Ricci Flow method in the existence problem of
the Kähler-Einstein metrics

LIU CHAO

1 Introduction

This note illustrates the Ricci flow method based on the Cao.H.D’s paper[1] and
Yau.S.T’s paper[4], and tries to explain the method in detail, especially in some
calculations. Jian Song and Weinkove’s note[9] used some other estimates to
obtain the result, this paper will explain some of their estimates as well. This
note was a seminar lecture note in 2022 summer when the author was giving
lectures on the geometry analysis seminar reasearching the Ricci flow method.
The part of the important zero order estimate is going to be added in a few
days. The level of the author is limited, if there are any errors, please do not
hesitate to advise. Any comments will be grateful.

2 The Ricci Flow equation

Let $M$ be a compact Kähler manifold with the Kähler dimension $n$ and the
Kähler metric $ds = g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. We will use the Einstein summation through
the whole article.

Let $R_{i\bar{j}} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{i\bar{j}})$ be the Ricci curvature of the corresponding
metric, and the Ricci Form, i.e. the $(1,1)$ tensor $\frac{\sqrt{-1}}{2\pi} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is closed so
we can define the cohomology class of it which is the first chern class $C_1(M)$ of
$M$.

We consider the Ricci Flow equation $\frac{\partial g_{i\bar{j}}}{\partial t} = -2R_{i\bar{j}} + \frac{\sqrt{-1}}{2\pi} T_{i\bar{j}}$ and its complex
version:

$$\frac{\partial \tilde{g}_{i\bar{j}}}{\partial t} = -\tilde{R}_{i\bar{j}} + T_{i\bar{j}}, \quad \tilde{g}_{i\bar{j}} = g_{i\bar{j}} \text{ at } t = 0 \quad (2.1)$$

where $\tilde{R}_{i\bar{j}}$ is the Ricci tensor of the $\tilde{g}_{i\bar{j}}$, $T_{i\bar{j}}$ is a representation of the first
Chern class $C_1(M)$, actually we can choose any representation satisfying our
requirement, such as a Kähler-Einstein metric.

Then our idea is: first prove the solution exists all the time, then prove when
t goes infinitely, $\tilde{g}_{i\bar{j}}$ converges to a definite $\tilde{g}_{i\bar{j}}(\infty)$ and hence $\frac{\partial \tilde{g}_{i\bar{j}}}{\partial t}$ converges to 0,
then we get $-\tilde{R}_{i\bar{j}} + T_{i\bar{j}} = 0$, then we get $T_{i\bar{j}}$ will be the Ricci tensor of $\tilde{g}_{i\bar{j}}(\infty)$
so the $\tilde{g}_{i\bar{j}}(\infty)$ is the metric we want.

However, the equation (2.1) is too abstract to solve, therefore, we replace
it. Due to $\frac{\sqrt{-1}}{2\pi} T_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is in the class of first chern class of $M$, so is
can assume \( \partial F \) equation \( \Delta \) cohomology group \tfrac{1}{2}(M, R) = \{ \partial \text{closed real (1,1 forms)} \}, \) so they deviate by a term which is in the kernel of the boundary operator \( d \), i.e.

\[
\frac{\partial^2 f}{\partial z_i \partial z_j} : T_{ij} - R_{ij} = \frac{\partial^2 f}{\partial z_i \partial z_j} .
\]

So we take \( t = 0 \) in the equation, similary, we can assume

\[ g_{ij} - g_{ij} = \frac{\partial^2 u}{\partial z_i \partial z_j} , \]

for \( u \in \text{C}^\infty(M \times [0, T]) \), \( T \) is nonegative and less or equal to infinity, in order to satisfy the initial condition, \( u(0) = 0 \), we caculatie the new scaler equation replacing \( g_{ij} \) by \( \tilde{g}_{ij} = g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j} : \)

The left hand side is: \( \frac{g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j}}{g_{ij}} = 0 + \frac{\partial^2 u}{\partial z_i \partial z_j} \cdot \frac{\partial^2 f}{\partial z_i \partial z_j} (\frac{\partial u}{\partial t} ) \)

The right hand side is: \(-R_{ij} + \frac{\partial^2 f}{\partial z_i \partial z_j} \) while we can calculate the Ricci tensor explicitly in terms of \( g_{ij} \) and \( \tilde{g}_{ij} : \)

\[ R_{ij} = -\frac{\partial^2}{\partial z_i \partial z_j} \log \det(g_{ij}) \]

so the right hand side is equal to:

\[-(\frac{\partial^2}{\partial z_i \partial z_j} \log \det(\tilde{g}_{ij}) - \frac{\partial^2}{\partial z_i \partial z_j} \log \det(g_{ij}) + \frac{\partial^2 f}{\partial z_i \partial z_j}) = \frac{\partial^2}{\partial z_i \partial z_j} ((\log \det(g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j}) - \log \det(g_{ij})) + \frac{\partial^2 f}{\partial z_i \partial z_j} \]

So we get:

\[ \frac{\partial^2}{\partial z_i \partial z_j} \frac{\partial u}{\partial t} = \frac{\partial^2}{\partial z_i \partial z_j} ((\log \det(g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j}) - \log \det(g_{ij})) + \frac{\partial^2 f}{\partial z_i \partial z_j} \]

Finally, \( \frac{\partial^2}{\partial z_i \partial z_j} (\frac{\partial u}{\partial t} \cdot \log \det(g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j}) + \log \det(g_{ij}) - f) = 0 \), the function \( (\frac{\partial u}{\partial t} \cdot \log \det(g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j}) + \log \det(g_{ij}) - f) \) satisfies this uniform elliptic equation \( \Delta F = 0 \), then by the strong maximum principle, the maximum and the minimum attain on the boundry, but \( M \) is compact, so only non-boundry points exist, then this function can only be a constant relative to \( \partial \bar{\partial} \), this means the term in the partials is equal to a smooth function relative to \( t : \)

\[ \frac{\partial u}{\partial t} = \log \det(g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j}) - \log \det(g_{ij}) + f \]

\( \phi(t) \) should satisfy the compatibility condition:

\[ \int_M e^{\frac{\partial u}{\partial t}} f dV = e^{\phi(t)} \text{Vol}(M) \]

Actually, this compatibility condition wants to say that the volume of the compact manifold stays invariant during the deformation of the metric \( \tilde{g}^{ij}(t) \).

From the equation above, \( \frac{\partial u}{\partial t} - f = \log \det(g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j}) - \log \det(g_{ij}) + f + \phi(t) \)
we take the exponential of the both sides and integrate them on \( M : \)

\[ \int_M e^{\frac{\partial u}{\partial t} - f} dV = \int_M e^{\log \det(g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j}) - \log \det(g_{ij}) + \phi(t)} dV \]

\[ \int_M e^{\frac{\partial u}{\partial t} - f} dV = e^{\phi(t)} \int_M e^{\log \det(g_{ij})} dV \]

While \( dV = \det(g_{ij}) \wedge_{i=1}^n (\sqrt{\frac{\partial^2}{\partial z_i \partial z_j}} d\bar{z}_i \wedge dz^j) = \sqrt{n} \]
\[ d\bar{V} = \det(\tilde{g}_{ij}) \wedge_{i=1}^n (\sqrt{\frac{\partial^2}{\partial z_i \partial z_j}} d\bar{z}_i \wedge dz^j) = \sqrt{n} \]
so the above equation changes to:

\[ \int_M e^{\frac{\partial u}{\partial t} - f} dV = e^{\phi(t)} \int_M d\bar{V} \]

\( d\bar{V} = e^{\phi(t)} \int_M dV = e^{\phi(t)} \int_M dV = e^{\phi(t)} \text{Vol}(M) \)

To prove the long time existence of the sultion of this parabolic equation, generally, we need to give up to third order estimate. In the proof of the existence and the uniform convergence, we will use these estimates.
3 Existence for the solution in all time

Actually, the goal equation is given initial value:
\[
\frac{\partial u}{\partial t} = \log \det(g_{ij} + \frac{\partial^2 u}{\partial x_i \partial x_j}) - \log \det(g_{ij}) + f
\]
\[u(x,t) = 0 \text{ when } t = 0.\]
And by the initial assumption, the solution exists in the interval (0,T] and the Kahler metric \(g_{ij} = g_{ij} + \frac{\partial^2 u}{\partial x_i \partial x_j}\) is positive definite so it's a Kahler metric for any \(t \in (0,T].\)

To prove the estimation, we need some notations. We differentiate the equation (3.1):
\[
\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = \frac{\partial}{\partial t}(\log \det(g_{ij} + \frac{\partial^2 u}{\partial x_i \partial x_j})) \quad \text{for } \frac{\partial}{\partial t}(f) = 0, \text{and } \frac{\partial}{\partial t}(\log \det(g_{ij})) = 0
\]
then
\[
\frac{\partial}{\partial t}(\log \det(g_{ij} + \frac{\partial^2 u}{\partial x_i \partial x_j})) = \frac{\partial}{\partial t}(\det(g_{ij}))
\]
\[= \frac{1}{\det(g_{ij})}\frac{\partial}{\partial t}g^{ij}\frac{\partial}{\partial t}g_{ij}\]
\[= \tilde{g}^{ij}\frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial x_i \partial x_j}) \quad \text{where } \tilde{g}^{ij} \text{ is the inverse of } \tilde{g}_{ij}. \]
It can not be ignored that the \(\tilde{g}^{ij}\frac{\partial}{\partial t}(\frac{\partial^2 u}{\partial x_i \partial x_j}) \) is a Einstein summation, \(\tilde{g}^{ij}\) is actually the \(((\tilde{g}_{ij})^{-1})^{ij}\).

So we get
\[
\frac{\partial}{\partial t}(\frac{\partial u}{\partial t}) = \tilde{g}^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(\frac{\partial u}{\partial t}) \quad \text{this is a parabolic equation, so by the maximum principle, at the time } t = 0, \text{we can consider this manifold with the initial metric is the bounder of the domain } M \times (0, T]. \text{ So we have}
\]
\[
\max_M |\frac{\partial u}{\partial t}| \leq \max_{t=0} |\frac{\partial u}{\partial t}| = \max_M |f|.
\]
Let \(\tilde{\Delta} = g_{ij} \frac{\partial^2}{\partial x_i \partial x_j}\) be the normalized Laplace of \(\tilde{g}_{ij}\), and \(\Delta = g_{ij} \frac{\partial^2}{\partial x_i \partial x_j}\) be the normalized Laplace of the \(g_{ij}\), \(\phi = \tilde{\Delta} - \frac{\partial^2}{\partial t^2}f\).

Zero order estimate

Let \(v = u - \frac{1}{\sqrt{Vol(M)}}\int_M u dV\) be the normalized of \(u\) such that \(\int_M v = 0\) then \(v\) satisfies the equation:
\[
\det(g_{ij} + \frac{\partial^2 v}{\partial x_i \partial x_j}) (\det(g_{ij}))^{-1} = e^F
\]
where \(F = \frac{\partial^2 u}{\partial t^2} - f\) in the paper of Yau, so we can directly use the calculation in Yau’s work, we get the following Lemma:

**Lemma 1**
\[\sup_{M \times (0, T]} |v| \leq C_2, \sup_{M \times (0, T]} \int_M |v| dV \leq C_3 \text{ for } C_2 \text{ and } C_3 \text{ constants}\]

**Remark** We are here to give a sketch proof of this zero order estimate following Yau’s method. We consider the Green’s function \(G(p, q)\) of the normalized Laplace operator of \(g_{ij}\), \(\Delta\) on \(M\), and let \(K\) be a constant which depends only on \(M\) such that \(G(p, q) + K \geq 0\). Then we find that
\[
\Delta v = \Delta u \quad \text{so } \tilde{g}_{ij} = g_{ij} + \frac{\partial^2 v}{\partial x_i \partial x_j} \quad \text{and } \int_M v = 0, \text{then by Yau’s work it shows}
\]
\[v(p) = -\int_M (G(p, q) + K) dq \quad \text{and } \sup_M \leq n \sup_{p \in M} \int_M (G(p, q) + K) dq, \text{while the right hand side depends only on } M, \text{hence we can consider it as a constant } C_2, \text{and we proceed by this estimate we have}
\]
\[\int_M |v| \leq \int_M |sup_{M} v| + \int_M |sup_{M} v| \leq (sup_{M} v) Vol(M) - \int_M v + (sup_{M} v) Vol(M) \leq 2n Vol(M) \sup_{p \in M} \int_M (G(p, q) + K) dq,
\]
similarly, the right hand side depends only on \(M\), hence we can also consider it as a constant \(C_3\).
First order estimate We want to give an estimate to $| \nabla v |$ by Schauder estimation, here $L = \Delta$ is an elliptic operator, and the equation $Lv = \Delta v$ is a uniformly elliptic equation (see G&T), then by Schauder estimate we have

$$||v||_{C^{2,\alpha}} = \sum_{|\gamma| \leq k} ||D^\gamma v||_{L^\infty} + \sum_{|\gamma| = k} ||D^\gamma v||_\alpha$$

$$\leq C(||Lv||_{C^0} + ||v||_{L^\infty})$$

$$= C(||\Delta v||_{L^\infty} + ||v||_{L^\infty})$$

$$= C(\sup_{M \times [0, T]} |\Delta v| + \sup_{M \times [0, T]} |v|),$$

while

$$\sup_{M \times [0, T]} |\nabla v| = ||\nabla v||_{L^\infty} \leq \sum_{|\gamma| \leq k} ||D^\gamma v||_{L^\infty} + \sum_{|\gamma| = k} ||D^\gamma v||_\alpha,$$

so

$$\sup_{M \times [0, T]} |\nabla v| \leq C(\sup_{M \times [0, T]} |\Delta v| + \sup_{M \times [0, T]} |v|).$$

where $||v||_{C^{2,\alpha}} = \sum_{|\gamma| \leq k} ||D^\gamma v||_{L^\infty} + \sum_{|\gamma| = k} ||D^\gamma v||_\alpha$ is the Hölder norms and $|v|_\alpha$ is the $\alpha$-Hölder constant on $\Omega$, i.e. $|f|_\alpha = \sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$.

Therefore, we must give an estimate to $\sup_{M \times [0, T]} |\Delta v|$ and $\sup_{M \times [0, T]} |v|$. 

Lemma 2

$\exists C_1, C_2 > 0$, such that $0 < n + \Delta u \leq C_1 e^{C_0(u - \inf_{M \times [0, T]} u)}$, for all $t \in [0, T)$

Proof.

Because $g_{ij} \sim g_{ij} = \delta_{ij}^u \frac{\partial u}{\partial x_j}$, so we get $g^{ij} \sim g^{ij} = g^{ij}(g_{ij} + \delta_{ij}^u \frac{\partial u}{\partial x_i}) = n + \Delta u$,

while $g_{ij}$ and $g^{ij}$ are both positive definite, so $g^{ij} \sim g^{ij}$ is positive definite,

$n + \Delta u$ is its trace so it’s positive.

As for the other inequality, first we learn from Yau’s work that:

If let $\bar{g}_{ij} = g_{ij} + \delta_{ij}^u \frac{\partial u}{\partial x_j}$ such that the equation holds

$$\operatorname{det}(g_{ij} + \delta_{ij}^u \frac{\partial u}{\partial x_j}) \operatorname{det}(g^{ij})^{-1} = e^F$$

then we get:

$$\Delta (e^{-C_F} (n + \Delta \varphi)) \geq e^{-C_F}(\Delta F - n^2 \inf_{i \neq l}(R_{i\bar{i}l})) - C_0 e^{-C \varphi} n(n + \Delta \varphi)$$

$$+ (C + \inf_{i \neq l}(R_{i\bar{i}l}))e^{-C_F} e^{\frac{n^2}{n+\Delta}} (n + \Delta \varphi)$$

where $R_{i\bar{i}l}$ is the bisectional curvature of the $g_{ij}$, $C_0$ a positive constant such that

$$C_0 + \inf_{i \neq l}(R_{i\bar{i}l}) > 0.$$
Then we assume for any $t \in (0, T)$, the function $(e^{-C_0} (n + \Delta u))$ achieves its maximum at $(p_0, t_0) \in M \times [0, t]$ and $t_0 > 0$, so at this point,
\[ \frac{\partial}{\partial t} (e^{-C_0} (n + \Delta u)) \leq 0 \]
then we get
\[ 0 \geq -(\Delta f + n^2 \inf_{R_{*ij}} | R_{*ij} |) - C_0 (n - \frac{\partial u}{\partial t})(n + \Delta u) \]
and from $\max M \left( \inf_{M} f \right)$ we get
\[ (n+\Delta u) \frac{n}{n+\Delta u} \left( \frac{\Delta f + n^2 \inf_{R_{*ij}} | R_{*ij} |}{C_0(n - \frac{\partial u}{\partial t})(n + \Delta u)} \right) \]
while the $(\Delta f + n^2 \inf_{R_{*ij}} | R_{*ij} |), C_0(n + \max M | f |)(C_0 + \inf_{R_{*ij}} | R_{*ij} |)$ are all independent of $t$, so there exists a constant $C'$ such that
\[ (n+\Delta u) \frac{n}{n+\Delta u} \leq C' + C' (n + \Delta u) \]
then there exists a constant $C_1$ such that
\[ (n+\Delta u) \frac{n}{n+\Delta u} \leq C_1, \text{ which means } (n + \Delta u) \leq C_1 \text{ for a constant } C_1 \text{ independent of } t. \]
Therefore, on $M \times [0, T]$ we have $(e^{-C_0} (n + \Delta u))$
\[ \leq C_1 e^{-C_0 u(p, t_0)} \text{ so } n+\Delta u \]
\[ \leq C_1 e^{C_0(u-u(p, t_0))} \]
\[ \leq C_1 e^{C_0(u-\inf_{M \times [0, T]} u)} \text{ and } C_0 \text{ and } C_1 \text{ are both independent of } t. \]

**Lemma 3**
There exists a constant $C_2$ so that $\sup_{M \times [0, T]} | v | < C_2$.

**Proof.**

Let $\omega = \sqrt[n]{\frac{1}{n+\Delta u}} g_{ij} dz^i \wedge d\bar{z}^j$, $\tilde{\omega} = \sqrt[n]{\frac{1}{n+\Delta u}} \hat{g}_{ij} d\tilde{z}^i \wedge d\bar{\tilde{z}}^j$, these are actually the Kähler forms of $g_{ij}$ and $\hat{g}_{ij}$, separately. And the volume forms
\[ dV = \det(g_{ij}) \wedge^n (\sqrt[n]{\frac{1}{n+\Delta u}} dz^i \wedge d\bar{z}^j) = \frac{\omega^n}{n!}, \]
\[ d\tilde{V} = \det(\hat{g}_{ij}) \wedge^n (\sqrt[n]{\frac{1}{n+\Delta u}} d\tilde{z}^i \wedge d\bar{\tilde{z}}^j) = \frac{\tilde{\omega}^n}{n!} \]
because of the equation (3.1) we get
\[ \log \det(g_{ij}) - \log \det(\hat{g}_{ij}) = \frac{\omega^2}{2} - f \]
then we get
\[ d\tilde{V} = (\sqrt[n]{\frac{1}{n+\Delta u}} d\tilde{z}^i \wedge d\bar{\tilde{z}}^j) = e^{\frac{n}{n+\Delta u} - f} dV \]
Therefore, for $p > 1$,
\[ -\frac{1}{n+\Delta u} \int_M \left( \frac{(v-p-1)}{p-1} \right) (\omega^n - \tilde{\omega}^{n}) = -\int_M \left( \frac{(v-p-1)}{p-1} \right) (dV - d\tilde{V}) \]
\[ = \int_M \left( \frac{(v-p-1)}{p-1} \right) (e^{\frac{n}{n+\Delta u} - f} - 1) dV \]
Because from Lemma 1 we know $v$ is bounded then we can renormalized it so that $v < 1$. Then on the other hand,
\[ -\int_M \left( \frac{(v-p-1)}{p-1} \right) (\omega^n - \tilde{\omega}^{n}) \]
\[ = \int_M \left( \frac{(v-p-1)}{p-1} \right) \left( \tilde{\omega} - \omega \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \]
\[ = \int_M \left( \frac{(v-p-1)}{p-1} \right) \left( \sqrt[2]{\frac{1}{n+\Delta u}} \partial \bar{\partial} \nabla \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \]
\[ = \int_M \left( \frac{(v-p-1)}{p-1} \right) \left( \sqrt[2]{\frac{1}{n+\Delta u}} \partial \bar{\partial} \nabla \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \]
\[ = \int_M \left( \frac{(v-p-2)}{p-1} \right) dv \left( \sqrt[2]{\frac{1}{n+\Delta u}} \partial \bar{\partial} \nabla \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \]
\[ = \frac{1}{2} \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \]
\[\int_M (-v)^p - (\frac{\sqrt{v}}{2}(\partial v \wedge \delta v)) \wedge \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1}\]

there we integral by part because \(v\) vanishes on the boundary of \(M\). Then the integral above
\[\geq \int_M (-v)^p - (\frac{\sqrt{v}}{2}(\partial v \wedge \delta v)) \wedge \omega^{n-1}\]
because each term in \(\frac{\sqrt{v}}{2}(\partial v \wedge \delta v) \wedge \tilde{\omega}^j \wedge \omega^{n-j-1}\) is nonnegative.

Then let \(|\nabla v|^2 = g^{ij} \frac{\partial v}{\partial x^i} \frac{\partial v}{\partial x^j}\) we have
\[\int_M (-v)^p |\nabla v|^2 \, dV \leq n! \int_M (-v)^{p-1} \left( e^{\frac{p}{p-1}f} - 1 \right) \, dV.\]

While because \((-v)^{p-2} |\nabla v|^2 = 4p-2 |\nabla(-v)^{\frac{p}{p-1}}|^2\) we replace the corresponding term in the inequality above:
\[\int_M 4p-2 |\nabla(-v)^{\frac{p}{p-1}}|^2 \, dV \leq n! \int_M (-v)^{p-1} \left( e^{\frac{p}{p-1}f} - 1 \right) \, dV,\]

while due to the compatibility condition, we get the term \(e^{\frac{p}{p-1}f} - 1\) is positive and bounded, therefore, the term above:
\[\leq C \frac{p^2}{p-1} \int_M (-v)^{p-1} \, dV\]
and by the norm \(|(-v)^{\frac{p}{p-1}}|_{L^1}\):
\[= \int_M |\nabla(-v)^{\frac{p}{p-1}}|^2 \, dV + \int_M (-v)^p \, dV\]
\[\leq \left( C \frac{p^2}{p-1} + 1 \right) \int_M (-v)^p \, dV\]
\[\leq C p \int_M (-v)^p \, dV,\] for \(p > 1\), there the \(C\) has changed but we still use it for a constant. When \(p = 1\) then we just replace \(\frac{(-v)^{p-1}}{p-1}\) by the term \(\log(-v)\) and the process works well.

\[\frac{1}{p-1} \int_M \log(-v) (\omega^n - \tilde{\omega}^n) = \int_M (-v)^{p-1} (\tilde{\omega} - \omega) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1}\]

\[= \int_M \log(-v) (\sqrt{\frac{p}{2}} (\partial v \wedge \delta v) \wedge \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1}\]
\[= \int_M d\log(-v) (\sqrt{\frac{p}{2}} (\partial v \wedge \delta v) \wedge \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1}\]
\[= -\int_M (-v)^{-1} d\log(-v) (\sqrt{\frac{p}{2}} (\partial v \wedge \delta v) \wedge \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1}\]
\[= -\int_M (-v)^{-1} (\sqrt{\frac{p}{2}} (\partial v \wedge \delta v) \wedge \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1}\]

And
\[\frac{1}{p-1} \int_M \log(-v) (\omega^n - \tilde{\omega}^n) = -\int_M \log(-v) (dV - \tilde{dV})\]
\[= \int_M \log(-v) \left( e^{\frac{p}{p-1}f} - 1 \right) \, dV\]
So \(\int_M (-v)^{p-1} |\nabla v|^2 \, dV \leq n! \int_M \log(-v) \left( e^{\frac{p}{p-1}f} - 1 \right) \, dV.\)
and \((-v^{-1}) |\nabla v|^2 = 4 |\nabla(-v)^{\frac{p}{p-1}}|^2\]
Then \(\|(-v)^{\frac{p}{p-1}}\|_{L^{p-1}}^2\):
\[= \int_M |\nabla(-v)^{\frac{p}{p-1}}|^2 \, dV + \int_M (-v)^{p-1} \, dV\]
\[\leq C \int_M \log(-v)^p \, dV + \int_M (-v)^p \, dV\]
\[\leq C p \int_M (-v)^p \, dV,\] for \(p > 1\) we assumed before and \(\log(-v)\) is slower than \((-v)\).
So the equality works well when \(p \geq 1\).

Because \(\int_M v \, dV = 0\), we learn from the Sobolov inequality(G&T[5] p155)
\[\|(-v)^{\frac{p}{p-1}}\|_{L^{p-1}} \leq C \| D((-v)^{\frac{p}{p-1}}) \|_{H^1}\]
two positive constant $A, B$ such that positive definite and Hermitian so each term is positive, therefore, there exists $\delta$ the product of these terms has an upper bound, so each term cannot go to the right hand side of the inequality induced above so it's independent of $t$.

Finally, we let $j$ increases to infinity then $\|v\|_{L^\infty} \leq C_4$ for $C_4$ a constant from the right hand side of the inequality induced above so it’s independent of $t$:

$$\sup_{M \times [0,T]} |v| \leq C_4,$$ this finishes the proof.

After the long and complex calculation, we can continue our estimate.

Because $\frac{1}{\nu(t(M))} \int_M u dV$ is not relative to $\partial \partial \bar{\bar{\bar{\partial}}}$, so we get:

$$0 < n + \Delta u = n + \Delta u \leq C_1 e^{C_0(u - inf f_{M \times [0,T]v})} = C_1 e^{C_0(u - inf f_{M \times [0,T]v})} \leq C_5$$

so the $\Delta u$ is bounded, the equality holds because the difference is not relevant to the constant $\frac{1}{\nu(t(M))} \int_M u dV$, then the constant $C_5$ appears because $\sup |v|$ is bounded.

The by the Schauder estimate we presented before

$$\sup_{M \times [0,T]} \|v\| \leq C_6 (\sup_{M \times [0,T]} |\Delta u| + \sup_{M \times [0,T]} |v|) \leq C_6 (\text{constant} + \text{constant})$$

by Lemma 2 and Lemma 3 then $\sup_{M \times [0,T]} \|\nabla v\| \leq C_7$. This finishes the first order estimate.

**Second order estimate**

From the estimate of $n + \Delta u$, in Yau’s work the choice of the metric (in page 348) on one hand make the matrix $(\delta + u_{ij})$ is positive definite and Hermitian so we know that $1 + u_{ij}$ is bounded above for any $i$, by the metric Yau chose,

$$\prod_{i=1}^m (1 + u_{ij}) = \sqrt{f}^{ij}$$

gives a lower estimate of the product has a upper bound, so each term can not get to the negative infinity, so they all have a lower bound, while the matrix $(\delta + u_{ij})$ is positive definite and Hermitian so each term is positive, therefore, there exists two positive constant $A, B$ such that

$$A \leq 1 + u_{ij} \leq B$$

**Third order estimate**

First let $S = \sum \bar{g}^{ij} \bar{g}^{jk} v_{i\bar{j}k} v_{\bar{s}\bar{t}}$. Actually in this disturbing definition, Cao said in his paper that he followed E. Calabi and Yau, Yau said he followed E. Calabi, but I have not read E. Calabi’s paper, I have only read Yau and Cao’s paper, so I do not know who I am following, that’s really interesting. To show respect to E. Calabi, I decide to write I’m following E. Calabi here too. I’ll read his work in a few minutes.
We note \(A \simeq B\) if \(|A - B| \leq C_1 \sqrt{S} + C_2\) for \(C_1\) and \(C_2\) are constants can be estimated, further more, we note \(A \simeq B\) if \(|A - B| \leq C_3 S + C_4 \sqrt{S} + C_5\) for \(C_3\) and \(C_4, C_5\) are constants can be estimated. And again by the metric Yau defined before, through totally 60 rows calculations in Yau’s appendix,
\[
\Delta S \simeq \sum (1 + v_{ik})^{-1} (1 + v_{kk})^{-1} (1 + v_{\alpha \beta})^{-1} \times \left\{ v_{ijk} - \sum v_{ibk} v_{\pi j}^{-1} v_{pjk} v_{\beta jk} + v_{pik} v_{\pi jk} v_{\beta jk} \right\}.
\]
And \(\Delta (\Delta v) \geq \sum (1 + v_{ik})^{-1} (1 + v_{ii})^{-1} |v_{ik} - C_6| - C_6^2\), where \(C_6\) is a constant which can be estimated, then without loss generality, we can choose a big \(C_7\) such that
\[
\Delta (S + C_7 \Delta v) \geq C_8 S - C_9, \quad \text{for } C_7, C_8, C_9, \text{ are positive constants can be estimated.}
\]
Then we assume \(p(t)\) is the maximum point of the funciton \(S + C_7 \Delta v\), we get
\[
0 \leq C_8 \ S - C_9; \quad \text{so } C_8 \ S \leq C_9;
\]
\[
C_8 (S + C_7 \Delta v) \leq C_9 + C_8 C_7 \Delta v.
\]
Due to the \(\Delta v\) is bounded from estimate before, this gives an estimate to \(\sup_{M \times (0,T)} \Delta (M + C_7 \Delta v)\) therefore, we have an estimate of \(\sup_{M \times [0,T]} S,\text{as Yau said, this gives an estimate to } v_{ikj} \text{ and similar term in terms of } g_{ik} \text{ and sup } |F|\),
\[
\text{sup} \ |\nabla F| \sup_{M \times [0,T]} |F_{ij}| \text{ and sup } \sup_{M \times [0,T]} |F_{i,j,k}|; \quad \text{sup } M \times [0,T]
\]
**Long time existence**

Finally we prove the proposition.

**Proposition of the long existence**

Assume \(u\) be the solution of the equation
\[
\frac{\partial u}{\partial t} = \log \det(g_{ij} + \frac{\partial^2 u}{\partial z \partial \bar{z}}) - \log \det(g_{ij}) + f ———-(3.1)
\]
where \(t \in [0,T]\) which is the maximum time interval. Let \(v\) be the normalization of \(u\):
\[
v = u - \frac{1}{V_{M \times [0,T]}} \int_M u dV. \quad \text{Then the } C_\infty \text{ norm of } v \text{ are uniformly bounded for any } t \in (0,T), \text{ which means } T = \infty. \text{ Then there exists a sequence } t_n \text{ increasing to infinity such that } v(x, t_n) \text{ converges in the topology generated by } C_\infty \text{ norm to a smooth function } v_\infty(x) \text{ on } M \text{ when } n \text{ increases to infinity.}
\]
**Proof**, 

Differentiate the equation(3.1):
\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial z \partial \bar{z}} = g^{ij} \frac{\partial}{\partial z} g_{ij} + \frac{\partial f}{\partial z} g_{ij} + \frac{\partial f}{\partial \bar{z}}
\]
so \(\frac{\partial}{\partial z} = g^{ij} \frac{\partial}{\partial z} g_{ij} - \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} g_{ij} + \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}\); we have the estimate of \(\frac{\partial u}{\partial t}\) and \(\Delta u\) so the coeffecients of the \(\phi\) are bounded, and by the \(C^{0,\alpha}\) norm is defined as: \(\|u\|_{C^{0,\alpha}} = \|u\|_{L^\infty} + \|u\|_\alpha\), according to our 0 order estimate, they are bounded in the \(C^{0,\alpha}\) norm; and these term are actually the Hölder coefficent so is has bounded \(C^{0,\alpha}\) norm. While the right hand side have similar estimate for \(C^{0,\alpha}\) norm for \(0 < \alpha < 1\) because the RHS only depends on the Kahler metric and \(f\) which are smooth. By interior regularity theory(see G& T section 6.4) which gives the \(C^{k+2,\alpha}\) estimate when then coeffecients and the nonhomogeneous term are of \(C^{k,\alpha}\) estimate, this \(\phi\) is elliptic and the RHS terms are smooth so they have \(C^{0,\alpha}\) estimate, then as the solution of this equation, \(\frac{\partial u}{\partial z}\) has the \(C^2\) estimate we proved before, then \(\frac{\partial u}{\partial \bar{z}}\) has a uniform \(C^{2,\alpha}\) estimate, similarly \(\frac{\partial^2 u}{\partial z \partial \bar{z}}\) also has this estimate, there the
C^{0,\alpha} norm estimate gives the C^{2,\alpha} norm estimate, so \frac{\partial u}{\partial z} and \frac{\partial u}{\partial \bar{z}} are uniformly bounded in C^{2,\alpha} norm estimate, then because \tilde{g}_{ij} = g_{ij} + \frac{\partial^2 u}{\partial z \partial \bar{z}}, then the C^{2,\alpha} estimate of \frac{\partial u}{\partial z} actually gives the C^{1,\alpha} estimate of the RHS and the coefficients of \odot which are determined by the metrics, therefore, the right hand side and the coefficients of \odot have uniform C^{1,\alpha} estimate. Use the Interior regularity theory again, then \frac{\partial u}{\partial z} and \frac{\partial u}{\partial \bar{z}} have another two orders more estimate...... Then we repeat use the theory, by iteration, then \frac{\partial u}{\partial t} has uniformly bounded C^{\infty} norm for any \tau \in (0,T), finally we choose a sequence of \tau goes into the infinity such that it has a subsequence \tau_n making \frac{\partial u}{\partial \tau} converge to a smooth function \frac{\partial u}{\partial \tau}, so that solution exists. And then because \frac{\partial u}{\partial \tau} is uniformly bounded referring to \tau, and as \tau goes into the infinity \frac{\partial u}{\partial \tau} can not blow up in finite time, this means that our estimates are independent of \tau then when solution metric get the upper bound of \tau, says T, then our estimates still works the estimates above are independent of \tau so if we choose \tau \in [0, T) then the solution also exists in [\tau_0, \tau_0 + \epsilon] for \epsilon independent of \tau, so we can use this \tilde{g}_{ij}(T) as the initial condition of the same equation choosing a new initial point and continue to deformation, so the process can continue to infinity because our estimates always work, so \frac{\partial u}{\partial \tau} exists for all time.

Up to here, the long time existence has finally been proved.

4 Argument for uniform convergence

At this time, since we have proved the existence of the long time solution, then to follow our idea, we need to show the uniform convergence of the solution and then let the time goes into the infinity to get the final conclusion.

\[ \frac{\partial u}{\partial t} = \log \det (g_{ij} + \frac{\partial^2 u}{\partial z \partial \bar{z}}) - \log \det (g_{ij}) + f \] (3.1)

and initial status: \frac{\partial u}{\partial t} = f(x) when \tau = 0. So in order to analyze \frac{\partial u}{\partial t}, we should first take care of this equation

\( (\Delta - \frac{\partial}{\partial \tau}) \frac{\partial u}{\partial \tau} = 0 \)

And following Yau’s work, Cao gave a modification of an important theory of this equation.

Theorem 4.1

We assume M be a compact manifold whose dimension is n, g_{ij}(t) be a family of Riemannian metrics on M, such that the following holds:

1. \exists constants C_1, C_2 positive and independent of \tau such that \n\nC_1 \ g_{ij}(0) \leq g_{ij}(\tau) \leq C_2 \ g_{ij}(0)
(2) \exists constants C_3 positive and independent of t such that
\[ \frac{\partial g_{ij}}{\partial t} |(t) \leq C_3 g_{ij}(0) \]
(3) \exists constants C_3 positive and independent of t such that
\[ R_{ij}(t) \geq -K g_{ij}(0) \]

Then we assume \( \phi \) is positive and satisfies the equation:
\[ (\Delta_x - \frac{\partial^2}{\partial t^2}) \phi = 0 \]
on \( M \times [0, \infty) \) where \( \Delta_x \) is the Laplace operator, then \( \forall \alpha > 1, \)
\[ \sup_{x \in M} \phi(x,t_1) \leq \inf_{x \in M} \phi(x,t_2) \left( \frac{2}{\alpha} \right)^{\frac{1}{2}} e^{\left( \frac{1}{\alpha} - t_1 \right)} \left( C_d^2 d^2 + C_2 C_3 (n+1)(t_2 - t_1) \right) \]
for \( d \) is the diameter of \( M \) measured by \( g_{ij}(0) \), i.e. \( d = \sup_{x,y \in M} g_{ij}(0)(x,y) \);
and
\[ A = \sup \| \nabla^2 \phi \|; \quad \text{and} \quad 0 < t_1 < t_2 < \infty. \]

Cao did not show the proof because the proof of the theorem is totally a tough work in Yau's paper. Now, we can use this conclusion, let \( F = \frac{\partial u}{\partial t} \), then by the maximum principle we still consider \( t=0 \) as the boundary of \( M \times [0, \infty) \) for this parabolic equation and \( t_2 > t_1 > 0 \), we get:
\[ \sup_{x \in M} F(x,t_2) < \sup_{x \in M} F(x,t_1) < \inf_{x \in M} F(x,t_2) > \inf_{x \in M} F(x,t_1) > \inf_{x \in M} f(x) \]
here \( t_2 > t_1 > 0 \) because we can always choose at \( t_2 \) the \( \partial u \) converges more than \( t_1 \).

**Remark** The conditions above also hold for \( \tilde{R}_{ij} \), i.e.
(1) \( \exists \) constants \( C_1, C_2 \) positive and independent of t such that
\[ C_1 \tilde{g}_{ij}(0) \leq \tilde{g}_{ij}(t) \leq C_2 \tilde{g}_{ij}(0) \]
(2) \( \exists \) constants \( C_3 \) positive and independent of t such that
\[ | \frac{\partial \tilde{g}_{ij}}{\partial t}(t) | \leq C_3 \tilde{g}_{ij}(0) \]
(3) \( \exists \) constants \( K \) positive and independent of t such that
\[ \tilde{R}_{ij}(t) \geq -K \tilde{g}_{ij}(0) \]

Actually, \( \tilde{g}_{ij}(0) = (g_{ij}(0) + \frac{\partial^2 u}{\partial x^2 \partial t}(0)) \), from the estimate before we know \( \frac{\partial^2 u}{\partial x^2 \partial t}(0) \) is bounded, then we can always choose a \( C_1 \leq \frac{(g_{ij}(0) + \frac{\partial^2 u}{\partial x^2 \partial t}(0))}{g_{ij}(0) + \frac{\partial^2 u}{\partial x^2 \partial t}(0)} \)
because \( g_{ij}(0) \leq C_3 \tilde{g}_{ij}(c) \) for some \( C \), similary, choose
\[ C_2 \geq \frac{(g_{ij}(t) + \frac{\partial^2 u}{\partial x^2 \partial t}(t))}{(g_{ij}(0) + \frac{\partial^2 u}{\partial x^2 \partial t}(0))} \]
, so the (1) holds;
then \( | \frac{\partial \tilde{g}_{ij}}{\partial t}(t) | \leq | \frac{\partial g_{ij}}{\partial t}(t) | + | \partial \tilde{u}(\tilde{t}) | \), while the estimate before tells that
\[ | \partial \tilde{u}(\tilde{t}) | \] are bounded so we can choose a \( C_3 \) such that
\[ \partial \tilde{u}(\tilde{t}) \geq C_3 | \partial \tilde{u}(\tilde{t}) | \] so (2) holds; (3) Vy the long time existence theorem,
\[ \tilde{g}_{ij}(t) = g_{ij} \] so when it holds for \( g_{ij} \) then in the convergence process, \( \tilde{g}_{ij}(t) \)
converges to \( g_{ij}(\infty) \), so whenever \( t \in [0, \infty) \), there exists such \( K \).

Then we define
\[ \varphi_n(x,t) = \sup_{x \in M} F(x,n-1) - F(x,n-1+t) \]
\[ \phi_n(x,t) = F(x,n-1+t) - \inf_{x \in M} F(x,n-1) \]
\[ \omega(t) = \sup_{x \in M} F(x,t) - \inf_{x \in M} F(x,t) \]

While because the sup and the inf are constants, and the \( d(n-1)+t = d(t) \) so the \( \varphi_n(x,t) \) and \( \phi_n(x,t) \) both satisfy the equation and the initial condition, and
by the inequality above, they are both positive. Then we take $t_1 = \frac{1}{2}$, $t_2 = 1$, then we use the Theorem 4.1 in $\phi_n(x,t)$ and $\phi_n(x,t)$ separately:

$$\sup_{x \in M} \phi_n(x,t) \leq \inf_{x \in M} \phi_n(x,t) \gamma,$$

where $\gamma = 2^2 e^{C^2 d^2 + \frac{1}{2} (\frac{\partial u}{\partial t} + C_2 C_3 (n + A))}$ is independent of $t$, then

$$\sup_{x \in M} F(x,n-1) - \inf_{x \in M} F(x,n-1) \leq \gamma (\sup_{x \in M} F(x,n-1) - \inf_{x \in M} F(x,n))$$

similarly,

$$\sup_{x \in M} F(x,n) - \inf_{x \in M} F(x,n) \leq \gamma (\sup_{x \in M} F(x,n) - \inf_{x \in M} F(x,n))$$

do not forget we have $\omega(t) = \sup_{x \in M} F(x,t) - \inf_{x \in M} F(x,t)$ so we add the two inequality together:

$$\omega(n-1) + \omega(n-1) \leq \gamma (\omega(n-1) - \omega(n))$$

because $\omega(n) \geq 0$ so $\omega(n-1) \leq \gamma (\omega(n-1) - \omega(n))$

then $(\gamma - 1) \omega(n-1) \geq \gamma \omega(n)$, so $\omega(n) \leq \delta \omega(n-1)$, for $\delta = \frac{\omega(1)}{\gamma} < 1$. We repeat this process and we get $\omega(n) \leq \delta \omega(n-1) \leq \delta^2 \omega(n-2) \leq \delta^3 \omega(n-3) \ldots$

finally,$\omega(n) \leq \delta^n \omega(0)$ for $\omega(0) = \sup_{x \in M} F - \inf_{x \in M} F$.

While because $\sup_{x \in M} F(x,t)$ gets smaller when $t$ gets larger, and $\inf_{x \in M} F(x,t)$ gets larger when $t$ gets larger, so $\omega(t)$ decreases when $t$ gets larger. Therefore, let $a = -\log(\delta)$, we can choose a constant $C_4$ independent of $t$ such that $\omega(t) \leq C_4 e^{-\alpha t}$.

Let $\phi(x,t) = \frac{\partial u}{\partial t} - \frac{1}{V_{vol}(M)} \int_M \frac{\partial u}{\partial t} d\bar{V}$, Then in order to show when $t$ goes into the infinity, we should analyze the behaviour of $\phi(x,t)$, if we can prove $\phi(x,t)$ goes into 0 then that means $\frac{\partial u}{\partial t}$ truly converges to some function.

We use $E = \frac{1}{2} \int_M \phi^2 d\bar{V}$ and we want to estimate $E$ in terms of $t$ to figure out how $\phi$ changes. While

$$d\bar{V} = \det(g_{ij}) \wedge_{i=1}^n \left( \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^i \right)$$

$$= \det(g_{ij}) + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \wedge_{i=1}^n \left( \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^i \right),$$

$$\frac{\partial}{\partial t} (d\bar{V})$$

$$= \left( \frac{\partial}{\partial t} \left( \det(g_{ij}) + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \right) \right) \wedge_{i=1}^n \left( \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^i \right)$$

$$= \frac{\partial g_{ij}}{\partial t} \frac{\partial}{\partial t} \det(g_{ij}) \wedge_{i=1}^n \left( \frac{\sqrt{-1}}{2} dz^i \wedge d\bar{z}^i \right)$$

$$= \frac{\partial g_{ij}}{\partial t} \frac{\partial}{\partial t} \det(g_{ij}) = \left( \frac{\partial}{\partial t} \log \det(g_{ij}) + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \right) d\bar{V}$$

$$\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} \right) d\bar{V} = \Delta \left( \frac{\partial u}{\partial t} \right) d\bar{V}.$$

Then we calculate

$$\frac{\partial^2 \phi}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{1}{V_{vol}(M)} \int_M \frac{\partial^2 u}{\partial t^2} d\bar{V} - \frac{V_{vol}(M)}{V_{vol}(M)} \int_M \frac{\partial u}{\partial t} \Delta \left( \frac{\partial u}{\partial t} \right) d\bar{V},$$

while

$$\frac{1}{V_{vol}(M)} \int_M \frac{\partial^2 u}{\partial t^2} d\bar{V}$$

$$= \frac{1}{V_{vol}(M)} \int_M \Delta \left( \frac{\partial u}{\partial t} \right) d\bar{V}$$

$$= \frac{1}{V_{vol}(M)} \int_M \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial t} \right) d\bar{V}$$

$$= \frac{\partial}{\partial t} \left( \frac{1}{V_{vol}(M)} \int_M \Delta \left( \frac{\partial u}{\partial t} \right) d\bar{V} \right) - \frac{1}{V_{vol}(M)} \int_M \frac{\partial u}{\partial t} \Delta \left( \frac{\partial u}{\partial t} \right) d\bar{V} = 0-0 = 0,$$

so $\frac{\partial^2 \phi}{\partial t^2}(x,t)$

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{1}{V_{vol}(M)} \int_M \frac{\partial u}{\partial t} \Delta \left( \frac{\partial u}{\partial t} \right) d\bar{V}$$

$$= \Delta \left( \frac{\partial u}{\partial t} \right) - \frac{1}{V_{vol}(M)} \int_M \frac{\partial u}{\partial t} \Delta \left( \frac{\partial u}{\partial t} \right) d\bar{V}.$$
\[
\begin{align*}
&= \int_M \left( \frac{\partial u}{\partial t} - \frac{1}{V_M} \int_M \frac{\partial u}{\partial t} \, dV \right) \left( \frac{\partial u}{\partial \gamma} - \frac{1}{V_M} \int_M \frac{\partial u}{\partial \gamma} \, dV \right) dV \\
&\quad + \frac{1}{2} \int_M \varphi^2 \left( \frac{\partial u}{\partial t} \right) dV \\
&\quad - \int_M \frac{\partial u}{\partial t} \, dV \\
&\quad + \frac{1}{2} \int_M \varphi^2 \left( \frac{\partial u}{\partial t} \right) dV \\
&\quad + \frac{1}{2} \int_M \varphi^2 \Delta \left( \frac{\partial u}{\partial t} \right) dV \\
&= \int_M \left( \frac{\partial u}{\partial t} \right) dV \\
&\quad - \int_M \frac{\partial u}{\partial t} \, dV \\
&\quad + \varphi^2 \left( \frac{\partial u}{\partial t} \right) dV \\
&\quad + \frac{1}{2} \int_M \varphi^2 \Delta \left( \frac{\partial u}{\partial t} \right) dV \\
&\quad \text{while because } \int_M \varphi^2 \Delta \left( \frac{\partial u}{\partial t} \right) dV \\
&\quad \text{just the same as the argument before,}
\end{align*}
\]

so \( \frac{du}{dt} = \int_M \frac{\partial u}{\partial t} \, dV \) + \( \frac{1}{2} \int_M \varphi^2 \Delta \left( \frac{\partial u}{\partial t} \right) dV \), there we use integrating by parts and M is compact so no boundary exists, due to \( \Delta = |\nabla|^2 \), so

\[
\begin{align*}
&\quad = \int_M \nabla \left( \frac{\partial u}{\partial t} \right) \cdot \nabla \left( \frac{\partial u}{\partial t} \right) dV \\
&\quad = \int_M |\nabla^2 u|^2 dV + \frac{1}{2} \int_M \varphi^2 \nabla \left( \frac{\partial u}{\partial t} \right) dV \\
&\quad = \int_M (-1 + \varphi) |\nabla^2 u|^2 dV \\
&\quad \text{for } |\nabla()|^2 = \tilde{g}^{ij}()_{ik}()_j \text{ is the square of the gradient.}
\end{align*}
\]

Because \( \sup_{x \in M} \varphi(x,t) \) is the difference between \( \varphi \) and its average, while \( \omega \) is the largest difference, ans we can choose \( t \) large enough such that \( \omega \) less than any value, so \( \exists t \) such that

\[
\sup_{x \in M} \varphi(x,t) < \omega(t) < \frac{1}{2}, \quad \text{then}
\]

\[
\begin{align*}
\frac{du}{dt} &= \int_M (-1 + \varphi) |\nabla^2 u|^2 dV \\
&\leq \frac{1}{2} \int_M |\nabla^2 u|^2 dV \\
&\quad \frac{1}{2} \int_M |\nabla^2 u|^2 dV \\
&\quad \frac{1}{2} \int_M |\nabla^2 u|^2 dV.
\end{align*}
\]

And by the definition of \( \varphi \), \( \int_M \varphi \, dV = 0 \), by the Poincare inequality

\[
|| \varphi ||_{L^p} \leq h ||D \varphi||_{L^p} \text{ then } \int_M \varphi^2 \, dV \leq h^2 \int_M |\nabla \varphi|^2 dV = h^2 \int_M |\nabla \varphi|^2 dV
\]

for \( h \) can be considered as the diameter of the domain, we get

\[
\int_M |\nabla \varphi|^2 \, dV \geq \lambda_1(t) \int_M \varphi^2 \, dV,
\]

where \( \lambda_1(t) \) is the first eigenvalue of \( \Delta \) at time \( t \).Then there exists a constant \( C_5 \) such that for any \( t \),

\[
\lambda_1(t) \geq C_5, \quad \text{so } \frac{du}{dt} = \frac{1}{2} \int_M |\nabla^2 \varphi|^2 dV \leq C_5 E.
\]

That’s really similar to an ordinary differential equation, and the exponential function is monotonic, so we can solve the equation and get the inequality :

\[
E \leq C_6 \, e^{-C_5 t}
\]

while \( dV \) is uniformly equivalent to \( dV \), so there exists constant \( C'_6 \) such that
\[ \int_M \varphi^2 dV \leq C_6 e^{-C_5t}, \] then we can finally prove the uniform convergence theorem.

**Theorem of uniform convergence**

Using the notation in the Proposition of long time existence, as \( t \) goes into the infinity, \( v(x,t) \) converges to the function \( v_\infty \) in \( C_\infty \) topology, therefore, as \( t \) goes into the infinity, \( \frac{\partial u}{\partial t} \) converges to a constant in \( C_\infty \) topology.

**Proof.**

First we prove \( v(x,s) \) is a Cauchy sequence in \( L_1 \) norm, as \( t \) goes into the infinity. For any \( 0 < s < s' \):

\[ \int_M | v(x,s) - v(x,s') | dV \]

\[ \leq \int_M \int_s^{s'} \left| \frac{\partial u}{\partial t}(x,t) \right| dt dV \]

\[ \leq \int_M \int_s^{s'} \left| \frac{\partial u}{\partial t}(x,t) \right| dt dV \]

\[ = \int_s^{s'} \int_M \left| \frac{\partial u}{\partial t}(x,t) \right| dV dt \]

\[ = \int_s^{s'} \int_M \left| \frac{\partial u}{\partial t} - \frac{1}{V_{\text{vol}(M)}} \int_M \frac{\partial u}{\partial t} dV \right| dV dt \]

\[ = \int_s^{s'} \int_M \left| \frac{\partial u}{\partial t} - \frac{1}{V_{\text{vol}(M)}} \int_M \frac{\partial u}{\partial t} dV + \frac{1}{V_{\text{vol}(M)}} \int_M \frac{\partial u}{\partial t} dV \right| dV dt \]

\[ \leq \int_M \int_s^{s'} \left| \varphi \right| dV dt + \int_s^{s'} \int_M \frac{1}{V_{\text{vol}(M)}} \left| \int_M \frac{\partial u}{\partial t} dV \right| dV dt \]

\[ \leq \inf_{x \in M} \left( \int_M \varphi^2 dV \right)^{1/2} dt + \sup_{x \in M} \left( \frac{1}{V_{\text{vol}(M)}} \int_M \frac{\partial u}{\partial t} dV \right)^{1/2} dt \]

\[ \leq \text{Vol}(M)^{1/2} \left( \int_s^{s'} \left( C_6 e^{-C_5t} \right)^{1/2} dt \right) + \text{Vol}(M) \left( \int_s^{s'} \left( C_4 e^{-at} \right) dt \right) \]

\[ = C_7 \int_s^{s'} e^{-C_5t/2} dt + C_8 \int_s^{s'} e^{-at} dt \]

while the above integral converges in \( s \), so if \( s \) goes into the infinity, the two integral can be very small, so this means \( v(x,s) \) is a Cauchy sequence in \( L_1 \) norm.

So by \( L_1 \) is complete, there exists a function \( v'_{\infty} (x) \) such that \( v(x,t) \) converges uniformly to \( v'_{\infty} (x) \), while in the long time existence theorem we know there exists a sequence \( t_k \) such that \( v(x,t_k) \) converges to \( v_\infty (x) \) there, so here \( v_\infty (x) = v'_{\infty} (x) \), so \( v(x,t) \) converges to \( v_\infty \) in \( L_1 \) norm. However, what we need is the convergence in \( C_\infty \) topology which can be expressed in the equation we study.

We prove it by contradiction. If \( \exists \ r > 0, \epsilon > 0 \) such that

\[ \forall N, \exists n>N, \text{such that} \ | v(x,t_n) - v_\infty (x) |_{C^r} > \epsilon, \text{while N can be chosen arbitrary, so we can find a sequence} \ t_n \text{such that} | v(x, t_n) - v_\infty (x) |_{C^r} > \epsilon, \]

but the sequence \( v(x,t_n) \) is bounded so there exists a subsequence \( v(x,t_{kn}) \) converges to \( \dot{v}_\infty \neq v_\infty (x) \) in \( C_\infty \) topology, but we know \( v(x,t_n) \) converges in \( L_1 \) norm to \( v_\infty (x) \), so it’s a contradiction. So finally \( v(x,t) \) converges to \( v_\infty (x) \) in \( C_\infty \) topology.

Then we consider the equation again:

\[ \frac{\partial u}{\partial t} = \log \det(g_{ij} + \frac{\partial^2 u}{\partial z^i \partial z^j}) - \log \det(g_{ij}) + f \quad (3.1) \]

because \( \partial \bar{\partial} v = \partial \bar{\partial} u \), so when \( t \) goes into the infinity,

\[ \log \det(g_{ij} + \frac{\partial^2 u}{\partial z^i \partial z^j}) - \log \det(g_{ij}) + f \text{ converges to} \]
$\log \det(g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j}) - \log \det(g_{ij}) + f$, which means $\frac{\partial g}{\partial t}$ converges to $\frac{\partial u}{\partial t}(x)$ in $C^\infty$ topology when $t$ goes into the infinity, while $\omega(t) = \sup_{x \in M} \frac{\partial u}{\partial t} - \inf_{x \in M} \frac{\partial u}{\partial t} \leq C_4 e^{-at}$, when $t$ goes into the infinity, the right hand side goes into 0, therefore, we can only gets $\frac{\partial u}{\partial t}$ converges to a constant.

5 The final Theorem

According to our idea: First prove the solution exists all the time, then prove when $t$ goes infinitely, $g_{ij}$ converges to a definite $\tilde{g}_{ij}(\infty)$ and hence $\frac{\partial u}{\partial t}$ converges to 0, then we get

$-R_{ij} + T_{ij} = 0$, then we get $T_{ij}$ will be the Ricci tensor of $\tilde{g}_{ij}(\infty)$ so the $\tilde{g}_{ij}(\infty)$ is the metric we want. Then we can finish the whole process.

Main Theorem

$M$ be a compact Kähler manifold with the Kähler metric $g_{ij}dz^i \wedge d\bar{z}^j$, $C_1(M)$ is the first Chern class of $M$, consider a presentation of it $\frac{\sqrt{-1}}{2\pi} T_{ij}dz^i \wedge d\bar{z}^j$, while from the initial metric $g_{ij}$, we consider an equation with changing $g_{ij}$:

$\frac{\partial g_{ij}}{\partial t} = -R_{ij} + T_{ij} : g_{ij} = g_{ij}$ at $t = 0$

then the equation exists a long time solution and $\tilde{g}_{ij}$ converges uniformly to another Kähler metric $\tilde{g}_{ij}$ which is in the same Kähler class of $g_{ij}$ such that $0 = -\tilde{R}_{ij} + T_{ij}$ then $\tilde{R}_{ij} = T_{ij}$, which means $T_{ij}$ is the Ricci tensor of $\tilde{g}_{ij}$.

Proof.

While the de Rahm cohomolgy class of the Kähler form $\frac{\sqrt{-1}}{2\pi} T_{ij}dz^i \wedge d\bar{z}^j$ is the first Chern class $C_1(M)$ of $M$, where the $R_{ij}$ is the Ricci curvature of the Kähler metric. Then because $\frac{\sqrt{-1}}{2\pi} T_{ij}dz^i \wedge d\bar{z}^j$ also represents $C_1(M)$, so

$T_{ij} - R_{ij} = \frac{\partial^2 f}{\partial z_i \partial z_j}$, where $f$ is a real-value smooth function on $M$. By the long time existence Theorem we know the equation

$\frac{\partial u}{\partial t} = \log \det(g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j}) - \log \det(g_{ij}) + f - \frac{\partial \tilde{g}_{ij}}{\partial t}$ (3.1)

$u(x,t) = 0$ when $t = 0$ exists a smooth solution $u(x,t)$ in all time such that $\tilde{g}_{ij}(t) - g_{ij} = \frac{\partial \tilde{g}_{ij}}{\partial t}$

Then from the uniformly convergence theorem we get that as $t$ goes into the infinity, $u(x,t)$ converges uniformly so $\tilde{g}_{ij}(\infty)$ converges in $C^\infty$ topology to $\tilde{g}_{ij}(\infty)$, and by

$\tilde{g}_{ij}(t) = g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j}$ so $\frac{\partial \tilde{g}_{ij}}{\partial t}$ converges uniformly to 0 for $\frac{\partial u}{\partial t}$ converges uniformly.

Then we differentiate the equation:

$\frac{\partial^2}{\partial z_i \partial z_j} (\frac{\partial u}{\partial t}) = \frac{\partial^2}{\partial z_i \partial z_j} ((\log \det(g_{ij} + \frac{\partial^2 u}{\partial z_i \partial z_j}) - \log \det(g_{ij}) + \frac{\partial^2 f}{\partial z_i \partial z_j})$

while we can calculate the Ricci tensor explicitly in terms of $g_{ij}$ and $\tilde{g}_{ij}$: $R_{ij} = -\frac{\partial^2 f}{\partial z_i \partial z_j}$ log det$(g_{ij})$, and $\frac{\partial \tilde{g}_{ij}}{\partial t} = \partial \tilde{\partial}(\frac{\partial u}{\partial t})$

so $\frac{\partial \tilde{g}_{ij}}{\partial t} = -\tilde{R}_{ij} + T_{ij}$, $\tilde{g}_{ij} = g_{ij}$ at $t = 0$
then let t goes into the infinity we have \( 0 = -\tilde{R}_{ij} + T_{ij} \) for \( \frac{\partial g_{ij}(\infty)}{\partial t} = 0 \), it’s a constant, so finally \( \tilde{R}_{ij}(\infty) = T_{ij} \). That’s the metric we want.

We prove a corollary as an application.

**Corollary**

If \( C_1(M) = 0 \), then we can deform the initial Kähler metric to a Ricci flat metric in the direction of negative Ricci tensor.

**Proof**

If \( C_1 M = 0 \), then \( \frac{\partial g_{ij}}{\partial t} = -\tilde{R}_{ij} \), \( g_{ij} = g_{ij} \) at \( t = 0 \), then by the theorem \( \tilde{g}^{ij} \) converges to \( \tilde{g}^{ij}(\infty) \), and \( 0 = -\tilde{R}_{ij}(\infty) \), which is the Ricci tensor of a Ricci flat metric.

### 6 The problem of existence of the Kähler-Einstein metric

Here we consider \( M \) a compact Kähler manifold with negative first Chern class \( C_1(M) \), if we want to find a Kähler-Einstein metric on \( M \) i.e. \( R = kg \). We consider the evolution function:

\[
\frac{\partial g_{ij}}{\partial t} = -\tilde{R}_{ij} - \tilde{g}^{ij}, \quad g_{ij} = g_{ij} \text{ at } t = 0,
\]

and \( g_{ij} \) is positive definite and represents the negative of the first Chern class, here we know that the first Chern class is the de Rahm cohomology class of the Ricci form which is a real \((1,1)\) form represented as: \( \rho(X,Y) = \frac{1}{2} \text{Rc}(JX,Y) \), and the first Chern class is negative if the tensor \( \text{Rc} \) is negative definite. Here we choose this evolution equation such that when \( t \) goes into the infinity, the Ricci tensor is negative definite for the Kähler metric is positive definite by the initial condition, that fits our assumption, then we derive the scalar equation:

\[
\tilde{g}_{ij}(t) = g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j},
\]

and \( \frac{\partial \tilde{g}_{ij}}{\partial t} = \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \)

\[
\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \left( \frac{\partial u}{\partial t} \right) = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij} + \frac{\partial u}{\partial z^i \partial \bar{z}^j}) - g_{ij} - \frac{\partial^2}{\partial z^i \partial \bar{z}^j} u,
\]

while \( g_{ij} \) represents the first Chern class of \( M \), so \( g_{ij} = -\tilde{R}_{ij} \), so the equation changes to:

\[
\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \left( \frac{\partial u}{\partial t} \right) = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij} + \frac{\partial u}{\partial z^i \partial \bar{z}^j}) - \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij}) - \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} u
\]

and similarly:

\[
\frac{\partial}{\partial t} \left( e^t \frac{\partial u}{\partial t} \right) = \Delta (e^t \frac{\partial u}{\partial t}) - \frac{\partial u}{\partial t}
\]

which means that

\[
\frac{\partial}{\partial t} \left( e^t \frac{\partial u}{\partial t} \right) = \Delta (e^t \frac{\partial u}{\partial t}) - \frac{\partial u}{\partial t}
\]

that’s actually a heat equation and we use the maximum principle again, which means

\[
| \frac{\partial u}{\partial t} | \leq | \frac{\partial u}{\partial t} |_{t=0} \text{ for when } t=0 \text{ it’s actually a boundary of } M \times [0,\infty), \text{ then}
\]

\[
| \frac{\partial u}{\partial t} |_{t=0} = | \log \det(g_{ij} + \frac{\partial u}{\partial z^i \partial \bar{z}^j}) - \log \det(g_{ij}) - u(0) + f | = | f | \text{ is bounded in } C^0, \text{ then there exists a constant } C > 0 \text{ such that } | e^t \frac{\partial u}{\partial t} | \leq C \text{ and then}
\]

\[
| \frac{\partial u}{\partial t} | \leq e^{-t} C \text{ which means the exponential decay of the } \frac{\partial u}{\partial t}.
\]

For \( s,t \geq 0, x \in M \),
\[ |u(x,s) - u(x,t)| = |\int_{t}^{s} \frac{\partial u}{\partial t}(x,m) \, dm| \leq \int_{t}^{s} |\frac{\partial u}{\partial t}(x,m)| \, dm \leq \int_{t}^{s} Ce^{-t} \, dm = C(e^{-t} - e^{-s}) \]

which means \(u(x,t)\) is a Cauchy sequence in the \(C^0(M)\), then \(u(t)\) converges uniformly in \(C^0(M)\) to some continuous function \(u_\infty\) on \(M\), then from above argument, we obtain the estimates below:

**Lemma 6.1**

1. \(\exists\) uniform constant \(C\) such that \(\forall t \in [0,\infty)\),
   \[ |\frac{\partial u}{\partial t}| \leq e^{-t} C, \]

2. \(\exists\) a continuous real-valued function \(u_\infty\) on \(M\) such that \(\forall t \in [0,\infty)\),
   \[ |u(t) - u_\infty|_{C^0(M)} \leq e^{-t} C, \]

3. \(u(t)|_{C^0(M)}\) is uniformly bounded for \(t \in [0,\infty)\)

Here \(\log \det(g_{ij} + \frac{\partial^2 u}{\overline{\partial z}_i \partial z_j}) = -\frac{\partial u}{\partial t} - \log \det(g_{ij}) - u + f\) and from estimate above we claim the RHS is uniformly bounded then the \(\tilde{g}^{ij}\) is uniformly bounded with a upper bound and a lower bound such that: there exists a uniformly constant \(C\) such that on \(M \times [0,\infty)\) and
\[ \frac{1}{C} g_{ij} \leq \tilde{g}_{ij} \leq C g_{ij}, \]
then we can actually prove the final theorem of the existence of the Kähler-Ricci flow, but the proof of this estimate refers to lots of canonical estimate of the general Kähler Ricci flow, hence in the following sections, we will give more estimates to complete the method.

### 7 More details from Kähler-Ricci Flow

In this section we will figure out some details and problems appeared in the former sections, mainly use some other estimates to figure the problem. First we prove the maximal existence time for the Kähler Ricci flow equation, then similar as before, we will give some important estimates for the normalized Mange-Ampere equation and the corresponding Kähler metric, finally we divide the problem into three condition: the first Chern class positive, negative, or equal to zero, and justify the long time convergence.

We consider the Kähler Ricci flow:
\[ \frac{\partial \omega}{\partial t} = -\text{Ric}(\omega), \quad \omega = \omega_0 \text{ when } t = 0 \]
where \(\omega\) is a family of Kähler form on \(M\), we will note \(\omega(t)\) as the solution of the equation. Then we take the cohomology class of the both sides:
\[ \frac{\partial [\omega]}{\partial t} = -C_1(M), \quad [\omega] = [\omega_0] \text{ when } t = 0 \]
That is an ordinary differential equation because we assume \(C_1(M)\) is known, and the great mathematician S.S.Chern has proved the \(C_1(M)\) is independent of the metric chosen. We solve it and get: \([\omega](t) = [\omega_0] - tC_1(M)\), then if the solution exists, we should get
Lemma 7.1

\[ T > 0, f(x,t) \text{ a smooth function on } M \times [0,T], \text{ if } f \text{ attains its maximum(minimum) at point } (x_0,t_0), \text{ then either } t_0 = 0 \text{ or at } (x_0,t_0) \text{ we have: } \frac{df}{dt} \geq 0 (\leq 0), \text{ } df = 0, \text{ } \sqrt{-T} \partial \bar{\partial} f \leq 0 (\geq 0). \]

Sketch of Proof. We know that if a smooth function attains its maximum at a point \((x_0,t_0)\), then it has zero first derivative and nonpositive Hessian at this point from high dimensional Taylor expansion, if \(t_0 > 0\), then \(f\) is nondecreasing at \(t_0\), so \(\frac{df}{dt} \geq 0\), while if \(t_0 = 0\), because it is the maximum, so we can only attains \(f\) is nonpositive from \(t_0\).
Back to the estimate. We should estimate the zero order of the solution \( \varphi(T) \). The argument is due to Song and Weinkove\[9\].

**Proposition 7.2**

\( \exists C \) constant such that \( \forall t \in [0, T_{\text{max}}) \), \( \| \varphi(t) \|_{C^0(M)} \leq C \).

*Proof.*

Consider \( \theta(t) = \varphi(t) - At \), where \( A \) is to be determined, because \( \varphi(0) = 0 \), so if we can choose an \( A \) to prove \( \theta \) attains its maximum at \( t = 0 \), then we can give a uniform estimate of \( \varphi(t) \).

The argument is due to Song and Weinkove\[9\].

**Proposition 7.3**

\( \exists C \) positive constant such that on \( M \times [0, T_{\text{max}}) \), \( \frac{1}{C} \Omega \leq \omega^n(t) \leq C \Omega \), equivalently, \( \| \frac{\partial}{\partial t} \|_{C^0} \) is uniformly bounded.

*Proof.*

\[
\frac{\partial}{\partial t} \log \omega^n(t) = \frac{\partial}{\partial t} (\log \omega^n(t) - \log \omega^n(0))
= \frac{\partial}{\partial t} \log \det g_{ij}
= g^{ij} \frac{\partial}{\partial t} g_{ij},
\]

but we take trace of the both sides of the equation \( \frac{\partial}{\partial t} \omega = -\text{Ric}(\omega) \); we get \( g^{ij} \frac{\partial}{\partial t} g_{ij} = -R \), where \( R \) is the scalar curvature, then \( \frac{\partial}{\partial t} \log \omega^n(t) = -R \).

While \( R = \text{tr}(\text{Ric}) \), so
uniformly \( C \) elliptic equation we use the maximum principle, \( Q \) is larger than its infimum of the boundary value at \( t=0 \) in \( M \).

As for the lower bound, take the derivative of the equation then we let \( Q = (T' - t) \frac{\partial \varphi}{\partial t} + \varphi + nt \), then we calculate

\[
\frac{\partial}{\partial t} Q = -\frac{\partial \varphi}{\partial t} + (T' - t) \frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial \varphi}{\partial t} + n, \quad \Delta Q = -(T' - t) \Delta \frac{\partial \varphi}{\partial t} + \Delta \varphi,
\]

hence \((\frac{\partial}{\partial t} - \Delta) Q \geq 0\), by the maximum principle \( R \mid R \leq C \). Integrating \( \frac{\partial}{\partial t} \log \omega^n(0) = R \), then we get \( \omega^n(t) \leq e^{C t} \omega^n(0) \). So we finish the upper bound.

For the lower bound, take the derivative of the equation

\[
\frac{\partial \varphi}{\partial t} = \log \left( \frac{\omega + \sqrt{\det \omega}}{\omega} \right) \text{ for } \varphi(0) = 0.
\]

Then we get

\[
\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial \varphi}{\partial t} \left[ T' - t \right] + \varphi + nt,
\]

we let \( Q = (T' - t) \frac{\partial \varphi}{\partial t} + \varphi + nt \), then we calculate

\[
\frac{\partial}{\partial t} Q = -\frac{\partial \varphi}{\partial t} + (T' - t) \frac{\partial^2 \varphi}{\partial t^2} + \frac{\partial \varphi}{\partial t} + n, \quad \Delta Q = -(T' - t) \Delta \frac{\partial \varphi}{\partial t} + \Delta \varphi,
\]

hence \((\frac{\partial}{\partial t} - \Delta) Q \geq 0\), by the maximum principle \( \omega^n(t) \leq e^{C t} \omega^n(0) \). So we finish the upper bound.

**Proposition 7.1**

There exists a uniform positive constant \( C \) such that on \( M \times [0, T_{max}] \),

\[
\frac{1}{T_{max}} \omega_0 \leq \omega \leq C \omega_0.
\]

**Proof.** First we estimate the lower bound. Due to the existence of the normal coordinates, we can always find a system of coordinates for the Hermitian metric \( g \) such that: \( (g_0)_{ij} = \delta_j^i \), and \( g^i_j = \lambda_i \delta^i_j \), for positive eigenvalue \( \lambda_i \). Then

\[
\operatorname{tr}_w \omega_0 = \sum_{i=1}^{n} \frac{1}{\lambda_i} \leq \frac{1}{(n-1)!} \left( \sum_{i=1}^{n} \lambda_i \right)^{n-1}
\]

\[
= \frac{1}{(n-1)!} (\operatorname{tr}_w \omega)^{n-1} \frac{\omega^n}{\omega_0} \leq C, \quad \text{because from Proposition 7.3 } \omega^n \text{ is uniformly bounded.}
\]
As for the upper bound, we prove there exists a uniform constant C such that $tr_{\omega_0} \omega \leq C$ on $M \times [0, T_{\text{max}}]$. We use the traditional method: find a useful quantity using coefficients to be determined, then we prove it satisfies an elliptic equation and use the maximum principle, that method has been used in many of our proof, but how to find a useful quantity is essential, this can only be attained by experience and attempt. We consider $Q = \log tr_{\omega_0} \omega - A\varphi$, where $A$ is a positive uniform constant to be determined. We fix $t \in (0, T_{\text{max}})$, and assume $Q$ attains its maximum at point $(x_0, t_0)$, with out loss generality let $t_0 > 0$.

First we estimate $(\frac{\partial}{\partial t} - \Delta) tr_{\omega_0} \omega$. Let $\hat{\omega}$ be a fixed Kähler form corresponding to the Kähler-Ricci metric $\hat{g}$ on $M$, and the same as before, $\omega$ the solution of the original Kähler-Ricci flow equation, we note $R_{ij}^p = g^{qp} g^{ij} R_{ijkl}$ is the curvature, $\nabla$ is the connection corresponding to $g$, $\hat{R}_{ij}^p = \hat{g}^{qp} \hat{g}^{ij} \hat{R}_{ijkl}$ is the curvature, $\hat{\nabla}$ is the connection corresponding to $\hat{\omega}$. Then the same as before, we use the normal coordinates for $\hat{g}$, then

$$\nabla tr_{\omega_0} \omega = g^{ik} \partial_k ( \hat{g}^{ji} g_{ij} ) = g^{ik} ( \partial_k ( \hat{g}^{ji} ) g_{ij} + \hat{g}^{ji} ( \partial_k \partial_j g_{ij} )$$

$$= g^{ik} \hat{R}_{ji}^k g_{ij} + \hat{g}^{ji} \hat{g}_{ip} \hat{g}_{jk} \partial_i \partial_j \partial_k g_{ij},$$

while $\frac{\partial}{\partial t} tr_{\omega_0} = \hat{g}^{ji} \hat{\nabla}_i g_{jk} \hat{\nabla}_j,$

we get $(\frac{\partial}{\partial t} - \Delta) tr_{\omega_0} \omega = g^{ik} \hat{R}_{ji}^k g_{ij} - \hat{g}^{ji} \hat{g}_{ip} \hat{g}_{jk} \hat{\nabla}_i \partial_j \partial_k g_{ij}.$

Then we use the formular above to calculate $(\frac{\partial}{\partial t} - \Delta) \log tr_{\omega_0} \omega$, then choose normal coordinates such that $g$ is diagonal, by Cauchy-Schwarz inequality we get

$$| \partial tr_{\omega_0} \omega |_{\hat{g}}^2 \leq (tr_{\omega_0} \omega) \sum_{i,j,k} g^{ii} g^{jj} g_{ij} g_{ij} \partial_k \partial_j g_{ij},$$

and then we define $C = -\inf_{x \in M} \hat{R}_{iijj}(x) | \partial z^1, \ldots, \partial z^n$, then calculate

$$\nabla tr_{\omega_0} \omega = \sum_{k,i} g^{ik} \hat{R}_{kii}^i g_{ii} \geq -C \sum_{k} g^{ik} g_{ii} = -C(\log tr_{\omega_0} \omega)$$(tr_{\omega_0} \omega), then use the equality above $(\frac{\partial}{\partial t} - \Delta) \log tr_{\omega_0} \omega \leq C tr_{\omega_0} \omega$.

This proof comes from [9], one can obtains more details from [9].

Then at point $(x_0, t_0)$ we use the estimate above.

$$0 \leq (\frac{\partial}{\partial t} - \Delta) Q = C_0 tr_{\omega_0} - A\varphi - A\varphi + A\Delta \varphi = tr_{\omega}(C_0 \omega_0 - A\omega_t) - A\varphi + A\Delta \varphi$$

do not forget

$$\Delta \varphi = tr_{\omega}(\chi^m \nabla \varphi) = tr_{\omega}(\omega - \omega_t) = n tr_{\omega} \omega_t,$$

for $C_0$ is a constant only depends on the lower bound of the bisectional curvature of $g_0$. Then we need choose $A$ large enough such that $A \omega_t - (C_0 + 1) \omega_0$ is Kähler on $M$, i.e. first it should be positive definite. Then $tr_{\omega}(A \omega_t - (C_0 + 1) \omega_0) \geq 0$, so
trω(−A ˆωt0+C0ω0) ≤ -trω0,
when at point (x0,t0),
0≤ trω(C0ω0 - A ˆωt0) - Alog∥ ̅n Trω∥ + An,
so trω0 + Alog∥ ̅n Trω∥ ≤ An,
then trω0 + Alog∥ ̅n Trω∥ ≤ C, for a constant C. Similarly, find a system of coordinates for the Hermitian metric g such that: (g0)ij = δij, and gij = λiδji, for positive eigenvalue λi, then it shows
\[ \sum \lambda_i \leq C \] due to λi being positive, so there is a uniform upper bound C for 1/λi + Alogλi ≤ C, while 1/λi is positive, we attain a uniform upper bound for Alogλi, hence a uniform upper bound for λi, i.e. λi ≤ C, then we attain an upper bound for trω0 at (x0,t0), then by the uniform bounded φ on M × [0,t′) when t′ ≤ T max and trω0 at (x0,t0), Q is uniformly bounded on M × [0,t′) when t′ ≤ T max, then use the uniformly bounded φ on [0, T max], we know that trω0 has a uniformly upper bound. Therefore, we have prove the uniformly bound for \[ \frac{∂φ}{∂t} = \log∥ ̅n Trω∥ \].

**Lemma 7.5**

If the solution ω(t) of the Kähler-Ricci flow equation before on M × [0, T) satisfies there ∃ a constant C0 such that : \[ \frac{1}{C0} ̅n ω ≤ ω ≤ C0 ̅n ω \].

Then for any positive integer m, there exists corresponding uniform constant Cm such that \[ ||ω(t)||_{C^m(g_0)} ≤ C_m \].

**Proof.**

If the solution ω(t) satisfies there ∃ a constant C0 such that : \[ \frac{1}{C0} ̅n ω ≤ ω ≤ C0 ̅n ω \], we prove first there exists constants C, C' depending only on C0 and ω0 such that:

\[ |∇g_n g| ^2 ≤ C \] and \[ (\frac{1}{C0} - Δ) |∇g_n g| ^2 ≤ \frac{1}{2} |R_{jkl} | ^2 + C' \];

then there exists constants C, C' depending only on C0 and ω0 such that:

\[ |R_{jkl} | ^2 ≤ C \] and \[ (\frac{1}{C0} - Δ) |R_{jkl} | ^2 ≤ - |∇R_{jkl} | ^2 + |∇R_{jkl} | ^2 + C' \], where \( \bar{\nabla} \) is the conjugate of \( \nabla \), then using the condition above we learn that ∃ uniform constants Cm for positive integer m such that \[ |∇R_{jkl} | ^2 ≤ C_m \], therefore using the conclusions of the above claim, for U an open subsets of M, for any compact subset K in U, positive integer m, there exists constants Cm depending only on ω0, K, U, and Cm such that \[ ||ω(t)||_{C^m(K,g_0)} ≤ C_m \], finally, from the above claim, we conclude the lemma.

Due to the much too long standard proof of this lemma, we give a sketch of proof, the remaining detail one can refer to [9], Theorem 2.13,2.14,2.15.

Now we prove the existence of the Kähler-Ricci flow solution.

**Theorem of the maximal existence time**

The Kähler-Ricci flow equation

\[ \frac{∂}{∂t} g = -\text{Ric}(g), \quad g = g_0 \] when \( t = 0 \)

has a unique solution in the maximal time \( t \in [0, T) \), then this solution exists for all time.

**Proof.**

By Proposition 7.4 and Lemma 7.5, we conclude the uniform C∞ estimate for \( ω(t) \) on \([0,T_{max})\), then as \( t \) goes into the \( T_{max} \), by Arzela-Ascoli Theorem and take countable diagonal subsequences, we obtain a solution \( g(T_{max}) \) while
g converges to it on \([0,T_{\text{max}}]\), then we use the same argument as solving the problem before, our estimates are independent of \(t\), then our estimates still works the estimates above are independent of \(t\), so if we choose \(t_0 \in [0,T]\) then the solution also exists in \([t_0,t_0+\epsilon]\) for \(\epsilon\) independent of \(t\), so we can use this \(\tilde{g}_{ij}(T)\) as the initial condition of the same equation choosing a new initial point and continue to deformation, which is contradict to the definition of \(T_{\text{max}}\), hence the process can continue to infinity because our estimates always work, so the solution exists for all time.

Then we complete the proof of the condition \(C_1(M) < 0\) which remains.

We assume \(C_1(M) < 0\), and \([\omega_0] = -C_1(M)\). We consider the normalized Kähler-Ricci flow:

\[
\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega) - \omega, \quad \omega = \omega_0 \quad \text{when} \ t = 0
\]

We use this normalized form to avoid the Kähler class \([\omega(t)]\) given by \((1+t)[\omega_0]\) diverges when \(t\) goes into the infinity. While let \(s = e^{t} - 1\), then by some simple calculation we know \(\omega(t)\) solves the normalized form equation is equivalent to \(\dot{\omega}(s) = e^{t} \omega(t)\) solves the original Kähler-Ricci flow equation

\[
\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega).
\]

Therefore, the estimates we have proved before can be used in the proof of the original problem. Using the same method as the former sections, we now prove the existence of the Kähler-Einstein metric problem.

**Theorem of existence of the Kähler – Einstein metric**

The solution to the equation

\[
\frac{\partial \tilde{g}_{ij}}{\partial t} = -\tilde{R}_{ij}, \quad \tilde{g}_{ij} = g_{ij} \quad \text{at} \ t = 0,
\]

converges in \(C^\infty\) to the unique Kähler-Einstein metric \(\tilde{g}_{ij}(\infty)\) which belongs to the negative first Chern class of \(M\).

**Proof.**

First it’s a Kähler Ricci flow equation which is a rescaling of \(\frac{\partial \tilde{g}_{ij}}{\partial t} = -\tilde{R}_{ij}\) : if \(u'(s)\) is a solution of the equation above, then \(u(t) = \frac{u'(e^s)}{e^s}\) for \(t = \log(s+1)\), where \(s \in [0,\infty)\), then it has a short time solution, and by the Theorem of maximal existence time, it has an all time solution, this is due to the parabolic equation theory, and since we now have an zero order estimate of \(\tilde{g}_{ij}\) by Lemma 7.5 and Lemma 6.1, and an estimate for \(\frac{\partial u}{\partial t}\) with its exponential decay by Lemma 6.1, then as the same argument as the Proposition before, we use Schauder estimate and Interior regularity theory to obtain a \(C^\infty\) estimate of \(\tilde{g}_{ij}\), then the \(C^0\) estimate of \(u(t)\) gives the uniform \(C^\infty\) estimate of it. While \(u(t)\) converges uniformly to \(u(\infty)\) continuously when \(t\) goes into the infinity by Lemma 6.1, we prove that \(u(t)\) converges to \(u(\infty)\) in the \(C^\infty\) sense by contradiction. If there exists an integer \(k\) and \(\epsilon\) positive, and a sequence \(t_i\) goes into the infinity such that

\[
||u(t_i) - u(\infty)||_{C^k(M)} \geq \epsilon \quad \text{for any positive integer } i,
\]

then because \(u(t_i)\) has uniform \(C^{k+1}\) bound then by Arzela-Ascoli Theorem, there exists a subsequence \(u(t_{i_k})\) converges to another limit, says \(u'(\infty)\) in the \(C^k\) sense, but

\[
||u(t_i) - u(\infty)||_{C^k(M)} \geq \epsilon \quad \text{implies } ||u'(\infty) - u(\infty)||_{C^k(M)} \geq \epsilon,
\]

Therefore, the estimates we have proved before can be used in the proof of the original problem. Using the same method as the former sections, we now prove the existence of the Kähler-Einstein metric problem.
so $u'(\infty) \neq u(\infty)$ which is contradic to the uniqueness of the uniformly convergence of $u(t)$. So we only get $u(t)$ converges to $u(\infty)$ in the $C^\infty$ sense. Then this presents that $\frac{\partial u}{\partial t}$ converges to 0 as $t$ goes into the infinity, because $u$ converges to a constant $u_\infty$ therefore, from $\tilde{g}_{ij}(\infty) = g_{ij} + \frac{\partial^2 u(\infty)}{\partial z_i \partial \bar{z}_j}$, we get $\tilde{g}_{ij}$ converges to a constant of $t$ when $t$ goes into the infinity, which means that $\frac{\partial \tilde{g}_{ij}(t)}{\partial t} = 0$, then

$$0 = -\tilde{R}_{ij}(\infty) - \tilde{g}_{ij}(\infty),$$

hence $\tilde{R}_{ij}(\infty) = -\tilde{g}_{ij}(\infty)$, so that’s the Kähler-Einstein metric $g$ we want.

And the uniqueness follows. If $g'$ is another Kähler-Einstein metric both belonging to the same negative first Chern class, then we can write:

$$g' = g + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi,$$

then by calculation $\text{Ric}(g') = \text{Ric}(g) - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi$, so

$$\frac{\log \det (g + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi)}{\log \det g} = \varphi + C$$

for $C$ a constant, then by the maximum principle of the function $\varphi + C$, the maximum and the minimum of $\varphi$ attains at the boundary, since by calculation as above argument this $\varphi + C$ satisfies the equation

$$\frac{\partial \varphi}{\partial t} = \log \det (g' + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi) - \log \det g - u + f$$

for $u \geq 0$ when $t=0$, so we get the maximum and the minimum of the $\varphi + C$ are 0, then the LHS is zero, then $g' = g$. The proof of uniqueness actually comes from Calabi. Finally, the proof is completed, and we just justify the deformation method works well in Kähler-Einstein metric existence problem by some important estimates from the Kähler Ricci flow equation.

**References**

1. Cao, H D. Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kahler manifolds. United States: N. p., 1986.
2. Chern, Shiing-shen. “Characteristic Classes of Hermitian Manifolds.” Annals of mathematics 47.1 (1946): 85–121. Web.
3. Richard S. Hamilton “Three-manifolds with positive Ricci curvature,” Journal of Differential Geometry, J. Differential Geom. 17(2), 255-306, (1982)
4. Yau, S.-T. (1978), On the ricci curvature of a compact Kähler manifold and the complex monge-ampère equation, I. Comm. Pure Appl. Math., 31: 339-411.
5. Gilbarg, David, and Neil S. Trudinger. Elliptic Partial Differential Equations of Second Order. 1st ed. 1977. Berlin, Germany, Springer-Verlag, 1977. Web.
6. Petersen, Peter. Riemannian Geometry. Third edition. Cham, Switzerland: Springer, 2016. Web.
7. Do Carmo, Manfredo Perdigao do. Riemannian Geometry, 2008. Print.
8. Elliptic Partial Differential Equations: Second Edition (Courant Lecture Notes) 2nd Edition. Han Qing, Lin Fang Hua,American Mathematical Society, 2011. Print.
9. Jian Song, and Ben Weinkove. “Lecture Notes on the Kähler-Ricci Flow.” arXiv:1212.3653 [math.DG]
10. PDE II Schauder estimate, Robert Hasslofer. Lecture notes, Toronto university https://www.math.toronto.edu/roberth/pde2/schauder_estimates.pdf
11. “The Ricci Flow; Techniques and Applications, Pt.1: Geometric Aspects.” SciTech Book News 31.2 (2007): n. pag. Print.
12. Calabi, Eugenio. "On Kähler Manifolds with Vanishing Canonical Class". Algebraic Geometry and Topology: A Symposium in Honor of Solomon Lefschetz, edited by Ralph Hartzler Fox, Princeton: Princeton University Press, 2015, pp. 78-89. https://doi.org/10.1515/9781400879915-006
13. Ben Weinkove, "The Kähler-Ricci flow on compact Kähler manifolds", arXiv:1502.06855 [math.DG]
14. Evans, Lawrence C. “Partial Differential Equations, Second edition.” (2010).