Rank $n$ swapping algebra

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Abstract

Inspired by the swapping algebra and the rank $n$ cross-ratio introduced by F. Labourie, to characterize moduli spaces of cross-ratios of rank $n$ and their natural Poisson structures, we introduce a quotient ring $\mathbb{Z}_n(\mathcal{P})$ equipped with the swapping Poisson bracket—the rank $n$ swapping algebra to characterize the moduli spaces of cross ratios in rank $n$. We prove that $\mathbb{Z}_n(\mathcal{P})$ inherits a Poisson structure from the swapping bracket. To consider the "cross-ratios" in the fraction ring of $\mathbb{Z}_n(\mathcal{P})$, by interpreting $\mathbb{Z}_n(\mathcal{P})$ by a geometric model, we prove that $\mathbb{Z}_n(\mathcal{P})$ is an integral domain.

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1. Introduction

In [AB83], Atiyah and Bott introduced a mapping class group invariant symplectic structure on the character variety of representations of closed surface groups in compact Lie group. Later on, Goldman [G84] generalized for noncompact Lie groups. The most interesting case is the largest component of the character variety of representations of closed surface groups in $\text{PSL}(n, \mathbb{R})$—Hitchin component [H92]. Recently in [L12], F. Labourie introduced the swapping algebra by pair of points on a circle to characterize weak cross ratios and its Poisson bracket on Hitchin component. He related, through the cross ratios, the swapping algebra to the Atiyah-Bott-Goldman symplectic structure on the universal (in genus) Hitchin component. He also related the swapping algebra to the Drinfeld-Sokolov reduction [DS85][S91] on the space of real opers with trivial holonomy. More generally, the swapping algebra is the natural Poisson algebra for all the moduli spaces of cross ratios for any rank. Here, the moduli space of cross ratios refers to the moduli space equipped with cross ratio like special functions and a cyclic order on points. As we can see for Hitchin component, the moduli space is the space of limit curves [L06], the special functions are the weak cross ratios, the cyclic order on points is induced from $\partial_{\infty}\pi_1(S)$.

In this article, to characterize the moduli spaces of cross ratios in each rank, inspired by the swapping algebra and the rank $n$ cross ratio, we introduce a quotient ring $\mathbb{Z}[P]/(P)$ equipped with the swapping Poisson bracket—the rank $n$ swapping algebra.

1.1 Swapping algebra

Let us recall the swapping algebra at the beginning. The swapping algebra is introduced by F. Labourie in [L12] through ordered pairs of points on a circle. Let $P$ be a cyclic subset of $S^1$, we represent an ordered pair $(r, x)$ of $P$ by the expression $rx$. Then we consider the associative commutative ring

$$\mathbb{Z}(P) := \mathbb{K}\langle\{xy\}_{x, y \in P}\rangle / \{xx\}_{\forall x \in P}$$

over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$), where $\{xy\}_{x, y \in P}$ are variables. Notably, $rx = 0$ in $\mathbb{Z}(P)$ if $r = x$. Then we equip $\mathbb{Z}(P)$ with a Poisson bracket.

**Definition 1.1 [Swapping bracket][L12]** The swapping bracket over $\mathbb{Z}(P)$ is defined by extending the following formula on generators to $\mathbb{Z}(P)$ by using Leibniz’s rule:

$$\{rx, sy\} = J(r, x, s, y) \cdot ry \cdot sx,$$

(1)

where $J(r, x, s, y)$ is the linking number of $ry$ and $sx$. (This is the case for $\alpha = 0$ in [L12].)

[Leibniz’s rule]

$$\{rx \cdot sy, tz\} = rx\{sy, tz\} + sy\{rx, tz\}.$$ 

(2)

In Section 2, we give more details on configurations of points with different linking numbers.

**Theorem 1.2 [F. Labourie][L12]** The swapping bracket is a Poisson bracket.

**Definition 1.3 [Swapping fraction algebra of $P$]** The swapping fraction algebra of $P$ is the total fraction $Q(P)$ of $\mathbb{Z}(P)$ equipped with the induced swapping bracket.

Since $\mathbb{Z}(P)$ is an integral domain, we consider the subring generated by the ”cross ratios”.

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**Definition 1.4** [Multifraction algebra of \( \mathcal{P} \)] Let \( x, y, z, t \) belong to \( \mathcal{P} \) so that \( x \neq t \) and \( y \neq z \). The **cross fraction** determined by \((x, y, z, t)\) is the element:

\[
[x, y, z, t] := \frac{xz}{xt} \cdot \frac{yt}{yz}. \tag{3}
\]

Let \( \text{CR}(\mathcal{P}) = \{[x, y, z, t] \in \mathcal{Q}(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P}, \ x \neq t, y \neq z\} \) be the set of all the cross fractions in \( \mathcal{Q}(\mathcal{P}) \). Let \( \mathcal{B}(\mathcal{P}) \) be the subring of \( \mathcal{Q}(\mathcal{P}) \) generated by \( \text{CR}(\mathcal{P}) \). Then, the **multifraction algebra of \( \mathcal{P} \)** is the ring \( \mathcal{B}(\mathcal{P}) \) equipped with the swapping bracket.

### 1.2 Rank \( n \) swapping algebra and main results

Inspired by the swapping algebra and the rank \( n \) cross-ratio, we introduce the rank \( n \) swapping algebra \( \mathcal{Z}_n(\mathcal{P}) \).

**Definition 1.5** [The rank \( n \) swapping ring \( \mathcal{Z}_n(\mathcal{P}) \)] For \( n \geq 2 \), let \( \mathcal{R}_n(\mathcal{P}) \) be the subring of \( \mathcal{Z}(\mathcal{P}) \) generated by

\[
\begin{align*}
D \in \mathcal{Z}_n(\mathcal{P}) & \mid D = \text{det} \begin{pmatrix} x_{1y_1} & \ldots & x_{1y_{n+1}} \\ \vdots & \ddots & \vdots \\ x_{n+1y_1} & \ldots & x_{n+1y_{n+1}} \end{pmatrix}, \forall x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in \mathcal{P},
\end{align*}
\]

Let \( \mathcal{Z}_n(\mathcal{P}) \) be the quotient ring \( \mathcal{Z}(\mathcal{P})/\mathcal{R}_n(\mathcal{P}) \).

For the well-definedness of the rank \( n \) swapping algebra, one of our main results of this article is the following

**Theorem 1.6** [Main result] For \( n \geq 2 \), \( \mathcal{R}_n(\mathcal{P}) \) is an ideal for the swapping bracket, thus \( \mathcal{Z}_n(\mathcal{P}) \) inherits a Poisson bracket from the swapping bracket.

By the above theorem, we have

**Definition 1.7** [Rank \( n \) swapping algebra of \( \mathcal{P} \)] The rank \( n \) swapping algebra of \( \mathcal{P} \) is the ring \( \mathcal{Z}_n(\mathcal{P}) \) equipped with the swapping bracket.

For the well-definedness of "cross ratios", another main result of this article is the following

**Theorem 1.8** [Main result] For \( n \geq 2 \), \( \mathcal{Z}_n(\mathcal{P}) \) is an integral domain.

**Definition 1.9** [Rank \( n \) swapping fraction algebra of \( \mathcal{P} \)] The rank \( n \) swapping fraction algebra of \( \mathcal{P} \) is the **total fraction ring** \( \mathcal{Q}_n(\mathcal{P}) \) of \( \mathcal{Z}_n(\mathcal{P}) \) equipped with the swapping bracket.

**Definition 1.10** [Rank \( n \) multifraction algebra of \( \mathcal{P} \)] Let \( x, y, z, t \) belong to \( \mathcal{P} \) so that \( x \neq t \) and \( y \neq z \). The **cross fraction** determined by \((x, y, z, t)\) is the element of \( \mathcal{Q}_n(\mathcal{P}) \):

\[
[x, y, z, t] := \frac{xz}{xt} \cdot \frac{yt}{yz}. \tag{4}
\]

Let \( \text{CR}_n(\mathcal{P}) = \{[x, y, z, t] \in \mathcal{Q}_n(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P}, \ x \neq t, y \neq z\} \) be the set of all the cross-fractions in \( \mathcal{Q}_n(\mathcal{P}) \). Let \( \mathcal{B}_n(\mathcal{P}) \) be the subring of \( \mathcal{Q}_n(\mathcal{P}) \) generated by \( \text{CR}_n(\mathcal{P}) \).

Then, the rank \( n \) multifraction algebra of \( \mathcal{P} \) is the ring \( \mathcal{B}_n(\mathcal{P}) \) equipped with the swapping bracket.

In the last section, we show the stability condition of \( \mathcal{Z}_n(\mathcal{P}) \) is related to hyperconvexity of a representation. We have a natural ring homomorphism from the rank \( n \) cross fractions to the weak cross ratios, but not injective. But when \( \pi_1(S) \) is trivial, it is injective.
1.3 Further discussions

In the upcoming papers, we relate the rank $n$ swapping algebra to the Fock-Goncharov coordinates [FG06][FG04] and to the discrete integrable system [FV93][SOT10][KS13].

We suggest that the rank $n$ swapping algebra is a very useful tool to understand the natural Poisson structure on moduli space of higher rank. It will be very interesting to study the relation between the rank $n$ swapping algebra with the cluster algebra, skein algebra, $\mathcal{W}$ algebra and their quantizations.

2. Swapping algebra

In this section, we recall some definitions about the swapping algebra created by F. Labourie. Our definitions here are based on Section 2 of [LT12].

2.1 Linking number of 4 points in the circle

**Definition 2.1** [Linking number] Let $(r, x, s, y)$ be a quadruple of 4 different points in the interval $[0, 1]$. Let $\sigma(\triangle) = -1, 0, 1$ whenever $\triangle < 0, \triangle = 0, \triangle > 0$ respectively. We call $J(r, x, s, y)$ the linking number of $(r, x, s, y)$, where

$$J(r, x, s, y) = \frac{1}{2} \cdot (\sigma(r-x) \cdot \sigma(r-y) \cdot \sigma(y-x) - \sigma(r-x) \cdot \sigma(s-x) \cdot \sigma(s-x)).$$

(5)

If $(r, x, s, y)$ is a quadruple of 4 points in the oriented circle $S^1$, the linking number of 4 points in the interval $S^1 \setminus o$ for $o \notin \{r, x, s, y\}$ does not depend on the choice of $o$. So, $J(r, x, s, y)$ is defined to be the linking number of 4 points in the circle $S^1$. We describe four cases in Figure 1.

2.2 Swapping algebra

Let $P$ be a finite subset of the circle $S^1$ provided with cyclic order. $K$ is a field ($\mathbb{C}$ or $\mathbb{R}$). We represent an ordered pair $(r, x)$ of $P$ by the expression $rx$.

**Definition 2.2** [Swapping ring of $P$] The swapping ring of $P$ is the ring $Z(P) := K[\{xy\}_{x,y \in P}]/\{xx\}_{\forall x \in P}$ over $K$, where $\{xy\}_{x,y \in P}$ are variables with values in $K$. 

---

**Figure 1.**

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Notably, \( rx = 0 \) if \( r = x \) in \( \mathcal{Z}(\mathcal{P}) \). Then we equip \( \mathcal{Z}(\mathcal{P}) \) with a Poisson bracket defined by F. Labourie in Section 2 of [L12].

**Definition 2.3** [SWAPPING BRACKET] The swapping bracket over \( \mathcal{Z}(\mathcal{P}) \) is defined by extending the following formula to \( \mathcal{Z}(\mathcal{P}) \) by using Leibniz’s rule and additive rule:

\[
\{rx, sy\} = J(r, x, s, y) \cdot ry \cdot sx.
\]  
(6)

(Here is the case for \( \alpha = 0 \) in Section 2 of [L12].)

Leibniz’s rule:

\[
\{rx \cdot sy, tz\} = rx\{sy, tz\} + sy\{rx, tz\}
\]  
(7)

for any \( rx, xy, tz \in \mathcal{P} \).

Additive rule:

\[
\{a + b, c\} = \{a, c\} + \{b, c\}
\]  
(8)

For any \( a, b, c \in \mathcal{Z}(\mathcal{P}) \).

**Theorem 2.4** [F. Labourie [L12].] The swapping bracket as above verifies the Jacobi identity. So the swapping bracket defines a Poisson structure on \( \mathcal{Z}(\mathcal{P}) \).

**Definition 2.5** [SWAPPING ALGEBRA OF \( \mathcal{P} \)] The swapping algebra of \( \mathcal{P} \) is \( \mathcal{Z}(\mathcal{P}) \) equipped with the swapping bracket.

### 2.3 Swapping fraction algebra, swapping multifraction algebra

**Definition 2.6** [CLOSED UNDER SWAPPING BRACKET] For a ring \( R \), if \( \forall a, b \in R \), we have \( \{a, b\} \in R \), then we say that \( R \) is closed under swapping bracket. Moreover, we say that \( R \) is equipped with the closed swapping bracket.

\( \mathcal{Z}(\mathcal{P}) \) is a integral domain, let \( \mathcal{Q}(\mathcal{P}) \) be the total fraction of \( \mathcal{Z}(\mathcal{P}) \). By Leibniz’s rule, since \( \{a, \frac{1}{b}\} = -\frac{\{a,b\}}{bx} \), the swapping bracket is well defined on \( \mathcal{Q}(\mathcal{P}) \). So we have

**Definition 2.7** [SWAPPING FRACTION ALGEBRA OF \( \mathcal{P} \)] The swapping fraction algebra of \( \mathcal{P} \) is \( \mathcal{Q}(\mathcal{P}) \) equipped with the induced swapping bracket.

**Definition 2.8** [CROSS FRACTION] Let \( x, y, z, t \) belong to \( \mathcal{P} \) so that \( x \neq t \) and \( y \neq z \). The cross fraction determined by \( (x, y, z, t) \) is the element of \( \mathcal{Q}(\mathcal{P}) \):

\[
[x, y, z, t] := \frac{xz}{xt} \cdot \frac{yt}{yz}.
\]  
(9)

Let \( \mathcal{CR}(\mathcal{P}) = \{[x, y, z, t] \in \mathcal{Q}(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P}, x \neq t, y \neq z\} \) be the set of all the cross-fractions in \( \mathcal{Q}(\mathcal{P}) \).

**Remark 2.9** Notice that the cross fractions verify the cross-ratio conditions [L07]:

- Symmetry: \([a, b, c, d] = [b, a, d, c]\),
- Normalisation: \([a, b, c, d] = 0\) if and only if \( a = c \) or \( b = d \),
- Normalisation: \([a, b, c, d] = 1\) if and only if \( a = b \) or \( c = d \),
- Cocycle identity: \([a, b, c, d] \cdot [a, b, d, e] = [a, b, c, e]\),
- Cocycle identity: \([a, b, d, e] \cdot [b, c, d, e] = [a, c, e, f]\).

Let \( \mathcal{B}(\mathcal{P}) \) be the subring of \( \mathcal{Q}(\mathcal{P}) \) generated by \( \mathcal{CR}(\mathcal{P}) \).
Proposition 2.10 \( B(\mathcal{P}) \) is closed under swapping bracket.

Proof. By Leibniz’s rule, \( \forall c_1, \ldots, c_n, d_1, \ldots, d_m \in \mathcal{Z}(\mathcal{P}) \)

\[
\{c_1 \cdots c_n, d_1 \cdots d_m\} = \sum_{i,j=1}^{n,m} \{c_i, d_j\},
\]

we have only to show that for any two elements \( \frac{xx}{xt} \cdot \frac{yt}{yz} \) and \( \frac{uw}{us} \cdot \frac{ws}{vw} \) in \( \mathcal{C} \mathcal{R}(\mathcal{P}) \), where \( x \neq t, y \neq z, u \neq s, v \neq w \), then \( \{\frac{xx}{xt} \cdot \frac{yt}{yz}, \frac{uw}{us} \cdot \frac{ws}{vw}\} \in B(\mathcal{P}) \). Let \( e_1 = xx, e_2 = \frac{yt}{yz}, e_3 = yt, e_4 = \frac{uw}{us} \), \( h_1 = uw, h_2 = \frac{1}{us}, h_3 = vs, h_4 = \frac{1}{vw} \). By the definition of the swapping bracket \( \{\frac{e_1}{e_1}, h_1\} \in \mathcal{C} \mathcal{R}(\mathcal{P}) \). Then by the Leibniz’s rule, we deduce that for any \( e, h \in \mathcal{Z}(\mathcal{P}) \),

\[
\frac{\{e, \frac{1}{h}\}}{e/h} = -\{e, h\}.
\]

So for any \( i, j = 1, 2, 3, 4 \), we have \( \{e_i, h_j\} \in \mathcal{C} \mathcal{R}(\mathcal{P}) \). \( e_1 e_2 e_3 e_4 \) and \( h_1 h_2 h_3 h_4 \) are also in \( \mathcal{C} \mathcal{R}(\mathcal{P}) \), so

\[
\{e_1 e_2 e_3 e_4, h_1 h_2 h_3 h_4\} = \sum_{i,j=1}^{4} \frac{\{e_i, h_j\}}{e_i \cdot h_j} \cdot (e_1 e_2 e_3 e_4 h_1 h_2 h_3 h_4) \in B(\mathcal{P}).
\]

Finally, we conclude that \( B(\mathcal{P}) \) is closed under swapping bracket. \( \square \)

Definition 2.11 [SWAPPING MULTIFRACTION ALGEBRA OF \( \mathcal{P} \)] The swapping multifraction algebra of \( \mathcal{P} \) is \( B(\mathcal{P}) \) equipped with the swapping bracket.

3. Rank \( n \) swapping algebra

In this section, we define the rank \( n \) swapping ring \( \mathcal{Z}_n(\mathcal{P}) \), then we prove one of our main results, says that \( \mathcal{Z}_n(\mathcal{P}) \) is compatible with the swapping bracket.

3.1 The rank \( n \) swapping ring \( \mathcal{Z}_n(\mathcal{P}) \)

In spirit of rank \( n \) cross-ratio in [L07] defined by F. Labourie, combining with the definition of the swapping ring above, we define the rank \( n \) swapping ring as follows.

Notation 3.1 Let

\[
\Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) = \det \begin{pmatrix} x_1 y_1 & \cdots & x_1 y_{n+1} \\ \vdots & \ddots & \vdots \\ x_{n+1} y_1 & \cdots & x_{n+1} y_{n+1} \end{pmatrix}.
\]

Definition 3.2 [THE RANK \( n \) SWAPPING RING \( \mathcal{Z}_n(\mathcal{P}) \)] For \( n \geq 2 \), let \( R_n(\mathcal{P}) \) be the ideal of \( \mathcal{Z}(\mathcal{P}) \) generated by

\[
\{D \in \mathcal{Z}(\mathcal{P}) \mid D = \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})), \forall x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1} \in \mathcal{P} \}.
\]

The rank \( n \) swapping ring \( \mathcal{Z}_n(\mathcal{P}) \) is the quotient ring \( \mathcal{Z}(\mathcal{P})/R_n(\mathcal{P}) \).

Remark 3.3 Decomposing the determinant \( D \) in first row, we have by induction that

\[
R_2(\mathcal{P}) \supseteq R_3(\mathcal{P}) \supseteq \cdots \supseteq R_n(\mathcal{P}).
\]
3.2 Swapping bracket over $\mathbb{Z}_n(P)$

Let us prove the first fundamental theorem of the rank $n$ swapping algebra.

**Theorem 3.4** [First main result] For $n \geq 2$, $R_n(P)$ is an ideal for the swapping bracket, thus $\mathbb{Z}_n(P)$ inherits a Poisson bracket from the swapping bracket.

**Proof.** The above theorem is equivalent to say $\forall h \in R_n(P), \forall f \in \mathbb{Z}_n(P)$, we have $\{f, h\} \in R_n(P)$, $h = \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))$.

**Lemma 3.5** Let $n \geq 2$. Let $x_1, ..., x_n+1, y_1, ..., y_{n+1}$ resp. different from each other in $P$, $a, b$ belong to $P$. Then $\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}$ belongs to $R_n(P)$.

In particular, $x_1, ..., x_l, y_1, ..., y_k$ are (strictly) on the right side of $ab$ and $x_{l+1}, ..., x_m, y_{k+1}, ..., y_p$ are on the left side of $ab$ as in Figure 2. Let

$$\Delta^R = \sum_{d=1}^l x_db \cdot \Delta((x_1, ..., x_{d-1}, a, x_{d+1}, ..., x_{n+1}), (y_1, ..., y_{n+1}))$$

$$- \sum_{d=1}^k ay_d \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{d-1}, b, y_{d+1}, ..., y_{n+1})), \tag{12}$$

$$\Delta^L = \sum_{d=l+1}^m x_db \cdot \Delta((x_1, ..., x_{d-1}, a, x_{d+1}, ..., x_{n+1}), (y_1, ..., y_{n+1}))$$

$$- \sum_{d=k+1}^p ay_d \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{d-1}, b, y_{d+1}, ..., y_{n+1})). \tag{13}$$

we obtain that

(i) when $a$ does not belong to $\{x_1, ..., x_{n+1}\}$ and $b$ does not belong to $\{y_1, ..., y_{n+1}\}$, $m = p = n + 1$, we have

$$\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \Delta^R = -\Delta^L; \tag{14}$$
(ii) when \( a \) coincides with \( x_{n+1} \), \( b \) does not belong to \( \{y_1, \ldots, y_{n+1}\} \), \( m = n, p = n + 1 \), we have
\[
\{ab, \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\} = \frac{1}{2} \cdot ab \cdot \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) + \Delta^R
\]
\[
= -\frac{1}{2} \cdot ab \cdot \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) - \Delta^L;
\]  

(iii) when \( a \) does not belong to \( \{x_1, \ldots, x_{n+1}\} \), \( b \) coincides with \( y_{n+1}, m = n + 1, p = n \), we have
\[
\{ab, \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\} = \frac{1}{2} \cdot ab \cdot \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) + \Delta^R
\]
\[
= \frac{1}{2} \cdot ab \cdot \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) - \Delta^L.
\]  

(iv) when a coincides with \( x_{n+1} \) and \( b \) coincides with \( y_{n+1}, m = p = n \), we have
\[
\{ab, \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\} = \Delta^R = -\Delta^L;
\]  

Remark 3.6 We sketch the main idea of the proof below. First, we represent \( ab \) by an arrow and \( \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\) by a complete bipartite graph to illustrate our calculations, then we consider the change of \( \{ab, \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}))\} \) when \( ab \) moves topologically in the complete bipartite graph plus a circle.

Proof. We suppose that \( x_1, x_2, \ldots, x_{n+1} \) (resp. \( y_1, y_2, \ldots, y_{n+1} \)) cyclically ordered on the circle with anticlockwise orientation.

We associate to \((x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1})\) the oriented complete bipartite graph \( K_{n+1, n+1} \) with the vertex sets \( X = \{x_1, x_2, \ldots, x_{n+1}\} \) and \( Y = \{y_1, y_2, \ldots, y_{n+1}\} \) in \( \mathcal{P} \), so that every vertex in \( X \) is adjacent to every vertex in \( Y \) by an arrow from \( x_i \) to \( y_j \), but there are no arrows within \( X \) or \( Y \). Let \( \Gamma \) be the set of all the subgraphs of \( K_{n+1, n+1} \) so that every vertex appear on exactly one edge.

Notation 3.7 Let \( \gamma \) be a oriented graph, let \( E(\gamma) \) be the set of all the arrows in \( \gamma \), for any \( e \) in \( E(\gamma), e = \bar{e}_+ e_- \) where \( e_+ (e_-) \) is said to be the origin (end respectively) of \( e \). If \( e = \bar{x_i y_j} \), we denote \( x_i y_j \) in \( \mathcal{Z}(\mathcal{P}) \) by \( e \) without loss of ambiguity.

For any \( \gamma \) in \( \Gamma \), we have exactly one \( \sigma \) in the permutation group of \( n + 1 \) elements \( S_{n+1} \), such that, for any \( e \) in \( \gamma \), we have \( x_i = e_+, y_j = e_-, \sigma(i) = j; \) for any \( \sigma \) in \( S_{n+1} \), we have exactly one \( \gamma \) in \( \Gamma \), such that for any \( e \) in \( \gamma \), we have \( x_i = e_+, y_j = e_-, \sigma(i) = j \). For example, the subgraph \( \gamma_0 \) with the arrows \( \{\bar{x_i y_j}\}_{i=1}^{n+1} \) corresponds to \( I \in S_n \). We identify the set \( \Gamma \) with the group \( S_{n+1} \) of permutations of \( \{1, \ldots, n+1\} \). We have
\[
\Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) = \sum_{\sigma \in S_{n+1}} (-1)^{\text{sign}(\sigma)} \prod_{i=1}^{n+1} x_i y_{\sigma(i)}.
\]  

Then \( \Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) \) can be written as the sum over \( \Gamma \):
\[
\Delta((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) = \sum_{\sigma \in \Gamma} (-1)^{\text{sign}(\sigma)} \prod_{e \in \sigma} e_+ e_-,
\]  

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where \( \text{sign}(\sigma) \) is the sign of the permutation in \( S_{n+1} \). By Leibniz’s rule, we have

\[
\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = \sum_{\sigma \in \Gamma} (-1)^{\text{sign}(\sigma)} \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \cdot \left( \frac{ab, e}{e} \right). \tag{20}
\]

**Notation 3.8** The arrow \( \rightarrow ab \) separates \( S^1 \) into disjoint unions

\[
\{a\} \cup \{b\} \cup I \cup J,
\]

where \( I \) is on the left side of \( \rightarrow ab \). Let

\[
d(a, b) := I \cap \{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\}.
\]

If a point \( u \) in \( S^1 \) does not coincide with any one of \( \{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\} \), we say that \( u \) is in general position.

We say that a point \( u \) (different from \( v \)) is next to \( v \) if \( \{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\} \\{u, v\} \) is included in the component of \( S^1 \setminus \{u, v\} \) which is on the left side of \( \rightarrow vu \).

Notice that every element of \( \{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\} \) appear only once in \( \prod_{i=1}^{n+1} x_i y_{\sigma(i)} \). We prove the lemma by induction on \( d(a, b) \). The following calculations are all based on the combinatorial model described above and the formula 20.

We start our induction with \( d(a, b) = 0 \), then there is no vertex of \( \{x_1, ..., x_{n+1}, y_1, ..., y_{n+1}\} \) on the left side of \( ab \).

(i) If \( a \) does not coincide with any \( x_i \) and \( b \) does not coincide with any \( y_j \), then for any \( x_s, y_t \), we have \( \{ab, x_s y_t\} = 0 \). By the formula 20, we have

\[
\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = 0 \in R_n(P).
\]

(ii) If \( a \) coincides with some \( x_i \) and \( b \) does not coincide with any \( y_j \) as in Figure 3, then for each \( \sigma \in \Gamma \), each \( e \in E(\sigma) \), we have \( \{ab, e\} = 0 \) except for the arrow \( e \) in \( \sigma \) so that \( e_+ = x_i, J(a, b, e_+, e_-) = -\frac{1}{2}. \) So \( \{ab, e\} = -\frac{1}{2} \cdot ab \cdot e. \) Hence in this case, By the formula 20, we deduce that

\[
\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} = -\frac{1}{2} ab \cdot \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1})) \in R_n(P).
\]
(iii) If $a$ does not coincide with any $x_i$, $b$ coincides with some $y_j$. Similarly as Case (ii), we have
\[
\{ab, \Delta((x_1, ..., x_{n+1}, (y_1, ..., y_{n+1})) = \frac{1}{2}ab \cdot \Delta((x_1, ..., x_{n+1}, (y_1, ..., y_{n+1})) \in \mathbb{R}_n(P).
\]

(iv) If $a$ coincides with some $x_i$, $b$ coincides with some $y_j$. By the formula (20), we have
\[
\{ab, \Delta((x_1, ..., x_{n+1}, (y_1, ..., y_{n+1}))

\begin{align*}
&= \sum_{\sigma \in \Gamma \text{ such that } ab \in \sigma} (-1)^{\text{sign}(\sigma)} \left( \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \frac{\{ab, e\}}{e} \right) \\
&\quad + \sum_{\sigma \in \Gamma \text{ such that } ab \notin \sigma} (-1)^{\text{sign}(\sigma)} \left( \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \frac{\{ab, e\}}{e} \right). \\
\end{align*}
\]

For $\sigma \in \Gamma$ such that $ab \in \sigma$, for any $e \in \sigma$, we have $J(a, b, x_i, y_j) = 0$. So the first part of the right hand side of the equation is zero.

For the second part of the right hand side of the equation, for each $\sigma$ such that $ab \notin \sigma$, we have $J(a, b, x_{\sigma-1}, y_j) = 0$ where $x_i = a$ and $y_j = b$, so
\[
\sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \frac{\{ab, e\}}{e} = \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) (J(a, b, x_i, y_{\sigma(i)}) + J(a, b, x_{\sigma-1}, y_j)) \cdot ab
\]

\[
= \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \left( -\frac{1}{2} + \frac{1}{2} \right) \cdot ab = 0 \in \mathbb{R}_n(P).
\]

Hence the lemma is valid for $d(a, b) = 0$.

We suppose that the lemma is true for $d(a, b) = k \geq 0$. When $\overrightarrow{ab}$ is the dotted arrow as in Figure 4 with $d(a, b) = k + 1$, we prove that $\{ab, \Delta((x_1, ..., x_{n+1}, y_1, ..., y_{n+1}))$ belongs to $\mathbb{R}_n(P)$ as follows.

(i) If $x_s$ is the next to the point $b$ as in Figure 4

(a) If $b$ is in general position. Let $b'$ be a point in general position next to $x_s$ on the other
side of $\overrightarrow{a x_s}$ as in Figure 4. Since there are $k$ vertices on the left side of $\overrightarrow{a b'}$, by hypothesis and the formula \[20\] we have

$$\{ab', \Delta(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1})\} = \sum_{\sigma \in \Gamma} (-1)^{\text{sign}(\sigma)} \left( \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \frac{\{ab', e\}}{e} \right)$$

\[23\]

Since the right hand side of the above formula is a polynomial in $ab'$, $x_1b', \ldots, x_{n+1}b'$, we denote the right hand side of the formula by $P(ab', x_1b', \ldots, x_{n+1}b')$. When we replace $ab', x_1b', \ldots, x_{n+1}b'$ by $ab, x_1b, \ldots, x_{n+1}b$ in $P(ab', x_1b', \ldots, x_{n+1}b')$, we obtain that

$$P(ab, x_1b, \ldots, x_{n+1}b) = \sum_{\sigma \in \Gamma} (-1)^{\text{sign}(\sigma)} \left( \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \frac{\mathcal{J}(a, b', e_+, e_-) \cdot ae_-e_+b}{e} \right)$$

\[24\]

Still, we have $P(ab, x_1b, \ldots, x_{n+1}b)$ belongs to $R_n(P)$. Then, by the formula \[23\] we have

$$\{ab, \Delta(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1})\} - P(ab, x_1b, \ldots, x_{n+1}b)$$

\[24\]

$$= \sum_{\sigma \in \Gamma} (-1)^{\text{sign}(\sigma)} \left( \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \frac{\mathcal{J}(a, b, e_+, e_-) - \mathcal{J}(a, b', e_+, e_-)) \cdot ae_-e_+b}{e} \right)$$

\[24\]

For $a$ is in any position, we always have:

For any $\sigma$ in $\Gamma$, $e$ in $\sigma$ such that $e_+ \neq x_s$,

$$\mathcal{J}(b', b, e_+, e_-) = 0$$

For any $\sigma$ in $\Gamma$, $e$ in $\sigma$ such that $e_+ = x_s$,

$$\mathcal{J}(b', b, e_+, e_-) = -1$$

Then we have

$$\sum_{\sigma \in \Gamma} (-1)^{\text{sign}(\sigma)} \left( \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \frac{\mathcal{J}(b', b, e_+, e_-)) \cdot ae_-e_+b}{e} \right)$$

\[25\]

$=-x_s b \cdot \Delta((x_1, \ldots, x_{s-1}, a, x_{s+1}, \ldots, x_{n+1}, (y_1, \ldots, y_{n+1})) \in R_n(P)$.

So $\{ab, \Delta((x_1, \ldots, x_{n+1}, (y_1, \ldots, y_{n+1}))\}$ belongs to $R_n(P)$ in this case.

(b) If $b$ coincides with $x_i$ for some $i$. Let $b''$ be a point in general position next to $b$ as in Figure 4. By the result of the case \[23\], we have

$$\{ab'', \Delta((x_1, \ldots, x_{n+1}, (y_1, \ldots, y_{n+1}))\} \in R_n(P).$$

Since $\{ab'', \Delta((x_1, \ldots, x_{n+1}, (y_1, \ldots, y_{n+1}))\}$ is a polynomial in $ab'', x_1b'', \ldots, x_{n+1}b''$, we denote it by $C(ab'', x_1b'', \ldots, x_{n+1}b'')$. When $ab'', x_1b'', \ldots, x_{n+1}b''$ is replaced by $ab, x_1b, \ldots, x_{n+1}b$ in $C(ab'', x_1b'', \ldots, x_{n+1}b'')$, we still have $C(ab, x_1b, \ldots, x_{n+1}b)$ belongs to $R_n(P)$.
similar argument, we have
\[
\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} - C(ab, x_1b, ..., x_n b)
\]
\[
= \sum_{\sigma \in \Gamma} (-1)^{sign(\sigma)} \left( \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \frac{J(a, b, e_+, e_-) - J(a, b', e_+, e_-), ae_-e_+ b}{e} \right)
\]
\[
= \sum_{\sigma \in \Gamma} (-1)^{sign(\sigma)} \left( \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \frac{J(b', b, e_+, e_-), ae_-e_+ b}{e} \right) .
\]

For \(a\) is in any position, we always have:
For any \(\sigma\) in \(\Gamma\), \(e\) in \(\sigma\) such that \(e_+ \neq x_i\),
\[J(b', b, e_+, e_-) = 0;\]
For any \(\sigma\) in \(\Gamma\), \(e\) in \(\sigma\) such that \(e_+ = x_i\),
\[J(b', b, e_+, e_-) = -\frac{1}{2} \]

By the above formula, we have
\[
\sum_{\sigma \in \Gamma} (-1)^{sign(\sigma)} \left( \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \frac{J(b', b, e_+, e_-), ae_-e_+ b}{e} \right)
\]
\[
= \sum_{\sigma \in \Gamma} (-1)^{sign(\sigma)} \left( \prod_{e' \in \sigma} e' \right) \frac{(-\frac{1}{2}), ay_\sigma(s), x_i b}{x_i y_\sigma(s)} \]
\[= 0. \]

So \(\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}\) belongs to \(R_n(\mathcal{P})\) in this case.

(c) If \(b\) coincides with \(y_i\) for some \(i\). Let \(b''\) be a point in general position next to \(b\) as in Figure [4]. Without loss of ambiguity, we mention that this \(b''\) is not the same as the case (b). By the result of the case \([a]\), we have
\[
\{ab'', \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} \in R_n(\mathcal{P}).
\]

Since \(\{ab'', \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\}\) is a polynomial in \(ab'', x_1b'', ..., x_{n+1} b''\), we denote it by \(D(ab'', x_1b'', ..., x_{n+1} b'')\). When \(ab'', x_1b'', ..., x_{n+1} b''\) is replaced by \(ab, x_1b, ..., x_{n+1} b\) in \(D(ab'', x_1b'', ..., x_{n+1} b'')\), we still have \(D(ab, x_1b, ..., x_{n+1} b)\) belongs to \(R_n(\mathcal{P})\). By the similar argument, we have
\[
\{ab, \Delta((x_1, ..., x_{n+1}), (y_1, ..., y_{n+1}))\} - D(ab, x_1b, ..., x_n b)
\]
\[
= \sum_{\sigma \in \Gamma} (-1)^{sign(\sigma)} \left( \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \frac{J(a, b, e_+, e_-) - J(a, b', e_+, e_-), ae_-e_+ b}{e} \right)
\]
\[
= \sum_{\sigma \in \Gamma} (-1)^{sign(\sigma)} \left( \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \frac{J(b', b, e_+, e_-), ae_-e_+ b}{e} \right) .
\]

For \(a\) is in any position, we always have:
For any \(\sigma\) in \(\Gamma\), \(e\) in \(\sigma\) such that \(e_- \neq y_i\),
\[J(b', b, e_+, e_-) = 0;\]
For any \(\sigma\) in \(\Gamma\), \(e\) in \(\sigma\) such that \(e_- = y_i\),
\[J(b', b, e_+, e_-) = \frac{1}{2} \]
Then we have
\[
\sum_{\sigma \in \Gamma} (-1)^{\text{sign}(\sigma)} \left( \sum_{e \in \sigma} \left( \prod_{e' \in \sigma} e' \right) \left( \mathcal{J}(b', b, e_+, e_-)ae_-e_+b \right) e \right) = \sum_{\sigma \in \Gamma} (-1)^{\text{sign}(\sigma)} \left( \prod_{e' \in \sigma} e' \right) \left( \frac{1}{2} ay_i \cdot x_{\sigma^{-1}(i)} b \right) x_{\sigma^{-1}(i)} y_i = \frac{1}{2} ay_i \cdot \Delta((x_1, ..., x_{n+1}),(y_1, ..., y_{n-1}, b, y_{i+1}, ..., y_{n+1})).
\]

So \{ab, \Delta((x_1, ..., x_{n+1}),(y_1, ..., y_{n+1}))\} belongs to \(R_n(\mathcal{P})\) in this case.

(d) For \(b\) coincides with certain point \(x\) such that \(x = x_i = y_j\), combining the cases (b) and (c), we have \{ab, \Delta((x_1, ..., x_{n+1}),(y_1, ..., y_{n+1}))\} belongs to \(R_n(\mathcal{P})\) by the similar arguments.

(ii) If \(x_s\) is replaced by certain \(y_t\) as in Figure 3 We have \{ab, \Delta((x_1, ..., x_{n+1}),(y_1, ..., y_{n+1}))\} belongs to \(R_n(\mathcal{P})\) by combining the above two cases.

We conclude that \{ab, \Delta((x_1, ..., x_{n+1}),(y_1, ..., y_{n+1}))\} in \(R_n(\mathcal{P})\) for any \(\Delta((x_1, ..., x_{n+1}),(y_1, ..., y_{n+1}))\) in \(R_n(\mathcal{P})\) and any \(ab\) in \(\mathcal{Z}(\mathcal{P})\).

Lemma 3.9 For any \(a, b, x_1, ..., x_{n+1}, y_1, ..., y_{n+1} \in \mathcal{P}\), we have
\[
\sum_{i=1}^{n+1} ay_i \cdot \Delta((x_1, ..., x_{n+1}),(y_1, ..., y_{i-1}, b, y_{i+1}, ..., y_{n+1})) = \sum_{i=1}^{n+1} x_i b \cdot \Delta((x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_{n+1}),(y_1, ..., y_{n+1})).
\]
\[
\sum_{i=1}^{n+1} ay_i \cdot \Delta((x_1, ..., x_{n+1}),(y_1, ..., y_{i-1}, b, y_{i+1}, ..., y_{n+1})) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} ay_i \cdot x_j b \cdot \Delta((x_1, ..., x_{j-1}, x_{j+1}, ..., x_{n+1}),(y_1, ..., y_{i-1}, y_{i+1}, ..., y_{n+1})).
\]

We develop the right hand side of Equation 30 in the row of \(a\), we have
\[
\sum_{i=1}^{n+1} x_i b \cdot \Delta((x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_{n+1}),(y_1, ..., y_{n+1})) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} x_i b \cdot ay_j \cdot \Delta((x_1, ..., x_{i-1}, x_{i+1}, ..., x_{n+1}),(y_1, ..., y_{j-1}, y_{j+1}, ..., y_{n+1})).
\]

Compare Equation 31 with Equation 32 we conclude that
\[
\sum_{i=1}^{n+1} ay_i \cdot \Delta((x_1, ..., x_{n+1}),(y_1, ..., y_{i-1}, b, y_{i+1}, ..., y_{n+1})) = \sum_{i=1}^{n+1} x_i b \cdot \Delta((x_1, ..., x_{i-1}, a, x_{i+1}, ..., x_{n+1}),(y_1, ..., y_{n+1})).
\]
Formulas \ref{eq:14} \ref{eq:15} \ref{eq:16} are proved by our induction procedures and the lemma above.

Remark 3.10  (i) For example, as in Figure 3, we have
\[ \{xz, \Delta((x,z,y),(z,x,t))\} = -xt \cdot \Delta((x,z,y),(z,x,z)) = 0. \] (34)

(ii) It is easy to notice that a coincides with some \( y_j \) and \( b \) coincides with some \( x_i \) can be combined into generic case. But the triviality of these cases are not due to the linking number, but due to the definition \( xx = 0 \) in \( \mathbb{Z}(P) \).

Finally, we conclude that the swapping bracket over \( \mathbb{Z}_n(P) \) is well defined for \( n \geq 2 \).

Definition 3.11 [Rank \( n \) Swapping Algebra of \( P \)] The rank \( n \) swapping algebra of \( P \) is the ring \( \mathbb{Z}_n(P) \) equipped with the swapping bracket.

4. Second main result \( \mathbb{Z}_n(P) \) is an integral domain

In this section, the field \( \mathbb{K} \) is \( \mathbb{R} \) or \( \mathbb{C} \). We want to know that if the cross fractions are well-defined in the fraction ring of \( \mathbb{Z}_n(P) \). To this end, we prove that \( \mathbb{Z}_n(P) \) is an integral domain through a geometry model considered by H. Weyl [W39] and C. De Concini and C. Procesi [CP76].

4.1 A geometry model for \( \mathbb{Z}_n(P) \)

Let us introduce a geometry model essentially associated to the weak cross ratios. Let \( M_{n,p} \) be the configuration space of \( p \) vectors in \( \mathbb{K}^n \) and \( p \) co-vectors in \( \mathbb{K}^{n^*} \). Here we denote \( a_i = (a_{i,l})_{l=1}^n, b_i = (b_{i,l})_{l=1}^n \). Let \( \mathbb{K}[M_{n,p}] \) be the polynomial ring generated by coordinates functions on \( M_{n,p} = (\mathbb{K}^n \times \mathbb{K}^{n^*})^p \).

Notation 4.1 We define the product between a vector \( a_i \) in \( \mathbb{K}^n \) and a co-vector \( b_j \) in \( \mathbb{K}^{n^*} \) by
\[ \langle a_i | b_j \rangle := b_j(a_i) = \sum_{k=1}^n a_{i,k} \cdot b_{j,k}. \] (35)
Let $GL(n, \mathbb{K})$ acts naturally on the vectors and the covectors by $g \circ a_i := g \cdot a_i$, $g \circ b_j := (g^{-1})^T \cdot b_j$ where $T$ is the transpose of the matrix. It induces a $GL(n, \mathbb{K})$ action on $\mathbb{K}[M_{n,p}]$, defined by:

- For any $g \in GL(n, \mathbb{K})$, $a, b \in \mathbb{K}[M_{n,p}]$, we have $g \circ (a + b) = g \cdot a + g \cdot b$;
- For any $g \in GL(n, \mathbb{K})$, $a, b \in \mathbb{K}[M_{n,p}]$, we have $g \circ (a \cdot b) = (g \cdot a) \cdot (g \cdot b)$.

Then $\mathbb{K}[M_{n,p}]$ is a $GL(n, \mathbb{K})$-module.

We denote the $GL(n, \mathbb{K})$ invariant ring of $\mathbb{K}[M_{n,p}]$ by $\mathbb{K}[M_{n,p}]^{GL(n,\mathbb{K})}$.

Let $B_{n\mathbb{K}}$ be the subring of $\mathbb{K}[M_{n,p}]$ generated by $\langle a_i | b_j \rangle_{i=1,j=1}^p$.

Since $\langle a_i | b_j \rangle \in \mathbb{K}[M_{n,p}]$ is invariant under $GL(n, \mathbb{K})$, we have $B_{n\mathbb{K}} \subseteq \mathbb{K}[M_{n,p}]^{GL(n,\mathbb{K})}$. Moreover, C. De Concini and C. Procesi proved that

**Theorem 4.2** [C. De Concini and C. Procesi [CP76]] $B_{n\mathbb{K}} = \mathbb{K}[M_{n,p}]^{GL(n,\mathbb{K})}$.

Since $\mathbb{K}[M_{n,p}]$ is an integral domain, we have

**Corollary 4.3** [C. De Concini and C. Procesi [CP76]] The subring $B_{n\mathbb{K}}$ is an integral domain.

H. Weyl describe $B_{n\mathbb{K}}$ as a quotient ring.

**Theorem 4.4** [H. Weyl [W39]] The relations in $B_{n\mathbb{K}}$ are generated by $R = \{ f \in B_{n\mathbb{K}} \mid f = \det \begin{pmatrix} \langle a_{i_1} | b_{j_1} \rangle & \cdots & \langle a_{i_1} | b_{j_{n+1}} \rangle \\ \cdots & \cdots & \cdots \\ \langle a_{i_{n+1}} | b_{j_1} \rangle & \cdots & \langle a_{i_{n+1}} | b_{j_{n+1}} \rangle \end{pmatrix}, \forall i_k, j_l = 1, \ldots, p \}$.

**Remark 4.5** In other words, let $W$ be the polynomial ring $\mathbb{K}[\{x_{i,j}\}_{i,j=1}^p]$, $r = \{ f \in W \mid f = \det \begin{pmatrix} x_{i_1,j_1} & \cdots & x_{i_1,j_{n+1}} \\ \cdots & \cdots & \cdots \\ x_{i_{n+1},j_1} & \cdots & x_{i_{n+1},j_{n+1}} \end{pmatrix}, \forall i_k, j_l = 1, \ldots, p \}$, let $T$ be the ideal of $W$ generated by $r$, then we have $B_{n\mathbb{K}} \cong W/T$.

Let us recall that $Z_n(\mathcal{P}) = Z(\mathcal{P})/R_n(\mathcal{P})$ is the rank $n$ swapping ring where $\mathcal{P} = \{x_1, \ldots, x_p\}$. When we identify $a_i$ with $x_i$ on the left and $b_i$ with $x_i$ on the right of the pairs of points in $Z_n(\mathcal{P})$, we obtain the main result of this subsection below.

**Theorem 4.6** Let $Z_n(\mathcal{P})$ be the rank $n$ swapping ring. Let $S_{n\mathbb{K}}$ be the ideal of $B_{n\mathbb{K}}$ generated by $\{ \langle a_i | b_i \rangle \}_{i=1}^p$, then $B_{n\mathbb{K}}/S_{n\mathbb{K}} \cong Z_n(\mathcal{P})$.

### 4.2 Proof of the second main result

**Theorem 4.7** [Second main result] For $n > 1$, $Z_n(\mathcal{P})$ is an integral domain.

**Proof.** Consider these $GL(n, \mathbb{K})$-modules:

(i) Let $L$ be the ideal of $\mathbb{K}[M_{n,p}]$ generated by $\{ \langle a_i | b_i \rangle \}_{i=1}^p$,

(ii) let $K_{n,p}$ be the quotient ring $\mathbb{K}[M_{n,p}]/L$.

---

1Thanks for the reference provided by J. B. Bost.
(iii) let $S_{n\mathbb{K}}$ be the ideal of $B_{n\mathbb{K}}$ generated by $\{\langle a_i | b_i \rangle \}_{i=1}^p$. 

There is an exact sequence of $GL(n, \mathbb{K})$-modules:

$$0 \rightarrow L \rightarrow \mathbb{K}[M_{n,p}] \rightarrow K_{n,p} \rightarrow 0.$$  \hspace{1cm} (36)

By Lie group cohomology \cite{CE48}, the exact sequence above induces the long exact sequence:

$$0 \rightarrow L^{GL(n, \mathbb{K})} \rightarrow \mathbb{K}[M_{n,p}]^{GL(n, \mathbb{K})} \rightarrow K_{n,p}^{GL(n, \mathbb{K})} \rightarrow H^1(GL(n, \mathbb{K}), L) \rightarrow ....$$  \hspace{1cm} (37)

By Weyl’s unitary trick, we have

**Proposition 4.10** Let $S$ be the finite subset $\{\langle a_i | b_i \rangle \}_{i=1}^p$. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Then

$$(\mathbb{K}[M_{n,p}] \cdot S)^{GL(n, \mathbb{K})} = \mathbb{K}[M_{n,p}]^{GL(n, \mathbb{K})} \cdot S.$$  

**Lemma 4.9** There is a ring homomorphism $\theta : B_{n\mathbb{K}}/S_{n\mathbb{K}} \rightarrow K_{n,p}^{GL(n, \mathbb{K})}$ induced from the exact sequence:

$$0 \rightarrow L^{GL(n, \mathbb{K})} \rightarrow \mathbb{K}[M_{n,p}]^{GL(n, \mathbb{K})} \rightarrow K_{n,p}^{GL(n, \mathbb{K})} \rightarrow H^1(GL(n, \mathbb{K}), L) \rightarrow ....$$  \hspace{1cm} (38)

is injective.

**Proof.** By Theorem \cite{4.2} we have $\mathbb{K}[M_{n,p}]^{GL(n, \mathbb{K})} = B_{n\mathbb{K}}$. By Proposition \cite{4.8} we have $L^{GL(n, \mathbb{K})} = (\mathbb{K}[M_{n,p}]\{\langle a_i | b_i \rangle \}_{i=1}^p)^{GL(n, \mathbb{K})} = B_{n\mathbb{K}}\{\langle a_i | b_i \rangle \}_{i=1}^p = S_{n\mathbb{K}}$. So the exact sequence becomes into:

$$0 \rightarrow S_{n\mathbb{K}} \rightarrow B_{n\mathbb{K}} \rightarrow K_{n,p}^{GL(n, \mathbb{K})} \rightarrow H^1(GL(n, \mathbb{K}), L) \rightarrow ...$$  \hspace{1cm} (39)

So there is an injective ring homomorphism $\theta$ from $B_{n\mathbb{K}}/S_{n\mathbb{K}}$ to $K_{n,p}^{GL(n, \mathbb{K})}$. \hfill $\square$

**Proposition 4.10** For $n > 1$, $K_{n,p}$ is an integral domain.

**Proof.** We proof the theorem by induction on the number of the vectors $p$. When $p = 1$, $K_{n,1} = \mathbb{K}[M_{n,p}]/\langle \{\sum_{k=1}^n a_{1,k} \cdot b_{1,k} \} \rangle$.

**Lemma 4.11** For $n > 1$, $\sum_{k=1}^n a_{1,k} b_{1,k}$ is an irreducible polynomial in the integral domain $\mathbb{K}[M_{n,p}]$.

**Proof.** Let us define the degree of a monomial in $\mathbb{K}[M_{n,p}]$ to be the sum of the degrees in all the variables. Let the degree of a polynomial $f$ in $\mathbb{K}[M_{n,p}]$ be the maximal degree of the monomials in $f$, denoted by $\text{deg}(f)$. Suppose that $\sum_{k=1}^n a_{1,k} b_{1,k}$ is a reducible polynomial in $\mathbb{K}[M_{n,p}]$, we have

$$\sum_{k=1}^n a_{1,k} b_{1,k} = g \cdot h,$$

where $g, h \in \mathbb{K}[M_{n,p}]$, $\text{deg}(g) > 0$ and $\text{deg}(h) > 0$. Since $\mathbb{K}[M_{n,p}]$ is a integral domain, $2 = \text{deg}(gh) = \text{deg}(g) + \text{deg}(h)$, so we have $\text{deg}(g) = \text{deg}(h) = 1$. Suppose that

$$g = \lambda_0 + \lambda_1 \cdot c_1 + ... + \lambda_r \cdot c_r,$$

$$h = \mu_0 + \mu_1 \cdot d_1 + ... + \mu_s \cdot d_s,$$

where $\lambda_1, ..., \lambda_r, \mu_1, ..., \mu_s$ are non zero elements in $\mathbb{K}$, $c_1, ..., c_r$ ($d_1, ..., d_s$ resp.) are different elements in $\{a_{1,k}, b_{1,k}\}_{k=1}^n$. Since there is no square in $g \cdot h$, we have

$$\{c_1, ..., c_r \} \cap \{d_1, ..., d_s \} = \emptyset$$

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and
\[ r \cdot s = n. \]
Since there are 2n variables in \( g \cdot h \), we have
\[ r + s = 2n. \]
Then \( r \cdot s \geq 2n - 1 > n = r \cdot s \), which is a contradiction. We conclude that \( \sum_{k=1}^{n} a_{1,k}b_{1,k} \) is an irreducible polynomial in \( K[M_{n,p}] \).

Since \( K[M_{n,p}] \) is an integral domain and \( \sum_{k=1}^{n} a_{1,k}b_{1,k} \) is an irreducible polynomial by Lemma 4.11, we obtain that \( K_{n,1} \) is an integral domain. Suppose that the theorem is true for \( p = m \geq 1 \). When \( p = m + 1 \),
\[ K_{n,m+1} = K \left[ \left\{ a_{i,k}, b_{i,k} \right\}_{i,k=1}^{m+1,n} / \left( \sum_{k=1}^{n} a_{i,k}b_{i,k} \right)_{i=1}^{m+1} \right] = K_{n,m} \left[ \left\{ a_{m+1,k}, b_{m+1,k} \right\}_{k=1}^{n} / \left( \sum_{k=1}^{n} a_{m+1,k}b_{m+1,k} \right) \right] , \] (40)
we have \( K_{n,m} \left[ \left\{ a_{m+1,k}, b_{m+1,k} \right\}_{k=1}^{n} \right] \) is an integral domain. \( \sum_{k=1}^{n} a_{m+1,k}b_{m+1,k} \) is an irreducible polynomial over \( K \left[ \left\{ a_{m+1,k}, b_{m+1,k} \right\}_{k=1}^{n} \right] \). Since \( a_{m+1,k}, b_{m+1,k} (k = 1, \ldots, n) \) are not variables that appear in \( K_{n,m} \), so \( \sum_{k=1}^{n} a_{m+1,k}b_{m+1,k} \) is an irreducible polynomial over \( K_{n,m} \left[ \left\{ a_{m+1,k}, b_{m+1,k} \right\}_{k=1}^{n} \right] \), so \( K_{n,m+1} \) is an integral domain.

We conclude that \( K_{n,p} \) is an integral domain for any \( p \geq 1 \). \( \square \)

By Proposition 4.10, \( K_{n,p} \) is an integral domain, we deduce that \( K_{n,p}^{GL(n,K)} \) is an integral domain. By Lemma 4.9, there is an injective ring homomorphism \( \theta \) from \( B_{nK}/S_{nK} \) to \( K_{n,p}^{GL(n,K)} \), so \( B_{nK}/S_{nK} \) is an integral domain. Moreover, by Theorem 4.6, \( Z_n(P) \cong B_{nK}/S_{nK} \), finally, we conclude that \( Z_n(P) \) is an integral domain. \( \square \)

Remark 4.12 \( Z_1(P) \) is not an integral domain, since
\[ D = xy, yz = \det \begin{pmatrix} xy & xz \\ yy & yz \end{pmatrix} \]
is zero in \( Z_1(P) \), but we have \( xy \) and \( yz \) are not zero in \( Z_1(P) \) whenever \( x \neq y, y \neq z \).

4.3 Rank \( n \) swapping fraction algebra of \( P \)

Definition 4.13 The rank \( n \) swapping fraction ring \( Q_n(P) \) is the total fraction ring of \( Z_n(P) \).

Similar to \( Q(P) \), we have the swapping bracket is well defined on \( Q_n(P) \).

Definition 4.14 The rank \( n \) swapping fraction algebra of \( P \) is the ring \( Q_n(P) \) equipped with the closed swapping bracket.
4.4 Rank \( n \) swapping multifraction algebra of \( \mathcal{P} \)

Let \( \mathcal{CR}_n(\mathcal{P}) = \{ [x, y, z, t] = \frac{xz}{xt} \cdot \frac{yt}{yz} \in \mathcal{Q}_n(\mathcal{P}) \mid \forall x, y, z, t \in \mathcal{P}, x \neq t, y \neq z \} \) be the set of all the cross fractions in \( \mathcal{Q}_n(\mathcal{P}) \).

**Definition 4.15** The rank \( n \) swapping multifraction ring \( \mathcal{B}_n(\mathcal{P}) \) is the subring of \( \mathcal{Q}_n(\mathcal{P}) \) generated by \( \mathcal{CR}_n(\mathcal{P}) \).

Similar to Proposition [2.10] we have

**Proposition 4.16** \( \mathcal{B}_n(\mathcal{P}) \) is closed under swapping bracket.

**Definition 4.17** The rank \( n \) swapping multifraction algebra of \( \mathcal{P} \) is the ring \( \mathcal{B}_n(\mathcal{P}) \) equipped with the closed swapping bracket.

5. Some observations

5.1 Hyperconvexity and stability

In this section, let \( S \) be a surface of negative Euler class.

**Definition 5.1** [Hitchin component [H92]] An \( n \)-Fuchsian homomorphism from \( \pi_1(S) \) to \( \text{PSL}(n, \mathbb{R}) \) is a homomorphism \( \rho = i \circ \rho_0 \), where \( \rho_0 \) is a discrete faithful homomorphism with values in \( \text{PSL}(2, \mathbb{R}) \) and \( i \) is the irreducible homomorphism from \( \text{PSL}(2, \mathbb{R}) \) to \( \text{PSL}(n, \mathbb{R}) \). A homomorphism is Hitchin if it may be deformed into an \( n \)-Fuchsian homomorphism. Hitchin component \( H_n(S) \) is the space of Hitchin homomorphisms up to adjoint action of \( \text{PSL}(n, \mathbb{R}) \).

**Definition 5.2** [Hyperconvex map] [L06] A continuous map \( \xi \) from a set \( \mathcal{P} \) to \( \mathbb{R}P^{n-1} \) is hyperconvex if for any pairwise distinct points \( (x_1, \ldots, x_p) \) with \( p \leq n \), the following sum is direct

\[
\xi(x_1) + \ldots + \xi(x_p).
\]

Let \( \partial_\infty \pi_1(S) \) be the boundary at infinity of \( \pi_1(S) \). When we fix an uniformisation of the universal cover of the surface \( S \) equipped with a complex structure, \( \partial_\infty \pi_1(S) \) can be identified with the real projective line \( \mathbb{R}P^1 \) as the boundary of \( \mathbb{H}^2 \).

**Definition 5.3** [\( n \)-hyperconvex] [L06] A homomorphism \( \rho \) from \( \pi_1(S) \) to \( \text{PSL}(n, \mathbb{R}) \) is \( n \)-hyperconvex, if there exists a \( \rho \)-equivariant hyperconvex map \( \xi \) from \( \partial_\infty \pi_1(S) \) to \( \mathbb{R}P^{n-1} \), namely \( \xi(\gamma x) = \rho(\gamma)\xi(x) \). Such a map is called the limit curve of the homomorphism.

**Theorem 5.4** [F. Labourie [L06], O. Guichard [Gu08]] Every homomorphism \( \rho \) from \( \pi_1(S) \) to \( \text{PSL}(n, \mathbb{R}) \) is Hitchin if and only if \( \rho \) is \( n \)-hyperconvex.

Let \( \text{PGL}(n, \mathbb{R}) \) acts canonically on \( (\mathbb{R}P^{n-1})^k \), recall the stability of Mumford.

**Definition 5.5** [Mumford [Mum94]] The stable point \( (z_1, \ldots, z_k) \) of \( (\mathbb{R}P^{n-1})^k \) is the point such that for every proper linear subspace \( L \) of \( \mathbb{R}P^{n-1} \):

\[
\frac{\text{number of points } z_i \text{ in } L}{k} < \frac{\dim L + 1}{n}
\]

(41)

By comparing the definition of \( n \)-hyperconvexity with the stability of Mumford, we obtain the main result of this subsection below.
Proposition 5.6 Let \( \rho \) be a Hitchin representation, let \( \xi \) be its associated \( n \)-hyperconvex map, let \( x_1, \ldots, x_k \in \partial_\infty \pi_1(S) \) be \( k \) different points where \( k \geq n \), then \( (\xi(x_1), \ldots, \xi(x_k)) \) is stable in \((\mathbb{R}^{[n-1]} )^k\).

5.2 Weak cross ratio

If \( \rho \) is \( n \)-hyperconvex, there is uniquely another \( \rho \)-equivariant hyperconvex map \( \xi^* \) from \( \partial_\infty \pi_1(S) \) to \( \mathbb{P}(\mathbb{R}^n)^* \) such that \( \xi(x) \in \ker(\xi^*(y)) \Leftrightarrow x = y \). Hence \( \rho \in H_n(S) \) is associated with a double limit curve \( (\xi, \xi^*) \) by Theorem 5.4.

Definition 5.7 [Weak cross ratio \[L07\]] Let \( (\xi, \xi^*) \) be a double limit curve. Let \( \tilde{\xi}, \tilde{\xi}^* \) be the lifts of \( \xi, \xi^* \) resp.) with values in \( \mathbb{R}^n \). The weak cross ratio \( B_{\xi, \xi^*} \) of 4 different points \( x, y, z, t \) in \( \partial_\infty \pi_1(S) \) is defined to be

\[
B_{\xi, \xi^*}(x, y, z, t) = \left\langle \tilde{\xi}(x), \tilde{\xi}^*(z) \right\rangle \cdot \left\langle \tilde{\xi}(y), \tilde{\xi}^*(t) \right\rangle, \tag{42}
\]

which is independent of the lifts \( \tilde{\xi} \) with values in \( \mathbb{R}^n \) and \( \tilde{\xi}^* \) with values in \( \mathbb{R}^{n*} \).

Remark 5.8 Similar to weak cross ratio is well defined for Hitchin component, cross fraction is well defined for the integral domain \( \mathbb{Z}(\mathcal{P}) \).

F. Labourie \[L12\] define a ring homomorphism \( I : \mathcal{B}(\mathcal{P}) \to C^\infty(H_n(S)) \) such that

\[
I \left(\frac{zx}{xt} \cdot \frac{yt}{yz}\right)(\rho) = B_{\xi, \xi^*}(x, y, z, t) \tag{43}
\]

for all \( \frac{zx}{xt}, \frac{yt}{yz} \in CR(\mathcal{P}) \). Since \( CR(\mathcal{P}) \) generates \( \mathcal{B}(\mathcal{P}) \) over \( \mathbb{R} \), the map above shall be extended to the whole ring \( \mathcal{B}(\mathcal{P}) \).

Since there is the swapping bracket on \( \mathcal{B}(\mathcal{P}) \), the ring homomorphism \( I \) induces a Poisson bracket \( \{\cdot, \cdot\}_I \) on \( I(\mathcal{B}(\mathcal{P})) \).

Definition 5.9 For any \( \alpha, \beta \in \mathcal{B}(\mathcal{P}) \), the Poisson bracket \( \{\cdot, \cdot\}_I \) on \( I(\mathcal{B}(\mathcal{P})) \) is

\[
\{I(\alpha), I(\beta)\}_I := I(\{\alpha, \beta\}). \tag{44}
\]

Definition 5.10 For \( n > 1 \), let \( i'_n : CR_n(\mathcal{P}) \to C^\infty(H_n(S)) \) be the map such that for any \( \frac{zx}{xt}, \frac{yt}{yz} \in CR_n(\mathcal{P}) \):

\[
i'_n \left(\frac{zx}{xt} \cdot \frac{yt}{yz}\right)(\rho) = B_{\xi, \xi^*}(x, y, z, t), \tag{45}
\]

where \( \rho \) is associated with the double limit curve \( (\xi, \xi^*) \).

By comparing the ring \( R_n(\mathcal{P}) \) with the rank \( n \) cross ratio condition \[L07\], we have

Proposition 5.11 For \( n > 1 \), the map \( i'_n \) extends to a ring homomorphism \( i_n : \mathcal{B}_n(\mathcal{P}) \to C^\infty(H_n(S)) \) where \( i_n|_{CR_n(\mathcal{P})} = i'_n \).

The ring homomorphism \( i_n \) is not injective.
5.3 Injectivity

The ring homomorphism $i_n$ is not injective at least due to the fact that $\frac{zx}{xt} \cdot \frac{yt}{yz} -yz \cdot \frac{zx}{xt} = 0$, now we consider a discrete version of $i_n$ where the corresponding $\pi_1(S)$ is identity.

Let us recall that $\mathcal{Z}_n(\mathcal{P}) = \mathcal{Z}(\mathcal{P})/R_n(\mathcal{P})$ is the rank $n$ swapping ring where $\mathcal{P} = \{x_1,\ldots,x_p\}$. Recall that $M_{n,p}$ is the configuration space of $p$ vectors $a_i$ in $\mathbb{K}^n$ and $p$ co-vectors $b_i$ in $\mathbb{K}^n\ast$. When we identify $x_i$ on the left with $a_i$ and $x_i$ on the right with $b_i$, we induce a map $j_n$ from $\mathcal{C}_n(\mathcal{P})$ to $\mathcal{C}^\infty(M_{n,p})$:

$$j_n \left( \frac{x_i x_k}{x_i x_l} \cdot \frac{x_j x_l}{x_j x_k} \right) (f) = \frac{\langle a_i | b_k \rangle}{\langle a_i | b_l \rangle} \cdot \frac{\langle a_j | b_l \rangle}{\langle a_j | b_k \rangle}$$

(46)

where $f = (a_1,\ldots,a_p,b_1,\ldots,b_p) \in M_{n,p}$.

Similar as Proposition 5.11, we have

**Proposition 5.12** For $n > 1$, the map $j_n$ extends to a ring homomorphism $k_n : \mathcal{B}_n(\mathcal{P}) \to \mathcal{C}^\infty(M_{n,p})$, where $k_n|_{\mathcal{C}_n(\mathcal{P})} = j_n$.

Moreover, the isomorphism between $\mathcal{B}_{n\mathbb{K}}/S_{n\mathbb{K}}$ and $\mathcal{Z}_n(\mathcal{P})$ induce the isomorphism between $k_n(\mathcal{B}_n(\mathcal{P}))$ and $\mathcal{B}_n(\mathcal{P})$. So we have

**Proposition 5.13** For $n > 1$, there exists a ring homomorphism $l_n$ from $k_n(\mathcal{B}_n(\mathcal{P}))$ to $\mathcal{B}_n(\mathcal{P})$ such that

$$l_n \circ k_n = Id_{\mathcal{B}_n(\mathcal{P})}.$$ 

**Corollary 5.14** For $n > 1$, $k_n$ is injective.

**Appendix**

The proof of Proposition 4.8

**Proof.** We prove the proposition in three steps. Firstly, we prove the proposition when we substitute $\mathcal{U}(n)$ for $\text{GL}(n,\mathbb{K})$; secondly, by Corollary 5.17, we prove the proposition for $\text{GL}(n,\mathbb{C})$; finally, by Corollary 5.17, we prove the proposition for $\text{GL}(n,\mathbb{R})$.

(i) Let $\mathcal{U}(n) = \{g \in \text{GL}(n,\mathbb{C}) \mid g \cdot \bar{g}^T = I\}$.

Let us prove that

$$\mathcal{C}[\mathcal{M}_p] \cdot S \subseteq \mathcal{C}[\mathcal{M}_p]^{\mathcal{U}(n)} \cdot S.$$ 

Of course, we have

$$(\mathcal{C}[\mathcal{M}_p] \cdot S)^{\mathcal{U}(n)} \supseteq \mathcal{C}[\mathcal{M}_p]^{\mathcal{U}(n)} \cdot S.$$ 

We now prove that $(\mathcal{C}[\mathcal{M}_p] \cdot S)^{\mathcal{U}(n)} \subseteq \mathcal{C}[\mathcal{M}_p]^{\mathcal{U}(n)} \cdot S$. Let $dg$ be a Haar measure on $\mathcal{U}(n)$. Let $x$ belongs to $(\mathcal{C}[\mathcal{M}_p] \cdot S)^{\mathcal{U}(n)}$. We represent $x$ by $\sum_{l=1}^k t_l \cdot s_l$ where $t_l \in \mathcal{C}[\mathcal{M}_p]$ and $s_l \in S$. Since $S \subseteq \mathcal{C}[\mathcal{M}_p]^{\text{GL}(n,\mathbb{C})} \subseteq \mathcal{C}[\mathcal{M}_p]^{\mathcal{U}(n)}$, for any $g \in \mathcal{U}(n)$, $g \circ s_l = s_l$. Thus we have

$$x = g \circ x = \sum_{l=1}^k (g \circ t_l) \cdot (g \circ s_l) = \sum_{l=1}^k (g \circ t_l) \cdot s_l.$$ 

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So
\[
g \circ x = \int_{U(n)} \sum_{l=1}^{k} (g \circ t_l) \cdot s_l \, dg = \sum_{l=1}^{k} \left( \int_{U(n)} g \circ t_l \, dg \right) \cdot s_l. \tag{47}
\]
Let
\[
b_l = \int_{U(n)} g \circ t_l \, dg.
\]
For any \(g_1\) in \(U(n)\), we have
\[
g_1 \circ b_l = \int_{U(n)} g_1 \circ (g \circ t_l) \, dg = \int_{U(n)} ((g_1 \cdot g) \circ t_l) \, dg = \int_{U(n)} ((g_1 \cdot g) \circ t_l) \, dg = b_l. \tag{48}
\]
So \(b_l\) belongs to \(\mathbb{C}[M_p]^{U(n)}\), hence \(x\) belongs to \(\mathbb{C}[M_p]^{U(n)} \cdot S\). Therefore, we have
\[
(\mathbb{C}[M_p] \cdot S)^{U(n)} \subseteq \mathbb{C}[M_p]^{U(n)} \cdot S.
\]
We conclude that
\[
(\mathbb{C}[M_p] \cdot S)^{U(n)} = \mathbb{C}[M_p]^{U(n)} \cdot S.
\]
(ii) Secondly, let us prove that
\[
(\mathbb{C}[M_p] \cdot S)^{GL(n, \mathbb{C})} = \mathbb{C}[M_p]^{GL(n, \mathbb{C})} \cdot S.
\]
Let \(\mathfrak{g}\) be the Lie algebra of the Lie group \(G\). Let \(\mathfrak{s}\) be a subset of \(\mathfrak{g}\). Let \(V\) be a subset of \(\mathbb{C}[M_p]\). Let
\[
\mathcal{V} = \{ a \in V \mid \left. \frac{d}{dt} \right|_{t=0} \exp(t \cdot h) \cdot a = 0 \quad \forall h \in \mathfrak{s} \}.
\]
Since for the groups \(U(n)\) and \(GL(n, \mathbb{C})\), the exponential map is surjective, for any subset \(V\) of \(\mathbb{C}[M_p]\), we have
\[
V^{U(n)} = V^{u(n)}
\]
and
\[
V^{GL(n, \mathbb{C})} = V^{\mathfrak{gl}(n, \mathbb{C})}.
\]
To prove that
\[
V^{\mathfrak{gl}(n, \mathbb{C})} = V^{u(n)},
\]
since
\[
V^{\mathfrak{gl}(n, \mathbb{C})} = V^{u(n) + i \cdot u(n)} = V^{u(n)} \cap V^{i \cdot u(n)},
\]
we only have to prove that
\[
V^{u(n)} = V^{i \cdot u(n)}.
\]

**Lemma 5.15** For any \(u \in \mathfrak{gl}(n, \mathbb{C})\), any \(v \in \mathbb{C}[M_p]\), we have
\[
\left. \frac{d}{dt} \right|_{t=0} (\exp (t \cdot i \cdot u) \circ (v)) = i \cdot \left. \frac{d}{dt} \right|_{t=0} (\exp (t \cdot u) \circ (v)). \tag{49}
\]
**Proof.** Let \(a_{i_1,j_1} \cdots a_{i_s,j_s} b_{k_1,l_1} \cdots b_{k_l,l_t}\) be a monomial in \(\mathbb{C}[M_p]\), where \(i_1, \ldots, i_s, k_1, \ldots, k_l \in \)
Replacing \( u \) by \( i \cdot u \) in the above formula, we have

\[
\left. \frac{d}{dt} \right|_{t=0} \exp \left( t \cdot (i \cdot u) \right) (a_{i_1,j_1} \cdots a_{i_s,j_s} \cdot b_{k_1,l_1} \cdots b_{k_t,l_t}) = \left. i \cdot \frac{d}{dt} \right|_{t=0} \exp \left( t \cdot u \right) (a_{i_1,j_1} \cdots a_{i_s,j_s} \cdot b_{k_1,l_1} \cdots b_{k_t,l_t}) \tag{51}
\]

Since for any \( v_1, \ldots, v_r \in \mathbb{C}[M_p] \), any \( \lambda_1, \ldots, \lambda_r \in \mathbb{C} \) and any \( h \in \mathfrak{gl}(n, \mathbb{C}) \), we have

\[
\left. \frac{d}{dt} \right|_{t=0} \exp \left( t \cdot h \right) (\lambda_1 \cdot v_1 + \ldots + \lambda_r \cdot v_r) = \sum_{k=1}^r \lambda_r \cdot \left. \frac{d}{dt} \right|_{t=0} \exp \left( t \cdot h \right) (v_k),
\]

we obtain that for any \( v \in \mathbb{C}[M_p] \),

\[
\left. \frac{d}{dt} \right|_{t=0} \exp \left( t \cdot (i \cdot u) \right) (v) = \left. i \cdot \frac{d}{dt} \right|_{t=0} \exp \left( t \cdot u \right) (v) \tag{52}
\]

\[ \square \]

**Remark 5.16** *This lemma largely depends on the group action of GL\( (n, \mathbb{K}) \) on \( \mathbb{K}[M_p] \), see Notation 4.7.*

**Corollary 5.17** *Let \( V \) be a subset of \( \mathbb{C}[M_p] \). Let \( \mathfrak{g} \) be a linear Lie algebra. Then

\[
V^\mathfrak{g} = V^{i \cdot \mathfrak{g}}.
\]

**Proof.** If \( v \in V^\mathfrak{g} \), then

\[
\left. \frac{d}{dt} \right|_{t=0} \exp \left( t \cdot u \right) (v) = 0
\]

for any \( u \in \mathfrak{g} \). By Lemma 5.15 we have

\[
\left. \frac{d}{dt} \right|_{t=0} \exp \left( t \cdot (i \cdot u) \right) (v) = 0
\]

for any \( u \in \mathfrak{g} \). So \( v \in V^{i \cdot \mathfrak{g}} \). Hence, we have \( V^\mathfrak{g} \subseteq V^{i \cdot \mathfrak{g}} \). Similarly, we have \( V^{i \cdot \mathfrak{g}} \supseteq V^\mathfrak{g} \). We conclude that

\[
V^\mathfrak{g} = V^{i \cdot \mathfrak{g}}.
\]

\[ \square \]
By Corollary 5.17, we obtain that
\[ V^u(n) = V^{i_u(n)}, \]
thus we have
\[ V^{\mathfrak{g}(n,\mathbb{C})} = V^{u(n)}. \]

Hence
\begin{align*}
(C[M_p] \cdot S)^{\text{GL}(n,\mathbb{C})} &= (C[M_p] \cdot S)^{\mathfrak{g}(n,\mathbb{C})} = (C[M_p] \cdot S)^{u(n)} = (C[M_p] \cdot S)^{U(n)} \\
&= C[M_p]^{U(n)} \cdot S = C[M_p]^{u(n)} \cdot S = C[M_p]^{\mathfrak{g}(n,\mathbb{C})} \cdot S = C[M_p]^{\text{GL}(n,\mathbb{C})} \cdot S. \tag{53}
\end{align*}

We conclude that
\[ (C[M_p] \cdot S)^{\text{GL}(n,\mathbb{C})} = C[M_p]^{\text{GL}(n,\mathbb{C})} \cdot S. \]

(iii) Finally, let us prove that
\[ (R[M_p] \cdot S)^{\mathfrak{g}(n,\mathbb{R})} = R[M_p]^{\text{GL}(n,\mathbb{R})} \cdot S. \]

Of course, we have
\[ (R[M_p] \cdot S)^{\mathfrak{g}(n,\mathbb{R})} \supseteq R[M_p]^{\text{GL}(n,\mathbb{R})} \cdot S. \]

To prove that
\[ (R[M_p] \cdot S)^{\mathfrak{g}(n,\mathbb{R})} \subseteq R[M_p]^{\text{GL}(n,\mathbb{R})} \cdot S, \]
firstly, we have
\[ (R[M_p] \cdot S)^{\mathfrak{g}(n,\mathbb{R})} \subseteq (R[M_p] \cdot S)^{\mathfrak{g}(n,\mathbb{R})}. \tag{54} \]

By Corollary 5.17, we have
\[ (R[M_p] \cdot S)^{\mathfrak{g}(n,\mathbb{R})} = (R[M_p] \cdot S)^{\mathfrak{g}(n,\mathbb{R}) + i \mathfrak{g}(n,\mathbb{R})} = (R[M_p] \cdot S)^{\mathfrak{g}(n,\mathbb{C})} = (R[M_p] \cdot S)^{\text{GL}(n,\mathbb{C})}. \tag{55} \]

On the other hand, by case 2, we have
\[ (C[M_p] \cdot S)^{\text{GL}(n,\mathbb{C})} = C[M_p]^{\text{GL}(n,\mathbb{C})} \cdot S = B_{\mathbb{C}} \cdot S. \]

When we restrict to the polynomials with real coefficients of the above equation, we have
\[ (R[M_p] \cdot S)^{\text{GL}(n,\mathbb{C})} = B_{\mathbb{R}} \cdot S = R[M_p]^{\text{GL}(n,\mathbb{R})} \cdot S. \tag{56} \]

By Equations 54, 55, we obtain that
\[ (R[M_p] \cdot S)^{\text{GL}(n,\mathbb{R})} \subseteq R[M_p]^{\text{GL}(n,\mathbb{R})} \cdot S. \]

We conclude that
\[ (R[M_p] \cdot S)^{\text{GL}(n,\mathbb{R})} = R[M_p]^{\text{GL}(n,\mathbb{R})} \cdot S. \]

Finally, we conclude that
\[ (\mathbb{K}[M_p] \cdot S)^{\text{GL}(n,\mathbb{K})} = \mathbb{K}[M_p]^{\text{GL}(n,\mathbb{K})} \cdot S \]
for \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \).

\[ \Box \]

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