EXTREMAL PRESENTATIONS FOR CLASSICAL LIE ALGEBRAS

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Abstract. The long-root elements in Lie algebras of Chevalley type have been well studied and can be characterized as extremal elements, that is, elements $x$ such that the image of $(ad x)^2$ lies in the subspace spanned by $x$. In this paper, assuming an algebraically closed base field of characteristic not 2, we find presentations of the Lie algebras of classical Chevalley type by means of minimal sets of extremal generators. The relations are described by simple graphs on the sets. For example, for $C_n$ the graph is a path of length $2n$, and for $A_n$ the graph is the triangle connected to a path of length $n - 3$.

1. Introduction

A nonzero element $x$ of a Lie algebra $L$ over a field $\mathbb{F}$ of characteristic not 2 is called extremal if $[x, [x, L]] \subseteq \mathbb{F}x$. Extremal elements are a well-studied class of elements in simple finite-dimensional Lie algebras of Chevalley type: they are the long root elements. In [CSUW01], Cohen, Steinbach, Ushirobira, and Wales have studied Lie algebras generated by extremal elements, in particular those of Chevalley type. The authors also find the minimum size of a set of generating extremal elements for the Lie algebras of Chevalley type and find such minimal generating sets of extremal elements explicitly. In the present paper, we also find such minimal generating sets of extremal elements explicitly for the four classical families of Lie algebras: those of type $A_n$, $B_n$, $C_n$, and $D_n$.

We will do this in a more geometrical setting and will find criteria for sets of extremal elements to generate Lie algebras of this type.

By Lemma 2.2, each Lie algebra generated by a pair of linearly independent extremal elements is in one of only three isomorphism classes: either the two-dimensional commutative Lie algebra, or the so-called Heisenberg Lie algebra $h$, or $\mathfrak{sl}_2$. Given a generating set $S$ of extremal elements, we examine the subalgebras generated by pairs of these elements. These give rise to graphs: the vertices correspond to the elements of $S$, and two vertices are adjacent if the corresponding extremal elements generate a three-dimensional algebra and nonadjacent if they commute. We will say that the Lie algebra generated by $S$ realizes this graph.

Following experiments using the GAP computer algebra system [GAP] and the GBNP package [GBNP] we conjectured one such graph for each Lie algebra of classical Chevalley type, depicted in Figures 1.1 up to 1.4.
In this paper we show that if a Lie algebra realizes one of the graphs in Figures 1.1 up to 1.4 then in the generic case it is isomorphic to the Lie algebra of the corresponding Chevalley type, in the following sense. Given a graph \( \Gamma \), we define a vector space \( V(\Gamma) \) parametrizing the Lie algebras that realize \( \Gamma \). Let \( X(\Gamma) \) be the subset of \( V(\Gamma) \) of values \( f \) for which the associated Lie algebra \( L(\Gamma,f) \) has maximal dimension among such algebras for the same graph \( \Gamma \). We will see in Lemma 5.10 that \( X(\Gamma) \) carries the structure of an affine variety. The following theorem and analogues for the three other families of Chevalley type Lie algebras will be our main results.

**Theorem 1.1.** Let \( n \geq 5 \). Let \( \Gamma_{D,n} \) be the graph of Figure 1.4. There is an open dense subset \( S \) of \( X(\Gamma_{D,n}) \) such that, if \( f \in S \), then \( L(\Gamma_{D,n},f) \) is isomorphic to the Lie algebra of type \( D_n \).

The graphs \( \Gamma_{A,n} \), \( \Gamma_{B,n} \), and \( \Gamma_{C,n} \) are subgraphs of \( \Gamma_{D,n} \), and as such induce extra commuting relations. Therefore, the \( D_n \) case is the most complicated one, from which we will deduce the conclusions for the other cases.

### 1.1. Contents and Strategy

In the rest of this section we will introduce some conventions and notation that we will use in this paper. In Section 2 we review some of the underlying theory. In Section 3 we show how to work with abstract Lie algebras that realize a given graph. We apply this to four proposed graphs \( \Gamma_{A,n} \), \( \Gamma_{B,n} \), \( \Gamma_{C,n} \) and \( \Gamma_{D,n} \) for the classical Chevalley types in Section 4. In Section 5, we extend this study to the parameter space \( V \) referred to above. In Theorems 6.10, 6.11, 6.18 and 6.19 we give concrete realizations of the Lie algebras of types \( A_n \), \( B_n \), \( C_n \) and \( D_n \) corresponding to the graphs from Figures 1.1 up to 1.4.

Finally, in Section 7 we prove the main results of this paper: we show that a Lie algebra \( L \) realizing one of the graphs \( \Gamma_{A,n} \), \( \Gamma_{B,n} \), \( \Gamma_{C,n} \) and \( \Gamma_{D,n} \) is in the generic case a quotient of the realization \( M \) found in Section 6. Since \( M \) is simple in most cases, it will follow that \( L \) and \( M \) are isomorphic. The only exception is \( A_n \) if \( p \mid n + 1 \).

This paper was inspired by the Masters thesis of the third author [Roo05] and reported on more extensively in the second author’s Ph.D. thesis [Pos07].

### 1.2. Conventions and notation

For the rest of this paper, \( F \) will be an algebraically closed field of characteristic not 2 and \( L \) will be a Lie algebra over \( F \).

Since we approach the matter from the angle of the generating sets of abstract extremal elements, we let \( n \) be the number of generating extremal elements. In Theorems Section 6 this will mean that we study, for example, the Lie algebra of type \( C_{n/2} \), defined over a vector space of dimension \( n \), where we have to assume that \( n \) is even. In that section, it would be more convenient to study the Lie algebra of type \( C_n \) instead, but we choose consistency over convenience and keep the meaning of \( n \) as the number of extremal generators.

If no confusion is possible, we write \( xy \) for \([x,y],\) and \( xyz \) for \([x,[y,z]]\); we will write \((xy)z\) for \([[x,y],z]].\) So, anticommutativity and the Jacobi identity will be written as

\[ xy = 0 \quad \text{(AC)} \]

and

\[ xyz + yzx + zxy = 0. \quad \text{(J)} \]

We will often work with long products of indexed elements. We use the following notation to make these products somewhat manageable. The general idea is that we put two numbers in the subscript with an operator consisting of one or two arrows in between, such as \( x_{5\|2} \): the first factor in the product is then indexed by the first number, after which we iterate adding (for up arrows) or subtracting (for down arrows) one (for single stroke arrows) or two (for double stroke arrows) to the index until we encounter the last number, where every step gives the next factor for this product. So the previous example \( x_{5\|2} \) is short for \( x_5 x_6 x_4 x_5 x_3 x_4 x_2 \).

In particular, there are four operators that we use, defined more precisely as follows. If \( i \leq j \), the notation \( x_{i\mid j} \) will mean \( x_i x_{i+1} x_{i+2} \cdots x_{j-1} x_j \), and \( x_{j\mid i} \) will mean \( x_j x_{j-1} x_{j-2} \cdots x_{i+1} x_i \). Furthermore, \( x_{j\|i} \) will mean \( x_j x_{j-1} x_{j-2} x_{j-3} \cdots x_{i+1} x_{i+2} x_i \), and similarly, \( x_{i\|j} \) will mean \( x_i x_{i-1} x_{i+1} x_{i+2} x_{i+3} \cdots x_{j-1} x_{j-2} x_j \).
We will also use constructions such as \( x_{i_1}x_{i_2} \cdots x_{i_k} \), which will mean \( x_3x_4x_5x_6x_4x_5x_3x_4x_2x_3x_1 \). Occasionally, it will be convenient to include in a set of monomials of the form, say, \( x_{j_1}x_{j_2} \) the case \( j = i - 1 \); this monomial will then simply be \( x_{i-2} \). So in this case \( x_{j_1} \) cannot be seen as a separate monomial.

We extend the notation to cover the case where we have a sequence \( i_1, \ldots, i_k \) of indices: then we write \( x_{i_k} \) for \( x_{i_1}x_{i_2} \cdots x_{i_{k-1}} \).

We say that a set of Lie algebra elements \( \{ x_i \mid i \in V \} \) realizes a given graph \( \Gamma = (V, E) \) if:

- each \( x_i \) is an extremal element of \( \langle x_j \mid j \in I \rangle_{\text{Lie}} \);
- vertices \( i \) and \( j \) are connected if and only if \( x_i \) and \( x_j \) do not commute.

We will sometimes also say that the Lie algebra \( \langle x_i \rangle_{\text{Lie}} \) realizes \( \Gamma \). Later in this paper it will be essential that each \( x_i \) is nonzero, which is implied by it being extremal.

1.3. Related results. In [DP08] Lie algebras realized by simply laced affine Dynkin diagrams were considered. There it was shown that in the generic case the Lie algebra is of the corresponding finite type. The diagram for \( A_n \) given there can be transformed into Figure 1.1 using a procedure similar to that described in Lemma 2.3. The diagram for \( D_n \) given in [DP08] is realized in a less straightforward manner, since Figure 1.4 has \( n + 1 \) vertices whereas its affine Dynkin diagram has \( n \) vertices.

Although the generators arising from our graphs and some of the Chevalley generators [Car72] are similar in the sense that they both correspond to long root elements, no direct relation is apparent.

2. Preliminaries

In this section, we will introduce a bilinear form defined on all Lie algebras generated by extremal elements, and recall some of its properties. None of these results are new; most can be found in e.g. [CSUW01] and thus we will omit most of the proofs. We will start by introducing a related family of linear functionals.

For extremal \( x \), let \( f_x : \mathcal{L} \to \mathbb{F} \) be the linear map defined by \( xxy = f_x(y)x \). Since \([,] \) is bilinear, this is indeed a linear map. We call \( f_x \) the extremal functional on \( x \).

**Lemma 2.1.** \( f_x(y) = f_y(x) \) for all extremal \( x, y \in \mathcal{L} \).

**Lemma 2.2.** Let \( \mathcal{L} = \langle x, y \rangle_{\text{Lie}} \) with \( x \) and \( y \) extremal and linearly independent. Then \( \mathcal{L} \) is isomorphic to the two-dimensional commutative Lie algebra, the Heisenberg algebra, or \( \mathfrak{sl}_2 \).

**Lemma 2.3.** If \( \mathcal{L} \) is generated by extremal elements, then it is linearly spanned by extremal elements.

**Lemma 2.4.** If \( \mathcal{L} \) is generated by extremal elements, the definition of \( f_x(y) \) can be extended to a unique bilinear form \( f(x, y) \) on \( \mathcal{L} \) with \( f(x, y) = f_x(y) \) if \( x \) is an extremal element. This bilinear form is associative and symmetric:

\[
\forall x, y, z : f(x, yz) = f(xy, z) \quad \text{(AS)}
\]

\[
\forall x, y : f(x, y) = f(y, x) \quad \text{(SM)}
\]

We call \( f \) the extremal form. We will use the following identities involving the extremal form, the first two of which go back to Premet and were first used in [Che89].

**Lemma 2.5.** If \( x, y, z \in \mathcal{L} \) and \( x \) extremal, then

\[
2(xxy)zx = f(x, y)zx + f(x, z)xy - f(x, y)xz, 
\]

\[
2xyxz = f(x, y)zx - f(x, z)xy - f(x, y)xz, 
\]

\[
f(x, yxz) = -f(x, z)f(x, y).
\]
Then we find Eq. (P2) as follows:

\[(xy)zx = ((xy)x)z + x(xy)z = -f(x, y)xz + x(xy)z,\]

and similarly,

\[(xy)zx = (xz)xy - ((xz)x)y - x(z)yz = f(x, z)xy - xxyz - x(xy)z.\]

Adding these two equations and applying anti-commutativity a few times, we obtain Eq. (P1).

Proof. By the Jacobi identity,

\[(xy)zx = ((xy)x)z + x(xy)z = -f(x, y)xz + x(xy)z,\]

and similarly,

\[(xy)zx = (xz)xy - ((xz)x)y - x(z)yz = f(x, z)xy - xxyz - x(xy)z.\]

Adding these two equations and applying anti-commutativity a few times, we obtain Eq. (P1). Then we find Eq. (P2) as follows:

\[2xyz = 2(xy)xz + 2yxzx = f(x, y)x + f(x, z)xy - f(x, y)xz - 2f(x, z)xy.\]

For Eq. (P3), we need the next lemma. The equation is then easily obtained as follows:

\[f(x, yxz) = f(x, y, xz) - f(y, xz) - f(y, x, xz) = -f(x, y)f(x, z).\]

3. The General Framework

In this section, we will establish a framework for dealing with Lie algebras generated by extremal elements where we prescribe the values of the extremal form. In the end, we prove Theorem 3.5 which shows that a certain parameter space for these prescribed values is an algebraic variety. The main objective of this section is to introduce the techniques for proving that theorem. In Section 3 we will use those techniques to prove similar theorems for a substantially smaller parameter space, but then for specific Lie algebra families.

Let \(n \in \mathbb{N}_+\) be fixed. Let \(\Gamma\) be a graph on \(n\) numbered vertices. Let \(\mathcal{F}\) be the free Lie algebra over \(\mathbb{F}\) on \(n\) generators \(x_1, \ldots, x_n\) with the standard grading. We will construct a quotient of \(\mathcal{F}\) where the projections of \(x_i\) in the quotient are extremal generators. Let

\[\mathcal{F}_\Gamma = \mathcal{F}/\langle x_ix_j \mid \{i, j\} \notin E(\Gamma)\rangle_{\text{Idl}}.\]

\(\mathcal{F}_\Gamma\) inherits the grading of \(\mathcal{F}\); this is possible because the ideal that is divided out is homogeneous with respect to the grading of \(\mathcal{F}\), in the sense that it is spanned by its intersections with the homogeneous components of \(\mathcal{F}\).

Let \(f = (f_1, \ldots, f_n)\) be an element of \(V(\Gamma) := (\mathcal{F}_\Gamma^*)^n\), so it consists of \(n\) functionals in the dual of \(\mathcal{F}_\Gamma\); we will make sure that \(f_i\) is the extremal functional \(f_{x_i}\) in the Lie algebra we will construct. To that end, define the ideal

\[I_{\Gamma, f} = \langle x_ix_j - f_i(y)x_i \mid y \in \mathcal{F}_\Gamma, 1 \leq i \leq n\rangle_{\text{Idl}}.\]

When taking \(f = 0\), we see that \(I_{\Gamma, 0}\) is homogeneous with respect to the standard grading of \(\mathcal{F}_\Gamma\). Let \(\mathcal{L}(\Gamma, f) = \mathcal{F}_\Gamma/I_{\Gamma, f}\) and let \(\xi_f: \mathcal{F}_\Gamma \to \mathcal{L}(\Gamma, f)\) be the natural projection. We will sometimes omit \(\xi_f\) if that does not stand in the way of clarity. Clearly each \(x_i\) is either an extremal element of \(\mathcal{L}(\Gamma, f)\) or zero, and \(f_i(y) = f(x_i, y)\). We find the following slight extension of Lemma 4.3 of [CSUW01].

Lemma 3.1. There is a finite list \(M_{\Gamma}\) of monomials in \(x_1, \ldots, x_n\) satisfying the following properties:

1. \(\xi_0(M_{\Gamma})\) is a basis of \(\mathcal{L}(\Gamma, 0)\),
2. if \(x_i, m \in M_{\Gamma}\), then \(m \in M_{\Gamma}\),
3. \(M_{\Gamma}\) contains all generators \(x_i\), and
4. \(\mathcal{L}(\Gamma, f) = \langle \xi_f(M_{\Gamma}) \rangle_f \) for all \(f \in V(\Gamma)\).

Proof. \(\mathcal{L}(\Gamma, 0)\) is a quotient of \(\mathcal{L}_n := \mathcal{L}(K_n, 0)\), where \(K_n\) is the complete graph. By Theorem 1 of Zel’mannov [Zel90] (or for characteristic 3, by Theorem 1 of Zel’mannov and Kostrikin [ZK90]), we know that \(\mathcal{L}_n\) is finite-dimensional. Hence we can find a finite set \(M_{\Gamma}\) of monomials in \(\mathcal{F}_\Gamma\) satisfying conditions 1, 2, and 3 by the following procedure. We start by setting \(M_{\Gamma}\) equal to \(\{x_1, \ldots, x_n\}\). This set is linearly independent because the free Abelian Lie algebra on \(n\) generators is a quotient of \(\mathcal{L}(\Gamma, 0)\), and the images of the \(x_i\) in it are linearly independent. Then we perform a number of rounds as follows. In each round we form the monomials \(x_im\), where \(x_i\) iterates over
the generators of $\mathcal{F}_\Gamma$ and $m$ iterates over the longest monomials in $\mathcal{M}_\Gamma$ so far. We select a subset of these such that its images under $\xi_0$ in $L(\Gamma, 0)$ are linearly independent of each other and of the images under $\xi_0$ of the elements in $\mathcal{M}_\Gamma$ so far, and add it to $\mathcal{M}_\Gamma$. Then we continue with the next round if we have added any new monomials this round. Since $L(\Gamma, 0)$ is finite-dimensional, this procedure terminates after finitely many steps.

Let $U = \langle \mathcal{M}_\Gamma \rangle \subseteq \mathcal{F}_\Gamma$. We now prove condition $\square$ by showing that $\xi_1(U) = L(\Gamma, f)$ for all $f \in V(\Gamma)$. Note that $I_{\Gamma, 0}$ is spanned by elements of the form $x_{ik_1}x_{i_1}r$, with $r$ a monomial in $\mathcal{F}_\Gamma$.

Clearly $\xi_1(U) \subseteq L(\Gamma, f)$. Suppose that it is a proper subset; then there are monomials $s \in \mathcal{F}_\Gamma$ such that $\xi_1(s) \notin \xi_1(U)$, whence $s \notin U$. Let $s$ be such a monomial of lowest degree. Since $\mathcal{F}_\Gamma = U + I_{\Gamma, 0}$, it is possible to express $s$ as a linear combination of monomials in $\mathcal{M}_\Gamma$ and monomials of the form $x_{ik_1}x_{i_1}r$. All these monomials have the same degree, because only the Jacobi identity and anticommutativity can be used for rewriting, in addition to homogeneous elements being 0. Let $t$ be a monomial of the form $x_{ik_1}x_{i_1}r$ such that $\xi_1(t) \notin \xi_1(U)$ and let $t_0 = x_{ik_1}$. Then

$$f_{i_1}(r)\xi_1(t_0) = \xi_1(t) \notin \xi_1(U),$$

so $f_{i_1}(r) \neq 0$ and $t_0 \notin U$. Since $\deg t_0 < \deg t = \deg s$, we have a contradiction. Hence $\xi_1(U) = L(\Gamma, f)$. \qed

Define $U = \langle \mathcal{M}_\Gamma \rangle \subseteq$ as in the preceding proof. Note that $\mathcal{F}_\Gamma = U + I_{\Gamma, f}$ for all $f$, not just for $f = 0$. Define $I$ and $m_i$ by letting $\mathcal{M}_\Gamma = \{m_i \mid i \in I\}$.

**Lemma 3.2.** For every monomial $m \in \mathcal{F}_\Gamma$, there exists a map $n_m: \mathcal{V}(\Gamma) \to U$, such that $n_m(f) = m$ (mod $I_{\Gamma, f}$) for all $f \in \mathcal{V}(\Gamma)$ and the following property holds. If $n_m(f) = \sum_{i \in I} \alpha_{m, i, f} m_i$, then $\alpha_{m, i, f}$, when regarded as a function in $i$ and $f$, is a polynomial function in the values of $f$ at monomials of degree less than $\deg m$.

**Proof.** Let $m = x_{ik_1}$ be a monomial in $\mathcal{F}_\Gamma$ of degree $k$. If $k = 1$, we put $n_m(f) = m$. We proceed by induction on $\deg m$. Since $\mathcal{F}_\Gamma = U + I_{\Gamma, 0}$, we can write $m$ as the sum of an element $u$ of $U$ and an element $w$ of $I_{\Gamma, 0}$; all monomials involved have the same degree, because only the Jacobi identity and anticommutativity can be used for rewriting, in addition to homogeneous elements being 0. If we prove that $n_u(f) = w$ (mod $I_{\Gamma, f}$) and that its coefficients $\alpha_{m, i, f}$ satisfy the polynomiality condition, then setting

$$n_m(f) = u + n_u(f) \quad (3.1)$$

will be sufficient to show that the lemma holds for $m$.

We may assume that $w$ is a single monomial. So the proof obligation reduces to the case where $m \in I_{\Gamma, 0}$. Then there exist $r, h \in \mathbb{N}_+$ such that $i_r = i_{r-1} = h$ and $m$ is thus of the form $x_{ik_1}x_{i_1}x_{i_r}x_{i_{r-1}}$. Hence, by the induction hypothesis,

$$m = f_h(x_{i_{r-1}})x_{ik_1} = f_h(x_{i_{r-1}})x_{ik_1}(f) \quad (\text{mod } I_{\Gamma, f}).$$

We choose

$$n_m(f) = f_h(x_{i_{r-1}})x_{ik_1}(f), \quad (3.2)$$

so that

$$\alpha_{m, j} = f_h(x_{i_{r-1}})\alpha_{x_{ik_1}, j, f}.$$ 

The coefficients $\alpha_{x_{ik_1}, j, f}$ are, by the induction hypothesis, polynomials in the values of $f$ at monomials of degree less than $k - r + 1 < k = \deg m$, so equations $\square$ and $\square$ define a map satisfying the conditions in the lemma. \qed

Note that we do not claim that $n_m$ is uniquely determined by these conditions. We choose a map $n_*$ as above and extend it to general elements of $\mathcal{F}_\Gamma$ by linearity.

Let $X(\Gamma) = \{f \mid \dim L(\Gamma, f) = \|\mathcal{M}_\Gamma\| \}$ and let $R: X(\Gamma) \to (U^*)^n$ be the map that restricts a functional to $U$.

**Lemma 3.3.** The restriction map $R$ is injective.
Proof. Let $f \in X(\Gamma)$. Then all $m_i \in M_\Gamma$ are linearly independent in $L(\Gamma, f)$, so $x_i \notin I_{\Gamma, f}$. Let $m$ be a monomial in $F_\Gamma$. We will show that $f_i(m)$ can be expressed in the values of $f_i$ on monomials in $M$. If $m \in M$, there is nothing to prove, so assume $m \notin M$. Since

$$x_i x_j m = f_i(m) x_i \quad (\text{mod } I_{\Gamma, f}),$$

and also

$$x_i x_j m = x_i x_j n_m(f) = f_i(n_m(f)) x_i \quad (\text{mod } I_{\Gamma, f}),$$

we find that $f_i(m) = f_i(n_m(f))$. Since $n_m(f)$ only depends on monomials of lower degree than $m$, we see that $f_i(m)$ can be expressed in the values of $f_i$ at monomials of lower degree than $m$. By induction on the degree of $m$, it can therefore be expressed ultimately in the values of $f_i$ on $M$, as we set out to prove.

Let $f, f' \in X(\Gamma)$ with $R(f) = R(f')$. Then $f_i$ and $f'_i$ agree on $U$, and thus on $F_\Gamma$, for all $i$. Hence $f = f'$.

Lemma 3.4. $R(X(\Gamma))$ is a closed subset of $(U^*)^n$.

Proof. For all $f \in V(\Gamma)$, let the bilinear anticommutative map $[\cdot, \cdot]: U \times U \to U$ be determined by

$$[v, w]_f = n_{[v, w]}(f).$$

If $f \in X(\Gamma)$, then

1. $[\cdot, \cdot]$ is a Lie multiplication (i.e. it satisfies the Jacobi identity),
2. $[x_i, [x_i, v]]_f = f_i(v) x_i$ for all $v \in U$ and all $i$,
3. $[x_i, x_j]_f = 0$ if nodes $i$ and $j$ are not connected by a line,
4. the Lie algebra $(U, [\cdot, \cdot])$ is generated by $x_1, \ldots, x_n$.

On the other hand, if all of the above conditions hold for a multiplication map $\mu$, then $(U, \mu)$ is a quotient of $L(\Gamma, f)$ of the same dimension, and hence isomorphic to $L(\Gamma, f)$. But these conditions are all polynomial in the values of $f$ on $U$: the Jacobi identity and conditions 2 and 3 are polynomial in a straightforward way, and condition 4 is always satisfied: for every $w = x_{i_{k+1}} \in M_\Gamma$, we have $w = [x_{i_k}, [x_{i_{k-1}}, \ldots, [x_{i_2}, x_{i_1}] \cdots]]_f$, since $n_w(f) = w$. So $R(X(\Gamma))$ is given as the zero set of a set of polynomial equations; thus it is closed.

Theorem 3.5. $X(\Gamma)$ carries a natural structure of an affine variety.

Proof. The restriction map $R$ is a continuous bijection of $X(\Gamma)$ with a Zariski closed subset of $(U^*)^n$. Clearly the restriction map $R$ is continuous. The preceding two lemmas show that it is injective and that its image is closed.

4. The monomials

In Figures 11 up to 14 we defined four graphs, $\Gamma_{A;n}$, $\Gamma_{B;n}$, $\Gamma_{C;n}$, and $\Gamma_{D;n}$, to be used for $\Gamma$ in the framework of Section 3. In this section, we will construct the basis $M_\Gamma$ of $U$ explicitly for each such $\Gamma$, resulting in Theorems 12, 13, 14, and 15.

Clearly, each of the algebras $F_\Gamma$ is defined by a subset of the four following relations.

$$x_i x_j = 0 \quad \text{for all } i, j \text{ with } |i - j| > 1, \{1, 3\} \neq \{i, j\} \neq \{n - 2, n\}, \quad \text{(R1)}$$

$$x_i x_3 = 0, \quad \text{(R2)}$$

$$x_{n-2} x_n = 0, \quad \text{(R3)}$$

$$x_{n-1} x_n = 0. \quad \text{(R4)}$$

We will use the following technical lemma:

Lemma 4.1. Let $a, b, n \in \mathbb{N}_+$, $i, j, k, l, m, i_1, \ldots, i_a, j_1, \ldots, j_b \in \{1, \ldots, n\}$. Let $\Gamma$ be one of the graphs $\Gamma_{A;n}$, $\Gamma_{B;n}$, $\Gamma_{C;n}$, and $\Gamma_{D;n}$, let $f \in V(\Gamma)$, let $x_i$ be the standard generators of $L(\Gamma, f)$, and let $t, u \in L(\Gamma, f)$. Furthermore, let $x_{i_a}$ commute with $x_{j_b}$ for all $p$ and $q$ and let $x_i$ commute with $x_j$. For Eq. 12 only, assume that $i < n - 2$. Then:
\[ x_j x_t = x_t x_j, \quad (Q1) \]
\[ x_i x_{i+1} x_{i+2} x_t = \frac{1}{2} \left( f(x_i, x_{i+1}, x_{i+2}, x_t) x_i - f(x_i, x_{i+2}, x_i x_{i+1}, x_t) - f(x_i, x_{i+1}, x_t x_i) x_{i+2} \right), \quad (Q2) \]
\[ x_k x_t x_m x_k t = \frac{1}{2} \left( f(x_k, x_m, t) x_k x_k + f(x_k, t) x_k x_k x_m - f(x_k, x_m) x_k x_k t + f(x_k, x_t) x_k x_k t \right) \]
\[ - f(x_k, x_m) x_k x_k t - f(x_k, x_t) x_k x_k t - f(x_k, x_m) x_k x_k t \]
\[ + f(x_k, x_t) x_k x_k t - f(x_k, x_t) x_k x_k t - f(x_k, x_m) x_k x_k t + f(x_k, x_t) x_k x_k t, \quad (Q3) \]
\[ f(u, x_k x_t x_m x_k t) = \frac{1}{2} \left( f(x_k, x_m, t) f(u, x_t x_k) + f(x_k, t) f(u, x_t x_k x_m) - f(x_k, x_m) f(u, x_t x_k) \right) \]
\[ + f(x_k, x_t) f(u, x_m) f(x_k, x_t) - f(x_k, x_t) f(u, x_k x_m) \]
\[ - f(x_k, x_t) f(u, x_k x_t) + f(x_k, x_t) f(u, x_k x_t) + f(x_k, x_t) f(u, x_k x_t) \]
\[ - f(x_k, x_t) f(u, x_k x_t) + f(x_k, x_t) f(u, x_k x_t) + f(x_k, x_t) f(u, x_k x_t) \]
\[ + f(x_k, x_t) f(u, x_k x_t) - f(x_k, t) f(u, x_m) x_k - f(x_k, t) f(u, x_m) x_k \]
\[ + f(x_k, x_t) f(u, x_k x_t), \quad (Q4) \]
\[ x_i, x_j, x_{j2} \cdots x_{j_n} = 0. \quad (Q5) \]

The proof of this lemma is straightforward, using the identities from Lemma 2.5 and the Jacobi identity.

4.1. Monomials for $\Gamma_{D,n}$. In the rest of this section, we will let $f \in \mathcal{V}(\Gamma_{D,n})$ be given and consider $\mathcal{L}(\Gamma_{D,n}, f)$, that is, only the relations in $[\mathcal{Q}]$ from page 6 are divided out. In Theorem 4.2, we will give a list of $2n^2 - n$ monomials in $x_1, \ldots, x_n$ and prove that they span $\mathcal{L}(\Gamma_{D,n}, f)$ linearly.

Note that the precise contents of this list is of little general importance. For example, the number of classes may be reduced using a more clever notation. However, one needs to fix such a list for the proof of Theorem 4.2 and the one presented below suffices.

**Theorem 4.2.** Let $\mathcal{M}_{\Gamma_{D,n}}$ be the set consisting of the following monomials:

- $y_{k,m}^1 = x_{k|1} x_{m}$, $n \geq k \geq m \geq 1$,
- $y_{k,m}^2 = x_{k|n-2} x_{m}$, $n - 2 \geq k > m \geq 1$,
- $y_{k,m}^3 = x_{k|n} x_{m|1}$, $n \geq k \geq m \geq 3$,
- $y_{k,m}^4 = x_{k|n-2} x_{m|1}$, $n - 2 \geq k \geq m \geq 3$,
- $y_{m}^5 = x_{n|2} x_{m}$, $n - 2 \geq m \geq 1$,
- $y_{k}^6 = x_{k|3} x_{1}$, $n \geq k \geq 3$,
- $y_{k}^8 = x_{k|n-2} x_{3|1}$, $n - 2 \geq k \geq 2$,
- $y_{m}^9 = x_{n|2} x_{m|1}$, $n - 2 \geq m \geq 3$,
- $y_{m}^{10} = x_{n|2} x_{m|1}$, $n - 2 \geq m \geq 3$,
- $y_{12}^{11} = x_{1|3} x_{n-2}$, $n \geq k \geq 1$,
- $y_{12}^{12} = x_{1|3} x_{n-2} x_{1}$, $n \geq k \geq 2$,
- $y_{13}^{13} = x_{n|2} x_{n-2}$, $n \geq k \geq 3$,
- $y_{14}^{14} = x_{n|2} x_{n-2} x_{3}$, $n \geq k \geq 3$,
- $y_{15}^{15} = x_{n|2} x_{n-2} x_{3}$, $n \geq k \geq 3$,
Let $x$ be a monomial in $x_1, \ldots, x_n$ of length $\ell$. Then $x$ is a linear combination of monomials $y \in M_{\Gamma, D, n}$ with length($y$) $\leq \ell$.

For convenience in the proof of Theorem 4.2, we extend the above definition: we define $y_{n,m}^2 = y_{n,m}^1$ and $y_{n-1,m}^3 = y_{n,m}^{\ell}$, for $m \leq n - 1$; and $y_{2,2}^1 = x_1$, $y_{2,2}^2 = x_1$, and $y_{n-1}^8 = y_{n}^7$.

**Proof.** Let $x$ be a monomial in $x_1, \ldots, x_n$ of length $\ell$. If $\ell = 1$ then $x \in M_{\Gamma, D, n}$. If $\ell = 2$, then either $x = 0$ or $x \in M_{\Gamma, D, n}$ or $-x \in M_{\Gamma, D, n}$ (using Eqs. (AC) and (K1)). We use induction on $\ell$ and may assume $\ell > 2$.

We have an $i \in \{1, \ldots, n\}$ and a monomial $y$ of length $\ell - 1$ such that $x = x_i y$. Moreover, because of the induction hypothesis we can assume $y \in M_{\Gamma, D, n}$. We will consider $x = x_i y^\ell$ for each of the seventeen classes in $M_{\Gamma, D, n}$ separately. In each case we will write $x$ as a linear combination of monomials of length at most $\ell$, where all monomials in the linear combination of length $\ell$ are members of $M_{\Gamma, D, n}$. By the induction hypothesis, this suffices to prove the theorem.

We will work modulo monomials of length at most $\ell - 1$, so because of extremality of $x_j$,

$$x_j x_j t = 0$$

whenever the left hand side occurs in a monomial of length $\ell$.

We show the cases $j = 1$ and $j = 3$ as an example, the other cases are very similar. For the complete proof, we refer to [Pos07].

**Case 1:** $j = 1, k \in \{1, \ldots, n\}$ and $m \in \{1, \ldots, k\}$. Since $\ell > 2$, we know that $m < k$. We distinguish the following sub-cases:

- If $i > k + 1$ and $k = n - 2$, then $x = y_{m}^5$.
- If $i > k + 1$ and $k \neq n - 2$, then $x = 0$ by Eq. (Q5).
- If $i = k + 1$, then $x = y_{k+1,m}^1$.
- If $i = k$, then extremality of $x_i$ shows that $x = 0$.
- If $i = k - 1$, then:

$$x = \begin{cases} x_i x_i x_{i+1} \overset{AC}{=} -x_i x_i x_{i+1} \overset{\Gamma}{=} 0, & \text{if } m = k - 1, \\ x_i x_i x_i y_{i-1,m}^1 \overset{\Gamma}{=} 0, & \text{otherwise}. \end{cases}$$

- If $i < k - 1$ and $i = n - 2$, then $x = y_{n-2,m}^2$.
- If $i < k - 1$ and $i \neq n - 2$, then $x = x_i x_{k|1}^m$. Applying Eq. (Q4) a sufficient number of times leads to one of the following situations:
  - If $i > m$, then $x = x_{k|1+i} x_i x_{i+1} x_{i+1} x_{i+1} x_{i+1} x_{i+1} x_{i+1} \overset{\Gamma}{=} 0$.
  - If $i = m$ and $i \neq 1$, then $x = x_{k|1+m+2} x_m x_{m+1} x_m \overset{\Gamma}{=} 0$.
  - If $i = m$ and $i = 1$, then $x = x_{k|1} x_1 x_3 x_2 x_1 \overset{\Gamma}{=} -x_{k|1} x_1 x_3 x_1 x_2 \overset{\Gamma}{=} 0$.
  - If $i = m - 1$ and $i \neq 1$, then $x = x_{k|1+m+1} x_m x_{m-1} x_m \overset{\Gamma}{=} -y_{k,m-1}$.
  - If $i = m - 1$ and $i = 1$, then $x = x_{k|1} x_1 x_3 x_2 \overset{\Gamma}{=} y_{k,3} - y_{k,1}$.
  - If $i < m - 1$ and $m \neq 3$, then $x = x_{k|1+m+1} x_m \overset{\Gamma}{=} 0$.
  - If $i < m - 1$ and $m = 3$, then $x = x_{k|1} x_1 x_3 \overset{AC}{=} -y_k^7$.

**Case 2:** $j = 3, k \in \{3, \ldots, n\}$ and $m \in \{3, \ldots, k\}$.

- If $i > k + 1$ and $k = n - 2$, then $x = y_{m}^3$.
- If $i > k + 1$ and $k \neq n - 2$, then $x = x_i x_{k|1+m+1} x_{m-1} \overset{\Gamma}{=} 0$.
- If $i = k + 1$, then $x = y_{3,m}^3$.
- If $i = k$, then

$$x = x_i y_{i,m}^3 = \begin{cases} x_i x_{i-1} x_i y_{i-1,i-1}^3 \overset{\Gamma}{=} 0, & \text{if } k = m, \\ x_i x_i y_{i-1,m}^3 \overset{\Gamma}{=} 0, & \text{otherwise}. \end{cases}$$
Theorem 4.3. Monomials for $n$ and show that these monomials $y_i$.

Case 4:

If $i = k - 1$, then we have

$$x = x_i y_{i+1,m} = \begin{cases} x_i x_i y_{i+1,i}^3 & \text{if } i + 1 = m, \\ x_i x_i y_{i+1,i}^3 = y_{i+1,i+1}^3 & \text{if } i + 1 = m + 1, \\ x_i x_i y_{i+1,i}^3 = 0, & \text{otherwise}. \end{cases}$$

If $i < k - 1$ and $i = n - 2$, then $x = y_{n-2m}^4$.

If $i < k - 1$ and $i \neq n - 2$, then $x = x_i x_k | m + 1 | x_m - 1 | y_{1}$. Repeated application of Eq. (Q1) leads to one of the following situations:

- If $i > m$, then $x = x_k | i + 2 | x_i y_{i+1,i}^3 = 0$.
- If $i = m$, then $x = x_k | m + 2 | x_m x_m + x_m - 1 | y_{1} = y_{k,m+1}^3$.
- If $i = m + 1$, then $x = x_k | m + 1 | x_m - 1 | y_{m-1}^3 = 0$.
- If $i < m - 1$, $m \neq 3$ and $i \neq 1$, then $x = x_k | m + 1 | x_m - 1 | y_{i+1,i}^3 = y_{i+1,i}^3 = 0$.
- If $i < m - 1$, $m \neq 3$ and $i = 1$, then $x = x_k | m + 1 | x_m - 1 | 4 x_{1} x_1 x_3 x_2 x_2 = 0$. This last identity follows from:

$$x_1 x_3 x_4 x_2 x_2 x_1 = x_1 x_3 x_4 x_2 x_2 x_1 + x_1 x_3 x_4 x_2 x_2 x_1 = 0.$$
Case 7: $y = y^{17}$. We find $y = x_n x_{n-1} x_{n-2} x_{n-3} x_{n-4} x_{n-5} x_{n-6} x_{n-7} x_{n-8} x_{n-9} x_{n-10} x_{n-11} x_{n-12} x_{n-13} x_{n-14} x_{n-15} x_{n-16} x_{n-17}$.

The following two theorems can be proved in exactly the same manner.

**Theorem 4.4.** Let $\mathcal{M}_{\Gamma_{A,n}}$ be the set consisting of the following monomials:

$$
\begin{align*}
\gamma_{k,m} &= x_{k+1}^m, \\
\gamma_{k,m}^3 &= x_{k+1}^{m+1} x_{m-1}^1, \\
\gamma_{k}^7 &= x_{k+1}^3 x_1,
\end{align*}
$$

$n \geq k \geq m \geq 1$, $n \geq k \geq m \geq 3$, $n \geq k \geq 3$.

Let $x$ be a monomial in $x_1, \ldots, x_n$ of length $l$. Then $x$ is a linear combination of monomials $y \in \mathcal{M}_{\Gamma_{A,n}}$ with length($y$) $\leq l$.

**Theorem 4.5.** Let

$$
\mathcal{M}_{\Gamma_{C,n}} = \{ y_{k,m} = x_{k+m} \mid n \geq k \geq m \geq 1 \}.
$$

Let $x$ be a monomial in $x_1, \ldots, x_n$ of length $l$. Then $x$ is a linear combination of monomials $y \in \mathcal{M}_{\Gamma_{C,n}}$ with length($y$) $\leq l$.

5. The Parameter Space

Recall that $X(\Gamma) = \{ f \mid \dim \mathcal{L}(\Gamma, f) = |\mathcal{M}_\Gamma| \}$. For $\Gamma \in \{ \Gamma_{A,n}, \Gamma_{B,n}, \Gamma_{C,n}, \Gamma_{D,n} \}$, we will find bijections $\psi_T$ from $X(\Gamma)$ to a vector space, such that $\mathcal{L}(\Gamma, f)$ is isomorphic to the Lie algebra of the corresponding Chevalley type if $\psi_T(f)$ is in a certain open dense subset of that vector space. In this section, we will be constructing the bijection. It will take some work to show that the map is injective; this will be the content of Lemmas 5.1, 5.2, 5.3 and 5.4. We will find the open dense subset in Section 5.

5.1. Parameters for $\Gamma_{D,n}$

**Lemma 5.1.** Let

$$
\psi_{\Gamma_{D,n}} : X(\Gamma_{D,n}) \to \mathbb{R}^{n+4}, f \mapsto (f_1(x_2), f_2(x_3), \ldots, f_{n-1}(x_n); f_1(x_3), f_1(x_2 x_3), f_{n-2}(x_n), f_{n-2}(x_{n-1} x_n), f_1(x_3 n-2 x_{n\mid 2})).
$$

Then $\psi_{\Gamma_{D,n}}$ is injective.

**Proof.** The values of all $f_i$ together determine the values of the extremal bilinear form on all of $\mathcal{F}_{\Gamma_{D,n}}$, since $f(x_i y, z) = f(x_i, y z) = f_i(y z)$. We will show that each value $f_i(y)$ for $y \in \mathcal{M}_{\Gamma_{D,n}}$ can be expressed in the values $f(z)$ in the theorem. To make this notationally convenient, let $\mathbb{F}_{D,n}$ be the rational function field obtained from $\mathbb{F}$ by extending it with $n+4$ symbols as follows. For every $f_i$ in the lemma, we extend $\mathbb{F}$ with the symbol $f_i(x)$ and assume that evaluating $f_i$ at $z$ yields this symbol $f_i(z) \in \mathbb{F}_{D,n}$. We will show that each value $f_i(y)$ is an element of $\mathbb{F}_{D,n}$ for all $i$ and all $y \in \mathcal{M}_{\Gamma_{D,n}}$.

Let $y = y^i \in \mathcal{M}_{\Gamma_{D,n}}$ of length $l$ and $i \in \{1, \ldots, n\}$. We will use induction on $l$. We consider the seventeen classes of monomials in $\mathcal{M}_{\Gamma_{D,n}}$ separately, and give cases $j = 1$ and $j = 3$ as an example. For the complete proof, we refer to [Pos07].

Let $x = x_i y_i = f(x_i, y_i) x_i$. It is sufficient to prove that $x = 0$ or $f(x_i, y_i) = 0$.

**Case 1:** $j = 1$, $k \in \{1, \ldots, n\}$, $m \in \{1, \ldots, k\}$. Note that $y_{k,m} = x_{k+m}$.

If $m = k$ then $y$ has length 1 and $f(x_i, y_i)$ is either 0 or in the list of values in the theorem. So assume that $k > m$.

- If $i > k + 1$ and $k = n - 2$, then:
  $$
  x = x_{n} x_{n-1} x_{n-2} x_{m+1} x_{m+1} - x_{m} x_{n} x_{n-2} | m+1 | \quad \text{and} \quad f(x_n, y_{n-2,m+1}) x_{m} x_{m+1} = 0.
  $$

- If $i > k + 1$ and $k < n - 2$, then:
  $$
  x = x_{k} x_{i} x_{y_{k-1,m}} - f(x_i, y_{k-1,m}) x_{k} x_{i} = 0.
  $$
Case 2:

- If $i = k + 1$, $m = k - 1$ and $i = n$, then $f(x_i, y_i) = f(x_n, x_{n-1}x_{n-2}) \in F_{D,n}$.
- If $i = k + 1$, $m = k - 1$ and $3 < i < n$, then
  $$f(x_i, y_i) = f(x_i, x_{i-1}x_{i-2}) = f(x_{i-1}, x_{i-3}x_{i-4}) = 0.$$ 
- If $i = k + 1$, $m = k - 1$ and $i = 3$ then $f(x_i, y_i) = f(x_i, x_2x_1) \in F_{D,n}$.
- If $i = k + 1$, $m < k - 1$ and $m > 1$, then
  $$x = -x_ix_{i+1} + x_{i-1}x_{i}x_{i+1} - x_mx_{i+1} - f(x_i, y_{i-1,m})x_m = 0.$$ 
- If $i = k + 1$, $m < k - 1$ and $m = 1$, then
  $$x = x_i + x_{i+1} + f(x_i, x_{i-1}x_{i+1}) = 0.$$ 
  Here we use the induction hypothesis for $f(x_i, y_{i-1,m})$.
- If $i = k - 1$ and $i = m$, then
  $$x = x_ix_{i+1}x_i = -x_ix_{i+1} = 0.$$ 
- If $i < k - 1$, $i = n - 2$ and $m = n - 1$, then
  $$f(x_i, y_i) = f(x_{n-2}, x_{n-1}x_{n-2}) \in F_{D,n}.$$ 
- If $i < k - 1$, $i = n - 2$ and $m = n - 2$, then
  $$f(x_i, y_i) = f(x_{n-2}, x_{n-1}x_{n-2}) = f(x_{n-2}, x_{n-1}x_{n-2}) = f(x_{n-2}, x_{n-1}x_{n-2}).$$ 
- If $i < k - 1$, $i = n - 2$ and $m \leq n - 3$, then
  $$f(x_i, y_i) = f(x_{n-2}, x_{n-1}x_{n-2}) = f(x_{n-2}, x_{n-1}x_{n-2}) = f(x_{n-2}, x_{n-1}x_{n-2}).$$ 
- If $i < k - 1$, $i < n - 2$ and $k > 3$, then
  $$x = x_ix_{k-1} = x_{k-1}x_{k-1} = f(x_i, y_{k-1,m}) = 0.$$ 
- If $i < k - 1$, $i < n - 2$ and $k = 3$ and $m = 2$, then $f(x_i, y_i) = f(x_i, x_3x_2) \in F_{D,n}$.
- If $i < k - 1$, $i < n - 2$ and $k = 3$ and $m = 1$, then we have:
  $$f(x_i, y_i) = f(x_i, x_3x_2x_1) = f(x_i, x_3x_2x_1) = f(x_i, x_3x_2x_1) = f(x_i, x_3x_2x_1) \in F_{D,n}.$$ 

Case 2: $j = 3$, $k \in \{3, \ldots, n\}$, $m \in \{3, \ldots, k\}$. Remember that $y^3_{k,m} = x_{k+m} + x_{m-1}$. 

- If $i > k + 1$ and $m = k$, then
  $$x = x_i + y^3_{k,k-1} = x_i + y^3_{k,k-1} = f(x_i, y^3_{k,k-1})x_i = 0.$$ 
- If $i > k + 1$ and $m < k$, then
  $$x = x_i + x_{k-1} = x_i + y^3_{k,k-1} = f(x_i, y^3_{k,k-1})x_i = 0.$$ 
- If $i = k + 1$, $m = k$ and $i = n$, then
  $$f(x_i, y_i) = f(x_n, x_{n-2}x_1) = f(x_n, x_{n-2}x_1) = f(x_n, x_{n-2}x_1) = f(x_n, x_{n-2}x_1).$$
Now we repeat the steps in the last line \(n - 6\) times to obtain

\[
f(x_i, y) = (-1)^n f(x_3, x_{4\mid n-2}x_{n\mid 4}x_3x_1)
\]

We will treat these terms separately.

\[
f(x_3, x_{4\mid n-2}x_{n\mid 4})(x_2x_3x_1)
\]

and

\[
f(x_3, x_{4\mid n-2}x_{n\mid 1})
\]

- If \(i = k + 1\), \(m = k\) and \(i < n\) then
  \[
x = x_i x_{i-1}y_{i-1,i-2}^3 x_{i-2}x_i y_{i-1,i-2}^3 \quad \implies \quad f(x_i x_{i-1}x_i y_{i-1,i-1}^3) \in \mathbb{F}_{D,n} x_{i-2}x_i = 0.
\]

- If \(i = k + 1\) and \(m < k\), then
  \[
x = x_i x_{i+1}x_{i-1}x_{i-2}x_i \quad \implies \quad f(x_i x_{i+1}x_{i-1}x_{i-2}x_i) \in \mathbb{F}_{D,n} x_{m-1}x_i = 0.
\]

- If \(i = k\) and \(m = k\), then
  \[
f(x_i, y) = f(x_i, x_i y_{i-1,i-1}^3) \in \mathbb{F}_{D,n}.
\]

- If \(i = k\) and \(m < k\), then
  \[
f(x_i, y) = f(x_i, x_i y_{i+1,i}^3) \in \mathbb{F}_{D,n}.
\]

- If \(i = k - 1\) and \(m = k\), then
  \[
f(x_i, y) = f(x_i, x_i y_{i-1,i}^3) \in \mathbb{F}_{D,n}.
\]

- If \(i = k - 1\) and \(m < k\), then
  \[
f(x_i, y) = f(x_i, x_{i+1}x_i y_{i-1,i}^3) \in \mathbb{F}_{D,n}.
\]

- If \(i = k - 2\), \(m = k\) and \(i > 1\), then
  \[
f(x_i, y) = f(x_i, x_{i+1}x_i y_{i-1,i}^3) \in \mathbb{F}_{D,n} + f(x_i, x_{i+1}x_i x_{i+2}x_i x_{i+1}x_i) \in \mathbb{F}_{D,n}.
\]

- If \(i = k - 2\), \(m = k\) and \(i = 1\), then
  \[
f(x_i, y) = f(x_i, x_2x_3x_1) \in \mathbb{F}_{D,n}.
\]

- If \(i = k - 3\), \(m = k\) and \(i > 1\), then
  \[
x = x_i x_{i+1}y_{k,k-1}^3 x_{k-1}x_i y_{k,k-1}^3 \quad \implies \quad f(x_i, y_{k,k-1}^3) x_{k-1}x_i = 0.
\]

- If \(i = k - 3\), \(m = k\) and \(i = 1\), then
  \[
f(x_i, y) = f(x_i, x_3x_4x_2x_3x_1) \in \mathbb{F}_{D,n}.
\]
• If \( i < k - 3 \) and \( m = k \), then
\[
x = x_i x_i x_{k-1} y_{k,k-1}^3 \quad (Q1)
\]
\[
x_k x_k x_{k-1} y_{k,k-1}^3 \quad (Q1)
\]
\[
f(x_i, y_{k,k-1}^3) x_{k-1} x_i = 0.
\]
• If \( i < k - 1 \), \( m < k \), \( i = n - 2 \) and \( m = n - 1 \), then
\[
f(x_i, y) = f(x_{n-2}, x_n x_{n-2} y_{n-1,n-2}^3) \quad (R3)
\]
\[
\in \mathbb{F}_{D,n}.
\]
• If \( i < k - 1 \), \( m < k \), \( i = n - 2 \) and \( m = n - 2 \), then
\[
f(x_i, y) = f(x_{n-2}, x_n x_{n-1} x_{n-3}^3) \quad (AS)
\]
\[
\in \mathbb{F}_{D,n},
\]
as proven earlier.
• If \( i < k - 1 \), \( m < k \), \( i = n - 2 \) and \( m < n - 2 \), then
\[
x = x_{n-2} x_{n-2} x_{n-1} x_{n-3}^3 \quad (Q1)
\]
\[
x_m x_{m-1} x_{n-2} x_{n-1} x_{n-3}^3 \quad (Q1)
\]
\[
f(x_{n-2}, y_{n,m-1}^3) x_{n-1} x_n = 0.
\]
• If \( i < k - 1 \), \( m < k \) and \( i < n - 2 \), then
\[
x = x_i x_i x_k y_{k-1,m}^3 \quad (Q1)
\]
\[
x_k x_k x_i y_{k-1,m}^3 \quad (Q1)
\]
\[
f(x_i, y_{k-1,m}^3) x_k x_i = 0.
\]

5.2. Parameters for \( \Gamma_{B,n} \).

Lemma 5.2. Let
\[
\psi_{\Gamma_{B,n}} : X(\Gamma_{B,n}) \to \mathbb{F}_{n+2}, f \mapsto (f_1(x_2), f_2(x_3), \ldots, f_{n-2}(x_{n-1}); f_1(x_3), f_1(x_2 x_3), f_{n-2}(x_n), f_1(x_3 x_{n-2} x_{n-2})).
\]

Then \( \psi_{\Gamma_{B,n}} \) is injective.

Proof. Remember that \( \mathcal{F}_{\Gamma_{B,n}} \) is a quotient of \( \mathcal{F}_{\Gamma_{D,n}} \). Hence relations between values of \( f_i \) that hold in \( \mathcal{F}_{\Gamma_{D,n}} \) hold in \( \mathcal{F}_{\Gamma_{B,n}} \) as well. This allows us to express \( f_i(y) \) in the values of Lemma 5.1. It then suffices to prove that \( f(x_n, x_{n-1}) \) and \( f(x_n, x_{n-1} x_{n-2}) \) are zero:

- \( f(x_n, x_{n-1}) = 0 \), because \( x_n x_{n-1} = 0 \)
- \( f(x_n, x_{n-1} x_{n-2}) = f(x_n, x_{n-1} x_{n-2}) = 0 \).

5.3. Parameters for \( \Gamma_{A,n} \).

Lemma 5.3. Let
\[
\psi_{\Gamma_{A,n}} : X(\Gamma_{A,n}) \to \mathbb{F}_{n+1}, f \mapsto (f_1(x_2), f_2(x_3), \ldots, f_{n-1}(x_n); f_1(x_3), f_1(x_2 x_3)).
\]

Then \( \psi_{\Gamma_{A,n}} \) is injective.

Proof. \( \mathcal{F}_{\Gamma_{A,n}} \) is a quotient of \( \mathcal{F}_{\Gamma_{D,n}} \), so by Lemma 5.1 and a reasoning similar to the one in the proof of Lemma 5.2 it suffices to show that \( f(x_n, x_{n-2}) \), \( f(x_n, x_{n-1} x_{n-2}) \) and \( f(x_1, x_{n-1} x_{n-2} x_{n-2}) \) are zero:

- \( f(x_n, x_{n-2}) = 0 \), because \( x_n x_{n-2} = 0 \)
- \( f(x_n, x_{n-1} x_{n-2}) = f(x_n, x_{n-2} x_{n-1}) = 0 \)
- \( f(x_1, x_{n-1} x_{n-2} x_{n-2}) = f(x_1 x_3, y_{1,2}^2) = 0 \).
5.4. Parameters for $\Gamma_{C; n}$.

Lemma 5.4. Let
\[ \psi_{\Gamma_{C; n}} : X(\Gamma_{C; n}) \to \mathbb{F}^{n-1}, f \mapsto (f_1(x_2), f_2(x_3), \ldots, f_{n-1}(x_n)). \]
Then $\psi_{\Gamma_{C; n}}$ is injective.

Proof. $\mathcal{F}_{\Gamma_{C; n}}$ is a quotient of $\mathcal{F}_{\Gamma_{A; n}}$, so by Lemma 5.3 and a reasoning similar to the one in the proof of Lemma 5.2 it suffices to show that $f(x_3, x_1)$ and $f(x_3, x_2, x_1)$ are zero:
- $f(x_3, x_1) = 0$, because $x_3x_3x_1 = 0$,
- $f(x_3, x_2, x_1) = -f(x_3, x_1, x_2) = f(x_3, x_1, x_2) = 0$. \hfill $\square$

Corollary 5.5. $\psi_{\Gamma}(X(\Gamma))$ is an algebraic variety for all $\Gamma \in \{\Gamma_{A; n}, \Gamma_{B; n}, \Gamma_{C; n}, \Gamma_{D; n}\}$.

6. REALIZATIONS OF THE FOUR CLASSICAL FAMILIES

In this section, we will find generating sets of extremal elements for the four classical families of Lie algebras, where these generating sets realize the graphs $\Gamma_{A; n}$, $\Gamma_{B; n}$, $\Gamma_{C; n}$ and $\Gamma_{D; n}$. We keep $n$ as the number of extremal generators and will see that these graphs correspond to the Lie algebras of type $A_{n-1}$, $B_{n-1}$, $C_{n/2}$ and $D_n$, respectively. In particular, the objective of this section will be the formulating and proving of Theorems 6.10, 6.11, 6.18 and 6.19, giving these explicit generators.

The extremal elements in Lie algebras of type $A_{n-1}$ or $C_{n/2}$ correspond to transvections, which will be discussed in Section 6.1. In the orthogonal Lie algebras, the extremal elements correspond to the Siegel transvections or Siegel transformations. These are examined in Section 6.2. In both sections, we first explore these elements in a general setting, and then discuss generators for the Lie algebras specifically.

6.1. Transvections. Let $n \in \mathbb{N}_+$, $x \in V = \mathbb{F}^n$, $h \in V^*$, and fix a basis $e_1, \ldots, e_n$ of $V$ and a corresponding dual basis $f_1, \ldots, f_n$ of $V^*$. We will see that the linear transformation $x \otimes h : v \mapsto h(v) x$ is an extremal element of $\mathfrak{sl}(V)$ if $h(x) = 0$ and $x, h$ nonzero. A transvection is a linear transformation of the form $1 + x \otimes h$ where $h(x) = 0$ and $x, h$ nonzero. We call $x$ the centre of the transvection and $h$ the axis. We then call $x \otimes h$ an infinitesimal transvection.

A transvection group is a group $\{1 + tx \otimes h \mid t \in \mathbb{F}\}$. The Lie algebra of a transvection group consists of the transvections $tx \otimes h$. We will use a result of McLaughlin [McL67] which classifies groups generated by transvection subgroups. This is a weaker version of a reformulation by Cameron and Hall, Theorem 2 from [CH91]:

Theorem 6.1. Let $G$ be a nontrivial group of linear transformations of the finite-dimensional $\mathbb{F}$-vector space $V$, which is generated by $\mathbb{F}$-transvection subgroups. If $V$ is spanned by a $G$-orbit on centres of these transvection subgroups, and $V^*$ is spanned by the axes, then one of the following holds:

1) $G = \text{SL}(V)$;
2) $G = \text{Sp}(V, B)$ for some symplectic form $B$.

We will need a tool to distinguish between $\text{SL}(V)$ and $\text{Sp}(V, B)$. This tool will be provided by analysis of the occurrence of Heisenberg subalgebras generated by pairs of extremal elements, further detailed in Lemma 6.8 and Corollary 6.9.

The following lemmas are easily seen to be true.

Lemma 6.2. $\mathfrak{sl}(V)$ is the span of the infinitesimal transvections in $\mathfrak{gl}(V)$.

Lemma 6.3. Infinitesimal transvections are extremal elements of $\mathfrak{sl}(V)$.

Lemma 6.4. All extremal elements in $\mathfrak{sl}(V)$ are infinitesimal transvections.

Lemma 6.5. Let $x \otimes h$ and $y \otimes k$ be two infinitesimal transvections and let $\mathcal{L} = \langle x \otimes h, y \otimes k \rangle_{\text{Lie}}$. Then the isomorphism class of $\mathcal{L}$ depends on the geometrical configuration of $\mathbb{F}x$, $\mathbb{F}y$, $\text{Ker} h$ and $\text{Ker} k$, as follows:
• If \( Fx = Fy \) and \( \text{Ker} \, h = \text{Ker} \, k \), then \( \mathcal{L} \) is one-dimensional.
• If either \( Fx = Fy \) or \( \text{Ker} \, h = \text{Ker} \, k \) but not both, then \( \mathcal{L} \) is two-dimensional.
  Assume for the other cases that \( Fx \neq Fy \) and \( \text{Ker} \, h \neq \text{Ker} \, k \).
• If \( Fx \subseteq \text{Ker} \, k \) and \( Fy \subseteq \text{Ker} \, h \), then \( \mathcal{L} \) is two-dimensional as in the preceding case.
• If either \( Fx \subseteq \text{Ker} \, k \) or \( Fy \subseteq \text{Ker} \, h \) but not both, then \( \mathcal{L} \) is isomorphic to the Heisenberg algebra.
• If \( Fx \nsubseteq \text{Ker} \, k \) and \( Fy \nsubseteq \text{Ker} \, h \), then \( \mathcal{L} \) is isomorphic to \( \mathfrak{sl}_2 \).

If \( x \otimes h \) and \( y \otimes k \) generate a Heisenberg algebra, we say that they form an extraspecial pair.

We will also realize \( \mathfrak{sp}(V) \), the Lie algebra of type \( C_{n/2} \), using infinitesimal transvections. In order to do this, let us assume that \( n \) is even and that we have a nondegenerate symplectic form \( B \). We will denote the matrix of \( B \) by \( B \) as well. For \( y \in V \), we write

\[
u(y) := y \otimes (v \mapsto B(y, v)): V \to V, v \mapsto B(y, v)y.
\]

Hence \( u(y) \) is an infinitesimal transvection. The following lemma is easily proven:

**Lemma 6.6.** The infinitesimal transvections in the Lie algebra of type \( C_{n/2} \) can all be written as \( u(y) \) for some \( y \).

Since \( u(y) \) is an extremal element in \( \mathfrak{sl}(V) \), it is also an extremal element in \( \mathfrak{sp}(V) \). The following two lemmas are equivalents of Lemmas 6.2 and 6.4 for \( \mathfrak{sp}(V) \).

**Lemma 6.7.** \( \mathfrak{sp}(V) \) is spanned by its infinitesimal transvections.

**Lemma 6.8.** All extremal elements in \( \mathfrak{sp}(V) \) are infinitesimal transvections.

**Proof sketch.** We may assume that \( B = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \). Let \( M \) be the matrix of an extremal element of \( \mathfrak{sp}(V) \), then \( M \) can be written as a block matrix \( \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & -M_{11} \end{pmatrix} \), where \( M_{12} \) and \( M_{21} \) are symmetric. For every matrix \( A \) also in \( \mathfrak{sp}(V) \), we have that

\[
[M, [M, A]] = M^2 A - 2MAM + AM^2 \in F M. \tag{6.1}
\]

We will mostly take for \( A \) a block matrix \( \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \) with \( A_{12} \) and \( A_{21} \) symmetric and \( A_{22} = -A_{11}^T \). However, substituting the \( 2n \times 2n \) identity matrix into the linear equation (6.1) yields a tautology, so we can add multiples of the \( n \times n \) identity matrices to \( A_{11} \) and \( A_{22} \) independently. We take the following values for \( A \).

• Take \( A_{11} = A_{12} = A_{22} = 0 \) and for \( A_{21} \) all matrices \( E_{ij} \) that has a one in position \( (i, j) \) and zeroes elsewhere. This shows that we can find a vector \( y_0 \in \mathbb{F}^n \) such that \( M_{12} = y_0 y_0^T \).

  Similarly, there is an \( y_1 \in \mathbb{F}^n \) with \( M_{21} = -y_1 y_1^T \).

• Take \( A_{11} = I \) and \( A_{12} = A_{21} = A_{22} = 0 \), then we find that

\[
M' := \begin{pmatrix}
2M_{12} & M_{11} - M_{12} M_{11}^T \\
-M_{11} M_{21} - M_{11}^T M_{21} & -2M_{12} \\
\end{pmatrix}
\]

is a multiple of \( M \). \tag{6.2}

If \( M' \) is nonzero, it follows fairly easily that \( M_{11} = -y_0 y_1^T \), and we are done. We continue with the case \( M' = 0 \), where one sees fairly easily that the products of pairs of distinct elements of \( \{M_{11}, M_{12}, M_{22}\} \) are zero.

• Take \( A_{12} = A_{21} = 0 \) and \( A_{22} = -A_{11}^T \). Using the fact that the products mentioned above are zero, we obtain that

\[
M'' := \begin{pmatrix}
[M_{11}, [M_{11}, A_{12}]] + 2M_{12} A_{11}^T M_{21} & -2(M_{11} A_{12} M_{11} + M_{12} A_{11}^T M_{11}^T) \\
-2(M_{21} A_{11} M_{11} + M_{21}^T A_{11} M_{11}) & -[M_{12}, [M_{11}, A_{11}]] - 2M_{21} A_{11} M_{12}
\end{pmatrix}
\]

is a multiple of \( M \). \tag{6.3}

We take for \( A_{11} \) the matrices \( E_{ij} \). The rest of the proof is easy if one distinguishes the case where \( y_0 = y_1 = 0 \) from the other case.

**Corollary 6.9.** \( \mathfrak{sp}(V) \) does not contain an extraspecial pair.

**Proof.** Let \( x \otimes h \) and \( y \otimes k \) form an extraspecial pair. Then we may assume that \( h(y) = 0 \neq k(x) \). But \( h(y) = B(x, y) = -B(y, x) = -k(x) \). \qed
We now proceed with the theorems giving the generating extremal elements.

**Theorem 6.10.** Suppose that $n$ is even. Let $B$ be the nondegenerate symplectic form determined by the matrix \( \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \). The transformations

\[
u(x_i) : v \mapsto B(x_i, v)x_i, \quad i = 1, \ldots, n,
\]

realize the graph $\Gamma_{C,n}$ for $n \geq 2$ if we take these values for $x_i$:

\[
x_{2\ell-1} = e_\ell \quad \text{for } 1 \leq \ell \leq n/2,
\]

\[
x_{2\ell} = e_{\ell+n/2} + e_{\ell+n/2+1} \quad \text{for } 1 \leq \ell < n/2,
\]

\[
x_n = e_n.
\]

The Lie algebra $\mathcal{L} = \langle u(x_i) \rangle_{\text{Lie}}$ is $\mathfrak{sp}_n$.

**Proof.** It is easy to check that the transformations $u(x_i)$ realize the graph. In order to prove that they generate $\mathfrak{sp}_n$, consider the group $G = (1 + tu(x_i) | t \in \mathbb{F}, 1 \leq i \leq n)_{\text{GP}}$, of which $\mathcal{L}$ is the Lie algebra. The action of $G$ on $\mathcal{L}$ is such that

\[
u(x_i)^{1+tu(y)} = (1 + tu(y))u(x)(1 - tu(y)) = u(x^{1+tu(y)}),
\]

as can be seen immediately by inspection. So if $B(x, y) \neq 0$, then

\[
(1 + B(x, y)^{-1}u(y))(1 + B(x, y)^{-1}u(x))(y) = x.
\]

This shows that the orbit of $G$ on $x_1$ spans $V$. Hence, using Theorem 6.1, either $G = \text{Sp}(V, B)$ or $G = \text{SL}(V)$. Thus $\mathcal{L}$ is either $\mathfrak{sp}_n$ or $\mathfrak{sl}_n$. But all given transformations are in $\mathfrak{sp}_n$. This can be verified by examining the matrices $A_i^2 M + MA_i$, where $M$ is the matrix of $B$ and $A_i$ is the matrix of $u(x_i)$; if we view elements of $V$ as column vectors, then $A_i = x_ix_i^T M$, so $A_i^2 = -Mx_ix_i^T$. Thus $A_i^2 M + MA_i = 0$, so $\mathcal{L} = \mathfrak{sp}_n$.

**Theorem 6.11.** The transformations $x_i \otimes h_i$ realize the graph $\Gamma_{A,n}$ for $n \geq 2$ if we take these values for $x$ and $h$:

\[
x_1 = e_1 - e_2 \quad h_1 = f_1 + f_2;
\]

\[
x_i = e_{i-1} + e_i \quad h_i = f_{i-1} - f_i \quad \text{for } 1 < i \leq n.
\]

The Lie algebra $\mathcal{L} = \langle x_i \otimes h_i \rangle_{\text{Lie}}$ is $\mathfrak{sl}_n$.

**Proof.** It is easy to check that the transformations $x_i \otimes h_i$ realize the graph. In order to prove that they generate $\mathfrak{sl}_n$, consider the group $G = (1 + t(x_i \otimes h_i) | t \in \mathbb{F}, 1 \leq i \leq n)_{\text{GP}}$, of which $\mathcal{L}$ is the Lie algebra. It is clear that the orbit of $G$ on $x_1$ spans $V$. Hence, using Theorem 6.1, either $G = \text{Sp}(V, B)$ or $G = \text{SL}(V)$. Hence $\mathcal{L}$ is either $\mathfrak{sp}_n$ or $\mathfrak{sl}_n$. Now consider $\exp(2 \text{ad} x_3 \otimes h_3)(x_1 \otimes h_1) = (x_1 - 2x_3) \otimes (h_1 - 2h_3)$. It forms an extraspecial pair with $x_2 \otimes h_2$, because $(h_1 - 2h_3)(x_2) = 0 \neq h_2(x_1 - 2x_3)$. By Corollary 6.9, $\mathcal{L} = \mathfrak{sl}_n$. □

### 6.2. Siegel transvections

In order to realize the two orthogonal types of algebras we use Siegel transvections. As an equivalent of Theorem 6.1 we will use the main theorem of Steinbach’s paper [Ste97], dealing with Siegel transvection groups in a similar way to how Theorem 6.1 deals with transvection groups. Steinbach’s main theorem is reprinted here in a weaker form as Theorem 6.10.

In this form, Theorem 6.10 also follows from the work by Liebeck and Seitz [LS94].

Let the dimension of $V$ be $2n$ or $2n - 1$. Let $B$ be a nondegenerate orthogonal bilinear form on $V$; we will denote the corresponding matrix by $B$ as well. We may assume that $B$ is either \( \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \) or \( \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \). Let $u, v \in V$ be linearly independent with $B(u, u) = B(u, v) = B(v, v) = 0$; in other words, $u$ and $v$ span an isotropic line. When we speak of lines in an orthogonal space we mean isotropic lines. Then

\[
S_{u,v} : V \rightarrow V, \quad x \mapsto x + B(u, x)v - B(v, x)u
\]

is known as the Siegel transvection determined by $u$ and $v$. If $S_{u,v}$ is a Siegel transvection, then we will call the map $T_{u,v} = S_{u,v} - 1$ an infinitesimal Siegel transvection. Note that $T_{u,v}$ is determined up to scalar multiples by the projective line containing $u$ and $v$. We call the group \( \{1 + tT_{u,v} | t \in \mathbb{F}\}_{\text{GP}} \) a Siegel transvection group.
Theorem 6.16. Let infinitesimal Siegel transvections are extremal elements of Lemma 6.13.

Lemma 6.12. $o(V)$ is spanned by infinitesimal Siegel transvections.

Lemma 6.13. Infinitesimal Siegel transvections are extremal elements of $o(V)$.

Lemma 6.14. For all $t \in F$ and $u, v, w, x \in V$, we have $\exp(t \text{ad } T_{u,v})T_{w,x} = T_{w + tT_{u,v}w, x + tT_{u,v}x}$.

In the following lemma, we will let “point” refer to projective points and “line” to projective isotropic lines.

Lemma 6.15. Let $\mathcal{L} = \langle T_{u,v}, T_{w,x} \rangle_{\text{Lie}}$, where $T_{u,v}$ and $T_{w,x}$ are two infinitesimal Siegel transvections. Then the isomorphism class of $\mathcal{L}$ depends on the geometrical configuration of the lines $\ell = \langle u, v \rangle_F$ and $m = \langle w, x \rangle_F$, as follows:

- If $\ell = m$, then $\mathcal{L}$ is one-dimensional.
- If $\ell \cap m$ is a point, then $\mathcal{L}$ is two-dimensional.
- If each point on $\ell$ is collinear to each point on $m$, then $\mathcal{L}$ is two-dimensional.
- If exactly one point on $\ell$ is collinear with all of $m$ and exactly one point on $m$ is collinear with all of $\ell$, then $\mathcal{L}$ is isomorphic to the Heisenberg algebra.
- If every point on $\ell$ is collinear with exactly one point of $m$ and every point on $m$ is collinear with exactly one point of $\ell$, then $\mathcal{L}$ is isomorphic to $\mathfrak{s}_2$.

These are all cases.

In the case where $\mathcal{L}$ is isomorphic to the Heisenberg algebra, we say that $\ell$ and $m$ form an extraspecial pair, just as in the previous section.

We focus our attention on the Siegel transvection groups now. Since the theorem of Steinbach is somewhat more involved than Theorem 6.1, we need to introduce some notation and terminology first.

Let $Y = \Omega(V, B)$ be the commutator subgroup of the orthogonal group $O(V, B)$. Then by [Ste97] section 1.1 $Y$ is generated by all Siegel transvection subgroups of $O(V, B)$. Let $G \neq 1$ be a subgroup of $Y$ generated by some of the Siegel transvection groups. For each Siegel transvection group $A \subseteq Y$, let $A^0 = A \cap G$; then either $A^0 = 1$ or $A^0 = A$ – this can easily be seen in the Lie algebra. This situation is a special case of the situation in the main theorem of [Ste97], reprinted here in a weaker version as Theorem 6.16, which tells us to which isomorphism classes such a group $G$ can belong.

Following [Ste97], we use the following abbreviations for different situations:

- **(O):** $G$ is an orthogonal group over the same vector space, viewed as a vector space over a smaller field. Also, if $B$ has maximal Witt index over a vector space of even dimension, then it contains the orthogonal group over a vector space of one dimension less where $B$ has a one-dimensional radical.
- **(E):** $G$ is a special case related to triality and $\dim V = 8$.
- **(ND):** $G$ is the special linear group on a vector space $V'$ such that $V$ has double the dimension of $V'$ and $B$ has maximal Witt index. $V$ is the direct sum of the space $V'$ where $G$ acts naturally and a space where it acts dually.
- **(I):** $G$ is a unitary group. The unitary space is then regarded as orthogonal space over the fixed field of the involutory automorphism.
- **(IQ):** $G$ is a unitary group over a quaternion division ring instead of a field.
- **(KC):** $G$ is a special case related to the Klein correspondence and $\dim V \in \{5, 6\}$.

Theorem 6.16. Let $G \neq 1$ be a subgroup of $Y = \Omega(V, q)$ generated by Siegel transvection groups $S_{u,v}$. Suppose these conditions are satisfied:

- **(H1)’** If $A, B$ are Siegel transvection subgroups of $Y$ that intersect $G$ nontrivially, and $(A, B)_{\text{Gp}} = SL_2(\mathbb{F})$, then $(A^0, B^0)_{\text{Gp}} = SL_2(\mathbb{F})$.
- **(H2)** All nilpotent normal subgroups of $G$ are contained in $Z(G)$ and in the commutator group $G'$, and it is not possible to decompose the set $\Sigma$ of Siegel transvection subgroups into two nonempty parts $\Sigma_1, \Sigma_2$ such that all groups from $\Sigma_1$ commute with all groups from $\Sigma_2$.
- **(H3)** There are three pairwise distinct commuting Siegel transvection subgroups $T$ of $G$ commuting with exactly two of these three subgroups.
Theorem 6.18. Let
\[ \text{for } x \in \text{sl}, \]
then one of the situations (O), (E), (ND), (I), (IQ) and (KC) holds.

\textbf{Theorem 6.17.} Suppose all conditions from Theorem 6.16 hold and the following extra conditions hold as well.

\begin{enumerate}[(X1)]  
  \item The dimension of \( V \) is greater than 8.
  \item There are extraspecial pairs in \( G \).
  \item There is no \( G \)-invariant decomposition of \( V \) into two equal-dimensional subspaces, such that each of these subspaces intersects every line corresponding to a Siegel transvection subgroup nontrivially.
  \item One of the Siegel transvection subgroups is isomorphic to \( \mathbb{F}^+ \).
  \item \( \langle u_i, v_i \rangle_{\mathbb{F}} = V \).
\end{enumerate}

Then \( G = Y \).

\textit{Proof.} Extra condition (X1) shows that we are not in situations (E) or (KC). By extra condition (X2), we must be in situation (ND) or (O). By extra condition (X3), we are not in situation (ND). Hence, we are in situation (O); \( G \) is an orthogonal group. Then condition (X5) shows that the field is all of \( \mathbb{F} \). Finally, because of extra condition (X4), \( G \) is all of \( Y \). \( \square \)

We define the basis of \( V \) in an order corresponding to the matrix of \( B \), as follows. Let \( k = n \) for \( D_n \) and \( k = n - 1 \) for \( B_n \). The basis of \( V \) consists of vectors \( \{ e_i \}_{i=1}^k \) spanning a maximal isotropic subspace, then of vectors \( \{ f_i \}_{i=1}^k \) spanning a maximal isotropic complement to \( \langle e_i \rangle_{\mathbb{F}} \) and with \( B(e_i, f_j) = \delta_{i,j} \), and if \( n \) is odd, finally a vector \( g \) with \( B(g, g) = 2 \). We can interpret the vectors and linear functionals from Section 6.14 in particular from Theorem 6.14 in this context as well: \( \{ f_i \} \) is still the dual basis of \( \{ e_i \} \). So we find the well-known isomorphic copy of \( \mathfrak{s}(W) \) in \( \mathfrak{o}(V) \), where \( W \) is a maximal isotropic subspace; the isomorphism is determined by sending \( x \otimes h \in \mathfrak{s}(W) \) to \( T_{x,h} \in \mathfrak{o}(V) \). We use the generating elements of \( \mathfrak{s}(W) \) in finding those for \( \mathfrak{o}(V) \), but with extra parameters for which the necessity will become apparent in Section 7.

\textbf{Theorem 6.18.} Let \( \alpha, \beta \in \mathbb{F} \) and write \( \kappa \) for \( \sqrt{1 + \beta} \). Let \( \lambda = \begin{cases} \frac{\alpha}{\alpha + 2}, & \text{if } n \text{ is odd}, \\ \frac{\alpha}{\alpha + 2}(1 + \beta + \kappa), & \text{if } n \text{ is even}. \end{cases} \)

Suppose that \( (\alpha + 2)\beta(\beta + 1) \neq 0 \) and that \( \lambda(2 - \beta + \lambda \beta) \neq 1 \). The transformations \( T_{u_i, v_i} \), for \( i \leq n \), realize the graph \( \Gamma_{D_n} \) for \( n \geq 5 \) if we take these values for \( u \) and \( v \):

\[ u_i = e_i - e_2 \quad \text{for } 1 < i < n, \]
\[ u_n = e_n - \beta f_n - e_n \quad \text{for } 1 < i < n, \]
where
\[ f = (0, 0, 1, 1, \ldots, 0) \quad \text{if } n \text{ is odd}, \]
\[ f = \frac{1}{1 + \beta + \kappa}(0, 0, -\kappa, -\kappa, \ldots, -\kappa, 1) \quad \text{if } n \text{ is even}; \]
\[ e = (0, 0, 1, \beta + \kappa, -1 - \beta - \kappa, \ldots, 1 + \beta + \kappa, (\beta + 1)(\kappa + 1)) \]
where we first write the coefficients of \( e_i \), then a bar (\( \bar{\phantom{1}} \)), then those of \( f_i \), and finally, if \( n \) is odd, the coefficient of \( g \). Then \( L = \langle T_{u_i, v_i} \rangle_{\mathfrak{l}ie} \) is \( \mathfrak{o}_{2n} \).

Note that \( B(f, f) = 0 \), and
\[ B(\bar{f}, u_i) = \begin{cases} 0, & \text{if } i \neq 3, \\ 1, & \text{if } i = 3; \end{cases} \]
\[ B(\bar{f}, v_i) = \begin{cases} 0, & \text{if } i \neq 3, \\ -1, & \text{if } i = 3 \text{ and } n \text{ is odd}, \\ \frac{\kappa}{1 + \beta + \kappa}, & \text{if } i = 3 \text{ and } n \text{ is even}. \end{cases} \]
Hence \( \lambda = -\frac{1}{D} B(\tilde{f}, v_3) \).

We will prove this theorem using Theorem 6.17. We will first state a similar theorem for the Lie algebra of type \( B_{n-1} \), Theorem 6.19, then state and prove some additional lemmas necessary to show that the conditions of Theorem 6.17 hold, and finally prove Theorems 6.18 and 6.19 on page 24.

**Theorem 6.19.** Let \( \gamma \in F \) be such that \( \gamma(\gamma + 1) \neq 0 \). The transformations \( T_{u_i,v_i} \) for \( i \in \{1, \ldots, n\} \) realize the graph \( \Gamma_{B_n} \) for \( n \geq 5 \) if we take these values for \( u \) and \( v \):

\[
\begin{align*}
  u_1 &= e_1 - e_2 & v_1 &= f_1 + f_2, \\
  u_i &= e_{i-1} + e_i & v_i &= f_i - f_{i-1} & \text{for } 1 < i < n, \\
  u_n &= \gamma e_{n-2} + f_{n-2} + \gamma e_{n-1} - f_{n-1} & v_n &= e_{n-2} - f_{n-2} + (1 - \gamma)e_{n-1} + g,
\end{align*}
\]

Then \( \mathcal{L} = \langle T_{u_i,v_i} \rangle_{\text{Lie}} \) is \( \mathfrak{so}_{2n-1} \).

The proof will be similar for \( D_n \) and \( B_{n-1} \), so let \( \mathcal{L} \) be one of the two algebras defined in Theorems 6.18 and 6.19 and let \( u_i \) and \( v_i \) be the corresponding vectors. Let \( G \) be the group generated by the Siegel transvection groups \( S_{u_i,v_i} \). We denote by \( \Sigma \) the orbit of \( G \) on the lines \( \langle u_i, v_i \rangle_F \) or, alternatively, on the projective infinitesimal Siegel transvections \( \mathbb{P}T_{u_i,v_i} \), or alternatively, on the Siegel transvection subgroups \( (1 + \mu T_{u_i,v_i}) | \mu \in F \), bijections between these three sets are given in Lemma 6.20. Before we start checking the conditions of Theorem 6.17, we will first verify that the action of \( G \) on these three classes of objects is equivalent and that \( G \) is transitive on \( \Sigma \). This is asserted by the following two lemmas, which are readily proven.

**Lemma 6.20.** The action of \( 1 + \mu T_{w,x} \) on \( \mathfrak{so}(V) \) by conjugation from the left is the same as the natural action of \( \exp(\mu \mathfrak{ad} T_{w,x}) \). Furthermore, the bijections sending \( \mathbb{P}T_{u,v} \in \Sigma \) to \( \langle u, v \rangle_F \) and \( (1 + \mu T_{u,v} | \mu \in F) \) commute with the action of \( G \).

**Lemma 6.21.** All \( T_{u_i,v_i} \) are in one orbit under \( G \).

**Lemma 6.22.** \( \Sigma \) contains an extraspecial pair.

**Proof.** We take \( \langle u_2, v_2 \rangle_F \) as the first line. For \( D_n \), the second line is (the line corresponding to)

\[
\exp(2 \mathfrak{ad} T_{u_3,v_3}) T_{u_1,v_1} = T_{u_1 + 2v_3, v_3 + 2(1 + \alpha)v_3 + 2(\alpha + 2)\lambda u_3};
\]

for \( B_{n-1} \), we specialize from these values by setting \( \alpha = \lambda = 0 \) and obtain as second line (the line corresponding to)

\[
\exp(2 \mathfrak{ad} T_{u_3,v_3}) T_{u_1,v_1} = T_{u_1 + 2v_3, v_3 + 2v_3}.
\]

\[\square\]

**Lemma 6.23.** \( \Sigma \) contains a copy of the set of infinitesimal transvections in \( \mathfrak{sl}_{n-1} \).

**Proof.** We will exhibit two sets of generators of different isomorphic copies of \( \mathfrak{sl}_{n-1} \). Let \( U \) be the vector space spanned by \( e_i \) for \( i < n \). If we are studying \( D_n \), then define

\[
v'_i = \begin{cases} 
  v_i - \lambda u_i, & i \in \{1, n\} \\
  v_i + \lambda u_i, & 1 < i < n;
\end{cases}
\]

for \( B_{n-1} \), substitute \( \lambda = 0 \) (whence \( v'_i = v_i \)), but let \( v''_n = u_n - \gamma v_n \). Additionally, for both \( B_{n-1} \) and \( D_n \) define \( u'_i = u_i \) except that \( u''_n = \frac{1}{1 + \gamma}(u_n + v_n) \) for \( B_{n-1} \). Then \( B(u'_i, u'_j) = B(v'_i, v'_j) = 0 \) for all \( \{i, j\} \neq \{n - 1, n\} \). Furthermore, \( u_i \otimes h_i = T_{u_i,v_i} = T_{u'_i, v'_i} \) as linear transformations, and \( U^* = \{ h_i | i < n \}_F \) is a vector space dual to \( U \) where the duality is provided by the form \( B \). The infinitesimal transvections \( u_i \otimes h_i \) generate \( \mathfrak{sl}(U) \) because of the same arguments that prove Theorem 6.11; the centres and axes of the transvections span \( U \) and its dual, respectively, and there is an extraspecial pair (both lines from Lemma 6.22 are in \( \mathfrak{sl}(U) \)). The group generated by all of the corresponding transvections is a subgroup of \( G \), and it is transitive on all transvections in \( \mathfrak{sl}(U) \). So in particular, the group of Siegel transvections is transitive on all infinitesimal Siegel transvections in \( \mathfrak{sl}(U) \).

Similarly, we can define \( \tilde{U} = \langle u'_1, u'_2, \ldots, u'_{n-2}, u'_{n} \rangle_F \) and \( \tilde{U}^* = \langle v'_1, v'_2, \ldots, v'_{n-2}, v'_{n} \rangle_F \), on which the transvections generate \( \mathfrak{sl}(\tilde{U}) \). \[\square\]
For $B_{n-1}$, there is a nontrivial intersection between $U + \hat{U}$ and $U^* + \hat{U}^*$: since
\[
\gamma(1 + \gamma) u'_n + v'_n = (1 + \gamma)u_n = (1 + \gamma)(\gamma u_n - v_n) = \gamma(1 + \gamma)u_{n-1} + (1 + \gamma)v'_{n-1},
\]
the intersection is spanned by $u'_n - u'_{n-1}$, which is $\gamma(1 + \gamma)$ times $(1 + \gamma)v'_{n-1} - v'_n$.

By the previous lemma, $\Sigma$ contains an isomorphic copy of $\mathfrak{s}_4$, which proves the following lemma:

**Lemma 6.24.** $\Sigma$ contains a 4-tuple $(T_a, T_b, T_c, T_d)$ of projectively distinct infinitesimal Siegel transvections such that $T_c$ and $T_d$ do not commute, but every other pair does.

**Lemma 6.25.** Let $T_a = T_{u_a, v_a} \in \Sigma$ and $T_b = T_{u_b, v_b} \in \Sigma$ satisfy $C_\Sigma(T_a) = C_\Sigma(T_b)$. Then $T_a = T_b$.

**Proof.** Since $G$ is transitive on $\Sigma$, we may assume that $T_a = T_{u_2, v_2} = T_{u_2', v_2'}$. Let us denote the subspace of $V$ perpendicular with respect to the bilinear form $B$ to a vector $u \in V$ by $u^\perp$, and similarly, let us denote the subspace of $V$ perpendicular to a subspace $S$ of $V$ by $S^\perp$.

Recall from Lemma 6.23 the definitions of $U$ and $U^\perp$. Pick a nonzero vector $u \in U$ which is perpendicular to $v'_2$, and let nonzero $v \in U^\perp$ be perpendicular to $u$ and to $u'_2$. Then the infinitesimal transvection $u \otimes v = T_{u, v}$ is in $\mathfrak{s}(U)$, hence in $\Sigma$, and it commutes with $T_{u_2, v_2}$. So $T_b$ should also commute with it. Then $(u_b, v_b)_F$ either intersects $(u, v)_F$, or $u_b$ and $v_b$ are both perpendicular to $u$ and $v$.

If $(u_b, v_b)_F$ does not intersect all $(u, v)_F$ for fixed $u$ and all nonzero $v \in S := u^\perp \cap u'_2 \cap U^\perp$, then $u_b$ and $v_b$ are perpendicular to $u$. Let us consider the case where $(u_b, v_b)_F$ intersects every such $(u, v)_F$. We will show that $u_b$ and $v_b$ are perpendicular to $u$ in this case as well. $S$ has dimension 1 or 2 in $U^\perp$ of dimension $n - 1$, so its dimension is at least 2. If dim $S > 2$, then $(u_b, v_b)_F$ must contain $u$ to intersect every $(u, v)_F$. In that case, $u_b$ and $v_b$ are certainly perpendicular to $u$. Hence assume dim $S = 2$ and $u \not\in (u_b, v_b)_F$. Then for different lines $(u, v)_F$, the intersection with $(u_b, v_b)_F$ is different. So the intersections span all of $(u, v)_F$. In particular, $u_b$ and $v_b$ themselves are on lines $(u, v)_F$. Thus both are perpendicular to $u$.

We see that $u_b$ and $v_b$ are perpendicular to all of $v'_2 \cap U$. Similarly, they are perpendicular to $v'_2 \cap \hat{U}$; that is, they are perpendicular to $S_u := v'_2 \cap (U + \hat{U})$. Dually, we see that $u_b$ and $v_b$ are perpendicular to $S_v := u'_2 \cap (U^* + \hat{U}^*)$.

For $B_{n-1}$, the intersection of $U + \hat{U}$ and $U^* + \hat{U}^*$ is spanned by $u'_n - u'_{n-1}$, which is perpendicular to both $u'_2$ and $v'_2$ for $D_n$, the intersection of $U + \hat{U}$ with $U^* + \hat{U}^*$ is empty. So $S_u + S_v$ is a 2n - 2-dimensional space for $D_n$ and to a 2n - 3-dimensional space for $B_{n-1}$. Since the form is nondegenerate, there is in both cases only a 2-dimensional space of vectors perpendicular to $S_u + S_v$. This space is $(u'_2, v'_2)_F$. Hence $u_b$ and $v_b$ are in $(u'_2, v'_2)_F$. □

**Lemma 6.26.** The graph $F(\Sigma)$ with vertex set $\Sigma$ and where two infinitesimal Siegel transvections are adjacent if they generate an algebra isomorphic to $\mathfrak{sl}_2$, is connected.

**Proof.** According to Lemma 2.13 of [Tim01], if $|\Sigma| > 1$, then $F(\Sigma)$ is connected if and only if $\Sigma$ is a conjugacy class in $G$ and $F(\Sigma)$ has an edge. Both of these conditions are fulfilled. □

**Lemma 6.27.** $G$ is a quasisimple group.

**Proof.** We use Lemma 2.14 of [Tim01]. A weaker version of it states that if the following conditions are satisfied:

- $|\Sigma| > 1$;
- the graph $F(\Sigma)$ from Lemma 6.23 is connected;
- there exists no pair $A \neq C \in \Sigma$ with $C_\Sigma(A) = C_\Sigma(B)$;
- $\Sigma$ contains an extraspecial pair;
- the elements of $\Sigma$ correspond to extremal Lie algebra elements;

then $G$ is a quasisimple group. Lemmas 6.22, 6.25, and 6.26 show that the three nontrivial conditions are fulfilled. □

The technical proof of the following lemma uses the previous lemmas and has been omitted here for brevity. It can be found in [Pos07].

**Lemma 6.28.** $G$ satisfies conditions (H2) from Theorem 6.16 and (X3) from Theorem 6.17.
Proof of Theorems 6.18 and 6.19. We intend to apply Theorem 6.17, so we will need to show that its conditions hold.

- Condition (H1') follows from the fact that for every Siegel transvection subgroup, we have $A^0 = 1$ or $A^0 = A$.
- Condition (H2) follows from Lemma 6.28.
- Condition (H3) follows from Lemma 6.24.
- Condition (X1) is clearly satisfied.
- Condition (X2) follows from Lemma 6.22.
- Condition (X3) follows from Lemma 6.22.
- Condition (X4) is true for every Siegel transvection subgroup.
- Condition (X5) is clearly satisfied.

7. The algebras nearly always correspond to these realizations

In this section, we show that a Lie algebra $L$ realizing one of the graphs $\Gamma_{B_n}$, $\Gamma_{B_{n-1}}$, $\Gamma_{C_n}$ and $\Gamma_{D_n}$, is in the generic case a quotient of the realization $M$ we found in the previous section. In order to see this, we find different sets of generators. For types $B_{n-1}$ and $D_n$, we will need the parameters $\alpha$, $\beta$ and $\gamma$ from Theorems 6.18 and 6.19 to have sufficient degrees of freedom to be able to make the sets of generators of $L$ and $M$ match up.

Since $M$ is simple in most cases, it will follow that $L$ and $M$ are isomorphic. The only exception is $A_{n-1}$ if $p \mid n$, as is well known from literature (see e.g. [Hum72] and [Str04]).

Theorem 7.1. Let $L = \langle x_1, \ldots, x_n \rangle_{\text{Lie}}$ be any Lie algebra realizing the graph $\Gamma_{C_n}$ from Figure 1.3. Suppose that $n$ is even and that $f(x_i, x_{i+1}) \neq 0$ for all $i$. Then $L$ is isomorphic to $\mathfrak{sp}_n$.

Proof. Let $M$ be the realization from Theorem 6.10. By that Theorem, $M = \mathfrak{sp}_n$. Denote the generators of $M$ realizing $\Gamma_{C_n}$ by $z_i$. Scale $x_i$ such that the extremal form is identical on both sets of generators. Then the map $\sigma: M \to L$ mapping each of the monomials $y_{k,m}$ in $z_i$ to the same monomial in $x_i$, is a Lie algebra homomorphism by Lemma 5.4. Hence $L$ is a quotient of $M$. But since $M$ is simple, $L \cong M$. □

Already for Lie algebras of type $A_{n-1}$, we need more degrees of freedom than can be obtained by just scaling the generators. The following lemma, which is based on Section 5 of [CSUW01], will be sufficient.

Lemma 7.2. Let $\pi, \rho, \sigma \in F$ all be nonzero. Let $x$, $y$, $z$ be extremal elements of $L$ such that $f(x,y)^2 \neq 2f(x,y)f(x,z)f(y,z)$ and $f(x,y) \neq 0 \neq f(y,z)$, and such that $x$ and $y$ commute with a set $S$ of elements of $L$. We can find extremal elements $x'$, $y'$ and $z'$ with the following properties:

- $\langle x, y, z \rangle_{\text{Lie}} = \langle x', y', z' \rangle_{\text{Lie}}$.
- $x'$ and $y'$ commute with $S$.
- $(f(x', y'), f(x', z'), f(y', z'), f(x', y'z')) = (\pi, \rho, \sigma, 0)$.

Proof. Let $s = f(x, yz)/(f(x, y)f(y, z))$. Let $\hat{x} = \exp(s \, \text{ad} \, y)(x)$. Then $\langle x, y, z \rangle_{\text{Lie}} = \langle \hat{x}, y, z \rangle_{\text{Lie}}$ (since $\exp(-s \, \text{ad} \, y)(\hat{x}) = x$) and

$$f(\hat{x}, y) = f(x, y),$$
$$f(\hat{x}, z) = f(x, z) - \frac{f(x, yz)^2}{2f(x, y)f(y, z)},$$
$$f(\hat{x}, yz) = 0.$$
Note that $f(\hat{x}, \hat{z}) \neq 0$. We drop the circumflex from now on and use $x$ to denote $\hat{x}$. Scale $x$, $y$, and $z$ to obtain $\check{x}$, $\check{y}$ and $\check{z}$:

\[
\check{x} = \sqrt{\frac{\pi \rho f(y, z)}{\sigma f(x, y)f(x, z)}},
\check{y} = \sqrt{\frac{\pi \sigma f(x, z)}{\rho f(x, y)f(y, z)}},
\check{z} = \sqrt{\frac{\rho \sigma f(x, y)}{\pi f(x, z)}}.
\]

Now all values of $f$ are as intended, possibly up to a factor of $-1$, and $f(\check{x}, \check{y})f(\check{x}, \check{z})f(\check{y}, \check{z}) = \pi \rho \sigma$.

So either all values of $f$ are exactly as intended (including sign), in which case we are done, or exactly two of them need their sign changed; say $f(\check{x}, \check{y})$ and $f(\check{x}, \check{z})$. Then let $x' = -\check{x}$, $y' = \check{y}$ and $z' = \check{z}$.

$y'$ commutes with $S$, since it is merely a scaled version of $y$. By the Jacobi identity, $[x, y]$ commutes with $S$, as well; hence $x' \in \langle x, [x, y] \rangle$ commutes with $S$.

**Theorem 7.3.** Let $\mathcal{L} = \langle x_1, \ldots, x_n \rangle_{\text{Lie}}$ be a realization of the graph $\Gamma_{A, n}$ in Figure 1.1. Suppose that the following Zariski-open conditions on the extremal form hold:

- $f(x_1, x_2 x_3)^2 \neq 2f(x_1, x_2)f(x_1, x_3)f(x_2, x_3)$,
- $f(x_i, x_{i+1}) \neq 0$ for all $i$.

Then:

- if $\text{char} \mathbb{F} = p > 0$ and $p \mid n$, then $\mathcal{L}$ is isomorphic to either $\mathfrak{sl}_n$ or its simple subalgebra of codimension $1$;
- otherwise, $\mathcal{L}$ is isomorphic to $\mathfrak{sl}_n$.

**Proof.** Let $\mathcal{M}$ be the realization from Theorem 6.11. By that theorem, $\mathcal{M} = \mathfrak{sl}_n$. Denote the generators of $\mathcal{M}$ realizing $\Gamma_{A, n}$ by $y_i$. We will exhibit a Lie algebra homomorphism from $\mathcal{M}$ to $\mathcal{L}$, showing that $\mathcal{L}$ is a quotient of $\mathcal{M}$. Since $\mathcal{L}$ cannot be one-dimensional, we then have the desired result.

Apply Lemma 7.2 with $(\pi, \rho, \sigma) = (1, 1, 1)$ and $(x, y, z) = (x_1, x_2, x_3)$. We obtain a new set of generators $x'_i$ of $\mathcal{L}$ that still realize $\Gamma_{A, n}$. Also apply Lemma 7.2 with $(x, y, z) = (y_1, y_2, y_3)$ and with the same values for $\pi$, $\rho$ and $\sigma$, obtaining new generators $y'_i$. Now for any pair of monomials in $y'_1$, $y'_2$ and $y'_3$, the extremal form on that pair is equal to the extremal form on the corresponding pair of monomials in $x'_1$, $x'_2$ and $x'_3$. For $i > 3$, inductively define $x'_i = f(y_{i-1}, y_i)f(x'_{i-1}, x_i)^{-1}x_i$ and $y'_i = y_i$. Now the extremal form is equal on all pairs of monomials given by $\psi_{\Gamma_{A, n}}(f)$, so the extremal form is identical. Then the desired Lie algebra homomorphism can be obtained by mapping $x'_i$ to $y'_i$.

**Theorem 7.4.** Let $\mathcal{L} = \langle x_1, \ldots, x_n \rangle_{\text{Lie}}$ be a realization of the graph $\Gamma_{D, n}$ in Figure 1.4. Then $\mathcal{L}$ is isomorphic to $\mathfrak{so}_2 n$, if the values of the extremal form satisfy a number of Zariski-open conditions.

These open conditions can be found as follows. If $n$ is odd,

- we have the condition $f(x_1, x_{3i} - x_{3i+2}) \neq 8$,
- furthermore, we define $\alpha$ by Eq. (7.5) and $\beta$ by Eq. (7.4);

if $n$ is even,

- we define $\alpha$ by Eq. (7.8) and $\beta$ by Eq. (7.7);

then the (other) conditions are

- $f(x_1, x_2 x_3)^2 \neq 2f(x_1, x_2)f(x_1, x_3)f(x_2, x_3)$,
- $f(x_n, x_{n-1} x_{n-2})^2 \neq 2f(x_n, x_{n-1})f(x_n, x_{n-2})f(x_{n-1}, x_{n-2})$,
- $f(x_i, x_{i+1}) \neq 0$ for all $i$,
- $(\alpha + 2)\beta(\beta + 1) \neq 0$,
- $\lambda(2 - \beta + \lambda \beta) \neq 1$. 

Now there are \( n \) Let Figure 7.2: Note that if Figure 1.4.

otherwise. For the generators that occur twice, viz. \( z \) multiply generators by a factor; in particular, for \( (\pi, \rho, \sigma) \) which the value of \( f \) on \( M \) is divided by \( (2, 2, 1) \). Now there are \( n - 5 \) pairs of elements left (on the “connecting line between the two triangles”), on which the value of \( f \) has not yet been adjusted, and additionally \( f(x_1, x_{3|n-2} x_{1|2}) \). We assume that \( f(x_{i-1}, x_i) \neq 0 \) for \( 4 \leq i \leq n - 3 \) and scale \( x_4 \) up to \( x_{n-3} \) such that \( f(x_{i-1}, x_i) = 2 \). This leaves \( f(x_{n-3}, x_{n-2}) \) and \( f(x_1, x_{3|n-2} x_{1|2}) \). The values of \( f \) other than \( f(x_1, x_{3|n-2} x_{1|2}) \) are as given in Figure 7.1.

We now perform a similar procedure on \( M \). Call the \( i \)th extremal generator \( z_i = T_{u_i, v_i} \). The values of \( f \) prior to any changes are as given in Figure 7.2. We perform the procedure of Lemma 7.2 on \( z_1, z_2 \) and \( z_3 \), with \( (\pi, \rho, \sigma) = (-8, 1, 2) \). Note that \( s = \alpha/4 \). In the first step, \( z_1 \) becomes

\[
z_1 = \frac{\alpha}{4} [z_1, z_2] - \frac{\alpha^2}{4} z_2 = T_{u_1, \frac{\alpha}{4} u_2, v_1 + \frac{\alpha}{4} v_2}.
\]

Now

\[
f(z_1, z_2) = -8,
\]

\[
f(z_2, z_3) = 2,
\]

so \( z_1 \) and \( z_3 \) are divided by \( (\alpha + 2)/\sqrt{2} \) and \( z_2 \) is multiplied with that same constant.

At the other end, we find that applying the procedure of Lemma 7.2 only entails scaling the generators by a factor; in particular, for \( (\pi, \rho, \sigma) = (2, 2, 1) \), we divide \( z_{n-1} \) and \( z_n \) by \( \sqrt{2} - 2 \) and multiply \( z_{n-2} \) by the same factor. Finally, to obtain a 2 for the value of \( f(z_{i-1}, z_i) \) where \( 4 \leq i \leq n - 3 \), we multiply \( z_i \) by \( (\alpha + 2)/\sqrt{2} \) for each \( i \) and divide it by that factor for odd \( i \). Hence

\[
f(z_{n-3}, z_{n-2}) = 2(\alpha + 4)^{(\alpha + 2)} / \sqrt{2} - 2 - \beta.
\]  

Now consider \( z = z_{3|n-2} z_{1|2} \). The generators involved have been scaled, but not changed otherwise. For the generators that occur twice, viz. \( z_3, z_4, \ldots, z_{n-2} \), their scaling factor affects \( z \) twice; the scaling factors of the other three \( (z_2, z_{n-1}, \text{ and } z_n) \) have an effect only once. In total, \( z \) was multiplied by \( (\alpha + 2)/\sqrt{2} \) by all these scalings, if \( n \) is odd, and divided by that constant if \( n \)
is even. We will now compute the value of \( z \) explicitly. This is easier if we temporarily forget all the scaling that occurred; so until further notice, we will use the values before scaling of the \( z_i \).

With induction it is easy to see that \( z_k|2 = (-1)^k(T_{u_k,v_2} + T_{u_2,v_k}) \) for \( 3 \leq k < n \). Then

\[
z_{n|2} = (-1)^{n-1}(\{z_n, T_{u_{n-1},v_2} + z_n, T_{u_2,v_{n-1}}\})
= (-1)^n(T_{u_n,v_2} - \beta T_{v_2,v_2} - T_{u_n,v_2} + T_{u_n,v_2} = (-1)^n(T_{u_n,u_2+v_2} - T_{u_2-\beta v_2,v_k}).
\]

We can again use induction to see that \( z_{k|n-2}z_{n|2} = (-1)^n(T_{u_k,u_2+v_2} - T_{u_2-\beta v_2,v_k}) \) for \( 4 \leq k \leq n-2 \). Finally, we compute

\[
z = z_{3|n-2}z_{n|2} = (-1)^n(\{z_3, T_{u_4,u_2+v_2} - z_3, T_{u_2-\beta v_2,v_4}\})
= (-1)^n(-T_{u_4,v_3} - T_{u_3,u_2+v_2} + T_{u_3,u_4} - T_{u_2-\beta v_2,v_3} + T_{u_1,v_4} - \beta T_{v_3,v_4})
= \begin{cases} (-1)^n(T_{u_3,u_2+v_2} + T_{u_3,u_2+v_2} + T_{u_3,u_2+v_2} - T_{u_2-\beta v_2,v_3})z_1, & \text{if } n \text{ is odd}, \\ 2(\alpha + 2)^n f(T_{u_1-\Phi u_2,v_1+\Phi v_2, T_{u_2-\beta v_2,u_4-\beta v_4,v_3} - T_{u_3,u_2+v_2+u_4+v_4} - T_{u_2-\beta v_2,v_3}z_1, & \text{if } n \text{ is even}. \\ \end{cases}
\]

A straightforward computation shows that

\[
f(T_{u_1-\Phi u_2,v_1+\Phi v_2, T_{u_2-\beta v_2,u_4-\beta v_4,v_3} - T_{u_3,u_2+v_2+u_4+v_4}) = 4\alpha(1 + \beta) + 8.
\]

We finish the proof by case distinction.

**Case 1:** Suppose that \( n \) is odd. Taking the previous equations together with Eq. (7.3), we need to choose \( \alpha \) and \( \beta \) such that

\[
f(x_1, x_{3|n-2}x_{n|2}) = 4\alpha(1 + \beta) + 8
\]

and

\[
f(x_{n-3}, x_{n-2}) = (2\alpha + 4)\sqrt{-1 - \beta}.
\]

If \( f(x_1, x_{3|n-2}x_{n|2}) = 8 \), then we choose \( \alpha = 0 \) and \( \beta = -1 - f(x_{n-3}, x_{n-2})^2/16 \), otherwise we use Eq. (1.1) to obtain

\[
1 + \beta = \frac{f(x_1, x_{3|n-2}x_{n|2}) - 8}{4\alpha},
\]

substituting this into Eq. (1.3), we obtain

\[
(\alpha + 2)^\frac{8 - f(x_1, x_{3|n-2}x_{n|2})}{\alpha} = f(x_{n-3}, x_{n-2})
\]

or

\[
\frac{\sqrt{\alpha} + 2}{\sqrt{\alpha}} = \frac{f(x_{n-3}, x_{n-2})}{\sqrt{8 - f(x_1, x_{3|n-2}x_{n|2})}}.
\]

This is a quadratic equation in \( \sqrt{\alpha} \) that can easily be solved to

\[
\alpha = \pm \left( f(x_{n-3}, x_{n-2}) \pm \sqrt{f(x_{n-3}, x_{n-2})^2 + 8f(x_1, x_{3|n-2}x_{n|2}) - 64} \right)^2.
\]

four solutions (some of which may coincide). Then \( \beta \) can be found from Eq. (7.4). Now all values of \( f \) are the same and hence \( L \) is a quotient of \( M \). Since \( M \) is simple, \( L = M \).
Theorem 7.5. Let $z$ be isomorphic to $\circ$. Let $\gamma$ be the realization defined in Theorem 6.19; we will specify the value of $\gamma$ later. Let $T_{u, \gamma}$ be the $i$th extremal generator of $M$. In the same manner as in the proof of Theorem 7.3, we change $x_i$ such that $f(x_i, x_j)$ is equal to those of $f(z_i, z_j)$ for $i, j \leq 3$, and such that $f(x_i, x_2 x_3) = f(z_i, z_2 z_3)$. By scaling $x_i$, we can assure that $f(x_{i-1}, x_i) = f(z_{i-1}, z_i)$ for $i < n$, and by scaling $x_n$ we can assure that $f(x_{n-2}, x_n) = f(z_{n-2}, z_n)$. Then what remains is assuring that $f(x_1, x_3\ldots x_{n-2} x_{n-1}) = f(z_1, z_3\ldots z_{n-2} z_{n-1})$.

Like in the proof of Theorem 7.3, we will do this by choosing the value of $\gamma$ appropriately. This requires explicitly constructing $z_{3\ldots n-2} z_{n-1}$. Again like before, for $3 \leq k < n$ we find with induction that $z_{n-2} = (\cdots (-1)^k T_{u_5, v_5} + T_{u_1, v_1})$. Then $z_{n-2} = (-1)^{n-1} (\cdots (-1)^k T_{u_5, v_5} + T_{u_1, v_1}) = (-1)^{n-1} T_{u_5, \gamma v_5+v_5}$. With induction we see that $z_{3\ldots n-2} z_{n-1} = (-1)^{n-1} T_{\gamma u_5 + v_5, \gamma v_5 + v_5}$ for $3 < k \leq n - 2$. Then $z_{3\ldots n-2} z_{n-1} = (-1)^{n-1} (\cdots (-1)^k T_{u_5, v_5} + T_{u_1, v_1}) = (-1)^{n-1} T_{\gamma u_5 + v_5, \gamma v_5 + v_5}$. Then $f(z_1, z_{3\ldots n-2} z_{n-1}) = 2(B(u_1, \gamma u_3 + v_3) B(v_1, \gamma u_2 + v_2 + \gamma v_3) B(u_1, \gamma u_2 + v_2 + \gamma u_4 + v_4))$.

So we choose $\gamma$ to be $(-1)^n 8 / 8$ times the value of $f(x_1, x_3\ldots x_{n-2} x_{n-1})$. Then $f$ is identical on $L$ and $M$, so $L$ is a quotient of $M$; since $M$ is simple, they are isomorphic.

For each of the four infinite families of Chevalley type Lie algebras, we have given a family of graphs such that a generic Lie algebra generated by a set of extremal elements realizing such a graph, is isomorphic to the corresponding Lie algebra.

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