A PROOF OF THE GRAHAM SLOANE CONJECTURE.

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Dedicated to the memory of Ronald Graham.

Abstract. We settle in the affirmative the Graham-Sloane conjecture.

1. Introduction.

A well-known conjecture in the area of graph labelings concerns the harmonious labelling. This labelling was introduced by Graham and Sloane [GS80] and was motivated by the study of additive bases. Given an Abelian group $\Gamma$ and a graph $G$, we say that a labelling $L : V(G) \to \Gamma$ is $\Gamma$-harmonious if the map $L' : E(G) \to \Gamma$ defined by $L'(u, v) = L(u) + L(v)$ is injective. In the case when $\Gamma$ is the group of integers modulo $n$ we omit it from our notation and simply call such a labelling harmonious. The Graham–Sloane [GS80] conjecture, better known as the Harmonious Labeling Conjecture (HLC), asserts that every tree admits a harmonious labeling. For a detailed survey of the extensive literature on graph labeling problems, see [Gal09]. Recently Montgomery, Pokrovskiy and Sudakov showed in [MPS19] that every $n$-vertex tree $T$ has an injective $\Gamma$-harmonious labelling for any Abelian group $\Gamma$ of order $n + o(n)$. In the present note we view harmonious labeling of graphs as special vertex labelings which results in a bijection between vertex labels and induced additive edge labels. Induced additive edge labels correspond to the residue classes modulo $n$ of the sum of integers assigned to the vertices spanning each edge. Our discussion is based upon a functional reformulation of the HLC. A rooted tree on $n > 0$ vertices is associated with a function

$$f \in \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$$

subject to

$$\forall i \in \mathbb{Z}/n\mathbb{Z}, \ f(0)(i) := i, \ \text{and } \forall k \geq 0, \ f^{(k+1)}(i) = f^{(k)}(f(i)) = f \left( f^{(k)}(i) \right).$$

In other words the function $f$ has a unique fixed point which is attractive over the whole domain of $f$. Every $f \in \mathbb{Z}/n\mathbb{Z}^{\mathbb{Z}/n\mathbb{Z}}$ has a corresponding functional directed graph denoted $G_f$ whose vertex, edge sets and automorphism group are respectively

$$V(G_f) := \mathbb{Z}/n\mathbb{Z}, \quad E(G_f) := \{(i, f(i)) : i \in \mathbb{Z}/n\mathbb{Z}\} \quad \text{and} \quad \text{Aut}(G_f).$$

When $f$ is subject to Eq. (1.1), the corresponding functional directed graph $G_f$ is a directed rooted tree with an additional loop edge placed at its fixed point. The edges of $G_f$ are oriented to ensure that every vertex has out-degree one. In other words each edge of the rooted tree is oriented to point towards the root of the tree (i.e. the fixed point). Applying the swap sink transformation to $G_f$ subject to Eq. (1.1) results in a rooted directed tree $G_{S(f,k)}$ associated with the function $S(f,k) \in \mathbb{Z}/n\mathbb{Z}^{\mathbb{Z}/n\mathbb{Z}}$. In other words the graph $G_{S(f,k)}$ differs from $G_f$ in the fact that its loop is relocated to a new vertex labeled $k$ and some edges are re-oriented to ensure that every vertex has out-degree one. For instance when function $f$ denotes the identically zero function

$$E(G_f) = \{(i, 0) : i \in \mathbb{Z}/n\mathbb{Z}\} \implies E(G_{S(f,1)}) = \{(0, 1), (1, 1)\} \cup \{(i, 0) : i \in \mathbb{Z}/n\mathbb{Z} \setminus \{0, 1\}\}.$$
Examples of induced edge labels associated with a functional directed graph $G_f$ of $f \in (\mathbb{Z}/n\mathbb{Z})^{2/n\mathbb{Z}}$ include:

- Induced additive edge labels given by $\{f(i)+i \mod n : i \in \mathbb{Z}_n\}$ and the graph $G_f$ is harmonious if there exists $\sigma \in S_n \subset (\mathbb{Z}/n\mathbb{Z})^{2/n\mathbb{Z}}$ such that
  \[
  \mathbb{Z}_n = \left\{ \sigma f \sigma^{-1}(i) + i \mod n : i \in \mathbb{Z}/n\mathbb{Z} \right\}.
  \]

- More general $\tau$-induced edge labels given by $\{\tau(f(i), i) : i \in \mathbb{Z}_n\}$, for some $\tau \in (\mathbb{Z}/n\mathbb{Z})^{2/n\mathbb{Z} \times 2/n\mathbb{Z}}$ and $G_f$ is $\tau$-Zen for if there exists $\sigma \in S_n \subset (\mathbb{Z}/n\mathbb{Z})^{2/n\mathbb{Z}}$ such that
  \[
  \mathbb{Z}_n = \left\{ \tau(\sigma f \sigma^{-1}(i), i) : i \in \mathbb{Z}/n\mathbb{Z} \right\}.
  \]

Our main result is a proof that when $n$ is odd, for all $f \in (\mathbb{Z}/n\mathbb{Z})^{2/n\mathbb{Z}}$ subject to $|f(n^{-1})(\mathbb{Z}/n\mathbb{Z})| = 1$, there exist $k \in \mathbb{Z}/n\mathbb{Z}$ such that
\[
|n| = \max_{\sigma \in S_n} \left| \left\{ \sigma S(f, k) \sigma^{-1}(i) + i : i \in \mathbb{Z}/n\mathbb{Z} \right\} \right|.
\]

Which settles in the affirmative the Graham-Sloane conjecture.

**Definition 1.** Let $f \in \mathbb{Z}_n^{2/n\mathbb{Z}}$ be subject to $|f(n^{-1})(\mathbb{Z}_n)| = 1$, then the set $\text{HaL}(G_f)$ denotes the subset of distinct functional directed graphs isomorphic to $G_f$ whose labeling is harmonious. Formally we write
\[
\text{HaL}(G_f) := \left\{ G_{\sigma f \sigma^{-1}} : \sigma \in S_n/\text{Aut}(G_f) \text{ and } n = \left| \left\{ \sigma f \sigma^{-1}(j) + j : j \in \mathbb{Z}/n\mathbb{Z} \right\} \right| \right\}.
\]

### 2. The Harmonious Invariance Group

The following expresses a necessary and sufficient condition for a functional directed graph to be harmonious.

**Proposition 2.** *(Harmonious Expansion)* Let $G_f$ denote the functional directed graph of $f \in (\mathbb{Z}/n\mathbb{Z})^{2/n\mathbb{Z}}$. Then $G_f$ is harmonious iff
\[
\exists \gamma, \sigma_\gamma \in S_n \text{ such that } f(i) = \sigma_\gamma^{-1}(\gamma \sigma_\gamma(i) - \sigma_\gamma(i)), \ \forall i \in \mathbb{Z}/n\mathbb{Z}.
\]

The subscript $\gamma$ notation for $\sigma_\gamma$, is meant to emphasize the dependence of the coset representative on the permutation parameter $\gamma$.

**Proof.** We prove only the forward direction for the converse easily follows. Recall that $G_f$ is harmonious if there exist $\gamma \in S_n$ and $\sigma_\gamma \in S_n/\text{Aut}(G_f)$ such that
\[
\sigma_\gamma f \sigma_\gamma^{-1}(i) + i = \gamma(i) \quad \forall i \in \mathbb{Z}/n\mathbb{Z},
\]

\[
\Rightarrow \quad \sigma_\gamma f \sigma_\gamma^{-1}(i) = \gamma(i) - i \quad \forall i \in \mathbb{Z}/n\mathbb{Z},
\]

\[
\Rightarrow \quad f(i) = \sigma_\gamma^{-1}(\gamma \sigma_\gamma(i) - \sigma_\gamma(i))
\]

\[\square\]

**Proposition 3.** *(Swap Sink Harmony)* Let $\text{GCD}(n, 2) = 1$, let $g \in (\mathbb{Z}/n\mathbb{Z})^{2/n\mathbb{Z}}$ subject to $|g(n^{-1})(\mathbb{Z}/n\mathbb{Z})| = 1$ and
\[
n - 1 = \max_{\sigma \in S_n} \left| \left\{ \sigma g \sigma^{-1} + i : i \in (\mathbb{Z}/n\mathbb{Z}) \setminus g(n^{-1})(\mathbb{Z}/n\mathbb{Z}) \right\} \right|.
\]
then there exist \( k \in \mathbb{Z}/n\mathbb{Z} \) such that

\[
n = \max_{\sigma \in S_n} \left| \{ \sigma^{(-1)} S(g, k) \sigma(i) + i : i \in \mathbb{Z}/n\mathbb{Z} \} \right|.
\]

**Proof.** The premise that the functional tree \( G_f \) is subject to the equality

\[
n - 1 = \max_{\sigma \in S_n} \left| \{ \sigma g \sigma^{(-1)} + i : i \in (\mathbb{Z}/n\mathbb{Z}) \setminus g^{(n-1)}(\mathbb{Z}/n\mathbb{Z}) \} \right|,
\]

implies that \( G_g \) can be relabeled such that its non-loop edges have distinct additive edge labels. For each such relabeling there is exactly one congruence class which does not occur as an edge label. Let us call that label \( l \). Since 2 is not a zero divisor of the ring \( \mathbb{Z}/n\mathbb{Z} \), relocating the loop edge at the vertex whose label correspond to the solution \( x \) to the equation

\[
x = l
\]

results in a harmoniously labeled graph as claimed. Thereby completing the proof. \( \square \)

**Proposition 4.** (Harmonious Right Invariant Group) Let the graph \( G_g \) associated with \( g \in (\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}/n\mathbb{Z}} \) be labeled such that for a subset \( T \subset \mathbb{Z}/n\mathbb{Z} \) we have

\[
|\{g(i) + i : i \in (\mathbb{Z}/n\mathbb{Z}) \setminus T\}| = n - |T|.
\]

Let \( c \) be an arbitrary element of the ring \( \mathbb{Z}/n\mathbb{Z} \) and the graph \( G_{g'} \) associated with the function

\[
g' = g(id + c), \ \forall i \in \mathbb{Z}/n\mathbb{Z},
\]

then

\[
|\{g'(i) + i : i \in (\mathbb{Z}/n\mathbb{Z}) \setminus T\}| = n - |T|.
\]

**Proof.** By the premise that for some \( T \subset \mathbb{Z}/n\mathbb{Z} \) we have

\[
n - |T| = |\{g(i) + i : i \in \mathbb{Z}/n\mathbb{Z} \setminus T\}|.
\]

by the same argument use to prove Prop. (1), there exist \( \gamma \in S_n \) such that

\[
g(i) = \gamma(i) - i
\]

\[
\Rightarrow g(i + c) = \gamma(i + c) - (i + c)
\]

\[
\Rightarrow (g \circ (id + c))(i) = \gamma(i + c) - c - i \quad \forall i \in \mathbb{Z}/n\mathbb{Z} \setminus T.
\]

\[
\Rightarrow g(i + c) = \left((id + c)^{(-1)} \circ \gamma \circ (id + c)\right)(i) - i
\]

From which it follows \(|\{g'(i) + i : i \in \mathbb{Z}/n\mathbb{Z} \setminus T\}| = (n - |T|)\) thereby completes the proof. \( \square \)

**Proposition 5.** (Harmonious Left Invariance Group) Let the graph \( G_g \) associated with \( g \in (\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}/n\mathbb{Z}} \) be labeled such that for a subset \( T \subset \mathbb{Z}/n\mathbb{Z} \) we have

\[
|\{g(i) + i : i \in \mathbb{Z}/n\mathbb{Z} \setminus T\}| = n - |T|.
\]

Let \( c \) be an arbitrary element of the ring \( \mathbb{Z}/n\mathbb{Z} \) and the graph \( G_{g''} \) associated with the function

\[
g'' = g + c, \ \forall i \in \mathbb{Z}/n\mathbb{Z},
\]

then

\[
|\{g''(i) + i : i \in \mathbb{Z}/n\mathbb{Z} \setminus T\}| = (n - |T|).
\]
Proof. By the premise that for some $T \subset \mathbb{Z}/nz$ we have
\[ n - |T| = |\{g(i) + i : i \in \mathbb{Z}/nz - T\}|. \]
by the same argument use to prove Prop. (1), there exist $\gamma \in S_n$ such that
\[ g(i) = \gamma(i) - i \]
\[ \Rightarrow g(i) + c = \gamma(i) - i \quad \forall i \in \mathbb{Z}/nz - T. \]
\[ \Rightarrow ((\text{id} + c) \circ g)(i) = (\text{id} + c) \circ \gamma(i) - i \]
\[ \Rightarrow g(i) + c = (\text{id} + c) \circ \gamma(i) - i \]
From which it follows $|\{g''(i) + i : i \in \mathbb{Z}/nz - T\}| = n - |T|$ thereby completes the proof. □

In Prop. (3) and (4) when $T$ is empty then we say that the $G_g$ is harmoniously labeled. It follows as corollary of Prop. (3) and (4) via an argument similar to the proof of Lagrange’s coset theorem that both the number of harmonious permutations on $n$ vertices and the number of harmoniously labeled permutations on $n$ vertices, must be divisible by $n$.

3. Useful facts about polynomials

Let $F(x), G(x) \in \mathbb{C}[x_0, \cdots, x_{n-1}]$, be multivariate polynomials which splits into irreducible factors of the form
\[ F(x) = \prod_{0 \leq i < m} (P_i(x))^{\alpha_i}, \quad G(x) = \prod_{0 \leq i < m} (P_i(x))^{\beta_i}, \]
where $\{\alpha_i, \beta_i : 0 \leq i < m\} \subset \mathbb{Z}_{\geq 0}$. Assume that each factor $P_i(x)$ is multilinear in the entries of $x$ and non-identically constant. Additionally, $P_i(x)$ has no common roots in the field of fractions $\mathbb{C}(x_0, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n-1})$, with any other factor in $\{P_j(x) : 0 \leq j \neq i < m\}$ for each $k \in \mathbb{Z}/nz$, then
\[ \text{LCM}(F(x), G(x)) := \prod_{0 \leq i < m} (P_i(x))^{\max(\alpha_i, \beta_i)}, \]
and
\[ \text{GCD}(F(x), G(x)) := \prod_{0 \leq i < m} (P_i(x))^{\min(\alpha_i, \beta_i)}. \]

Definition 6. By the quotient remainder theorem an arbitrary $H(x) \in \mathbb{C}[x_0, \cdots, x_{n-1}]$ admits an expansion of the form
\[ H(x) = \sum_{l \in \mathbb{Z}/nz} q_l(x) ((x))^n - 1) + \sum_{g \in \mathbb{Z}/nz} H\left(\omega^{g(0)}, \cdots, \omega^{g(i)}, \cdots, \omega^{g(n-1)}\right) \prod_{k \in \mathbb{Z}/nz} \left(\prod_{j \in \mathbb{Z}/nz \setminus \{g(k)\}} \left(\frac{x_k - \omega^j}{\omega^{g(k)} - \omega^j}\right)\right), \]
where $\omega = \exp\left\{\frac{2\pi \sqrt{-1}}{n}\right\}$. Incidentally, the canonical representative of the congruence class
\[ H(x) \mod \left\{\left(\frac{x_k^n - 1}{k \in \mathbb{Z}/nz}\right)\right\}, \]
is defined as the unique polynomial of degree at most $(n - 1)$ in each variable whose evaluations matches exactly evaluations of $H(x)$ over the lattice $\Omega^n$ where
\[ \Omega := \{\omega^k : k \in \mathbb{Z}/nz\}. \]
The canonical representative is explicitly expressed as

\[
(3.1) \quad \sum_{g \in \mathbb{Z}/n\mathbb{Z}} H \left( \omega^{g(0)}, \ldots, \omega^{g(i)}, \ldots, \omega^{g(n-1)} \right) \prod_{k \in \mathbb{Z}/n} \left( \prod_{j_k \in \mathbb{Z}/n \setminus \{g(k)\}} \left( \frac{x_k - \omega^{j_k}}{\omega^{g(k)} - \omega^{j_k}} \right) \right),
\]

The canonical representative of \( H(x) \) modulo algebraic relations \( \{(x_i)^n - 1 : i \in \mathbb{Z}/n\mathbb{Z}\} \), is thus obtained via Lagrange interpolation over the integer lattice \( \Omega^n \) as prescribed by Eq. (3.1). Alternatively, the canonical representative is obtained as the final remainder resulting from performing \( n \) Euclidean divisions irrespective of the order in which distinct divisors are successively taken from the set \( \{(x_i)^n - 1 : i \in \mathbb{Z}/n\mathbb{Z}\} \).

**Proposition 7** (Determinantal Certificate). For \( f \in (\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}/n\mathbb{Z}} \), subject to \( |f^{(n-1)}(\mathbb{Z}/n\mathbb{Z})| = 1 \) we have

\[
n - 1 = \max_{\sigma \in S_n} \left| \sigma f^{(-1)}(i) + i : i \in (\mathbb{Z}/n\mathbb{Z}) \right| \quad \text{if and only if}
\]

\[
0 \neq \text{LCM} \begin{cases} \prod_{0 \leq i < j < n} (x_j - x_i), & 0 \leq i < j < n \\ \prod_{i, j \in (\mathbb{Z}/n\mathbb{Z}) \setminus f^{(n-1)}(\mathbb{Z}/n\mathbb{Z})} (x_{f(j)}x_j - x_{f(i)}x_i) \end{cases} \mod \left\{ (x_k)^n - 1 \right\}_{k \in \mathbb{Z}/n\mathbb{Z}}
\]

**Proof.** The LCM in the claim of the proposition is well defined since both polynomials

\[
\prod_{0 \leq i < j < n} (x_j - x_i) \quad \text{and} \quad \prod_{i, j \in (\mathbb{Z}/n\mathbb{Z}) \setminus f^{(n-1)}(\mathbb{Z}/n\mathbb{Z})} (x_{f(j)}x_j - x_{f(i)}x_i),
\]

split into irreducible multilinear factors. Given that we are reducing modulo algebraic relations

\[
(x_k)^n \equiv 1, \forall k \in \mathbb{Z}/n\mathbb{Z},
\]

the canonical representative of the congruence class is completely determined by evaluations of the dividend at lattice points taken from \( \Omega^n \) where

\[
\Omega := \{ \omega^k : k \in \mathbb{Z}/n\mathbb{Z} \}
\]
as prescribed by Eq. (3.1). This ensures a discrete set of roots for the canonical representative of the congruence class. On the one hand, the LCM polynomial construction vanishes when we assign to \( x \) a lattice point in \( \Omega^n \), only if one of the irreducible multilinear factors vanishes at the chosen evaluation point. On the other hand, one of the factor of the polynomial construction vanishes at a lattice point only if either two distinct vertex variables say \( x_i \) and \( x_j \) are assigned the same label (we see this from the vertex Vandermonde determinant factor) or alternatively if two distinct edges are assigned the same induced additive edge label (we see this from the edge Vandermonde determinant factor). The proof of sufficiency follows from the observation that the only possible roots over \( \Omega^n \) to the multivariate polynomial

\[
\text{LCM} \begin{cases} \prod_{0 \leq i < j < n} (x_j - x_i), & 0 \leq i < j < n \\ \prod_{i, j \in (\mathbb{Z}/n\mathbb{Z}) \setminus f^{(n-1)}(\mathbb{Z}/n\mathbb{Z})} (x_{f(j)}x_j - x_{f(i)}x_i) \end{cases} \mod \left\{ (x_k)^n - 1 \right\}_{k \in \mathbb{Z}/n\mathbb{Z}}
\]
arise from vertex label assignments \( x \in \Omega^n \) in which either distinct vertex variables are assigned the same label or distinct edges are assigned the same induced additive edge label. Consequently, the congruence identity

\[
0 \equiv \text{LCM} \left\{ \prod_{0<i<j<n} (x_j - x_i), \quad \prod_{0<i<j<n} (x_{f(j)}x_j - x_{f(i)}x_i) \right\} \mod \left\{ (x_k)^n - 1 \right\}_{k \in \mathbb{Z}/n\mathbb{Z}},
\]

implies that \( n - 1 > \max_{\sigma \in S_n} \left| \{ \sigma f \sigma^{-1} (i) + i : i \in (\mathbb{Z}/n\mathbb{Z}) \setminus f^{(n-1)}(\mathbb{Z}/n\mathbb{Z}) \} \right| \). Furthermore, the proof of necessity follows from the fact that every non-vanishing assignment to the vertex variables entries of \( x \) in the polynomial

\[
\text{LCM} \left\{ \prod_{0<i<j<n} (x_j - x_i), \quad \prod_{0<i<j<n} (x_{f(j)}x_j - x_{f(i)}x_i) \right\}
\]

describes a Harmonious labeling by Prop. (3).

\[ \square \]

**Definition 8.** Let \( P(x) \in \mathbb{C}[x_0, \cdots, x_{n-1}] \), we denote by \( \text{Aut}\{P(x)\} \) the stabilizer subgroup of \( S_n \subset \mathbb{Z}_n^n \) of \( P(x) \) with respect to permutation of the variable entries of \( x \).

**Proposition 9 (Stabilizer).** For an arbitrary \( f \in \mathbb{Z}_n^n \), let \( P_f(x) \in \mathbb{C}[x_0, \cdots, x_{n-1}] \) be defined such that

\[
P_f(x) = \prod_{0\leq i\neq j<n} (x_j - x_i) \prod_{0\leq i\neq j<n} (x_{f(j)}x_j - x_{f(i)}x_i),
\]

then

\[ \text{Aut}\{P_f(x)\} = \text{Aut}(G_f). \]

**Proof.** Note that for all \( \gamma \in S_n \), which fixes the loop edge we have

\[
\prod_{0\leq i\neq j<n} (x_j - x_i) \prod_{0\leq i\neq j<n} (x_{f(j)}x_j - x_{f(i)}x_i) = \prod_{0\leq i\neq j<n} (x_{\gamma(j)} - x_{\gamma(i)}) \prod_{0\leq i\neq j<n} (x_{f_\gamma(j)}x_{\gamma(j)} - x_{f_\gamma(i)}x_{\gamma(i)}),
\]

Consequently for all \( \sigma \in \text{Aut}(G_f) \) we have

\[
\prod_{0\leq i\neq j<n} (x_{\sigma(j)} - x_{\sigma(i)}) \prod_{0\leq i\neq j<n} (x_{\sigma f(j)}x_{\sigma(j)} - x_{\sigma f(i)}x_{\sigma(i)}).
\]
For this purpose, we associate with an arbitrary $g \in (\mathbb{Z}/n\mathbb{Z})^{2^n}$, subject to $|g^{(n-1)}(\mathbb{Z}/n\mathbb{Z})| = 1$, a multivariate polynomial $P_g(x) \in \mathbb{Q}[x_0, \cdots, x_{n-1}]$ given by

$$P_g(x) = \prod_{0 \leq i \neq j < n} (x_j - x_i) \prod_{0 \leq i \neq j < n} (x_g(j)x_j - x_g(i)x_i).$$

It also follows that for every permutation representative $\sigma \in (S_n/\text{Aut}(G_f)) \setminus \text{Aut}(G_f)$ we have

$$\prod_{0 \leq i \neq j < n} (x_{\sigma(j)} - x_{\sigma(i)}) \prod_{0 \leq i \neq j < n} (x_{\sigma(j)}x_{\sigma(j)} - x_{\sigma(i)}x_{\sigma(i)}) \neq P_f(x).$$

We now state and prove our composition lemma

**Lemma 10** (Composition Lemma). Let $f \in (\mathbb{Z}/n\mathbb{Z})^{2^n}$, subject to $|f^{(n-1)}(\mathbb{Z}/n\mathbb{Z})| = 1$ such that $\text{Aut}(G_f) \subset \text{Aut}(G_{f(2)})$, then

$$(n - 1) = \max_{\sigma \in S_n} \left| \left\{ \sigma f^{(2)}(\sigma^{(-1)}(i) + i : i \in (\mathbb{Z}/n\mathbb{Z}) \setminus f^{(n-1)}(\mathbb{Z}/n\mathbb{Z}) \right\} \right| \leq \max_{\sigma \in S_n} \left| \left\{ \sigma f^{(1)}(\sigma^{(-1)}(i) + i : i \in (\mathbb{Z}/n\mathbb{Z}) \setminus f^{(n-1)}(\mathbb{Z}/n\mathbb{Z}) \right\} \right|.$$  

Proof. We show that

$$\text{LCM} \left\{ \prod_{0 \leq i \neq j < n} (x_j - x_i), \prod_{0 \leq i \neq j < n} (x_{f^{(2)}(j)x_j} - x_{f^{(2)}(i)x_i}) \right\} \neq 0 \mod \left\{ \frac{1}{k^1 - 1} \right\},$$

implies that

$$\text{LCM} \left\{ \prod_{0 \leq i \neq j < n} (x_j - x_i), \prod_{0 \leq i \neq j < n} (x_{f(j)x_j} - x_{f(i)x_i}) \right\} \neq 0 \mod \left\{ \frac{1}{k^1 - 1} \right\}.$$  

For this purpose, we associate with an arbitrary $g \in (\mathbb{Z}/n\mathbb{Z})^{2^n}$, subject to $|g^{(n-1)}(\mathbb{Z}/n\mathbb{Z})| = 1$, a multivariate polynomial $P_g(x) \in \mathbb{Q}[x_0, \cdots, x_{n-1}]$ given by

$$P_g(x) = \prod_{0 \leq i \neq j < n} (x_j - x_i) \prod_{0 \leq i \neq j < n} (x_g(j)x_j - x_g(i)x_i).$$
We prove the claim by contradiction. By our premise,

\[ n - 1 = \max_{\sigma \in S_n} \left| \{ \sigma f(2) \sigma^{-1} (i) + i : i \in \mathbb{Z}/n\mathbb{Z} \setminus f(n-1)(\mathbb{Z}/n\mathbb{Z}) \} \right| \quad \text{and} \quad \text{Aut}(G_f) \subseteq \text{Aut}(G_{f(2)}) . \]

By definition,

\[ P_f (x) = \prod_{0 \leq i \neq j < n} (x_j - x_i) \prod_{0 \leq i \neq j < n} x_j x_i - x_{f(i)} x_{f(j)} . \]

By telescoping we have

\[
P_f (x) = \prod_{0 \leq i \neq j < n} (x_j - x_i) \prod_{0 \leq i \neq j < n} \left( x_{f(2)(j)} x_j - x_{f(2)(i)} x_i + x_{f(2)(j)} x_{f(i)} - x_{f(2)(i)} x_{f(j)} \right) \]

\[
= \prod_{0 \leq i \neq j < n} (x_j - x_i) \prod_{0 \leq i \neq j < n} \left( x_{f(2)(j)} x_j - x_{f(2)(i)} x_i \right) \sum_{k_{ij} \in \{0, 1\}} \prod_{i \neq j} k_{ij} \left( (x_{f(j)} - x_{f(2)(j)}) x_j - (x_{f(i)} - x_{f(2)(i)}) x_i \right)^{k_{ij}} \times \left( x_{f(2)(j)} x_{f(j)} - x_{f(2)(i)} x_{f(i)} \right)^{k_{ij}} \times \left( (x_{f(j)} - x_{f(2)(j)}) x_j - (x_{f(i)} - x_{f(2)(i)}) x_i \right)^{1-k_{ij}} \]

Let \( g \in \mathbb{Z}/n\mathbb{Z} \) be subject to \(|g(n-1)(\mathbb{Z}/n\mathbb{Z})| = 1 \) and let

\[ \kappa_g := \left\{ G_{g \sigma^{-1}} : \sigma \in S_n/\text{Aut}(G_g) \right\} \]

We now will analyze the residue of \( \sum_{\sigma \in S_n/\text{Aut}(G_{f(2)})} P_{\sigma f(2)}(x) \) over a collection of moduli. Specifically, we’d like to pick a set of relations that are identically constant for symmetric polynomials and correspond to setting the \( x_i \)’s to pairwise distinct roots of unity. We pick as our basis the power sum polynomials

\[ p_k(x) = p_k(x_0, x_1, \ldots, x_{n-1}) = \sum_{i \in \mathbb{Z}_n} (x_i)^k \equiv \begin{cases} 0 & \text{if } 0 \leq k \leq n - 1 \\ n & \text{if } k = n \end{cases} \]

This gives us the pairwise distinct substitution of roots of unity for \( x_i \)’s. To see this, recall the elementary symmetric function basis is defined

\[ \prod_{i \in \mathbb{Z}_n} (\lambda - x_i) = \sum_{0 \leq k \leq n} e_k(x) \lambda^{n-k} \]
The Newton-Girard formulae state that

\[ e_k(x) = \frac{1}{k} \sum_{0 \leq \ell \leq k} (-1)^{\ell-1} e_{k-\ell}(x) p_{\ell}(x) \]

As a result of our moduli, we have that

\[ e_k(x) = \begin{cases} 
0 & \text{if } 0 < k < n \\
(-1)^{n-1} & \text{if } k = n \text{ (recall that } n \text{ is even)}
\end{cases} \]

As a result, the residue of our moduli is equivalent to substituting in the roots of \( \lambda^n - 1 \). Indeed,

\[ \prod_{i \in \mathbb{Z}_n} (\lambda - x_i) = \sum_{0 \leq k \leq n} e_k(x) \lambda^{n-k} \]

\[ = \lambda^n - 1 \]

Now summing over a conjugation orbit of arbitrarily chosen coset representatives, (where we are careful to select only one coset representative per left coset of \( \text{Aut}(G_{f(2)}) \)), yields the equality for some non-zero constant \( C_{f(2)} \)

\[ \left( \sum_{\sigma \in \mathbb{S}_n / \text{Aut}(G_{f(2)})} P_{\sigma f(\sigma(-1))} \right) \equiv C_{f(2)} (-1)^{\binom{n}{2}} n^n + \]

\[ (-1)^{\binom{n}{2}} n^n \sum_{\sigma \in \mathbb{S}_n / \text{Aut}(G_{f(2)})} \prod_{k_{ij} \in \{0,1\}} \left( x_{\sigma f(\sigma(-1)(j))} x_j - x_{\sigma f(\sigma(-1)(i))} x_i \right)^{k_{ij} \times} \]

\[ \left( (x_{f(j)} - x_{f(2)(j)}) x_j - (x_{f(i)} - x_{f(2)(i)}) x_i \right)^{1-k_{ij}} \mod \left\{ \sum_{i \in \mathbb{Z}/n\mathbb{Z}} (x_i)^k \right\} \]

Since \( \text{Aut}(G_f) \subseteq \text{Aut}(G_{f(2)}) \), it follows that the polynomial

\[ \sum_{\sigma \in \mathbb{S}_n / \text{Aut}(G_{f(2)})} \prod_{k_{ij} \in \{0,1\}} \left( x_{\sigma f(\sigma(-1)(j))} x_j - x_{\sigma f(\sigma(-1)(i))} x_i \right)^{k_{ij} \times} \]

\[ \left( (x_{f(j)} - x_{f(2)(j)}) x_j - (x_{f(i)} - x_{f(2)(i)}) x_i \right)^{1-k_{ij}} \]

does not lie in the ring of symmetric polynomial in the entries of \( x \). Consequently there must be at least one evaluation point \( x \) subject to the constraints

\[ \left\{ \begin{array}{c}
0 = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} (x_i)^k \\
\text{where } 0 < k < n \\
n = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} (x_i)^n
\end{array} \right\} , \]

for which the evaluation of the said polynomial depend on the choice of coset representatives. For otherwise, evaluations of the said polynomial would be independent of choices of coset representatives. In which case the
polynomial obtained by summing over the conjugation orbit of coset representatives would be symmetric and thereby contradict the premise \( \text{Aut}(G_f) \neq \text{Aut}(G_f^{(2)}) \). We conclude that 
\[
P_f(x) \not\equiv 0 \mod \left\{ \frac{(x_k)^n - 1}{k \in \mathbb{Z}/n\mathbb{Z}} \right\}.
\]

Note that the premise \( \text{Aut}(G_f) \subset \text{Aut}(G_f^{(2)}) \) incurs no loss of generality when \( n > 2 \). For we see that if \( f \) is not identically constant and \( \text{Aut}(G_f) = \text{Aut}(G_f^{(2)}) \) then there exists \( k \in \mathbb{Z}/n\mathbb{Z} \) such that \( \text{Aut}(G_{S(f,k)}) \subset \text{Aut}(G_{S(f,k)^{(2)}}) \).

For instance, take \( k \) to be a vertex at edge distance \( 2 \) from a leaf node. Crucially, for all \( k \in \mathbb{Z}/n\mathbb{Z} \)
\[
\max_{\sigma \in S_n} \left| \left\{ \sigma S(f,k)\sigma^{(-1)}(i) + i : i \in (\mathbb{Z}/n\mathbb{Z}) \setminus S(f,k)^{(n-1)}(\mathbb{Z}/n\mathbb{Z}) \right\} \right| = \max_{\sigma \in S_n} \left| \left\{ \sigma f\sigma^{(-1)}(i) + i : i \in (\mathbb{Z}/n\mathbb{Z}) \setminus f^{(n-1)}(\mathbb{Z}/n\mathbb{Z}) \right\} \right|.
\]

5. THE HARMONIOUS LABELING THEOREM

Equipped with the composition lemma, we settle in the affirmative the Graham–Sloane conjecture.

**Theorem 11.** For all \( f \in (\mathbb{Z}/n\mathbb{Z})^{(\mathbb{Z}/n\mathbb{Z})} \) subject to \( |f^{(n-1)}(\mathbb{Z}/n\mathbb{Z})| = 1 \), there exist \( k \in \mathbb{Z}/n\mathbb{Z} \)
\[
n = \max_{\sigma \in S_n} \left| \left\{ \sigma S(f,k)\sigma^{(-1)}(i) + i : i \in \mathbb{Z}/n\mathbb{Z} \right\} \right|.
\]

**Proof.** It suffices to show that for all \( f \) subject to \( |f^{(n-1)}(\mathbb{Z}/n\mathbb{Z})| = 1 \) we have
\[
n - 1 = \max_{\sigma \in S_n} \left| \left\{ \sigma f\sigma^{(-1)}(i) + i : i \in \mathbb{Z}/n\mathbb{Z} \setminus f^{(n-1)}(\mathbb{Z}/n\mathbb{Z}) \right\} \right|.
\]

This latter claim follows by repeatedly applying the composition lemma. For we know that for any such function \( f \) the iterate \( f^{(2^{\lfloor \log_2(n-1) \rfloor})} \) is necessarily identically constant. Recall that the graph of identically constant functions in \((\mathbb{Z}_n)^{\mathbb{Z}_n}\) are all harmoniously labeled. \(\square\)

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