Feigin-Fuchs Representations for Nonequivalent Algebras of $N = 4$ Superconformal Symmetry

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Abstract

The $N = 4$ SU(2)$_k$ superconformal algebra has the global automorphism of SO(4) $\approx$ SU(2)×SU(2) with the left factor as the Kac-Moody gauge symmetry. As a consequence, an infinite set of independent algebras labeled by $\rho$ corresponding to the conjugate classes of the outer automorphism group SO(4)/SU(2)=SU(2) are obtained à la Schwimmer and Seiberg. We construct Feigin-Fuchs representations with the $\rho$ parameter embedded for the infinite set of the $N = 4$ nonequivalent algebras. In our construction the extended global SU(2) algebras labeled by $\rho$ are self-consistently represented by fermion fields with appropriate boundary conditions.
Two dimensional conformal and superconformal field theories have become the fundamental subject for study in pursuing superstring theories in particle physics or two dimensional critical phenomena in statistical physics. The underlying superconformal algebras with \( N=0,1,2,3 \) and 4 have been much studied, and their representation theories have been investigated to a great extent.

In particular it is well recognized nowadays that the so-called Feigin-Fuchs (FF) representations (or the Coulomb-gas representations) \([1]\) are very important and almost inevitably required tools for investigating the representation theories of the conformal and superconformal algebras. By now we have established the FF representations of the superconformal algebras with higher number of supercharges \([2, 3, 4, 5, 6]\), up to \( N=4 \) \([7, 8, 9]\).

On the other hand, the spectral flows resulting from the \textit{inner} automorphisms of the conformal and superconformal algebras with \( N=2,3 \) and 4 were first recognized by Schwimmer and Seiberg \([10]\), and their remarkable implications on the representation theories of the algebras have been discussed by many people \([11, 12, 13]\).

In their same paper \([10]\) was presented the mechanism through which the truly different types of algebras arise for each \( N \). In general, a given algebra has a global automorphism group \( G \). Then, the different types of algebras are obtained by imposing boundary conditions on the generators of the given algebra and are labeled by the conjugate classes of \( G \). However, some of the \textit{twists} introduced in this manner can be removed by operating the local gauge transformations on the algebras which are reflected by the presence of the Kac-Moody subalgebras. The truly independent algebras are then labeled by those conjugate classes of the global automorphisms not contained in the local automorphisms. In other words, the independent algebras correspond to the conjugate classes of the \textit{outer} automorphism group of the algebra, while the \textit{inner} automorphisms give equivalent algebras.
In the present paper we shall study the $N = 4$ SU(2)$_k$ superconformal algebra from this algebraic point of view. The $N = 4$ SU(2)$_k$ superconformal algebra has the global SU(2) as well as the local SU(2) Kac-Moody symmetries embedded. Consequently the algebra exhibits the global automorphism group structure of SO(4)$\approx$SU(2)$\times$SU(2), resulting in the infinite set of algebras through periodicity conditions imposed on the generators. Removing some of the *twists* by use of the local SU(2) gauge transformations, the truly independent algebras are obtained which are labeled by $\rho$ corresponding to the conjugate classes of the *outer* automorphism group SO(4)/SU(2)$=SU(2)$.

In order to make the paper self-contained and also to fix our notations, we shall first summarise the known results which are relevant to our study. Then, we shall construct the FF representations of the truly independent algebras being labeled by the $\rho$ parameter. The representation theories along this line has not been fully investigated so far. The attempt to construct unitary representations of the $\rho$-extended algebras has been challenged [14], but has remained to be unsuccessful. Our construction of the FF representations allow one to study not only the unitary, but also the nonunitary representations of the infinite set of the $\rho$-extended algebras.

The $N=4$ SU(2)$_k$ superconformal algebra is defined by the form of the operator product expansions (OPE) among operators given by the energy-momentum tensor $L(z)$, the SU(2)$_k$ local nonabelian generators $T^i(z)$, and the iso-doublet and iso-antidoublet supercharges $G^a(z)$ and $\bar{G}_a(z)$ [7, 8, 15]:

$$
L(z)L(w) \sim \frac{3k}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w},
$$

$$
T^i(z)T^j(w) \sim \frac{k\eta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}\eta_{kl}T^l(w)}{z-w},
$$

$$
T^i(z)G^a(w) \sim -\frac{1}{2}(\sigma^i)^a_b G^b(w) + \frac{1}{2}G^a(w)(\sigma^i)^b_a \frac{\partial_w G^a(w)}{z-w},
$$

$$
L(z)G^a(w) \sim \frac{\partial_w G^a(w)}{(z-w)^2} + \frac{\partial_w G^a(w)}{z-w},
$$

$$
G^a(z)G^b(w) \sim 0,
$$

$$
\bar{G}_a(z)\bar{G}_b(w) \sim 0,
$$

$$
L(z)G^a(w) \sim \frac{3}{2}G^a(w) + \frac{\partial_w G^a(w)}{z-w},
$$

$$
L(z)\bar{G}_a(w) \sim \frac{3}{2}\bar{G}_a(w) + \frac{\partial_w \bar{G}_a(w)}{z-w}.
$$
\[ G^a(z) \bar{G}_b(w) \sim \frac{4k \delta^a_b}{(z-w)^3} - \frac{4(\sigma^i)^a_b \eta_{ij} T^j(w)}{(z-w)^2} - \frac{2(\sigma^i)^a_b \eta_{ij} \partial_w T^j(w)}{z-w} + \frac{2 \delta^a_b L(w)}{z-w} , \] (1)

where \( i = 0, \pm \) denote SU(2) triplets in the diagonal basis, while the superscripts (subscripts) \( a = 1, 2 \) label SU(2) doublet (antidoublet) representations. The symmetric tensors \( \eta^{ij} = \eta^{ji} = \eta_{ij} \) are defined in the diagonal basis as \( \eta^{++} = 1 \), while the antisymmetric tensors \( \epsilon^{ijk} = -\epsilon^{jik} = -\epsilon^{ikj} \) are similarly defined as \( \epsilon^{+-} = -i \), etc., and otherwise zero. The Pauli matrices are given by \( \sigma^\pm = (\sigma^1 \pm i \sigma^2)/\sqrt{2} \), \( \sigma^0 = \sigma^3 \). The corresponding notations in terms of the isospin raising and lowering by a half unit for the iso-doublet and iso-antidoublet fermionic operators are given by

\[
\begin{align*}
(G^a(z)) &= (G^1(z), G^2(z)) = (G^-(z), G^+(z)), \\
(\bar{G}_a(z)) &= (\bar{G}_1(z), \bar{G}_2(z)) = (\bar{G}_+(z), \bar{G}_-(z)).
\end{align*}
\] (2)

The symmetric delta function \( \delta^a_b = \delta^b_a \) has the standard meaning with \( \delta^{++} = 1 \), \( \delta^{+-} = 0 \), etc. The \( N=4 \) algebra of Eq.(1) has the global SU(2) symmetry, whose zero-mode generators we hereby denote as \( S_0 = (S_0^+, S_0^-, S_0^0) \). The generators \( S_0 \) are not included in the \( N=4 \) algebra, but are introduced here to classify the \( N=4 \) states:

\[
\begin{align*}
[S_0^i, S_0^j] &= i \epsilon^{ijk} \eta_{kl} S_0^l , \\
[S_0^i, \hat{G}^a(z)] &= -\frac{1}{2} (\sigma^i)^a_b \hat{G}^b(z) , \quad [S_0^i, \bar{G}_a(z)] = \frac{1}{2} \bar{G}_b(z)(\sigma^i)^b_a , \\
[S_0^i, T^j(z)] &= [S_0^i, L(z)] = 0 ,
\end{align*}
\] (3)

where the doublet and antidoublet combinations of supercharges, \( \hat{G}^a \) and \( \bar{G}_a \) \( (a = 1, 2) \), under the global SU(2) symmetry are given by

\[
\begin{align*}
(\hat{G}^a(z)) &= \left( \hat{G}^1(z), \hat{G}^2(z) \right) = \left( \hat{G}_+(z), \hat{G}_-(z) \right) , \\
(\bar{G}_a(z)) &= \left( \bar{G}^1(z), \bar{G}^2(z) \right) = \left( \bar{G}^-(z), \bar{G}^+(z) \right) .
\end{align*}
\] (4)
The automorphism group for the above $N=4$ algebra is $SO(4) \cong SU(2) \times SU(2)$, therefore the conjugate classes are characterized by two rotation angles, $2\pi \eta$ and $2\pi \rho$.

The corresponding boundary conditions are given by

$$
T^i(z) = e^{-2i\pi(i\eta)} T^i(e^{2i\pi} z),
$$

$$
G^a(z) = -e^{i\pi(\rho-a\eta)} G^a(e^{2i\pi} z),
$$

$$
\bar{G}^a(z) = -e^{-i\pi(\rho+a\eta)} \bar{G}^a(e^{2i\pi} z).
$$

(5)

where we have used the following notation: $i\eta = (\pm \eta, 0)$ for $i = (\pm, 0)$ and $a\eta = \pm \eta$ for $a = \pm$.

The local symmetry is just the SU(2) gauge symmetry which is the left factor of $SO(4) \cong SU(2) \times SU(2)$. We have therefore the inner automorphism specified by the local parameter $\alpha(z)$:

$$
L(z) \rightarrow L(z) + i \frac{d\alpha(z)}{dz} T^0(z) - \frac{k}{4} \left( \frac{d\alpha(z)}{dz} \right)^2,
$$

$$
T^0(z) \rightarrow T^0(z) + \frac{k}{2} \frac{d\alpha(z)}{dz}, \quad T^\pm(z) \rightarrow e^{\pm i\alpha(z)} T^\pm(z),
$$

$$
G^\mp(z) \rightarrow e^{\mp i\frac{\alpha(z)}{2}} G^\mp(z), \quad \bar{G}^\pm(z) \rightarrow e^{\pm i\frac{\alpha(z)}{2}} \bar{G}^\pm(z).
$$

(6)

One can therefore gauge away the $\eta$ phase by choosing $\alpha(z) = i\eta \log z$.

Since the algebra for each choice of $\eta$ is equivalent to each other through the inner automorphism mentioned above, we may only consider a typical value of $\eta$ like $\eta = 0$ or $\eta = 1$. The value $\eta = 0$ in Eq.(4) corresponds to the Ramond (R) sector, while that of $\eta = 1$ to the Neveu-Schwarz (NS) one. In the following we shall consider the NS case for simplicity, unless stated otherwise.

As the result of taking $\eta = 1$ we end up with the simpler periodicity conditions specified by $\rho$ only:

$$
T^\pm(z) = T^\pm(e^{2i\pi} z),
$$

(6)
expansions for the generators of the NS sector:

\[
G^\pm(z) = e^{i\pi \rho} G^\mp(e^{2i\pi} z), \\
\bar{G}_\pm(z) = e^{-i\pi \rho} \bar{G}_\mp(e^{2i\pi} z).
\]  

(7)

Corresponding to the boundary conditions Eq.(7), we have the following Fourier mode expansions for the generators of the NS sector:

\[
L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \\
T^i(z) = \sum_{n \in \mathbb{Z}} T^i_n z^{-n-1}, \\
G^a(z) = \sum_{n \in \mathbb{Z}} G^a_{n+\frac{1}{2}(1+\rho)} z^{-n-\frac{1}{2}(1+\rho)-\frac{3}{2}}, \\
\bar{G}_a(z) = \sum_{n \in \mathbb{Z}} \bar{G}_{a,n+\frac{1}{2}(1-\rho)} z^{-n-\frac{1}{2}(1-\rho)-\frac{3}{2}} = \sum_{n \in \mathbb{Z}} \bar{G}_{a,n-\frac{1}{2}(1+\rho)} z^{-n+\frac{1}{2}(1+\rho)-\frac{3}{2}}.
\]  

(8)

Thus we conclude that one can gauge away the \( \eta \) phase but not the \( \rho \) phase, which implies that all algebras which differ by the value of the parameter \( \eta \) are equivalent, whereas the remaining infinite set of the algebras labeled by the \( \rho \) parameter are all nonequivalent to each other.

For the convenience of our later use, we shall give here in terms of the Fourier components the infinite set of nonequivalent \( N=4 \) SU(2)\(_k\) superconformal algebras which are labeled by the continuous parameter \( 0 \leq \rho < 2 \):

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{k}{2} m(m^2 - 1) \delta_{m+n,0}, \\
[T^i_m, T^j_n] = i \epsilon^{ijk} \eta_{kl} T^k_{m+n} + \eta^{ij} \frac{k}{2} m \delta_{m+n,0}, \quad [L_m, T^i_n] = -n T^i_{m+n}, \\
[T^i_m, G^a_{n+\frac{1}{2}(1+\rho)}] = \frac{1}{2} (\sigma^i)^a_b G^b_{m+n+\frac{1}{2}(1+\rho)}, \\
[T^i_m, \bar{G}_{a,n-\frac{1}{2}(1+\rho)}] = \frac{1}{2} \bar{G}_{b,m+n-\frac{1}{2}(1+\rho)} (\sigma^i)^b_a, \\
[L_m, G^a_{n+\frac{1}{2}(1+\rho)}] = \left( \frac{1}{2} m - n - \frac{1}{2}(1+\rho) \right) G^a_{m+n+\frac{1}{2}(1+\rho)}, \\
[L_m, \bar{G}_{a,n-\frac{1}{2}(1+\rho)}] = \left( \frac{1}{2} m - n + \frac{1}{2}(1+\rho) \right) \bar{G}_{a,m+n-\frac{1}{2}(1+\rho)}, \\
\{G^a_{m+\frac{1}{2}(1+\rho)}, G^b_{n+\frac{1}{2}(1+\rho)}\} = 0, \quad \{\bar{G}_{a,m-\frac{1}{2}(1+\rho)}, \bar{G}_{b,n-\frac{1}{2}(1+\rho)}\} = 0, \\
\{G^a_{m+\frac{1}{2}(1+\rho)}, \bar{G}_{b,n-\frac{1}{2}(1+\rho)}\} = 2 \delta^a_b L_{m+n} - 2 (m - n + \rho + 1) (\sigma^i)^a_b \eta_{ij} T^j_{m+n}.
\]
\[+\delta_{ab} \frac{k}{2} \left( 4 \left( m + \frac{1}{2}(1 + \rho) \right)^2 - 1 \right) \delta_{m+n,0}, \quad (9)\]

where \(m\) and \(n\) run over integers. The modings for the extended SU(2) global generators \(S_\rho = (S^+, S^-, S^0)\) are now given so that we have

\[
\begin{align*}
[S^+, S^-] &= S^0 + \frac{\rho}{2}, \\
[S_\rho, G^a_{n+\frac{1}{2}(1+\rho)}] &= 0, \\
[S^+_\rho, \bar{G}^a_{a,n+\frac{1}{2}(1-\rho)}] &= \frac{1}{\sqrt{2}} \epsilon_{ab} G^b_{n+\frac{1}{2}(1+\rho)}, \\
[S^-_\rho, G^a_{n+\frac{1}{2}(1+\rho)}] &= \frac{1}{\sqrt{2}} \epsilon_{ab} \bar{G}^b_{b,n+\frac{1}{2}(1-\rho)}, \\
[S^0, G^a_{n+\frac{1}{2}(1+\rho)}] &= \frac{1}{2} G^a_{n+\frac{1}{2}(1+\rho)}, \\
[S^0, \bar{G}^a_{a,n+\frac{1}{2}(1-\rho)}] &= -\frac{1}{2} G^a_{a,n+\frac{1}{2}(1-\rho)}, \\
[L_n, S^\pm_{\rho}] &= \mp \rho S^\pm_{n+\rho}, \\
[L_n, S^0_{\rho}] &= [T^i_n, S^\pm_{\rho}] = [T^i_n, S^0_{\rho}] = 0, \quad (10)
\end{align*}
\]

where the antisymmetric tensors are defined as \(\epsilon_{12} = -\epsilon_{21} = -\epsilon^{12} = \epsilon^{21} = 1,\) otherwise zero. The c-number term \(\rho/2\) in the right hand side of the first equality signals the presence of an anomaly when \(\rho \neq 0\). Let us notice that, as this anomaly and the equality before the last in Eq.(10) show, the global SU(2) \(\times\) SU(2) symmetry is broken down to SU(2) \(\times\) U(1) when \(\rho \neq 0\). By the way we point out that the presence of the anomaly term just mentioned was overlooked in the paper by Yu [14].

Here we take the raising operators to be

\[
L_n \quad (n > 0), \\
T^i_n \quad (n > 0 \text{ or } i = + \text{ and } n = 0), \\
G^a_{n+\frac{1}{2}(1+\rho)} \quad (n \geq 0), \\
\bar{G}^a_{a,n+\frac{1}{2}(1-\rho)} \quad (n \geq 0), \\
\delta_{\rho,0} S^+_\rho,
\]

the Cartan subalgebra to be \(\{L_0, T^0_n, k; S^0\}\), and the lowering operators to be the remaining generators. Then we define a highest weight representation (hwrep) of the
algebras Eqs. (9) and (10) to be one containing a highest weight state (hws) vector 
\( |h, j; s\rangle \) such that
\[
L_0 |h, j; s\rangle = h |h, j; s\rangle ,
\]
\[
T_0^0 |h, j; s\rangle = j |h, j; s\rangle ,
\]
\[
S_0^0 |h, j; s\rangle = s |h, j; s\rangle ,
\]
(12)
and
\[
X_+ |h, j; s\rangle = 0 ,
\]
(13)
for all raising operators \( X_+ \).

Now we present our construction of the FF representations for the infinite set of 
the \( N=4 \) SU(2)\(_k\) algebras with \( \rho \). We use four real bosons \( \varphi_\alpha(z) \) \( (\alpha = 1, 2, 3, 4) \), and 
four real fermions forming a pair of complex fermion doublet \( \gamma^a(z) \) and antidoublet 
\( \bar{\gamma}_a(z) \) \( (a = 1, 2 \text{ or } \pm) \) under SU(2)\(_k\):
\[
\begin{pmatrix} \gamma^a(z) \\ \gamma^\alpha(z) \end{pmatrix} = \begin{pmatrix} \gamma^1(z) \\ \gamma^2(z) \end{pmatrix} = \begin{pmatrix} \gamma^+(z) \\ \gamma^-(z) \end{pmatrix} ,
\]
\[
\begin{pmatrix} \bar{\gamma}_a(z) \\ \bar{\gamma}_\alpha(z) \end{pmatrix} = \begin{pmatrix} \bar{\gamma}_1(z) \\ \bar{\gamma}_2(z) \end{pmatrix} = \begin{pmatrix} \bar{\gamma}_+(z) \\ \bar{\gamma}_-(z) \end{pmatrix} ,
\]
(14)
whose propagators are given by
\[
\langle \varphi_\alpha(z) \partial \varphi_\beta(w) \rangle = \langle \partial \varphi_\beta(w) \varphi_\alpha(z) \rangle = \frac{\delta_{\alpha\beta}}{z-w} ,
\]
\[
\langle \bar{\gamma}_a(z) \gamma^b(w) \rangle_\rho = -\langle \gamma^b(w) \bar{\gamma}_a(z) \rangle_\rho = \frac{\delta_{a,b}}{z-w} \left( \frac{z-w}{\rho} \right)^\frac{1}{2} .
\]
(15)
Here we note that in accordance with the periodicity conditions Eq.(7) the above 
fermion pairs have been taken to satisfy the following boundary conditions
\[
\gamma^a(z) = e^{i\pi \rho} \gamma^a(e^{2i\pi} z) , \quad \bar{\gamma}_a(z) = e^{-i\pi \rho} \bar{\gamma}_a(e^{2i\pi} z) .
\]
(16)
As a result the gamma pairs have the Fourier mode expansions given by
\[
\gamma^a(z) = \sum_{n \in \mathbb{Z}} \gamma^a_{n+\frac{1}{2}(1+\rho)} z^{-n+\frac{1}{2}(1+\rho)-\frac{1}{2}} ,
\]
\[
\bar{\gamma}_a(z) = \sum_{n \in \mathbb{Z}} \bar{\gamma}_a_{n+\frac{1}{2}(1-\rho)} z^{-n-\frac{1}{2}(1-\rho)-\frac{1}{2}} .
\]
(17)
Corresponding to the transformation properties Eqs. (2) and (4) of the supercharges, the following rearranged pairs of the fermion fields \[16\] are considered to transform as a doublet \(\hat{\gamma}^a(z)\) and an antidoublet \(\bar{\hat{\gamma}}_a(z)\) \((a = 1, 2)\) pairs under the global SU(2) symmetry:

\[
\begin{align*}
(\hat{\gamma}^a(z)) &= \left( \begin{array}{c}
\hat{\gamma}_1(z) \\
-\hat{\gamma}_2(z)
\end{array} \right) = \left( \begin{array}{c}
\tilde{\gamma}^+(z) \\
-\tilde{\gamma}^-(z)
\end{array} \right), \\
(\bar{\hat{\gamma}}_a(z)) &= \left( \begin{array}{c}
\gamma^1(z), -\tilde{\gamma}_2(z)
\end{array} \right) = \left( \begin{array}{c}
\gamma^- (z), -\tilde{\gamma}^-(z)
\end{array} \right).
\end{align*}
\]

Here we shall define an extended normal-ordering operation for non-zero \(\rho\) such that our formalism of the FF representations be given self-consistently for any value of \(\rho\). The \(\rho\) parameter is only relevant to fermion fields, so the following definition of normal-ordering is sufficient for our present purpose:

\[
\begin{align*}
\bar{\hat{\gamma}}_a(z)\gamma^b(w) \equiv & \bar{\hat{\gamma}}_a(z)\gamma^b(w) - \langle \bar{\hat{\gamma}}_a(z)\gamma^b(w) \rangle_{\rho,\text{div}}, \\
\bar{\hat{\gamma}}_a(z)\partial \gamma^b(w) \equiv & \bar{\hat{\gamma}}_a(z)\partial \gamma^b(w) - \langle \bar{\hat{\gamma}}_a(z)\partial \gamma^b(w) \rangle_{\rho,\text{div}}, \\
\bar{\hat{\gamma}}_a(z)\gamma^b(w) \equiv & \bar{\hat{\gamma}}_a(z)\gamma^b(w) - \langle \partial \bar{\hat{\gamma}}_a(z)\gamma^b(w) \rangle_{\rho,\text{div}}, \\
\bar{\hat{\gamma}}_a(z)\partial \gamma^b(w) \equiv & \bar{\hat{\gamma}}_a(z)\partial \gamma^b(w) - \langle \partial \bar{\hat{\gamma}}_a(z)\partial \gamma^b(w) \rangle_{\rho,\text{div}},
\end{align*}
\]

(19)

where the notation \(\langle \cdots \rangle_{\rho,\text{div}}\) in a given expression \(\langle \cdots \rangle_{\rho}\) stands for the divergent contribution when the limit \(w \to z\) is taken in that expression. They are simply given by the following \(\rho\)-independent expressions:

\[
\begin{align*}
\langle \bar{\hat{\gamma}}_a(z)\gamma^b(w) \rangle_{\rho,\text{div}} &= \frac{\delta_a^b}{z-w}, \\
\langle \bar{\hat{\gamma}}_a(z)\partial \gamma^b(w) \rangle_{\rho,\text{div}} &= \frac{\delta_a^b}{(z-w)^2}, \\
\langle \partial \bar{\hat{\gamma}}_a(z)\gamma^b(w) \rangle_{\rho,\text{div}} &= -\frac{\delta_a^b}{(z-w)^2}, \\
\langle \partial \bar{\hat{\gamma}}_a(z)\partial \gamma^b(w) \rangle_{\rho,\text{div}} &= -\frac{2\delta_a^b}{(z-w)^3}.
\end{align*}
\]

(20)

The above definition of extended normal-ordering satisfies required properties like the
antisymmetry for fermion fields whose few examples are

\[ \gamma^a(z) \gamma^b(w) = - \gamma^b(w) \gamma^a(z), \quad \gamma^a(z) \partial \gamma^b(w) = - \partial \gamma^b(w) \gamma^a(z), \]  

such that a Wick contraction can be performed in a consistent manner.

In Eq.\((19)\), note the difference from the usual definition of normal-ordering which

is given for example by

\[ : \bar{\gamma}^a(z) \gamma^b(w) : \equiv \bar{\gamma}^a(z) \gamma^b(w) - \langle \bar{\gamma}^a(z) \gamma^b(w) \rangle_{\rho}, \]

where \(\cdots :\) implies that in the product \(\cdots\) of operators all creation operators stand to the left of all annihilation operators. Note also that we naturally have

\[ \gamma^a(z) \gamma^b(w) = : \gamma^a(z) \gamma^b(w) :, \quad \bar{\gamma}^a(z) \bar{\gamma}^b(w) = : \bar{\gamma}^a(z) \bar{\gamma}^b(w) :, \]

while the difference between the two definitions Eqs.\((19)\) and \((22)\) is just given by a finite \(\rho\) dependent c-number term. In particular we have

\[ \bar{\gamma}^a(z) \gamma^b(z) = : \bar{\gamma}^a(z) \gamma^b(z) : + \frac{\partial}{2z} \delta^a_b, \]

\[ \bar{\gamma}^a(z) \gamma^b(z) - \bar{\gamma}^a(z) \partial \gamma^b(z) = : \left( \bar{\gamma}^a(z) \gamma^b(z) - \bar{\gamma}^a(z) \partial \gamma^b(z) \right) : + \frac{\rho^2}{4z^2} \delta^a_b, \]

when the limit \(w \to z\) is taken. Multiplying a fermion field \(\gamma^c(w)\) or \(\bar{\gamma}^c(w)\) to the first equality in the above and performing the Wick contraction to the operator products on both sides, one can also prove that the equalities

\[ \gamma^a(z) \gamma^b(z) \gamma^c(z) = - \gamma^b(z) \bar{\gamma}^a(z) \gamma^c(z), \]

\[ \gamma^a(z) \gamma^b(z) \bar{\gamma}^c(z) = - \gamma^b(z) \bar{\gamma}^a(z) \bar{\gamma}^c(z), \]

hold in the limit \(w \to z\).
With our definition $\tilde{x}(\cdots)^{x}_{x}$ of normal-ordering one can perform the OPE calculations in a compact manner for any values of $0 \leq \rho < 2$ as you would do in the usual Neveu-Schwarz case with $\rho = 0$. One should only keep in mind that $: (\cdots) :$ in the NS case with $\rho = 0$ be replaced by $\tilde{x}(\cdots)^{x}_{x}$ everywhere in the course of the calculations, so that one can easily keep track of the $\rho$ dependent extra terms which show up as the differences between the two definitions of normal-ordering. Only at the end of the calculations one may use the relations like Eqs.(24) and (25) given above to transform the obtained results with $\tilde{x}(\cdots)^{x}_{x}$ into more familiar expressions with $: (\cdots) :$ having $\rho$-dependent extra terms added.

Now we first consider the SU(2)$_{\hat{k}}$ Kac-Moody subalgebras with level $\hat{k}$. The generators are given in terms of the first three bosons by [7, 17]

\[
J^0(z) = i\sqrt{\frac{\hat{k}}{2}} \partial \varphi_3(z),
\]
\[
J^{\pm}(z) = :i\sqrt{\frac{\hat{k}+2}{2}} \partial \varphi_1(z) \pm i\sqrt{\frac{\hat{k}}{2}} \partial \varphi_2(z) : e^{\pm i\sqrt{\frac{2}{\hat{k}+2}}(\varphi_3(z) - i\varphi_2(z))}.
\] (26)

The corresponding contribution to the energy-momentum tensor is then given in the Sugawara form as [8]

\[
\frac{1}{\hat{k}+2} \sum_{i,j=\pm,0} :J^i(z)\eta_{ij}J^j(z) := -\frac{1}{2} \sum_{\alpha=1}^3 :\left(\partial \varphi_\alpha(z)\right)^2 : +i\frac{\tau}{2} \partial^2 \varphi_1(z),
\] (27)

where

\[
\tau \equiv \sqrt{\frac{2}{\hat{k}+2}}.
\] (28)

The total energy-momentum tensor $L(z)$ is obtained by adding the contribution from the fourth boson and from the fermion doublets to Eq.(27):

\[
L(z) = -\frac{1}{2} \sum_{\alpha=1}^4 :\left(\partial \varphi_\alpha(z)\right)^2 : + \left(\frac{i\tau}{2} \partial^2 \varphi_1(z) - i\kappa \partial^2 \varphi_4(z)\right)
\]
\[
+ \frac{1}{2} \times \left(\partial \gamma(z) \cdot \gamma(z) - \gamma(z) \cdot \partial \gamma(z)\right)^x
\]
\[
= -\frac{1}{2} \sum_{\alpha=1}^4 :\left(\partial \varphi_\alpha(z)\right)^2 : + \left(\frac{i\tau}{2} \partial^2 \varphi_1(z) - i\kappa \partial^2 \varphi_4(z)\right).
\]
\[ + \frac{1}{2} \left[ : (\partial \bar{\gamma}(z) \cdot \gamma(z) - \bar{\gamma}(z) \cdot \partial \gamma(z)) : + \frac{\rho^2}{2z^2} \right] \] (29)

with the parameter
\[ \kappa \equiv \frac{i}{2} \overline{k} \tau \, , \quad k \equiv \hat{k} + 1 \, . \] (30)

The last line of Eq.(29) is obtained by use of the second identity in Eq.(24).

We also define the total SU(2)\(_k\) Kac-Moody currents \(T^i(z)\) \((i = \pm, 0)\) with level \(k = \hat{k} + 1\) by adding the fermionic contribution to \(J^i(z)\):

\[ T^i(z) = J^i(z) + \frac{1}{2} \bar{\gamma}(z) \sigma^i \gamma(z) \] (31)

where the first identity in Eq.(24) was used to get the last expression.

Finally the \(N = 4\) supercurrents \(G^a(z)\) and \(\bar{G}_a(z)\) \((a = 1, 2\) or \(\pm)\) in our Feigin-Fuchs representations are given by

\[
G^a(z) = \gamma^a(z)i\partial \varphi_4(z) - 2\kappa \partial \gamma^a(z) - i\tau J^i(z) \eta_{ij} \left( \sigma^j \gamma(z) \right)^a \\
\quad + i\tau \left( \bar{\gamma}(z) \cdot \gamma(z) \right) \gamma^a(z) \]  

\[
= \gamma^a(z)i\partial \varphi_4(z) - 2\kappa \partial \gamma^a(z) - i\tau J^i(z) \eta_{ij} \left( \sigma^j \gamma(z) \right)^a \\
\quad + i\tau \left[ : (\bar{\gamma}(z) \cdot \gamma(z)) \gamma^a(z) : + \frac{\rho}{2z} \gamma^a(z) \right] ,
\]

\[
\bar{G}_a(z) = \bar{\gamma}_a(z)i\partial \varphi_4(z) - 2\kappa \partial \bar{\gamma}_a(z) + i\tau J^i(z) \eta_{ij} \left( \bar{\sigma}^j \gamma(z) \right)_a \\
\quad - i\tau \left( \bar{\gamma}(z) \cdot \gamma(z) \right) \bar{\gamma}_a(z) \]  

\[
= \bar{\gamma}_a(z)i\partial \varphi_4(z) - 2\kappa \partial \bar{\gamma}_a(z) + i\tau J^i(z) \eta_{ij} \left( \bar{\sigma}^j \gamma(z) \right)_a \\
\quad - i\tau \left[ : (\bar{\gamma}(z) \cdot \gamma(z)) \bar{\gamma}_a(z) : + \frac{\rho}{2z} \bar{\gamma}_a(z) \right] ,
\] (32)

where Eq.(24) was used to obtain the second expressions of \(G^a(z)\) and \(\bar{G}_a(z)\).

It is to be pointed out here that, when \(\rho = 0\), the Feigin-Fuchs representations of Eqs.(29),(31),(32) presented above are just reduced as they should be to the form of those first constructed [7] by one (S.M.) of the present authors.
Now, the generators $S^i_{\rho} = (S^+_\rho, S^-_{-\rho}, S^0_0)$ or $S^i_{i\rho}$ \((i = \pm, 0)\) of the extended SU(2) global symmetry defined in Eq.(10) can be constructed as

\[
S^i_{i\rho}(z) \equiv \frac{1}{2\pi i} \oint s^i_{i\rho}(z) z^{i\rho} \, dz ,
\]

\[
s^i_{i\rho}(z) \equiv \frac{1}{2} \epsilon^{i\rho}_{\sigma\tau} \gamma^\sigma(z) \gamma^\tau(z) ,
\]

where we note the following notation: \(i\rho = (\rho, -\rho, 0)\) for each \(i = (+, -, 0)\). To be more explicit, we have for \(s^\rho_{\rho}(z) = (s^\rho_{\rho}(z), s^-_{-\rho}(z), s^0_0(z))\) or \(s^i_{i\rho}(z)\)

\[
s^\rho_{\rho}(z) = -\frac{1}{\sqrt{2}} \sqrt{\gamma^{-}(z) \gamma^{+}(z)} = -\frac{1}{2\sqrt{2}} \epsilon_{ab} : \gamma^a(z) \gamma^b(z) : ,
\]

\[
s^-_{-\rho}(z) = -\frac{1}{\sqrt{2}} \sqrt{\gamma^{-}(z) \gamma^{+}(z)} = -\frac{1}{2\sqrt{2}} \epsilon_{ab} : \gamma^a(z) \gamma^b(z) : ,
\]

\[
s^0_0(z) = \frac{1}{2\sqrt{2}} \sqrt{\gamma^{-}(z) \gamma^{+}(z)} = -\frac{1}{2} : \gamma(z) \cdot \gamma(z) : -\frac{\rho}{2} z ,
\]

with the mode expansion of \(\gamma^a(z)\) and \(\gamma^a(z)\) given by Eq.(17).

With this representation of Eq.(34) we actually generate the \(\rho\)-extended SU(2) local algebra

\[
[S^i_{m+i\rho}, S^j_{n+j\rho}] = i \epsilon^{ijk} \eta_{kl} S^k_{m+n+k\rho} + \eta^{ij} \frac{1}{2} (m + i\rho) \delta_{m+n,0} ,
\]

with the following mode expansion:

\[
s^i_{i\rho}(z) = \sum_{n \in \mathbb{Z}} S^i_{n+i\rho} z^{-n-i\rho} .
\]

By putting \(m = n = 0\) in Eq.(35) we obtain the \(\rho\)-extended algebra of the SU(2) global symmetry given in Eq.(11).

With Eqs.(29) and (34) we also have the OPE relation

\[
L(z) s^i_{i\rho}(w) \sim \partial_w \left( \frac{1}{z-w} s^i_{i\rho}(w) \right) \quad (i = +, -, 0) ,
\]

which reproduces the last two commutation relations between \(L_n\) and \(S^i_{i\rho}\) in Eq.(11) if it is transformed into Fourier modes. Thus we have

\[
[L_n, S^i_{i\rho}] = -i \rho S^i_{n+i\rho} \quad (i = +, -, 0) .
\]

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Now we can construct highest weight states (hws) explicitly in terms of the four bosons $\varphi_\alpha(z)$ ($\alpha = 1, 2, 3, 4$) and four fermions $\gamma^a(z), \bar{\gamma}_a(z)$ ($a = 1, 2$). The mode expansions for the latter are given by Eq. (17), whereas those for the former are given by

$$\varphi_\alpha(z) = q_\alpha - ip_\alpha \log z + i \sum_{n \in \mathbb{Z}, n \neq 0} \frac{\varphi_{\alpha,n}}{n} z^{-n} \quad (\alpha = 1, 2, 3, 4).$$

(39)

The commutators for the Fourier modes are

$$[\varphi_{\alpha,m}, \varphi_{\beta,n}] = m \delta_{\alpha\beta} \delta_{m+n,0}, \quad [q_\alpha, p_\beta] = i \delta_{\alpha\beta},$$

$$[\varphi_{\alpha,n}, q_\beta] = [\varphi_{\alpha,n}, p_\beta] = 0,$$

$$\{\gamma^a_{m+\frac{1}{2}(1+\rho)}, \bar{\gamma}^b_{n-\frac{1}{2}(1+\rho)}\} = \delta^a_b \delta_{m+n,0},$$

$$\{\gamma^a_{m+\frac{1}{2}(1+\rho)}, \gamma^b_{n+\frac{1}{2}(1+\rho)}\} = \{\bar{\gamma}_{a,m-\frac{1}{2}(1+\rho)}, \bar{\gamma}^b_{n-\frac{1}{2}(1+\rho)}\} = 0.$$  

(40)

The ground state vacuum $|0\rangle$ is now defined by

$$\varphi_{\alpha,n}|0\rangle = p_\alpha|0\rangle = 0 \quad (n > 0),$$

$$\gamma^a_{m+\frac{1}{2}(1+\rho)}|0\rangle = \bar{\gamma}_{a,m+\frac{1}{2}(1-\rho)}|0\rangle = 0 \quad (m \geq 0).$$

(41)

With the vertex operators

$$V(t, j, j_0; z) = :e^{it\varphi_4(z)}V_{j,j_0}(z):,$$

$$V_{j,j_0}(z) = :e^{-ij\tau_1(z)}e^{ij_0\sqrt{\hat{k}^2(\varphi_3(z)-\varphi_2(z))}}:,$$

(42)

which were first introduced by one (S.M.) of the present authors [3, 4], a primary state with conformal dimension $h_\rho(t, j)$ and SU(2) spin $(j, j_0)$ is represented as

$$|h_\rho(t, j), j_0\rangle \sim V(t, j, j_0; z = 0)|0\rangle,$$

(43)

where the conformal weight $h_\rho(t, j)$ is given by

$$h_\rho(t, j) \equiv \frac{t^2}{2} + \kappa t + \frac{\tau^2}{2} j(j + 1) + \frac{\rho^2}{4},$$

$$= \frac{1}{2}(t + \kappa)^2 + \frac{\tau^2}{2}(j + \frac{1}{2})^2 + \frac{\hat{k}}{4} + \frac{\rho^2}{4}.$$  

(44)
Note also that the following OPE relations hold\textsuperscript{[7, 8]}:

\begin{align*}
J^0(z) V_{j,j_0}(w) & \sim \frac{j_0}{z-w} V_{j,j_0}(w), \\
\sqrt{2} J^\pm(z) V_{j,j_0}(w) & \sim \frac{-j \pm j_0}{z-w} V_{j,j_0 \pm 1}(w).
\end{align*}

(45)

Thus a hws vector $|h_\rho, j; s_\rho\rangle$ is now given as

\begin{equation}
|h_\rho = h_\rho(t, j), j; s_\rho = -\frac{\rho}{2}\rangle \equiv \langle h_\rho(t, j), j_0 = j| \\
\sim V(t, j, j_0 = j; z = 0)|0\rangle.
\end{equation}

(46)

Here we have some remarks in order. First, as to the realization of the boundary conditions Eq.(7) by our FF representations, we note that the bosonic part $J^i(z)$ of the SU(2)\textsubscript{k} Kac-Moody currents $T^i(z)$ satisfies the periodicity equations

\begin{align*}
J^0(z) & = J^0(e^{2i\pi} z), \\
J^\pm(z) & = J^\pm(e^{2i\pi} z) e^{\mp 2i\pi \sqrt{2} \hat{k}(p_3 - ip_2)},
\end{align*}

(47)
as operator identities, while, as is clear from our vertex operator expressions Eq.(42), the momentum eigenvalues always satisfy

\begin{equation}
p_3 - ip_2 = 0
\end{equation}

(48)
for any conformal states spanned on the primary state Eq.(43) in our FF representation. Thus the boundary conditions Eq.(7) is guaranteed to hold in a nontrivial manner for our FF representations.

Secondly, the “charge” operator $\hat{C}$ considered by Kent and Riggs\textsuperscript{[18]} is defined here as

\begin{equation}
\hat{C} \equiv 2S_0^0 + \rho.
\end{equation}

(49)

In particular we have for the hws

\begin{equation}
\hat{C}|h_\rho, j; s_\rho\rangle = 0.
\end{equation}

(50)
Lastly, the irreducible hwreps of the \( \rho \)-extended \( N = 4 \) algebras Eq.(10) can be constructed as quotients of its Verma modules \[19\], which we denote by \( V(h_{\rho}, j, k; \rho) \). Here we define the ordering of numbers by writing \((p, r) < (q, s)\) if \( p < q \) or both \( p = q \) and \( r < s \). Then, we have that if \( x < y \), \( n_x < n_y \) and \((p_x, r_x) < (q_y, s_y)\). With this notation of ordering, the Verma module is spanned by a basis of the states of the form

\[
\left( \prod_{i=1}^{j_0} (L_{n_i})^\ell_{n_i} \right) \left( \prod_{i=(\pm,0)}^{j_i,0} T_{m_{j_i}}^i \right) \left( \prod_{a=\pm}^{k_{a,0}} G_{p_{k_a} + \frac{1}{2}(1+\rho)}^{a} \right) \times \left( \prod_{b=\pm}^{l_{b,0}} \bar{G}_{q_{l_b} + \frac{1}{2}(1-\rho)}^{b} \right) |h_{\rho}, j; s_\rho\rangle , \tag{51}
\]

where only lowering operators appear in the products, and the modes \( n_i, m_{j_i}, p_{k_a}, q_{l_b} \) are negative integers while the powers \( \ell_{n_i}, t_{m_{j_i}} \) are positive integers. Defining

\[
N \equiv \sum_{i=1}^{j_0} (-n_i)\ell_{n_i} + \sum_{i=(\pm,0)}^{j_i,0} (-m_{j_i})t_{m_{j_i}} + \sum_{a=\pm}^{k_{a,0}} (-p_{k_a} - \frac{1}{2}) + \sum_{b=\pm}^{l_{b,0}} (-q_{l_b} - \frac{1}{2}) ,
\]

\[
K \equiv \sum_{i=(\pm,0)}^{j_i,0} it_{m_{j_i}} + \sum_{a=\pm}^{k_{a,0}} \frac{1}{2} a + \sum_{b=\pm}^{l_{b,0}} \frac{1}{2} b ,
\]

\[
C \equiv \sum_{a=\pm}^{k_{a,0}} 1 + \sum_{b=\pm}^{l_{b,0}} (-1) - \sum_{b=\pm}^{l_{b,0}} , \tag{52}
\]

we find that the state Eq.(51) has \( L_0 \)-eigenvalue \( h_{\rho} + N - \frac{1}{2}C \rho \), \( T_0^0 \)-eigenvalue \( j + K \), and \( \hat{C} \)-eigenvalue \( C \). The Verma module splits into simultaneous eigenspaces of \( L_0 \), \( T_0^0 \) and \( \hat{C} \), whose splitting can be written as

\[
V(h_{\rho}, j, k; \rho) = \bigoplus_{N \geq 0}^{K \in \frac{1}{2}Z} V_{N,K,C}(h_{\rho}, j, k; \rho) \tag{53}
\]
In conclusion, we have considered the infinite number of nonequivalent algebras of \( N = 4 \) SU(2)\(_k\) superconformal symmetry labeled by the \( \rho \) parameter which specifies the global boundary conditions for the generators of the \( N = 4 \) algebra. We have presented the FF representations of the \( \rho \)-extended \( N = 4 \) algebras in terms of four bosons and a pair of complex fermion doublets. The generators of the \( \rho \)-extended SU(2) global symmetry are explicitly constructed in terms of the fermion pairs. The formalism of the hwrep’s of the \( \rho \)-extended \( N = 4 \) SU(2)\(_k\) superconformal algebras has been given in our FF representations with \( \rho \).

One of the authors (S.M) would like to thank Professor Tsuneo Uematsu for discussions.
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