On special geometry of the moduli space of string vacua with fluxes

Boyu Hou, Sen Hu, Yanhong Yang

1 Department of physics, Northwest University, Xi’an, Shannxi,
2 Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026
† shu@ustc.edu.cn

Abstract: In this paper we construct a special geometry over the moduli space of type II string vacua with both NS and RR fluxes turning on. Depending on what fluxes are turning on we divide into three cases of moduli space of generalized structures. They are respectively generalized Calabi-Yau structures, generalized Calabi-Yau metric structures and $\mathcal{N} = 1$ generalized string vacua. It is found that the $\text{dd}^c$ lemma can be established for all three cases. With the help of the $\text{dd}^c$ lemma we identify the moduli space locally as a subspace of $d_H$ cohomologies. This leads naturally to the special geometry of the moduli space. It has a flat symplectic structure and a Kähler metric with the Hitchin functional (modified if RR fluxes are included) the Kähler potential. Our work is based on previous works of Hitchin and recent works of Graña-Louis-Waldram, Goto, Gualtieri, Yi Li and Tomasiello. The special geometry is useful in flux compactifications of type II string theories.

Keywords: Generalized structure, string vacua, special geometry.
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1. Introduction

String compactifications of flux backgrounds are of great interests in recent years. It is hoped that in the general background more realistic models can be constructed, moduli stability problem of inflation models can be tracked and more general black hole attractors can be obtained, among other things. There have been much developments in this direction. For recent reviews see [1], [2].

The notion of generalized Calabi-Yau manifolds was introduced by Hitchin in [3] and was further studied by Gualtieri [4]. Hitchin [3] studies special geometry of the moduli space of generalized complex structures. Their work attracts great interests of string theorists and generated much works in the past years.

In string compactifications a crucial step is to establish special geometry of the moduli space of generalized Calabi-Yau manifolds. Special geometry of generalized Calabi-Yau manifolds which incorporate all fluxes was proposed and established by Graña-Louis-Waldram in [21], [22]. In this paper we shall give an alternative treatment of the moduli space of string vacua of generalized structures which will lead naturally to such a special geometry.

Depending on what fluxes are turning on we divide into three cases of moduli space of generalized structures. They are respectively generalized Calabi-Yau structures, generalized Calabi-Yau metric structures and \( \mathcal{N} = 1 \) generalized string vacua.

To incorporate Neveu-Schwarz fluxes we change exterior derivative \( d \) to \( d + H \wedge \). To include RR fluxes we introduce local sources, as anticipated from the no-go theorem. We shall work on a non-compact manifolds with divisors consisting of generalized complex submanifolds (with respect to the integrable generalized complex structure) and orientifolds as local sources.

The moduli space of generalized structures can be defined as the space of generalized structures over a manifold modulo diffeomorphisms and exact one forms. Deformations of generalized manifolds were studied by Li Yi [15] and Goto [16]. They construct a complex of Lie algebroid and prove unobstructed-ness for the moduli space of generalized structures over a manifold. We borrow a crucial argument from Goto that the period map is injective whenever the \( \partial \bar{\partial} \) lemma is true. Such a lemma is provided by Gualtieri for twisted generalized Kähler manifolds. We extend the \( \partial \bar{\partial} \) lemma for all \( \mathcal{N} = 1 \) generalized vacua. This would help us to identify the moduli space as co-chains represented by pure spinors.

As long as we have a good identification of the moduli space it leads naturally to the special geometry of the moduli space. We shall prove that there is a natural symplectic form, a complex structure and a Kähler metric with logarithm of Hitchin functional as its potential function. We modify the Hitchin functional so that RR fluxes arise as source terms. The critical points of the new Hitchin functional are precisely the super-symmetric solutions of type II strings with all fluxes turning on.
Those are the basis of special geometry for the moduli space of super-symmetric solutions.

2. Geometry of $G$ structures

2.1 Moving frames: $G$ structures

In the study of geometry it is important to find out interesting geometric structures to work with. It turns out that $G$ structures and subsequently generalized $G$ structures are of great interests from both geometric and physical points of view.

Given a manifold $M$ we consider the frame bundle of the tangent space. The frame bundle is a bundle with fiber $Gl(n, \mathbb{R})$. A $G$ structure is a subbundle of the frame bundle whose fiber is $G$. We also say that we have the structure group reduction to $G$.

For example, if we are given a metric $g$ then we may consider all orthonormal frames. This gives a subbundle whose fiber is $O(n, \mathbb{R})$. If the manifold is orientable we would have the structure group reduction to $Sl(n, \mathbb{R})$.

If we are given an almost complex structure, i.e. $J : TM \rightarrow TM, J^2 = -Id$, then we have $T \otimes C = T^{1,0} \oplus T^{0,1}$. This way we have the structure group reduction $Gl(n, C) \subset Gl(2n, \mathbb{R})$.

The existence of an $G$ structure is a topological problem. It turns out that the interesting geometric objects are those $G$ structures which are integrable. Here is a definition of integrability of a $G$ structure $[5]$:

A $G$ structure is called holonomic or integrable if: Under the Levi-Civita parallelism permissible frames remains permissible.

For example, an almost complex structure is integrable iff $\nabla J = 0$.

An integrable $G$ structure is a manifold with special holonomy with the holonomy group $G$. Those manifolds of special holonomy has been classified by M. Berger into seven classes provided that they are not homogeneous manifolds.

1) $G = SO(n)$, Orientable Riemannian manifolds;
2) $G = U(n)$, Kähler manifolds.

There are five classes of manifolds of special holonomy on which there exists a non-zero spinor $\epsilon$ such that $\nabla \epsilon = 0$:
3) $G = SU(n)$, Calabi-Yau manifolds;
4) $G = Sp(n)$, hyper-Kähler manifolds;
5) $G = Sp(n) \times Sp(1)$, Quaternionic Kähler manifolds;
6) $G = G_2$, seven dimensional manifolds;
7) $G = Spin(7)$, eight dimensional manifolds.

One may relax the integrability condition of a Levi-Civita connection to a connection with non-vanishing torsion. This gives Chern connection and Bismut connection. A connection $\nabla + T$ is compatible with the metric if and only if $T$ is an anti-symmetric
tensor. It turns out that the torsion gives NS fluxes in string vacua. In recent years several authors got new solutions of string vacua with NS fluxes. See [7, 34] and references in for more details.

2.2 Special geometry over the moduli space of Calabi-Yau manifolds

A complex manifold of $SU(3)$ holonomy is called a Calabi-Yau manifold. It appears in the compactification of string theory. The key property of a Calabi-Yau manifold is the existence of a non-vanishing spinor which is parallel with respect to the spin connection $\nabla$:

$$\nabla_\mu \epsilon = 0.$$ 

By Yau’s theorem, any Kähler manifold $M$ with $c_1(TM) = 0$ is a Calabi-Yau manifold. For such a manifold, there exists a unique Ricci flat metric with a given $(1,1)$ closed form $\omega$, such that $\int_M \omega \wedge \omega > 0$.

The Calabi-Yau structure (the above Ricci flat metric) depends on a symplectic structure $\omega$ and a complex structure and the moduli space of Calabi-Yau structures is:

$$M = \mathcal{M}_K \times \mathcal{M}_C \subset H^{1,1}(M) \times H^{2,1}(M).$$

It turns out that the moduli space itself is a Kähler manifold with the Kähler metric. We call this the special geometry of the moduli space of Calabi-Yau manifolds. The metric is:

$$ds^2 = \frac{1}{V} \int_{M^6} g^{\bar{a}b} g^{\bar{c}d} (\delta g_{ac} \delta g_{bd} + (\delta g_{ad} \delta g_{cb} - \delta B_{ad} \delta B_{cb}) d^6 x.$$ 

It is a Kähler metric with Kähler potential:

$$e^{K^{2,1}} = -i \int \Omega \wedge \bar{\Omega},$$

$$e^{K^{1,1}} = -i \int \omega \wedge \omega \wedge \omega.$$ 

Much of developments in string theory depend on this special geometry. For example one may use special geometry to study variation of Hodge structures of Calabi-Yau manifolds. This gives a topological B model. One may study black hole attractors. And of course it is important to study string compactifications through the special geometry. And one may study mirror symmetry. The problem is given a Calabi-Yau manifold $M_1$ find a mirror Calabi-Yau manifold $M_2$, such that

$$\mathcal{M}_K(M_1) = \mathcal{M}_C(M_2).$$
3. Generalized structures and string vacua with fluxes

In the study of string theory we want to know string vacua with fluxes turning on. It turns out that the setting of generalized geometry is a good framework to construct new string vacua. The purpose of this paper is to establish a mathematical foundation of special geometry for such new string vacua. We start with some elementary definitions.

3.1 Spinors and bispinors

Let \((V, q)\) be a vector space with a quadratic form \(q\), then we have a Clifford algebra:

\[
Cl(V, q) = \oplus_{r \geq 0} V^\otimes r / \{uv + vu = q(u, v)Id\}.
\]  

(3.1)

It is known that, depending on whether the dimension of \(V\) is odd or even, there are one or two irreducible representations of \(Cl(V, q)\):

\[
\rho : Cl(V, q) \rightarrow \text{End}(\Delta).
\]  

(3.2)

Here \(\Delta\) is the space of spinors. Now we consider a special case. Let \(T\) be a vector space and let \(V = T \oplus T^*\), where \(T^*\) is the dual space of \(T\). There is a natural pairing between \(T\) and \(T^*\). So we have a quadratic form:

\[
< X + \xi, Y + \eta > = \frac{1}{2} (\xi(Y) + \eta(X)).
\]  

(3.3)

Now consider the Clifford algebra \(Cl(V, <, >)\). We find that the exterior algebra of \(T^*\) is the natural space of spinors for this Clifford algebra, i.e. if we consider

\[
(X + \xi) \omega = i_X \omega + \xi \wedge \omega,
\]  

(3.4)

we would have a natural representation:

\[
\rho : Cl(V, <,>) \rightarrow \text{End}(\Lambda^* T^*).
\]  

(3.5)

Given any \((T, g)\), where \(g\) is a quadratic form over \(T\), we have a Clifford algebra. Let \(\gamma^\mu\) be a basis of the Clifford algebra. It satisfies:

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}Id.
\]  

(3.6)

The representation of \(Cl(T, g)\) gives rise to \((T, g)\) spinors. We need to imbed \((T, g)\) into \((T \oplus T^*)\) isometrically. This can be done as follows:

\[
X \rightarrow X + i_X g \pm i_X b,
\]  

(3.7)

where \(b\) is any anti-symmetric two form. Let us consider anti-symmetrized \(\gamma\)-matrices,
\gamma^{\mu_1 \cdots \mu_p} = \gamma^{[\mu_1 \cdots \mu_p]}.

(3.8)

The precise relationship of \((T, g)\) spinors and \((T \oplus T^*, <, >)\) spinors is given by the so-called bi-spinor construction. Let \(\epsilon_1, \epsilon_2\) be two \((T, g)\) spinors, then

\[ \Sigma_{\mu_1 < \mu_2 < \cdots < \mu_p} \bar{\epsilon}_2 \gamma^{\mu_1 \cdots \mu_p} \epsilon_1 e_{\mu_1} \wedge \cdots \wedge e_{\mu_p} \]

(3.9)

is an element of \(\Lambda^*(T^*)\), here \(\{e_{\mu}\}_{\mu=1,\ldots,n}\) is a basis of \(T^*\).

This gives rise to a map which is actually an isomorphism, we call it the bispinor construction:

\[ \Delta \otimes \Delta \cong \Lambda^* T^*, \ (\epsilon_1, \epsilon_2) \rightarrow \Sigma_{0 \leq \mu_1 < \mu_2 < \cdots < \mu_p \leq n} \bar{\epsilon}_2 \gamma^{\mu_1 \cdots \mu_p} \epsilon_1 e_{\mu_1} \wedge \cdots \wedge e_{\mu_p}. \]  

(3.10)

### 3.2 Almost generalized complex structures

**Definition:** An almost generalized complex structure over a vector space \(T \oplus T^*\) is an endomorphism:

\[ \mathcal{J} : T \oplus T^* \rightarrow T \oplus T^*, \quad \text{where} \quad \mathcal{J}^2 = -Id, \quad <\mathcal{J}. , \mathcal{J}. > = < . , . >. \]  

(3.11)

We have then the maximal isotropic space:

\[ L = \{ \mathcal{J} = i \} \subset (T \oplus T^*)_{\mathbb{C}}, \]

(3.12)

such that

\[ (T \oplus T^*)_{\mathbb{C}} = L \oplus \bar{L}, \quad L \cap \bar{L} = \{0\}. \]  

(3.13)

We also call such a structure an almost Dirac structure. Recall that \(\Lambda^* T^*\) is the space of spinors for \((T \oplus T^*, <, >)\), via the map:

\[ (X + \xi) . \omega = i_X \omega + \xi \wedge \omega. \]

(3.14)

\(\Lambda^{ev/od} T^*\) of even and odd degrees are two irreducible representations of the spin group. If there is a \(\rho \in \Lambda^* T^*\), such that

\[ L = \{ X + \xi | (X + \xi) . \rho = 0 \} \]

(3.15)

we call \(\rho\) a pure spinor. Given a pure spinor \(\rho\) from Proposition 2.16 we have that \(\rho + i \hat{\rho} = 2\varphi\) is a stable spinor. The condition that \(L \cap \bar{L} = \{0\}\) is precisely the condition that \((\varphi, \bar{\varphi}) \neq 0\). We see that a pure spinor, modulo multiplication of a nonzero constant, determines a maximal isotropic space of an almost generalized complex structure. Each maximal isotropic subbundle \(L\) corresponds to a sub line bundle of pure spinors \(U\).
We can parameterize the space of generalized $G$ structures by using stable spinors. For us the most important cases are six and seven dimensional. The moduli space of almost generalized $SU(3) \times SU(3)$ structures over a vector space is:

$$\mathcal{M} = SO(6,6)/SU(3,3) = U_\rho / \mathbb{C}^*,$$

where $U_\rho$ is the space of stable spinors. The moduli space of almost generalized $G_2 \times G_2$ structures over a vector space is:

$$\mathcal{M} = SO(7,7)/G_2 \times G_2 = U_\rho / \mathbb{C}^*, \quad \text{(3.17)}$$

where $U_\rho$ is the space of stable spinors.

To pass the definition of an almost generalized $G$ structure over a vector space to a manifold we need the notion of bundles. Let $M^6$ be a manifold, over the manifold we have a bundle:

$$U_\rho / \mathbb{C}^* \to \mathcal{E} \to M^6.$$  

The space of almost generalized $G$ structures is the space of sections of the above bundle.

**Remark:** It is purely a topological condition whether an almost generalized $G$ structure exists. A necessary condition is the existence of a stable spinor. This condition is also a sufficient condition for dimensions 6 and 7.

In general we define geometric structures as an orbit of $B(V)$ structures. Let $V$ be an $n$-dimensional vector space with a dual space $V^*$. The conformal group $Cpin(V \oplus V^*)$ of $V \oplus V^*$ acts on direct sums of exterior algebra $\oplus^i \Lambda^* V^*$. Let $\Phi = (\phi_1,...,\phi_l)$ be an element of the direct sum $\oplus^i \Lambda^* V^*$ and $B(V)$ the orbit of $Cpin(V \oplus V^*)$ through $\Phi$. We fix the orbit $B(V)$ and goes to a manifold $M^n$. The orbit $B(V)$ yields the orbit in $\oplus^i \Lambda^* T_x M^*$ for each point $x \in \mathcal{M}^n$. We then have a fibre bundle $B(M)$ by

$$B(M) := \cup_{x \in M^n} B(T_x M) \to M^n.$$ 

The set of global sections of $B(M)$ is denoted by $\mathcal{E}_B(M)$. We define a $B(V)$-structure by a $d_H$-closed section of $\mathcal{E}_B(M)$. In the next section we shall explain that the $d_H$-closed condition is precisely the integrability condition of the underlying almost structure.
3.3 Integrability of an almost generalized structure

Given an almost generalized complex structure whether it is integrable or not is of great interests. We will see that integrable generalized $G$ structures are closely related to Type II string vacua.

**Definition: (Integrability of an almost generalized complex structure)**

Let $J$ be an almost generalized complex structure and $L$ the maximal isotropic space associated to $J$. We say that $J$ is integrable if $L$ is Courant involutive, i.e. $L$ is closed with respect to the Courant bracket:

$$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi).$$  \hspace{1cm} (3.19)

This condition can be expressed in terms of pure spinors.

The Clifford algebra $CL(T \oplus T^*)$ is filtered as follows:

$$\mathbb{R} = CL^0 < CL^2 < ... < CL^{2n} = CL^+(T \oplus T^*)$$  \hspace{1cm} (3.20)

$$T \oplus T^* = CL^1 < CL^3 < ... < CL^{2n-1} = CL^-(T \oplus T^*),$$  \hspace{1cm} (3.21)

where $CL^l$ is spanned by products of numbers of not more than $l$ elements of $T \oplus T^*$. The Clifford multiplication respects this graded filtration structure.

Suppose we have a trivialization of the pure sub line bundle $U$ with a nonzero section $\rho$. We may decompose the space of differential forms by Clifford multiplication on $\rho$:

$$U = U_0 < U_2 < ... < U_{2n} = \Lambda^{ev/od}T^* \otimes \mathbb{C},$$  \hspace{1cm} (3.22)

$$L^* U = U_1 < U_3 < ... < U_{2n-1} = \Lambda^{od/ev}T^* \otimes \mathbb{C},$$  \hspace{1cm} (3.23)

where $U_k$ is defined as $CL^k U, k = 1, ..., 2n, U_k$ are eigenspaces of eigenvalue $-ik$ of $\mathcal{J}$ acting on forms through spin representation.

**Theorem:** \[4\] The almost Dirac structure $L$ is Courant involutive if and only if the exterior derivative $d$ satisfies

$$d(C^\infty(U)) \subset C^\infty(U_1),$$  \hspace{1cm} (3.24)

i.e. $L$ is involutive if and only if, for any local trivialization $\rho$ of $U$, there exists a section $X + \xi \in C^\infty((T \oplus T^*) \otimes \mathbb{C})$ such that

$$d\rho = i_X \rho + \xi \wedge \rho.$$  \hspace{1cm} (3.25)

This condition is invariant under rescaling of $\rho$ by a smooth function.

We can also define the $\bar{\partial}$ operator:
\[ \bar{\partial} = \pi_{k+1} d : C^\infty(U_k) \to C^\infty(U_{k+1}), \]  
(3.26) \[
\partial = \pi_{k-1} d : C^\infty(U_k) \to C^\infty(U_{k-1}), \]  
(3.27)

here \( \pi_k \) is the projection operator from \( \Lambda \cdot T^* \) to \( U_k \). Then the integrability condition is equivalent to \( d = \partial + \bar{\partial} \).

**Definition:** (Twisted almost generalized \( G \) structure) We may twist an almost generalized complex structure by a closed three form \( H \) and the Courant bracket is generalized to:

\[ [X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi) + i_X i_Y H. \]  
(3.28)

A twisted almost generalized \( G \) structure is integrable if and only if that \( L \) is closed with respect to the generalized Courant bracket. In terms of spinors the integrability condition is then

\[ d_H(C^\infty(U)) \subset C^\infty(U_1), \]  
(3.29)

where \( d_H = d + H \wedge \). Similarly one defines \( \bar{\partial}_H = \pi_{k+1} d_H \) and \( \partial_H = \pi_{k-1} d_H \). Then the integrability condition of a twisted almost generalized complex structure is equivalent to \( d_H = \partial_H + \bar{\partial}_H \).

3.4 String vacua with fluxes turning on

The supersymmetric transformations for type II theories contain two ten-dimensional Majorona-Weyl spinor parameters \( \epsilon^{1,2} \). The ten-dimensional manifold is topologically a Minkowski space times an internal six-dimensional manifold. The ten-dimensional spinors can be decomposed into spinors in four dimensions times internal spinors. We are interested in backgrounds preserving four-dimensional \( \mathcal{N} = 1 \) supersymmetry and there is a single four dimensional conserved spinors. We write

\[ \epsilon^1 = \chi_+ \otimes \eta_1^- + \chi_- \otimes \eta_1^+, \]

\[ \epsilon^2 = \chi_+ \otimes \eta_2^- + \chi_- \otimes \eta_2^+, \]

for type IIA and

\[ \epsilon^i = \chi_+ \otimes \eta_+^i + \chi_- \otimes \eta_-^i, \quad i = 1, 2 \]

for type IIB with \( \chi_+ \) any four-dimensional spinor and \( \chi_- \) being its Majorana conjugate. We have \( (\eta_+^i)^* = \eta_-^i \) so that \( \epsilon^i \) are Majorana in ten dimensions.

Given two spinors we can produce two pure spinors by bispinor construction:
They are compatible pure spinors in the sense that they have exactly three common annihilators.

If we have one pure spinor we would have structure group reduction $SU(3, 3) \subset SO(6, 6)$. When we have two compatible pure spinors we would have further structure group reduction $SU(3) \times SU(3) \subset SO(6, 6)$. Thus $\Phi_\pm$ defines an almost $SU(3) \times SU(3)$ structure on $T \oplus T^*$. Preserved supersymmetry imposes differential conditions on the Clifford(6) spinors. As a consequence, the pure Clifford(6,6) spinors, as formal sums of differential forms by bispinor construction, have to obey differential conditions. In order to preserve $\mathcal{N} = 1$ supersymmetry, the conditions are \cite{10, 11, 12, 13}:

$$(d + H \wedge)\Phi_1 = 0, (d + H \wedge)\Phi_2 = *F,$$

where $F = F_0 + F_2 + F_4 + F_6 + F_8$ (for type IIA) or $F = F_1 + F_3 + F_5 + F_7 + F_9$ (for type IIB) are Ramond-Ramond fluxes consisting of formal sums of even or odd degree. Furthermore the fluxes $H, F$ obey Bianchi identities: $dH = 0$, $d_H F = \delta(\cup_i D_i)$, where $D_i$ are generalized submanifolds or orientifolds.

In case of $RR = 0$ we would have $\mathcal{N} = 2$ string vacua.

There are $\mathcal{N} = 1$ string vacua with NS fluxes only. We have one pure spinor and it gives a $SU(3)$ structure with an almost complex structure and an almost symplectic structure. Those structures give a $(3,0)$ form $\Omega$ and a $(1,1)$ form $J$. Preserving supersymmetry implies they satisfy the following equations \cite{1}:

$$(d + H \wedge)(e^{2\phi} \Omega) = 0, e^{2\phi} d(e^{-2\phi} J) = *H, d(e^{2\phi} J^2) = 0.$$ 

4. The moduli space of generalized string vacua

In this section we shall give a proper definition of the moduli space of generalized string vacua. Depending on what fluxes are turning on we shall divide them into several cases.

4.1 Generalized Calabi-Yau structure

Let $\Phi = \Sigma_{i=1}^l \phi_i$ be a sum of forms from a pure spinor and $B(V)$ the orbit of $C\text{pin}(V \oplus V^*)$ through $\Phi$. We fix the orbit $B(V)$ and go to a manifold $M$. The orbit $B(V)$ yields the orbit in $\bigoplus_{i=1}^l \Lambda^* T^*_x M$ for each point $x \in M^n$. We then have a fibre bundle $B(M)$ by

$$B(M) := \cup_{x \in M^n} B(T_x M) \rightarrow M.$$
The set of global sections of $\mathcal{B}(M)$ is denoted by $\mathcal{E}_B(M)$. The space of generalized Calabi-Yau structures is:

$$\mathcal{M} = \{ \Phi \in \mathcal{E}_B(M) | d\Phi = 0 \}.$$  \hspace{1cm} (4.1)

We need to mode out the action of diffeomorphisms isotopic to identity and the action of exact two forms. They form a space by taking a semi-product:

$$0 \to \Omega^1(M) \to Diff_0(M) \to Diff(M) \to 0.$$  \hspace{1cm} (4.2)

Diffeomorphisms isotopic to identity are generated by vector fields $X$. In addition to action of diffeomorphisms we also have action of $B$ fields. We have the action of $X + \xi$ on $\Phi$ as:

$$L_{X+\xi} \Phi = d(i_X \Phi + \xi \wedge \Phi).$$

The universal moduli space of generalized Calabi-Yau structures is defined as:

$$\tilde{\mathcal{M}} = \mathcal{M} / \tilde{Diff}(M^6)$$  \hspace{1cm} (4.3)

In fact the above universal moduli space is the usual Universal moduli space for the case of complex structures over a Riemann surface. We may define the moduli space of generalized Calabi-Yau structures as:

$$\mathcal{M} = \tilde{\mathcal{M}} / Diff\tilde{f}(M^6)$$  \hspace{1cm} (4.4)

where $Diff\tilde{f}(M^6)$ is the full group of diffeomorphisms together with a semi product of closed two forms with integer coefficients.

### 4.2 $\mathcal{N} = 2$ Type II string vacua

To incorporate Nervu-Schwarz fluxes we simply twist the exterior derivative $d$ by $d_H = d + H \wedge$. There is a subtlety here when we try to generalize the Lie derivative with respect to $F = X + \xi \in T \oplus T^*$. The generalization is:

$$\mathcal{L}_F \Phi = F.d_H \Phi + d_H(F.\Phi).$$  \hspace{1cm} (4.5)

The reason to do such a modification is that $H$ changes under a diffeomorphism. It changes $H$ to $H + d(i_X H)$. We knew that $H$ is the field strength of a $B$ field which defines a gerbe background. $B$ field changes to $B + i_X H$ under a diffeomorphism generated by $X$.

It is consistent with the fact that a diffeomorphism transforms integrable structures to integrable structures. For an integrable twisted generalized $G$ structure defined by $\Phi$ we have $d_H \Phi = 0$, then we have $\mathcal{L}_F \Phi = d_H(F.\Phi)$. So under a diffeomorphism generated by $F, \Phi$ stays in the same $d_H$-cohomology class.
The space of $\mathcal{N} = 2$ string vacua is:

$$\tilde{\mathcal{M}} = \{ \Phi = (\Phi_1, \Phi_2) \in \mathcal{E}(M^6) | d_H \Phi_1 = 0, d_H \Phi_2 = 0, ||\Phi_1 = \Phi_2|| \}/Diff(M^6). \quad (4.6)$$

The moduli space of $\mathcal{N} = 2$ string vacua is:

$$\mathcal{M} = \tilde{\mathcal{M}}/Diff(M^6) \quad (4.7)$$

where $Diff(M^6)$ is the full group of diffeomorphisms together with a semi product of closed two forms with integer coefficients.

The above structure is the same as generalized Calabi-Yau metric structure. A generalized Calabi-Yau metric geometry is defined by a generalized Kähler structure $(\bar{J}_1, \bar{J}_2)$ where each generalized complex structure has holomorphically trivial canonical bundle, i.e. their canonical line bundles have non-vanishing closed sections $\rho_1, \rho_2 \in C^\infty(\Lambda^2 T^* \otimes \mathbb{C})$. We also require that the lengths of these sections are related by a constant, i.e. $(\rho_1, \bar{\rho}_1) = c(\rho_2, \bar{\rho}_2)$.

A generalized Calabi-Yau metric structure may also be twisted by a three form $H$, by requiring that $(\bar{J}_1, \bar{J}_2)$ is an $H$-twisted generalized Kähler structure defined by $d_H$-closed forms $\rho_1, \rho_2$ satisfying the above length constraint.

### 4.3 $\mathcal{N} = 1$ Type II string vacua

If we turn on all fluxes the equations for $\mathcal{N} = 1$ vacua of generalized structures are:

$$d_H \Phi_1 = 0,$$

$$d_H \Phi_2 = *F.$$

We also have the Bianchi identity $d_H F = \delta$, where $\delta$ is supported over magnetic sources which are usually orientifolds and generalized submanifolds of the manifold. To be specific, we consider the case that the $\delta$ function is supported on union of generalized submanifolds. We formulate the problem as Hodge systems over a non-compact manifold with divisors union of generalized complex submanifolds.

We then have the space of $\mathcal{N} = 1$ string vacua as:

$$\tilde{\mathcal{M}} = \{ (\Phi_1, \Phi_2) \in \mathcal{E}(M^6) | d_H \Phi_1 = 0, d_H \Phi_2 = *F \}/Diff(M^6). \quad (4.8)$$

The moduli space of $\mathcal{N} = 1$ type II string vacua is:

$$\mathcal{M} = \tilde{\mathcal{M}}/Diff(M^6) \quad (4.9)$$

where $Diff(M^6)$ is the full group of diffeomorphisms together with a semi product of closed two forms with integer coefficients.
5. Deformations of generalized structures

5.1 Deformations of generalized Calabi-Yau structures

We now describe deformations of generalized Calabi-Yau structures by constructing a complex. Recall that

\[ CL^{k+1} = \Lambda^k(T \oplus T^*) / \{ u.v + v.u = 2 < u, v > Id \}. \]  \hspace{1cm} (5.1)

Let \( E^k(M) = CL^{k+1}.\Phi \). Since \( \Phi \) is integrable we have that \( d\Phi = 0 \). In general this implies that \( dE^k \subset E^{k+1} \). We then have a complex:

\[ \ldots \to E^0(M) \overset{d}{\to} E^1(M) \overset{d}{\to} E^2(M) \to \ldots \]  \hspace{1cm} (5.2)

From [16] we knew that \( H^1 = Ker d^2 / Im d^1 \) is the space of infinitesimal deformations of the generalized complex structures.

We also have a chain map of the above complex into de Rham complex. Since the forms of pure spinors are closed we have the period map:

\[ p : H^1 \to H^*(M). \]  \hspace{1cm} (5.3)

**Definition (Elliptic):** An orbit of generalized structures \( B(V) \) is elliptic if the deformation complex is an elliptic complex.

**Definition (Topological structure):** \( B(V) \) is a topological structure if the period map \( p_k \) is injective for \( k = 1, 2 \).

From the following lemma [16] we have that if the \( dd^J \)-lemma is true then the above period map is injective. Hence the generalized structure is a topological structure.

\[ dd^J \text{- Lemma} : \] We say that a manifold have \( dd^J \) property if for any exact form \( \alpha = d\tau \) which is also \( d^J \) closed form then it can be written as \( d\tau = dd^J \beta \).

**Lemma:** Suppose that the \( dd^J \) lemma is true for a manifold \( M \), then for any form \( d\tau \) satisfying \( d^J d\tau = 0 \), there exists \( X + \xi \) so that \( d\tau = d(i_X \Phi + \xi \wedge \Phi) \).

**Remark:** The construction here is also true for twisted exterior derivative \( d_H = d + H \wedge \). The above lemma is true for a twisted generalized \( G \) structure as long as the \( d_H d^J_H \) lemma is true. The proof is in the same way as in [16].

**Remark:** A consequence of the above lemma is that the effect of actions of diffeomorphisms and one forms is the same as taking cohomology. This would mean that the generalized structure is a topological structure. This way we get a very simple description of the moduli space by simply taking cohomologies of the formal sum of a pure spinor.

5.2 Deformations of generalized Calabi-Yau metrical structures

The equations for \( \mathcal{N} = 2, RR = 0 \) string vacua are:
\[ d_H \Phi_1 = 0, d_H \Phi_2 = 0, ||\Phi_1|| = ||\Phi_2||. \]

Those are called twisted generalized Calabi-Yau metric structures and they describe all \((2,2)\) nonlinear sigma models. The Universal moduli space of twisted generalized Calabi-Yau metric structures is:

\[ \tilde{M} = \{ (\Phi_1, \Phi_2) | d_H \Phi_1 = 0, d_H \Phi_2 = 0, ||\Phi_1|| = ||\Phi_2|| \} / Diff_0(M) \times \Omega^1. \]

There is a mapping class group \(MCG = Diff(M) / Diff_0(M) \times H^2(M; \mathbb{Z})\) acting on \(\mathcal{M}\). The true moduli space of \(N = 2\) vacua is \(\mathcal{M} = \tilde{M} / MCG\).

Each pure spinor gives a generalized complex structure. We see that a generalized Calabi-Yau structure must be a generalized Kähler structure.

**Definition: (Twisted Generalized Kähler structure)** A twisted generalized Kähler structure consists of two commuting twisted generalized complex structures \(\mathcal{J}_1, \mathcal{J}_2\) such that \(\mathcal{G} = -\mathcal{J}_1 \mathcal{J}_2\) is positive definite over \(T \oplus T^*\).

Since \(\mathcal{G}^2 = I, T \oplus T^* = C_+ \oplus C_-\), where \(C_\pm = \{ \mathcal{G} = \pm 1 \}\). We have that \(<, > |_{C_+} > 0, <, > |_{C_-} < 0, <, > |_{C_+}\) gives a metric over \(C_+\).

Let \(* = a_1...a_n\) be the product in \(CL(C_+), CL(T \oplus T^*)\) of an oriented orthonormal basis for \(C_+\). This volume element acts on the differential forms via the spin representation. When \(b = 0\), it gives the Hodge star operator \(* : \Lambda T^* \rightarrow \Lambda T^*\). In general we have \(*^2 = (-1)^{n(n-1)/2}\). Let \(\alpha\) be a \(k\) form, define \(\sigma \alpha = (-1)^{k(k-1)/2} \alpha\). Over \(\Lambda^*T^*\) we have an inner product:

\[ h(\alpha, \beta) = \int_M < \alpha, \sigma(*) \beta >, \]

which we call the Born-Infeld inner product. We will calculate the adjoint of the twisted exterior derivative \(d_H\) with respect to this inner product.

The \(d_Hd_H^t\) lemma for a twisted generalized Kähler manifold is established in [18]. We reproduce it here for convenience. This will be crucial for us to imbed the moduli space into the space of de Rham cohomologies.

For each twisted generalized complex structure \(\mathcal{J}\) we have decomposition of forms into

\[ U = U_0 < U_2 < ... < U_{2n} = \Lambda^{ev/od}T^* \otimes \mathbb{C}, \]

\[ L^* U_0 = U_1 < U_3 < ... < U_{2n-1} = \Lambda^{od/ev}T^* \otimes \mathbb{C} \]

where \(U_k\) is an invariant eigenspace of \(\mathcal{J}\) with eigenvalue \(-ik\) acting on forms through spin representation. A generalized complex structure \(\mathcal{J}\) preserves the canonical pairing between tangent and cotangent spaces. It is then an element of \(so(n, n)\). It then acts on the space of forms \(\Lambda^*T^*\) by spin representation.
With respect to the second twisted generalized complex structure which commutes with the first twisted generalized complex structure we have further decomposition:

\[ U_k = U_{k,|k|^{-n}} \oplus U_{k,|k|^{-n+2}} \oplus \cdots \oplus U_{k,n-|k|}, \quad U_{p,q} = U_p^1 \cap U_q^2. \] (5.4)

It gives the Hodge decomposition. For a twisted generalized Kähler manifold, we have that

\[ d_H = \delta_+ + \delta_- + \bar{\delta}_+ + \bar{\delta}_-, \] (5.5)
\[ \bar{\partial}_1 = \bar{\delta}_+ + \bar{\delta}_-, \quad \bar{\partial}_2 = \bar{\delta}_+ + \delta_- . \] (5.6)

Gualtieri announced the following generalization of Kähler identities [18]:

\[ \bar{\delta}_+^* = -\delta_+, \quad \bar{\delta}_-^* = \delta_- . \] (5.7)

Here \( \bar{\delta}_+^* = *\delta_+**^{-1} \) is the adjoint. Those identities would imply the \( d_Hd_H^I \) – lemma.

From [10], a generalized Calabi-Yau metric structure can be described by two generalized \( Sl(n,C) \) structures. An \( Sl(n,C) \) structure is a complex form of type \( (n,0) \), \( \Omega_V \) with respect to a complex structure \( J \).

Conversely for each \( Sl(n,C) \) structure \( \Omega_V \) we define a complex subspace \( \ker \Omega_V \) by \( \ker \Omega_V = \{ v \in V_C| i_v \Omega_V = 0 \} \). We have then \( V_C = \ker \Omega_V \oplus \ker \bar{\Omega}_V \). This determines a complex structure \( J \). In other words an \( Sl(n,C) \) structure \( \Omega_V \) gives a complex structure \( J \) on \( V \).

Let \( \Omega_V \) be an \( Sl(n,C) \) structure and \( \omega_V \) a real two form on \( V \). We define a bilinear form \( g \) by \( g(u,v) = \omega(Ju,v) \). A pair \( (\Omega_V, \omega_V) \) is a Calabi-Yau structure on \( V \) if the followings hold:

1) \( \Omega_V \wedge \omega_V = 0 \),
2) \( \Omega \wedge \bar{\Omega} = c_n \omega^n \),
3) The corresponding bi-linear form \( g \) is positive-definite.

Given Gualtieri’s \( d_Hd_H^I \) lemma and by slightly modifying the proof of Goto we have:

**Theorem:** The twisted generalized Calabi-Yau orbit is an elliptic and topological orbit. This would imply that the moduli space can be embedded into the space of twisted cohomologies locally.

### 5.3 Deformations of generalized structures of \( \mathcal{N} = 1 \) vacua

The equations for \( \mathcal{N} = 1 \) vacua of generalized structures are:

\[ d_H \Phi_1 = 0, \]
\[ d_H \Phi_2 = \ast F. \]

We also have the Bianchi identity \( d_H F = \delta \), where \( \delta \) is supported over magnetic sources \([24]\). To be specific, we consider the case that the \( \delta \) function is supported on union of generalized submanifolds with respect to the first generalized complex structure. We formulate the equations as Hodge systems over a non-compact manifold with divisors union of generalized complex submanifolds, i.e. \( M = \bar{M} - \cup_i D_i \). We further assume that \( D_i \) are divisors with normal crossing, i.e. locally \( D_i \) is described by equations \( z_1 \ldots z_r = 0 \) in the case of complex structure with respect to complex coordinates, or by \( p_1 = c_1, \ldots, p_r = c_r \) in the case of symplectic structure with respect to a polarization. By a generalized Darboux theorem, a generalized complex structure is locally a product of complex and symplectic structures.

Over \( M \) we need to choose a proper functional space to work with. It turns out that the proper functional space consists of forms which are square integrable and their exterior derivatives are square integrable \([19]\).

Over this functional space the Hodge systems are soluble. The key step is to construct a Green current which gives fundamental solutions of the Hodge systems. This can be done over this functional space. Let \( d_H^* \) be the adjoint of \( d_H \). We then have Laplacian \( \Delta_H = d_H d_H^* + d_H^* d_H \). The Green current is a solution of the equation \( \Delta_H G = \delta(\cup_i D_i) \). The Green current satisfies equivalent equations: \( d_H G = 0, d_H \ast G = \ast F \). We have \( d_H(\Phi_2 - \ast G) = 0 \) so that \( \Phi_2 - \ast G \) represents an integrable generalized complex structure \( J'_2 \). We see that the Green current represents a co-chain of \( d_H \)-cohomology which represent the union of divisors.

Alternatively, the Green current can be constructed by solving the Dirac equation \( D G = F \), where \( D = d_H + d_H^* \). We see then \( D(\Phi_2 - \ast G) = 0 \).

\( J_1 \) and \( J'_2 \) commutes. This follows from that the divisors are generalized submanifolds with respect to \( J_1 \). Hence the divisors are invariant with respect to \( J_1 \). The Green current is then invariant with respect to \( J_1 \).

For the first generalized complex structure coming from \( \Phi_1 \), we have decomposition of forms as eigenspaces of the action by \( J_1 \) in the spin representation. For the second generalized complex structure coming from \( \Phi_2 \) it is not integrable. We can replace it by \( \Phi_2 - \ast G \) which is a new integrable generalized complex structure \( J'_2 \). We have decomposition of forms as eigenspaces of the action by \( J'_2 \) in the spin representation. \( J_1 \) and \( J'_2 \) commutes. So we have Hodge decomposition over \( M \) by using generalized complex structures \( J_1 \) and \( J'_2 \).

We may decompose the twisted exterior operator \( d_H \) into four operators \( \delta_+ + \delta_- + \delta_+ + \delta_- \) with \( \partial_1 = \delta_+ + \delta_- \) and \( \partial_2 = \delta_+ + \delta_- \). Here the decomposition is respect to \( J_1 \) and \( J'_2 \).

We still have the generalized Kähler identities: \( \delta^* = -\delta_+ + \delta_- \). This would imply the \( \partial_H \bar{\partial}_H \) lemma. Proof of the generalized Kähler identities is to compute
adjoint of the $\delta_+$ with respect to the Born-Infeld metric. We can perform similar computations as for Kähler manifolds. The conditions of square integrable of forms and their exterior derivatives make all the computations possible.

Hence the period map is injective so the universal moduli space of $\mathcal{N} = 1$ generalized vacua can be imbedded into the space of cohomologies. We have two pure spinors. For $\Phi_1$ we have $d_H\Phi_1 = 0$ and the moduli would be just the $d_H$-cohomology classes of $\Phi_1$. For $\Phi_2$ we have $d_H\Phi_2 = *F$ and the moduli would be the $d_H$-cohomology classes of the co-chain represented by $\Phi_2$. All this follows from the $\partial_H \bar{\partial}_H$ lemma.

There is again a mapping class group $\text{MCG}$ acting on $\mathcal{M}$. Here we need to modify the definition of $H^2(M, \mathbb{Z})$ to $H^2_c(M, \mathbb{Z})$ of cohomology with compact support. And $\text{Diff}_c(M)$ consists of diffeomorphisms leaving divisors invariant.

Finally we have the true moduli space of $\mathcal{N} = 1$ generalized vacua. $\text{MCG}$ induces an action on the space of cohomologies which gives a local system. The flat connection of such a local system is the usual Gauss-Manin connection. The usual Picard-Fuchs equations may follow from this and thus give a basis of topological B model.

6. Special geometry of the moduli space of generalized string vacua

Over the moduli space we have a special geometry which means that we have a flat symplectic structure, an integrable complex structure and a Kähler metric. In this section we shall construct such a special geometry (see also [3], [21, 22]).

6.1 The symplectic structure over $\mathcal{M}$

We define a symplectic structure over $\mathcal{M}$:

$$\omega(\Phi_1, \Phi_2) = \int_{M^6} <\Phi_1, \Phi_2>.$$  

(6.1)

Here we have the Mukai Pairing:

$$<\Phi_1, \Phi_2> = \sum_p (-1)^\frac{p+1}{2} \int \Phi_{1,p} \wedge \Phi_{2,n-p}.\quad (6.2)$$

To show that it defines a symplectic structure over the moduli space we may check that the integral depends on $d_H$-cohomology classes only. This follows from a formular proved by Hitchin:

$$\int_M \sigma(\hat{\alpha}) \wedge d_H \beta = - \int_M \sigma(d_H \hat{\alpha}) \wedge \beta.\quad (6.3)$$
Since we are integrating it over the manifold it is also diffeomorphism invariant. Actually the Clifford action of the tangent bundle and the cotangent bundle stays in the same $d_H$ cohomology classes.

Since the definition only depends on integrating forms and not depends on anything else it is a constant form so we have $d\omega = 0$. According to Darboux’s theorem around each point we can take a special Darboux coordinates so that

\[ \omega = \Sigma dx^K \wedge dy_K. \]  

(6.4)

Those coordinates are very useful in constructing special geometries.

\section*{6.2 The complex structure over $\mathcal{M}$}

We knew that stable spinors can be decomposed into sum of pure spinors:

\[ \Phi = \rho + \hat{\rho}. \]  

(6.5)

We have an involution map:

\[ X : \rho = \phi + \bar{\phi} \rightarrow \hat{\rho} = -i\phi + i\bar{\phi}, \]  

(6.6)

The differential of $X$ defines an almost complex structure over stable spinors because $DX.DX = -Id$. Actually $J = DX$ defines an integrable complex structure over $\mathcal{M}$. We can prove this by using Darboux coordinates. Consider the dual basis of $dx^K, dy_K$ of the tangent space we can express the symplectic form as:

\[ \omega(\Phi, \bar{\Phi}) = \Sigma_K (Z^K F_K - Z^K \bar{F}_K). \]  

(6.7)

One can show that $Z^K, F_K$ are two independent complex coordinates.

\section*{6.3 Hitchin functional and the Kähler metric over $\mathcal{M}$}

By using the symplectic structure $\omega$ and the complex structure $J$ we may define a symmetric two tensor $g(.,.) = \omega(., J.)$. We shall see that this gives rise a metric. This metric is actually Kähler and its Kähler potential is the Hitchin functional.

Recall that the generalized Hitchin functional is:

\[ H(\phi, F) = \int_{\mathcal{M}} -i < \phi, \hat{\phi} > + < F, \eta >, \]  

(6.8)

where $\phi = \phi_0 + d_H \eta$ is a stable spinor and $F$ is a formal sum of forms which represent the Ramond-Ramond fluxes. It is a function over $\mathcal{M}$ because it depends only on the $d_H$ cohomology class and it is a diffeomorphic invariant.

From the definition of the Hitchin functional we see that $\hat{\rho} = \delta H / \delta \rho$. Since the derivative of the map $X : \rho \rightarrow \hat{\rho}$ gives rise to the complex structure $J$ we have that the metric $\omega(J, .)$ is the same as the one using the Hitchin functional:
\[ g_{\alpha \beta} = \frac{\partial^2 H}{\partial \chi^\alpha \partial \bar{\chi}^\beta} \quad (6.9) \]

Here \( \partial \chi_\alpha \) is a basis of the holomorphic tangent space of the moduli space. This metric is actually Kähler and the Kähler potential of the metric is: \( K = - \log H \).

We finally have:

\[ e^{-K(\Phi)} = H(\Phi) = i \omega(\Phi, \bar{\Phi}) = \sum_K i(\bar{Z}^K F_K - Z^K \bar{F}_K). \quad (6.10) \]

This generalize a formular for the moduli space of complex structures of Calabi-Yau manifold:

\[ K^{2,1} = - \log(i \int \Omega \wedge \bar{\Omega}), K^{1,1} = - \log(i \int_M \omega \wedge \omega \wedge \omega). \quad (6.11) \]

7. Examples

Here are a few examples of generalized Calabi-Yau structures and twisted generalized Kähler structures. There are more examples in the literature.

1) Nilmanifolds

A nilmanifold is a homogeneous space \( M = G/\Gamma \), where \( G \) is a simply-connected nilpotent real Lie group and \( \Gamma \) is a lattice of maximal rank in \( G \). The nilmanifold can be described by giving the differentials of a set \( \{e_1, e_2, ..., e_6\} \) of linearly independent left-invariant 1-forms. In the nilmanifold literature one uses the array \( (0, 0, 0, 12, 13, 14 + 35) \) to describe a nilmanifold with de Rham complex generated by 1-forms \( \{e_1, e_2, ..., e_6\} \) and such that \( de_1 = de_2 = de_3 = 0 \), while \( de_4 = e_1 \wedge e_2, de_5 = e_1 \wedge e_3, \) and \( de_6 = e_1 \wedge e_4 + e_3 \wedge e_5 \). For six dimensional nilmanifolds there are 5 classes of nilmanifold which admit no known complex or symplectic structure. They are:

\[ (0, 0, 12, 13, 14 + 23, 34 + 52); \]
\[ (0, 0, 0, 12, 13, 14 + 35); \]
\[ (0, 0, 0, 0, 12, 15 + 34). \]

Those five families admit generalized complex structures. In each case, the canonical bundle is holomorphically trivial. Hence they are examples of generalized Calabi-Yau structures. For more details, see [1], [25].

2) Homogeneous space
Any compact even-dimensional Lie group admits left- and right-invariant complex structure $J_L, J_R$, and that if the group is semi-simple, they can be chosen to be Hermitian with respect to the bi-invariant metric induced from the Killing form $<,>$. We would have a 3-form $H, H(X,Y,Z) = <[X,Y],Z>$. It turns out that $(<,>, J_L, J_R)$ forms an H-twisted generalized Kähler structure, see [4].

3) Connected sum of $S^3 \times S^3$

$S^3 \times S^3$ is a manifold with two complex structures. However they are not a Kähler manifold since $h^{1,1} = 0$. Consider $S^3$ as a copy of $SU(2)$. The Lie algebra of $su(2)$ are Pauli matrices $\sigma_1, \sigma_2, \sigma_3$. We have then two complex structures. One of them is given by:

$$\omega = e^{i\pi/4}(\sigma^1 + i\sigma^2) \wedge (\hat{\sigma}^1 - i\hat{\sigma}^2) \wedge (\sigma^3 + i\hat{\sigma}^3),$$
$$\Omega = \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \sigma^3.$$

The other complex structure is:

$$\omega = e^{i\pi/4}(\sigma^1 + i\hat{\sigma}^1) \wedge (\sigma^2 + i\hat{\sigma}^2) \wedge (\sigma^3 + i\hat{\sigma}^3),$$
$$\Omega = \sigma^1 \wedge \hat{\sigma}^1 + \sigma^2 \wedge \hat{\sigma}^2 + \sigma^3 \wedge \hat{\sigma}^3.$$

We may take connected sums of $S^3 \times S^3$. It is found that manifolds of $k \geq 2$ copies of $S^3 \times S^3$ satisfy $\partial \bar{\partial}$ Lemma property ([27]). Such manifolds appears in the smoothing of Calabi-Yau manifolds.

8. Conclusions and prospects

The importance of flux compactifications have been realized for a variety of problems in string theory and in its applications to field theory and cosmology. Systematic studies are carried out in recent years. New geometric structures such as generalized G structures arises and it fits naturally to describe string vacua with fluxes turning on.

In string compactifications we need to know deformations of a given string vacua and the special geometry they obey. As a first step we need to identify the moduli space of vacua and its tangent space. It turns out there are several cases depending on what fluxes are turning on. They are respectively generalized Calabi-Yau, generalized Calabi-Yau metric structures and the most general case of $\mathcal{N} = 1$ generalized vacua with both NN fluxes and RR fluxes turning on. In this paper we give definitions of moduli spaces for those cases respectively and identify their tangent spaces.

It turns out that the crucial step is an elementary lemma called the $\partial \bar{\partial}$ lemma which implies the $dd^c$ lemma. Those lemmas play key roles to show that the period map is injective so that the moduli space can be imbedded into the space of de Rham cohomologies.
As soon as one identifies the space of vacua one can establish a special geometry on the moduli space of vacua. One confirms that the Hitchin functional appears naturally as the potential of the Kähler metric over the moduli space.

There are a number of directions one can follow naturally. There are works to identify the open string moduli \([28, 29, 30, 31, 32]\). It would be interesting to consider the full moduli space and to identify superpotential.

It is interesting to study dualities among different kinds of compactifications. In heterotic string compactifications the equations of supersymmetric solutions were derived by Strominger \([35]\). There are much works recently in finding new solutions and applications to heterotic string compactifications \([36, 37, 38, 39]\). It would be interesting to study dualities of flux vacua of heterotic strings with flux vacua of type II strings. It is also interesting to study supersymmetric solutions in M theory \([42, 43]\).

There is an string duality of vacua between type IIA and type IIB called mirror symmetry. There are a number of interesting proposals for generalized geometry that the mirror symmetry is simply interchanging two generalized complex structures \([50, 48]\). Since mirror symmetry acts on topological models one needs to study such models for the generalized geometries. It would be interesting to extend works of \([53, 54]\) for the setting of generalized geometry.

Our study can be considered as a study of variations of Hodge structures. It then raises many problems to extend the work on variations of Hodge structures over Kähler manifolds to generalized Kähler manifolds. Since we have the crucial \(dd^c\) lemma we would expect many works extends to the more general case. See \([55, 56]\) for works on variations of Hodge structures.

The study of flux compactifications are limited largely due to the lack of examples. There are quite a few examples appeared in recent years. It is still lacking a theorem like Calabi-Yau theorem for the category of generalized complex geometry. For this even a proper generalization of Calabi’s conjecture would be very interesting.

There are many applications in physics. For example, there are some works on generalized black hole attractor mechanisms, works on gauge-gravity correspondence and works on moduli stabilization especially for inflation models. We expect that the work on flux compactifications would help substantially on all those problems.

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A. Mukai pairing and the symplectic structure

Definition A.1. Mukai pairing
Let \( x \mapsto x^T \) be the main antiautomorphism of the Clifford algebra \( CL(V \oplus V^*) \), i.e. that determines by the tensor map \( v_1 \otimes v_2 \otimes \ldots \otimes v_k \mapsto v_k \otimes v_{k-1} \otimes \ldots \otimes v_1 \). The Mukai pairing is defined to be

\[
(\cdot, \cdot)_M : \wedge V^* \otimes \wedge V^* \to \det V^*, (s, t)_M = [s^T \wedge t]_m, \, m = \dim V.
\] (A.1)

Mukai pairing has the following properties:

\[
(x \cdot s, t)_M = (s, x^T \cdot t)_M, \forall x \in CL(V \oplus V^*). \tag{A.2}
\]

Proposition A.2. The Mukai pairing is invariant under the identity component of \( \text{Spin} \):

\[
(g \cdot s, g \cdot t)_M = (s, t)_M, \forall g \in \text{Spin}_0(V \oplus V^*). \tag{A.3}
\]

Therefore the Mukai pairing defines a \( \text{Spin}_0 \) invariant bilinear form on \( S = \wedge V^* \otimes (\det V)^{1/2} \).

When \( m = 6 \), the Mukai pairing is skew symmetric. We shall begin to study the algebra by working over the complex numbers, using \( \text{Spin}(12, \mathbb{C}) \) instead of \( \text{Spin}(6, 6) \) and a complex six-dimensional vector space \( V \). In this dimension the bilinear form on each of the 32-dimensional spin spaces \( S^\pm \) is skew symmetric, and so these are symplectic representations. We shall fix \( S = S^\pm \).

A symplectic action of a Lie group \( G \) on a vector space \( S \) defines a moment map

\[
\overline{\pi} : S \to g^* \otimes \wedge^6 V^*
\] (A.4)
given by

\[
\overline{\pi}(\rho) = \frac{1}{2}(\sigma(a)\rho, \rho)_M, \forall a \in g, \rho \in S. \tag{A.5}
\]

Here \( G = \text{Spin}(V \oplus V^*) \), \( g = \text{spin}(V \oplus V^*) \cong \text{so}(12, \mathbb{C}) \), \( \sigma : g \to \text{End}(S) \) is the representation of Lie algebras and \( a \in g \). And we identify \( g \) with \( g^* \) by the inner product

\[
(\cdot, \cdot) : g \otimes g \to \mathbb{C}, \, (X, Y) \to \text{tr}(XY). \tag{A.6}
\]

Definition A.3. Let \( \overline{\pi} \) be the moment map for the spin representation \( S \) of \( \text{Spin}(12, \mathbb{C}) \), then

\[
\overline{\pi}(\rho) = \text{tr} \overline{\pi}(\rho)^2 \tag{A.7}
\]
is an invariant quartic function on \( S \).

This quartic has a close relationship with pure spinors:
Proposition A.4. For $\rho \in S, \overline{\rho} \neq 0$ if and only if $\rho = \alpha + \beta$, where $\alpha, \beta$ are pure spinors and $(\alpha, \beta)_M \neq 0$. The spinors $\alpha, \beta$ are unique up to ordering.

It is proved by Hitchin [3]. From the proof, we have

$$\overline{q}(\alpha + \beta) = 3(\alpha, \beta)_M^2, \overline{p}^2 = \frac{1}{48}q(\rho)I.$$  \hspace{1cm} (A.8)

Let us consider a real 6−dim vector space $W$, $S = \wedge^6 W^*$. $q(\rho) \neq 0$ implies there are two possibilities: $\alpha$ and $\beta$ are both real, or $\beta = \overline{\alpha}$, denoted by $\overline{q}(\rho) > 0$, $\overline{q} < 0$.

(Note: Let $L$ is a real one-dimensional vector space, $u \in L \otimes L$, we say $u > 0$ if $u = s \otimes s$ for some $s \in L$; $u < 0$ if $-u > 0$.)

Let us fix an orientation $\epsilon \in \wedge^6 W^*$ on $W$, define a symplectic form on $\wedge^\bullet W^*$:

$$\omega : \wedge^\bullet W^* \times \wedge^\bullet W^* \to \mathbb{R},$$  \hspace{1cm} (A.9)

such that $\omega(\rho_1, \rho_2)\epsilon = (\rho_1, \rho_2)_M$.

Re-define the moment map $\mu : S \to g^*$ by $\rho \mapsto \mu(\rho)(a) = \frac{1}{2}\omega(\sigma(a)\rho, \rho)$. And let $q(\rho) = tr\mu(\rho)^2$. It is easy to see that $\overline{p} = \mu \otimes \epsilon$, $\overline{q}(\rho) = q(\rho)\epsilon^2$.

Consider the open set

$$U = \{\rho \in S : q(\rho) < 0\} \hspace{1cm} (A.10)$$

acted on transitively by the real group $\mathbb{R}^* \times \text{Spin}(6, 6)$.

Definition A.5. Define a homogeneous function of degree 2 on $U$: $\phi(\rho) = \sqrt{-q(\rho)/3}$.

Note from the above proposition we can write $\rho = \varphi + \bar{\varphi}$ for a pure spinor $\varphi$ such that

$$i\phi(\rho) = \omega(\phi, \bar{\phi}).$$  \hspace{1cm} (A.11)

Proposition A.6. Let $X$ be the Hamiltonian vector field on $U$ defined by the function $\phi$ using the constant symplectic form on $U \subset S$. Describe the vector field on the open set $U$ in the vector space $S$ as a function $X : U \to S$. Then

$\star$ $X(\rho) = \dot{\rho}$ where $\rho + i\dot{\rho} = 2\varphi$;

$\star$ $X$ generates the circle action $\varphi \mapsto e^{-i\theta}\varphi$.

$\star$ the derivative $DX : U \to \text{End}(S)$ defines an integrable almost complex structure $J$ on $U$.

By the definition of Hamiltonian vector field, we have the derivative at $\rho$ of $\phi$ is a linear map can be written as

$$D\phi(\dot{\rho}) = \omega(\dot{\rho}, \dot{\rho});$$  \hspace{1cm} (A.12)

The second derivative

$$D^2\phi(\dot{\rho}_1, \dot{\rho}_2) = \omega(DX\dot{\rho}_1, \dot{\rho}_2) = \omega(J\dot{\rho}_1, \dot{\rho}_2)$$  \hspace{1cm} (A.13)
Suppose $M$ is a compact oriented 6–manifold with volume form $\epsilon$, and $\rho$ is a sum of forms, either odd or even, which lies at each point of $M$ in the open subset $U$ described above. Such a form is called stable. We can then define a volume functional
\[ V(\rho) = \int_M \phi(\rho) \epsilon. \] (A.14)

Let $H$ be a closed 3-form on $M$, define an operator on forms:
\[ d_H \alpha = d\alpha + H \wedge \alpha. \] (A.15)

It is easy to see that $d_H^2 = 0$. We can define $d_H$–cohomology. Consider the variational problem in a fixed $d_H$–cohomology class. We have

**Theorem A.7.** A $d_H$–closed stable form $\rho \in \wedge^{ev/od}(M)$ is a critical point of $V(\rho)$ in its $d_H$–cohomology class if and only if $d_H(\hat{\rho}) = 0$.

*Proof:* by computation, we have $\int_M (\hat{\rho}, d_H \alpha)_M = -\int_M (d_H \hat{\rho}, \alpha)_M$.

At a critical point of $V$, the Hessian $H$ is
\[ H(d_H \alpha_1, d_H \alpha_2) = -\int_M (d_H J d_H \alpha_1, \alpha_2)_M. \] (A.16)

**B. Hitchin’s functional and the variational principle**

Given a stable spinor $\rho$ we define Hitchin’s functional as:
\[ H(\rho) = \int_M (\sigma(\rho) \wedge J \rho)_{\text{top}}. \] (B.1)

We then consider an variational problem of $H(\rho)$ with a fixed $d_H$-cohomology with $d_H = d + H \wedge$. The Euler-Lagrange equations are:
\[ d_H \rho = 0, d_H (J \rho) = 0. \] (B.2)

From [13] those equations are precisely the integrability conditions of a twisted almost generalized $G$ structure.

To incorporate the Ramond-Ramond fluxes we generalize Hitchin’s functional as:
\[ H(\rho) = \int_M (\sigma(\rho) \wedge J \rho)_{\text{top}} + (\eta \wedge F)_{\text{top}}, \] (B.3)
where $\rho = \rho_0 + d_H \eta, d_H \rho_0 = 0, d_H (J \rho_0) = 0$.

The Euler-Lagrange equations of $H(\rho)$ for a fixed $d_H$-cohomology class are [40]:
\[ d_H \rho = 0, d_H (J \rho) = *F. \] (B.4)

Those equations are precisely the supersymmetry equations with all fluxes turning on.
References

[1] M. Graña, Flux compactifications in string theory: A comprehensive review, Phys.Rept. 423 (2006) 91-158, hep-th/0509003.

[2] M. Douglas and S. Kachru, Flux compactification, hep-th/0610102.

[3] N. Hitchin, Generalized Calabi-Yau manifolds, Quart.J.Math.Oxford Ser. 54 (2003) 281-308, math.DG/0209099.

[4] M. Gualtieri, Generalized complex geometry, D.Phil thesis, Oxford University, math.DG/0401221.

[5] S. S. Chern, The geometry of $G$ structures, Selected papers, vol 3, 23-75, Springer-Verlag.

[6] J. P. Gauntlett, D. Martelli, S. Pakis, D. Waldram, G-Structures and Wrapped NS5-Branes, Commun.Math.Phys. 247 (2004) 421-445, arXiv:hep-th/0205050.

[7] J. P. Gauntlett, D. Martelli and D. Waldram, Superstrings with intrinsic torsion, Phys.Rev. D69 (2004) 086002, hep-th/0302158.

[8] A. Martelli and J. Sparks, G-structures, Fluxes and Calibrations in M-Theory, Phys.Rev. D68 (2003) 085014, hep-th/0306225.

[9] K. Becker, M. Becker and J. Schwarz, String Theory and M Theory, Cambridge University Press, 2007.

[10] U. Linstrom, R. Minasian, A. Tomasiello and M. Zabzine, Generalized complex manifolds and supersymmetry, Commun.Math.Phys. 257 (2005) 235-256, hep-th/0405085.

[11] M. Graña, R. Minasian, M. Petrini and A. Tomasiello, Supersymmetric backgrounds from generalized Calabi-Yau manifolds, JHEP 0408 (2004) 046, hep-th/0406137.

[12] M. Graña, R. Minasian, M. Petrini and A. Tomasiello, Generalized structures of $\mathcal{N}=1$ vacua, JHEP 0511 (2005) 020, hep-th/0505212.

[13] F. Witt, Generalized $G_2$ manifolds, math.DG/0411642.

[14] C. Jeschek, F. Witt, Generalised geometries, constrained critical points and Ramond-Ramond fields, math.DG/0510131.

[15] Li Yi, On deformations of generalized complex structures, the generalized Calabi-Yau case, hep-th/0508030.

[16] R. Goto, On deformations of Generalized Calabi-Yau, hyper-Kähler, $G_2$ and $Spin(7)$ structures I, math.DG/0512211.
[17] R. Goto, Deformations of generalized complex and generalized Kahler structures, arXiv:0705.2495.

[18] M. Gualtieri, Generalized geometry and the Hodge decomposition, math.DG/0409093.

[19] G. de Rham, Differentiable Manifolds, Springer-Verlag, 1984.

[20] A. Tomasiello, Reformulating supersymmetry with a generalized Dolbeault operator, hep-th/0704.2613.

[21] M. Graña, J. Louis and D. Waldram, Hitchin functionals in N=2 Supergravity, JHEP 0601 (2006), hep-th/0505264.

[22] M. Graña, J. Louis and D. Waldram, $SU(3) \times SU(3)$ compactification and mirror duals of magnetic fluxes, hep-th/0612237.

[23] N. Halmagyi and A. Tomasiello, Generalized Kähler potential from supergravity, hep-th/0708.1032.

[24] G. Villasdoro, F. Zwizner, On general flux backgrounds with localized sources, hep-th/0710.2551.

[25] G. Cavalcanti and M. Gualtieri, Generalized complex structures on nilmanifolds, math.DG/0404451.

[26] Henrique Bursztyn, Gil R. Cavalcanti, Marco Gualtieri, Generalized Kahler and hyper-Kahler quotients, arXiv:math/0702104.

[27] J. Gutowski, S. Ivanov and G. Papadopoulos, Deformations of generalized Calibrations and compact non-Kähler manifolds with vanishing first Chern class, Asian Journal of Mathematics 7 (2003), 39-80, math.DG/0205012.

[28] P. Koerber and L. Martucci, From ten to four and back again: how to generalize the geometry, JHEP 0708 (2007) 059, hep-th/0707.1038.

[29] P. Koerber and L. Martucci, Deformations of calibrated D-branes in flux generalized complex manifolds, JHEP 0612 (2006) 062, hep-th/0610044.

[30] P. Koerber, L. Martucci, Warped generalized geometry compactifications, effective theories and non-perturbative effects, arXiv:0803.3149.

[31] P. Koerber and D. Tsimpis, Supersymmetric sources, integrability and generalized-structure compactifications, JHEP 0708 (2007) 082, hep-th/0706.1244.

[32] J. Evslin and L. Martucci, D-brane networks in flux vacua, generalized cycles and calibrations, hep-th/0703129.

[33] L. Martucci, D-branes on general N=1 backgrounds: superpotentials and D-terms, JHEP 0606 (2006) 033, arXiv:hep-th/0602129.
[34] Luca Martucci, Paul Smyth, Supersymmetric D-branes and calibrations on general
N=1 backgrounds, JHEP 0511 (2005) 048, arXiv:hep-th/0507099.

[35] A. Strominger, Superstrings with torsion, Nuclear Physics B, 274(1986), no. 2,
253-284.

[36] Ji-Xiang Fu, Shing-Tung Yau, The theory of superstring with flux on non-Kähler
manifolds and the complex Monge-Ampere equation, hep-th/0604063

[37] Katrin Becker, Melanie Becker, Ji-Xiang Fu, Li-Sheng Tseng, Shing-Tung Yau,
Anomaly Cancellation and Smooth Non-Kähler Solutions in Heterotic String Theory,
Nucl.Phys. B751 (2006) 108-128, hep-th/0604137

[38] Melanie Becker, Li-Sheng Tseng, Shing-Tung Yau, Moduli Space of Torsional
Manifolds, hep-th/0612290

[39] Sergey Grigorian, Shing-Tung Yau, Local geometry of the $G_2$ moduli space,
hep-th/08020723

[40] Claus Jeschek, Frederik Witt, Generalised $G_2$-structures and type IIB superstrings,
JHEP 0503 (2005) 053, arXiv:hep-th/0412280.

[41] Florian Gmeiner, Frederik Witt, Calibrations on spaces with GxG-structure,
Fortsch.Phys. 55 (2007) 727-730, arXiv:hep-th/0701109.

[42] D. Tsimpis, M-theory on eight-manifolds revisited: N=1 supersymmetry and
generalized Spin(7) structures, JHEP 0604 (2006) 027, hep-th/0511047.

[43] C. M. Hull, Generalized Geometry for M-Theory, hep-th/0701203.

[44] Paulo Pires Pacheco, Daniel Waldram, M-theory, exceptional generalised geometry
and superpotentials, arXiv:0804.1362.

[45] Peter Kaste, Ruben Minasian, Alessandro Tomasiello, Supersymmetric M-theory
compactifications with fluxes on seven-manifolds and G-structures, JHEP 0307 (2003)
004, arXiv:hep-th/0303127.

[46] Sebastien Gurrieri, Jan Louis, Andrei Micu, Daniel Waldram, Mirror Symmetry in
Generalized Calabi-Yau Compactifications, Nucl.Phys. B654 (2003) 61-113,
arXiv:hep-th/0211102.

[47] A. Tomasiello, Topological mirror symmetry with fluxes, JHEP 0506 (2005) 067,
arXiv:hep-th/0502148.

[48] U. Lindstrom, M. Rocek, I. Ryb, R. von Unge and M. Zabzine, T duality and
Generalized Kähler Geometry, hep-th/0707.1696.

[49] S. Fidanza, R. Minasian, A. Tomasiello, Mirror symmetric SU(3)-structure manifolds
with NS fluxes, Commun.Math.Phys. 254 (2005) 401-423, arXiv:hep-th/0311122.
[50] R. D’Auria, S. Ferrara, M. Trigiante, On the supergravity formulation of mirror symmetry in generalized Calabi-Yau manifolds, hep-th/0701247.

[51] S. Bellucci, S. Ferrara, R. Kallosh, A. Marrani, Extremal Black Hole and Flux Vacua Attractors, arXiv:0711.4547.

[52] S. Ferrara, K. Hayakawa, A. Marrani, Erice Lectures on Black Holes and Attractors, arXiv:0805.2498.

[53] E. Witten, Quantum Background Independence In String Theory, hep-th/9306122.

[54] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes, Commun.Math.Phys. 165 (1994) 311-428, hep-th/9309140.

[55] C. Voisin, Hodge theory and complex algebraic geometry, two volumes, Cambridge University Press 2002.