NONUNIFORM ALMOST REDUCIBILITY OF NONAUTONOMOUS LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We prove that a linear nonautonomous differential system with nonuniform hyperbolicity on the half line can be written as diagonal system with a perturbation which is small enough. Moreover we show that the diagonal terms are contained in the nonuniform exponential dichotomy spectrum. For this purpose we introduce the concepts of nonuniform almost reducibility and nonuniform contractibility which are generalization of this notions originally defined in a uniform context.

1. Introduction

Given a linear operator $T$, to find an ordered basis in which $T$ assumes an especially simple form is a classic problem in linear algebra. When working in finite dimensional spaces, this problem has a strong relation with the study of the dynamics of a linear differential equation

\begin{equation}
\dot{x} = A(t)x.
\end{equation}

1.1. Autonomous and Nonautonomous contexts. When $A(t) = A$, the knowledge of the real part of eigenvalues of $A$ allows us to construct the stable and unstable invariant manifolds and the resulting canonical form of $A$ gives some insights into the form of the solutions of (1).

In a nonautonomous context, the problem of finding a simpler form of the matrix $A(t)$ and as a consequence to study the qualitative behavior of (1) is a more delicate task. In fact, contrarily to the autonomous case, the eigenvalues analysis does not always allow any conclusion over the stability of the solutions, and thus alternative focusing must be considered.

A first approach in this direction was given by G. Floquet [9], which established that a periodic system can be transformed into a constant coefficients system. Floquet’s result can be seen as an example of the properties of kinematical similarity and reducibility, which refers that a linear system (1) can be transformed into

\begin{equation}
\dot{y} = B(t)y
\end{equation}

through a Lyapunov transformation $x = L(t)y$.

The problem to obtain a simpler form to (1) has been tackled by using the concept of reducibility by O. Perron in [15], which proves that (1) can be reduced...
via unitary transformation to a system \(^{(2)}\) where \(B(t)\) has a triangular form whose diagonal coefficients are real. Moreover, under subtle technical conditions it can be proved that \(B(t)\) has a block–triangular form consisting of blocks whose diagonal coefficients are real.

We have mentioned that an eigenvalues–based approach has several shortcomings and is not an adequate tool to cope with stability issues in the nonautonomous framework. A tool that emulates the role of the eigenvalues in this context was developed in terms of the property of uniform exponential dichotomy (a type of nonautonomous hyperbolicity), namely, the Sacker–Sell spectrum associated to \(^{(1)}\), which is the set

\[
\sigma(A) = \{\lambda \in \mathbb{R}: \dot{x} = (A(t) - \lambda I)x \text{ has not uniform exponential dichotomy on } J \subset \mathbb{R}\}.
\]

This spectrum plays a fundamental role in a better localization of diagonal terms when the system \(^{(1)}\) can be transformed to a diagonal one. In fact, B. F. Bylov in \(^{(1)}\) introduced the notion of almost reducibility, i.e., reducibility with a negligible error and proved that any linear system is almost reducible to some diagonal system with real coefficients. Later, F. Lin in \(^{(13)}\) improves the Bylov’s result by proving that the diagonal coefficients are contained in the Sacker–Sell Spectrum. Moreover, F. Lin proved that this spectrum is the minimal compact set where the diagonal terms belong, this phenomenon is known as the contractibility of a linear system.

We emphasize that these concepts of reducibility and almost reducibility also have a vast literature as well as in the uniform hyperbolicity \(^{(5}, \, (12)\) or in Schrödinger operators \(^{(10)}\).

1.2. Structure and novelty of the article. The section 2 introduces the concepts of nonuniformly almost reducible and nonuniformly contractible systems, both notions are the generalizations of the ideas of almost reducibility and contractibility previously mentioned. Instead of using the set \(\sigma(A)\), we use the spectrum of the nonuniform exponential dichotomy introduced in \(^{(8}, \, (20)\). The main result of section 3 states that if the linear system \(^{(1)}\) verifies a subtle condition of nonuniform hyperbolicity on \(J = \mathbb{R}^+_0\), then this system is uniformly contracted to the spectrum of nonuniform exponential dichotomy (a formal definition will be given later). The section 4 deals with preparatory Lemmas to obtain the main result, which is proved in the section 5. Finally, in section 6, we give an application of our principal theorem.

Our main result is a generalization of the Lin’s work. In spite that its proof follows the lines of \(^{(13)}\), it is worth to stress that, compared with the uniform case, the nonuniform behavior of the solutions of \(^{(1)}\) combined with our restriction to \(J = \mathbb{R}^+_0\) arises technical subtleties and bulky technicalities (namely conditions \((C1)–(C4)\) in the section 5) in order to obtain the desired result, which deserve interest on itself.

2. Preliminaries

We consider the linear system \(^{(1)}\) with \(x\) as a column vector of \(\mathbb{R}^n\) and the matrix function \(t \mapsto A(t) \in \mathbb{R}^{n \times n}\) with the following properties:

\begin{itemize}
  \item[(P1)] For \(\mu, M > 0\), \(\|A(t)\| < M \exp(\mu t)\) for any \(t \in \mathbb{R}^+_0\).
\end{itemize}
(P2) The evolution operator $\Phi(t, s)$ of (1) has a nonuniformly bounded growth \((20)\), namely, there exist constants $K_0 \geq 1$, $a \geq 0$ and $\varepsilon \geq 0$ such that
\[
\|\Phi(t, s)\| \leq K_0 \exp(a|t-s|+\varepsilon s), \quad t, s \in \mathbb{R}_0^+,
\]
where $\|\cdot\|$ denotes a matrix norm and
\[
x(t) = \Phi(t, s)x(s), \quad \Phi(t, s)\Phi(s, \tau) = \Phi(t, \tau), \quad \text{for all } t, s, \tau \in \mathbb{R}_0^+.
\]

The purpose of this article is to study the nonuniform contractibility or nonuniform almost reducibility to a diagonal system. Namely, the $\delta$-nonuniform kinematical similarity of (1) to
\[
\dot{y} = U(t)y,
\]
when $U(t) = C(t) + B(t)$, $C(t)$ is a diagonal matrix and $B(t)$ has smallness properties which will be explained later.

**Definition 1.** (20) The system (1) is nonuniformly kinematically similar (resp. $\delta$–nonuniformly kinematically similar with a fixed $\delta > 0$) to (3) if there exist an invertible transformation $S(t)$ (resp. $S_\delta(t)$) and $\upsilon \geq 0$ satisfying
\[
\|S(t)\| \leq M_\upsilon \exp(\upsilon t) \quad \text{and} \quad \|S^{-1}(t)\| \leq M_\upsilon \exp(\upsilon t)
\]
or respectively
\[
\|S(\delta, t)\| \leq M_{\upsilon, \delta} \exp(\upsilon t) \quad \text{and} \quad \|S^{-1}(\delta, t)\| \leq M_{\upsilon, \delta} \exp(\upsilon t),
\]
such that the change of coordinates $y(t) = S^{-1}(t)x(t)$ (resp. $y(t) = S_\delta^{-1}(t)x(t)$) transforms (1) into (3), where
\[
U(t) = S^{-1}(t)A(t)S(t) - S^{-1}(t)\dot{S}(t),
\]
for any $t \in \mathbb{R}_0^+$.

**Remark 1.** Nonuniform kinematical similarity preserves nonuniformly growth bounded. In fact, if (1) and (3) are nonuniform kinematically similar through of the function $S(\cdot)$ and their respective evolution operators are $\Phi_1(t, s)$ and $\Phi_2(t, s)$, then by lemma 3.1 in [20] we have the equality
\[
\Phi_1(t, s)S(s) = S(t)\Phi_2(t, s) \quad \text{for all } t, s \in \mathbb{R}_0^+,
\]
and if $\|\Phi_1(t, s)\| \leq K_0 \exp(a|t-s|+\varepsilon |s|)$, then we have
\[
\|\Phi_2(t, s)\| \leq \|S^{-1}(t)\| \|S(s)\| \|\Phi_1(t, s)\| \leq M_\upsilon \exp(\upsilon t)M_\upsilon \exp(\upsilon s)K_0 \exp(a|t-s|+\varepsilon s),
\]
and finally, we obtain that
\[
\|\Phi_2(t, s)\| \leq M_\upsilon^2 K_0 \exp((\upsilon + a)|t-s| + (2\upsilon + \varepsilon)s).
\]

As we said previously, the concept of almost reducibility was introduced by B. F. Bylov in the continuous context. A discrete version of this notion was given by Á. Castañeda and G. Robledo (see [7]).

Now we introduce the definition of *nonuniformly almost reducible* which is a version of the previous concept in the nonuniform framework.
Definition 2. The system (1) is nonuniformly almost reducible to
\[ \dot{y} = C(t)y, \]
if for any \( \delta > 0 \) and \( \varepsilon \geq 0 \), there exists a constant \( K_{\delta, \varepsilon} \geq 1 \) such that (1) is \( \delta \)-nonuniformly kinematically similar to
\[ \dot{y} = [C(t) + B(t)]y, \quad \text{with} \quad \|B(t)\| \leq \delta K_{\delta, \varepsilon} \]
for any \( t \in \mathbb{R}_+^+ \).

In the case when \( C(t) \) is a diagonal matrix, if \( K_{\delta, \varepsilon} = 1 \) it is said that (1) is almost reducible to a diagonal system and it was proved in [4] that any continuous linear system satisfies this property and the components of \( C(t) \) are real numbers.

The concept of almost reducibility to diagonal system was rediscovered and improved by F. Lin in [13], who introduces the concept of contractibility in the continuous context, while in the discrete case was proposed by Á. Castañeda and G. Robledo in [7]. In this paper we introduce its nonuniform version.

Definition 3. The system (1) is nonuniformly contracted to the compact subset \( E \subset \mathbb{R} \) if is nonuniformly almost reducible to a diagonal system
\[ \dot{y} = \text{Diag}(C_1(t), \ldots, C_n(t))y, \]
where \( C_i(t) \in E \), for any \( t \in \mathbb{R}_+^+ \).

It is worth emphasize that while Bylov’s result only says that the diagonal components are real numbers, Lin’s definition provides explicit localization properties, as the fact that a compact set is contractible if it is the minimal compact set such that the system (1) can be contracted.

In the continuous and discrete cases, the concept of contractibility has been applied in some results of topological equivalence and almost topological equivalence respectively (see [14], [6]). The major contribution of [13] is to prove that the contractible set of a linear system (1) is its Sacker and Sell spectrum (see [17]). Mimicing the construction of the Sacker and Sell spectrum, S. Siegmund in [18] define the nonuniform spectrum \( \Sigma(A) \) (a formal definition will be given later). To the best of knowledge there no exists result in the nonuniform framework and the purpose of this article is to obtain condition for the nonuniform contractibility of (1) to \( \Sigma(A) \) by following some lines of Lin’s and Castañeda–Robledo’s works.

3. Main result: Nonuniform almost reducibility to diagonal systems and nonuniform spectrum.

3.1. Dichotomy and nonuniform spectrum. In this section we recall the concept of nonuniform exponential dichotomy introduced by L. Barreira and C. Valls in [2] and its associated spectrum with some properties.

Definition 4. (2, 8, 20) The system (1) has a nonuniform exponential dichotomy on \( J \subset \mathbb{R} \) if there exist an invariant projector \( P(\cdot) \), constants \( K \geq 1 \), \( \alpha > 0 \) and \( \varepsilon \geq 0 \), with \( \varepsilon < \alpha \) such that
\[
\|\Phi(t, s)P(s)\| \leq K \exp(-\alpha(t-s) + \varepsilon|s|), \quad t \geq s, \quad t, s \in J,
\]
\[
\|\Phi(t, s)(I - P(s))\| \leq K \exp(\alpha(t-s) + \varepsilon|s|), \quad t \leq s, \quad t, s \in J.
\]

Remark 2. We have the following comments with respect to this nonuniform dichotomy:
In the definition of nonuniform exponential dichotomy the condition \( \varepsilon < \alpha \) appears, for technical reasons, in [8] and [20].

It is considered a projector \( P(t) \) that satisfies the equation
\[
P(t)\Phi(t, s) = \Phi(t, s)P(s)
\]
and it is invariant in the next sense
\[
\dim(Ker(P(t))) = \dim(Ker(P(s))),
\]
for all \( t, s \in J \).

**Definition 5.** ([8], [20]) The nonuniform spectrum (also called nonuniform exponential dichotomy spectrum) of (1) is the set \( \Sigma(A) \) of \( \lambda \in \mathbb{R} \) such that the systems
\[
\dot{x} = [A(t) - \lambda I]x
\]
have not nonuniform exponential dichotomy on \( \mathbb{R}_+ \).

**Remark 3.** The evolution operator of (6) is \( \Phi_\lambda(t, s) = \exp(-\lambda(t - s))\Phi(t, s) \). Moreover, if \( \lambda \notin \Sigma(A) \), then there exist constants \( K \geq 1, \alpha > 0, \varepsilon > 0 \), with \( \varepsilon < \alpha \) and an invariant projector \( P(t) \) such that
\[
\|
\exp(-\lambda(t - s))\Phi(t, s)P(s)\|
\leq\ K \exp(-\alpha(t - s) + \varepsilon |s|), \quad t \geq s,
\]
\[
\|
\exp(\lambda(t - s))\Phi(t, s)(I - P(s))\|
\leq\ K \exp(\alpha(t - s) + \varepsilon |s|), \quad t \leq s.
\]

**Remark 4.** If \( \lambda \notin \Sigma(A) \), then \( \lambda \) belongs to the resolvent set of \( A \), which is denoted by \( \rho(A) \).

The following result allows us to give a better description of the spectrum if the evolution operator has a nonuniformly bounded growth.

**Proposition 1.** ([1], [11], [18], [20]) If the evolution operator of (7) satisfies (P2), its nonuniform spectrum \( \Sigma(A) \) is the union of \( m \) compact intervals where \( 0 < m \leq n \), namely,
\[
\Sigma(A) = \bigcup_{i=1}^{m} [a_i, b_i],
\]
with \( -\infty < a_1 \leq b_1 < \ldots < a_m \leq b_m < +\infty \).

**Remark 5.** Notice that in [11] Theorem 5] this result is done in the discrete framework with \( J = \mathbb{N} \). While that the rest of references are immersed in the continuous context. For more details see [11] Theorem 5.12, [18] Theorem 3.1 and [20] Theorem 1.2.

The following result allows characterizing the nonuniformly bounded growth of the evolution operator associated to (1) from subtle hypothesis about its nonuniform spectrum.

**Proposition 2.** Suppose that the system (7) has spectrum \( \Sigma(A) = [a, b] \), then its evolution operator \( \Phi(t, s) \) satisfies (P2).

**Proof.** Let \( \gamma, \lambda \in \rho(A) \) such that \( \gamma < a \leq b < \lambda \), then we have the system
\[
\dot{x} = (A(t) - \gamma I)x
\]
has a nonuniform exponential dichotomy with projector \( P(t) = 0 \). On the other hand, the system
\[
\dot{x} = (A(t) - \lambda I)x
\]
has a nonuniform exponential dichotomy with projector $P(t) = I$.

Then there exist $\alpha_1, \alpha_2, > 0, \varepsilon_1, \varepsilon_2 \geq 0$, $K_1, K_2 \geq 1$ such that satisfies
\[
\|\Phi(t, s)\| \leq K_1 \exp((\gamma + \alpha_1)(t - s) + \varepsilon_1 s) \quad (t \leq s),
\]
\[
\|\Phi(t, s)\| \leq K_2 \exp((\lambda - \alpha_2)(t - s) + \varepsilon_2 s) \quad (t \geq s).
\]

Now we define $a = \max \{0, -\gamma - \alpha_1, \lambda - \alpha_2\}, \varepsilon = \max \{\varepsilon_1, \varepsilon_2\}$ and $K = \max \{K_1, K_2\}$
then we conclude that
\[
\|\Phi(t, s)\| \leq K \exp(a|t - s| + \varepsilon s) \quad (t, s \in \mathbb{R}_0^+).
\]

\[\square\]

**Remark 6.** If $\lambda \notin \Sigma(A)$, it follows from Definition 5 that (6) has a nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with projector $P_\lambda$. Nevertheless, it is interesting to note that:

a) $\text{Rank}(P_\lambda)$ is constant for any $\lambda \in (b_{i-1}, a_i)$ $(i \in \{1, \ldots, m\})$.

b) If $\lambda_i \in (b_{i-1}, a_i)$ and $\lambda_{i+1} \in (b_i, a_{i+1})$, then $\text{Rank}(P_{\lambda_i}) < \text{Rank}(P_{\lambda_{i+1}})$.

c) $\text{Rank}(P_\lambda) = 0$ for any $\lambda \in (-\infty, a_i)$ and $\text{Rank}(P_\lambda) = n$ for any $\lambda \in (b_m, +\infty)$.

### 3.2. Main result.** The main goal of this article is prove the following result.

**Theorem 1.** If (P1)-(P2) are satisfied, then $\Box$ is nonuniformly contracted to $\Sigma(A)$.

In the next section we will give the technical results which allow us prove this Theorem.

### 4. Preparatory results.

The nonuniform kinematical similarity between $\Box$ and $\Box$ will be denoted by $A \cong U$. Let us recall that nonuniform kinematical similarity is an equivalence relation having several properties.

**Lemma 1.** If $A \cong B$, then $A - \lambda I \cong B - \lambda I$ for any $\lambda \in \mathbb{R}$

**Proof.** If $A \cong B$ by the transformation $y(t) = S^{-1}(t)x(t)$, then $S(t)$ satisfies
\[
B(t) = S^{-1}(t)A(t)S(t) - S^{-1}(t)\dot{S}(t).
\]
It is straightforward see that
\[
(B(t) - \lambda I) = S^{-1}(t)(A(t) - \lambda I)S(t) - S^{-1}(t)\dot{S}(t),
\]
then $A(t) - \lambda I \cong B(t) - \lambda I$. \[\square\]

**Lemma 2.** If $A \cong B$, then $\Sigma(A) = \Sigma(B)$.

**Proof.** Let $\lambda \in \rho(A)$ then the system
\[
\dot{x} = [A(t) - \lambda I]x
\]
have a nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with invariant projector $P$.

We define $\Phi_B(t, s) = S^{-1}(t)\Phi_A(t, s)S(t)$ as the evolution operator associated to system
\[
\dot{y} = [B(t) - \lambda I]y
\]
which has $Q(t) = S^{-1}(t)P(t)S(t)$ as an invariant projector.
This fact combined with the submultiplicative property of norms and the estimates for $S$ and $S^{-1}$ allows to prove that if $t \geq s$ (the case $t \leq s$ can be proved similarly),
\[
\|\Phi_B(t, s)\exp(-\lambda(t-s))Q(s)\| \leq \|S^{-1}(t)\| \|\Phi_A(t, s)\exp(-\lambda(t-s))P(s)\| \|S(s)\|
\]
\[
\leq M_\epsilon \exp(vt)K \exp(-\alpha(t-s) + \epsilon s)M_\epsilon \exp(\epsilon s).
\]

Finally, if $\alpha > \epsilon + 3\nu$, then $\lambda \in \rho(B)$. To prove the other contention, we use the fact that $\cong$ is an equivalence relation.

**Proposition 3.** If $\Sigma(A) \subset [a, b]$ and $\lambda > b$ (resp. or $\lambda < a$) the system
\[
\dot{x} = (A(t) - \lambda I)x
\]
has a nonuniform exponential dichotomy with projector $P(t) = I$ (resp. with projector $P(t) = 0$).

**Proof.** By (P2) we have that the evolution operator satisfies
\[
\|\Phi(t, s)\| \leq K_0 \exp(L|t-s| + \bar{\epsilon}|s|), \quad t, s \in \mathbb{R}_0^+.
\]

We consider $\lambda > b$. Let $h = \max\{L + 1 + \bar{\epsilon}, \lambda + 1 + \bar{\epsilon}\}$ and
\[
\Phi_h(t, s) = \Phi(t, s)\exp(-h(t-s)).
\]
Then
\[
\|\Phi_h(t, s)\| = \|\Phi(t, s)\|\exp(-h(t-s)) \leq K_0 \exp(L(t-s) + \bar{\epsilon}|s| - h(t-s)), \quad (t \geq s).
\]
Now we define $\alpha = h - L > \bar{\epsilon}$ and the previous equation becomes
\[
\|\Phi_h(t, s)\| \leq K_0 \exp(-\alpha(t-s) + \bar{\epsilon}|s|) \quad (t \geq s),
\]
which implies that the system
\[
\dot{x} = (A(t) - hI)x
\]
has a nonuniform exponential dichotomy with projector $P(t) = I$ and $[\lambda, h] \subset \rho(A)$.

By using Remark 6 the system
\[
\dot{x} = (A(t) - \lambda I)x
\]
also has a nonuniform exponential dichotomy with projector $P(t) = I$.

For $\lambda < a$ the proof is similar considering $h = \min\{-L + 1 + \bar{\epsilon}, \lambda - 1 - \bar{\epsilon}\}$. □

The following result has been proved by X. Zhang and J. Chu et al. using the condition (P2).

**Proposition 4.** ([19]) If the system [17] satisfies (P1)–(P2) then its spectrum is as in [19] and there exist $m+2$ matrix functions $B_i : \mathbb{R} \to \mathbb{R}^{N_i \times N_i}$ such that
\[
\|B_i(t)\| \leq M_i \exp(\mu_i t) \text{ with } \mu_i, M_i > 0
\]
where $\Sigma(B_i) = [a_i, b_i]$ with $i \in \{1, \ldots, m\}$ and $\Sigma(B_0) = \Sigma(B_{m+1}) = \emptyset$ such that [17] is nonuniformly kinematically similar to
\[
\dot{y} = \text{Diag}(B_0(t), B_1(t), \ldots, B_m(t), B_{m+1}(t))y.
\]

**Remark 7.** In our case the blocks $B_0(t)$ and $B_{m+1}(t)$ are omitted due to their dimensions $N_0$ and $N_{m+1}$ respectively are 0 (for more details, see [19] Theorem 3.2)).
In [16] it is introduced the concept of diagonal significance which is fundamental for obtain the almost reducibility in the the case of exponential dichotomy [7, Proposition 4] in a discrete context.

We point out that in [3] the concept of diagonal significance is studied in the continuous framework. In our case this condition it is not necessary. Moreover, in the case of nonuniform exponential dichotomy the condition of diagonal significance is still open.

**Proposition 5.** Let $C(t)$ be an upper triangular $n \times n$-matrix function such that $\Sigma(C) = [a, b]$, then

$$\bigcup_{i=1}^{n} \Sigma(c_{ii}) \subset \Sigma(C),$$

where $c_{ii}(t)$ are the diagonal coefficients of $C(t)$.

**Proof.** We will prove that $\bigcup_{i=1}^{n} \Sigma(c_{ii}) \subset \Sigma(C)$. Let $\lambda \notin \Sigma(C) = [a, b]$ such that $\lambda > b$. By Proposition 3, we have that the upper triangular system

$$\dot{x} = (C(t) - \lambda I)x,$$

has nonuniform exponential dichotomy with projector $P(t) = I$. That is, the evolution operator of (11), namely $\Phi_{\lambda}(t, s)$, satisfies

$$\|\Phi_{\lambda}(t, s)\| \leq K_{\lambda} \exp(-\alpha_{\lambda}(t - s) + \varepsilon_{\lambda}|s|) \quad (t \geq s).$$

Now for each $i \in \{1, \ldots, n\}$, we have the following estimate

$$\exp\left(\int_{s}^{t} (c_{ii}(r) - \lambda)dr\right) \leq \|\Phi_{\lambda}(t, s)\| \leq K_{\lambda} \exp(-\alpha_{\lambda}(t - s) + \varepsilon_{\lambda}|s|) \quad (t \geq s),$$

and we conclude that the diagonal systems

$$\dot{x}_i = (c_{ii}(t) - \lambda)x_i$$

has a nonuniform exponential dichotomy with projector $P(t) = 1$ (scalar systems), which implies that $\lambda \notin \bigcup_{i=1}^{n} \Sigma(c_{ii})$.

The case $\lambda < a$ can be proved analogously, thus $\bigcup_{i=1}^{n} \Sigma(c_{ii}) \subset \Sigma(C)$. □

5. PROOF OF MAIN RESULTS.

5.1. **Proof of Theorem 1.** The proof will be made in several steps:

**Step 1:** [11] is nonuniform kinematically similar to an upper triangular system:

By Proposition 1, there exists a positive integer $m \leq n$ such that:

$$\Sigma(A) = \bigcup_{i=1}^{m} [a_i, b_i], \quad \text{with} \ -\infty < a_1 \leq b_1 < \ldots < a_m \leq b_m < +\infty.$$

The Proposition 4 says that [11] is nonuniform kinematically similar to [13], where $B_i(t)$ are matrix function of order $n_i \times n_i$ satisfying [13] and $\Sigma(B_i) = [a_i, b_i]$ with $i \in \{1, \ldots, m\}$. Now, by using the method of QR factorization, we know that, for each $i \in \{1, \ldots, m\}$, the systems

$$\dot{x}_i = B_i(t)x_i$$

are kinematically similar (see Definition 1 with $\nu = 0$) to

$$\dot{y}_i = D_i(t)y_i,$$
where $D_i(t)$ is a upper triangular $n_i \times n_i$-matrix function such that
\[ \|D_i(t)\| \leq \mathcal{N}_i \exp(\varepsilon t) \text{ and } \Sigma(D_i) = [a_i, b_i] \]

**Step 2): Nonuniform exponential dichotomy of scalar differential equation:** From now on, the diagonal terms of the upper triangular matrix $D_i$ described in (13) will be denoted by $d^{(i)}_{rr}(t)$ where $r$ is a fixed element of $\{1, \ldots, m\}$. Now, by Proposition 5, we have
\[ \bigcup_{r=1}^{n_i} \Sigma(d^{(i)}_{rr}) \subset \Sigma(D_i). \]

By Proposition 3, for any $\delta > 0$ there exists $M_\delta = \frac{\delta}{2} > 0$ such that the scalar differential equation
\[ \dot{x} = \left[ d^{(i)}_{rr}(t) - (a_i - M_\delta) \right] x \]
has a nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with projector $P(t) = 0$ and
\[ \dot{x} = \left[ d^{(i)}_{rr}(t) - (b_i + M_\delta) \right] x \]
has a nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with projector $P(t) = 1$. In consequence, there exist $\alpha > 0$, $\varepsilon > 0$, $\beta > 1$ such that
\[ |\exp(\Phi(t, s))| \leq \beta \exp(\alpha(t - s) + \varepsilon s) \quad (t \leq s), \]
\[ |\exp(\Psi(t, s))| \leq \beta \exp(-\alpha(t - s) + \varepsilon s) \quad (t \geq s), \]
where
\[ \exp(\Phi(t, s)) = \exp\left( \int_s^t (d^{(i)}_{rr}(\tau) - (a_i - M_\delta)) d\tau \right), \]
and
\[ \exp(\Psi(t, s)) = \exp\left( \int_s^t (d^{(i)}_{rr}(\tau) - (b_i + M_\delta)) d\tau \right), \]
are the evolution operators of (14) and (15) respectively.

**Step 3): Upper and lower bounds for (16):** For any fixed $i \in \{1, \ldots, m\}$, there exist two functions $c^{(i)}_r$ and $\lambda^{(i)}_r$ such that
\[ a_i \leq c^{(i)}_r(t) \leq b_i \quad \text{and} \quad |\lambda^{(i)}_r(t)| \leq M_\delta \quad \text{for any } t \in \mathbb{R}_0^+ \]
and there exist $\bar{\Delta}, v \geq 0$ verifying
\[ \left| \int_0^t [d^{(i)}_{rr}(\tau) - (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau))] d\tau \right| \leq \bar{\Delta} + vt, \quad \text{if } t \geq 0 \]
for any $r \in \{1, \ldots, n_i\}$.

We will construct a strictly increasing and unbounded sequence of real numbers $\{T^{(i)}_l\}_{l=0}^{+\infty}$ satisfying $T^{(i)}_0 = 0$ such that the function $c^{(i)}_r$, $\lambda^{(i)}_r: \mathbb{R}_0^+ \to \mathbb{R}$ defined by:
\[ c^{(i)}_r(t) = \begin{cases} a_i & \text{if } t \in [T^{(i)}_q, T^{(i)}_{q+1}) \quad (q = 0, 2, 4, \ldots) \\ b_i & \text{if } t \in [T^{(i)}_{q+1}, T^{(i)}_{q+2}) \end{cases} \]
and

\[
\chi^{(i)}_t(t) = \begin{cases} 
-M_\delta & \text{if } t \in [T_q^{(i)}, T_{q+1}^{(i)}] \\
M_\delta & \text{if } t \in [T_{q+1}^{(i)}, T_{q+2}^{(i)}]
\end{cases} \quad (q = 0, 2, 4, \ldots)
\]

satisfy properties (17) and (18) on \( \mathbb{R}_0^+ \).

It is straightforward to see that (17) is always satisfied. In order to verify (18), we interchange \( t \) by \( s \) in the first inequality of (16), then we have:

\[
\begin{align*}
\Phi(t, s) & \geq \alpha(t - s) - \varepsilon t - \ln(\beta) & (t \geq s), \\
\Psi(t, s) & \leq -\alpha(t - s) + \varepsilon s + \ln(\beta) & (t \geq s).
\end{align*}
\]

By using induction, we will verify that there exists a sequence \( \{T_t^{(i)}\}_{t=0}^{+\infty} \) satisfying (18). First, by the Proposition 2 combined with the fact that

\[
\exp\left( \int_s^t d_{t'}^{(i)}(\tau)d\tau \right) \leq \|\Phi_{D_i}(t, s)\|
\]

where \( \Phi_{D_i}(t, s) \) is the evolution operator of the system (13) then there exist constants \( \bar{a} \geq 0, \bar{\varepsilon} \geq 0 \) and \( K \geq 1 \) such that satisfies the following

\[
\int_s^t d_{t'}^{(i)}(\tau)d\tau \leq \bar{a}|t - s| + \bar{\varepsilon}s + \ln(K).
\]

Then, using the equation (20) we have

\[
\Phi(t, s) \leq \bar{a}(t - s) + \bar{\varepsilon}s + \ln(K) + |a_i|(t - s) + M_\delta(t - s).
\]

On the other hand, by (19) we obtain

\[
\int_s^t d_{t'}^{(i)}(\tau)d\tau \geq \alpha(t - s) - \varepsilon t + a_i(t - s) - M_\delta(t - s) - \ln(\beta),
\]

and using the last expression we deduce

\[
\Psi(t, s) \geq (\alpha - (b_i - a_i) - M_t^{(i)})(t - s) - \varepsilon t - \ln(\beta).
\]

By the equations (19), (21) and (23) we have the following

\[
\begin{align*}
\Phi(t, s) & \geq (\alpha - \varepsilon)(t - s) - \varepsilon s - \ln(\beta) & (t \geq s), \\
\Phi(t, s) & \leq (\bar{a} + |a_i| + M_\delta + \bar{\varepsilon})(t - s) + \bar{\varepsilon}s + \ln(K) & (t \geq s).
\end{align*}
\]

\[
\begin{align*}
\Psi(t, s) & \leq -(\alpha - \varepsilon)(t - s) + \varepsilon s + \ln(\beta) & (t \geq s), \\
\Psi(t, s) & \geq (\varepsilon - (b_i - a_i) - M_\delta)(t - s) - \varepsilon s - \ln(\beta) & (t \geq s).
\end{align*}
\]

Now we will introduce constants and conditions that allow us to obtain the desired result (this conditions are inherent in the nonuniform framework).

Let \( N, \xi, p, \xi, \bar{p} \in \mathbb{R} \) constants that satisfy:

(C1) \( 0 < N < \min\{\alpha - \varepsilon, \bar{a} + |a_i| + M_\delta + \bar{\varepsilon}\} \).

(C2) \( \max\{\ln(K), \ln(\beta)\} < p = -\bar{p} \).

(C3) \( 0 \leq \max\{\bar{\varepsilon}, \varepsilon\} \leq -\bar{\xi} \leq \xi \).
If \( s = 0 \) in the first inequality of (24) we obtain
\[
\Phi(t, 0) \geq (\alpha - \varepsilon)t - \ln(\beta), \quad t \geq 0,
\]
which implies that \( \Phi(t, 0) \) is unbounded in \( \mathbb{R}^+ \), since \( \alpha > \varepsilon \). In consequence, given \( N, \xi, p \in \mathbb{R} \), there exists \( T_1^{(i)} > 0 \) such that
\[
\Phi(T_1^{(i)}, 0) = N(T_1^{(i)} - 0) + \xi + p, \quad \Phi(t, 0) < N(t - 0) + \xi + p \quad (0 \leq t < T_1^{(i)}).
\]

Then we consider the value \( \bar{T}_1^{(i)} + \bar{p} \) and
\[
T_2^{(i)} = \min \left\{ \omega \in \mathbb{R}^+ : \Psi(\omega, T_2^{(i)}) = -N(T_1^{(i)} - 0) - \xi - p \right\},
\]
with \( T_2^{(i)} > T_1^{(i)} \). Now we will calculate the slope of the line that joins the points \( \bar{T}_1^{(i)} + \bar{p} \) and \( -N(T_1^{(i)} - 0) - \xi - p \), which we will denote by \( \bar{N} \). Moreover, \( \bar{N} \) satisfies the following technical condition
\[
\text{(C4)} \quad \max \left\{ -(\alpha - \varepsilon), -(\varepsilon + (b_i - a_i) + 2M_b) \right\} < \bar{N}.
\]

Then we have
\[
\bar{N} = \frac{-N(T_1^{(i)} - 0) - \xi - p - (\bar{T}_1^{(i)} + \bar{p})}{T_2^{(i)} - T_1^{(i)}}.
\]

Due to the conditions (C1) and (C3), we have that \( \bar{N} \leq 0 \). In this way, we consider the straight \( \bar{N}(t - T_1^{(i)}) + \bar{T}_1^{(i)} + \bar{p} \).

Based on the above and the equation (25), if \( s = T_1^{(i)} \) then there exists \( T_2^{(i)} > T_1^{(i)} \) such that
\[
\Psi(T_2^{(i)}, T_1^{(i)}) = \bar{N}(T_2^{(i)} - T_1^{(i)}) + \bar{T}_1^{(i)} + \bar{p},
\]
\[
\Psi(t, T_1^{(i)}) > N(t - T_1^{(i)}) + \bar{T}_1^{(i)} + \bar{p} \quad (T_1^{(i)} \leq t < T_2^{(i)}).
\]

By (24) and (26) we obtain that for \( t \in [0, T_1^{(i)}] \)
\[
-\varepsilon t - \ln(\beta) - Nt - \xi t - p \leq \int_0^t (d_i^{(j)}(\tau) - (c_i^{(j)}(\tau) + \lambda_i^{(j)}(\tau)))d\tau \leq -\bar{N}t - \bar{\xi}t - \bar{p}.
\]

In fact, we have
\[
-\varepsilon t - \ln(\beta) - (-\bar{N}t - \bar{\xi}t - \bar{p}) \leq -\varepsilon t - \ln(\beta),
\]
by the equation (24)
\[
-\varepsilon t - \ln(\beta) \leq \int_0^t (d_i^{(j)}(\tau) - (a_i - M_b))d\tau,
\]
then by the equation (26)
\[
\int_0^t (d_i^{(j)}(\tau) - (a_i - M_b))d\tau < N(t - 0) + \xi 0 + p
\]
and finally,
\[
N(t - 0) + \xi 0 + p \leq Nt + \xi t + p.
\]

On the other hand, from the equations (25) and (27), we have for \( t \in [T_1^{(i)}, T_2^{(i)}] \)
\[ \varepsilon t + \ln(\beta) + N t + \xi t + p \geq \int_{t_1}^{t} (d_{rr}^{(i)}(\tau) - (c_i^{(i)}(\tau) + \lambda_{i}^{(i)}(\tau))) d\tau \geq -(-N t - \varepsilon t - \bar{p}). \]

Similar to the previous, we have that
\[ \varepsilon t + \ln(\beta) + N t + \xi t + p \geq \varepsilon t + \ln(\beta), \]
by the equation \((25)\) we have
\[ \varepsilon t + \ln(\beta) \geq \int_{t_1}^{t} (d_{rr}^{(i)}(\tau) - (b_i + M_{b})) d\tau, \]
and then by the equation \((27)\)
\[ \int_{t_1}^{t} (d_{rr}^{(i)}(\tau) - (b_i + M_{b})) d\tau > \bar{N}(t - T_1^{(i)}) + \xi T_1^{(i)} + \bar{p} \geq -(-N t - \varepsilon t - \bar{p}). \]

Thus for \(t \in [0, T_2^{(i)})\)
\[ \left| \int_{0}^{t} (d_{rr}^{(i)}(\tau) - (c_i^{(i)}(\tau) + \lambda_{i}^{(i)}(\tau))) d\tau \right| \leq 2\varepsilon t + 2 \ln(\beta) + 2 \max \left\{ N t + \xi t + p, -\bar{N} t - \bar{\xi} t - \bar{p} \right\}. \]

As inductive hypothesis, we will assume that there exists \(2m + 1\) numbers
\[ 0 = T_0^{(i)} < T_1^{(i)} < T_2^{(i)} < \cdots < T_{2m-1}^{(i)} < T_{2m}^{(i)} \]
such that \((17)\) is satisfied and for \(t \in [0, T_{2m}^{(i)})\)
\[ \left| \int_{0}^{t} (d_{rr}^{(i)}(\tau) - (c_i^{(i)}(\tau) + \lambda_{i}^{(i)}(\tau))) d\tau \right| \leq 2\varepsilon t + 2 \ln(\beta) + 2 \max \left\{ N t + \xi t + p, -\bar{N} t - \bar{\xi} t - \bar{p} \right\}. \]

By using the first inequality of \((24)\) and considering \(s = T_{2m}^{(i)}\), we have that
\[ \Phi(t, T_{2m}^{(i)}) \geq (\alpha - \varepsilon)(t - T_{2m}^{(i)}) - \varepsilon T_{2m}^{(i)} - \ln(\beta) \]
is unbounded for any \(t > T_{2m}^{(i)}\). Then, there exists \(T_{2m+1}^{(i)} > T_{2m}^{(i)}\) such that
\[
\begin{align*}
\Phi(T_{2m+1}^{(i)}, T_{2m}^{(i)}) &= N(T_{2m+1}^{(i)} - T_{2m}^{(i)}) + \xi T_{2m}^{(i)} + p, \\
\Phi(t, T_{2m}^{(i)}) &< N(t - T_{2m}^{(i)}) + \xi T_{2m}^{(i)} + p \quad (T_{2m}^{(i)} \leq t < T_{2m+1}^{(i)}).
\end{align*}
\]

Now we consider the value \(\bar{\xi} T_{2m+1}^{(i)} + \bar{p}\) and
\[ T_{2m+2}^{(i)} = \min \left\{ \omega \in \mathbb{R}_+ : \Psi(\omega, T_{2m+1}^{(i)}) = -N(T_{2m+1}^{(i)} - T_{2m}^{(i)}) - \xi T_{2m}^{(i)} - p \right\}, \]
with \(T_{2m+2}^{(i)} > T_{2m+1}^{(i)}\).

As before, let \(\bar{N}\) be the slope of the line joining the points
\[ \bar{\xi} T_{2m+1}^{(i)} + \bar{p} \quad \text{and} \quad -N(T_{2m+1}^{(i)} - T_{2m}^{(i)}) - \xi T_{2m+1}^{(i)} - p. \]

By the conditions \((C1), (C2)\) and \((C3)\) we have \(\bar{N} \leq 0\). In this way, we consider the straight
\[ \bar{N}(t - T_{2m+1}^{(i)}) + \bar{\xi} T_{2m+1}^{(i)} + \bar{p}. \]
Combining the above straight and the equation (25), if \( s = T_{2m+1}^{(i)} \) then there exists \( T_{2m+2}^{(i)} > T_{2m+1}^{(i)} \) such that

(29) \[
\begin{aligned}
\Psi(T_{2m+2}^{(i)}, T_{2m+1}^{(i)}) &= N(T_{2m+2}^{(i)} - T_{2m+1}^{(i)}) + \bar{\xi}T_{2m+1}^{(i)} + \bar{p}, \\
\Psi(t, T_{2m+1}^{(i)}) &> N(t - T_{2m+1}^{(i)}) + \xi T_{2m+1}^{(i)} + \bar{p},
\end{aligned}
\]

for \( T_{2m+1}^{(i)} \leq t < T_{2m+2}^{(i)} \).

Now we will prove that for \( t \in [0, T_{2m+2}^{(i)}) \) we obtain

\[
\left| \int_0^t (d^{(i)}_{rr}(\tau) - (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau)))d\tau \right| \leq 2\varepsilon t + 2\ln(\beta) + 2 \max \{ Nt + \xi t + p, -\bar{N}t - \bar{\xi}t - \bar{p} \}.
\]

By inductive hypothesis, we have proved the case in which \( t \in [0, T_{2m}^{(i)}) \). If \( t \in [T_{2m}^{(i)}, T_{2m+1}^{(i)}) \) we have

\[
\left| \int_0^t (d^{(i)}_{rr}(\tau) - (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau)))d\tau \right| = \left| \int_{T_{2m}^{(i)}}^t (d^{(i)}_{rr}(\tau) - (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau)))d\tau \right|
\]

\[
= \left| \int_{T_{2m}^{(i)}}^t (d^{(i)}_{rr}(\tau) - (a_i - M_\delta))d\tau \right|.
\]

By the equations (24) and (28), as before we obtain that for \( t \in [T_{2m}^{(i)}, T_{2m+1}^{(i)}) \)

\[
-\varepsilon t - \ln(\beta) - Nt - \xi t - p \leq \int_{T_{2m}^{(i)}}^t (d^{(i)}_{rr}(\tau) - (a_i - M_\delta))d\tau \leq -\bar{N}t - \bar{\xi}t - \bar{p}.
\]

In the case \( t \in [T_{2m+1}^{(i)}, T_{2m+2}^{(i)}) \), we have

\[
\left| \int_0^t (d^{(i)}_{rr}(\tau) - (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau)))d\tau \right| = \left| N(T_{2m+1}^{(i)} - T_{2m}^{(i)}) + \xi T_{2m}^{(i)} + p \right|
\]

\[
+ \int_{T_{2m}^{(i)}}^t (d^{(i)}_{rr}(\tau) - (b_i + M_\delta))d\tau.
\]

Then, by the equations (25) and (29), we have that for \( t \in [T_{2m+1}^{(i)}, T_{2m+2}^{(i)}) \) is satisfied that

\[
\varepsilon t + \ln(\beta) + Nt + \xi t + p \geq N(T_{2m+1}^{(i)} - T_{2m}^{(i)}) + \xi T_{2m}^{(i)} + p
\]

\[
+ \int_{T_{2m}^{(i)}}^t (d^{(i)}_{rr}(\tau) - (b_i + M_\delta))d\tau
\]

\[
\geq -(-\bar{N}t - \bar{\xi}t - \bar{p}).
\]

In fact, (25) and (29) implies that

(30) \[
\varepsilon t + \ln(\beta) \geq \int_{T_{2m+1}^{(i)}}^t (d^{(i)}_{rr}(\tau) - (b_i + M_\delta))d\tau \geq N(t - T_{2m+1}^{(i)}) + \xi T_{2m+1}^{(i)} + \bar{p}.
\]

Now considering the two previous inequalities separately in (30), we obtain

\[
\varepsilon t + \ln(\beta) + N(T_{2m+1}^{(i)} - T_{2m}^{(i)}) + \xi T_{2m}^{(i)} + p \geq \int_0^t (d^{(i)}_{rr}(\tau) - (b_i + M_\delta))d\tau
\]

\[
\geq -(-\bar{N}t - \bar{\xi}t - \bar{p}).
\]
which implies

\[ \int_{0}^{t} (d_{r}^{(i)}(\tau) - (b_{i} + M_{\delta}))d\tau \geq \bar{N}(t - T_{2m+1}^{(i)}) + \xi T_{2m+1}^{(i)} + \bar{p} + N(T_{2m+1}^{(i)} - T_{2m}^{(i)}) + \xi T_{2m}^{(i)} + p. \]

Then for the first inequality we have

\[ \varepsilon t + \ln(\beta) + Nt + \xi t + p \geq \varepsilon t + \ln(\beta) + N(T_{2m+1}^{(i)} - T_{2m}^{(i)}) + \xi T_{2m}^{(i)} + p \]

and on the other hand for the second inequality we have

\[ \bar{N}(t - T_{2m+1}^{(i)}) + \xi T_{2m+1}^{(i)} + \bar{p} + N(T_{2m+1}^{(i)} - T_{2m}^{(i)}) + \xi T_{2m}^{(i)} + p \geq - (\bar{N}t - \bar{\xi}t - \bar{p}). \]

Therefore, for \( t \in [0, T_{2m+2}^{(i)}] \)

\[ \left| \int_{0}^{t} (d_{r}^{(i)}(\tau) - (c_{r}^{(i)}(\tau) + \lambda_{r}^{(i)}(\tau)))d\tau \right| \leq 2\varepsilon t + 2\ln(\beta) + 2 \max \left\{ Nt + \xi t + p, -\bar{N}t - \bar{\xi}t - \bar{p} \right\}. \]

Finally, we will prove that \( T_{m}^{(i)} \to +\infty \) as \( m \to +\infty \). For that, first of all, we have by the equations (24) and (28):

\[ N(T_{2m+1}^{(i)} - T_{2m}^{(i)}) + \xi T_{2m}^{(i)} + p = \int_{T_{2m}^{(i)}}^{T_{2m+1}^{(i)}} (d_{r}^{(i)}(\tau) - (a_{i} - M_{\delta}))d\tau \]

\[ \leq (\bar{a} + |a_{i}| + M_{\delta} + \varepsilon - N)(T_{2m+1}^{(i)} - T_{2m}^{(i)}), \]

which implies

\[ (\xi - \varepsilon)T_{2m}^{(i)} + p - \ln(K) \leq (\bar{a} + |a_{i}| + M_{\delta} + \varepsilon - N)(T_{2m+1}^{(i)} - T_{2m}^{(i)}). \]

By the conditions (C1) and (C2), we have the following

\[ 0 < \frac{p - \ln(K)}{\bar{a} + |a_{i}| + M_{\delta} + \varepsilon - N} \leq T_{2m+1}^{(i)} - T_{2m}^{(i)}. \]

On the other hand, in view of the equations (25) and (29) we have

\[ \bar{N}(T_{2m+2}^{(i)} - T_{2m+1}^{(i)}) + \xi T_{2m+1}^{(i)} + \bar{p} = \int_{T_{2m+1}^{(i)}}^{T_{2m+2}^{(i)}} (d_{r}^{(i)}(\tau) - (b_{i} + M_{\delta}))d\tau \]

\[ \geq (-\varepsilon - (b_{i} - a_{i}) - 2M_{\delta})(T_{2m+2}^{(i)} - T_{2m+1}^{(i)}) \]

\[ -\varepsilon T_{2m+1}^{(i)} - \ln(\beta), \]

which implies

\[ (\xi + \varepsilon)T_{2m+1}^{(i)} + \bar{p} + \ln(\beta) \geq -(\varepsilon + (b_{i} - a_{i}) + 2M_{\delta} + \bar{N})(T_{2m+2}^{(i)} - T_{2m+1}^{(i)}). \]

Similarly, the conditions (C2) and (C4) allow us to ensure that

\[ 0 < \frac{-\bar{p} - \ln(\beta)}{\varepsilon + (b_{i} - a_{i}) + 2M_{\delta} + \bar{N}} \leq T_{2m+2}^{(i)} - T_{2m+1}^{(i)}. \]

So the above allows us to obtain the existence of \( c_{r}^{(i)}(t), \lambda_{r}^{(i)}(t) \) defined on \( \mathbb{R}_{0}^{+} \) verifying (17) and finally:

\[ \left| \int_{0}^{t} (d_{r}^{(i)}(\tau) - (c_{r}^{(i)}(\tau) + \lambda_{r}^{(i)}(\tau)))d\tau \right| \leq \Delta + vt \quad (t \in \mathbb{R}_{0}^{+}), \]
with $\Delta \geq 0$ and $v = v_\varepsilon$, defined by
\begin{equation}
(31) \quad v = \max \{ 2(\varepsilon + N + \xi), 2(\varepsilon - N - \xi) \}.
\end{equation}

From our definition of $c^{(i)}_r(t)$ and $\lambda^{(i)}_r(t)$ we know that are piecewise continuous. Therefore, there exists continuous functions $\bar{c}^{(i)}_r(t)$, $\bar{\lambda}^{(i)}_r(t)$ satisfying
\begin{equation}
(32) \quad a_i \leq \bar{c}^{(i)}_r(t) \leq b_i \quad \text{and} \quad |\bar{\lambda}^{(i)}_r(t)| \leq M_\delta \quad \text{for any} \ t \in \mathbb{R}_0^+.
\end{equation}

and
\[
\int_0^t \left| (c^{(i)}_r(\tau) + \lambda^{(i)}_r(\tau)) - (\bar{c}^{(i)}_r(\tau) + \bar{\lambda}^{(i)}_r(\tau)) \right| d\tau \leq 1,
\]
thus
\[
\left| \int_0^t [d^{(i)}_r(\tau) - (\bar{c}^{(i)}_r(\tau) + \bar{\lambda}^{(i)}_r(\tau))] d\tau \right| \leq \tilde{\Delta} + vt
\]
with $\tilde{\Delta} = \Delta + 1$.

As a consequence of this result, we construct the $n_i \times n_i$ matrix:
\[
L_i(t) = \text{Diag}(\mu_1(t), \ldots, \mu_{n_i}(t)),
\]
where for any $r \in \{1, \ldots, n_i\}$, $\mu_r$ are defined by
\[
\mu_r(t) = \exp \left( \int_0^t (d^{(i)}_r(\tau) - (\bar{c}^{(i)}_r(\tau) + \bar{\lambda}^{(i)}_r(\tau))) d\tau \right),
\]
and we conclude that
\[
\|L_i(t)\| \leq \Omega \exp(vt) \quad \text{and} \quad \|L^{-1}_i(t)\| \leq \Omega \exp(vt) \quad \text{for any} \ t \in \mathbb{R}_0^+,
\]
with $\Omega = \exp(\tilde{\Delta})$.

\textit{Step 4): The systems} $[13]$ \textit{can be nonuniformly contracted to} $[a_i, b_i]$, \textit{for any} $i = 1, \ldots, m$: \textit{The system} $[13]$ \textit{is nonuniform kinematically similar to}
\begin{equation}
(33) \quad \dot{z}_i = \Lambda_i(t)z_i,
\end{equation}
with $y_i(t) = L_i(t)z_i(t)$, where $\Lambda_i(t) = L_i^{-1}(t)D_i(t)L_i(t) - L_i^{-1}(t)\dot{L}_i(t)$ is a $n_i \times n_i$ matrix whose rs-coefficient is defined by
\[
\{\Lambda_i(t)\}_{rs} = \begin{cases} 
    c^{(i)}_r(t) + \bar{\lambda}^{(i)}_s(t) & \text{if} \ r = s, \\
    \frac{\mu_r(t)}{\mu_s(t)}d^{(i)}_r(t) & \text{if} \ 1 \leq r < s \leq n_i, \\
    0 & \text{if} \ 1 \leq s < r \leq n_i.
\end{cases}
\]

We observe that $|d^{(i)}_r(t)| \leq K_1 \exp(\kappa_1 t)$ with $K_1 > 0$, for $1 \leq r < s \leq n_i$ and by the equation (31), we have $\frac{\mu_r(t)}{\mu_s(t)} \leq K_2 \exp(\kappa_2 t)$ with $K_2 > 1$ and $\kappa_2 = 2v$ with $v$ as in (31), then
\begin{equation}
(34) \quad |\{\Lambda_i(t)\}_{rs}| \leq K_2 K_1 \exp(\kappa t),
\end{equation}
where $\kappa = \kappa_\varepsilon = \kappa_1 + \kappa_2$.

Let us define the transformation
\[
z_i(t) = R_i(t)w_i(t),
\]
with
\[
R_i(t) = \text{Diag}(\exp(-\kappa M_\delta K_\delta t), \eta \exp(-2\kappa M_\delta K_\delta t), \ldots, \eta^{n_i-1} \exp(-n_i \kappa M_\delta K_\delta t)),
\]

\textit{where} $\eta = 1$.
and we also define $K_\delta$ such that $K_\delta M_\delta \geq 1$ and

$$(35) \quad 0 < \eta < \frac{M_\delta}{M_\delta + K_1 K_2}. $$

Now, we can see that (33) and (13) are $\delta$-nonuniform kinematically similar to $\dot{w}_i = \Gamma_i(t)w_i$, where the $rs$-coefficient of $\Gamma_i(t)$ is

$$\{\Gamma_i(t)\}_{rs} = \begin{cases} 
\{\Lambda_i(t)\}_{rs} + r \kappa M_\delta K_\delta & \text{if } r = s, \\
\eta^{s-r} \{\Lambda_i(t)\}_{rs} \exp(-(s-r)\kappa M_\delta K_\delta t) & \text{if } 1 \leq r < s \leq n_i, \\
0 & \text{if } 1 \leq s < r \leq n_i.
\end{cases}$$

Let us observe that $\Gamma_i(t)$ can be written as follows:

$$\Gamma_i(t) = \tilde{C}_i(t) I + \tilde{B}_i(t),$$

where $\tilde{C}_i(t) = Diag(\tilde{c}_1(t), \ldots, \tilde{c}_{n_i}(t))$ and the $rs$-coefficient of $\tilde{B}_i(t)$ is defined by

$$\{\tilde{B}_i(t)\}_{rs} = \begin{cases} 
\{\lambda_i(t)\}_{rs} + r \kappa M_\delta K_\delta & \text{if } r = s, \\
\eta^{s-r} \frac{\mu_\delta(t)\mu_\delta(t)^j(t)}{\mu_\delta(t)} \exp(-(s-r)\kappa M_\delta K_\delta t) & \text{if } 1 \leq r < s \leq n_i, \\
0 & \text{if } 1 \leq s < r \leq n_i.
\end{cases}$$

By (32) and (34), we can verify that

$$\|\tilde{B}_i(t)\| \leq M_\delta [1 + r \kappa K_\delta] + K_1 K_2 [\eta + \eta^2 + \cdots + \eta^{n_i}] \leq M_\delta [1 + n_i \kappa K_\delta] + K_1 K_2 \frac{\eta}{1 - \eta}. $$

Recall that $M_\delta = \frac{\delta}{m}$ and by using (35) it follows that $\|\tilde{B}_i(t)\| \leq \frac{\delta}{m} K_\delta \varepsilon$, where $K_\delta \varepsilon = 2 + n_i \kappa K_\delta$.

Thus, for any $i \in \{1, \ldots, m\}$ the system (13) is $\tilde{\delta}$-nonuniform kinematically similar (with $\tilde{\delta} = \frac{\delta}{m}$) to

$$\tilde{\omega}_i = [\tilde{C}_i(t) + \tilde{B}_i(t)]w_i,$$

where

$$\tilde{c}_j(t) \in [a_i, b_i] = \Sigma(B_i), \quad j \in \{1, \ldots, n_i\} \quad \text{and} \quad \|\tilde{B}_i(t)\| \leq \frac{\delta}{m} K_\delta \varepsilon.$$

Finally, (13) is nonuniformly contracted to $\Sigma(B_i)$.

Step 5): The system (1) can be nonuniformly contracted to $\Sigma(A)$: By using the previous result, we can see that (1) is $\delta$-nonuniform kinematically similar to

$$\dot{w} = [C(t) + B(t)]w,$$

where

$$C(t) = Diag(\tilde{C}_1(t), \ldots, \tilde{C}_m(t)) \quad \text{and} \quad B(t) = Diag(\tilde{B}_1(t), \ldots, \tilde{B}_m(t)).$$

In consequence, note that

$$C(t) \subset \bigcup_{i=1}^{m} [a_i, b_i] = \Sigma(A) \quad \text{and} \quad \|B(t)\| \leq \delta K_\delta \varepsilon.$$
Finally, the system (1) is nonuniformly contracted to $\Sigma(A)$. 

\[\Box\]

Remark 8. The inequalities (24) and (25) show that the functions $\Phi(t, s)$ and $\Psi(t, s)$ are bounded. This bounds not necessarily cross to graph this functions, thus the conditions (C1)–(C4) allows us to find straights that cross it to least once. Moreover this procedure enable us to construct $\{T_i\}_{l=0}^{+\infty}$ which is the sequences of first crossing time of the graph of this function $\Phi(t, s)$ and $\Psi(t, s)$ with that straights. We have proved that this sequence of crossing time has not accumulations points.

6. Application of the main result

Example 1. Let us consider the scalar differential equation studied in [8, pp. 547]

\[(36) \quad \dot{x} = A(t)x, \quad \text{with} \quad A(t) = \lambda_0 + at \sin(t) \quad \text{and} \quad |\lambda_0| > 3|a|, \quad \lambda_0 < a < 0.\]

It is straightforward to verify that the example can be adapted to the case $\mathbb{R}_0^+$ and therefore the spectrum of (36) is $\Sigma(A) = [\lambda_0 + a, \lambda_0 - a]$.

We claim that (36) is nonuniformly contracted to $\Sigma(A)$. Indeed, given a fixed $\delta > 0$ and $\varepsilon = 2|a|$, we consider the matrix function $t \to S(t) \in \mathbb{R}^{n \times n}$ defined by

\[S(t) = \exp\left(\frac{\varepsilon}{2} t \cos(t) - \delta \sin(t)\right),\]

and we can verify that (36) is $\delta$-nonuniformly kinematically similar to

\[\dot{y} = (C(t) + B(t))y, \quad \text{with} \quad C(t) = \lambda_0 \quad \text{and} \quad B(t) = -\delta \cos(t) \left(1 + \frac{\varepsilon}{2}\right).\]

The claim follows since $C(t) \in [\lambda_0 + a, \lambda_0 - a]$ and $\|B(t)\| \leq \delta K_{\delta, \varepsilon}$, where $K_{\delta, \varepsilon} = \frac{\varepsilon}{2\delta}$.

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