The Local Potential Approach to frustrated antiferromagnets

Shunsuke Yabunaka\textsuperscript{1} and Bertrand Delamotte\textsuperscript{2}

\textsuperscript{1}Yukawa Institute for Theoretical Physics, The Kyoto University, Kitashirakawa Oiwake-Cho, 606-8502 Kyoto, Japan
\textsuperscript{2}Sorbonne Universités, UPMC Univ Paris 06, LPTMC, CNRS UMR 7600, F-75005, Paris, France

Abstract

We revisit the critical behavior of classical frustrated systems using the nonperturbative renormalization group (NPRG) equation. Our study is performed within the local potential approximation of this equation to which is added the flow of the field renormalization. Our flow equations are functional to avoid possible artifacts coming from field expansions which consists in keeping only a limited number of coupling constants. We present a simple numerical method to follow the fixed point solution of our equations by changing gradually the dimension $d$ and the number $N$ of spin-components. We explain in details the advantage of this method as well as the numerical difficulties we encounter, which become severe close to $d = 2$. The function $N_c(d)$ separating the regions of first and second order in the $(d, N)$ plane is computed for $d$ between 4 and 2.2. Our results confirm what was previously found within cruder approximation of the NPRG equation and contradict both the fixed dimension perturbative approach and the results obtained within the conformal bootstrap approach.

PACS numbers: 75.10.Hk, 05.10.Cc, 12.38.Lg
I. INTRODUCTION

The critical behavior of antiferromagnetic frustrated systems is still a debated question forty years after the first studies of these systems [1, 2]. The key difference between frustrated and nonfrustrated systems is that the order parameter is a vector in the nonfrustrated case and a matrix in the other cases. When frustration originates from the geometry of the system as in Stacked triangular Antiferromagnets (STA), the symmetry of the Hamiltonian is $O(N) \otimes O(2)$ for $N$-component spins and the order parameter is a rectangular $N \times 2$ matrix [3]. Depending on $N$ and the dimension $d$ of space, the nature of the phase transition changes, being first order for low values of $N$ and dimensions close to four and second order otherwise. One of the key questions is thus the determination of the line $N_c(d)$ separating the first and second order regions. It turns out that the value of $N_c(d = 3)$ is certainly close to 3 and its precise determination is crucial to know whether the transition is first or second order for the systems realized in nature that are either Ising, XY or Heisenberg. Numerical simulations of several frustrated antiferromagnets such as XY and Heisenberg STA show unambiguously that the transition is first order for these systems [11–14]. However, depending on the theoretical approach considered, the determination of $N_c(d)$ varies much when $d \lesssim 3.3$ and, as a result, it is not yet settled whether all $O(N) \otimes O(2)$ symmetric systems undergo first order phase transitions in $d = 3$ for $N \leq 3$. The two-dimensional physics of the XY and Heisenberg systems is also debated because the relevance of topological defects is not yet understood, in particular the possibility that they trigger a phase transition at finite temperature [9, 15–19].

The different theoretical approaches tackling with the problem of the calculation of $N_c(d)$ can be roughly divided into two classes: the perturbative and the nonperturbative renormalization group (NPRG) calculations. The class of perturbative calculations can be again divided into several different subclasses depending on whether they are performed directly in $d = 3$ (at six loops) [7, 8, 10] or in an $\epsilon$- or pseudo-$\epsilon$-expansion (respectively at six and five loops) [6]. In the latter case, the value of $N_c(d = 3)$ is systematically found larger than 3 (of order 6) as it is also the case for the NPRG calculations that find $N_c(d = 3) \simeq 5.1$ [11, 26–28, 32]. On the contrary, the perturbative calculation performed directly in $d = 3$ at six loops yields a fixed point for $N = 2$ and 3 and thus predicts that several $O(N) \otimes O(2)$ symmetric systems should undergo a second order phase transition.
Recently, a completely different method based on the conformal bootstrap has been used to study matrix models in $d = 3$ and in particular the $O(N) \otimes O(2)$ frustrated systems $^{20, 21}$. A critical behavior has been found in the Heisenberg case with exponents in good agreement with those of the six-loop fixed dimension approach. This approach has the advantage of being unbiased by convergence problems since it is not based on series expansions, contrary to RG methods and, when applied to the ferromagnetic $O(N)$ models, it leads to an extremely accurate determination of the critical exponents, at least when it is truncated at large orders $^{22–25}$.

The situation of the NPRG approach, that we re-examine here, is therefore the following. Either the conclusions drawn from its results are correct and then both the fixed dimension perturbative RG approach and the conformal bootstrap are wrong or, conversely, it is wrong (together with the $\epsilon$-expansion approaches) and this implies that the approximations used are too drastic to reproduce the correct physics. In both cases, something very unusual is at work because the methodologies that have been used in these studies lead in many cases to correct and accurate results.

As for the NPRG, which is based on an exact RG equation, the approximations used so far to tackle with frustrated systems consists in performing a derivative expansion $^{37}$ and a field expansion of the Gibbs free energy $^{1, 28, 32}$. The rationale behind this choice is (i) that the critical behavior of thermodynamic quantities such as the specific heat or the susceptibility for instance are dominated by long wavelength fluctuations which justifies expanding the correlation functions in their momenta (derivative expansion) and (ii) that the impact of the $n$-point functions with $n$ large on the RG flow of the zero or two-point functions should be small (field-expansion). It is the aim of this article to eliminate one source of inaccuracy of the NPRG approach, the field expansion, which is known to be inaccurate at low dimensions even for simple models such as the ferromagnetic $O(N)$ models $^{1}$. The price to pay to get rid of this approximation is to work functionally, that is, to follow the RG flow of functions of the fields instead of a limited number of coupling constants. In the case of nonfrustrated systems, this is relatively simple since the $O(N)$ symmetry implies that all functions involved in the RG flows depend on the fields only through the unique $O(N)$-invariant: $\rho = \vec{\phi}^2$. For frustrated systems, there exists two $O(N) \otimes O(2)$ invariants and the resulting flow equations are partial differential equations that are rather involved. We show in this article how to simplify the numerical problem and point out why the numerical difficulties are so severe at
low dimensions that our method does no longer work when approaching \( d = 2 \). We provide the results thus obtained for the curve \( N_c(d) \) between \( d = 4 \) and \( d = 2.2 \). Our results confirm what was previously found within a NPRG approximation involving a field expansion of the potential and the \( \epsilon \)-approaches and thus contradict both the fixed-dimension perturbative approach and the results obtained with the conformal bootstrap.

II. THE MODEL

As the archetype of frustrated spin systems, we employ the Stacked Triangular Antiferromagnets (STA). This system is composed of two-dimensional triangular lattices that are piled-up in the third direction. At each lattice site \( i \), is defined a \( N \)-component vector \( \mathbf{S}_i \) of modulus 1. The Hamiltonian of this system is given by

\[
H = \sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j. \tag{1}
\]

The sum \( \langle ij \rangle \) runs on all pairs of nearest neighbor spins. The coupling constants \( J_{ij} \) are given by \( J_\perp \) for a pair of sites inside a plane and \( J_\parallel \) between planes. We assume that the interactions inside a plane are antiferromagnetic: \( J_\perp \) is positive.

The long distance effective theory for the STA has been derived by Yosefin and Domany\(^3\). The order parameter consists of the \( N \times 2 \) matrix \( \Phi = (\phi_1, \phi_2) \) that satisfies

\[
\phi_i \cdot \phi_j = \delta_{ij} \tag{2}
\]

for \( i, j = 1, 2 \). Then, the effective Hamiltonian in the continuum is given by

\[
H = \int d^d \mathbf{x} \left( \frac{1}{2} \left[ (\partial \phi_1)^2 + (\partial \phi_2)^2 \right] \right). \tag{3}
\]

The constraint \( \phi_i \cdot \phi_j = \delta_{ij} \) for \( i, j = 1, 2 \) can be replaced by a soft potential \( U(\phi_1, \phi_2) \) whose minima are given by \( \phi_i \cdot \phi_j = \text{const} \times \delta_{ij} \) and the Ginzburg-Landau-Wilson Hamiltonian for STA reads

\[
H = \int d^d \mathbf{x} \left( \frac{1}{2} \left[ (\partial \phi_1)^2 + (\partial \phi_2)^2 \right] + U(\phi_1, \phi_2) \right). \tag{4}
\]

Instead of \( \phi_i \), it is convenient to work with the invariants of the \( O(N) \times O(2) \) group that can be chosen as:

\[
\rho = \text{Tr} (\Phi \Phi) = \phi_1^2 + \phi_2^2, \quad \tau = \frac{1}{2} \text{Tr} (\Phi \Phi - \rho/2)^2 = \frac{1}{4} (\phi_1^2 - \phi_2^2)^2 + (\phi_1 \phi_2)^2. \tag{5}
\]
With this choice, the ground state configuration corresponds to \( \rho = \text{const.} \) and \( \tau = 0 \). Up to the fourth order \( U(\rho, \tau) \) can be written as

\[
U(\rho, \tau) = \frac{\lambda}{2} (\rho - \kappa)^2 + \mu \tau, \tag{6}
\]

where \( \lambda \) and \( \mu \) are positive coupling constants. A typical ground state in terms of \( \Phi \) is given by \( \Phi_{\alpha,i} = \sqrt{\kappa/2} \delta_{\alpha,i} \), that is:

\[
\Phi_{\min} \equiv \begin{pmatrix}
\sqrt{\frac{\kappa}{2}} & 0 \\
0 & \sqrt{\frac{\kappa}{2}} \\
\vdots & \vdots \\
0 & 0
\end{pmatrix}. \tag{7}
\]

### III. THE NONPERTURBATIVE RENORMALIZATION GROUP EQUATION

The NPRG method is based on Wilson’s idea of integrating statistical fluctuations step by step. In this paper, we employ the effective average action method as an implementation of the NPRG in continuum space \([33–36]\).

The first step is to introduce a \( k \)-dependent partition function \( Z_k \) in the presence of sources:

\[
Z_k[J_i] = \int \mathcal{D}\phi_i \exp \left( -H[\phi_i] - \Delta H_k[\phi_i] + J_i \cdot \phi_i \right), \tag{8}
\]

where \( J_i \cdot \phi_i = \sum_{i=1}^{2} \int_x J_i(x) \cdot \phi_i(x) \), and \( \Delta H_k = \sum_{i=1}^{2} \phi_i(x) R_k(x-y) \phi_i(y) \). The idea underlying the effective average action is that in \( Z_k \) only the fluctuations of large wave-numbers (the rapid modes) compared to \( k \) are integrated over while the others (the slow modes) are frozen by the \( \Delta H_k \) term. As \( k \) is decreased, more and more modes are integrated until they are all when \( k = 0 \). The function \( R_k(q^2) \), which is the Fourier transform of \( R_k(x) \), plays the role of separating rapid and slow modes: It almost vanishes for \( |q| > k \) so that the rapid modes are summed over and is large (of order \( k^2 \)) below \( k \) so that the fluctuations of the slow modes are frozen. We define as usual \( W_k[J_i] = \ln Z_k[J_i] \). Thus, the order parameter \( \varphi_j(x) \) at scale \( k \) is defined by

\[
\varphi_i(x) = \langle \phi_i(x) \rangle = \frac{\delta W_k[J_i]}{\delta J_i(x)}. \tag{9}
\]

The running effective average action \( \Gamma_k[\varphi_i] \) is defined as the (modified) Legendre transform of \( W_k \):

\[
\Gamma_k[\varphi_i] = -W_k[J_i] + J_i \cdot \varphi_i - \Delta H_k[\varphi_i] \tag{10}
\]
where $J_i$ is defined such that Eq. (9) holds for fixed $\varphi_i$. From this definition one can show that

$$\begin{cases} \Gamma_{k=\Lambda} \simeq H, \\ \Gamma_{k=0} = \Gamma \end{cases},$$

(11)

where the cutoff $\Lambda$ is the inverse of the lattice spacing $a$. Equations (11) imply that $\Gamma_k$ interpolates between the Hamiltonian of the system when no fluctuation has been summed over, that is, when $k = \Lambda$, and the Gibbs free energy $\Gamma$ when they have all been integrated, that is, when $k = 0$. We define the variable $t$, called “RG time”, by $t = \ln (k/\Lambda)$. The exact flow equation for $\Gamma_k$ reads [33, 34]:

$$\partial_t \Gamma_k[\varphi_i] = \frac{1}{2} \text{Tr} \int_{x,y} \partial_x R_k(x - y) \left( \frac{\delta^2 \Gamma_k[\varphi_i]}{\delta \varphi^\alpha_i (x) \delta \varphi^\alpha'_{i'} (y)} + R_k(x - y) \delta_{i,i'} \delta_{\alpha,\alpha'} \right)^{-1},$$

(12)

for $\alpha, \alpha' = 1, 2, \cdots N$ and $i, i' = 1, 2$.

IV. TRUNCATIONS OF THE NPRG EQUATION

It is generally not possible to solve exactly the above flow equation (12) and approximations are required in practice. In this paper, we employ the approximation of lowest level in the derivative expansion dubbed the local potential approximation (LPA) and some of its refinements.

Within the LPA, $\Gamma_k$ is approximated by a series expansion in the gradient of the field, truncated at its lowest non trivial order:

$$\Gamma_k[\varphi_i] = \int d^d x \left( \frac{1}{2} \left[ (\partial \varphi_1)^2 + (\partial \varphi_2)^2 \right] + U_k(\rho, \tau) \right),$$

(13)

Only a potential term $U_k(\rho, \tau)$ is thus retained in this approximation which is accurate as long as the impact of the renormalization of the derivative terms on the flow of the potential is small. This is most probably the case when the anomalous dimension is small and $d > 2$. The next level of approximation consists in including in the approximation a running field renormalization $Z_k$

$$\Gamma_k = \int_x \left\{ U_k(\rho, \tau) + \frac{1}{2} Z_k \left( (\partial \varphi_1)^2 + (\partial \varphi_2)^2 \right) \right\},$$

(14)

This approximation has been used in [1, 28–32] where the function $U_k(\rho, \tau)$ was further expanded in powers of the invariants $\rho$ and $\tau$. This is what we improve here to avoid any
artifact coming from this field truncation. This approximation, that we call LPA', yields the one-loop result obtained within the $\epsilon$-expansion in $d = 4 - \epsilon$ and also, in the $O(N)$ case, the one-loop result of the $\epsilon = d - 2$ expansion of the nonlinear sigma model. Although the situation is a little more involved in our case, it is very probable that the LPA' is very accurate close to $d = 2$ and our numerical results confirm this, see the following. Our approach is therefore at least a clever interpolation between the results obtained either in $d = 4$ or $d = 2$.

The $k$-dependent effective potential $U_k (\rho, \tau)$ is defined by

$$\Omega U_k (\rho, \tau) = \Gamma_k [\varphi]$$

(15)

where $\varphi, i = 1, 2$ are constant fields and $\Omega$ is the volume of the system. The running field renormalization $Z_k$ is set to one in LPA: $Z_k^{\text{LPA}} = 1$, which leads to a vanishing anomalous dimension: $\eta = 0$. In LPA’ calculations, the anomalous dimension $\eta$ is obtained from the flow of $Z_k$ since it can be shown that at criticality:

$$Z_{k \to 0} \sim \left( \frac{k}{\Lambda} \right)^{-\eta}.$$  

(16)

The flows of $U_k$ and $Z_k$ have been derived in [1, 28, 32] and we recall them for completeness in Appendix A. These flows are rather complicated and their numerical integration suffers from all the inherent difficulties of the nonlinear partial differential equations.

The first difficulty comes from the choice of variables. It is tempting to work with the invariants $\rho$ and $\tau$ defined above because the symmetry of the problem is encoded in the very definition of the variables and any smooth function of these variables corresponds to a function that has the right symmetry. However, $\rho$ and $\tau$ satisfy $\frac{1}{4} \rho^2 \geq \tau \geq 0$ and it is not easy to deal with this constraint numerically because the domain where the variables $\rho$ and $\tau$ live is nontrivial. Thus, we define another set of variables $\psi_i$ which is numerically more convenient. For any $\varphi_1$ and $\varphi_2$, it can be proven that there exists $O_1 \in O(N)$ and $O_2 \in O(2)$ such that the matrix $M \equiv O_1 \Psi O_2$, where $N \times 2$ matrix $\Psi$ is defined as $\Psi = (\varphi_1, \varphi_2)$, becomes “diagonal”, namely,

$$M \equiv \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$  

(17)
Because of the $O(N) \times O(2)$ symmetry of the model, we conclude that $U_k(\Psi) = U_k(M)$. This fact shows that we can parametrize the order parameter space using $\psi_1$ and $\psi_2$, instead of $\varphi_1$ and $\varphi_2$. The $O(N) \times O(2)$ invariants $\rho$ and $\tau$ are expressed in terms of $\psi_1$ and $\psi_2$ as

$$\rho = \psi_1^2 + \psi_2^2$$
$$\tau = \frac{1}{4} (\psi_1^2 - \psi_2^2)^2.$$  

From the definitions (18) we find that the symmetries of the original problem imply:

$$U_k(\psi_1, \psi_2) = U_k(-\psi_1, \psi_2) = U_k(\psi_1, -\psi_2) = U_k(\psi_2, \psi_1).$$  

Thus, to solve the flow equations, it is sufficient to consider the region $\psi_2 \geq \psi_1 \geq 0$. This triangular domain is much more convenient from a numerical point of view than the parabolic domain $\frac{1}{4} \rho^2 \geq \tau \geq 0$ for the invariants $\rho$ and $\tau$.

At criticality, the $k$-dependent effective action is attracted towards the fixed point solution of the NPRG equation once it is expressed in terms of the dimensionless renormalized fields $\tilde{\psi}_i$ and a dimensionless local potential $\tilde{U}_k(\tilde{\psi}_i)$. We thus define the dimensionless and renormalized quantities:

$$\tilde{\psi}_i = (Z_k k^{2-d})^{1/2} \psi_i$$
$$\tilde{U}_k(\tilde{\psi}_i) = k^{-d} U_k(\psi_i).$$  

The flow equation for $\tilde{U}_k$ is given by Eq. (A5) in Appendix A. The critical exponent $\nu$ of the correlation length is obtained from the relevant eigenvalue of the linearized flow around the fixed point solution and $\eta$ from the flow of $Z_k$. The other critical exponents can be deduced from these ones by scaling relations.

The scaled $O(N) \times O(2)$ invariants $\tilde{\rho}$ and $\tilde{\tau}$ are defined by $\tilde{\rho} = Z_k k^{2-d} \rho$, $\tilde{\tau} = Z_k^2 k^{2(2-d)} \tau$, and the potential and couplings by $\tilde{U}_k(\tilde{\rho}, \tilde{\tau}) = k^{-d} U_k(\rho, \tau)$, $y = q^2 / k^2$, $R_k(q^2) = Z_k k^2 y r(y)$. Notice that as said above $Z_k$ does not reach a fixed point but $\eta_k$, defined by $\eta_k = -d \log Z_k / d \log k$, does: $\eta_{k \to 0} \to \eta$ at criticality with $\eta$ the anomalous dimension of the fields.

V. NUMERICAL METHODS

A. The fixed point

From a numerical point of view, there are two possibilities for finding fixed points when they exist. The first is to dynamically integrate the flow. In this case, the problem is to
find the critical surface which is usually done by dichotomy on the temperature. Once it is found, the fixed point is (approximately) reached since it is attractive on the critical surface. The other method is to look directly for the solution of the fixed point equation (coupled with Eq. (A3)): \( \partial_t \tilde{U}^*(\tilde{\psi}_i) = 0 \). This is what we do here. The advantage of this method is three-fold: (i) The numerical scheme is much simpler than integrating the flow; (ii) several numerical instabilities occurring during the integration of the flow are avoided; (iii) the critical exponents are easily obtained from the diagonalization of the RG flow around the fixed point. We show in the following that although this scheme works very well in dimension \( d = 3 \), numerical difficulties appear in dimensions close to \( d = 2 \) that make almost impossible to study the physics of frustrated systems in this dimension, at least with our numerical scheme.

The basic idea of this scheme is simple. It consists in solving the fixed point equations for \( \tilde{U}^* \) on a grid in \((\psi_1, \psi_2)\) space, taking into account the symmetries (19) of this space. We introduce a cut-off field value \( \tilde{\psi}_{\text{max}} \) and consider the domain \( D: \tilde{\psi}_{\text{max}} \geq \tilde{\psi}_2 \geq \tilde{\psi}_1 \geq 0 \). We then discretize \( D \) on a square lattice with mesh size \( \Delta \tilde{\psi} = \tilde{\psi}_{\text{max}} / (N_p - 1) \), where \( N_p \) is the number of lattice points on the axis \( \psi_1 = 0 \). The lattice points are given by \((i \Delta \tilde{\psi}, j \Delta \tilde{\psi})\) for integers \( i \) and \( j \) that satisfy \( 0 \leq i \leq j \leq N_p - 1 \). We define \( \tilde{U}_t(i, j) \equiv \tilde{U}_t(i \Delta \psi, j \Delta \psi) \) to alleviate the notation.

The fixed point equation for the potential is a differential equation. We transform it into a set of algebraic equations by discretizing the derivatives of \( \tilde{U} \). We give below some details about this procedure because all our numerical problems come from the boundary of the domain \( D \), precisely at the points where the discretization involves exceptional cases.

The formulae for the derivatives \( \tilde{U}_t(i, m) (i, j) \) for \( l, m = 0, 1, 2 \) are constructed as follows:

1. In the bulk region \((0 \leq i \leq j \leq N_p - 3)\): \( U_t^{(1,0)} \) and \( U_t^{(2,0)} \) as well as \( U_t^{(0,1)} \) and \( U_t^{(0,2)} \) are computed with five points. \( U_t^{(1,1)} \) is computed with the nine points \( \tilde{U}_t(i, j), \tilde{U}_t((i + 1), (j + 1)), \tilde{U}_t((i + 1), (j - 1)), \tilde{U}_t((i - 1), (j + 1)) \), \( \tilde{U}_t((i - 1), (j - 1)), \tilde{U}_t((i + 2), (j + 2)), \tilde{U}_t((i + 2), (j - 2)), \tilde{U}_t((i - 2), (j + 2)), \tilde{U}_t((i - 2), (j - 2)) \). The formulae are exact up to \((\Delta \psi)^3\). Notice that for points on the two borders of \( D \) defined either by \( \tilde{\psi}_1 = 0 \) or \( \tilde{\psi}_1 = \tilde{\psi}_2 \), the derivatives of \( \tilde{U} \) involve points outside \( D \). By using (19), we can compute these values of \( \tilde{U} \) from those that are inside \( D \). This is one of the advantage of the choice of variables \((\psi_1, \psi_2)\) compared to the choice \((\rho, \tau)\): The derivatives on the two borders \( \tilde{\psi}_1 = 0 \) and \( \tilde{\psi}_1 = \tilde{\psi}_2 \) can be computed in the same way as in the bulk.

2. On the boundary of the domain \( D \) corresponding to the large field region, \( j = \)
\(N_p - 2, N_p - 1\), we compute the derivatives in the \(\psi_1\) direction \(U^{(1,0)}(i, j)\) and \(U^{(2,0)}(i, j)\) in the same way as in \((1)\), that is, as in the bulk. The formulae for \(U^{(0,1)}(i, j)\) and \(U^{(0,2)}(i, j)\) are constructed with the five quantities \(\tilde{U}_t(i, j')\) for \(j' = N_p - 5, \ldots, N_p - 1\) and are exact at order \((\Delta \psi)^2\). The formula for \(U^{(1,1)}(i, N_p - 1)\) for \(0 \leq i \leq N_p - 2\) involves the six values \(\tilde{U}_t(i + 1, j'), \tilde{U}_t(i - 1, j')\) for \(j' = N_p - 3, N_p - 2, N_p - 1\) and is exact at order \((\Delta \psi)\). Finally, for \(U^{(1,1)}(N_p - 1, N_p - 1)\) we use twelve points in the region \(N_p - 4 \leq i \leq j \leq N_p - 1\) and the formula is exact at order \((\Delta \psi)^2\).

Notice that we have increased the precision of the derivatives on the boundary of the domain \(D\) corresponding to the large field region in order to test the robustness of our results with respect to the choice of discretization and to try to reduce numerical problems when \(d\) is close to 2. In all cases studied we did not find any significant changes. In particular, the scheme is not more stable when the number of points chosen to compute the derivatives is increased.

Once the derivatives are discretized, the fixed point equation \(\partial_t \tilde{U}^* (\psi_1, \psi_2) = 0\) becomes a set of coupled algebraic equations for \(g^*_{i,j} \equiv \tilde{U}(i, j)\). We look for a solution to these equations by a Newton’s-like method. One of the difficulty of this method is the huge number of unknowns and the possibility for Newton’s method to get lost in the very complicated landscape of extrema of the set of equations to be solved. The way out of this difficulty is to deform continuously a solution of the problem.

Our strategy in this paper is to follow the fixed point potential \(\tilde{U}^*\) by changing the dimension \(d\) and the number of spin components \(N\) gradually starting from \(d = 3.9\) and \(N = 22\) where the field-expansion method provides a good approximation of the fixed point potential. We use as an initial condition of Newton’s method:

\[
\tilde{U}^{*, \text{init}} (\tilde{\psi}_1, \tilde{\psi}_2) = \frac{\tilde{\lambda}^*}{2} (\tilde{\rho} - \tilde{\kappa}^*)^2 + \tilde{\mu}^* \tilde{\tau} \tag{21}
\]

and \(\eta = 0\). The parameters \(\tilde{\lambda}^*, \tilde{\kappa}^*\) and \(\tilde{\mu}^*\) are determined by performing a field-expansion of the LPA equation on \(\tilde{U}\) at order four in the fields and solving the fixed point equation for these parameters in \(d = 3.9\) and for \(N = 22\). As expected, we find four fixed points: the Gaussian and the \(O(2N)\) fixed points as well as a once-unstable fixed point \(C_+\) driving the phase transition and \(C_-\) that corresponds to a tricritical fixed point. Once an approximation of \(C_+\) is found with the truncation of Eq. \((21)\), we use it as the initial condition of Newton’s method for the full potential equation (supplemented by \(\eta\)) and we easily find \(\tilde{U}^*\). Then, we
move in the \((d, N)\) plane by little steps using as new initial condition what was found for
the previous value of \(d\) and/or \(N\) studied. The fixed potential potential deforms smoothly
and the Newton’s method always works properly this way.

B. The line \(N_c(d)\)

The line \(N_c(d)\) separates in the \((d, N)\) plane the region where the phase transition is of
second order and the region where it is of first order. When \(N\) is lowered at fixed \(d\), this line
corresponds to the locus of points where \(C_+\) disappears by collapsing with \(C_-\). There are
two possibilities to determine \(N_c(d)\). Either we decrease \(N\) at fixed \(d\) and look for the value
of \(N\) where \(C_+\) is no longer found and then repeat the same procedure by decreasing \(d\). Or
we compute the smallest eigenvalue of the flow around the fixed point \(C_+\) corresponding to
an irrelevant direction and look for the value of \(N\) where it vanishes. This eigenvalue is a
measure of the speed of the flow on the RG trajectory joining \(C_+\) and \(C_-\) and this speed
goes to 0 when the fixed points collapse. This second method is much more accurate and
less demanding than the first one and we therefore use it.

For each value of \((d, N)\) studied, we thus compute the eigenvalues of the stability matrix
\(\Theta (\{i, j\}, \{i', j'\})\) defined as

\[
\Theta (\{i, j\}, \{i', j'\}) = \left. \frac{\partial g_{\{i,j\}} (t)}{\partial g_{\{i',j'\}} (t)} \right|_{g_{i,j}^*}
\] (22)

where we consider \(\{i, j\}\) and \(\{i', j'\}\) as (super-)indices. Since the RG time \(t = \log k/\Lambda\) is
negative, a negative (positive) eigenvalue of the matrix \(\Theta\) corresponds to a relevant (irrel-
levant) eigendirection around the fixed point. We sort the eigenvalues as \(\sigma_0 (= -d) < \sigma_1 <
\cdots < \sigma_{i-1} < \sigma_i < \cdots\). Note that the above stability matrix around any fixed point solution
has a trivial relevant eigendirection corresponding to the constant shift \(g_{i,j} = g_{i,j}^* + \text{const}\)
with the eigenvalue \(\sigma_0 = -d\), which can be easily seen from Eq. \(A5\). Hereafter, this trivial
eigenvalue is omitted when we discuss the stability of a fixed point. The critical exponent \(\nu\)
is given by \(\nu = -1/\sigma_1\) and the smallest positive eigenvalue we are interested in is \(\sigma_2\).

C. Numerical instabilities

For each dimension \(d\) and value of \(N\) we have to make sure that our results are converged.
Once the choice of discretization of the derivatives has been made, there are two parameters
Table I: Several of the most relevant eigenvalues around the $C_+$ fixed point for $N = 5$ and $d = 3$. The minimum of the potential corresponds to $\tilde{\psi}_{\min} = 3.96$ and we have chosen $\tilde{\psi}_{\max} = 9$. The physical eigenvalues are given on the first line for each value of $N_p$ and the others, that are spurious, on the second line. For $N_p = 61$, the eigenvalues $-0.464 \pm 34.8i$ are relevant since their real part is negative. This eigenvalue disappears when increasing $N_p$.

| $N_p$  | Eigenvalues                                           |
|--------|-------------------------------------------------------|
| 61     | $-3, -1.45, 0.218, 0.827, 1.99, 2.79$                |
|        | $-0.464 \pm 34.8i, 0.250 \pm 30.9i, 1.07 \pm 27.6i$  |
| 81     | $-3, -1.45, 0.218, 0.827, 1.99, 2.79$                |
|        | $0.059 \pm 47.9i, 0.868 \pm 43.5i, 1.76 \pm 39.9i$  |
| 101    | $-3, -1.45, 0.218, 0.827, 1.99, 2.79$                |
|        | $0.704 \pm 61.05i, 1.627 \pm 56.3i$                 |

that can be tuned: the values of $\tilde{\psi}_{\max}$ and of the mesh size $\Delta \tilde{\psi} = \tilde{\psi}_{\max} / (N_p - 1)$. The potential $\tilde{U}^*$ shows a minimum at $\tilde{\psi}_1 = \tilde{\psi}_2 = \tilde{\psi}_{\min}$ and we have observed that $\tilde{\psi}_{\max}$ should be at least 1.5 times larger than $\tilde{\psi}_{\min}$ to get values of $N_c(d)$ converged with an accuracy of less than 1%. We have also observed that the smaller the dimension, the smaller $\Delta \tilde{\psi}$ must be to get converged results. This last point has two origins. First, at small $d$ the fixed point potential is steep at large fields because it behaves as $\left(\tilde{\psi}_1^2 + \tilde{\psi}_2^2\right)^{1/4}$ and a small mesh size is necessary to accurately describe the shape of $\tilde{U}^*$. Second, if $N_p$ is too small, we find that even far away from $d = 2$, say $d = 3$, several eigenvalues corresponding to relevant eigendirections appear in the spectrum and spoil the degree of stability of the fixed point $C_+$. These eigenvalues are clearly spurious because their values change considerably when either $\Delta \tilde{\psi}$ is decreased or $\tilde{\psi}_{\max}$ is increased whereas the complementary set of eigenvalues, the physical ones, remain unchanged up to the sixth digit, see Table I. We observe that as $\Delta \tilde{\psi}$ is decreased, these spurious eigenvalues systematically disappear (or, at least, get a very large real part which makes them highly irrelevant). The conclusion of this study is that for each $d$, a sufficiently large $N_p$ should be chosen so that the set of first most relevant eigenvalues is converged as for their numbers and values. We find that in $d = 3$, $N_p = 101$ is sufficient to get fully converged results while leading to numerically feasible calculations. We also find that as $d$ approaches 2, “large” values of $\tilde{\psi}_{\max}$ favor the presence of spurious
Figure 1: The curve $N_c(d)$. The crosses correspond to the calculation performed in this article either with the LPA or LPA’. The two continuous curves correspond to the five-loop results obtained within the $\epsilon$-expansion resummed either by assuming that $N_c(d = 2) = 2$ in resummation 2 or assuming nothing about the value of $N_c(d = 2) = 2$ in resummation 1.

eigenvalues that can only be eliminated by increasing $N_p$. It turns out that around $d = 2.4$, very large values of $N_p$, such as $N_p = 200$, would be necessary to avoid spurious eigenvalues and that decreasing $d$ would impose to increase $N_p$ in a prohibitive way. We have been able to compute $N_c(d)$ down to $d = 2.2$ by computing directly the value of $N$ where no fixed point $C_+$ is found with Newton’s method but we have not been able to go below this dimension.

VI. NUMERICAL RESULTS AND CONCLUSION

We have checked by varying all parameters ($\bar{\psi}_{\text{max}}$ and $\Delta \psi$) that our results are fully converged in $d = 3$ from a numerical viewpoint both at the LPA and LPA’ levels. They are also converged down to $d \approx 2.4$ and are less reliable in $d = 2.2$ at the LPA’ level although we are not able to give a quantitative estimate of the impact of our numerical errors on the value of $N_c(d)$ in this dimension. We show our determination of $N_c(d)$ in Fig. 1 together with the results obtained from the $\epsilon$-expansion at five loops.

For $d = 3$, our results confirm the previous results obtained either by the NPRG [1, 26–28, 32] or the $\epsilon$-expansion approaches [4–6]. The comparison between the LPA and LPA’ results
strongly suggests that neglecting the effect of the derivative terms on the determination of $N_c(d)$ plays a minor role in $d = 3$. Moreover, $N_c(d = 3)$ increases between the LPA and LPA’ and becomes closer to the results obtained with the $\epsilon$-expansion, which is expected. It seems therefore very difficult to imagine that $N_c(d = 3)$ could be smaller than 3.

Let us also emphasize that the only Monte Carlo simulation that still finds a second order transition for a value of $N$ below our value of $N_c(d = 3)$, that is, for $N \leq 4$, has been performed for $N = 2$ by Calabrese et al. [10] on a discretization of the Ginzburg-Landau model Eqs. (4), (6). They found that depending on the values of $\lambda$ and $\mu$, the transition is of first or second order: At fixed $\lambda$ and small $\mu$, the transition is of second order whereas it is of first order at large $\mu$. Since nonuniversal quantities, such as phase diagrams [42, 43], can be accurately computed from the integration of the NPRG flow equations, it is possible to estimate the magnitude of the correlation length $\xi_c$ at the transition within the LPA’ by initializing the flow with the data corresponding to the simulations. By varying these data as well as the cut-off function $R_k(q)$, it is found that $\xi_c$ is always finite (since there is no fixed point) but very large, typically larger than 2000 lattice spacings [44]. From a numerical point of view, there is no doubt that such a large correlation length makes impossible to decide in favor of a second or a (very weak) first order phase transition since in both cases the physics will look the same at the scale of the lattice size which was at most 120 lattice spacings in the numerical simulations. We conclude that this Monte Carlo result does not contradict our conclusion that $N_c(d = 3) \simeq 5$.

This result shows unambiguously that if our result is wrong, the origin of the problem can only be found by including the renormalization of the functions in front of the derivative terms. However, considering that the anomalous dimension is small for these systems when they undergo a second order phase transition, that is, for $N > N_c$, this hypothesis seems very doubtful. We therefore suggest that it is useless to study the order two of the derivative expansion in these models that, most probably, would bring only minor modifications as compared to the present study. We also suggest that only the Blaizot-Mendez-Wshebor approach [38, 40], where the full momentum dependence of the two-point functions is retained as well as the full field-dependence of the potential $\tilde{U}$ could lead to a very accurate determination of $N_c(3)$.

As for the approach to $d = 2$, we find a remarkable agreement between our results and what was found within the $\epsilon$-expansion. Two resummations of the $\epsilon$-expansion were
performed by the authors of [6], either by assuming that $N_c(d = 2) = 2$ or by letting free the value of $N_c(d = 2)$. This agreement is not very surprising because we expect the LPA' to be accurate around $d = 2$ for $N > 3$ (it is one-loop exact in the nonfrustrated case). Notice that our results are not precise enough to determine unambiguously the value of $N_c(d = 2)$ although it seems clear that it cannot be very different from 2. It is therefore very unlikely that $N_c(d = 2) > 3$ and our results show that the $C_+$ fixed point must exist for all dimensions larger than two in the Heisenberg case. Since the NPRG flow reproduces the low-temperature expansion of the nonlinear sigma model around $d = 2$, we conclude that the critical behavior of frustrated systems in $d = 2 + \epsilon$ is driven for $N = 3$ by the fixed point $C_+$ corresponding to a critical temperature of order $\epsilon$ in agreement with Mermin-Wagner theorem. Since we find no other once-unstable fixed point, we conclude that our study rules out the possibility of having a finite temperature fixed point in $d = 2$ for $N = 3$ contrary to what was found at five loops in a fixed dimension RG calculation [9].

To conclude, we have presented a rather simple method to compute the fixed point properties of matrix models describing frustrated systems without having recourse to a field expansion of the free energy $\Gamma$ (but keeping a derivative expansion of $\Gamma$). This is especially important in low dimensions where the field expansion is known to fail. In dimension $d = 3$, our results fully confirm what was previously found within less accurate NPRG calculations that involved field truncations on top of the derivative expansion [1, 28, 32]. In dimension $d = 2$, more stable numerical schemes are still needed to study the physics of topological excitations in frustrated systems (that are of different natures than in nonfrustrated systems) and we believe that the present work is the first step in this direction.

VII. ACKNOWLEDGMENT

This work was supported in part by a Grant-in-Aid for Young Scientists (B) (15K17737), Grants-in-Aid for Japan Society for Promotion of Science (JSPS) Fellows (Grants Nos. 241799 and 263111), the JSPS Core-to-Core Program "Non-equilibrium dynamics of soft matter and information".
Appendix A: The nonperturbative renormalization group flow equations and the anomalous dimension

Throughout this paper we employ the following $R_k(q^2)$, which is useful for analytical treatments [41]:

$$R_k(q^2) = Z_k(k^2 - q^2) \Theta(k^2 - q^2), \quad (A1)$$

where $Z_k$ is defined as

$$Z_k = \left( \frac{\partial}{\partial p^2} \left( \frac{\delta^2 \Gamma_k}{\delta \phi_1^3(p) \delta \phi_1^3(-p)} \right) \right)_{p=0, \text{min}}, \quad (A2)$$

where the field values are set to the minimum of $U_k$ given by Eq. (7). Here the Fourier transform $\phi_1^3(p)$ is defined as $\phi_1^3(p) = \int d^d x \phi_1^3(x) \exp(-i x \cdot q)$.

Then, the running anomalous dimension $\eta_k = -k \partial_k Z_k$ is given, at the level of LPA', by

$$\eta_k = 64 \tilde{\kappa} v_d \frac{\tilde{\rho}}{d} \left( \frac{1}{1 + 2 \tilde{U}_k^{(1,0)'}} \right)^2 \times \left( 2 \left( \frac{\tilde{U}_k^{(2,0)'}}{1 + 2 \tilde{U}_k^{(1,0)'}} + \frac{4 \tilde{U}^{(2,0)'}}{1 + 2 \tilde{U}_k^{(1,0)'}} + 4 \tilde{U}_k^{(1,0)'} \right) \right)^2 \frac{\tilde{\rho}}{\tilde{\tau}}, \quad (A3)$$

where we set $\tilde{\rho} = \tilde{\kappa}$ and $\tilde{\tau} = 0$. The derivatives $\tilde{U}_k^{(i,j)'}$ with respect to the invariants $\tilde{\rho}$ and $\tilde{\tau}$, and $v_d$ are defined as

$$\tilde{U}_k^{(i,j)'} = \frac{\partial^{i+j} \tilde{U}_k}{\partial \tilde{\rho}^i \partial \tilde{\tau}^j}, \quad v_d = \frac{1}{2^{d+1} \pi^{d/2} \Gamma \left( \frac{d}{2} \right)}.$$

The scaled nonperturbative renormalization group flow equation for the potential $\tilde{U}_k$ is
given by

\[
\partial_t \tilde{U}_k = -d\tilde{U}_k + \frac{1}{2}(-2 + d + \eta_k) \left( \tilde{\psi}_1 \tilde{U}_k^{(1,0)} + \tilde{\psi}_2 \tilde{U}_k^{(0,1)} \right)
+ \frac{4(2 + d - \eta_k)}{d(2 + d)} v_d
\times \left( \frac{\tilde{\psi}_1 - \tilde{\psi}_2}{\tilde{\psi}_1 - \tilde{\psi}_2 - \tilde{U}_k^{(1,0)} + \tilde{U}_k^{(0,1)}} + \frac{\tilde{\psi}_1 + \tilde{\psi}_2}{\tilde{\psi}_1 + \tilde{\psi}_2 + \tilde{U}_k^{(0,1)} + \tilde{U}_k^{(1,0)}} \right)
+ (N - 2) \left( \frac{\tilde{\psi}_2}{\tilde{\psi}_2 + \tilde{U}_k^{(0,1)}} + \frac{\tilde{\psi}_1}{\tilde{\psi}_1 + \tilde{U}_k^{(1,0)}} \right)
+ \frac{2 + \tilde{U}_k^{(0,2)} + \tilde{U}_k^{(0,2)}}{1 - \left( \tilde{U}_k^{(1,1)} \right)^2 + \tilde{U}_k^{(2,0)} + \tilde{U}_k^{(0,2)} \left( 1 + \tilde{U}_k^{(2,0)} \right)}.
\] (A4)

Here, to simplify the notation, we have defined another kind of derivatives \( \tilde{U}_k^{(i,j)} \) with respect to \( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \) as

\[
\tilde{U}_k^{(i,j)} \equiv \frac{\partial^{i+j} \tilde{U}_k}{\partial \tilde{\psi}_1^i \partial \tilde{\psi}_2^j}.
\] (A5)

In our calculations, we use the rescaled potential \( v_d^{-1} \tilde{U}_k \) and fields \( (v_d)^{-1/2} \tilde{\psi}_i \) for \( i = 1, 2 \) in such a way that \( v_d \) disappears in Eqs (A3) and (A4).

[1] B. Delamotte, D. Mouhanna, and M. Tissier, Phys.Rev. B 69, 134413 (2004).
[2] H. Kawamura, Journal of Physics: Condensed Matter 10, 4707 (1998).
[3] M. Yosefin and E. Domany, Phys. Rev. B 32, 1778 (1985).
[4] D. R. T. Jones, A. Love, and M. A. Moore. J. Phys. C 9, 743 (1976).
[5] D. Bailin, A. Love, and M. A. Moore. J. Phys. C 10, 1159 (1977).
[6] P. Calabrese and P. Parruccini, Nucl. Phys. B 679, 568 (2004).
[7] A. Pelissetto, P. Rossi, and E. Vicari, Phys. Rev. B 64, 140414 (2001).
[8] P. Calabrese, P. Parruccini, and A. I. Sokolov, Phys. Rev. B 66, 180403 (2002).
[9] P. Calabrese, E. V. Orlov, P. Parruccini, and A. I. Sokolov, Phys. Rev. B 67, 024413 (2003).
[10] P. Calabrese, P. Parruccini, A. Pelissetto, and E. Vicari, Phys. Rev. B 70, 174439 (2004).
[11] D. Loison and K. D. Schotte. Eur. Phys. J. B 5, 735 (1998).
[12] D. Loison and K. D. Schotte. Eur. Phys. J. B, 14, 125 (2000).
[13] M. Itakura, J. Phys. Soc. Jpn. 72, 74 (2003).
[14] V. Thanh Ngo and H. T. Diep, J. Appl. Phys. 103, 07C712 (2008).
[15] H. Kawamura and S. Miyashita, J. Phys. Soc. Jpn. 53, 4138 (1984).
[16] M. Wintel, H. U. Everts, and W. Apel, Europhys. Lett. 25, 711 (1994).
[17] W. Stephan and B. W. Southern, Phys. Rev. B 61, 11514 (2000).
[18] M. Caffarel, P. Azaria, B. Delamotte, and D. Mouhanna, Phys. Rev. B 64, 014412 (2001).
[19] P. Azaria, B. Delamotte, F. Delduc, and T. Jolicoeur, Nucl. Phys. B 408, 485 (1993).
[20] Yu. Nakayama and T. Ohtsuki, Phys. Rev. D 89, 126009, (2014).
[21] Yu. Nakayama and T. Ohtsuki, Phys. Rev. D 91, 021901, (2015).
[22] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi Phys. Rev. D 86, 025022, (2012).
[23] F. Kos, D. Poland, D. Simmons-Duffin, JHEP 11, 109, (2014).
[24] F. Kos, D. Poland, D. Simmons-Duffin, JHEP 06, 091, (2014).
[25] F. Kos, D. Poland, D. Simmons-Duffin, A. Vichi, JHEP 11, 106 (2016).
[26] G. Zumbach, Phys. Rev. Lett., 71, 2421 (1993).
[27] G. Zumbach, Nucl. Phys. B, 413, 771 (1994).
[28] M. Tissier, B. Delamotte, and D. Mouhanna, Phys. Rev. Lett. 84, 5208 (2000).
[29] M. Tissier, D. Mouhanna, and B. Delamotte, Phys. Rev. B 61, 15327 (2000).
[30] M. Tissier, B. Delamotte, and D. Mouhanna, Int. J. Mod. Phys. A 16, 2131 (2001).
[31] M. Tissier, B. Delamotte, and D. Mouhanna, Phys. Rev. B 67, 134422 (2003).
[32] B. Delamotte, M. Dudka, D. Mouhanna, S. Yabunaka, Phys. Rev. B 93, 064405 (2016).
[33] C. Wetterich, Nucl. Phys. B 352, 529 (1991).
[34] C. Wetterich, Phys. Lett. B 301, 90 (1993).
[35] T. R. Morris, Int. J. Mod. Phys. A 9, 2411 (1994).
[36] U. Ellwanger, Z. Phys. C 58, 619 (1993).
[37] J. Berges, N. Tetradis, and C. Wetterich. Phys. Rep. 363, 223 (2002).
[38] J.-P. Blaizot, R. Mendez-Galain, and N. Wschebor. Phys. Lett. B 632, 571 (2006).
[39] J.-P. Blaizot, R. Mendez-Galain, and N. Wschebor. Phys. Rev. E 74, 051116 (2006).
[40] J.-P. Blaizot, R. Mendez-Galain, and N. Wschebor. Phys. Rev. E 74, 051117 (2006).
[41] D. F. Litim, Nucl. Phys. B 631, 128 (2002).
[42] T. Machado, N. Dupuis, Phys. Rev. E 82, 041128 (2010).
[43] L. Canet, H. Chaté, B. Delamotte, Phys. Rev. Lett. 92, 255703 (2004).
[44] T. Debelhoir and N. Dupuis, private communication.