The Arithmetic Site
Le Site Arithmétique

Alain Connes\textsuperscript{a,}, Caterina Consani\textsuperscript{b,1}

\textsuperscript{a}Collège de France, 3 rue d’Ulm, Paris F-75005 France; I.H.E.S. and Ohio State University
\textsuperscript{b}The Johns Hopkins University Baltimore, MD 21218 USA

Abstract

We show that the non-commutative geometric approach to the Riemann zeta function has an algebraic geometric incarnation: the “Arithmetic Site”. This site involves the tropical semiring $\overline{\mathbb{N}}$ viewed as a sheaf on the topos $\mathbb{N}^\times$ dual to the multiplicative semigroup of positive integers. We realize the Frobenius correspondences in the square of the “Arithmetic Site”.

1. Introduction

We unveil the “Arithmetic Site” as a ringed topos deeply related to the non-commutative geometric approach to RH. The topos is the presheaf topos $\mathbb{N}^\times$ of functors from the multiplicative semigroup $\mathbb{N}^\times$ of positive integers to the category of sets. The structure sheaf is a sheaf of semirings of characteristic 1 and (as an object of the topos) is the tropical semiring $\overline{\mathbb{N}} := (\mathbb{N} \cup \infty, \inf, +)$, $\mathbb{N} = \mathbb{Z}_{\geq 0}$, on which the semigroup $\mathbb{N}^\times$ acts by multiplication. We prove that the set of points of the arithmetic site $(\mathbb{N}^\times, \overline{\mathbb{N}})$ over $\mathbb{N}$.
the maximal compact subring $[0, 1]_{\text{max}} \subset \mathbb{R}_+^{\text{max}}$ of the tropical semifield is the non-commutative space $\mathbb{Q} \backslash \mathbb{A}_Q/\mathbb{Z}^*$ quotient of the adèle class space of $\mathbb{Q}$ by the action of the maximal compact subgroup $\hat{\mathbb{Z}}^*$ of the idele class group. In [5, 6] it was shown that the action of $\mathbb{R}_+^*$ on $\mathbb{Q} \backslash \mathbb{A}_Q/\mathbb{Z}^*$ yields the counting distribution whose Hasse-Weil zeta function is the complete Riemann zeta function. This result is now applied to the arithmetic site to show that its Hasse-Weil zeta function is the complete Riemann zeta function. The action of $\mathbb{R}_+^*$ on $\mathbb{Q} \backslash \mathbb{A}_Q/\mathbb{Z}^*$ indeed corresponds to the action of the Frobenius automorphisms $F_{\lambda} \in \text{Aut}(\mathbb{R}_+^{\text{max}})$, $\lambda \in \mathbb{R}_+^*$, on the points of $(\hat{\mathbb{N}}^*, \hat{\mathbb{N}})$ over $[0, 1]_{\text{max}} \subset \mathbb{R}_+^{\text{max}}$. The square of the arithmetic site over the semifield $\mathbb{B} = ((0, 1), \text{max}, \times)$ has an unreduced and reduced version. In both cases the underlying topos is $\hat{\mathbb{N}}^{\times 2}$. The structure sheaf in the unreduced case is $\hat{\mathbb{N}} \otimes_{\mathbb{B}} \mathbb{N}$ and in the reduced case is the multiplicatively cancellative semiring canonically associated to $\mathbb{N} \otimes_{\mathbb{B}} \mathbb{N}$. We determine this latter semiring as the semiring $\text{Conv}_\gamma(\mathbb{N} \times \mathbb{N})$ of Newton polygons with the operations of convex hull of the union and sum.

On both versions there is a canonical action of $\mathbb{N}^{\times 2}$ by endomorphisms $F_{n,m}$. By composing this action with the diagonal (given by the product $\mu$) one obtains the Frobenius correspondences $\Psi(\lambda) = \mu \circ F_{n,m}$ for rational values $\lambda = n/m$. The Frobenius correspondences $\Psi(\lambda)$ for arbitrary positive real numbers $\lambda$ are realized as curves in the square obtained from the rational case using diophantine approximation. Finally we determine the composition law of these correspondences and show that it is given by the product law in $\mathbb{R}_+^*$ with a subtle nuance in the case of two irrational numbers whose product is rational.

This note provides the algebraic geometric space underlying the non-commutative approach to RH. It gives a geometric framework reasonably suitable to transpose the conceptual understanding of the Weil proof in finite characteristic as in [7]. This translation would require in particular an adequate version of the Riemann-Roch theorem in characteristic 1.

2. The arithmetic site

Given a small category $C$ we denote by $\hat{C}$ the topos of contravariant functors from $C$ to the category of sets. We let $\mathbb{N}^*$ be the category with a single object $\ast$, $\text{End}(\ast) = \mathbb{N}^*$.

**Definition 2.1** We define the arithmetic site $(\hat{\mathbb{N}}^*, \hat{\mathbb{N}})$ as the topos $\hat{\mathbb{N}}^*$ endowed with the structure sheaf $\hat{\mathbb{N}} := (\mathbb{N} \cup \infty, \text{inf}, +)$ viewed as a semiring in the topos.

Notice that $\hat{\mathbb{N}}^* \simeq \text{Sh}(\mathbb{N}^*, J)$, where $J$ is the chaotic topology on $\mathbb{N}^*$ ([1] Exposé IV, 2.6).

2.1. The points of the topos $\hat{\mathbb{N}}^*$

A point of a topos $\mathcal{T}$ is defined as a geometric morphism from the topos of sets to $\mathcal{T}$ ([1, 8]).

**Theorem 2.2** (i) The category of points of the topos $\hat{\mathbb{N}}^*$ is canonically equivalent to the category of totally ordered groups isomorphic to non-trivial subgroups of $(\mathbb{Q}, \mathbb{Q}_+)$, and injective morphisms of ordered groups.

(ii) Let $\mathbb{A}_f$ be the ring of finite adèles of $\mathbb{Q}$. The space of isomorphism classes of points of $\hat{\mathbb{N}}^*$ is canonically isomorphic to the double quotient $\mathbb{Q}_+^* \backslash \mathbb{A}_f/\mathbb{Z}^*$ where $\mathbb{Q}_+^*$ acts by multiplication on $\mathbb{A}_f$.

We denote by $\mathbb{F} = \mathbb{Z}_{\text{max}}$ the semifield of fractions of the semiring $\hat{\mathbb{N}}$.

**Corollary 2.3** The category of points of the topos $\hat{\mathbb{N}}^*$ is equivalent to the category of algebraic extensions of the semifield $\mathbb{F} = \mathbb{Z}_{\text{max}}$ i.e. of extensions: $\mathbb{F} \subset K \subset \mathbb{F} = \mathbb{Q}_{\text{max}}$. The morphisms are the injective morphisms of semifields.
2.2. The structure sheaf $\mathbb{N}$

The next result provides an explicit description of the semiring structure inherited automatically by the stalks of the sheaf $\mathbb{N}$ on the topos $\mathbb{F}^\vee$.

**Theorem 2.4** At the point of the topos $\mathbb{F}^\vee$ associated to the intermediate semifield $\mathbb{F} \subset K \subset \mathbb{F} = \mathbb{Q}_{\text{max}}^\vee$ the stalk of the structure sheaf $\mathcal{O} := \mathbb{N}$ is the semiring $\mathcal{O}_K := \{ r \in K \mid r \vee 1 = 1 \}$ where $\vee$ denotes addition.

2.3. The points of the arithmetic site $(\mathbb{F}^\vee, \mathbb{N})$ over $[0, 1]_{\text{max}}$

The following definition provides the notion of point of the arithmetic site over a local semiring.

**Definition 2.5** Let $R$ be a local semiring. Then a morphism $f : \text{Spec}(R) \to (\mathbb{F}^\vee, \mathbb{N})$ is a pair of a point $p$ of $\mathbb{F}^\vee$ and a local morphism of semirings $f_p : \mathcal{O}_p \to R$.

The next crucial statement determines the interpretation of the space underlying the non-commutative geometric approach to RH in terms of algebraic geometry.

**Theorem 2.6** The points of the arithmetic site $(\mathbb{F}^\vee, \mathbb{N})$ over the maximal compact subring $[0, 1]_{\text{max}} \subset R_{\text{max}}^\vee$ of the tropical semifield form the quotient $\mathbb{Q}^\vee \backslash \mathbb{A}_Q / \mathbb{Z}^\vee$ of the ad\'ele class space of $\mathbb{Q}$ by the action of $\hat{\mathbb{Z}}^\vee$. The action of the Frobenius automorphisms $\text{Fr}_\lambda \in \text{Aut}([0, 1]_{\text{max}}^\vee)$ on these points corresponds to the action of the id\'ele class group (mod $\mathbb{Z}^\vee$) on the above quotient of the ad\'ele class space.

Notice that the quotient $\mathbb{Q}^\vee \backslash \mathbb{A}_Q / \mathbb{Z}^\vee$ is the disjoint union of the following two spaces:

(i) $\mathbb{Q}^\vee \backslash \mathbb{A}_f / \hat{\mathbb{Z}}^\vee$ is the space of ad\'ele classes whose archimedean component vanishes. The corresponding points of the arithmetic site $(\mathbb{F}^\vee, \mathbb{N})$ are those which are defined over $\mathbb{B}$; they are given by the points of $\mathbb{F}^\vee$ (Theorem 2.2).

(ii) $\mathbb{Q}^\vee \backslash (\mathbb{A}_f / \hat{\mathbb{Z}}^\vee) \times \mathbb{R}^+_1$ is the space of ad\'ele classes whose archimedean component does not vanish. It is in canonical bijection with rank one subgroups of $\mathbb{R}$ through the map

$$(a, \lambda) \mapsto \lambda H_a, \quad \forall a \in \mathbb{A}_f / \hat{\mathbb{Z}}^\vee, \lambda \in \mathbb{R}^+_1, \quad H_a := \{ q \in \mathbb{Q} \mid qa \in \mathbb{Z} \}.$$ 

2.4. Hasse-Weil formula for the Riemann zeta function

In order to count the number of fixed points of the Frobenius action on points of $(\mathbb{F}^\vee, \mathbb{N})$ over $[0, 1]_{\text{max}}$ we let $\partial_u \xi(x) = \xi(u^{-1}x)$ be the scaling action of the id\'ele class group $G = \text{GL}_1(\mathbb{A}_Q) / \text{GL}_1(\mathbb{Q})$ on functions on the ad\'ele class space $\mathbb{A}_Q / \mathbb{Q}^\vee$ and use the trace formula ([3, 4, 9]) in the form ($\Sigma_Q = \text{places of } \mathbb{Q}, d^u$ multiplicative Haar measure)

$$\text{Trdistr} \left( \int_G h(u) \partial_u d^u u \right) = \sum_{v \in \Sigma_Q} \int_{Q_v^\vee} \frac{h(u^{-1})}{1 - u} d^u u. \quad (1)$$

We apply (1) to test functions of the form $h(u) = g(|u|)$ where the support of $g$ is contained in $(1, \infty)$ and $|u|$ is the module. The invariance of $h$ under the kernel $\hat{\mathbb{Z}}^\vee$ of the module $G \to \mathbb{R}^+_1$ corresponds at the geometric level to taking the quotient of the ad\'ele class space by the action of $\mathbb{Z}^\vee$. Using Theorem 2.6 and [6], §2, one obtains the counting distribution $N(u)$, $u \in [1, \infty)$ associated to the Frobenius action on points of $(\mathbb{F}^\vee, \mathbb{N})$ over $[0, 1]_{\text{max}}$.

**Theorem 2.7** The zeta function $\zeta_N$ associated by the equation

$$\frac{\partial_u \zeta_N(s)}{\zeta_N(s)} = - \int_1^\infty N(u) u^{-s} d^u u$$

(2)
to the counting distribution $N(u)$ is the complete Riemann zeta function $\zeta_Q(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$.

In [5], equation (2) was shown (following a suggestion made in [11]) to be the limit, when $q \to 1$, of the Hasse-Weil formula for counting functions over finite fields $\mathbb{F}_q$.

3. The square of the arithmetic site

3.1. The unreduced square $(\overline{\mathbb{N}}^{r+2}, \mathbb{N} \otimes \mathbb{N})$

Given a partially ordered set $J$, we let $\text{Sub}_J(J)$ be the set of subsets $E \subset J$ which are hereditary, i.e. such that $x \in E \implies y \in E, \forall y \supset x$. Then $\text{Sub}_J(J)$ endowed with the operation $E \oplus E' := E \cup E'$ is a $\mathbb{B}$-module. We refer to [10] for the general treatment of tensor products of semi-modules.

**Proposition 3.1** (i) Let $\mathbb{N} \times \mathbb{N}$ be endowed with the partial order $(a, b) \leq (c, d) \iff a \leq c \& b \leq d$. Then one has a canonical isomorphism of $\mathbb{B}$-modules $\mathbb{N} \otimes \mathbb{N} \simeq \text{Sub}_J(\mathbb{N} \times \mathbb{N})$.

(ii) There exists on the $\mathbb{B}$-module $S = \overline{\mathbb{N}} \otimes \overline{\mathbb{N}}$ a unique bilinear multiplication such that, using multiplicative notation where $q$ is a formal variable, one has

$$
(q^a \otimes q^b)(q^c \otimes q^d) = q^{a+c} \otimes q^{b+d}.
$$

(iii) The multiplication (3) turns $\overline{\mathbb{N}} \otimes \overline{\mathbb{N}}$ into a semiring of characteristic 1.

(iv) The following formula defines an action of $\mathbb{N}^r \times \mathbb{N}$ by endomorphisms on $\mathbb{N} \otimes \mathbb{N}$

$$
\text{Fr}_{n,m}(\sum q^a \otimes q^b) := \sum q^{na} \otimes q^{nb}.
$$

**Definition 3.2** The unreduced square $(\overline{\mathbb{N}}^{r+2}, \mathbb{N} \otimes \mathbb{N})$ of the arithmetic site $(\overline{\mathbb{N}}^r, \mathbb{N})$ is the topos $\overline{\mathbb{N}}^{r+2}$ with the structure sheaf $\mathbb{N} \otimes \mathbb{N}$, viewed as a semiring in the topos.

3.2. The Frobenius correspondences

The product in the semiring $\overline{\mathbb{N}}$ yields a morphism $\mu : (\overline{\mathbb{N}} \otimes \overline{\mathbb{N}}) \to \overline{\mathbb{N}}$, given on simple tensors by $\mu(q^a \otimes q^b) = q^{a+b}$.

**Proposition 3.3** (i) The range of the morphism $\mu \circ \text{Fr}_{n,m} : \overline{\mathbb{N}} \otimes \overline{\mathbb{N}} \to \overline{\mathbb{N}}$ only depends, up to canonical isomorphism, on the ratio $r = n/m$. Assuming that $n, m$ are relatively prime, this range contains the ideal

$$
\{q^a \mid a \geq (n-1)(m-1)\} \subset \overline{\mathbb{N}}.
$$

(ii) Let $r = n/m, \ q \in (0, 1)$ and let $m_r : \overline{\mathbb{N}} \otimes \overline{\mathbb{N}} \to \mathbb{R}^{\text{max}}$ be given by

$$
m_r(\sum (q^{ni} \otimes q^{m_i})) = q^a, \ a = \inf(ni + m_i).
$$

Up to canonical isomorphism of their ranges, the morphisms $\mu \circ \text{Fr}_{n,m}$ and $m_r$ are equal.

Proposition 3.3 (ii) allows one to extend the definition of the Frobenius correspondence to arbitrary positive real numbers.

**Proposition 3.4** (i) Let $\lambda \in \mathbb{R}^+ \& q \in (0, 1)$ then the following formula defines a homomorphism

$$
\mathcal{F}(\lambda, q) : \overline{\mathbb{N}} \otimes \overline{\mathbb{N}} \to \mathbb{R}^{\text{max}}, \quad \mathcal{F}(\lambda, q)(\sum (q^{ni} \otimes q^{m_i})) = q^a, \ a = \inf(n \lambda i + m_i).
$$

(ii) The semiring $\mathcal{R}(\lambda) := \text{Im}(\mathcal{F}(\lambda, q))$ is independent, up to canonical isomorphism, of $q \in (0, 1)$.

(iii) The semirings $\mathcal{R}(\lambda)$ and $\mathcal{R}(\lambda')$ are isomorphic if and only if $\lambda' = \lambda$ or $\lambda' = 1/\lambda$. 

4
3.3. The reduced square \((\mathbb{N}^\times 2, \text{Conv}_\geq (\mathbb{N} \times \mathbb{N}))\)

Let \(R\) be a semiring without zero divisors and \(\iota : R \to \text{Frac}R\) the canonical morphism to the semifield of fractions. It is not true in general that \(\iota\) is injective (cf. [2]). We shall refer to \(\iota(R)\) as the reduced semiring of \(R\).

**Definition 3.5** We let \(\text{Conv}_\geq (\mathbb{N} \times \mathbb{N})\) be the set of closed convex subsets \(C\) of the quadrant \(Q := \mathbb{R}^+ \times \mathbb{R}^+\), such that (i) \(C + Q = C\) and (ii) the extreme points \(\partial C\) belong to \(\mathbb{N} \times \mathbb{N} \subset Q\).

The set \(\text{Conv}_\geq (\mathbb{N} \times \mathbb{N})\) is a semiring for the operations of convex hull of the union and sum.

**Proposition 3.6** (i) The semiring \(\text{Conv}_\geq (\mathbb{N} \times \mathbb{N})\) is multiplicatively cancellative.
(ii) The homomorphism \(\gamma : \mathbb{N} \otimes B \mathbb{N} \to \text{Sub}_\geq (\mathbb{N} \times \mathbb{N}) \to \text{Conv}_\geq (\mathbb{N} \times \mathbb{N})\) given by convex hull is the same as the homomorphism \(\iota : \mathbb{N} \otimes B \mathbb{N} \to \iota(\mathbb{N} \otimes B \mathbb{N})\).
(iii) Let \(R\) be a multiplicatively cancellative semiring and \(\rho : \mathbb{N} \otimes B \mathbb{N} \to R\) a homomorphism of semirings such that \(\rho^{-1}(\{0\}) = \{0\}\). Then there exists a unique semiring homomorphism \(\rho' : \text{Conv}_\geq (\mathbb{N} \times \mathbb{N}) \to R\) such that \(\rho = \rho' \circ \gamma\).

**Definition 3.7** The reduced square \((\widehat{\mathbb{N}}^\times 2, \text{Conv}_\geq (\mathbb{N} \times \mathbb{N}))\) of the arithmetic site \((\widehat{\mathbb{N}}^\times, \widehat{\mathbb{N}})\) is the topos \(\widehat{\mathbb{N}}^\times 2\) with the structure sheaf \(\text{Conv}_\geq (\mathbb{N} \times \mathbb{N})\), viewed as a semiring in the topos.

4. Composition of Frobenius correspondences

4.1. Reduced correspondences

**Definition 4.1** A reduced correspondence over the arithmetic site \((\widehat{\mathbb{N}}^\times, \widehat{\mathbb{N}})\) is given by a triple \((R, \ell, r)\) where \(R\) is a multiplicatively cancellative semiring, \(\ell, r : \widehat{\mathbb{N}} \to R\) are semiring morphisms such that
\(\ell^{-1}(\{0\}) = \{0\}, \quad r^{-1}(\{0\}) = \{0\}\) and that \(R\) is generated by \(\ell(N) r(N)\).

By construction, cf. Proposition 3.4, the Frobenius correspondence gives a reduced correspondence:

\[
\Psi(\lambda) := (R, \ell(\lambda), r(\lambda)), \quad R := \mathcal{R}(\lambda), \quad \ell(\lambda)(q^n) := \mathcal{F}(\lambda, q)(q^n \otimes 1), \quad r(\lambda) := \mathcal{F}(\lambda, q)(1 \otimes q^n) \tag{5}
\]

By (4) one gets that the elements of \(\mathcal{R}(\lambda)\) are powers \(q^n\) where \(\alpha \in \mathbb{N} + \lambda \mathbb{N}\) and that the morphisms \(\ell(\lambda)\) and \(r(\lambda)\) are described as follows:

\[
\ell(\lambda)(q^n)q^\alpha = q^{\alpha + n \lambda}, \quad r(\lambda)(q^n)q^\alpha = q^{\alpha + n}
\]

4.2. The composition of the correspondences \(\Psi(\lambda) \circ \Psi(\lambda')\)

The composition \(\Psi(\lambda) \circ \Psi(\lambda')\) of the Frobenius correspondences is obtained as the left and right action of \(N\) on the reduced semiring of the tensor product \(\mathcal{R}(\lambda) \otimes \mathcal{R}(\lambda')\). In order to state the general result we introduce a variant \(\text{Id}_{\ell}\) of the identity correspondence. We let \(\text{Germ}_{\ell=0}(\mathbb{R}^{\max}_{\ell})\) be the semiring of germs of continuous functions from a neighborhood of 0 ∈ \(\mathbb{R}\) to \(\mathbb{R}^{\max}_{\ell}\), endowed with the pointwise operations. Let \(\mathbb{N}_{\ell}\) be the the sub-semiring of \(\text{Germ}_{\ell=0}(\mathbb{R}^{\max}_{\ell})\) generated, for fixed \(q \in (0, 1)\), by \(q\) and \(\text{Fr}_{1+\epsilon}(q) = q^{1+\epsilon}\). \(\mathbb{N}_{\ell}\) is independent, up to canonical isomorphism, of the choice of \(q \in (0, 1)\).

**Definition 4.2** The tangential deformation of the identity correspondence is given by the triple \((\mathbb{N}_{\ell}, \ell_{\epsilon}, r_{\epsilon})\) where \(\ell_{\epsilon}(q^n) := \text{Fr}_{1+\epsilon}(q^n)\) and \(r_{\epsilon}(q^n) := q^n\), \(\forall n \in \mathbb{N}\).

**Theorem 4.3** Let \(\lambda, \lambda' \in \mathbb{R}^{\ell}_{\ell}\) such that \(\lambda \lambda' \notin \mathbb{Q}\). The composition of the Frobenius correspondences is then given by

\[
\Psi(\lambda) \circ \Psi(\lambda') = \Psi(\lambda \lambda')
\]

The same equality holds if \(\lambda\) and \(\lambda'\) are rational. When \(\lambda, \lambda'\) are irrational and \(\lambda \lambda' \in \mathbb{Q}\),

\[
\Psi(\lambda) \circ \Psi(\lambda') = \Psi(\lambda \lambda') \circ \text{Id}_{\ell} = \text{Id}_{\ell} \circ \Psi(\lambda \lambda')
\]

where \(\text{Id}_{\ell}\) is the tangential deformation of the identity correspondence.

**References**

[1] M. Artin, A. Grothendieck, J-L. Verdier, eds. (1972), SGA4, LNM 269-270-305, Berlin; New York: Springer-Verlag.
[2] D. Castella, Algèbres de polynômes tropicaux, Annales mathématiques Blaise Pascal 20 (2013), 301–330.
[3] A. Connes, Trace formula in noncommutative geometry and the zeros of the Riemann zeta function, Selecta Math. (N.S.) 5 (1999), no. 1, 29–106.
[4] A. Connes, M. Marcolli, Noncommutative Geometry, Quantum Fields, and Motives, Colloquium Publications, Vol.55, American Mathematical Society, 2008.
[5] A. Connes, C. Consani, Schemes over \(F_1\) and zeta functions, Compositio Mathematica 146 (6), (2010) 1383–1415.
[6] A. Connes, C. Consani, From monoids to hyperstructures: in search of an absolute arithmetic, in Casimir Force, Casimir Operators and the Riemann Hypothesis, de Gruyter (2010), 147–198.
[7] A. Grothendieck, Sur une note de Mattuck-Tate J. reine angew. Math. 200, 208-215 (1958).
[8] S. Mac Lane, I Moerdijk, Sheaves in geometry and logic. A first introduction to topos theory. Corrected reprint of the 1992 edition. Universitext. Springer-Verlag, New York, 1994.
[9] R. Meyer, On a representation of the idèle class group related to primes and zeros of \(L\)-functions. Duke Math. J. Vol.127 (2005), N.3, 519–595.
[10] B. Pareigis, H. Rohrl, Remarks on semimodules, arXiv:1305.5331v2 [mathRA] (2013).
[11] C. Soulé, Les variétés sur le corps à un élément. Mosc. Math. J. 4 (2004), no. 1, 217–244.