Some positivity results of the curvature on the group corresponding to the incompressible Euler equation with Coriolis force

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October 19, 2021

Abstract

In this article, we investigate the geometry of a central extension \( \hat{\mathcal{D}}_\mu(S^2) \) of the group of volume-preserving diffeomorphisms of the 2-sphere equipped with the \( L^2 \)-metric, whose geodesics correspond solutions of the incompressible Euler equation with Coriolis force. In particular, we calculate the Misiołek curvature of this group. This value is related to the existence of a conjugate point and its positivity directly implies the positivity of the sectional curvature.

Keywords: inviscid fluid flow, diffeomorphism group, conjugate point, Coriolis force, curvature, central extension.

MSC2020; Primary 35Q35; Secondary 35Q31.

1 Introduction

The incompressible Euler equation on a Riemannian manifold \( M \) is given by

\[
\begin{align*}
\frac{\partial u}{\partial t} + \nabla u \cdot u &= -\nabla p, \\
\text{div} \, u &= 0, \\
|t=0| &= u_0.
\end{align*}
\]

(1.1)

Its solutions correspond to geodesics on the group \( \mathcal{D}_\mu(M) \) of volume-preserving diffeomorphisms of \( M \) with \( L^2 \)-metric \( \langle \cdot, \cdot \rangle \), which was discovered by V. I. Arnol’d [1]. In the case of \( M = \mathbb{T}^2 \), the flat torus, G. Misiołek calculated the second variation of a geodesic corresponding to a certain stationary solution \( X \) of (1.1)

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and showed that the existence of a conjugate point along it. Moreover, he also revealed the importance of the value

\[ MC_{X,Y}^g := -\| [X, Y] \|_2^2 - \langle [X, [X, Y]], Y \rangle \]

where \( Y \in \mathfrak{g} \) and \( \mathfrak{g} \) is the space of divergence-free vector fields. Namely, he essentially proved Fact 2.1 which states that the \( MC_{X,Y}^g > 0 \) ensures the existence of a conjugate point on the group. We call this important value \( MC_{X,Y}^g \) the Misio\'lek-curvature and want to study when it is positive or negative. We note that the existence of conjugate points along a geodesic is related to some stability of corresponding solution in this context.

For \( s \geq 1 \), define a 2-dimensional manifold \( M_s \) by

\[ M_s := \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = s^2(1 - z^2) \}. \]

Note that \( M_s = S^2 \) if \( s = 1 \). In this article, we calculate the Misio\'lek curvature in the case of the incompressible Euler equation with Coriolis force \( az \) (\( a > 0 \)) on \( M_s \):

\[
\begin{align*}
\frac{\partial u}{\partial t} + \nabla u &= az \ast (u) - \text{grad} p, \\
\text{div} u &= 0, \\
u|_{t=0} &= u_0,
\end{align*}
\]

where \( \ast \) is the Hodge operator. (This equation is sometimes used as the model of flows on the earth, see [3].) In this case, solutions correspond to geodesics on the central extension \( \hat{D}_\mu(M_s) \) of the group of volume-preserving diffeomorphisms of \( M_s \) (see Section 3), whose tangent space at the identity is identified with \( \mathfrak{g} \oplus \mathbb{R} \).

Our main result is the Misio\'lek curvature \( MC_{\mathfrak{g} \oplus \mathbb{R}} \) of this group in the direction to a west-facing zonal flow:

**Definition 1.1.** We call a vector field \( Z \) on \( M_s \) a zonal flow if \( Z \) has the form

\[ Z = F(z)(x \partial_y - y \partial_x) \]

for some function \( F \). Moreover, if \( F \leq 0 \), we call \( Z \) a west-facing zonal flow.

Note that any zonal flow is a stationary solution of (1.1) and (1.2). Then, our main results are the following:

**Theorem 1.2.** Let \( Z \) be a nonzero west-facing zonal flow and \( a \in \mathbb{R}_{>0} \). Then we have

\[ MC_{\mathfrak{g} \oplus \mathbb{R}}^{Z,a} > MC_{Z,Y}^g \]

for any \( (Y, b) \in \mathfrak{g} \oplus \mathbb{R} \).

Note that the definition of the west-facing zonal flow is just for simplicity. We can easily generalize it.
Corollary 1.3. Suppose $s > 1$. Let $Z$ be a nonzero west-facing zonal flow whose support is contained in $M_s \setminus \{(0,0,\pm 1)\}$ and $a \in \mathbb{R}_{>0}$. Then, there exists $Y \in g$ satisfying $MC_{(Z,a),(Y,b)} > 0$ for any $b \in \mathbb{R}$.

The theorem states that for any west-facing zonal flow $Z$, the Misio\l ek curvature of $Z$ regarded as a solution of (1.2) is greater than the Misio\l ek curvature regarded as a solution of (1.1). We note that the positivity of the Misio\l ek curvature directly implies the positivity of the sectional curvature on the corresponding group (see Definition B.4 and Lemma B.6 in Appendix). Thus, the corollary implies the positivity of the sectional curvature on $\hat{D}_{\mu}(M_s)$ under some support condition.

Our motivation of this study is the existence of a stable multiple zonal jet flow on Jupiter whose mechanism is not yet well understood. See [8, Section 1] and references therein for more explanations and related studies.

Acknowledgment. The authors are very grateful to G. Misio\l ek for the very fruitful discussion. Research of TT was partially supported by Grant-in-Aid for JSPS Fellows (20J00101), Japan Society for the Promotion of Science (JSPS). Research of TY was partially supported by Grant-in-Aid for Scientific Research B (17H02860, 18H01136, 18H01135 and 20H01819), Japan Society for the Promotion of Science (JSPS).

2 Misio\l ek curvature

In this section, we define the Misio\l ek curvature and explain its importance. We refer to [7, 8].

Let $G$ be a (infinite-dimensional) Lie group with right-invariant metric $\langle , \rangle$, and $g$ the Lie algebra of $G$. Then, we define the Misio\l ek curvature $MC_{X,Y} := MC^g_{X,Y}$ by

$$MC_{X,Y} := -||[X,Y]||^2 - \langle X, [[X,Y],Y] \rangle. \quad (2.3)$$

The first importance of this value is that the positivity of $MC$ directly implies that of the curvature (see Definition B.3 and Lemma B.5 in Appendix). Note that this formula of $MC$ seems to be simpler than the general formula of the curvature on the group with right-invariant metric (see Lemma B.2). The second and main importance of $MC$ is Fact 2.1 given below. In [7], this fact is proved for the case that $G$ is the group $D^{\mu}_{\mu}((T^2)$ of volume-preserving $H^\mu$-diffeomorphisms of the 2-dimensional flat torus $T^2$. (For the case $D^{\mu}_{\mu}(M)$, where $M$ is a compact $n$-dimensional Riemannian manifold, see also [8].) The essential point of the proof in [7] is the fact that the inverse function theorem holds for the Riemannian exponential map $\exp : T_xD^{\mu}_{\mu}(M) \to D^{\mu}_{\mu}(M)$. Here, we say that the inverse function theorem holds for $\exp$ if $\exp$ is isometry near $X \in T_xD^{\mu}_{\mu}(M)$ whenever the differential of $\exp$ is an isomorphism at $X$. Thus, we obtain the following:

Fact 2.1. Suppose that there exists the (Riemannian) exponential map $\exp : g \to G$ and the inverse function theorem holds for $\exp$. Let $X \in g$ be a stationary
solution of the Euler-Arnol’d equation. Suppose that there exists \( Y \in g \) satisfying \( MC_{X,Y} > 0 \). Then, there exists a point conjugate to the identity element \( e \in G \) along the geodesic corresponding to \( X \) on \( G \).

Fact 2.1 states that the positivity of the Misiołek curvature ensures that the existence of a conjugate point.

3 Central extension of volume-preserving diffeomorphism group

In this section, we briefly recall about the central extension of the volume-preserving diffeomorphism group by a Lichnerowicz cocycle. Our main references are \([6, 9]\).

Let \((M, g)\) be a compact \( n \)-dimensional Riemannian manifold and \( \mathcal{D}_\mu(M) \) the group of volume-preserving \( C^\infty \)-diffeomorphisms with the \( L^2 \)-metric:

\[
\langle X, Y \rangle := \int_M g(X, Y) \mu,
\]

where \( \mu \) is the volume form. We write \( g \) for the space of divergence-free vector fields on \( M \), which is identified with the tangent space of \( \mathcal{D}_\mu(M) \) at the identity element.

For a closed 2-form \( \eta \), we define a Lichnerowicz 2-cocycle \( \Omega \) on \( g \) by

\[
\Omega(X, Y) := \int_M \eta(X, Y).
\]

If \( \eta \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R}) \), this cocycle integrates the group \( \mathcal{D}_{\mu}^{\text{ex}}(M) \) of exact volume preserving diffeomorphism. This group coincides with the identity component of \( \mathcal{D}_\mu(M) \) if \( H^{n-1}(M, \mathbb{R}) = 0 \). Thus, in such a case, there exists a central extension \( \hat{\mathcal{D}}_\mu(M) \) of the identity component of \( \mathcal{D}_\mu(M) \), whose tangent space at the identity is \( g \oplus \mathbb{R} \) and its Lie bracket and inner product are given by

\[
[(X, a), (Y, b)] = ([X, Y], \Omega(X, Y)),
\]

\[
\langle (X, a), (Y, b) \rangle = \langle X, Y \rangle + ab.
\]

Take a \((n-2)\)-form \( B \) satisfying \( \eta = \iota_B(\mu) \), or equivalently,

\[
\Omega(X, Y) = \int_M \eta(X, Y) \mu = \int_M \mu(B, X, Y) = \int_M g(B \times X, Y),
\]

where \( B \times X := \ast(B \wedge X) \). Note that this can be rewritten as

\[
\Omega(X, Y) = \langle P(B \times X), Y \rangle,
\]

where \( P : \mathfrak{X}(M) \to g \) is the projection to the divergence-free part. Then, the Euler-Arnol’d equation of \( \hat{\mathcal{D}}_\mu(M) \) is

\[
\frac{\partial u}{\partial t} = -\nabla u + au \times B - \grad p.
\]
Remark 3.1. This formula is slightly different from [7] because the sign convention differs. In order to clarify this, we summarize our conventions in Appendix.

We summarize the contents of this section in dim \( M = 2 \).

Proposition 3.2. Suppose that \( \dim M = 2 \), \( \eta \in H^2(M, \mathbb{Z}) \) and \( H^1(M, \mathbb{R}) = 0 \). Then there exists a central extension group \( \hat{D}_\mu(M) \) of the identity component of \( D_\mu(M) \), whose Euler-Arnol’d equation is

\[
\frac{\partial u}{\partial t} = -\nabla_u u + au \ast B - \text{grad } p.
\] (3.10)

Moreover, the Misiolek curvature \( MC(x,a),(y,b) := MC_{\mathbb{R} \oplus \mathbb{R}}(x,a),(y,b) \) is given by

\[
MC(x,a),(y,b) = -||[X,Y]||^2 - \langle [X,[[X,Y],Y]] - \Omega(X,Y)^2 - a\Omega([X,Y],Y)
\]

\[
MC_{XY} - \Omega(X,Y)^2 - a\Omega([X,Y],Y)
\] (3.11)

for \( (x,a),(y,b) \in g \oplus \mathbb{R} \).

Proof. Note that \( u \times B = u \ast B \) by \( \dim M = 2 \). Moreover, the assertion of the Misiolek curvature follows from the definition, (3.6) and (3.7). \( \square \)

4 \( M = M_s \) case

In this section, we apply the results in Section 3 to the case \( M = M_s \). Recall

\[ M_s := \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = s^2(1 - z^2) \} \]

Proposition 4.1. Let \( B := z \) and \( \eta := z\mu \). Then, there exists a group \( \hat{D}_\mu(M_r) \) whose Euler-Arnol’d equation is (3.11).

Proof. Note that \( \eta = 0 \in H^2(M_s, \mathbb{Z}) \) because

\[
\int_{M_s} z\mu = 0.
\]

Thus, the proposition follows from Proposition 3.2 \( \square \)

Take a “spherical coordinate” of \( M_s \):

\[
\phi := \phi_s : (-d,d) \times (-\pi, \pi) \rightarrow M_s \quad (r, \theta) \mapsto (c_1(r) \cos \theta, c_1(r) \sin \theta, c_2(r))
\]

in such a way that \( c_2(0) = 0, c_1(r) > 0, \dot{c}_2(r) > 0 \), and that \( c_1^2 + c_2^2 = 1 \). Note that \( (c_1,c_2,d) = (\cos(r),\sin(r),\pi/2) \) in the case of \( s = 1 \ (M_1 = S^2) \). Then, we obtain

\[
g(\partial_r, \partial_r) = 1, \quad g(\partial_r, \partial_\theta) = 0, \quad g(\partial_\theta, \partial_\theta) = c_1^2
\]

and

\[
\mu = c_1(r) d\theta \wedge dr.
\]
This implies
\[ *\partial_r = -\frac{\partial_\theta}{c_1}, \quad *d\theta = -c_1 d\theta, \quad *\partial_\theta = c_1 \partial_r, \quad *d\theta = \frac{dr}{c_1}, \]
and
\[ \langle X, Y \rangle = \int_{-d}^{d} \int_{-\pi}^{\pi} \left( X_1 Y_1 + X_2 Y_2 c_1^2 \right) c_1 d\theta dr \]
for \( X = X_1 \partial_r + X_2 \partial_\theta \) and \( Y = Y_1 \partial_r + Y_2 \partial_\theta \), which are elements of \( \mathfrak{g} \). Moreover, we have
\[ \text{grad } f = \partial_r f \partial_r + c_1^{-2} \partial_\theta f \partial_\theta, \]
\[ \text{div } u = (\partial_r + c_1^{-1} \partial_\theta c_1) u_1 + \partial_\theta u_2 \]
for a function \( f \) on \( M \) and \( u = u_1 \partial_r + u_2 \partial_\theta \). Recall that we call a vector field \( Z \) on \( M \) a zonal flow if \( Z \) has the form
\[ Z = F(r) \partial_\theta \]
for some function, which depends only on the variable \( r \). Moreover, if \( F \leq 0 \), we call \( Z \) a west-facing zonal flow.

**Lemma 4.2.** Let \( Z = F(r) \partial_\theta \) be a zonal flow. Then, we have
\[ \Omega(Z, Y) = 0 \]
\[ \Omega(Y, [Y, Z]) = \int_{-d}^{d} \int_{-\pi}^{\pi} c_1^2 Y_1^2 F \partial_r c_2 dr d\theta. \]
for \( Y = Y_1 \partial_r + Y_2 \partial_\theta \in \mathfrak{g} \).

**Proof.** Recall that \( B = z = c_2(r) \) and that
\[ \Omega(Z, Y) = \langle P(B \times Z), Y \rangle = \langle P(B * Z), Y \rangle. \]
The last equality follows from \( \dim M = 2 \). On the other hand,
\[ B * Z = c_1(r) c_2(r) F(r) \partial_r. \]
This expression implies that there exists a function \( f \) satisfying \( \text{grad } f = B * Z \). Thus we have \( P(B * Z) = 0 \) which implies the first equality.

For the second equality, we have
\[ [Y, Z] = -F \partial_\theta Y_1 \partial_r + (Y_1 \partial_r F - F \partial_\theta Y_2) \partial_\theta \]
\[ *BY = B \left( c_1 Y_2 \partial_r - \frac{Y_1}{c_1} \partial_\theta \right). \]
Moreover,
\[
\Omega(Y, [Y, Z]) = (P(\ast BY), [Y, Z])
\]
\[
= (\ast BY, [Y, Z])
\]
\[
= \int_{-\pi}^{\pi} B (-Fc_1Y_2\partial_\theta Y_1 - c_1Y_1 (Y_1\partial_r F - F\partial_\theta Y_2)) \ c_1 \ drd\theta
\]
\[
= \int_{-\pi}^{\pi} B (Fc_1^2(Y_2\partial_\theta Y_1 - Y_1\partial_\theta Y_2) + c_1^2Y_1^2\partial_r F) \ drd\theta.
\]
This is equal to
\[
= - \int_{-\pi}^{\pi} B (Fc_1^2Y_1^2\partial_r F - 2c_1^2Y_1F\partial_r Y_2) \ drd\theta
\]
\[
= - \int_{-\pi}^{\pi} B (c_1^2Y_1^2\partial_r Y_2 + 2c_1^2Y_1F - 2c_1^2Y_1F\partial_\theta Y_2) \ drd\theta.
\]
Recall that
\[
\text{div} \ Y = \partial_r Y_1 + \frac{\partial_r c_1}{c_1} Y_1 + \partial_\theta Y_2.
\]
Thus, divergence-freeness of \( Y \) implies
\[
\Omega(Y, [Y, Z]) = - \int_{-\pi}^{\pi} B (c_1^2Y_1^2\partial_r F + 2c_1^2Y_1F (\partial_r Y_1 + \frac{\partial_r c_1}{c_1} Y_1)) \ drd\theta
\]
\[
= - \int_{-\pi}^{\pi} B (c_1^2Y_1^2\partial_r F + c_1^2\partial_r (Y_1^2)F + \partial_r (c_1^2) Y_1^2 F) \ drd\theta.
\]
By the Stokes theorem, this is equal to
\[
= \int_{-\pi}^{\pi} \partial_r Bc_1^2Y_1^2 F \ drd\theta.
\]
This completes the proof.

\[\square\]

**Corollary 4.3.** Let \( Z = F(r)\partial_\theta \) be a zonal flow. Then, we have
\[
MC_{(Z,a)}(Y,b) = MC_{Z,Y} - a \int_{-\pi}^{\pi} c_1^2Y_1^2 F \sqrt{1 - c_1^2} \ drd\theta
\]
for \( Y = Y_1\partial_r + Y_2\partial_\theta \in g. \)
Proof. This is a consequence of (3.11), Lemma 4.2 and the definition of $c_1, c_2$. Note that $\Omega([Z, Y], Y) = \Omega(Y, [Y, Z])$ by the fact that $\Omega$ is a 2-cocycle. □

Proof of Theorem 1.2 Corollary 4.3 and (3.11) imply the theorem. □

Proof of Corollary 1.3. This follows from Theorem 1.2 and Fact 4.4. □

Fact 4.4 ([8, Thm. 1.2]). Let $s > 1$. Then, for any zonal flow $Z$ on $M_s$ whose support is contained in $M_s \setminus \{(0, 0, \pm 1)\}$, there exists $Y \in \mathfrak{g}$ satisfying $MC_{Z, Y} > 0$.

5 Final remark

Note that we do not know whether $\hat{D}_\mu(M_s)$, whose existence is guaranteed by Proposition 1.1, satisfies the assumption of Fact 2.1. Therefore, we cannot conclude the existence of a conjugate point on $\hat{D}_\mu(M_s)$. This is still under intensive research.

A Appendix

A.1 Sign conventions

In this subsection, we briefly derive the formula (3.9) in order to clarify our sign conventions. Therefore all contents in this section are known.

We refer to [4, Section 46] or [9, Section 2].

A.1.1 Right-invariant Maurer-Cartan form

Let $G$ be a (possibly infinite-dimensional) Lie group and $\mathfrak{g}$ its Lie algebra.

Definition A.1. The right-invariant Maurer-Cartan form $\omega$ is the $\mathfrak{g}$-valued 1-form on $G$ defined by

$$\omega_g(X) := r_g^{-1} X \in \mathfrak{g},$$

where $r_g^{-1}$ is the differential of the right translation map $R_g^{-1}(h) := hg^{-1}$.

For $X \in \mathfrak{g}$, we write $X^R$ for the right-invariant vector fields on $G$ with $X^R(e) = X$. Note that

$$[X^R, Y^R] = -[X, Y]^R.$$  \hspace{1cm} (A.12)

Lemma A.2. For $X, Y \in \mathfrak{g}$, we have

$$\omega([X^R, Y^R]) = [\omega(X^R), \omega(Y^R)].$$

Proof. By (A.12), we have

$$\omega([X^R, Y^R]) = \omega(-[X, Y]^R) = -[X, Y]^R = [X^R, Y^R] = [\omega(X^R), \omega(Y^R)]$$

which completes the proof. □
Lemma A.3. For smooth vector fields $U, V$ on $G$, we have
\[ d\omega(U, V) = -[\omega(U), \omega(V)]. \]

Proof. Recall that
\[ d\omega(U, V) = U(\omega(V)) - V(\omega(U)) - \omega([U, V]). \]
In the case of $U = X^R$ and $V = Y^R$, we have
\[ d\omega(X^R, Y^R) = -\omega([X^R, Y^R]) = -[\omega(X^R), \omega(Y^R)]. \]
This equation is the one as $g$-valued 2-forms. Thus this holds for any $U, V$.

Corollary A.4. For smooth vector fields $U, V$ on $G$, we have
\[ [\omega(U), \omega(V)] = U(\omega(V)) + V(\omega(U)). \]

Proof. This is obvious by preceding lemma.

A.1.2 Euler-Arnol’d equation

Let $G$ be a (possibly infinite-dimensional) Lie group with right-invariant metric $\langle \cdot, \cdot \rangle$ and $g = T_e G$ the Lie algebra. Define $\langle [\cdot, \cdot]^*, \cdot \rangle$ by
\[ \langle [X, Y]^*, Z \rangle = \langle Y, [X, Z] \rangle \]
if it exists. We always assume the existence of $\langle [\cdot, \cdot]^*, \cdot \rangle$ in this article.

Lemma A.5. Let $\eta$ be a curve on $G$. Define a curve $c : [0, t_0] \to g$ by $c(t) := r_{\eta^{-1}}(\dot{\eta})$. Then, $\eta$ is a geodesic if and only if $c$ satisfies
\[ \partial_t c = [c, c]^*. \quad (A.13) \]
Moreover, for a curve $c : [0, t_0] \to g$ satisfying $(A.13)$, there exists a geodesic $\eta$ on $G$ satisfying $c(t) = r_{\eta^{-1}}(\dot{\eta})$ if $G$ is regular in the sense of [2, Def. 7.6].

Proof. Consider the energy function of a curve $\eta$ on $G$:
\[ E(\eta) = \frac{1}{2} \int_0^t \langle \dot{\eta}, \dot{\eta} \rangle dt \]
\[ = \frac{1}{2} \int_0^t ||r_{\eta^{-1}}(\dot{\eta})|| dt. \]
For a proper variation $\eta_s$ of $\eta$, define $c_s(t) := r_{\eta^{-1}}(\dot{\eta}) = \omega(\dot{\eta}) \in g$, $X_s(t) := \partial_s \gamma_s(t)$, and $x_s(t) := \omega(X_s)$ where $\omega$ is the right-invariant Maurer-Cartan form. Then, the first variation is
\[ \partial_s E(\eta_s) = \int_0^t \langle c_s, \partial_s c_s \rangle dt. \]
Corollary A.4 implies
\[
\partial_s c_s = X_s(\omega(\dot{\eta})) = [\omega(\dot{\eta}), \omega(X_s)] + \dot{\eta}(\omega(X_s)) = [c_s, x_s] + \partial_t x_s.
\]
Thus,
\[
\partial_s E(\eta_s) = \int_0^t \langle c_s, [c_s, x_s] + \partial_t x_s \rangle dt
\]
\[
= \int_0^t \langle [c_s, c_s]^* - \partial_t c_s, x_s \rangle dt.
\]
This completes the proof.

**Definition A.6.** We define the Euler-Arnol’d equation of $G$ by
\[
u_t = [u, u]^*.
\]
for $u : [0, t_0] \to \mathfrak{g}$. In other words
\[
u_t = -\text{ad}_u^* u,
\]
where $\text{ad}_u v = [u, v]$ and $\langle \text{ad}_u v, w \rangle = \langle v, \text{ad}_u^* w \rangle$.

**A.1.3 Euler-Arnol’d equation of $D_\mu(M)$**

Let $D_\mu(M)$ be the group of volume-preserving $C^\infty$-diffeomorphisms of an $n$-dimensional compact Riemannian manifold $(M, g)$ with right-invariant Riemannian metric
\[
\langle X, Y \rangle := \int_M g(X, Y)\mu,
\]
where $X, Y \in \mathfrak{g} := T_e D_\mu(M)$. Let $P : \mathfrak{X}(M) \to \mathfrak{g}$ be the projection to the divergence-free part, where $\mathfrak{X}(M)$ is the space of vector fields.

**Lemma A.7.** For $X, Y \in \mathfrak{g}$, we have
\[
\nabla^M_X Y^R = -(P(\nabla^M_X Y))^R,
\]
where $\nabla^M$ is the Levi-Civita connection on $M$.

**Proof.** The Koszul formula and the right-invariance imply
\[
2\langle \nabla^M_X Y^R, Z^R \rangle = \langle [X^R, Y^R], Z^R \rangle - \langle [X^R, Z^R], Y^R \rangle - \langle [Y^R, Z^R], X^R \rangle
\]
\[
= -\langle [X, Y]^R, Z^R \rangle + \langle [X, Z]^R, Y^R \rangle + \langle [Y, Z]^R, X^R \rangle
\]
\[
= -\langle [X, Y], Z \rangle + \langle [X, Z], Y \rangle + \langle [Y, Z], X \rangle
\]
This completes the proof by the Koszul formula on $M$ and the right-invariance of $\langle \cdot, \cdot \rangle$. \qed
Lemma A.8. For $X \in \mathfrak{g}$, we have

$$[X, X]^* = -P(\nabla^M_X X).$$

Proof. By the Koszul formula, we have

$$2\langle \nabla^M_X X, Z \rangle = -2\langle \nabla_X X^R, Z^R \rangle = -(\langle [X^R, X^R], Z^R \rangle + \langle [X^R, Z^R], X^R \rangle + \langle [X^R, Z^R], X^R \rangle) = 2\langle X^R, [X^R, Z^R] \rangle = -2\langle X^R, [X, Z]^R \rangle = -2\langle X, [X, Z] \rangle.$$ 

This completes the proof. \hfill \square

Corollary A.9. The Euler-Arnol’d equation of $\mathcal{D}_\mu(M)$ is

$$u_t = -\nabla^M_u u - \text{grad} \, p.$$ 

A.1.4 Lichnerowicz 2-cocycle

Let $\mathcal{D}_\mu(M)$ be the groups of volume-preserving diffeomorphisms of a compact $n$-dimensional Riemannian manifold $M$, and $\mathfrak{g} := T_e \mathcal{D}_\mu(M)$, which is identified with the space of divergence-free vector fields on $M$. For a closed 2-form $\eta$, define a skew-symmetric bilinear form $\Omega$ on the Lie algebra $\mathfrak{g}$ by

$$\Omega(X, Y) := \int_M \eta(X, Y).$$

Lemma A.10. The form $\Omega$ defines a 2-cocycle on $\mathfrak{g}$, namely, it satisfies the “Jacobi identity”

$$\Omega([X, Y], Z) + \Omega([Z, X], Y) + \Omega([Y, Z], X) = 0$$

for any $X, Y, Z \in \mathfrak{g}$.

Proof. Because $\eta$ is closed, there exists a one-form $\alpha$ on $M$ such that $\eta = d\alpha$. Recall the formula of the exterior derivative of 1-form:

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]).$$

This implies

$$\Omega([X, Y], Z) = \int_M ([X, Y](\alpha(Z)) - Z(\alpha([X, Y])) - \alpha([[X, Y], Z])) \mu$$

$$= \int_M \alpha([[X, Y], Z]) \mu.$$ 

Note that the second equality follows from

$$\int_M X(f) \mu = \int_M g(\text{grad} \, f, X) \mu = 0$$

\[11\]
for any \( f \in C^\infty(M) \) and a divergence-free vector field \( X \). Thus, we have
\[
\Omega([[X, Y], Z]) + \Omega([[Z, X], Y]) + \Omega([[Y, Z], X])) \mu = 0
\]
by the Jacobi identity of \([,]\). This completes the proof. \( \square \)

This Lemma allows us to endow \( \hat{g} := g \oplus \mathbb{R} \) with a Lie algebra structure defined by
\[
[(X, a), (Y, b)] := ([X, Y], \Omega(X, Y)).
\]
Take \( B \in C^\infty(\wedge^{n-2}TM) \) satisfying \( \iota_B \mu = \eta \). Then, we have
\[
\Omega(X, Y) = \int_M \iota_B \mu(X, Y) = \int_M \mu(B, X, Y) = \int_M g(B \times X, Y) = \langle P(B \times X), Y \rangle,
\]
where \( B \times X = \star(B \wedge X) \) (see (A.19) and Lemma A.18) and \( P : \mathfrak{X}(M) \to g \) is the projection to the divergence-free part.

**Lemma A.11.** For \((X, a), (Y, b) \in \hat{g}\), we have
\[
[(X, a), (Y, b)]^* = ([X, Y]^* + bP(B \times X), 0).
\]

**Proof.**
\[
\langle [(X, a), (Y, b)]^*, (Z, c) \rangle = \langle (Y, b), [(X, a), (Z, c)] \rangle = \langle (Y, b), ([X, Z], \Omega(X, Z)) \rangle = \langle Y, [X, Z] \rangle + b\Omega(X, Z) = \langle [[X, Y]^*, Z] + bP(B \times X), Z \rangle.
\]
This completes the proof. \( \square \)

Thus, we have

**Theorem A.12.** The Euler-Arnol’d equation of \( \hat{g} \) is
\[
\frac{du}{dt} = [u, u]^* + aP(B \times u),
\]
or equivalently,
\[
\frac{du}{dt} = -\nabla_u u + a(B \times u) - \text{grad } p.
\]
A.1.5 Formulae on Riemannian manifold

In this subsection, for the convenience of readers, we briefly summarize formulae on Riemannian manifold. Let \((M, g)\) be a \(n\)-dimensional Riemannian manifold and \(\mu\) the volume form. Write \(\mathfrak{X}^p(M)\) for the space of \(p\)-vector fields on \(M\), and \(\mathcal{E}^q(M)\) for the space of \(q\)-forms on \(M\).

**Definition A.13.** Define \(\flat : \mathfrak{X}^1(M) \to \mathcal{E}^1(M)\) and \(\sharp : \mathcal{E}^1(M) \to \mathfrak{X}^1(M)\) by

\[
V^{\flat} := g(V, \cdot) \in \mathcal{E}^1(M), \\
g(\alpha^{\sharp}, \cdot) = \alpha \in \mathcal{E}^1(M)
\]

for \(V \in \mathfrak{X}^1(M)\) and \(\alpha \in \mathcal{E}^1(M)\). We extend these isomorphisms to \(\flat : \mathfrak{X}^p(M) \to \mathcal{E}^p(M)\) and \(\sharp : \mathcal{E}^p(M) \to \mathfrak{X}^p(M)\) for any \(p \in \mathbb{Z}\).

**Definition A.14.** Define \(\langle \, , \rangle_{\mathfrak{X}} : \mathfrak{X}^p(M) \otimes_{C^\infty(M)} \mathfrak{X}^q(M) \to C^\infty(M)\) by

\[
\langle V, W \rangle_{\mathfrak{X}} := \iota_V(W^{\flat}),
\]

where \(\iota\) is the interior derivative. Similarly, define \(\langle \, , \rangle_{\mathcal{E}} : \mathcal{E}^p(M) \otimes_{C^\infty(M)} \mathcal{E}^q(M) \to C^\infty(M)\) by

\[
\langle \alpha, \beta \rangle_{\mathcal{E}} := \iota_{\alpha^{\sharp}}(\beta).
\]

**Lemma A.15.** Let \(V, W \in \mathfrak{X}^1(M)\). Then, we have

\[
\langle V, W \rangle_{\mathfrak{X}} = g(V, W).
\]

**Proof.** By Definition A.13 and A.14 we have

\[
\langle V, W \rangle_{\mathfrak{X}} = \iota_V(W^{\flat}) = W^{\flat}(V) = g(W, V).
\]

This completes the proof.

**Definition A.16.** We define the Hodge star operator \(* : \mathfrak{X}^p(M) \to \mathfrak{X}^{n-p}(M)\) and \(* : \mathcal{E}^p(M) \to \mathcal{E}^{n-p}(M)\) by

\[
V \wedge * W = \langle V, W \rangle_{\mathfrak{X}} \mu^{\sharp} \quad \text{for any } V \in \mathfrak{X}^p(M),
\]

\[
\alpha \wedge (* \beta) = \langle \alpha, \beta \rangle_{\mathcal{E}} \mu \quad \text{for any } \alpha \in \mathcal{E}^p(M).
\]

Note that

\[
*^2 \alpha = (-1)^{n-1} \alpha \quad \text{for } \alpha \in \mathcal{E}^1(M).
\]

Moreover, applying \(\mu\) to A.15, we have

\[
\mu(V \wedge * W) = \langle V, W \rangle_{\mathfrak{X}}.
\]

**Definition A.17.** For \(X \in \mathfrak{X}^1(M)\) and \(B \in \mathfrak{X}^{n-2}(M)\), define

\[
B \times X := *(B \wedge X) \quad \in \mathfrak{X}^1(M).
\]
Lemma A.18. For $X, Y \in \mathfrak{X}^1(M)$ and $B \in \mathfrak{X}^{n-2}(M)$, we have
\[
\int_M \iota_B(\mu)(X,Y)\mu = \int_M g(B \times X,Y)\mu.
\]

Proof. By the definition, we have
\[
\int_M \iota_B(\mu)(X,Y)\mu = \int_M \mu(B,X,Y)\mu.
\]

On the other hand,
\[
\int_M g(B \times X,Y)\mu = \int_M \langle Y, B \times X \rangle \mu = \int_M \mu(Y \wedge \ast (B \wedge X))\mu = (-1)^{n-1} \int_M \mu(Y \wedge B \wedge X)\mu = \int_M \mu(B,X,Y)\mu.
\]

The third and fifth equalities follow from \([A.17]\) and \([A.18]\), respectively. \qed

B Appendix II

B.1 Misiolek curvature

In this section, we briefly recall the Misiolek curvature. We refer to \([7, 8]\).

B.1.1 Curvature on group with right-invariant metric

In this subsection, we recall the formulae concerning to a group with right-invariant metric $\langle \cdot, \cdot \rangle$. Thus, all contents in this subsection are known.

Let $G$ be a (possibly infinite-dimensional) Lie group with a right-invariant metric $\langle \cdot, \cdot \rangle$, and $\mathfrak{g}$ its Lie algebra. Define
\[
\langle [X,Y], Z \rangle = \langle Y, [X,Z]^* \rangle.
\]

Lemma B.1. For $X, Y \in \mathfrak{g}$, we have
\[
2\nabla_X^n Y^R = (-[X,Y] + [X,Y]^* + [Y,X]^*)^R
\]

where $\nabla$ is the right-invariant Levi-Civita connection on $G$. In particular,
\[
\nabla_X^n X^R = [X,X]^R.
\]
Proof. The Koszul formula and the right-invariance imply
\[
2\langle \nabla_{X^R} Y^R, Z^R \rangle = \langle \langle X^R, Y^R \rangle, Z^R \rangle - \langle \langle X^R, Z^R \rangle, Y^R \rangle - \langle \langle Y^R, Z^R \rangle, X^R \rangle
\]
\[= -\langle \langle X, Y \rangle^R, Z^R \rangle + \langle \langle X, Z \rangle^R, Y^R \rangle + \langle \langle Y, Z \rangle^R, X^R \rangle \]
\[= -\langle [X, Y], Z \rangle + \langle [X, Z], Y \rangle + \langle [Y, Z], X \rangle \]
\[= \langle -[X, Y] + [X, Y]^* + [Y, X]^*, Z \rangle. \]
This completes the proof. \qed

For the simplicity, put
\[
A^\pm(X, Y) = [X, Y]^* \pm [Y, X]^*. \]
Define
\[
R(X^R, Y^R) = \nabla_{X^R} \nabla_{Y^R} - \nabla_{Y^R} \nabla_{X^R} - \nabla_{[X^R, Y^R]}.
\]

Lemma B.2 ([2 Thm. 2.1 in IV. §2]). For \(X, Y \in \mathfrak{g}\), we have
\[
4\langle R(X^R, Y^R) Y^R, X^R \rangle
= -4\langle [Y, Y]^*, [X, X]^* \rangle + ||A^+(X, Y)||^2 - 3||X, Y||^2 - 2\langle [X, Y], A^-(X, Y) \rangle.
\]
Proof. By the property of the Levi-Civita connection and the right-invariance, we have
\[
4\langle \nabla_{X^R} \nabla_{Y^R} Y^R, X^R \rangle
= 4X^R \langle \nabla_{Y^R} Y^R, X^R \rangle - 4\langle \nabla_{Y^R} Y^R, \nabla_{X^R} X^R \rangle
\]
\[= -4\langle [Y, Y]^*, [X, X]^* \rangle.
\]
Similarly,
\[
-4\langle \nabla_{Y^R} \nabla_{X^R} Y^R, X^R \rangle
= -4Y^R \langle \nabla_{X^R} Y^R, X^R \rangle + 4\langle \nabla_{X^R} Y^R, \nabla_{Y^R} X^R \rangle
\]
\[= \langle (-[X, Y] + A^+(X, Y))^R, (-[X, Y] + A^+(X, Y))^R \rangle
\]
\[= ||A^+(X, Y)||^2 - ||[X, Y]||^2. \]
Moreover,
\[
-4\langle \nabla_{[X^R, Y^R]} Y^R, X^R \rangle
= 4\langle \nabla_{[X, Y]^R} Y^R, X^R \rangle
\]
\[= 2\langle (-[X, Y], Y) + [[X, Y], Y]^* + [Y, [X, Y]]^*, X \rangle
\]
\[= 2\langle [X, Y], [Y, X]^* \rangle + 2\langle [Y, [X, Y], X] \rangle + 2\langle [X, Y], [Y, X] \rangle
\]
\[= 2\langle [X, Y], [Y, X]^* \rangle - 2||[X, Y]^*||^2
\]
\[= -2\langle [X, Y], A^-(X, Y) \rangle - 2||X, Y||^2. \]
This completes the proof. \qed
B.1.2 Definition of Misiołek curvature

Let $G$ be a (possibly infinite-dimensional) Lie group with a right-invariant metric $\langle \cdot, \cdot \rangle$, $\mathfrak{g}$ its Lie algebra. Define

$$
\langle [X, Y], Z \rangle = \langle Y, [X, Z]^* \rangle,
$$

$$
R(X^R, Y^R) = \nabla_{X^R} \nabla_{Y^R} - \nabla_{Y^R} \nabla_{X^R} - \nabla_{[X^R, Y^R]},
$$

$$
A^\pm(X, Y) = [X, Y]^* \pm [Y, X]^*.
$$

Then we have

$$
2\nabla_{X^R} Y^R = -[X, Y]^R + A^+(X, Y)^R.
$$

Note that $[X, X]^* = 0$ implies that $X$ is a stationary solution of the Euler-Arnol’d equation of $G$.

Lemma B.3. Let $X \in \mathfrak{g}$ satisfying $[X, X]^* = 0$ and $\eta$ a geodesic corresponding to $X \in T_e G$. For $Y \in \mathfrak{g}$ and $f \in C^\infty(G)$ with $f(e) = f(t_o) = 0$ for some $t_o > 0$, define a vector field $\tilde{Y}$ along $\eta$ by $\tilde{Y}(\eta(t)) = f(t) \cdot Y^R(\eta(t))$. Then the second variation $E''$ of the energy function of $\eta$ is

$$
E''(\eta)(\tilde{Y}, \tilde{Y}) = \int_{t_o}^{t_o} \left( \dot{f}^2 ||Y||^2 - f^2 (\langle R(X^R, Y^R) Y^R, X^R \rangle - ||\nabla_{X^R} Y^R||^2) \right) dt
$$

where $\dot{f} := X^R f$.

Proof. Note that $\dot{\eta} = X^R$ by the assumption. The general formula for the second variation of the energy function implies

$$
E''(\eta)(\tilde{Y}, \tilde{Y}) = \int_{t_o}^{t_o} ||\nabla_{X^R} \tilde{Y}||^2 dt - \int_{t_o}^{t_o} \langle R(X^R, \tilde{Y}), \tilde{Y}, X^R \rangle dt.
$$

For the first term, we have

$$
\nabla_{X^R} \tilde{Y} = \dot{f} Y^R + f \nabla_{X^R} Y^R.
$$

Thus,

$$
||\nabla_{X^R} \tilde{Y}||^2 = \dot{f}^2 ||Y||^2 + 2f \dot{f} \langle Y, \nabla_{X^R} Y^R \rangle + f^2 ||\nabla_{X^R} Y^R||^2.
$$

Then, $\langle Y^R, \nabla_{X^R} Y^R \rangle = -\langle \nabla_{X^R} Y^R, Y^R \rangle$ implies

$$
||\nabla_{X^R} \tilde{Y}||^2 = \dot{f}^2 ||Y||^2 + f^2 ||\nabla_{X^R} Y^R||^2.
$$

On the other hand,

$$
\langle R(X^R, \tilde{Y}) Y^R, X^R \rangle = f^2 \langle R(X^R, Y^R) Y^R, X^R \rangle.
$$

This completes the proof. \qed
**Definition B.4.** Let \( X \in \mathfrak{g} \) satisfying \([X, X]^* = 0\). Define the Misiolek curvature \( MC_{X,Y} := MC_{X,Y}^G \) by

\[
MC_{X,Y} = (R(X^R, Y^R)Y^R, X^R) - ||\nabla_{X^R} Y^R||^2
\]

for \( Y \in \mathfrak{g} \).

**Theorem B.5.** Let \( X \in \mathfrak{g} \) satisfying \([X, X]^* = 0\) and \( \eta \) a geodesic corresponding to \( X \in T_eG \). Suppose \( Y \in \mathfrak{g} \) satisfies \( MC_{X,Y} > 0 \). For \( s > 0 \), define

\[
t_s := \pi ||Y|| \sqrt{\frac{s}{MC_{X,Y}}} \in \mathbb{R}_{>0}
\]

\[
f_s(t) := \sin \left( \frac{t}{||Y||} \sqrt{\frac{MC_{X,Y}}{s}} \right) \quad \in C^\infty(\mathbb{R}_{\geq 0})
\]

and a vector field \( \tilde{Y} \) along \( \eta \) by \( \tilde{Y}(\eta(t)) = f_s(t)Y^R(\eta(t)) \). Then, the second variation \( E'' \) of the energy function of \( \eta \) is

\[
E''(\eta)(\tilde{Y}, \tilde{Y}) = \pi \frac{1}{2} (1 - s)||Y||\sqrt{\frac{MC_{X,Y}}{s}}.
\]

In particular, \( E''(\eta)(\tilde{Y}, \tilde{Y}) < 0 \) if \( 0 < s < 1 \).

**Proof.** For simplicity, set \( M := MC_{X,Y} \). By Lemma B.3, we have

\[
E''(\eta)(\tilde{Y}, \tilde{Y}) = \int_0^{s} \left( \frac{f_s^2 ||Y||^2 - f_s^2 M}{s} \right) dt
\]

\[
= \int_0^{s} \left( \frac{M \cos^2 \left( \frac{t}{||Y||} \sqrt{\frac{M}{s}} \right) - M \sin^2 \left( \frac{t}{||Y||} \sqrt{\frac{M}{s}} \right) }{s} \right) dt
\]

\[
= M \int_0^{s} \left( \frac{1}{s} \cos^2(x) - \sin^2(x) \right) \frac{dx}{\sqrt{\frac{M}{s}}}
\]

\[
= \frac{\pi}{2} (1 - s)||Y||\sqrt{\frac{M}{s}}.
\]

This completes the proof. \( \square \)

**Lemma B.6.** Let \( X \in \mathfrak{g} \) satisfying \([X, X]^* = 0\) and \( Y \in \mathfrak{g} \). Then, we have

\[
MC_{X,Y} = -||[X, Y]||^2 - ([X, Y], Y), X).
\]

**Proof.** By Lemma B.2, we have

\[
4(R(X^R, Y^R)Y^R, X^R) = ||A^+(X, Y)||^2 - 3||[X, Y]||^2 - 2([X, Y], A^{-}(X, Y)).
\]

On the other hand,

\[
-4||\nabla_{X^R} Y^R||^2 = -||[X, Y] + A^+(X, Y)||
\]

\[
= -||[X, Y]||^2 + 2([X, Y], A^+(X, Y)) - ||A^+(X, Y)||^2.
\]

This completes the proof. \( \square \)
B.1.3 Misiolek curvature of $D_\mu(M)$

Let $D_\mu(M)$ be the group of volume-preserving $C^\infty$-diffeomorphisms of a compact $n$-dimensional manifold $M$ with the $L^2$ right-invariant metric:

$$ (X, Y) := \int_M g(X, Y) \mu. $$

Here $X, Y \in \mathfrak{g} = T_e G$, which is identified with the space of divergence-free vector fields.

**Lemma B.7.** Let $X \in \mathfrak{g}$ satisfying $[X, X]^* = 0$ and $Y \in \mathfrak{g}$. Then, we have

$$ MC_{X,Y} = \langle \nabla^M X \[X, Y\] + \nabla^M_{[X,Y]} X, Y \rangle, $$

where $\nabla^M$ is the Levi-Civita connection on $M$.

**Proof.** By the Koszul formula, we have

$$ 2\langle \nabla^M_X [X, Y], Y \rangle = \langle [X, [X, Y]], Y \rangle - \langle [X, Y], [X, Y] \rangle - \langle [[X, Y], Y], X \rangle, $$

$$ 2\langle \nabla^M_{[X,Y]} X, Y \rangle = \langle [[X, Y], X], Y \rangle - \langle [X, Y], [X, Y] \rangle - \langle X, [X, Y], [X, Y] \rangle. $$

Thus Lemma [1.3] implies the lemma. $\square$

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