SOME CANCELLATION RESULTS OF PROJECTIVE MODULES OVER POLYNOMIAL ALGEBRAS

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ABSTRACT. Let $A$ be an affine algebra over $\mathbb{F}_p$ of dimension $d$. Let $P$ be a projective $A[T_1, ..., T_n]$-module of rank $d$. Then $P$ is cancellative.

1. INTRODUCTION

All the rings are assumed to be commutative Noetherian with $1(\neq 0)$ and all projective modules are finitely generated with constant rank function. Let $A$ be a ring of finite (Krull) dimension $d$. Let $P$ be a projective $A$-module of rank $n$. $P$ is said to be cancellative if $P \oplus A^k \cong Q \oplus A^k$ for some integer $k \geq 1$ and some projective $A$-module $Q$, implies that $P \cong Q$. By a classical result of Bass [2] it is well known that if $n > d$ then $P$ is cancellative. Therefore, the study of the cancellation property becomes interesting whenever the equality $n = d$ holds. Let $\text{Um}(P)$ be the set of all elements $p \in P$ such that there exists a $A$-linear surjection $\phi_p : P \to A$ with the property $\phi_p(p) = 1$. It is well documented in the literature that the study of the cancellation problem (whenever $n = d$) turns out to be the study of the orbit space $\text{Um}(P \oplus R)/\text{Aut}(P \oplus R)$.

Let $P$ be a projective $A[T]$-module of rank $d + 1$. Then by a result due to Plumstead [8] it is known that $P$ is cancellative. We denote $\text{Um}_{d+1}(A[T])$ by the set $\text{Um}((A[T])^{d+1})$ and any element of the set $\text{Um}_{d+1}(A[T])$ is called a unimodular row in $A[T]$ of length $d+1$. In [11], Suslin proved that any unimodular row $v(T_1, ..., T_n) \in \text{Um}_{d+2}(A[T_1, ..., T_n])$ can be transformed to the unimodular vector $(1, 0, ..., 0)$ by an elementary matrix, whenever $d \geq 2$. That is, the group $E_{d+2}(A[T_1, ..., T_n])$, generated by all the elementary matrices, acts transitively on $\text{Um}_{d+2}(A[T_1, ..., T_n])$.

In this paper we give a criterion [Proposition 2.2] for the cancellation problem over polynomial rings. As a corollary we prove that if $R$ is an affine $\mathbb{F}_p$-algebra of dimension $d$, then any unimodular row in $R[T_1, ..., T_n]$ of length $d + 1$ is completable. In fact we have proved that projective $R[T_1, ..., T_n]$-modules of rank $d$ are cancellative. As a consequence we prove a splitting criterion of projective $R[T]$-modules of rank $d$ in

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terms of its generic section, whenever \( d \) is even. With some extra hypothesis [please see Theorem 2.6], we prove that the free module \((R[T_1, ..., T_d])^d\) is cancellative, where \( R \) is a \( \mathbb{F}_p \)-algebras of dimension \( d \).

Let \( R \) be an affine \( \mathbb{F}_p \)-algebra of dimension \( d \). In section 4, we ask whether the completion of unimodular rows in \( R[T_1, ..., T_n] \) of length \( d + 1 \), can be made elementarily? In Theorem 4.2, we show that an affirmative answer of the above question will imply that the free module \((R[T_1, ..., T_n])^{d-1}\) is cancellative, with an extra assumption \( p \geq d \geq 4 \). This is an ongoing project.

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2. CANCELLATION OF RANK \( d \) PROJECTIVE MODULES

Lemma 2.1. Let \( A \) be a ring and \( J \subset \text{Jac}(A) \) be an ideal. Let \( v \in \text{Um}_n(A) \). If \( v \) is elementarily completable going modulo \( J \) then \( v \) is elementarily completable in the ring \( A \).

Proof. Let ‘bar’ denote going modulo \( J \). Let \( \epsilon \in \text{E}_n(A/J) \) be such that \( ve = e_1 \). Since the canonical map \( \text{E}_n(A) \rightarrow \text{E}_n(A/J) \) is surjective there exists \( \alpha_1 \in \text{E}_n(A) \) such that \( \bar{\alpha}_1 = \epsilon \). Therefore, we get \( v \alpha_1 = (1 + j_1, j_2, ..., j_n) \), where \( j_i \in J \) for \( i = 1, ..., n \). Since \( j_1 \in J \subset \text{Jac}(A) \), the element \( 1 + j_1 \) is invertible in the ring \( A \). Hence we can find \( \alpha_2 \in \text{E}_n(A) \) such that \( (1 + j_1, j_2, ..., j_n) \alpha_2 = e_1 \). Let us define \( \alpha = \alpha_1 \alpha_2 \in \text{E}_n(A) \). Then we get \( v\alpha = e_1 \).

Notation. Let \( A \) be a ring and \( u, v \in \text{Um}_n(A) \). We write \( u \sim_{\text{E}_n(A)} v \) if and only if there exists an elementary matrix \( \epsilon \in \text{E}_n(A) \) such that \( ue = v \).

Proposition 2.2. Let \( A \) be a commutative Noetherian ring of dimension \( d \) which contains a field. Let \( v(T_1, ..., T_n) \in \text{Um}_{d+1}(A[T_1, ..., T_n]) \). Suppose that there exists a non-zero divisor \( s \in A \) such that the following holds:

1. \( A_s \) is smooth;
2. There exists \( \beta \in \text{SL}_{d+1}(A) \) such that \( v(0, ..., 0) \beta = e_1 \);
3. \( v(0, ..., 0) \) is completable to an elementary matrix in \( A/ < s > A \), where ‘bar’ denote going modulo \( < s > A \).

Then \( v(T_1, ..., T_n) \) is completable to an invertible matrix.

Proof. For \( d < 2 \) the result is trivial, therefore, we may assume that \( d \geq 2 \). Let \( T = (T_1, ..., T_n) \). Without loss of generality we may assume that \( A \) is reduced. Note that if \( s \) is a unit then \( A \) is smooth therefore, using [6] and [13, Theorem 3.3] we can find
\( \theta \in E_{d+1}(A[T]) \) such that \( v(T)\theta = v(0) \). Since \( v(0) \) is completable to an invertible matrix the result follows. Therefore, we may assume that \( s \) is not a unit. Let \( B = A/ < s > A \). Then \( \dim(B) \leq d - 1 \). Note that in the polynomial ring \( B[T], \) using [8] and [11] Theorem 2.6 we can find \( \tau \in E_{d+1}(B[T]) \) be such that \( v(T)\tau = \tau_1 \).

Let \( R = (A[T])_{1+s}A[T]. \) Then \( s > R \subset \text{Jac}(R) \) and \( R/ < s > R = B[T] \). Using [11] Theorem 2.6 and the hypothesis (3), we get \( \tau(T) \sim_{E_{d+1}(R/sR)} \tau_1 \sim_{E_{d+1}(R/sR)} \tau(0). \) Since \( s > R \subset \text{Jac}(R), \) by Lemma 2.1 we have \( v(T) \sim_{E_{d+1}(R)} e_1 \sim_{E_{d+1}(R)} v(0). \) Therefore, we can find \( f(T) \in A[T] \) such that \( \alpha_1' \in E_{d+1}(A[T])_{1+sf(T)} \) and \( v(T)\alpha_1' = v(0) \) in the ring \( A[T]_{1+sf(T)}. \) Let \( \alpha_1 = \alpha_1'((\beta)_{1+sf(T)}). \) Then we get \( v(T)\alpha_1 = e_1. \)

Note that since \( A_s \) is smooth by \([6]\) there exists \( \alpha_2(T)' \in SL_{d+1}(A_s[T]) \) such that \( v(T)\alpha_2(T)' = v(0). \) Moreover, replacing \( \alpha_2(T)' \) by \( \alpha_2(T)'(\alpha_2(0))^{-1} \) we may assume that \( \alpha_2(0)' = 1_{d+1}. \) Therefore, using [13] Theorem 3.3 we get \( \alpha_2(T)' \in E_{d+1}(A_s[T]) \) keeping the fact \( v(T)\alpha_2(T)' = v(0) \) unchanged. Let \( \alpha_2 = \alpha_2(T)'(\beta)_s. \) Then we get \( v(T)\alpha_2 = e_1. \)

Let \( t = 1+sf(T) \in A[T]. \) Let \( \eta = (\alpha_2)_t^{-1}(\alpha_1)_s. \) Then note that \( \eta = (\beta)_s^{-1}(\alpha_2(T))_t^{-1}(\alpha_1')_s(\beta)_st = (\beta)_s^{-1}(\alpha_2(T'))_t^{-1}(\alpha_1')_s(\beta)_st \in E_{d+1}(A[T]_{st}), \) since \( E_{d+1}(A[T]_{st}) \) is a normal subgroup of \( GL_{d+1}(A[T]_{st}). \) Observe that \( e_1\eta = e_1(\alpha_2)_t^{-1}(\alpha_1)_s = v(T)(\alpha_1)_s = e_1. \)

Claim. There exists \( \eta_1 \in E_{d+1}(A[T]_t) \) and \( \eta_2 \in E_{d+1}(A[T]_s) \) such that \( \eta = (\eta_2)_t(\eta_1)_s \) and \( e_1\eta_i = e_1 \) for \( i = 1, 2. \)

Proof of the claim. Since \( \eta \in E_{d+1}(A[T]_{st}) \) (in particular \( \eta \) is isotopic to identity) there exists \( \eta(X) \in E((\alpha_1)_s^2[A[T]_{st}]/[X]) \) such that the followings hold:

(i) \( \eta(0) = I_{d+1}; \)
(ii) \( \eta(1) = \eta; \)
(iii) \( e_1\eta(X) = e_1. \)

Using Quillen’s splitting lemma [9] Theorem 1, paragraph 2], for \( g = (s)^N \) with large \( N, \) we get

\[
\eta(X) = (\eta(Y)\eta(gX)^{-1})_t(\eta(gX))_s,
\]

with \( \eta(Y)\eta(gX)^{-1} \in E_{d+1}((\alpha_1)_s[A[T]_{st}]/[X]) \) and \( \eta(gX) \in E_{d+1}((\alpha_1)_s[A[T]_{st}]/[X]). \) Since \( e_1\eta(X) = e_1, \) this gives us \( e_1\eta(Y) = e_1 \) and \( e_1\eta(Y)\eta(gX)^{-1} = e_1. \) Let us define \( \eta_1 := \eta(g) \in E_{d+1}(A[T]_t) \) and \( \eta_2 := \eta(1)\eta(g)^{-1} \in E_{d+1}(A[T]_s). \) Therefore, we have \( \eta = (\eta_2)_t(\eta_1)_s \) and \( e_1\eta_i = e_1, \) for \( i = 1, 2. \) This completes the proof of the claim.

We define \( \sigma_t = \alpha_1\eta_1^{-1} \in SL_{d+1}(A[T]_t) \) and \( \sigma_s = \alpha_2\eta_2 \in SL_{d+1}(A[T]_s). \) Then note that \( (\sigma_t)_s = (\sigma_s)_t \) (as \( \eta = (\alpha_2)_t^{-1}(\alpha_1)_s = (\eta_2)_t(\eta_1)_s \)). Using [3] Proposition 2.2, page no 211 there exists a unique \( \alpha \in SL_{d+1}(A[T]) \) such that \( (\alpha)_s = \sigma_s \) and \( (\alpha)_t = \sigma_t. \)

Now note that \( v(T)\sigma_s = v(T)\alpha_2\eta_2 = e_1\eta_2 = e_1 \) and \( v(T)\sigma_t = v(T)\alpha_1\eta_1^{-1} = e_1\eta_1^{-1} = e_1. \) Since \( s > A[T] \) and \( t > A[T] = A[T], \) no maximal ideal \( m \subset A[T] \) can contain
both $s$ and $t$ simultaneously. Therefore, we either have $(\alpha)_m = (\alpha_m)_s = (\alpha_s)_m = (\sigma_s)_m$
or $(\alpha)_m = (\alpha_m)_t = (\alpha_t)_m = (\sigma_t)_m$. Hence a local checking ensures that $v(T) \alpha = e_1$.
This completes the proof.

Theorem 2.3. Let $A$ be an affine algebra over $\mathbb{F}_p$ of dimension $d$. Let $v(T_1, ..., T_n) \in Um_{d+1}(A[T_1, ..., T_n])$be a unimodular row of length $d + 1$. Then there exists $\alpha \in SL_{d+1}(A[T_1, ..., T_n])$ such that$v(T_1, ..., T_n)\alpha = e_1$.

Proof. Let $T = (T_1, ..., T_n)$. Note that if $d < 2$ then the result follows trivially. Therefore,we may assume that $d \geq 2$. In view of Proposition 2.2, it is enough to show that allthe hypotheses of Proposition 2.2 are satisfied. Note that without loss of generality we may assume that $A$ is reduced. Since $A$ is reduced affine algebra over $\mathbb{F}_p$, the ideal ofsingular locus has height $\geq 1$. Therefore, there exists a non-zero divisor $s \in A$ such that$A_s$ smooth. Using [12, Corollary 17.3] we can find $\beta \in E_{d+1}(A) \subset SL_{d+1}(A)$ such that$v(0)\beta = e_1$. Since $\beta \in E_{d+1}(A)$, the condition (3) of Proposition 2.2 is also satisfied. □

The next theorem is a “projective” version of the above theorem. The proof followsby mimicking the arguments given in the proof of Theorem 2.3 with some small adjustments. We shall sketch the proof for the sake of completeness.

Theorem 2.4. Let $A$ be an affine algebra over $\mathbb{F}_p$ of dimension $d$. Let $P$ be a projective$A[T_1, ..., T_n]$–module of rank $d$. Let $(a(T_1, ..., T_n), p) \in Um(A[T_1, ..., T_n] \oplus P)$. Then thereexists $\sigma \in Aut (A[T_1, ..., T_n] \oplus P)$ such that $\sigma(a(T_1, ..., T_n), p) = (1, 0)$. In particular, $P$is cancellative.

Proof. Let $T = (T_1, ..., T_n)$. Without loss of generality we may assume that $A$ is reduced. Let $S$be the set of all non-zero divisors in $A$, then note that $P \otimes S^{-1}A[T]$ is a free module. Therefore, there exists a non-zero divisor $s_1 \in A$ such that $P_{s_1}$ is a free module. Notethat if $s_1$ is a unit then this will imply that $P$ is free. In this case the proof followsfrom Theorem 2.3. Therefore, without loss of generality we may assume that $s_1$ is not a unit.

Since the ideal of singular locus of $A$ has height $\geq 1$, there exists a non-zero divisor(which is not a unit) $s_2 \in A$ such that $A_{s_2}$ is smooth. Let $s = s_1s_2$. Then note that wehave the followings:

1. $A_s$ is smooth;
2. $P_s$ is free;
3. $s$ is a non-zero divisor but not a unit in $A$.

Therefore, in the ring $A_s[T]$, using the argument used in the proof of Theorem 2.3,we can find $\alpha_1 \in E_{d+1}(A_s[T] \oplus P_s)$ such that $\alpha_1(a(T), p) = (1, 0)$.

Let $B = A/ < s > A$ and $R = (A[T])_{1+<s>\mathbb{F}[T]}$. Then $\dim(B) \leq d - 1$. Let ’bar’denote going modulo $< s >$. Note that in the polynomial ring $B[T]$, using [7] Theorem
2.6] we can find \( \tau \in E(B[T] \oplus \overline{\mathcal{P}}) \) be such that \( \tau(\overline{a}(T), \overline{p}) = \tau_1 \). Note that \( < s > R \subset Jac(R) \). Also observe that \( R/ < s > R = B[T] \) and \( \overline{\mathcal{P}} = P \otimes R/ < s > R \). Since the canonical map \( E(R \oplus (P \otimes R)) \rightarrow E((R/ < s > R) \oplus (P \otimes R/ < s > R)) = E(B[T] \oplus \overline{\mathcal{P}}) \), we can find a lift \( \epsilon \in E(R \oplus (P \otimes R)) \) of \( \tau \). Then note that \( \epsilon(a(T), p) = (1 + s \lambda, sq) \), for some \( \lambda \in R \) and \( q \in P \otimes R \). Since \( s \in Jac(R) \), \( 1 + s \lambda \in R^n \). Therefore, there exists \( \epsilon' \in E(R \oplus (P \otimes R)) \) such that \( \epsilon'(1 + s \lambda, sq) = (1, 0) \). Let \( \alpha_2 = \epsilon \epsilon' \in E(R \oplus (P \otimes R)) \). Then we have \( \alpha_2(a(T), p) = (1, 0) \).

Now following the same patching technique used in Proposition 2.2, the result follows. \( \square \)

**Corollary 2.5.** Let \( R \) be an affine algebra over \( \overline{\mathcal{F}}_p \), of dimension \( d \). Let \( A = S^{-1} R \), where \( S \subset R \) be a multiplicatively closed set such that \( \dim(A) = d \). Let \( v(T_1, ..., T_n) \in Um_{d+1}(A[T_1, ..., T_n]) \) be an unimodular row of length \( d + 1 \). Then there exists \( \alpha(T_1, ..., T_n) \in SL_{d+1}(A[T_1, ..., T_n]) \) such that \( e_1 \alpha(T_1, ..., T_n) = v(T_1, ..., T_n) \).

Proof. Let \( T = (T_1, ..., T_n) \). Using direct limit argument there exists an element \( s \in R \) such that \( v(T) \in Um_{d+1}(R[T]) \). Since \( \dim(A) = d \) this implies that \( \dim(R_s) = d \). Therefore, the proof follows from Theorem 2.3 \( \square \)

The next result is an analogy of Theorem 2.3 over finite fields with some extra hypothesis.

**Theorem 2.6.** Let \( R \) be an affine algebra over a finite field \( \overline{\mathcal{F}}_p \) of dimension \( d \). Moreover assume that there exists a non-zero divisor \( s \in R \) such that \( R_s \) is smooth. Let \( v(T_1, ..., T_n) \in Um_{d+1}(R[T_1, ..., T_n]) \) be an unimodular row of length \( d+1 \). Then there exists \( \alpha \in SL_{d+1}(A[T_1, ..., T_n]) \) such that \( v(T_1, ..., T_n) \alpha = e_1 \).

Proof. Let \( T = (T_1, ..., T_n) \). Note that if \( d < 2 \) then the result follows trivially. Therefore, we may assume that \( d \geq 2 \). In view of Proposition 2.2, it is enough to show that all the hypotheses of Proposition 2.2 are satisfied. Note that by the hypothesis, condition (1) of Proposition 2.2 is already satisfied. Using [12 Corollary 17.3] we can find \( \beta \in E_{d+1}(A) \subset SL_{d+1}(A) \) such that \( v(0) \beta = e_1 \). Since \( \beta \in E_{d+1}(A) \), the condition (3) of Proposition 2.2 is also satisfied. \( \square \)

3. APPLICATION: SPLITTING CRITERION OF PROJECTIVE MODULES OF EVEN RANK

**Lemma 3.1.** Let \( A \) be an affine algebra over \( \overline{\mathcal{F}}_p \) of dimension \( d \geq 2 \). Moreover assume that \( d \) is even. Then \( E^d(A[T]) \cong E_0^d(A[T]) \).

Proof. Let \( I \subset A[T] \) be an ideal such that \( \mu(I) = \text{ht}(I) = d \). Note that since the canonical map \( E^d(A[T]) \rightarrow E_0^d(A[T]) \) is surjective it is enough to show that any local orientation
\begin{align*}
I = & \langle f_1, \ldots, f_d \rangle + I^2 \text{ has a lift. Let } \omega_1 \text{ be the local orientation of } I \text{ induced by the set of generators } \{f_1, \ldots, f_d\}. \\
\text{Let } I = & \langle g_1, \ldots, g_d \rangle. \text{ Let } \omega_2 \text{ be the local orientation of } I \text{ induced by the set of generators } \{g_1, \ldots, g_d\}. \text{ Then note that } (I, \omega_2) = 0 \text{ in the } d\text{-th Euler class group } E^d(A[T]). \text{ Let } '\text{bar}' \text{ denote going modulo } I. \text{ As a } A[T]/I \text{-module two sets of generators of } I/I^2 \text{ must differ by some invertible matrix } \alpha \in \text{ GL}_d(A[T]/I) \text{ that is } (\overline{f}_1, \ldots, \overline{f}_d) = (\overline{g}_1, \ldots, \overline{g}_d). \text{ Let } \det(\alpha) = \overline{\pi} \in (A[T]/I)^* \text{ then we have } (I, \pi^{-1}\omega_2) = (I, \omega_1). \text{ We get } b \in A[T] \text{ such that } ab - 1 \in I. \text{ Then note that } (b, g_1, g_2, \ldots, g_d) \in \text{ Um}_{d+1}(A[T]). \text{ By Theorem 2,3 the unimodular row } (b, g_1, g_2, \ldots, g_d) \text{ is completable to an invertible matrix.} \\
\text{We can follow the arguments as in [10, Proposition, page 957, second proof]} \text{ to conclude that there is a matrix } \sigma \in M_d(A[T]) \text{ with determinant } a^{d-1} \text{ modulo } I \text{ such that, if } (g_1, \ldots, g_d)\sigma = (F_1, \ldots, F_d), \text{ then } I = \langle F_1, \ldots, F_d \rangle. \text{ Let } \omega_1' = \text{ the local orientation of } I \text{ induced by the set of generators } \{F_1, \ldots, F_d\}. \text{ Then note that we have the followings } \text{0} = (I, \omega_1') = (I, \pi^{d-1}\omega_2) \text{ in } E^d(A[T]); \\
\text{Only remaining is to show that } f_i - F_i \in I^2. \text{ To show this note that it is enough to show that } (I, \omega_1) = 0 \text{ in } E^d(A[T]). \\
\text{As } d \text{ is even we have } (I, \omega_1) = (I, \pi^{-1}\omega_2) = (I, \pi^{d-1}\omega_2) \text{ (by [3, Proposition 4.9])} = (I, \omega_1') = 0. \text{ This completes the proof.} \qedhere
\end{align*}

**Theorem 3.2.** Let \( R \) be a \( d(\geq 2) \) dimensional affine algebra over \( \mathbb{F}_p \). Moreover assume that \( d \) is even. Let \( P \) be a projective \( R[T] \)-module with trivial determinant of rank \( d \). Let \( I \subset R[T] \) be an ideal of height \( d \) such that there is a surjection \( \phi : P \twoheadrightarrow I \). Then \( P \) has a unimodular element if and only if \( \mu(I) = d \).

Proof. Suppose that \( P \) has a unimodular element. Then it is proved in [11]. Therefore, we may assume that \( \mu(I) = d \). Let \( \chi \) be a trivialization \( P \). Let \( \omega \) be a Euler cycle induced by the triplet \((P, \chi, \phi)\). Since \( \mu(I) = d \) using Lemma 3.1 we get \( (I, \omega) = 0 \) in \( E^d(A[T]) \). Therefore, by [11] \( P \) has a unimodular element. This completes the proof. \( \square \)

4. A QUESTION AND SOME OF ITS CONSEQUENCE

**Question 4.1.** Let \( A \) be an affine algebra over \( \mathbb{F}_p \) of dimension \( d \geq 2 \). Let \( v(T_1, \ldots, T_n) \in \text{ Um}_{d+1}(A[T_1, \ldots, T_n]) \) be an unimodular row of length \( d + 1 \). Then does there exist \( \epsilon \in E_{d+1}(A[T_1, \ldots, T_n]) \) such that \( v(T_1, \ldots, T_n)\epsilon = e_1 \)?

If the above question has an affirmative answer then we have the following interesting consequence:

**Theorem 4.2.** Let \( A \) be an affine algebra over \( \mathbb{F}_p \) of dimension \( d \geq 4 \) and \( \frac{1}{(d-1)!} \in A \). Let \( v(T_1, \ldots, T_n) \in \text{ Um}_{d}(A[T_1, \ldots, T_n]). \) Assume that Question 4.1 has an affirmative answer. Then there exists \( \alpha \in \text{ SL}_d(A[T_1, \ldots, T_n]) \) such that \( v(T_1, \ldots, T_n)\alpha = e_1 \).
Proof. Let $T = (T_1, \ldots, T_n)$. Without loss of generality we may assume that $A$ is reduced. Since the ideal of singular locus has height $\geq 1$, we can find a non-zero divisor $s \in A$ such that $A_s$ is smooth. By [4, Theorem 3.5], there exists $\beta \in \text{SL}_d(A)$ such that $v(0)\beta = e_1$.

Using [8] and [13, Theorem 3.3], there exists $\alpha'_2 \in E_d(A_s[T])$ such that $v(T)\alpha'_2 = v(0)$. Let $\alpha_2 = \alpha'_2(\beta)_s \in \text{SL}_d(A_s[T])$. Then note that $v(T)\alpha_2 = e_1$.

Let $R = (A[T])_{1+s<s>A[T]}$ and $B = A/ < s > A$. Then $B$ is an affine $\mathbb{F}_p$-algebra of dimension $\leq d - 1$. Note that $< s > R \subset \text{Jac}(R)$ and $R/ < s > R = B[T]$. Let ‘bar’ denote going modulo $< s > R$. Then by our hypothesis $\pi(T) \sim_{E_d(R/< s > R)} \pi_1$. Note that $B$ is affine algebra over $\mathbb{F}_p$ of dimension $\leq d - 1$ therefore, $\pi_1 \sim_{E_{d+1}(B)} \pi(0)$. Since $B \subset B[T] = R/ < s > R$ therefore, $E_d(B) \subset E_d(B[T]) = E_d(R/ < s > R)$. Hence we get $\pi_1 \sim_{E_{d+1}(R/< s > R)} \pi(0)$. Since $< s > R \subset \text{Jac}(R)$, by Lemma 2.1, there exists $\epsilon' \in E_d(R)$ such that $v(T)\epsilon' = v(0)$. Let $\alpha_1 = \epsilon'(\beta)_{1+s<s>A[T]}$. Then we get $\alpha_1 \in \text{SL}_d(R)$ and $v(T)\alpha_1 = e_1$. There exists $f(T) \in A[T]$ such that $\alpha_1 \in \text{SL}_d((A[T])_{1+s<sf(T)})$ and $v(T)\alpha_1 = e_1$.

Let $t = 1+sf(T)$ and $\eta = (\alpha_2^{-1}\alpha_1)_s = (\beta)^{-1}((\alpha'_2^{-1})t\epsilon'_s)(\beta)_s$. Note that $(\alpha_2^{-1})_t\epsilon'_s \in E_d(R_{st})$. Since $E_d(R_{st})$ is a normal subgroup of $\text{GL}_d(R_{st})$, the matrix $(\beta)^{-1}((\alpha'_2^{-1})t\epsilon'_s)(\beta)_s \in E_d(R_{st})$. Now using the same patching technique used in the proof of Proposition 2.2, the result follows.

\[ \square \]

\section*{References}

[1] S. Banerjee and M. K. Das. Splitting criteria of projective modules on polynomial extensions over various base rings. June 2022.

[2] H. Bass. K-theory and stable algebra. IHES, 22(1):5–60, dec 1964.

[3] M. K. Das. A criterion for splitting of a projective module in terms of its generic sections. International Mathematics Research Notices, 2021(13):10073–10099, jun 2019.

[4] A. M. Dhorajia and M. K. Keshari. A note on cancellation of projective modules. Journal of Pure and Applied Algebra, 216(1):126–129, 2012.

[5] T. Y. Lam. Serre’s Problem on Projective Modules. Springer Berlin Heidelberg, 2006.

[6] H. Lindel. On the Bass-Quillen conjecture concerning projective modules over polynomial rings. Inventiones Mathematicae, 65(2):319–323, jun 1981.

[7] H. Lindel. Unimodular elements in projective modules. Journal of Algebra, 172(2):301–319, mar 1995.

[8] B. Plumstead. The conjectures of Eisenbud and Evans. American Journal of Mathematics, 105(6):1417–1433, 1983.

[9] D. Quillen. Projective modules over polynomial rings. Inventiones Mathematicae, 36:167–171, 1976.

[10] R. Sridharan. Non-vanishing sections of algebraic vector bundles. Journal of Algebra, 176(3):947–958, sep 1995.

[11] A. A. Suslin. On the structure of the special linear group over polynomial rings. Mathematics of the USSR-Izvestiya, 11(2):221–238, apr 1977.
[12] L. N. Vaseršteĭn and A. A. Suslin. Serre’s problem on projective modules over polynomial rings, and algebraic K-theory. *Mathematics of the USSR-Izvestiya*, 10(5):937–1001, oct 1976.

[13] T. Vorst. The general linear group of polynomial rings over regular rings. *Communications in Algebra*, 9(5):499–509, jan 1981.

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