Enhanced First and Zeroth Order Variance Reduced Algorithms for Min-Max Optimization

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Abstract

Min-max optimization captures many important machine learning problems such as robust adversarial learning and inverse reinforcement learning, and nonconvex-strongly-concave min-max optimization has been an active line of research. Specifically, a novel variance reduction algorithm SREDA was proposed recently by Luo et al. (2020) to solve such a problem, and was shown to achieve the optimal complexity dependence on the required accuracy level \(\epsilon\). Despite the superior theoretical performance, the convergence guarantee of SREDA requires stringent initialization accuracy and an \(\epsilon\)-dependent stepsize for controlling the per-iteration progress, so that SREDA can run very slowly in practice. This paper develops a novel analytical framework that guarantees the SREDA’s optimal complexity performance for a much enhanced algorithm SREDA-Boost, which has less restrictive initialization requirement and an accuracy-independent (and much bigger) stepsize. Hence, SREDA-Boost runs substantially faster in experiments than SREDA. We further apply SREDA-Boost to propose a zeroth-order variance reduction algorithm named ZO-SREDA-Boost for the scenario that has access only to the information about function values not gradients, and show that ZO-SREDA-Boost outperforms the best known complexity dependence on \(\epsilon\). This is the first study that applies the variance reduction technique to zeroth-order algorithm for min-max optimization problems.

1 Introduction

Min-max optimization has attracted significant growth of attention in machine learning as it captures several important machine learning models and problems including generative adversarial networks (GANs) Goodfellow et al. (2014), robust adversarial machine learning Madry et al. (2018), imitation learning Ho and Ermon (2016), etc. Min-max optimization typically takes the following form

\[
\min_{x \in \mathbb{R}^{d_1}} \max_{y \in \mathbb{R}^{d_2}} f(x, y), \quad \text{where} \quad f(x, y) \triangleq \begin{cases} 
E[F(x, y; \xi)] & \text{(online case)} \\
\frac{1}{n} \sum_{i=1}^{n} F(x, y; \xi_i) & \text{(finite-sum case)}
\end{cases}
\]

where \(f(x, y)\) takes the expectation form if data samples \(\xi\) are taken in an online fashion, and \(f(x, y)\) takes the finite-sum form if a dataset of training samples \(\xi_i\) for \(i = 1, \ldots, n\) are given in advance. This paper focuses on the nonconvex-strongly-concave min-max problem, in which \(f(x, y)\) is nonconvex with respect to \(x\) for all \(y \in \mathbb{R}^{d_2}\), and \(f(x, y)\) is \(\mu\)-strongly concave with respect to \(y\) for all \(x \in \mathbb{R}^{d_1}\). The problem then takes the following equivalent form:

\[
\min_{x \in \mathbb{R}^{d_1}} \left\{ \Phi(x) \triangleq \max_{y \in \mathbb{R}^{d_2}} f(x, y) \right\}.
\]

The objective function \(\Phi(\cdot)\) in eq. (2) is nonconvex in general, and hence algorithms for solving eq. (2) are expected to attain an approximate (i.e., \(\epsilon\)-accurate) first-order stationary point. The convergence of deterministic algorithms for solving eq. (2) has been established in Jin et al. (2019); Lu et al. (2020); Nouiehed...
et al. (2019); Thekumparampil et al. (2019). SGD-type of stochastic algorithms have also been proposed to solve such a problem more efficiently, including SGDmax Jin et al. (2019), PGSM Rafique et al. (2018), and SGDA Lin et al. (2019), which respectively achieve the overall complexity of $O(\kappa^3\epsilon^{-4}\log(1/\epsilon))$, $O(\kappa^3\epsilon^{-4})$, and $O(\kappa^3\epsilon^{-4})$.

Furthermore, several variance reduction methods have been proposed for solving eq. (2) for the nonconvex-strongly-concave case. PGSVRG Rafique et al. (2018) adopts a proximally guided SVRG method and achieves the overall complexity of $O(\kappa^3\epsilon^{-4})$ for the online case and $O(\kappa^3\epsilon^{-4})$ for the finite-sum case. Wai et al. (2019) converted the value function evaluation problem to a specific min-max problem and applied SAGA to achieve the overall complexity of $O(\kappa\epsilon^{-2})$ for the finite-sum case. More recently, Luo et al. (2020) proposed a novel nested-loop algorithm named Stochastic Recursive Gradient Descent Ascent (SREDA), which adopts SARAH/SPIDER-type Fang et al. (2018); Nguyen et al. (2017a) of recursive variance reduction method (originally designed for solving the minimization problem) for designing gradient estimators to update both $x$ and $y$. Specifically, $x$ takes the normalized gradient update in the outer-loop and each update of $x$ is followed by an entire inner-loop updates of $y$. Luo et al. (2020) showed that SREDA achieves an overall complexity of $O(\kappa^3\epsilon^{-3})$ for the online case in eq. (1), which attains the optimal dependence on $\epsilon$ Arjevani et al. (2019). For the finite-sum case, SREDA achieves the complexity of $O(\kappa^2\sqrt{n}\epsilon^{-2} + n + (n + k)\log(\kappa/\epsilon))$ for $n \geq \kappa^2$, and $O((\kappa^2 + \kappa n)\epsilon^{-2})$ for $n \leq \kappa^2$.

Despite the superior theoretical performance of SREDA, two important issues of SREDA may substantially degrade its practice performance. (1) SREDA has a stringent requirement on the initialization accuracy $\zeta = \kappa^{-2}\epsilon^2$ in order to guarantee the complexity performance. It hence requires $O(\kappa^2\epsilon^{-2}\log(\kappa/\epsilon))$ gradient estimations in the initialization to attain such an accuracy, which is costly and dependent on the accuracy $\epsilon$. (2) The convergence of SREDA requires a very small per-iteration increment $\|x_{t+1} - x_t\|_2 = O(\epsilon/\kappa\ell)$, which is guaranteed by normalized gradient descent with an accuracy-dependent stepsize $\alpha_t = O(\min\{\epsilon/(\kappa\ell \|v_t\|_2), 1/(\kappa\ell)\})$. Due to the choice of $\epsilon$-dependent stepsize, SREDA can run very slowly in practice.

• Thus, the first focus of this paper is on designing an enhanced SREDA algorithm, which has more computationally efficient initialization, takes an accuracy-independent (and hence large) constant stepsize, and still retains the superior complexity performance of SREDA. Providing the convergence guarantee for such an enhanced algorithm requires to devise a new analysis framework that goes significantly further beyond that in Luo et al. (2020).

In many machine learning scenarios, min-max optimization problems need to be solved without the access of the gradient information, but only the function values, e.g., in multi-agent reinforcement learning with bandit feedback Wei et al. (2017); Zhang et al. (2019) and robotics Bogunovic et al. (2018); Wang and Jegelka (2017). This motivates the design of zeroth-order (i.e., gradient-free) algorithms. For nonconvex-strongly-concave min-max optimization, Liu et al. (2019) studied a constrained problem and proposed ZO-min-max algorithm that achieves the computational complexity of $O((d_1 + d_2)\epsilon^{-6})$. Wang et al. (2020) designed ZO-SGDA and ZO-SGDMSA, where ZO-SGDMA achieves the best known query complexity of $O((d_1 + d_2)\kappa^2\epsilon^{-4}\log(1/\epsilon))$ among the zeroth-order algorithms for this problem. All of the above studies on zeroth-order algorithms are of SGD-type, and no efforts have been made on developing variance reduction zeroth-order algorithms for nonconvex-strongly-concave min-max optimization to further improve the query complexity.

• The second focus of this paper is on applying the enhanced SREDA algorithm that we develop to design a zeroth-order variance reduced algorithm for nonconvex-strongly-concave min-max problems, which outperforms existing stochastic algorithms.

1.1 Main Contributions

The first contribution of this paper lies in proposing an enhanced SREDA, which we call as SREDA-Boost, for solving nonconvex-strongly-concave min-max problems. SREDA-Boost achieves the same state-of-the-art complexity order as SREDA (see Table 1), but improves SREDA with the following two additional advantages. (1) For the initialization, SREDA-Boost requires only the accuracy of $\zeta = \kappa^{-1}$ for initialization, which is much less stringent than that of $\zeta = \kappa^{-2}\epsilon^2$ required by SREDA. It thus saves the computational complexity for initialization by a factor of $O(\kappa\epsilon^{-3})$ compared with SREDA. (2) For the update of parameter, SREDA-Boost allows the stepsize $\alpha = O(1/(\kappa\ell))$, which is accuracy independent, and much larger than

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1The constant $\kappa = \ell/\mu$, where $\mu$ is the strong concavity parameter of $f(x, \cdot)$, and $\ell$ is the Lipschitz constant of the gradient of $f(x, y)$ as defined in Assumption 2. Typically, $\kappa$ is much larger than one.
the stepsize $\alpha = \mathcal{O}(\min\{\epsilon/(\kappa\ell \|v_1\|_2), 1/(\kappa\ell)\})$ adopted by SREDA. Hence, SREDA-Boost can run much faster than SREDA as demonstrated by our experiments in Section 5, because it makes a considerably larger progress per iteration.

The convergence analysis requires nontrivial technical developments beyond that for SREDA in Luo et al. (2020) in order to guarantee that SREDA-Boost enjoys the same computational complexity but under much more relaxed requirements for initialization and accuracy independent stepsize. Specifically, the main challenge for such analysis lies in bounding two inter-connected stochastic error processes: tracking error and gradient estimation error (see Section 3.2 for their formal definitions). In the analysis of SREDA in Luo et al. (2020), the initialization and stepsize requirements help substantially to bound the two errors separately at each iteration so that the convergence follows. In contrast, this is not applicable to SREDA-Boost due to the enhanced initialization and stepsize. Hence, central to our new analysis framework are the developments of three novel steps: bounding the two error processes accumulatively over the entire algorithm execution, decoupling these two inter-related stochastic error processes, and establishing each of their relationships with the accumulative gradient estimators.

Based on SREDA-Boost, the second contribution of this paper lies in proposing the zeroth-order variance reduced algorithm ZO-SREDA-Boost for nonconvex-strongly-concave min-max optimization when the gradient information is not accessible. For the online case, we show that ZO-SREDA-Boost achieves an overall query complexity of $\mathcal{O}((d_1 + d_2)\kappa^3\epsilon^{-3})$, which outperforms the best known complexity (achieved by ZO-SGDMSA Wang et al. (2020)) in the case with $\epsilon \leq \kappa^{-1}$. For the finite-sum case, we show that ZO-SREDA-Boost achieves an overall query complexity of $\mathcal{O}((d_1 + d_2)(\kappa^2\sqrt{\kappa}\epsilon^{-2} + n) + d_2(\kappa^2 + \kappa n)\log(\kappa))$ when $n \geq \kappa^2$, and $\mathcal{O}((d_1 + d_2)(\kappa^2 + \kappa n)\kappa\epsilon^{-3})$ when $n \leq \kappa^2$. This is the first study that applies the variance reduction method for zeroth-order nonconvex-strongly-concave min-max optimization.

### 1.2 Related Work

Due to the vast amount of studies on min-max optimization and the variance reduced algorithms, we include below only the studies that are highly relevant to this work.

Variance reduction methods for min-max optimization are highly inspired by those for conventional minimization problems, including SAGA Defazio et al. (2014); Reddi et al. (2016), SVRG Allen-Zhu (2017); Allen-Zhu.

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**Table 1: Comparison of stochastic algorithms for nonconvex-strongly-concave min-max problems**

| Type | Algorithm | Stepsize | Initialization Complexity | Overall Complexity |
|------|-----------|----------|---------------------------|--------------------|
| FO   | SGDmax Jin et al. (2019) | $\mathcal{O}(\kappa^{-1}\ell^{-1})$ | N/A | $\mathcal{O}(\kappa^3\epsilon^{-4}\log(\frac{1}{\epsilon}))$ |
|      | SGDA Lin et al. (2019) | $\mathcal{O}(\kappa^{-2}\ell^{-1})$ | N/A | $\mathcal{O}(\kappa^3\epsilon^{-4})$ |
|      | PGSMR Rafique et al. (2018) | $\mathcal{O}(\kappa^{-2})$ | N/A | $\mathcal{O}(\kappa^3\epsilon^{-4})$ |
|      | PGSVRG Rafique et al. (2018) | $\mathcal{O}(\kappa^{-2})$ | N/A | $\mathcal{O}(\kappa^3\epsilon^{-4})$ |
|      | SREDA Luo et al. (2020) | $\mathcal{O}(\min\{\frac{\epsilon}{\kappa^2\ell^2}, \frac{1}{\kappa^2}\})$ | $\mathcal{O}(\kappa^2\epsilon^{-2}\log(\frac{1}{\epsilon}))$ | $\mathcal{O}(\kappa^3\epsilon^{-3})$ |
|      | SREDA-Boost | $\mathcal{O}(\kappa^{-1}\ell^{-1})$ | $\mathcal{O}(\kappa\log(\kappa))$ | $\mathcal{O}(\kappa^3\epsilon^{-3})$ |
| ZO   | ZO-min-max Liu et al. (2019) | $\mathcal{O}(\kappa^{-1}\ell^{-1})$ | N/A | $\mathcal{O}(\kappa^3\epsilon^{-3})$ |
|      | ZO-SGDA Wang et al. (2020) | $\mathcal{O}(\kappa^{-4}\ell^{-1})$ | N/A | $\mathcal{O}(\kappa^3\epsilon^{-4})$ |
|      | ZO-SGDMSA Wang et al. (2020) | $\mathcal{O}(\kappa^{-1}\ell^{-1})$ | N/A | $\mathcal{O}(\kappa^3\epsilon^{-4}\log(\frac{1}{\epsilon}))$ |
|      | ZO-SREDA-Boost | $\mathcal{O}(\kappa^{-1}\ell^{-1})$ | $\mathcal{O}(\kappa\log(\kappa))$ | $\mathcal{O}(\kappa^3\epsilon^{-3})$ |

1 "FO" stands for "First-Order", and "ZO" stands for "Zeroth-Order".

2 We include only the stepsize for updating $x_t$ for comparison.

3 The complexity for first-order algorithms refer to the total gradient computations to attain an $\epsilon$-stationary point, and for zeroth-order algorithms refers to the total function value queries.

4 We include only the complexity in the online case in the table, because many studies did not cover the finite-sum case. We comment on the finite-case in Sections 3.2 and 4.2.

5 We define $d = d_1 + d_2$. 

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"Algorithm" | "Initialization Complexity" | "Overall Complexity"
and Hazan (2016); Johnson and Zhang (2013), SARAH Nguyen et al. (2017a,b, 2018), SPIDER Fang et al. (2018), SpiderBoost Wang et al. (2019), etc. But the convergence analysis for min-max optimization is much more challenging, and is typically quite different from their counterparts in minimization problems. For strongly-convex-strongly-concave min-max optimization, Palaniappan and Bach (2016) applied SVRG and SAGA to the finite-sum case and established a linear convergence rate, and Chavdarova et al. (2019) proposed SVRE later to obtain a better bound. When the condition number of the problem is very large, Luo et al. (2019) proposed a proximal point iteration algorithm to improve the performance of SAGA. For some special cases, Du et al. (2017); Du and Hu (2019) showed that the linear convergence rate of SVRG can be maintained without the strongly-convex or strongly concave assumption. Yang et al. (2020) applied SVRG to study the min-max optimization under the two-sided Polyak-Lojasiewicz condition. Nonconvex-strongly-concave min-max optimization is the focus of this paper. As we discuss at the beginning of the introduction, the SGD-type algorithms have been developed and studied, including SGDmax Jin et al. (2020), PGSVRG Rafique et al. (2018), the SAGA-type algorithm for min-max optimization Wai et al. (2019), and SREDA Luo et al. (2020). Particularly, SREDA has been shown in Luo et al. (2020) to achieve the optimal complexity dependence on $\epsilon$. This paper proposes the SREDA-Boost algorithm, and improves the convergence guarantee analysis with considerably different technical developments. Which allows SREDA-Boost to have a large and accuracy-independent stepsize and require less computational cost for initialization compared to SREDA. While SGD-type zeroth-order algorithms have been studied for min-max optimization, such as Menickelly and Wild (2020); Roy et al. (2019) for convex-concave min-max problems and Liu et al. (2019); Wang et al. (2020) for nonconvex-strongly-concave min-max problems, variance reduced algorithms have not been developed for zeroth-order min-max optimization so far. This paper proposes the first such an algorithm named ZO-SREDA-Boost for nonconvex-strongly-concave min-max optimization, and established its complexity performance that outperforms the existing comparable algorithms (see Table 1).

2 Notation and Preliminaries

In this paper, we use $\|\cdot\|_2$ to denote the Euclidean norm of vectors. For a finite set $S$, we denote its cardinality as $|S|$. For a positive integer $n$, we denote $[n] = \{1, \cdots, n\}$. We assume that the min-max problem eq. (2) satisfies the following assumptions, which have also been adopted by Luo et al. (2020) for SREDA. We slightly abuse the notation $\xi$ below to represent the random index in both the online and finite-sum cases, where in the finite-sum case, $\xi_\ell[\cdot]$ is with respect to the uniform distribution over $\{\xi_1, \cdots, \xi_n\}$.

**Assumption 1.** The function $\Phi(\cdot)$ is lower bounded, i.e., we have $\Phi^* = \inf_{x \in \mathbb{R}^d} \Phi(x) > -\infty$.

**Assumption 2.** The component function $F$ has an averaged $\ell$-Lipschitz gradient, i.e., for all $(x, y), (x', y') \in \mathbb{R}^d \times \mathbb{R}^d$, we have $\mathbb{E}_\xi[\|\nabla F(x, y; \xi) - \nabla F(x', y'; \xi)\|_2^2] \leq \ell^2(\|x - x'\|_2^2 + \|y - y'\|_2^2)$.

**Assumption 3.** The function $f$ is $\mu$-strongly-concave in $y$ for any $x \in \mathbb{R}^d$, and the component function $F$ is concave in $y$, i.e., for any $x \in \mathbb{R}^d$, $y, y' \in \mathbb{R}^d$ and $\xi$, we have $f(x, y) \leq f(x, y') + \langle \nabla_y f(x, y'), y - y' \rangle - \frac{\mu}{2}\|y - y'\|_2^2$, and $F(x, y; \xi) \leq F(x, y'; \xi) + \langle \nabla_y F(x, y'; \xi), y - y' \rangle$.

**Assumption 4.** The gradient of each component function $F(x, y; \xi)$ has a bounded variance, i.e., there exists a constant $\sigma > 0$ such that for any $(x, y) \in \mathbb{R}^{d_1 \times d_2}$, we have $\mathbb{E}_\xi[\|\nabla F(x, y; \xi) - \nabla f(x, y)\|_2^2] \leq \sigma^2 < \infty$.

Since $\Phi$ is nonconvex in general, it is NP-hard to find its global minimum. The goal here is to develop stochastic gradient algorithms that output an $\epsilon$-stationary point as defined below.

**Definition 1.** The point $\bar{x}$ is called an $\epsilon$-stationary point of the differentiable function $\Phi$ if $\|\nabla \Phi(\bar{x})\|_2 \leq \epsilon$, where $\epsilon$ is a positive constant.
3 SREDA-Boost: First-order Variance Reduction Algorithm

3.1 SREDA-Boost Algorithm

We first introduce the SREDA algorithm proposed in Luo et al. (2020), and then propose SREDA-Boost as an enhanced algorithm.

SREDA (see Option I in Algorithm 1) utilizes the variance reduction techniques proposed in SARAH Nguyen et al. (2017a) and SPIDER Fang et al. (2018) for minimization problems to construct the gradient estimator recursively for min-max optimization. Specifically, the parameters $x_t$ and $y_t$ are updated in a nested loop fashion: each update of $x_t$ in the outer-loop is followed by $(m + 1)$ updates of $y_t$ over one entire inner loop. Furthermore, the outer-loop updates of $x_t$ is divided into epochs for variance reduction. Consider a certain outer-loop epoch $t = \{ (n_t - 1)q, \ldots, n_t q - 1 \}$ ($1 \leq n_t < \lceil T/q \rceil$ is a positive integer). At the beginning of such an epoch, the gradients are evaluated with a large batch size $S_1$ (see line 6 in Algorithm 1). Then, for each subsequent outer-loop iteration, an inner loop of ConcaveMaximizer (see Algorithm 2) recursively updates the gradient estimators for $\nabla_x f(x, y)$ and $\nabla_y f(x, y)$ with a small batch size $S_2$. Note that although the inner loop does not update $x$, the gradient estimator $\nabla_x f(x, y)$ is updated in the inner loop. With such a variance reduction technique, SREDA outperforms all previous algorithms for nonconvex-strongly-concave min-max problems (see Table 1), and was shown to achieve the optimal dependency on $\epsilon$ in complexity Luo et al. (2020).

Algorithm 1 SREDA and SREDA-Boost

1: Input: $x_0$, initial accuracy $\zeta$, learning rate $\alpha$, $\beta = O(\frac{1}{\zeta})$, batch size $S_1, S_2$ and periods $q, m$
2: Option I (SREDA): $\zeta = \kappa^{-3} \epsilon^2$; Option II (SREDA-Boost): $\zeta = \kappa^{-1}$
3: Initialization: $y_0 = iSARAH(-f(x_0, \cdot), \zeta)$ (see Appendix B.2 for iSARAH(\cdot))
4: for $t = 0, 1, \ldots, T - 1$ do
5: if mod$(t, q) = 0$ then draw $S_1$ samples $\{\xi_1, \ldots, \xi_{S_1}\}$
6: $v_t = \frac{1}{S_1} \sum_{i=1}^{S_1} \nabla_x F(x_t, y_t, \xi_i), \quad u_t = \frac{1}{S_2} \sum_{i=1}^{S_2} \nabla_y F(x_t, y_t, \xi_i)$
7: else
8: $v_t = \tilde{v}_{t-1, \tilde{m}_{t-1}}, \quad u_t = \tilde{u}_{t-1, \tilde{m}_{t-1}}$
9: end if
10: Option I (SREDA): $\alpha_t = \min\{\frac{\kappa}{\|v_t\|_2}, \frac{1}{S_1}\} O(\frac{1}{\zeta})$; Option II (SREDA-Boost): $\alpha_t = \alpha = O(\frac{1}{\epsilon})$
11: $x_{t+1} = x_t - \alpha_t v_t$
12: $y_{t+1} = \text{ConcaveMaximizer}(t, m, S_2)$
13: end for
14: Output: $\hat{x}$ chosen uniformly at random from $\{x_t\}_{t=0}^{T-1}$

Algorithm 2 ConcaveMaximizer$(t, m, S_2)$

1: Initialization: $\tilde{x}_{t-1} = x_t, \tilde{y}_{t-1} = y_t, \tilde{x}_{t, 0} = x_{t+1}, \tilde{y}_{t, 0} = y_t, \tilde{v}_{t-1} = v_t, \tilde{u}_{t-1} = u_t$
2: Draw $S_2$ samples $\{\xi_1, \ldots, \xi_{S_2}\}$
3: $\tilde{v}_{t, 0} = \tilde{v}_{t-1} + \frac{1}{S_2} \sum_{i=1}^{S_2} \nabla_x F(\tilde{x}_{t, 0}, \tilde{y}_{t, 0}, \xi_i)$
4: $\tilde{u}_{t, 0} = \tilde{u}_{t-1} + \frac{1}{S_2} \sum_{i=1}^{S_2} \nabla_y F(\tilde{x}_{t, 0}, \tilde{y}_{t, 0}, \xi_i)$
5: $\tilde{x}_{t, 1} = \tilde{x}_{t, 0}, \quad \tilde{y}_{t, 1} = \tilde{y}_{t, 0} + \beta \tilde{u}_{t, 0}$
6: for $k = 1, 2, \ldots, m + 1$ do
7: Draw $S_2$ samples $\{\xi_1, \ldots, \xi_{S_2}\}$
8: $\tilde{v}_{t, k} = \tilde{v}_{t, k-1} + \frac{1}{S_2} \sum_{i=1}^{S_2} \nabla_x F(\tilde{x}_{t, k}, \tilde{y}_{t, k}, \xi_i)$
9: $\tilde{u}_{t, k} = \tilde{u}_{t, k-1} + \frac{1}{S_2} \sum_{i=1}^{S_2} \nabla_y F(\tilde{x}_{t, k}, \tilde{y}_{t, k}, \xi_i)$
10: $\tilde{x}_{t, k+1} = \tilde{x}_{t, k}, \quad \tilde{y}_{t, k+1} = \tilde{y}_{t, k} + \beta \tilde{u}_{t, k}$
11: end for
12: Output: $y_{t+1} = \tilde{y}_{t, \tilde{m}_t}$ with $\tilde{m}_t$ chosen uniformly at random from $\{0, 1, \ldots, m\}$

Although SREDA achieves a desirable performance in theory, two issues can substantially slow down its practical performance. (a) Its initialization $y_0$ needs to satisfy a stringent and accuracy-dependent requirement $\mathbb{E}[\|\nabla_y f(x_0, y_0)\|_2^2] \leq \kappa^{-2} \epsilon^2$ (see line 2 in Algorithm 1), which requires as large as $O(\kappa^2 \epsilon^{-2} \log(\kappa/\epsilon))$ stochastic
gradient computations Luo et al. (2020). This is quite costly. (b) SREDA uses an ɛ-dependent stepsize and applies normalized gradient descent, so that each outer-loop update makes very small progress given by \( \|x_{i+1} - x_i\|_2 = O(\epsilon/(\kappa \ell)) \). This prevents SREDA from running fast. By following the analysis of SREDA, it appears that such choices for initialization and stepsize are necessary to obtain the guaranteed convergence rate.

In this paper, we propose SREDA-Boost (see Option II in Algorithm 1) that enhances SREDA over the above two issues. (a) SREDA-Boost relaxes the initialization requirement to be \( \mathbb{E}[\|\nabla_y f(x_0, y_0)\|_2^2] \leq \kappa^{-1} \), which requires only \( O(\kappa \log \kappa) \) gradient computations. This improves the computational cost upon SREDA by a factor of \( O(\kappa \epsilon^{-2}) \). (b) SREDA-Boost adopts a much larger and ɛ-independent stepsize \( \alpha_t = O(1/(\kappa \ell)) \) for \( x_t \) so that each outer-loop update can make much bigger progress than SREDA. As our experiments in Section 5 demonstrate, SREDA-Boost runs much faster than SREDA. The main reason that SREDA-Boost can take the above advantageous design is due to the new analysis framework that we develop (see Section 3.2), which provably guarantees that SREDA-Boost still achieves the same optimal complexity order as SREDA even under the much relaxed conditions on the initialization and the stepsize.

### 3.2 Convergence Analysis of SREDA-Boost

The following theorem provides the computational complexity of SREDA-Boost for finding a first-order stationary point of \( \Phi(\cdot) \) with ɛ accuracy.

**Theorem 1.** Apply SREDA-Boost to solve the online case of the problem eq. (1). Suppose Assumptions 1-4 hold. Let \( \zeta = \kappa^{-1}, \alpha = O(\kappa^{-1} \ell^{-1}), \beta = O(\ell^{-1}), q = O(\epsilon^{-1}), m = O(\kappa), S_1 = O(n^2 \kappa \epsilon^{-2}) \) and \( S_2 = O(n \kappa \epsilon^{-1}) \). Then for \( T \) to be at least at the order of \( O(\kappa \epsilon^{-2}) \), Algorithm 1 outputs \( \hat{x} \) that satisfies

\[
\mathbb{E}[\|\nabla \Phi(\hat{x})\|_2] \leq \epsilon
\]

with stochastic gradient complexity \( O(\kappa^3 \epsilon^{-3}) \).

Furthermore, SREDA-Boost is also applicable to the finite-sum case of the problem eq. (1) by replacing the large batch \( S_1 \) of samples used in line 6 of Algorithm 1 with the full set of samples.

**Corollary 1.** Apply SREDA-Boost described above to solve the finite-sum case of the problem eq. (1). Suppose Assumption 1-4 hold. Under appropriate parameter settings given in Appendix B.4, the overall gradient complexity to attain an \( \epsilon \)-stationary point is \( O(n \kappa \sqrt{\epsilon - 2} + n + (n + \kappa) \log(n)) \) for \( n \geq \kappa^2 \), and \( O((\kappa^2 + \kappa n) \epsilon^{-2}) \) for \( n \leq \kappa^2 \).

Theorem 1 and Corollary 1 indicate that SREDA-Boost achieves the same gradient computational complexity as SREDA in Luo et al. (2020), but under more relaxed initialization and a much bigger and accuracy-independent stepsize \( \alpha \).

The convergence analysis of SREDA-Boost in Theorem 1 is very different from the proof of SREDA in Luo et al. (2020). At a high level, such analysis mainly focuses on bounding two inter-related errors: **tracking error** \( \delta_t = \mathbb{E}[\|\nabla_y f(x_t, y_t)\|_2^2] \) that captures how well the output \( y_t \) of the inner loop approximates the optimal point \( y^*(x_t) \) for a given \( x_t \), and **gradient estimation error** \( \Delta_t = \mathbb{E}[\|v_t - \nabla_x f(x_t, y_t)\|_2^2 + \|u_t - \nabla_y f(x_t, y_t)\|_2^2] \) that captures how well the stochastic gradient estimators approximate the true gradients. In the analysis of SREDA in Luo et al. (2020), the stringent requirements for initialization and stepsize and the ɛ-dependent normalized gradient descent update substantially help to bound both errors \( \delta_t \) and \( \Delta_t \) separately at the ɛ level for each iteration so that the convergence bound follows. In contrast, this is not applicable to SREDA-Boost which has relaxed and accuracy-independent initialization and stepsize. Hence, we develop a novel analysis framework to bound the accumulative errors \( \sum_{t=0}^{T-1} \delta_t \) and \( \sum_{t=0}^{T-1} \Delta_t \) over the entire algorithm execution, and then decouple these two inter-related stochastic error processes and establish their relationships with the accumulative gradient estimators \( \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2] \). We next provide a sketch of the proof for Theorem 1 to further illustrate our ideas.

**Proof Sketch of Theorem 1.** The proof of Theorem 1 consists of the following five steps.

**Step 1:** We establish the induction relationships for the tracking error and gradient estimation error upon one outer-loop update for SREDA-Boost. Namely, we develop the relationship between \( \delta_t \) and \( \delta_{t-1} \) as well as that between \( \Delta_t \) and \( \Delta_{t-1} \).
Step 2: We provide the bounds on the inter-related accumulative errors $\sum_{t=0}^{T-1} \Delta_t$ and $\sum_{t=0}^{T-1} \delta_t$ over the entire execution of the algorithm.

Step 3: We decouple the bounds on $\sum_{t=0}^{T-1} \Delta_t$ and $\sum_{t=0}^{T-1} \delta_t$ in Step 2 from each other, and establish their separate relationships with the accumulative gradient estimators $\sum_{i=0}^{T-1} E[\|v_i\|^2]$.

Step 4: We bound $\sum_{i=0}^{T-1} E[\|v_i\|^2]$, and further cancel out the impact of $\sum_{t=0}^{T-1} \Delta_t$ and $\sum_{t=0}^{T-1} \delta_t$ by exploiting Step 3.

Step 5: We establish the convergence bound on $E[\|\nabla \Phi(\hat{x})\|_2]$ based on the bounds on its estimators $\sum_{i=0}^{T-1} E[\|v_i\|^2]$ and the two error bounds $\sum_{t=0}^{T-1} \Delta_t$, and $\sum_{t=0}^{T-1} \delta_t$. \hfill $\square$

The analysis of SREDA-Boost for min-max problems is inspired by that for SpiderBoost in Wang et al. (2019) for minimization problems, but the analysis here is much more challenging due to the complicated mathematical nature of min-max optimization. Specifically, SpiderBoost needs to handle only one type of the gradient estimation error, whereas SREDA-Boost requires to handle two strongly coupled errors in min-max problems. Hence, the novelty for analyzing SREDA-Boost mainly lies in bounding and decoupling the two errors in order to characterize their impact on the convergence bound.

4 ZO-SREDA-Boost: Zeroth-Order Variance Reduction Algorithm

In this section, we study the min-max problem when the gradient information is not available, but only function values can be used for designing algorithms. Based on the first-order SREDA-Boost algorithm, we first propose the zeroth-order variance reduced algorithm called ZO-SREDA-Boost and then provide the convergence analysis for such an algorithm.

4.1 ZO-SREDA-Boost Algorithm

The ZO-SREDA-Boost algorithm (see Algorithm 4 in Appendix C.1) shares the same update scheme as SREDA-Boost, but makes the following changes.

(1) In line 3 of SREDA-Boost, instead of using iSARAH, ZO-SREDA-Boost utilizes a zeroth-order algorithm and satisfies $\zeta$

\begin{equation}
\end{equation}

Apply ZO-SREDA-Boost in Algorithm 4 to solve the online case of the problem eq. (1).

4.2 Convergence Analysis of ZO-SREDA-Boost

The following theorem provides the query complexity of ZO-SREDA-Boost for finding a first-order stationary point of $\Phi(\cdot)$ with $\epsilon$ accuracy.

Theorem 2. Apply ZO-SREDA-Boost in Algorithm 4 to solve the online case of the problem eq. (1). Suppose Assumptions I-4 hold. Let $\zeta = \kappa^{-1}$, $\alpha = \mathcal{O}(\kappa^{-1} \ell^{-1})$, $\beta = \mathcal{O}(\ell^{-1})$, $q = \mathcal{O}(\epsilon^{-1})$, $m = \mathcal{O}(\kappa)$, $S_1 = \mathcal{O}(\kappa^2 \epsilon^{-2})$, $S_{2,x} = \mathcal{O}(d_1 \kappa \epsilon^{-1})$, $S_{2,y} = \mathcal{O}(d_2 \kappa \epsilon^{-1})$, $\delta = \mathcal{O}(d_1^2 + d_2^2)^{0.5} \kappa^{-1} \ell^{-1}$, $\mu_1 = \mathcal{O}(d_1^{1.5} \kappa^{-2.5} \ell^{-1} \epsilon)$ and $\mu_2 = \mathcal{O}(d_2^{1.5} \kappa^{-2.5} \ell^{-1})$. Then for $T$ to be at least at the order of $\mathcal{O}(\kappa \epsilon^{-2})$, Algorithm 4 outputs $\hat{x}$ that satisfies

$$E[\|\nabla \Phi(\hat{x})\|_2] \leq \epsilon$$

with the overall function query complexity $\mathcal{O}(d_1 + d_2) \kappa^3 \epsilon^{-3}$. 
Furthermore, ZO-SREDA-Boost is also applicable to the finite-sum case of the problem eq. (1), by replacing the large batch sample \( S_1 \) used in line 6 of Algorithm 4 with the full set of samples.

**Corollary 2.** Apply ZO-SREDA-Boost described above to solve the finite-sum case of the problem eq. (1). Suppose Assumptions 1-4 hold. Under appropriate parameter settings given in Appendix C.6, the function query complexity to attain an \( \epsilon \)-stationary point is \( O((d_1 + d_2)(\sqrt{n\kappa^2} \epsilon^{-2} + n) + d_2(\kappa^2 + \kappa n)\log(\kappa)) \) for \( n \geq \kappa^2 \), and \( O((d_1 + d_2)(\kappa^2 + \kappa n)\epsilon^{-2}) \) for \( n \leq \kappa^2 \).

Theorem 2 and Corollary 2 provide the first convergence analysis and the query complexity for the variance-reduced zeroth-order algorithms for min-max optimization. These two results indicate that the query complexity of ZO-SREDA-Boost matches the optimal dependence on \( \epsilon \) of the first-order algorithm SREDA-Boost in Theorem 1 and Corollary 1. The dependence on \( d_1 \) and \( d_2 \) typically arises in zeroth-order algorithms due to the estimation of gradients with dimensions \( d_1 \) and \( d_2 \). Furthermore, in the online case, ZO-SREDA-Boost outperforms the best known query complexity dependence on \( \epsilon \) among the existing zeroth-order algorithms by a factor of \( O(1/\epsilon) \). Including the conditional number \( \kappa \) into consideration, SREDA-Boost outperforms the best known query complexity achieved by ZO-SGDMA in the case with \( \epsilon \leq \kappa^{-1} \) (see Table 1). Furthermore, Corollary 2 provides the first query complexity for the finite-sum zeroth-order min-max problems.

As a by-product, our analysis of ZO-SREDA-Boost also yields the convergence rate and the query complexity (see Lemma 21) for ZO-iSARAH for the conventional minimization problem, which provides the first complexity result for the zeroth-order recursive variance reduced algorithm SARAH/SPIDER for strongly convex optimization (see Appendix C.4 for detail).

## 5 Experiments

Our experiments focus on two types of comparisons. First, we compare SREDA-Boost with SREDA to demonstrate the practical advantage of SREDA-Boost. Second, we compare our proposed zeroth-order variance reduction algorithm ZO-SREDA-Boost with the other existing zeroth-order stochastic algorithms and demonstrate the superior performance of ZO-SREDA-Boost.

Our experiments solve a distributionally robust optimization problem, which is commonly used for studying min-max optimization Lin et al. (2019); Rafique et al. (2018). We conduct the experiments on three datasets from LIBSVM Chang and Lin (2011). The details of the problem and the datasets are provided in Appendix A.

**Comparison between SREDA-Boost and SREDA:** We set \( \epsilon = 0.001 \) for both algorithms. For SREDA, we set \( \alpha = \min\{\epsilon / \|v_t\|_2, 0.005\} \) as specified by the algorithm, and for SREDA-Boost, we set \( \alpha = 0.005 \) as the algorithm allows. It can be seen in Figure 1 that SREDA-Boost enjoys a much faster convergence speed than SREDA due to the allowance of a large stepsize.

**Comparison among zeroth-order Algorithms:** We compare the performance of our proposed ZO-SREDA-Boost with that of two existing stochastic algorithms ZO-SGDA Wang et al. (2020) and ZO-SGDMSA Wang et al. (2020) designed for nonconvex-strongly-concave min-max problems. For ZO-SGDA and ZO-SGDMSA, as suggested by the theorem, we set the mini-batch size \( B = Cd_1/\epsilon^2 \) and \( B = Cd_2/\epsilon^2 \) for updating.
the variables $x$ and $y$, respectively. For ZO-SREDA-Boost, based on our theory, we set the mini-batch size $B = Cd_1/\epsilon$ and $B = Cd_2/\epsilon$ for updating the variables $x$ and $y$, and set $S_1 = n$ for the large batch, where $n$ is the number of data samples in the dataset. We set $C = 0.1$ and $\epsilon = 0.1$ for all algorithms. We further set the stepsize $\eta = 0.01$ for ZO-SREDA-Boost and ZO-SGDMSA. Since ZO-SGDA is a two time-scale algorithm, we set $\eta = 0.01$ as the stepsize for the fast time scale, and $\eta/\kappa^3$ as the stepsize for slow time scale (based on the theory) where $\kappa^3 = 10$. It can be seen in Figure 2 that ZO-SREDA-Boost substantially outperforms the other two algorithms in terms of the function query complexity (i.e., the running time).

![Figure 2: Comparison of function query complexity among three algorithms.](image)

6 Conclusion

In this work, we have proposed enhanced variance reduction algorithms, which we call SREDA-Boost and ZO-SREDA-Boost, for solving nonconvex-strongly-concave min-max problems. In specific, SREDA-Boost requires less initialization effort and allows a large stepsize. Moreover, The complexity of SREDA-Boost and ZO-SREDA-Boost achieves the best complexity dependence on the targeted accuracy among their same classes of algorithms. We have also developed a novel analysis framework to characterize the convergence and computational complexity for the variance reduction algorithms. We expect such a framework will be useful for studying various other stochastic min-max problems such as proximal, momentum, and manifold optimization.

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Supplementary Materials

A Specifications of Experiments

The distributionally robust optimization problem is formulated as follows:

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} \sum_{i=1}^{n} y_i f_i(x) - r(y),$$

where $\mathcal{X} = \{x \in \mathbb{R}^d\}$, $\mathcal{Y} = \{y \in \mathbb{R}^n | \sum_{i=1}^{n} y_i = 1, y_i \geq 0, i = 1, \ldots, n\}$, $r(y) = 10 \sum_{i=1}^{n} (y_i - 1/n)^2$, $f_i(x) = \phi(l(x))$ where $\phi(\theta) = 2 \log(1 + \theta^2)$, $l(x; s, z) = \log(1 + \exp(-zx^\top s))$, and $(s, z)$ are the feature and label pair of a data sample. It can be seen that the problem is a min-max problem with $d_1 = d$ and $d_2 = n$.

Since the distributionally robust optimization aims at an unbalance dataset, we pick the samples from the original dataset and set the ratio between the number of negative labeled samples and the number of positive labeled samples to be 1:4. Since the maximization of $y$ is a constrained optimization problem, we incorporate a projection step after updates of $y$ for all algorithms.

The details of the datasets used for the comparison between SREDA and SREDA-Boost are listed in Table 2.

| Datasets   | # of samples | # of features | # Pos: # Neg |
|-----------|--------------|---------------|--------------|
| mushrooms | 2000         | 112           | 1:4          |
| w8a       | 5000         | 300           | 1:4          |
| a9a       | 8000         | 123           | 1:4          |

The details of the datasets used for the comparison among zeroth-order algorithms are listed in Table 3.

| Datasets   | # of samples | # of features | # Pos: # Neg |
|-----------|--------------|---------------|--------------|
| mushrooms | 200          | 112           | 1:4          |
| w8a       | 100          | 300           | 1:4          |
| a9a       | 150          | 123           | 1:4          |

B Convergence Analysis of SREDA-Boost

B.1 Preliminaries

We first provide useful inequalities in convex optimization Nesterov (2013); Polyak (1963) and auxiliary lemmas from Fang et al. (2018); Luo et al. (2020).

Lemma 1 (Nesterov (2013),Polyak (1963)). Suppose $h(\cdot)$ is convex and has $\ell$-Lipschitz gradient. Then, we have

$$\langle \nabla h(w) - \nabla h(w'), w - w' \rangle \geq \frac{1}{\ell} \| \nabla h(w) - \nabla h(w') \|^2_2. \quad (3)$$

Lemma 2 (Nesterov (2013),Polyak (1963)). Suppose $h(\cdot)$ is $\mu$-strongly convex and has $\ell$-Lipschitz gradient. Let $w^*$ be the minimizer of $h$. Then for any $w$ and $w'$, we have the following inequalities hold.

$$\langle \nabla h(w) - \nabla h(w'), w - w' \rangle \geq \frac{\mu \ell}{\mu + \ell} \| w - w' \|^2_2 + \frac{1}{\mu + \ell} \| \nabla h(w) - \nabla h(w') \|^2_2. \quad (4)$$
\[ \|\nabla h(w) - \nabla h(w')\|_2 \geq \mu \|w - w'\|_2, \] (5)
\[ 2\mu (h(w) - h(w')) \leq \|\nabla h(w)\|_2^2. \] (6)

The following structural lemma developed in Lin et al. (2019) provides further information about \( \Phi \) for nonconvex-strongly-concave min-max optimization.

**Lemma 3** (Lin et al. (2019), Lemma 3.3). Under Assumption 2 and Assumption 3, the function \( \Phi(\cdot) = \max_{y \in \mathbb{R}^{d_2}} f(\cdot, y) \) is \((k + 1)\ell\)-gradient Lipschitz and \( \nabla \Phi(x) = \nabla_x f(x, y(x)) \) is \( \kappa \)-Lipschitz, where \( y(x) = \text{argmin}_{y \in \mathbb{R}^{d_2}} f(\cdot, y) \).

We let \( L \triangleq (1 + \kappa)\ell \) denote the Lipschitz constant of \( \nabla \Phi(x) \).

**Lemma 4** (Fang et al. (2018), Lemma 2). Suppose Assumption 4 hold. For any \((x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}\) and sample batch \( \{\xi_1, \cdots, \xi_S\} \), let \( v = \frac{1}{S} \sum_{i=1}^{S} \nabla_x F(x, y, \xi_i) \) and \( u = \frac{1}{S} \sum_{i=1}^{S} \nabla_y F(x, y, \xi_i) \). We have
\[
\mathbb{E}[\|v - \nabla_x f(x, y)\|_2^2] + \mathbb{E}[\|u - \nabla_y f(x, y)\|_2^2] \leq \frac{\sigma^2}{S}.
\]

**Lemma 5** (Fang et al. (2018), Lemma 1). Let \( \mathcal{V}_t \) be an estimator of \( \mathcal{B}(z_t) \) as
\[
\mathcal{V}_t = \mathcal{B}_{S_1}(z_t) - \mathcal{B}_{S_2}(z_{t-1}) + \mathcal{V}_{t-1},
\]
where \( \mathcal{B}_{S_t} = \frac{1}{S_t} \sum_{i \in S_t} \mathcal{B}_i \) satisfies
\[
\mathbb{E}[\mathcal{B}_t(z_t) - \mathcal{B}_t(z_{t-1})|z_0, \cdots, z_{t-1}] = \mathbb{E}[\mathcal{V}_t - \mathcal{V}_{t-1}|z_0, \cdots, z_{t-1}].
\]
For all \( k = 1, \cdots, K \), we have
\[
\mathbb{E}[\|\mathcal{V}_t - \mathcal{V}_{t-1} - (\mathcal{B}_{S_1}(z_t) - \mathcal{B}_{S_1}(z_{t-1}))\|_2^2] \leq \frac{1}{S_t} \mathbb{E}[\|\mathcal{B}_t(z_t) - \mathcal{B}_t(z_{t-1})\|_2^2 |z_0, \cdots, z_{t-1}],
\]
and
\[
\mathbb{E}[\|\mathcal{V}_t - \mathcal{B}(z_t)|z_0, \cdots, z_{t-1}\|_2^2] \leq \mathbb{E}[\|\mathcal{V}_{t-1} - \mathcal{B}(z_{t-1})\|_2^2] + \frac{1}{S_t} \mathbb{E}[\|\mathcal{B}_t(z_t) - \mathcal{B}_t(z_{t-1})\|_2^2 |z_0, \cdots, z_{t-1}].
\]
Furthermore, if \( \mathcal{B}_t \) is \( L \)-Lipschitz continuous in expectation, we have
\[
\mathbb{E}[\|\mathcal{V}_t - \mathcal{B}(z_t)|z_0, \cdots, z_{t-1}\|_2^2] \leq \mathbb{E}[\|\mathcal{V}_{t-1} - \mathcal{B}(z_{t-1})\|_2^2] + \frac{L^2}{S_t} \mathbb{E}[\|z_t - z_{t-1}\|_2^2 |z_0, \cdots, z_{t-1}].
\]

**B.2 Initialization by iSARAH**

We present the detailed procedure of iSARAH in Algorithm 3, which is used to initialize \( y_0 \) in SREDA-Boost (line 3 of Algorithm 1). We consider the following convex optimization problem:
\[
\min_{w \in \mathbb{R}^d} \mathbb{E}_\xi[P(w; \xi)], \quad \text{(7)}
\]
where \( P \) is average \( \ell \)-gradient Lipschitz and convex, \( p \) is \( \mu \)-strongly convex, and \( \xi \) is a random vector.

**Algorithm 3** iSARAH

1: **Input:** \( \tilde{w}_0 \), learning rate \( \gamma > 0 \), inner loop size \( I \), batch size \( B_1 \) and \( B_2 \)
2: **for** \( t = 1, 2, \ldots, T \) **do**
3: \( w_0 = \tilde{w}_{t-1} \)
4: **draw** \( B_1 \) samples \( \{\xi_1, \cdots, \xi_{B_1}\} \)
5: \( v_0 = \frac{1}{B_1} \sum_{i=1}^{B_1} \nabla P(w_0, \xi_i) \)
6: \( w_1 = w_0 + \gamma v_0 \)
7: **for** \( k = 1, 2, \ldots, I - 1 \) **do**
8: **draw** minibatch sample \( M = \{\xi_1, \cdots, \xi_{B_1}\} \)
9: \( v_k = v_{k-1} + \frac{1}{B_2} \sum_{i=1}^{B_2} \nabla P(w_k, \xi_i) - \frac{1}{B_2} \sum_{i=1}^{B_2} \nabla P(w_{k-1}, \xi_i) \)
10: \( w_{k+1} = w_k + \gamma v_k \)
11: **end for**
12: \( \tilde{w}_t \) chosen uniformly at random from \( \{w_k\}_{k=0}^{I} \)
13: **end for**
We have the following convergence result by using iSARAH to solve the problem in eq. (7).

**Lemma 6 (Nguyen et al. (2018), Corollary 4).** Consider Algorithm 3. Set \( \gamma = \Theta(\ell^{-1}) \), \( B_1 = \Theta(\epsilon^{-1}) \), \( B_2 = 1 \), \( I = \Theta(\kappa) \) and \( T = \Theta(\log \frac{1}{\epsilon}) \). We have

\[
\mathbb{E}[\|\nabla p(\tilde{w}_T)\|_2^2] \leq \epsilon,
\]

with the total sample complexity given by \( \mathcal{O}((\kappa + \frac{1}{\epsilon}) \log \frac{1}{\epsilon}) \).

Moreover, Algorithm 3 can be slightly modified to solve the minimization problem in eq. (7) in the finite-sum setting, in which

\[
p(w) = \frac{1}{n} \sum_{i=1}^n P(w, \xi_i).
\]  (8)

By replacing the large batch sample \( S_1 \) used line 4 in Algorithm 3 with the full set of samples, we obtain the so-called SARAH algorithm Nguyen et al. (2017a). The following lemma characterizes the convergence result of SARAH to solve eq. (8).

**Lemma 7 (Nguyen et al. (2018), Corollary 2).** Consider Algorithm 3. Set \( \gamma = \Theta(\ell^{-1}) \), \( B_2 = 1 \), \( I = \Theta(\kappa) \) and \( T = \Theta(\log \frac{1}{\epsilon}) \). We have

\[
\mathbb{E}[\|\nabla p(\tilde{w}_T)\|_2^2] \leq \epsilon,
\]

with the total sample complexity given by \( \mathcal{O}((\kappa + n) \log \frac{1}{\epsilon}) \).

**B.3 Proof of Theorem 1**

Throughout the paper, let \( n_t = \lceil t/q \rceil \) such that \( (n_t - 1)q \leq t \leq n_t q - 1 \). Without loss of generality, we assume \( \epsilon \leq 1 \) since \( \epsilon \) is typically very small. Define \( \Delta_t = \mathbb{E}[\|\nabla_x f(x_t, y_t) - v_t\|_2^2] + \mathbb{E}[\|\nabla_y f(x_t, y_t) - u_t\|_2^2], \Delta_{t,k} = \mathbb{E}[\|\nabla_x f(\tilde{x}_{t,k}, \tilde{y}_{t,k}) - \tilde{v}_{t,k}\|_2^2] + \mathbb{E}[\|\nabla_y f(\tilde{x}_{t,k}, \tilde{y}_{t,k}) - \tilde{u}_{t,k}\|_2^2], \) and \( \delta_t = \mathbb{E}[\|\nabla_y f(x_t, y_t)\|_2^2]. \)

We start our proof by a few supporting lemmas. The following lemma is a slightly modified version of Lemma 4 of Luo et al. (2020). The steps in the proof of Lemma 4 of Luo et al. (2020) does not yield their desired result.

**Lemma 8 (Modified version of Lemma 4 of Luo et al. (2020)).** Consider Algorithm 2. For all \( 1 \leq t \leq m, \beta \leq \frac{1}{2\ell} \) and \( S_2 \geq 2(\kappa + 1)\ell\beta \). We have

\[
\mathbb{E}[\|\tilde{u}_{t,k}\|_2^2 | \mathcal{F}_{t,k}] \leq a \|\tilde{u}_{t,k-1}\|_2^2
\]

where \( a = 1 - \frac{\mu\ell\beta}{\nu+\ell} \).

Our Lemma 8 has the conditional number \( a = 1 - \frac{\mu\ell\beta}{\nu+\ell} \), which is slightly larger than \( 1 - \frac{2\mu\ell\beta}{\nu+\ell} \) given in Lemma 4 of Luo et al. (2020). The convergence analysis of SREDA in Luo et al. (2020) still holds but with \( a = 1 - \frac{\mu\ell\beta}{\nu+\ell} \).

**Proof.** The update of Algorithm 2 implies that

\[
\mathbb{E}[\|\tilde{u}_{t,k}\|_2^2 | \mathcal{F}_{t,k}]
= \|\tilde{u}_{t,k-1}\|_2^2 + 2\mathbb{E}[\langle \tilde{u}_{t,k-1}, \nabla_y G(\tilde{y}_{t,k}) - \nabla_y G(\tilde{y}_{t,k-1}) \rangle | \mathcal{F}_{t,k}] + \mathbb{E}[\|\nabla_y G(\tilde{y}_{t,k}) - \nabla_y G(\tilde{y}_{t,k-1})\|_2^2 | \mathcal{F}_{t,k}]
= \|\tilde{u}_{t,k-1}\|_2^2 + \frac{2}{\beta} \mathbb{E}[\langle \tilde{y}_{t,k} - \tilde{y}_{t,k-1}, \nabla_y g(\tilde{y}_{t,k}) - \nabla_y g(\tilde{y}_{t,k-1}) \rangle | \mathcal{F}_{t,k}] + \mathbb{E}[\|\nabla_y G(\tilde{y}_{t,k}) - \nabla_y G(\tilde{y}_{t,k-1})\|_2^2 | \mathcal{F}_{t,k}]
\leq \|\tilde{u}_{t,k-1}\|_2^2 - \frac{2}{\beta} \left( \frac{\mu}{\mu + \epsilon} \|\tilde{y}_{t,k} - \tilde{y}_{t,k-1}\|_2^2 + \frac{1}{\mu + \ell} \|\nabla_y g(\tilde{y}_{t,k}) - \nabla_y g(\tilde{y}_{t,k-1})\|_2^2 \right)
+ \mathbb{E}[\|\nabla_y G(\tilde{y}_{t,k}) - \nabla_y G(\tilde{y}_{t,k-1})\|_2^2 | \mathcal{F}_{t,k}]
\]
We then provide the following lemma to characterize the induction relationships for different estimation error terms, which can be obtained directly from the proof of (Luo et al., 2020, Lemma 5 in Section C).

**Lemma 9.** Suppose Assumption 1-4 hold. Let $\beta \leq \frac{1}{T}$. The following hold

\[
\Delta_t \leq \tilde{\Delta}_{t-1,0} + \frac{\ell^2 \beta^2}{S_2(1-a)} \mathbb{E}[\|\tilde{u}_{t-1,0}\|^2],
\]  

\[
\delta_t \leq \frac{2}{\mu \beta (m+1)} \left( \mathbb{E}[\|\nabla_y f(x_t, y_{t-1}) - \nabla_y f(x_{t-1}, y_{t-1})\|^2] + \delta_{t-1} \right) + \frac{\ell \beta}{2 - \ell \beta} \mathbb{E}[\|\tilde{u}_{t-1,0}\|^2] + \tilde{\Delta}_{t-1,0},
\]

\[
\tilde{\Delta}_{t,0} \leq \Delta_t + \frac{\ell^2}{S_2} \mathbb{E}[\|x_{t+1} - x_t\|^2],
\]

\[
\mathbb{E}[\|\tilde{u}_{t-1,0}\|^2] \leq 3 \left( \tilde{\Delta}_{t-1,0} + \mathbb{E}[\|\nabla_y f(x_t, y_{t-1}) - \nabla_y f(x_{t-1}, y_{t-1})\|^2] + \delta_{t-1} \right).
\]

**Proof.** eq. (10) can be obtained from the second inequality of eq. (23) in Luo et al. (2020) together with Lemma 8 as a correct version of Lemma 4 in Luo et al. (2020). eq. (11) can be obtained from the second inequality in the derivation of upper bound of "$\delta_{k+1}$" in the page 22 of Luo et al. (2020). eq. (12) can be obtained from the second equality of eq. (22) in Luo et al. (2020). eq. (13) can be obtained from the first inequality of eq. (24) in Luo et al. (2020).

We then provide the following lemma to characterize the induction relationships for $\Delta_t$ and $\delta_t$.

**Lemma 10.** Suppose Assumption 1-4 hold. Let $\beta \leq \frac{1}{T}$. The following hold:

\[
\Delta_t \leq \left( 1 + \frac{3\ell^2 \beta^2}{S_2(1-a)} \right) \Delta_{t-1} + \frac{\ell^2 \beta^2}{S_2} \left( 1 + \frac{6\ell^2 \beta^2}{1-a} \right) \mathbb{E}[\|v_{t-1}\|^2] + \frac{3\ell^2 \beta^2}{S_2(1-a)} \delta_{t-1},
\]

\[
\delta_t \leq \left( \frac{2}{\mu \beta (m+1)} + \frac{3\ell \beta}{2 - \ell \beta} \right) \delta_{t-1} + \left( \frac{2\ell^2 \alpha^2}{\mu \beta (m+1)} + \frac{6\ell^2 \beta^2}{2 - \ell \beta} + \ell^2 \alpha^2 \right) \mathbb{E}[\|v_{t-1}\|^2] + \frac{2 + 2\ell \beta}{2 - \ell \beta} \Delta_{t-1}.
\]

**Proof.** To prove eq. (14), we proceed as follows:

\[
\Delta_t \overset{(i)}{\leq} \tilde{\Delta}_{t-1,0} + \frac{\ell^2 \beta^2}{S_2(1-a)} \mathbb{E}[\|\tilde{u}_{t-1,0}\|^2]
\]

\[
\overset{(ii)}{\leq} \tilde{\Delta}_{t-1,0} + \frac{\ell^2 \beta^2}{S_2(1-a)} \left( \tilde{\Delta}_{t-1,0} + \mathbb{E}[\|\nabla_y f(x_t, y_{t-1}) - \nabla_y f(x_{t-1}, y_{t-1})\|^2] + \delta_{t-1} \right)
\]

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Then, Algorithm 1 outputs
\[ \hat{x} \]
with at most
\[ O(\kappa^3\epsilon^{-3}) \]
where (i) follows from eq. (10), (ii) follows from eq. (13), (iii) follows from eq. (15), (iv) follows from eq. (12), and (v) follows from the fact that \( S_2 \geq 1 \).

To prove eq. (15), we proceed as follows:

\[
\delta_t \leq \frac{2}{\mu \beta(m + 1)} \left( \ell^2 \alpha^2 E[\|v_t - 1\|_2^2] + \delta_{t-1} \right) + \frac{\ell^2 \beta}{2 - \ell \beta} E[\|\tilde{u}_{t-1,0}\|_2^2] + \Delta_{t-1,0}
\]

\[
\leq \frac{2}{\mu \beta(m + 1)} \left( \ell^2 \alpha^2 E[\|v_t - 1\|_2^2] + \delta_{t-1} \right) + \frac{3 \ell \beta}{2 - \ell \beta} \left( \Delta_{t-1,0} + \ell^2 \alpha^2 E[\|v_t - 1\|_2^2] + \delta_{t-1} \right) + \Delta_{t-1,0}
\]

\[
= \left( \frac{2}{\mu \beta(m + 1)} + \frac{3 \ell \beta}{2 - \ell \beta} \right) \delta_{t-1} + \left( 1 + \frac{3 \ell \beta}{2 - \ell \beta} \right) \Delta_{t-1,0} + \left( \frac{2 \ell^2 \alpha^2}{\mu \beta(m + 1)} + \frac{3 \ell^3 \beta \alpha^2}{2 - \ell \beta} \right) E[\|v_t - 1\|_2^2]
\]

\[
\leq \frac{2}{\mu \beta(m + 1)} + \frac{3 \ell \beta}{2 - \ell \beta} \Delta_{t-1} + \left( \frac{2 \ell^2 \alpha^2}{\mu \beta(m + 1)} + \frac{3 \ell^3 \beta \alpha^2}{2 - \ell \beta} \right) E[\|v_t - 1\|_2^2]
\]

where (i) follows from eq. (11) and the fact that \( \|\nabla_y f(x_t, y_{t-1}) - \nabla_y f(x_{t-1}, y_{t-1})\|_2 \leq \ell \alpha \|v_t - 1\|_2^2 \), (ii) follows from eq. (13), (iii) follows from eq. (12), and (iv) follows from the fact that \( S_2 \geq 1 \).

We restate Theorem 1 as follows to include the specifics of the parameters.

**Theorem 3** (Restatement of Theorem 1). Let Assumption 1-4 hold and apply SREDA-Boost in Algorithm 1 to solve the problem in eq. (1) with the following parameter choices:

\[
\zeta = \frac{1}{\kappa}, \quad \alpha = \frac{1}{10(\kappa + 1)\ell}, \quad \beta = \frac{2}{13\ell}, \quad q = \frac{2}{13(1 + \kappa)\epsilon}, \quad m = 52\kappa - 1,
\]

\[
S_1 = \frac{9366\sigma^2\kappa^2}{\epsilon^2}, \quad S_2 = \frac{\kappa}{\epsilon}, \quad T = \max \left\{ \frac{3345\kappa}{\epsilon^2}, 6600(1 + \kappa)\ell \frac{(\Phi(x_0) - \Phi^*)}{\epsilon^2} \right\}.
\]

Then, Algorithm 1 outputs \( \hat{x} \) that satisfies

\[
E[\|\nabla \Phi(\hat{x})\|_2^2] \leq \epsilon
\]

with at most \( \mathcal{O}(\kappa^3\epsilon^{-3}) \) stochastic gradient evaluations.

**Proof of Theorem 1/Theorem 3.** By Lemma 3, the objective function \( \Phi \) is \( L \)-smooth, which implies that

\[
\Phi(x_{t+1}) \leq \Phi(x_t) - \alpha(\nabla_x \Phi(x_t), v_t) + \frac{L\alpha^2}{2} \|v_t\|_2^2
\]

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one outer-loop update for SREDA-Boost. Namely, we develop the relationship between Rearranging eq. (18) and summing over \(i\) over the entire execution of the algorithm.

\[
\begin{align*}
\Phi(x_t) &+ \alpha \left\| \nabla_x \Phi(x_t) - v_t \right\|^2
\leq \Phi(x_t) + \frac{\alpha}{2} \| v_t \|^2 + \frac{\alpha}{2} \| \nabla_v \Phi(x_t) - v_t \|^2 - \left( \frac{\alpha}{2} - \frac{L^2 \alpha^2}{2} \right) \| v_t \|^2,
\end{align*}
\]

(17)

where (i) follows from the fact that \((-1) \langle \nabla_x \Phi(x_t) - v_t, v_t \rangle \leq \frac{1}{2} \left( \left\| \nabla_x \Phi(x_t) - v_t \right\|^2 + \frac{1}{2} \| v_t \|^2 \right)\), and (ii) follows from the fact that

\[
\left\| \nabla_x \Phi(x_t) - \nabla_x f(x_t, y_t) \right\|^2 = \left\| \nabla_x f(x_t, y^*(x_t)) - \nabla_x f(x_t, y_t) \right\|^2 \leq \ell^2 \left\| y^*(x_t) - y_t \right\|^2.
\]

Taking expectation on both sides of eq. (17) yields

\[
\begin{align*}
\mathbb{E}[\Phi(x_{t+1})] \leq & \mathbb{E}[\Phi(x_t)] + \alpha \kappa^2 \mathbb{E}[\| \nabla_y f(x_t, y_t) \|^2] + \alpha \mathbb{E}[\| \nabla_x f(x_t, y_t) - v_t \|^2] - \left( \frac{\alpha}{2} - \frac{L^2 \alpha^2}{2} \right) \mathbb{E}[\| v_t \|^2]
\end{align*}
\]

Rearranging eq. (18) and summing over \(t = \{0, \cdots, T - 1\} \) yield

\[
\left( \frac{\alpha}{2} - \frac{L^2 \alpha^2}{2} \right) \sum_{t=0}^{T-1} \mathbb{E}[\| v_t \|^2] \leq \Phi(x_0) - \mathbb{E}[\Phi(x_T)] + \alpha \kappa^2 \sum_{t=0}^{T-1} \delta_t + \alpha \Delta_t.
\]

(19)

Then, we proceed the proof in the following five steps.

**Step 1.** We establish the induction relationships for the tracking error and gradient estimation error upon one outer-loop update for SREDA-Boost. Namely, we develop the relationship between \(\delta_t\) and \(\delta_{t-1}\), as well as that between \(\Delta_t\) and \(\Delta_{t-1}\), which are captured in Lemma 10.

**Step 2.** Based on Step 1, we provide the bounds on the inter-related accumulative errors \(\sum_{t=0}^{T-1} \Delta_t\) and \(\sum_{t=0}^{T-1} \delta_t\) over the entire execution of the algorithm.

We first consider \(\sum_{t=0}^{T-1} \Delta_t\). For any \((n_T - 1)q \leq t < p < T - 1\), we apply eq. (14) recursively to obtain the following inequality

\[
\begin{align*}
\Delta_t &\leq \left( 1 + \frac{3 \ell^2 \beta^2}{S_2(1 - a)} \right) \sum_{t=0}^{p-1} \left( 1 + \frac{3 \ell^2 \beta^2}{S_2(1 - a)} \right) \sum_{t=0}^{t' - 1} \left( 1 + \frac{6 \ell^2 \beta^2}{S_2(1 - a)} \right) \delta_{t'}
\leq \left( 1 + \frac{3 \ell^2 \beta^2}{S_2(1 - a)} \right) \sum_{t=0}^{p-1} \left( 1 + \frac{6 \ell^2 \beta^2}{S_2(1 - a)} \right) \delta_{t'}
\leq \left( 1 + \frac{3 \ell^2 \beta^2}{S_2(1 - a)} \right) \sum_{t=0}^{p-1} \left( 1 + \frac{6 \ell^2 \beta^2}{S_2(1 - a)} \right) \delta_{t'}
\leq 2 \Delta_t + \frac{2 \ell^2 \beta^2}{S_2(1 - a)} \sum_{t=0}^{p-1} \mathbb{E}[\| v_t \|^2] + \frac{6 \ell^2 \beta^2}{S_2(1 - a)} \sum_{t=0}^{p-1} \delta_t,
\end{align*}
\]

(20)

where (i) follows from the fact that

\[
\left( 1 + \frac{3 \ell^2 \beta^2}{S_2(1 - a)} \right)^{p-t'} \leq \left( 1 + \frac{3 \ell^2 \beta^2}{S_2(1 - a)} \right)^{q} \leq 1 + \frac{3 \ell^2 \beta^2}{S_2(1 - a)} \sum_{t=0}^{p-1} \mathbb{E}[\| v_t \|^2] + \frac{6 \ell^2 \beta^2}{S_2(1 - a)} \sum_{t=0}^{p-1} \delta_t.
\]

(20)
Taking summation of eq. (23) over where \(\sum\) follows from Bernoulli’s inequality Li and Yeh (2013)

\[
(1 + c)^r \leq 1 + \frac{rc}{1 - (r - 1)c} \quad \text{for} \quad c \in \left[-1, \frac{1}{r - 1}\right), \quad r > 1, \quad (21)
\]

(iii) follows from the fact that \(q = (1 - a)S_2\) and (iv) follows from the fact that \(\beta = \frac{2}{13}\).

Letting \(t' = (n_T - 1)q\) and taking summation of eq. (20) over \(t = \{(n_T - 1)q, \ldots, T - 1\}\) yield

\[
\Delta_t \leq 2(T - (n_T - 1)q)\Delta_{(n_T - 1)q} + \frac{2\alpha^2\ell^2}{S_2} \left(1 + \frac{6\ell^2\beta^2}{1 - a}\right)\sum_{t=(n_T-1)q}^{T-1} \sum_{t=(n_T-1)q}^{t-1} \mathbb{E}[\|v_t\|_2^2] \\
+ \frac{6\ell^2\beta^2}{S_2(1 - a)} \sum_{t=(n_T-1)q}^{T-2} \delta_t,
\]

\[
\leq 2(T - (n_T - 1)q)\sigma_T^2 S_1 + \frac{2\alpha^2\ell^2q}{S_2} \left(1 + \frac{6\ell^2\beta^2}{1 - a}\right)\sum_{t=(n_T-1)q}^{T-2} \mathbb{E}[\|v_t\|_2^2] \\
+ \frac{6\ell^2\beta^2q}{S_2(1 - a)} \sum_{t=(n_T-1)q}^{T-2} \delta_t,
\]

where (i) follows from the fact that \(\Delta_{(n_T-n)q} \leq \sigma_T^2 S_1^2\) for all \(n \leq n_T\) (following Lemma 4),

\[
\sum_{t=(n_T-1)q}^{T-1} \sum_{t=(n_T-1)q}^{t-1} \mathbb{E}[\|v_t\|_2^2] \leq q \sum_{t=(n_T-1)q}^{T-2} \mathbb{E}[\|v_t\|_2^2],
\]

and

\[
\sum_{t=(n_T-1)q}^{T-1} \sum_{t=(n_T-1)q}^{t-1} \delta_t \leq q \sum_{t=(n_T-1)q}^{T-2} \delta_t.
\]

Applying steps similar to those in eq. (22) for \(p = \{(n_T - n_t)q, \ldots, (n_T - n_t + 1)q - 1\}\) (where \(n_t\) is an integer that satisfies \(2 \leq n_t < n_T\)) yields

\[
\sum_{t=(n_T-n_t)q}^{(n_T-n_t+1)q-1} \Delta_t \leq 2\sigma_T^2 S_1 + \frac{2\alpha^2\ell^2q}{S_2} \left(1 + \frac{6\ell^2\beta^2}{1 - a}\right)\sum_{t=(n_T-n_t)q}^{(n_T-n_t+1)q-1} \mathbb{E}[\|v_t\|_2^2] \\
+ \frac{6\ell^2\beta^2q}{S_2(1 - a)} \sum_{t=(n_T-n_t)q}^{(n_T-n_t+1)q-1} \delta_t,
\]

Taking summation of eq. (23) over \(n_t = \{2, \ldots, n_T\}\) and combing with eq. (22) yield

\[
\sum_{t=0}^{T-1} \Delta_t \leq \frac{2\sigma_T^2 T}{S_1} + \frac{2\alpha^2\ell^2q}{S_2} \left(1 + \frac{6\ell^2\beta^2}{1 - a}\right) \sum_{t=0}^{T-2} \mathbb{E}[\|v_t\|_2^2] + \frac{6\ell^2\beta^2q}{S_2(1 - a)} \sum_{t=0}^{T-2} \delta_t \\
\leq \frac{2\sigma_T^2 T}{S_1} + 4\alpha^2\ell^2 \sum_{t=0}^{T-2} \mathbb{E}[\|v_t\|_2^2] + \frac{1}{5} \sum_{t=0}^{T-2} \delta_t.
\]

Then we consider the upper bound on \(\sum_{t=0}^{T-1} \delta_t\). Since \(m = \frac{8}{\mu_3} - 1\) and \(\beta = \frac{2}{13}\), eq. (15) implies

\[
\delta_t \leq \frac{1}{2} \delta_{t-1} + \frac{7}{4} \ell^2\alpha^2 \mathbb{E}[\|v_{t-1}\|_2^2] + \frac{5}{4} \Delta_{t-1},
\]

(25)
for all $t \geq 1$. Applying eq. (25) recursively yields

$$\delta_t \leq \frac{1}{2^t} \delta_0 + \frac{7}{4}\ell^2\alpha^2 \sum_{t=0}^{T-1} \frac{1}{2^t} E[\|v_t\|_2^2] + \frac{5}{4} \sum_{t=0}^{T-1} \frac{1}{2^t} \Delta_t. \quad (26)$$

Taking the summation of eq. (26) over $t = \{0, 1, \cdots, T - 1\}$ yields

$$\sum_{t=0}^{T-1} \delta_t \leq \delta_0 + \frac{7}{4}\ell^2\alpha^2 \sum_{t=0}^{T-1} \frac{1}{2^t} E[\|v_t\|_2^2] + \frac{5}{4} \sum_{t=0}^{T-1} \frac{1}{2^t} \Delta_t \leq 2\delta_0 + \frac{7}{2}\ell^2\alpha^2 \sum_{t=0}^{T-2} E[\|v_t\|_2^2] + \frac{5}{2} \sum_{t=0}^{T-2} \Delta_t. \quad (27)$$

**Step 3.** We decouple the bounds on $\sum_{t=0}^{T-1} \Delta_t$ and $\sum_{t=0}^{T-1} \delta_t$ in Step 2 from each other, and establish their separate relationships with the accumulative gradient estimators $\sum_{t=0}^{T-1} E[\|v_t\|_2^2]$.

Substituting eq. (27) into eq. (24) yields

$$\sum_{t=0}^{T-1} \Delta_t \leq \frac{2\sigma^2 T}{S_1} + \frac{2}{5} \delta_0 + 5\alpha^2 \ell^2 \sum_{t=0}^{T-2} E[\|v_t\|_2^2] + \frac{1}{2} \sum_{t=0}^{T-2} \Delta_t,$$

which implies

$$\sum_{t=0}^{T-1} \Delta_t \leq \frac{4\sigma^2 T}{S_1} + \frac{4}{5} \delta_0 + 10\alpha^2 \ell^2 \sum_{t=0}^{T-2} E[\|v_t\|_2^2]. \quad (28)$$

Substituting eq. (28) into eq. (27) yields

$$\sum_{t=0}^{T-1} \delta_t \leq \frac{10\sigma^2 T}{S_1} + 4\delta_0 + 30\alpha^2 \ell^2 \sum_{t=0}^{T-2} E[\|v_t\|_2^2]. \quad (29)$$

**Step 4.** We bound $\sum_{t=0}^{T-1} E[\|v_t\|_2^2]$, and further cancel out the impact of $\sum_{t=0}^{T-1} \Delta_t$ and $\sum_{t=0}^{T-1} \delta_t$ by exploiting Step 3.

Substituting eq. (28) and eq. (29) into eq. (19) yields

$$\left(\frac{\alpha}{2} - \frac{L\alpha^2}{2}\right) \sum_{t=0}^{T-1} E[\|v_t\|_2^2] \leq \Phi(x_0) - E[\Phi(x_T)] + \left(10\alpha^2 + 4\right) \frac{\alpha\sigma^2 T}{S_1} + \left(4\kappa^2 + \frac{4}{5}\right) \alpha\delta_0$$

$$+ 10\alpha^3 \ell^2 \left(3\kappa^2 + 1\right) \sum_{t=0}^{T-2} E[\|v_t\|_2^2] \leq \Phi(x_0) - E[\Phi(x_T)] + \frac{14\alpha\kappa^2 \sigma^2 T}{S_1} + 5\kappa^2 \alpha\delta_0 + 40\alpha^3 L^2 \sum_{t=0}^{T-2} E[\|v_t\|_2^2], \quad (30)$$

where $(i)$ follows from the fact that $L = (1 + \kappa)\ell$ and $\kappa > 1$. Rearranging eq. (30), we have

$$\left(\frac{\alpha}{2} - \frac{L\alpha^2}{2} - 40L^2\alpha^3\right) \sum_{t=0}^{T-1} E[\|v_t\|_2^2] \leq \Phi(x_0) - E[\Phi(x_T)] + \frac{14\alpha\kappa^2 \sigma^2 T}{S_1} + 5\kappa^2 \alpha\delta_0. \quad (31)$$

Since $\alpha = \frac{1}{10L}$, we obtain

$$\frac{\alpha}{2} - \frac{L\alpha^2}{2} - 40L^2\alpha^3 = \frac{1}{200L}. \quad (32)$$
Substituting eq. (32) into eq. (31) and applying Assumption 1 yield

\[
\sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|^2] \leq 200L(\Phi(x_0) - \Phi^*) + \frac{280\kappa^2 \sigma^2 T}{S_1} + 100\kappa^2 \delta_0.
\] (33)

**Step 5.** We establish the convergence bound on \( \mathbb{E}[\|\nabla \Phi(\hat{x})\|^2] \) based on the bounds on its estimators \( \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|^2] \) and the two error bounds \( \sum_{t=0}^{T-1} \Delta_t \), and \( \sum_{t=0}^{T-1} \delta_t \).

Observe that

\[
\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_t)\|^2] \leq \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_t) - \nabla_x f(x_t, y_t) + \nabla_x f(x_t, y_t) - v_t + v_t\|^2]
\]

\[
\leq 3 \sum_{t=0}^{T-1} \left[ \mathbb{E}[\|\nabla \Phi(x_t) - \nabla_x f(x_t, y_t)\|^2] + \mathbb{E}[\|\nabla_x f(x_t, y_t) - v_t\|^2] + \mathbb{E}[\|v_t\|^2] \right]
\]

\[
\leq 3 \sum_{t=0}^{T-1} \left( \kappa^2 \mathbb{E}[\|f(x_t, y_t)\|^2] + \mathbb{E}[\|\nabla_x f(x_t, y_t) - v_t\|^2] + \mathbb{E}[\|v_t\|^2] \right)
\]

\[
\leq 3\kappa^2 \sum_{t=0}^{T-1} \delta_t + 3 \sum_{t=0}^{T-1} \Delta_t + 3 \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|^2].
\] (34)

Substituting eq. (28), eq. (29) and eq. (33) into eq. (34), and using the fact that \( \kappa \geq 1 \) yield

\[
\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_t)\|^2] \leq 42\kappa^2 \frac{\sigma^2 T}{S_1} + 15\kappa^2 \delta_0 + 11 \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|^2]
\]

\[
\leq 2200L(\Phi(x_0) - \Phi^*) + \frac{3122\kappa^2 \sigma^2 T}{S_1} + 1115\kappa^2 \delta_0.
\] (35)

Recall \( L = (1 + \kappa)\ell \). Then, eq. (35) implies that

\[
\mathbb{E}[\|\nabla \Phi(\hat{x})\|^2] = \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_t)\|^2]
\]

\[
\leq 2200(1 + \kappa)\ell \frac{\Phi(x_0) - \Phi^*}{T} + \frac{3122\kappa^2 \sigma^2}{S_1} + 1115 \frac{\kappa^2 \delta_0}{T}.
\] (36)

If we let \( \delta_0 = \frac{1}{\kappa} \), \( T = \max\{\frac{3345\kappa}{(1 + \kappa)\ell} \frac{\Phi(x_0) - \Phi^*}{\sigma^2} \}, S_1 = \frac{9366\kappa^2 \sigma^2}{(1 + \kappa)\ell}, S_2 = \frac{5}{\kappa} \) and \( q = (1 - a)S_2 \), then we have

\[
\mathbb{E}[\|\nabla \Phi(\hat{x})\|^2] \leq \sqrt{\mathbb{E}[\|\nabla \Phi(\hat{x})\|^2]} \leq \epsilon.
\]

We define \( T_0 \) as the sample complexity of iSARAH to achieve the accuracy \( \mathbb{E}[\|\nabla_y f(x_0, y_0)\|^2] \leq \frac{1}{\kappa} \). Lemma 6 implies that \( T_0 = \mathcal{O}(\kappa \log(\kappa)) \). Then, the total sample complexity is given by

\[
T \cdot S_2 \cdot m + \left[ \frac{T}{q} \right] \cdot S_1 + T_0 \leq \Theta\left( \frac{\kappa}{\epsilon^2} \cdot \kappa \cdot \kappa \right) + \Theta\left( \frac{\kappa}{\epsilon} \cdot \kappa \cdot \kappa \right) + \mathcal{O}(\kappa \log(\kappa))
\]

\[
= \mathcal{O}\left( \frac{\kappa^3}{\epsilon^3} \right),
\]

which completes the proof. \qed

**B.4 Proof of Corollary 1**

In the finite-sum case, recall that we have

\[
f(x, y) = \frac{1}{n} \sum_{i=1}^{n} F(x, y; \xi_i).
\]
Here we modify Algorithm 5 by replacing the large batch sample used in line 6 of Algorithm 1 with the full gradient and using SARAH Nguyen et al. (2017a) as initialization.

**Case 1:** \( n > \kappa^2 \)

In the finite-sum case, due to the utilization of the full gradient every \( q \) steps, we have \( S_1 = n \) and \( \Delta_{(n_T-n)q} = 0 \) for all \( n \leq n_T \). Then following steps similar to those from eq. (22) to eq. (36), we obtain

\[
E[\|\nabla \Phi(\hat{x})\|_2^2] \leq 2200(1 + \kappa)\ell\frac{\Phi(x_0) - \Phi^*}{T} + 1115\kappa^2\delta_0 \frac{1}{T}.
\]

If we let \( \delta_0 = \frac{1}{\kappa}, T = \max\left(\frac{2330}{\epsilon^2}, 4400(1 + \kappa)\ell\frac{\Phi(x_0) - \Phi^*}{\epsilon^2}\right) \), \( S_2 = \sqrt{n} \), and \( q = \lfloor (1 - a)S_2 \rfloor \), then we have

\[
E[\|\nabla \Phi(\hat{x})\|_2] \leq \sqrt{E[\|\nabla \Phi(\hat{x})\|_2^2]} \leq \epsilon.
\]

We define \( T_0 \) as the sample complexity of SARAH to achieve the accuracy \( E[\|\nabla_y f(x, y_0)\|_2^2] \leq \frac{1}{\kappa} \). Lemma 7 implies that \( T_0 = \mathcal{O}((n + \kappa) \log(\kappa)) \). Then, the total sample complexity is given by

\[
T \cdot S_2 \cdot m + \left\lceil \frac{T}{q} \right\rceil \cdot S_1 + T_0 \leq \Theta\left(\frac{\kappa^2}{\epsilon^2} \cdot \sqrt{n} \cdot \kappa \right) + \Theta\left(\left\lfloor \frac{\kappa^2}{\epsilon^2} \cdot \sqrt{n} \right\rfloor \cdot n \right) + \mathcal{O}((n + \kappa) \log(\kappa)) = \mathcal{O}(\kappa^2 \sqrt{n} \epsilon^{-2} + n) + \mathcal{O}((n + \kappa) \log(\kappa)).
\]

**Case 2:** \( n \leq \kappa^2 \)

In this case, we let \( q = 1 \) and \( S_2 = 1 \). Then, we have \( \Delta_t = 0 \) for all \( 0 \leq t \leq T - 1 \). Since the analysis of \( \delta_t \) does not depend on the value of \( S_2 \), eq. (25) still holds, which implies

\[
\delta_t \leq \frac{1}{2}\delta_{t-1} + \frac{7}{4}\ell^2\alpha^2 E[\|v_{t-1}\|_2^2].
\]

Following steps similar to those from eq. (25)-27 yields

\[
\sum_{t=0}^{T-1} \delta_t \leq 2\delta_0 + \frac{7}{2}\ell^2\alpha^2 \sum_{t=0}^{T-2} E[\|v_t\|_2^2].
\]

Substituting eq. (37) into eq. (18) yields

\[
\left(\frac{\alpha}{2} - \frac{L\alpha^2}{2}\right) \sum_{t=0}^{T-1} E[\|v_t\|_2^2] \leq \Phi(x_0) - \mathbb{E}[\Phi(x_T)] + 2\alpha\kappa^2\delta_0 + \frac{7}{2}\ell^2\kappa^2\alpha^3 \sum_{t=0}^{T-2} E[\|v_t\|_2^2]
\]

\[
\leq \Phi(x_0) - \mathbb{E}[\Phi(x_T)] + 2\alpha\kappa^2\delta_0 + \frac{7}{2}L^2\alpha^3 \sum_{t=0}^{T-2} E[\|v_t\|_2^2],
\]

where (i) follows from the fact that \( L = (1 + \kappa)\ell \). Rearranging eq. (38), we have

\[
\left(\frac{\alpha}{2} - \frac{L\alpha^2}{2} - \frac{7}{2}L^2\alpha^3\right) \sum_{t=0}^{T-1} E[\|v_t\|_2^2] \leq \Phi(x_0) - \mathbb{E}[\Phi(x_T)] + 2\alpha\kappa^2\delta_0.
\]

Let \( \alpha = \frac{1}{4L} \). We have

\[
\frac{\alpha}{2} - \frac{L\alpha^2}{2} - \frac{7}{2}L^2\alpha^3 = \frac{5}{128L}\geq 0.
\]

Combining eq. (40) and eq. (39) and using Assumption 1 yield

\[
\sum_{t=0}^{T-1} E[\|v_t\|_2^2] \leq 26L(\Phi(x_0) - \Phi^*) + 14\kappa^2\delta_0.
\]
Recalling eq. (34), we have
\[ \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_t)\|^2] \leq 3\kappa^2 \sum_{t=0}^{T-1} \delta_t + 3 \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|^2]. \] (42)

Substituting eq. (37) and eq. (41) into eq. (42) yields
\[ \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_t)\|^2] \leq 6\kappa^2 \delta_0 + 4 \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|^2] \leq 62\kappa^2 \delta_0 + 104L(\Phi(x_0) - \Phi^*). \] (43)

Recall that \( L = (1 + \kappa)\ell \). Then eq. (43) implies that
\[ \mathbb{E}[\|\nabla \Phi(\hat{x})\|^2] = \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_t)\|^2] \leq \frac{62\kappa^2 \delta_0}{T} + \frac{104(\kappa + 1)\ell(\Phi(x_0) - \Phi^*)}{T}. \] (44)

Let \( \delta_0 \leq \frac{1}{\kappa} \) and \( T = \max\{\frac{124\kappa}{\epsilon^2}, \frac{208(\kappa + 1)(\Phi(x_0) - \Phi^*)}{\epsilon} \} \). Then we have
\[ \mathbb{E}[\|\nabla \Phi(\hat{x})\|^2] \leq \sqrt{\mathbb{E}[\|\nabla \Phi(\hat{x})\|^2]} \leq \epsilon. \]

The total sample complexity is given by
\[ T \cdot S_2 \cdot m + \left\lfloor \frac{T}{q} \right\rfloor \cdot S_1 + T_0 \leq \Theta \left( \frac{\kappa}{\epsilon^2} \cdot 1 \cdot \kappa \right) + \Theta \left( \frac{\kappa}{\epsilon^2} \cdot n \right) + \mathcal{O}((n + \kappa) \log(k)) = \mathcal{O}(\kappa^2 + \kappa n \epsilon^{-2}). \]

C Convergence Analysis of Zeroth-Order SREDA-Boost

C.1 ZO-SREDA-Boost Algorithm

Algorithm 4 ZO-SREDA-Boost

1: Input: \( x_0 \), initial accuracy \( \zeta \), learning rate \( \alpha = \Theta(\frac{1}{\kappa}) \), \( \beta = \Theta(\frac{1}{\ell}) \), batch size \( S_1 \), \( S_2 \) and periods \( q, m \).
2: Initialization: \( y_0 = \text{ZO-iSARAH}(-f(x_0), \zeta, \mu_2) \) (detailed in Algorithm 6 in Appendix C.4)
3: for \( t = 0, 1, \ldots, T - 1 \) do
4:  if \( \text{mod}(k, q) = 0 \) then
5:       draw \( S_1 \) samples \( \{\xi_1, \ldots, \xi_{S_1}\} \)
6:       \( v_t = \frac{1}{S_1} \sum_{i=1}^{S_1} \sum_{j=1}^{d_1} F(x_t + \delta_{x,j} x_t, y_t, \xi_i) - F(x_t - \delta_{x,j} x_t, y_t, \xi_i) e_j \)
7:     \( u_t = \frac{1}{2} \sum_{i=1}^{S_1} \sum_{j=1}^{d_2} F(x_t, y_t + \delta_{y,j} y_t, \xi_i) - F(x_t, y_t - \delta_{y,j} y_t, \xi_i) e_j \)
8:     where \( e_j \) denotes the vector with \( j \)-th natural unit basis vector.
9:  else
10:     \( v_t = \tilde{v}_{t-1, \tilde{m}_{t-1}}, u_t = \tilde{u}_{t-1, \tilde{m}_{t-1}} \)
11: end if
12: \( x_{t+1} = x_t - \alpha v_t \)
13: \( y_{t+1} = \text{ZO-ConcaveMaximizer}(t, m, S_2, x_{t+1}) \)
14: end for
15: Output: \( \hat{x} \) chosen uniformly at random from \( \{x_t\}_{t=0}^{T-1} \)
Algorithm 5 ZO-ConcaveMaximizer$(t, m, S_{2,x}, S_{2,y})$

1: **Initialization:** $\tilde{x}_{t-1} = x_t$, $\tilde{y}_{t-1} = y_t$, $\tilde{x}_{t,0} = x_{t+1}$, $\tilde{y}_{t,0} = y_t$, $\tilde{v}_{t,-1} = v_t$, $\tilde{u}_{t,-1} = u_t$
2: Draw minibatch sample $\mathcal{M}_x = \{\xi_1, \ldots, \xi_{S_{2,x}}\}$, $\mathcal{M}_1 = \{v_1, \ldots, v_{S_{2,x}}\}$ and $\mathcal{M}_2 = \{\omega_1, \ldots, \omega_{S_{2,x}}\}$, and $\mathcal{M}_y = \{\xi_1, \ldots, \xi_{S_{2,y}}\}$, $\mathcal{M}_1 = \{v_1, \ldots, v_{S_{2,y}}\}$ and $\mathcal{M}_2 = \{\omega_1, \ldots, \omega_{S_{2,y}}\}$
3: $v_{t,0} = \tilde{v}_{t,-1} + G(\tilde{x}_{t,0}, \tilde{y}_{t,0}, \nu_{M_{1,x}}, \xi_{M_{1}}) - G(\tilde{x}_{t,-1}, \tilde{y}_{t,-1}, \nu_{M_{1,x}}, \xi_{M_{1}})$
4: $u_{t,0} = \tilde{u}_{t,-1} + H(\tilde{x}_{t,0}, \tilde{y}_{t,0}, \omega_{M_{2,y}}, \xi_{M_{2}}) - H(\tilde{x}_{t,-1}, \tilde{y}_{t,-1}, \omega_{M_{2,y}}, \xi_{M_{2}})$
5: $\tilde{x}_{t,1} = \tilde{x}_{t,0}$
6: $\tilde{y}_{t,1} = \tilde{y}_{t,0} + \beta \tilde{u}_{t,0}$
7: for $k = 1, 2, \ldots, m + 1$ do
8: Draw minibatch sample $\mathcal{M}_x = \{\xi_1, \ldots, \xi_{S_{2,x}}\}$, $\mathcal{M}_1 = \{v_1, \ldots, v_{S_{2,x}}\}$ and $\mathcal{M}_2 = \{\omega_1, \ldots, \omega_{S_{2,x}}\}$, and $\mathcal{M}_y = \{\xi_1, \ldots, \xi_{S_{2,y}}\}$, $\mathcal{M}_1 = \{v_1, \ldots, v_{S_{2,y}}\}$ and $\mathcal{M}_2 = \{\omega_1, \ldots, \omega_{S_{2,y}}\}$
9: $\tilde{v}_{t,k} = \tilde{v}_{t,k-1} + G_{\mu_1}(\tilde{x}_{t,k}, \tilde{y}_{t,k}, \nu_{M_{1,x}}, \xi_{M_{1}}) - G_{\mu_1}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1}, \nu_{M_{1,x}}, \xi_{M_{1}})$
10: $\tilde{u}_{t,k} = \tilde{u}_{t,k-1} + H_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}, \omega_{M_{2,y}}, \xi_{M_{2}}) - H_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1}, \omega_{M_{2,y}}, \xi_{M_{2}})$
11: $\tilde{x}_{t,k+1} = \tilde{x}_{t,k}$
12: $\tilde{y}_{t,k+1} = \tilde{y}_{t,k} + \beta \tilde{u}_{t,k}$
13: end for
Output: $y_{t+1} = \tilde{y}_{t,m}$ with $\tilde{m}_t$ chosen uniformly at random from $\{0, 1, \ldots, m\}$

C.2 Preliminaries

Consider a function $h(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}$. Let $\nu$ be a $d$-dimensional standard Gaussian random vector and $\mu > 0$ be the smoothing parameter. Then a smooth approximation of $h(\cdot)$ is defined as $h_\nu(x) = \mathbb{E}_\nu[h(x + \tau \nu)]$. We have the following lemmas.

**Lemma 11** (Nesterov and Spokoiny (2017), Section 2). If $h(\cdot)$ is convex, then $h_\nu(\cdot)$ is also a convex function.

**Lemma 12** (Ghadimi and Lan (2013), Section 3.1). If $h(\cdot)$ has $\ell$-Lipschitz gradient, then $h_\nu(\cdot)$ also has $\ell$-Lipschitz gradient.

**Lemma 13** (Nesterov and Spokoiny (2017), Theorem 1). If $h(\cdot)$ has $\ell$-Lipschitz gradient, then for all $x \in \mathbb{R}^d$, we have $|h(x) - h_\tau(x)| \leq \frac{\tau^2}{2} \ell^2(d + 3)^3$.

**Lemma 14** (Nesterov and Spokoiny (2017), Lemma 3). If $h(\cdot)$ has $\ell$-Lipschitz gradient, then $\|\nabla_x h_\tau(x) - \nabla_x h(x)\|_2^2 \leq \frac{\tau^2}{4} \ell^2(d + 3)^3$.

**Lemma 15.** Suppose Assumption 2 and Assumption 4 hold. Suppose mod$(t, q) = 0$, and let $\epsilon(S_1, \delta) = \mathbb{E}[\|v_1 - \nabla_x f_{\mu_1}(x_t, y_t)\|_2^2] + \mathbb{E}[\|u_t - \nabla_y f_{\mu_2}(x_t, y_t)\|_2^2]$. Then, we have

$$
\epsilon(S_1, \delta) \leq \frac{(d_1 + d_2)^2 \ell^2 \delta^2}{2} + \frac{4\sigma^2}{S_1} + \frac{\mu_1^2}{2} \ell^2(d_1 + 3)^3 + \frac{\mu_2^2}{2} \ell^2(d_2 + 3)^3.
$$

**Proof.** B.56 and B.57 in Fang et al. (2018) imply that

$$
\mathbb{E}[\|v_1 - \nabla_x f(x_t, y_t)\|_2^2] \leq \frac{d_1 \ell^2 \delta^2}{2} + \frac{2\sigma^2}{S_1}, \quad (45)
$$

and

$$
\mathbb{E}[\|u_t - \nabla_y f(x_t, y_t)\|_2^2] \leq \frac{d_2 \ell^2 \delta^2}{2} + \frac{2\sigma^2}{S_1}. \quad (46)
$$

Then we proceed as follows:

$$
\begin{align*}
& \mathbb{E}[\|v_1 - \nabla_x f_{\mu_1}(x_t, y_t)\|_2^2] + \mathbb{E}[\|u_t - \nabla_y f_{\mu_2}(x_t, y_t)\|_2^2] \\
& \leq 2\mathbb{E}[\|v_1 - \nabla_x f(x_t, y_t)\|_2^2] + 2\mathbb{E}[\|u_t - \nabla_y f(x_t, y_t)\|_2^2] \\
& \quad + 2\mathbb{E}[\|\nabla_x f_{\mu_1}(x_t, y_t) - \nabla_x f(x_t, y_t)\|_2^2] + 2\mathbb{E}[\|\nabla_y f_{\mu_2}(x_t, y_t) - \nabla_y f(x_t, y_t)\|_2^2]
\end{align*}
$$
\[(i) \leq 2\mathbb{E}\left[\|v_t - \nabla_x f(x_t, y_t)\|^2\right] + 2\mathbb{E}\left[\|u_t - \nabla_y f(x_t, y_t)\|^2\right] + \frac{\mu_1^2}{2} \ell^2(d_1 + 3)^3 + \frac{\mu_2^2}{2} \ell^2(d_2 + 3)^3 \]

\[(ii) \leq (d_1 + d_2)\ell^2\delta^2 + \frac{8\sigma^2}{\delta^2} + \frac{\mu_1^2}{2} \ell^2(d_1 + 3)^3 + \frac{\mu_2^2}{2} \ell^2(d_2 + 3)^3,\]

where \((i)\) follows from Lemma 14, and \((ii)\) follows from eq. (45) and eq. (46).

We denote

\[G_{\mu_1}(x, y, \nu, \xi) = \frac{F(x + \mu_1 \nu, y, \xi) - F(x, y, \xi)}{\mu_1}\]

and

\[H_{\mu_2}(x, y, \omega, \xi) = \frac{F(x, y + \mu_2 \omega, \xi) - F(x, y, \xi)}{\mu_2}\]

as unbiased estimators of \(\nabla_x f_{\mu_1}(x, y)\) and \(\nabla_y f_{\mu_2}(x, y)\), respectively. Then we have the following lemma.

**Lemma 16.** Suppose Assumption 2 holds, and suppose \(u_1\) and \(u_2\) are standard Gaussian random vector, i.e., \(\nu_1 \sim N(0,1_{d_1})\) and \(\omega_1 \sim N(0,1_{d_2})\). Then, we have

\[\mathbb{E}\left[\|G_{\mu_1}(x, y, \nu, \xi) - G_{\mu_1}(x', y, \nu, \xi)\|^2\right] \leq 2(d_1 + 4)\ell^2 \|x - x'\|^2_2 + 2\mu_1^2(d_1 + 6)^3\ell^2,\]

\[\mathbb{E}\left[\|G_{\mu_1}(x, y, \nu, \xi) - G_{\mu_1}(x, y', \nu, \xi)\|^2\right] \leq 2(d_1 + 4)\ell^2 \|y - y'\|^2_2 + 2\mu_1^2(d_1 + 6)^3\ell^2,\]

and

\[\mathbb{E}\left[\|H_{\mu_2}(x, y, \nu, \xi) - H_{\mu_2}(x', y, \nu, \xi)\|^2\right] \leq 2(d_2 + 4)\ell^2 \|x - x'\|^2_2 + 2\mu_2^2(d_2 + 6)^3\ell^2,\]

\[\mathbb{E}\left[\|H_{\mu_2}(x, y, \nu, \xi) - H_{\mu_2}(x, y', \nu, \xi)\|^2\right] \leq 2(d_2 + 4)\ell^2 \|y - y'\|^2_2 + 2\mu_2^2(d_2 + 6)^3\ell^2.\]

**Proof.** The proof is similar to that of Lemma 3 in Fang et al. (2018). Here we provide the proof for completeness. We only show how to upper bound the term \(\mathbb{E}\left[\|G_{\mu_1}(x, y, \nu, \xi) - G_{\mu_1}(x', y, \nu, \xi)\|^2\right]\) here. Then, the upper bounds on the remaining three terms can be obtained by following similar steps. We have that

\[\mathbb{E}\left[\|G_{\mu_1}(x, y, \nu, \xi) - G_{\mu_1}(x, y', \nu, \xi)\|^2\right] \leq 2\mathbb{E}\left[\left\|\frac{F(x + \mu_1 \nu, y, \xi) - F(x, y, \xi)}{\mu_1} - \frac{F(x + \mu_1 \nu, y', \xi) - F(x, y', \xi)}{\mu_1}\right\|_2^2\right]\]

\[= \mathbb{E}\left[\left\|\frac{F(x + \mu_1 \nu, y, \xi) - F(x, y, \xi)}{\mu_1} - \frac{F(x + \mu_1 \nu, y', \xi) - F(x, y', \xi)}{\mu_1}\right\|_2^2\right] + \mathbb{E}\left[\left\|\nabla_x F(x, y, \xi) - \nabla_x F(x, y', \xi)\right\|_2^2\right]

\[\leq 2\mathbb{E}\left[\left\|\frac{F(x + \mu_1 \nu, y, \xi) - F(x, y, \xi) - \nabla_x F(x, y, \xi) \cdot \mu_1 \nu}{\mu_1}\right\|_2^2\right]

\[+ \mathbb{E}\left[\left\|\nabla_x F(x, y, \xi) - \nabla_x F(x, y', \xi)\right\|_2^2\right].\]
Consider Algorithm 5. For any given $\nu_0$, we define 

$$F(x, y, \xi, \nu) = \langle \nabla_y F(x, y, \xi), \nu \rangle.$$

We also define

$$\nu_i = \mu_i \nu_0.$$

Lemma 17

Let $\nu_i = \mu_i \nu_0$ for any given $\nu_0$, we define 

$$F(x, y, \xi, \nu) = \langle \nabla_y F(x, y, \xi), \nu \rangle.$$ 

Consider Algorithm 5. For any given $\nu_0$, we define

$$F(x, y, \xi, \nu) = \langle \nabla_y F(x, y, \xi), \nu \rangle.$$ 

We have

$$F(x, y, \xi, \nu) = \langle \nabla_y F(x, y, \xi), \nu \rangle.$$ 

Lemma 18

Let $\nu_i = \mu_i \nu_0$ for any given $\nu_0$, we define 

$$F(x, y, \xi, \nu) = \langle \nabla_y F(x, y, \xi), \nu \rangle.$$ 

Lemma 19

Let $\nu_i = \mu_i \nu_0$ for any given $\nu_0$, we define 

$$F(x, y, \xi, \nu) = \langle \nabla_y F(x, y, \xi), \nu \rangle.$$ 

C.3 Useful Properties for Zeroth-Order Concave Maximizer

In this section, we show some properties for the zeroth-order concave maximizer Algorithm 5. For simplicity, for any given $t \geq 0$, we define $g_t(y) = -f(x_t+1, y)$ and $g_{t+1}(y) = -f_{t+1}(x_t+1, y)$. Lemma 11 and Lemma 12 imply that $g_t(\cdot)$ is $\mu$-strongly convex and has $\ell$-Lipschitz continuous gradient; (ii) follows because

$$E[\|\langle (a, \nu) \rangle \nu_0 \|^2] \leq (d_1 + 4) \|a\|^2,$$

obtained from (33) in Nesterov and Spokoiny (2017), and (iii) follows because $E[\|\nu_0 \|^2] \leq (d_1 + 6)^2$ in (17) of Nesterov and Spokoiny (2017).

Lemma 17 (Lemma 9 of Luo et al. (2020)). Consider Algorithm 5. We have

$$\sum_{k=0}^{m} E[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,k})\|^2] \leq \frac{2}{\beta} E[\nabla g_{t, \mu_2}(\tilde{y}_{t,0}) - g_{t, \mu_2}(\tilde{y}_{t,m+1})] + \sum_{k=0}^{m} E[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,k}) - \tilde{u}_{t,k}\|^2].$$

Lemma 18 (Lemma 11 of Luo et al. (2020)). Consider Algorithm 5 with any $\beta \leq \frac{2}{\ell}$ and $k \geq 1$. We have

$$E[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,k}) - \tilde{u}_{t,k}\|^2] \leq E[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,0}) - \tilde{u}_{t,0}\|^2] + \frac{\ell \beta}{2 - \ell \beta} E[\|\tilde{u}_{t,0}\|^2].$$

Lemma 19. Consider Algorithm 5. For any $k \geq 1$ and $\beta \leq \frac{1}{4}$, we have

$$E[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,m})\|^2] \leq \frac{2}{\beta \mu \rho} E[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,0})\|^2] + \|\nabla g_{t, \mu_2}(\tilde{y}_{t,0}) - \tilde{u}_{t,0}\|^2] + \frac{\ell \beta}{2 - \ell \beta} E[\|\tilde{u}_{t,0}\|^2]$$

$$+ \frac{2}{\beta (m+1)} \left( \frac{\mu_2}{4 \mu} \ell^2 (d_2 + 3) + \mu_2 \ell d_2 \right).$$
Proof. Taking summation of the result of Lemma 18 over \( t = \{0, \ldots, m\} \) yields

\[
\sum_{k=0}^{m} \mathbb{E}[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,k}) - \tilde{u}_{t,k}\|_2^2] \leq (m + 1)\mathbb{E}[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,0}) - \tilde{u}_{t,0}\|_2^2] + \frac{\ell \beta (m + 1)}{2 - \ell \beta} \mathbb{E}[\|\tilde{u}_{t,0}\|_2^2].
\]  

(47)

Combining eq. (47) with Lemma 17 yields

\[
\sum_{k=0}^{m} \mathbb{E}[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,k})\|_2^2] \leq \frac{2}{\beta (m + 1)} \mathbb{E}[g_{t, \mu_2}(\tilde{y}_{t,0}) - g_{t, \mu_2}(\tilde{y}_{t,m+1})] + (m + 1)\mathbb{E}[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,0}) - \tilde{u}_{t,0}\|_2^2]
\]

\[
+ \frac{\ell \beta (m + 1)}{2 - \ell \beta} \mathbb{E}[\|\tilde{u}_{t,0}\|_2^2].
\]  

(48)

Dividing both sides of eq. (48) yields

\[
\mathbb{E}[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,m})\|_2^2] \leq \frac{2}{\beta (m + 1)} \mathbb{E}[g_{t, \mu_2}(\tilde{y}_{t,0}) - g_{t, \mu_2}(\tilde{y}_{t,m+1})] + \mathbb{E}[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,0}) - \tilde{u}_{t,0}\|_2^2]
\]

\[
+ \frac{\ell \beta}{2 - \ell \beta} \mathbb{E}[\|\tilde{u}_{t,0}\|_2^2].
\]  

(49)

We can bound the term \( \mathbb{E}[g_{t, \mu_2}(\tilde{y}_{t,0}) - g_{t, \mu_2}(\tilde{y}_{t,m+1})] \) as follows:

\[
\mathbb{E}[g_{t, \mu_2}(\tilde{y}_{t,0}) - g_{t, \mu_2}(\tilde{y}_{t,m+1})]
\]

\[
= \mathbb{E}[g_t(\tilde{y}_{t,0}) - g_t(\tilde{y}_{t,m+1})] + \mathbb{E}[g_{t, \mu_2}(\tilde{y}_{t,0}) - g_t(\tilde{y}_{t,0})] + \mathbb{E}[g_{t, \mu_2}(\tilde{y}_{t,m+1}) - g_t(\tilde{y}_{t,m+1})]
\]

\[
\leq \mathbb{E}[g_t(\tilde{y}_{t,0}) - g_t(\tilde{y}_{t,m+1})] + \mathbb{E}[g_{t, \mu_2}(\tilde{y}_{t,0}) - g_t(\tilde{y}_{t,0})] + \mathbb{E}[g_{t, \mu_2}(\tilde{y}_{t,m+1}) - g_t(\tilde{y}_{t,m+1})]
\]

\[
\leq (i) \leq \mathbb{E}[g_t(\tilde{y}_{t,0}) - g_t(\tilde{y}_{t,m+1})] + \mu_2^2 \ell d_2
\]

\[
\leq \mathbb{E}[g_t(\tilde{y}_{t,0}) - \bar{g}_t] + \mu_2^2 \ell d_2
\]

\[
\leq (ii) \leq \frac{1}{2\mu} \mathbb{E}[\|\nabla g_t(\tilde{y}_{t,0})\|_2^2] + \mu_2^2 \ell d_2
\]

\[
\leq \frac{1}{\mu} \mathbb{E}[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,0})\|_2^2] + \frac{\mu_2^2}{\mu} \ell^2 (d_2 + 3)^3 + \mu_2^2 \ell d_2
\]

\[
\leq (iii) \leq \frac{1}{\mu} \mathbb{E}[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,0})\|_2^2] + \frac{\mu_2^2}{4\mu} \ell^2 (d_2 + 3)^3 + \mu_2^2 \ell d_2.
\]  

(50)

where (i) follows from Lemma 13, (ii) follows from eq. (6) in Lemma 2, and (iii) follows from Lemma 14. Substituting eq. (50) into eq. (49) yields

\[
\mathbb{E}[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,m})\|_2^2] \leq \frac{2}{\beta \mu (m + 1)} \mathbb{E}[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,0})\|_2^2] + \mathbb{E}[\|\nabla g_{t, \mu_2}(\tilde{y}_{t,0}) - \tilde{u}_{t,0}\|_2^2] + \frac{\ell \beta}{2 - \ell \beta} \mathbb{E}[\|\tilde{u}_{t,0}\|_2^2]
\]

\[
+ \frac{2}{\beta (m + 1)} \left( \frac{\mu_2^2}{4\mu} \ell^2 (d_2 + 3)^3 + \mu_2^2 \ell d_2 \right),
\]

which completes the proof. \(\square\)

**Lemma 20.** Consider Algorithm 5. Let \( S_{2,y} \geq 16\epsilon(d_2 + 4)\ell \beta \) and \( \beta \leq \frac{1}{6\ell} \). For any \( t > 0 \), we have

\[
\sum_{k=0}^{m} \mathbb{E}[\|\tilde{u}_{t,k}\|_2^2] \leq \frac{1}{1 - b} \mathbb{E}[\|\tilde{u}_{t,0}\|_2^2] + \frac{m + 1}{1 - b} \left[ \frac{2\mu_2^2 \ell \kappa}{\beta} (d_2 + 3)^3 + 7\mu_2^2 (d_2 + 6)^3 \ell^2 \right],
\]

where \( b = 1 - \frac{\beta \mu \ell}{2(\mu + \ell)} \).

Proof. The update of Algorithm 5 implies that

\[
\mathbb{E}[\|\tilde{u}_{t,k}\|_2^2 | F_{t,k}]
\]
\[
\begin{align*}
&= \|\tilde{u}_{t,k-1}\|^2 + 2E[\langle \tilde{u}_{t,k-1}, H_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}, \omega_{M_2}, \xi_{M}) - H_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1}, \omega_{M_2}, \xi_{M})\rangle | F_{t,k}] \\
&\quad + E[\|H_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}, \omega_{M_2}, \xi_{M}) - H_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1}, \omega_{M_2}, \xi_{M})\|^2 | F_{t,k}] \\
&= \|\tilde{u}_{t,k-1}\|^2 + \frac{2}{\beta} \langle \tilde{y}_{t,k} - \tilde{y}_{t,k-1}, \nabla_y f_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}) - \nabla_y f_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1})\rangle \\
&\quad + E[\|H_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}, \omega_{M_2}, \xi_{M}) - H_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1}, \omega_{M_2}, \xi_{M})\|^2 | F_{t,k}] \\
&\leq \|\tilde{u}_{t,k-1}\|^2 + \frac{2}{\beta} \left( \frac{\mu \ell}{\mu + \ell} \right) \|	ilde{y}_{t,k} - \tilde{y}_{t,k-1}\|^2 + \frac{1}{\mu + \ell} \|
abla_y f_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}) - \nabla_y f_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1})\|^2 \\
&\quad + \frac{2}{\beta} \left( \frac{\mu \ell}{4(\mu + \ell)} \right) \|	ilde{y}_{t,k} - \tilde{y}_{t,k-1}\|^2 + \frac{\mu + \ell}{\mu \ell} \|
abla_y f_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}) - \nabla_y f_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1})\|^2 \\
&\quad + E[\|H_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}, \omega_{M_2}, \xi_{M}) - H_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1}, \omega_{M_2}, \xi_{M})\|^2 | F_{t,k}] \\
&\leq \left( 1 - \frac{\beta \mu \ell}{\mu + \ell} \right) \|	ilde{u}_{t,k-1}\|^2 - \frac{2}{\beta(\mu + \ell)} \|
abla_y f_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}) - \nabla_y f_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1})\|^2 \\
&\quad + 2E[\|H_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}, \omega_{M_2}, \xi_{M}) - H_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1}, \omega_{M_2}, \xi_{M}) - \nabla_y f_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}) + \nabla_y f_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1})\|^2 | F_{t,k}] \\
&\quad + 6E[\|
abla_y f_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}) - \nabla_y f_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1})\|^2 | F_{t,k}] \\
&\quad + 6E[\|
abla_y f_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1}) - \nabla_y f_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1})\|^2 | F_{t,k}] \\
&\leq \left( 1 - \frac{\beta \mu \ell}{\mu + \ell} \right) \|	ilde{u}_{t,k-1}\|^2 - \frac{2}{\beta(\mu + \ell)} \|
abla_y f_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}) - \nabla_y f_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1})\|^2 \\
&\quad + \frac{2}{\beta \mu \ell} E[\|H_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}, \omega_i, \xi_i) - H_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1}, \omega_i, \xi_i)\|^2 | F_{t,k}] \\
&\quad + 3\mu \ell^2 (d_2 + 3)^3 + \frac{\mu \ell^2 (\mu + \ell)}{\beta \mu} (d_2 + 3)^3 \\
&\quad \leq \left( 1 - \frac{\beta \mu \ell}{\mu + \ell} \right) \|	ilde{u}_{t,k-1}\|^2 + \frac{2}{\beta \mu \ell} E[\|H_{\mu_2}(\tilde{x}_{t,k}, \tilde{y}_{t,k}, \omega_i, \xi_i) - H_{\mu_2}(\tilde{x}_{t,k-1}, \tilde{y}_{t,k-1}, \omega_i, \xi_i)\|^2 | F_{t,k}] \\
&\quad + 3\mu \ell^2 (d_2 + 3)^3 + \frac{\mu \ell^2 (\mu + \ell)}{\beta \mu} (d_2 + 3)^3 
\end{align*}
\]
We present the detailed procedure of ZO-iSARAH in Algorithm 6, which is used to initialize Taking summation of eq. (52) over \( P \) where

\[
\psi \quad \Psi(\psi, \psi_{M1}, \psi_{M}) = \frac{1}{|\mathcal{M}|} \sum_{i \in |\mathcal{M}|} \frac{P(\psi + \tau \psi, \xi) - P(\psi, \xi)}{\tau} \psi_i,
\]

where \( \psi \sim N(0, 1_d) \) independently across the index \( i \).
Algorithm 6 ZO-iSARAH

1: **Input:** \( \bar{w}_0 \), learning rate \( \gamma > 0 \), inner loop size \( I \), batch size \( B_1 \) and \( B_2 \)
2: **for** \( t = 1, 2, ..., T \) **do**
3: \( w_0 = \bar{w}_{t-1} \)
4: draw \( B_1 \) samples \( \{\xi_{1}, \ldots, \xi_{B_1}\} \)
5: \( v_0 = \frac{1}{B_1} \sum_{i=1}^{B_1} \sum_{j=1}^{d} \frac{P(w_0 + 2 \epsilon_j, \xi_i) - P(w_0 - 2 \epsilon_j, \xi_i)}{2^{\epsilon_j}} \epsilon_j \)
6: where \( \epsilon_j \) denotes the vector with \( j \)-th natural unit basis vector.
7: \( w_1 = w_0 + \gamma v_0 \)
8: **for** \( k = 1, 2, ..., I - 1 \) **do**
9: Draw minibatch sample \( M = \{\xi_1, \ldots, \xi_{B_2}\} \) and \( M_1 = \{\psi_1, \ldots, \psi_{B_2}\} \)
10: \( v_k = v_{k-1} + \Psi_r(w_k, \psi_{M_1}, \xi_M) - \Psi_r(w_{k-1}, \psi_{M_1}, \xi_M) \)
11: \( w_{k+1} = w_k + \gamma v_k \)
12: **end for**
13: \( \bar{w}_1 \) chosen uniformly at random from \( \{w_k\}_{k=0}^{T-1} \)
14: **end for**

We have the following convergence result by using ZO-iSARAH to solve problem eq. (53).

**Lemma 21.** Consider Algorithm 6. Set \( \gamma = \frac{2}{\delta}, \ B_1 = \frac{25\sigma^2}{\delta}, \ B_2 = d, \ I = 36\kappa - 1, \ T = \log_2 \frac{5\|\nabla p_r(\bar{w}_0)\|_2^2}{\epsilon}, \delta = \frac{2^{0.5}}{5\ell d^2}, \) and \( \tau = \min\{\frac{2^{0.5}}{3I(d+3)^3}, \sqrt{\frac{2^2}{5\ell d}}\} \). Then, we have

\[
\mathbb{E}[\|\nabla p_r(\bar{w}_T)\|_2^2] \leq \epsilon,
\]

with the total sample complexity given by \( \mathcal{O} ((\kappa^2 + \frac{1}{\epsilon}) \log \frac{1}{\epsilon}) \).

**Proof:** Following steps similar to those in Lemmas 17-19, at \( t \)-th outer-loop iteration, we can obtain the following convergence result of inner loop:

\[
\begin{align*}
\mathbb{E}[\|\nabla p_r(\bar{w}_t)\|_2^2] &\leq \frac{2}{\gamma I(I+1)} \mathbb{E}[\|\nabla p_r(w_0)\|_2^2] + \mathbb{E}[\|\nabla p_r(w_0) - v_0\|_2^2] + \frac{\ell \gamma}{2 - \ell \gamma} \mathbb{E}[\|v_0\|_2^2] \\
&+ \frac{2}{\gamma(I+1)} \left( \frac{\tau^2 d^2}{4\mu} + \cfrac{\ell d^2 (d+3)^3 + \tau^2}{d} \right) \\
&\leq \left( \frac{2}{\gamma \mu (I+1)} + \frac{2\ell}{2 - \ell \gamma} \right) \mathbb{E}[\|\nabla p_r(w_0)\|_2^2] + \left( 1 + \frac{2\ell}{2 - \ell \gamma} \right) \mathbb{E}[\|\nabla p_r(w_0) - v_0\|_2^2] \\
&+ \frac{2}{\gamma (I+1)} \left( \frac{\tau^2 d^2 (d+3)^3 + \tau^2}{4\mu} \right). (55)
\end{align*}
\]

Then, following steps similar to those in Lemma 15, we can obtain

\[
\mathbb{E}[\|\nabla p_r(w_0) - v_0\|_2^2] \leq \frac{2\sigma^2}{B_1} + \frac{d \ell^2 \sigma^2}{2} + \frac{\tau^2 d^2}{2}. (56)
\]

Letting \( \gamma = \frac{2}{\delta}, \ I = 36\kappa - 1, \) substituting eq. (56) into eq. (55), and recalling the fact that \( w_T = \bar{w}_t \) and \( w_0 = \bar{w}_{t-1} \) yield

\[
\mathbb{E}[\|\nabla p_r(\bar{w}_t)\|_2^2] \leq \frac{1}{2} \mathbb{E}[\|\nabla p_r(\bar{w}_{t-1})\|_2^2] + \frac{5\sigma^2}{2} + \frac{5d \ell^2 \sigma^2}{8} + 11\tau^2 \frac{d^2}{16} \frac{d + 3}{d} + \frac{\tau^2}{4} \ell \mu d. (57)
\]

Applying eq. (57) iteratively from \( t = T \) to 0 yields

\[
\mathbb{E}[\|\nabla p_r(\bar{w}_T)\|_2^2] \leq \frac{1}{2^T} \mathbb{E}[\|\nabla p_r(\bar{w}_0)\|_2^2] + \frac{5\sigma^2}{2B_1} \sum_{t=0}^{T-1} \frac{1}{2^T}
\]

30
Algorithm 4 is given by

\[
\text{To solve the problem in eq. (59), we slightly modify Algorithm 6 by replacing line 5 with the full gradient.}
\]

The total sample complexity is given by

\[
T \cdot (I \cdot B_2 + d \cdot B_1) = \mathcal{O} \left( d \left( \kappa + \frac{1}{\epsilon} \right) \log \left( \frac{1}{\epsilon} \right) \right).
\]

**Extension to finite-sum case:** ZO-isSARAH is also applicable for strongly-convex optimization in the finite-sum case, which takes the form given by

\[
\text{Letting } T = \log_2 \frac{5\|\nabla p_r(\tilde{w}_0)\|_2^2}{\epsilon}, B_1 = \frac{5\sigma^2}{\epsilon}, \delta = \frac{2^{0.5}}{d\kappa\tau}, \text{ and } \tau = \min\{\frac{2^{0.5}}{d(3+3)^{1.5}}, \sqrt{\frac{2\kappa}{d\mu\tau}}\}, \text{ we have}
\]

\[
E[\|\nabla p_r(\tilde{w}_T)\|_2^2] \leq \epsilon.
\]

The total sample complexity is given by

\[
T \cdot (I \cdot B_2 + d \cdot B_1) = \mathcal{O} \left( d \left( \kappa + n \right) \log \left( \frac{1}{\epsilon} \right) \right). \tag{60}
\]

Let \( P(\cdot; \xi) = -F(x_0, \cdot; \xi) \). Then we can conclude that the sample complexity for the initialization of Algorithm 4 is given by \( \mathcal{O}(d_{2}\kappa \log(\kappa)) \) in the online case, and given by \( \mathcal{O}(d_{2}(\kappa + n) \log(\kappa)) \) in the finite-sum case.

**C.5 Proof of Theorem 2**

We define \( \Delta'_t = E[\|\nabla_x f_{\mu_1}(x_t, y_t) - v_t\|_2^2] + E[\|\nabla_y f_{\mu_2}(x_t, y_t) - u_t\|_2^2], \Delta'_t = \frac{1}{d} \sum_{i=1}^{d} \Delta'_{i,t}, \) and \( E[\|\nabla_x f_{\mu_1}(x_{t,k}, y_{t,k}) - \tilde{v}_{t,k}\|_2^2], E[\|\nabla_y f_{\mu_2}(x_{t,k}, y_{t,k}) - \tilde{u}_{t,k}\|_2^2], \) and \( \delta'_t = E[\|\nabla f_{\mu}(x_t, y_t)\|_2^2] \). In this section, we establish the following lemmas to characterize the relationship between \( \Delta_t, \Delta'_t, \) and \( \delta_t, \delta'_t, \) and the recursive relationship of \( \Delta'_t \) and \( \delta'_t, \) which are crucial for the analysis of Theorem 2.

**Lemma 22.** Suppose Assumption 2 hold. Then, for any \( 0 \leq t \leq T - 1, \) we have

\[
\Delta_t \leq 2\Delta'_t + \frac{\mu_1^2}{2} \ell^2(d_1 + 3)^3 + \frac{\mu_2^2}{2} \ell^2(d_2 + 3)^3,
\]

and

\[
\delta_t \leq 2\delta'_t + \frac{\mu_2^2}{2} \ell^2(d_2 + 3)^3.
\]
Proof. For the first inequality, we have
\[
\Delta_t = \mathbb{E}[\|\nabla_x f(x_t, y_t) - v_t\|^2_2] + \mathbb{E}[\|\nabla_y f(x_t, y_t) - u_t\|^2_2] \\
= \mathbb{E}[\|\nabla_x f_{\mu_2}(x_t, y_t) - v_t - \nabla_x f(x_t, y_t) - \nabla_x f_{\mu_1}(x_t, y_t)\|^2_2] \\
+ 2\mathbb{E}[\|\nabla_y f_{\mu_2}(x_t, y_t) - u_t + \nabla_y f(x_t, y_t) - \nabla_y f_{\mu_2}(x_t, y_t)\|^2_2] \\
\leq 2\Delta_t + 2\mathbb{E}[\|\nabla_y f_{\mu_2}(x_t, y_t) - u_t\|^2_2] + 2\mathbb{E}[\|\nabla_y f(x_t, y_t) - \nabla_y f_{\mu_2}(x_t, y_t)\|^2_2],
\]
where \((i)\) follows from Lemma 14. For the second inequality, we have
\[
\delta_t = \mathbb{E}[\|\nabla_y f(x_t, y_t)\|^2_2] = \mathbb{E}[\|\nabla_y f_{\mu_2}(x_t, y_t) + \nabla_y f(x_t, y_t) - \nabla_y f_{\mu_2}(x_t, y_t)\|^2_2] \\
\leq 2\delta_t + 2\mathbb{E}[\|\nabla_y f_{\mu_2}(x_t, y_t)\|^2_2] + 2\mathbb{E}[\|\nabla_y f(x_t, y_t) - \nabla_y f_{\mu_2}(x_t, y_t)\|^2_2] \\
\leq 2\delta_t + \frac{\mu_y^2}{2} \ell^2(d_2 + 3)^3,
\]
where \((i)\) follows from Lemma 14.

We provide the following two lemmas to characterize the relationship between \(\delta_t\) and \(\delta_{t-1}\) as well as that between \(\Delta_t\) and \(\Delta_{t-1}\).

**Lemma 23.** Suppose Assumptions 2 hold. Then, we have
\[
\Delta_t' \leq \frac{1}{1 - \beta} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \Delta_{t-1}' + \frac{6\ell^2 \beta^2}{1 - \beta} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \delta_{t-1}' \\
+ 2\ell^2 \alpha^2 \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \left( 1 + \frac{9\ell^2 \beta^2}{1 - \beta} \right) \mathbb{E}[\|v_{t-1}\|^2_2] + \pi_\Delta(d_1, d_2, \mu_1, \mu_2),
\]
where \(b = 1 - \frac{\beta \mu_y^2}{\alpha(d_2 + 3)^3}\) and
\[
\pi_\Delta(d_1, d_2, \mu_1, \mu_2) = \frac{2\ell^2 \beta^2}{1 - \beta} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \left( 6\ell^2 \left[ \frac{\mu_y^2(d_1 + 6)^3}{S_{2,x}} + \frac{\mu_y^2(d_2 + 6)^3}{S_{2,y}} \right] + (m + 1) \left( 2\frac{\mu_y^2 \kappa}{\beta} (d_2 + 3)^3 \right) \\
+ \frac{7\mu_y^2(d_2 + 6)^3 \ell^2}{S_{2,y}} \right) \frac{2(m + 2) \mu_y^2(d_1 + 6)^3 \ell^2}{S_{2,x}} + \frac{2(m + 2) \mu_y^2(d_2 + 6)^3 \ell^2}{S_{2,y}}.
\]
Moreover, if we let \(\beta = \frac{2}{3\beta_0}, \ m = 10\beta_0 - 1, \ S_{2,x} \geq 5600(d_1 + 4)\) and \(S_{2,y} \geq 5600(d_2 + 4)\), then we have
\[
\pi_\Delta(d_1, d_2, \mu_1, \mu_2) \leq \kappa^3 \ell^2 [\mu_y^2(d_1 + 6)^3 + \mu_y^2(d_2 + 6)^3].
\]

Proof. We proceed as follows:
\[
\Delta_t' = \Delta_{t-1}' + \delta_{t-1}' \\
= \mathbb{E}[\|\nabla_x f_{\mu_1}(\hat{x}_{t-1}, \hat{m}_{t-1}, \hat{y}_{t-1}, \hat{m}_{t-1}) - \hat{v}_{t-1}, \hat{m}_{t-1}\|^2_2] \\
\leq \mathbb{E}[\|\nabla_x f_{\mu_1}(\hat{x}_{t-1}, \hat{m}_{t-1}, \hat{y}_{t-1}, \hat{m}_{t-1}) - \hat{v}_{t-1}, \hat{m}_{t-1}\|^2_2] \\
+ \frac{1}{S_{2,x}} \mathbb{E}[\|G_{\mu_1}(\hat{x}_{t-1}, \hat{m}_{t-1}, \hat{y}_{t-1}, \hat{m}_{t-1}, \nu_t, \xi_t) - G_{\mu_1}(\hat{x}_{t-1}, \hat{m}_{t-1}, \hat{y}_{t-1}, \hat{m}_{t-1}, \nu_t, \xi_t)\|^2_2] \\
\leq \mathbb{E}[\|\nabla_x f_{\mu_1}(\hat{x}_{t-1}, \hat{m}_{t-1}, \hat{y}_{t-1}, \hat{m}_{t-1}) - \hat{v}_{t-1}, \hat{m}_{t-1}\|^2_2]
\]

\[32\]
where (i) follows from Lemma 5, and (ii) follows from Lemma 16. Applying eq. (61) recursively yields
\[
\mathbb{E}\left[\|\nabla_x f_{\mu_1}(\tilde{x}_{t-1}, \tilde{m}_{t-1}, \tilde{y}_t-1, \tilde{m}_{t-1}) - \tilde{v}_{t-1, \tilde{m}_{t-1}}\|^2\right] \\
\leq \mathbb{E}\left[\|\nabla_x f_{\mu_1}(\tilde{x}_{t-1,0}, \tilde{y}_t-1,0) - \tilde{v}_{t-1,0}\|^2\right] + \frac{2(d_1 + 4)\ell^2 \beta^2}{S_{2,x}} \sum_{k=0}^{\tilde{m}_{t-1}-1} \mathbb{E}[\|\tilde{u}_{t-1,k}\|^2] \\
+ \frac{2\tilde{m}_{t-1}\mu_1^2(d_1 + 6)^3\ell^2}{S_{2,x}} \\
\leq \mathbb{E}\left[\|\nabla_x f_{\mu_1}(\tilde{x}_{t-1,0}, \tilde{y}_t-1,0) - \tilde{v}_{t-1,0}\|^2\right] + \frac{2(d_1 + 4)\ell^2 \beta^2}{S_{2,x}} \sum_{k=0}^{m} \mathbb{E}[\|\tilde{u}_{t-1,k}\|^2] \\
+ \frac{2(m + 1)\mu_1^2(d_1 + 6)^3\ell^2}{S_{2,x}}. \tag{62}
\]
Similarly, we obtain
\[
\mathbb{E}\left[\|\nabla_y f_{\mu_2}(\tilde{x}_{t-1,0}, \tilde{y}_t-1,0) - \tilde{u}_{t-1,0}\|^2\right] \\
\leq \mathbb{E}\left[\|\nabla_y f_{\mu_2}(\tilde{x}_{t-1,0}, \tilde{y}_t-1,0) - \tilde{u}_{t-1,0}\|^2\right] + \frac{2(d_2 + 4)\ell^2 \beta^2}{S_{2,y}} \sum_{k=0}^{m} \mathbb{E}[\|\tilde{u}_{t-1,k}\|^2] \\
+ \frac{2(m + 1)\mu_2^2(d_2 + 6)^3\ell^2}{S_{2,y}}. \tag{63}
\]
Combining eq. (62) and eq. (63) yields
\[
\Delta_{t} \leq \Delta_{t-1} + \left(\frac{2(d_1 + 4)\ell^2 \beta^2}{S_{2,x}} + \frac{2(d_2 + 4)\ell^2 \beta^2}{S_{2,y}}\right) \sum_{k=0}^{m} \mathbb{E}[\|\tilde{u}_{t-1,k}\|^2] \\
+ \frac{2(m + 1)\mu_1^2(d_1 + 6)^3\ell^2}{S_{2,x}} + \frac{2(m + 1)\mu_2^2(d_2 + 6)^3\ell^2}{S_{2,y}}. \tag{64}
\]
For \(\Delta_{t-1,0}\), we obtain
\[
\Delta_{t-1,0} = \mathbb{E}[\|\nabla_x f_{\mu_1}(\tilde{x}_{t-1,0}, \tilde{y}_t-1,0) - \tilde{v}_{t-1,0}\|^2] + \mathbb{E}[\|\nabla_y f_{\mu_2}(\tilde{x}_{t-1,0}, \tilde{y}_t-1,0) - \tilde{u}_{t-1,0}\|^2] \\
\leq \mathbb{E}[\|\nabla_x f_{\mu_1}(\tilde{x}_{t-1,0}, \tilde{y}_t-1,0) - \tilde{v}_{t-1,0}\|^2] + \mathbb{E}[\|\nabla_y f_{\mu_2}(\tilde{x}_{t-1,0}, \tilde{y}_t-1,0) - \tilde{u}_{t-1,0}\|^2] \\
+ \frac{1}{S_{2,x}} \mathbb{E}[\|G(\tilde{x}_{t,0}, \tilde{y}_{t,0}, \nu_i, \xi_i) - G(\tilde{x}_{t-1,0}, \tilde{y}_{t-1,0}, \nu_{M_t}, \xi_i)\|^2] \\
+ \frac{1}{S_{2,y}} \mathbb{E}[\|H(\tilde{x}_{t,0}, \tilde{y}_{t,0}, \nu_i, \xi_i) - H(\tilde{x}_{t-1,0}, \tilde{y}_{t-1,0}, \nu_{M_t}, \xi_i)\|^2] \\
\leq \Delta_{t-1} + \left(\frac{2(d_1 + 4)\ell^2 \alpha^2}{S_{2,x}} + \frac{2(d_2 + 4)\ell^2 \alpha^2}{S_{2,y}}\right) \mathbb{E}[\|v_{t-1}\|^2] \\
+ \frac{2\mu_1^2(d_1 + 6)^3\ell^2}{S_{2,x}} + \frac{2\mu_2^2(d_2 + 6)^3\ell^2}{S_{2,y}}. \tag{65}
\]
where (i) follows from Lemma 5 and (ii) follows from Lemma 16. Substituting eq. (65) into eq. (64) yields
\[
\Delta_{t} \leq \Delta_{t-1} + \left(\frac{2(d_1 + 4)\ell^2 \alpha^2}{S_{2,x}} + \frac{2(d_2 + 4)\ell^2 \alpha^2}{S_{2,y}}\right) \mathbb{E}[\|v_{t-1}\|^2] \\
+ \left(\frac{2(d_1 + 4)\ell^2 \beta^2}{S_{2,x}} + \frac{2(d_2 + 4)\ell^2 \beta^2}{S_{2,y}}\right) \sum_{k=0}^{m} \mathbb{E}[\|\tilde{u}_{t-1,k}\|^2]
\]
\begin{align*}
&+ \frac{2(m + 2)\mu_1^2(d_1 + 6)^3\ell^2}{S_{2,x}} + \frac{2(m + 2)\mu_3^2(d_2 + 6)^3\ell^2}{S_{2,y}} \\
&\leq \Delta'_t + \left( \frac{2(d_1 + 4)\ell^2\alpha^2}{S_{2,x}} + \frac{2(d_2 + 4)\ell^2\alpha^2}{S_{2,y}} \right) \mathbb{E}[\|v_{t-1}\|_2^2]
&+ \frac{2\ell^2\beta^2}{S_{2,x}} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \left[ \mathbb{E}[\|\tilde{u}_{t,0}\|_2^2] + (m + 1) \left( \frac{2\mu_2^2\beta}{\beta} (d_2 + 3)^3 + 7\mu_2^2(d_2 + 6)^3\ell^2 \right) \right]. \tag{66}
\end{align*}

where (i) follows from Lemma 20. We next bound the term \(\mathbb{E}[\|\tilde{u}_{t-1,0}\|_2^2]\) as follows:

\begin{align*}
\mathbb{E}[\|\tilde{u}_{t-1,0}\|_2^2] &= \mathbb{E}[\|\tilde{u}_{t-1,0} - \nabla_y f_{\mu_2}(x_t, y_{t-1})\|_2^2] + 3\mathbb{E}[\|\nabla_y f_{\mu_2}(x_t, y_{t-1})\|_2^2]
&\leq 3\Delta'_t + 3\Delta'_t + 3\Delta'_t + 3\Delta'_t + 3\Delta'_t + 3\Delta'_t
&\leq 3\Delta'_t + 3\Delta'_t + \left[ 3 + \frac{6(d_1 + 4)}{S_{2,x}} + \frac{6(d_2 + 4)}{S_{2,y}} \right] \alpha^2\ell^2\mathbb{E}[\|v_{t-1}\|_2^2] + 6\ell^2 \left[ \frac{\mu_1^2(d_1 + 6)^3}{S_{2,x}} + \frac{\mu_3^2(d_2 + 6)^3}{S_{2,y}} \right] \\
&\leq 3\Delta'_t + 3\Delta'_t + 3\Delta'_t + 9\alpha^2\ell^2\mathbb{E}[\|v_{t-1}\|_2^2] + 6\ell^2 \left[ \frac{\mu_1^2(d_1 + 6)^3}{S_{2,x}} + \frac{\mu_3^2(d_2 + 6)^3}{S_{2,y}} \right] \tag{67}
\end{align*}

where (i) follows from Lemma 12, and (ii) follows from eq. (65), and (iii) follows from the fact that \(S_{2,x} \geq 2(d_1 + 4)\) and \(S_{2,y} \geq 2(d_2 + 4)\). Substituting eq. (67) into eq. (66) yields

\begin{align*}
\Delta'_t &\leq \left[ 1 + \frac{6\ell^2\beta^2}{1 - b} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \right] \Delta'_t + \frac{6\ell^2\beta^2}{1 - b} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \Delta'_t
&+ \frac{2\ell^2\alpha^2}{1 - b} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \left[ 1 + \frac{6\ell^2\beta^2}{1 - b} \mathbb{E}[\|v_{t-1}\|_2^2] \right]
&+ \frac{2\ell^2\alpha^2}{1 - b} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \left[ \frac{6\ell^2 \left[ \frac{\mu_1^2(d_1 + 6)^3}{S_{2,x}} + \frac{\mu_3^2(d_2 + 6)^3}{S_{2,y}} \right]}{(m + 1) \left( \frac{2\mu_2^2\beta}{\beta} (d_2 + 3)^3 + 7\mu_2^2(d_2 + 6)^3\ell^2 \right)} \right]
&+ \frac{7\mu_2^2(d_2 + 6)^3\ell^2}{S_{2,x}} + \frac{2(m + 2)\mu_3^2(d_2 + 6)^3\ell^2}{S_{2,y}} \\
&\leq \left[ 1 + \frac{6\ell^2\beta^2}{1 - b} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \right] \Delta'_t + \frac{6\ell^2\beta^2}{1 - b} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \Delta'_t
&+ \frac{2\ell^2\alpha^2}{1 - b} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \left[ 1 + \frac{6\ell^2\beta^2}{1 - b} \mathbb{E}[\|v_{t-1}\|_2^2] + \pi_\Delta(d_1, d_2, \mu_1, \mu_2). \tag{68}
\end{align*}

\textbf{Lemma 24.} Suppose Assumptions 2-3 hold. Let \(S_{2,x} \geq 2d_1 + 8\) and \(S_{2,y} \geq 2d_1 + 8\). Then, we have

\begin{align*}
\Delta'_t &\leq \left( \frac{4}{\beta\mu(m + 1)} + \frac{3\ell\beta}{2 - \beta} \right) \Delta'_t + \frac{2 + 2\ell\beta}{2 - \ell\beta} \Delta'_t + \left( \frac{4\ell^2\alpha^2}{\beta\mu(m + 1)} + 2\ell^2\alpha^2 + \frac{9\ell^2\beta \alpha^2}{2 - \ell\beta} \right) \mathbb{E}[\|v_{t-1}\|_2^2] \\
&+ \pi_\Delta(d_1, d_2, \mu_1, \mu_2),
\end{align*}

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where

\[
\pi_\delta(d_1, d_2, \mu_1, \mu_2) = \frac{2\ell^2(2 + 2\ell\beta)}{2 - \ell\beta} \left( \frac{\mu_2^2(d_1 + 6)^3}{S_{2,x}} + \frac{\mu_2^2(d_2 + 6)^3}{S_{2,y}} \right) + \frac{2}{\beta(m + 1)} \left( \frac{\mu_2^2}{4\mu} \ell^2(d_2 + 3)^3 + \mu_2^2\ell d_2 \right).
\]

Furthermore, if we let \( \beta = \frac{2}{13\ell^2}, m = 104\kappa - 1 \), then we have

\[
\pi_\delta(d_1, d_2, \mu_1, \mu_2) = \frac{5}{2} \mu_2^2\ell^2(d_1 + 6)^3 + 3\mu_2^2\ell^2(d_2 + 6)^3 + \frac{1}{8}\mu_2^2\ell d_2.
\]

**Proof.** Using the result in Lemma 19, and recalling in Algorithm 4 that \( \nabla g_{t,\mu_2}(\hat{y}_{t,\hat{m}}) = \nabla_y f(x_t, y_t) \) and \( \nabla g_{t,\mu_2}(\hat{y}_{t,0}) = \nabla_y f_{\mu_2}(x_{t+1}, y_t) \), we have

\[
\delta_{t+1}' \leq \frac{2}{\beta\mu(m + 1)} \mathbb{E}[\|\nabla_y f_{\mu_2}(x_{t+1}, y_t)\|^2] + \mathbb{E}[\|\nabla g_{t,\mu_2}(\hat{y}_{t,0}) - \hat{u}_{t,0}\|^2] + \frac{\ell\beta}{2 - \ell\beta} \mathbb{E}[\|\hat{u}_{t,0}\|^2] \\
+ \frac{2}{\beta\mu(m + 1)} \mathbb{E}[\|\nabla_y f_{\mu_2}(x_t, y_t)\|^2] + \frac{4}{\beta\mu(m + 1)} \mathbb{E}[\|\nabla g_{t,\mu_2}(x_t, y_t)\|^2] + \frac{\ell\beta}{2 - \ell\beta} \mathbb{E}[\|\hat{u}_{t,0}\|^2] \\
+ \frac{2}{\beta\mu(m + 1)} \left( \frac{\mu_2^2}{4\mu} \ell^2(d_2 + 3)^3 + \mu_2^2\ell d_2 \right) + \frac{4}{\beta\mu(m + 1)} \mathbb{E}[\|v_t\|^2] \\
+ \frac{\ell\beta}{2 - \ell\beta} \left[ 3\Delta_t' + 3\delta_t' + 9\ell^2\alpha^2 \mathbb{E}[\|v_t\|^2] + 6\ell^2 \left( \frac{\mu_2^2(d_1 + 6)^3}{S_{2,x}} + \frac{\mu_2^2(d_2 + 6)^3}{S_{2,y}} \right) \right] \\
+ \frac{2}{\beta(m + 1)} \left( \frac{\mu_2^2}{4\mu} \ell^2(d_2 + 3)^3 + \mu_2^2\ell d_2 \right) \\
= \left( \frac{4}{\beta\mu(m + 1)} + \frac{3\ell\beta}{2 - \ell\beta} \right) \delta_t' + \frac{2 + 2\ell\beta}{2 - \ell\beta} \Delta_t' + \left( \frac{4\ell^2\alpha^2}{\beta\mu(m + 1)} + 2\ell^2\alpha^2 + \frac{9\ell^3\beta\alpha^2}{2 - \ell\beta} \right) \mathbb{E}[\|v_t\|^2] \\
+ \frac{2\ell^2(2 + 2\ell\beta)}{2 - \ell\beta} \left( \frac{\mu_2^2(d_1 + 6)^3}{S_{2,x}} + \frac{\mu_2^2(d_2 + 6)^3}{S_{2,y}} \right) + \frac{2}{\beta(m + 1)} \left( \frac{\mu_2^2}{4\mu} \ell^2(d_2 + 3)^3 + \mu_2^2\ell d_2 \right) \\
\leq \left( \frac{4}{\beta\mu(m + 1)} + \frac{3\ell\beta}{2 - \ell\beta} \right) \delta_t' + \frac{2 + 2\ell\beta}{2 - \ell\beta} \Delta_t' + \left( \frac{4\ell^2\alpha^2}{\beta\mu(m + 1)} + 2\ell^2\alpha^2 + \frac{9\ell^3\beta\alpha^2}{2 - \ell\beta} \right) \mathbb{E}[\|v_t\|^2] \\
+ \pi_\delta(d_1, d_2, \mu_1, \mu_2),
\]

(69)

where (i) follows from eq. (65) and eq. (67), and from the fact that \( S_{2,x} \geq 2d_1 + 8 \) and \( S_{2,y} \geq 2d_2 + 8 \). The proof is complete by shifting the index in eq. (69) from \( t \) to \( t - 1 \).

We restate Theorem 2 as follows to include the specifics of the parameters.
Theorem 4 (Restate of Theorem 2 with parameter specifics). Let Assumptions 1, 2, 4, and 3 hold and apply ZO-SREDA-Boost in Algorithm 4 to solve the problem in eq. (1) with the following parameters:

\[
\begin{align*}
\zeta &= \frac{1}{\kappa}, & \alpha &= \frac{1}{24(\kappa + 1)\ell}, & \beta &= \frac{2}{13\ell}, & q &= \frac{2800\kappa}{13\kappa(\kappa + 1)}, \\
m &= 104\kappa - 1, & S_{2,x} &= \frac{5600(d_1 + 4)\kappa}{\epsilon}, & S_{2,y} &= \frac{5600(d_2 + 4)\kappa}{\epsilon}, \\
S_1 &= \frac{4032\sigma^2\kappa^2}{c^2}, & T &= \max\{1728(\kappa + 1)\ell, \frac{\Phi(x_0) - \Phi^*}{\sigma^2}, \frac{810\kappa}{\sigma^2}\}, \\
\delta &= \frac{711\kappa\ell \sqrt{d_1 + d_2}}{2}, & \mu_1 &= \frac{71\kappa^2.5\ell(d_1 + 6)^{1.5}}{2}, & \mu_2 &= \frac{71\kappa^2.5\ell(d_2 + 6)^{1.5}}{2}.
\end{align*}
\]

Algorithm 4 outputs \( \hat{x} \) such that

\[
\mathbb{E}[\|\nabla \Phi(\hat{x})\|_2] \leq \epsilon
\]

with at most \( O((d_1 + d_2)\kappa^3\epsilon^{-3}) \) function queries.

Proof. Recalling from eq. (18), we have

\[
\begin{align*}
\mathbb{E}[\Phi(x_{t+1})] &\leq \mathbb{E}[\Phi(x_t)] + \alpha \kappa^2 \delta_t + \alpha \Delta_t - \left( \frac{\alpha}{2} - \frac{L\alpha^2}{2} \right) \mathbb{E}[\|v_t\|_2^2] \\
&\leq \mathbb{E}[\Phi(x_t)] + 2\alpha \kappa^2 \delta_t' + 2\alpha \Delta_t' - \left( \frac{\alpha}{2} - \frac{L\alpha^2}{2} \right) \mathbb{E}[\|v_t\|_2^2] \\
&\quad + \frac{\mu_2 \alpha (\kappa^2 + 1)}{2} \ell^2(d_2 + 3)^3 + \frac{\mu_1 \alpha}{2} \ell^2(d_1 + 3)^3,
\end{align*}
\]

(70)

where (i) follows from Lemma 22. Rearranging eq. (70) and taking the summation over \( t = \{0, 1, \cdots, T - 1\} \) yield

\[
\left( \frac{\alpha}{2} - \frac{L\alpha^2}{2} \right) \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2] \leq \Phi(x_0) - \mathbb{E}[\Phi(x_T)] + 2\alpha \kappa^2 \sum_{t=0}^{T-1} \delta_t' + 2\alpha \sum_{t=0}^{T-1} \Delta_t' \\
&\quad + \alpha T \pi(d_1, d_2, \mu_1, \mu_2).
\]

(71)

Note that in eq. (71) we define

\[
\pi(d_1, d_2, \mu_1, \mu_2) = \frac{\mu_2^2 (\kappa^2 + 1)}{2} \ell^2(d_2 + 3)^3 + \frac{\mu_1^2}{2} \ell^2(d_1 + 3)^3.
\]

(72)

Then we proceed to prove Theorem 2/Theorem 4 in the following five steps.

Step 1. We establish the induction relationships for the tracking error and gradient estimation error with respect to the Gaussian smoothed function upon one outer-loop update for SREDA-Boost. Namely, we develop the relationship between \( \Delta_t' \) and \( \delta_{t-1}' \) as well as that between \( \Delta_t' \) and \( \delta_{t-1}' \), which are captured in Lemma 23 and Lemma 24.

Step 2. Based on Step 1, we provide the bounds on the inter-related accumulative errors \( \sum_{t=0}^{T-1} \Delta_t' \) and \( \sum_{t=0}^{T-1} \delta_t' \) over the entire execution of the algorithm.

We first consider \( \sum_{t=0}^{T-1} \Delta_t' \), for any \( n(T - 1) \leq t' < T - 1 \). Applying the inequality in Lemma 23 recursively, we obtain the following bound

\[
\Delta_t' \leq \left[ 1 + 6\ell^2 \beta \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \right] \Delta_{t-1}' + 6\ell^2 \beta \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \delta_{t-1}' \\
+ 2\ell^2 \alpha \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \left( 1 + \frac{9\ell^2 \beta^2}{1 - b} \right) \mathbb{E}[\|v_{t-2}\|_2^2] + \pi_{\Delta}(d_1, d_2, \mu_1, \mu_2, S_2)
\]

\[
\leq \left[ 1 + 6\ell^2 \beta \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \right]^{t-t'} \Delta_t'.
\]

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where (i) follows from the fact that
\[
\left[1 + \frac{6\ell^2\beta^2}{1-b} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \right]^{p-t'} \leq \left[1 + \frac{6\ell^2\beta^2}{1-b} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \right]^q \leq \left[1 + \frac{6\ell^2\beta^2}{1-b} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \right] \leq 2,
\]
where (ii) follows from the Bernoulli’s inequality Li and Yeh (2013) (eq. (21)), and (iii) follows from the fact that \( q = (1-b) \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right)^{-1}, \beta = \frac{2}{1-M}, \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) < 1, \) and \( b = 1 - \frac{\beta(M + \tau)}{2(M + \tau)}, \) which further implies that
\[
\frac{6\ell^2\beta^2}{1-b} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \leq \frac{6\ell^2\beta^2}{1-b} \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) = \frac{6\ell^2\beta^2}{1-6\ell^2\beta^2} < 1.
\]
Letting \( t' = (nT - 1)q \) and taking summation of eq. (73) over \( t = \{(nT - 1)q, \cdots, T-1\} \) yield
\[
\sum_{t=(nT-1)q}^{T-1} \Delta'_t \leq 2(T-(nT-1)q)\Delta'_{(nT-1)q} + 6\ell^2\beta^2 \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \sum_{t=(nT-1)q}^{T-1} \sum_{t=(nT-1)q}^{T-1} \delta'_p
+ 2\ell^2\alpha^2 \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \left(1 + 9\ell^2\beta^2 \right) \sum_{t=(nT-1)q}^{T-1} \sum_{t=(nT-1)q}^{T-1} \mathbb{E}[\|v_p\|_2^2]
+ 2(T-(nT-1)q)\pi_{\Delta}(d_1, d_2, \mu_1, \mu_2, S_2)
\leq 2(T-(nT-1)q)\epsilon(S_1, \delta) + 6\ell^2\beta^2 \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \sum_{t=(nT-1)q}^{T-2} \delta'_t
+ 2\ell^2\alpha^2 \left( \frac{d_1 + 4}{S_{2,x}} + \frac{d_2 + 4}{S_{2,y}} \right) \left(1 + 9\ell^2\beta^2 \right) \sum_{t=(nT-1)q}^{T-2} \mathbb{E}[\|v_t\|_2^2]
+ 2(T-(nT-1)q)\pi_{\Delta}(d_1, d_2, \mu_1, \mu_2)
= 2(T-(nT-1)q)\epsilon(S_1, \delta) + 6\ell^2\beta^2 \sum_{t=(nT-1)q}^{T-2} \delta'_t
+ 2\ell^2\alpha^2(1-b) \left(1 + 9\ell^2\beta^2 \right) \sum_{t=(nT-1)q}^{T-2} \mathbb{E}[\|v_t\|_2^2]
\]
Applying eq. (77) recursively from
and in follows from the fact that \( t_n \leq 1 + 2(1 + 9t_0^2) \sum_{t=(nT-1)q}^{T-2} \delta_1' \sum_{t=(nT-1)q}^{T-2} \mathbb{E}[\|v_t\|_2^2] \)

\[ + 2(T - (nT - 1)q) \pi_{\Delta}(d_1, d_2, \mu_1, \mu_2) \]

\[ \leq 2(T - (nT - 1)q) \epsilon(S_1, \delta) + 6t_0^2 \beta^2 \sum_{t=(nT-1)q}^{T-2} \delta_1' + 2t_0^2 \alpha^2 \sum_{t=(nT-1)q}^{T-2} \mathbb{E}[\|v_t\|_2^2] \]

\[ + 2(T - (nT - 1)q) \pi_{\Delta}(d_1, d_2, \mu_1, \mu_2) \]

\[ \leq 2(T - (nT - 1)q) \epsilon(S_1, \delta) + \frac{1}{7} \sum_{t=(nT-1)q}^{T-2} \delta_1' + 3t_0^2 \alpha^2 \sum_{t=(nT-1)q}^{T-2} \mathbb{E}[\|v_t\|_2^2] \]

\[ + 2(T - (nT - 1)q) \pi_{\Delta}(d_1, d_2, \mu_1, \mu_2), \]

where (i) follows from the fact that \( \Delta'_{(nT-n)q} \leq \epsilon(S_1, \delta) \) for all \( n \leq n_T \) (following from Lemma 4),

\[ \sum_{t=(nT-1)q}^{T-1} \sum_{p=(nT-1)q}^{t-1} \delta_p' \leq q \sum_{t=(nT-1)q}^{T-2} \delta_1', \]

and

\[ \sum_{t=(nT-1)q}^{T-1} \sum_{p=(nT-1)q}^{t-1} \mathbb{E}[\|v_t\|_2^2] \leq q \sum_{t=(nT-1)q}^{T-2} \mathbb{E}[\|v_t\|_2^2], \]

and in (ii) we use the fact that \( \beta = \frac{2}{13\mu} \). Applying steps similar to those in eq. (22) for iterations over \( t = \{(nT - n_t)q, \cdots, (nT - n_t + 1)q - 1\} \) yields

\[ \sum_{t=(nT-1)q}^{(nT-n_t+1)q-1} \Delta_t' \leq 2q \epsilon(S_1, \delta) + \frac{1}{7} \sum_{t=(nT-1)q}^{(nT-n_t+1)q-1} \delta_1' + 3t_0^2 \alpha^2 \sum_{t=(nT-1)q}^{(nT-n_t+1)q-1} \mathbb{E}[\|v_t\|_2^2] + 2q \pi_\Delta(d_1, d_2, \mu_1, \mu_2). \]

Taking summation of eq. (75) over \( n = 2, \cdots, n_T \) and combing with eq. (74) yield

\[ \sum_{t=0}^{T-1} \Delta_t' \leq 2T \epsilon(S_1, \delta) + \frac{1}{7} \sum_{t=0}^{T-1} \delta_1' + 3t_0^2 \alpha^2 \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2] + 2T \pi_\Delta(d_1, d_2, \mu_1, \mu_2). \]

Then we consider the upper bound on \( \sum_{t=0}^{T-1} \delta_t' \). Since \( m = \frac{16}{\mu \beta} + 1 \) and \( \beta = \frac{2}{13\mu} \), Lemma 24 implies

\[ \delta_t' \leq \frac{1}{2} \delta_0' + \frac{5}{4} \sum_{p=0}^{t-1} \frac{1}{2p} \Delta_p' + 3t_0^2 \alpha^2 \sum_{p=0}^{t-1} \frac{1}{2p} \mathbb{E}[\|v_p\|_2^2] + \pi_\delta(d_1, d_2, \mu_1, \mu_2). \]

Applying eq. (77) recursively from \( t \) to 0 yields

\[ \delta_t' \leq \frac{1}{2^t} \delta_0' + \frac{5}{2^t} \sum_{p=0}^{t-1} \frac{1}{2p} \Delta_p' + 3t_0^2 \alpha^2 \sum_{p=0}^{t-1} \frac{1}{2p} \mathbb{E}[\|v_p\|_2^2] + \pi_\delta(d_1, d_2, \mu_1, \mu_2), \]

Taking the summation of eq. (78) over \( t = \{0, 1, \cdots, T - 1\} \) yields

\[ \sum_{t=0}^{T-1} \delta_t' \leq \delta_0' + \frac{5}{2} \sum_{t=0}^{T-1} \frac{1}{2p} \Delta_p' + 3t_0^2 \alpha^2 \sum_{t=0}^{T-2} \frac{1}{2p} \mathbb{E}[\|v_p\|_2^2] \]

\[ + \pi_\delta(d_1, d_2, \mu_1, \mu_2) \sum_{t=0}^{T-1} \frac{1}{2p} \]

\[ \leq 2\delta_0' + \frac{5}{2} \sum_{t=0}^{T-2} \Delta_t' + 6t_0^2 \alpha^2 T \mathbb{E}[\|v_0\|_2^2] + 2T \pi_\delta(d_1, d_2, \mu_1, \mu_2). \]
Step 3. We decouple the bounds on $\sum_{t=0}^{T-1} \Delta'_t$ and $\sum_{t=0}^{T-1} \delta'_t$ in Step 2 from each other, and establish their separate relationships with the accumulative gradient estimators $\sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2]$.

Substituting eq. (79) into eq. (76) yields

$$\sum_{t=0}^{T-1} \Delta'_t \leq 2T\epsilon(S_1, \delta) + \frac{2}{7} \delta'_0 + 4\alpha^2 \ell^2 \sum_{t=0}^{T-2} \mathbb{E}[\|v_t\|_2^2] + \frac{5}{14} \sum_{t=0}^{T-2} \Delta'_t \quad + 2T\pi_\Delta(d_1, d_2, \mu_1, \mu_2) + \frac{2}{7} T\pi_\delta(d_1, d_2, \mu_1, \mu_2),$$

which implies

$$\sum_{t=0}^{T-1} \Delta'_t \leq 4T\epsilon(S_1, \delta) + \frac{1}{2} \delta'_0 + 7\alpha^2 \ell^2 \sum_{t=0}^{T-2} \mathbb{E}[\|v_t\|_2^2] \quad + \frac{1}{2} T\pi_\Delta(d_1, d_2, \mu_1, \mu_2) + 4T\pi_\delta(d_1, d_2, \mu_1, \mu_2). \tag{80}$$

Substituting eq. (80) into eq. (79) yields

$$\sum_{t=0}^{T-1} \delta'_t \leq 10T\epsilon(S_1, \delta) + 4\delta'_0 + 24\alpha^2 \ell^2 \sum_{t=0}^{T-2} \mathbb{E}[\|v_t\|_2^2] \quad + 10T\pi_\Delta(d_1, d_2, \mu_1, \mu_2) + 4T\pi_\delta(d_1, d_2, \mu_1, \mu_2). \tag{81}$$

Step 4. We bound $\sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2]$, and further cancel out the impact of $\sum_{t=0}^{T-1} \Delta'_t$ and $\sum_{t=0}^{T-1} \delta'_t$ by exploiting Step 3.

Substituting eq. (80) and eq. (81) into eq. (71) yields

$$\left(\frac{\alpha}{2} - \frac{L\alpha^2}{2}\right) \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2] \leq \Phi(x_0) - \mathbb{E}[\Phi(x_T)] + (20\kappa^2 + 8)\alpha T\epsilon(S_1, \delta) + (8\kappa^2 + 1)\alpha \delta'_0 + (48\kappa^2 + 14)\alpha^3 \ell^2 \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2] \quad + (20\kappa^2 + 1)\alpha T\pi_\Delta(d_1, d_2, \mu_1, \mu_2) + (8\kappa^2 + 8)\alpha T\pi_\delta(d_1, d_2, \mu_1, \mu_2) + \alpha T\pi(d_1, d_2, \mu_1, \mu_2) \quad (i) \leq \Phi(x_0) - \mathbb{E}[\Phi(x_T)] + 28\kappa^2 \alpha T\epsilon(S_1, \delta) + 9\kappa^2 \alpha \delta'_0 + 62\alpha^3 L^2 \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2] \quad + 21\kappa^2 \alpha T\pi_\Delta(d_1, d_2, \mu_1, \mu_2) + 16\kappa^2 \alpha T\pi_\delta(d_1, d_2, \mu_1, \mu_2) + \alpha T\pi(d_1, d_2, \mu_1, \mu_2), \tag{82}$$

where (i) follows from the fact that $L = (1 + \kappa)\ell$ and $\kappa > 1$. Rearranging eq. (82), we have

$$\left(\frac{\alpha}{2} - \frac{L\alpha^2}{2} - 62L^2 \alpha^3\right) \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2] \leq \Phi(x_0) - \mathbb{E}[\Phi(x_T)] + 28\kappa^2 \alpha T\epsilon(S_1, \delta) + 9\kappa^2 \alpha \delta'_0 \quad + 21\kappa^2 \alpha T\pi_\Delta(d_1, d_2, \mu_1, \mu_2) + 16\kappa^2 \alpha T\pi_\delta(d_1, d_2, \mu_1, \mu_2) + \alpha T\pi(d_1, d_2, \mu_1, \mu_2). \tag{83}$$

Since $\alpha = \frac{1}{4\ell L}$, we obtain

$$\frac{\alpha}{2} - \frac{L\alpha^2}{2} - 62L^2 \alpha^3 = \frac{214}{13824L} \geq \frac{1}{72L}. \tag{84}$$

Substituting eq. (84) into eq. (83) and applying Assumption 1 yield

$$\sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2] \leq 72L(\Phi(x_0) - \Phi^*) + 84\kappa^2 T\epsilon(S_1, \delta) + 27\kappa^2 \delta'_0.$$
\[ + 63 \kappa^2 T\pi_\Delta(d_1, d_2, \mu_1, \mu_2) + 48 \kappa^2 T\pi_\delta(d_1, d_2, \mu_1, \mu_2) + 3 T\pi(d_1, d_2, \mu_1, \mu_2). \]  

(85)

**Step 5.** We establish the convergence bound on \( \mathbb{E}[\|\nabla \Phi(\hat{x})\|_2] \) based on the bounds on its estimators \( \sum_{t=0}^{T-1} \mathbb{E}[v_t^2] \) and the two error bounds \( \sum_{t=0}^{T-1} \Delta_t \) and \( \sum_{t=0}^{T-1} \delta'_t \).

Recall eq. (34) we have

\[
\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_t)\|_2^2] \leq 3 \kappa^2 \sum_{t=0}^{T-1} \delta_t + 3 \sum_{t=0}^{T-1} \Delta_t + 3 \sum_{t=0}^{T-1} \mathbb{E}[v_t^2] \\
\leq 6 \kappa^2 \sum_{t=0}^{T-1} \delta'_t + 6 \sum_{t=0}^{T-1} \Delta'_t + 3 \sum_{t=0}^{T-1} \mathbb{E}[v_t^2] + 3 T\pi(d_1, d_2, \mu_1, \mu_2) \tag{86}
\]

where \( (i) \) follows from Lemma 22. Substituting eq. (80), eq. (81) and eq. (85) into eq. (86) yields

\[
\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_t)\|_2^2] \\
\leq (60 \kappa^2 + 24) T\epsilon(S_1, \delta) + (24 \kappa^2 + 3) \delta'_0 + (60 \kappa^2 + 3) T\pi_\Delta(d_1, d_2, \mu_1, \mu_2) \\
+ (24 \kappa^2 + 24) T\pi_\delta(d_1, d_2, \mu_1, \mu_2) + (144 \kappa^2 \alpha^2 \ell^2 + 42 \alpha^2 \ell^2 + 3) \sum_{t=0}^{T-1} \mathbb{E}[v_t^2] \\
+ 3 T\pi(d_1, d_2, \mu_1, \mu_2) \tag{87}
\]

where \( (i) \) follows from the fact that \( \kappa > 1 \), \( L = (\kappa + 1) \ell \) and \( \alpha = \frac{1}{3T} \), and \( (ii) \) follows from eq. (85). Recall \( L = (1 + \kappa) \ell \). Then, eq. (87) implies that

\[
\mathbb{E}[\|\nabla \Phi(\hat{x})\|_2^2] \\
\leq 288 L(\Phi(x_0) - \Phi^*) + 420 \kappa^2 T\epsilon(S_1, \delta) + 135 \kappa^2 \delta'_0 + 315 \kappa^2 T\pi_\Delta(d_1, d_2, \mu_1, \mu_2) \\
+ 240 \kappa^2 T\pi_\delta(d_1, d_2, \mu_1, \mu_2) + 15 T\pi(d_1, d_2, \mu_1, \mu_2). \tag{88}
\]

Recalling Lemma 15, we have

\[
\epsilon(S_1, \delta) \leq \frac{(d_1 + d_2) \ell^2 \delta^2}{2} + \frac{4 \sigma^2}{S_1} + \frac{\mu_1^2}{2} \ell^2 (d_1 + 3)^3 + \frac{\mu_2^3}{2} \ell^2 (d_2 + 3)^3.
\]

If we let \( \delta'_0 \leq \frac{1}{\kappa}, T = \max\{1728(\kappa + 1)^2(\frac{\ell(x_0) - \Phi^*}{\sqrt{\kappa^2 + 3}}), \frac{180}{\kappa^2 \ell^2}, \frac{4032 \kappa^2 \ell^2}{\kappa^2 + 3}, S_1 = \frac{40320 \kappa^2 \ell^2}{\kappa^2 + 3}, \mu_1 = \frac{40320 \kappa^2 \ell^2}{\kappa^2 + 3}, \mu_2 = \frac{40320 \kappa^2 \ell^2}{\kappa^2 + 3}\} \), and further let \( \delta = \frac{\sqrt{144 \kappa^2 \alpha^2 \ell^2 + 42 \alpha^2 \ell^2 + 3} S_1}{\frac{180}{\kappa^2 \ell^2}} = \frac{144 \kappa^2 \alpha^2 \ell^2}{\kappa^2 + 3} \), according to the definition of \( \epsilon(S_1, \delta) \) (Lemma 15), \( \pi_\Delta(d_1, d_2, \mu_1, \mu_2) \) (Lemma 23), \( \pi_\delta(d_1, d_2, \mu_1, \mu_2) \) (Lemma 24) and \( \pi(d_1, d_2, \mu_1, \mu_2) \) (eq. (72)), then we have \( 420 \kappa^2 \epsilon(S_1, \delta) \leq \frac{\epsilon^2}{6} \), and

\[
315 \kappa^2 \pi_\Delta(d_1, d_2, \mu_1, \mu_2) + 240 \kappa^2 \pi_\delta(d_1, d_2, \mu_1, \mu_2) + 15 \pi(d_1, d_2, \mu_1, \mu_2) \leq \frac{\epsilon^2}{2},
\]

which implies

\[
\mathbb{E}[\|\nabla \Phi(\hat{x})\|_2] \leq \sqrt{\mathbb{E}[\|\nabla \Phi(\hat{x})\|_2^2]} \leq \epsilon.
\]
We also let $S_{2,x} = \frac{5600(d_1 + 4)\kappa}{\epsilon}$, $S_{2,y} = \frac{5600(d_2 + 4)\kappa}{\epsilon}$ and $q = \frac{2800\kappa}{14(\kappa + 1)}$. Then, the total sample complexity is given by

$$T \cdot (S_{2,x} + S_{2,y}) \cdot m + \left[ \frac{T}{q} \right] \cdot S_1 \cdot (d_1 + d_2) + T_0$$

$$\leq \Theta \left( \frac{\kappa}{\epsilon^2} \cdot \frac{(d_1 + d_2)\kappa}{\epsilon} \cdot \kappa \right) + \Theta \left( \frac{\kappa^2}{\epsilon^2} \cdot (d_1 + d_2) \right) + \Theta (d_2\kappa \log(\kappa))$$

$$= \mathcal{O} \left( \frac{(d_1 + d_2)\kappa^3}{\epsilon^3} \right),$$

which completes the proof. □

C.6 Proof of Corollary 2

In the finite-sum case, recall that

$$f(x, y) \triangleq \frac{1}{n} \sum_{i=1}^{n} F(x, y; \xi_i).$$

Here we modify Algorithm 5 by replacing the mini-batch update used in line 6 of Algorithm 4 with the following update using all samples:

$$v_t = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{d_1} F(x_t + \delta e_j, y_t, \xi_i) - F(x_t - \delta e_j, y_t, \xi_i) \frac{2\delta}{\epsilon} e_j,$$

$$u_t = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{d_2} F(x_t, y_t + \delta e_j, \xi_i) - F(x_t, y_t - \delta e_j, \xi_i) \frac{2\delta}{\epsilon} e_j,$$

where $e_j$ denotes the $j$-th canonical unit basis vector. In this case, if $\text{mod}(k, q) = 0$, then we have

$$\epsilon(S_1, \delta) \leq \frac{(d_1 + d_2)\ell^2\delta^2}{2} + \frac{\mu_1^2}{2} \ell^2(d_1 + 3)^3 + \frac{\mu_2^2}{2} \ell^2(d_2 + 3)^3. \quad (89)$$

Case 1: $n \geq \kappa^2$

Substituting eq. (89) into eq. (88), it can be checked easily that under the same parameter settings for $\delta_0^*, T$, $\delta$, $\mu_1$ and $\mu_2$ in Theorem 2, we have

$$\mathbb{E}[||\nabla \Phi(\hat{x})||_2] \leq \sqrt{\mathbb{E}[||\nabla \Phi(\hat{x})||^2_2]} \leq \epsilon.$$

Then, let $S_{2,x} = 5600(d_1 + 4)\kappa\sqrt{n}$, $S_{2,y} = 5600(d_2 + 4)\kappa\sqrt{n}$ and $q = \frac{2800\kappa}{14(\kappa + 1)}$. Recalling the sample complexity result of ZO-iSARSH in the finite-sum case in Appendix C.4, we have $T_0 = \mathcal{O}(d_2(\kappa + n) \log(\kappa))$. The total sample complexity is given by

$$T \cdot (S_{2,x} + S_{2,y}) \cdot m + \left[ \frac{T}{q} \right] \cdot S_1 \cdot (d_1 + d_2) + T_0$$

$$\leq \Theta \left( \frac{\kappa}{\epsilon^2} \cdot \frac{(d_1 + d_2)\kappa}{\epsilon} \cdot \sqrt{n} \cdot \kappa \right) + \Theta \left( \frac{\kappa^2}{\epsilon^2} \cdot n \cdot (d_1 + d_2) \right) + \Theta (d_2(\kappa + n)\kappa \log(\kappa))$$

$$= \mathcal{O} \left( (d_1 + d_2)(\sqrt{n}\kappa^2\epsilon^{-2} + n) \right) + \mathcal{O}(d_2(\kappa^2 + \kappa n) \log(\kappa)).$$

Case 2: $n \leq \kappa^2$

In this case, we let $S_{2,x} = 56(d_1 + 4) + 420$, $S_{2,y} = 56(d_2 + 4) + 420$ and $q = 1$. Then we have

$$\Delta_t' \leq \epsilon = \frac{(d_1 + d_2)\ell^2\delta^2}{2} + \frac{\mu_1^2}{2} \ell^2(d_1 + 3)^3 + \frac{\mu_2^2}{2} \ell^2(d_2 + 3)^3, \quad \text{for all} \quad 0 \leq t \leq T - 1. \quad (90)$$
Given the value of $S_{2,x}$ and $S_{2,y}$, it can be checked that the proofs of Lemma 20 and Lemma 24 still hold. Following from the steps similar to those from eq. (71) to eq. (79), we obtain

$$
\sum_{t=0}^{T-1} \delta'_t \leq 2 \delta'_0 + \frac{5}{2} T \epsilon_{\Delta} + 6 \ell^2 \alpha^2 \sum_{t=0}^{T-2} \mathbb{E}[\|v_t\|_2^2] + 2 T \pi_\delta(d_1, d_2, \mu_1, \mu_2).
$$

(91)

Substituting eq. (90) and eq. (91) into eq. (71) yields

$$
\left(\frac{\alpha}{2} - \frac{L \alpha^2}{2}\right) \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2]
\leq \Phi(x_0) - \mathbb{E}[\Phi(x_T)] + 4 \alpha \kappa^2 \delta'_0 + 7 \alpha \kappa^2 T \epsilon_{\Delta} + 12 L^2 \alpha^3 \sum_{t=0}^{T-2} \mathbb{E}[\|v_t\|_2^2]
+ 4 \alpha \kappa^2 T \pi_\delta(d_1, d_2, \mu_1, \mu_2) + \alpha T \pi(d_1, d_2, \mu_1, \mu_2),
$$

(92)

where in (i) we use $L = (1 + \kappa) \ell$. Rearranging eq. (92) yields

$$
\left(\frac{\alpha}{2} - \frac{L \alpha^2}{2} - 12 L^2 \alpha^3\right) \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2]
\leq \Phi(x_0) - \mathbb{E}[\Phi(x_T)] + 4 \alpha \kappa^2 \delta'_0 + 7 \alpha \kappa^2 T \epsilon_{\Delta} + 4 \alpha \kappa^2 T \pi_\delta(d_1, d_2, \mu_1, \mu_2) + \alpha T \pi(d_1, d_2, \mu_1, \mu_2).
$$

(93)

Letting $\alpha = \frac{1}{32 L}$, we obtain

$$
\frac{\alpha}{2} - \frac{L \alpha^2}{2} - 12 L^2 \alpha^3 = \frac{1}{32 L}.
$$

(94)

Substituting eq. (93) into eq. (94) and applying Assumption 1 yield

$$
\sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2] \leq 32 L (\Phi(x_0) - \Phi^*) + 16 \kappa^2 \delta'_0 + 28 \kappa^2 T \epsilon_{\Delta} + 16 \kappa^2 T \pi_\delta(d_1, d_2, \mu_1, \mu_2)
+ 4 T \pi(d_1, d_2, \mu_1, \mu_2).
$$

(95)

Substituting eq. (95) and eq. (90) into eq. (86) yields

$$
\sum_{t=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_t)\|_2^2]
\leq 6 \kappa^2 \sum_{t=0}^{T-1} \delta'_t + 6 T \epsilon_{\Delta} + 3 \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2] + 3 T \pi(d_1, d_2, \mu_1, \mu_2)
\leq 12 \kappa^2 \delta'_0 + 21 \kappa^2 T \epsilon_{\Delta} + 4 \sum_{t=0}^{T-1} \mathbb{E}[\|v_t\|_2^2] + 12 \kappa^2 T \pi_\delta(d_1, d_2, \mu_1, \mu_2) + 3 T \pi(d_1, d_2, \mu_1, \mu_2)
\leq 128 L (\Phi(x_0) - \Phi^*) + 76 \kappa^2 \delta'_0 + 133 \kappa^2 T \epsilon_{\Delta} + 76 \kappa^2 T \pi_\delta(d_1, d_2, \mu_1, \mu_2) + 19 T \pi(d_1, d_2, \mu_1, \mu_2).
$$

(96)

Recall that $L = (1 + \kappa) \ell$. Then, eq. (96) implies

$$
\mathbb{E}[\|\nabla \Phi(\hat{x})\|_2^2] \leq 128 (\kappa + 1) \ell \frac{\Phi(x_0) - \Phi^*}{T} + 133 \kappa^2 \epsilon_{\Delta} + \frac{76 \kappa^2 \delta'_0}{T} + 76 \kappa^2 \pi_\delta(d_1, d_2, \mu_1, \mu_2)
+ 19 \pi(d_1, d_2, \mu_1, \mu_2).
$$

If we let $\delta'_0 \leq \frac{1}{T}, T = \max \{640(\kappa + 1)\ell \frac{\Phi(x_0) - \Phi^*}{\epsilon_{\Delta}}, \frac{380 \kappa}{\epsilon_{\Delta}}\}$, and let $\mu_1, \mu_2$ and $\delta$ follow the same setting in Theorem 2, then we have

$$
\mathbb{E}[\|\nabla \Phi(\hat{x})\|_2^2] \leq \sqrt{\mathbb{E}[\|\nabla \Phi(\hat{x})\|_2^2]} \leq \epsilon.
$$

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Recall the sample complexity result of ZO-iSARSH in the finite-sum case in Appendix C.4. Then, we have $T_0 = O(d_2(\kappa + n) \log(\kappa))$. The total sample complexity is given by

$$T \cdot (S_{x} + S_{y}) \cdot m + \left\lceil \frac{T}{q} \right\rceil \cdot S_1 \cdot (d_1 + d_2) + T_0 \leq \Theta\left(\frac{\kappa}{\epsilon^2} \cdot (d_1 + d_2) \cdot \kappa\right) + \Theta\left(\left\lfloor \frac{\kappa}{\epsilon^2} \right\rfloor \cdot n \cdot (d_1 + d_2)\right) + \Theta(d_2(\kappa + n) \log(\kappa))$$

$$= O\left((d_1 + d_2)(\kappa^2 + \kappa n)\epsilon^{-2}\right).$$