ENTROPIC SOLUTION OF THE INNOVATION CONJECTURE OF T. KAILATH

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Abstract. On a general filtered probability space, for a given signal $U_t = B_t + \int_0^t \dot{u}_s ds$, we prove that the filtration of $U$ is equal to the filtration of its innovation process $Z$ if and only if

$$
H(Z(\nu)|\mu) = \frac{1}{2} E_{\nu}\left[ \int_0^1 |E_P[\dot{u}_s]|^2 ds \right]
$$

where $d\nu = \exp(-\int_0^1 E_P[\dot{u}_s][U_s]dZ_s - \frac{1}{2} \int_0^1 |E_P[\dot{u}_s][U_s]|^2 ds)dP$ in case the density has expectation one, otherwise we give a localized version of the same strength with a sequence of stopping times of the filtration of $U$.

Keywords: Invertibility, entropy, Girsanov theorem, almost sure invertibility

1. Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \in [0, 1]), P)$ be a probability space satisfying the usual conditions and denote by $(W, H, \mu)$ the classical Wiener space, i.e., $W = C_0([0, 1], \mathbb{R}^d)$, $H$ is the corresponding Cameron-Martin space consisting of $\mathbb{R}^d$-valued absolutely continuous functions on $[0, 1]$ with square integrable derivatives. Denote by $(B_t, t \in [0, 1])$ the filtration of the canonical Wiener process, completed w.r.t. $\mu$-negligeable sets. The question that we address in this paper is the following: assume that $U : \Omega \to W$ is a map of the following form:

$$
U(\omega)(t) = U_t(\omega) = B_t(\omega) + \int_0^t \dot{u}_s(\omega)ds,
$$

where $B = (B_t, t \in [0, 1])$ is an Brownian motion on $\Omega$, $(s, \omega) \to \dot{u}_s(\omega)$ is an $\mathbb{R}^d$-valued map belonging to the space $L^2(\Omega; H)$ which consists of the elements of $L^2([0, 1] \times \Omega, \mathcal{B}([0, 1]) \otimes \mathcal{F}, dt \otimes dP)$ which are $(\mathcal{F}_s, s \in [0, 1])$-adapted for almost all $s \in [0, 1]$. Let us define the innovation process $Z$ associated to $U$ as to be

$$
Z_t = U_t - \int_0^t E_P[\dot{u}_s|\mathcal{U}_s]ds,
$$

where $(\mathcal{U}_t, t \in [0, 1])$ is the filtration generated by $U$. It is well-known that $Z$ is a $P$-Brownian motion w.r.t. $(\mathcal{U}_t, t \in [0, 1])$. $Z$ is naturally adapted to $(\mathcal{U}_t, t \in [0, 1])$, this means that the information obtained via $Z$ is included in the information obtained from $U$. F.Frost [4] and T. Kailath [5] have conjectured that in practical situations the converse of this observation is also true. In [2], V.A. Beneš has remarked that this conjecture holds if and only if there is a hidden process which is a strong solution of a certain stochastic differential equation from which one can construct the initial system. This conjecture has also been proved under restrictive supplementary hypothesis (cf. [1]) where $\dot{u}$ is independent of the Brownian motion $B$. The main objection to these works lies in the fact that the condition of [2] is unverifiable from the observed data, hence numerically it is not useful, the second one uses a hypothesis of independence which is too strong to be encountered in the engineering applications. In this work we give a necessary and sufficient condition in the most general case using the entropic characterization of the almost sure invertibility of adapted perturbations of identity (API in short). Let us explain it briefly for the reader to understand the
idea and the difference from the other works: for simplicity, assume that
\[
(1.1) \quad E_P \left[ \exp \left( - \int_0^1 E_P [\dot{u}_s | \mathcal{U}_s] dZ_s - \frac{1}{2} \int_0^1 |E_P [\dot{u}_s | \mathcal{U}_s]|^2 ds \right) \right] = 1,
\]
and denote by \( \rho(-\delta_Z \hat{u}) \) the Girsanov exponential inside the above expectation, here we use the notation \( \delta \) to denote the divergence operator on the Wiener space, which coincides with the Itô integral if the Lebesgue density of the vector field is adapted (cf., for example [8, 9]). Define a new measure \( \nu \) as
\[
d\nu = \rho(-\delta_Z \hat{u}) dP.
\]
Then the observation process \( \mathcal{U} \) is adapted to the filtration of the innovation process \( Z \) up to negligible sets if and only if we have
\[
H(Z(\nu)|\mu) = \frac{1}{2} E_\nu [\hat{u}|^2] = \frac{1}{2} E_\nu \int_0^1 |E_P [\dot{u}_s | \mathcal{U}_s]|^2 ds,
\]
where \( Z(\nu) \) denotes the push forward of the measure \( \nu \) under \( Z \) and \( H(Z(\nu)|\mu) \) is the relative entropy of \( Z(\nu) \) w.r.t. the Wiener measure \( \mu \), i.e.,
\[
H(Z(\nu)|\mu) = \int_W \frac{dZ(\nu)}{d\mu} \log \frac{dZ(\nu)}{d\mu} d\mu.
\]
As it is clear, the verification of this condition, namely the equality of the entropy to the total kinetic energy of \( \hat{u} \), requires only the knowledge about the observation process \( \mathcal{U} \), however the calculation of the relative entropy may be time consuming. In fact, as it follows from Theorem 2, all these results are valid when one works causally with time, in other words, they hold also when one works on the time interval \([0, t] \), for \( t \geq 0 \) since they are restictable even to the random time intervals. The final result says that we can also suppress the hypothesis (1.1) using a sequence of \((\mathcal{U}_t, t \in [0, 1]) \) stopping times.

2. Characterization of the invertible shifts on the canonical space

We begin with the definition of the notion of almost sure invertibility with respect to a measure. This notion is extremely important since it makes the things work. Let us note that in this section all the expectations and conditional expectations are taken w.r.t. the Wiener measure \( \mu \).

**Definition 1.** Let \( T : W \to W \) be a measurable map

- \( T \) is called called \((\mu, T \mu)-\)almost surely left invertible if there exists a measurable map \( S : W \to W \) such that and \( S \circ T = I_W \) \( \mu \)-a.s.
- Moreover, in this case it is trivial to see that \( T \circ S = I_W T \mu \)-a.s., where \( T \mu \) denotes the image of the measure \( \mu \) under the map \( T \).
- If \( T \mu \) is equivalent to \( \mu \), then we say in short that \( T \) is \( \mu \)-a.s. invertible.
- Otherwise, we may say that \( T \) is \((\mu, T \mu)\)-invertible in case precision is required or just \( \mu \)-a.s. left invertible and \( S \) is called the \( \mu \)-left inverse of \( T \).

The following theorem has been proved in [12], for the reader’s convenient we give a short and different proof:

**Theorem 1.** For any \( u \in L^2(\mu, H) \), we have the following inequality
\[
H(U \mu|\mu) \leq \frac{1}{2} E \int_0^1 |\dot{u}_s|^2 ds,
\]
where \( H(U \mu|\mu) \) is the relative entropy of the measure \( U \mu \) w.r.t. \( \mu \).
**Proof:** Let $L$ be the Radon-Nikodym density of $U \mu$ w.r.t. $\mu$. For any $0 \leq g \in C_b(W)$, using the Girsanov theorem, we have

$$E[g \circ U] = E[g \circ L] \geq E[g \circ U \circ \rho(-\delta u)] ,$$

hence

$$L \circ U E[\rho(-\delta u)|U] \leq 1 ,$$

$\mu$-a.s. Consequently, using the Jensen inequality

$$H(U\mu|\mu) = E[L \log L] = E[\log L \circ U] \leq -E[\log \rho(-\delta u)|U] \leq -E[\log \rho(-\delta u)] = \frac{1}{2} E \int_0^1 |\dot{u}_s|^2 ds .$$

**Theorem 2.** Assume that $U = I_W + u$ is an API, i.e., $u \in L^2_2(\mu, H)$ such that $s \rightarrow \dot{u}(s, w)$ is $\mathcal{F}_s$-measurable for almost all $s$. Then $U$ is almost surely left invertible with a left inverse $V$ if and only if

$$H(U\mu|\mu) = \frac{1}{2} E[|u|^2_H] = \frac{1}{2} E \int_0^1 |\dot{u}_s|^2 ds ,$$

i.e., if and only if the entropy of $U\mu$ is equal to the energy of the drift $u$.

**Proof:** Due to Theorem [1] the relative entropy is finite as soon as $u \in L^2_2(\mu, H)$. Let us suppose now that the equality holds and let us denote by $L$ the Radon-Nikodym derivative of $U\mu$ w.r.t. $\mu$. Using the Itô representation theorem, we can write

$$L = \exp \left(-\int_0^1 \dot{v}_s dW_s - \frac{1}{2} \int_0^1 |\dot{v}_s|^2 ds \right) ,$$

$U\mu$-almost surely. Let $V = I_W + v$, as described in [3], from the Itô formula and Paul Lévy’s theorem, it is immediate that $V$ is an $U\mu$-Wiener process, hence

$$E[L \log L] = \frac{1}{2} E[L |v|^2_H] .$$

Now, for any $f \in C_b(W)$, we have from the Girsanov theorem

$$E[f \circ U] = E[f L] \geq E[f \circ U L \circ \rho(-\delta u)]$$

consequently

$$L \circ U E[\rho(-\delta u)|U] \leq 1 ,$$

$\mu$-a.s. Let us denote $E[\rho(-\delta u)|U] \by \hat{\rho}$. We have then $\log L \circ U + \log \hat{\rho} \leq 0 \mu$-a.s. Taking the expectation w.r.t. $\mu$ and the Jensen inequality give

$$H(U\mu|\mu) = E[L \log L] \leq -E[\log \hat{\rho}] \leq -E[\log \rho(-\delta u)] = \frac{1}{2} E[|u|^2_H] .$$

Since log is a strictly concave function, the equality $E[\log \hat{\rho}] = E[\log \rho(-\delta u)]$ implies that $\rho(-\delta u) = \hat{\rho}$ $\mu$-a.s. Hence we obtain

$$E[L \log L + \log \rho(-\delta u)] = E[\log(L \circ U \rho(-\delta u))] = 0 ,$$

since $L \circ U \rho(-\delta u) \leq 1 \mu$-a.s., we should have

$$L \circ U \rho(-\delta u) = 1$$

(2.3)
\[ 0 = \left( \int_0^1 \dot{v}_s dW_s \right) \circ U + \frac{1}{2} |v \circ U|^2_H + \delta u + \frac{1}{2} |u|^2_H \]
\[ = \delta(v \circ U) + \delta u + (v \circ U, u)_H + \frac{1}{2} (|u|^2_H + |v \circ U|^2_H) \]
(2.4)
\[ = \delta(v \circ U + u) + \frac{1}{2} |v \circ U + u|^2_H \]
\[ 0 = \left( \int_0^1 \dot{v}_s dW_s \right) \circ U + \frac{1}{2} |v \circ U|^2_H + \delta u + \frac{1}{2} |u|^2_H \]
\[ = \delta(v \circ U) + \delta u + (v \circ U, u)_H + \frac{1}{2} (|u|^2_H + |v \circ U|^2_H) \]
(2.4)
\[ = \delta(v \circ U + u) + \frac{1}{2} |v \circ U + u|^2_H \]
\mu\text{-a.s.}

Combining the exponential representation of \( L \) with the relation (2.3) implies
\[ 0 = \left( \int_0^1 \dot{v}_s dW_s \right) \circ U + \frac{1}{2} |v \circ U|^2_H + \delta u + \frac{1}{2} |u|^2_H \]
\[ = \delta(v \circ U) + \delta u + (v \circ U, u)_H + \frac{1}{2} (|u|^2_H + |v \circ U|^2_H) \]
(2.4)
\[ = \delta(v \circ U + u) + \frac{1}{2} |v \circ U + u|^2_H \]
\mu\text{-a.s.}

From the relation (2.2) it follows that \( v \circ U \in L^2_n(\mu, H) \), hence taking the expectations of both sides of (2.4) w.r.t. \( \mu \) is licit and this implies \( v \circ U + u = 0 \) \( \mu \text{-a.s.} \), which means that \( V = I_W + v \) is the \( \mu \)-left inverse of \( U \).

To show the necessity, let us denote by \( (L_t, t \in [0, 1]) \) the martingale
\[ L_t = E[L|\mathcal{F}_t] = E \left[ \frac{dU_\mu}{d\mu}|\mathcal{F}_t \right] \]
and let
\[ T_n = \inf \left( t : L_t < \frac{1}{n} \right) . \]
Since \( U \circ V = I_W \) \( (U \mu)\text{-a.s.} \), \( V \) can be written as \( V = I_W + v (U \mu)\text{-a.s.} \) and that \( v \in L^0_n(U \mu, H) \), i.e.,
\[ v(t, w) = \int_0^1 \dot{v}_s(w) ds, \quad \text{\dot{v} is adapted to the filtration (\( \mathcal{F}_t \)) completed w.r. to } U \mu \text{ and } \int_0^1 |\dot{v}_s|^2 ds < \infty \]
\( (U \mu)\text{-a.s.} \). Since \( \{ t \leq T_n \} \subset \{ L > 0 \} \) and since on this latter set \( \mu \) and \( U \mu \) are equivalent, we have
\[ \int_0^{T_n} |\dot{v}_s|^2 ds < \infty \]
\( \mu\text{-almost surely.} \)

Consequently the inequality
\[ E_\mu[\rho(-\delta v^n)] \leq 1 \]
holds true for any \( n \geq 1 \), where \( v^n(t, w) = \int_0^1 1_{[0, T_n]}(s, w) \dot{v}_s(w) ds \). By positivity we also have
\[ E_\mu[\rho(-\delta v^n)1_{\{L > 0\}}] \leq 1 . \]
Since \( \lim_n T_n = \infty(U \mu)\text{-a.s.} \), we also have \( \lim_n T_n = \infty \) \( \mu\text{-a.s.} \) on the set \( \{ L > 0 \} \) and the Fatou lemma implies
\[ E_\mu[\rho(-\delta v)1_{\{L > 0\}}] = E_\mu[\lim_n \rho(-\delta v^n)1_{\{L > 0\}}] \leq \liminf_n E_\mu[\rho(-\delta v^n)1_{\{L > 0\}}] \leq 1 . \]
for any \( n \geq 1 \). From the identity \( U \circ V = I_W \) \( (U \mu)\text{-a.s.} \), we have \( v + u \circ V = 0 \) \( (U \mu)\text{-a.s.} \), hence \( v \circ U + u = 0 \) \( \mu\text{-a.s.} \). An algebraic calculation gives immediately
\[ \rho(-\delta v) \circ U \rho(-\delta u) = 1 \]
\mu\text{-a.s.}

Now applying the Girsanov theorem to API \( U \) and using the relation (2.6), we obtain
\[ E[\rho \circ U] = E[\rho L] = E \left[ f \circ U(\rho(-\delta v)1_{\{L > 0\}}) \circ U \rho(-\delta u) \right] \leq E \left[ \rho(\rho(-\delta v)1_{\{L > 0\}}) \right] , \]
for any positive \( g \in C_b(W) \) (note that on the set \( \{ L > 0 \} \) \( \rho(-\delta v) \) is perfectly well-defined w.r.t. \( \mu \)). Therefore
\[ L \leq \rho(-\delta v)1_{\{L > 0\}} \]
\mu\text{-a.s.}

Now, this last inequality, combined with the inequality (2.5) entails that
\[ L = \rho(-\delta v)1_{\{L > 0\}} \]
\mu\text{-a.s., hence
\[ L \circ U \rho(-\delta u) = 1 \]
μ-a.s. To complete the proof it suffices to remark then that
\[ H(U \mu | \mu) = E[\log L] = E[\log L \circ U] = E[-\log \rho(-\delta u)] = \frac{1}{2} E[|u|^2_H]. \]

The following result comes almost for free:

**Theorem 3.** Assume that \( U = I_W + u \) is an API which is μ-a.s. left invertible, let \( \tau \) be any stopping time such that \( u^\tau(t, w) = u(t \wedge \tau(w), w) \) such that
\[ E[\rho(-\delta u^\tau)] = 1. \]

Then \( U^\tau = I_W + u^\tau \) is μ-a.s. invertible, in other words there exists some API, say \( V^\prime \), such that \( V^\prime \circ U^\tau = U^\tau \circ V^\prime = I_W \) μ-a.s.

**Proof:** Since \( E[\rho(-\delta u^\tau)] = 1 \), \( U^\tau \mu \) is equivalent to the Wiener measure \( \mu \), hence its Radon-Nikoyym density can be written as
\[ \frac{dU^\tau \mu}{d\mu} = \rho(-\delta \xi). \]

From the Girsanov theorem it follows that
\[ (2.7) \quad \rho(-\delta \xi) \circ U^\tau E[\rho(-\delta u^\tau)|U^\tau] = 1 \]
μ-a.s. Let \( z \) be the innovation process of \( U^\tau \), which is defined as \( z_t = U^\tau_t - \int_0^t E[\dot{u}^\tau_s | \mathcal{U}^\tau_s] ds \), where \( (\mathcal{U}^\tau_s, s \in [0,1]) \) denotes the filtration corresponding to \( U^\tau \). Applying the Girsanov theorem again, this time using the Brownian motion \( z \) (cf. [12] for the details), we find that
\[ E[\rho(-\delta u^\tau)|U^\tau] = \exp \left( -\int_0^1 E[\dot{u}^\tau_s | \mathcal{U}^\tau_s] ds - \frac{1}{2} \int_0^1 |E[\dot{u}^\tau_s | \mathcal{U}^\tau_s]|^2 ds \right). \]

This relation, combined with the equation (2.7) gives the relation
\[ \dot{\xi}_t \circ U^\tau + E[\dot{u} 1_{[0,\tau]}(t) | \mathcal{U}^\tau_t] = 0 \]
dt × dμ-a.s. Besides, for any \( A \in L^\infty(\mu) \), we have
\[ E[A E[\dot{u} 1_{[0,\tau]}(t) | \mathcal{U}^\tau_t]] = E[E[A | \mathcal{U}^\tau_t] \dot{u} 1_{[0,\tau]}(t)] = E[E[A | \mathcal{U}_t] \dot{u} 1_{[0,\tau]}(t)] = E[A E[\dot{u}^\tau_t | \mathcal{U}_t]] = E[A \dot{u}^\tau_t], \]
where the last equality follows from the left invertibility of \( U \). Hence we obtain
\[ \dot{\xi}_t \circ U^\tau + \dot{u} 1_{[0,\tau]}(t) = \dot{\xi}_t \circ U^\tau + \dot{u}^\tau_t = 0 \]
dt × dμ-a.s., which is equivalent to μ-a.s. invertibility of \( U^\tau \).
\[ \square \]
3. The case of a general probability space

The following result is essential for the proof of the conjecture where we use the notations explained in the introduction and we differentiate carefully the Wiener measure $\mu$ and the probability $P$ as well as the respective expectations and conditional expectations to avoid any ambiguity.

**Theorem 4.** Let $U = B + u = B + \int_0^t \dot{u}_s ds$ be an adapted perturbation of identity mapping $\Omega$ to $W$ with $u \in L^2(P, H)$. Then

$$H(U(P)|\mu) = \frac{1}{2} E_P[|u|^2_H]$$

if and only if there exists some $v : W \to H$ (of the form $v = \int_0^t \dot{v}_s ds$) with $\dot{v}$ adapted ds-a.s. to the filtration $(B_t(W))$ such that

$$U(\omega) = B(\omega) - v \circ U(\omega),$$

which implies in particular that $B = Z$, where $Z$ is the innovation process associated to $U$, in other words $U$ is a solution of the following stochastic differential equation

$$dU_t = -\dot{v} \circ U dt + dB_t.$$

**Proof:** Note first that $U$ is not necessarily a strong solution. Let us now prove the necessity: Since $U$ is an API, $U(P)$ is absolutely continuous w.r.t. the Wiener measure $\mu$, let $l$ be the corresponding Radon-Nikodym derivative. We can represent it as a Girsanov exponential $U(P)$-a.s., i.e., we have

$$l = \frac{dU(P)}{d\mu} = \rho(-\delta v)$$

$$= \exp \left( \int_0^1 \dot{v}_s dW_s - \frac{1}{2} \int_0^1 |\dot{v}_s|^2 ds \right),$$

$U(P)$-a.s., where $(W_t)$ is the canonical Wiener process. For any positive $f \in C_b(W)$, it follows from the Girsanov theorem

$$E_P[f \circ U] = E_\mu[f l] \geq E_P[f \circ U l \circ U \rho(-\delta B u)],$$

where

$$\rho(-\delta B u) = \exp \left( \int_0^1 \dot{u}_s dB_s - \frac{1}{2} \int_0^1 |\dot{u}_s|^2 ds \right).$$

This inequality, which is valid for any positive, measurable $f$, implies that

$$l \circ U E_P[\rho(-\delta B u)|U] \leq 1$$

$P$-a.s. Therefore

$$H(U(P)|\mu) = E_\mu[l \log l] = E_P[\log l \circ U]$$

$$\leq -E_P[\log E_P[\rho(-\delta B u)|U]] \leq \frac{1}{2} E_P[|u|^2_H].$$

The equality hypothesis $H(U(P)|\mu) = \frac{1}{2} E_P[|u|^2_H]$ and the strict convexity of the function $x \to -\log x$ imply that

$$l \circ U \rho(-\delta B u) = 1$$

$P$-a.s. Therefore

$$1 = \rho(-\delta v) \circ U \rho(-\delta B)$$

$$= \exp -\left( (\delta v) \circ U + \frac{1}{2} |v \circ U|^2_H + \delta_B u + \frac{1}{2} |u|^2_H \right)$$

$$= \exp -\left( \delta_B (v \circ U) + (v \circ U, u)_H + \frac{1}{2} |v \circ U|^2_H + \delta_B u + \frac{1}{2} |u|^2_H \right),$$

which implies that

$$\delta_B (u + v \circ U) + \frac{1}{2} |v \circ U + u|^2_H = 0$$
P-a.s. Since \( E_P[|v \circ U|^2_H] = E_\mu[|v|^2_H] = 2E_\mu[|\log l|] \), it follows that \( v \circ U + u = 0 \) P-a.s. Note that we can write

\[
U = B + u = Z + \hat{u}, \quad \hat{u} = \int_0^1 E_P[\hat{u}_s|U_s]ds,
\]

since \( u = -v \circ U \), \( \hat{u} \) is adapted to the filtration of \( U \), therefore \( B = Z \).

**Sufficiency:** If \( U = B - v \circ U \), then \( Z = B \) and \( v \circ U + u = 0 \). Let \( l \) denote again the Radon-Nikodym derivative of \( U(P) \) w.r.t. \( \mu \), as before we can write \( l = \rho(-\delta_x) U(P) \)-a.s., for some \( \xi : W \to H \) such that \( \xi = \int_0^1 \hat{\xi}_s ds \), \( \int_0^1 |\hat{\xi}_s|^2 ds < \infty U(P) \)-a.s. and \( \hat{\xi} \) is \( B_s(W) \)-measurable \( ds \)-a.s. Using the Girsanov theorem as above, we find that

\[
l \circ U E_P[\rho(-\delta_B u)|U] \leq 1
\]

but the hypothesis implies that \( \rho(-\delta_B u) \) is \( U \)-measurable, it then follows that

\[
\delta_B(u + \xi \circ U) + \frac{1}{2}|\xi \circ U + u|^2_H \leq 0
\]

P-a.s. Since \( E_P[|\xi \circ U|^2_H] = 2H(U(P)|\mu) < \infty \), it follows that \( \xi \circ U = v \circ U \) P-a.s. Consequently

\[
H(U(P)|\mu) = E_\mu[|\log l|] = E_P[|\log l \circ U|] = -E_P[\log \rho(-\delta_B u)] = \frac{1}{2} E_P[|u|^2_H]
\]

and this completes the proof.

Theorem \( \text{4} \) says that \( U = B + u \) with \( u \in L^0_0(P,H) \), is the weak solution of the SDE

\[
dU_t = dB_t - \hat{v}_t \circ U dt
\]

if and only if we have the equality between the entropy \( H(U(P)|\mu) \) and the total kinetic energy of \( u \) qw.r.t. to the probability \( P \). A natural question is: when this solution is strong? The following theorem gives the answer:

**Theorem 5.** Assume that \( U \) is a weak solution of the SDE

\[
dU_t = dB_t - \hat{v}_t \circ U dt,
\]

with the hypothesis that \( v \circ U \in L^0_0(P,H) \), define the sequence of stopping times \( (t_n, n \geq 1) \) as

\[
t_n = \inf \left( t : \int_0^t |\hat{v}_s|^2 ds > n \right)
\]

let

\[
\hat{u}_n(t, \omega) = -\hat{v}_t \circ U(\omega) 1_{[0, t_n \circ V]}
\]

and let \( U^n = B + u_n \) where \( u_n(t, \omega) = \int_0^t \hat{u}_n(s, \omega) ds \). Define a new probability \( Q_n \) as \( dQ_n = \rho(-\delta_B(u_n)) dP \). Then

\[
H(B(Q_n)|\mu) = \frac{1}{2} E_{Q_n}[|u_n|^2_H],
\]

for any \( n \geq 1 \) if and only if \( U \) is a strong solution.

**Proof:** **Necessity:** Since under \( Q_n \), \( U_n \) is a Brownian motion and the hypothesis combined with Theorem \( \text{4} \) implies that \( v_n \circ U \) is measurable w.r.t. the filtration of \( B \) up to \( Q_n \)-negligeable sets, since \( Q_n \) is equivalent to \( P \), it follows that \( v_n \circ U \) is adapted to the same filtration completed with \( P \)-negligeable sets. Since \( \lim_{n \to \infty} v_n \circ U = v \circ U \), \( U \) is also adapted to the \( P \)-completion of the filtration of \( B \), hence \( U \) is a strong solution of the above SDE.

**Sufficiency:** If \( U \) is a strong solution, then it is of the form \( U = \hat{U}(B) = B - v \circ \hat{U}(B) \) and \( \hat{U} : W \to W \) has a \( \mu \)-a.s. left inverse \( V = I_W + v \). Since \( Q_n \) is equivalent to \( P \), we have also \( U = \hat{U}(B) Q_n \)-a.s. Moreover \( B = U_n + v_n \circ U \) and \( v_n \circ U \) is adapted to the filtration of \( B \) up to \( Q_n \)-negligeable sets for any \( n \geq 1 \). Due to Theorem \( \text{4} \) this is equivalent to the equality

\[
H(B(Q_n)|\mu) = \frac{1}{2} E_{Q_n}[|u_n|^2_H],
\]
for any $n \geq 1$. \hfill \Box

4. Proof of the Innovation Conjecture

We are now at a position to give the proof of the conjecture. We shall do it in two steps using
the notations explained in the introduction. The first step is with a supplementary hypothesis to
explain clearly the idea, the second one is in full generality.

We have the relation

$$U = B + u = Z + \hat{u},$$

we shall denote by $\mathcal{Z}(t, t \in [0,1])$ the filtration generated by the innovation process $Z$. We use also
the notation

$$\rho(-\delta Z \hat{u}) = \exp\left(-\int_0^1 E_P[\hat{u}_s|\mathcal{U}_s]dZ_s - \frac{1}{2} \int_0^1 |E_P[\hat{u}_s|\mathcal{U}_s]|^2ds\right).$$

First we give a proof with a supplementary hypothesis which will be suppressed at the final proof:

**Proposition 1.** Assume that

$$E_P[\rho(-\delta Z \hat{u})] = 1,$$

denote then by $\nu$ the probability defined by $d\nu = \rho(\delta Z \hat{u})dP$. Then $\mathcal{U}_t = \mathcal{Z}_t$ for any $t \geq 0$
up to negligeable sets and $\hat{u} = v \circ Z$, with $v \in L^0(\mu, H)$ with $\hat{v}_s$ being $\mathcal{B}_s(W)$-measurable $ds$-almost surely,
if and only if

$$H(Z(\nu)|\mu) = \frac{1}{2} E_\nu[|\hat{u}|_H^2].$$

**Proof:** From Paul Lévy’s Theorem, $U$ is a Brownian motion under the measure $\nu$ and $Z = U - \hat{u}$. Then Theorem 4
says that (replacing $B$ by $U$ and $P$ by $\nu$), $\hat{u}$ is a functional of $Z$ and that $s \to E_P[\hat{u}_s|\mathcal{U}_s]$ is adapted to the filtration
$(\mathcal{Z}_s, s \in [0,1])$ ds-a.s. Hence $U$ is $Z$-measurable. Moreover, the same theorem implies the existence of some $v \in L^0(\mu, H)$
which is defined as

$$\frac{dZ(\nu)}{d\mu} = \rho(\delta v)$$

such that $\hat{u} = v \circ Z$. \hfill \Box

Now we are ready to give the full proof:

**Theorem 6.** Let $T_n = \inf(t : \int_0^t |E_P[\hat{u}_s|\mathcal{U}_s]|^2ds > n)$, define

$$\hat{u}_n(t, \omega) = \hat{u}(t \land T_n, \omega)$$

$$U_n = Z + \hat{u}_n.$$

Then $\mathcal{Z}_t = \mathcal{U}_t$ for any $t \geq 0$ up to negligeable sets and $\hat{u} = \hat{u} \circ Z$ with some $\hat{u} \in L^0(\mu, H)$ if and only if
we have

$$H(Z(\nu_n)|\mu) = \frac{1}{2} E_{\nu_n}[|\hat{u}_n|_H^2]$$

for any $n \geq 1$, where $d\nu_n = \rho(-\delta Z \hat{u}_n)dP$, and

$$\rho(-\delta Z \hat{u}_n) = \exp\left(-\int_0^{T_n} E_P[\hat{u}_s|\mathcal{U}_s]dZ_s - \frac{1}{2} \int_0^{T_n} |E_P[\hat{u}_s|\mathcal{U}_s]|^2ds\right).$$

**Proof:** **Sufficiency:** Under the measure $\nu_n$, $U_n$ is a Brownian motion and $Z = U_n - \hat{u}_n$. It follows from Theorem 4
that $\hat{u}_n$ is $(\mathcal{Z}_t, \nu_n)$-adapted if and only if the relation (4.3) holds true. Since $\nu_n$ is equivalent to $P$, $U_n$ is also
$(\mathcal{Z}_t, P)$-adapted for any $n \geq 1$, since $U_n \to U$ in $L^0(P, W)$, $U$ is also
$(\mathcal{Z}_t, P)$-adapted.

**Necessity:** Assume that $U$ is $(\mathcal{Z}_t, P)$-adapted, then it is also $(\mathcal{Z}_t, \nu_n)$-adapted since $\nu_n \sim P$ for any $n \geq 1$. Hence $U_n$ is also
$(\mathcal{Z}_t, \nu_n)$-adapted for any $n \geq 1$ and this is equivalent to the relation (4.3) for any $n \geq 1$. \hfill \Box
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