Existence and multiplicity of solutions for a prescribed mean-curvature problem with critical growth

Giovany M. Figueiredo ∗
Universidade Federal do Pará
Faculdade de Matemática
CEP: 66075-110 Belém - Pa, Brazil.
e-mail: giovany@ufpa.br
and
Marcos T. O. Pimenta† ‡
Faculdade de Ciências e Tecnologia
Universidade Estadual Paulista - Unesp
19060-900, Presidente Prudente - SP, Brazil.
e-mail: pimenta@fct.unesp.br

Abstract
In this work we study an existence and multiplicity result for the following prescribed mean-curvature problem with critical growth

\[
\begin{aligned}
- \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) &= \lambda |u|^{q-2} u + |u|^{2^*-2} u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where Ω is a bounded smooth domain of \( \mathbb{R}^N \), \( N \geq 3 \) and \( 1 < q < 2 \). In order to employ variational arguments, we consider an auxiliary problem which is proved to have infinitely many solutions by genus theory. A clever estimate in the gradient of the solutions of the modified problem is necessary to recover solutions of the original one.

2010 Mathematics Subject Classification: 35J93, 35J62, 35J20.
Key words: prescribed mean-curvature problem, critical exponent, variational methods.

*Supported by PROCAD/CASADINHO: 552101/2011-7, CNPq/PQ 301242/2011-9 and CNPQ/CSF 200237/2012-8
†Supported by FAPESP: 2012/20160-0
‡Corresponding author
1 Introduction

In this work we deal with questions of existence and multiplicity of solutions for quasilinear problems with nonlinearity of Brézis-Nirenberg type (see [4])

\[
\begin{aligned}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) &= \lambda |u|^{q-2} u + |u|^{2^* - 2} u & \text{in } \Omega \\
\quad u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain, \( \lambda > 0, 1 < q < 2 \) and \( 2^* = \frac{2N}{N-2} \).

This kind of problem has applications not just to describe a surface given by \( u(x) \), whose mean curvature is described by the right hand side of \((P_\lambda)\), but also in capillarity theory where when the nonlinearity is replaced by \( \kappa u \), the resultant equation describe the equilibrium of a liquid surface with constant surface tension in a uniform gravity field (see p. 262 in [15]).

Problems like \((P_\lambda)\) has been intensively studied over the last decades. In the work [6], the authors studied a related subcritical problem in which they obtained positive solutions. In the recent work [3], Bonheure, Derlet and Valeriola have studied a purely subcritical version of \((P_\lambda)\), where they proved the existence and multiplicity of nodal \( H^1_0(\Omega) \) solutions, to sufficiently large values of \( \lambda \). They overcame the difficulty in working in the \( BV(\Omega) \) space, which is the natural functional space to treat \((P_\lambda)\), by doing a truncation in the degenerate part of the mean-curvature operator in order to make possible construct a variational framework in the Sobolev space \( H^1_0(\Omega) \). Nevertheless, this truncation requires sharp estimates on the gradient of the solutions, in order to prove that the solutions of the modified problem in fact are solutions of the original one.

When \( \Omega = \mathbb{R}^N \) and the nonlinearity is substituted by \( u^q \), i.e., the Gidas-Spruck analogue for the mean-curvature operator, Ni and Serrin in [10] [11] has proved that if \( 1 < q < \frac{N}{N-2} \) no positive solution exist, while for \( q \geq 2^* - 1 \) there exist infinitely many solutions. In the range \( \frac{N}{N-2} < q < 2^* - 1 \) some contributions has been given by Clément et al in [3] and by Del Pino and Guerra in [7], where in the latter the authors prove that many positive solutions do exist if \( q < 2^* - 1 \) is sufficiently close to \( 2^* - 1 \).

Still in the case \( \Omega = \mathbb{R}^N \) but with nonlinearity given by \( \lambda u + u^p \), Peletier and Serrin in [12] succeed in proving the existence of positive radial solutions when \( \lambda < 0 \) is small enough and \( p \) is subcritical. In the case \( \lambda > 0 \), they stated there is no regular solution to that problem no matter how much small or large \( p \) is.

In this work, because of the boundedness of \( \Omega \), we prove a result in a strike opposition of that in [12], in which we obtain the existence of infinitely many regular solutions of \((P_\lambda)\), for small enough \( \lambda > 0 \). More specifically, we prove the following result.

**Theorem 1.1.** If \( 1 < q < 2 \), then there exists \( \lambda^* > 0 \) such that if \( 0 < \lambda < \lambda^* \), \((P_\lambda)\) has infinitely many solutions. Moreover, if \( u_\lambda \) is a solution of \((P_\lambda)\), then
$u_{\lambda} \in H_{0}^{1}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$, and
\[
\lim_{\lambda \to 0} \|u_{\lambda}\| = \lim_{\lambda \to 0} \|u_{\lambda}\|_{\infty} = \lim_{\lambda \to 0} \|\nabla u_{\lambda}\|_{\infty} = 0,
\]
where $\| \cdot \|$ is the Sobolev norm in $H_{0}^{1}(\Omega)$.

Our approach follows the main ideas of Bonheure et al in [3], in order to make possible consider a related modified problem in $H_{0}^{1}(\Omega)$. Afterwards, to get solutions of the modified problem we apply Krasnoselskii genus theory in the same way that Azorero and Alonso in [1]. Finally, we use the Moser iteration technique and a regularity result by Lewy and Stampacchia in [14] to get decay in $\lambda$ of the gradient of the solutions, which will imply that the solutions of the modified problem in fact are solutions of the original one.

The paper is organized as follows. In the second section we present the auxiliary problem and the variational framework. In the third one we make a brief review of Genus theory. In the fourth we prove some technical results which imply on the existence of infinitely many solutions to the auxiliary problem. The last one is dedicated to present the proof of the main result, which consists in estimates in $L^{\infty}(\Omega)$ norm of the gradient of solutions.

2 The auxiliary problem and variational framework

Let us consider $r \geq 0$ and $\delta > 0$ and a function $\eta \in C^{1}([r,r+\delta])$ such that
\[
\eta(r) = \frac{1}{\sqrt{1 + r}}, \quad \eta(r + \delta) = \frac{1}{\sqrt{1 + r + \delta}},
\]
\[
\eta'(r) = -\frac{1}{2\sqrt{(1 + r)^{3}}} \quad \text{and} \quad \eta'(r + \delta) = 0.
\]

Now we define
\[
a(t) := \begin{cases} 
\frac{1}{\sqrt{1 + t}}, & \text{if } 0 \leq t \leq r, \\
\frac{1}{\sqrt{1 + r + \delta}}, & \text{if } r \leq t \leq r + \delta, \\
\frac{1}{1 + r + \delta}, & \text{if } t \geq r + \delta.
\end{cases}
\]

Note that $a \in C^{1}([0,\infty))$ is decreasing and $K_{0} \leq a(t) \leq 1$ for $t \in [0,\infty)$. Let us fix $r > 0$ such that
\[
\frac{2}{2\nu} < K_{0} < 1. \tag{2.1}
\]

The proof of the Theorem 1.1 is based on a careful study of solutions of the following auxiliary problem
\[
(T_{\lambda}) \quad \begin{cases} 
-\text{div}(a(|\nabla u|^{2})\nabla u) = \lambda |u|^q - 2u + |u|^{2^{*} - 2}u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
We say that \( u \in H_0^1(\Omega) \) is a weak solution \((T_\lambda)\) if it verifies
\[
\int_\Omega a(|\nabla u|^2) \nabla u \nabla \phi \, dx = \lambda \int_\Omega |u|^{q-2} u \phi \, dx + \int_\Omega |u|^{2^* - 2} u \phi \, dx,
\]
for all \( \phi \in H_0^1(\Omega) \). Let us consider \( H_0^1(\Omega) \) with its usual norm \( \|u\| = \left( \int_\Omega |\nabla u|^2 \right)^{1/2} \) and define the \( C^1 \)-functional \( I_\lambda : H_0^1(\Omega) \to \mathbb{R} \) by
\[
I_\lambda(u) = \frac{1}{2} \int_\Omega A(|\nabla u|^2) \, dx - \frac{\lambda}{q} \int_\Omega |u|^q \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx,
\]
where \( A(t) = \int_0^t a(s) \, ds \). Note that
\[
I'_\lambda(u) \phi = \int_\Omega a(|\nabla u|^2) \nabla u \nabla \phi \, dx - \lambda \int_\Omega |u|^{q-2} u \phi \, dx - \int_\Omega |u|^{2^* - 2} u \phi \, dx,
\]
for all \( \phi \in H_0^1(\Omega) \) and then, critical points of \( I_\lambda \) are weak solutions of \((T_\lambda)\).

In order to use variational methods, we first derive some results related to the Palais-Smale compactness condition.

We say that a sequence \( (u_n) \subset H_0^1(\Omega) \) is a \((PS)_{c_\lambda}\) sequence for \( I_\lambda \) if
\[
I_\lambda(u_n) \to c_\lambda \quad \text{and} \quad \|I'_\lambda(u_n)\|_{H^{-1}(\Omega)} \to 0, \quad \text{as} \quad n \to \infty \tag{2.2}
\]
where
\[
c_\lambda = \inf_{\pi \in \Gamma} \max_{t \in [0,1]} I_\lambda(\pi(t)) > 0
\]
and
\[
\Gamma := \{ \pi \in C([0,1], H_0^1(\Omega)) : \pi(0) = 0, \ I_\lambda(\pi(1)) < 0 \}.
\]

If \( (2.2) \) implies the existence of a subsequence \( (u_{n_j}) \subset (u_n) \) which converges in \( H_0^1(\Omega) \), we say that \( I_\lambda \) satisfies the Palais-Smale condition on the level \( c_\lambda \).

### 3 Genus theory

We start by considering some basic facts on the Krasnoselskii genus theory that we will use in the proof of Theorem 1.1.

Let \( E \) be a real Banach space. Let us denote by \( \mathfrak{A} \) the class of all closed subsets \( A \subset E \setminus \{0\} \) that are symmetric with respect to the origin, that is, \( u \in A \) implies \( -u \in A \).

**Definition 3.1.** Let \( A \in \mathfrak{A} \). The Krasnoselskii genus \( \gamma(A) \) of \( A \) is defined as being the least positive integer \( k \) such that there is an odd mapping \( \phi \in C(A, \mathbb{R}^k) \) such that \( \phi(x) \neq 0 \) for all \( x \in A \). When such number does not exist we set \( \gamma(A) = \infty \). Furthermore, by definition, \( \gamma(\emptyset) = 0 \).
In the sequel we will establish only the properties of the genus that will be used through this work. More informations on this subject may be found in [9].

**Theorem 3.2.** Let \( E = \mathbb{R}^N \) and \( \partial \Omega \) be the boundary of an open, symmetric and bounded subset \( \Omega \subset \mathbb{R}^N \) such that \( 0 \in \Omega \). Then \( \gamma(\partial \Omega) = N \).

**Corollary 3.3.** \( \gamma(S^{N-1}) = N \).

**Proposition 3.4.** If \( K \in \mathfrak{A}, \ 0 \notin K \) and \( \gamma(K) \geq 2 \), then \( K \) has infinitely many points.

## 4 Technical results

The genus theory requires that the functional \( I_\lambda \) is bounded from below. Since this is not the case, it is necessary to work with a related functional, which will be done employing some ideas contained in [1].

In the light of the Proposition 3.4 it seems to be useful prove that the set of critical points of the related functional has genus greater than 2, in order to obtain infinitely many solutions of \((T_\lambda)\).

Let’s gonna present the way in which we truncate the function \( I_\lambda \). From (2.1) and Sobolev’s embedding, we get

\[
I_\lambda(u) \geq K_0 \frac{1}{2} ||u||^2 - \frac{\lambda}{qS_q^{q/2}} ||u||^q - \frac{1}{2^{*}S^{2^{*}/2}} ||u||^{2^{*}} = g(||u||^2),
\]

where \( S \) and \( S_q \) are, respectively, the best constants of the Sobolev’s embeddings \( H^1_0(\Omega) \hookrightarrow L^2(\Omega) \) and \( H^1_0(\Omega) \hookrightarrow L^q(\Omega) \) and

\[
g(t) = K_0 t - \frac{\lambda}{qS_q^{q/2}} t^{q/2} - \frac{1}{2^{*}S^{2^{*}/2}} t^{2^{*}/2}. \tag{4.1}
\]

Hence, there exists \( \tau_1 > 0 \) such that, if \( \lambda \in (0, \tau_1) \), then \( g \) attains its positive maximum.

Let \( R_0 \leq R_1 \) the roots of \( g \). We have that \( R_0 = R_0(\tau_1) \) and the following result holds:

**Lemma 4.1.**

\[
R_0(\tau_1) \to 0 \quad \text{as} \quad \lambda \to 0. \tag{4.2}
\]

**Proof:** From \( g(R_0(\tau_1)) = 0 \) and \( g'(R_0(\tau_1)) > 0 \), we have

\[
\frac{K_0}{2} R_0(\tau_1) = \frac{\lambda}{qS_q^{q/2}} R_0(\tau_1)^{q/2} + \frac{1}{2^{*}S^{2^{*}/2}} R_0(\tau_1)^{2^{*}/2} \tag{4.3}
\]

and

\[
\frac{K_0}{2} > \frac{\lambda}{2qS_q^{q/2}} R_0(\tau_1)^{(q-2)/2} + \frac{1}{2S^{2^{*}/2}} R_0(\tau_1)^{(2^{*}-2)/2}, \tag{4.4}
\]
for all \( \lambda \in (0, \tau_1) \). From (4.3) we conclude that \( R_0(\tau_1) \) is bounded. Suppose that \( R_0(\tau_1) \to R_0 > 0 \) as \( \lambda \to 0 \). Then,

\[
\frac{K_0}{2} = \frac{1}{2^{*}S^{2*/2}}R_0(\tau_1)^{2^*/2}
\]

and

\[
\frac{K_0}{2} \geq \frac{1}{2^{*}S^{2*/2}}R_0(\tau_1)^{2^*/2},
\]

which is a contradiction, because \( 2^* > 2 \). Therefore \( R_0 = 0 \).

We consider \( \tau_1 \) such that \( R_0 \leq r \) and we modify the functional \( I_\lambda \) in the following way. Take \( \phi \in C^\infty([0, +\infty)) \), \( 0 \leq \phi \leq 1 \) such that \( \phi(t) = 1 \) if \( t \in [0, R_0] \) and \( \phi(t) = 0 \) if \( t \in [R_1, +\infty) \). Now, we consider the truncated functional

\[
J_\lambda(u) = \frac{1}{2} \int_\Omega A(|\nabla u|^2) \, dx - \frac{\lambda}{q} \int_\Omega |u|^q \, dx - \phi(||u||^2) \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx.
\]

Note that \( J_\lambda \in C^1(H_0^1(\Omega), \mathbb{R}) \) and, as in (4.1),

\[
J_\lambda(u) \geq \gamma(||u||^2),
\]

where

\[
\gamma(t) = \frac{K_0}{2} t - \frac{\lambda}{qS^{q/2}} t^{q/2} - \phi(t) \frac{1}{2^*S^{2*/2}} t^{2^*/2}.
\]

Let us remark that if \( ||u||^2 \leq R_0 \), then \( J_\lambda(u) = I_\lambda(u) \) and if \( ||u||^2 \geq R_1 \), then \( J_\lambda(u) = \frac{1}{2} \int_\Omega A(|\nabla u|^2) \, dx - \frac{\lambda}{q} \int_\Omega |u|^q \, dx \), which implies that \( J_\lambda \) is coercive and hence bounded from below.

Now we show that \( J_\lambda \) satisfy the local Palais-Smale condition. For this, we need the following technical result, which is analogous of Lemma 4.2 in [1].

**Lemma 4.2.** Let \( (u_n) \subset H_0^1(\Omega) \) be a bounded sequence such that

\[
I_\lambda(u_n) \to c_\lambda \text{ and } I'_\lambda(u_n) \to 0.
\]

If

\[
c_\lambda < \left( \frac{K_0}{2} - \frac{1}{2^*} \right) K_0^{(N-2)/2} S^{N/2}
\]

then up to a subsequence \( (u_n) \) is strongly convergent in \( H_0^1(\Omega) \).

**Proof:** Taking a subsequence, we may suppose that

\[
|\nabla u_n|^2 \to |\nabla u|^2 + \sigma \quad \text{and} \quad |u_n|^{2^*} \to |u|^{2^*} + \nu \quad \text{in the weak* sense of measures.}
\]
Using the concentration compactness-principle due to Lions (cf. [8, Lemma 2.1]), we obtain an at most countable index set $\Lambda$, sequences $(x_i) \subset \Omega$, $(\mu_i), (\sigma_i), (\nu_i) \subset [0, \infty)$, such that

$$\nu = \sum_{i \in \Lambda} \nu_i \delta_{x_i}, \quad \sigma \geq \sum_{i \in \Lambda} \sigma_i \delta_{x_i}, \quad \text{and} \quad S \nu_i^{2/2} \leq \sigma_i,$$

for all $i \in \Lambda$, where $\delta_{x_i}$ is the Dirac mass at $x_i \in \Omega$.

Now we claim that $\Lambda = \emptyset$. Arguing by contradiction, assume that $\Lambda \neq \emptyset$ and fix $i \in \Lambda$. Consider $\psi \in C_0^\infty(\Omega, [0, 1])$ such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\Omega \setminus B_2(0)$ and $|\nabla \psi|_{\infty} \leq 2$. Defining $\psi_\rho(x) := \psi((x - x_i)/\rho)$ where $\rho > 0$, we have that $(\psi_\rho u_n)$ is bounded. Thus $I'_\lambda(u_n)(\psi_\rho u_n) \to 0$, that is,

$$\int_{\Omega} a(|\nabla u_n|^2)u_n \nabla u_n \nabla \psi_\rho \, dx + \int_{\Omega} a(|\nabla u_n|^2)\psi_\rho |\nabla u_n|^2 \, dx = \lambda \int_{\Omega} |u_n|^q \psi_\rho \, dx + \int_{\Omega} \psi_\rho |u_n|^{2*} \, dx + o_n(1).$$

Since $\text{supp}(\psi_\rho) \subset B_{2\rho}(x_i)$, we obtain

$$\left| \int_{\Omega} u_n \nabla u_n \nabla \psi_\rho \, dx \right| \leq \int_{B_{2\rho}(x_i)} |\nabla u_n| |u_n \nabla \psi_\rho| \, dx.$$

By Hölder inequality and the fact that the sequence $(u_n)$ is bounded in $H_0^1(\Omega)$ we have

$$\left| \int_{\Omega} u_n \nabla u_n \nabla \psi_\rho \, dx \right| \leq C \left( \int_{B_{2\rho}(x_i)} |u_n \nabla \psi_\rho|^2 \, dx \right)^{1/2}.$$

By the Dominated Convergence Theorem $\int_{B_{2\rho}(x_i)} |u_n \nabla \psi_\rho|^2 \, dx \to 0$ as $n \to +\infty$ and $\rho \to 0$. Thus, we obtain

$$\lim_{\rho \to 0} \left[ \lim_{n \to \infty} \int_{\Omega} u_n \nabla u_n \nabla \psi_\rho \, dx \right] = 0.$$

Since $0 < K_0 \leq a(t) \leq 1$, for all $t \in \mathbb{R}$, we get

$$\lim_{\rho \to 0} \lim_{n \to \infty} \left[ \int_{\Omega} a(|\nabla u_n|^2)u_n \nabla u_n \nabla \psi_\rho \, dx \right] = 0.$$

Moreover, similar arguments applies in order to obtain

$$\lim_{\rho \to 0} \lim_{n \to \infty} \left[ \int_{\Omega} \psi_\rho |u_n|^q \, dx \right] = 0.$$
Thus, we have
\[ K_0 \int_\Omega \psi \varrho d\sigma \leq \int_\Omega \psi \varrho d\nu + o_\varrho(1). \]

Letting \( \varrho \to 0 \) and using standard theory of Radon measures, we conclude that \( K_0 \sigma_I \leq \nu_i \). It follows from (4.7) that
\[ \sigma_i \geq K_0^{(N-2)/2} S^{N/2}. \tag{4.8} \]

Now we shall prove that the above expression cannot occur, and therefore the set \( \Lambda \) is empty. Indeed, if for some \( i \in \Lambda \) (4.8) hold, then
\[
\begin{align*}
c_\lambda &= I_\lambda(u_n) - \frac{1}{2^s} I_\lambda'(u_n) u_n + o_n(1) \\
&\geq \left( K_0 - \frac{1}{2^s} \right) \int_\Omega |\nabla u_n|^2 \, dx - \lambda \left( \frac{1}{q} - \frac{1}{2^s} \right) \int_\Omega |u_n|^q \, dx.
\end{align*}
\]

By H"older’s inequality and Sobolev’s embedding we obtain
\[
\begin{align*}
c_\lambda &\geq \left( K_0 - \frac{1}{2^s} \right) K_0^{(N-2)/2} S^{N/2} + \left( K_0 - \frac{1}{2^s} \right) \frac{1}{S^{2^s/2}} \int_\Omega |u|^2^s \, dx \\
&\quad - \lambda \left( \frac{1}{q} - \frac{1}{2^s} \right) |\Omega|^{(2^*-q)/2^s} \left( \int_\Omega |u|^{2^*} \, dx \right)^{q/2^s}.
\end{align*}
\]

Note that
\[
f(t) = \left( K_0 - \frac{1}{2^s} \right) \frac{1}{S^{2^s/2}} t^{2^s} - \lambda \left( \frac{1}{q} - \frac{1}{2^s} \right) |\Omega|^{(2^*-q)/2^s} t^q
\]
is a continuous function that attains its absolute minimum, for \( t > 0 \), at the point
\[
o_0 = \left[ \frac{q}{2^s} \lambda \left( \frac{1}{q} - \frac{1}{2^s} \right) |\Omega|^{(2^*-q)/2^s} \left( \frac{K_0}{2^s} \right)^{-1} S^{2^s/2} \right]^{1/(2^*-q)}.
\]
Then,
\[ c_\lambda \geq (\frac{K_0}{2} - \frac{1}{2^s})K_0^{(N-2)/2} S^{N/2} + (\frac{K_0}{2} - \frac{1}{2^s}) \frac{1}{S^{2s/2}} \alpha_0^2 - \lambda \left( \frac{1}{q} - \frac{1}{2^s} \right) |\Omega| \frac{(2^s-q)}{2^s} \alpha_0^q. \]

So

\[ c_\lambda \geq (\frac{K_0}{2} - \frac{1}{2^s})K_0^{(N-2)/2} S^{N/2} \]

Thus, up to a subsequence,\[ J(u) = \int_\Omega |\nabla u|^2 \] we obtain
\[ \lambda (1 - \frac{1}{2^s}) |\Omega| \frac{(2^s-q)}{2^s} \frac{1}{S^{2s/2}} \left( \frac{K_0}{2} - \frac{1}{2^s} \right) \alpha_0^q \]

which is a contradiction. Thus \( \Lambda \) is empty and it follows that \( u_n \to u \) in \( L^2(\Omega) \).

Thus, we conclude that

\[ v_n = I_{\lambda}(u_n)u_n - I_{\lambda}(u)u + o_n(1) = o_n(1). \]

By the Lemma 4.2 we conclude that, there exists \( \tau_2 > 0 \) such that, for all \( \lambda \in (0, \tau_2) \) we get

\[ (\frac{K_0}{2} - \frac{1}{2^s})K_0^{(N-2)/2} S^{N/2} \]

and, hence, if \( (u_n) \) is a bounded sequence such that \( I_{\lambda}(u_n) \to c, I_{\lambda}'(u_n) \to 0 \) with \( c < 0 \), then \( (u_n) \) has a convergent subsequence.

Lemma 4.3. If \( J_\lambda(u) < 0 \), then \( \|u\|^2 < R_0 \leq r \) and \( J_\lambda(v) = I_\lambda(v) \), for all \( v \) in a small enough neighborhood of \( u \). Moreover, \( J_\lambda \) verifies a local Palais-Smale condition for \( c < 0 \).

Proof: Since \( \pi(\|u\|^2) \leq J_\lambda(u) < 0 \), then \( \|u\|^2 < R_0 \leq r \). By the choice of \( \tau_1 \) in 4.2 we have that \( J_\lambda(u) = I_\lambda(u) \). Moreover, since \( J_\lambda \) is continuous, we conclude that \( J_\lambda(v) = I_\lambda(v) \), for all \( v \in B_{R_0/2}(0) \). Besides, if \( (u_n) \) is a sequence such that \( J_\lambda(u_n) \to c \) and \( J_\lambda'(u_n) \to 0 \) as \( n \to \infty \), then for \( n \) sufficiently large \( I_{\lambda}(u_n) = J_\lambda(u_n) \to c < 0 \) and \( I_{\lambda}'(u_n) = J_\lambda'(u_n) \to 0 \) as \( n \to \infty \). Since \( J_\lambda \) is coercive, we get that \( (u_n) \) is bounded in \( H^1_0(\Omega) \). From Lemma 4.2 for all \( \lambda \in (0, \tau_2) \), we obtain

\[ c < 0 < (\frac{K_0}{2} - \frac{1}{2^s})K_0^{(N-2)/2} S^{N/2} \]

and

\[ \lambda (1 - \frac{1}{2^s}) |\Omega| \frac{(2^s-q)}{2^s} \left( \frac{K_0}{2} - \frac{1}{2^s} \right) \alpha_0^q \]

Thus, up to a subsequence,
and hence, up to a subsequence \((u_n)\) is strongly convergent in \(H^1_0(\Omega)\).

Now, we construct an appropriate minimax sequence of negative critical values.

**Lemma 4.4.** Given \(k \in \mathbb{N}\), there exists \(\epsilon = \epsilon(k) > 0\) such that
\[
\gamma(J^\epsilon) \geq k,
\]
where \(J^\epsilon = \{ u \in H^1_0(\Omega) : J_\lambda(u) \leq -\epsilon \}\).

**Proof:** Consider \(k \in \mathbb{N}\) and let \(X_k\) be a \(k\)-dimensional subspace of \(H^1_0(\Omega)\). Since in \(X_k\) all norms are equivalent, there exists \(C(k) > 0\) such that
\[
-C(k)||u||^q \geq -\int_\Omega |u|^q \, dx,
\]
for all \(u \in X_k\).

We now use the inequality above to conclude that
\[
J_\lambda(u) \leq \frac{1}{2} ||u||^2 - \frac{C(k)}{q} ||u||^q = ||u||^q \left( \frac{1}{2} ||u||^{2-q} - \frac{C(k)}{q} \right),
\]
for all \(u \in X_k\).

Considering \(R > 0\) sufficiently small, there exists \(\epsilon = \epsilon(R) > 0\) such that
\[
\lambda u \left( S_R \right) < -\epsilon < 0,
\]
for all \(u \in S_R = \{ u \in X_k; ||u|| = R \}\). Since \(X_k\) and \(\mathbb{R}^k\) are isomorphic and \(S_R\) and \(S^{k-1}\) are homeomorphic, we conclude from Corollary 3.3 that \(\gamma(S_R) = \gamma(S^{k-1}) = k\). Moreover, once that \(S_R \subset J^\epsilon\) and \(J^\epsilon\) is symmetric and closed, we have
\[
k = \gamma(S_R) \leq \gamma(J^\epsilon).
\]

We define now, for each \(k \in \mathbb{N}\), the sets
\[
\Gamma_k = \{ C \subset H : C \text{ is closed, } C = -C \text{ and } \gamma(C) \geq k \},
\]
\[
K_c = \{ u \in H : J_\lambda(u) = 0 \text{ and } J_\lambda(u) = c \}
\]
and the number
\[
c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J_\lambda(u).
\]

**Lemma 4.5.** Given \(k \in \mathbb{N}\), the number \(c_k\) is negative.

**Proof:** From Lemma 4.4, for each \(k \in \mathbb{N}\) there exists \(\epsilon > 0\) such that \(\gamma(J^\epsilon) \geq k\). Moreover, \(0 \notin J^\epsilon\) and \(J^\epsilon \in \Gamma_k\). On the other hand
\[
\sup_{u \in J^\epsilon} J_\lambda(u) \leq -\epsilon.
\]
Hence,
\[-\infty < c_k = \inf_{C \in \Gamma_k} \sup_{u \in C} J_\lambda(u) \leq \sup_{u \in J_\lambda^\ast} J_\lambda(u) \leq -\epsilon < 0.\]

The next Lemma allows us to prove the existence of critical points of $J_\lambda$.

Lemma 4.6. If $c = c_k = c_{k+1} = \ldots = c_{k+r}$ for some $r \in \mathbb{N}$, then there exists $\lambda^* > 0$ such that
\[\gamma(K_c) \geq r + 1,\]
for $\lambda \in (0, \lambda^*)$.

Proof: Since $c = c_k = c_{k+1} = \ldots = c_{k+r} < 0$, for $\lambda^* = \min\{\tau_1, \tau_2\}$ and for all $\lambda \in (0, \lambda^*)$, from Lemma 4.2 and Lemma 4.3, we get that $K_c$ is compact. Moreover, $K_c = -K_c$. If $\gamma(K_c) \leq r$, there exists a closed and symmetric set $U$ with $K_c \subset U$ such that $\gamma(U) = \gamma(K_c) \leq r$. Note that we can choose $U \subset J_\lambda^0$ because $c < 0$. By the deformation lemma [2] we have an odd homeomorphism $\eta : H \to H$ such that $\eta(J_\lambda^{c+\delta} - U) \subset J_\lambda^{c-\delta}$ for some $\delta > 0$ with $0 < \delta < -c$. Thus, $J_\lambda^{c+\delta} \subset J_\lambda^0$ and by definition of $c = c_{k+r}$, there exists $A \in \Gamma_{k+r}$ such that $\sup_{u \in A} < c + \delta$, that is, $A \subset J_\lambda^{c+\delta}$ and
\[\eta(A - U) \subset \eta(J_\lambda^{c+\delta} - U) \subset J_\lambda^{c-\delta}.\]
But $\gamma(A - U) > \gamma(A) - \gamma(U) \geq k$ and $\gamma(\eta(A - U)) \geq \gamma(A - U) \geq k$. Then $\eta(A - U) \in \Gamma_k$ which contradicts (4.9).

5 Proof of Theorem 1.1

If $-\infty < c_1 < c_2 < \ldots < c_k < \ldots < 0$ with $c_i \neq c_j$, once each $c_k$ is a critical value of $J_\lambda$, we obtain infinitely many critical points of $J_\lambda$ and then, $(T_\lambda)$ has infinitely many solutions.

On the other hand, if $c_k = c_{k+r}$ for some $k$ and $r$, then $c = c_k = c_{k+1} = \ldots = c_{k+r}$ and from Lemma 4.6 there exists $\lambda^* > 0$ such that
\[\gamma(K_c) \geq r + 1 \geq 2\]
for all $\lambda \in (0, \lambda^*)$. From Proposition 3.4 $K_c$ has infinitely many points, that is, $(T_\lambda)$ has infinitely many solutions.

Let $\lambda^*$ be as in Lemma 4.4 and, for $\lambda \in (0, \lambda^*)$, let $u_\lambda$ be a solution of $(T_\lambda)$. Thus $J_\lambda(u_\lambda) = I_\lambda(u_\lambda) < 0$. Hence,
\[\|u_\lambda\|^2 \leq R_0,\]
which together with (4.2) implies that
\[ \lim_{\lambda \to 0} \| u_\lambda \| = 0. \] (5.1)

Now we use the Moser iteration technique in order to prove that there exists a constant positive \( C \), independent on \( \lambda \) such that
\[ \| u_\lambda \|_\infty \leq C \| u_\lambda \|. \] (5.2)

Using (5.2) we can conclude that \( \lim_{\lambda \to 0} \| u_\lambda \|_\infty = 0 \). (5.3)

In order to save notation, from now on we denote \( u_\lambda \) by \( u \). In what follows, we fix \( R > R_1 > 0 \), \( R > 1 \) and take a cut-off function \( \eta_R \in C_0^\infty(\Omega) \) such that \( 0 \leq \eta_R \leq 1 \), \( \eta_R \equiv 0 \) in \( B_R^c \), \( \eta_R \equiv 1 \) in \( B_{R_1} \), and \( |\nabla \eta_R| \leq C/R \), where \( B_R \subset \Omega \) and \( C > 0 \) is a constant.

Let \( h(t) = \lambda t^{q-1} + t^{2^* - 1} \). Thus
\[ |h(t)| \to 0 \quad \text{as} \quad t \to 0 \]
and
\[ \frac{|h(t)|}{t^{2^* - 1}} \to 1 \quad \text{as} \quad t \to \infty. \]

Thus, for all \( \delta > 0 \) there is \( C_\delta(\lambda) > 0 \) such that
\[ h(t) \leq \delta + C_\delta(\lambda)t^{2^* - 1}. \] (5.4)

Moreover, for \( \lambda \in [0, \lambda_0] \), \( C_\delta(\lambda) \) can be chosen uniformly in \( \lambda \) in such a way that (5.4) holds independently of \( \lambda \). For each \( L > 0 \), define
\[ u_L(x) = \begin{cases} u(x), & \text{if} \quad u(x) \leq L \\ L, & \text{if} \quad u(x) \geq L. \end{cases} \]

with \( \sigma > 1 \) to be determined later. In the course of this proof, \( C_1, C_2, \ldots \), denote constants independent of \( \lambda \).

Taking \( z_L \) as a test function we obtain
\[ I'(u)z_L = 0. \]

More specifically,
\[ \int_\Omega a(|\nabla u|^2) \nabla u \nabla z_L = \lambda \int_\Omega u^{q-1} z_L + \int_\Omega u^{2^*-1} z_L. \]

12
Hence
\[ K_0 \int_\Omega \nabla u \nabla z_L \leq \int_\Omega h(u) z_L. \]

By (5.4) we obtain
\[ \int_\Omega \nabla u \nabla z_L \leq \delta K_0^{-1} \int_\Omega z_L + K_0^{-1} C_\delta \int_\Omega u^{2^*-1} z_L. \]

Let us fix \( \delta > 0 \) small enough in such a way that
\[ \int_\Omega \nabla u \nabla z_L \leq C \int_\Omega u^{2^*-1} z_L. \]

Using \( z_L \) we obtain
\[ \int_\Omega \eta^2 R u^2 (\sigma - 1) |\nabla u|^2 \ dx \leq - \int_\Omega \eta R u u^2 (\sigma - 1) \nabla \eta R \nabla u \ dx + \int_\Omega \eta^2 R u^2 (\sigma - 1) \ dx, \]
and the definition of \( u_L \) implies
\[ -2(\sigma - 1) \int_\Omega u_L^{2(\sigma - 3)} u \nabla u \nabla u_L \leq 0. \]

Thus
\[ \int_\Omega \eta^2 R u^2 (\sigma - 1) |\nabla u|^2 \ dx \leq + \int_\Omega \eta R u u^2 (\sigma - 1) |\nabla \eta R|^2 |\nabla u| \ dx + \int_\Omega \eta^2 R u^2 (\sigma - 1) \ dx. \]

Taking \( z_L \) as a test function and using (5.4), we obtain
\[ \int_\Omega \eta^2 R u^2 (\sigma - 1) |\nabla u|^2 \ dx \leq C_1 \int_\Omega \eta R u u^2 (\sigma - 1) |\nabla \eta R|^2 |\nabla u| \ dx + C_1 \int_\Omega \eta^2 R u^2 (\sigma - 1) \ dx. \]

Fixing \( \tau > 0 \) and using Young’s inequality, we obtain
\[ \int_\Omega \eta^2 R u^2 (\sigma - 1) |\nabla u|^2 \ dx \leq C_1 \int_\Omega \left( \tau \eta R |\nabla u|^2 + C \tau u^2 |\nabla \eta R|^2 \right) u^2 (\sigma - 1) \ dx + C_1 \int_\Omega \eta^2 R u^2 (\sigma - 1) \ dx. \]

Choosing \( \tau \leq 1/4 \), it follows that
\[ \int_\Omega \eta^2 R u^2 (\sigma - 1) |\nabla u|^2 \ dx \leq C_2 \left( \int_\Omega u^2 (\sigma - 1) |\nabla \eta R|^2 \ dx + \int_\Omega \eta^2 R u^2 (\sigma - 1) \ dx \right), \]
On the other hand, we get
\[ S \| w_L \|_{L^2(\Omega)}^2 \leq \int_{\Omega} \left| \nabla (\eta_R u L^{-\alpha}) \right|^2 \]
\[ \leq \int_{\Omega} |u|^2 u L^2(\sigma-1) |\nabla \eta_R|^2 + \int_{\Omega} \eta_R^2 |\nabla (u u L^{-\alpha})|^2. \]

But
\[ \int_{\Omega} \eta_R^2 |\nabla (u u L^{-\alpha})|^2 = \int_{\{ |u| \leq L \}} \eta_R^2 |\nabla (u u L^{-\alpha})|^2 + \int_{\{ |u| > L \}} \eta_R^2 |\nabla (u u L^{-\alpha})|^2 \]
\[ = \int_{\{ |u| \leq L \}} \eta_R^2 |\nabla u|^2 + \int_{\{ |u| > L \}} \eta_R^2 u L^2(\sigma-1) |\nabla u|^2 \]
\[ \leq \sigma^2 \int_{\Omega} \eta_R^2 u^2 L^2(\sigma-1) |\nabla u|^2, \]
and therefore
\[ \| w_L \|_{L^2(\Omega)}^2 \leq C_3 \sigma^2 \left( \int_{\Omega} |u|^2 u L^2(\sigma-1) |\nabla \eta_R|^2 + \int_{\Omega} \eta_R^2 u L^2(\sigma-1) |\nabla u|^2 \right). \]

From this and [35],
\[ \| w_L \|_{L^2(\Omega)}^2 \leq C_4 \sigma^2 \left( \int_{\Omega} |u|^2 u L^2(\sigma-1) |\nabla \eta_R|^2 + \int_{\Omega} \eta_R^2 u^2 L^2(\sigma-1) \right), \quad (5.6) \]
for all \( \sigma > 1 \). The above expression, the properties of \( \eta_R \) and \( u_L \leq u \), imply that
\[ \| w_L \|_{L^2(\Omega)}^2 \leq C_4 \sigma^2 \int_{B_R} \left( |u|^2 |\nabla \eta_R|^2 + |u|^{2^* - 2} |u|^2 \right). \quad (5.7) \]

If we set
\[ t := \frac{2^* 2^*}{2(2^* - 2)} > 1, \quad \alpha := \frac{2t}{t-1} < 2^*, \quad (5.8) \]
we can apply Hölder’s inequality with exponents \( t/(t-1) \) and \( t \) in [37] to get
\[ \| w_L \|_{L^2(\Omega)}^2 \leq C_4 \sigma^2 \| u \|_{L^{2^* \alpha}(B_R)}^{2^* \alpha} \left( \int_{B_R} |\nabla \eta_R|^{2^*} \right)^{1/t} + C_4 \sigma^2 \| u \|_{L^{2^* \alpha}(B_R)}^{2^* \alpha} \left( \int_{B_R} |u|^{2^* (2^* / 2)} \right)^{1/t}. \quad (5.9) \]
Since \( \eta_R \) is constant on \( B_R \cup B_R^c \) and \( |\nabla \eta_R| \leq C/R \), we conclude that
\[ \int_{B_R} |\nabla \eta_R|^{2^*} = \int_{B_R \setminus B_{R_1}} |\nabla \eta_R|^{2^*} \leq \frac{C_5}{R^{2^* - N}} \leq C_5. \quad (5.10) \]
We have used \( R > 1 \) and \( 2t = \frac{2^*}{2} N > N \) in the last inequality.
Claim. There exist a constants $K > 0$ independent on $\lambda$ such that,
\[
\int_{\Omega} |u|^{2^*(2^*/2)} \leq K.
\]

Assuming the claim is true, we can use (5.9) and (5.10) to conclude that
\[
\|w_L\|_{L^{2^*}(\Omega)}^2 \leq C_6 \sigma^2 \|u\|_{L^2^*(B_R)}^{2^*_\sigma}.
\]
Since
\[
\|u_L\|_{L^2^*(B_R)}^{2^*_\sigma} = \left( \int_{B_R} u_L^{2^*_\sigma} \right)^{2/2^*} \leq \left( \int_{\Omega} \eta_R^{2} |u|^{2^*_\sigma} u^{2^*(\sigma-1)} \right)^{2/2^*} = \|w_L\|_{L^{2^*}(\Omega)}^2 \leq C_6 \sigma^2 \|u\|_{L^2^*(B_R)}^{2^*_\sigma},
\]
we can apply Fatou’s lemma in the variable $L$ to obtain
\[
\|u\|_{L^{2^*}(B_R)} \leq C_7 \sigma^{1/\sigma} \sigma^{1/\sigma} \|u\|_{L^2^*(B_R)},
\]
whenever $|u|^\sigma \in L^1(B_R)$. Here, $C_7$ is a positive constant independent on $R$. Iterating this process, for each $k \in \mathbb{N}$, it follows that
\[
\|u\|_{L^{2^*}(B_R)} \leq C_7^{\sum_{i=1}^{k} \sigma^{-i}} \sigma^{\sum_{i=1}^{m} \sigma^{-i}} \|u\|_{L^2^*(\Omega)}^i.
\]
Since $\Omega$ can be covered by a finite number of balls $B_R^l$, we have that
\[
\|u\|_{L^{2^*}(\Omega)} \leq \sum_{j=finite} \|u\|_{L^{2^*}(B_R^j)} \leq \sum_{j=finite} C_7^{\sum_{i=1}^{k} \sigma^{-i}} \sigma^{\sum_{i=1}^{m} \sigma^{-i}} \|u\|_{L^2^*(\Omega)}^i.
\]
Since $\sigma > 1$, we let $k \to \infty$ to get
\[
\|u\|_{L^\infty(\Omega)} \leq K_2 \|u\|,
\]
for some $K_2 > 0$ independent on $\lambda$.

It remains to prove the claim. From (5.10)
\[
\|w_L\|_{L^{2^*}(\Omega)}^2 \leq C_9 \sigma^2 \left( \int_{\Omega} |u|^2 u_L^{2(\sigma-1)} |\nabla \eta_R|^2 + \int_{\Omega} \eta_R^2 |u|^2 u_L^{2(\sigma-1)} \right),
\]
We set $\sigma := 2^*/2$ in (5.6) to obtain
\[
\|w_L\|_{L^{2^*}(\Omega)}^2 \leq C_{10} \left( \int_{\Omega} |u|^2 u_L^{(2^*/2)-2} |\nabla \eta_R|^2 + \int_{\Omega} \eta_R^2 |u|^2 u_L^{(2^*/2)-2} |u|^{(2^*/2)-2} \right).
\]

By Hölder’s inequality with exponents $2^*/2$ and $2^*/(2^* - 2)$ we get
\[
\|w_L\|_{L^{2^*}(\Omega)}^2 \leq C_{10} \int_{\Omega} |u|^2 u_L^{(2^*-2)}|\nabla \eta_R|^2 
+ C_{10} \left( \int_{B_R} (\eta_R |u| u_L^{(2^*-2)/2})^{2^*/2} \right) \|u\|_{L^{2^*}(\Omega)}^{2^*/2}.
\]
From (5.1) and recalling that $\eta_R u u_L^{(2^*-2)/2} = w_L$, $u_L \leq u$ and $\nabla \eta_R$ is bounded, we obtain
\[
\|w_L\|_{L^{2^*}(\Omega)}^2 \leq C_{11} \int_{\Omega} |u|^2 u_L^{(2^*-2)}|\nabla \eta_R|^2 \leq C_{11} \int_{\Omega} |u|^{2^*} \leq C_{12}.
\]
The definition of $\eta_R$ and $w_L$ and the above inequality imply that
\[
\left( \int_{B_R} |u|^2 u_L^{(2^*-2)/2} \right)^{2^*/2} \leq |w_L|_{L^{2^*}(\Omega)}^2 \leq C_{12}.
\]
Using Fatou’s lemma in the variable $L$, we have
\[
\int_{B_R} |u|^{2^*(2^*/2)} \leq K := C_{12}^{2^*/2}.
\]
Since $\Omega$ can be covered by a finite number of balls $B_R^j$, we have that
\[
\int_{\Omega} |u|^{2^*(2^*/2)} \leq \sum_j \int_{B_R} |u|^{2^*(2^*/2)} \leq K_3,
\]
for some $K_3 > 0$.

In order to estimate $\|\nabla u_\lambda\|_\infty$, we make use of the following result by Stampacchia in [14]

**Lemma 5.1.** Let $A(\eta)$ a given $C^1$ vector field in $\mathbb{R}^N$, and $f(x,s)$ a bounded Carathéodory function in $\Omega \times \mathbb{R}$. Let $u \in H^1_0(\Omega)$ be a solution of
\[
\int_{\Omega} (A(|\nabla u|)\nabla \varphi + f(x,u)\varphi) = 0,
\]
for all $\varphi \in H^1_0(\Omega)$. Assume that there exist $0 < \nu < M$ such that
\[
\nu |\xi|^2 \leq \frac{\partial A_i}{\partial \eta_j}(\nabla u)\xi_i \xi_j, \quad \text{and} \quad \left| \frac{\partial A_i}{\partial \eta_j}(\nabla u) \right| \leq M, \quad (5.12)
\]
for all $i, j = 1, ..., N$ and $\xi \in \mathbb{R}^N$. Then $u \in W^{2,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, for all $\alpha \in (0,1)$ and $p > 1$. Moreover
\[
\|u\|_{1,\alpha} \leq C(\nu, M, \Omega)\|f(\cdot, u)\|_\infty. \quad (5.13)
\]
16
By the definition of $a$, for $r$ small enough (5.12) hold. This, together with the fact that $\|u_\lambda\|_\infty$ is bounded allow us to apply the last result. Then (5.3) implies that
\[
\|u\|_{1,\alpha} \leq \lambda \|u\|_\infty^{q-1} + \|u\|_\infty^{\frac{2^*-1}{2}} = o(\lambda),
\] as $\lambda \to 0$.

Then, there exists $\lambda^* > 0$ such that $\lambda \in (0, \lambda^*)$ implies that $\|\nabla u\|_\infty \leq r$ and hence, $u_\lambda$ is a solution of $(P_\lambda)$.

References

[1] J. G. Azorero and I. P. Alonso, *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*, Trans. Amer. Math. Soc., vol 323 N. 2(1991) 877-895.

[2] V. Benci, *On critical points theory for indefinite functionals in the presence of symmetric*, Trans. Amer. Math. Soc. 274 (1982)533-572.

[3] D. Bonheure, Ann Derlet, S. Valeriola, *On the multiplicity of nodal solutions of a prescribed mean curvature problem*, Mathematische Nachrichten (2013) 1-15.

[4] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. 36 (1983) 437-477.

[5] P. Clément, Raúl Manásevich and E. Mitidieri *On a modified capillary equation*, J. Diff. Equation 124 (1996) 343-358.

[6] C. Coffman and W. Ziemer, *A prescribed mean curvature problem on domains without radial symmetry*, SIAM J. Math Anal. 22 (1991) no. 4 982-990.

[7] M. Del Pino and I. Guerra, *Ground states of a prescribe mean curvature equation*, J. Diff. Equation 241 (2007) 112-129.

[8] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case*, Rev. Mat. Iberoamericana 1 (1985)145-201.

[9] M.A. Krasnolselskii, *Topological methods in the theory of nonlinear integral equations*, Mac Millan, New York, 1964.

[10] W. Ni and J. Serrin, *Existence and non-existence theorems for ground state for quasilinear partial differential equations*, Att. Convergi Lincei 77 (1985) 231-257.

[11] W. Ni and J. Serrin, *Non existence theorems for quasilinear partial differential equations*, Rend. Circ. Mat. Palermo 8 (1985) 171-185.
[12] L. Peletier, J. Serrin, *Ground states for the prescribed mean curvature equation*, Proc. American Math. Society 100 (1987) 694-700.

[13] P. Pucci and G. Serrin, *Critical exponents and critical dimensions for polyharmonic operators*, J. Math. Pures Appl. 69 (1990) 55-83.

[14] G. Stampacchia and H. Lewy, *On existence and smoothness of solutions of some non-coercive variational inequalities*, Arch. Rational Mech. Anal. 41 (1971) no. 4, 241-253.

[15] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Springer (1998).