Non-Abelian Discrete Flavor Symmetries on Orbifolds

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Abstract

We study non-Abelian flavor symmetries on orbifolds, $S^1/Z_2$ and $T^2/Z_3$. Our extra dimensional models realize $D_N$, $\Sigma(2N^2)$, $\Delta(3N^2)$ and $\Delta(6N^2)$ including $A_4$ and $S_4$. In addition, one can also realize their subgroups such as $Q_N$, $T_7$, etc. The $S_3$ flavor symmetry can be realized on both $S^1/Z_2$ and $T^2/Z_3$ orbifolds.
1 Introduction

Non-Abelian discrete flavor symmetries play an important role in particle physics. In particular, several non-Abelian discrete flavor symmetries such as $S_3$, $S_4$, $A_4$, $D_N$, $Q_N$, $T_7$, $\Delta(27)$ and $\Delta(54)$, have been used to derive lepton mass matrices with the large mixing angles as well as quark mass matrices. (See e.g [1–12] and also see for reviews [13, 14].) In addition to quark/lepton mass matrices, non-Abelian discrete flavor symmetries are also important to control supersymmetry (SUSY) breaking terms such as squark/slepton masses and A-terms. (See e.g. [15, 16].)

Some of non-Abelian discrete symmetries are symmetries of geometrical solids. Hence, we expect that non-Abelian discrete flavor symmetries would be originated from extra dimensional theories and superstring theories. Indeed, orbifolds have certain geometrical symmetries. Thus, field theories on orbifolds can realize non-Abelian discrete flavor symmetries and localized modes on fixed points of orbifolds correspond to certain reducible/irreducible representations [17, 18].

In general, it is not possible to realize desired nontrivial non-Abelian groups and representations purely by geometrical symmetry of the orbifold. However there is another room that we can assign different $Z_N$ charges to the fields of different elements in a multiplet. This results in enhanced non-Abelian discrete symmetries. Thus our classification is characterized as the following properties:

1. We specify the geometric distribution. On special locations, like the fixed points of toroidal orbifolds, fields are democratically distributed. They would have the symmetry of the solid or in general a permutation symmetry $S_N$ in the low energy Lagrangian.

2. Since fields are complex valued, we collect such fields on fixed points to form multiplets and allow complex representation, which is not necessary irreducible.

With these, we study how far the symmetry can go. The second assignment easily mimics the symmetry from stringy selection rules. For example, localized (twisted) strings at orbifold fixed points have definite $Z_N$ charges, which control the selection rules of allowed couplings. Combinations between such $Z_N$ symmetries and geometrical symmetries enhance flavor symmetries. Then, in heterotic string orbifold models we can realize non-Abelian discrete flavor symmetries larger than geometrical symmetries of orbifolds such as $D_4$ and $\Delta(54)$ [16,19,20].

It is important to extend the above analysis. We study field theories on orbifolds. Inspired by heterotic string orbifold models, we assume $Z_N$ charges for modes localized on fixed points. Only limited kinds of $Z_N$ charges can be realized in heterotic string orbifold models. That is the reason why there are a strong restriction on realization of non-Abelian discrete flavor symmetries in heterotic orbifold models. However, here we consider localized modes with generic $Z_N$ charges. Those localized modes would have symmetries, which are combinations between $Z_N$ symmetries and geometrical symmetries

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1 Similar symmetries can be realized in magnetized/intersecting brane models [21].
of orbifolds. Then, those localized modes correspond to definite reducible/irreducible representations, while the bulk modes would correspond to (trivial) singlets in our theory like heterotic orbifold models.

We are interested in three-generation models. Thus, we consider orbifolds with two or three fixed points, because the number of fixed points corresponds to the number of generations. For the orbifold with two fixed points, we expect one generation may originate, e.g. from a bulk field. Hence, we restrict ourselves to $S^1/Z_2$ and $T^2/Z_3$ orbifolds, which have two and three fixed points, respectively. (See for geometrical aspects of other orbifolds, e.g. [22].) The $T^2/Z_4$ orbifold also has two fixed points of the same deficit angles, which has the $Z_2$ permutation symmetry, and would lead to the same results as ones on the $S^1/Z_2$ orbifold. Also note that the orbifold with a single fixed point does not lead to non-Abelian discrete flavor symmetries in our approach. Thus, we do not consider such orbifolds like, $T^2/Z_6$. Using $S^1/Z_2$ and $T^2/Z_3$ orbifolds, we study which flavor symmetries can appear and which representations can be realized.

This paper is organized as follows. In section 2 and 3, we consider $S^1/Z_2$ and $T^2/Z_3$ orbifolds, respectively, and study which flavor symmetries appear and which representations can realized. Section 4 is devoted to conclusion and discussion.

2 $S^1/Z_2$ orbifold

Here, we consider the $S^1/Z_2$ orbifold, which has two fixed points. We denote fields localized at these two fixed points by $\phi_1$ and $\phi_2$. The orbifold has the geometrical $Z_2$ symmetry, which permutes these two fixed points. Such a permutation symmetry acts on two localized fields as

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$  (1)

on the basis $(\phi_1, \phi_2)^T$.

Here, we assume the Abelian $Z_N$ symmetry and the fields, $\phi_1$ and $\phi_2$, have $Z_N$ charges, $q_1$ and $q_2$. That is, such a $Z_N$ symmetry is represented as

$$A = \begin{pmatrix} \rho^{q_1} & 0 \\ 0 & \rho^{q_2} \end{pmatrix},$$  (2)

on the basis $(\phi_1, \phi_2)^T$, where $\rho = \exp(2\pi i/N)$.

The full flavor symmetry is a closed algebra including the generators $A$ and $B$. Such a symmetry would become non-Abelian symmetry, although each of the geometrical symmetry $B$ and the $Z_N$ symmetry $A$ corresponds to an Abelian symmetry. For example, when $\rho^{q_1} = 1$ and $\rho^{q_2} = -1$, such a symmetry is realized in heterotic string on the $S^1/Z_2$ orbifold and that corresponds to the $D_4$ symmetry [16][19][20]. The two fields $(\phi_1, \phi_2)^T$ correspond to a $D_4$ doublet $2$. Here we consider more generic values of $Z_N$ charges, $q_i$. We can think of different signs in each element of $B$, but they would be redundant.
Z\textsubscript{N} charges of \((\phi_1, \phi_2)\) & Representation of \(D_N\) \\ \hline \((k, -k) \pmod{N}\) & \(2_k\) \\ \hline \((0, 0)\) & \(1_+ + 1_-\) \\

Table 1: \(D_{N=\text{odd}}\) representations of the fields localized on the \(S^1/Z_2\) orbifold.

2.1 \(S_3\) and \(D_N\)

First, let us consider the model with \(q_1 = -q_2 \pmod{N}\), i.e.

\[
A = \begin{pmatrix} \rho' & 0 \\ 0 & \rho'^{-1} \end{pmatrix},
\]

where \(\rho' = \exp(2\pi i q_1/N)\). The generators \(A\) and \(B\) satisfy

\[
B^2 = I, \quad BAB = A^{-1},
\]

where \(I\) denotes the identity. Furthermore, when \(q_1 = 1\), we have

\[
A^N = I.
\]

Their closed algebra is \(D_N\), which has \(2N\) elements. All of \(D_N\) elements are written as \(A^m B^k\) with \(m = 0, \cdots, N-1\) and \(k = 0, 1\). When \(N = \text{odd}\), the \(D_N\) group has two singlets \(1_+\) and \((N-1)/2\) doublets \(2_k\) \((k = 1, \cdots, (N-1)/2)\) as irreducible representations. When \(N = \text{even}\), there are four singlets \(1_{\pm}\) and \((N/2 - 1)\) doublets, \(2_k\) \((k = 1, \cdots, (N/2 - 1))\).

When \(N = \text{odd}\), the localized fields \((\phi_1, \phi_2)\) with \(Z_N\) charges \((k, -k)\) correspond to the doublets \(2_k\), where \(k \neq 0 \pmod{N}\). When the localized fields \((\phi_1, \phi_2)\) have trivial \(Z_N\) charges, i.e. \(k = 0 \pmod{N}\), the generator \(A\) is represented trivially, i.e. \(A = I\). Then, we can use the basis \((\phi_1 + \phi_2, \phi_1 - \phi_2)\) such that the generator \(B\) is diagonalized as \(\text{diag}(1, -1)\). That is, these fields correspond to two singlets, \(1_+\) and \(1_-\). Note that the combination of \(1_+\) and \(1_-\) appears by the fields localized on two fixed points, but one of them can not appear without the other singlet. These results are shown in Table \(|1|\).

Note that \(S_3\) is isomorphic to \(D_3\). Thus, the \(S_3\) flavor symmetry is included in the above analysis and the localized fields with the \(Z_3\) charges \((1, 2)\) and \((0, 0)\) on two fixed points correspond to the doublet \(2\) and the combination \(1_+ + 1_-\), respectively, where the generator \(A\) is represented as \(A = \text{diag}(\omega, \omega^2)\) and \(A = \text{diag}(1, 1)\) with \(\omega = \exp(2\pi i/3)\). In the next section, we study another type of realization of \(S_3\) on the \(T^2/Z_3\) orbifold, where the fields localized on three fixed points of \(T^2/Z_3\) correspond to the combination, e.g. \(1_+ + 2\). Thus, we can realize the \(S_3\) flavor symmetry on both \(S^1/Z_2\) and \(T^2/Z_3\), but combinations of representations appearing on fixed points are different from each other.

Similarly, we can study the model with \(N = \text{even}\). The localized fields \((\phi_1, \phi_2)\) with \(Z_N\) charges \((k, -k)\) correspond to the doublets \(2_k\), where \(k \neq 0 \pmod{N/2}\). The fields with \(k = 0 \pmod{N}\) has the same situation as those for \(N = \text{odd}\) in the above case. In
addition, when $k = N/2$, the generator $A$ is represented as $A = -I$. Thus, we can use the field basis such that the generator $B$ is diagonalized as $\text{diag}(1,1)$. As a result, the localized fields $(\phi_1, \phi_2)$ with the trivial $Z_N$ charges $k = 0$ correspond to $1_{++}$ and $1_{--}$. On the other hand, the localized fields $(\phi_1, \phi_2)$ with the $Z_N$ charges $k = N/2 \ (\text{mod } N)$ correspond to $1_{+-}$ and $1_{-+}$. Here, signs in the subscripts mean as follows. The first sign denotes the eigenvalue of $B$ and the second sign denotes the eigenvalues of $AB$. Note that only the combination $1_{++} + 1_{--}$ or $1_{+-} + 1_{-+}$ appears as the fields localized on two fixed points, but other combinations or only one singlet can not appear. These results are shown in Table 2.

Here, we comment on bulk fields. The generator $B$ can be identified as the $Z_2$ reflection of $S^1/Z_2$. In general, bulk fields, e.g. vector and spinor fields, have $Z_2$ even and odd zero-modes. Those $Z_2$ even and odd zero-modes would correspond to a trivial singlet and a non-trivial singlet with $B = -1$. However, $Z_2$ odd zero-modes have no direct couplings with localized modes, while $Z_2$ even zero-modes can couple. Hence, among bulk modes, only trivial singlet fields would be useful in 4D effective field theory, at least in simple models. At any rate, a trivial singlet appears as a bulk field. For example, for $D_N$ with $N = odd$, we have claimed that whenever singlets appear on two fixed points, they always appear as the combination $1_+ + 1_-$. Here we can write mass terms between $1_+$ of the bulk and localized modes. Then, below such a mass scale, the light modes include only the trivial singlet $1_-$. This implies that any combinations of $1_+, 1_-$ and $2$ can appear on the $S^1/Z_2$ orbifold. Hence, we can realize any $S_3$ models and $D_N$ models with $N = odd$ on the $S^1/Z_2$ orbifold.

A similar comment is also available for $D_N$ with $N = even$. That is, the bulk fields, which would play a role, correspond to trivial singlets, and such trivial singlets have $D_N$-invariant mass terms with trivial singlets on the fixed points. Also these comments would be applicable to other flavor symmetries, which will be discussed in the following sections.

### 2.2 $\Sigma(2N^2)$

Next, let us consider more generic assignment of $Z_N$ chargers, i.e. $q_1 \neq -q_2 \ (\text{mod } N)$. In this case, the closed algebra corresponds to $\Sigma(2N^2)$, which is isomorphic to $(Z_N \times Z_N) \rtimes Z_2$. Non-trivial singlets of bulk fields may play an important role in complicated models, but here we do not consider such a possibility further.
The localized fields \((\phi_1, \phi_2)\) with \(Z_N\) charges \((m, n)\) for \(m \neq n\) correspond to doublets of \(\Sigma(2N^2)\), \(2_{m,n}\). On the other hand, when the localized fields \((\phi_1, \phi_2)\) have \(Z_N\) charges, \((m, m)\), the \(Z_N\) generator can be represented by \(\rho^n I\). For these fields, we can use the field basis diagonalizing the generator \(B\), i.e. \(\phi_1 \pm \phi_2\), where the generator \(B\) is represented by \(B = \pm 1\). Thus, these fields correspond to the combination of two singlets, \(1_{+m} + 1_{-m}\). These results are shown in Table 3.

The \(\Sigma(2N^2)\) groups include several subgroups and such subgroups can also be realized on the \(S^1/Z_2\) orbifold. One subgroup of \(\Sigma(2N^2)\) is the \(D_N\) group, and its realization on the \(S^1/Z_2\) orbifold has been studied in the previous section. Another non-trivial subgroup of \(\Sigma(2N^2)\) for \(N = \text{even}\) is the \(Q_N\) group. We define \(\tilde{A} = A^{-1}A'\) and \(\tilde{B} = BA^{N/2}\). By use of the algebra (7), it is found that they satisfy the following algebraic relations,

\[
\tilde{A}^N = I, \quad \tilde{B}^2 = \tilde{A}^{N/2}, \quad \tilde{B}^{-1}\tilde{A}\tilde{B} = \tilde{A}^{-1}. \tag{9}
\]

These are the algebraic relations of \(Q_N\) generators. Indeed, the closed algebra of \(\tilde{A}\) and \(\tilde{B}\) corresponds to \(Q_N\) and all of \(Q_N\) elements can be written by \(\tilde{A}^m\tilde{B}^k\) with \(m = 0, 1 \cdots, N - 1\) and \(k = 0, 1\). Thus, a proper breaking of \(\Sigma(2N^2)\) would lead to the \(Q_N\) flavor symmetry [14]. For example, a vacuum expectation value of the singlet scalar \(1_{-m}\) with \(m = \text{odd}\) would break the symmetries generated by \(B\) and \(A'\), but the above \(Q_N\) symmetry would remain. Then, the doublet \(2_{m+N/2,m}\) of \(\Sigma(2N^2)\) becomes \(2_k\) of \(Q_N\), while the doublet \(2_{m+k,m}\) of \(\Sigma(2N^2)\) decomposes to two singlets \(1_{+-} + 1_{--}\) in \(Q_N\).
Table 4: Decomposition of $\Sigma(2N^2)$ representations to $Q_N$ representations.

Furthermore, the singlets, $1_{+m}$ and $1_{-m}$, of $\Sigma(2N^2)$ correspond to the $Q_N$ singlets, $1_{-}$ and $1_{++}$, respectively. These results are summarized in Table 4.

Similarly, we could obtain other subgroups, e.g. $(Z_4 \times Z_2) \rtimes Z_2$, which is a subgroup of $\Sigma(32)$ [14].

3 $T^2/Z_3$ orbifold

Here, we consider the $T^2/Z_3$ orbifold, which has three fixed points. We denote fields localized at these three fixed points by $\phi_i$ ($i = 1, 2, 3$).

3.1 $S_3$

The $T^2/Z_3$ orbifold has a large geometrical symmetry compared with the symmetry of $S^1/Z_2$. First of all, there is a cyclic permutation symmetry among three fixed points. Such a cyclic permutation symmetry acts on three localized fields as

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

(10)

on the basis $(\phi_1, \phi_2, \phi_3)^T$. Obviously, this is the generator of the $Z_3$ symmetry, i.e. $B^3 = I$. In addition, the $T^2/Z_3$ orbifold has a reflection symmetry, where we exchange two fixed points each other with fixing the other fixed point. One of such reflections can be represented by

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

(11)

on the basis $(\phi_1, \phi_2, \phi_3)^T$. Being $C^2 = I$ and $CBC = B^{-1}$, the closed algebra of $B$ and $C$ correspond to $S_3$. Thus, the $S_3$ flavor symmetry can be realized on the $T^2/Z_3$ orbifold. The fields localized on three fixed points correspond to the combination of a singlet $1_+$ and a doublet $2$ of $S_3$. Alternatively, we may assume that the generator $C$ is represented by

$$C = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

(12)
on the basis \((\phi_1, \phi_2, \phi_3)^T\). In this case, the flavor symmetry is the same as \(S_3\), but the fields localized on three fixed points correspond to the combination of \(1\) and a doublet \(2\) of \(S_3\).

### 3.2 \(A_4\) and \(\Delta(3N^2)\)

Similar to the previous section, here, we assume that the localized fields \(\phi_i\) \((i = 1, 2, 3)\) have \(Z_N\) charges, \(q_i\), under an additional \(Z_N\) symmetry. Then, such a \(Z_N\) symmetry is represented as

\[
A = \begin{pmatrix}
\rho^{q_1} & 0 & 0 \\
0 & \rho^{q_2} & 0 \\
0 & 0 & \rho^{q_3}
\end{pmatrix},
\]

on the basis \((\phi_1, \phi_2, \phi_3)^T\), where \(\rho = \exp(2\pi i/N)\). For example, the \(Z_3\) symmetry with \(q_1 = 0, q_2 = 1\) and \(q_3 = 2\) can be realized in heterotic string models on the \(T^2/Z_3\) orbifold. Here, we consider more generic values of \(Z_N\) charges \(q_i\).

First, let us consider the model with \(q_1 = -q_3 = 1\) and \(q_2 = 0\), i.e.

\[
A = \begin{pmatrix}
\rho^{-1} & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

on the basis \((\phi_1, \phi_2, \phi_3)^T\), where \(\rho = \exp(2\pi i/N)\). For the moment, we assume that low-energy effective field theory has the cyclic permutation symmetry corresponding to \(B\), but not the \(Z_2\) reflection symmetry corresponding to \(C\). Hence, we study the closed algebra including \(B\) and \(A\) for Eq. (14). For example, we find \(B^{-1}AB = A'\), where

\[
A' = \begin{pmatrix}
\rho^{-1} & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Indeed, their closed algebra corresponds to \(\Delta(3N^2)\), which is isomorphic to \((Z_N \times Z_N') \rtimes Z_3\) \([9, 14]\). All of \(\Delta(3N^2)\) elements are written as \(B^kA^mA^n\) for \(k = 0, 1, 2\) and \(m, n = 0, 1, \ldots, N - 1\), and explicitly they are represented by

\[
\begin{pmatrix}
\rho^m & 0 & 0 \\
0 & \rho^n & 0 \\
0 & 0 & \rho^{-m-n}
\end{pmatrix}, \quad
\begin{pmatrix}
0 & \rho^m & 0 \\
0 & 0 & \rho^n \\
\rho^{-m-n} & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & \rho^m \\
\rho^n & 0 & 0 \\
0 & \rho^{-m-n} & 0
\end{pmatrix},
\]

where \(m, n = 0, 1, \ldots, N - 1\).

Let us study representations of localized fields under \(\Delta(3N^2)\) with smaller values of \(N\). The smallest and non-trivial model corresponds to \(N = 2\) and we obtain \(\Delta(12)\), which is isomorphic to \(A_4\). In this model, we assume that \(A\) in Eq. (14) generates the \(Z_2\) symmetry. The fields localized on three fixed points have the \(Z_2\) charges, \((q_1, q_2, q_3) = (1, 0, 1)\). They
correspond to an $A_4$ triplet 3. In addition, there can appear the localized fields, all of which have $Z_2$-even charges, $(q_1, q_2, q_3) = (0, 0, 0)$. On those localized fields, the generator $A$ is represented trivially as $A = I$. Thus, we can use the field basis diagonalizing the generator $B$, i.e.

$$
\phi_1 + \omega^n \phi_2 + \omega^{2n} \phi_3,
$$

for $n = 0, 1, 2$ and $\omega = e^{2\pi i/3}$. In this basis, the generator $B$ is represented by $B = \omega^n$ for $n = 0, 1, 2$. These correspond to three types of $A_4$ singlets, 1, 1’ and 1”. Note that singlets appear only in the combination, $1 + 1' + 1''$ for the localized fields. On the other hand, a trivial singlet 1 can appear as bulk modes. These results are shown in Table 5. For example, in the $A_4$ lepton model by Altarelli and Feruglio [3], three families of left-handed and right-handed charged leptons are assigned with 3 and $1 + 1' + 1''$, respectively, while Higgs fields correspond to trivial singlets 1. In addition, all of flavon fields correspond to trivial singlets 1 and triplets 3. Such representations can be realized on the $T^2/Z_3$ orbifold model with proper $Z_2$ charges.

Next, we consider the model with $N = 3$, where we obtain $\Delta(27)$ $^3$. In this model, we assume that $A$ in Eq. (14) generates the $Z_3$ symmetry. Then, the fields localized on three fixed points have the $Z_3$ charges, $(q_1, q_2, q_3) = (1, 0, 2)$, and they correspond to a triplet $3_1$. In addition, there are also the localized fields with $Z_3$ charges, $(q_1, q_2, q_3) = (2, 0, 1)$ and they correspond to another triplet $3_2$. Moreover, there are also the localized fields with $Z_3$ charges, $(q_1, q_2, q_3) = (m, m, m)$, for $m = 0, 1, 2$. On these fields, the generator $A$ is represented by $A = \omega^m I$ for $m = 0, 1, 2$. For these fields, we can use the field basis diagonalizing the generator $B$, i.e. $\phi_1 + \omega^n \phi_2 + \omega^{2n} \phi_3$ for $n = 0, 1, 2$. In this basis, the generator $B$ is represented by $B = \omega^n$ for $n = 0, 1, 2$. These correspond to nine singlets of $\Delta(27)$, $1_{nm}$, on which the generators $A$ and $B$ are represented by $A = \omega^m$ and $B = \omega^n$. Note that only the combinations $1_{0m} + 1_{1m} + 1_{2m}$ appear on three fixed points. These results are shown in Table 6.

| $Z_2$ charges of $(\phi_1, \phi_2, \phi_3)$ | Representation of $A_4$ |
|------------------------------------------|------------------------|
| (1, 0, 1)                               | 3                      |
| (0, 0, 0)                               | $1 + 1' + 1''$         |

Table 5: $A_4$ representations of the fields localized on the $T^2/Z_3$ orbifold.

Similarly, we can study the $T^2/Z_3$ orbifold models with $Z_N$ charges for $N > 3$. The $\Delta(3N^2)$ flavor symmetry can be realized and certain representations can be obtained on fixed points such as all of triplets and certain combinations of singlets. Furthermore, subgroups of $\Delta(3N^2)$ can also be realized. For example, the $T_7$ flavor symmetry, which is isomorphic to $Z_7 \times Z_3$ $^{7,8,14}$, can be realized by the generator $A = \text{diag}(\rho, \rho^2, \rho^4)$ with $\rho = \exp(2\pi i/7)$ and the generator $B$. Other subgroups of $\Delta(3N^2)$ can also be realized.

$^3$The $\Delta(27)$ symmetry can be realized in magnetized/intersecting brane models $^{21}$. 

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Table 6: $\Delta(27)$ representations of the fields localized on the $T^2/Z_3$ orbifold.

| $Z_3$ charges of ($\phi_1, \phi_2, \phi_3$) | Representation of $\Delta(27)$       |
|------------------------------------------|-------------------------------------|
| (1, 0, 2)                                | $3_1$                               |
| (2, 0, 1)                                | $3_2$                               |
| (0, 0, 0)                                | $1_{00} + 1_{10} + 1_{20}$          |
| (1, 1, 1)                                | $1_{01} + 1_{11} + 1_{21}$          |
| (2, 2, 2)                                | $1_{02} + 1_{12} + 1_{22}$          |

Table 7: $S_4$ representations of the fields localized on the $T^2/Z_3$ orbifold.

| $Z_2$ charges of ($\phi_1, \phi_2, \phi_3$) | Representation of $S_4$ with $C$ in Eq. (11) | Representation of $S_4$ with $C$ in Eq. (12) |
|------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| (1, 0, 1)                                | $3$                                           | $3'$                                          |
| (0, 0, 0)                                | $1 + 2$                                       | $1' + 2$                                      |

3.3 $S_4$ and $\Delta(6N^2)$

Here, let us study the model, where the geometrical symmetry includes both $B$ of Eq. (10) and $C$ of Eq. (11) and fields also have $Z_N$ charges corresponding to $A$ of Eq. (14). Their closed algebra corresponds to $\Delta(6N^2)$, which is isomorphic to $(Z_N \times Z_N) \rtimes S_3$ \cite{11,14}. All of $\Delta(6N^2)$ elements are written as

$$B^k C^\ell A^m A^m,$$

for $k = 0, 1, 2$, $\ell = 0, 1$ and $m, n = 0, 1, \cdots, N - 1$.

Let us study representations of localized fields under $\Delta(6N^2)$ with smaller values of $N$ as the previous section. Here, we start with $N = 2$ by using the generator $A$ in Eq. (14) as the $Z_2$ symmetry. Then, we obtain $\Delta(24)$, which is isomorphic to $S_4$. The fields localized on three fixed points have $Z_2$ charges $(1, 0, 1)$. They correspond to a triplet $3$ of $S_4$. The localized fields with $Z_2$ charges $(0, 0, 0)$ correspond to the combinations, $1+2$ of $S_4$. When we use the generator $C$ of Eq. (12) instead of Eq. (11), then the three localized fields with $Z_2$ charges $(1, 0, 1)$ and $(0, 0, 0)$ correspond to another triplet $3'$ and the combination $1' + 2$, respectively. These results are shown in Table 7. The bulk fields would appear as a trivial singlets $1$. Such a singlet may have a mass term with the trivial singlet of the combinations of the three localized fields, $1+2$. Then, below such a mass scale, the doublets $2$ remain as light modes. Furthermore, the doublets between the combinations, $1+2$ and $1'+2$ may have mass terms. Below such a mass scale, only the non-trivial singlet $1'$ remains as a light mode. Hence, any combinations of $1$, $1'$, $2$, $3$ and $3'$ can appear on the $T^2/Z_3$ orbifold. Then, we can realize any combinations of all $S_4$ representations. Therefore, we can realize any $S_4$ models on the $T^2/Z_3$ orbifold.

Next, we consider the model with $N = 3$. In this model, the generator $A$ can be represented by $A = \text{diag}(\omega, 1, \omega^2)$, and the full flavor symmetry corresponds to $\Delta(54)$.\
Indeed, this flavor symmetry can be realized in heterotic string theory on the $T^2/Z_3$ orbifold [20]. For the generator $C$ in Eq. (11), the localized fields with the $Z_3$ charges $(0, k, 2k)$ ($k = 1, 2$) on the three fixed points correspond to a triplet $3_{1(k)}$ of $\Delta(54)$. In addition, the localized fields $(\phi_1, \phi_2, \phi_3)$ with the trivial $Z_3$ charge $q_i = 0$ for $i = 1, 2, 3$ correspond to a trivial singlet $1_+$ and a doublet $2_1$. Here, the singlet is the linear combination, $\phi_1 + \phi_2 + \phi_3$, while the doublet corresponds to

\[
(\phi_1 + \omega \phi_2 + \omega^2 \phi_3, \; \phi_1 + \omega^2 \phi_2 + \omega \phi_3).
\]  

(19)

When the generator $C$ is represented by Eq. (12), the localized fields with the $Z_3$ charges $(0, k, 2k)$ ($k = 1, 2$) on the three fixed points correspond to a triplet $3_{2(k)}$, and the localized fields $(\phi_1, \phi_2, \phi_3)$ with the trivial $Z_3$ charge $q_i = 0$ correspond to a non-trivial singlet $1_-$ and a doublet $2_1$. Similarly, we can study the $T^2/Z_3$ orbifold models with $Z_N$ charges for $N > 3$, where the $\Delta(6N^2)$ flavor symmetry can be realized.

### 3.4 Larger symmetries

We can realize larger flavor symmetries by using generic values of $Z_N$ charges, i.e. the generator $A$ of Eq. (13) with generic values of $q_1, q_2$ and $q_3$. For example, the combination between such a generator $A$ and the generator $B$ would lead to the flavor symmetry, $\Sigma(3N^3)$. All of the $\Sigma(3N^3)$ elements are written as

\[
\begin{pmatrix}
\rho^\ell & 0 & 0 \\
0 & \rho^m & 0 \\
0 & 0 & \rho^n
\end{pmatrix}, \quad
\begin{pmatrix}
0 & \rho^m & 0 \\
0 & 0 & \rho^n \\
\rho^\ell & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & \rho^n \\
\rho^\ell & 0 & 0 \\
0 & \rho^m & 0
\end{pmatrix}.
\]

(20)

Here, the group for $N = 2$ is isomorphic to $Z_2 \times \Delta(12)$ and $Z_2 \times A_4$. Thus, the non-trivial group with the smallest $N$ is $\Sigma(81)$ [8, 14].

Furthermore, we can include the generator $C$ of Eq. (11) in the above algebra. Then, the $Z_3$ part generated by $B$ is replaced by $S_3$ generated by $B$ and $C$, and the total symmetry becomes larger. All elements are written by

\[
\begin{pmatrix}
\rho^\ell & 0 & 0 \\
0 & \rho^m & 0 \\
0 & 0 & \rho^n
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & \rho^m \\
0 & \rho^n & 0 \\
\rho^\ell & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & \rho^n & 0 \\
\rho^\ell & 0 & 0 \\
0 & 0 & \rho^m
\end{pmatrix},
\]

(21)

in addition to the above $\Sigma(3N^3)$ elements (20).

### 4 Conclusion

We have studied non-Abelian discrete flavor symmetries on the $S^1/Z_2$ and $T^2/Z_3$ orbifolds. We have introduced $Z_N$ charges for fields localized on orbifold fixed points and classified the possible symmetries. By combining $Z_N$ symmetries and geometrical symmetries of the orbifolds, we can realize $D_N$, $\Sigma(2N^2)$, $\Delta(3N^2)$ and $\Delta(6N^2)$ including $S_3$, $A_4$, $S_4$. Their
subgroups are also obtained such as $Q_N, T_7$, etc. Furthermore, larger flavor symmetries such as $\Sigma(3N^3)$ are also possible. Thus, our models provide with geometrical setups of models using these flavor symmetries. In general, certain combinations of representations are allowed to appear. However, any combinations of representations can appear, e.g. for $S_3$ and $S_4$ in low-energy effective theory by using certain mass terms. Although we have constraints on combinations of representations in $A_4$ models, our setups could fit assignments of the model by Altarelli and Feruglio and other models. Therefore, our orbifold setups could realize several interesting models.

To derive realistic quark/lepton masses and mixing angles, we have to break non-Abelian discrete flavor symmetries along certain directions. Orbifolds are also useful to break flavor symmetries by boundary conditions \[23\,27\]. Furthermore, anomalies of non-Abelian discrete symmetries would also lead to an important constraint \[14, 28, 30\]. We leave these studies as future work.

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