DESCENT OF CONTINUOUS FIELDS OF HILBERT SPACES

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ABSTRACT. We prove a descent theorem for continuous fields of Hilbert spaces along uniformly finite, proper mappings of locally compact Hausdorff spaces. As part of the proof, we compute the completely bounded norm of the ‘flip’ map on the Haagerup tensor product $C_0Y_1 \otimes_{C_0X} C_0Y_2$ associated to a pair of continuous mappings $Y_1 \to X \leftarrow Y_2$.

The basic problem of descent, for a given category of spaces (e.g., topological spaces or schemes) and a category of geometric objects defined on those spaces (e.g., vector bundles or sheaves), is as follows: given a map of spaces $\mu : Y \to X$ we wish to characterise, in terms of compatibility conditions along the fibres of $\mu$, those geometric objects on $Y$ that are pullbacks of geometric objects on $X$. The compatibility conditions can typically be expressed in several different ways, of which one of the more transparently geometric is Grothendieck’s notion of descent data ([Gro60], cf. [BLR90, Chapter 6]). Another way to express these conditions is through the language of comonads, coalgebras, and comodules ([BR70], cf. [Bor94, Chapter 4]). In [Cri19] we showed how the comonadic approach to descent can be made to work when the ‘spaces’ under consideration are $C^*$-algebras, and the ‘geometric objects’ are Hilbert $C^*$-modules. Our results relied, both for their proofs and their formulation, on aspects of operator-space theory—most notably, the Haagerup tensor product—that fall outside the usual purview of Hilbert $C^*$-module theory. In this paper we shall prove that the main result of [Cri19] can, in a special case, be reformulated entirely in terms of Hilbert $C^*$-modules.

The setting will be as follows. Our spaces are locally compact Hausdorff spaces (or equivalently, commutative $C^*$-algebras), and our geometric objects are continuous fields of Hilbert spaces (equivalently, Hilbert $C^*$-modules). Each continuous map of topological spaces $\mu : Y \to X$ yields a pullback functor on $C^*$-modules, $\mu^* : \text{CM}(C_0X) \to \text{CM}(C_0Y)$, and the problem is to characterise the image of this functor. When the map $\mu$ is proper, [Cri19, Theorem 5.6] gives a solution to this problem in terms of comodules and the Haagerup tensor product, and our goal here is to translate this solution into the more directly geometric language of descent data: following [Gro60], we define a descent datum for $\mu$ to be a pair $(\mathcal{E}, \varphi)$ consisting of a Hilbert $C^*$-module $\mathcal{E}$ over $C_0Y$, and a unitary isomorphism $\varphi : \pi_1^* \mathcal{E} \to \pi_2^* \mathcal{E}$ between the pullbacks of $\mathcal{E}$ to
Y ×_X Y along the two coordinate projections, satisfying a cocycle condition. (See Definition [1.21]) If X is a point and Y is two points, for example, then a descent datum is a unitary isomorphism between two Hilbert spaces. In general a descent datum for μ gives a collection of unitary isomorphisms among the Hilbert spaces in each fibre of μ.

In the parallel algebraic setting of sheaves on schemes, the identification of descent data with comodules is a matter of rather straightforward algebra (see [Bor94 4.7]), relying on the identification of Cartesian products of spaces with tensor products of function algebras, and on the commutativity of the tensor product. Two significant obstacles prevent our applying the same line of argument to the C^*-algebraic setting: the Haagerup tensor product typically does not satisfy identities like _C_0Y ⊗_h C_0X _C_0Y ∼ C_0(Y ×_X Y); and it is typically non-commutative. The main technical contribution of this paper is to prove that both of these obstacles are the same, and to characterise those maps μ for which the obstacles vanish. The condition is simple to state: μ should be uniformly finite, in the sense that its fibres μ^(-1)x should all be finite of uniformly bounded cardinality. Restricting to this class of maps, and thus removing the obvious obstacles, we show that the relationship between geometric and comonadic descent goes through in much the same way as in the algebraic setting, modulo some attention to Hermitian structures. Thus the comonadic descent theorem of [Cri19] becomes a ‘geometric’ descent theorem.

We do not know whether our descent theorem, in its geometric formulation, remains valid without the uniform finiteness hypothesis; certainly any proof of such a generalisation would have to be quite different to the one given here. The condition of uniform finiteness is rather restrictive, but interesting examples do exist. For instance, if G is a complex semisimple Lie group then the Dixmier-Douady theory [DD63] combined with a theorem of Wallach [Wal71] gives a Morita equivalence _C_*G ∼ _C_0(_h*/W) between the reduced group C^*-algebra of G and the commutative C^*-algebra of Weyl-invariant continuous functions on the dual of a Cartan subalgebra _h_ of the Lie algebra of G. This is explained in [PP83]. Using this Morita equivalence, our descent theorem says that the functor of tempered parabolic restriction (cf. [CCH16]) from G to H = exp _h_ gives an equivalence between CM(_C_*G) and the category of descent data for the (uniformly finite) quotient mapping _h*/W.

The rest of the paper is organised as follows. In Section 1 we define the C^*-category Des_μ of descent data associated to a continuous map μ : Y → X, and we formulate our main theorem (Theorem 1.3): if μ is proper, surjective, and uniformly finite then the canonical comparison functor D : CM(_C_0X) → Des_μ is a unitary equivalence. In Section 2 we establish the necessary technical results on the Haagerup tensor product _C_0Y_1 ⊗_h _C_0X _C_0Y_2 associated to a pair of continuous maps _Y_1 → X ← _Y_2; the main results of this section, Propositions 2.3 and 2.7 may be of interest independently of the descent problem. Finally, in Section 3 we combine the results of Section 2 and of [Cri19] to prove the descent theorem.
1. Descent

**Hilbert C⁺-modules.** For a C⁺-algebra B we let CM(B) denote the C⁺-category of right Hilbert C⁺-modules over B, with adjointable B-module maps as morphisms. We briefly recall (cf. [Lan95] for details) that an object in CM(B) is a right B-module ℰ, equipped with an inner product ⟨ ⟩ : ℰ × ℰ → B satisfying the B-valued analogues of the axioms for a Hilbert-space inner product. The morphisms in CM(B) are those B-linear maps t : ℰ₁ → ℰ₂ for which there exists a map t⁺ : ℰ₂ → ℰ₁ satisfying ⟨te₁|e₂⟩ = ⟨e₁|t⁺e₂⟩ for all e₁ ∈ ℰ₁ and e₂ ∈ ℰ₂. Such a morphism is called a unitary if t⁺ = t⁻¹. All Hilbert C⁺-modules are nondegenerate, in the sense that every element of ℰ has the form eb for some e ∈ ℰ and b ∈ B. Note that when B = C₀Y is commutative, the category CM(B) is unitarily equivalent to the category of continuous fields of Hilbert spaces over Y [Fak72].

If ρ : A → M(B) is a nondegenerate ⋆-homomorphism from a C⁺-algebra A to the multiplier algebra of a C⁺-algebra B—with nondegeneracy meaning that every element of B has the form ρ(a)b for some a ∈ A and some b ∈ B—then we consider the ⋆-functor

$$\rho : \text{CM}(A) \rightarrow \text{CM}(B), \quad \mathcal{F} \mapsto \mathcal{F} \otimes_A^* \mathcal{B}, \quad t \mapsto t \otimes \text{id}_B$$

where B is made into a left A-module via ρ, and where \( \otimes^* \) denotes the Hilbert C⁺-module tensor product; we recall that the B-valued inner product on \( \mathcal{F} \otimes_A^* \mathcal{B} \) is defined by \( (f \otimes b)f' \otimes b' : = b^*\rho((f | f')b'). \)

**Descent data.** Let μ : Y → X be a continuous map of locally compact Hausdorff spaces. Pullback of functions along μ gives a nondegenerate ⋆-homomorphism μ⁺ : C₀X → M(C₀Y), and an associated functor

$$\mu^* : \text{CM}(C₀X) \rightarrow \text{CM}(C₀Y), \quad \mathcal{F} \mapsto \mathcal{F} \otimes_{C₀X}^* C₀Y, \quad t \mapsto t \otimes \text{id}_B$$

of pullback of continuous fields of Hilbert spaces. Our goal is to characterise the image of this functor.

Let Y ×ᵢ Y := \{ (yᵰ, yᵧ) ∈ Y × Y | μ(yᵰ) = μ(yᵧ) \}, equipped with the product topology, and let πᵰ and πᵧ denote the coordinate projections Y ×ᵢ Y → Y. Similarly, for each pair i < j with i, j = 1, 2, 3, we let πᵢⱼ denote the projection

$$πᵢⱼ : Y ×ᵢ Y ×ᵢ Y \rightarrow Y ×ᵢ Y, \quad πᵢⱼ(yᵰ, yᵧ, yⱼ) := (yᵰ, yⱼ).$$

To compactify the notation we shall use the following abbreviations:

$$A := C₀X, \quad B := C₀Y, \quad C := C₀(Y ×ᵢ Y), \quad D := C₀(Y ×ᵢ Y ×ᵢ Y).$$

Pullback along each πᵢ gives a nondegenerate ⋆-homomorphism π⁺ᵢ : B → M(C), and a ⋆-functor π⁺ᵢ : CM(B) → CM(C). In the same way, each πᵢⱼ gives rise to a ⋆-functor π⁺ᵢⱼ : CM(C) → CM(D). The equalities π₁μ = π₂μ, π₁π₁₂ = π₁π₂₃, π₁π₁₂ = π₁π₁₃, and π₂π₁₃ = π₂π₂₃ yield unitary isomorphisms of functors

$$\mu^* \pi^*_i \cong \mu^* \pi^*_i, \quad \pi^*_i \pi^*_j \cong \pi^*_i \pi^*_j, \quad \pi^*_i \pi^*_j \cong \pi^*_i \pi^*_j, \quad \pi^*_i \pi^*_j \cong \pi^*_i \pi^*_j,$$

which we shall often apply without further comment.
**Definition 1.1.** A descent datum for the map $\mu : Y \to X$ is a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E} \in \text{CM}(B)$ is a $C^*$-module over $B = C_0Y$, and where $\varphi : \pi_1^* \mathcal{E} \to \pi_2^* \mathcal{E}$ is a unitary isomorphism of Hilbert $C^*$-modules over $C = C_0(Y \times_X Y)$ making the diagram commute. (This last condition will be referred to as the cocycle condition.) A morphism of descent data $(\mathcal{E}_1, \varphi_1) \to (\mathcal{E}_2, \varphi_2)$ is an adjointable mapping of Hilbert $C^*$-modules $t : \mathcal{E}_1 \to \mathcal{E}_2$ satisfying

$$\pi_2^*(t) \varphi_1 = \varphi_2 \pi_1^*(t).$$

Equipped with these morphisms, the descent data for $\mu$ form a $C^*$-category, which we denote by $\text{Des} \mu$.

**Definition 1.2.** For each $\mathcal{F} \in \text{CM}(C_0X)$ we let $\varphi_{\mathcal{F}} : \pi_1^* \mu^* \mathcal{F} \to \pi_2^* \mu^* \mathcal{F}$ be the canonical unitary isomorphism coming from the equality $\mu \pi_1 = \mu \pi_2$. The assignment $\mathcal{F} \mapsto (\mu^* \mathcal{F}, \varphi_{\mathcal{F}})$ extends to a $*$-functor $D : \text{CM}(C_0X) \to \text{Des} \mu$, defined on morphisms by $D(t) := \mu^*(t)$.

Here is our main result.

**Theorem 1.3.** Let $\mu : Y \to X$ be a proper, surjective, uniformly finite mapping of locally compact Hausdorff spaces. The functor $D : \text{CM}(C_0X) \to \text{Des} \mu$ is a unitary equivalence of $C^*$-categories.

Our strategy for proving Theorem 1.3 is to reformulate both sides of the purported equivalence using the Haagerup tensor product, and then to import the standard algebraic arguments. On the left-hand side we use the results of [Cri19] to replace $\text{CM}(C_0X)$ by a category of comodules over the Sweedler-type coalgebra $C_0Y \otimes_{C_0X} C_0Y$. On the right-hand side, we shall use the results of the next section to replace the $C^*$-algebra $C_0(Y \times_X Y)$ by the Haagerup tensor product $C_0Y \otimes_{C_0X}^h C_0Y$.

**2. Commutativity of the Haagerup tensor product**

In this section, which can be read independently of the rest of the paper, we study the Haagerup tensor product of a pair of commutative $C^*$-algebras over a common $*$-subalgebra.

**The Haagerup tensor product.** Let us first establish some notation and recall some basic facts about the Haagerup tensor product, referring to [BLM04] for details. Let $B_1$ and $B_2$ be $C^*$-algebras, and let $A$ be a $C^*$-algebra equipped with nondegenerate $*$-homomorphisms $A \to M(B_1)$ and $A \to M(B_2)$ into the multiplier algebras of $B_1$ and $B_2$. We use these homomorphisms to regard $B_1$ and
$B_2$ as $A$-bimodules. For all positive integers $n$ and $m$ we consider the 'external' matrix product

$$\left( M_{n,m} \otimes B_1 \right) \otimes \left( M_{m,n} \otimes B_2 \right) \xrightarrow{D \otimes E \rightarrow D \otimes E} M_n \otimes (B_1 \otimes_A B_2)$$

$$(d \otimes b_1) \otimes (e \otimes b_2) := de \otimes (b_1 \otimes b_2).$$

Here $M_{n,m}$ means $n \times m$ complex matrices, and $M_n = M_{n,n}$. We use an undecorated $\otimes$ to mean the algebraic tensor product over $\mathbb{C}$; subscripts will indicate tensor products over other algebras, and superscripts will indicate completions.

We equip each $M_n \otimes B$ with its $C^*$-algebra norm, and we equip $M_{n,m} \otimes B$ with the norm that it inherits as a subspace of $M_{\max(n,m)} \otimes B$. The Haagerup seminorm on $M_n \otimes (B_1 \otimes_A B_2)$ is defined by

$$\|F\|_h := \inf \left\{ \|D\| \cdot \|E\| \mid D \in M_{n,m} \otimes B_1, E \in M_{m,n} \otimes B_2, F = D \otimes E \right\}.$$ 

The separated completion of $B_1 \otimes_A B_2$ in the $n = 1$ Haagerup seminorm is called the Haagerup tensor product, denoted by $B_1 \otimes_A^h B_2$. The Haagerup norms extend to the spaces $M_n \otimes (B_1 \otimes_A^h B_2)$, and these norms collectively give $B_1 \otimes_A^h B_2$ the structure of an operator space.

Whenever $\rho_1 : B_1 \to C$ and $\rho_2 : B_2 \to C$ are $\ast$-homomorphisms satisfying $\rho_1(b_1a)\rho_2(b_2) = \rho_1(b_1)\rho_2(ab_2)$ for all $b_1 \in B_1$, $b_2 \in B_2$, and $a \in A$, we obtain a map

$$[\rho_1, \rho_2] : B_1 \otimes_A^h B_2 \to C, \quad b_1 \otimes b_2 \mapsto \rho_1(b_1)\rho_2(b_2)$$

satisfying, for all $n \geq 1$ and all $F \in M_n \otimes (B_1 \otimes_A B_2)$,

$$\|([\text{id}_{M_n} \otimes [\rho_1, \rho_2]](F))\|_{M_n \otimes C} \leq \|F\|_h.$$ 

In other words, the map $[\rho_1, \rho_2]$ is completely contractive.

**Relationship between $C_0(Y \otimes_{\mathcal{C}_0(X)}^h C_0(Y \times X Y)$**. Now let $Y_1$, $Y_2$, and $X$ be locally compact Hausdorff topological spaces, and let $\mu_1 : Y_1 \to X$ and $\mu_2 : Y_2 \to X$ be continuous maps. Pullback of functions along $\mu_i$ gives a non-degenerate $\ast$-homomorphism $\mu_i^* : C_0X \to M(C_0 Y_i)$, and using these homomorphisms we form the Haagerup tensor product $C_0 Y_1 \otimes_{\mathcal{C}_0(X)}^h C_0 Y_2$. Our goal in this section is to relate this tensor product to the $C^*$-algebra $C_0(Y_1 \times_X Y_2)$. Note that $C_0(Y_1 \times_X Y_2) \cong C_0 Y_1 \otimes_{\mathcal{C}_0(X)}^{\min} C_0 Y_2$, the quotient of the minimal $C^*$-algebra tensor product by the ideal generated by elements of the form $b_1 a \otimes b_2 - b_1 \otimes ab_2$.

Consider the coordinate projections $\pi_i : Y_1 \times_X Y_2 \to Y_i$. Pullback along $\pi_i$ gives a non-degenerate $\ast$-homomorphism $\pi_i^* : C_0Y_i \to M(C_0(Y_1 \times_X Y_2))$. Since $\mu_1 \pi_1 = \mu_2 \pi_2$, the homomorphisms $\pi_i^*$ induce, as in (2.1), a completely contractive map

$$[\pi_1^*, \pi_2^*] : C_0 Y_1 \otimes_{\mathcal{C}_0(X)}^h C_0 Y_2 \to M(C_0(Y_1 \times_X Y_2)),$$

$$[\pi_1^*, \pi_2^*](b_1 \otimes b_2) : (y_1, y_2) \mapsto b_1(y_1)b_2(y_2).$$

In this case the map $[\pi_1^*, \pi_2^*]$ actually has image contained in $C_0(Y_1 \times_X Y_2)$. 


Proposition 2.3. Let \( \mu_1 : Y_1 \to X \) and \( \mu_2 : Y_2 \to X \) be continuous mappings of locally compact Hausdorff spaces, and suppose that function \( x \mapsto \min_{i=1,2} \mu_i^{-1} x \) is uniformly bounded on \( X \). Then the map

\[
[\pi_1^* \pi_2^*] : C_0(Y_1 \otimes_{C_0 X} C_0 Y_2) \to C_0(Y_1 \times X Y_2)
\]

is a completely bounded isomorphism. Explicitly,

\[
\|[\pi_1^* \pi_2^*]\|_{cb} \leq 1 \quad \text{and} \quad \|[\pi_1^* \pi_2^*]^{-1}\| \leq \sqrt{\max_{x \in X} \min_{i=1,2} \mu_i^{-1} x}.
\]

(We shall later prove that the reverse implication also holds: see Corollary 2.10.)

Our proof of Proposition 2.3 relies on a couple of preliminary observations about the Haagerup norm.

Lemma 2.4. Let \( \mu_i : Y_i \to X \) (for \( i = 1, 2 \)) be continuous maps of locally compact Hausdorff spaces. For each \( x \in X \) let \( i_{i,x} : \mu_i^{-1} x \to Y_i \) be the inclusion map, and consider the linear map

\[
t_{i,x}^* \otimes t_{2,x}^* : C_0 Y_1 \otimes_{C_0 X} C_0 Y_2 \to C_0(\mu_1^{-1} x) \otimes C_0(\mu_2^{-1} x).
\]

For each \( n \geq 1 \) and each \( F \in M_n \otimes (C_0 Y_1 \otimes_{C_0 X} C_0 Y_2) \) we have

\[
\|F\|_h = \sup_{x \in X} \|(\text{id}_{M_n} \otimes t_{1,x}^* \otimes t_{2,x}^*) F\|_h.
\]

Proof: The Haagerup tensor product \( C_0 Y_1 \otimes_{C_0 X} C_0 Y_2 \) embeds completely isometrically in the free-product C*-algebra \( C_0 Y_1 \ast_{C_0 X} C_0 Y_2 \), via the map \( b_1 \otimes b_2 \mapsto b_1 \ast b_2 \) [Pis02, Theorem 5.13]. In every irreducible representation of this free product the algebra \( C_0 X \) acts centrally, and thus by a character \( a \mapsto a(x) \). So every irreducible representation of \( C_0 Y_1 \ast_{C_0 X} C_0 Y_2 \) factors through some \( t_{1,x}^* \ast t_{2,x}^* \). Taking the supremum over all of the irreducible representations thus gives the desired formula for \( \|F\|_h \).

The other ingredient in our proof of Proposition 2.3 is the following well-known fact:

Lemma 2.5. For all \( n, m \in \mathbb{N} \), all C*-algebras \( A \), and all \( F \in M_n \otimes \mathbb{C}^m \otimes A \), we have

\[
\|F\|_{\text{min}} \leq \|F\|_h \leq \sqrt{m}\|F\|_{\text{min}}
\]

where \( h \) indicates the Haagerup norm on \( M_n \otimes (\mathbb{C}^m \otimes^h A) \), and min indicates the C*-algebra norm. The same bounds hold for \( F \in M_n \otimes A \otimes \mathbb{C}^m \), with \( h \) indicating the Haagerup norm on \( M_n \otimes (A \otimes^h \mathbb{C}^m) \).

The bound \( \|F\|_{\text{min}} \leq \|F\|_h \) holds for tensor products of arbitrary operator spaces [BLM04, 1.5.13]. Since we were not immediately able to find a reference for the other inequality, let us give a proof.
Proof. Take an arbitrary element

\[ F = \sum_{i,j=1}^{n} \sum_{k=1}^{m} e_{i,j} \otimes e_{k} \otimes a_{i,j,k} \in M_n \otimes C^m \otimes A \]

where as usual \( e_{i,j} \) denotes the matrix with 1 in the \( i,j \) position and zeros everywhere else, and \( e_1, \ldots, e_m \in C^m \) are the standard basis elements. Factor \( F \) as \( D \otimes E \), where

\[ D := \sum_{i=1}^{n} \sum_{k=1}^{m} e_{i,1} \otimes e_{i,k} \otimes e_k \in M_{1,n} \otimes M_{n,m} \otimes C^m \quad \text{and} \]

\[ E := \sum_{i,j=1}^{n} \sum_{k=1}^{m} e_{i,1} \otimes a_{i,j,k} \in M_{1,n} \otimes M_{n,m} \otimes A. \]

(Here \( D \otimes E \) is defined by using the Kronecker product to identify \( M_{1,n} \otimes M_{m,m} \cong M_{m,m} \otimes M_{n,m} \) and \( M_{n,1} \otimes M_{m,m} \cong M_{m^2,m} \).)

We have \( E^*E = \sum_{k=1}^{m} F_k \otimes F_k \), where \( F_k = \sum_{i,j=1}^{n} e_{i,j} \otimes a_{i,j,k} \in M_n \otimes A \). Since \( D D^* \) is the identity in \( M_n \otimes C^m \), this computation gives the bound

\[ \|F\|_2^2 \leq \|E\|_2^2 \leq m \sup_k \|F_k\|_{M_n \otimes A}^2 = m\|F\|_{\min}. \]

This proves the result for \( C^m \otimes A \), and taking operator-space adjoints gives the result for \( A \otimes C^m \).

Proof of Proposition 2.2 For each \( x \in X \) and \( i = 1, 2 \) let \( \pi_i : \mu_i^{-1}x \times \mu_i^{-1}x \to \mu_i^{-1}x \) be the projection onto the \( i \)th coordinate. Consider the commuting diagram

\[ \begin{array}{ccc}
C_0 Y_1 \otimes h C_0 \mu_1 x & \xrightarrow{[\pi_1^* \pi_2^*]} & C_0(Y_1 \times_X Y_2) \\
\cap_x \sqcup \, \cap \, & \text{[2.6]} & \\
\cap_x C(\mu_1^{-1}x) \otimes h C(\mu_2^{-1}x) & \xrightarrow{[\cap_x \pi_1^* \pi_2^*]} & \cap_x C(\mu_1^{-1}x \times \mu_2^{-1}x) \\
\end{array} \]

where \( \square \) denotes the \( \ell^\infty \) product of operator spaces. Lemma 2.4 implies that the left-hand vertical arrow in [2.6] is a complete isometry. The right-hand vertical arrow is an injective \( \ast \)-homomorphism, hence it too is a complete isometry. If the quantity \( \min_i \#\mu_i^{-1}x \) is uniformly bounded then Lemma 2.5 implies that the bottom horizontal arrow is a completely bounded isomorphism of operator spaces: indeed, this map is a complete contraction, and its inverse has \( \text{cb} \) norm bounded by \( \sqrt{\max_x \min_i \#\mu_i^{-1}x} \). Hence the commutativity of the diagram ensures that \( [\pi_1^* \pi_2^*] \) is a completely bounded isomorphism onto a closed subspace of \( C_0(Y_1 \times_X Y_2) \), and an application of the Stone-Weierstrass theorem shows that this subspace is all of \( C_0(Y_1 \times_X Y_2) \). Since the vertical arrows in [2.6] are complete isometries, the previously noted bounds on the \( \text{cb} \) norms of the
If the function \( \tau \) satisfies that the \( \tau \) map implies that \( \text{min} \) over \( n \) of the operator norms of the maps \( \tau \) and \( \tau \) is complete bounded, since \( \tau \) is conjugate via the cb isomorphism \( [\pi_1^*, \pi_2^*] \) to the \( \ast \)-isomorphism \( \sigma^*: C_0(Y_1 \times X Y_2) \to C(Y_2 \times X Y_1) \) of pullback along the homeomorphism \( \sigma(y_2, y_1) = (y_1, y_2) \). We shall now prove the converse: if \( \tau \) is completely bounded then \( \text{min} \) over \( n \) of \( \#\mu_i^{-1} x \) is uniformly bounded. While not strictly necessary for the proof of Theorem [1.2] this result is of interest inasmuch as it demonstrates the necessity of the uniform finiteness hypothesis for our line of argument: the theorem may hold with weaker hypotheses, but a very different proof would be required.

We shall compute the completely bounded norm \( \|\tau\|_{\text{cb}} \): that is, the supremum over \( n \) of the operator norms of the maps

\[
\text{id}_{M_n} \otimes \tau : M_n \otimes (C_0 Y_1 \otimes C_0 Y_2) \to M_n \otimes (C_0 Y_2 \otimes C_0 Y_1).
\]

The computation is inspired by, but distinct from, Tomiyama’s computation of the cb norm of the transpose map \([\text{Tom}83]\); cf. Remark [2.11].

**Proposition 2.7.** The completely bounded norm of the flip map \( \tau \) associated to a pair of continuous maps \( \mu_1 : Y_1 \to X \) and \( \mu_2 : Y_2 \to X \) is given by

\[
\|\tau\|_{\text{cb}} = \sqrt{\text{max} \text{ min} \#\mu_i^{-1} x}.
\]

This includes the assertion that if one side of the equation is infinite, then so is the other.

We shall prove the two inequalities in Proposition [2.7] separately, in Lemmas [2.8] and [2.9].

**Lemma 2.8.** \( \|\tau\|_{\text{cb}} \geq \sqrt{\text{max} \text{ min} \#\mu_i^{-1} x} \). In particular, if the right-hand side is infinite then so is \( \|\tau\|_{\text{cb}} \).

**Proof.** Fix a point \( x \in X \), and suppose that \( \#\mu_i^{-1} x \geq m \) for \( i = 1, 2 \). We will prove that \( \|\text{id}_{M_m} \otimes \tau\| \geq \sqrt{m} \).

Let \( y_1, \ldots, y_m \in Y_1 \) and \( z_1, \ldots, z_m \in Y_2 \) be distinct points with \( \mu_1(y_i) = \mu_2(z_i) = x \) for all \( i \). Let \( b_1, \ldots, b_m \in C_0 Y_1 \) and \( c_1, \ldots, c_m \in C_0 Y_2 \) be a collection of functions with the following properties:

\[
\|b_i\| = \|c_i\| = b_i(y_i) = c_i(z_i) = 1 \text{ for all } i; \text{ and } b_i b_j = c_i c_j = 0 \text{ when } i \neq j.
\]
Let $\omega \in \mathbb{C}$ be a primitive $m$th root of 1, and consider the following element of $M_m \otimes (C_0Y_1 \otimes_{c_0X} C_0Y_2)$:

$$F := \sum_{i,j,k=1}^m \omega^{k(j-i)}e_{i,j} \otimes b_j \otimes c_k.$$ 

We first claim that $\|F\|_h \leq \sqrt{m}$. To show this we write $F = D \otimes E$ where $D \in M_{1,m} \otimes M_{m,m} \otimes C_0Y_1$ and $E \in M_{m,1} \otimes M_{m,m} \otimes C_0Y_2$ are defined by

$$D := \sum_{i,j,k=1}^m \omega^{-kj}e_{i,j} \otimes e_{i,k} \otimes b_j, \quad E := \sum_{j,k=1}^m \omega^k e_{j,1} \otimes e_{k,j} \otimes c_k.$$ 

We have

$$\|D\|^2 = \|DD^*\| = \left\| \sum_{i,j=1}^m e_{i,j} \otimes |b_j|^2 \right\| = m \left\| \sum_{j=1}^m |b_j|^2 \right\| = m,$$

while

$$\|E\|^2 = \|E^*E\| = \left\| \sum_{j,k=1}^m e_{j,j} \otimes |c_k|^2 \right\| = \left\| \sum_{k=1}^m |c_k|^2 \right\| = 1,$$

giving $\|F\|_h \leq \|D\| \cdot \|E\| \leq \sqrt{m}$.

Now let $\rho_1 : C_0Y_1 \to M_m$ and $\rho_2 : C_0Y_2 \to M_m$ be the $*$-homomorphisms

$$\rho_1(b) := \sum_{i=1}^m b(y_i)e_{i,i} \quad \text{and} \quad \rho_2(c) := U\left(\sum_{i=1}^m c(z_i)e_{i,i}\right)U^*,$$

where $U := m^{-1/2}\sum_{i,j=1}^m \omega^{ij}e_{i,j}$ (the unitary Fourier transform in $M_m$). A straightforward computation shows that $(\text{id}_{M_m} \otimes [\rho_2 \rho_1] \tau)(F) = \sum_{i,j} e_{i,j} \otimes e_{i,j}$, which is $m$ times an orthogonal projection. Thus (2.2) gives

$$\|(\text{id}_{M_m} \otimes \tau) F\|_h \geq \|(\text{id}_{M_m} \otimes [\rho_2 \rho_1] \tau)(F)\|_{M_m \otimes M_m} = m \geq \sqrt{m}\|F\|_h. \quad \square$$

**Lemma 2.9.** $\|\tau\|_{cb} \leq \sqrt{\max_i \min_i \# \mu_i^{-1}}$. 

**Proof.** Assume that the quantity $m = \max_i \min_i \# \mu_i^{-1}x$ is finite. Using the isomorphism established in Proposition 2.3 we can write the flip on $C_0Y_1 \otimes_{c_0X} C_0Y_2$ as the composite

$$C_0Y_1 \otimes_{c_0X} C_0Y_2 \xrightarrow{[\pi_1 \pi_2]} C_0(Y_1 \times X_2) \xrightarrow{\sigma^*} C_0(Y_2 \times X_1) \xrightarrow{(\pi_1^{-1} \pi_2)} C_0Y_2 \otimes_{c_0X} C_0Y_1$$

where $\sigma^*$ is the isomorphism of $C^*$-algebras induced by the homeomorphism $\sigma(\pi_2, \pi_2) = (\pi_1, \pi_2)$. The norm bounds established in Proposition 2.3 thus give

$$\|\tau\|_{cb} \leq \|[\pi_2 \pi_2^{-1}]\|_{cb} \cdot \|\sigma^*\|_{cb} \cdot \|[\pi_1 \pi_2]^{-1}\|_{cb} \leq \sqrt{m}. \quad \square$$

**Corollary 2.10.** The following are equivalent for continuous maps $\mu_1 : Y_1 \to X$ and $\mu_2 : Y_2 \to X$:

(a) The canonical map $[\pi_1 \pi_2] : C_0Y_1 \otimes_{c_0X} C_0Y_2 \to C_0(Y_1 \times X_2)$ is a completely bounded isomorphism.
(b) The flip \( \tau : C_0Y_1 \otimes^h_{C^\omega X} C_0Y_2 \to C_0Y_2 \otimes^h_{C^\omega X} C_0Y_1 \) is completely bounded.

(c) \( \max_{x \in X} \min_{i=1,2} \mu_i^{-1} x < \infty \).

**Proof.** Proposition \[2.7\] implies that (b) and (c) are equivalent. Proposition \[2.3\] shows that (c) implies (a). To show that (a) implies (b), we first take operator-space adjoints in (a) to conclude that \([\pi_2^* \pi_1^*]\) is also a completely bounded isomorphism. Then, as in the proof of Lemma \[2.9\], we write \( \tau = [\pi_2^* \pi_1^*]^{-1} \sigma^*[\pi_1^* \pi_2^*] \) and conclude that \( \tau \) is completely bounded.

**Remark 2.11.** Let \( B \) be a \( C^* \)-algebra and consider the flip \( \tau \) on \( B \otimes^h B \). Simple algebraic manipulations show that

\[ \|\text{id}_{M_n} \otimes \tau\| = \|T \otimes \text{id}_{B \otimes^h B}\| \]

where \( T \) is the transpose map on \( M_n \). We have an isometric embedding \( M_n \otimes (B \otimes^h B) \to M_n \otimes (B \otimes B) \), and it is interesting to compare the transpose maps on these two matrix spaces. Take \( B = \mathbb{C}^n \). On the one hand, Proposition \[2.7\] shows that the transpose map on \( M_n \otimes (B \otimes^h B) \) has norm bounded by \( \sqrt{n} \) (and equal to \( \sqrt{n} \) when \( n \geq m \)). On the other hand, Tomyama’s results \[Tom83\] show that if \( m \geq 3 \) then the transpose map on \( M_n \otimes (B \otimes B) \) has norm \( n \) (because \( B \otimes B \) surjects on to the \( C^* \)-algebra of \( \text{PSL}(2, \mathbb{Z}) \), and hence has irreducible representations of arbitrarily large dimension).

3. **Descent, concluded**

Throughout this section we assume the hypotheses of Theorem \[1.3\]: \( \mu : Y \to X \) is a proper, surjective, uniformly finite continuous mapping of locally compact Hausdorff spaces, and the notation \( \pi, \pi_{ij}, A, B, C, D, \) etc., is as in Section \[1\]. In this section we shall recall the main result of \[Cri19\], which gives an alternative description of the category \( \text{CM}(C_0X) \); then we shall use the results of Section \[2\] to give an alternative description of the category \( \text{Des} \mu \); and finally we shall prove Theorem \[1.3\].

**Hilbert \( C^* \)-comodules.** First some further preliminaries related to the Haagerup tensor product. In this section we consider Haagerup tensor products not just of \( C^* \)-algebras, but of *operator modules* over \( C^* \)-algebras; see \[BLM04\] for the theory. The most important example will be the following: if \( \mathcal{F} \) is a Hilbert \( C^* \)-module over \( A \), then \( \mathcal{F} \) is in a canonical way an operator module over \( A \), and over each \( C^* \)-subalgebra of \( A \); and the Haagerup tensor product \( \mathcal{F} \otimes^h A B \) is completely isometrically isomorphic to the \( C^* \)-module tensor product \( \mathcal{F} \otimes^h A B \). A very special feature of the Haagerup tensor product over \( C^* \)-algebras is that if \( t : \mathcal{R}_1 \to \mathcal{R}_2 \) and \( s : \mathcal{L}_1 \to \mathcal{L}_2 \) are completely isometric maps of right and of left (respectively) operator \( A \)-modules, then the map \( t \otimes s : \mathcal{R}_1 \otimes^h A \mathcal{L}_1 \to \mathcal{R}_2 \otimes^h A \mathcal{L}_2 \) is also completely isometric.

Next some more notation. For each right operator module \( \mathcal{E} \) over \( B \) we let \( m_\mathcal{E} : \mathcal{E} \otimes^h A B \to \mathcal{E} \) be the completely contractive multiplication map, \( e \otimes b \mapsto eb \).

For each nondegenerate right operator module \( \mathcal{F} \) over \( A \) we define \( \eta_\mathcal{F} : \mathcal{F} \to \mathcal{F} \otimes^h A B \) to be the completely isometric \( A \)-linear map defined by \( \eta_\mathcal{F}(f) := f \otimes 1 \).
In case $B$ is not unital, the meaning of $f \otimes 1$ is as follows: we embed $B$ into any unitisation $B^+$, and consider the induced completely isometric embedding $\mathcal{F} \otimes_A^h B \hookrightarrow \mathcal{F} \otimes_A^h B^+$. The element $f \otimes 1 \in \mathcal{E} \otimes_A^h B^+$ actually lies in the submodule $\mathcal{F} \otimes_A^h B$ because we may use nondegeneracy to write $f = f^t a$ for some $a \in A$, whence $f \otimes 1 = f^t \otimes \mu^t(a) \in \mathcal{F} \otimes_A^h B$. Finally, for each $\mathcal{E} \in \text{CM}(B)$ we consider the products

$$\langle \ | \rangle : \mathcal{E} \times \mathcal{E} \otimes_A B \rightarrow B \otimes_A^h B, \quad \langle e | e' \otimes b \rangle := \langle e|e' \rangle \otimes b, \text{ and}$$

$$\langle \ | \rangle : \mathcal{E} \otimes_A^h B \times \mathcal{E} \rightarrow B \otimes_A^h B, \quad \langle e \otimes b|e' \rangle := b^* \otimes \langle e|e' \rangle.$$

**Definition 3.1.** The $C^*$-category $CC(B \otimes_A^h B)$ of Hilbert $C^*$-comodules over $B \otimes_A^h B$ is defined as follows. An object of this category is a pair $(\mathcal{E}, \delta)$ consisting of a Hilbert $C^*$-module $\mathcal{E} \in \text{CM}(B)$, together with a completely bounded $B$-linear map $\delta : \mathcal{E} \rightarrow \mathcal{E} \otimes_A^h B$ making the following two diagrams commute:

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\delta} & \mathcal{E} \otimes_A^h B \\
\downarrow & & \downarrow \\
\mathcal{E} \otimes_A^h B & \xrightarrow{\delta \otimes \text{id}_B} & \mathcal{E} \otimes_A^h B \otimes_A^h B \\
& \downarrow{\eta_e \otimes \text{id}_B} & \downarrow{\delta \otimes \text{id}_B} \\
& \mathcal{E} \otimes_A^h B \otimes_A^h B & \xrightarrow{\text{id} \otimes \text{id}_B} & \mathcal{E} \otimes_A^h B \otimes_A^h B \\
\end{array}$$

and also satisfying $\langle e|\delta(e') \rangle = \langle \langle \delta(e)|e' \rangle \rangle$ for all $e, e' \in \mathcal{E}$. A morphism $(\mathcal{E}_1, \delta_1) \rightarrow (\mathcal{E}_2, \delta_2)$ in this category is an adjointable map of Hilbert $C^*$-modules $t : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ satisfying $\delta_2 t = (t \otimes \text{id}_B) \delta_1$.

Given a $C^*$-module $\mathcal{F} \in \text{CM}(A)$, we define a comodule structure $\delta_{\mathcal{F}}$ on the pullback $\mu^* \mathcal{F} \in \text{CM}(B)$ as follows:

$$\delta_{\mathcal{F}} : \mu^* \mathcal{F} = \mathcal{F} \otimes_A^h B \xrightarrow{\eta_x \otimes \text{id}_B} \mathcal{F} \otimes_A B \otimes_A^h B = \mu^* \mathcal{F} \otimes_A^h B,$$

The assignment $\mathcal{F} \mapsto (\mu^* \mathcal{F}, \delta_{\mathcal{F}})$ extends to a $*$-functor $S : \text{CM}(A) \rightarrow \text{CC}(B \otimes_A^h B)$, given on morphisms by $S(t) := \mu^*(t)$.

**Theorem 3.2** ([Cri19, Theorem 5.6]). The functor $S : \text{CM}(A) \rightarrow \text{CC}(B \otimes_A^h B)$ is a unitary equivalence of $C^*$-categories.  

**Reduced descent data.** We have defined a descent datum for $\mu$ to be an isomorphism of $C$-modules $\varphi : \mathcal{E} \otimes_B^* C_1 \rightarrow \mathcal{E} \otimes_B^* C_2$ with certain additional properties; here we are writing $C_i$ to indicate $C$ considered as a left $B$-module via the homomorphism $\pi^*_i$. Since $\mu$ is assumed to be uniformly finite, Proposition 2.3 implies that the map $[\pi^*_1, \pi^*_2] : B \otimes_A^h B \rightarrow C_1$ is a completely bounded isomorphism of left $B$-modules. The map $\sigma^+ : C_2 \rightarrow C_1$ (pullback along the flip $(y_1, y_2) \mapsto (y_2, y_1)$) is likewise a completely bounded isomorphism of left $B$-modules, and using these two isomorphisms we obtain from the map $\varphi$ a completely bounded $A$-linear isomorphism $\Psi(\varphi) : \mathcal{E} \otimes_A^h B \rightarrow \mathcal{E} \otimes_A^h B$, defined as
the following composition:

\[(3.3) \quad \psi(\varphi): \mathcal{E} \otimes^h_A B \xrightarrow{\cong} \mathcal{E} \otimes^h_B B \otimes^h_A B \xrightarrow{\text{id}_{\mathcal{E}} \otimes \varphi \otimes \text{id}_{\mathcal{E}}} \mathcal{E} \otimes^h_A C_1 \xrightarrow{\varphi} \ldots \]

\[\ldots \rightarrow \mathcal{E} \otimes^h_B C_2 \xrightarrow{\text{id}_{\mathcal{E}} \otimes \varphi \otimes \text{id}_{\mathcal{E}}} \mathcal{E} \otimes^h_B C_1 \xrightarrow{\text{id}_{\mathcal{E}} \otimes \varphi \otimes \text{id}_{\mathcal{E}}} \mathcal{E} \otimes^h_B B \otimes^h_A B \rightarrow \mathcal{E} \otimes^h_A B.\]

(The first and the last arrows are the canonical isomorphisms \(\mathcal{E} \otimes^h_B B \otimes^h_A B \xrightarrow{\cong} \mathcal{E}.)

Applying Proposition [2.3] to the maps \(\mu \pi_1: Y \times_X Y \rightarrow X\) and \(\mu: Y \rightarrow X\) gives a three-variable analogue \(B \otimes^h_A B \otimes^h_B B \xrightarrow{\cong} D\) of the isomorphism \([\pi_1^* \pi_2^*],\) and using this isomorphism it is a routine matter to translate the defining properties of a descent map \(\varphi\) (i.e., \(C^\ast\)-linearity, the cocycle condition, and unitarity) into corresponding properties of \(\Psi(\varphi)\). To state the result we need one more piece of notation: for each \(C^\ast\)-module \(\mathcal{E} \in \text{CM}(B)\) we consider the inner product

\[\langle \langle \rangle \rangle: (\mathcal{E} \otimes^h_A B) \times (\mathcal{E} \otimes^h_B B) \rightarrow \mathcal{E} \otimes^h_B B, \quad \langle \langle e \otimes b | e' \otimes b' \rangle \rangle := \langle e | e' \rangle \otimes b^h b'.\]

Recall that \(\tau: B \otimes^h_A B \rightarrow B \otimes^h_A B\) denotes the flip, \(\tau(b_1 \otimes b_2) = b_2 \otimes b_1\). This map is completely bounded by Corollary [2.10].

**Definition 3.4.** A reduced descent datum for \(\mu\) is a pair \((\mathcal{E}, \psi)\), where \(\mathcal{E} \in \text{CM}(B)\) is a Hilbert \(C^\ast\)-module over \(B\), and where \(\psi: \mathcal{E} \otimes^h_B B \rightarrow \mathcal{E} \otimes^h_A B\) is a completely bounded \(A\)-linear isomorphism satisfying the following conditions:

1. the automorphisms \(\psi \otimes \text{id}_B\) and \(\text{id}_\mathcal{E} \otimes \tau\) of \(\mathcal{E} \otimes^h_B B \otimes^h_A B\) satisfy the braid relation:

\[(\psi \otimes \text{id}_B)(\text{id}_\mathcal{E} \otimes \tau)(\psi \otimes \text{id}_B) = (\text{id}_\mathcal{E} \otimes \tau)(\psi \otimes \text{id}_B)(\text{id}_\mathcal{E} \otimes \tau).\]

2. \(\psi(m_{\mathcal{E}} \otimes \text{id}_B) = (\text{id}_\mathcal{E} \otimes m_B)(\psi \otimes \text{id}_B)(\text{id}_\mathcal{E} \otimes \tau)\) as maps \(\mathcal{E} \otimes^h_B B \otimes^h_A B \rightarrow \mathcal{E} \otimes^h_A B\).

3. \(\langle \langle \psi(w) | \psi(w') \rangle \rangle = \tau(\langle \langle w | w' \rangle \rangle)\) for all \(w, w' \in \mathcal{E} \otimes^h_B B\).

A morphism \(t: (\mathcal{E}_1, \psi_1) \rightarrow (\mathcal{E}_2, \psi_2)\) of reduced descent data is an adjointable map of Hilbert \(C^\ast\)-modules \(t: \mathcal{E}_1 \rightarrow \mathcal{E}_2\) satisfying \(\psi_2(t \otimes \text{id}_B) = (t \otimes \text{id}_B)\psi_1\). We write \(\text{Des}'\) \(\mu\) for the \(C^\ast\)-category of reduced descent data.

**Lemma 3.5.** If \((\mathcal{E}, \psi)\) is a reduced descent datum then \(\psi^2 = \text{id}_{\mathcal{E} \otimes^h_B B}\).

**Proof.** The braid relation, together with the fact that \(\tau^2 = \text{id}_{B \otimes^h_A B}\), implies that \(\psi^2 \otimes \text{id}_B\) is the identity on \(\mathcal{E} \otimes^h_B B \otimes^h_A B\). Now the diagram

\[
\begin{array}{ccc}
\mathcal{E} \otimes^h_A B & \xrightarrow{\psi^2} & \mathcal{E} \otimes^h_B B \\
\downarrow{\eta_{\mathcal{E} \otimes^h_A B}} & & \downarrow{\eta_{\mathcal{E} \otimes^h_B B}} \\
\mathcal{E} \otimes^h_B B \otimes^h_A B & \xrightarrow{\psi^2 \otimes \text{id}_B} & \mathcal{E} \otimes^h_A B \otimes^h_B B
\end{array}
\]

is commutative, and the vertical arrows are complete isometries. \(\square\)

**Lemma 3.6.** The assignment \((\mathcal{E}, \varphi) \mapsto (\mathcal{E}, \Psi(\varphi))\), together with the identity map on morphisms, gives a unitary isomorphism of \(C^\ast\)-categories \(R: \text{Des}\ \mu \cong \text{Des}'\ \mu\).
The composite functor $RD : \text{CM}(A) \to \text{Des'} \mu$ is given on objects by $RD(\mathcal{F}) = (\mu^* \mathcal{F}, \text{id}_\mathcal{F} \otimes \tau)$, and on morphisms by $RD(\tau) = \mu^*(\tau)$.

**Proof.** The fact that $\varphi$ is $C$-linear and satisfies the cocycle condition if and only if $\Psi(\varphi)$ satisfies conditions (1) and (2) in Definition 3.4 is proved exactly as in the parallel algebraic setting; we omit the details. Supposing these equivalent conditions to be satisfied, let us explain why $\varphi$ is unitary if and only if $\psi := \Psi(\varphi)$ satisfies condition (3) in Definition 3.4.

A computation with elements of the form $z = e \otimes \pi_1^*(b_1) \pi_2^*(b_2)$ and $z' = e' \otimes \pi_1^*(b_1') \pi_2^*(b_2')$ shows that

\[
\langle z|z' \rangle_{C_1} = [\pi_1^* \pi_2^*] (\langle \Phi(z)|\Phi(z') \rangle) \quad \text{for all } z, z' \in \mathcal{E} \otimes_B^* C_1.
\]

Then a computation with elements of the form $z = e \otimes c$ and $z' = e' \otimes c'$ shows that

\[
\sigma^*(\langle z|z' \rangle)_{C_2} = (\langle \text{id}_c \otimes \sigma^* \rangle \circ (\text{id}_c \otimes \sigma^*) \circ (\langle z'|z \rangle)_{C_1} \quad \text{for all } z, z' \in \mathcal{E} \otimes_B^* C_2.
\]

Putting together (3.7) and (3.8), we find for all $z, z' \in \mathcal{E} \otimes_B^* C_1$ that

\[
\langle \varphi(z)|\varphi(z') \rangle_{C_2} = \sigma^* (\langle \text{id}_{c'} \otimes \sigma^* \rangle \circ (\text{id}_{c'} \otimes \sigma^*) \circ (\langle z'|z \rangle)_{C_1} = [\pi_1^* \pi_2^*] (\langle \Phi(\text{id}_c \otimes \sigma^* \circ \varphi)(z)|\Phi(\text{id}_c \otimes \sigma^* \circ \varphi)(z') \rangle) = [\pi_1^* \pi_2^*] (\langle \psi(\Phi(z))|\psi(\Phi(z')) \rangle).
\]

Comparing this with (3.7) shows that $\langle \varphi(z)|\varphi(z') \rangle_{C_2} = \langle z|z' \rangle_{C_1}$ for all $z, z' \in \mathcal{E} \otimes_B^* C_1$ if and only if $\tau (\langle \psi(w)|\psi(w') \rangle) = (\langle w|w' \rangle)$ for all $w, w' \in \mathcal{E} \otimes_A^h B$. In other words, $\varphi$ is unitary if and only if $\psi$ satisfies the condition (3) of Definition 3.4.

It follows from this that the assignment $R : (\mathcal{E}, \varphi) \mapsto (\mathcal{E}, \Psi(\varphi))$ (and the identity map on morphisms) is a unitary isomorphism of $C^*$-categories $\text{Des} \mu \to \text{Des'} \mu$. The formula for $RD$ follows from a straightforward computation. \qed

**Proof of Theorem 1.3.** Theorem 3.2 and Lemma 3.6 reduce the proof of Theorem 1.3 to the following lemma:

**Lemma 3.9.** There is a unitary equivalence of $C^*$-categories $T : \text{CC}(B \otimes_A^h B) \xrightarrow{\cong} \text{Des'} \mu$ making the diagram

\[
\begin{array}{ccc}
\text{CM}(A) & \xrightarrow{S} & \text{Des'} \mu \\
\text{CC}(B \otimes_A^h B) & \xrightarrow{T} & \end{array}
\]

commute.

**Proof.** Given a comodule $(\mathcal{E}, \delta) \in \text{CC}(B \otimes_A^h B)$, let $\psi_{\delta} : \mathcal{E} \otimes_A^h B \to \mathcal{E} \otimes_A^h B$ be the following completely bounded $A$-linear map:

\[
\psi_{\delta} : \mathcal{E} \otimes_A^h B \otimes_{A} \text{id}_{\mu} \to \mathcal{E} \otimes_A^h B \otimes_{A} \text{id}_{\mu} \mapsto \mathcal{E} \otimes_A^h B \otimes_{A} \text{id}_{\mu} \mapsto \mathcal{E} \otimes_A^h B.
\]
Theorem 3.2 implies that there is a module $\mathcal{F} \in \text{CM}(A)$ and a unitary $B$-linear isomorphism $u : \mathcal{E} \xrightarrow{\cong} \mathcal{F} \otimes_A B$ satisfying $\delta = (u^* \otimes \text{id}_B)(\eta_{\mathcal{F}} \otimes \text{id}_B)u$. A routine computation then shows that

$$\psi_\delta = (u^* \otimes \text{id}_B)(\text{id}_\mathcal{E} \otimes \tau)(u \otimes \text{id}_B).$$

Since $(\mathcal{F} \otimes_A B, \text{id}_\mathcal{F} \otimes \tau) = \text{RD}(\mathcal{F})$ is a reduced descent datum, (3.10) ensures that the same is true of $(\mathcal{E}, \psi_\delta)$. Thus the assignment $(\mathcal{E}, \psi) \mapsto (\mathcal{E}, \psi_{\delta})$ (and the identity map on morphisms) determines a $\ast$-functor $T : \text{CC}(B \otimes_A^h B) \to \text{Des}' \mu$. Putting $u = \text{id}_{\mathcal{F} \otimes_A B}$ into (3.10) shows that $TS = \text{RD}$.

We will show that $T$ is a unitary isomorphism of $C^*$-categories, by constructing an inverse. Given a reduced descent datum $(\mathcal{E}, \psi) \in \text{Des}' \mu$, let $\delta_\psi : \mathcal{E} \to \mathcal{E} \otimes_A B$ be the completely bounded map defined as the following composition:

$$\delta_\psi : \mathcal{E} \xrightarrow{\eta_\mathcal{E}} \mathcal{E} \otimes_A B \xrightarrow{\psi} \mathcal{E} \otimes_A B.$$

Algebraic computations as in [Bor94, p.246] show that this map $\delta_\psi$ is $B$-linear, and makes the two diagrams in Definition 3.1 commute. Let us show that $\delta_\psi$ satisfies the Hermitian condition $\langle e | \delta_\psi(e') \rangle = \langle \eta_\mathcal{E}(e) | \psi(e') \rangle$ for all $e, e' \in \mathcal{E}$.

We first observe that the inner products $\langle \cdot | \cdot \rangle$, $\langle \langle \cdot | \cdot \rangle$, and $\langle \langle \rangle | \langle \rangle \rangle$ are related by the formulas

$$\langle e | w \rangle = \langle \eta_\mathcal{E}(e) | w \rangle \quad \text{and} \quad \langle w | e \rangle = \tau(\langle w | \eta_\mathcal{E}(e) \rangle)$$

for all $e \in \mathcal{E}$ and $w \in \mathcal{E} \otimes_A^h B$; this follows immediately from the definitions upon writing $e = e'\mu^*(a)$ and $w = e'' \otimes b$. We then use the Hermitian property of $\psi$ (property (3) from Definition 3.4) and the fact that $\psi^2 = \text{id}$ (Lemma 3.5) to compute, for $e, e' \in \mathcal{E}$,

$$\langle e | \delta_\psi(e') \rangle = \langle \eta_\mathcal{E}(e) | \psi_\left(\eta_\mathcal{E}(e')\right) \rangle = \tau(\langle \psi_\left(\eta_\mathcal{E}(e)\right) \eta_\mathcal{E}(e') \rangle) = \langle \psi_\left(\eta_\mathcal{E}(e)\right) \eta_\mathcal{E}(e') \rangle.$$ 

Thus the pair $(\mathcal{E}, \delta_\psi)$ is an object in $\text{CC}(B \otimes_A^h B)$, and the assignment $(\mathcal{E}, \psi) \mapsto (\mathcal{E}, \delta_\psi)$ (and the identity on morphisms) yields a $\ast$-functor $\text{Des}' \mu \to \text{CC}(B \otimes_A^h B)$. To see that this functor is inverse to $T$, one shows by a computation as in [Bor94, p.248] that $\psi_{\delta_\psi} = \psi$ for all reduced descent data $(\mathcal{E}, \psi)$, and that $\delta_{\psi_\delta} = \delta$ for all comodules $(\mathcal{E}, \delta)$. This completes the proof of Lemma 3.9 and of Theorem 1.3.

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