Optimal control of linear time-varying systems using the Chebyshev wavelets
(a comparative approach)
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This paper extends the application of continuous Chebyshev wavelet expansions to find the optimal solution of linear time-varying systems using two different approaches. By using the product of two time functions together with the operational matrix of integration, the system of state equations are changed into a set of algebraic equations which can be solved using a digital computer. In addition, the Chebyshev wavelets are more successful to find the optimal solution of linear time-varying systems when compared with the other existing mentioned algorithms. Finally, the main feature of this paper over similar possible works is that the use of the Lagrange multipliers approach gives more accurate results in comparison with the results of the Riccati approach. The given examples support these claims.

Keywords: time-varying systems; Chebyshev wavelets; Lagrange multipliers approach; Riccati approach; optimal control

1. Introduction
Orthogonal functions and polynomials series such as Walsh series (Chen & Hsiao, 1975), block-pulse functions (Cheng & Hsu, 1982), Laguerre series (Clement, 1982), Chebyshev polynomials (Chou & Horng, 1985), Legendre polynomials (Paraskevopoulos, 1985), Taylor series (Mouroutos & Sparis, 1985) and Fourier series (Samavat & Rashidi, 1995) have received much attention in dealing with various problems in control theory. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus it greatly simplifies the problem.

In recent years, wavelets have found their way into many different fields of science and engineering. Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. The wavelet theory is a relatively new and an emerging area in mathematical research. It has been applied in a wide range of engineering science; particularly, wavelets are successfully used in signal analysis for waveform representation and segmentations, identification, optimal control and many other applications. Wavelets permit the accurate representation of a variety of functions and operators. Particular concentration has been given to the application of Haar wavelets (Chen, Lai, & Ho, 2006; Hsiao, 2004), sine–cosine wavelets (Nasehi, Samavat, & Vali, 2008), Legendre wavelets (Ebrahimi, Vali, Samavat, & Gharaveisi, 2009; Razzaghi & Yousefi, 2001; Wang, 2008; Yousefi, 2006) and Chebyshev wavelets (Babolian & Fattahzadeh, 2007a, 2007b; Nasehi, 2009).

Since the optimal solution of time-varying systems is not as easy as those of time-invariant systems, much effort has been devoted to the optimal solution of these systems. The polynomial approximations of time functions have been taken by many researchers to solve control problems (Chang & Lee, 1986; Hwang & Chen, 1985; Razzaghi, 1990). Optimizations of time-varying systems have been of considerable interest to control engineers in the recent years (Hsiao & Wang, 1999; Wang, 2007).

For the first time, this paper uses the continuous Chebyshev wavelets to find the optimal solution of linear time-varying systems using two different approaches. Furthermore, it is shown that if we use the Lagrange multipliers approach, we get more accurate results when compared with the results of the Riccati approach. Some numerical examples (Tables 3 and 5) are given to confirm the results. Finally, as given in the tables, the proposed method shows more accurate results when compared with some of the existing mentioned references. Therefore, we may conclude that the results of Chang and Lee (1986) and Hsiao and Wang (1999) are modified.

2. Preliminaries and problem statement
2.1. Chebyshev wavelets
Chebyshev wavelets \( \Phi_{n,m}(k,n,m,t) \) have four arguments, \( n = 1, 2, \ldots, 2^{k-1}, k \) can assume any positive integer,
\( m \) is the degree of Chebyshev polynomials of the first kind and \( t \) denotes the time (Babolian & Fattahzadeh, 2007)

\[
\Phi(t)_{n,m} = \begin{cases} 
2^{\frac{n}{2}} \tilde{T}_m(2^{\frac{n}{2}}t - 2n + 1), & n - 1 \leq \frac{n}{2^{k-1}} < t < \frac{n}{2^{k-1}}, \\
0, & \text{otherwise},
\end{cases}
\]

(1)

where

\[
\tilde{T}_m(t) = \begin{cases} 
\frac{1}{\sqrt{2}}, & m = 0, \\
\sqrt{2} T_m(t), & m > 0,
\end{cases}
\]

(2)

and \( m = 0, 1, \ldots, M - 1, n = 1, 2, \ldots, 2^{k-1} \). In Equation (2) the coefficients are used for orthonormality. Here, \( T_m(t) \) are Chebyshev polynomials of the first kind which are orthogonal with respect to the weight function \( \omega(t) = 1/\sqrt{1 - t^2} \), on the interval \([-1, 1]\), and satisfy the following recursive equation:

\[
\begin{align*}
T_0(t) &= 1, & T_1(t) &= t, \\
T_{m+1}(t) &= 2tT_m(t) - T_{m-1}(t).
\end{align*}
\]

(3)

We should note that in dealing with Chebyshev wavelets, the weight function \( \omega(t) = \omega(2t - 1) \) has to be dilated and translated as follows:

\[
\omega(t) = \omega(2^k t - 2n + 1).
\]

(4)

Remark The time interval \([0, 1)\) in Chebyshev wavelets can be extended to an arbitrary interval \([0, t_f]\) as follows (Nasehi, 2009):

\[
\Phi(t)_{n,m} = \begin{cases} 
2^{\frac{n}{2}} \tilde{T}_m(2^{\frac{n}{2}}t - 2n + 1) \cdot t_f \times \frac{n - 1}{2^{k-1}} \leq t < t_f, \\
0, & \text{otherwise}
\end{cases}
\]

(5)

A time function \( f(t) \) that is square integrable on the time interval \( t \in [0, t_f] \) may be expanded by Chebyshev wavelets as follows:

\[
f(t) = \sum_{n=1}^{N} \sum_{m=0}^{M-1} C_{nm} \phi_{nm}(t) = C^T \Phi(t),
\]

(6)

where \( C \) and \( \Phi \) are \( NM \times 1 \) matrices and given by

\[
C = [c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{NM-1}]^T,
\]

(7)

\[
\Phi(t) = [\phi_{00}(t), \phi_{11}(t), \ldots, \phi_{1M-1}(t), \phi_{20}(t), \ldots, \phi_{2M-1}(t), \ldots, \phi_{NM-1}(t)]^T.
\]

(8)

2.2. The operational matrix of backward integration

We obtain the backward integrals of Chebyshev wavelets \( \Phi(t) \) which can be represented as

\[
\int_0^t \Phi(\tau) d\tau = G \Phi(t).
\]

(9)

2.3. The operational matrix of forward integration

We obtain the forward integrals of Chebyshev wavelets \( \Phi(t) \), which may be represented as

\[
\int_0^t \Phi(\tau) d\tau = P \Phi(t),
\]

(10)

where \( P \) is the \( NM \times NM \) operational matrix of forward integration and is given as (Babolian & Fattahzadeh, 2007)

\[
P = \frac{1}{2^k} \begin{bmatrix} F & S & \cdots & S \\ 0 & F & \cdots & S \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F \end{bmatrix}^{MN \times MN}_{MN \times MN}
\]

In Equation (10) \( F \) and \( S \) are \( M \times M \) matrices given by

\[
F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ -2\sqrt{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2} \left( \frac{1 - (-1)^{m+1}}{m+1} - \frac{1 - (-1)^{m-1}}{m-1} \right) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\]

and

\[
S = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{4} & \frac{1}{4} & \cdots \\ \frac{\sqrt{2}}{3} & 0 & 1 & \frac{1}{4} & \cdots \\ \frac{\sqrt{2}}{3} & -\frac{1}{2} & 0 & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\sqrt{2}}{2} \left( \frac{1 - (-1)^{M-1}}{M-1} - \frac{1 - (-1)^{M-2}}{M-2} \right) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\
\end{bmatrix}
\]
2.4. The product operational matrix

The product of two Chebyshev wavelet function vectors can be approximated by truncated $m$-term Chebyshev wavelets as follows (Babolian & Fattahzadeh, 2007):

$$C^T \Phi(t) \Phi(t)^T \cong \Phi^T(t) \tilde{C} \tag{12}$$

where $C$ is given in Equation (7) and $\Phi(t)$ can be obtained similar to Equation (8). Also, $\tilde{C}$ is an $(NM) \times (NM)$ matrix and is given as (Babolian & Fattahzadeh, 2007)

$$\tilde{C} = \begin{bmatrix}
    c_{00} & c_{01} & c_{02} & \cdots & c_{0N} \\
    c_{10} & c_{11} + \frac{1}{\sqrt{2}} c_{12} & c_{12} + \frac{1}{\sqrt{2}} c_{13} & \cdots & c_{1N} \\
    c_{20} & \frac{1}{\sqrt{2}} (c_{21} + c_{23}) & \frac{1}{\sqrt{2}} (c_{22} + c_{24}) & \cdots & c_{2N} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    c_{m0} & \frac{1}{\sqrt{2}} (c_{m1} + c_{m3}) & \frac{1}{\sqrt{2}} (c_{m2} + c_{m4}) & \cdots & c_{mN} \\
    c_{mN} & \frac{1}{\sqrt{2}} (c_{(m-1)1} + c_{(m-1)3}) & \frac{1}{\sqrt{2}} (c_{(m-1)2} + c_{(m-1)4}) & \cdots & c_{MN} \\
\end{bmatrix}$$

where

$$
\mu = \begin{cases}
    M - 2, & M \text{ even}, \\
    M - 1, & M \text{ odd},
\end{cases}
$$

$$
v = \begin{cases}
    M/2, & M \text{ even}, \\
    (M - 1)/2, & M \text{ odd},
\end{cases}
$$

$$
\tilde{C} = \text{diag}(\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_i),
$$

\(i = 1, 2, \ldots, 2^{k-1}\).

2.5. Problem statement

Consider linear time-varying systems characterized by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) \text{ is specified}, \tag{14}$$

where the state variable $x(t)$ and the input variable $u(t)$ are $b$-vector and $r$-vector, respectively. The time-varying coefficient matrices $A(t)$ and $B(t)$ are $b \times b$ and $b \times r$ matrices, respectively.

The problem is to find the optimal control $u(t)$ and the corresponding state trajectory $x(t)$, satisfying Equation (14) while minimizing the quadratic performance index

$$J = \frac{1}{2} \int_0^T \left[ x^T(t)Q(t)x(t) + u^T(t)R(t)u(t) \right] dt \tag{15}$$

by using two different approaches. In Equation (15), $Q(t)$ is a positive-semi-definite $b \times b$ matrix and $R(t)$ is a positive-definite $r \times r$ matrix.

3. Approximation of the system dynamics

We approximate the system dynamics as follows (Marzban & Razzaghi, 2004): let

$$x(t) = [x_1(t), x_2(t), \ldots, x_b(t)]^T, \tag{16}$$

$$u(t) = [u_1(t), u_2(t), \ldots, u_r(t)]^T, \tag{17}$$

$$\hat{\Phi}(t) = I_b \otimes \Phi(t), \tag{18}$$

$$\hat{\Phi}_1(t) = I_r \otimes \Phi(t), \tag{19}$$

where $I_b$ and $I_r$ are $b$- and $r$-dimensional identity matrices, respectively. Also, $\otimes$ denotes the Kronecker product. $\hat{\Phi}(t)$ and $\Phi(t)$ are $bMN \times b$ and $rMN \times r$ matrices, respectively. Assume that $x_i(t)$ and each $u_j(t), i = 1, 2, \ldots, b, j = 1, 2, \ldots, r$, can be written in terms of Chebyshev wavelet functions as

$$x_i(t) = \Phi^T(t)x_i, \tag{20}$$

$$u_j(t) = \Phi^T(t)u_j. \tag{21}$$

Using Equations (16)–(19), we have

$$x(t) = \hat{\Phi}^T(t)x \tag{22}$$

$$u(t) = \hat{\Phi}_1^T(t)u \tag{23}$$
where $X$ and $U$ are vectors of order $bMN \times 1$ and $rMN \times 1$, respectively, given by

\begin{align}
X &= [X_1, X_2, \ldots, X_T]^T, \\
U &= [U_1, U_2, \ldots, U_T]^T.
\end{align}

Similarly, we have

\begin{equation}
x(0) = \Phi^T(t)d
\end{equation}

where $d$ is a vector of order $bMN \times 1$ given by

\begin{equation}
d = [d_1, d_2, \ldots, d_b]^T.
\end{equation}

We now expand $A(t)$ and $B(t)$ by Chebyshev wavelet functions as follows:

\begin{align}
A(t) &= [A_{10}, A_{11}, \ldots, A_{1M-1}, \ldots, A_{N0}, A_{N1}, \ldots, A_{NM-1}]^T \\
\Phi(t) &= A^T\Phi(t) \\
B(t) &= [B_{10}, B_{11}, \ldots, B_{1M-1}, \ldots, B_{N0}, B_{N1}, \ldots, B_{NM-1}]^T \\
\Phi(t) &= B^T\Phi(t)
\end{align}

where $A^T$ and $B^T$ have dimensions of $b \times bMN$ and $b \times rMN$, respectively.

Now, we have

\begin{align}
A(t)x(t) &= A^T\Phi(t)\Phi^TX = \Phi^TA^TX \\
B(t)u(t) &= B^T\Phi(t)\Phi^TU = \Phi^TB^TU
\end{align}

where $\tilde{A}$ and $\tilde{B}$ can be calculated similarly to the matrix $\tilde{C}$ in Equation (13). Furthermore

\begin{equation}
\int_0^t \Phi^T(t')dt' = (I_b \otimes \Phi^T(t))(I_b \otimes P^T) = \Phi^T(t)\tilde{P}^T
\end{equation}

By integrating Equation (14) from 0 to $t$ and using Equations (22)–(32), we have

\begin{equation}
\Phi^T(t)X - \Phi^T(t)d = \Phi^T(t)\tilde{P}^T\tilde{A}^TX + \Phi^T(t)\tilde{P}^T\tilde{B}^TU
\end{equation}

or

\begin{equation}
Z^* = (\tilde{P}^T\tilde{A}^T - I)X + \tilde{P}^T\tilde{B}^TU + d = 0,
\end{equation}

where $I$ is the $bMN$-dimensional identity matrix.

4. **An optimal control problem**

4.1. **The Lagrange approach**

4.1.1. The performance index approximation

By substituting Equations (22) and (23) into Equation (15), we obtain

\begin{equation}
J = \frac{1}{2}X^T \left[ \int_0^T \dot{\Phi}(t)Q(t)\dot{\Phi}(t)^Tdt \right]X \\
+ \frac{1}{2}U^T \left[ \int_0^T \dot{\Phi}_1(t)R(t)\dot{\Phi}_1(t)^Tdt \right]U
\end{equation}

Equation (34) can be computed more efficiently by writing $J$ as

\begin{equation}
J = \frac{1}{2}X^T \left[ \int_0^T \Phi(t)\Phi^T(t) \otimes Q(t)dt \right]X \\
+ \frac{1}{2}U^T \left[ \int_0^T \Phi(t)\Phi^T(t) \otimes R(t)dt \right]U.
\end{equation}

For problems with time-varying performance index, $Q(t)$ and $R(t)$ are functions of time, and

\begin{equation}
\int_0^T \Phi(t)\Phi^T(t) \otimes Q(t)dt, \int_0^T \Phi(t)\Phi^T(t) \otimes R(t)dt
\end{equation}

can be evaluated numerically.

4.1.2. Solution of the optimization problem

The optimal control problem has been reduced to a parameter optimization problem which can be stated as follows. Find $x$ and $u$ so that $J(x, u)$ is minimized subject to the constraint given in Equation (33)

\begin{equation}
J^*(x, u, \lambda) = J(x, u) + \lambda^T Z^*,
\end{equation}

where the vector $\lambda$ represents the unknown Lagrange multipliers, then the necessary conditions for optimization are given by

\begin{align}
\frac{\partial}{\partial x}J^*(x, u, \lambda) &= 0, \\
\frac{\partial}{\partial u}J^*(x, u, \lambda) &= 0,
\end{align}

\begin{equation}
\frac{\partial}{\partial \lambda}J^*(x, u, \lambda) = 0.
\end{equation}

4.2. **The Riccati approach**

It is well known that the optimal control variable $u(t)$ can be given as (Hwang & Chen, 1985)

\begin{equation}
u(t) = \tilde{R}^{-1}(t)\tilde{B}^T(t)p(t),
\end{equation}

where the adjoint variable $p(t)$, a $b$-vector, satisfies the following canonical equation:

\begin{equation}
\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = F(t) \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}, \quad \begin{bmatrix} x(0) \end{bmatrix} \text{ is specified}
\end{equation}

\begin{equation}
p(t_f) = 0
\end{equation}
where
\[
F(t) = \begin{bmatrix}
A(t) & B(t)R^{-1}(t)B^T(t) \\
Q(t) & -A^T(t)
\end{bmatrix} \in \mathbb{R}^{2b \times 2b}.
\] (42)

Letting \( \lambda(t_f, t) \) be the state transition matrix of Equation (41), it can be decomposed into the following form:
\[
\lambda(t_f, t) = \begin{bmatrix}
\lambda_{11}(t_f, t) & \lambda_{12}(t_f, t) \\
\lambda_{21}(t_f, t) & \lambda_{22}(t_f, t)
\end{bmatrix}.
\] (43)

Then, from the relation
\[
\dot{\lambda}(t_f, t) x(t) + \lambda(t_f, t) p(t) = 0.
\] (45)

Therefore
\[
p(t) = -\lambda_{22}^{-1}(t_f, t) \lambda_{21}(t_f, t) x(t).
\] (46)

Substituting Equation (46) into Equation (40), we obtain the optimal control law
\[
u(t) = -k(t)(t),
\] (47)

with the time-varying gain \( k(t) \)
\[
k(t) = R^{-1}(t)B^T(t)\lambda_{22}^{-1}(t_f, t)\lambda_{21}(t_f, t).
\] (48)

Since \( R(t) \) and \( B(t) \) are given, it is necessary to find only the matrices \( \lambda_{21}(t_f, t) \) and \( \lambda_{22}(t_f, t) \) to determine \( k(t) \).

Differentiating Equation (44) with respect to \( t \) yields
\[
\frac{d}{dt} \begin{bmatrix}
\lambda(t_f, t) x(t) \\
\lambda(t_f, t) p(t)
\end{bmatrix} = \dot{\lambda}(t_f, t) \begin{bmatrix}
x(t) \\
p(t)
\end{bmatrix} + \lambda(t_f, t) \begin{bmatrix}
\dot{x}(t) \\
\dot{p}(t)
\end{bmatrix} = 0.
\] (49)

Rearranging the above equation and substituting Equation (41) into it, we obtain
\[
\dot{\lambda}(t_f, t) \begin{bmatrix}
x(t) \\
p(t)
\end{bmatrix} = -\lambda(t_f, t) F(t) \begin{bmatrix}
x(t) \\
p(t)
\end{bmatrix}.
\] (50)

Since Equation (50) is true for all values of \( t \) on the time interval \([0, t_f]\), we have
\[
\dot{\lambda}(t_f, t) = -\lambda(t_f, t) F(t), \quad \lambda(t_f, t) = I.
\] (51)

Now, the Chebyshev wavelet approximations are used to solve for \( \lambda(t_f, t) \).

Integrating Equation (51) backward from \( t_f \) to \( t \) gives
\[
\lambda(t_f, t) = I = \int_{t_f}^{t} \lambda(t_f, t) F(t) dt,
\] (52)

where \( \lambda(t_f, t_f) = I \) has been used.

Suppose all elements of matrices \( \lambda(t_f, t) = [\lambda_{ij}(t_f, t)]_{2b \times 2b} \) and \( F(t) = [F_{ij}(t)]_{2b \times 2b} \) are square inerrable on the time interval \([0, t_f]\), then the Chebyshev wavelet approximations of \( \lambda(t_f, t) \) and \( F(t) \) are
\[
\lambda(t_f, t) = \begin{bmatrix}
\eta_{11}(t_f) \Phi_{11}(t) & \eta_{12}(t_f) \Phi_{12}(t) & \cdots & \eta_{12b}(t_f) \Phi_{12b}(t) \\
\eta_{21}(t_f) \Phi_{21}(t) & \eta_{22}(t_f) \Phi_{22}(t) & \cdots & \eta_{22b}(t_f) \Phi_{22b}(t) \\
\vdots & \vdots & \ddots & \vdots \\
\eta_{21}(t_f) \Phi_{21}(t) & \eta_{22}(t_f) \Phi_{22}(t) & \cdots & \eta_{22b}(t_f) \Phi_{22b}(t)
\end{bmatrix}.
\] (53)

and
\[
F(t) = \begin{bmatrix}
F_{11}(t) \Phi_{11}(t) & F_{12}(t) \Phi_{12}(t) & \cdots & F_{12b}(t) \Phi_{12b}(t) \\
F_{21}(t) \Phi_{21}(t) & F_{22}(t) \Phi_{22}(t) & \cdots & F_{22b}(t) \Phi_{22b}(t) \\
\vdots & \vdots & \ddots & \vdots \\
F_{21}(t) \Phi_{21}(t) & F_{22}(t) \Phi_{22}(t) & \cdots & F_{22b}(t) \Phi_{22b}(t)
\end{bmatrix}.
\] (54)

where \( \eta_{ij}(t_f) \) and \( F_{ij}(t) \), \( i, j = 1, 2, \ldots, 2b \), are the Chebyshev wavelet coefficient vectors of \( \lambda_{ij}(t_f, t) \) and \( F_{ij}(t) \), respectively
\[
\eta_{ij}(t_f) = [\eta_{ij0} \eta_{ij1} \cdots \eta_{ij,2b-1}],
\] (55)

\[
F_{ij}(t_f) = [F_{ij0} F_{ij1} \cdots F_{ij,2b-1}],
\] (56)

Therefore
\[
\lambda(t_f, t) F(t) = \begin{bmatrix}
\sum_{j=1}^{2b} \eta_{1j}^T(t_f) F_{1j}(t) \Phi_{1j}(t) \\
\sum_{j=1}^{2b} \eta_{2j}^T(t_f) F_{2j}(t) \Phi_{2j}(t)
\end{bmatrix}.
\] (57)

Substituting Equations (53) and (57) into Equation (52) and applying Equation (9), and equating the coefficients of \( \Phi_{NM}(t) \) of both sides of Equation (53) give
\[
\eta_{ik}(t_f) = -\sum_{j=1}^{2b} \eta_{ikj}^T(t_f) F_{ikj}(t) G_{NM} \quad \text{for } i \neq k,
\] (58)
coefficient vectors algebraic equations that can be used to solve the unknown

$$\begin{array}{ccccccccc}
0.0 & 0.966 & 0.962 & 0.967 & 0.964 & 0.966 & 1.297 & 0.969 & 0.969 \\
0.1 & 0.950 & 0.950 & 0.953 & 0.952 & 0.952 & 1.112 & 0.954 & 0.954 \\
0.2 & 0.908 & 0.910 & 0.912 & 0.911 & 0.912 & 0.940 & 0.911 & 0.911 \\
0.3 & 0.841 & 0.843 & 0.844 & 0.844 & 0.843 & 0.843 & 0.843 & 0.843 \\
0.4 & 0.749 & 0.750 & 0.751 & 0.752 & 0.753 & 0.753 & 0.753 & 0.753 \\
0.5 & 0.636 & 0.637 & 0.638 & 0.638 & 0.638 & 0.638 & 0.638 & 0.638 \\
0.6 & 0.506 & 0.508 & 0.509 & 0.509 & 0.509 & 0.509 & 0.509 & 0.509 \\
0.7 & 0.368 & 0.370 & 0.373 & 0.372 & 0.372 & 0.372 & 0.372 & 0.372 \\
0.8 & 0.231 & 0.234 & 0.236 & 0.235 & 0.235 & 0.235 & 0.235 & 0.235 \\
0.9 & 0.107 & 0.108 & 0.108 & 0.108 & 0.108 & 0.108 & 0.108 & 0.108 \\
1.0 & 0.007 & 0.007 & 0.001 & 0.004 & 0.003 & -0.026 & 0.000 & 0.000 \\
\end{array}\)
The values of cost functions for both cases are given in Table 4. Using the procedure given in Equation (4.1), the optimal solution of linear time-varying systems subject to a quadratic cost function. The method reduces the differential equation into a set of algebraic equations which is very convenient for digital computation. Illustrative examples show that only a small number of terms are required to obtain accurate approximations. Hence, the present method does not require large computer memory. We may therefore conclude that this paper presents an efficient and simple method for the optimal solution of linear time-varying systems. Finally, as it is shown by Tables 3 and 5, the Lagrange approach gives more accurate results when compared with the results of the Riccati approach. The following might be the reason: the Lagrange approach uses fewer approximations. (In the Riccati approach, in order to find \( J \), we should find \( k(t) \) first and then look for the values of \( x \) and \( u \), but in the Lagrange approach we can directly solve for \( x \) and \( u \)).

### 6. Conclusion

The Chebyshev wavelets and its operational matrices together with the product operational matrix have been used to find the optimal solution of linear time-varying systems subject to a quadratic cost function. The method reduces the differential equation into a set of algebraic equations which is very convenient for digital computation. Illustrative examples show that only a small number of terms are required to obtain accurate approximations. Hence, the present method does not require large computer memory. We may therefore conclude that this paper presents an efficient and simple method for the optimal solution of linear time-varying systems. Finally, as it is shown by Tables 3 and 5, the Lagrange approach gives more accurate results when compared with the results of the Riccati approach. The following might be the reason: the Lagrange approach uses fewer approximations. (In the Riccati approach, in order to find \( J \), we should find \( k(t) \) first and then look for the values of \( x \) and \( u \), but in the Lagrange approach we can directly solve for \( x \) and \( u \)).

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