On a gauge invariant description of soliton dynamics

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We present important elements of a gauge and diffeomorphism invariant formulation of the moduli space approximation to soliton dynamics. We argue that explicit velocity-dependent modifications are determined entirely from gauge and diffeomorphism invariance. We illustrate the formalism for the case of a Yang-Mills theory on a curved spacetime background.

1 Introduction

Initiated by the work of Manton [1], the geodesic approximation for soliton dynamics and scattering has been developed and applied in various contexts, ranging from BPS monopoles of Yang-Mills theory, to abelian vortices, lump solutions in $CP$-models, as well as extremal black holes. The principal idea in all these situations is to approximate the classical dynamics of solitons by their geodesic motion in the space of static/stationary solutions (moduli space). In some cases, a general (albeit implicit) formula for the metric on the moduli space was given [2, 3]. Such an expression is lacking for the case of gravitation, though the moduli space metric is known, for example, for some particular extremal Reissner-Nordström black holes (see e.g. [4–8]). Our interest in this question is related to attempts to understand the moduli space geometry for the more complicated black hole solutions discussed in [9].

Let us first remind the reader briefly how the geodesic approximation is derived in the simplest setting, namely for a theory without gauge invariance. Consider, e.g. the Lagrangian of a non-linear sigma model with potential

$$L = \int d^3x \left( \frac{1}{2} g_{IJ} \partial_t \phi^I \partial_t \phi^J - V[\phi, \partial_m \phi] \right)$$

(1)

where $m = 1, 2, 3$ labels the spatial components (the number of spacetime dimensions is not crucial for what follows). We assume that this theory has static solutions which can be parametrized by a number of continuous integration constants, $X^a$, which we call collective coordinates. These could, for example, parametrize the positions of separated lumps in multi-soliton solutions. For the purpose of this note it suffices that the solutions can be encoded in time-independent functions $\phi^I(\vec{x}, X^a)$, which characterize completely the continuous variety of extrema of the potential. The geodesic approximation is effected by truncating the fields to $\hat{\phi}^I(t, \vec{x}) = \phi^I(\vec{x}, X(t))$, where the collective coordinates can depend on time. The caret indicates that these are the fields that will be reinserted into the action, obtaining an action $S[\phi, \partial_m \phi]$ for the collective coordinates. Upon adopting Hamilton’s principle one then derives the equations of motion for the $X^a(t)$. In the case of (1) this yields the equations for geodesic motion of a particle in moduli space, with corresponding metric

$$G_{ab}(X) = \int d^3x \ g_{IJ}(\phi(\vec{x}, X)) \ \partial_a \phi^I(\vec{x}, X) \ \partial_b \phi^J(\vec{x}, X).$$

(2)
As is well-known, the symmetries of the static solutions of the underlying field theory are reflected in corresponding symmetry features of the moduli space. We refrain from elaborating on this. Because we have adopted a Lorentz frame once we specify the static solutions, Lorentz boosts have no role to play in the moduli space description.

2 Gauge theory

In the case of a gauge theory the static solutions are in general subject to a class of residual gauge transformations that do not involve the time variable. This implies that these solutions are still ambiguous and the corresponding gauge degeneracy has to be modded out when extracting the correct moduli space deformations that do not involve the time variable. This implies that these solutions are still ambiguous and in the case of a gauge theory the static solutions are in general subject to a class of residual gauge transformations. Obviously, these depend on the collective coordinates, which themselves are gauge invariant. However, it is unclear whether this description will lead to a gauge invariant and gauge independent moduli space metric. This question is hard to answer, also in view of the fact that it is difficult to respect the gauge conditions when reintroducing time. See, for instance, the examples discussed in [2, 3], where the initial gauge conditions are modified by velocity-dependent terms. Hence we will pursue a covariant approach, in which none of the residual (i.e. time-independent) gauge transformations are fixed. We will then argue that the above mentioned velocity-dependent modifications follow from gauge covariance and are uniquely determined within the geodesic approximation.

For concreteness, let us discuss a Yang-Mills theory minimally coupled to a scalar field \( \phi \) in the adjoint representation of the gauge group,

\[
S_{\text{YM}} = \int dt \, d^3x \, \text{Tr} \left[ \frac{1}{4} F_{\mu
u} F^{\mu\nu} + \frac{1}{2} D_\mu \phi \, D^\mu \phi \right] - V(\phi) .
\]

The static configurations are subject to residual gauge transformations. Obviously, these depend on \( \vec{x} \), but in addition they can also depend on the collective coordinates \( X^a \), so that inequivalent solutions (characterized by different values for the \( X^a \)) may be subject to different gauge transformations. This implies that we are necessarily dealing with an extended base space parametrized by the coordinates \((\vec{x}^m, X^a)\). To define parallel transport in this extended bundle, we need connections \((A_m, A_a)\), where \( A_a(\vec{x}, X) \) is a new connection field, which for the moment is left undetermined. The Yang-Mills connections that appear in (3) are denoted by \( A_m(\vec{x}, X) \) and \( A_t(\vec{x}, X) \). Under the residual gauge transformations with parameter \( \Lambda(\vec{x}, X) \), the fields transform according to

\[
\begin{align*}
\delta \phi &= [\Lambda, \phi] , \\
\delta A_t &= [\Lambda, A_t] , \\
\delta A_m &= \partial_m \Lambda - [A_m, \Lambda] , \\
\delta A_a &= \partial_a \Lambda - [A_a, \Lambda] .
\end{align*}
\]

Covariant translations of the fields induced by shifts of the collective coordinates, take the form [2, 3]

\[
\begin{align*}
\delta_{\text{cov}} \phi &= \delta X^a \left( \partial_a \phi - [A_a, \phi] \right) = \delta X^a D_a \phi , \\
\delta_{\text{cov}} A_m &= \delta X^a \partial_a A_m - D_m(\delta X^a A_a) = \delta X^a F_{am} , \\
\delta_{\text{cov}} A_t &= \delta X^a \left( \partial_a A_t - [A_a, A_t] \right) = \delta X^a D_a A_t , \\
\delta_{\text{cov}} A_a &= \delta X^b \partial_b A_a - D_a(\delta X^b A_b) = \delta X^b F_{ba} ,
\end{align*}
\]

where \( F_{am} \) and \( F_{ab} \) are the nonabelian field strengths, which are tensors in the extended space. In addition we have \( F_{mt} = -F_{tm} = D_m A_t \) and \( F_{at} = -F_{ta} = D_a A_t \).

This concludes the discussion of the space of static solutions. Subsequently we reintroduce a dependence on time through the collective coordinates, \( X^a \to X^a(t) \), which implies that the residual gauge transformations will also become time-dependent, \( \Lambda \to \bar{\Lambda} = \Lambda(\vec{x}, X(t)) \). Note that the transformation property...
for the geodesic lift of the scalar field, \( \tilde{\phi} = \phi(\tilde{x}, X(t)) \), remains the same, even when the gauge parameter depends on time through \( X^a(t) \). The same conclusion holds for the other fields. However, to ensure that the (residual) gauge invariance is maintained under these extended transformations one needs to identify a proper connection \( \hat{\Lambda} \) in order to define a covariant time derivative. The required expression is a modification of the original connection \( A_t(\tilde{x}, X) \) and reads

\[
\hat{\Lambda}_t = A_t + \dot{X}^a A_a .
\]

(6)

With this modification, \( \hat{D}_t \tilde{\phi} = \partial_t \tilde{\phi} - [\hat{\Lambda}_t, \tilde{\phi}] \) transforms covariantly under the gauge transformations with parameters \( \Lambda \). Note that here \( \partial_t = \dot{X}^a \partial_a \), as the time dependence resides in \( X^a \). Hence we obtain

\[
\hat{D}_t \tilde{\phi} = \dot{X}^a D_a \tilde{\phi} - [A_t, \tilde{\phi}] .
\]

(7)

The corresponding field strengths follow from constructing the commutators of the covariant derivatives,

\[
\hat{F}_{mt} = F_{mt} + F_{mb} \dot{X}^b , \quad \hat{F}_{at} = F_{at} + F_{ab} \dot{X}^b .
\]

(8)

Hence the requirement of gauge invariance leads to explicit velocity-dependent modifications; these organize themselves into pullback terms to the worldline in moduli space. Note that, at this point, there are no velocity-dependent modifications to the other components of the gauge potentials.

Covariant translations induced by shifts \( X^a(t) \rightarrow X^a(t) + \delta X^a(t) \) now involve arbitrary functions \( \delta X^a(t) \) and this aspect requires some care. Here we only note that the correct result for the covariant translation of (6), \( \delta_{\text{cov}} \hat{\Lambda}_t = \delta X^a \hat{F}_{at} \), does not involve terms proportional to \( \delta \dot{X}^a \). This result follows from varying \( X^a(t) \) in the definition of \( \hat{\Lambda}_t \) and adding a gauge transformation with parameter \( \delta X^a A_a \); the variation is consistent with the generalized Leibniz rule, \( \delta_{\text{cov}} (\hat{D}_t \tilde{\phi}) = \hat{D}_t (\delta_{\text{cov}} \tilde{\phi}) - (\delta_{\text{cov}} \hat{\Lambda}_t) \tilde{\phi} \). This relation is the same as for the underlying field theory, and it is crucial for unambiguously identifying covariant field variations \( \delta_{\text{cov}} \) with the variations associated with the moduli action principle. Similar results apply for the variations of the various field strengths.

Replacing \( F_{\mu\nu}, D_\mu \) and \( \phi \) in (3) by \( \hat{F}_{\mu\nu}, \hat{D}_\mu \) and \( \tilde{\phi} \), respectively, and dropping the (constant) contribution from the potential terms, one obtains the following moduli action:

\[
S[X(t)] = \int dt \left( \frac{1}{2} \hat{G}_{ab}(X) \dot{X}^a \dot{X}^b - J_a(X) \dot{X}^a \right) ,
\]

(9)

where

\[
\hat{G}_{ab}(X) = - \int d^3x \, \text{Tr} \left[ F_{am} F_{bm} + D_a \phi D_b \phi \right] ,
\]

\[
J_a(X) = - \int d^3x \, \text{Tr} \left[ A_t \left( D_m F_{ma} + [\phi, D_a \phi] \right) \right] .
\]

(10)

This result is invariant under residual gauge transformations and covariant under moduli-space diffeomorphisms (similar results were obtained in [2, 3]). However, it still depends (apart from on the static solutions) on the extra connection \( A_a \), which can be eliminated in a gauge-invariant fashion by use of its equation of motion (valid for any \( \delta A_a \))

\[
\dot{X}^a \dot{X}^b \int d^3x \, \text{Tr} \left[ \delta A_a \left( D_m F_{mb} + [\phi, D_b \phi] \right) \right] = 0 .
\]

(11)

The friction term \( J_a \dot{X}^a \) did not contribute to (11), since its variation is proportional to the (static) \( A_t \)-equation of motion. Furthermore, it vanishes in the effective action, once the constraint (11) on \( A_a \) is imposed.
Upon partial integration and comparison with (5) one observes that (11) is the well-known orthogonality condition [1, 10]

$$\int \! \! d^3x \; \text{Tr} \left[ (\delta_{\text{cov}} A_m) (\delta_{\text{gauge}} A_m) + (\delta_{\text{cov}} \phi) (\delta_{\text{gauge}} \phi) \right] = 0, \quad (12)$$

which ensures that the geodesic motion corresponding to $\delta_{\text{cov}}$ is orthogonal to the gauge orbits. In this connection observe that the moduli space metric $G_{ab}(X)$ can be written as

$$G_{ab}(X) = -\int \! \! d^3x \; \text{Tr} \left[ \frac{\delta_{\text{cov}} A_m}{\delta X^a} \frac{\delta_{\text{cov}} A_m}{\delta X^b} + \frac{\delta_{\text{cov}} \phi}{\delta X^a} \frac{\delta_{\text{cov}} \phi}{\delta X^b} \right]. \quad (13)$$

Since the constraint (11) is a covariant equation for $A_a$, we may solve for $A_a$ and reinsert it into the expression for the metric $G_{ab}$ without affecting gauge invariance. In principle, the above framework can be used for more general Lagrangians, including Lagrangians that contain terms of higher order in the field strengths.

### 3 Gravitational background

For theories invariant under spacetime diffeomorphisms no analogous approach has been worked out so far. In this section we present the case of a gauge theory coupled to a stationary gravitational background, taking the residual spacetime diffeomorphisms into account. This is a modest step towards a more complete treatment of theories with gravity and it will reveal the presence of additional velocity-dependent corrections. We start from a stationary metric in adapted coordinates, such that its components are time independent. Together with the gauge fields, the metric is determined as a stationary solution of some underlying field theory which is assumed to depend on a number of collective coordinates $X^\alpha$. The residual gauge transformations are now extended to include the following residual diffeomorphisms,

$$t \to t + \xi^\ell(x, X^a), \quad x^m \to x^m + \xi^m(x, X), \quad X^\alpha \to X^\alpha + \xi^\alpha(X), \quad (14)$$

where, for completeness, we also included arbitrary moduli space diffeomorphisms. Obviously, the latter are independent of the spacetime coordinates, so that $\partial_\alpha \xi^\alpha = 0$. Under residual diffeomorphisms a scalar and the time component $A_t$ of a gauge field transform according to\(^1\),

$$\delta \xi^\phi = -\xi^M \partial_M \phi, \quad \delta \xi A_t = -\xi^M \partial_M A_t, \quad (15)$$

whereas $A_M = (A_m, A_a)$ transforms according to

$$\delta \xi^A_M = -\xi^N \partial_N A_M - \partial_M \xi^\ell X^\ell A_t. \quad (16)$$

Observe that the above result implies that the $A_m, A_a$ and $A_t$ mix in a nontrivial way. It can be shown that this is required by the closure of the combined algebra of residual spacetime diffeomorphisms, moduli space diffeomorphisms and gauge transformations (always subject to the condition $\partial_\alpha \xi^\alpha = 0$).

In the adapted coordinates that we use, the line element reads

$$ds^2 = g_{tt}(dt + \sigma_m dx^m)^2 + g_{mn} dx^m dx^n, \quad (17)$$

where $g_{tt}, \sigma_m$ and $g_{mn}$ depend on $x^m$ and $X^\alpha$ and transform under (14). In particular we note the behaviour of $\sigma_m$ under the transformations (14),

$$\delta \xi \sigma_m = -\partial_m \xi^\ell - \xi^M \partial_M \sigma_m - \partial_n \xi^n \sigma_n, \quad (18)$$

\(^1\) As before $m, n, \ldots$ denote spatial indices and $a, b, \ldots$ label the moduli space coordinates. The spacetime indices, denoted by $\mu, \nu, \ldots$ comprise the spatial indices $m, n, \ldots$ and the time index $t$, whereas the indices $M, N, \ldots$ comprise the indices of the base manifold of the stationary configurations and thus cover both $m, n, \ldots$ and $a, b, \ldots$. 
so that \( \sigma_m \) transforms as a gauge field with respect to \( \xi^t \)-transformations. Just as in the gauge theory case, we must introduce extra connection components in order to define parallel transport in the bundle over the extended base space parametrized by \((x^m, X^a)\). These extra fields are denoted by \( \sigma_a \) and \( V^m_a \) and are associated with the transformation parameters \( \xi^t \) and \( \xi^m \), respectively. Under (14) they transform according to

\[
\begin{align*}
\delta \xi \sigma_a &= -\partial_a \xi^t - \xi^M \partial_M \sigma_a - \partial_a \xi^M \sigma_M, \\
\delta \xi V^m_a &= -\partial_a \xi^m - \xi^M \partial_M V^m_a - \partial_a \xi^b V^m_b + V^m_a \partial_a \xi^m.
\end{align*}
\]

These new fields and their transformation rules have an elegant geometrical interpretation in terms of an extended block-triangular vielbein field

\[
E_{\Omega \Xi} = \begin{pmatrix} e^\mu_\nu & \varnothing \\ e^a_\mu & e^b_a \end{pmatrix}
\]

where \( \Xi = \nu, b \), and the underlined indices refer to the corresponding tangent space. Here \( e^\mu_\nu \) is the spacetime vielbein, such that \( e^\mu_\mu e^\nu_\mu \) equals the inverse spacetime metric corresponding to the stationary line element (17), and \( e^a_\nu(X) \) is some reference vielbein in moduli space; the off-diagonal block contains the new fields \( \sigma_a \) and \( V^m_a \).

\[
e^t_a = -e^b_a (\sigma_b - V^m_b \sigma_m), \quad e^m_a = -e^b_a V^m_b.
\]

With the exception of \( e^a_\nu(X) \) all components of the vielbein depend on both \( x^m \) and \( X^a \). This functional dependence is preserved by the residual coordinate transformations (14) owing to the block-triangular form of the vielbein. The tangent space rotations acting on the vielbein decompose into local Lorentz transformations, which may depend on both \( x^m \) and \( X^a \), and \( X^a \)-dependent orthogonal transformations of the moduli tangent space. It is often convenient to impose a gauge choice on the spacetime vielbein \( e^\mu_\nu \), but this is not needed below.

The covariant translations induced by shifts of the moduli now include the residual gauge transformations and diffeomorphisms with field-dependent parameters, analogous to (5). For a scalar field we thus obtain

\[
\delta_{\text{cov}} \phi = \delta X^a [D_a - \sigma_a D_t - V^m_a (D_a - \sigma_a D_t)] \phi,
\]

where the covariant derivatives contain the gauge connections. The covariant time derivative is just given by \( D_t \phi = -[A_t, \phi] \), because there is no dependence on time. This result takes the form of a linear combination of field-dependent diffeomorphisms and gauge transformations. The correctness of this formula can be verified by requiring that \( \delta_{\text{cov}} \phi \) transforms precisely as \( \phi \) itself, using the various transformation rules given above. Making use of the extended vielbein, the result (22) can be written as follows:

\[
\delta_{\text{cov}} \phi = \delta X^a D_a \phi,
\]

which makes it obvious that (22) has the required properties as we have expressed the result in terms of a tangent-space derivative.

Naturally, this result can be extended to other fields, but in those cases one may need (dependent) spin and affine connections. There is no obstacle for doing this, but we prefer not to enter into the details of their construction here. Apart from this extension we have dealt with the stationary solutions and the structure of the corresponding moduli space.

Reintroducing time by letting the collective coordinates become time dependent, proceeds in the same way as for the gauge theory, except that matters are rather more subtle. Knowing that the time component
The gauge field does acquire a velocity-dependent term, one must introduce the following velocity-dependent modifications for all the gauge field components in order to uniformly preserve the transformation rules:

\[
\hat{A}_t = A_t + \dot{X}^a A_a^2, \quad \hat{A}_m = A_m + \sigma_m \dot{X}^a A_a, \quad \hat{A}_a = A_a + \sigma_a \dot{X}^b A_b. \tag{24}
\]

We emphasize that \(\dot{X}^a A_a\) takes a complicated form:

\[
\dot{X}^a A_a = \dot{X}^a \left[ A_a - \sigma_a A_t - V^a_n (A_n - \sigma_n A_t) \right]. \tag{25}
\]

The derivatives of \(\phi\) thus have the form

\[
\hat{D}_t \hat{\phi} = \dot{X}^a D_a \phi - [A_t, \phi], \quad \hat{D}_M \hat{\phi} = D_M \phi + \sigma_M \dot{X}^a D_a \phi. \tag{26}
\]

The new connections (24) transform in an unusual fashion under gauge transformations. Indeed, the resulting geometry with the velocity-dependent terms is rather complicated and involves nontrivial torsion. There exists a well-defined tensor calculus that allows a systematic construction of covariant quantities such as the ones listed above. The complications are also reflected in the field strengths. As a first step we have constructed the following expressions for the field strengths, which transform covariantly with respect to both gauge transformations and diffeomorphisms, and contain velocity-dependent terms:

\[
\hat{F}_{Mt} = F_{Mt} + (F_{Mb} - \sigma_M F_b^t) \dot{X}^b, \quad \hat{F}_{MN} = F_{MN} - 2 \sigma_{[M} F_{N]b} \dot{X}^b. \tag{27}
\]

Obviously, these results are an extension of (8) and include nontrivial corrections due to the gravitational background. They will receive appropriate additive modifications by terms which are separately consistent with the symmetries. Their form is fixed by other requirements, to which we have already been alluding in the text (c.f. the paragraph following equation (8)). One is that they have a role to play in the covariant translations \(\delta_{\text{cov}}\) that we have discussed before, and another one concerns the validity of a generalized Leibniz rule. For the purpose of this exposition we will neglect these modifications and we will assume that (27) is complete; a discussion of these subtle issues is relegated to a separate publication.

What remains is to substitute the new field strengths and the covariant derivatives into the action (3), which is now covariantized with respect to diffeomorphisms by including the spacetime metric corresponding to (17). In this way, the moduli action takes the form

\[
S[X(t)] = \int dt \left( \frac{1}{2} G_{\dot{\alpha} \dot{\beta}}(X) \dot{X}^\alpha \dot{X}^\beta - J_\alpha(X) \dot{X}^\alpha \right), \tag{28}
\]

where

\[
G_{\dot{\alpha} \dot{\beta}} = - \int d^3 x \sqrt{g / |g_{tt}|} \text{Tr} \left[ g^{mn} \left( F_{\dot{\alpha} m} - \sigma_m F_{\dot{\alpha} t} \right) \left( F_{\dot{\beta} n} - \sigma_n F_{\dot{\beta} t} \right) + D_{\dot{\alpha}} \phi D_{\dot{\beta}} \phi \right], \tag{29}
\]

where \(g_{mn}\) and \(g_{tt}\) have been defined in (17), \(g = \det(g_{mn})\), and \(g^{mn}\) is the inverse of \(g_{mn}\). Note that the \(\sigma_m\)-terms appear only to build up \(\xi^\alpha\)-invariant combinations and the integral is in fact fully invariant under gauge transformations and diffeomorphisms. The linear term in the moduli action is analogous to the one presented in the gauge theory case and we refrain from further comment. The discussion of the constraints is premature in view of the fact that one should also include the Einstein-Hilbert term. It will be interesting to analyze the final result of this approach in the context of the results of [5].
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