On Aharoni’s rainbow generalization of the Caccetta-Häggkvist conjecture

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Abstract

For a digraph $G$ and $v \in V(G)$, let $\delta^+(v)$ be the number of out-neighbors of $v$ in $G$. The Caccetta-Häggkvist conjecture states that for all $k \geq 1$, if $G$ is a digraph with $n = |V(G)|$ such that $\delta^+(v) \geq k$ for all $v \in V(G)$, then $G$ contains a directed cycle of length at most $\lceil n/k \rceil$. In [2], Aharoni proposes a generalization of this conjecture, that a simple edge-colored graph on $n$ vertices with $n$ color classes, each of size $k$, has a rainbow cycle of length at most $\lceil n/k \rceil$. In this paper, we prove this conjecture if each color class has size $\Omega(k \log k)$. 
1 Introduction and preliminaries

A graph or digraph is simple if there are no loops or parallel edges. For a simple digraph $G$ and a vertex $v \in V(G)$, let $\delta^+(v)$ denote the number of out-neighbors of $v$ in $G$. A famous conjecture in graph theory is the following, due to Caccetta and Häggkvist [1]:

**Conjecture 1.1 (Caccetta-Häggkvist)** Suppose $n, k$ are positive integers, and let $G$ be a simple digraph on $n$ vertices with $\delta^+(v) \geq k$ for all $v \in V(G)$; then $G$ contains a directed cycle of length at most $\lceil n/k \rceil$.

For a graph $G$ and a function $c : E(G) \to \{1, \ldots, |V(G)|\}$, a rainbow cycle (with respect to $c$) is a cycle $C$ in $G$ such that for all $e, f \in E(C)$ with $e \neq f$, we have $c(e) \neq c(f)$. We will refer to $c$ as a coloring of the edges of $G$. We say that $c$ has color classes of size at least $k$ for $k \in \mathbb{N}$ if $|c^{-1}(i)| \geq k$ for all $i \in \{1, \ldots, |V(G)|\}$.

In [2], Aharoni proposes a generalization of Conjecture 1.1:

**Conjecture 1.2 ([2])** Let $n, k$ be positive integers, and let $G$ be a simple graph on $n$ vertices. Let $c : E(G) \to \{1, \ldots, n\}$ be a coloring of the edges of $G$ with color classes of size at least $k$; then $G$ has a rainbow cycle of length at most $\lceil n/k \rceil$.

In a recent paper, Devos et al. [4] prove that Conjecture 1.2 is true for $k = 2$:

**Theorem 1.3 ([4])** Let $G$ be a simple graph on $n$ vertices, and let $c$ be a coloring of the edges of $G$ with color classes of size at least $2$; then there exists a rainbow cycle of length at most $\lceil n/2 \rceil$.

We also make use of the following results due to Bollobás and Szemerédi [3] and Shen [5], respectively. The first deals with the girth of a simple graph, while the second is an approximate result for Conjecture 1.1. In this paper, log denotes the logarithm with base 2.

**Theorem 1.4 ([3])** For all $n \geq 4$ and $k \geq 2$, if $G$ is a simple graph on $n$ vertices with $n + k$ edges, then $G$ contains a cycle of length at most

$$\frac{2(n + k)}{3k}(\log k + \log \log k + 4).$$

**Theorem 1.5 ([5])** Let $G$ be a simple digraph with $\delta^+(v) \geq k$ for all $v \in V(G)$. Then $G$ contains a directed cycle of length at most $\lceil n/k \rceil + 73$.

2 Main result

Our main result is the following:

**Theorem 2.1** Let $k > 1$ be an integer, and let $G$ be a graph. Let $c$ be a coloring of the edges of $G$ with color classes of size at least $301k \log k$. Then $G$ contains a rainbow cycle of length at most $\lceil n/k \rceil$.

*Note that $c$ is not required to be a proper edge-coloring.*
Proof. We proceed by induction on the number of vertices. Let \( f(k) = 7k \log k \), and let \( G \) be a graph on \( n \) vertices. Let \( c \) be a coloring of the edges of \( G \) with color classes of size at least \( 43f(k) \). Suppose for a contradiction that there is no rainbow cycle of length at most \( \lfloor n/k \rfloor \). Note that \( G \) has at least \( 43f(k)n \) edges, and therefore, \( n > 43f(k) \).

For \( v \in V(G) \) and \( i \in \{1, \ldots, n\} \), we say that \( i \) is dominant at \( v \) if \( v \) is incident with at least \( 7f(k) \) edges \( e \) such that \( c(e) = i \). We call a vertex \( v \in V(G) \) color-dominating if there exists \( i \in \{1, \ldots, n\} \) such that \( i \) is dominant at \( v \). We call a color \( i \in \{1, \ldots, n\} \) vertex-dominating if there exists a vertex \( v \in V(G) \) such that \( i \) is dominant at \( v \). Let us say that \( H \subseteq V(G) \) is nice if

- for every vertex-dominating color \( i \in \{1, \ldots, n\} \), there is a vertex \( v \in V(G) \setminus H \) such that \( i \) is dominant at \( v \); and

- there are at most \( |H| \) colors \( i \in \{1, \ldots, n\} \) such that \( i \) is not vertex-dominating and for all \( e \in c^{-1}(i) \), at least one end of \( e \) is in \( H \).

(Claim 1) \( \) If there is a nice set \( H \subseteq V(G) \) with \( 6f(k) \leq |H| < n \), then there is a nice set \( H' \subseteq V(G) \) with \( |H'| = \lfloor 6f(k) \rfloor \).

We remove vertices from \( H \) one-by-one such that the remaining set is nice. Suppose that we have removed \( j \geq 0 \) vertices from \( H \), leaving a nice set \( H_j \) with \( |H_j| > \lfloor 6f(k) \rfloor \). Let \( C_j \) be the set of colors \( i \in \{1, \ldots, n\} \) which are not vertex-dominating and also do not have an edge \( e \) with \( c(e) = i \) such that both ends of \( e \) are in \( V(G) \setminus H_j \). From the definition of a nice set, we know \( |C_j| \leq |H_j| \). If \( |C_j| < |H_j| \), then removing any vertex from \( H_j \) gives a smaller nice set. So, we may assume that \( |C_j| = |H_j| \). If there is a color \( i \) in \( C_j \) and an edge \( e = uv \in c^{-1}(i) \) with \( v \in H_j \) and \( u \in G \setminus H_j \), then \( H_j \setminus \{v\} \) is nice. If there is no such \( i \in C_j \), then for every color \( i \in C_j \), all edges in \( c^{-1}(i) \) have both their ends in \( H_j \). Now applying induction to the subgraph of \( G \) with vertex set \( H_j \) and edge set \( c^{-1}(C_j) \) gives a rainbow cycle of length at most \( \lfloor n/k \rfloor \) in \( G \), a contradiction. This proves Claim 1.

(Claim 2) \( \) There is a nice set \( H' \subseteq V(G) \) with \( |H'| = \lfloor 6f(k) \rfloor \).

For each vertex-dominating color \( i \), we pick a vertex \( v_i \) such that \( i \) is dominant at \( v_i \), and let \( S \) be the set of these vertices \( v_i \). Let \( H = V(G) \setminus S \). Note that \( H \) is nice; thus by Claim 1, we may assume that either \( |H| < 6f(k) \) or \( |H| = n \).

We first consider the case when \( |H| = n \). Since \( 43f(k) \geq 2 \), Theorem 1.3 guarantees the existence of a rainbow cycle \( K \) of length at most \( n/2 + 1 \) in \( G \). Let \( H' = V(G) \setminus V(K) \). Then \( H' \) is nice, and \( n > |H'| \geq n/2 - 1 \geq 6f(k) \); so by Claim 1, \( G \) contains a nice set of size \( \lfloor 6f(k) \rfloor \).

Now we may assume that \( |H| < 6f(k) \). We construct a digraph \( G' \) with \( V(G') = S \), and for all \( i, j \) with \( v_i, v_j \in S \), there is an arc \( v_i \rightarrow v_j \) if \( v_i v_j \in E(G) \) and \( c(v_i v_j) = i \). Every vertex \( v_i \) is incident with at least \( 7f(k) \) edges \( e \) with \( c(e) = i \), and since \( |H| < 6f(k) \), there are at least \( f(k) \) edges \( e = v_i u \) with \( c(e) = i \) and \( u \in S \). Therefore, \( \delta^+(G') \geq f(k) \).

Now, we claim \( n/f(k) + 74 \leq n/k \), which is equivalent to \( 74kf(k) \leq n(f(k) - k) \). Since \( k \geq 2 \), we have \( \log(k) \geq 117/301 \), and thus \( 74kf(k) \leq 43f(k)(f(k) - k) \leq n(f(k) - k) \), as claimed.

Then, by applying Theorem 1.5 to \( G' \) we obtain a directed cycle \( K \) of length at most \( \lceil n/f(k) \rceil + 73 \leq \lceil n/k \rceil \) in \( G' \). The edges of \( G \) that correspond to arcs of \( K \) form a rainbow cycle of length at most \( \lceil n/k \rceil \) in \( G \), a contradiction. This proves Claim 2.
Let $H \subseteq V(G)$ be a nice set with $|H| = \lceil 6f(k) \rceil$. Then there exists $H' \subseteq H$ (Claim 3) such that $|H'| \geq \lceil 2f(k) \rceil$ and such that for at least $n - \lceil f(k) \rceil + 1$ colors $i$, at least one edge $e \in c^{-1}(i)$ has both ends in $V(G) \setminus H'$.

Let $C$ be the set of colors $i$ which are not vertex-dominating and for which no edge of $c^{-1}(i)$ has both ends in $V(G) \setminus H$. Since $H$ is nice, it follows that $|C| \leq |H| = \lceil 6f(k) \rceil$. Let $D \subseteq C$ be the set of colors $i \in C$ such that there is a vertex $v \in H$ which is incident with all edges in $c^{-1}(i)$ that have one end in $H$ and the other in $V(G) \setminus H$. We claim that $|D| \leq \lceil f(k) \rceil - 1$. Indeed, for each color $i \in D$, there are at least $\lceil 36f(k) \rceil$ edges in $c^{-1}(i)$ with both ends in $H$ since $i$ is not vertex-dominating. If $|D| > \lceil f(k) \rceil - 1$, then we obtain more than $(f(k) - 1)(36f(k))$ edges with both ends in $H$. Now, since $k \geq 2$, we have $f(k) \geq 72/23$, and it follows that:

$$\left(f(k) - 1\right)(36f(k)) \geq \frac{49f(k)^2}{2} \geq \frac{(6f(k) + 1)^2}{2} \geq \frac{|H|^2}{2}$$

which gives a contradiction. Thus, $|D| \leq \lceil f(k) \rceil - 1$.

Next, we claim there exists $H' \subseteq H$ such that $|H'| = \lceil 2f(k) \rceil$ and such that for all $i \in \{1, \ldots, n\} \setminus D$, there is an edge $e \in c^{-1}(i)$ with both ends in $V(G) \setminus H'$. To see this, we construct a graph $J$ with vertex set $H$ and the following set of edges. For each $i \in C \setminus D$, we choose two vertices $v^i_1, v^i_2 \in H$, each incident with an edge in $c^{-1}(i)$ whose other end is in $V(G) \setminus H$; we know from the definition of $D$ that this is possible. Now, the graph $J$ has $|H|$ vertices and at most $|H|$ edges, and so $J$ has a stable set $H' \subseteq V(J)$ of size at least $|V(J)|/3 \geq 2f(k)$; and so $|H'| \geq \lceil 2f(k) \rceil$.

Now, for every color $i \in C \setminus D$, $V(G) \setminus H'$ contains at least one of $v^i_1, v^i_2$, and therefore, there is an edge in $c^{-1}(i)$ with both ends in $V(G) \setminus H'$. Moreover, for every $i \in \{1, \ldots, n\} \setminus C$, either $i$ dominates a vertex $v$ in $V(G) \setminus H \subseteq V(G) \setminus H'$ (and so, since $|H'| < 7f(k)$, there is an edge in $c^{-1}(i)$ incident with $v$ whose other end is not in $H'$); or there is an edge in $c^{-1}(i)$ with both ends in $V(G) \setminus H \subseteq V(G) \setminus H'$. Thus, for at least $n - |D| \geq n - \lceil f(k) \rceil + 1$ colors $i$, at least one edge in $c^{-1}(i)$ has both ends in $V(G) \setminus H'$. This proves Claim 3.

By combining Claim 2 and Claim 3, we conclude that there exists $H' \subseteq V(G)$ with $|H'| \geq \lceil 2f(k) \rceil$, and such that for at least $n - \lceil f(k) \rceil + 1$ colors $i$, at least one edge in $c^{-1}(i)$ has both ends in $V(G) \setminus H'$. Let $H''$ be a subgraph of $G$ with vertex set $V(G) \setminus H'$, obtained by taking exactly one edge in $c^{-1}(i)$ with both ends in $V(G) \setminus H'$ for all $i \in \{1, \ldots, n\}$ which have such an edge. It follows that $|E(H'')| \geq |V(H'')| + \lceil f(k) \rceil$.

Now, we claim that $\frac{2(n + f(k))}{3(f(k))}(\log \log(f(k)) + \log(f(k)) + 4) \leq \frac{n}{f(k)}$. Using $f(k) < n/43$, it suffices to show:

$$\frac{88(\log \log(f(k)) + \log(f(k)) + 4)}{129} \leq 7\log(k)$$

Let $g(k) = 7\log(k) - \frac{88}{129}(\log \log(f(k)) + \log(f(k)) + 4)$. We have that $g(2) > 0$, and for $k \geq 2$ we have:

$$f(k)g'(k)\ln(2) = 49\log(k) - \frac{88}{129}f'(k)\left(\frac{1}{\log(f(k))\ln(2)} + 1\right) > 0$$
since for \( k \geq 2 \) we have:

\[
f'(k) \left( \frac{1}{\log(f(k)) \ln(2)} + 1 \right) < (7 + 7\log(k))(3) \leq 49\log(k)
\]

So \( g'(k) > 0 \) for \( k \geq 2 \), and it follows that \( g(k) \geq 0 \) for \( k \geq 2 \), as desired.

Then, Theorem 1.4 gives a rainbow cycle of length at most \( 2(n+f(k))(\log\log(f(k)) + \log(f(k)) + 4) \leq \lceil \frac{n}{k} \rceil \), a contradiction. This proves Theorem 2.1.

We have an immediate corollary which gives us a result for the case of \( \Omega(n \log n) \) color classes each of size \( k \):

\textbf{Corollary 2.2} Let \( k \) be a positive integer and let \( G \) be a simple graph on \( n \) vertices. Let \( c : E(G) \rightarrow \{1, \ldots, t\} \) with \( t \geq 303n \log n \), and with \( |c^{-1}(i)| \geq k \) for all \( i \in \{1, \ldots, t\} \). Then \( G \) contains a rainbow cycle in \( G \) of length at most \( \lceil n/k \rceil \).

\textbf{Proof.} Note that \( t \geq 303n \log n \geq 303n \log k \). Since \( 303n \log k \geq n[301 \log k] \), we can partition \( \{1, \ldots, t\} \) into \( n \) parts, each of size at least \( \lceil 301 \log k \rceil \); that is, there is a function \( f : \{1, \ldots, t\} \rightarrow \{1, \ldots, n\} \) such that \( |f^{-1}(i)| \geq \lceil 301 \log k \rceil \) for all \( i \in \{1, \ldots, n\} \). By Theorem 2.1, applied to \( G \) and \( f \circ c \), we obtain a rainbow cycle of length at most \( \lceil n/k \rceil \) in \( G \) with respect to \( f \circ c \), which is also rainbow with respect to \( c \). This proves Corollary 2.2.

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