On bipartite graphs of defect at most 4

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Abstract

We consider the bipartite version of the degree/diameter problem, namely, given natural numbers \( \Delta \geq 2 \) and \( D \geq 2 \), find the maximum number \( N^b(\Delta, D) \) of vertices in a bipartite graph of maximum degree \( \Delta \) and diameter \( D \). In this context, the Moore bipartite bound \( M^b(\Delta, D) \) represents an upper bound for \( N^b(\Delta, D) \).

Bipartite graphs of maximum degree \( \Delta \), diameter \( D \) and order \( M^b(\Delta, D) \) – called Moore bipartite graphs – have turned out to be very rare. Therefore, it is very interesting to investigate bipartite graphs of maximum degree \( \Delta \geq 2 \), diameter \( D \geq 2 \) and order \( M^b(\Delta, D) - \epsilon \) with small \( \epsilon > 0 \); that is, bipartite \((\Delta, D, -\epsilon)\)-graphs. The parameter \( \epsilon \) is called the defect.

This paper considers bipartite graphs of defect at most 4, and presents all the known such graphs. Bipartite graphs of defect 2 have been studied in the past; if \( \Delta \geq 3 \) and \( D \geq 3 \), they may only exist for \( D = 3 \). However, when \( \epsilon > 2 \) bipartite \((\Delta, D, -\epsilon)\)-graphs represent a wide unexplored area.

The main results of the paper include several necessary conditions for the existence of bipartite \((\Delta, D, -4)\)-graphs; the complete catalogue of bipartite \((3, D, -\epsilon)\)-graphs with \( D \geq 2 \) and \( 0 \leq \epsilon \leq 4 \); the complete catalogue of bipartite \((\Delta, D, -\epsilon)\)-graphs with \( \Delta \geq 2 \), \( 5 \leq D \leq 187 \) (\( D \neq 6 \)) and \( 0 \leq \epsilon \leq 4 \); and a proof of the non-existence of all bipartite \((\Delta, D, -4)\)-graphs with \( \Delta \geq 3 \) and odd \( D \geq 5 \).

Finally, we conjecture that there are no bipartite graphs of defect 4 for \( \Delta \geq 3 \) and \( D \geq 5 \), and comment on some implications of our results for the upper bounds of \( N^b(\Delta, D) \).
Keywords: Moore bipartite bound; Moore bipartite graph; Degree/diameter problem for bipartite graphs; defect; repeat.

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1 Introduction

Due to the diverse features and applications of interconnection networks, it is possible to find many interpretations of network “optimality” in the literature. Here we are concerned with the following; see \cite{8, pp. 18}, \cite{10, pp. 168}, and \cite{16, pp. 91].

An optimal network contains the maximum possible number of nodes, given a limit on the number of connections attached to a node and a limit on the distance between any two nodes of the network.

This interpretation has attracted network designers and the research community in general due to its implications in the design of large interconnection networks. In graph-theoretical terms, this interpretation leads to the degree/diameter problem (the problem of finding the largest possible number of vertices in a graph with given maximum degree and diameter). If the graphs in question are subject to further restrictions such as being bipartite, planarity and/or transitivity, we can state the degree/diameter problem for the classes of graphs under consideration.

In this paper we will consider only bipartite graphs, and in this case, the degree/diameter problem can be stated as follows.

Degree/diameter problem for bipartite graphs: Given natural numbers $\Delta \geq 2$ and $D \geq 2$, find the largest possible number $N^b(\Delta, D)$ of vertices in a bipartite graph of maximum degree $\Delta$ and diameter $D$.

Note that $N^b(\Delta, D)$ is well defined for $\Delta \geq 2$ and $D \geq 2$. An upper bound for $N^b(\Delta, D)$ is given by the Moore bipartite bound $M^b(\Delta, D)$, defined below:

$$M^b(\Delta, D) = 2 \left( 1 + (\Delta - 1) + \cdots + (\Delta - 1)^{D-1} \right).$$

Bipartite graphs of degree $\Delta$, diameter $D$ and order $M^b(\Delta, D)$ are called Moore bipartite graphs. Moore bipartite graphs are rare; for $\Delta = 2$ they are the cycles of length $2D$, while for $\Delta \geq 3$ Moore bipartite graphs exist only for diameters 2, 3, 4 and 6; see \cite{9}. Therefore, we are interested in studying the existence or otherwise of bipartite graphs of given maximum degree $\Delta$, diameter $D$ and order $M^b(\Delta, D) - \epsilon$ for $\epsilon > 0$;
that is, bipartite \((\Delta, D, -\epsilon)\)-graphs, where the parameter \(\epsilon\) is called the def\(ect\). For notational convenience, we consider Moore bipartite graphs as having defect \(\epsilon = 0\).

Only a few values of \(N_b(\Delta, D)\) are known at present. With the exception of \(N_b(3, 5) = M_b(3, 5) - 6\), settled in [11], the other known values of \(N_b(\Delta, D)\) are those for which there is a Moore bipartite graph. The paper [14] combined with [5, 6] almost settled the case of bipartite graphs of defect 2; if \(\Delta \geq 3\) and \(D \geq 3\), then such graphs may only exist for \(D = 3\) and certain values of \(\Delta\). Bipartite \((\Delta, D, -\epsilon)\)-graphs with \(\epsilon > 2\) have been rarely considered in the literature so far.

In this paper we consider bipartite \((\Delta, D, -4)\)-graphs with \(\Delta \geq 2\) and \(D \geq 3\). By using combinatorial approaches we obtain several important results about bipartite graphs of defect 4, including several necessary conditions for the existence of bipartite \((\Delta, D, -4)\)-graphs; the complete catalogue of bipartite \((3, D, -\epsilon)\)-graphs with \(D \geq 2\) and \(0 \leq \epsilon \leq 4\); the complete catalogue of bipartite \((\Delta, D, -\epsilon)\)-graphs with \(\Delta \geq 2, 5 \leq D \leq 187, (D \neq 6)\) and \(0 \leq \epsilon \leq 4\); and a proof of the non-existence of all bipartite \((\Delta, D, -4)\)-graphs with \(\Delta \geq 3\) and odd \(D \geq 5\). Finally, we conjecture that there are no bipartite graphs of defect 4 for \(\Delta \geq 3\) and \(D \geq 5\).

The main results in this paper do not apply to bipartite \((\Delta, D, -4)\)-graphs with \(\Delta \geq 4\) and \(D = 3, 4\). Some of our assertions, however, do offer a partial characterisation of all bipartite \((\Delta, D, -4)\)-graphs with \(\Delta \geq 3\) and \(D \geq 3\). At the time of writing the paper we do not foresee a conclusive way to take on the diameters 3 and 4. To deal with such graphs it would be necessary to either find different ideas or complement some of the ones presented here. Section 6.1 contains further comments on such diameters.

2 Notation and Terminology

The terminology and notation used in this paper is standard and consistent with that used in [7], so only those concepts that can vary from text to text will be defined.

All graphs considered are simple. The vertex set of a graph \(\Gamma\) is denoted by \(V(\Gamma)\), and its edge set by \(E(\Gamma)\). The difference between the graphs \(\Gamma\) and \(\Gamma'\), denoted by \(\Gamma - \Gamma'\), is the graph with vertex set \(V(\Gamma) - V(\Gamma')\) and edge set formed by all the edges with both endvertices in \(V(\Gamma) - V(\Gamma')\).

The set of neighbours of a vertex \(x\) in \(\Gamma\) is denoted by \(N(x)\). For an edge \(e = \{x, y\}\) we write \(e = xy\), or alternatively \(x \sim y\). The set of edges in a graph \(\Gamma\) joining a vertex \(x\) in \(X \subseteq V(\Gamma)\) to a vertex \(y\) in \(Y \subseteq V(\Gamma)\) is denoted by \(E(X, Y)\); for simplicity, we write \(E(x, Y)\) rather than \(E(\{x\}, Y)\).

A path of length \(k\) is called a \(k\)-path, and cycle of length \(k\) is called a \(k\)-cycle. A path from a vertex \(x\) to a vertex \(y\) is denoted by \(x - y\). Whenever we refer to paths we mean shortest paths. We will use the following notation for subpaths of a path \(P = x_0x_1 \ldots x_k\): \(x_iP x_j = x_i \ldots x_j\), where \(0 \leq i \leq j \leq k\). The distance between a vertex \(x\) and a vertex \(y\) is denoted by \(d(x, y)\).
The union of three independent paths of length \( D \) with common endvertices is denoted by \( \Theta_D \). In a graph \( \Gamma \) a vertex of degree at least 3 is called a branch vertex of \( \Gamma \).

### 3 Known bipartite \((\Delta, D, -\epsilon)\)-graphs with \( \Delta \geq 2 \), \( D \geq 2 \) and \( 0 \leq \epsilon \leq 4 \)

For \( \Delta = 2 \) the Moore bipartite graphs are the cycles on \( 2D \) vertices, while for \( D = 2 \) and each \( \Delta \geq 3 \) they are the complete bipartite graphs of degree \( \Delta \) and order \( 2\Delta \). For \( D = 3, 4, 6 \) Moore bipartite graphs of degree \( \Delta \) have been constructed only when \( \Delta - 1 \) is a prime power \([1]\). Furthermore, Singleton \([15]\) proved that the existence of a Moore bipartite graph of diameter 3 is equivalent to the existence of a projective plane of order \( \Delta - 1 \). The question of whether Moore bipartite graphs of diameter 3, 4 or 6 exist for other values of \( \Delta \) remains open, and represents one of the most famous problems in combinatorics. For other values of \( D \geq 2 \) and \( \Delta \geq 3 \) there are no Moore bipartite graphs (see \([15, 9]\)).

When \( \Delta = 2 \) or \( D = 2 \) bipartite \((\Delta, D, -\epsilon)\)-graphs with \( \epsilon \geq 1 \) can be obtained by simple observation. For a given \( D \geq 2 \) there is only one bipartite \((2, D, -\epsilon)\)-graph with \( \epsilon \geq 1 \): the path of length \( D \), which has defect \( \epsilon = D - 1 \). For a given \( \Delta \geq 2 \) there are exactly \( \Delta - 1 \) bipartite \((\Delta, 2, -\epsilon)\)-graphs with \( \epsilon \geq 1 \); they are the complete bipartite graphs with partite sets of size \( \Delta \) and \( \Delta - \epsilon \), where \( 1 \leq \epsilon \leq \Delta - 1 \). Therefore, from now on we assume \( \Delta \geq 3 \) and \( D \geq 3 \).

We continue with some conditions for the regularity of bipartite \((\Delta, D, -\epsilon)\)-graphs, which were obtained in \([5]\).

**Proposition 3.1** ([5]) *For \( \epsilon < 1 + (\Delta - 1) + (\Delta - 1)^2 + \ldots + (\Delta - 1)^{D-2} \), \( \Delta \geq 3 \) and \( D \geq 3 \), a bipartite \((\Delta, D, -\epsilon)\)-graph is regular.*

**Proposition 3.2** ([5]) *For \( \epsilon < 2 \left( (\Delta - 1) + (\Delta - 1)^3 + \ldots + (\Delta - 1)^{D-2} \right) \), \( \Delta \geq 3 \) and odd \( D \geq 3 \), a bipartite \((\Delta, D, -\epsilon)\)-graph is regular.*

By Propositions 3.1 and 3.2 bipartite \((\Delta, D, -\epsilon)\)-graphs with \( \Delta \geq 3 \), \( D \geq 3 \) and \( \epsilon \leq 3 \) must be regular, implying the non-existence of such graphs for \( \Delta \geq 3 \), \( D \geq 3 \) and \( \epsilon = 1, 3 \). In the same way, bipartite \((\Delta, D, -4)\)-graphs with \( \Delta \geq 3 \) and \( D \geq 4 \) and bipartite \((\Delta, 3, -4)\)-graphs with \( \Delta \geq 4 \) must be regular.

For \( \Delta \geq 3 \) and \( D \geq 3 \), the only known bipartite \((\Delta, D, -2)\)-graphs are depicted in Fig. 1. Recall that such graphs do not exist when \( \Delta \geq 3 \) and \( D \geq 4 \); see \([5, 6, 14]\).

Bipartite \((3, 3, -4)\)-graphs may be irregular. Figure 2 depicts all such graphs, which were obtained by using the program *geng* from the package *nauty* written by McKay \([13]\). The unique bipartite \((3, 4, -4)\)-graph is shown in Fig. 3.
Figure 1: (a) the unique bipartite $(3, 3, -2)$-graph and (b) the unique bipartite $(4, 3, -2)$-graph.

Figure 2: All the bipartite $(3, 3, -4)$-graphs.

Figure 3: The unique bipartite $(3, 4, -4)$-graph.
Figure 4: All the bipartite $(4, 3, -4)$-graphs.

Figure 5: The only known bipartite $(5, 3, -4)$-graph.
All the bipartite \((4, 3, -4)\)-graphs are depicted in Fig. 4. These graphs were obtained computationally by Meringer \[12\] using the program \textit{genreg}. An alternative description of the graph in Fig. 4 (b) was communicated to the second author by Charles Delorme: take \(\mathbb{Z}/22\mathbb{Z}\) as the vertex set of the graph, and for each even \(x\), add the edges \(\{x, x + 1\}, \{x, x - 1\}, \{x, x + 7\}\), and \(\{x, x + 11\}\).

The only known bipartite \((5, 3, -4)\)-graph is depicted in Fig. 5; this graph was independently found by Charles Delorme and by the first author. Charles Delorme described this graph as follows: take \(\mathbb{Z}/38\mathbb{Z}\) as its vertex set, and for each even \(x\), add the edges \(\{x, x - 1\}, \{x, x + 1\}, \{x, x + 5\}, \{x, x + 13\}\), and \(\{x, x + 23\}\).

4 Preliminary Results

From now on, when referring to a class of regular bipartite graphs, we prefer the symbol \(d\) to \(\Delta\) to denote the maximum degree of a graph. However, if the graph class also involves irregular graphs, we use the symbol \(\Delta\). Recall that, unless \(d = 3\) and \(D = 3\), a bipartite \((d, D, -4)\)-graph with \(d \geq 3\) and \(D \geq 3\) must be regular. Therefore, when referring to a regular bipartite \((d, D, -4)\)-graph with \(d \geq 3\) and \(D \geq 3\), we are actually referring to any bipartite \((d, D, -4)\)-graph with \(d \geq 3\) and \(D \geq 3\) other than the ones exhibited in Fig. 2 (c) and (d).

In a bipartite \((d, D, -4)\)-graph we call a cycle of length at most \(2D - 2\) a short cycle.

**Proposition 4.1** The girth of a regular bipartite \((d, D, -4)\)-graph \(\Gamma\) with \(d \geq 3\) and \(D \geq 3\) is \(2D - 2\). Furthermore, any vertex \(x\) of \(\Gamma\) lies on the short cycles specified below and no other short cycle, and we have the following cases:

- **x is contained in exactly three \((2D - 2)\)-cycles.** Then
  
  (i) \(x\) is a branch vertex of one \(\Theta_{D-1}\), or

- **x is contained in two \((2D - 2)\)-cycles.** Then
  
  (ii) \(x\) lies on exactly two \((2D - 2)\)-cycles, whose intersection is an \(\ell\)-path with \(\ell \in \{0, \ldots, D - 1\}\).

Each case is considered as a type. For instance, a vertex satisfying case (i) is called a vertex of Type (i).

**Proof.** Let \(xy\) be an edge of \(\Gamma\). Let us use the standard decomposition for a bipartite graph of even girth with respect to the edge \(xy\) [3]. For \(0 \leq i \leq D - 1\), the sets \(X_i\) and \(Y_i\) are defined as follows:
\[ X_i = \{ z \in V(\Gamma) | d(x, z) = i, d(y, z) = i + 1 \} \]
\[ Y_i = \{ z \in V(\Gamma) | d(y, z) = i, d(x, z) = i + 1 \}. \]

The decomposition of \( \Gamma \) into the sets \( X_i \) and \( Y_i \) is called the standard decomposition for a graph of even girth with respect to the edge \( xy \). Since \( \Gamma \) is bipartite, its girth is even and \( X_i \cap Y_j = \emptyset \) for \( 0 \leq i, j \leq D - 1 \).

Claim 1 \( g(\Gamma) = 2D - 2 \).

Proof of Claim 1. Since the assertion is trivial for \( D = 3 \), we suppose that \( g(\Gamma) \leq 2D - 4 \) for \( D \geq 4 \). Assume that the edge \( xy \) lies on a cycle of length \( g(\Gamma) \). Then, \( |X_i| = |Y_i| = (d - 1)i \) for \( 1 \leq i \leq \frac{g(\Gamma)}{2} - 1 \), and

\[
|X_{D-2}| \leq ((d - 1)^{D-3} - 1)(d-1) + d - 2 = (d-1)^{D-2} - 1 \\
|Y_{D-2}| \leq ((d - 1)^{D-3} - 1)(d-1) + d - 2 = (d-1)^{D-2} - 1 \\
|X_{D-1}| \leq ((d - 1)^{D-2} - 1)(d-1) \\
|Y_{D-1}| \leq ((d - 1)^{D-2} - 1)(d-1).
\]

Therefore,

\[
|V(\Gamma)| = \sum_{i=0}^{D-1} |X_i| + \sum_{i=0}^{D-1} |Y_i| \leq 2 \left( 1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{D-3} \right) + \\
+ 2(d-1)^{D-2} - 2 + 2(d-1)^{D-1} - 2(d-1) = \\
= 2 \left( 1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{D-1} \right) - 2(d-1) - 2 = \\
= M_b(d, D) - 2d,
\]

which is a contradiction. Hence, \( g(\Gamma) \geq 2D - 2 \). If \( g(\Gamma) = 2D \) then the order of \( \Gamma \) would be at least \( M_b(d, D) \) \[2\]. Thus, \( g(\Gamma) = 2D - 2 \) and the claim follows. \( \square \)

We now proceed to prove the second part of the proposition.

For a given vertex \( x \), we use again the standard decomposition for a bipartite graph with respect to an edge \( xy \) in \( \Gamma \). Suppose that there are at least three edges joining vertices at \( X_{D-2} \) to vertices at \( Y_{D-2} \); that is, \( |E(X_{D-2}, Y_{D-2})| \geq 3 \). In such case

\[
|X_{D-1}| \leq (d-1)^{D-1} - 3, \\
|Y_{D-1}| \leq (d-1)^{D-1} - 3,
\]
and therefore

\[ |V(\Gamma)| = \sum_{i=0}^{D-1} |X_i| + \sum_{i=0}^{D-1} |Y_i| \leq 2 \left( 1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{D-2} \right) + \\
+ 2 \left( (d-1)^{D-1} - 3 \right) = \\
= 2 \left( 1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{D-1} \right) - 6 = \\
= M_{d,D}^b - 6, \]

which is a contradiction. Consequently, \( 0 \leq |E(X_{D-2}, Y_{D-2})| \leq 2. \)

Suppose that \( |E(X_{D-2}, Y_{D-2})| = 2 \). If the two edges are both incident to a common vertex of \( Y_{D-2} \) then \( x \) is of Type (i), otherwise \( x \) is of Type (ii).

If instead \( |E(X_{D-2}, Y_{D-2})| = 1 \) then \( |E(X_{D-2}, Y_{D-1})| = |E(Y_{D-2}, Y_{D-1})| = (d-1)^{D-1} - 1. \) Since \( |X_{D-1}| = |Y_{D-1}| = (d-1)^{D-1} - 2, \) there is a vertex \( u \in X_{D-1} \) such that \( |E(u, X_{D-2})| = 2. \) Therefore, it follows (ii).

Finally, if \( |E(X_{D-2}, Y_{D-2})| = 0 \) then both types may occur. Indeed, if there is a vertex \( u \in X_{D-1} \) such that \( |E(u, X_{D-2})| = 3 \) then \( x \) is of Type (i) (this case can only occur if \( d \geq 4 \)), otherwise there must exist two vertices \( u, v \in X_{D-1} \) such that \( |E(u, X_{D-2})| = |E(v, X_{D-2})| = 2, \) in which case \( x \) is of Type (ii). This completes the proof of the proposition. \( \square \)

We continue with the following observation, which will be implicitly used throughout the paper:

**Observation 4.1** Let \( \Gamma = (V_1 \cup V_2, E) \) (the sets \( V_1 \) and \( V_2 \) are called partite sets) be any bipartite graph of even (odd) finite diameter \( D. \) The distance between a vertex \( u \in V_1 \) and any vertex \( v \in V_2 \) \((w \in V_1)\) is at most \( D - 1. \)

In virtue of Proposition 4.1 we define the following concepts:

If two short cycles \( C^1 \) and \( C^2 \) are non-disjoint we say that \( C^1 \) and \( C^2 \) are neighbours.

For a vertex \( x \) lying on a short cycle \( C, \) we denote by \( \text{rep}^C(x) \) the vertex \( x' \) in \( C \) such that \( d(x, x') = D - 1. \) We say \( x' \) is the repeat of \( x \) in \( C \) and vice versa, or simply that \( x \) and \( x' \) are repeats in \( C. \) Alternatively, and more generally, we say that \( x' \) is a repeat of \( x \) with multiplicity \( m_x(x') \) \((1 \leq m_x(x') \leq 2)\) if there are exactly \( m_x(x') + 1 \) different paths of length \( D - 1 \) from \( x \) to \( x'. \) Proposition 4.1 tells us that a vertex in \( \Gamma \) may have a repeat of multiplicity 2. Accordingly, we denote by \( \text{Rep}(x) \) the multiset of the repeats of a vertex \( x \) in \( \Gamma. \)

The concept of repeat can be easily extended to paths. For a path \( P = x - y \) of length at most \( D - 2 \) contained in a short cycle \( C, \) we denote by \( \text{rep}^C(P) \) the path \( P' \subset C \) defined as \( \text{rep}^C(x) - \text{rep}^C(y). \) We say that \( P' \) is the repeat of \( P \) in \( C \) and vice versa, or simply that \( P \) and \( P' \) are repeats in \( C. \)
Often our arguments revolve around the identification of the elements in the set $S_x$ of short cycles containing a given vertex $x$; we call this process saturating the vertex $x$. A vertex $x$ is called saturated if the elements in $S_x$ have been completely identified. The following lemma will help us in this cycle identification process.

**Lemma 4.1 (Saturating Lemma)** Let $C$ be a $(2D - 2)$-cycle in a regular bipartite $(d, D, -4)$-graph $\Gamma$ with $d \geq 3$ and $D \geq 3$, and $\alpha, \alpha'$ two vertices in $C$ such that $\alpha' \in \text{rep}_C(\alpha)$. Let $\gamma$ be a neighbour of $\alpha$ not contained in $C$, and $\mu_1, \mu_2, \ldots, \mu_{d-2}$ the neighbours of $\alpha'$ not contained in $C$. Suppose there is no short cycle in $\Gamma$ containing the edge $\alpha \sim \gamma$ and intersecting $C$ at a path of length greater than $D - 3$.

Then, in $\Gamma$ there exist a vertex $\mu \in \{\mu_1, \mu_2, \ldots, \mu_{d-2}\}$ and a short cycle $C^1$ such that $\gamma$ and $\mu$ are repeats in $C^1$, and $C \cap C^1 = \emptyset$.

**Proof.** Let $\alpha'_1, \alpha'_2$ be the neighbours of $\alpha'$ contained in $C$.

For $1 \leq i \leq d - 2$, consider the path $P^i = \gamma - \mu_i$. As $g(\Gamma) = 2D - 2$, $P^i$ cannot go through $\alpha'_1$ or $\alpha'_2$. If $P^i$ went through certain $\mu_j$ ($j \neq i$), then a cycle $\gamma P^i \mu_j \alpha'^1 \alpha \gamma$ would either have length smaller than $2D - 2$ or be a short cycle intersecting $C$ at a $(D - 1)$-path. Consequently, $P^i$ must go through one of the neighbours of $\mu_i$ other than $\alpha'$, and must be a $(D - 1)$-path; see Fig. 6 (a). In addition, $V(P^i \cap C) = \emptyset$.

![Figure 6: Auxiliary figure for Lemma 4.1](image)

Let $\rho$ be one of the neighbours of $\gamma$ other than $\alpha$, not contained in any of the paths $P^i$ (there is at least one of such vertices). Consider a path $Q = \rho - \alpha'$. If $Q$ went through $\alpha'_1$, then the closed walk $\rho Q \alpha'_1 C \alpha \gamma \rho$ would either contain a cycle of length smaller than $2D - 2$ or be a short cycle intersecting $C$ at a $(D - 2)$-path. Consequently, $Q$ must go through a certain $\mu_k$ ($1 \leq k \leq d - 2$) and $V(Q \cap C) = \{\alpha'\}$ (Fig. 6 (b)). Note that $Q$ must be a $(D - 1)$-path, and that $V(Q \cap P^k) = \{\mu_k\}$; otherwise there would be a cycle in $\Gamma$ of length smaller than $2D - 2$. 
Thus, we have obtained a short cycle $C^1 = \gamma \rho Q \mu_k P^k \gamma$ such that $\gamma$ and $\mu_k$ are repeats in $C^1$, and $C \cap C^1 = \emptyset$. By setting $\mu = \mu_k$ the lemma follows. \hfill $\square$

**Corollary 4.1** Let $\alpha, \gamma$ be vertices in a regular bipartite $(d, D, -4)$-graph $\Gamma$ with $d \geq 3$ and $D \geq 3$ such that $\gamma \in N(\alpha)$. Then, for every $\alpha' \in \text{Rep}(\alpha)$ it follows that $N(\alpha')$ contains a repeat of $\gamma$.

**Proof.** Let $C$ be a short cycle containing $\alpha$ and $\alpha'$. If the vertex $\gamma$ is contained in $C$ or the edge $\alpha \gamma$ belongs to a short cycle in $\Gamma$ intersecting $C$ at a path of length $D - 2$ or $D - 1$, then the corollary trivially follows. If we instead assume that $\gamma \notin C$ and there is no short cycle in $\Gamma$ containing the edge $\alpha \gamma$ and intersecting $C$ at a path of length greater than $D - 3$, then the corollary follows from the Saturating Lemma. \hfill $\square$

### 4.1 Repeats of Cycles

In this section we extend the concept of repeat to short cycles; see the Repeat Cycle Lemma.

**Lemma 4.2 (Repeat Cycle Lemma)** Let $C$ be a short cycle in a regular bipartite $(d, D, -4)$-graph $\Gamma$ with $d \geq 3$ and $D \geq 3$, $\{C^1, C^2, \ldots, C^k\}$ the set of neighbours of $C$, and $I_i = C^i \cap C$ for $1 \leq i \leq k$. Suppose at least one $I_j$, for $j \in \{1, \ldots, k\}$, is a path of length smaller than $D - 2$. Then there is an additional short cycle $C'$ in $\Gamma$ intersecting $C^i$ at $I'_i = \text{rep}^{C^i}(I_i)$, where $1 \leq i \leq k$.

**Proof.** Observe that, according to our premises and Proposition 4.1, $k \geq 3$ and $I_i \cap I_j = \emptyset$ for $1 \leq i < j \leq k$.

We assume that the denotation of the neighbours $C^1, C^2, \ldots, C^k$ of $C$ and the corresponding intersection paths $I_1 = x_1 - y_1, I_2 = x_2 - y_2, \ldots, I_k = x_k - y_k$ is such that $C = x_1 I_1 y_1 x_2 I_2 y_2 \ldots x_k I_k y_k x_1$. For $1 \leq i \leq k$, we also denote the endvertices of $I'_i$ by $x'_i$ and $y'_i$, where $x'_i = \text{rep}^{C^i}(x_i)$ and $y'_i = \text{rep}^{C^i}(y_i)$ (see Fig. 4 (a)).

For $1 \leq i \leq k$, consider the cycles $C^i$ and $C^{(i \mod k) + 1}$.

First suppose that $I_i$ is a path of length smaller than $D - 2$. Since $y_i$ is saturated, there cannot be a short cycle in $\Gamma$, other than $C$, containing the edge $y_i \sim x^{(i \mod k) + 1}$. Since $I_i$ is a path of length smaller than $D - 2$, we apply the Saturating Lemma (mapping $C^i$ to $\bar{C}$, $y_i$ to $\alpha$, $y'_i$ to $\alpha'$ and $x^{(i \mod k) + 1}$ to $\gamma$) and obtain an additional short cycle $C^1$ in $\Gamma$ such that $x^{(i \mod k) + 1}$ is a repeat in $C^1$ of a neighbour $v \notin C^i$ of $y'_i$, and $C^1 \cap C = \emptyset$. Since $x^{(i \mod k) + 1}$ is saturated, we have that necessarily $C^1 = C^{(i \mod k) + 1}$, which in turn implies $v = x^{(i \mod k) + 1}$. In other words, it follows that $y'_i \sim x^{(i \mod k) + 1} \in E(\Gamma)$.

If instead $I_i$ is a $(D - 2)$-path then $I^{(i \mod k) + 1}$ must be a path of length smaller than $D - 2$. Therefore, we can apply the above reasoning and deduce that $x^{(i \mod k) + 1}_i \sim y'_i \in E(\Gamma)$.

In this way we obtain a subgraph $\Upsilon = \bigcup_{i=1}^{k} (I'_i \cup y'_i \sim x^{(i \mod k) + 1}) = x'_1 I'_1 y'_1 x'_2 I'_2 y'_2 \ldots x'_k I'_k y'_k x'_1$ intersecting $C^i$ at $I'_i$ for $1 \leq i \leq k$ (see Fig. 4 (b), where part of the subgraph $\Upsilon$ is highlighted in bold).

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Therefore, we have that \( z \) would have at least two neighbours in \( \Upsilon \). Therefore, \( x \) neighbours of \( C^{\ell} \) in \( \Gamma \) containing \( Q_{4} \). \( \Upsilon \) contains no cycle. Since \( \Upsilon \) is clearly connected it is therefore a tree.

Let \( i \) must be a \((i = 5) \) see Fig. 8 (I) for Proposition 4.1, \( \Upsilon \) contains no cycle. Since \( \Upsilon \) is clearly connected it is therefore a tree.

Let \( z \in C^{\ell} \) be an arbitrary leaf in \( \Upsilon \). If the repeat path \( I'_{\ell} = x_{\ell}^{'} - y_{\ell}^{'} \) had length greater than 0, then \( z \) would have at least two neighbours in \( \Upsilon \). Therefore, \( I_{\ell} = C \cap C^{\ell} \) contains exactly one vertex, and thus, \( x_{\ell} = y_{\ell} \) and \( z = x_{\ell} = y_{\ell}^{'} \).

Recall we do addition modulo \( k \) on the subscripts of the vertices and the superscripts of the cycles.

Since \( x_{\ell}^{'} \sim y_{\ell-1}^{'} \) and \( x_{\ell}^{'} \sim x_{\ell+1}^{'} \) are edges in \( \Upsilon \), it holds that \( y_{\ell-1}^{'} \) and \( x_{\ell+1}^{'} \) denote the same vertex. Let \( u_{\ell-1}^{'} \), \( v_{\ell-1}^{'} \) be the neighbours of \( y_{\ell-1}^{'} \) in \( C^{\ell-1} \); \( u_{\ell+1}^{'} \), \( v_{\ell+1}^{'} \) the neighbours of \( x_{\ell+1}^{'} \) in \( C^{\ell+1} \), and \( u_{\ell}, v_{\ell} \) the neighbours of \( x_{\ell} \) in \( C^{\ell} \). We have that \( V(C^{\ell-1} \cap C^{\ell+1}) = \{y_{\ell-1}^{'}\} \), otherwise there would be a third short cycle in \( \Gamma \) containing \( x_{\ell} \). In particular, the vertices in \( \{u_{\ell-1}^{'}, v_{\ell-1}^{'}, u_{\ell+1}^{'}, v_{\ell+1}^{'}, x_{\ell}^{'}\} \) are pairwise distinct and \( d \geq 5 \). See Fig. 8 (a) and (b) for two drawings of this situation.

Let \( t_{1}, t_{2}, \ldots, t_{d-4} \) denote the vertices in \( N(x_{\ell}) - \{y_{\ell-1}, x_{\ell+1}, u_{\ell}, v_{\ell}\} \); see Fig. 8 (c). Consider a path \( Q^{i} = t_{i} - y_{\ell-1}^{'} \). Recall that \( Q^{i} \) has length at most \( D-1 \). Since \( x_{\ell} \) cannot be contained in a further short cycle, \( Q^{i} \) must be a \((D - 1)\)-path and go through a neighbour of \( y_{\ell-1}^{'} \) not contained in \( \{u_{\ell-1}^{'}, v_{\ell-1}^{'}, u_{\ell+1}^{'}, v_{\ell+1}^{'}, x_{\ell}^{'}\} \).

Therefore, we have that \( d \geq 6 \) and, by the pigeonhole principle, that there are two paths \( Q^{r} \) and \( Q^{s} \) containing a common neighbour of \( y_{\ell-1}^{'} \). In this way, \( x_{\ell} \) would be contained in a third short cycle, a
As a result, we conclude that the repeat graph $\Upsilon$ of $C$ is indeed a $(2D - 2)$-cycle $C'$ as claimed. This completes the proof of Claim 1, and thus, of the lemma.

While not of primary interest, it is not difficult to prove now that the cycles $C_1, C_2, \ldots, C_k$ in the previous lemma are pairwise disjoint.

We call the aforementioned cycle $C'$ the repeat of the cycle $C$ in $\Gamma$, and denote it by $\text{rep}(C)$. Next some simple consequences of the Repeat Cycle Lemma follow.

**Corollary 4.2 (Repeat Cycle Uniqueness)** If a short cycle $C$ has a repeat cycle $C'$ then $C'$ is unique.
**Corollary 4.3 (Repeat Cycle Symmetry)** If $C' = \text{rep}(C)$ then $C = \text{rep}(C')$.

**Corollary 4.4** Let $C$ and $C^1$ be two short cycles in a regular bipartite $(d, D, -4)$-graph $\Gamma$ with $d \geq 3$ and $D \geq 3$ which intersect at a path $I$ of length smaller than $D - 2$, and let $I' = \text{rep}^{C^1}(I)$. Then, the repeat cycle of $C$ intersects $C^1$ at $I'$.

**Corollary 4.5 (Handy Corollary)** Let $C$ be a short cycle in a regular bipartite $(d, D, -4)$-graph $\Gamma$ with $d \geq 3$ and $D \geq 3$, and $x, x'$ repeat vertices in $C$. Let $C^1$ and $C^2$ be the other short cycles containing $x$ and $x'$, respectively. Suppose that $I = C^1 \cap C$ is a path of length smaller than $D - 2$. Then, setting $y = \text{rep}^{C^1}(x)$ and $y' = \text{rep}^{C^2}(x')$, we have that $y$ and $y'$ are repeat vertices in the repeat cycle of $C$.

**Proof.** We denote the $k$ neighbour cycles of $C$ as $E^1, E^2, \ldots, E^k$ and their respective intersection paths with $C$ as $I_1 = x_1 - y_1, I_2 = x_2 - y_2, \ldots, I_k = x_k - y_k$ in such way that $C = x_1 I_1 y_1 x_2 I_2 y_2 \ldots x_k I_k y_k x_1$. For $1 \leq j \leq k$, we also denote $I'_j = x'_j - y'_j$, where $x'_j = \text{rep}^{E^1}(x_j)$ and $y'_j = \text{rep}^{E^j}(y_j)$.

Obviously, for some $r, s$ ($1 \leq r, s \leq k$) we have that $C^1 = E^r$, $C^2 = E^s$, $x \in I_r$, $x' \in I_s$, $y \in I'_r$, and $y' \in I'_s$. We may assume $r < s$. By the Repeat Cycle Lemma, the vertices $y$ and $y'$ belong to the repeat cycle $C'$ of $C$. Then the paths $x I_r y_r x_{r+1} I_{r+1} y_{r+1} \ldots x_{s-1} I_{s-1} y_{s-1} x_s I'_s y'_s C$ and $y I'_r y'_r x'_{r+1} I'_{r+1} y'_{r+1} \ldots x'_{s-1} I'_{s-1} y'_{s-1} x'_s I'_s y'_s C'$ are both $(D - 1)$-paths in $\Gamma$, and the corollary follows. $\Box$

**Proposition 4.2** The set $S(\Gamma)$ of short cycles in a bipartite $(d, D, -4)$-graph $\Gamma$ with $d \geq 3$ and $D \geq 3$ can be partitioned into sets $S_{D-1}(\Gamma)$, $S_{D-2}(\Gamma)$ and $S_{D-3}(\Gamma)$, where

- $S_{D-1}(\Gamma)$ is the set of short cycles whose intersections with neighbour cycles are $(D - 1)$-paths;
- $S_{D-2}(\Gamma)$ is the set of short cycles whose intersections with neighbour cycles are $(D - 2)$-paths; and
- $S_{D-3}(\Gamma)$ is the set of short cycles whose intersections with neighbour cycles are paths of length at most $D - 3$.

**Proof.** If $\Gamma$ is one of the non-regular graphs in Fig. 2, the result trivially follows. We then assume that $\Gamma$ is regular.

Let $C$ be a short cycle in $\Gamma$. If $C$ is contained in a $\Theta_{D-1}$ then, according to Proposition 4.1, all the intersections of $C$ with its neighbour cycles are $(D - 1)$-paths, in which case $C \in S_{D-1}(\Gamma)$.

Now suppose that, for some short cycle $C^1$, $P_1 = C \cap C^1$ is a path of length $D - 2$. Note that all vertices in $P_1$ are saturated. Let $v$ be an arbitrary vertex in $P_1$, $v' = \text{rep}^C(v)$, and $C^2$ the short cycle other than $C$ containing $v'$. Suppose that $P_2 = C \cap C^2$ is not a $(D - 2)$-path. Then clearly $P_2$ cannot be a $(D - 1)$-path, so it has length at most $D - 3$. But according to Corollary 4.3, the cycle $\text{rep}(C^2)$ intersects
at exactly $\text{rep}^C(P_2)$, a proper subpath of $P_1$. This implies that $\text{rep}(C^2)$ is a third short cycle containing the vertex $v$, a contradiction. Consequently, the intersections of $C$ with its (exactly two) neighbour cycles are $(D - 2)$-paths, and $C \in S_{D-2}(\Gamma)$.

Finally, if there is a short cycle intersecting $C$ at a path of length at most $D - 3$ then, by the above reasoning, the intersections of $C$ with all of its neighbour cycles are paths of length at most $D - 3$, and $C \in S_{D-3}(\Gamma)$.

The preceding result could be stated alternatively in term of vertices as follows:

**Proposition 4.3** The set $V(\Gamma)$ of vertices in a regular bipartite $(d, D, -4)$-graph $\Gamma$ with $d \geq 3$ and $D \geq 3$ can be partitioned into sets $V_{D-1}(\Gamma), V_{D-2}(\Gamma)$ and $V_{D-3}(\Gamma)$, where

- $V_{D-1}(\Gamma)$ is the set of vertices contained in cycles of $S_{D-1}(\Gamma)$;
- $V_{D-2}(\Gamma)$ is the set of vertices contained in cycles of $S_{D-2}(\Gamma)$;
- $V_{D-3}(\Gamma)$ is the set of vertices contained in cycles of $S_{D-3}(\Gamma)$;

and $S_{D-1}(\Gamma), S_{D-2}(\Gamma), S_{D-3}(\Gamma)$ are defined as in Proposition 4.2.

5 Main results

5.1 Non-existence of subgraphs isomorphic to $\Theta_{D-1}$

**Theorem 5.1** A bipartite $(d, D, -4)$-graph $\Gamma$ with $d \geq 3$ and $D \geq 5$ does not contain a subgraph isomorphic to $\Theta_{D-1}$.

**Proof.** Suppose that $\Gamma$ has a subgraph $\Theta$ isomorphic to $\Theta_{D-1}$, with branch vertices $a$ and $b$. Let $p_1, p_2, p_3, p_4$ and $p_5$ be as in Fig. 9(a), and let $q_1$ be one of the neighbours of $p_1$ not contained in $\Theta$.

Since all vertices of $\Theta$ are saturated, there cannot be a short cycle in $\Gamma$ containing any of the incident edges of $p_1, p_2, p_3, p_4$ or $p_5$ which are not contained in $\Theta$. According to this and by applying the Saturating Lemma, there is an additional short cycle $D^1$ in $\Gamma$ such that $q_1$ and one of the neighbours of $p_2$ not contained in $\Theta$ (say $q_2$) are repeats in $D^1$, and $D^1 \cap \Theta = \emptyset$. Analogously, in $\Gamma$ there is an additional short cycle $D^2$ such that $q_2$ and one of the neighbours of $p_3$ not contained in $\Theta$ (say $q_3$) are repeats in $D^2$, and $D^2 \cap \Theta = \emptyset$; an additional short cycle $D^3$ such that $q_3$ and one of the neighbours of $p_4$ not contained in $\Theta$ (say $q_4$) are repeats in $D^3$, and $D^3 \cap \Theta = \emptyset$; and an additional short cycle $D^4$ such that $q_4$ and one of the neighbours of $p_5$ not contained in $\Theta$ (say $q_5$) are repeats in $D^4$, and $D^4 \cap \Theta = \emptyset$. See Fig. 9(b).
Note that \( D^1 \cap D^2 \) is a path of length at most 2 < \( D - 2 \); otherwise for some vertex \( t \in D^1 \cap D^2 \) the closed walk \( tD^3q_1p_1bp_3q_5D^2t \) would contain a cycle of length at most \( 2D - 2 \) to which the vertex \( b \) would belong, a contradiction. For similar reasons, the intersection paths \( D^2 \cap D^3 \) and \( D^3 \cap D^4 \) all have length at most 2, with \( 2iD - 2 \). We now apply the Handy Corollary. By mapping the cycle \( D^2 \) to \( C \), the vertex \( q_2 \) to \( x \), the vertex \( q_3 \) to \( x' \), the cycle \( D^1 \) to \( C^1 \), the cycle \( D^3 \) to \( C^2 \), the vertex \( q_1 \) to \( y \) and the vertex \( q_4 \) to \( y' \), we obtain that \( q_1 \) and \( q_4 \) are repeat vertices in the repeat cycle of \( D^2 \). Therefore, since \( q_4 \in D^4 \), it follows that \( D^2 \) and \( D^4 \) are repeat cycles and \( q_1 = q_5 \). In this way, there would be a cycle \( q_1p_1bp_5q_5 \) in \( \Gamma \) of length 4 < \( 2D - 2 \) (since \( D \geq 5 \)), a contradiction to the fact that \( g(\Gamma) = 2D - 2 \). \( \square \)

**Proposition 5.1** The number \( N_{2D-2} \) of short cycles in a bipartite \((d, D, -4)\)-graph \( \Gamma \) with \( d \geq 3 \) and \( D \geq 5 \) is given by the expression \( \frac{2x\left(1+(d-1)+\ldots+(d-1)^{D-1}\right)-4}{D-1} \).

**Proof.** By Theorem 5.1, \( \Gamma \) does not contain a subgraph isomorphic to \( \Theta_{D-1} \). Then, according to Proposition 4.1, every vertex of \( \Gamma \) is contained in exactly two short cycles. We then count the number \( N_{2D-2} \) of short cycles of \( \Gamma \). Since the order of \( \Gamma \) is \( 2 \times (1 + (d - 1) + \ldots + (d - 1)^{D-1}) - 4 \), we have that

\[
N_{2D-2} = \frac{2x\left(1+(d-1)+\ldots+(d-1)^{D-1}\right)-4}{2D-2} = \frac{2x\left(1+(d-1)+\ldots+(d-1)^{D-1}\right)-4}{D-1},
\]

and the proposition follows. \( \square \)
5.2 Non-existence results on bipartite \((d, D, -4)\)-graphs

Since the number of short cycles in a graph \(\Gamma\) must be an integer, the expression obtained for \(N_{2D-2}\) in Proposition \[\text{5.1}\] already suffices to prove the non-existence of bipartite \((d, D, -4)\)-graphs for infinitely many pairs \((d, D)\).

Consider first the case in which \(D - 1 = p^\vartheta\) is an odd prime power. Let \(G = \{1, 2, \ldots, p - 1\}\) be the multiplicative group of the field \(\mathbb{Z}/p\mathbb{Z}\), let \(d - 1 \not\equiv 0, 1 \pmod{p}\), and let \(H\) be the cyclic subgroup of \(G\) generated by \(d - 1\). We observe that the sum of the elements of \(H\) is null \((\pmod{p})\). Furthermore, since the order of \(H\) divides the order of \(G\), it must also divide \(p^\vartheta - 1 = D - 2\). Thus, we have

\[
2 \times (1 + (d - 1) + \ldots + (d - 1)^{D-1}) - 4 \equiv \begin{cases} 
-2 \pmod{p} & \text{if } d - 1 \equiv 0, 1 \pmod{p}, \\
2(d - 1) - 2 \pmod{p} & \text{if } d - 1 \not\equiv 0, 1 \pmod{p}.
\end{cases}
\]

Therefore, it immediately follows

**Corollary 5.1** There is no bipartite \((d, D, -4)\)-graph with \(d \geq 3\) and \(D \geq 5\) such that \(D - 1\) is an odd prime power.

More generally, if \(p\) is an odd prime factor of \(D - 1\) and \(D - 1 \equiv r \pmod{p - 1}\), then

\[
2 \times (1 + (d - 1) + \ldots + (d - 1)^{D-1}) - 4 \equiv \begin{cases} 
-2 \pmod{p} & \text{if } d - 1 \equiv 0, 1 \pmod{p}, \\
\frac{2(d-1)^r - 1}{d-2} - 4 \pmod{p} & \text{if } d - 1 \not\equiv 0, 1 \pmod{p};
\end{cases}
\]

**Corollary 5.2** There is no bipartite \((d, D, -4)\)-graph with \(d \geq 3\) and \(D \geq 6\) such that \(d - 1 \equiv 0, 1 \pmod{p}\), where \(p\) is an odd prime factor of \(D - 1\).

It is also possible to examine completely the case of some small odd prime factors of \(D - 1\). For example, it is not difficult to verify that, if \(D - 1 = 3k\) then 3 does not divide \(2 \times (1 + (d - 1) + \ldots + (d - 1)^{D-1}) - 4\); thus,

**Corollary 5.3** There is no bipartite \((d, D, -4)\)-graph with \(d \geq 3\) and \(D \geq 5\) such that \(D - 1 \equiv 0 \pmod{3}\).

Now we turn to structural arguments to obtain other non-existence results.

**Lemma 5.1** Any two non-disjoint short cycles in a bipartite \((d, D, -4)\)-graph \(\Gamma\) with \(d \geq 3\) and \(D \geq 7\) intersect at a path of length smaller than \(D - 2\).
**Proof.** Since $\Gamma$ does not contain a graph isomorphic to $\Theta_{D-1}$, it is only necessary to prove here that any two non-disjoint short cycles in $\Gamma$ cannot intersect at a path of length $D - 2$.

Suppose, by way of a contradiction, that there are two short cycles $C^1$ and $C^2$ in $\Gamma$ intersecting at a path $I_1$ of length $D - 2$. According to Proposition 4.2, $C^2$ is intersected by exactly two short cycles, namely $C^1$ and $C^3$, at two independent $(D - 2)$-paths. By repeatedly applying this reasoning and considering $\Gamma$ is finite, we obtain a maximal length sequence $C^1, C^2, C^3, \ldots, C^m$ of pairwise distinct short cycles in $\Gamma$ such that $C^i$ intersects $C^{i+1}$ at a path $I_i$ of length $D - 2$ $(1 \leq i \leq m - 1)$, and $C^i \cap C^j = \emptyset$ for any $i, j \in \{1, \ldots, m\}$ such that $2 \leq |i - j| \leq m - 2$.

Let us denote the paths $I_1 = x_1 - y_1, \ldots, I_{m-1} = x_{m-1} - y_{m-1}$ in such way that, for $1 \leq i \leq m - 2$, $x_i \sim x_{i+1}$ and $y_i \sim y_{i+1}$ are edges in $\Gamma$. Also, let $x_0 \in N(x_1) \cap (C^1 - I_1)$, $y_0 \in N(y_1) \cap (C^1 - I_1)$, $x_m \in N(x_{m-1}) \cap (C^m - I_{m-1})$, and $y_m \in N(y_{m-1}) \cap (C^m - I_{m-1})$; see Fig. 10 (a). Set $I_0 = x_0 - y_0$ and $I_m = x_m - y_m$. Since the sequence $C^1, C^2, C^3, \ldots, C^m$ is maximal and all the vertices in $I_1, \ldots, I_{m-1}$ are saturated, it follows that $I_0 = I_m$, and we have either $x_0 = x_m$ and $y_0 = y_m$ (Fig. 10 (b)), or $x_0 = y_m$ and $y_0 = x_m$ (Fig. 10 (c)).

If $x_0 = x_m$ and $y_0 = y_m$, then $m \geq 2D$; otherwise the cycle $x_1x_2 \ldots x_my_1x_1$ would have length at most $2D - 2$, contradicting the saturation of $x_1$. If conversely $x_0 = y_m$ and $y_0 = x_m$ then $m \geq D$; otherwise the cycle $x_1x_2 \ldots x_my_1y_2 \ldots y_mx_1$ containing $x_1$ would have length at most $2D - 2$, a contradiction as well. For our purposes, it is enough to state $m \geq D \geq 7$ in any case.

Let $p_1$ be the neighbour of $y_1$ on $I_1$, and $p_{i+1} = \text{rep}^{C^i+1}(p^i)$ for $1 \leq i \leq 4$. Also, let $q_1$ be a neighbour of $p_1$ not contained in $I_1$; see Fig. 11 (a).

Since all vertices on $I_1$ are saturated, the edge $q_1 \sim p_1$ cannot be contained in a further short cycle. We apply the Saturating Lemma (by mapping $C^2$ to $\mathcal{C}$, $p_1$ to $\alpha$, $p_2$ to $\alpha'$, and $q_1$ to $\gamma$), and obtain in $\Gamma$ an additional short cycle $D^1$ such that $q_1$ and one of the neighbours of $p_2$ not contained in $I_2$ (say $q_2$) are repeats in $D^1$, and $D^1 \cap C^2 = \emptyset$. Analogously, for $2 \leq i \leq 4$ we obtain an additional short cycle $D^i$ in $\Gamma$ such that $q_i$ and a neighbour of $p_{i+1}$ not contained in $I_{i+1}$ (say $q_{i+1}$) are repeats in $D^i$, and $D^i \cap C^{i+1} = \emptyset$; see Fig. 11 (b).

For $i = 1$ or $3$, $D^i \cap D^{i+1}$ cannot be a $(D - 2)$-path; otherwise for some vertex $t_i \in D^i \cap D^{i+1}$, there would be a cycle $q_1p_1y_1y_{i+1}y_{i+2}p_{i+2}q_{i+2}D_{i+1}t_iD_iq_i$ of length at most $6 + D - 4 + D - 4$ (since $D - 2 \geq 5$), a contradiction to the fact that $p_i$ is saturated and $g(\Gamma) = 2D - 2$. Analogously, $D^2 \cap D^3$ cannot be a $(D - 2)$-path.

We now apply the Handy Corollary. By mapping the cycles $D^2$ to $\mathcal{C}$, $D^1$ to $C^1$ and $D^3$ to $C^2$, and the vertices $q_2$ to $x$, $q_3$ to $x'$, $q_1$ to $y$, and $q_4$ to $y'$, it follows that the vertices $q_1$ and $q_4$ are repeat vertices in the repeat cycle of $D^2$. Since $q_4 \in D^4$, $D^2$ and $D^4$ are repeat cycles and $q_5 = q_1$. In this way, we obtain a cycle $q_1p_1y_1y_2y_3y_4y_5p_5q_5$ in $\Gamma$ of length $8 < 2D - 2$, a contradiction.
This completes the proof of the lemma.  

\textbf{Theorem 5.2} There are no bipartite \((d, D, -4)\)-graphs for \(d \geq 3\) and odd \(D \geq 5\).

\textbf{Proof.} The case \(D = 5\) can be easily discarded by using Proposition \ref{prop:5.1}, so we assume \(D \geq 7\).

Suppose there is a bipartite \((d, D, -4)\)-graph \(\Gamma\) with \(d \geq 3\) and odd \(D \geq 7\). According to Lemma \ref{lem:5.1}, any two non-disjoint short cycles in \(\Gamma\) intersect at a path of length smaller than \(D - 2\), which means that every short cycle \(C\) in \(\Gamma\) has a repeat cycle \(C'\) (by the Repeat Cycle Lemma). Because of the uniqueness and symmetry of repeat cycles, the number \(N_{2D-2}\) of short cycles in \(\Gamma\) must be even.

However, since \(D\) is odd, the number \(N_{2D-2} = \frac{2 \times (1 + (d-1) + \ldots + (d-1)^{D-1})}{D-1} - 4\) of short cycles in \(\Gamma\) is odd, a contradiction.  

\[\square\]
Furthermore, using Theorem 5.2 and Proposition 5.1 we have the following result.

**Theorem 5.3** There is no bipartite \((d, D, -4)\)-graph with \(d \geq 3\) and \(5 \leq D \leq 187\).

### 5.3 Non-existence of bipartite \((3, D, -4)\)-graphs with \(D \geq 5\)

In this section we complete the catalogue of bipartite \((3, D, -4)\)-graphs. Specifically, we prove the non-existence of bipartite \((3, D, -4)\)-graphs with even \(D \geq 6\).

**Lemma 5.2** Any two non-disjoint short cycles in a bipartite \((3, D, -4)\)-graph \(\Gamma\) with \(d \geq 3\) and \(D \geq 7\) intersect at a path of length smaller than \(D - 3\).

**Proof.** By Lemma 5.1 it is only necessary to prove here that any two short cycles \(C\) and \(C^1\) in \(\Gamma\) cannot intersect at a path \(I = x - y\) of length \(D - 3\). We proceed by contradiction. Let \(x'\) and \(y'\) be the
repeat vertices of $x$ and $y$ in $C^1$, respectively. By Corollary 4.4, the repeat cycle $C'$ of $C$ intersects $C^1$ at $I' = x' - y'$ (the repeat path of $I$ in $C^1$); see Fig. 12. If we denote by $z$ the neighbour of $x$ on $C^1 - C'$, then we have that the other short cycle containing $z$ would also contain at least one of the vertices in $\{x, y'\}$, which contradicts the fact that $x$ and $y'$ are both saturated. □

![Figure 12: Auxiliary figure for Lemma 5.1](image)

**Theorem 5.4** There are no bipartite $(3, D, -4)$-graphs with even $D \geq 6$.

**Proof.** Recall that Theorem 5.3 covers the case $D = 6$.

Let $\Gamma$ be a bipartite $(3, D, -4)$-graph with even $D \geq 8$, $C^0$ a short cycle in $\Gamma$, and $x_0, x'_0$ two repeat vertices in $C^0$. Let $x_1$ and $x'_1$ be the neighbours of $x_0$ and $x'_0$, respectively, not contained in $C^0$. According to the Saturating Lemma, there is an additional short cycle $C^1$ containing $x_1$ and $x'_1$ such that $C^0 \cap C^1 = \emptyset$. Let $y_1$ be one of the neighbours of $x_1$ contained in $C^1$, and $y'_1 = \text{rep}^{-1}(y_1)$. Denote by $x_2$ and $x'_2$ the neighbours of $y_1$ and $y'_1$, respectively, not contained in $C^1$. Again by the Saturating Lemma, there is an additional short cycle $C^2$ such that $x'_2 = \text{rep}^{-1}(x_2)$ and $C^1 \cap C^2 = \emptyset$. Since $d = 3$, we may assume that the other short cycle $C$ containing $x_0$ also contains $x_1$ and a neighbour of $x_0$ in $C_0$ (say $y_0$). We first prove that $C^0 \cap C = y_0 x_0$.

**Claim 1.** $C^0 \cap C = y_0 x_0$.

**Proof of Claim 1.** Let $y_0, z_0, y_2, y'_2$ and $z'_2$ be as in Fig. 13 (a).

Consider a path $P = x'_2 - y_0$. If $y'_1 \in P$, then $P$ would go through a neighbour of $y'_1$ contained in $C^1$ and there would be a cycle in $\Gamma$ of length at most $2D - 4$. Therefore, we may assume $y'_2 \in P$. If $\{x'_2, y'_2, z'_2\} \subset V(P \cap C^2)$ then there would be a short cycle intersecting the cycle $C^2$ at a path of length $D - 3$, a contradiction to Lemma 5.2. Similarly, we have that $z_0 \notin P$. Also, $P$ must be a $(D - 1)$-path and $x_0 \notin P$; otherwise there would be a short cycle intersecting the cycle $C^2$ at a path of length $D - 2$, a contradiction to Lemma 5.1.
For $3 \leq i \leq D/2 + 1$, let $x_i$ and $x'_i$ be the neighbours of $y_{i-1}$ and $y'_{i-1}$, respectively, not contained in $C^{i-1}$, and let $C^i$ be (in virtue of the Saturating Lemma) the additional short cycle disjoint from $C^{i-1}$ which contains $x_i$ and $x'_i$. Since $P$ must go through $x'_i$, we denote by $y'_i$ the neighbour of $x'_i$ on $P \cap C^i$ and set $y_i = \text{rep}^{C_i}(y'_i)$. We now show that, if $i \neq D/2 + 1$, $P \cap C^i = x'_i y'_i$. Assume the contrary; that is, $P \cap C^i = x'_iy'_iz'_i$ (since $g(\Gamma) = 2D - 2$, $|V(P \cap C^i)| \leq 3$). In such case, there would be a short cycle $y_0Py'_ix_1y_{i-1}x_{i-1}y_{i-2}x_{i-2} \ldots y_1x_1x_0y_0$ intersecting $C^i$ at a path of length $D - 3$, contradicting Lemma 5.2 (see Figures 13 (b) and 14). Consequently, $P \cap C^i = x'_iy'_i$ and $P$ must go through a neighbour of $y'_i$ not
contained in $C^i$.

In this way, for $3 \leq i \leq D/2 + 1$ we have $d(y_0, y'_i) = d(y_0, y'_{i-1}) - 2 = D - 2(i - 1)$, which means that $d(y_0, y'_{D/2+1}) = 0$. Since the cycle $C^{D/2+1}$ contains the vertices $y_0 \in C^0 \cap C \cap P$ and $x'_{D/2+1} \in P - C^0$, we have $C^{D/2+1} = C$, which implies that $C^0 \cap C = y_0x_0$. \hfill $\Box$

As the selection of $C^0$ and $C$ was arbitrary, basically as a corollary of Claim 1 we have:

**Claim 2.** Any two non-disjoint short cycles in $\Gamma$ intersect at an edge.

Finally, suppose that $C^0$ and $C$ intersect at $y_0x_0$, as stated by Claim 1. Let $y'_0$ be the repeat vertex of $y_0$ in $C^0$; then, by Corollary 4.4, the repeat cycle $C'$ of $C$ intersects $C^0$ at $y'_0x'_0$ (the repeat path of $y_0x_0$ in $C^0$). Setting $Q = x_0C^0y'_0 = x_0w_1 \ldots w_{D-3}y'_0$, we have $Q$ is a path of length $D - 2$ with saturated endvertices (see Fig. 15). Therefore, by Claim 2, there exists a sequence $F^1, \ldots, F^{D/2-2}$ of short cycles such that $F^i \cap C^0 = w_{2i-1}w_{2i}$. However, since $D$ is even, the other short cycle containing $w_{D-3}$ would also contain one of the vertices in $\{w_{D-4}, y'_0\}$, which contradicts the fact that $w_{D-4}$ and $y'_0$ are both saturated. This completes the proof of Theorem 5.4. \hfill $\Box$

![Figure 15: Auxiliary figure for Theorem 5.4.](image)

Combining Theorems 5.2 and 5.4 we have that the only bipartite $(3, D, -4)$-graphs with $D \geq 2$ are those depicted in Figures 2 and 3, completing in this way the catalogue of such graphs.

### 6 Conclusions

The main results obtained in this paper are summarised below.

First we stated important structural properties of bipartite $(d, D, -4)$-graphs with $d \geq 3$ and $D \geq 3$. We found necessary conditions for the existence of bipartite $(d, D, -4)$-graphs with $d \geq 3$ and $D \geq 5$, which allowed us to prove the non-existence of such graphs for infinitely many pairs $(d, D)$; this included the case in which $D - 1$ is an odd prime power, and the case in which $D - 1 \equiv 0 \pmod{3}$. Afterwards, we went on to proving that bipartite $(d, D, -4)$-graphs for $d \geq 3$ and odd $D \geq 5$ do not exist.
We completed the catalogue of bipartite \((\Delta, D, -4)\)-graphs with maximum degree \(\Delta \geq 2\) and diameter 
\(5 \leq D \leq 187\), which in turn completed the catalogue of bipartite \((\Delta, D, -\epsilon)\)-graphs with \(\Delta \geq 2\), 
\(5 \leq D \leq 187\), \(D \neq 6\) and \(0 \leq \epsilon \leq 4\).

**Catalogue of bipartite \((d, D, \epsilon)\)-graphs with \(\Delta \geq 2\), \(5 \leq D \leq 187\) and \(\epsilon = 0, 2\).** In the case of \(\epsilon = 0\),
for \(5 \leq D \leq 187\) and \(d = 2\) the only Moore bipartite graphs are the \(2D\)-cycles, whereas for \(D = 6\) and \(d \geq 3\) they are incidence graphs of generalised polygons. For other values of \(d \geq 3\), \(5 \leq D \leq 187\) and \(\epsilon = 0\) there are no Moore bipartite graphs.

In the case of \(\epsilon = 2\) the results of [14] combined with [5, 6] showed that there are no such graphs.

**Catalogue of bipartite \((\Delta, D, -4)\)-graphs with \(\Delta \geq 2\) and \(5 \leq D \leq 187\).** The path of length 5 is the only such graph.

Another important result of the paper is the completion of the catalogue of bipartite \((3, D, -\epsilon)\)-graphs with \(D \geq 2\) and \(0 \leq \epsilon \leq 4\).

**Catalogue of bipartite \((3, D, 0)\)-graphs with \(D \geq 2\).** The cubic Moore bipartite graphs are the complete bipartite graph \(K_{3,3}\) for \(D = 2\), the unique incidence graph of the projective plane of order 2 for \(D = 3\), the unique incidence graph of the generalised quadrangle of order 2 for \(D = 4\), and the unique incidence graph of the generalised hexagon of order 2 for \(D = 6\).

**Catalogue of bipartite \((3, D, -2)\)-graphs with \(D \geq 2\).** There are only two non-isomorphic \((3, D, -2)\)-graphs with \(D \geq 2\); a unique bipartite \((3, 2, -2)\)-graph (the claw graph), and a unique \((3, 3, -2)\)-graph, which is depicted in Fig. [1](a).

**Catalogue of bipartite \((3, D, -4)\)-graphs with \(D \geq 2\).** There exist no bipartite \((3, 2, -4)\)-graphs. When the diameter is 3, there are four non-isomorphic bipartite \((3, 3, -4)\)-graphs; all of them are shown in Figure [2]. For diameter 4, there is a unique bipartite \((3, 4, -4)\)-graph, which is depicted in Fig. [3].

The results of this paper, combined with those of [11], assert that there are no bipartite \((3, D, -4)\)-graphs with \(D \geq 5\), outcome that gives an alternative proof of the optimality of the known bipartite \((3, 5, -6)\)-graph (see [4]).

### 6.1 Bipartite \((d, D, -4)\)-graphs with \(d \geq 4\) and \(D = 3, 4\)

The main results in this paper did not include bipartite \((d, D, -4)\)-graphs with \(d \geq 4\) and \(D = 3, 4\). However, we believe that the structural properties of these graphs provided in Section [4] could bear more conclusive results on such diameters.
For instance, by using Proposition 4.1, Lemma 4.1 (Saturating Lemma), Lemma 4.2 (Repeat Cycle Lemma), Proposition 4.2, and Proposition 4.3, we were able to prove analytically the uniqueness of the two bipartite (4,3,−4)-graphs depicted in Fig. 4. We also think there should be no major difficulty to complete as well –in a very similar manner– the catalogue of bipartite (5,3,−4)-graphs, which has so far as a unique element, the graph in Fig. 5.

Unfortunately, the final ideas used in the paper cannot be easily extended to cover bipartite \((d,D,−4)\)-graphs with \(d \geq 4\) and \(D = 3, 4\). With our current approach we cannot have Theorem 5.1 for \(D = 3, 4\). In Theorem 5.1 the intersection paths \(D^1 \cap D^2\), \(D^2 \cap D^3\) and \(D^3 \cap D^4\) have length at most 2, and for us to apply the the Repeat Cycle Lemma we need the lengths of such paths to be less than \(D − 2\). Indeed, the graph in Fig. 4(a) offers a good illustration of this. Even if we had Theorem 5.1 something similar would occur with Lemma 5.1; see the graphs in Fig. 4(b) and Fig. 5.

6.2 Remarks on the upper bound for \(N^b(d,D)\)

Our results improve the upper bound on \(N^b_{d,D}\) for many combinations of \(d\) and \(D\). Recall that a bipartite \((d,D,−5)\)-graph \(\Gamma\) with \(d \geq 3\) and \(D \geq 5\) must be regular (by Proposition 3.1) and thus cannot exist. Indeed, the upper bound of \(M^b(d,D) − 6\) for \(N^b(d,D)\) has been established in the following cases:

- \(d \geq 3\) and \(D \geq 8\) such that \(D − 1\) is an odd prime power.
- \(d \geq 3\) and \(D \geq 7\) such that \(d − 1 \equiv 0, 1\) (mod \(p\)), where \(p\) is an odd prime factor of \(D − 1\).
- \(d \geq 3\) and \(D \geq 5\) such that \(D \equiv 1, 3, 4, 5\) (mod 6).
- \(d \geq 3\) and \(5 \leq D \leq 187\) \((D \neq 6\).
- \(d = 3\) and any \(D \geq 5\) \((D \neq 6\).

Finally, we feel that the next conjectures are valid.

**Conjecture 6.1** There is no bipartite \((d,D,−4)\)-graph with \(d \geq 3\) and \(D \geq 5\).

**Conjecture 6.2** For natural numbers \(d \geq 3\) and \(D \geq 5\) such that \(D \neq 6\), \(N^b(d,D) \leq M^b(d,D) − 6\).

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