BOUNDING THE HIGH ORDER EDGE EXPANSION VIA HIGH ORDER RADIUS

TALI KAUFMAN AND IZHAR OPPENHEIM

Abstract. We define the notion of cone radius of a simplicial complex that generalizes the known notion of radius of a graph. We show that for symmetric simplicial complexes, the cone radius can be used to give a lower bound on the coboundary expansion of the complex. The coboundary expansion of a complex is the high order analogue of a Cheeger constant or edge expansion in a graph. We then define the notion of filling constants of a complex and use them to bound the cone radius and therefore the coboundary expansion. Our paper gives the first general criterion for obtaining high order expanders in the geometric sense, i.e., high dimensional expanders that satisfy a generalized notion of edge expansion. We further present simplicial complexes that we conjecture to satisfy our criterion and form new coboundary expanders.

1. Introduction

High dimensional expansion is a vibrant emerging field that has found applications to PCPs [DK17] and property testing [KLL], to counting problems and matroids [ALGTY], and to list decoding [DHK+19]. We refer the reader to [Lub17] for a recent survey.

The term high dimensional expander means a simplicial complex that have expansion properties that are analogous to expansion in a graph. We recall that a $n$-dimensional simplicial complex $X$ is a hypergraph whose maximal hyperedges are of size $n+1$, and which is closed under containment. Namely, for every hyperedge $\tau$ (called a face) in $X$, and every $\eta \subset \tau$, it must be that $\eta$ is also in $X$. In particular, $\emptyset \in X$. For example, a graph is a 1-dimensional simplicial complex.

Nevertheless, the question of what is a high dimensional expander is still unclear. There is a spectral definition of high dimensional expanders that generalizes the spectral definition of expander graphs and a geometric definition that generalizes the notion of edge expansion (or Cheeger constant) of a graph. For a graph the spectral and the geometric definitions of expansion are known to be equivalent (via the celebrated Cheeger inequality) while in high dimensions the spectral and geometric definitions are known to be NOT equivalent (see for instance [KKL16]).

As of now, the only known criterion that implies high dimensional expansion in the geometric sense is given in [LMM16] and applies only to simplicial complexes with a very specific structure. This work provides, for the first time, a general criterion for high dimensional expansion in the geometric sense. Moreover, we present new candidates for coboundary expanders for which this criterion should be applicable.

Let us recall the geometric notion of expansion in graphs known as the edge expansion or Cheeger constant of a graph:

Definition 1.1 (Cheeger constant of a graph). For a graph $X = (V, E)$:

$$h(X) := \min_{A \neq \emptyset, V} \frac{|E(A, \bar{A})|}{\min\{m(A), m(A)\}},$$

where $m(A)$ denotes the number of edges incident to $A$. This definition generalizes the Cheeger constant of a graph to higher dimensions. It captures the notion of high order edge expansion, which is analogous to the notion of edge expansion in graphs.
where for a set of vertices $U \subseteq V$, $m(U)$ denotes the sum of the degrees of the vertices in $U$.

The generalization of the Cheeger constant to higher dimensions originated in the works of Linial, Meshulam and Wallach (LMW09, MW09) and independently in the work of Gromov (Gro10) and is now known as **coboundary expansion** (the exact definition is given in Section 2 below).

As noted above, unlike the case of graphs, in simplicial complexes a high dimensional version of Cheeger inequality does not hold. Thus, there is a need to develop machinery in order to prove coboundary expansion that does not rely on spectral arguments. For graphs such machinery is available, under the assumptions that the graph has a large symmetry group. A discussion regarding the Cheeger constant of symmetric graphs appear in [Chu97, Section 7.2] and in particular, the following theorem is proven there:

**Theorem 1.2.** [Chu97, Theorem 7.1] Let $X$ be a finite connected graph such that there is a group $G$ acting transitively on the edges of $X$. Denote $h(X)$ to be the Cheeger constant of $X$ and $D$ to be the diameter of $X$. Then $h(X) \geq \frac{1}{2D}$.

**Remark 1.3.** Note that the inequality stated in the Theorem does not hold without the assumption of symmetry. For instance, let $X_N$ by the graph that is the ball of radius $N$ in the $3$-regular infinite tree. Then the diameter of $X$ is $2N+1$ and $h(X_N)$ is of order $O(N)$. The goal of this paper is to generalize Theorem 1.2 to the setting of symmetric simplicial complexes. This is done in two steps: First, we prove a bound on the coboundary expansion of the complex for symmetric complexes that is based on a high dimensional generalization of the diameter (or more precisely of the radius) of the complex. Second, we show how the high dimensional analogue of the radius could be bounded using filling constants of the complex.

In order to specify our exact criterion for bounding the coboundary expansion we need to introduce the boundary and co-boundary maps

### 1.1. Boundary and Co-Boundary maps.

The boundary and co-boundary maps are crucial for defining the generalized Cheeger constant in higher dimensions and for getting a lower bound on its size, but before we can define them, we need to introduce some terminology.

Let $X$ be an $n$-dimensional simplicial complex. A simplicial complex $X$ is called **pure** if every face in $X$ is contained in some face of size $n + 1$. The set of all $k$-faces of $X$ is denoted $X(k)$, and we will be using the convention in which $X(−1) = \{\emptyset\}$.

Let us denote by $C_k(X) = C_k(X, F_2)$ the $F_2$-vector space with basis $X(k)$ (or equivalently, the $F_2$-vector space of subsets of $X(k)$), and $C^k(X) = C^k(X, F_2)$ the $F_2$-vector space of functions from $X(k)$ to $F_2$.

The **boundary map** $\partial_k : C_k(X, F_2) \to C_{k−1}(X, F_2)$ is:

$$\partial_k(F) = \sum_{G \subseteq F, |G| = |F|−1} G,$$

where $F \in X(k)$, and the **coboundary map** $d_k : C^k(X, F_2) \to C^{k+1}(X, F_2)$ is:

$$d_k(\phi)(G) = \sum_{F \subseteq G, |F| = |G|−1} \phi(F),$$

where $\phi \in C^k$ and $G \in X(k + 1)$.

For $A \in C_k(X)$ and $\phi \in C^k(X)$, we denote

$$\phi(A) = \sum_{\tau \in A} \phi(\tau),$$
Thus, for \( \phi \in C^k(X) \) and \( A \in C_{k+1}(X) \)

\[
(4) \quad (d_k \phi)(A) = \phi(\partial_{k+1} A)
\]

We sometimes refer to \( k \)-chains as subsets of \( X(k) \), e.g., the 0-chain \( \{u\} + \{v\} \) will be sometimes referred to as the set \( \{\{u\}, \{v\}\} \). For \( A \in C_k(X) \), we denote \( |A| \) to be the size of \( A \) as a set.

Well known and easily calculated equations are:

\[
(5) \quad \partial_k \circ \partial_{k+1} = 0 \text{ and } d_{k+1} \circ d_k = 0
\]

Thus, if we denote: 
\[ B_k(X) = B_k(X,F_2) = \text{Image}(\partial_{k+1}) \] the space of \( k \)-boundaries.
\[ Z_k(X) = Z_k(X,F_2) = \text{Ker}(\partial_{k+1}) \] the space of \( k \)-cycles.
\[ B^k(X) = B^k(X,F_2) = \text{Image}(d_{k-1}) \] the space of \( k \)-coboundaries.
\[ Z^k(X) = Z^k(X,F_2) = \text{Ker}(d_k) \] the space of \( k \)-cocycles.

We get from (5)

\[
(6) \quad B_k(X) \subseteq Z_k(X) \subseteq C_k(X) \text{ and } B^k(X) \subseteq Z^k(X) \subseteq C^k(X).
\]

Define the quotient spaces \( \tilde{H}_k(X) = Z_k(X)/B_k(X) \) and \( \tilde{H}^k(X) = Z^k(X)/B^k(X) \), the \( k \)-homology and the \( k \)-cohomology groups of \( X \) (with coefficients in \( F_2 \)).

1.2. Generalized notion of diameter in higher dimensions. Following we define a generalized notion of diameter (or radius) of a simplicial complex called a cone. We will later show that in symmetric simplicial complexes a bound on the cone can be used to lower bound the high order Cheeger constant of the complex, known as the coboundary expansion of the complex.

Definition 1.4 (Cone function). Let \( X \) be a pure \( n \)-dimensional simplicial complex. Let \(-1 \leq k \leq n-1\) be a constant and \( v \) be a vertex of \( X \). A \( k \)-cone function with apex \( v \) is a linear function \( \text{Cone}^v_k : \bigoplus_{j=-1}^k C_j(X) \to \bigoplus_{j=-1}^k C_{j+1}(X) \) defined inductively as follows:

1. For \( k = -1 \), \( \text{Cone}^{v}_{-1}(\emptyset) = \{v\} \).
2. For \( k \geq 0 \), \( \text{Cone}^v_k(\bigoplus_{j=-1}^{k-1} C_j(X)) \) is a \((k-1)\)-cone function with apex \( v \) and for every \( A \in C_k(X) \), \( \text{Cone}^v_k(A) \in C_{k+1}(X) \) is a \((k+1)\)-chain that fulfills the equation
\[
\partial_{k+1} \text{Cone}^v_k(A) = A + \text{Cone}^v_k(\partial_k A).
\]

Observation 1.5. By linearity, the condition that
\[
\partial_{k+1} \text{Cone}^v_k(A) = A + \text{Cone}^v_k(\partial_k A), \forall A \in C^k(X)
\]
is equivalent to the condition:
\[
\partial_{k+1} \text{Cone}^v_k(\tau) = \tau + \text{Cone}^v_k(\partial_k \tau), \forall \tau \in X(k).
\]

Example 1.6 (0-cone example). Let \( X \) be an \( n \)-dimensional simplicial complex. Fix some vertex \( v \) in \( X \). By definition, for every \( \{u\} \in X(0) \), \( \text{Cone}^v_0(\{u\}) \) is a 1-chain such that \( \partial_0 \text{Cone}^v_0(\{u\}) = \{u\} + \{v\} \).

If the 1-skeleton of \( X \) is connected, we can define \( \text{Cone}^v_0(\{u\}) \) to be a 1-chain that consists of a sum of edges that form a path between \( \{u\} \) and \( \{v\} \). If the 1-skeleton is not connected, a 0-cone function does not exist: for \( \{u\} \in X(0) \) that is not in the connected component of \( \{v\} \), \( \text{Cone}^v_0(\{u\}) \) cannot be defined. Assuming that the 1-skeleton of \( X \) is connected, we note that the construction of \( \text{Cone}^v_0 \) is usually not unique: different choices of paths between \( \{u\} \) and \( \{v\} \) give different 0-cone functions.
Example 1.7 (1-cone example). Let $X$ be an $n$-dimensional simplicial complex. Assume that the 1-skeleton of $X$ is connected and define a $0$-cone function as in the example above and define $\text{Cone}_0^v$ on $C_0(X)$ as that $0$-cone function. We note that for every $\{u, w\} \in X(1)$, $\{u, w\} + \text{Cone}_0^v(\{u\}) + \text{Cone}_0^v(\{w\})$ forms a closed path, i.e., a 1-cycle, in $X$. If $H_1(X) = 0$, we can deduce that $\{u, w\} + \text{Cone}_0^v(\{u\}) + \text{Cone}_0^v(\{w\})$ is a boundary. Therefore, for every $\{u, w\} \in X(1)$, we can choose $\text{Cone}_0^v(\{u, w\})$ such that

$$\partial_2 \text{Cone}_0^v = \{u, w\} + \text{Cone}_0^v(\{u\}) + \text{Cone}_0^v(\{w\}).$$

Definition 1.8 (Cone radius - Main Definition). Let $X$ be an $n$-dimensional simplicial complex. Assume that there is a group $G$ given that they are “building-like”, i.e., that they have have sub-

The main result of this paper is that in a symmetric simplicial complex $X$, $\text{Vol}(\text{Cone}_k^v) = \max_{\tau \in X(k)} |\text{Cone}_k^v(\tau)|$. Define the $k$-th cone radius of $X$ to be

$$\text{Crad}_k(X) = \min\{\text{Vol}(\text{Cone}_k^v) : \{v\} \in X(0), \text{Cone}_k^v \text{ is a } k\text{-cone function}\}.$$ 

If $k$-cone functions do not exist, we define $\text{Crad}(X) = \infty$.

Remark 1.9. The reason for the name “cone radius” is that in the case where $k = 0$, $\text{Crad}_0(X)$ is exactly the (graph) radius of the 1-skeleton of $X$. Indeed, for $k = 0$, choose $\{v\} \in X(0)$ such that for every $\{v'\} \in X(0)$,

$$\max_{\{u\} \in X(0)} \text{dist}(v, u) \leq \max_{\{u\} \in X(0)} \text{dist}(v', u),$$

where $\text{dist}$ denotes the path distance. For such a $\{v\} \in X(0)$, define $\text{Cone}_0^v(\{u\})$ to be the edges of a shortest path between $v$ and $u$. By our choice of $v$, it follows that $\text{Vol}(\text{Cone}_0^v)$ is the radius of the one-skeleton of $X$ and we leave it to the reader to verify that this choice gives $\text{Crad}_0(X) = \text{Vol}(\text{Cone}_0^v)$.

1.3. Bounding the high order Cheeger constant by high order diameter.

The main result of this paper is that in a symmetric simplicial complex $X$, the $k$-th cone radius gives a lower bound on the $k$-order Cheeger constant of the complex $\text{Exp}_k^X(\chi)$ known as the $k$-coboundary expansion of $X$.

Theorem 1.10 (Main Theorem). Let $X$ be a pure finite $n$-dimensional simplicial complex. Assume that there is a group $G$ of automorphisms of $X$ acting transitively on $X(n)$. For every $0 \leq k \leq n - 1$, if $\text{Crad}_k(X) < \infty$, then $\text{Exp}_k^X(\chi) \geq 1$. Theorem 1.10 stated above generalizes a result of of Lubotzky, Meshulam and Mozes in which coboundary expansion was proven for symmetric simplicial complexes given that they are “building-like”, i.e., that they have have sub-complexes that behave (in some sense) as apartments in a Bruhat-Tits building.

1.4. Bounding the high order diameter by the filling constants of the complex.

Once we realize that the $k$-th cone radius of the complex can be used to bound the generalized Cheeger constant of the complex, we need to find a way to bound the cone radius. In order to do so, we define what we call the “filling constants” of the complex.

In a nutshell, the filling constant measure for a given $k$-cycle $B$, how large is a $k + 1$-cochain $A$ that satisfy $\partial_{k+1}A = B$. The filling constants will be small if $|A|$ is not much larger than $|B|$. They will be infinite if there is no $A$ such that $\partial_{k+1}A = B$.

In order to give the precise definition, we will need the following notation:
Theorem 1.12. Let \( M \) be the constants defined in Proposition 1.11. Assume that there is a group \( G \) of automorphisms of \( X \) acting transitively on \( X(n) \). Fix \( 0 \leq k \leq n - 1 \),

\[
M_k = \text{Fill}_k((k + 1)M_{k-1} + 1).
\]

If \( \text{Sys}_j(X) > (j + 1)M_{j-1} + 1 \) for every \( 0 \leq j \leq k \), then \( \text{Crad}_k(X) \leq M_k \).

Combining this proposition with Theorem 1.10, we deduce the following:

Theorem 1.12. Let \( X \) be a pure finite \( n \)-dimensional simplicial complex. Let \( M_k \) be the constants defined in Proposition 1.11. Assume that there is a group \( G \) of automorphisms of \( X \) acting transitively on \( X(n) \). Fix \( 0 \leq k \leq n - 1 \). If \( \text{Sys}_j(X) > (j + 1)M_{j-1} + 1 \) for every \( 0 \leq j \leq k \), then \( \text{Exp}_k(X) \geq \frac{1}{(k+1)M_k} \).

As seen in Examples 1.6, 1.7 above, for a simplicial complex \( X \) and \( k \geq 0 \), a \( k \)-cone function may not exist and if it exists it may not be unique. The existence of a \( k \)-cone function turns out to be equivalent to vanishing of (co)homology:

Proposition 1.13. Let \( X \) be a finite \( n \)-dimensional simplicial complex and \( 0 \leq k \leq n - 1 \). There exists a \( k \)-cone function (with some apex) if and only if \( \tilde{H}_j(X) = \tilde{H}^j(X) = 0 \) for every \( 0 \leq j \leq k \).

Remark 1.14. We note that the takeaway from combining this Proposition with Proposition 1.11 is that assuming only large systoles (when compared to the filling constants) implies no systoles at all.

1.5. Conjectured new coboundary expanders. In [KO18], the authors constructed new examples of local spectral expanders using the coset structure of elementary matrix groups. In Section 3 we introduce a similar construction of simplicial complexes and conjecture that these examples are new examples of coboundary expanders. We conjecture that we can bound their filling constants and hence Theorem 1.12 will allow us to bound their coboundary expansion.

1.6. Organisation of the paper. This paper is organized as follows: In Section 2 we introduce the exact definition of coboundary expansion. In Section 3, we prove Theorem 1.10 namely, we prove that for symmetric simplicial complexes, the cone radius can be used to bound the coboundary expansion. In Section 4 we prove Proposition 1.11 that shows that filling constants of the complex can be used to bound the cone radius. In Section 5 we show that the existence of a cone function is equivalent to the vanishing of (co)homology, i.e., we prove Proposition 1.13. Last, in Section 6 we define simplicial complexes that are conjecturally new examples of coboundary expander (but are not “building-like” in any obvious way).

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2. The coboundary expansion

Below, we explain all relevant terminology needed to define coboundary expansion:

1. The function \( w : \bigcup_{n=1}^{n} X(n) \rightarrow \mathbb{R}_+ \) is defined as
   \[
   \forall \tau \in X(n), w(\tau) = \frac{|\{ \sigma \in X(n) : \tau \subseteq \sigma \}|}{(n+1)|X(n)|}.
   \]
   We note that \( \sum_{\tau \in X(n)} w(\tau) = 1. \)

2. For every \( \phi \in C^k(X) \), \( w(\phi) \) is defined as
   \[
   w(\phi) = \sum_{\tau \in \supp(\phi)} w(\tau).
   \]

3. For every \( 0 \leq k \leq n-1 \), define the following \( k \)-expansion constants:
   \[
   \Exp_{B^k(X)} = \min \left\{ \frac{w(d_k \phi)}{\min_{\psi \in B^k(X)} w(\phi + \psi)} : \phi \in C^k(X) \setminus B^k(X) \right\}.
   \]

After these notations, we can define coboundary expanders:

Definition 2.1. Let \( \varepsilon > 0 \) be a constant. We say that \( X \) is an \( \varepsilon \)-coboundary expander if for every \( 0 \leq k \leq n-1 \), \( \Exp_{B^k(X)} \geq \varepsilon. \)

Remark 2.2. We leave it for the reader to verify that in the case where \( X \) is a graph, i.e., the case where \( n = 1 \), \( \Exp_{B^k(X)} \) is exactly the Cheeger constant of \( X. \)

3. Cone radius as a bound on coboundary expansion

Lemma 3.1. For \( -1 \leq k \leq n-1 \) and a \( k \)-cone function \( \Cone{v}_k \) with apex \( v \). Define the contraction operator \( t_{\Cone{v}_k} \):
   \[
   t_{\Cone{v}_k} : \bigoplus_{j=-1}^{k} C^{j+1}(X) \rightarrow \bigoplus_{j=-1}^{k} C^{j}(X)
   \]
as follows: for \( \phi \in C^{j+1}(X) \) and \( A \in C_j(X) \), we define
   \[
   (t_{\Cone{v}_k} \phi)(A) = \phi(\Cone{v}_k(A)).
   \]

Then for every \( \phi \in C^k(X), \)
   \[
t_{\Cone{v}_k} d_k \phi = \phi + d_{k-1} t_{\Cone{v}_k} \phi.
   \]

Proof. Let \( A \in C_k(X) \), then
   \[
t_{\Cone{v}_k} d_k \phi(A) = (d_k \phi)(\Cone{v}_k(A))
   \]
   \[
   = \phi(d_k A + (\Cone{v}_k(A)))
   \]
   \[
   = \phi(A + (\Cone{v}_k(A)))
   \]
   \[
   = \phi(A) + (t_{\Cone{v}_k} \phi)(d_k A)
   \]
   \[
   = \phi(A) + (d_{k-1} t_{\Cone{v}_k} \phi)(A),
   \]
as needed.

Naively, it might seem that this Lemma gives a direct approach towards bounding the coboundary expansion: if one could find is some constant \( C = C(n,k,\Crad_k(X)) \) such that \( w(t_{\Cone{v}_k} d_k \phi) \leq C w(d_k \phi) \), then for every \( \phi \),
   \[
   \frac{w(d_k \phi)}{\min_{\psi \in B^k(X)} w(\phi + \psi)} \geq \frac{1}{C} \frac{w(t_{\Cone{v}_k} d_k \phi)}{w(\phi + d_{k-1} t_{\Cone{v}_k} \phi)} = \frac{1}{C}.
   \]

However, by Remark 1.3, we note that without symmetry, the existence of a \( k \)-cone function cannot give an effective bound on the coboundary expansion.
Our proof strategy below is to improve on this naive idea by using the symmetry of $X$: we will show that for a group $G$ that acts on $X$, the group $G$ also acts on $k$-cone functions and we will denote this action by $\rho$. We then show that when $G$ acts transitively on $X(n)$, we can average the action on the $k$-cone function that realizes the cone radius and deduce that

$$\frac{1}{|G|} \sum_{g \in G} w(t_{\rho(g)}(\text{Cone}_k \eta)) \leq \binom{n+1}{k+1} \text{Crad}_k(X) w(d_k \phi).$$

Thus, using an averaged version of the naive argument above will get a bound on the coboundary expansion.

We start by defining an action on $k$-cone functions. Assume that $G$ is a group acting simplicially on $X$. For every $g \in G$ and every $k$-cone function $\text{Cone}_k^v$ define

$$(\rho(g) \cdot \text{Cone}_k^v)(A) = g \cdot (\text{Cone}_k^v(g^{-1} \cdot A)), \forall A \in \bigoplus_{j=-1}^k \partial_j(X).$$

**Lemma 3.2.** For $g \in G$, $-1 \leq k \leq n-1$ and a $k$-cone function $\text{Cone}_k^v$ with apex $v$, $\rho(g) \cdot \text{Cone}_k^v$ is a $k$-cone function with apex $g \cdot v$ and $\text{Vol}(g \cdot \text{Cone}_k^v) = \text{Vol}(\text{Cone}_k^v)$. Moreover, $\rho$ defines an action of $G$ on the set of $k$-cone functions.

**Proof.** If we show that $g \cdot \text{Cone}_k^v$ is a $k$-cone function the fact that $\text{Vol}(g \cdot \text{Cone}_k^v) = \text{Vol}(\text{Cone}_k^v)$ will follow directly from the fact that $G$ acts simplicially.

The proof that $\rho(g) \cdot \text{Cone}_k^v$ is a $k$-cone function is by induction on $k$. For $k = -1$,

$$(\rho(g) \cdot \text{Cone}_{v,-1}^v)(\emptyset) = g \cdot \text{Cone}_{v,-1}^v(g^{-1} \cdot \emptyset) = g \cdot \text{Cone}_{v,-1}^v(\emptyset) = g \cdot \{v\} = \{g \cdot v\},$$

then $\rho(g) \cdot \text{Cone}_{v,-1}^v$ is a $(-1)$-cone function with an apex $g \cdot v$.

Assume the assertion of the lemma holds for $k-1$. Thus, $\rho(g) \cdot \text{Cone}_k^v|_{\bigoplus_{j=-1}^{k-1} \partial_j(X)}$ is a $(k-1)$-cone function with an apex $g \cdot v$ and, by Observation 1.3, we are left to check that for every $\tau \in X(k)$,

$$\partial_{k+1}(\rho(g) \cdot \text{Cone}_k^v)(\tau) = \tau + g \cdot \text{Cone}_k^v(\partial_k \tau).$$

Note that the $G$ acts simplicially on $X$ and thus the action of $G$ commutes with the $\partial$ operator. Therefore, for every $\tau \in X(k)$,

$$\partial_{k+1}(\rho(g) \cdot \text{Cone}_k^v)(\tau) = \partial_{k+1}(g \cdot (\text{Cone}_k^v(g^{-1} \cdot \tau)))$$

$$= g \cdot (\partial_{k+1} \text{Cone}_k^v(g^{-1} \cdot \tau))$$

$$= g \cdot (g^{-1} \cdot \tau + \text{Cone}_k^v(\partial_k g^{-1} \cdot \tau))$$

$$= \tau + g \cdot \text{Cone}_k^v(g^{-1} \cdot \partial_k \tau)$$

$$= \tau + g \cdot \text{Cone}_k^v(\partial_k \tau).$$

The fact that $\rho$ is an action is straight-forward and left for the reader. □

Applying our proof strategy above, will lead us to consider the constant $\theta(\eta)$ defined in the Lemma below.

**Lemma 3.3.** Assume that $G$ is a group acting simplicially on $X$ and that this action is transitive on $n$-simplices. Let $0 \leq k \leq n-1$ and assume that $\text{Crad}_k(X) < \infty$. Fix $\text{Cone}_k^v$ to be a $k$-cone function such that $\text{Crad}_k(X) = \text{Vol}(\text{Cone}_k^v)$. For every $\eta \in X(k+1)$, denote

$$\theta(\eta) = \frac{1}{w(\eta)|G|} \sum_{g \in G} \sum \{w(\tau) : \tau \in X(k), \eta \in (\rho(g) \cdot \text{Cone}_k^v)(\tau)\}.$$

Then for every $\eta \in X(k+1)$, $\theta(\eta) \leq \binom{n+1}{k+1} \text{Crad}_k(X)$. 
Proof. Fix some \( \eta \in X(k+1) \). First, we note that \( G \) acts transitively on \( X(n) \) and therefore \( \bigcup_{g \in G} \{g.\sigma : \sigma \in X(n), \eta \subseteq \sigma \} = X(n) \). This yields that
\[
|X(n)| \leq \frac{|G|}{|G_\eta|} |\{ \sigma \in X(n) : \eta \subseteq \sigma \}|
\]
and therefore
\[
|G_\eta| \leq |G| \binom{n+1}{k+1} w(\eta).
\]
Second, we note that for every \( g \in G \), and every \( \eta \in X(k+1) \),
\[
\eta \in (\rho(g).\Cone_k^\eta)(\tau) \iff \eta \in g.(\Cone_k^\eta(g^{-1}.\tau))
\]
\[
\iff g^{-1}.\eta \in \Cone_k^\eta(g^{-1}.\tau).
\]
Thus,
\[
\theta(\eta) = \frac{1}{w(\eta)|G|} \sum_{g \in G} \sum_{\tau \in X(k)} \{ w(\tau) : \tau \in X(k), \eta \in (\rho(g).\Cone_k^\eta)(\tau) \}
\]
\[
= \frac{1}{w(\eta)|G|} \sum_{g \in G} \sum_{\tau \in X(k)} \{ w(\tau) : \tau \in X(k), g^{-1}.\eta \in \Cone_k^\eta(g^{-1}.\tau) \}
\]
\[
= \frac{1}{w(\eta)|G|} \sum_{g \in G} \sum_{\tau \in X(k)} w(\tau) \sum_{g \in G} g^{-1}.\eta \in \Cone_k^\eta(\tau) \frac{1}{|G|}
\]
\[
\leq \frac{1}{w(\eta)} \sum_{\tau \in X(k)} w(\tau) \Cone_k^\eta(\tau) \frac{|G_\eta|}{|G|}
\]
\[
\leq \frac{1}{w(\eta)} \sum_{\tau \in X(k)} w(\tau) \Cone_k^\eta(\tau) \binom{n+1}{k+1} w(\eta)
\]
\[
= \binom{n+1}{k+1} \Cone_k(X) \sum_{\tau \in X(k)} w(\tau) = \binom{n+1}{k+1} \Cone_k(X),
\]
as needed. \( \square \)

We turn now to prove Theorem 3.10

Proof. Assume that \( G \) is a group acting simplicially on \( X \) such that the action is transitive on \( X(n) \). Let \( 0 \leq k \leq n-1 \) and assume that \( \Cone_k(X) < \infty \). For \( \phi \in C^k(X) \), we denote
\[
|\phi| = \{ \phi + \psi : \psi \in B^k(X) \}, \text{ and } w(|\phi|) = \min_{\phi' \in |\phi|} w(\phi').
\]
Thus, we need to prove that for every \( \phi \in C^k(X) \),
\[
\frac{1}{\binom{n+1}{k+1}} \Cone_k(X) \cdot w(|\phi|) \leq w(d_k \phi),
\]
or equivalently,
\[
|G| w(|\phi|) \leq |G| w(d_k \phi) \left( \binom{n+1}{k+1} \Cone_k(X) \right).
\]

Fix \( \Cone_k^\eta \) to be a \( k \)-cone function such that \( \Cone_k(X) = \Vol(\Cone_k^\eta) \). By Lemma 3.2 for every \( g \in G \), \( \rho(g) \). \( \Cone_k^\eta \) is a \( k \)-cone function.
In the notation of Lemma 3.1, for every \( g \in G \), denote \( \iota_g = \iota_{\rho(g)} \cdot \text{Cone}_k^v \). By Lemma 3.1, for every \( g \in G \), \( w(\phi) \leq w(\iota_g d_k \phi) \) and therefore
\[
|G||w(\phi)| \leq \sum_{g \in G} w(\iota_g d_k \phi) = \sum_{g \in G} \sum_{\jmath \in \{1, \ldots, M \}} \sum_{\tau \in \text{supp}(\iota_g d_k \phi)} \{ u(\tau) : \tau \in X(k), d_k \phi(\rho(g) \cdot \text{Cone}_k^v(\tau)) = 1 \}
\]
\[
\sum_{g \in G} \sum_{\tau \in X(k), \eta \in \text{supp}(d_k \phi)} \{ u(\tau) : \tau \in X(k), \eta \in \rho(g) \cdot \text{Cone}_k^v(\tau) \}
\]
\[
\sum_{\eta \in \text{supp}(d_k \phi)} |G||w(\eta) \theta(\eta)| \leq |G||w(d_k \phi)| \left( \binom{n + 1}{k + 1} \text{Crad}_k(X) \right),
\]
as needed.

\[\square\]

4. Using filling constants to bound the cone radius

The aim of this section is to prove Proposition 1.11 that establishes a connection between filling constants of the complex and the cone radius.

**Lemma 4.1.** Let \( X \) be an \( n \)-dimensional simplicial complex, \( 0 \leq k \leq n - 1 \) and \( \{v\} \in X(0) \). If \( \text{Cone}_{k-1}^v \) is a \((k-1)\)-cone function, then for every \( \tau \in X(k) \), \( \tau + \text{Cone}^v_{k-1}(\partial_k \tau) \in Z_k(X) \).

**Proof.** For \( k = 0 \), we note that for every \( \{u\} \in X(0) \),
\[
\text{Cone}^v_{(0)}(\partial_0 \{u\}) = \text{Cone}^v_{(0)}(\emptyset) = \{v\}
\]
and thus
\[
\partial_0(\{u\} + \text{Cone}^v_{(0)}(\partial_0 \{u\})) = \partial_0(\{u\} + \{v\}) = 2\emptyset = 0.
\]
Assume that \( k > 0 \), then by the definition of the cone function, for every \( \tau \in X(k-1) \),
\[
\partial_k(\tau + \text{Cone}^v_{k-1}(\partial_k \tau)) = \partial_k \tau + \partial_k \text{Cone}^v_{k-1}(\partial_k \tau)
\]
\[
= \partial_k \tau + \partial_k \tau + \text{Cone}^v_{k-1}(\partial_{k-1} \partial_k \tau)
\]
\[
= 2\partial_k \tau + \text{Cone}^v_{k-1}(0)
\]
\[
= 0.
\]

\[\square\]

**Proof of Proposition 1.11.** Let \( M_k, 0 \leq k \leq n - 1 \) be the constants of Proposition 1.11. Fix \( 0 \leq k \leq n - 1 \) and assume that for every \( 0 \leq j \leq k \), \( \text{Sys}_j(X) > (j + 1)M_{j-1} + 1 \). We will show that under these conditions \( \text{Crad}_k(X) \leq M_k \).

The proof is by an inductive construction of a \( j \)-cone function \( \text{Cone}^v_j \) with volume \( \leq M_j \) for every \( -1 \leq j \leq k \). The construction is as follows: fix some \( \{v\} \in X(0) \) and define \( \text{Cone}^v_{(0)}(\emptyset) = \{v\} \). Then \( \text{Vol(Con}_{(0)}^v) = 1 = M_{-1} \) as needed.

Let \( 0 \leq j \leq k \) and assume that \( \text{Cone}^v_{j-1} \) is defined such that \( \text{Vol(Con}_{j-1}^v) \leq M_{j-1} \). We define \( \text{Cone}^v_j(\emptyset) = \text{Cone}^v_{j-1} \) and we are left to define \( \text{Cone}^v_j(\tau) \) for every \( \tau \in X(j) \). Fix some \( \tau \in X(j) \). By our induction assumption,
\[
|\tau + \text{Cone}^v_{j-1}(\partial_j \tau)| \leq 1 + \sum_{\alpha \in X(j-1), \alpha \subseteq \tau} |\text{Cone}^v_{j-1}(\alpha)| \leq 1 + (j + 1) \text{Vol(Con}_{j-1}^v) \leq (j + 1)M_{j-1} + 1.
\]
We assumed that $\text{Sys}_j(X) > (j + 1)M_{j-1} + 1$ and thus $\tau + \text{Cone}_{j-1}^\tau(\partial_j \tau) \notin Z_j(X) \setminus B_j(X)$. By Lemma 4.1 for any $\tau \in X(j)$, $\tau + \text{Cone}_{j-1}^\tau(\partial_j \tau) \in Z_j(X)$ and therefore we deduce that $\tau + \text{Cone}_{j-1}^\tau(\partial_j \tau) \in B_j(X)$, i.e., there is $A \in C_{j+1}(X)$ such that $\partial_{j+1} A = \tau + \text{Cone}_{j-1}^\tau(\partial_j \tau)$.

Also, we can choose this $A$ to be minimal in the sense that for every $A' \in C_{k+1}(X)$, if $\partial_{j+1} A' = \tau + \text{Cone}_{j-1}^\tau(\partial_j \tau)$, then $|A| \leq |A'|$. For such a minimal $A$, define $\text{Cone}_j^\tau(\tau) = A$. Since we chose $A$ to be minimal, it follows that $|\text{Cone}_j^\tau(\tau)| \leq \text{Fill}_j(\tau + \text{Cone}_{j-1}^\tau(\partial_j \tau)) \leq \text{Fill}_j((j+1)M_{j-1} + 1) = M_j$.

Thus, $\text{Vol}(\text{Cone}_j^\tau) \leq M_j$ as needed. \hfill \qed

5. The existence of a cone function and vanishing of (co)homology

Here we prove Proposition 1.13 that states that the existence of a $k$-cone function is equivalent to the vanishing of homology / cohomology (we recall that by the universal coefficient theorem the vanishing of the $j$-th homology with coefficients in $\mathbb{F}_2$ is equivalent to the vanishing of the $j$-th cohomology with coefficients in $\mathbb{F}_2$).

Proof of Proposition 1.13. Let $X$ be a finite $n$-dimensional simplicial complex and $0 \leq k \leq n - 1$.

Assume first that for every $0 \leq j \leq k$, $\widetilde{H}_j(X) = 0$. Then by definition, for every $0 \leq j \leq k$, $\text{Sys}_j(X) = \infty$. Thus conditions of Proposition 1.11 are fulfilled trivially and as a result there exists a $k$-cone function.

In the other direction, assume there exists a $k$-cone function $\text{Cone}_k^\tau$. Let $\phi \in Z^k(X)$. Then $d_0 \phi = 0$ and thus in the notation of Lemma 3.1 $\iota_{\text{Cone}_k^\tau} d_k \phi = 0$. By Lemma 3.1 it follows that $\phi + d_{k-1} \iota_{\text{Cone}_k^\tau} \phi = 0$, and therefore $\phi = d_{k-1} \iota_{\text{Cone}_k^\tau} \phi \in B^k(X)$.

Thus, $Z^k(X) = B^k(X)$ and $\widetilde{H}^k(X) = 0$.

By definition, the existence of a $k$-cone function implies the existence of a $j$-cone function for every $0 \leq j \leq k$ and therefore the above argument shows that $\widetilde{H}^j(X) = 0$ for every $0 \leq j \leq k$.

6. Conjectured new coboundary expanders

Fix $n \in \mathbb{N}$ and $q > 2$ to be a prime power. Denote $\mathbb{F}_q[t]$ to be the ring of polynomials with coefficients in $\mathbb{F}_q$. For $1 \leq i, j \leq n + 1$, $i \neq j$ and $r \in \mathbb{F}_q[t]$, let $e_{i,j}(r)$ be the $(n+1) \times (n+1)$ (elementary) matrix with $1$’s along the main diagonal, $r$ in the $(i,j)$ entry and 0’s in all the other entries.

Define the group $G$ to be a subgroup of $(n+1) \times (n+1)$ invertible matrices with entries in $\mathbb{F}_q[t]$ in generated by the set $\{e_{i,i+1}(a + bt) : a, b \in \mathbb{F}_q, 1 \leq i \leq n\}$. More explicitly, an $(n+1) \times (n+1)$ matrix $A$ is in $G$ if and only if

$$A(i,j) = \begin{cases} 1 & i = j \\ 0 & i > j \\ a_0 + a_1 t + \ldots + a_{j-i} t^{j-i} & i < j, a_0, \ldots, a_{j-i} \in \mathbb{F}_q \end{cases},$$

(observe that all the matrices in $G$ are upper triangular).


For $1 \leq i \leq n$, define a subgroup $K_{\{i\}} < G$ as

$$K_{\{i\}} = \langle e_{j,j+1}(a + bt) : j \in \{1, \ldots, n+1\} \setminus \{i\}, a, b \in \mathbb{F}_q \rangle.$$ 

Define $X$ to be the following $n$-dimensional complex:

- The vertices of $X$ are the cosets $\bigcup_{i=1}^n \{gK_{\{i\}} : g \in G\}$.
- Two vertices $gK_{\{i\}}$ and $g'K_{\{i'\}}$ are connected by an edge if $gK_{\{i\}} \cap g'K_{\{i'\}} \neq \emptyset$ and $i \neq i'$.
- If the vertices $g_0K_{\{i_0\}}, \ldots, g_kK_{\{i_k\}}$ are pairwise connected by edges, then $\{g_0K_{\{i_0\}}, \ldots, g_kK_{\{i_k\}}\}$ is a $k$-dimensional simplex in $X$, i.e., $X$ is the clique complex of its one-skeleton.

**Conjecture 6.1.** For every $q > 2$ and every $n \geq 1$, the complex $X$ defined above is a coboundary expander and the constants $\text{Exp}_k^X$ that can be bounded independently of $q$.

We note that as in [KO18], we can show that the group $G$ acts transitively on $X(n)$ and we hope that we will be able to bound the filling constant of $X$ using facts regarding the group presentation of $G$. If that is indeed the case, we can apply Theorem 1.12 to this new family of examples and prove that they are in fact coboundary expanders.

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Department of Computer Science, Bar-Ilan University, Ramat-Gan, 5290002, Israel
E-mail address: kaufmant@mit.edu

Department of Mathematics, Ben-Gurion University of the Negev, Be’er Sheva 84105, Israel
E-mail address: izharo@bgu.ac.il