Stability analysis of Cu – C₆H₉NaO₇ and Ag – C₆H₉NaO₇ nanofluids with effect of viscous dissipation over stretching and shrinking surfaces using a single phase model

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A B S T R A C T

A mathematical analysis is performed to study the flow and heat transfer phenomena of Casson based nanofluid with effects of the porosity parameter and viscous dissipation over the exponentially permeable stretching and shrinking surface. The considered nanofluid comprises Casson as a base fluid that contains silver (Ag) and copper (Cu) solid nanoparticles. The system of the nonlinear governing partial differential equations (PDEs) are converted into ordinary differential equations (ODEs) by applying similarity transformation. The obtained ODEs are solved by using shooting technique in Maple software. Numerically obtained results reveal dual solutions for various values of pertinent parameters. Due to occurrence of dual solutions, the stability analysis is done in order to find stable solution. Positive signs of smallest eigenvalues point out that the first solution is stable and second unstable. The variation of the velocity and the temperature profiles with coefficient of the skin friction and the Nusselt number are shown graphically. Both temperature profiles and its boundary layer thicknesses increase as volume fraction of nanoparticles of Ag and Cu are increased in the Casson fluid. Velocity profiles and corresponding boundary layer thicknesses decrease by suspension of nanoparticles of silver and copper, whereas the silver Ag nanoparticles show the greater rate of heat transfer enhancement as compared to copper Cu nanoparticles when suspended in Casson fluid.

1. Introduction

It is worth to note that we are living in the period of the science and technology. While at present, the most modern rapid growing technology is a nanotechnology that possess huge applications in different fields such as in transportation, atomic reactors, electronics and biomedicine. After discovery of carbon nanotubes by Morinobu Endo [1] and the potential of their importance have been achieved a great importance to the various fields of research. The nanofluid is one of the interesting topics of nano-science. It may be defined as, the suspension of the nano-meter size particles with 1nm–100nm in the common fluid is named as nanofluid [3]. The nanoparticles that are mostly used in preparation of the nanofluids are ceramics (Al₂O₃, CuO), pure metals (Au, Ag, Cu, etc), nitrides (AlN, SiN), metal carbides (SiC), nano-tubes of carbon, non-metals of Graphite, coated of metals (Al + Al₂O₃, Cu + C) and some functionalized nanoparticles Masuda, Ebata & Teramae [4]. Whereas the base fluids includes water, oils, bio-fluids, lubricants, ethylene, acetone, decent, polymer solutions and other coolants.

Choi and Eastman [5] were the first to introduce such a new kind of the modern fluid. It is observed the material that is mostly used to prepare the nanofluids are ceramic particles. Because of these particles are more stable in solution as compared to others [6]. The ceramics can be divided into three different classes, such as oxides (i.e. zirconia and alumina), non-oxides (i.e. silicide's, nitrides and carbides) and mixtures of non-oxides and oxides. Each one of them possess unique material property. The nanofluids prepared by the combination of pure metal and alloy with common base fluid possess high thermal conductivity as compare to prepared by oxides, but these fluids are seen in less quantity in literature. The alloy nanofluids are prepared by the inert gas condensation process or by the mechanical alloying. The carbon based nanofluids have large intrinsic thermal conductivity. The examples of nanofluids prepared from carbon are carbon nanotubes that are single wall nanotubes, multi wall nanotubes, and ultra-dispersed diamond in the different fluids. Thermal

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conductivity performance in nanofluids have been reported up to 150% by Choi et al. [7] and critical heat flux enhancement up to 200% by You et al. [8] when compared to the corresponding properties and critical heat flux values of common base fluids.

The study about the flow over stretching and shrinking surfaces have taken much importance due to their huge applications in engineering and industries like capillary effect in the small pores, hydraulic properties of the agricultural clay soil, rising of the shrinking balloons, the shrinking-swell behavior and shrinking films. Miklavec and Wang [9] studied different types of the flows on shrinking surfaces first time. Flow is more resisted in shrinking surfaces as compare to the stretching surfaces. Since the flow is probably not going to occur in the shrinking surface easily, it presents sufficient suction on the boundary to conflict vorticity in the boundary layer. Recently, many researchers considered the stretching/shrinking surfaces such as Lund et al. [10], Salleh et al. [11], Ahmad et al. [12], Dero et al. [13], Ismail et al. [14], Selimefendigil et al. [15]. Hayat et al. [16] investigated the Tiwari and Das’s model on the exponentially stretching/shrinking surfaces such as Lund et al. [10], Salleh et al. [11], Ahmad et al. [12]. Recently, many researchers considered the stretching/shrinking surfaces such as Lund et al. [10], Salleh et al. [11], Ahmad et al. [12]. The well-known examples of the Casson fluid are blood and chocolate.

In this study, there have been extended the work of Hatami et al. [19] with multiple solutions for Ag and Cu – CuH2NaO2 nanofluids over a stretching/shrinking surface. To best of author’s knowledge, the stability analysis in occurrence of the multiple solutions in Ag – CuH2NaO2 and Cu – CuH2NaO2 nanofluids flow over the exponential permeable stretching/shrinking surface is not examined by other researchers in past years. The aim of present research is to find multiple solutions along stability analysis and to find the effect of silver and copper nanoparticles in Casson fluid flow. The fluid is vital for enhancement of the heat transfer and numerous techniques are introduced to improve it. Rohni et al. [20] claimed that execution of conventional fluid in the upgrade of heat transfer is extremely poor. Considering the conclusion of all previous studies, this study considered non-Newtonian model. The motivation behind this work is that the Tiwari and Das’s model has many applications for Casson based nanofluids and to consider all multiple solutions.

There is hoped that this study will provide a fruitful help to the researchers which are interested to work in field of the nanotechnology theoretically and practically.

2. Problem formulation

Consider steady two-dimensional incompressible flow and heat transfer of Casson base nanofluid over the exponentially stretching and shrinking surface with viscous dissipations in the porous medium as shown in Figure 1. It is supposed the rheological equation of the state to the isotropic and the Casson fluid flow is written as by Nakamura and Sawada [21]:

\[
\tau_{ij} = \begin{cases} 
\mu_b + P_{nf} \frac{\tau}{\sqrt{2\pi}} & \pi > \pi_c, \\
\mu_b + P_{nf} \frac{\tau}{\sqrt{2\pi}} & \pi < \pi_c.
\end{cases}
\]

where plastic dynamical viscosity relative to non-Newtonian fluid that acts like solid if shear stress is lesser than yield stress, It shows deformation when yield stress becomes lesser than shear stress. Recent study about Casson fluid flow can be found in [17, 18]. The well-known examples of the Casson fluids are blood and chocolate.

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The velocity components are defined as
\[ \frac{\partial \psi}{\partial x} v = \frac{\partial \psi}{\partial y} u \]
By applying above transformations, Eqs. (3) and (4) along the boundary conditions (5) take the following forms:

\[ \left(1 + \frac{1}{\beta}\right) f' + \left(1 - \varphi + \varphi \left(\rho_f/\rho_s\right)\phi\right)(1 - \varphi)^2 f - 2f' - f' = 0 \]

subject to boundary condition:
\[ f(0) = S; f'(0) = \lambda; \theta(0) = 1; f'(\infty) \to 0; \theta(\infty) \to 0 \] (9)

Moreover, \( \gamma = \frac{\eta}{1 + \eta} \) is the porosity parameter, \( Pr = \frac{\eta/\rho_f c_p}{k_f} \) is Prandtl number, \( Ec = \frac{\omega_k}{\nu_r} \) is Eckert number and \( S = -\sqrt{\frac{\lambda s}{\mu_f}} \) is suction/injection parameter.

The skin-friction coefficient \( C_f \) and the local Nusselt number \( N_u \) is defined as:

\[ C_f = \frac{\mu_f}{\rho u_t^2} \quad N_u = \frac{-x k_f \frac{\partial \psi}{\partial y} \left|_{y=0} \right.}{k_f (T_e - T_m)} \]

Using Eq. (6) in above relations, we get:

\[ C_f(R_e) = \frac{1}{(1 - \infty)^{1/2}} f'(0); \quad N_u(R_e) = \frac{k_f}{k_f (T_e - T_m)} \] (10)

where \( R_e = 2LU_\infty e^{2\gamma/\beta} \) is a local Reynolds number.

3. Stability analysis

According to requirement of stability analysis, there is need to convert Eqs. (3) and (4) in unsteady case in order to perform stability analysis of solutions. Thus, we have:

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\rho u}{\rho_v K} \]

\[ \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\partial T}{\partial y} - \frac{\partial T}{\partial y} \] (12)

where \( t \) is the time. Now, a new dimensionless variable \( \tau \) is introduced, then equation [6] can be written as follows:

\[ \psi = \sqrt{2BIU_\infty e^{2\gamma/\beta}} \left(\eta, \tau\right); \quad \theta(\eta) = \frac{T - T_m}{T_m - T_m} \quad \eta = \sqrt{\frac{U_\infty}{2BI}} e^{2\gamma/\beta}; \quad \tau = \frac{U_\infty}{2BI} \] (13)

By applying Eq. (13) in Eqs. (11) and (12), we get:

\[ \left(1 + \frac{1}{\beta}\right) \frac{\partial \psi}{\partial y} + \left(1 - \varphi + \varphi \left(\rho_f/\rho_s\right)\phi\right)(1 - \varphi)^2 \left(1 - \varphi\right) \left(1 + \frac{1}{\beta}\right) \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial y} \]

subject to boundary conditions:

\[ f(0, \tau) = S; \frac{\partial f(0, \tau)}{\partial \eta} = \lambda; \theta(0, \tau) = 1; \]

\[ \frac{\partial f(\eta, \tau)}{\partial \eta} \to 0; \theta(\eta, \tau) \to 0; \quad \text{as} \quad \eta \to \infty \] (16)

To check the stability of solutions of \( f(\eta, \tau) = f_0(\eta) \) and \( \theta(\eta) = \theta_0(\eta) \), that satisfy boundary value problems (7) and (9) we have:

\[ f(\eta, \tau) = f_0(\eta) + e^{-\epsilon \tau} F(\eta, \tau) \]

\[ \theta(\eta, \tau) = \theta_0(\eta) + e^{-\epsilon \tau} G(\eta, \tau) \] (17)

Where both the \( F(\eta, \tau) \) and \( G(\eta, \tau) \) are the small relative with the \( f_0(\eta) \) and \( \theta_0(\eta) \) respectively, \( \epsilon \) is the unknown eigenvalue that provides an infinite set of the eigenvalues \( \epsilon_1 < \epsilon_2 < \epsilon_3 \ldots \). The stability of solution depends upon the sign of the smallest eigenvalue \( \epsilon_1 \). If the value of \( \epsilon_1 \) is positive that means that flow is the stable and there takes place the initial decay, conversely, the negative value of \( \epsilon_1 \) shows the unstable flow which indicates initial growth of disturbance. By substituting Eq. (17) in Eqs. (14), (15), and (16), we get:

\[ \left(1 + \frac{1}{\beta}\right) F_0 + \left(1 - \varphi + \varphi \left(\rho_f/\rho_s\right)\phi\right)(1 - \varphi)^2 \left(1 + \frac{1}{\beta}\right) \frac{\partial F_0}{\partial \eta} - \frac{\partial F_0}{\partial \eta} = 0 \] (18)

\[ \frac{k_f}{k_f} \frac{\partial^2 G_0}{\partial \eta^2} + \left(1 - \varphi + \varphi \left(\rho_f/\rho_s\right)\phi\right)(1 - \varphi)^2 \left(1 + \frac{1}{\beta}\right) \frac{\partial G_0}{\partial \eta} = 0 \] (19)

subject to boundary conditions:

\[ F_0(0) = 0; \quad F_0(\eta) \to 0; \quad G_0(\eta) = 0 \]

\[ \theta_0(\eta) \to 0; \quad \text{as} \quad \eta \to \infty \] (20)

To achieve stability analysis of the solution, Eqs. (18) and (19) of linear eigenvalue problem along the boundary conditions (20) have been solve by applying BVP4c solver function in the MATLAB. Further, we kept the values of following parameters as \( \beta = 2; \varphi = 0.1; Pr = 10; Ec = 0.5 \) and vary the values of \( \lambda \) and \( S \).
4. Result and discussion

In present paper, two dimensional Casson base nanofluids prepared by two different kinds of the solid nanoparticles of Silver (Ag) and Copper (Cu) are studied. The effects of the solid volume fractions (φ) are examined in a range of 0 ≤ φ ≤ 0.2 with value of the Pr = 10. The thermo-physical properties of the base fluid and nanoparticles are presented in Table 1.

| Physical properties | ρ (kg/m³) | C_p (J/kg·K) | k (W/m·K) |
|---------------------|-----------|--------------|-----------|
| CsH_3NO_3           | 989       | 4175         | 0.6376    |
| Cu                  | 8933      | 385          | 401       |
| Ag                  | 10500     | 235          | 429       |

Table 1. Thermo-physical properties of base fluid (Casson) and nanoparticles.

| Table 2. Comparison of results for f''(0) and −θ'(0) for Pr = 0.7, λ = −1, Ec = 0 and φ = 0 |
|---------------------------------------------------------------|
| Sudipta Ghosh1 and Swati Mukhopadhyay [22]                     |
| First solution ___ Second solution ___                         |
| f''(0)             − θ'(0)         f''(0)             − θ'(0)         |
| 2.39082            1.77124         -0.97223          0.84832     |

| Present study       |
|---------------------|
| First solution ___   Second solution ___                      |
| f''(0)             − θ'(0)         f''(0)             − θ'(0)         |
| 2.39095            1.77271         -0.97362          0.85083     |

Figure 2. Graph of f''(0) with different values of λ and S for Ag – CsH_3NO_3 working fluid.

Figure 3. Graph of f''(0) with different values for λ and S for Cu – CsH_3NO_3 working fluid.

Figure 4. Graph of θ'(0) with different values of λ and S for Ag – CsH_3NO_3 working fluid.

Figure 5. Graph of θ'(0) with different values of λ and S for Cu – CsH_3NO_3 working fluid.
The numerical results of Eqs. (7) and (8) subjecting to the boundary conditions (9) are obtained by applying the shooting technique with Maple software. Dual solutions are found for the skin friction coefficient \(f''(0)\) and local Nusselt number \(-\theta'(0)\) for the different applied parameters at the different initial guesses. In order to validate the obtained numerical results, the present results are compared to the results obtained by Ghosh & Mukhopadhyay [22] that are shown in Table 2. The comparison shows very good agreement in both solutions. Furthermore, to clarify the silent features concerned to the flow and the heat transfer phenomena, the obtained numerical results are shown in Figures 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, and 21, graphically.

Figure 6. Graph of \(f''(0)\) with different values of suction \(S\) for different values of \(\varphi\) in \(\text{Cu} - \text{C}_6\text{H}_9\text{NaO}_7\) working fluid.

Figure 7. Graph of \(f''(0)\) with different values of suction \(S\) for different values of \(\varphi\) in \(\text{Ag} - \text{C}_6\text{H}_9\text{NaO}_7\) working fluid.

Figure 8. Graph of rate of heat transfer \(-\theta'(0)\) with different values of suction \(S\) for different values of \(\varphi\) in \(\text{Cu} - \text{C}_6\text{H}_9\text{NaO}_7\) working fluid.

Figure 9. Graph of rate of heat transfer \(-\theta'(0)\) with different values of suction \(S\) for different values of \(\varphi\) in \(\text{Ag} - \text{C}_6\text{H}_9\text{NaO}_7\) working fluid.

Figure 10. Graph of heat transfer rate with the different values of \(\varphi\) for \(\text{Cu}\) and \(\text{Ag} - \text{C}_6\text{H}_9\text{NaO}_7\) working fluid.
collocation polynomial provides a $C_1$ continuous solution that is fourth order accurate uniformly in $\frac{1}{2}a/b$. Mesh selection and error control are based on the residual of the continuous solution. The smallest eigenvalues shown in Table 3 express the first (second) solution is the stable (unstable) and the physically feasible (unrealizable) because smallest eigenvalue is positive (negative).

Figures 2, 3, 4, and 5 demonstrate the change in skin friction coefficient $C_f$ and Nusselt number $Nu$ for solid nanoparticles Ag and Cu Casson base nanofluids with variation of stretching/shrinking parameter $\lambda$ at various values of the suction parameter $S$, keeping values of the other parameters $Pr = 10$, $Ec = 0.5$, $\gamma = 0.5$, $j = 1$, and $\phi = 0.1$ fixed. Furthermore, these Figs also show the stretching and shrinking parameter is one of the main reason behind the existence of the multiple (dual) solutions in this problem. It is observed that there are two solutions for case of the $\lambda > \lambda_c$, unique solution for $\lambda = \lambda_c$ whereas in case of the $\lambda < \lambda_c$ no solution exits. It should be noted that the $\lambda_c$ indicates critical point of the parameter $\lambda$ where solution exists. The critical values of the $\lambda$ are $\lambda_c = -1.00876$, $-1.3539$, $-1.7613$ for Ag-Casson base fluid and $\lambda_c = -0.9683$, $-1.28247$, $-1.65721$ for Cu-Casson base fluid for $S = 2$, 2.4, 2.8 respectively that are demonstrated in Figures 2, 3, 4, and 5 of the paper.

Figures 2 and 3 show the increasing value of suction parameter $S$ along the variation of stretching/shrinking parameter $\lambda$, decreases rate of skin friction coefficient $f'(0)$ for the stretching case $\lambda > 0$ and opposite result is examined for the shrinking case when $\lambda < 0$ in first solution for both Ag and Cu Casson base nanofluids. Whereas in the second solution, increasing value of suction parameter $S$ decreases the value of skin friction coefficient $f'(0)$ throughout the flow over both stretching and shrinking surfaces. The effect of the suction parameter $S$ along variation of $\lambda$ at the Nusselt number or rate of heat transfer $-\theta'(0)$ for Ag and Cu Casson base nanofluids is depicted in Figures 4 and 5 respectively. From these Figures it is clearly examined that the increment in the value of suction parameter $S$ increase the heat transfer rate in first solutions and
reverse to it, in the second solutions heat transfer rate decrease. Same type of findings about skin friction coefficient $f'(0)$ and heat transfer rate $-\theta'(0)$ along the suction parameter $S$ for the different values of nanoparticles volume fraction $\phi$ for the Ag and Cu Casson base nanofluid are shown in Figures 6, 7, 8, and 9. For particular parameters, these Figs represents the regions of no solutions for $S < S_1$, $S < S_2$ and $S < S_3$ and, dual solutions for $S_3 < S$, $S_2 < S$ and $S_3 < S$. Figures 6 and 7 show the increasing the quantity of the solid volumetric fraction $\phi$ of the silver Ag and copper Cu nanoparticles in the based fluid with respect to suction parameter $S$, the skin friction rate increases in the first solution whereas in the second solution it decreases. This is because of the increasing rate of the solid nanoparticles volume fraction increases the thermal conductivity of the Casson fluid that leads to further thinning of the thickness of the velocity boundary layer. Figures 8 and 9 illustrate the rate of the heat transfer at various values of solid volume fraction $\phi$ of nanoparticles for Ag and Cu Casson base nanofluid respectively, along the variation of suction parameter $S$. The results of the both Figs clarify that the increasing rate of volume fraction $\phi$ of both types of nanoparticles in Casson fluid as the rate of heat transfer decreases. Figure 10 indicates the differentiation in heat transfer rate with suspension of the Ag and Cu nanoparticles in Casson fluid due to variation of $S$. From the Figure it is observed that Ag nanoparticles show more rate of heat enhancement compared to copper Cu nanoparticles in Casson fluid. Whereas in Figure 11 same type of the findings are observed when suction parameter
$S$ is replaced by the volumetric fraction parameter $\phi$. Figures 12 and 13 illustrate the effect of Casson parameter $\beta$ at velocity profiles of Cu and Ag Casson base nanofluids for two values of porosity parameter $\gamma$ ($\gamma = 0.5$ and $\gamma = 0$) respectively. From Figure 12, it can be observed the velocity and boundary layer thickness decrease in first solution that indicates the magnitude of velocity is greater in the Casson based fluid as compared to viscous fluids. But in second solution, at start, it is increasing but after a certain point it is decreasing as in the first solution where the value of $\beta$ is increased for both values of parameter $\gamma$. In case of silver Ag nanoparticles suspended in Casson fluid, Figure 13 also shows the trend of velocity and boundary layer thickness of the Ag-Casson base nanofluid remained same as it is already observed in Cu-Casson nanofluid. Similar observation mentioned by Gangadhar et al. [24]. The effect of Casson parameter $\beta$ for two different values of the porosity parameter $\gamma$ ($\gamma = 0.5$ and $\gamma = 0$) on the velocity profile of the Cu and Ag Casson base nanofluid are shown in Figure 14. From this Fig, it is observed that the flow of the Cu-Casson nanofluid is comparatively faster than the flow of Ag-Casson base nanofluid in first solution but in second solution at start, the flow of Ag Casson base nanofluid is slightly faster as compared to Cu Casson base nanofluid but after a point result show same trend as it is already observed in the first solution by decreasing the numerical value of $\beta$. Figure 15 indicates the temperature profile of Cu-Casson base nanofluid, where the increasing value of Casson parameter $\beta$, the temperature of the flowing nanofluid decreases in the first solution clearly for both values of porosity parameter $\gamma$. In the second solution it is reverse for $\gamma = 0$ but for $\gamma = 0.5$ the temperature increases at start while after a point reverse result is observed. Furthermore, in Figure 16 the temperature profile of Ag Casson base nanofluid with increasing value of Casson parameter $\beta$ also show the same trend of the result as it is shown in temperature profile of the Cu suspended nanofluid. Fig. 17 indicates the combined result of the temperature profiles for Ag and Cu Casson base nanofluids. This show a slight difference in thermal enhancement in variation of the nanoparticles in base fluid, the Cu-Casson nanofluid show nearly more heat transfer enhancement as compare to the Ag base nanofluid in first solution. Whereas, in second solution at start, the heat transfer enhancement of Ag-Casson base nanofluid is noticed more than Cu-Casson base nanofluid but after a point result remained same as it is observed in first solution. Figures 18 and 19 show that in both solutions, the temperature and thermal boundary layer thickness increases as Eckert number ($Ec$) increases in Cu and Ag Casson based nanofluids respectively, whether the porosity exists ($\gamma = 0.5$) or not ($\gamma = 0$). The main reason behind rising of thermal boundary layer thickness is rising of the viscous resistance at increasing value of the $Ec$. As a results, more heat energy is accumulated within the fluid particles near to the boundary. It can be clearly examined from Figures 20 and 21 that the greater value of the Prandtl number ($Pr$) decrease the temperature of Cu and Ag Casson based nanofluids in both solutions respectively, for both values of the porosity parameter $\gamma$. Generally, the greater Prandtl number possesses lower thermal diffusivity. Therefore, an increase in Prandtl number will
decrease the thermal boundary layer thickness and the temperature of the nanofluid.

5. Conclusion

In present paper, the boundary layer flow and heat transfer of the Casson based nanofluid with two metallic nanoparticles copper (Cu) and silver (Ag) over the exponential permeable stretching/shrinking surface is examined. A single phase fluid model proposed by the Tiwari and Das [20] is used. The dual solutions are found for particular values of pertinent parameters. Stability analysis is performed to find the stable solution. The main findings are given below:

1. The dual solutions are found for both stretching and shrinking surface.
2. The increasing rate of the solid volume fraction of copper (Cu) and silver (Ag) nanoparticles in Casson fluid enhance rate of skin friction and decline rate of the heat transfer in both solutions.
3. The silver (Ag) nanoparticles show more rate of heat transfer enhancement comparatively to copper (Cu) nanoparticles in Casson fluid.
4. The Eckert number enhances the heat transfer rate whereas Prandtl number declines the rate of the heat transfer in both solutions.
5. Casson parameter declines the heat transfer rate in first solution and the reverse result is seen in second solution.

Declarations

Author contribution statement

Sumera Dero: Analyzed and interpreted the data; Wrote the paper.
Azizah Mohd Rohni: Conceived and designed the analysis.
Azizan Saaban: Contributed analysis tools or data; Wrote the paper.

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Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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