Better Synchronizability Predicted by Crossed Double Cycle

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In this brief report, we propose a network model named crossed double cycles, which are completely symmetrical and can be considered as the extensions of nearest-neighboring lattices. The synchronizability, measured by eigenratio $R$, can be sharply enhanced by adjusting the only parameter, crossed length $m$. The eigenratio $R$ is shown very sensitive to the average distance $L$, and the smaller average distance will lead to better synchronizability. Furthermore, we find that, in a wide interval, the eigenratio $R$ approximately obeys a power-law form as $R \sim L^{-1.5}$.

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Synchronization is observed in a variety of natural, social, physical and biological systems[1], and has found applications in a variety of field including communications, optics, neural networks and geophysics.[2] The large networks of coupled dynamical systems that exhibit synchronized state are subjects of great interest. In the early stage, the corresponding studies are restricted to either the regular networks[8, 9], or the random ones[10, 11]. However, recent empirical studies have demonstrated that many real-life networks can not be treated as regular or random networks. The most important two of their common statistic characteristics are called small-world effect[12] and scale-free property[13]. Therefore, very recently, most of the studies about network synchronization focus on complex networks, and find that the networks of small-world effect and scale-free property may be easier to synchronize than regular lattices[14-16].

One of the ultimate goals in studying network synchronization is to understand how the network topology affects the synchronizability. In the simplest case (see below), the network synchronizability can be well measured by the eigenratio $R$[17, 18, 19, 20], thus the above question degenerates to understanding the relationship between network structure and its eigenvalues. Since there are countless topological characters for networks, a natural question is addressed: what is the most important factor by which the synchronizability of the system is mainly determined? Some previous works indicated the average distance $L$[21] is one of the key factors. However, the consistent conclusion have not been achieved[19, 22, 23, 24, 25]. Another extensively studied one is the network heterogeneity, which can be measured by the variance of degree distribution or betweenness distribution[26, 27]. Some detailed comparisons among various networks have been done, indicating the network synchronizability will be better with smaller heterogeneity[24, 28, 29]. However, a well-known counterexample is the regular networks with homogeneous structure while displaying very poor synchronizability. Because of the networks used for comparing in previous studies are of both varying average distances and degree variances, the strict and clear conclusions can not be achieved. In addition, some researchers deem that the more intrinsic ingredient leading to better synchronizability is the randomness[30], that is to say, the intrinsic reason making small-world and scale-free networks having better synchronizability than regular ones is their random structures. Therefore, if one wants to clearly show how $L$ affects the network synchronizability, he should investigate the networks of different $L$ but the same degree variance. And if he wants to assert that it is not the randomness but smaller (or longer) $L$ resulting in the better synchronizability, the deterministic networks are required.

In this brief report, we proposed a deterministic network model named Crossed Double Cycles (CDCs for short). The CDCs are of degree variance equal to zero, and by adjusting the only parameter $m$, named crossed length, the average distance of CDCs can be changed. By using this ideal model, we demonstrate that the smaller $L$
transitivity networks are completely symmetric, which
\[ \theta \] is vertex-transitivity, that is to say, for any two vertices 
\[ C \] and 
\[ N \] shown in Fig. 1.

For \( m \) two vertices connecting by a crossed edge are of distance 
\( a \) one-dimensional lattice with periodic boundary con-

FIG. 2: (Color online) The average distance of the CDCs. The
black squares, red circles, blue triangles and green pentagons
represent the case of \( N = 1000, 2000, 3000 \) and \( 4000 \), re-
respectively. The smaller solid symbols and larger hollow symbols
represent the simulation and analytic results, respectively.

will result in better synchronizability, and provide a useful
method to enhance the synchronizability of nearest-
neighbor coupling networks.

In network language \[31\], the cycle \( C_N \) denotes a net-
work consisting of \( N \) vertices \( x_1, x_2, \ldots, x_N \). These \( N \)
vertices is arranged as a ring, and the nearest two vertices
are connected. Hence, \( C_N \) has \( N \) edges connecting the
vertices \( x_1 x_2, x_2 x_3, \ldots, x_{N-1} x_N \), and \( x_N x_1 \). The CDCs,
denoted by \( G(N, m) \), can be constructed by adding two
edges, called crossed edges, to each vertex in \( C_N \). The
two vertices connecting by a crossed edge are of distance
\( m \) in \( C_N \). For example, the network \( G(N, 3) \) can be con-
structed from \( C_N \) by connecting \( x_1 x_4, x_2 x_5, \ldots, x_{N-1} x_2 \),
and \( x_N x_3 \). And the network \( G(N, 2) \) is isomorphic\[32\] to
a one-dimensional lattice with periodic boundary condi-
tions wherein each vertex connects to its nearest and
next-nearest neighbors. A sketch map of \( G(20, 4) \) is
shown in Fig. 1.

Clearly, all the vertices in \( G(N, m) \) are of degree 4, thus
the degree variance is equal to 0. Furthermore, \( G(N, m) \)
is vertex-transitivity, that is to say, for any two vertices 
\( x \) and \( y \) in \( G(N, m) \), there exists an automorphism map-
ing \( \theta : V(G) \rightarrow V(G) \) such that \( y = \theta(x) \). The vertex-
transitivity networks are completely symmetric, which
are of particular practicability in the design of topolog-
ical structures of data memory allocation and multiple
processor systems \[33\].

Denote \( L(k) \) the average distance of \( C_{k+1} \), we have
\[ L(k) = \begin{cases} \frac{k+2}{(k+1)^2}, & k \text{ is even} \\ \frac{4k}{k+1}, & k \text{ is odd} \end{cases} \] (1)

For \( N \gg m \), we assume \( N \) can be exactly divided by \( m \)
and denote \( k = \frac{N}{m} \). Since \( G(N, m) \) is vertex-transitivity,

the average distance of \( G(N, m) \) is equal to the average
distance between vertex \( x_1 \) to all other vertices. The
network \( G(N, m) \) contains \( k \) end-to-end \( C_{m+1} \) as
\[ x_1 x_2 \cdots x_{m+1}, x_{m+1} x_{m+2} \cdots x_{2m+1}, \ldots, x_{N+m-1} \cdots x_N x_1 \].
Then, go from the vertex \( x_1 \) to a certain vertex \( x_i \) can
be divided to two processes. Firstly, travel through the
crossed edges to the nearest vertex that belongs to \( C_i \);
and denote \( m \) as the cases \( m = 2, m = 3, \) and \( m = 4 \), respectively.
The corresponding curves represent the analytical solutions.

FIG. 3: (Color online) Eigenvalues. The network size \( N = 4000 \) if fixed, and the black squares, red circles and green tri-
angles denote the numerical results of eigenvalues for the cases
\( m = 2, m = 3, \) and \( m = 4 \), respectively. The corresponding
curves represent the analytical solutions.

FIG. 4: (Color online) The smallest nonzero eigenvalue \( \gamma_1(\theta_i) \)
vs \( m \). The black squares, red circles and green triangles denote
the cases \( N = 10000, N = 250000, \) and \( N = 1000000 \),
respectively. The solid lines are of slope 2 for comparison.
The inset shows the cut-off point \( m_c \) as a function of network
size.
path from \( x_1 \) to \( x_{10} \) in \( G(20,4) \) is \( x_1 \rightarrow x_5 \rightarrow x_9 \rightarrow x_{10} \). The first two edges are the crossed edges and identified by red lines in Fig. 1, and the last edge is in the cycle \( x_9x_{10}x_{11}x_{12}x_{13} \) with \( N = 1000, 2000, 3000, 4000 \) and \( m \leq 100 \), which accurately agree with the analytic ones.

In succession, we investigate the changes of CDCs’ synchronizability with \( m \). Consider \( N \) identical dynamical systems (oscillators) with the same output function, which are located on the vertices of a network and coupled linearly and symmetrically with neighbors connected by edges of the network. The coupling fashion ensures the synchronization manifold an invariant manifold, and the dynamics can be locally linearized near the synchronous state. The state of the \( i \)th oscillator is described by \( \mathbf{x}^i \), and the set of equations of motion governing the dynamics of the \( N \) coupled oscillators is

\[
\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) + \sigma \sum_{j=1}^{N} G_{ij} \mathbf{H}(\mathbf{x}^j),
\]

where \( \dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) \) governs the dynamics of individual oscillator, \( \mathbf{H}(\mathbf{x}^j) \) is the output function and \( \sigma \) is the coupling strength. The \( N \times N \) Laplacian \( \mathbf{G} \) is given by

\[
G_{ij} = \begin{cases} 
  k_i & \text{for } i = j \\
  -1 & \text{for } j \in \Lambda_i, \\
  0 & \text{otherwise}
\end{cases}
\]

Because of the positive semidefinite of \( \mathbf{G} \), all the eigenvalues of it are nonnegative reals and the smallest eigenvalue \( \theta_0 \) is always zero, for the rows of \( \mathbf{G} \) have zero sum. Thus, the eigenvalues can be ranked as \( \theta_0 \leq \theta_1 \leq \cdots \leq \theta_{N-1} \). The ratio of the maximum eigenvalue \( \theta_{N-1} \) to the smallest nonzero one \( \theta_1 \) is widely used to measure the synchronizability of the network\([17,17]\), if the eigenratio \( R = \theta_{N-1}/\theta_1 \) satisfies

\[
R < \alpha_2/\alpha_1,
\]

we say the network is synchronizable. The right-hand side of this inequality depends only on the dynamics of individual oscillator and the output function\([11]\), while the eigenratio \( R \) depends only on the Laplacian \( \mathbf{G} \). \( R \) indicates the synchronizability of the network, the smaller it is the better synchronizability and vice versa. In this brief report, for universality, we will not address a particular dynamical system, but concentrate on how the network topology affects eigenratio \( R \).

Since the Laplacian for any CDC is shift invariant, the eigenvalues can be calculated from a discrete Fourier transform of a row of the Laplacian matrix\([20]\). Denote \( \gamma_i \) (\( i = 0,1,\cdots,N-1 \)) the eigenvalue, and it reads

\[
\gamma_i = 2 \left( 1 - \cos \frac{2\pi i}{N} \right).
\]

Figure 3 shows the numerical results of eigenvalues, which accurately agree with the analytic solutions. Clearly, for odd \( m \) and even \( N \), the maximal eigenvalue is \( \theta_{N-1} = \gamma_{N/2} = 8 \). However, if \( m \) is even or \( N \) is odd, the maximal eigenvalue \( \theta_{N-1} \) is smaller than 8 and can not be expressed sententiously. The smallest nonzero eigenvalue \( \theta_1 \) equals \( \gamma_1 \), and can be approximately obtained under the condition \( N \gg m \)

\[
\theta_1 = \gamma_1 \approx \frac{4\pi^2}{N^2}(1 + m^2).
\]

Figure 4 reports \( \gamma_1 \) as a function of \( m \), which scales as \( m^2 \), as predicted by Eq. (7), before reaching a cut-off point \( m_c \). The numerical value of \( m_c \) is also shown in the inset of Fig. 4, which accurately obeys the form \( m_c = \sqrt{N} \).

Because of the cut-off in \( \theta_1 \), the fluctuations in \( \theta_{N-1} \) for even \( m \) and the relatively complex relationship between \( m \) and \( L \), we can not obtain a straightforward expression to comprehensively depict the relationship between \( R \) and \( L \). In Fig. 5, we only report the numerical results about how the average distance affects the network synchronizability. One can see clearly, the network synchronizability is very sensitive to the average distance: as the increase of \( L \), the eigenratio \( R \) sharply spans more than three magnitudes. And the network synchronizability is remarkably enhanced by reducing \( L \). When the crossed length \( m \) is not very small or very large (comparing with \( N \)), the networks with the same average distance have approximately the same synchronizability no matter what the network size is. More interesting, the calculated results indicate that the eigenratio \( R \) approximately obeys

FIG. 5: (Color online) \( R \) vs \( L \). The black squares, red circles, blue triangles and green pentagons represent the case of \( N = 1000, 2000, 3000 \) and \( 4000 \), respectively. The inset shows the same data in log-log plot, indicating that the eigenratio \( R \) approximately obeys a power-law form as \( R \sim L^{1.5} \). The solid line is of slope 1.5 for comparison.
a power-law form as $R \sim L^{1.5}$ in a wide interval of $L$ (see the inset of Fig. 5).

To sum up, we propose an ideal network model, and investigate its synchronizability. The results indicate that the average distance is an important factor affecting the network synchronizability greatly. The smaller average distance will lead to better synchronizability. This is similar to the communication systems, wherein the average distance is one of the most important parameters to measure the transmission delay (or time delay) encountered by a message travelling through the network from its source to destination, and the smaller average distance means higher efficiency for homogeneous networks. Very recently, by numerical studies, some authors think that there may exist some common features between dynamics on communication networks (traffic and diffusion) and network synchronization\cite{20, 23, 24, 25}. Since in the former dynamics shorter $L$ will lead to greater throughput and fast spread, the underlying common features provide a possible explanation why shorter average distance corresponding to better synchronizability.

The CDCs are natural extensions of the lattice of nearest neighbors, they are symmetric and with better synchronizability, thus have great potential in the applications for designing of topological structures of distributed processing systems, local area networks, data memory allocation and data alignment in single instruction multiple data processors\cite{28}. In fact, the processor network of one kind of the earliest parallel processing computers is $G(16, 4)$\cite{23}. Besides in synchronization, the cross method also has been applied in communication systems. For example, the crossed cubes have much larger throughput than hypercubes thus are widely used in designing parallel computing networks\cite{28, 40}

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