(s_l - s_q))^{n(l, q)} ds_1 \ldots ds_{2p} \lesssim \prod_{q=1}^{2p} \int_{-t}^{t} |\varphi^\varepsilon(s)|^{\xi_q} ds.
\]

Since \(\tau_q + \xi_q = k_q\) by Hölders inequality,
\[
\int_{-t}^{t} |\varphi^\varepsilon(s)|^{\tau_q} ds \int_{-t}^{t} |\varphi^\varepsilon(s)|^{\xi_q} ds \leq 2t \int_{-t}^{t} |\varphi^\varepsilon(s)|^{k_q} ds.
\]

Therefore,
\[
\left(\prod_{q=1}^{2p} \sum_{l>q, l \in \Gamma_q} (\varphi^\varepsilon(s_l - s_q))^{n(l, q)} ds_1 \ldots ds_{2p}\right)^2 \lesssim \prod_{q=1}^{2p} \left( t \int_{-t}^{t} |\varphi^\varepsilon(s)|^{k_q} ds\right).
\]

By Lemma 2.4 we obtain, for each \(q \in \{1, \ldots, N\}\),
\[
\alpha(\varepsilon, H^\ast(k_q))^2 t \int_{-t}^{t} |\varphi^\varepsilon(s)|^{k_q} ds \lesssim t^{2H^\ast(k_q)\ve_1},
\]

hence, the first part of the lemma follows.

For \(G_i\) and \(G_j\) we obtain as in Equation (3.1), using the fact that \(\varphi^\varepsilon > 0\) and thus we may enlarge our integration area,
This leads to Assumption 2.10 (1), which restricts the $G_i$’s to the class of functions whose coefficients in the Hermite expansion decay sufficiently fast.

By monotonicity of $H^*$ and the fact that $k_i \geq m_{G_i}$ and $k_i' \geq m_{G_j}$,

$$\left(\prod_{l=1}^{p} t^{H^*(k_l)} \cdot t^{H^*(k_l')} \right)^{\frac{1}{p}} \leq t^{H^*(m_{G_i}) \vee \frac{1}{2} + H^*(m_{G_j}) \vee \frac{1}{2}},$$

concluding the proof. □

For functions not belonging to a finite chaos we must count the number of graphs in the computation and need some assumptions. Let $M(k_1, \ldots, k_{2p})$ denote the cardinality of admissible graphs with $2p$ nodes with respectively $(k_1, \ldots, k_{2p})$ edges. In [1aq77] it was shown that

$$M(k_1, k_2, \ldots, k_{2p}) \leq \prod_{l=1}^{2p} (2p - 1)^{\frac{k_l}{2}} \sqrt{k_l}.$$ 

This leads to Assumption 2.10 (1), which restricts the $G_i$’s to the class of functions whose coefficients in the Hermite expansion decay sufficiently fast.

**Proposition 3.2** Suppose that each $G_k$ satisfies Assumption 2.10. Then, one has for $i, j \in \{1, \ldots, N\}$,

$$\left\|\alpha_i(\varepsilon)\alpha_j(\varepsilon) \int_0^t \int_0^\delta G_i(y_{s}^\varepsilon)G_j(y_{s}^\varepsilon) dr ds \right\|_{L^p(\Omega)} \lesssim t^{H^*(m_{G_i}) \vee \frac{1}{2} + H^*(m_{G_j}) \vee \frac{1}{2}}.$$ 

Consequently, $X^\varepsilon$ is tight in $C^0$ for $\gamma \in \left(\frac{1}{3}, \frac{1}{2} - \frac{1}{\min_{k\leq n} p_k}\right)$. 

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Proof. As above using \( g^\varepsilon > 0 \) and \( \prod_{p=1}^\infty t^{H^+ (k_p)} \leq t^{p(H^+ (m_{\tau}) + H^+ (m_{\epsilon}) + \frac{\varepsilon}{2})} \),
\[
\|\mathcal{X}_{0,t}^{i,j,\varepsilon}\|^p_{L_p(\Omega)} 
\leq \alpha_i(\varepsilon)^p \alpha_j(\varepsilon)^p \sum_{k_1,\ldots,k_p=m_{\tau}}^\infty \sum_{k_1',\ldots,k_p'=m_{\tau}}^\infty \prod_{i=1}^p |c_i(k_i, c_j, k_i')| \int_{[0,\varepsilon]}^t E \left( \prod_{i=1}^p |H_{k_i}(y^\varepsilon_i)H_{k_i'}(y^\varepsilon_i)| \right) \, dr_1 \, ds_1.
\]
\[
= \alpha_i(\varepsilon)^p \alpha_j(\varepsilon)^p \sum_{k_1,\ldots,k_p=m_{\tau}}^\infty \sum_{k_1',\ldots,k_p'=m_{\tau}}^\infty \prod_{i=1}^p |c_i(k_i, c_j, k_i')| I(\varepsilon, 2p, \Gamma)
\leq t^{p(H^+ (m_{\tau}) + H^+ (m_{\epsilon}) + \frac{\varepsilon}{2})} \sum_{k_1,\ldots,k_p=m_{\tau}}^\infty \sum_{k_1',\ldots,k_p'=m_{\tau}}^\infty \prod_{i=1}^p |c_i(k_i, c_j, k_i')| M(k_1, \ldots, k_p, k_1', \ldots, k_p')
\leq t^{p(H^+ (m_{\tau}) + H^+ (m_{\epsilon}) + \frac{\varepsilon}{2})} \sum_{k_1,\ldots,k_p=m_{\tau}}^\infty \sum_{k_1',\ldots,k_p'=m_{\tau}}^\infty \prod_{i=1}^p |c_i(k_i, c_j, k_i')| k_1! \ldots k_p! (2p - 1)
\leq \frac{1}{m_{\tau}} - \frac{1}{m_{\tau} + \varepsilon}.
\]

By the chaos decay assumption, these sums are finite and this completes the proof for the required moment bounds. Finally, using Theorem 2.7 and Proposition 3.2, we can conclude the tightness of \( X^\varepsilon \) in \( C^\gamma \), where \( \gamma \in \left( \frac{1}{2}, \frac{1}{2} - \frac{1}{\min m_{\tau} p_\tau} \right) \), by an application of Lemma 5.6.

\[\square\]

3.2 Young integral case (functional non-CLT in rough topology)

**Lemma 3.3** Assume Assumption 2.10. Then,
\[
(X^\varepsilon, X^{i,j,\varepsilon})_{\{i,j\in\{1,\ldots,N\}; i\neq j\neq n\}},
\]
converges in finite dimensional distributions to \( (X, X^{i,j}) \), where \( X^{i,j} = \int_0^t X^{i}_s dX^{j}_s \) and these integrals are well defined as Young integrals.

**Proof.** By Assumption 2.10 and Theorem 2.7, each component of \( X^\varepsilon \) converges in a Hölder space. Furthermore, by Assumption 2.10(2) there exist numbers \( \eta \) and \( \tau \), with \( \eta + \tau > 1 \), such that the Hölder regularity of the limits corresponding to a Wiener process, are bounded below by \( \eta \), and the ones corresponding to a Hermite process bounded from below by \( \tau \). Therefore, taking the integrals
\[
\alpha_i(\varepsilon) \alpha_j(\varepsilon) \int_0^t \int_0^s G_j(y^\varepsilon_s)G_i(y^\varepsilon_s) \, ds \, dr = \int_0^t X^{i,\varepsilon}_s dX^{j,\varepsilon}_s
\]
is a continuous and well-defined operation from \( C^{\eta} \times C^{\tau} \rightarrow C^{\tau} \) or \( C^{\tau} \times C^{\eta} \rightarrow C^{\eta} \), thus weak convergence in \( C^{\eta} \) follows. Let \( F \) denote the continuous map such that, for \( i, j \) with \( i \neq j \neq n \), \( X^{i,j,\varepsilon} = F(X^\varepsilon)^{i,j} \). Now, set
\[
\tilde{F} = \text{id} \times F
\]
\[
\tilde{F}(X^\varepsilon) = (X^\varepsilon, F(X^\varepsilon)) = (X^\varepsilon, X^{i,j})_{\{i,j\in\{1,\ldots,N\}; i\neq j\neq n\}},
\]
which by the above is a continuous function. Thus, by an application of the continuous mapping theorem we can conclude the lemma. \( \square \)

**Remark 3.4** Note that by the moment bounds obtained in Theorem 2.7 and Proposition 3.2, the joint convergence takes place in better Hölder spaces.

Now it is left to deal with the parts of the natural rough path lift involving two Wiener scaling terms, this is carried out in the next section.
3.3 Itô integral case (functional CLT in rough topology)

We proceed to establish the convergence of the iterated integrals where both components belong to the high Hermit rank case.

**Remark 3.5** We further assume $H^*(m_k) < 0$ for each $k$ which gives rise to a Wiener scaling. Thus, we do not obtain Logarithmic terms and therefore work with the $\frac{1}{\sqrt{\varepsilon}}$ scaling from here on. Furthermore, in this case $\alpha(\varepsilon) \int_0^\varepsilon G(y_s^k) ds$ equals $\sqrt{\varepsilon} \int_0^\varepsilon G(y_s^k) ds$ in law and for simplicity we will work with the latter in this chapter.

From here onwards in this section, we take $k, i, j \leq n$. Thus, both $G_i$ and $G_j$ give rise to Wiener processes. Recall that,

$$X^{k,\varepsilon}_t = \sqrt{\varepsilon} \int_0^t G_k(y_s) ds.$$

By Theorem 3.7, $(X^{i,\varepsilon}, X^{j,\varepsilon}) \to (W^i, W^j)$, where $W^i$ and $W^j$ denote Wiener processes with covariances as specified in Theorem 3.7 weakly. We now want to show that the convergence of the following integral

$$\int_0^\varepsilon X^{i,\varepsilon}_s dX^{j,\varepsilon}_s = \varepsilon \int_0^\varepsilon \int_0^s G_i(y_r)G_j(y_s) dr ds$$

$$= I_1(\varepsilon) + I_2(\varepsilon).$$

We will show that $I_1(\varepsilon) \to \int_0^\varepsilon W^i_s dW^j_s$ weakly, where the integral is understood in the Itô-sense, and $I_2(\varepsilon) \to t A^{i,j}$ in probability for some constants $A^{i,j}$. For this we aim to use [KP91] Theorem 2.2, hence, we need to approximate $X^{k,\varepsilon}$ by a suitable martingale, see also [BC17]. For any $L^2(\mu)$ function $U$, in particular for the $G_k$'s, one would have liked to work with the stationary process,

$$\Phi_U(t) = \int_t^\infty U(y_r) dr$$

and use it to define $L^2(\Omega)$-martingale differences, see [KV86]. This unfortunately does not posses good enough integrability properties, thus, as in [BC17], we instead define

$$\hat{U}(k) := \int_{k-1}^\infty \mathbf{E}(U(y_r) | \mathcal{F}_k) dr.$$  

(3.6)

Since $y$ is stationary, we do have $(\hat{U} \circ \tau)(k) = \hat{U}(k + 1)$, where $\tau$ is the shifting operator on sequences. To show that $\hat{U}$ posses the desired integrability properties is a bit more involved. We will show that that there exists a local independent decomposition of the fractional Ornstein-Uhlenbeck process as follows: for every $t$ there exists a decomposition, $y_t = \tilde{y}_t + \hat{y}_t^k$, such that the first term $\tilde{y}_t^k$ is $\mathcal{F}_k$ measurable, $\hat{y}_t^k$ is independent of $\mathcal{F}_k$, where $\mathcal{F}_k$ is the filtration generated by the driving fractional Brownian motion up to time $k$. Both terms are Gaussian processes. This is given in section 3.5. To proceed further we also need a couple of lemmas.

**Lemma 3.6** For $x, y, a, b \in \mathbb{R}$ such that $a^2 + b^2 = 1$,

$$H_m(ax + by) = \sum_{j=0}^m \binom{m}{j} a^j b^{m-j} H_j(x) H_{m-j}(y).$$  

(3.7)

**Lemma 3.7** Let $H \in (0, 1) \setminus \left\{ \frac{1}{2} \right\}$. Set $a_t = \|\hat{y}_t^k\|_{L^2(\Omega)}$. Then,

$$\mathbf{E}[H_m(y_t) | \mathcal{F}_k] = (a_t)^m H_m \left( \frac{\hat{y}_t^k}{a_t} \right).$$
Proof. Let $y_t = \tilde{y}_t^k + \hat{y}_t^k$ denote the local independent decomposition of the fOU from (5.5) and set $b_t = \|\hat{y}_t^k\|_{L^2(\Omega)}$. By the independence of $\tilde{y}_t^k$ and $\hat{y}_t^k$, we obtain
\[
1 = \|y_t\|_{L^2(\Omega)}^2 = \|\tilde{y}_t^k\|_{L^2(\Omega)}^2 + \|\hat{y}_t^k\|_{L^2(\Omega)}^2 = (a_t)^2 + (b_t)^2.
\]
Now we decompose $H_m(y_t)$ using the above identity and obtain,
\[
H_m(y_t) = H_m(\tilde{y}_t^k + \hat{y}_t^k) = H_m\left(a_t \left(\frac{\tilde{y}_t^k}{a_t}\right) + b_t \left(\frac{\hat{y}_t^k}{b_t}\right)\right).
\]
By construction $\frac{\tilde{y}_t^k}{a_t}$ and $\frac{\hat{y}_t^k}{b_t}$ are standard Gaussian random variables, together with the fact that $\hat{y}_t^k$ is measurable with respect to $\mathcal{F}_k$ this leads to,
\[
E[H_m(y_t)|\mathcal{F}_k] = \sum_{j=0}^{m} \binom{m}{j} (a_t)^j (b_t)^{m-j} H_j\left(\frac{\tilde{y}_t^k}{a_t}\right) E[H_{m-j}\left(\frac{\hat{y}_t^k}{b_t}\right)|\mathcal{F}_k] = (a_t)^m H_m\left(\frac{\tilde{y}_t^k}{a_t}\right),
\]
where we used the fact that $E\left[H_j\left(\frac{\hat{y}_t^k}{b_t}\right)\right]$ vanishes for any $j \geq 1$, $\hat{y}_t^k$ is independent of $\mathcal{F}_k$, and $H_0 = 1$. \qed

Proposition 3.8 If $U \in L^2(\mu)$ has Hermite rank $m$, then
\[
\|\tilde{U}(k)\|_{L^2(\Omega)}^2 \leq \|U\|_{L^2(\mu)}^2 \int_{k-1}^{\infty} \int_{k-1}^{\infty} \left(E(\tilde{y}_s^k \tilde{y}_r^k)\right)^m dr ds. \tag{3.8}
\]
In particular $\{\tilde{U}(k)\}_{k \geq 1}$ is bounded in $L^2(\Omega)$ if $H \in (0,1) \setminus \\{\frac{1}{2}\}$ and $U$ has Hermite rank $m$ such that $H^+(m) < 0$.

Proof. The ‘in particular’ part of the assertion follows from the statement that if $H \in (0,1) \setminus \{\frac{1}{2}\}$ and $U$ has Hermite rank $m$ such that $H^+(m) < 0$, then $\int_{k-1}^{\infty} \int_{k-1}^{\infty} \left(E(\tilde{y}_s^k \tilde{y}_r^k)\right)^m dr ds < \infty$, see Proposition 3.20. Due to the lack of the strong mixing property, the proof for this is lengthy and independent of the error estimates here and therefore postponed to section 5.5.

We go ahead proving the identity. Starting with the definition of $\tilde{U}$ and the Hermite expansion $U = \sum_{q=m}^{\infty} c_q H_q$, we compute the $L^2(\Omega)$ norm as follows:
\[
\|\tilde{U}(k)\|_{L^2(\Omega)} = \int_{k-1}^{\infty} \int_{k-1}^{\infty} \sum_{q=m}^{\infty} \sum_{j=m}^{\infty} c_q c_j E\left(E[H_q(y_s)|\mathcal{F}_k] E[H_j(y_r)|\mathcal{F}_k]\right) dr ds
\]
\[
= \int_{k-1}^{\infty} \int_{k-1}^{\infty} \sum_{q=m}^{\infty} (c_q)^2 E\left((a_s)^q (a_r)^q H_q\left(\frac{\tilde{y}_s^k}{a_s}\right) H_q\left(\frac{\tilde{y}_r^k}{a_r}\right)\right) dr ds
\]
\[
= \int_{k-1}^{\infty} \int_{k-1}^{\infty} \sum_{q=m}^{\infty} (c_q)^2 q! (a_s)^q (a_r)^q E\left(\frac{\tilde{y}_s^k \tilde{y}_r^k}{a_s a_r}\right)^q dr ds
\]
\[
= \int_{k-1}^{\infty} \int_{k-1}^{\infty} \sum_{q=m}^{\infty} (c_q)^2 q! \left(E(\tilde{y}_s^k \tilde{y}_r^k)\right)^q dr ds \leq \|U\|_{L^2(\mu)} \int_{k-1}^{\infty} \int_{k-1}^{\infty} \left(E(\tilde{y}_s^k \tilde{y}_r^k)\right)^m dr ds.
\]
The desired conclusion follows from the summability of $\sum_{q=m}^{\infty} (c_q)^2 q!$, which is $\|U\|_{L^2(\mu)}^2$. \qed
With this we may define two families of $L^2(\Omega)$ martingales.

**Corollary 3.9** Given $U, V \in L^2(\mu)$ such that there Hermite ranks $m_U$ and $m_V$ satisfy $H^*(m_U) < 0$ and $H^*(m_V) < 0$, then, the process $(M_k, k \geq 1)$, where

$$M_k := \sum_{j=1}^{k} \left( \hat{U}(j) - E(\hat{U}(j)|\mathcal{F}_{j-1}) \right),$$

is an $\mathcal{F}_k$-adapted $L^2(\Omega)$ martingale with shift covariant martingale difference. The same holds for

$$N_k := \sum_{j=1}^{k} \left( \hat{V}(j) - E(\hat{V}(j)|\mathcal{F}_{j-1}) \right).$$

We can now formulate the main result of this sub-section.

**Proposition 3.10** Let $U, V, M$ and $N$ be as in Corollary 3.9 then there exists a function $E_r(\epsilon)$ converging to zero in probability as $\epsilon \to 0$, such that

$$\epsilon \int_0^t \int_0^s U(y_s)V(y_r)drds = \epsilon \sum_{k=1}^{[\frac{t}{\epsilon}]} (M_{k+1} - M_k) N_k + t\gamma + E_r(\epsilon), \quad (3.9)$$

where

$$\gamma = \int_0^\infty E(U(y_s)V(y_0))ds.$$

The proof for this is given in the rest of the section. Afterwards we show that $\epsilon \sum_{k=1}^{[\frac{t}{\epsilon}]} (M_{k+1} - M_k) N_k$ converges to the relevant Itô integrals of the limits of $\sqrt{\epsilon} \int_0^{[\frac{t}{\epsilon}]} U(y_r)dr$ and $\sqrt{\epsilon} \int_0^{[\frac{t}{\epsilon}]} V(y_r)dr$.

**Lemma 3.11** The stationary Ornstein-Uhlenbeck process is ergodic.

**Proof.** A stationary Gaussian process is ergodic if its spectral measure has no atom, see [CFSS82, Sam06]. The spectral measure $F$ of a stationary Gaussian process is obtained from Fourier transforming its correlation function and $\varphi(\lambda) = \int_{\mathbb{R}} e^{ix\lambda} dF(x)$. According to [CKM03]:

$$\varphi(s) = \frac{\Gamma(2H + 1) \sin(\pi H)}{2\pi} \int_{\mathbb{R}} e^{isx} \frac{|x|^{1-2H}}{1+x^2} dx, \quad (3.10)$$

so the spectral measure is absolutely continuous with respect to the Lebesgue measure with spectral density $s(x) = e^{\frac{|x|^{1-2H}}{1+x^2}}$.

For $k \in \mathbb{N}$, we define the $\mathcal{F}_k$-adapted processes:

$$I(k) = \int_{k-1}^{k} U(y_s)ds = \Phi_U(k - 1) - \Phi_U(k)$$

$$J(k) = \int_{k-1}^{k} V(y_s)ds = \Phi_V(k - 1) - \Phi_V(k).$$
Remark 3.12 We note the following useful identities. For $k \in \mathbb{N}$

\[ \hat{U}(k) = I(k) + E[\hat{U}(k + 1) | \mathcal{F}_k], \quad (3.11) \]
\[ M_{k+1} - M_k = I(k) + \hat{U}(k + 1) - \hat{U}(k), \quad (3.12) \]
\[ \sum_{j=1}^{k} I(j) = \int_{0}^{k} U(y_r)dr = M_k - \hat{U}(k) + \hat{U}(1) - M_1. \]

and similarly for $V$ and $N$.

Henceforth in this section we set $L = L(\varepsilon) = \lfloor \frac{1}{\varepsilon} \rfloor$.

Lemma 3.13 There exists a function $E_{R_1}(\varepsilon)$, which converges to zero in probability as $\varepsilon \to 0$, such that

\[ \varepsilon \int_{0}^{1} \int_{0}^{s} U(y_s)V(y_r)drds = \varepsilon \sum_{k=1}^{L} \int_{1}^{k} J(l) + t \int_{0}^{1} \int_{0}^{s} E(U(y_s)V(y_r))drds + E_{R_1}(\varepsilon) \quad (3.13) \]

Proof. Let us divide the integration region $0 \leq r \leq s \leq \frac{1}{\varepsilon}$ into several parts,

\[ \int_{0}^{L} \int_{0}^{s} U(y_s)V(y_r)drds + \int_{L}^{1} \int_{0}^{s} U(y_s)V(y_r)drds. \]

The second term is of order $o(\varepsilon)$ since $\| \int_{L}^{1} U(y_s)ds \|_{L^2(\Omega)}$ is bounded by stationarity of $y_r$ and Theorem 2.7 see also [GL20]. Furthermore, the term $\| \sqrt{\varepsilon} \int_{0}^{1} V(y_r)dr \|_{L^2(\Omega)}$ is bounded by $\frac{1}{\sqrt{\varepsilon}}$. We compute for the remaining part,

\[ \int_{0}^{L} \int_{0}^{s} U(y_s)V(y_r)drds = \sum_{k=1}^{L} \int_{1}^{k} U(y_s)\left( \int_{0}^{k-1} V(y_r)dr + \int_{k-1}^{s} V(y_r)dr \right) ds \]
\[ = \sum_{k=1}^{L} \int_{1}^{k} U(y_s)ds \int_{0}^{k-1} V(y_r)dr + \sum_{k=1}^{L} \int_{\{1 \leq r \leq s \leq k\}} U(y_s)V(y_r)drds \]
\[ = \sum_{k=1}^{L} I(k) \sum_{l=1}^{k-1} J(l) + \sum_{k=1}^{L} \int_{\{1 \leq r \leq s \leq k\}} U(y_s)V(y_r)drds. \]

The stochastic process $Z_k = \int_{\{1 \leq r \leq s \leq k\}} U(y_s)V(y_r)drds$ is shift invariant and the shift operator is ergodic with respect to the probability distribution on the path space generated by the IOU process, hence, by Birkhoff’s ergodic theorem,

\[ \frac{1}{L} \sum_{k=1}^{L} Z_k (\varepsilon \to 0) \to E_{Z_1} = \int_{0}^{1} \int_{0}^{s} E(U(y_s)V(y_r))drds. \]

This completes the proof. \qed

Lemma 3.14 The following converges in probability:

\[ \lim_{\varepsilon \to 0} \left( \varepsilon \sum_{k=1}^{L} I(k) \sum_{l=1}^{k-1} J(l) - \varepsilon \sum_{k=1}^{L} (M_{k+1} - M_k)N_k \right) = t \int_{0}^{1} \int_{0}^{s} E(U(y_s)V(y_r))drds. \]
Proof. A. Following [BC17] and using the identities of Remark 3.12 we obtain:

\[
\sum_{k=1}^{L} \left( I(k) \sum_{l=1}^{k-1} J(l) - (M_{k+1} - M_k)N_k \right) \\
= \sum_{k=1}^{L} I(k) \left( N_k - \hat{V}(k) + \hat{V}(1) - N_1 \right) - \left( I(k) + \hat{U}(k+1) - \hat{U}(k) \right)N_k \\
= \sum_{k=1}^{L} -I(k) \hat{V}(k) + \sum_{k=1}^{L} I(k)(\hat{V}(1) - N_1) - \sum_{k=1}^{L} (\hat{U}(k+1) - \hat{U}(k))N_k.
\]

Firstly, by the shift invariance of the summands below and Birkhoff’s ergodic theorem we obtain

\[
-\varepsilon \sum_{k=1}^{L} I(k)\hat{V}(k) \longrightarrow (-t) \mathbb{E}[I(1)\hat{V}(1)] = (-t)\mathbb{E} \left( \int_0^1 U(y_r)dr \int_0^\infty V(y_s)ds \right), \quad (3.14)
\]

Next, since \(\hat{V}(1) - N_1 = \mathbb{E}[\hat{V}(1) | \mathcal{F}_0]\),

\[
\mathbb{E} \left| \varepsilon \sum_{k=1}^{L} I(k)(\hat{V}(1) - N_1) \right|^2 = \mathbb{E} \left| \varepsilon \int_0^L U(y_r) dr \mathbb{E}[\hat{V}(1) | \mathcal{F}_0] \right|^2 \\
\leq \varepsilon^2 \mathbb{E}[\hat{V}(1)]^2 \int_0^L \int_0^L \mathbb{E}[U(y_r)U(y_s)] ds dr,
\]

which by Lemma 2.3 and expanding into Hermite polynomials converges to 0 as \(\varepsilon \to 0\).

B. It remains to discuss the convergence of

\[
\varepsilon \sum_{k=1}^{L} (\hat{U}(k+1) - \hat{U}(k))N_k.
\]

We change the order of summation to obtain the following decomposition

\[
\sum_{k=1}^{L} (\hat{U}(k+1) - \hat{U}(k))N_k \\
= \sum_{k=1}^{L} (\hat{U}(k+1) - \hat{U}(k)) \left[ \sum_{j=1}^{k-1} (N_{j+1} - N_j) + N_1 \right] \\
= \sum_{j=1}^{L-1} (N_{j+1} - N_j) \sum_{k=j+1}^{L} (\hat{U}(k+1) - \hat{U}(k)) + \sum_{k=1}^{L} (\hat{U}(k+1) - \hat{U}(k))N_1 \\
= \sum_{j=1}^{L-1} (N_{j+1} - N_j) \hat{U}(L+1) - \sum_{j=1}^{L-1} (N_{j+1} - N_j) \hat{U}(j+1) + \left( \hat{U}(L+1) - \hat{U}(1) \right)N_1.
\]

We may now apply Birkhoff’s ergodic theorem to the first term, taking \(\varepsilon \to 0\),

\[
\lim_{\varepsilon \to 0} -\varepsilon \sum_{j=1}^{L-1} (N_{j+1} - N_j) \hat{U}(L+1) = 0,
\]

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in probability. By the same ergodic theorem, the second term converges to $-t \mathbb{E}\left(\hat{U}(2)(N_2 - N_1)\right)$ in probability. By Proposition 3.8, $\hat{U}(j)$ is bounded in $L^2(\Omega)$, hence, for the third term we obtain,

$$\varepsilon \left| \left(\hat{U}(L + 1) - \hat{U}(1)\right) N_1 \right|_{L^2(\Omega)} \lesssim \varepsilon.$$  

Overall we end up with

$$\lim_{\varepsilon \to 0} (-\varepsilon) \sum_{k=1}^{L} (\hat{U}_{k+1} - \hat{U}(k)) N_k = t \mathbb{E}\left(\hat{U}(2)(N_2 - N_1)\right),$$

(3.15)

where the convergence is in probability, hence,

$$\lim_{\varepsilon \to 0} \left( \epsilon \sum_{k=1}^{L} \sum_{l=1}^{k} I(k)J(l) - \varepsilon \sum_{k=1}^{L} (M_{k+1} - M_k)N_k \right) = t \mathbb{E}\left[\hat{U}(2)(N_2 - N_1) - I(1)\hat{V}(1)\right].$$  

(3.16)

C. We look for a better expression of the limit in (3.16). Firstly by Corollary 3.9, we have

$$(N_2 - N_1) = \hat{V}(2) - \mathbb{E}[\hat{V}(2)|\mathcal{F}_1] = \hat{V}(2) - \mathbb{E}[\hat{V}(2)|\mathcal{F}_1]$$

$$= \hat{V}(2) - \hat{V}(1) + \int_0^1 V(y_s) \, ds.$$  

Using this and $I(1) = \int_0^1 U(y_s) \, ds = \hat{U}(1) - \mathbb{E}[\hat{U}(2)|\mathcal{F}_1]$, we compute

$$\mathbb{E}\left(\hat{U}(2)(N_2 - N_1) - I(1)\hat{V}(1)\right)$$

$$= \int_1^{\infty} \int_0^1 \mathbb{E}(U(y_s)V(y_r)) \, drds + \mathbb{E}\left(\hat{U}(2)\left(\hat{V}(2) - \hat{V}(1)\right) - \left(\hat{U}(1) - \mathbb{E}[\hat{U}(2)|\mathcal{F}_1]\right)\hat{V}(1)\right).$$

Since $\hat{V}(1)$ is $\mathcal{F}_1$ measurable,

$$\mathbb{E}\left(\hat{U}(2)\left(\hat{V}(2) - \hat{V}(1)\right) - \left(\hat{U}(1) - \mathbb{E}[\hat{U}(2)|\mathcal{F}_1]\right)\hat{V}(1)\right),$$

vanishes by the shift covariance of $\hat{U}(k)\hat{V}(k)$. This concludes the proof of the lemma. \hfill \Box

**Proof of Proposition 3.10**

Combining (3.13), Lemma 3.14 and Lemma 3.15 we have

$$\varepsilon \int_0^s \int_0^s U(y_s)V(y_r) \, drds$$

$$= \varepsilon \sum_{k=1}^{L} J(l) + t \int_0^1 \int_0^s \mathbb{E}(U(y_s)V(y_r)) \, drds + \mathbb{E}r_1(\varepsilon)$$

$$= \varepsilon \sum_{k=1}^{L} (M_{k+1} - M_k)N_k + t \int_0^\infty \int_0^1 \mathbb{E}(U(y_s)V(y_r)) \, drds$$

$$+ t \int_0^1 \int_0^s \mathbb{E}(U(y_s)V(y_r)) \, drds + \mathbb{E}r_1(\varepsilon) + \mathbb{E}r_2(\varepsilon)$$

$$= \varepsilon \sum_{k=1}^{L} (M_{k+1} - M_k)N_k + t \int_0^\infty \mathbb{E}(U(y_s)V(y_0)) \, du + \mathbb{E}r_1(\varepsilon) + \mathbb{E}r_2(\varepsilon),$$

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where we used stationarity, \( E(U(y_s)V(y_r)) = E(U(y_{s-r})V(y_0)) \), and the change of variables \( u = s + r, \ v = s - r \), leading to the identity,

\[
\left( \int_0^1 \int_0^s + \int_1^t \int_0^1 \right) E(U(y_s)V(y_r))drds = -\frac{1}{2} \int_0^s \int_{u}^{u-2} E(U(y_v)V(y_0))dudv = \int_0^\infty E(U(y_v)V(y_0))dv.
\]

This completes the proof of Proposition 3.10.

**Proposition 3.15** Let \( X^{W,\varepsilon} = (X^{1,\varepsilon}, \ldots, X^{n,\varepsilon}) \), then

\[
(X^{W,\varepsilon}, X^{W,\varepsilon}) = (X^{W,\varepsilon}, \int_0^t X^{W,\varepsilon} dX^{W,\varepsilon}) \to (X^W, \int_0^t X^W dX^W + tA),
\]

jointly in finite dimensional distributions, where the integration is understood in the Itô sense and \( A \) is as in Theorem 2.

**Proof.** For each \( X^k \) we first define the martingales \( M^k \) as in Corollary 3.9 for \( U = G_k \). Then, we define the Càdlàg martingales as follows

\[
M_t^{k,\varepsilon} = \sqrt{\varepsilon} M_t^k(\frac{y}{\varepsilon}).
\]

Using the identity (3.12) we obtain,

\[
M_t^{k,\varepsilon} = \sqrt{\varepsilon} \sum_{q=1}^{[\frac{t}{\varepsilon}]} (M_{q+1}^k - M_q^k) + \sqrt{\varepsilon} M_t^k = \sqrt{\varepsilon} \sum_{q=1}^{[\frac{t}{\varepsilon}]} G_k(y_{q\varepsilon})ds + \sqrt{\varepsilon} \hat{G}_k \left( \left[ \frac{t}{\varepsilon} \right] \right) - \sqrt{\varepsilon} \hat{G}_k(1) + \sqrt{\varepsilon} M_t^k.
\]

Since \( \hat{G}_k \) is \( L^2(\Omega) \) bounded, the joint convergence \( (M_1^{1,\varepsilon}, \ldots, M_n^{n,\varepsilon}) \to X^W \) in finite dimensional distributions follows from Theorem 2.7. Next by Proposition 3.10

\[
\left( \int_0^t X_s^W dX_s^W \right)^{i,j} = \varepsilon \sum_{q=1}^{[\frac{t}{\varepsilon}]} (M_{q+1}^j - M_q^j)M_t^i + tA^{i,j} + \text{Er}(\varepsilon) \int_0^t M_s^{i,\varepsilon} dM_s^{j,\varepsilon} + tA^{i,j} + \text{Er}(\varepsilon),
\]

where the integration is understood in the Itô sense and \( \text{Er}(\varepsilon) \to 0 \) in probability. The joint convergence of \( (M^{k,\varepsilon}, \int_0^t M_s^{i,\varepsilon} dM_s^{j,\varepsilon})_{i,j,k \leq n} \) in finite dimensional distributions follows, as for each \( k \), \( E\left( M_t^{k,\varepsilon} \right)^2 \lesssim t + o(\varepsilon) \), by an application of Theorem 2.2 in [KP91], which states that given a sequence of jointly convergent martingales bounded in \( L^2(\Omega) \), then these martingales also converge jointly with their Itô integrals. This concludes the proof.

We summarise this section with the following more general statement, which follows from the proofs above:

**Remark 3.16** Let \( y_t \) be a stationary and ergodic stochastic process with stationary measure \( \pi \), \( G_k \in L^2(\pi) \), \( X_t^{k,\varepsilon} = \sqrt{\varepsilon} \int_0^t G_k(y_s)ds \) such that \( X^\varepsilon = (X_1^{1,\varepsilon}, \ldots, X_{n,\varepsilon}) \) converges, as \( \varepsilon \to 0 \), to a Wiener process \( X \). Suppose that \( \left( \int_{k-1}^\infty E(G_s(y_r)F_k)dr, k \geq 1 \right) \) is \( L^2(\Omega) \) bounded. Then, \( X^\varepsilon = (X^\varepsilon, X^\varepsilon) \), the canonical rough path lift of \( \left( X^\varepsilon \right) \), converges to \( (X, X + (t-s)A) \), the Stratonovich lift of \( X \). (By this we mean that \( X^\varepsilon_{t,s} = \int_0^t X^\varepsilon_s dX^\varepsilon_s \), where the integral is understood in the Itô sense and \( A \) denotes the corresponding Stratonovich correction.)
3.4 Proof of Theorem [B]

Now we are ready to conclude the convergence of $X^\varepsilon$ weakly in the rough path topology. We assume that $G_k \in L^{p_k}(\mu)$ satisfy the integrability conditions specified in Assumption [2.10]. Fix $H \in (0, 1) \setminus \{\frac{1}{2}\}$ and a final time $T$. Let, for $t \in [0, T]$,

$$X^t_i := \left(\alpha_1(\varepsilon) \int_0^t G_1(y^\varepsilon_s)ds, \ldots, \alpha_N(\varepsilon) \int_0^t G_N(y^\varepsilon_s)ds\right)$$

$$A^{i,j} = \begin{cases} \int_0^\infty \mathbb{E}(G_i(y_0)G_j(y_0))ds, & \text{if } i, j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem [2.7], $X^\varepsilon$ has a limit which we denote by $X$. Set further,

$$X^i_{u,t} := \alpha_i(\varepsilon) \alpha_j(\varepsilon) \int_u^t \int_s^t G_i(y^\varepsilon_s)G_j(y^\varepsilon_r)drds,$n

$$X^{i,j} = \int_0^t X^i_{u,t}dX^j_s,$$ where the second integral is to be understood in the Itô-sense if two Wiener processes appear, and in the Young sense otherwise. We will prove below that, as $\varepsilon \to 0$,

$$X^\varepsilon = (X^\varepsilon, X^\varepsilon) \rightharpoonup X = (X, X + A(t - s)),$$ weakly in $C^\gamma([0, T], \mathbb{R}^N)$ for $\gamma \in (\frac{1}{2}, \frac{1}{2} - \frac{1}{\min_i \mu_i})$.

**Proof.** Firstly, we recall that by Theorem [2.7] the basic processes converge jointly and the limiting Wiener process, $X^W$, is independent of the limiting Hermite process, $X^Z$. In Proposition [3.15] we showed that the high Hermite rank components, $X^{W,\varepsilon}$, converge jointly with their iterated integrals $\int_0^t X^{W,\varepsilon}dX^{W,\varepsilon}$. In Lemma [3.3] we proved that the first order processes together with the lifts for which $i \lor j > n$ converge jointly and in particular these lifts are continuous functionals of $(X^{W,\varepsilon}, X^{Z,\varepsilon})$. By the continuous dependence of $\int_0^t X^{W,\varepsilon}dX^{W,\varepsilon}$, where $i \lor j > n$, we may leave out these iterated integrals, it is sufficient to show joint convergence of $(X^{W,\varepsilon}, \int_0^t X^{W,\varepsilon}dX^{W,\varepsilon}, X^{Z,\varepsilon})$. This vector is tight in $C^\gamma \times C^\gamma$ by Theorem [2.7] Proposition [3.2] and application of Kolmogorov’s Theorem (see also Lemma [5.6] below), hence we may chose a converging subsequence. We now want to identify the limiting distribution. As we have already established convergence of the marginals $(X^{W,\varepsilon}, \int_0^t X^{W,\varepsilon}dX^{W,\varepsilon})$ and $(X^{Z,\varepsilon}, X^Z)$, by independence of $X^W$ and $X^Z$ their limiting distribution is just given by the product measure between $(X^W, \int_0^t X^WdX^W)$ and $X^Z$. The choice of subsequence was arbitrary, hence each subsequence has the same limit and the whole sequence converges. This concludes the proof. \qed

3.5 Proof of the conditional integrability of fOU

The aim of this section is to prove that $\sup_k \int_{k-1}^k \int_{k-1}^k \mathbb{E}(\tilde{g}_s^k \tilde{g}_r^k)^m drds$ is indeed finite, for which we restrict ourselves to the case $H \in (0, 1) \setminus \{\frac{1}{2}\}$ and $H^*(m) < 0$. We first compute the conditional expectations of $\mathbb{E}(G(y_t)|F_k)$ where $G \in L^2(\mu)$.

In [Ha05b] it was show that a fractional Brownian motion has a locally independent decomposition: for any $k < t$, $B_t - B_k = \tilde{B}_t^k + \tilde{B}_k^k$, in which the smooth process $\tilde{B}_t^k$ is adapted to $F_k$ and the rough part $\tilde{B}_k^k$ is independent of $F_k$. This is given by a Mandelbrot Van-Ness representation, indeed, up to a multiplicative factor,

$$\tilde{B}_t^k = \int_{-\infty}^{k} (t-r)^{H-\frac{1}{2}} - (k-r)^{H-\frac{1}{2}}dW_r, \quad \tilde{B}_k^k = \int_{k}^{t} (t-r)^{H-\frac{1}{2}}dW_r.$$
Furthermore, it was shown that the filtration generated by the fractional Brownian motion is the same as the one generated by the two-sided Wiener process \( W_t \). We now prove such a decomposition for the fractional Ornstein-Uhlenbeck process.

**Lemma 3.17** Let \( \mathcal{F}_s \) be the filtration generated by \( B^H_t \). For \( k < t \) define

\[
\tilde{y}^k_t = \left( \int_{-\infty}^k e^{-(t-r)} dB_r + \int_k^t e^{-(t-r)} dB^k_r \right), \quad \tilde{y}^k_t = \int_k^t e^{-(t-r)} dB^k_r.
\]

Then, \( \tilde{y}^k_t \) is independent of \( \mathcal{F}_k \). \( \tilde{y}^k_t \) \( \in \mathcal{F}_k \), both are Gaussian random variables and \( y_t = \tilde{y}^k_t + \tilde{y}^k_t \). (In case \( t \leq k \) we set \( \tilde{y}^k_t = y_t \).)

**Proof.** Splitting the integral and using, \( B_r - B_k = \bar{B}^k_r + \bar{B}^k_r \), we obtain,

\[
y_t = \int_{-\infty}^k e^{-(t-r)} dB_r = \int_{-\infty}^k e^{-(t-r)} dB_r + \int_k^t e^{-(t-r)} dB_r - \int_k^t e^{-(t-r)} dB_r
\]

\[
= \left( \int_{-\infty}^k e^{-(t-r)} dB_r + \int_k^t e^{-(t-r)} dB^k_r \right) + \int_k^t e^{-(t-r)} dB^k_r
\]

where the first term \( \tilde{y}^k_t \) is \( \mathcal{F}_k \) measurable and \( \tilde{y}^k_t \) is independent of \( \mathcal{F}_k \). \( \square \)

**Lemma 3.18** Let \( \tau > -1 \). For any \( k < s \),

\[
\int_k^s e^{-(s-v)} (v-k)^\tau \, dv \lesssim 1 \land (s-k)^\tau.
\]

The \( \lesssim \) sign indicate that the constant on the right hand side is independent of \( k \) and \( s \).

**Proof.** We may assume that \( s > 4k + 4 \), otherwise the integral is finite as the exponential term can be estimated by 1 and as \( \tau > -1 \), the singularity is integrable. Splitting the integral into two regions \( \int_{k+1}^{s+1} + \int_{k+1}^{s} \), we have \( \int_{k+1}^{s+1} e^{-(s-v)} (v-k)^\tau \, dv \lesssim e^{-(s-k-1)} \) and furthermore using integration by parts,

\[
\int_{k+1}^{s+1} e^{-(s-v)} (v-k)^\tau \, dv
\]

\[
= (s-k)^\tau - e^{-(s-k-1)} - \tau \left( \int_{k+1}^{s+1} e^{-(s-v)} (v-k)^{\tau-1} \, dv \right)
\]

\[
\lesssim (s-k)^\tau - e^{-(s-k-1)} \lesssim (s-k)^\tau - e^{\frac{s-k}{2}} (v-k)^{\tau}\big|_{k+1}^{s+1} - (v-k)^{\tau}\big|_{k+1}^{s+1}
\]

\[
\lesssim (s-k)^\tau + e^{\frac{s-k}{2}} + (s-k)^{\frac{s}{2}} \lesssim (s-k)^\tau.
\]

This gives the required estimate. \( \square \)

**Lemma 3.19** For \( t \geq k-1 \) the following estimate holds, \( \| \tilde{y}^k_t \|_{L^2(\Omega)} \lesssim 1 \land |t-k|^{H-1} \).
Proof. Firstly, as \( \|y_k\|_{L^2(\Omega)} = 1 \) we also obtain \( \|\hat{y}_k\|_{L^2(\Omega)} \leq 1 \). Thus, it is only left to consider the behaviour when \( t \) becomes large. Using the above Lemma we obtain

\[
\|\hat{y}_k\|_{L^2(\Omega)} = \|e^{-(t-k)}y_k + \int_k^t e^{-(t-s)} d\hat{B}_k\|_{L^2(\Omega)} \leq e^{-(t-k)} \|y_k\|_{L^2(\Omega)} + \int_k^t e^{-(t-s)} \|\hat{B}_s\|_{L^2(\Omega)} ds
\]

\[
\leq e^{-(t-k)} + \int_k^t e^{-(t-s)} |s - k|^{\frac{1}{2} - \mu - 1} ds \lesssim |t - k|^{\frac{1}{2} + \mu - 1}.
\]

\[\square\]

**Proposition 3.20** Given \( H \in (0, 1) \setminus \{\frac{1}{2}\} \) and suppose that \( H^*(m) < 0 \). Then,

\[
\sup_k \int_{k-1}^\infty \int_{k-1}^\infty (E(\frac{y_k - L^2(\Omega)}{2}))^m dt ds < \infty.
\]

**Proof.** As

\[
\int_{k-1}^\infty \int_{k-1}^\infty (E(\frac{y_k - L^2(\Omega)}{2}))^m dt ds \leq \int_{k-1}^\infty \int_{k-1}^\infty (\|\frac{y_k - L^2(\Omega)}{2}\|_{L^2(\Omega)})^m dt ds
\]

\[
= \left( \int_{k-1}^\infty \|\frac{y_k - L^2(\Omega)}{2}\|_{L^2(\Omega)} dt \right)^2,
\]

it is sufficient to show finiteness of \( \int_{k-1}^\infty (E(\frac{y_k - L^2(\Omega)}{2}))^m dt \). By Lemma 3.19 we obtain,

\[
\int_{k-1}^\infty (E(\frac{y_k - L^2(\Omega)}{2}))^m dt \leq \int_{k-1}^\infty 1 \wedge |t - k|^{\frac{1}{2} + \mu - 1} dt.
\]

This expression is finite if \( m(\frac{1}{2} - \mu) < -1 \). As \( H^*(m) = m(\frac{1}{2} - \mu) + 1 < 0 \) this concludes the proof. \[\square\]

## 4 Multi-scale homogenisation theorem

For Hermite polynomials we have the hypercontractivity estimate:

\[
\|H_k\|_{L^{2q}(\mu)} \leq (2q - 1)^{\frac{1}{2}} \sqrt{E(H_k)^2} = (2q - 1)^{\frac{1}{2}} \sqrt{k}!.
\]

Consequently, if an \( L^2(\mu) \) function \( G = \sum_{l=0}^\infty c_l H_l \) satisfies the fast chaos decay condition with parameter \( q \), then, \( \|G\|_{L^{2q}(\mu)} \leq \sum_{l=0}^\infty |c_l| \|H_l\|_q < \infty \). We used \( \sum_{l=0}^\infty |c_l| \sqrt{l} (2q - 1)^{\frac{1}{2}} < \infty \). Thus, \( G \in L^{2q}(\mu) \). Observe that \( \frac{1}{2} - \frac{1}{2q} > \frac{1}{4} \), a condition needed for the convergence in \( \mathcal{C} \), is equivalent to the condition \( q > 3 \). Also, if \( G \) satisfies the decay condition with \( q > 1 \), then \( G \) has a continuous representation. Indeed, we have

\[
|e^{-x^2/2} H_k(x)|_\infty \leq 1.0865 \sqrt{k}!,
\]

see [AS84 pp787], the polynomials in [AS84] are orthogonal with respect to \( e^{-x^2} dx \) and one should take care with the convention. Thus the power series \( e^{-x^2} \sum_{l=0}^\infty c_l H_l \) converges uniformly in \( x \), the limit \( G \) is continuous.

**Remark 4.1** If \( G \) satisfies the fast chaos decay condition with parameter \( q > 1 \), then \( G \) has a representation in \( L^{2q} \cap \mathcal{C} \), with which we will work from here on.
We reformulate the main theorem here where it is proved.

**Theorem 4.2** Let $H \in (0, 1) \setminus \{\frac{1}{2}\}$, $f_k \in C^3_b(\mathbb{R}^d, \mathbb{R}^d)$, and $G_k$ satisfy Assumption 2.10. Set $f = (f_1, \ldots, f_N)$, then, the following statements hold.

1. The solutions $x^\varepsilon_t$ of (1.2) converge weakly in $C^\gamma$ on any finite interval and for any $\gamma \in (\frac{1}{3}, \frac{1}{2} - \frac{1}{\min_{k \leq n} p_k})$.
2. The limit solves the rough differential equation
   \[
   dx_t = f(x_t) dX_t \quad x_0 = x_0. \tag{4.1}
   \]
   Here $X = (X, X_{s,t} + (t - s)A)$ is a rough path over $\mathbb{R}^N$, as specified in Theorem B.
3. Equation (4.1) is equivalent to the stochastic equation below:
   \[
   dx_t = \sum_{k=1}^{n} f_k(x_t) \circ dX^k_t + \sum_{l=n+1}^{N} f_l(x_t) dX^l_t, \quad x_0 = x_0,
   \]
   where $(X^1, \ldots, X^n)$ and $(X^{n+1}, \ldots, X^N)$ are independent, the $\circ$ denotes Stratonovich integral, and the other integrals are Young integral.

**Proof.** We want to formulate our slow/fast random differential equation as a family of rough differential equations such that the drivers converge in the rough path topology. Using the continuity of the solution map, we obtain weak convergence of the solutions to a rough differential equation. Results in rough path theory relate this rough differential equation to usual Stratonovich/Young equations, this is explained in §5.1.2 where we introduce the notations from rough differential equations, see also [FV10, FH14, LCL07]. We define $F : \mathbb{R}^d \to L(\mathbb{R}^N, \mathbb{R}^d)$ as follows:

\[
F(x)(u_1, \ldots, u_N) = \sum_{k=1}^{N} u_k f_k(x).
\]

If we further set

\[
G^\varepsilon = \left(\alpha_1(\varepsilon)G_1, \ldots, \alpha_N(\varepsilon)G_N\right),
\]

we may then write our slow equation as

\[
\dot{x}_t^\varepsilon = F(x_t^\varepsilon) G^\varepsilon(y_t^\varepsilon).
\]

For the rough path $X^\varepsilon = (X, X^\varepsilon)$ defined by (2.10,2.11), we may rewrite equation (1.2) as a rough differential equation with respect to $X^\varepsilon$:

\[
dx_t^\varepsilon = F(x_t^\varepsilon) dX^\varepsilon(t).
\]

with covariance as specified in Theorem 2.7 and Theorem B.

By Theorem B $X^\varepsilon$ converges to $X = (X, X + (t - s)A)$ in $\mathcal{D}^\gamma$ where $\gamma \in (\frac{1}{3}, \frac{1}{2} - \frac{1}{\min_{k \leq n} p_k})$ on every finite interval. Since $\gamma > \frac{1}{3}$ by Assumption 2.10, we may apply the continuity theorem for rough differential equations, Theorem 5.3, to conclude that the solutions converge to the solutions of the rough differential equation

\[
\dot{x}_t = F(x_t) dX_t.
\]

Since $F$ belongs to $C^3_b$, this is well posed as a rough differential equation. We completed the proof for the convergence. To show the independence of $X^k$ for $k \leq n$ from the other processes we observe that by Assumption 2.10 the terms of $X^\varepsilon_{i,j}$ for which at least $i > n$ or $j > n$ do not contribute in the limit, hence, we conclude the proof by Theorem B. \qed
The purpose of the appendix is to explain the notation we used from rough path theory. We include the theorems needed for proving the tightness theorem and the homogenisation theorem. Finally we explain how to interpret the effective rough differential equations (1.1) with Itô integrals and Young integrals, and hope this self-contained material will be useful for those not familiar with the rough path theory.

5 Appendix

The theorems needed for proving the tightness theorem and the homogenisation theorem. Finally we explain how to interpret the effective rough differential equations (1.1) with Itô integrals and Young integrals, and hope this self-contained material will be useful for those not familiar with the rough path theory.

5.1 Some rough path theory

If \( X \) and \( Y \) are Hölder continuous functions on \([0, T]\) with exponent \( \alpha \) and \( \beta \) respectively, such that \( \alpha + \beta > 1 \), the Young integration theory enables us to define \( \int_0^T Y \, dX \) via limits of Riemann sums \( \sum_{[u,v]} Y_u (X_v - X_u) \), where \( \mathcal{P} \) denotes a partition of \([0, T]\). Furthermore \( (X, Y) \mapsto \int_0^T Y \, dX \) is a continuous map. Thus, for \( X \in C^{\frac{\alpha}{2} + \epsilon} \), one can make sense of a solution \( Y \) to the Young integral equation \( dY_t = f(Y_s) \, dX_s \), given enough regularity on \( f \). If \( f \in C^2 \), the solution is continuous with respect to both the driver \( X \) and the initial data, see [You36]. In the case of \( X \) having Hölder continuity less or equal to \( \frac{1}{2} \), this fails and one cannot define a pathwise integration for \( \int X \, dX \) by the above Riemann sum anymore. Rough path theory provides us with a machinery to treat less regular functions by enhancing the process with a second order process, giving a better local approximation, which then can be used to enhance the Riemann sum and show it converges. If \( X \) is a Brownian motion and taking a dyadic approximation, then, the usual Riemann sum converges in probability to the Itô integral. The enhanced Riemann sum, however, provides a better approximation and defines a pathwise integral agreeing with the Itô integral provided the integrand belongs to both domains of integration. Their domains of integration are quite different, the first uses an additional adaptedness condition and requires arguably less regularity than the second. We restrict ourselves to the case where \( X_t \) is a continuous path over \([0, T]\), which takes values in \( \mathbb{R}^d \). A rough path of regularity \( \alpha \in (\frac{1}{2}, \frac{1}{2}) \), is a pair of process \( \mathbb{X} = (X_t, \mathbb{X}_{s,t}) \) where \( (\mathbb{X}_{s,t}) \in \mathbb{R}^{d \times d} \) is a two parameter stochastic processes satisfying the following algebraic conditions: for \( 0 \leq s < u < t \leq T \),

\[
\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t},
\]

(Chen’s relation)

where \( X_{s,t} = X_t - X_s \), and \( (X_{s,u} \otimes X_{u,t})^{i,j} = X_{s,u}^i X_{u,t}^j \) as well as the following analytic conditions,

\[
\|X_{s,t}\| \lesssim |t-s|^\alpha, \quad \|\mathbb{X}_{s,t}\| \lesssim |t-s|^{2\alpha}.
\]  

(5.1)

The set of such paths will be denoted by \( C^\alpha([0, T]; \mathbb{R}^d) \). The so called second order process \( \mathbb{X}_{s,t} \) can be viewed as a possible candidate for the iterated integral \( \int_s^t X_{s,u} \, dX_u \).

Remark 5.1 Using Chen’s relation for \( s = 0 \) one obtains

\[
\mathbb{X}_{u,t} = \mathbb{X}_{0,t} - \mathbb{X}_{0,u} - X_{0,u} \otimes X_{u,t},
\]

thus one can reconstruct \( \mathbb{X} \) by knowing the path \( t \to (X_{0,t}, \mathbb{X}_{0,t}) \).

Given a path \( X \), which is regular enough to define its iterated integral, for example \( X \in C^1([0, T]; \mathbb{R}^d) \), we define its natural rough path lift to be given by

\[
\mathbb{X}_{s,t} := \int_s^t X_{s,u} \, dX_u.
\]

It is now an easy exercise to verify that \( \mathbb{X} = (X, \mathbb{X}) \) satisfies the algebraic and analytic conditions (depending on the regularity of \( X \)), by which we mean Chen’s relation and (5.1). Note that given any function \( F \)
Given two rough paths \( X \) and \( Y \) we may define, for \( \alpha \in \left( \frac{1}{3}, \frac{1}{2} \right) \), the distance

\[
g_\alpha(X, Y) = \sup_{s \neq t} \frac{\|X_{s,t} - Y_{s,t}\|}{|t - s|^{\alpha}} + \sup_{s \neq t} \frac{\|X_{s,t} - Y_{s,t}\|}{|t - s|^{\alpha}}. \tag{5.2}
\]

This defines a complete metric on \( \mathcal{C}^\alpha([0, T]; \mathbb{R}^d) \), this is called the inhomogeneous \( \alpha \)-Hölder rough path metric. We are also going to make use of the norm like object

\[
\|X\|_\alpha = \sup_{s \neq t \in [0, T]} \frac{\|X_{s,t}\|}{|t - s|^{\alpha}} + \sup_{s \neq t \in [0, T]} \frac{\|X_{s,t}\|}{|t - s|^{\alpha}}, \tag{5.3}
\]

where we denote for any two parameter process \( X \) a semi-norm:

\[
\|X\|_{2\alpha} := \sup_{s \neq t \in [0, T]} \frac{\|X_{s,t}\|}{|t - s|^{2\alpha}}.
\]

Given a path \( X \), as the second order process \( X \) takes the role of an iterated integral, another sensible conditions to impose is the chain rule (or integration by parts formulae) leading to the following definition.

**Definition 5.2** A rough path \( X \) satisfying the following condition,

\[
S_{ym}(X_{s,t})^{(i)} = \frac{1}{2} (X_{s,t}^{(i)} + X_{s,t}^{(i)}) = \frac{1}{2} X_{s,t}^{(i)} \otimes X_{s,t}^{(j)} \tag{5.4}
\]

is called a geometric rough path. The space of all of geometric rough paths of regularity \( \alpha \) is denoted by \( \mathcal{C}^\alpha([0, T]; \mathbb{R}^d) \) and forms a closed subspace of \( \mathcal{C}^\alpha([0, T]; \mathbb{R}^d) \).

Furthermore, one can show that if a sequence of \( C^1([0, T], \mathbb{R}^d) \) paths \( X^n \) converges in the rough path metric to \( X \), then \( X \) is a geometric rough path. To obtain a geometric rough path from a Wiener process, as \( \int_0^t W_s \circ dW_s = \frac{W^2_t}{2} \), one has to enhance it with its Stratonovich integral, \( W_{s,t} = \int_0^s (W_r - W_s) \circ dW_r \), up to an antisymmetric part.

Given a rough path \( X \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d) \), we may define the integral \( \int_0^T Y dX \) for suitable paths \( Y \in C^\alpha([0, T], L(\mathbb{R}^d, \mathbb{R}^m)) \), which admit a Gubinelli derivative \( Y' \in C^\alpha([0, T], L(\mathbb{R}^d \times d, \mathbb{R}^m)) \) with respect to \( X \), meaning \( Y_{s,t} = Y'_{s,t} X_{s,t} + R_{s,t} \), where the two parameter function \( R \) satisfies \( \|R\|_{2\alpha} < \infty \). The pair \( Y := (Y, Y') \) is said to be a controlled rough path, their collection is denoted by \( D_X^{2\alpha} \). The remainder term for the case \( Y = f(X) \) with \( f \) smooth is the remainder term in the Taylor expansion. This is done by showing that the enhanced Riemann sums \( \sum_{[s,t] \in \mathcal{F}} Y_s X_{s,t} + Y'_{s,t} X_{s,t} \), converge as the partition size is going to zero, and the limit is defined to be \( \int Y dX \). Given \( Y \in D_X^{2\alpha} \), then \( (\int Y dX, Y) \in D_X^{2\alpha} \), and the map \( (X, Y) \mapsto (\int Y dX, Y) \) is continuous with respect to \( (X, Y) \in \mathcal{C}^{\alpha} \) and \( Y \in D_X^{2\alpha} \).

With this theory of integration one can study the equation,

\[
dY = f(Y)dX.
\]

However, unlike in the theory of stochastic differential equations one now has continuous dependence on the noise \( X \). We now state the precise theorem for our application, see also \( \text{[Lyo94]} \).

**Theorem 5.3** \( \text{[FHT14]} \) Let \( Y_0 \in \mathbb{R}^m \), \( \beta \in \left( \frac{1}{3}, 1 \right) \), \( f \in C^\beta_\alpha(\mathbb{R}^m, L(\mathbb{R}^d, \mathbb{R}^m)) \) and \( X \in \mathcal{C}^\beta([0, T], \mathbb{R}^d) \). Then, the differential equation

\[
Y_t = Y_0 + \int_0^t f(Y_s)dX_s \tag{5.5}
\]
has a unique solution which belongs to $C^\beta$. Furthermore, the solution map $\Phi_f : \mathbb{R}^d \times \mathcal{C}^\beta([0, T], \mathbb{R}^d) \to D^\beta_X([0, T], \mathbb{R}^m)$, where the first component is the initial condition and the second component the driver, is continuous.

As continuous maps preserve weak convergence to show weak convergence of solutions to rough differential equations

$$dY^\varepsilon = f(Y^\varepsilon) dX^\varepsilon,$$

it is enough to establish weak convergence of the rough paths $X^\varepsilon$ in the topology defined by the rough metric. Obtaining convergence in this topology follows the convergence of the finite dimensional distributions of the rough paths $X^\varepsilon$ plus tightness in the space of rough paths with respect to that topology.

5.1.1 Tightness of rough paths

The following lemma can be obtained via an Arzela-Ascoli argument, for details see [FH13,FV10].

**Lemma 5.4** Let $0$ denote the rough path obtained from the $0$ path enhanced with a $0$ second order process, then, for $\gamma > \gamma' > \frac{1}{3}$, the sets $\{ X \in \mathcal{C}^\gamma : \varrho_\gamma(X, 0) < R, X(0) = 0 \}$ are compact in $\mathcal{C}^{\gamma'}$.

**Lemma 5.5** Let $\theta \in (0, 1)$, $\gamma \in \left(\frac{1}{3}, \frac{2}{3} - \frac{1}{p}\right)$ and $X^\varepsilon = (X^\varepsilon, X^\varepsilon)$ such that

$$\|X^\varepsilon_{s,t}\|_{L^p(W)} \lesssim |t-s|^\theta, \quad \|X^\varepsilon_{s,t}\|_{L^2(W)} \lesssim |t-s|^{2\theta},$$

then,

$$\sup_{\varepsilon \in (0, 1]} \mathbb{E}(\|X^\varepsilon\|_\gamma)^p < \infty.$$

**Proof.** The proof is based on a Besov-Hölder embedding, for details we refer to [FV10,CFK+19].

**Lemma 5.6** Let $X^\varepsilon$ be a sequence of rough paths, $\gamma \in \left(\frac{1}{3}, \frac{2}{3} - \frac{1}{p}\right)$, such that $X(0) = 0$, and

$$\sup_{\varepsilon \in (0, 1]} \mathbb{E}(\|X^\varepsilon\|_\gamma)^p < \infty,$$

then $X^\varepsilon$ is tight in $\mathcal{C}^{\gamma'}$ for every $\frac{1}{3} < \gamma' < \gamma$.

**Proof.** Choose $\alpha \in (\gamma', \gamma)$, as $\varrho_\alpha(X, 0) \leq \|X\|_\alpha + \|X\|_\alpha^2$ we obtain

$$\mathbb{P}(\varrho_\alpha(X^\varepsilon, 0) > R) \leq \frac{\mathbb{E}(\varrho_\alpha(X^\varepsilon, 0))^{\frac{\gamma'}{\gamma}}}{R^{\frac{\gamma'}{\gamma}}} \leq \frac{\mathbb{E}(\|X\|_\alpha + \|X\|_\alpha^2)^{\frac{\gamma'}{\gamma}}}{R^{\frac{\gamma'}{\gamma}}} \lesssim \frac{C}{R^{\frac{\gamma'}{\gamma}}}.$$

This proves the claim by Lemma 5.4.

5.1.2 Interpreting the effective dynamics by classical equations

We now explain what the limiting equation means in the classical sense. Our set up is the following.

**Assumption 5.7** Let $X_t = (X_t^W, X_t^Z)$, where $X_t^W$ is a $n$-dimensional possibly correlated Wiener process and $X_t^Z$ a $N - n$-dimensional Hermite process. The two components $X_t^W$ and $X_t^Z$ are independent, we set

$$2A := \begin{pmatrix} \text{Cov}(X_t^W) & 0 \\ 0 & 0 \end{pmatrix}. $$

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We write $A_{i,j}$ for the components of $A$. We are concerned with the classical interpretation of the rough differential equation $\dot{x}_t = F(x_t) dX_t$, where $F: \mathbb{R}^d \to \mathbb{L}(\mathbb{R}^N, \mathbb{R}^d)$ is a $C^2_b$ map, $X = (X_1, X_2 \cdots)$ is a $C^1$ map, and $Y = (Y_1, Y_2 \cdots)$ is a $C^2$ map. By assumption 2.10 (2) the terms containing $X_n$ as a linear combination of a standard differential equation in the controlled rough path space is interpreted as Itô integrals if $i, j \leq n$, otherwise as Young integrals.

We show that the rough differential equation (4.1) is really the same as the equations given in part 3 of Theorem 4.2. Without loss of generality, we will assume our solution is defined on the interval $[0, 1]$. According to Theorem 8.4 in [FH14], see also [Lyo94, FV10], there exists a unique solution to our rough differential equation in the controlled rough path space $D^{2\alpha}_X([0, 1]; \mathbb{R}^d)$, where $\alpha > \frac{1}{2}$. The solution exists globally in time and the full controlled process is given by $(x_s, F(x_s))$, which means $x_{s,t} = F(x_s)X_{s,t} + R_{s,t}$, where $\|R\|_{2\alpha} < \infty$. By Lemma 7.3 in [FH14] given a controlled rough path $(Y, Y') \in D^{2\alpha}_X([0, 1]; \mathbb{R}^d)$ and a function $\phi \in C^2_b$, then $(\phi(Y), \phi(Y'))$ is also a controlled rough path in $D^{2\alpha}_X([0, 1]; \mathbb{R}^d)$, where $\phi(Y)' = D\phi(Y)Y'$. In our case $F \in C^2_b$, thus, $(F(x_s), DF(x_s)F(x_s)) \in D^{2\alpha}_X([0, 1]; \mathbb{R}^d)$ and

$$x_t - x_0 = \int_0^t F(x_u) dX_u = x_0 + \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} F(x_u)X_{u,v} + DF(x_u)F(x_u)X_{u,v} + DF(x_u)F(x_u)(v - u)A_{i,j}$$

In components these are just, for $l = 1, \ldots, d$,

$$x_t^l - x_0^l + \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \sum_{k=1}^N F_k(x_u)X_{u,v}^k + \sum_{l' = 1}^d \sum_{i,j=1}^n DF(x_u)^{l',i}F_j^l(x_u)X_{u,v}^{i,j} + DF(x_u)^{l',i}F_j^l(x_u)(v - u)A_{i,j}$$

By assumption 2.10 (2) the terms containing $X_{i,j}^{i,j}$, where $i \vee j > n$, do not contribute to the limit, hence we may neglect them, see also Lemma 4.2 in [FH14]. We will drop these terms and use $A_{i,j} = 0$ with only $i \vee j > n$. Let

$$I_1(\mathcal{P}) = \sum_{[u,v] \in \mathcal{P}} \sum_{k=1}^n F_k(x_u)X_{u,v}^k + \sum_{l' = 1}^d \sum_{i,j=1}^n DF(x_u)^{l',i}F_j^l(x_u)X_{u,v}^{i,j} + DF(x_u)^{l',i}F_j^l(x_u)(v - u)A_{i,j}.$$

$$I_2(\mathcal{P}) = \sum_{[u,v] \in \mathcal{P}} \sum_{k=n+1}^N F_k(x_u)X_{u,v}^k.$$

Now $I_2(\mathcal{P})$ gives rise the classical Young integrals $\int F(x_u) dX_u$. For $I_1$ we write $X^W = (X_1, \ldots, X^n)$ as a linear combination of a standard $n$ dimensional Wiener $W$. Let $U$ be given such that $U^T U = 2A$ so $X^W = UW$. Then $X_{u,v}^{i,j} = A_{u,v}^{i,j} W_{u,v}^{i,j}$, where $W_{u,v}^{i,j}$ denotes the Itô lift of $W$, $(W_{u,v}^{i,j} = \int_u^v X_{u,r}^i dX_r^j)$. We
obtain,
\[
I_1(P) = \sum_{[u,v] \in P} \sum_{k=1}^n F^k_u(x_u) \sum_{q=1}^{U^{k,q}} W^{q}_{u,v} + \sum_{l'=1}^d \sum_{i,j=1}^n \sum_{i,j=1}^n D F(x_u)^{l',i} F^l_j (x_u) 2 A^{i,j}_{u,v} W^{i,j}_{u,v} + D F(x_u)^{l',i} F^l_j (x_u)(v-u) A^{i,j}.
\]

Now, by Proposition 3.5 and Theorem 9.1 in [FH14] \( \lim_{|P| \to 0} I_1(P) \) coincides almost surely with the proclaimed Stratonovich integrals as the term \( D F(x_u)^{l',i} F^l_j (x_u)(v-u) A^{i,j} \) corresponds exactly the Stratonovich correction. We may now conclude our explanation.

Finally, we conclude the paper with a question.

**Open Problem.** For Theorem A and B to hold, the only restriction on the Hermite rank of the functions \( G_k \) comes from the lack of integral bound (3.3). We can only prove this bound when \( H^+(m) \in [0, \frac{1}{2}] \). Our question is: Can one lift the restriction \( H^+(m) < 0 \), and still obtain the bound (3.3)? A proposal for obtaining this is to depart from the Hölder path approach used here and take on the p-variation rough path formulation instead. In [CFK+19], the authors have improve their regularity assumption from their previous work by using the p-variation rough path formulation instead of the Hölder one. They were studying the diffusive homogenisation problem, for this they managed to include \( p = \frac{1}{6} \).

**References**

[AS84] Milton Abramowitz and Irene A. Stegun, editors. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York; John Wiley & Sons, Inc., New York, 1984. Reprint of the 1972 edition, Selected Government Publications.

[BC17] I. Bailleul and R. Catellier. Rough flows and homogenization in stochastic turbulence. *J. Differential Equations*, 263(8):4894–4928, 2017.

[BH02] Samir Ben Hariz. Limit theorems for the non-linear functional of stationary Gaussian processes. *J. Multivariate Anal.*, 80(2):191–216, 2002.

[BM83] Peter Breuer and Péter Major. Central limit theorems for nonlinear functionals of Gaussian fields. *J. Multivariate Anal.*, 13(3):425–441, 1983.

[CFK+19] Ilya Chevyrev, Peter K. Friz, Alexey Korepanov, Ian Melbourne, and Huilin Zhang. Multiscale systems, homogenization, and rough paths. In *Probability and Analysis in Interacting Physical Systems*, 2019.

[CFS82] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai. *Ergodic theory*, volume 245 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982. Translated from the Russian by A. B. Sosinskii.

[CKM03] Patrick Cheridito, Hideyuki Kawaguchi, and Makoto Maejima. Fractional Ornstein-Uhlenbeck processes. *Electron. J. Probab.*, 8:no. 3, 14, 2003.
[LH19] Xue-Mei Li and Martin Hairer. Averaging dynamics driven by fractional brownian motion. arXiv:1902.11251 To appear in the Annals of Probability., 2019.

[LOV00] C. Landim, S. Olla, and S. R. S. Varadhan. Asymptotic behavior of a tagged particle in simple exclusion processes. Bol. Soc. Brasil. Mat. (N.S.), 31(3):241–275, 2000.

[LS] Xue-Mei Li and J. Sieber. Slow/fast system with fractional environment and dynamics. In preparation.

[Lyo94] Terry Lyons. Differential equations driven by rough signals. I. An extension of an inequality of L. C. Young. Math. Res. Lett., 1(4):451–464, 1994.

[MT07] Makoto Maejima and Ciprian A. Tudor. Wiener integrals with respect to the Hermite process and a non-central limit theorem. Stoch. Anal. Appl., 25(5):1043–1056, 2007.

[NP05] David Nualart and Giovanni Peccati. Central limit theorems for sequences of multiple stochastic integrals. The Annals of Probability, 33(1):177–193, 2005.

[PK74] G. C. Papanicolaou and W. Kohler. Asymptotic theory of mixing stochastic ordinary differential equations. Comm. Pure Appl. Math., 27:641–668, 1974.

[PT17] Vladas Pipiras and Murad S. Taqqu. Long-range dependence and self-similarity. Cambridge Series in Statistical and Probabilistic Mathematics, [45]. Cambridge University Press, Cambridge, 2017.

[Sam06] Gennady Samorodnitsky. Long range dependence. Found. Trends Stoch. Syst., 1(3):163–257, 2006.

[Taq75] Murad S. Taqqu. Weak convergence to fractional Brownian motion and to the Rosenblatt process. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 31:287–302, 1975.

[Taq77] Murad S. Taqqu. Law of the iterated logarithm for sums of non-linear functions of Gaussian variables that exhibit a long range dependence. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 40(3):203–238, 1977.

[Tay21] G. I. Taylor. Diffusion by Continuous Movements. Proc. London Math. Soc. (2), 20(3):196–212, 1921.

[You36] L. C. Young. An inequality of the Hölder type, connected with Stieltjes integration. Acta Math., 67(1):251–282, 1936.