Entanglement Spectrum as a Probe for the Topology of a Spin-Orbit Coupled Superconductor

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The classification of electron systems according to their topology has been at the forefront of condensed matter research in recent years. It has been found that systems of the same symmetry, previously thought of as equivalent, may in fact be distinguished by their topological properties. Moreover, the non-trivial topology found in some insulators and superconductors has profound physical implications that can be observed experimentally and can potentially be used for applications. However, characterizing a system’s topology is not always a simple task, even for a theoretical model. When translation and other symmetries are present in a quadratic model the topological invariants are readily defined and easily calculated in a variety of symmetry classes. However, once interactions or disorder come into play the task becomes difficult, and in many cases prohibitively so. The goal of this paper is to suggest alternatives to the topological invariants which are based on the entanglement spectrum and entanglement entropy. Using quadratic models of superconductors we demonstrate that these entanglement properties are sensitive to changes in topology. We choose quadratic models since the topological phase diagram can be mapped using the topological invariants and then compared to the entanglement entropy/spectrum features. This work sets the stage for learning about topology in interacting and disordered systems through their entanglement properties.

I. INTRODUCTION

Over the past several years the study of topology in condensed matter systems has become a topic of great interest. While the topological properties of quantum Hall systems where studied since the 80’s[1], topological systems with time reversal symmetry were only predicted[2,3] and realized[4] recently. The introduction of topology into the discussion of solid-state phenomena has revolutionized the classification of materials. For instance, two insulating states in the same dimension and symmetry class, formerly thought of as being equivalent, could have a different topology and are not the same state of matter. This classification is also supported by the direct physical implications of non-trivial topology, namely localized modes on system boundaries[5,6]. These modes are current carrying states on sample surfaces and Majorana fermions in vortex cores of topological superconductors.

In light of the above, it is desirable to assign a label which carries the information about the topology to any system of interest. This is easy to do in a non-interacting system with translation invariance as it is described by a periodic, quadratic Hamiltonian. This label is the ‘topological invariant’, which is an integer number, related to Berry curvature of the system. Loosely speaking, the invariant measures the phase winding of single particle states as the momentum is scanned in the Brillouin zone. For example, in a two dimensional superconductor with broken time reversal symmetry, such as the model in the following discussion, the topological invariant is a Chern number. The Chern number is the integral of the Berry curvature over the Brillouin zone. Calculating the Chern number requires knowing the single particle wave function at any point in the Brillouin zone and the presence of additional symmetries (like mirror or particle-hole) simplify the procedure greatly.

The ease with which one can evaluate the topological invariants in a non-interacting, clean system, unfortunately, does not carry over to dirty and/or interacting systems. While breaking of translation invariance by disorder prevents the use of lattice momentum, interactions invalidate the notion of a single particle wave function altogether. K-theory classification[7,8,9] itself is valid in non-interacting dirty systems and there are formal ways of evaluating the topological invariants for interacting systems. This could be based on flux insertion, similar to Laughlin’s argument for quantum Hall systems[10] or using Green’s function[11,12]. However, these methods are not easy to implement, especially in situations where the ground state (only) is found numerically and given as a superposition of configurations. It is therefore desired to devise an alternative way of distinguishing a topological state from a trivial one in the presence of disorder and interactions and in a way that utilizes the ground state only, without requiring the full spectrum (or Green’s function). In this work we adopt the entanglement entropy and the entanglement spectrum as calculation tools which contain information about the topology of the underlying system. We test these quantities on non-interacting, clean models which display trivial and topological phases. This allows us to calculate the topological invariants directly and compare our results with the classification suggested by the entanglement.

Some pioneering work on topology and entanglement has been carried out in recent years with an emphasis on topological order[13]. We would like to distinguish the current models of interest from the ones displaying topological order. Our topological superconductor is a


The entanglement entropy and spectrum will be defined in the next section. However, before presenting the formal definitions, let us simply note that these include dividing the system in some way to subsystems A and B and tracing out degrees of freedom associated with subsystem B. Therefore, it is to be expected that the entanglement entropy will depend on the length of the boundary between the two subsystems. Indeed, it is generally true that the leading term in the entropy is linear in this length. This is referred to as the area law. Additionally, there can be sub-leading terms with non-trivial dependence on the length of the boundary. For example, there are logarithmic terms in the presence of corners and long range order\(^\text{15}\) and, very interestingly, there are constant sub-leading terms associated with topological order\(^\text{19,20}\). For topological insulators and superconductors (symmetry protected topological states) there are no sub-leading terms associated with topology\(^\text{19}\).

The main results of this paper are as follows. (i) When varying model parameters such that the system changes its topology, the derivatives of the entanglement entropy with respect to model parameters are sharply peaked at the transition. This finding is consistent with a previous observation by Oliveira\(^\text{19}\) on a different model. (ii) The topological superconductors studied here do not violate the area law mentioned above. Any effort to find violations of the area law in this model using a ‘finite’ partition to A and B subsystems are overwhelmed by finite size effects. We therefore adopt a corner-less partition where we establish that any possible violations would have to be due to corners\(^\text{21}\). (iii) Looking at the entanglement spectrum (to be defined later) of the corner-less partition also allows us to establish a connection between the low energy states in the entanglement spectrum of a partition with a prevaling edge and the low energy states of a physical system with an edge. (iv) In a topological superconductor, a topological phase transition can be seen in the entanglement spectrum by varying the partition. This result is similar to that of Hsieh\(^\text{14}\), who looked at topological insulators. Briefly, we find that in the superconductor, if the state is of non-trivial topology, the gap in the entanglement spectrum can be closed by tuning the partition in the proper way, without changing any of the model parameters.

The rest of this paper is organized as follows: In the next section we introduce the entanglement spectrum, entanglement entropy, and our model framework. In Section \[\text{III}\] we present and discuss our results of the entanglement entropy in parameter space while in Section \[\text{IV}\] we study the entanglement entropy as a function of system size. Section \[\text{V}\] contains our partition tuning study and concluding remarks are presented in Section \[\text{VI}\].

### II. MODEL AND METHODS

#### A. The Reduced Density Matrix, Entanglement Spectrum and Entanglement Entropy

We start by defining the reduced density matrix, the entanglement spectrum, and the entanglement entropy. We also discuss how they are obtained relatively simply in a non-interacting system. Starting from a ground state $|\psi\rangle$ one defines the reduced density matrix by dividing the system into two parts, A and B. The reduced density matrix\(^\text{22}\) of subsystem A is then given by

$$
\rho_A = \text{Tr}_B (|\psi\rangle \langle \psi|) \equiv \frac{e^{-H_A}}{Z_A},
$$

where the trace is over all configurations of subsystem B and the above equation serves as the definition of $H_A$, the entanglement Hamiltonian. The entanglement spectrum is defined as the set of eigenvalues $\{E_i\}$ of the entanglement Hamiltonian, $H_A$. $Z_A = \text{Tr}_A (e^{-H_A})$ is the partition function. The entanglement entropy (EE) we choose to work with is the von-Neuman entropy defined by:

$$
S_A = \text{Tr} (\rho_A \log \rho_A)
$$

We now specialize our discussion to the system at hand: a quadratic system with superconductivity. In order to calculate the entanglement spectrum of a system with pairing we appeal to the fact that the entanglement spectrum of a quadratic system is completely determined by its correlations. To do this we generalize a method proposed in Refs.\(^\text{23,24}\). We briefly review the main steps of the method here, adjusted to the case of a superconductor. Consider a state $|\psi\rangle$ which is the ground state of some quadratic Hamiltonian. $|\psi\rangle$ is a Slater determinant of single particle states and therefore obeys Wick’s theorem. Now let us consider averages $C_{i,j} = \langle c_i^\dagger c_j \rangle$ where $i,j$ are both in subsystem $A$. This average must be completely determined by the reduced density matrix $\rho_A$. Moreover, since $|\psi\rangle$ is a determinant all averages must obey Wick’s theorem. Therefore when we write averages like $\langle O_A \rangle = \text{Tr} (\rho_A O_A)$, where $O_A$ is any local operator in subsystem $A$, the trace must also obey Wick’s theorem. From this it follows that $\rho_A$ is an exponent of a quadratic entanglement Hamiltonian. Further, if $|\psi\rangle$ is a ground state with some pairing (i.e. a BCS like wave function) then the anomalous averages $\langle c_i^\dagger c_j \rangle$ must be non-zero. From this it follows that $H_A$ must also contain pairing.

The considerations above lead us to write a general
form for $H_A$ as follows

$$H_A = \sum_{i,j \in A} \left( c^\dagger_{i} h_{i,j} c_{j} + \frac{1}{2} \left( c^\dagger_{i} \Delta_{i,j} c_{j} + \text{h.c.} \right) \right)$$

(3)

where $i, j$ label both site and spin in subsystem $A$. The above Hamiltonian can be written as $H_A = \psi \dagger \mathcal{H} \psi$ where

$$\psi = (c_1 \ldots c_N, c_{1}^\dagger \ldots c_{N}^\dagger)^T.$$ 

The BdG Hamiltonian $\mathcal{H}$ obeys particle-hole symmetry and thus it can be diagonalized as $\mathcal{H} = WDW^\dagger$ where $D = \text{diag}(E_1, \ldots, E_N)$ with $E_i > 0 \ \forall i$ and

$$W = \begin{pmatrix} u & v^\dagger \\ v & u^\dagger \end{pmatrix}.$$ 

(4)

where $u$ and $v$ are matrices in position and spin space. If we now define the correlation matrix

$$G = \begin{pmatrix} \langle c_{i} c_{j}^\dagger \rangle & \langle c_{i} c_{j} \rangle \\ \langle c_{i}^\dagger c_{j} \rangle & \langle c_{i}^\dagger c_{j}^\dagger \rangle \end{pmatrix}.$$ 

(5)

and calculate the averages in terms of traces over $\rho_A$, one can show that $G$ can be represented as $G = WGW^\dagger$ where $G = \text{diag}(I - fI)$ with $f = \text{diag}(n_f(E_1), \ldots, n_f(E_N))$ with $n_f(x) = 1/(1 + e^{x})$. We now make the observation that $G$ and $\mathcal{H}$ are diagonalized by the same transformation. Therefore if we define the first $N$ eigenvalues of $G$ as $\zeta_i = 1 - f(E_i)$ then the entanglement spectrum is obtained by $E_i = \ln \left( \frac{\zeta_i}{1 - \zeta_i} \right)$. Thus the entanglement spectrum is obtained via the following program. Using a ground state $|\psi\rangle$ we calculate $G_{i,j}$ for $i, j$ in subsystem $A$, diagonalize the matrix $G$ and then use its eigenvalues to obtain the entanglement spectrum.

Using the relation between the entanglement entropy and the entanglement Hamiltonian in Eq. (2) and $\zeta_i = 1 - f(E_i)$, we find

$$S_A = - \sum_i (\zeta_i \ln \zeta_i + (1 - \zeta_i) \ln (1 - \zeta_i)),$$

(6)

which is just the entropy of a free fermionic gas with energies $E_i$. For a vanishing correlation length, as expected for an insulator, the entropy has the form

$$S_A = \alpha L - \gamma + O(1/L),$$

(7)

where $L$ is the length of the partition between the two sub-systems. The first term, proportional to $L$ is referred to as the area law and the sub-leading term $\gamma$ is called the ‘topological entanglement entropy’.[19] This term only depends on the topology of the ground state and is thus universal. Since the entanglement Hamiltonian of a $2+1d$ topological system is related to the Hamiltonian of $1+1d$ conformal field theory,[19] one could obtain the above expression by taking the large $L$ limit of the CFT partition function.

For our bulk model we expect $\gamma$ to be zero,[12,23] since our topological state is a symmetry protected one. The assumption of a vanishing correlation length $\xi$ is justified, as long as the characteristic length of each subsystem is large compared to $\xi$. Thus, for a general partition, this limit is inappropriate due to the presence of corners and one gets,

$$\alpha \rightarrow \alpha(\xi), \quad -\gamma \rightarrow -\gamma(L, \xi)$$

(8)

This can then lead to sub-leading terms contributing to the entanglement entropy. In the following sections we find that any deviation from the area law is related to the introduction of corners in the partition and vanish for a corner-less partition.

### B. Quadratic Hamiltonian with Pairing

In our model we look at quadratic states with $p$-wave or $f$-wave pairing. These pairing states are the result adding momentum-spin locking (via spin-orbit coupling) to systems which otherwise tend to pair in the singlet $s$- or $d$-wave channel.[20,21] These systems have translational invariance and can thus be diagonalized in momentum space and therefore their Chern number (the relevant topological invariant) can be calculated exactly. This means, conveniently, that the topological phase diagram is known. We can therefore use this to analyze the results given by the entanglement spectrum and entanglement entropy.

The model we consider is as follows

$$H = T + H_{SO} + H_{SC},$$

(9)

where,

$$T = - \sum_{i,j,\sigma} c^\dagger_{i,\sigma} t_{i,j} c_{j,\sigma},$$

(10)

is the tight binding kinetic energy where $t_{i,j}$ are the hopping amplitudes. Here we take $t_{i,j} = t_{i-j}$ and define its Fourier transform as $\epsilon_k$. For our purposes we choose a simple model of hopping on nearest neighbours which leads to $\epsilon_k = -2t(\cos k_x + \cos k_y)$. Next,

$$H_{SC} = \sum_{k} (\epsilon_k c^\dagger_k \Delta_k c_{-k,+} + h.c.),$$

(11)

where $\Delta_k$ is the superconducting order parameter. In what follows when we refer to the $s$-wave model we mean an order parameter of the form $\Delta_k = \Delta_0$ while $d + id$-wave symmetry means we have used $\Delta_k = \Delta_1 (\cos(k_x) - \cos(k_y)) + i \Delta_2 \sin(k_x) \sin(k_y)$. Please note that the spin orbit coupling may drive the system into the topological $p$- or $f$-wave channels although the order parameter is unchanged. This is due to the projection on the spin-orbit coupled band[23]. The spin-orbit coupling term takes the form

$$H_{SO} = \sum_k \Psi_{k}^\dagger \mathcal{H}_k \Psi_{k},$$

(12)
where \( \Psi_k = (c_{k,\uparrow}, c_{k,\downarrow})^T \), \( \mathcal{H}_k = \mathbf{d}_k \cdot \vec{\sigma} \) (with \( \vec{\sigma} \) a vector of Pauli matrices acting on the spin). \( \mathbf{d}_k \) could in principle take any form which is convenient to describe spin-orbit coupling. Here we choose \( \mathbf{d}_k = (A \sin k_x, A \sin k_y, 2B(\cos k_x + \cos k_y - 2) + M) \) (\( A, B \) and \( M \) are material parameters which describe the various spin-orbit coupling and Zeeman strengths). This choice resembles the spin-orbit coupling term used by Bernevig, Hughes and Zhang\(^{10}\) in the description of 2d topological insulators.

The Hamiltonian \([9]\) satisfies

\[
U_C H^*(-k)U_C^{-1} = -H(k),
\]

where \( U_C \) is a unitary operator \( \sigma_y \otimes I_2 \) in the basis of \((\psi_k, \psi_{-k}^\dagger)\). Since \( U_C^2 U_C = -\mathbb{1}_4 \), this topological superconductor belongs to Class C\(^{12,26,27}\).

One can block diagonalize this hamiltonian by a unitary transformation and the topological number is given by a doubled Chern number. Defining \( \xi_k = c_k - \mu \) the Chern number is given by\(^{12,26,27}\)

\[
C_1 = \frac{1}{i\pi} \log \left[ \frac{Q(0,0)Q(\pi,\pi)}{Q(\pi,0)Q(0,\pi)} \right],
\]

where \( Q(k) = \text{sgn}(|\Delta_k|^2 + \xi_k^2 - d_k^2) \). For our particular model we have \( Q(0,\pi) = Q(\pi,0) \), regardless of parameters. We are therefore left with

\[
C_1 = \frac{1}{i\pi} \log \left( \text{sgn}\left[ (|\Delta_0|^2 + \xi_0^2 - d_0^2)(|\Delta_{\pi}|^2 + \xi_{\pi}^2 - d_{\pi}^2) \right] \right),
\]

where \( Q = (\pi,\pi) \). Using the above formulation we can map the topological phase diagram of the superconductor described by the Hamiltonian \( H \).

III. TOPOLOGICAL PHASE BOUNDARY AND THE ENTANGLEMENT ENTROPY

When plotting the entanglement entropy and its derivatives with respect to the model’s spin-orbit coupling parameters we see the following intriguing property. The topological phase boundaries of our model coincide with “kinks” in the entanglement entropy. That is, there’s a change in behavior of the entanglement entropy at the transition from a trivial superconductor to a topological superconductor or vice versa. Moreover, these kinks lead to a very strong peak in the derivative of the entanglement entropy with respect to material parameters. To make a rather loose analogy with standard thermodynamic variables, the transition appears to be a second order phase transition. A similar property was found in a spin-orbit coupled triplet superconductor in Ref.\(^{10}\).

In general, phase transitions between states of different topology but the same symmetry are not characterized by an order parameter. The entanglement entropy in this case serves as a substitute to a thermodynamic potential and exhibits a kink at the transition. One may expect that exactly at the transition the bulk gap should close, giving rise to that kink.

In Fig. 1 we present a cut through the phase diagram, where only the spin-orbit coupling parameter \( B \) is changed. In panel (a) we see that the behavior of the entanglement entropy changes abruptly at about \( B = 0.6t \). This change is more apparent in the derivative of \( S_A \) in panel (b). Checking with the Chern number calculated above, we expect a topological phase transition for this choice of parameters at \( B = 0.6t \); precisely where this peak occurs. \( B < 0.6t \) and \( B > 0.6t \) correspond to the trivial and the topological phases respectively. One might expect that a trivial phase has a smaller value of the entanglement entropy than that of a topological phase because of the absence of the mid-gap entanglement states.

![FIG. 1: Plot of the entanglement entropy across the phase boundary for the d-wave system. Figure (a) shows the entanglement entropy \( S_A \) for subregion A a square of side length 12, figure (b) gives \( \frac{\partial S_A}{\partial B} \) for the same geometry and figure (c) plots the bulk energy gap as a function of \( B \). In the figure we have fixed \( \mu = 0, A = 0.25t, M = 0.8t, \Delta_1 = 0.8t \) and \( \Delta_2 = 0.4t \). \( B/t = 0.6 \) is the critical point and \( B/t < 0.6 \) \((B/t > 0.6)\) corresponds to the trivial (topological) phase. Notice that the entanglement entropy takes larger values in the trivial phases. This result is different than that of the s-wave topological superconductor shown in Fig. 2.](image_url)
However, the entanglement entropy of the d-wave superconductor shows the opposite result: the trivial phase has a larger value of the entanglement entropy. This suggests that in general the leading term of the entanglement entropy cannot be used alone to distinguish trivial phases from topological phases. However, it does change abruptly at the transition. In Fig. 1 we have plotted the bulk gap of our full (unpartitioned) system. The most noticeable feature of the gap is that it closes at $B = 0.6t$, as is necessary for a topological phase transition. One may also note that the maximum value of $S_A$ occurs around $B = 0.48t$. While we are presently not certain about the origin of this maximum, we may speculate that it is related to some correlation length increase which approaches the system size at $B = 0.48t$, before the true transition at $B = 0.6t$.

To further explore this behavior we plot $S_A$ and its relevant partial derivatives in parameter space and compare its behavior to the expected phase boundaries. First we explore this for a d-wave superconductor. We fix $\mu = 0, A = 0.25t, \Delta_1 = 0.8t$ and $\Delta_2 = 0.4t$ and explore $M - B$ space. For this specific choice of parameters and focusing on positive values of $B$, we expect a topological phase boundary along the line $B/t = M/8t + 0.5$. We have generated data for $S_A$, $\partial S_A/\partial M$, and $\partial S_A/\partial B$ for this particular choice of parameters, these data are presented in Fig. 2.

Studying Fig. 2, we see a fundamental change in the behavior of the entanglement entropy across the phase boundary line $B/t = M/8t + 0.5$. The entropy is large in the trivial phase ($B/t < (M/8t+0.5)$) and then decreases to a lower and much slower changing value across the phase boundary line. This sudden change is more transparent in the derivatives of the entanglement entropy as panels (b) and (c). We see in both of these figures that the derivatives are comparatively small away from the phase boundary lines and increase substantially as these critical points are approached. The exact position of the peak in the derivatives is better seen in the $B$ derivative, as the phase boundary is rather shallow along lines of fixed $B$ which limits our resolution in the $M$ derivative data. Focusing on the plot of $\partial S_A/\partial B$, one can see a line that is formed by looking for the maximum value of $\partial S_A/\partial B$ for a given value of $M$. Fitting this line gives, to 3 decimal places, a slope of 0.125 and an intercept of 0.500, providing a rather convincing case that $\partial S_A/\partial B$ is peaked along the line $B/t = M/8t + 0.5$.

To further study these peaks and also to provide evidence that this behavior isn’t unique to the d-wave system, we have also studied the parameter space dependence of $S_A$ in a system with s-wave superconductivity. Here we have chosen parameters such that we make as close a connection as possible with the model of Sau et al in Ref. [31]. We therefore set $B = 0$ and define $\tilde{\mu} = \mu + 4t$. In this case our model reduces to that of Ref. [31] when the continuum limit is taken.

Using $B = 0$, $\tilde{\mu} = \mu + 4t$, Eq. (15) and assuming $64t^2 > -\Delta^4_1 + M^2 - \tilde{\mu}^2 + 16t\tilde{\mu}$, the Chern number is simplified to

$$C_{1,s} = \frac{\log(\text{sgn} [\Delta_1^4 + \tilde{\mu}^2 - M^2])}{4\pi},$$

where the subscript $s$ denotes s-wave. It then follows that if $\Delta_1^4 + \tilde{\mu}^2 - M^2 < 0$ the system is topological. Thus the topological phase boundary is defined by the equation $\Delta_1^4 + \tilde{\mu}^2 = M^2$.

We choose to fix $\tilde{\mu} = 0$ and study the resulting behavior in the $M$-$\Delta_0$ plane. According to the Chern number we should see phase boundaries at $\Delta_0 = \pm |M|$. Indeed, we
FIG. 3: Plot of the entanglement entropy and its derivatives in $M - \Delta_0$-space for an s-wave system. From left to right we have the entropy, (a) $S_A$, (b) $\frac{\partial S_A}{\partial M}$, and (c) $\frac{\partial S_A}{\partial \Delta_0}$. In all figures we have picked a subregion $A$ a square of side length 20 and fixed $\mu = 4t$, $A = 0.25t$ and $B = 0$. $|\Delta_0| = |M|$ is the phase transition line and $|\Delta_0| > |M|$ ($|\Delta_0| < |M|$) corresponds to the trivial (topological) phase. Notice that the entanglement entropy take larger values in the topological phase, which is opposite to the $d$-wave case.

see strong indications of a phase boundary along this line. This behavior isn’t overtly obvious in the entanglement entropy in Fig. 3a, however upon taking derivatives of the data with respect to $M$ and $\Delta_0$ it becomes more apparent. This can be seen in Figs. 3b and 3c, where strong peaks appear along the lines $\Delta_0 = M$ and $\Delta_0 = -M$. Thus we have a second clear indication that $S_A$ changes its behavior across topological phase transition. Comparing to Fig. 2, this demonstration has come from not only a different order parameter symmetry but also from varying a different parameter.

IV. FUNCTIONAL DEPENDENCE OF THE ENTANGLEMENT ENTROPY

The study of the functional dependence of the entanglement entropy $S_A$ on the ‘surface area’ of a partition $A$ enables one to make conclusions about the ground state of the system. Violations of the area law have been studied extensively for a variety of different models in different dimensions (see [32] for a review) and depend on the particular model and ground state under investigation. An example of this in two-dimensional fermionic models can be found in Ref. [18]. This work shows that in particular models with a spontaneously broken continuous symmetry, the Goldstone mode causes the entanglement entropy to have a sub-leading corner correction proportional to $\ln L$, where $L$ is the circumference of the partition. Additionally, for two-dimensional critical fermionic models, one also expects a logarithmic term, not associated with corners [25].

One difficulty in analyzing the area law is that the circumference of the partition in a lattice model is not uniquely defined. In our calculations we chose the boundary as the line that divides the distance between the outer layer of the partition and the first layer of the complement into half.

As the sub-leading nature of these corrections makes it very hard to see them directly in the entanglement entropy, we look at the quantity

$$S_{\text{sub}}(L) = LS_{L+1} - (L + 1)S_L,$$  \hfill (17)

in which the leading linear term is eliminated. In the case of only a constant sub-leading term and in the limit of large $L$, $S_{\text{sub}} \propto \text{const}$. For a logarithmic term, the behavior is $S_{\text{sub}} \propto \ln L$, whereas for a power law we have $S_{\text{sub}} \propto L^\eta$ for some exponent $\eta$. We will also study the

FIG. 4: Upper panel: Schematic plot of the different shapes used for the partition of the system. From left to right: square, cross and (reflected) L-shape. Lower panel: Plot of the linear coefficient in the entanglement entropy for $d$-wave coupling for $M = 0$, $\mu = 0$, $\Delta_1 = 0.8t$, $\Delta_2 = 0.4t$, and $A = 0.25t$ by varying $B$ for a square (red plus), an L-shaped (green cross), a cross shaped partition (blue star) as well as the left right partition (pink square). For this $M$-value the critical point is at $B = 0.5$. 

**FIGURE 3**: Plot of the entanglement entropy and its derivatives in $M - \Delta_0$-space for an s-wave system. From left to right we have the entropy, (a) $S_A$, (b) $\frac{\partial S_A}{\partial M}$, and (c) $\frac{\partial S_A}{\partial \Delta_0}$. In all figures we have picked a subregion $A$ a square of side length 20 and fixed $\mu = 4t$, $A = 0.25t$ and $B = 0$. $|\Delta_0| = |M|$ is the phase transition line and $|\Delta_0| > |M|$ ($|\Delta_0| < |M|$) corresponds to the trivial (topological) phase. Notice that the entanglement entropy take larger values in the topological phase, which is opposite to the $d$-wave case.

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**FIGURE 4**: Upper panel: Schematic plot of the different shapes used for the partition of the system. From left to right: square, cross and (reflected) L-shape. Lower panel: Plot of the linear coefficient in the entanglement entropy for $d$-wave coupling for $M = 0$, $\mu = 0$, $\Delta_1 = 0.8t$, $\Delta_2 = 0.4t$, and $A = 0.25t$ by varying $B$ for a square (red plus), an L-shaped (green cross), a cross shaped partition (blue star) as well as the left right partition (pink square). For this $M$-value the critical point is at $B = 0.5$. 

**FIGURE 3**: Plot of the entanglement entropy and its derivatives in $M - \Delta_0$-space for an s-wave system. From left to right we have the entropy, (a) $S_A$, (b) $\frac{\partial S_A}{\partial M}$, and (c) $\frac{\partial S_A}{\partial \Delta_0}$. In all figures we have picked a subregion $A$ a square of side length 20 and fixed $\mu = 4t$, $A = 0.25t$ and $B = 0$. $|\Delta_0| = |M|$ is the phase transition line and $|\Delta_0| > |M|$ ($|\Delta_0| < |M|$) corresponds to the trivial (topological) phase. Notice that the entanglement entropy take larger values in the topological phase, which is opposite to the $d$-wave case.

see strong indications of a phase boundary along this line. This behavior isn’t overtly obvious in the entanglement entropy in Fig. 3a, however upon taking derivatives of the data with respect to $M$ and $\Delta_0$ it becomes more apparent. This can be seen in Figs. 3b and 3c, where strong peaks appear along the lines $\Delta_0 = M$ and $\Delta_0 = -M$. Thus we have a second clear indication that $S_A$ changes its behavior across topological phase transition. Comparing to Fig. 2, this demonstration has come from not only a different order parameter symmetry but also from varying a different parameter.

**IV. FUNCTIONAL DEPENDENCE OF THE ENTANGLEMENT ENTROPY**

The study of the functional dependence of the entanglement entropy $S_A$ on the ‘surface area’ of a partition $A$ enables one to make conclusions about the ground state of the system. Violations of the area law have been studied extensively for a variety of different models in different dimensions (see [32] for a review) and depend on the particular model and ground state under investigation. An example of this in two-dimensional fermionic models can be found in Ref. [18]. This work shows that in models with a spontaneously broken continuous symmetry, the Goldstone mode causes the entanglement entropy to have a sub-leading corner correction proportional to $\ln L$, where $L$ is the circumference of the partition. Additionally, for two-dimensional critical fermionic models, one also expects a logarithmic term, not associated with corners [25].

One difficulty in analyzing the area law is that the circumference of the partition in a lattice model is not uniquely defined. In our calculations we chose the boundary as the line that divides the distance between the outer layer of the partition and the first layer of the complement into half.

As the sub-leading nature of these corrections makes it very hard to see them directly in the entanglement entropy, we look at the quantity

$$S_{\text{sub}}(L) = LS_{L+1} - (L + 1)S_L,$$  \hfill (17)

in which the leading linear term is eliminated. In the case of only a constant sub-leading term and in the limit of large $L$, $S_{\text{sub}} \propto \text{const}$. For a logarithmic term, the behavior is $S_{\text{sub}} \propto \ln L$, whereas for a power law we have $S_{\text{sub}} \propto L^\eta$ for some exponent $\eta$. We will also study the
dependence of the entanglement entropy on the geometry of the partition. To this end, we will look at a square partition, a cross shaped partition and an L-shaped partition (see the upper panel of Fig. 4). As suggested earlier by other authors\cite{21}, this constant is an effect of the corners, where the dimensions of the partition are of the order of the correlation length. Thus, we would expect to find a constant ratio of the constant of a cross (L-shaped) partition with the constant of a square partition to be 3 (1.5). And indeed, throughout the trivial phase (far away from the critical point), we find the ratios of the constants to be $c_{\text{cross}}/c_{\text{square}} \approx 3$, and $c_{\text{cross}}/c_{\ell 4} \approx 1.5$, as expected for a system with zero topological entanglement entropy, $\gamma$. The topological phase, unfortunately, is not reachable in this approach due to finite size effects.

Near a topological phase boundary one must exercise caution when analyzing the functional dependence of $S_A$ on the system boundary size $L$. As the system nears the phase boundary the correlation length grows and so finite size effects become very large. For partitions such as those in Fig. 4, these finite size effects become important as we are technically limited to modest sized subsystems by the computational time and memory required to diagonalize the matrix $G$ in subsystem $A$. Using a reasonable amount of memory limits our system size to have a maximum side length in the range 50 – 60. Thus, when the correlation length is large we do not have the ability to make our subsystem large enough to see the finite size effects subside. If one is not careful one could misinterpret the finite size effects in this region as some sort of non-trivial subleading contribution to $S_A$, such as log $L$ or $L^\gamma$. Note that some authors (e.g. Ding et al\cite{25}) find area law violations in the topological phase. We suspect that these may be associated with corners and will vanish in a corner-less geometry.

To further illustrate our observation that any subleading terms to $S_A$ for our system are artifacts of corners and at the same time show just how important finite size effects become with an increased correlation length, we have looked at the entanglement entropy of a ‘corner-less’ partition. We obtain such a partition by defining system $A$ to be a ring on our torus. If the torus dimensions are $L \times L_4$ where $L_4$ is the longer dimension wrapped around the doughnut hole then our ring dimensions are $L \times l$ and we take $l = L/4$. The boundary of $A$ is then varied by varying the entire system size. Besides having no corners this partition has the advantage that translation symme-
try along the ring’s azimuthal direction is conserved.

Our results for this type of subsystem are illustrated in Fig. 5(b). The first striking feature we note is that $S_{\text{sub}}$ converges to zero for large $L$ for all parameter choices. This leads to the conclusion that any subleading terms we have seen above must be a result of corners and subsequently that all subleading behavior beyond the area law for $S_A$ is zero. This is consistent with our observation that the topological entanglement entropy for this system should always be zero.

The second purpose of Fig. 5(b) is to illustrate the importance of finite size effects when looking at area laws for spin-singlet superconductors. We see that as the spin-orbit parameter $B$ is increased $S_{\text{sub}}$ acquires a damped oscillatory behavior as a function of $L$. For larger $B$ the amplitude and decay length of these oscillations increase.

The way in which $L$ is changed for this partition requires changes both the boundary length of subsystem $A$ and the total system size. Thus inherent in $S_{\text{sub}}$ are both finite size effects from the fact that $S_A$ depends on the total system size (for smaller lattice sizes, before the thermodynamic limit is reached) and finite size effects from non-area law behavior in $S_A$. The system size required to overcome these effects increases with $B$. We see that even for $B = 0.7t$ a very large system size is required before finite size effects vanish. This system size is certainly unreachable using partitions with corners, such as those in Fig. 4.

Another indication for finite size effects can be seen in the inset of Fig. 5(b), where we show the subleading correction to the entanglement entropy right at the critical point. It displays oscillatory behavior with a very large amplitude which increases with $L$. At this point, the correlation length diverges and finite size effects have an even more drastic effect.

Let us discuss another interesting characteristic of the corner-less partition introduced above. This partition introduces an artificial boundary into the system and therefore we are able to probe boundary physics in a bulk model by looking at the entanglement spectrum of subsystem $A$. As a first look at this let us think about a simple $s$-wave model (whose topological phase is a $p$-wave superconductor). If we were to introduce a boundary into the $s$-wave system we would expect to see a zero energy edge mode when the Chern number of the system is 1 and no edge mode when the Chern number is zero. We can see this same physics in the bulk model by looking at the entanglement spectrum of the $A$ subsystem. To illustrate this we have plotted the spectrum in the trivial phase and in the topological phase by properly changing parameters. Our results are presented in Fig. 6. We see quite unmistakably the development of a zero mode upon crossing into the topological region. This zero mode is localized on the boundary of subsystem $A$, as is shown in the inset of Fig. 6.

Finally we explore the edge physics of the bulk $d$-wave model (whose topological phase is a $p$- or $f$-wave superconductor). The solution of a $d$-wave system with an edge results in a spectrum which is slightly more complicated than the one above for the $s$-wave case. From a topological standpoint one expects to see an even number (odd number) of zero energy states when the topology of the system is trivial (non-trivial). We have compared the low-lying states in our corner-less partition entanglement spectrum to those of a physical system with a boundary found, for example, in Reference 31. We find consistency between the two with respect to the number of zero energy states, their position in $k$ space as well as their low energy dispersion. A representative example of our results is shown in Fig. 7. Our choice of parameters is such that the low lying states of this plot should be compared with those of Fig. 5II of Ref. 31.

Another interesting feature of the data in Figs. 6 and 7 is the nature of the eigenstate itself, both at and away from zero energy. At lower energies the wave functions are very localized on the edges of the system, with localization length increasing with energy. This state, however, does not become truly delocalized at any energy.

A second interesting feature of the eigenstates comes from studying the $E = 0$ states and, in particular, looking for Majorana modes. We note that in our finite lattice it is futile to look for a single Majorana state, as these modes must come in pairs in a finite system. The next best option is to look for pairs of Majorana states that are spatially separated and reside on opposite sides of the
partition.
To begin we notice that the entanglement spectrum exhibits particle hole symmetry, that is to say if $|\psi\rangle$ is an eigenstate with energy $E$ then $(A|\psi\rangle)^T$ is an eigenstate with energy $-E$, where $A = I \otimes \sigma_x$ with $\sigma_x$ acting on Nambu space and $I$ is the identity on a space of lattice sites and spin. This leads to the observation that at $E = 0$, $|\psi\rangle$ and $(A|\psi\rangle)^*$ are degenerate eigen states. All eigenstates at $E = 0$ are highly localized on the boundary of the system, an example of this is the state plotted in the inset of Fig. [6]. Looking for Majorana zero energy states then becomes a task of looking for linear combinations of $|\psi\rangle$ and $(A|\psi\rangle)^*$ that give states localized at opposite ends of the system and obey the following condition: given two generic linear combinations

$$|\phi_{M,i}\rangle = \alpha_{1,i}|\psi\rangle + \alpha_{2,i}(A|\psi\rangle)^* = (u_i, v_i)^T,$$

where $u_i$ and $v_i$ are themselves vectors (each with dimension of one half the dimension of subsystem $A$) we require $u_i = v_i^*$.

As an example we have studied the gap closure in the d-wave spectrum in Fig. [7a] at $k_y = 0$ in detail. Our numerical results give two states with very small (approximately $10^{-12}$). Treating these two states as degenerate it is possible to form two linear combinations of them which we denote $|M_1\rangle$ and $|M_2\rangle$. We have plotted the density of these states in Figs. [7b] and [7c]. These density of these states on a lattice site $n$ is defined as

$$n_i(n) = |u_i^\dagger(n)|^2 + |u_i^\dagger(n)|^2 + |v_i^\dagger(n)|^2 + |v_i^\dagger(n)|^2,$$

where $u_i(n)$ is the $n^{th}$ entry in the vector $u_i$ and it itself a 2-component object (spin-up and spin-down) and the label $i = 1, 2$ denotes which state we are interested in. Note that this definition is also used in the inset of Fig. [6]. The two combinations $|M_1\rangle$ and $|M_2\rangle$ are highly localized on the respective boundaries of subsystem $A$. Averaging the modulus of the difference between $u_i$ and $v_i^*$ of both of these states over every lattice site (and spin projection) in subsystem $A$ gives a result which is of order $10^{-5}$. Thus these two states are localized on different boundaries and (to a high numerical precision) satisfy the Majorana condition $u_i = v_i^*$.

V. PARTITION INDUCED GAP CLOSURE

Another way to look for non-trivial topology in the system is the analysis of the entanglement spectrum (ES). As we’ve seen above, it is important to specify what kind of partition is used and which information is to be extracted from the ES. For example, by partitioning a gapped system into a left and a right part, the low entanglement spectrum is similar to the excitation spectrum near a physical boundary, as seen above. Thus, such a partition provides information about edge states. This seems natural since the entanglement entropy is dependent on the boundary $L$ of the partition and measures the entanglement across the boundary. In fact, states in the ES that are connected to bulk degrees of freedom tend to lie very high in the spectrum of such a partition and barely contribute to the entanglement entropy.

Nonetheless, one may extract information about the bulk by defining ‘extensive partitions’. These partitions divide the system into two parts such that the boundary between the two extends throughout the whole system in every direction. Thus, the partition forms a new superlattice which is in itself periodic. By using this kind of partition, the low energy part of the ES resembles the physical spectrum of the bulk.

As an example for such an extensive partition, one can imagine partitioning the system into a number of small ‘islands’ $B$ and the remaining ‘sea’ $A$, as shown in Fig. 8a. If we start in a topologically non-trivial ground state $|\psi\rangle$ and imagine $B$ to be the empty set, we can see that the ground state for subsystem $A$ will be topologically non-trivial as it is identical to the full ground state $|\psi\rangle$. When we now increase the number of small islands in subsystem $B$, the ground state of $B$ will be a simple product state and thus be trivial, while the ground state for $A$ will still be topologically non-trivial. Additionally, both of these ground states will be gapped. We can now imagine that we further increase the number of islands and move them closer together such that they touch each other on part of the edges thus turning subsystem $B$ into the sea and subsystem $A$ into isolated islands. In this case, $B$ will have a topologically non-trivial ground state while $A$ is trivial. Thus, at some point between where the entire system is $A$ and the case where $A$ is a set of islands, there must exist a quantum phase transition where the topology of system $A$ changes. The case made in [17] is that this transition must be facilitated by a closing of the gap in the entanglement spectrum of subsystem $A$ and this gap closing must occur at a ‘symmetric tuning point’. This point is exactly where subsystems $A$ and $B$ can be transformed into each other (i.e., $A \to B$ and $B \to A$) by a symmetry operator.

In Fig. 9, we have sketched an extensive partition while the symmetric point is shown in Fig. 8a. In both cases, for a $d$-wave as well as an $s$-wave SC, the ES in the asymmetric case is gapped, as can be seen for the case of a $d$-wave SC in Fig. 9 independent of the phase the system is in.

When reaching the symmetric limit and when the model parameters are such that the full system is topologically non-trivial, the ES gap closes as expected from our earlier discussion. It is important to note that the gap closing is solely caused by the change of the partition while the model parameters remain unchanged. Using the fact that the symmetric partition forms a superlattice, one can define $k$-vectors with respect to the superlattice and arrange the states in the ES with momentum. The ES can then be compared to the spectrum of an unpartitioned system, whose parameters are set to the critical point. The result can be seen in the lower part.
VI. CONCLUSION

In this paper we showed that the entanglement entropy and entanglement spectrum can be useful in detecting the phase topological transition between phases of the same symmetry and different topology. We specialize to the case of superconductivity with spin orbit coupling where models of both s-wave and d-wave order parameter have a topologically non-trivial and trivial phases. We have analyzed the dependence of a bipartite partition on the circumference of the partition and found a dependence of the form \( S(L) = \alpha L + \ldots \), where the first term is the celebrated area law and the dots stand for sub-leading terms. The coefficient \( \alpha \) was found to have a sharp kink right at the phase transition such that it captures the transition very clearly. In the trivial phase, the only sub-leading term was found to be a constant caused by corner effects. Meanwhile, the topological phase is not easily classified using a finite system (due to finite-size effects) and we must defer to a corner-less system where the area law is obeyed. We conclude that any non-area law contributions in the finite system must be due to corners. As expected, throughout all phases the topological entanglement entropy, \( \gamma \), was found to be zero.

Another signature of the topology of the system can be found by looking at the entanglement spectrum. Depending on the choice of partitioning one may obtain different topological properties of the entanglement Hamiltonian. A phase transition between the topological and the trivial phase can be seen as a gap closure in the entanglement spectrum. This is obtained by changing the extensive partitioning while leaving the physical parameters unchanged. This property is related to the non-trivial topology of the underlying state. Moreover, this finding implies that one has to apply special care when using the entanglement spectrum to extract information about the ground state of a physical system as it can undergo a phase transition while the physical system does not.

The method applied in this study relies on the fact that our Hamiltonian is quadratic. This is needed in order to ensure that the entanglement Hamiltonian is quadratic as well and can be diagonalized by the algorithm applied here. It would be highly desirable to extend this to inter-
acting systems, where an analytic expression for the topological invariant of the system might not be available. Furthermore for systems without translation invariance, the entanglement entropy might be able to give an indication of a phase transition where the usual methods for computing a topological invariant cannot be applied.

It is important to note the versatility of this approach; it can be applied to all quadratic models with or without translational invariance where in the latter case the system sizes are limited by computational power. The use of various forms of partitions leads to a consistent picture of the different topological phases of a system, as shown for a spin-orbit coupled superconductor with \( d + id\)- and \( s\)-wave coupling.

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The charged topological entanglement entropy, which is a universal sub-leading term of the charged entanglement entropy, distinguishes symmetry protected topological phases from trivial phases. On the other hand, as we show below, the leading term is sensitive to the topology, as might be expected, since the entanglement entropy is closely related to correlations.