Beyond the semi-classical description of black hole evaporation

Renaud Parentani†
Laboratoire de Physique Théorique, 
UMR CNRS 8627,
Université Paris-Sud,
91405 Orsay Cedex, France.
February 1, 2008

Abstract:
In the semi-classical treatment, i.e. in a classical black hole geometry, Hawking quanta emerge from trans-Planckian configurations because of scale invariance. There is indeed no scale to stop the blue-shifting effect encountered in the backward propagation towards the event horizon. On the contrary, when taking into account the gravitational interactions neglected in the semi-classical treatment, a UV scale stopping the blue-shift could be dynamically engendered. To show that this is the case, we use a non-perturbative treatment based on the large-$N$ limit, where $N$ is the number of matter fields. In this limit, the semi-classical treatment is the leading contribution. Non-linear gravitational effects appear in the next orders and in the first of these, the effects are governed by the two-point correlation function of the energy-momentum tensor evaluated in the vacuum. Taking this correlator into account, backward propagated modes are dissipated at a distance from the horizon $\propto G\kappa$ when measured in a freely falling frame. ($G$ is Newton’s constant and $\kappa$ the surface gravity.) This result can be also obtained by considering light propagation in a stochastic ensemble of metrics whose fluctuations are determined by the above correlator.

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*An earlier version of this work was published in Proceedings[21] in 2002. We postponed submitting it to the arXiv in the hope of improving the evaluation of radiative corrections. We have recently modified the text, corrected some mistakes, and added remarks on this difficult point which still needs work – that hopefully someone will take over.

†E-mail: Renaud.Parentani@th.u-psud.fr
1 Introduction

In his original derivation [1], Hawking considered the propagation of quantum radiation in a fixed background metric, that of a collapsing star. This means that the metric is once for all determined by the energy of the collapsing star. It is therefore unaffected by the quantum processes under examination. In this approximation, the radiation field satisfies a linear equation (in the absence of matter interactions). One then finds that infalling and outgoing field configurations are completely uncorrelated near the black hole horizon. This is explicitized by the fact that the connected part of the two-point correlation function of $T_{uu}$ and $T_{vv}$, the energy-momentum tensor of outgoing and infalling configurations,

$$\langle T_{vv}(x) T_{uu}(x') \rangle_C = \langle T_{vv}(x) T_{uu}(x') \rangle - \langle T_{vv}(x) \rangle \langle T_{uu}(x') \rangle,$$  

vanishes in the (Unruh) vacuum. Nevertheless, Hawking radiation is pair creation. This is perfectly consistent with eq. (1) since the pairs are composed of two outgoing quanta, one of each side of the event horizon. The external ones form the asymptotic flux whereas their partners propagate towards the singularity at $r = 0$. Upon tracing over these inner configurations, one gets an incoherent flux described by a thermal density matrix [3].

From this fixed background description, one may go one step further by performing a mean field approximation, i.e. by including the metric change determined by Einstein’s equations driven by the expectation value $\langle T_{\mu\nu} \rangle$. One then finds that this expectation value is regular [6, 7, 8, 3]. This guarantees that the black hole will adiabatically evaporate while keeping the regularity of the near horizon geometry. This regularity in turn implies that the infalling and outgoing configurations will stay uncorrelated. Therefore, the correlation function (1) still vanishes in the semi-classical treatment.

This adiabatic evolution would provide a reliable starting point for including perturbatively radiative corrections were another feature of black hole physics not present. Namely, the field configurations giving rise to Hawking quanta possess arbitrary high (trans-Planckian) frequencies near the horizon [9, 10, 11, 3]. When measured by free falling (FF) observers at $r$, the frequency of an outgoing quantum of asymptotic energy $\lambda$ grows as

$$\Omega \propto \frac{\lambda}{1 - 2M/r}.$$  

This implies that wave packets centered along the null outgoing geodesic $u$ had FF frequency growing as $\Omega \propto \lambda e^{\kappa u}$ when they emerged from the collapsing star. Unlike processes characterized

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$^1$In this paper we consider only s-waves and neglect the residual potential barrier which partially back-scatter some Hawking quanta thereby inducing some correlations. These however play no role in what follows and shall be ignored. In the next equations, $v$ and $u$ are radial advanced and retarded null coordinates. In the Schwarzschild geometry, they are given by $v = t + r^*$, $u = t - r^*$ where $r^* = r + 2M \ln(r/2M - 1)$ is the tortoise coordinate [2].

$^2$The non-vanishing character of the correlation function $\langle T_{uu} T_{uu} \rangle_C$ across the horizon demonstrates that the partners of Hawking quanta are inside outgoing configurations. An explicite calculation gives [4, 5]

$$\langle T_{uu}(u) T_{uu}(u'_L) \rangle_C \propto |u - u'_L + i\pi/\kappa|^{-4},$$  

where $\kappa u_L = \ln(\kappa U_K)$ is a null coordinate for the inside configurations. $U_K$ is the usual Kruskal retarded time; it vanishes on the horizon, and is positive inside the hole. $\kappa = 1/4M$ is the surface gravity. (It fixes Hawking temperature $T_H = \kappa/2\pi$. We work in Planck units: $c = \hbar = M_{Planck} = 1$.) The smooth maximum of this two-point function for opposite points, i.e. $u = u'_L$ or $U_K = -U'_K$, is a direct consequence of the fact that each pair is indeed composed of two outgoing quanta leaving on either side.
by a typical energy scale, the relation $\Omega \propto \lambda e^{\kappa u}$ shows that black hole evaporation rests, in this scenario, on arbitrary high frequencies. This conclusion drawn from the analysis of wave packets is confirmed by the study of the non-diagonal matrix elements of $T_{\mu\nu}$. These characterize the fluctuations of the flux around its mean value. As shown in [5, 11, 3], unlike the expectation value (the diagonal part) which is regular and of the order of $M^{-4}$, these matrix elements are singular on the horizon, i.e., their Fourier content is characterized by FF frequencies which grow according to eq. (3).

This growth is not an artifact of a bad coordinate choice. Indeed, as emphasized by 't Hooft [12], and explained below in Section 3, gravitational interactions between the configurations giving rise to Hawking quanta and in-falling quanta also grow according to eq. (3). This questions the validity of the semi-classical treatment and the vanishing of eq. (1).

In questioning the validity of the semi-classical description, two issues should be distinguished, see e.g. section 3.7 in [3]. First, there is the question of low frequency $O(\kappa)$ modifications of the asymptotic properties of Hawking radiation, and secondly, that of high frequency modifications of the near horizon physics. Since all thermo-dynamical reasonings indicate that the asymptotic properties (namely thermality governed by $\kappa$ and stationarity) should be preserved, the problem is to reconcile this robustness with the radical change of the near horizon physics which is needed to tame the growth of gravitational interactions. This is not an easy problem: Indeed, the perturbative analysis of near horizon interactions performed in [13] leads to back-reaction effects which grow like $\Omega$ in eq. (3). This threatens the stationarity of the flux and therefore questions the choice of the treatment which is adequate to go beyond the semi-classical approximation.

As a first step towards a full quantum gravitational treatment, inspired by [12]-[18], we proposed[19, 20, 21] a non-perturbative treatment of the interactions occurring in the FF vacuum based on $\langle T_{\mu\nu} T_{\mu\nu} \rangle_C$, the correlation function of in-falling configurations. As discussed in more detail in what follows, this treatment emerges in a large $N$ limit, where $N$ is the number of copies of the quantum field. In physical terms, in this limit, in-falling configurations act as an environment for the outgoing quanta and their gravitational interactions express themselves in terms of a stochastic ensemble of metric fluctuations. The statistical properties of the latter are determined by the fluctuations of the $N$ fields in the FF vacuum. Their main effect is to dissipate the outgoing modes traced backwards towards the horizon at the locus where their FF frequency (3) reaches $1/\bar{\sigma}$, i.e., at $r = 2M + \bar{\sigma}$ for modes of Killing frequency equal to $\kappa$.

In our model, the gravitationally induced length-scale is

$$\bar{\sigma} \propto \frac{\kappa}{M_{Planck}^2} \sqrt{N \ln(\Lambda/\kappa)}.$$  (4)

In the above $\Lambda$ is a high frequency cut-off whose value requires further study to be fixed. It is

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3In section 9 of the review[12], one reads “Any decomposition of Hilbert space in terms of mutually non-interacting field quanta will be hopelessly inadequate in this (near horizon) region.”

4Added comment. When presented in meetings, this result was skeptically received on the basis that $\bar{\sigma}$ is much shorter than the Planck length for macroscopic black holes. Indeed one has $\bar{\sigma} \propto L_{Planck}/GM_{bh}$. However it should be pointed out that $\bar{\sigma}$, which is the proper distance between $2M$ and $2M + \bar{\sigma}$ measured in a FF frame, actually corresponds to a proper distance of the order of $L_{Planck}$ when measured along surfaces of constant (Killing) time. It should be also recalled that $r$ is defined by the square root of the area of 2-spheres/4\pi, and is therefore coordinate invariant. On the contrary, the notion of distance from the horizon depends on the choice of the time slices used. In this sense our effective propagation law does not singles out a preferred frame. This should be compared with the procedure involved when using an a priori given dispersion relation, see [22]-[27]. I am grateful to Ted Jacobson for discussions on this.
important to notice that in spite of these dissipative effects, the gravitational interactions do not significantly affect the asymptotic properties of Hawking radiation. As shown in [18], the asymptotic corrections scale indeed as $(\kappa\bar{\sigma})^2$, hence they are negligible for macroscopic black holes. It should also be noticed that the dissipative effects break the 2D local [9] Lorentz invariance.

An unsolved question concerns the range of validity of our treatment. This is a complicated question whose final answer requires a better understanding of quantum gravity. Let us nevertheless make some remarks. First, this question closely follows that concerning the validity of the semi-classical treatment. Secondly, our analysis indicates that the semi-classical treatment fails before our treatment. ‘Before’ should be understood radially, given the blue shift effect encountered in the backward propagation of outgoing configurations. What emerges is a kind of Russian doll structure in which quantum gravity progressively dominates the physics.

Far away from the hole $(r - 2M \gg 2M)$ one has outgoing thermal (on shell) radiation. In a first intermediate regime $(\bar{\sigma} \ll r - 2M \ll 2M)$ the propagation of outgoing modes is still governed by the d’Alembertian but observers at fixed $r$ and free falling ones perceive quanta differently (in that the locally defined FF vacuum no longer agrees with the Killing (Boulware) vacuum). It is in this regime (well described by the semi-classical treatment) that Hawking radiation gets established, see eqs. (76-83) in [5]. This description based on modes stops to be valid when reaching a Jacobson’s time-like boundary $[10]$, at $r \simeq 2M + \bar{\sigma}$, when outgoing modes get progressively entangled to the infalling configurations, thereby loosing their ‘mode’ quality.

The principle aim of this paper is to analyze this transitory regime. Deeper in $r$, one has some unknown regime governed by Planckian physics. This physics presumably also occurs around us but stays well hidden inside its Planckian husk in the absence of a good microscope.

Below we discuss two aspects. We first explain why we adopted a treatment based on the large $N$ limit, and then we further discuss the relation between this treatment and the dispersive physics involved in acoustic black holes. These two subsections can be skipped in a first reading.

1.1 The choice of the treatment: the large-$N$ limit.

In order to compute/estimate quantum gravitational corrections to Hawking radiation in the absence of a theory of quantum gravity, one should adopt an approximative treatment allowing to compute radiative corrections to some physical quantities. In this paper, we have chosen a statistical treatment based on a large-$N$ limit, where $N$ is the number of copies of the quantum matter field. The reason for this choice are the following.

First, the semi-classical treatment is the leading contribution in the large-$N$ limit. This is most simply understood in a path integral approach [29]: By performing a saddle point approximation in the evaluation the one-loop effective action, one verifies that the location of the saddle is determined by the semi-classical Einstein equations. Hence, the semi-classical

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3In the vicinity of a black hole horizon, free propagation of $s$-waves is governed by a 2D Lorentz (and scale) invariance in the $u,v$ plane, see Section 2. When including radiative corrections, the dressed propagator looses this property. This is similar to the fact that the self-energy of an electron immersed in a thermal bath of photons is not Lorentz invariant either[28]. In both cases, the integrands governing loop corrections are not Lorentz invariant, more on this in Section 5.

4The validity of the semi-classical treatment has been often questioned in rather general terms. However, a significant answer requires to find physical quantities (i.e., matrix elements of observables) which are incorrectly evaluated in this treatment and to propose improved expressions for the same quantities in order to see the discrepancy. We shall provide an explicit exemple in Section 4.
treatment is the mean field approximation (Hartree) in which the \(N\) copies of the radiation field propagate in a self-consistent classical geometry: a solution of Einstein equations driven by \(N\) times \(\langle T_{\mu\nu} \rangle\), the mean value of the energy-momentum tensor of one field. This result can be also understood from a diagrammatic point of view in the following way. One first verifies that the expectation value of any observable can be expanded as a double series in \(G\) and \(N\) in which the power of \(N\) is always smaller or equal to that of \(G\). One then verifies that the semi-classical value of this observable corresponds to the resummed series containing all terms governed only by the one-point function \(\langle T_{\mu\nu} \rangle\). Moreover, all these graphs are 1-particle reducible and their weight is a positive power of \(GN\). This analysis furnishes an alternative proof that the semi-classical treatment is the leading contribution and that it corresponds to a mean field treatment, see the Appendix for more detail.

This diagrammatic analysis naturally leads to inquire about the next series. Upon having first summed up the leading series in powers of \(GN\), one encounters a next series containing positive powers of \(G^2N\). This second series is governed by the two-point function \(\langle T_{\mu\nu}(x)T_{\alpha\beta}(x') \rangle_C\), the ‘specific heat’ of the radiation field. The second reason of having chosen the large \(N\) limit is that non-perturbative effects can be obtained by resumming this second series\(^{[19]}\).

The third reason which has led us to choose a treatment based on a statistical basis arises from the trans-Planckian problem. Indeed, as previously discussed, the unbounded growth of frequencies encountered near the event horizon seems to invalidate perturbative treatments of radiative corrections, see \([13]\) for an exemple of the divergences encountered.

Let us briefly explain what is the nature of the trans-Planckian problem and why radiative corrections induced by quantum gravity might give some important effects when applied to Hawking radiation (or to quantum effects induced by the presence of an event horizon). When studying the origin of Hawking quanta one faces a difficulty which is specific to horizon physics: The configurations giving rise to Hawking quanta are characterized by ultra-high frequencies when measured by infalling observers near the horizon \([30, 9, 12, 5]\). Indeed, in the semi-classical treatment, the use of free fields propagating in a classical background leads to unbounded frequencies as a direct consequence of the structure of the outgoing null geodesics near the horizon. Therefore, any ultra-violet scale which would signal the breakdown of the semi-classical treatment will be inevitably reached.

This reasoning is nicely illustrated by considering sound propagation in an acoustic geometry which possesses a horizon: One first finds that the propagation is dramatically modified with respect to the standard propagation (governed by the d’Alembertian) when the FF frequency \(\Omega\) reaches \(\Omega_c\), the UV scale (the inverse inter-atomic distance). In particular, for all non-linear dispersion relations, one finds that the focussing effectively stops when this new scale is reached. Secondly, this dramatic modification of the near horizon propagation leaves no imprint on the asymptotic properties of Hawking radiation when the inverse inter-atomic distance is well separated \([22, 23]\) from \(\kappa\), the surface gravity of the hole.

Our aim is to show that similar results are obtained when computing non-perturbatively the gravitational effects driven by the connected two-point function \(\langle T_{\mu\nu}T_{\alpha\beta} \rangle_C\). We shall find that the trans-Planckian correlations which existed in the semi-classical treatment are washed out when the \(r - 2M\) reaches \(\bar{\sigma}\), the new length scale which plays the role of the inter-atomic distance. Moreover, as already mentioned, this washing-out mechanism leaves the asymptotic properties of Hawking radiation unaffected: the thermal flux receives corrections which scale like \((\kappa\bar{\sigma})^2\) and which are therefore negligible for large black holes.
1.2 The lesson from acoustic black hole physics.

For the interested reader, we further discuss the relationships between our approach and the physics of sonic black holes. As just explained, the appealing feature of these models is to provide both a simple explanation (in terms of adiabaticity which essentially follows from scale separation $\Omega_c \gg \kappa$) for the robustness of the asymptotic properties of the flux, and a simple physical reason (a modified dispersion relation) which eradicates the ultra-high frequencies. (It should be pointed out that a similar trans-Planckian problem arises in inflationary models when studying the origin of the spectrum of primordial energy density fluctuations [25, 32, 33]. In that case as well, scale separation and regularity of the metric are sufficient conditions to guarantee that the properties of the spectrum are unmodified [34].)

Besides the robustness of the IR properties, the main outcome of these considerations is that a new universality has emerged: for all dispersion relations but the linear one, the blue shifting effect stops. Therefore, the never ending blue shifting effect obtained by using the linear (scale-less and non-dispersive) relation $\Omega = p$ now appears as an isolated and unstable behaviour. Thus, instead of asking:

_is Hawking radiation robust against modifying the dispersion relation?_

we are led to question[35]:

_is $\Omega = p$ robust against radiative corrections?_

or it is simply an artifact of free field theory?

These considerations suggest that quantum gravity should engender a new UV scale when evaluating radiative corrections in a black hole geometry. This new scale would then break the scale invariance of free field propagation and prevent the appearance of trans-Planckian frequencies. To verify this conjecture, one must determine the physical effects induced by the non-linearities engendered by gravitational interactions. When this is done, one can make contact with sound wave physics[37] by determining how phenomenologically describe by an effective linear equation (i.e., a non-trivial dispersion relation) the dissipative/dispersive effects induced by non-linearities.

In this paper we shall implement the second question by computing how the radiative corrections driven by $\langle T_{\nu\nu} T_{\nu\nu} \rangle$ modify the two-point function of out-going configurations. We shall see that $\Omega = p$ is not robust in that it provides a good approximation of the effective propagation law only when the FF frequency $\Omega$ obeys $\Omega \ll 1/\bar{\sigma}$ where $\bar{\sigma}$ is given in eq. (4).
2 The Model

For simplicity, we shall consider only $s$-waves propagating in spherically symmetric space times. For definiteness, the background metric is taken to be that resulting from the collapse of a null shell of mass $M_0$ which propagates along the null ray $v = 0$. Inside the shell, for $v < 0$, the geometry is Minkowski and described by

$$ds^2 = -(1 - \frac{2M(v)}{r})dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

(5)

with $M = 0$. Outside the shell, the metric is Schwarzschild and given by eq. (5) with $M(v) = M_0$. As we shall see, this choice of the collapsing metric will have no influence in what follows since the near horizon vacuum interactions are stationary.

As we see, the near horizon vacuum interactions are stationary.

\[ J^+ \]

**Figure 1: Penrose diagram of the background geometry.** The light-like infalling shell propagates along $v = 0$. The other continuous line emerging from $J^- = v = v_H$, the radial light ray which forms the event horizon after having bounced on $r = 0$. The dashed line represents a characteristic of the configurations $\phi_-$ which are responsible for Hawking quanta. The dotted line represents a partner’s characteristic. For quanta reaching $J^+$ at late $u$, both of these characteristics are extremely close to $v_H$, see eq. (7). The configurations $\phi_+$ have support for $v > 0$ and are always infalling. In this paper, we study the interactions between $\phi_-$ and $\phi_+$ which occur outside the star, in the near horizon empty region, when the (initial) state of $\phi_+$ and $\phi_-$ is vacuum.

To identify the degrees of freedom involved in these interactions, we first analyze the global properties of radial modes, when working in the geometric optic approximation, i.e. when working with $\partial_u \partial_v \phi = 0$. (In the exact d’Alembertian, see eq. (10), there is a potential around $r = 3M$ which induces partial reflection, a phenomenon irrelevant for our purposes.) The ingoing massless waves fall into two classes according to their support on $J^-$, the light-like past infinity. The waves in the first class have support only for $v < 0$, inside the shell, and will be noted $\phi_-$. They propagate inward in the flat geometry till $r = 0$ where they bounce off and become outgoing configurations, see Figure 1. The relationship between the value of $u$ of the geodesic which originates from $v$ on $J^-$ is \[ 18 \]:

$$V(u) = -4M(1 + e^{-ku}),$$

(6)
The first class is thus divided in two subsectors: For $v < -4M$, the reflected waves cross the in-falling shell with $r > 2M$ and reach the asymptotic region whereas those for $0 > v > -4M$ cross the shell with $r < 2M$ and propagate in the trapped region till the singularity. The separating light ray $v_H = -4M$ becomes the future horizon $u = \infty$ after bouncing off at $r = 0$. The configurations which form the second class live outside the shell, have support only for $v > 0$ and are noted $\phi_+$. They propagate in the static Schwarzschild geometry, are always in-falling and cross the horizon towards the singularity.

In Hawking’s derivation of black hole radiation, the field operator obeys the d’Alembert equation. Hence the above classical properties apply: The configurations for $v < v_H$ give rise to the asymptotic quanta, those for $v_H < v < 0$ to their partners \[5\] whereas the configurations described by $\phi_+$ play no role in the asymptotic radiation. This follows from the asymptotic ($\kappa u \gg 1$) behaviour of the relation $V(u)$

$$V(u) - v_H \propto e^{-\kappa u}. \quad (7)$$

As shown in \[1\], this exponential is responsible for the thermal radiation at temperature $\kappa/2\pi$. It also shows that Hawking quanta emerge from trans-Planckian frequencies on $\mathcal{J}^-$ since $\omega dV = \lambda du$ (where $\omega = i\partial_u$ on $\mathcal{J}^-$ and $\lambda = i\partial_u$ on $\mathcal{J}^+$) gives $\omega \propto \lambda e^{\kappa u}$. Finally it fixes the correlations between the asymptotic quanta and their partners, see eq. \[2\]. These follow from the fact that, on $\mathcal{J}^-$ and in the vacuum, the rescaled field $\phi = \sqrt{4\pi r^2} \chi$ (where $\chi$ is the 4D $s$-wave) satisfies

$$\langle \phi(v) \phi(v') \rangle = \int_0^\infty \frac{d\omega}{4\pi \omega} e^{-i\omega(v-v')} = -\frac{1}{4\pi} \ln(v-v' - i\epsilon) + \text{constant}. \quad (8)$$

Since this equation is valid for all $v, v'$ one might think that there also exist correlations between $\phi_-$ and $\phi_+$. However, for late Hawking quanta, they effectively vanish since these quanta and their partners emerge from configurations which are localized extremely close to $v_H$ as indicated in eq. \[7\]. For a description of the other properties of Hawking radiation, we refer to the review \[3\].

In brief, in the absence of gravitational interactions, $\phi_-$ and $\phi_+$ are effectively two independent fields. By independent we mean that by sending quanta described by wave packets built with $\phi_+$, there is no induced emission of Hawking quanta. Indeed, in order to get induced emission \[31\] at time $u$, one should send $\phi_-$ quanta localized close to $v_H$ as indicated in eq. \[7\] and correspondingly characterized by high frequencies $\omega \propto \lambda e^{\kappa u} \gg \kappa$.

Let us now analyze more closely how these properties translate in Fock space. When evaluated in the background eq. \[5\], the action of $\phi$ is

$$S_\phi^\phi = -\int dv \, dr \left[ \partial_u \phi \partial_v \phi + \left(1 - \frac{2M}{r} \right) \frac{(\partial_r \phi)^2}{2} \right], \quad (9)$$

with $M(v) = 0$ for $v < 0$ and $M(v) = M_0$ for $v > 0$. Being interested in the near horizon physics, we have dropped the potential term of $s$-waves, $(2M_0/r^3)\phi^2$, since it does not affect the near horizon propagation. This can be seen by using the double null coordinate system $u = v - 2r^*, v$. Using them, the 4D d’Alembertian reads

$$\left[ \partial_u \partial_v - \left(1 - \frac{2M_0}{r} \right) \frac{l(l+1)}{r^2} + \frac{2M_0}{r^3} \right] \phi_l = 0 \quad (10)$$

where $\phi_l$ is the rescaled mode of angular momentum $l$. Thus, as emphasized in \[14\], the propagation of waves (at fixed angular momentum and even for an arbitrary mass) effectively obeys a
2D conformal invariance in the near horizon geometry. This is confirmed by the fact that, classically, the trace of 2D part of $T_{\mu\nu}$ vanishes independently of the equations of motion. Thus, in our model for s-waves, $T_{\mu\nu}$ has only two q-number components, $T_{vw} = (\partial_v \phi)^2$ and $T_{uu} = (\partial_u \phi)^2$.

The 2D conformal invariance also implies that the Fock space is composed of tensorial products of in-falling states (on which $\phi_+$ acts) and outgoing states. In a Schroedinger language this means that an initially factorized state (i.e., $|\Psi\rangle = |\Psi_+\rangle \otimes |\Psi_-\rangle$) remains factorized when its evolution is governed by eq. (9). This factorizability explains the absence of induced emission when adding $\phi_+$ quanta. In a Heisenberg language, it tells us that any matrix element of $\phi$ is a combination of matrix elements of $\phi_-$ and $\phi_+$ evaluated separately. This implies in particular that the connected part of the two-point correlation eq. (11) identically vanishes for all factorized states. This applies to the “Unruh” vacuum, the state describing Hawking radiation. Physically, the vanishing of eq. (11) means that the fluctuations of $T_{vw}$ and $T_{uu}$ around their mean values are completely uncorrelated. It should be clear that this absence of quantum correlations is precisely what is contested by t’Hooft, see footnote 3.

Finally, we mention that, in spite of this absence of correlations, the mean value of $T_{vw}$ and $T_{uu}$ are related to each other by energy conservation through the 2D trace anomaly [36]. However, this third component of $T_{\mu\nu}$ does not fluctuate: it is a c-number. Hence it cannot play any role in the gravitational interactions between the field operators $\phi_-$ and $\phi_+$.

3 The gravitational interactions between $\phi_-$ and $\phi_+$

The aim of this Section is to obtain the dominant part of the action governing the gravitational interactions between $\phi_-$ and $\phi_+$. In the next Sections, we shall study the consequences of these interactions with particular emphasis on the correlations they induce.

The generating functional governing our matter-gravity system is

$$Z = \int D\phi \ Dh \ e^{i\left[S^{\phi}_{g+h} + S_{h,g}\right]}.$$  

$Z$ is the change of the metric with respect to the background $g$ discussed above and $S_{h,g}$ is the Einstein-Hilbert action. (We only consider metric fluctuations in the Ricci flat region outside the infalling matter forming the hole.) $S^{\phi}_{g+h}$ is the action of $\phi$ propagating in the fluctuating geometry $g + h$.

When the metric fluctuations are spherically symmetric, $h$ can be characterized by two functions: $\nu, \mu$. Moreover both are completely determined by the energy-momentum tensor of $\phi$. This is like the longitudinal part of the electro-magnetic field which is constrained to follow charge density fluctuations, by Gauss’ law. The line element in the fluctuating metric can be

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3 This invariance leads directly to the trans-Planckian problem: the steady production rate of outgoing quanta arises from an integral over in-frequencies $\omega$ whose measure is that of a 2D massless field. Explicitly one obtains that the thermal distribution is multiplied by $d\omega/\omega = \kappa du$ since $\omega \propto e^{\kappa u}$, see [3, 13].
written as $[18]$

$$
\text{d} s^2 = e^\psi \left[ - \left( 1 - \frac{2M}{r} \right) \text{d} v^2 + 2 \text{d} v \text{d} r \right] + r^2 \text{d} \Omega_2^2.
$$

(13)

where $\tilde{M} = M_0 + \mu(v, r)$ for $v > 0$.

In this new metric, the matter action is the same as in equation $[9]$

$$
S_{g+h}^\phi = - \int \text{d} v \text{d} r \left[ \partial_v \phi \partial_r \phi + \left( 1 - \frac{2\tilde{M}}{r} \right) \frac{(\partial_v \phi)^2}{2} \right].
$$

(14)

The new mass function $\tilde{M}$ incorporates the only relevant metric change $\mu$. Indeed, $S_{g+h}^\phi$ is independent of $\psi$, thereby recovering the 2D conformal invariance mentioned earlier.

Our aim is to work out the first order corrections due to the gravitational interactions between $\phi_-$ and $\phi_+$. To this end only quadratic terms in $h$ should be kept in $S_{h,g}$. The Gaussian integration over $h$ can be performed (this is equivalent to solve the linearized Einstein’s equations). It gives rise to a self-interacting field theory described by

$$
Z = \int \mathcal{D} \phi \, e^{i S_{g+h}^\phi + i S_{\text{int}}}.
$$

(15)

By construction $S_{\text{int}}$ is a non-local quadratic form of the energy-momentum tensor of $\phi$.

To identify the relevant part of $S_{\text{int}}$, we first recall that $T_{\mu\nu}$ has only two fluctuating components, thanks to the 2D conformal invariance. Thus, in a perturbative treatment (such as the interacting picture) one obtains two types of interaction terms only. First one has self-interaction terms depending on $\phi_-$ or $\phi_+$ separately. These terms do not destroy the factorisability of the theory and will not be considered in what follows. Secondly, one has a cross term coupling $\phi_-$ to $\phi_+$. This term will inevitably break the factorisability of the theory into the $\pm$ sectors. Therefore, the two-point function will no longer vanish. Let us analyze this term in more detail.

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8 This line element differs from that used by Bardeen:

$$
\text{d} s^2 = e^\psi \left[ -e^\psi \left( 1 - \frac{2M_0 + 2\mu_B}{r} \right) \text{d} v^2 + 2 \text{d} v \text{d} r \right] + r^2 \text{d} \Omega_2^2.
$$

The $\psi$ function is the same whereas, to first order in $\psi$ and $\mu_B$, the mass fluctuation $\mu = \mu_B - \psi(r - 2M_0)/2$. The usefulness of our choice is that $\psi$ no longer affects the null geodesics. When expressing $T_{\mu\nu}$ in terms of the null fluxes $T_{v v}, T_{u u}$, Einstein’s equations give

$$
\partial_v \mu_B = T_{v v} - T_{u u}, \quad \partial_r \psi = 4T_{u u}/(r - 2M).
$$

(12)

9 In the $t, r$ coordinate system, i.e., when $g_{tt} = 0$, $S_{\text{int}}$ is given by a linearized version (see equation (90) in [5]) of the BCMN Hamiltonian.

10 The validity of this simplification (also adopted in [12, 14, 15]) requires further analysis. On-shell, the $\phi_+$ contribution to $S_{\text{int}}$ vanishes. This can be understood from the fact that the Vaidya metric (2) is an exact solution for any classical infalling massless flux $T_{v v}(v)$. The $\phi_- \phi_-$ contribution to $S_{\text{int}}$ is more tricky to handle in the advanced coordinates $v, r$. The reason is that infalling geodesics are affected by the presence of an outgoing flux $T_{u u}$ (as clearly seen when using the coordinates $u, r$). This modification translates in $v, r$ into a deformation of the description of outgoing geodesics $u = u(v, r)$ and it is this effect which is responsible for the $\phi_- \phi_-$ contribution to $S_{\text{int}}$. Let us finally notice that a non-perturbative treatment of the self-interactions of $\phi_-$ has been developed in [39]. It leads to small effects $O(\kappa/M)$ and induces no damping of the waves when approaching the horizon.
Since infalling configurations obey $\partial_r \phi_+ = 0$ even in the presence of gravitational interactions, the cross term coupling $\phi_-$ to $\phi_+$ is, see eq. (14),

$$S_{\text{int}} = G \int_0^\infty dr \int_0^\infty dv \frac{\mu_+(v)}{r} (\partial_r \phi_-)^2,$$

$$= G \int_0^\infty dr \int_0^\infty dv \frac{\mu_+(v)}{r} (du/dr)^2 T_{uu}.$$

(16)

where $G$ is Newton’s constant. We have re-introduced it in the front of $\mu$ to read more easily the order of the interactions between $\phi_-$ and $\phi_+$ in the forthcoming equations. $\mu_+(v)$ is the mass fluctuation driven by $\phi_+$. Einstein’s equations constrain it to be

$$\mu_+(v) = \int_0^v dv' T_{vv}(v') = \int_0^v dv' (\partial_{v'} \phi_+)^2.$$

(17)

(The reader might be surprised by the fact that we are using on-shell fields $\phi_+ \pm$ in $S_{\text{int}}$, i.e. that we have used the equations of motion. In principle indeed, only the off-shell field $\phi$ should be used in the action. However, when calculating perturbatively lowest order corrections in $G$, this amounts to use eq. (16) as it stands.)

We are now in position to show that the gravitational interactions between $\phi_+$ and $\phi_-$ diverge on the horizon. To this end, let us consider two classical fluxes described respectively by $T_{vv} = \omega \delta(v - v_0)$ and $T_{uu} = \lambda \delta(u - u_0)$. $\omega$ and $\lambda$ are the asymptotic energies measured on $\mathcal{J}^-$ and $\mathcal{J}^+$ respectively, and $v_0$ and $u_0$ are such that the two spherical shells meet at $r_0$ in the near horizon geometry, for $r_0 - 2M \ll 2M$. In this case, using $r_0 - 2M \simeq 2Me^{\kappa(v_0 - u_0)}$, one obtains

$$S_{\text{int}} \simeq 4G \frac{\omega \lambda}{r_0/2M - 1}.$$

(18)

The action $S_{\text{int}}$ diverges as $r_0 \rightarrow 2M$ like the FF frequency $\Omega$ did in (3). The difference with (3) is that $S_{\text{int}}$ is a scalar. Hence the divergence in eq. (18) is coordinate (gauge) invariant.

To get an estimate of where the gravitational interactions become strong, i.e., can no longer be ignored, let us consider two shells whose asymptotic energy is Hawking temperature, i.e. $\omega = \lambda = \kappa$. The condition $S_{\text{int}} = 1$ is reached for

$$r_0/2M - 1 = 4G \kappa^2.$$

(19)

The proper distance ($r_0 - 2M \simeq 1/M$) is much smaller than Planck length, see footnote 4. This simple estimate will be recovered in Section 6 when considering radiative corrections in the vacuum. We should emphasize this last point: even though our approach closely follows that of [12] (it can be considered as an $s$-wave reduction of it) we shall not study the interactions between $\phi_+$ and $\phi_-$ quanta. Rather we shall focus on the residual interactions when the state of $\phi_+$ is vacuum. For earlier attempts in this direction, we refer to [14, 15]. Before analyzing these second quantized effects, it is instructive to solve two preparatory exercises with on-shell fluxes.

In the first we shall show that $S_{\text{int}}$ acts only as a shift operator of the asymptotic value of $u$. In spite of this simplicity, in the second exercise, we show that $S_{\text{int}}$ nevertheless engenders an entanglement which prevents the factorisability of the states into $\pm$ sectors. This provides an explicit example of a quantum effect induced by eq. (16) which cannot be described in the semi-classical treatment.
4 Non-vacuum gravitational effects

4.1 Classical interactions and shifts in \( u \)

We first consider the following problem: Given two classical field configurations: \( \phi_0^+ \) specified on \( J^- \) for \( v > 0 \) and \( \phi_0^- \) on \( J^+ \), what is the value of the field amplitude \( \phi \) near the horizon when taking into account the gravitational interactions of eq. (16)?

Because of the 2D conformal invariance, \( \phi \) still decomposes as \( \phi_+ + \phi_- \). Then, since \( \partial_v \phi_+ = 0 \) is exact in our gauge wherein \( v \) stays light-like, \( \phi_+(v) = \phi_0^+(v) \) to all orders in \( G \). Thus the infalling flux of energy is unaffected by the energy carried by \( \phi_- \) and it is given by its initial value on \( J^- \): \( T_{\nu\nu} = (\partial_v \phi_0^+)^2 \). Hence, \( \mu_+ \) of eq. (17) acts as a given metric change in the equation of motion of \( \phi_- \):

\[
\left[ 2\partial_v + \left( 1 - \frac{2M_0 + 2G\mu_+(v)}{r} \right) \partial_r \right] \phi_- = 0. \tag{20}
\]

Since this equation is linear in \( \phi_- \) and first order in the space-time derivatives, its exact solution is

\[
\phi_-(v, r) = \phi_0^-(u_\mu(v, r)), \tag{21}
\]

where \( u_\mu(v, r) \) gives (in the coordinate system of eq. (13)) the outgoing null geodesic in the modified metric characterized by \( M_0 + G \mu_+(v) \).

To determine the function \( u_\mu(v, r) \), one notices that it also obeys eq. (20) with the (final) boundary condition that it converges to the un-modified value \( u_0(v, r) = v - 2r \) for \( r \to \infty \). Hence, to first order in \( G \), the change \( \delta u = u_\mu - u_0 \) is determined by a non-homogeneous equation. Using the fact that \( 2\partial_v + (1 - 2M_0/r)\partial_r \) defines \( 2\partial_v|_{u_0} \) (by definition of \( u_0(v, r) = \text{constant} \)), \( \delta u \) obeys

\[
\partial_v|_{u_0} \delta u = G \frac{\mu_+}{r} \partial_r|_{u_0} u_0. \tag{22}
\]

The solution is

\[
\delta u(v)|_{u_0} = G \int_v^\infty dv' \frac{2\mu_+(v')}{r(v')|_{u_0} - 2M_0}, \tag{23}
\]

where \( r(v)|_{u_0} \) is obtained by inverting \( u_0(v, r) = v - 2r^* \). As in the action (16) the integral in eq. (23) is dominated by the near horizon region when the denominator \( r - 2M_0 \ll 2M_0 \). A tiny infalling energy flux \( \mu_+ \ll M \) can therefore induce an arbitrary large change of \( u \).

The lesson we got from eq. (21) is that the eikonal approximation is exact: the scattered value of the field amplitude is given by its asymptotic value evaluated along the modified characteristic \( u_\mu(v, r) \). Thus, classically, the gravitational interactions encoded in eq. (16) only induce a shift of the argument of field. They do not induce non trivial non-linearities in the field amplitude in that \( \mu_+ \) can be computed irrespectively of the value of \( \phi_- \). The origin of this miracle is the 2D conformal invariance.

4.2 Entanglement between \( \phi_- \) and \( \phi_+ \)

In spite of this absence of non-linearities in field amplitude, we shall now prove that the quantum evolution governed by the action \( S_0^\phi + S_{int} \) dynamically engenders entanglement between the otherwise uncorrelated \( \pm \) sectors. To this end, we consider the evolution of an initially factorized wave function

\[
|\Psi^m\rangle = |\Psi_+^m\rangle \otimes |\Psi_-^m\rangle. \tag{24}
\]
The infalling part $|\Psi^+_{in}\rangle$ is specified on $\mathcal{J}^-$ for $v > 0$. To clearly exhibit the entanglement, we choose $|\Psi^+_{in}\rangle$ to be a superposition of two well defined and separated states:

$$|\Psi^+_{in}\rangle = A |\Psi^+_{in,a}\rangle + B |\Psi^+_{in,b}\rangle.$$  \hfill (25)

The two kets are normalized and orthogonal to each other: $\langle \Psi^+_{in,i} | \Psi^+_{in,j} \rangle = \delta_{ij}$. Thus $A, B$ are probability amplitudes obeying $|A|^2 + |B|^2 = 1$. By well defined and separated we mean that the two fluxes associated with each component, $\langle T^a_{vw} \rangle \equiv \langle \Psi^+_{in,a} | T^a_{vw} | \Psi^+_{in,a} \rangle$ with $i = a, b$, are well localized in $v$ and separated from each other.

The other piece of the initial ket, $|\Psi^{iS}_{in}\rangle$, is specified on $v < 0$. After reflection on $r = 0$, it determines the state of outgoing configurations. For the moment we do not need to further specify its quantum state.

Having specified the initial state, we study the quantum dynamics (in the interacting picture). The evolution operator $e^{iS_{int}}$ acting on the initial state $|\Psi^{iS}_{in}\rangle$ gives two uncorrelated evolutions weighted by $A$ and $B$:

$$|\Psi\rangle \equiv e^{iS_{int}}|\Psi^{iS}_{in}\rangle = e^{iS_{int}} \left( A |\Psi^+_{in,a}\rangle + B |\Psi^+_{in,b}\rangle \right) \otimes |\Psi^-_{in}\rangle,$$

$$= A e^{iS_{int}} \left( |\Psi^+_{in,a}\rangle \otimes |\Psi^-_{in}\rangle \right) + B e^{iS_{int}} \left( |\Psi^+_{in,b}\rangle \otimes |\Psi^-_{in}\rangle \right).$$ \hfill (26)

One should thus study each piece separately. This is nothing but the expression of the superposition principle.

In each state, the equation $\partial_\ell \phi_+ = 0$, now viewed as an Heisenberg equation, tells us that the evolution in the $+$ sector is trivial, as in classical terms. To study the evolution of the $-$ sector, we perform the approximation which consists in neglecting the fluctuations of $T_{vw}$ in each infalling state $|\Psi^+_{in,i}\rangle$. In this approximation, the evolution operator $e^{iS_{int}}$ becomes a c-number for infalling configurations and acts only on outgoing configurations. One thus has

$$|\Psi\rangle = A \left( |\Psi^+_{in,a}\rangle \otimes e^{iS_{int}^a} |\Psi^-_{in}\rangle \right) + B \left( |\Psi^+_{in,b}\rangle \otimes e^{iS_{int}^b} |\Psi^-_{in}\rangle \right).$$ \hfill (27)

In the $A$-weighted state, the operator $e^{iS_{int}^a}$ governs the propagation of $\phi_-$ in the $a$-modified metric characterized by the mass function $M_a = M_0 + G \mu^a_+$ with

$$\mu^a_+(v) = \int_0^v dv' \langle \Psi^+_{in,a} | \hat{T}^a_{vw}(v') |\Psi^+_{in,a}\rangle,$$ \hfill (28)

whereas in the $B$ state, one finds the $b$-modified metric characterized by $\mu^b_+(v)$. The evolution in each case is thus governed by the time-ordered product of "its" evolution operator acting on same initial $|\Psi_{in}\rangle$. Explicitly, the two Hamiltonians acting on $|\Psi^+_{in}\rangle$ are given by eq. (16) with the corresponding the c-number metric changes $\mu^+_{ij}, i = a, b$. In the above treatment, we took into account the first moment of $T_{vw}$, $\langle T_{vw} \rangle$, the mean value in each state. We indeed neglected the higher moments which govern the fluctuations in each state. In doing so, we accounted for the fluctuations of $T_{vw}$ which are due to the fact that the infalling state is a superposition, but only those. This will be clarified below.

The entanglement induced by $S_{int}$ acts, as usual, as a measurement. For the interested reader we recommend the very instructive reading of Chapter IV. in [40]. Consider, as in that chapter, the Stern-Gerlach experiment wherein the center of mass motion of an electron in a magnetic field is governed by its spin projection along that field. The mapping from that situation to
the present one is as follows. The two kets representing the spin projections are here played by the two infalling states $|\Psi_{+}^{m_{1},1}\rangle$. The center of mass wave function is played by the outgoing wave function $|\Psi_{-}\rangle$ and the interaction hamiltonian is $S_{int}$ of eq. (16). The analogy works quite well when the initial outgoing wave function $|\Psi_{0}\rangle$ also describes a localized flux. Then, its “image” on $\mathcal{J}^{+}$ would be either a spot at $u_{0} + \delta u_{a}$ with probability $|A|^{2}$, or one at $u_{0} + \delta u_{b}$ with probability $1 - |A|^{2}$. The location $u_{0}$ is that of the spot when the gravitational interactions are ignored, whereas the values of the shifts $\delta u_{a}, \delta u_{b}$ are given by eq. (23), evaluated along the unperturbed geodesic $u_{0}$, fed with the mass changes $\mu_{a}^{2}$ and $\mu_{b}^{2}$ respectively, and with the lower value $v$ set to 0.

This quantum result should be compared with what would have been obtained by applying the semi-classical treatment to the entire wave function (rather than to each sector separately). In that case, the change in the common location is driven the mean mass change

$$\bar{\mu}_{\pm}(v) = |A|^{2} \mu_{a}(v) + |B|^{2} \mu_{b}(v). \quad (29)$$

Therefore, the semi-classical treatment incorrectly predicts a single spot on $\mathcal{J}^{+}$ located at the “mean” position $u_{0} + \delta u$ with $\delta u = |A|^{2} \delta u_{a} + |B|^{2} \delta u_{b}$.

In the above example, the validity of the semi-classical treatment rests on the possibility of neglecting the fluctuations of the $\delta u_{-}$-operator-valued shift $\bar{\delta} u$. The dispersion about the mean quantifies their importance. One finds

$$\langle \delta u_{a} \delta u_{a} \rangle_{C} = \langle (\delta u_{a} - \langle \delta u_{a} \rangle)^{2} = \langle \delta u_{a} \rangle^{2} - \langle \delta u_{a} \rangle^{2},$$

$$= |AB|^{2} \langle \delta u_{a}(u_{0}) - \delta u_{a}(u_{0}) \rangle^{2},$$

$$= 4G^{2} \int_{0}^{\infty} \int_{0}^{\infty} dv_{1} dv_{2} \frac{\langle \bar{\mu}_{\pm}(v_{1}) \bar{\mu}_{\pm}(v_{2}) \rangle_{C}}{(r(v_{1})|u_{0} - 2M_{0})(r(v_{2})|u_{0} - 2M_{0})}. \quad (30)$$

To get the second line we have used the results of the above two paragraphs. From eq. (30) one verifies that the semi-classical treatment is valid iff

$$\frac{\langle \delta u_{a} \delta u_{a} \rangle_{C}}{\langle \delta u_{a} \rangle^{2}} = |AB|^{2} \frac{\langle \delta u_{a} - \delta u_{a} \rangle^{2}}{(\delta u)^{2}} \ll 1. \quad (32)$$

It is negligible either when $|AB|^{2} \to 0$, that is, when one of the infalling flux is rarely found, or when the two shifts in $u$ are close to their mean $^{11}$

To get eq. (31) we simply have used the definition of $\delta u$, eq. (23), and that of connected correlation functions, see (1). Eq. (31) displays the relation between the dispersion of $\delta u$ and the correlation function of the metric fluctuation $\mu(v)$ in the quantum state one is dealing with, here given in eq. (25).$^{12}$ Because of the horizon, the importance of the

$^{11}$We refer to [50] [53] for a treatment of the solutions of the Wheeler-deWitt equation in which the gravitational back-reaction is computed for each component of the matter wave function separately. This radically differs from the usual mean field treatment in which the gravitational back-reaction is determined at once.[50] Only the first treatment respects the superposition principle. Indeed the back-reaction in the second case contains the probabilities to find each component, exactly like in $\delta u = |A|^{2} \delta u_{a} + |B|^{2} \delta u_{b}$. Hence the limitations of the validity of the second (the semi-classical treatment) are not intrinsic to the gravitational dynamics.

$^{12}$We remind the reader that to get [53] we have neglected the fluctuations of $\mu$ in each infalling state, i.e. only the dispersion induced by the $A-B$ superposition has so far been taken into account. The aim of the next Section is to incorporate the inherent fluctuations which are present in any infalling state including the vacuum. This explains the title of the present Section: Non-vacuum gravitational effects.
dispersion is determined by two things. On the one hand, the fluctuations of $\delta u$ are driven by the fluctuations of the mass function $\mu = \int dv \hat{T}_{vv}$ which are themselves determined by those of the infalling flux $\hat{T}_{vv}$, given the state of infalling configurations. The connected correlation function $\langle T_{vv} T_{vv} \rangle_C$ quantifies their dispersion. On the other hand, the mass fluctuations are amplified by the denominators in eq. (31). Therefore, even if the relative importance of $\langle T_{vv} T_{vv} \rangle_C$ with respect to the square mean $\langle T_{vv} \rangle^2$ is small, the semi-classical treatment can give incorrect predictions when the fluctuations are sufficiently amplified by the denominators. Instead, in the absence of amplification, the mean theory provides a reliable approximation, unless of course if $\mu_a(v) - \mu_b(v)/M \simeq 1$, but in this case the $a, b$ branches completely decouple/decohere, and one should consider each possibility separately.

In brief, the crucial points we have reached are the following. First, unlike what one encounters in usual circumstances, i.e., without an event horizon, tiny fluctuations of $T_{vv}$ might give rise to large shifts in $u$ because of the amplification due to the growth of the gravitational interactions when the configurations meet near an event horizon. Second, these fluctuations break the factorizability of the theory into the $\phi^+, \phi^-$ sectors. Third, the quantity which governs the validity of the semi-classical scenario is $\langle T_{vv} T_{vv} \rangle_C$, the connected two point function of $T_{vv}$.

The aim of Section 5 is to extend the analysis of eq. (31) when the infalling configurations are in the vacuum, i.e. in a coherent state as opposed to a superposition as in (25). In that Section, the (inherent, minimal, and divergent) vacuum fluctuations will be the only contribution to $\langle T_{vv} T_{vv} \rangle_C$.

### 4.3 Relationship with former treatments

Before considering vacuum effects, it is also interesting to relate eq. (15) to the former treatments of black hole evaporation discussed in the literature: Hawking’s approach [1] and the semi-classical treatment.

Hawking’s approach formulated in a fixed geometry is recovered by putting $G\mu_+ = 0$ in eq. (15). Then $Z$ factorizes as $Z_+ \otimes Z_-$ (when ignoring the trace anomaly) and $\phi_-$ is a free outgoing field propagating in the background geometry $g$. Thus $\phi_+$ drops out from all matrix elements built with the operator $\phi_-$. It should be emphasized that the trans-Planckian problem encountered in Hawking’s approach directly follows from this factorisability. Indeed it is the absence of gravitational coupling between the $+ \text{ and } -$ sectors which permits the unbounded growth of frequencies when probing, near the horizon, configurations specified on $J^+ [5, 18]$.

The semi-classical treatment [6, 7, 8] can be obtained from the path integral formalism eq. (11) by first integrating over $\phi$ at fixed $h$ and then searching for the classical extremum of $h$. In this approach, by construction, the fluctuations of $h$ and $T_{\mu \nu}$ are neglected. Thus the near horizon propagation of $\phi$ is governed by a single but now self-consistent metric governed by mean $\langle \mu_+(v) \rangle$. This mean evolution characterizes by the shrinking of the horizon area according to

$$\frac{d \langle \mu_+(v) \rangle}{dv} = \langle T_{vv} \rangle |_{r = r_{\text{horizon}} = 2M}.$$  \hspace{1cm} (33)

When working in the vacuum eq. (3), the (properly subtracted [3]) expectation of $T_{vv}$ is

$$\langle T_{vv}(v) \rangle |_{r = 2M(v)} = -\frac{\pi}{12} \left( \frac{\kappa(v)}{2\pi} \right)^2 ,$$  \hspace{1cm} (34)

where $\kappa(v) = 1/4M(v)$ with $M(v) = M_0 + G(\mu_+(v))$. This flux has the opposite value of a 2D thermal flux. The only change with respect to the fixed background approach of Hawking
is the replacement of $M_0$ by $M(v)$. Thus the propagation of out-going configurations is hardly affected by the evaporation as long as it is slow, i.e., as long as $M(v) \gg M_{\text{Planck}}$.

Therefore, in the semi-classical scenario, the trans-Planckian problem stays as in Hawking’s approach: The coupling between $\phi_-$ and the mean change $\langle \mu_+ \rangle$ is incapable to provide a taming mechanism since it does not open new interacting channels. To solve this problem clearly requires to take into account the fluctuating character of the interactions between $\phi_-$ and $\phi_+$, i.e., the possibility of entangling their wave functions, as in the quantum mechanical exercise presented above.

5 Modified two-point function

Our aim is to determine how the (dressed) two-point function of $\phi_-$

$$G_-(x_2;x_1) = \frac{\int D\phi_- (x_2) \phi_- (x_1) e^{iS_g+iS_{\text{int}}}}{Z},$$

where $Z$ is given in eq. (15), is affected by the gravitational interactions encoded in $S_{\text{int}}$ when the infalling configurations are in their vacuum state.

To evaluate eq. (35) beyond the semi-classical treatment, one should adopt some rules to cope with the UV divergences. Even at one loop, there are several inequivalent graphs which result from the various Wick contractions. To extract consistently the contribution of the simplest ones, we propose to consider $N$ copies of $\phi$. The calculation of eq. (35) can then be achieved in two different (but equivalent) approaches. The first consists integrating first over the $N-1 \simeq N$ spectator copies not appearing in the numerator in so as to determine the (linear term in $N$ of the) influence functional (IF) governing the effective dynamics of $\phi$. The other consists in developing $e^{iS_{\text{int}}}$ in both integrands of eq. (35) in powers of $S_{\text{int}}$ so as to engender the (connected) graphs governing the radiative corrections. Then in the large-$N$ limit the graphs can be classified into infinite series according to the relative power between $G$ and $N$, see the Appendix for the details. The leading series (in $GN$) reproduces the semi-classical treatment. The next series of graphs, those weighted by powers of $G^2 N$, all correspond to the graphs engendered by the term in the influence functional which is linear in $G^2 N$.

The usefulness of the IF approach is that non-linear (in $G^2 N$) modifications of the propagation of outgoing configurations are easily taken into account through this influence functional. The same results can of course be reached from the diagrams approach at the cost of re-summing the corresponding infinite subsets of graphs. It is in the identification of these infinite subsets that the large $N$ limit finds its justification. For a schematic description of these diagrammatic aspects, we refer to the Appendix. In what follows, we shall pursue with the IF approach.

When computing the lowest order corrections to the effective action in eq. (35), we can use eq. (8), the ‘free’ propagator of $\phi_+$. This approximation concerning degrees of freedom not directly involved in the matrix elements (i.e., which factorized out in the absence of interactions) is a common procedure both in quantum field theory where it gives the vacuum contribution, see chapter 9 in [41], and in statistical mechanics (e.g., the polaron, chapter 11). In our case, in this approximation, the IF gives rise to a non-local action which is a sum of terms containing

\[^{13}\] In would also be interesting to determine if the higher order terms in powers of $G$ will become operator-valued [14] in $\phi_-$, thereby obtaining a situation analogous to that of transition amplitudes when enlarging the phase space so as to take into account recoil effects [17].
\[(\partial_r \phi_-)^2\] and kernels given by the Wick contractions of \(T_{vv}\) evaluated with eq. (8). To order \(G^2N\), one obtains\(^{14}\)

\[S_{IF} = i\langle S_{int}S_{int} \rangle_+ = iG^2N \int d^2x \int d^2x' (rr')^{-1} (\partial_r \phi_-)^2 \langle \mu_+(v)\mu_+(v') \rangle (\partial_r \phi_-)^2, \quad (36)\]

where \(\langle \ \rangle_+\) means that the expectation value applies to \(\phi_+\) only. Using eq. (8), the connected two-point function is

\[\langle T_{vv}(v) T_{vv}(v') \rangle_C = \frac{1}{16\pi^2 (v-v'-i\epsilon)^4}. \quad (37)\]

Then, eq. (17) gives

\[\langle \mu_+(v)\mu_+(v') \rangle = \frac{1}{96\pi^2 (v-v'-i\epsilon)^2} \int_0^\infty d\omega \omega \exp[-i\omega(v-v')]. \quad (38)\]

This equation gives the mean fluctuations driven by one \(\phi_+\) field in the unperturbed \((G = 0)\) vacuum state, see \[17\] for a analysis of the 4D two-point function of induced metric fluctuations in Minkowski vacuum, see also \[48\] for a analysis of metric fluctuations in the near horizon region.

It should be noticed that \(\langle \mu_+(v)\mu_+(v') \rangle\) is not real. This follows from the quantum vacuum which contains only positive frequencies when hit by \(\phi\), see eq. (8). Notice that eq. (37) gives the “un-subtracted” value of the connected two-point function. As shown in \[43\], the counter-terms which lead to the renormalized one-point function, eq. (34), provide divergent contributions to eq. (37) which tame its singular behaviour as \(v \rightarrow v'\), and make it a well-defined distribution\[17\].

Keeping only eq. (36) in the IF (or by summing the corresponding infinite set of Feynman graphs, see the Appendix) is equivalent to work with a stochastic (i.e., a classically given, albeit not real) and Gaussian ensemble of metric fluctuations. By equivalent we mean that the graphs obtained from the Gaussian integration in the stochastic treatment are in one to one correspondence of those of the \(G^2N\)-truncated quantum treatment. Therefore all matrix elements of \(\phi_-\), such as eq. (35), can be computed from the stochastic theory. In brief, as far as the propagation of \(\phi_-\) is concerned, neglecting the time ordering in the evolution operator, and to lowest order in \(G\), the functional integration over \(\phi_+\) in eq. (35) defines an effective stochastic ensemble of metric fluctuations governed by eq. (38).

In this case, all the techniques developed in \[18\] apply. In what follows we shall present schematically the main results and we refer to this work for details. The key point is the following. Because of the Gaussianity of the ensemble, one can obtain non-linear corrections to eq. (35) from the fluctuating characteristics of eq. (20), i.e., the outgoing null geodesics \(u_\mu(v,r)\), the non-trivial solutions of \(ds^2 = 0\) in the fluctuating metric eq. (13). In this we recovered that

\(^{14}\)In the above expression, we did not take into account the time ordered character of the evolution operator \(e^{iS_{int}}\). It would be interesting to determine the consequences of this neglect. We refer to App. A of \[45\] for a discussion of this point in the context of atomic transitions. We also refer to the recent work of Weinberg\[40\] wherein the 1-loop radiative corrections to the two-point function determining the power spectrum in a inflationary context are computed (in a large \(N\) limit as well) taking properly into account the time ordering. In \[21\], we tried to improve the calculation of \[19\] by taking it into account. We obtained an extra divergence in eq. (44) which induces dispersive effects. However the meaning of this extra term is unclear as it seems to correspond to a tadpole contribution which should be subtracted.
there is no non-linearities in the field amplitude: as in Section 4 the non-linearities in \( G \) only occur through the characteristics.

To determine the effects engendered these metric fluctuations, it is instructive to analyze the backward in time propagation of configurations representing asymptotic Hawking quanta. In particular it will reveal the role played by shift in \( u \) studied in section 4.1. To this end, we consider the Fourier transform, performed on \( J^+ \), of the \( \text{in} - \text{in} \) Green function (that obtained by taking the expectation value in the initial vacuum on \( J^- \)):

\[
G_-(\lambda; v, r) \equiv \int du_1 e^{i\lambda u_1} G_-(v_1 = \infty, u_1; v, r). \tag{39}
\]

In the unperturbed metric, \( \mu = 0 \), i.e. ignoring \( S_{int} \) in eq. (35), we get

\[
G_-(\lambda; v, r) \propto e^{i\lambda u(v, r)}, \tag{40}
\]

where \( u(v, r) = u_0(v, r) = v - 2r^* \) is the unperturbed null characteristic. Hence near the horizon, the Fourier transform of \( G_- \) behaves as

\[
e^{-i\lambda u_0(v, r)} \simeq e^{-i\lambda v} (r - 2M_0)^{i\kappa \lambda}. \tag{41}
\]

It possesses an infinite number of oscillations as \( r \to 2M_0 \) with a radial momentum given by

\[
\hat{p}_r e^{-i\lambda u_0(v, r)} = -i\partial_r e^{-i\lambda u_0(v, r)} = \frac{2\lambda}{r/2M_0 - 1} e^{-i\lambda u_0(v, r)} \equiv p_0(v, r) e^{-i\lambda u_0(v, r)}. \tag{42}
\]

A FF observer will attribute to this wave a FF frequency that grows as in eq. (3) because \( dr\big|_v \propto d\tau \), where \( \tau \) is his proper time. In fact \( p_r \) should be interpreted as a frequency rather than a momentum (its sign fixes that of the Klein-Gordon inner product \([23, 26]\)).

Taking into account the first order change in \( u \), see eq. (23), in any Gaussian ensemble of infalling metric fluctuations, the Fourier transform of the ensemble average of the \( \text{in} - \text{in} \) Green function of eq. (39) will be governed by the averaged waves given by

\[
\langle \langle e^{-i\lambda u_0(v, r)} \rangle \rangle = e^{-i\lambda u_0(v, r)} e^{-\frac{1}{2}\sigma_\Lambda^2 \langle \langle \delta u(v) \delta u(v) \rangle \rangle}. \tag{43}
\]

In the above the averaging acts on the fluctuating quantity \( \mu_+(v) \).

We now focus on the particular (Gaussian) ensemble wherein the fluctuations are those induced by \( N \) fields in the vacuum, i.e. they are governed by \( N \) times the expression of eq. (38). Then, using the fact that \( r_0(v)\big|_{u_0 = 2M_0} \approx 2M_0 e^{\kappa(v-u_0)} \), one obtains

\[
\langle \delta u(v)\big|_{u_0}, \delta u(v)\big|_{u_0} \rangle = \frac{G^2 N}{(r/2M_0 - 1)^2} \int_0^\Lambda d\omega \frac{\kappa^2 \omega}{3 \kappa^2 + \omega^2} = \frac{1}{(r/2M_0 - 1)^2}. \tag{44}
\]

The spread \( \sigma_\Lambda \) governs the damping of the backward propagated waves. It is equal to

\[
\sigma_\Lambda = G\kappa \sqrt{N \ln(\Lambda/\kappa)/3}, \tag{45}
\]

when the hard UV cut-off \( \Lambda \) satisfies \( \Lambda \gg \kappa \). We have introduced \( \Lambda \) to define the two integrals over \( \omega \). Notice that \( \Lambda \) is a Lorentz scalar in the sense that it is the energy of an s-wave in its rest frame in a stationary and spherically symmetric background.
The important result of eq. (44) is that \( \bar{\sigma} \) is not proportional to \( \Lambda \) even though \( \langle \mu^2 \rangle \simeq \Lambda^2 \). Notice indeed that \( \bar{\sigma} \) hardly depends on the value of \( \Lambda \) since
\[
\bar{\sigma}_{\Lambda=4M_0} = \sqrt{2} \bar{\sigma}_{\Lambda=\text{Planck}}.
\]
This insensitivity follows from the fact that high frequencies (\( \omega \gg \kappa \)) are damped by the integration over \( \nu' \) in eq. (23). The frequencies \( \omega \simeq \kappa \) dominate the contribution to \( \bar{\sigma} \) in eq. (44), see (46) for a similar result in an inflationary context. However they are not sufficiently damped to give a finite result. Therefore, in the simplified treatment we are using, the value of \( \Lambda \) must be chosen from the outset.

Instead, in an improved treatment of the regularization of the divergences in eq. (44), we believe that this ambiguity will be resolved. That is, when using the properly subtracted two-point function \( \langle T_{\mu\nu} T_{\alpha\beta} \rangle \), only \( \omega \simeq \kappa \) should contribute to the \( \bar{\sigma} \)'s. The reason is the following: in the UV domain, all expressions become Minkowskian in character and hence cannot contribute to Lorentz breaking effects such as those engendered by \( \bar{\sigma} \).

The second result of eq. (44) is that \( \langle \delta u \delta u \rangle \) diverges as \( r \to 2M_0 \). Thus the correlations between asymptotic quanta and early configurations, which existed in a given background as shown in eq. (11), are washed out by the metric fluctuations once \( r - 2M_0 \sim \bar{\sigma}_\Lambda \). The reason of this loss of coherence is that the state of \( \phi_- \) becomes correlated to that of \( \phi_+ \) [12, 14]. Physically, this loss of coherence implies that induced emission [31] no longer exists when the threshold energy (measured in the FF frame) \( 1/\bar{\sigma} = 4M_0 \) is reached [15]. Phenomenologically this loss can be viewed as a dissipation of outgoing waves, and, as in condensed matter [37, 16], it can be described by an effective dispersion relation. Explicitly, using the notations of [26, 27], we get an effective relation \( \Omega^2 = F^2(p) \), where \( \Omega \) and \( p \) are respectively the frequency and the radial momentum measured in the FF frame, and where \( F^2 \) behaves as
\[
F^2(p) = p^2 (1 - 2i \bar{\sigma}^2 \kappa p) = p^2 (1 - 2i \frac{p}{p_c}).
\]
To obtain this, it suffices to rewrite the r.h.s. of (43) as \( e^{-i\lambda_0} e^{-\bar{\sigma}^2 \mu_0^2 / 2} \) and to work in the momentum representation. Then with the help of eqs. (123, 124) in [26] the identification of the function \( F \) is straightforward. It should be noticed that the UV momentum scale \( p_c \) which weighs the cubic term is \( p_c = (\bar{\sigma}^2 \kappa)^{-1} = (1/\bar{\sigma})(L_{\text{Planck}})^{-2} = M_{\text{Pl}} (L_{\text{Planck}})^{-3} \), and not \( p_c = 1/\bar{\sigma} \) as one might have thought. This mismatch of UV scales illustrates that it is probably meaningless to try to identify a well defined dispersion relation when starting from the result: the modified propagation of eq. (43) which unequivocally states that backward propagated modes are dissipated when their FF momentum reaches \( 1/\bar{\sigma} \). The reason why this identification fails is that the modified propagation arises from the non-trivial properties of the background and not from short-distance physics as it is the case in condensed matter.

We should further explain the physical relevance of these results. To this end, one must identify the matrix elements of \( \phi_- \) which are sensitive to the metric fluctuations (and governed by the ensemble averaged waves eq. (43)) and those which are not. The simplest example of an

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15: In spite of this ambiguity, we still believe that the \( \omega \simeq \kappa \) dominance and the associated robustness of \( \bar{\sigma} \) give credit to the validity of (45). This is conforted by the fact that \( \bar{\sigma} \) also governs the modifications of the asymptotic properties of Hawking radiation (45).

16: An interesting and unsolved question raised by this loss is whether new correlations are induced by the gravitational interactions as the same time as the old ones are washed out. This phenomenon of replacement of correlations has been clearly derived in a slightly different context in [47].
operator which is sensitive is the Fourier transform of the in-in two-point function in eq. (39). On one hand, the phase of the wave function used in the Fourier transform is not affected by the metric fluctuations. On the other hand, the location $v, r$ of the second field operator is also unaffected by the metric fluctuations. However the two-point correlation function is sensitive to the metric fluctuations encountered from $J^+$ to $v, r$\footnote{One might wonder if the effects we are describing are not induced by the choice of working at fixed $u$ or at fixed $v, r$. To avoid misunderstanding, we recall that Green functions have no physical meaning per se, rather they are elements which appear in integrals describing transition amplitudes (for a discussion of this point in a quantum gravitational context see Section 2 in \cite{50}). Having made this remark, one verifies that $u$ is a physically meaningful coordinate on $J^+$ by noticing that $du|_r = dt$ where $dt$ is the proper time of a particle detector on $J^+$. Thus when additional quantum mechanical systems are coupled to the radiation field, the matrix elements governing transition amplitudes will have, in their integrand, phase factors behaving like $e^{-i\lambda u}$ in any coordinate system. Similarly, upon questioning what an infalling observer might see when crossing the horizon, the dependence in $v, r$ is meaningful since $dr|_r \propto -d\tau$ where $d\tau$ is his proper time.}. It is this (unusual, see below) discrepancy between the phase at each point which explains why the ensemble averaged waves of eq. (33) govern the above Fourier transform of the two-point function.

Instead usual expectation values, such as for instance the in–in Green function with two points evaluated at fixed $u$ on $J^+$ (or two nearby points close to the horizon), are not severely affected by the metric fluctuations because these expectation values are not governed by the ensemble averaged waves \footnote{This clearly illustrates that the physics seen by infalling observers completely differs from that reconstructed from observers at large distance from the hole. This is similar to what was advocated in \cite{14}. However, the fact that the FT transformed on $J^+$ is governed by eq. (43) indicates that the near horizon physics is unaccessible and therefore lost to remote observers. Thus it seems that these two descriptions cannot not obey the ‘complementarity principle’ \cite{14}. Indeed, it appears from our analysis that in the interacting FF vacuum early out going configurations are entangled to infalling ones. Let us recall that by complementary to each other, it was meant that the two descriptions are both complete, like the position and momentum representations of the same vector state in quantum mechanics.}. The reason is that the ensemble average is performed after having computed the $\phi_-$ expectation value for each member of the ensemble. (This is not a choice: the stochastic ensemble is merely a tool to reproduce quantum mechanical expectation values. Its quantum origin fixes the rules of ensemble averaging without ambiguity.). In our case, this implies that the shift eq. (23) affects coherently the phase at each point, see Section IV.A in \cite{18}. This guarantees that the shift drops out in the coincidence point limit. This cancellation in turn guarantees that the asymptotic properties are (almost) unaffected since the Green function possesses the usual Hadamard singularity \cite{42}. By almost we mean that the corrections scale like $(\kappa\vec{\sigma})^2$ and thus are order $1/M^4$. It is important to point out that it is again the dynamically induced scale $\vec{\sigma}_A$ and not the UV cut-off $\Lambda$ which governs these corrections.

We would like to further discuss the fact that the metric fluctuations strongly affect the correlations between configurations specified on $J^+$ and near the horizon without modifying the short distance behaviour of the Green function\footnote{It is this (or two nearby points close to the horizon), are not severely affected by the metric fluctuations because these expectation values are not governed by the ensemble averaged waves \cite{18}.}. The radical difference of the impact of vacuum gravitational interactions follows from the fact that any asymptotic measurement will always be governed by projectors on some out-states, i.e. states with a definite particle content defined on $J^+$, to probe the physics. Therefore, these measurements will always be of the in – out type since the Heisenberg state of the field is specified (prepared) before the collapse. It is this two-states formalism giving rise to non-diagonal matrix elements \cite{5} (exactly like in a $S$-matrix formulation \cite{12}) which is at the origin of the difference: The metric fluctuations cannot coherently affect configurations specified in the ‘ket’ on $J^-$ and in the ‘bra’ on $J^+$, hence the coherence is lost. On the contrary, measurements performed by infalling observers...
only probe the near horizon behaviour of the Green function. Hence the coherence is maintained for them.

6 Conclusions

We have studied the effects induced by the gravitational interactions governed by eq. (16). Even though we worked out only the lowest order in $G (\bar{\sigma} \Lambda \propto G)$ we believe that our main result is robust. We see no reason for higher order terms to suppress the entanglement of $\phi_-$ with $\phi_+$ so as to give $\bar{\sigma}_\Lambda = 0$ thereby recovering trans-Planckian correlations. Indeed higher order modifications to eq. (37) and eq. (38) should be of the type $(G \omega^2)^n$ and therefore will not affect the low frequency behaviour of eq. (38) thereby leaving the effective spread $\bar{\sigma}_\Lambda$ essentially untouched. Moreover, when considering the effects of higher angular momentum modes, as indicating by [15], $\bar{\sigma}$ should be larger than our estimate based on s-modes because the effects of higher angular momenta should add incoherently.

In brief, given that gravitational interactions grow without bound near the horizon, we claim that the entanglement of $\phi_-$ with $\phi_+$ is unavoidable and universal. (By universal we mean that a similar entanglement would be also found when considering the coupling of $\phi$ to other quantum fields such as massive ones.) The entanglement will then prevent the unbounded growth of frequencies encountered in the free field theory and will be accompanied by the reorganization of the description of vacuum. That is, when $r \to 2M$, the usual states of the free field theory which give rise to the notion of on-shell asymptotic particles provide bad approximations of the true eigenstates (albeit still characterized by the Killing energy $\lambda$ since the situation is stationary) of the interacting theory. It is the growing discrepancy which leads to the ‘dissipation’ of the amplitude in eq. (43).

We also claim that the radiative corrections encoding dissipation should be finite because only low frequency $\omega \simeq \kappa$ infalling configurations contribute to them. Indeed, in the UV regime for both infalling and outgoing configurations, the expressions coincide with those evaluated in the tangent plane and are Minkowskian in character. Hence the high frequency regime cannot contribute to the effects which break Lorentz invariance. (This still needs to be confirmed by an explicit calculation.)

We would like to conclude this work by several remarks on related aspects of Quantum Gravity and Black Hole Physics.

First, we point out the similarity between the above effects induced by the metric fluctuations representing gravitational interactions in the vacuum and those attributed [51] to ‘foam’. By foam we mean quantum gravitational configurations which radically affect the smoothness of

\footnote{Added comment: We can not exclude the possibility that, when properly renormalized, the 1-loop radiative corrections governed by (29) will give $\bar{\sigma}_\lambda = 0$. In fact, the self-energy of an electron in a thermal bath of photons (as a thermal bath of gravitons) induces no dissipative effects at one loop, see [28] and refs. therein. Nevertheless the naive reasoning that loop corrections in a thermal bath induce dissipative effects ‘becomes’ correct at two-loop level. These effects are indeed propotional to $e^2$ where $e$ is the electric charge. Thus, in the case one finds $\bar{\sigma}_\lambda = 0$ at one loop, one still needs to confront the question whether this result be found at any loop level, to all order in $G$ (for a deep reason, e.g. because we are dealing with the true interacting vacuum), or shall $\bar{\sigma}_\Lambda \neq 0$ at some higher loop level (as in a heat bath, because the integrands are no longer 2D Lorentz invariant), thereby validating the present conclusions.}

Motivated by the present work, we have recently constructed an interacting QFT model which is exactly solvable and which explicitly displays this phenomenon [49], namely the growing discrepancy between the usual modes and the true eigenstates manifests itself through the dissipation of the two-point function.
space-time at short distance. (They might arise from gravitational instantons [52] or stringy effects [53].) In all cases, the replacement of free field propagation in a fixed background by the appropriate interacting model might lead to very similar (universal?) deviations when analyzing the departure from the free field description that all models possess at large distances. Therefore, these first deviations might be described by some effective mesoscopic theory of space-time properties which would essentially signal the existence of a minimal resolution length [54], the equivalent of our $\bar{\sigma}$, in the otherwise local field theory.

Secondly, we conjecture that $\bar{\sigma}$ (properly computed so as to include the contribution of higher angular momentum modes) should also be the length scale which enters in the entanglement description of the black hole entropy [55]. We recall that when using free field in a given space time, the entanglement entropy diverges due to the unbounded character of the reservoir of high energy modes. To get a finite entropy density per unit area, some cut-off should be introduced. We propose that the cut-off defining the black hole entropy should be the dynamically induced length-scale $\bar{\sigma}(N)$, i.e., the length scale at which correlations between configurations on $J^+$ and the near horizon region get lost when $N$ quantum fields contribute to the entropy and therefore to the near horizon gravitational interactions. We refer the interested reader to our recent work[27] wherein we study the entanglement entropy in the presence of dispersion defined in a FF frame such as that of eq. (5). In that work, it is shown that one obtains an entanglement entropy which scales as the horizon area per Planck length square only when the UV cutoff (which specifies at which scale measured in the FF frame the propagation ceases to be the usual Lorentz invariant one) scales as in our model, i.e. when the UV cutoff scales as $1/\bar{\sigma} \propto M_{\text{Planck}}^2/\kappa$.

Acknowledgements

I wish to thank D. Arteaga, R. Balbinot, C. Barrabès, R. Brout, L. Ford, V. Frolov, L. Garay, B.L. Hu, T. Jacobson, S. Liberati, S. Massar, E. Verdaguer and G. Volovik for useful discussions. The topics here explained have been presented and discussed in the following meetings: the 6-th and 7-th Peyresq meetings (France) on Quantum Spacetime, Brane Cosmology and Stochastic Effective Theories, the Artificial Black Holes meeting held in Rio de Janeiro, the Black Hole III conference held in Canada and the ESF-COSLAB workshop held in London in July 2001. I wish to thank the organizers of these meetings. This work was supported by the NATO Grant CLG.976417.

Finally I am grateful to D. Campo for a critical reading of the revised manuscript.

7 Appendix: The large-$N$ limit

We briefly mention several interesting features of the large $N$ limit which illuminate the problems we addressed.

The semi-classical description of quantum processes occurring in a curved background is based on the following equations

\[ G_{\mu\nu} = 8\pi G \langle \Psi | T_{\mu\nu} | \Psi \rangle, \quad (48) \]
\[ \Box g^{\phi} = 0. \quad (49) \]

In eq. (49) the field operator propagates in the classical geometry $g = g^{\Psi}_{\mu\nu}$ which is a solution
of eq. (48) driven by the expectation value $\langle \Psi | T_{\mu\nu} | \Psi \rangle$ evaluated in the state $| \Psi \rangle$ using eq. (49). In this sense, eq. (48, 49) is a self-consistent (Hartree) approximation.

It is quite reasonable that this approximation correctly predicts the time evolution of certain quantities in certain circumstances, e.g., the rate of mass loss of a large black hole. However, the criteria which characterize the validity range of the predictions obtained from eq. (48,49) are not known. An obstacle in finding these criteria is the identification of the “small parameter(s)” which control the deviations between the exact evolution and the semi-classical one.

A rather formal answer to these questions is provided by considering a large $N$ limit, where $N$ is the number of copies of the $\phi$ field. The simplest way to understand why the above equations govern the large $N$ limit is by considering the path integral approach of matrix elements. By duplicating $N$ times $\phi$ in eq. (11) and first integrating over them with a fixed metric $h$, the 1-loop effective action for $h$ contains an overall prefactor $N$ when replacing Newton’s constant $G$ by $G/N$. Therefore, in a large $N$ limit, the path integral over $h$ can be evaluated by a saddle point approximation. The stationary phase condition then gives rise to eq. (48) where the expectation value of $T_{\mu\nu}$ is that of one field. The spread around the saddle point scales like $N^{-1/2}$.

In this approach, the validity of the semi-classical equations apparently relies on a statistical argument, as mean quantities emerge in the thermodynamic limit. The weakness of this argument is the absence of role played by the hierarchy of the length scales governing the processes under examination. This is unlike what is found when a Born-Oppenheimer treatment is applied to quantum gravity [58].

More interestingly the large $N$ limit makes also predictions beyond the semi-classical equations. For instance, in the limit $N \to \infty$ with $GN$ fixed, the short distance behaviour of the graviton propagator is modified, see [43, 17, 44]. In Minkowski vacuum, these modifications are of course Lorentz invariant. However, in a thermal bath or a curved background, the correction terms will no longer possess the Lorentz invariant form. Hence they can induce the effects we are seeking: a dynamically induced scale which breaks the (local) Lorentz invariance that the un-interacting theory possessed. Moreover, only low energies (i.e., energies comparable to the temperature) contribute to this new scale because in the UV limit all expressions tend to their Minkowski vacuum, and hence Lorentz-invariant, form. Hence they shouldn’t be any additional UV divergences in the expressions giving rise to the new scale, as it is the case for the corrections to the self-energy of an electron immersed in a thermal bath.

To further strengthen the relations with what we did in Section 6, it is instructive to see how the semi-classical treatment emerges from eq. (35) viewed as generating perturbatively the connected Feynman diagrams when expanding $e^{iS_{\text{int}}}$ in powers of $S_{\text{int}}$. In this description, one finds that the Green function is a double sum of powers of $N$ and $G$ which possess the following properties.

- The power of $N$ is equal or inferior to that of $G$.
- The semi-classical treatment corresponds to the leading series: the set of graphs weighted by $(GN)^n$. All graphs are one-particle reducible and are governed by the one-point function $\langle T_{\mu\nu} \rangle$. Upon summing this series, one identically recovers the Green function evaluated in the ‘mean’ geometry $g^\Psi_{\mu\nu}$, the solution of eq. (48).
- Having summed up the leading series in $(GN)^n$, our treatment corresponds to the next series: the set of graphs weighted by $(G^2 N)^n$. All graphs are two-particle reducible and
are governed by the connected two-point function $\langle T_{\mu\nu}T_{\alpha\beta}\rangle_C$. Upon summing this new series, one obtains the Green function evaluated in the stochastic ensemble governed by eq. (38). This second series should also be related to the use of the above mentioned large-$N$ modified graviton propagator in the place of the unperturbed one.

In brief, $N$ is a parameter which organizes the double sum of graphs into a sum of non-perturbative series. When the former series have been summed up, the $m$-th series contains all powers of $(G^mN)$, and is governed by the $m$-th correlation function of $T_{\mu\nu}$.

The physical question raised by these results is the following: given the dimensionality of $G = l^2_{\text{Planck}}$, can one infer that high orders in $m$ (the relative power of $G$ with respect to that of $N$) become relevant only for high (Planckian) energies? We conjecture that this is the case: the sorting out of graphs in terms of $m$ is effectively an expansion in the energy of the processes involved in the matrix element under consideration. This is what seems to emerge from our analysis. As indicated by eqs. (43, 44), the semiclassical description of the correlations breaks down when $r - 2M \approx \tilde{\sigma}$, i.e., when the FF energy of a mode $\Omega \approx p_r = \lambda/(r/2M - 1)$, see eq. (42), reaches the new scale $1/\tilde{\sigma}$. We thus find, as in [58], that the validity of the semi-classical equations relies on a hierarchy of energy scales. Indeed, for a large black hole, $\tilde{\sigma} \ll M$ even when $N = 1$. Thus $N$ is not necessary to justify the validity of the semi-classical description. It is rather a useful parameter which helps sorting out the different contributions in radiative corrections. In particular, it allows to discard the self-interacting (non-planar) graphs which are more difficult to evaluate (since they cannot be expressed in terms of metric fluctuations in the present case) because their power of $N$ is smaller those weighted by $(G^mN)^n$.

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21 Notice that $\tilde{\sigma}$ scales as $N^{-1/2}(GN/M)$. This writing expresses that in the large $N$ limit at fixed $GN$, i.e., with a $N$-independent Hawking flux, $\tilde{\sigma} \rightarrow 0$ like $N^{-1/2}$, thereby verifying that in this limit the scale which signals the breakdown of the semi-classical description indeed vanishes.
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