On particular diameter bounds for integral point sets in
higher dimensions *

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Abstract. An integral point set \( P \) is a set of \( n \) points in the \( m \)-dimensional Euclidean space \( \mathbb{R}^m \) with pairwise integral distances, such that \( P \) is not contained in a \( (m - 1) \)-dimensional hyperplane. In the present paper we discuss some classes of planar integral point sets \( (m = 2) \); we mostly focus on the subsets of a union of two straight lines: facher, rails, scissors and pyramid sets. We use rails and pyramid sets to construct lower bounds for a diameter of integral point sets in higher dimensions.

1 Introduction

The study of integral point sets belongs to the classical analysis. It is motivated by both pure theoretical aspects [1,2] and applications [3].

Definition 1. A planar integral point set (PIPS) is a set \( P \) of non-collinear points in the plane \( \mathbb{R}^2 \) such that for any pair of points \( P_1, P_2 \in P \) the Euclidean distance \( |P_1P_2| \) between points \( P_1 \) and \( P_2 \) is integral.

Definition 1 can be generalized as follows:

Definition 2. An integral point set \( P \) is a set of \( n \) points in the \( m \)-dimensional Euclidean space \( \mathbb{R}^m \) with pairwise integral distances, such that \( P \) is not contained in a \( (m - 1) \)-dimensional hyperplane.

How can we characterize an integral point set? First of all, we can look at its dimension; then, we can find its cardinality, which is always finite [4,5]; finally, we can compute the diameter of a finite point set, which is naturally defined as follows:

Definition 3. The diameter of an integral point set \( P \) is defined as

\[
\text{diam}(P) = \max_{P_1, P_2 \in P} |P_1P_2|. \tag{1}
\]

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Every integral point set has also a characteristic\[6,7\].

**Definition 4.** The characteristic of a PIPS \(\mathcal{P}\) is a squarefree number \(q\) such that the area of any triangle \(M_1M_2M_3, \{M_1, M_2, M_3\} \subset \mathcal{P}\), is commensurable with \(\sqrt{q}\).

The following notion was introduced in \[8\].

**Definition 5.** The function \(d(m,n)\) is the minimal possible diameter of an integral point set \(\mathcal{P}\) of \(n\) points in \(m\)-dimensional Euclidean space \(\mathbb{R}^m\).

The computation of exact values of \(d(m,n)\) is a difficult problem. In 2003, Solymosi proved \[9\] that \(d(2,n) \geq cn\) for some constant \(c\); now it is known \[10\] that \(c > 5/11\). The most recent advances for higher dimensions can be found in \[11\]. So, even finding the estimates is not easy.

A long list of known bounds for \(d(m,n)\) and some exact values can be found in \[8\, Theorem 1\] or in \[12\]. Below we will discuss the following ones presented at \[8\]:

\[
\begin{align*}
    d(m,2m+1) &\leq 8 \\
    d(m,2m+2) &\leq 13 \\
    d(m,3m) &\leq 109
\end{align*}
\]

and the following theorem \[8\, Theorem 2.1]\).

**Theorem 1.** Let \(\mathcal{P}\) be a planar integral point set consisting of \(n - 2\) points on the line \(l_1\) and two points \(P_1\) and \(P_2\) on a parallel line \(l_2\) with distance \(r\) between \(l_1\) and \(l_2\). If there exist positive integers \(v, w\) with \(f^2 + v^2 = w^2\) and \(v < 2r\), where \(|P_1P_2| = f\), then

\[
d(m,n+2(m-2)) \leq \max(w, \text{diam}(\mathcal{P}))
\]

In this paper we further study the various features of IPS. In Section 2, we discuss the classification of planar integral points sets; in Sections 3 and 4, we present some upper bounds for \(d(m,n)\) based on planar integral point sets of special types and provide some general constructions of such bounds. All the bounds, like \(2)–(4)\), are of the form \(d(m,km+p) \leq c_{k,p}\).

## 2 Classification of planar integral point sets

### 2.1 Integral point sets situated on two straight lines

**Definition 6.** A planar integral point sets of \(n\) points with \(n - 1\) points on a straight line is called a facher set.

Most of the examples of planar integral point sets turns to be facher sets. In \[13\], facher sets with characteristic 1 are called semi-crabs. For 9 \(\leq n \leq 122\), the diameter \(d(2,n)\) is attained on a facher point set \[14\].

Non-facher integral point sets situated on two straight lines can be further classified in the following three cases:

**Definition 7.** A planar integral point set situated on two parallel straight lines is called a rails set.
Among the rails sets, sets with 2 points on one line and all the other on another line prevail. In Section 3 below, some examples of such sets are discussed; they include an example of rails sets with 2 points on one line and 38 point on the other line (Fig. 38) and an example of rails set with 4 points on one line and 4 points on the other (Fig. 40). We still do not know whether there are rails set with 4 points on one line and 5 points on the other; and the same for rails sets with 3 points on one line and 9 points on the other.

For the sake of brevity, we will hereafter use the notation $\sqrt{p/q} \ast \{(x_1, y_1), ..., (x_n, y_n)\}$ from [12,15,16], which means that each abscissa is multiplied by $1/q$ and each ordinate is multiplied by $\sqrt{p/q}$, that is

$$\sqrt{p/q} \ast \{(x_1, y_1), ..., (x_n, y_n)\} = \left\{\left(\frac{x_1}{q}, \frac{y_1\sqrt{p}}{q}\right), ..., \left(\frac{x_n}{q}, \frac{y_n\sqrt{p}}{q}\right)\right\}.$$  

On Fig. 1 we show the following rails set with 3 points on one line and 8 points on the other:

$$\mathcal{P}_{3,8} = \sqrt{255255}/2 \ast \{(1767; -3); (2791; -3); (4071; -3); (-306; 0); (0; 0); (1798; 0); (2304; 0); (2760; 0); (3534; 0); (4040; 0); (4558; 0)\}$$

It’s notable that $\text{char } \mathcal{P}_{3,8} = 255255 = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ is a product of the first 6 odd primes, and the distances between the three points on the lower line are $512 = 2^9$, $640 = 2^7 \cdot 5$ and $512 + 640 = 1152 = 2^7 \cdot 3^2$.

Figure 1: PIPS of cardinality 11 and diameter 2423

Rails sets with 4 and 4 points on the lines are somewhat rather common; an example of such a set is shown on Fig. 2

$$\sqrt{770}/1 \ast \{(0; 0); (380; 0); (731; 0); (1111; 0); (354; -12); (377; -12); (734; -12); (757; -12)\}$$

Definition 8. A planar integral point set situated on two perpendicular straight lines is called a cross set.

Every cross set has characteristic 1; in [13], cross sets with only 2 points out of one of the lines that have two axes of symmetry are called crabs.

Definition 9. A planar integral point set situated on two straight lines that are not parallel nor perpendicular, is called a scissors set.

There is an important subclass of scissors sets.
Definition 10. A scissors set with an axis of symmetry, which is the angle bisector for the straight lines, is called a pyramid set.

Examples of pyramid sets are given in Section 4 on Fig. 44 and Fig. 45. We don’t have any examples of pyramid sets with cardinality more than 9.

It still remains unknown whether a scissors set can have another axis of symmetry; on Fig. 4 the following scissors set with central symmetry is shown:

$$\sqrt{91}/1 \ast \{(0; 0); (792; 120); (-792; -120);$$

$$(\pm 256; 0); (\pm 646; 0); (\pm 693; 0); (\pm 1152; 0); (\pm 1242; 0); (\pm 1682; 0)\}$$

2.2 Other integral point sets

Definition 11. A planar integral point set that is situated on a circle is called a circular point set.

Circular sets are very important examples of integral point sets [17–19]; for example, the best known upper bound for \(d(2, n)\) is attained on a circular set.

Definition 12. A planar integral point set that is situated on the conjunction of a circle with its center, is called a centered-circular point set.

Remark 1. Every centered-circular point set has characteristic 1. Indeed, let \(P = \{M_0, M_1, M_2, M_3, ..., M_k\}\) be a centered-circular point set with \(M_0\) being the center. by the Law of sines, in the triangle \(M_1M_2M_3\)

$$\sin \angle M_1M_2M_3 = \frac{|M_1M_3|}{2R},$$

(6)
where \( R = |M_0M_1| \) is the radius of the triangle’s circumcircle. Then the area of the triangle \( M_1M_2M_3 \) is

\[
S = \frac{1}{2} |M_2M_1| \cdot |M_2M_3| \cdot \sin \angle M_1M_2M_3 = \frac{1}{2} |M_2M_1| \cdot |M_2M_3| \cdot \frac{|M_1M_3|}{2|M_0M_1|} \in \mathbb{Q}.
\] (7)

Figure 3: A centered-circular PIPS of cardinality 17 and diameter 2210

Figure 4: A scissors PIPS of cardinality 15 and diameter 3364

On Fig. 3 the following centered-circular PIPS is shown:

\[
\sqrt{\frac{1}{1105}} \cdot \{(0; 0); (\pm 121025; 0); (113953; 1215696); (-113953; -1215696);
(341887; 1172184); (-341887; -1172184); (680225; 1014000); (-680225; -1014000);
(782977; -936936); (-782977; 936936); (859775; -867000); (-859775; 867000);
(1073057; 582624); (-1073057; -582624); (1198975; 231000); (-1198975; -231000); \}
\]

Most of the known planar integral point sets fall into these six classes; however, there are some sophisticated examples which do not [20–22]. A lower bound for the diameter of a PIPS with no collinear triples (so-called PIPS in semi-general position) is presented in [23].

### 3 Bounds based on rails sets

Let \( i = \overline{1,k} \) denote the enumeration of all \( i \) from 1 to \( k \). Theorem [8, Theorem 2.1] can be generalized as follows:

**Theorem 2.** Let \( \mathcal{P} \) be a planar integral point set consisting of \( n - k \) points on the line \( l_1 \) and \( k \) points \( P_1, P_2, \ldots, P_k \) on a parallel line \( l_2 \) with distance \( r \) between \( l_1 \) and \( l_2 \). If there exist positive integers \( v, w_{ij} \), with \( f_{ij}^2 + v^2 = w_{ij}^2 \) and \( v < 2r \), where \( i = \overline{1,k-1}, j = i + 1,k, |P_iP_j| = f_{ij} \), then

\[
d(m, n + k(m - 2)) \leq \max(w_{1k}, \text{diam}(\mathcal{P}))
\] (8)
Below we give the examples of planar integral point sets and the corresponding estimates of the function $d(m, n)$ for $n = 2m + k$, $3 \leq k \leq 36$.

Below $f$ and $w$ are those from Theorem $1$.

- $\mathcal{P} = \sqrt{315}/2 \times \{(13, 1), (29, 1), (0, 0), (10, 0), (16, 0), (26, 0), (32, 0)\}$ (Figure 5)
  - $f = 8, v = 6, w = 10, \text{diam}(\mathcal{P}) = 17$,
  - which gives $d(m, 2m + 3) \leq 17$.
- $\mathcal{P} = \sqrt{315}/2 \times \{(13, 1), (29, 1), (0, 0), (10, 0), (16, 0), (26, 0), (32, 0), (42, 0)\}$ (Figure 6)
  - $f = 8, v = 6, w = 10, \text{diam}(\mathcal{P}) = 21$,
  - which gives $d(m, 2m + 4) \leq 21$.
- $\mathcal{P} = \sqrt{70}/1 \times \{(-44, 12), (78, 0), (62, 0), (55, 0), (10, 0)\}$ (Figure 7)
  - $f = 88, v = 66, w = 110, \text{diam}(\mathcal{P}) = 158$,
  - which gives $d(m, 2m + 5) \leq 158$.
- $\mathcal{P} = \sqrt{70}/1 \times \{(-44, 12), (78, 0), (62, 0), (55, 0), (10, 0)\}$ (Figure 8).
  - $f = 88, v = 66, w = 110, \text{diam}(\mathcal{P}) = 158$,
  - which gives $d(m, 2m + 6) \leq 158$.
- $\mathcal{P} = \sqrt{70}/2 \times \{(99, 2), (207, 0), (145, 0), (63, 0), (25, 0), (297, 0)\}$ (Figure 9).
  - $f = 99, v = 20, w = 101, \text{diam}(\mathcal{P}) = 252$,
  - which gives $d(m, 2m + 7) \leq 252$.
- $\mathcal{P} = \sqrt{70}/2 \times \{(99, 24), (207, 0), (145, 0), (63, 0), (25, 0)\}$ (Figure 10).
  - $f = 99, v = 20, w = 101, \text{diam}(\mathcal{P}) = 297$,
  - which gives $d(m, 2m + 8) \leq 297$.
- $\mathcal{P} = \sqrt{19019}/2 \times \{(200, 3), (873, 0), (615, 0), (377, 0), (215, 0), (23, 0), (-1273, 0)\}$ (Figure 11).
  - $f = 200, v = 45, w = 205, \text{diam}(\mathcal{P}) = 1073$,
  - which gives $d(m, 2m + 9) \leq 1073$. 

Figure 5: PIPS of cardinality 7 and diameter 17

Figure 6: PIPS of cardinality 8 and diameter 21
\[ P = \sqrt{19019} / 2 \times \{ (\pm 200, 3), (\pm 1273, 0), (\pm 873, 0), (\pm 615, 0), (\pm 377, 0), (\pm 215, 0), (\pm 23, 0) \} \] (Figure 12).

\( f = 200, \ v = 45, \ w = 205, \ \text{diam}(P) = 1273, \)

which gives \( d(m, 2m + 10) \leq 1273. \)
\( \mathcal{P} = \sqrt{385/2} \times \{(\pm 1105, 48), (\pm 1587, 0), (\pm 1269, 0), (\pm 763, 0), (\pm 623, 0), (\pm 529, 0), (\pm 339, 0), (-2189, 0)\} \) (Figure 13).

\( f = 1105, v = 300, w = 1145, \text{diam}(\mathcal{P}) = 1888, \) which gives \( d(m, 2m + 11) \leq 1888. \)

\( f = 1748, v = 336, w = 1780, \text{diam}(\mathcal{P}) = 3224, \) which gives \( d(m, 2m + 13) \leq 3224. \)
\[ P = \sqrt{\frac{154}{1}} \times \{(\pm 874, 60), (\pm 1848, 0), (\pm 1376, 0), (\pm 1036, 0), (\pm 899, 0), (\pm 849, 0), (\pm 613, 0), (\pm 576, 0), (\pm 100, 0)\} \] (Figure 16).

\[ f = 1748, \ v = 336, \ w = 1780, \ \text{diam}(P) = 3696, \]

which gives \( d(m, 2m + 14) \leq 3696. \)

\[ P = \sqrt{\frac{154}{1}} \times \{(\pm 874, 60), (\pm 1848, 0), (\pm 1376, 0), (\pm 1036, 0), (\pm 899, 0), (\pm 849, 0), (\pm 613, 0), (\pm 576, 0), (\pm 100, 0), (\pm 3293, 0)\} \] (Figure 17).

\[ f = 1748, \ v = 336, \ w = 1780, \ \text{diam}(P) = 5141, \]

which gives \( d(m, 2m + 15) \leq 5141. \)

\[ P = \sqrt{\frac{154}{1}} \times \{(\pm 874, 60), (\pm 3293, 0), (\pm 1848, 0), (\pm 1376, 0), (\pm 1036, 0), (\pm 899, 0), (\pm 849, 0), (\pm 613, 0), (\pm 576, 0), (\pm 100, 0)\} \] (Figure 18).

\[ f = 1748, \ v = 336, \ w = 1780, \ \text{diam}(P) = 6586, \]

which gives \( d(m, 2m + 16) \leq 6586. \)
\[
\mathcal{P} = \sqrt{154}/1 \times \{(\pm 2622, 180), (\pm 6148, 0), (\pm 5544, 0), (\pm 4128, 0), (\pm 3108, 0), (\pm 2697, 0), (\pm 2547, 0), (\pm 1839, 0), (\pm 1728, 0), (\pm 904, 0), (\pm 300, 0)\} \text{ (Figure 20).}
\]

\[ f = 5244, v = 1008, w = 5340, \text{ diam}(\mathcal{P}) = 12296, \]

which gives \( d(m, 2m + 18) \leq 12296. \)

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Figure 19: PIPS of cardinality 21 and diameter 11692

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\[
\mathcal{P} = \sqrt{154}/1 \times \{(\pm 5244, 360), (\pm 12296, 0), (\pm 11088, 0), (\pm 8579, 0), (\pm 8256, 0), (\pm 6216, 0), (\pm 5394, 0), (\pm 5094, 0), (\pm 3678, 0), (\pm 3456, 0), (\pm 1808, 0), (\pm 600, 0), (\pm 19758, 0)\} \text{ (Figure 23).}
\]

\[ f = 10488, v = 1015, w = 10537, \text{ diam}(\mathcal{P}) = 32054, \]

which gives \( d(m, 2m + 21) \leq 32054. \)
Figure 22: PIPS of cardinality 24 and diameter 19758

\[ \mathcal{P} = \sqrt{\frac{154}{1}} \times \{ (\pm 5244, 360), (\pm 19758, 0), (\pm 12296, 0), (\pm 11088, 0), (\pm 8579, 0), (\pm 8256, 0), (\pm 6216, 0), (\pm 5394, 0), (\pm 5094, 0), (\pm 3678, 0), (\pm 3456, 0), (\pm 1808, 0), (\pm 600, 0) \} \] (Figure 24).

\[ f = 10488, \ v = 1015, \ w = 10537, \ diam(\mathcal{P}) = 39516, \]

which gives \( d(m, 2m + 22) \leq 39516. \)

Figure 23: PIPS of cardinality 25 and diameter 32054

\[ \mathcal{P} = \sqrt{\frac{154}{1}} \times \{ (\pm 36708, 2520), (\pm 116058, 0), (\pm 86072, 0), (\pm 77616, 0), (\pm 60053, 0), (\pm 57792, 0), (\pm 43512, 0), (\pm 37758, 0), (\pm 35658, 0), (\pm 25746, 0), (\pm 24192, 0), (\pm 12656, 0), (\pm 4200, 0), (\pm 138306, 0) \} \] (Figure 25).

\[ f = 73416, \ v = 2710, \ w = 73466, \ diam(\mathcal{P}) = 254364, \]

which gives \( d(m, 2m + 23) \leq 254364. \)

Figure 24: PIPS of cardinality 26 and diameter 39516

\[ \mathcal{P} = \sqrt{\frac{154}{1}} \times \{ (\pm 36708, 2520), (\pm 138306, 0), (\pm 116058, 0), (\pm 86072, 0), (\pm 77616, 0), (\pm 60053, 0), (\pm 57792, 0), (\pm 43512, 0), (\pm 37758, 0), (\pm 35658, 0), (\pm 25746, 0), (\pm 24192, 0), (\pm 12656, 0), (\pm 4200, 0) \} \] (Figure 26).

\[ f = 73416, \ v = 2710, \ w = 73466, \ diam(\mathcal{P}) = 276612, \]

which gives \( d(m, 2m + 24) \leq 276612. \)

\[ \mathcal{P} = \sqrt{\frac{154}{1}} \times \{ (\pm 36708, 2520), (\pm 138306, 0), (\pm 116058, 0), (\pm 86072, 0), (\pm 77616, 0), (\pm 60053, 0), (\pm 57792, 0), (\pm 43512, 0), (\pm 37758, 0), (\pm 35658, 0), (\pm 25746, 0), (\pm 24192, 0), (\pm 12656, 0), (\pm 4200, 0), (\pm 199863, 0) \} \] (Figure 27).

\[ f = 73416, \ v = 2710, \ w = 73466, \ diam(\mathcal{P}) = 338169, \]

which gives \( d(m, 2m + 25) \leq 338169. \)
Figure 25: PIPS of cardinality 27 and diameter 254364

Figure 26: PIPS of cardinality 28 and diameter 276612

Figure 27: PIPS of cardinality 29 and diameter 338169

\[ P = \sqrt{\frac{154}{1}} \times \{(\pm 36708, 2520), (\pm 199863, 0), (\pm 138306, 0), (\pm 116058, 0), (\pm 86072, 0), (\pm 77616, 0), (\pm 60053, 0), (\pm 57792, 0), (\pm 43512, 0), (\pm 37758, 0), (\pm 35658, 0), (\pm 25746, 0), (\pm 24192, 0), (\pm 12656, 0), (\pm 4200, 0)\} \text{ (Figure 28).} \]

\[ f = 73416, \ v = 2710, \ w = 73466, \ \text{diam}(P) = 399726, \]

which gives \( d(m, 2m + 26) \leq 399726. \)

Figure 28: PIPS of cardinality 30 and diameter 399726

\[ P = \sqrt{\frac{154}{1}} \times \{(\pm 580290, 0), (\pm 430360, 0), (\pm 388080, 0), (\pm 300265, 0), (\pm 288960, 0), (\pm 217560, 0), (\pm 188790, 0), (\pm 183540, 12600), (\pm 178290, 0), (\pm 120960, 0), (\pm 63280, 0), (\pm 21000, 0), (\pm 11224, 0), (-1033912, 0), (-999315, 0), (-691530, 0)\} \text{ (Figure 29).} \]

\[ f = 367080, \ v = 3206, \ w = 367094, \ \text{diam}(P) = 1614202, \]

which gives \( d(m, 2m + 27) \leq 1614202. \)

\[ P = \sqrt{\frac{154}{1}} \times \{(\pm 691530, 0), (\pm 580290, 0), (\pm 430360, 0), (\pm 388080, 0), (\pm 300265, 0), (\pm 288960, 0), (\pm 217560, 0), (\pm 188790, 0), (\pm 183540, 12600), (\pm 178290, 0), (\pm 120960, 0), (\pm 63280, 0), (\pm 21000, 0), (\pm 11224, 0), (-1033912, 0), (-999315, 0), (-999315, 0)\} \text{ (Figure 30).} \]

\[ f = 367080, \ v = 3206, \ w = 367094, \ \text{diam}(P) = 1725442, \]
which gives \( d(m, 2m + 28) \leq 1725442 \).

\[ P = \sqrt{154}/1 \times \{(\pm 999315, 0), (\pm 691530, 0), (\pm 580290, 0), (\pm 430360, 0), (\pm 388080, 0), (\pm 300265, 0), (\pm 288960, 0), (\pm 217560, 0), (\pm 188790, 0), (\pm 183540, 12600), (\pm 178290, 0), (\pm 128730, 0), (\pm 120960, 0), (\pm 63280, 0), (\pm 21000, 0), (\pm 11224, 0), (\pm 1033912, 0)\} \ (\text{Figure 31}). \]

\( f = 367080, v = 3206, w = 367094, \ \text{diam}(P) = 2033227, \)

which gives \( d(m, 2m + 29) \leq 2033227 \).

\[ P = \sqrt{154}/1 \times \{(\pm 3997260, 0), (\pm 2766120, 0), (\pm 2321160, 0), (\pm 1721440, 0), (\pm 1552320, 0), (\pm 1201060, 0), (\pm 1155840, 0), (\pm 870240, 0), (\pm 755160, 0), (\pm 734160, 50400), (\pm 713160, 0), (\pm 514920, 0), (\pm 483840, 0), (\pm 380175, 0), (\pm 253120, 0), (\pm 84000, 0), (\pm 44896, 0), (\pm 4135648, 0)\} \ (\text{Figure 32}). \]
\[ f = 1468320, \ v = 12824, \ w = 1468376, \ \text{diam}(\mathcal{P}) = 8132908, \]
which gives \( d(m, 2m + 31) \leq 8132908. \)

Figure 33: PIPS of cardinality 35 and diameter 8132908

- \( \mathcal{P} = \sqrt{154}/1 \times \{(\pm 4135648, 0), (\pm 3997260, 0), (\pm 2766120, 0), (\pm 2321160, 0), \\
  (\pm 1721440, 0), (\pm 1552320, 0), (\pm 1201060, 0), (\pm 1155840, 0), (\pm 870240, 0), \\
  (\pm 755160, 0), (\pm 734160, 50400), (\pm 713160, 0), (\pm 514920, 0), (\pm 483840, 0), \\
  (\pm 308175, 0), (\pm 253120, 0), (\pm 84000, 0), (\pm 44896, 0)\} \) (Figure 34).

\[ f = 1468320, \ v = 12824, \ w = 1468376, \ \text{diam}(\mathcal{P}) = 8271296, \]
which gives \( d(m, 2m + 32) \leq 8271296. \)

Figure 34: PIPS of cardinality 36 and diameter 8271296

- \( \mathcal{P} = \sqrt{154}/1 \times \{(\pm 12991095, 0), (\pm 8989890, 0), (\pm 7543770, 0), (\pm 5594680, 0), \\
  (\pm 5045040, 0), (\pm 3903445, 0), (\pm 3756480, 0), (\pm 3694845, 0), (\pm 2828280, 0), \\
  (\pm 2454270, 0), (\pm 2386020, 163800), (\pm 2317770, 0), (\pm 1673490, 0), (\pm 1572480, 0), \\
  (\pm 822640, 0), (\pm 484680, 0), (\pm 273000, 0), (\pm 145912, 0), \\
  (-13440856, 0)\} \) (Figure 35).

\[ f = 4772040, \ v = 41678, \ w = 4772222, \ \text{diam}(\mathcal{P}) = 26431951, \]
which gives \( d(m, 2m + 33) \leq 26431951. \)

Figure 35: PIPS of cardinality 37 and diameter 26431951

- \( \mathcal{P} = \sqrt{154}/1 \times \{(\pm 13440856, 0), (\pm 12991095, 0), (\pm 8989890, 0), (\pm 7543770, 0), \\
  (\pm 5594680, 0), (\pm 5045040, 0), (\pm 3903445, 0), (\pm 3756480, 0), (\pm 3694845, 0), \\
  (\pm 2828280, 0), (\pm 2454270, 0), (\pm 2386020, 163800), (\pm 2317770, 0), (\pm 1673490, 0), \\
  (\pm 1572480, 0), (\pm 822640, 0), (\pm 484680, 0), (\pm 273000, 0), (\pm 145912, 0), \\
  (\pm 13440856, 0)\} \) (Figure 36).

\[ f = 4772040, \ v = 41678, \ w = 4772222, \ \text{diam}(\mathcal{P}) = 26881712, \]
which gives \( d(m, 2m + 34) \leq 26881712. \)
Remark 2. In all the examples above, the maximum in the right-hand side of (5) is attained on the diameter of the system. The following example shows that the maximum can be attained on \( w \):

\[ d(m, 2m + 35) \leq 105727804. \]
• $P = \sqrt{2}/1 \ast \{(\pm 76, 60), (\pm 62, 0), (\pm 29, 0), (\pm 6, 0)\}$ (Figure 39).

  $f = 152, v = 114, w = 190, \text{diam}(P) = 162$.

  We can obtain the estimate $d(m, 2m + 2) \leq 190$ (which is weaker than the one given in (3)).

Guy gives [24, D 20] a PIPS of 8 points located on two parallel lines. The points of the set form a rectangle (Figure 41):

• $P = \sqrt{1}/1 \ast \{(0, \pm 60), (182, \pm 60), (209, \pm 60), (391, \pm 60)\}$ (Figure 40),

  where $n = 8, \text{diam}(P) = 409$. Using the “blowing up” construction, we obtain

  $$d(m, 4m) \leq 409$$  \hspace{1cm} (9)

  Figure 43 shows an example for $m = 3$.

4 Bounds based on pyramid sets

**Theorem 3.** Let $P$ be a planar integral point set consisting of $k$ points on the line $l_1$ and $k$ points on the line $l_2$. Besides, these points are symmetric with respect to one of the bisectors of the angles formed by the intersection of lines $l_1$ and $l_2$, then

**d**($m, (k - \alpha)m + \alpha$) $ \leq \text{diam}(P)$,  \hspace{1cm} (10)
where
\[ \alpha = \begin{cases} 
1, & \text{when the intersection point } \in \mathcal{P}, \\
0, & \text{when the intersection point } \notin \mathcal{P}.
\end{cases} \]

Remark 3. The angles can be acute or obtuse. Figure 42 shows that the intersection angle of the lines \( l_1 \) and \( l_2 \) cannot be equal to \( \pi/2 \).

Below we give the examples of planar integral point sets and the corresponding estimates of the function \( d(m, n) \) for \( n = 3m + 1 \) and \( n = 4m + 1 \).

\[ \mathcal{P} = \sqrt{3}/2 \star \{(±56, 0), (14, 0), (-34, 0), (-10, 24), (-21, 35), (35, -21)\} \quad \text{(Figure 44)}, \]

where \( n = 7 \), \( \text{diam}(\mathcal{P}) = 56 \). Using the “blowing up” construction, we obtain
\[ d(m, 3m + 1) \leq 56 \quad (11) \]
The resulting estimate improves the estimate for \( d(m,3m) \), which is presented in \[6\]. Figure 46 shows an example for \( m = 3 \).

\[ P = \sqrt{39/8} \times \{(\pm 5040, 0), (1911, 315), (2352, 0), (944, 0), (336, 0), (2940, -420), (2044, 220), (735, 1155)\} \] (Figure 45),

where \( n = 9 \), \( \text{diam}(P) = 1260 \). Using the “blowing up” construction, we obtain

\[ d(m,4m + 1) \leq 1260 \quad (12) \]

Figure 47 shows an example for \( m = 3 \).

However, we have to admit that both bounds which employ pyramid sets are indeed weaker than the one

\[ d(m, m^2 + m) \leq 17 \quad (13) \]

proved in \[8\], so the constructions of bounds based on pyramid sets tend to be only for the conceptual purpose.

5 Final remarks

All the discussed PIPSs were obtained through a combination of computer search and intuition of the authors; so, the further search may lead to better bounds employing the same constructions.

There is still no general construction for a rails or scissors PIPS of arbitrary cardinality. For rails PIPSs, we can conjecture that there exists a set of arbitrary cardinality, with 2 points on
one line and all the rest on the other; on the other hand, it is still unknown whether there are any rails PIPSs with 4 and 5 points on the lines.

As for today, we have found pyramid PIPS of cardinality at most 9.

The source code can be obtained at https://gitlab.com/Nickkolok/ips-algo

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References

1. Ascher K., Braune L., Turchet A. The Erdős–Ulam problem, Lang’s conjecture and uniformity // Bulletin of the London Mathematical Society. — 2020. — Vol. 52, no. 6. — Pp. 1053–1063. — arXiv: 1809.02037.

2. Solymosi J., de Zeeuw F. On a question of Erdős and Ulam // Discrete & Computational Geometry. — 2010. — Vol. 43, no. 2. — Pp. 393–401. — arXiv: 0806.3095.

3. Kurz S. Enumeration of Integral Tetrahedra // Journal of Integer Sequences. — 2007. — Vol. 10, no. 2. — P. 3.
4. Anning N. H., Erdős P. Integral distances // Bulletin of the American Mathematical Society. — 1945. — Vol. 51, no. 8. — Pp. 598–600. — DOI: 10.1090/S0002-9904-1945-08407-9

5. Erdős P. Integral distances // Bulletin of the American Mathematical Society. — 1945. — Vol. 51, no. 12. — P. 996. — DOI: 10.1090/S0002-9904-1945-08490-0

6. Kemnitz A. Punktmengen mit ganzzahligen Abständen. — 1988.

7. Kurz S. On the characteristic of integral point sets in $\mathbb{E}^n$ // Australasian Journal of Combinatorics. — 2006. — Vol. 36. — Pp. 241–248. — arXiv: math/0511704

8. Kurz S., Laue R. Bounds for the minimum diameter of integral point sets // Australasian Journal of Combinatorics. — 2007. — Vol. 39. — Pp. 233–240. — arXiv: 0804.1296

9. Solymosi J. Note on integral distances // Discrete & Computational Geometry. — 2003. — Vol. 30, no. 2. — Pp. 337–342. — DOI: 10.1007/s00454-003-0014-7

10. Avdeev N. On existence of integral point sets and their diameter bounds // Australasian Journal of Combinatorics. — 2020. — Vol. 77, no. 1. — Pp. 100–116. — arXiv: 1906.11926 — URL: https://ui.adsabs.harvard.edu/abs/2019arXiv190611926A/abstract

11. Nozaki H. Lower bounds for the minimum diameter of integral point sets // Australasian Journal of Combinatorics. — 2013. — Vol. 56. — Pp. 139–143.

12. Авдеев Н. Н., Семёнов Е. М. Множества точек с целочисленными расстояниями на плоскости и в евклидовом пространстве // Математический форум (Итоги науки. Юг России). — 2018. — Pp. 217–236.

13. Antonov A. R., Kurz S. Maximal integral point sets over $\mathbb{Z}^2$ // International Journal of Computer Mathematics. — 2008. — Vol. 87, no. 12. — Pp. 2653–2676. — DOI: 10.1080/00207160902993636 — arXiv: 0804.1280

14. Kurz S., Wassermann A. On the minimum diameter of plane integral point sets // Ars Combinatoria. — 2011. — Vol. 101. — Pp. 265–287. — arXiv: 0804.1307

15. Авдеев Н. Н. Научно-исследовательские публикации, 2018. — Pp. 492–498.

16. Harborth H., Kemnitz A., Möller M. An upper bound for the minimum diameter of integral point sets // Discrete & Computational Geometry. — 1993. — Vol. 9, no. 4. — Pp. 427–432. — DOI: 10.1007/bf02189331

17. Piepmeyer L. The maximum number of odd integral distances between points in the plane // Discrete & Computational Geometry. — 1996. — Vol. 16, no. 1. — Pp. 113–115. — DOI: 10.1007/bf02711135
19. Bat-Ochir G. On the number of points with pairwise integral distances on a circle // Discrete Applied Mathematics. — 2018. — Vol. 254. — Pp. 17–32. — DOI: 10.1016/j.dam.2018.07.004.

20. Kreisel T., Kurz S. There are integral heptagons, no three points on a line, no four on a circle // Discrete & Computational Geometry. — 2008. — Vol. 39, no. 4. — Pp. 786–790. — DOI: 10.1007/s00454-007-9038-6.

21. Constructing 7-clusters / S. Kurz [et al.] // Serdica Journal of Computing. — 2014. — Vol. 8, no. 1. — Pp. 47–70. — arXiv: 1312.2318.

22. Avdeev N., Momot E., Zvolinsky A. On particular examples of planar integral point sets and their classification. — arXiv: 2102.12462 [math.CO].

23. Avdeev N. On diameter bounds for planar integral point sets in semi-general position. — 2019. — arXiv: 1907.09331 — URL: https://ui.adsabs.harvard.edu/abs/2019arXiv190709331A/abstract.

24. Guy R. Unsolved problems in number theory. Vol. 1. — Springer Science & Business Media, 2013. — DOI: 10.1007/978-1-4757-1738-9.