Coxeter Group Actions on Interacting Particle Systems

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Abstract

We provide a conceptual proof of the color–position symmetry of colored ASEP by relating it to the actions of Coxeter groups. The group action (and hence the color–position symmetry) also applies to more general interacting particle systems, such as the colored ASEP\((q,j)\) or systems with open boundary conditions. As an application, we find the asymptotics of the expected positions of second–class particles in the ASEP\((q,j)\).

1 Introduction

In a recent paper [BB19], the authors prove a color–position symmetry for a inhomogeneous ASEP, generalizing previous results for homogeneous ASEP [AAV11] and TASEP [AHR09]. They remark on the need for a “conceptual understanding of why it should hold.” Here, we show that the color–position symmetry follows through a Coxeter group action on the state space of the interacting particle system. In particular, as observed in [Kua19], the left action of a Coxeter group can be viewed as permuting the positions, while the right action can be viewed as permuting the colors – color–position symmetry amounts to the commutation between the two actions.

As a by–product of this approach, it will be shown that the colored (or multi–species) ASEP\((q,j)\) (introduced in [Kua17], generalizing the single–single model introduced in [CGRS16]) also satisfies a color–position symmetry. Additionally, by using the type \(B\) Coxeter group, it will be shown that the colored ASEP\((q,j)\) with open boundary conditions also satisfies color–position symmetry, and has \(q\)–exchangeable stationary measures. By combining the color–position symmetry with hydrodynamics of the single–species ASEP\((q,j)\), we find the expected positions of second–class particles in the ASEP\((q,j)\).

Shortly before this manuscript was completed, the author was informed of a preprint [Buf20] which takes a similar approach to color–position symmetry with asymptotic applications, but using Hecke algebras.

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2 Preliminaries

2.1 Coxeter Groups

We recall some background knowledge about Coxeter groups. The statements here are from [BB05] and [Car85].

A Coxeter group, or Coxeter system, consists of a pair \((W,S)\) where \(W\) is a group and \(S\) is a set of generators. The relations are given by \(s_i s_j s_i \cdots = s_j s_i s_j \cdots\) for any \(s_i, s_j \in S\), where \(m_{ij}\) are some positive integers. In this paper, we will consider two examples of Coxeter groups. The first is the \(A_{N-1}\) Coxeter group, which is defined by setting \(S = \{s_1, \ldots, s_{N-1}\}\) and defining

\[
m_{ij} = \begin{cases} 
2, & \text{if } |i - j| > 1, \\
3, & \text{if } |i - j| = 1, \\
1, & \text{if } i = j.
\end{cases}
\]

Concretely, the \(A_{N-1}\) Coxeter group is isomorphic to the symmetric group \(S_N\) on \(N\) letters, and \(s_i\) is the transposition \((i \ i + 1)\).
The $B_C N$ Coxeter group has generating set denoted $S = \{s_0, \ldots, s_{N-1}\}$ with

$$m_{ij} = \begin{cases} 2, & \text{if } |i - j| > 1, \\ 3, & \text{if } |i - j| = 1, \\ 1, & \text{if } i = j. \end{cases}$$

and

$$m_{0i} = m_{i0} = \begin{cases} 2, & \text{if } i > 0, \\ 4, & \text{if } i = 1, \\ 1, & \text{if } i = 0. \end{cases}$$

An explicit presentation of the type $B_C$ Coxeter group is the wreath product $S_2 \wr S_N$, where $S_N$ acts on $(S_2)^N$ by permuting the coordinates. From this description, it is clear that $|W| = 2^N N!$. Another description is that $W$ is the group of permutations on $\{-N, \ldots, -1, 1, \ldots, N\}$ which preserve the pairs $(-i, i)$. In this description, the elements of $S$ can be described by the permutations $s_0 = (-1 1)$ and $s_i = (i i + 1)(-i - i - 1)$ for $1 \leq i \leq N - 1$.

Every $w \in W$ can be written as a word $s_{i_1} \cdots s_{i_k}$ with letters in $S$; let $l(w)$ be the length of the shortest such word. For $S_N$, viewed as the symmetric group on $N$ letters, this length function is

$$l(w) = |\{1 \leq i < j \leq N : w(i) > w(j)\}|$$

For the $B_C N$ Coxeter group, represented as a subgroup of $S_{2N}$, the length function is

$$l(w) = \frac{1}{2} |\{-N \leq i < j \leq N : w(i) > w(j)\}| + \frac{1}{2} |\{1 \leq i \leq N : w(i) < 0\}|.$$

Given a subset $I \subseteq S$, let $W_I$ denote the subgroup of $W$ generated by $I$. The pair $(W_I, I)$ is itself a Coxeter group, and such subgroups are called parabolic subgroups. The length function on $W_I$ is simply the restriction of the length function on $W$. For any parabolic group $H \leq W$, every left coset of $H$ has a unique element of minimal length. Let $D_H \subset W$ denote the set of these distinguished left coset representatives. Furthermore, every $w \in W$ can be written uniquely as a product $w = x b$, where $x \in D_H, b \in H$ and $l(w) = l(x) + l(b)$.

This also extends to double cosets. Given any two parabolic subgroups $H', H \leq W$, every double coset of $H'$ and $H$ has a unique element of minimal length. Let $D_{H' \cap H} = D_{H'} \cap D_H$ denote the set of these distinguished double coset representatives. One might expect that every $w \in W$ can be written uniquely as a product $w = axb$, where $a \in H', x \in D_{H' \cap H}, b \in H$ and $l(w) = l(a) + l(x) + l(b)$, but this turns out not to be true. Instead, one must require that $a \in H' \cap D_K$, where $K$ is another parabolic subgroup that depends on $w$, $H'$ and $H$. The details of $K$ will not be needed here.

Given any subset $Y \subset W$, let $Y(q)$ be the Poincaré series of a Coxeter group $W$, defined by

$$Y(q) = \sum_{w \in Y} q^{l(w)}.$$

We will call a probability measure $\mathbb{P}$ on $Y$ the $q$-exchangeable probability measure if

$$\mathbb{P}(y) = \frac{q^{l(y)}}{Y(q)}.$$

If $Y = W$, then the normalizing constant has the simple expression

$$W(q) = \prod_{i=1}^n [e_i + 1]_q,$$

where the $e_i$ are the exponents of $W$ and $[m]_q = 1 + q + \ldots + q^{m-1} = (1 - q^m)/(1 - q)$ is a $q$-deformed integer. In the case of the $B_C N$ Coxeter group, the exponents are $1, 3, \ldots, 2N - 1$. In the case of the $A_{N-1}$ Coxeter group, the exponents are $1, 2, \ldots, N - 1$. 

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2.2 Interacting Particles

Let us first describe the state space of the interacting particle system.

Fix two sequences of positive integers \( \mathbf{m} = (m_x : x \in \{1, \ldots, L\}) \) and \( \mathbf{N} = (N_1, \ldots, N_n) \) which both sum to \( N \). Let \( S(\mathbf{N}, \mathbf{m}) \) denote the set of all particle configurations on the lattice \( \{1, \ldots, L\} \) where exactly \( m_x \) particles occupy site \( x \), subject to the constraint that there are exactly \( N_i \) particles of species \( i \) or \(-i\). More specifically, an element of \( S(\mathbf{N}, \mathbf{m}) \) can be written as

\[
\bar{k} = (k^{(i)}_x : x \in \{1, \ldots, L\}, i \in \{-n, \ldots, -1, 1, \ldots, n\})
\]

The non-negative integer \( k^{(i)}_x \) represents the number of particles of species \( i \) at vertex site \( x \). We require that

\[
\sum_{i=-n}^{n} k^{(i)}_x = m_x, \quad \sum_{x=1}^{L} (k^{(i)}_x + k^{(-i)}_x) = N_i
\]

Let \( S^{(0)}(\mathbf{N}, \mathbf{m}) \) denote the set of particle configurations, subject to the constraint that there are exactly \( N_i \) particles of species \( i-1 \) or \(-(i-1)\). In other words, we allow there to be particles of species 0, and

\[
\sum_{i=-n}^{n} k^{(i)}_x = m_x, \quad \sum_{x=1}^{L} (k^{(i)}_x + k^{(-i)}_x) = N_i \quad \text{for all} \quad 2 \leq i \leq n, \quad \sum_{x=1}^{L} k^{(0)}_x = N_1.
\]

Let \( S_1(\mathbf{N}, \mathbf{m}) \) denote the set of \( \bar{k} \) satisfying

\[
k^{(i)}_1 = k^{(-i)}_1, \quad \sum_{i=-n}^{n} k^{(i)}_1 = 2m_1, \quad \sum_{i=-n}^{n} k^{(i)}_x = m_x, \quad \text{for} \quad x \geq 2, \quad \frac{1}{2} (k^{(1)}_1 + k^{(-1)}_1) + \sum_{x=2}^{L} (k^{(1)}_x + k^{(-1)}_x) = N_{i-1}
\]

In words, every particle configuration of \( S_1(\mathbf{N}, \mathbf{m}) \) can be constructed by placing exactly \( m_x \) particles at site \( x \), with a total of \( N_i \) particles of species \( i \) or \(-i\), and then “cloning” every particle of species \( i \) at 1 with a particle of species \(-i\). Similarly to before, let \( S_1^{(0)}(\mathbf{N}, \mathbf{m}) \) denote the same state space, but allowing particles of species 0 and fixing exactly \( N_i \) particles of species \( i-1 \) or \(-(i-1)\).

Now we proceed to defining the dynamics of the multi–species ASEP\((q, \mathbf{m})\), which was introduced in [Kna17], generalizing the single–species model in [CGRS10]. Suppose that the current state is a particle configuration \( \bar{k} \), which can be in any of the four sets described above. The jump rate for a particle of species \( i \) at lattice site \( x \) to switch places with a particle of species \( j < i \) at lattice site \( x+1 \) is

\[
\frac{q^{\sum_{x=x_1}^{x_2} k^{(x)}_x} [k^{(j)}_{x+1}]_q}{[m_x]_q} - \frac{q^{\sum_{x=x_1}^{x_2} k^{(x)}_x} [k^{(j)}_{x+1}]_q}{[m_{x+1}]_q},
\]
and the jump rate for a particle of species $i$ at lattice site $x + 1$ to switch places with a particle of species $j < i$ at lattice site $x$ is

$$q \cdot \frac{q^{\sum_{x > j} k_x^{(j)}}}{[m_x]_q} \cdot \frac{q^{\sum_{x < i} k_{x+1}^{(i)}}}{[m_{x+1}]_q}.$$ 

The jump rates of the multi–species ASEP$(q, \vec{m})$ can be related to the jump rates of multi–species ASEP. Let $\mathbf{1}$ denote the sequence $(1, 1, \ldots, 1)$. Define a map $\Lambda : S(\mathbf{N}, \mathbf{m}) \to S(\mathbf{N}, \mathbf{1})$ as a product $\Lambda = \bigotimes_{x \in \mathbb{Z}} \Lambda(x)$, where each $\Lambda(x)$ splits the $m_x$ particles at lattice site $x$ randomly along $m_x$ sites, in such a way that the image of each $\Lambda(x)$ is $q$–exchangeable. Define the map $\Phi : S(\mathbf{N}, \mathbf{1}) \to S(\mathbf{N}, \mathbf{m})$ by fusing adjacent lattice sites. See Figure 5. The maps $\Lambda$ and $\Phi$ can be similarly defined on $S_1, S^{(0)}, S^{(0)}_1$.

If $L$ denotes the generator of the multispecies ASEP and $\hat{L}$ denotes the generator of the multi–species ASEP, then

$$\hat{L} = \Lambda L \Phi. \quad (1)$$

This was observed in Theorem 4.2(ii) of [Kua].

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3.1 Particle Configurations as Double Coset Representatives

We now show how particle configurations can be viewed as double coset representatives. A similar construction was made in [Kua19]. First consider the case when $W$ is the $A_{N-1}$ Coxeter system, that is, when $W = S_N$. Fix two sequences of positive integers $\mathbf{m} = (m_x : x \in \{1, \ldots, L\})$ and $\mathbf{N} = (N_1, \ldots, N_n)$

![Diagram](image)

Figure 5: The maps $\Lambda$ and $\Phi$. 

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which both sum to $N$. Let $S(N,m) \subset S(N,m)$ be the subset consisting solely of particles with positive species number.

Let 1 denote the sequence $(1,1,\ldots,1)$. There are two natural bijections from $\theta_1, \theta_2$ from $W$ to $\tilde{S}(1,1)$. For each $w \in W$, let $\theta_1(w)$ be the particle configuration where there is a particle of species $w^{-1}(x)$ at lattice site $x$ for $1 \leq x \leq N$. Let $\theta_2(w)$ be the particle configuration where there is a particle of species $w(x)$ at lattice site $x$ for $1 \leq x \leq N$.

Recalling that $D_{H',H} = D_{H'} \cap D_H$, there are two natural chains of surjections $W \rightarrow D_H \rightarrow D_{H',H}$ and $W \rightarrow D_{H'} \rightarrow D_{H',H}$, which map an element to its coset representative. There are also chains of surjections $\tilde{S}(1,1) \rightarrow \tilde{S}(1,N) \rightarrow \tilde{S}(m,N)$. The first projection $\Pi_N : \tilde{S}(1,1) \rightarrow \tilde{S}(1,N)$ is defined by (here, set $N_0 = 0$ and $N_{[i,j]} = N_i + N_{i+1} + \ldots + N_j$ for convenience)

$$(\Pi_N(k))^{(1)}_x = k^{(N[1,i])}_x + k^{(N[i,i]-1)}_x + \ldots + k^{(N[1,i-1]+1)}_x, \quad \text{for } 1 \leq i \leq n.$$ 

In words, $\Pi_N$ is the projection which “forgets” the distinction between particles of species $N_{i-1} + 1, \ldots, N_i$. This is also called the “color-blind” projection. The second projection is $\Phi_m : \tilde{S}(1,N) \rightarrow \tilde{S}(m,N)$ defined by (again setting $m_{[i,j]} = m_i + \ldots + m_j$)

$$\Phi_m(k)_x = k^{(m_{[1,x+1]})}_x + k^{(m_{[1,x]})}_{m_{[1,x]}} + \ldots + k^{(m_{[1,x]+1])}_{m_{[1,x]+1}},$$

In words, $\Phi_m$ is the projection which “fuses” the vertex sites $m_{[1,x]+1}, \ldots, m_{[1,x+1]}$.

These sequences of surjections can be related to each other in the following way:

**Proposition 3.1.** Set $H = S(N) = S(N_1) \times \cdots \times S(N_n) \leq S_N$ and $H' = S(m) = S(m_1) \times \cdots \times S_{m_k} \leq S_N$.

For any $\sigma \in D_H$, the set $\theta_1(\sigma H) \subset S(1,1)$ is a fiber of $\Pi_N$. Thus, there is a bijection $\theta'_1 : D_H \rightarrow S(1,N)$ defined by $\Pi^{-1}_N(\theta'_1(\sigma)) = \theta_1(\sigma H)$. For any $\tau \in D_{H',H}$, the set $\theta_2(\tau H')$ is a fiber of $\Phi_m$, and thus there is a bijection $\theta''_1 : D_{H',H} \rightarrow S(m,N)$ defined by $\Phi^{-1}_m(\theta''_1(\tau)) = \theta_2(\tau H')$.

Similarly, for any $\sigma \in D_H^{-1}$, the set $\theta_2(H\sigma) \subset S(1,1)$ is a fiber of $\Pi_N$. Thus, there is a bijection $\theta''_2 : D_H^{-1} \rightarrow S(1,N)$ defined by $\Pi^{-1}_N(\theta''_2(\sigma)) = \theta_2(H\sigma)$. For any $\tau \in D_{H',H'}$, the set $\theta''_2(\tau H')$ is a fiber of $\Phi_m$, and thus there is a bijection $\theta'_2 : D_{H,H'} \rightarrow S(m,N)$ defined by $\Phi^{-1}_m(\theta'_2(\tau)) = \theta'_2(\tau H')$.

This can be viewed in the commutative diagram

```
  W       \theta_1       S(1,1)       \theta_2       W
      |                           |  \Pi  |
D_H  \theta'_1 \rightarrow S(1,N) \leftarrow \theta''_1 D_H^{-1}
     \Phi_m
D_{H',H} \theta'_2 \rightarrow S(m,N) \leftarrow \theta''_2 D_{H',H'}
```

**Proof.** We only prove the statements about $\theta_1, \theta'_1, \theta''_1$, since the proof for $\theta_2, \theta'_2, \theta''_2$ are similar.

By definition, $\theta_1(\sigma H)$ is the set of all configurations where there is a particle of species $b^{-1}\sigma^{-1}(x)$ at lattice site $x$, where $b^{-1}$ ranges over $H = S(N)$. The projection $\Pi$ maps all species $b^{-1}\sigma^{-1}(x)$ to the same species, so $\theta_1(\sigma H)$ is a fiber of $\Pi$.

Similarly, $\theta_2(\tau H')$ is the set of all configurations where there is a particle of species $\tau^{-1}a^{-1}(x)$ at lattice site $x$, where $a$ ranges over $H' = S(m)$. This is equivalent to having a particle of species $\tau^{-1}(x)$ at lattice site $a(x)$. Since $\Phi_m$ fuses the lattice sites $a(x)$ together, this shows that $\theta_2(\tau H')$ is a fiber of $\Phi$.

We now proceed to an analogous result for the case when $W$ is of type $BC_N$. In that case, there is the additional possibility that $s_0$ is in the parabolic subgroups. Given $N = (N_1,\ldots,N_n)$ such that $N_1 + \ldots + N_n = N$, let $S(N) = S(N_1) \times \cdots \times S(N_n) \leq S_N \leq W$, and let $S^{(0)}(N) \leq W$ be generated by $S(N)$ and $s_0$. 


Proposition 3.2. Fix two sequences of integers $N = (N_1, \ldots, N_n)$ and $m = (m_1, \ldots, m_L)$, where each sequence sums to $N$. Then there is the commutative diagram of sets

\[
\begin{array}{ccc}
W & \sim & S(1,1) \\
\downarrow & & \downarrow \\
D_H & \sim & B \\
\downarrow & & \downarrow \\
D_{H',H} & \sim & A \\
\downarrow & & \downarrow \\
D_{H'} & \sim & W
\end{array}
\]

(where the $\sim$ indicates a bijection) in the following cases:

(a) When the parabolic subgroups are $H' = S(m)$ and $H = S(N)$, the partition $\Pi$ is

$$\{\pm\{N_1 + \ldots + N_{i-1} + 1, \ldots, N_1 + \ldots + N_i\} : 1 \leq i \leq n\},$$

and $B = S(N,1), A = S(N,m)$.

(b) When the parabolic subgroups are $H' = S(m)$ and $H = S^{(0)}(N)$, the partition $\Pi$ is

$$\{\pm\{N_1 + \ldots + N_{i-1} + 1, \ldots, N_1 + \ldots + N_i\} : 2 \leq i \leq n\} \bigcup \{N_1, \ldots, -1,1, \ldots, N_1\},$$

and $B = S^{(0)}(N,1), A = S^{(0)}(N,m)$.

(c) When the parabolic subgroups are $H' = S^{(0)}(m)$ and $H = S(N)$, the partition $\Pi$ is

$$\{\pm\{N_1 + \ldots + N_{i-1} + 1, \ldots, N_1 + \ldots + N_i\} : 1 \leq i \leq n\},$$

and $B = S(N,1), A = S^{(0)}_1(N,m)$.

(d) When the parabolic subgroups are $H' = S^{(0)}(m)$ and $H = S^{(0)}(N)$, the partition $\Pi$ is

$$\{\pm\{N_1 + \ldots + N_{i-1} + 1, \ldots, N_1 + \ldots + N_i\} : 2 \leq i \leq n\} \bigcup \{N_1, \ldots, -1,1, \ldots, N_1\},$$

and $B = S^{(0)}(N,1), A = S^{(0)}_1(N,m)$.

Proof. This is identical to the proof of Proposition 3.1. $\square$

3.2 Dynamics and stationary measures

Define a Markov operator from $\Lambda$ from $D_{H',H}$ to $D_H$. Each $w \in D_{H',H}$ will be mapped to a random element in the coset $wH$. More specifically,

$$P(\Lambda(w) = wb) = \frac{q^{(b)}}{H(q)} \text{ for all } b \in H(q).$$

If $W = A_{N-1}$, we know that the ASEP($q,m$) jump rates are given by the formula $\hat{L} = \Lambda L \Phi$. We will now see the analogous construction for the type $BC_N$ Coxeter group.

Away from the boundary, the evolution on $D_{H',H}$ will be the same as a multi–species ASEP($q,m$), but at the boundary the jump rates will differ, depending on whether or not $H'$ contains $s_0$. We will consider two cases separately:

Case 1:

First, consider when $s_0 \notin H'$. In this case, $\Lambda$ is the same $\Lambda$ as in the ASEP($q,m$). Given particles at lattice site 1, we consider a “mirror” set of particles at lattice site 0, consisting of the same particles
at lattice site 1, but with the negative of the species numbers. A particle of species \(j\) at lattice site 1 is replaced with a particle of species \(-j\) at rate

\[
q \sum_{r<j} R_{4 \to 3} \left[ \frac{[k(j)]_{q}}{[m_1]_q} \right] \quad \text{if } j < 0,
\]

\[
q \cdot q \sum_{r<j} R_{4 \to 3} \left[ \frac{[k(j)]_{q}}{[m_1]_q} \right] \quad \text{if } j > 0.
\]

These are essentially the jump rates of the ASEP\((q, m)\) with no contribution from the lattice site 0.

**Case 2:**

Now suppose \(s_0 \in H'\). In this case, particles cannot enter or exit the lattice at the left boundary. The system evolves as an ASEP\((q, m')\), where \(m' = (2m_1, m_2, \ldots, m_L)\), with one key difference: in order to maintain that the particles at lattice site 1 consist of pairs of species \((j, -j)\), we mandate that when a particle of species \(j\) at 1 and a species of particle \(i\) at 2 switch places, the particle at species \(-j\) at 1 is replaced with a particle of species \(-i\) instantaneously.

**Proposition 3.3.** The \(q\)-exchangeable measures are stationary under the two processes defined above.

**Proof.** By construction, the generator of the process can be defined as \(\Lambda L\Phi\), where \(L\) is the generator of multi–species ASEP. Since \(\Lambda, L\) and \(\Phi\) all preserve \(q\)-exchangeability, this means that \(q\)-exchangeable measures are stationary. \(\square\)

### 3.3 Color Position Symmetry

For any Coxeter system \((W, S)\), let \(\mathbb{C} [W]\) denote the group algebra of \(W\). For any \(s \in S\), define the linear map \(L_{s,x}\) on \(\mathbb{C} [W]\) by

\[
L_{s,x} (w) = \left\{ \begin{array}{ll}
(1 - x) w + x w, & l(sw) > l(w) \\
(1 - qx) w + q x w, & l(sw) < l(w)
\end{array} \right.
\]

Fix an arbitrary set of elements \(s_1, \ldots, s_n\) and parameters \(x_1, \ldots, x_n\) Define the coefficients \(f_n(w \to \pi)\) by

\[
L_{s_i_1, x_1} \cdots L_{s_i_n, x_n} w = \sum_{\pi \in W} f_n(w \to \pi) \pi.
\]

Similarly, define the coefficients \(\tilde{f}_n(w \to \pi)\) by

\[
L_{s_i_1, x_1} \cdots L_{s_i_n, x_n} w = \sum_{\pi \in W} \tilde{f}_n(w \to \pi) \pi.
\]

Note that the ordering of the \(t_j, x_j\) are reversed. In [BB19], it is shown that for all \(\pi \in S_N\),

\[
f_n(e \to \pi) = \tilde{f}_n(e \to \pi^{-1}). \tag{2}
\]

In the context of ASEP, \(2\) can be viewed as a color–position symmetry.

As seen above, the color and position permutations are given an algebraic interpretation. Namely, a permutation of the colors is viewed as the left action of \(S_N\) on itself, while a permutation of the positions is viewed as the right action of \(S_N\) on itself. In light of this interpretation, it makes sense to view \(w s, x w\) as a left action, with \(w L_{s,x}\) as its corresponding right action (the definition of \(L_{s,x}\) will be given later).

Here are some heuristics to see the color–position symmetry through left and right actions. If

\[
L_{s,x} e = e L_{s,x}, \tag{3}
\]

then we would expect

\[
L_{s_i_1, x_1} \cdots L_{s_i_n, x_n} e = e L_{s_i_1, x_1} \cdots L_{s_i_n, x_n} \tag{4}
\]

Note that the \(s_i, x_k\) are applied to \(e\) in reverse orders on both sides of \(4\). Equation \(4\) could then be used to prove \(2\). However, this heuristic implicitly uses the associativity of the left and right actions:

\[
(L_{s_i, x_1} w) L_{s_j, x_2} = L_{s_i, x_1} (w L_{s_j, x_2}), \tag{5}
\]
which is not immediately obvious. For instance, it would be false if \( q \) were allowed to take different values.

Now define the right action

\[
  w \tilde{L}_{s,x} = \begin{cases} (1-x)w + xws, & l(ws) > l(w), \\ (1-qx)w + qxws, & \text{else}. \end{cases}
\]

Note that (3) holds immediately. Define the coefficients \( \tilde{g}_n(w \to \pi) \) by

\[
  w \tilde{L}_{s_{i_1} \cdots s_{i_n}, x_1} = \sum_{w \in W} \tilde{g}_n(w \to \pi) \pi.
\]

**Theorem 3.4.** The coefficients \( \tilde{f}_n \) and \( \tilde{g}_n \) are related via

\[
  \tilde{g}_n(\pi \to \pi) = \tilde{f}_n(\pi \to \pi^{-1}).
\]

If (4) holds, then so does (2).

**Proof.** The definition of \( \tilde{g}_n \) is the same as the definition of \( \tilde{f}_n \), except that the multiplication is applied on the right instead of on the left. Thus \( \tilde{g}_n(\pi \to \pi) = \tilde{f}_n(\pi \to \pi^{-1}) \).

If (4) holds, then \( f_n(\pi \to \pi) = \tilde{g}_n(\pi \to \pi) \), which implies (2).

For two transpositions \( s_i, s_j \), we say that \((s_i, s_j)\)–associativity holds for \( w \in S_N \) if for any \( x_1, x_2, \)

\[
  (L_{s_{i_1}, x_1})^n \tilde{L}_{s_{i_2}, x_2} = L_{s_{i_1}, x_1}(w \tilde{L}_{s_{i_2}, x_2})
\]

**Lemma 3.5.** For any \( s_i, s_j \in S \) and any parameters \( x_1, x_2 \), we have that \((s_i, s_j)\)–associativity holds for \( w \) for the following three cases:

1. \( l(s_i w) > l(w), \quad l(ws_j) > l(w), \quad l(s_i ws_j) > l(ws_j), \quad l(s_i ws_j) > l(s_i w), \)
2. \( l(s_i w) < l(w), \quad l(ws_j) > l(w), \quad l(s_i ws_j) < l(ws_j), \quad l(s_i ws_j) > l(s_i w), \)
3. \( l(s_i w) > l(w), \quad l(ws_j) < l(w), \quad l(s_i ws_j) > l(ws_j), \quad l(s_i ws_j) < l(s_i w), \)
4. \( l(s_i w) < l(w), \quad l(ws_j) < l(w), \quad l(s_i ws_j) < l(ws_j), \quad l(s_i ws_j) < l(s_i w), \)

**Proof.** In the first case, it can be checked that both sides equal

\[
  (1 - x_1)(1 - x_2)w + (1 - x_1)x_2ws_j + x_1(1 - x_2)s_iw + x_1x_2s_1ws_j.
\]

In the second case, it can be checked that both sides equal

\[
  (1 - qx_1)(1 - x_2)w + (1 - qx_1)x_2ws_j + qx_1(1 - x_2)s_iw + qx_1x_2s_1ws_j.
\]

In the third case, it can be checked that both sides equal

\[
  (1 - x_1)(1 - qx_2)w + (1 - x_1)x_2ws_j + x_1(1 - qx_2)s_iw + qx_1x_2s_1ws_j.
\]

In the fourth case, it can be checked that both sides equal

\[
  (1 - qx_1)(1 - qx_2)w + (1 - qx_1)x_2ws_j + qx_1(1 - qx_2)s_iw + q^2x_1x_2s_1ws_j.
\]

A priori, there are 16 cases that need to be checked. However, if \( l(s_i w) > l(w) \) and \( l(ws_j) > l(w) \), then \( l(s_i w) = l(ws_j) = l(w) + 1 \), which means that either \( l(s_i w) = l(ws_j) > l(s_i ws_j) \) or \( l(s_i w) = l(ws_j) < l(s_i ws_j) \). Similarly, if \( l(s_i w) < l(w) \) and \( l(ws_j) < l(w) \), then either \( l(s_i w) = l(ws_j) < l(s_i ws_j) \) or \( l(s_i w) = l(ws_j) > l(s_i ws_j) \). If \( l(s_i w) > l(w) \) and \( l(ws_j) < l(w) \), then \( l(s_i w) = l(w) + 1 \) and \( l(ws_j) = l(w) - 1 \), which means that \( l(s_i ws_j) = l(w) \), forcing \( l(s_i ws_j) > l(ws_j) \) and \( l(s_i ws_j) < l(s_i w) \). Similarly, \( l(s_i w) < l(w) \) and \( l(ws_j) > l(w) \) imply that \( l(s_i ws_j) < l(ws_j) \) and \( l(s_i ws_j) > l(s_i w) \). Thus, there are only six cases total, with four of them checked in the lemma. The remaining two are:

\[
  \begin{align*}
  l(s_i w) > l(w), \quad l(ws_j) > l(w), \quad l(s_i ws_j) < l(ws_j), \quad l(s_i ws_j) < l(s_i w), \\
  l(s_i w) < l(w), \quad l(ws_j) < l(w), \quad l(s_i ws_j) > l(ws_j), \quad l(s_i ws_j) > l(s_i w).
  \end{align*}
\]
Examples of $w$ in each of the two cases are (respectively) $s_j s_i$, and $s_i s_j s_i$, where $s_i = s_j = (j \cdot j + 1)$ and $s_j = s_{j+1} = (j + 1 \cdot j + 2)$. One can check, however, that $(L_{s_{j+1}} x w) L_{s_j x_2} = L_{s_j x_1} (w L_{s_{j+1} x_2})$ does not hold for $w = s_i s_j s_i$.

Let us refer to the four cases in Lemma 3.5 as cases 1, 2, 3, 4 respectively, and the remaining two cases as cases 5, 6. The proof for case 5 requires a slightly more delicate proof. In particular, we recall the exchange condition: if $t$ is a transposition and $w = s_i \cdots t_r$ is an arbitrary element of $S_N$ such that $l(t w) < l(w)$, then there exists $h \in \{1, \ldots, r\}$ such that $s_{t_1} \cdots t_{h-1} = s_t \cdots t_h$. As a consequence, $t w = s_t \cdots t_h \cdots t_r$, where the $t_h$ indicates that $t_h$ has been deleted. Similarly, if $l(t w) < l(w)$ then $w t = s_t \cdots t_h \cdots t_r$ for some $1 \leq k \leq r$.

**Lemma 3.6.** Fix $s_i, s_j$ and suppose that $w$ falls into case 5. Then

$$\begin{align*}
(L_{s_{j+1}} x w) \tilde{L}_{s_{j+2} x_2} &= (1 - x)(1 - y)w + (1 - x)yzs_j x + xqys_i ws_j \\
L_{s_{j+1}} (w \tilde{L}_{s_{j+2} x_2}) &= (1 - x)(1 - y)w + x(1 - y) s_i w + (1 - xq) y ws_j + xqys_i ws_j
\end{align*}
$$

If $w$ falls into case 6, then

$$\begin{align*}
(L_{s_{j+1}} x w) \tilde{L}_{s_{j+2} x_2} &= (1 - qx_1)(1 - qx_2)w + (1 - qx_1) q x_2 ws_j + qx_1(1 - x_2) s_i w + x_2 s_i ws_j \\
L_{s_{j+1}} (w \tilde{L}_{s_{j+2} x_2}) &= (1 - qx_1)(1 - qx_2)w + qx_1(1 - qx_2) s_i w + q(1 - x_1) x_2 ws_j + qx_1 x_2 s_i ws_j
\end{align*}
$$

**Proof.** This follows from a direct calculation.

A priori, the two terms in (6) are not equal to each other. It turns out, however, that they actually are equal.

**Lemma 3.7.** Fix $s_i, s_j$ and suppose that $w$ falls into case 5 or case 6. Then $s_i w = ws_j$, and furthermore

$$(L_{s_{j+1}} x w) \tilde{L}_{s_{j+2} x_2} = L_{s_{j+1}} (w \tilde{L}_{s_{j+2} x_2})$$

**Proof.** Suppose that case 5 holds; then by assumption $l(s_i w) > l(w)$ and $l(s_j ws_j) < l(ws_j)$. Write $w = s_{i_1} \cdots s_{i_k}$. By the exchange condition, $s_{i_1} \cdots s_{i_k} s_j$ equals either $s_{i_1} \cdots s_{i_h} \cdots s_{i_k} s_j$ (for some $h \in \{1, \ldots, r\}$) or equals $s_{i_1} \cdots s_{i_r}$. However, the former situation cannot hold because by assumption $l(s_i w) > l(w)$. Thus $s_i ws_j = w$, or equivalently $s_i w = ws_j$. By (6) and

$$(1 - x)y + x(1 - qy) = x(1 - y) + (1 - qx)y,$$

the identity holds.

Now suppose that case 6 holds: then by assumption $l(s_i w) < l(w)$ and $l(s_j ws_j) < l(ws_j)$, and $l(s_i ws_j) = l(w)$. Let $s_{i_1} \cdots s_{i_l}$ be a reduced expression for $w$. By the exchange condition, there exists $h \in \{1, \ldots, l\}$ such that

$$s_{i_1} s_{i_2} \cdots s_{i_{h-1}} = s_{i_1} \cdots s_{i_h}.$$

Set $\tilde{w} = s_i ws_j$ and let $s_{j_1} \cdots s_{j_t}$ be the reduced expression for $\tilde{w}$ (by assumption, $l(w) = l(\tilde{w})$) given by

$$j_1 = i_1, \ldots, j_{h-1} = i_{h-1}, j_h = i_{h+1}, \ldots, j_{r-1} = i_1, s_{j_t} = s_j.$$

This is a reduced expression for $\tilde{w}$ because of (9).

Note that $l(s_i \tilde{w}) = l(ws_j) < l(s_i ws_j) = l(w)$. By the exchange condition, either $s_i \tilde{w}$ equals the permutation $s_{i_1} \cdots s_{i_{k}} \cdots s_{i_{h}} \cdots s_{i_{t}} s_j$ for some $h, k$, or $s_i \tilde{w} = s_{i_1} \cdots s_{i_{h}} \cdots s_{i_{j_t}}$. The former equality would imply that $l(s_i \tilde{w} s_j) = l(w) = l - 2$, which contradicts $l(w) = l$. Therefore the latter equality holds, which then implies $\tilde{w} = s_{i_1} s_{i_2} \cdots s_{i_{h}} s_j$. By (6), this implies that $\tilde{w} = s_{i_1} \cdots s_{i_l} = w$, which means that $s_i w = ws_j$, as claimed. The identity then holds by (7) and (8).

**Remark 1.** In the language of Hecke algebras, the color–position symmetry can be viewed as the commutativity of left and right actions; see, for example, the proposition in section 7.2 of [Hum92]. Note that the approach of [Buf20] describes color–position symmetry in terms of an involution on the Hecke algebra, rather than in terms of left and right actions.
As stated, Theorem 3.4 is purely an algebraic statement. It can also be stated in terms of interacting particles. Before doing so, we first describe a coupling between the multi–species ASEP(q, j) and its time reversal. This involves the graphical representation of an interacting particle system (see e.g. [Lig05]). In the simplest case of the single–species ASEP, we have independent Poisson processes \{\mathcal{P}(x)\}_{x \in \mathbb{Z}} each on state space \mathbb{R}_{\geq 0} with rates \(c(x) = 1\). At each point from \(\mathcal{P}(x)\), apply the Markov operator \(L_{(x, x+1), t}\); this corresponds to an update of the particle configuration. With this description, an ASEP during times \(t \in [0, T]\) can be coupled with a time–reversed ASEP by mapping every \(\mathcal{P}(x)\) to \(T - \mathcal{P}(x)\). Note that this coupling works for any initial conditions on ASEP and its time reversal.

The coupling between multi–species ASEP(q, m) and its time reversal is similar; instead of applying the Markov operator \(w_{(x, x+1), t}\), randomly swap a particle at \(x\) with a particle at \(x + 1\) that is consistent with the multi–species ASEP(q, m). By (1), this random swap is equal to \(\Lambda\) this follows once we show \(\Lambda\) and \(\Phi\).

**Theorem 3.8.** Fix \(H = S(N)\) and \(H' = S(m)\). Let \(\tilde{k}_t \in S(N, m)\) evolve as a multi–species ASEP(q, m) with initial condition \(\tilde{k}_0\), and set \(w_t \in D_{H', H}\) be defined by \(w_t = (\theta'_q)^{-1}(\tilde{k}_t)\). Let \(\tilde{l}_t \in S(N, m)\) evolve as a time–reversed multi–species ASEP(q, m) with initial condition \(\theta'_q(c)\). Let \(v_t \in D_{H, H'}\) be defined by \(v_t = (\theta'_q)^{-1}(\tilde{l}_t)\).

Then, at any time \(t\), the distribution of \(w_t\) is the same as the distribution of \((w_0)^{-1}v_t\)^{-1}.

**Proof.** Because this proof is similar to Theorem 3.1 of [BB19], which is itself similar the proof of Theorem 1.4 of [AV11], we keep the proof short.

Let \(\iota : W \to W\) map every \(w\) to its inverse \(w^{-1}\). Use the same symbol to denote the restriction to any subset of \(W\). By Theorems [3.4] and Proposition 3.1 (Proposition 3.2 for open boundary conditions), this follows once we show

\[ \iota\Lambda = \Lambda, \quad \iota L_{s,x} = \tilde{L}_{s,x}, \quad \iota \Phi = \Phi. \]

This follows immediately from the definitions. \(\square\)

### 3.4 Asymptotic Application

Before finding asymptotic applications of color–position symmetry, we first need hydrodynamics and local statistics for the single–species ASEP(q, m). The key idea is to use a duality for ASEP(q, j) and previously known results for ASEP. First we recall the results for ASEP, which go back to [BF87] and [AV87].

Define the density profile

\[ \rho(x, t) = \mathbb{E}[k_x(t)], \]

which by definition takes values in \([0, m_x]\). Here, we consider the step initial condition, where \(k_x = m_x 1_{x \leq 0}\).

**Theorem 3.9.** Take all \(m_x = 1\) (i.e. the ASEP case), and let \(\{x(t) : t \geq 0\}\) be a collection of integers such that \(\lim_{t \to \infty} x(t)/t = y \in \mathbb{R}\). Then

\[ \lim_{t \to \infty} \rho_{\text{step}}(x(t), t) = d(y) := \begin{cases} 0, & y \geq 1 - q \\ \frac{1}{q}(1 - \frac{y}{1 - q}), & -(1 - q) < y < 1 - q, \\ 1, & y \leq -(1 - q). \end{cases} \]

Now let us turn to the analog of these results for ASEP(q, j).

**Proposition 3.10.** Take all values of \(m_x\) to equal a fixed \(m\). Let \(\{x(t) : t \geq 0\}\) be a collection of integers such that \(\lim_{t \to \infty} x(t)/t = y \in \mathbb{R}\). Then

\[ \lim_{t \to \infty} \rho_{\text{step}}(x(t), t) = md(y), \]

where \(d(y)\) is the function in Theorem 3.9.

**Proof.** We recall some results and notation from [CGRS16]. Given a particle configuration \(\eta\), let

\[ N_i(\eta) = \sum_{x \geq i} \eta_x. \]
Let $|\eta|$ denote the number of particles in $\eta$ (possibly infinite). Let $x(t)$ denote a single random walker evolving under ASEP$(q, m/2)$, which has the same distribution as letting $\tilde{x}(t)$ evolve as a single random walker under ASEP$(q^m, 1/2)$. Then, according to Lemma 3.1 of [CGRS10], there is the duality result

$$E_q[q^{2N_t(\eta(t))}] = q^{2|\eta|} - \sum_{k=-\infty}^{r-1} q^{-2mk}E_k \left[ q^{2mx(t)}(1-q^{2m})q^{2N_x(t)}(\eta) \right],$$

where $\eta(t)$ evolves as ASEP$(q, m/2)$. Then, for step initial conditions $\eta^{\text{step}}$,

$$E_{q^{\text{step}}}[q^{2N_t(\eta(t))}] = -\sum_{k=-\infty}^{r-1} q^{-2mk}E_k \left[ q^{2mx(t)}(1-q^{2m})q^{2N_x(t)}(\eta)1_{x(t)\leq 0} \right] = -\sum_{k=-\infty}^{r-1} q^{-2mk}E_k \left[ (1-q^{2m})1_{x(t)\leq 0} \right] = (q^{2m} - 1) \sum_{k=-\infty}^{r-1} q^{-2mk}P_k(x(t) < 0)$$

Since $x(t)$ has the same evolution as ASEP$(q^m, 1/2)$, we see that the parameter $m$ only affects the quantity $E_{q^{\text{step}}}[q^{2N_t(\eta(t))}]$ by replacing $q$ with $q^m$. Thus, $N_t(\eta(t)) = m N_t(\tilde{\eta}(t))$, where $\tilde{\eta}(t)$ evolves as the usual ASEP. Combined with Theorem 3.9 this shows that $\rho_{\text{step}}(x(t), t) = md(y)$.

We have the following applications to the distribution of a second class particle in the ASEP$(q, m/2)$. Consider a deformed step initial condition, where there are $m$ first class particles at lattice sites to the left at 0, and no particles to the right of 0, and $m$ second class particles at 0. Let $\tilde{t}_1^{(i)}(t) \leq \ldots \leq \tilde{t}_m^{(i)}(t)$ denote the positions of the second class particles at time $t$, and as before let $\tilde{\rho}(x, t)$ denote the density profile (of the first class particles).

**Theorem 3.11.** For any lattice site $x$ and any time $t$,

$$E[\{j : \tilde{t}_j^{(i)}(t) \leq x\}] = \rho(0, t),$$

where the process on the right-hand-side begins with the shifted step initial condition $\tilde{k}_y = 1_{y \leq x} m$.

**Proof.** Once color–position symmetry is proven, this is similar to the proof of Theorem 4.1 of [BB19]. The process on the left-hand-side can be coupled with a process which has initial condition consisting of $m(x-1) + 1, m(x-1) + 2, \ldots, mx$ at lattice site $x$. The probability on the left-hand-side is then the probability of the species 0 particle being located at a site which is $\leq x$. By color–position symmetry, this is the probability of a particles of species $\leq x$ being located at 0.

Combining this with Proposition 3.10 we have:

**Corollary 3.12.** As $t \to \infty$,

$$E[\{j : \tilde{t}_j^{(i)}(t) \leq yt\}] \to \begin{cases} m, & y \geq 1 - q, \\ m(1 - \frac{1}{1-q} - 1), & -q < y < 1 - q, \\ 0, & y \leq -(1-q). \end{cases}$$

We note that this corollary is motivated by Theorem 5.2 of [BB19].

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