Power Error Locating Pairs
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Abstract

We present a new decoding algorithm based on error locating pairs and correcting an amount of errors exceeding the half minimum distance. This algorithm applies to Reed–Solomon codes, algebraic geometry codes and any other code benefiting from an error correcting pair. When applied to Reed–Solomon or algebraic geometry codes, the algorithm’s decoding radius turns out to be the same as Sudan’s or power decoding algorithm. Similarly to power decoding it boils down to linear algebra, returns at most one solution and might fail in some rare cases. On the other hand it presents the advantage to have a smaller space and time complexity than power decoding.

Key words: Error correcting codes; Reed–Solomon codes; algebraic geometry codes; decoding algorithms; power decoding; error correcting pairs.

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Introduction

Algebraic codes such as Reed–Solomon codes or algebraic geometry codes are of central interest in coding theory because, compared to random codes, these structured codes benefit from polynomial time decoding algorithms that can correct a significant amount of errors. The decoding of Reed Solomon and algebraic geometry codes is a fascinating topic at the intersection of algorithms, computer algebra, complexity theory and theoretical algebra.

Decoding of Reed–Solomon codes

Reed Solomon codes benefit from a very nice algebraic structure coming from univariate polynomial algebras. Thanks to this structure, one can easily prove that they are maximum distance separable (MDS). In addition, one can design an efficient unique decoding algorithm based on the resolution of a so-called key equation [29] 6 and correcting up to half the minimum distance. This decoding algorithm is sometimes referred to as Berlekamp–Welch algorithm in the literature.

In the late nineties, two successive breakthroughs due to Sudan [27] and Guruswami and Sudan [7] permitted to prove that Reed–Solomon codes and algebraic geometry codes can be decoded in polynomial time with an asymptotic radius reaching the so-called Johnson bound [10]. These algorithms have decoding radii exceeding the half minimum distance at the price that they may return a list of codewords instead of a single one. This drawback has actually a very limited impact since in practice, the list size is almost always less than or equal to 1 (see [13] for further details). Note that decoding Reed–Solomon codes beyond the Johnson bound remains a fully open problem: it is proved in [8] that the

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maximum likelihood decoding problem for Reed–Solomon codes is NP–hard but the possible existence of a theoretical limit between the Johnson bound and the covering radius under which decoding is possible in polynomial time remains an open question.

All the previously described decoders are worst case, i.e. correct any corrupted codeword at distance less than or equal to some fixed bound \( t \). On the other side, some probabilistic algorithms may correct more errors at the cost of some rare failures. For instance, it is known for a long time that the classical Berlekamp–Welch algorithm applied to interleaved Reed–Solomon is a probabilistic decoder reaching the channel capacity \[22\] when the number of interleaved codewords tends to infinity. Inspired by this approach Bostert et. al. \[23\] proposed a probabilistic decoding algorithm for decoding genuine Reed–Solomon codes by interleaving the received word and some of its successive powers with respect to the component wise product. This algorithm has been called power decoding in the sequel. A striking feature of this power decoding is that it has the same decoding radius as Sudan algorithm, moreover an improvement of the algorithm due to Rosenkilde \[21\] permits to reach Guruswami Sudan radius, that is to say the Johnson bound.

However, compared to Sudan algorithm which is worst case and returns always the full list of codewords at bounded distance from the received word, the power decoding algorithm returns at most one element and might fail. The full analysis of its failure probability and the classification of failure cases is still an open problem but practical experimentations give evidences that this failure probability is very low.

Decoding of algebraic geometry codes

All the previously described decoding algorithms for Reed–Solomon codes have natural extensions to algebraic geometric codes at the cost of a slight deterioration of the decoding radius which is propotional to the curve’s genus. The problem of decoding algebraic geometry codes motivated hundreds of articles in the three last decades. The story starts in the late 80’s with and article of Justesen, Larsen, Jensen, Havemose and Høholdt \[11\] proposing a Berlekamp–Welch like algorithm for codes from plane curves. The algorithm has then be extended to arbitrary curves by Skorobogatov and Vladuț \[25\]. The original description of these algorithms was strongly based on algebraic geometry. However, subsequently, Pellikaan proposed an abstract description of these algorithms expurgated from the formalism of algebraic geometry. This description was based on an object called error correcting pair \[18, 19\]. An error correcting pair for a code \( C \) is a pair of codes \( A, B \) such that the space \( A \ast B \) spanned by the component wise products of words of \( A \) and \( B \) is contained in \( C^\perp \). The existence of such a pair together with some conditions on the dimension of \( A \) and the dual distance of \( B \) provide an efficient decoding algorithm essentially based on linear algebra. This approach provides a unified framework to describe algebraic decoding for Reed–Solomon codes, algebraic geometry codes and some cyclic codes \[3, 4\].

Concerning list decoding, Sudan and Guruswami Sudan algorithms extend naturally to algebraic geometry codes. Sudan algorithm has been extended to algebraic geometry codes by Shokrollahi and Wasserman \[24\] and Guruswami Sudan original article \[7\] treated the list decoding problem for both Reed–Solomon and algebraic geometry codes. Similarly, the power decoding algorithm generalizes to such codes. As far as we know, no reference provides such a generalisation in full generality, however, such an algorithm is presented for the case of Hermitian curves in \[16\] and the generalisation to arbitrary curves is rather elementary and sketched in Appendix A.

For further details on unique and list decoding algorithms for algebraic geometry codes, we refer the reader to the excellent survey papers \[9, 1\].

Our contribution

In summary, on the one hand, Reed–Solomon and algebraic geometry codes benefit from a Berlekamp–Welch like unique decoding algorithm, a Sudan–like list decoding algorithm and a power decoding. On the other hand, they both benefit from a unique decoding based on error correcting pairs. This overview is summarized by Figure 1. In the present article we fill the gap by proposing a new algorithm in the spirit of error correcting pairs algorithms but permitting to correct errors beyond the half minimum distance. Similarly as the power decoding algorithm this algorithm returns at most one element but might fail sometimes. Another common feature with power decoding is that our algorithm essentially
boils down to linear algebra with an advantage compared to power decoding: the linear system we have
to solve is smaller and hence permits to have a better space and time complexity.

\[ t \leq \left\lfloor \frac{d-1}{2} \right\rfloor \]  
Berlekamp-Welch  
Error Correcting Pairs

\[ t > \left\lfloor \frac{d-1}{2} \right\rfloor \]  
Sudan  
Power Decoding  
?

Figure 1: Existing decoding algorithms for Reed–Solomon and algebraic geometry codes

Outline of the article

Notation and prerequisites are recalled in Section 1. In Section 2, we present some well–known decoding algorithms for algebraic codes with a particular focus on Reed–Solomon codes. Namely, a presentation of Berlekamp–Welch algorithm, Sudan algorithm, Power decoding and Error correcting pairs algorithm is given. Our contribution, the power error locating pairs algorithm is presented in Section 3 in the case of Reed–Solomon codes and in Section 4 for algebraic geometry codes.

1 Notation and prerequisites

In this article, we work over a finite field \( F_q \). Given a positive integer \( n \), the Hamming weight on \( F_q^n \) is denoted as

\[ \forall x = (x_1, \ldots, x_n) \in F_q^n, \quad w(x) \overset{\text{def}}{=} |\{i \in \{1, \ldots, n\} \mid x_i \neq 0\}| \]

and the Hamming distance \( d(\cdot, \cdot) \) as

\[ \forall x, y \in F_q^n, \quad d(x, y) \overset{\text{def}}{=} w(x - y). \]

The support of a vector \( a \in F_q^n \) is the subset \( \text{supp}(a) \subseteq \{1, \ldots, n\} \) of indexes \( i \) satisfying \( a_i \neq 0 \).

Given a code \( C \subseteq F_q^n \), the minimum distance of \( C \) is denoted by \( d(C) \) or \( d \) when there is no ambiguity on the code. In addition, we recall the notions of puncturing and shortening codes and introduce notation for these notions.

Definition 1.1. Given \( J = \{j_1, \ldots, j_s\} \subseteq \{1, \ldots, n\} \) and \( x = (x_1, \ldots, x_n) \in F_q^n \), we denote by \( x_J \) the projection of \( x \) on the coordinates in \( J \) and by \( Z(x) \) the complement of the support of \( x \) in \( \{1, \ldots, n\} \). Namely,

\[ x_J \overset{\text{def}}{=} (x_{j_1}, \ldots, x_{j_s}) \quad \text{and} \quad Z(x) \overset{\text{def}}{=} \{i \in \{1, \ldots, n\} \mid x_i = 0\} \]

Moreover, for \( A \subseteq F_q^n \), we define

(i) \( A_J \overset{\text{def}}{=} \{a_J \mid a \in A\} \subseteq F_q^{|J|} \) (puncturing);

(ii) \( A(J) \overset{\text{def}}{=} \{a \in A \mid a_J = \mathbf{0}\} \subseteq F_q^n \) (shortening);

(iii) \( Z(A) \overset{\text{def}}{=} \{i \in \{1, \ldots, n\} \mid a_i = 0 \quad \forall a \in A\} \) (complementary of the support).

1.1 Decoding problems

The purpose of the present article is to introduce a new algorithm extending the Error Correcting Pairs algorithm (see §2.4) and permitting to correct errors beyond half the minimum distance. This is possible at one cost: the algorithm might fail sometimes. First, let us list several decoding problems which will be discussed in the sequel.
Problem 1 (Bounded decoding). Let \( C \subseteq \mathbb{F}_q^n \), \( y \in \mathbb{F}_q^n \) and \( t \in \{0, \ldots, n\} \). Return (if exists) a word \( c \in C \) such that \( d(c, y) \leq t \).

When \( t \leq \lfloor (d(C) - 1)/2 \rfloor \), then the solution, if exists, is unique and the corresponding problem is sometimes referred to as the unambiguous decoding problem. For larger values of \( t \), the set of codewords at distance less than or equal to \( t \) from \( y \) might have more than one element. To solve the bounded decoding problem in such a situation, various decoders exist. Some of them return the closest codeword (if unique) other ones return the whole list of codewords at distance less than or equal to \( t \). This is the point of the following problem.

Problem 2 (List decoding). Let \( C \subseteq \mathbb{F}_q^n \) be a code, \( y \in \mathbb{F}_q^n \) and \( t \in \{0, \ldots, n\} \). Return the full list of codewords \( c \in \mathbb{F}_q^n \) such that \( d(c, y) \leq t \).

Definition 1.2. For an algorithm solving one of these problems, the largest possible \( t \) such that the algorithm succeeds is referred to as the decoding radius of the algorithm.

To conclude this section, let us state an assumption which we suppose to be satisfied for any decoding problems considered in the sequel.

Assumption 1. In the following, given a code \( C \) and a positive integer \( t \), when considering one of the above decoding problems, we always suppose that the received vector \( y \in \mathbb{F}_q^n \) is of the form \( y = c + e \) where \( c \in C \) and \( w(e) \leq t \). Equivalently, we always suppose that the bounded decoding problem has at least one solution.

1.2 Reed–Solomon codes

The space of polynomials with coefficients in \( \mathbb{F}_q \) and degree less than \( k \) is denoted by \( \mathbb{F}_q[X]_{< k} \). Given an integer \( n \geq k \) and a vector \( \mathbf{e} \in \mathbb{F}_q^n \) whose entries are distinct, the Reed–Solomon code of length \( n \) and dimension \( k \) is the image of the space \( \mathbb{F}_q[X]_{< k} \) by the map

\[
ev_{\mathbf{e}} : \left\{ \begin{array}{c}
\mathbb{F}_q[X] \\
f
\end{array} \right\} \rightarrow \left\{ \begin{array}{c}
\mathbb{F}_q^n \\
f(x_1), \ldots, f(x_n)
\end{array} \right\}.
\]

This code is denoted by \( \text{RS}_q[k, \mathbf{e}] \) or \( \text{RS}_q[k] \) when there is no ambiguity on the vector \( \mathbf{e} \). That is:

\[
\text{RS}_q[k] \overset{\text{def}}{=} \{(f(x_1), \ldots, f(x_n)) \mid f \in \mathbb{F}_q[X]_{< k}\} = \text{ev}_{\mathbf{e}} (\mathbb{F}_q[X]_{< k}).
\]

Such a code has length \( n \) dimension \( k \) and minimum distance \( d = n - k + 1 \). In this article, for the sake of simplicity, we focus on the case \( n = q \), i.e. the so-called full-support Reed–Solomon codes. This context is much more comfortable for duality since we can assert that

\[
\text{RS}_q[k] = \text{RS}_q[n - k].
\]

In the general case, the above statement remains true by replacing Reed–Solomon codes by generalised Reed–Solomon codes. Actually, the results of the present article extend straightforwardly to generalised Reed–Solomon codes at the cost of slightly more cumbersome notation. For this reason, we fix from now on \( \mathbf{x} \in \mathbb{F}_q^n \) with distinct entries and \( n = q \).

1.3 Algebraic geometry codes

In what follows, by curve we always mean a smooth projective geometrically connected curve defined over \( \mathbb{F}_q \). Given such a curve \( \mathcal{X} \), a divisor \( G \) on \( \mathcal{X} \) and a sequence \( \mathcal{P} = (P_1, \ldots, P_n) \) of rational points of \( \mathcal{X} \) avoiding the support of \( G \), one can define the code

\[
C_L(\mathcal{X}, \mathcal{P}, G) \overset{\text{def}}{=} \{(f(P_1), \ldots, f(P_n)) \mid f \in L(G)\},
\]

where \( L(G) \) denotes the Riemann Roch space associated to \( G \). When \( \deg G < n \), such a code has dimension \( k \geq \deg G + 1 - g \) where \( g \) denotes the genus of \( \mathcal{X} \) and minimum distance \( d > n - \deg G \). We refer the reader to [28, 26] for further details on algebraic geometry, function fields and algebraic geometry codes.
1.4 Star product of words and codes

The space \( \mathbb{F}_q^n \) is a product of \( n \) fields and hence has a natural structure of ring. We denote by \( \ast \) the component wise product of vectors

\[
(a_1, \ldots, a_n) \ast (b_1, \ldots, b_n) \overset{\text{def}}{=} (a_1 b_1, \ldots, a_n b_n).
\]

Given a vector \( a \in \mathbb{F}_q^n \), the \( i \)-th power \( a^i \) of \( a \) is defined as \( a^i \overset{\text{def}}{=} (a_1^i, \ldots, a_n^i) \).

Remark 1.1. This product should not be confused with the canonical inner product in \( \mathbb{F}_q^n \) defined by \( \langle a, b \rangle = \sum_{i=1}^n a_i b_i \). Note that these two operations are related by the following adjunction property

\[
\langle a \ast b, c \rangle = \langle a \ast b, c \rangle.
\]

In particular \( a \ast b \in c^\perp \iff a \ast c \in b^\perp \).

Given two codes \( A, B \subseteq \mathbb{F}_q^n \), the star product \( A \ast B \) is the code spanned by the products \( a \ast b \) for \( a \in A \) and \( b \in B \). Algebraic codes satisfy peculiar properties with respect to this operation.

Proposition 1.1 (Star product of Reed–Solomon codes). Let \( x \in \mathbb{F}_q^n \) be a vector with distinct entries and \( k, k' \) be positive integers such that \( k + k' - 1 \leq n \). Then,

\[
\text{RS}_q[x, k] \ast \text{RS}_q[x, k'] = \text{RS}_q[x, k + k' - 1].
\]

Remark 1.2. Actually Proposition 1.1 holds true even if \( k + k' - 1 > n \) but in this situation the right-hand side becomes \( \mathbb{F}_q^n \).

Proposition 1.2 (Star product of AG codes). Let \( X \) be a curve of genus \( g \), \( \mathcal{P} = (P_1, \ldots, P_n) \) be a sequence of rational points of \( X \) and \( G, G' \) be two divisors of \( X \) such that \( \deg G \geq 2g \) and \( \deg G' \geq 2g + 1 \). Then,

\[
\mathcal{C}_L(X, \mathcal{P}, G) \ast \mathcal{C}_L(X, \mathcal{P}, G') = \mathcal{C}_L(X, \mathcal{P}, G + G').
\]

Proof. This is a direct consequence of [15, Theorem 6]. For instance, see [2, Corollary 9]. \( \square \)

2 Former decoding algorithms for Reed Solomon and algebraic geometry codes

It is known that several different decoding algorithms have been designed for Reed-Solomon and algebraic geometry codes. In particular, depending on the algorithm, we are able to solve either Problem 1 up to the half minimum distance, or Problem 2 up to the Johnson bound.

In this section, we recall several classical unique and list decoding algorithm for Reed–Solomon codes. For all of them an extension to algebraic geometry codes is known. Recall that, whenever we discuss a decoding problem we suppose Assumption 1 to be satisfied, i.e. we suppose that the decoding problem has at least one solution. Hence, we can write

\[
y = c + e,
\]

for some \( c \in C \) and \( e \in \mathbb{F}_q^n \) with \( w(e) = t \). Note that since \( C = \text{RS}_q[k] \), the codeword \( c \) can be written as the evaluation of a polynomial \( f(x) \) with \( \deg(f) < k \). The vector \( e \) is called the error vector. Moreover, we define

\[
I \overset{\text{def}}{=} \text{supp}(e) = \{ i \in \{1, \ldots, n\} \mid e_i \neq 0 \}.
\]

Hence we have \( t = w(e) = |I| \).
2.1 Berlekamp-Welch algorithm

Berlekamp-Welch algorithm boils down to a linear system based on \( n \) key equations. In this case the decoding radius is given by a sufficient condition, i.e. if there is a solution, the algorithm will find it.

**Definition 2.1.** Given \( y, e \) and \( I \) as above we define

- the **locator polynomial** as \( \Lambda(X) \overset{\text{def}}{=} \prod_{i \in I} (X - x_i) \);
- \( N(X) \overset{\text{def}}{=} \Lambda(X)f(X) \).

Hence, for any \( i \in \{1, \ldots, n\} \), the polynomials \( \Lambda \) and \( N \) verify

\[
\Lambda(x_i)y_i = N(x_i). \tag{2}
\]

The aim of the algorithm is then to solve the following

**Key Problem 1.** Find a pair of polynomials \((\lambda, \nu)\) such that \( \deg(\lambda) \leq t, \deg(\nu) \leq t + k - 1 \) and

\[
\forall i = 1, \ldots, n, \quad \lambda(x_i)y_i = \nu(x_i). \tag{S_{BW}}
\]

\( S_{BW} \) is a linear system of \( n \) equations in \( 2t + k + 1 \) unknowns and we know that the pair \((\Lambda, \Lambda f)\) is in its solutions space. The following result proves that, for certain values of \( t \), actually it is not necessary to find exactly that solution to solve the decoding problem.

**Lemma 2.1.** Let \( t \leq \left\lfloor \frac{d-1}{2} \right\rfloor \). If \((\lambda, \nu)\) is a nonzero solution of \((S_{BW})\), then \( \lambda \neq 0 \) and \( f = \frac{\nu}{\lambda} \).

For the proof we refer for instance to [12, Theorem 4.2.2]. We can finally write the algorithm (see Algorithm 1). Its correctness is entailed by Lemma 2.1 whenever \( t \leq \left\lfloor \frac{d-1}{2} \right\rfloor \), that is, the decoding radius of Berlekamp-Welch algorithm is \( t = \left\lfloor \frac{d-1}{2} \right\rfloor \).

**Algorithm 1** Berlekamp-Welch Algorithm

**Inputs:** \( y, k, t \)

**Output:** \( f \in \mathbb{F}_q[X]_{< k} \) such that \( ev_x(f) = y \)

1: \( (\lambda, \nu) \leftarrow \text{random nonzero element in the solutions space of } S_{BW} \)
2: return \( f = \frac{\nu}{\lambda} \)

2.2 Sudan algorithm

Sudan algorithm is a list decoding algorithm that is, given a positive integer \( t \), it returns the list of any possible codeword \( z \) such that \( d(y, z) \leq t \). It can be seen as an extension of Berlekamp-Welch algorithm. To see that, let us write the key equations of Berlekamp-Welch algorithm in an equivalent form. First, we define

\[
Q(X, Y) \overset{\text{def}}{=} \Lambda(X)Y - N(X).
\]

This polynomial fulfills \( Q(x_i, y_i) = 0 \) for all \( i \in \{1, \ldots, n\} \). Hence, if \( t = \left\lfloor \frac{d-1}{2} \right\rfloor \), the problem faced in Berlekamp-Welch algorithm is equivalent to the following interpolation problem.

**Key Problem 2.** Given \( y \in \mathbb{F}_q^n \), find \( Q(X, Y) = Q_0(X) + Q_1(X)Y \) such that

(i) \( Q(x_i, y_i) = 0 \) for all \( i = 1, \ldots, n; \)

(ii) \( \deg(Q_j) < n - t - j(k - 1) \) for \( j = 0, 1 \).

**Remark 2.1.** Since for \( t \leq \left\lfloor \frac{d-1}{2} \right\rfloor \) Key Problems 1 and 2 are equivalent, the solution spaces of the corresponding linear systems are equal.
If this problem admits a nonzero solution \( Q(X,Y) \), then we can easily recover \( f \) as \(-\frac{Q}{Q_1}\) (it will be entailed by Remark 2.1 and Lemma 2.2). Sudan’s idea consists in defining a higher degree polynomial \( Q(X,Y) = Q_0(X) + Q_1(X)Y + \cdots + Q_\ell(X)Y^\ell \) and generalise Key Problem 2 as follows:

**Key Problem 3.** Given \( y \in \mathbb{F}_q^n \), find \( Q(X,Y) = Q_0(X) + \cdots + Q_\ell(X)Y^\ell \) such that

\[ (i') \quad Q(x_i, y_i) = 0 \text{ for all } i = 1, \ldots, n; \]

\[ (i'') \quad \deg(Q_j) < n - t - j(k - 1) \text{ for } j = 0, \ldots, \ell. \]

As for Key Problem 1, this problem reduces to solve a linear system \( S_{\text{Sud}} \) whose unknowns are the coefficients of the polynomial \( Q \) and the equations are given by \((i)\) and \((ii)\). Once we find such a polynomial \( Q(X,Y) \neq 0 \), we would like to recover our solutions, but this time we no longer have a link between the shape of \( Q \) and the locator polynomial \( \Lambda \). Though the following result shows that all solutions show up in the roots of \( Q \) with respect to the variable \( Y \).

**Lemma 2.2.** Let \( Q \) fulfill \((i)\) and \((ii)\) for some \( t \). If \( f(X) \in \mathbb{F}_q[X]_{<k} \) is such that \( d(\text{ev}_x(f), y) \leq t \), then \((Y - f(X))Q(X,Y)\).

For the proof, see [27]. If Key Problem 3 has a nonzero solution \( Q(X,Y) \), we look for the factors of \( Q \) with degree 1 in \( Y \) (in Berlekamp-Welch case, we had exactly one factor alike). The algorithm consists then in two main parts (see Algorithm 2): an interpolation problem and a root finding problem. In the root finding step it is not necessary to compute the complete factorisation of \( Q \), but Gao-Shokrollahi algorithm [5] can be used instead and that can be done in \( O(n^2\ell^3) \) operations in \( \mathbb{F}_q \). The main cost will be then that of the interpolation step consisting in solving the linear system \( S_{\text{Sud}} \), that is \( O(n^3\ell) \) operations using Gaussian elimination. Note that the interpolation step can be solved efficiently using Kötter’s algorithm [14].

**Algorithm 2 Sudan Algorithm**

**Inputs:** \( y, k, t \)

**Output:** \([g \in \mathbb{F}_q[X]_{<k} | d(\text{ev}_x(g), y) \leq t]\)

1. \( L \leftarrow \text{a nonzero solution of } S_{\text{Sud}} \)
2. \( L \leftarrow [g(X) \text{ such that } (Y - g(X))Q(X,Y)] \)
3. \( \text{return } [g(X) \in L \text{ such that } \deg(g) < k \text{ and } d(\text{ev}_x(g), y) \leq t] \)

**Decoding Radius**

First, let us observe that Key Problem 3 gives a constraint on \( t \). Indeed, from \((ii)\), we need to have

\[ t < n - \ell(k - 1). \]

Moreover, Lemma 2.2 implies that a sufficient condition for the algorithm to work, is the existence of a nonzero solution \( Q \) for the system \( S_{\text{Sud}} \). Hence a sufficient condition for that, is to have more unknowns than equations. By this condition we find

\[ t \leq \frac{2n\ell - k\ell(\ell + 1) + \ell(\ell + 1) - 2}{2(\ell + 1)}. \]

**2.3 Power Decoding algorithm**

In the previous section, we have seen that Sudan algorithm returns the list of any possible solution. In particular, given \( y \in \mathbb{F}_q^n \) and \( t \in \mathbb{N} \), let us suppose there exist two codewords \( c_1, c_2 \in C \) such that

\[ d(y, c_1) = d(y, c_2) = \min_{c \in C} d(y, c) \leq t. \] (3)

We refer to such a case as to a *worst* case, that is, a case where it is not possible to choose the best solution to the decoding problem since the closest codeword is not unique. The answer of a decoding
algorithm to worst cases depends clearly on the characteristics of the algorithm. Sudan algorithm treats worst cases naturally by returning the list of all possible solutions, that is, it does not have to choose between \( c_1 \) and \( c_2 \). Some other algorithm could even fail. It turns out that worst cases are very rare and actually, most of the times the list of solutions will be of size one.

This is the reason why designing an algorithm such as the Power Decoding that returns at most one solution at the cost of some failures in some rare cases such as \( g \) makes sense. An advantage is given by the cost. Indeed, in Sudan algorithm we have a linear system and a root-finding step while in the Power Decoding algorithm it suffices to solve a linear system.

Power decoding is inspired from a decoding algorithm for interleaved Reed–Solomon codes. It consists in considering several “powers” (with respect to the star product) of the received vector \( y \) in order to have more relations to work on. Given the vector \( y = c + e \) we want to correct, we consider the \( i \)-th powers \( y^i \) of \( y \) for \( i = 1, \ldots, \ell \) (see § 1.3 for the definition of \( y^i \)). In this section, we only present the case \( \ell = 2 \) for simplicity. We have

\[
\begin{align*}
y &= c + e \\
y^2 &= c^2 + 2ce + e^2.
\end{align*}
\]

We then rename \( e \) by \( e^{(1)} \) and \( 2c \cdot e + e^2 \) by \( e^{(2)} \) and get

\[
\begin{align*}
y &= c + e^{(1)} \\
y^2 &= c^2 + e^{(2)}.
\end{align*}
\]

One can see \( y^2 \) as a perturbation of a word \( c^2 \in C^2 = RS_q[2k - 1] \) by the error vector \( e^{(2)} \). Hence \( y^2 \) is an instance of another decoding problem. In addition, we have the following elementary result.

**Proposition 2.3.** It holds \( \text{supp}(e^{(2)}) \subseteq \text{supp}(e^{(1)}) \).

The previous result, whose proof is left to the reader is the key of power decoding. It asserts that on \( y \) and \( y^2 \), the errors are localised at the same positions. More precisely, error positions on \( y^2 \) are error positions on \( y \). Hence, we are in the error model of interleaved codes: equations (5) and (7) can be regarded as a decoding problem for the interleaving of two codewords with errors at most at \( t \) positions. Therefore, errors can be decoded simultaneously using the same error locator polynomial. To do that, we consider the error locating polynomial \( \Lambda(X) = \prod_{i \in I}(X - x_i) \) as for Berlekamp-Welch algorithm and the polynomials

\[
N_1 \overset{\text{def}}{=} \Lambda f, \quad N_2 \overset{\text{def}}{=} \Lambda f^2.
\]

Thanks to Remark 2.3 it is possible to write the key equations

\[
\begin{align*}
\Lambda(x_i)y_{i1} &= N_1(x_i) \quad \forall i = 1, \ldots, n \\
\Lambda(x_i)y_{i2} &= N_2(x_i) \quad \forall i = 1, \ldots, n.
\end{align*}
\]

The Power Decoding algorithm consists in solving the following problem.

**Key Problem 4.** Given \( y \in F_q^n \) and \( t \in \mathbb{N} \), find \( (\lambda, \nu_1, \nu_2) \) which fulfill

\[
\begin{align*}
\Lambda(x_i)y_{i1} &= \nu_1(x_i) \quad \forall i = 1, \ldots, n \\
\Lambda(x_i)y_{i2} &= \nu_2(x_i) \quad \forall i = 1, \ldots, n.
\end{align*}
\]

with \( \text{deg}(\lambda) \leq t, \text{deg}(\nu_j) \leq t + j(k - 1) \) for \( j = 1, 2 \).

The vector \( (\lambda, \Lambda f, \Lambda f^2) \) is a solution of the linear system \( \mathcal{S}_{Po} \). Though, at the moment there is no guaranteed method to recover it among all others solutions. We only know that, if \( g \) is a polynomial such that \( \text{deg}(g) < k \) and \( d(e_{\text{ev}}(g), y) \leq t \), then there exists an error locator polynomial \( \Gamma \) of the error with respect to \( g \), such that the vector

\[
(\Gamma, \Gamma g, \Gamma g^2)
\]

is solution of \( \mathcal{S}_{Po} \). Among all solutions like that, we want to pick the one that gives the closest codeword, that is the one such that \( \text{deg}(\Gamma) \) is minimal (see pt.1 in Algorithm 3).
However, bound (10) is actually much larger than the decoding radius. We look then for a stricter bound which gives correctness of the algorithm is \( t < n \).

In the same way, we would have (0, 0, \( \pi \)) \( \in \text{Sol} \) if \( t + 2k - 2 \geq n \). Hence, a necessary condition for the correctness of the algorithm is \( t < n - 2k + 2 \). For a larger \( \ell \) this condition becomes
\[
t < n - \ell(k - 1).
\]

Remark 2.3. To compute the decoding radius of the Power Decoding algorithm we look for a condition on the size of the system \( (S_{Po}) \). Note that the algorithm gives one solution or none, hence there cannot be a sufficient condition for the correctness of the algorithm as soon as \( t > \frac{d}{2} \). For this reason, we look for a necessary condition for the system \( (S_{Po}) \) to have a solution space of dimension 1.

2.3.1 Decoding radius and complexity

Since we know that \( (\Lambda, \Lambda f, \Lambda f^2) \) is a solution for \( (S_{Po}) \), the algorithm works if the solution space of \( (S_{Po}) \) has dimension one.

Remark 2.4. The condition the solution space of \( (S_{Po}) \) has dimension one is sufficient for the algorithm to succeed but is not necessary. The solution space could have a larger dimension and, in such a situation, the algorithm might succeed if the solution \( (\lambda, \nu_1, \nu_2) \) where \( \lambda \neq 0 \) is monic with the smallest possible degree equals to \( (\Lambda, \Lambda f, \Lambda f^2) \).

First, let us define the polynomial
\[
\pi(X) = \prod_{i=1}^{n}(X - x_i)
\]
and consider the bounds in Key Problem 4 on the degrees of \( \nu_1 \) and \( \nu_2 \). If \( t + k - 1 \geq n \), then we would have
\[
(0, \pi, 0) \in \text{Sol}.
\]
In the same way, we would have (0, 0, \( \pi \)) \( \in \text{Sol} \) if \( t + 2k - 2 \geq n \). Hence, a necessary condition for the correctness of the algorithm is \( t < n - 2k + 2 \). For a larger \( \ell \) this condition becomes
\[
t < n - \ell(k - 1).
\]
However, bound (10) is actually much larger than the decoding radius. We look then for a stricter bound on \( t \). Another necessary condition to have a solution space of dimension one, is:
\[
\#\text{unknowns} \leq \#\text{equations} + 1,
\]
which gives \( t \leq \frac{2n - 3k + 1}{3} \). The same process can be used for a general \( \ell \) and we obtain the following decoding radius
\[
t \leq \frac{2n\ell - k\ell + 1 + \ell(\ell - 1)}{2(\ell + 1)}.
\]
About the cost, we have to solve a linear system of \( n\ell \) equations in \( O(n\ell) \) unknowns, hence the complexity is in \( O(n^\omega \ell^2) \) operations in \( \mathbb{F}_q \), where \( \omega \) denotes the cost of linear algebra. Using Gaussian elimination, we can take \( \omega = 3 \).

Remark 2.5. Note that bound (10) does not give a necessary condition for the success of the algorithm. Indeed, it just gives a necessary condition to have a solution space of dimension one, while the algorithm could work even if the solution space dimension is bigger. See § 2.3.2.

Algorithm 3 Power Decoding Algorithm

**Inputs:** \( y, t, k \)

**Output:** some \( g \in \mathbb{F}_q[X]_{<k} \) such that \( d(ev_a(g), y) \leq t \) or failure

1. \( (\lambda, \nu_1, \nu_2) \leftarrow \) a nonzero solution of \( (S_{Po}) \) with \( \lambda \) of smallest possible degree
2. if \( \lambda | \nu_1 \text{ and } \lambda | \nu_2 \text{ and } \left( \frac{\nu_1}{\lambda} \right)^2 = \frac{\nu_2}{\lambda} \) then \( g \leftarrow \frac{\lambda}{\nu_1} \)
3. if \( d(ev_a(g), y) \leq t \) then return \( g \).
4. return failure

**Remark 2.2.** As for Sudan algorithm’s problem (see Key Problem 3), Key Problem 1 can be regarded as a generalisation of Key Problem 1. Though this time, the root-finding part is avoided, while the linear system becomes bigger in both unknowns and equations. In particular these \( 2n \) equations we obtain, consist in the key equations for \( y \) and the key equations for \( y^2 \), that is two simultaneous decoding problems. Indeed, the important aspect is that these two decoding problems share the locator polynomial \( \Lambda \). Hence, by adding \( n \) relations, we only add \( \text{deg}(N_2) + 1 \) unknowns instead of \( \text{deg}(N_2) + t + 2 \).

**Remark 2.4.** The condition the solution space of \( (S_{Po}) \) has dimension one is sufficient for the algorithm to succeed but is not necessary. The solution space could have a larger dimension and, in such a situation, the algorithm might succeed if the solution \( (\lambda, \nu_1, \nu_2) \) where \( \lambda \neq 0 \) is monic with the smallest possible degree equals to \( (\Lambda, \Lambda f, \Lambda f^2) \).

First, let us define the polynomial
\[
\pi(X) = \prod_{i=1}^{n}(X - x_i)
\]
and consider the bounds in Key Problem 4 on the degrees of \( \nu_1 \) and \( \nu_2 \). If \( t + k - 1 \geq n \), then we would have
\[
(0, \pi, 0) \in \text{Sol}.
\]
In the same way, we would have (0, 0, \( \pi \)) \( \in \text{Sol} \) if \( t + 2k - 2 \geq n \). Hence, a necessary condition for the correctness of the algorithm is \( t < n - 2k + 2 \). For a larger \( \ell \) this condition becomes
\[
t < n - \ell(k - 1).
\]
2.3.2 About failure cases

If failure cases turn out to be rare in practice, their classification is still unknown but in some particular situations. In what follows, we discuss some situations where failures may occur. In the literature, we only know that they depend uniquely on the error $e$ and not on the codeword affected by $e$. Moreover there exist some upper bounds for the probability that the algorithm fails for particular choices of the power $t$ (see [21, 20, 23]). Clearly, if the space of solutions of $\mathcal{S}_\text{Po}$ has dimension 1, then, under Assumption [11] the algorithm succeeds. On the other hand, we can see further, the algorithm might succeed even when the dimension of the solutions spaces of $\mathcal{S}_\text{Po}$ is larger than 1. Actually, the following notion is more crucial in the failure analysis than the dimension of the solution space.

**Definition 2.2.** Given $y \in \mathbb{F}_q^n$ and $t \in \mathbb{N}$, let $\text{Sol} \subseteq \mathbb{F}_q[X]_{\leq t} \times \mathbb{F}_q[X]_{\leq t+k-1} \times \mathbb{F}_q[X]_{\leq t+2k-2}$ be the solution space for the linear system $\mathcal{S}_\text{Po}$. We define

$$\mathcal{M}_\text{Sol} \overset{\text{def}}{=} \langle \text{Sol} \rangle_{\mathbb{F}_q[x]} \subseteq \mathbb{F}_q[X]^3$$

the $\mathbb{F}_q[x]$-module generated by $\text{Sol}$.

**Proposition 2.4.** Suppose Assumption [11] to hold. Therefore, there exists $f \in \mathbb{F}_q[x]_{< k}$ is such that $d(\text{ev}_x(f), y) \leq t$. If $\text{rk}(\mathcal{M}_\text{Sol}) = 1$, then for any $(\lambda, \nu_1, \nu_2) \in \mathcal{M}_\text{Sol}$ it holds

$$f = \frac{\nu_1}{\lambda}, \quad f^2 = \frac{\nu_2}{\lambda}.$$

Moreover, if $\Lambda$ is the locator polynomial associated to $f$, then $\deg(\Lambda) = \min\{\deg(\lambda) \mid (\lambda, \nu_1, \nu_2) \in \text{Sol}\}$.

This result is easy to prove and turns out to be very helpful. Indeed even if $t$ fulfills bound [11], generally the dimension of the solution space is actually often bigger than 1. Though, since $\text{rk}(\mathcal{M}_\text{Sol}) = 1$ almost always, the algorithm works in most of the cases.

**Proposition 2.5.** Given $y \in \mathbb{F}_q^n$ and under Assumption [11] if there exist more than one solution to Problem [2] then $\text{rk}(\mathcal{M}_\text{Sol}) > 1$.

**Proof.** Suppose that we have two solutions to Problem [2]

$$c_f = \text{ev}_x(f), \quad c_g = \text{ev}_x(g) \quad \text{with} \quad f \neq g.$$

Since $c_f$ and $c_g$ are both solutions for Problem [2] there exist locator polynomials $\Lambda, \Gamma \in \mathbb{F}_q[x]$ with $\deg(\Lambda), \deg(\Gamma) \leq t$ and

$$\langle \Lambda f, \Lambda f^2 \rangle, \quad \langle \Gamma g, \Gamma g^2 \rangle \in \text{Sol}.$$

We get $\text{rk}(\mathcal{M}_\text{Sol}) > 1$ since $f \neq g$ entails that $(1, f, f^2)$ and $(1, g, g^2)$ are $\mathbb{F}_q[x]$-independent.

When $\text{rk}(\mathcal{M}_\text{Sol}) > 1$, two situations may occur

- The vector $(\lambda, \nu_1, \nu_2) \in \text{Sol}$ with $\lambda \neq 0$ is monic with minimal degree is such that $\nu_1 = \lambda f$ and $\nu_2 = \lambda f^2$ where $\deg f < k$. Then, $\text{ev}_x(f)$ provides the unique closest codeword;
- else the algorithm fails.

**Remark 2.6.** If $c_f = \text{ev}_x(f), \ c_g = \text{ev}_x(g)$ are both solution of the decoding problem and

$$d(c_f, y) = d(c_g, y). \quad (12)$$

Then, if we denote the respective locators by $\Lambda, \Gamma$, the vectors $(\Lambda, \Lambda f, \Lambda f^2)$ and $(\Gamma, \Gamma g, \Gamma g^2)$ are in $\text{Sol}$ and, by [12], $\deg \Lambda = \deg \Gamma$. Thus, some $\mathbb{F}_q$–linear combination of these vectors provide another vector $(\lambda, \nu_1, \nu_2) \in \text{Sol}$ with $\deg \lambda < \deg \Lambda$ which might not correspond to any codeword. In such a situation the algorithm fails.
2.4 The Error Correcting Pairs algorithm

The Error Correcting Pairs (ECP) algorithm has been designed by Pellikaan [19]. Its formalism gives an abstract description of a decoding algorithm originally arranged for Algebraic-Geometry codes [25] and whose description required notions of algebraic geometry. In his work, Pellikaan simplified the instruments needed in the original decoding algorithm and made the algorithm applicable to any linear codes benefiting from a certain elementary structure called error correcting pair and defined in Definition 2.3 below. Given a code \( C \) and a received vector \( y = c + e \) where \( c \in C \) and \( w(e) \leq t \) for some positive integer \( t \), the ECP algorithm consists in two steps:

1. find \( J \subseteq \{1, \ldots, n\} \) such that \( J \supseteq I \), where \( I \) denotes the support of \( e \);
2. recover the nonzero values of \( e \).

As said before, these steps can be solved if the code has an error correcting pair and \( t \) small enough.

**Definition 2.3.** Given a linear code \( C \subseteq \mathbb{F}_q^n \), a pair \((A, B)\) of linear codes, with \( A, B \subseteq \mathbb{F}_q^n \) is called \( t \)-error correcting pair for \( C \) if

- \((ECP1)\) \( A \ast B \subseteq C^\perp \);
- \((ECP2)\) \( \dim(A) > t \);
- \((ECP3)\) \( d(B^\perp) > t \);
- \((ECP4)\) \( d(A) + d(C) > n \).

**Remark 2.7.** One can observe that, thanks to Remark 1.1, it holds \( A \ast B \subseteq C^\perp \iff A \ast C \subseteq B^\perp \).

Since this notion does not look very intuitive, an example of error correcting pair for Reed-Solomon codes is given further in §2.5 and an interpretation of the various hypotheses above are given in light of this example is given in §2.6. For now, we want to explain more precisely how the ECP algorithm works.

### 2.4.1 First step of the Error Correcting Pair algorithm

In Step (11) of the ECP algorithm, we wish to find a set which contains \( I \). A good candidate for \( J \) could then be \( Z(A(I)) \), indeed the following result (see [19]), entails that \( I \subseteq Z(A(I)) \subseteq \{1, \ldots, n\} \) (see Definition 1.1).

**Proposition 2.6.** If \( \dim(A) > t \), then \( A(I) \neq 0 \).

Though, since we do not know \( I \), we do not have any information about \( A(I) \). That is why a new vector space is introduced

\[ M_1 \overset{\text{def}}{=} \{a \in A \mid \langle a \ast y, b \rangle = 0 \ \forall b \in B\}. \quad (13) \]

The key of the algorithm is in the following result.

**Theorem 2.7.** Let \( y = c + e \), \( I = \supp(e) \) and \( M_1 \) as above. If \( A \ast B \subseteq C^\perp \), then

1. \( A(I) \subseteq M_1 \subseteq A \);
2. if \( d(B^\perp) > t \), then \( A(I) = M_1 \);
3. if \( A \ast B \subseteq C^\perp \), then \( Z(M_1) \) is non trivial and contains \( I \). Therefore, Step (11) of the algorithm consists in computing \( J = Z(M_1) \).
2.4.2 Second step of Error Correcting Pair algorithm

We now show that Step (2) reduces to a linear system depending on $J$ and the syndrome of $y$. First, some notations are needed.

Notation. Let $\mathcal{M}_{n,m}(\mathbb{F}_q)$ be the space of the matrices over $\mathbb{F}_q$ with $n$ rows and $m$ columns. Let $H \in \mathcal{M}_{n,m}(\mathbb{F}_q)$, and $H'$ its columns. Given $J \subseteq \{1, \ldots, m\}$, we denote by $H_J$ the submatrix of $H$ whose columns are the $H'$'s with $j \in J$.

Suppose we have computed $J \supseteq I$ in Step (1) of the algorithm. Consider a full rank–parity check matrix $H$ for $C$. The vector $e_I$ satisfies $H_J \cdot e_I^T = H \cdot y^T$. and we want to then recover $e_I$ by solving the linear system

$$H_J \cdot e = H \cdot y.$$  \hspace{1cm} (14)

Though, à priori, the solution may not be unique. In particular the only (ECH1), (ECI2) and (ECI3) in Definition 2.3 do not entail the correctness of the algorithm.

Lemma 2.8. If $d(A) + d(C) > n$, $\dim(A) > t$ and $J = Z(M_1)$, then $|J| < d(C)$.

Proof. By Proposition 2.6 there exists $a \in A(I) \setminus \{0\}$. Now, since $d(A) + d(C) > n$, we get

$$|J| = |Z(M_1)| \leq |Z(a)| = n - w(a) \leq n - d(A) < d(C).$$

\hfill $\square$

Theorem 2.9. Given $y \in \mathbb{F}_q^n$ and $J \subseteq \{1, \ldots, n\}$ with $t = |J| < d(C)$, then there exists at most one solution for (14).

This is a well-known result of coding theory and it is easy to prove. This theorem, together with Lemma 2.8 entail that if $J$ contains the support of the error, $e_J$, the unique solution to system (14). Then, the second step of the algorithm consists in finding $e_J$ by solving system (14) and recovering $e$ from $e_J$ imposing $e_i = 0$ for all $i \notin J$. The entire algorithm is described in Algorithm 4.

\begin{algorithm}
\caption{Error Correcting Pairs Algorithm}
\begin{algorithmic}
\Input{c, A, B, C}
\Output{e \in C such that $y = c + e$ for some $e \in \mathbb{F}_q^n$ with $w(e) \leq t$}
1: compute $M_1 = \{a \in A \mid \langle a, y, b \rangle = 0 \ \forall b \in B\}$ (linear system)
2: $J \leftarrow Z(M_1)$
3: if system (14) does not have solution then\Return failure
4: $e_J \leftarrow$ solution of (14)
5: recover $e$ from $e_J$
6: $\Return e = y - e$
\end{algorithmic}
\end{algorithm}

Theorem 2.10. Let $C \subseteq \mathbb{F}_q^n$ be a linear code. If there exists a $t$-error correcting pair for $C$, then for all $y \in \mathbb{F}_q^n$ such that

$$y = c + e,$$

with $c \in C$ and $w(e) \leq t$, the ECP algorithm recovers $c$ in $O(n^2)$ operations in $\mathbb{F}_q$.

Proof. See [19]. \hfill $\square$

It is straightforward to see that the algorithm returns a unique solution and that a sufficient condition for the algorithm to correct $t$ errors, is the existence of an error correcting pair with parameter $t$. This consideration, together with Theorem 2.10 leads to the following result.

Corollary 2.11. If a linear code $C$ has a $t$-error correcting pair, then

$$t \leq \left\lfloor \frac{d(C) - 1}{2} \right\rfloor.$$
2.5 Error Correcting Pairs for Reed-Solomon codes

It is difficult to find an error correcting pair for a code. Indeed, the existence of an ECP for a given code relies on the existence of a pair \((A, B)\) of codes, both having a sufficiently large dimension and satisfying \(A \ast B \subseteq C^\perp\). This is actually a very restrictive condition. Among the codes for which an ECP exists, there are Reed-Solomon codes. Indeed, given \(C = \text{RS}_q[k]\), we consider the pair \((A, B)\) where

\[
A = \text{RS}_q[t + 1], \quad B^\perp = \text{RS}_q[t + k].
\]  

(15)

Recall that thanks to Proposition 1.1, we have

\[
A \ast C = \text{RS}_q[t + k].
\]

We will see in §3 that the following lemma is the main part of the Error Correcting Pairs algorithm we are going to work on, in order to design a more powerful algorithm.

Lemma 2.12. Given \(B\) as above, it holds

\[
d(B^\perp) > t \iff t \leq \frac{d(C) - 1}{2}.
\]

(16)

Proposition 2.13. The pair \((A, B)\) of (15) is a t-error correcting pair for \(C\) for any \(t \leq \frac{d(C) - 1}{2}\).

Proof. We have to prove that (17) is a necessary and sufficient condition for (EC1)–(4) in Definition 2.3 to hold. First of all, by Lemma 2.12 we have (EC1). Moreover \(\dim(A) = t + 1 > t\) by definition of \(A\) and this gives (EC2). By Proposition 1.1 as seen above, the codes \(A, B, C\) verify \(A \ast C = B^\perp\), then by Remark 1.4 we obtain \(A \ast B \subseteq C^\perp\). Finally, it is easy to see that \(d(A) + d(C) > n \iff t < d(C)\). Hence if \(t \leq \frac{d(C) - 1}{2}\), (EC1)–(4) hold and conversely. 

Remark 2.8. In §3 we are going to work with structures which are more elementary than error correcting pairs, that is, we will only require (EC1), (EC2) and (EC3) to hold. Note that, given \(A\) and \(B\) as in (15), the three hold if and only if \(t < d(C)\).

2.6 ECP and Berlekamp–Welch key equations

The example of Reed–Solomon also permits to understand the rationale behind EPC’s in light of Berlekamp–Welch algorithm. Indeed, we now show that the choice of \(M_1\) we made in the ECP algorithm, if one looks at the key equations of Berlekamp–Welch algorithm, appears to be really natural. Let us consider \(C = \text{RS}_q[k]\) and the pair \((A, B)\) we defined in §2.5. We can write (2) for any \(i \in \{1, \ldots, n\}\) using the star product in this way

\[
(\Lambda(x_1), \ldots, \Lambda(x_n)) \ast y = (N(x_1), \ldots, N(x_n)).
\]

From that, we can deduce

- \((N(x_1), \ldots, N(x_n)) \in \text{RS}_q[t + k] = B^\perp;\)
- \((A(x_1), \ldots, A(x_n)) \in \text{RS}_q[t + 1(I)] = A(I);\)
- Moreover \((\Lambda(x_1), \ldots, \Lambda(x_n)) \in \{a \in A \mid \langle a \ast y, b \rangle = 0 \ \forall b \in B\}.\)

In other words, the vector \((\Lambda(x_1), \ldots, \Lambda(x_n))\) belongs to the space \(A(I)\) we are looking for in the ECP algorithm. Moreover it fulfills a property which characterises a space \(M_1 \supseteq A(I)\), that is exactly the space we define in the ECP algorithm and that turns to be equal to \(A(I)\) under certain conditions.
3 Power Error Locating Pairs algorithm

We now present the Power Error Locating Pairs (PELP) algorithm. First, we face the case of Reed-Solomon codes. Indeed, for these codes, the generalisation of the Error Correcting Pairs algorithm is much more intuitive. The idea we used, leans for some aspects on the Power Decoding algorithm. In particular, similarly to Power Decoding algorithm, PELP algorithm depends on a new parameter $\ell$ we call power. The changes with respect to the basic algorithm mainly consist in requiring a new structure, less complex than error correcting pairs, called error locating pair and in generalising the object $M_1$ defined in (13). Once we benefit from this new algorithm for Reed-Solomon codes, we show that actually the objects introduced can be defined for any code with an error locating pair. An example of application of the Power Error Locating Pairs algorithm to other codes will be given in §4.

3.1 PELP algorithm for Reed-Solomon codes

Let us now present the Power Error Locating Pairs (PELP) algorithm for Reed-Solomon codes. For simplicity we first present the case with power $\ell = 2$ and afterwards, we show how this idea can be adapted to a larger $\ell$.

3.1.1 The case $\ell = 2$

The idea is to preserve the structure we defined in §2.6. That is, for a Reed-Solomon code $C = \text{RS}_q[k]$ we consider the pair $(A,B)$ where

$$A = \text{RS}_q[t + 1] \quad B^\perp = \text{RS}_q[k + t].$$

We have seen in Theorem 2.10 that if $t \leq \left\lfloor \frac{d(C) - 1}{2} \right\rfloor$, then it is possible to correct $t$ errors by using the Error Correcting Pairs algorithm. What we want to do, is to increase the amount of errors we correct beyond the quantity $\frac{d(C) - 1}{2}$. We now show that, for such a values of $t$, some steps in the Error Correcting Pairs algorithm are not going to work. The aim of this section is to modify these steps someway.

First, we observe that if $t > \left\lfloor \frac{d(C) - 1}{2} \right\rfloor$, then the pair $(A,B)$ is no longer a $t$-error correcting pair. In fact, by Lemma 2.12

$$d(B^\perp) > t \iff t \leq \frac{d(C) - 1}{2}.$$  

We decide then to work without the property $d(B^\perp) > t$, that is, we relax the conditions we asked in the Error Correcting Pairs algorithm.

**Definition 3.1** (Pellikaan, [18]). Given a linear code $C \subseteq \mathbb{F}_q^n$, a pair of linear codes $(A,B)$ with $A, B \subseteq \mathbb{F}_q^n$ is a $t$-error locating pair for $C$ if

- $(ELP1)$ $A \ast B \subseteq C^\perp$;
- $(ELP2)$ $\dim(A) > t$;
- $(ELP3)$ $d(A) + d(C) > n$.

Note that, by Remark 2.8 the pair $(A,B)$ in (18) is a $t$-error locating pair for $C$ if $t < d(C)$.

**Remark 3.1.** In the ECP algorithm, we used the property $d(B^\perp) > t$ in Theorem 2.7. This result guaranteed the equality $M_1 = A(I)$, where $M_1$ is the set we defined in (13).

$$M_1 = \{a \in A \mid \langle a \ast y, b \rangle = 0 \quad \forall b \in B\}.$$

Since now we are working with an error locating pair, we only have $A(I) \subseteq M_1$, so $M_1$ could be too large to give any information on $I$.

The previous remark gives the motivation for our strategy: first, we define a new set $M$, such that $A(I) \subseteq M \subseteq M_1$. Second, we look for some conditions in order to have $A(I) = M$. Under these conditions it will be possible to use the Error Correcting Pairs algorithm with the only difference that we find $M$ instead of $M_1$. 

14
Construction of \( M \)

In §2.6, we have seen that it is possible to link the definition of \( M \) to the key equations of Berlekamp-Welch algorithm. Now, we would like to have a clue for the choice of \( M \) by looking at the key equations of another algorithm, that is the Power Decoding algorithm. It is possible to write (8) in this way

\[
e_{\mathbf{x}}(\Lambda) \ast \mathbf{y} = e_{\mathbf{x}}(N_1)\\
e_{\mathbf{x}}(\Lambda) \ast \mathbf{y}^2 = e_{\mathbf{x}}(N_2),
\]

where \( e_{\mathbf{x}} \) is the evaluation map introduced in (1). Hence, we can deduce

\begin{itemize}
  \item \( e_{\mathbf{x}}(N_1) \in \operatorname{RS}_q[t+k] = B^\perp \);
  \item \( e_{\mathbf{x}}(N_2) \in \operatorname{RS}_q[t+2k-1] = B^\perp \ast C \);
  \item \( e_{\mathbf{x}}(\Lambda) \in \operatorname{RS}_q[t+1](I) = A(I) \);
  \item \( e_{\mathbf{x}}(\Lambda) \in M_1 \cap M_2 \),
\end{itemize}

where

\[
M_1 := \{ a \in A \mid \langle a \ast y, b \rangle = 0 \ \forall b \in B \},
M_2 := \{ a \in A \mid \langle a \ast y^2, v \rangle = 0 \ \forall v \in (B^\perp \ast C)^\perp \}.
\]

Therefore, we define

\[
M := M_1 \cap M_2.
\]

This choice seems quite reasonable, since \( M_1 \) is the set defined for the Error Correcting Pairs algorithm. Moreover we have the following result.

**Proposition 3.1.** The set \( M := M_1 \cap M_2 \) fulfills \( A(I) \subseteq M \subseteq M_1 \subseteq A \).

**Proof.** We have to prove that \( A(I) \subseteq M \), that is \( A(I) \subseteq M_1, M_2 \). First, we already know that \( A(I) \subseteq M_1 \) by Theorem 2.7(1). About \( M_2 \), we adapt the proof of Theorem 2.7(1). First, note that we have

\[
A \ast B \subseteq C^\perp \iff A \ast C \subseteq B^\perp \implies A \ast C^2 \subseteq B^\perp \ast C.
\]

Given \( a \in A(I) \), for any \( v \in (B^\perp \ast C)^\perp \) we obtain

\[
\langle a \ast y^2, v \rangle = \langle a \ast c^2, v \rangle + \langle a \ast e^{(2)}, v \rangle = 0.
\]

In (21) we used the bilinearity, while (22) holds because of (20) and the fact that the supports of \( e^{(2)} \) and \( a \) are complementary (see Proposition 2.3).

The idea is now to use the rest of the Error Correcting Pairs algorithm without modifications (see Algorithm 5).

**Algorithm 5** Power Error Locating Pairs Algorithm

**Inputs:** \( y, A, B, C \)

**Output:** \( c \in C \) such that \( y = y + e \) for some \( e \in F_q^n \) with \( w(e) \leq t \)

1. compute \( M = M_1 \cap M_2 \) (linear system)
2. \( J \leftarrow Z(M) \)
3. if system (13) does not have a solution then
   \( \text{return failure} \)
4. \( e^{(1)}_J \leftarrow \text{nonzero solution of (13)} \)
5. recover \( e^{(1)} \) from \( e^{(1)}_J \)
6. \( \text{return } c = y - e^{(1)} \)

**Remark 3.2.**

(i) the set \( M \) can be defined for any code which benefits from an error locating pair. Therefore, for any code alike, Proposition 3.1 holds and it is possible to run the algorithm;

(ii) this algorithm works whenever \( M = A(I) \). Indeed in such a case, it is possible to apply Lemma 2.8 and Theorem 2.9 as showed for the Error Correcting Pairs algorithm.
Decoding Radius and conditions to have \( M = A(I) \)

Any condition, either sufficient or necessary, to have \( M = A(I) \), will give us a bound on \( t \), that is, a decoding radius for the algorithm. Of course, it will depend on the pair \((A,B)\) we consider and in particular on the code \( C \). Hence, in this paragraph we find a condition to have \( M = A(I) \) for \( C = \text{RS}_q[k] \), that is, a decoding radius for the PELP algorithm for Reed-Solomon codes.

Since the second part of the algorithm equals the second part of the Error Correcting Pairs algorithm, it gives one solution or none (see Lemma 2.7). Therefore, for \( t > \frac{d(C) - 1}{2} \), the existence of an error locating pairs is not a sufficient condition for the algorithm to recover any possible solution while for \( t \leq \frac{d(C) - 1}{2} \), by Theorem 2.10 the existence of an error correcting pair is a sufficient condition to recover the (unique) solution.

Since \( t > \frac{d(C) - 1}{2} \), there could be more than one solution to the decoding problem and hence we do not look for a sufficient condition to have \( A(I) = M \) depending on a bound \( t \) for \( t \). Therefore, we now focus on a necessary condition for that instead.

**Lemma 3.2.** The following statements are equivalent:

(i) \( M = A(I) \);

(ii) \( M(I) = M \);

(iii) \( M_I = \{0\} \).

**Proof.** First, notice that \( A(I) = M(I) \) by Proposition 3.1. Hence, (i) and (ii) are equivalent. We now prove (ii) \( \iff \) (iii). It is easy to see that, if \( M(I) = M \), the projection on \( I \) of every element of \( M \), is the zero vector and conversely. \( \square \)

Thank to this lemma, if we find a necessary condition for \( M_I = \{0\} \), then we find a necessary condition for \( M = A(I) \) and conversely. For this reason, we now study the object \( M_I \). We remind that \( e^{(1)} \) and \( e^{(2)} \) are the error vectors for \( y \) and \( y^2 \), that is, \( y = c + e^{(1)} \) and \( y^2 = c^2 + e^{(2)} \).

**Theorem 3.3.** It holds \( M_I = A_I \cap (e^{(1)} * B)^\perp \cap (e^{(2)} * (B^\perp * C)^\perp)^\perp \).

**Proof.** Observe that for the two objects \( (e^{(1)} * B) \) and \( (e^{(2)} * (B^\perp * C)) \) the operations “projection on \( I \)” and “\( \perp \)” commute, since \( \text{supp}(e^{(1)}) \) and \( \text{supp}(e^{(2)}) \) are both included in \( I \). Now, let us characterise the elements of \( A \) that are in \( M_1 \) and \( M_2 \). Given \( a \in A \), we get

\[
\begin{align*}
\text{if } a \in M_1 & \iff \langle a * y, b \rangle = 0 \quad \forall b \in B \\
& \iff \langle a, e^{(1)} * b \rangle = 0 \quad \forall b \in B \\
& \iff a_I \in (e^{(1)} * B)^\perp_I.
\end{align*}
\]

For the second equivalence we used the property \( A * B \subseteq C^\perp \) which is equivalent to \( A * C \subseteq B^\perp \) (see Remark 2.7) while in the third, we used \( \text{supp}(e^{(1)}) = I \). In the same way, it is possible to prove that given \( a \in A \), it holds

\[
\begin{align*}
\text{if } a \in M_2 & \iff a_I \in (e^{(2)} * (B^\perp * C)^\perp)^\perp_I.
\end{align*}
\]

The result comes now easily. \( \square \)

For Reed-Solomon codes it is possible to simplify the result of Theorem 3.3. It is indeed possible to prove the following result.

**Lemma 3.4.** Let \( C \subseteq \mathbb{F}_q^n \) be a linear code and \( t \in \mathbb{N} \). The following facts are equivalent:

- \( d(A^\perp) > t \);
- \( A_I = \mathbb{F}_q^t \) for any \( I \subseteq \{1, \ldots, n\} \) with \( |I| = t \).

\(^1\)If we had that, then we could have a situation where \( t \) verifies this sufficient condition and there are two solutions with two error supports \( I_1 \) and \( I_2 \). Hence it would hold \( M = A(I_1) = A(I_2) \) which is not always verified.
Hence, since $A = RS[n,t+1]$ is MDS, its projection (puncturing) onto any set of $t$ coordinates is isomorphic to $\mathbb{F}_q^n$. Hence, we can write

$$M_I = (e^{(1)} * B)^I \cap \left( e^{(2)} * (B^\perp * C)^I \right)^I.$$ 

Therefore, looking for a necessary condition to have $M_I = \{0\}$, we get

$$M_I = \{0\} \implies \dim((e^{(1)} * B)^I) + \dim((e^{(2)} * (B^\perp * C)^I)^I) \leq t \quad \quad (23)$$

$$\implies \dim(B) + \dim((B^\perp * C)^I) \geq t. \quad \quad (24)$$

Suppose first that $(B^\perp * C)^I = \{0\}$ that is $B^\perp * C = \mathbb{F}_q^n$. In this situation, $M_I$ depends only on $e^{(1)}$ and our algorithm will not be able to correct more errors than the usual Error Correcting Pairs algorithm. Therefore, we need to suppose that

$$B^\perp * C = RS_q[n,t+2] \subset \mathbb{F}_q^n. \quad \quad (25)$$

This last assertion is equivalent to

$$t \leq n - 2k + 1. \quad \quad (26)$$

By that, we can deduce that a necessary condition for the correctness of the algorithm. Assuming $(26)$, or equivalently $(25)$, then inequality $(24)$ becomes

$$(n - k - t) + (n - t - 2k + 1) \geq t$$

which yields the following decoding radius

$$t \leq \frac{2n - 3k + 1}{3}. \quad \quad (27)$$

Remark 3.3. The decoding radii $(26)$ and $(27)$, correspond respectively to the decoding radii $(10)$ and $(11)$ of the Power Decoding algorithm for $\ell = 2$.

### 3.2 The general case: $\ell \geq 2$

It is possible to generalise the algorithm for $\ell \geq 2$. Indeed, as for the case $\ell = 2$, the only thing that changes is the choice of $M$:

$$M := \bigcap_{i=1}^{\ell} M_i \quad M_i = \{ a \in A \mid \left< a * y^i, v \right> = 0 \quad \forall v \in (B^\perp * C^{i-1})^I \}. \quad \quad (28)$$

As before, it is possible to motivate the definition of $M$ by looking at the key equations of the Power Decoding algorithm with power $\ell$, but this definition does not depend on Reed-Solomon codes. Therefore, again, this algorithm can be runned on any code which has an ELP.

#### 3.2.1 Decoding Radius

The decoding radius for the general case is given again by a condition to have $M_I = \{0\}$ or equivalently $M = A(I)$. We study again the particular case of Reed-Solomon codes, for which we have

$$M_I = (e^{(1)} * B)^I \cap \bigcap_{i=2}^{\ell} (e^{(i)} * (B^\perp * C^{i-1})^I)^I,$$

where $e^{(i)} = \sum_{h=1}^i \sum_{j=0}^{h} c^j e^{(i)h-j}$ is such that $y^i = c^i + e^{(i)}$ for all $i = 1, \ldots, \ell$. We generalise the necessary condition $(23)$ by

$$\dim\left( (e^{(1)} * B)^I \right) + \dim\left( \bigcap_{i=2}^{\ell} (e^{(i)} * (B^\perp * C^{i-1})^I)^I \right) \leq t \quad \quad (29)$$

Now we want to simplify this bound.
Remark 3.4.

- Given a vector space $V$ with $\dim(V) = t$ and $A_1, \ldots, A_n \subseteq V$, it holds
  \[ \dim\left( \bigcap_{i=1}^{n} A_i^\perp \right) \geq t - \sum \dim(A_i); \]

- it is easy to see that $\dim((e^{(1)} \ast B)_{\mathcal{I}}) \leq \dim(B)$ and
  \[ \dim((e^{(2)} \ast (B^\perp \ast C_i^{-1})_{\mathcal{I}})) \leq \dim((B^\perp \ast C_i^{-1})_{\mathcal{I}}) \quad \forall i = 2, \ldots, \ell. \]

As in the case $\ell = 2$, we need to exclude the case where $B^\perp \ast C_i^{-1} = F_q^\parallel$ for some $i \in \{2, \ldots, \ell\}$. By Proposition 1.1 and Remark 1.2, we have
\[ \forall i \in \{1, \ldots, \ell\}, \quad B^\perp \ast C_i^{-1} = \text{RS}_q[t + (i - 1)(k - 1) + 1]. \]
Therefore we suppose
\[ t < n - \ell(k - 1). \]

Then, by Remark 3.3, bound (20) entails
\[ t \leq \frac{2n\ell - k\ell(\ell + 1) + \ell(\ell - 1)}{2(\ell + 1)}. \]

We observe that, as for the case $\ell = 2$, this decoding radius is the same decoding radius as Power Decoding algorithm’s (11). The difference is that for the Error Correcting Pairs algorithm, it is given by a necessary condition for the nullity of a certain intersection of spaces, while for the Power Decoding algorithm, it derives from a necessary condition on the size of a linear system. These two algorithms differ in complexity too. Indeed, the Power Error Locating Pairs algorithm consists mainly in two linear systems, one to find $e^{(1)}$ and one to find $M$, which is the largest one. It is a system of $O(n\ell)$ equations in $t + 1 = O(n)$ unknowns. Using Gaussian elimination, this gives the cost $O(n^3\ell^3)$ which represents an improvement, compared to the cost $O(n^3\ell^3)$ of the Power Decoding algorithm. This difference is motivated by the fact that in the Power Decoding algorithm we look for $\ell + 1$ polynomials $(\lambda, \nu_1, \ldots, \nu_\ell)$, while in the Power Error Locating Pairs algorithm we only look for one polynomial, namely $\lambda$, that is, the system has much less unknowns.

3.3 Failure cases, comparison with power decoding

In the previous subsection we have seen that the Power Decoding algorithm and the Power Error Locating Pairs algorithm have the same decoding radius for Reed Solomon codes. In fact, for such codes, it holds an even stronger result. Indeed, we now prove that these two algorithms fail in the same cases. In order to do that, we specify what we mean by “failure case” and present some examples.

3.3.1 Comparison of failure cases

In order to speak about failure cases, as in the previous sections, we still work under Assumption 1.

**PowD** Let $\text{Sol}$ and $\mathcal{M}_{\text{Sol}}$ be as in Definition 2.2. We know that, as long as $\text{rk}(\mathcal{M}_{\text{Sol}}) = 1$, there exists only one solution and the algorithm finds it. Now, even if $\text{rk}(\mathcal{M}_{\text{Sol}}) > 1$, sometimes the algorithm could find a good solution. Indeed the Power Decoding algorithm selects the element in $\text{Sol}$ with the $\lambda$ of the smallest degree (see Algorithm 3 pt.1) and it can happen that this element gives the solution we look for. In a first analysis, we treat as failure cases every situation where $\text{rk}(\mathcal{M}_{\text{Sol}}) > 1$. The treatment of the cases where $\text{rk}(\mathcal{M}_{\text{Sol}}) > 1$ and the Power Decoding succeeds is discussed further in § 3.3.3.

**PELP** In the Power Error Locating Pairs algorithm we know that, given $y = c + e$ and $I = \text{supp}(e)$, if $M_I = \{0\}$, then the algorithm works. By failure case for this algorithm, we mean a case where, given $J$ from the first step of the algorithm, the system
\[ H_J \cdot u^T = H \cdot y^T \]

does not have any solution. In other words, for any solution $e$ with $I = \text{supp}(e)$, we have $M_I \neq \{0\}$. 

Theorem 3.5. Let \( y = \mathbb{F}_q^n \) and \( t \) a positive integer, suppose se run both the Power Decoding algorithm and the Power Error Locating Pairs algorithm with the same \( t \). Then \( \text{rk}(M_{Sol}) \geq 1 \) if and only if for any solution \( e \), then \( M_I \neq \{0\} \) where \( I = \text{supp}(e) \).

Proof. For the sake of simplicity, we provide the proof in the case \( t = 2 \). The proof in the general case is easy to deduce at the cost of heavier notation.

First let us observe that there is a one-to-one correspondence between the elements of the spaces \( \text{Sol} \) and \( M \). In particular, we have a map

\[
\varphi \overset{\text{def}}{=} \begin{cases} \text{Sol} & \longrightarrow M \\ (\lambda, \nu_1, \nu_2) & \longmapsto \text{ev}_\lambda(\nu). \end{cases}
\]

Indeed since this vector belongs to \( \text{Sol} \), it holds

\[
\begin{align*}
\lambda(x_i)y_i &= \nu_1(x_i) & i &= 1, \ldots, n \\
\lambda(x_i)y_i^2 &= \nu_2(x_i) & i &= 1, \ldots, n.
\end{align*}
\]

(33)

As we have seen by defining \( M \) in subsection 3.1.1 these two conditions are equivalent to the statement

\[
\text{ev}_\lambda(\nu) \in M_1 \cap M_2 = M.
\]

Conversely, given \( a \in M \), there exists \( \lambda \in \mathbb{F}_q[X] \) with \( \deg(\lambda) < t + 1 \) such that \( \text{ev}_\lambda(\nu) = a \). Moreover, since \( a \in M_1 \), we have

\[
a \ast y \in B^\perp = \mathbb{R}_q[t + k], \quad a \ast y^2 \in B^\perp \ast C = \mathbb{R}_q[t + 2k - 1].
\]

Thus, there exist \( \nu_1, \nu_2 \in \mathbb{F}_q[X] \) with \( \deg(\nu_1) < t + k \), \( \deg(\nu_2) < t + 2k - 1 \) and

\[
\text{ev}_\lambda(\nu_1) = a \ast y, \quad \text{ev}_\lambda(\nu_2) = a \ast y^2.
\]

We can then define another map

\[
\psi \overset{\text{def}}{=} \begin{cases} M & \longrightarrow \text{Sol} \\
 a & \longmapsto (\lambda, \nu_1, \nu_2) \end{cases},
\]

(35)

where \( \lambda, \nu_1 \) and \( \nu_2 \) are the polynomials associated to \( a \) as before. It is easy to prove that, under the condition

\[
t < n - 2(k - 1)
\]

it holds \( \varphi \circ \psi = \text{Id}_M \) and \( \psi \circ \varphi = \text{Id}_{\text{Sol}} \).

(\( \Rightarrow \)) Let \( \text{rk}(M_{Sol}) > 1 \). We want to prove that there exists \( a \in M \) such that \( a_I \neq 0 \). Let \( c = \text{ev}_\lambda(f) \) with \( \deg(f) < k \) such that \( y = c + e \) for a certain \( e \in \mathbb{F}_q^n \). Let \( I = \text{supp}(e) \) and \( \Lambda(x) = \prod_{i \in I}(x - x_i) \). We have

\[
(\Lambda, \Lambda f, \Lambda f^2) \in \text{Sol}.
\]

Moreover, by hypothesis there exists another vector \((\lambda, \nu_1, \nu_2) \in \text{Sol} \) which is \( \mathbb{F}_q[X] \)-independent with \((\Lambda, N_1, N_2)\). In particular \( \text{ev}_\lambda(\lambda) = \varphi((\lambda, \nu_1, \nu_2)) \in M \). We now prove that \( \text{ev}_\lambda(\lambda) \neq 0 \). If \( \text{ev}_\lambda(\lambda) = 0 \), then \( \Lambda|\lambda \) and hence there exists \( h \in \mathbb{F}_q[X] \) such that \( \lambda = h\Lambda \) and Consequently,

\[
(\lambda, \nu_1, \nu_2) - h \cdot (\Lambda, \Lambda f, \Lambda f^2) = (0, \nu_1 - \lambda f, \nu_2 - \lambda f^2) \in \mathcal{M}_{Sol}.
\]

Therefore, its entries satisfy

\[
\forall i \in \{1, \ldots, n\}, \quad \begin{cases} 0 &= \nu_1(x_i) - \lambda(x_i)f(x_i) \\
0 &= \nu_2(x_i) - \lambda(x_i)f(x_i)\end{cases}
\]

and hence \( \nu_1 - \lambda f \) and \( \nu_2 - \lambda f \) have \( n \) roots while their respective degrees are less than or equal to \( k - 1 + t \) and \( 2k - 2 + t \). From \( \text{R.G.} \), these two polynomials are zero, which contradicts the assumption that the vectors \((\Lambda, \Lambda f, \Lambda f^2)\) and \((\lambda, \nu_1, \nu_2)\) are \( \mathbb{F}_q[X] \)-independent.
Let us suppose that \( \text{rk}(M_{Sol}) = 1 \). Since we are working under Assumption 1 by Proposition 2.4 there exists a solution \( f \) and, given \( \Lambda \) its locator polynomial, the module \( M_{Sol} \) is generated by the vector \( (\Lambda, \Lambda f, \Lambda f^2) \). Let \( e \) be the error vector associated to \( f \) and \( I = \text{supp}(e) \). For any \( (\lambda, \nu_1, \nu_2) \in \text{Sol} \), since \( \Lambda|\lambda \), we get \( (e_{\nu}(\lambda))_f = 0 \). That is, \( M_f = \{0\} \).

As consequence of this result, we have the following link between the solutions of the two algorithms.

**Corollary 3.6.** Given \( y \in \mathbb{F}_q^n \) suppose we run the Power Decoding algorithm and the Power Error Locating Pairs algorithm on \( y \) and a certain value of \( t \). If the Power Error Locating Pairs algorithm returns a solution with a corresponding error vector \( e \) with \( I = \text{supp}(e) \), and let \( \Lambda \) be its locator polynomial

\[
\Lambda(X) = \prod_{i \in I} (X - x_i),
\]

then, \( \text{rk}(M_{Sol}) = 1 \). In particular, there exists \( f \in \mathbb{F}_q[X]_{<k} \) such that \( (\Lambda, \Lambda f, \Lambda f^2) \in \text{Sol} \) and \( \deg(\Lambda) = \min\{\deg(\lambda) \mid (\lambda, \nu_1, \nu_2) \in \text{Sol} \} \).

**Proof.** Here again, we prove the result in the case \( \ell = 2 \) for the sake of simplicity. Let \( J = Z(M) \) in the Error Locating Pairs algorithm. Since \( M_J = \{0\} \), by Theorem 3.5 we have that \( \text{rk}(M_{Sol}) = 1 \). Finally the minimality of the degree of \( \Lambda \) is a consequence of the very definition of the error locator polynomial.

**3.3.2 A particular failure case for the Power Error Locating Pairs algorithm**

We now present a failure case for any \( t > \frac{d-1}{2} \). The following example will systematically lead to a failure for the PELP algorithm we introduced while the Power Decoding might succeed. We finish this section by presenting an additional routine to add in the PELP algorithm thanks to which the failure cases of the Power Decoding and those of the PELP are exactly the same.

**Remark 3.5.** Given \( C = \text{RS}_q[k] \) and \( t \) which fulfills

\[
\frac{d(C) - 1}{2} < t < d(C),
\]

(37)

it is always possible to exhibit some \( y, c, c', e, e' \in \mathbb{F}_q^n \) with \( w(e), w(e') \leq t \), such that \( c, c' \in C \) and

\[
y = c + e = c' + e'.
\]

(38)

Indeed, let us consider \( e' = 0 \) and \( c \in C \) such that \( w(c) = d(C) \). Without loss of generality we can suppose that \( c = (c_1, \ldots, c_{d(C)}, 0, \ldots, 0) \). Then, we consider the error vectors \( e = (-c_1, \ldots, -c_t, 0, \ldots, 0) \) and \( e' = (0, \ldots, 0, c_{t+1}, \ldots, c_{d(C)}, 0, \ldots, 0) \). By these choices, (38) holds. Moreover, we get \( w(e) = t \) and

\[
w(e') = d(C) - t \leq t \text{ by hypothesis (37)}.
\]

**Proposition 3.7.** If \( y = c + e = c' + e' \) as in Remark 3.5, the PELP algorithm fails.

**Proof.** Let \( I \) and \( I' \) be as in Remark 3.5. Since we have two solutions, by Proposition 2.5, \( \text{rk}(M_{Sol}) > 1 \) and by Theorem 3.5 we have \( M_I, M_{I'} \neq \{0\} \). Hence, the algorithm fails.

One could wonder if it is possible to recover the space \( A(I \cup I') \) somehow (that is, not using the set \( M \)). Indeed, in this case, \( J = Z(A(I \cup I')) \supseteq I \cup I' \) and maybe it would be possible to work on the solution space of the system

\[
H_{11} \cdot u^T = H \cdot y^T
\]

to find the solutions \( c \) and \( c' \). Though, the following result shows that actually this path is not effective.

**Proposition 3.8.** If \( y = c + e = c' + e' \) with \( I = \text{supp}(e) \) and \( I' = \text{supp}(e') \), then \( A(I \cup I') = \{0\} \).

**Proof.** Let us suppose that \( A(I \cup I') \neq \{0\} \). Therefore there exists \( a \in A(I \cup I') \setminus \{0\} \) and it holds \( |Z(a)| \geq |I \cup I'| \), hence \( w(a) \leq n - |I \cup I'| \). Moreover \( d(c, c') \leq |I \cup I'| \). Hence, we obtain the contradiction

\[
n < d(A) + d(C) < w(a) + d(c, c') \leq n.
\]

(39)
3.3.3 A possible repair

In the previous example, if \( w(e') < w(e) \), then for the Power decoding algorithm the module \( M_{Sol} \) would have rank > 1 but the algorithm might succeed when returning the vector \( (\lambda, \nu_1, \nu_2) \in Sol \) where \( \lambda \) has the smallest possible degree. The corresponding \( \lambda \) being the error locator \( \Lambda' \) for \( e' \).

In this situation, the Power Error Locating Pairs algorithm will fail. However, reconsidering the proof of Theorem 3.5 one sees that \( M \) is the image of the map \( \varphi \) defined in (32). Then, take a \( \mathbf{a}_1, \ldots, \mathbf{a}_s \) basis of \( M \). By interpolation one deduces a family of polynomials \( \lambda_1, \ldots, \lambda_s \in \mathbb{F}_q[X] \leq t \) since \( M \subseteq A = \mathbb{R}^n \mathbb{F}_q[t+1] \). The nonzero polynomial of the smallest possible degree in the space spanned by \( \lambda_1, \ldots, \lambda_s \) can be computed by Gaussian elimination and will be equal to \( \Lambda' \) if the Power Decoding succeeded. Using this trick it is possible to have exactly the same failure cases for the Power Decoding and the Power Error Correcting Pairs algorithm.

3.4 Tests

In practice the Power Error Locating Pairs algorithm works quite well. We runned the same tests in Sage with \( \ell = 2 \) on this algorithm together with Sudan algorithm and the Power Decoding algorithm.

3.4.1 Equivalence of failure cases

In §3.3.1 we showed that, given \( M_{Sol} \) as in Definition 2.2 and \( M \) as in (28), \( \text{rk}(M_{Sol}) > 1 \) if and only if \( Z(M) \) does not contain any support of a possible solution. Though, we know that even if \( \text{rk}(M_{Sol}) > 1 \), the Power Decoding algorithm could still find a solution, while for the PELP algorithm there is not any chance unless we use the additional routine presented in §3.3.3.

In Table 1 we present the results of testing these algorithms 200 times for any choice of parameters \( q, n, k \) and \( t \) (here we only consider \( \ell = 2 \)) and for random choices of the error vectors. In particular \( \text{Failure}_{\text{PowD}} \) and \( \text{Failure}_{\text{PELP}} \) indicate the amount of failure cases obtained respectively from the Power Decoding algorithm and the Power Error Locating Pairs algorithm. Remember that it holds

\[ \text{Failure}_{\text{PowD}} \leq \text{Failure}_{\text{PELP}}, \]

with equality if we use the trick indicated in §3.3.3. Note that this additional routine has not been implemented, therefore, for our implementation, the PELP algorithm might fail more often than the Power Decoding. Actually, such failure cases are so rare that it has not been possible to observe one in our experiments.

We indicate with (\( * \)) the values of \( t \) which equal the decoding radius of the algorithms

\[ \frac{2n\ell - k\ell(\ell + 1) + t(\ell - 1)}{2(\ell + 1)}. \]  \hspace{1cm} (40)

Looking at Table 1 it is clear that failure cases are indeed very rare if the error is randomly generated (we only registered 4 failure cases over 5200 tests). In this cases, both algorithm failed. Moreover, note that these failure cases only showed up for \( t \) equal to the decoding radius.

What happened in Sudan algorithm? In these failure cases, the roots of the Sudan polynomial were a polynomial (which was the right solution \( f \)) and a rational function \( g_1 \). Now, we found that both the vectors

\( (1, f, f^2), \quad (g_1^2, g_1g_2, g_2) \)

had a multiple in \( Sol \). It is easy to see that they are \( \mathbb{F}_q[X] \)-linearly independent.

What happens in Sudan algorithm generally? In the tests where Power Decoding algorithm and PELP algorithm did not fail, Sudan still gave a polynomial \( f \) and a rational function. Though only the vector \( (1, f, f^2) \) had a multiple in \( Sol \).
Failure cases comparison

| \(q\) | \(n\) | \(k\) | \(\lfloor \frac{d-1}{2} \rfloor\) | \(t\) | Failure\(_{PowD}\) | Failure\(_{PELP}\) |
|---|---|---|---|---|---|---|
| 211 | 200 | 50 | 75 | 80 | 0 | 0 |
| 211 | 200 | 50 | 75 | 83\(^{(*)}\) | 0 | 0 |
| 211 | 200 | 40 | 80 | 87 | 0 | 0 |
| 211 | 200 | 40 | 80 | 93\(^{(*)}\) | 0 | 0 |
| 211 | 200 | 30 | 85 | 94 | 0 | 0 |
| 211 | 200 | 30 | 85 | 103\(^{(c)}\) | 0 | 0 |
| 211 | 199 | 60 | 69 | 71 | 0 | 0 |
| 211 | 199 | 60 | 69 | 72 | 0 | 0 |
| 211 | 199 | 60 | 69 | 73\(^{(c)}\) | 0 | 0 |
| 211 | 199 | 50 | 74 | 80 | 0 | 0 |
| 211 | 199 | 50 | 74 | 82 | 0 | 0 |
| 211 | 199 | 50 | 74 | 83\(^{*}\) | 0 | 0 |
| 211 | 199 | 40 | 79 | 88 | 0 | 0 |
| 211 | 199 | 40 | 79 | 92 | 0 | 0 |
| 211 | 199 | 40 | 79 | 93\(^{(*)}\) | 1 | 1 |
| 256 | 250 | 80 | 85 | 86 | 0 | 0 |
| 256 | 250 | 80 | 85 | 87\(^{(c)}\) | 2 | 2 |
| 256 | 250 | 70 | 90 | 95 | 0 | 0 |
| 256 | 250 | 70 | 90 | 96 | 0 | 0 |
| 256 | 250 | 70 | 90 | 97\(^{(c)}\) | 0 | 0 |
| 256 | 250 | 60 | 95 | 100 | 0 | 0 |
| 256 | 250 | 60 | 95 | 106 | 0 | 0 |
| 256 | 250 | 60 | 95 | 107\(^{(c)}\) | 1 | 1 |
| 256 | 250 | 50 | 100 | 110 | 0 | 0 |
| 256 | 250 | 50 | 100 | 116 | 0 | 0 |
| 256 | 250 | 50 | 100 | 117\(^{(c)}\) | 0 | 0 |

Table 1: Failure cases comparison

3.4.2 Some other failure cases

As seen in §3.3.2, even if it is rare to have a failure case with randomly generated parameters, it is easy to generate them. We now present an even easier failure case, by using homogeneous errors.

**Definition 3.2.** Let \(\alpha \in \mathbb{F}_q\). An vector \(e \in \mathbb{F}_q^n\) is \(\alpha\)-homogeneous if, for any \(i \in \{1, \ldots, n\}\), \(e_i \neq 0\) if and only if \(e_i = \alpha\).

Let \(\mathbf{e}\) be an \(\alpha\)-homogeneous error vector, \(f \in \mathbb{F}_q[X]_{<k}\) and \(y = ev_x(f) + \mathbf{e}\).

**Remark 3.6.** It is easy to see that if \(d(ev_x(f), y) = t\), then \(d(ev_x(f + \alpha), y) = n - t\).

If \(n - t > t, f + \alpha\) is not a solution to the decoding problem. Hence, intuitively, the Power Decoding algorithm and the PELP algorithm would not “see” \(f + \alpha\) as a solution. Though, in the tests we made, both the algorithms failed for \(t > \frac{d-1}{2}\). Indeed, it seems that as soon as \(t > \frac{d-1}{2}\), both the vectors

\[(1, f, f^2), \quad (1, f + \alpha, (f + \alpha)^2)\]

have a multiple in \(\text{Sol}\).

**What happens in Sudan algorithm?** In these homogeneous error cases, if \(t \leq \frac{d-1}{2}\), Sudan polynomial has only one polynomial root \(f\). For \(\ell = 2\) the other root is a rational function. In particular, even if there are other roots of Sudan polynomial, only the vector \((1, f, f^2)\) has a multiple in \(\text{Sol}\). Though, as \(t > \frac{d-1}{2}\), both the polynomials \(f\) and \(f + \alpha\) show up among the roots of Sudan polynomial and, as seen before, in all the tests both have a multiple in \(\text{Sol}\).
4 Power Error correcting pairs algorithm for algebraic geometry codes

As said previously, the Power Error Locating Pairs algorithm can be run on any code with an ELP. We have seen that Reed-Solomon codes belong to this class of codes and now we will show that, generally, all the algebraic geometry codes with specific parameters do.

We denote by $k$ and $d$ respectively the dimension and the minimum distance of the code and by $g$ the genus of the curve $X$. Moreover we denote by $D$ the divisor $P_1 + \cdots + P_n$ and $W$ the divisor $(\omega)$ where $\omega \in \Omega(X)$ is a differential such that $v_{P_i}(\omega) = -1$ and $res_{P_i}(\omega) = 1$ for any $i \in \{1, \ldots, n\}$. We introduce an extra divisor $F$ on $X$ and introduce the codes.

$$A = C_L(X, \mathcal{P}, F) \quad B = C_L(X, \mathcal{P}, D + W - F - G).$$  \hspace{1cm} (41)

This pair of divisors is our candidate to be an error locating pair.

4.1 Decoding Radius

In order to find the decoding radius of the Power Error Correcting Pairs algorithm for algebraic geometry codes, we follow the same path as for Reed-Solomon codes. Again, we want to find a necessary condition to have $M_1 I = \{0\}$.

We analyse the case $\ell = 2$ for simplicity (it is easy to generalise what we are going to see). In the following part we need additional conditions on the parameters of the code and on the degree of the divisor $F$ to be able to compute the decoding radius. The following result is one of the reasons to ask for these conditions.

**Theorem 4.1.** Let $E, F$ be two divisors for $X$. Then, if $\deg(E) \geq 2g$ and $\deg(F) \geq 2g + 1$, it holds

$$L(E)L(F) = L(E + F).$$

**Proof.** See [15, Theorem 6]. \hfill \Box

**Corollary 4.2.** Given $B$ as in (41), if $\deg(G) \geq 2g$ and $\deg(F) \geq 1$, it holds

$$B^\perp * C = C_L(X, \mathcal{P}, F + 2G).$$

Now we can start to look for the decoding radius. As for Reed-Solomon codes, we have

$$(M_1 \cap M_2)_I = (e^{(1)} * B)^I \cap (e^{(2)} * (B^\perp * C)^I)^I \cap A_I.$$

In order to use the same necessary condition as in §3 we want $A_I = \mathbb{F}_q^n$. We then ask for

**Additional Condition 1.** $\deg(F) \geq t + 2g$.

Indeed, it is easy to verify that under Additional Condition 1 by Lemma 3.1 we obtain $A_I = \mathbb{F}_q^n$. Let us fix then the value of $\deg(F) = 2g + t$. Now we can consider, as for Reed-Solomon codes, the necessary condition

$$\dim(B) + \dim((B^\perp * C)^I) \geq t.$$  \hspace{1cm} (43)

We need to know the exact dimension of these spaces. First, we want to write $B^\perp * C$ as an Algebraic Geometry code, that is, we want to use Theorem 4.1. We ask then for the following

**Additional Condition 2.** $\deg(G) \geq 2g$.

Now, we want to know the dimension of $B$ and $(B^\perp * C)^I$, hence we impose some conditions on the degree of the divisor $F$.

To know the dimension of $B$ it suffices to have $t < n - \deg(G) - 2g$, while for $(B^\perp * C)^I$, we need $t < n - 2 \deg(G) - 2g$. \footnote{Remember that if $C = C_L(X, \mathcal{P}, G)$ with $2g - 2 < \deg(G) < n$, then $\dim(C) = \deg(G) - g + 1$.}
Proposition 4.3. If \( \deg(F) = t + 2g < n - \deg(G) \), \( \deg(G) \geq 2g \) and \( 1 - \deg(G) \leq t \leq n - 2 \deg(G) - 2g \), then \( C = C_L(X, P, G) \) admits an ELP as in (41) and for \( \ell = 2 \), the decoding radius is

\[
t \leq \frac{2n - 3 \deg(G) - 2}{3} - \frac{2}{3}g.
\]

Proof. Condition (ELP1) is obviously satisfied by the codes \( A, B \) defined in (41). The condition \( \deg(F) = t + 2g \) asserts that \( \dim A > t \) i.e. Condition (ELP2). Finally, Condition (ELP3) is a consequence of the condition \( \deg(F) < n - \deg(G) \), which indeed entails \( d(A) + d(C) \geq 2n - \deg(F) - \deg(G) > n \).

The decoding radius is directly deduced from (43).

The decoding radius can be computed even for arbitrary values of \( \ell \). To do that we just have to impose \( 1 - \deg(G) \leq t \leq n - \ell \deg(G) - 2g \):

\[
t \leq \frac{2n \ell - \ell (\ell + 1) \deg(G)}{2(\ell + 1)} - g - \frac{g - \ell}{\ell + 1}.
\]

4.2 Comparison with decoding radii of other algorithms for algebraic geometry codes

We can now compare this decoding radius with the decoding radii of Sudan algorithm and the Power Decoding algorithm for algebraic geometry codes. We have (see [24, Theorem 2.1] and Appendix A):

\[
\begin{align*}
\text{Sudan} & \quad t \leq \frac{2n \ell - \ell (\ell + 1) \deg(G)}{2(\ell + 1)} - g - \frac{1}{\ell + 1} \\
\text{PowD} & \quad t \leq \frac{2n \ell - \ell (\ell + 1) \deg(G)}{2(\ell + 1)} - g - \frac{\ell}{\ell + 1}.
\end{align*}
\]

First, we notice that for algebraic geometry codes, the Power Decoding algorithm and the Power Error Locating Pairs algorithm no longer have the same decoding radius, but the second one is larger. Furthermore, note that if

\( g > \ell - 1 \),

the decoding radius of PELP algorithm (45) is even larger than Sudan algorithm decoding radius.

Conclusion

We proposed a new decoding algorithm that can be applied to any code family benefiting from an error locating pair. For Reed–Solomon and algebraic geometry codes, this algorithm the same decoding radius as the Power decoding, which is almost equal to that of Sudan algorithm. A comparison of the features of these various algorithm is given in Table 4.2.

Such an algorithm returns at most one solution and might fail in some very rare cases. Computer aided simulations confirm the rarity of these cases Finally, we observe that Power decoding and our new algorithm share many inputs leading to a failure. For this reason, we hope that our new algorithm could be helpful in the future for a further analysis of the failure cases of Power decoding, which remains an open problem.

| \( t \) | BW | ECP | Sudan | PowD | PELP |
|-------|----|-----|-------|------|------|
| \( \geq \frac{n + 1}{2} \) | yes | yes | yes | yes | yes |
| \( > \frac{n + 1}{2} \) | no | no | yes | yes | yes |
| May fail | no | no | no | yes | yes |
| Worst case | yes | yes | yes | no | no |

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[19] R. Pellikaan. On decoding by error location and dependent sets of error positions. Discrete Math., 106–107:369–381, 1992.
We now show how the Power Decoding algorithm adapts to Algebraic Geometry codes. Let $C = C_L(X, P, G)$ and $y = c + e \in F_q^n$ the word we want to correct, where $c \in C$. We have then $c = ev_P(f)$ with $f \in L(G)$.

Furthermore, as in the previous sections, we suppose that $w(e) = t$ and we denote the support of $e$ by $I$.

We keep the same idea we used in the version of the algorithm for Reed-Solomon codes. Indeed, let us suppose to have a $\Lambda \in F_q(X)$ such that $\Lambda(P_i) = 0$ for all $i \in I$. Then, given $\ell \in \mathbb{N}$ we get

$$\Lambda(P_i) y_j^i = \Lambda(P_i) f^j(P_i) \quad \forall \quad i = 1, \ldots, n \quad j = 1, \ldots, \ell. \quad (46)$$

We would like then to find $\Lambda$ as above. It is easy to see that such a $\Lambda$ has to be searched in $L(F)$ for a certain $F$ such that $\deg(F) \geq t$ (we will give a better bound for that soon). It is possible to see $(\Lambda, f)$ as a solution of

$$\lambda(P_i) y_j^i = \lambda(P_i) g^j(P_i) \quad \forall \quad i = 1, \ldots, n \quad j = 1, \ldots, \ell, \quad (47)$$

that is, a system of $n\ell$ equations whose unknowns are the coordinates of $\lambda$ and $g$ in the basis of respectively $L(F)$ and $L(G)$. System (47) is not linear in the unknowns though, hence we linearise it by considering a new function $\nu_j := \lambda g^j$ for any equation. For all $j \in \{1, \ldots, \ell\}$, we get

$$\nu_j \in L(F)L(jG) \subseteq L(F + jG).$$

In order to use Theorem 4.1 let us fix $\deg(F) = t + 2g$ and suppose $\deg(G) \geq 2g + 1$. We get then the following problem.

**Key Problem 5.** Given $y \in F_q^n$ and $t \in \mathbb{N}$, look for $\lambda, \nu_1, \ldots, \nu_\ell \in F_q^n(X)$ such that
\( \lambda \in L(F) \) with \( \deg(F) = t + 2g; \)

\( \nu_j \in L(F + jG) \) for all \( j = 1, \ldots, \ell; \)

\( \lambda(x_i) y_i = \nu_j(x_i) \) for all \( i = 1, \ldots, n \) and \( j = 1, \ldots, \ell. \)

Therefore, even this case, the Power Decoding algorithm consists in solving a linear system and we will just consider a solution different from the zero vector.

**Decoding Radius**  
As in the case of Reed-Solomon codes, we would like to have a solution space of dimension one. A necessary condition for that, is:

\[
\#\text{unknowns} \leq \#\text{equations} + 1. \tag{48}
\]

Now, the number of equations is \( n\ell \). For the number of unknowns, we need to know the dimension of the spaces \( L(F + jG) \) for all \( j = 1, \ldots, \ell \). The bounds we have set in the hypothesis give

\[ \dim(L(F + jG)) = t + g + j \deg(G) + 1. \]

Hence by condition (48) we get the following decoding radius

\[ t \leq \frac{2n\ell - \ell(\ell + 1) \deg(G)}{2(\ell + 1)} - g - \frac{\ell}{\ell + 1}. \tag{49} \]

**Remark A.1.** This bound is not a sufficient condition to have a solution, but it is not even a necessary condition. In fact, as for the Power Decoding algorithm for Reed-Solomon codes, we could find a good solution even for a bigger value of \( t \) and on the other hand the algorithm could fail even if \( t \) fulfills condition (49).