Generalised Cesaro Convergence, Root Identities and the Riemann Hypothesis

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November 9, 2011

Abstract

We extend the notion of generalised Cesaro summation/convergence developed in [1] to the more natural setting of what we call “remainder” Cesaro summation/convergence and, after illustrating the utility of this approach in deriving certain classical results, use it to develop a notion of generalised root identities. These extend elementary root identities for polynomials both to more general functions and to a family of identities parametrised by a complex parameter $\mu$. In so doing they equate one expression (the derivative side) which is defined via Fourier theory, with another (the root side) which is defined via remainder Cesaro summation. For $\mu \in \mathbb{Z}_{\leq 0}$ these identities are naturally adapted to investigating the asymptotic behaviour of the given function and the geometric distribution of its roots. For the gamma function we show that $\Gamma$ satisfies the generalised root identities and use them to constructively deduce Stirling’s theorem. For the Riemann zeta function, $\zeta$, the implications of the generalised root identities for $\mu = 0, -1$ and $-2$ are explored in detail; in the case of $\mu = -2$ a symmetry of the non-trivial roots is broken and allows us to conclude, after detailed computation, that the Riemann hypothesis must be false. In light of this, some final direct discussion is given of areas where the arguments used throughout the paper are deficient in rigour and require more detailed justification. The conclusion of section 1 gives guidance on the most direct route through the paper to the claim regarding the Riemann hypothesis.

1 Introduction

This paper expands the concept of generalised Cesaro convergence introduced in [1] and uses this to introduce a notion of generalised root identities. In turn these identities are used to investigate both the asymptotic behaviour and the geometric distribution of roots and poles of the gamma function, $\Gamma$, and the Riemann zeta function, $\zeta$.

In section 2.1 we introduce the core notion of “remainder” Cesaro summation/convergence, where the partial-sum function analysed via a Cesaro scheme is now on a contour relative to chosen point $z_0$, rather than always being on
As an initial example we show that the Hurewicz zeta function, \( \zeta_H \), is defined naturally in this context, and that well-known relationships among Bernoulli polynomials follow naturally within this remainder Cesaro approach.

In section 2.2 this is extended in the usual way to a notion of remainder Cesaro products and, when applied to the elementary case of \( f(z) = z \), is used to give a Cesaro definition of the Gamma function, \( \Gamma \), at least after ensuring that the geometric location of terms in the remainder sum/product is respected in the Cesaro analysis.

In section 2.3 we demonstrate the utility of this remainder Cesaro approach, and in particular its geometric intuitiveness when coupled with the dilation-invariance of Cesaro convergence, in analysing three example results. The first is an elementary derivation of the duplication formulae for \( \Gamma \); the second is the derivation of the functional equations for \( \Gamma \) and \( \zeta \), by exploiting symmetries in the Cesaro context to introduce bi-directional sums which are apriori periodic and thus amenable to Fourier series analysis; and the third is a short derivation of an interesting integral identity for \( \zeta_H \).

In section 3 we then turn to the question of root identities, and in particular the possibility of generalising the well-known identities relating roots and coefficients of polynomials to some more general class of functions.

In section 3.1 a family of such identities, for arbitrary \( z_0 \in \mathbb{C} \) and \( \mu \in \mathbb{Z}_{\geq 1} \), are initially derived on a formal, heuristic basis. In general, these equate a quantity defined in terms of the derivatives of a function (the “derivative side”) with a quantity defined in terms of the roots of the function (the “root side”). The latter quantity in fact includes equally contributions from roots (with multiplicity \( M \geq 1 \)), poles (with \( M \leq -1 \)) and even branch points (with e.g. \( M = \frac{1}{2} \)) of a function; all of which cases are thus subsumed thereafter under the label of “generalised roots”. A natural equivalence is then established between satisfying the root identities at a single \( z_0 \) \( \forall \mu \in \mathbb{Z}_{\geq 1} \), and satisfying them at arbitrary \( z_0 \in \mathbb{C} \) (at least an open neighbourhood) for \( \mu = 1 \) alone.

Unfortunately it is readily seen that most non-polynomial functions do not satisfy these generalised root identities. However, at least for those where the discrepancy is sufficiently well-behaved (as a function of \( z_0 \)), we show that it is possible to find an “equivalent” function with the same root-set which does satisfy these identities. The construction of this equivalent function by iterative removal of obstructions to the successive (\( \mu = 1, 2, 3, \ldots \)) root identities at a given fixed \( z_0 \) is outlined.

In section 3.2, however, we verify that in practice many natural and important functions do already satisfy these identities with no, or minimal, adjustment by considering three examples which will become primary focuses of attention in this article. In the first we show that \( f(z) = \cos \left( \frac{\pi z}{2} \right) \) satisfies the identities for all \( z_0 \in \mathbb{C} \) and all \( \mu \in \mathbb{Z}_{\geq 1} \) and we note that this leads naturally to evaluation of some values of \( \zeta \), for example Euler’s famous evaluation of \( \zeta(2) = \frac{\pi^2}{6} \). In the second we verify that \( \Gamma(z + 1) \) also satisfies the root identities for all \( z_0 \in \mathbb{C} \) and all \( \mu \in \mathbb{Z}_{\geq 1} \), albeit in the case of \( \mu = 1 \) that a renormalisation is required on the root side to handle uniformly the ln-divergences which arise for any \( z_0 \).
and which do not have Cesaro limits. In the third we find that \( \zeta \) alone does not satisfy the root-identity for \( \mu = 1 \), but see by experimentation using the first 100,000 non-trivial zeros that the obstruction in this case appears to be a constant, \(-\frac{1}{2}\ln \pi\), independent of \( s_0 \). This suggests that instead \( \pi^{-\frac{s}{2}}\zeta(s) \) does satisfy the \( \mu = 1 \) root identity for all \( s_0 \) and we find that in fact this is true and reflects a well-known explicit formula for \( \zeta \) which represents the Hadamard product formula for \( \xi(s) \). Thus, in this case, the generalised root identity for \( \mu = 1 \) is equivalent to the Hadamard product formula; for \( \mu \in \mathbb{Z}_{\geq 2} \) the extra factor \( \pi^{-\frac{s}{2}} \) no longer contributes on either the derivative or root sides and so \( \zeta \) itself then satisfies these identities for all \( s_0 \) directly.

In section 3.3 we then turn to the possibility of further generalising our root identities by allowing \( \mu \) to be not just a positive integer, but \( \mu \in \mathbb{C} \) arbitrary, with the cases of \( \mu \in \mathbb{Z}_{\leq 0} \) being of particular interest. We begin by discussing extensively how to even make sense of the identities in the case of arbitrary complex \( \mu \). On the root side a generalised convergence scheme - always Cesaro in this paper - is required to make sense of what are now usually divergent sums, and we find that it is crucial that the summands, \( \frac{M_i}{(s_0 - r_i)^\mu} \), in this Cesaro context be added in geometrically at the shifted points \( z_0 - r_i \). On the derivative side the notion of \( \left( \frac{d}{dz} \right)^\mu \) now becomes problematic; we note various basic properties it should have and then define it consistently with these via Fourier theory, in a manner familiar from pseudodifferential operators.

As a first example of the value of this generalisation to arbitrary complex \( \mu \) we give a short derivation showing that, for our example of \( f(z) = \cos(\frac{\pi z}{2}) \), \( f \) satisfying the generalised root identities for all \( z_0 \) and all \( \mu \in \mathbb{C} \) is equivalent to the functional equation for \( \zeta \). With this as indication of the utility of this extension, we then discuss these generalised identities in the particular cases of \( \mu \in \mathbb{Z}_{\leq 0} \) where the \( \Gamma(\mu) \) factor becomes singular. For the remainder of the paper, in fact, we shall ignore the case of arbitrary \( \mu \in \mathbb{C} \) and focus exclusively on \( \mu \in \mathbb{Z}_{\leq 0} \). In this case we show how to interpret our derivative-side definition via a distributional interpretation of the expressions \( \frac{\xi(z)}{\Gamma(\mu + 1)} \) and note that in fact it is this singular behaviour that generally makes calculation of the derivative sides more tractable for \( \mu \in \mathbb{Z}_{\leq 0} \) than for arbitrary \( \mu \in \mathbb{C} \) since it allows omission of hard-to-compute finite integral contributions, leaving only these distributional contributions. We then discuss at some length why, for a function \( f \) satisfying the generalised root identities, the identities for \( \mu \in \mathbb{Z}_{\leq 0} \) should naturally give us much more information than the corresponding cases of \( \mu \in \mathbb{Z}_{\geq 1} \) both about the asymptotic behaviour of \( f \) and about the asymptotic and geometric distribution of its roots. We conclude section 3.3 with a quick overview of the initial formal Fourier theory results that we will use in the remainder of the paper, which is concerned with analysing the implications of the generalised root identities for our remaining two key examples, \( \Gamma(z + 1) \) and \( \zeta(s) \).

In section 3.4 we consider the case of \( \Gamma(z + 1) \). We first verify that \( \Gamma(z + 1) \) does in fact continue to satisfy the root identities for all \( \mu \in \mathbb{Z}_{\leq 0} \) by explicit calculation of both the root sides (using Cesaro) and derivative sides (using Fourier and distributional calculations) in these cases. Then, as an example
of the asymptotic information contained within the identities for $\mu \in \mathbb{Z}_{\leq 0}$ we show how, for this example of $\Gamma$, we can use them to deduce Stirling’s theorem describing the asymptotics of $\Gamma(z + 1)$. We do this in an engineering fashion by successively using the identities for $\mu = 0, -1, -2 \ldots$ to construct the next order in the Stirling asymptotics. We show, using the generating function which defines the Bernoulli numbers, why the Stirling formula in turn leaves the root identities for $\mu \in \mathbb{Z}_{\geq 1}$ in tact, without need of further correction terms to remove obstructions in these cases.

Finally, we conclude section 3.4 with some discussion of why our derivation of Stirling’s theorem in fact implies a “two-sided” result giving Stirling-like asymptotics also as $z \to -\infty$, in addition to the actual Stirling asymptotics as $z \to +\infty$. We use the functional equation for $\Gamma$ to verify that this is indeed the case, but with specific additional “one-sided” asymptotic contributions as $z \to -\infty$, albeit ones which did not derail our derivation above.

In section 4 we then turn to consideration of the generalised root identities for $\mu \in \mathbb{Z}_{\leq 0}$ for $\zeta$. In section 4.1 we consider the derivative sides of these identities and demonstrate that in fact they are all identically zero for all $s_0$ when $\mu \in \mathbb{Z}_{\leq 0}$. This is done by using the Euler product formula for $\zeta$ together with a basic property of $\left(\frac{d}{ds}\right)^{\mu}$ to in fact derive an explicit formula for the derivative side as a sum over primes $p$.

In section 4.2, which then takes up the bulk of the remainder of the paper, we consider the cases of $\mu = 0, \mu = -1$ and $\mu = -2$ in turn on the root sides with a view to seeing whether these root sides can likewise be seen to be identically zero for all $s_0$ in spite of not knowing the exact location of the non-trivial roots; and indeed whether any asymptotic or geometric information can be gleaned on the location of these non-trivial zeros from the requirement that the root-sides be zero in these cases.

In section 4.2.1 we consider the case of $\mu = 0$, in section 4.2.2 the case of $\mu = -1$, and in section 4.2.3 the case of $\mu = -2$. In all cases the evaluation of the root-sides proceeds by considering first the trivial roots, T, then the simple pole and finally the non-trivial roots, NT.

As expected, the majority of the effort is in the computations for the contributions from NT. Here we rely on the explicit Riemann-von Mangoldt formula for $\hat{N}(T)$, the function which counts non-trivial roots with imaginary parts between 0 and $T$; this expresses $N(T)$ as the sum of three pieces $N(T) = \hat{N}(T) + S(T) + \frac{1}{\pi} \delta(T)$ where $\hat{N}(T)$ is the leading asymptotic part of $N(T)$, $S(T)$ is the famous argument of the Riemann zeta function, and $\delta(T)$ is a completely explicit function giving the negative order asymptotics of $N(T)$. The calculations of the root-side contribution from NT always thus further subdivide into calculations of the Cesaro limits for the terms arising from each of these three pieces. In carrying out these Cesaro computations we confirm that it is absolutely critical when $\mu \in \mathbb{Z}_{\leq 0}$ both (a) to respect the location of the roots in determining the geometric placement of summands and (b) to independently carry out computations for the NT-roots above and below the real axis using separate parameters $T$ and $\tilde{T}$ (and associated geometric complex variables
z and \( \tilde{z} \), but to recombine these into a 2-d partial sum function (of \( z \) and \( \tilde{z} \)) before taking limits. Failure to do so leads to incorrect evaluation and paradoxes involving apparent inconsistency between the results for \( \mu = 0 \) and \( \mu = -1 \).

For \( \mu = 0 \) in section 4.2.1 we verify that we do indeed have the root-side identically zero for all \( s_0 \), at least after assigning Cesaro limit 0 to a residual 2-d log-term, \( \ln (\tilde{z}) \) (with this latter seemingly ad hoc calculation either following from a suitable 2-d Cesaro analysis, or alternatively to be treated as a consequence of the root identity for \( \mu = 0 \), allowing it at least to be used confidently for the subsequent cases of \( \mu = -1 \) and \( \mu = -2 \) where it also arises).\(^1\)

For \( \mu = -1 \) in section 4.2.2 we likewise verify that we do indeed have the root-side identically zero for all \( s_0 \), at least subject to an estimate on \( S(T) \) which is known to hold conditional on the Riemann hypothesis (RH).

For \( \mu = -2 \) in section 4.2.3, however, we find that a symmetry of the NT-roots is broken. For \( \mu = 0 \) and \( \mu = -1 \) the fact that any non-trivial roots off the critical line occur in mirror pairs meant that the net contributions from any such pairs to our Cesaro calculations was exactly as for a corresponding double-root on the critical line, and so no explicit terms expressing potential contributions from departures from the critical line arose. But for \( \mu = -2 \) there is such a term, \( X_\epsilon \), and after carrying out the analogous, albeit more intricate, computations for the root side in this case we find that the value of the root-side for \( \mu = -2 \) is given by \( -\frac{1}{2} + X_\epsilon \).

In order for the \( \mu = -2 \) root identity to hold, this would need to be identically zero and this allows us to conclude section 4.2.3 by inferring, as a consequence, that the Riemann hypothesis must be false.

Given the nature of this claim, we end by including a final section, section 4.3, itemising and discussing the admittedly many areas where the arguments in the paper up to this point have been informal, heuristic or otherwise in need of more rigorous treatment. We nonetheless note that, while our emphasis throughout the paper is on developing and applying calculational tools rather than on formal rigour, we believe the nature of the resulting computations is sufficiently suggestive that we are convinced of their correctness, and certainly convinced that they warrant further analysis to try to fill the gaps.

Finally we note that, for those wishing to focus as rapidly as possible on the calculations for \( \zeta \), section 2.3 and the discussion of Stirling’s theorem in section 3.4 may be omitted en route.

1.1 Notation

In this paper we use the definition of Bernoulli numbers as given recursively by

\[
B_0 = 1 \quad \text{and} \quad B_n = \sum_{j=0}^{n} \binom{n}{j} B_j \quad \forall n \geq 2
\]

\(^1\)We discuss this issue further in section 4.3 and in [8] we return to address it again and justify why assigning this 2d Cesaro limit the value 0 is legitimate.
which implies \( B_1 = \frac{1}{2} \), \( B_2 = \frac{1}{6} \), \( B_4 = \frac{1}{30} \), \( B_6 = \frac{1}{42} \) and so on. For \( n \geq 1 \) we have \( B_{2n+1} = 0 \) and the even-index Bernoulli numbers are related to the values of \( \zeta \) at negative odd integers by

\[
B_{2n} = \frac{2}{2n} \zeta(-2n + 1).
\]

A generating function for the \( B_n \) is given by

\[
te^t - 1 = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n
\]

Bernoulli polynomials are then defined by

\[
B_n(x) := \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j} = \sum_{j=0}^{n} \binom{n}{j} B_{n-j} x^j
\]

and are monic polynomials satisfying \( B_n(0) = B_n \). Since \( B_n(0) = B_n(1) \) we alternatively let \( \tilde{B}_n(x) \) be the periodic, period-1 extensions of the Bernoulli polynomials from \([0, 1]\) to all of \( \mathbb{R} \). For \( n \geq 3 \) these are differentiable and satisfy \( \frac{d}{dx} \tilde{B}_n(x) = n \tilde{B}_{n-1}(x) \), and this relationship continues to hold for \( n = 2 \) except at the integer points (of measure 0) where \( \tilde{B}_1(x) \) is defined to have value 0. The first few \( B_n(x) \) and \( \tilde{B}_n(x) \) are given by

\[
\begin{align*}
B_1(x) &= x - \frac{1}{2}, \quad \tilde{B}_1(x) = \{x\} - \frac{1}{2}, \\
B_2(x) &= x^2 - x + \frac{1}{6}, \quad \tilde{B}_2(x) = \{x\}^2 - \{x\} + \frac{1}{6}, \\
B_3(x) &= x^3 - \frac{3}{2} x^2 + \frac{1}{2} x, \quad \tilde{B}_3(x) = \{x\}^3 - \frac{3}{2} \{x\}^2 + \frac{1}{2} \{x\}
\end{align*}
\]

and so on where \( \{x\} = x - \text{Floor}(x) \) is the fractional part of \( x \).

We denote by \( b_n(k) \) the polynomials in \( k \in \mathbb{Z}_{>0} \) given by the sums of \((n-1)^{st}\) powers:

\[
b_n(k) := \sum_{j=1}^{k} j^{n-1}
\]

which turn out to be closely related to the Bernoulli polynomials.

We take the Fourier transform and inverse Fourier transform to be defined by

\[
\mathcal{F}[f](\xi) = \int_{-\infty}^{\infty} f(x) \cdot e^{-ix\xi} \, dx \quad \text{and} \quad \mathcal{F}^{-1}[g](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi) \cdot e^{ix\xi} \, d\xi
\]

Finally we denote the odd Heaviside function by \( \tilde{H}_0(x) \) so that
\[ \hat{H}_0(x) = \begin{cases} \frac{1}{2}, & x > 0 \\ -\frac{1}{2}, & x < 0 \end{cases} \]

while \( H^+_0(x) \) denotes the standard Heaviside function

\[ H^+_0(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \]

## 2 Remainder Convergence

### 2.1 Remainder Cesaro Summation

In [1] the notion of Cesaro convergence was recast in terms of a Cesaro operator

\[ P[f](x) := \frac{1}{2} \int_x^\infty f(t) \, dt \]

(on functions \( f : \mathbb{R} \to \mathbb{C} \)) and extended by consideration of the eigenvalues and eigenfunctions of \( P \) so that a classically divergent function \( f \) has generalised Cesaro limit \( L \) if \( q(P)[f](x) \to L \) as \( x \to \infty \) classically for some regular polynomial in \( P \), \( q(P) \) (regularity being equivalent to having \( q(1) = 1 \)). Since the eigenfunctions of \( P \), namely \( x^\rho \) with eigenvalue \( \frac{1}{\rho^{1+}} \) \((\rho \in \mathbb{C})\), arise naturally in the Euler-McLaurin asymptotic expansion for the partial sums of the Riemann zeta series, \( \sum_{n=1}^{\infty} n^{-s} \), generalised Cesaro summation can be used (see [1]) to derive the analytic continuation of \( \zeta \) to \( \text{Re}(s) \leq 1 \).

While various “schemes” were considered in [1], for \( \zeta \) the cleanest extension arose from a geometric Cesaro approach in which a partial sum function (of a real variable) is obtained by adding in each term \( n^{-s} \) at the point \( z = n \).

It is more natural, however, to view this analysis in [1] as a special case of remainder Cesaro summation in which contributions occur at points located relative to an initial point \( z_0 \in \mathbb{C} \). Specifically, given a function \( f : \mathbb{C} \to \mathbb{C} \) we wish to define its strict remainder sum at \( z_0 \in \mathbb{C} \) by

\[ R_+(f)(z_0) := \sum_{n=1}^{\infty} f(z_0 + n) \]

where we want to understand the sum on the RHS in a generalised Cesaro sense along the horizontal contour \( \gamma : t \mapsto z_0 + t \). Adapting the working definition in lemma 4 in [1], we do this by adopting the following working definition of Cesaro convergence along a contour:

**Definition 1:** Suppose \( \gamma : [0, \infty) \to \mathbb{C} : t \mapsto \gamma(t) \) is a contour parametrised by arc length \( t \), starting at \( z_0 \) with \( \gamma(t) \to \infty \) as \( t \to \infty \), and suppose \( f \) is a function on \( \gamma \) which can be written as

\[ f(t) = \sum_{j=1}^{n} a_j(\gamma(t))^{\rho_j}(\ln(\gamma(t)))^{m_j} + R(t) \]
for some finite collection of constants $a_j \in \mathbb{C}$, $\rho_j \in \mathbb{C}\setminus\{0\}$ and $m_j \in \mathbb{Z}_{\geq 0}$ and some remainder function $R(t)$. If there exists $n \in \mathbb{Z}_{>0}$ such that $P^n[R](t) \to L$ as $t \to \infty$, then we say that $f$ has generalised Cesaro limit $L$ along $\gamma$, where $P$ is now the averaging operator (in $t$) along $\gamma$ defined by

$$P[h](t) = \frac{1}{t} \int_0^t h(u) \, du$$

Notes on Definition 1: (i) In the rest of this paper the only contours we shall need to consider are either horizontal rays parallel to the real axis ($\gamma : t \mapsto z_0 \pm t$) or vertical rays parallel to the imaginary axis ($\gamma : t \mapsto z_0 \pm it$), which simplifies interpretation. For these cases it is not hard to see uniqueness of limits in definition 1.

(ii) It is critical in this definition, and will be crucial throughout this paper, that the functions $z^{\rho_j}(\ln z)^{m_j}$ that we “throw away” (i.e. assign generalised Cesaro limit 0 for $\rho_j \neq 0$) are functions of the geometric variable $z = \gamma(t)$, not just the arc-length parameter $t$; we continue to think of these as Cesaro eigenfunctions and generalised eigenfunctions in this setting even though this association is now loose given the necessary definition of $P$ in terms of $t$ rather than $z$. Of course, in the case where $\rho_j \notin \mathbb{Z}$ the need for this distinction disappears, at least when $\gamma$ is a simple ray as per (i), since in this case a straightforward Taylor expansion allows expression of each $z^{\rho_j}(\ln z)^{m_j}$ as a linear combination of functions of the form $t^{\rho_i}(\ln t)^{m_i}$ with all $\rho_i \neq 0$, and vice-versa. When $\rho_j$ is an integer, however, this distinction becomes pivotal in order to avoid pure powers of $\ln$, which correspond to Cesaro “eigenfunctions” with eigenvalue 1 and thus have no generalised Cesaro limit (see [1], section 2).

(iii) In relation to this last remark, recall the key point that Cesaro definition of generalised convergence is fundamentally intended as a tool for constructive analytic continuation (in $z_0$ and some parameter $\mu \in \mathbb{C}$ which drives the values of the $\rho_j$’s in the situation being analysed). As such, the above distinction could be rephrased as saying that when all $\rho_j \notin \mathbb{Z}$ the definition 1 can be recast, at least for $\gamma$ a ray, purely in terms of the parameter $t$ (and then fully justified rigorously in terms of genuine eigenfunctions and eigenvalues of a regular polynomial in $P_t$); but when $\mu$ is such that there is a $\rho_j \in \mathbb{Z}_{\geq 0}$ we find that we need to distinguish between $z$ and $t$ as per the discussion in (ii) in order to obtain the correct analytic continuation across this value of $\mu$.

(iv) When $\gamma$ is just a horizontal ray then, for $z_0$ real, definition 1 devolves simply to the case considered in [1] and so simply represents a natural extension of the definition in [1] to achieve the desired constructive analytic continuation from $\mathbb{R}$ to $\mathbb{C}$ just mentioned. The case of vertical rays can be viewed similarly. While the motivating example of remainder summation in (6) leads to a partial sum function with evenly spaced jumps, the function $f$ in definition 1 obviously need not have any jumps or even arise from a summation process, or even if it does, may arise in a way where the spacing between summands is not even.

(v) We will return briefly to discuss definition 1 and the above observations (i)-(iv) further in the final section of the paper, by which stage the intervening
calculations will hopefully have both demonstrated the utility of this definition and clarified the meaning and significance of some of these comments (which may seem obscure at present). However, since our primary focus in this paper is on calculation, we defer further discussion here and simply conclude by summarising definition 1 into the following “working recipe” for calculation of generalised Cesaro limits along contours, as it will be used throughout the paper to perform the constructive analytic continuation (in \( z_0 \) and \( \mu \)) just mentioned:

(a) first remove linear combinations of Cesaro eigenfunctions and generalised eigenfunctions, but critically doing so geometrically as eigenfunctions in \( z = \gamma(t) \), rather than simply \( t \), and then

(b) apply a suitable power of the Cesaro averaging operator “along the contour” (i.e. averaging in the contour arc-length \( t \)).

Returning now to the case of remainder summation in (6), here we form the contour \( \gamma : t \mapsto z_0 + t \), form the partial sum function \( s_f(z_0, t) := \sum_{n \leq t} f(z_0 + n) \) and use definition 1 to obtain \( R_+[f](z_0) \) as \( \lim_{z^{-s} \to \infty} f(z^{-s}) = \lim_{z^{-s} \to \infty} f(z_0^{-s}) \), where \( z := z_0 + t \) and we have used the notation for Cesaro analysis.

In similar fashion we define \( R_{+0}[f](z) := \lim_{n \to \infty} f(z + n) \), \( R_0[f](z) := \lim_{n \to \infty} f(z - n) \), \( R_{+0_0}[f](z) := \lim_{n \to \infty} f(z + n) \) and so forth. In the last case with a bi-directional sum, the definition is via two independent Cesaro sums, for \( R_{+0} \) and \( R_{-0} \) respectively.

In the case of \( f(z) = z^{-s} \) we see at once that \( \zeta(s) = R_+[\tilde{z}^{-s}](0)^2 \) and the analysis referred to in [1] consisted of detailing the generalised Cesaro interpretation of this remainder summation at 0 for arbitrary \( s \). For \( z \neq 0 \), \( R_+[^{-s}](0) \) in turn yields the Hurewicz zeta function \( \zeta_H(z,s) = \sum_{n=1}^{\infty} (z+n)^{-s} \) via identical Cesaro analysis.

As a first simple example of the flexibility allowed by letting \( z \in \mathbb{C} \) and moving to the remainder Cesaro viewpoint, consider in this context the polynomials in \( k \in \mathbb{Z} \) defined in section 1.1 by \( b_n(k) := \sum_{j=1}^{k} j^{n-1} \). These are related by formal differentiation of the discrete summation variable, \( k \), namely

$$\frac{d}{dk}(b_n(k)) = (n-1) \cdot \{ b_{n-1}(k) - \zeta(2-n) \} \quad (7)$$

from which they may be readily recursively generated and shown to be connected to the Bernoulli polynomials \( B_n(x) \). This formal differentiation may be rigourised in the remainder Cesaro approach by naturally defining \( b_n(z) \), for arbitrary \( z \), as

$$b_n(z) = \sum_{j=1}^{z} j^{-s} := R_+[\tilde{z}^{-s}](0) - R_+[^{-s}](z) = \zeta(s) - \zeta_H(z, s) \quad (8)$$

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2The notation here means, of course, \( R_+[f](0) \) where \( f(z) = z^{-s} \); we shall use this notational shorthand extensively.

3Here \( s \) is playing the role of the variable \( \mu \) in the discussion above; with \( z_0 = 0 \) held fixed the Cesaro convergence is used to analytically continue the expression \( R_+[^{-s}](0) \) from its classical region of convergence when \( \Re(s) > 1 \) to all \( s \in \mathbb{C} \).
where \( s = -(n - 1) \). Utilising the fact that we may now rigorously differentiate with respect to \( z \) and that \( \frac{d}{dz} \zeta_H(z, s) = -s\zeta_H(z, s + 1) \) (after noting that summation and differentiation may be commuted in this case in a careful Cesaro analysis) we immediately obtain the claimed relationship on setting \( z = k \).

A second way in which remainder summation immediately expands the scope of analysis is that \( R_+ \) (and \( R_- \) etc) are now operators on functions and thus amenable to spectral analysis.

### 2.2 Remainder Cesaro Products

The notion of a Cesaro remainder product follows naturally via exponentiation of the remainder sum of the logarithm:

\[
\prod_R [f](z) := \exp(R_+[\ln(f)](z)) \tag{9}
\]

For example, for the identity function \( f(z) = z \), we have \( \prod_R [\tilde{z}](z) = \exp(R_+[\ln(\tilde{z})](z)) \) and the calculation of \( R_+[\ln(\tilde{z})](z) \) by Cesaro means follows readily from the Euler-McLaurin sum formula. Specifically, for any given \( z_0 \), using the notation from [1] where \( \alpha \in [0, 1) \) and recalling that \( z \ln z \) and \( z \) are generalised eigenfunctions of \( P \) where \( z = z_0 + k + \alpha \), we have

\[
\sum_{j=1}^{k} \ln(z_0 + j) = \int \ln(z_0 + t) \, dt + \frac{1}{2} \ln(z_0 + k) + C_{z_0} + o(1)
\]

\[
= (z_0 + k + \frac{1}{2}) \ln(z_0 + k) - (z_0 + k) + C_{z_0} + o(1)
\]

\[
\sim (z_0 + k + \alpha) \ln(z_0 + k + \alpha) - (z_0 + k + \alpha) + C_{z_0}
\]

Thus

\[
\prod_R [\tilde{z}](z_0) = \exp(C_{z_0}) \tag{10}
\]

where

\[
C_{z_0} = \lim_{k \to \infty} \left\{ \sum_{j=1}^{k} \ln(z_0 + j) - (z_0 + k + \frac{1}{2}) \ln(z_0 + k) + (z_0 + k) \right\} \tag{11}
\]

It turns out this example leads directly to the \( \Gamma \) function:

**Lemma 1:** The \( \Gamma \) function is given directly in terms of remainder Cesaro products by

\[
\Gamma(z + 1) = \frac{\prod_R [\tilde{z}](0)}{\prod_R [\tilde{z}](z)} \tag{12}
\]
or equivalently

\[
\ln(\Gamma(z + 1)) = R_+[\ln \tilde{z}](0) - R_+[\ln \tilde{z}](z) = \frac{1}{2}\ln(2\pi) - R_+[\ln \tilde{z}](z) \quad (13)
\]

This is readily proven by, for example, comparing Taylor series for the two sides of (13), but we omit details here. Note, however, that it is critical in the above Cesaro analysis that summands are introduced geometrically at \(z_0 + j\) and Cesaro eigenfunctions are then viewed w.r.t. this geometric variable \(z = z_0 + j + \alpha\), rather than simply the summation index \(j\); this need to respect the geometric location of summands in the Cesaro setting will be crucial later in this paper.

Note that the defining relation for the \(\Gamma\) function, \(\Gamma(z + 1) = z\Gamma(z)\), is self-evident from this definition, as is the fact that \(\Gamma(k + 1) = k!\) for \(k \in \mathbb{Z}\). Other basic identities for \(\Gamma\) (and for other functions) likewise follow directly from geometric considerations within the Cesaro viewpoint.

### 2.3 Geometric Effects Within the Remainder Cesaro Approach

To conclude section 2 we briefly give three examples to illustrate the way geometry can be exploited in the remainder Cesaro framework.

**Example 1 - Duplication formulae:** The well-known duplication formulae for \(\Gamma\) state that for any \(n \in \mathbb{Z}_{\geq 1}\)

\[
(2\pi)^{\frac{n+1}{2}}\Gamma(z + 1) = n^{z+\frac{1}{2}}\Gamma\left(\frac{z + 1}{n}\right) \cdot \Gamma\left(\frac{z + 2}{n}\right) \cdot \ldots \cdot \Gamma\left(\frac{z + n}{n}\right) \quad (14)
\]

Using the Cesaro definitions in (12) and (13) this follows almost immediately from the dilation invariance of Cesaro summation (see Appendix 5.1). For, taking logarithms, (14) is equivalent to

\[
\frac{(n-1)}{2}\ln(2\pi) + \ln(\Gamma(z + 1)) = (z + \frac{1}{2})\ln n + \sum_{j=1}^{n} \ln(\Gamma\left(\frac{z}{n} - \frac{(n-j)}{n} + 1\right))
\]

which, in light of (13), is equivalent to

\[
-R_+[\ln \tilde{z}](z) = \left(z + \frac{1}{2}\right)\ln n - \sum_{j=1}^{n} R_+[\ln \tilde{z}](\frac{z}{n} - \frac{(n-j)}{n})
\]

But, by dilation invariance, if we dilate each remainder Cesaro sum on the RHS by \(n\) (i.e. place each summand \(\ln(z - \frac{(n-j)}{n} + k)\) at \(z - (n-j) + kn\)) and note that \(\ln(z - \frac{(n-j)+kn}{n}) = \ln(z - (n-j) + kn) - \ln n\), then the summands now intersperse perfectly (see figure 1 in Appendix 5.3) to yield

\[
RHS = \left(z + \frac{1}{2}\right)\ln n - R_+[\ln \tilde{z} - \ln n](z)
\]
The result then follows on recalling that \( \ln n \) is a constant and therefore \( R_+[\ln n](z) = (-z - \frac{1}{2}) \ln n \) (which itself follows immediately on setting \( s = 0 \) in (8) and using the fact that \( \zeta(0) = -\frac{1}{2} \) and \( b_1(k) = k \)).

It is clear from the structure of the above proof that an analogous duplication identity will hold for any function defined by remainder summation \( R_+[f](z) \) (or corresponding product) as long as \( f(n \tilde{z}) \) can be related readily to \( f(\tilde{z}) \). It is trivial for example that

\[
\zeta_H(z, s) = n^{-s} \sum_{j=1}^{n} \zeta_H \left( \frac{z}{n} - \frac{(n-j)}{n}, s \right) \tag{15}
\]

**Example 2 - Functional equations:** The functional equation for \( \Gamma \) states that

\[
\Gamma(z) \cdot \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} \tag{16}
\]

or equivalently that

\[
\ln(\Gamma(z)) + \ln(\Gamma(1 - z)) = \ln \pi - \ln(\sin(\pi z)) \tag{17}
\]

This can be proven in the remainder Cesaro framework using (13) by converting the pair of remainder sums on the LHS in (17) into a bi-directional sum, \( R_+[\ln \tilde{z}](1 - z_0) \) and then noting that this is automatically periodic with period 1 and thus amenable to Fourier analysis. In more detail, rewriting (13) as

\[
\ln(\Gamma(z)) = \frac{1}{2} \ln(2\pi) - R_+[\ln \tilde{z}](z),
\]

consider the diagram in figure 2 in Appendix 5.3, for a given choice of \( z_0 \).

Now in the case shown, for \( \text{Im}(z_0) > 0 \), and using \(-\pi < \theta \leq \pi \) as the principal branch of \( \ln \), we have \( \ln(j - z_0) = \ln(z_0 - j) - i\pi \) for all \( j \). Let \( z_1 := ((k - z_0) + \alpha) \) be the variable in the partial sum, \( s_1 \), for the Cesaro computation of \( R_+[\ln \tilde{z}](1 - z_0) \) and let \( z_2 := ((z_0 - k) - \alpha) \) be the corresponding variable in the partial sum, \( s_2 \), for computing \( R_-[\ln \tilde{z}](z_0) \), so that we have

\[
R_+[\ln \tilde{z}](1 - z_0) = \operatorname{Clim}_{z_1 \to \infty} s_1(z_1) \quad \text{and} \quad R_-[\ln \tilde{z}](z_0) = \operatorname{Clim}_{z_2 \to -\infty} s_2(z_2)
\]

Then

\[
s_1(z_1) = \sum_{j=1}^{k} \ln(j - z_0) = \sum_{j=1}^{k} \ln(z_0 - j) - i\pi \sum_{j=1}^{k} 1 = s_2(z_2) - i\pi k
\]

Now we know from (11) that

\[
s_1(z_1) = (-z_0 + k + \frac{1}{2}) \ln(-z_0 + k) - (-z_0 + k) + C_{-z_0} + o(1) \sim C_{-z_0}
\]
Therefore
\[ s_2(z_2) = -(z_0 - k - \frac{1}{2}) \ln(z_0 - k) - i\pi + (z_0 - k) + C_{-z_0} + i\pi k + o(1) \]

But
\[ -z_2 \ln z_2 = -(z_0 - k - \alpha) \ln(z_0 - k - \alpha) \]
\[ = -(z_0 - k) \cdot \left\{ \ln(z_0 - k) - \frac{\alpha}{(z_0 - k)} \right\} + \alpha \ln(z_0 - k) + o(1) \]
\[ \approx -(z_0 - k) \ln(z_0 - k) + \frac{1}{2} \ln(z_0 - k) + \frac{1}{2} \]

Thus
\[ s_2(z_2) \approx -z_2 \ln z_2 + z_2 + C_{-z_0} + i\pi(z_0 - \alpha) \]
\[ \approx C_{-z_0} + i\pi(z_0 - \frac{1}{2}) \]
\[ = i\pi(z_0 - \frac{1}{2}) + \frac{1}{2} \ln(2\pi) - \ln(\Gamma(1 - z_0)) \]

and on adding \( s_1 \) and \( s_2 \) we have
\[ R_{+,0,-} \[ \ln \tilde{z}(z_0) = i\pi(z_0 - \frac{1}{2}) + \ln(2\pi) - \ln(\Gamma(1 - z_0)) - \ln(\Gamma(z_0)) \] \]

(18)

We see that (17) follows if and only if
\[ R_{+,0,-} \[ \ln \tilde{z}(z) = \ln(\sin(\pi z)) + i\pi(z - \frac{1}{2}) + \ln 2 \]
\[ \approx \ln(\sin(\pi z)) = \ln \left( \frac{-e^{-i\pi z}}{2i} \right) \left( 1 - e^{2\pi i z} \right) = -i\pi z + \frac{i\pi}{2} - \ln 2 - \sum_{n=1}^{\infty} \frac{e^{2\pi inz}}{n} \]

this in turn follows if and only if
\[ R_{+,0,-} \[ \ln \tilde{z}(z) = -\sum_{n=1}^{\infty} \frac{e^{2\pi inz}}{n} \] \]

(19)

Now, by definition, \( R_{+,0,-} \[ \ln \tilde{z}(z) \) is a periodic function with period 1, expressible as a Fourier series
\[ R_{+,0,-} \[ \ln \tilde{z}(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi inz} \]

and so we see that (19) is equivalent to verifying that the Fourier coefficients in this expansion, \( a_n \), satisfy
\[ a_n = \begin{cases} -\frac{1}{n}, & n \in \mathbb{Z}_{>0} \\ 0, & n \in \mathbb{Z}_{\leq0} \end{cases} \]
But

\[
\begin{align*}
  a_n &= \int_0^1 R_{+0,-}[\ln z](z) \cdot e^{-2\pi inz} \, dz \\
  &= \int_0^\infty \ln z \cdot e^{-2\pi inz} \, dz \\
  &= \int_{-\infty}^\infty \ln |z| \cdot e^{-2\pi inz} \, dz + i\pi \int_{-\infty}^0 e^{-2\pi inz} \, dz
\end{align*}
\]

In a Cesaro analysis the second term here gives

\[
\text{Clim}_{x \to \infty} \frac{-i\pi}{2\pi in} (1 - e^{-2\pi inx}) = \begin{cases} 
  -\frac{1}{2n}, & n \in \mathbb{Z} \neq 0 \\
  0, & n = 0
\end{cases}
\]

while the first term, on performing integration by parts and using a Cesaro calculation to justify ignoring the boundary terms, gives

\[
\frac{1}{2\pi in} \int_{-\infty}^\infty \frac{1}{z} \cdot e^{-2\pi inz} \, dz
\]

Now the Fourier transform of the constant function 1 is \(2\pi \delta_0(\xi)\) and multiplication by \(z\) corresponds to \(i \frac{d}{d\xi}\), so division by \(z\) corresponds to \(-i \frac{d}{d\xi}\) and we see that this last term integrates to a Heaviside function

\[
-\frac{2\pi i}{2\pi in} \cdot \begin{cases} 
  -\frac{1}{2}, & n \in \mathbb{Z}_{<0} \\
  0, & n = 0 \\
  \frac{1}{2}, & n \in \mathbb{Z}_{>0}
\end{cases}
\]

Combining terms, this yields at once the required form for \(a_n\) and the functional equation for \(\Gamma\) follows as claimed from (19).

We see that the relationship between values of \(\Gamma\) at \(z\) and \(1 - z\) expressed in the functional equation (16) arises naturally from the geometry of the remainder Cesaro approach shown in figure 2, together with the Fourier-amenability of the resulting periodic bi-directional sum \(R_{+0,-}\). As such it is again the case that an analogous functional equation should hold for any function defined, like \(\Gamma\), by a remainder Cesaro sum. This is indeed the case, for example, for the Hurewicz zeta function \(\zeta_H(z,s)\). While the Fourier leg of the analysis is difficult to perform in this case for arbitrary \(z\), for the particular case of \(z = 0\) it can be accomplished relatively directly and the functional equation for the Riemann zeta function is then derived, namely

\[
\zeta(1 - s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)
\]
For reasons of space, however, we omit details here, particularly in light of the fact that we will derive the functional equation for $\zeta$ in section 3.3 by alternative means arising from our introduction of generalised root identities. Note, however, that in the derivation using bi-directional sums and Fourier theory the connection between values at $s$ and $1-s$ arises from the Fourier theory rather than from the Cesaro geometry as it did in the example of $\Gamma$.

**Example 3 - An integration relationship for $\zeta_H$ and $\zeta$:** In [1] it was shown that if $x = k + \alpha$ then we have

$$\lim_{k \to \infty} k^\rho = \begin{cases} (-1)^\rho \frac{1}{\rho+1} & , \quad \rho \in \mathbb{Z}_{\geq 0} \\ 0 & , \quad \text{else} \end{cases}$$

and that this combined with the Euler-McLaurin sum formula leads to the derivation of the values of $\zeta$ as Cesaro limits of partial sums $\sum_{n=1}^k n^{-s}$. Since $\int_{-1}^0 k^\rho \, dk = (-1)^\rho \frac{1}{\rho+1}$, so when $s \in \mathbb{Z}_{\leq 0}$ we obtain

$$\zeta(s) = \int_{-1}^0 \sum_{n=1}^k n^{-s} \, dk$$

and in light of (8) this can be re-expressed as saying that

$$\int_{-1}^0 \zeta_H(z, s) \, dz = 0 \quad \text{for } s \in \mathbb{Z}_{\leq 0} \quad (21)$$

In this last example we show how (21) in fact follows for arbitrary $s \neq 1$ as a consequence again of geometric considerations in the remainder Cesaro approach together with dilation-invariance arguments similar to those deployed in example 1 for the $\Gamma$-function duplication formulae.

**Lemma 2:** We have

$$\int_{-1}^0 \zeta_H(z, s) \, dz = 0 \quad \forall s \neq 1 \quad (22)$$

**Proof:** Suppose initially that $Re(s) < 1$. Using right Riemann sums, the integral on the LHS becomes

$$\int_{-1}^0 \zeta_H(z, s) \, dz = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \zeta_H(-\frac{j}{n}, s) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} R_+ [\tilde{z}^{-s}(-\frac{j}{n})]$$
But dilating by $n$ and noting that \((\tilde{z})^{-s} = n^s \tilde{z}^{-s}\) and that the summation points after dilation all interleave to precisely fill up the positive integers, it follows from Cesaro dilation-invariance that this becomes

\[
\lim_{n \to \infty} n^{s-1} R_+ [\tilde{z}^{-s}](0) = \lim_{n \to \infty} n^{s-1} \zeta(s) = 0
\]

This proves (22) for $Re(s) < 1$. At $s = 1$ the integral is not Cesaro convergent but for all other $s$ the result then follows by analytic continuation.\(^4\)

Having introduced and illustrated applications of remainder Cesaro summation, we now turn to the primary focus of this paper, in which it plays a key role, namely generalised root identities.

### 3 Generalised Root Identities

#### 3.1 Introduction

For a polynomial $p(z) = \sum a_n z^n$ it is trivial that the roots are related to the coefficients by $\sum_{\text{roots } r_i} r_i = -a_n^{-1}$, and similar identities for $\sum_{\text{roots } r_i} r_i^m$, $m \in \mathbb{Z}_{>0}$. These identities can be recast in a way potentially applicable to more general functions, for example entire functions with Taylor series $\sum_{n=1}^{\infty} a_n z^n$, as relations involving the reciprocals of the roots as follows: Suppose $f(z) = \prod (z - r_i)^{M_i}$, then $\ln(f(z)) = \sum M_i \ln(z - r_i)$ and therefore in general for any $\mu \in \mathbb{Z}_{>0}$ and any $z_0$

\[
\frac{1}{\Gamma(\mu)} \left( \frac{d}{dz} \right)^\mu (\ln(f(z))) \big|_{z=z_0} = (-1)^\mu \sum_{\{\text{roots } r_i\}} \frac{M_i}{(z_0 - r_i)^\mu} \tag{23}
\]

In this formula note that nothing requires $M_i \in \mathbb{Z}_{>0}$. The multiplicities, $M_i$, may be negative integers ($r_i$ a pole) or indeed arbitrary complex numbers ($r_i$ a branch point). We shall continue to use the term “roots”, or sometimes “generalised roots”, to cover all these possibilities.

Note also that while (23) gives a family of identities for $\mu = 1, 2, 3, \ldots$ at any given $z_0$, this is of course equivalent to knowing the single identity for $\mu = 1$ at general $z_0$. For if we have (23) for all $\mu \in \mathbb{Z}_{>1}$ at $z_0$ then for $\mu = 1$ at any

\(^4\)Note that the reasoning here likewise immediately yields a family of special values of $\zeta_H$: for example, for any prime $p$, $\sum \zeta_H(-\frac{1}{p^n}, s) = (p^s - 1) \zeta(s)$ where the sum is over $1 < j < p$; for $p, q$ prime, $\sum \zeta_H(-\frac{1}{pq}, s) = ((pq)^s - p^s - q^s + 1) \zeta(s)$ where the sum is over $1 < j < pq$, $j$ coprime to $pq$, and so on.
$z_0 + h$ in (23) we have

\[
LHS = - \frac{d}{dz} (\ln(f(z))) \bigg|_{z=z_0+h} = - \frac{d}{dz} (\ln(\tilde{f}(z))) \bigg|_{z=z_0} \text{ where } \tilde{f}(z) = f(z + h)
\]

\[
= - \frac{d}{dz} \left( e^{h \frac{d}{dz} (\ln(f(z)))} \right) \bigg|_{z=z_0}
\]

\[
= - \sum_{j=0}^{\infty} \frac{h^j}{\Gamma(j+1)} \left( \frac{d}{dz} \right)^{j+1} (\ln(f(z))) \bigg|_{z=z_0}
\]

\[
= \sum_{j=0}^{\infty} (-1)^j h^j \sum_{\{\text{roots } r_i \} \text{ of } f} \frac{M_i}{(z_0 - r_i)^{j+1}}
\]

\[
= - \sum_{\{\text{roots } r_i \} \text{ of } f} \frac{M_i}{(z_0 + h - r_i)} = RHS
\]

and vice-versa.

Unfortunately, of course, while both sides of (23) now make sense for more general functions, the identity is generally untrue for non-polynomials. For example, for $f(z) = e^z$, $f$ has no roots so the RHS of (23) is zero while the LHS is identically 1 at $\mu = 1$ for any $z_0$.

However, suppose $f$ fails (23) at $\mu = 1$ with error function

\[
g(z) = -\frac{f'(z)}{f(z)} + \sum_{\{\text{roots } r_i \} \text{ of } f} \frac{M_i}{(z - r_i)}
\]

and $g$ entire. Then if we set $h(z) = e^{G(z)} f(z)$ where $G'(z) = g(z)$ it follows from the fact that the root-sets of $f$ and $h$ are identical that in (23) for $h(z)$ we have

\[
LHS = - \frac{d}{dz} (\ln(h(z))) = - \frac{d}{dz} \left( G(z) + \ln(f(z)) \right)
\]

\[
= -g(z) - \frac{f'(z)}{f(z)} = - \sum_{\{\text{roots } r_i \} \text{ of } h} \frac{M_i}{(z - r_i)} = RHS
\]

Thus, even though $f$ fails the generalised root identity (23), there is a unique (up to an overall scalar), nowhere-zero entire function $e^{G(z)}$ whose product with $f(z)$ yields a function with the same root-set satisfying the identity.

In terms of equivalence classes, if we set two functions equivalent, $f \sim h$, if and only if there exists an entire, nowhere-zero function, $k(z)$, with $k(0) = 1$ such that $h(z) = k(z) f(z)$, then within any equivalence class there is a unique representative which satisfies the generalised root identities (23).

This can be viewed in a different way by thinking of $z_0$ fixed and $\mu = 1, 2, 3, \ldots$ successively. Taking $z_0 = 0$ for example, if the LHS and RHS of (23)
differ by \( a_1 \) for \( \mu = 1 \) then we can remove this “obstruction” by multiplying \( f \) by \( e^{a_1 z} \) since this leaves the root side undisturbed and contributes the required \( a_1 \) to the derivative side. Similarly, if the obstruction of the \( \mu = n \) identity at \( z_0 = 0 \) is \( a_n \) then multiplying \( f \) by \( \exp(\frac{a_n z^n}{n!}) \) again leaves the root side of (23) unchanged but contributes precisely the required correction of \( a_n \) to the \( \mu = n \) identity, while contributing nothing to any of the other identities with \( \mu \in \mathbb{Z} \neq n \) and thus leaving these identities undisturbed. In this way, working successively through \( \mu = 1, 2, 3, \ldots \) we can correct each obstruction and produce a new function \( h(z) = \exp(\sum a_n z^n/n!) f(z) \) which has the same generalised root-set as \( f \) and does satisfy the root identities (23) for all \( \mu \in \mathbb{Z} \geq 1 \).

Note that if the error function, \( g(z) \), obstructing the \( \mu = 1 \) identity for \( f \) is not entire, then the situation becomes more complex.

### 3.2 Some Examples

In reality the root identities (23) hold without any need for adjustment for many well-known functions. We next briefly consider three examples.

**Example 1** - \( f(z) = \cos(\frac{\pi z}{2}) \): Here the roots of \( f \) are all simple roots of multiplicity 1 at the points \( \pm (2k - 1), k \in \mathbb{Z}_{>0} \), and so for \( \mu = 1 \) at arbitrary \( z_0 \) in (23), we have

\[
\text{RHS} = -\sum_{k=1}^{\infty} \frac{2z_0}{z_0^2 - (2k - 1)^2}, \quad \text{while} \quad \text{LHS} = \frac{\pi}{2} \tan\left(\frac{\pi z_0}{2}\right)
\]

These are equal by a well-known identity, and so \( f(z) = \cos(\frac{\pi z}{2}) \) satisfies the generalised root identities (23) for arbitrary \( z_0 \) and all \( \mu \in \mathbb{Z}_{\geq 1} \).

Since the roots of \( f \) are closely related to the positive integers, the RHS of (23) leads easily to values of \( \zeta(n), n \in \mathbb{Z}_{>0} \). In particular, for \( \mu = 2 \) the root identity at \( z_0 = 0 \) immediately captures Euler’s famous formula that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \). This is an example of an “inverse” use of the generalised root identities, where a function of interest, \( \zeta \), arises naturally from the root side of the identities for a known function, \( f(z) = \cos(\frac{\pi z}{2}) \), and information about \( \zeta \) is then deduced by considering the derivative side of the identities for \( f \). We shall return to this example, along these lines, in the next section.

**Example 2** - \( \Gamma(z+1) \): The roots of \( \Gamma(z+1) \) are all simple poles (\( M_i = -1 \forall i \)) at \( z = -1, -2, -3, \ldots \). Thus for \( \mu = 1 \) and arbitrary \( z_0 \) in (23), we have

\[
\text{RHS} = \sum_{n=1}^{\infty} \frac{1}{z_0 + n}
\]

This is divergent for all \( z_0 \) and is not even Cesaro summable because the partial sums involve the generalised Cesaro eigenfunction with eigenvalue 1, \( \ln z \). How-
ever, since this Cesaro-obstruction is uniform, it can be “renormalised” away uniformly as

\[ \text{RHS} = \sum_{n=1}^{\infty} \left( \frac{1}{z_0 + n} - \frac{1}{n} \right) + \gamma \]  

(24)

where \( \gamma \approx 0.577 \) is Euler’s constant and arises because the difference between the \( \ln z \) divergence we are removing and the partial sums of \( \sum \frac{1}{n} \) approaches \( \gamma \) in the limit (\( \lim_{N \to \infty} (\sum_{n=1}^{N} \frac{1}{n} - \ln N) = \gamma \)).

But (24) is in turn a well-known expression for \(-\frac{\Gamma'(z+1)}{\Gamma(z+1)}\big|_{z=z_0}\) and thus we see that, after renormalisation of the \( \ln \) Cesaro divergence, \( \Gamma(z+1) \) does satisfy the generalised root identity (23) for \( \mu = 1 \) and arbitrary \( z_0 \) (note that the expression in (24) retains the required simple poles at \( \mathbb{Z}_{\leq 0} \)).

The need to perform this renormalisation adjustment arises only for the case of \( \mu = 1 \). For \( \mu = 2 \) the RHS of (23) is \(-\sum_{n=1}^{\infty} \frac{1}{(z_0+n)^2} \), which is classically convergent for all \( z_0 \) and clearly equals the the LHS, namely \(-\frac{d^2}{ds^2}(\ln(\Gamma(z+1)))|_{z=z_0}\), on differentiating the expression for \(-\frac{\Gamma'(z+1)}{\Gamma(z+1)}\big|_{z=z_0} \) in (24); likewise for \( \mu = 3, 4, \ldots \).

Thus overall \( \Gamma(z+1) \) satisfies the generalised root identities (23) for arbitrary \( z_0 \) and all \( \mu \in \mathbb{Z}_{\geq 1} \), albeit after requiring an additional renormalisation when \( \mu = 1 \) to uniformly remove the Cesaro non-amenable \( \ln \)-divergences in this case.

It is easy to see that the same continues to hold true for the case of general \( \Gamma(az+b) \).

Example 3 - \( \zeta(s) \): The root set of \( \zeta(s) \) consists of the trivial zeros at \( s = -2, -4, \ldots \) (with \( M_i = 1 \)), the simple pole at \( s = 1 \) (with \( M_i = -1 \)) and the famous non-trivial zeros in the critical strip \( 0 < \Re(s) < 1 \) (for which \( M_i > 0 \) are unknown in general). We let \( T \) denote the set of trivial roots and \( NT \) the set of non-trivial roots, which we will also denote by \( \rho_i \) rather than \( r_i \).

If the Riemann hypothesis is true, then the \( \rho_i \) occur in conjugate pairs solely on the critical line \( s = \frac{1}{2} \); if not then some occur in quadruples via reflection in the real axis and critical line: \( \rho_i, \bar{\rho}_i, 1-\rho_i, \) and \( 1-\bar{\rho}_i \).

Of course, the roots in \( NT \) are not known exactly and thus tackling the root side of the generalised root identities (23) directly is difficult. However, working empirically first, we can take a list of, say, the first 100,000 non-trivial zeros and use them to test experimentally whether \( \zeta \) seems to satisfy (23) for \( \mu = 1 \). In doing this, however, a renormalisation analogous to the last example will have to be carried out to handle the trivial zeros, \( T \); considering the case \( s_0 = 0 \) initially, this is equivalent to setting \( \sum_{T} \frac{1}{|\nu-r_i|} = \frac{2}{\pi} \) (the \( \frac{1}{2} \) factor arising naturally since \( T \) only covers the negative even integers). Thus, taking initially only the truncated non-trivial root-set \( \bar{NT} \) given, for example, by Odlyzko in [2] and consisting of the first 100,000 non-trivial zeros (actually 200,000 including
conjugates) we find that the root side of (23) for \( \mu = 1 \) at \( s_0 = 0 \) becomes

\[
RHS \approx -\left\{ \frac{\gamma}{2} + 1 + \sum_{NT} \frac{1}{(0 - \rho_i)} \right\} \approx -1.2655342
\]

(on performing the last computation numerically to obtain \( \sum_{NT} \frac{1}{\rho_i} \approx 0.0230737 \)).

By contrast the derivative side in (23) gives

\[
-\frac{c(0)}{\zeta(0)} = \ln(2\pi) \approx -1.8378771.
\]

At once we see that \( \zeta \) does not directly satisfy the generalised root identities for \( \mu = 1 \) at \( s_0 = 0 \), with the obstruction in this instance being, numerically, \( 0.5723429 \approx 0.5723649 = \frac{1}{2} \ln \pi \).

Next consider \( s_0 = \frac{1}{2} \). Here, by the symmetry outlined above, \( \sum_{NT} \frac{1}{(s - \rho_i)} = 0 \) and so

\[
RHS = -\left\{ \sum_{\rho} \frac{1}{(s - \rho)} - \frac{1}{(s - i\pi)} \right\}.
\]

In this case the calculation of the renormalised value of the first sum is best done by noting that the trivial zeros occur at the same locations as the poles of \( \Gamma(\frac{s}{2} + 1) \) and thus, up to an overall factor of -1, this sum can be evaluated using the generalised root identity for \( \Gamma(\frac{s}{2} + 1) \) at \( s_0 = \frac{1}{2} \), namely as

\[
\frac{1}{2} \Gamma(\frac{s}{4} + \frac{1}{2}) = -\frac{1}{2} \left\{ \sum_{n=1}^{\infty} \left( \frac{1}{(n+\frac{1}{2})} - \frac{1}{n} \right) + \gamma \right\}.
\]

We thus obtain

\[
RHS = -\frac{\gamma}{2} + \frac{\pi}{4} - \frac{3}{2} \ln 2 \quad \text{while} \quad LHS = -\frac{c(\frac{s}{4} + \frac{1}{2})}{\zeta(\frac{s}{2})} = -\frac{\gamma}{2} - \frac{\pi}{4} - \frac{3}{2} \ln 2 - \frac{1}{2} \ln(\pi),
\]

so that again we obtain the same value of \( \frac{1}{2} \ln(\pi) \) for the obstruction.

This suggests, along the lines discussed before, that while \( \zeta \) does not directly satisfy the generalised root identities (23) for \( \mu = 1 \), the function \( \pi^{-\frac{s}{2}} \zeta(s) \) may well do so.

This conjecture can be expressed in a different way. Recall that the functional equation for \( \zeta \) given in (20) can be re-expressed as simply

\[
\xi(s) = \xi(1-s) \quad \text{where} \quad \xi(s) := (1-s)\Gamma\left(\frac{s}{2} + 1\right)\pi^{-\frac{s}{2}}\zeta(s)
\]

Since the generalised root identities (23) hold for \( (1-s) \) and \( \Gamma(\frac{s}{2} + 1) \) (as discussed in example 2), and since a product of functions satisfying these identities will also satisfy these identities, so the conjecture that \( \pi^{-\frac{s}{2}} \zeta(s) \) satisfies them for \( \mu = 1 \) and arbitrary \( s_0 \) is equivalent to having \( \xi \) satisfy (23) for \( \mu = 1 \) and arbitrary \( s_0 \).

In fact this conjecture turns out to be true. In light of the discussion of \( \Gamma \) in the previous example it is equivalent to the following corollary which can be found in [4, pg35]:

**Theorem 1:** Let \( NT \) denote the set of non-trivial zeros of \( \zeta \), and \( NT_+ \) denote the subset with imaginary part > 0. Then, with sums understood to include multiplicities, we have

\[
\frac{\zeta'(s)}{\zeta(s)} = \left\{ \begin{array}{l}
\frac{1}{\pi} + \frac{1}{s-1} + \frac{1}{2} \frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} \\
-(2s-1) \sum_{\rho \in NT_+} \frac{1}{(s-\rho)(s-(1-\rho))} - \frac{1}{2} \ln(\pi)
\end{array} \right\}
\]
Thus, overall, both $\pi^{-\frac{1}{2}}\zeta(s)$ and $\xi(s)$ satisfy the generalised root identities (23) for $\mu = 1$ and arbitrary $s_0$, and hence also for arbitrary $\mu \in \mathbb{Z}_{>1}$. Since the factor $\pi^{-\frac{1}{2}}$ only contributes to the root identities when $\mu = 1$, note that for $\mu \in \mathbb{Z}_{>1}$ the root identities will in fact be satisfied directly by $\zeta$ (which can of course be readily spot-checked experimentally).

Note that, as discussed in [4], this corollary is in fact the expression of the Hadamard product formula for $\xi$ (which is a holomorphic integral function of order 1). We thus see that the generalised root identity (23) for $\mu = 1$ is in this case equivalent to the Hadamard product formula for $\xi$. Since the identities for $\mu \in \mathbb{Z}_{>1}$ are just derivatives of the $\mu = 1$ identity, they will contain no additional information, but we turn now to considering cases of more general $\mu$, in particular $\mu \in \mathbb{Z}_{\leq 0}$, in the hope that they will lead to new tools with additional content beyond the Hadamard identity.

### 3.3 Further Generalisation of the Root Identities

In (23) our current root identities constitute a generalisation from the case of polynomials, relating derivatives of $\ln(f(z))$ at $z_0$ to sums of integer powers of shifted reciprocals of (generalised) roots of $f$. It is natural, however, to ask whether they can be further generalised by allowing $\mu$ to be an arbitrary complex number rather than just a positive integer; that is

$$\frac{-1}{\Gamma(\mu)} \left( \frac{d}{dz} \right)^\mu (\ln(f(z)))|_{z=z_0} = e^{i\pi \mu} \sum_{\{\text{roots } r_i\}} \frac{M_i}{(z_0 - r_i)^\mu}, \quad \mu \in \mathbb{C} \quad (27)$$

The LHS here we call the derivative side of the root identity and denote by $d_f(z_0, \mu)$ (or just $d(z_0, \mu)$ if the context is clear). The RHS is the root side and is denoted by $r_f(z_0, \mu)$ (or just $r(z_0, \mu)$), and (27) is interpreted as asserting identity of these two functions of two complex variables. Of course, in attempting such an extension the question immediately arises of how to interpret either side of (27):

(a)(i) To overcome the potential divergence of the sum on the root side (e.g. when $\Re(\mu)$ becomes negative) the RHS must be interpreted via a generalised convergence scheme, which analytically continues the RHS from its region of convergence in the $\mu$-plane. In all cases in this paper this will be a generalised Cesaro scheme.

(a)(ii) In implementing this Cesaro approach, the sum on the root-side will need to be interpreted geometrically. Specifically, each term $\frac{M_i}{(z_0 - r_i)^\mu}$ must be added in at the shifted point $z_0 - r_i$ itself in the complex plane (rather than, for example, always being added in at $r_i$ itself). Thus, as $z_0$ varies, not only do the summands vary, but their locations move too. We shall emphasise this by writing $\sum_{\{z_0-\text{roots } r_i\}}$ on the RHS in (27).

(b)(i) On the derivative side in (27), the interpretation of $\left( \frac{d}{dz} \right)^\mu$ also becomes problematic. Two facts that should hold in any such definition, however, are
that

\[ \left( \frac{d}{dz} \right)^\mu (a^z)|_{z=z_0} = a^{z_0}(\ln a)^\mu \]  

(28)

and

\[ \left( \frac{d}{dz} \right)^\mu (z^\rho)|_{z=0} = \begin{cases} 
\Gamma(\rho + 1), & \rho = \mu \\
0, & \text{else}
\end{cases} \]  

(29)

We shall formalise the definition of the LHS of (27) in a manner consistent with these criteria below. First, however, we demonstrate the extra power obtained from this extension of the generalised root identities to arbitrary complex \( \mu \) by showing how the full functional equation for \( \zeta \), given in (20), follows very simply from applying them to the function \( f(z) = \cos(\frac{\pi z}{2}) \) considered before in example 1 in section 2.1. For in this case, taking \( z_0 = 0 \) in (27) we have

\[ \text{RHS} = r_f(0, \mu) = e^{i\pi \mu}(1 + e^{-i\pi \mu})(1 - 2^{-\mu})\zeta(\mu) \]

while for \( \mu \neq 1 \) we have

\[ \text{LHS} = d_f(0, \mu) = -\frac{1}{\Gamma(\mu)} \left( \frac{d}{dz} \right)^\mu \left( \ln \left( \frac{e^{z^\mu} + e^{-z^\mu}}{2} \right) \right)|_{z=0} \]

\[ = -\frac{1}{\Gamma(\mu)} \left( \frac{d}{dz} \right)^\mu \left\{ -\frac{i\pi z}{2} + \ln(1 + e^{i\pi z}) \right\}|_{z=0} \]

\[ = -\frac{1}{\Gamma(\mu)} \left( \frac{d}{dz} \right)^\mu \left\{ e^{i\pi z} - \frac{e^{2i\pi z}}{2} + \frac{e^{3i\pi z}}{3} - \cdots \right\}|_{z=0} \]

\[ = -\frac{e^{i\frac{\pi}{2} \pi^\mu}}{\Gamma(\mu)} \left\{ 1^{\mu-1} - 2^{\mu-1} + 3^{\mu-1} - \cdots \right\} \]

\[ = -\frac{e^{i\frac{\pi}{2} \pi^\mu}}{\Gamma(\mu)} (1 - 2^\mu)\zeta(1 - \mu) \]

so that setting \( d_f(0, \mu) = r_f(0, \mu) \) we obtain

\[ \zeta(1 - \mu) = 2^{1-\mu}\pi^{-\mu}\cos(\frac{\pi \mu}{2})\Gamma(\mu)\zeta(\mu) \]

which is precisely (20). Thus the functional equation for \( \zeta \) is equivalent to \( \cos(\frac{\pi z}{2}) \) satisfying the generalised root identities (27) (after also extending easily to the case of \( \mu = 1 \)).

With this example as a motivation, and (28) and (29) as guidance, we formalise the meaning of the complex derivative \( \left( \frac{d}{dz} \right)^\mu \) in the LHS of (27) via Fourier theory, in a manner familiar from linear partial differential operators, as follows:

(b)(ii) Writing \( f \) as the inverse Fourier transform of its Fourier transform, namely

\[ g(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x)e^{i(z-x)\xi}dx\xi \]  

22
we define \( \left( \frac{d}{dz} \right)^\mu g(z) \big|_{z=z_0} \) by

\[
\left( \frac{d}{dz} \right)^\mu g(z) \big|_{z=z_0} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} (i\xi)^\mu g(x)e^{i(z_0-x)\xi}dx d\xi
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^\mu F[g](\xi)e^{iz_0\xi}d\xi
\]

(30)

With (a)(i), (a)(ii) and (b)(ii) the generalised root identities (27) now have a well-defined meaning for arbitrary complex \( \mu \). We have considered their interpretation for \( f(z) = \cos(\frac{\pi z}{2}) \) and will examine them next for \( \Gamma(z+1) \) and \( \zeta(s) \), but first we make several general observations about these identities and their potential uses:

(i) While the cases of \( \mu \in \mathbb{Z}_>0 \) give information on the sums of integer powers of shifted reciprocals of roots, (27) gives a much more sensitive relationship between the distribution of these roots and the behaviour of the derivatives of the log of the function.

(ii) In particular, for \( \mu = 0 \), (27) should give information about the Cesaro count of the roots (adding \( M_i \) at each point \((z_0-r_i)\)), while for \( \mu \in \mathbb{Z}_{<0} \) we get information regarding the Cesaro sums of first, second and higher powers of these roots. In particular, taking \( \text{Re}(\mu) \) sufficiently negative should give information about the asymptotic distribution of the roots and the cases \( \mu \in \mathbb{Z}_{\leq0} \) will be of particular interest.

(iii) For \( \mu \in \mathbb{Z}_{\leq0} \), however, it is immediately clear that further care will need to be taken in (27) owing to the poles of \( \Gamma \) in the factor \( \frac{1}{\Gamma(\mu)} \) on the LHS. At first glance these appear to make the \( df(z_0,\mu) \) always zero whenever \( \mu \in \mathbb{Z}_{\leq0} \) which seems problematic (e.g. in interpreting the case \( \mu = 0 \) as yielding a count of roots). Thus, in general, we will need to interpret (27) distributionally, and for \( \mu \in \mathbb{Z}_{\leq0} \) the distributional result we shall rely on in this paper is the following fact (see Hörmander’s treatment in [3, pp74]) regarding the function

\[
x_+^a := \begin{cases} x^a, & x > 0 \\ 0, & x < 0 \end{cases}
\]

and its normalised counterpart \( \chi_+^a := \frac{1}{\Gamma(a+1)} x_+^a \):

**Lemma 3:** For \( n \in \mathbb{Z}_{\geq0} \) we have

\[
\chi_+^{-n} = \delta_0^{(n-1)}(x)
\]

where \( \delta_0 \) is the usual delta distribution.

(iv) The fact that (27) naturally gives (Cesaro) asymptotic information about the distribution of the roots of \( f \) makes it a natural tool with which to investigate this asymptotic behaviour. Another way of looking at this is to compare with the case of (23) in which \( \mu \) was restricted to \( \mathbb{Z}_{\geq0} \). From an experimental point of view, since for any positive integer \( \mu \) the RHS of (23) or (27)

23
diverges as \( z_0 \) approaches any finite root \( r_i \), so the cases of such positive integer \( \mu \) give a tool for detecting the location of finite roots by looking at the dominant behaviour of these identities as we let \( z_0 \) approach such a location. They give little immediate guidance, however, about how the root locations behave asymptotically as they tend to \( \infty \).

By contrast, for \( \mu = 0 \) the RHS in (27) is clearly insensitive to the location of finite roots, but its Cesaro calculation depends critically on the asymptotic location of the roots as they approach \( \infty \); and the same is qualitatively true for \( \mu = -1, -2, -3, \ldots \).

In part this in turn reflects the non-locality of the derivatives \( \left( \frac{d}{dz} \right)^\mu \) on the LHS whenever \( \mu \notin \mathbb{Z}_{\geq 0} \), which can briefly be seen by considering an alternative approach to defining such derivatives, namely: let \( T_h \) be the operator of translation by \( h \) (i.e. \( T_h [g](z) = g(z+h) \)). Since \( \frac{d}{dz} = \lim_{h \to 0} \left\{ \frac{-1}{h}(1 - T_h) \right\} \), so in general formally

\[
\left( \frac{d}{dz} \right)^\mu = \lim_{h \to 0} \left\{ e^{i\pi\mu} h^{-\mu} (1 - T_h)^\mu \right\}
\]

\[
= \lim_{h \to 0} \left\{ e^{i\pi\mu} h^{-\mu} \{1 - \left( \frac{\mu}{1} \right) T_h + \left( \frac{\mu}{2} \right) T_h^2 - \ldots\} \right\}
\]

It is only for \( \mu \in \mathbb{Z}_{\geq 1} \) that this series definition truncates and leaves a purely local resulting function; for all other \( \mu \) the resulting function, \( \left( \frac{d}{dz} \right)^\mu (g(z)) \), depends on \( g(z) \) as \( z \to \infty \) and so is intrinsically non-local.\(^5\)

(v) In using the Fourier definition of the LHS in (27) we shall use the standard Fourier relationships repeatedly, namely

\[
\mathcal{F} [xf(x)](\xi) = i\mathcal{F} [f'](\xi) \quad \& \quad \mathcal{F} [f'(x)](\xi) = i\xi \mathcal{F} [f](\xi)
\]

\[
\mathcal{F} [f(x)e^{ia\xi}](\xi) = \mathcal{F} [f(\xi - a)] \quad \& \quad \mathcal{F} [f(x+a)](\xi) = \mathcal{F} [f](\xi)e^{i\alpha \xi}
\]

Since \( \mathcal{F} [1] = 2\pi \delta_0(\xi) \) it follows that, respecting oddness/evenness, we have

\[
\mathcal{F} \left[ \frac{1}{x} \right] = -2\pi i \hat{H}_0(\xi) = -2\pi i \cdot (H_0^+(\xi) - \frac{1}{2})
\]

and

\[
\mathcal{F} \left[ \frac{1}{x^2} \right] = -2\pi \hat{H}_0(\xi) \cdot \xi = -2\pi (H_0^+(\xi) - \frac{1}{2}) \cdot \xi
\]

and in general

\[
\mathcal{F} \left[ \frac{1}{x^\rho} \right] = 2\pi e^{-i\pi \rho} \frac{\hat{H}_0(\xi) \xi^{\rho-1}}{\Gamma(\rho)} = 2\pi e^{-i\pi \rho} \frac{(H_0^+(\xi) - \frac{1}{2}) \xi^{\rho-1}}{\Gamma(\rho)} \quad (32)
\]

\(^5\)For example for \( \mu = -1 \) we get \( \left( \frac{d}{d\xi} \right)^{-1} = \lim_{h \to 0} \{-h + T_h + T_h^2 + \ldots\} \) which in the limit yields the left Riemann sum definition of \(- \int_x^\infty g(u)du \).
Also, inverting the relationship for Fourier transform of a derivative and applying it to the formula for $\mathcal{F}[\frac{1}{x}]$ above, we have

$$\mathcal{F}[\ln x] = -2\pi \frac{\hat{H}_0(\xi)}{\xi} = -2\pi \left( \frac{H_0^+(\xi) - \frac{1}{\xi}}{\xi} \right)$$  \hspace{1cm} (33)$$

and

$$\mathcal{F}[x \ln x - x] = 2\pi i \frac{\hat{H}_0(\xi)}{\xi^2} = 2\pi i \left( \frac{H_0^+(\xi) - \frac{1}{\xi}}{\xi^2} \right) \quad \& \ldots$$  \hspace{1cm} (34)$$

with corresponding results for higher anti-derivatives on the LHS. It is these latter identities that will be used most directly in what follows.

From the point of view of calculation on the derivative side of the root identities (27), note that $H_0^+(\xi) \cdot \xi^a = \xi^a$, so that, for $a \in \mathbb{Z}_{<0}$, the first term in the brackets in these expressions leads naturally to use of the Hörmander result (31). The second half of these expressions, $\frac{1}{2} \xi^a$, leads, under inverse Fourier transform, either to $\delta$-function type contributions (for $a \in \mathbb{Z}_{\geq0}$) which may be ignored for $z_0 \neq 0$, or to Heaviside-type contributions in $z_0$ (for $a \in \mathbb{Z}_{<0}$) whose finite values for any $z_0$ cancel to 0 under the factor $\frac{1}{\Gamma(\mu)}$ on the derivative side when $\mu \in \mathbb{Z}_{\leq0}$. As such, for our calculations when $\mu \in \mathbb{Z}_{<0}$ in the root identities for $\Gamma$ in the next section we may effectively ignore these second-half contributions and elide the distinction between $\hat{H}_0(\xi) \cdot \xi^a$ and $\xi^a$.

We now take the two examples of $\Gamma(z+1)$ and $\zeta(s)$, considered in the previous section for $\mu \in \mathbb{Z}_{>0}$, and reconsider them for arbitrary $\mu \in \mathbb{Z}_{<0}$. We devote the rest of this section to the consideration of $\Gamma$, and then turn in the next section to focus exclusively on the case of $\zeta$.

### 3.4 Example 2 - The Case of $\Gamma(z + 1)$:

For $\Gamma(z + 1)$, on the root side of (27) we clearly have at once

$$r_\Gamma(z_0, \mu) = e^{i\pi \mu} \sum_{\{z_0 - \text{roots } r_i\}} \frac{M_i}{(z_0 - r_i)^\mu} = -e^{i\pi \mu} \zeta_H(z_0, \mu)$$  \hspace{1cm} (35)$$

When $z_0 = 0$ this reduces to the zeta function

$$r_\Gamma(0, \mu) = -e^{i\pi \mu} \zeta(\mu)$$  \hspace{1cm} (36)$$

and since $\zeta_H(z_0, \mu) = \zeta(\mu) - \sum_{j=1}^{z_0} j^{-\mu}$, so for example at $\mu = 0, -1, -2, -3, \ldots$ we get

---

6Technically in fact $\mathcal{F}[\ln x] = -2\pi \frac{\hat{H}_0(\xi)}{\xi} - 2\pi \gamma \delta(\xi)$ with the extra term arising from the constant of integration; however we ignore the additional $\delta(\xi)$ term here since it will not contribute to the derivative side of the root identities at $\mu \in \mathbb{Z}_{<0}$ in any of our remaining calculations in this paper on account of the factor $\frac{1}{\Gamma(\mu)}$ whose denominator diverges at such $\mu$. 

25
\[ r_{\Gamma}(z_0, 0) = z_0 + \frac{1}{2} \]
\[ r_{\Gamma}(z_0, -1) = -\frac{1}{2}z_0^2 - \frac{1}{2}z_0 - \frac{1}{12} \]
\[ r_{\Gamma}(z_0, -2) = \frac{1}{3}z_0^3 + \frac{1}{2}z_0^2 + \frac{1}{6}z_0 \]
\[ r_{\Gamma}(z_0, -3) = -\frac{1}{4}z_0^4 - \frac{1}{2}z_0^3 - \frac{1}{4}z_0^2 + \frac{1}{120} \]  

(37)

and in general for \( n \in \mathbb{Z}_{>0} \)
\[ r_{\Gamma}(z_0, -n) = (-1)^n\{ b_{n+1}(z_0) - \zeta(-n) \} = -\frac{1}{n+1}B_{n+1}(-z_0) \]  

(38)

On the derivative side, noting \( (\frac{d}{dz})^\mu \) \( (\frac{d}{dx})^\mu \) \( (\frac{d}{dx})^\mu \) and recalling the identity in (24) we have
\[
d_{\Gamma}(z_0, \mu) = -\frac{1}{\Gamma(\mu)}(\frac{d}{dz})^\mu (\ln(\Gamma(z + 1)))|_{z = z_0} \\
= -\frac{1}{2\pi} \frac{1}{\Gamma(\mu)} \int_{-\infty}^{\infty} (i\xi)^{\mu-1} \left\{ - \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+x} \right) - \gamma \right\} e^{i(z_0-x)\xi} dx d\xi
\]

For all except \( \mu = 1 \) we can omit the \( \delta_0(\xi) \) terms arising from the \( \frac{1}{n} \) and \( \gamma \) terms here and thus, by the Fourier identities canvassed in (v), we obtain
\[
d_{\Gamma}(z_0, \mu) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^{\infty} (i\xi)^{\mu-1} \sum_{n=1}^{\infty} -i\hat{H}_0(\xi)e^{in\xi} e^{iz_0\xi} d\xi \\
= \frac{1}{\Gamma(\mu)} \int_{0}^{\infty} (i\xi)^{\mu-1} \left( \frac{e^{i\xi} - 1}{e^{i\xi} - 1} \right) e^{iz_0\xi} d\xi
\]
on ignoring the contributions from the \( \frac{1}{\xi} \) term in \( \hat{H}_0(\xi) = H_0^+(\xi) - \frac{1}{\xi} \) as just discussed. Thus
\[
d_{\Gamma}(z_0, \mu) = \frac{i}{\Gamma(\mu)} \int_{0}^{\infty} (i\xi)^{\mu-2} e^{i(z_0+1)\xi} \left\{ 1 - \frac{(i\xi)}{2i} + \frac{(i\xi)^2}{3!} \right\} \left\{ 1 + \left( \frac{(i\xi)}{2i} + \frac{(i\xi)^2}{3!} + \ldots \right) + \right\} d\xi \\
= \frac{i}{\Gamma(\mu)} \int_{0}^{\infty} (i\xi)^{\mu-2} \left\{ 1 + i \left( \frac{(z_0 + \frac{1}{2})}{(z_0 + \frac{1}{2})} \right) \xi + \left( \frac{(z_0 + \frac{1}{2})}{(z_0 + \frac{1}{2})} \right) \xi^2 \right\} d\xi \\
= \frac{i}{\Gamma(\mu)} \int_{0}^{\infty} (i\xi)^{\mu-2} \left\{ 1 + \left( \frac{1}{2} \right) \xi + \left( \frac{1}{2} \right) \xi^2 \right\} d\xi
\]
Now suppose $\mu \in \mathbb{Z}_{\leq 0}$. Working distributionally as discussed earlier in (iii), since for $\mu \in \mathbb{Z}_{\leq 0}$ we have $\frac{\xi^{\mu - 2}}{\Gamma(\mu - 1)} = \delta_0^{(1-\mu)}(\xi)$ (by (31)), so for $\mu = 0$ we have immediately $LHS_{z_0,0} = z_0 + \frac{1}{2}$, in agreement with the first result from (37). Similarly, for $\mu = -1, -2$ and $-3$, after disentangling the impact of the $\frac{1}{\mu - 1}$ factor and the factorial arising from differentiation, we get agreement with the formulae in (37) (each arising as a scalar multiple of the appropriate coefficient of $\xi^{1-\mu}$ in the bracket); and in general

$$\forall \mu \in \mathbb{Z}_{\leq 0} \quad d\gamma(z_0, \mu) = r\gamma(z_0, \mu)$$

so that $\Gamma(z + 1)$ does satisfy the generalised root identities (27) for all $\mu \in \mathbb{Z}_{\leq 0}$.

Thus, overall, $\Gamma(z + 1)$ satisfies the generalised root identities (27) for arbitrary $z_0$ and $\mu \in \mathbb{Z}$; and in fact, as noted for $\mu \in \mathbb{Z}_{>0}$ in a previous section, it is easy to adapt the above arguments to verify that this remains true for $\Gamma(az + b)$ for any $a, b \in \mathbb{C}$.

To conclude our analysis of $\Gamma(z + 1)$ we illustrate the power of these new root identities for $\mu \in \mathbb{Z}_{\leq 0}$ to give information about asymptotic behaviour. We do this by constructively deducing Stirling’s theorem from successive consideration of the identities for $\mu = 0, -1, -2, \ldots$, namely:

**Stirling’s Theorem:** As $z \to +\infty$ we have

$$\Gamma(z + 1) = \sqrt{2\pi} z^{z + \frac{1}{2}} e^{-z} e^{J(z)}$$

where $J(z)$ has the asymptotic expansion

$$J(z) = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n-1) \cdot 2n} \frac{1}{z^{2n-1}} = \frac{1}{12} \frac{1}{z} - \frac{1}{360} \frac{1}{z^3} + \ldots$$

Equivalently, as $z \to +\infty$, we have asymptotically

$$\ln(\Gamma(z + 1)) = (z + \frac{1}{2}) \ln z - z + J(z)$$

To deduce this consider first the generalised root identity for $\mu = 0$. By (37) we have seen (from a Cesaro count of roots on the RHS) that this means

$$\lim_{\mu \to 0} \frac{-1}{\Gamma(\mu)} \left( \frac{d}{dz} \right)^\mu (\ln(\Gamma(z + 1))) \bigg|_{z=z_0} = \frac{1}{2}$$

As we move through $\mu = -1, -2, \ldots$ the corresponding polynomials in (37) are successive integrals of this (as one would expect from each additional application of $\left( \frac{d}{dz} \right)^{-1}$, i.e. integration, on the LHS), with the only new information each time being the value of the integration constant, namely $(-1)^{\mu+1} \zeta(\mu)$ (by (38)). Our requirement is thus to understand (a) what factors must occur in $\ln(\Gamma(z + 1))$ in order for the terms $z_0$ and $\frac{1}{2}$ to arise on the LHS in (42) when $\mu = 0$, and then (b) how the successive integration constants just identified can be made to appear
on the derivative side of the root identities (27) when \( \mu = -1, -2, \ldots \) by the inclusion of further terms without disturbing the identities already examined.

Now considering the definition in (30) we see that integer powers of \( z_0 \) will arise when the action of \( \frac{1}{2\pi i} \int_{-\infty}^{\infty} (i\xi)^\mu \mathcal{F}[g](\xi) \cdot d\xi \) on \( e^{iz_0\xi} \) consists of differentiation to some integer order. In light of (31) this means we require \( \xi^\mu \mathcal{F}[g](\xi) = \chi_+^{n}(\xi) = \frac{\Gamma(1-n)}{\Gamma(n)} \mathcal{H}_0(\xi) \) for some \( n \in \mathbb{Z}_{>0} \). We thus require \( \mathcal{F}[g](\xi) \) itself to be of the form \( C \frac{\mathcal{H}_0(\xi)}{\xi^n} \) for some suitable \( C \) and \( \rho \) (ignoring the contribution from the \( \frac{1}{2} \) in \( \tilde{H}_0(\xi) = \mathcal{H}_0^+(\xi) - \frac{1}{2} \) as usual). In particular it is clear that when \( \mu = 0 \), in order to get a constant term \( \frac{1}{2} \) we need to have \( \mathcal{F}[g](\xi) = \frac{1}{2} \mathcal{H}_0(\xi) \) (so that we get \( \frac{1}{2} \chi_+^{-1} \)), i.e. \( \frac{1}{2} \delta_0(\xi) \), acting on \( e^{iz_0\xi} \); while to get a term \( z_0 \) we need to have \( \mathcal{F}[g](\xi) = \frac{\mathcal{H}_0(\xi)}{\xi} \) (so that we get \( \chi_+^{-2} \), i.e. \( -\delta_0(\xi) \), acting on \( e^{iz_0\xi} \)). But, comparing with (32), it is clear that this means we need to take \( g(x) = (x \ln x) + \frac{1}{2} \ln x \) and so \( \frac{1}{2} \ln x = 1 \) must appear in \( \ln(\Gamma(z+1)) \).

These in turn generate the terms \( \frac{1}{2} \chi_+^{-1} \), \( \frac{1}{2} \delta_0(\xi) \), \( \frac{1}{2} \delta_0(\xi) \), \( \frac{1}{2} \delta_0(\xi) \), and so include an extra factor of \( e^{iz_0\xi} \) does not disturb the existing identity at \( \mu = 0 \) since there it leads to a finite Cesaro integral which is cancelled to 0 by the \( \frac{1}{\Gamma(\mu)} \) factor in the \( \mu = 0 \) identity.

So far we thus have contributions \( z + \frac{1}{2} e^{-1} e^{\frac{\pi a}{4 \pi + 2}} \) in \( \Gamma(z+1) \), thereby ensuring the \( \mu = 0 \) and \( \mu = -1 \) identities are satisfied. In the same way, working inductively, for \( \mu = -n \) the existing terms generated up to \( \mu = -n+1 \) will integrate to yield the polynomial \( (-1)^n b_{n+1}(z_0) \) in the expression (38) for the \( \mu = -n \) root identity (after adjusting for the change in \( \frac{1}{\Gamma(\mu)} \) factor on the derivative side) and we need to introduce a further term \( e^{\frac{\pi a}{4 \pi + 2}} \) in order to match the new constant term \( -(-1)^n \zeta(-n) \) which is all that remains unaccounted for. But then taking logs yields \( g(z) = \frac{a}{2\pi i} \) with \( \mathcal{F}[g](\xi) = 2\pi ae^{-i\frac{\pi}{4\pi + 2}} \mathcal{H}_0(\xi) e^{\xi^{n-1}} \) and so \( -\frac{1}{\Gamma(\mu)} \frac{1}{2\pi i} (i\xi)^\mu \mathcal{F}[g](\xi) = -a \cdot \frac{1}{\Gamma(\mu)} \frac{1}{2\pi i} = -an \delta_0(\xi) \) after simplifying the two \( \Gamma \) terms by cancellation (or alternatively using the functional identity (16)) and retaining only the \( H_0(\xi) \) term from \( \tilde{H}_0(\xi) \) as usual. Acting on \( e^{iz_0\xi} \), we thus need to take \( a = (-1)^n \zeta(-n) \) in order to match the required constant term.

When \( n \) is even (i.e. \( \mu = -2, -4, \ldots \)) we are at a trivial zero of \( \zeta \) and \( a = 0 \), so that there are no even order reciprocal terms in the asymptotic expansion for \( J(z) \); when \( n = 2k - 1 \) is odd we know that \( \zeta(-2k-1) = -\frac{B_{2k}}{2k} \) so that
yields an integrand of derivative side of the root identity, after the usual passage from the coefficient

\[ \text{ln} C \]

identities for \( \Gamma(\mu) \) for \( \mu \in \mathbb{Z}_{\leq 0} \) respectively to deduce integral (because log-divergent) in its action on \( e^{\pi z} \). Letting \( z = k \to \infty \) through the positive integers, taking logs and noting that \( J(k) \to 0 \) as \( k \to \infty \) we get

\[ \ln C = \lim_{k \to \infty} \left\{ \ln(k!) - \left( (k + \frac{1}{2}) \ln k - k \right) \right\} = \lim_{k \to \infty} \left( \sum_{j=1}^{k} \ln j - \left( (k + \frac{1}{2}) \ln k - k \right) \right) \]

But this last expression was evaluated in [1] (in the course of calculating \( \zeta'(0) \)) as precisely \( \frac{1}{2} \ln(2\pi) \) and so \( C = \sqrt{2\pi} \) as required.

This completes the constructive derivation of the expression in (39) purely from the root identities for \( \mu \in \mathbb{Z}_{\leq 0} \) satisfied by \( \Gamma(z + 1) \). To complete the proof of Stirling's theorem it remains only to verify that no further terms of the form \( e^{\pi z n} \), \( n \in \mathbb{Z}_{\geq 1} \), are required in order to also satisfy the root identities for \( \mu \in \mathbb{Z}_{\geq 1} \), i.e. that the expression \( \sqrt{2\pi} z^{z+\frac{1}{2}} e^{-z} e^{J(z)} \) already satisfies all these root identities for \( \Gamma(z + 1) \) without requiring the healing of any obstructions.

For \( \mu = 1 \) we immediately encounter a problem from the term \( z^{\frac{1}{2}} e^{-\frac{1}{2}} \); on the derivative side of the root identity, after the usual passage from \( H_0 \) to \( H_0^+ \) this yields an integrand of \( \frac{H^+_0(\xi)}{\xi} \cdot (i\xi) = -H^+_0(\xi) \) and this leads to a Cesaro divergent integral (because log-divergent) in its action on \( e^{i\zeta_0 \xi} \) since we no longer have the coefficient \( \frac{1}{\Gamma(1)} \) to normalise this into a \( \delta \)-function. However, this merely mirrors the corresponding Cesaro divergence of \( \sum \frac{1}{n!} \) on the root side, for which we needed to perform the renormalisation described earlier in verifying the \( \mu = 1 \) root identity; a corresponding renormalisation is of course required on the derivative side.

With this in mind we proceed with the other terms: \( z^{\frac{1}{2}} \) gives integrand contribution \( -\frac{1}{2} H^+_0(\xi) \), and each term \( e^{\frac{i\pi}{2k(2k-1)}} \cdot (-i)^{2k-1} \frac{\xi^{2k-2}}{(2k-2)!} (i\xi) = -i^{2k} \frac{B_{2k}}{(2k)!} \xi^{2k-1} \) (by (32)). Thus, for \( \mu = 1 \), on the derivative side of the root identity we get

\[
- \frac{1}{\Gamma(1)} \int_0^\infty \left\{ -\frac{1}{\xi} \frac{i}{2} - \sum_{k=1}^\infty \frac{B_{2k}}{(2k)!} \xi^{2k-1} \right\} e^{i\zeta_0 \xi} d\xi
= i \int_0^\infty \left\{ \frac{1}{i\xi} + \frac{i}{2} + i \sum_{j=1}^\infty \frac{B_{2k}}{(2k)!} (i\xi)^{2k-1} \right\} e^{i\zeta_0 \xi} d\xi
\]
Recalling the generating function for the Bernoulli numbers in (2) we see that this becomes

\[ i \int_{0}^{\infty} \left\{ \frac{1}{e^{i\xi} - 1} + 1 \right\} e^{iz_0 \xi} d\xi = i \int_{0}^{\infty} \left\{ -\sum_{j=1}^{\infty} e^{ij\xi} \right\} e^{iz_0 \xi} d\xi \]

\[ = -i \int_{0}^{\infty} \left\{ \sum_{j=1}^{\infty} e^{i(z_0+j)\xi} \right\} d\xi \]

\[ = -i \sum_{j=1}^{\infty} \left( \frac{-1}{i(z_0+j)} \right) = \sum_{j=1}^{\infty} \frac{1}{z_0 + j} \]

where the last step follows from a direct Cesaro computation of the integrals. Since this formally equals the root side of the \( \mu = 1 \) root identity we see that, (up to renormalisation on both sides), \( \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} e^{J(z)} \) does indeed satisfy the root identity for \( \Gamma(z+1) \) without requiring any further correction terms of the form \( e^{az} \).

For \( \mu = 2 \) and higher, no renormalisation issues arise. At \( \mu = 2 \), on the derivative side, \( z^2 e^{-z} \) contributes \( \frac{-1}{2z_0} \), \( z^2 \) contributes \( \frac{1}{2z_0} \), \( e^{-z} \) makes no contribution and \( e^{\mu \frac{B_{2k}}{(z_0 + j)^{2k}}} \) contributes \( -B_{2k} \frac{1}{(z_0 + j)^{2k}} \) after 2\( k \)-fold Cesaro integration by parts. We thus end up with \( \frac{1}{z_0} + \frac{1}{2z_0} - \sum_{m=1}^{\infty} \frac{B_{2m}}{z_0^{2m+1}} \) which is a well-known expression for \( -\frac{d^2}{dz^2} (\ln(\Gamma(z+1))) \mid_{z=z_0} \). Since we have already verified that this equals the root side of the generalised root identity for \( \Gamma(z+1) \) (i.e. that \( \Gamma(z+1) \) does satisfy the \( \mu = 2 \) root identity) so it follows again that \( \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} e^{J(z)} \) satisfies the \( \mu = 2 \) root identity without need of any further correction factor \( e^{az^2} \). Since this is true for arbitrary \( z_0 \) and without need of renormalisation, so it follows also for all \( \mu \in \mathbb{Z}_{\geq 3} \) and thus Stirling’s theorem is finally proven.

One comment is worth noting regarding this constructive argument. Stirling’s theorem actually expresses the asymptotic behaviour of \( \Gamma(z+1) \) only as \( z \to +\infty \). The argument above, however, is naturally “two-sided” and suggests that \( \Gamma(z+1) \) is asymptotically given by \( \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} e^{J(z)} \) also as \( z \to -\infty \), since in picking out the form of functions in \( \xi \) so as to match the root side of our identities on the derivative side, we have had to call on functions (like \( z \ln z - z, \ln z, \frac{1}{z} \) etc) which are defined on all of \( \mathbb{R} \), not just \( (0, \infty) \) or \( (-\infty, 0) \).

To check this, we can use the logarithmic version of the functional equation for \( \Gamma \) in (17) to deduce the asymptotic behaviour of \( \ln(\Gamma(z+1)) \) as \( z \to -\infty \) from its known behaviour as \( z \to \infty \) from Stirling’s theorem. We have, as \( z \to -\infty \),

\[ \ln(\Gamma(z+1)) = (\ln \pi + i\pi) - \ln(\sin(\pi z)) - \ln(\Gamma(-z)) \]

and, by Stirling’s theorem,
\[-\ln(\Gamma(-z)) = -\ln(\Gamma(|z|)) = -\ln(\Gamma(|z| - 1 + 1))\]

\[= -\left\{ (|z| - \frac{1}{2}) \ln(|z| - 1) - (|z| - 1) + \frac{1}{12} \frac{1}{(|z| - 1)^2} + \ldots \right\} \]

\[= \left\{ (z + \frac{1}{2}) \left\{ \ln|z| - \frac{1}{|z|} - \frac{1}{2 |z|^2} - \frac{1}{3 |z|^3} - \ldots \right\} - (z + 1) \right\} \]

\[= \left\{ -(z + 1) + \frac{1}{12} \frac{1}{z} \left[ 1 - \frac{1}{z} + \frac{1}{z^2} - \ldots \right] \right\} \]

\[= -(z + \frac{1}{2}) \cdot i\pi + \left\{ (z + \frac{1}{2}) \ln z - z + \frac{1}{12} \frac{1}{z} \right\} \]

after simplification. Thus, overall, as \(z \to -\infty\), we have

\[\ln(\Gamma(z + 1)) = \left\{ (z + \frac{1}{2}) \ln z - z + \frac{1}{12} \frac{1}{z} \right\} - (z + \frac{1}{2}) \cdot i\pi \]

\[+ (\ln \pi + i\pi) - \ln(\sin(\pi z)) \quad (43)\]

We see that, as predicted, we do have the same asymptotic behaviour as per Stirling’s theorem also when \(z \to -\infty\), except that in addition we have the linear and constant terms shown \((z + \frac{1}{2}) \cdot i\pi\) and \((\ln \pi + i\pi)\) respectively and the term \(\ln(\sin(\pi z))\) which contains a countable collection of ln-divergences at the negative integer points.

The reason our above derivation of Stirling’s theorem nonetheless worked in spite of these extra one-sided terms as \(z \to -\infty\) is that, owing to the \(\frac{1}{\Gamma(\mu)}\) factors on the derivative side of our root identities, the contributions made by these terms on this derivative side for \(\mu \in \mathbb{Z}_{\leq 0}\) are 0. We shall omit calculations demonstrating this here, but note that in general terms this reflects the fact that it is asymptotic behaviour within a log-scale, rather than local divergences (even a countable collection of them), that matter in the computation of the derivative side of the root identities for \(\mu \in \mathbb{Z}_{\leq 0}\).
4 The Generalised Root Identities for $\zeta$

In the case of $\zeta$, of course, we do not know the location of the non-trivial roots in advance and so our perspective must change. Unlike for $\Gamma$, where we calculated values for the root-sides of the generalised root identities by Cesaro means and used these to infer information about asymptotic behaviour of $\Gamma$ from the derivative sides (e.g. Stirling’s theorem), we now aim to calculate the derivative sides of the generalised root identities for $\mu \in \mathbb{Z}_{\leq 0}$ and thereby to investigate the location of the roots of $\zeta$ by utilising Cesaro methods on the corresponding root sides, noting in particular that these Cesaro methods naturally depend critically on the geometric location of summands.

4.1 The Derivative Sides of the Root Identities for $\zeta$

In this case, for $\zeta$, it turns out that there is no need to do detailed calculations on the derivative side using the Fourier definition; all that is required is the property (28) of $\left( \frac{d}{ds} \right)^n$. Recall the Euler product formula for $\zeta$, namely

$$\zeta(s) = \prod_{\text{pprime}} (1 - p^{-s})^{-1}$$

which is convergent for $\Re(s) > 1$. It follows that

$$\ln(\zeta(s)) = - \sum_{\text{pprime}} \ln(1 - p^{-s})$$

and so, in light of property (28), we have that for $\Re(s_0) > 1$ the derivative side of the root identity for $\zeta$ at $\mu$ is given by

$$d_\zeta(s_0, \mu) = - \frac{e^{i\pi \mu}}{\Gamma(\mu)} \sum_{\text{pprime}} (\ln p)^\mu \left\{ p^{-s_0} + 2^{\mu-1} p^{-2s_0} + 3^{\mu-1} p^{-3s_0} + \ldots \right\}$$

This expression is clearly convergent for arbitrary $\mu$ for $\Re(s_0) > 1$ and so gives an expression for the derivative sides of the root identities for $\zeta$ for arbitrary $\mu \in \mathbb{C}$ and $\Re(s_0) > 1$, which can then be extended also to $\Re(s_0) \leq 1$ by unique analytic continuation.

Since, for arbitrary $\Re(s_0) > 1$ the sum in (46) converges to some finite value as $\mu \to n$ for any $n \in \mathbb{Z}_{\leq 0}$, and since $\Gamma$ has simple poles at all the non-positive integers, so a corollary of the result in (46) is the following:
Lemma 4: When $\mu \in \mathbb{Z}_{\leq 0}$ the derivative sides of the generalised root identities for $\zeta$ are all identically zero as functions of $s_0$; that is

$$d(s_0, \mu) = 0 \quad \forall s_0 \quad \forall \mu = 0, -1, -2, \ldots$$

(47)

Since $\zeta$ satisfies the generalised root identities, we should therefore have also

$$r_\zeta(s_0, \mu) = 0 \quad \forall s_0 \quad \forall \mu = 0, -1, -2, \ldots$$

(48)

4.2 The Root Sides of the Root Identities for $\zeta$ for $\mu = 0, -1$ and $-2$

We now try to calculate the root sides of the generalised root identities for $\zeta$ for the cases $\mu = 0, -1$ and $-2$ in order to verify (48) in these cases.

4.2.1 Case (a): $\mu = 0$:

First we consider the trivial roots $T$. For $s_0 = 0$ we get the Cesaro sum of $1's$ placed at the points $2, 4, 6, \ldots$. By dilation invariance, this equals the Cesaro sum of $1's$ placed at the points $1, 2, 3, \ldots$ and this is familiar from [1] as giving $\zeta(0) = -\frac{1}{2}$. For $s_0 = 2k$ this becomes instead the Cesaro sum of $1's$ placed at the points $2k + 2, 2k + 4, 2k + 6, \ldots$ which thus immediately leads to the value $-\frac{1}{2} - k$, and so in general

$$\sum_{\{s_0 - T\}} M_i(s_0 - r_i)^0 = -\frac{1}{2} - \frac{1}{2} s_0$$

(49)

Next, from the simple pole at $s = 1$ we get contribution $-1$ (since $M_i = -1$ here).

Now, for the contribution from the non-trivial roots, $NT$, in the critical strip we recall the explicit form of the Riemann-von Mangoldt counting function, $N(T)$, which counts roots $\rho_i = \beta_i + i\gamma_i$ with $0 < \gamma_i < T$. By Karatsuba-Korolev [5], this has the explicit form

$$N(T) = \tilde{N}(T) + S(T) + \frac{1}{\pi} \delta(T)$$

(50)

where

$$\tilde{N}(T) = \frac{T}{2\pi} \ln\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8}$$

(51)

and $S(T)$ is the famous argument of the zeta function (see e.g. [5] for formal definition), and

---

7The reasoning in going from $s_0 = 2k$ to arbitrary $s_0$ to derive (49) here is deliberately heuristic, for reasons of brevity, but the result & similar deductions for the trivial root contributions when $\mu = -1$ and $\mu = -2$ are easy to deduce rigorously by Cesaro means if desired.
\[ \delta(T) = \frac{T}{4} \ln \left( 1 + \frac{1}{4T^2} \right) + \frac{1}{4} \tan^{-1} \left( \frac{1}{2T} \right) - \frac{T}{2} \int_0^\infty \left( \frac{\frac{1}{2} - \{u\}}{(u + \frac{1}{2})^2 + \left( \frac{\{u\}}{2} \right)^2} \right) du \tag{52} \]

with \( \{u\} := u - \text{Floor}(u) \) being the saw-tooth function which rises linearly from 0 to 1 on each integer interval \([k, k+1)\).

Clearly for the \( \mu = 0 \) root identity we are interested in finding the Cesaro limit of \( N(T) \) as \( T \to \infty \), at least after combining with the corresponding calculation for the roots below the real axis; i.e., writing \( \tilde{T} \) for the parameter tracing our count of roots with negative imaginary parts, we have

\[ \sum_{\{s_0 - NT\}} M_i(s_0 - \rho_i)^0 = \text{Clim}\{N(T) + N(\tilde{T})\} \tag{53} \]

Now, by [5] we know that \( \delta(T) = O(\frac{1}{T}) \), and by either [5] or Gordon-Sabeh in [6] we also know that if we define \( S_1(T) := \int_0^T S(t)dt \) in the usual way, then

\[ S_1(T) = O(\ln T) \tag{54} \]

so that \( P[S](T) = \frac{S_1(T)}{T} \to 0 \) and \( \text{Clim} S(T) = 0 \). Thus (53) reduces to

\[ \sum_{\{s_0 - NT\}} M_i(s_0 - \rho_i)^0 = \text{Clim}\{\tilde{N}(T) + \tilde{N}(\tilde{T})\} \tag{55} \]

As we have seen, however, in considering this Cesaro limit, we need to be adding in 1 for each root counted, not at the root \( \rho_i \) itself but at \( s_0 - \rho_i \), and this geometry will be critical in our computations, as will the need to combine \( \tilde{N}(T) \) and \( \tilde{N}(\tilde{T}) \) before taking Cesaro limits (in particular we will shortly need this to resolve an apparent paradox between results for \( \mu = 0 \) and \( \mu = -1 \)). To reflect this geometry we thus write

\[ z = s_0 - \left( \frac{1}{2} + iT \right) \quad \text{and} \quad \tilde{z} = s_0 - \left( \frac{1}{2} - iT \right) \tag{56} \]

so that

\[ T = i(z - (s_0 - \frac{1}{2})) \quad \text{and} \quad \tilde{T} = -i(\tilde{z} - (s_0 - \frac{1}{2})) \tag{57} \]

and therefore

\[ \ln T = \frac{i \pi}{2} + \ln z - \frac{(s_0 - \frac{1}{2})}{z} - \frac{1}{2} \frac{(s_0 - \frac{1}{2})^2}{z^2} - \frac{1}{3} \frac{(s_0 - \frac{1}{2})^3}{z^3} - \ldots \tag{58} \]

and

\[ \ln \tilde{T} = \frac{-i \pi}{2} + \ln \tilde{z} - \frac{(s_0 - \frac{1}{2})}{\tilde{z}} - \frac{1}{2} \frac{(s_0 - \frac{1}{2})^2}{\tilde{z}^2} - \frac{1}{3} \frac{(s_0 - \frac{1}{2})^3}{\tilde{z}^3} - \ldots \tag{59} \]
Here we are for the moment assuming that, at least for purposes of calculation of Cesaro limits, we may treat the parameters \( z \) and \( \tilde{z} \) as running purely along the critical line in enumerating the non-trivial zeros. We shall discuss this assumption (which would of course be justified if the Riemann hypothesis is true, and may be justified in any case by symmetry of roots around the critical line) further later.

Thus we have

\[
\frac{T}{2\pi} \ln\left(\frac{T}{2\pi}\right) = \frac{i}{2\pi}(z - (s_0 - \frac{1}{2})) \left\{ \ln\left(\frac{z}{2\pi}\right) + \frac{\pi}{2} - \frac{(s_0 - \frac{1}{2})}{z} - \frac{1}{2} \frac{(s_0 - \frac{1}{2})^2}{z^2} \right\} - \frac{1}{3} \frac{(s_0 - \frac{1}{2})^3}{z^3} - \ldots
\]

Thus we have

\[
\frac{T}{2\pi} \ln\left(\frac{T}{2\pi}\right) = \frac{i}{2\pi}(z - (s_0 - \frac{1}{2})) \left\{ \ln\left(\frac{z}{2\pi}\right) - \frac{\pi}{2} \left(\frac{z}{2\pi}\right) - \frac{i}{2\pi}(s_0 - \frac{1}{2}) \ln\left(\frac{z}{2\pi}\right) \right. \\
\left. + \left(\frac{(s_0 - \frac{1}{2})}{4} - \frac{i}{2\pi}\frac{(s_0 - \frac{1}{2})}{2\pi}\right) + o(1) \right. \\
\]

and therefore

\[
\frac{T}{2\pi} \ln\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + 7\frac{7}{8} = \frac{i}{2\pi}(z - (s_0 - \frac{1}{2})) \left\{ \ln\left(\frac{z}{2\pi}\right) - \left(\frac{\pi}{2} + i\right)\left(\frac{z}{2\pi}\right) - \frac{i}{2\pi}(s_0 - \frac{1}{2}) \ln\left(\frac{z}{2\pi}\right) \right. \\
\left. + \left(\frac{1}{4}s_0 + \frac{3}{4}\right) + o(1) \right. \\
\]

We see that taking care to place the NT roots relative to \( s_0 \) by considering \( z \) rather than \( T \) introduces the extra terms of \( \frac{1}{2}s_0 \) and \( -\frac{1}{8} \) (which cancels the \( \frac{7}{8} \) to \( \frac{3}{4} \)) in the constant term. If we add an equivalent constant-term contribution from \( \tilde{N}(\tilde{T}) \), arising from the roots in NT below the real axis, we get a constant term of \( \frac{1}{2}s_0 + \frac{3}{4} \), precisely as required to cancel the contributions from the trivial roots in (49) and from the simple pole and leave the root side of the \( \mu = 0 \) root identity with value 0 as hoped - i.e.

\[
r_\zeta(s_0, 0) = \sum_{\{s_0-\text{roots of } \zeta\}} M_i(s_0 - r_i)^0 = 0
\]

in agreement with (48).

However, there is a subtlety here and we must be careful. If we consider (60) alone then, recalling the dilation invariance of Cesaro limits (so that \( z \mapsto \frac{z}{2\pi} \) leaves limits unchanged) and the fact that \( z \ln z \) and \( z \) are both pure Cesaro eigenfunctions (with eigenvalue \( \frac{1}{2} \)) and thus have generalised Cesaro limit 0, it follows that

\[
\text{Clim}_{z \to \infty} \left\{ i\left(\frac{z}{2\pi}\right) \ln\left(\frac{z}{2\pi}\right) - \left(\frac{\pi}{2} + i\right)\left(\frac{z}{2\pi}\right) \right\} = 0 \\
\]

(61)
But the term \(-\frac{i}{2\pi}(s_0 - \frac{1}{2})\ln(\frac{z}{\tilde{z}})\) has a pure log-divergence which means that, on its own, it has no generalised Cesaro limit. Here, however, we see again the importance of being careful in both the geometric placement of roots and in considering the sum of \(\hat{N}(T)\) and \(\hat{N}(\tilde{T})\) before taking the Cesaro limit, with \(\tilde{T}\) critically being treated as independent of \(T\). Specifically, if we carefully mimic the calculations for (60) for \(\hat{N}(\tilde{T})\) in terms of \(\tilde{z}\) instead, taking account of the sign changes in (57) and (59) vis-a-vis the corresponding results for \(z\), we end up with the corresponding relation

\[
\frac{T}{2\pi}\ln\left(\frac{T}{2\pi}\right) - \frac{\tilde{T}}{2\pi} + \frac{7}{8} = -i\left(\frac{\tilde{z}}{2\pi}\right)\ln\left(\frac{\tilde{z}}{2\pi}\right) - \left(\frac{\pi}{2} - i\right)\left(\frac{\tilde{z}}{2\pi}\right) + i\left(\frac{1}{2}\right)\ln\left(\frac{\tilde{z}}{2\pi}\right)
\]

\[
+\left(\frac{1}{4}s_0 + \frac{3}{4}\right) + o(1)
\]  

(62)

In (55) we thus have, on noting (61) and its analogue for \(\tilde{z}\), that

\[
\sum_{\{s_0 - NT\}} M_i(s_0 - \rho_i)^0 = \frac{1}{2}s_0 + \frac{3}{2} - Clim_{z,\tilde{z} \to \infty} \left\{ \frac{i}{2\pi}(s_0 - \frac{1}{2})\ln\left(\frac{z}{\tilde{z}}\right) \right\}
\]  

(63)

As discussed, on combining with the results for \(T\) and for the simple pole, we will thus have the desired result that, for \(\mu = 0\),

\[
r_\zeta(s_0, 0) = \sum_{\{s_0 - roots of \zeta\}} M_i(s_0 - r_i)^0 = 0 \quad \forall s_0
\]

if and only if we have that

\[
Clim_{z,\tilde{z} \to \infty} \ln\left(\frac{z}{\tilde{z}}\right) = 0
\]  

(64)

But this is true in spite of the non-Cesaro-convergence of either \(\ln z\) or \(\ln \tilde{z}\) in the 1-d Cesaro setting because, as a 2-d function, \(\ln(z)\) can be ascribed a Cesaro limit and this limit is 0. To see this we first observe that \(f(z, \tilde{z}) = \ln(z)\) is a direct eigenfunction of the 2-d Cesaro operator, \(P_{2d}\), with eigenvalue 1, since

\[
P_{2d}[\ln\left(\frac{z}{\tilde{z}}\right)] = P_{2d}[\ln z] - P_{2d}[\ln \tilde{z}] = (\ln z - 1) - (\ln \tilde{z} - 1) = \ln\left(\frac{z}{\tilde{z}}\right)
\]

As such it can have a 2-d Cesaro limit, although the value of this limit is not directly clear. But then further note that since \(\ln(z)\) and \(\ln(\tilde{z})\) should have the same limit, by symmetry, yet \(\ln(z) = -\ln(\tilde{z})\), so this limit must in fact be 0.

It follows at last that we have verified (48) and the generalised root identity for \(\zeta\) in the case \(\mu = 0\) by direct computation on the root side from the known form of \(N(T)\) in (56) - (59). This confirms that in this case the Cesaro count of all roots (T, pole and NT) of \(\zeta\) is identically 0 for all \(s_0\) and has been achieved by showing that
\[
\sum_{\{s_0 - NT\}} M_i(s_0 - \rho_i)^0 = \frac{1}{2}s_0 + \frac{3}{2}
\]  

(65)

in a Cesaro sense.\(^8\)

In passing we acknowledge that the above argument regarding (64) is somewhat heuristic. This is discussed further in section 4.3 and [8], but we do note that, if desired, we could change perspective and instead view (64) as an implication of \( \zeta \) satisfying the \( \mu = 0 \) root identity (rather than seeking to use (64) to verify this). This would at least then allow us to legitimately use (64) again in our next calculations for the \( \mu = -1 \) and \( \mu = -2 \) root identities, where it reappears.

We thus turn now to the case of \( \mu = -1 \). We will see that a naive calculation on the root side leads to an apparent paradox unless we are careful in again taking account of the geometric locations of \( z \) and \( \tilde{z} \) and of their independence. But doing so we can simultaneously resolve this paradox and verify that, on the root-side of the \( \mu = -1 \) root identity for \( \zeta \) we also have

\[
r_\zeta(s_0, -1) = \sum_{\{s_0 - \text{roots of } \zeta\}} M_i(s_0 - r_i) = 0 \quad \forall s_0
\]

(66)

as claimed in (48), at least modulo an estimate on \( S(T) \) which certainly holds conditional on RH.

4.2.2 Case (b): \( \mu = -1 \):

In this case, for the trivial roots \( T \), for \( s_0 = 0 \) we get the Cesaro sum of \( 2, 4, 6, \ldots \) placed at the points \( 2, 4, 6, \ldots \). By dilation invariance, this equals the Cesaro sum of \( 2, 4, 6, \ldots \) placed at the points \( 1, 2, 3, \ldots \) and this is familiar from [1] as giving \( 2\zeta(-1) = -\frac{1}{6} \). For \( s_0 = 2k \) this becomes instead the Cesaro sum of \( 2k + 2, 2k + 4, 2k + 6, \ldots \) placed at the points \( 2k + 2, 2k + 4, 2k + 6, \ldots \) which thus immediately leads to the value \(-\frac{1}{6} - k^2 - k\), and so in general

\[
\sum_{\{s_0 - T\}} M_i(s_0 - r_i)^1 = -\frac{1}{4}s_0^2 - \frac{1}{2}s_0 - \frac{1}{6}
\]

(67)

From the simple pole at \( s = 1 \) we get contribution

\[
\sum_{\{s_0 - \text{pole}\}} M_i(s_0 - r_i)^1 = -1 \cdot (s_0 - 1) = 1 - s_0
\]

(68)

Thus, overall, from the combination of the trivial roots and the simple pole we get a total contribution to the root side of the \( \mu = -1 \) root identity for \( \zeta \) of

\[
\sum_{\{s_0 - T \cup \text{pole}\}} M_i(s_0 - r_i)^1 = -\frac{1}{4}s_0^2 - \frac{3}{2}s_0 + \frac{5}{6}
\]

(69)

\(^8\)Note as an aside that it follows trivially from the fact that the count of non-trivial roots in (65) is generically non-integral that there must be infinitely many non-trivial roots of \( \zeta \).
Turning now to the contribution from the non-trivial roots we have

\[
\sum_{\{s_0-NT\}} M_i(s_0 - \rho_i)^1 = \sum_{\{s_0-NT\}} M_i(s_0 - \beta_i - i\gamma_i) \\
= \sum_{\{s_0-NT\}} M_i(s_0 - \beta_i) - i \cdot \sum_{\{s_0-NT\}} M_i\gamma_i \\
= (s_0 - \frac{1}{2}) \sum_{\{s_0-NT\}} M_i - i \cdot \sum_{\{s_0-NT\}} M_i\gamma_i \\
= \left( \frac{1}{2} s_0^2 + \frac{5}{4} s_0 - \frac{3}{4} \right) - i \cdot \sum_{\{s_0-NT\}} M_i\gamma_i
\]

(70)

where here we have invoked the symmetry of NT roots w.r.t. the critical line to take \((s_0 - \beta_i)\) outside the summation as \((s_0 - \frac{1}{2})\) and then utilised result (65) from the previous \(\mu = 0\) computation. In these terms, the paradox alluded to earlier amounts to claiming naively that, because NT roots also occur in conjugate pairs symmetrically w.r.t. the real axis, so for real \(s_0\) their contributions should cancel the remaining sum on the RHS in (70) to 0, thus leaving

\[
\sum_{\{s_0-NT\}} M_i(s_0 - \rho_i)^1 = \left( \frac{1}{2} s_0^2 + \frac{5}{4} s_0 - \frac{3}{4} \right) - i \cdot \sum_{\{s_0-NT\}} M_i\gamma_i
\]

This does not cancel the contribution in (69) to 0 \(\forall s_0\) and would thus leads to a contradiction of (48) and the generalised root identity for \(\zeta\) when \(\mu = -1\).

However this paradox is resolved by recalling that \(z\) and \(\tilde{z}\) must be treated as independent; and since the contributions to the sum in (70) from roots above the real axis and from roots below it are both individually classically divergent, we must work very carefully within a Cesaro framework to deduce the true Cesaro sum of \(\sum_{\{s_0-NT\}} M_i\gamma_i\), rather than rely on naive pairwise cancellations. When we do this we will find that in fact the contribution from \(\sum_{\{s_0-NT\}} M_i\gamma_i\) is exactly as required to validate that, overall, on the root side of the \(\mu = -1\) root identity for \(\zeta\) we do have

\[
r_\zeta(s_0, -1) = \sum_{\{s_0-\text{roots of } \zeta\}} M_i(s_0 - r_i)^1 = 0 \quad \forall s_0
\]

and thus (48) and the generalised root identities remain true for \(\mu = -1\).

To see this, first note that for roots above the real axis we have the partial sum function of \(\sum M_i\gamma_i\) given by \(\int_0^T u dN(u)\) and similarly for roots below the axis. Thus, after accounting for the difference in sign of \(\gamma_i\) in the two cases, we get in (70) that

\[
\sum_{\{s_0-NT\}} M_i(s_0 - \rho_i)^1 = \left( \frac{1}{2} s_0^2 + \frac{5}{4} s_0 - \frac{3}{4} \right) - i \lim_{\tilde{z} \to \infty} \left\{ \int_0^T u dN(u) - \int_0^{\tilde{z}} udN(u) \right\}
\]

(71)

Now we shall adopt the notation that
\[ S_0(T) := S(T), \quad S_1(T) = \int_0^T S_0(t)dt, \quad S_2(T) = \int_0^T S_1(t)dt, \ldots \]

and adapt it identically to define

\[ N_0(T) := N(T), \quad \text{and} \quad N_i(T) = \int_0^T N_{i-1}(t)dt, \quad \forall i \in \mathbb{Z}_{>0}, \]

\[ \hat{N}_0(T) := \hat{N}(T), \quad \text{and} \quad \hat{N}_i(T) = \int_0^T \hat{N}_{i-1}(t)dt, \quad \forall i \in \mathbb{Z}_{>0}, \]

\[ \delta_0(T) := \delta(T), \quad \text{and} \quad \delta_i(T) = \int_0^T \delta_{i-1}(t)dt, \quad \forall i \in \mathbb{Z}_{>0} \]

for the functions introduced in (56).

In this notation, in (71) we get (on noting that all of \( N, \hat{N}, S, \delta \) have finite limits as \( t \to 0^+ \), with \( N(0) = 0 \)) that

\[ \int_0^T u \, dN(u) = T N_0(T) - N_1(T) \]

\[ = \{ T \hat{N}_0(T) - \hat{N}_1(T) \} + \{ T S_0(T) - S_1(T) \} + \frac{1}{\pi} \left\{ T \delta_0(T) - \delta_1(T) \right\} \]

(72)

and similarly for \( \int_0^T \tilde{u} \, dN(\tilde{u}) \). Now

(i) Suppose we have the estimate that

\[ S_2(T) = o(T) \]

(73)

Then since, in general,

\[ P[S_n](T) = \frac{S_{n+1}(T)}{T} \]

(74)

it would follow immediately that \( P[S_1](T) \to 0 \) as \( T \to \infty \) so that

\[ \lim S_1(T) = 0 \]

(75)

Also, using integration by parts and noting \( S_1(0) = 0 \) by definition, we would have

\[ P[tS_0(t)](T) = \frac{1}{T} \int_0^T tS_0(t)dt = S_1(T) - \frac{1}{T} \int_0^T S_1(t)dt \]

\[ = S_1(T) - P[S_1](T) \]

\[ \lesssim 0 \]

(76)
Thus, combining (75) and (76) it follows in (72) that, conditional on the estimate (73), we have no contribution from the $S(T)$-related terms, i.e. $\text{Clim}(TS_0(T) - S_1(T)) = 0$, and thus

$$\int_0^T udN(u) = \{T\hat{N}_0(T) - \hat{N}_1(T)\} + \frac{1}{\pi} \{T\delta_0(T) - \delta_1(T)\} \quad (77)$$

and similarly for $\int_0^T \tilde{u}dN(\tilde{u})$.

Unfortunately, an unconditional estimate of the form (73) is not known for $S_2(T)$. We do know, however, that a much stronger estimate for all $S_n(T)$ is in fact known, conditional on the Riemann hypothesis (RH), namely (see Titchmarsh in [7, pg 354])

$$S_n(T) = O\left( \frac{\ln T}{(\ln \ln T)^{n+1}} \right) \quad \forall n \in \mathbb{Z}_{\geq 0} \quad (78)$$

As such, we will proceed here on the assumption of (73) and (77) and thus end up obtaining a calculation of $r_{\zeta}(s_0, -1)$ conditional on (73), which is itself implied by RH.

(ii) From the definition of $\hat{N}(T) := \frac{T}{2\pi} \ln\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{7}{8}$ we clearly have

$$T\hat{N}(T) = 2\pi \left\{ \left( \frac{T}{2\pi} \right)^2 \ln\left(\frac{T}{2\pi}\right) - \left( \frac{T}{2\pi} \right)^2 + \frac{7}{8} \left( \frac{T}{2\pi} \right) \right\} \quad (79)$$

and

$$\hat{N}_1(T) = \int_0^T \left\{ \frac{t}{2\pi} \ln\left(\frac{t}{2\pi}\right) - \frac{t}{2\pi} + \frac{7}{8} \right\} dt$$

$$= 2\pi \left\{ \frac{1}{2} \left( \frac{T}{2\pi} \right)^2 \ln\left(\frac{T}{2\pi}\right) - \frac{3}{4} \left( \frac{T}{2\pi} \right)^2 + \frac{7}{8} \left( \frac{T}{2\pi} \right) \right\} \quad (80)$$

Thus

$$T\hat{N}_0(T) - \hat{N}_1(T) = 2\pi \left\{ \frac{1}{2} \left( \frac{T}{2\pi} \right)^2 \ln\left(\frac{T}{2\pi}\right) - \frac{1}{4} \left( \frac{T}{2\pi} \right)^2 \right\} \quad (81)$$

and in light of (56)-(59) we get
\[ T\tilde{N}_0(T) - \tilde{N}_1(T) \]
\[ = 2\pi \left\{ \frac{1}{2} \left( -\frac{1}{2\pi^2} (z - (s_0 - \frac{1}{2})) \right)^2 : \left\{ \ln \frac{z}{2\pi} + i\frac{z}{2} - \frac{(s_0 - \frac{1}{2})}{z} - \frac{1}{2} \left( \frac{(s_0 - \frac{1}{2})}{z^2} \right)^2 \right\} \right\} \]
\[ = 2\pi \left\{ -\frac{1}{2} \left( \frac{z}{2\pi} \right)^2 \ln \left( \frac{z}{2\pi} \right) + \left( i\frac{z}{2\pi} + \frac{1}{2} \right) \left( \frac{z}{2\pi} \right)^2 + \left( \frac{(s_0 - \frac{1}{2})}{(2\pi)^2} \right) \left( \frac{z}{2\pi} \right) \ln \left( \frac{z}{2\pi} \right) \right\} \]
\[ + \left( \frac{1}{2} \left( \frac{(s_0 - \frac{1}{2})}{(2\pi)^2} \right) \cdot i\frac{z}{2} - \frac{1}{2} \left( \frac{(s_0 - \frac{1}{2})}{(2\pi)^2} \right)^2 \right) + O(1) \]

In similar fashion, we have

\[ \tilde{T}\tilde{N}_0(\tilde{T}) - \tilde{N}_1(\tilde{T}) \]
\[ = 2\pi \left\{ \frac{1}{2} \left( \frac{z}{2\pi} \right)^2 \ln \left( \frac{z}{2\pi} \right) + \left( i\frac{z}{2\pi} + \frac{1}{2} \right) \left( \frac{z}{2\pi} \right)^2 + \left( \frac{(s_0 - \frac{1}{2})}{(2\pi)^2} \right) \left( \frac{z}{2\pi} \right) \ln \left( \frac{z}{2\pi} \right) \right\} \]
\[ + \left( \frac{1}{2} \left( \frac{(s_0 - \frac{1}{2})}{(2\pi)^2} \right) \cdot i\frac{z}{2} - \frac{1}{2} \left( \frac{(s_0 - \frac{1}{2})}{(2\pi)^2} \right)^2 \right) + O(1) \]

It follows, on taking Cesaro limits after combining \( T\tilde{N}_0(T) - \tilde{N}_1(T) \) and \( \tilde{T}\tilde{N}_0(\tilde{T}) - \tilde{N}_1(\tilde{T}) \) in (71) (and recalling \( \text{Clim} z^2 \ln z = 0 = \text{Clim} z \ln z = \ldots \)), that we have

\[ \text{Clim}_{z, \tilde{z} \to \infty} \left\{ [T\tilde{N}_0(T) - \tilde{N}_1(T)] \right\} = 2\pi \left\{ \left( \frac{(s_0 - \frac{1}{2})}{(2\pi)^2} \right) \cdot (-i\frac{z}{4} - i\frac{\tilde{z}}{4}) \right\} \]
\[ + \text{Clim}_{z, \tilde{z} \to \infty} \left( -\frac{1}{2} \left( \frac{(s_0 - \frac{1}{2})}{(2\pi)^2} \right) \ln \left( \frac{z}{2\pi} \right) \right) \]
\[ = -\frac{i}{4} (s_0 - \frac{1}{2})^2 \]

on again invoking the 2-d Cesaro argument used before to justify taking \( \text{Clim} \ln \left( \frac{z}{2\pi} \right) = 0 \) as per (64).
(iii) This leaves just the terms from \( \delta_0(T) \) and \( \delta_1(T) \) in (71) and (77) to resolve in order to complete the calculation of the root side of the \( \mu = -1 \) root identity.

Using the definition of \( \delta(T) \) in (52) we note from [5] that

\[
\int_0^\infty \frac{\left( \frac{1}{2} - \{u\} \right)}{(u + \frac{1}{4})^2 + (\frac{T}{2})^2} \, du = O \left( \frac{1}{T^2} \right)
\]

and in fact

\[
\delta(T) = \frac{a_1}{T} + \frac{a_3}{T^3} + \ldots
\]  

It follows that, in order to understand the Cesaro limit of \( T\delta(T) \) we need to first calculate \( a_1 \), which is given by

\[
a_1 = \frac{1}{16} + \frac{1}{8} - \frac{1}{2} \lim_{T \to \infty} \left\{ T^2 \int_0^\infty \frac{\left( \frac{1}{2} - \{u\} \right)}{(u + \frac{1}{4})^2 + (\frac{T}{2})^2} \, du \right\}
\]  

with the initial terms arising from the obvious expansions of \( \ln \left( 1 + \frac{1}{4T^2} \right) \) and \( \tan^{-1} \left( \frac{1}{2T} \right) \). To calculate the last term in (86), although it is a classical limit, we will use a formal Cesaro computation and will likewise use such Cesaro methods in all calculations related to \( \delta \) in the remainder of this paper. Since these methods are formal and non-rigorous (see further discussion in section 4.3) we note immediately, however, that all results related to \( \delta \) are in fact rigorously verified, either theoretically or numerically, in Appendix 5.3. We start with the partial integral and apply a formal Taylor series expansion of the integrand:

\[
\int_0^{k+\alpha} \frac{\left( \frac{1}{2} - \{u\} \right)}{(u + \frac{1}{4})^2 + (\frac{T}{2})^2} \, du = \frac{4}{T^2} \int_0^{k+\alpha} \frac{\left( \frac{1}{2} - \{u\} \right)}{1 + \frac{4(u + \frac{1}{4})^2}{T^2}} \, du
\]

\[
= \frac{4}{T^2} \left\{ \int_0^{k+\alpha} \left( \frac{1}{2} - \{u\} \right) \, du \right\}
\]

\[
= \frac{4}{T^2} \left\{ -\frac{4}{T^2} \int_0^{k+\alpha} (\frac{1}{2} - \{u\}) (u + \frac{1}{4})^2 \, du \right\}
\]

\[
= \frac{4}{T^2} \int_0^{\alpha} (\frac{1}{2} - \tilde{a}) \, d\tilde{a} + O \left( \frac{1}{T^4} \right)
\]

\[
= \frac{4}{T^2} \left( \frac{1}{2} \alpha - \frac{1}{2} \alpha^2 \right) + O \left( \frac{1}{T^4} \right)
\]

On recalling from [1] that \( \lim_{k \to \infty} \alpha^n = \frac{1}{n+1} \), in (86) we therefore get that we have

\[
a_1 = \frac{3}{16} - \frac{1}{12} = \frac{1}{48}
\]

and thus, in (85), we have

\[
42
\]
\[ \delta(T) = \frac{1}{48} T + \frac{a_3}{T^3} + \ldots \] (88)

This suffices to allow computation of the Cesaro limit of \( T \delta_0(T) \), and would in fact suffice to allow computation of the contribution from \( \delta \)-terms in (77) and (71) (since the constant of integration terms from \( \delta_1(T) \) in the \( T \) and \( \tilde{T} \) contributions will cancel); but since we will need \( \delta_2(T) \) in detail for the \( \mu = -2 \) root identity later, we shall calculate \( \delta_1(T) \) more carefully here anyway.

From (52) we have

\[ \delta_1(T) = \int_0^T \frac{t}{4} \ln \left( 1 + \frac{1}{4t^2} \right) \, dt + \frac{1}{4} \int_0^T \tan^{-1}\left( \frac{1}{2t} \right) \, dt \]

\[ -\frac{1}{2} \int_0^T t \int_0^\infty \frac{(\frac{1}{2} - \{u\})}{(u + \frac{1}{4})^2 + (\frac{1}{2})^2} \, du \, dt \] (89)

Now

\[ \int_0^T \frac{t}{4} \ln \left( 1 + \frac{1}{4t^2} \right) \, dt = \frac{1}{8} T^2 \ln \left( 1 + \frac{1}{4T^2} \right) + \frac{1}{4} \int_0^T \frac{t}{1 + 4t^2} \, dt \]

\[ = \frac{1}{8} T^2 \ln \left( 1 + \frac{1}{4T^2} \right) + \frac{1}{32} \ln (1 + 4T^2) \]

and

\[ \frac{1}{4} \int_0^T \tan^{-1}\left( \frac{1}{2t} \right) \, dt = \frac{1}{4} T \cdot \tan^{-1}\left( \frac{1}{2T} \right) - \frac{1}{4} \int_0^T \frac{t}{1 + 4t^2} \cdot \frac{1}{2} \cdot \frac{-1}{t^2} \, dt \]

\[ = \frac{1}{4} T \cdot \tan^{-1}\left( \frac{1}{2T} \right) + \frac{1}{2} \int_0^T \frac{t}{1 + 4t^2} \, dt \]

\[ = \frac{1}{4} T \cdot \tan^{-1}\left( \frac{1}{2T} \right) + \frac{1}{16} \ln(1 + 4T^2) \]

In the final term, on reversing the order of integration we get

\[ \frac{1}{2} \int_0^\infty (\frac{1}{2} - \{u\}) \int_0^T \frac{t}{(u + \frac{1}{4})^2 + \frac{T^2}{4}} \, dt \, du \]

\[ = \int_0^\infty (\frac{1}{2} - \{u\}) \left[ \ln \left( (u + \frac{1}{4})^2 + \frac{T^2}{4} \right) \right]_0^T \, du \]

\[ = \left\{ \int_0^\infty (\frac{1}{2} - \{u\}) \ln \left( (u + \frac{1}{4})^2 + \frac{T^2}{4} \right) \, du \right\} \]

\[ - 2 \int_0^\infty (\frac{1}{2} - \{u\}) \ln(u + \frac{1}{4}) \, du \]

43
Now the first of these two integrals can be expressed asymptotically in $T$ as

$$\int_0^\infty \left( \frac{1}{2} - \{u\} \right) \left\{ 2 \ln \left( \frac{T}{2} \right) + 4 \left( \frac{u + \frac{1}{4}}{T^2} \right)^2 - \ldots \right\} \, du$$

and a Cesaro argument identical to the one just used in deriving (87) (as one would expect) shows this is equal to

$$\frac{1}{6} \ln(T) - \frac{1}{6} \ln 2 + O\left( \frac{1}{T^2} \right)$$

while the second of the integrals is a constant which we label $A$, i.e.

$$A := \int_0^\infty \left( \frac{1}{2} - \{u\} \right) \ln(u + \frac{1}{4}) \, du$$

(90)

and do not calculate further at this point.\(^9\)

Thus, overall, combining terms we get finally in (89) that

$$\begin{align*}
\delta_1(T) &= \left\{ \frac{1}{8} T^2 \ln \left(1 + \frac{1}{4T^2}\right) + \frac{3}{32} \ln \left(1 + 4T^2\right) + \frac{1}{4} T \cdot \tan^{-1} \left( \frac{1}{2T} \right) \\
&\quad - \int_0^\infty \left( \frac{1}{2} - \{u\} \right) \ln \left( (u + \frac{1}{4})^2 + \frac{T^2}{4} \right) \, du \\
&\quad + 2 \int_0^\infty \left( \frac{1}{2} - \{u\} \right) \ln(u + \frac{1}{4}) \, du \right\} \\
&= \frac{1}{48} \ln T + C_1 + O\left( \frac{1}{T^2} \right)
\end{align*}$$

(91)

where

$$C_1 = \frac{5}{32} + \frac{17}{48} \ln 2 + 2A$$

(93)

Combining (88) and (92) in (77) we then get

$$\begin{align*}
\frac{1}{\pi} \left\{ T \delta_0(T) - \delta_1(T) \right\} &= \frac{1}{\pi} \left\{ -\frac{1}{48} \ln(T) + \left( \frac{1}{48} - C_1 \right) + O\left( \frac{1}{T^2} \right) \right\} \\
&= \frac{1}{\pi} \left\{ -\frac{1}{48} \left[ \ln z + i\pi \frac{\pi}{2} \right] + \left( \frac{1}{48} - C_1 \right) + O\left( \frac{1}{z} \right) \right\}
\end{align*}$$

(94)

and similarly

$$\begin{align*}
\frac{1}{\pi} \left\{ T \tilde{\delta}_0(T) - \tilde{\delta}_1(T) \right\} &= \frac{1}{\pi} \left\{ -\frac{1}{48} \left[ \ln \tilde{z} - i\pi \frac{\pi}{2} \right] + \left( \frac{1}{48} - C_1 \right) + O\left( \frac{1}{\tilde{z}} \right) \right\}
\end{align*}$$

(95)

\(^9\)We show in Appendix 5.3 that the integral defining $A$ does indeed have a well-defined Cesaro value $A \simeq -0.104$.
It follows thus in (71) that

\[
\lim_{z, \tilde{z} \to \infty} \begin{cases} \frac{1}{\pi} \{ T \delta_0(T) - \delta_1(T) \} \\ \frac{1}{\pi} \{ \tilde{T} \delta_0(\tilde{T}) - \delta_1(\tilde{T}) \} \end{cases} = \lim_{z, \tilde{z} \to \infty} \begin{cases} -\frac{i}{48} - \frac{1}{48\pi} \ln \left( \frac{z}{\tilde{z}} \right) \end{cases} = -\frac{i}{48}
\]  

(96)
on invoking the same 2-d Cesaro argument as before.

Finally, combining (77), (84) and (96) in (71) we conclude that the root side of the \( \mu = -1 \) root identity for \( \zeta \) has contribution from NT of

\[
\sum_{\{s_0 \text{ - NT}\}} M_i(s_0 - \rho_i)^1 = \frac{1}{4} s_0^2 + \frac{3}{2} s_0 + \left( -\frac{3}{4} - \frac{1}{16} - \frac{1}{48} \right)
\]

\[
= \frac{1}{4} s_0^2 + \frac{3}{2} s_0 - \frac{5}{6}
\]

(97)

and on combining with (69) this yields that, overall,

\[
r_\zeta(s_0, -1) = \sum_{\{s_0 \text{ - roots of } \zeta\}} M_i(s_0 - r_i)^1 = 0 \quad \forall s_0
\]

(98)

That is, we have verified as promised that, at least conditional on estimate (73) for \( S_2(T) \), which is itself implied by RH, the root side of the \( \mu = -1 \) root identity for \( \zeta \) is indeed identically 0 \( \forall s_0 \) as claimed in (48).

Hence we now turn to the case of \( \mu = -2 \) and again try to perform a careful Cesaro calculation of the root side of the \( \mu = -2 \) root identity for \( \zeta \) to see whether any insight can be gleaned from the claim in (48) that this, likewise, must be identically equal to 0.

4.2.3 Case (c): \( \mu = -2 \):

In this case, for the trivial roots \( T \), for \( s_0 = 0 \) we get the Cesaro sum of 4, 16, 36, \ldots placed at the points 2, 4, 6, \ldots. By dilation invariance, this equals the Cesaro sum of 4, 16, 36, \ldots placed at the points 1, 2, 3, \ldots and this is familiar from [1] as giving \( 4\zeta(-2) = 0 \). For \( s_0 = 2k \) this becomes instead the Cesaro sum of \( (2k+2)^2, (2k+4)^2, (2k+6)^2, \ldots \) placed at the points \( 2k+2, 2k+4, 2k+6, \ldots \) which thus immediately leads to the value \( 0 - 4 \sum_{j=1}^{k} j^2 = -\frac{4}{3} k^3 - 2k^2 - \frac{2}{3} k \), and so in general

\[
\sum_{\{s_0 \text{ - } T\}} M_i(s_0 - r_i)^2 = -\frac{1}{6} s_0^3 - \frac{1}{2} s_0^2 - \frac{1}{3} s_0
\]

(99)

From the simple pole at \( s = 1 \) we get contribution

45
\[
\sum_{\{s_0 - \text{pole}\}} M_i(s_0 - r_i)^2 = -1 \cdot (s_0 - 1)^2 = -s_0^2 + 2s_0 - 1 \quad (100)
\]

Thus, overall, from the combination of the trivial roots and the simple pole we get a total contribution to the root side of the \(\mu = -2\) root identity for \(\zeta\) of

\[
\sum_{\{s_0 - T \cup \text{pole}\}} M_i(s_0 - r_i)^2 = -\frac{1}{6}s_0^3 - \frac{3}{2}s_0^2 + \frac{5}{3}s_0 - 1 \quad (101)
\]

Turning now to the contribution from the non-trivial roots we have

\[
\sum_{\{s_0 - \text{NT}\}} M_i(s_0 - \beta_i)^2 = \sum_{\{s_0 - \text{NT}\}} M_i(s_0 - \beta_i - i\gamma_i)^2
\]

\[
= \sum_{\{s_0 - \text{NT}\}} M_i(s_0 - \beta_i)^2 - 2i \sum_{\{s_0 - \text{NT}\}} M_i(s_0 - \beta_i)\gamma_i
\]

\[
- \sum_{\{s_0 - \text{NT}\}} M_i\gamma_i^2 \quad (102)
\]

Writing \(\beta_i = \frac{1}{2} + \epsilon_i\) the first of these sums becomes

\[
\sum_{\{s_0 - \text{NT}\}} M_i(s_0 - \beta_i)^2 = \sum_{\{s_0 - \text{NT}\}} M_i(s_0 - \frac{1}{2})^2 + \sum_{\{s_0 - \text{NT}\}} M_i\epsilon_i^2
\]

\[
= (s_0 - \frac{1}{2})^2 (\frac{1}{2}s_0 + \frac{3}{2}) + X_\epsilon \quad (103)
\]

where

\[
X_\epsilon := \sum_{\{s_0 - \text{NT}\}} M_i\epsilon_i^2 = C\text{lim} \left\{ \int_0^T \epsilon_i^2(t)dN(t) + \int_{\tilde{T}} \epsilon_i^2(\tilde{t})dN(\tilde{t}) \right\} \quad (104)
\]

on noting that the cross-term in \(\epsilon_i^1\) vanishes because the non-trivial roots occur in mirror pairs either side of the critical line (and this holds identically for any given \(T\) so there are no Cesaro concerns with this cancellation).

The sum in the second term becomes, on noting again that in going from \(T\) to \(T + dT\) any NT roots off the critical line occur in mirror pairs:

\[
\sum_{\{s_0 - \text{NT}\}} M_i(s_0 - \beta_i)\gamma_i = (s_0 - \frac{1}{2}) \sum_{\{s_0 - \text{NT}\}} M_i\gamma_i
\]

\[
= (s_0 - \frac{1}{2}) \cdot i \cdot (-\frac{1}{4}s_0^2 + \frac{1}{4}s_0 - \frac{1}{12}) \quad (105)
\]

on using our results just computed for the non-trivial roots from the case of the \(\mu = -1\) root identity (specifically (84) and (96)).
And the sum in the third term becomes

\[
\sum_{\{s_0 - NT\}} M_i \gamma_i^2 = \lim_{z, \tilde{z} \to \infty} \left\{ \int_0^T t^2 dN(t) + \int_0^T \tilde{t}^2 dN(\tilde{t}) \right\}
\]

Thus, overall, in (70) we get, after simplification,

\[
\sum_{\{s_0 - NT\}} M_i (s_0 - \rho_i)^2 = \left\{ \begin{array}{ll}
\left( \frac{7}{4} s_0^2 - \frac{43}{24} s_0 + \frac{11}{24} \right) \\
- \lim_{z, \tilde{z} \to \infty} \left\{ \int_0^T t^2 dN(t) + \int_0^T \tilde{t}^2 dN(\tilde{t}) \right\} + \epsilon
\end{array} \right\}
\]

(106)

Now we have

\[
\int_0^T t^2 dN(t) = T^2 N_0(T) - 2 \int_0^T tN_0(t) \, dt
\]

\[
= T^2 N_0(T) - 2TN_1(T) + 2 \int_0^T N_1(t) \, dt
\]

\[
= T^2 N_0(T) - 2TN_1(T) + 2N_2(T)
\]

on noting \( N_1(0) = N_2(0) = 0 \); i.e.

\[
\int_0^T t^2 dN(t) = T^2 N_0(T) - 2TN_1(T) + 2N_2(T)
\]

(108)

and similarly for \( \int_0^T \tilde{t}^2 dN(\tilde{t}) \).

Recalling the original expression for \( N(T) = \tilde{N}(T) + S(T) + \frac{1}{\tau} \delta(T) \) it follows in (107) that we have
Now

(i) As in the case of \( \mu = -1 \), suppose here that we have an estimate

\[
S_3(T) = o(T) \tag{110}
\]

which, as before, would certainly follow from the much stronger estimates (78) if RH is true. Then we would have that \( \frac{1}{T} \int_0^T S_2(t) \, dt = \frac{S_2(T)}{T} = o(1) \) so that \( P[S_2](T) \to 0 \) and thus \( \text{Clim} \, S_2(T) = \text{Clim} \, S_2(\tilde{T}) = 0 \); and, given the capacity to turn Cesaro limits of \( T^2 S_0 \) and \( T S_1 \) into Cesaro limits of \( S_2 \) (in the same fashion as per the argument in (76)), it would follow once again that there was no contribution in (109) from the \( S \)-terms.

As in the case of \( \mu = -1 \), no such estimate (110) is in fact known unconditionally, but we shall proceed here on the assumption of (110), so that

The contribution of the \( S \)-terms to (109) equals 0 \( \tag{111} \)

and hence continue on towards a calculation of \( r_\zeta(s_0, -2) \) conditional on (110), which is itself implied by RH.

(ii) From our earlier computation of \( \tilde{N}_1(T) \) in (80), on letting \( u = \frac{T}{2\pi} \), we have

\[
\tilde{N}_2(T) = (2\pi)^2 \int_0^{\frac{T}{2\pi}} \frac{1}{2} u^2 \ln u - \frac{3}{4} u^2 + \frac{7}{8} u \, du
\]

\[
= (2\pi)^2 \left\{ \frac{1}{6} \left( \frac{T}{2\pi} \right)^3 \ln \left( \frac{T}{2\pi} \right) - \frac{11}{36} \left( \frac{T}{2\pi} \right)^3 + \frac{7}{16} \left( \frac{T}{2\pi} \right)^2 \right\} \tag{112}
\]
and therefore, on combining (57), (80) and (112) and simplifying, we get that

\[ T^2 \tilde{N}_0(T) - 2T \tilde{N}_1(T) + 2 \tilde{N}_2(T) = (2\pi)^2 \left\{ \frac{1}{3} \left( \frac{T}{2\pi} \right)^3 \ln \left( \frac{T}{2\pi} \right) - \frac{1}{9} \left( \frac{T}{2\pi} \right)^3 \right\} \]

(113)

Thus

\[ T^2 \tilde{N}_0(T) - 2T \tilde{N}_1(T) + 2 \tilde{N}_2(T) \]

\[ = (2\pi)^2 \left\{ -\frac{i}{3} \left( \frac{(z-(s_0-\frac{1}{2}))^3}{(2\pi)^3} \right) \left[ \ln \frac{z}{2\pi} + i \frac{\pi}{2} - \frac{(s_0-\frac{1}{2})}{z} \right] \right\} \]

(114)

and similarly

\[ \tilde{T}^2 \tilde{N}_0(\tilde{T}) - 2\tilde{T} \tilde{N}_1(\tilde{T}) + 2 \tilde{N}_2(\tilde{T}) \]

\[ = (2\pi)^2 \left\{ \frac{i}{3} \left( \frac{((\tilde{z}-(s_0-\frac{1}{2}))^3}{(2\pi)^3} \right) \left[ \ln \frac{\tilde{z}}{2\pi} - i \frac{\pi}{2} - \frac{(s_0-\frac{1}{2})}{z} \right] \right\} \]

(115)

Thus, bearing in mind our usual observations that \( \text{Clim} z^3 \ln z = 0 \) etc, we have, after omission of these terms and simplification, that

\[ \text{Clim}_{z,\tilde{z}\to\infty} \left\{ \left\{ T^2 \tilde{N}_0(T) - 2T \tilde{N}_1(T) + 2 \tilde{N}_2(T) \right\}^2 + \left\{ \tilde{T}^2 \tilde{N}_0(\tilde{T}) - 2\tilde{T} \tilde{N}_1(\tilde{T}) + 2 \tilde{N}_2(\tilde{T}) \right\}^2 \right\} \]

\[ = \text{Clim}_{z,\tilde{z}\to\infty} \left\{ -\frac{i}{6} (s_0-\frac{1}{2})^3 + \frac{i}{3} (s_0-\frac{1}{2})^3 \cdot \frac{1}{2\pi} \ln \left( \frac{\tilde{z}}{2\pi} \right) \right\} \]

\[ = -\frac{1}{6} (s_0-\frac{1}{2})^3 \]

(116)
on invoking the same 2-d Cesaro arguments as before.

(iii) From (91) we have

\[
\delta_2(T) = \begin{cases} \frac{1}{8} \int_0^T t^2 \ln \left(1 + \frac{1}{4t^2}\right) dt + \frac{3}{32} \int_0^T \ln \left(1 + 4t^2\right) dt \\ + \frac{1}{4} \int_0^T t \cdot \tan^{-1} \left(\frac{1}{2t}\right) dt \\ - \int_0^T \int_0^\infty \left(\frac{1}{2} - \{u\}\right) \ln \left((u + \frac{1}{4})^2 + t^2\right) du dt + 2AT \end{cases}
\]

(117)

where \(A\) is defined as per (90). Now

(a) \[
\frac{1}{8} \int_0^T t^2 \ln \left(1 + \frac{1}{4t^2}\right) dt = \frac{1}{8} \cdot \frac{T^3}{3} \ln \left(1 + \frac{1}{4T^2}\right) + \frac{1}{12} \int_0^T \frac{t^2}{1 + 4t^2} dt \\
= \frac{1}{8} \cdot \frac{T^3}{3} \ln \left(1 + \frac{1}{4T^2}\right) + \frac{1}{48} T - \frac{1}{96} \tan^{-1}(2T)
\]

(118)

(b) \[
\frac{3}{32} \int_0^T \ln \left(1 + 4t^2\right) dt = \frac{3}{32} T \ln \left(1 + 4T^2\right) - \frac{3}{4} \int_0^T \frac{t^2}{1 + 4t^2} dt \\
= \frac{3}{32} T \ln \left(1 + 4T^2\right) - \frac{3}{16} T + \frac{3}{32} \tan^{-1}(2T)
\]

(119)

(c) \[
\frac{1}{4} \int_0^T t \cdot \tan^{-1} \left(\frac{1}{2t}\right) dt = \frac{1}{8} T^2 \tan^{-1} \left(\frac{1}{2T}\right) + \frac{1}{16} \int_0^T \frac{4t^2}{1 + 4t^2} dt \\
= \frac{1}{8} T^2 \tan^{-1} \left(\frac{1}{2T}\right) + \frac{1}{16} T - \frac{1}{32} \tan^{-1}(2T)
\]

(120)

and

(d) Reversing the order of integration we get

\[
\int_0^\infty \left(\frac{1}{2} - \{u\}\right) \int_0^T \ln \left((u + \frac{1}{4})^2 + t^2\right) dt du \\
= \int_0^\infty \left(\frac{1}{2} - \{u\}\right) \left\{ T \ln \left((u + \frac{1}{4})^2 + \frac{T^2}{4}\right) - 2 \int_0^T \frac{t^2}{t^2 + 4(u + \frac{1}{4})^2} dt \right\} du
\]

50
\[ \int_0^\infty \left( \frac{1}{2} - \{u\} \right) \left\{ T \ln \left( (u + \frac{1}{4})^2 + \frac{T^2}{4} \right) - 2T \right\} du \]

\[ = \int_0^\infty \left( \frac{1}{2} - \{u\} \right) \left\{ T \ln \left( (u + \frac{1}{4})^2 + \frac{T^2}{4} \right) - 2T \right\} du \]

\[ = \int_0^\infty \left( \frac{1}{2} - \{u\} \right) \left\{ T \ln \left( (u + \frac{1}{4})^2 + \frac{T^2}{4} \right) - 2T \right\} du \]

\[ = \int_0^\infty \left( \frac{1}{2} - \{u\} \right) \left\{ T \ln \left( (u + \frac{1}{4})^2 + \frac{T^2}{4} \right) - 2T \right\} du \]

\[ = \left\{ \begin{array}{l}
2 \{ \int_0^\infty (\frac{1}{2} - \{u\})du \} \ln T \\
-(2 + 2 \ln 2) \{ \int_0^\infty (\frac{1}{2} - \{u\})du \} T \\
+2\pi \{ \int_0^\infty (\frac{1}{2} - \{u\}) \} (u + \frac{1}{4})du + O(\frac{1}{T}) \end{array} \right\} \]

Now

(d)(i) We saw before that \( \lim_{k \to \infty} \int_0^k (\frac{1}{2} - \{u\})du = \frac{1}{12} \) so that, in Cesaro terms,

\[ \int_0^\infty (\frac{1}{2} - \{u\})du = \frac{1}{12} \] (122)

and

(d)(ii) Similarly

\[ \int_0^k (\frac{1}{2} - \{u\}) \cdot (u + \frac{1}{4})du \]

\[ = \sum_{j=0}^{k-1} \int_0^1 (\frac{1}{2} - \hat{\alpha})(j + \hat{\alpha} + \frac{1}{4})d\hat{\alpha} + \int_0^\alpha (\frac{1}{2} - \hat{\alpha})(k + \hat{\alpha} + \frac{1}{4})d\hat{\alpha} \]

\[ = \sum_{j=0}^{k-1} \int_0^1 (\frac{1}{2} - \hat{\alpha})(\hat{\alpha}^2) d\hat{\alpha} + (k + \frac{1}{4}) \int_0^\alpha (\frac{1}{2} - \hat{\alpha})d\hat{\alpha} + \int_0^\alpha (\frac{1}{2} - \hat{\alpha})d\hat{\alpha} \]

\[ = -\frac{1}{12} k (k + \frac{1}{4}) \left( \frac{1}{2} \alpha - \frac{1}{2} \alpha^2 \right) + \frac{1}{4} \alpha^2 - \frac{1}{3} \alpha^3 \]

\[ = -\frac{1}{2} k^2 + \frac{1}{2} k \alpha - \frac{1}{12} k (k + \frac{1}{4}) \left( \frac{1}{2} \alpha - \frac{1}{2} \alpha^2 \right) + \frac{1}{4} \alpha^2 - \frac{1}{3} \alpha^3 \]

\[ = -\frac{1}{2} k^2 + \frac{1}{2} k \alpha - \frac{1}{12} k^2 + \left( -\frac{1}{3} \alpha^3 + \frac{1}{8} \alpha^2 + \frac{1}{8} \alpha^3 \right) \] (123)

But, extending lemma 12 in [1], we have in fact (see proof in Appendix 5.4) that

**Lemma 6:** If \( f(k + \alpha) = k^n \alpha^r \) then
Applying this in (123) we thus get, after simplification,

\[
\text{Clim}_{k \to \infty} \int_0^{k+\alpha} \left( \frac{1}{2} - \{u\} \right) \cdot (u + \frac{1}{4}) du = \frac{1}{48}
\]

i.e. in Cesaro terms

\[
\int_0^\infty \left( \frac{1}{2} - \{u\} \right) \cdot (u + \frac{1}{4}) du = \frac{1}{48}
\]  

(125)

Then, combining (122) and (125) in (121) it follows that we have, up to \(O(\frac{1}{T})\),

\[
\int_0^T \int_0^\infty \left( \frac{1}{2} - \{u\} \right) \ln \left( (u + \frac{1}{4})^2 + \frac{t^2}{4} \right) du dt = \frac{1}{6} T \ln T - \frac{1}{6} (1 + \ln 2) T + \frac{\pi}{24} (126)
\]

and therefore, finally, combining (118), (119), (120) and (126) in (117) we get, after simplification,

\[
\delta_2(T) = \frac{1}{48} T \ln T + BT - \frac{\pi}{64} + O(\frac{1}{T})
\]  

(127)

where

\[B = \frac{13}{96} + \frac{17}{48} \ln 2 + 2A\]  

(128)

Combining (127) with our earlier expressions for \(\delta_0(T)\) and \(\delta_1(T)\) in (88) and (92) it follows that we have, in the \(\delta\)-terms in (109), that

\[
\frac{1}{\pi} \left\{ T^2 \delta_0(T) - 2T \delta_1(T) + 2\delta_2(T) \right\} = \frac{1}{\pi} \left\{ \frac{1}{38} T - \frac{1}{32} T \ln T - \frac{1}{12} T - \frac{\pi}{32} \right\} + O(\frac{1}{T})
\]  

(129)

where

\[
D = \frac{1}{48} - 2C_1 + 2B = -\frac{1}{48}
\]

and similarly for \(\frac{1}{\pi} \left\{ \tilde{T}^2 \delta_0(T) - 2\tilde{T} \delta_1(T) + 2\delta_2(T) \right\}\).

On taking \(\text{Clim}_{z,\tilde{z} \to \infty}\) we thus get contribution from these \(\delta\)-terms in (109) of

\[
\frac{1}{\pi} \text{Clim}_{z,\tilde{z} \to \infty} \left\{ Di \cdot (z - (s_0 - \frac{1}{2})) - Di \cdot (\tilde{z} - (\tilde{s}_0 - \frac{1}{2})) - \frac{\pi}{16} \right\} = -\frac{1}{16}
\]  

(130)
on noting $\lim_{z \to \infty} z = \lim_{\tilde{z} \to \infty} \tilde{z} = 0$ in the usual way.

Finally, combining our results from (111) (conjectural or conditional), (116) and (130) in (109), we obtain in (107) that we have

$$\sum_{\{s_0 - NT\}} M_i(s_0 - \rho_i)^2 = \begin{cases} \left( \frac{7}{4} s_0^2 - \frac{43}{24} s_0 + \frac{11}{24} \right) \\ + \frac{1}{6} (s_0 - \frac{1}{2})^3 + \frac{1}{16} + X_\epsilon \end{cases}$$

$$= \frac{1}{6} s_0^3 + \frac{3}{2} s_0^2 - \frac{5}{3} s_0 + \frac{1}{2} + X_\epsilon$$

(131)

And then, finally, combining (131) in turn with (101) we find that, at least conditional on estimate (110) for $S_3(T)$, which is itself implied by the RH, the root side of the $\mu = \frac{-2}{2}$ root identity for $\zeta$ is given by

$$r_\zeta(s_0, -2) = \sum_{\{s_0 - \text{roots of } \zeta\}} M_i(s_0 - r_i)^2 = -\frac{1}{2} + X_\epsilon \quad \forall s_0$$

(132)

From this and the fact that, according to (48) this should equal 0 identically, we can make the following claim (modulo the rigour of the preceding computations):

**Result 1:** The Riemann hypothesis (RH) is false - that is, there exist non-trivial zeros of $\zeta$ off the critical line $\Re(s) = \frac{1}{2}$ in the critical strip.

**Proof:** We argue by contradiction. If RH were true, then

(i) $X_\epsilon = 0$ trivially

(ii) The conditional results regarding Cesaro limits of $S_2(T)$, $S_1(T)$, and $S_0(T)$ noted in (111) all actually do hold, by the explicit estimates, conditional on RH, given in [7, pg 354], and

(iii) All the NT roots $\rho_i$ are of the form exactly $\frac{1}{2} + i\gamma_i$ ($\gamma_i \in \mathbb{R}$) and so the use of $z$ and $\tilde{z}$ exactly to parametrise the counting functions $N(T)$ and $N(\tilde{T})$ in $\mathbb{C}$ becomes legitimate, since these really do then become step functions on the critical line.

But then, in (132) we obtain that

$$r_\zeta(s_0, -2) = \sum_{\{s_0 - \text{roots of } \zeta\}} M_i(s_0 - r_i)^2 = -\frac{1}{2} \quad \forall s_0$$

which contradicts the requirement, from the fact that $\zeta$ satisfies the generalised root identities (and in particular the $\mu = -2$ root identity), that $r_\zeta(s_0, -2)$ should be identically zero for all $s_0$ as per (48).
4.3 Discussion of Issues Requiring Further Attention

The emphasis in this paper has been on the development of certain methods pertaining to remainder Cesaro summation/convergence, and on calculation using these methods. Consequently, there has been a conscious de-emphasis on formal rigour and detailed proof. In light of this, and especially of the claim in result 1 that such computations imply the Riemann hypothesis is false, we now conclude by trying to clarify which claims are in fact rigorously supported and where there are gaps requiring more detailed justification.

(1) The first major issue requiring more rigorous justification is that, in the case of $\zeta$, we have simply asserted that because $\zeta$ satisfies the generalised root identity for $\mu = 1$ for all $s_0$ (modulo the $\pi^{-\frac{1}{2}}$ obstruction factor), so it must (like $\Gamma$) continue to satisfy these identities for all $s_0$ for $\mu = 0, -1, -2, \ldots$. In part this reflects the belief that, having verified the $\mu = 1$ identity, so that roughly $-\frac{\zeta'(s_0)}{\zeta(s_0)} = \sum_{\{s_0\text{-roots of }\zeta\}} \frac{M_i}{(s_0 - r_i)}$ for arbitrary $s_0$ (the more rigorous statement including the renormalisation correction and $\pi^{-\frac{1}{2}}$ term is (26)), so further differentiation by $\left(\frac{d}{ds}\right)^{-1}$ should imply that the generalised root identities hold for $\zeta$ also at arbitrary $\mu$, and in particular $\mu = 0, -1, -2, \ldots$. But as well as being heuristic this skates over the possibility that there exist functions which act as obstructions for the root identities for $\mu = 0, -1, -2, \ldots$ in the same way as the functions $e^{\pi z^2}$ act as obstructions for the root identities for $\mu \in \mathbb{Z}_{\geq 1}$. Since $\pi^{-\frac{1}{2}}\zeta(s)$ does already satisfy the $\mu = 1$ identity for arbitrary $s_0$ (and hence for $\mu = 2, 3, \ldots$), so this is the question of whether, for any given $n \in \mathbb{Z}_{\leq 0}$, there exists a function, $g(s)$ which fails to satisfy the root identities for $\mu = n$ but satisfies them all at arbitrary $s_0$ for $\mu = n + 1, 2, 3, \ldots$ (since we need to leave these identities undisturbed). We have not addressed this question at all in this paper although, based on considerable experimentation, we believe that the non-existence of such additional obstructions is a reasonable conjecture (as well as one that seems quite distinct from RH).

Another way of thinking of this issue is that we know from [1] (e.g. the discussion of removable singularities for discrete Cesaro schemes discussed in [1], section 4.3) that the values obtained by Cesaro methods at individual parameter values (e.g. $\mu = 0, -1$ and $-2$) are only reliable if they arise from Cesaro analysis in an open region around these values which represents a proper analytic continuation (e.g. in $\mu$) to these regions. Since we have only actually verified the root identities $d_\zeta(s_0, \mu) = r_\zeta(s_0, \mu)$ for $\zeta$ at the isolated values $\mu \in \mathbb{Z}_{\geq 1}$, we have not in fact even established a starting open region in $\mathbb{C}$ where the root identities are true, let alone systematically extended to regions encompassing the critical points $\mu = 0, -1$ and $-2$. In a separate paper ([8]), however, we address this deficiency by showing numerically that it appears that the generalised root identities for $\zeta$ are indeed satisfied for all real $\mu > 1$, where both $d_\zeta(s_0, \mu)$ and $r_\zeta(s_0, \mu)$ are given by classically convergent expressions; and furthermore that they appear to continue to be satisfied both for values $0 < \mu < 1$ where Cesaro divergences need to be removed on the root side and for values $-1 < \mu < 0$ where Cesaro divergences not only need to be removed but a further Cesaro
averaging of the resulting residual partial-sum function applied. For \( \mu \) further to the left (i.e. \( \mu < -1 \)) the numerical difficulties in verifying the root identities then become too significant without major effort beyond that attempted in [8] owing to the need to perform additional applications of the Cesaro averaging operator \( P \), but nonetheless these numerical results in [8] seem to us to be strongly suggestive that \( \zeta \) does indeed satisfy the generalised root identities for all \( \mu \in \mathbb{R} \) (and hence all \( \mu \in \mathbb{C} \) by analytic continuation) as desired.

In addition to this we would also argue that the fact that we have computed the root sides explicitly when \( \mu = 0 \) and \( \mu = -1 \) and thereby demonstrated that \( \zeta \) does satisfy the generalised root identities at these points (modulo estimate (73) in the case of \( \mu = -1 \)) is strongly suggestive on its own that \( \zeta \) should continue to satisfy these identities at \( \mu = -2 \), which is the key to result 1; and we believe this especially because it seems to us that the nature of the calculation in the \( \mu = 0 \) and \( \mu = -1 \) cases is non-obvious and itself suggestive of the aptness of the Cesaro approach.

(2) A second major issue is that we have not, in fact, rigorously validated the definition of generalised Cesaro convergence along a contour given in definition 1, i.e. we have not rigorously shown that this definition, and in particular the distinction between removing “eigenfunctions” in the geometric variable \( z = \gamma(t) \) but then just averaging in arc-length \( t \), guarantees correct analytic continuation in the situations in which we have applied it (e.g. in \( z_0 \) and \( \mu \) on the root-side, \( r(z_0, \mu) \), of root identities). The numerous examples in the paper where this is confirmed by calculation (e.g. the definition of \( \Gamma \), the root identities for \( \Gamma \) at \( \mu \in \mathbb{Z} \leq 0 \) and for \( \zeta \) at \( \mu = 0 \) and \( \mu = -1 \)) give strong “experimental” evidence that this is so, but no rigorous argument has been given. Briefly, such an argument would likely proceed along the following lines, illustrated here in the context of the simpler case of the root side of the root identities for \( \Gamma(z+1) \) (but easily adaptable to the case of \( \zeta \)):

(a) Initially restrict to \( z_0 \) real so that for \( r_T(z_0, \mu) \) we are back in the setting of partial sum functions on \([0, \infty)\) handled rigorously in [1] via regular polynomials, \( q(P) \), in \( P \). When \( \mu \) is real and \( \mu > 1 \) the series defining \( r_T(z_0, \mu) \) is classically convergent, and when \( \mu < 1 \) the “removable” Cesaro eigenfunctions are all in fact naturally functions (e.g. \( z^\rho \) etc) in the geometric variable \( z = z_0 + k + \alpha \)

(b) Thus the rigorous methods of [1] should establish, for all \( z_0 \) real, the correct analytic continuation of \( r_T(z_0, \mu) \) from \( \mu > 1 \) to all real \( \mu \). Analytic extension of \( r_T(z_0, \mu) \) to \( z_0 \) off the real line (holding \( \mu \) real still initially) is then uniquely determined and seemingly must involve still “removing” the same divergent terms in the variable \( z \), which is no longer real but has the same geometric meaning, namely \( z = z_0 + k + \alpha \) as described in definition 1

(c) Finally, analytic continuation to all \( \mu \) off the real line is then also uniquely determined (although it is no longer necessarily still interpretable in terms of Cesaro convergence if \( z_0 \) is not real; note that in this paper we have never actually considered non-real \( \mu \)).

With respect to (b), note that the fact that we remove “eigenfunctions” such as \( z^\rho \) in the geometric variable \( z \) (rather than, say, \( t^\rho \) in \( t = k + \alpha \)) means that as \( \mu \) varies and causes \( \rho \) to pass through positive integer values \( 1, 2, \ldots \) we do
not instantaneously pick up discontinuous contributions to the Cesaro limits at these \( \mu \)-values (as we would if we were removing “eigenfunctions” in \( t \)). This observation bears directly on the remarks (ii) and (iii) made in commenting upon definition 1 in section 2.1, and is especially relevant for why we hope that we can still trust the Cesaro evaluations in this paper for \( \Gamma \) and \( \zeta \) at non-positive integer values without need for correction of removable singularities.

Note finally, in passing, that the adaptation of the above scheme of argument to the case of \( \zeta \) would involve negligible change for handling the trivial roots, and would merely require an adaptation of the approach in [1] from functions on \([0, \infty)\) to functions on the positive and negative imaginary axes for handling the \( NT^+ \) and \( NT^- \) components of the non-trivial roots.

(3) For \( \zeta \) in the cases of \( \mu = 0, -1 \) and \( -2 \) we have relied on a claim that we can ascribe a 2d Cesaro limit of 0 to \( \text{Clim}_{z, \tilde{z} \to \infty} \ln \left( \frac{z}{\tilde{z}} \right) \). In fact, along with a parameter (say \( w \)) for the trivial roots, \( T \), the calculations for \( r_{\zeta}(s_0, \mu) \) at \( \mu = 0, -1, -2, \ldots \) are all really 3d Cesaro calculations, but we can treat the trivial root calculations independently and it is only in relation to the claim that \( \text{Clim}_{z, \tilde{z} \to \infty} \ln \left( \frac{z}{\tilde{z}} \right) = 0 \) that we are obliged to consider the two variables \( z \) and \( \tilde{z} \) simultaneously. The argument for this claim (in (64)) is, however, clearly non-rigorous (particularly since we have not formalised the 2d notion, on a pair of contours, being used). Nonetheless, we make the following two observations about this claim here:

(a) First we reiterate that, as noted in section 4.2.1, for purposes of our claim re RH this 2d Cesaro limit could alternatively be taken as an implication of the \( \mu = 0 \) root identity and then applied without caveat to the \( \mu = -1 \) and \( \mu = -2 \) cases on that basis, and

(b) Secondly, in [8], as part of explicitly considering the Cesaro treatment of the non-trivial root contributions to \( r_{\zeta}(s_0, \mu) \) in a neighbourhood of 0, we provide strong numerical evidence to justify taking \( \text{Clim}_{z, \tilde{z} \to \infty} \ln \left( \frac{z}{\tilde{z}} \right) = 0 \) that \( r_{\zeta}(s_0, \mu) \to 0 \) as \( \mu \) approaches 0 from below. Moreover we give a precise conjecture as to how this \( \ln \left( \frac{z}{\tilde{z}} \right) \) divergence may be arising naturally in the limit as \( \mu \to 0 \) (essentially from the contributions from \( NT^+ \) and \( NT^- \) each individually diverging as \( \mu \to 0 \) but in opposite directions so that the overall \( NT \)-contribution remains finite) and likewise give strong numerical evidence that this conjecture is true. This would validate fully that taking \( \text{Clim}_{z, \tilde{z} \to \infty} \ln \left( \frac{z}{\tilde{z}} \right) = 0 \) at \( \mu = 0 \) (and presumably similarly at \( \mu = -1 \) and \( \mu = -2 \)) is the correct thing to do in order to ensure correct analytic continuation.

(4) Lastly, in order to focus on calculation, in numerous areas throughout the paper we have been somewhat loose, for example in:

(a) Adopting a heuristic approach in applying Fourier theory to perform calculations on the derivative side in some examples of the generalised root identities, and indeed even in regards to the Fourier definition of \( \left( \frac{d}{dt} \right)^\mu \) on the derivative side of these identities in the first place,

(b) Interpreting integrals, \( \int_0^\infty \) or \( \int_{-\infty}^\infty \), in a Cesaro way as required (effectively applying a universal Cesaro viewpoint throughout the paper).
(c) Utilising Taylor series expansions without always being careful regarding their domains of convergence, and

(d) Using formal Cesaro methods to obtain the required parts of the asymptotic expansions of $\delta, \delta_1$ and $\delta_2$ in the calculations of the root sides of the root identities for $\zeta$ at $\mu = 0, -1$ and $-2$.

For instance, in analysing $\delta, \delta_1$ and $\delta_2$ as mentioned in (d), despite the fact that these are all classically well-defined functions and thus should be calculable without recourse to Cesaro methods, we instead began by applying a Taylor series expansion

$$\frac{1}{1 + \frac{4(u + \frac{1}{4})^2}{T^2}} = 1 - \frac{4(u + \frac{1}{4})^2}{T^2} + \frac{16(u + \frac{1}{4})^4}{T^4} - \ldots$$

inside an integral, $\int_{u=0}^{u=\infty}$ (namely the integral term in the definition of $\delta$), where for any fixed $T$ this expansion is not classically valid for most of the domain of integration $0 < u < \infty$. We nonetheless proceeded by working formally term by term in descending powers of $T$, using Cesaro methods (specifically lemma 6) to evaluate the resulting divergent integrals in $u$ which then form the coefficients.

And, for $\delta$, we thereby ended up with result (88), which can readily be extended by keeping the next lower order terms in $T$ to give a more detailed expansion

$$\delta(T) = \frac{1}{48} \frac{1}{T} + \frac{7}{5760} \frac{1}{T^3} + \frac{31}{80640} \frac{1}{T^5} + o\left(\frac{1}{T^5}\right)$$

and similar results for $\delta_1(T)$ and $\delta_2(T)$.

Notwithstanding this “looseness,” however, we believe that all the calculational claims in this paper are solid. In particular, in relation to the listed areas (a) - (d) and how they bear on our calculations for $\zeta$, note that:

(i) On the derivative side of the root identities for $\zeta$ at arbitrary $\mu$ we have only ever relied on property (28) of $\left(\frac{d}{ds}\right)^\mu \zeta(s)$ and not needed to explicitly invoke Fourier and distributional arguments\(^{10}\), and

(ii) As noted in section 4.2.2, all the claims regarding $\delta, \delta_1$ and $\delta_2$ used in calculating the root sides of the root identities for $\zeta$ at $\mu = 0, -1$ and $-2$ are in fact true and have been verified both numerically and by alternative rigorous derivation (see Appendix 5.3)\(^{11}\).

Overall, despite the issues (1)-(4) flagged in this section (some of which are further addressed in [8]), we believe, both for the variety of reasons mentioned in

\(^{10}\)We have in fact attempted a derivation of lemma 4 using Fourier and distributional methods. The arguments rely on identifying the asymptotic behaviour of $\zeta(s)$ as $s \to \infty$ and hence also as $s \to -\infty$ via the functional equation; to handle the $\Gamma$-style asymptotic behaviour that thereby arises as $s \to -\infty$ we adapt our arguments regarding Stirling’s theorem in section 3.4 to this “one-sided” setting (i.e. Stirling asymptotics only as $s \to -\infty$ not in both directions). Details are omitted here however.

\(^{11}\)The reason we have nonetheless adopted the formal Cesaro approach in analysing these functions in the paper itself is partly for reasons of brevity, partly in order to keep a Cesaro perspective throughout the paper and partly because we believe that the fact that these Cesaro calculations work successfully despite their formalism is in itself interesting.
the discussion in this section and on the basis of the results derived throughout
the paper in a range of examples, that the methods and results of this paper
are correct.

5 Appendices

5.1 Dilation Invariance of Cesaro Convergence

The inverse of the Cesaro operator \( P[f](x) := \frac{1}{x} \int_{0}^{x} f(t) \, dt \) is easily seen to be
given by

\[
P^{-1} = x \frac{d}{dx} + 1
\]

(as discussed in [1]). On the other hand, the generator of the dilation group
\( D_r : x \mapsto rx, r > 0 \), is given by \( x \frac{d}{dx} \), since for \( \epsilon \) very small we have

\[
f((1 + \epsilon)x) = f(x) + f'(x) \cdot x \epsilon + O(\epsilon^2)
\]

so that

\[
D_{1+\epsilon} = 1 + \epsilon \cdot x \frac{d}{dx} + O(\epsilon^2)
\]

It follows at once that dilations commute with the application of \( P \) and so
Cesaro limits of functions must be dilation-invariant.

A similar, though more involved, argument shows that, in fact, Cesaro limits
are also invariant under rescaling \( (S_r : x \mapsto x', r > 0) \), but this has not been
utilised in this paper and so we omit details here.

5.2 Figures

The first two of the following diagrams have been referenced in the body of the
paper.

Figure1: Cesaro Dilation Invariance and the Duplication Formulae for \( \Gamma \)
Figure 2: Bi-directional Remainder Summation and the Functional Equation for $\Gamma$
The final figure below depicts relationships among the concepts and results of this paper, and linkages with other well-known results. It is included in order to provide a survey of the avenues investigated in the paper, and to illustrate why they may provide a new perspective on certain issues, in particular the way they combine in leading to the claim made in Result 1.

Figure 3: Connections Among the Results of this Paper
5.3 Rigorous Verification of Formal Cesaro Results for $\delta$, $\delta_1$ and $\delta_2$

In the course of evaluating the root side of the root identities for $\zeta$ when $\mu = -1$ and $\mu = -2$ we used the following results regarding $\delta_0 (= \delta)$, $\delta_1$ and $\delta_2$ which we derived using formal Cesaro means:

$$\delta(T) = \frac{1}{48} + \frac{7}{5760} T + \frac{31}{80640} T^3 + o\left(\frac{1}{T^5}\right) \tag{135}$$

$$\delta_1(T) = \frac{1}{48} \ln T + \left\{ \frac{5}{32} + \frac{17}{48} \ln 2 + 2A \right\} - \frac{7}{11520} \frac{1}{T^2} - \frac{31}{322560} \frac{1}{T^4} + o\left(\frac{1}{T^3}\right) \tag{136}$$

and

$$\delta_2(T) = \frac{1}{48} T \ln T + \left\{ \frac{13}{96} + \frac{17}{48} \ln 2 + 2A \right\} T - \frac{\pi}{64} + \frac{7}{11520} \frac{1}{T} - \frac{31}{967680} \frac{1}{T^3} + o\left(\frac{1}{T^3}\right) \tag{137}$$

where here $A$ is the constant given by the Cesaro value of the integral

$$A = \int_0^\infty \left( \frac{1}{2} - \{u\} \right) \ln(u + \frac{1}{4}) \, du \tag{138}$$

These implied the relationships
\[
T\delta_0(T) - \delta_1(T) = -\frac{1}{48}\ln T - \left\{\frac{13}{96} + \frac{17}{48}\ln 2 + 2A\right\} + \frac{21}{11520}\frac{1}{T^2} + \frac{155}{322560}\frac{1}{T^4} + o\left(\frac{1}{T^5}\right)
\]

and

\[
T^2\delta_0(T) - 2T\delta_1(T) + 2\delta_2(T) = -\frac{1}{48}T - \frac{\pi}{32} + \frac{21}{5760}\frac{1}{T} + \frac{31}{48384}\frac{1}{T^3} + o\left(\frac{1}{T^5}\right)
\]

which constituted the \(\delta\)-term contributions to the roots sides of the \(\mu = -1\) and \(\mu = -2\) root identities for \(\zeta\).

We now show that these results are all in fact valid despite their heuristic derivation in the body of the paper. We start with a proof of the asymptotic expansion for \(\delta\) in result (135).

**Proof of Asymptotic Expansion for \(\delta\):** Recall that

\[
\delta(T) = \frac{T}{4} \ln \left(1 + \frac{1}{4T^2}\right) + \frac{1}{4} \tan^{-1}\left(\frac{1}{2T}\right) - \frac{T}{2} \int_0^\infty \frac{\{\frac{1}{2} - \{u\}\}}{(u + \frac{1}{4})^2 + \left(\frac{T}{2}\right)^2} \, du
\]

Using that \(\ln(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \ldots\), we first get the following expansion, which converges classically if \(T > \frac{1}{2}\):

\[
\frac{T}{4} \ln \left(1 + \frac{1}{4T^2}\right) = \frac{1}{16} \frac{1}{T} - \frac{1}{128} \frac{1}{T^3} + \frac{1}{768} \frac{1}{T^5} - \ldots
\]

Similarly, using \(\tan^{-1}(t) = t - \frac{t^3}{3} + \frac{t^5}{5} - \ldots\) we get the following expansion which also converges classically if \(T > \frac{1}{2}\):

\[
\frac{1}{4} \tan^{-1}\left(\frac{1}{2T}\right) = \frac{1}{8} \frac{1}{T} - \frac{1}{96} \frac{1}{T^3} + \frac{1}{640} \frac{1}{T^5} - \ldots
\]

Next, write the final term in the expression for \(\delta\) as

\[
\int_0^\infty g_T(u) \tilde{B}_1(u) \, du
\]

where

\[
g_T(u) = \frac{T}{(u + \frac{1}{4})^2 + \left(\frac{T}{2}\right)^2}
\]

and \(\tilde{B}_1(u) = \{u\} - \frac{1}{2}\) is the first “periodic Bernoulli function” as defined in section 1.1. Using the fact that \(\frac{d}{du}\tilde{B}_n(u) = n\tilde{B}_{n-1}(u)\) for \(n \geq 2\) (except at the integer points when \(n = 2\) which is a set of measure 0), we can write the integral (141) as
\[ \frac{1}{2} \int_0^\infty g_T(u) \frac{d}{du} \hat{B}_2(u) \, du = \frac{1}{2} g_T(u) \hat{B}_2(u) \bigg|_{u=0} - \frac{1}{2} \int_0^\infty g_T'(u) \hat{B}_2(u) \, du \]

and since \( g_T(u) \to 0 \) as \( u \to \infty \) and \( \hat{B}_2(0) = B_2 \) this is

\[ -\frac{B_2}{2} g_T(0) - \frac{1}{2} \int_0^\infty g_T'(u) \hat{B}_2(u) \, du \]

\[ = -\frac{B_2}{2} g_T(0) - \frac{1}{6} \int_0^\infty g_T'(u) \frac{d}{du} \hat{B}_3(u) \, du \]

Now \( \hat{B}_3(0) = B_3 = 0 \) and it is easy to check that \( g_T'(u) \to 0 \) as \( u \to \infty \), so integrating by parts again, the integral (141) becomes

\[ -\frac{B_2}{2} g_T(0) + \frac{1}{6} \int_0^\infty g_T''(u) \hat{B}_3(u) \, du \]

\[ = -\frac{B_2}{2} g_T(0) + \frac{1}{24} \int_0^\infty g_T''(u) \frac{d}{du} \hat{B}_4(u) \, du \]

which, since \( g_T''(u) \to 0 \) as \( u \to \infty \) and \( \hat{B}_4(0) = B_4 \), becomes in turn

\[ -\frac{B_2}{2} g_T(0) - \frac{B_4}{24} g_T''(0) - \frac{1}{24} \int_0^\infty g_T'''(u) \hat{B}_4(u) \, du \]

Continuing in the same way, noting that \( g_T^{(r)}(u) \to 0 \) as \( u \to \infty \) and that \( \hat{B}_k(0) = B_k \) with \( B_{2l+1} = 0 \) for all \( l \geq 1 \), we obtain the following expression for the integral (141)

\[
\int_0^\infty g_T(u) \hat{B}_1(u) \, du = \left\{ \begin{array}{l}
-\frac{B_2}{2} g_T(0) - \frac{B_4}{24} g_T''(0) - \frac{B_6}{6} g_T^{(4)}(0) \\
+ \frac{1}{24} \int_0^\infty g_T^{(6)}(u) \hat{B}_7(u) \, du
\end{array} \right\}
\]

(142)

which could be extended further if desired.

In this case we have
\[ g_T'(u) = -\frac{T(u + \frac{1}{2})}{((u + \frac{1}{2})^2 + (\frac{T}{2})^2)^2} = O\left(\frac{1}{T^3}\right) \]

\[ g_T''(u) = \frac{T(3(u + \frac{1}{2})^2 - (\frac{T}{2})^2)}{((u + \frac{1}{2})^2 + (\frac{T}{2})^2)^3} = O\left(\frac{1}{T^3}\right) \]

\[ g_T'''(u) = \frac{12T(- (u + \frac{1}{2})^3 + (u + \frac{1}{2})(\frac{T}{2})^2)}{((u + \frac{1}{2})^2 + (\frac{T}{2})^2)^4} = O\left(\frac{1}{T^5}\right) \]

\[ g_T^{(4)}(u) = \frac{12T(5(u + \frac{1}{2})^4 - 10(u + \frac{1}{2})^2(\frac{T}{2})^2 + (\frac{T}{2})^4)}{((u + \frac{1}{2})^2 + (\frac{T}{2})^2)^5} = O\left(\frac{1}{T^5}\right) \]

\[ g_T^{(5)}(u) = \frac{15T(-24(u + \frac{1}{2})^5 + 80(u + \frac{1}{2})^3(\frac{T}{2})^2 - 24(u + \frac{1}{2})(\frac{T}{2})^4)}{((u + \frac{1}{2})^2 + (\frac{T}{2})^2)^6} = O\left(\frac{1}{T^7}\right) \]

\[ g_T^{(6)}(u) = \frac{45T(56(u + \frac{1}{2})^6 - 280(u + \frac{1}{2})^4(\frac{T}{2})^2 + 168(u + \frac{1}{2})^2(\frac{T}{2})^4 - 8(\frac{T}{2})^6)}{((u + \frac{1}{2})^2 + (\frac{T}{2})^2)^7} = O\left(\frac{1}{T^7}\right) \]

But it follows from the expression for \( g_T^{(6)}(u) \) that in (142) we have

\[ \int_0^\infty g_T^{(6)}(u) \hat{B}_T(u) \, du = o\left(\frac{1}{T^3}\right) \]

which we shall show by demonstrating that

\[ I_T := T^5 \int_0^\infty g_T^{(6)}(u) \hat{B}_T(u) \, du \]

satisfies \( I_T \to 0 \) as \( T \to \infty \). To see this, note from the expression for \( g_T^{(6)}(u) \) that

\[ |g_T^{(6)}(u)| \leq \frac{CT}{((u + \frac{1}{2})^2 + (\frac{T}{2})^2)^4} \]

and so

\[ I_T = \int_0^\infty h_T(u) \hat{B}_T(u) \, du \]

where \( h_T(u) \leq \frac{C_u}{T^2} \) for a constant \( C_u \) depending only on \( u \). Moreover,

\[ T^5|g_T^{(6)}(u)| \leq \frac{CT^6}{((u + \frac{1}{2})^2 + (\frac{T}{2})^2)^4} \]

and differentiating the right hand side with respect to \( T \), we see that, for fixed \( u \), the right hand side is maximised when \( T = C(u + \frac{1}{2}) \). This maximum \( h(u) \) is
a constant multiple of \( \frac{1}{(\gamma + \frac{1}{2})^3} \). Hence \( \int_0^\infty h(u)\,du < \infty \) and since \( h_T(u) \tilde{B}_T(u) \) is at most a constant multiple of \( h(u) \), so we can apply the Dominated Convergence Theorem and see that \( I_T \to 0 \) as \( T \to \infty \). Thus in (142) we have that the integral (141) is given by

\[
\int_0^{k+\alpha} (\frac{1}{2} - \{u\}) \ln(u + \frac{1}{4}) \,du
\]

\[
\sum_{j=0}^{k-1} \int_0^{\frac{1}{2} - \tilde{a}} (\ln j + (\frac{\tilde{a} + \frac{1}{2}}{j}) - \frac{1}{2} (\frac{\tilde{a} + \frac{1}{2}}{j})^2 + \ldots) \,d\tilde{a}
\]

\[
+ \int_{\alpha}^{\frac{1}{2} - \tilde{a}} (\ln k + (\frac{\tilde{a} + \frac{1}{2}}{k}) - \frac{1}{2} (\frac{\tilde{a} + \frac{1}{2}}{k})^2 + \ldots) \,d\tilde{a}
\]

\[
\sum_{j=0}^{k-1} (\frac{1}{j} - \tilde{a})(\frac{\tilde{a} + \frac{1}{2}}{j}) \,d\tilde{a}
\]

\[
+(\frac{k}{2} \alpha - \frac{1}{2} \alpha^2) \ln k + C + O(\frac{1}{k})
\]

\[
- \frac{1}{12} \sum_{j=0}^{k-1} \frac{1}{j} + (\frac{1}{2} \alpha - \frac{1}{2} \alpha^2) \ln k + C + O(\frac{1}{k})
\]

where \( \gamma \approx 0.577 \) is Euler’s constant. But since \( \int_0^1 (\tilde{a}^2 - \tilde{a} + \frac{1}{4}) \,d\tilde{a} = 0 \) it follows easily that \( P((\tilde{a}^2 - \tilde{a} + \frac{1}{4}) \ln \tilde{k}) (k + \alpha) \to 0 \) as \( k \to \infty \). Hence we have that the partial integral \( \int_0^{k+\alpha} (\frac{1}{2} - \{u\}) \ln(u + \frac{1}{4}) \,du \) does have a generalised Cesaro limit as \( k \to \infty \) and in fact we have
\[
A = C - \frac{1}{12} \gamma
\]

\[
= \lim_{k \to \infty} \sum_{j=0}^{k-1} \int_0^1 \left( \frac{1}{2} - \tilde{\alpha} \right) \left\{ \ln(j + \tilde{\alpha} + \frac{1}{4}) - \ln j - \left( \frac{\tilde{\alpha} + \frac{1}{2}}{j} \right) \right\} d\tilde{\alpha} - \frac{1}{12} \gamma
\]

(143)

which we may calculate numerically to get the above approximate value.

This leaves just the verification of the asymptotic expressions (136) and (137) for \( \delta_1 \) and \( \delta_2 \). Fully rigorous derivations of these have also been obtained along the lines of the derivation above for \( \delta \), after obtaining recurrence relations and certain preliminary estimates for the integrals

\[
I_{r,m} := \int_0^\infty \frac{B_{2r-1}(u)}{((u + \frac{1}{4})^2 + (\frac{r}{2})^2)^m} du
\]

For the sake of brevity, however, we omit details here. Instead, as an alternative we note that the expressions in (135)-(137) for \( \delta_0 \), \( \delta_1 \) and \( \delta_2 \) can also all be readily verified numerically by writing the integral in the definition of \( \delta \) as a convergent sum over \( j \) of integrals from \( j \) to \( j + 1 \); since on each such unit interval the integral is easy to perform exactly (and likewise for the further integrations for \( \delta_1 \) and \( \delta_2 \)), so the resulting expressions for \( \delta_0 \), \( \delta_1 \) and \( \delta_2 \) are comprised entirely of elementary functions and convergent sums of the same, allowing thorough numerical verification of the claimed asymptotic expansions. Such sample numerical confirmations, implemented via VBA code, can be found in the XL2007 spreadsheet “delta_0_1_2_AsymptoticsFinal2.xlsx” made available with this paper.

5.4 Proof of Lemma 6

In order to perform our calculations related to \( \delta \) in our derivations of the values of \( r(\xi, s_0, -1) \) and \( r(\xi, s_0, -2) \) we invoked the result (124) given as lemma 6, namely that

\[
\lim_{k \to \infty} k^n \alpha^r = \frac{1}{n + r + 1}
\]

(144)

In these cases the specific claims required (that \( \lim_{k \to \infty} \alpha^r = \frac{1}{r+1} \), \( \lim_{k \to \infty} k \alpha^2 = -\frac{1}{4} \), \( \lim_{k \to \infty} k \alpha = -\frac{1}{2} \), and \( \lim_{k \to \infty} k = -\frac{1}{2} \)) all involve only \( n = 0 \) or \( n = 1 \) and are readily checked independently by direct calculation. However, since we regard lemma 6 as an independently interesting result in the Cesaro framework, we include its general proof here now.

Writing \( x = k + \alpha \) as usual, let

\[
f_{n,r}(x) := k^n \alpha^r \quad \text{and} \quad g_{n,r}(x) := x^n \alpha^r
\]

66
When \( n = 0 \) note that \( f_{0,r}(x) = g_{0,r}(x) \) and it is trivial that they both converge in a generalised Cesaro sense to \( \frac{1}{r+1} \), i.e.

\[
\lim_{x \to \infty} f_{0,r}(x) = \lim_{x \to \infty} g_{0,r}(x) = \frac{1}{r + 1} \tag{145}
\]

We first prove the following additional lemma:

**Lemma 7:**

\[
f_{n,r}(x) \xrightarrow{C} \frac{(-1)^n}{n + r + 1} \quad \text{if and only if} \quad g_{n,r}(x) \xrightarrow{n \in \mathbb{Z}_{>0}} 0 \tag{146}
\]

**Proof of Lemma 7:** On the one hand, if \( k^n \alpha^r \xrightarrow{C} \frac{(-1)^n}{n + r + 1} \) for all \( n \in \mathbb{Z}_{>0} \) then

\[
x^n \alpha^r = (k + \alpha)^n \alpha^r = \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \alpha^{j+r} \xrightarrow{C} \sum_{j=0}^{n} \binom{n}{j} (-1)^n \frac{1}{n + r + 1} = 0
\]

on recognising the binomial expansion of \((1-1)^n\). On the other, if \( x^n \alpha^r \xrightarrow{C} 0 \) for all \( n \in \mathbb{Z}_{>0} \) then \( k^n \alpha^r = (x-\alpha)^n \alpha^r = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} \cdot (-1)^j \cdot \alpha^{j+r} \xrightarrow{C} \frac{1}{n + r + 1} \) on using the fact that only the \( j = n \) term is non-zero and invoking (145). Thus the result of lemma 7 holds in both directions.

To complete the proof of lemma 6 we then deduce the pair of Cesaro limits for \( f_{n,r} \) and \( g_{n,r} \) given in lemma 7 simultaneously by induction on \( n \).

When \( n = 0 \) we have result (145) and when \( n = 1 \) we have

\[
P[f_{1,r}](x) = \frac{1}{x} \int_0^{k+\alpha} f_{1,r}(t) \, dt
\]

\[
= \frac{1}{x} \left\{ \sum_{j=0}^{k-1} \frac{j}{r+1} + k \cdot \frac{\alpha^{r+1}}{r+1} \right\}
\]

\[
= \frac{1}{x(r+1)} \left\{ \frac{1}{2} (x-\alpha)(x-\alpha-1) + (x-\alpha)\alpha^{r+1} \right\}
\]

\[
\xrightarrow{C} \frac{1}{2(r+1)} \left\{ x - (2\alpha + 1) + 2\alpha^{r+1} \right\} + o(1)
\]

\[
\xrightarrow{C} \frac{1}{2(r+1)} \left\{ -2 + \frac{2}{r+2} \right\} = -\frac{1}{r+2}
\]

on invoking (145) and recalling that \( x \) is an eigenfunction of \( P \) with eigenvalue \( \frac{1}{2} \) so that \( \lim_{x \to \infty} x = 0 \). This verifies lemma 6 and the formulae in (146) when \( n = 1 \) as our base case.

Now, for the inductive step, suppose that \( g_{j,r}(x) \xrightarrow{C} 0 \quad \forall 1 \leq j \leq n - 1 \) and for all \( r \geq 0 \). Then in similar fashion to the working for the \( n = 1 \) case we have
Now in the second term here, writing

\[
\frac{k^n \cdot \alpha^{r+1}}{x} = \frac{(x - \alpha)^n \cdot \alpha^{r+1}}{x} = \frac{1}{x} \sum_{l=0}^{n} \binom{n}{l} x^l (-\alpha)^{n-l} \alpha^{r+1}
\]

it follows by the inductive hypothesis that the only term with non-zero Cesaro limit is the \( l = 1 \) term, giving overall contribution to (147) of

\[
\frac{1}{r+1} \cdot \binom{n}{1} (-1)^{n-1} \alpha^{n+r} \lesssim (-1)^{n-1} \frac{n}{(r+1)(n+r+1)}
\]

For the first term, recall that

\[
b_{n+1}(k-1) = \frac{1}{n+1} (B_{n+1}(k) - B_{n+1}) \]

and so we get overall contribution to (147) of

\[
\frac{1}{x(r+1)} \sum_{l=1}^{n+1} \binom{n+1}{l} B_{n+1-l} (x - \alpha)^l
\]

\[
= \frac{1}{(r+1)(n+1)} \sum_{l=1}^{n+1} \binom{n+1}{l} B_{n+1-l} \left\{ \sum_{j=0}^{l} \binom{l}{j} x^j (-\alpha)^{l-j} \right\}
\]

If \( l \leq n \) the inductive hypothesis applies to all except the \( j = 1 \) term, and when \( l = n+1 \) it applies to all except the \( j = 1 \) and \( j = n+1 \) terms. In the last case of \( l = j = n+1 \), however, the resulting function is a pure power of \( x \) with generalised Cesaro limit 0 in any case, and so the only contributions across the board come from the \( j = 1 \) terms, namely

\[
\frac{1}{(r+1)(n+1)} \sum_{l=1}^{n+1} \binom{n+1}{l} B_{n+1-l} \cdot l \cdot (-1)^{l-1} \alpha^{l-1} \lesssim
\]

\[
\frac{1}{(r+1)(n+1)} \sum_{l=1}^{n+1} \binom{n+1}{l} B_{n+1-l} \cdot (-1)^l
\]

\[
= \frac{1}{(r+1)(n+1)} \{ B_{n+1}(-1) - B_{n+1} \}
\]

\[
= \frac{(-1)^n}{(r+1)(n+1)} (n+1) = \frac{(-1)^n}{(r+1)}
\]
Combining the contributions (148) & (149) in (147) it follows finally that

\[ P[f_{n,r}](x) \rightarrow C \frac{(-1)^n}{(r+1)} + \frac{(-1)^{n-1}n}{(r+1)(n+r+1)} = \frac{(-1)^n}{(n+r+1)} \]

and this completes the inductive step, thus completing the proof of lemma 6 and the equivalent formulae in lemma 7.

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