A Necessary and Sufficient Condition on the Weyl Manifolds Admitting a Semi Symmetric Non-Metric Connection to be S-Concircular

Assist. Prof. Dr. Füsun NURCAN BAŞTAN
The Department of Mathematics,
Faculty of Science and Letters,
Marmara University, TURKEY.
funal@marmara.edu.tr

November 14, 2014

Abstract

The object of this paper is to obtain the concircular curvature tensor of the semi symmetric non-metric connection on the Weyl manifold and to give a necessary and sufficient condition for a semi symmetric non-metric connection to be S-concircular.

Keywords: Weyl manifolds, semi symmetric non-metric connection, concircular curvature tensor, S-concircular connection.

Mathematical Subject Classification :53A40

1 Introduction

An n-dimensional manifold which has a symmetric connection $\nabla$ and a conformal metric tensor $g$ satisfying the compatibility condition

$$\nabla g = 2 (T \otimes g)$$

where $T$ is a 1-form is called a Weyl space which is denoted by $W_n(g, T)$, (see [1]). In local coordinates, the compatibility condition is given by

$$\nabla_k g_{ij} - 2g_{ij} T_k = 0$$

where $T_k$ is a complementary covariant vector field. Such a Weyl manifold will be denoted by $W_n(g_{ij}, T_k)$. If $T_k = 0$ or $T_k$ is gradient, a Riemannian manifold is obtained.
In [1], under the transformation of the metric tensor $g_{ij}$ in the form of

\[ \tilde{g}_{ij} = \lambda^2 g_{ij} \] (1.2)

$T_k$ changes by

\[ \tilde{T}_k = T_k + \partial_k (\ln \lambda), \]

where $\lambda$ is a scalar function defined on $W_n$.

The coefficients $\Gamma^i_{jk}$ of the symmetric connection $\nabla$ on the Weyl manifold are defined by

\[ \Gamma^i_{jk} = \{ i \}_{jk} - g^{im} (g_{mj} T_k + g_{mk} T_j - g_{jk} T_m) \] (1.3)

where $\{ i \}_{jk}$'s are the coefficients of the Levi-Civita connection, (see [1]).

In [1], the coefficients $\Gamma^i_{jk}$ and the curvature tensor $R^h_{ijk}$ of the symmetric connection $\nabla$ change by

\[ \Gamma^i_{jk} \ast = \Gamma^i_{jk} + \delta^i_j P_k + \delta^i_k P_j - g_{jk} P^i \] (1.4)

and

\[ R^h_{ijk} \ast = R^h_{ijk} + 2 \delta^h_i \nabla_{[j} P_{k]} + \delta^h_j P_i - \delta^h_k P_j + g_{ij} g^{hr} P_k - g_{jk} g^{hr} P_i \] (1.5)

where $T^i - T^i \ast = P_i$ and $P_{ij} = \nabla_j P_i - P_i P_j + \frac{1}{2} g_{ij} g^{kr} P_k P_r$ under conformal mapping $g_{ij}^\ast = g_{ij}$.

The conformal curvature tensor $C^h_{ijk}$ and the concircular curvature tensor $Z^h_{ijk}$ of the symmetric connection $\nabla$ on the Weyl manifold are given by

\[ C^h_{mijk} = R^h_{mijk} - \frac{1}{n} g_{mi} R^r_{rjk} + \frac{1}{n-2} (g_{mj} R_{ik} - g_{mk} R_{ij} - g_{ij} R_{mk} + g_{ik} R_{mj}) \\
- \frac{1}{n(n-2)} (g_{mj} R^r_{rkj} - g_{mk} R^r_{rjk} - g_{ij} R^r_{rkm} + g_{ik} R^r_{rjm}) \] (1.6)

and

\[ Z^h_{mijk} = R^h_{mijk} - \frac{R}{n(n-1)} (g_{mk} g_{ij} - g_{mj} g_{ik}), \] (1.7)

where $R^h_{ijk}$, $R_{ij}$ and $R$ denote the curvature tensor, Ricci tensor and scalar curvature tensor of $\nabla$, respectively, (see [2], [3]).

In [4], V.Murgescu defined the coefficients $\Gamma^i_{jk}$ of a generalized connection $\nabla^i$ on the Weyl manifold by

\[ \Gamma^i_{jk} = \Gamma^i_{jk} + a_{jkh} g^{hi} \] (1.8)

where

\[ a_{jkh} = g_{jr} \Omega^r_{kh} + g_{rk} \Omega^r_{jh} + g_{rh} \Omega^r_{jk} \] (1.9)
and $\Gamma_{ijk}^i$'s are the coefficients of the symmetric connection $\nabla$.

By choosing

$$\Omega_{ijk}^i = \delta_j^i a_k - \delta_k^i a_j$$

in (1.9), the coefficients $\Gamma_{ijk}^i$'s of a semi symmetric non-metric connection $\nabla$ on the Weyl manifold are obtained by

$$\Gamma_{jk}^i = \Gamma_{jk}^i + \delta_j^i S_j - g_{jk} S^i$$

(1.10)

(see [3]). In (1.10), $S_i = -2a_i$ where $a_i$ is an arbitrary covariant vector field.

The following results are also obtained in [5]:

The torsion tensor $T_{jk}^i$ with respect to the semi symmetric connection $\nabla$ is

$$T_{jk}^i = \delta_j^i S_j - \delta_j^i S_k$$

(1.11)

The curvature tensor $R(X, Y) Z$ of the semi symmetric non-metric connection $\nabla$ on the Weyl manifold is defined by

$$R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

In local coordinates, above equation becomes

$$R_{ijk}^h = \partial_j \Gamma_{ik}^h - \partial_i \Gamma_{jk}^h + \Gamma_{rj}^h \Gamma_{ik}^r - \Gamma_{rk}^h \Gamma_{ij}^r$$

(1.12)

By means of (1.10) and (1.12), the relation between the curvature tensors $R_{ijk}^h$ and $\mathcal{R}_{ijk}^h$ of $\nabla$ and $\mathcal{\nabla}$, respectively, is obtained as

$$\mathcal{R}_{ijk}^h = R_{ijk}^h + \delta_j^i S_{ik} - \delta_j^i S_{ik} + g_{ij} g^{hr} S_{rk} - g_{ik} g^{hr} S_{rj}$$

(1.13)

where

$$S_{ij} = S_{i,j} - S_i S_j + \frac{1}{2} g_{ij} g^{kr} S_{kr}$$

(1.14)

and $S_{i,j}$ denotes the covariant derivative of $S_i$ with respect to the symmetric connection $\nabla$.

Transvecting (1.13) by $g_{mh}$ and contracting on the indices $h$ and $k$ in the same equation give

$$\mathcal{R}_{mijk} = R_{mijk} + g_{mk} S_{ij} - g_{mj} S_{ik} + g_{ij} S_{mk} - g_{ik} S_{mj}$$

(1.15)

and

$$\mathcal{R}_{ij} = R_{ij} + (n - 2) S_{ij} + S g_{ij}$$

(1.16)

where $S = g^{mk} S_{mk}$, respectively.

The scalar curvatures $R$ and $\mathcal{R}$ of the connections $\nabla$ and $\mathcal{\nabla}$, respectively, are related by

$$\mathcal{R} = R + 2(n - 1) S$$

(1.17)

The curvature tensor of the semi-symmetric connection $\mathcal{\nabla}$ has the following properties:
a $\overline{R}_{mijk} + \overline{R}_{mikj} = 0$,

b $\overline{R}_{mijk} + \overline{R}_{imjk} = 4g_{mi}\nabla_j [k T_j]$,

c $\overline{R}_{rjk} = R_{rjk} = 2R_{[kj]} = 2n\nabla_{[k T_j]}$

d $\overline{R}_{mijk} + \overline{R}_{mikj} + \overline{R}_{mkij} = 2 (g_{mi}\nabla_j [k S_j] + g_{mj}\nabla_i [S_k] + g_{mk}\nabla_j [j S_i])$.

The conformal curvature tensor $\overline{C}_{mijk}$ of $\nabla$ is given by

$$
\overline{C}_{mijk} = \overline{R}_{mijk} - \frac{1}{n}g_{mi}\overline{R}_{rjk} + \frac{1}{n-2} \{g_{mj}\overline{R}_{rk} - g_{mk}\overline{R}_{rij} - g_{ij}\overline{R}_{rkm} + g_{ik}\overline{R}_{rjm}\} 
$$

$\overline{R}_{rjk} = \overline{R}_{ijk} = 2\overline{R}_{[kj]} = 2n\overline{\nabla}_j [k T_j]$,

$$
\overline{R}_{mijk} + \overline{R}_{mikj} + \overline{R}_{mkij} = 2 (g_{mi}\nabla_j [k S_j] + g_{mj}\nabla_i [S_k] + g_{mk}\nabla_j [j S_i]).
$$

The conformal curvature tensors $C_{mijk}$ and $\overline{C}_{mijk}$ of the connections $\nabla$ and $\nabla$ are related by

$$
\overline{C}_{mijk} = C_{mijk}
$$

The projective curvature tensor $\overline{W}_{mijk}$ of $\nabla$ is in the form of

$$
\overline{W}_{mijk} = \overline{R}_{mijk} + \frac{g_{mi}}{n+1} \{ (\overline{R}_{jk} - \overline{R}_{kj}) + 2(n-1)\overline{\nabla}_j [S_k]\}
$$

$$
+ \frac{1}{n^2-1} \{g_{mj}\overline{H}_{ik} - g_{mk}\overline{H}_{ij}\}
$$

where

$$
\overline{H}_{ij} = n\overline{R}_{ij} + \overline{R}_{ji} + 2(n-1)\overline{\nabla}_j [S_i].
$$

The projective curvature tensors $W_{mijk}$ and $\overline{W}_{mijk}$ of the connections $\nabla$ and $\nabla$ are related by the equation

$$
W_{mijk} = W_{mijk} + \frac{2}{n+1}g_{mi}\nabla_j [S_k] + \frac{1}{n^2-1} (g_{mk}K_{ij} - g_{mj}K_{ik}) + g_{ij}S_{mk} - g_{ik}S_{mj}
$$

where

$$
K_{ij} = nS_{ij} + S_{ji} + (n+1)S_{gi}.
$$

2 Weyl manifolds admitting a semi symmetric non-metric connection under concircular mapping

Let $\sigma : (W_n, g_{ij}, T_k, S_k) \to (W_n^*, g_{ij}^*, T_k^*, S_k^*)$ be a conformal mapping given by $g_{ij}^* = g_{ij}$. In $[5]$, according to this mapping, the coefficients $\Gamma^*_{jk}$ and the curvature tensor $\overline{R}_{jk}^*$ of the semi symmetric connection $\nabla^*$ change by:

$$
\Gamma^*_{jk} = \Gamma_{jk} + \delta^*_{jk} P_k + \delta^*_{jk} (P_j - Q_j) - g_{jk} (P^* - Q^*)
$$
where \( P_j = T_j - T_j^\ast \), \( Q_j = S_j - S_j^\ast \) and

\[
\overline{\underline{R}}^h_{ijk} = \underline{R}^h_{ijk} + 2\delta^h_i (\nabla_j P_k + P_j S_k) + \delta^h_k W_{ij} - \delta^h_i W_{ik} + g_{ij} g^{hr} W_{rk} \]
\[
- g_{ik} g^{hr} W_{rj} + 2 g^{sr} P_s Q_r (\delta^h_j g_{ik} - \delta^h_k g_{ij})
\]

(2.2)

where

\[
W_{ij} = P_{ij} - Q_{ij} + 2 P_{ij} (Q_{ij})
\]
\[
P_{ij} = P_{ij} - P_i S_j,
\]
\[
Q_{ij} = Q_{ij} - Q_i S_j,
\]

Since a conformal mapping which \( W_{ij} = \phi g_{ij} \) changes a geodesic circle into a geodesic circle, it is called concircular mapping by means of [6].

Let \( \sigma \) be a concircular mapping, that is, \( P_{ij} \) is symmetric. Then, (2.2) can be rewritten as follows:

\[
\overline{\underline{R}}^h_{ijk} = \underline{R}^h_{ijk} + 2 (\phi - g^{sr} P_s Q_r) (\delta^h_k g_{ij} - \delta^h_i g_{jk})
\]

(2.3)

By contracting on \( h \) and \( k \) in (2.3),

\[
\overline{\underline{R}}^r_{ij} = \underline{R}^r_{ij} + 2 (n - 1) (\phi - g^{sr} P_s Q_r) g_{ij}
\]

(2.4)

Transvecting (2.4) by \( g^{ij} = g^{ij}_* \), yields

\[
\overline{\underline{R}}^r = \underline{R} + 2 n (n - 1) (\phi - g^{sr} P_s Q_r)
\]

(2.5)

If the expression \( 2 (\phi - g^{sr} P_s Q_r) = \overline{\underline{R}}^r / n (n - 1) \) obtained from (2.5) is substituted in (2.3),

\[
\overline{\underline{R}}^h_{ijk} = \underline{R}^h_{ijk} + \frac{\overline{\underline{R}}^r - \underline{R}}{n (n - 1)} (\delta^h_k g_{ij} - \delta^h_i g_{jk})
\]

is arranged as

\[
\overline{\underline{R}}^h_{ijk} = \frac{\overline{\underline{R}}^r}{n (n - 1)} (\delta^h_k g_{ij} - \delta^h_i g_{jk}) = \underline{R}^h_{ijk} - \frac{\underline{R}^r}{n (n - 1)} (\delta^h_k g_{ij} - \delta^h_i g_{jk}).
\]

If the concircular curvature tensor \( \overline{\underline{Z}}^h_{ijk} \) is defined by

\[
\overline{\underline{Z}}^h_{ijk} = \underline{Z}^h_{ijk} - \frac{\underline{Z}^r}{n (n - 1)} (\delta^h_k g_{ij} - \delta^h_i g_{jk}),
\]

(2.6)

it is invariant under the concircular transformation, i.e.

\[
\overline{\underline{Z}}^h_{ijk} = \underline{Z}^h_{ijk}.
\]
First transvecting (2.6) by $g_{mh}$ and then contracting on the indices $h$ and $k$ in (2.6), the equations

$$
\mathcal{Z}_{mijk} = \mathcal{R}_{mijk} - \frac{\mathcal{R}}{n(n-1)} (g_{mk}g_{ij} - g_{mj}g_{ik})
$$

and

$$
\mathcal{Z}_{ij} = \mathcal{R}_{ij} - \frac{\mathcal{R}}{n} g_{ij}
$$

are obtained.

**Lemma 1** The concircular curvature tensor of the semi symmetric connection $\nabla$ has the following properties:

a. $\mathcal{Z}_{mijk} + \mathcal{Z}_{mikj} = 0$,

b. $\mathcal{Z}_{mijk} + \mathcal{Z}_{imjk} = 4g_{mi}\nabla[kT_j]$,

c. $\mathcal{Z}_{rjk} = \mathcal{R}_{rjk}$,

d. $\mathcal{Z}_{mijk} + \mathcal{Z}_{mjki} + \mathcal{Z}_{mkij} = 0$.

The concircular curvature tensors $\mathcal{Z}^h_{ijk}$ and $\mathcal{Z}^h_{ijk}$ of $\nabla$ and $\nabla$, respectively, are related by

$$
\mathcal{Z}^h_{ijk} = Z^h_{ijk} + \delta^h_{k}S_{ij} - \delta^h_{j}S_{ik} + g_{ij}g^{mh}S_{mk} - g_{ik}g^{mh}S_{mj} - \frac{2}{n}S(\delta^h_{k}g_{ij} - \delta^h_{j}g_{ik})
$$

by substituting (1.7), (1.13) and (1.17) in (2.6).

Transvecting (2.9) by $g_{mh}$ and contracting on $h$ and $k$ in the same equation give

$$
\mathcal{Z}_{mijk} = Z_{mijk} + g_{mk}S_{ij} - g_{mj}S_{ik} + g_{ij}S_{mk} - g_{ik}S_{mj} - \frac{2}{n}S(g_{mk}g_{ij} - g_{mj}g_{ik})
$$

and

$$
\mathcal{Z}_{ij} = Z_{ij} + (n-2)S_{ij} - \frac{(n-2)}{n}g_{ij}S
$$

3 Semi symmetric non-metric S-Concircular Connection

In [7], Liang defined semi symmetric recurrent metric connection which is S-concircular on the Riemannian manifolds. In this paper, a semi symmetric non-metric S-concircular connection on the Weyl manifold is defined by as follows:
Definition 2  If the semi symmetric non-metric connection $\nabla$ satisfies the condition given by

$$S_{ij} = \nabla_j S_i - S_i S_j + \frac{1}{2} g_{ij} g^{rs} S_r S_s = \beta g_{ij}$$

where $\beta$ is a smooth function on the Weyl manifold, then it is called $S$-concircular.

Suppose that the concircular curvature tensors $Z_{mijk}$ and $\overline{Z}_{mijk}$ of symmetric and semi symmetric non-metric connections $\nabla$ and $\overline{\nabla}$, respectively, be the same. Then

$$\overline{R}_{mijk} - \frac{R}{n(n-1)} (g_{mk} S_{ij} - g_{mj} S_{ik}) = R_{mijk} - \frac{R}{n(n-1)} (g_{mk} S_{ij} - g_{mj} S_{ik}) .$$

(3.1)

By using (1.15) in (3.1), we get

$$g_{mk} S_{ij} - g_{mj} S_{ik} + g_{ij} S_{mk} - g_{ik} S_{mj} = \frac{\overline{R} - R}{n(n-1)} (g_{mk} S_{ij} - g_{mj} S_{ik})$$

(3.2)

By transvecting (3.2) by $g^{mk}$, it is obtained as

$$(n-2) S_{ij} + S g_{ij} = \frac{\overline{R} - R}{n} g_{ij} .$$

By (1.17),

$$S_{ij} = \frac{\overline{R} - R}{2n(n-1)} g_{ij}$$

(3.3)

which states that $\overline{\nabla}$ is $S$-concircular.

Conversely, suppose that $\overline{\nabla}$ is $S$-concircular. By using $S_{ij} = \beta g_{ij}$, from Definition 2 in (2.10), it is obtained as

$$Z_{mijk} = Z_{mijk} .$$

In the view of the above results, we can state the following theorem:

Theorem 3 The necessary and sufficient condition for the semi symmetric non-metric connection $\nabla$ to be $S$-concircular is that the concircular curvature tensors $Z_{mijk}$ and $\overline{Z}_{mijk}$ of the connections $\nabla$ and $\overline{\nabla}$, respectively, coincide.

References

[1] Norden, A., Affinely connected spaces, GRMFL, Moscow (in Russian), 1976.
[2] Miron, R., Mouvements conformes dans les espaces $W_n$, Tensor N.S., 1968, 19, 33-41.
[3] Özdeğer, A., Şentürk, Z., Generalized Circles in Weyl Spaces and their conformal mapping, Publ. Math. Debrecen, 2002, 60, 1-2, 75-87.
[4] Murgescu, V., Espaces de Weyl a torsion et leurs representations conformes, Ann. Sci. Univ. Timisoara, 1968, 221-228.

[5] Unal, F.&Uysal, A., Weyl Manifolds with semi-symmetric connections, Mathematical and Computational Applications, 2005, Vol.10, No:3.

[6] Yano, K., Concircular Geometry I. Concircular Transformations, Mathematical Institute, Tokyo Imperial University, 1940, 195-200.

[7] Liang, Y., On semi symmetric recurrent metric S-concircular connections, Journal of Mathematical Study, 1994, 104-108.