CONVERGENCE OF A SPECTRAL REGULARIZATION OF A TIME-REVERSED
REACTION-DIFFUSION PROBLEM WITH HIGH-ORDER SOBOLEV-GEVREY
SMOOTHNESS

VO ANH KHOA

Abstract. The present paper analyzes a spectral regularization of a time-reversed reaction-diffusion problem with globally and locally Lipschitz nonlinearities. This type of inverse and ill-posed problems arises in a variety of real-world applications concerning heat conduction and tumour source localization. In accordance with the weak solvability result for the forward problem, we focus on the inverse problem with high-order Sobolev-Gevrey smoothness and with Sobolev measurements. As expected from the well-known results for the linear case, we prove that this nonlinear spectral regularization possesses a logarithmic rate of convergence in a high-order Sobolev norm. The proof can be done by the verification of variational source condition; this way validates such a fine strategy in the framework of inverse problems for nonlinear partial differential equations. Ultimately, we study a semi-discrete version of the regularization method for a class of reaction-diffusion problems with non-degenerate nonlinearity. The convergence of this iterative scheme is also investigated.

1. Introduction

1.1. Statement of the inverse problem. Here we are interested in a time-reversed reaction-diffusion model, denoted by (B), with nonlinear source terms. Let $\Omega = [0, \ell]^d$ be a cube of $\mathbb{R}^d$ for $d \in \mathbb{N}$. In this context, we consider a population density $u = u(x,t)$ where $(x,t) \in Q_T := \Omega \times (0,T)$ with $T > 0$, obeying the following evolution equation:

$$u_t + Au = F(u) \quad \text{in} \; Q_T,$$

associated with the periodic boundary conditions, i.e. $u(x + e_i, t, \cdot) = u(x, \cdot)$ for $1 \leq i \leq d$ where $e_i$ denotes the standard basis vector for $\mathbb{R}^d$.

Here, $A := I - \Delta$ involves the linear second-order differential operator and thus accounts for the anisotropic diffusion of the population. The nonlinearity $F$ indicates either the deterministic reaction rate or the proliferation rate for some mechanism processes. Eventually, we complete the time-reversed model by the final condition

$$u(x, t = T) = g_T(x) \quad \text{in} \; \Omega.$$

Together with the periodic boundary condition, (1.1) and (1.2) structure our time-reversed reaction-diffusion model. In principle, the problem (B) is well known to be severely ill-posed and has been investigated in a wide range of real-world applications; see e.g. [14, 10, 5] and references cited therein for an overview of recent results and existing models. Taking into account tumour models (see e.g. [10]), the physical meaning behind this problem (B) is locating the tumour source

2000 Mathematics Subject Classification. 35R30, 65J20, 45L05, 65N15, 35B65.

Key words and phrases. Inverse reaction-diffusion problem, spectral regularization, variational source condition, error estimates, Sobolev-Gevrey smoothness, iterations.
by recovering the initial density of the tumour cells. In other words, the inverse problem we want to investigate in this paper is seeking the initial value \( u(x, t = 0) = g_0(x) \) in a regular tissue \( \Omega \), provided that (1.1) and (1.2) are satisfied.

Naturally, we impose here the standard measurement on the final data (1.2), which reads as
\[
\|g_T - g_T^f\|_{L^2(\Omega)} \leq \varepsilon,
\]
where \( \varepsilon > 0 \) represents the deterministic noise level. In practice, one attempts to get the initial function \( g_0(x) \) from this measured data \( g_T^f \) using some potential regularization.

Remark 1. As noteworthy examples for the nonlinearity \( F(u) \) described in Model (B), we take into account sigmoidal laws in population dynamics of cancer. The modest and simplest tumor growth is the logistic law, which can be generalized by the von Bertalanffy law with \( f(u) = au - bu^{N+1} \) where \( a, b \) and \( N \) are specific non-negative numbers depending on every model. We also know that if the growth rate \( N \) decays exponentially, the logistic growth turns out to be the so-called Gompertz law \( F(u) = au - bu \log u \) whenever \( u \) satisfies some additional information to avoid the singularity of the logarithmic form. In more complex scenarios (e.g. two-species models), one usually agrees with the de Pillis-Radunskaya law standing for the fractional kill rate of tumor-specific effector cells, which reads as \( F(u) = au^\sigma/(b+u^\sigma) \). This is, furthermore, analogous to the generalized Michaelis–Menten law in enzyme kinetics. Another example can also be the Frank-Kamenetskii model in combustion theory, governed by the Arrhenius law of the form \( F(u) = \exp(au) \). Accordingly, we see that investigating Model (B) can be helpful in many areas of science and engineering.

1.2. Organization of the paper. In the following, we denote by \( H^p_\# \) for \( p \in \mathbb{N} \) the Hilbert spaces equipped with the standard norms and with the periodic boundary conditions posed in the domain. For \( \sigma \geq 0 \), we define the Gevrey classes \( G^{p/2}_\sigma = D(A^{p/2}e^{\sigma A^{1/2}}) \) where the operator \( A \) is defined in (1.1). These are also Hilbert spaces with respect to the inner product
\[
\langle v, w \rangle_{G^{p/2}_\sigma} = \sum_{j \in \mathbb{Z}^d} v_j \cdot \bar{w}_j \left( 1 + |j|^2 \right)^p e^{2\sigma(1+|j|^2)^{1/2}},
\]
and then with the corresponding norm
\[
\|v\|_{G^{p/2}_\sigma} = \left( \sum_{j \in \mathbb{Z}^d} |v_j|^2 \left( 1 + |j|^2 \right)^p e^{2\sigma(1+|j|^2)^{1/2}} \right)^{1/2}.
\]

With this setting, it is worth mentioning the weak solvability of the forward problem of (B) where \( F \) is real analytic. Note that cf. [2], the real analyticity of \( F \) means that if it can be represented by \( F(u) = \sum_{j=0}^{\infty} a_j u^j \) for \( a_j \in \mathbb{R} \), then the corresponding majorising series \( \sum_{j=0}^{\infty} |a_j| u^j \) is convergent for any \( u \in \mathbb{R} \).

Theorem 2. [2] Theorem 1] Assume the initial data \( g_0 \in H^p_\#(\Omega) \) for \( p > d/2 \) and the nonlinearity \( F \) is real analytic. Then there exists a time \( T^* > 0 \) such that the forward model of Model (B) has a unique regular solution \( u \) in the sense that \( u \in C\left([0, T^*]; H^p_\#(\Omega)\right) \cap L^2(0, T^*; D(A)) \) satisfying \( u_t \in L^2\left(0, T^*; L^2(\Omega)\right) \) and
\[
\langle u_t, v \rangle_{(H^1(\Omega))^*, H^1(\Omega)} + \langle A^{1/2} u, A^{1/2} v \rangle_{L^2(\Omega)} + \langle F(u), v \rangle_{L^2(\Omega)} = 0,
\]
for all $v \in H^1_#(\Omega)$ and for a.e. $t \in (0, T^*)$. Furthermore, this regular solution satisfies $u(\cdot, t) \in G^{p/2}_t(\Omega)$ for $t \in [0, T^*)$.

As a by-product of Theorem 2, we can show that $u(\cdot, t) \in G^{p/2}_t(\Omega)$ for all $t \geq T^*$ and then $F(u) \in G^{p/2}_t(\Omega)$ by virtue of the Taylor series expansion of $F$. Therefore, in this work we assume our final time of observation $T$ such that $T < T^*$. Since $G^{p/2}_t(\Omega) \subset H^p(\Omega)$, it is reasonable to assume in (B) that

$$g_T, g^T \in H^p(\Omega).$$

This paper is devoted to the convergence analysis of a modified cut-off regularization of the inverse problem (B). In this regard, we comply with the weak solvability of the forward model to take into account the Gevrey-Sobolev source conditions for the inverse problem. In other words, together with Assumption (3.4) we make use of the following source condition:

$$u \in C\left([0, T^*]; H^p_#(\Omega)\right) \cap L^2\left((0, T); D(A)\right) \text{ and } u(\cdot, t) \in G^{p/2}_t(\Omega) \text{ for } t \in [0, T),$$

It is worth mentioning that in the forward process, we obtain the solution in $C\left([0, T^*]; H^p_#(\Omega)\right)$ with $p > d/2$, which indicates the fact that the solution belongs to $C\left([0, T]; L^\infty(\Omega)\right)$. Instead of working with the real analyticity of $F$, we can further apply the mean value theorem to get

$$|F(u) - F(v)| \leq \left(\sup_{|w| \leq M} \left|\frac{\partial F}{\partial w}(w)\right|\right) |u - v|,$$

where for $u, v \in C\left([0, T]; L^\infty(\Omega)\right)$ we have denoted by

$$M = 2 \max \left\{\|u\|_{C([0,T];L^\infty(\Omega))}, \|v\|_{C([0,T];L^\infty(\Omega))}\right\} > 0.$$

To this end, we will thus exploit Assumption (1.6) through the analysis of Section 2. We also assume that there exists a continuous function $L(M) > 0$ such that

$$\sup_{|w| \leq M} \left|\frac{\partial F}{\partial w}(w)\right| \leq L(M).$$

Naturally, observing the applications mentioned in Remark 1 we have

- for the von Bertalanffy law: $L(M) = |a| + |b| (N + 1) M^N$;
- for the Gompertz law: $L(M) = |a| + |b| M$;
- for the de Pillis-Radunskaya law: $L(M) = |a| M^{N-1}$ for $b \neq 0$;
- for the Arrhenius law: $L(M) = |a|$ if $a < 0$ and $L(M) = ae^a M$ if $a \geq 0$.

In this work, all the constants $C$ used here are independent of the measurement error $\varepsilon$. Nonetheless, their precise values may change from line to line and even within a single chain of estimates. On the other side, we use either the superscript or the subscript $\varepsilon$ to accentuate the possible dependence of the present error on the constants.

In the first step, we derive the error bounds when $L(M) = C$ is independent of $M$. Our proof relies on the way we verify a variational source condition. This interestingly contributes to one of their first applications to the convergence analysis of regularization of inverse problems for nonlinear PDEs. In this approach, we prove the usual logarithmic-type rate of convergence of the nonlinear spectral regularization. We remark that during the time evolving backward in the open set $(0, T)$, the nonlinear scheme yields the asymptotic Hölder rate. When $L(M)$ essentially depends on $M$, the convergence is slower due to the phenomenal growth of the quantity $L(M)$ involved in the
Lipschitz property (1.3). Technically, the impediment of this growth, albeit its high-impact on the structure of the variational source condition, can be solved using a careful choice of a cut-off function for the nonlinearity $F$. In this sense, the regularized solution is sought in a proper open set decided by the measurement parameter $\varepsilon$. Eventually, a slower logarithmic-type convergence is also expected, which extends the results in [8] [17] [15] and references cited therein. Typically, this extension illuminates that the Gevrey regularity conventionally restricted to the convergence analysis in [17] [15] is applicable when solving a class of time-reversed PDEs. We also add that this work partly completes the gap of verifying variational source conditions for a class of nonlinear PDEs, which is still questioned in [8].

In the second theme, our concentration moves to a derivation of an iteration-based version of the nonlinear spectral scheme with a stronger version of (1.6). We end up with the convergence of the approximate scheme by exploring the choice of the so-called stabilization constant, which depends not only on the number of iterations, but also on the measurement $\varepsilon$.

Prior to the setting of our approach and to closing this section, we introduce what the variational source condition concerns in principle as it is exploited in the skeleton of our proof.

1.3. Background of the variational source condition. Consider the ill-posed operator equation $T(f) = g$ where $T$ maps from $D(T) \subset X$ to $Y$ with $X$ and $Y$ being Hilbert spaces. In this Hilbert setting, we denote by $f^1 \in D(T)$ the exact solution and by $g^\varepsilon \in Y$ the noisy data satisfying $\|T(f^1) - g^\varepsilon\|_Y \leq \varepsilon$, where $\varepsilon > 0$ represents the deterministic noise level (cf. e.g (1.3)). There are several stable approximations to regularize such equations. One of the most effective methods is Tikhonov regularization in which the exact solution is approximated by a solution of the minimization problem, viz.

$$f_\alpha^\varepsilon \in \arg\min_{f \in D(T)} \left\{ \|T(f) - g^\varepsilon\|_Y^2 + \alpha \|f\|_X^2 \right\} =: \mathcal{R}_\alpha(g^\varepsilon),$$

for some regularization parameter $\alpha := \alpha(\varepsilon) > 0$.

In regularization theory, one can prove that $\mathcal{R}_\alpha(g^\varepsilon) \xrightarrow{\alpha \searrow 0^+} T^{-1}$ for suitable choices of $\alpha$ in some adequate topology. Furthermore, people usually want to specify rates of convergence, which turns out to be the question of finding the worst case error, since such rates are arbitrarily slow in general. Denote by $\psi : [0, \infty) \to [0, \infty)$ an index function if it is continuous and monotonically increasing with $\psi(0) = 0$. In this sense, one aims to:

Find (usually compact) subspaces $K \subset X$ such that there exists an index function $\psi$ satisfying for all $f \in K$, it holds

$$\sup \left\{ \|\mathcal{R}_\alpha(g^\varepsilon) - f\|_X : \|T(f) - g^\varepsilon\|_Y \leq \varepsilon \right\} \leq \psi(\varepsilon).$$

Nevertheless, we often need additional a priori assumptions on the exact solution $f^1$ to gain the speed of convergence, which are called source conditions. In the literature, the former type of such conditions singles out that $f^1 = \psi \left( T^* \left[ f^1 \right] * T^* \left[ f^1 \right] \right) w$ for some $w \in X$, renowned as the spectral source condition in the community of inverse and ill-posed problems for several years; see e.g. [11] for a prevailing background of source conditions. Here $\psi$ could be of the Hölder and logarithmic types, depending on every situation. More recently, a novel formulation for source conditions has been derived in the sense of variational inequality, which reads as

$$2 \langle f^1, f^1 - f \rangle_X \leq \frac{1}{2} \|f - f^1\|_X^2 + \psi \left( \|T(f) - T(f^1)\|^2_Y \right) \text{ for all } f \in D(T) \subset X. \quad (1.9)$$

Cf. [8], it is in particular called the variational source condition. Essentially, we can benefit from this source condition not only to simplify proofs for convergence rates in Hilbert frameworks, but also to extensively work in some Banach setting and in more general models with distinctive types, depending on every situation. More recently, a novel formulation for source conditions has been derived in the sense of variational inequality, which reads as

$$2 \langle f^1, f^1 - f \rangle_X \leq \frac{1}{2} \|f - f^1\|_X^2 + \psi \left( \|T(f) - T(f^1)\|^2_Y \right) \text{ for all } f \in D(T) \subset X. \quad (1.9)$$

Cf. [8], it is in particular called the variational source condition. Essentially, we can benefit from this source condition not only to simplify proofs for convergence rates in Hilbert frameworks, but also to extensively work in some Banach setting and in more general models with distinctive
measurements (cf. e.g. [9, 4, 12]). Compared to the spectral source condition, it does not require the \( \text{Fréchet derivative} \) \( T' \) and would thus be helpful in certain applications where the forward operator is not sufficiently smooth. Most importantly, variational source conditions have been shown to be well-adapted to bounded linear operators in Hilbert spaces; see [3]. On top of that, the variational source condition combined with some nonlinearity condition has been formulated in e.g. [16] to treat nonlinear operators.

Even though there have been enormous advantages over the spectral source conditions, variational source conditions could barely be verified in particular models in terms of partial differential equations. So far, with the aid of complex geometrical optics solutions the variational source conditions could barely be verified in particular models in terms of partial differential equations. Even though there have been enormous advantages over the spectral source conditions, variational source conditions could barely be verified in particular models in terms of partial differential equations. Even though there have been enormous advantages over the spectral source conditions,.

In line with the characterizations of variational source conditions hold true for all \( r \in J \):

\[
\begin{align*}
(C_1) \quad & \| f^t - P_r f^t \|_X \leq \kappa (r), \\
(C_2) \quad & \langle f^t, P_r (f^t - f) \rangle_X \leq \sigma (r) \| T (f^t) - T (f) \|_X + c \kappa (r) \| f^t - f \|_X \quad \text{for any} \ f \in \mathcal{D} (T) \text{ with} \\
& \| f - f^t \|_X \leq 4 \| f^t \|_X.
\end{align*}
\]

Then \( f^t \) satisfies the variational source condition (1.9) with

\[
\psi (\delta) := 2 \inf_{r \in J} \left[ (c + 1)^2 |\kappa (r)|^2 + \sigma (r) \sqrt{\delta} \right],
\]

and \( \psi \) is a concave index function if \( \inf_{r \in J} \kappa (r) = 0 \).

2. Spectral cut-off projection

2.1. Settings of the cut-off approach. In line with the characterizations of variational source conditions, in this section we verify and improve the cut-off projection that has been postulated in [15] [18], using Assumptions (1.3), (1.4), (1.5), (1.6), (1.7). The nature of this approach is that one solves the problem in a finite dimensional subspace, which turns out to be a well-posed problem effectively controlled by the measurement error.

We denote by \( \{ E_\lambda \} \) the spectral family of the positive operator \( A \) defined in the backward problem (B) and the function \( \lambda \to \| E_\lambda v \| \) is called the \textit{spectral distribution function} of \( v \in L^2 (\Omega) \).

Thereby, with \( C_\varepsilon > 0 \) being a cut-off parameter that will be chosen appropriately, we introduce the \textit{spectral cut-off projection} \( E_{\Lambda_\varepsilon} := 1_{(0, \Lambda_\varepsilon)} (A) \) where \( 1_{(0, \Lambda_\varepsilon)} \) denotes the characteristic function of the interval \( (0, \Lambda_\varepsilon] \) with \( [\Lambda_\varepsilon] = C_\varepsilon \). To establish the stable approximate problem of (B), we adapt the cut-off projection to both the nonlinearity \( F \) and the final data \( g^T \). When doing so, we characterize the abstract Gevrey class \( G_{\sigma / 2}^p \) for \( \sigma \geq 0 \) and \( p \in \mathbb{N} \) as

\[
G_{\sigma / 2}^p = \left\{ v \in L^2 (\Omega) : \int_0^\infty \lambda^p e^{2\sigma \lambda} d \| E_\lambda v \|^2 < \infty \right\},
\]
equipped with the norm
\[ \|v\|_{G^p/2}^2 = \int_0^\infty \lambda^{p/2} d\|E\lambda v\|^2 < \infty. \]

Henceforward, for each \( \varepsilon > 0 \) we take into account the following approximate problem, denoted by \((\mathcal{B}_\varepsilon)\),
\[
\begin{cases}
u_t + Au = \int_0^\infty \lambda^{p/2} dE\lambda F (u_t) & \text{ in } Q_T, \\
u (T) = \int_0^\infty \lambda^{p/2} dE\lambda g_T & \text{ in } \Omega.
\end{cases}
\]

We define a mildly weak solution of \((\mathcal{B}_\varepsilon)\) (or a mild solution for short) to be a continuous mapping \(u_\varepsilon : [0, T] \to H^p (\Omega)\) obeying the integral equation
\[
(2.1) \quad u_\varepsilon (t) = \int_0^\infty e^{(T-t)\lambda}\lambda^{p/2} dE\lambda g_T - \int_t^T \int_0^\infty e^{(s-t)\lambda}\lambda^{p/2} dE\lambda F (u_\varepsilon) (s) \, ds.
\]
Observe that in (2.1) we actually obtain an integral equation of the form \(u_\varepsilon (t) = \mathcal{G} (u_\varepsilon) (t)\) where \(\mathcal{G}\) is completely formulated by the right-hand side, mapping from \(C ([0, T] : H^p (\Omega))\) onto itself. Therefore, the existence and uniqueness results for (2.1) can be done by the standard fixed-point argument, requiring that the number of fixed-point iterations must be larger than the stability magnitude usually involved in the approximate problem. Since these results are standard, we, for simplicity, refer the reader to the concrete reference [17] for the detailed notion of proof.

Observe again from (2.1) that we can define the cut-off operator \(\mathcal{P}_\varepsilon \in L (H^p (\Omega))\) for the solution \(u (\cdot, t_0)\) by fixing \(t = t_0\). In particular, it is an orthogonal projection given by
\[
(2.2) \quad \mathcal{P}_\varepsilon u (\cdot, t_0) = \int_0^\infty e^{(T-t_0)\lambda}\lambda^{p/2} dE\lambda g_T - \int_0^T \int_0^\infty e^{(s-t_0)\lambda}\lambda^{p/2} dE\lambda F (u) (s) \, ds.
\]
We also remark that in the same spirit of (2.1) the nonlinear ill-posed operator \(\mathcal{T}\) (see Subsection 1.3) can be written in the following closed form:
\[
(2.3) \quad g_T = \mathcal{T} g_0 := \int_0^\infty e^{-T\lambda}\lambda^{p/2} dE\lambda g_0 + \int_0^T \int_0^\infty e^{(s-T)\lambda}\lambda^{p/2} dE\lambda F (u) (s) \, ds,
\]
in which we define that \(\mathcal{T} : \mathcal{D} (\mathcal{T}) \to H^p_\# (\Omega)\) with \(\mathcal{D} (\mathcal{T}) = H^p (\Omega)\).

2.2. **Verification of the cut-off approach: high-order Sobolev-Gevrey smoothness.** From now on, we aim to verify a modified version of the spectral cut-off regularization \(\mathcal{P}_\varepsilon\) in a high-order smoothness setting. As delved into the nonlinear case in [15], if we impose the Gevrey smoothness on \(u (\cdot, t)\) for all \(t \in [0, T]\), the verification of this approach is straightforward. We thereupon obtain the convergence for \(t \geq 0\); compared to the standard cut-off method which yields the convergence only for \(t > 0\) in the nonlinear case. This regularity assumption is very strong and it does not seem applicable as discussed in the analysis of the forward model (cf. Theorem 2), saying that the Gevrey smoothness can only be available for \(t > 0\). It turns out that we need some modification of this cut-off projection in this paper. On the whole, we use the following scheme:

- Since \(u (\cdot, t)\) satisfies the Gevrey smoothness \(G^{p/2}_t (\Omega)\) for \(t > 0\), we keep solving the backward problem \(\mathcal{T} u (t_0) = g_T\) for \(t_0 > 0\) by the projection \(\mathcal{P}_\varepsilon\) constructed from (2.1).
- Assume that \(u_t (\cdot, t) \in H^p (\Omega)\). We find \(t_\varepsilon \in (0, T)\) such that \(u (t_\varepsilon)\) is an approximation of \(g_0\) in \(H^p (\Omega)\).

The central point of this modification is that this time we use the Sobolev smoothness on \(u_t\) in the neighborhood of \(t = 0\) to avoid the Gevrey smoothness on \(g_0\). This assumption is consistent with the fact that \(u_t \in L^2 (0, T; L^2 (\Omega))\) obtained in Theorem 2 and thus attainable by the inclusion...
SPECTRAL REGULARIZATION OF A TIME-REVERSED REACTION-DIFFUSION PROBLEM

7

C(0, T; H^p(Ω)) ⊂ L^2(0, T; L^2(Ω)). Note that the Sobolev smoothness imposed on both g_0 and u(·, t) in H^p(Ω) with p > d/2 is essential for the presence of the nonlinearity F.

2.2.1. Case 1: L(M) = C independent of M.

Theorem 3. Let p > d/2. Assume that (1.8), (1.9), (1.10), (1.11), (1.12) hold. Then for t ∈ (0, T) the variational source condition (1.9) holds true for the operator (1.3) with

\[ u \in C^1(0, T) \]

Proof. Here we provide proof of the solution smoothness (C_1) (cf. Subsection 1.3 whenever u(·, t) ∈ C^{p/2}_t for p > d/2 and t ∈ (0, T). In this sense, we have

\[ \|u(·, t) - \mathcal{P}_e u(·, t)\|^2_{H^2(Ω)} = \int_0^∞ \lambda_κ^2 d\|E_λ u(·, t)\|^2 \]

\[ \leq e^{-2C_κ t} \int_0^∞ \lambda_κ^2 e^{2C_κ t} d\|E_λ u(·, t)\|^2 \leq e^{-2C_κ t} \|u(·, t)\|^2_{C^{p/2}(Ω)}, \]

which implies that \( \kappa_t(\mathcal{E}_e) \) decays with the rate \( e^{-C_κ t} \).

Concerning the degree of ill-posedness (C_2), we employ the equivalent formulation of the variational source condition (1.9):

\[ \|P_r(f^1 - f)\|_{\mathcal{X}} \leq \sigma(r) \|T(f^1) - T(f)\|_{\mathcal{X}} \]

for any \( f \in \mathcal{D}(T) \subset \mathcal{X} \).

Notice from (1.3) that the data measurement is not as smooth as the regularity assumption we imposed. Henceforward, for any given \( \bar{u} \in C([0, T]; H^p(Ω)) \) \( \{0\} \) it follows from (2.1) that

\[ \|P_e(u(·, t) - \bar{u}(·, t))\|^2_{H^2(Ω)} \leq 2C_κ \int_0^∞ e^{2(T-t)C_κ} d\|E_λ (g_T^e - \bar{g}_T^e)\|^2 \]

\[ + 2 \int_t^T \int_0^∞ e^{2(s-t)C_κ} \lambda_κ^2 d\|E_λ F(u - \bar{u})(s)\|^2 ds, \]

where we have denoted by \( \bar{g}_T^e \) the corresponding final data of \( \bar{u} \).

In (2.6), applying the globally Lipschitz continuity of \( F \) and then multiplying the resulting estimate by the exponential weight \( e^{2TC_κ} \) we find that

\[ e^{2TC_κ} \|P_e(u(·, t) - \bar{u}(·, t))\|^2_{H^2(Ω)} \leq 2C_κ e^{2TC_κ} \|g_T^e - \bar{g}_T^e\|^2_{L^2(Ω)} \]

\[ + 2L^2 \int_t^T e^{2sC_κ} \|P_e(u(·, s) - \bar{u}(·, s))\|^2_{H^2(Ω)} ds. \]

With the aid of the Gronwall inequality, we get

\[ \|P_e(u(·, t) - \bar{u}(·, t))\|^2_{H^2(Ω)} \leq 2C_κ e^{2(T-t)(C_κ + L^2)} \|g_T^e - \bar{g}_T^e\|^2_{L^2(Ω)}, \]
yielding $c = 0$ and that $\sigma (C_\varepsilon)$ increases with the rate $C_\varepsilon e^{(T-t)}(C_\varepsilon + L^2)$ in $(C_2)$.

Therefore, we conclude that the solution $u (\cdot, t)$ satisfies the variational source condition (1.9) with

$$\psi_t (\delta) := 2 \inf_{C_\varepsilon > 0} \left[ ||u (\cdot, t)||_{G_\varepsilon}^2 e^{-2C_\varepsilon t} + C_\varepsilon e^{(T-t)}(C_\varepsilon + L^2)^2 \right],$$

then these two terms in the infimum are equal for $\delta = C_\varepsilon e^{-2(T-t)}C_\varepsilon e^{2(T-t)L^2} ||u (\cdot, t)||_{G_\varepsilon}^2$. With this, we take the logarithm on both sides and use the elementary inequality that $\log (a) > 1 - a^{-1}$ for any $a > 0$ to rule out the choice of $C_\varepsilon$. Thus, we can find some $C > 0$ independent of $\delta$ such that

$$C_\varepsilon = \frac{C}{T + t + \frac{2}{\varepsilon}} \log \frac{e^{\frac{2}{\varepsilon} + (T-t)L^2} ||u (\cdot, t)||_{G_\varepsilon}}{\delta^{1/2}},$$

and plugging this into (2.7), we obtain the index function (2.4). On top of that, cf. [4] the error bound (2.5) itself holds for $t \in (0, T)$.

As a byproduct of Theorem 5 and in accordance with the argument in [4], the minimizers of the Tikhonov functional (1.8) satisfy the convergence rate (2.5) if $\alpha$ can be chosen such that $-\frac{1}{2\alpha} \in \partial (\psi)$ ($4c^2$), where $\partial (\psi)$ denotes the subdifferential of $-\psi$. By this means, we compute

$$\frac{2(t+2)}{\alpha},$$

indicating that $\alpha \searrow 0^+$ as the measurement error tends to 0. In addition, we obtain the following conditional stability estimate.

**Corollary 4.** For $t \in (0, T)$, let $\tilde{u} (\cdot, t)$ and $\tilde{u} (\cdot, t)$ be two solutions obtained from the integral equation (2.7). Under the assumptions of Theorem 5 the following estimate holds

$$||\tilde{u} (\cdot, t) - \tilde{u} (\cdot, t)||_{H^p (\Omega)} \leq C \left( \frac{||\bar{g}_T - \bar{g}_T||_{L^2 (\Omega)} e^{p+2(t+2)L^2} ||u (\cdot, t)||_{G_\varepsilon}^2}{\delta^{1/2}} \right),$$

where $\bar{g}_T$ and $\bar{g}_T$ are final conditions corresponding to $\tilde{u} (\cdot, t)$ and $\tilde{u} (\cdot, t)$, respectively.

Our modified scheme now brings into play its own feature: pointing out an approximation candidate of our solution $g_0$. In fact, it is clear to see that (2.5) is not convergent when $t = 0$. Our scheme to approximate the initial density $g_0$ is very simple as we merely need to compute the “data” $u (\cdot, t_\varepsilon)$ that have been solved through the standard spectral scheme $P_\varepsilon$. In the following, we not only show the existence of such $t_\varepsilon$, but also obtain a rigorous $\varepsilon$-dependent admissible set that contains it, directly proving the fact that $t_\varepsilon \searrow 0^+$ as $\varepsilon \searrow 0^+$. Mathematically, one can establish an orthogonal projection and then mimic the way we gain Theorem 5 to deduce the rate of convergence. Since, again, this step is very trivial and somewhat self-contained by the help of the triangle inequality, our proof below follows the conventional way. Thus, the expression of the index function is clear through the derivation of the convergence rate.

**Theorem 5.** Let $p > d/2$. Suppose that $u_t (\cdot, t) \in H^p (\Omega)$ for $t \in (0, T)$. Let $u_0 (\cdot, t)$ be the unique solution obtained from the projection $P_\varepsilon \in \mathcal{L} (H^p (\Omega))$ in (2.7). Then there always exists a sufficiently small $\varepsilon$-dependent time $t_\varepsilon > 0$ such that

$$||u_0 (t_\varepsilon) - u (0)||_{H^p (\Omega)} \leq \frac{C (T + \frac{2}{\varepsilon})}{\sqrt{(T + \frac{2}{\varepsilon} + 1)^2 + 4 (T + \frac{2}{\varepsilon}) \log (\varepsilon^{-1}) + T + \frac{2}{\varepsilon} - 1}}.$$
Proof. Using \((2.5)\) and the triangle inequality, for any \(t_\varepsilon \in (0, T)\), we find the following estimate :

\[
\|u_\varepsilon (t_\varepsilon) - u (0)\|_{H^p (\Omega)} \leq \|u_\varepsilon (t_\varepsilon) - u (t_\varepsilon)\|_{H^p (\Omega)} + \|u (t_\varepsilon) - u (0)\|_{H^p (\Omega)}
\]

\[
\leq C \varepsilon^{\mu+\frac{t_\varepsilon}{2}} + t_\varepsilon \|u|\|_{H^p (\Omega)}.
\]

This upper bound can be done if we can find the infimum \(\frac{1}{2} \inf_{t_\varepsilon > 0} \left( \varepsilon^{\frac{t_\varepsilon}{t_\varepsilon+t_\varepsilon+T} + t_\varepsilon} \right)\) for some \(t_\varepsilon \in (0, T)\). This indicates that we need to solve the following algebraic problem:

\[
\varepsilon^{\frac{t_\varepsilon}{t_\varepsilon+t_\varepsilon+T} + t_\varepsilon} = t_\varepsilon,
\]

expecting that \(t_\varepsilon > 0\) is sufficiently small. Taking the logarithm on both sides of this equation and using the standard inequality \(\log (a) > 1 - a^{-1}\) for any \(a > 0\), we have the following inequality:

\[
t_\varepsilon^2 (\log \varepsilon - 1) - \left( T + \frac{p}{2} - 1 \right) t_\varepsilon + T + \frac{p}{2} > 0.
\]

Due to the fact that \([T + \frac{p}{2} - 1]^2 + 4 (T + \frac{p}{2}) (1 - \log \varepsilon) > 0\) and \(\log \varepsilon < 1\), we deduce that

\[
t_\varepsilon \in \left( \frac{-b - \sqrt{b^2 + 4 (b + 1) (1 - \log \varepsilon)}}{2 (\log \varepsilon - 1)}, \frac{-b + \sqrt{b^2 + 4 (b + 1) (1 - \log \varepsilon)}}{2 (\log \varepsilon - 1)} \right),
\]

where we have denoted by \(b := T + \frac{p}{2} - 1\). Notice that taking \(\varepsilon \searrow 0^+\), the rationalizing technique gives the following limit:

\[
\lim_{\varepsilon \to 0^+} \left( \frac{-b + \sqrt{b^2 + 4 (b + 1) (1 - \log \varepsilon)}}{2 (\log \varepsilon - 1)} \right) = \lim_{\varepsilon \to 0^+} \frac{b^2 - b^2 - 4 (b + 1) (1 + \log (\varepsilon^{-1}))}{2 (1 + \log (\varepsilon^{-1})) (b - \sqrt{b^2 + 4 (b + 1) (1 + \log (\varepsilon^{-1}))})} = \lim_{\varepsilon \to 0^+} \frac{2 (b + 1)}{b - \sqrt{b^2 + 4 (b + 1) (1 + \log (\varepsilon^{-1}))}} = 0,
\]

and similarly,

\[
\lim_{\varepsilon \to 0^+} \left( \frac{-b - \sqrt{b^2 + 4 (b + 1) \log (\varepsilon^{-1})}}{2 (\log \varepsilon - 1)} \right) = \lim_{\varepsilon \to 0^+} \frac{2 (b + 1)}{b + \sqrt{b^2 + 4 (b + 1) (1 + \log (\varepsilon^{-1}))}} = 0.
\]

Hence, it follows from \((2.9)\) that the upper bound we gain would be of the form in \((2.8)\). \(\Box\)

2.2.2. Case 2: \(L (M)\) dependent of \(M\). When \(L (M)\) depends on \(M\), our nonlinear spectral regularization \(P_\varepsilon\) may be no longer convergent as the boundedness of the regularized solution is not well-controlled; cf. \((2.4)\). This means that the quantity \(M\) now has to be dependent of \(\varepsilon\), saying that \(M = M_\varepsilon\), when getting involved in the scheme \(P_\varepsilon\). This \(\varepsilon\) dependence impacts on the structure of the index function and further on the whole rate of convergence in Theorem 3.

From now on, for some constant \(\ell > 0\) and for \(w \in \mathbb{R}\) we introduce the cut-off function of \(F\), denoted by \(F_\ell\), as follows:

\[
F_\ell (w) := \begin{cases} F (\ell) & \text{if } w \geq \ell, \\ F (w) & \text{if } |w| \leq \ell, \\ F (-\ell) & \text{if } w \leq -\ell. \end{cases}
\]
Following this way, we modify the regularization scheme (2.2) by

\[
\bar{P}_\varepsilon u (\cdot, t_0) = \int_0^\infty e^{T\lambda_\varepsilon} \lambda_\varepsilon^{p/2} dE_\lambda \mathcal{G}_T - \int_{t_0}^T \int_0^\infty e^{s\lambda_\varepsilon} \lambda_\varepsilon^{p/2} dE_\lambda F_M (u) (s) ds.
\]

and therefore, it enables us to derive the convergence rate under a suitable choice of $M_\varepsilon$. Note that our cut-off function $F_M$ (2.10) possesses the similar property in (1.6), i.e.

\[
|F_M (u) - F_M (v)| \leq L (M_\varepsilon) |u - v|.
\]

**Theorem 6.** Under the assumptions of Theorem 3, we choose $M_\varepsilon > 0$ such that for $t \in (0, T)$

\[
L^2 (M_\varepsilon) \leq \frac{1}{2 (t - T)} \log (\varepsilon^\beta) \quad \text{for } \beta \in \left(0, \frac{1}{2}\right).
\]

Then the variational source condition (1.9) holds true for the operator (2.3) with

\[
\psi_t (\delta) := C \left( \frac{\delta^{1-2\beta}}{e^p \|u (\cdot, t)\|_{G_\varepsilon}^{p/2}} \right)^{\frac{t}{T+t+\frac{p}{2}}}
\]

Consequently, choosing the cut-off parameter

\[
C_\varepsilon = \frac{C}{T + t + \frac{p}{2}} \log \frac{e^p \|u (\cdot, t)\|_{G_\varepsilon}^{p/2}}{\varepsilon^{1/2 - \beta}},
\]

the orthogonal projection $\bar{P}_\varepsilon \in \mathcal{L} (H^p (\Omega))$ defined in (2.11) is convergent with the rate

\[
\|u_\varepsilon (\cdot, t) - u (\cdot, t)\|_{H^p (\Omega)} \leq C_\varepsilon (1 - 2\beta)^{1/4 (t+\varepsilon^\beta)}.
\]

**Proof.** Proof of this theorem is straightforward. In fact, starting from the argument (2.7) we know in this case that $\delta = C_\varepsilon e^{2(\varepsilon^\beta) - 2(t + t + \frac{p}{2}) L^2 (M_\varepsilon)} \|u (\cdot, t)\|_{G_\varepsilon}^{p/2}$. Thus, the choice of $M_\varepsilon$ in (2.12) is apparent. This then results in the form (2.13) of the index function. Hence, the proof is complete. \(\square\)

**Remark 7.** Based on the result obtained in Theorem 6, we remark the following:

- As is well-known in continuous population models for single species, we suppose the non-linearity in the form of

\[
F (u) = u (1 - u) - \frac{u^2}{1 + u^2},
\]

as a prominent example of the growth-and-predation rate for the spruce budworm which critically defoliated the balsam fir in Canada (cf. [14]). In this circumstance, it is easy to get $L (M) = 1 + 4M > 0$ and hence, we can choose that

\[
M_\varepsilon \leq \frac{1}{4} \left( \sqrt{\frac{1}{2 (t - T)}} \log (\varepsilon^\beta) - 1 \right),
\]

working with the concrete assumption $\varepsilon < e^{2(t-T)/3}$. 

This result can be further extended to multiple-species cases. Indeed, we consider a time evolution of the concentration of \( L \in \mathbb{N} \) chemical components or constituents (molecules, radicals, ions) in a reaction network (cf. e.g. [1]) which reads as

\[
\sum_{l=1}^{L} \alpha(l,r) X_l \rightarrow \sum_{l=1}^{L} \eta(l,r) X_l \quad \text{for} \quad r = \overline{1,R},
\]

where \( \alpha(l,r) \geq 0 \) and \( \eta(l,r) \geq 0 \) are the stoichiometric coefficients or molecularities. In a constructive manner, the mass action kinetic deterministic model of this reaction is governed by the following system of PDEs:

\[
\partial_t c_l = \Delta c_l + \sum_{r=1}^{R} (\eta(l,r) - \alpha(l,r)) \prod_{l=1}^{L} c_l^{\alpha(l,r)} \quad \text{for} \quad l = \overline{1,L},
\]

where \( c_l \) is viewed as the concentration of the \( l \)-th component at time \( t \in [0,T] \). In this regard, we obtain the vector-valued reaction-diffusion equation (2.11) in the form of \( \mathbf{u} + \mathbf{A} \mathbf{u} = \mathbf{F}(\mathbf{u}) \) by denoting the vector of concentrations \( \mathbf{u}(t) = (c_1(t), \ldots, c_L(t)) \in \mathbb{R}^L \) and

\[
\mathbf{u}^\alpha = \begin{pmatrix} u^{\alpha(1,1)}, \ldots, u^{\alpha(L,1)} \end{pmatrix} \quad \text{for} \quad u^{\alpha(l,r)} = \prod_{l=1}^{L} c_l^{\alpha(l,r)},
\]

\[
\text{diag}(\mathbf{z}) = \begin{bmatrix} z_1 & 0 & \cdots & \cdots & 0 \\ 0 & z_2 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & 0 & z_{L-1} \\ 0 & \cdots & \cdots & 0 & z_L \end{bmatrix} \quad \text{for} \quad \mathbf{z} \in \mathbb{R}^L,
\]

with \( \mathbf{F}(\mathbf{u}) = (\eta - \alpha) \text{diag}(\mathbf{u}^\alpha) \) where \( \eta \) and \( \alpha \) are the vectors of the stoichiometric coefficients. By this way Theorem 4 can be applied.

Last but not least, we state the convergence rate at \( t = 0 \) by combining the results of Theorems 5 and 6.

**Theorem 8.** Under the assumptions of Theorem 5, let \( u_\epsilon(\cdot,t) \) for \( t \in (0,T) \) be the unique solution obtained from the projection \( \mathcal{P}_\epsilon \in \mathcal{L}(H^p(\Omega)) \) in (2.11). Then there always exists a sufficiently small \( \epsilon \)-dependent time \( t_\epsilon > 0 \) such that

\[
\|u_\epsilon(t_\epsilon) - u(0)\|_{H^p(\Omega)} \leq \frac{C(T + \frac{\epsilon}{2})}{\sqrt{(T + \frac{\epsilon}{2} + 1)^2 + 4(T + \frac{\epsilon}{2})(1 - 2\beta)\log(\epsilon^{-1}) + T + \frac{\epsilon}{2} - 1}}.
\]

**Proof.** In the same manner as in proof of Theorem 5 we prove the target estimate (2.16) by seeking the infimum \( \frac{1}{\epsilon} \inf_{t_\epsilon > 0} \left( \frac{t_\epsilon (1 - 2\beta)}{r + t_\epsilon + \frac{\epsilon}{2}} + t_\epsilon \right) \). Solving the algebraic problem

\[
\frac{t_\epsilon (1 - 2\beta)}{r + t_\epsilon + \frac{\epsilon}{2}} = t_\epsilon,
\]

we are led to the following inequality

\[
t_\epsilon^2 ((1 - 2\beta) \log \epsilon - 1) - \left( T + \frac{\epsilon}{2} - 1 \right) t_\epsilon + T + \frac{\epsilon}{2} > 0.
\]
Hereby, we find the admissible interval for \( t_\varepsilon \), as follows:
\[
t_\varepsilon \in \left( \frac{-b - \sqrt{b^2 + 4(b + 1)(1 - \log \varepsilon)}}{2((1 - 2\beta)\log \varepsilon - 1)}, \frac{-b + \sqrt{b^2 + 4(b + 1)(1 - \log \varepsilon)}}{2((1 - 2\beta)\log \varepsilon - 1)} \right),
\]
where \( b = T + \frac{\rho}{2} - 1 \) is recalled.

Therefore, it is evident to obtain the rate (2.15). We complete the proof of the theorem. □

3. Convergence of an iterative scheme

In this section, we reduce ourselves to the case that the nonlinearity \( F \) does not degenerate, i.e. \( F(0) = 0 \) and there exist positive constants \( L_0 \) and \( L_1 \) such that
\[
0 < L_0 \leq \sup_{|w| \leq M} \frac{\partial F}{\partial w}(w) \leq L_1.
\]

Note that we now focus on solving the regularized solution in the open set \((0, T)\) since at \( t = 0 \) we only need to compute the approximation at \( t = t_\varepsilon \). Let \( 1 \leq N \in \mathbb{N} \) and take \( \omega = T/N \). In this regard, we put
\[
t_n = T - n\omega \quad \text{for} \quad n = 1, \ldots, N.
\]

This setting allows us to seek a numerical solution \( u^{r,n}_\varepsilon(x) \approx u_\varepsilon(x, t_n) \) for \( r \in \mathbb{N} \) in the equivalent mesh-width in \( t \). The function \( u_\varepsilon(x, t_n) \) is the semi-discrete solution of the nonlinear scheme (2.11) under scrutiny in the previous part. Starting from the projection \( \overline{P}_\varepsilon \) with the cut-off function \( \bar{F}_{M_\varepsilon} \) used in (2.11), the iterative scheme is designed by
\[
(K + 1) u^{r+1,n}_\varepsilon = K u^{r,n}_\varepsilon + \int_0^\infty e^{(T-t_n)\lambda} \chi_{C_{\varepsilon}}^{\rho/2} dE_{\lambda_{\varepsilon}} \bar{F}_{T} - \int_0^\infty \gamma_\varepsilon(t_n) \chi_{C_{\varepsilon}}^{\rho/2} dE_{\lambda_{\varepsilon}} F_{M_\varepsilon} (u^{r,n}_\varepsilon),
\]
with the initial guesses \( u^{0,0}_\varepsilon \equiv \bar{F}_{T} \) and \( u^{0,n}_\varepsilon \equiv 0 \). In addition, we have denoted by \( \gamma_\varepsilon(t) := \int_t^T e^{(s-t)\lambda} \rho_{s} ds \leq C_{\varepsilon}^{-1} \left( e^{(T-t)\lambda} - 1 \right) =: \gamma_\varepsilon \) for \( t \in (0, T) \).

The presence of the so-called stabilization constant \( K > 0 \) is to guarantee the convergence of the scheme, somehow hindered by the Lipschitz nonlinearity \( F \). In fact, we wish to designate an unconditional numerical scheme for the projection \( \overline{P}_\varepsilon \) in the sense that the number of discretizations \( N \) becomes free-to-choose by a suitable choice of \( K \). It is worth mentioning that the sequence \( \{u^{r,n}_\varepsilon\}_{r \in \mathbb{N}} \) is well-defined in the Sobolev space \( H^p(\Omega) \) for each \( \varepsilon > 0 \) and thus the existence and uniqueness are self-contained by virtue of the linearity of the scheme. Note that the function \( \gamma_\varepsilon \) is decreasing in the time argument because of \( \gamma_\varepsilon(t_{n+1}) \geq \gamma_\varepsilon(t_n) \) for any \( 0 \leq t_{n+1} \leq t_n \leq T \), whilst it is transparently increasing in the argument \( \lambda_{\varepsilon} \).

**Theorem 9.** Under the assumptions of Theorem 2 let \( \{u^{r,n}_\varepsilon\}_{r \in \mathbb{N}} \) be the solution of the scheme (3.2). Then by choosing
\[
K := K(n, \varepsilon) = \max \left\{ \gamma_\varepsilon(t_n) L_1, e^{TC_{\varepsilon}} \right\} > 0 \quad \text{for} \quad n \in \mathbb{N},
\]
this sequence is uniformly bounded in \( H^p(\Omega) \).

**Proof.** In view of the decomposition
\[
u^{r,n}_\varepsilon = \int_0^\infty \lambda^{\rho/2} dE_{\lambda_{\varepsilon}} u^{r,n}_\varepsilon + \int_0^\infty \lambda^{\rho/2} dE_{\lambda_{\varepsilon}} u^{r,n}_\varepsilon,
\]
we can find the upper bound in $H^p$-norm of $u_{\varepsilon}^{r+1,n}$ as follows:

$$
(K + 1) \left\| u_{\varepsilon}^{r+1,n} \right\|_{H^p(\Omega)} \leq e^{(T-t_n)C_\varepsilon} \left\| g_T^\varepsilon \right\|_{H^p(\Omega)} + K \int_{C_\varepsilon} \lambda^{p/2} dE_\lambda u_{\varepsilon}^{r,n} + \int_0^\infty \lambda^{p/2} \left( h_{M_\varepsilon} \right) (\lambda_{\varepsilon}) dE_\lambda u_{\varepsilon}^{r,n} \tag{3.4}
$$

where we have denoted by $h_{M_\varepsilon} \left[ y^j \right] := K y^2 - \gamma_{\varepsilon} (t_j) F_{M_\varepsilon} \left[ y^j \right]$ for $y = \left( y^j \right)_{0 \leq j \leq N}$.

At this stage, we remark that $h' \left[ y^j \right] = K - \gamma_{\varepsilon} (t_j) F'_{M_\varepsilon} \left( y^j \right)$ and therefore, it holds $|h'| \leq K - \gamma_{\varepsilon} (t_j) L_0$ for a.e. $y^j \in \mathbb{R}$. By the mean value theorem together with the fact that $F(0) = 0$, we estimate that

$$
(K + 1) \left\| u_{\varepsilon}^{r+1,n} \right\|_{H^p(\Omega)} \leq e^{(T-t_n)C_\varepsilon} \left\| g_T^\varepsilon \right\|_{H^p(\Omega)} + K \int_{C_\varepsilon} \lambda^{p/2} dE_\lambda u_{\varepsilon}^{r,n} + \int_0^\infty \lambda^{p/2} \left( K - \gamma_{\varepsilon} (t_n) L_0 \right) dE_\lambda u_{\varepsilon}^{r,n} \leq e^{(T-t_n)C_\varepsilon} \left\| g_T^\varepsilon \right\|_{H^p(\Omega)} + (K - \gamma_{\varepsilon} (t_n) L_0) \left\| u_{\varepsilon}^{r,n} \right\|_{H^p(\Omega)} + \bar{\gamma}_{\varepsilon} (t_n) L_0 \int_{C_\varepsilon} \lambda^{p/2} dE_\lambda u_{\varepsilon}^{r,n}.
$$

By the choice of $C_\varepsilon$, we gain

$$
\lim_{\varepsilon \to 0^+} \int_{C_\varepsilon} \lambda^{p/2} dE_\lambda u_{\varepsilon}^{r,n} = \lim_{\varepsilon \to 0} \int_{C_\varepsilon} \lambda^{p/2} dE_\lambda u_{\varepsilon}^{r,n} = 0.
$$

Thus, we now enjoy the choice of $K$ in (3.3) to rule out that there exists $\mu \in (0, 1)$ independent of $r, n$ and $\varepsilon$ such that

$$
\left\| u_{\varepsilon}^{r+1,n} \right\|_{H^p(\Omega)} \leq \mu \left( \left\| g_T^\varepsilon \right\|_{H^p(\Omega)} + \int_{C_\varepsilon} \lambda^{p/2} dE_\lambda u_{\varepsilon}^{r,n} + \left\| u_{\varepsilon}^{r,n} \right\|_{H^p(\Omega)} \right).
$$

By induction, we obtain

$$
\left\| u_{\varepsilon}^{r,n} \right\|_{H^p(\Omega)} \leq C \sum_{j=1}^r \mu^j \left\| g_T^\varepsilon \right\|_{H^p(\Omega)},
$$

which enables us to state that the scheme (3.2) is bounded in $H^p(\Omega)$ for any $r, n$ and $\varepsilon$. Hence, we complete the proof of the theorem.

Using the Banach-Alaoglu theorem, the uniform bound deduced in the proof of Theorem 9 indicates that we can extract a further subsequence (which we relabel with the same indexes if necessary) such that $u_{\varepsilon}^{r,n} \to u_0^r$ weakly in $H^p(\Omega)$ as $r \to \infty$. Furthermore, thanks to the Banach-Saks theorem we know that this subsequence also admits another subsequence such that the so-called Cesàro mean is strongly convergent to $u_0^r$ in $H^p(\Omega)$. In this sense, we can write

$$
\left( \frac{1}{R} \sum_{r=0}^R u_{\varepsilon}^{r,n} - u_0^r \right) \to 0 \quad \text{as} \quad R \to \infty.
$$

Define $u_{\varepsilon}^{R,n} := \frac{1}{R} \sum_{r=0}^R u_{\varepsilon}^{r,n}$ for $R \in \mathbb{N}$. Our next step is to find the rate of convergence of the sequence $\left\{ u_{\varepsilon}^{R,n} \right\}_{R \in \mathbb{N}}$ acquired by (3.3). By this way, we not only prove that $u_{\varepsilon}^{r,n} \to u_0^r$ strongly
in $H^p(\Omega)$, but also show that $u^\varepsilon_n$ is identically the semi-discrete solution of the nonlinear scheme \[(2.11)\].

**Theorem 10.** Under the assumptions of Theorem 9 let $u_\varepsilon(\cdot, t)$ for $t \in (0, T)$ be the unique solution obtained from the projection $P_\varepsilon \in L(H^p(\Omega))$ in \[(2.11)\]. Let $\{u_\varepsilon^{r,n}\}_{r \in \mathbb{N}}$ be the solution of the iterative scheme \[(3.6)\] and let $\{w^R_{\varepsilon,n}\}_{R \in \mathbb{N}}$ be the Cesàro mean of $u_\varepsilon^{r,n}$ with respect to $r$. Then for $r, n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\bar{\mu} \in (0, 1)$ independent of $r, n$ and $\varepsilon$ such that the following error bound holds

$$
\|u_\varepsilon^{r,n} - u_\varepsilon(\cdot, t_n)\|_{H^p(\Omega)} \leq C \bar{\mu}^r,
$$

for $r$ sufficiently large. Moreover, for $R \in \mathbb{N}$ it holds

$$
\|u_\varepsilon^{R,n} - u_\varepsilon(\cdot, t_n)\|_{H^p(\Omega)} \leq C \left( \frac{\mu}{R} \right)^R.
$$

**Proof.** Define $v_\varepsilon^{r+1,n} := u_\varepsilon^{r+1,n} - w_\varepsilon^{R,n} \in H^p(\Omega)$. To gain the convergence rate, we compute the difference equation:

\[(3.6)\]

$$(R + 1) v_\varepsilon^{r+1,n} = \sum_{r=0}^{R+1} u_\varepsilon^{r,n} - \sum_{r=0}^R u_\varepsilon^{r,n} + (R + 1) \left( \frac{1}{R + 1} - \frac{1}{R} \right) \sum_{r=0}^R u_\varepsilon^{r,n}$$

$$= u_\varepsilon^{R+1,n} - \frac{1}{R} \sum_{r=0}^R u_\varepsilon^{r,n} = \frac{1}{R} \sum_{r=0}^R (u_\varepsilon^{R+1,n} - u_\varepsilon^{r,n}).$$

From now onward, we define $U_\varepsilon^{r+1,n} := u_\varepsilon^{r+1,n} - u_\varepsilon^{r,n} \in H^p(\Omega)$. Following the same way we have done in the proof of Theorem 9 the function $U_\varepsilon^{r+1,n}$ is expressed as

$$(K + 1) U_\varepsilon^{r+1,n} = K U_\varepsilon^{r,n} - \int_0^\infty \gamma_\varepsilon(t_n) \lambda_\varepsilon^{p/2} dE_{\lambda_\varepsilon}(u_\varepsilon^{r,n})$$

$$+ \int_0^\infty \gamma_\varepsilon(t_n) \lambda_\varepsilon^{p/2} dE_{\lambda_\varepsilon} F_{\lambda_\varepsilon}(u_\varepsilon^{r-1,n}).$$

With the aid of the decomposition

$$U_\varepsilon^{r,n} = \int_{C_\varepsilon}^\infty \lambda_\varepsilon^{p/2} dE_{\lambda_\varepsilon} U_\varepsilon^{r,n} + \int_0^\infty \lambda_\varepsilon^{p/2} dE_{\lambda_\varepsilon} U_\varepsilon^{r,n},$$

the function $U_\varepsilon^{r+1,n}$ can be bounded from above in the $H^p$-norm by

$$(K + 1) \| U_\varepsilon^{r+1,n} \|_{H^p(\Omega)} \leq K \int_{C_\varepsilon}^\infty \lambda_\varepsilon^{p/2} dE_{\lambda_\varepsilon} U_\varepsilon^{r,n}$$

$$+ \int_0^\infty \lambda_\varepsilon^{p/2} h_{M_\varepsilon}(\lambda_\varepsilon) dE_{\lambda_\varepsilon} u_\varepsilon^{r,n} - \int_0^\infty \lambda_\varepsilon^{p/2} h_{M_\varepsilon}(\lambda_\varepsilon) dE_{\lambda_\varepsilon} u_\varepsilon^{r-1,n},$$

where we have recalled that $h_{M_\varepsilon}[y^j] := K y^j - \gamma_\varepsilon(t_n) F_{\lambda_\varepsilon}[y^j]$ for $y = (y^j)_{0 \leq j \leq N}$.

Henceforward, we arrive at

$$(K + 1) \| U_\varepsilon^{r+1,n} \|_{H^p(\Omega)} \leq (K - \gamma_\varepsilon(t_n) L_0) \| U_\varepsilon^{r,n} \|_{H^p(\Omega)} + \gamma_\varepsilon(t_n) L_0 \int_{C_\varepsilon}^\infty \lambda_\varepsilon^{p/2} dE_{\lambda_\varepsilon} U_\varepsilon^{r,n}$$

$$\leq K \| U_\varepsilon^{r,n} \|_{H^p(\Omega)},$$

for $r$ sufficiently large.
by virtue of the fact already known that \(|h| \leq K - \gamma \varepsilon (t_j) L_0\) for a.e. \(y^j \in \mathbb{R}\) under the choice of \(K\) in (3.3). Hereby, choosing \(\bar{\mu} = K (K + 1)^{-1} \in (0, 1)\) independent of \(r, n\) and \(\varepsilon\) we can conclude that

\[
\|U^{r+1,n}_\varepsilon\|_{H^p(\Omega)} \leq \bar{\mu} \|U^{r,n}_\varepsilon\|_{H^p(\Omega)},
\]

which, by mathematical induction, leads to

\[
\|U^{r,n}_\varepsilon\|_{H^p(\Omega)} \leq C \bar{\mu}^r \|g\|_{H^p(\Omega)}.
\]

Eventually, by the back-substitution of the function \(U^{r,n}_\varepsilon\) this means that

\[
\|u^{r,l,n}_\varepsilon - u^{r,n}_\varepsilon\|_{H^p(\Omega)} \leq \bar{\mu}^{r+l} \|u^{r-1,n}_\varepsilon - u^{r,n}_\varepsilon\|_{H^p(\Omega)} + \ldots + \bar{\mu}^r \|u^{1,n}_\varepsilon - u^{0,n}_\varepsilon\|_{H^p(\Omega)} \leq \frac{\bar{\mu}^r (1 - \bar{\mu})}{1 - \bar{\mu}} \|u^{1,n}_\varepsilon\|_{H^p(\Omega)},
\]

which proves the fact that the sequence \(\{u^{r,n}_\varepsilon\}_{r \in \mathbb{N}}\) is Cauchy in \(H^p(\Omega)\). Consequently, there exists \(\bar{u}_\varepsilon \in H^p(\Omega)\) to which \(u^{r,n}_\varepsilon\) is strongly convergent as \(r \to \infty\). In addition, it follows from (3.8) that when \(l \to \infty\),

\[
\|u^{r,n}_\varepsilon - \bar{u}_\varepsilon\|_{H^p(\Omega)} \leq C \bar{\mu}^r,
\]

and thus, for \(r\) sufficiently large we obtain the convergence of the source term, i.e. \(F(u^{r,n}_\varepsilon) \to F(\bar{u}_\varepsilon)\) strongly in \(H^p(\Omega)\) as \(r \to \infty\), whenever \(L_1\) is dependent of \(M\) or not; see again the choice (2.12) to decide how big the iteration step \(r\) needs to be.

Collectively, we have proved that \(\bar{u}_\varepsilon\) is identically the semi-discrete solution of the nonlinear regularization (2.11) and further, it coincides the function \(u^n\) derived from the weak convergence above. Now, it suffices to close the proof of the theorem by combining (3.8) and (3.9). Essentially, we have

\[
\|u^{R+1,n}_\varepsilon\|_{H^p(\Omega)} \leq \frac{1}{R (R + 1)} \sum_{r=0}^{R} \|u^{R+1,n}_\varepsilon - u^{r,n}_\varepsilon\|_{H^p(\Omega)} \leq \frac{C \bar{\mu}}{R + 1}.
\]

Similar to (3.9), the Cesàro mean is strongly convergent to \(u^n\) with the rate

\[
\|u^{R,n}_\varepsilon - u^n\|_{H^p(\Omega)} \leq C \left(\frac{\bar{\mu}}{R}\right)^R.
\]

Hence, we complete the proof of the theorem. \(\square\)

**Corollary 11.** Under the assumptions of Theorems 6 and 10 one has the following error estimate:

\[
\|u^{r,n}_\varepsilon - u (\cdot, t_0)\|_{H^p(\Omega)} \leq C \left(\frac{\bar{\mu}^r + \varepsilon \bar{\mu}^{r+1}}{R}\right)^n \quad \text{for } n = 1, N - 1.
\]

**Remark 12.** When \(L_1\) is independent of \(M\), the assumption (3.1) can also be found in some examples, which also aids the applicability of the global Lipschitz case in Subsection 2.2. Observe what have been enlisted in Subsection 1.2. It is immediate to see that the Michaelis–Menten law (\(N = 1\) in the de Pillis–Radunskaya law) with \(F (u) = au / (b + u)\) for \(a, b > 0\) gives

\[
\sup_{|w| \leq M} \frac{\partial F}{\partial w} (w) = \sup_{|w| \leq M} \frac{ab}{(b + w)^2} \in \left[\frac{ab}{(b + M)^2}, \frac{a}{b}\right].
\]
4. Discussions

We have studied a nonlinear spectral regularization to solve a semi-linear backward parabolic equation. The scheme significantly modifies the cut-off method developed in [15, 18, 8] so that it not only fits the nonlinear context under consideration, but also handles certain smoothness of the solution to the forward model in an appropriate manner. In this fashion, our proposed method is convergent in a Hölder-type rate for $t \in (0, T)$, decreasing backwards in time, and in a logarithmic-type rate for $t = 0$. It is worth mentioning that the strong convergence obtained in $H^p$ may allow us to gain the convergence of the regularized solution on the boundary with the same rates by the standard trace theorem.

We have also studied the convergence of an iterative method for this nonlinear scheme. To gain the strong convergence, this approximation works with the non-degeneracy of $F$, which is a certainly stronger condition than those met in the analysis of the nonlinear scheme. Essentially, we see that the property of the nonlinearity $F$ plays a pretty much important role in deciding the convergence of the numerical scheme, as postulated in Theorems [9] and [10]. This also points out the most difficult issue in solving inverse problems for nonlinear PDEs; compared to the linear cases investigated so far. One may think that the presence of Theorem [9] seems unnecessary (and so is the largeness of the stabilization constant $K$ taken in (3.3)) since it is clear that the strong convergence of the numerical scheme has already been obtained in Theorem [10]. Nevertheless, it can be understood that we have depicted a general procedure to verify the convergence of numerical regularization schemes in the future topics. In fact, such boundedness (as the stability analysis) obtained in Theorem [9] orientates towards the strong convergence of the Cesàro mean. In some sense, this unravels the possibility that the strong convergence of the numerical sequence is not obtainable, generally hindered by the property of $F$. The choice of $K$ can also be very helpful because the discretization as well as the number of iterations become more effectively economical. In the near future, we wish to understand deeper numerical issues caught in particularly complex networks as presented in Remark [7].

The results of this paper can initiate the convergence analysis of the other classes of nonlinear backward PDEs using the strategy of verifying variational source conditions. One can also attempt to achieve the strong convergence result in the Besov spaces for regularization of the present backward model ($B$) in the unbounded domain as it is in agreement with the well-posedness of the forward problem (see e.g. [13]). Observe that although the variational source condition theory does not require the fact that the ill-posed operator $T$ admits the Fréchet derivative, it is self-contained in this framework. Thus, this time we are allowed to not only derive the variational source condition from the spectral source condition, but also apply the iteration-based regularized Gauss-Newton method. The convergence analysis of this method for the nonlinear backward PDEs should also be considered in the forthcoming works.

Acknowledgments

The author thanks Prof. Dr. Mohammad Kazemi (Charlotte, USA) for his support of the author’s research career.

References

[1] P. Erdi and J. Tóth. *Mathematical Models of Chemical Reactions: Theory and Applications of Deterministic and Stochastic Models*. Nonlinear Science. Princeton University Press, 1989.

[2] A. B. Ferrari and E. S. Titi. Gevrey regularity for nonlinear analytic parabolic equations. *Communications in Partial Differential Equations*, 23(1–2):1–16, 1998.
[3] J. Flemming, B. Hofmann, and P. Mathé. Sharp converse results for the regularization error using distance functions. *Inverse Problems*, 27(2):025006, 2011.

[4] M. Grasmair. Generalized Bregman distances and convergence rates for non-convex regularization methods. *Inverse Problems*, 26(11):115014, 2010.

[5] D. N. Hao, N. V. Duc, and N. V. Thang. Backward semi-linear parabolic equations with time-dependent coefficients and local Lipschitz source. *Inverse Problems*, 34(5):055010, 2018.

[6] B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators. *Inverse Problems*, 23(3):987–1010, 2007.

[7] T. Hohage and F. Weidling. Verification of a variational source condition for acoustic inverse medium scattering problems. *Inverse Problems*, 31(7):075006, 2015.

[8] T. Hohage and F. Weidling. Characterizations of variational source conditions, converse results, and maxisets of spectral regularization methods. *SIAM Journal on Numerical Analysis*, 55(2):598–620, 2017.

[9] T. Hohage and F. Werner. Convergence rates for inverse problems with impulsive noise. *SIAM Journal on Numerical Analysis*, 52(3):1203–1221, 2014.

[10] R. Jaroudi, G. Baravdish, B. T. Johansson, and F. Åström. Numerical reconstruction of brain tumours. *Inverse Problems in Science and Engineering*, pages 1–21, 2018.

[11] B. Kaltenbacher, A. Neubauer, and O. Scherzer. *Iterative Regularization Methods for Nonlinear Ill-Posed Problems*. Radon Series on Computational and Applied Mathematics. Berlin, Boston: De Gruyter, 2008.

[12] C. König, F. Werner, and T. Hohage. Convergence rates for exponentially ill-posed inverse problems with impulsive noise. *SIAM Journal on Numerical Analysis*, 54(1):341–360, 2016.

[13] C. Miao and B. Zhang. The Cauchy problem for semilinear parabolic equations in Besov spaces. *Houston Journal of Mathematics*, 30(3):829–878, 2004.

[14] J. D. Murray. Continuous Population Models for Single Species. In *Interdisciplinary Applied Mathematics*, volume 17, pages 1–43. Springer, New York, 1993.

[15] P. T. Nam. An approximate solution for nonlinear backward parabolic equations. *Journal of Mathematical Analysis and Applications*, 367(2):337–349, 2010.

[16] O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier, and F. Lenzen. *Variational Methods in Imaging*. Applied Mathematical Sciences. Springer, New York, 2008.

[17] N. H. Tuan, V. V. Au, V. A. Khoa, and D. Lesnic. Identification of the population density of a species model with nonlocal diffusion and nonlinear reaction. *Inverse Problems*, 33(5):055019, 2017.

[18] N. H. Tuan, L. D. Thang, V. A. Khoa, and T. Tran. On an inverse boundary value problem of a nonlinear elliptic equation in three dimensions. *Journal of Mathematical Analysis and Applications*, 426(2):1232–1261, 2015.