Unitary Invariants and Classification of Four-Qubit States via Negativity Fonts

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To detect and quantify entanglement of composite quantum systems is a challenge taken up with great zeal by theorists and experimentalists alike. On the way, from the elegant bipartite separability criterion of Peres [1] up to classification schemes for four qubit states [2–11], several useful entanglement measures and invariants have been found [12–30]. Two qubit entanglement is quantified by concurrence [31], which for a pure state is equal to global negativity [17, 32]. Entanglement of a three qubit state due to three-body quantum correlations is quantified by three tangle [13]. For the most general three qubit state, the difference of squared global negativity and three tangle is a measure of two qubit correlations and satisfies CKW inequality [12]. A natural question is, which polynomial function of the coefficients quantifies entanglement due to four-body correlations? Can we write an invariant analogous to global negativity for two qubits and three tangle for three qubits to quantify four-body correlations?

Invariant theory describes invariant properties of homogenous polynomials under general linear transformations. If we write a qubit state in multilinear form, we can find the set of invariants of the form in terms of state coefficients \( a_{i_1i_2...i_N} \) by using standard methods, as has been done in \([4,5,18]\). One may then investigate the properties of all invariants in the set. Our general aim, however, is to construct those polynomial invariants that quantify entanglement due to \( K \)-body correlations in an \( N \)-qubit \((N \geq K)\) pure state. This is done by constructing \( N \)-qubit invariants from multivariate forms with \((K − 1)\)-qubit invariants as coefficients instead of \( a_{i_1i_2...i_N} \). In particular, the invariant that quantifies entanglement due to \( N \)-body correlations is obtained from a biform having as coefficients the \( N − 1 \) qubit invariants. The term \( N \)-body correlations refers, strictly, to correlations of the type present in an \( N \)-qubit GHZ state. The advantage of our approach [21, 22] is twofold. Firstly, we can choose to construct invariants that contain information about entanglement of a part of the system. Secondly, since the form of \( N \)-qubit invariants is directly linked to the underlying structure of the composite system state, it can throw light on the suitability of a given state for a specific information processing task. Local unitary invariance and the notion of negativity fonts are used as the principle tools to identify \( K \)-qubit invariants in an \( N \)-qubit state. Negativity fonts are the elementary units of entanglement in a quantum superposition state. Determinants of negativity fonts are linked to matrices obtained from state operator through selective partial transposition [33, 34]. In this article, we obtain analytical expressions for polynomial invariants of degree 8, 12, and 24 for \( N = 4 \) states. One of the four qubit invariants is found to be non zero on states with four-body quantum correlations and zero on separable states as well as on states with entanglement due to two and three body correlations. It is analogous to three qubit invariant used to define three tangle [12], and can likewise be used to construct an entanglement monotone to quantify four-body correlations.

To obtain four qubit invariants that quantify four qubit quantum correlations, we follow a sequence of steps as given below:

1. Identify two qubit invariants for a given pair in a three qubit state.
2. Obtain a quadratic equation with two qubit invariants for a given pair of qubits as coefficients. Discriminant of the form is the three qubit invariant written in terms of two qubit invariants.
3. Identify two qubit invariants in a four qubit state. Select three qubits and write three qubit invariants for these in a four qubit state. We identify five invariants, including two invariants analogous to ones known for a three qubit state.
4. A local unitary on fourth qubit yields transformation equations for three qubit invariants. Proper unitaries can reduce the number of three qubit invariants in the set to four. The process of finding such local unitaries yields a quartic equation from which four qubit invariants are obtained. Since the invariants in a larger Hilbert space...
are written in terms of relevant invariants in subspaces, it is possible to differentiate the invariants that quantify three-body quantum correlations from those that quantify four-body quantum correlations.

In principle the process can be carried on to higher number of qubits. Polynomial invariants introduced by Luque and Thibon [18] got geometrical meaning in the work of Levay [27]. We point out the relation of our four qubit invariants with invariants in [18] and [27].

Polynomial invariants that identify the nature of correlations in a state are useful to apply classification criteria proposed in [11] to four qubit states. Two multi qubit pure states are equivalent under stochastic local operations and classical communication (SLOCC) [14] if one can be obtained from the other with some probability using only local operations and classical communication amongst different parties. SLOCC equivalence is the central point in four qubit state classification into nine families in [2]. Borsten et al. [9] have invoked the black-hole–qubit correspondence to derive the classification of four-qubit entanglement. However, it has been found that the number of four qubit SLOCC entanglement classes is much larger [8]. The main result of Lamata et al. [7] is that each of the eight genuine in-equivalent entanglement classes contains a continuous range of strictly non equivalent states, although with similar structure. O. Viehmann et al. [10] select a set of generators for the SL(2, C)⊗4 -invariant polynomials or tangles and classify the eight families of ref. [7] using tangle patterns. In our classification scheme using correlation based criterion [11], multipartite states within the same class have same type of correlations but may have different number and type of negativity fonts in canonical state (all the states may not be SLOCC equivalent). In section IV, we calculate the relevant invariants for SLOCC families [2] and re-classify the states on the basis of number and nature of negativity fonts with non-zero determinants. The polynomial invariants used to classify the states in our scheme quantify correlations generated by distinct interaction types. Intuitively, this information should be extremely useful to quantum state engineering. Negativity font analysis can be a helpful tool to optimize the subsystem interactions to tailor the invariant dynamics for a specific quantum information processing task. A minor point that will be discussed relates to the controversy regarding the family $L_{abc}$ which is pointed out in ref. [28] to be a subclass of $L_{abc}$ with $(a = c)$, while in [8] it has been shown that $L_{ab}$ and $L_{abc}$ belong to distinct SLOCC classes.

I. NEGATIVITY FONTS AND TWO QUBIT INVARIANTS

In this section, we briefly review the concepts of global partial transpose [1], global negativity [17, 32], $K$–way partial transpose [35] and $K$–way negativity fonts [21, 22]. We also identify those two qubit invariants which determine the entanglement of a pair of qubits in a three qubit state.

A general N-qubit pure state may be written as

$$|\Psi^{A_1, A_2, \ldots, A_N}\rangle = \sum_{i_1 i_2 \ldots i_N} a_{i_1 i_2 \ldots i_N} |i_1 i_2 \ldots i_N\rangle,$$

(1)

where $|i_1 i_2 \ldots i_N\rangle$ are the basis vectors spanning $2^N$ dimensional Hilbert space, and $A_p$ is the location of qubit $p$. The coefficients $a_{i_1 i_2 \ldots i_N}$ are complex numbers. The local basis states of a single qubit are labelled by $i = 0, 1$, where $m = 1, \ldots, N$. The global partial transpose of an $N$ qubit state $\tilde{\rho} = |\Psi^{A_1, A_2, \ldots, A_N}\rangle \langle \Psi^{A_1, A_2, \ldots, A_N}|$ with respect to qubit at location $p$ is constructed from the matrix elements of $\tilde{\rho}$ through

$$\langle i_1 i_2 \ldots i_N | \tilde{\rho}^{T_A_p} | j_1 j_2 \ldots j_N\rangle = \langle i_1 i_2 \ldots i_{p-1} i_{p+1} \ldots i_N | \tilde{\rho} | j_1 j_2 \ldots j_{p-1} j_{p+1} \ldots j_N\rangle.$$

(2)

To construct a $K$–way partial transpose [35], every matrix element $\langle i_1 i_2 \ldots i_N | \tilde{\rho} | j_1 j_2 \ldots j_N\rangle$ is labelled by a number $K = \sum_{m=1}^{N} (1 - \delta_{i_m, j_m})$, where $\delta_{i_m, j_m} = 1$ for $i_m = j_m$, and $\delta_{i_m, j_m} = 0$ for $i_m \neq j_m$. Matrix elements of state operator with a given $K$ represent $K$–way coherences present in the state. Local operations on a quantum superposition transform $K$–way coherences to $K \pm 1$ way coherences. The $K$–way partial transpose of $\tilde{\rho}$ with respect to subsystem $p$ for $K > 2$ is obtained by selective transposition such that

$$\langle i_1 i_2 \ldots i_N | \tilde{\rho}^{T_{A_p}} | j_1 j_2 \ldots j_N\rangle = \langle i_1 i_2 \ldots i_{p-1} i_{p+1} \ldots i_N | \tilde{\rho} | j_1 j_2 \ldots j_{p-1} j_{p+1} \ldots j_N\rangle,$$

if $\sum_{m=1}^{N} (1 - \delta_{i_m, j_m}) = K$, and $\delta_{i_p, j_p} = 0$

(3)

and

$$\langle i_1 i_2 \ldots i_N | \tilde{\rho}^{T_{A_p}} | j_1 j_2 \ldots j_N\rangle = \langle i_1 i_2 \ldots i_N | \tilde{\rho} | j_1 j_2 \ldots j_N\rangle,$$

if $\sum_{m=1}^{N} (1 - \delta_{i_m, j_m}) \neq K$.

(4)
while
\[
\langle i_1 i_2 \ldots i_N | \hat{\rho}^T_K | j_1 j_2 \ldots j_N \rangle = \langle i_1 i_2 \ldots i_p-1 i_p i_p+1 \ldots i_N | \hat{\rho} | j_1 j_2 \ldots j_p-1 i_p j_p+1 \ldots j_N \rangle,
\]
if \( \sum_{m=1}^N (1 - \delta_{i_m,j_m}) = 1 \) or \( 2 \), and \( \delta_{i_p,j_p} = 0 \) \( (5) \)

and
\[
\langle i_1 i_2 \ldots i_N | \hat{\rho}^T_K | j_1 j_2 \ldots j_N \rangle = \langle i_1 i_2 \ldots i_N | \hat{\rho} | j_1 j_2 \ldots j_N \rangle,
\]
if \( \sum_{m=1}^N (1 - \delta_{i_m,j_m}) \neq 1 \) or \( 2 \). \( (6) \)

One can verify that global partial transpose may be expanded as
\[
\hat{\rho}^{T_K} = \sum_{K=2}^N \hat{\rho}^{T_K} - (N - 2) \hat{\rho}. \quad (7)
\]

Negativity of \( \hat{\rho}^{T_K} \), defined as \( N^{A_\nu} = (\| \hat{\rho}^{T_K} \| - 1) \), where \( \| \hat{\rho} \| \) is the trace norm of \( \hat{\rho} \), arises due to all possible negativity fonts present in \( \hat{\rho}^{T_K} \). Since \( \hat{\rho} \) is a positive operators, global negativity depends on the negativity of \( K \)-way partially transposed operators with \( K \geq 2 \).

To understand the concept of a negativity font in the context of an \( N \)-qubit system, consider the state
\[
\Phi^{A_1 A_2 \ldots A_N} = \sum_{i_1 i_2 \ldots i_N} \lambda_{i_1 i_2 \ldots i_N} | i_1 i_2 \ldots i_N \rangle \langle i_1 i_2 \ldots i_N |,
\]

with \( K = \sum_{m=1}^N (1 - \delta_{i_m,j_m}) \) and \( \delta_{i_1 j_1} = 0 \). The state \( \Phi^{A_1 A_2 \ldots A_N} \) is the product of a \( K \)-qubit GHZ-like state with \( N - K \) qubit product state. Let \( \hat{\sigma}^{T_{A_1}}_K \) be the \( K \)-way partial transpose of \( \hat{\sigma}_K = \left| \Phi^{A_1 A_2 \ldots A_N}_K \right\rangle \left\langle \Phi^{A_1 A_2 \ldots A_N}_K \right| \) with respect to qubit \( A_1 \). If \( \hat{\rho} \) is a pure state given by \( \hat{\rho} = | \Psi^{A_1 A_2 \ldots A_N} \rangle \left\langle \Psi^{A_1 A_2 \ldots A_N} | \right. \), then \( \hat{\sigma}^{T_{A_1}}_K \) is a \( 4 \times 4 \) sub-matrix of \( \hat{\rho}^{T_{A_1}}_K \) and \( \hat{\rho}^{T_{A_1}}_K \) with negative eigenvalue given by
\[
\lambda^* = -\det \left[ \begin{array}{cc} a_{i_1 i_2 \ldots i_N} & a_{j_1+1 j_2 \ldots j_N} \\ a_{i_1+1 i_2 \ldots i_N} & a_{j_1 j_2 \ldots j_N} \end{array} \right].
\]

The matrix \( \left[ \begin{array}{cc} a_{i_1 i_2 \ldots i_N} & a_{j_1+1 j_2 \ldots j_N} \\ a_{i_1+1 i_2 \ldots i_N} & a_{j_1 j_2 \ldots j_N} \end{array} \right] \) is referred to as a \( K \)-way negativity font \( 21, 22 \). A symbol used to represent a negativity font, must identify the qubits that appear in \( K \) qubit GHZ-like state. Therefore we split the set of \( N \) qubits with their locations and local basis indices given by, \( T = \{ (A_1)_{i_1} (A_2)_{i_2} \ldots (A_N)_{i_N} \} \), into two subsets, with \( S_{1,T} \) containing qubits with local basis indices satisfying \( \delta_{i_m,j_m} = 0 \) \( (i_m \neq j_m) \), and \( S_{2,T} \) having qubits for which \( \delta_{i_m,j_m} = 1 \) \( (i_m = j_m) \). To simplify the notation, we represent by \( s_{1,T} \), the sequence of local basis indices for qubits in \( S_{1,T} \). A specific negativity font is therefore represented by
\[
\hat{\rho}^{T_{A_1}}_{S_{2,T}} = \left[ \begin{array}{cc} a_{i_1 i_2 \ldots i_N} & a_{j_1+1 j_2 \ldots j_N} \\ a_{i_1+1 i_2 \ldots i_N} & a_{j_1 j_2 \ldots j_N} \end{array} \right].
\]

A nonzero determinant \( D^{T_{A_1}}_{S_{2,T}} = \det \left( \hat{\rho}^{T_{A_1}}_{S_{2,T}} \right) \) ensures that \( \hat{\sigma}^{T_{A_1}}_K \) is negative. A measurement on the state of a qubit with index in \( S_{1,T} \) reduces \( \hat{\sigma}_K \) to a separable state, whereas, measuring the state of a qubit in \( S_{2,T} \) does not change the negativity of \( \hat{\sigma}^{T_{A_1}}_K \). Elementary negativity fonts that quantify the negativity of \( \rho^{T_{A_1}}_K \) for \( p \neq 1 \) are defined in an analogous fashion. The determinant of a \( K \)-way negativity font detects \( K \)-body quantum correlations in an \( N \) qubit state. For even \( K \), proper combinations of determinants of \( K \)-way negativity fonts are found to be invariant under the action of local unitary operations on \( K \) qubits \( 22 \).

For a two qubit state negative eigenvalue of partial transpose is the invariant that distinguishes between the separable and entangled states. Global negativity of \( | \Psi^{A_1 A_2} \rangle = \sum a_{i_1 i_2} | i_1 i_2 \rangle \) is determined by \( I^{A_1 A_2}_2 = | a_q a_{11} - a_q a_{10} | \), which is invariant under \( U^{A_1} \otimes U^{A_2} \). Here \( U^{A_i} \) is a local unitary operator that acts on qubit \( A_i \). The subscript on \( I^{A_1 A_2}_2 \)
refers to two-body correlations. A two qubit state therefore has a single negativity font $\nu^{00} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$. In a general three qubit state,

$$\ket{\Psi^{A_1A_2A_3}} = \sum a_{i_1i_2i_3} \ket{i_1i_2i_3},$$

the number of two-qubit invariants, for a selected pair of qubits, is three. For the pair $A_1A_2$, for example, these are determinants of 2-way negativity fonts defined as

$$D_{(A_3)i_3}^{00} = \det \begin{bmatrix} a_{00i_3} & a_{01i_3} \\ a_{10i_3} & a_{11i_3} \end{bmatrix}, \ i_3 = 0, 1,$$

and the difference $(D_{(A_3)i_3}^{00} - D_{(A_3)i_3}^{01}) = (D_{(A_3)i_3}^{00} + D_{(A_3)i_3}^{01})$, where

$$D_{(A_3)i_3}^{01} = \det \begin{bmatrix} a_{0i_20} & a_{0i_2+1,1} \\ a_{1i_20} & a_{1i_2+1,1} \end{bmatrix}, \ i_2 = 0, 1,$$

is determinant of a three-way negativity font.

II. THREE-BODY CORRELATIONS AND THREE QUBIT INVARIANTS

Our method was applied in ref. [21], to construct three-tangle [13] and a degree two four qubit invariant which is a function of determinants of 4-way negativity fonts. To clarify the process, we review the three qubit case and show that by using three qubit invariants one may classify three qubit entangled states into states with (i) three and two body correlations, (ii) states with only three body correlations and iii) a set of states with only two body correlations. Class (i) states are the most general states. Class (ii) states with GHZ type entanglement have the form

$$\ket{\Psi^{A_1A_2A_3}} = a_{i_1i_2i_3} \ket{i_1i_2i_3} + a_{i_1i_1i_2+1,i_3+1} \ket{i_1 + 1, i_2 + 1, i_3 + 1},$$

and Class (iii) contains W-like entangled states and bi-separable states of three qubits. First of all, we write down the transformation equation for two qubit invariant $D_{(A_3)i_3}^{00}$ to obtain the invariant which quantifies three-body correlations. The form of this invariant is later used to identify three qubit invariants in four qubit states. In the absence of three-body correlations, modified transformation equations yield three qubit invariants that quantify two body correlations in a three qubit state.

Under a local unitary $U^{A_3} = \frac{1}{\sqrt{1+|x|^2}} \begin{bmatrix} 1 & -x^* \\ x & 1 \end{bmatrix}$, $D_{(A_3)i_3}^{00}$ transforms as

$$\left( D_{(A_3)i_3}^{00} \right)' = \frac{1}{1 + |x|^2} \left( D_{(A_3)i_3}^{00} + (x)^2 D_{(A_3)i_3}^{00} + x \left( D_{(A_3)i_3}^{00} + D_{(A_3)i_3}^{00} \right) \right),$$

such that

$$\left( N_{A_3}^{A_1, A_2} \right)^2 = \left| D_{(A_3)i_3}^{00} \right|^2 + \left| D_{(A_3)i_3}^{00} \right|^2 + 2 \left( \frac{D_{(A_3)i_3}^{00} + D_{(A_3)i_3}^{00}}{2} \right)^2$$

is a three qubit invariant. If the pair of qubits $A_1A_2$ is entangled then $N_{A_3}^{A_1, A_2} \neq 0$. We can verify that global negativity of $\rho_G^{T^{A_1}}$ is given by

$$\left( N_{G}^{A_1} \right)^2 = 4 \left( N_{A_3}^{A_1, A_2} \right)^2 + 4 \left( N_{A_2}^{A_1, A_3} \right)^2,$$

where

$$
\left( N_{A_2}^{A_1, A_3} \right)^2 = \left| D_{(A_3)i_3}^{00} \right|^2 + \left| D_{(A_3)i_3}^{00} \right|^2 + 2 \left( \frac{D_{(A_3)i_3}^{00} - D_{(A_3)i_3}^{00}}{2} \right)^2.
$$

The discriminant of $\left( D_{(A_3)i_3}^{00} \right)' = 0$, yields three qubit invariant

$$I_3^{A_1, A_2, A_3} = \left( D_{(A_3)i_3}^{00} + D_{(A_3)i_3}^{00} \right)^2 - 4D_{(A_3)i_3}^{00} D_{(A_3)i_3}^{00}.$$
which is a polynomial invariant of degree four in coefficients $a_{i_1i_2i_3i_4}$. The subscript in $I_{3}^{A_1A_2A_3}$ refers to three-body correlations of the type present in a three qubit GHZ state. The terms $D^{000}_{(A_3)_0}, D^{010}_{(A_3)_1}$, and $D^{000}_{(A_3)_1}$ vanish on a product state of qubits $A_1$ and $A_2$. On the state

$$\langle \Psi^{A_1A_2}, \Psi^{A_3} \rangle = \sum_{i_1i_2i_3} a_{i_1i_2i_3} |i_1i_2i_3\rangle (b_0 |0\rangle + b_1 |1\rangle), \quad (i_m = 0, 1), \quad (15)$$

with $D^{00} \neq 0$, we have $D^{00}_{(A_3)_0} = (b_0)^2 D^{00}_{(A_3)_1} = (b_1)^2 D^{00}_{(A_3)_0}$, and $D^{000}_{(A_3)_0} = b_0 b_1 D^{00}_{(A_3)_0}$ as such $I_{3}^{A_1A_2A_3} = 0$. Modulus of $I_{3}^{A_1A_2A_3}$, quantifies the entanglement of qubits $A_1A_2A_3$ due to three body correlations. Three tangle $\tau_3$, $\tau_3 = 4 |I_{3}^{A_1A_2A_3}|$, is a well known entanglement monotone.

For a general three qubit state with $I_{3}^{A_1A_2A_3} = 0$, determinants of two-way fonts transform as

$$\left( D^{00}_{(A_3)_0} \right)' = \frac{1}{1 + |x|^2} \left( x^* \sqrt{D^{00}_{(A_3)_1} - \sqrt[\sqrt[2]{2}]{D^{00}_{(A_3)_0}}} \right)^2,$$

$$\left( D^{00}_{(A_3)_1} \right)' = \frac{1}{1 + |x|^2} \left( x \sqrt{D^{00}_{(A_3)_0} + \sqrt[\sqrt[2]{2}]{D^{00}_{(A_3)_1}}} \right)^2,$$

therefore

$$N_{A_3}^{A_1A_2} = \left| \left( D^{00}_{(A_3)_0} \right)' \right| + \left| \left( D^{00}_{(A_3)_1} \right)' \right| = \left| D^{00}_{(A_3)_0} \right| + \left| D^{00}_{(A_3)_1} \right|, \quad \text{(16)}$$

is a three qubit invariant. In other words if $I_{3}^{A_1A_2A_3} = 0$ then $N_{A_3}^{A_1A_2}$ quantifies two body correlations of the pair $A_1A_2$. One can verify that $\left| D^{00}_{(A_m)_0} \right| + \left| D^{00}_{(A_m)_1} \right|$ $(m = 1, 2, 3)$, are three qubit invariants in this case. The sum of product invariants

$$I_{2}^{A_1A_2A_3} = 3 \sum_{i,j=1}^{3} \left( \left| D^{00}_{(A_3)_0} \right| + \left| D^{00}_{(A_3)_1} \right| \right) \left( \left| D^{00}_{(A_j)_0} \right| + \left| D^{00}_{(A_j)_1} \right| \right),$$

$$= 3 \left( N_{A_1}^{A_2A_3} N_{A_1}^{A_3} + N_{A_2}^{A_2A_3} N_{A_2}^{A_3} + N_{A_2}^{A_2A_3} N_{A_3}^{A_2A_3} \right)$$

(17)

detects W-like tripartite entanglement. It is zero on bi-separable states for which only one of the three $N_{A_m}^{A_iA_j} = \left| D^{00}_{(A_m)_i} \right| + \left| D^{00}_{(A_m)_j} \right|$ $(i \neq j \neq m)$ is non zero and one on a three qubit W-state. Major classes of three qubits states are uniquely defined by values of polynomial invariants $4 \left| I_{3}^{A_1A_2A_3} \right|^{2} - 4 \left| I_{3}^{A_1A_2A_3} \right| + I_{2}^{A_1A_2A_3}$.

III. FOUR-BODY CORRELATIONS AND FOUR-QUBIT INVARIANTS

Four qubit states live in the Hilbert space $C^2 \otimes C^2 \otimes C^2 \otimes C^2$ with a distinct subspace for each set of three qubits. If there were no four body correlations, three qubit invariants $I_{3}^{A_iA_jA_k}$ $(ii = 0, 1)$, may determine the entanglement of a four qubit state. In general, additional three qubit invariants that depend also on four-way negativity fonts exist. For a selected set of three qubits, three qubit invariants constitute a five dimensional space and are easily found by the action of a local unitary on the fourth qubit. To write down transformation equations for three qubit invariants, first of all, we identify two qubit invariants.

In the most general four qubit state

$$\langle \Psi^{A_1A_2A_3A_4} \rangle = \sum_{i_1i_2i_3i_4} a_{i_1i_2i_3i_4} |i_1i_2i_3i_4\rangle, \quad (i_m = 0, 1), \quad \text{(18)}$$

when state of qubit $A_1$ is transposed, we are looking at entanglement of qubit $A_1$ with rest of the system. Qubit $A_1$ may have pairwise entanglement with qubits $A_2$, $A_3$, or $A_4$. For a given pair, there are four two-way two qubit invariants (the remaining pair of qubits being in state |00⟩, |10⟩, |01⟩ or |11⟩). For example, the determinants of two-way negativity fonts for the pair $A_1A_2$, written as

$$D^{00}_{(A_3)_i} = \det \begin{vmatrix} a_{00i3i4} & a_{01i3i4} \\ a_{10i3i4} & a_{11i3i4} \end{vmatrix}, \quad (i_3, i_4 = 0, 1), \quad \text{(19)}$$
are invariant with respect to unitaries on qubits $A_1$ and $A_2$. Three-way coherences generate two qubit invariants $D^{(000)}_{(A_4)_{i_4}} - D^{(010)}_{(A_4)_{i_4}}$ ($i_4 = 0, 1$), and $D^{(000)}_{(A_3)_{i_3}} - D^{(010)}_{(A_3)_{i_3}}$ ($i_3 = 0, 1$), for the pair $A_1 A_2$. Here determinants of three-way fonts for $\{A_1 A_2 A_3\}$ and $\{A_1 A_2 A_4\}$, respectively, are defined as

\[
D^{012}_{(A_4)_{i_4}} = \det \begin{bmatrix}
 a_{0120i_4} & a_{0121,1i_4} \\
 a_{1120i_4} & a_{1121,1i_4}
\end{bmatrix}, \quad (i_2, i_4 = 0, 1),
\]

and

\[
D^{012}_{(A_3)_{i_3}} = \det \begin{bmatrix}
 a_{01230i_3} & a_{0123,1i_3} \\
 a_{11230i_3} & a_{1123,1i_3}
\end{bmatrix}, \quad (i_2, i_3 = 0, 1).
\]

If four-way negativity fonts are present, then additional $A_1 A_2$ invariants, $D^{0000} - D^{0100}$ and $D^{0001} - D^{0101}$, are to be considered. Determinants of four-way negativity fonts are given by

\[
D^{012}_{(A_i)_{i_4}} = \det \begin{bmatrix}
 a_{0120i_4} & a_{0121,1i_4} & a_{0122,1i_4} & a_{0123,1i_4} \\
 a_{1120i_4} & a_{1121,1i_4} & a_{1122,1i_4} & a_{1123,1i_4}
\end{bmatrix}, \quad (i_2, i_4 = 0, 1).
\]

Degree two four qubit invariant

\[
I_4 = (D^{0000} + D^{0011} - D^{0010} - D^{0001}),
\]

obtained in $[21]$ is the same as invariant $H$ of degree two in ref. $[18]$. The entanglement monotone, $\tau_4 = 4 |I_4|$, was called four-tangle in analogy with three tangle $[13]$. In $[22]$ our method was successfully applied to derive degree two $N$–qubit invariants for even $N$ and degree four invariants for odd $N$ in terms of determinants of negativity fonts. It was also shown that one may use the method to construct degree four invariants for odd $N$ in terms of determinants of negativity fonts.

Presently, we focus on the set $A_1 A_2 A_3$ of three qubits in state $|\Psi_{A_1 A_2 A_3 A_4}\rangle$ (Eq. $[18]$ ) viewed as

\[
|\Psi_{A_1 A_2 A_3 A_4}\rangle = |\Psi_{(A_4)_{i_4}}\rangle = |0\rangle + |\Psi_{(A_4)_{i_4}}\rangle |1\rangle,
\]

where

\[
|\Psi_{(A_4)_{i_4}}\rangle = \sum_{i_1 i_2 i_3} a_{i_1 i_2 i_3 i_4} |i_1 i_2 i_3 i_4\rangle; \quad (i_4 = 0, 1).
\]

Three qubit invariants

\[
\left(I_3^{(A_1 A_2 A_3)}\right)_{(A_4)_{i_4}} = \left(D^{000}_{(A_4)_{i_4}} + D^{001}_{(A_4)_{i_4}}\right)^2 - 4D^{00}_{(A_3)_{0}(A_4)_{i_4}} D^{00}_{(A_3)_{1}(A_4)_{i_4}}; \quad i_4 = 0, 1,
\]

quantify GHZ state like three-way correlations in three qubit subspace $C^2 \otimes C^2 \otimes C^2$. Continuing the search for a four qubit invariant that detects four qubit correlations, we examine the transformation of three qubit invariant $\left(I_3^{(A_1 A_2 A_3)}\right)_{(A_4)_{i_4}}$ under $U^{A_4} = \frac{1}{\sqrt{1+|y|^2}} \begin{bmatrix} 1 & -y^* \\ y & 1 \end{bmatrix}$. The resulting transformation equation is

\[
\left(I_3^{(A_1 A_2 A_3)}\right)'_{(A_4)_{i_4}} = \frac{1}{\left(1+|y|^2\right)^2} |y|^4 \left[I_3^{(A_1 A_2 A_3)}\right]_{(A_4)_{i_4}} + 4y^3 P^{A_1 A_2 A_3}_{(A_4)_{i_4}} + 6y^2 T^{A_1 A_2 A_3}_{(A_4)_{i_4}} + 4y P^{A_1 A_2 A_3}_{(A_4)_{i_4}} + \left[I_3^{(A_1 A_2 A_3)}\right]_{(A_4)_{i_4}},
\]

where

\[
T^{A_1 A_2 A_3}_{(A_4)_{i_4}} = \frac{1}{6} \left(D^{0000} + D^{0011} + D^{0010} + D^{0001}\right)^2
- \frac{2}{3} \left(D^{0000}_{(A_3)_{0}} + D^{0011}_{(A_3)_{0}}\right) \left(D^{0000}_{(A_3)_{1}} + D^{0011}_{(A_3)_{1}}\right)
+ \frac{1}{3} \left(D^{0000}_{(A_4)_{0}} + D^{0011}_{(A_4)_{0}}\right) \left(D^{0000}_{(A_4)_{1}} + D^{0011}_{(A_4)_{1}}\right)
- \frac{2}{3} \left(D^{0000}_{(A_3)_{0}(A_4)_{i_4}} + D^{0011}_{(A_3)_{0}(A_4)_{i_4}} + D^{0000}_{(A_3)_{1}(A_4)_{i_4}} + D^{0011}_{(A_3)_{1}(A_4)_{i_4}}\right),
\]
with global negativity, one may define a four qubit invariant of degree four, 
while the discriminant reads as

\[
I = \frac{1}{2} \left( D_{(A_4)}^{0001} + D_{(A_4)}^{0011} \right) \left( D_{(A_4)}^{0000} + D_{(A_4)}^{0010} + D_{(A_4)}^{0011} \right) 
- \left( D_{(A_3)}^{0001} \left( D_{(A_3)}^{0000} + D_{(A_3)}^{0010} \right) + D_{(A_3)}^{0000} \left( D_{(A_3)}^{0001} + D_{(A_3)}^{0011} \right) \right). 
\] (28)

Discriminant of a quartic equation, \( y^4a - 4by^3 + 6y^2c - 4dy + f = 0 \), in variable \( y \) is \( \Delta = S^3 - 27T^2 \) where \( S = 3c^2 - 4bd + af \), and \( T = ace - ad^2 - b^2f + 2bdc - c^3 \) (cubic invariant), are polynomial invariants. When a selected \( U^{A_4} \) results in \( \left( \left( \psi_{A_1A_2A_3} \right)_{(A_4)} \right)' = 0 \) (Eq. (26)), the associated polynomial invariant is

\[
I_{(4,8)}^{A_1A_2A_3A_4} = 3 \left( T_{A_4}^{A_1A_2A_3} \right)^2 - 4P_{(A_3)}^{A_1A_2A_3}P_{(A_4)}^{A_1A_2A_3} 
+ \left( I_{3}^{A_1A_2A_3} \right)_{(A_3)} \left( I_{3}^{A_1A_2A_3} \right)_{(A_4)}, 
\] (29)

which is a four qubit invariant of degree eight expressed in terms of three qubit invariants for \( A_1A_2A_3 \). In order to distinguish between degree 2 invariant \( I_4 \) and the new invariant, degree of the invariant has been added to the subscript. By construction, the four qubit invariant \( I_{(4,8)}^{A_1A_2A_3A_4} \) is a combination of three qubit \((A_1A_2A_3)\) invariants.

It is easily verified that on a state which is a product of \( |\psi^{A_1A_2A_3} = \sum_{i_1i_2i_3}a_{i_1i_2i_3} |i_1i_2i_3 \rangle \) with \( I_{3}^{A_1A_2A_3} \neq 0 \), and \( \psi^{A_4} = d_0|0 \rangle + d_1|1 \rangle \), we obtain

\[
\left( I_{3}^{A_1A_2A_3} \right)_{(A_4)} = \left( I_{3}^{A_1A_2A_3} \right)_{(A_3)} = T_{A_4}^{A_1A_2A_3} = P_{(A_3)}^{A_1A_2A_3} = P_{(A_4)}^{A_1A_2A_3}, 
\] (30)

leading to \( I_{(4,8)}^{A_1A_2A_3A_4} = 0 \). Likewise, \( I_{(4,8)}^{A_1A_2A_3A_4} \) vanishes on product state \( |\psi^{A_1A_2} \rangle |\psi^{A_3A_4} \rangle \), where \( |\psi^{A_1A_2} \rangle = \sum_{i_1i_2}a_{i_1i_2} |i_1i_2 \rangle \) and \( |\psi^{A_3A_4} \rangle = \sum_{i_3i_4}b_{i_3i_4} |i_3i_4 \rangle \). Besides that \( I_{(4,8)}^{A_1A_2A_3A_4} = 0 \) on a four qubit W-like state, and all entangled states with only three and two-body correlations, as seen in section IV.

The cubic invariant associated with Eq. (25) is

\[
J^{A_1A_2A_3A_4} = \det \begin{bmatrix} I_{3}^{A_1A_2A_3} & P_{(A_4)}^{A_1A_2A_3} & T_{A_4}^{A_1A_2A_3} \\ P_{(A_3)}^{A_1A_2A_3} & T_{A_4}^{A_1A_2A_3} & P_{(A_4)}^{A_1A_2A_3} \\ T_{A_4}^{A_1A_2A_3} & P_{(A_3)}^{A_1A_2A_3} & I_{3}^{A_1A_2A_3} \end{bmatrix}, 
\] (31)

while the discriminant reads as

\[
\Delta = \left( I_{(4,8)}^{A_1A_2A_3A_4} \right)^3 - 27 \left( J^{A_1A_2A_3A_4} \right)^2. 
\] (32)

Since there are four ways in which a given set of three qubits may be selected, \( \Delta \) can be expressed in terms of different sets of three qubit invariants. In addition (Eq. (26)) also leads to

\[
\left( N_{A_4}^{A_1A_2A_3} \right)^2 = \left( I_{3}^{A_1A_2A_3} \right)_{(A_4)}^2 + \left( I_{3}^{A_1A_2A_3} \right)_{(A_3)}^2 
+ 6 \left| T_{A_4}^{A_1A_2A_3} \right|^2 + 4 \left| P_{(A_4)}^{A_1A_2A_3} \right|^2 + 4 \left| P_{(A_4)}^{A_1A_2A_3} \right|^2, 
\] (33)

which is a four qubit invariant analogous to \( \left( N_{A_3}^{A_1A_2} \right)^2 \) (Eq. (11)) for three qubit states. In general, one can construct an invariant \( N_{A_i}^{A_jA_k} \) \((i \neq j \neq k \neq l)\) for a selected three qubit subsystem \( A_iA_jA_k \) of four qubit state. In analogy with global negativity, one may define a four qubit invariant of degree four,

\[
\left( N_{A_4}^{A_1A_2A_3} \right)^2 = 16 \left( N_{A_4}^{A_1A_2A_3} \right)^2 + 16 \left( N_{A_3}^{A_1A_2A_4} \right)^2 + 16 \left( N_{A_2}^{A_1A_3A_4} \right)^2, 
\] (34)

which detects bipartite entanglement of qubit \( A_i \) with subsystem \( A_2A_3A_4 \) due to three and four body quantum correlations. If \( I_{(4,8)}^{A_1A_2A_3A_4} = 0 \), but at least two of the \( N_{A_i}^{A_jA_k} \) are finite, then 4-partite entanglement can be due to
three and two body correlations. In this case the invariant that detects entanglement may be defined as

\[
N^{A_1A_2A_3A_4}_{(4,8)} = 16N^A_1N^A_2N^A_3N^A_4 + 16 \left( N^A_2N^A_3N^A_4A_1 + N^A_1N^A_3N^A_4A_2 + N^A_2N^A_1N^A_4A_3 + N^A_2N^A_3N^A_1A_4 \right) N^A_1N^A_2N^A_3N^A_4 + 16 \left( N^A_2N^A_3N^A_4A_1 + N^A_1N^A_3N^A_4A_2 + N^A_2N^A_1N^A_4A_3 + N^A_2N^A_3N^A_1A_4 \right) N^A_1N^A_2N^A_3N^A_4.
\]

(35)

On the other hand, if we have a state on which all \(N^A_{i+}N^A_{i-}\) are zero, then the quantities \(I_{(4,8)}^{A_1A_2A_3A_4} = \sum_{i,j} D^{(0)}_{(A_1)_i(A_2)_j} \) \((p \neq q \neq r \neq s = 1 \to 4)\), turn out to be four qubit invariants. A different class of entangled states is obtained if only one of the \(N^A_{i+}N^A_{i-}\) is non zero along with a finite \(I_{(4,8)}^{A_1A_2A_3}\). In section II we noted that \(I_{2}^{A_1A_2A_3}\) detects W-like entanglement of qubits. Likewise, when \(I_{(4,8)}^{A_1A_2A_3A_4} = N^{A_1A_2A_3A_4}_{(4,8)} = 0\), the invariant

\[
I^{A_1A_2A_3A_4}_{(2,6)} = \frac{3}{2} \left( I^{A_1A_2A_3}_2 + I^{A_1A_4}_2 + I^{A_2A_3}_2 \right) + \frac{3}{2} \left( I^{A_1A_2A_4}_2 + I^{A_1A_3}_2 + I^{A_2A_4}_2 \right)
\]

(36)
detects W-like four qubit entanglement. Here \(I^{A_1A_2A_3}_2 = \frac{3}{2} I^{A_1A_2A_3}_2 I^{A_1A_4}_2, \) \((p \neq q \neq r \neq s = 1 \to 4)\), is the invariant that detects W-like entanglement of qubits \(A_pA_qA_r\) in a four qubit state.

In ref. four qubit invariants have been obtained in terms of coefficients having geometrical significance. A comparison of Eq. (56) of ref. with our Eq. (26), indicates that their set of invariants \((I_1, I_2, I_3, I_4)\) may be expressed in terms of our three qubit invariants, though they are not exactly the same. A method equivalent to method of Schlaflis has been used to arrive at Eq. (22)

\[
R(t) = c_0t^4 + 4c_1t^3 + 6c_2t^2 + 4c_3t + c_4
\]

by Luque and Thibon. Then higher degree invariants are expressed in terms of \(c_i\) coefficients and computer algebra relates these to basic four qubit invariants. Since for \(t_0 = 1\), expression for \(R(t)\) has the same form as Eq. (22), a direct correspondence can be established between \(c_i\) coefficients and our three qubit invariants. Such a comparison establishes a neat connection of our invariants with projective geometry approach and classical invariant theory concepts.

IV. INVARIANTS AND CLASSIFICATION OF FOUR-QUBIT STATES

Decomposition of global partial transpose \(\rho_G^T_{Ap}\) of four qubit state \(\Psi^{A_1A_2A_3A_4}\) with respect to qubit \(A_p\) in terms of \(K\)-way partially transposed operators (Eq. 4) reads as

\[
\rho_G^T_{Ap} = \sum_{K=2}^{4} \rho_K^T_{Ap} - 2\rho.
\]

(37)

When a state has only \(K\)-way coherences, we have \(\rho_G^T_{Ap} = \rho_K^T_{Ap}\), for a selected set of \(K\) qubits. For a given qubit, the number of \(K\)-way negativity fonts in a \(K\)-way partially transposed matrix varies from 0 to 4. Local unitary operations can be used to annihilate the negativity fonts that is obtain a state for which determinants of selected negativity fonts are zero. The process leads to canonical state which is a state written in terms of minimum number of local basis product states. In ref. we proposed a classification scheme in which an entanglement class is characterized by the minimal set of \(K\)-way \((2 \leq K \leq 4)\) partially transposed matrices present in the expansion of global partial transpose of the canonical state. Seven possible ways in which the global partial transpose (GPT) of a four qubit canonical state may be decomposed correspond to seven major entanglement classes that is class I. \(\rho_G^T_{Ap} = \sum_{K=2}^{4} \rho_K^T_{Ap} - 2\rho_c\), II. \(\rho_G^T_{Ap} = (\rho_c)^T_{Ap} + (\rho_c)^T_{Ap} - \rho_c\), III. \(\rho_G^T_{Ap} = (\rho_c)^T_{Ap} + (\rho_c)^T_{Ap} - \rho_c\), IV. \(\rho_G^T_{Ap} = (\rho_c)^T_{Ap} + (\rho_c)^T_{Ap} - \rho_c\), V. \(\rho_G^T_{Ap} = (\rho_c)^T_{Ap} + (\rho_c)^T_{Ap} - \rho_c\), VI. \(\rho_G^T_{Ap} = (\rho_c)^T_{Ap} + (\rho_c)^T_{Ap} - \rho_c\), VII. \(\rho_G^T_{Ap} = (\rho_c)^T_{Ap} + (\rho_c)^T_{Ap} - \rho_c\). Of these, six classes contain states with four-partite entanglement, while class VI with \(\rho_G^T_{Ap} = (\rho_3)^T_{Ap}\) has only three qubit entanglement. Each major class contains sub-classes depending on the number and type of negativity fonts in global partial transpose of the canonical state. Table lists the decomposition of \((\rho_G^T_{Ap})\), invariants \(I^{A_1A_2A_3A_4}_{(4,8)}\),
TABLE I: Decomposition of \((\rho_c)_{G}^T_{\Delta}\), invariants \(I^{A_1A_2A_3}_{(A_4)}\), \(D^{A_1A_2A_3}_{A_4}\), \(\Delta\), and \(N_{K\text{-way}}\) \((K = 2,3,4)\) in canonical state, for seven classes of four qubit entangled states

| Class | Decomposition of \((\rho_c)_{G}^T_{\Delta}\) | \(I^{A_1A_2A_3}_{(A_4)}\) | \(D^{A_1A_2A_3}_{A_4}\) | \(\Delta\) | \(N_{2\text{-way}}\) | \(N_{3\text{-way}}\) | \(N_{4\text{-way}}\) |
|-------|---------------------------------|-----------------|-----------------|---------|---------|---------|---------|
| I     | \(\sum_{k=2}^{\infty} (\rho_c)_{K}^T_{\Delta_k} - 2\rho_c\) | \(\neq 0\)     | \(\neq 0\)     | \(\neq 0\) | \(\geq 1\) | \(\geq 1\) | \(\geq 1\) |
| II    | \((\rho_c)_{2}^T_{\Delta_2} + (\rho_c)_{3}^T_{\Delta_3} - \rho_c\) | \(\neq 0\)     | \(\neq 0\)     | \(0\)    | \(\geq 1\) | \(0\)    | \(\geq 1\) |
| III   | \((\rho_c)_{3}^T_{\Delta_3} + (\rho_c)_{2}^T_{\Delta_2} - \rho_c\) | \(\neq 0\)     | \(\neq 0\)     | \(\geq 1\) | \(0\)    | \(\geq 1\) | \(0\)    |
| IV    | \((\rho_c)_{3}^T_{\Delta_3} + (\rho_c)_{2}^T_{\Delta_2} - \rho_c\) | \(\neq 0\)     | \(\neq 0\)     | \(\neq 0\) | \(\geq 1\) | \(0\)    | \(\geq 1\) |
| V     | \((\rho_c)_{3}^T_{\Delta_3} + (\rho_c)_{2}^T_{\Delta_2} - \rho_c\) | \(0\)          | \(\neq 0\)     | \(\neq 0\) | \(\geq 1\) | \(0\)    | \(\geq 1\) |
| VI    | \((\rho_c)_{3}^T_{\Delta_3} + (\rho_c)_{2}^T_{\Delta_2} - \rho_c\) | \(0\)          | \(\neq 0\)     | \(\neq 0\) | \(\geq 1\) | \(0\)    | \(\geq 1\) |
| VII   | \((\rho_c)_{3}^T_{\Delta_3} + (\rho_c)_{2}^T_{\Delta_2} - \rho_c\) | \(0\)          | \(\neq 0\)     | \(\neq 0\) | \(\geq 1\) | \(0\)    | \(\geq 1\) |

\(D^{A_1A_2A_3}_{A_4}\), \(\Delta\), and \(N_{K\text{-way}}\) \((K = 2,3,4)\) in canonical state, for different classes of four qubit entangled states. Here \(D^{A_1A_2A_3}_{A_4} = \left( N^{A_1A_2A_3}_{A_4} \right)^2 - 2 \left| I^{A_1A_2A_3}_{(A_4)} \right|^2\) is a measure of residual three-way correlations between qubits \(A_1A_2A_3\) and \(N_{K\text{-way}}\) \((K = 2,3,4)\) is the number of \(K\)-way negativity fonts in a state.

A four qubit state with a single four way negativity font

\[ | \Psi_{ab} \rangle = a \left( |0000\rangle + |1111\rangle \right) + b \left( |1101\rangle + |1110\rangle + |0011\rangle \right), \]

is an example of class I states. Three qubit invariants for the state are \(I^{A_1A_2A_3}_{(A_4)} = a^2b^2\), \(P^{A_1A_2A_3}_{(A_4)} = \frac{1}{2}a^3b\), \(I^{A_1A_2A_3}_{(A_4)} = b^4\), \(P^{A_1A_2A_3}_{(A_4)} = -\frac{1}{2}a^2b^2\), and \(T^{A_1A_2A_3}_{(A_4)} = \frac{1}{6} (a^4 - 2ab^3)\). Four qubit invariants are found to be \(I^{A_1A_2A_3}_{(A_4)} = \frac{1}{12} (a^4 + 4ab^3)^2\), \(D^{A_1A_2A_3}_{A_4} \neq 0\), and \(\Delta \neq 0\). A representative of class II states with \(\rho_{G}^T_{\Delta} = \rho_4^T_{\Delta} + \rho_3^T_{\Delta} - \rho_2^T_{\Delta}\) is, \(| \Psi_a \rangle = a \left( |0000\rangle + |1111\rangle \right) + |1110\rangle\). The state is SLOCC equivalent to GHZ state, however it deserves a distinct status since on removal of qubit \(A_4\) it has residual three way coherences.

Invariants for class III states

\[
G_{abcd} = \frac{a + d}{2} \left( |0000\rangle + |1111\rangle \right) + \frac{a - d}{2} \left( |1100\rangle + |0011\rangle \right) + \frac{b + c}{2} \left( |1010\rangle + |0101\rangle \right) + \frac{b - c}{2} \left( |0110\rangle + |1001\rangle \right),
\]

\[ (38) \]

\[
L_{abc2} = \frac{a + b}{2} \left( |0000\rangle + |1111\rangle \right) + \frac{a - b}{2} \left( |1100\rangle + |0011\rangle \right) + c \left( |1010\rangle + |0101\rangle \right) + |0110\rangle,
\]

\[ (39) \]

\[
L_{a2b2} = a \left( |0000\rangle + |1111\rangle \right) + b \left( |0101\rangle + |1010\rangle \right) + |0110\rangle + |0011\rangle,
\]

\[ (40) \]

and

\[
L_{a203} = a \left( |0000\rangle + |1111\rangle \right) + |0101\rangle + |0110\rangle + |0011\rangle,
\]

\[ (41) \]

of ref. [2] with \((\rho_c)_{G}^T_{\Delta} = (\rho_c)_{4}^T_{\Delta_4} + (\rho_c)_{2}^T_{\Delta_2} - \rho_c\) are listed in Table III. All three way coherences are convertible to two way coherences as such three-way negativity fonts have zero determinants. Four qubit entanglement occurs due to four-way and two-way coherences. For all these states, the invariants \(P^{A_1A_2A_3}_{(A_4)}\) and \(P^{A_1A_3}_{(A_4)}\) are identically zero. In Table III for states in family \(G_{abcd}\), three qubit invariants used for the set \(A_1A_2A_3\) are

\[
T^{A_1A_2A_3}_{(A_4)} = \frac{1}{6} (A - 2B), \quad \left( I^{A_1A_2A_3}_{(A_4)} \right)_{(A_4)} = \left( I^{A_1A_2A_3}_{(A_4)} \right)_{(A_4)} = B,
\]

where

\[
A = (a^2 - b^2) (d^2 - c^2), \quad B = \frac{1}{4} (a^2 - d^2) (b^2 - c^2).
\]

\[ (42) \]
TABLE II: Invariants for class III states $G_{abcd}$, $L_{abc2}$, $L_{a2b2}$ and $L_{a20_{00\overline{1}}}$ with $(\rho_c)^{T_A}_{G} = (\rho_c)^{T_A}_{4} + (\rho_c)^{T_A}_{2} - \tilde{\rho}_c$. $A$ and $B$ in column II are as defined in Eq. (42).

| Invariant | $G_{abcd}$ | $L_{abc2}$ | $L_{a2b2}$ | $L_{a20_{00\overline{1}}}$ |
|-----------|-----------|-----------|-----------|----------------|
| $(N_{4}^{A_1A_2A_3})^2$ | $\frac{1}{5} |A - 2B|^2 + 2 |B|^2$ | $\frac{1}{5} |a^2 - c^2| (b^2 - c^2)^2$ | $\frac{1}{5} |a^2 - b^2|^2$ | $\frac{1}{5} |a|^2$ |
| $I_{A_1A_2A_3}^{(4,8)}$ | $\frac{1}{72} (A - 2B)^2 + B^2$ | $\frac{1}{72} (a^2 - c^2)^2 (b^2 - c^2)^2$ | $\frac{1}{72} (a^2 - b^2)^4$ | $\frac{1}{72} a^8$ |
| $D_{A_1A_2A_3}^{(4,8)}$ | $\neq 0$ | $|c(a^2 - b^2)|^2$ | 0 | 0 |
| $\Delta$ | $\neq 0$ | 0 | 0 | 0 |

For states $G_{a\overline{0}0\overline{0}}$ and $G_{00\overline{0}a}$, $\Delta = 0$. For states $L_{abc2}$, with $(T_{A_1A_2A_3})_A = \frac{1}{6} (a^2 - c^2)(b^2 - c^2)$, $I_{A_1A_2A_3}^{(4,8)} = c(a^2 - b^2)$, the value $I_{A_1A_2A_3}^{(4,8)} = 0$ results in $\Delta = 0$. A comparison of states $L_{abc2}$ with $a = c$ and $L_{ab3}$ shows that the states are not SLOCC equivalent [8] because the number of negativity fonts is not equal. However, since four qubit correlations are null $I_{A_1A_2A_3}^{(4,8)} = 0$ for $L_{abc2}$ with $a = c$ as well as $L_{ab3}$, these are subclasses of the same major class in correlation type based classification, partially supporting the result of [28].

The families of states $L_{ab3}$ and $L_{a3}$ of ref. [2] have a similar global partial transpose composition. The value of degree two invariant is $I_4 = \frac{3a^2 + b^2}{2}$ for $L_{ab3}$ and $I_4 = 2a^2$ for $L_{a3}$ indicating that four-way coherences are present. However, for the set of qubits $A_1A_2A_3$, only non zero three qubit invariant is $I_{A_1A_2A_3}^{(4,8)} = \frac{a^2 - b^2}{2}$ for $L_{ab3}$ and $I_{A_1A_2A_3}^{(4,8)} = -4a^2$ for $L_{a3}$. A finite $I_4$ but zero $I_{A_1A_2A_3}$ indicates that the superposition contains a product of two qubit entangled states. Four partite entanglement may, in this case, be detected by products

$(N_{A_1}^{A_2A_3}) (N_{A_2}^{A_1A_3}).$

The states in families $L_{a2b2}$ and $L_{a20_{00\overline{1}}}$ [2] have $(N_{A_4}^{A_2A_3})^2 = 2I_{A_1A_2A_3}^{(4,8)}$. The states in $L_{a2b2}$ and $L_{a20_{00\overline{1}}}$ differ from each other in the number of two way negativity fonts with non-zero determinants. Only non zero three tangle for the states $G_{abba}$ is $T_{A_4}^{A_1A_2A_3}$. The states $G_{a00a}$ and $G_{00\overline{0}a}$, with $(\rho_c)^{T_A}_{G} = (\rho_c)^{T_A}_{4}$ belong to class IV in classification scheme based on correlation type. For these states only non-zero three tangle is $T_{A_4}^{A_1A_2A_3}$, therefore $I_{A_1A_2A_3}^{(4,8)} = 0$, $\Delta = 0$ and $D_{A_1A_2A_3}^{(4,8)} = 0$.

The global partial transpose has composition, $(\rho_c)^{T_A}_{G} = (\rho_c)^{T_A}_{3} + (\rho_c)^{T_A}_{2} - \tilde{\rho}_c$, for class V states $L_{0}\otimes\Gamma$ and $L_{0}\otimes\Gamma$. In both cases $I_{A_1A_2A_3}^{(4,8)} = 0$, while the product $\left(N_{A_4}^{A_1A_2A_3}\right) \left(N_{A_3}^{A_1A_2A_4}\right) \neq 0$. Two states differ in the the number of two-way negativity fonts with non-zero determinants. Only non-zero invariant for Class VI state $L_{0}\otimes\Gamma^{(4,8)}[2]$ with $(\rho_c)^{T_A}_{G} = (\rho_c)^{T_A}_{3} = (I_{A_3}^{A_1A_2A_3})^{(A_4)}$. The state has only three qubit entanglement. Class VII with $(\rho_c)^{T_A}_{G} = (\rho_c)^{T_A}_{2}$ contains four qubit states with W-type entanglement represented by $L_{a=0b=0}$ and separable states with entangled qubit pairs, for example $G_{aaaa}$.

The polynomial invariant $I_{A_1A_2A_3}^{(4,8)}$ is non-zero on states $|\Psi_{ab}\rangle$, $G_{abcd}$, $L_{abc2}$, $L_{a2b2}$, $L_{a20_{00\overline{1}}}$, $G_{a00a}$ and $G_{00\overline{0}a}$ and vanishes on states $L_{ab3}$, $L_{a3}$, $L_{0}\otimes\Gamma$, $L_{0}\otimes\Gamma$, $L_{0}\otimes\Gamma^{(4,8)}$ and $G_{aaaa}$. We define an entanglement monotone to quantify four qubit correlations as

$$\tau_{(4,8)} = 4 \left| 12I_{A_1A_2A_3A_4}^{(4,8)} \right|^\dagger,$$

which is one on states with maximal entanglement due to four-body correlations, finite on all states with entanglement due to four-body correlations and zero otherwise. The subscript $(4,8)$ is carried on from $I_{A_1A_2A_3A_4}^{(4,8)}$. One can verify that on four qubit GHZ state

$$|GHZ\rangle = \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle)$$
as well as cluster states \[38, 39\]

\[|C_1\rangle = \frac{1}{2} (|0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle)\]

\[|C_2\rangle = \frac{1}{2} (|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle),\]

\[|C_3\rangle = \frac{1}{2} (|0000\rangle + |1010\rangle + |0101\rangle - |1111\rangle),\]

\[\tau_{(4,8)} = 1 \quad \text{and} \quad \left(N^A_{4_A}A_3\right)^2 = 2 \left| I^{A_1A_2A_3A_4}_{(4,8)} \right|. \]

So what is different in cluster states? We recall the invariants \(J^{A_iA_j}\) from \[22\], the invariants that detect entanglement of a selected pair, \(A_iA_j\), of qubits in a four qubit state. For a GHZ state \(J^{A_iA_j} = \frac{1}{4}\), for \((i \neq j) = 1\) to 4, while for a cluster state all \(J^{A_iA_j}\) \[22\], do not have the same value. In canonical form, GHZ has a single four-way negativity font, while a cluster state has two four-way negativity fonts besides also having two-way negativity fonts (state reduction does not destroy all the coherences).

Another state proposed through a numerical search in ref. \[40\] to be a maximally entangled state is

\[|\Phi\rangle = \frac{1}{2} (|0000\rangle + |1101\rangle) + \frac{1}{\sqrt{8}} (|1011\rangle + |0011\rangle + |0110\rangle - |1110\rangle),\]

However, on this state

\[T^A_{4_A}A_2A_3 = \left( I^A_{4_A}A_2A_3 \right)_{(A_4)_0} = \left( I^A_{4_A}A_2A_3 \right)_{(A_4)_1} = \frac{1}{32}, \]

\[\left( P^A_{4_A}A_2A_3 \right)_{(A_4)_0} = \left( P^A_{4_A}A_2A_3 \right)_{(A_4)_1} = 0, \]

therefore \(I^{A_1A_2A_3A_4}_{(4,8)} = \frac{1}{256}\), and \(\tau_{(4,8)} = \sqrt{\frac{4}{3}}\). On two excitation four qubit Dicke state

\[|\Psi_D\rangle = \frac{1}{\sqrt{6}} (|0011\rangle + |1100\rangle + |0101\rangle + |1010\rangle + |0110\rangle + |1010\rangle),\]

we have, \(\tau_{(4,8)} = \frac{5}{6}\), while it is zero on four qubit W-state

\[|W\rangle = \frac{1}{2} (|0000\rangle + |1100\rangle + |1010\rangle + |1010\rangle).\]

Four tangle \(\tau_4\) also vanishes on W–like state of four qubits, however, it fails to vanish on product of two qubit entangled states. Contrary to \(\tau_{(4,8)}\), a non zero \(\tau_4\) does not ensure four-partite entanglement. On four qubit state

\[|HS\rangle = \frac{1}{\sqrt{6}} \left( |0011\rangle + |1100\rangle + \exp\left(\frac{i2\pi}{3}\right) |1010\rangle + |0110\rangle \right) \]

\[+ \frac{1}{\sqrt{6}} \exp\left(\frac{i4\pi}{3}\right) |1001\rangle + 0110\rangle, \quad (44)\]

conjectured to have maximal entanglement in ref. \[41\], we have \(D^{00}_{(A_3)_0(A_4)_1} = D^{00}_{(A_3)_1(A_4)_0} = \frac{1}{6}\), and for 4–way negativity fonts \(D^{0011} = \frac{1}{6}, D^{0001} = \frac{11}{12} (1 - i\sqrt{3})\), and \(D^{0010} = \frac{1}{12} (1 + i\sqrt{3})\). Therefore

\[T^A_{4_A}A_2A_3 = \left( I^A_{4_A}A_2A_3 \right)_{(A_4)_0} = \left( I^A_{4_A}A_2A_3 \right)_{(A_4)_1} = 0, \]

\[\left( P^A_{4_A}A_2A_3 \right)_{(A_4)_0} = \left( P^A_{4_A}A_2A_3 \right)_{(A_4)_1} = 0, \]
leading to $\tau_{(4,8)} = 0$. However, the invariant $I^{A_1 A_2 A_3 A_4}_{(2,6)}$ takes value $\frac{27}{64}$ on the four qubit $|W\rangle$ state. It reflects the fact that a measurement on the state of a qubit, in $|HS\rangle$ always leaves the three remaining qubits in a three qubit W-state, whereas a similar measurement on a $|W\rangle$ state yields a mixture of three qubit W-state with three qubits in a separable state.

The choice $I^{A_1 A_2 A_3 A_4}_{(4,8)}$ to quantify four qubit correlations is also supported by the conclusions of [12], where for a selected set of four qubit states, generator $S$ of ref. [18] has been shown to have the same parameter dependence as optimized Bell type inequalities and a combination of global negativity and 2-qubit concurrences.

To summarize, degree 8, 12 and 24 four qubit invariants, expressed in terms of three qubit invariants, have been obtained. One can continue the process to higher number of qubits. Commonly, multivariate forms in terms of state coefficients $a_{i_1 i_2 \ldots i_N}$ are used to obtain polynomial invariants for qubit systems. Our strategy is to write multivariate forms with relevant $K-$qubit invariants as coefficients. The advantage of our technique is that relevant invariants in a larger Hilbert space are easily related to invariants in sub spaces such as to the structure of the quantum state at hand. Construction of polynomial invariants for states other than the most general state is a great help in classification of states. Our method can be easily applied to determine the invariants for any given state. Entanglement monotone that quantifies four qubit correlations can be used to quantify correlations in pure and mixed (via convex roof extension) four qubit states.

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