ABSTRACT. We extend the Besicovitch-Federer projection theorem to transversal families of mappings. As an application we show that on a certain class of Riemann surfaces with constant negative curvature and with boundary, there exist natural 2-dimensional measures invariant under the geodesic flow having 2-dimensional supports such that their projections to the base manifold are 2-dimensional but the supports of the projections are Lebesgue negligible. In particular, the union of complete geodesics has Hausdorff dimension 2 and is Lebesgue negligible.

1. Introduction

A pair of pants $S$ is a 2-sphere minus three points endowed with a metric of constant curvature $-1$ in such a way that the boundary consists of three closed geodesics of length $a$, $b$ and $c$ called the cuffs. The metric is uniquely determined by these three lengths. (For more details, see e.g. [H].) For each point $x$ in $S$, write $\Omega_x$ for the set of unit tangent vectors $v \in T^1_x S$ such that the geodesic ray $\gamma_v(t), t \geq 0$, with initial condition $(x, v)$ never meets the boundary $\partial S$ of $S$. The set $\Omega_x$ is a Cantor set of dimension $\delta = \delta(a, b, c)$. The number $\delta$ is an important geometric invariant of the pair of pants $S$: it is the critical exponent of the Poincaré series of $\pi_1(S)$ and the topological entropy of the geodesic flow on $T^1 S$ (cf. [S2]). We will recall in Section 3 why the function $(a, b, c) \mapsto \delta$ is real analytic. In particular, the function $a \mapsto \delta(a, a, a)$ is continuous from $(0, \infty)$ onto the open set $(0, 1)$. In a very similar setting, McMullen ([Mc]) gives asymptotics for $1 - \delta(a, a, a)$ when $a \to 0$ and for $\delta(a, a, a)$ when $a \to \infty$.

We are interested in the set

$$C(S) := \{x \in S \mid \text{there exists } v \in T^1_x S \text{ such that } v \in \Omega_x \text{ and } -v \in \Omega_x\}. \hspace{1cm} (1.1)$$
In other words, \( C(S) \) is the set of points in complete geodesics in \( S \). Let
\[
D(S) := \{(x, v) \in T^1S \mid x \in C(S), v \in \Omega_x, -v \in \Omega_x \}
\]
be the subset of \( T^1S \) where the geodesic flow is defined for all \( t \in \mathbb{R} \). Clearly, \( \Pi(D(S)) = C(S) \), where \( \Pi : T^1S \to S, \Pi(x, v) = x \), is the canonical projection.

We write \( L^l \) and \( H^s \) to denote the \( l \)-dimensional Lebesgue measure and the \( s \)-dimensional Hausdorff measure. For the Hausdorff dimension we use the notation \( \text{dim}_H \).

We consider the following theorem:

**Theorem 1.1.** With the above notation,
- \( L^2(C(S)) > 0 \) provided that \( \delta > 1/2 \) and
- \( \text{dim}_H C(S) = 1 + 2\delta \) and \( L^2(C(S)) = 0 \) provided that \( \delta \leq 1/2 \).

It is known that \( \text{dim}_H(D(S)) = 1 + 2\delta \) (see Section 3). Ledrappier and Lindenstrauss proved in [LL] (see [JJL] for a different proof) that \( \Pi \) does not diminish the Hausdorff dimension of a measure which is invariant under the geodesic flow.

The new part of our result is when \( \delta \) is exactly \( 1/2 \). In that case, [LL] implies that \( \text{dim}_H C(S) = 2 \), and we sharpen this by proving that \( C(S) \) is Lebesgue negligible.

The main technical part of our paper is the following extension of Besicovitch-Federer projection theorem to transversal families of maps. (For the definition of transversality, see Definition 2.4.) We believe that Theorem 1.2 is of independent interest (see for example [OS]), and therefore we verify it in a more general setting than needed for the purpose of proving Theorem 1.1.

**Theorem 1.2.** Let \( E \subset \mathbb{R}^n \) be \( H^m \)-measurable with \( H^m(E) < \infty \). Assume that \( \Lambda \subset \mathbb{R}^l \) is open and \( \{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m\}_{\lambda \in \Lambda} \) is a transversal family of maps. Then \( E \) is purely \( m \)-unrectifiable, if and only if \( H^m(P_\lambda(E)) = 0 \) for \( L^l \)-almost all \( \lambda \in \Lambda \).

In [HJLL] we showed that on any Riemann surface with (variable) negative curvature there exist 2-dimensional measures which are invariant under the geodesic flow and have singular projections with respect to \( L^2 \). The measures are supported by the whole unit tangent bundle \( T^1S \) and they are singular with respect to \( H^2 \) on \( T^1S \). However, the measures constructed in this paper have 2-dimensional supports and they are absolutely continuous with respect to \( H^2 \) on \( T^1S \). Thus their singularity is due to the projection.

The paper is organized as follows: In Section 2 we introduce the notation and prove Theorem 1.2. In Section 3 we recall basic properties of the geodesic flow on a pair of pants and prove Theorem 1.1 as an application of Theorem 1.2.

### 2. Projections

In this section we prove Theorem 1.2 as a consequence of several lemmas. In the case of orthogonal projections in \( \mathbb{R}^n \), one can find a proof for the “only if”-part of Theorem 1.2 in [Ma, Chapter 18] or in [E, Chapter 3.3]. The main idea of our proof is same as that of [Ma], but, due to our more general setting, some
modifications are naturally needed – the major ones being in Lemma 2.5. For the convenience of the reader we give the main arguments. In fact, our approach simplifies slightly the corresponding arguments in [Ma].

In this section \( \Lambda \subset \mathbb{R}^l \) is open and \( l,m \) and \( n \) are integers with \( m \leq l \) and \( m < n \). The closed ball with radius \( r \) centred at \( x \) is denoted by \( B(x,r) \). As in [Ma], a non-negative, subadditive set function vanishing for the empty set is called a measure. We start by defining cones around preimages of points with respect to Lipschitz continuous mappings.

**Definition 2.1.** Let \( \lambda \in \Lambda \) and let \( P_{\lambda} : \mathbb{R}^n \to \mathbb{R}^m \) be Lipschitz continuous. For all \( a \in \mathbb{R}^n, 0 < s < 1 \) and \( r > 0 \), we define

\[
X(a, \lambda, s) := \{ x \in \mathbb{R}^n | |P_{\lambda}(x) - P_{\lambda}(a)| < s|x - a| \}
\]
and

\[
X(a, r, \lambda, s) := X(a, \lambda, s) \cap B(a,r).
\]

The following lemma is an analogue of [Ma, Corollary 15.15].

**Lemma 2.2.** Suppose that \( E \subset \mathbb{R}^n \) is purely \( m \)-unrectifiable. Let \( \delta > 0 \) and \( \lambda \in \Lambda \). Defining

\[
E_{1,\delta}(\lambda) := \{ a \in E | \limsup_{s \to 0} \sup_{0 < r < \delta} (rs)^{-m} \mathcal{H}^m(E \cap X(a, r, \lambda, s)) = 0 \},
\]
we have \( \mathcal{H}^m(E_{1,\delta}(\lambda)) = 0 \).

**Proof.** Replacing \( Q_V \) by \( P_\lambda \) in [Ma] Lemmas 15.13 and 15.14 and observing that the Lipschitz constant of \( Q_V \) is one, the proof of [Ma, Corollary 15.15] works in our setting. Here \( Q_V \) is the projection onto the orthogonal complement \( V^\perp \) of an \( m \)-plane going through the origin. \( \square \)

Next we consider the analogue of [Ma] Lemmas 18.3 and 18.4 in our setting. The proof of [Ma, Lemma 18.3] relies on the fact that \( Q_V(\{ x \in B(a,r) | |Q_V(x - a)| < s|x - a| \} = U(Q_V(a),rs) \cap V^\perp \) where \( U(z,r) \) is the open ball with centre at \( z \) and with radius \( r \). Note that this does not hold when \( Q_V \) is replaced by \( P_\lambda \). However, the proof given in [F] Lemma 3.3.9 works in our setting.

**Lemma 2.3.** Let \( E \subset \mathbb{R}^n \) with \( \mathcal{H}^m(E) < \infty, \delta > 0 \) and \( \lambda \in \Lambda \). Defining

\[
E_{2,\delta}(\lambda) := \{ a \in E | \limsup_{s \to 0} \sup_{0 < r < \delta} (rs)^{-m} \mathcal{H}^m(E \cap X(a, r, \lambda, s)) = \infty \}
\]
and

\[
E_3(\lambda) := \{ a \in E | \#(E \cap P_\lambda^{-1}(P_\lambda(a))) = \infty \},
\]
we have \( \mathcal{H}^m(P_\lambda(E_{2,\delta}(\lambda))) = 0 \) and \( \mathcal{H}^m(P_\lambda(E_3(\lambda))) = 0 \).

**Proof.** The first claim can be verified in the same way as [F] Lemma 3.3.9 and the latter one follows from [Ma, Theorem 7.7]. \( \square \)

Throughout the rest of this section we assume that the family \( \{ P_\lambda : \mathbb{R}^n \to \mathbb{R}^m \}_{\lambda \in \Lambda} \) is transversal. We use a slight variant of the \( \beta = 0 \) case of the definition of \( \beta \)-transversality given in [PS, Definition 7.2].
Definition 2.4. Let $\Lambda \subset \mathbb{R}^l$ be open. A family of maps $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m\}_{\lambda \in \Lambda}$ is transversal if it satisfies the following conditions for each compact set $K \subset \mathbb{R}^n$:

1. The mapping $P : \Lambda \times K \to \mathbb{R}^m$, $(\lambda, x) \mapsto P_\lambda(x)$, is continuously differentiable and twice differentiable with respect to $\lambda$.
2. For $j = 1, 2$ there exist constants $C_j$ such that the derivatives with respect to $\lambda$ satisfy

$$\|D_j^\lambda P(\lambda, x)\| \leq C_j$$

for all $(\lambda, x) \in \Lambda \times K$.
3. For all $\lambda \in \Lambda$ and $x, y \in K$ with $x \neq y$, define

$$T_{x,y}(\lambda) := \frac{P_\lambda(x) - P_\lambda(y)}{|x - y|}.$$ Then there exists a constant $C_T > 0$ such that the property

$$|T_{x,y}(\lambda)| \leq C_T$$

implies that

$$\det \left( D_\lambda T_{x,y}(\lambda) (D_\lambda T_{x,y}(\lambda))^T \right) \geq C_T^2.$$ (4) There exists a constant $C_L$ such that

$$\|D_\lambda^2 T_{x,y}(\lambda)\| \leq C_L$$

for all $\lambda \in \Lambda$ and $x, y \in K$ with $x \neq y$.

Next we verify the analogue of [Ma, Lemma 18.9].

Lemma 2.5. Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable with $\mathcal{H}^m(E) < \infty$ and let $\delta > 0$. For $\mathcal{L}^l$-almost all $\lambda \in \Lambda$ we have for $\mathcal{H}^m$-almost all $a \in E$ either

$$\limsup_{s \to 0} \sup_{0 < r < \delta} (rs)^{-m} \mathcal{H}^m(E \cap X(a, r, \lambda, s)) = 0 \quad (2.1)$$

or

$$\limsup_{s \to 0} \sup_{0 < r < \delta} (rs)^{-m} \mathcal{H}^m(E \cap X(a, r, \lambda, s)) = \infty \quad (2.2)$$

or

$$(E \setminus \{a\}) \cap P_{\lambda}^{-1}(P_\lambda(a)) \cap B(a, \delta) \neq \emptyset. \quad (2.3)$$

Proof. Given $\delta > 0$ and $a \in E$, we prove that for $\mathcal{L}^l$-almost all $\lambda \in \Lambda$ either (2.1), (2.2) or (2.3) holds. Then the claim follows by Fubini’s theorem. The measurability arguments needed for applying Fubini’s theorem are similar as those in [F, Lemma 3.3.2]. We may clearly suppose that $E \subset K$ for some compact $K \subset \mathbb{R}^n$, and furthermore, by [Ma, Theorem 1.10] $E$ may be assumed to be $\sigma$-compact.

Fix $a \in E$, $\lambda_0 \in \Lambda$ and $0 < \delta < \delta_0$ such that $B(\lambda_0, 2\delta_0) \subset \Lambda$. Let $V \subset \mathbb{R}^l$ be an $m$-dimensional linear subspace and let $V_{\lambda_1} = V + \lambda_1$ for all $\lambda_1 \in \Lambda$. For all $\lambda_1 \in B(\lambda_0, \delta_0)$, define a measure $\Psi_{V_{\lambda_1}}$ on $B(\lambda_0, 2\delta_0) \cap V_{\lambda_1}$ by

$$\Psi_{V_{\lambda_1}}(A) := \sup_{0 < r < \delta} r^{-m} \mathcal{H}^m(E \cap B(a, r) \cap V_{\lambda_1}(A))$$
for all \( A \subset B(\lambda_0, 2\delta_0) \cap V_{\lambda_1} \), where
\[
L_{V_{\lambda_1}}(A) := \bigcup_{\lambda \in A} P^{-1}_\lambda(P\lambda(a)).
\]

The set
\[
C_{V_{\lambda_1}} := \{ \lambda \in B(\lambda_0, 2\delta_0) \cap V_{\lambda_1} \mid (E \setminus \{a\}) \cap L_{V_{\lambda_1}}(\{\lambda\}) \cap B(a, \delta) \neq \emptyset \}
\]
is \( \mathcal{H}^m \)-measurable. This follows from the fact that it is \( \sigma \)-compact which can be seen as follows: defining a continuous function
\[
\tilde{P} : (B(\lambda_0, 2\delta_0) \cap V_{\lambda_1}) \times \mathbb{R}^n \to \mathbb{R}^m, \quad \tilde{P}(\lambda, x) := P\lambda(x) - P\lambda(a),
\]
and \( \sigma \)-compact sets
\[
S_1 := \{(\lambda, x) \in (B(\lambda_0, 2\delta_0) \cap V_{\lambda_1}) \times \mathbb{R}^n \mid \tilde{P}(\lambda, x) = 0\}
\]
and
\[
S_2 := S_1 \cap (B(\lambda_0, 2\delta_0) \times ((E \setminus \{a\}) \cap B(a, \delta))),
\]
we conclude that \( C_{V_{\lambda_1}} = \Pi_\lambda(S_2) \), where \( \Pi_\lambda : \Lambda \times \mathbb{R}^n \to \Lambda \) is the projection \( \Pi_\lambda(\lambda, x) = \lambda \). Thus \( C_{V_{\lambda_1}} \) is \( \sigma \)-compact.

Let \( D_{V_{\lambda_1}} := (B(\lambda_0, 2\delta_0) \cap V_{\lambda_1}) \setminus C_{V_{\lambda_1}} \). From the definitions of \( \Psi_{V_{\lambda_1}} \) and \( C_{V_{\lambda_1}} \), we deduce that \( \Psi_{V_{\lambda_1}}(D_{V_{\lambda_1}}) = 0 \). Now [Ma Theorem 18.5] implies that for \( \mathcal{H}^m \)-almost all \( \lambda \in B(\lambda_0, \delta_0) \cap V_{\lambda_1} \) either
\[
\limsup_{t \downarrow 0} t^{-m}\Psi_{V_{\lambda_1}}(B(\lambda_0, 2\delta_0) \cap V_{\lambda_1} \cap B(\lambda, t)) = 0 \tag{2.4}
\]
or
\[
\limsup_{t \downarrow 0} t^{-m}\Psi_{V_{\lambda_1}}(B(\lambda_0, 2\delta_0) \cap V_{\lambda_1} \cap B(\lambda, t)) = \infty \tag{2.5}
\]
or
\[
\lambda \in C_{V_{\lambda_1}}. \tag{2.6}
\]

Applying Fubini’s theorem we see that for \( \mathcal{L}^l \)-almost all \( \lambda \in B(\lambda_0, \delta_0) \) either (2.4), (2.5) or (2.6) holds with \( V_{\lambda_1} \) replaced by \( V_\lambda \). (The measurability proofs needed here can be dealt with in a similar manner as those in [F Lemma 3.3.3].) Note that here the exceptional set of \( \mathcal{L}^l \)-measure zero depends on the \( m \)-plane \( V \). Hence it is sufficient to find a finite collection of linear \( m \)-planes \( V^1, \ldots, V^k \subset \mathbb{R}^l \) and \( C > 0 \) such that for all \( \lambda \in B(\lambda_0, \delta_0) \)
\[
\bigcup_{j=1}^k B(a, r) \cap L_{V_{\lambda_1}^j}(B(\lambda_0, 2\delta_0) \cap V_{\lambda_1}^j \cap B(\lambda, C^{-1}s)) \setminus \{a\} \subset X(a, r, \lambda, s)
\]
and for every small enough \( s > 0 \). Indeed, by [IJN Lemma 3.3] there are \( C > 0 \) and \( s_0 > 0 \) such that for any \( 0 < s < s_0 \) and for any \( x \in X(a, r, \lambda, s) \) there exists an \( m \)-dimensional coordinate plane \( W \) such that \( x \in L_{W}(B(\lambda_0, 2\delta_0) \cap W_{\lambda} \cap \).
Then for every $a \in \Lambda$, assume that $B$ is a linear subspace of $\mathbb{R}^l$. Finally, the first inclusion is true for any $m$-plane since, by transversality, $\|D_{\lambda}T_{x,a}(\lambda)\|$ is bounded. 

For the “if”-part of Theorem 2.2, we need the following lemma.

**Lemma 2.6.** Assume that $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m\}_{\lambda \in \Lambda}$ is a transversal family of mappings. Then for every $a \in \mathbb{R}^n$, for every $m$-dimensional $C^1$-submanifold $S \subset \mathbb{R}^n$ containing $a$ and for $\mathcal{L}^1$-almost all $\lambda \in \Lambda$ there exist $\gamma > 0$ and $r > 0$ such that $|P_\lambda(x) - P_\lambda(y)| \geq \gamma|x - y|$ for all $x, y \in B(a, r) \cap S$.

**Proof.** We begin by showing that $P_\lambda$ is a submersion, that is, $D_\lambda P_\lambda(a)$ has rank $m$ at every point $a \in \mathbb{R}^n$. Here $D_\lambda P_\lambda$ is the derivative of $P_\lambda$ with respect to $x$.

Let $\lambda_0 \in \Lambda$ and let $D_\lambda P_\lambda(a) \subset \mathbb{R}^n$ be the kernel of $D_\lambda P_\lambda(a)$. By [JN, Lemma 3.2], Definition 2.4 implies that for any unit vector $e \in \ker D_\lambda P_\lambda(a)$ one can find an $m$-dimensional plane $V^e \subset \mathbb{R}^l$ such that the mapping $g^e : V^e_0 \cap \Lambda \to \mathbb{R}^m$, defined as $g^e(\lambda) := D_\lambda P_\lambda(a)(e)$, is a diffeomorphism (onto its image) on a small neighbourhood of $\lambda_0$. Furthermore, the parallelepiped $Dg^e(\lambda_0)([-1, 1]^m)$ is uniformly thick – by this we mean that the lengths of the edges and the angles between the edges are bounded from below by a constant which is independent of $\lambda_0 \in \Lambda$, $e \in \ker D_\lambda P_\lambda(a)$ and $a \in K$ for any fixed compact $K \subset \mathbb{R}^n$.

Since $D_\lambda P_\lambda(a)$ is continuous in $\lambda$ and $\dim \ker D_\lambda P_\lambda(a) \geq n - m$ for all $\lambda \in \Lambda$, there exist $e \in \ker D_\lambda P_\lambda(a)$ such that $e = \lim_{\lambda \to \lambda_0} e_\lambda$, where $e_\lambda \in \ker D_\lambda P_\lambda(a)$. Define a function $f^e : V^e_0 \cap \Lambda \to \mathbb{R}^m$ by

$$f^e(\lambda) := e - \text{proj}_{\ker D_\lambda P_\lambda(a)}(e),$$

where $\text{proj}_{\ker}$ is the orthogonal projection onto $V \subset \mathbb{R}^n$. Observe that $g^e(\lambda) = D_\lambda P_\lambda(a)(f^e(\lambda))$. The fact that $Dg^e(\lambda_0)([-1, 1]^m)$ is uniformly thick implies that the same is true for $Df^e(\lambda_0)([-1, 1]^m)$.

Assuming that $\dim \ker D_\lambda P_\lambda(a) > n - m$ there are at most $m - 1$ directions perpendicular to $\ker D_\lambda P_\lambda(a)$. Thus $Df^e(\lambda_0)([-1, 1]^m)$ intersects $\ker D_\lambda P_\lambda(a)$ in a set containing a line segment of positive length. In particular, there is a unit vector $v \in V^e$ satisfying $Df^e(\lambda_0)(v) \in \ker D_\lambda P_\lambda(a)$ which, in turn, gives the contradiction $Dg^e(\lambda_0)(v) = 0$ and completes the proof that $P_\lambda$ is a submersion.

We proceed by verifying that for every $a \in \mathbb{R}^n$ and for every $m$-dimensional linear subspace $W \subset \mathbb{R}^n$ we have $\ker D_\lambda P_\lambda(a) \cap W = \{0\}$ for $\mathcal{L}^1$-almost all $\lambda \in \Lambda$.

Fix $\lambda_0 \in \Lambda$ such that $\ker D_\lambda P_\lambda(a) \cap W = U$ with $\dim U = k$, where $1 \leq k \leq m$. Clearly, it is sufficient to prove that there is $\delta > 0$ such that $\ker D_\lambda P_\lambda(a) \cap W = \{0\}$ for $\mathcal{L}^1$-almost all $\lambda \in B(\lambda_0, \delta)$. Let $e_1, \ldots, e_k$ be an orthonormal basis for $U$ and let $M := \langle W \cup \ker D_\lambda P_\lambda(a) \rangle$ be the subspace spanned by $W$ and $\ker D_\lambda P_\lambda(a)$. Observe that $k = \dim M$. For all $i = 1, \ldots, k$, consider the functions $f^{e_i}$ defined above. Since $P_\lambda$ is a submersion for all $\lambda$, we see that $\ker D_\lambda P_\lambda(a)$ tends to $\ker D_\lambda P_\lambda(a)$ as $\lambda \to \lambda_0$. Thus $Df^{e_i}(\lambda_0)([-1, 1]^m)$ is perpendicular to $\ker D_\lambda P_\lambda(a)$ for all $i = 1, \ldots, k$. In particular, for each $i$ there is a $k$-dimensional plane $W^{e_i} \subset V^{e_i}$ such that $Df^{e_i}(\lambda_0)(W^{e_i}) = M$. This implies the existence of $v \in \mathbb{R}^l$ such
that $Df^{c_1}(\lambda_0)v, \ldots, Df^{c_k}(\lambda_0)v$ are linearly independent. Hence, for a sufficiently small $\varepsilon > 0$ we have $\ker D_xP_\lambda(a) \cap W = \{0\}$ for $\mathcal{L}^1$-almost all $\lambda \in B(\lambda_0, \varepsilon) \cap \{v\}_{\lambda_0}$.

Theorem 1.2.

By continuity, there exists $\delta > 0$ such that this is valid if we replace $\lambda_0$ by any $\lambda_1 \in B(\lambda_0, \delta)$. Finally, Fubini’s theorem implies that $\ker D_xP_\lambda(a) \cap W = \{0\}$ for $\mathcal{L}^1$-almost all $\lambda \in B(\lambda_0, \delta)$.

The claim follows by choosing $W = T_aS$ and using the fact that since $P_\lambda$ is a smooth submersion it is locally a fibration (see [GHL, Remark 1.92]).

Now we are ready to prove the generalization of the Besicovitch-Federer projection theorem.

**Proof of Theorem 1.2.** The proof of the “only if”-part of Theorem 1.2 is similar to the one given in [Ma, p. 257-258]. Indeed, defining $E_{1,\delta}(\lambda)$ and $E_{2,\delta}(\lambda)$ as in Lemmas 2.2 and 2.3, setting

$$E_{3,\delta}(\lambda) := \{a \in E | (E \setminus \{a\}) \cap P_\lambda^{-1}(P_\lambda(a)) \cap B(a, \delta) \neq \emptyset\},$$

and applying Lemmas 2.2, 2.3 and 2.5, we conclude, as in [Ma, p. 257-258], that the claim holds.

To prove the “if”-part of the theorem, assume to the contrary that there is an $m$-rectifiable $F \subset E$ with $\mathcal{H}^m(F) > 0$. According to [F, Theorem 3.2.29], there exist $m$-dimensional $C^1$-submanifolds $S_1, S_2, \ldots \subset \mathbb{R}^n$ such that $\mathcal{H}^m(F \setminus \bigcup_{i=1}^\infty S_i) = 0$.

Fixing $i$ and letting $a$ be a density point of $F \cap S_i$, Lemma 2.6.2 implies the existence of $r > 0$ and $\varepsilon > 0$ such that for $\mathcal{L}^i$-almost all $\lambda \in \Lambda$ we have $|P_\lambda(x) - P_\lambda(y)| \geq \gamma|x - y|$ for all $x, y \in B(a, r) \cap S_i$. This in turn gives that $\mathcal{H}^m(P_\lambda(F)) \geq \gamma^m\mathcal{H}^m(F \cap B(a, r) \cap S_i) > 0$ for $\mathcal{L}^i$-almost all $\lambda \in \Lambda$ which is a contradiction.

**Remark 2.7.** In the “only if”-part of the previous proof we did not use the assumption that the mapping $(\lambda, x) \mapsto P_\lambda(x)$ is continuously differentiable in $x$ (see Definition 2.4). It is sufficient to suppose that it is Lipschitz continuous. The differentiability in the second coordinate is needed only for the “if”-part of Theorem 1.2.

3. Dynamics of the geodesic flow

3.1. Pairs of pants and right angle octagons. The contents of this subsection and the following one are standard, see e.g. [Se]. Suppose $S$ is a pair of pants with cuff lengths $a, b$ and $c$ (See Figure 1). The seams of $S$ are the shortest geodesic segments connecting the cuffs. Consider the seam connecting the cuffs $a$ and $c$ and code by $\beta$ and $\overline{\beta}$ the two sides of this seam. Analogously, consider the seam connecting the cuffs $b$ and $c$ and code by $\alpha$ and $\overline{\alpha}$ its two sides. If we cut $S$ along these two seams, we obtain a hyperbolic octagon with right angles. We label the four sides of this octagon corresponding to the cut seams by the code of the part of $S$ inside the octagon. The $c$ cuff is cut into two geodesics of length $c/2$, which we label as $c_1$ and $c_2$. We see consecutively the labels $\alpha, b, \overline{\alpha}, c_1, \overline{\beta}, a, \beta$ and $c_2$ on the sides of the octagon (up to possibly exchanging the role of $\alpha$ and $\overline{\alpha}, \beta$ and $\overline{\beta}$, or $c_1$ and $c_2$). Let $R$ be a copy of the octagon inside the hyperbolic space $\mathbb{H}^2$. 

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For $\tau = \alpha, \overline{\alpha}, \beta$ or $\overline{\beta}$, let $\varphi_\tau$ be the Möbius transformation sending the geodesic $\tau$ on the geodesic $\overline{\tau}$ (with the convention that $\overline{\overline{\tau}} = \tau$) and the half-plane separated by the complete extension of $\tau$ containing $R$ onto the half-plane separated by the complete extension of $\overline{\tau}$ not containing $R$. We have $\varphi_\tau = \varphi_{\overline{\tau}}^{-1}$ for all $\tau$. The union of $S$ and its boundary $\partial S$ is obtained from the closure of $R$ by identifying the sides $\alpha$ and $\overline{\alpha}$ using $\varphi_\alpha$ and by identifying $\beta$ and $\overline{\beta}$ using $\varphi_\beta$. Moreover, the geodesics extending the $\tau$ sides do not intersect one another, and therefore, by the classical ping-pong argument, $\varphi_\alpha$ and $\varphi_\beta$ generate a free group $G$. The images of the interior of $R$ by $G$ are disjoint and the region containing $R$ and delimited by the four extensions of the $\tau$ geodesics is a fundamental domain for $G$. For all $g \in G$, label the geodesic sides of $gR$ by the image of the labelling of the geodesic sides of $R$. In a consistent way, each geodesic segment of the form $g\tau$ has two opposite labels corresponding to the two images of $R$ that it separates.

We say that a geodesic $\gamma$ in $T^1\mathbb{H}^2$ starts from $R$ if $\gamma(0) \in \partial R$ and there is some $t > 0$ with $\gamma(t) \in R$. Let $\gamma$ be a geodesic starting from $R$. It corresponds to a geodesic in $C(S)$ (recall the definition (1.1)), if and only if it never cuts the sides of $G(R)$ labelled as $a, b, c_1$ or $c_2$. In other words, $\gamma$ intersects only $\tau$ geodesics. Record the interior label of these geodesics successively as $\omega_n, n \in \mathbb{Z}, \omega_0$ being the label of the side by which the geodesic $\gamma$ enters $R$. This sequence is called the cutting sequence of $\gamma$. The cutting sequence of any geodesic in $C(S)$ is a reduced infinite word in $\alpha, \overline{\alpha}, \beta$ and $\overline{\beta}$, where reduced means that the succession $\tau \overline{\tau}$ is not permitted. Since two infinite geodesics in $\mathbb{H}^2$ with distinct supports are not at a bounded distance from each other, any cutting sequence is the cutting sequence of a unique geodesic. The boundary geodesics correspond to the reduced words $(\alpha)^\infty, (\overline{\alpha})^\infty, (\beta)^\infty, (\overline{\beta})^\infty, (\alpha \overline{\beta})^\infty$ and $(\overline{\alpha} \beta)^\infty$.

Consider the four disjoint complete geodesics in $\mathbb{H}^2$ extending the segments $\alpha, \overline{\alpha}, \beta$ and $\overline{\beta}$ of the previous subsection. Each of them cut $S^1$, the circle at infinity, into two intervals. Write $A, \overline{A}, B$ and $\overline{B}$ for the interval separated from $R$ by the geodesic $\alpha, \overline{\alpha}, \beta$ and $\overline{\beta}$, respectively. Let $\varphi$ be defined on each $T$ ($T = A, \overline{A}, B, \overline{B}$) by the corresponding Möbius transformation $\varphi_\tau$. The mapping $\varphi$ is expanding.
(see [Se]) and \( \varphi(T) = S^1 \setminus \text{int } T \), where the interior of a set in \( S^1 \) is denoted by \( \text{int} \).

In particular, \( \varphi(T) \) contains the three intervals different from \( T \).

We define the boundary expansion of a point \( \xi \in S^1 \). If \( \xi \) does not belong to \( \text{int}(A \cup \overline{A} \cup B \cup \overline{B}) \), stop here. Otherwise, let \( \xi_0 = \alpha, \overline{\alpha}, \beta \) or \( \overline{\beta} \) accordingly. Apply then the procedure to \( \varphi(\xi) \) and iterate. Every point has an empty, finite or infinite sequence of symbols attached, which is called its boundary expansion. Boundary expansions are reduced words in \( \alpha, \overline{\alpha}, \beta \) and \( \overline{\beta} \). The set of points with an infinite boundary expansion is a Cantor subset \( \Omega \subset S^1 \). For a geodesic \( \gamma \) starting in \( R \), the positive part of the coding sequence is the boundary expansion of the limit point \( \gamma(+\infty) \). Similarly, the sequence \( \overline{\omega}_0, \overline{\omega}_{-1}, \overline{\omega}_{-2}, \ldots \) is the boundary expansion of \( \gamma(-\infty) \). This defines a one-to-one correspondence \( \Psi \) between cutting sequences of geodesics starting from \( R \) and the set \[
(\Omega \times \Omega)^* = \{ (\xi, \eta) \in \Omega \times \Omega | \xi_0 \neq \eta_0 \},
\]

namely, \( \Psi(\omega) = (\xi, \eta) \) where \( \xi_i = \omega_{i+1} \) and \( \eta_j = \overline{\omega_j} \) for \( i, j = 0, 1, \ldots \).

Clearly, if \( \{\omega_n\}_{n \in \mathbb{Z}} \) is the cutting sequence of the geodesic \( \gamma \), the shifted sequence \( \{\omega'_n\}_{n \in \mathbb{Z}}, \omega'_n = \omega_{n+1} \) is associated to the geodesic \( \gamma(\cdot + \ell) \), where \( \ell \) is the first positive time \( t \) when \( \gamma(t) \) is not in \( R \). Consider the mapping

\[
\Phi : \{ (\xi, \eta, s) | (\xi, \eta) \in (\Omega \times \Omega)^*, 0 < s < \ell(\Psi^{-1}(\xi, \eta)) \} \rightarrow T^1 \mathbb{H}^2
\]

which associates to \( (\xi, \eta, s) \) the point \( (x, v) \in T^1 \mathbb{H}^2 \) such that the geodesic \( \gamma \) with initial condition \( (x, v) \) satisfies \( \gamma(+\infty) = \xi, \gamma(-\infty) = \eta \) and \( \gamma(-s) \) is entering into \( R \). The mapping \( \Phi \) is a restriction of the usual chart of \( T^1 \mathbb{H}^2 \) given by \( (S^1 \times S^1)^* \times \mathbb{R} \). Its image is a subset of \( T^1 R \) which is identified with \( NW = D(S) \cup T^1(\partial S) \) (recall (1.2)). Metric properties of \( NW \), and consequently those of \( C(S) \), will be read from metric properties of \( \Omega \) through this Lipschitz mapping \( \Phi \). Moreover, from the above symbolic representation, we see that \( NW \) is the nonwandering set of the geodesic flow on \( T^1 S \cup T^1(\partial S) \). The geodesic flow, restricted to \( D(S) \cup T^1(\partial S) \), is therefore represented by a suspension over the set of reduced words with suspension function \( \ell(\omega) \), where \( \ell(\omega) \) is the time spent in \( R \) by the geodesic with cutting sequence \( \omega \).

### 3.2. Markov repellers

We use properties of Markov repellers as established by Bowen and Ruelle [R1]. A Markov repeller is an expanding piecewise \( C^{1+\alpha} \) map of the real line into itself with a finite family of disjoint intervals \( A_i, i \in J \), such that if \( f(A_i) \) intersects the interior of some \( A_j \), then \( f(A_i) \) contains \( A_j \). The set of points which remain in \( \bigcup_{j \in J} A_j \) under applications of all the iterates \( f^n, n \in \mathbb{N} \), is a Cantor set \( X \). The set \( X \) is invariant under \( f \). For any \( f \)-invariant probability measure \( \mu \) on \( X \) consider the metric entropy \( h_\mu(f) \). For any continuous function \( g \) on \( X \), define the pressure \( P(g) \) by

\[
P(g) := \sup_\mu \left\{ h_\mu(f) + \int_X g \, d\mu \right\},
\]
where $\mu$ varies over all $f$-invariant probability measures on $X$. Assume that $f$ is topologically transitive. Then there exists a unique $s$ with $0 < s < 1$ such that $P(-s\ln|f'|) = 0$. The number $s$ is both the Hausdorff dimension and the packing dimension of $X$. More precisely, there exists a unique $f$-invariant probability measure $\mu_0$ on $X$ such that

$$h_{\mu_0} - s \int_X \ln |f'| \, d\mu_0 = 0.$$ 

The measure $\mu_0$ is Ahlfors $s$-regular on $X$: for all $\varepsilon$ small enough and for all $x \in X$ the ratio $\mu(B(x, \varepsilon))\varepsilon^{-s}$ is bounded away from 0 and infinity. In particular, $0 < \mathcal{H}^s(X) < \infty$.

Since the Patterson measure $\nu_0$ is also Ahlfors regular [SI, Section 3], the measures $\nu_0$ and $\mu_0$ are mutually absolutely continuous with bounded densities. The geodesic flow invariant measure $m$ constructed in [SI, Section 4] (called the Bowen-Margulis-Patterson-Sullivan measure) has support $D$, is the measure of maximal entropy $s$ for the geodesic flow on $T^1S$ and has dimension $1 + 2s$.

Finally, if $(a, b, c) \mapsto f_{a,b,c}$ is a real analytic family of piecewise $C^{1+\varepsilon}$ expanding mappings, then the function $(a, b, c) \mapsto \dim_{\mathcal{H}}(X)$ is real analytic as well (see for example [K2, Corollary 7.10 and Section 7.28]).

3.3. **Proof of Theorem 1.1** For fixed $a$, $b$ and $c$, consider the set $\Omega_{a,b,c} \subset S^1$ of the previous subsection. It is a transitive Markov repeller for the mapping $\varphi_{a,b,c}$. The mapping $\varphi_{a,b,c}$ is given by a piecewise Möbius transformation, and therefore, it belongs to a semi-algebraic variety of piecewise analytic mappings. Moreover, $(a, b, c) \mapsto \varphi_{a,b,c}$ is real analytic, and thus the function $(a, b, c) \mapsto \delta(a, b, c) = \dim_{\mathcal{H}}(\Omega_{a,b,c})$ is real analytic. In particular, there is a two-dimensional submanifold of values $a$, $b$ and $c$ such that $\delta(a, b, c) = 1/2$.

**Proposition 3.1.** Assume that $\delta(a, b, c) = 1/2$. Then the nonwandering set $NW$ is purely 2-unrectifiable and has positive and finite 2-dimensional Hausdorff measure.

**Proof.** It is enough to consider $D(S)$ since $T^1(\partial S)$ is 1-dimensional. Recalling that $\tau\pi$ is a forbidden word for $\xi \in \Omega$, the above discussion implies that $D(S) = \cup_{i=1}^n U_i$ and each $U_i$ is Lipschitz equivalent to an open subset of $\Omega \times \Omega \times I$, where $I$ is a real interval. Since the measure $\mu_0$ is Ahlfors 1/2-regular on $\Omega$, the measure $\mu_0 \times \mu_0 \times \mathcal{L}^1$ is Ahlfors 2-regular on $\Omega \times \Omega \times I$. Therefore $\dim_{\mathcal{H}}(\Omega \times \Omega \times I) = 2$ and $0 < \mathcal{H}^2(\Omega \times \Omega \times I) < \infty$. Thus $\dim_{\mathcal{H}}(D(S)) = 2$ and $0 < \mathcal{H}^2(D(S)) < \infty$. For the first claim it is enough to notice that $\Omega \times \Omega$ is purely 1-unrectifiable, since it is a product of two Cantor sets of dimension $1/2$ [Ma, Example 15.2]. Thus the product $\Omega \times \Omega \times I$ is purely 2-unrectifiable, and so is $D(S)$. \hfill \Box

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** In [JJL, Section 3] it is shown that locally there exist an open set $U \subset T^1S$, bi-Lipschitz mappings $\psi_1 : U \to I^3$ and $\psi_2 : I^2 \to \Pi(U)$ and a
smooth mapping \( P : I^3 \to I^2 \) such that

\[
\Pi|_U = \psi_2 \circ P \circ \psi_1,
\]

where \( I \subset \mathbb{R} \) is the open unit interval. The mapping \( P \) is defined by \( P(y_1, y_2, t) = (P_t(y_1, y_2), t) \), where \( \{P_t : I^2 \to I\} \) is a transversal family of smooth mappings.

By Proposition 3.1, the set \( \psi_1(D(S) \cap U) = E \times I \) is purely 2-rectifiable. Thus \( E \subset I^2 \) is purely 1-rectifiable. Furthermore, \( P(E \times I) = \bigcup_{t \in I} P_t(E) \times \{t\} \). By Theorem 1.2,

\[
\mathcal{H}^1(P_t(E)) = 0 \text{ for } \mathcal{L}^1\text{-almost all } t \in I,
\]
giving \( \mathcal{H}^2(P(E \times I)) = 0 \) by Fubini’s theorem. This implies that

\[
\mathcal{H}^2(\Pi(D(S) \cap U)) = \mathcal{H}^2((\psi_2 \circ P \circ \psi_1)(D(S) \cap U))
\]

\[
= \mathcal{H}^2(\psi_2(P(E \times I))) = 0,
\]
since \( \psi_2 \) is a bi-Lipschitz mapping. The claim follows from the fact that \( T^1S \) can be covered by countably many open sets \( U \).

Corollary 3.2. The Bowen-Margulis-Patterson-Sullivan measure \( m \) is 2-dimensional, its support \( \text{spt } m = D(S) \) is 2-dimensional and \( \mathcal{L}^2(\text{spt } (\Pi, m)) = 0 \).

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