Classification of homogeneous functors in manifold calculus

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Abstract

For any object $A$ in a simplicial model category $\mathcal{M}$, we construct a topological space $\hat{A}$ which classifies linear functors whose value on an open ball is equivalent to $A$. More precisely for a manifold $M$, and $\mathcal{O}(M)$ its poset category of open sets, weak equivalence classes of such functors $\mathcal{O}(M) \to M$ are shown to be in bijection with homotopy classes of maps $[M, \hat{A}]$. The result extends to higher degree homogeneous functors. At the end we explain a connection to a classification result of Weiss.

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References

1 Introduction

Let $M$ be a manifold and let $\mathcal{O}(M)$ be the poset of open subsets of $M$. In order to study the space of smooth embeddings of $M$ inside another manifold, Goodwillie and Weiss [5, 15] introduced the theory of manifold calculus, which is one incarnation of calculus of functors. One can define manifold calculus as the study of contravariant functors from $\mathcal{O}(M)$ to Top, the category of spaces. Being a calculus of functors, its philosophy is to take a functor $F$ and replace it by its Taylor tower $\{T_k(F) \to T_{k-1}(F)\}_{k \geq 1}$, which converges to the original functor in good cases, very much like the approximation of a function by its Taylor series. The functor $T_kF$ is the polynomial approximation to $F$ of degree $\leq k$. The “difference” between $T_kF$ and $T_{k-1}F$, or more precisely the homotopy fiber of the canonical map $T_kF \to T_{k-1}F$, belongs to a class of objects called homogeneous functors of degree $k$. In [15, Theorem 8.5], Weiss proves a deep result about the classification of homogeneous functors of degree $k$. Specifically, he shows that any such functor is equivalent to a functor constructed out of a fibration $p: Z \to F_k(M)$ over the unordered configuration space of $k$ points in $M$, with a preferred section (germ) near the fat diagonal of $M$.

In this paper we classify homogeneous functors of degree $k$ from $\mathcal{O}(M)$ into any simplicial model category $\mathcal{M}$. Such functors are determined by their values on disjoint unions of $k$ balls [14, Lemma 6.5]. Let $\mathcal{F}_{kA}(\mathcal{O}(M); M)$ denote the category of homogeneous functors $F: \mathcal{O}(M) \to \mathcal{M}$ of degree $k$ such that $F(U) \simeq A$ for any $U$ diffeomorphic to the disjoint union of exactly $k$ open balls (see [14, Definition 6.2]). Let $\mathcal{F}_{kA}(\mathcal{O}(M); M)/we$ denote the collection of weak equivalence classes of such functors. For spaces $X$ and $Y$, we let $[X,Y]$ be the standard notation for the set of homotopy classes of maps from $X$ to $Y$. We classify objects of $\mathcal{F}_{kA}(\mathcal{O}(M); M)$ not through fibrations, but instead by maps from $F_k(M)$ to a certain topological space. Specifically, we have the following, which is the main result of this paper.

**Theorem 1.1.** Let $\mathcal{M}$ be a simplicial model category, and let $A \in \mathcal{M}$. Then there is a topological space $\hat{A}$ such that for any manifold $M$,

(i) if $k = 1$, there is a bijection

$$\mathcal{F}_{1A}(\mathcal{O}(M); \mathcal{M})/we \cong [M, \hat{A}].$$

(ii) If $k \geq 2$ and $\mathcal{M}$ has a zero object, there is a bijection

$$\mathcal{F}_{kA}(\mathcal{O}(M); \mathcal{M})/we \cong [F_k(M), \hat{A}].$$

One may ask the following natural questions.

1. How is $\hat{A}$ constructed?
2. What do we know about \( \hat{A} \)?

3. How is our classification related to that of Weiss?

To answer the first question, let \( \hat{\Delta}^n, n \geq 0 \), denote the poset whose objects are nonempty subsets of \( \{0, \ldots, n\} \), and whose morphisms are inclusions. We construct \( \mathcal{C}_A \subseteq \mathcal{M} \), a small subcategory consisting of a certain collection of fibrant-cofibrant objects of \( \mathcal{M} \) that are weakly equivalent to \( A \). The morphisms of \( \mathcal{C}_A \) are the weak equivalences between its objects (see Definition 3.2). Define \( \hat{A}_\bullet \) as the simplicial set whose \( n \)-simplices are contravariant functors \( \hat{\Delta}^n \rightarrow \mathcal{C}_A \) that are required to be fibrant with respect to the injective model structure on \( \mathcal{C}_A^{\Delta^+} \). Face maps are defined in the standard way, while degeneracies are more intricate (see Definition 3.8). It turns out that \( \hat{A}_\bullet \) is a Kan complex. We define \( \hat{A} \) as the geometric realization of \( \hat{A}_\bullet \).

For the second question, we do not know that much about \( \hat{A} \). By definition it is connected, and we believe its fundamental group is the group of (derived) homotopy automorphisms of \( A \). Further computations seem hard.

Regarding the third question, let us consider Weiss’ result as mentioned above. In addition to this, he proves that the fiber \( p^{-1}(S) \) of the fibration \( p : Z \rightarrow F_k(M) \) that classifies a homogeneous functor \( E : \mathcal{O}(M) \rightarrow \text{Top} \) of degree \( k \) is homotopy equivalent to \( E(U_S) \), where \( U_S \) is a tubular neighborhood of \( S \) so that \( U_S \) is diffeomorphic to a disjoint union of \( k \) open balls \[15\]. So the classification of objects of \( \mathcal{F}_{kA}(\mathcal{O}(M); \text{Top}) \) amounts to the classification of fibrations over \( F_k(M) \) with a section near the fat diagonal and whose fiber is \( A \). In the case \( k = 1 \), the fat diagonal is empty and we are just looking at fibrations over \( M \) whose fiber is \( A \).

It is well known that there is a classifying space for such fibrations, namely \( BHautA \) where \( HautA \) denotes the topological/simplicial monoid of (derived) homotopy automorphisms of \( A \). If \( k > 1 \) and \( \mathcal{M} = \text{Top}_\ast \), the category of pointed spaces, one has a similar classifying space for the objects of \( \mathcal{F}_{kA}(\mathcal{O}(M); \text{Top}_\ast) \). For a general simplicial model category \( \mathcal{M} \) we believe our classifying space, \( \hat{A} \), is homotopy equivalent to \( BHautA \), but we do not know how to prove this. We also believe there should be another approach (which does not involve \( \hat{A} \)) to try to show that \( \mathcal{F}_{kA}(\mathcal{O}(M); \mathcal{M})/we \) is in one-to-one correspondence with homotopy classes of maps \( F_k(M) \rightarrow BHautA \). We will say more about all this in Section 8.

One may use Theorem 1.1 to set up the concept of characteristic classes or invariants of homogeneous functors though we do not know whether \( \hat{A} \) is homotopy equivalent to \( BHautA \). Let \( F \in \mathcal{F}_{kA}(\mathcal{O}(M); \mathcal{M}) \) and let \( f : F_k(M) \rightarrow \hat{A} \) denote the classifying map of \( F \). One can define the characteristic classes or invariants of \( F \) as the cohomology classes \( f^*(H^\bullet(\hat{A})) \subseteq H^\bullet(F_k(M)) \). If two functors are weakly equivalent, then by Theorem 1.1 they are homotopic and therefore they are equal in cohomology. It would be interesting to see what kind of characteristic classes one could recover using our approach, or if other more traditional classifying spaces can be seen as special cases of our construction.

**Strategy of the proof of Theorem 1.1** To prove the first part, we need four intermediate results. Let \( \mathcal{T}^M \) be a triangulation of \( M \), that is, a simplicial complex homeomorphic to \( M \) together with a homeomorphism \( \mathcal{T}^M \rightarrow M \). There is no need for the link condition (which says that the link of any simplex is a piecewise-linear sphere). Associated with \( \mathcal{T}^M \) is the poset \( \mathcal{U}(\mathcal{T}^M) \subseteq \mathcal{O}(M) \), which was introduced in [13 Section 4.1] and recalled in Definition 4.1. That poset is one of the key objects of this paper as it enables us to connect many categories. Associated with \( \mathcal{U}(\mathcal{T}^M) \) is the poset \( \mathcal{B}_\mathcal{U}(\mathcal{T}^M) \subseteq \mathcal{O}(M) \) defined as

\[
\mathcal{B}_\mathcal{U}(\mathcal{T}^M) = \{ B \text{ diffeomorphic to an open ball such that } B \subseteq U_\sigma \text{ for some } \sigma \}\.
\]

It turns out that \( \mathcal{B}_\mathcal{U}(\mathcal{T}^M) \) is a basis for the topology of \( M \). For a subposet \( \mathcal{S} \subseteq \mathcal{O}(M) \), we denote by \( \mathcal{F}_A(\mathcal{S}; \mathcal{M}) \) the category of isotopy functors \( F : \mathcal{S} \rightarrow \mathcal{M} \) such that \( F(U) \) is weakly equivalent to \( A \) for any \( U \) diffeomorphic to an open ball. The first result we need is Lemma 6.5 from [14] which says that the categories \( \mathcal{F}_{1A}(\mathcal{O}(M); \mathcal{M}) \) and \( \mathcal{F}_A(\mathcal{B}_\mathcal{U}(\mathcal{T}^M); \mathcal{M}) \) are weakly equivalent (in the sense of [14 Definition 6.3]), that is,

\[
\mathcal{F}_{1A}(\mathcal{O}(M); \mathcal{M}) \simeq \mathcal{F}_A(\mathcal{B}_\mathcal{U}(\mathcal{T}^M); \mathcal{M}) .
\]
Looking closer at the proof of Lemma 6.5 from [14], one can see that the hypothesis that \( \mathcal{M} \) has a zero object is not needed when \( k = 1 \). Since \( \mathcal{U}(\mathcal{T}^M) \) is a very good cover (in the sense of [13] Definition 4.1) of \( M \), we have the following which can be proved along the lines of [13] Proposition 4.7.

\[
\mathcal{F}_A(\mathcal{B}_C(\mathcal{T}^M); \mathcal{M}) \simeq \mathcal{F}_A(\mathcal{U}(\mathcal{T}^M); \mathcal{M}).
\]  

(See also Proposition 4.7.) As we can see, (1.1) and (1.2) are just “local versions” of some results from [13, 14]. The following two technical results are new and proved using model category techniques. To state them, let \( \mathcal{C}_A \) as above. Let \( \mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A) \) denote the category of isotopy functors from \( \mathcal{U}(\mathcal{T}^M) \) to \( \mathcal{C}_A \). By definition, there is an obvious functor \( \phi : \mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A) \rightarrow \mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{M}) \). Though we do not define a functor in the other direction, we succeed to prove that the localization of \( \phi \) is an equivalence of categories. That is,

\[
\mathcal{F}_A(\mathcal{U}(\mathcal{T}^M); \mathcal{M})[W^{-1}] \simeq \mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A)[W^{-1}].
\]  

(See Proposition 4.11.) To get (1.3), we show that the localization of \( \phi \) is essentially surjective and fully faithful, the essentially surjectivity being the most difficult part. The final result we need is stated as follows. Let \( \hat{\mathcal{A}} \) as above. One can associate to \( \mathcal{T}^M \) a canonical simplicial set denoted \( \hat{\mathcal{T}}^M \). We have the bijection

\[
\mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A)/\text{we} \cong \left[ \hat{\mathcal{T}}^M, \hat{\mathcal{A}} \right].
\]  

(See Proposition 6.11.) To get (1.4) we construct explicit maps between the sets involved. The hardest part is to show that those maps are well defined. Defining \( \hat{\mathcal{A}} \) as above, and noticing that the geometric realization of \( \hat{\mathcal{T}}^M \) is \( M \), one deduces Theorem 1.1-(i) from (1.1)-(1.4).

The second part of Theorem 1.1 is an immediate consequence of the first part and the following weak equivalence, which is [14] Theorem 1.3.

\[
\mathcal{F}_kA(\mathcal{O}(M); \mathcal{M}) \simeq \mathcal{F}_kA(\mathcal{O}(F_k(M)); \mathcal{M}).
\]  

Theorem 1.1 has many hypotheses including the following: \( \mathcal{M} \) is a simplicial model category and \( \mathcal{M} \) has a zero object. Note that the two underlined terms are not needed to prove (1.2)-(1.4).

Outline The plan of the paper is as follows (see also the Table of Contents at the beginning of the paper). In Section 2 we prove basic results we will use later. Section 3 defines the simplicial set \( \hat{\mathcal{A}} \), and proves Proposition 3.16 which says that \( \hat{\mathcal{A}} \) is a Kan complex. First we construct a small category \( \mathcal{C}_A \subseteq \mathcal{M} \) out of a model category \( \mathcal{M} \) and an object \( A \in \mathcal{M} \). Next we construct a specific fibrant replacement functor \( \mathcal{R} : \mathcal{C}_A^\to \rightarrow \mathcal{C}_A^\to \), which is essentially used to define degeneracy maps of \( \hat{\mathcal{A}} \). In Section 4 we prove Proposition 4.14 or (1.3). In Sections 5 and 6 we prove Proposition 6.11 or (1.4). Section 7 is dedicated to the proof of the main result of this paper: Theorem 1.1. Finally, in Section 8 we state a conjecture saying how our classification is related to that of Weiss.

Convention, notation, etc. These will be as in [13] Section 2, with the following additions. Throughout this paper the letter \( M \) stands for a second-countable smooth manifold. The only place we need \( M \) to be second-countable is Section 6.2. We write \( \mathcal{M} \) for a model category [8] Definition 1.1.4, while \( A \) is an object of \( \mathcal{M} \). As part of the definition, the factorizations in \( \mathcal{M} \) are functorial. Wherever necessary, additional conditions on \( \mathcal{M} \) will be imposed. In the sequel, the term “simplicial complex” means geometric simplicial complex. We write \( [n] \) for the set \{0, \ldots, n\}, \( [n]_i \) for \([n]\setminus\{i\}\), and \( [n]_{ij} \) for the set \([n]\setminus\{i,j\}\). Also let \( \{a_0, \ldots, a_i \} \) denote the set \( \{a_0, \ldots, a_i\}\setminus\{a_i\} \). If \( \mathcal{S} \) is a small category and \( \mathcal{C} \) is a subcategory of \( \mathcal{M} \), we write \( \mathcal{C}^{\mathcal{S}} \) for the category of contravariant functors from \( \mathcal{S} \) to \( \mathcal{C} \). An object of that category is called \( \mathcal{S}-\text{diagram} \) or just \( \text{diagram} \) in \( \mathcal{C} \). As usual, weak equivalences in \( \mathcal{C}^{\mathcal{S}} \) are natural transformations which are objectwise weak equivalences.

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2 Basic results and recollections

As we said in the introduction, this section presents some basic results we will use later. It is organized as follows. In Section 2.1 we define the posets \( \tilde{\Delta}^n \), \( \partial \tilde{\Delta}^n \), and \( \partial \tilde{\Delta}^n \), and endow the collection \( \{\tilde{\Delta}^n\}_{n \geq 0} \) with a natural cosimplicial structure. In Section 2.2 we endow the category of \( \tilde{\Delta}^n \)-diagrams with the injective model structure and discuss some basic results about fibrant diagrams. In Section 2.3 we recall some classical facts about the localization of categories.

2.1 The posets \( \tilde{\Delta}^n \), \( \partial \tilde{\Delta}^n \) and \( \partial \tilde{\Delta}^n \)

Definition 2.1. For \( n \geq 0 \), define \( \tilde{\Delta}^n \) to be the poset whose objects are nonempty ordered subsets \( \alpha = \{a_0, \cdots, a_s\} \) of \( \{0, \cdots, n\} \). Of course the order on \( \alpha \) is induced by the natural order of the set \( \{0, \cdots, n\} \).

Morphisms of \( \tilde{\Delta}^n \) are inclusions.

Anytime we write \( \alpha = \{a_0, \cdots, a_s\} \), it will always mean \( a_0 \leq \cdots \leq a_s \).

Remark 2.2. The poset \( \tilde{\Delta}^n \) has distinguished morphisms, namely

\[
d^i : \{a_0, \cdots, \hat{a_i}, \cdots, a_s\} \to \{a_0, \cdots, a_s\}, \quad 0 \leq i \leq s.
\]

One can check that every morphism of \( \tilde{\Delta}^n \) can be written as a composition of \( d^i \)’s.

Example 2.3. The following diagram is the poset \( \tilde{\Delta}^2 \).

Varying \( n \) we get the collection \( \tilde{\Delta}^* = \{\tilde{\Delta}^n\}_{n \geq 0} \), which turns out to be endowed with a natural cosimplicial structure defined as follows. Let \( \Delta \) be the category whose objects are sets of the form \( [n] = \{0, \cdots, n\}, n \geq 0 \), endowed with the natural order, and whose morphisms are non-decreasing maps. Let \( d^i : [n] \to [n+1] \) and \( s^k : [n+1] \to [n] \) be the special morphisms of \( \Delta \) (see [4, Section I.1]). It is well known that \( d^i \) and \( s^k \) satisfy the cosimplicial identities.

Definition 2.4. (i) Define a functor \( d^i : \tilde{\Delta}^n \to \tilde{\Delta}^{n+1}, 0 \leq i \leq n + 1 \), as \( d^i(\{a_0, \cdots, a_s\}) := \{d^i(a_0), \cdots, d^i(a_s)\} \).

(ii) Define a functor \( s^k : \tilde{\Delta}^{n+1} \to \tilde{\Delta}^n, 0 \leq k \leq n \), as \( s^k(\{a_0, \cdots, a_s\}) := \{s^k(a_0), \cdots, s^k(a_s)\} \).

On morphisms, define \( d^i \) and \( s^k \) in the obvious way.
The following proposition is straightforward.

**Proposition 2.5.** The functors $d^i$ and $s^k$ we just defined satisfy the cosimplicial identities. That is, $\tilde{\Delta}^\bullet = \{\tilde{\Delta}^n\}_{n \geq 0}$ is a cosimplicial category.

**Definition 2.6.** (i) For $i \in \{0, \cdots, n\}$, define $\partial^i \tilde{\Delta}^n \subseteq \tilde{\Delta}^n$ to be the full subposet whose objects are nonempty subsets of $[n]_i$.

(ii) Define $\partial \tilde{\Delta}^n := \bigcup_{i=0}^n \partial^i \tilde{\Delta}^n$. Equivalently, $\partial \tilde{\Delta}^n$ is the full subposet of $\tilde{\Delta}^n$ whose objects are nonempty proper subsets of $\{0, \cdots, n\}$.

By the definitions, one can easily see that there is a canonical isomorphism between $\partial^i \tilde{\Delta}^n$ and $\tilde{\Delta}^{n-1}$. That is, $\partial^i \tilde{\Delta}^n \cong \tilde{\Delta}^{n-1}$. Similarly, one has $\partial^i \tilde{\Delta}^n \cap \partial^j \tilde{\Delta}^n \cong \tilde{\Delta}^{n-2}$. We will make these identifications throughout the paper.

### 2.2 Model category structure on $\mathcal{M} \tilde{\Delta}^n$

**Definition 2.7.** Let $T$ be a simplicial complex. Define $P(T)$ to be the poset whose objects are (non-degenerate) simplices of $T$. Given two objects $\alpha, \sigma \in P(T)$, there is a morphism $d^{\alpha \sigma} : \alpha \rightarrow \sigma$ if and only if $\alpha$ is a face of $\sigma$.

**Remark 2.8.** From Definition 2.7, it is clear that $P(T)$ is isomorphic to $\tilde{\Delta}^n$ when $T$ is the standard geometric $n$-simplex. That is, $P(\tilde{\Delta}^n) \cong \tilde{\Delta}^n$. The same remark holds for the poset $\partial \tilde{\Delta}^n$ introduced in Definition 2.6. That is, $P(\partial \tilde{\Delta}^n) \cong \partial \tilde{\Delta}^n$.

**Definition 2.9.** Let $T$ and $P(T)$ be as in Definition 2.7.

(i) For an object $\sigma \in P(T)$, define $\partial \sigma$ as the simplicial complex whose simplices are nonempty proper faces of $\sigma$.

(ii) Let $F \in \mathcal{M} P(T)$. The matching object of $F$ at $\sigma \in P(T)$, denoted $M_\sigma(F)$, is defined as

$$M_\sigma(F) := \lim_{\alpha \in P(\partial \sigma)} F(\alpha).$$

(iii) For $F \in \mathcal{M} P(T)$ and $\sigma \in P(T)$, the canonical map $F(\sigma) \rightarrow M_\sigma(F)$, provided by the universal property of limit, is called the matching map of $F$ at $\sigma$.

**Proposition 2.10.** Let $T$ and $P(T)$ as above. There exists a model structure on the category $\mathcal{M} P(T)$ of $P(T)$-diagrams in $\mathcal{M}$ such that weak equivalences and cofibrations are objectwise. Furthermore, a map $F \rightarrow G$ is a (trivial) fibration if and only if the induced map $F(\sigma) \rightarrow G(\sigma) \times_{M_\sigma(G)} M_\sigma(F)$ is a (trivial) fibration for all $\sigma \in P(T)$.

**Proof.** This follows from two facts. The first one is the fact that the poset $P(T)$ is clearly a direct category in the sense of [8] Definition 5.1.1. So any $F \in \mathcal{M} P(T)$ is an inverse diagram[3](remember that for us diagram means contravariant functor). The second fact is [8] Theorem 5.1.3].

The category of diagrams, $\mathcal{M} P(T)$, will be always endowed with this model structure.

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1“face” will always mean face of any dimension. For instance one possible choice of $\alpha$ is $\sigma$ itself.

2An inverse diagram is either a covariant functor out of a small inverse category [8] Definition 5.1.1] or a contravariant functor out of a small direct category.
Remark 2.11. It is not difficult to see that a diagram $F: P(\Delta^n) \rightarrow \mathcal{M}$ is fibrant if and only if all of its $k$-faces, $0 \leq k \leq n$, are fibrant $P(\Delta^k)$-diagrams. By $k$-face of $F$, we mean the diagram $F \circ P(\tau)$, where $\tau: \Delta^k \rightarrow \Delta^n$ is the canonical inclusion map.

Proposition 2.12. Let $\mathcal{T}$ be a finite simplicial complex, and let $P(\mathcal{T})$ be as in Definition 2.7. Let $F: P(\mathcal{T}) \rightarrow \mathcal{M}$ be a fibrant diagram. Then the limit $\lim_{\alpha \in P(\mathcal{T})} F(\alpha)$ is a fibrant object of $\mathcal{M}$.

Proof. Since $F$ is fibrant, it follows that for every $\sigma \in P(\mathcal{T})$, $F(\sigma)$ is fibrant. Furthermore the limit of the diagram $F$ is the same as its homotopy limit again because $F$ is fibrant. Applying now [7, Theorem 18.5.2], we get the desired result.

Definition 2.13. Recall the poset $\partial \Delta^n$ from Definition 2.6. For $n \geq 1$ and $0 \leq k \leq n$, define $\Delta^n_k \subseteq \Delta^n$ as the full subposet whose objects are nonempty proper subsets of $[n]$ except $[n]_k$. That is, $\text{ob}(\Delta^n_k) = \text{ob}(\partial \Delta^n) \setminus [n]_k$.

Proposition 2.14. Let $F: \Delta^n_k \rightarrow \mathcal{M}$ be a fibrant diagram in which every morphism is a weak equivalence. Then for any $\alpha' \in \Delta^n_k$, the canonical projection $p_{\alpha'}: \lim_{\alpha \in \Delta^n_k} F(\alpha) \rightarrow F(\alpha')$ is a weak equivalence.

Proof. By inspection, one can see that the indexing category $\Delta^n_k$ is contractible. One can then apply the dual of [8, Corollary 1.18] to get the desired result.

2.3 Localization of categories

The aim of this section is to recall some classical results about the localization of categories we need. Our main references are [3] and [8]. The material of this section will be used in Section 4.3.

Let $\mathcal{E}$ be a model category and let $\mathcal{D} \subseteq \mathcal{E}$ be a full subcategory (not necessarily a model subcategory) of $\mathcal{E}$. One should think of $\mathcal{E}$ as the diagram category $\mathcal{E}^{\mathcal{D}(\mathcal{T}^M)}$, and $\mathcal{D}$ as the category $\mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C})$ that will be introduced in Definition 4.8. We denote by $\mathcal{W}_D$ the class of weak equivalences of $\mathcal{E}$ that lie in $\mathcal{D}$. We will use the standard notation $\mathcal{D}[\mathcal{W}_D^{-1}]$ for the localization of $\mathcal{D}$ with respect to $\mathcal{W}_D$. Roughly speaking, $\mathcal{D}[\mathcal{W}_D^{-1}]$ has the same objects as $\mathcal{D}$, and morphisms of $\mathcal{D}[\mathcal{W}_D^{-1}]$ are strings $(f_1, \cdots, f_n)$ of composable arrows where $f_i$ is either an arrow of $\mathcal{D}$ or the formal inverse $w^{-1}$ of an arrow $w$ of $\mathcal{W}_D$.

Recall the notion of cylinder object for $X \in \mathcal{E}$, denoted $X \times I$, from [8, Definition 1.2.4], and let $i_0$ and $i_1$ be the canonical maps from $X$ to $X \times I$. The following result is straightforward.

Proposition 2.15. Assume that for any $X \in \mathcal{D}$ there is a cylinder object $X \times I$ for $X$ in $\mathcal{D}$. Then one has $i_0 = i_1$ in the category $\mathcal{D}[\mathcal{W}_D^{-1}]$.

For $f, g: X \rightarrow Y$ in $\mathcal{D}$, if $f$ is homotopic to $g$ (see [8, Definition 1.2.4]), we will write $f \sim g$.

Proposition 2.16. [8, Corollary 1.2.6] Let $f, g: X \rightarrow Y$ be morphisms of $\mathcal{E}$. Assume $X$ cofibrant and $Y$ fibrant. If $f \sim g$ then there is a left homotopy $H: X \times I \rightarrow Y$ from $f$ to $g$ using any cylinder object $X \times I$.

The following proposition is also straightforward.

Proposition 2.17. Assume that $\mathcal{D}$ is closed under taking fibrant and cofibrant replacements, and let $X, Y \in \mathcal{D}$.

(i) Then there is a natural isomorphism

$$
\text{Hom}_{\mathcal{D}}(QX,RY)/ \sim \cong \text{Hom}_{\mathcal{D}[\mathcal{W}_D^{-1}]}(X,Y).
$$

\[ \theta' \]

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If $X$ and $Y$ are both fibrant and cofibrant, the map $\text{Hom}_D(X,Y) \xrightarrow{\varphi} \text{Hom}_{D[W_D^{-1}]}(X,Y)$ induced by the canonical functor $D \to D[W_D^{-1}]$ is surjective.

Proof. Part (i) follows from [8, Theorem 1.2.10]. Part (ii) comes from the fact that the following canonical triangle commutes and the map $\pi$ is surjective.

$$\begin{array}{ccc}
\text{Hom}(X,Y) & \xrightarrow{\varphi} & \text{Hom}_{D[W_D^{-1}]}(X,Y) \\
\downarrow{\pi} & & \downarrow{\varphi} \\
\text{Hom}(X,Y)/\sim & \cong & \text{Hom}(X,Y)/\sim
\end{array}$$

$\square$

3 The simplicial set $\hat{A}_\bullet$

In this section $\mathcal{M}$ is a model category, and $A$ is an object of $\mathcal{M}$. The goal here is to construct a simplicial set $\hat{A}_\bullet$, which classifies homogeneous functors in manifold calculus, out of $\mathcal{M}$ and $A$. We also show that $\hat{A}_\bullet$ is a Kan complex (see Proposition 3.16). We need $\hat{A}_\bullet$ to be a Kan complex because of (1.4), which involves homotopy classes of simplicial maps into $\hat{A}_\bullet$. This section is organized as follows. In Section 3.1 we define a small category $\mathcal{C}_A$, which will play the role of the target category for many functors including $n$-simplices of $\hat{A}_\bullet$. In Section 3.2 we define an explicit fibrant replacement functor $R: \mathcal{C}_A^{\Delta^n} \to \mathcal{C}_A^{\Delta^n}$, which will be essentially used to define degeneracy maps of $\hat{A}_\bullet$. Lastly, in Section 3.3 we define $\hat{A}_\bullet$ and prove Proposition 3.16.

3.1 The category $\mathcal{C}_A$

In Section 3.3 we will define an $n$-simplex of $\hat{A}_\bullet$ as a contravariant functor from $\tilde{\Delta}^n$ to a subcategory $\mathcal{C}$ of $\mathcal{M}$ that satisfies certain conditions. To guarantee that the collection of all $n$-simplices is actually a set, we need $\mathcal{C}$ to be small. This section defines a small category $\mathcal{C}_A$, which will play the role of $\mathcal{C}$.

For each morphism $f: X \to Y$ of $\mathcal{M}$, choose two functorial factorizations:

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\sim & & \sim \\
\downarrow{V_f} & & \downarrow{W_f}
\end{array}$$

We construct $\mathcal{C}_A$ by induction. Let $QRA$ be a fibrant-cofibrant replacement of $A$. Define $\mathcal{C}_A^0 := \{QRA\}$, the full subcategory of $\mathcal{M}$ with a single object. Assume we have defined a full subcategory $\mathcal{C}_A^{i-1} \subseteq \mathcal{M}, i \geq 1$, and recall the poset $\partial\Delta^n$ from Definition 2.6. Let $X \in \mathcal{C}_A^{i-1}$ and let $F: \partial\Delta^n \to \mathcal{C}_A^{i-1}$ be a fibrant diagram with respect to the injective model structure we described in Proposition 2.10. Let $\phi: X \to \lim_{\alpha \in \partial\Delta^n} F(\alpha)$.
be a morphism of $\mathcal{M}$, and consider the functorial factorization

$$
\xymatrix{
X \ar[r]^-{\phi} \ar[d]_-{\sim}^{\tau} & \lim_{\alpha \in \partial \Delta^n} F(\alpha) \ar[dr]^-{p} \\
Z_{(X,F,\phi)} & & X
}
$$

(3.1)

where $Z_{(X,F,\phi)} := V_\phi$. Also consider the functorial factorization to the canonical map $g: X \coprod X \to X$:

$$
\xymatrix{
X \coprod X \ar[r]^-{g} \ar[d]_-{\sim}^{\pi} & X \\
Z_X & & X
}
$$

(3.2)

where $Z_X := W_g$. Now recall the poset $\lozenge_k^n$ from Definition 2.13. Let $R\text{Hom}(\partial \Delta^n; C_A^{i-1})$ denote the set of fibrant objects of $(C_A^{i-1})^{\partial \Delta^n}$, and let $\bar{R}\text{Hom}(\lozenge_k^n, C_A^{i-1})$ denote the set of fibrant diagrams $F: \lozenge_k^n \to C_A^{i-1}$ sending every morphism to a weak equivalence. Define three full subcategories $E_A^i$, $H_A^i$, and $D_A^i$ of $\mathcal{M}$ as

$$
\text{ob}(E_A^i) := \left\{ Z_{(X,F,\phi)} \mid X \in C_A^{i-1}, F \in R\text{Hom}(\partial \Delta^n; C_A^{i-1}), \phi \in \text{Hom}(X, \lim_{\alpha \in \partial \Delta^n} F(\alpha)) \right\},
$$

and

$$
\text{ob}(H_A^i) := \left\{ Z_X \mid X \in C_A^{i-1} \right\},
$$

and

$$
\text{ob}(D_A^i) = \left\{ Q\text{lim}_{\lozenge_k^n} F \mid F \in \bar{R}\text{Hom}(\lozenge_k^n, C_A^{i-1}), 0 \leq k \leq n \right\},
$$

where $Q(-)$ stands of course for the cofibrant replacement functor. Also define the full subcategory $A_A^i \subseteq \mathcal{M}$ as

$$
\text{ob}(A_A^i) := \left\{ R\text{colim}_{\Psi} F \mid F \in \bar{Q}\text{Hom}(\Psi, C_A^{i-1}) \right\},
$$

where

- $R(-)$ is the fibrant replacement functor.
- $\Psi$ is the poset whose objects are $\emptyset, \{1\}, \cdots, \{n\}$ for some positive integer $n$ and whose morphisms are inclusions.
- $\bar{Q}\text{Hom}(\Psi, C_A^{i-1})$ is the set of covariant functors $F: \Psi \to C_A^{i-1}$ such that $F(f)$ is an acyclic cofibration for every morphism $f$ of $\Psi$.

**Definition 3.1.** [9, Section 3] A diagram of shape $\Psi$, denoted by a collection of maps $\{f_1, \cdots, f_n\}$ in $\mathcal{M}$ with same domain, is called menorah.

**Definition 3.2.** Let $\mathcal{M}$ be a model category, and let $A \in \mathcal{M}$. Define $C_A \subseteq \mathcal{M}$ as the full subcategory

$$
C_A := \bigcup_{i \geq 0} C_A^i,
$$

where $C_A^i := \left\{\right.$

- $\{QRA\}$ if $i = 0$
- $E_A^i \cup H_A^i \cup D_A^i \cup A_A^i \cup C_A^{i-1}$ if $i \geq 1$

**Remark 3.3.** From the definition, there are five kinds of objects in $C_A$. 

9
The distinguished object, QRA, will be used in Lemma 4.11 to prove that a certain functor is essentially surjective.

Objects of the form \(Z_{(X,F,\phi)}\) will be used in Section 3.2, Lemmas 4.11, 5.4, and other places.

Objects of the form \(Z_X\) will be used in Lemma 4.13 to prove that a certain functor is faithful.

Objects of the form \(Q\lim_{\Lambda^n_k} F\) will be used in Proposition 3.16 to prove that \(\hat{A}_\ast\) is a Kan complex.

Lastly, objects of the form \(R\colim_{\Psi} F\) will be used in Section 3.3 to define the degeneracy maps of \(\hat{A}_\ast\).

The category \(C_A\) has nice features given by the following two propositions.

**Proposition 3.4.** The collection of objects of \(C_A\) is a set. That is, the category \(C_A\) is small.

**Proof.** By induction, it is easy to show that for all \(i\) the collection of objects of \(C_A^i\) is a set. This implies the proposition.

**Proposition 3.5.** Every object of \(C_A\) is fibrant, cofibrant, and weakly equivalent to \(A\).

**Proof.**

(i) To see that every object of \(C_A\) is fibrant, we will proceed by induction on \(i\). If \(i = 0\) then the statement is clearly true since QRA is fibrant. Assume that every object of \(C_A^{i-1}\) is fibrant, and let \(Z \in C_A^i\). If \(Z\) is equal to \(Z_{(X,F,\phi)}\) as in (3.1) then \(Z\) is fibrant since the map \(p\) is a fibration and its target is fibrant (by Proposition 2.12). If \(Z\) is equal to \(Z_X\) as in (3.2), the same argument applies. If \(Z\) is equal to \(Q\lim_{\Lambda^n_k} F\) for some \(F \in R\text{Hom}(\Lambda^n_k, C_A^{i-1})\), then \(Z\) is fibrant since \(\lim_{\Lambda^n_k} F\) is fibrant (by Proposition 2.12). Objects of the form \(Z = R\colim_{\Psi} F\) are obviously fibrant.

(ii) Similarly, one can show by induction that every object of \(C_A\) is cofibrant. Objects of the form \(R\colim_{\Psi} F\) deserve a special attention. They are cofibrant basically because every morphism of the diagram \(F\) is an acyclic cofibration, and thanks to the shape of \(F\), one can compute its colimit by taking “successive pushouts”. In fact, one can easily show that the map from \(F(\emptyset)\) to \(colim_{\Psi} F\) is an acyclic cofibration (by using the fact that the pushout of a cofibration is again a cofibration, and the pushout of a weak equivalence along a cofibration is a weak equivalence). This implies that \(colim_{\Psi} F\) is cofibrant since \(F(\emptyset)\) is cofibrant by the induction hypothesis.

(iii) We proceed again by induction to prove that every object of \(C_A\) is weakly equivalent to \(A\). The base case is obvious. Assume that the statement holds for \(i - 1\). Let \(Z \in C_A^i\). If \(Z = Z_{(X,F,\phi)}\) or \(Z = Z_X\), then \(Z\) is weakly equivalent to \(A\) by the induction hypothesis and the fact that the maps \(\tau\) and \(\pi\) from (3.1) and (3.2) respectively are both weak equivalences. Now assume that \(Z = Q\lim_{\Lambda^n_k} F\). Since \(Q(-)\) is the cofibrant replacement functor, we have the weak equivalence \(\hat{A}_\ast \xrightarrow{\sim} \lim_{\Lambda^n_k} F\). Furthermore the limit \(\lim_{\Lambda^n_k} F\) is weakly equivalent to \(A\) by the induction hypothesis and the fact that the map from the limit of \(F\) to each piece of the diagram is a weak equivalence. This latter fact comes from Proposition 2.14. Lastly, objects of the form \(Z = \colim_{\Psi} F\) are weakly equivalent to \(A\) since the map \(F(\emptyset) \to \colim_{\Psi} F\) is a weak equivalence as explained in the previous part.

This ends the proof.
3.2 Specific fibrant replacement functor $\mathcal{R} : \mathcal{A}^n \to \mathcal{A}^n$

On the category $\mathcal{A}^n$, consider the model structure described in Proposition 2.10. For our purposes, we need to construct a specific fibrant replacement functor $\mathcal{R} : \mathcal{A}^n \to \mathcal{A}^n$ that has nice properties (see Proposition 3.6 below). The idea is to first take the fibrant replacement of $0$-simplices, then $1$-simplices, and so on. So we need to proceed by induction on $n$.

- For $n = 0$, define $\mathcal{R} : \mathcal{A}^n \to \mathcal{A}^n$ as $\mathcal{R}(F) := F$. This makes sense since a diagram $F : \Delta^0 \to \mathcal{A}$ consists of a single object, $F(\{0\})$, which is fibrant thanks to Proposition 3.5.

- Let $n = 1$, and let $F : \Delta^1 \to \mathcal{A}$ be an object of $\mathcal{A}^n$. We want to define a fibrant diagram $\mathcal{R}(F) : \Delta^1 \to \mathcal{A}$. If $F$ is fibrant, define $\mathcal{R}(F) := F$. If $F$ is not fibrant, define $\mathcal{R}(F)$ as follows. Set $F = \{X_0 \leftarrow X_{01} \to X_1\}$, where $X_i := F(%\{i\})$ and $X_0 := f(%\{0,1\})$. Consider the map $\phi : X_0 \to \lim_{\alpha \in \partial \Delta^1} F(\alpha)$ provided by the universal property of limit. Also consider the projection $p_{\alpha} : \lim_{\alpha \in \partial \Delta^1} F(\alpha) \to F(\alpha')$, $\alpha' \in \partial \Delta^1$. Replacing $X$ by $X_{01}$ and $n$ by $1$ in (3.1), we get $\Delta_{01} := Z(X_{01}, F, \phi) \in \mathcal{A}$ together with the map $\overline{p} : \Delta_{01} \to \lim_{\alpha \in \partial \Delta^1} F(\alpha)$. Define $\mathcal{R}(F)$ as

$$\mathcal{R}(F) := \left\{ X_0 \xleftarrow{p_{\alpha} \overline{p}} \Delta_{01} \xrightarrow{p_{\alpha} \overline{p}} X_1 \right\},$$

where $\alpha_0 = \{0\}$ and $\alpha_1 = \{1\}$. Note that by definition the functor $\mathcal{R}$ has the following property:

$$\mathcal{R}(Fd^i) = \mathcal{R}(F)d^i, \quad \text{for all } d^i : \Delta^0 \to \Delta^1.$$  

- Assume we have defined $\mathcal{R} : \mathcal{A}^k \to \mathcal{A}^k$ for all $k \leq n - 1$. Also assume that $\mathcal{R}$ has the following property:

$$\mathcal{R}(Fd^i) = \mathcal{R}(F)d^i, \quad \text{for all } d^i : \Delta^{n-2} \to \Delta^{n-1}. \quad (3.3)$$

Let $F \in \mathcal{A}^n$. We need to define $\mathcal{R}(F) : \Delta^n \to \mathcal{A}$. If $F$ is fibrant, define $\mathcal{R}(F) := F$. Assume $F$ is not fibrant, and define $\mathcal{R}(F)$ as follows. The idea is to first take the fibrant replacement of all $(n-1)$-faces of $F$ using the induction hypothesis (this will produce a new functor $\mathcal{F} : \Delta^n \to \mathcal{A}$), and then substitute $\mathcal{F}([n])$ by the appropriate $Z$ in $\mathcal{A}$. So we will proceed in two steps.

- Construction of $\mathcal{F}$. For $0 \leq j \leq n$, define $\mathcal{F}^j : \partial \Delta^n \to \mathcal{A}$ as $\mathcal{F}^j := \mathcal{R}(Fd^j)$. By using (3.3), and the definitions it is straightforward to see that $\mathcal{F}^j$ agrees with $\mathcal{F}^j$ on the intersection $\partial \Delta^n \cap \partial \Delta^n$. This allows us to define $\mathcal{F} : \partial \Delta^n \to \mathcal{A}$ on the boundary of $\Delta^n$ as $\mathcal{F}(\alpha) := \mathcal{F}^j(\alpha)$ if $\alpha \in \partial \Delta^n$. Since the functor $\mathcal{F}$ is a fibrant replacement of $F/\partial \Delta^n$, there is a weak equivalence $\eta$ from the latter to the former. Define $\mathcal{F}([n]) := F([n])$. For $\alpha \in \partial \Delta^n$, $d^n[n] : \alpha \to [n]$ a morphism of $\Delta^n$, define $\mathcal{F}(d^n[n])$ as the composition

$$F([n]) \xrightarrow{F(d^n[n])} F(\alpha) \xrightarrow{\eta(\alpha)} \mathcal{F}(\alpha),$$

where $\eta(\alpha)$ is the component of the natural transformation $\eta$ at $\alpha$. This completes the definition of $\mathcal{F} : \Delta^n \to \mathcal{A}$.

- The boundary of $\mathcal{F}$, that is $\mathcal{F} |_{\partial \Delta^n}$, is certainly fibrant, but $\mathcal{F}$ itself might not be fibrant as the matching map $\phi : \mathcal{F}([n]) \to \lim_{\alpha \in \partial \Delta^n} \mathcal{F}(\alpha)$ might not be a fibration. To fix this, consider the object $Z(\mathcal{F}([n]), \mathcal{F}, \phi) \in \mathcal{A}$, which comes equipped with $p : Z(\mathcal{F}([n]), \mathcal{F}, \phi) \to \lim_{\alpha \in \partial \Delta^n} \mathcal{F}(\alpha)$ (see (3.1)). Define

$$\mathcal{R}(F)(\partial \Delta^n) := \mathcal{F}, \quad \text{and} \quad \mathcal{R}(F)([n]) := Z(\mathcal{F}([n]), \mathcal{F}, \phi).$$

For $\alpha' \in \partial \Delta^n$, $d^n[n] : \alpha' \to [n]$, define $\mathcal{R}(F)(d^n[n]) := p_{\alpha'} \overline{p}$, where $p_{\alpha'} : \lim_{\alpha \in \partial \Delta^n} \mathcal{F}(\alpha) \to \mathcal{F}(\alpha')$ is the canonical projection as usual. This completes the definition of $\mathcal{R}(F)$, which is indeed fibrant and has the required property by construction.
Since the factorization (3.1) is functorial, a simple induction argument on \( n \) shows that the assignment \( F \mapsto R(F) \) is functorial. That functor has nice properties given by the following.

**Proposition 3.6.** Let \( F: \tilde{\Delta}^n \to C_A \) be a contravariant functor.

(i) If \( F \) is fibrant then \( R(F) = F \).

(ii) If \( \tau: \tilde{\Delta}^k \to \tilde{\Delta}^n \) is injective, then \( R(F\tau) = R(F)\tau \).

(iii) If \( \tau: \tilde{\Delta}^k \to \tilde{\Delta}^n \) is injective and \( F\tau \) is fibrant, then \( R(F)\tau = F\tau \), equivalently \( R(F)|\tilde{\Delta}^k = F|\tilde{\Delta}^k \). In other words, if any face of \( F \) is fibrant, it appears in \( R(F) \).

**Proof.** This follows immediately from the construction of \( R \). \( \square \)

### 3.3 The simplicial set \( \hat{A}_\bullet \)

To prove the main result of this paper (that is, Theorem 1.1), we need to construct a simplicial set \( X_\bullet \) with the following properties.

(A) \( X_\bullet \) is a Kan complex.

(B) There is a pair

\[
\begin{array}{ccc}
\mathcal{F}(U(T^M); C_A) & \xrightarrow{\Lambda} & \text{Hom}(T^M; X_\bullet) \\
\downarrow \Theta & & \downarrow \text{sSet} \\
\end{array}
\]

of maps (for the meaning of \( \mathcal{F}(U(T^M); C_A) \), see Definition 4.8) that satisfies the following four conditions:

(i) \( \Lambda(F) \) is homotopic to \( \Lambda(F') \) whenever \( F \) is weakly equivalent to \( F' \);

(ii) \( \Theta(f) \) is weakly equivalent to \( \Theta(f') \) whenever \( f \) is homotopic to \( f' \);

(iii) \( \Theta \Lambda(F) \) is weakly equivalent to \( F \) for any \( F \);

(iv) \( \Lambda \Theta(f) \) is homotopic to \( f \) for any \( f \).

The natural candidate for \( X_\bullet \) is the simplicial set \( \hat{A}_\bullet \) defined as follows. An \( n \)-simplex of \( \hat{A}_\bullet \) is just a contravariant functor \( \sigma: \tilde{\Delta}^n \to C_A \). The simplicial structure of \( \hat{A}_\bullet \) is the one induced by the cosimplicial structure of \( \tilde{\Delta}^\bullet \) (see Proposition 2.5). The issue with the simplicial \( \hat{A}_\bullet \) is that it might not satisfy (A), and whether the other conditions are satisfied can depend on how \( \Lambda \) is defined. We can show that \( \hat{A}_\bullet \) satisfies (B)- (i) for some \( \Lambda \), and (B)- (iv) for another \( \Lambda \), but we do not know how to prove that these two conditions are met for the same \( \Lambda \). For the natural \( \Lambda \) (with \( X_\bullet = \hat{A}_\bullet \)), we can easily prove that \( \Lambda \Theta = \text{id} \) and \( \Theta \Lambda = \text{id} \), but we do not know how to prove that it satisfies (B) - (i).

We now define \( \hat{A}_\bullet \). In proposition 3.16 below we will prove that \( \hat{A}_\bullet \) is a Kan complex. In the upcoming sections, we will show that \( \hat{A}_\bullet \) meets the remaining conditions.

**Definition 3.7.** Let \( C_A \) be the category from Definition 3.2 and let \( \tilde{\Delta}^a \) be the poset from Definition 2.1. Define \( \hat{A}_n \) as the collection of contravariant functors \( \sigma: \tilde{\Delta}^n \to C_A \) satisfy the following two conditions:

(a) \( \sigma \) sends every morphism to a weak equivalence;

(b) \( \sigma \) is a fibrant \( \tilde{\Delta}^n \)-diagram.
Note that by Proposition 3.5 for all \( x \in \tilde{\Delta}^n \), \( \sigma(x) \) is weakly equivalent to \( A \). Also note that each \( \hat{A}_n \) is a set since the category \( \Delta^n \) is small by definition, and \( C_A \) is small as well (by Proposition 3.4).

**Definition 3.8.** Recall the functor \( d^i: \tilde{\Delta}^{n-1} \to \tilde{\Delta}^n \) from Definition 2.4.

(i) Define the face map \( d_i: \hat{A}_n \to \hat{A}_{n-1}, 0 \leq i \leq n \), as \( d_i(\sigma) := \sigma d^i \).

(ii) The degeneracy maps are defined in Section 3.3.1 below.

Intuitively, \( d_i(\sigma) \) can be defined “geometrically” as follows. Thinking of \( \sigma \in \hat{A}_n \) as \( \sigma(\tilde{\Delta}^n) \), the object \( d_i(\sigma) \) is nothing but the \((n-1)\)-face of \( \sigma \) “opposite” to the vertex \( \sigma(\{i\}) \). For instance, for \( n = 2 \), consider the diagram obtained by applying \( \sigma \) to the diagram from Example 2.3 For \( i = \{0,1\} \), one has

\[
d_0(\sigma) = \left\{ \begin{array}{c}
\sigma(\{1\}) \\
\sigma(\{1,2\}) \\
\sigma(\{2\})
\end{array} \right\},
\]

\[
d_1(\sigma) = \left\{ \begin{array}{c}
\sigma(\{0\}) \\
\sigma(\{0,2\}) \\
\sigma(\{2\})
\end{array} \right\}.
\]

**3.3.1 Definition of the degeneracies** \( s_j: \hat{A}_n \to \hat{A}_{n+1} \)

First we need the following.

**Definition 3.9.** Let \( k \geq 1 \). Consider the set of sequences \( (i_1, \ldots, i_k) \) of length \( k \) where \( i_p \) is an non-negative integer for every \( p \). Define on that set the equivalence relation \( \sim \) generated by

\[
(i_1, \ldots, i_{p-1}, i_p, \ldots, i_k) \sim (i_1, \ldots, i_p + 1, i_{p-1}, \ldots, i_k), \ i_{p-1} \leq i_p
\]

This means that when \( i_{p-1} \leq i_p \), we switch \( i_{p-1} \) and \( i_p \), then we add 1 to \( i_p \), the other numbers remaining unchanged.

So two sequences \( (i_1, \ldots, i_k) \) and \( (r_1, \ldots, r_k) \) are in relation if one can be obtained from the other by using (3.4) as many times as needed. Given a simplicial set \( X \) and two sets \( X_{m_0} \) and \( X_{m_k} \), we assign to each sequence \( (i_1, \ldots, i_k) \), with \( 0 \leq i_{k-p+1} \leq m_{p-1} \) for all \( p \), a sequence of degeneracy maps \( (s_{i_k}, \ldots, s_{i_1}) \) between \( X_{m_0} \) and \( X_{m_k} \):

\[
X_{m_0} \xrightarrow{s_{i_k}} \cdots \xrightarrow{s_{i_1}} X_{m_k}
\]

So \( (i_1, \ldots, i_k) \sim (r_1, \ldots, r_k) \) amounts to saying that the composite \( s_{i_k} \cdots s_{i_1} \) can be obtained from \( s_{r_k} \cdots s_{r_1} \) by using the simplicial identity \( s_is_j = s_{j+1}s_i, i \leq j \), as many times as needed.

We now define \( s_j: \hat{A}_n \to \hat{A}_{n+1}, 0 \leq j \leq n \) by induction on \( n \) as follows. Recall the functor \( R \) from Section 3.2. Also recall the functor \( s^j: \tilde{\Delta}^{n+1} \to \tilde{\Delta}^n \) from Definition 2.4. Given a simplex \( \sigma \in \hat{A}_n \), the composite \( \sigma s^j \) is not a priori an element of \( \hat{A}_{n+1} \) as it might fail to be fibrant. Actually, \( \sigma s^j \) is a \((n+1)\)-simplex of \( \hat{A}_n \), the simplicial set we defined at the beginning of Section 3.3.1. We still denote the face maps of \( \hat{A}_n \) by \( d_i \).

For \( n = 0, s_0: \hat{A}_0 \to \hat{A}_1 \) is defined as \( s_0(\sigma) := R(\sigma s^0) \), where \( s^0: \tilde{\Delta}^1 \to \tilde{\Delta}^0 \).

Let \( n \geq 1 \).

(IH1) Suppose that for every \( m \leq n \) we have defined the degeneracy maps \( s_j: \hat{A}_{m-1} \to \hat{A}_m, 0 \leq j \leq m-1 \).

(IH2) Assume that the simplicial identities
\[
\begin{align*}
\frac{d_is_j}{d_is_j} = s_{j-1}d_i & \quad \text{if } i < j \\
\frac{d_is_j}{d_is_j} = sjd_{i-1} & \quad \text{if } i > j + 1 \\
\frac{s_is_j}{s_is_j} = d_{j+1}s_j & \quad \text{if } i \leq j
\end{align*}
\]

are satisfied for all \(s_j: \hat{A}_{m-1} \to \hat{A}_m, 0 \leq j \leq m - 1, m \leq n\).

(IH3) Assume that for all \(m \leq n\), for all \(\sigma \in \hat{A}_m\), there are weak equivalences

\(\bullet\) \(d_i(\sigma s^j) \sim s_{j-1}(d_i\sigma)\), if \(i < j\), and

\(\bullet\) \(d_i(\sigma s^j) \sim s_j(d_{i-1}\sigma)\), if \(i > j + 1\).

(IH4) Suppose that for every \(m_0 \leq \cdots \leq m_{k+1} \leq n + 1\), for any sequences \((i_0, \cdots, i_k)\) and \((r_0, \cdots, r_k)\), with \(0 \leq i_k - p, r_k - p \leq m_p\) for all \(p\), such that \((i_0, \cdots, i_k) \sim (r_0, \cdots, r_k)\), there exists an acyclic cofibration

\[
\phi_{\sigma s^j}^{m_k+1}: \lambda s_i \cdots s_i' s_i''([m_k+1]) \sim (s_{r_1} \cdots s_{r_k} \lambda) s''([m_k+1]) \tag{3.5}
\]

for every non-degenerate simplex \(\lambda \in \hat{A}_{m_0}\).

Define \(s_j: \hat{A}_n \to \hat{A}_{n+1}\) as follows. Let \(\sigma \in \hat{A}_n\). Then there exists a unique non-degenerate simplex \(\lambda \in \hat{A}_{m_0}\) and a sequence \((s_{i_1}, \cdots, s_{i_k})\) between \(\hat{A}_{m_0}\) and \(\hat{A}_n\) such that \(\sigma = s_{i_1} \cdots s_{i_k} \lambda\). (Note that if \(\sigma\) is itself non-degenerate, then \(m_0 = n, s_{i_1} = \cdots = s_{i_k} = id\), and \(\lambda = \sigma\).) Consider the composite \(\sigma s^j\).

\(\bullet\) If \(i < j\), by (IH3), there exists a weak equivalence \(d_i(\sigma s^j) \sim s_{j-1}(d_i\sigma)\). This allows us to replace the face \(d_i(\sigma s^j)\) by \(s_{j-1}(d_i\sigma)\).

\(\bullet\) Similarly, if \(i > j + 1\), replace the face \(d_i(\sigma s^j)\) by \(s_j(d_{i-1}\sigma)\).

\(\bullet\) This defines a contravariant functor \(\hat{\sigma}s^j: \hat{\Delta}^{n+1} \to C_A\) as \(\hat{\sigma}s^j([n + 1]) := \sigma s^j([n + 1])\), and

\[
\hat{\sigma}s^j|\partial\hat{\Delta}^{n+1} := \begin{cases} 
  s_{j-1}(d_i\sigma) & \text{if } i < j \\
  s_j(d_{i-1}\sigma) & \text{if } i > j + 1 \\
  \sigma & \text{if } i \in \{j, j + 1\}.
\end{cases}
\]

On morphisms of \(\hat{\Delta}^{n+1}\), define \(\hat{\sigma}s^j\) in the obvious way.

\(\bullet\) We now replace \(\hat{\sigma}s^j([n + 1])\) by an object \(Y \in C_A\) defined as follows. Let \(i_0 := j\). Recalling Definition 3.1, define the menorah \(D: \Psi \to C_A\) as the collection of morphisms from \(3.5\) with \(m_k + 1 = n + 1\). That is,

\[
D := \{\phi_{r_0, \cdots, r_k}^{n+1} \mid (r_0, \cdots, r_k) \sim (i_0, \cdots, i_k)\}. \tag{3.6}
\]

Thanks to Lemma 3.11 below, there exists a natural transformation \(\text{colim}_\Psi D \sim \hat{\sigma}s^j|\partial\hat{\Delta}^{n+1}\) given by the universal property of colimit (here \(\text{colim}_\Psi D\) is viewed as the constant diagram). Taking the fibrant replacement (objectwise) of that morphism, we get a weak equivalence

\[
\varphi: R\text{colim}_\Psi D \to R\hat{\sigma}s^j|\partial\hat{\Delta}^{n+1} = \hat{\sigma}s^j|\partial\hat{\Delta}^{n+1}
\]

in \(C_A\). (Remember \(R\text{colim}_\Psi D\) is an object of \(C_A\) by Definition 3.2.) Define \(Y := R\text{colim}_\Psi D\). The map \(\varphi\) gives rise to a contravariant functor \(\hat{\sigma}s^j: \hat{\Delta}^{n+1} \to C_A\) defined as \(\hat{\sigma}s^j([n + 1]) := Y\) and \(\hat{\sigma}s^j|\partial\hat{\Delta}^{n+1} := \hat{\sigma}s^j|\partial\hat{\Delta}^{n+1}\).

**Definition 3.10.** The \((n + 1)\)-simplex \(s_j(\sigma)\) is defined as \(s_j(\sigma) := R(\hat{\sigma}s^j)\).

By construction, it is straightforward to show that the hypothesis (IH2), (IH3), and (IH4) are verified if one replaces \(n\) by \(n + 1\).

**Lemma 3.11.** Consider the menorah \(D\) above \(3.6\).
(i) For any sequence \((r_0, \cdots, r_k)\) such that \((r_0, \cdots, r_k) \sim (i_0, \cdots, i_k)\), there exists a natural transformation
\[
\psi_r: (s_{r_1} \cdots s_{r_k} \lambda)s^{i_0}([n + 1]) \longrightarrow \widetilde{\sigma^{i_0}}|\partial \Delta^{n+1},
\]
which is a weak equivalence.

(ii) In order to lighten the notation, we denote \(\phi_r := \phi^{i_0+1}_{r_0 \cdots r_k}\). For any other sequence \((t_0, \cdots, t_k) \sim (i_0, \cdots, i_k)\), the following square commutes for every \(x \in \partial \Delta^{n+1}\).

\[
\begin{align*}
\lambda s^{i_k} \cdots s^{i_0}([n + 1]) & \xrightarrow{\phi_r} (s_{t_1} \cdots s_{t_k} \lambda)s^{i_0}([n + 1]) \\
\phi_i & \sim \\
(s_{t_1} \cdots s_{t_k} \lambda)s^{i_0}([n + 1]) & \xrightarrow{\psi_r} \gamma_\sigma(x)
\end{align*}
\]

Proof. We begin with the first part. Let \(0 \leq l \leq n + 1\). We need to define
\[
\psi_r: (s_{r_1} \cdots s_{r_k} \lambda)s^{i_0}([n + 1]) \longrightarrow (d_l \sigma s^{i_0})(x)
\]
for every \(x \in \partial^l \Delta^{n+1}\). We define \(\psi_r\) when \(x = [n + 1]_l \cong [n]\). Then for the other values of \(x\), \(\psi_r\) is defined as the obvious composition. We will leave to the reader to check that the map \(\psi_r\) is indeed a natural transformation. Let us treat only the case \(l < i_0\), the cases \(l > j + 1\) and \(l \in \{j, j + 1\}\) being similar. First of all, define \(d_l s_{t_0} s_{t_1} \cdots s_{t_k} \lambda([n]) := s_{i_0 - 1}d_l s_{t_1} \cdots s_{t_k} \lambda([n])\) and \(d_l s_{r_0} s_{r_1} \cdots s_{r_k} \lambda([n]) := s_{r_0'}d_l s_{r_1} \cdots s_{r_k} \lambda([n])\), where \((r_0', l', r_1, \cdots, r_k) \sim (r_0, r_1, \cdots, r_k)\). On the one side we have
\[
d_l \sigma s^{i_0}([n]) = s_{i_0 - 1}d_l s_{t_1} \cdots s_{t_k} \lambda([n]) \text{ by definition}
\]
\[
= d_l s_{t_0} s_{t_1} \cdots s_{t_k} \lambda([n]) \text{ by definition}
\]
\[
= d_l s_{r_0} s_{r_1} \cdots s_{r_k} \lambda([n]) \text{ by hypothesis and (IH2)}
\]
\[
= s_{r_0'} \cdots s_{r_k} d_l \nu \lambda([n]) \text{ by definition and (IH2)}
\]
On the other side, we have the weak equivalences
\[
(s_{r_1} \cdots s_{r_k} \lambda)s^{i_0}([n + 1]) \xrightarrow{\sim} d_l(\sigma s_{r_0} \cdots \sigma s_{r_k})([n]) \xrightarrow{\sim} s_{r_0'} s_{r_1'} \cdots s_{r_k'} d_l \nu \lambda([n])
\]
The first one is obvious, while the second is nothing but (IH3). Using (IH2), we get that \(s_{r_0'} s_{r_1'} \cdots s_{r_k'} d_l \nu \lambda([n]) = s_{r_0'} s_{r_1'} \cdots s_{r_k'} d_l \nu \lambda([n])\). We thus obtain the required map
\[
\psi_r: (s_{r_1} \cdots s_{r_k} \lambda)s^{i_0}([n + 1]) \longrightarrow s_{r_0'} s_{r_1'} \cdots s_{r_k'} d_l \nu \lambda([n]) = d_l(\sigma s^{i_0}([n])
\]
If \(d_l\) happens to disappear when using (IH2) or \(d_l \nu \lambda\) happens to be a degenerate simplex, we define \(\psi_r\) is in a similar fashion.

The second part is straightforward. This ends the proof. \qed

Remark 3.12. (i) From the definition, it is straightforward to see that \(s_j(\sigma) = \mathcal{R}(\sigma s^j)\) whenever \(\sigma\) and all of its faces are non-degenerate.

(ii) A natural question one may ask is to know why we do not define all \(s_j\)'s by the simple formula \(\mathcal{R}(\sigma s^j)\). The reason is the fact that if we do so, the simplicial identity \(s_i s_j = s_{j+1} s_i, i \leq j\), won't hold when \(i < j\).
Example 3.13. Consider the 1-simplex

\[ \sigma = \{ X_0 \xrightarrow{d_1} X_{01} \xrightarrow{d_0} X_1 \} , \]

where \( X_i := \sigma \{ i \} , X_{01} := \sigma \{ 0, 1 \} \), and \( d_i := \sigma (d^i) \). We have

\[ \sigma s^0 = \]

One can notice that the 1-face \( \{ X_0 \xrightarrow{id} X_{01} \xrightarrow{id} X_0 \} \) is not fibrant as the matching map \((id, id) : X_0 \rightarrow X_0 \times X_0\) is the diagonal map, which is not a fibration. The two other 1-faces are both equal to \( \sigma \), which is fibrant. Since one face is not fibrant, it follows that the whole diagram \( \sigma s^0 \) is not fibrant as well. If \( \sigma \) is non-degenerate, by applying \( R \) to \( \sigma s^0 \) we get \( s_0 \sigma = R(\sigma s^0) = \)

Proposition 3.14. The collection \( \hat{A}_\bullet = \{ \hat{A}_n \}_{n \geq 0} \) endowed with \( d_i : \hat{A}_n \rightarrow \hat{A}_{n-1} \) and \( s_j : \hat{A}_n \rightarrow \hat{A}_{n+1} \) forms a simplicial set.

Proof. The simplicial identity \( d_i d_j = d_{j-1} d_i , i < j \), follows from the definition and Proposition 2.5 while the other four simplicial identities follow immediately from the construction of \( s_j \).

Remark 3.15. For every \( n \geq 0 \), for every \( \sigma, \sigma' \in \hat{A}_n \), for any natural transformation \( \beta : \sigma \rightarrow \sigma' \), there exist two natural transformations \( d_i \beta : d_i(\sigma) \rightarrow d_i(\sigma) \) and \( s_j \beta : s_j(\sigma) \rightarrow s_j(\sigma') \) for all \( i, j \). This is straightforward to prove by induction on \( n \).

3.3.2 Proving that \( \hat{A}_\bullet \) is a Kan complex

Proposition 3.16. The simplicial set \( \hat{A}_\bullet \) is a Kan complex.

Proof. Let \( n \geq 0 \), and let \( k \in \{ 0, \ldots , n \} \). Consider \( n (n-1) \)-simplices \( \sigma_0 , \ldots , \sigma_k , \ldots , \sigma_n \) of \( \hat{A}_\bullet \) such that \( d_i(\sigma_j) = d_{j-1}(\sigma_i) , i < j \), and \( i, j \neq k \). Our goal is to construct an \( n \)-simplex \( \sigma \in \hat{A}_n \) such that \( d_i \sigma = \sigma_i \) for all \( i \neq k \). We first need to construct an intermediate functor \( \sigma : \hat{\Delta}^n \rightarrow \mathcal{C}_A \).
Recall the poset $\Lambda^n_k$ from Definition 2.13. For $i \in \{0, \cdots, n\}, i \neq k$, define $\sigma: \Lambda^n_k \rightarrow C_A$ as $\sigma(\alpha) := \sigma_i(\alpha)$ for $\alpha \in \partial^i \Delta_n \cong \Delta^{n-1}$. It is straightforward to see that this is well defined on the intersection $\partial^i \Delta_n \cap \partial^j \Delta_n$. Now define

$$\sigma([n]) := Q \lim_{\alpha \in \Lambda^n_k} \sigma(\alpha), \quad \sigma([n]_k) := \sigma([n]), \quad \text{and} \quad \sigma(d^k) := \text{id},$$

where $d^k: [n]_k \rightarrow [n]$ is a morphism of $\Delta^n$. For $\alpha' \in \Lambda^n_k$, and $d^{\alpha'}[n]: \alpha' \rightarrow [n]$ a morphism of $\Delta^n$, define $\sigma(d^{\alpha'}[n])$ as the composition

$$Q \lim_{\alpha \in \Lambda^n_k} \sigma(\alpha) \xrightarrow{\sim} \lim_{\alpha \in \Lambda^n_k} \sigma(\alpha) \xrightarrow{p_{\alpha'}} \sigma(\alpha'),$$

where $p_{\alpha'}$ is the canonical projection as usual. Define $\sigma := R(\sigma)$. By construction and Proposition 2.14 the functor $\sigma: \Delta^n \rightarrow C_A$ thus defined belongs to $\Delta_n$. Moreover, using the properties of $R$ from Proposition 3.6 one has $d_i \sigma = \sigma_i$ for all $i \neq k$. This proves the proposition.

4. The functor categories $\mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{M})$ and $\mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A)$

From now on, we let $\mathcal{T}^M$ denote a triangulation of $M$. Let $\mathcal{C}_A \subseteq \mathcal{M}$ be the small subcategory constructed in Section 3.1. In this section we recall an important poset $\mathcal{U}(\mathcal{T}^M)$ and introduce two categories: $\mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{M})$ and $\mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A)$. The first one is the category of isotopy functors $\mathcal{U}(\mathcal{T}^M) \rightarrow \mathcal{M}$ that send every object to an object weakly equivalent to $A$, while the second is the category of isotopy functors from $\mathcal{U}(\mathcal{T}^M)$ to $\mathcal{C}_A$. By the definitions there is an inclusion functor $\phi: \mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A) \hookrightarrow \mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{M})$. The goal of this section is to prove (1.3) or Proposition 4.14 below, which says that the localization of $\phi$ with respect to weak equivalences is an equivalence of categories. We begin with the definition of $\mathcal{U}(\mathcal{T}^M)$.

4.1 The poset $\mathcal{U}(\mathcal{T}^M)$

The poset $\mathcal{U}(\mathcal{T}^M)$ was introduced by the authors in [13, Section 4.1]. We recall its definition, and refer the reader to [13, Section 4.1] for more explanation.

Definition 4.1. First take two barycentric subdivisions of $\mathcal{T}^M$, and then define $U_\sigma$ as the interior of the star of $\sigma$. An object of $\mathcal{U}(\mathcal{T}^M)$ is defined to be $U_\sigma, \sigma \in \mathcal{T}^M$. There is a morphism $U_\sigma \rightarrow U_{\sigma'}$ if and only if $\sigma$ is a face of $\sigma'$. In other words, morphisms of $\mathcal{U}(\mathcal{T}^M)$ are just inclusions.

The poset $\mathcal{U}(\mathcal{T}^M)$ is related to the poset $P(\mathcal{T}^M)$ we introduced in Definition 2.7 as follows.

Remark 4.2. From the definitions, one has a canonical isomorphism $\mathcal{U}(\mathcal{T}^M) \xrightarrow{\cong} P(\mathcal{T}^M), U_\sigma \mapsto \sigma$. When $\mathcal{T}^M = \Delta^n$, there is an isomorphism $\mathcal{U}(\Delta^n) \cong \Delta^n$, where $\Delta^n$ is the poset from Definition 2.1.

Remark 4.3. (i) It is clear that every morphism of $\mathcal{U}(\mathcal{T}^M)$ is a composition of $d^i$’s, where

$$d^i: U_{\{v_0, \cdots, \hat{v}_a, \cdots, v_n\}} \rightarrow U_{\{v_0, \cdots, v_n\}} \quad (4.1)$$

(ii) It is also clear that every morphism of $\mathcal{U}(\mathcal{T}^M)$ is an isotopy equivalence since the inclusion of one open ball of $M$ inside another one is always an isotopy equivalence [2, Chapter 8].

4.2 The functor categories $\mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{M})$ and $\mathcal{F}(\mathcal{B}_1; \mathcal{M})$

Here we prove (1.2) or Proposition 4.7 below. We begin with a couple of definitions.
Definition 4.4. Define $\mathcal{B}_{\mathcal{U}(T^M)}$ to be the collection of all subsets $B$ of $M$ diffeomorphic to an open ball such that $B$ is contained in some $U_\sigma \in \mathcal{U}(T^M)$.

Certainly $\mathcal{B}_{\mathcal{U}(T^M)}$ is a basis for the topology of $M$ that contains $\mathcal{U}(T^M)$. Often $\mathcal{B}_{\mathcal{U}(T^M)}$ will be thought of as the poset whose objects are $B \in \mathcal{B}_{\mathcal{U}(T^M)}$, and whose morphisms are inclusions.

Definition 4.5. Let $S \subseteq \mathcal{O}(M)$ be a subcategory. Define $\mathcal{F}_A(S; \mathcal{M})$ to be the category whose objects are isotopy functors (see [14, Definition 4.4]) $F : S \rightarrow \mathcal{M}$ such that for every $U \in S$, $F(U)$ is weakly equivalent to $A$. Of course morphisms are natural transformations.

We are mostly interested in the case when $S = \mathcal{U}(T^M)$ or $S = \mathcal{B}_{\mathcal{U}(T^M)}$.

Notation 4.6. If two categories $\mathcal{C}$ and $\mathcal{D}$ are weakly equivalent in the sense of [14, Definition 6.3], we write $\mathcal{C} \simeq \mathcal{D}$.

Proposition 4.7. The category $\mathcal{F}_A(\mathcal{U}(T^M); \mathcal{M})$ is weakly equivalent to the category $\mathcal{F}_A(\mathcal{B}_{\mathcal{U}(T^M)}; \mathcal{M})$. That is,

$$\mathcal{F}_A(\mathcal{U}(T^M); \mathcal{M}) \simeq \mathcal{F}_A(\mathcal{B}_{\mathcal{U}(T^M)}; \mathcal{M}).$$

Proof. Define $\varphi : \mathcal{F}_A(\mathcal{B}_{\mathcal{U}(T^M)}; \mathcal{M}) \rightarrow \mathcal{F}_A(\mathcal{U}(T^M); \mathcal{M})$ as the restriction to $\mathcal{U}(T^M)$. That is, $\varphi(G) := G|\mathcal{U}(T^M)$. To define a functor $\psi$ in the other way, let $F : \mathcal{U}(T^M) \rightarrow \mathcal{M}$ be an object of $\mathcal{F}_A(\mathcal{U}(T^M); \mathcal{M})$. For $B \in \mathcal{B}_{\mathcal{U}(T^M)}$ define $\psi(F)(B) := F(U_\sigma_B)$, where $U_\sigma_B$ is provided by the axiom $(C_3)$ from [14, Definition 4.1]. From the same axiom, one can define $\psi(F)$ on morphisms in the standard way. If $\beta : F \rightarrow F'$ is a morphism of $\mathcal{F}_A(\mathcal{U}(T^M); \mathcal{M})$, define $\psi(\beta)(B) := \beta[U_\sigma_B]$. It is straightforward to see that $\varphi$ and $\psi$ preserve weak equivalences. It is also straightforward to check that $\phi \psi = id$ and $\psi \phi \sim id$. This proves the proposition.

4.3 Proving that $\mathcal{F}_A(\mathcal{U}(T^M); \mathcal{M}) \simeq \mathcal{F}(\mathcal{U}(T^M); \mathcal{C}_A)$

The goal here is to prove [13, Proposition 4.14] below, which says that the localization of $\mathcal{F}_A(\mathcal{U}(T^M); \mathcal{M})$ is equivalent to the localization of a certain small category $\mathcal{F}(\mathcal{U}(T^M); \mathcal{C}_A)$ that we now define.

Definition 4.8. (i) Define $\mathcal{F}(\mathcal{U}(T^M); \mathcal{C}_A)$ as the category of isotopy functors (see [14, Definition 4.4]) from $\mathcal{U}(T^M)$ to $\mathcal{C}_A$. To simplify the notation, we will often write $\mathcal{K}$ for $\mathcal{F}(\mathcal{U}(T^M); \mathcal{C}_A)$ in this section. That is, $\mathcal{K} := \mathcal{F}(\mathcal{U}(T^M); \mathcal{C}_A)$. Weak equivalences of $\mathcal{K}$ are natural transformations which are objectwise weak equivalences. We denote the class of weak equivalences of $\mathcal{K}$ by $\mathcal{W}_\mathcal{K}$.

(ii) Define $\mathcal{L} := \mathcal{F}_A(\mathcal{U}(T^M); \mathcal{M})$ (see Definition 4.3). We denote the class of weak equivalences of $\mathcal{L}$ by $\mathcal{W}_\mathcal{L}$.

By Definition 4.8 and Propsoition 3.5, one has $\mathcal{K} \subseteq \mathcal{L}$.

Remark 4.9. The subcategory $\mathcal{K} \subseteq \mathcal{M}(\mathcal{U}(T^M))$ might not be a model category on its own right as it might not be closed under factorizations or under taking small limits. The same remark applies to $\mathcal{L}$.

Nevertheless $\mathcal{K}$ and $\mathcal{L}$ are closed under certain things described in Proposition 4.10 below. Before we state it, we need to introduce a functor. Let $n$ be the dimension of $M$. Consider the fibrant replacement functor $\mathcal{R} : \mathcal{C}_A^{\Delta^n} \rightarrow \mathcal{C}_A^{\Delta^n}$ from Section 3.2. According to Remark 4.2, one has an isomorphism $\mathcal{U}(\Delta^n) \cong \Delta^n$. This enables us to regard $\mathcal{R}$ as a functor $\mathcal{R} : \mathcal{C}_A^{\mathcal{U}(\Delta^n)} \rightarrow \mathcal{C}_A^{\mathcal{U}(\Delta^n)}$. From the definition of $\mathcal{R}$, and the fact that the simplicial complex $T^M$ can be built up by gluing together the $\Delta^n$'s, the functor $\mathcal{R}$ can be extended in the obvious way to a functor that we denote

$$\overline{\mathcal{R}} : \mathcal{C}_A^{\mathcal{U}(T^M)} \rightarrow \mathcal{M}^{\mathcal{U}(T^M)}.$$ (4.2)
Proposition 4.10.  
(i) For every $F \in \mathcal{K}$, $\overline{RF}$ belongs to $\mathcal{K}$.

(ii) The category $\mathcal{L}$ is closed under taking cofibrant and fibrant replacements.

(iii) For every $F \in \mathcal{K}$ (respectively $G \in \mathcal{L}$) there exists a cylinder object $F \times I$ in $\mathcal{K}$ (respectively $G \times I$ in $\mathcal{L}$)

So it makes sense to talk about homotopy in $\mathcal{K}$ and $\mathcal{L}$ provided that the source is cofibrant and the target is fibrant.

Proof of Proposition 4.10.  
Parts (i) and (ii) follow from the definitions. Regarding (iii), let $F \in \mathcal{K}$ and let $U \in \mathcal{U}(\mathcal{M})$. Consider (3.2) with $F(U)$ in place of $X$, and define $F \times I: \mathcal{U}(\mathcal{M}) \rightarrow \mathcal{C}_A$ on objects as $(F \times I)(U) := Z_{F(U)}$, and in the obvious way on morphisms. Since weak equivalences and cofibrations are both objectwise, it follows that $F \times I$ is a cylinder object for $F$. Certainly $F \times I$ belongs to $\mathcal{K}$. A similar construction can be performed in $\mathcal{L}$.

Before we state and prove the main result (Proposition 4.14) of this section, we need three preparatory lemmas. We will use the notation and terminology from Section 2.3.

Lemma 4.11. Consider the categories $\mathcal{K}$ and $\mathcal{L}$ from Definition 4.8. Then the functor

$$\phi: \mathcal{K}[\mathcal{L}_{\mathcal{K}}^{-1}] \rightarrow \mathcal{L}[\mathcal{L}_{\mathcal{L}}^{-1}], \quad F \mapsto F,$$

induced by the inclusion functor $\mathcal{K} \hookrightarrow \mathcal{L}$ is essentially surjective.

Proof. Let $n = \dim \mathcal{M}$ as above. Since the simplicial complex $\mathcal{T}^M$ can be built up by gluing together the $\Delta^n$’s, it is enough to prove the lemma when $\mathcal{T}^M = \Delta^n$. Set $\mathcal{K}_n = \mathcal{F}(\mathcal{U}(\Delta^n); \mathcal{C}_A)$ and $\mathcal{L}_n = \mathcal{F}(\mathcal{U}(\Delta^n); \mathcal{M})$. The idea of the proof is to proceed by induction on $n$ by showing that for all $n \geq 0$, for all $F \in \mathcal{L}_n[\mathcal{L}_{\mathcal{L}}^{-1}]$, there exist $\overline{G} \in \mathcal{K}_n[\mathcal{K}_{\mathcal{K}}^{-1}]$ and a zigzag of weak equivalences $F \xrightarrow{\sim} RF \xleftarrow{\sim} QRF \xrightarrow{\sim} \overline{G}$.

For $n = 0$, the standard geometric simplex $\Delta^0$ has only one vertex, say $v$. So $\mathcal{U}(\Delta^0) = \{U_v\}$. Let $F: \mathcal{U}(\Delta^0) \rightarrow \mathcal{M}$ be an object of $\mathcal{L}_0[\mathcal{L}_{\mathcal{L}}^{-1}]$. Define $\overline{G}: \mathcal{U}(\Delta^0) \rightarrow \mathcal{C}_A$ as $\mathcal{G}(U_v) := \mathcal{QRA}$. By the definition of $\mathcal{L}_0$, the object $F(U_v)$ is weakly equivalent to $\mathcal{QRA}$. So $QRF(U_v)$ is also weakly equivalent to $\mathcal{QRA}$, that is, there is a zigzag of weak equivalences

$$QRF(U_v) \xrightarrow{\sim} RF \xleftarrow{\sim} QRF \xrightarrow{\sim} \overline{G}.$$  

Using standard techniques from model categories and the fact that the objects $QRF(U_v)$ and $\mathcal{QRA}$ are both fibrant and cofibrant, one can replace (4.4) by a direct morphism $\overline{g}: QRF(U_v) \xrightarrow{\sim} \mathcal{QRA} = \mathcal{G}(U_v)$. This proves the base case.

Assume that the statement is true for all $k \leq n - 1$, and let $F \in \mathcal{L}_n[\mathcal{L}_{\mathcal{L}}^{-1}]$. We need to find $\overline{G} \in \mathcal{K}_n[\mathcal{K}_{\mathcal{K}}^{-1}]$ and a weak equivalence $\overline{g}: QRF \xrightarrow{\sim} \overline{G}$. First of all let $\Delta^n = \{v_0, \ldots, v_n\}$, and define $\mathcal{U}(\Delta^n) \subseteq \mathcal{U}(\Delta^n)$ as the full subposet whose objects are $U_{v_\sigma}$’s with $\sigma$ be a simplex of the boundary of $\Delta^n$. By the induction hypothesis there exist an isotopy functor $\mathcal{G}: \mathcal{U}(\Delta^n) \rightarrow \mathcal{C}_A$ and a natural transformation $\beta: QRF \xrightarrow{\sim} G$. From the base case, there is a weak equivalence $g: QRF(U_{v_0, \ldots, v_n}) \xrightarrow{\sim} QRA$. Since $\mathcal{QRA}$ and $QRF(U_{v_0, \ldots, v_n})$ are both fibrant and cofibrant, it follows by Proposition 2.17 that $g$ admits a homotopy inverse, say $f$. Let $H: QRF(U_{v_0, \ldots, v_n}) \times I \rightarrow QRF(U_{v_0, \ldots, v_n})$ denote a homotopy from $fg$ to $id$. Now define $G': \mathcal{U}(\Delta^n) \rightarrow \mathcal{C}_A$ as

$$G'[\mathcal{U}(\Delta^n)] := G, G'(U_{v_0, \ldots, v_n}) := \mathcal{QRA}, \quad \text{and} \quad G'(d') := \beta[U_{v_0, \ldots, v_i, \ldots, v_n}] \circ F(d') \circ f,$$

where $d'$ is a face map of $\Delta^n$.

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where \(d'\) is the map from [4.11]. On the compositions we define \(G'\) in the obvious way (that is, \(G'(a \circ b) := G'(b) \circ G'(a)\)). It is straightforward to see that \(G'\) is a contravariant functor. One may take \(G\) to be \(G'\) and \(\bar{\beta}\) to be \(\beta'\), where \(\beta': F \to G'\) is defined as \(\beta\) on \(\partial K_i(\Delta^n)\) and \(g\) on \(U_{(v_0, \ldots, v_n)}\). The issue with that definition is the fact that the square involving \(G'(d')\), \(F(d')\), \(g\) and \(\beta[U_{(v_0, \ldots, \tilde{v}_i, \ldots, v_n)}]\) is only commutative up to homotopy. To fix this, consider the following commutative diagram.

![Diagram](image)

In that diagram the square involving \(g, \mathcal{H}, i_0\) and \(p\) is induced by the commutative square

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{g} & \mathcal{H} \\
\downarrow i_0 & & \downarrow p \\
\mathcal{H} & \xrightarrow{g} & \mathcal{H}
\end{array}
\]

while \(\mathcal{P}r\) is the factorization of \(p\) such that \(Z_{(QRA, G', p)}\) belongs to \(\mathcal{C}_A\), and \(\mathcal{H}\) is given by the lifting axiom. Now define \(\mathcal{G}: U(\Delta^n) \to \mathcal{C}_A\) as

\[
\mathcal{G}\partial K_i(\Delta^n) := G', \quad \mathcal{G}(U_{(v_0, \ldots, v_n)}) := Z_{(QRA, G', p)} \quad \text{and} \quad \mathcal{G}(d') := P_{U_{\sigma_i}} \circ \mathcal{P},
\]

where \(\sigma_i := (v_0, \ldots, \tilde{v}_i, \ldots, v_n)\) and \(P_{U_{\sigma_i}}: \lim_{U_{\sigma_i}} G'(U_{\sigma_i}) \to G'(U_{\sigma_i})\) is the canonical projection. Also define

\[
\bar{\beta}(\partial K_i(\Delta^n)) := \beta \quad \text{and} \quad \bar{\beta}[U_{(v_0, \ldots, v_n)}] := \mathcal{H} \circ i_1,
\]

where \(i_1: QRF(U_{(v_0, \ldots, v_n)}) \to QRF(U_{(v_0, \ldots, v_n)}) \times I\) is the canonical inclusion. It is straightforward to see that \(\bar{\beta}\) is a weak equivalence. This proves the lemma. 

**Lemma 4.12.** Let \(\mathcal{K}\) and \(\mathcal{L}\) be as in Lemma 4.11. Then the functor \(\phi\) from (4.3) is full.

**Proof.** Let \(F_1, F_2 \in \mathcal{K}[W_{\mathcal{K}}^{-1}]\). We need to show that the canonical map

\[
\phi_{F_1, F_2}: \text{Hom}_{\mathcal{K}}(F_1, F_2) \to \text{Hom}_{\mathcal{L}}(\phi(F_1), \phi(F_2))
\]

induced by \(\phi\) is surjective. To do this, let \(f_1: QF_1 \to F_1\) be a cofibrant replacement of \(F_1\). One can take \(QF_1 = F_1\) and \(f_1 = id\) by Propositions 2.10, 3.5. Also let \(f_2: F_2 \to \mathcal{K}F_2\) be a fibrant replacement of \(F_2\), which lies in \(\mathcal{K}\) thanks to Proposition 4.10. Consider the following commutative square.

\[
\begin{array}{ccc}
\text{Hom}(F_1, F_2) & \xrightarrow{\phi_{F_1, F_2}} & \text{Hom}(\phi(F_1), \phi(F_2)) \\
\downarrow \theta & & \downarrow \theta' \\
\text{Hom}(F_1, \mathcal{K}F_2) & \xrightarrow{\phi_{F_1, \mathcal{K}F_2}} & \text{Hom}(\phi(F_1), \mathcal{K}F_2) \sim \text{Hom}(F_1, \mathcal{K}F_2)
\end{array}
\]
In that square \( \theta \) is defined as the string \( \theta(f) := (f, f_2^{-1}) \), \( \pi \) is the canonical surjection, and the isomorphism \( \theta' \) comes from Proposition 2.17. The equality comes from the fact that \( \mathcal{K} \) is a full subcategory of \( \mathcal{L} \) by definition. Since the composition \( \theta' \circ \pi \) is surjective, and since the square commutes, it follows that \( \phi_{F_1,F_2} \) is a surjective map. This ends the proof.

Lemma 4.13. Let \( \mathcal{K} \) and \( \mathcal{L} \) be as in Lemma 4.11. Then the functor \( \phi \) from (4.3) is faithful.

Proof. Let \( F_1, F_2 \in \mathcal{K}[\mathcal{W}_{\mathcal{K}}^{-1}] \). As we mentioned in the proof of Lemma 4.12 \( QF_1 = F_1 \) and \( \mathcal{R}F_2 \) belongs to \( \mathcal{K} \). We need to show that the map \( \phi_{F_1,F_2} \) from (4.5) is injective. Consider the diagram (4.5), and let \( \eta_0, \eta_1 \in \text{Hom} (F_1, F_2) \) such that \( \phi_{F_1,F_2}(\eta_0) = \phi_{F_1,F_2}(\eta_1) \). This latter equality implies that \( \eta_0 \) is homotopic to \( \eta_1 \), where \( \eta_0 := \theta^{-1}(\phi_{F_1,F_2}(\eta_1)) \). By Propositions 2.16 4.10, there is a left homotopy \( H : F_1 \times I \rightarrow \mathcal{R}F_2 \) for some cylinder object \( F_1 \times I \in \mathcal{L} \) for \( F_1 \). The key point of the proof is the fact that one can always choose \( F_1 \times I \) in \( \mathcal{K} \) thanks to Proposition 4.10. Now consider the following commutative diagram in \( \mathcal{K} \).

\[
\begin{array}{ccc}
F_1 & \xrightarrow{i_0} & F_1 \\
\downarrow \quad \eta_0 & & \downarrow \quad \eta_1 \\
F_1 \times I & \xrightarrow{H} & \mathcal{R}F_2 \\
\downarrow \quad i_1 & & \downarrow \quad \quad \\
F_1 & & 
\end{array}
\]

Since \( i_0 = i_1 \) in \( \mathcal{K}[\mathcal{W}_{\mathcal{K}}^{-1}] \) by Proposition 2.15, it follows that \( \eta_0 = \eta_1 \) in \( \mathcal{K}[\mathcal{W}_{\mathcal{K}}^{-1}] \), which implies \( \eta_0 = \eta_1 \). This proves the lemma.

Proposition 4.14. Let \( \mathcal{K} \) and \( \mathcal{L} \) be as in Definition 4.8. Then the inclusion \( \mathcal{K} \hookrightarrow \mathcal{L} \) induces an equivalence of categories between the localizations \( \mathcal{K}[\mathcal{W}_{\mathcal{K}}^{-1}] \) and \( \mathcal{L}[\mathcal{W}_{\mathcal{L}}^{-1}] \).

Proof. This follows immediately from Lemmas 4.11 4.12 and 4.13 as it is well known that a fully faithful and essentially surjective functor is an equivalence of categories.

5 The map \( \Lambda : \mathcal{F}(\mathcal{U}(T^M); \mathcal{C}_A) \rightarrow [T^M_\bullet, \hat{A}_\bullet] \)

Recall the category \( \mathcal{F}(\mathcal{U}(T^M); \mathcal{C}_A) \) (which was called \( \mathcal{C} \) for convenience in Section 4.3) from Definition 4.8. It turns out that it is a small category since \( \mathcal{U}(T^M) \) and \( \mathcal{C}_A \) are both small. In this section and the next one, we will view \( \mathcal{F}(\mathcal{U}(T^M); \mathcal{C}_A) \) as a set. Choose a well-order on the set of vertices of \( T^M \), and let \( T^M_\bullet \) denote the canonical associated simplicial set. Recall the simplicial set \( \hat{A}_\bullet \) from Section 3.3, and let \( \text{Hom}(T^M_\bullet, \hat{A}_\bullet) \) denote the set of simplicial maps \( T^M_\bullet \rightarrow \hat{A}_\bullet \). The goal of this section is to define a map \( \Lambda : \mathcal{F}(\mathcal{U}(T^M); \mathcal{C}_A) \rightarrow \text{Hom}(T^M_\bullet, \hat{A}_\bullet) \) and prove Proposition 5.2 below, which says that \( \Lambda(F) \) is homotopic to \( \Lambda(F') \) whenever \( F \) is weakly equivalent to \( F' \). The notion of homotopy between two simplicial maps we use is the one from [4] Section 1.6.

We begin with the construction of \( \Lambda \). Define \( \Lambda(F) := f : T^M_\bullet \rightarrow \hat{A}_\bullet, \sigma \mapsto f_\sigma \) as follows. First consider the fibrant replacement functor \( \mathcal{R} \) from (4.2). Let \( n \geq 0 \), and let \( \sigma = \langle v_0, \ldots, v_n \rangle \in T^M_n \). Depending on the fact that \( \sigma \) is degenerate or not we need to deal with two cases.

- If \( \sigma \) is non-degenerate, define \( f_\sigma : \hat{A}_n^{\sigma} \rightarrow \mathcal{C}_A \) on objects as \( f_\sigma(\langle a_0, \ldots, a_s \rangle) := \mathcal{R}(F)(U(v_{a_0}, \ldots, v_{a_s})) \).
- On morphisms, it is enough to define \( f_\sigma \) on the generators \( d^i \)'s from Remark 2.2. For a morphism \( \theta : U(v_{a_0}, \ldots, \hat{a}_i, \ldots, v_{a_s}) \leftarrow U(v_{a_0}, \ldots, v_{a_s}) \) of \( \mathcal{U}(T^M) \), define \( f_\sigma(d^i) := \mathcal{R}(F)(\theta^i) \).
If \( \sigma \) is degenerate, a classical result on simplicial sets claims the existence of a unique non-degenerate simplex \( \lambda \) of some degree \( p \leq n \) and a unique non-decreasing surjection \( s: [n] \rightarrow [p] \) such that \( \sigma = T^M_n(s)(\lambda) \). Define \( f_\sigma := \tilde{A}_n(s)(f_\lambda) \). Here \( T^M_n \) and \( \tilde{A}_n \) are indeed viewed as contravariant functors from the standard simplicial category \( \Delta \) to sets.

Certainly \( f_\sigma \) belongs to \( \tilde{A}_n \) and \( f: T^M_n \rightarrow \tilde{A}_n \) is a simplicial map. This completes the definition of \( \Lambda \).

Now on the set \( \mathcal{F}(\mathcal{U}(T^M_n); \mathcal{C}_A) \) define the equivalence relation “we” as \( F \ we \ F' \) if and only if \( F \) is weakly equivalent to \( F' \), that is, there is a zigzag of weak equivalences between \( F \) and \( F' \). Let \( [T^M_n, \tilde{A}_n] \) be the set of homotopy classes of simplicial maps from \( T^M_n \) to \( \tilde{A}_n \).

**Definition 5.1.** Consider the map \( \Lambda: \mathcal{F}(\mathcal{U}(T^M_n); \mathcal{C}_A) \rightarrow \text{Hom}(T^M_n, \tilde{A}_n) \) and the equivalence relation “we” we just defined. Thanks to Proposition 5.2 below \( \Lambda \) passes to the quotient. Define \( \Lambda: \mathcal{F}(\mathcal{U}(T^M_n); \mathcal{C}_A) \) to be the resulting quotient map.

**Proposition 5.2.** Let \( F, F' \in \mathcal{F}(\mathcal{U}(T^M_n); \mathcal{C}_A) \). Assume that there is a zigzag of weak equivalences between \( F \) and \( F' \). Then \( \Lambda(F) \) is homotopic to \( \Lambda(F') \).

**Definition 5.3.** Let \( f, f': T^M_n \rightarrow \tilde{A}_n \) be simplicial maps.

(i) A morphism \( \beta: f \rightarrow f' \) consists of a collection \( \beta = \{ \beta_\sigma: f_\sigma \rightarrow f'_\sigma \}_{n \geq 0, \sigma \in T^M_n} \) of natural transformations such that

\[
d_i \beta_\sigma = \beta_{d_i \sigma} \quad \text{and} \quad s_k \beta_\sigma = \beta_{s_k \sigma} \quad \text{for all} \quad \sigma, i, k.
\]

(see Remark 3.15)

(ii) We say that \( \beta \) is a weak equivalence if for every \( n \geq 0 \), for every \( \sigma \in T^M_n \), \( \beta_\sigma \) is a weak equivalence of \( \Delta^n \)-diagrams.

**Lemma 5.4.** Let \( f, f': T^M_n \rightarrow \tilde{A}_n \) be simplicial maps. Assume that there is a morphism \( \beta: f \rightarrow f' \) which is a weak equivalence (see Definition 5.3). Then \( f \) is homotopic to \( f' \).

Idea of proof. For \( n \geq 0 \), we let \( \Delta[n] \) be the standard simplicial model for \( \Delta^n \). To prove Lemma 5.4 we need to construct a homotopy \( H: T^M_n \times \Delta[1] \rightarrow \tilde{A}_n \) between \( f \) and \( f' \). So, for all \( n \geq 0, \sigma \in T^M_n, t \in \Delta[1]_n \), we need to define \( H(\sigma, t) \in \tilde{A}_n \). To get a better idea of the construction of \( H(\sigma, t) \), let us treat a specific example. Let \( n = 2, \sigma = v_{012} := (v_0, v_1, v_2) \) and \( t = 011 \). We need the following equation, which comes from the fact that \( f \) is a simplicial map and the definition of \( d_i: \tilde{A}_n \rightarrow \tilde{A}_{n-1} \).

\[
f_{d_i \sigma}([n-1]) = f_\sigma([n]_i), \quad \sigma \in T^M_n.
\]

Recalling the functor \( \mathcal{R} \) from Section 3.2 define \( H(v_{012}, 011) := \mathcal{R}(\overline{H}) \), where \( \overline{H} \) is the diagram from 5.3 defined as follows.
Depending on the value of $H(v_0, 0) := f_{v_0}([0])$ and $H(v_1, 1) := f_{v_1}([1])$. The diagram $H(v_{01}, 01) := f_{v_{01}}([1])$ is defined as the composition

$$f_{v_{01}}([2]) \xrightarrow{f_{v_{01}}(d^2)} f_{v_{01}}([1]) = f_{v_0}(1),$$

where the equality comes from (5.1). The map $d_0 : H(v_{012}, 011) \rightarrow H(v_{12}, 11)$ is defined as the composition

$$f_{v_{012}}([2]) \xrightarrow{f_{v_{012}}(d^0)} f_{v_{012}}([1, 2]) = f_{v_{12}}([1]) \xrightarrow{\beta_{v_{12}} [1]} f'_{v_{12}}([1]).$$

The other morphisms are defined in a similar fashion. The diagram $H$ thus obtained commutes thanks to (5.1) and the fact that for every $\sigma, \beta_\sigma$ is a natural transformation. Note that there is a weak equivalence

$$f_{v_{012}}([2]) \rightarrow \text{fibrant replacement of the boundary of } (5.3),$$

where $f_{v_{012}}([2])$ is viewed as the constant diagram at $f_{v_{012}}([2])$.

**Proof of Lemma 5.4.** First of all recall the posets $\partial \widetilde{\Delta}^n$ and $\partial^i \Delta^n$ from Definition 2.6. Our goal is to define a simplicial map $H : T^M_0 \times \Delta[1] \rightarrow \tilde{A}\bullet$ making (??) commute. We will proceed by induction on the skeletons $\text{sk}_n(T^M_0 \times \Delta[1])$. More precisely, we will show that for all $n \geq 0$, there exist a simplicial map $H_n : \text{sk}_n(T^M_0 \times \Delta[1]) \rightarrow \tilde{A}\bullet$ and a weak equivalence

$$\eta_{n+1} : f_\sigma([n+1]) \xrightarrow{\sim} H^\sigma_{t(n+1)} \quad \forall \sigma \in T^M_{n+1}, \forall t \in \Delta[1]_{n+1},$$

where $H^\sigma_{t(n+1)} : \partial \widetilde{\Delta}^{n+1} \rightarrow C\sigma$, defined as

$$H^\sigma_{t(n+1)}|_{\partial \widetilde{\Delta}^{n+1}} := H_n(d_i \sigma, d_i t),$$

is a fibrant diagram and $f_\sigma([n+1])$ is viewed as the constant functor at $f_\sigma([n+1])$.

- On the 0-skeleton, define $H_0 : \text{sk}_0(T^M_0 \times \Delta[1]) \rightarrow \tilde{A}\bullet$ as $H_0(v, 0) := f_v$ and $H_0(v, 1) := f'_v$.

Let $\sigma = (v_0, v_1) \in T^M_0$ and let $t \in \Delta[1]_1$. Consider the poset

$$\widetilde{\Delta}^1 = \left\{ 0 \xrightarrow{d^1} 0, 1 \xleftarrow{d^0} 1, \right\},$$

and the functor $H^\sigma_{t_1} : \partial \widetilde{\Delta}^1 \rightarrow C\sigma$ from (5.5). We need to define a natural transformation $\eta_1 : f_\sigma([1]) \xrightarrow{\sim} H^\sigma_{t_1}$. Depending on the value of $t \in \{00, 11, 01\}$, we need to deal with three cases.

- If $t = 00$ then $H^\sigma_{t_1} : \partial \widetilde{\Delta}^1 \rightarrow C\sigma$ is given by $H^\sigma_{t_1}([0]) = f_{v_0}([0])$ and $H^\sigma_{t_1}([1]) = f_{v_1}([0])$. Note that by (5.2) one has $f_{v_0}([0]) = f_\sigma([0])$ and $f_{v_1}([0]) = f_\sigma([1])$. Now define

  $$\eta_1([0]) : f_\sigma([0, 1]) \rightarrow f_{v_0}([0]) \quad \text{and} \quad \eta_1([1]) : f_\sigma([0, 1]) \rightarrow f_{v_1}([0])$$

  as $\eta_1([0]) := f_\sigma(d^1)$ and $\eta_1([1]) := f_\sigma(d^0)$.

- If $t = 11$ then $H^\sigma_{t_1}$ is given by $H^\sigma_{t_1}([0]) = f'_{v_0}([0])$ and $H^\sigma_{t_1}([1]) = f'_{v_1}([0])$. By hypothesis there is a weak equivalence $\beta_\sigma : f_\sigma \xrightarrow{\sim} f'_\sigma$. Define $\eta_1([0])$ and $\eta_1([1])$ as

  $$\eta_1([0]) := \beta_\sigma([0]) \circ f_\sigma(d^1) \quad \text{and} \quad \eta_1([1]) := \beta_\sigma([1]) \circ f_\sigma(d^0).$$

- If $t = 01$ then $H^\sigma_{t_1}([0]) = f_{v_0}([0])$ and $H^\sigma_{t_1}([1]) = f'_{v_1}([0])$. Define

  $$\eta_1([0]) := f_\sigma(d^1) \quad \text{and} \quad \eta_1([1]) := \beta_\sigma([1]) \circ f_\sigma(d^0).$$
Clearly, in each of the above cases, the natural transformation \( \eta \) is a weak equivalence and the diagram \( H^\sigma_{1t} \) is fibrant.

- Assume that the statement is true for all \( k \leq n - 1 \). To prove it for \( k = n \), there are two things to do. The first one is to define \( H_n : \text{sk}_n(T^M_M \times \Delta[1]) \to \tilde{A}_n \), and the second is to get (5.4). To define \( H_n \), let \( \sigma \in T^M_M \) and let \( t \in \Delta[1]_n \). Define \( H_n(\sigma, t) : \tilde{\Delta}^n \to C_A \) as

\[
H_n(\sigma, t) := \begin{cases} 
  f_{\sigma} & \text{if } t = 0 \cdots 0 =: t^n_0 \\
  f'_{\sigma} & \text{if } t = 1 \cdots 1 =: t^{n+1}_1 \\
  \psi_n & \text{otherwise},
\end{cases}
\]

where \( \psi_n : \tilde{\Delta}^n \to C_A \) is defined as follows. By the induction hypothesis, there is a weak equivalence \( \eta_n : f_\sigma([n]) \xrightarrow{\sim} H^\sigma_{nt} \) with \( H^\sigma_{nt} \) fibrant. Consider the following factorization of the limit of \( \eta_n \) such that \( Z \in C_A \).

\[
f_\sigma([n]) \xrightarrow{\lim(\eta_n)} \lim_{\alpha \in \tilde{\Delta}^n} H^\sigma_{nt} \xrightarrow{\theta_\sigma} Z 
\]

Now define \( \psi_n |_{\partial \tilde{\Delta}^n} := H_{n-1}(d_1 \sigma, d_1 t) \), \( \psi_n([n]) := Z \) and \( \psi_n(d^n) := p_i \circ p \),

where \( p_i : \lim_{\alpha \in \tilde{\Delta}^n} H^\sigma_{nt} \to H^\sigma_{nt}([n]_i) \) is the canonical projection as usual. It is straightforward to check that \( \psi_n \) belongs to \( \tilde{A}_n \).

To get (5.4), let \( \alpha \in T^M_{n+1} \) and let \( t \in \Delta[1]_{n+1} \). We need to deal with three cases.

- If \( t = t^n_0+2 \), define \( \eta_{n+1}([n + 1]_i) := f_\alpha(d^n) \), where \( d^n : [n + 1]_i \to [n + 1] \) is the inclusion map.
- If \( t = t^{n+1}_1 \) we define \( \eta_{n+1}([n + 1]_i) := \beta_\alpha([n + 1]_i) \circ f_\alpha(d^n) \).
- If \( t \not\in \{t^n_0+2, t^{n+1}_1\} \), one has \( \eta_{n+1}([n + 1]_i) := \theta_{d_{1,\alpha}} \circ f_\alpha(d^n) \), where \( \theta_{d_{1,\alpha}} \) is the map from (5.6).

On the other objects of \( \partial \tilde{\Delta}^{n+1} \), define \( \eta_{n+1} \) as the obvious compositions. Clearly \( H^\sigma_{nt} \) is fibrant. It is straightforward to check that \( \eta_{n+1} \) is a natural transformation which is a weak equivalence. It is also straightforward to check that the map \( H \) thus defined is a homotopy from \( f \) to \( f' \). This proves the lemma.

Now we can prove the main result of this section.

**Proof of Proposition 5.5.** Let \( \eta : F \xrightarrow{\sim} F' \) be a weak equivalence of \( \mathcal{F}(\mathcal{U}(T^M_M), C_A) \). Set \( \Lambda(F) = f \) and \( \Lambda(F') = f' \). For \( n \geq 0 \) and \( \sigma = (v_0, \ldots, v_n) \in T^M_M \), if \( \sigma \) is non-degenerate, define \( \beta_\sigma : f_\sigma \to f'_\sigma \) as \( \beta_\sigma([a_0, \ldots, a_n]) := (\mathcal{R})_\sigma[\lambda(v_{a_0}, \ldots, v_{a_n})] \). If \( \sigma \) is degenerate, there exist a unique non-degenerate simplex \( \lambda \) and a sequence of degeneracy maps \( s_i \), \( i = 1, \ldots, s_k \) such that \( \sigma = s_1 \cdots s_k(\lambda) \). By Remark 3.15, we have a map \( \varphi : s_1 \cdots s_k(f_\lambda) \to s_1 \cdots s_k(f'_\lambda) \) induced by \( \lambda : f_\lambda \to f'_\lambda \). Define \( \beta_\sigma := \varphi \). By definition \( \beta_\sigma \) is a weak equivalence for all \( \sigma \). It is straightforward to check that the collection \( \{\beta_\sigma\}_\sigma \) satisfies (5.1).

Applying Lemma 5.4, we have that \( f \) is homotopic to \( f' \). Now assume that there is a zigzag \( F \xrightarrow{\sim} \cdots \xrightarrow{\sim} F' \). Applying the first part to each map of that zigzag, and taking the inverse homotopy associated to each backward arrow, we have a homotopy between \( \Lambda(F) \) and \( \Lambda(F') \). This proves the proposition.
6 The map $\overline{\Theta}: [\mathcal{T}_*^M, \hat{A}_*] \to \mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A)/we$

In Section 5 or more precisely in Definition 5.1 we defined a map $\overline{\Theta}: \mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A)/we \to [\mathcal{T}_*^M, \hat{A}_*]$. The goal of this section is to construct its inverse, and thus get (1.4). We continue to use the well-order on the set of vertices of $\mathcal{T}^M$ we chose in the previous section.

Definition 6.1. Define a map $\Theta: \text{Hom}(\mathcal{T}_*^M, \hat{A}_*) \to \mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A)$ as $\Theta(f) := F$, where $F: \mathcal{U}(\mathcal{T}^M) \to \mathcal{C}_A$ is defined on objects as $F(U_\sigma) := \hat{f}_\sigma([n])$ for $\sigma = (v_{\sigma_0}, \ldots, v_{\sigma_n})$. On morphisms, it is enough to define $F$ only on $d^i$’s from Remark 4.3. Define $F(d^i) := \hat{f}_{\sigma}(n_i) \to \hat{f}_{\sigma}([n])$.

Definition 6.2. Thanks to Proposition 6.3 below, the map $\Theta$ passes to the quotient. Define $\overline{\Theta}: [\mathcal{T}_*^M, \hat{A}_*] \to \mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A)/we$ to be the resulting quotient map.

Proposition 6.3. Let $f, f': \mathcal{T}_*^M \to \hat{A}_*$ be two simplicial maps such that $f$ is homotopic to $f'$. Then there exists a zigzag of weak equivalences $\Theta(f) \sim \cdots \sim \Theta(f')$ in $\mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A)$.

The proof of Proposition 6.3 occupies the section and will be given at the end. Our strategy goes through two big steps. In the first one we prove the result when the poset $\mathcal{U}(\mathcal{T}^M)$ is finite. In the case where $\mathcal{U}(\mathcal{T}^M)$ is infinite, the idea of the proof is to write $F = \Theta(f)$ as the homotopy limit of a certain diagram $E_1F_1 \leftarrow E_2F_2 \leftarrow \cdots$, where $F_i$ is the restriction of $F$ to a finite subposet $\mathcal{U}(\mathcal{T}^M) \subseteq \mathcal{U}(\mathcal{T}^M)$ and $E_iF_i$ is the right Kan extension of $F_i$ along the inclusion $\mathcal{U}(\mathcal{T}^M) \to \mathcal{U}(\mathcal{T}^M)$. Using the fact that $F_i$ is weakly equivalent to $F'_i$ by the first step, one can deduce the proposition.

This section is organized as follows. In Section 6.1 we prove Lemma 6.5 below, which says that if $f, f': \mathcal{T}_*^M \to \hat{A}_*$ are homotopic, then $\Theta(f)$ is weakly equivalent to $\Theta(f')$ provided that $\mathcal{T}^M$ has finite number of simplices. Since the proof of that lemma is technical, we begin with a special case: $\mathcal{T}^M = \Delta^1$. Section 6.2 deals with the case where $\mathcal{T}^M$ has infinite number of simplices.

6.1 Case where $\mathcal{T}^M$ has finite number of simplices

Lemma 6.4. Let $\mathcal{T}^M = \Delta^1$, and let $f, f': \mathcal{T}_*^M \to \hat{A}_*$ be simplicial maps that are homotopic. Then there exists a zigzag of weak equivalences $\Theta(f) \sim \cdots \sim \Theta(f')$ in $\mathcal{F}(\mathcal{U}(\mathcal{T}^M); \mathcal{C}_A)$.

Proof. Let $H$ be a homotopy between $f$ and $f'$. For the sake of simplicity, we will often use the notation $v_{0 \ldots n} := \langle v_0, \ldots, v_n \rangle$ in this proof and the next one. Consider the poset $\mathcal{U}(\Delta^1) = \left\{ U_{v_0} \xrightarrow{d^i} U_{v_{01}} \xleftarrow{d^i} U_{v_1} \right\}$. Also consider Figure 1 which is a subdivision of $\Delta^1 \times [0, 1]$ into two 2-simplices, namely $\langle (v_0, 0), (v_0, 1), (v_1, 1) \rangle$ and $\langle (v_0, 0), (v_1, 0), (v_1, 1) \rangle$.

We now explain the algorithm that produces functors out of $H$ and Figure 1. Define the baricenter of $\sigma_0 := ((v_0, 0), (v_1, 0), (v_1, 1))$ as the pair $(v_{01}, 001)$, and that of $\sigma_1 := ((v_0, 0), (v_0, 1), (v_1, 1))$ as the pair $(v_{001}, 011)$. In general, the baricenter of $\langle (v_0, j_0), (v_1, j_1), (v_2, j_2) \rangle$ is defined to be $\langle v_{011}, j_0, j_1, j_2 \rangle$. Applying the homotopy $H$ to those baricenters, we get the commutative diagram (6.1) below in which we make the following simplifications at the level of notation. We write $H(v_{011}, 001)$ for $H(v_{011}, 001)\{0, 1, 2\}$, $H(v_{01}, 01)$ for $H(v_{01}, 01)\{0, 1\}$, and so on. Also we write $d_i$ for $H(\cdot, -)(d^i)$. 

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Now define the bottom of (6.1) as a functor $F_0^L : U(\Delta^1) \rightarrow C_A$, where the letter “$L$” stands for “lower”. Specifically, we have

$$F_0^L(U_{v_0}) := H(v_0, 0), \quad F_0^L(U_{v_{01}}) := H(v_{01}, 00), \quad F_0^L(U_{v_1}) := H(v_1, 0), \quad \text{and} \quad F_0^L(d^i) := d_i.$$  

Certainly that functor is the same as $F$. The functor associated to the barycenter of the simplex $\sigma_0$ is defined as (“$B$” stands for “barycenter”)

$$F_0^B := \left\{ \begin{array}{c} H(v_0, 0) \xleftarrow{d_1 d_2} H(v_{011}, 001) \xrightarrow{d_0} H(v_{11}, 01) \end{array} \right\},$$

while the functor corresponding to its upper face (which is the same as the functor associated to the lower face of $\sigma_1$) is defined as

$$F_0^U = F_1^L := \left\{ \begin{array}{c} H(v_0, 0) \xleftarrow{d_1} H(v_{01}, 01) \xrightarrow{d_0} H(v_1, 1) \end{array} \right\}.$$

Here “$U$” stands for “upper” of course. Lastly, the functor corresponding to the barycenter of the 2-simplex $\sigma_1$ is defined as

$$F_1^B := \left\{ \begin{array}{c} H(v_{00}, 01) \xleftarrow{d_2} H(v_{001}, 011) \xrightarrow{d_0 d_0} H(v_1, 1) \end{array} \right\}.$$
and that associated to its upper face, denoted \( F^U \), is defined as the top of \([6.1]\). Clearly one has the following zigzag of weak equivalences, which are all natural since \([6.1]\) is commutative.

\[
F = F_0^L \xrightarrow{(id,d_2,d_3)} \sim F_0^B \xrightarrow{(id,d_1,d_0)} \sim F_0^U = F_1^L \xrightarrow{(d_1,d_1,id)} \sim F_1^B \xrightarrow{(d_0,d_0,id)} \sim F_1^U = F'.
\]

This ends the proof.

**Lemma 6.5.** Let \( f, f' : T^M_\bullet \rightarrow \hat{A}_\bullet \) be simplicial maps that are homotopic. Assume that \( T^M_\bullet \) has finite number of simplices. Then there exists a zigzag of weak equivalences \( \Theta(f) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \Theta(f') \) in \( \mathcal{F}(\mathcal{U}(T^M_\bullet); \mathcal{C}_A) \).

**Proof.** Since the simplicial complex \( T^M_\bullet \) has finite number of simplices, one can assume without loss of generality that it is a subcomplex of \( \Delta^n \) for some \( n \). Set \( F = \Theta(f) \) and \( F' = \Theta(f') \), and let \( H : T^M_\bullet \times \Delta[1] \rightarrow \hat{A}_\bullet \) be a homotopy from \( f \) to \( f' \). We need to construct a zigzag of weak equivalences between the functors \( F, F' : \mathcal{U}(T^M_\bullet) \rightarrow \mathcal{C}_A \). Following the special case above, the idea is to first subdivide \( \Delta^n \times [0, 1] \) into \( (n + 1) \) \((n + 1)\)-simplices in a suitable way. Each \((n + 1)\)-simplex will produce three functors (one for the upper face, one for the lower face, and one for the barycenter), and two natural transformations like \( \bullet \xrightarrow{\sim} \bullet \xrightarrow{\sim} \bullet \).

Let us consider the prism \( \Delta^n \times I, I = [0, 1] \), and set

\[
\Delta^n \times \{ t \} = \langle (v_0, t), \ldots, (v_n, t) \rangle, \quad t \in \{0, 1\}.
\]

We will sometimes write \( v_j \) for \( (v_j, 0) \). For \( 0 \leq i \leq n \), define an \((n + 1)\)-simplex

\[
\sigma_i := \langle (v_0, 0), \ldots, (v_{n-i}, 0), (v_{n-i}, 1), \ldots, (v_n, 1) \rangle.
\]

Certainly \( \Delta^n \times [0, 1] \) is the union of simplices \( \sigma_i, 0 \leq i \leq n \), each intersecting the next in an \( n \)-simplex face. As we said earlier, each \( \sigma_i \) produces three functors \( F_i^L, F_i^B, F_i^U : \mathcal{U}(T^M_\bullet) \rightarrow \mathcal{C}_A \) that we now define. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{U}(T^M_\bullet) & \xrightarrow{\Psi_{T^M_\bullet}} & \mathcal{P}(T^M_\bullet) \\
\phi^U_i \downarrow & & \downarrow \phi^B_i \\
\mathcal{P}(\sigma_i) \cap \mathcal{P}(T^M_\bullet \times I) & \xrightarrow{\iota} & \mathcal{P}(T^M_\bullet \times I) \\
\phi^L_i \downarrow & & \downarrow \phi \\
\mathcal{C}_A \end{array}
\]

defined as follows.

- \( P(-) \) is the construction from Definition \([2.7]\).
- The functor \( \Psi_{T^M_\bullet} \) is defined as \( \Psi_{T^M_\bullet}(U_\lambda) := \lambda \). Clearly this is an isomorphism of categories. The functor \( \Phi \) is defined as \( \Phi := \Psi_{T^M_\bullet \times I}^{-1} \).
- \( \iota \) is just the inclusion functor.
- Defining \( \overline{\mathcal{H}} \). Recall the map \( \Theta \) from Definition \([6.1]\) and let us denote it by \( \Theta_{T^M_\bullet} \). Then one has the map

\[
\Theta_{T^M_\bullet \times I} : \text{Hom}_{\mathbf{Set}}((T^M_\bullet \times I)_*, \hat{A}_*) \rightarrow \mathcal{F}(\mathcal{U}(T^M_\bullet \times I); \mathcal{C}_A).
\]

The functor \( \overline{\mathcal{H}} \) is defined as \( \overline{\mathcal{H}} := \Theta_{T^M_\bullet \times I}(H) \).

- We now define the functor \( \phi^B_i : \mathcal{P}(T^M_\bullet) \rightarrow \mathcal{P}(\sigma_i) \times \mathcal{P}(T^M_\bullet \times I) \). Let \( \lambda = \langle v_{a_0}, \ldots, v_{a_s} \rangle \) be an object of \( \mathcal{P}(T^M_\bullet) \). We need to deal with four cases.
- If \( n - i = a_j \) for some \( j \), define
\[
\phi_i^B(\lambda) := \langle (v_{a_0}, 0), \ldots, (v_{a_j}, 0), (v_{a_j+1}, 1), \ldots, (v_{a_z}, 1) \rangle.
\]
- If \( a_j < n - i < a_{j+1} \) for some \( j \), define
\[
\phi_i^B(\lambda) := \langle (v_{a_0}, 0), \ldots, (v_{a_j}, 0), (v_{a_{j+1}}, 1), \ldots, (v_{a_z}, 1) \rangle.
\]
- If \( n - i < a_j \) for all \( j \), define
\[
\phi_i^B(\lambda) := \langle (v_{a_0}, 1), \ldots, (v_{a_z}, 1) \rangle.
\]
- If \( n - i > a_j \) for all \( j \), define
\[
\phi_i^B(\lambda) := \langle (v_{a_0}, 0), \ldots, (v_{a_z}, 0) \rangle.
\]
- Define \( \phi_i^L \) as
\[
\phi_i^L(\lambda) := \langle (v_{a_0}, 0), \ldots, (v_{a_j}, 0), (v_{a_j+1}, 1), \ldots, (v_{a_z}, 1) \rangle
\]
if \( n - i = a_j \) for some \( j \), and \( \phi_i^L(\lambda) := \phi_i^B(\lambda) \) otherwise.
- Define \( \phi_i^U \) as
\[
\phi_i^U(\lambda) := \langle (v_{a_0}, 0), \ldots, (v_{a_j}, 0), (v_{a_{j+1}}, 1), \ldots, (v_{a_z}, 1) \rangle
\]
if \( n - i = a_j \) for some \( j \), and \( \phi_i^U(\lambda) := \phi_i^B(\lambda) \) otherwise.

For \( X \in \{B, L, U\} \), define \( F_i^X \) as the composite
\[
F_i^X := \mathcal{H} \circ i \circ \phi_i^X \circ \psi.
\]

From the definitions, there are two natural transformations \( \alpha_i : \phi_i^L \rightarrow \phi_i^B \) and \( \beta_i : \phi_i^U \rightarrow \phi_i^B \) which are nothing but the inclusions (that is, for every \( \lambda \in P(\mathcal{T}^M_i) \), the components of \( \alpha_i \) and \( \beta_i \) at \( \lambda \) are the obvious inclusions). These give rise to a zigzag \( F_i^L \leftarrow F_i^B \rightarrow F_i^U \) (remember \( \mathcal{H} \) is contravariant). Each map from that zigzag is a weak equivalence since the functor \( \mathcal{H} \) sends every morphism to a weak equivalence by the definitions. Clearly, one has \( F_0^L = F^L \), \( F_0^U = F^U \), and \( F_i^L = F_{i+1}^L \) for all \( i \). This ends the proof. \( \square \)

### 6.2 Case where \( \mathcal{T}^M \) has infinite number of simplices

In this section we assume that \( \mathcal{T}^M \) has infinitely many simplices. Since \( M \) is second-countable by assumption (see the introduction), it follows that \( \mathcal{T}^M \) has countably many simplices. This enables us to choose a family

\[
\mathcal{T}^M_i \subseteq \cdots \subseteq \mathcal{T}^M_i \subseteq \mathcal{T}^M_{i+1} \subseteq \cdots \subseteq \mathcal{T}^M
\]

of subcomplexes of \( \mathcal{T}^M \) satisfying the following two conditions:

(a) Each \( \mathcal{T}^M_i \) has finitely many simplices;

(b) \( P(\mathcal{T}^M_{i+1}) = P(\mathcal{T}^M_i) \cup \{z\} \), where \( z \notin P(\mathcal{T}^M_i) \) and for every simplex \( x \) of \( \mathcal{T}^M_i \), \( z \) is not a face of \( x \).

Let \( \{\mathcal{U}(\mathcal{T}^M_i)\}_{i \geq 1} \) be the corresponding family of subposets of \( \mathcal{U}(\mathcal{T}^M) \). (Certainly, one has \( \bigcup_i \mathcal{U}(\mathcal{T}^M_i) = \mathcal{U}(\mathcal{T}^M) \).) Let \( R_i : \mathcal{U}(\mathcal{T}^{M_i}) \rightarrow \mathcal{U}(\mathcal{T}^M) \) and \( R_{ij} : \mathcal{U}(\mathcal{T}^{M_j}) \rightarrow \mathcal{U}(\mathcal{T}^{M_i}) \) denote the inclusion functors. Consider the pair

\[
R^*_i : \mathcal{M}(\mathcal{T}^M) \leftrightarrow \mathcal{M}(\mathcal{T}^M_i) : E_i,
\]
where $R_i^*$ is the restriction functor (that is, $R_i^*(F) := F|\mathcal{U}(\mathcal{T}_i^M)$), and $E_i$ is nothing but the right Kan extension functor along $R_i$, that is,

$$E_i(F)(x) := \lim_{y \to x, y \in \mathcal{U}(\mathcal{T}_i^M)} F(y) = \lim_{\mathcal{U}(\mathcal{T}_i^M) \ni x} \mathbb{D}_i.$$  \hfill (6.2)

Here $\mathbb{D}_i : \mathcal{U}(\mathcal{T}_i^M) \downarrow x \to \mathcal{M}$ is the functor that sends $(y \to x)$ to $F(y)$. Clearly $E_i$ is right adjoint to $R_i^*$. Similarly, we have a diagram $\mathbb{D}_{ij}$, and a pair of adjoint functors

$$R_{ij}^* : \mathcal{M}(\mathcal{T}^M_i) \longrightarrow \mathcal{M}(\mathcal{T}^M_j) : E_{ij},$$

where $E_{ij}(F)(x)$ is of course the limit of $\mathbb{D}_{ij}$.

Before we prove Proposition 6.3, we need five lemmas.

**Lemma 6.6.** Let $j \leq i$. Then

(i) For every $F \in \mathcal{M}(\mathcal{T}^M_j)$, $E_i(E_{ij}(F))$ is naturally isomorphic to $E_j(F)$. That is, $E_i(E_{ij}(F)) \equiv E_j(F)$.

(ii) Let $F \in \mathcal{M}(\mathcal{T}^M)$ be a fibrant diagram. Then the diagram $\mathbb{D}_i$ above as well as $\mathbb{D}_{ij}^*$ is fibrant for any $x \in \mathcal{U}(\mathcal{T}^M)$.

**Proof.** Part (i) follows immediately from the definitions. To see part (ii), let $x \in \mathcal{U}(\mathcal{T}^M)$. Then, by Definition 4.1, there exists a unique simplex $\sigma_x \in \mathcal{T}^M$ such that $x = U(\sigma_x)$. Associated with $\sigma_x$ is the poset $\mathcal{U}(\sigma_x) = \{U_{\lambda} \mid \lambda \subseteq \sigma_x\}$. Clearly we have $\mathcal{U}(\mathcal{T}^M_i) \downarrow x = \mathcal{U}(\mathcal{T}^M_j) \cap \mathcal{U}(\sigma_x)$. So $\mathbb{D}_i$ is nothing but the restriction of $F$ to $\mathcal{U}(\mathcal{T}^M_j) \cap \mathcal{U}(\sigma_x)$, which is definitely fibrant since $F$ is fibrant by assumption. Likewise, one can show that $\mathbb{D}_{ij}$ is fibrant. \hfill $\square$

Let $\Theta$ be the map that appears in Proposition 6.3 and let $f, f' : \mathcal{T}^*_M \to \hat{\mathcal{A}}_*$ be simplicial maps that are homotopic. Consider the categories $\mathcal{M}$ and $\mathcal{C}_A$ as in Definition 3.3 and set $F = \Theta(f)$ and $F' = \Theta(f')$. By the definitions, the functors $F, F' \in \mathcal{C}_A(\mathcal{T}^M) \subseteq \mathcal{M}(\mathcal{T}^M)$ are both fibrant and cofibrant. For convenience, we will regard these as functors into $\mathcal{M}$. Define $F_i := F|\mathcal{U}(\mathcal{T}^M_i)$ and $F_i' := F'|\mathcal{U}(\mathcal{T}^M_i)$.

**Lemma 6.7.** For every $i$ there exist $\mathcal{F}_i \in \mathcal{M}(\mathcal{T}^M_i)$ and two weak equivalences $F_i \xleftarrow{\alpha_i} \mathcal{F}_i \xrightarrow{\alpha'_i} F'_i$ satisfying the following two conditions: (a) $\mathcal{F}_i$ is cofibrant. (b) The map $(\alpha_i, \alpha'_i) : \mathcal{F}_i \to F_i \times F'_i$ is a fibration.

**Proof.** Consider the map $\Theta_i : \text{Hom}(\mathcal{T}^*_M, \hat{\mathcal{A}}_*) \to \mathcal{F}(\mathcal{U}(\mathcal{T}^M_i); \mathcal{C}_A)$ defined in the same way as $\Theta$, and let $f_i = f|\mathcal{T}^*_M$ and $f'_i = f'|\mathcal{T}^*_M$. Then it is clear that $f_i$ is homotopic to $f'_i$, $\Theta_i(f_i) = F_i$, and $\Theta_i(f'_i) = F'_i$. Applying Lemma 6.5 we get a zigzag

$$F_i \overset{\sim}{\leftarrow} \ast \overset{\sim}{\leftarrow} \cdots \overset{\sim}{\leftarrow} \ast \overset{\sim}{\leftarrow} F'_i \tag{6.3}$$

Using now the fact that $F_i$ and $F'_i$ are both fibrant and cofibrant, and some standard techniques from model categories, one can construct out of $\mathcal{F}_i$ a short zigzag (of the form indicated in the lemma) which has the required properties. \hfill $\square$

**Lemma 6.8.** Let $j \leq i$, and let $\alpha_i$ and $\alpha'_i$ be as in Lemma 6.7. Then there exists a map $\mathcal{F}_j \leftarrow R_{ij}^* \mathcal{F}_i$ making the following diagram commute.

$$\begin{array}{ccc}
R_{ij}^* F_i \cong \mathcal{F}_i & \xrightarrow{\sim} & R_{ij}^* \mathcal{F}_i \\
\downarrow \alpha_i & & \downarrow \alpha'_i \\
F_j & \cong & F'_j
\end{array} \tag{6.4}$$
Proof. Since $\alpha_j$ is a fibration by Lemma [6.7] and since $F_j$ is cofibrant, by the lifting axiom, there exists a map $\alpha_j^{-1}: F_j \to \overline{F_j}$ making the obvious square commute. From the construction of the zigzag [6.3] (look closer at the proof of Lemma 6.5), the following square commutes up to homotopy.

\[
\begin{array}{ccc}
R_{ij}^*F_i & \xrightarrow{R_{ij}^*(\alpha_i)R_{ij}^*(\alpha_i^{-1})} & R_{ij}^*F_i' \\
\downarrow{id} & & \downarrow{id} \\
F_j & \xrightarrow{\alpha_j\alpha_j^{-1}} & F_j'.
\end{array}
\]

Since $\alpha_j'$ is a fibration, and since $R_{ij}^*\overline{F_i}$ is cofibrant (this is because $\overline{F_i}$ is cofibrant by Lemma 6.7 and cofibrations are objectwise), the lifting axiom guarantees the existence of a map $R_{ij}^*\overline{F_i} \xrightarrow{\phi} \overline{F_j}$ that makes the righthand square of the following diagram commute.

\[
\begin{array}{ccc}
R_{ij}^*F_i & \xrightarrow{\sim} & R_{ij}^*\overline{F_i} \\
\downarrow{id} & & \downarrow{\phi} \\
F_j & \xrightarrow{\sim} & \overline{F_j} \\
\downarrow{\alpha_j} & & \downarrow{\alpha_j'} \\
\overline{F_j} & \xrightarrow{\sim} & \overline{F_j}'.
\end{array}
\]

Combining this with the fact that the square (6.5) commutes up to homotopy, we deduce that the lefthand square commutes up to homotopy as well. By Lemma 6.9 below, one can then replace $\phi$ by a map $R_{ij}^*\overline{F_i} \xrightarrow{\overline{\psi}} \overline{F_j}$ that makes the whole diagram commute.

**Lemma 6.9.** Consider the following diagram in a model category $\mathcal{M}$.

\[
\begin{array}{ccc}
A_0 & \xleftarrow{f_0} & B & \xrightarrow{f_1} & A_1 \\
\downarrow{g_0} & & \downarrow{g} & & \downarrow{g_1} \\
D_0 & \xleftarrow{f'_0} & C & \xrightarrow{f'_1} & D_1.
\end{array}
\]

Assume that each square commutes up to homotopy. Also assume that $B$ is cofibrant. If the map $(f'_0, f'_1): C \to D_0 \times D_1$ is a fibration, then there exists $\overline{g}$ homotopic to $g$ that makes the whole diagram commute.

Proof. Since each square commutes up to homotopy, there exists a homotopy $H: B \times I \to D_0 \times D_1$ from $(g_0f_0, g_1f_1)$ to $(f'_0, f'_1) \circ g$, which fits into the following commutative diagram.

\[
\begin{array}{ccc}
B & \xrightarrow{g} & C \\
\downarrow{i_1} & & \downarrow{(f'_0, f'_1)} \\
B & \xleftarrow{i_0} & B \times I & \xrightarrow{H} & D_0 \times D_1.
\end{array}
\]

The canonical inclusion $i_1$ is an acyclic cofibration since $B$ is cofibrant [3, Lemma 4.4]. The map $(f'_0, f'_1)$ is a fibration by assumption, and the map $\psi$ is provided by the lifting axiom. Now define $\overline{g} = \psi \circ i_0$. It is straightforward to check that $\overline{g}$ does the work. \qed

We still need an important lemma. From the definition of the right Kan extension [6.2], there is a canonical map $\eta_i: E_i F_i \to E_{i-1} F_{i-1}$ induced by the universal property of limit. These maps fit into the covariant diagram $\mathbb{E}: \mathbb{N} \to \mathcal{M}^{id(T^{<\omega})}$ defined by $\mathbb{E}(i) = E_i F_i$ and $\mathbb{E}(i \to (i - 1)) = \eta_i$. Here $\mathbb{N}$ is viewed as the poset \{1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots\}. 

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Lemma 6.10. The canonical map \( \kappa : F \longrightarrow \varprojlim_{\mathbb{N}} E \) is a weak equivalence.

Proof. We begin by claiming that for every \( x \in \mathcal{U}(T^M) \), there exists \( s \in \mathbb{N} \) such that \( F(x) \cong E_r F_r(x) \) for all \( r \geq s \). To see this, let \( x \in \mathcal{U}(T^M) \). Since \( \bigcup \mathcal{U}(T^M_i) = \mathcal{U}(T^M) \), there exists \( s \in \mathbb{N} \) such that \( x \in \mathcal{U}(T^{M_s}) \). Let \( r \geq s \). Since the sequence \( \{ \mathcal{U}(T^{M_i}) \}_i \) is increasing, the indexing category \( \mathcal{U}(T^{M_r}) \downarrow x \) has a terminal object, namely \( x \longrightarrow x \). This implies (by the definition (6.2) of \( E_r \)) that the canonical map \( F(x) \longrightarrow E_r F_r(x) \) is an isomorphism. Thanks to that isomorphism, we have \( F \cong \lim E_r F_r \). To end the proof, it suffices to show that the diagram \( E : \mathbb{N} \longrightarrow \mathcal{M}^{U(T^M)} \) is fibrant. By Proposition 2.10, we have to show that the matching map of \( E \) at \( i \) is a fibration for any \( i \). This is the case for \( i = 1 \) since \( E_1 F_1 \) is fibrant (because of the fact that \( F \) is fibrant). Let \( i \geq 2 \). Then the matching map at \( i \) is nothing but the canonical map \( \eta_i : E_i F_i \longrightarrow E_{i-1} F_{i-1} \).

Looking at the definition of a fibration (see Proposition 2.10) in \( \mathcal{M}^{U(T^M)} \), we need to show that the map \( p_x \) from the following commutative diagram is a fibration for every \( x \in \mathcal{U}(T^M) \) to conclude that \( \eta_i \) is a fibration.

![Diagram](chart)

So let \( x \in \mathcal{U}(T^M) \). Let \( z \in \mathcal{U}(T^M) \) such that \( \mathcal{U}(T^{M_i}) = \mathcal{U}(T^{M_{i-1}}) \cup \{ z \} \). We need to deal with two cases depending on the fact that \( x = z \) or \( x \neq z \).

- If \( x = z \), then \( \theta_{x(i-1)} \) is an isomorphism. Since the pullback of an isomorphism is an isomorphism, it follows that \( \lambda_x \) is an isomorphism. Since \( \theta_{x_1} \) is exactly the matching map of \( E_i F_i \) at \( z = x \), the map \( \theta_{x_1} \) is a fibration. Thus \( p_x \) is a fibration.
- Assume that \( x \neq z \). We have two cases. If there is no map from \( z \) to \( x \), then by the definitions, we have \( E_{i-1} F_{i-1}(y) = E_i F_i(y) \) for all \( y \rightarrow x \). This implies that \( M_x(\eta_i), \beta_x, (\eta)_x \), and \( p_x \) are isomorphisms. If there is a map \( z \rightarrow x \), one can see that \( \theta_{x(i-1)} \) and \( \theta_{x_1} \) are both isomorphisms since \( x \notin \mathcal{U}(T^{M_{i-1}}) \) and \( x \notin \mathcal{U}(T^{M_i}) \). This implies that \( \lambda_x \) and \( p_x \) are also isomorphisms.

We thus obtain the desired result. \( \square \)

We are now ready to prove the main result of this section: Proposition 6.3.

Proof of Proposition 6.3. We need to show that \( \Theta(f) = F \) and \( \Theta(f') = F' \) are weakly equivalent in \( \mathcal{F}(\mathcal{U}(T^M); C_A) \). By Proposition 4.4.14, it is enough to show that \( \phi F \cong \phi F' \) in \( \mathcal{F}_A(\mathcal{U}(T^M); \mathcal{M}) \) (see Definition 4.3.8). Here \( \phi : \mathcal{F}(\mathcal{U}(T^M); C_A) \longrightarrow \mathcal{F}_A(\mathcal{U}(T^M); \mathcal{M}) \) is the obvious functor defined by \( \phi(G) = G \). From now on, we will regard \( F \) and \( F' \) as objects of \( \mathcal{F}_A(\mathcal{U}(T^M); \mathcal{M}) \). As before, we let \( F_i \) (respectively \( F'_i \)) denote the restriction of \( F \) (respectively \( F' \)) to \( \mathcal{U}(T^{M_i}) \). Taking the adjoint to (6.4), we get (6.7), which is of course a commutative diagram.

![Diagram](chart)

(6.7)
From Lemma 6.6-(ii), it is easy to see why the maps $E_{ij}(\alpha_j)$ and $E_{ij}(\alpha_j')$ are both weak equivalences. Applying now the functor $E_i$ to (6.7), and using Lemma 6.6-(i), we get

$$E_i F_I \overset{\sim}{\leftarrow} E_i \overline{F}_I \overset{\sim}{\rightarrow} E_i F'_I$$

where $\beta_i := E_i(\alpha_i)$. This gives rise to two weak equivalences: $E \overset{\sim}{\leftarrow} \overline{E} \overset{\sim}{\rightarrow} E'$, where $\overline{E} : \mathbb{N} \rightarrow \mathcal{M}^{\mathcal{U}(\mathcal{T}_M)}$ is the obvious functor defined by $\overline{E}(i) = E_i \overline{F}_I$. Recalling the map $\kappa$ from Lemma 6.10, we have the zigzag

$$F \overset{\sim}{\leftarrow} \text{holim}_E \mathbb{N} \overset{\sim}{\leftarrow} \text{holim}_N \mathbb{E} \overset{\sim}{\rightarrow} \text{holim}_N \mathbb{E} \overset{\sim}{\leftarrow} F'_I,$$

which completes the proof. \[\square\]

We close this section with the following result.

**Proposition 6.11.** The map $\bar{X}$ from Definition 5.7 is a bijection. That is,

$$\mathcal{F}(\mathcal{U}(\mathcal{T}_M); \mathcal{C}_A) \xrightarrow{\bar{\kappa}} \mathbb{X} \xrightarrow{\kappa} [\mathcal{T}_M, \hat{\mathcal{A}}].$$

In fact the inverse of $\bar{X}$ is the map $\overline{\Theta}$ from Definition 6.2.

**Proof.** From the definitions, it is easy to see that $\Theta\Lambda(F) \simeq F$ for every $F \in \mathcal{F}(\mathcal{U}(\mathcal{T}_M); \mathcal{C}_A)$. So $\Theta \circ \Lambda \circ \overline{\Theta} = \id$. On the other hand, let $f : \mathcal{T}_M \rightarrow \hat{\mathcal{A}}$ be a simplicial map. By construction $\Theta(f)$ is fibrant since $f : \Delta^n \rightarrow \mathcal{C}_A$ is fibrant for any $n \geq 0, \sigma \in \mathcal{T}_M^n$. Therefore $\Theta \circ \Lambda \circ \overline{\Theta} = \id$, which completes the proof. \[\square\]

## 7 Proof of the main result

We now have all ingredients to prove Theorem 1.1, which is the main result of this paper. Roughly speaking it classifies a class of functors called homogeneous (see Definition 6.2).

**Definition 7.1.** Define $\mathcal{F}_{kA}(\mathcal{O}(\mathcal{M}); \mathcal{M})$ as the category of homogeneous functors $F : \mathcal{O}(\mathcal{M}) \rightarrow \mathcal{M}$ of degree $k$ such that $F(U) \simeq A$ for every $U$ diffeomorphic to the disjoint union of exactly $k$ open balls.

**Lemma 7.2.** Let $\mathcal{M}$ be a simplicial model category that has a zero object, and let $\mathcal{F}_{kA}(\mathcal{O}(\mathcal{M}); \mathcal{M})$ be as in Definition 7.1. Let $\mathcal{T}_{F_k}(\mathcal{M})$ be a triangulation of $F_k(\mathcal{M})$, and let $\mathcal{F}_{A}(\mathcal{U}(\mathcal{T}_{F_k}(\mathcal{M})); \mathcal{M})$ be the category from Definition 4.5. Then the categories $\mathcal{F}_{kA}(\mathcal{O}(\mathcal{M}); \mathcal{M})$ and $\mathcal{F}_{A}(\mathcal{U}(\mathcal{T}_{F_k}(\mathcal{M})); \mathcal{M})$ are weakly equivalent.

**Proof.** By using the same approach as that we used to prove Theorem 1.3 & Lemma 6.5, one has

$$\mathcal{F}_{kA}(\mathcal{O}(\mathcal{M}); \mathcal{M}) \simeq \mathcal{F}_{1A}(\mathcal{O}(F_k(\mathcal{M})); \mathcal{M}),$$

and

$$\mathcal{F}_{1A}(\mathcal{O}(F_k(\mathcal{M})); \mathcal{M}) \simeq \mathcal{F}_{1A}(\mathcal{B}(\mathcal{U}(\mathcal{T}_{F_k}(\mathcal{M})); \mathcal{M}),$$

One may ask the question to know whether the objects that appear in diagrams (6.7), (6.8), and (6.9) belong to $\mathcal{F}_{A}(\mathcal{U}(\mathcal{T}_M); \mathcal{M})$. The answer is yes. This is straightforward by using Lemma 6.6-(ii) and some classical properties of homotopy limit.
where \( \mathcal{F}_A(\mathcal{B}_U(\mathcal{T}_{F_k(M)}); \mathcal{M}) \) is the category from Definition 4.5. Furthermore, by Proposition 4.7, we have
\[
\mathcal{F}_A(\mathcal{B}_U(\mathcal{T}_{F_k(M)}); \mathcal{M}) \simeq \mathcal{F}_A(\mathcal{U}(\mathcal{T}_{F_k(M)}); \mathcal{M}).
\] (7.3)
Combining (7.1), (7.2), and (7.3), we get the desired result.

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** For the first part, we refer the reader to the introduction. The second part is proved as follows. Let \( \mathcal{T}_{F_k(M)} \) as above. From Lemma 7.2 and [14, Remark 6.4], we deduce that the localizations of \( \mathcal{F}_{pA}(\mathcal{O}(M); \mathcal{M}) \) and \( \mathcal{F}_A(\mathcal{U}(\mathcal{T}_{F_k(M)}); \mathcal{M}) \) are equivalent in the classical sense. Furthermore, by Proposition 4.14, we have that the localization of the latter category is equivalent to the localization of \( \mathcal{F}(\mathcal{U}(\mathcal{T}_{F_k(M)}); \mathcal{C}_A) \). This implies that weak equivalence classes of \( \mathcal{F}_{pA}(\mathcal{O}(M); \mathcal{M}) \) are in one-to-one correspondence with weak equivalence classes of \( \mathcal{F}(\mathcal{U}(\mathcal{T}_{F_k(M)}); \mathcal{C}_A) \). That is,
\[
\mathcal{F}_{pA}(\mathcal{O}(M); \mathcal{M})/\text{we} \cong \mathcal{F}(\mathcal{U}(\mathcal{T}_{F_k(M)}); \mathcal{C}_A)/\text{we}.
\] (7.4)
Applying Proposition 6.11, we get
\[
\mathcal{F}(\mathcal{U}(\mathcal{T}_{F_k(M)}); \mathcal{C}_A)/\text{we} \cong \left[ \mathcal{T}_{F_k(M)}, \hat{A} \right].
\] (7.5)
Define \( \hat{A} \) to be the geometric realization of \( \hat{A} \). That is, \( \hat{A} := [\hat{A}] \). Since \( |\mathcal{T}_{F_k(M)}| \cong F_k(M) \), it follows that \( \left[ \mathcal{T}_{F_k(M)}, \hat{A} \right] \cong [F_k(M), \hat{A}] \). This proves the theorem.

8 How our classification is related to that of Weiss

In this section we briefly recall the Weiss classification of homogeneous functors, and we explain a connection to our classification result (Theorem 1.1). As usual, we let \( \text{Top} \) (respectively \( \text{Top}_p \)) denote the category of spaces (respectively pointed spaces).

We begin with Weiss’ result about the classification of homogeneous functors. Let \( p: Z \to F_k(M) \) be a fibration. Define \( F: \mathcal{O}(M) \to \text{Top} \) as \( F(U) = \Gamma(p; F_k(U)) \), the space of sections of \( p \) over \( F_k(U) \). It turns out that \( F \) is polynomial of degree \( \leq k \) (see [15, Example 7.1]). Define another functor \( G: \mathcal{O}(M) \to \text{Top} \) as follows. Let \( M^k/\Sigma_k \) denote the orbit space of the action of the symmetric group \( \Sigma_k \) on the \( k \)-fold product \( M^k \). Let \( \Delta_k M \) be the complement of \( F_k(M) \) in \( M^k/\Sigma_k \). (The space \( \Delta_k M \) is the so-called *fat diagonal* of \( M \).) Define \( G(U) := \text{hocolim}_{N \in N} \Gamma(p; F_k(U) \cap N) \), where \( N \) stands for the poset of neighborhoods of \( \Delta_k M \).

The space \( G(M) \) should be thought of as the space of sections near the fat diagonal of \( M \). Let \( \eta: F \to G \) be the canonical map induced by the inclusions \( F_k(U) \cap N \subseteq F_k(U) \). It turns out that \( \eta \) is nothing but the canonical map \( T_k F \to T_{k-1} F \) since \( T_{k-1} F \) is equivalent to \( G \) (see [15, Propositions 7.5 and 7.6]). Selecting a point \( s \) in \( G(M) \), if one exists, we define \( E_{p,s}: \mathcal{O}(M) \to \text{Top} \) as the homotopy fiber of \( \eta \) over \( s \), that is, \( E_{p,s} := \text{hofiber}(\eta) \). It follows from [15, Example 8.2] that \( E_{p,s} \) is homogeneous of degree \( k \). Weiss’ classification says that every homogeneous functor can be constructed in this way. Specifically, we have the following result.

**Theorem 8.1.** [15, Theorem 8.5] Let \( E: \mathcal{O}(M) \to \text{Top} \) be homogeneous of degree \( k \). Then there is a (levelwise) homotopy equivalence \( E \to E_{p,s} \) for a fibration \( p: Z \to F_k(M) \) with a section \( s \) near the fat diagonal of \( M \).

Weiss also describes the fiber of \( p \) in terms of \( E \).
Proposition 8.2. \[\text{[13 Proposition 8.4 and Theorem 8.5]}\] Let $E$ be as in Theorem 8.1 and suppose $E$ is classified by a fibration $p: Z \to F_k(M)$. Then the fiber $p^{-1}(S)$ over $S \in F_k(M)$ is homotopy equivalent to $E(U_S)$, where $U_S$ is a tubular neighborhood of $S$, so that $U_S$ is diffeomorphic to the disjoint union of $k$ open balls.

Combining Theorem 8.1 and Proposition 8.2, we get that the classification of the objects of $\mathcal{F}_k\mathcal{A}(\mathcal{O}(M);\text{Top})$ (see Definition 7.1) amounts to the classification of fibrations over $F_k(M)$ with a section $s$ near the fat diagonal, and whose fiber is homotopy equivalent to $A$. Note that in the case $k = 1$ the fat diagonal is empty, and we just look at fibrations over $M$ with fiber $A$.

We now explain a connection to our classification result. As usual, $\mathcal{M}$ is an arbitrary simplicial model category, and $A$ is an object of $\mathcal{M}$. Assume that $A$ is both fibrant and cofibrant. Let $\text{haut}A$ be the simplicial monoid of self weak equivalences $A \xrightarrow{\sim} A$. Define $\text{Bhaut}A := \overline{\text{haut}}A$, where $\overline{\text{W}}$ is the functor from [11, p. 87] or [1] p. 269. (Note that $\overline{\text{W}}$ lands in the category of simplicial sets.) We still denote by $\text{Bhaut}A$ the geometric realization of $\text{Bhaut}A$. So depending on the context, $\text{Bhaut}A$ should be interpreted as either a simplicial set or a topological space. We have the following two conjectures.

Conjecture 8.3. Let $\mathcal{M}$ be a simplicial model category, and let $A \in \mathcal{M}$ be an object which is both fibrant and cofibrant. Let $\hat{A}$ be the simplicial set from Section 8.3 constructed out of $A$, and let $\hat{A}$ be its geometric realization. Then $\hat{A}$ is weakly equivalent to $\text{Bhaut}A$. That is, $\hat{A} \simeq \text{Bhaut}A$.

Conjecture 8.4. Let $\mathcal{M}$ and $A$ be as in Conjecture 8.3. Then for any manifold $M$,

1. if $k = 1$, there is a bijection $\mathcal{F}_1\mathcal{A}(\mathcal{O}(M);\mathcal{M})/\text{we} \cong [M, \text{Bhaut}A]$,
2. if $k \geq 2$ and $\mathcal{M}$ has a zero object, there is a bijection $\mathcal{F}_k\mathcal{A}(\mathcal{O}(M);\mathcal{M})/\text{we} \cong [F_k(M), \text{Bhaut}A]$.

One way to get the latter is to prove Conjecture 8.3 and then use our Theorem 1.1. When $\mathcal{M}$ is the category of spaces, we have that $\text{Bhaut}A$ classifies fibrations with fiber homotopy equivalent to $A$ (see [10 Corollary 9.5]).

We wanted to prove Conjecture 8.4 and state it as the main result of this paper, but we couldn’t find a way to do it. We tried another interesting approach, which does not involve $\hat{A}$ at all, that we now explain.

8.1 Another attempt to demonstrate Conjecture 8.4

We discuss the case $k = 1$; the cases $k \geq 2$ can be treated similarly. In the introduction we provided the proof of the first part of Theorem 1.1 that can be summarized as

$\mathcal{F}_1\mathcal{A}(\mathcal{O}(M);\mathcal{M})/\text{we} \cong \mathcal{F}(\mathcal{U}(\mathcal{T})^M;\mathcal{C}_A)/\text{we} \cong [M, \hat{A}]$.

For the purposes of the new approach, we need to replace $\mathcal{C}_A$ with a larger category $\mathcal{C}'_A$ defined as follows. The objects of $\mathcal{C}'_A$ are the objects of $\mathcal{M}$ which are connected to $A$ by a zigzag of weak equivalences. The morphisms of $\mathcal{C}'_A$ are the weak equivalences between its objects. By definition $\mathcal{C}'_A$ is connected. Note that the category $\mathcal{C}'_A$ is nothing but what Dwyer and Kan call special classification complex of $A$ (see [2, Section 2.2] where they use the notation sc$A$ instead). The usefulness of $\mathcal{C}'_A$ is due to the fact that its (enriched) nerve (which is denoted below by $d\mathcal{NC}^\prime_A$) has the homotopy type of $\text{Bhaut}A$ [2 Proposition 2.3], and depends only on the weak equivalence class of $A$.

Using the same approach as that we used to prove Theorem 1.1 one can show that there is a bijection

$\mathcal{F}_1\mathcal{A}(\mathcal{O}(M);\mathcal{M})/\text{we} \cong \mathcal{F}(\mathcal{U}(\mathcal{T})^M;\mathcal{C}'_A)/\text{we}$.
This makes sense if and only if $F_{\lambda A}(\mathcal{O}(M); \mathcal{M})$ is viewed as the category of homogeneous functors of degree 1 whose morphisms are natural transformations which are (objectwise) weak equivalences. The next step is to try to see whether $F(\mathcal{U}(T^M); C_A')/we$ is in bijection with $[M, \text{Bhaut} A]$. For this, consider the diagram

$$
\begin{array}{c}
F(\mathcal{U}(T^M); C_A')/we \xrightarrow{\tilde{N}} \text{Hom}(\tilde{N}\mathcal{U}(T^M), \tilde{N}C_A')/ \cong \xleftarrow{\alpha \cong} \text{Hom}(N\mathcal{U}(T^M), NC_A')/ \cong \\
\text{Hom}(\tilde{N}\mathcal{U}(T^M), \tilde{N}C_A')/ \cong \xrightarrow{\psi_*} \text{Hom}(d\tilde{N}\mathcal{U}(T^M), d\tilde{N}C_A')/ \cong \\
[M, d\tilde{N}C_A'] \cong [M, \text{Bhaut} A],
\end{array}
$$

where

- $\mathcal{U}(T^M)$ is viewed as a simplicially enriched category with the constant simplicial set at $\text{Hom}(u, v)$ for every $u, v \in \mathcal{U}(T^M)$. The simplicial enrichment of $C_A' \subseteq \mathcal{M}$ is of course induced by that of $\mathcal{M}$. Note that since the source, $\mathcal{U}(T^M)$, is discrete, we have that every functor $F: \mathcal{U}(T^M) \to C_A'$ is a simplicial functor in the sense that $F$ respects the simplicial enrichment;

- $s^2\text{Set}$ is the category of bisimplicial sets, and $\tilde{N}$ is the bisimplicial nerve functor defined as follows. For a simplicially enriched category $A$, $\tilde{N} A$ is the simplicial object in simplicial sets whose $k$-simplices are given by

$$
(\tilde{N} A)_k = \prod_{(a_0, \ldots, a_k) \in A^{k+1}} \text{Hom}(a_i, a_{i+1}),
$$

where $\text{Hom}(-, -)$ stands for the simplicial hom-set functor. It turns out that $\tilde{N}$ is fully faithful. The equivalence relation “$\cong$” that appears in $\text{Hom}(\tilde{N}\mathcal{U}(T^M), \tilde{N}C_A')/ \cong$ is the one generated by homotopies. That is, two bisimplicial maps $f, g: \tilde{N}\mathcal{U}(T^M) \to \tilde{N}C_A'$ are in relation with respect to $\cong$ if they are connected by a zigzag $f \leftarrow f_1 \leftarrow \cdots \leftarrow f_n \rightarrow g$ where $f_i \rightarrow f_j$ means that there is a homotopy from $f_i \rightarrow f_j$;

- $N$ is the ordinary/discrete nerve functor, $\text{sSet}$ is the category of simplicial sets as usual, and the equivalence relation “$\cong$” that appears in $\text{Hom}(N\mathcal{U}(T^M), NC_A')/ \cong$ is generated by homotopies in the same sense as before. The isomorphism $\alpha$ is defined in the standard way.

- $d$ is the diagonal functor $d: s^2\text{Set} \to \text{sSet}$, and $\psi_*$ is the map induced by the canonical map $\psi: NC_A' \to d\tilde{N}C_A'$. Note that by definition $d\tilde{N}\mathcal{U}(T^M) = N\mathcal{U}(T^M)$.

- $\mathcal{R}$ is a fibrant replacement functor $\mathcal{R}: \text{sSet} \to \text{sSet}$, $\sim$ is the usual homotopy relation, and $\varphi_*$ is the map induced by the fibrant replacement $\varphi: d\tilde{N}C_A' \xrightarrow{\sim} \mathcal{R}d\tilde{N}C_A'$;

- $|−|$ is of course the geometric realization functor, and $|\varphi|_*$ is the map induced by $|\varphi|$. Note that $|N\mathcal{U}(T^M)|/we$ is homeomorphic to $M$;

- The final bijection comes from the fact that $|d\tilde{N}C_A'| \cong \text{Bhaut} A$ as mentioned above.

This is another candidate for a possible proof of Conjecture 8.4, but the problem here is that we do not know how to show that $\psi_*$ and $\varphi_*$ are both isomorphisms.
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