TOWARDS CHARACTERISING POLYNOMIALITY OF $\frac{1 - q^b}{1 - q^a} \binom{n}{m}$
AND APPLICATIONS

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Abstract. In this note we shall give conditions which guarantee that $\frac{1 - q^b}{1 - q^a} \binom{n}{m} \in \mathbb{Z}[q]$ holds. We shall provide a full characterisation for $\frac{1 - q^b}{1 - q^a} \binom{ka}{m} \in \mathbb{Z}[q]$. This unifies a variety of results already known in literature. We shall prove new divisibility properties for the binomial coefficients and a new divisibility result for a certain finite sum involving the roots of the unity.

1. Introduction

Throughout, let $\mathbb{N}$ denote the set of positive integers, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ be the set of nonnegative integers, and let $\mathbb{Z}$ denote the set of integers. Accordingly, let $\mathbb{Z}[q]$ denote the set of polynomials in $q$ with coefficients in $\mathbb{Z}$ and let $\mathbb{N}_0[q]$ be the set of polynomials in $q$ with coefficients in $\mathbb{N}$. Recall that for a complex number $q$ and a complex variable $x$, the $q$-shifted factorials are given by $(x; q)_0 = 1$, $(x; q)_n = \prod_{i=0}^{n-1} (1 - xq^i)$, $(x; q)_\infty = \lim_{n \to \infty} (x; q)_n = \prod_{i=0}^{\infty} (1 - xq^i)$ and the $q$-binomial coefficients are given for any $m, n \in \mathbb{N}_0$ by

$$\binom{n}{m} = \begin{cases} \frac{(x; q)_n}{(x; q)_m (x; q)_{n-m}}, & \text{if } n \geq m \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Andrews [2] introduced the function

$$A(n, j) = \frac{1 - q^n}{1 - q^j} \binom{n}{j},$$

which, for our purposes, we extend as follows.

Definition 1. For $a \in \mathbb{N}$ and $b, m, n \in \mathbb{N}_0$, let

$$A(b, a; n, m) = \frac{1 - q^b}{1 - q^a} \binom{n}{m}, \quad a \in \mathbb{N}, \ b, m, n \in \mathbb{N}_0.$$ 

We say that $A(b, a; n, m)$ is reduced (or in reduced form) if $a \leq n < 2a$ and $0 \leq m < a$. Writing $m = ua + r$ and $n = va + s$ with $0 \leq r < a$ and $a \leq s < 2a$, it is clear that the reduced form of $A(b, a; n, m)$ is $A(b, a; s, r)$.

Remark 1. By Guo and Krattenthaler [6, Lemma 5.1], if $b \leq a$ and $A(b, a; n, m) \in \mathbb{Z}[q]$, then $A(b, a; n, m) \in \mathbb{N}_0[q]$.

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Slightly modifying [3, Theorem 5], we shall show that $A(b, a; n, m) \in \mathbb{N}_0[q]$ if and only if $A(b, a; s, r) \in \mathbb{N}_0[q]$. More specifically, we have:

**Theorem 1.** Let $a \in \mathbb{N}$ and $b, m, n \in \mathbb{N}_0$ such that $m \leq n$. Then

$$A(b, a; n, m) \in \mathbb{Z}[q] \text{ if and only if } A(b, a; n + la, m + ka) \in \mathbb{Z}[q]$$

for all integers $k, l$ such that $0 \leq m + ka \leq n + la$.

By Theorem 1 and Remark 1 we have:

**Corollary 1.** Let $a \in \mathbb{N}$ and $b, m, n \in \mathbb{N}_0$ such that $b \leq a$ and $m \leq n$. Then

$$A(b, a; n, m) \in \mathbb{N}_0[q] \text{ if and only if } A(b, a; n + la, m + ka) \in \mathbb{N}_0[q]$$

for all integers $k, l$ such that $0 \leq m + ka \leq n + la$.

Andrews [2, Theorem 2] gave the following characterisation:

**Corollary 2.** Let $a_\mathbb{N} \text{ and } a_\mathbb{M}$ be nonnegative integers such that $a > 0$ and $n a \geq m$. Then $A(1, n; na, m) \in \mathbb{Z}[q]$ if and only if $\gcd(n, m) = 1$.

Moreover, by Guo and Krattenthaler [6, Theorem 3.2] we have:

$$A(\gcd(a, b), a + b; a + b, a) \in \mathbb{N}_0[q].$$

Notice that the functions in (1), (2), and (3) are of type $A(b, a; n, m)$ with $a \mid n$. So, it is natural to ask for conditions guaranteeing the statement $A(b, a; na, m) \in \mathbb{N}_0[q]$ to hold. To this end, we have the following characterisation.

**Theorem 2.** Let $a, b, m, n$ be nonnegative integers such that $a > 0$ and $n a \geq m$. Then $A(b, a; na, m) \in \mathbb{Z}[q]$ if and only if $\gcd(a, m) \mid b$.

Combining Remark 1 with Theorem 2 we have the following consequence.

**Corollary 2.** Let $a, b, m, n$ be nonnegative integers such that $a > 0$, $b \leq a$ and $n a \geq m$. Then $A(b, a; na, m) \in \mathbb{N}_0[q]$ if and only if $\gcd(a, m) \mid b$.

Further, Guo and Krattenthaler [6, Theorem 3.1] showed that all of the functions

$$A(1, 6n - 1; 12n, 3n), A(1, 6n - 1; 12n, 4n), A(1, 30n - 1; 60n, 6n)$$

$$A(1, 30n - 1; 120n, 40n), A(1, 30n - 1; 120n, 45n), A(1, 66n - 1; 3300n, 88n)$$

are in $\mathbb{N}_0[q]$.

**Remark 2.** To investigate the polynomiality of $A(1, a; n, m)$ we may assume by virtue of Theorem 1 that $A(1, a; n, n - m)$ is reducible, i.e. $n = a + r$ and $n - m = a - s$ with $0 \leq r < a$ and $0 \leq s < a$. In this case we have $m = r + s$ and so, we may assume that $n = a + r$ and $n + a \geq m \geq r$.

Observe that the reduced forms of all of the functions listed in (4) have the form $A(1, a; a + r, m)$ with $r \leq m$. We have the following unifying argument.
Theorem 3. Let $a \in \mathbb{N}$, let $a > r \in \mathbb{N}_0$, let $n = a + r$, and let $m \in \mathbb{N}_0$ such that $n \geq m \geq r$. If $\gcd(a, m) = 1$ and $\gcd(a, m - j) \mid n$ for all $j = 1, \ldots, r$, then $A(1, a; n, m) \in \mathbb{N}_0[q]$. 

For instance, applying Theorem 3 to $a = 6n - 1$, $r = 2$, and $m = 3n$ gives that $A(1, 6n - 1; 12n, 3n) \in \mathbb{N}_0[q]$ and applying Theorem 3 to $a = 30n - 1$, $r = 4$, and $m = 45n$ gives that $A(1, 30n - 1; 120n, 45n) \in \mathbb{N}_0[q]$. One can check the polynomiality of the other functions listed in (4) in a similar way.

An important application of the function $A(1, a; n, m)$ is the fact that whenever it is a polynomial in $\mathbb{Z}[q]$ and $\gcd(a, b) = 1$, then $a \mid \binom{n}{m}$. Our next result deals with divisibility properties for the binomial coefficients.

Theorem 4. If $a$ and $n$ are nonnegative integers such that $a \geq 3$, then

(a) $((a - 1)n + 1) \mid \gcd\left(\binom{(a - 1)^2n - 1}{(a - 1)n}, \binom{a(a - 1)n}{2(a - 1)n + 1}\right)$,

(b) $((a - 1)n - 1) \mid \gcd\left(\binom{(a - 1)^2n - 1}{(a - 1)n - 2}, \binom{a(a - 1)n - 2}{2(a - 1)n - 3}\right)$.

Finally, by a result of Gould [5] we have for any nonnegative integers $N$ and $M < n$

\begin{equation}
\sum_{j \geq 0} \binom{N + mn}{M + jn} = \frac{1}{n} \sum_{j=1}^{n} w^{-jM} (1 + w^j)^{N + mn},
\end{equation}

where $w = e^{2\pi i/n}$ is a primitive $n$th root of unity. In particular, this implies that

$\sum_{j=1}^{n} w^{-jM} (1 + w^j)^{N + mn}.$

We have the following generalisation.

Theorem 5. If $A(1, n; N, M) \in \mathbb{Z}[q]$, then for any nonnegative integer $m$ we have

$n^2 \sum_{j=1}^{n} w^{-jM} (1 + w^j)^{N + mn},

where $w = e^{2\pi i/n}$ is a primitive $n$th root of unity.

2. Proof of Theorem 1

The implication from the right to the left is clear. Assume now that $A(b, a; n, m) \in \mathbb{Z}[q]$.

By the well-known identity

$q^M - 1 = \prod_{d \mid M} \Phi_d(q),

where $\Phi_d(q)$ is the $d$-th cyclotomic polynomial in $q$, we obtain

$A(b, a; n, m) \equiv \prod_{d=2}^{n} \Phi_d(q)^{e_d},

where

$e_d = \chi(d \mid b) - \chi(d \mid a) + \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{n - m}{d} \right\rfloor.$
with \( \chi(S) = 1 \) if \( S \) is true and \( \chi(S) = 0 \) if \( S \) is false. As \( A(b, a; n, m) \in \mathbb{Z}[q] \) and \( \Phi_d(q) \) is irreducible for any \( d \) we must have \( e_d \geq 0 \) for all \( d = 2, \ldots, n \). As to \( A(b, a; n + la, m + ka) \), we have

\[
A(b, a; n + la, m + ka) = \prod_{d=2}^{n+la} \Phi_d(q)^{e_d},
\]

where

\[
e_d = \chi(d \mid b) - \chi(d \mid a) + \left\lfloor \frac{n + la}{d} \right\rfloor - \left\lfloor \frac{m + ka}{d} \right\rfloor - \left\lfloor \frac{n - m + (l - k)a}{d} \right\rfloor.
\]

Then clearly \( e_d \geq 0 \) unless \( d \mid a \). But if \( d \mid a \), then

\[
\left\lfloor \frac{n + la}{d} \right\rfloor - \left\lfloor \frac{m + ka}{d} \right\rfloor - \left\lfloor \frac{n - m + (l - k)a}{d} \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{n - m}{d} \right\rfloor
\]

and therefore \( e_d \geq 0 \) by assumption, implying that \( A(b, a; n + la, m + ka) \) is a polynomial in \( q \).

3. Proof of Theorem 2

Suppose that \( \gcd(a, m) = g \nmid b \) and that \( A(b, a; na, m) \in \mathbb{Z}[q] \). Then clearly \( A(b, g; na, m) \in \mathbb{Z}[q] \) and so, by Theorem 1 we have

\[
A(b, g; na, 0) = \frac{1 - q^b}{1 - q^g} \in \mathbb{Z}[q],
\]

which is impossible as \( g \nmid b \). Assume now that \( \gcd(a, m) \mid b \). Then just as before, we have

\[
A(b, a; na, m) = \prod_{d=2}^{na} \Phi_d(q)^{e_d},
\]

where

\[
e_d = \chi(d \mid b) - \chi(d \mid a) + \left\lfloor \frac{na}{d} \right\rfloor - \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{na - m}{d} \right\rfloor.
\]

Then \( e_d \geq 0 \) unless \( d \mid a \). But if \( d \mid a \), then

\[
e_d = \chi(d \mid b) - 1 - \left( \left\lfloor \frac{m}{d} \right\rfloor + \left\lfloor \frac{-m}{d} \right\rfloor \right).
\]

**Case 1:** \( d \mid m \). Then \( d \mid \gcd(a, m) \) and so also \( d \mid b \). From these facts and the identity \( \chi(d \mid b) - 1 + 1 \geq 0 \) we conclude that \( e_d = 0 \).

**Case 2:** \( d \nmid m \). Then \( \lfloor m/d \rfloor + \lfloor -m/d \rfloor = -1 \) and so, \( e_d = \chi(d \mid b) - 1 + 1 \geq 0 \).

This completes the proof.

4. Proof of Theorem 3

Proceeding as before, we have

\[
A(1, a; n, m) = \prod_{d=2}^{n} \Phi_d(q)^{e_d},
\]

with

\[
e_d = -\chi(d \mid a) + \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{n - m}{d} \right\rfloor.
\]
Then $e_d \geq 0$ unless $d \mid a$. Let $2 \leq d \mid a$. Suppose that there is some $j = 1, \ldots, r$ such that $d \mid \gcd(a, m - j)$. Then $d \mid n$ but $d \nmid m$ and we get

\[
e_d = -1 - \left\lfloor \frac{m}{d} \right\rfloor + \left\lfloor \frac{-m}{d} \right\rfloor = -1 - (-1) \geq 0.
\]

Suppose now that $d \nmid m - 1, \ldots, d \nmid m - r$. Then

\[
\left\lfloor \frac{m}{d} \right\rfloor + \left\lfloor \frac{r - m}{d} \right\rfloor = \left\lfloor \frac{m}{d} \right\rfloor + \left\lfloor \frac{-m}{d} \right\rfloor = -1
\]

and so,

\[
e_d = -1 + \left\lfloor \frac{a + r}{d} \right\rfloor - \left( \left\lfloor \frac{m}{d} \right\rfloor + \left\lfloor \frac{a + r - m}{d} \right\rfloor \right)
= -1 + \frac{r}{d} - \left( \left\lfloor \frac{m}{d} \right\rfloor + \left\lfloor \frac{r - m}{d} \right\rfloor \right)
= -1 + \frac{r}{d} - (-1)
\geq 0,
\]

implying that $A(1, a; n, m) \in \mathbb{Z}[q]$. The fact that $A(1, a; a + r, m) \in \mathbb{N}_0[q]$ is a consequence of Remark [4].

5. **Proof of Theorem [4]**

(a) Let $a \geq 3$ and $n$ be nonnegative integers. From the evident fact

\[
A(an + 1, n + 1; an, n) = \left\lfloor \frac{an + 1}{n + 1} \right\rfloor \in \mathbb{Z}[q]
\]

and Theorem [4] we get

\[
A(an + 1, n + 1; an - n - 1, n) \in \mathbb{Z}[q] \quad \text{and} \quad A(an + 1, n + 1; an, n + n + 1) \in \mathbb{Z}[q],
\]

from which we find

\[
(n + 1)(an + 1) \gcd \left( \left( \frac{(a - 1)n - 1}{n} \right), \left( \frac{an}{2n + 1} \right) \right).
\]

Letting $n := (a - 1)m$ we have that $\gcd(n + 1, an + 1) = 1$ and so the previous divisibility implies

\[
((a - 1)m + 1) \gcd \left( \left( \frac{(a - 1)m - 1}{(a - 1)m - 1} \right), \left( \frac{a(a - 1)m}{2(a - 1)m + 1} \right) \right),
\]

as desired.

(b) Follows similarly by applying Theorem [4] to the fact

\[
A(an - 1, n - 1; an - 2, n - 2) = \left\lfloor \frac{an - 1}{n - 1} \right\rfloor \in \mathbb{Z}[q].
\]
6. Proof of Theorem 5

Suppose that \( A(1, n; N, M) \in \mathbb{Z}[q] \). Then by virtue of Theorem 1 we have

\[
A(1, n; N + mn, M + jn) \in \mathbb{Z}[q]
\]

for all nonnegative integers \( j \) such that \( M + jn \leq N + mn \). It follows with the help of Gould’s identity

\[
n \mid \sum_{j \geq 0} \binom{N + mn}{N + jn} = \frac{1}{n} \sum_{j=1}^{n} w^{-j} M (1 + w^j)^{N + mn},
\]

from which the desired divisibility follows.

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