Resolution for hybrid logics

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Abstract. We describe a resolution method and a procedure to transform formulae of some pure hybrid logics into their clausal form.

Keywords: hybrid logic, resolution, clause.

Introduction

Before applying the resolution method to the formulas of classical logic we transform them into clausal form. Well-known transformation methods of the formulas of classical logic are not suitable for neither modal nor hybrid logics. Transformation of formulas of hybrid logics needs a different approach. In [6, 7] Mints et al. describe transformation of formulae into their clausal form for modal logics $S_4$ and $S_5$. A modal literal is defined as formula of the form $l$, $\Box l$ or $\Diamond l$, where $l$ is a propositional literal. A modal clause is a disjunction of modal literals. We prove that for every modal logic formula $F$ there exist clauses $D_1, \ldots, D_n$ and a propositional literal $l$ such that sequent $\vdash F$ is derivable in sequent calculus $S_4$ (and, accordingly, $S_5$) if and only if sequent $\Box D_1, \ldots, \Box D_n, l \vdash$ is derivable. This transformation is the basis for the resolution calculus for modal logic $S_4$ presented in [7]. $F$ is a tautology if and only if an empty clause is derivable from the set $\{\Box D_1, \ldots, \Box D_n, l\}$. The paper [10] describes a procedure to transform formulae of hybrid logic $\mathcal{H}(\Diamond)$ over transitive and reflexive frames into their clausal form. This paper shows how we can transform formulae of hybrid logics $\mathcal{H}(\Diamond, \downarrow)$, $\mathcal{H}(\Diamond, \exists)$, $\mathcal{H}(\Diamond, E)$ into their clausal form. We also describe resolution method for the hybrid logics $\mathcal{H}(\Diamond, \downarrow)$, $\mathcal{H}(\Diamond, E)$. For more information about hybrid logic and its properties see [2, 3, 4, 5].

1 Transformation

A literal of hybrid logic $\mathcal{H}(\Diamond)$ is a formula of the form $l$, $\Box l$, $\Diamond l$, $\Diamond l$ (where $l$ is a propositional variable, nominal or their negation; $i$ is nominal). In addition, $\forall x l$, $\exists x l$ are also literals in the logic $\mathcal{H}(\Diamond, \exists)$, $A l$, $E l$ — in the logic $\mathcal{H}(\Diamond, E)$, $\downarrow x l$ — in the logic $\mathcal{H}(\Diamond, \downarrow)$.

A clause is a formula of the form $L$, $\Box L$, $\Diamond L$ (where $L$ is a disjunction of hybrid literals). In addition, a formula of the form $\Diamond l, \forall x L$ is clause of logic $\mathcal{H}(\Diamond, \exists)$ and a formula $\Diamond l, \downarrow x L$ is clause of logic $\mathcal{H}(\Diamond, \downarrow)$.

First of all, consider a formula of logic $\mathcal{H}(\Diamond)$. We transform a subformula of the form $\Diamond G$ into clause $\Diamond l \lor a$ (where $a$ is a new propositional variable). We
transform other subformulas similarly as described in [2, 3, 10]. If any subformula
occur in the scope of modal operators □, ◦ we write @z in the front. In the following
we assume that the variable z is reserved, that is, z does not occur in the formulas
under consideration.

Assume we want to know whether a sequent ⊢ F has a deduction in the sequent
calculus H presented in [9, 8]. We denote all the possible subformulas, with the
exception of nominals, by the new propositional variables a, b, c, . . . . To the goal
formula F assign letter a. Assume the formula F is in negation normal form, that is,
the formula contains logical connectives only from the list ¬, ∨, ∧ and the negation
symbol appears only in front of nominals. Recall that in this paper we transform
only formulas of pure hybrid logic. Obtainable after transformation clauses contain
propositional variables and nominals. Formula F has one of the forms: 1. G ∧ H,
2. G ∨ H, 3. □ H, 4. ◦ H, 5. @H. Suppose the variable b assigned to the formulas
G, □H, ◦H, @H and variable c assigned to formula H.

In the first case formula is derivable if and only if ¬a, a ∨ ¬b, a ∨ ¬c ⊢ is derivable.
In the second case if sequent ¬a, a ∨ ¬b, a ∨ ¬c ⊢ is derivable.

We add new rule
\[ \frac{Γ, @zG, @z(u ∨ G)}{Γ, @z(u ∨ G), @z¬u} \]
to the calculus H and denote them by H′. In the third case ⊢ F is derivable in the
calculus H iff ¬a, @z(a ∨ ◦¬b) ⊢ is derivable in calculus H′. Where u is a nominal.
In addition, we can apply only the rule @z to the formulas beginnig with @z.

In the fourth case, if sequent ¬a, @z(a ∨ □¬b) ⊢ is derivable in calculus H′. In the
fifth case, if sequent ¬a, a ∨ @z¬b ⊢ is derivable.

We apply the transformation to subformulas b, c and its components as long as
it possible. We will say that list of obtained clauses D1, . . . , Dn corresponds to
formula F. We get the following result.

**Theorem 1.** For any formula F of logic H(@) a sequent ⊢ F is derivable in H iff a
sequent D1, . . . , Dn ⊢ corresponding to F is derivable in H′.

**Example 1.** F = □◦(i ∨ j) ∧ ◦¬j.

Let the letters e, c, f, d, b, a denote the subformula of, respectively ¬j, ◦¬j, i ∨ j,
◦(i ∨ j), □◦(i ∨ j), F.

The following list of clauses corresponds to formula F: ¬a, a∨¬b∨¬c, @z(b∨□¬d),
@z(d ∨ □¬f), @z(f ∨ ¬j), @z(f ∨ ¬j), @z(e ∨ ¬c), @z(e ∨ j).

The case of logic H(@, E) is treated in a similar manner.

**Example 2.** F = □E◦(i ∨ j) ∧ A◦¬j. Let the letters e, c, c′, f, d, d′, b, a denote the
subformula of, respectively ¬j, ◦¬j, A◦¬j, i ∨ j, ◦(i ∨ j), E◦(i ∨ j), □E◦(i ∨ j), F.

The following list of clauses corresponds to formula F: ¬a, a ∨ ¬b ∨ ¬c′, @z(b ∨
□¬d′), @z(d′ ∨ A¬d), @z(d ∨ □¬f), @z(f ∨ ¬i), @z(f ∨ ¬j), @z(c′ ∨ E¬c), @z(e ∨ □¬c),
@z(e ∨ j).

The new propositional variables become in logics H(@, 3), H(@, ↓) functions of
nominal variables.
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Examples.
1. Logic $\mathcal{H}(\oplus, 3)$. We transform formula $\Box \exists x \Diamond (x \lor j) \land \forall y \Diamond \neg y$.

We introduce new variables $a, b, c, d, e, f, c'$ (where $z$ is reserved) to get a set of clauses.

Consider the following list of subformulas: $\neg y, \Diamond \neg y, \forall y \Diamond \neg y, x \lor j, \Diamond (x \lor j), \exists x \Diamond (x \lor j), \Box \exists x \Diamond (i \lor j), F$ and denote them respectively by $e, e', f, d, a, d, a$.

The following list of clauses corresponds to formula $F$: $\neg a, a \lor \neg b \lor \neg c', @z(b \lor \Diamond \neg d'), @z(d' \lor \forall x \neg d(x)), @z \forall x(d(x) \lor \Diamond \neg f(x)), @z \forall x(f(x) \lor \neg x), @z \forall x(f(x) \lor \neg j), @z(c' \lor \exists y \neg c(y)), @z \forall y(c(y) \lor \Box \neg e(y)), @z \forall y(e(y) \lor y)$.

2. Logic $\mathcal{H}(\oplus, \downarrow)$. We transform formula $\Box \downarrow x. \Diamond (x \lor j) \land \neg j$.

We introduce new variables $a, b, c, d, e, f, d'$ (where $z$ is reserved).

We have the following list of subformulas: $\neg j, \Diamond \neg j, x \lor j, \Diamond (x \lor j), \Box \downarrow x. \Diamond (x \lor j), F$ and denote them respectively by $e, c, f, d'$. The latter four clauses form a set.

The following list of clauses corresponds to formula $F$: $\neg a, a \lor \neg b \lor \neg c, @z(b \lor \Box \neg d'), @z(d' \lor \downarrow x. \neg d(x)), @z \downarrow x. (d(x) \lor \Box \neg f(x)), @z \downarrow x.(f(x) \lor \neg x), @z \downarrow x.(f(x) \lor \neg j)$.

2 Resolution method

We describe the rules of the resolution method for logics $\mathcal{H}(\oplus), \mathcal{H}(\oplus, \downarrow)$ and $\mathcal{H}(\oplus, E)$. We do not write the rules of the resolution method for a logic $\mathcal{H}(\oplus, 3)$ because we do not know skolemization in this logic. Rules for logic $\mathcal{H}(\oplus)$.

$z$ is reserved, $n$ is new.

\[
\begin{align*}
\frac{\@a(F \lor G)}{\@a F \lor \@a G} & \quad \frac{\@a i \lor H}{\@a i \lor \forall G} & \quad \frac{\@a i \lor \neg G}{\@a i \lor \forall G} & \quad \frac{\@a i \lor \neg u}{\@a i \lor \forall G} \\
\frac{\@a G \lor H}{\@a G \lor \forall H} & \quad \frac{\@a i \lor G \lor H}{\@a i \lor \forall G \lor H} \quad \frac{\@a i \lor \neg G \lor H}{\@a i \lor \forall G \lor H}
\end{align*}
\]

The logic $\mathcal{H}(\oplus, \downarrow)$ also contains a rule:

\[
\frac{\@a i \lor \neg G \lor H}{\@a i \lor \forall G \lor H}
\]

Logic $\mathcal{H}(\oplus, E)$ contains two new rules:

\[
\frac{\@a i \lor EF \lor H}{\@a i \lor \forall F \lor H} \quad \frac{\@a i \lor \neg F \lor H}{\@a i \lor \forall F \lor H}
\]

Theorem 2. A formula $\neg F$ of logics $\mathcal{H}(\oplus, \downarrow), \mathcal{H}(\oplus, E)$ is derivable in sequent calculus $H$ if and only if empty clause is derivable from set of clauses corresponding to formula $F$.

Proof. The described rules are an adaptation of the resolution method [1] to the clauses under consideration. ☐
Conclusions

The described transformation produces clauses of very simple form. Resolution method become much simpler.

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