Phase-Shift Plateaus in the Sagnac Effect for Matter Waves

M. C. Kandes,1, 2 R. Carretero-González,1, 3 and M. W. J. Bromley1, 4

1 Computational Science Research Center, San Diego State University, San Diego, CA 92182, USA
2 Institute of Mathematical Sciences, Claremont Graduate University, Claremont, CA, 91711, USA
3 Department of Mathematics and Statistics, San Diego State University, San Diego, CA 92182, USA
4 School of Mathematics and Physics, The University of Queensland, Brisbane, Qld, 4072, Australia

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We simulate ultracold Sagnac atom interferometers using quantum-mechanical matter wavepackets, e.g. Bose-Einstein condensates, that counter-propagate within a rotating ring-trap. We find that the accumulation of the relative phase difference between wavepackets, i.e. the matter wave Sagnac effect, is manifested as discrete phase jumps. These plateaus result from three effects; that the atoms should be initially trapped at rest with respect to the rotating frame, that they counter-propagate with the same group velocities in the rotating frame, and that the imaging is performed in the rotating frame. We show that the plateaus persist in the presence of nonlinear atom-atom interactions, and in atoms undergoing various rotations, and thus will occur during matter wavepacket experiments. We also introduce the simplest possible Sagnac atom interferometry scheme which relies on wavepacket dispersion around a ring-trap.

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The Sagnac effect was first demonstrated experimentally for light one hundred years ago by French physicist Georges Sagnac [1] and, in recent years, atoms have begun to exhibit a rotation measurement sensitivity able to go beyond that of light-based systems [2]. The Sagnac effect is an interference phenomena of waves encountered in rotating frames of reference that can be used to measure absolute rotations with respect to an inertial frame [3], such that the phase-shifts are well-known, well-tested, and are given by [4]

\[ \Delta_{\text{matter}} = \frac{2m}{\hbar} \left( \vec{\Omega} \cdot \vec{A} \right), \quad \Delta_{\text{light}} = \frac{4\pi}{\lambda_{\text{light}} c} \left( \vec{\Omega} \cdot \vec{A} \right). \]  

(1)

where \( m \) is the particle mass, \( \hbar \) is the reduced Planck constant, \( c \) is the speed of light, and \( \lambda_{\text{light}} \) is the wavelength of the light. In both cases the shift is proportional to the magnitudes of the areal vector \( \vec{A} \) and the angular velocity vector \( \vec{\Omega} \) (see Fig. 1(a)), however, it is (to first-order) independent of the velocity of the rotating matter waves [5]. The optical Sagnac effect has not only been used to address fundamental scientific questions (e.g. as in the Michelson-Gale-Pearson experiment [6]), but it has also found technological applications. In particular, the Sagnac effect is the fundamental basis of rotation measurements made with fiber-optic and ring-laser gyroscopes [7], which are currently deployed in many high-precision inertial navigation systems.

The Sagnac effect has also been observed for matter waves, including neutrons [8], neutral atoms [9], electrons [10] and superfluids [11]. In this paper we ignore the (translated) advice of Malykin, who stated “quantum-mechanical calculations of the imaginary part of a wave function are not at all necessary to compute the phase-shift of de Broglie counter-propagating waves in a rotating ring interferometer, attributable to the Sagnac effect” [3]. Instead, we present a theoretical scheme and calculations to elucidate how the Sagnac effect is manifested in localized matter waves. The basic setup, schematically shown in Fig. 1(b), involves trapping a BEC, splitting it into two counter-propagating waves within a ring-shaped atom trap, and allowing them to collide. Instead of a linear accumulation of the phase-difference with time, we find perfectly-step-like jumps of the phase-shift between wavepacket collisions, with the magnitude of the phase jumps being exactly predicted.

* http://www.smp.uq.edu.au/people/brom/
by the Sagnac formula. These plateaus will be observed in rotating BEC experiments in the various geometries shown in Fig. 1(a).

The motivation for developing Sagnac systems with atoms is the potential improvement, \( \Delta_{\text{matter}}/\Delta_{\text{light}} \approx 10^{11} \) over photons (near optical wavelengths) \(^{[12]}\). A state-of-the-art atom-based Sagnac interferometer is currently able to achieve a precision comparable to ring-laser-based gyroscopes \(^{[2]}\). This was based on thermal atoms in a non-portable 2 m long vacuum chamber with an enclosed area of \( \approx 30 \text{ mm}^2 \), i.e. an effective circle of radius 3 mm.

While there are competing technologies developing compact sensors \(^{[11]}\), ultracold gaseous atoms in a vacuum chamber offer a flexible scaling of the geometry \(^{[13]}\), and thus could explore systematics on the fly. A BEC-based rotation sensor would enable a relatively compact sensor with long experiment times \(^{[4]}\) and the use of common mode noise rejection by making the atoms enclose loops, e.g. using guided BEC configurations \(^{[14]}\) and portable atom chips \(^{[15]}\). Such a sensor could be used on Earth \(^{[4]}\), but it would also be feasible to send it into space \(^{[16]}\), in the search for physics beyond the standard model \(^{[17]}\). The best attempt so far at the measurement of the rotation of the Earth in a BEC experiment \(^{[18]}\) enabled \( \approx 1 \text{ mm} \) (at the limit for being able to observe the Earth’s rotation), but was unable to do so due to technical challenges.

Apart from schemes where a localised BEC is split and traverses a guided path around the interferometer \(^{[18]}\), there have been schemes proposed to use the interference due to the counter-rotation of BEC modes \(^{[19]}\) which are yet to be realised experimentally. There has also been recent experimental progress in this direction with a Sagnac measurement using sound-waves in a BEC in a ring-trap \(^{[20]}\). The key idea that differentiates such systems from the present paper is that their propagating BEC modes fill up the ring-trap and hence they experience a linear accumulation of the phase shift, i.e. \( \vec{A} = \int_0^t \frac{d\vec{L}}{dt} \, dt \), that can be extracted at any measurement time. In contrast, for wavepackets we need to wait until a collision.

I. 1-D THOUGHT EXPERIMENT

We start with the geometry of case-(i) of Fig. 1(a) to walk through the thought experiment of Fig. 1(b). The rotation here has cylindrical \((\rho, \phi, z)\) symmetry about the axis passing through the ring centre and normal to the plane in which the ring lies. The ring-trap potential we choose has harmonic confinement in the transverse \((\rho)\) and \(z\) directions, with \( V_{\text{ring}}(\rho, \phi, z) = \frac{\hbar^2}{2m} \left( \frac{\rho^2}{R^2} + (\rho - R)^2 \right) \) with oscillator units used throughout this paper (see Appendix A). Here we approximate this setup as a 1-D ring since the rotation term in the Hamiltonian is \(-\vec{\Omega} \cdot \vec{L} = -\Omega \vec{L}_z = i \Omega \hbar \partial_{\phi_0}\), and so we choose ansätze \(\Psi(\rho, \phi, z, t) = \psi_1 \psi_2 \psi(\phi, t)\). The \(\psi_1\) and \(\psi_2\) are frozen-in-time Gaussians based on atom traps of frequency \(\omega_\perp = 1\), with all of the 1-D ring dynamics then described by \(\psi(\phi, t)\). Four 1-D calculations are shown in the spacetime plots of Fig. 2 which depict the time-dependent probability density for atom/s in a ring-trap of radius \(R = 10\) which is rapidly rotating at \(\Omega = 0.0150\) (in osc. units, e.g. at 1.5 Hz). The first two plots involve (linear) Schrödinger physics with a single atom, \(N = 1\) (see Appendix A) thus \(g_0 = 0\), the third plot has the nonlinear interactions through a strong 1-D nonlinear coupling constant \(g_1 = g_3/(2\pi\beta_1^2) \approx 10\) (in osc. units, e.g. \(\approx 1000\) \(^{87}\text{Rb}\) atoms). The first three plots in Fig. 2 highlight the four operations (hereafter numbered I, II, III and IV) that our thought experiment demands, which we now discuss sequentially.

(I) The wavepacket requires an initial state. The wavefunction is initially loaded within a ring of radius \(R\) and localised for \(t < 0\) at an angle \(\phi_0\) by an ad-

![FIG. 2. (colour online) Spacetime plot of the probability density for counter-propagating wavepackets (as viewed in the rotating frame) initially trapped with \(\omega_\perp = 0.1\) in an \(\Omega = 0.0150\) rotating 1-D ring of radius \(R = 10\), with (a) \(L_0 = 0\) and \(g_1 = 0\), (b) \(L_0 = \Omega R^2\) and \(g_1 = 0\), and (c) \(L_0 = \Omega R^2\) and \(g_1 = 10\). All units given in oscillator units. Symmetric angular kicks have been applied at \(t = 0\) with \(L_k = 10\). Plot (d) shows a dispersion-based \((L_k = 0)\) atom interferometer with \(\omega_\perp = 1\), \(L_0 = \Omega R^2\), \(g_1 = 0\). The highest probability densities are coloured and the lowest densities are white. The vertical black lines indicate the azimuthal positions \(\pm \pi R \approx 15.7\) and \(\pm \frac{3}{2} \pi R \approx 47.1\) on the ring. The horizontal black lines indicate the times, \(\tau_k = 10\pi\) osc. units, at which two symmetrically counter-propagating classical particles with angular momentum \(L_k = 10\) would undergo 1, 2, 3 collisions in the ring.](image-url)
dential potential, \( V_{\text{trap}}(\phi, t < 0) = \frac{1}{2} \omega_\phi^2 R^2 (\phi - \phi_0)^2 \), by, for example, lingering a laser-induced ring potential longer around \( \phi_0 \). This breaks the rotational invariance of the Hamiltonian, with solutions \( \psi_0(\phi) = C \exp(-\frac{1}{2} \omega_\phi R^2 (\phi - \phi_0)^2) \exp(iL_0(\phi - \phi_0)) \). Here \( C \) is the normalisation, while \( L_0 \) is the initial angular momentum of the wavepacket which depends on what reference frame one considers the atom/s are initially at rest with respect to. For example, Fig. 2(a) is a simulation with the atom initially at rest with respect to the inertial frame (i.e. \( L_0 = 0 \)). In contrast, however, if the atom is at rest with respect to the co-rotating frame of the ring, as seen in Fig. 2(b), then \( L_0 = L_\Omega = \Omega R^2 \). However, at what speed should the atom trap itself be rotating? Since in a general Sagnac experiment the apparatus will rotate while the relative phase-shift between them accumulates with time, which is called the Sagnac effect [see Appendix C for our derivation of Eq. (1)]. For the systems translates with time, which is called the Sagnac effect [see Appendix C for our derivation of Eq. (1)].

For the artificial case of Fig. 2(a) we find that the time-dependent phase-shift accumulates linearly with time, exactly at the rate predicted by Eq. (1) as one might expect from previous studies of Sagnac atom interferometry. For the case depicted in Fig. 2(b) we find that the time-dependent phase-shift response undergoes discrete phase jumps between the classical collision times at multiples of \( t = \tau_k \approx 31.4 \). We also observe that the magnitude of the observed phase jumps are dependent only upon the magnitude of the angular velocity of the ring (\( \Omega \)) and not dependent on the kick (\( L_k \)). Performing the same analysis for the nonlinear case, as per Fig. 2(c), we continue to observe perfectly flat plateaus. Essentially, despite the extra dispersion, this shows that the Sagnac effect remains independent of the (group) velocity of the counter-propagating atoms (see Ref. [3]). Without any dispersion, the interference fringes during the collisions are seen to be perfectly stationary in the rotating frame as a consequence of having the split wavepacket with \( L_0 = L_\Omega \) [see Appendix D for a derivation].

II. DISPERSION-BASED SAGNAC INTERFEROMETER

We now introduce a simplified, dispersion-based, Sagnac atom interferometer by removing step (II) from the system, i.e. the self-induced expansion of a single wavepacket will also realise an interferometer. This is seen in 1-D calculations in Fig. 2(d), where we use a tighter initial trap, with \( \omega_\phi = 1 \), to enhance the dispersion. The interference pattern is again seen to exhibit a constant (3\( \pi \)) phase-shift on the opposite side of the ring from the release. Since no splitting kick is...
Our calculations (Fig. 3) show that for counter-propagating single-particle wavepackets split with a kick \( L_k = 10 \) where (a) \( L_0 = 0, g_l = 0 \), (b) \( L_0 = \Omega R^2, g_l = 0 \), and (c) \( L_0 = \Omega R^2, g_l = 10 \). The different curves correspond to \( R = 10 \) rings rotating at various rates \( \Omega \) as indicated in the top panel, all extracted using Fourier components with \( 2\phi_{0} = 20 \).

Plot (d) shows a single dispersion-based atom interferometer for one rotation rate \( \Omega = 0.0010 \) with \( \omega_{\phi} = 1, L_0 = \Omega R^2, g_l = 0 \), and no kick \((L_k = 0)\). Plot (d) shows two sets of phase-shifts that are simultaneously extracted from each calculation using effective momenta \( L_k' = 5 \) and \( 10 \). We set \( \Delta(t) = 0 \) when there is not enough signal (see Appendix B). The \( \times \)-symbols denote the predicted Sagnac shifts using Eq. (1).

For \( L_k = 0 \), there is no dominant frequency to extract phase-shifts at. Nonetheless, we are able to observe phase-shift plateaus due to the spectrum of frequencies that compose a wavepacket, with each frequency observed in the interference (see Fig. 2(d)). The phase-shifts are shown in Fig. 3(d) for \( \Omega = 0.0010 \), where we have simultaneously extracted the phase-shifts from the same interference patterns using two effective \( L_k' = 5 \) and \( L_k' = 10 \) momenta in the Fourier analysis. As each \( L_k' \) momentum component propagates at a different (group) velocity around the ring, each has its own particular collision time \( (\tau_{k'}) \), around which one is able to image the atoms and extract the Sagnac phase-shift.

We now remove the \( \psi_{\rho} \) confinement and show that plateaus persist in 2-D calculations (neglecting gravity), corresponding to cases (ii) and (iii) of Fig. 1(a). These require the ansätze \( \Psi(x, y, z, t) \approx \psi_{\rho}(x, y, t) \), given a ring-trap \( V_{\text{ring}}(\rho, \phi, z) = \frac{1}{2} \omega_{\perp}^2 z^2 + \omega_{\rho} (\rho - R)^2 \) and a localising initial potential \( V_{\text{trap}}(\phi, t < 0) = \frac{1}{2} \omega_{\phi}^2 (\phi - \phi_0)^2 \). We choose a \( R = 10 \) ring with confinement strengths: \( \omega_{\perp} = 1 \) (which sets the length scale), \( \omega_{\rho} = 10 \) (for tight radial confinement), and \( \omega_{\phi} = 1 \) (for angular localisation). We break the rotational symmetry by placing the rotation ‘off-axis’ at \((x_0, y_0) = (-40, 0)\) (see Appendix A), with the ring centred at \((x, y) = (0, 0)\) in Cartesian-based calculations. An initial state as per case (ii), located near \((x_0, y_0) = (10, 0)\), has an initial phase gradient across the wavefunction that is similar to the 1-D calculations in Fig. 2(b).

As the wavefunction disperses with time, we integrate the 2-D density onto a 1-D ring and extract the phase-shifts (at two different effective momenta \( L_k' = 5 \) and \( 10 \)), which clearly exhibits plateaus as shown in Fig. 4(a). We take this further in Fig. 4(b), which realises case (iii) in Fig. 1(a), with the initial state localised with \( \phi_0 = \frac{1}{2} \pi \), i.e. at \((x_0, y_0) = (0, 10)\). This is a different thought experiment to that setup in Fig. 1(b). The wavepacket dispersion is initially along the \( x \)-axis, i.e. into/away from the center of the rotation (effectively \( L_0 = 0 \) around the direction of the ring). The dynamics are complicated, however, the phase-shifts do briefly touch into plateaus around the ‘collision’ points. Note that for case (iii)-systems, as the rotation rate is increased, the plateaus become increasingly distorted due to the dynamics induced by the angular asymmetry of the initial state (due to the centrifugal force), which is not experienced for a case (i) or (ii) system [see Appendix B and Ancillary videos].

The final case that we consider is the 3-D calculation of Fig. 1(a) case (iv). We choose to align the rotating...
frame such that the $x > 0$ axis aligns to the local north, $z > 0$ aligns away from the origin of the rotation (at $(x_0, y_0, z_0) = (0, 0, -40)$), while $y > 0$ aligns westward (away from the rotation). The angular velocity is chosen as $\tilde{\Omega} = \Omega \left[ \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right]$ (with $\Omega = 0.0010$). Our $R = 10$ ring-trap potential, again with $\omega_\perp = 1$ and $\omega_\rho = 10$, has $\vec{A} = [0, 0, \pi 10^2]$. Our angular harmonic trap with $\omega_\phi = 1$ and $\phi_0 = \frac{3}{4} \pi$ localises the wave for $t < 0$ near $(x_0, y_0, z_0) = (-10, 0, 0)$. After the release into the 3-D ring at $t = 0$, in Fig. 1(c), we again see plateaus around the ‘collision’ times of the different (angular) momentum components, of step-sizes that are in perfect agreement with Eq. (1), given the reduced product $\Omega \cdot \vec{A} = \Omega A/\sqrt{2}$.

In conclusion, we have described how the Sagnac phase shift accumulates in time as observed through a set of numerical simulations modeling the dynamics of a matter wave Sagnac interferometer. In the end, Malykin was correct [3], and one can mostly ignore the complex nature of the Sagnac effected matter wave. However, by considering a thought experiment, where the wavepackets are initially at rest with respect to the waveframe such that the $x > 0$ axis aligns to the local north, $z > 0$ aligns away from the origin of the rotation (at $(x_0, y_0, z_0) = (0, 0, -40)$), while $y > 0$ aligns westward (away from the rotation). The angular velocity is chosen as $\tilde{\Omega} = \Omega \left[ \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right]$ (with $\Omega = 0.0010$). Our $R = 10$ ring-trap potential, again with $\omega_\perp = 1$ and $\omega_\rho = 10$, has $\vec{A} = [0, 0, \pi 10^2]$. Our angular harmonic trap with $\omega_\phi = 1$ and $\phi_0 = \frac{3}{4} \pi$ localises the wave for $t < 0$ near $(x_0, y_0, z_0) = (-10, 0, 0)$. After the release into the 3-D ring at $t = 0$, in Fig. 1(c), we again see plateaus around the ‘collision’ times of the different (angular) momentum components, of step-sizes that are in perfect agreement with Eq. (1), given the reduced product $\Omega \cdot \vec{A} = \Omega A/\sqrt{2}$.

In conclusion, we have described how the Sagnac phase shift accumulates in time as observed through a set of numerical simulations modeling the dynamics of a matter wave Sagnac interferometer. In the end, Malykin was correct [3], and one can mostly ignore the complex nature of the Sagnac effected matter wave. However, by considering a thought experiment, where the wavepackets are initially at rest with respect to the rotating frame, we have found that the phase shift accumulates in apparent discrete phase jumps, and proposed a simple dispersion-based Sagnac atom interferometer. One advantage of these observations for experimentalists is that the phase shift is insensitive to the exact time that the measurement is performed (although maximum contrast will be at the peak of the collision). This measurement insensitivity will also likely apply if the experiment involves a recombination laser pulse followed by a time-of-flight expansion to extract the number of atoms in the two output ports [13]. Future work will involve examining the robustness of the plateaus to further perturbations than in the systems examined here, and the inclusion of finite-temperature effects [24].

Our results may guide the design of high-precision rotation sensors, especially for measurements aimed at the San Diego Chapter of the ARCS Foundation, the San Diego State University, Cymer Incorporated and the Inamori Foundation, as well as the Australian Research Council through a Future Fellowship (FT100100905). Thanks to B.M. Fahy for 3-D parallel code development and S.A. Haine for instructive conversations.

Appendix A: Modelling the atoms

To model a rotating ultracold Bose gas we rely on the nonlinear Schrödinger Equation (NLSE) [27],

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_r(r, t) - \tilde{\Omega} \cdot \vec{L} + g_3|\psi|^2 \right] \psi. \quad (A1)$$

As per (number conserving) theory [28], this describes the evolution of a single atom. $f|\psi|^2 dr = 1$, while the nonlinear coupling constant, $g_3 = 4\pi \hbar^2 a_s/(N-1)/m$, characterises the short-range pairwise interactions between it and the other $(N - 1)$ bosons in the gas. This depends on the s-wave atom-atom scattering length, $a_s$, of two interacting bosons. Note that alternative treatments give $g_3 \propto N$, and then the NLSE is known as the Gross-Pitaevskii equation (GPE) [27, 28]. The rotating configuration of the potential, $V_r(r, t)$, here will generally be time-independent in some reference frame, but we do abruptly switch between potentials at $t = 0$. The transformation to the rotating reference frame is achieved with angular momentum operators $\hat{L} = [\hat{L}_x, \hat{L}_y, \hat{L}_z]$. Despite its simplicity, Eq. (A1) accounts for the inertial forces that may act on the atom in the rotating reference frame — namely, the centrifugal force, the Coriolis force (and, if $\tilde{\Omega}$ was time-dependent, the Euler force).

We use the Crank-Nicolson finite-differences method, employing split-operator algorithms [29] for 2-D and 3-D parallelisation. For 2-D and 3-D calculations we centre the ring at $(x, y) = (0, 0)$ on a Cartesian-grid at $(x, y) = (0, 0)$ with generalised rotation operators e.g. $-\Omega \hat{L}_z = i\hbar \Omega \left( (x - x_0) \frac{2}{m} - (y - y_0) \frac{2}{F} \right)$ where $(x_0, y_0)$ is the origin of the rotation. We imprint a Gaussian-based initial wavefunction with the correct phase gradient to match the rotation, and use imaginary time propagation to determine the initial state of the 2-D and 3-D systems that is stationary in the rotating frame.

We rescale Eq. (A1) in harmonic oscillator units, which rescales energies, lengths and times by $\hbar \omega_\perp$, $\beta_\perp = \sqrt{\hbar/m \omega_\perp}$ and $\omega_\perp^{-1}$, respectively, where $\omega_\perp$ is some natural angular frequency of the system. For example, for a gas of $^{87}$Rb with $\omega_\perp = 2\pi \times 100$ Hz then, for our results presented in this paper, lengths are in units of $\beta_\perp \approx 1.1 \mu m$ and time is in units of $1.6$ ms. The rotation of the Earth, for example, is then $\Omega \approx 10^{-7}$ osc. units.

Appendix B: Extracting phase-shifts

Our phase-shifts are obtained from the density snapshots with a Fourier transform algorithm [30]. This algorithm numerically computes the phase-shift by approximating it as

$$\Delta(t) \approx \tan^{-1} \left\{ \mathcal{F}_{2k_x} \left[ |\psi(t)|^2 \right] \right\}. \quad (B1)$$

The $\mathcal{F}$ is the Fourier transform of the atom/s probability density computed at the dominant frequency of the
interference, here twice the wavenumber, \( k_0 \), of the colliding modes (\( 2k_0 \equiv 2L_k/R \) in osc. units). We use a numerical threshold such that \( \Delta = 0 \) for \( |\mathcal{F}_{2k_0}| < 10^{-10} \). Note that we ‘unwrap’ \( \Delta(t) \) by adding/subtracting appropriate multiples of \( \pi \) (e.g. along a plateau of value \( \frac{1}{2}\pi \) occasionally the algorithm returns \(-\frac{3}{2}\pi\)).

Appendix C: Derivation of Sagnac phase-shift

We present here an illuminating derivation of the matter-wave Sagnac formula of Eqn. (1) of our paper, based on the thought experiment as presented in our paper in Fig. (1). This gives the same answer as generalised treatments of the Sagnac effect as seen via semiclassical methods [9] [10] [12]. It is also more illustrative than derivations based on straightforward substitutions to convert the optical Sagnac effect into a matter wave expression [9]. Our thought experiment is different to a textbook treatment of the problem [22], and in our proof we utilise the group and phase velocities in the inertial frame to demonstrate the Sagnac phase accumulation.

We begin our derivation by assuming a matter wave packet is initially localised in a potential undergoing uniform circular motion when viewed from an inertial frame of reference, with the potential located at a radius \( R \) from the axis of rotation and traversing its circular path with an angular velocity \( \Omega \), e.g. see Fig. 1(a). As a result of being trapped in this rotating potential, in the inertial frame the wave packet acquires an initial group velocity and wavenumber (tangent to the circular motion) of

\[
v_{g0} = R\Omega, \quad k_0 = \frac{mR\Omega}{\hbar}.
\] (C1)

This initial wave packet is then split into a superposition of two counter-propagating wave packets that traverse the ring-trap, which also has radius \( R \). The out-going wave packets have group velocities

\[
v_{g\pm} = \frac{\hbar k_0}{m},
\] (C2)

in the inertial frame, where \( \hbar \) is the reduced Planck constant, and \( m \) is the particle mass, such that

\[
k_{\pm} = (k_0 \pm k),
\] (C3)

are the wavenumbers of the out-going wave packets given that \( k \) is the wavenumber symmetrically imparted to the wave packets during the splitting process. Assuming that the wave packets are free particles as they traverse the ring-trap in opposite directions, then they have phase velocities in the inertial frame and wavelengths of

\[
v_{p\pm} = \frac{v_{g\pm}}{2}, \quad \lambda_{\pm} = \frac{2\pi}{k_0 \pm k}.
\] (C4)

It then follows straightforwardly that the phase of each wave packet advances in time as

\[
\varphi_{\pm}(t) = \frac{2\pi}{\lambda_{\pm}} v_{g\pm} t = \frac{2\pi v_{g\pm}}{\lambda_{\pm}} t = 2R\Omega kt,
\] (C5)

where \( n(t) \) is the number of wavelengths the phase has shifted relative to each wave packets peak. Thus, the relative phase difference between the two counter-propagating wave packets is given by

\[
\Delta(t) = \varphi_+(t) - \varphi_-(t) = 2\pi \left( \frac{v_{p+}}{\lambda_+} - \frac{v_{p-}}{\lambda_-} \right) t = 2R\Omega kt,
\] (C6)
i.e. the phase difference accumulates linearly with time. If we now compute the relative phase difference at the time when the peaks of the wave packets would first coincide in the ring-trap, i.e.

\[
\tau_k = \frac{\pi m R}{\hbar} \Omega,
\] (C7)

(which is also when two classical particles would collide), we finally find that

\[
\Delta(t = \tau_k) = \frac{2\pi m}{\hbar} \Omega A,
\] (C8)

where \( A = \pi R^2 \). This is the expected Sagnac phase-shift.

Appendix D: Derivation of motion of fringes

Next, we give a simple analysis showing that the interference of the counter-propagating wave packets produces an interference pattern that rotates with the group velocity of the initially trapped wave packet, \( v_{g0} \), in the inertial frame. Thus, when viewed in the rotating frame, the fringes appear stationary.

Each wave packet is assumed to counter-propagate about the ring-trap as a time-dependent, free-particle Gaussian wave packet of the form

\[
\psi_{\pm}(x, t) = Ce^{i\frac{\hbar}{m}(x - v_{p\pm} t)} e^{-\frac{(x - v_{g\pm} t)^2}{2\sigma^2}},
\] (D1)

where \( C \) is some normalization constant and \( \sigma \) is the width of the wave packets. In general, both \( C \) and \( \sigma \) are time-dependent, and have complex components. To obtain our simple proof, however, we assume they are fixed and real, effectively ignoring the effects of dispersion on the interference (as seen in Figs. 2(b,c)).

Computing the probability density of a superposition of these analytic wave packets, we find

\[ |\psi_+(x, t) + \psi_-(x, t)|^2 \] (D2)
We see here that the moving Gaussian envelopes form a background for the interference pattern which travels at a velocity $v_{g0}$ in the inertial frame (and thus appears stationary in the rotating frame as per Figs. 2(b,c,d)). This is what gives rise to the apparent plateaus as measured in the rotating frame. The phase shift is still accumulating during the collision, but that accumulation gives rise to a motion in the interference fringes that exactly matches the rotation. We also note that the interference pattern has a wavenumber of $2k$ which we rely on in the phase-shift extraction algorithm.

Appendix E: Ancillary - Videos

Three videos are available as Ancillary arXiv Files. These are animations of 2-D calculations of our dispersion-based Sagnac atom interferometers with initial states shown in Fig. 1(a) corresponding to cases (i), (ii), and (iii). These calculations are shown in the rotating frame, with a fast rotation rate of $\Omega = 0.0150$, along with a weak angular trap with $\omega_\phi = 0.1$, chosen to enhance the visualisation of the phase structures, especially for $t < 0$. This $\Omega$ is an order of magnitude faster than the calculations shown in Figs. 3 and 4, however, it is the same as the rate chosen for the 1-D calculations shown in Fig. 2. Thus, the expected phase-shift at the first collision is $\Delta(t) = 3\pi$, and thus $\Delta(t) = 6\pi$ at the second. For video (i) we thus have the centre of the rotation at $(x_{t\Omega}, y_{t\Omega}) = (0, 0)$, and for (ii) and (iii) we have $(x_{t\Omega}, y_{t\Omega}) = (-40, 0)$. To limit to a $240 \times 240$ pixel video resolution, the axis labels are not shown. The range shown spans $x, y \in [-15, 15]$ to enclose the ring of radius $R = 10$ osc. units.

The videos display both parts of the wavefunction, $\Psi(x, y, t) = A(x, y, t) \exp(i\phi(x, y, t))$, at once. Firstly the contours denote lines of equal probability density ($A^2$, chosen at $10^{-4}$, $10^{-2}$ and $10^{0}$), and secondly coloured by the local phase of the wavefunction (i.e. $\phi \in [0, 2\pi]$). The initial part of the videos show the Gaussian initial states as they are evolved in imaginary time to establish the true (rotating) ground state of the systems with both of their $V_{ring}$ and $V_{trap}$ turned on (note the lines of equal phase pointing towards $(x_{t\Omega}, y_{t\Omega})$). Once that has converged, we commence the real time calculation (at $t = 0$) by turning $V_{trap}$ off, and the wavepackets slowly disperse around the ring, with the animations ending at $t = 100$ osc. units.

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