BIFURCATION OF CRITICAL PERIODS OF A QUINTIC SYSTEM

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ABSTRACT. We investigate the critical period bifurcations of the system
\[ \dot{x} = ix + x\overline{x}(ax^3 + bx^2\overline{x} + \overline{x}\overline{x}^2 + d\overline{x}^3) \]
studied in [6]. We prove that at most three critical periods can bifurcate from any nonlinear center of the system.

1. INTRODUCTION

Consider a system of ordinary differential equations on \( \mathbb{R}^2 \) of the form
\[
\begin{align*}
\dot{u} &= -v + P(u, v), \\
\dot{v} &= u + Q(u, v),
\end{align*}
\]
where \( u \) and \( v \) are real unknown functions and \( P \) and \( Q \) are polynomials without constant and linear terms. The singularity at the origin of system (1.1) is either a center or a focus. In a neighborhood of a center the so-called period function \( T(r) \) gives the least period of the periodic solution passing through the point with coordinates \((u, v) = (r, 0)\) inside the period annulus of the center.

If \( T(r) \) is constant in a neighbourhood of the origin, then the center at the origin is called isochronous. For a center that is not isochronous any value \( r > 0 \) for which \( T'(r) = 0 \) is called a critical period. The problem of critical period bifurcations is aimed on estimation of the number of critical periods that can arise near the center under small perturbations. It was investigated for the first time by Chicone and Jacobs [2] in 1989 for quadratic systems and some Hamiltonian systems. After that, many studies were devoted to the problem (see, e.g., [1] [5] [7] [10] [12] [15] [16] [17] [18] [19] [20] [21] and references given there). One of difficulties in investigations of this problem is that before studying the critical periods bifurcation for a polynomial system one should resolve the center problem for the system, that is, find all systems in the family with a center at the origin.

Studies of the center problem are usually simpler if one considers the problem in the complex setting. To perform a complexification we can make the substitution
x = u + iv obtaining from (1.1) the complex differential equation

\[ \dot{x} = ix - \sum_{j+k=1}^{n-1} a_{jk}x^j \bar{x}^k. \]  

(1.2)

Adjoining to (1.2) its complex conjugate and considering \( \bar{a}_{jk} \) as a new parameter \( b_{kj} \) and \( \bar{x} \) as a distinct unknown function \( y \) we obtain the system

\[ \dot{x} = ix - \sum_{j+k=1}^{n-1} a_{jk}x^j y^k = ix + \tilde{P}(x, y), \]

\[ \dot{y} = -iy + \sum_{j+k=1}^{n-1} b_{kj}x^j y^k = -iy + \tilde{Q}(x, y). \]  

(1.3)

This system is called the complexification of (1.1) and it is equivalent to (1.2) when \( y = \bar{x} \) and \( b_{kj} = \bar{a}_{jk} \).

By Poincaré-Lyapunov theorem system (1.1) has a center at the origin if and only if it admits in a neighbourhood of the origin an analytic first integral of the form

\[ \Phi(u, v) = u^2 + v^2 + \text{h.o.t.}, \]

which is equivalent to the existence of a first integral of the form

\[ \Psi(x, \bar{x}) = x\bar{x} + \text{h.o.t.} \]

for system (1.2).

Thus, extending the notion of center from real systems to systems (1.3) it is said that complex system (1.3) has a center at the origin if in a neighbourhood of the origin it admits an analytic first integral of the form

\[ \Psi(x, y) = xy + \sum_{j+k=3}^{\infty} \Psi_{jk} x^j y^k. \]  

(1.4)

Since to each \( a_{jk} \) in the first equation of (1.3) corresponds the parameter \( b_{kj} \) in the second equation of (1.3), system (1.3) has \( 2\ell \) parameters. We denote the ordered \( 2\ell \)-tuple of the parameters of (1.3) by \( (a, b) \); that is,

\[ (a, b) = (a_{p_1q_1}, \ldots, a_{p_\ell q_\ell}, b_{q_\ell p_\ell}, \ldots, b_{q_1 p_1}), \]  

(1.5)

and we use the notation \( \mathbb{C}[a, b] \) for the ring of polynomials in the variables \( a_{p_1 q_1}, \ldots, a_{p_\ell q_\ell}, b_{q_\ell p_\ell}, \ldots, b_{q_1 p_1} \) over \( \mathbb{C} \).

Recently, García, Llibre and Maza study limit cycle bifurcations near a center or a focus at the origin of the quintic system written in the complex form as the equation

\[ \dot{x} = i x + x \bar{x}(ax^3 + bx^2 \bar{x} + cx^2 \bar{x}^2 + dx^3), \]

which, in order to use the notation similar to the one in (1.2), we write as the complex equation

\[ \dot{x} = i(x - a_{31} x^4 \bar{x} - a_{22} x^3 \bar{x}^2 - a_{13} x^2 \bar{x}^3 - a_{04} x \bar{x}^4). \]  

(1.6)

In this paper we study critical period bifurcations from the center at the origin of system (1.6). We first describe a way to compute the period function of system (1.2) using the normal form of its complexification (1.3). Then we prove that at most three critical periods can bifurcate from any nonlinear center of the system.
2. Preliminaries

To study critical period bifurcations of system (1.6) we have to compute a series expansion of the period function $T(r)$ of the system. One possibility is to pass to polar coordinates. This way is geometrically and theoretically straightforward, however it is not computationally efficient since one needs to compute integrals of trigonometric polynomials, and this is a difficult task in the case of polynomials of high degree.

Another possible computational approach relies on calculations of Poincaré-Dulac normal form of the complexification (1.3). We briefly remind it following [14] and [5].

As it is well-known system (1.1) has an isochronous center at the origin if and only if the system is linearizable. Thus, the real systems (1.1), which parameters after the complexification are in

$$V \subset \mathbb{C}^2,$$

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if the system is linearizable. Thus, the real systems (1.1), which parameters after the complexification are in

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where $g_{kk}$ is a polynomial in the coefficients of system (1.3). The polynomial $g_{kk}$ is called the $k$-th focus quantity. Clearly, system (1.3) with fixed coefficients $(a^*, b^*)$ has a center at the origin if and only if $g_{kk} \equiv 0$ for all $k \in \mathbb{N}$. We call the ideal

$$\mathcal{B} := \langle g_{kk} : k \in \mathbb{N} \rangle \subset \mathbb{C}[a, b]$$

the Bautin ideal of system (1.3). The variety of $\mathcal{B}$, $\mathcal{V} = \mathcal{V}(\mathcal{B})$, is called the center variety. We will also use the ideal generated by the first $K$ focus quantities, which we denote

$$\mathcal{B}_K := \langle g_{kk} : k = 1, \ldots, K \rangle \subset \mathbb{C}[a, b].$$

Let us denote

$$G = Y_1 + Y_2,$$
$$H = Y_1 - Y_2.$$

It is easy to see that the origin is a center for (1.3) if and only if $G \equiv 0$, in which case $H$ has purely imaginary coefficients and the distinguished normalizing transformation converges. We also define

$$\tilde{H}(w) = -\frac{1}{2} i H(w).$$

When (1.3) is the complexification of a real system one can recover the real system by replacing every occurrence of $y_2$ by $\bar{y}_1$ in each equation of (2.2). In such case, performing the transformation $y_1 = re^{i\varphi}$ we obtain from (2.2) the equations for $\dot{r}$ and $\dot{\varphi}$ as follows:

$$\dot{r} = \frac{1}{2r} (\dot{y}_1 \bar{y}_1 + y_1 \dot{\bar{y}}_1) = 0,$$
$$\dot{\varphi} = \frac{i}{2r^2} (y_1 \dot{\bar{y}}_1 - \dot{y}_1 \bar{y}_1) = 1 + \tilde{H}(r^2).$$

We write the function $\tilde{H}$ as

$$\tilde{H}(w) = \sum_{k=1}^{\infty} \tilde{H}_{2k+1} w^k.$$

The integration of the second equation in (2.4) gives the least period of the periodic solution of (1.3) passing through the point with coordinates $(r, 0)$ as

$$T(r) = \frac{2\pi}{1 + \tilde{H}(r^2)} = 2\pi \left(1 + \sum_{k=1}^{\infty} p_{2k}(a, \bar{a}) r^{2k}\right)$$

for some coefficients $p_{2k}$. The center at the origin of system (1.6) corresponding to a parameter $a^*$ is isochronous if and only if $p_{2k}(a^*, \bar{a}^*) = 0$ for $k \geq 1$.

It is easy to see that $p_{2k}$ are polynomials in the parameters $a$, $\bar{a}$ of system (1.2). We can extend the polynomial functions $p_{2k}(a, \bar{a})$ to the set of parameters $(a, b)$ setting in (2.4) $y_2$ instead of $\bar{y}_1$. Then instead of (2.5) we obtain the function

$$T(r, a, b) = 2\pi \left(1 + \sum_{k=1}^{\infty} p_{2k}(a, b) r^{2k}\right),$$

which coincides with the period function (2.5) when $b = \bar{a}$. 


We call the polynomial \( p_{2k}(a, b) \) in \([2,6]\) the \( k\)-th isochronicity quantity. Using \([2,5]\) and the formula for the inversion of series the first three polynomials \( p_{2k} \) are computed as:

\[
p_2 = -\bar{H}_3 = \frac{i}{2} (Y_1^{(2,1)} - Y_2^{(1,2)})
\]

\[
p_4 = -\bar{H}_5 + (\bar{H}_3)^2 = \frac{i}{2} (Y_1^{(3,2)} - Y_2^{(2,3)}) - \frac{1}{4} (Y_1^{(2,1)} - Y_2^{(1,2)})^2,
\]

\[
p_6 = -\bar{H}_7 + 2\bar{H}_3\bar{H}_5 - (\bar{H}_3)^3
\]

\[= \frac{i}{2} (Y_1^{(4,3)} - Y_2^{(3,4)}) - \frac{1}{2} (Y_1^{(2,1)} - Y_2^{(1,2)}) (Y_1^{(3,2)} - Y_2^{(2,3)})
\]

\[\quad - \frac{i}{8} (Y_1^{(2,1)} - Y_2^{(1,2)})^3.
\]

(2.7)

Since values of the isochronicity quantity \( p_{2k} \) are of interest only on the center variety, we should work with the equivalence class \([p_{2k}]\) of \( p_{2k} \) in the coordinate ring \( \mathbb{C}[V_\phi] \) of the center variety, which can be viewed as the set of equivalence classes of polynomials \( \mathbb{C}[a, b] \) by \( V_\phi \). That is, for polynomials \( f, g \in \mathbb{C}[a, b] \),

\[ [f] = [g] \quad \text{in} \quad \mathbb{C}[V_\phi] \]

if and only if

\[ f - g \equiv 0 \quad \text{on} \quad V_\phi. \]

We denote

\[ P = \langle p_{2k} : k \in \mathbb{N} \rangle \subset \mathbb{C}[a, b] \quad \text{and} \quad \bar{P} = \langle [p_{2k}] : k \in \mathbb{N} \rangle \subset \mathbb{C}[V_\phi], \]

and for \( K \in \mathbb{N} \),

\[ P_K = \langle p_2, \ldots, p_{2K} \rangle \quad \text{and} \quad \bar{P}_K = \langle [p_2], \ldots, [p_{2K}] \rangle. \]

The ideal \( P \) is called the isochronicity ideal.

Finally, we remind that given a Noetherian ring \( R \) and an ordered set

\[ B = \{b_1, b_2, \ldots\} \subset R, \]

we construct a basis \( M_I \) of the ideal \( I = \langle b_1, b_2, \ldots \rangle \) as follows:

(a) initially set \( M_I = \{b_p\} \), where \( b_p \) is the first non-zero element of \( B \);

(b) sequentially check successive elements \( b_j \), starting with \( j = p + 1 \), adding

\[ b_j \quad \text{to} \quad M_I \quad \text{if and only if} \quad b_j \notin \langle M_I \rangle \]

The cardinality of \( M_I \) is called the Bautin depth of \( I \).

3. AN UPPER BOUND FOR CRITICAL PERIODS BIFURCATING FROM CENTERS OF SYSTEM \([1,6]\)

Along with system \([1,6]\) we consider its complexification

\[
\dot{x} = ix(1 - a_{31}x^3y - a_{22}x^2y^2 - a_{13}xy^3 - a_{04}y^4),
\]

\[
\dot{y} = -iy(1 - b_{40}x^4 - b_{31}x^3y - b_{22}x^2y^2 - b_{13}xy^3).
\]

(3.1)

Our study is based on the following theorem which is an immediate corollary of \([5\]

Theorem 5.2 and Remark 5.3).

**Theorem 3.1.** Suppose that for the complexification \([1,3]\) of the family \([1,2]\):

(a) \( V_\phi = \mathbf{V}(P_K) \cap V_\phi \),
(b) the Bautin depth (i.e., the cardinality of the minimal basis) of $\bar{P}_K$ in $\mathbb{C}[V_\varphi]$ is $m$, and

(c) a primary decomposition of $P_K + \sqrt{R}$ can be written $R \cap N$ where $R$ is the intersection of the ideals in the decomposition that are prime and $N$ is the intersection of the remaining ideals in the decomposition.

Then for any system of family (1.2) corresponding to $(\alpha^*, \tilde{\alpha}^*) \in V_\varphi \setminus V(N)$, at most $m - 1$ critical periods bifurcate from a center at the origin.

Thus, to estimate the number of bifurcating critical periods of system (3.1) we have to know the center and linearizability varieties of the system.

First we note that it follows from Corollary 3.4.6 in [14] that for system (3.1) the focus quantities $g_{2k+1,2k+1}$ are zero polynomials. Using the results of [9] we can easily prove the following statement.

**Proposition 3.2.** The center variety of system (3.1) is defined by the seven first non-zero focus quantities,

$$V(\mathcal{B}) = V(\mathcal{B}_{14}),$$

where $\mathcal{B}_{14} = \langle g_{2,2}, g_{4,4}, g_{6,6}, g_{8,8}, g_{10,10}, g_{12,12}, g_{14,14} \rangle$, and it consists of four components defined by the following prime ideals:

- $I_1 = \langle a_{22} - b_{22}, a_{31}a_{13} - b_{13}b_{31}, b_{31}^2a_{04} - a_{13}^2b_{40}, a_{31}b_{31}a_{04} - b_{13}a_{13}b_{40}, a_{31}^2a_{04} - b_{13}^2b_{40} \rangle$,
- $I_2 = \langle b_{40}, b_{31}, a_{13}, b_{22}, a_{22}, b_{13} \rangle$,
- $I_3 = \langle a_{04}, b_{31}, a_{13}, b_{22}, a_{22}, a_{31} \rangle$,
- $I_4 = \langle a_{22} - b_{22}, 3b_{13} - a_{13}, 3a_{31} - b_{31} \rangle$.

**Proof.** Using the algorithm in [14, Chapter 3] and a Mathematica code similar to the one given in [14, Fig. 6.1 of Appendix] we computed the focus quantities $g_{2,2}, g_{4,4}, \ldots, g_{14,14}$ (since the expressions are long, we do not present them here, but one can easily compute them using any available computer algebra system). Then, using the routine `minAssGTZ`, which is based on the algorithm of [8], of the computer algebra system SINGULAR [3] we found that the minimal associate primes of $\mathcal{B}_{14}$ are the prime ideals $I_1, \ldots, I_4$ in the statement of the theorem.

By the results of [9] if the parameters $(a, b)$ of system (3.1) are from one of the varieties $V(I_1), \ldots, V(I_4)$, then the corresponding systems have a center. This means that (3.2) holds.

Note, that taking into account that $V(\mathcal{B})$ is a complex variety, from (3.2) we obtain that the radical of $\mathcal{B}$ coincides with the radical of $\mathcal{B}_{14}$, that is,

$$\sqrt{\mathcal{B}} = \sqrt{\mathcal{B}_{14}}.$$
\[ Y_{1}^{(7,6)} = (4a_{13}a_{22}a_{31} + a_{22}a_{31}b_{13} - 6a_{13}a_{31}b_{22} + a_{31}b_{13}b_{22} - 11a_{13}a_{22}b_{31} - 6a_{04}a_{31}b_{31} + 2a_{22}b_{13}b_{31} - 3a_{13}b_{22}b_{31} - 10a_{04}b_{31}^2 - 12a_{13}b_{40} - 5a_{04}b_{22}b_{40} - 2a_{04}b_{22}b_{40})/4; \]
\[ Y_{2}^{(6,7)} = i(-a_{22}a_{31}b_{13} - 2a_{13}a_{31}b_{22} - a_{31}b_{13}b_{22} + 3a_{13}a_{22}b_{31} + 6a_{22}b_{13}b_{31} + 11a_{13}b_{22}b_{31} - 4b_{13}b_{22}b_{31} + 12a_{04}b_{31}^2 + 10a_{13}b_{40} + 2a_{04}a_{22}b_{40} + 6a_{13}b_{13}b_{40} + 5a_{04}b_{22}b_{40})/4; \]
\[ Y_{1}^{(9,8)} = i(132a_{13}a_{22}a_{31} + 36a_{13}a_{22}a_{31} + 108a_{04}a_{22}a_{31} - 150a_{22}a_{31}b_{13} - 6a_{13}a_{31}b_{13} + 6a_{13}b_{13} + 192a_{13}a_{22}a_{31}b_{22} - 72a_{04}a_{31}b_{22} + 204a_{22}b_{13}b_{31} + 24a_{22}b_{13}b_{31} + 24a_{13}a_{22}b_{31} + 6a_{13}b_{31}^2 - 132a_{13}b_{22}b_{31} - 267a_{04}a_{31}b_{31} - 24a_{22}b_{13}b_{31} + 18a_{13}b_{22}b_{31} - 162a_{13}b_{31}^2 - 384a_{04}a_{22}b_{31} - 18a_{13}b_{13}b_{31} - 132a_{04}a_{22}b_{31} - 396a_{13}b_{22}b_{31} - 78a_{04}a_{22}b_{31} - 200a_{04}a_{13}b_{31}b_{40} + 27a_{13}a_{22}b_{13}b_{40} - 180a_{13}b_{22}b_{40} - 48a_{04}a_{22}b_{31}b_{40} - 80a_{04}a_{13}b_{31}b_{40} - 56a_{04}b_{13}b_{40}b_{40} - 48a_{04}b_{22}b_{40}^2)/48; \]
\[ Y_{2}^{(8,9)} = i(-18a_{22}a_{31}b_{13} - 6a_{13}a_{31}b_{13} - 6a_{13}b_{13}^2 - 24a_{13}a_{22}b_{31}b_{22} - 204a_{22}b_{13}b_{31}b_{22} + 150a_{31}b_{13}b_{22} - 18a_{13}a_{22}b_{31} - 90a_{04}a_{22}a_{31}b_{31} + 324a_{22}b_{13}b_{31} + 24a_{13}a_{22}b_{31}b_{31} + 6a_{13}b_{13}b_{31}^2 - 132a_{13}b_{22}b_{31} + 27a_{04}a_{31}b_{31} - 192a_{13}b_{22}b_{31} + 198a_{13}b_{22}b_{31}^2 - 132a_{13}b_{22}b_{31} + 162a_{13}b_{31}^2 + 180a_{04}a_{22}b_{31}^2 + 126a_{13}b_{13}b_{31}^2 - 36a_{13}b_{31}^2 + 396a_{04}a_{22}b_{31}^2 + 132a_{13}a_{22}b_{40} + 56a_{04}a_{13}a_{31}b_{40} + 267a_{13}b_{22}b_{13}b_{40} + 72a_{22}b_{13}b_{31}b_{40} + 384a_{13}b_{22}b_{40} + 48a_{04}a_{22}b_{22}b_{40} - 192a_{13}b_{13}b_{22}b_{40} - 108a_{13}b_{22}b_{40} + 78a_{04}a_{22}b_{40}^2 + 80a_{04}a_{13}b_{31}b_{40} + 200a_{04}b_{13}b_{31}b_{40} + 48a_{04}b_{22}b_{40}^2)/48. \]

Then, using (2.7) for the calculation of \( p_{4} \) and computing the series expansions (2.6) in order to find \( p_{8}, p_{12} \), and \( p_{16} \) we obtain the first four non-zero reduced isochronicity quantities (by the reduced quantities we mean the polynomials obtained in such way that in formulas (2.7) and their extensions to any \( p_{2k} \) terms containing the highest order coefficients of the normal form are taking into account; it is sufficient to work with the reduced quantities since the other terms of \( p_{2k} \) are in the ideal \( \langle p_{2}, \ldots, p_{2k-2} \rangle \) of system (3.1) as follows:

\[ p_{4} = \frac{1}{2}(a_{22} + b_{22}); \]
\[ p_{8} = -\frac{1}{2}(a_{31}b_{13} - a_{31}a_{13} - b_{13}b_{31} - 3a_{13}b_{31} - 2a_{04}b_{40}); \]
\[ p_{12} = \frac{1}{8}(-4a_{13}a_{22}a_{31} - 2a_{22}a_{31}b_{13} + 4a_{13}a_{31}b_{22} - 2a_{31}b_{13}b_{22} + 14a_{13}a_{22}b_{31} + 6a_{04}a_{31}b_{31} + 4a_{22}b_{13}b_{31} + 14a_{13}b_{22}b_{31} - 4b_{13}b_{22}b_{31} + 22a_{04}b_{31}^2 + 22a_{13}b_{40} + 7a_{04}a_{22}b_{40} + 6a_{13}b_{13}b_{40} + 7a_{04}b_{22}b_{40}); \]
\( p_{16} = \frac{1}{48} (-66a_{13}a_{22}a_{31} - 18a_{12}^2a_{31} - 54a_{04}a_{22}a_{31} + 66a_{22}a_{31}b_{13} - 6a_{13}^2b_{13} \\
- 108a_{12}a_{22}a_{31}b_{22} + 36a_{04}a_{13}^2b_{22} - 204a_{22}a_{31}b_{13}b_{22} + 150a_{13}a_{31}b_{22} \\
+ 66a_{31}b_{13}b_{22} + 90a_{13}a_{22}^2b_{31} + 72a_{13}a_{31}b_{31} - 141a_{04}a_{22}a_{31}b_{31} \\
+ 150a_{13}a_{22}b_{13}b_{31} + 24a_{13}a_{31}b_{13}b_{31} + 132a_{13}a_{22}b_{22}b_{31} + 120a_{04}a_{31}b_{22}b_{31} \\
- 108a_{22}a_{13}b_{22}b_{31} + 90a_{13}b_{22}^2b_{31} - 66b_{13}b_{22}^2b_{31} + 162a_{13}^2b_{22}^2 + 282a_{04}a_{22}b_{31}^2 \\
+ 72a_{13}b_{13}b_{31}^2 - 18a_{12}b_{31}^2 + 264a_{04}b_{22}b_{31}^2 + 264a_{13}a_{22}b_{40} + 39a_{04}a_{22}b_{40} \\
+ 128a_{04}a_{13}a_{31}b_{40} + 120a_{13}a_{22}b_{13}b_{40} + 36a_{22}b_{13}^2b_{40} + 282a_{13}^2b_{22}b_{40} \\
+ 48a_{04}a_{22}b_{22}b_{40} - 141a_{13}b_{22}b_{40} - 54b_{13}b_{22}b_{40} + 39a_{04}b_{22}b_{40} \\
+ 800a_{04}a_{13}b_{31}b_{40} + 128a_{04}b_{13}b_{31}b_{40} + 48a_{13}^2b_{40}).
\)

We now look for the linearizability of system (3.1).

**Proposition 3.3.** For system (3.1),
\[
V_{\mathcal{L}} = V(\mathcal{Y}_8) = V_{\mathcal{E}} \cap V(P_8).
\]

**Proof.** Using the routine minAssGTZ of SINGULAR we found that the minimal associate primes of ideals \( \langle \mathcal{Y}_8 \rangle \) and \( \langle \mathcal{B}_{14}, P_8 \rangle \) are the same. Namely, they are the ideals:
\[
Q_1 = \langle b_{40}, b_{31}, a_{13}, b_{22}, a_{22}, b_{13} \rangle, \quad Q_2 = \langle b_{40}, b_{31}, b_{22}, a_{22}, a_{31} \rangle, \\
Q_3 = \langle b_{40}, a_{04}, b_{22}, a_{22}, b_{13} - 3a_{13}, a_{31} - 3b_{31} \rangle, \\
Q_4 = \langle b_{40}, a_{04}, b_{22}, a_{22}, b_{13} + a_{13}, a_{31} + b_{31} \rangle, \\
Q_5 = \langle a_{04}, a_{13}, b_{22}, a_{22}, a_{13} \rangle, \quad Q_6 = \langle a_{04}, b_{31}, a_{13}, b_{22}, a_{22}, a_{31} \rangle.
\]

By the results of [13] systems with the coefficients from the varieties of these ideals are linearizable. This proves (3.3). \(\square\)

We now can estimate the number of critical periods near a center at the origin of system (1.6).

**Theorem 3.4.** At most 3 critical periods bifurcate from nonlinear centers of system (1.6).

**Proof.** By Proposition 3.3 part (a) of Theorem 3.1 holds with \( K = 8 \). We then check that in \( \mathbb{C}[V_{\mathcal{E}}] \):
\[
[p_8] \not\in \langle [p_4] \rangle, \quad [p_{12}] \not\in \langle [p_4], [p_8] \rangle, \quad [p_{16}] \not\in \langle [p_4], [p_8], [p_{12}] \rangle.
\]

(3.4)

To this end, with the routine radical of the computer algebra system SINGULAR we compute the radical of the Bautin ideal \( \mathcal{B} = \mathcal{B}_{14} \) denoted \( \mathcal{R}_{14} \), that is,
\[
\mathcal{R}_{14} = \sqrt{\mathcal{B}_{14}}
\]

(one can also compute \( \mathcal{R}_{14} \) using the routine intersect of SINGULAR and the ideals \( I_1 - I_4 \) given in the statement of Proposition 3.2 since it is follows from the proof of Proposition 3.2 that \( \mathcal{R}_{14} = \cap_{k=1}^4 I_k \)). Then with the reduce of SINGULAR we check that for \( k = 2, 3, 4 \) the remainder of the division of the polynomial \( p_{4k} \) by a Groebner basis of the ideal
\[
\langle p_4, \ldots, p_{4(k-1)} \rangle, \mathcal{R}_{14}
\]
is nonzero. That means, that \([3.4]\) holds, which, in turn, yields that the Bautin depth of \(P_8\) in \(\mathbb{C}[V]\) is 4.

Then, with the routine \texttt{primdecGTZ} \cite{SINGULAR} of SINGULAR we have computed the primary decomposition of the ideal
\[
Q = (P_8, R_{14})
\]
and found that
\[
Q = \cap_{k=1}^{13}Q_k,
\]
where \(Q_1, \ldots, Q_6\) are prime ideals given in the statement of Proposition \(3.2\).

\(Q_7, \ldots, Q_{13}\) are some ideals defined by many polynomials (for these reason we do not present them here, however the interested reader can easily compute \(Q\) and the primary decomposition \(Q = \cap_{k=1}^{13}Q_k\) with an appropriate computer algebra system using the ideals \(P_8\) and \(I_1-I_4\) presented above) whose associate primes are:
\[
\begin{align*}
\sqrt{Q_7} &= \langle b_{40}, b_{31}, b_{22}, a_{22}, b_{13} - 3a_{13}, a_{31} \rangle, \\
\sqrt{Q_8} &= \langle b_{40}, b_{31}, b_{22}, a_{22}, b_{13} + a_{13}, a_{31} \rangle, \\
\sqrt{Q_9} &= \langle a_{04}, a_{13}, b_{22}, a_{22}, b_{13} - 3a_{31} \rangle, \\
\sqrt{Q_{10}} &= \langle a_{04}, a_{13}, b_{22}, a_{22}, b_{13}, a_{31} + b_{31} \rangle, \\
\sqrt{Q_{11}} &= \langle b_{40}, b_{31}, a_{13}, b_{22}, b_{13}, a_{31} \rangle, \\
\sqrt{Q_{12}} &= \langle a_{04}, a_{13}, b_{22}, a_{22}, a_{31} \rangle, \\
\sqrt{Q_{13}} &= \langle b_{40}, a_{04}, b_{31}, a_{13}, b_{22}, b_{13}, a_{31} \rangle.
\end{align*}
\]

Thus, \(\sqrt{Q_k} = Q_k\) for \(k = 1, \ldots, 6\) and \(\sqrt{Q_k} \neq Q_k\) for \(k = 7, \ldots, 13\); that is, the ideals \(R\) and \(N\) from the statement of Theorem \(3.1\) are
\[
R = \cap_{k=1}^{6} Q_k \quad \text{and} \quad N = \cap_{k=7}^{13} Q_k.
\]

To find systems \((1.6)\) whose coefficients are in the variety of the ideal \(N\) we perform as follows. Let \(T_s = \sqrt{Q_{s+6}}\) for \(s = 1, \ldots, 7\). Using the \texttt{intersect} of SINGULAR we compute the ideal \(T = \cap_{k=1}^{7} T_k\) and find that
\[
T = \langle a_{22}, b_{22}, a_{04}b_{40}, a_{13}b_{40}, b_{13}b_{40}, a_{04}b_{31}, a_{04}a_{31}, a_{13}b_{31}, \\
&b_{13}b_{31}, a_{13}a_{31}, -3a_{13}^2 - 2a_{13}b_{13} + b_{13}^2, a_{31}b_{13}, a_{31}^2, -2a_{31}b_{31} - 3b_{31}^2 \rangle.
\]

Clearly, \(V(N) = V(T)\) in \(\mathbb{C}^8\).

Since in the case when \((3.1)\) is a complexification of the real system the parameters \(a_{ks}\) and \(b_{ks}\) are complex conjugate we perform the change of variables
\[
\begin{align*}
a_{31} &= A_{31} + iB_{31}, & b_{13} &= A_{31} - iB_{31}, \\
a_{22} &= A_{22} + iB_{22}, & b_{22} &= A_{22} - iB_{22}, \\
a_{13} &= A_{13} + iB_{13}, & b_{31} &= A_{13} - iB_{13}, \\
a_{04} &= A_{04} + iB_{04}, & b_{40} &= A_{04} - iB_{04},
\end{align*}
\]
where \(A_{ks}, B_{ks}\) are real parameters. Substituting these values into the ideal \(T\) and computing a Groebner bases of the obtained ideal in the ring
\[
\mathbb{Q}[A_{04}, A_{13}, A_{22}, A_{31}, B_{04}, B_{13}, B_{22}, B_{31}]
\]
we find the ideal
\[ T_R = \langle A_{22}, B_{22}, (B_{13} - B_{31})(3B_{13} + B_{31}), A_{31}^2 + B_{31}^2, \\ A_{31}B_{13} + A_{13}B_{31}, 3A_{13}B_{13} + 2A_{31}B_{13} + A_{31}B_{31}, \\ A_{13}A_{31} - B_{13}B_{31}, 3A_{13}^2 + 2B_{13}B_{31} + B_{31}^2, A_{31}B_{04} + A_{04}B_{31}, \\ - A_{13}B_{04} + A_{04}B_{13}, A_{04}A_{31} - B_{04}B_{31}, A_{04}A_{13} + B_{04}B_{13}, A_{04}^2 + B_{04}^2 \rangle. \] (3.5)
The basis of \( T_R \) contains the polynomials
\[ A_{22}, B_{22}, A_{31}^2 + B_{31}^2, A_{04}^2 + B_{04}^2. \]
Since \( A_{31}, B_{31}, A_{04}, B_{04} \) are real parameters we conclude that
\[ A_{22} = B_{22} = A_{31} = B_{31} = A_{04} = B_{04} = 0. \] (3.6)
Substituting the values from (3.6) into polynomials of the ideal \( T_R \) given in (3.5) we find that also
\[ A_{13} = B_{13} = 0. \]
It means that the only system of the form (1.6) whose parameters are in the variety of the ideal \( N \) is the linear system (1.2), that is, the system \( \dot{x} = ix \). Thus, by Theorem 3.1 at most 3 critical periods bifurcate from non-linear isochronous centers of system (1.6).

\[ \square \]

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References

[1] X. Chen, V. G. Romanovski, W. Zhang; Critical periods of perturbations of reversible rigidly isochronous centers, J. Differential Equations 251 (2011), 1505–1525.
[2] C. Chicone, M. Jacobs; Bifurcation of critical periods for plane vector fields, Trans. Amer. Math. Soc. 312 (1989), 433–486.
[3] W. Decker, G.-M. Greuel, G. Pfister, H. Schönenmann; SINGULAR 3-1-6—A Computer Algebra System for Polynomial Computations. http://www.singular.uni-kl.de, 2012.
[4] W. Decker, S. Laplagne, G. Pfister, H. A. Schönemann; primdec.lib SINGULAR 3-1-6 library for computing the primary decomposition and radical of ideals, 2010.
[5] B. Ferrèc, V. Levandovskyy, V. G. Romanovski, D. S. Shafer; Bifurcation of critical periods of polynomial systems. J. Differential Equations 259 (2015), 3825–3853.
[6] I.A. García, J. Llibre, S. Maza; The Hopf cyclicity of the centers of a class of quintic polynomial vector fields, J. Differential Equations 258 (2015), 1990-2009.
[7] A. Gasull, C. Liu, J. Yang; On the number of critical periods for planar polynomial systems of arbitrary degree, J. Differential Equations 249 (2010), 684-692.
[8] P. Gianni, B. Trager, G. Zacharias; Gröbner bases and primary decomposition of polynomials, J. Symbolic Comput. 6 (1988), 146–167.
[9] J. Gine, V. G. Romanovski; Integrability conditions for Lotka-Volterra planar complex quintic systems, Nonlinear Anal. Real World Appl. 11 (2010), 2100-2105.
[10] M. Grau, J. Villadelprat; Bifurcation of critical periods from Pleshkan’s isochrones, J. Lond. Math. Soc. (2) 81 (2010), 142–160.
[11] C. Liu, M. Han; Bifurcation of critical periods from the reversible rigidly isochronous centers, Nonlinear Anal. 95 (2014), 388–403.
[12] Y.-R. Liu, J. Li, W. Huang; *Singular Point values, Center Problem, and Bifurcations of Limit Cycles of Two Dimensional Differential Autonomous Systems*. Beijing: Science Press, 2008.

[13] V. G. Romanovski, X. Chen, Z. Hu; *Linearizability of linear systems perturbed by fifth degree homogeneous polynomials*, J. Phys. A: Math. Theor. **40** (2007), 5905-5919.

[14] V. G. Romanovski, D. S. Shafer; *The Center and Cyclicity Problems: A Computational Algebra Approach*. Boston: Birkhäuser, 2009.

[15] C. Rousseau, B. Toni; *Local bifurcation of critical periods in vector fields with homogeneous nonlinearities of the third degree*, Canad. Math. Bull. **36** (1993), 473–484.

[16] C. Rousseau, B. Toni; *Local bifurcation of critical periods in the reduced Kukles system*, Canad. Math. Bull. **49** (1997), 338–358.

[17] J. Villadelprat; *Bifurcation of local critical periods in the generalized Loud’s system*, Appl. Math. Comput. **218** (2012), 6803–6813.

[18] Z. Wang, X. Chen, W. Zhang; *Local bifurcation of critical periods in a generalized 2D LV system*, Appl. Math. Comput. **214** (2009), 17–25.

[19] Q. Xu, W. Huang; *The center conditions and local bifurcation of critical periods for a Liénard system*, Appl. Math. Comput. **217** (2011), 6637-6643.

[20] P. Yu, M. Han, J. Zhang; *Critical periods of third-order planar Hamiltonian systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **20** (2010), 2213–2224.

[21] L. Zou, X. Chen, W. Zhang; *Local bifurcations of critical periods for cubic Liénard equations with cubic damping*, J. Comput. Appl. Math. **222** (2008), 404–410.

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