A NONLOCAL WEICKERT TYPE PDE APPLIED TO
MULTI-FRAME SUPER-RESOLUTION

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(Communicated by Gang Bao)

ABSTRACT. In this paper, we propose a nonlocal Weickert type PDE for the
multiframe super-resolution task. The proposed PDE can not only preserve
singularities and edges while smoothing, but also can keep safe the texture
much better. This PDE is based on the nonlocal setting of the anisotropic
diffusion behavior by constructing a nonlocal term of Weickert type, which is
known by its coherence enhancing diffusion tensor properties. A mathematical
study concerning the well-posedness of the nonlocal PDE is also investigated
with an appropriate choice of the functional space. This PDE has demonstrated
its efficiency by combining the diffusion process of Perona-Malik in the flat
regions and the anisotropic diffusion of the Weickert model near strong edges, as
well as the ability of the non-local term to preserve the texture. The elaborated
experimental results give a great insight into the effectiveness of the proposed
nonlocal PDE compared to some PDEs, visually and quantitatively.

1. Introduction. The need for high-resolution (HR) images in various practical
applications continues to rise day after day. Indeed, to see or detect fine details in a
given image, the resolution plays a critical role; more the resolution is large more the
image details are clear. For instance, in medical imaging, HR magnetic resonance
imaging (MRI) must be at high resolution to facilitate the analysis or diagnosis
made by doctors. Otherwise, to recognize a vehicle license plate or criminal face,
the police need to zoom some parts of the image or video, which requires a large
resolution. However, the hardware limitations and the very expensive price that
hardware components require leads to the use of mathematical algorithms. Among
these algorithms, we find the super-resolution (SR) theory which considered as a
powerful image processing tool to enhance and increase the quality of a degraded
image [13, 38, 1, 10, 6].

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The main purpose of the SR techniques is to build HR images through an estimation of the unknown pixel values from given low-resolution (LR) images. The super-resolution techniques are separated into two families, namely: the single SR and the multi-frame SR. Concerning the single frame SR technique, the missing high frequency information in the LR image during the acquisition step is estimated through a considerable number of training set images which complete the resolution of the LR image [15, 18, 39]. While for the multi-frame SR technique, the main objective is to guess the missing information in a given LR image by combining different information that other LR images contain [11, 20, 26, 29, 35, 24]. Several multi-frame SR techniques have been investigated in various image-processing tools, especially in the medical imaging [12, 14, 41].

The multi-frame super-resolution have been intensively studied in order to improve its three main steps, namely: the registration, the fusion and the restoration, as denoted in [30]. After the seminal frequency domain approach proposed by Tsai et al. [34], several similar techniques have been proposed to improve the multi-frame SR problems [19]. However, these approaches are very limited by the considered image observation model, while real problems are much more sophisticated. Then, the interpolation methods were proposed with less contribution since they not take into consideration blur and noise. In order to overcome the difficulties encountered by the frequency domain and interpolation methods, the SR image reconstruction algorithms based on the spatial domain are considered. Currently, the regularization functions are widely considered in the SR context and are showing their robustness and adaptive capacity in preserving image edges. The SR algorithms are formulated using an optimization problem with a fidelity term which measures the difference between the LR frames and the obtained HR one, and a regularization term that imposes some prior knowledge on the desired HR image. One of the widely-used prior functions was the Total Variation (TV) regularization [32, 33, 28] which gives promising results, preserving then image features like edges. The simplest choice of this function in the SR process was the Bilateral Total Variation (BTV) [10], which is a generalized discrete form of TV regularization. This prior function is constructed by replacing every pixel with a weighted average of its neighborhood. Even if the BTV term preserves edges and smooth areas, it produces artificial edges in the flat surfaces. Based on the advantages of the TV and BTV norms, a combined term was also introduced in [23] which increases the quality of the restoration step of the SR process. A more robust SR model was proposed using a spatially weighted TV model [40], where the authors introduce a spatial information indicator that identifies the spatial properties of the image region based on the difference curvature coefficient. This can provide the necessary information to preserve sharp edges and avoids smoothing flat areas. A more recent SR approach based on Huber-Norm using Bregman distances was proposed in [24] with more consistency against contrast loss while strong edges and contours are well preserved with less blur.

On another side, the Euler-Lagrange equation associated to nonlinear PDEs has also been treated in the super-resolution context with pleasant results. One of consistent nonlinear PDEs was proposed by Maiseli et al. [27], it takes the advantages of perona-Malik equation and the TV norm. This adaptive diffusion-based PDE can efficiently preserve image features but it suffers from the blurring effect. A more robust PDE was proposed by El Mourabit et al. [9] which takes into consideration the coherence-enhancing property and avoids blur much better. However, when the
blur and noise levels are too high, the obtained HR image still contains some artefacts, in particular the blur. Recently, a nonlinear fourth order PDE that preserves singularities and tiny edges. This PDE is based on anisotropic diffusion behavior by imposing some constraints to the Weickert coherence enhancing diffusion tensor in order to adjust the diffusion process near edges and avoid the destruction of corners. In fact, this PDE combines between the diffusion process of Perona-Malik in the flat regions and the anisotropic diffusion of the Weickert model near strong edges.

Otherwise, nonlocal approaches were also presented to preserve well the texture. One of the famous work was presented by Potter et al., where a Non-Local Means regularization method (NLM) [31] is introduced using the redundant structures in the image. Using the same principle, the nonlocal total variation (TV) [43] regularization term was also proposed which aims to preserve the constant regions in the texture. Recently, a nonlocal Laplace regularization approach was proposed and have shown very promising results [21] but didn’t take into consideration the contrast changes. Another improved nonlocal approach was also studied in [21], where the authors presented a nonlocal form of the BTV term for $p = 2$. Even if these approaches are succeeded to preserve texture, they still suffer from the staicasing effect and falls when the noise level are high in the LR images. A more robust PDE is then desired to take into account the image texture and also the intensity changes in the image.

The main contribution of this paper consists of reducing the blur and noise without destroying the obtained HR image using a nonlocal PDE. Since the Weickert term is known for its efficiency in avoiding blur during the restoration process, we propose an alternative nonlocal second-order PDE which generalizes the proposed one in [9] to the nonlocal setting. This equation can keep a balance between the modified Weickert filter [37] and the efficiency of the nonlocal operators in preserving texture. In fact, the weakness of the classical second-order PDE which is characterized by the appearance of blur in the smooth areas is diverted by the introduction of the gradient of piecewise constant functions, while using nonlocal operators can preserve much better the texture. In addition, we propose a mathematical analysis of the proposed nonlocal multi-frame SR PDE including the well-posedness of the solution. As a result, the proposed PDE can efficiently preserve flat regions, texture and sharp edges.

This paper is organized as follows. In section 2, we present the multi-frame super-resolution problem setting. Then, we introduce the theoretical analysis of the proposed SR method. After, we give a brief discretization part of the proposed nonlocal PDE. Finally, in section 5, we present some real and synthetic results, while we also give a comparison with some competitive methods.

2. Problem setting. Inspired by the success of the nonlocal models in the texture preservation for numerous image processing tasks, we present in this paper a generalization of the local-Weickert-type PDEs [37, 9, 22] for multiframe super-resolution, to the nonlocal configuration. This will preserve the image texture and increase the sharpness of the restored HR image. In this section, we present our main nonlocal multi-frame super-resolution model. For that, we start by introducing some useful definitions and notations of the nonlocal operators. Then, we describe the proposed nonlocal PDE and finally, we give the functional framework for the nonlocal model.

2.1. Notations. Now, we give the definitions of some necessary nonlocal operators. Let $u$ be a scalar function defined from $\mathbb{R}^n \to \mathbb{R}$ and let $\nu$ and $\sigma$ be two vector
functions defined from $\mathbb{R}^n \times \mathbb{R}^n$ in $\mathbb{R}^d$ (with $n$ and $d$ are in $\mathbb{N}^*$) such that
\[ \alpha(x, y) = -\alpha(y, x), \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n. \]

In the following, we denote by $u \cdot v \in \mathbb{R}$ the inner product of two vectors $u, v \in \mathbb{R}^d$. The action of the nonlocal divergence operator $\mathcal{D}$ on $\nu$ is defined as follows:
\[ \mathcal{D}(\nu)(x) := \int_{\mathbb{R}^n} (\nu(x, y) + \nu(y, x)) \alpha(x, y) dy, \quad \forall x \in \mathbb{R}^n, \quad (1) \]
The nonlocal gradient operator $\mathcal{D}^*$, which is the adjoint operator of $\mathcal{D}$, is given by:
\[ \mathcal{D}^*(u)(x, y) = -(u(y) - u(x)) \alpha(x, y), \quad \forall x, y \in \mathbb{R}^n. \quad (2) \]

These nonlocal operators have been elaborated by analogy to the classical local operators [8].

We consider a bounded open domain $\Omega$ of $\mathbb{R}^n$ and its corresponding interaction domain $\Omega_I \subset \mathbb{R}^n$ defined as:
\[ \Omega_I := \{ y \in \mathbb{R}^n \setminus \Omega : \alpha(x, y) \neq 0 \text{ for some } x \in \Omega \}. \]
this domain contains all the points outside of $\Omega$ that interact with the points in $\Omega$. The following figure describes four of the possible configurations for $\Omega$ and $\Omega_I$.

![Figure 1. Some possible configurations for $\Omega$ and $\Omega_I$.](image)

The interaction operator $\mathcal{N}(\nu) : \Omega_I \rightarrow \mathbb{R}$ whose action on $\nu$ is equivalent to the Neumann condition in the local case, is defined by:
\[ \mathcal{N}(\nu)(x) := -\int_{\Omega \cup \Omega_I} (\nu(x, y) + \nu(y, x)) \alpha(x, y) dy, \quad \forall x \in \Omega_I. \quad (3) \]

In the following subsection, we define the proposed nonlocal multi-frame super-resolution model.

2.2. The nonlocal multi-frame SR problem. Multi-frame SR aims to produce a visually pleasing high-resolution (HR) image given from several low-resolution (LR) images of the same scene and it is always a challenging task in the image restoration framework. In practice, low-cost imaging sensors give rise to several image degradations such as motion blur and noise. We assume that all the frames are taken under the same environmental conditions using the same sensor. The envisaged mathematical super-resolution degradation model is given as follow:
\[ Y_k = DF_kHX + V_k, \quad \forall k = 1, 2, ..., m, \quad (4) \]
As mentioned above, we intend to give a nonlocal model generalizing the SR model presented in [9]. In other words, we propose the following nonlocal evolution problem

$$\begin{cases}
\frac{\partial X}{\partial t} + D(\psi(J_p(-D^*(X_\sigma)))D^*(X)) = \frac{1}{m} \sum_{k=1}^{m} (DF_k H)^t (DF_k H X - Y_k), & \text{in } \Omega \times ]0, T[, \\
\mathcal{N}(\psi(J_p(-D^*(X_\sigma)))D^*(X)) = 0, & \text{in } \Omega, \\
X(x, 0) = X_0, & \text{in } \Omega,
\end{cases}$$

(5)

where \(\Omega\) is considered with piecewise smooth boundary and satisfy the interior cone condition. The \(\Omega_I\) and \(\Omega \cup \Omega_I\) have the same properties and \(J_p\) is the structure tensor defined by:

$$J_p(-D^*(X_\sigma)) = K_p \ast ((-D^*(X_\sigma)) \otimes (-D^*(X_\sigma))),$$

where

$$D^*(X_\sigma) := D^*(K_\sigma \ast \tilde{X}),$$

with \(\tilde{X}\) is a linear and continuous extension of \(X\) to \(\mathbb{R}^n\), \(K_p\) and \(K_\sigma\) are two Gaussian kernels such as \(K_p(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)\) and \(K_\sigma(x) = \frac{1}{\sqrt{4\pi\sigma^2}} \exp\left(-\frac{x^2}{4\sigma^2}\right)\) respectively.

The function \(\psi\) is the diffusion tensor calculated using the eigenvalues \(\lambda_1, \lambda_2\), the eigenfunctions \(w_1, w_2\) of \(J_p\) and it is given as follows

$$\psi := \begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} f_1(\lambda_1, \lambda_2) & 0 \\ 0 & f_2(\lambda_1, \lambda_2) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}^t.$$  

(6)

While the functions \(f_1\) and \(f_2\) are defined by

$$\begin{cases}
f_1(\lambda_1, \lambda_2) = \exp\left(-\frac{\lambda_1}{k_1}\right), \\
f_2(\lambda_1, \lambda_2) = \exp\left(-\frac{\lambda_2}{k_2}\right)(1 - \exp\left(-\frac{\lambda_2}{k_2}\right)),
\end{cases}$$

(7)

where \(k_1\) and \(k_2\) are two threshold parameters defining the diffusion with respect to the directions \(w_1\) and \(w_2\) respectively. This choice allows a remarkable reduction of noise, especially in uniform zones and smooth edges.

2.3. **Functional framework.** The nonlocal energy space needed for the analysis of the proposed nonlocal PDE is given as follow:

$$V(\Omega \cup \Omega_I) := \left\{ u \in L^2(\Omega \cup \Omega_I) \text{ such that } |||u||| < \infty \right\},$$

equipped with the following norm

$$|||u||| := \left(\frac{1}{2} \int_{\Omega \cup \Omega_I} \int_{\Omega \cup \Omega_I} D^*(u)(x, y), D^*(u)(x, y) dydx \right)^{\frac{1}{2}}.$$
The dual of the space $V(\Omega \cup \Omega_I)$ is denoted by $V^*(\Omega \cup \Omega_I)$ and it is equipped by the following norm

$$
|||f|||_{V^*(\Omega \cup \Omega_I)} := \sup_{\varphi \in V(\Omega \cup \Omega_I)} \left|\left| \langle f, \varphi \rangle_{V^*(\Omega \cup \Omega_I), V(\Omega \cup \Omega_I)} \right| \right|, \quad \forall f \in V^*(\Omega \cup \Omega_I).
$$

After introducing the functional framework, we give some compactness results of the space $V$. For this purpose, we assume that the domains $\Omega$, $\Omega_I$ and $\Omega \cup \Omega_I$ are bounded with piecewise smooth boundary and satisfy the interior cone condition and that the kernel $\gamma(x, y) = \alpha(x, y).\alpha(x, y)$ satisfies the following assumptions

$A_1$ : For all $x \in \Omega \cup \Omega_I$

$$
\begin{cases}
\gamma(x, y) \geq \gamma_0 > 0, & \forall y \in B_{\varepsilon/2}(x), \\
\gamma(x, y) = 0, & \forall y \in (\Omega \cup \Omega_I) \setminus B_{\varepsilon}(x).
\end{cases}
$$

(8)

$A_2$ : There exist $s \in (0, 1)$ and positive constants $\gamma_s$ and $\gamma^s$ such that, for all $x \in \Omega$

$$
\frac{\gamma_s}{|y - x|^{n+2s}} \leq \gamma(x, y) \leq \frac{\gamma^s}{|y - x|^{n+2s}}, \quad \forall y \in B_{\varepsilon}(x),
$$

where $B_{\varepsilon}(x) := \{ y \in \Omega \cup \Omega_I : |y - x| \leq \varepsilon \}$, $\gamma_0$ and $\varepsilon$ are given positive constants.

Based on these assumptions, Du et al. [8] proved that the nonlocal energy space $V(\Omega \cup \Omega_I)$ is equivalent to the fractional-order Sobolev space $H^s(\Omega \cup \Omega_I)$ and that is compactly embedded in $L^2(\Omega \cup \Omega_I)$. In [8], one can find strong results that we will use in the following section such as the nonlocal Green’s identities and the nonlocal integration by parts formula.

Before investigating the well-posedness of the solution for the proposed nonlocal PDE (5), we give the following assumptions denoted by $(H)$ :

$H_1$ : The function $\alpha$ is in $L^2(\Omega \cup \Omega_I \times \Omega \cup \Omega_I)$ (the antisymmetric function appearing in the nonlocal operators $\mathcal{D}$ and $\mathcal{D}^*$).

$H_2$ : The function $\psi$ is Lipschitz, positive-definite matrix and coercive with the coercivity constant is $\beta$.

$H_3$ : The function $Y_k \in L^2(0, T; L^2(\Omega))$, $k = 1, \ldots, m$ and $X_0 \in L^2(\Omega)$.

Now we precise in which sense we want to solve the nonlocal SR problem. The following definition is obtained using the nonlocal Green’s first identity [8] :

**Definition 2.1.** Under the assumptions $(H)$, we say that $X$ is a weak solution of the proposed nonlocal problem (5) if it satisfies

$$
X \in L^2(0, T; V(\Omega \cup \Omega_I)) \text{ with } \frac{\partial X}{\partial t} \in L^2(0, T; V^*(\Omega \cup \Omega_I))
$$

such that

$$
\begin{cases}
\left( \frac{\partial X}{\partial t}, \varphi \right)_{V^*(\Omega \cup \Omega_I), V(\Omega \cup \Omega_I)} + \int_{\Omega \cup \Omega_I} \int_{\Omega \cup \Omega_I} (\psi(J_p(-\mathcal{D}^*(X_\sigma))), \mathcal{D}^*(X_\sigma)) \mathcal{D}^*(\varphi) dy dx \\
- \frac{1}{m} \int_{\Omega} \sum_{k=1}^{m} (DF_k H)^t D F_k H X \varphi dx = - \frac{1}{m} \int_{\Omega} \sum_{k=1}^{m} (DF_k H)^t Y_k \varphi dx, \quad \forall \varphi \in L^2(0, T; V(\Omega \cup \Omega_I)).
\end{cases}
$$

(9)
3. Analysis of the nonlocal SR problem. This section is devoted to proving the existence and uniqueness of a weak solution to the problem (5). For this reason, we need to present and prove the following useful lemma that will be used later.

**Lemma 3.1.** Assuming that $X \in L^\infty(0,T;L^2(\Omega \cup \Omega_I))$. Then, we have the following inequality

$$\| J_\rho(-D^*(X_\sigma)) \|_{L^\infty(0,T;L^2(\Omega_1 \times \Omega_\Omega_I))} \leq C \| X \|_{L^\infty(0,T;L^2(\Omega \cup \Omega_I))}. \quad (10)$$

Also, for all $X^1, X^2 \in L^\infty(0,T;L^2(\Omega \cup \Omega_I))$. We have the following inequality

$$\| J_\rho(-D^*(X^1_\sigma)) - J_\rho(-D^*(X^2_\sigma)) \|_{L^\infty(0,T;L^2(\Omega_1 \times \Omega_\Omega_I))} \leq C \| X^1 - X^2 \|_{L^\infty(0,T;L^2(\Omega \cup \Omega_I))}. \quad (11)$$

**Proof.** Using the analogy between the proof of inequality (10) and inequality (11), we will prove only the first one.

Let's prove the inequality (10). Since $\Omega \cup \Omega_I$ is a Lipschitz domain, there is a bounded linear extension operator for $L^2(\Omega \cup \Omega_I)$ (see [3])

$$P : L^2(\Omega \cup \Omega_I) \rightarrow L^2(\mathbb{R}^2)$$

such that $PX |_{\Omega \cup \Omega_I} = X$ for all $X \in L^2(\Omega \cup \Omega_I)$.

We can then define the convolution

$$X_\sigma = K_\sigma * \tilde{X},$$

and we find

$$\| J_\rho(-D^*(X_\sigma)) \|_{L^\infty(\Omega \cup \Omega_I)} \leq \| J_\rho(-D^*(X_\sigma)) \|_{L^\infty(\mathbb{R}^2)} \leq \| K_\rho \|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} D^*(X_\sigma), D^*(X_\sigma)' dxdy$$

$$\leq \| K_\rho \|_{L^\infty(\mathbb{R}^2)} \| D^*(X_\sigma) \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \leq C_1 \| D^*(X_\sigma) \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}.$$

From the definition of the nonlocal gradient, we get

$$\| D^*(X_\sigma) \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} = \| D^*(K_\sigma * \tilde{X}) \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \leq \| (K_\sigma * \tilde{X})(y) - (K_\sigma * \tilde{X})(x) \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \leq C_2 \| \alpha \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \| K_\sigma * \tilde{X} \|_{L^2(\mathbb{R}^2)} \leq C_2 \| \alpha \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \| K_\sigma \|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} \| \tilde{X} \|_{L^2(\mathbb{R}^2)} \leq C_3 \| X \|_{L^2(\Omega_\Omega_I)}. \quad (13)$$

Then

$$\| J_\rho(-D^*(X_\sigma)) \|_{L^\infty(0,T;L^2(\Omega_1 \times \Omega_\Omega_I))} \leq C \| X \|_{L^\infty(0,T;L^2(\Omega \cup \Omega_I))}. \quad \square$$

The following lemma gives some a priori estimations of the solution.

**Lemma 3.2.** Assume that assumptions (H) are satisfied, then there exists positive constants $M_i$ for $i = 1, 2, 3$, such that the weak solution of the nonlocal problem (9) satisfies the following estimations

$$\| X \|_{L^\infty(0,T;L^2(\Omega \cup \Omega_I))} \leq M_1,$$
\[ \| X \|_{L^2(0,T;V(\Omega,\Omega_t))} \leq M_2, \]
\[ \left\| \frac{\partial X}{\partial t} \right\|_{L^2(0,T;V^*(\Omega,\Omega_t))} \leq M_3. \]

**Proof.** Taking \( \varphi = X \) in the variational problem (9), and integrating on \( t \in (0,T) \) with \( \tau \in (0,T] \), we get
\[
\frac{1}{2} \| X(\tau) \|_{L^2(\Omega,\Omega_t)}^2 + \int_0^\tau \int_{\Omega,\Omega_t} (\psi(J_\rho(-D^*(X_\sigma))).D^*(X)).D^*(X)dydxdt \\
= \frac{1}{m} \int_0^\tau \int_{\Omega} \sum_{k=1}^m (DF_kH)^tDF_kHX^2dx dt \\
- \frac{1}{m} \int_0^\tau \int_{\Omega} \sum_{k=1}^m (DF_kH)^tY_kXdx dt + \frac{1}{2} \| X(0) \|_{L^2(\Omega,\Omega_t)}^2.
\]

Using the fact that \( \psi(J_\rho) \) is coercive and \( \beta > 0 \) is the constant of coercivity, we find
\[
\frac{1}{2} \| X(\tau) \|_{L^2(\Omega,\Omega_t)}^2 + \beta \int_0^\tau \| X \|^2 dt \leq \frac{1}{m} \int_0^\tau \int_{\Omega} \sum_{k=1}^m (DF_kH)^tDF_kHX^2dx dt \\
- \frac{1}{m} \int_0^\tau \int_{\Omega} \sum_{k=1}^m (DF_kH)^tY_kXdx dt + \frac{1}{2} \| X(0) \|_{L^2(\Omega,\Omega_t)}^2.
\]

Since we have
\[
\frac{1}{m} \int_0^\tau \int_{\Omega} \sum_{k=1}^m (DF_kH)^tDF_kHX^2dx dt \leq \eta \| X \|_{L^2(0,T;L^2(\Omega))}^2,
\]
and by using H"older inequality, we obtain
\[
\frac{1}{2} \| X(\tau) \|_{L^2(\Omega,\Omega_t)}^2 + \beta \int_0^\tau \| X \|^2 dt \leq \eta \| X \|_{L^2(0,T;L^2(\Omega,\Omega_t))}^2 + \frac{1}{m} \sum_{k=1}^m \| DF_kH \|_{L^\infty(\Omega)} \| Y_k \|_{L^2(0,T;L^2(\Omega,\Omega_t))} \| X \|_{L^2(0,T;L^2(\Omega,\Omega_t))} \\
+ \frac{1}{2} \| X(0) \|_{L^2(\Omega,\Omega_t)}^2.
\]

Using Young inequality and Gronwall inequality, we have
\[
\max_{0 \leq t \leq T} \| X \|_{L^2(\Omega,\Omega_t)}^2 \leq \exp((2\eta + 1)T)(\| X(0) \|_{L^2(\Omega,\Omega_t)}^2) \\
+ \frac{1}{m^2} \sum_{k=1}^m \| DF_kH \|_{L^\infty(\Omega)} \| Y_k \|_{L^2(0,T;L^2(\Omega))}^2 = M_1.
\]

From the equation (14), we have
\[
\beta \int_0^\tau \| X \|^2 dt \leq \eta \| X \|_{L^2(0,T;L^2(\Omega,\Omega_t))}^2 + \frac{1}{m} \sum_{k=1}^m \| DF_kH \|_{L^\infty(\Omega)} \| Y_k \|_{L^2(0,T;L^2(\Omega,\Omega_t))} \| X \|_{L^2(0,T;L^2(\Omega,\Omega_t))} \\
+ \frac{1}{2} \| X(0) \|_{L^2(\Omega,\Omega_t)}^2,
\]
by using
\[
\max_{0 \leq t \leq T} \| X \|_{L^2(\Omega \cup \Omega_I)}^2 \leq M_1.
\]
and Young inequality, we find
\[
\| X \|_{L^2(0,T;V(\Omega \cup \Omega_I))}^2 \leq \frac{1}{\beta} \left( \frac{1 + 2\eta}{2} M_1 T + \frac{1}{2m^2} \sum_{k=1}^m \| \nabla F_k H \|_{L^\infty(\Omega)} \| Y_k \|_{L^2(0,T;L^2(\Omega))} \right)
\]
\[
= M_2.
\]
Let’s show now the estimation of \( \frac{\partial X}{\partial t} \). From the weak formulation (9), we have
\[
\left| \left( \frac{\partial X}{\partial t}, \varphi \right)_{V^*(\Omega \cup \Omega_I), V(\Omega \cup \Omega_I)} \right| \leq \int_{\Omega \cup \Omega_I} \int_{\Omega \cup \Omega_I} (\psi(J_\rho(-D^*(X))) \cdot D^*(\varphi) - D^*(\varphi)dydx) \]
\[
+ \frac{1}{m} \int_{\Omega} \sum_{k=1}^m (DF_k H)^t DF_k H \varphi dx \]
\[
+ \frac{1}{m} \int_{\Omega} \sum_{k=1}^m (DF_k H)^t Y_k \varphi dx \quad \forall \varphi \in L^2(0,T;V(\Omega \cup \Omega_I)),
\]
using the assumption \((H_2)\) and Hölder inequality, we obtain
\[
\| \frac{\partial X}{\partial t} \|_{V^*(\Omega \cup \Omega_I)} \leq C \| J_\rho(-D^*(X)) \|_{L^\infty(\Omega \cup \Omega_I \times \Omega \cup \Omega_I)} \| X \| \]
\[
+ \frac{1}{m} \sum_{k=1}^m \| (DF_k H)^t DF_k H \|_{L^\infty(\Omega)} \| X \|_{L^2(\Omega \cup \Omega_I)}
\]
\[
+ \frac{1}{m} \sum_{k=1}^m \| DF_k H \|_{L^\infty(\Omega)} \| Y_k \|_{L^2(\Omega \cup \Omega_I)}.
\]
Integrating on \( t \in (0,T) \) and using the lemma 3.1, we get
\[
\| \frac{\partial X}{\partial t} \|_{L^2(0,T;V^*(\Omega \cup \Omega_I))} \leq M_3,
\]
which concludes the proof.

The following theorem shows the existence and uniqueness of a solution to the proposed nonlocal PDE.

**Theorem 3.3.** Under the assumptions \((H)\), the problem (5) admits a unique weak solution.

**Proof.** **Existence:** For the proof of existence, we use the classical Schauder fixed point theorem [42]. To define the fixed point operator, we introduce first these spaces
\[
V(0,T) = \left\{ v \in L^2(0,T;V(\Omega \cup \Omega_I)), \frac{\partial v}{\partial t} \in L^2(0,T;V^*(\Omega \cup \Omega_I)) \right\},
\]
which is a Hilbert space equipped with the norm

\[ \| v \|_{V(0,T)} = \| v \|_{L^2(0,T;V(\Omega,\Omega_I))} + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(0,T;V^*(\Omega,\Omega_I))}, \]

and

\[ V_0 = \{ v \in V(0,T), \| v \|_{L^\infty(0,T;L^2(\Omega,\Omega_I))} \leq M_1, \| v \|_{L^2(0,T;V(\Omega,\Omega_I))} \leq M_2, \]
\[ \left\| \frac{\partial v}{\partial t} \right\|_{L^2(0,T;V^*(\Omega,\Omega_I))} \leq M_3, \text{ and } v(0) = X_0 \}

which is a nonempty, convex and weakly compact subset of \( V(0,T) \). The estimations introduced in the functional space \( V_0 \) are deduced from the lemma 3.2.

Schauder’s fixed point operator is given as follows

\[ F : V_0 \to V_0 \]
\[ v \mapsto X_v \]

where \( X_v \) is the solution associated to \( v \), for the following problem

\[
\begin{cases}
\left( \frac{\partial X}{\partial t}, \varphi \right)_{V^*(\Omega,\Omega_I),V(\Omega,\Omega_I)} + \int_{\Omega} \int_{\Omega_I} \left( \psi(J_{\rho}(-D^*(v_n))) \cdot D^*(X) \right) \cdot \varphi \, dy \, dx \\
= \frac{1}{m} \sum_{k=1}^{m} (DF_k H)(X - Y_k) \varphi, & \forall \varphi \in L^2(0,T;V(\Omega,\Omega_I)), \\
X(0) = X_0. 
\end{cases}
\]

(15)

which is now linear in \( X \). The well-posedness of a similar nonlocal evolution problem such as the one in (15) was proven in [7] as consequence of Galerkin-type arguments. Which allows to conclude the existence of a unique solution \( X_v \in V(0,T) \) to the problem (15).

The existence of a weak solution for the problem (9) is equivalent to the existence of a fixed point for the operator \( F \). For this reason, we apply Schauder’s fixed point theorem (see [42]), which requires only to prove that the mapping \( F \) is weakly continuous.

Let \( (v_n)_{n \in \mathbb{N}} \) be a sequence in \( V_0 \) such that \( v_n \xrightarrow{n \to \infty} v \) and \( X_n = F(v_n) \), we should prove that \( X_n = F(v_n) \xrightarrow{n \to \infty} X_v = F(v) \). Since \( (v_n)_{n \in \mathbb{N}} \) is a sequence in \( V_0 \) and using the same proof as in lemma 3.2, we deduce the existence of a subsequence still denoted \( (v_n)_{n \in \mathbb{N}} \) such that:

\[
\begin{align*}
\frac{\partial X_n}{\partial t} & \xrightarrow{n \to \infty} \frac{\partial X}{\partial t} & \text{in} & & L^2(0,T;V^*(\Omega \cup \Omega_I)) \\
X_n & \xrightarrow{n \to \infty} X & \text{in} & & L^2(0,T;L^2(\Omega \cup \Omega_I)) \\
D^*(X_n) & \xrightarrow{n \to \infty} D^*(X) & \text{in} & & (L^2(0,T;L^2(\Omega \cup \Omega_I)))^2 \\
v_n & \xrightarrow{n \to \infty} v & \text{in} & & L^2(0,T;L^2(\Omega \cup \Omega_I)) \\
\psi(J_{\rho}(-D^*(v_n))) & \xrightarrow{n \to \infty} \psi(J_{\rho}(-D^*(v))) & \text{in} & & L^2(0,T;L^2(\Omega \cup \Omega_I)) \\
X_n(0) & \xrightarrow{n \to \infty} X(0) & \text{in} & & V^*(\Omega \cup \Omega_I) 
\end{align*}
\]

Using these convergences and by the uniqueness of the solution of (15), we have

\[ X_n = F(v_n) \xrightarrow{n \to \infty} X = X_v = F(v). \]
Which proves that $F$ is weakly continuous. Then, by applying Schauder’s fixed point theorem \cite{42}, the operator $F$ admits a fixed point solution to the problem (9).

**Uniqueness:** In order to show that the solution of problem (9) is unique, we consider two different solutions $X_1$ and $X_2$ to the problem (9). By subtracting both variational formulations of $X_1$ and $X_2$, and taking $\varphi = X_1 - X_2$ we obtain:

$$
\frac{\partial X_1}{\partial t} - \frac{\partial X_2}{\partial t}, X_1 - X_2) V^*(\Omega \cup \Omega), V(\Omega \cup \Omega),
$$

$$
\int_{\Omega \cup \Omega} \int_{\Omega \cup \Omega} \left( \psi(J_\rho(-D^*(X_{1t})))D^*(X_1) - \psi(J_\rho(-D^*(X_{2t})))D^*(X_2) \right).
$$

$$
D^*(X_1 - X_2) dy dx
$$

$$
= \frac{1}{m} \int_{\Omega} \left( DF_k H \right) (DF_k H (X_1 - X_2))^2 dx.
$$

Then, we get

$$
\frac{\partial X_1}{\partial t} - \frac{\partial X_2}{\partial t}, X_1 - X_2) V^*(\Omega \cup \Omega), V(\Omega \cup \Omega),
$$

$$
\int_{\Omega \cup \Omega} \int_{\Omega \cup \Omega} \left( \psi(J_\rho(-D^*(X_{1t})))D^*(X_1) - \psi(J_\rho(-D^*(X_{2t})))D^*(X_2) \right) dy dx
$$

$$
= \frac{1}{m} \int_{\Omega} \sum_{k=1}^{m} (DF_k H)^2 (DF_k H (X_1 - X_2))^2 dx
$$

$$
+ \int_{\Omega \cup \Omega} \int_{\Omega \cup \Omega} \left( \psi(J_\rho(-D^*(X_{1t})))D^*(X_1) - \psi(J_\rho(-D^*(X_{2t})))D^*(X_2) \right) dy dx
$$

$$
D^*(X_2 - X_1) dy dx
$$

which gives

$$
\frac{1}{2} \frac{d}{dt} \left( \| X_1(t) - X_2(t) \|^2_{2,\Omega \cup \Omega} \right) + \beta \int_{\Omega \cup \Omega} \int_{\Omega \cup \Omega} \left( X_1(t) - X_2(t) \right) dy dx
$$

$$
\leq \| \psi(J_\rho(-D^*(X_{1t}))) - \psi(J_\rho(-D^*(X_{2t}))) \|_{L^\infty(\Omega \times \Omega, \Omega \times \Omega)} \| X_1(t) - X_2(t) \| + \eta \| X_1(t) - X_2(t) \|^2_{2,\Omega \cup \Omega}
$$

then, by using lemma 3.1 and applying Young’s inequality for an enough small $\epsilon$, we get

$$
\frac{1}{2} \frac{d}{dt} \left( \| X_1(t) - X_2(t) \|^2_{2,\Omega \cup \Omega} \right) + \left( \beta - \frac{\epsilon}{2} \right) \| X_1(t) - X_2(t) \|^2
$$

$$
\leq \frac{M^2}{2\epsilon} \| X_1(t) - X_2(t) \|^2_{2,\Omega \cup \Omega} \| X_1(t) - X_2(t) \|^2 + \eta \| X_1(t) - X_2(t) \|^2_{2,\Omega \cup \Omega}
$$

Since $X_1(0) = X_2(0) = X_0$ and using Gronwall’s inequality we obtain the desired uniqueness.

4. Discretization. In the following, we are interested in estimating a high-resolution image $X$ from a low-resolution input $X_0$ (obtained from a bilinear interpolation of the image $Y_1$) as a solution to the problem:
For every two point function \( q \) approximation:

\[
\psi \text{ where } \sum_{i,j \in N} (X(i) - X(j))(\alpha_1(i, j) \alpha_2(i, j)) = 0 \quad \text{in } \Omega \times [0, T]
\]

The discrete iterative scheme of the problem (16) is then given by:

\[
\begin{cases}
\frac{\partial X}{\partial t} + D(\psi D^*(X)) = \frac{1}{m} \sum_{k=1}^{m} (DF_k H)^t(DF_k HX - Y_k) \\
N(\psi D^*(X)) = 0 \quad \text{in } \Omega \times [0, T] \\
X(x, 0) = X_0 \quad \text{in } \Omega
\end{cases}
\]

(16)

where \( \psi = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \) is the diffusion tensor which is calculated by the structure tensor \( J_p(-D^*(X_0)) \).

More precisely, \( a, b \) and \( c \) are the components of the tensor \( \psi \) defined by

\[
\begin{pmatrix} a & b \\ b & c \end{pmatrix} = f_1(\lambda_1, \lambda_2)w_1w_1^T + f_1(\lambda_1, \lambda_2)w_2w_2^T.
\]

(17)

We denote by \( X(i) \) the value of a pixel \( i \) in the image \( (1 \leq i \leq N) \) or in the interaction domain \( (N_1 + 1 \leq i \leq N_2) \), \( N_i = \{ j : |i - j| \leq r \} \) is the neighbors set of the pixel \( i \).

The discretization of the time derivative is given using a forward difference approximation:

\[
\frac{\partial X}{\partial t}(i) = \frac{X^{n+1}(i) - X^n(i)}{\tau}
\]

(18)

where \( \tau \) is the time step size.

Let \( \alpha(i, j) = \begin{pmatrix} \alpha_1(i, j) \\ \alpha_2(i, j) \end{pmatrix} \) denote the discrete version of \( \alpha(x, y) \), we define for every one point function \( X \in \mathbb{R} \), the discrete gradient approximation:

\[
D^*(X)(i, j) = \begin{pmatrix} (X(i) - X(j))\alpha_1(i, j) \\ (X(i) - X(j))\alpha_2(i, j) \end{pmatrix}
\]

(19)

where \( (1 \leq i \leq N_1) \) or in the interaction domain \( (N_1 + 1 \leq i \leq N_2) \), \( N_i = \{ j : |i - j| \leq r \} \) is the neighbors set of the pixel \( i \).

For every two point function \( q = (q_1, q_2) \in \mathbb{R}^2 \), we define the discrete divergence approximation:

\[
D(q)(i) = \sum_{j \in N_i} ((q_1(i, j) + q_1(j, i))\alpha_1(i, j) + (q_2(i, j) + q_2(j, i))\alpha_2(i, j))
\]

(20)

The discrete iterative scheme of the problem (16) is then given by:

\[
X^{n+1} = X^n + \tau \left( \sum_{j \in N_i} 2(X^n(j) - X^n(i))(a(\alpha_1(i, j))^2 + 2b\alpha_1(i, j)\alpha_2(i, j) + c(\alpha_2(i, j))^2) \right) \\
+ \tau \left( \frac{1}{m} \sum_{k=1}^{m} (DF_k H)^t(DF_k HX^n(i) - Y_k(i)) \right) \quad \text{for } i \in \{1, ..., N_1\}, n \geq 0
\]

\[
\sum_{j \in \{1, ..., N_1\}} 2(X^n(i) - X^n(j))(a(\alpha_1(i, j))^2 + 2b\alpha_1(i, j)\alpha_2(i, j) + c(\alpha_2(i, j))^2) \\
= 0 \quad \text{for } i \in \{N_1 + 1, ..., N_2 \}, n \geq 0
\]

\[
X^0(i) = X_0(i) \quad \text{for } i \in \{1, ..., N_1\}
\]

(21)
Finally, we summarize the proposed steepest descent algorithm for our new nonlocal SR model as follows:

**Inputs:** The LR sequence \( Y_k \); \( X_0 \) the interpolate image of \( Y_1 \), the steepest descent parameter \( \tau \).
we choose a small parameter \( \epsilon > 0 \);

**The procedure:**
\[
X^{n+1} = X^n + \tau \left( \sum_{j \in N_i} 2(X^n(j) - X^n(i))(a(\alpha_1(i,j))^2 + 2b\alpha_1(i,j)\alpha_2(i,j) + c(\alpha_2(i,j))^2) \right) \\
+ \tau \left( \frac{1}{m} \sum_{k=1}^m (DF_kH)^t(DF_kHX^n(i) - Y_k(i)) \right) \quad \text{for} \ i \in \{1, \ldots, N_\Omega \}, \ n \geq 0 \\
+ \sum_{j \in \{1, \ldots, N_\Omega \}} 2(X^n(i) - X^n(j))(a((\alpha_1(i,j))^2 + 2b\alpha_1(i,j)\alpha_2(i,j) + c(\alpha_2(i,j))^2) \\
\sum_{j \in \{1, \ldots, N_\Omega \}} = 0 \quad \text{for} \ i \in \{N_\Omega + 1, \ldots, N_\Omega \}, \ n \geq 0 \\
X^0(i) = X_0(i) \quad \text{for} \ i \in \{1, \ldots, N_\Omega \} \\
\text{Stopping criterion:} \frac{\|X_{n+1} - X_n\|_{L^2(\Omega)}}{\|X_n\|_{L^2(\Omega)}} < \epsilon.
\]

**Output:** The HR image \( X \)

**Algorithm 1:** Steepest descent algorithm

5. **Numerical experiments.** In this section, we present several experimental results to show the performance of the proposed nonlocal SR methods. We note that all of the following experiments are achieved using Matlab R2013a running on a Laptop with 3.2 GHz Intel Core i7 CPU and 8G RAM memory. For the Algorithm computation, the used stopping criteria is the classical relative error between two successive iterations, such as:
\[
\frac{\|X_{n+1} - X_n\|_{L^2(\Omega)}}{\|X_n\|_{L^2(\Omega)}} \leq 10^{-5}.
\]  
(22)

The results are separated into two subsections: the first one contains simulated tests representing some results in the denoising framework, while the second part is mainly dedicated to real experiments where the aim is to recover the clean image from video sequences.

5.1. **Experiments on simulated images.** In this subsection, we evaluate the performance of the proposed SR approach. Firstly, we consider the denoising context to show the efficiency of the nonlocal Weickert regularization term compared to other classical nonlocal terms. For that, we select three images with smoothly varying areas, constant regions, and tiny edges. We compare the proposed regularization term with the nonlocal means (NLM) [2] and the nonlocal TV (NTV) [25]. We note that we used the \( L^1 \) norm as a fidelity term for all the comparative methods. Concerning the computational efficiency, we set \( k_1 \in \{35, 50, 120\} \), \( k_2 \in \{40, 66, 130\} \), \( \rho = 1.5 \) (for \( J_\rho \)) and the iteration number \( N = 2000 \). The weighted function \( \alpha \) is chosen in the discrete form such as:
\[
\alpha_1(i,j) = \alpha_2(i,j) = \frac{1}{|y(j) - x(i)|^s + \eta(i,j)}, \ \text{with} \ 0 < s < 1,
\]
where \(1 \leq i \leq N\) and \(N_j = \{j : |i - j| \leq r\}\) is the neighbors set of the pixel \(i\), while \(\eta\) is the repulsive function between two patches defined by:

\[
\eta(i,j) = \exp\left(-\frac{||((\Gamma_x(X))_i - (\Gamma_y(X))_j)||^2}{\sigma^2}\right).
\]

\(||.||\) is the Euclidean distance. We denote by \(\Gamma_x(X)\) the patch of size \(r \times r\) around a pixel \(x \in \Omega\), it is given by

\[
\Gamma_x(X)(t) = \Gamma(x+t), \quad \text{for} \quad t \in \left[-\frac{r-1}{2}, \ldots, \frac{r-1}{2}\right],
\]

where \(r\) is an odd integer, usually chosen as follow \(r = 5, 7\) or \(11\). In the following experiment results, we fix \(r = 7\), \(s = 0.6\), \(n = 2\) the image dimension and \(\sigma = 40\) the spatial proximity parameter. For the first experiment, we consider the “Square” image which is contaminated by a Gaussian noise with zero mean and variance \(\sigma^2 = 0.03\). In Fig. 2, we show the recovered clean image through the image on the top left (noisy image) using our approach compared with the other nonlocal methods. For the second experiment (the “Cercle” image), we increase the noise level which is taken with zero mean and variance \(\sigma^2 = 0.04\). The restored image using different methods is shown in Fig. 3. We follow the same thing for the third experiment, where the Gaussian noise is considered with zero mean and variance \(\sigma^2 = 0.045\). The obtained clean image is illustrated in Fig. 4 compared to other methods. In the three cases, we can observe that the proposed nonlocal model outperforms the other methods. Note that we select the parameters of the compared methods according to the best PSNR value.

In the second part of the experiments, we consider the super-resolution context. We present then three simulated tests, where we consider a variety of noise and blur levels. For these tests, we simulate 20 synthetic LR images from the original images of Butterfly, Penguin and Build, such that: each LR image is translated, blurred by a Gaussian low-pass filter with a \(2 \times 2\) and a standard deviation of 2. After, the blurred frames are down-sampled vertically and horizontally by a factor of \(r = 4\) and finally, a Gaussian noise was added with different standard deviations \(\sigma^2 = 0.03, 0.04, 0.05\), respectively. To demonstrate the robustness of the proposed SR method, we compare it with some relevant SR approaches, such as the SR method using a nonlinear PDE (NPDE) \([27]\), the nonlinear tensor PDE (NT-PDE) \([9]\), the nonlocal Laplacian PDE (NLPDE) \([21]\), the edge preserving fourth order PDE (FOPDE) \([22]\) and the Tensor Approximation With Laplacian Scale (TALS) \([5]\). Figs. 5 - 7 show the obtained HR image for the three simulated tests compared with the other SR methods. We can see, once again that for the three cases, the proposed nonlocal PDE does better than the others, especially in the case of high noise level, which confirms the robustness of the proposed method against different outliers.

Furthermore, to better see the performance of the proposed SR compared to the others, a quantitative evaluation is used. In fact, we used two classical metrics: the peak-signal-to-noise ratio (PSNR) and the mean structure similarity (SSIM) \([36]\). In Table 1, we present the PSNR values related to the previous tests and other classical images with different noise levels, while in Table 2 we illustrated the associated SSIM values. Once again, we can see that the proposed nonlocal PDE is always with the best PSNR and SSIM values. Concerning the execution time of the proposed method, it is relatively high compared to the other methods since computing the functions values \(f_1\) and \(f_2\) needs much more CPU time.
Figure 2. The denoising result of the restored “Square” image with Gaussian noise (parameter $\sigma^2 = 0.03$).

Figure 3. The denoising result of the restored “Cercle” image with the Gaussian noise (parameter $\sigma = 0.04$).
We end the simulated part by two tests to show how the proposed nonlocal PDE can cope with the machine learning based approaches, such as: super-resolution with convolutional neural networks (SRCN) [17], based on Shock Filter and Non Local Means (SBM3D) [16] with the BM3D [4] for congruity and low-rank fusion.
combined with sparse coding (LRSC) [44]. Note that the used parameters for these methods are chosen as described in the respective papers. We use two images with different sizes and content such as one is less-textured image *Cash-box* and a much textured one *Satellite*. Note that in this part, our aim is to measure the robustness of the proposed method against noise where the decimation factor is high. For that, we construct $n = 50$ synthetic LR images from the original images such that: each frame is slightly deformed, blurred by a Gaussian low-pass filter with a $3 \times 3$ and a standard deviation of 1.5 for both tests. Then, the blurred frames are down-sampled vertically and horizontally by a factor of $r = 4$ and Gaussian noise was added with standard deviations $\sigma^2 = 0.03$ and $\sigma^2 = 0.04$, respectively. The recovered image for the two tests are presented in Figs. 8 and 9. We can obviously see that the proposed nonlocal PDE is more accurate and recover much better the
Table 1. The PSNR results of different SR methods for selected images. Note that we used a benchmark of 30 images in our tests and present only ten in this table.

| Image  | $\sigma^2$ noise | NPDE | NTPDE | NLPDE | FOPDE | TALS | Our |
|--------|------------------|------|-------|-------|-------|------|-----|
| Butterfly | $\sigma^2 = 0.03$ | 29.33 | 29.11 | 30.12 | 31.02 | 31.06 | 32.54 |
| Penguin | $\sigma^2 = 0.04$ | 27.88 | 28.29 | 28.52 | 29.04 | 29.21 | 29.90 |
| Build | $\sigma^2 = 0.05$ | 25.33 | 25.06 | 25.83 | 26.52 | 27.44 | 28.03 |
| Barbara | $\sigma^2 = 0.06$ | 24.66 | 25.60 | 26.42 | 26.74 | 28.03 | 28.10 |
| Pirate | $\sigma^2 = 0.02$ | 30.12 | 30.87 | 31.02 | 31.86 | 32.10 | 32.53 |
| Lena | $\sigma^2 = 0.04$ | 29.44 | 30.12 | 30.06 | 31.17 | 30.54 | 31.50 |
| Cameraman | $\sigma^2 = 0.01$ | 30.06 | 30.97 | 30.52 | 31.11 | 31.74 | 32.86 |
| Baboon | $\sigma^2 = 0.02$ | 30.10 | 30.22 | 31.08 | 31.16 | 31.99 | 32.40 |
| Fly | $\sigma^2 = 0.03$ | 29.44 | 29.70 | 30.16 | 30.22 | 30.76 | 31.05 |
| Horses | $\sigma^2 = 0.01$ | 29.88 | 30.44 | 30.77 | 30.49 | 31.42 | 32.20 |

Table 2. The SSIM results of different SR methods for selected images. Note that we used a benchmark of 30 images in our tests and present only ten in this table.

| Image  | $\sigma^2$ noise | NPDE | NTPDE | NLPDE | FOPDE | TALS | Our |
|--------|------------------|------|-------|-------|-------|------|-----|
| Butterfly | $\sigma^2 = 0.03$ | 0.822 | 0.829 | 0.839 | 0.877 | 0.888 | 0.901 |
| Penguin | $\sigma^2 = 0.04$ | 0.785 | 0.796 | 0.806 | 0.828 | 0.843 | 0.882 |
| Build | $\sigma^2 = 0.05$ | 0.683 | 0.694 | 0.714 | 0.730 | 0.764 | 0.800 |
| Barbara | $\sigma^2 = 0.06$ | 0.626 | 0.660 | 0.691 | 0.674 | 0.708 | 0.755 |
| Pirate | $\sigma^2 = 0.02$ | 0.839 | 0.847 | 0.855 | 0.891 | 0.886 | 0.922 |
| Lena | $\sigma^2 = 0.04$ | 0.760 | 0.775 | 0.793 | 0.811 | 0.826 | 0.829 |
| Cameraman | $\sigma^2 = 0.01$ | 0.881 | 0.902 | 0.933 | 0.948 | 0.942 | 0.967 |
| Baboon | $\sigma^2 = 0.02$ | 0.826 | 0.836 | 0.842 | 0.859 | 0.870 | 0.889 |
| Fly | $\sigma^2 = 0.03$ | 0.781 | 0.767 | 0.807 | 0.819 | 0.826 | 0.867 |
| Horses | $\sigma^2 = 0.01$ | 0.869 | 0.890 | 0.917 | 0.923 | 0.948 | 0.968 |

image features, especially, in the second example where the image is more textured and where the noise level is much higher.

5.2. Experiments on real data. In this subsection, we show the SR results for real videos data. The three sequences used are: “Text”, “Bar-code” and “Wheel” videos, which are known in the SR experiments as challenging sequences. We select the first twenty LR sequences from each video and we recover an HR image for each video by upscaling the LR version using a scale of 4. Without clue of the camera’s PSF, we assume that the blur is Gaussian with a kernel size $7 \times 7$ and standard deviation equal to 1.5 for the Text video, Gaussian with a kernel size $13 \times 13$ and standard deviation equal to 2 for the Bar-code video, while it is assumed to be Gaussian with kernel size $11 \times 11$ and standard deviation equal to 1.5 for “Wheel” video. Note that we use the approach in [21], which is based on a non-parametric registration to estimate the motion between the LR frames for our method and the
Figure 8. The super-resolution results of the sequence “Cashbox”. The obtained PSNR values for these methods are depicted such as: SRCN (32.77), LRSC (33.27), SBM3D (33.88), Our (33.79).

comparable ones. The reconstructed images by different methods are illustrated in the Figs. 10, 11 and 12. We can clearly observe that the restored image by our SR method is visually better than the others.

6. Conclusion. In this paper, we introduced a novel nonlocal PDE of Weickert type for the multiframe super-resolution enhancement. The well-posedness of the model is firstly ensured in a suitable Banach space. A noticeable advantage of this model is its ability to avoid straicsing effect together with a more clean image with remarkable control over the desired smoothing and sharpening effect. A remaining question is how to define a general choice of the function \( \alpha \)? This will be treated as another inverse problem with some assumptions on this function.

Acknowledgments. The authors are thankful to the reviewers for their helpful suggestions. Their feedback had improved the paper quality and presentation compared to the first draft.

Compliance with ethical standards.

- Funding: This research was entirely funded by the respective institutions of the authors.
- Conflict of interest: The authors declare that they have no conflict of interest.
Figure 9. The super-resolution results of the sequence “Satellite”. The obtained PSNR values for these methods are depicted such as: SRCN (25.11), LRSC (25.29), SBM3D (25.58), Our (26.08).

Figure 10. The super-resolution results of the video sequence “Text”.
Figure 11. The super-resolution results of the video sequence “Bar-code” image.

Figure 12. The super-resolution results of the video sequence “Wheel”.

REFERENCES

[1] S. Baker and T. Kanade, Limits on super-resolution and how to break them, *IEEE Transactions on Pattern Analysis & Machine Intelligence*, (2002), 1167–1183.

[2] T. Brox, O. Kleinschmidt and D. Cremers, Efficient nonlocal means for denoising of textural patterns, *IEEE Transactions on Image Processing*, 17 (2008), 1083–1092.
[3] A.-P. Calderón, Lebesgue spaces of differentiable functions and distributions, *In Proc. Sympos. Pure Math.*, 4 (1961), 33–49.

[4] K. Dabov, A. Foi, V. Katkovnik and K. Egiazarian, Image denoising by sparse 3-d transform-domain collaborative filtering, *IEEE Transactions on Image Processing*, 16 (2007), 2080–2095.

[5] W. Dong, T. Huang, G. Shi, Y. Ma and X. Li, Robust tensor approximation with laplacian scale mixture modeling for multi-frame image and video denoising, *IEEE Journal of Selected Topics in Signal Processing*, 12 (2018), 1435–1448.

[6] W. Dong, L. Zhang, G. Shi and X. Wu, Image deblurring and super-resolution by adaptive sparse domain selection and adaptive regularization, *IEEE Transactions on Image Processing*, 20 (2011), 1838–1857.

[7] Q. Du, M. Gunzburger, R. B. Lehoucq and K. Zhou, Analysis and approximation of nonlocal diffusion problems with volume constraints, *SIAM Review*, 54 (2012), 667–696.

[8] Q. Du, M. Gunzburger, R. B. Lehoucq and K. Zhou, A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws, *Mathematical Models and Methods in Applied Sciences*, 23 (2013), 493–540.

[9] I. El Mourabit, M. El Rhabi, A. Hakim, A. Laghrib and E. Moreau, A new denoising model for multi-frame super-resolution image reconstruction, *Signal Processing*, 132 (2017), 51–65.

[10] S. Farsiu, D. Robinson, M. Elad and P. Milanfar, Advances and challenges in super-resolution, *International Journal of Imaging Systems and Technology*, 14 (2004), 47–57.

[11] S. Farsiu, M. D. Robinson, M. Elad and P. Milanfar, Fast and robust multiframe super resolution, *IEEE Transactions on Image Processing*, 13 (2004), 1327–1344.

[12] M. Fernández-Suárez and A. Y. Ting, Fluorescent probes for super-resolution imaging in living cells, *Nature Reviews Molecular Cell Biology*, 9 (2008), 929–943.

[13] W. T Freeman, T. R. Jones and E. C. Pasztor, Example-based super-resolution, *IEEE Computer Graphics and Applications*, 22 (2002), 56–65.

[14] B. Huang, H. Babcock and X. Zhuang, Breaking the diffraction barrier: Super-resolution imaging of cells, *Cell*, 143 (2010), 1047–1058.

[15] D. G. S. Bagon Michal Irani, Super-resolution from a single image, *In Proceedings of the IEEE International Conference on Computer Vision, Kyoto, Japan*, pages 349–356, 2009.

[16] K. Iwamoto, T. Yoshida and M. Ikehara, Super-resolutions based on shock filter and non local means with bm3d for congruity, *In TENCON 2014-2014 IEEE Region 10 Conference*, pages 1–6. IEEE, 2014.

[17] A. Kappeler, S. Yoo, Q. Dai and A. K. Katsaggelos, Video super-resolution with convolutional neural networks, *IEEE Transactions on Computational Imaging*, 2 (2016), 109–122.

[18] K. In Kim and Y. Kwon, Single-image super-resolution using sparse regression and natural image prior, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 32 (2010), 1127–1133.

[19] A. Laghrib, A. Ben-Loghfyry, A. Hadri and A. Hakim, A nonconvex fractional order variational model for multi-frame image super-resolution, *Signal Processing: Image Communication*, 67 (2018), 1–11.

[20] A. Laghrib, M. Ezzaki, M. El Rhabi, A. Hakim, P. Monasse and S. Raghay, Simultaneous deconvolution and denoising using a second order variational approach applied to image super resolution, *Computer Vision and Image Understanding*, 168 (2018), 50–63.

[21] A. Laghrib, A. Ghazdali, A. Hakim and S. Raghay, A multi-frame super-resolution using diffusion registration and a nonlocal variational image restoration, *Computers & Mathematics with Applications*, 72 (2016), 2535–2548.

[22] A. Laghrib, A. Hadri and A. Hakim, An edge preserving high-order pde for multi-frame image super-resolution, *Journal of the Franklin Institute*, 356 (2019), 5834–5857.

[23] A. Laghrib, A. Hakim and S. Raghay, A combined total variation and bilateral filter approach for image robust super resolution, *EURASIP Journal on Image and Video Processing*, 2015 (2015), 19.

[24] A. Laghrib, A. Hakim and S. Raghay, An iterative image super-resolution approach based on bregman distance, *Signal Processing: Image Communication*, 58 (2017), 24–34.

[25] X. Liu and L. Huang, A new nonlocal total variation regularization algorithm for image denoising, *Mathematics and Computers in Simulation*, 97 (2014), 224–233.

[26] B. Maiseli, C. Wu, J. Mei, Q. Liu and H. Gao, A robust super-resolution method with improved high-frequency components estimation and aliasing correction capabilities, *Journal of the Franklin Institute*, 351 (2014), 513–527.
A NONLOCAL WEICKERT TYPE PDE

[27] B. Jacob Maiseli, N. Ally and H. Gao, A noise-suppressing and edge-preserving multiframe super-resolution image reconstruction method, Signal Processing: Image Communication, 34 (2015), 1–13.

[28] A. Marquina and S. J. Osher, Image super-resolution by tv-regularization and bregman iteration, Journal of Scientific Computing, 37 (2008), 367–382.

[29] D. Mitze, T. Pock, T. Schoenemann and D. Cremers, Video super resolution using duality based tv-l 1 optical flow, In Joint Pattern Recognition Symposium, pages 432–441. Springer, 2009.

[30] S. C. Park, M. K. Park and M. G. Kang, Super-resolution image reconstruction: A technical overview, IEEE Signal Processing Magazine, 20 (2003), 21–36.

[31] M. Protter, M. Elad, H. Takeda and P. Milanfar, Generalizing the nonlocal-means to super-resolution reconstruction, IEEE Transactions on Image Processing, 18 (2009), 36–51.

[32] L. I. Rudin, S. Osher and E. Fatemi, Nonlinear total variation based noise removal algorithms, Physica D: Nonlinear Phenomena, 60 (1992), 259–268.

[33] F. Shi, J. Cheng, L. Wang, P.-T. Yap and D. Shen, Ltv: Mr image super-resolution with low-rank and total variation regularizations, IEEE Transactions on Medical Imaging, 34 (2015), 2459–2466.

[34] R. Tsai, Multiframe image restoration and registration, Advance Computer Visual and Image Processing, 1 (1984), 317–339.

[35] M. Unger, T. Pock, M. Werlberger and H. Bischof, A convex approach for variational super-resolution, In Joint Pattern Recognition Symposium, pages 313–322. Springer, 2010.

[36] Z. Wang, E. P. Simoncelli and A. C. Bovik, Multiscale structural similarity for image quality assessment, In The Thirty-Seventh Asilomar Conference on Signals, Systems & Computers, 2 (2003), 1398–1402. IEEE.

[37] J. Weickert, Coherence-enhancing diffusion filtering, International Journal of Computer Vision, 31 (1999), 111–127.

[38] J. Yang, J. Wright, T. S. Huang and Y. Ma, Image super-resolution via sparse representation, IEEE Transactions on Image Processing, 19 (2010), 2861–2873.

[39] W. Yang, X. Zhang, Y. Tian, W. Wang, J.-H. Xue and Q. Liao, Deep learning for single image super-resolution: A brief review, IEEE Transactions on Multimedia, 21 (2019), 3106–3121.

[41] F. C. Zanacchi, Z. Lavagnino, M. P. Donnorso, A. Del Bue, L. Furia, M. Faretta and A. Diaspro, Live-cell 3d super-resolution imaging in thick biological samples, Nature Methods, 8 (2011), 1047–1049.

[42] E. Zeidler, Nonlinear Functional Analysis Vol.1: Fixed-Point Theorems, Springer-Verlag Berlin and Heidelberg GmbH and Co. K, springer edition, 1986.

[43] WL Zeng and XB Lu, A robust variational approach to super-resolution with nonlocal tv regularisation term, The Imaging Science Journal, 61 (2013), 268–278.

[44] X. Zhu, P. Jin, X. Wang and N. Ai, Multi-frame image super-resolution reconstruction via low-rank fusion combined with sparse coding, Multimedia Tools and Applications, 78 (2019), 7143–7154.

Received December 2019; revised May 2020.

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