On Minimum Area Homotopies of Normal Curves in the Plane

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Abstract

In this paper, we study the problem of computing a homotopy from a planar curve $C$ to a point that minimizes the area swept. The existence of such a minimum homotopy is a direct result of the solution of Plateau’s problem. Chambers and Wang studied the special case that $C$ is the concatenation of two simple curves, and they gave a polynomial-time algorithm for computing a minimum homotopy in this setting. We study the general case of a normal curve $C$ in the plane, and provide structural properties of minimum homotopies that lead to an algorithm. In particular, we prove that for any normal curve there exists a minimum homotopy that consists entirely of contractions of self-overlapping sub-curves (i.e., consists of contracting a collection of boundaries of immersed disks).

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1 Introduction

The theory of minimal surfaces has been extensively studied by many mathematicians and the existence of such surfaces with a given boundary, known as Plateau’s problem, has been proven by Rado and Douglas [5, 9, 10, 13, 14]. In this work, we address the related problem of computing a minimum homotopy that minimizes the homotopy area of a normal curve in the plane. Chambers and Wang [3] have defined the notion of minimum homotopy area to measure the similarity between two simple curves that share the same start and end points. Many continuous deformations, i.e., homotopies, between the two curves exist, but a minimum-area homotopy is a deformation that minimizes the total area swept. Chambers and Wang provided a dynamic programming algorithm to compute such a minimum homotopy in polynomial time.

Here, we study the more general task of computing the minimum homotopy area of an arbitrary closed curve being contracted to a point; see Figure 1 for an example of such a minimum homotopy. This generalizes the Chambers and Wang setting. One application would be to measure the similarity of two non-simple curves (where we create a closed loop by concatenating the two curves).
Any normal homotopy can be described in terms of the combinatorial changes it incurs on the curve, and can thus be characterized by a sequence of homotopy moves \cite{7} which are projections of the well-known Reidemeister moves for knots \cite{1}. In this paper, we provide structural insights for minimum-area homotopies. One of the key ingredients is the use of self-overlapping curves \cite{2, 6, 11, 15, 16}. These curves are the boundaries of immersed disks and they have a natural interior. An algorithm with a polynomial runtime has been given in \cite{15} to detect whether a given normal curve is self-overlapping or not and to find the interior of the curve in case it is self-overlapping. We show that the minimum homotopy area for a self-overlapping curve is equal to its winding area, the integral of the winding numbers over the plane.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{A minimum homotopy is given as a sequence of homotopy moves. The initial curve is self-overlapping.}
\end{figure}

For a general normal curve, we show that a minimum homotopy can be obtained by contracting a sequence of self-overlapping subcurves that are based at intersection points of the curve. This structural theorem reduces the space of homotopies to a finite candidate set. In a preprint \cite{12}, Nie provides an abstract algebraic construction for computing the minimum homotopy. He reduces the problem to computing the weighted cancellation distance on elements of the fundamental group induced by the planar embedding, and this distance can be computed in polynomial-time using dynamic programming. However, our approach is quite different and geometric in nature.

Our results not only solve the problem but also relate minimum homotopy to an interesting class of curves.

\section{Preliminaries}

In this section, we introduce the concepts of normal curves and homotopy moves, which we use throughout the paper.

\subsection{Normal Curves}

A \textit{closed curve} is a continuous map \( C : [0, 1] \to \mathbb{R}^2 \) with \( C(0) = C(1) \). Let \( [C] \) denote the image of this map. We call a closed curve (\textit{piecewise}) regular if it is (piecewise) differentiable and its left and right derivatives never vanish. Note that any regular curve \( C \) is an immersion of the unit circle \( S \) into \( \mathbb{R}^2 \). For a piecewise regular curve \( C \), we call a point \( p \in [C] \) an \textit{intersection point} if \( C^{-1}(p) \) consists of more than a single point. Without loss of generality, we assume that \( C(t) \neq C(0) \) for any \( t \in (0, 1) \).

An intersection point \( p \in [C] \) is called a \textit{simple crossing point} if there exist \( t_1, t_2 \in [0, 1] \), with \( t_1 \neq t_2 \), such that \( C^{-1}(p) = \{t_1, t_2\} \) and if the tangent vectors at \( t_1 \) and \( t_2 \) exist and are linearly independent. In other words, a crossing point is simple if the intersection is transverse. A piecewise regular curve \( C \) is called \textit{normal} if it contains only a finite number of
intersection points and these are all simple crossing points. For a normal curve \( C \), we define the complexity of \( C \) as the number of simple crossing points. We set \( P_C = \{p_0, p_1, \ldots, p_n\} \) where \( p_0 = C(0) \) and \( p_i \) is an intersection point of \( C \) for \( i > 0 \).

Each normal curve \( C \) naturally corresponds to a planar embedded directed graph \( G = (V,E) \). The vertex set \( V = \{0, 1, \ldots, n\} \) represents the set of simple crossing points \( P_C = \{p_0, p_1, \ldots, p_n\} \), including the base point \( p_0 \). A directed edge \((i,j) \in E\) represents a direct connection along the curve from \( p_i \) to \( p_j \). We call two normal curves \( C_1 \) and \( C_2 \) combinatorially equivalent if their induced planar embedded graphs are isomorphic.

Note that each face of this planar embedded graph corresponds to a maximal connected component of \( \mathbb{R}^2 \setminus [C] \) whose boundary consists of a union of edges of the graph. Let \( C \) be a regular curve and let \( f_0, f_1, \ldots, f_k \) be the faces of the induced graph defined by the image of \( C \). For each \( x \in \mathbb{R}^2 \setminus [C] \), the \textit{winding number} of \( C \) at \( x \), which we denote as \( \text{wn}(x,C) \), is defined as the signed number of times that the curve ‘wraps around’ \( x \)\(^3\). Notice that the winding number is constant on each face \( f \). Thus, the winding area of a face \( \text{wn}(f,C) \) is well-defined. For all \( x \in [C] \), we define the winding area to be zero.

For a point \( p_0 \in \mathbb{R}^2 \), let \( \mathcal{C}_{p_0} \) denote the set of all normal curves with start point \( p_0 \), including the constant curve at \( p_0 \). In the following, we only consider normal curves. Such an assumption is justified, as Whitney proved that any regular curve can be approximated with a normal curve that is obtained from an arbitrarily small deformation\(^{\text{[17]}}\).

The \textit{Whitney index} \( \text{Wh}(C) \) of a regular normal curve \( C \) is defined to be the winding number of the derivative \( C' \) about the origin. Note that, by definition of a regular curve, the derivative \( C' \) also defines a closed curve and \( (0,0) \not\in [C'] \). The well-known Whitney-Graustein theorem\(^{\text{[17]}}\) states that two regular curves are regularly homotopic if and only if they have the same Whitney index. For a piecewise regular closed curve \( C \), we set \( \text{Wh}(C) = \text{Wh}(_{\text{reg}}) \), where \(_{\text{reg}} \) is a regular curve approximating \( C \), obtained by smoothing the corners, i.e., the non-differentiable points, of \( C \) in an arbitrarily small deformation.

### 2.2 Homotopies and Homotopy Moves

Let \( p_0 \in \mathbb{R}^2 \). A \textit{homotopy} between two curves \( C_1, C_2 \in \mathcal{C}_{p_0} \) is a continuous map \( H : [0, 1]^2 \to \mathbb{R}^2 \) such that \( H(0,t) = C_1(t) \), \( H(1,t) = C_2(t) \), and \( H(s,0) = p_0 = H(s,1) \) for all \( (s,t) \in [0, 1]^2 \). A homotopy \( H \) between \( C_1 \) and \( C_2 \) is denoted as \( C_1 \overset{H}{\Rightarrow} C_2 \). Since \( \mathbb{R}^2 \) is simply connected, any two curves in \( \mathcal{C}_{p_0} \) are homotopic. In particular, any curve \( C \in \mathcal{C}_{p_0} \) is homotopic to the constant curve \( p_0 \).

We concatenate two homotopies \( C_1 \overset{H_1}{\Rightarrow} C_2 \) and \( C_2 \overset{H_2}{\Rightarrow} C_3 \), denoted \( H_1 + H_2 =: H \), where the new homotopy is given as \( H(s,t) = H_1(2s,t) \) for \( t \in [0, \frac{1}{2}] \) and \( H(s,t) = H_2(2s - 1) \) for \( t \in [\frac{1}{2}, 1] \). Notice that \( H \) is a homotopy from \( C_1 \) to \( C_3 \). Similarly, for a sequence of homotopies \( \{C_k \overset{H_i}{\Rightarrow} C_{i+1}\}_{i=1}^k \), we denote their concatenation \( C_1 \overset{H}{\Rightarrow} C_{k+1} \) with \( H = \sum_{i=1}^k H_i \).

Let \( C \overset{H}{\Rightarrow} p_0 \) be a homotopy. Consider an intermediate curve \( \tilde{C} \) of the homotopy. For each \( p \in \tilde{C} \), if \( p \) is not an intersection point, then \( p \) neighbors two faces of \( \mathbb{R}^2 \setminus [\tilde{C}] \). We can use the orientation of the curve to define one face to be the \textit{left face} and the other to be the \textit{right face}. We call \( H \) \textit{left sense-preserving} if for any non-intersection point \( p = H(s,t) \in [H] \), the point \( H(s + \epsilon, t) \) lies on or to the left of the oriented curve \( H(s, \cdot) \) for each \( s \) and \( t \). Similarly, \( H \) is \textit{right sense-preserving} if \( H(s + \epsilon, t) \) always lies on or to the right of \( H(s, \cdot) \).

As we deform normal curves using homotopies, we necessarily encounter non-normal curves. In order to stay within a nice family of curves, we define a curve to be \textit{almost normal} if it has a finite number of intersection points, which are either simple crossing points, triple points, or non-transverse (tangential) crossing points. We call a homotopy \( H \)
from $C_1$ to $C_2$ a normal homotopy if each intermediate curve is (piecewise) regular, and either normal or almost normal, with only a finite set of them being almost normal.

Any normal homotopy $C \xrightarrow{H} p_0$ can be captured by a sequence of homotopy moves which are similar to Reidemeister moves for knots. There are three types of such moves: the $I_a$- and $I_b$-moves destroy/contract and create self-loops; the $\Pi_a$- and $\Pi_b$-moves destroy and create regions defined by a double-edge of the corresponding graph; and the $\text{III}$-moves invert a triangle. See Figure 2. Without loss of generality, we assume that at each event time point there is only a single homotopy move. Any piecewise differentiable homotopy can be approximated by a normal homotopy [8].

3 Minimum Homotopy Area

In this section, we define minimum homotopy area and give basic properties of minimum homotopies. We introduce self-overlapping curves and decompositions of curves.

3.1 Definition of Minimum Homotopy Area

Let $C_1, C_2 \in \mathcal{C}_{p_0}$ be two curves and $C_1 \xrightarrow{H} C_2$ be a homotopy. Let $E_H: \mathbb{R}^2 \to \mathbb{Z}$ be the function that assigns to each $x \in \mathbb{R}^2$ the number of connected components of $H^{-1}(x)$. In other words, $E_H$ counts how many times the intermediate curves $H(s)$ sweep over $x$. The homotopy area $\text{AREA}(H)$ of $H$ is defined as the integral of $E_H$ over the plane:

$$\text{AREA}(H) = \int_{\mathbb{R}^2} E_H(x) \, dx.$$  

Since addition distributes over the integral and since $E_{H_1+H_2}(x) = E_{H_1}(x) + E_{H_2}(x)$, the area is additive: $\text{AREA}(H_1 + H_2) = \text{AREA}(H_1) + \text{AREA}(H_2)$.

We define the minimum homotopy area between $C_1$ and $C_2$, denoted as $\sigma(C_1, C_2)$, as the infimum homotopy area over all piecewise differentiable homotopies between $C_1$ and $C_2$:

$$\sigma(C_1, C_2) = \inf_H \text{AREA}(H).$$

In this paper, we are interested in the special case where $C_2$ is the constant curve. Hence, we define the minimal homotopy area of a single curve $C \in \mathcal{C}_{p_0}$ to denote the minimal nullhomotopy of the curve $C$, hence we write $\sigma(C) := \sigma(C, p_0)$. We note here that $\sigma(C)$ is well-defined, since $\sigma(C, p) = \sigma(C, p_0)$ for all $p \in [C]$.

A minimum homotopy $H$ is a homotopy that realizes the above infimum. The existence of minimum homotopies is a result of Douglas’ work on the solution of Plateau’s problem; see [10, Theorem 7].
Lemma 1 (Splitting a Minimal Homotopy). Let \( C_1 \xrightarrow{H_1} C_2 \) and \( C_2 \xrightarrow{H_2} C_3 \) be two homotopies and \( H = H_1 + H_2 \). If \( C_1 \xrightarrow{H} C_3 \) is a minimum homotopy, then we have:

1. The sub-homotopies \( H_1 \) and \( H_2 \) are also minimum.
2. \( \sigma(C_1, C_3) = \sigma(C_1, C_2) + \sigma(C_2, C_3) \).
3. If \( C_2 \xrightarrow{H_2'} C_3 \) is another minimum homotopy, so is \( C_1 \xrightarrow{H'} C_3 \) where \( H' = H_1 + H_2' \).

Proof. Let \( H = H_1 + H_2 \) be a minimal homotopy such that \( C_1 \xrightarrow{H_1} C_2 \) and \( C_2 \xrightarrow{H_2} C_3 \). For the sake of contradiction, assume that \( H_1 \) is not minimum. Then, there exists a homotopy \( C_1 \xrightarrow{H_1'} C_2 \) such that \( \text{Area}(H_1') < \text{Area}(H_1) \). Define \( H' = H_1' + H_2 \), and observe that \( H' \) is a homotopy \( C_1 \xrightarrow{H'} C_3 \) such that

\[
\text{Area}(H') = \text{Area}(H_1') + \text{Area}(H_2) < \text{Area}(H_1) + \text{Area}(H_2) = \text{Area}(H).
\]

However, the homotopy \( H \) was minimum, so we have a contradiction. Similarly, we can show that \( H_2 \) must be minimum, which proves Part 1 of this Lemma.

Since \( H, H_1, \) and \( H_2 \) are minimal (from Part 1), we know that \( \sigma(C_1, C_3) = \text{Area}(H) \), \( \sigma(C_1, C_2) = \text{Area}(H_1) \), and \( \sigma(C_2, C_3) = \text{Area}(H_2) \). Putting this together, we conclude \( \sigma(C_1, C_3) = \text{Area}(H) = \text{Area}(H_1) + \text{Area}(H_2) = \sigma(C_1, C_2) + \sigma(C_2, C_3) \). This proves Part 2.

Let \( C_2 \xrightarrow{H_2'} C_3 \) be a minimal homotopy, and let \( H' = H_1 + H_2' \). Then, we know that \( \text{Area}(H') = \text{Area}(H_1) + \text{Area}(H_2') \). Since \( H_2 \) and \( H_2' \) are both minimal, we have \( \text{Area}(H_2) = \text{Area}(H_2') \), and so \( \text{Area}(H') = \text{Area}(H_1) + \text{Area}(H_2) = \text{Area}(H) \), thus proving Part 3 since \( H \) is minimum.

3.2 Winding Area

The winding number defines a function \( wn(\cdot, C) : \mathbb{R}^2 \to \mathbb{Z} \), where \( wn(x, C) \) is the winding number of \( C \) around the point \( x \). We define the winding area \( W(C) \) of \( C \) as the integral:

\[
W(C) = \int_{\mathbb{R}^2} |wn(x, C)| \, dx.
\]

Let \( f_0, f_1, \ldots, f_k \) be the faces of \( C \), where \( f_0 \) is the outer face. Since \( wn(\cdot, C) \) is constant at each face of the curve and \( wn(f_0, C) = 0 \), we obtain the following formula:

\[
W(C) = \sum_{i=1}^{k} |wn(f_i, C)| \cdot \text{Area}(f_i).
\]

For example, consider the curve in Figure 3a. Here, we have \( W(C) = 2\text{Area}(f_2) + \text{Area}(f_1) \), which is equal to the minimum homotopy area of the curve. In general, the winding area is a lower bound for the minimum homotopy area. This has been proved by Chambers and Wang for a special class of curves [13], but the same proof applies to our more general setting, which gives us the following lemma.

Lemma 2 (Winding Area Lower Bound). For any normal curve \( C \), we have \( W(C) \leq \sigma(C) \).

For some curves, the winding area is equal to the minimum homotopy area as in Figure 3a. In Section 3.3, we define a class of curves for which the winding area equals the homotopy area. In general, however, this equality does not hold, as is illustrated in Figure 3b.

A direct consequence of Lemma 2 is the following.
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Figure 3 On the left, a minimum homotopy is given for the curve $C$, where $\sigma(C) = \text{Area}(H) = 2\text{Area}(f_3) + \text{Area}(f_1)$. Notice that $wn(f_1, C) = 1$, $wn(f_2, C) = 2$, and $wn(f_3, C) = 0$.

Corollary 3. If there exists a homotopy $C \xrightarrow{H} p_0$ such that $\text{Area}(H) = W(C)$, then $H$ is minimum and $\sigma(C) = W(C)$.

More generally, we have the following theorem.

Theorem 4 (Sense-Preserving Homotopy Area). If $C \xrightarrow{H} p_0$ is a sense-preserving homotopy, then $H$ is minimum and $W(C) = \sigma(C) = \text{Area}(H)$. Similarly, if $\text{Area}(H) = W(C)$, then $H$ is sense-preserving.

Proof. Without loss of generality, assume that $H$ is left sense-preserving. Consider $H(s, \cdot)$ as $s$ ranges over $[0, 1]$. Let $x \in \mathbb{R}^2 - [C]$. Then, each time $x \in [H(s, \cdot)]$, the winding number at $x$ decreases by one. Hence, $E_H(x) = wn(x, C)$ and $W(C) = \text{Area}(H)$, or $H$ is a minimum homotopy and $W(C) = \sigma(C)$ by Corollary 3.

On the other hand, if $H$ is not sense-preserving, then there is a region $R$ that is swept by edges moving left and edges moving right. Hence, if $x \in R$, $E_H(x) > |wn(x, C)|$. In other words $\text{Area}(H) > W(C)$.

3.3 Self-Overlapping Curves and $k$-Boundaries

Chambers and Wang introduced the notion of consistent winding numbers to describe a class of curves for which the homotopy area and the winding area are equal. In this subsection, we identify a more general class of closed curves for which the same equality is satisfied.

A regular normal curve $C \in \mathfrak{C}_{p_0}$ is self-overlapping if there exists an immersion of the two-disk $F : D^2 \to \mathbb{R}^2$ such that $|C| = F|_{\partial D^2}$. If $C$ is not a regular normal curve, then we call $C$ self-overlapping if there exists an arbitrarily-close approximation $\tilde{C}$, which is a regular normal curve that is self-overlapping. The image $F(D^2)$ is called the interior of $C$.

Self-overlapping curves have been investigated in [2, 11, 15, 16]. A dynamic programming algorithm for testing whether a given curve is self-overlapping has been given in [15]; the runtime of this algorithm is cubic in the number of vertices of the input polygon. Examples of self-overlapping curves are given in Figure 1, Figure 5, and Figure 3a. The curve in Figure 3b is an example of a curve which is not self-overlapping. In the following theorem, we prove that the homotopy area equals the winding area for self-overlapping curves.

Theorem 5 (Winding Area Equality for Self-Overlapping Curves). If $C \in \mathfrak{C}_{p_0}$ is a self-overlapping curve, then $\sigma(C) = W(C)$.

Proof. A straight-line deformation retract $r_s : D^2 \to D^2$ from the unit disk $r_0(\cdot) = D^2$ to a point $r_1(\cdot) = q_0 \in \mathbb{S}^1 = \partial D^2$ induces a minimum-area homotopy $H$ from $\mathbb{S}^1$ to $q_0$, for which $E_H(x) = 1$ if $x \in D^2$ and $E_H(x) = 0$ otherwise.
Now, let $F : D^2 \rightarrow \mathbb{R}^2$ be an immersion of the disk such that $F(q_0) = p_0$ and the curve $C = F|_{\gamma_1}$ is normal. Then, the composition $f_0 = F \circ r_0$ is a deformation retract of the immersed disk $f_0 = F(D^2)$ to $f_1 = p_0$. Restricting this map to $S^1$, we obtain a homotopy $H(s, t) = f_s(e^{2i\pi t})$ for which $H(0, \cdot) = C$ and $H(1, \cdot) = p_0$. Moreover, we have $E_H(x) = |F^{-1}(x)| = wn(x, C)$; a proof of this can be found in [4]. In other words, the homotopy $H$ satisfies $\text{Area}(H) = W(C)$. Hence, by Corollary 3, we conclude that $H$ is a minimum homotopy with $\sigma(C) = \text{Area}(H) = W(C)$.

If $C$ is regular and normal, then the homotopy defined in the proof of Theorem 5 is regular. Furthermore, the intermediate curves eventually become simple loops with Whitney number \pm 1. Hence, by the Whitney-Graustein Theorem [17], we know that $\text{Wh}(C) = \pm 1$ for a self-overlapping curve $C$. We call a self-overlapping curve positive if $\text{Wh}(C) = 1$, and otherwise we call it negative. Observe that, by definition, the Jacobian of an immersion of the disk is either always positive or always negative. Hence, a self-overlapping curve is positive (or negative) if it can be extended to an immersion whose Jacobian is always positive (resp., negative). We summarize this with the following lemma:

**Lemma 6 (Equivalent Properties for Self-Overlapping Curves).** The following statements are equivalent for a regular self-overlapping curve $C$:

- $\text{Wh}(C) = 1$
- An immersion $S \rightarrow C$ can be extended to an immersion of $D^2$ whose Jacobian is always positive.
- The winding numbers $wn(C, p)$ are non-negative for all $p \in \mathbb{R}^2$.
- If $C \xrightarrow{H} p$ is a minimum homotopy for some $p \in [C]$, then $H$ is left sense-preserving.

**Observation 7.** The Whitney index of a curve is invariant under II- and III-moves, but not under I-moves. Moreover, a regular homotopy uses only II- and III-moves, hence the complexity of a self-overlapping curve (defined above to be the number of simple crossing points) is always even.

**Definition 8 (Decomposition).** A decomposition of a normal curve $C$ is a set $\Gamma = \{\gamma_i\}_{i=1}^l$ of closed subcurves of $C$ such that:

- Each $\gamma_i$ is self-overlapping.
- For each $i \neq j \in \{1, 2, \ldots, l\}$, $[\gamma_i] \cap [\gamma_j] \in P_C$.
- $\sqcup_{i=1}^l [\gamma_i] = [C]$.

Such a decomposition always exists for the following reason. Each curve contains a self-overlapping loop. If we remove this self-overlapping loop from the curve, we still have a closed curve which contains another self-overlapping loop. Continuing this process, we decompose the curve into self-overlapping loops. An example of a decomposition of a curve is given in Figure 11. For each $\gamma \in \Gamma$, we define the root $p \in P_C$ of $\gamma$ as follows when $p_0 \notin \gamma$: $p := C(\inf\{t \in [0, 1] : C(t) \in [\gamma]\})$. If $p_0 \in \gamma$, then we define the root of $\gamma$ to be the root of the complement $C \setminus \gamma$.

For any decomposition, there exists an ordering $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ such that the root of $\gamma_i$ does not appear in $\gamma_j$ for any $j \geq i$. Thus, the decomposition $\Gamma$ defines a homotopy $H_{\Gamma}$, which can be obtained by contracting each subloop $\gamma_i \in \Gamma$ to its roots, as in Theorem 5 starting from the last subcurve $\gamma_k$ to the first subcurve $\gamma_1$. If $C \in \mathcal{C}_{p_0}$ admits a decomposition $\{\gamma_1, \ldots, \gamma_k\}$, where each $\gamma_i$ is positive, we call $C$ a $k$-boundary. These curves have been investigated by Titus [16], where he calls such curves interior boundaries. He also gives an algorithm to detect whether a given curve is a $k$-boundary.
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Observation 9. If \( C \) is a \( k \)-boundary with decomposition \( \Gamma = \{ \gamma_1, \gamma_2, \ldots, \gamma_k \} \), then 
\[
\text{Area}(H_\Gamma) = \sum_{i=1}^{k} W(\gamma_i) = \sum_{i=1}^{k} \sigma(\gamma_i).
\]
By Corollary 3 we conclude that \( H_\Gamma \) is a well-behaved minimum homotopy, as defined in the next section, and \( \sigma(C) = W(C) = \text{Area}(H_\Gamma) \). In addition, if \( C \) is a two-boundary with decomposition \( \{ \gamma_1, \gamma_2 \} \), then \( \sigma(C) = \sigma(\gamma_1) + \sigma(\gamma_2) \).

We call the curve \( C \) a \((k\)-boundary if the inverse of the curve \( C^{-1} \) is a \( k \)-boundary. Such curves admit a decomposition where each self-overlapping subloop is negative. More generally, we observe that 
\[
\text{WH}(C) = \sum_{i=1}^{l} \text{WH}(\gamma_i) \quad \text{and} \quad \text{wn}(x, C) = \sum_{i=1}^{l} \text{wn}(x, \gamma_i)
\]
for each point \( x \) in the plane. Hence, \( W(C) \leq \sum_{i=1}^{l} W(\gamma_i) \) and \( \sigma(C) \leq \text{Area}(H_\Gamma) = \sum_{i=1}^{l} \sigma(\gamma_i) \).

Theorem 10 (Minimum Homotopy Decomposition). Let \( C \) be a self-overlapping curve. If \( \Gamma \) is a decomposition with \( |\Gamma| > 1 \), then the induced homotopy \( H_\Gamma \) is not minimum. Likewise, if \( C \) is a \( k \)-boundary and \( \Gamma \) is a decomposition of \( C \) with \( |\Gamma| > k \), then \( H_\Gamma \) cannot be minimum.

Proof. We prove the base case for a proof by induction. Let \( C \) be positive and self-overlapping, and let \( \Gamma = \{ \gamma_1, \gamma_2, \ldots, \gamma_\ell \} \) be a decomposition with \( \ell > 1 \). Then, there exists a negative self-overlapping subcurve \( \gamma \in \Gamma \) and a positive self-overlapping subcurve \( \tilde{\gamma} \in \Gamma \), since \( 1 = \text{WH}(C) = \sum_{i=1}^{l} \text{WH}(\gamma_i) \) and \( \text{WH}(\gamma_i) = \pm 1 \). Observe that the induced homotopy should be right sense-preserving on \( \gamma \) and left sense-preserving on \( \tilde{\gamma} \). In other words, the total homotopy is not sense-preserving. Thus, by Theorem 4 \( \text{Area}(H_\Gamma) > W(C) \). This implies that \( H_\Gamma \) is not minimum by Theorem 5. The second half of the proof follows from a simple inductive argument.

4 Construction of a Minimum Homotopy

In this section, we prove our main theorem which states that each normal curve \( C \) admits a decomposition \( \Gamma \) such that the induced homotopy \( H_\Gamma \) is minimum.

4.1 Well-behaved Minimum Homotopies

Let \( C \in \mathfrak{C}_{p_0} \) be a curve and let \( P_C = \{ p_0, p_1, \ldots, p_n \} \) be its set of simple crossing points. Let \( C \xrightarrow{H} p_0 \) be a homotopy. Observe that when we perform the homotopy, each simple crossing point moves continuously following the intersection points of intermediate curves until those simple crossing points are eliminated. Assume for now that \( H \) does not create new simple crossing points, i.e., it does not contain any \( b \)-moves. Then, each simple crossing point \( p_j \in P_c \) is eliminated via either a \( I_a \) or a \( I_b \)-move. We call a index \( j \) an anchor index if \( p_j \) is eliminated via a \( I_a \)-move; in this case, we call \( p_j \) the corresponding anchor point.

Similarly, when a new intersection point \( p_j \) is created by a \( b \)-move, \( j \) is called an anchor index, if it is later eliminated by a \( I_a \)-move.

For a homotopy \( C \xrightarrow{H} p_0 \), we define \( A_H = \{ j : p_j \text{ is an anchor point} \} \). We order \( A_H \) according to the time the vertices are destroyed. Notice that \( p_0 \) is always an anchor point since the last move for each homotopy is a \( I_b \)-move which contracts an intermediate curve which is a simple loop.

At first glance, one may think that minimum homotopies should only decrease the complexity of the graph of the curve, and that \( b \)-moves increase the complexity. Naturally, one may conjecture that each curve has a minimum homotopy without any \( b \)-moves. However, there are curves for which this is not true. Consider for example the Milnor curve shown in Figure 5. For this curve, any minimum homotopy has to contain a \( I_b \)-move. (This curve is,
For this homotopy, $p_j$ is an anchor point, since it is removed with a $I_b$-move. On the other hand, $p_k$ and $p_l$ are not anchor points as they are removed with a $II_a$-move.

In fact, self-overlapping, and a minimum homotopy that is indicated by the shading sweeps an area equal to the winding area. In the following, we show that these particular $II_b$-moves do not create any complications. Let $C_1 \xrightarrow{H_1} C_2 \xrightarrow{H_2} p_0$ be a homotopy such that $H_1$ consists of a single $II_b$-move, thus creating two new intersection points. Then, we say that the $II_b$-move is significant if either of the intersection points that is created by the move is an anchor point of $H_2$. Intuitively, a significant $II_b$-move makes a structural impact itself, while an insignificant $II_b$-move is only an intermediate move that allows a portion of the curve to pass over another portion. We call a minimum homotopy well-behaved if it does not contain any $I_b$-moves or significant $II_b$-moves.

This curve is self-overlapping and it does not admit a minimum homotopy without $II_b$-moves. The shading indicates an immersion that also defines a minimum homotopy.

Lemma 11. Let $C \xrightarrow{H} p_0$ be a minimum homotopy which has a single anchor point $p_0$, then $H$ is sense-preserving.

The proof of Lemma 11 is identical to the proof of Lemma 3.2 in [3].

Lemma 12. Let $C_1 \xrightarrow{H_1} C_2 \xrightarrow{H_2} p_0$ be a minimum homotopy, where $H_1$ consists of a single homotopy move that is either a $II_a$-move, an insignificant $II_b$-move or a $III$-move. Then, the curve $C_1$ is a positive self-overlapping curve if and only if $C_2$ is a positive self-overlapping curve.

Proof. Notice that both $H_1$ and $H_2$ are left sense-preserving. Divide the disk $D^2$ into two regions, $W$ and $E$ with a line segment $L$. The different homotopy moves induce regions $W'$, $E'$ and curve segment $L'$ in $C_1$ as shown in Figure 5, and since $Wh(C_1) = 1$ the region $W'$ always lies in the interior of $C_1$.

Now, if $C_1$ is positive self-overlapping, we can find an immersion $F : D^2 \to \mathbb{R}^2$ that maps $W$ to $W'$ and $L$ to $L'$, and the restriction of $F$ to $E$ gives an immersion whose boundary is $C_2$. Similarly, if $C_2$ is self-overlapping, then there is an immersion $G : E \to \mathbb{R}^2$ that maps $L$ to $L'$. We can extend $G$ to $D^2$ by mapping $W$ to $W'$ so that $\partial W = L$ is mapped to $\partial W' = L'$. The extended immersion sends the boundary of the disk to $C_1$.

The theorem below follows from the previous two lemmas, and is illustrated in Figure 1.
Here, we map the region $W$ of the disk to corresponding region $W'$ under the three types of homotopy moves induced by a deformation retraction on $D^2$.

**Theorem 13.** Let $C$ be a normal curve with a minimum homotopy $C \xrightarrow{H} p_0$. If $H$ is well-behaved and has a single anchor point $p_0$, then $C$ is self-overlapping. Furthermore, each intermediate curve is self-overlapping.

**Proof.** Notice that before the last homotopy move of $H$, the intermediate curve is a simple loop. Simple loops are self-overlapping. Hence by Lemma 12 each intermediate curve is self-overlapping. Therefore, $C$ is itself self-overlapping.

For a well-behaved homotopy with more than one anchor point, we have the following:

**Theorem 14.** Let $C$ be a normal curve which admits a well-behaved minimum homotopy $C \xrightarrow{H} p_0$. Then, there is a corresponding decomposition $\Gamma$ of $C$ such that $\text{Area}(H) = \text{Area}(H_\Gamma)$. Hence, $H_\Gamma$ is also a minimum homotopy for $C$.

**Proof.** Let $C \xrightarrow{H} p_0$ be a well-behaved homotopy. If $|A_H| = 1$, then $C$ is self-overlapping by Theorem 13. In other words, $\Gamma = \{C\}$ and the theorem follows. Hence, we assume that $|A_H| > 1$. Let $A_H = \{i_1, i_2, \ldots, i_k\}$. Consider the first anchor index $i_1$. Let $\gamma_1$ be the subcurve of $C$ based at the intersection $p_{i_1} \in [C]$. Since $H$ restricted to $\gamma_1$ only has $i_1$ as an anchor index, it follows from Theorem 13 that $\gamma_1$ is self-overlapping. We define $H_1$ to be the homotopy that contracts $\gamma_1$ linearly as in Theorem 5. We denote the remaining curve $C_1$. Analogously, we consider $i_2$ and its corresponding subcurve $\gamma_2$ of $C_1$, which is also a subcurve of $C$. Then we define $H_2$ by contracting $\gamma_2$ in a similar fashion to obtain a curve which we denote $C_2$.

Continuing this way, we are left with a self-overlapping curve $\gamma_k = C_{k-1}$ based at $p_0$ which we can contract to the point $p_0$ in a similar way. Hence, we constructed a decomposition $\Gamma = \{\gamma_i, i = 1, 2, \ldots, k\}$ of $C$. The homotopy $H_\Gamma$ sweeps each point of the plane no more than $H$ does. In other words, $\text{Area}(H_\Gamma) = \text{Area}(H)$ and $H_\Gamma$ is also a minimum homotopy.

An immediate corollary of Theorem 14 is the following:

**Corollary 15.** A curve $C$ is a $k$-boundary if and only if it admits a left sense-preserving homotopy with $k$ anchor points.

**Proof.** If $C$ is a $k$-boundary, then $C$ admits a decomposition with $k$ positive self-overlapping subcurves. Contracting each of them to the corresponding roots gives a left sense-preserving homotopy with $k$ anchor points. On the other hand, if the homotopy is left sense-preserving with $k$ anchor points, then it decomposes the curve into $k$ positive self-overlapping subcurve.
4.2 Main Theorem

To prove our main theorem, we show that there exists a well-behaved minimum homotopy, i.e., a homotopy that does not contain any I_b-move or a significant II_b-move. This is done by taking an arbitrary minimum homotopy and eliminating any such b-moves one by one starting from the last one. Hence, our main theorem follows from Theorem 14.

Now, we prove a technical lemma. Let \( C \) be a curve and let \( \gamma = \{ C(t) : t \in [t^*, t^{**}] \} \) be a simple subloop with the root \( p = C(t^*) = C(t^{**}) \). Denote \( [C \setminus \gamma] = \{ C(t) : t \in [0, 1] \setminus (t^*, t^{**}) \} \). In other words, \( C \setminus \gamma \) is the curve obtained from \( C \) by removing the simple loop \( \gamma \).

Lemma 16 (Decomposing Self-Overlapping Curves). Let \( C \) be a positive self-overlapping curve. If there is a simple subloop \( \gamma \) of \( C \) which is negative, i.e., \( WH(\gamma) = -1 \), then the curve obtained by contracting \( \gamma \) via a I_a-move is a two-boundary.

Proof. Let \( p \in [C] \) be the root of \( \gamma \), and let \( C_1 \) be the curve obtained from \( C \) by contracting \( \gamma \) via a I_a-move. We observe that \( WH(C_1) = WH(C) - WH(\gamma) = 1 - (-1) = 2 \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{The complete set of normal curves with exactly two intersection points, up to (planar) graph isomorphism. The first four curves are non-self-overlapping; whereas, the rightmost curve is self-overlapping.}
\end{figure}

First, consider the case where \( C \) has exactly two intersection points. In this case, there are five unique normal curves, up to planar graph isomorphism. As shown in Figure 7, only one of these curves (the rightmost) is self-overlapping.

Let \( \gamma \) be the unique simple negative subloop of \( C \), and let \( C_1 \) be obtained from \( C \) by a single I_a-move that contracts \( \gamma \). We illustrate in Figure 8 that the curve \( C_1 \) is the union of two closed positive curves, which can be contracted to \( p \) with a left sense-preserving homotopy: first, contract the outer curve to the remaining intersection point and then contract the inner curve to the root of \( \gamma \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8}
\caption{The curve \( C \) satisfies the assumptions of Lemma 16 when \( C \) has exactly two intersection points. Contracting \( \gamma \) with a \( I_a \)-move yields the curve \( C_1 \).}
\end{figure}

For an arbitrary self-overlapping curve, we consider an immersion \( F : D^2 \to \mathbb{R}^2 \), where the boundary of the disk is mapped to \([C]\). Let \( \theta \subset \partial D^2 \) have the image \( F(\theta) = [\gamma] \). Let \( p \) be the root of \( \gamma \), and let \( q \) be any other point in \( \gamma \) whose preimage is \( q' \in \theta \). We obtain a homotopy \( H \) from \( C \) to \( q \) by retracting the disk \( D^2 \) to the point \( q' \) in such a way that the homotopy fixes \( \gamma \) until an intermediate curve \( \tilde{C} \) is left with only two intersections. We
know that such a homotopy exists by the following argument: at the end of the homotopy, a simple curve is contracted to a point. The last move before this would either be a I₁-a- or I₁₁-a-move. However, since self-overlapping curves must have at least two intersection points by Observation 7, the last move cannot be a I₁-a-move since all intermediate curves induced from a deformation retraction of $D^2$ are necessarily self-overlapping.

Notice that the intermediate curve $C$ is necessarily the unique (up to graph isomorphism) normal self-overlapping curve encountered above. Let $\tilde{\gamma}$ be the loop isomorphic to $\gamma$ in Figure 8. The curve $\tilde{C}_1$ obtained from $\tilde{C}$ by contracting $\tilde{\gamma}$ with a $I_0$-move is a two-boundary that can contract to the root $p$ of $\tilde{\gamma}$ via a left sense-preserving homotopy $\tilde{H}_1$. We now extend this to a left sense-preserving homotopy from $C_1$ to $p$, where $C_1$ is obtained from $C$ by contracting $\gamma$. Let $C_1 \xrightarrow{H_1} \tilde{C}_1 \xrightarrow{H_2} p$ be the sub-homotopy of $H$ connecting $C_1$ to $\tilde{C}$. Since $\tilde{H}$ is induced from a deformation retraction of $D^2$, the homotopy must be sense-preserving. And since we know that $\tilde{H}_1$ is left-sense preserving, we know that $H$ must be left sense-preserving (otherwise the minimal homotopy of $C_1$ would be right sense-preserving as there is only one positive self-overlapping curve with two simple crossing points). Finally, we compose these two homotopies: $C_1 \xrightarrow{H_1} \tilde{C}_1 \xrightarrow{H_2} p$, which is a left sense-preserving nullhomotopy with two anchor points. Hence, by Corollary 15, the curve obtained by contracting $\gamma$ via a $I_0$-move is a two-boundary.

Using Lemma 16, we prove Lemmas 17, 18, and 19 below which provide the key ingredients for proving our main theorem.

> **Lemma 17.** Let $C_0 \xrightarrow{H_1} C_1 \xrightarrow{H_2} p_0$ be a minimum homotopy, where $H_1$ consists of a single $I_0$-move and $H_2$ is well-behaved. If $C_1$ is a positive self-overlapping curve, then $C_0$ is a two-boundary and $H_1 + H_2$ can be replaced by a well-behaved minimum homotopy.

**Proof.** We observe that a negative loop is oriented clockwise and a positive loop is oriented counter-clockwise. Hence, a left sense preserving homotopy expands the negative loop and increases the area of the interior face. Similarly, a left sense-preserving homotopy contacts a positive loop and decreases the area of the interior face.

Now, since $H_2$ is well-behaved and $C_1$ is self-overlapping, we know by Theorem 13 that $H_2$ has one anchor point. Since no contraction happened in $H_1$, we know that $H_1 + H_2$ has only one anchor point and the homotopy is left sense-preserving by Lemma 11. This implies that $H_1$ creates a negative loop, since the loop is expanding by the homotopy when it is created for the first time. Thus, by Lemma 16, $C_0$ is a two-boundary.

A similar approach is used to eliminate significant $I_0$-moves.

> **Lemma 18 (Existence of a Well-Behaved Minimum Homotopy).** Let $C_0 \xrightarrow{H_1} C_1 \xrightarrow{H_2} p_0$ be a minimum homotopy where $H_1$ consists of a single significant $I_0$-move and $H_2$ is well-behaved. If $C_1$ is a two-boundary, then $C_0$ is also a two-boundary. Furthermore, $H_1 + H_2$ can be replaced by a well-behaved minimum homotopy $C_0 \xrightarrow{H_0} p_0$.

**Proof.** (Note: Here, we give a sketch of the proof and leave the technical details to the full version of this paper. We note where details are omitted below.) Since $H_1$ is a single significant $I_0$-move, then we know that one of the two crossing points created is an anchor point. Let’s call that point $p_k$. Therefore, we have three potential cases, each of which is illustrated in Figure 9.

In Case 1, we notice that splitting at $p_k$ results in only one curve, hence a contradiction (since if $p_k$ were an anchor point, $p_k$ would be the root of two curves that form a decomposition of $C_1$).
Proof. Since a well-behaved minimum homotopy, $\gamma$ must be a two-boundary $H$. We apply Lemma 18 to by Theorem 14. Let $H \Gamma$ argument for Case 1 required for Case 2. Let $H \Gamma$ follows $p$ to $C$ to $p$ and $k$ to $H_1$ remains; we will call this intersection point $p_k$.

In Case 2, since $\gamma$ and $\gamma'$ are both closed curves, this implies that there must exist at least one more intersection point in addition to $p_t$ (recalling that $p_k$ can be perturbed away). Let $A$ be the set of intersection points between $\gamma$ and $\gamma'$ that are also simple crossing points of $C_0$. Let $p_i \in A$, and notice that there are two curves from $p_i$ to $p_k$ and two curves from $p_k$ to $p_i$ such that the union of these four curves is $C$. Define curves $\alpha_i$ and $\beta_i$ such that $\alpha_i$ follows $\gamma$ from $p_k$ to $p_i$ then $\gamma'$ from $p_i$ to $p'_k = p_k$ and $\beta_i$ follows $\gamma'$ from $p'_k = p_k$ to $p_i$ and then $\gamma$ from $p_i$ to $p_k$. If $C_1$ is a two-boundary, then there exists a $p_i$ such that the curves $\alpha_i$ and $\beta_i$ that map to $\alpha_i$ and $\beta_i$ under $H_1$ are positive self-overlapping. The proof of the existence of such an $i$ is quite technical, and is deferred to the full version of this paper.

In Case 3, we can have two subcases: first, if $\gamma \cap \gamma' = p_k$, then we let $A$ be the set of intersection crossing points of $\gamma$, and we can find a $p_i \in A$ using a similar technical argument as required for Case 2. Second, if $|\gamma \cap \gamma'| > 1$, then we follow an argument identical to the argument for Case 2.

Now, we define a homotopy $H$ by first contracting $\tilde{\alpha}_i$ to $p_k$, and then contracting $\tilde{\beta}_i$ to $p_0$. By Theorem 14, we conclude that $H$ is a minimal homotopy and $C_0$ is a two-boundary.

The following lemma generalizes Lemmas 17 and 18 by removing additional assumptions on $H_2$.

Lemma 19. Let $C_0 \xrightarrow{H_1} C_1 \xrightarrow{H_2} p_0$ be a minimum homotopy where $H_1$ consists of a single $I_b$-move or significant $\Pi_b$-move, and $H_2$ is well-behaved. Then, $H_1 + H_2$ can be replaced by a well-behaved minimum homotopy.

Proof. Since $H_2$ is already well-behaved, there is a corresponding decomposition $\Gamma_1$ of $C_1$ by Theorem 14. Let $\gamma$ be the self-overlapping curve containing the newly created loop by $H_1$. Without loss of generality, assume that $\gamma$ is positive. If $H_1$ consists of a single $I_b$-move, then we apply Lemma 17 to $\gamma$. The remainder of $H_2$ remains well-behaved.

If $H_1$ consists of a significant $\Pi_b$-move, then it can be shown by case analysis that there must be a two-boundary $\gamma'$ that contains $\gamma$, otherwise the $\Pi_b$-move is not significant or the homotopy not minimum. We apply Lemma 18 to $\gamma'$ and the remainder of $H_2$ remains well-behaved.

We are now ready to prove our main theorem.

Theorem 20 (Main Theorem). Let $C$ be a normal curve. Then:
Any minimum homotopy \( C \xrightarrow{H} p_0 \) can be replaced with a well-behaved minimum homotopy.

There exists a decomposition \( \Gamma \) such that the induced homotopy \( H_\Gamma \) is minimum.

**Proof.** We split \( H \) into a sequence of subhomotopies \( C \xrightarrow{H_1} C_1 \xrightarrow{H_2} C_2 \ldots \xrightarrow{H_k} p_0 \), \( H = \sum_{i=1}^{k} H_i \) where each subhomotopy consists of a single homotopy move. If none of the moves is an \( I_b \)-move or a significant \( II_b \)-move, then the homotopy is already well-behaved. Otherwise, let \( C_{j-1} \xrightarrow{H_j} C_j \) be the subhomotopy containing the last such \( b \) move. Then, we can replace \( H_j + H_{j+1} + \ldots + H_k \) with a well-behaved homotopy, \( \tilde{H}_j \), using Lemma 19. The new homotopy \( H_1 + H_2 + \ldots + \tilde{H}_j \) is a minimum homotopy which has one less such \( b \) move. Removing such \( b \) moves one by one, we obtain a well-behaved minimum homotopy.

The second part of the theorem follows from the first part and Theorem 14. ▶

It follows that there is a minimum homotopy \( H \) such that \( E_H \) is constant on each face.

**Corollary 21.** Let \( C \) be a normal curve and let \( f_0, f_1, \ldots, f_k \) be the set of faces of \( C \) where \( f_0 \) is the exterior face. Then, there exists a minimum homotopy \( C \xrightarrow{H} p_0 \) such that \( E_H \) is constant on the faces of \( C \), i.e., if two points \( x, y \in \mathbb{R}^2 \) are in the same face, then \( E_H(x) = E_H(y) \). Hence, if we set \( E_H(f_i) = E_H(x_i) \) where \( x_i \in f_i \), then \( E_H(f_0) = 0 \) and

\[
\sigma(C) = \sum_{i=1}^{k} E_H(f_i) \cdot \text{Area}(f_i).
\]

### 4.3 Algorithm

Let \( C \in \mathcal{C}_{p_0} \) be a normal curve and let \( P_C \) be the set of intersection points of \( C \). Recall that by Theorem 20, there exists a minimum homotopy for \( C \) that consists of contractions of self-overlapping subcurves to anchor points. We can therefore check each intersection point to see if it might serve as an anchor point. If \( i \in P_C \) is an intersection point of \( C \), then it breaks \( C \) into two subcurves that we denote with \( C_{i,1} \) and \( C_{i,2} \). The following recursive formula naively checks all possible ways to break the curves along their intersection points:

\[
\sigma(C) = \begin{cases} 
W(C), & \text{if } C \text{ is self-overlapping} \\
\min_{i \in P_C} \sigma(C_{i,1}) + \sigma(C_{i,2}), & \text{otherwise}
\end{cases}
\]

Using this formula we split \( C \) at each intersection point, take the best split and proceed recursively. In the worst case, this recursive algorithm takes exponential time in \(|P_C|\).

### 5 Conclusions

We have shown that normal curves admit minimum homotopies that are composed of contractions of self-overlapping curves. At this stage, we have a straight-forward exponential algorithm to compute a minimum homotopy. But, we are optimistic that our structural main theorem lays the foundation for developing a polynomial-time algorithm. In fact, undergraduates Parker Evans and Andrea Burns have developed a tool to visualize minimum-area homotopies (http://www.cs.tulane.edu/~carola/research/code.html), which has led us to insights on which we can base an efficient dynamic programming algorithm.

Another problem to consider is to find a minimum homotopy between any two normal curves with the same end point not just between a curve and its endpoint. For some pair of curves \( C_1 \) and \( C_2 \), the minimum homotopy area between these curves is equal to the minimum homotopy area of the curve \( C_1 \circ C_2^{-1} \). Also, we can extend the minimum...
homotopy problem to curves on other surfaces. We hope to address all these problems in future work.

6 Acknowledgments

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A  Metric Space

**Theorem 22 (Metric Space).** Define $\mathcal{C}_{p_0}^+ = \{ C \in \mathcal{C}_{p_0} \mid \text{Wh}(C) \geq 0 \} / \sim$, where $C_1 \sim C_2$ if $|C_1| = |C_2|$. Then $(\mathcal{C}_{p_0}^+, \sigma)$ is a metric space.

**Proof.** First, we must show that $\sigma$ is well-defined. If $C_1, C_2 \in \mathcal{C}_{p_0}^+$ and if $C_1 \sim C_2$, where $C_1(\phi(t)) = C_2(t)$ for some function $\phi$, then we have a homotopy $C_1 \sim C_2$ such that $H(s,t) = C_1(s\phi(t) + (1-s)t)$. For this homotopy, we have $E_H(x) = 1$ if $x \in [C_1]$ and $E_H(x) = 0$ otherwise. Since the curve has zero measure, we have $\text{Area}(H) = 0$, which gives us $\sigma(C_1, C_2) = 0$. Similarly, if $C_1 \sim C_2$ and $C'_1 \sim C'_2$ then $\sigma(C_1, C'_1) = \sigma(C_2, C'_2)$. Hence, $\sigma$ is well-defined.

To finish this proof, we must show that $\sigma$ satisfies the metric space identities (the identity of indiscernibles, symmetry, and subadditivity). Clearly, $\sigma(C_1, C_2) = 0$ if and only if $C_1 \sim C_2$. If $C_1 \sim C_2$ is a homotopy, then $C_2 \sim C_1$ is a homotopy with $H^{-1}(s,t) = H(1-s,t)$ and $\text{Area}(H) = \text{Area}(H^{-1})$. Hence, $\sigma(C_1, C_2) = \sigma(C_2, C_1)$. Finally, if $C_1 \sim H_1, C_2 \sim H_2, C_3$ and $C_1 \sim C_3$ are minimum homotopies, then $\sigma(C_1, C_3) = \text{Area}(H_3) \leq \text{Area}(H_1 + H_2) = \text{Area}(H_1) + \text{Area}(H_2) = \sigma(C_1, C_2) + \sigma(C_2, C_3)$. Thus, we conclude that $(\mathcal{C}_{p_0}^+, \sigma)$ is a metric space.

B  Examples

In this section, we apply our main theorem to calculate a minimum homotopy for the curves in Figure 3b and Figure 11. We say that a set of vertices $A = \{i_1, \ldots, i_k\}$ is valid if there exists a decomposition $\Gamma$ whose set of roots corresponds to the intersection points $\{p_{i_1}, \ldots, p_{i_k}\}$.

**Example B.1.** We check whether the curve in Figure 3b is self-overlapping or not. It is not self-overlapping. However, notice that splitting the curve into two at either intersection point 2 or 3 creates two self-overlapping curves. Hence, there are two different ways to decompose the curve. $\{0, 2\}$ corresponds to the first decomposition $\Gamma_1$ and $\{0, 3\}$ corresponds to the second decomposition $\Gamma_2$. Let $H_1$ be the homotopy obtained from $\Gamma_1$, and let $H_2$ be the other homotopy, see Figure 11. We compute $\text{Area}(H_1) = 2\text{Area}(f_2 \cup f_1) + \text{Area}(f_4)$ and $\text{Area}(H_2) = 2\text{Area}(f_3 \cup f_1) + \text{Area}(f_4)$. The smallest of them, in this case $H_2$, is the minimum homotopy and $\sigma(C) = \text{Area}(H_2)$. On the other hand, $W(C) = 2\text{Area}(f_1) + \text{Area}(f_4)$. This shows that $\sigma(C) > W(C)$.

**Example B.2.** For the curve in Figure 11, there are 22 different possible decompositions. We list the valid sets of vertices as follows: $A_1 = \{0, 3, 9\}$, $A_2 = \{0, 3, 10\}$, $A_3 = \{0, 1, 2, 3, 9\}$, $A_4 = \{0, 1, 2, 3, 10\}$, $A_5 = \{0, 3, 4, 5, 7\}$, $A_6 = \{0, 3, 4, 5, 8\}$, $A_7 = \{0, 3, 5, 7, 9\}$, $A_8 = \{0, 3, 5, 7, 10\}$, $A_9 = \{0, 3, 5, 8, 9\}$, $A_{10} = \{0, 3, 5, 8, 10\}$, $A_{11} = \{0, 1, 2, 3, 4, 5, 7\}$, $A_{12} = \{0, 1, 2, 3, 4, 5, 8\}$, $A_{13} = \{0, 1, 2, 3, 5, 7, 9\}$, $A_{14} = \{0, 1, 2, 3, 5, 7, 10\}$, $A_{15} = \{0, 1, 2, 3, 5, 8, 9\}$, $A_{16} = \{0, 1, 2, 3, 5, 8, 10\}$, $A_{17} = \{0, 3, 4, 5, 6, 7, 8\}$, $A_{18} = \{0, 3, 4, 5, 6, 7, 9\}$, $A_{19} = \{0, 3, 5, 6, 7, 8, 10\}$, $A_{20} = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $A_{21} = \{0, 1, 2, 3, 5, 6, 7, 8, 9\}$, $A_{22} = \{0, 1, 2, 3, 5, 6, 7, 8, 10\}$.

Among these valid sets, the least area is swept by the homotopy obtained from the set $A_5$. Hence, the minimum homotopy area is equal to $\sigma(C) = 2\text{Area}(f_2 \cup f_4 \cup f_7 \cup f_9 \cup f_{11}) + \text{Area}(f_1 \cup f_6 \cup f_{10})$. Notice that the winding area is equal to $W(C) = 2\text{Area}(f_2 \cup f_4 \cup f_9) + \text{Area}(f_1 \cup f_6 \cup f_{10})$, i.e., $\sigma(C) > W(C)$.
Figure 10 Two well-behaved minimum homotopies for the curve in Figure 3b. Here, $H_2$ sweeps less area since $\text{Area}(f_3) < \text{Area}(f_2)$.

Figure 11 The set of anchor indices of the minimum homotopy for this curve is $A_H = \{0, 3, 4, 5, 7\}$. These vertices decompose the curve into five self-overlapping subcurves, the red curve with root $p_7$, the blue curve with root $p_5$, the green curve with root $p_4$ and the cyan curve with root $p_3$ and the purple curve with root $p_0$. The homotopy is obtained by contracting first the red curve, then the blue curve, then the green curve, then the cyan curve and finally the purple curve to the corresponding roots $p_7$, $p_5$, $p_4$, $p_3$ and $p_0$ respectively.
C  An Application

Let $\alpha, \beta$ be two open curves sharing the same end-points $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. We can concatenate $\alpha$ and $\beta$ to create a closed curve $C_{\alpha, \beta}$, where $C_{\alpha, \beta}(t) = \alpha(2t)$ for $t \in \left[0, \frac{1}{2}\right]$ and $C_{\alpha, \beta}(t) = \beta(2 - 2t)$ for $t \in \left[\frac{1}{2}, 1\right]$. We assume that $C_{\alpha, \beta}$ is a normal curve, or else we apply a small deformation as discussed previously.

We define the minimum homotopy area between $\alpha$ and $\beta$ as the minimum homotopy area of $C_{\alpha, \beta}$, and we denote $\sigma(\alpha, \beta)$. In other words, $\sigma(\alpha, \beta) = \sigma(C_{\alpha, \beta})$.

If two curves $\alpha$ and $\beta$ do not share the same endpoints, we create a closed curve by joining the endpoints via straight lines and define the minimum homotopy area between $\alpha$ and $\beta$ as the minimum homotopy area of this closed curve. See Figure 12.

Minimum homotopy area can be used to measure the distance between two plane graphs, in particular maps created from a set of GPS trajectories.

Let $G_1 = (V_1, E_1, w_1)$ and $G_2 = (V_2, E_2, w_2)$ be two connected, weighted plane graphs. We say that a vertex $\tilde{v} \in V_2$ of $G_2$ is an associate of $v$ if

$$\|v - \tilde{v}\| = \min_{w \in V_2} \|v - w\|$$

In other words $\tilde{v}$ is the closest vertex of $G'$ to $v$. We denote it by $v \sim \tilde{v}$. For any pair $(u, v) \in V_1 \times V_1$, $u \neq v$, let $\{p_\alpha(u, v)\}_{\alpha \in I}$ be the set of all shortest paths from $u$ to $v$ and $\{q_\beta(u, v)\}_{\beta \in J}$ be the set of all shortest paths from an associate $\tilde{u}$ of $u$ to an associate $\tilde{v}$ of $v$. Here $p_\alpha(u, v)$ is a path $G_1$ and $q_\beta(u, v)$ is a path in $G_2$. Let $C_{\alpha, \beta}(u, v)$ denote the concatenation of $p_\alpha(u, v)$ and $q_\beta(u, v)$ as defined previously and

$$\sigma(u, v) = \min_{\alpha, \beta} \sigma(C_{\alpha, \beta}(u, v)).$$

And, finally, we define the homotopy area distance between two graphs $G_1$ and $G_2$ as

$$\sigma(G_1, G_2) = \frac{1}{n(n - 1)} \sum \sigma(u, v f)$$

where the summation is taken for each different pair of vertices $u, v \in V_1$ and $n = |V_1|$.