EXPOZENTIAL DECAY OF CONNECTIVITY AND UNIQUENESS IN PERCOLATION ON FINITE AND INFINITE GRAPHS

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Summary. We give an upper bound for the uniqueness transition on an arbitrary locally finite graph \( G \) in terms of the limit of the spectral radii \( \rho[H(G_t)] \) of the non-backtracking (Hashimoto) matrices for an increasing sequence of subgraphs \( G_t \subseteq G_{t+1} \) which converge to \( G \). With the added assumption of strong local connectivity for the oriented line graph (OLG) of \( G \), connectivity on any finite subgraph \( G' \subseteq G \) decays exponentially for \( p < (\rho[H(G')])^{-1} \).

Introduction. Percolation is widely used in network theory applications, yet formation of an infinite cluster is not sufficient to ensure high likelihood that an arbitrary pair of selected sites are connected, since the percolation cluster may not be unique. In this work we give upper bounds on the connectivity in site percolation on finite and infinite graphs in terms of the corresponding non-backtracking (Hashimoto) matrices, and related bounds for the uniqueness transition.

Definitions. For a graph \( G \) with the vertex set \( V = V(G) \) and edge set \( E \), we also consider the set of arcs (directed edges) \( A(G) \). The Hashimoto\([6]\) matrix \( H = H(G) \) is the adjacency matrix of the oriented line graph of \( G \). For any pair of arcs \{a, b\} \( \in A \), \( H_{a,b} = 1 \) iff \{a, b\} form a non-backtracking walk of length two, i.e., the head of \( a \) coincides with the tail of \( b \), but \( b \) is not the reverse of \( a \).

In site percolation on a connected undirected graph \( G \), each vertex is chosen to be open with the fixed probability \( p \), independent from other vertices. We focus on a subgraph \( G' \subseteq G \) induced by all open vertices on \( G \). For each vertex \( v \) on \( G' \), let \( C(v) \subseteq G' \) be the connected component of \( G' \) which contains the vertex \( v \), otherwise \( C(v) = \emptyset \). Denote\([9]\) by

\[
\theta_v \equiv \theta_v(G, p) = \Pr(C(v) = \infty),
\]

the probability that \( C(v) \) is infinite. If \( C(v) \) is infinite, for some \( v \), we say that percolation occurs. The percolation transition occurs at the critical probability \( p_c = \sup_v \{ p : \theta_v = 0 \} \). Similarly, introduce the local susceptibility,

\[
\chi_v \equiv \chi_v(G, p) = \mathbb{E}(|C(v)|),
\]

the expected cluster size connected to \( v \), and the associated critical value \( p_T = \inf \{ p : \chi_v = \infty \} \). Generally, \( p_c \leq p_T \); on quasitransitive graphs the two thresholds coincide\([8]\). A third critical value, \( p_u \), corresponds to a transition associated with the number of infinite clusters. For \( p > p_u \) there can be only one infinite cluster and in general \( p_u \geq p_c \). This inequality is strict on non-amenable graphs\([2]\). The uniqueness phase can be characterized by the connectivity,

\[
\tau_{u,v} \equiv \tau_{u,v}(G, p) = \Pr(u \in C(v)),
\]

the probability that vertices \( u \) and \( v \) are in the same cluster. Indeed, if the percolating cluster is unique, for \( p > p_u \), the connectivity is bounded from below, \( \tau_{u,v} \geq \theta_u \theta_v \).

For any non-negative matrix \( H \) (finite or infinite) we define \( p \)-norm growth,

\[
gr_p H \equiv \sup_v \left\{ \lambda > 0 : \lim_{m \to \infty} \frac{\|e_v^T H^m e_v\|_p}{\lambda^m} = 0 \right\},
\]

and a similarly defined \( \overline{gr}_p H \) using limit superior. Here \( e_v \) is a vector with the only non-zero element at \( v \) equal to one. We note that for any finite graph, \( gr_p H = \overline{gr}_p H = \rho(H) \). Moreover, if \( H \) is the Hashimoto matrix associated with a tree \( T \), \( \|H^m e_v\|_1 \) is the number of sites reachable in \( m \) non-backtracking steps from the arc \( v \in A(T) \). Then, \( gr_1 H \) is exactly the growth of the tree\([7]\), and \( gr_1 H \) is the uniformly limited growth\([1]\). Furthermore, on a tree, \( gr_2 H = (gr_1 H)^{1/2} \) is the point spectral radius\([7]\). More generally, for any graph \( G \), \( gr_2 H \) gives an upper bound for the spectral radius \( \rho_2(H) \) of \( H \) treated as an operator on \( l^2(A) \); it satisfies the following inequalities

\[
(gr_1 H)^{1/2} \leq \rho_2(H) \leq gr_2 H \leq gr_1 H,
\]

where the rightmost inequality is strict if \( G \) is non-amenable.

Results. We prove the following bounds:

**Theorem 1.** Consider site percolation on a locally finite graph \( G \) characterized by the Hashimoto matrix \( H \). Then

\[
p_T \geq 1/\overline{gr}_1 H, \quad p_c \geq 1/\gr_1 H.
\]
The first inequality is obtained by evaluating a union bound for $\chi_v$ over all non-backtracking walks starting with $v$ [5, 4]; the second by using the bound on the percolation transition on a graph in terms of the transition on the universal cover[3]. The following connectivity bound follows directly from the alternative definition of $p_c(e) = \lim_{m \to \infty} \|H_m^{\infty}\|_F^{1/m}$:

**Theorem 3.** Consider site percolation on a finite graph $G$ whose OLG is locally strongly $\ell$-connected. Let $H$ be the Hashimoto matrix of $G$. Then, if $\lambda \equiv p \rho(H) < 1$, the connectivity between any pair of vertices satisfies

$$\tau_{i,j} \leq \max(\deg i, \deg j) \frac{1 + [\rho(H)]^\ell}{1 - \lambda} A^{k(i,j)}.$$  

Moreover, for any locally-finite graph $G$ whose OLG is locally strongly $\ell$-connected, we have:

**Theorem 4.** Consider an increasing sequence of subgraphs $G_t \subset G_{t+1} \subset G$ convergent to a locally-finite graph $G$. The following limit exists

$$\rho_0 = \lim_{t \to \infty} \rho(H_t) \leq \rho_c(G).$$  

The upper bound is saturated, $\rho_0 = \rho_c(G)$, if the OLG of $G$ is locally strongly $\ell$-connected.

The same parameter $\rho_0$ also defines a lower bound on the uniqueness transition:

**Theorem 5.** For a locally finite graph $G$, the uniqueness transition satisfies $p_u \geq 1/\rho_0$.

This follows from a bound on the expected number of self-avoiding cycles passing through a given arc, and the related analysis of cluster stability[4].

**Example 1.** A degree-$d$ infinite tree $T_d$ can be obtained as a limit of an increasing sequence of its subgraphs, $t$-generation trees $G_t = T_d(t)$. We have $\rho(H_t) = 0$ for any $t$, thus $\rho_0 = 0$, consistent with the known fact that there is no uniqueness phase for percolation on $T_d$.

**Conclusions.** We give lower bounds for all three transitions usually associated with site percolation on infinite graphs. We also identify a region of $p$ where connectivity decays exponentially with the distance. For certain graphs with many short cycles, we give an improved upper bound on connectivity’s exponential falloff with the distance, with explicitly specified parameters.

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