Between the conjectures of Pólya and Turán

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Abstract
This paper is concerned with the constancy in the sign of
\( L(X, \alpha) = \sum_{n \leq X} \lambda(n) n^{\alpha} \), where \( \lambda(n) \) the Liouville function. The non-positivity of \( L(X, 0) \) is the Pólya conjecture, and the non-negativity of \( L(X, 1) \) is the Turán conjecture — both of which are false. By constructing an auxiliary function, evidence is provided that \( L(X, \frac{1}{2}) \) is the best contender for constancy in sign. The core of this paper is the conjecture that \( L(X, \frac{1}{2}) \leq 0 \) for all \( X \geq 17 \): this has been verified for \( X \leq 300,001 \).

1 Introduction

Let \( \lambda(n) \) denote the Liouville function defined as \( \lambda(n) = (-1)^{\Omega(n)} \), where \( \Omega(n) \) counts, with multiplicity, the number of prime factors of \( n \). It is well-known (see, e.g. [8, Ch. I]) that, for \( \Re(s) > 1 \)
\[
\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}. \tag{1}
\]
Pólya [6] showed that, if \( L(X) = \sum_{n \leq X} \lambda(n) \) is of a constant sign (for \( X \) sufficiently large), then this implies the Riemann hypothesis. This lead to the

Conjecture (The Pólya Conjecture).
\[
\sum_{n \leq X} \lambda(n) \leq 0,
\]
for all \( X \geq 2 \).

Similarly, Turán [9] showed that the Riemann hypothesis is true if the sum
\[
1^n \text{Actually, Turán, in his work with the partial sums of the zeta-function, } U_\nu(s) = \sum_1^n \nu^{-s}, \text{ showed that the Riemann hypothesis is true if } U_\nu(s) \text{ has no zeroes in the region } \sigma > 1. \text{ This was shown to be false by Montgomery [5] in 1983. The non-vanishing of } U_\nu(s) \text{ in } \sigma > 1 \text{ is equivalent to } \sum_1^n \lambda(n) n^{-s} \geq 0, \text{ for } X \text{ sufficiently large, for all } \sigma \geq 1 \text{ — see the remarks after [2].}
\]
\[ T(X) = \sum_{n \leq X} \frac{\lambda(n)}{n} \text{ is of constant sign (for } X \text{ sufficiently large). This lead to the}

\textbf{Conjecture (The Turán Conjecture).}

\[ \sum_{n \leq X} \frac{\lambda(n)}{n} \geq 0, \]

\textit{for all } \( X \geq 0. \)

In 1958 Haselgrove [3] showed that both of these conjectures are false, in spite of rather extensive numerical verification. The first value of \( X \) (other than \( 1! \)) for which \( L(X) > 0 \) is \( X = 906, 105, 257 \), which was proved by Tanaka [7]; the least \( X \) for which \( T(X) < 0 \) is greater than \( 7 \cdot 10^{13} \), and was calculated explicitly by Borwein, Ferguson and Mossinghof [1].

Here the following sum will be considered

\[ L(X, \alpha) = \sum_{n \leq X} \frac{\lambda(n)}{n^\alpha}, \]

where \( \alpha \geq 0. \) The case \( \alpha > 1 \) can be dispensed with easily. For a fixed \( \alpha > 1 \) one can consider the Euler product of (1), viz.

\[ \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^\alpha} = \prod_p \left( \frac{1 - \frac{1}{p^\alpha}}{1 - \frac{1}{p^\alpha}} \right) = \prod_p \left( 1 + \frac{1}{p^\alpha} \right)^{-1}. \quad (2) \]

This is increasing with \( \alpha \), whence one can use partial summation to show there is a sufficiently large number, \( X_0 \) (which depends on \( \alpha \)) such that, for \( X > X_0 \) the sum \( L(x, \alpha) \geq 0. \)

Of interest, therefore, are in intermediate cases \( 0 < \alpha < 1. \) The approach taken here follows that taken by Haselgrove [op. cit.] and is based on a theorem of Ingham [4]. Suppose one proposes the ‘\( \alpha \) + Conjecture’, which says that, for \( X \) sufficiently large, \( L(X, \alpha) \geq 0. \) To disprove such a conjecture one creates an auxiliary function which approximates \( L(X, \alpha) \), whence, if this auxiliary function were negative in a certain region, then \( L(X, \alpha) \) would likely be negative there as well. The locating of such a value (or the determining of an interval in which such a value lies) establishes the falsity of the \( \alpha \) + Conjecture.

Despite the retrograde approach, it will be shown in the next section that the creating of this auxiliary function gives an insight into the expected behaviour of \( L(X, \alpha) \). It is difficult to predict, \textit{ab initio}, those values of \( \alpha \) for which there is a preponderance of negative (or positive) values of \( L(X, \alpha) \); a naive computation of various values of \( \alpha \) for, say, \( X \leq 100 \) is not very illuminating.

\textsuperscript{2}Fortunately, there is none of the customary ambiguity with the exclamation mark and factorial sign, nor for that matter, with the marking of this footnote.
2 The work of Ingham

Let \( F(s, \alpha) = \int_0^\infty A(u, \alpha) e^{-su} \, du \), where \( A(u, \alpha) \) is absolutely integrable over every finite interval \( 0 \leq u \leq U \), and suppose the integral is convergent in the half-plane \( \sigma > \frac{1}{2} - \alpha \geq 0 \). Write \( \rho_n = \frac{1}{2} + i\gamma_n \) for the \( n \)th zero of \( \zeta(s) \), where \( \gamma_n = -\gamma_n \) and define \( \gamma_0 = 0 \). Suppose \( F(s, \alpha) \) has poles at \( s = \rho_n - \alpha \) for \(-N \leq n \leq N\), with residues labeled \( r_n = r_n(\alpha) \). Define, for \( \sigma > \frac{1}{2} - \alpha \), the function \( F^*(s, \alpha) = \sum_{-N}^{N} \frac{r_n}{s-(\rho_n-\alpha)} \). Thus the function \( F(s, \alpha) - F^*(s, \alpha) \) is regular in the region \( \sigma > \frac{1}{2} - \alpha \). If one writes \( A^*(u) = e^{(1/2-\alpha)u} \sum_{n} r_n e^{i\gamma_n u} \), then it is easily seen that \( F^*(s) = \int_0^\infty A^*(u) e^{-su} \, du \). Then we have the following

**Theorem** (Ingham). For a fixed \( T \), \( \sigma \rightarrow \infty \)

\[
\lim_{u \to \infty} e^{(\alpha-1/2)u} A(u, \alpha) \leq \lim_{u \to \infty} A^*_T(u, \alpha) \leq \lim_{u \to \infty} e^{(\alpha-1/2)u} A(u, \alpha), \tag{3}
\]

where \( A^*_T(u, \alpha) \) is a smoothed trigonometric polynomial defined as

\[
A^*_T(u, \alpha) = r_0 + 2R \sum_{0 < \gamma_n < T} \left( 1 - \frac{\gamma_n}{T} \right) r_n e^{i\gamma_n u}.
\]

**Proof.** See Ingham [op. cit.]

For the present purpose take \( A(u) = L(e^u, \alpha) \) — see (1). If one were to find a value of \( u \) and \( T \) for which \( A^*_T(u) > 0 \) then \( L(e^u, \alpha) > 0 \), with a similar property holding in the other direction, courtesy of (3). Henceforth \( A^*_T(u) \), which will of course depend of \( \alpha \), will be called the ‘auxiliary polynomial’.

3 Application of Ingham’s Theorem

For \( \sigma > 1 - \alpha \), one can use partial summation on (1) to write

\[
\frac{\zeta(2(\alpha + s))}{\zeta(\alpha + s)} = \lim_{X \to \infty} \frac{\left( \sum_{n \leq X} \frac{\lambda(n)}{n^\alpha} \right)}{X^s} + s \int_1^X \left( \sum_{n \leq t} \frac{\lambda(n)}{n^\alpha} \right) t^{-(s+1)} \, dt \tag{4}
\]

It was first shown by Landau that, on the Riemann hypothesis, \( \sum_{n \leq X} \lambda(n) = O(X^{1/2+\epsilon}) \). Thus the integral on the right side of (4) converges and defines an analytic function for \( \sigma > \frac{1}{2} - \alpha \), whence it follows by analytic continuation that

\[
F(s, \alpha) := \frac{\zeta(2(\alpha + s))}{s\zeta(\alpha + s)} = \int_1^\infty \left( \sum_{n \leq t} \frac{\lambda(n)}{n^\alpha} \right) t^{-(s+1)} \, dt \tag{5}
\]

Possible poles of \( F(s, \alpha) \) are at \( s = 0 \), at \( s = \rho_n - \alpha \) and at \( s = \frac{1}{2} - \alpha \). When \( \alpha = 1 \) there is no pole at \( s = 0 \), and when \( \alpha < \frac{1}{2} \) the pole at \( s = 0 \) lies outside the half-plane \( \sigma > \frac{1}{2} - \alpha \). Thus, when \( \frac{1}{2} < \alpha < 1 \),

\[
\text{Res } (F(s, \alpha); s = 0) = \lim_{s \to 0} \frac{\zeta(2s)}{\zeta(s)} = 1.
\]

3
When $\alpha = \frac{1}{2}$, writing $\zeta(s) = (s-1)^{-1} + \gamma + O(s-1)$, where $\gamma$ is Euler’s constant, gives
\[
\text{Res } (F(s, \alpha); s = 0) = \frac{\gamma}{\zeta\left(\frac{1}{2}\right)}.
\]
Lastly, assuming the simplicity of the zeroes,
\[
\text{Res } (F(s, \alpha); s = \rho_n - \alpha, n \neq 0) = \frac{\zeta(2\rho_n)}{(\rho_n - \alpha)\zeta'(\rho_n)}
\]
Thus the auxiliary polynomial becomes
\[
A_T^*(u, \alpha) = r_0(\alpha) + 2\Re \sum_{0 < \gamma_n < T} \frac{\zeta(2\rho_n)}{(\rho_n - \alpha)\zeta'(\rho_n)} \left\{ 1 - \frac{\gamma_n}{T} \right\} e^{i\gamma_n u},
\]
where,
\[
r_0(\alpha) = \begin{cases} 
\{(1 - 2\alpha)\zeta\left(\frac{1}{2}\right)\}^{-1}, & \text{if } 0 \leq \alpha < \frac{1}{2}, \alpha = 1 \\
\{(1 - 2\alpha)\zeta\left(\frac{1}{2}\right)\}^{-1} + 1, & \text{if } \frac{1}{2} < \alpha < 1 \\
\gamma/\zeta\left(\frac{1}{2}\right), & \text{if } \alpha = \frac{1}{2}.
\end{cases}
\]
Thus the $\alpha = 0, 1$ cases of (7) give, respectively, with the Pólya and the Turán auxiliary polynomials. Since $\zeta\left(\frac{1}{2}\right) < 0$, the auxiliary polynomial $A_T^*(u, 0)$ consists of a negative term followed by a sum of oscillating terms. One might suspect therefore that the Pólya conjecture ‘should’ be true, because the terms in the sum should not reinforce one another sufficiently often.

The leading term in $A_T^*(u, \alpha)$ decreases without limit as $\alpha \uparrow \frac{1}{2}$. Thus, one might suspect that, say, $L(X, 0.499)$ might be positive for all sufficiently large $X$. Computations of this sort do not yield the ‘expected’ result, perhaps due to the presence of the $p_n - \alpha$ term in the denominator of (7).

Note though, when $\alpha = \frac{1}{2}$, the summands in (6) are of the form
\[
\Re \frac{\zeta(1 + 2i\gamma_n)}{\gamma_n\zeta'(\frac{1}{2} + i\gamma_n)} \theta_n,
\]
where $|\theta_n| \leq 1$. Fujii [2] has proved that $\zeta'(\frac{1}{2} + i\gamma_n)$ is real positive in the mean. On the Riemann hypothesis, Littlewood (see, e.g. [8, Ch. XIV]) showed that $\zeta(1 + t) = O(\log \log t)$. Thus, not only should the summands in (6) be small, but they are at their smallest when $\alpha = \frac{1}{2}$. This suggests that perhaps the strongest case for a constancy in sign of the sum $L(X, \alpha)$ is made when $\alpha = \frac{1}{2}$. Using Mathematica, I have checked that the sum $L(X, 1/2)$ is negative for all $17 \leq X \leq 300,001$. Of interest would be the investigation of

**The $\alpha = 1/2$ Conjecture.**

\[
L(X, 1/2) = \sum_{1 \leq n \leq X} \frac{\lambda(n)}{n^{1/2}} \leq 0,
\]
for all $X \geq 17$. 
References

[1] Ferguson R. Borwein, P. and M. J. Mossinghoff. Sign changes in sums of the Liouville function. *Mathematics of Computation*, 77(263):1681–1694, 2008.

[2] A. Fujii. On a conjecture of Shanks. *Proceedings of the Japan Academy, Series A*, 70:109–114, 1994.

[3] C.B. Haselgrove. A disproof of a conjecture of Pólya. *Mathematika*, 5:141–145, 1958.

[4] A. E. Ingham. On two conjectures in the theory of numbers. *American Journal of Mathematics*, 64(1):313–319, 1942.

[5] H. L. Montgomery. *Zeros of approximations to the zeta function in “Studies in Pure Mathematics: to the memory of Paul Turán”*. Birkhäuser, Basel, 1983.

[6] G. Pólya. Verschiedene bemerkungen zur zahlentheorie. *Jahresber. Deutsch. Math.-Verein*, 28:31–40, 1919.

[7] M. Tanaka. A numerical investigation on cumulative sum of the Liouville function. *Tokyo Journal of Mathematics*, 3(1):187–189, 1980.

[8] E. C. Titchmarsh. *The Theory of the Riemann zeta-function*. Oxford Science Publications. Oxford University Press, Oxford, 2nd edition, 1986.

[9] P. Turán. On some approximative dirichlet-polynomials in the theory of the zeta-function of riemann. *Danske Vid. Selsk. Math.-Fys. Medd.*, 24(17):1–36, 1948.