Quantum Hamiltonian identification from measurement time traces

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Precise identification of parameters governing quantum processes is a critical task for quantum information and communication technologies. In this work we consider a setting where system evolution is determined by a parameterized Hamiltonian, and the task is to estimate these parameters from temporal records of a restricted set of system observables (time traces). Based on the notion of system realization from linear systems theory we develop a constructive algorithm that provides estimates of the unknown parameters directly from these time traces. We illustrate the algorithm by applying it to a one-dimensional spin chain model with variable couplings.

The promise of quantum technologies for tasks such as computation, communication, and metrology is motivating the construction of devices that are precisely engineered at the nanoscale, and whose quantum dynamics are exceptionally well characterized and controlled \cite{1}. The fragility and sensitivity of typical quantum devices make achieving such objectives extremely challenging, and significant research efforts over the past two decades have focused on addressing these challenges.

Process tomography is the most generally applied technique for characterizing an unknown quantum dynamical process \cite{1,2}. However, all variants of process tomography are very resource demanding, e.g., in the required number of measurements settings and number of input state preparations. In addition, it is often unsuitable in resource-constrained situations where one may only have measurement access to certain observables or subsystems; e.g., see Fig. \ref{fig:1}. Furthermore, process tomography does not utilize often available partial information about the system. One such common scenario is when the structure of a dynamical model can be obtained from underlying physics and what is to be determined are some unknown parameters in the model. This is the quantum version of parameter estimation in classical system sciences.

In this work, we consider a new approach to quantum parameter estimation. Whereas process tomography typically measures a complete basis of system observables at one time instant, we ask what can be achieved if a temporal record of a small set of system observables is collected? We refer to such a successive record of observable expectations as an \textit{measurement time trace}, and develop a method that enables information about dynamical parameters to be extracted from such time traces. Our method takes into account \textit{a priori} information and fits naturally into resource constrained situations, and as such we expect that it will be very experimentally relevant and feasible. Note that because our scheme utilizes a time trace, it can identify the generator of dynamics (e.g., a Hamiltonian) as opposed to the dynamical map (e.g., a unitary at a fixed time), which is typically what process tomography achieves. This is advantageous since in physically realistic scenarios the generator of dynamics is more compactly specified than the map. This will be discussed in more detail below.

Several authors have considered parameter estimation from various types of time-dependent measurement records \cite{3–11}. Devitt \textit{et al.} presented a scheme to identify any two-qubit Hamiltonian from the temporal evolution of the concurrence measure of entanglement \cite{3,4}. Cole \textit{et al.} used Fourier analysis to identify a single qubit Hamiltonian from one measurement observable \cite{5}. Subsequent work by Burgarth \textit{et al.} \cite{6,7} and Di Franco \textit{et al.} \cite{8} generalized this approach to estimate the coupling strengths in a many-qubit network from measurements on a small part of the network. Recently, Burgarth \textit{et al.} presented a framework for quantum system identification based on input/output information and formalized the notion of equivalence between system realizations \cite{12}.

Our approach differs from existing work on quantum system identification in two critical aspects. Firstly, we develop a constructive algorithm for identification

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{A spin (or qubit) lattice as an example illustrating the type of system considered in this work. The spins interact with each other through nearest-neighbor or long-range couplings and certain local observables are measurable for a subset of the spins (circled above). The task is to identify the parameters defining the Hamiltonian of the interconnected system from a time trace of expectation values of these observables.}
\end{figure}

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of arbitrary Hamiltonian quantum dynamics that takes advantage of available prior knowledge of the system (e.g., network structure or partial knowledge of parameters). The technique can also be employed when such prior information is absent. Secondly, in contrast to most existing system identification schemes, we do not require state tomography of a restricted set of subsystems, but rather develop a technique that produces parameter estimates based only on the collected measurement time traces.

Setup – We consider the task of identifying the Hamiltonian of an unknown quantum dynamical process. Assume that the dimension of the system is finite and known, and that the dynamical process can be prepared at some well-characterized initial states. Further, we assume that the dynamical evolution of the process is unitary (no decoherence). This condition can be relaxed and the approach will be extended to the non-unitary case in a future publication.

A parametrized form of the Hamiltonian governing the quantum dynamical process can be written as,

\[ H = \sum_{m=1}^{M} a_m(\theta) X_m, \]

where \( \theta \) is a vector consisting of unknown parameters, \( a_m \in \mathbb{R} \) are some known functions of \( \theta \), and \( X_m \) are known Hermitian operators [21]. Assume that the dimension of the quantum process is \( N \), and thus \( iH \in \mathfrak{su}(N) \), i.e. the Lie algebra consisting of all the \( N \times N \) skew-Hermitian matrices. An orthonormal basis of \( N^2 - 1 \) matrices \( \{iX_m\} \) can be chosen for \( \mathfrak{su}(N) \), where the Hilbert-Schmidt inner product is defined as \( \langle iX_m, iX_n \rangle = \text{tr}(X_m^\dagger X_n) \), and hence \( a_m = \text{tr}(HX_m) \). For example, \( \frac{1}{2} \sigma_1^x \otimes \sigma_2^z \) form a basis for the two-qubit algebra \( \mathfrak{su}(4) \), where \( \sigma_\alpha \), \( \sigma_\beta \) can be Pauli matrices \( \sigma_x \), \( \sigma_y \), \( \sigma_z \), or the identity matrix \( I_2 \), and superscripts label the qubits [22]. The numbers \( C_{jkl} \) such that

\[ [iX_j, iX_k] = \sum_{l=1}^{N^2-1} C_{jkl}(iX_l), \quad j, k = 1, \ldots, N^2 - 1, \]

are the structure constants of the Lie algebra \( \mathfrak{su}(N) \) with respect to this basis. Each element \( X_m \) is Hermitian and thus can be considered an observable for the system. Furthermore, we can consider that \( a_m \) as our unknown parameters, because solving for \( \theta \) from \( a_m \) is simply an algebraic problem.

Note that in Eq. (1), typically \( M \ll N^2 - 1 \) because of physical constraints on system energy, locality, and weight of interactions. For instance, the Hamiltonian for the spin lattice system in Fig. 1 contains only weight-one and weight-two basis elements \( X_m, X_{m\pm 1} \), and furthermore, the weight-two interactions might be restricted to only being between nearest-neighbor spins on the lattice. By utilizing measurement time traces our identification algorithm can estimate the process at the Hamiltonian level where there are only \( M \) unknown parameters. In contrast, process tomography generally does not consider time traces and therefore must estimate the process at the unitary level where there are in general \( N^2 - 1 \) unknown parameters.

Observable dynamics – The dynamics of the expectation value of an observable \( X_k \), written as \( x_k = \langle \psi | X_k | \psi \rangle \), can be derived as

\[ \frac{d}{dt} x_k = \sum_{l=1}^{N^2-1} \left( \sum_{m=1}^{M} C_{mkl} a_m \right) x_l. \]

Collecting the \( x_k \) in a vector \( x \in \mathbb{R}^{N^2-1} \), we obtain a linear equation describing the complete dynamics:

\[ \frac{d}{dt} x = Ax, \quad x(0) = \langle \psi(0) | X_k | \psi(0) \rangle, \]

where the matrix \( A \in \mathbb{R}^{(N^2-1)\times(N^2-1)} \) has elements \( A_{kl} = \sum_{m=1}^{M} C_{mkl} a_m \). Using the symmetries of the structure constants, it can be shown that \( A^T = -A \). The vector \( x \), often called the coherence vector [13], is a complete representation of the quantum state. Eq. (3) explicitly describes the quantum dynamics as a linear time invariant (LTI) system and hence it enables application of results from classical linear systems theory.

Typically, some observable expectation values may be easily measured, i.e., some elements in \( x \) can be tracked as a function of time. A physically motivated example of such a scenario is where local observables of a collection of spins are measured, see Fig. 1. Denote the set of independent measured observables by \( \mathcal{M} \).

The vector \( x \) is a vector of length \( p \) that contains the indices of the observables. We assume that the measured observables belong to the chosen \( \mathfrak{su}(N) \) basis for simplicity, since we can always redefine the basis for \( \mathfrak{su}(N) \) to make this the case. Generally, \( p \ll N^2 - 1 \).

In the following we will use time traces of the measured observables to identify the unknown Hamiltonian parameters. To this end, we first need to derive the dynamical equation governing the time evolution of these observables. Parallel to the study of controllability in classical nonlinear systems theory [14], we give a constructive procedure to obtain the closed dynamics for these observables. For the Hamiltonian in Eq. (1), let \( \Delta = \{ X_m \}_{m=1}^{M} \). Define an iterative procedure as

\[ G_0 = \mathcal{M}, \quad \text{and} \]

\[ G_i = [G_{i-1}, \Delta] \cup G_{i-1}, \]

where \([G_{i-1}, \Delta] := \{ [g, h] : g \in G_{i-1}, h \in \Delta \} \) [23]. In geometric control theory, the sequence of \( G_i \) are referred to as the filtration associated to \( \Delta \) [14]. Since \( \mathfrak{su}(N) \) is finite, this iteration will saturate at a maximal set \( G \) after finite steps, and we refer to this set as the accessible set. Intuitively, the set \( G \) contains the elements of the system that couple to the measured observables. Then, writing
all the $x_k$ with $X_k \in \mathcal{G}$ in a vector $x_a$ of dimension $K \leq N^2 - 1$, the dynamics for this vector is given by

$$\frac{d}{dt} x_a = \tilde{A} x_a,$$  \hspace{1cm} (5)

where $\tilde{A}$ is a $K \times K$ sub-matrix of $A$, i.e., only the elements necessary to describe the evolution of the subset of observable averages collected in $x_a$.

Identification algorithm – It is clear that the parameter $a_m$ is accessible from the time trace of $M$ only if it participates in the dynamical equation (5). Then a necessary condition for the identifiability of $a_m$ is that it be present in the matrix $\tilde{A}$, because if this were not satisfied, then there is no way to infer its value from examining the observables in $M$. In order to estimate these accessible parameters we utilize the notion of a system realization constructed from the measurement time traces. In linear systems theory there are many methods for constructing a realization of a linear dynamical system based on measurement results [15], and in the following we adapt one of these, the eigenstate realization algorithm (ERA) [10], for the purposes of Hamiltonian parameter estimation.

The estimation setting we consider is the following. Suppose we have access to the expectation values of the observables in $M$ at regular time instants $j \Delta t$ for some sampling period $\Delta t$. Denote these values as $\{y(j \Delta t)\}$, and they may have to be collected from averaging measurements on several runs of the experiment under the same initial state. Note that $y(j \Delta t)$ is the output of the following discretized form of Eq. (5):

$$x_a(j + 1) = \tilde{A}_d x_a(j), \hspace{1cm} y(j) = C x_a(j),$$  \hspace{1cm} (6)

where for brevity of notation we use $x_a(j) \equiv x_a(j \Delta t)$ and $y(j) \equiv y(j \Delta t)$, and $\tilde{A}_d = e^{\Delta t}$. The $p \times K$ matrix $C$ picks up the entries in $x_a(j)$ that correspond to expectation values of elements of $M$. Also assume that the system is prepared at a fixed, known initial state $x(0)$, and the corresponding initial state for Eq. (6) is $x_a(0)$. Then these relations can be solved easily to obtain an explicit form for the outputs:

$$y(j) = C \tilde{A}_d^j x_a(0).$$  \hspace{1cm} (7)

We will comment more on the choice of initial state or states later. Our objective is to use the collected data $\{y(j)\}_{j=0}^M$ to identify the Hamiltonian parameters $\{a_m\}_{m=1}^M$.

The first stage of the estimation algorithm is to construct a minimal realization of the system based on input/output information. This is achieved by ERA in three steps, as follows.

**Step 1:** Collect the measured data into an $r p \times s$ matrix (generalized Hankel matrix) as:

$$H_{rs}(k) = \begin{bmatrix} y(k) & y(k + t_1) & \cdots & y(k + t_{s-1}) \\ y(j_1 + k) & y(j_1 + k + t_1) & \cdots & y(j_1 + k + t_{s-1}) \\ \vdots & \vdots & \ddots & \vdots \\ y(j_{r-1} + k) & y(j_{r-1} + k + t_1) & \cdots & y(j_{r-1} + k + t_{s-1}) \end{bmatrix}$$

with arbitrary integers $j_i \ (i = 1, \ldots, r - 1)$ and $t_i \ (l = 1, \ldots, s - 1)$.

**Step 2:** Find the singular value decomposition (SVD) of $H_{rs}(0)$ as

$$H_{rs}(0) = P \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} Q^T = [P_1 \ P_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix},$$

where $P \in \mathbb{R}^{rp \times rp}$, $Q \in \mathbb{R}^{rs \times s}$ are both orthonomal, and $\Sigma$ is a diagonal matrix with the non-zero singular values of $H_{rs}(0)$ determined up to numerical accuracy $\epsilon$, i.e., $\Sigma_{ii} > \epsilon$ for all $i \leq n_\Sigma$ where $n_\Sigma$ is the dimension of $\Sigma$. The matrices $P_1$, $P_2$, $Q_1$, $Q_2$ are partitions with compatible dimensions.

**Step 3:** Form a realization of the system (6) as:

$$\hat{A}_d = \Sigma^{-\frac{1}{2}} P_1^T H_{rs}(1) Q_1 \Sigma^{-\frac{1}{2}}, \hspace{1cm} \hat{C} = E_p^T P_1 \Sigma^{-\frac{1}{2}},$$

where $E_p = [I_p, 0_p, \ldots, 0_p]$. The pair $(\hat{A}_d, \hat{C})$ forms a realization of Eq. (6) in the sense that they generate the same output:

$$y(j) = \hat{C} \hat{A}_d^j x_a(0) = \hat{C} \tilde{A}_d^j \dot{x}(0), \hspace{1cm} \text{for all } j \geq 0,$$  \hspace{1cm} (8)

provided that $\dot{x}(0) \equiv \Sigma^{\frac{1}{2}} Q_1^T e_1$, where $e_1$ is the first column of $I_s$.

This completes the specification of the ERA algorithm. It results in a realization of the dynamical system in the form of the triple $(\hat{A}_d, \hat{C}, \dot{x}(0))$. In general $(\hat{A}_d, \hat{C}, \dot{x}(0)) \neq (\hat{A}_d, C, x_a(0))$, but it can be shown that ERA is a correct algorithm in the sense that given sufficient data the transfer function for any finite dimensional model can be exactly reproduced in the regime of negligible measurement noise [17].

In order to estimate the Hamiltonian parameters using this realization, we let $\hat{A} = \log \hat{A}_d/\Delta t$, and by expanding the matrix logarithm function in a power series and combining that with Eq. (8), we obtain the equalities

$$\hat{C} \tilde{A}_d^j x_a(0) = \hat{C} \tilde{A}^j \dot{x}(0), \hspace{1cm} \text{for all } j \geq 0.$$  \hspace{1cm} (9)

Note that the left hand side of Eq. (9) is completely determined by the measured data, and the right hand side is a set of polynomials in the Hamiltonian parameters $a_m$ with maximum order $j$. Given sufficient measurement data and enough equations from Eq. (9) (i.e., large enough $j$), we can reconstruct the Hamiltonian parameters by solving these polynomial equations.
We emphasize the non-trivial difference between Eqns. \ref{eq:7} and \ref{eq:9}. Having access to the time trace \( y(j) \), one could simply try to solve the set of equations in Eq. \ref{eq:4} directly, and this does not require our algorithm. However, since \( \tilde{A}_d \) is a transcendental function of \( u_m \), determining the parameters this way is usually infeasible. Instead, we have utilized ERA and formulated a new relationship in Eq. \ref{eq:9}, from which parameter estimation only requires solving polynomial equations. When \( j \) is large these equations can be complex; however, there are many established techniques for solving large systems of polynomial equations \cite{18}. Furthermore, these equations can also be cast as an optimization problem for the unknown parameters, which may be more effective if the system of equations is overdetermined, or if there is noise in the measurement results.

A judicious choice for the initial state is crucial to this identification scheme. For instance, if \( x_n \) is zero or an eigenvector of \( \tilde{A} \), it leads to no sensitivity in the output to any of the unknown parameters. Care must be taken to avoid such degenerate cases. In fact, running the algorithm with multiple initial states leads to more polynomial equations with low order and thus helps to solve these equations more efficiently.

This system identification algorithm can result in multiple, non-unique estimates of the unknown parameters, all of which satisfy the input/output relations captured by Eq. \ref{eq:9}. This is because several system Hamiltonians can generate the same map between an input state and measurement time trace, and hence are equivalent from an input/output perspective \cite{12}. In the event that the algorithm results in multiple parameter estimates and more specification is needed, one has to appeal to prior information, or add resources such as additional input states or observable time traces.

The rank of Hankel matrix, \( n_{\Sigma} \), is the size of the re-constructed realization, and is an informative parameter. If \( n_{\Sigma} < K \), this means that the original dynamical systems lacks complete controllability or observability. An obvious way in which this can happen is if, for example, some coupling parameters for a network of qubits are actually zero and thus part of the network is decoupled from the portion being measured. That part of the system is then irrelevant for the dynamics captured in the Hankel matrix and is non-identifiable from the measured observables.

**Example** – Consider the following Hamiltonian for a one-dimensional chain of \( n \) qubits:

\[
H = \sum_{k=1}^{n} \frac{\omega_k}{2} \sigma_k^{z} + \sum_{k=1}^{n-1} \delta_k (\sigma_k^{z} \sigma_{k+1}^{+} + \sigma_k^{+} \sigma_{k+1}^{z}) . \tag{10}
\]

This Hamiltonian is often used as a model for a spin “wire” that enables quantum state transfer \cite{19}. Suppose that only one end of the spin chain is observable, and choose \( \langle \sigma_z^i \rangle \) as the observable that is tracked. Calculating the filtration per Eq. \ref{eq:4} yields the accessible set as \( G = \{ \sigma_z^1, \sigma_y^1 \} \cup \{ \sigma_z^{k-1} \sigma_z^k, \sigma_z^{k-1} \} \}_{k=2}^n \). The system matrix \( \tilde{A} \) is \( 2n \times 2n \) and has the following simple structure

\[
\tilde{A} = \begin{bmatrix}
0 & \omega_1 & 0 & -\delta_1 \\
-\omega_1 & 0 & \delta_1 & 0 & 0 \\
0 & -\delta_1 & 0 & \omega_2 & 0 \\
\vdots & \delta_1 & 0 & -\omega_2 & \ddots & 0 \\
0 & \vdots & \vdots & \ddots & \ddots & -\delta_{n-1} \\
0 & 0 & -\delta_{n-1} & 0 & \omega_n & 0 \\
\delta_{n-1} & 0 & -\omega_n & 0 & 0 & 0
\end{bmatrix}
\]

with \( x_n = [\tilde{x}_1, \tilde{y}_1, ..., \tilde{x}_n, \tilde{y}_n] \), where \( \tilde{x}_1 = \langle \sigma_z^1 \rangle, \tilde{y}_1 = \langle \sigma_y^1 \rangle \) and \( \tilde{x}_k \equiv \langle \sigma_z^1 \cdots \sigma_z^{k-1} \sigma_z^k \rangle, \tilde{y}_k \equiv \langle \sigma_z^1 \cdots \sigma_z^{k-1} \sigma_y^k \rangle \) for \( k \geq 2 \). In this basis \( C = [1, 0, 0, ..., 0] \). All parameters in Eq. \ref{eq:10} appear in \( \tilde{A} \), and therefore the necessary condition for identifying all parameters is satisfied for an estimation strategy that uses only time traces of \( \langle \sigma_z^1 \rangle \).

Choose two initial states as \( \frac{|0\rangle + |1\rangle}{\sqrt{2}} |0\cdots0\rangle \) and \( \frac{|0\rangle + |1\rangle}{\sqrt{2}} |0\cdots0\rangle \), and the corresponding coherence vectors \( x_n \) are \( [1, 0, 0, ..., 0]^T \) and \( [0, 1, 0, ..., 0]^T \), respectively. Running the identification algorithm with these two initial states results in two realizations \( (\tilde{A}_1, \tilde{C}_1, x_1(0)) \) and \( (\tilde{A}_2, \tilde{C}_2, x_2(0)) \), and the first few informative polynomial equations obtained from Eq. \ref{eq:9} are:

\[
\omega_1 = \tilde{C}_2 \tilde{A}_2 \tilde{x}_2(0) \\
\omega_1^2 + \delta_1^2 = -\tilde{C}_1 \tilde{A}_1^2 \tilde{x}_1(0) \\
\omega_1^3 + \delta_1^2(2\omega_1 + \omega_2) = -\tilde{C}_2 \tilde{A}_2^2 \tilde{x}_2(0) \\
\omega_1^4 + \delta_1^2(3\omega_1^2 + 2\omega_1 \omega_2 + \omega_2^2 + \delta_1^2 + \delta_2^2) = \tilde{C}_1 \tilde{A}_1^3 \tilde{x}_1(0) .
\]

These equations can be solved successively for the unknown parameters \( \omega_k \) and \( \delta_k \). Note that the coupling strengths, \( \delta_k \) only occur to even order in these equations, which means that measuring the local observable \( \langle \sigma_z^1 \rangle \) can only determine these coupling parameters up to a sign. Additional measurements or prior information are required to determine the sign. In the Supplementary Information we simulate time traces for this model with \( n = 3 \) qubits, and solve the polynomial equations above to explicitly demonstrate the parameter estimation algorithm.

**Conclusion** – We have developed an algorithm to identify the unknown parameters of a quantum Hamiltonian from the time traces of a set of system observables, which naturally takes into account prior information and restrictions on measurement access. In this work we have assumed that the measurement record is noiseless. Most experimental measurement time traces will be noisy and we are currently investigating the robustness of the algorithm to such noise. Another direction for future work
is the generalization of this algorithm to parameter estimation for open quantum systems governed by Lindblad evolution [20], in which case the evolution of the coherence vector is described by an affine time-invariant system of equations [13].

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[21] We set ℏ = 1 and therefore the am have units 1/s.
[22] In the following we will omit the tensor product when writing multi-qubit Pauli operators for brevity.
[23] The weight of a multi-qubit Pauli operator is the number of non-identity terms in the tensor product.
[24] We do not need to keep track of multiplicative constants, only the operators generated by these commutators.
SUPPLEMENTARY INFORMATION

In this Supplementary Information we explicitly demonstrate our parameter estimation algorithm for the spin chain example in the main text, with \( n = 3 \) qubits.

Consider the following Hamiltonian:

\[
H = \sum_{k=1}^{3} \frac{\omega_k}{2} \sigma_z^k + \sum_{k=1}^{2} \delta_k \left( \sigma_+^k \sigma_-^{k+1} + \sigma_-^k \sigma_+^{k+1} \right),
\]

with true parameter values \( \omega_1 = 1.3, \omega_2 = 2.4, \omega_3 = 1.7, \delta_1 = 0.3, \delta_2 = 0.2 \) (all parameters have units \( 1/s \)).

Recall that we choose two initial states \( |0\rangle + |1\rangle \sqrt{2} |0\rangle \) and \( |0\rangle + |1\rangle \sqrt{2} |0\rangle \), and the measurement time trace is of \( \langle \sigma_z^1 \rangle \). Fig. 2 shows measurement time traces for the two initial states when simulated for \( T = 20s \) with \( \Delta t = 0.1s \).

![Figure 2](image)

**FIG. 2:** Measurement time traces for observable \( \langle \sigma_z^1 \rangle \) for the spin chain example with \( n = 3 \) qubits. The black circles show the measurement trace for initial state \( |0\rangle + |1\rangle \sqrt{2} |0\rangle \) and the magenta crosses show the measurement trace for initial state \( |0\rangle + |1\rangle \sqrt{2} |0\rangle \).

Using this data we construct the Hankel matrix \( H_{rs}(0) \) with \( r = 100, s = 100 \) and all \( j_i = 1 \) and \( t_l = 1 \). Then, performing the remaining ERA steps we obtain two realizations \( (\hat{A}_{d1}, \hat{C}_1, \hat{x}_1(0)) \) and \( (\hat{A}_{d2}, \hat{C}_2, \hat{x}_2(0)) \), one for each initial state. Both realizations have the same dimension as the original \( \hat{A}_i; i.e., n_{SC} = 6 \).

For the parameter estimation stage of the algorithm we need to formulate five polynomial equations (since there are five unknown parameters in this system) after taking the logarithm of the system matrix realizations. The five lowest order polynomials resulting from the two realizations are:

\[
\begin{align*}
\omega_1 & = \hat{C}_2 \hat{A}_2 \hat{x}_2(0) = 1.3 \\
\omega_1^2 + \delta_1^2 & = -\hat{C}_1 \hat{A}_2^2 \hat{x}_1(0) = 1.78 \\
\omega_1^3 + \delta_1^2 (2\omega_1 + \omega_2) & = -\hat{C}_2 \hat{A}_2^3 \hat{x}_2(0) = 2.647 \\
\omega_1^4 + \delta_1^2 (3\omega_1^2 + 2\omega_1 \omega_2 + \omega_2^2) + \delta_1^4 & = \hat{C}_1 \hat{A}_1^4 \hat{x}_1(0) = 4.4041 \\
\delta_1^5 (3\omega_1 + 2\omega_2) + \delta_1^2 (\delta_1^2 (2\omega_1 + \omega_2 + \omega_3) + 4\omega_1^3 + 3\omega_2 \omega_1^2 + 2\omega_2^2 \omega_1 + \omega_2^3) + \omega_1^5 & = \hat{C}_2 \hat{A}_1^5 \hat{x}_2(0) = 8.2942
\end{align*}
\]

Note that the odd order polynomials of the first realization and the even order polynomials of the second realization are not informative because both sides of Eq. (9) in the main text are zero in these cases. These equations can be solved successively, and doing so results in the following estimates for the parameters:

\[
\hat{\omega}_1 = 1.3, \hat{\omega}_2 = 2.4, \hat{\omega}_3 = 1.7, \hat{\delta}_1 = \pm 0.3, \hat{\delta}_2 = \pm 0.2.
\]

The estimates exactly match the true parameters, except for the indeterminate sign for the coupling parameters. As discussed in the main text, this uncertainty in the sign is a result of the equivalency of systems under some input/output maps, and cannot be resolved unless additional measurements and/or initial states are introduced.