ALMOST EVERYWHERE CONVERGENCE OF SPLINE SEQUENCES

BY

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ABSTRACT
We prove the analogue of the Martingale Convergence Theorem for polynomial spline sequences. Given a natural number $k$ and a sequence $(t_i)$ of knots in $[0,1]$ with multiplicity $\leq k - 1$, we let $P_n$ be the orthogonal projection onto the space of spline polynomials in $[0,1]$ of degree $k - 1$ corresponding to the grid $(t_i)_{i=1}^n$. Let $X$ be a Banach space with the Radon–Nikodým property. Let $(g_n)$ be a bounded sequence in the Bochner–Lebesgue space $L_X^1[0,1]$ satisfying

$$g_n = P_n(g_{n+1}), \quad n \in \mathbb{N}.$$

We prove the existence of $\lim_{n \to \infty} g_n(t)$ in $X$ for almost every $t \in [0,1]$. Already in the scalar valued case $X = \mathbb{R}$ the result is new.

1. Introduction

In this paper we prove a convergence theorem for splines in vector valued $L^1$-spaces. By way of introduction we consider the analogous convergence theorems for martingales with respect to a filtered probability space $(\Omega, (\mathcal{A}_n), \mu)$. We first review two classical theorems for scalar valued martingales in $L^1 = L^1(\Omega, \mu)$. See Neveu [6].

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(M1) Let \( g \in L^1 \). If \( g_n = \mathbb{E}(g | \mathcal{A}_n) \) then \( \|g_n\|_1 \leq \|g\|_1 \) and \( (g_n) \) converges almost everywhere and in \( L^1 \).

(M2) Let \( (g_n) \) be a bounded sequence in \( L^1 \) such that \( g_n = \mathbb{E}(g_{n+1} | \mathcal{A}_n) \). Then \( (g_n) \) converges almost everywhere and \( g = \lim g_n \) satisfies \( \|g\|_1 \leq \sup \|g_n\|_1 \).

Next we turn to vector valued martingales. We fix a Banach space \( X \) and let \( L^1_X = L^1_X(\Omega, \mu) \) denote the Bochner–Lebesgue space. The Radon–Nikodým property (RNP) of the Banach space \( X \) is intimately tied to martingales in Banach spaces. We refer to the book by Diestel and Uhl [3] for the following basic and well known results.

(M3) Let \( g \in L^1_X \). If \( g_n = \mathbb{E}(g | \mathcal{A}_n) \) then \( \|g_n\|_{L^1_X} \leq \|g\|_{L^1_X} \). The sequence \( (g_n) \) converges almost everywhere in \( X \) and in \( L^1_X \). (This holds for any Banach space \( X \).)

(M4) Let \( (g_n) \) be a bounded sequence in \( L^1_X \) such that \( g_n = \mathbb{E}(g_{n+1} | \mathcal{A}_n) \). If the Banach space \( X \) satisfies the Radon–Nikodým property, then \( (g_n) \) converges almost everywhere in \( X \) and \( g = \lim g_n \) satisfies \( \|g\|_{L^1_X} \leq \sup \|g_n\|_{L^1_X} \). Moreover, the \( L^1_X \)-density of the \( \mu \)-absolutely continuous part of the vector measure

\[
\nu(E) = \lim_{n \to \infty} \int_E g_n d\mu, \quad E \in \cup \mathcal{A}_n
\]

determines \( g = \lim g_n \).

(M5) Conversely, if \( X \) fails to satisfy the Radon–Nikodým property, then there exists a filtered probability space \( (\Omega, (\mathcal{A}_n), \mu) \) and bounded sequence in \( L^1_X(\Omega, \mu) \) satisfying \( g_n = \mathbb{E}(g_{n+1} | \mathcal{A}_n) \) such that \( (g_n) \) fails to converge almost everywhere in \( X \).

In the present paper we establish a new link between probability (almost sure convergence of martingales, the RNP) and approximation theory (projections onto splines in \([0, 1]\)).

We review the basic setting pertaining to spline projections. (See, for instance, [12], [9], [11].) So, fix an integer \( k \geq 2 \), and let \( (t_i) \) be a sequence of grid points in \((0, 1)\) where each \( t_i \) occurs at most \( k - 1 \) times. We emphasize that in contrast to [9], in the present paper we don’t assume that the sequence of grid points is dense in \((0, 1)\).
Let $S_n$ denote the space of splines on the interval $[0, 1]$ of order $k$ (degree $k-1$) corresponding to the grid $(t_i)_{i=1}^n$. Let $\lambda$ denote Lebesgue measure on the unit interval $[0, 1]$. Let $P_n$ be the orthogonal projection with respect to $L^2([0, 1], \lambda)$ onto the space of splines $S_n$. By Shadrin’s theorem [12], $P_n$ admits an extension to $L^1([0, 1], \lambda)$ such that

$$\sup_{n \in \mathbb{N}} \|P_n : L^1([0, 1], \lambda) \to L^1([0, 1], \lambda)\| < \infty.$$ 

Assuming that the sequence $(t_i)$ is dense in the unit interval $[0, 1]$, the second named author and A. Shadrin [9] proved—in effect—that for any $g \in L^1_X([0, 1], \lambda)$ the sequence $g_n = P_n g$ converges almost everywhere in $X$. The vector valued version of [9] holds true without any condition on the underlying Banach space $X$. Thus the paper [9] established the spline analogue of the martingale properties (M1) and (M3)—under the restriction that $(t_i)$ is dense in the unit interval $[0, 1]$.

Our main theorem—extending [9]—shows that the vector valued martingale convergence theorem has a direct counterpart in the context of spline projections. Theorem 1.1 gives the spline analogue of the martingale properties (M2) and (M4). The first step in the proof of Theorem 1.1 consists in showing that the restrictive density condition on $(t_i)$ may be lifted from the assumptions in [9].

**Theorem 1.1 (Spline Convergence Theorem):** Let $X$ be a Banach space with RNP and $(g_n)$ be a sequence in $L^1_X$ with the properties

1. $\sup_n \|g_n\|_{L^1_X} < \infty$,
2. $P_m g_n = g_m$ for all $m \leq n$.

Then, $g_n$ converges $\lambda$-a.e. to some $L^1_X$ function.

Already in the scalar case $X = \mathbb{R}$ Theorem 1.1 is a new result. In the course of its proof we intrinsically describe the pointwise limit of the sequence $(g_n)$. At the end of Section 6 we formulate a refined version of Theorem 1.1 employing the tools we developed for its proof. This includes an explicit expression of $\lim g_n$ in terms of $B$-splines.

We point out that only under significant restrictions on the geometry of the grid points $(t_i)$, is it true that the spline projections $P_n$ are Calderon–Zygmund operators (with constants independent of $n$). See [4].
Our present paper should be seen in context with the second named author’s work [7], where Burkholder’s martingale inequality
\[ \left\| \sum \pm (\mathbb{E}(f|A_n) - \mathbb{E}(f|A_{n-1})) \right\|_{L^p(\Omega,\mu)} \leq C_p \|f\|_{L^p(\Omega,\mu)} \]
was given a counterpiece for spline projections as follows:
\[ \left\| \sum \pm (P_n(g) - P_{n-1}(g)) \right\|_{L^p([0,1])} \leq C_p \|g\|_{L^p([0,1])}, \]
where \(1 < p < \infty\), and \(C_p \sim p^2/(p-1)\). The corresponding analogue for vector valued spline projections is still outstanding. (See however [5] for a special case.)

**Organization.** The presentation is organized as follows. In Section 2 we collect some important facts and tools used in this article. Section 3 treats the convergence of \(P_n g\) for \(L_1^X\)-functions \(g\). Section 4 contains special spline constructions associated to the point sequence \((t_i)\). In Section 5 we give a measure theoretic lemma that is subsequently employed and may be of independent interest in the theory of splines. Finally, in Section 6 we give the proof of the Spline Convergence Theorem.

### 2. Preliminaries

#### 2.1. Basics about vector measures.
We refer to the book [3] by J. Diestel and J. J. Uhl for basic facts on martingales and vector measures. Let \((\Omega, \mathcal{A})\) be a measure space and \(X\) a Banach space. Every \(\sigma\)-additive map \(\nu : \mathcal{A} \to X\) is called a vector measure. The variation \(|\nu|\) of \(\nu\) is the set function
\[ |\nu|(E) = \sup_{\pi} \sum_{A \in \pi} \|\nu(A)\|_X, \]
where the supremum is taken over all partitions \(\pi\) of \(E\) into a finite number of pairwise disjoint members of \(\mathcal{A}\). If \(\nu\) is of bounded variation, i.e., \(|\nu|(\Omega) < \infty\), the variation \(|\nu|\) is \(\sigma\)-additive. If \(\mu : \mathcal{A} \to [0, \infty)\) is a measure and \(\nu : \mathcal{A} \to X\) is a vector measure, \(\nu\) is called \(\mu\)-continuous, if
\[ \lim_{\mu(E) \to 0} \nu(E) = 0 \]
for all \(E \in \mathcal{A}\).
Definition 2.1: A Banach space $X$ has the **Radon–Nikodým property** (RNP) if for every measure space $(\Omega, \mathcal{A})$, for every positive measure $\mu$ on $(\Omega, \mathcal{A})$ and for every $\mu$-continuous vector measure $\nu$ of bounded variation, there exists a function $f \in L^1_X(\Omega, \mathcal{A}, \mu)$ such that

$$\nu(A) = \int_A f \, d\mu, \quad A \in \mathcal{A}. $$

**Theorem 2.2** (Lebesgue decomposition of vector measures): Let $(\Omega, \mathcal{A})$ be a measure space, $X$ a Banach space, $\nu : \mathcal{A} \to X$ a vector measure and $\mu : \mathcal{A} \to [0, \infty)$ a measure. Then, there exist unique vector measures $\nu_c, \nu_s : \mathcal{A} \to X$ such that

1. $\nu = \nu_c + \nu_s$,
2. $\nu_c$ is $\mu$-continuous,
3. $x^*\nu_s$ and $\mu$ are mutually singular for each $x^* \in X^*$.

If $\nu$ is of bounded variation, $\nu_c$ and $\nu_s$ are of bounded variation as well,

$$|\nu|(E) = |\nu_c|(E) + |\nu_s|(E)$$

for each $E \in \mathcal{A}$ and $|\nu_s|$ and $\mu$ are mutually singular.

The following theorem provides the fundamental link between convergence of vector valued martingales and the RNP of the underlying Banach space $X$. See Diestel–Uhl [3, Theorem V.2.9]. It is the point of reference for our present work on convergence of spline projections.

**Theorem 2.3** (Martingale convergence theorem): Let $(\Omega, \mathcal{A})$ be a measure space and $\mu : \mathcal{A} \to [0, \infty)$ a measure. Let $(\mathcal{A}_n)$ be a sequence of increasing sub-$\sigma$-algebras of $\mathcal{A}$. Let $X$ be a Banach space, let $(g_n)$ be a bounded sequence in $L^1_X(\Omega, \mathcal{A}_n, \mu)$ such that $g_n = \mathbb{E}(g_{n+1}|\mathcal{A})$, and let

$$\nu(E) = \lim_{n \to \infty} \int_E g_n \, d\mu, \quad E \in \bigcup_{n=1}^\infty \mathcal{A}_n.$$ 

Let $\nu = \nu_c + \nu_s$ denote the Lebesgue decomposition of $\nu$ with respect to $\mu$. Then $\lim_{n \to \infty} g_n$ exists almost everywhere with respect to $\mu$ if and only if $\nu_c$ has a Radon–Nikodým derivative $f \in L^1_X(\Omega, \mu)$. In this case

$$\lim_{n \to \infty} g_n = \mathbb{E}(f|\mathcal{A}_\infty),$$

where $\mathcal{A}_\infty$ is the $\sigma$-algebra generated by $\bigcup_{n=1}^\infty \mathcal{A}_n$. 
Let $X$ be a Banach space, let $v \in L^1(\Omega, \mathcal{A}, m)$ and $x \in X$. We recall that $v \otimes x : \Omega \to X$ is defined by

$$v \otimes x(\omega) = v(\omega)x$$

and that

$$L^1(\Omega, \mathcal{A}, m) \otimes X = \text{span}\{v_i \otimes x_i : v_i \in L^1(\Omega, \mathcal{A}, m), x_i \in X\}.$$ 

The following lemmata are taken from [10].

**Lemma 2.4**: For any Banach space $X$, the algebraic tensor product

$$L^1(\Omega, \mathcal{A}, m) \otimes X$$

is a dense subspace of the Bochner–Lebesgue space $L^1_X(\Omega, \mathcal{A}, m)$.

**Lemma 2.5**: Given a bounded operator $T : L^1(\Omega, \mathcal{A}, m) \to L^1(\Omega', \mathcal{A}', m')$ there exists a unique bounded linear map $\tilde{T} : L^1_X(\Omega, \mathcal{A}, m) \to L^1_X(\Omega', \mathcal{A}', m')$ such that

$$\tilde{T}(\varphi \otimes x) = T(\varphi)x, \quad \varphi \in L^1(\Omega, \mathcal{A}, m), \ x \in X.$$ 

Moreover, $\|\tilde{T}\| = \|T\|$.

**Lemma 2.6**: Let $X_0$ be a separable closed subspace of a Banach space $X$. Then, there exists a sequence $(x_n^*)$ in the unit ball of the dual $X^*$ of $X$ such that

$$\|x\| = \sup_n |x_n^*(x)|, \quad x \in X_0.$$ 

2.2. Tools from Real Analysis. We use the book by E. Stein [13] as our basic reference to Vitali’s covering lemma and weak-type estimates for the Hardy–Littlewood maximal function.

**Lemma 2.7** (Vitali covering lemma): Let $\{C_x : x \in \Lambda\}$ be an arbitrary collection of balls in $\mathbb{R}^d$ such that $\sup\{\text{diam}(C_x) : x \in \Lambda\} < \infty$. Then, there exists a countable subcollection $\{C_x : x \in J\}, J \subset \Lambda$ of balls from the original collection that are disjoint and satisfy

$$\bigcup_{x \in \Lambda} C_x \subset \bigcup_{x \in J} 5C_x.$$ 

Vitali’s covering lemma implies weak type estimates for the Hardy–Littlewood maximal function.
Theorem 2.8: Let \( f \in L^1_X \) and \( Mf(t) := \sup_{I \ni t} \frac{1}{\lambda(I)} \int_I \|f(s)\|_X \, ds \) be the Hardy–Littlewood maximal function. Then \( M \) satisfies the weak type estimate
\[
\lambda(\{Mf > u\}) \leq C \|f\|_{L^1_X}, \quad u > 0,
\]
where \( C > 0 \) is an absolute constant.

2.3. Spline spaces. Denote by \( |\Delta_n| \) the maximal mesh width of the grid \( \Delta_n = (t_i)_{i=1}^n \) augmented with \( k \) times the boundary points \( \{0, 1\} \). Recall that \( P_n \) is the orthogonal projection operator onto the space \( S_n \) of splines corresponding to the grid \( \Delta_n \), which is a conditional expectation operator for \( k = 1 \).

For the following, we introduce the notation \( A(t) \lesssim B(t) \) to indicate the existence of a constant \( c > 0 \) that only depends on \( k \) such that \( A(t) \leq cB(t) \), where \( t \) denotes all explicit or implicit dependences that the expressions \( A \) and \( B \) might have. As is shown by A. Shadrin, the sequence \( (P_n) \) satisfies \( L^1 \) estimates as follows:

**Theorem 2.9** ([12]): The orthogonal projection \( P_n \) admits a bounded extension to \( L^1 \) such that
\[
\sup_n \|P_n : L^1 \rightarrow L^1\| \lesssim 1.
\]

By Lemma 2.5, the operator \( P_n \) can be extended to the vector-valued \( L^1 \) space \( L^1_X \) with the same norm so that for all \( \varphi \in L^1 \) and \( x \in X \), we have
\[
P_n(\varphi \otimes x) = (P_n\varphi)x.
\]
We also have the identity

\[
(2.1) \quad \int_0^1 P_n g(t) \cdot f(t) \, d\lambda(t) = \int_0^1 g(t) \cdot P_n f(t) \, d\lambda(t), \quad g \in L^1_X, \; f \in L^\infty,
\]
which is just the extension of the fact that \( P_n \) is selfadjoint on \( L^2 \).

Fix \( f \in C[0,1] \). Consider the \( k \)th forward differences of \( f \) given by
\[
D^k_h f(t) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(t + jh).
\]
The \( k \)th modulus of smoothness of \( f \) in \( L^\infty \) is defined as
\[
\omega_k(f, \delta) = \sup_{0 \leq h \leq \delta} \sup_{0 \leq t \leq 1 - kh} |D^k_h f(t)|,
\]
where \( 0 \leq \delta \leq 1/k \). We have \( \lim_{\delta \to 0} \omega_k(f, \delta) = 0 \) for any \( f \in C[0,1] \). Any continuous function can be approximated by spline functions satisfying the following quantitative error estimate.
Theorem 2.10 ([11, Theorem 6.27]): Let \( f \in C[0, 1] \). Then,

\[ d(f, S_n)_{\infty} \lesssim \omega_k(f, |\Delta_n|), \]

where \( d(f, S_n)_{\infty} \) is the distance between \( f \) and \( S_n \) in the sup-norm. Therefore, if \( |\Delta_n| \to 0 \), we have \( d(f, S_n)_{\infty} \to 0 \).

Denote by \((N_i^{(n)})_i\) the B-spline basis of \( S_n \) normalized such that it forms a partition of unity and by \((N_i^{(n)})_i^*\) its corresponding dual basis in \( S_n \). Observe that

\[ P_n f(t) = \sum_i \langle f, N_i^{(n)} \rangle N_i^{(n)*}(t), \quad f \in L^2. \]

Since the B-spline functions \( N_i^{(n)} \) are contained in \( C[0, 1] \), we can also insert \( L^1 \)-functions as well as measures in the above formula.

If we set \( a_{ij}^{(n)} = \langle N_i^{(n)*}, N_j^{(n)*} \rangle \), we can expand the dual B-spline functions as a linear combination of B-spline functions with those coefficients:

\[ N_i^{(n)*} = \sum_j a_{ij}^{(n)} N_j^{(n)}. \]

Moreover, for \( t \in [0, 1] \) denote by \( I_n(t) \) a smallest grid point interval of positive length in the grid \( \Delta_n \) that contains the point \( t \). We denote by \( i_n(t) \) the largest index \( i \) such that \( I_n(t) \subset \text{supp} N_i^{(n)} \). Additionally, denote by \( h_{ij}^{(n)} \) the length of the convex hull of the union of the supports of \( N_i^{(n)} \) and \( N_j^{(n)} \).

With this notation, we can give the following estimate for the numbers \( a_{ij}^{(n)} \) and, a fortiori, for \( N_i^{(n)*} \):

Theorem 2.11 ([9]): There exists \( q \in (0, 1) \) depending only on the spline order \( k \), such that the numbers \( a_{ij}^{(n)} = \langle N_i^{(n)*}, N_j^{(n)*} \rangle \) satisfy the inequality

\[ |a_{ij}^{(n)}| \lesssim \frac{q^{i-j}}{h_{ij}^{(n)}}, \]

and therefore, in particular, for all \( i, \)

\[ |N_i^{(n)*}(t)| \lesssim \frac{q^{i-i_n(t)}}{\max \left( \lambda(I_n(t)), \lambda(\text{supp} N_i^{(n)}) \right)}, \quad t \in [0, 1]. \]

Proof. The first inequality is proved in [9] and the second one is an easy consequence of the first one inserted in formula (2.2) for \( N_i^{(n)*} \).
An almost immediate consequence of this estimate is the following pointwise maximal inequality for $P_n g$:

**Theorem 2.12 ([9]):** For all $g \in L^1_X$,

$$
\sup_n \|P_n g(t)\|_X \lesssim M g(t), \quad t \in [0, 1],
$$

where

$$
M g(t) = \sup_{I \ni t} \frac{1}{\lambda(I)} \int_I \|g(s)\|_X \, ds
$$

denotes the Hardy–Littlewood maximal function.

This result and Theorem 2.10 combined with Theorem 2.8 imply the a.e. convergence of $P_n g$ to $g$ for any $L^1$-function $g$ provided that the point sequence $(t_i)$ is dense in the unit interval $[0, 1]$; cf. [9].

As the spline spaces $S_n$ form an increasing sequence of subspaces of $L^2$, we can write the B-spline function $N^{(n)}_i$ as a linear combination of the finer B-spline functions $(N^{(n+1)}_j)$. The exact form of this expansion is given by Böhm’s algorithm [1] and it states in particular that the following result is valid:

**Proposition 2.13:** Let $f = \sum_i \alpha_i N^{(m)}_i \in S_m$ for some $m$. Then, there exists a sequence $(\beta_i)$ of coefficients so that

$$
\sum \beta_i N^{(m+1)}_i
$$

and, for all $i$, $\beta_i$ is a convex combination of $\alpha_{i-1}$ and $\alpha_i$.

By induction, an immediate consequence of this result is

**Corollary 2.14:** For any positive integers $n \geq m$ and any index $i$, the B-spline function $N^{(m)}_i$ can be represented as

$$
N^{(m)}_i \equiv \sum_j \lambda_j N^{(n)}_j,
$$

with coefficients $\lambda_j \in [0, 1]$ for all $j$.

In the following theorem it is convenient to display explicitly the order $k$ of the B-splines $N^{(n)}_i = N^{(n)}_{i,k}$. The relation between the sequences $(N^{(n)}_{i,k})_i$ and $(N^{(n)}_{i,k-1})_i$ is given by well known recursion formulae, for which we refer to [2]. See also [11].
Theorem 2.15: Let \([a, b] = \text{supp } N_{i,k}^{(n)}\). Then, the B-spline function \(N_{i,k}^{(n)}\) of order \(k\) can be expressed in terms of two B-spline functions of order \(k - 1\) as follows:

\[
N_{i,k}^{(n)}(t) = \frac{t-a}{\lambda(\text{supp } N_{i,k-1}^{(n)})} N_{i,k-1}^{(n)}(t) + \frac{b-t}{\lambda(\text{supp } N_{i+1,k-1}^{(n)})} N_{i+1,k-1}^{(n)}(t).
\]

3. Convergence of \(P_n g\)

As we are considering arbitrary sequences of grid points \((t_i)\) which are not necessarily dense in \([0, 1]\), as a first stage in the proof of the Spline Convergence Theorem, we examine the convergence of \(P_n g\) for \(g \in L^1\).

We first notice that \(P_n g\) converges in \(L^1\). Indeed, this is a consequence of the uniform boundedness of \(P_n\) on \(L^1\) as we will now show. Observe that for \(g \in L^2\), we get that if we define \(S_\infty\) as the \(L^2\) closure of \(\bigcup S_n\) and \(P_\infty\) as the orthogonal projection onto \(S_\infty\),

\[
\|P_n g - P_\infty g\|_{L^2} \to 0.
\]

Next, we show that this definition of \(P_\infty\) can be extended to \(L^1\) functions \(g\). So, let \(g \in L^1\) and \(\varepsilon > 0\). Since \(L^2\) is dense in \(L^1\), we can choose \(f \in L^2\) with the property \(\|g - f\|_1 < \varepsilon\). Now, choose \(N_0\) sufficiently large that for all \(m, n > N_0\), we have \(\|(P_n - P_m)f\|_2 < \varepsilon\). Then, we obtain

\[
\|(P_n - P_m)g\|_{L^1} \leq \|(P_n - P_m)(g - f)\|_{L^1} + \|(P_n - P_m)f\|_{L^1} \\
\leq 2C\varepsilon + \|(P_n - P_m)f\|_{L^2} \\
\leq (2C + 1)\varepsilon
\]

for a constant \(C\) depending only on \(k\) by Theorem 2.9. This means that \(P_n g\) converges in \(L^1\) to some limit that we will again call \(P_\infty g\). It actually coincides with the operator \(P_\infty\) on \(L^2\) and satisfies the same \(L^1\) bound as the sequence \((P_n)\). Summing up we have

\[
\|P_n g - P_\infty g\|_{L^1} \to 0,
\]

for any \(g \in L^1\). Applying Lemma 2.5 to \((P_n - P_\infty)\) we obtain the following vector valued extension. For any Banach space \(X\)

\[
\|P_n g - P_\infty g\|_{L^1_X} \to 0,
\]

for \(g \in L^1_X\).
The next step is to show pointwise convergence of $P_n g$ for continuous functions $g$. We define $U$ to be the complement of the set of all accumulation points of the given knot sequence $(t_i)$. This set $U$ is open, so it can be written as a disjoint union of open intervals

$$U = \bigcup_{j=1}^{\infty} U_j.$$  

**Lemma 3.1:** Let $g \in C[0,1]$. Then, $P_n g$ converges pointwise a.e. to $P_\infty g$ with respect to Lebesgue measure.

**Proof.** We first show that on each interval $U_j$, $P_n g$ converges locally uniformly. Let $A \subset U_j$ be a compact subset. Then the definition of $U_j$ implies that

$$s := \inf \{ \lambda(I_n(t)) : t \in A, n \in \mathbb{N} \}$$

is positive. Observe that of course, since in particular $g \in L^1[0,1]$, the sequence $P_n g$ converges in $L^1$. Therefore, for $\varepsilon > 0$, we can choose $M$ so large that for all $n, m \geq M$, $\|P_n g - P_m g\|_{L^1} \leq \varepsilon s$. We then estimate by Theorem 2.11 for $n \geq m \geq M$ and $t \in A$:

$$|(P_n - P_m)g(t)| = |P_n(P_n - P_m)g(t)|$$

$$\leq \sum_i \langle (P_n - P_m)g, N_i^{(n)} \rangle N_i^{(n)*}(t)$$

$$\lesssim \sum_i \frac{q|\mathbf{i} - \mathbf{i}_n(t)|}{\lambda(I_n(t))} \|P_n - P_m\|_{L^1(\text{supp } N_i^{(n)})}$$

$$\leq \|P_n - P_m\|_{L^1([0,1])} \sum_i \frac{q|\mathbf{i} - \mathbf{i}_n(t)|}{s}$$

$$\lesssim \frac{\|P_n - P_m\|_{L^1([0,1])}}{s} \leq \varepsilon,$$

so $P_n g$ converges uniformly on $A$.

If $t \in U^c$, we can assume that on both sides of $t$, there is a subsequence of grid points converging to $t$, since if there is a side that does not have a sequence of grid points converging to $t$, the point $t$ would be an endpoint of an interval $U_j$ and the union over all endpoints of $U_j$ is countable and therefore a Lebesgue zero set. Let $\varepsilon > 0$ and let $\ell$ be such that

$$q^\ell \|g\|_{L^\infty} \leq \varepsilon.$$
We choose $M$ so large that for any $m \geq M$ on each side of $t$ there are $\ell$ grid points of $\Delta_m$ and each of those grid point intervals has the property that the length is $< \delta$ with $\delta > 0$ being such that $\omega_k(g, \delta) < \varepsilon$, where $\omega_k$ is the $k$th modulus of smoothness. With this choice, by Theorem 2.10 there exists a function $f \in S_M$ with $\|f\|_{L^\infty} \lesssim \|g\|_{L^\infty}$ that approximates $g$ well on the smallest interval $B$ that contains $\ell - k$ grid points to the left of $t$ and $\ell - k$ grid points to the right of $t$ in $\Delta_M$ in the sense that

$$
(3.2) \quad \|f - g\|_{L^\infty(B)} \lesssim \omega_k(g, \delta) \leq \varepsilon.
$$

Therefore, we can write for $n, m \geq M$

$$(P_n - P_m)g(t) = P_n(g - f)(t) + P_m(f - g)(t).$$

Next, estimate $P_n(g - f)(t)$ for $n \geq M$ by Theorem 2.11:

$$|P_n(g - f)(t)| = \left| \sum_i \langle g - f, N_i^{(n)} \rangle N_i^{(n)*}(t) \right|
\lesssim \sum_i \|g - f\|_{L^\infty(supp(N_i^{(n)}))} \lambda(supp N_i^{(n)}) \frac{q|i - i_n(t)|}{\lambda(supp N_i^{(n)})}
= \sum_i q|i - i_n(t)| \|g - f\|_{L^\infty(supp N_i^{(n)})}.$$  

In estimating the above series we distinguish two cases for the value of $i$:

$$|i - i_n(t)| \leq \ell - 2k \quad \text{and} \quad |i - i_n(t)| > \ell - 2k.$$

Using $\|g - f\|_{L^\infty(supp N_i^{(n)})} \leq \|g - f\|_{L^\infty(B)}$ and (3.2) we get

$$\sum_{i:|i - i_n(t)| \leq \ell - 2k} q|i - i_n(t)| \|g - f\|_{L^\infty(supp N_i^{(n)})} \lesssim \varepsilon.$$  

Using $\|g - f\|_{L^\infty(supp N_i^{(n)})} \lesssim \|g\|_{L^\infty}$ and (3.1) gives

$$\sum_{i:|i - i_n(t)| > \ell - 2k} q|i - i_n(t)| \|g - f\|_{L^\infty(supp N_i^{(n)})} \lesssim \varepsilon.$$  

This yields $|P_n(g - f)(t)| \lesssim \varepsilon$ for $n \geq M$ and therefore $P_ng(t)$ converges as $n \to \infty$.  

The following theorem establishes the spline analogue of the martingale results (M1) and (M3). The role of Lemma 3.1 in the proof given below is to free the main theorem in [9] from the restriction that the sequence of knots $(t_i)$ is dense in $[0,1]$.  

Theorem 3.2: Let $X$ be any Banach space. For $f \in L^1_X$, there exists $E \subset [0,1]$ with $\lambda(E) = 0$ such that

$$\lim_{n \to \infty} P_n f(t) = P_\infty f(t),$$

for any $t \in [0,1] \setminus E$.

Proof. The proof uses standard arguments involving Lemma 3.1, Theorems 2.12 and 2.8. (See [9].)

Step 1 (The scalar case): Fix $v \in L^1$ and $\ell \in \mathbb{N}$. Put

$$A^{(\ell)}(v) = \bigcap_N \bigcup_{m,n \geq N} \{t : |P_n v(t) - P_m v(t)| > 1/\ell\}.$$ 

By Lemma 3.1, for any $u \in C[0,1]$,

$$\lambda(A^{(\ell)}(v)) = \lambda(A^{(\ell)}(v - u)).$$

Let $P^*(v - u)(t) = \sup_n |P_n (v - u)(t)|$. Clearly we have

$$\lambda(A^{(\ell)}(v - u)) \leq \lambda(\{t : 2P^*(v - u)(t) \geq 1/\ell\}).$$

By Theorem 2.12 $P^*$ is dominated pointwise by the Hardy–Littlewood maximal function and the latter is of weak type 1-1. Hence

$$\lambda(\{t : P^*(v - u)(t) \geq 1/\ell\}) \lesssim \ell \|v - u\|_{L^1}.$$ 

Now fix $\varepsilon > 0$. Since $C[0,1]$ is dense in $L^1$, there exists $u \in C[0,1]$ such that $\|v - u\|_{L^1} \leq \varepsilon/\ell$. Thus, we have obtained $\lambda(A^{(\ell)}(v)) < \varepsilon$ for any $\varepsilon > 0$, or $\lambda(A^{(\ell)}(v)) = 0$. It remains to observe that

$$\lambda(\{t : P_n v(t) \text{ does not converge}\}) = \lambda\left(\bigcup_{\ell} A^{(\ell)}(v)\right) = 0.$$ 

Step 2 (Vector valued extension): Let $g_m = v_m \otimes x_m$ where $v_m \in L^1$ and $x_m \in X$ and let $g \in L^1 \otimes X$ be given as

$$g = \sum_{m=1}^{M} g_m.$$ 

Applying Step 1 to $v_m$ shows that $P_n g(t)$ converges in $X$ for $\lambda$-almost every $t \in [0,1]$. Taking into account that $L^1 \otimes X$ is dense in $L^1_X$, we may now repeat
the argument above to finish the proof. Details are as follows: Fix \( f \in L^1_X \) and \( \ell \in \mathbb{N} \). Put
\[
A^{(\ell)}(f) = \bigcap_N \bigcup_{m,n \geq N} \{ t : \| P_n f(t) - P_m f(t) \|_X > 1/\ell \}.
\]
Then
\[
\lambda(\{ t : P_n f(t) \text{ does not converge in } X \}) = \lambda \left( \bigcup_\ell A^{(\ell)}(f) \right).
\]
It remains to prove that \( \lambda(A^{(\ell)}(f)) = 0 \). To this end observe that for \( g \in L^1 \otimes X \) we have \( \lambda(A^{(\ell)}(f)) = \lambda(A^{(\ell)}(f - g)) \). Define the maximal function
\[
P^*(f - g)(t) = \sup_n \| P_n (f - g)(t) \|_X.
\]
Clearly we have
\[
\lambda(A^{(\ell)}(f - g)) \leq \lambda(\{ t : 2P^*(f - g)(t) \geq 1/\ell \}).
\]
By Theorem 2.12 and the weak type 1-1 estimate for the Hardy–Littlewood maximal function,
\[
\lambda(\{ t : P^*(f - g)(t) \geq 1/\ell \}) \lesssim \ell \| f - g \|_{L^1_X}.
\]
Fix \( \varepsilon > 0 \) and choose \( g \in L^1 \otimes X \) such that \( \| f - g \|_{L^1_X} \leq \varepsilon / \ell \). This gives \( \lambda(A^{(\ell)}(f)) \lesssim \varepsilon \) for any \( \varepsilon > 0 \), proving that \( \lambda(A^{(\ell)}(f)) = 0 \).

4. B-spline constructions

Recall that we defined \( U \) to be the complement of the set of all accumulation points of the sequence \( (t_i) \). This set \( U \) is open, so it can be written as a disjoint union of open intervals
\[
U = \bigcup_{j=1}^\infty U_j.
\]
Observe that, since a boundary point \( a \) of \( U_j \) is an accumulation point of the sequence \( (t_j) \), there exists a subsequence of grid points converging to \( a \). Let
\[
B_j := \{ a \in \partial U_j : \text{there is no sequence of grid points contained in } U_j \text{ that converges to } a \}
\]
Now we set \( V_j := U_j \cup B_j \) and \( V := \bigcup_j V_j \).
Consider an arbitrary interval $V_{j_0}$ and set

$$a = \inf V_{j_0}, \quad b = \sup V_{j_0}.$$ 

We define the sequences $(s_j)$ and $(s_j^{(n)})$—rewritten in increasing order with multiplicities included—to be the points in $(t_j)$ and $(t_j)^n_{j=1}$, respectively, that are contained in $V_{j_0}$. If $a \in V_{j_0}$, the sequence $(s_j)$ is finite to the left and we extend the sequences $(s_j)$ and $(s_j^{(n)})$ so that the point $a$ is contained by them $k$ times and they are still increasing. Similarly, if $b \in V_{j_0}$, the sequence $(s_j)$ is finite to the right and we extend the sequences $(s_j)$ and $(s_j^{(n)})$ so that the point $b$ is contained by them $k$ times and they are still increasing. Observe that if $a \notin V_{j_0}$ or $b \notin V_{j_0}$, the sequence $(s_j)$ is infinite to the left or infinite to the right, respectively. We choose the indices of the sequences $(s_j)$ and $(s_j^{(n)})$ so that for fixed $j$ and $n$ sufficiently large, we have $s_j = s_j^{(n)}$. Let $(\bar{N}_j)$ and $(\bar{N}_j^{(n)})$ be the sequences of B-spline functions corresponding to the sequences $(s_j)$ and $(s_j^{(n)})$, respectively. Observe that the choice of the sequences $(s_j)$ and $(s_j^{(n)})$ implies for all $j$ that $\bar{N}_j \equiv \bar{N}_j^{(n)}$ if $n$ is sufficiently large. Let $(\bar{N}_j^{(n)})$ be the sequence of those B-spline functions from Section 2 whose supports intersect the set $V_{j_0}$ on a set of positive Lebesgue measure, but do not contain any of the points $\partial U_{j_0} \setminus B_{j_0}$ and, without loss of generality, we assume that this sequence is enumerated in such a way that the starting index and ending index coincide with those of the sequence $(\bar{N}_j^{(n)})_j$. Then, the relation between $(\bar{N}_j^{(n)})_j$ and $(\bar{N}_j)_j$ is given by the following lemma:

**Lemma 4.1:** For all $j$, the sequence of functions $(\bar{N}_j^{(n)}1_{V_{j_0}})$ converges uniformly to some function that coincides with $\bar{N}_j$ on $U_{j_0}$.

**Proof.** If the support of $\bar{N}_j^{(n)}$ is a subset of $V_{j_0}$ for sufficiently large $n$, the sequence $n \mapsto \bar{N}_j^{(n)}$ is eventually constant and coincides by definition with $\bar{N}_j$. In the other case, this follows by the recursion formula (Theorem 2.15) for B-splines and observing that for piecewise linear B-splines, this is clear. 

In view of the above lemma, we may assume that $\bar{N}_i$ coincides with the uniform limit of the sequence $(\bar{N}_i^{(n)}1_{V_{j_0}})$. Define $(\bar{N}_i^{(n)\ast})$ to be the dual B-splines to $(\bar{N}_i^{(n)})$. For $t \in [0,1]$ denote by $\bar{I}_n(t)$ a smallest grid point interval of positive length in the grid $(s_j^{(n)})$ that contains the point $t$. We denote by $\bar{i}_n(t)$ the largest index $i$ such that $\bar{I}_n(t) \subset \text{supp } \bar{N}_i^{(n)}$. Additionally, denote by $\bar{h}_i^{(n)}$
the length of the convex hull of the union of the supports of $\bar{N}_i^{(n)}$ and $\bar{N}_j^{(n)}$. Similarly, we let $\bar{I}(t)$ denote a smallest grid point interval of positive length in the grid $(s_j)$ containing $t \in [0,1]$. We denote by $\bar{i}(t)$ the largest index $i$ such that $\bar{I}(t) \subset \text{supp} \bar{N}_i$. Next, we identify dual functions to the sequence $(\bar{N}_j)$:

**Lemma 4.2:** For each $j$, the sequence $\bar{N}_j^{(n)*}$ converges uniformly on each interval $[s_i, s_{i+1}]$ to some function $\bar{N}^*_j$ that satisfies

1. $\langle \bar{N}^*_j, \bar{N}_i \rangle = \delta_{ij}$ for all $i$,
2. for all $t \in U_{j_0}$,

$$|\bar{N}^*_j(t)| \lesssim \frac{q|j-\bar{i}(t)|}{\lambda(\bar{I}(t))},$$

where $q \in (0,1)$ is given by Theorem 2.11.

**Proof.** We fix the index $j$, the point $t \in U_{j_0}$ and $\varepsilon > 0$. Next, we choose $M$ sufficiently large so that for all $m \geq M$ and all $\ell$ with the property $|\ell - \bar{i}(t)| \leq L$ we have

$$s^{(m)}_{\ell} = s_\ell,$$

where $L$ is chosen so that $q^L/\lambda(\bar{I}(t)) \leq \varepsilon$ and $|j - \bar{i}(t)| \leq L - k$. For $n \geq m \geq M$, we can expand the function $\bar{N}_j^{(m)*}$ in the basis $(\bar{N}_i^{(n)*})$ and write

$$\bar{N}_j^{(m)*} = \sum_i \alpha_{ji} \bar{N}_i^{(n)*}.$$  

We now turn to estimating the coefficients $\alpha_{ji}$ defined by equation (4.2). Observe that for $\ell$ with $|\ell - \bar{i}(t)| \leq L - k$, we have

$$\bar{N}_\ell^{(m)} \equiv \bar{N}_\ell^{(n)},$$

and therefore, for such $\ell$,

$$\delta_{j\ell} = \langle \bar{N}_j^{(m)*}, \bar{N}_\ell^{(m)} \rangle = \langle \bar{N}_j^{(m)*}, \bar{N}_\ell^{(n)} \rangle = \sum_i \alpha_{ji} \langle \bar{N}_i^{(n)*}, \bar{N}_\ell^{(n)} \rangle = \alpha_{j\ell},$$

which means that the expansion (4.2) takes the form

$$\bar{N}_j^{(m)*} = \bar{N}_j^{(n)*} + \sum_{\ell:|\ell - \bar{i}(t)| > L - k} \alpha_{j\ell} \bar{N}_\ell^{(n)*}. $$
Next we show that $|\alpha_{j\ell}|$ is bounded by a constant independently of $j, \ell$ and $m, n$. Recall that $\bar{h}^{(m)}_{ij}$ denotes the length of the smallest interval containing $\text{supp } \bar{N}^{(m)}_i \cup \text{supp } \bar{N}^{(m)}_j$.

By Theorem 2.11 applied to the matrix $(\bar{a}^{(m)}_{ij}) = (\langle \bar{N}^{(m)*}_i, \bar{N}^{(m)*}_j \rangle)$, we get

$$|\alpha_{j\ell}| = |\langle \bar{N}^{(m)*}_j, \bar{N}^{(n)}_{\ell} \rangle| = \left| \sum_i \bar{a}^{(m)}_{ij} \bar{N}^{(m)}_i, \bar{N}^{(n)}_{\ell} \right|$$

$$\lesssim \sum_i q^{|i-j|} \bar{h}^{(m)}_{ij} \langle \bar{N}^{(m)}_i, \bar{N}^{(n)}_{\ell} \rangle$$

$$\leq \sum_i q^{|i-j|} \bar{h}^{(m)}_{ij} \lambda(\text{supp } \bar{N}^{(m)}_i)$$

$$\leq \sum_i q^{|i-j|} \lesssim 1.$$

This can be used to obtain an estimate for the difference between $\bar{N}^{(m)*}_j(t)$ and $\bar{N}^{(n)*}_j(t)$ by inserting it into (4.3) and applying again Theorem 2.11

$$|(\bar{N}^{(m)*}_j - \bar{N}^{(n)*}_j)(t)| \leq \sum_{\ell : |\ell - \bar{i}(t)| > L-k} |\alpha_{j\ell}| |\bar{N}^{(n)*}_{\ell}(t)|$$

$$\lesssim \sum_{\ell : |\ell - \bar{i}(t)| > L-k} \frac{q^{L|\ell - \bar{i}(t)|}}{\lambda(I_n(t))} \lesssim \frac{q^L}{\lambda(I(t))} \leq \varepsilon.$$

This finishes the proof of the convergence part. Estimate (4.1) now follows from the corresponding estimate for $\bar{N}^{(n)*}_j$ in Theorem 2.11.

Now, we turn to the proof of property (1). Let $j, i$ be arbitrary. Choose $M$ sufficiently large so that for all $n \geq M$, we have $\bar{N}_i \equiv \bar{N}^{(n)}_i$ on $U_{j_0}$, therefore,

$$|\langle \bar{N}^*_j, \bar{N}_i \rangle - \delta_{ij}| = |\langle \bar{N}^*_j, \bar{N}_i \rangle - \langle \bar{N}^{(n)*}_j, \bar{N}^{(n)}_i \rangle|$$

$$= |\langle \bar{N}^*_j - \bar{N}^{(n)*}_j, \bar{N}^{(n)}_i \rangle|$$

$$\leq \|\bar{N}^*_j - \bar{N}^{(n)*}_j\|_{L^\infty(\text{supp } \bar{N}^{(n)}_i)} \cdot \lambda(\text{supp } \bar{N}^{(n)}_i),$$

which, by the local uniform convergence of $\bar{N}^{(n)*}_j$ to $\bar{N}^*_j$, tends to zero. ■
5. A measure estimate

Let $\sigma$ be a measure defined on the unit interval. Recall that $P_n(\sigma)$ is defined by duality. In view of Theorem 2.11, localized and pointwise estimates for $P_n(\sigma)$ are controlled by terms of the form

$$\sum_{i,j} \frac{q|^{i-j|}}{h_{ij}^{(n)}} |\sigma|(\text{supp } N_i^{(n)}) N_j^{(n)}.$$ 

Subsequently, the following lemma will be used to show that $P_n(\sigma)$ converges a.e. to zero, for any measure $\sigma$ singular to the Lebesgue measure.

**Lemma 5.1:** Let $F_r$ be a Borel subset of $V^c$ and $\theta$ a positive measure on $[0,1]$ with $\theta(F_r) = 0$ so that for all $x \in F_r$, we have

$$\limsup_n b_n(x) > 1/r,$$

where $b_n(x)$ is a positive function satisfying

$$b_n(x) \lesssim \sum_{i,j} \frac{q|^{i-j|}}{h_{ij}^{(n)}} \theta(\text{supp } N_i^{(n)}) N_j^{(n)}(x), \quad x \in F_r.$$ 

Then,

$$\lambda(F_r) = 0.$$ 

**Proof.** First observe that we can assume that each point in $F_r$ can be approximated from both sides with points of the sequence $(t_i)$, since the set of points in $V^c$ for which this is not possible is a subset of $\bigcup_j \partial V_j$ and therefore of Lebesgue measure zero.

**Step 1:** For an arbitrary positive number $\varepsilon$, by the regularity of $\theta$, we can take an open set $U_\varepsilon \subset [0,1]$ with $U_\varepsilon \supset F_r$ and $\theta(U_\varepsilon) \leq \varepsilon$. Then, for $x \in F_r$, we choose a ball $B_x \subset U_\varepsilon$ with center $x$, define $s_m(x) = \{j : N_j^{(m)}(x) \neq 0\}$ and calculate

$$b_m(x) \lesssim \sum_{i,j} \frac{q|^{i-j|}}{h_{ij}^{(m)}} \theta(\text{supp } N_i^{(m)}) N_j^{(m)}(x) \lesssim \sum_{j \in s_m(x)} \sum_i \frac{q|^{i-j|}}{h_{ij}^{(m)}} \theta(\text{supp } N_i^{(m)})$$

$$\lesssim \max_{j \in s_m(x)} \sum_i \frac{q|^{i-j|}}{h_{ij}^{(m)}} \theta(\text{supp } N_i^{(m)})$$

$$= C \max_{j \in s_m(x)} (\Sigma_{1,j}^{(m)} + \Sigma_{2,j}^{(m)}),$$
Define \( N_i^{(m)} \) where
\[
\Sigma_{1,j}^{(m)} := \sum_{i \in \Lambda_1^{(m)}} \frac{q^{i-j}}{h_{ij}^{(m)}} \theta(\text{supp } N_i^{(m)}), \quad \Sigma_{2,j}^{(m)} := \sum_{i \in \Lambda_2^{(m)}} \frac{q^{i-j}}{h_{ij}^{(m)}} \theta(\text{supp } N_i^{(m)})
\]
and
\[
\Lambda_1^{(m)} = \{ i : \text{supp } N_i^{(m)} \subset B_x \}, \quad \Lambda_2^{(m)} = (\Lambda_1^{(m)})^c.
\]

**STEP 2:** Next, we show that it is possible to choose \( m \) sufficiently large to have
\[
\Sigma_{2,j}^{(m)} \leq 1/(2C) \quad \text{for all } j \in s_m(x).
\]

To do that, let \( j_m \in s_m(x) \) and observe that
\[
\Sigma_{2,j_m}^{(m)} = \sum_{i \in \Lambda_2^{(m)}} \frac{q^{i-j_m} \theta(\text{supp } N_i^{(m)})}{h_{ij_m}^{(m)}} \leq \sum_{i \in \Lambda_2^{(m)}} \frac{q^{i-j_m} \theta(\text{supp } N_i^{(m)})}{d(x, \text{supp } N_i^{(m)})} =: A_{2,j_m}^{(m)}
\]
where \( d(x, \text{supp } N_i^{(m)}) \) denotes the Euclidean distance between \( x \) and \( \text{supp } N_i^{(m)} \).

Now, for \( n > m \) sufficiently large, we get
\[
A_{2,j_n}^{(n)} \leq \sum_{i \in \Lambda_2^{(m)}} \sum_{\ell \in \Lambda_2^{(n)}, \supp N_i^{(n)} \subset \text{supp } N_i^{(m)}} \frac{q^{i-j_m} \theta(\text{supp } N_i^{(n)})}{d(x, \text{supp } N_i^{(n)})}.
\]

Define \( L_{n,m} \) to be the cardinality of the set \( \{ t_i : m < i \leq n \} \cap B_x \cap [0, x] \) and \( R_{n,m} \) the cardinality of \( \{ t_i : m < i \leq n \} \cap B_x \cap [x, 1] \). Put
\[
K_{n,m} = \min\{ L_{n,m}, R_{n,m} \}.
\]

The term \( A_{2,j_n}^{(n)} \) admits the following upper bound:
\[
q^{K_{n,m}} \sum_{i \in \Lambda_2^{(m)}, \supp N_i^{(n)} \subset \text{supp } N_i^{(m)}} \frac{q^{i-j_m}}{d(x, \text{supp } N_i^{(n)})} \quad \theta(\text{supp } N_i^{(n)}) \leq q^{K_{n,m}} \sum_{i \in \Lambda_2^{(m)}, \supp N_i^{(n)} \subset \text{supp } N_i^{(m)}} \frac{q^{i-j_m}}{d(x, \text{supp } N_i^{(m)})} \theta(\text{supp } N_i^{(m)}) = q^{K_{n,m}} A_{2,j_m}^{(m)}
\]
Since \( x \) can be approximated by grid points from both sides, \( \lim_{n \to \infty} K_{n,m} = \infty \), and we can choose \( m \) sufficiently large to guarantee
\[
\Sigma_{2,j}^{(m)} \leq A_{2,j}^{(m)} \leq \frac{1}{2Cr}.
\]

**Step 3:** Next, we show that for any \( x \in F_r \), there exists an open interval \( C_x \subset B_x \) such that \( \theta(C_x) / \lambda(C_x) \geq 1/(2Cr) \).

By Step 2 and the fact that \( \limsup b_n(x) > 1/r \) for \( x \in F_r \), there exists an integer \( m \) and an index \( j_0 \in s_m(x) \) with
\[
\Sigma_{1,j_0}^{(m)} \geq \frac{1}{2Cr},
\]
which means that
\[
\frac{1}{2Cr} \leq \sum_{i \in \Lambda_i^{(m)}} \frac{q^{i-j_0}}{h_{ij_0}^{(m)}} \theta(\text{supp } N_i^{(m)})
\leq \sum_{i \in \Lambda_i^{(m)}} \frac{q^{i-j_0}}{h_{ij_0}^{(m)}} \theta(\text{conv}(\text{supp } N_i^{(m)} \cup \text{supp } N_{j_0}^{(m)})),
\]
where \( \text{conv}(A) \) denotes the convex hull of the set \( A \). Since \( \sum_{i \in \Lambda_i^{(m)}} q^{i-j_0} \lesssim 1 \), there exists a constant \( c \) depending only on \( q \) and an index \( i \) with \( \text{supp } N_i^{(m)} \subset B_x \) and
\[
\frac{\theta(\text{conv}(\text{supp } N_i^{(m)} \cup \text{supp } N_{j_0}^{(m)}))}{h_{ij_0}^{(m)}} \geq \frac{c}{2Cr},
\]
which means that there exists an open interval \( C_x \) with \( x \in C_x \subset B_x \) with the property \( \theta(C_x) / \lambda(C_x) \geq c/(2Cr) \).

**Step 4:** Now we finish with a standard argument using the Vitali covering lemma (Lemma 2.7): there exists a countable collection \( J \) of points \( x \in F_r \) such that \( \{C_x : x \in J\} \) are disjoint sets and
\[
F_r \subset \bigcup_{x \in F_r} C_x \subset \bigcup_{x \in J} 5C_x.
\]

Combining this with Steps 1–3, we conclude that
\[
\lambda(F_r) \leq \lambda \left( \bigcup_{x \in J} 5C_x \right) \leq 5 \sum_{x \in J} \lambda(C_x) \leq \frac{10Cr}{c} \sum_{x \in J} \theta(C_x) \leq \frac{10Cr}{c} \theta(U_\varepsilon) \leq \frac{10Cr}{c} \varepsilon.
\]
Since this inequality holds for all \( \varepsilon > 0 \), we get that \( \lambda(F_r) = 0 \).
6. Proof of the Spline Convergence Theorem

In this section, we prove the Spline Convergence Theorem \[1.1\] For \(f \in S_m\), a consequence of (2.1) is

\[
\int_0^1 g_n(t) \cdot f(t) \, d\lambda(t) = \int_0^1 g_n(t) \cdot P_m f(t) \, d\lambda(t) = \int P_m g_n(t) \cdot f(t) \, d\lambda(t) = \int_0^1 g_m(t) \cdot f(t) \, d\lambda(t), \quad n \geq m.
\]

This means in particular that for all \(f \in \bigcup_n S_n\), the limit of \(\int_0^1 g_n(t) \cdot f(t) \, d\lambda(t)\) exists, so we can define the linear operator

\[T : \bigcup S_n \to X, \quad f \mapsto \lim_n \int_0^1 g_n(t) \cdot f(t) \, d\lambda(t).\]

By Alaoglu’s theorem, we may choose a subsequence \(k_n\) such that the bounded sequence of measures \(\|g_{k_n}\|_X \, d\lambda\) converges in the weak*-topology to some scalar measure \(\mu\). Then, as each \(f \in \bigcup_n S_n\) is continuous,

\[(6.1) \quad \|Tf\|_X \leq \int_0^1 |f(t)| \, d\mu(t), \quad f \in \bigcup S_n.
\]

We let \(W\) denote the \(L^1([0,1],\mu)-\)closure of \(\bigcup_n S_n\). By (6.1), the operator \(T\) extends to \(W\) with norm bounded by 1.

We set

\[(P_n T)(t) := \sum_i (TN_i^{(n)}(t))N_i^{(n)*}(t)\]

which is well defined. Moreover,

\[(P_n T)(t) = \sum_i (TN_i^{(n)}(t))N_i^{(n)*}(t) = \sum_i \lim_m \int g_m N_i^{(n)}(t) \, d\lambda \cdot N_i^{(n)*}(t) = \sum_i \langle g_n, N_i^{(n)}(t) \rangle N_i^{(n)*}(t) = (P_n g_n)(t) = g_n(t).
\]

Thus we verify a.e. convergence of \(g_n\), by showing a.e. convergence of \(P_n T\) below.
Lemma 6.1: For all \( f \in \bigcup S_n \), the function \( f \mathbb{1}_{V_j} \) is contained in \( W \) and also \( f \mathbb{1}_V \) is contained in \( W \). Additionally, on the complement of \( V = \bigcup V_j \), the \( \sigma \)-algebra \( \mathcal{F} = \{ A \in \mathcal{B} : \mathbb{1}_A \in W \} \) coincides with the Borel \( \sigma \)-algebra \( \mathcal{B} \), i.e.,

\[
V^c \cap \mathcal{F} = V^c \cap \mathcal{B}.
\]

Proof. Since \( W \) is a linear space, it suffices to show the assertion for each B-spline function \( N^{(m)}_i \) contained in some \( S_m \). By Corollary 2.14, it can be written as a linear combination of finer B-spline functions \( (n \geq m) \)

\[
N^{(m)}_i = \sum_{\ell} \lambda^{(n)}_{\ell} N^{(n)}_{\ell},
\]

where each coefficient \( \lambda^{(n)}_{\ell} \) satisfies the inequality \( |\lambda^{(n)}_{\ell}| \leq 1 \). We set

\[
h_n := \sum_{\ell \in \Lambda_n} \lambda^{(n)}_{\ell} N^{(n)}_{\ell},
\]

where the index set \( \Lambda_n \) is defined to contain precisely those indices \( \ell \) so that \( \text{supp} \, N^{(n)}_{\ell} \) intersects \( V_j \) but does not contain any of the points \( \partial U_j \setminus B_j \). The function \( h_n \) is contained in \( S_n \) and satisfies \( |h_n| \leq 1 \). Observe that \( \text{supp} \, h_n \subset O_n \) for some open set \( O_n \) and \( h_n \equiv N^{(m)}_i \) on some compact set \( A_n \subset V_j \) that satisfy \( O_n \setminus A_n \downarrow \emptyset \) as \( n \to \infty \) and thus

\[
\|N^{(m)}_i \mathbb{1}_{V_j} - h_n\|_{L^1(\mu)} \lesssim \mu(O_n \setminus A_n) \to 0.
\]

This shows that \( N^{(m)}_i \mathbb{1}_{V_j} \in W \).

Since \( \mu \) is a finite measure, \( \lim_n \mu(\bigcup_{j \geq n} V_j) = 0 \), and therefore, \( f \mathbb{1}_V = f \mathbb{1}_{\bigcup_j V_j} \) is also contained in \( W \).

Similarly, we see that the collection \( \mathcal{F} = \{ A \in \mathcal{B} : \mathbb{1}_A \in W \} \) is a \( \sigma \)-algebra. So, in order to show \( V^c \cap \mathcal{F} = V^c \cap \mathcal{B} \) we will show that for each interval \( (c, d) \) contained in \([0, 1] \), we can find an interval \( I \in \mathcal{F} \) with the property \( V^c \cap (c, d) = V^c \cap I \). By the same reasoning as in the approximation of \( N^{(m)}_i \mathbb{1}_{V_j} \) by finer spline functions, we can give the following sufficient condition for an interval \( I \) to be contained in \( \mathcal{F} \): if for all \( a \in \{ \inf I, \sup I \} \) we have either

\[
a \in I \text{ and there exists a seq. of grid points conv. from outside of } I \text{ to } a
\]
or

\[
a \notin I \text{ and there exists a seq. of grid points conv. from inside of } I \text{ to } a,
\]
then \( I \in \mathcal{F} \). Let now \((c, d)\) be an arbitrary interval and assume first that \( c, d \notin \bigcup_j \partial U_j \). For arbitrary points \( x \in [0, 1] \), define

\[
I(x) := \begin{cases} 
V_j, & \text{if } x \in U_j, \\
\emptyset, & \text{otherwise}.
\end{cases}
\]

Then, by the above sufficient criterion, the set \( I = (c, d) \setminus (I(c) \cup I(d)) \) is contained in \( \mathcal{F} \). Moreover, \( V^c \cap (c, d) = V^c \cap I \) and this shows that

\[
(c, d) \cap V^c \in \mathcal{F} \cap V^c.
\]

In general, since the set \( \bigcup_j \partial U_j \) is countable, we can find sequences \( c_n \geq c \) and \( d_n \leq d \) with \( c_n, d_n \notin \bigcup_j \partial U_j \), \( c_n \to c \), \( d_n \to d \), and

\[
(c, d) \cap V^c = \left( \bigcup_n (c_n, d_n) \right) \cap V^c \in \mathcal{F} \cap V^c,
\]

since \( \mathcal{F} \cap V^c \) is a \( \sigma \)-algebra. This shows the fact that \( \mathcal{F} \cap V^c = \mathcal{B} \cap V^c \).

\[ \text{Proof of Theorem 1.1} \]

\[ \text{Part 1 } t \in V^c: \]

By Lemma 6.1, we can decompose

\[
g_n(t) = (P_nT)(t) = \sum_i T(N_i^{(n)})N_i^{(n)*}(t)
\]

\[
= \sum_i T(N_i^{(n)}1_{V^c})N_i^{(n)*}(t) + \sum_i T(N_i^{(n)}1_{V^c^c})N_i^{(n)*}(t)
\]

\[
=: \Sigma_1^{(n)}(t) + \Sigma_2^{(n)}(t).
\]

\[ \text{Part 1.a } \Sigma_1^{(n)}(t) \text{ for } t \in V^c: \]

We will show that \( \Sigma_1^{(n)}(t) \) converges to zero a.e. on \( V^c \). This is done by defining the measure

\[
\theta(E) := \mu(E \cap V), \quad E \in \mathcal{B},
\]

and

\[
F_r = \{ t \in V^c : \limsup_n \| \Sigma_1^{(n)}(t) \|_X > 1/r \} \subset V^c.
\]

Observe that \( \theta(F_r) = 0 \) and, by (6.1) and Theorem 2.1,

\[
\| \Sigma_1^{(n)}(t) \|_X \lesssim \sum_{i,j} g_{i,j}^{[i-j]} \theta(\text{supp} N_i^{(n)}N_j^{(n)}(t)), \quad t \in F_r,
\]

which allows us to apply Lemma 5.1 on \( F_r \) and \( \theta \) to get \( \lambda(F_r) = 0 \) for all \( r > 0 \), i.e., \( \Sigma_1^{(n)}(t) \) converges to zero a.e. on \( V^c \).
PART 1.b $\Sigma_2^{(n)}(t)$ for $t \in V^c$:

Let $B_{V^c} = V^c \cap B$. Thus $B_{V^c}$ is the restriction of the Borel $\sigma$-algebra $B$ to $V^c$. In this case, we define the vector measure $\nu$ of bounded variation on $(V^c, B_{V^c})$ by

$$\nu(A) := T(1_A), \quad A \in B_{V^c}.$$

Here we use the second part of Lemma 6.1 to guarantee that the right hand side is defined and (6.1) ensures $|\nu| \leq \mu$. Apply Lebesgue decomposition Theorem 2.2 to get

$$d\nu = g d\lambda + d\nu_s$$

where $g \in L^1_X$ and $|\nu_s|$ is singular to $\lambda$. Observe that for all $f \in \bigcup S_n$, we have

$$\int f d\nu = T(f 1_{V^c}).$$

Indeed, this holds for indicator functions by definition and each $f \in \bigcup S_n$ can be approximated in $L^1(\mu)$ by linear combinations of indicator functions. Therefore, (6.3) is established, since both sides of (6.3) are continuous in $L^1(\mu)$. So,

$$\sum_{i,j} \left| a_{i,j}^{(n)} \right| \int N_i^{(n)} d(x^* \circ \nu_s) \cdot N_j^{(n)}(t) \lesssim \sum_{i,j} \left| a_{i,j}^{(n)} \right| \left| x^* \circ \nu_s \right|(\text{supp } N_i^{(n)}) \cdot N_j^{(n)}(t),$$

we can apply Lemma 5.1 to $F_{r,x^*}$ and the measure $\theta(B) = |x^* \circ \nu_s|(B \cap V^c)$ to obtain $\lambda(F_{r,x^*}) = 0$. Since the closure $X_0$ in $X$ of the set $\{P_n \nu_s(t) : t \in [0,1], n \in \mathbb{N}\}$
is a separable subspace of \(X\), by Lemma 2.6, there exists a sequence \((x^*_n)\) of elements in \(X^*\) such that for all \(x \in X_0\) we have \(\|x\| = \sup_n |x^*_n(x)|\). This means that we can write

\[
F := \{ t \in A : \limsup_n \| P_n \nu_s(t) \| > 0 \} = \bigcup_{n, \ell = 1}^{\infty} F_{\ell, x^*_n},
\]

and thus \(\lambda(F) = 0\), which shows that \(P_n \nu_s\) tends to zero almost everywhere on \(V^c\) with respect to Lebesgue measure.

**Part 2** \(t \in V\):

Now, we consider \(t \in V\) or, more precisely, \(t \in U\). This makes no difference for considering a.e. convergence since the difference between \(V\) and \(U\) is a Lebesgue zero set. We choose the index \(j_0\) such that \(t \in U_{j_0}\) and, based on the location of \(t\), we decompose (using Lemma 6.1)

\[
g_n(t) = P_n T(t) = \sum_i T(N_i^{(n)}) N_i^{(n)*}(t)
\]

\[
= \sum_i T(N_i^{(n)} 1_{V_{j_0}}) \cdot N_i^{(n)*}(t) + \sum_i T(N_i^{(n)} 1_{V^c_{j_0}}) \cdot N_i^{(n)*}(t)
\]

\[=: \Sigma_1^{(n)}(t) + \Sigma_2^{(n)}(t).\]

**Part 2.a** \(\Sigma_1^{(n)}(t)\) for \(t \in U_{j_0}\):

We now consider

\[
\Sigma_1^{(n)} = \sum_i T(N_i^{(n)} 1_{V_{j_0}}) N_i^{(n)*}(t), \quad t \in U_{j_0},
\]

and perform the construction of the B-splines (\(\tilde{N}_j\)) and their dual functions (\(\tilde{N}_j^*\)) corresponding to \(V_{j_0}\) described in Section 4. Define the function

\[
(6.4) \quad u(t) := \sum_j T(\tilde{N}_j) \tilde{N}_j^*(t), \quad t \in U_{j_0},
\]

and first note that \(\tilde{N}_j \in W\) since by Lemma 4.1 it is the uniform limit of the functions \((N_j^{(n)} 1_{V_{j_0}})\), which, in turn, are contained in \(W\) by Lemma 6.1. Therefore, \(T(\tilde{N}_j)\) is defined. Moreover, the series in (6.4) converges pointwise for \(t \in U_{j_0}\), since \(\lambda(\tilde{I}(t)) > 0\), the sequence \(j \mapsto \tilde{N}_j^*(t)\) admits a geometric decay estimate by (4.1) and the inequality \(\|T(\tilde{N}_i)\|_X \leq \mu(\text{supp} \tilde{N}_i)\). If one additionally notices that (4.1) implies the estimate \(\|\tilde{N}_j^*\|_{L^1} \lesssim 1\) we see that the convergence in (6.4) takes place in \(L^1_X\) as well. This implies \(\langle u, \tilde{N}_i \rangle = T(\tilde{N}_i)\) for all \(i\) by Lemma 4.2.
Next, we show that if for all \( n \), \( (a_i) \) and \( (a_i^{(n)}) \) are sequences in \( X \) so that for all \( i \) we have \( \lim_n a_i^{(n)} = a_i \), and \( \sup_i \|a_i\|_X + \sup_{i,n} \|a_i^{(n)}\|_X \leq 1 \), it follows that

\[
\lim_n \sum_i (a_i^{(n)} - a_i) N_i^{(n)*}(t) = 0, \quad t \in U_{j_0}.
\]

Indeed, let \( \varepsilon > 0 \), the integer \( L \) be such that \( q^L \leq \varepsilon \cdot \inf_n \lambda(I_n(t)) \) and \( M \) be sufficiently large that for all \( n \geq M \) and all \( i \) with \( |i - i_n(t)| \leq L \), we have \( \|a_i^{(n)} - a_i\|_X \leq \varepsilon \cdot \inf_n \lambda(I_n(t)) \). Then, by Theorem 2.11

\[
\left \| \sum_i (a_i^{(n)} - a_i) N_i^{(n)*}(t) \right \|_X \leq \sum_i \|a_i^{(n)} - a_i\|_X \frac{q^{|i-i_n(t)|}}{\lambda(I_n(t))}
\]

\[
= \left( \sum_{i:|i-i_n(t)| \leq L} + \sum_{i:|i-i_n(t)| > L} \right) \|a_i^{(n)} - a_i\|_X \frac{q^{|i-i_n(t)|}}{\lambda(I_n(t))}
\]

\[
\lesssim \sum_{i:|i-i_n(t)| \leq L} \varepsilon q^{|i-i_n(t)|} + \sum_{i:|i-i_n(t)| > L} \frac{q^{|i-i_n(t)|}}{\lambda(I_n(t))} \lesssim \varepsilon.
\]

We now use these remarks to show that

\[
\lim_n \|\Sigma_1^{(n)}(t) - P_n u(t)\|_X = 0, \quad t \in U_{j_0}.
\]

Indeed, since \( \langle u, \bar{N}_i \rangle = T(\bar{N}_i) \) for all \( i \),

\[
\Sigma_1^{(n)}(t) - P_n u(t) = \sum_i \left( T(N_i^{(n)} 1_{V_{j_0}}) - \langle u, N_i^{(n)} \rangle \right) N_i^{(n)*}(t)
\]

\[
= \sum_i \left( T(N_i^{(n)} 1_{V_{j_0}}) - T(\bar{N}_i) \right) N_i^{(n)*}(t)
\]

\[
+ \sum_i \left( \langle u, \bar{N}_i \rangle - \langle u, N_i^{(n)} \rangle \right) N_i^{(n)*}(t).
\]

Now, observe that for all \( i \), we have \( T(N_i^{(n)} 1_{V_{j_0}}) \rightarrow T(\bar{N}_i) \) and \( \langle u, N_i^{(n)} \rangle \rightarrow \langle u, \bar{N}_i \rangle \) since by Lemma 4.1 \( N_i^{(n)} \) converges uniformly to \( \bar{N}_i \) on \( V_{j_0} \) and \( u \in L^1 \). Moreover, all the expressions \( T(N_i^{(n)} 1_{V_{j_0}}), T(\bar{N}_i), \langle u, N_i^{(n)} \rangle \) are bounded in \( i \) and \( n \). As a consequence, we can apply (6.5) to both of the sums in the above display to conclude that

\[
\lim_n \|\Sigma_1^{(n)}(t) - P_n u(t)\|_X = 0, \quad t \in U_{j_0}.
\]

But we know that \( P_n u(t) \) converges a.e. to \( u(t) \) by Theorem 3.2, this means that also \( \Sigma_1^{(n)}(t) \) converges to \( u \) a.e.
Part 2.b $\Sigma_2^{(n)}(t)$ for $t \in U_{j_0}$:

We show that

$$\Sigma_2^{(n)}(t) = \sum_i T(N_i^{(n)} 1_{V_{j_0}^c}) \cdot N_i^{(n)*}(t)$$

converges to zero for $t \in U_{j_0}$. Let $\varepsilon > 0$ and set $s = \inf_n \lambda(I_n(t))$, where we recall that $I_n(t)$ is the grid interval in $\Delta_n$ that contains the point $t$. Since $s > 0$ we can choose an open interval $O$ with the property

$$\mu(O \setminus V_{j_0}) \leq \varepsilon s.$$

Then, due to the fact that $t \in U_{j_0}$, we can choose $M$ sufficiently large that both intervals $(\inf O, t)$ and $(t, \sup O)$ contain $L$ points of the grid $\Delta_M$ where $L$ is such that $q^L \leq \varepsilon s / \mu([0, 1])$. Thus, we estimate for $n \geq M$ by (6.1) and Theorem 2.11

$$\|\Sigma_2^{(n)}(t)\|_X \leq \sum_i \mu(\text{supp } N_i^{(n)} \cap V_{j_0}^c) \frac{q|i-i_n(t)|}{\lambda(I_n(t))} \leq \frac{1}{s} \left( \sum_{i: \text{supp } N_i^{(n)} \cap O \neq \emptyset} + \sum_{i: \text{supp } N_i^{(n)} \subset O} \right) \left( \mu(\text{supp } N_i^{(n)} \cap V_{j_0}^c) q|i-i_n(t)| \right) \lesssim \frac{1}{s} (\mu([0, 1]) q^L + \mu(O \setminus V_{j_0})) \lesssim \varepsilon.$$

This proves that $\Sigma_2^{(n)}(t)$ converges to zero for $t \in U_{j_0}$.

By looking at the above proof and employing the notation therein, we have actually proved the following, explicit form of the Spline Convergence Theorem:

Theorem 6.2: Let $X$ be a Banach space with RNP and $(g_n)$ be a sequence in $L_X^1$ with the properties

1. $\sup_n \|g_n\|_{L_X^1} < \infty$,
2. $P_m g_n = g_m$ for all $m \leq n$.

Then, $g_n$ converges a.e. to the $L_X^1$-function

$$g 1_{V^c} + \sum_{j_0} \sum_j T(\tilde{N}_{j_0,j}) \tilde{N}_{j_0,j}^* 1_{V_{j_0}}.$$ 

Here, $g$ is defined by (6.2), and for each $j_0$, $(\tilde{N}_{j_0,j})$ and $(\tilde{N}_{j_0,j}^*)$ are the B-splines and their dual functions constructed in Section 4 corresponding to $V_{j_0}$.
Remark 6.3: In order to emphasize the pivotal role of the set $V$ and its complement we note that the proof of Theorem 6.2 implies the following: If $(g_n)$ be a sequence in $L^1_X$ such that

1. $\sup_n \|g_n\|_{L^1_X} < \infty$,
2. $P_m g_n = g_m$ for all $m \leq n$,

and if $\lambda(V^c) = 0$, then, without any condition on the Banach space $X$, $g_n$ converges a.e. to

$$\sum_j \sum_{j_0} T(N_{j_0,j}) N^*_{j_0,j} 1_{U_{j_0}}.$$

Remark 6.4: Based on the results of the present paper, an intrinsic spline characterization of the Radon–Nikodým property in terms of splines was obtained by the second named author in [8]. The result in [8] establishes the full analogy between spline and martingale convergence.

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