Resonant algebras and gravity

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Abstract

The $S$-expansion framework is analyzed in the context of a freedom in closing the multiplication tables for the abelian semigroups. Including the possibility of the zero element in the resonant decomposition, and associating the Lorentz generator with the semigroup identity element, leads to a wide class of the expanded Lie algebras introducing interesting modifications to the gauge gravity theories. Among the results, we find all the Maxwell algebras of type $B_m$, $C_m$, and the recently introduced $D_m$. The additional new examples complete the resulting generalization of the bosonic enlargements for an arbitrary number of the Lorentz-like and translational-like generators. Some further prospects concerning enlarging the algebras are discussed, along with providing all the necessary constituents for constructing the gravity actions based on the obtained results.

Keywords: $S$-expansion, resonant Lie algebras, Maxwell algebra, semigroup expansion

1. Introduction

Following the recent introduction in [1] of the Maxwell-like $D_m$ family of algebras, a question has been raised of finding other examples, which may be interesting in the context of gravity or supergravity. The so-called $S$-expansion procedure [2–4], by the use of the abelian semigroups, introduces a very general and quite convenient description for this kind of algebraic enlargement. Starting with the anti-de Sitter (AdS) algebra, one can reproduce the Maxwell algebra [5, 6] introduced in the 70s, and the algebra proposed by Soroka–Soroka [7, 8], and generalize them into two Maxwell families [3, 4], which following the notation of [1], we denote as $B_m$ and $C_m$, with the subindex indicating $(m − 1)$ different types of the generators. A new $D_m$ family can also be cast into this scheme, but the way to achieve this was not straightforward, and only the presence of the direct sums in these algebras allowed for the generalization to an arbitrary number $m$.

As we will see, all these families (and many other examples) can arise quite naturally by concerning the freedom in closing the algebras. This will be achieved through closing the
particular semigroups obeying the so-called resonant decomposition, but now with the possibility of a zero element along with the semigroup identity element associated with the Lorentz generator.

To some extent, it could be possible to find the new algebras at the level of generators, simply by completing the commutators to fulfill the Jacobi identities. Eventually however, to construct the gravity actions it would still be necessary to make a transition to the $S$-expansion to obtain the invariant tensors. Naturally, from the semigroup multiplication tables one can easily read off the explicit algebra of the generators (see appendix A). Moreover, such a framework allows us to present the resulting schemes in a very compact and illustrative way.

The work is organized as follows: we start with the standard $S$-expansion setup and then modify some of the conditions. In section 3 we reproduce all known Maxwell algebraic families and obtain a general scheme for the new ones. Sections 4 and 5 will be devoted to a discussion of the non-standardly enlarged algebras, whereas section 6 will offer details concerning the construction of the gravity actions.

2. Resonant decomposition with zero and identity elements

Suitable conditions required to provide reasonable grounds for theories of gravity constructed as gauge theories begin with the commutator of the Lorentz generator $J_{ab}$ with itself and the generator of the translations $P_a$. This becomes essential in gauging the algebra because to obtain the field transformations and construct the curvature two-form $F = dA + A \wedge A = \frac{1}{2} F_{\mu \nu} (x) T_M d\xi^\mu \wedge d\xi^\nu$, for a given gauge parameter $\Theta = \Theta^M (x) T_M$ and a connection one-form $A = A^M (x) T_M d\xi^M$, one needs to evaluate

$$\delta \Theta = - \partial_\mu \Theta + \Theta \partial_\mu + [A_\mu, A_\nu].$$

Indeed, for the AdS-valued connection, $A = \frac{1}{2} \omega^{ab} \tilde{J}_{ab} + \frac{1}{2} \pi^P \tilde{P}$ with group indices $a, b = 1, \ldots, D$, the Riemann curvature and torsion two-forms are associated with the following commutators:

$$[\tilde{J}_{ab}, \tilde{J}_{cd}] = \eta_{bc} \tilde{J}_{ad} - \eta_{ac} \tilde{J}_{bd} - \eta_{bd} \tilde{J}_{ac} + \eta_{ac} \tilde{J}_{bd} \rightarrow R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega^b_c,$$

$$[\tilde{J}_{ab}, \tilde{P}_c] = \eta_{bc} \tilde{P}_a - \eta_{ac} \tilde{P}_b \rightarrow T^a = d\omega^a + \omega^a_b \wedge \omega^b_c.$$  

(3)

Altogether with the last piece,

$$[\tilde{P}_a, \tilde{P}_b] = 0 \quad \text{or} \quad [\tilde{P}_a, \tilde{P}_b] = \tilde{J}_{ab} \quad \text{or} \quad [\tilde{P}_a, \tilde{P}_b] = - \tilde{J}_{ab},$$

(4)

this obviously corresponds to the Poincaré (ISO), anti-de Sitter (AdS) and de Sitter (dS) group of symmetries, which brings the contribution of zero or $\pm \frac{1}{2} \omega^{ab} \wedge \omega^b_c$ to the Lorentz part of the curvature. The conventions used throughout this paper, due to possible future supergravity applications, will only account for the first two cases (Poincaré and AdS).

In the $S$-expansion method [2, 3] the new algebras are derived from the AdS by using a particular choice of the abelian semigroup. The procedure starts from the decomposition of the original algebra $g$ into subspaces,

$$g = so(D - 1, 2) = so(D - 1, 1) \oplus \frac{so(D - 1, 2)}{so(D - 1, 1)} = V_0 \oplus V_1,$$

(5)
where $V_0$ is spanned by the Lorentz generator $\tilde{J}_{ab}$ and $V_1$ by the AdS translation generator $\tilde{P}_a$. The subspace structure can then be written as

$$[V_0, V_0] \subset V_0, \quad [V_0, V_1] \subset V_1, \quad [V_1, V_1] \subset V_0. \quad (6)$$

If we define $S = \{\lambda_0, \lambda_1, \ldots\}$ as an abelian semigroup with the multiplication law being associative and commutative, then the Lie algebra $G = S \times \mathfrak{g}$ is called the $S$-expanded algebra of $\mathfrak{g}$. For the resonant subset decomposition $S = S_0 \cup S_1$, with

$$S_0 = \{\lambda_{2i}\} \quad \text{and} \quad S_1 = \{\lambda_{2i+1}\} \quad \text{for } i, j = 0, 1, 2, \ldots, \quad (7)$$

the new algebra will be spanned by the $\{J_{ab,(i)}, P_{a,(j)}\}$, where the new generators are related to the original $so(D - 1, 2)$ ones through

$$J_{ab,(i)} = \lambda_{2i} \tilde{J}_{ab} \quad \text{and} \quad P_{a,(j)} = \lambda_{2j+1} \tilde{P}_a. \quad (8)$$

This decomposition satisfies

$$S_0 \cdot S_0 \subset S_0, \quad S_0 \cdot S_1 \subset S_1, \quad S_1 \cdot S_1 \subset S_0 \quad (9)$$

and is called resonant since it has the same structure as (6). The final form of the algebra will depend on the number of elements and particular law of semigroup multiplication. Such a setup for $S = \{\lambda_0\}^N$ with the algebra subindex $m = N + 2$, a priori does not contain the zero element. One needs to include it as the $\lambda_{N+1}$ element and apply the so-called $S$-reduction process \cite{2–4} to finally obtain the Maxwell $\mathfrak{B}_M$ family and make the connection with an expansion by the means of the Maurer–Cartan forms \cite{9}.

Let us now do things differently and just start with the minimal set of elements with some minimal multiplication rules and then check what the freedom is in closing them to form a semigroup useful in the context of gravity. Naturally, to provide the most general treatment, the resonant decomposition from equation (9) gets extended to the eventuality of having an explicit zero element\(^1\) in $S_0 = S_0 \cup \{0_0\}$ and $S_1 = S_1 \cup \{0_1\}$. This absorbing element, for which $0_2 \lambda_k = \lambda_k 0_2 = 0_2$, acting on a generator corresponds to $0_2 \mathfrak{T}_M = 0$.

Explicitly marking the resonant decomposition in the multiplication tables can help us fix the missing entries. Thus, the subsequent separation in the outcome of the multiplications will be indicated by $\bar{S}_0$ and $\bar{S}_1$ describing, respectively, the elements later associated with the Lorentz-like and translation-like generators.

There will be, nevertheless, an important exception. We are going to introduce the semigroup identity element, chosen to be $\lambda_0$, and relate it to the Lorentz generator. Therefore, here we effectively deal with the monoids. Mind that then $\lambda_0$ preserves all the elements, so the zero element is excluded from the results of $\lambda_0$ times, whatever the element other than $0_2$. That alone already provides two starting conditions related to the commutators $[\tilde{J}_{ab}, \tilde{J}_{cd}]$ and $[\tilde{J}_{ab}, \tilde{P}_c]$:

$$\lambda_0 \lambda_0 = \lambda_0, \quad (10)$$

$$\lambda_0 \lambda_1 = \lambda_1, \quad (11)$$

which were needed for the right definitions of the curvature and torsion in equations (2) and (3).

After putting all of that into the commutative multiplication table (but keeping in mind that the $\lambda_0$ row and column come with the restriction pointed out above)

\(^1\)In such situations, though, we will refrain from presenting the additional row and column corresponding to the $0_2$ element.
only the last entry is left to be determined. Since $\lambda_1 \lambda_0 \in S_0 = \{ 0_5, \lambda_0 \}$, there are clearly two choices for closing it

|     | ISO | AdS |
|-----|-----|-----|
| $\lambda_0$ | $\lambda_0$ | $\lambda_0$ |
| $\lambda_1$ | $\lambda_0$ | $\lambda_1$ |
| $\lambda_0$ | $\lambda_1$ | $0_S$ |

which effectively correspond to the Poincaré and AdS algebras, as the commutator

$$[P_a, P_b] = \lambda_0 \lambda_1 [\bar{P}_a, \bar{P}_b] = \lambda_0 \lambda_1 \bar{J}_{ab}$$

leads to $[P_a, P_b] = 0$ and $[P_a, P_b] = J_{ab}$. However, this is not the only possibility!

3. Maxwell algebras

We notice that the $\lambda_1 \lambda_0$ multiplication could be closed by another element $\lambda_2 \in S_0$. Obviously, this extends our table by another row and column with the additional multiplications to be determined. One can show that $\lambda_0 \lambda_2 = \lambda_0 (\lambda_1 \lambda_0) = (\lambda_0 \lambda_1) \lambda_0 = \lambda_2$, thus

Now it is easy to see that the unknown entries will be expressed by the powers of the $\lambda_0$ element. Because the resonant decomposition forces $\lambda_1^2 \in S_0$, we can immediately establish, for the available list of elements, the two possible associative tables for $\lambda_1^2 = 0_S$ and $\lambda_1^2 = \lambda_1$ along with the straightforward identification of the corresponding $\lambda_2^2 = 0$ and $\lambda_2^2 = \lambda_2$
Remarkably, these tables represent the Maxwell algebra of type $\mathcal{B}_4$ and $\mathcal{C}_4 \equiv \text{AdS} \oplus \text{Lorentz}$. In both cases, the Lorentz $J_{ab} = J_{ab,(0)} = \lambda_0 J_{ab}$ and translation $P_a = P_{a,(0)} = \lambda_0 P_a$ generators are equipped with the new generator $Z_{ab} = J_{ab,(1)} = \lambda_3 J_{ab}$ (see appendix A for more details). Besides the starting papers [5–7], both algebras have been studied in various contexts, starting with the deformations and dynamical realizations [10], use in the BF gravity models [11, 12], and the exploitation of the Maxwell symmetry in the emergence of the cosmological constant term [13]. Further applications include a bimetric and cosmology context [14], construction of the Brans–Dicke theory [15], the WZW model [16], and finding their supersymmetric extensions [8, 17–19].

Interestingly, the $\mathcal{C}_4$ algebra under the specific change of basis can be rewritten as a direct sum of the AdS and Lorentz algebras (check appendix B). Such a feature leads to the question of whether a commutator of any physical Lorentz generator with any other generator could give zero. In this paper, we decide to extend a behavior known from $[J, J] \sim \eta J$ and $[J, P] \sim \eta P$ in further cases, leaving any discussion concerning other options for the future.

In analogy, the previously discussed multiplication of $\lambda_0 \lambda_2 = \lambda_2 \lambda_1 = \lambda_3$ could also lead to the new element $\lambda_0 \lambda_3 = (\lambda_0 \lambda_4) \lambda_2 = \lambda_3$. The table obtained in this way immediately gives three patterns due to the outcome of $\lambda \lambda \lambda \lambda \in \{0, \lambda_0, \lambda_2\}$. To better emphasize the resulting structure, let us stop coloring the resonant decomposition and now color the different types of semigroup elements

\[
\begin{array}{cccc}
\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\
\lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\
\lambda_1 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\
\lambda_2 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\
\lambda_3 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\
\end{array}
\]

(17)

These three examples share the same part up to the main anti-diagonal (with the simple multiplication law $\lambda_0 \lambda_3 = \lambda_0 + \lambda_3$), whereas further we are going to have:

- all zeros,
- anti-diagonal pattern,
- chessboard pattern.

In this way we have found semigroups corresponding to the Maxwell algebras of type $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, which are spanned by four kinds of generators $\{J_{ab}, P_a, Z_{ab}, R_a\}$. The associated field content $\{\omega^{ab}, k^{a}, k^{ab}, J^i\}$, through the expanded curvatures and modifications of the invariant
tensors, non-trivially affects the final form of the Lagrangians. This was exploited in [20, 21] to establish a relation in 5D between Chern–Simons (CS) gravity and the Einstein–Hilbert action through the $\mathfrak{B}_5$ algebra. After generalization to the arbitrary odd dimensions, the same was achieved in even dimensions for Born–Infeld (BI) gravity and GR [22, 23]. Algebra $\mathfrak{C}_5$ was recently analyzed in a similar context to obtain the so-called pure Lovelock (PL) action [24], which instead of the full Lanczos–Lovelock series contains only the cosmological constant term and a single $p$ power polynomial term in the Riemann curvature (in 5D it can be either $RR$ or $Re^3$). The last $\mathfrak{D}_5$ algebra introduced in [1] admits the direct sum of the $AdS \oplus$ Poincaré algebras (see appendix B), which effectively leads to the CS/BI action expressed as the sum of two independent pieces.

Ultimately, we arrive at the arbitrary number of elements/generators. For algebra with $m = 6$ one fixes $\lambda_0 \lambda_3 = \lambda_1^3$ as a new element, thus considering $\lambda_1^5 \in \{0, \lambda_1, \lambda_3\}$ produces the tables:

$\mathfrak{B}_6$ : $\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5$

$\mathfrak{C}_6$ : $\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5$

$\mathfrak{D}_6$ : $\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5$

For a larger number of elements it will be possible to find more schemes. Indeed, starting from $m = 7$ we can find another family $\mathfrak{E}_m$, which structurally resembles a lot of the $\mathfrak{C}_m$ algebra:

$\mathfrak{B}_7$ : $\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6$

$\mathfrak{C}_7$ : $\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6$

$\mathfrak{D}_7$ : $\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6$

(19)
The total number of possible patterns depends on \( m \) and is equal to the integer part of \( \left[ \frac{m+1}{2} \right] \), which comes from fixing the outcome of \( \lambda_0^{m-1} \) as being one of \( \{0, \lambda_0, \lambda_2, \ldots, \lambda_m\} \) when \( m = \text{odd} \) or \( \{0, \lambda_1, \lambda_3, \ldots, \lambda_{m-3}\} \) when \( m = \text{even} \). Excluding the \( \mathfrak{B}_m \) family (the scheme with 0), it is possible to capture all other Maxwell families in the single multiplication law

\[
\lambda_a \lambda_b = \begin{cases} 
\lambda_{a+b}, & \text{for } \alpha + \beta \leq m - 2 \\
\lambda_\gamma, & \text{for } \alpha + \beta > m - 2
\end{cases}
\]  

(20)

where element \( \lambda_m \), being directly under the main anti-diagonal, determines the particular type of algebra through

\[
\gamma = (\alpha + \beta - (m - 1)) \mod ((m - 1) - \rho) + \rho.
\]  

(21)

The family \( \mathfrak{D}_{m \geq 5} \), with the maximal element indicated by \( \rho = m - 3 \), reproduces the chessboard scheme, where the rest of the families realize an anti-diagonal arrangement differing only in the starting element. Regarding the parity, and not exceeding the range fixed by the existence of \( \mathfrak{D}_m \), the algebra of type \( \mathfrak{C}_{m \geq 3} \) will be generated by \( \rho = 0 \) (\( m = \text{odd} \)) and \( \rho = 1 \) (\( m = \text{even} \)). Analogously, if \( m \) is large enough, \( \rho = 2 \) and \( \rho = 3 \) will generate \( \mathfrak{E}_{m \geq 7} \), whereas \( \rho = 4 \) and \( \rho = 5 \) will generate \( \mathfrak{F}_{m \geq 9} \), and so on. Finally, the \( \mathfrak{B}_m \) family is retrieved as the Inönü–Wigner contraction of all these algebras in the limit of the dimensionless parameter \( \mu \rightarrow \infty \) scaling generators \( P_{a,0} \rightarrow \mu P_{a,0}, J_{ab,1} \rightarrow \mu^2 J_{ab,1}, P_{a,1} \rightarrow \mu^3 P_{a,1}, \ldots \)

4. Poincaré-like, AdS-like, and Maxwell-like algebras

It is important to note that in our derivation we have missed quite an interesting branch of enlargements, which comes from the possibility of adding the new element \( \lambda_2 \) independently to the table, while still having \( \lambda_0 \lambda_2 \) equal to 0 or \( \lambda_0 \). This goes beyond the procedure presented earlier, as the new elements will not be related to the powers of \( \lambda_2 \). Then, to obtain the proper tables there is nothing else but to laboriously analyze the semigroup associativity. Considering the general setup, we see the emergence of the three basic families containing the Poincaré, the AdS and the Maxwell algebras as the subalgebra:

\[
\begin{array}{ccc}
\lambda_0 & \lambda_1 & \lambda_2 \\
\lambda_0 & 0 & \lambda_2 \\
\lambda_1 & 0 & \lambda_2 \\
\lambda_2 & 0 & \lambda_2
\end{array}
\]  

(22)

Of course this generalization could go on further, i.e. \( \lambda_1 \lambda_1 = \lambda_4 \), then \( \lambda_1 \lambda_1 = \lambda_6 \), and so on, but we will not explore such scenarios here.

An explicit check of the associativity (which can be done through a dedicated tool at the website [www.resonantalgebras.wordpress.com](http://www.resonantalgebras.wordpress.com)) shows that it is possible to construct:
• 4 × Poincaré-like algebras, which we could denote as type $B_4$, $\tilde{B}_4$, $\tilde{C}_4$, and $C_4 \equiv ISO \oplus Lorentz$:

\[
\begin{array}{cccccccccccc}
B_4 & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & \tilde{B}_4 & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & \tilde{C}_4 & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & C_4 & $\lambda_0$ & $\lambda_1$ & $\lambda_2$\\
\lambda_0 & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & \lambda_0 & $\lambda_1$ & $\lambda_2$ & \lambda_0 & $\lambda_1$ & $\lambda_2$ & \lambda_0 & $\lambda_1$ & $\lambda_2$ & \lambda_0 & $\lambda_1$ & $\lambda_2$ \\
\lambda_1 & $\lambda_1$ & 0 & 0 & \lambda_1 & 0 & 0 & \lambda_1 & 0 & 0 & \lambda_1 & 0 & 0 & \lambda_1 & 0 & 0 \\
\lambda_2 & $\lambda_2$ & 0 & 0 & \lambda_2 & 0 & 0 & \lambda_2 & 0 & 0 & \lambda_2 & 0 & 0 & \lambda_2 & 0 & 0 \\
\end{array}
\]

(23)

• no AdS-like algebra (the associativity is not fulfilled in any configuration)
• 2 × Maxwell-like algebras of type $\mathcal{B}_4$ and $\mathcal{C}_4$, already introduced in the previous section:

\[
\begin{array}{cccccccc}
\mathcal{B}_4 & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & \mathcal{C}_4 & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ \\
\lambda_0 & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & \lambda_0 & $\lambda_1$ & $\lambda_2$ \\
\lambda_1 & $\lambda_1$ & $\lambda_2$ & 0 & \lambda_1 & $\lambda_2$ & 0 \\
\lambda_2 & $\lambda_2$ & 0 & 0 & \lambda_2 & 0 & 0 \\
\end{array}
\]

(24)

Further enlargement, coming with the new translational $R_a = P_{a(t)} = \lambda_3 \tilde{P}_d$ generator, brings about a much richer collection of algebras:

• 17 × Poincaré-like
• 3 × AdS-like of type $B_5$, $C_5$, and $D_5 \equiv AdS \oplus AdS$

\[
\begin{array}{cccccccccccccccc}
\mathcal{B}_5 & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & $\lambda_3$ & \mathcal{C}_5 & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & $\lambda_3$ & \mathcal{D}_5 & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & $\lambda_3$ & $\lambda_3$ \\
\lambda_0 & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & $\lambda_3$ & \lambda_0 & $\lambda_1$ & $\lambda_2$ & $\lambda_3$ & \lambda_0 & $\lambda_1$ & $\lambda_2$ & $\lambda_3$ & $\lambda_3$ \\
\lambda_1 & $\lambda_1$ & $\lambda_0$ & $\lambda_2$ & $\lambda_3$ & \lambda_1 & $\lambda_0$ & $\lambda_2$ & $\lambda_3$ & \lambda_1 & $\lambda_0$ & $\lambda_2$ & $\lambda_3$ & $\lambda_3$ \\
\lambda_2 & $\lambda_2$ & $\lambda_3$ & 0 & 0 & \lambda_2 & $\lambda_3$ & 0 & 0 & \lambda_2 & $\lambda_3$ & 0 & 0 & $\lambda_3$ \\
\lambda_3 & $\lambda_3$ & $\lambda_2$ & 0 & 0 & \lambda_3 & $\lambda_2$ & 0 & 0 & \lambda_3 & $\lambda_2$ & 0 & 0 & $\lambda_3$ \\
\end{array}
\]

(25)

• 10 × Maxwell-like, three of which, $\mathcal{B}_5$, $\mathcal{C}_5$, $\mathcal{D}_5$, have already been derived in a previous section
although surprisingly there are five others with the 05 (of which only one is presented below) and an additional two without the zero elements

Normal, calligraphic and gothic labels distinguish, respectively, the Poincaré-, AdS-, and Maxwell-like algebras. Note that the elements of the semigroup $C_5$, just like any other representative of this family when $m = \text{odd}$, correspond to the cyclic group $Z_4$ (in an arbitrary case $\mathbb{Z}_m$). In turn, one of the AdS-like tables, schematically denoted as $C_5$, corresponds to the Klein group given by $Z_2 \times Z_2$. This appeared in the $S$-expansion context in [25], but some special conditions, like introducing minus signs to the definitions of the generators, were necessary to reach the final goal.

Explicit tables for the algebras labeled by $m = 3, 4, 5, 6$, with or without the zero elements, as well as the tool checking the associativity, can be found at resonantalgebras.wordpress.com.

Before going any further, it should be pointed out that in gravity applications, the spin connection should be associated with the semigroup identity (the Lorentz generator), but there might still be an ambiguity related to what the true vielbein is. Switching the labels between $e^a$ and $h^a$, and interchanging the second row and column with the fourth row and column, causes an isomorphism between the last AdS-like table in (25) and the last Maxwell-like example in (27), as well as between the first AdS-like and some other Poincaré-like table. Moreover, the same could be said about the $4 \times$Poincaré-like tables and the $4 \times$Maxwell-like ones containing the zero elements. To avoid confusion while constructing the gravity theory, and so as not to complicate the classification of the algebras, we are always going to relate a vielbein $e^a$ to $\lambda_0$ and $P_a$, just like the spin connection $\omega^{ab}$ will always be related to $\lambda_0$ and the Lorentz generator $J_{ab}$. 

\begin{center}
\begin{tabular}{c|cccc}
$\mathcal{B}_5$ & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & $\lambda_3$ \\
$\lambda_0$ & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & $\lambda_3$ \\
$\lambda_1$ & $\lambda_1$ & $\lambda_2$ & $\lambda_3$ & 05 \\
$\lambda_2$ & $\lambda_2$ & $\lambda_3$ & 05 & 05 \\
$\lambda_3$ & $\lambda_3$ & 05 & 05 & 05 \\
\end{tabular}
\end{center}

(26)

\begin{center}
\begin{tabular}{c|cccc}
$\mathcal{C}_5$ & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & $\lambda_3$ \\
$\lambda_0$ & $\lambda_0$ & $\lambda_1$ & $\lambda_2$ & $\lambda_3$ \\
$\lambda_1$ & $\lambda_1$ & $\lambda_2$ & $\lambda_3$ & 05 \\
$\lambda_2$ & $\lambda_2$ & $\lambda_3$ & 05 & 05 \\
$\lambda_3$ & $\lambda_3$ & 05 & 05 & 05 \\
\end{tabular}
\end{center}

(27)
5. Non-standard enlargement

It is worth mentioning one more overlooked generalization, which lies in the enlargement order. In principle, there could be two schemes for adding the new generators:

- $J_{ab}, P_a$ and then $Z_{ab}$
- $J_{ab}, P_a$ and then $R_a$

but only the first one has been explored in the literature. Of course, including the additional generator $R_a$ in the first scheme, as in a standard enlargement, and adding $Z_{ab}$ to the second, gives the same result, as $\{J_{ab}, P_a, R_a, Z_{ab}\} \equiv \{J_{ab}, P_a, Z_{ab}, R_a\}$. However, for yet another generator we return to the same starting situation. A minimal setup with only three nonzero elements $\{\lambda_0, \lambda_1, \lambda_3\}$ shows that the semigroup conditions used in this paper are only fulfilled in one case, given below under the label $I$. Loosening up the requirement of the identity element for a moment, so $\lambda_0$ does not necessarily preserve anything other than $\lambda_0$ and $\lambda_1$, brings two more tables:

|   | $\lambda_0$ | $\lambda_1$ | $\lambda_3$ |
|---|-------------|-------------|-------------|
| I | $\lambda_0$ | $\lambda_0$ | $\lambda_1$ |
|   | $\lambda_1$ | $\lambda_3$ | $\lambda_3$ |
| II| $\lambda_0$ | $\lambda_0$ | $\lambda_1$ |
|   | $\lambda_1$ | $\lambda_3$ | $\lambda_3$ |

The related curvatures are defined as:

$$F_I = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{\ell} (e^{\alpha a} + \omega_\beta^a e^{\beta b}) P_a + \frac{1}{\ell} (e^{\alpha a} + \omega_\beta^b e^{\beta b}) R_a,$$

(28)

$$F_{II} = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{\ell} (e^{\alpha a} + \omega_\beta^a e^{\beta b} + \omega_\beta^b e^{\beta b}) P_a + \frac{1}{\ell} d e^{\alpha a} R_a,$$

(29)

$$F_{III} = \frac{1}{2} (R^{ab} + \frac{1}{\ell^2} (e^{\alpha a} + h^e)(e^{\beta b} + h^b)) J_{ab} + \frac{1}{\ell} (e^{\alpha a} + \omega_\beta^a e^{\beta b} + \omega_\beta^b e^{\beta b}) P_a + \frac{1}{\ell} d e^{\alpha a} R_a.$$

(30)

The last case opens the interesting possibility of an extra contribution to the cosmological constant, with a price to pay in the modification of the torsion and possible impact on the boundary. It would be very interesting to be able to understand the implications of giving away a preservation of the $\lambda_0, \lambda_1$ outcome as well, which would naturally have even more serious consequences for the translations and the definition of the torsion. Such associative tables read

|   | $\lambda_0$ | $\lambda_1$ | $\lambda_3$ |
|---|-------------|-------------|-------------|
| I | $\lambda_0$ | $\lambda_0$ | $\lambda_1$ |
|   | $\lambda_1$ | $\lambda_3$ | $\lambda_3$ |
| II| $\lambda_0$ | $\lambda_0$ | $\lambda_1$ |
|   | $\lambda_1$ | $\lambda_3$ | $\lambda_3$ |

effectively causing the reversion of the labels of $P_a \leftrightarrow R_a$ and the role of the fields $e^{\alpha a} \leftrightarrow h^e$. Preliminary inspection of the tables with even more elements does not seem to lead to the realization of the mixed situations where $\lambda_0 \lambda_1 = \lambda_3$ and $\lambda_0 \lambda_3 = \lambda_1$. All these types of enlargements
are justified within the $S$-expansion framework, but it might be better to leave this for a more detailed analysis in the future, as it goes beyond the use of the monoid requirement used in this work.

6. Gauge transformations, curvatures and gravity actions

In this section, we focus on the gravity actions based on the found symmetries. To this end, we provide the necessary constituents and prescriptions concerning the extended field content, curvatures, transformations and invariant tensors. In general, any type of the resonant algebras shown here can be used in the arbitrary dimension. Looking at the types of generators $\{J_{ab(i)}, P_{a(i)}\} = \{J_{ab}, P_a, Z_{ab}, R_{ab}, \ldots\}$, depending on the parity of $m$, there will either be the same amount of $J$-like and $P$-like generators or there will be one more of the former. Thus, being careful with the range of indices, we introduce separate $i, p = 0, 1, \ldots$ and $j, q = 0, 1, \ldots$ to take this into account. For a chosen algebra one builds the algebra-valued one-form connection,

$$A = \frac{1}{2} \epsilon^{ab} J_{ab(i)} + \frac{1}{\ell} \epsilon^{a(i)} P_{a(i)} ,$$

where group indices $a, b$ run from 1 to the spacetime dimension $D$. Then the zero-form gauge parameter,

$$\Theta = \frac{1}{2} N^{ab(i)} J_{ab(i)} + \epsilon^{a(i)} P_{a(i)} ,$$

(31)

defines the infinitesimal gauge transformation according to

$$\delta \omega \xi = \delta \omega \lambda \lambda = -\frac{1}{\ell} (\epsilon^{a(i)} \xi^{b(i)} - \epsilon^{b(i)} \xi^{a(i)}) \lambda_{2j,2q} + \lambda_{2j+1,2q} + \lambda_{2j,2q} + \lambda_{2j+1,2q}$$

(32)

$$\delta \omega \xi \lambda \lambda = -\frac{1}{\ell} (\epsilon^{a(i)} \Phi^{b(i)} + \epsilon^{b(i)} \Phi^{a(i)}) \lambda_{2j,2q} + \lambda_{2j+1,2q} + \lambda_{2j,2q} + \lambda_{2j+1,2q}$$

(33)

$$\delta \xi \lambda \lambda = -\frac{1}{\ell} (\epsilon^{a(i)} \Phi^{b(i)} + \epsilon^{b(i)} \Phi^{a(i)}) \lambda_{2j,2q} + \lambda_{2j+1,2q} + \lambda_{2j,2q} + \lambda_{2j+1,2q}$$

(34)

To obtain the final form, it is necessary to choose a particular semigroup multiplication law. The identity element $\lambda_0$ reproduces the standard transformation part, but it gets a new contribution every time $\lambda_0$ again appears as a result of the multiplication between some other elements

$$\delta \omega \mu \mu = -D_{\mu}^{ab} \lambda_{ab} + \frac{1}{\ell} (\epsilon^{a} \xi^{b} - \epsilon^{b} \xi^{a}) + \ldots$$

$$\frac{1}{\ell} \delta \omega \mu \mu = -D_{\mu}^{ab} \lambda_{ab} + \frac{1}{\ell} (\epsilon^{a} \xi^{b} - \epsilon^{b} \xi^{a}) + \ldots$$

(35)

Naturally, the transformations of other fields become even more rearranged going from one algebra to another. The same happens for the curvature two-form

$$F = \frac{1}{2} F^{ab(i)} J_{ab(i)} + \frac{1}{\ell} F^{a(i)} P_{a(i)} = \frac{1}{2} F^{ab(i)} \lambda_{2j,2q} J_{ab} + \frac{1}{\ell} F^{a(i)} \lambda_{2j,2q} + \lambda_{2j+1,2q}$$

(36)

which results from
\[
F = \frac{1}{2} \left( \omega^{\mu \nu (i)} \epsilon_{ij} \lambda_{2j} + \omega^{\mu \nu \alpha (i)} \epsilon_{ijk} \lambda_{2j} \lambda_{2p} + \frac{1}{\ell^2} \epsilon^{\alpha \beta (i)} \epsilon^{k \lambda (q)} \lambda_{2j+1} \lambda_{2q+1} \right) \tilde{P}_{ij}
\]

(37)

One could try to construct the action by hand to get the full invariance or give away a part of the symmetries, for example, keeping only the Lorentz invariance like in [13]. The deformed BF theory for 4D [11, 12] offers nice control on encoding the particular invariance through the additional auxiliary two-forms. In odd dimensions, the Chern–Simons theory (CS) [26] by the construction assures the action invariance due to the full local gauge symmetry (local Lorentz and AdS boosts). In even dimensions, one needs to use the Born–Infeld (BI) action with the invariance due to the Lorentzian subalgebra [22, 23]. The remarkable advantage of the $S$-expansion method lies in the fact that it automatically provides the invariant tensor for the $S$-expanded algebra $\mathfrak{g} = S \times \mathfrak{g}$ in terms of the original $\mathfrak{g} = \text{AdS}$ tensor. For the $(2n+1)$-dimensional expanded Chern–Simons gravity action [2] it is defined as follows:

\[
\left\langle J_{a_1 a_2 \ldots a_{2n+1}} \right\rangle = \sigma_{2j+1} \delta_{a_1 a_2 \ldots a_{2n+1}} \frac{2^n}{n+1} \epsilon_{a_1 a_2 \ldots a_{2n+1}}.
\]

(38)

Through its index, an arbitrary dimensionless constant $\sigma_{\alpha}$ depending on the algebra, introduces specific components of the invariant tensor. Effectively, for a chosen set of generators, index $\alpha$ in

\[
\left\langle (\lambda_{2k_1} \cdots \lambda_{2k_n} \lambda_{2k_{n+1}}) J_{a_1 a_2 \ldots a_{2n+1}} \right\rangle = \sigma_{\alpha} \frac{2^n}{n+1} \epsilon_{a_1 a_2 \ldots a_{2n+1}},
\]

(39)

will simply be determined as an outcome of the multiplication of the related $\lambda$s

\[
\lambda_{2k_1} \cdots \lambda_{2k_n} \lambda_{2k_{n+1}} = \lambda_{\alpha}.
\]

(40)

Obviously, the presence of $O_3$ acting on the generators causes the whole $\left\langle \cdots \right\rangle = 0$.

Similarly, in even $2n$-dimensions, an invariant tensor for the Born–Infeld action [22] is given as

\[
\left\langle (\lambda_{2k_1} \cdots \lambda_{2k_n}) J_{a_1 a_2 \ldots a_{2n}} \right\rangle = \sigma_{\alpha} \frac{2^{n-1}}{n} \epsilon_{a_1 a_2 \ldots a_{2n}}.
\]

(41)

The $S$-expansion now concerns the Lorentz algebra $\mathfrak{so}(2n-1, 1)$ and uses $S_0 = \{0, \lambda_{19}, \lambda_{21}, \ldots\}$ as the semigroup. In this case an invariance is only achieved for the subalgebra $\mathcal{L} = \mathfrak{S}_0 \times \mathfrak{v}_6$.

Both BI and CS prescriptions were thoroughly analyzed in various dimensions and for many different algebras. Without going into much detail concerning the factors or constants, let us now just briefly highlight the main features of the 5D CS and 4D BI actions based on all algebras with $m = 4$ and 5. Focusing only on the purely gravity content ($\omega$, $e$), the final CS terms depend on the chosen algebra and belong to the following sectors of the invariant tensor:

| 5D CS | Poincaré-like | Maxwell | AdS-like | Maxwell-like |
|-------|--------------|---------|---------|-------------|
| Terms | $m = 4$ and $5$ | $\mathfrak{B}_4$ | $\mathfrak{C}_4$ | $m = 5$ | $\mathfrak{B}_5$ | $\mathfrak{C}_5$ | $\mathfrak{D}_5$ | Tables from (27) | Remainder of the tables with $0_1$ |
| $RRe$ | $\sigma_1$ | $\sigma_1$ | $\sigma_1$ | $\sigma_1$ | $\sigma_1$ | $\sigma_1$ | $\sigma_1$ | $\sigma_1$ |
| $Re^3$ | 0 | 0 | $\sigma_1$ | $\sigma_1$ | $\sigma_3$ | $\sigma_3$ | $\sigma_3$ | $\sigma_1$ | 0 |
| $e^5$ | 0 | 0 | $\sigma_1$ | $\sigma_1$ | 0 | $\sigma_1$ | $\sigma_3$ | $\sigma_1$ | 0 |
The extra field content naturally comes with its own set of $\sigma_\alpha$ constants. The most nontrivial shift of the various terms into different invariant tensor components happens for the Maxwell $\mathfrak{B}_5$, $\mathfrak{C}_5$, $\mathfrak{D}_5$ algebras. This was used in [20, 21] as a starting point to find the Einstein–Hilbert term as a special limit of the CS theory, and recently further explored in [24] in the context of the pure Lovelock theory.

Looking at an analogous table for the purely gravitational terms for the 4D Born–Infeld actions, we notice similar relations:

| $m = 4$ and 5 | $\mathfrak{B}_4$ | $\mathfrak{C}_4$ | $\mathfrak{B}_5$ | $\mathfrak{C}_5$ | $\mathfrak{D}_5$ | Tables from (27) | Remainder of the tables with $0_i$ |
|---------------|------------------|------------------|------------------|------------------|------------------|--------------------------|-----------------------------|
| $RR$          | $\sigma_0$       | $\sigma_0$       | $\sigma_0$       | $\sigma_0$       | $\sigma_0$       | $\sigma_0$       | $\sigma_0$               |
| $Re^2$        | $0$              | $\sigma_2$       | $\sigma_2$       | $\sigma_2$       | $\sigma_2$       | $\sigma_2$       | $\sigma_2$               |
| $e^2$         | $0$              | $0$              | $\sigma_0$       | $\sigma_0$       | $\sigma_0$       | $\sigma_0$       | $\sigma_0$               |

The full construction of the 4D BI actions, based on $\mathfrak{B}_4$ and $\mathfrak{C}_4$, was given in detail in [3]. Besides repeating the same procedure for the other cases, it would be especially interesting to use the $\mathfrak{B}_4$ and $\mathfrak{C}_4$ algebras from (23) and mimic the Einstein–Hilbert action with the cosmological constant term through the additional constraint $D^a k^{ab} + k^{ac} \wedge k^b = \frac{1}{3} e^a \wedge \omega^b$.

For purely gravity content, it could be regarded as generic behavior, not only for $\mathfrak{C}_5$ in 5D, but more generally, that the $\mathfrak{C}_D$ algebra in any odd $D \geq 5$ dimension puts the maximal (dimensionally continued Euler) and minimal (cosmological constant) CS terms into the same invariant sector. Similarly, the requirement of vanishing the cosmological constant term from the CS Lagrangian but preserving all the others implies using the $\mathfrak{B}_m$ family. Analogously, we can show the same for the BI maximal and minimal terms for $\mathfrak{C}_5$ in 4D, and more generally the $\mathfrak{C}_{D-1}$ algebra in arbitrary even $D \geq 4$ dimensions. For the CS/BI limit, a unique characteristic of the $\mathfrak{D}_m$ family could lead to the Einstein–Hilbert with the $\Lambda$ term, but only at the level of the action. This would be done by switching off the extra fields and forcing the part of the invariant tensor associated with the maximal term to vanish by setting the particular dimensionless constants $\sigma_\alpha$ to zero.

However, the case of pure Lovelock gravity teaches us that the same sectors are not the whole story: sometimes it is necessary to obtain the relative sign difference between the terms to admit a proper vacuum and solutions. Nevertheless, for a sufficient amount of different types of generators—and thus high enough $m$ value—one can ultimately assure that each of the terms we are interested in is able to come with a separate arbitrary constant. This allows us to explicitly encode any relation between them. Mind that this kind of immersion is still not totally unrestricted. For instance, although $\mathfrak{C}_l$ finally assured in [24] the desired PL action limit from 5D CS, it still failed with a dynamical limit. Even though we can demand that the extra fields vanish, their variations bring the additional highly nontrivial conditions on the $\omega, e$ terms, and further special identification among the extra fields is required to overcome this issue. Maybe use of the other algebra, offering different extra field configurations, could help to improve this result.

In the 4D non-geometrical construction, which uses the Hodge $\ast$ star operator, it is possible to introduce the $R(\omega + k) \wedge \ast R(\omega + k) = F(\omega) \wedge \ast F(\omega)$ term motivated by the direct sum algebras $\mathfrak{C}_4 \equiv ISO \oplus Lorentz$ and $\mathfrak{C}_4 \equiv AdS \oplus Lorentz$ (see [14, 19] and the appendix B). The final action can then have the form of GR (related to the first piece of algebra: the Poincaré or AdS) coupled with the Yang–Mills term for the gauge group related to the second piece of algebra. Such an action is invariant due to the transformations generated by both
the Lorentz-like generators $L_{ab} = Z_{ab}$ and $N_{ab} = J_{ab} - Z_{ab}$. Given in [14], a possible link to the biconnection/bitetrad/bimetric theories could now be studied in the much more general framework of other algebras.

A detailed discussion concerning all the mentioned applications will be left to future work.

7. Summary

Imposing specific conditions on a wide class of the $S$-expanded resonant algebras, introduced in [27, 28], not only assured consistent grounds for many valuable and interesting algebras, but it also limited the overwhelming vastness of algebraic examples. We find that expansions for the abelian semigroups with zero and the identity, simultaneously obeying the resonant decomposition, are well defined and straightforwardly applicable in the context of gravity.

Among the results, we have found all known Maxwell families of types $B, C, D$ and managed to generalize their description to the new examples. The new enlargements containing the Poincaré and $AdS$ as the subalgebras put it all in greater perspective, and in the end allow for a complete generalization of the bosonic enlargements with an arbitrary number of Lorentz-like and translational-like generators.

The wide class of algebras derived here delivers a nontrivial extension to the construction of gauge gravity theories. Certainly, there is still a lot of work required to find a satisfactory interpretation of the extra field content, a description of the new symmetries and understand their consequences. The motivation is not very different from supersymmetry—that is, to extend the general notion accommodating extra symmetries/fields and go beyond the starting setup. Some applications related to the dark energy/matter and cosmological constant term were considered, but it would also be very interesting to look at the supersymmetric versions of the algebras presented here. Some of them have already been analyzed in [2, 9, 29–31]. New examples could help us understand even better the underlying relations between various supergravity theories.

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Appendix A. Algebra from the multiplication table

For a given semigroup multiplication table it is possible to instantly read off the corresponding Lie algebra. The expanded generators can be schematically collected in a similar commutation table, of course keeping in mind particular structural constants belonging to the commutators of $[X, X], [X, X]$ and $[X, X]$. Taking as an example the original Maxwell algebra, where $0_{\mathcal{T}_{H}} = 0$, we see that
with the generators defined as

\[ J_{ab} = \lambda_0 \tilde{J}_{ab}, \quad P_a = \lambda_1 \tilde{P}_a, \quad Z_{ab} = \lambda_2 \tilde{J}_{ab}. \]  

(A.2)

The commutation relations can be directly read off from the table on the right, or explicitly derived using the original AdS algebra given in equations (2)–(4),

\[
\begin{align*}
[J_{ab}, J_{cd}] &= \lambda_0 \lambda_0 (\eta_{ab} J_{cd} - \eta_{ac} J_{bd} - \eta_{ad} J_{bc} - \eta_{bd} J_{ac}) = \eta_{bc} J_{ad} - \eta_{bd} J_{ac} + \eta_{ab} J_{bc} - \eta_{ac} J_{bd}, \\
[J_{ab}, Z_{cd}] &= \lambda_0 \lambda_2 (\eta_{ab} J_{cd} - \eta_{ac} J_{bd} - \eta_{ad} J_{bc} - \eta_{bd} J_{ac}) = \eta_{bc} Z_{ad} - \eta_{bd} Z_{ac} + \eta_{ab} Z_{bc} - \eta_{ac} Z_{bd}, \\
[Z_{ab}, Z_{cd}] &= \lambda_2 \lambda_2 (\eta_{ab} J_{cd} - \eta_{ac} J_{bd} - \eta_{ad} J_{bc} - \eta_{bd} J_{ac}) = 0, \\
[J_{ab}, P_a] &= \lambda_0 \lambda_1 (\eta_{ab} \tilde{P}_a - \eta_{ac} \tilde{P}_b) = \eta_{bc} P_a - \eta_{bd} P_b, \\
[Z_{ab}, P_a] &= \lambda_2 \lambda_1 (\eta_{ab} \tilde{P}_a - \eta_{ac} \tilde{P}_b) = 0, \\
[P_a, P_b] &= \lambda_1 \lambda_1 \tilde{J}_{ab} = Z_{ab}. 
\end{align*}
\]

(A.3)

Appendix B. Generalized change of basis

The generalized change of basis presented in [1] originates from the result of [8], where the generators are reorganized to bring together \( L_{ab} = Z_{ab} \) and \( L_a = P_a \), and to introduce a shift in the definition of \( N_{ab} = J_{ab} - Z_{ab} \). Then, the two algebra tables labeled by the subindex \( m = 4 \), being Poincaré-like \( C_4 \) and Maxwell \( \mathfrak{C}_4 \),

\[
\begin{align*}
C_4 & \quad J_{..} \quad P \quad Z \quad \mathfrak{C}_4 & \quad J_{..} \quad P \quad Z, \\
J_{..} & \quad J_{..} \quad P \quad Z \quad J_{..} & \quad J_{..} \quad P \quad Z, \\
P & \quad P \quad 0 \quad P \quad P & \quad P \quad P \quad P \quad Z \quad P, \\
Z_{..} & \quad Z_{..} \quad P \quad Z \quad Z_{..} & \quad Z_{..} \quad P \quad Z \quad Z_{..} \quad P \quad Z \quad Z_{..} \quad P \quad Z, \\
\end{align*}
\]

by means of \([N_{..}, L_{..}] = 0\) and \([N_{..}, L_{..}] = 0\), happen to be equivalent to

\[
\begin{align*}
C_4 & \quad L_{..} \quad L \quad N_{..} \quad \mathfrak{C}_4 & \quad L_{..} \quad L \quad N_{..}, \\
L_{..} & \quad L \quad 0 \quad L \quad L \quad L \quad 0, \\
L_{..} & \quad 0 \quad 0 \quad L \quad L \quad L \quad 0, \\
N_{..} & \quad 0 \quad 0 \quad N \quad N \quad 0 \quad 0 \quad N. 
\end{align*}
\]

(B.2)
obviously forming the direct sums of the subalgebras: \( \text{Poincaré} \oplus \text{Lorentz} \) and \( \text{AdS} \oplus \text{Lorentz} \). The same can be repeated for the case of \( m = 5 \). Indeed, three particular algebras

\[
\begin{array}{cccccccc}
D_5 & J & P & Z & R & D_5 & J & P & Z & R & D_5 & J & P & Z & R \\
J & J & P & Z & R & J & J & P & Z & R & J & J & P & Z & R \\
P & P & 0 & R & 0 & P & P & J & R & Z & P & P & Z & R & Z \\
Z & Z & R & Z & R & Z & Z & R & Z & R & Z & Z & R & Z & R \\
R & R & 0 & R & 0 & R & R & Z & R & Z & R & R & Z & R & Z \\
\end{array}
\]

(B.3)

with the redefinitions \( L_{ab} = Z_{ab} \) and \( L_a = R_a \) along with \( N_{ab} = J_{ab} - Z_{ab} \) and \( N_a = P_a - R_a \) are equivalent to

\[
\begin{array}{cccccccc}
D_5 & L & N & N & N & D_5 & L & N & N & N & D_5 & L & N & N & N \\
L & L & 0 & 0 & L & L & L & 0 & 0 & L & L & L & 0 & 0 \\
L & L & 0 & 0 & L & L & L & 0 & 0 & L & L & L & 0 & 0 \\
N & 0 & 0 & N & N & N & 0 & 0 & N & N & N & 0 & 0 & N & N \\
N & 0 & 0 & N & 0 & N & 0 & 0 & N & N & N & 0 & 0 & N & 0 \\
\end{array}
\]

(B.4)

clearly forming the direct sums of \( \text{Poincaré} \oplus \text{Poincaré}, \text{AdS} \oplus \text{AdS} \) and \( \text{AdS} \oplus \text{Poincaré} \). The explicit prescription for the change of basis in a general case can be found in [1].

Mind that in the algebraic sums above one deals with \( L_{ab} \) and \( N_{ab} \), which represent a peculiar kind of the Lorentz generator that commutes with other generators. This is very different from what happens to the standard \( J_{ab} \) outcomes, and it leave us with the question of whether these Lorentz generators could be physical, or they are just a mere artifact of the mathematical description. From a formal point of view, introducing an identity element might not exhaust all the possibilities, as the semigroup equivalents of the algebraic redefinitions above, along with other examples (see www.resonantalgebras.wordpress.com) containing the zero elements, still seem to represent valid algebras with possible use in the gravity models.

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