The slice-independent gauge-fixed superstring chiral measure in genus 2 derived in the earlier papers of this series for each spin structure is evaluated explicitly in terms of theta-constants. The slice-independence allows an arbitrary choice of superghost insertion points $q_1, q_2$ in the explicit evaluation, and the most effective one turns out to be the split gauge defined by $S_3(q_1, q_2) = 0$. This results in expressions involving bilinear theta-constants $M$. The final formula in terms of only theta-constants follows from new identities between $M$ and theta-constants which may be interesting in their own right. The action of the modular group $Sp(4, \mathbb{Z})$ is worked out explicitly for the contribution of each spin structure to the superstring chiral measure. It is found that there is a unique choice of relative phases which insures the modular invariance of the full chiral superstring measure, and hence a unique way of implementing the GSO projection for even spin structure. The resulting cosmological constant vanishes, not by a Riemann identity, but rather by the genus 2 identity expressing any modular form of weight 8 as the square of a modular form of weight 4. The degeneration limits for the contribution of each spin structure are determined, and the divergences, before the GSO projection, are found to be the ones expected on physical grounds.

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1 Introduction

This paper is the fourth of a series whose goal is to establish unambiguous, explicit formulas for superstring multiloop amplitudes. In [1], a summary of the main results was presented. In [2], a gauge-fixed expression for the superstring partition function in genus 2 was obtained, which is invariant under variations of the gauge slice $\chi_\alpha$ by local reparametrizations and local supersymmetry. Specializing to Dirac-like gauge slices of the form $\chi_\alpha(z) = \delta(z, q_\alpha)$, the gauge-fixed expression was shown directly in [3] to be independent of the points $q_\alpha$. For the superstring partition function, the measure on moduli space is of the form

$$A = \int (\det \text{Im}\Omega)^{-5} d\mu(\Omega) \wedge \overline{d\mu(\Omega)} \quad (1.1)$$

Here the chiral superstring measure $d\mu(\Omega)$ is given by

$$d\mu(\Omega) = \sum_\delta \eta_\delta d\mu[\delta](\Omega) \quad (1.2)$$

with $\eta_\delta$ the relative phases between different spin structures, yet to be determined, and $d\mu[\delta](\Omega)$ the contribution to the superstring measure of each even spin structure $\delta$. Explicit expressions for $d\mu[\delta](\Omega)$ have been obtained in [3] in terms of $\vartheta$-functions, but not yet in terms of $\vartheta$-constants.

The purpose of the present paper is to show that, exploiting the invariance of the gauge-fixed expression from the points $q_\alpha$, the chiral measure $d\mu[\delta](\Omega)$ can actually be rewritten remarkably simply in terms of $\vartheta$-constants. In fact, we show that

$$d\mu[\delta](\Omega) = \frac{\vartheta^4[\delta](0, \Omega) \Xi_6[\delta](\Omega)}{16\pi^6 \Psi_{10}(\Omega)} \prod_{I \leq J} d\Omega_{IJ} \quad (1.3)$$

where $\Psi_{10}$ is the well-known genus 2 modular form of modular weight 10

$$\Psi_{10}(\Omega) \equiv \prod_\epsilon \vartheta^2[\epsilon](0, \Omega) \quad (1.4)$$

The product in $\Psi_{10}$ is over all 10 even spin structure $\epsilon$. The object $\Xi_6[\delta](\Omega)$ is the following spin structure dependent combination of $\vartheta$ constants, and is of modular weight 6

$$\Xi_6[\delta](\Omega) \equiv \sum_{1 \leq i < j \leq 3} \langle \nu_i | \nu_j \rangle \prod_{k=4,5,6} \vartheta^4[\nu_i + \nu_j + \nu_k](0, \Omega) \quad (1.5)$$

In the definition of $\Xi_6[\delta]$, the even spin structure $\delta$ is written as a sum of three distinct odd spin structure $\delta = \nu_1 + \nu_2 + \nu_3$, and $\nu_4, \nu_5, \nu_6$ denote the remaining 3 distinct odd spin structures. Finally, the signature of a pair $\kappa, \lambda$ of spin structures (even or odd) is defined by

$$\langle \kappa | \lambda \rangle \equiv \exp\{4\pi i (\kappa' \lambda'' - \kappa'' \lambda')\} \quad (1.6)$$
We would like to stress that $\Xi_6[\delta](\Omega)$ is not a modular form, since the spin structure $\delta$ transforms under modular transformations. More precisely, it turns out that modular transformations map the spin structure $\delta \rightarrow \tilde{\delta}$, the period matrix $\Omega \rightarrow \tilde{\Omega}$, and the chiral superstring measure $d\mu[\delta]$ as follows,

$$
\tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbb{Z})
$$

$$
d\mu[\tilde{\delta}](\tilde{\Omega}) = \det(C\Omega + D)^{-5}d\mu[\delta](\Omega)
$$

(1.7)

A crucial feature of superstring theory in the RNS formulation [4] is the imposition of the Gliozzi-Scherk-Olive (GSO) projection [5], which is to be carried out independently in the two chiral sectors. This is done by summing over all spin structures the chiral amplitudes for given spin structure with an assignment of relative phases. This phase assignment must be consistent with modular invariance. From the above modular transformation laws of chiral measure, it follows that there is a unique modular invariant choice of relative phases, namely $\eta_\delta = 1$ for each even spin structure $\delta$. Since $d\mu(\Omega) = \sum_\delta d\mu[\delta](\Omega)$ is now by construction a modular form, we can identify by examining its degeneration behavior, using the classification of modular forms of Igusa [6, 7, 8]. We find that the chiral superstring measure $d\mu(\Omega)$ itself vanishes,

$$
\sum_\delta d\mu[\delta](\Omega) = 0
$$

(1.8)

and this implies in turn that the two-loop cosmological constants for both Type II and heterotic strings vanish pointwise on moduli space.

1.1 The Starting Point of the Derivation

The starting point for this paper is the gauge-fixed expression for the superstring chiral measure $d\mu[\delta](\Omega)$ on moduli space obtained in [3] for each spin structure $\delta$,

$$
d\mu[\delta](\Omega) = \prod_{I \leq J} d\Omega_{IJ} \int \prod_\alpha d\zeta^\alpha A[\delta](\Omega, \zeta)
$$

$$
A[\delta](\Omega, \zeta) = \mathcal{Z}\left\{1 + \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 + \mathcal{X}_4 + \mathcal{X}_5 + \mathcal{X}_6\right\}
$$

(1.9)

Here $\mathcal{Z}$ is the chiral matter and superghost correlation function

$$
\mathcal{Z} = \frac{\langle \prod_{a=1}^3 b(p_a) \prod_{a=1}^2 \delta(\beta(q_a)) \rangle}{\det \omega_I \omega_J(p_a)}
$$

(1.10)

where $p_a$ are 3 arbitrary points on the worldsheet $M$, and $\omega_I$ are holomorphic 1-forms with the usual normalization conditions on a canonical basis $(A_I, B_I)_{I=1}^3$ of homology cycles

$$
\oint_{A_I} \omega_J = \delta_{IJ}, \quad \oint_{B_I} \omega_J = \Omega_{IJ}.
$$

(1.11)
The terms $\mathcal{X}_i$ are given by

$$\mathcal{X}_1 + \mathcal{X}_6 = \frac{\zeta_1 \zeta_2}{16\pi^2} \left[ -10 S_5(q_1, q_2) \partial q_1 \partial q_2 \ln E(q_1, q_2) \\
- \partial q_1 G_2(q_1, q_2) \partial \psi^*_1(q_2) + \partial q_2 G_2(q_2, q_1) \partial \psi^*_2(q_1) \\
+ 2G_2(q_1, q_2) \partial \psi^*_1(q_2) f_{3/2}^{(1)}(q_2) - 2G_2(q_2, q_1) \partial \psi^*_2(q_1) f_{3/2}^{(2)}(q_1) \right]$$  \hspace{1cm} (1.12)

$$\mathcal{X}_2 = \frac{\zeta_1 \zeta_2}{16\pi^2} \omega_I(q_1, q_2) S_6(q_1, q_2) \left[ \partial_I \partial_J \ln \frac{\vartheta[\delta](0)^5}{\vartheta[\delta](D)} + \partial_I \partial_J \ln \vartheta(D_b) \right]$$

$$\mathcal{X}_3 = \frac{\zeta_1 \zeta_2}{8\pi^2} S_6(q_1, q_2) \sum_a \varpi_a(q_1, q_2) \left[ B_2(p_a) + B_3/2(p_a) \right]$$

$$\mathcal{X}_4 = \frac{\zeta_1 \zeta_2}{8\pi^2} S_6(q_1, q_2) \sum_a \left[ \partial_{p_a} \partial_{q_1} \ln E(p_a, q_1) \varpi^*_a(q_2) + \partial_{p_a} \partial_{q_2} \ln E(p_a, q_2) \varpi^*_a(q_1) \right]$$

$$\mathcal{X}_5 = \frac{\zeta_1 \zeta_2}{16\pi^2} \sum_a \left[ S_6(p_a, q_1) \partial_{p_a} S_6(p_a, q_2) - S_6(p_a, q_2) \partial_{p_a} S_6(p_a, q_1) \right] \varpi_a(q_1, q_2).$$

Here, the quantity $\partial \psi^*_1(q_2)$ is a tensor, given by a multiplicative formula

$$\partial \psi^*_1(q_2) = \frac{\vartheta[\delta](2q_2 - 2\Delta)}{\vartheta[\delta](q_1 + q_2 - 2\Delta) E(q_1, q_2) \sigma(q_1)^2} \frac{\sigma(q_2)^2}.$$

(1.13)

Finally, $\varpi_a$ are finite-dimensional determinants of holomorphic forms defined by

$$\varpi_a(u, v) = \frac{\det \omega_I \omega_J(p_b \{u, v; a\})}{\det \omega_I \omega_J(p_b)},$$

$$\omega_I \omega_J(p_b \{u, v; a\}) = \begin{cases} \omega_I \omega_J(p_b) & \text{if } b \neq a \\ \frac{1}{2} (\omega_I(u) \omega_J(v) + \omega_J(v) \omega_I(u)) & \text{if } b = a \end{cases}.$$ (1.14)

All other quantities, such as the holomorphic 2-forms $B_2(w)$, $B_{3/2}(w)$, the holomorphic 1-forms $\varpi^*_a(w)$ and the meromorphic 1-forms $f_{3/2}^{(a)}(w)$ in (1.9) and (1.12) were defined in III, and will not be repeated here since their precise form will not be needed here.

### 1.2 Key Steps of the Procedure

All terms in the gauge-fixed expression of (1.9) and (1.12) can be written explicitly in terms of $\vartheta$-functions. However, although the expression (1.9) has been shown to be independent of the points $p_a, q_a$, its apparent dependence on $p_a, q_a$ is a major impediment to understand its modular transformations, and ultimately, to its effective evaluation. Our main task is then to recast (1.3) in terms of $\vartheta$-constants. The key steps in this procedure are the following.

- **The split gauge condition.** The first delicate but very useful choice is that of the points $q_a$. In view of the appearance of the Szegö kernel $S_6(q_1, q_2)$ as an overall factor in
the terms $X_2, X_3,$ and $X_4,$ it is very convenient to require the points $q_1, q_2$ to satisfy the following condition

$$S_3(q_1, q_2) = 0. \tag{1.15}$$

which we shall refer to as the *split gauge condition*. This condition is also natural from another viewpoint. Recall that the genus 2 period matrix $\Omega_{IJ}$ and the super period matrix $\hat{\Omega}_{IJ}$ are related by (see \cite{9, 10})

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int d^2 z \int d^2 w \, \omega_I(z) \chi^+_z S_6(z, w) \chi^+_w \omega_J(w). \tag{1.16}$$

For $\chi^+_z = \sum_{\alpha=1}^{2} \zeta^{\alpha} \delta(z, q_{\alpha}),$ the condition $S_6(q_1, q_2) = 0$ implies that $\Omega_{IJ} = \hat{\Omega}_{IJ}.$ With (1.13), we have $X_2 = X_3 = X_4 = 0,$ and it remains to evaluate only $X_1 + X_6$ and $X_5.$

- **The hyperelliptic representation.** The difficulty with the gauge choice (1.13) is that the two points $q_1$ and $q_2$ are now related in a complicated moduli and spin structure dependent way. Fortunately, in genus 2, it turns out that the relation (1.13) between $q_1$ and $q_2$ can be solved explicitly, by making use of the hyperelliptic representation. In this representation, the surface is described as a double cover of the complex plane, with three branch cuts supported at six branch points $u_i$, $i = 1, \cdots, 6.$ There is a one-to-one map between the six odd spin structures $\nu_i$ and the branch points $u_i$; there is also a one-to-one map between the 10 even spin structures and partitions of the set of six branch points into two disjoint sets of 3 branch points each.

- **The points $p_a, a = 1, 2, 3,$ at branch points.** The correlation function $Z$ is by itself independent of the ghost insertion points $p_a,$ so there is a great flexibility in setting $p_a$ at various special points of the Riemann surface $\Sigma.$ It is very useful to make a further gauge choice and put the three points $p_a$ at the three branch points (or the complementary three branch points) of the partition of branch points associated with spin structure $\delta$ in the hyperelliptic representation of $\Sigma.$ With this choice and the explicit hyperelliptic solution to the split gauge condition $S_6(q_1, q_2) = 0,$ we find

$$X_1 + X_6 = 0. \tag{1.17}$$

This choice also leads to an explicit formula for $Z \delta,$ and hence for the chiral measure $d\mu[\delta],$ where the residual dependence on the points $q_1, q_2$ explicitly cancels out. The resulting formula is expressed entirely in terms of the following *bilinear $\vartheta$-constants*

$$\mathcal{M}_{\nu_1 \nu_2} \equiv \mathcal{M}_{\nu_1 \nu_2}(\Omega) \equiv \partial_1 \vartheta[\nu_1](0, \Omega) \partial_2 \vartheta[\nu_2](0, \Omega) - \partial_2 \vartheta[\nu_1](0, \Omega) \partial_1 \vartheta[\nu_2](0, \Omega) \tag{1.18}$$

and it is

$$d\mu[\delta](\Omega) = \prod_{I \leq J} d\Omega_{IJ} \, \vartheta[\delta](0)^4 \cdot \frac{\langle \nu_1 | \nu_2 \rangle \mathcal{M}_{\nu_1 \nu_2}^4 + \langle \nu_2 | \nu_3 \rangle \mathcal{M}_{\nu_2 \nu_3}^4 + \langle \nu_3 | \nu_1 \rangle \mathcal{M}_{\nu_3 \nu_1}^4}{16\pi^2 \mathcal{M}_{\nu_1 \nu_2}^2 \mathcal{M}_{\nu_2 \nu_3}^2 \mathcal{M}_{\nu_3 \nu_1}^2}. \tag{1.19}$$

*The split gauge condition is more properly a one complex parameter family of gauge conditions, as for example the point $q_1$ may be chosen freely, leaving a two-fold solution for $q_2.$
Relation between bilinear $\vartheta$-constants and $\vartheta$-constants. The bilinear $\vartheta$-constants $M_{\nu,\nu}$ still involve derivatives of $\vartheta$-functions. We can however establish a very powerful identity which expresses $M_{\nu,\nu}$ directly in terms of $\vartheta$-constants

$$M_{\nu,\nu} = \pm \pi^2 \prod_{k \neq i,j} \vartheta[\nu_i + \nu_j + \nu_k](0, \Omega)$$

This identity is proven by noticing that both sides have modular weight 2, that the right hand side vanishes only on the boundary of moduli space and that the asymptotic behaviors of both sides at the boundary of moduli space match. With this identity, we obtain the desired expression (1.3) for $d\mu[\delta]$.

Relative phase factors associated with the GSO projection. Under modular transformations, the expressions $M_{\nu,\nu}$ transform covariantly into each other. By working out the effect of each generator of $SL(4,\mathbb{Z})$, we can show that the unique choice leading to a modular form for the sum over spin structures is when the relative phase factors for $d\mu[\delta]$ are all $\eta_\delta = 1$.

The Cosmological Constant. The expression

$$\sum_{\delta} \vartheta^4[\delta](0, \Omega) \Xi_6[\delta](\Omega)$$

entering into (1.8) is a modular form of weight 8. As we noted before, the ring of modular forms in genus 2 has been identified completely by Igusa [6, 7, 8]. Modular forms of weight 8 must be proportional to the square of the unique form of weight 4, $\Psi_4(\Omega)^2$. Now $\Psi_4(\Omega)$ does not vanish along the degeneration divisor, where the Riemann surface degenerates into two disjoint tori. One shows that (1.21) vanishes, since it tends to 0 along this divisor. More precisely, upon using the Riemann relations for genus 2, (1.21) may be rearranged in the following way,

$$\sum_{\delta} \vartheta^4[\delta](0, \Omega) \Xi_6[\delta](\Omega) = 2 \sum_{\delta} \vartheta^{16}[\delta] - \frac{1}{2} \left( \sum_{\delta} \vartheta^8[\delta] \right)^2$$

an expression which is known to vanish, but not just by the use of the Riemann relations.

This paper is organized as follows. In Section §2, we recall some basic facts about hyperelliptic Riemann surfaces and their spin structures. We also establish some important formulas for the sequel, including Szegö kernel identities, the relations between the Szegö kernel and the Green’s function on 2-forms, and Thomae-type identities for the constants $M_{\nu,\nu}$. Section §3 is devoted to the reduction of the full gauge-fixed expression (1.9) to the expression (1.19) purely in terms of $M_{\nu,\nu}$. The steps include the evaluation of $Z$ by setting the $p_a$’s at different branch points, the proof of the vanishing of $X_1 + X_6$, the explicit evaluation of $ZX_5$, using Thomae-formulae for $M_{\nu,\nu}$, and the concrete realization of the gauge condition $S_\delta(q_1, q_2) = 0$ in the hyperelliptic representation. In Section §4,
we show how our considerations lead heuristically to the key identity relating $M_{\nu_i\nu_j}$ to $\vartheta$-constants. Assuming this relation for the moment, we derive the formula \((1.3)\). Section §5 is devoted to the proof of the required $\vartheta$ identities, namely the identity between $M_{\nu_i\nu_j}$ and $\vartheta$-constants mentioned above, and another identity, the $M$ product formula. For this, we need to examine in detail the degeneration behavior of $M_{\nu_i\nu_j}$, both when the degenerating cycle is or is not a separating cycle. In Section §6, we begin by determining explicitly the effect of modular transformations on the $\Xi_6[\delta](\Omega)$, with particular care about the phases. The net outcome is a proof that the “zero relative phases” is the only possible choice leading to modular invariance for the superstring chiral measure, so that there is one and exactly one way of implementing the GSO projection. The vanishing of the cosmological constant is established by examining the degeneration limits of $d\mu(\Omega)$. In Section §7, the chiral measure for the heterotic string is shown to follow at once, and the heterotic cosmological constant is shown to vanish. In this section, we take the opportunity to show how our methods also give readily the well-known chiral measure for the bosonic string, by direct evaluation and without appealing this time to Igusa’s classification of genus 2 modular forms. In Section §8, we verify the consistency of the chiral measure $d\mu[\delta](\Omega)$ with the degeneration behavior expected on physical grounds. We do find the expected tachyon and massless intermediate state divergences.
2 Genus 2 Riemann Surfaces

We begin by collecting fundamental facts about genus 2 Riemann surfaces, their spin structures, their holomorphic and meromorphic differentials and modular properties, both in the $\vartheta$-function formulation and in the hyperelliptic representation.

2.1 $\vartheta$-Characteristics and Spin Structures

On a Riemann surface $\Sigma$ of genus 2, there are 16 spin structures, of which 6 are odd (usually denoted by the letter $\nu$) and 10 are even (usually denoted by the letter $\delta$). Each spin structure $\kappa$ (even or odd) can be identified with a $\vartheta$-characteristic $\kappa = (\kappa', \kappa'')$, where $\kappa', \kappa'' \in \{0, \frac{1}{2}\}^2$, and we shall represent those here by column matrices. The parity of the spin structure $\kappa$ is the same as the parity of the integer $4\kappa' \cdot \kappa''$. The corresponding $\vartheta$-function is an entire function, defined by

$$\vartheta[\kappa](\zeta, \Omega) \equiv \sum_{n \in \mathbb{Z}^2} \exp\{\pi i(n + \kappa')\Omega(n + \kappa') + 2\pi i(n + \kappa')(\zeta + \kappa'')\}$$

which is even or odd depending on the parity of the spin structure. It is convenient to list here the following useful periodicity relations for $\vartheta[\kappa](\zeta, \Omega)$, in which $M, N \in \mathbb{Z}^2$

$$\vartheta[\kappa](\zeta + M + \Omega N, \Omega) = \vartheta[\kappa](\zeta, \Omega) \exp\{-i\pi N \Omega N - 2\pi i N(\zeta + \kappa'') + 2\pi i \kappa' M\}$$

$$\vartheta[\kappa', N, \kappa'' + M](\zeta, \Omega) = \vartheta[\kappa', \kappa''](\zeta, \Omega) \exp\{2\pi i \kappa' M\}$$

The standard $\vartheta$-function is defined by $\vartheta(\zeta, \Omega) = \vartheta[0](\zeta, \Omega)$. We have the following relation between $\vartheta$-functions with different characteristics

$$\vartheta[\kappa](\zeta, \Omega) = \vartheta(\zeta + \kappa'' + \Omega \kappa', \Omega) \exp\{\pi i \kappa' \Omega \kappa' + 2\pi i \kappa'(\zeta + \kappa'')\}$$

For each odd spin structure $\nu$ we have $\vartheta[\nu](0, \Omega) = 0$. For each even spin structure $\delta$ one defines the particularly important $\vartheta$-constants,$$
\vartheta[\delta] \equiv \vartheta[\delta](0, \Omega).$$

Every genus 2 surface admits a hyperelliptic representation, given by a double cover of the complex plane with three quadratic branch cuts supported by 6 branch points, which we shall denote $u_i, i = 1, \cdots, 6$. The full surface $\Sigma$ is obtained by gluing together two copies of $\mathbb{C}$ along, for example, the cuts from $u_{2j-1}$ to $u_{2j}, 1 \leq j \leq 3$. The surface is then parametrized by

$$s^2 = \prod_{i=1}^{6} (x - u_i)$$

\[\text{It is customary to introduce a local coordinate system } z(x) = (x, s(x)), \text{ which is well-defined also at the branch points. Throughout, the formulas in the hyperelliptic representation will be understood in this way. However, to simplify notation, the local coordinate } z(x) \text{ will not be exhibited explicitly.}\]
In the hyperelliptic representation, there is another convenient way of identifying spin structures. Each spin structure can be viewed then as a partition of the set of branch points \(u_i, i = 1, \cdots, 6\) into two disjoint subsets, in the following way.

\[
\begin{align*}
\nu \text{ odd} & \iff \text{branch point } u_i \\
\delta \text{ even} & \iff \text{partition } A \cup B, \quad A = \{u_{i_1}, u_{i_2}, u_{i_3}\}, \; B = \{u_{i_4}, u_{i_5}, u_{i_6}\}
\end{align*}
\]  

where \((i_1, i_2, i_3, i_4, i_5, i_6)\) is a permutation of \((1, 2, 3, 4, 5, 6)\).

A typical explicit relation between even spin structures as identified with \(\vartheta\)-characteristics and spin structures as identified with partitions \(\{u_i\} = A \cup B\) of the set of branch points in the hyperelliptic representation is a Thomae formula for \(\vartheta\)-constants

\[
\vartheta[\delta] = c \prod_{i<j \in A} (u_i - u_j)^2 \prod_{k<l \in B} (u_k - u_l)^2 \quad c = \det\left( \int_{A_i} x^{J-1} dx \right)
\]  

Here \(c\) is a spin structure \(\delta\) independent quantity \([11, 12]\).

The signature assignment between (even or odd) spin structures \(\kappa\) and \(\lambda\) is defined by

\[
\langle \kappa | \lambda \rangle \equiv \exp\{4\pi i (\kappa' \lambda'' - \kappa'' \lambda')\}
\]  

and has the following properties,

- If \(\nu_1\) and \(\nu_2\) are odd then
  \[
  \langle \nu_1 | \nu_2 \rangle = +1 \iff \nu_1 - \nu_2 \text{ even}
  \]
  \[
  \langle \nu_1 | \nu_2 \rangle = -1 \iff \nu_1 - \nu_2 \text{ odd}
  \]  

- If \(\nu_1, \nu_2\) and \(\nu_3\) are odd and all distinct, then
  \[
  \langle \nu_1 | \nu_2 \rangle \langle \nu_2 | \nu_3 \rangle \langle \nu_3 | \nu_1 \rangle = -1
  \]

2.2 Relations between even and odd spin structures

There exists in genus 2 a simple relation between even and odd spin structures, i.e. between even and odd \(\vartheta\)-characteristics. We shall need this relation extensively and thus discuss it in detail. To see this type of relation as explicitly as possible, it is very convenient to choose a homology basis. In the next subsection, we shall exhibit the behavior of spin structure and \(\vartheta\)-characteristics under modular transformations.

The odd spin structures may be labeled as follows:\(^\dagger\)

\[
\begin{align*}
2\nu_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} & \quad 2\nu_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} & \quad 2\nu_5 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
2\nu_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} & \quad 2\nu_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \quad 2\nu_6 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\end{align*}
\]  

\(^\dagger\)The pairs for which \(\langle \nu_i | \nu_j \rangle = -1\) are 14, 16, 23, 25, 35, 46; all others give \(\langle \nu_i | \nu_j \rangle = +1\).
while the even ones may be labeled by,

\[
2\delta_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad 2\delta_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad 2\delta_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad 2\delta_4 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad 2\delta_5 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad 2\delta_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad 2\delta_7 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad 2\delta_8 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad 2\delta_9 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\]

The first relation between even and odd \( \vartheta \)-characteristics states that the sum of all odd spin structures is a specific double period,

\[
\nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 + \nu_6 = 4\delta_0.
\]  

(2.13)

The second relation makes it clear that every even spin structure \( \delta \) may be viewed as a partition of the set of 6 branch points into two disjoint sets of 3 branch points,

\[
\begin{align*}
\nu_1 + \nu_2 + \nu_3 &= \delta_7 + 2\nu_3 \\
\nu_1 + \nu_2 + \nu_5 &= \delta_3 + 2\nu_5 \\
\nu_1 + \nu_3 + \nu_4 &= \delta_8 + 2\nu_3 \\
\nu_1 + \nu_3 + \nu_6 &= \delta_9 + 2\nu_3 \\
\nu_1 + \nu_4 + \nu_6 &= \delta_1 + 2\delta_0 \\
\nu_1 + \nu_4 + \nu_5 &= \delta_2 + 2\nu_6 \\
\nu_1 + \nu_5 + \nu_6 &= \delta_4 + 2\nu_5
\end{align*}
\]

(2.14)

Thus, each even spin structure \( \delta \) can be written as \( \delta = \nu_{i_1} + \nu_{i_2} + \nu_{i_3} \), where the \( \nu_{i_a}, a = 1, 2, 3 \) are odd and pairwise distinct. The mapping \( \{\nu_{i_1}, \nu_{i_2}, \nu_{i_3}\} \rightarrow \delta \) is 2 to 1, with \( \nu_{i_1} + \nu_{i_2} + \nu_{i_3} \) and its complement \( \nu_1 + \cdots + \nu_6 - (\nu_{i_1} + \nu_{i_2} + \nu_{i_3}) \) corresponding to the same even spin structure, in view of (2.13).

### 2.3 The Action of Modular Transformations

Modular transformations \( M \) form the infinite discrete group \( Sp(4, \mathbb{Z}) \), defined by

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
\]  

(2.15)

where \( A, B, C, D \) are integer valued \( 2 \times 2 \) matrices and the superscript \( ^t \) denotes transposition. The group is generated by the following elements

\[
\begin{align*}
M_i &= \begin{pmatrix} I & B_i \\ 0 & I \end{pmatrix} \\
S &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \\
\Sigma &= \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} \\
\sigma &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
T &= \begin{pmatrix} \tau_+ & 0 \\ 0 & \tau_- \end{pmatrix} \\
\tau_+ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
\tau_- &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}
\end{align*}
\]  

(2.16)
To exhibit the action of the modular group on 1/2 characteristics $\kappa$ (even or odd), it is convenient to assemble the 1/2 characteristics into a single column of 4 entries and the action of the modular group is then given by

$$\left(\begin{array}{c}
\tilde{\kappa}' \\
\tilde{\kappa}''
\end{array}\right) = \left(\begin{array}{cc}
D & -C \\
-B & A
\end{array}\right) \left(\begin{array}{c}
\kappa' \\
\kappa''
\end{array}\right) + \frac{1}{2} \text{diag}(CD^T)$$

Here and below, $\text{diag}(M)$ of a $n \times n$ matrix $M$ is an $1 \times n$ column vector whose entries are the diagonal entries on $M$. With the above generators, this action of the modular group on characteristics reduces to the following expressions, (mod 1), (returning to our previous notation for the characteristics as $2 \times 2$ matrices)

$$M_i (\kappa' | \kappa'') = \left(\begin{array}{c}
\kappa' \\
\kappa'' + B_i \kappa'
\end{array}\right) + \frac{1}{2} \text{diag}(B_i)$$

$$S (\kappa' | \kappa'') = (\kappa'' | \kappa')$$

$$\Sigma (\kappa' | \kappa'') = (\sigma \kappa' | \sigma \kappa'')$$

$$T (\kappa' | \kappa'') = (\tau_+ \kappa' | \tau_+ \kappa'')$$

On the period matrix, the transformation acts by

$$\tilde{\Omega} = (A\tilde{\Omega} + B)(C\tilde{\Omega} + D)^{-1}$$

while on the Jacobi $\vartheta$-functions, the action is given by

$$\vartheta[\tilde{\kappa}](\{(C\tilde{\Omega} + D)^{-1}\}^t \zeta, \tilde{\Omega}) = \epsilon(\kappa, M) \det(C\tilde{\Omega} + D)^{\frac{1}{2}} e^{i\pi \zeta t(C\tilde{\Omega}+D)^{-1}C\zeta} \vartheta[\kappa](\zeta, \Omega)$$

where $\kappa = (\kappa' | \kappa'')$ and $\tilde{\kappa} = (\tilde{\kappa}' | \tilde{\kappa}'')$. The phase factor $\epsilon(\kappa, M)$ depends upon both $\kappa$ and the modular transformation $M$ and obeys $\epsilon(\kappa, M)^8 = 1$. We shall be most interested in the modular transformations of $\vartheta$-constants $\vartheta^4[\delta]$ and thus in even spin structures $\delta$ and the fourth powers of $\epsilon$, which are given by

$$\epsilon^4(\delta, M_i) = \exp\{4\pi i\delta' B_i \delta''\} \quad i = 1, 2$$

$$\epsilon^4(\delta, M_3) = \epsilon^4(\delta, S) = \epsilon^4(\delta, \Sigma) = \epsilon^4(\delta, T) = 1$$

A convenient way of establishing these values is by first analyzing the case of the shifts $M_i$, whose action may be read off from the definition of the $\vartheta$-function,

$$\epsilon(\delta, M_i) = \exp\{-i\pi \delta' B_i \delta'' - i\pi \delta' \text{diag}(B_i)\}$$

and then of the transformations $S$, $\Sigma$ and $T$ by letting the surface undergo a separating degeneration $\Omega_{12} \to 0$, and using the sign assignments of genus 1 $\vartheta$-functions. The non-trivial entries for $\epsilon^4$ are listed in Table 2.
Table 1: Modular transformations of odd spin structures

| $\nu$ | $M_1$ | $M_2$ | $M_3$ | $S$ | $\Sigma$ | $T$  |
|-------|-------|-------|-------|-----|---------|-----|
| $\nu_1$ | $\nu_3$ | $\nu_1$ | $\nu_3$ | $\nu_1$ | $\nu_2$ | $\nu_3$ |
| $\nu_2$ | $\nu_2$ | $\nu_4$ | $\nu_4$ | $\nu_2$ | $\nu_1$ | $\nu_6$ |
| $\nu_3$ | $\nu_1$ | $\nu_3$ | $\nu_1$ | $\nu_5$ | $\nu_4$ | $\nu_1$ |
| $\nu_4$ | $\nu_4$ | $\nu_2$ | $\nu_6$ | $\nu_3$ | $\nu_6$ | $\nu_4$ |
| $\nu_5$ | $\nu_5$ | $\nu_6$ | $\nu_5$ | $\nu_4$ | $\nu_5$ | $\nu_2$ |
| $\nu_6$ | $\nu_6$ | $\nu_6$ | $\nu_5$ | $\nu_4$ | $\nu_6$ | $\nu_2$ |

Table 2: Modular transformations of even spin structures

| $\sum_i \nu_i$ | $\delta$ | $M_1$ | $M_2$ | $M_3$ | $S$ | $\Sigma$ | $T$  | $\epsilon^4(\delta, M_1)$ | $\epsilon^4(\delta, M_2)$ |
|----------------|---------|-------|-------|-------|-----|---------|-----|-----------------|-----------------|
| $\nu_1 + \nu_4 + \nu_6$ | $\delta_1$ | $\delta_3$ | $\delta_2$ | $\delta_1$ | $\delta_1$ | $\delta_1$ | $+$ | $+$ |
| $\nu_1 + \nu_2 + \nu_6$ | $\delta_2$ | $\delta_4$ | $\delta_1$ | $\delta_2$ | $\delta_3$ | $\delta_4$ | $+$ | $+$ |
| $\nu_1 + \nu_3 + \nu_5$ | $\delta_3$ | $\delta_1$ | $\delta_4$ | $\delta_7$ | $\delta_2$ | $\delta_3$ | $+$ | $+$ |
| $\nu_1 + \nu_4 + \nu_5$ | $\delta_4$ | $\delta_3$ | $\delta_4$ | $\delta_9$ | $\delta_4$ | $\delta_2$ | $+$ | $+$ |
| $\nu_1 + \nu_2 + \nu_4$ | $\delta_5$ | $\delta_6$ | $\delta_6$ | $\delta_2$ | $\delta_7$ | $\delta_5$ | $+$ | $-$ |
| $\nu_1 + \nu_5 + \nu_6$ | $\delta_6$ | $\delta_5$ | $\delta_6$ | $\delta_8$ | $\delta_8$ | $\delta_6$ | $+$ | $-$ |
| $\nu_1 + \nu_2 + \nu_3$ | $\delta_7$ | $\delta_7$ | $\delta_8$ | $\delta_3$ | $\delta_5$ | $\delta_9$ | $-$ | $+$ |
| $\nu_1 + \nu_3 + \nu_4$ | $\delta_8$ | $\delta_8$ | $\delta_7$ | $\delta_6$ | $\delta_6$ | $\delta_0$ | $-$ | $+$ |
| $\nu_1 + \nu_3 + \nu_6$ | $\delta_9$ | $\delta_9$ | $\delta_0$ | $\delta_4$ | $\delta_9$ | $\delta_7$ | $-$ | $-$ |
| $\nu_1 + \nu_3 + \nu_5$ | $\delta_0$ | $\delta_9$ | $\delta_0$ | $\delta_0$ | $\delta_0$ | $\delta_8$ | $-$ | $-$ |

2.4 The Riemann relations

At various times, we shall make use of the Riemann relations. They may be expressed as the following quadrilinear sum over all spin structures

$$\sum_\kappa \langle \kappa | \lambda \rangle \vartheta[\kappa](\zeta_1) \vartheta[\kappa](\zeta_2) \vartheta[\kappa](\zeta_3) \vartheta[\kappa](\zeta_4) = 4 \vartheta[\lambda](\zeta'_1) \vartheta[\lambda](\zeta'_2) \vartheta[\lambda](\zeta'_3) \vartheta[\lambda](\zeta'_4)$$

where the signature symbol $\langle \kappa | \lambda \rangle$ was introduced in (2.8). There is one Riemann relation for each spin structure $\lambda$. We have the following relations between the vectors $\zeta$ and $\zeta'$, expressed in terms of a matrix $\Lambda$, which satisfies $\Lambda^2 = I$ and $2\Lambda$ has only integer entries,

$$\begin{pmatrix} \zeta'_1 \\ \zeta'_2 \\ \zeta'_3 \\ \zeta'_4 \end{pmatrix} = \Lambda \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{pmatrix} \quad \Lambda = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

(2.24)
In the special case where $\zeta = \zeta' = 0$, only even spin structures $\kappa = \delta$ contribute to the sum and we have one Riemann identity for each odd spin structure $\lambda = \nu$ on the Riemann constants
\[
\sum_\delta \langle \nu | \delta \rangle \vartheta[\delta]^4(0, \Omega) = 0 \tag{2.25}
\]
For later use, we shall list these 6 equations in the basis of characteristics introduced previously. We make use of the standard abbreviation [8]
\[
(i) = \vartheta[\delta_i]^4, \quad i = 0, 1, \cdots, 9 \tag{2.26}
\]
and find
\[
\begin{align*}
+ \ \nu_1 : \quad & (1) - (2) + (3) - (4) - (5) - (6) + (7) - (8) - (9) + (0) = 0 \\
- \ \nu_2 : \quad & (1) + (2) - (3) - (4) + (5) - (6) - (7) - (8) - (9) + (0) = 0 \\
- \ \nu_3 : \quad & (1) - (2) + (3) - (4) - (5) - (6) - (7) + (8) + (9) - (0) = 0 \\
+ \ \nu_4 : \quad & (1) + (2) - (3) - (4) - (5) + (6) - (7) - (8) + (9) - (0) = 0 \\
- \ \nu_5 : \quad & (1) - (2) - (3) + (4) - (5) + (6) + (7) - (8) - (9) - (0) = 0 \\
+ \ \nu_6 : \quad & (1) - (2) - (3) + (4) + (5) - (6) - (7) + (8) - (9) - (0) = 0
\end{align*}
\]
There exists one linear relation between these 6 equations, obtained by summing all six after multiplication by the sign factor appearing to the left of each $\nu_i$.

### 2.5 Holomorphic and Meromorphic Forms for Genus 2

As in (2.3), we represent the genus 2 Riemann surface $\Sigma$ by a double sheeted cover of the plane, given by the equation
\[
s^2 = (x - u_1)(x - u_2)(x - u_3)(x - u_4)(x - u_5)(x - u_6)
\]
It is convenient to label each branch point $u_{\nu_i}$ by the unique corresponding odd spin structure $\nu_i$, using the Abel map, and the Riemann constants $\Delta_I$ with base point $z_0$ (see for example Appendix A of [2]),
\[
(\nu_i)_I = \int_{z_0}^{u_{\nu_i}} \omega_I - \Delta_I \tag{2.27}
\]
A separation of the branch points into a partition $A = \{u_1, u_2, u_3\}$ and $B = \{u_4, u_5, u_6\}$ represents the choice of an even spin structure $\delta \equiv \nu_1 + \nu_2 + \nu_3$. Which spin structure this is may be inferred from the assignment of an odd spin structure $\nu_i$ to each of the branch points and then using the above relation expressing uniquely any even spin structure $\delta$ in terms of a partition of the six odd spin structures into two groups of 3. It is convenient to define the following functions for the partition associated with a spin structure $\delta$,
\[
\begin{align*}
r_A(x) &= (x - u_1)(x - u_2)(x - u_3) \\
r_B(x) &= (x - u_4)(x - u_5)(x - u_6) \tag{2.28}
\end{align*}
\]
The hyperelliptic representation of \( \Sigma \) may be recast in the form
\[
s^2 = r_A(x)r_B(x).
\]

In terms of these quantities, we have the following representation of holomorphic differentials. For even spin structure, there are no holomorphic 1/2 differentials. We denote by \( \omega_{\nu_i}(x) \), the unique (up to normalization) holomorphic 1-forms with double zero at the branch point \( u_i \) associated with an odd spin structure \( \nu_i \). Given its uniqueness properties, \( \omega_{\nu_i}(x) \) may be readily identified both in its \( \vartheta \)-function and hyperelliptic forms,
\[
\omega_{\nu_i}(z) \equiv \vartheta(z)\vartheta[\nu](0) \equiv \mathcal{N}_{\nu_i}(x-u_i) \frac{dx}{s(x)} \tag{2.29}
\]
where \( \mathcal{N}_{\nu_i} \) is a moduli and \( \nu_i \) dependent normalization factor. Finally, we denote by \( \psi_A(x) \) (respectively \( \psi_B(x) \)) the unique (up to normalization) holomorphic 3/2 forms all of whose three zeros are at the three points of the partition \( A \) (respectively \( B \)). We have the following explicit formulas,
\[
\psi_A(x) \equiv r_A(x)^{1/2} \left( \frac{dx}{s(x)} \right)^{3/2}
\psi_B(x) \equiv r_B(x)^{1/2} \left( \frac{dx}{s(x)} \right)^{3/2} \tag{2.30}
\]
The \( \vartheta \)-function form of these quantities may be deduced from their definition (2.30) and from the expressions for the square roots of the 1-forms \( \omega_{\nu_i} \) in (2.29). While the square root of each \( \omega_{\nu_i} \) is double valued on a surface with even spin structure \( \delta \), the square roots of the products of three \( \omega_{\nu_i} \) with all three \( \nu_i \) spanning either the \( A \) partition or the \( B \) partition associated with \( \delta \) is single valued, and proportional to either \( \psi_A \) or \( \psi_B \). Their precise normalizations involve \( \mathcal{N}_{\nu} \) and will not be needed here.

Finally, the only meromorphic form we shall need explicitly is the Szegő kernel \( S_\delta(z, w) \) for even spin structure \( \delta \). Its \( \vartheta \)-function form is standard,
\[
S_\delta(z, w) = \frac{\vartheta[\delta](z-w)}{\vartheta[\delta](0)E(z, w)} \tag{2.31}
\]
and its hyperelliptic form may be found in \[11\],
\[
S_\delta(z, w) = \frac{1}{2} \left[ \frac{r_A(x)r_B(y)}{s(x)} \right]^{1/2} + \left[ \frac{r_A(y)r_B(x)}{s(y)} \right]^{1/2} \left( \frac{dx}{s(x)} \frac{dy}{s(y)} \right)^{1/2} \tag{2.32}
\]
Recall that the notation subsumes local coordinates \( z = (x, s(x)) \) and \( w = (y, s(y)) \) which distinguish between the two sheets of the surface \( \Sigma \). As expected, the Szegő kernel has singularities only when \( z = w \), i.e., when \( x = y \) and the points \( z \) and \( w \) are on the same sheet. This is because the numerator above can be rewritten as
\[
\left[ \frac{r_A(x)r_B(y)}{s(x)} \right]^{1/2} + \left[ \frac{r_A(y)r_B(x)}{s(y)} \right]^{1/2} = \left( \frac{r_B(y)}{r_B(x)} \right)^{1/2} \left( \frac{r_B(x)}{r_B(y)} \right)^{1/2} \tag{2.33}
\]
so that it vanishes when \( x = y \), and \( z \) and \( w \) are on different sheets.
2.6 The split gauge condition \( S_\delta(q_1, q_2) = 0 \)

An important advantage of the hyperelliptic representation is that the split gauge condition \( S_\delta(q_1, q_2) = 0 \) of (1.13) can be solved for, essentially explicitly. In fact, it is equivalent to the cancellation of the numerator factor,

\[
[r_A(q_1) r_B(q_2)]^\frac{1}{2} + [r_A(q_2) r_B(q_1)]^\frac{1}{2} = 0 \iff \psi_A(q_1) \psi_B(q_2) + \psi_A(q_2) \psi_B(q_1) = 0
\]

\[
\Rightarrow r_A(q_1) r_B(q_2) = r_A(q_2) r_B(q_1).
\]

(2.34)

Given \( q_1 \), the last line is a degree 3 polynomial in \( q_2 \), with 3 roots. However, one root \( q_2 = q_1 \) does not actually correspond to a zero of \( S_\delta \) since it is neutralized by the denominator factor. Thus, two solutions remain, as is expected.

2.7 Ghost insertion points \( p_a \) at branch points

The hyperelliptic representation also allows us to make a special choice for the ghost insertion points \( p_a \). We consider now a fixed even spin structure \( \delta \) on a surface of genus 2. Let \( \nu_1, \nu_2 \) and \( \nu_3 \) be the three odd spin structures such that \( \delta \equiv \nu_1 + \nu_2 + \nu_3 \). In view of our previous discussion on spin structures, each odd spin structure is uniquely associated with one of the branch points, denoted \( u_{\nu_i} \), and the even spin structure \( \delta \) corresponds to a partition \( A \cup B \) of the set of branch points into two disjoint subsets \( A = \{ u_{\nu_i}; i = 1, 2, 3 \} \) and \( B = \{ u_{\nu_i}; i = 4, 5, 6 \} \) of 3 points each. We place the points \( p_i, i = 1, 2, 3 \), at the three branch points \( u_{\nu_i} \) in one of the subsets of the partition, say the subset \( A \). To keep the notation symmetric, we shall denote the points \( u_{\nu_i} \) in the \( B \)-set by \( p_i, i = 4, 5, 6 \), so that

\[
p_i = u_{\nu_i}, \quad i = 1, 2, 3, 4, 5, 6.
\]

(2.35)

We shall continue to use the subscript \( a \) in \( p_a \) to denote only the three \( p \)'s in the \( A \)-set.

This very special choice produces a remarkable simplification in the form of the \( b, c \) ghost Green’s function \( G_2(z, w) = G_2(z, w; p_a) \), whose definition

\[
\nabla_z^{(2)} G_2(z, w; p_a) = 2\pi \delta(z, w)
\]

\[
\nabla_w^{(-1)} G_2(z, w; p_a) = -2\pi \delta(z, w) + 2\pi \sum_a \phi_a^{(2)*}(z) \delta(w, p_a)
\]

(2.36)

involves the points \( p_a \). The holomorphic 2-forms \( \phi_a^{(2)*}(z) \) are normalized by \( \phi_a^{(2)*}(p_b) = \delta_{ab} \).

Viewed as a 2-form in \( z \), \( G_2(z, w; p_a) \) is meromorphic with a simple pole at \( z = w \) and three zeros at \( p_a \). Having chosen the points \( p_a \) at branch points of the \( A \)-partition of the even spin structure \( \delta \), the holomorphic 3/2 form \( \psi_A(z) \) now has its 3 zeros precisely at the points \( p_a \) and has no other zeros. Thus, \( G_2(z, w; p_a)/\psi_A(z) \) is a meromorphic 1/2 form with a single pole at \( z = w \), and must therefore be proportional to the Szegö kernel \( S_\delta(z, w) \). Using these arguments, we readily find,

\[
G_2(z, w; p_a) = S_\delta(z, w) \frac{\psi_A(z)}{\psi_A(w)}
\]

(2.37)
The formula may also be proven directly from the well-known (see Appendix A of [2]) \( \varphi \)-function expressions of both sides.

2.8 The bilinear \( \varphi \)-Constants \( \mathcal{M}_{\nu_i \nu_j} \)

As an even more important application, we derive additional Thomae-type formulas which will play a central role in the sequel. These are formulas for the key bilinear \( \varphi \)-constants \( \mathcal{M}_{\nu_i \nu_j} \) defined as follows

\[
\mathcal{M}_{\nu_i \nu_j} \equiv \partial_1 \varphi[\nu_i](0, \Omega) \partial_2 \varphi[\nu_j](0, \Omega) - \partial_2 \varphi[\nu_i](0, \Omega) \partial_1 \varphi[\nu_j](0, \Omega) \tag{2.38}
\]

We shall often abbreviate \( \partial_1 \varphi[\nu] \equiv \partial_1 \varphi[\nu](0, \Omega) \). We continue to use the correspondence between odd spin structures \( \nu_i \) and branch points \( p_i, i = 1, \ldots, 6 \), introduced in the preceding subsection. We consider the holomorphic 1-form \( \omega_{\nu_i}(z) \) with a double zero at the branch point \( p_i \), given (2.29). Since \( \omega_{\nu_i}(p_i) = 0 \), we have

\[
\omega_I(p_i) \partial_I \varphi[\nu_i] = 0. \tag{2.39}
\]

First, evaluate the following scalar ratio, (for \( i \neq j, k \))

\[
\frac{\omega_{\nu_j}(p_i)}{\omega_{\nu_k}(p_i)} = \frac{\omega_I(p_i) \partial_I \varphi[\nu_j]}{\omega_I(p_i) \partial_I \varphi[\nu_k]} \tag{2.40}
\]

For genus 2, the \( \omega_I(p_i) \) may be eliminated because the formula is homogeneous in them and their ratio is given by the vanishing of \( \omega_I(p_i) \partial_I \varphi[\nu_i] \). The result can be expressed in terms of \( \mathcal{M}_{\nu_i \nu_j} \) as

\[
\frac{\mathcal{N}_{\nu_j}(p_i - p_j)}{\mathcal{N}_{\nu_k}(p_i - p_k)} = \frac{\omega_{\nu_j}(p_i)}{\omega_{\nu_k}(p_i)} = \frac{\mathcal{M}_{\nu_i \nu_j}}{\mathcal{M}_{\nu_i \nu_k}} \tag{2.41}
\]

Taking the cross ratio of four branch points (with \( i, l \neq j, k \), the normalization factors \( \mathcal{N}_{\nu_i} \) cancel out and we get the desired identity

\[
\frac{p_i - p_j}{p_i - p_k} \cdot \frac{p_k - p_l}{p_j - p_l} = \frac{\mathcal{M}_{\nu_i \nu_j} \mathcal{M}_{\nu_k \nu_l}}{\mathcal{M}_{\nu_i \nu_k} \mathcal{M}_{\nu_j \nu_l}} \tag{2.42}
\]

This is clearly a Thomae-type formula, relating \( \varphi \)-constants to rational expressions of branch points. The existence of two Thomae-type formulas, (2.7) and (2.42) suggests that there should be a direct relation between the bilinear \( \varphi \)-constants \( \mathcal{M}_{\nu_i \nu_j} \) and standard \( \varphi[\delta] \) constants. Such a relation indeed exists and will be discussed in detail in §5.
3 The Chiral Measure via Bilinear $\vartheta$-Constants

In this section, we choose the split gauge $S_5(q_1, q_2) = 0$ for the points $q_1$ and $q_2$, and place all three ghost insertion points $p_a$'s at the three branch points of the $A$-partition associated with the even spin structure $\delta$, as in (2.33). With these choices, all $X_i$, except $X_5$ will vanish, and the product of the overall factor $Z$ with $X_5$ may be recast into a simple final expression involving only the bilinear $\vartheta$-constants $M_{\nu, \nu_3}$, and the standard $\vartheta[\delta]^4$ constants.

3.1 The Chiral Partition Function Overall Factor

We begin by evaluating the overall factor $Z$ of (1.10), which is the matter and superghost chiral partition function. Using the expressions for the ghost and superghost correlators established in [13], we obtain

$$Z = \frac{\langle \prod_a b(p_a) \prod_{a} \delta(\beta(q_a)) \rangle}{\det \omega_I \omega_J(p_a)} = \frac{\vartheta[\delta](0) \vartheta(D_b) \prod_{a<b} E(p_a, p_b) \prod_{a} \sigma(p_a)^2}{Z^{15} \vartheta[\delta](D_\beta) E(q_1, q_2) \prod_{a} \sigma(q_a)^2 \det \omega_I \omega_J(p_a)}$$  (3.1)

where the chiral scalar partition function $Z$ is defined by

$$Z^3 = \frac{\vartheta(\sum_I z_I - w_0 - \Delta) \prod_{I<J} E(z_I, z_J) \prod_I \sigma(z_I)}{\sigma(w_0) \prod_I E(z_I, w_0) \det \omega_I(z_J)}$$  (3.2)

for any triplet of distinct points $z_1, z_2, w_0$. Recall that

$$D_b = p_1 + p_2 + p_3 - 3\Delta \quad D_\beta = q_1 + q_2 - 2\Delta.$$  (3.3)

Notice that $Z$ is independent of the three points $p_a$, but does depend upon the points $q_1, q_2$. Upon evaluating this quantity, we may thus choose $p_a$ any way we want, while the $q$'s have to be the same as the ones used throughout. In particular, just in this calculation, it is convenient not to choose the $p_a$ as we did in (2.35).

In the expression (3.1), and in one factor $Z^3$ of its denominator (leaving another factor $Z^{12}$ untouched), we set $z_1 = p_1$, $z_2 = p_2$. Also, we place $p_3$ at a branch point, labeled by odd spin structure $\nu_3$, $p_3 = \Delta + \nu_3$, but leave the points $p_1$ and $p_2$ arbitrary. As a result, this $Z^3$ factor in the denominator will cancel the factor $\vartheta(D_b)$ in the numerator, up to an exponential factors arising from a shift by an integral period in the $\vartheta$-function,

$$\vartheta(p_1 + p_2 + p_3 - 3\Delta) = C \cdot \vartheta(p_1 + p_2 - p_3 - \Delta)$$

$$C = -\exp\{-4\pi i \nu_3'(p_1 + p_2 - 2\Delta)\}$$  (3.4)

so that we have

$$Z = C \cdot \frac{\vartheta[\delta](0) E(p_1, p_2)^2 \sigma(p_1)^2 \sigma(p_2)^2 \det \omega_I(p_1, p_2)}{Z^{12} \vartheta[\delta](D_\beta) E(q_1, q_2) \sigma(q_1)^2 \sigma(q_2)^2 \det \omega_I \omega_J(p_a)}$$  (3.5)
Next, we let \( p_2 \to p_1 \), keeping the point \( p_1 \) still arbitrary, so that the determinants simplify as follows

\[
\frac{\det \omega_f(p_1, p_2)}{\det \omega_f(p_a)} \to -\left( \omega_1(p_3)\omega_2(p_1) - \omega_1(p_1)\omega_2(p_3) \right)^{-2} \tag{3.6}
\]

and we are left with

\[
Z = -C \cdot \frac{\partial[\delta](0)^5 E(p_1, p_3)^4 \sigma(p_1)^4 \sigma(p_3)^4}{Z^{12}\partial[\delta](D_\beta) E(q_1, q_2) \sigma(q_1)^2 \sigma(q_2)^2 \omega(v_1(p_3)\omega_2(p_1) - \omega(v_1(p_1)\omega_2(p_3))^2} \tag{3.7}
\]

Next, we evaluate the remaining factors of \( Z \) in the following way. We let \( z_1 = p_1 \) and \( z_2 = p_3 \), so that

\[
Z^3 = \frac{\partial(p_1 + p_3 - w_0 - \Delta) E(p_1, p_3) \sigma(p_1) \sigma(p_3)}{\sigma(w_0) E(p_1, w_0) E(p_3, w_0) \det \omega_f(p_1, p_3)} \tag{3.8}
\]

and now set \( p_1 = \Delta + \nu_1 \) as we already had \( p_3 = \Delta + \nu_3 \). We obtain two different but equivalent formulas by letting \( w_0 \to p_3 \) or \( w_0 \to p_1 \) respectively

\[
Z^3 = -C_1 \frac{\omega\nu_1(p_3) \sigma(p_1)}{\omega(v_1(p_3)\omega_2(p_1) - \omega(v_1(p_1)\omega_2(p_3))} \tag{3.9}
\]

with

\[
C_i = \exp\{-i\pi \nu_i^* \Omega_\nu_i - 2\pi i \nu_i / \nu_i^* \} \tag{3.10}
\]

In evaluating \( Z \), we use the first formula for one \( Z^0 \) factor, while the second formula for the second \( Z^6 \) factor and we get

\[
Z = -\frac{C}{C_1^2 C_3^2} \cdot \frac{\partial[\delta](0)^5 E(p_1, p_3)^4 \sigma(p_1)^2 \sigma(p_3)^2 \omega_1(p_3)\omega_2(p_1) - \omega_1(p_1)\omega_2(p_3))^2}{\partial[\delta](D_\beta) E(q_1, q_2) \sigma(q_1)^2 \sigma(q_2)^2 \omega\nu_1(p_3) \omega\nu_3(p_1)^2} \tag{3.11}
\]

The ratio

\[
\frac{\omega_1(p_3)\omega_2(p_1) - \omega_1(p_1)\omega_2(p_3)}{\omega\nu_1(p_3) \omega\nu_3(p_1)} \tag{3.12}
\]

may be easily computed because it involves only the ratios \( \omega_2 / \omega_1(p_1) \) and \( \omega_2 / \omega_1(p_3) \), and they are known from the fact that \( \omega\nu_1(p_1) = \omega\nu_3(p_3) = 0 \),

\[
\frac{\omega_2(p_1)}{\omega_1(p_1)} = -\frac{\partial \partial[\nu_1]}{\partial[\nu_1]} \quad \frac{\omega_2(p_3)}{\omega_1(p_3)} = -\frac{\partial \partial[\nu_3]}{\partial[\nu_3]} \tag{3.13}
\]

and we find

\[
\frac{\omega_1(p_3)\omega_2(p_1) - \omega_1(p_1)\omega_2(p_3)}{\omega\nu_1(p_3) \omega\nu_3(p_1)} = \frac{1}{M_{\nu_1\nu_3}} \tag{3.14}
\]

Putting all together, we have

\[
Z = -\frac{C}{C_1^2 C_3^2} \cdot \frac{\partial[\delta](0)^5 E(p_1, p_3)^4 \sigma(p_1)^2 \sigma(p_3)^2}{\partial[\delta](D_\beta) E(q_1, q_2) \sigma(q_1)^2 \sigma(q_2)^2} \frac{1}{M_{\nu_1\nu_3}} \tag{3.15}
\]

We stress that at this moment, no choice for the \( q_\alpha \) has been made as yet. Instead of trying to simplify this factor further, we shall rather start combining it with the \( \mathcal{X} \) terms.
3.2 Vanishing of $X_1 + X_6$ in Split Gauge and $p_a$ at Branch Points

Combining the split gauge condition $S_\delta(q_1, q_2) = 0$ on the points $q_1$ and $q_2$ while placing the ghost insertion points $p_a$ at the branch points belonging to the $A$-partition of the even spin structure $\delta$ produces drastic simplifications. From the expression for $X_1 + X_6$ in [13] and [14], it is clear that the basic ingredients are $G_2(q_1, q_2)$, $\partial_q G_2(q_1, q_2)$ and $\partial_{q_2} G_2(q_2, q_1)$, since the first line already vanishes in split gauge. Actually, with this choice of points $p_a$, the ghost Green’s function $G_2(z, w; p_a)$ is given by (2.37) and thus we readily have in split gauge that

$$G_2(q_1, q_2) = 0$$

(3.16)

As a result, $\partial_q G_2(q_1, q_2)$ transforms as a tensor, which we now evaluate,

$$\partial_q G_2(q_1, q_2) = \partial_q S_\delta(q_1, q_2) \frac{\psi_A(q_1)}{\psi_A(q_2)}$$

(3.17)

To evaluate $\partial_q S_\delta(q_1, q_2)$, subject to the condition $S_\delta(q_1, q_2) = 0$, is usually not so easy, but the calculation is feasible in the hyperelliptic representation.

We begin by proving the following useful formula, valid when $S_\delta(q_1, q_2) = 0$,

$$\partial_q S_\delta(q_1, q_2) = \partial \omega^*_1(q_2) \partial \psi^*_2(q_1)$$

(3.18)

where $\omega^*_a$ and $\psi^*_a$ are the holomorphic 1-forms and 3/2-forms with normalizations at the points $q_\beta$, given by $\omega^*_a(q_\beta) = \psi^*_a(q_\beta) = \delta_{a\beta}$. To show (3.18), one begins by calculating $\omega^*$ and $\psi^*$ in the hyperelliptic representation.

$$\omega^*_1(q) = \frac{q - q_2}{q_1 - q_2} \cdot \frac{dq}{s(q_1)} \cdot \frac{s(q)}{dq}$$

$$\psi^*_2(q) = \frac{r_A(q) \frac{1}{2} r_B(q_2)^{\frac{1}{2}} + r_A(q_2) \frac{1}{2} r_B(q)^{\frac{1}{2}}}{2 s(q_2)} \left( \frac{dq}{s(q)} \right)^{3/2}$$

(3.19)

Using the condition $S_\delta(q_1, q_2) = 0$, the derivatives needed in the formula may be readily evaluated as well

$$\partial \omega^*_1(q_2) = \frac{1}{q_1 - q_2} \left( \frac{dq}{s(q_1)} \right)$$

$$\partial \psi^*_2(q_1) = \frac{1}{2} \partial \ln s(q_1) \left( \frac{r_A(q_1)^{\frac{1}{2}} r_B(q_2)^{\frac{1}{2}}}{s(q_2)} \right) \left( \frac{dq}{s(q_1)} \right)^{3/2} dq_1$$

$$\partial q_1 S_\delta(q_1, q_2) = \frac{1}{2} \partial \ln s(q_1) \left( \frac{r_A(q_1)^{\frac{1}{2}} r_B(q_2)^{\frac{1}{2}}}{q_1 - q_2} \right) \left( \frac{dq}{s(q_1)} \right)^{3/2} \left( \frac{dq}{s(q_2)} \right)^{\frac{1}{2}}$$

(3.20)

The above formula (3.18) now follows immediately.
Combining (3.17) with (3.18), and substituting the result into $\mathcal{X}_1 + \mathcal{X}_6$ of (1.12), the term $\mathcal{X}_1 + \mathcal{X}_6$ is found to reduce to

$$
\mathcal{X}_1 + \mathcal{X}_6 = \frac{\zeta^1 \zeta^2}{16\pi^2} \partial \psi^*_1(q_2) \partial \psi^*_2(q_1) \left[ \partial \omega^*_2(q_1) \frac{\psi_A(q_2)}{\psi_A(q_1)} - \partial \omega^*_1(q_2) \frac{\psi_A(q_1)}{\psi_A(q_2)} \right]
$$

(3.21)

It remains to evaluate the terms within the brace. We shall do so by using the hyperelliptic representation. We begin by considering (for any points $q_1$ and $q_2$, not necessarily in split gauge) the expressions

$$
\partial \omega^*_1(q_2) = \frac{1}{q_1 - q_2} (dq_2)^2 \cdot s(q_1)
$$

and

$$
\partial \omega^*_2(q_1) = \frac{1}{q_2 - q_1} (dq_1)^2 \cdot s(q_2)
$$

(3.22)

Upon taking their ratio, we obtain

$$
\frac{\partial \omega^*_1(q_2)}{\partial \omega^*_2(q_1)} = -\frac{(dq_2)^2 / s(q_2)^2}{(dq_1)^2 / s(q_1)^2}.
$$

(3.23)

Clearly, this object is the ratio of a holomorphic 3-form evaluated at $q_2$ and evaluated at $q_1$. As a function of $q_2$, this 3-form has its 6 simple zeros at all 6 branch points, and by inspection, it may be rewritten as

$$
\frac{\partial \omega^*_1(q_2)}{\partial \omega^*_2(q_1)} = -\frac{\psi_A(q_2) \psi_B(q_2)}{\psi_A(q_1) \psi_B(q_1)}.
$$

(3.24)

Note that this formula was derived without assuming any relation between $q_1$ and $q_2$.

Next, recall the relation $\psi_A(q_1) \psi_B(q_2) + \psi_A(q_2) \psi_B(q_1) = 0$, which was derived in (2.34) and holds whenever $S_6(q_1, q_2) = 0$. We use this relation to rewrite the above ratio as

$$
\frac{\partial \omega^*_1(q_2)}{\partial \omega^*_2(q_1)} = \frac{\psi_A(q_2)^2}{\psi_A(q_1)^2}.
$$

(3.25)

It is trivially seen that this relation makes $\mathcal{X}_1 + \mathcal{X}_6 = 0$ when $S_6(q_1, q_2) = 0$.

### 3.3 First Evaluation of $\mathcal{X}_5$

The overall factor $Z$ did not depend upon the points $p_a$ and so they may be taken to be anything. In particular, in combining $Z$ with $\mathcal{X}_5$, we shall let the points $p_1$ and $p_3$ in the expression for $Z$ depend upon the term labeled by $a$ in the following way: $p_1 \rightarrow p_b$ and $p_3 \rightarrow p_a$ where $b \neq a$. Starting from

$$
\mathcal{X}_5 = \frac{\zeta^1 \zeta^2}{16\pi^2} \sum_a \omega_a(q_1, q_2) \frac{\psi[\delta](q_1 + q_2 - 2p_a) E(q_1, q_2)}{\psi[\delta](0) E(q_1, p_a)^2 E(q_2, p_a)^2}
$$

(3.26)
using the fact that

\[ C' \equiv \frac{\vartheta[\delta](q_1 + q_2 - 2p_a)}{\vartheta[\delta](q_1 + q_2 - 2\Delta)} \]
\[ = \exp\{-4\pi i\nu'_a\Omega_b + 4\pi iv'_a(\delta'' + q_1 + q_2 - 2\Delta) - 4\pi i\delta\nu'_a\} \]  

(3.27)

and the following two relations (established by using the explicit representations of a ratio of \(\sigma\)-functions, see Appendix A of [2] for more details),

\[ \frac{E(q_a, p_a)^2}{E(p_b, p_a)^2} \frac{\sigma(q_a)^2}{\sigma(p_b)^2} = \frac{\omega_{\nu_a}(q_a)}{\omega_{\nu_a}(p_b)} \exp\{4\pi i\nu'_a(q_a - p_b)\} \quad \alpha = 1, 2 \]  

(3.28)

we find

\[ \mathcal{Z}\mathcal{X}_5 = \frac{\zeta_1}{16\pi^2} \frac{\vartheta[\delta](0)^4}{\vartheta[\delta](0)^4} \sum_a \frac{\varpi_a(q_1, q_2)}{\omega_{\nu_a}(q_1)\omega_{\nu_a}(q_2)} \cdot \frac{\sigma(p_a)^2\omega_{\nu_a}(p_a)^2}{\sigma(p_b)^2} \cdot \frac{C}{\mathcal{M}_{\nu_a \nu_b}^2} \]  

(3.29)

where \(C\) collects all the exponential factors and is given by

\[ C = -\frac{CC'}{C^2_b} \exp\{-4\pi i(q_1 + q_2 - 2p_b)\} \]
\[ = \exp\{2\pi i\nu'_b\Omega_b - 2\pi i\nu'_a\Omega'_a + 4\pi i(\nu'_a\delta'' - \nu'_a\delta')\} \]  

(3.30)

with \(C_a\) given in (3.10). Furthermore, we have

\[ \frac{\sigma(p_a)}{\sigma(p_b)} = \frac{\vartheta(r + s - p_a - \Delta)E(p_b, r)E(p_b, s)}{\vartheta(r + s - p_b - \Delta)E(p_a, r)E(p_a, s)} = -\frac{\omega_{\nu_b}(p_a)}{\omega_{\nu_b}(p_b)} \frac{C_b}{C_a} \]  

(3.31)

so that

\[ \frac{C}{\sigma(p_b)^2} = \frac{\omega_{\nu_b}(p_a)^2}{\omega_{\nu_b}(p_b)^2} \]  

(3.32)

where we recall from (2.8) that the signature \(\langle \nu | \delta \rangle\) of two spin structures is defined by

\[ \langle \nu | \delta \rangle \equiv \exp 4\pi i\{\nu'_a\delta'' - \nu'_a\delta'\} \].

Putting all together, we obtain a first formula for \(\mathcal{Z}\mathcal{X}_5\)

\[ \mathcal{Z}\mathcal{X}_5 = \frac{\zeta_1}{16\pi^2} \frac{\vartheta[\delta](0)^4}{\vartheta[\delta](0)^4} \sum_a \frac{\varpi_a(q_1, q_2)}{\omega_{\nu_a}(q_1)\omega_{\nu_a}(q_2)} \cdot \frac{\omega_{\nu_b}(p_a)^2}{\mathcal{M}_{\nu_a \nu_b}^2} \langle \nu_a | \delta \rangle \]  

(3.33)

As a result of (2.11), this expression is in fact independent of the choice of the point \(p_b\) and its associated spin structure \(\nu_b\).

### 3.4 Good Formulas for \(\varpi_a\)

The expression for \(\mathcal{Z}\mathcal{X}_5\) will now be rendered more explicit by deriving better formulas for \(\varpi_a\). We set \(p_a = \Delta + \nu_a\), \(a = 1, 2, 3\) where \(\delta \equiv \nu_1 + \nu_2 + \nu_3\) and use the holomorphic 1-forms with double zeros at \(p_a\), \(\omega_{\nu_a}(z)\), which obey one linear dependence relation

\[ A_1\omega_{\nu_1}(z) + A_2\omega_{\nu_2}(z) + A_3\omega_{\nu_3}(z) = 0 \]  

(3.34)
with $A_a \neq 0$. The coefficients $A_a$ will be identified later. We begin with the denominator of the determinants for $\varpi_a(q_1, q_2)$,

$$D = \begin{vmatrix}
    \omega_{l_1} \omega_{l_1} (p_1) & \omega_{l_2} \omega_{l_3} (p_1) & \omega_{l_1} \omega_{l_2} (p_1) \\
    \omega_{l_1} \omega_{l_1} (p_2) & \omega_{l_2} \omega_{l_3} (p_2) & \omega_{l_1} \omega_{l_2} (p_2) \\
    \omega_{l_1} \omega_{l_1} (p_3) & \omega_{l_2} \omega_{l_3} (p_3) & \omega_{l_1} \omega_{l_2} (p_3)
\end{vmatrix} \quad (3.35)$$

To the first column, add the $A_2/A_1$ times the third column; to the second column, add $A_1/A_2$ times the third column and use the linear dependence relation. Now using the fact that $\omega_{l_a}(p_a) = 0$, we readily find a nicely factorized expression

$$D = -\frac{A_3^2}{A_1 A_2} \omega_{l_1} (p_2) \omega_{l_1} (p_3) \omega_{l_2} (p_1) \omega_{l_2} (p_2) \omega_{l_1} (p_1) \omega_{l_3} (p_2) \quad (3.36)$$

Next, we look at the numerator for $\varpi_1(q_1, q_2)$, which is

$$D \varpi_1(q_1, q_2) = \begin{vmatrix}
    \omega_{l_1} \omega_{l_1} (q_1) & \omega_{l_2} \omega_{l_3} (q_1) & \frac{1}{2} \omega_{l_1} \omega_{l_2} (q_1) + \frac{1}{2} \omega_{l_1} \omega_{l_2} (q_1) \\
    \omega_{l_1} \omega_{l_1} (q_2) & \omega_{l_2} \omega_{l_3} (q_2) & \omega_{l_1} \omega_{l_2} (q_2) \\
    \omega_{l_1} \omega_{l_1} (q_3) & \omega_{l_2} \omega_{l_3} (q_3) & \omega_{l_1} \omega_{l_2} (q_3)
\end{vmatrix}$$

We perform exactly the same linear combinations as we did on the denominator and after some simplifications find the following results

$$\varpi_1(q_1, q_2) = \frac{\omega_{l_2} (q_1) \omega_{l_3} (q_2) + \omega_{l_3} (q_2) \omega_{l_1} (q_1)}{2 \omega_{l_2} (p_1) \omega_{l_3} (p_1)}$$

$$\varpi_2(q_1, q_2) = \frac{\omega_{l_2} (q_1) \omega_{l_1} (q_2) + \omega_{l_3} (q_2) \omega_{l_1} (q_1)}{2 \omega_{l_3} (p_2) \omega_{l_3} (p_2)}$$

$$\varpi_3(q_1, q_2) = \frac{\omega_{l_2} (q_1) \omega_{l_3} (q_2) + \omega_{l_3} (q_2) \omega_{l_1} (q_1)}{2 \omega_{l_1} (p_3) \omega_{l_3} (p_3)} \quad (3.37)$$

For later use, we calculate the coefficients $A_i$ up to an overall factor by letting $z = p_a$ for all three $a = 1, 2, 3$, and we find

$$M_{\nu_2 \nu_3} \omega_{l_1} (z) + M_{\nu_1 \nu_2} \omega_{l_2} (z) + M_{\nu_1 \nu_3} \omega_{l_3} (z) = 0 \quad . \quad (3.38)$$

### 3.5 Elimination of $\varpi_a$

We return to the expression (3.33) obtained for $\mathcal{Z} \mathcal{X}_5$. Recall that this expression was independent of the choice of $p_1$. For definiteness, we concentrate on the $p$-dependence of, for example, the term $a = 1$. It is given by

$$\frac{\omega_{l_2} (p_1)^2}{\omega_{l_2} (p_1) \omega_{l_3} (p_1) M_{\nu_1 \nu_0}^2} \langle \nu_1 | \delta \rangle \quad (3.39)$$

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Without loss of generality, let us take $b = 2$, so that we need the ratio
\[
\frac{\omega_{\nu_2}(p_1)}{\omega_{\nu_3}(p_1)} = \frac{M_{\nu_1\nu_2}}{M_{\nu_1\nu_3}}
\]
(3.40)
of (2.41), and we get
\[
\frac{\omega_{\nu_b}(p_1)^2}{\omega_{\nu_2}(p_1)\omega_{\nu_3}(p_1)} \frac{\langle \nu_1 | \delta \rangle}{M_{\nu_1\nu_b}^2} = \frac{\langle \nu_1 | \delta \rangle}{M_{\nu_1\nu_2} M_{\nu_1\nu_3}}
\]
(3.41)
The expression for the spin structure $\delta = \nu_1 + \nu_2 + \nu_3$ may be used to simplify the exponential further
\[
\langle \nu_1 | \delta \rangle = \exp 4\pi i \{ \nu'_1 \nu''_3 + \nu_1 \nu''_3 - \nu''_1 \nu'_3 \} = \langle \nu_1 | \nu_2 \rangle \langle \nu_1 | \nu_3 \rangle
\]
(3.42)
and we notice that this product of signatures factorizes along with the $M$ factors, so that
\[
\frac{\omega_{\nu_b}(p_1)^2}{\omega_{\nu_2}(p_1)\omega_{\nu_3}(p_1)} \frac{\langle \nu_1 | \delta \rangle}{M_{\nu_1\nu_b}^2} = \frac{\langle \nu_1 | \nu_2 \rangle}{M_{\nu_1\nu_2}} \frac{\langle \nu_1 | \nu_3 \rangle}{M_{\nu_1\nu_3}}
\]
(3.43)
Finally, putting all together, we obtain a second formula for $\mathcal{Z} \mathcal{X}_5$,
\[
\mathcal{Z} \mathcal{X}_5 = \frac{\zeta^4 \zeta_s^2}{32\pi^2} \theta[\delta](0)^4 \left[ + \frac{\omega_{\nu_2}(q_1)\omega_{\nu_3}(q_2)}{\omega_{\nu_2}(q_1)\omega_{\nu_3}(q_2)} \frac{\langle \nu_1 | \nu_2 \rangle}{M_{\nu_1\nu_2}} \frac{\langle \nu_1 | \nu_3 \rangle}{M_{\nu_1\nu_3}} \\
+ \frac{\omega_{\nu_2}(q_1)\omega_{\nu_3}(q_2)}{\omega_{\nu_2}(q_1)\omega_{\nu_3}(q_2)} \frac{\langle \nu_2 | \nu_3 \rangle}{M_{\nu_2\nu_3}} \frac{\langle \nu_1 | \nu_1 \rangle}{M_{\nu_1\nu_1}} \\
+ \frac{\omega_{\nu_2}(q_1)\omega_{\nu_3}(q_2)}{\omega_{\nu_2}(q_1)\omega_{\nu_3}(q_2)} \frac{\langle \nu_3 | \nu_1 \rangle}{M_{\nu_3\nu_1}} \frac{\langle \nu_3 | \nu_2 \rangle}{M_{\nu_3\nu_2}} \right] + (q_1 \leftrightarrow q_2)
\]
(3.44)
Although it may appear longer, this expression is much more explicit than the earlier one in (3.33).

3.6 The $M$ Product Formula

The split gauge condition $S_\delta(q_1, q_2) = 0$ on the points $q_\alpha$ still allows for a one-parameter family of choices. The full amplitude $\mathcal{Z} \sum_{\alpha=1}^6 \mathcal{X}_\alpha$ was shown to be independent of the points $q_\alpha$. Therefore, it remains independent of any residual choice of points $q_1, q_2$ satisfying $S_\delta(q_1, q_2) = 0$. Actually, it is instructive to perform a further consistency check and verify this residual independence on $q_1$ and $q_2$ in split gauge of the term $\mathcal{Z} \mathcal{X}_5$, to which the full amplitude reduces in this gauge. In the process of doing so, we shall come across new $\theta$-function identities which will play a key role in the sequel.

3.6.1 The hyperelliptic representation

Since the variables $q_1$ and $q_2$ are related by $S_\delta(q_1, q_2) = 0$, we must consider their simultaneous variation. It is convenient to do this in the hyperelliptic representation, where
the dependence on all points is expressed via rational functions. The final expression for $\mathcal{Z} \mathcal{X}_5$ in (3.44) is not manifestly a combination of cross ratios and so does not manifestly admit an expression in terms of rational functions on the hyperelliptic curve. This is easily remedied by using the formula (3.40). The first term in (3.44) may be recast in the form

$$
\frac{\omega_{v_2}(q_1)\omega_{v_3}(q_2)}{\omega_{v_1}(q_1)\omega_{v_1}(q_2)} \cdot \frac{\langle \nu_1 | \nu_2 \rangle \langle \nu_1 | \nu_3 \rangle}{M_{v_1 v_2} M_{v_1 v_3}}
$$

Carrying out similar manipulations on the other two terms, and symmetrizing in $q_1$ and $q_2$, we may recast the result in the following form

$$
\mathcal{Z} \mathcal{X}_5 = \frac{\zeta^4}{32 \pi^2 M_{v_1 v_2}^2 M_{v_2 v_3}^2 M_{v_3 v_1}^2} \cdot \mathcal{R}
$$

where $\mathcal{R}$ is given by

$$
\mathcal{R} = + \langle \nu_2 | \nu_3 \rangle \mathcal{M}^4 \frac{p_3 - p_2}{p_3 - p_2} \cdot \frac{p_2 - p_1}{p_2 - p_1} \left\{ \frac{q_1 - p_2}{q_1 - p_1} \cdot \frac{q_2 - p_3}{q_2 - p_1} + \frac{q_2 - p_2}{q_2 - p_1} \cdot \frac{q_1 - p_3}{q_1 - p_1} \right\}
$$

$$
+ \langle \nu_3 | \nu_1 \rangle \mathcal{M}^4 \frac{p_1 - p_2}{p_1 - p_3} \cdot \frac{p_2 - p_3}{p_2 - p_3} \left\{ \frac{q_1 - p_3}{q_1 - p_2} \cdot \frac{q_2 - p_1}{q_2 - p_3} + \frac{q_2 - p_3}{q_2 - p_3} \cdot \frac{q_1 - p_1}{q_1 - p_3} \right\}
$$

$$
+ \langle \nu_1 | \nu_2 \rangle \mathcal{M}^4 \frac{p_2 - p_1}{p_2 - p_1} \cdot \frac{p_1 - p_3}{p_1 - p_2} \left\{ \frac{q_1 - p_1}{q_1 - p_3} \cdot \frac{q_2 - p_2}{q_2 - p_3} + \frac{q_2 - p_1}{q_2 - p_3} \cdot \frac{q_1 - p_2}{q_1 - p_3} \right\}
$$

This formula has all the symmetry properties manifest.

### 3.6.2 Absence of singularities

The combination $\mathcal{R}$, and thus the full amplitude $\mathcal{Z} \mathcal{X}_5$ in this gauge, appears to exhibit poles when $q_\alpha \rightarrow p_1, p_2, p_3$. These poles must of course cancel since the expression should be independent of the $q_\alpha$, in split gauge. We begin by checking that such poles indeed cancel. We may let $q_1 \rightarrow p_1$, without loss of generality. Then, it follows from the split gauge condition that $q_2$ must tend either to $p_2$ or to $p_3$, namely to one of the other two points in the same $A$-partition of the spin structure $\delta$ as $p_1$ belongs to. Choosing $q_2 \rightarrow p_2$, it is convenient to parametrize this joint solution for $q_1$ and $q_2$ in the following way

$$
q_1 = p_1 + F_1 t^2 \\
q_2 = p_2 + F_2 t^2
$$

(3.48)
where \( F_1 \) and \( F_2 \) depend on \( p_i \) and on \( t^2 \). For small \( t^2 \), as we are using in the vicinity of the branch points, \( F_{1,2} \) effectively reduce to their value for \( t = 0 \), which are given by solving the equation (2.34) recalled below,

\[
    r_A(p_1 + F_1 t^2)\frac{4}{3} r_B(p_2 + F_2 t^2)\frac{5}{3} + r_B(p_1 + F_1 t^2)\frac{4}{3} r_A(p_2 + F_2 t^2)\frac{5}{3} = 0. \tag{3.49}
\]

Using the fact that \( r_A(p_1) = r_A(p_2) = 0 \), we get

\[
    F_1 r'_A(p_1) r_B(p_2) = F_2 r'_A(p_2) r_B(p_1) \tag{3.50}
\]

The pole contributions in (3.47) arise only from the first term in the brace of the first line and the first term in the brace of the second line, and is given by

\[
    R \bigg|_{\text{pole}} = -\langle \nu_2 | \nu_3 \rangle \mathcal{M}_{\nu_2 \nu_3}^4 \frac{(p_3 - p_1)(p_2 - p_1)}{(p_3 - p_2) F_1 t^2} - \langle \nu_3 | \nu_1 \rangle \mathcal{M}_{\nu_3 \nu_1}^4 \frac{(p_1 - p_2)(p_3 - p_2)}{(p_3 - p_1) F_2 t^2}. \tag{3.51}
\]

Proving the vanishing of this quantity is equivalent to showing that the ratio of the two terms on the right hand side equals \(-1\). The ratio equals

\[
    -\frac{\langle \nu_2 | \nu_3 \rangle \mathcal{M}_{\nu_2 \nu_3}^4 \frac{(p_3 - p_1)^2}{(p_3 - p_2)^2}}{\langle \nu_3 | \nu_1 \rangle \mathcal{M}_{\nu_3 \nu_1}^4} \cdot \frac{F_2}{F_1}
    = +\langle \nu_1 | \nu_2 \rangle \frac{\mathcal{M}_{\nu_2 \nu_3}^4}{\mathcal{M}_{\nu_3 \nu_1}^4} \frac{(p_3 - p_1)^2}{(p_3 - p_2)^2} \frac{r'_A(p_1) r_B(p_2)}{r'_A(p_2) r_B(p_1)}
    = -\langle \nu_1 | \nu_2 \rangle \frac{\mathcal{M}_{\nu_2 \nu_3}^4}{\mathcal{M}_{\nu_3 \nu_1}^4} \frac{(p_3 - p_1)^3}{(p_3 - p_2)^3} \prod_{i=4,5,6} \frac{p_2 - p_i}{p_1 - p_i}
\]

This expression may be written purely in terms of the \( \mathcal{M} \)-functions by using again the cross-ratio formula

\[
    \frac{p_3 - p_1}{p_3 - p_2} \cdot \frac{p_i - p_1}{p_i - p_2} = \frac{\mathcal{M}_{\nu_3 \nu_1} \mathcal{M}_{\nu_3 \nu_i}}{\mathcal{M}_{\nu_2 \nu_1} \mathcal{M}_{\nu_2 \nu_i}} \tag{3.53}
\]

and we find that the condition for the cancellation of the pole is

\[
    \langle \nu_1 | \nu_2 \rangle \prod_{i=3,4,5,6} \frac{\mathcal{M}_{\nu_2 \nu_i}}{\mathcal{M}_{\nu_1 \nu_i}} = 1 \tag{3.54}
\]

This identity is written completely in terms of \( \vartheta \)-functions. We shall refer to it as the \( \mathcal{M} \) product formula. In view of our discussion, the \( \mathcal{M} \) product formula follows at once from the independence of \( \mathcal{M} \mathcal{X}_5 \) from any choice of \( q_1, q_2 \) satisfying \( S_5(q_1, q_2) = 0 \). We shall presently give a direct proof of it, – up to overall signs – using the classical Thomae formula. Later, we shall obtain an explicit formula for the bilinear \( \vartheta \)-constant \( \mathcal{M}_{\nu_i \nu_j} \) itself, which will imply the full product identity including signs.
3.6.3 Proof of the $\mathcal{M}$ product formula – up to overall signs

We translate the desired identity – up to overall signs – into the hyperelliptic representation, using the normalizations of holomorphic Abelian differentials with double zeros appearing in the Thomae formula. The starting point is formula (2.41), adapted here to the spin structures $\nu_1$, $\nu_2$ and $\nu_3$,

$$\frac{\omega_{\nu_2}(p_i)}{\omega_{\nu_1}(p_i)} = \frac{\mathcal{M}_{\nu_2\nu_i}}{\mathcal{M}_{\nu_1\nu_i}} = \frac{N_{\nu_2}(p_i - p_2)}{N_{\nu_1}(p_i - p_1)}$$

with the normalization factors related by (see also [12])

$$\frac{N_{\nu_2}^4}{N_{\nu_1}^4} = \pm \prod_{i \neq 1,2} \frac{p_1 - p_i}{p_2 - p_i}$$

(3.55)

Taking the product over $i = 3, 4, 5, 6$, we have

$$\prod_{i = 3,4,5,6} \frac{\mathcal{M}_{\nu_2\nu_i}}{\mathcal{M}_{\nu_1\nu_i}} = \pm \frac{N_{\nu_2}^4}{N_{\nu_1}^4} \prod_{i = 3,4,5,6} \frac{p_2 - p_i}{p_1 - p_i} = \pm 1.$$  

(3.56)

This proves the identity up to a ± sign.

3.7 The Chiral Measure in terms of $\mathcal{M}_{\nu_1\nu_2}$

The expression (3.44) which we have obtained so far for $Z\mathcal{X}_5$ mixes both $\vartheta$-functions (as encoded in $\mathcal{M}_{\nu_1\nu_2}$) and the hyperelliptic representation (as encoded in the branch points). We proceed now to simplify it further. The strategy is to eliminate directly any reference to the points $q_\alpha$, using the relation (3.54). Multiplying numerator and denominator by $\mathcal{M}_{\nu_1\nu_3}^4$, and using the cross ratio formula in terms of $\mathcal{M}$’s, we deduce that

$$\frac{\mathcal{M}_{\nu_1\nu_1}^4}{\mathcal{M}_{\nu_2\nu_3}^4} = \langle \nu_1 | \nu_2 \rangle \frac{(p_3 - p_1)^3}{(p_3 - p_2)^3} \prod_{i = 4,5,6} \frac{(p_i - p_1)^3}{(p_1 - p_i)^3}$$

$$\frac{\mathcal{M}_{\nu_1\nu_2}^4}{\mathcal{M}_{\nu_2\nu_3}^4} = \langle \nu_1 | \nu_3 \rangle \frac{(p_1 - p_2)^3}{(p_3 - p_2)^3} \prod_{i = 4,5,6} \frac{(p_i - p_1)^3}{(p_1 - p_i)^3}.$$  

(3.57)

The relation (2.34) between $q_1$ and $q_2$ may be re-expressed in terms of cross ratios of points $p_i$, $i = 1, \cdots, 6$ and $q_1$ and $q_2$,

$$\frac{(q_1 - p_1)(q_1 - p_2)(q_1 - p_3)(q_2 - p_4)(q_2 - p_5)(q_2 - p_6)}{(q_2 - p_1)(q_2 - p_2)(q_2 - p_3)(q_1 - p_4)(q_1 - p_5)(q_1 - p_6)} = 1$$

(3.58)

so that the expression for $\mathcal{R}$, the ratios of various $\mathcal{M}^4$ and the relation between $q_1$ and $q_2$ may all be expressed in terms of cross ratios only and are thus Möbius invariant.
Using Möbius invariance, we may set \( p_1 = \infty, p_2 = 0 \) and \( p_3 = 1 \). The relation between \( q_1 \) and \( q_2 \) simplifies and may be expressed in terms of the symmetric polynomials

\[
B_1 = p_4 + p_5 + p_6 \\
B_2 = p_4 p_5 + p_5 p_6 + p_6 p_4 \\
B_3 = p_4 p_5 p_6
\]

via the following equation

\[
q_1^2 q_2^2 - q_1 q_2 (q_1 + q_2) + (B_1 - B_2) q_1 q_2 + B_3 (q_1 + q_2) - B_3 = 0.
\]

Similarly, the ratios of \( \mathcal{M}'s \) may be expressed in these terms

\[
\frac{\mathcal{M}^4_{\nu_1 \nu_2 \nu_3}}{\mathcal{M}^4_{\nu_2 \nu_3}} = \langle \nu_1 | \nu_2 \rangle B_3 \\
\frac{\mathcal{M}^4_{\nu_2 \nu_3}}{\mathcal{M}^4_{\nu_2 \nu_3}} = \langle \nu_1 | \nu_3 \rangle (1 - B_1 + B_2 - B_3).
\]

With the help of these expressions for the \( B_i \), the relation between \( q_1 \) and \( q_2 \) may be expressed solely in terms of the \( \mathcal{M}'s \),

\[
\langle \nu_2 | \nu_3 \rangle \mathcal{M}^4_{\nu_2 \nu_3} + \frac{\langle \nu_1 | \nu_2 \rangle}{(1 - q_1)(1 - q_2)} \mathcal{M}^4_{\nu_1 \nu_2} + \frac{\langle \nu_3 | \nu_1 \rangle}{q_1 q_2} \mathcal{M}^4_{\nu_3 \nu_1} = 0.
\]

Finally, the expression for \( \mathcal{R} \) also simplifies considerably and we have

\[
\mathcal{R} = + \langle \nu_2 | \nu_3 \rangle \mathcal{M}^4_{\nu_2 \nu_3} \left\{ q_1 (1 - q_2) + q_2 (1 - q_1) \right\} \\
+ \langle \nu_3 | \nu_1 \rangle \mathcal{M}^4_{\nu_3 \nu_1} \left\{ \frac{q_1 - 1}{q_1 q_2} + \frac{q_1 - 1}{q_1 q_2} \right\} \\
+ \langle \nu_1 | \nu_2 \rangle \mathcal{M}^4_{\nu_1 \nu_2} \left\{ - \frac{q_2}{(1 - q_1)(1 - q_2)} - \frac{q_1}{(1 - q_1)(1 - q_2)} \right\}
\]

To see how the cancellation of \( g \)-dependence comes about, multiply (3.62) by a factor \((q_1 + q_2 - 2 - 2q_1 q_2)\) and regroup terms as in the expression for \( \mathcal{R} \). We find

\[
0 = + \langle \nu_2 | \nu_3 \rangle \mathcal{M}^4_{\nu_2 \nu_3} \left\{ -2 + q_1 (1 - q_2) + q_2 (1 - q_1) \right\} \\
+ \langle \nu_3 | \nu_1 \rangle \mathcal{M}^4_{\nu_3 \nu_1} \left\{ -2 + \frac{q_1 - 1}{q_1 q_2} + \frac{q_1 - 1}{q_1 q_2} \right\} \\
+ \langle \nu_1 | \nu_2 \rangle \mathcal{M}^4_{\nu_1 \nu_2} \left\{ -2 - \frac{q_2}{(1 - q_1)(1 - q_2)} - \frac{q_1}{(1 - q_1)(1 - q_2)} \right\}
\]

This implies

\[
\mathcal{R} = 2 \langle \nu_1 | \nu_2 \rangle \mathcal{M}^4_{\nu_1 \nu_2} + 2 \langle \nu_2 | \nu_3 \rangle \mathcal{M}^4_{\nu_2 \nu_3} + 2 \langle \nu_3 | \nu_1 \rangle \mathcal{M}^4_{\nu_3 \nu_1}
\]

so that our final answer is

\[
\mathcal{Z}_5 = \frac{\zeta^4 \zeta^2}{16 \pi^2} \cdot \vartheta[\delta](0)^4 \cdot \frac{\langle \nu_1 | \nu_2 \rangle \mathcal{M}^4_{\nu_1 \nu_2} + \langle \nu_2 | \nu_3 \rangle \mathcal{M}^4_{\nu_2 \nu_3} + \langle \nu_3 | \nu_1 \rangle \mathcal{M}^4_{\nu_3 \nu_1}}{\mathcal{M}^2_{\nu_1 \nu_2} \mathcal{M}^2_{\nu_2 \nu_3} \mathcal{M}^2_{\nu_3 \nu_1}}
\]

This expression is entirely in terms of \( \vartheta \)-functions alone. Given that \( \mathcal{M}_{\nu_i \nu_j} \) and \( \vartheta[\delta](0)^4 \) both have modular weight 2, the whole combination has modular weight -2, as expected.
4 The Chiral Measure in terms of $\vartheta$-Constants

In order to identify the measure factor derived above as a modular form we begin by searching for formulas expressing $M_{\nu_i \nu_j}$ in terms of $\vartheta$-constants. Modular forms of any weight are then related to $\vartheta$-constants by standard expressions.

4.1 A Key $\vartheta$-Constants and Bilinear $\vartheta$-Constants Identity

It turns out that there is a remarkable identity giving the bilinear $\vartheta$-constant $M_{\nu_i \nu_j}$ in terms of $\vartheta$-constants. Here we shall show how the existence of such an identity can be conjectured from all the other identities which we have obtained so far. Our derivation here is only up to signs, but the sign can be determined later by degeneration arguments.

We continue to use the standard notation of the previous section where the branch points $p_i$ are associated with odd characteristics $\nu_i$. Consider the Thomae-type formula (2.42) for the bilinear $\vartheta$-constant

$$M_{\nu_1 \nu_2} M_{\nu_3 \nu_4} = \frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)}$$

and compare this with the standard form for the Thomae formula given earlier in (2.7) in terms of $\vartheta$-constants $\vartheta[\delta] \equiv \vartheta[\delta](0, \Omega)$,

$$\vartheta[\delta]^8 = c \prod_{a < b} (p_{ia} - p_{ib})^2 (p_{ja} - p_{jb})^2$$

where $c$ depends on moduli but is independent of the spin structure. The even spin structure $\delta$ may be identified with the partition of the branch points into two groups

$$\delta \sim \{p_{i_1}, p_{i_2}, p_{i_3}\} \cup \{p_{j_1}, p_{j_2}, p_{j_3}\}.$$ (4.2)

Taking the ratio of two such expressions for different $\delta$ allows one to cancel out the spin structure independent factor $c$. For example,

$$\delta_1 \sim \{p_1, p_2, p_3\} \cup \{p_4, p_5, p_6\}$$
$$\delta_2 \sim \{p_1, p_2, p_4\} \cup \{p_3, p_5, p_6\}$$ (4.3)

yields a fully determined expression of cross-ratios, which may be recast in terms of $M_{\nu_i \nu_j}$'s,

$$\frac{\vartheta[\nu_1 + \nu_2 + \nu_3]^8}{\vartheta[\nu_1 + \nu_2 + \nu_4]^8} = \frac{(p_1 - p_3)^2(p_2 - p_3)^2(p_4 - p_5)^2(p_4 - p_6)^2}{(p_1 - p_4)^2(p_2 - p_4)^2(p_3 - p_5)^2(p_3 - p_6)^2}$$

$$= \frac{M^2_{\nu_1 \nu_2} M^2_{\nu_3 \nu_4} M^2_{\nu_2 \nu_3} M^2_{\nu_4 \nu_5}}{M^2_{\nu_1 \nu_2} M^2_{\nu_3 \nu_6} M^2_{\nu_2 \nu_4} M^2_{\nu_4 \nu_5}}$$ (4.4)
Interchanging the roles of \( p_2 \) and \( p_5 \) and taking the ratio of both, we find

\[
\frac{\vartheta[\nu_1 + \nu_2 + \nu_3]^8 \vartheta[\nu_1 + \nu_4 + \nu_5]^8}{\vartheta[\nu_1 + \nu_2 + \nu_4]^8 \vartheta[\nu_1 + \nu_3 + \nu_5]^8} = \frac{\mathcal{M}^4_{\nu_1 \nu_2 \nu_3} \mathcal{M}^4_{\nu_1 \nu_4 \nu_5}}{\mathcal{M}^4_{\nu_2 \nu_3 \nu_4} \mathcal{M}^4_{\nu_3 \nu_4 \nu_5}} \tag{4.5}
\]

Making use of the \( \mathcal{M}_{\nu_i \nu_j} \) product formula, \[(3.54)\] we get

\[
\frac{\mathcal{M}^4_{\nu_1 \nu_2}}{\mathcal{M}^4_{\nu_2 \nu_3}} = \prod_{i=4,5,6} \frac{\mathcal{M}^4_{\nu_1 \nu_2 \nu_i}}{\mathcal{M}^4_{\nu_2 \nu_3 \nu_i}}
\]

\[
= \frac{\vartheta[\nu_1 + \nu_2 + \nu_4]^16 \vartheta[\nu_1 + \nu_2 + \nu_5]^16 \vartheta[\nu_1 + \nu_2 + \nu_6]^16 \vartheta[\nu_1 + \nu_3 + \nu_4]^16 \vartheta[\nu_1 + \nu_3 + \nu_5]^16 \vartheta[\nu_1 + \nu_3 + \nu_6]^16}{\vartheta[\nu_2 + \nu_3 + \nu_4]^16 \vartheta[\nu_2 + \nu_3 + \nu_5]^16 \vartheta[\nu_2 + \nu_3 + \nu_6]^16}. \tag{4.6}
\]

To arrange this result in a more symmetrical form, we multiply numerator and denominator by the same factor \( \vartheta[\nu_1 + \nu_2 + \nu_3]^16 \), so that

\[
\frac{\mathcal{M}^4_{\nu_1 \nu_2}}{\mathcal{M}^4_{\nu_2 \nu_3}} = \frac{\vartheta[\nu_1 + \nu_2 + \nu_4]^16 \vartheta[\nu_1 + \nu_2 + \nu_5]^16 \vartheta[\nu_1 + \nu_2 + \nu_6]^16 \vartheta[\nu_1 + \nu_3 + \nu_4]^16 \vartheta[\nu_1 + \nu_3 + \nu_5]^16 \vartheta[\nu_1 + \nu_3 + \nu_6]^16}{\vartheta[\nu_2 + \nu_3 + \nu_4]^16 \vartheta[\nu_2 + \nu_3 + \nu_5]^16 \vartheta[\nu_2 + \nu_3 + \nu_6]^16}. \tag{4.7}
\]

This equation is solved by the following simple guess for the identity between bilinear \( \vartheta \)-constants and standard \( \vartheta \)-constants that we were looking for, (up to a 16-th root of unity)

\[
\mathcal{M}_{\nu_1 \nu_2} \sim \prod_{i=3,4,5,6} \vartheta[\nu_1 + \nu_2 + \nu_i] \tag{4.8}
\]

Since both sides transform covariantly under the modular group and have modular weight 2, the factor of proportionality must be a modular function. In subsection §5.2, we shall prove that the factor is constant and equal to \( \pm \pi^2 \). To prove this, we shall use the fact that the rhs vanishes only at the boundary of moduli space, and that the behavior of both sides at the boundary of moduli space coincides, so that the factor of proportionality must be a modular function holomorphic on (compactified) moduli space and thus constant. Before we give this lengthy proof in section §5, we show in the next subsection that this formula gives us the final result for the chiral superstring measure on moduli.

### 4.2 The Chiral Measure in terms of \( \vartheta \)-Constants

We now make use of the crucial formula \[(4.8)\] to recast the expression for the chiral superstring measure on moduli, derived in \[(3.66)\], solely in terms of standard \( \vartheta \)-constants. First, we note that each \( \mathcal{M}^2_{\nu_i \nu_j} \) has an overall common factor of \( \vartheta[\delta]^2 = \vartheta[\nu_1 + \nu_2 + \nu_3]^2 \),

\[
\mathcal{M}^2_{\nu_1 \nu_2} = \pi^4 \vartheta[\delta]^2 \prod_{k=3,4,5} \vartheta[\nu_1 + \nu_2 + \nu_k]^2. \tag{4.9}
\]

It follows that

\[
\frac{\langle \vartheta \rangle_{\nu_1 \nu_2} \mathcal{M}^4_{\nu_1 \nu_2} + \langle \vartheta \rangle_{\nu_2 \nu_3} \mathcal{M}^4_{\nu_2 \nu_3} + \langle \vartheta \rangle_{\nu_3 \nu_1} \mathcal{M}^4_{\nu_3 \nu_1}}{\mathcal{M}^2_{\nu_1 \nu_2} \mathcal{M}^2_{\nu_2 \nu_3} \mathcal{M}^2_{\nu_3 \nu_1}} = \frac{1}{\pi^4 \vartheta_{10}} \sum_{1 \leq i < j \leq 3} \langle \vartheta \rangle_{\nu_i \nu_j} \prod_{k=4,5,6} \vartheta[\nu_i + \nu_j + \nu_k]^4 \tag{4.10}
\]
Here, Ψ_{10} is the weight 10 modular form for genus 2,

\[ \Psi_{10}(\Omega) \equiv \prod_{\delta} \vartheta[\delta]^2(0, \Omega) \]

and the product is taken over all even spin structures δ.

### 4.3 Mirror property

The quantity

\[ \sum_{1 \leq i < j \leq 3} \langle \nu_i | \nu_j \rangle \prod_{k=4,5,6} \vartheta[\nu_i + \nu_j + \nu_k]^4 \]  \tag{4.11} \]

appears to depend not only upon the spin structure δ = \{\nu_1, \nu_2, \nu_3\} ∪ \{\nu_4, \nu_5, \nu_6\} but on the actual triplet chosen to represent δ. However, the odd spin structures \nu_1, \nu_2, \nu_3 resulted from the choice of the points \( p_a \), and the complete independence of the points \( p_a \) established in [3] guarantees that (4.11) is independent of the actual triplet chosen. Thus, (4.11) must be invariant under the mirror transformation \{\nu_1, \nu_2, \nu_3\} ↔ \{\nu_4, \nu_5, \nu_6\}.

Here, we shall provide a direct proof of this mirror property. Given the transitive action of modular transformations on spin structures, it suffices to show its validity for any single spin structure. Consider the case \nu_1, \nu_2, \nu_3 with the basis of (2.14), i.e. δ_7, so then the following quantity should vanish

\[ Q = \sum_{1 \leq i < j \leq 3} \langle \nu_i | \nu_j \rangle \prod_{k=4,5,6} \vartheta[\nu_i + \nu_j + \nu_k]^4 - \sum_{4 \leq i < j \leq 6} \langle \nu_i | \nu_j \rangle \prod_{k=1,2,3} \vartheta[\nu_i + \nu_j + \nu_k]^4. \] \tag{4.12} \]

Using again the standard abbreviation \( (i) = \vartheta[\delta_i]^4 \), \( Q \) takes the form,

\[ Q = (8)(9)(0) - (1)(4)(6) + (2)(3)(5) - (2)(4)(9) - (5)(6)(8) + (1)(3)(0) \] \tag{4.13} \]

To show that this quantity vanishes using the Riemann relations, given at the end of subsection §2.4, we produce the following linear superpositions of the Riemann relations first: \nu_1 ± \nu_2, \nu_3 ± \nu_4 and \nu_5 ± \nu_6. Retaining the equation \nu_1 - \nu_2 as defining \( (7) = \vartheta[\delta_7]^4 \) and eliminating \( (7) \) from all other equations and omitting the linearly dependent equation, we are left with 4 Riemann relations that do not involve \( (7) \). We may cast these in the form where they express \( (1), (2), (5), (0) \) in terms of \( (3), (4), (6), (8), (9) \),

\[
\begin{align*}
(1) & = +(3) + (8) + (9) \\
(2) & = +(3) - (6) + (8) \\
(5) & = +(3) - (4) + (9) \\
(0) & = -(3) + (4) + (6)
\end{align*}
\tag{4.14}
\]

The expression for \( Q \) now becomes

\[
Q = (8)(9)[-(3) + (4) + (6)] - [(3) + (8) + (9)](4)(6) + [(3) - (6) + (8)](3)(4)(9) - [(3) - (4) + (9)](6)(8) + [(3) + (8) + (9)](3)(-3) + (4) + (6)]
\tag{4.15}
\]
This sum may be seen to cancel term by term and $Q = 0$.

### 4.4 The final formula

In summary, we have then established the main formulas of the present paper,

$$\mathcal{A}[\delta] = \mathcal{Z} + \frac{\zeta_1 \zeta_2}{16\pi^6} \frac{\vartheta^4[\delta](0, \Omega) \Xi_6[\delta](\Omega)}{\Psi_{10}(\Omega)}$$  \hspace{1cm} (4.16)

$$\Xi_6[\delta](\Omega) \equiv \sum_{1 \leq i < j \leq 3} \langle \nu_i | \nu_j \rangle \prod_{k=4,5,6} \vartheta[\nu_i + \nu_j + \nu_k]^4(0, \Omega)$$  \hspace{1cm} (4.17)

These formulas imply the ones for the chiral measure $d\mu[\delta](\Omega)$ stated in the Introduction. Due to the mirror property established in the preceding subsection, $\Xi_6[\delta](\Omega)$ depends only upon $\delta$. It has modular weight 6, but it is not a modular form. Its precise modular transformation properties will be derived in section §6. The above formula thus gives the contribution of each even spin structure $\delta = \nu_1 + \nu_2 + \nu_3$ to the chiral superstring measure entirely in terms of $\vartheta$-constants. Determining the modular covariant assignments of the GSO projection phase factors will be carried out in section §6.
5 Degenerations – Proofs of $\vartheta$-constant Relations

In this section, we shall derive three key results. The first result, in subsection § 5.1 below, is to obtain degeneration limits of $\vartheta$-constants (and their derivatives) in the separating and non-separating cases. The second result, in subsection § 5.2 below, is to prove the bilinear $\vartheta$-constant relation between the bilinear $\vartheta$ constants $M_{\nu_i \nu_j}$ and ordinary $\vartheta$ constants,

$$M_{\nu_i \nu_j} \equiv \partial_1 \vartheta[\nu_i] \partial_2 \vartheta[\nu_j] - \partial_1 \vartheta[\nu_j] \partial_2 \vartheta[\nu_i] = \pm \pi^2 \prod_{k \neq i,j} \vartheta[\nu_i + \nu_j + \nu_k]. \quad (5.1)$$

The $\pm$ sign on the rhs is not intrinsic, since it will change under interchange of $i \leftrightarrow j$ as well as under the addition of certain complete periods to the $\nu_k$. However, in a given basis of $\nu_k$, the sign is uniquely fixed and will be determined below. Finally, the third result, in subsection § 5.3 below, is to prove the $M$ product Formula,

$$\langle \nu_i | \nu_j \rangle \prod_{k \neq i,j} \frac{M_{\nu_i \nu_k}}{M_{\nu_j \nu_k}} = 1. \quad (5.2)$$

Relation (5.1) is proven using the fact that both sides transform covariantly under the modular group, so that the factor of proportionality between the left and right sides, à priori, must be a modular function, independent of the spin structures. Since genus 2 $\vartheta$-constants (for even spin structures) can vanish only at the boundary of moduli space, the ratio of the lhs by the rhs must be a modular function that is holomorphic on the inside of moduli space. Using the degeneration limits of subsection § 5.1, we shall show that this modular function tends to 1 at both separating and non-separating boundary components of moduli space, and must therefore be a constant. The asymptotics then uniquely determines this constant. Relation (5.2) is an immediate consequence of relation (5.1), with the precise sign assignments.

5.1 Degenerations

We begin by fixing a canonical homology basis of 1-cycles $A_I, B_I$, such that $\#(A_I, B_J) = \delta_{IJ}$ for $I, J = 1, 2$. In this homology basis, we parametrize the period matrix $\Omega$ by the complex variables $\tau_1, \tau_2$ and $\tau$,

$$\Omega = \begin{pmatrix} \tau_1 & \tau \\ \tau & \tau_2 \end{pmatrix} \quad (5.3)$$

and decompose the (even or odd) spin structures $\kappa$ accordingly,

$$\kappa = \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} \quad \begin{cases} \kappa_1 = (\kappa'_1 | \kappa''_1) & \kappa'_1, \kappa''_1 = 0, 1/2 \\ \kappa_2 = (\kappa'_2 | \kappa''_2) & \kappa'_2, \kappa''_2 = 0, 1/2 \end{cases} \quad (5.4)$$

We may picture the genus 2 surface as two tori joined by a cylinder. Torus 1 has homology basis $A_1, B_1$, spin structure $\kappa_1$ and modulus $\tau_1$, while torus 2 has homology basis $A_2, B_2,$
spin structure $\kappa_2$ and modulus $\tau_2$. The parameter $\tau$ characterizes the (trivial from the genus 2 homological point of view) cylinder that joins both tori.

We shall need to work out degenerations of $\vartheta$-functions and it will turn out to be convenient to have available the $\vartheta$-function in the above variables, and $z = (z_1, z_2)$

$$\vartheta[\kappa](z, \Omega) = \sum_{m,n \in \mathbb{Z}} \exp\left\{ i\pi(m + \kappa'_1)^2 \tau_1 + 2\pi i(m + \kappa'_1)(z_1 + \kappa''_1) \\
+ i\pi(n + \kappa'_2)^2 \tau_2 + 2\pi i(n + \kappa'_2)(z_2 + \kappa''_2) \\
+ 2\pi i\tau(m + \kappa'_1)(n + \kappa'_2) \right\}$$

There are two possible degenerations (up to the action of the modular group): the separating degeneration, $\tau \to 0$, keeping $\tau_1$ and $\tau_2$ fixed; and the non-separating degeneration, $\tau_2 \to +i\infty$, keeping $\tau_1$ and $\tau$ fixed. We discuss each case in turn below.

The resulting asymptotics may be expressed in terms of genus 1 $\vartheta$-functions, which will be denoted by $\vartheta_1$. We begin by recalling the definition, mainly in order to fix our conventions

$$\vartheta_1[\kappa_i](z_i, \tau_i) \equiv \sum_{m \in \mathbb{Z}} \exp\left\{ i\pi(m + \kappa'_i)^2 \tau_i + 2\pi i(m + \kappa'_i)(z_i + \kappa''_i) \right\} \quad i = 1, 2 \quad (5.6)$$

The $\vartheta_1$ function satisfies the heat equation

$$\partial^2_{z_i} \vartheta_1[\kappa_i](z_i, \tau_i) = 4\pi i \partial_\tau \vartheta_1[\kappa_i](z_i, \tau_i) \quad i = 1, 2 \quad (5.7)$$

a product relation,

$$\vartheta'_i[\nu_0](0, \tau_1) = -\pi \vartheta_1[\mu_1] \vartheta_1[\mu_3] \vartheta_1[\mu_5](\tau_1) = -2\pi \eta(\tau_1)^3$$

$$\vartheta'_i[\nu_0](0, \tau_2) = -\pi \vartheta_1[\mu_2] \vartheta_1[\mu_4] \vartheta_1[\mu_5](\tau_2) = -2\pi \eta(\tau_2)^3 \quad (5.8)$$

and a doubling equation,

$$\vartheta_1[\nu_0](2z, \tau) \eta(\tau)^3 = \vartheta_1[\nu_0](z, \tau) \prod_{i=1,3,5} \vartheta_1[\mu_i](z, \tau) \quad (5.9)$$

where $\kappa_i, i = 1, 2$, stand for any genus 1 spin structures while the spin structures $\mu_1$, $\mu_3$, $\mu_5$ are the three distinct even spin structures (same for $\mu_2$, $\mu_4$ and $\mu_6$).

### 5.1.1 Separating degeneration, $\tau \to 0$ keeping $\tau_{1,2}$ fixed

The $\vartheta$-function has the following Taylor expansion around $\tau = 0$, given in terms of genus 1 $\vartheta$-functions, which we denote $\vartheta_1$ for clarity,

$$\vartheta[\kappa](z, \Omega) = \sum_{p=0}^{\infty} \frac{1}{p!} \left( \frac{\tau}{2\pi i} \right)^p \partial^p_{z_1} \vartheta_1[\kappa_1](z_1, \tau_1) \partial^p_{z_2} \vartheta_1[\kappa_2](z_2, \tau_2) \cdot \quad (5.10)$$

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As general functions of $z_1$ and $z_2$, all $p$ contribute. Specializing to $\vartheta$-constants, or derivatives thereof (such as in $\mathcal{M}_{\nu_0}$), only even or odd $p$ contribute depending on the parity of the genus 1 spin structures $\kappa_1$ and $\kappa_2$. We denote the unique genus 1 odd spin structure by $\nu_0 \equiv [\frac{11}{22}]$, and any even spin structure by $\mu$. Furthermore, we use the heat equation of (5.7) satisfied by $\vartheta_1$ to express double $z_i$-derivatives in terms of single $\tau_i$ derivatives, We then have the following cases: the $\vartheta$-constants for even spin structure are given by

$$
\vartheta \left[ \frac{\mu_1}{\mu_2} \right] (0, \Omega) = \sum_{p=0}^{\infty} \frac{(2\tau)^{2p}}{(2p)!} \partial_{\tau_1}^p \vartheta_1[\mu_1](0, \tau_1) \partial_{\tau_2}^p \vartheta_1[\mu_2](0, \tau_2)
$$

$$
\vartheta \left[ \frac{\nu_0}{\nu_0} \right] (0, \Omega) = \frac{1}{4\pi i} \sum_{p=0}^{\infty} \frac{(2\tau)^{2p+1}}{(2p + 1)!} \partial_{\tau_1}^p \vartheta_1'[\nu_0](0, \tau_1) \partial_{\tau_2}^p \vartheta_1'[\nu_0](0, \tau_2)
$$

while the first derivatives $\vartheta$-constants for odd spin structure are given by

$$
\partial_1 \vartheta \left[ \frac{\nu_0}{\mu} \right] (0, \Omega) = \sum_{p=0}^{\infty} \frac{(2\tau)^{2p}}{(2p)!} \partial_{\tau_1}^p \vartheta_1'[\nu_0](0, \tau_1) \partial_{\tau_2}^p \vartheta_1[\mu](0, \tau_2)
$$

$$
\partial_2 \vartheta \left[ \frac{\nu_0}{\mu} \right] (0, \Omega) = \sum_{p=0}^{\infty} \frac{(2\tau)^{2p+1}}{(2p + 1)!} \partial_{\tau_1}^p \vartheta_1'[\nu_0](0, \tau_1) \partial_{\tau_2}^p \vartheta_1[\mu](0, \tau_2)
$$

$$
\partial_1 \vartheta \left[ \frac{\mu}{\nu_0} \right] (0, \Omega) = \sum_{p=0}^{\infty} \frac{(2\tau)^{2p+1}}{(2p + 1)!} \partial_{\tau_1}^p \vartheta_1[\mu](0, \tau_1) \partial_{\tau_2}^p \vartheta_1'[\nu_0](0, \tau_2)
$$

$$
\partial_2 \vartheta \left[ \frac{\mu}{\nu_0} \right] (0, \Omega) = \sum_{p=0}^{\infty} \frac{(2\tau)^{2p}}{(2p)!} \partial_{\tau_1}^p \vartheta_1[\mu](0, \tau_1) \partial_{\tau_2}^p \vartheta_1'[\nu_0](0, \tau_2)
$$

The leading asymptotics of the $\vartheta$-constants may now be read off,

$$
\vartheta \left[ \frac{\mu_1}{\mu_2} \right] (0, \Omega) = \vartheta_1[\mu_1](0, \tau_1) \vartheta_1[\mu_2](0, \tau_2) + \mathcal{O}(\tau^2)
$$

$$
\vartheta \left[ \frac{\nu_0}{\nu_0} \right] (0, \Omega) = -2\pi i \eta(\tau_1)^3 \eta(\tau_2)^3 + \mathcal{O}(\tau^3)
$$

whence follows the leading asymptotics of the modular form $\Psi_{10}(\Omega) = \prod_{\delta} \vartheta[\delta]^2$,

$$
\Psi_{10}(\Omega) = -(2\pi \tau)^2 \cdot 2^{12} \cdot \eta(\tau_1)^{24} \eta(\tau_2)^{24} + \mathcal{O}(\tau^4)
$$

and the limits of the objects $\Xi_6[\delta](\Omega)$,

$$
\Xi_6 \left[ \frac{\mu_1}{\mu_2} \right] (\Omega) = -2^8 \cdot \langle \mu_1 | \nu_0 \rangle \langle \mu_2 | \nu_0 \rangle \eta(\tau_1)^{12} \eta(\tau_2)^{12} + \mathcal{O}(\tau^2)
$$

$$
\Xi_6 \left[ \frac{\nu_0}{\nu_0} \right] (\Omega) = -3 \cdot 2^8 \cdot \eta(\tau_1)^{12} \eta(\tau_2)^{12} + \mathcal{O}(\tau^2)
$$

(5.15)
5.1.2 Non-separating degeneration, $\tau_2 \to +i\infty$ keeping $\tau_1$ and $\tau$ fixed

The genus 2 $\vartheta$-function (5.3) manifestly has a Taylor expansion in powers of $q^{1/2}$, where we introduce the standard abbreviation

$$q \equiv \exp\{i\pi \tau_2\}$$  \hspace{1cm} (5.16)

We shall be interested only in the leading asymptotics as $q \to 0$. The behavior of the $\vartheta$-function in this limit depends upon the value of $\kappa'_2$, following the same notation as in (5.4). This component of the spin structure is singled out because it is associated with the torus with modulus $\tau_2$ that is degenerating to a wire in the limit $q \to 0$.

We begin by examining the behavior of the ordinary $\vartheta$-constants and even characteristics. If $\kappa'_2 = 0$, only the $n = 0$ terms in (5.3) will contribute to the leading asymptotics, while for $\kappa'_2 = 1/2$, both $n = 0, -1$ will contribute. The leading asymptotics is now easily retained from (5.3),

$$\vartheta \left[ \begin{array}{c} \mu \\ \mu_j \end{array} \right] (0, \Omega) = \vartheta_1[\mu](0, \tau_1) + O(q) \quad \text{all even } \mu, \text{ and } j = 2, 4$$

$$\vartheta \left[ \begin{array}{c} \mu \\ \mu_6 \end{array} \right] (0, \Omega) = 2q^{1/2} \vartheta_1[\mu]\left(\frac{5}{2}, \tau_1\right) + O(q^{5/4}) \quad \text{all even } \mu$$

$$\vartheta \left[ \begin{array}{c} \nu_0 \\ \nu_0 \end{array} \right] (0, \Omega) = 2i q^{1/4} \vartheta_1[\nu_0]\left(\frac{3}{2}, \tau_1\right) + O(q^{5/4})$$  \hspace{1cm} (5.17)

Using this asymptotic behavior, we readily calculate that of the modular form $\Psi_{10}(\Omega)$,

$$\Psi_{10}(\Omega) = -2^{12} q^2 \eta(\tau_1)^{18} \vartheta_1[\nu_0]^2(\tau, \tau_1) + O(q^3)$$  \hspace{1cm} (5.18)

Notice that $q$ enters here via $q^2$ only, as expected from the modular covariance of $\Psi_{10}$.

The asymptotics of the objects $\Xi_6[\delta]$ necessary to determine the asymptotics of the measure up to and including order $O(q)$ are given by (we abbreviate $\vartheta_1[\mu] \equiv \vartheta_1[\mu](0, \tau_1)$),

$$\Xi_6 \left[ \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right] = -\Xi_6 \left[ \begin{array}{c} \mu_1 \\ \mu_4 \end{array} \right] = 16 q \vartheta_1[\mu_3] \vartheta_1[\mu_5](\vartheta_1[\mu_1] - \vartheta_1[\nu_0]) \frac{(\tau, \tau_1)^4}{(\frac{5}{2}, \tau_1)^4} + O(q^2)$$

$$\Xi_6 \left[ \begin{array}{c} \mu_3 \\ \mu_2 \end{array} \right] = -\Xi_6 \left[ \begin{array}{c} \mu_3 \\ \mu_4 \end{array} \right] = 16 q \vartheta_1[\mu_1] \vartheta_1[\mu_5](\vartheta_1[\mu_3] - \vartheta_1[\nu_0]) \frac{(\tau, \tau_1)^4}{(\frac{5}{2}, \tau_1)^4} + O(q^2)$$

$$\Xi_6 \left[ \begin{array}{c} \mu_5 \\ \mu_2 \end{array} \right] = -\Xi_6 \left[ \begin{array}{c} \mu_5 \\ \mu_4 \end{array} \right] = 16 q \vartheta_1[\mu_1] \vartheta_1[\mu_3] \frac{(\vartheta_1[\mu_1] - \vartheta_1[\nu_0]) \frac{(\tau, \tau_1)^4}{(\frac{5}{2}, \tau_1)^4}} + O(q^2)$$

$$\Xi_6 \left[ \begin{array}{c} \mu_i \\ \mu_6 \end{array} \right] = O(q) \quad i = 1, 3, 5$$  \hspace{1cm} (5.19)

$$\Xi_6 \left[ \begin{array}{c} \nu_0 \\ \nu_0 \end{array} \right] = -16 q \sum_{i=1,3,5} \langle \mu_i | \nu_0 \rangle \vartheta_1[\mu_i] \vartheta_1[\mu_i] \frac{(\tau, \tau_1)^4}{(\frac{5}{2}, \tau_1)^4} + O(q^2)$$

Next, we derive the asymptotics for the first derivatives of $\vartheta$ evaluated on odd spin structures $\nu$ which enter into $\mathcal{M}_{\nu\nu'}$, for example. This is easily done by inspecting (5.4).
Differentiating by $\partial_1$ brings down in the sum of (5.3) a factor of $2\pi i(m + \kappa_1')$ while differentiating by $\partial_2$ brings down a factor of $2\pi i(n + \kappa_2')$. The asymptotic behavior depends upon the value of $\kappa_2'$. When $\kappa_2' = 0$, the $n = 0$ term is leading in the $\partial_1$ derivative, but cancels out in the $\partial_2$ derivative where the leading contribution comes from the $n = \pm 1$ terms instead. We have the following limits

\begin{align*}
\partial_1 \vartheta \left[ \frac{\nu_0}{\mu_i} \right] (0, \Omega) &= -2\pi \eta(\tau_1)^3 + \mathcal{O}(q) & i &= 2, 4 \\
\partial_2 \vartheta \left[ \frac{\nu_0}{\mu_i} \right] (0, \Omega) &= 4\pi i q (-2\nu_i') \vartheta_1[\nu_0](\tau, \tau_1) + \mathcal{O}(q^2) & i &= 2, 4 \\
\partial_1 \vartheta \left[ \frac{\nu_0}{\mu_6} \right] (0, \Omega) &= 4q^{\frac{1}{2}} \frac{\partial}{\partial \tau} \vartheta_1[\nu_0] \left( \frac{\tau}{2}, \tau_1 \right) + \mathcal{O}(q^{5/4}) \\
\partial_2 \vartheta \left[ \frac{\nu_0}{\mu_6} \right] (0, \Omega) &= 2\pi q^{\frac{1}{2}} \vartheta_1[\mu](\tau, \tau_1) + \mathcal{O}(q^{5/4}) \\
\partial_1 \vartheta \left[ \frac{\mu}{\nu_0} \right] (0, \Omega) &= 4iq^{\frac{1}{2}} \frac{\partial}{\partial \tau} \vartheta_1[\mu](\tau, \tau_1) + \mathcal{O}(q^{5/4}) & \text{all even } \mu \\
\partial_2 \vartheta \left[ \frac{\mu}{\nu_0} \right] (0, \Omega) &= -2\pi q^{\frac{1}{2}} \vartheta_1[\mu](\tau, \tau_1) + \mathcal{O}(q^{5/4}) & \text{all even } \mu & (5.20)
\end{align*}

5.2 Proof of the bilinear $\vartheta$-constant relation

We shall now prove the bilinear $\vartheta$-constant relation of (5.1) and determine the multiplicative $\pm$ sign in the formula for later use in subsection §5.3. Since the sign is not intrinsic, it is necessary to fix a definite basis of characteristics. In the separating degeneration limit, every odd spin structure descends to a spin structure assignment on the two resulting tori which is odd on one torus while even on the other. Two distinct cases emerge, depending on whether the genus 2 spin structures $\nu_i$ and $\nu_j$ in $M_{\nu_i, \nu_j}$ descend to the odd spin structure on opposite tori (first case below) or on the same torus (second case below).

5.2.1 Separating Degenerations : First Case

Let us now check the proposed formula for the first case,

\begin{equation}
\nu_1 = \begin{bmatrix} \mu_1 \\ \nu_0 \end{bmatrix} \quad \nu_2 = \begin{bmatrix} \nu_0 \\ \mu_2 \end{bmatrix} \tag{5.21}
\end{equation}

where $\mu_1$ and $\mu_2$ are any two even genus 1 spin structures (not necessarily taking the expressions of the Table (2.11)). We work to the two lowest orders $\tau^0$ and $\tau^2$, (lowest order is sufficient, but it is interesting to also have the next to leading order available),

\begin{align*}
\partial_1 \vartheta [\nu_1](0, \Omega) &= 2\tau \partial_1 \vartheta_1[\mu_1](\tau_1) \vartheta'_1[\nu_0](\tau_2) \\
\partial_2 \vartheta [\nu_1](0, \Omega) &= \vartheta_1[\mu](\tau_1) \vartheta'_1[\nu_0](\tau_2) + 2\tau^2 \partial_1 \vartheta_1[\mu_1](\tau_1) \partial_2 \vartheta'_1[\nu_0](\tau_2) \\
\partial_1 \vartheta [\nu_2](0, \Omega) &= \vartheta'_1[\nu_0](\tau_1) \vartheta_1[\mu_2](\tau_2) + 2\tau^2 \partial_1 \vartheta_1'[\nu_0](\tau_1) \partial_2 \vartheta_1[\mu_2](\tau_2) \\
\partial_2 \vartheta [\nu_2](0, \Omega) &= 2\tau \vartheta'_1[\nu_0](\tau_1) \partial_2 \vartheta_1[\mu_2](\tau_2) \tag{5.22}
\end{align*}
The remaining 4 odd spin structures are

\[
\begin{align*}
\nu_3 &= \begin{bmatrix} \mu_3 \\ \nu_0 \end{bmatrix}, \\
\nu_4 &= \begin{bmatrix} \nu_0 \\ \mu_4 \end{bmatrix}, \\
\nu_5 &= \begin{bmatrix} \mu_5 \\ \nu_0 \end{bmatrix}, \\
\nu_6 &= \begin{bmatrix} \nu_0 \\ \mu_6 \end{bmatrix}
\end{align*}
\] (5.23)

where \(\mu_1, \mu_3\) and \(\mu_5\) are three distinct even genus 1 spin structures and \(\mu_2, \mu_4\) and \(\mu_6\) are also three distinct even genus 1 spin structures. They satisfy

\[
\begin{align*}
\mu_1 + \mu_3 + \mu_5 &= \nu_0 \\
\mu_2 + \mu_4 + \mu_6 &= \nu_0.
\end{align*}
\] (5.24)

The quantity \(\mathcal{M}_{\nu_1 \nu_2}\) may now be easily expressed to this order

\[
\mathcal{M}_{\nu_1 \nu_2} = -\vartheta_1[\mu_1](\tau_1) \vartheta'_1[\nu_0](\tau_2) \vartheta'_1[\nu_0](\tau_1) \vartheta_1[\mu_2](\tau_2) + 4\tau^2 \partial_{\tau_1} \vartheta_1[\mu_1](\tau_1) \vartheta'_1[\nu_0](\tau_2) \vartheta'_1[\nu_0](\tau_1) \partial_{\tau_2} \vartheta_1[\mu_2](\tau_2) - 2\tau^2 \vartheta_1[\mu_1](\tau_1) \vartheta'_1[\nu_0](\tau_2) \partial_{\tau_1} \vartheta'_1[\nu_0](\tau_1) \partial_{\tau_2} \vartheta_1[\mu_2](\tau_2) - 2\tau^2 \vartheta'_1[\nu_0](\tau_1) \vartheta_1[\mu_2](\tau_2) \partial_{\tau_1} \vartheta_1[\mu_1](\tau_1) \partial_{\tau_2} \vartheta'_1[\nu_0](\tau_2)
\] (5.25)

We make use of the well-known identity for genus one \(\vartheta\)-functions, (5.8). Substituting these expressions, we see that the \(\tau_1\) derivative terms on \(\vartheta[\mu_1]\) cancel, as well as the \(\tau_2\) derivatives on \(\vartheta[\mu_2]\). One is left with

\[
\mathcal{M}_{\nu_1 \nu_2} = -\pi^2 \vartheta_1[\mu_1]^2 \vartheta_1[\mu_3] \vartheta_1[\mu_5](\tau_1) \cdot \vartheta_1[\mu_2]^2 \vartheta_1[\mu_4] \vartheta_1[\mu_6](\tau_2) - 2\pi^2 \tau^2 \vartheta_1[\mu_1] \partial_{\tau_1} \vartheta_1[\mu_1](\tau_1) \vartheta_1[\mu_2](\tau_1) \vartheta_1[\mu_5](\tau_1) \cdot \vartheta_1[\mu_2]^2 \vartheta_1[\mu_4] \vartheta_1[\mu_6](\tau_2) - 4\pi^2 \tau^2 \vartheta_1[\mu_1]^2 \partial_{\tau_1} \vartheta_1[\mu_3] \vartheta_1[\mu_5](\tau_1) \cdot \vartheta_1[\mu_2]^2 \vartheta_1[\mu_4] \vartheta_1[\mu_6](\tau_2)
\]

We now wish to compare these asymptotics with the product of \(\vartheta\)-constants for even characteristics, appearing on the rhs of (5.1). Worked out to the same order, the \(\vartheta\)-constants behave as,

\[
\vartheta \left[ \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} \right] (0, \Omega) = \vartheta_1[\kappa_1](\tau_1) \vartheta_1[\kappa_2](\tau_2) + 2\tau^2 \partial_{\tau_1} \vartheta_1[\kappa_1](\tau_1) \partial_{\tau_2} \vartheta_1[\kappa_2](\tau_2)
\] (5.26)

Up to the addition of complete periods, this assignment of even characteristics is precisely what corresponds to the bilinear \(\vartheta\) relation that we seek to prove.

\[
\begin{align*}
\nu_1 + \nu_2 + \nu_3 &= \begin{bmatrix} \nu_0 + \mu_1 + \mu_3 \\ 2\nu_0 + \mu_2 \end{bmatrix} \equiv \begin{bmatrix} \mu_5 \\ \mu_2 \end{bmatrix} \\
\nu_1 + \nu_2 + \nu_4 &= \begin{bmatrix} 2\nu_0 + \mu_1 \\ \nu_0 + \mu_2 + \mu_4 \end{bmatrix} \equiv \begin{bmatrix} \mu_1 \\ \mu_6 \end{bmatrix} \\
\nu_1 + \nu_2 + \nu_5 &= \begin{bmatrix} \nu_0 + \mu_1 + \mu_5 \\ 2\nu_0 + \mu_2 \end{bmatrix} \equiv \begin{bmatrix} \mu_3 \\ \mu_2 \end{bmatrix} \\
\nu_1 + \nu_2 + \nu_6 &= \begin{bmatrix} 2\nu_0 + \mu_1 \\ \nu_0 + \mu_2 + \mu_6 \end{bmatrix} \equiv \begin{bmatrix} \mu_1 \\ \mu_4 \end{bmatrix}
\end{align*}
\] (5.27)
The effects of the full periods $2\nu_0$ cancel out and we are left with our final formula,

$$\mathcal{M}_{\nu_1 \nu_2} = -\pi^2 \partial \left[ \begin{array}{c} \mu_3 \\ \mu_2 \\ \mu_5 \\ \mu_4 \\ \mu_1 \\ \mu_6 \end{array} \right] \partial \left[ \begin{array}{c} \mu_3 \\ \mu_2 \\ \mu_5 \\ \mu_4 \\ \mu_1 \\ \mu_6 \end{array} \right](0, \Omega) \quad (5.28)$$

It must be stressed here that this formula is exact, including a proper assignment of the overall (non-intrinsic but important) sign. In the expressions for the superstring chiral measure, only $\mathcal{M}^2_{\nu_1 \nu_2}$ will enter and the non-intrinsic sign will be unimportant.

### 5.2.2 Separating Degenerations : Second Case

Let us now check the proposed formula for the second type

$$\nu_1 = \left[ \begin{array}{c} \mu_1 \\ \nu_0 \end{array} \right] \quad \nu_3 = \left[ \begin{array}{c} \mu_3 \\ \nu_0 \end{array} \right] \quad (5.29)$$

where $\mu_1$ and $\mu_3$ are any two distinct even genus 1 spin structures. We work to lowest orders in $\tau$,

$$\partial_1 \vartheta[\nu_1](0, \Omega) = 2\tau \partial_1 \vartheta[\mu_1](\tau_1) \vartheta'[\nu_0](\tau_2)$$

$$\partial_2 \vartheta[\nu_1](0, \Omega) = \vartheta[\mu_1](\tau_1) \vartheta'[\nu_0](\tau_2)$$

$$\partial_1 \vartheta[\nu_3](0, \Omega) = 2\tau \partial_1 \vartheta[\mu_3](\tau_1) \vartheta'[\nu_0](\tau_2)$$

$$\partial_2 \vartheta[\nu_3](0, \Omega) = \vartheta[\mu_3](\tau_1) \vartheta'[\nu_0](\tau_2) \quad (5.30)$$

The remaining 4 odd spin structures are

$$\nu_2 = \left[ \begin{array}{c} \nu_0 \\ \mu_2 \end{array} \right] \quad \nu_4 = \left[ \begin{array}{c} \nu_0 \\ \mu_4 \end{array} \right] \quad \nu_5 = \left[ \begin{array}{c} \mu_5 \\ \nu_0 \end{array} \right] \quad \nu_6 = \left[ \begin{array}{c} \nu_0 \\ \mu_6 \end{array} \right] \quad (5.31)$$

where, as before, $\mu_1$, $\mu_3$ and $\mu_5$ are three distinct even genus 1 spin structures and $\mu_2$, $\mu_4$ and $\mu_6$ are also three distinct even genus 1 spin structures. They satisfy

$$\mu_1 + \mu_3 + \mu_5 = \nu_0$$

$$\mu_2 + \mu_4 + \mu_6 = \nu_0 \quad (5.32)$$

The expression for $\mathcal{M}_{\nu_1 \nu_3}$ is then given by

$$\mathcal{M}_{\nu_1 \nu_3} = 2\tau \left( \partial_1 \vartheta[\mu_1][\mu_3] \vartheta[\nu_0] \right)(\tau_1) \cdot \vartheta'[\nu_0](\tau_2)^2 \quad (5.33)$$

$$= 2\pi^2 \tau \left( \partial_1 \vartheta[\mu_1][\mu_3] \vartheta[\nu_0] \right)(\tau_1) \cdot \vartheta[\mu_2]^2 \vartheta[\mu_4]^2 \vartheta[\mu_6]^2(\tau_2)$$

Furthermore, the expansion to lowest order for the even spin structure $\vartheta$-constants is

$$\vartheta \left[ \begin{array}{c} \sigma_1 \\ \sigma_2 \end{array} \right](0, \Omega) = \vartheta[\mu_1](0, \tau_1) \vartheta[\mu_2](0, \tau_2)$$

$$\vartheta \left[ \begin{array}{c} \nu_0 \\ \mu_0 \end{array} \right](0, \Omega) = \frac{2\tau}{4\pi i} \vartheta[\nu_0](0, \tau_1) \vartheta'[\nu_0](0, \tau_2) \quad (5.34)$$

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To make contact between these two formulas, we need the following equation between derivatives of ratios of genus one theta constants

$$\partial_1 \ln \frac{\vartheta_1[\mu_1]}{\vartheta_1[\mu_3]} = \frac{i\pi}{4} \vartheta_1[\mu_5]^4 \cdot \sigma(\mu_1, \mu_3)$$

(5.35)

where $\sigma$ obeys $\sigma(\mu_1, \mu_3) = -\sigma(\mu_3, \mu_1)$ and is given by

$$\sigma([10], [00]) = \sigma([00], [01]) = \sigma([10], [01]) = +1.$$ 

(5.36)

Thus, we have the expression

$$\partial_1 \vartheta_1[\mu_1] \vartheta_1[\mu_3] - \partial_1 \vartheta_1[\mu_3] \vartheta_1[\mu_1] = \frac{i\pi}{4} \vartheta_1[\mu_1] \vartheta_1[\mu_3] \vartheta_1[\mu_5]^4(\tau_1) \sigma(\mu_1, \mu_3)$$

(5.37)

Using this formula to eliminate the derivative terms in the expression for $M_{\nu_1, \nu_3}$ and using the product formula for $\vartheta'_1[\nu_0]$, we get

$$M_{\nu_1, \nu_3} = -\pi^2 \sigma(\mu_1, \mu_3) \vartheta\left[\begin{array}{c} \nu_0 \\ \nu_0 \end{array}\right] \vartheta\left[\begin{array}{c} \mu_5 \\ \mu_2 \end{array}\right] \vartheta\left[\begin{array}{c} \mu_5 \\ \mu_4 \end{array}\right] \vartheta\left[\begin{array}{c} \mu_5 \\ \mu_6 \end{array}\right] (0, \Omega)$$

(5.38)

which is our final formula for the second case. Up to the addition of complete periods, (whose effects cancel out completely) this assignment of even characteristics is precisely what corresponds to the product formula that we proposed to prove.

$$\nu_1 + \nu_3 + \nu_2 = \left[\begin{array}{c} \nu_0 + \mu_1 + \mu_3 \\ 2\nu_0 + \mu_2 \end{array}\right] \equiv \left[\begin{array}{c} \mu_5 \\ \mu_2 \end{array}\right]$$

$$\nu_1 + \nu_3 + \nu_4 = \left[\begin{array}{c} \nu_0 + \mu_1 + \mu_3 \\ 2\nu_0 + \mu_4 \end{array}\right] \equiv \left[\begin{array}{c} \mu_5 \\ \mu_4 \end{array}\right]$$

$$\nu_1 + \nu_3 + \nu_5 = \left[\begin{array}{c} \mu_1 + \mu_3 + \mu_5 \\ 3\nu_0 \end{array}\right] \equiv \left[\begin{array}{c} \nu_0 \\ \nu_0 \end{array}\right]$$

$$\nu_1 + \nu_3 + \nu_6 = \left[\begin{array}{c} \nu_0 + \mu_1 + \mu_3 \\ 2\nu_0 + \mu_6 \end{array}\right] \equiv \left[\begin{array}{c} \mu_5 \\ \mu_6 \end{array}\right]$$

(5.39)

For both cases a completely intrinsic formula may be obtained by squaring the above

$$M_{\nu_1, \nu_3}^2 = \pi^4 \prod_{i \neq 1, 3} \vartheta[\nu_1 + \nu_3 + \nu_i]^2 (0, \Omega)$$

(5.40)

and so this formula is universally valid.
5.2.3 Non-Separating Degenerations : First Case

Next, we show that the same formulas with the same signs are reproduced in the non-separating degeneration limit. Recall that the first case corresponded to the spin structure assignments

\[
\nu_i = \begin{bmatrix} \mu_i \\ \nu_0 \end{bmatrix}, \quad \nu_j = \begin{bmatrix} \nu_0 \\ \mu_j \end{bmatrix}
\]

where \(\mu_i\) and \(\mu_j\) are any even genus 1 characteristics. Actually, when analyzing non-separating degenerations, this case itself falls into two subcases; \(\mu_i\) can be any even spin structure, but the cases \(j = 2, 4\) and \(j = 6\) generate different asymptotic behaviors and must be treated differently. For concreteness, and without loss of generality, we make definite assignments:

\(\mu_i = \mu_1, \mu_j = \mu_2\) for the first subcase, while \(\mu_i = \mu_1, \mu_j = \mu_6\) for the second subcase. All other 5 cases in this class are analogous to one of these two.

The asymptotics of \(M_{\nu_i \nu_j}\) is easily computed in both cases, using (5.20), and we find,

\[
M_{\nu_1 \nu_2} = -4\pi^2 q^{1/2} \eta(\tau_1)^3 \vartheta_1[\mu_1](\frac{\tau}{2}, \tau_1) \]

\[
M_{\nu_1 \nu_6} = -8\pi^2 q^{1/2} \left\{ \frac{\partial}{\partial \tau} \vartheta_1[\mu_1](\frac{\tau}{2}, \tau_1) \vartheta_1[\nu_0](\frac{\tau}{2}, \tau_1) - \frac{\partial}{\partial \tau} \vartheta_1[\nu_0](\frac{\tau}{2}, \tau_1) \vartheta_1[\mu_1](\frac{\tau}{2}, \tau_1) \right\}
\]

This result needs to be compared with the limit of the product of \(\vartheta\)-constants for these spin structure assignments. The corresponding even spin structures occurring in the product were already determined in (5.27), but we now need to adapt the notation to that of the current situation. The products of the corresponding genus two \(\vartheta\)-functions as well as their asymptotic behaviors are given by the following limits,

\[
\vartheta \left[ \begin{bmatrix} \mu_1 \\ \mu_4 \end{bmatrix} \vartheta \left[ \begin{bmatrix} \mu_1 \\ \mu_6 \end{bmatrix} \vartheta \left[ \begin{bmatrix} \mu_3 \\ \mu_2 \end{bmatrix} \vartheta \left[ \begin{bmatrix} \mu_5 \\ \mu_2 \end{bmatrix} (0, \Omega) = 2q^{1/2} \vartheta_1[\mu_1](\frac{\tau}{2}, \tau_1) \prod_{i=1,3,5} \vartheta_1[\mu_i](0, \tau_1)
\right. \vartheta_1[\mu_3](\frac{\tau}{2}, \tau_1) \vartheta_1[\mu_5](\frac{\tau}{2}, \tau_1) \right) \right)
\]

Using the product formula (5.8), it is manifest that the subcase \(\nu_1 \nu_2\) is consistent with the following equality,

\[
M_{\nu_1 \nu_2} = -\pi^2 \vartheta \left[ \begin{bmatrix} \mu_1 \\ \mu_4 \end{bmatrix} \vartheta \left[ \begin{bmatrix} \mu_1 \\ \mu_6 \end{bmatrix} \vartheta \left[ \begin{bmatrix} \mu_3 \\ \mu_2 \end{bmatrix} \vartheta \left[ \begin{bmatrix} \mu_5 \\ \mu_2 \end{bmatrix} (0, \Omega) \right. \right. \right) \right)
\]

which is indeed the precise same form as we had obtained for the separating degenerations in (5.28), including the non-intrinsic sign.

Comparison for the subcase \(\nu_1 \nu_6\) is more complicated because the limits of \(M_{\nu_1 \nu_6}\) and the product are rather different looking. The required genus 1 identity does not appear to be familiar when expressed in terms of \(\vartheta\)-functions, but it is well known when translated.
into the Jacobian elliptic functions \( sn(u), cn(u) \) and \( dn(u) \). The correspondence between the functions \( sn(u), cn(u), dn(u) \) and \( \vartheta \)-functions is given by\(^{5}\)

\[
\begin{align*}
    sn(u) &= -\frac{\vartheta_1[\mu_1](0)\vartheta_1[\nu_0](v)}{\vartheta_1[\mu_3](0)\vartheta_1[\mu_5](v)} \quad &dn(u) &= +\frac{\vartheta_1[\mu_3](0)\vartheta_1[\mu_1](v)}{\vartheta_1[\mu_1](0)\vartheta_1[\mu_3](v)} \\
    cn(u) &= +\frac{\vartheta_1[\mu_3](0)\vartheta_1[\mu_5](v)}{\vartheta_1[\mu_5](0)\vartheta_1[\mu_3](v)} \quad &u &\equiv v\tau\vartheta_1[\mu_1](0)^2 \\
\end{align*}
\] (5.45)

Here, the modular parameter has been suppressed. The standard derivative formulas for Jacobian elliptic functions \( (sn'(u) = cn(u)dn(u),
 cn'(u) = -sn(u)dn(u) \) and \( dn'(u) = -k^2 sn(u)cn(u) \) together with the standard quadratic relations between these functions produce the following relations,

\[
\begin{align*}
    \frac{\partial}{\partial u} \ln sn(u) &= \frac{cn(u)}{sn(u)dn(u)} \quad &\frac{\partial}{\partial u} \ln \frac{sn(u)}{dn(u)} &= \frac{dn(u)}{sn(u)cn(u)} \\
    \frac{\partial}{\partial u} \ln sn(u) &= \frac{cn(u)dn(u)}{sn(u)} \\
\end{align*}
\] (5.46)

Translating these formulas into \( \vartheta \)-functions using (5.45), and changing variables from \( u \) to \( v \) gives

\[
\frac{\partial}{\partial v} \ln \frac{\vartheta_1[\nu_0](v)}{\vartheta_1[\mu_1](v)} = -\pi \vartheta_1[\mu_1]^2(0) \frac{\vartheta_1[\mu_3](v)\vartheta_1[\mu_5](v)}{\vartheta_1[\nu_0](v)\vartheta_1[\mu_1](v)} \\
\] (5.47)

valid for any even spin structure \( \mu_1 \) and its two distinct partners \( \mu_3 \) and \( \mu_5 \). Multiplying both sides by \( \vartheta_1[\nu_0](v)\vartheta_1[\mu_1](v) \) yields,

\[
\frac{\partial}{\partial v} \vartheta_1[\nu_0](v)\vartheta_1[\mu_1](v) - \frac{\partial}{\partial v} \vartheta_1[\mu_1](v)\vartheta_1[\nu_0](v) = -\pi \vartheta_1[\mu_1]^2(0)\vartheta_1[\mu_3](v)\vartheta_1[\mu_5](v) \\
\] (5.48)

Replacing \( v = \tau/2 \), and restoring the genus 1 modulus dependence, we finally get the desired formula,

\[
\frac{\partial}{\partial \tau} \vartheta_1[\nu_0](\frac{\tau}{2},\tau_1)\vartheta_1[\mu_1](\frac{\tau}{2},\tau_1) - \frac{\partial}{\partial \tau} \vartheta_1[\mu_1](\frac{\tau}{2},\tau_1)\vartheta_1[\nu_0](\frac{\tau}{2},\tau_1) = -\frac{\pi}{2} \vartheta_1[\mu_1]^2(0)\vartheta_1[\mu_3](\frac{\tau}{2},\tau_1)\vartheta_1[\mu_5](\frac{\tau}{2},\tau_1) \\
\] (5.49)

This reproduces the following formula

\[
M_{\nu_1\nu_6} = -\pi^2 \vartheta \left[ \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right] \vartheta \left[ \begin{array}{c} \mu_1 \\ \mu_4 \end{array} \right] \vartheta \left[ \begin{array}{c} \mu_3 \\ \mu_6 \end{array} \right] \\
\] (5.50)

in agreement with the general formula proposed and with the sign of the separating case.

\(^{5}\)A convenient reference is [14]. We notice, however, that the precise correspondence between our notations for \( \vartheta \)-functions and those of [14] involves subtle signs, given by \( \vartheta_1[\mu_1](v) = +\vartheta_3(v), \vartheta_1[\mu_3](v) = +\vartheta_4(v), \vartheta_1[\mu_5](v) = +\vartheta_2(v), \vartheta_1[\nu_0](v) = -\vartheta_3(v) \).
5.2.4 Non-Separating Degenerations: Second Case

Finally, we show that the same formulas with the same signs are also reproduced in the non-separating degeneration limit for the second case. Here, we must further distinguish between two subcases.

The first subcase has both genus 1 even spin structures on the degenerating torus,

\[ \nu_i = \begin{bmatrix} \nu_0 \\ \mu_i \end{bmatrix}, \quad \nu_j = \begin{bmatrix} \nu_0 \\ \mu_j \end{bmatrix}, \quad i, j = 2, 4, 6 \]

and we have (the case \( M_{\nu_4\nu_6} \) is analogous to \( M_{\nu_2\nu_6} \))

\[ M_{\nu_2\nu_4} = 16i\pi^2 q\eta(\tau_1)^3 \vartheta_1[\nu_0](\tau, \tau_1) \]
\[ M_{\nu_2\nu_6} = -4i\pi^2 q^2 \eta(\tau_1)^3 \vartheta_1[\nu_0](\frac{\tau}{2}, \tau_1) \] (5.51)

For these two assignments, the product of \( \vartheta \)-constants is given by

\[ \vartheta_1[\nu_0] \vartheta_1[\mu_1] \vartheta_1[\mu_3] \vartheta_1[\mu_5] (0, \Omega) = 16i\eta(\tau_1)^3 \vartheta_1[\nu_0](\tau, \tau_1) \]
\[ \vartheta_1[\nu_0] \vartheta_1[\mu_1] \vartheta_1[\mu_3] \vartheta_1[\mu_5] (0, \Omega) = -4i\eta(\tau_1)^3 \vartheta_1[\nu_0](\frac{\tau}{2}, \tau_1) \] (5.52)

Both are manifestly in agreement with the relations obtained for separating degenerations.

The second subcase has both genus 1 odd spin structures on the degenerating torus,

\[ \nu_i = \begin{bmatrix} \mu_i \\ \nu_0 \end{bmatrix}, \quad \nu_j = \begin{bmatrix} \mu_j \\ \nu_0 \end{bmatrix}, \quad i, j = 1, 3, 5 \]

For this subcase, we have

\[ M_{\nu_i\nu_j} = -8i\pi q^{\frac{1}{2}} \left( \frac{\partial}{\partial \tau} \vartheta_1[\mu_i](\frac{\tau}{2}, \tau_1) \vartheta_1[\mu_j](\frac{\tau}{2}, \tau_1) - \frac{\partial}{\partial \tau} \vartheta_1[\mu_j](\frac{\tau}{2}, \tau_1) \vartheta_1[\mu_i](\frac{\tau}{2}, \tau_1) \right) \] (5.54)

and the product formula (here, \( k \neq i, j \))

\[ \vartheta_1[\nu_0] \vartheta_1[\mu_1] \vartheta_1[\mu_3] \vartheta_1[\mu_5] (0, \Omega) = 4i\eta(\tau_1)^{\frac{3}{2}} \vartheta_1[\nu_0](\tau, \tau_1) \] (5.55)

The genus 1 identities needed here are obtained by working out the following combinations of derivatives

\[ \frac{\partial}{\partial u} \ln \frac{dn(u)}{cn(u)} = -\frac{\vartheta_1[\mu_5]}{\vartheta_1[\mu_1]} \frac{sn(u)cn(u)}{dn(u)} \]
\[ \frac{\partial}{\partial u} \ln \frac{dn(u)}{sn(u)} = -\frac{sn(u)dn(u)}{cn(u)} \]
\[ \frac{\partial}{\partial u} \ln \frac{dn(u)}{cn(u)} = \frac{\vartheta_1[\mu_3]}{\vartheta_1[\mu_1]} \frac{sn(u)}{cn(u)dn(u)} \] (5.56)

\[ \frac{\partial}{\partial u} \ln \frac{dn(u)}{sn(u)} = \frac{sn(u)dn(u)}{cn(u)} \] (5.57)
and translating these equations into \( \vartheta_1 \) functions using (5.43),

\[
\frac{\partial}{\partial \tau} \vartheta_1[\mu_i]\left(\frac{\tau}{2}, \tau_1\right) \vartheta_1[\mu_j]\left(\frac{\tau}{2}, \tau_1\right) - \frac{\partial}{\partial \tau} \vartheta_1[\mu_j]\left(\frac{\tau}{2}, \tau_1\right) \vartheta_1[\mu_i]\left(\frac{\tau}{2}, \tau_1\right)
\]

\[
= +\sigma(\mu_i, \mu_j) \pi_2 \vartheta_1[\mu_k]\left(0, \tau_1\right) \vartheta_1[\nu_0]\left(\frac{\tau}{2}, \tau_1\right) (5.58)
\]

where the signature \( \sigma(\mu_1, \mu_2) \) was introduced in (5.36). Using these relations, we again recover the formula established in the case of separating degenerations.

### 5.3 Proof of the \( \mathcal{M} \) Product Formula

As we had stressed, the \( \mathcal{M}_{\nu_i \nu_j} \) product formula follows from the independence of the chiral measure from the points \( q_0 \). Here, we give a direct and independent proof of the identity using only \( \vartheta \)-function results. Again, we must split this treatment into two cases according to the form of the spin structures, since we need the detailed sign assignments.

We begin with the first case (5.21), and need to prove the formula

\[
\langle \nu_1 | \nu_2 \rangle \prod_{i=3,4,5,6} \frac{\mathcal{M}_{\nu_2 \nu_i}}{\mathcal{M}_{\nu_1 \nu_i}} = 1 .
\]

(5.59)

It is convenient to use the abbreviations,

\[
(00) = \vartheta \left[ \begin{array}{c} \nu_0 \\ \nu_0 \end{array} \right] (\Omega) \quad (ij) = \vartheta \left[ \begin{array}{c} \mu_i \\ \mu_j \end{array} \right] (\Omega).
\]

(5.60)

Then, we have the following expressions for the \( \mathcal{M}_{\nu_i \nu_j} \)'s entering the product

\[
\mathcal{M}_{\nu_1 \nu_3} = -\pi^2(00)(52)(54)(56) \cdot \sigma(\mu_1, \mu_3)
\]

\[
\mathcal{M}_{\nu_1 \nu_4} = -\pi^2(34)(54)(12)(16)
\]

\[
\mathcal{M}_{\nu_1 \nu_5} = -\pi^2(00)(32)(34)(36) \cdot \sigma(\mu_1, \mu_5)
\]

\[
\mathcal{M}_{\nu_1 \nu_6} = -\pi^2(36)(56)(12)(14)
\]

(5.61)

as well as

\[
\mathcal{M}_{\nu_2 \nu_3} = +\pi^2(12)(52)(34)
\]

\[
\mathcal{M}_{\nu_2 \nu_4} = -\pi^2(00)(16)(36)(56) \cdot \sigma(\mu_2, \mu_4)
\]

\[
\mathcal{M}_{\nu_2 \nu_5} = +\pi^2(12)(32)(54)(56)
\]

\[
\mathcal{M}_{\nu_2 \nu_6} = -\pi^2(00)(14)(34)(54) \cdot \sigma(\mu_2, \mu_6)
\]

(5.62)

Putting all factors together, we find that all the factors of \( \pi \) and \( (ij) \) precisely cancel one another, so that we are left with

\[
\prod_{i=3,4,5,6} \frac{\mathcal{M}_{\nu_2 \nu_i}}{\mathcal{M}_{\nu_1 \nu_i}} = \frac{\sigma(\mu_2, \mu_4) \cdot \sigma(\mu_2, \mu_6)}{\sigma(\mu_1, \mu_3) \cdot \sigma(\mu_1, \mu_5)}
\]

(5.63)
It remains to evaluate the product of \( \sigma \)'s. By inspection of all cases, it is easy to see that one has

\[
\sigma(\mu_1, \mu_3) \cdot \sigma(\mu_1, \mu_5) = -\langle \mu_1 | \nu_0 \rangle
\]

\[
\sigma(\mu_2, \mu_4) \cdot \sigma(\mu_2, \mu_6) = -\langle \mu_2 | \nu_0 \rangle
\] (5.64)

and that

\[
\langle \nu_1 | \nu_2 \rangle = \langle \mu_1 | \nu_0 \rangle \langle \mu_2 | \nu_0 \rangle
\] (5.65)

which proves the formula for the first case.

For the second case (5.29), we need to prove the formula

\[
\langle \nu_1 | \nu_3 \rangle \prod_{i=2,4,5,6} \frac{M_{\nu_3 \nu_i}}{M_{\nu_1 \nu_i}} = 1.
\] (5.66)

We use the same abbreviation as for the first case. Then, we have the following expressions for the \( M_{\nu_i \nu_j} \)'s entering the product

\[
M_{\nu_1 \nu_2} = -\pi^2(32)(52)(14)(16)
\]

\[
M_{\nu_1 \nu_4} = -\pi^2(34)(54)(12)(16)
\]

\[
M_{\nu_1 \nu_5} = -\pi^2(00)(32)(34)(36) \cdot \sigma(\mu_1, \mu_5)
\]

\[
M_{\nu_1 \nu_6} = -\pi^2(36)(56)(12)(14)
\] (5.67)

as well as

\[
M_{\nu_3 \nu_2} = -\pi^2(12)(52)(34)(36)
\]

\[
M_{\nu_3 \nu_4} = -\pi^2(14)(54)(32)(36)
\]

\[
M_{\nu_3 \nu_5} = -\pi^2(00)(12)(14)(16) \cdot \sigma(\mu_3, \mu_5)
\]

\[
M_{\nu_3 \nu_6} = -\pi^2(16)(56)(32)(34)
\] (5.68)

Putting all factors together, we find that all the factors of \( \pi \) and of \((ij)\) precisely cancel one another, and we are left with

\[
\prod_{i=2,4,5,6} \frac{M_{\nu_3 \nu_i}}{M_{\nu_1 \nu_i}} = \frac{\sigma(\mu_3, \mu_5)}{\sigma(\mu_1, \mu_5)} = \sigma(\mu_3, \mu_5) \sigma(\mu_1, \mu_5)
\] (5.69)

The last product of \( \sigma \)'s is easily re-expressed, by inspection of all possible cases

\[
\sigma(\mu_3, \mu_5) \sigma(\mu_1, \mu_5) = \langle \nu_1 | \nu_3 \rangle
\] (5.70)

and this proves the formula for the second case.
The correct degrees of freedom of the superstring are obtained only after enforcing the GSO projection. In the string path integral formulation, this corresponds to summing the contributions to the chiral measure of each spin structure. Now, due to chiral splitting, the contribution of each spin structure $\delta$ is only determined up to a global phase factor, which is independent of the moduli, but may depend on $\delta$. The issue in enforcing the GSO projection is then to determine the relative phases between the contributions of various spin structures. The key criterion that the relative phases must satisfy is modular invariance for the full chiral measure.

6.1 The GSO Projection

It was shown in subsection §4.3 that $\Xi_6[\delta](\Omega)$, defined earlier by

$$\Xi_6[\delta](\Omega) \equiv \sum_{1 \leq i < j \leq 3} \langle \nu_i | \nu_j \rangle \prod_{k=4,5,6} \vartheta[\nu_i + \nu_j + \nu_k]^4(0, \Omega)$$

depends only on $\delta$ and not on the particular triplet of points in the partition used to represent $\delta$. The modular transformation properties of $\Xi_6[\delta](\Omega)$ can be now read off from Table 2. It turns out that they are closely related to those of $\vartheta[\delta]^4(0, \Omega)$ and given by

$$\vartheta[\tilde{\delta}]^4(0, \tilde{\Omega}) = \epsilon(\delta, M)^4 \det(C\Omega + D)^2 \vartheta[\delta]^4(0, \Omega)$$

$$\Xi_6[\tilde{\delta}](\tilde{\Omega}) = \epsilon(\delta, M)^4 \det(C\Omega + D)^6 \Xi_6[\delta](\Omega)$$

so that the product of the two transforms as

$$\vartheta[\tilde{\delta}]^4(0, \tilde{\Omega}) \Xi_6[\tilde{\delta}](\tilde{\Omega}) = \det(C\Omega + D)^8 \vartheta[\delta]^4(0, \Omega) \Xi_6[\delta](\Omega)$$

where there are NO SIGNS in front of the right hand side.

To enforce the GSO projection, we have to sum over spin structures the chiral measures $d\mu[\delta](\Omega)$ with suitable relative phases $\eta_{\delta}$. The criterion is that the total chiral measure $d\mu(\Omega) = \sum_{\delta} \eta_{\delta} d\mu[\delta](\Omega)$ has to lead to a full measure $[\prod]_I$ invariant under $Sp(4, \mathbb{Z})$. Now recall that a modular form $\Psi_k(\Omega)$ of weight $k$ is a holomorphic function of $\Omega_{IJ}$ which transforms as follows under $Sp(4, \mathbb{Z})$

$$\Psi_k(\Omega) = \det(C\Omega + D)^k \Psi_k(\Omega)$$

Since $\det \text{Im} \Omega$ and the measure $\prod_{I \leq J} d\Omega_{IJ}$ transform under $Sp(4, \mathbb{Z})$ as

$$\det \text{Im} \tilde{\Omega} = |\det(C\Omega + D)|^{-2} \det \text{Im} \Omega$$

$$\prod_{I \leq J} d\tilde{\Omega}_{IJ} = \det(C\Omega + D)^{-3} \prod_{I \leq J} d\Omega_{IJ}$$

(6.5)
we find that the holomorphic coefficient\( \frac{d\mu(\Omega)}{\prod_{I \leq J} d\Omega_{IJ}} \) must be a modular form of weight \(-2\)

\[
\frac{d\mu(\tilde{\Omega})}{\prod_{I \leq J} d\tilde{\Omega}_{IJ}} = \det(C\Omega + D)^{-2} \frac{d\mu(\Omega)}{\prod_{I \leq J} d\Omega_{IJ}} \tag{6.6}
\]

Returning now to the expression for \(d\mu[\delta](\Omega)\) in terms of \(\Xi_6[\delta](\Omega)\), we observe that the expressions \(\vartheta^4[\delta](\Omega)\Xi_6[\delta](\Omega)\) for each spin structure \(\delta\) are not modular forms, because they transform into each other. However, in view of the preceding phases (6.3) for modular transformations, there is a unique choice of relative phases which will make the sum of spin structures into a modular form. It is obtained by taking all the relative phase factors to be the same, say \(+1\). We obtain in this way the following modular form of weight 8

\[
\Upsilon_8(\Omega) = \sum_\delta \vartheta[\delta]^4(\Omega)\Xi_6[\delta](\Omega). \tag{6.7}
\]

The resulting chiral superstring measure is then

\[
d\mu(\Omega) = \frac{1}{16\pi^6} \prod_{I \leq J} d\Omega_{IJ} \Upsilon_8(\Omega)\Psi_{10}^{-1}(\Omega)
\]

with the modular form \(\Upsilon_8(\Omega)\Psi_{10}^{-1}(\Omega)\) of weight \(-2\) as coefficient, as desired.

Notice that there is a unique way of constructing this modular form, as opposed to, say, a mere superposition of \(\vartheta[\delta]^4\), where there would be 6 independent choices. For example, had \(\Xi_6[\delta](\Omega)\) been independent of \(\delta\) (and hence been already a modular form \(\Xi_6(\Omega)\) of weight 6, instead of merely transforming covariantly into \(\Xi_6[\tilde{\delta}]\)), we would have had the following six independent choices

\[
\Xi_6(\Omega) \sum_\delta \langle \nu | \delta \rangle \vartheta[\tilde{\delta}](\Omega) \tag{6.8}
\]

for each odd spin structure \(\nu\). They would all lead to a modular form of weight 8, which is identically 0, by the Riemann identity.

### 6.2 The Ring of Genus 2 Modular Forms

It may be helpful to summarize here some basic facts about modular forms in genus \(h = 2\). The ring of modular forms in genus 2 has been identified by Igusa ([10], [11], [12]) as a polynomial ring with generators \(\Psi_4\), \(\Psi_6\), \(\Psi_{10}\) and \(\Psi_{12}\). Of particular interest to us are \(\Psi_4\), \(\Psi_6\), and \(\Psi_{10}\). We have encountered \(\Psi_{10} = \prod_\delta \vartheta^2[\delta](\Omega)\) before, while \(\Psi_4(\Omega)\) and \(\Psi_6(\Omega)\) are defined respectively by

\[
\Psi_4(\Omega) \equiv \sum_\delta \vartheta^8[\delta](0, \Omega) \tag{6.9}
\]

and by

\[
\Psi_6(\Omega) = \frac{1}{4} \sum_{syz(\delta_1, \delta_2, \delta_3)} \pm \vartheta[\delta_1]^4 \vartheta[\delta_2]^4 \vartheta[\delta_3]^4 \tag{6.10}
\]
Here, the sum is taken over syzygous triples of even characteristics $\delta_1$, $\delta_2$ and $\delta_3$ which are defined to satisfy
\[
\langle \delta_1 | \delta_2 \rangle \langle \delta_1 | \delta_3 \rangle \langle \delta_1 | \delta_3 \rangle = 1
\] (6.11)
while antisyzygous triples have $-1$ instead. Of the 120 triples of pairwise distinct even spin structures, 60 are syzygous and 60 anti-syzygous, and these characterizations are invariant under the action of the modular group. The expression (6.10) looks deceptively similar to $\Xi_6[\delta](\Omega)$, but it is of course different. It does not depend on a spin structure and is a genuine modular form. Had it taken the place of $\Xi_6[\delta](\Omega)$ in the expression (1.9) for $d\mu[\delta](\Omega)$, there would have been no unique way of enforcing the GSO projection.

6.3 Vanishing of the Cosmological Constant

Now that the full chiral measure for the superstring has been determined, we can turn to the evaluation of the string cosmological constant for the Type II superstrings, which is given by
\[
\Lambda_{\text{II}} = \int \det^{-5} \text{Im} \Omega \left| \frac{\prod_{I<J} d\Omega_{IJ}}{16\pi^6} \right|^2 \left| \frac{\Upsilon_8(\Omega)}{\Psi_{10}(\Omega)} \right|^2
\] (6.12)
We shall show that the cosmological constant $\Lambda_{\text{II}}$ vanishes by showing that the modular form $\Upsilon_8(\Omega)$ vanishes identically in $\Omega$. It is very instructive to do this in two different ways.

The first proof of the vanishing of $\Upsilon_8(\Omega)$ is obtained by exploiting the fact that $\Upsilon_8$ is a modular form of weight 8. Recall that there is in genus 2 a unique modular form $\Psi_4$ of weight 4 given by (6.9). The sum in (6.9) is over all even spin structures $\delta$. By the classification result of Igusa quoted above, any modular form of weight 8 must be proportional to $\Psi_4^2$, so that $\Upsilon_8(\Omega) = r \Psi_4(\Omega)^2$, with $r$ a constant. Since $r$ is independent of $\Omega$, it may be evaluated in the separating degeneration limit of §5.1.1. It is well-known that the limit $\tau \to 0$ of $\Psi_4$ is non-vanishing; from (5.13), it is given by
\[
\Psi_4(\Omega) = \left( \sum_{\mu_1} \vartheta_1[\mu_1](0, \tau_1)^4 \right) \left( \sum_{\mu_2} \vartheta_1[\mu_2](0, \tau_2)^4 \right) + \mathcal{O}(\tau^2)
\] (6.13)
where the summations are over all genus 1 even spin structures $\mu_1$ and $\mu_2$. The limit of $\Upsilon_8$ may be obtained from the limits of $\vartheta^4[\delta] \Xi_6[\delta]$, derived in turn from (5.13) and (5.13),
\[
\vartheta \left[ \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right] \Xi_6 \left[ \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right] = -2^8 \langle \mu_1 | \nu_0 \rangle \langle \mu_2 | \nu_0 \rangle \vartheta_1[\mu_1]^4(0, \tau_1) \vartheta_1[\mu_2]^4(0, \tau_2) \eta(\tau_1)^{12} \eta(\tau_2)^{12} + \mathcal{O}(\tau^2)
\]
\[
\vartheta \left[ \begin{array}{c} \nu_0 \\ \nu_0 \end{array} \right] \Xi_6 \left[ \begin{array}{c} \nu_0 \\ \nu_0 \end{array} \right] = -3 \cdot 2^{12} \pi^4 \eta(\tau_1)^{24} \eta(\tau_2)^{24} + \mathcal{O}(\tau^6)
\] (6.14)
Clearly the last case above tends to 0 as $\tau \to 0$, and will not contribute in this limit. The summation over all even spin structures thus reduces to a summation over all genus 1 even
spin structures $\mu_1$ and $\mu_2$, and the limit is given by

$$\Upsilon_8(\Omega) = -2^8 \eta(\tau_1)^2 \eta(\tau_2)^2 \prod_{i=1,2} \left( \sum_{\mu_i} \langle \mu_i | \nu_0 \rangle \vartheta_1[\mu_i](0, \tau_i) \right)^4 + \mathcal{O}(\tau^2) \quad (6.15)$$

but this vanishes by the genus 1 Riemann identity

$$\sum_{\mu_i} \langle \mu_i | \nu_0 \rangle \vartheta_1[\mu_i](0, \tau_i)^4 = 0 \quad (6.16)$$

Thus, the constant $r$ vanishes and therefore we have $\Upsilon_8(\Omega) = 0$ identically. As a result, the cosmological constant vanishes for both the Type II and the heterotic string theories to two loop order.

The second proof of the vanishing of $\Upsilon_8(\Omega)$ is constructed by using the Riemann relations to recast $\Upsilon_8(\Omega)$ in terms of the modular form $\Psi_8^4(\Omega)$ and the modular form $\Psi_8(\Omega) \equiv \sum_\delta \vartheta[\delta]^{16}(0, \Omega)$ (6.17)

and exploiting the well-known relation $4 \Psi_8 = \Psi_4^2$. To compute $\Upsilon_8$ in terms of $\Psi_4^2$ and $\Psi_8$, we operate as follows. We write out the sum over $\delta$ as a sum over triples $\delta = \nu_l + \nu_m + \nu_n$, taking into account the mirror property,

$$\Upsilon_8(\Omega) = \frac{1}{2} \sum_{1 \leq i < j < k \leq 6} \sum_{l < m \in \{i,j,k\}} \langle \nu_l | \nu_m \rangle \prod_{n \neq l,m} \vartheta[\nu_l + \nu_m + \nu_n]^4 \quad (6.18)$$

Given that a pair $(l, m)$ belongs to 4 different triples, we may rewrite this sum simply as a sum over all pairs $(m, n)$, as follows

$$\Upsilon_8(\Omega) = 2 \sum_{l < m} \langle \nu_l | \nu_m \rangle \prod_{n \neq l,m} \vartheta[\nu_l + \nu_m + \nu_n]^4 \quad (6.19)$$

There are 15 such pairs, giving rise to 15 different contributions to $\Upsilon_8$,

$$\frac{1}{2} \Upsilon_8 = - (1)(4)(5)(8) - (1)(2)(6)(9) + (2)(3)(5)(7) + (2)(3)(6)(8) + (3)(4)(5)(9) - (1)(4)(6)(7) + (2)(4)(7)(9) + (5)(6)(7)(8) - (1)(3)(8)(9) - (1)(3)(7)(0) + (2)(4)(8)(0) + (5)(6)(9)(0) + (7)(8)(9)(0) - (1)(2)(5)(0) + (3)(4)(6)(0) \quad (6.20)$$

Next, we use the Riemann bilinear relations $R[\nu] = \sum_\delta \langle \nu | \delta \rangle \vartheta[\delta]^4 = 0$, each of which is associated with an odd spin structure $\nu$. Each of the 15 terms of $\Upsilon_8$ is in one to one correspondence with one of the 15 linear combinations $R[\nu_i] - \langle \nu_i | \nu_j \rangle R[\nu_j]$ of pairs $\nu_i$ and...
Indeed, for any given pair, there are precisely 4 non-zero terms in each the Riemann relation
\[ R[\nu_i] - \langle \nu_i | \nu_j \rangle R[\nu_j], \]
determined by the non-vanishing of the quantity
\[ \langle \delta | \nu_i \rangle - \langle \delta | \nu_j \rangle \langle \nu_j | \nu_i \rangle = \begin{cases} 2\langle \nu_j | \nu_k \rangle & \text{if } \delta = \nu_i + \nu_j + \nu_k \\ 0 & \text{if } \delta = \nu_i + \nu_k + \nu_j \end{cases} \]  
(6.21)

where \( \nu_i, \nu_j, \nu_k \) and \( \nu_l \) are all distinct. The case where \( \delta \) is the sum of three \( \nu \)'s, all distinct from \( \nu_i \) and \( \nu_j \) is equivalent to the first case above using the mirror property.

There are two cases, depending upon whether \( (1) \) enters the product or not. The corresponding Riemann identities are then of the form
\[ 0 = (1) - (i) - (j) - (k) \]
and
\[ 0 = (h) + (i) - (j) - (k) \]  
(6.22)

for \( h, i, j, k \) pairwise distinct and different from 1, and such that the corresponding products \( (1)(i)(j)(k) \) and \( (h)(i)(j)(k) \) occur in the sum \( \Upsilon_8 \). By placing two terms on the lhs and the other two terms on the rhs and squaring both sides, one gets
\[ + 2(1)(i) + 2(j)(k) = (1)^2 + (i)^2 - (j)^2 - (k)^2 \]
\[ -2(h)(i) + 2(j)(k) = (h)^2 + (i)^2 - (j)^2 - (k)^2 \]  
(6.23)

Taking again the square, we obtain
\[ \pm 8(g)(i)(j)(k) = (g)^4 + (i)^4 + (j)^4 + (k)^4 - 2(g)^2(i)^2 - 2(g)^2(j)^2 \]
\[ -2(g)^2(k)^2 - 2(i)^2(j)^2 - 2(i)^2(k)^2 - 2(j)^2(k)^2 \]  
(6.24)

where the sign on the left is + for \( g = 1 \) and \(-\) for \( g = h \neq 1 \). Using this formula and summing all terms in \( \Upsilon_8 \), we readily get
\[ \Upsilon_8(\Omega) = 2\Psi_8(\Omega) - \frac{1}{2}\Psi_4(\Omega)^2 \]  
(6.25)
and this vanishes by the well-known identity \( 4\Psi_8(\Omega) = \Psi_4(\Omega)^2 \). However, this identity does not follow from the Riemann relations alone.

In conclusion, we observe that the preceding mechanism for enforcing the GSO projection and producing a vanishing cosmological constant provides yet another distinction with the many earlier efforts to treat supermoduli \cite{15, 16, 17, 18, 19, 20, 21, 22} and resolve ambiguities \cite{23, 24, 27} in superstring multiloop amplitudes. In particular, the earlier proposals based on the picture-changing operator Ansatz \cite{15} inserted at various special points had all relied only on Riemann identities to insure both modular invariance and the vanishing of the cosmological constant \cite{26, 27, 28}. \[49\]
7 The Bosonic and Heterotic Strings

To complete our treatment of closed orientable string theories, we shall discuss the chiral measures of the purely bosonic string in 26 dimensions and of the heterotic strings in 10 dimensions.

7.1 The chiral bosonic string measure

A closed expression in terms of modular forms for the bosonic measure was obtained in [31] by exploiting the constraints of modular invariance and the behavior at the boundary of moduli space known on physical grounds. Here, as another illustration of the methods in this paper, we shall present a first principles derivation following the calculational scheme used for the superstring. The starting point is the chiral measure of [13, 9], when bosonic ghosts are inserted at points $p_a$, $a = 1, 2, 3$ and the period matrix is used as moduli coordinates on the surface,

$$d\mu_B(\Omega) = Z_B(\Omega)d\Omega_{11}d\Omega_{12}d\Omega_{22}$$  \hspace{1cm} (7.1)

where the chiral matter-ghost partition function is given by

$$Z_B = \frac{\vartheta(\sum a p_a - 3\Delta) \prod_{a<b} E(p_a, p_b) \prod_a \sigma(p_a)^3}{Z^{27} \det\omega_I \omega_J(p_a)}$$  \hspace{1cm} (7.2)

The chiral partition function is completely independent of the ghost insertion points $p_a$, as may be checked by matching the zeros of the numerator and denominator in $Z_B$. We recall the expression for the chiral partition function $Z$ of a single chiral boson,

$$Z = \frac{\vartheta(z_1 + z_2 - w - \Delta) E(z_1, z_2) \sigma(z_1) \sigma(z_2)}{E(z_1, w) E(z_2, w) \sigma(w) \det\omega_I (z_I)}$$  \hspace{1cm} (7.3)

where the points $z_1$, $z_2$ and $w$ are arbitrary generic points.

We now wish to evaluate the quantity $Z_B$ as a modular form. We proceed as follows. First, we place the three ghost insertion points at three distinct branch points. Each branch point uniquely corresponds to an odd spin structure and the partition of three odd spin structures fixes a unique even spin structure, which we shall denote by $\delta$,

$$\delta = \nu_1 + \nu_2 + \nu_3 \hspace{1cm} p_a = \Delta + \nu_a \hspace{1cm} a = 1, 2, 3$$  \hspace{1cm} (7.4)

Second, we choose the points $z_1$, $z_2$ and $w$ in (7.3) to coincide with the points $p_a$, which may be done in three different ways. Multiplying these three different ways together, and expressing the points $p_a$ in terms of the odd spin structures inside the $\vartheta$-functions, we have

$$Z^9 = \frac{\vartheta(\nu_1 + \nu_2 - \nu_3) \vartheta(\nu_2 + \nu_3 - \nu_1) \vartheta(\nu_3 + \nu_1 - \nu_2) \sigma(p_1) \sigma(p_2) \sigma(p_3)}{E(p_1, p_2) E(p_2, p_3) E(p_3, p_1) \det\omega_I (p_1, p_2) \det\omega_I (p_2, p_3) \det\omega_I (p_3, p_1)}$$  \hspace{1cm} (7.5)
Third, we use the following remarkable identity:

\[ \det \omega I \omega J (p_a) = -\det \omega I (p_1, p_2) \det \omega J (p_2, p_3) \det \omega I (p_3, p_1) \]  

(7.6)

which may be derived by noticing that the $3 \times 3$ determinant on the left hand side is of the Vandermonde form and the right hand side is its product representation. Using (7.5) for one power $Z^9$ in the denominator of (7.7), the determinants cancel and we are left with

\[ Z_B = \frac{\vartheta (\nu_1 + \nu_2 - \nu_3) \prod_{a \neq b} E(p_a, p_b) \prod_a \sigma (p_a)^2}{Z^{18} \vartheta (\nu_1 + \nu_2 - \nu_3) \vartheta (\nu_2 + \nu_3 - \nu_1) \vartheta (\nu_3 + \nu_1 - \nu_2)} \]  

(7.7)

Next, we produce an alternative expression for $Z^3$. Using (3.14) to eliminate the finite-dimensional determinant in $Z^3$ in favor of $\mathcal{M}$, we obtain

\[ Z^3 = \frac{\vartheta (\nu_a + \nu_b - \nu_c) E(p_a, p_b) \sigma (p_a) \sigma (p_b) E(p_a, p_c) E(p_b, p_c) \sigma (p_c) \omega_{\nu_a} (p_a) \omega_{\nu_b} (p_b) \omega_{\nu_c} (p_c) \cdot \mathcal{M}_{\nu_a \nu_b} \cdot \mathcal{M}_{\nu_a \nu_c}}{\prod_{a \neq b} \omega_{\nu_a} (p_b)} \]  

(7.8)

where $c \neq a, b$. Again, given the points $p_a, a = 1, 2, 3$, there are 3 different ways of expressing $Z^3$ this way. Multiplying together all 3 expressions, we get

\[ Z^9 = \frac{\vartheta (\nu_1 + \nu_2 - \nu_3) \vartheta (\nu_2 + \nu_3 - \nu_1) \vartheta (\nu_3 + \nu_1 - \nu_2) \sigma (p_1) \sigma (p_2) \sigma (p_3) \prod_{a \neq b} \omega_{\nu_a} (p_b) \mathcal{M}_{\nu_1 \nu_2} \mathcal{M}_{\nu_2 \nu_3} \mathcal{M}_{\nu_3 \nu_1}}{E(p_1, p_2) E(p_2, p_3) E(p_3, p_1) \prod_{a \neq b} \omega_{\nu_a} (p_b) \mathcal{M}_{\nu_1 \nu_2} \mathcal{M}_{\nu_2 \nu_3} \mathcal{M}_{\nu_3 \nu_1}} \]  

(7.9)

Using this expression to eliminate the remaining factor of $Z^{18}$ in (7.7), we get

\[ Z_B = \frac{\vartheta (\nu_1 + \nu_2 + \nu_3) E(p_1, p_2)^4 E(p_2, p_3)^4 E(p_3, p_1)^4 \prod_{a < b} \omega_{\nu_a} (p_b)^4 \prod_{a \neq b} \omega_{\nu_a} (p_b)^4}{\vartheta (\nu_1 + \nu_2 - \nu_3)^3 \vartheta (\nu_2 + \nu_3 - \nu_1)^3 \vartheta (\nu_3 + \nu_1 - \nu_2)^3 \mathcal{M}_{\nu_1 \nu_2}^2 \mathcal{M}_{\nu_2 \nu_3}^2 \mathcal{M}_{\nu_3 \nu_1}^2} \]  

(7.10)

Evaluating the prime forms, we use their form in terms of $\vartheta$-functions at odd spin structure and we choose to evaluate $E(p_a, p_b)$ with the help of the third odd spin structure $\nu_c$, associated with the third point $p_c, c \neq a, b$. This way, one obtains

\[ E(p_a, p_b)^2 \omega_{\nu_c} (p_a) \omega_{\nu_c} (p_b) = \vartheta [\nu_c] (\nu_a - \nu_b)^2 \]  

(7.11)

Using this result for each of the prime form factors above, we obtain an expression for $Z_B$ in terms of $\vartheta$-constants and bilinear $\vartheta$-constants $\mathcal{M}$ only,

\[ Z_B = \frac{\vartheta (\nu_1 + \nu_2 + \nu_3) \vartheta [\nu_3] (\nu_1 - \nu_2)^4 \vartheta [\nu_1] (\nu_2 - \nu_3)^4 \vartheta [\nu_2] (\nu_3 - \nu_1)^4}{\vartheta (\nu_1 + \nu_2 - \nu_3)^3 \vartheta (\nu_2 + \nu_3 - \nu_1)^3 \vartheta (\nu_3 + \nu_1 - \nu_2)^3 \mathcal{M}_{\nu_1 \nu_2}^2 \mathcal{M}_{\nu_2 \nu_3}^2 \mathcal{M}_{\nu_3 \nu_1}^2} \]  

(7.12)

*The sign factor in this formula corresponds to ordering the pairwise index $IJ$ on the lhs as follows 11, 12, 22; and the single index $I$ on the rhs as follows 1, 2.
Now, all the $\vartheta$-constants in the above relation are proportional to $\vartheta[\delta](0, \Omega)$, with the following proportionality exponential factors,

$$
\vartheta(\nu_1 + \nu_2 + \nu_3) = C[\delta] \vartheta[\delta](0, \Omega)
$$

$$
\vartheta(\nu_a + \nu_b - \nu_c) = C_a[\delta] \vartheta[\delta](0, \Omega)
$$

$$
\vartheta[\nu_a](\nu_b - \nu_c) = \tilde{C}_a[\delta] \vartheta[\delta](0, \Omega)
$$

(7.13)

The exponential factors $C[\delta], C_a[\delta]$ and $\tilde{C}_a[\delta]$ are easily determined; suffice it to know here that the combination that enters into (7.12) reduces to 1 since

$$
C[\delta] \tilde{C}_1[\delta] = \pi \psi_{10}(\Omega)
$$

in view of the property (4.9) of $M$ and the definition (1.4) of the modular form $\psi_{10}(\Omega)$. Notice that the measure $d\mu_B(\Omega)$ is a modular form of weight $-13$, as indeed expected for the bosonic string in 26 dimensions.

### 7.2 The Heterotic String Measure and Cosmological Constant

The heterotic string amplitudes and measure are constructed by assembling, at fixed internal momenta $p_I^\mu$, the chiral amplitude and measure for the left moving (holomorphic) part of the superstring with the chiral amplitude for the right moving (anti-holomorphic) part of the bosonic string, of which 16 dimensions have been compactified on the Cartan tori of $\text{Spin}(32)/\mathbb{Z}_2$ or of $E_8 \times E_8$, [30]. For the two-loop measure, the contribution of the compactified bosons produces a winding contribution which is given by an extra factor of the $\text{Spin}(32)/\mathbb{Z}_2$ or $E_8 \times E_8$ root lattice $\vartheta$-functions. These are modular forms of weight 8 and must thus be proportional to $\psi_{8}(\Omega)$. Thus, the two-loop right chiral heterotic string measure is given by

$$
d\mu_H(\bar{\Omega}) = \psi_{8}(\Omega) \frac{1}{\pi^{12} \psi_{10}(\Omega)} d\bar{\Omega}_1 d\bar{\Omega}_2 d\bar{\Omega}_3
$$

(7.15)

In the fermionic realization, the same factor arises by removing 16 chiral bosons (i.e. multiplying by a factor of $\bar{Z}^{16}$) and adding in 32 Majorana-Weyl fermions (i.e. multiplying by the associated chiral partition function $\psi_{8}(\Omega)/\bar{Z}^{16}$. The heterotic string cosmological constant is given by integration of the product of the measures,

$$
\Lambda_H = \int d\mu_H(\bar{\Omega})d\mu(\Omega) = \int \det^{-5} \text{Im} \Omega \left| \prod_{I \leq J} d\Omega_{IJ} \right|^2 \frac{\psi_{8}(\Omega)}{16\pi^6 \psi_{10}(\Omega)} \psi_{8}(\Omega)
$$

(7.16)

Which also vanishes in view of the fact that $\gamma_8 = 0$. 

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8 Asymptotic Behavior of the Measure

In this last section, we derive the asymptotic behaviors of the chiral superstring measure at fixed even spin structure as the genus 2 surface degenerates, and we interpret the leading behaviors in terms of physical processes. Two types of degenerations may be distinguished. In a given canonical homology basis, we may write the period matrix as

\[ \Omega = \begin{pmatrix} \tau_1 & \tau \\ \tau & \tau_2 \end{pmatrix} \] (8.1)

The two degenerations are then: separating degeneration as \( \tau \to 0 \), with \( \tau_1 \) and \( \tau_2 \) kept fixed; and non-separating degeneration as \( \tau_2 \to +i\infty \), with \( \tau_1 \) and \( \tau \) kept fixed. We investigate these two types of degenerations of the measure in turn. (The degenerations for the bosonic string are well-known [32] and may be recovered from (7.14).)

8.1 Separating Degenerations

We shall be interested in the asymptotics of the measure \( d\mu[\delta](\Omega) \) to leading order. Therefore, we shall need the leading asymptotics given in (5.13), (5.14) and (5.15). Putting all together, we find the following asymptotic behavior of the chiral superstring measure,

\[ d\mu \left[ \frac{\mu_1}{\mu_2} \right] (\Omega) = \frac{1}{2^{10} \pi^8 \tau^2} \langle \mu_1 | \nu_0 \rangle \langle \mu_2 | \nu_0 \rangle \frac{\partial_1[\mu_1]^4(\tau_1) \partial_2[\mu_2]^4(\tau_2)}{\eta(\tau_1)^{12} \eta(\tau_2)^{12}} \ d\tau_1 \ d\tau_2 \ d\tau + O(\tau^0) \]

\[ d\mu \left[ \frac{\nu_0}{\nu_0} \right] (\Omega) = \frac{3\tau^2}{2^6 \pi^4} \ d\tau_1 \ d\tau_2 \ d\tau + O(\tau^0) \] (8.2)

The physical interpretation of the divergences is as follows.

- The measure for the single spin structure \( \delta_0 \) with R sectors on each genus 1 component behaves in a completely regular way as \( \tau \to 0 \). Neither intermediate tachyon nor massless particles arise, and this is as expected.
- Viewed as a measure on the genus 1 components, the limiting measure \( d\mu[\delta_0] \) is also completely regular if we further let \( \tau_2 \to +i\infty \) in the R sector, again as expected.
- The measure for the remaining 9 spin structures \( \delta_i, \ i = 1, \ldots, 9 \), with NS sectors on each genus 1 component exhibits an intermediate tachyon pole in the form \( d\tau/\tau^2 \), as expected. It also exhibits a massless pole, which may be seen by combining the left and right measures and expanding the measure factor \( (\det\text{Im}\Omega)^{-5} \) for small \( \tau \), as expected.
- Viewed as a measure on the genus 1 components, the limiting measure \( d\mu[\delta_i] \) further exhibits a tachyon divergence as \( \tau_2 \to +i\infty \), as expected.
Performing a partial sum over all even spin structures on a single genus 1 component cancels the tachyon divergence as expected as well,

\[
\sum_{i=1,3,5} d\mu_{\left[\frac{\mu_i}{\mu_k}\right]}(\Omega) = \sum_{j=2,4,6} d\mu_{\left[\frac{\mu_l}{\mu_j}\right]}(\Omega) = 0 \quad k = 2, 4, 6; \ l = 1, 3, 5
\]

8.2 Non-Separating Degenerations

We shall be interested in the singular behavior as \(q \to 0\) of the measure \(d\mu[\delta](\Omega)\). Therefore, it suffices to use the asymptotics of the \(\vartheta\)-functions, of \(\Psi_{10}(\Omega)\) and of the objects \(\Xi_6[\delta](\Omega)\) obtained in Section §5,

\[
\begin{align*}
\langle \mu_2 \rangle &= -d\mu_{\left[\frac{\mu_1}{\mu_4}\right]} = \frac{\vartheta_{11}[\mu_1](\tau, \tau_1)^4 + \vartheta_{10}[\nu_0](\tau, \tau_1)^4}{2^8 \pi^6 \cdot q \cdot \eta(\tau_1)^6 \vartheta_{11}[\nu_0](\tau, \tau_1)^2} \ d\tau_1 \ d\tau_2 \ d\tau + \mathcal{O}(q^0) \\
\langle \mu_3 \rangle &= -d\mu_{\left[\frac{\mu_2}{\mu_4}\right]} = \frac{-\vartheta_{11}[\mu_2](\tau, \tau_1)^4 + \vartheta_{10}[\nu_0](\tau, \tau_1)^4}{2^8 \pi^6 \cdot q \cdot \eta(\tau_1)^6 \vartheta_{11}[\nu_0](\tau, \tau_1)^2} \ d\tau_1 \ d\tau_2 \ d\tau + \mathcal{O}(q^0) \\
\langle \mu_5 \rangle &= -d\mu_{\left[\frac{\mu_2}{\mu_4}\right]} = \frac{-\vartheta_{11}[\mu_3](\tau, \tau_1)^4 + \vartheta_{10}[\nu_0](\tau, \tau_1)^4}{2^8 \pi^6 \cdot q \cdot \eta(\tau_1)^6 \vartheta_{11}[\nu_0](\tau, \tau_1)^2} \ d\tau_1 \ d\tau_2 \ d\tau + \mathcal{O}(q^0) \\
\langle \nu_0 \rangle &= d\mu_{\left[\frac{\mu_i}{\mu_6}\right]} = \mathcal{O}(q^0) \quad i = 1, 3, 5
\end{align*}
\]

The physical interpretation of the divergences is as follows.

- The \(1/q\) divergence signals the tachyon passing along the \(B_2\) cycle (i.e. traversing the \(A_2\) cycle) of the second genus 1 component of the degeneration.

- Summing the measures for spin structures \(\mu_2 = [00]\) and \(\mu_4 = [012]\) on the genus 1 component with canonical homology cycles \(A_2\) and \(B_2\) corresponds to a partial GSO projection in the NS sector of the long \(B_2\) cylinder. This partial projection eliminates the tachyon in the \(B_2\) loop and as the cycle \(B_2\) becomes infinitely long the tachyon cancels out, as expected.

- The remaining even spin structures (last line in (8.3)) correspond to Ramond states moving along the \(B_2\) cycle and therefore have no poles, as expected.

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