Extending Hybrid CSP with Probability and Stochasticity

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Abstract. Probabilistic and stochastic behavior are omnipresent in computer controlled systems, in particular, so-called safety-critical hybrid systems, because of fundamental properties of nature, uncertain environments, or simplifications to overcome complexity. Tightly intertwining discrete, continuous and stochastic dynamics complicates modelling, analysis and verification of stochastic hybrid systems (SHSs). In the literature, this issue has been extensively investigated, but unfortunately it still remains challenging as no promising general solutions are available yet. In this paper, we give our effort by proposing a general compositional approach for modelling and verification of SHSs. First, we extend Hybrid CSP (HCSP), a very expressive and process algebra-like formal modeling language for hybrid systems, by introducing probability and stochasticity to model SHSs, which is called stochastic HCSP (SHCSP). To this end, ordinary differential equations (ODEs) are generalized by stochastic differential equations (SDEs) and non-deterministic choice is replaced by probabilistic choice. Then, we extend Hybrid Hoare Logic (HHL) to specify and reason about SHCSP processes. We demonstrate our approach by an example from real-world.

1 Introduction

Probabilistic and stochastic behavior are omnipresent in computer controlled systems, such as safety-critical hybrid systems, because of uncertain environments, or simplifications to overcome complexity. For example, the movement of aircrafts could be influenced by wind; in networked control systems, message loss and other random effects (e.g., node placement, node failure, battery drain, measurement imprecision) may happen.

Stochastic hybrid systems (SHSs) are systems in which discrete, continuous and stochastic dynamics tightly intertwine. As many of SHSs are safety-critical, a thorough validation and verification activity is necessary to enhance the quality of SHSs and, in particular, to fulfill the quality criteria mandated by the relevant standards. But modeling, analysis and verification of SHSs is difficult and challenging. An obvious research line is to extend hybrid automata [10], which is the most popular model for traditional hybrid systems, by adding probability and stochasticity. Then, verification of SHSs can be done naturally through reachability analysis, either by probabilistic model-checking [12,13,19,20,6], or by simulation i.e., statistical model-checking [15,22]. Along this line, several different notions of stochastic hybrid automata have been proposed [11,23,19,8,20,6], with the difference on where to introduce randomness. One option is to replace deterministic jumps by probability distribution over deterministic
jumps. Another option is to generalize differential equations inside a mode by stochastic differential equations. Stochastic hybrid systems comprising stochastic differential equations have been investigated in [13,5,1]. More general models can be obtained by mixing the above two choices, and by combining them with memoryless timed probabilistic jumps [4], with a random reset function for each discrete jump [6]. An overview of this line can be found in [4].

To model complex systems, some compositional modelling formalisms have been proposed, e.g., HMODEST [7] and stochastic hybrid programs [17]. HCSP due to He, Zhou, et al. [9,21] is an extension of CSP [12] by introducing differential equations to model continuous evolution and three types of interruptions (i.e., communication interruption, timeout and boundary condition) to model interactions between continuous evolutions and discrete jumps in HSs. The extension of CSP to probabilistic setting has been investigated by Morgan et al. [16]. In this paper, we propose a compositional approach for modelling and verification of stochastic hybrid systems. First, we extend Hybrid CSP (HCSP), a very expressive and process algebra-like modeling language for hybrid systems by introducing probability and stochasticity, called stochastic HCSP (SHCSP), to model SHSs. In SHCSP, ordinary differential equations (ODEs) are generalized to stochastic differential equations (SDEs), and non-deterministic choice is replaced by probabilistic choice. Different from Platzer’s work [17], SHCSP provides more expressive constructs for describing hybrid systems, including communication, parallelism, interruption, and so on.

Probabilistic model-checking of SHSs does not scale, in particular, taking SDEs into account. For example, it is not clear how to approximate the reachable sets of a simple linear SDEs with more than two variables. Therefore, existing verification techniques based on reachability analysis for SHSs are inadequate, and new approaches are expected. As an alternative, in [17], Platzer for the first time investigated how to extend deductive verification to SHSs. Inspired by Platzer’s work, for specifying and reasoning about SHCSP process, we extend Hybrid Hoare Logic [14], which is an extension of Hoare logic [11] to HSs, to SHSs. Comparing with Platzer’s work, more computation features of SHSs, and more expressive constructs such as concurrency, communication and interruption, can be well handled in our setting. We demonstrate our approach by modeling and verification of the example of aircraft planning problem from the real-world.

2 Background and Notations

Assume that $\mathcal{F}$ is a $\sigma$-algebra on set $\Omega$ and $P$ is a probability measure on $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, P)$ is called a probability space. We here assume that every subset of a null set (i.e., $P(A) = 0$) with probability 0 is measurable. A property which holds with probability 1 is said to hold almost surely (a.s.). A filtration is a sequence of $\sigma$-algebras $\{\mathcal{F}_t\}_{t \geq 0}$ with $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$ for all $t_1 < t_2$. We always assume that a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ has been completed to include all null sets and is right-continuous.

Let $\mathcal{B}$ represent the Borel $\sigma$-algebra on $\mathbb{R}^n$, i.e. the $\sigma$-algebra generated by all open subsets. A mapping $X : \Omega \rightarrow \mathbb{R}^n$ is called $\mathbb{R}^n$-valued random variable if for each $B \in \mathcal{B}$, we have $X^{-1}(B) \in \mathcal{F}$, i.e. $X$ is $\mathcal{F}$-measurable. A stochastic process $X$ is a function
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A system in Stochastic HCSP (SHCSP) consists of a finite set of sequential processes in parallel which communicate via channels synchronously. Each sequential process is represented as a collection of stochastic processes, each of which arises from the interaction of discrete computation and stochastic continuous dynamics modeled by stochastic differential equations.

Let $Proc$ represent the set of SHCSP processes, $\Sigma$ the set of channel names. The syntax of SHCSP is given as follows:

$$P ::= \text{skip} \mid x ::= e \mid ch?x \mid ch!e \mid P; Q \mid B \rightarrow P \mid P^* \mid P \parallel_p Q \mid \langle ds = b dt + \sigma dW & B \rangle \mid \langle ds = b dt + \sigma dW & B \rangle \geq \|_{t \in T} (\omega_t \cdot ch_i*) \rightarrow Q_i$$

$S ::= P \mid S \parallel S$

Here $ch, ch_i \in \Sigma, ch_i*$ stands for a communication event, e.g. $ch?x$ or $ch!e$, $x$ is a variable, $B$ and $e$ are Boolean and arithmetic expressions, $P, Q, Q_i \in Proc$ are sequential processes, $p \in [0, 1]$ stands for the probability of the choice between $P$ and $Q$, $s$ for a vector of continuous variables, $b$ and $\sigma$ for functions of $s$, $W$ for the Brownian motion process. At the end, $S$ stands for a system, i.e., a SHCSP process.

As defined in the syntax of $P$, the processes in the first line are original from HCSP, while the last two lines are new for SHCSP. The individual constructs can be understood intuitively as follows:

- $\vdash \vdash$ denotes a communication event, e.g., $\vdash \vdash ?x$, or $\vdash \vdash !e$.
- $x$ is a variable.
- $\parallel_p$ denotes a parallel composition with probability $p$.
- $\langle ds = b dt + \sigma dW \rangle$ represents the stochastic differential equation (SDE) to model stochastic continuous evolution, which is of the form $dX_t = b(X_t)dt + \sigma(X_t)dW_t$, where $W_t$ is a Brownian motion. In which, the drift coefficient $b(X_t)$ determines how the deterministic part of $X_t$ changes with respect to time and the diffusion coefficient $\sigma(X_t)$ determines the stochastic influence to $X_t$ with respect to the Brownian motion $W_t$. Obviously, any solution to an SDE is a stochastic process.
- **skip**, the assignment \( x := e \), the sequential composition \( P; Q \), and the alternative statement \( B \rightarrow P \) are defined as usual.
- \( ch?x \) receives a value along channel \( ch \) and assigns it to \( x \).  
- \( ch!e \) sends the value of \( e \) along channel \( ch \). A communication takes place when both the sending and the receiving parties are ready, and may cause one side to wait.
- The repetition \( P^* \) executes \( P \) for some finite number of times.
- \( P \upharpoonright_p Q \) denotes probabilistic choice. It behaves as \( P \) with probability \( p \) and as \( Q \) with probability \( 1 - p \).
- \( \langle ds = bdt + \sigma dW & B \rangle \) specifies that the system evolves according to the stochastic process defined by the stochastic differential equation \( ds = bdt + \sigma dW \). As long as the boolean expression \( B \), which defines the domain of \( s \), turns false, it terminates. We will later use \( d(s) \) to return the dimension of \( s \).
- \( \langle ds = bdt + \sigma dW & B \rangle \ast \bigcup_{i \in I} (\omega_i \cdot ch_i \ast \rightarrow Q_i) \) behaves like \( \langle ds = bdt + \sigma dW & B \rangle \), except that the stochastic evolution is preempted as soon as one of the communications \( ch_i \ast \) takes place, after that the respective \( Q_i \) is executed. \( I \) is supposed to be finite and for each \( i \in I \), \( \omega_i \in \mathbb{Q}^+ \) represents the weight of \( ch_i \ast \). If one or more communications are ready at the same time, say they are \( \{ch_j \ast\}_{j \in J} \) with \( J \subseteq I \) and \( |J| \geq 1 \), then \( ch_j \) is chosen with the probability \( \frac{\omega_j}{\sum_{j \in J} \omega_j} \), for each \( j \in J \). If the stochastic dynamics terminates before a communication among \( \{ch_i \ast\}_I \) occurring, then the process terminates without communicating.
- \( S_1 \parallel S_2 \) behaves as if \( S_1 \) and \( S_2 \) run independently except that all communications along the common channels connecting \( S_1 \) and \( S_2 \) are to be synchronized. The processes \( S_1 \) and \( S_2 \) in parallel can neither share variables, nor input nor output channels.

### 3.1 A Running Example

We use SHCSP to model the aircraft position during the flight, which is inspired from \([18]\).

Consider an aircraft that is following a flight path consisting of a sequence of line segments at a fixed altitude. Ideally, the aircraft should fly at a constant velocity \( v \) along the nominal path, but due to the wind or cloud disturbance, the deviation of the aircraft from the path may occur. For safety, the aircraft should follow a correction heading to get back to the nominal path as quickly as possible. On one hand, the correction heading should be orthogonal to the nominal path for the shortest way back, but on the other hand, it should also go ahead to meet the destination. Considering these two objectives, we assume the correction heading always an acute angle with the nominal path.

Here we model the behavior of the aircraft along one line segment. Without loss of generality, we assume the segment is along \( x \)-axis, with \( (x_s, 0) \) as the starting point and \( (x_e, 0) \) as the ending point. When the aircraft deviates from the segment with a vertical distance greater than \( \lambda \), we consider it enters a dangerous state. Let \( (x_s, y_0) \) be the initial position of the aircraft in this segment, then the future position of the aircraft \( (x(t), y(t)) \) is governed by the following SDE:

\[
\begin{bmatrix}
\frac{dx(t)}{dt} \\
\frac{dy(t)}{dt}
\end{bmatrix} = v \begin{bmatrix}
\cos(\theta(t)) \\
\sin(\theta(t))
\end{bmatrix} dt + dW(t)
\]
where $\theta(t)$ is the correction heading and is defined with a constant degree $\frac{\pi}{4}$ when the aircraft deviates from the nominal path:

$$\theta(t) = \begin{cases} 
-\frac{\pi}{4} & \text{if } y(t) > 0 \\
0 & \text{if } y(t) = 0 \\
\frac{\pi}{4} & \text{if } y(t) < 0 
\end{cases}$$

Define $B$ be $x_s \leq x \leq x_e$, the movement of the aircraft described above can be modelled by the following SHCSP process $P_{Air}$:

$$x = x_s; y = y_0; \langle [dx, dy]^T = v[\cos(\theta(t)), \sin(\theta(t))]^T dt + dW(t)&B \rangle$$

### 4 Operational Semantics

Before giving operational semantics, we introduce some notations first.

**System Variables** In order to interpret SHCSP processes, we use non-negative reals $\mathbb{R}^+$ to model time, and introduce a global clock $\text{now}$ as a system variable to record the time in the execution of a process. A **timed communication** is of the form $\langle ch.c, b \rangle$, where $ch \in \Sigma$, $c \in \mathbb{R}$ and $b \in \mathbb{R}^+$, representing that a communication along channel $ch$ occurs at time $b$ with value $c$ transmitted. The set $\Sigma \times \mathbb{R} \times \mathbb{R}^+$ of all timed communications is denoted by $T\Sigma$. The set of all timed traces is

$$T\Sigma^* = \{ \gamma \in T\Sigma^* \mid \text{if } \langle ch_1.c_1, b_1 \rangle \text{ precedes } \langle ch_2.c_2, b_2 \rangle \text{ in } \gamma, \text{ then } b_1 \leq b_2 \}.$$ 

If $C \subseteq \Sigma$, $\gamma |\_C$ is the projection of $\gamma$ onto $C$ such that only the timed communications along channels of $C$ in $\gamma$ are preserved. Given two timed traces $\gamma_1, \gamma_2$, and $X \subseteq \Sigma$, the **alphabetized parallel** of $\gamma_1$ and $\gamma_2$ over $X$, denoted by $\gamma_1 \parallel\_X \gamma_2$, results in the following set of timed traces

$$\{ \gamma \mid \gamma |\_\{(\Sigma(\gamma_1) \cup \Sigma(\gamma_2))\} = \epsilon, \gamma |\_\Sigma(\gamma_1) = \gamma_1, \gamma |\_\Sigma(\gamma_2) = \gamma_2 \text{ and } \gamma |\_X = \gamma_1 |\_X = \gamma_2 |\_X \},$$

where $\Sigma(\gamma)$ stands for the set of channels that occur in $\gamma$.

To model synchronization of communication events, we need to describe their readiness. Because a communication itself takes no time when both parties get ready, thus, at a time point, multiple communications may occur. In order to record the execution order of communications occurring at the same time point, we prefix each communication readiness a timed trace that happened before the ready communication event. Formally, each **communication readiness** has the form of $\gamma.ch?$ or $\gamma.ch!$, where $\gamma \in T\Sigma^*_\leq$. We denote by $RDY$ the set of communication readiness in the sequel.

Finally, we introduce two system variables, $rdy$ and $tr$, to represent the ready set of communication events and the timed trace accumulated at the considered time, respectively. In what follows, we use $\text{Var}(P)$ to represent the set of process variables of $P$, plus the system variables $\{rdy, tr, now\}$ introduced above, which take values respectively from $\mathbb{R} \cup RDY \cup T\Sigma^*_\leq \cup \mathbb{R}^+$, denoted by $Val$. 


States and Functions To interpret a process $P \in \text{Proc}$, we define a state $ds$ as a mapping from $\text{Var}(P)$ to $\text{Val}$, and denote by $\mathcal{D}$ the set of such states. Because of stochasticity, we introduce a random variable $\rho : \Omega \rightarrow \mathcal{D}$ to describe a distribution of all possible states. In addition, we introduce a stochastic process $H : \text{Intv} \times \Omega \rightarrow \mathcal{D}$ to represent the continuous flow of process $P$ over the time interval $\text{Intv}$, i.e., state distributions on the interval. In what follows, we will abuse state distribution as state if not stated otherwise.

Given two states $\rho_1$ and $\rho_2$, we say $\rho_1$ and $\rho_2$ are parallelable iff for each $\omega \in \Omega, \text{Dom}(\rho_1(\omega)) \cap \text{Dom}(\rho_2(\omega)) = \{\text{rdy}, \text{tr}, \text{now}\}$ and $\rho_1(\omega)(\text{now}) = \rho_2(\omega)(\text{now})$. Given two parallelable states $\rho_1$ and $\rho_2$, paralleling them over $X \subseteq \Sigma$ results in a set of new states, denoted by $\rho_1 \uplus \rho_2$, any of which $\rho$ is given by

$$
\rho(\omega)(v) \overset{\text{def}}{=} \begin{cases} 
\rho_1(\omega)(v) & \text{if } v \in \text{Dom}(\rho_1(\omega)) \setminus \text{Dom}(\rho_2(\omega)), \\
\rho_2(\omega)(v) & \text{if } v \in \text{Dom}(\rho_2(\omega)) \setminus \text{Dom}(\rho_1(\omega)), \\
\rho_1(\omega)(\text{now}) & \text{if } v = \text{now}, \\
\gamma, \text{ where } \gamma \in \rho_1(\omega)(\text{tr}) \cap \rho_2(\omega)(\text{tr}) & \text{if } v = \text{tr}, \\
\rho_1(\omega)(\text{rdy}) \cup \rho_2(\omega)(\text{rdy}) & \text{if } v = \text{rdy}.
\end{cases}
$$

It makes no sense to distinguish any two states in $\rho_1 \uplus \rho_2$, so hereafter we abuse $\rho_1 \uplus \rho_2$ to represent any of its elements. $\rho_1 \uplus \rho_2$ will be used to represent states of parallel processes.

Given a random variable $\rho$, the update $\rho[v \rightarrow e]$ represents a new random variable such that for any $\omega \in \Omega$ and $x \in \text{Var}$, $\rho[v \rightarrow e](\omega)(x)$ is defined as the value of $e$ if $x$ is $v$, and $\rho(\omega)(x)$ otherwise. Given a stochastic process $X : [0, d] \times \Omega \rightarrow \mathcal{D}(s)$, for any $t$ in the domain, $\rho[s \rightarrow X_t]$ is a new random variable such that for any $\omega \in \Omega$ and $x \in \text{Var}$, $\rho[s \rightarrow X_t](\omega)(x)$ is defined as $X(t, w)$ if $x$ is $s$, and $\rho(\omega)(x)$ otherwise.

At last, we define $H^\rho_d$ as the stochastic process over interval $[\rho(\text{now}), \rho(\text{now}) + d]$ such that for any $t \in [\rho(\text{now}), \rho(\text{now}) + d]$ and any $\omega$, $H^\rho_d(t, \omega) = \rho(\text{now} \rightarrow t)(\omega)$, and moreover, $H^\rho_s,X_d$ as the stochastic process over interval $[\rho(\text{now}), \rho(\text{now}) + d]$ such that for any $t \in [\rho(\text{now}), \rho(\text{now}) + d]$ and any $\omega$, $H^\rho_s,X_d(t, \omega) = \rho(\text{now} \rightarrow t, \text{rdy} \rightarrow \emptyset, s \rightarrow X_t)(\omega)$.

4.1 Operational Semantics

Each transition relation has the form of $(P, \rho) \xrightarrow{\alpha} (P', \rho')$, where $P$ and $P'$ are processes, $\alpha$ is an event, $\rho$, $\rho'$ are states, $H$ is a stochastic process. It expresses that starting from initial state $\rho$, $P$ evolves into $P'$ by performing event $\alpha$, and ends in state $\rho'$ and the execution history of $\alpha$ is recorded by continuous flow $H$. When the transition is discrete and thus produces a flow on a point interval (i.e., current time now), we will write $(P, \rho) \xrightarrow{\alpha} (P', \rho')$ instead of $(P, \rho) \xrightarrow{\alpha} (P', \rho', \{\rho(\text{now}) \mapsto \rho'\})$. The label $\alpha$ represents events, which can be an internal event like skip, assignment, or a termination of a continuous etc, uniformly denoted by $\tau$, or an external communication event $\text{ch} \; \text{c}$ or $\text{ch}? \; \text{c}$, or an internal communication $\text{ch} \; \text{c}$, or a time delay $d$ that is a positive real number. We call the events but the time delay discrete events, and will use $\beta$ to range over them. We define the dual of $\text{ch} \; \text{c}$ (denoted by $\text{ch} \; \text{c}$) as $\text{ch}\; \text{c}$, and vice versa, and define $\text{comm}(\text{ch} \; \text{c}, \text{ch}? \; \text{c})$ or $\text{comm}(\text{ch}? \; \text{c}, \text{ch} \; \text{c})$ as the communication $\text{ch} \; \text{c}$. In the operational
semantics, besides the timed communications, we will also record the internal events that have occurred till now in \( tr \).

For page limit, we present the semantics for the new constructs of SHCSP in the paper in Table 1. The semantics for the rest is same to HCSP, which can be found in Appendix. The semantics for probabilistic choice is given by rules (PCho-1) and (PCho-2): it is defined with respect to a random variable \( U \) which distributes uniformly in \([0, 1]\), such that for any sample \( \omega \), if \( U(\omega) \leq p \), then \( P \) is taken, otherwise, \( Q \) is taken. In either case, it is assumed that an internal action happened. A stochastic dynamics can continuously evolve for \( d \) time units if \( B \) always holds during this period, see (Cont-1). In (Cont-1), the variable \( X \) solves the stochastic process and the ready set keeps unchanged, reflected by the flow \( H_{d}^{p-s,X} \). The stochastic dynamics terminates at a point whenever \( B \) turns out false at a neighborhood of the point (Cont-2). Communication interrupt evolves for \( d \) time units if none of the communications \( ch_i \ast \) is ready (IntP-1), or is interrupted to execute \( ch_i j \ast \) whenever \( ch_i j \ast \) occurs first (IntP-2), or terminates immediately in case the continuous terminates before any communication happening (IntP-3).

The following theorem indicates that the semantics of SHCSP is well defined.

**Theorem 1.** For each transition \((P, \rho) \xrightarrow{\alpha} (P', \rho', H)\), \( H \) is an almost surely càdlàg process and adapted to the completed filtration \((\mathcal{F}_{t})_{t \geq 0}\) (generated by \( \rho \), the Brownian motion \((B_{s})_{s \leq t}\), the weights \( \{\omega_i\}_{i \in I} \) and uniform \( U \) process) and the evolving time from \( P \) to \( P' \), denoted by \( \Delta(P, P') \), is a Markov time.

**Proof.** The proof of this theorem can be found in Appendix.

## 5 Assertions and Specifications

In this section, we define a specification logic for reasoning about SHCSP programs. We will first present the assertions including syntax and semantics, and then the specifications based on Hoare triples. The proof system will be given in next section.

### 5.1 Assertion Language

The assertion language is essentially defined by a first-order logic with emphasis on the notion of explicit time and the addition of several specific predicates on occurrence of communication traces and events. Before giving the syntax of assertions, we introduce three kinds of expressions first.

\[
\begin{align*}
h & ::= \varepsilon \mid \langle ch.E, T \rangle \mid h \cdot h \mid h^* \\
E & ::= c \mid x \mid f^k(E_1, ..., E_k) \\
T & ::= o \mid now \mid u^l(T_1, ..., T_l)
\end{align*}
\]

\( h \) defines trace expressions, among which \( \langle ch.E, T \rangle \) represents that there is a value \( E \) transmitted along channel \( ch \) at time \( T \). \( E \) defines value expressions, including a value constant \( c \), a variable \( x \), or arithmetic value expressions. \( T \) defines time expressions, including a time constant \( o \), system variable \( now \), or arithmetic time expressions.
U is a random variable distributed uniformly in \([0, 1]\), \(U(\omega) \leq p\) (PCho-1)

\[
\begin{array}{l}
(P \cup_p Q, \rho) \xrightarrow{d} (P, \rho|tr \mapsto tr \cdot \langle \tau, \text{now} \rangle) \\
(P \cup_p Q, \rho) \xrightarrow{d} (Q, \rho|tr \mapsto tr \cdot \langle \tau, \text{now} \rangle)
\end{array}
\]

\(X : [0, d) \times \Omega \rightarrow \mathbb{R}^{d(n)}\) is the solution of \(ds = bdt + \sigma dW \land \forall t \in [0, d], \forall \omega, p(now \mapsto \text{now} + t, s \mapsto X_1)(\omega)(B) = T\) (Cont-1)

\[
\begin{array}{l}
(ds = bdt + \sigma dW & B) \Rightarrow \exists \omega. \rho(\omega)(B) = F \text{ or } (X : [0, d) \times \Omega \rightarrow \mathbb{R}^{d(n)} \text{ is the solution of } ds = bdt + \sigma dW, \\
\exists \varepsilon > 0 \forall t \in (0, \varepsilon) \exists \omega. p(now \mapsto \text{now} + t, s \mapsto X_1)(\omega)(B) = F
\end{array}
\]

\[
(ds = bdt + \sigma dW & B, \rho) \xrightarrow{d} (\epsilon, \rho|tr \mapsto tr \cdot \langle \tau, \text{now} \rangle)
\]

\(\{ch_i \ast * Q_i, \rho\} \xrightarrow{d} \langle ch_i \ast * Q_i, \rho_i, H \rangle, \forall i \in I\) (IntP-1)

\[
\begin{array}{l}
(ds = bdt + \sigma dW & B) \Rightarrow (\forall \omega \in I \omega_i \cdot ch_i \ast * Q_i), \rho) \xrightarrow{d} (\langle ds = bdt + \sigma dW & B \rangle \Rightarrow (\forall \omega \in I \omega_i \cdot ch_i \ast * Q_i), \\
\langle \rho'[\text{rdy} \mapsto \cup j \in I \rho'_j(\text{rdy})], \text{H}[\text{rdy} \mapsto \cup j \in I \rho'_j(\text{rdy})] \rangle
\end{array}
\]

\(\{ch_i \ast * Q_i\}_{1 \leq i \leq n} \text{ get ready simultaneously while others not}\)

\[
\begin{array}{l}
U(\omega) < \sum_{k=1}^{n} \omega_k \text{ and } (ch_j \ast * Q_j, \rho) \xrightarrow{d} (Q_j, \rho')
\end{array}
\]

\[
\begin{array}{l}
(ds = bdt + \sigma dW & B) \Rightarrow (\forall \omega \in I \omega_i \cdot ch_i \ast * Q_i), \rho) \xrightarrow{d} (Q_j, \rho')
\end{array}
\]

\[
\begin{array}{l}
(ds = bdt + \sigma dW & B) \Rightarrow (\forall \omega \in I \omega_i \cdot ch_i \ast * Q_i), \rho) \xrightarrow{d} (\epsilon, \rho')
\end{array}
\]

Table 1. The semantics of new constructs of SHCSP

The categories of the assertion language include terms, denoted by \(\theta, \theta_1 \text{ etc.}\), state formulas, denoted by \(S, S_1 \text{ etc.}\), formulas, denoted by \(\varphi, \varphi_1 \text{ etc.}\), and probability formulas, denoted by \(P \text{ etc.}\), which are given by the following BNFs:

\[
\begin{array}{l}
\theta ::= E \mid T \mid h \mid tr \\
S ::= \bot \mid R^n(\theta_1, ..., \theta_n) \mid h.ch? \mid h.ch! \mid \neg S \mid S_1 \lor S_2 \\
\varphi ::= \bot \mid S \at T \mid \neg \varphi \mid \varphi_1 \lor \varphi_2 \mid \forall v. \varphi \mid \forall t. \varphi \\
P ::= P(\varphi) \lor p \mid \neg P \mid P \lor P
\end{array}
\]

The terms \(\theta\) include value, time and trace expressions, plus trace variable \(tr\). The state expressions \(S\) include false (denoted by \(\bot\)), truth-valued relation \(R^n\) on terms, readiness, and logical combinations of state formulas. In particular, the readiness \(h.ch?\) or
h.ch! represents that the communication event ch? or ch! is enabled, and prior to it, the sequence of communications recorded in h has occurred. The formulas $\varphi$ include false, a primitive $S$ at $T$ representing that $S$ holds at time $T$; and logical combinations of formulas ($v, t$ represent logical variables for values and time resp.). For time primitive, we have an axiom that $(S_1 \text{ at } T \wedge S_2 \text{ at } T) \iff (S_1 \wedge S_2)$ at $T$. We omit all the other axiom and inference rules for the formulas, that are same to first-order logic. The probability formula $P$ has the form $P(\varphi) \propto p$, where $\propto \in \{<, \leq, >, \geq\}$, $p \in \mathbb{Q} \cap [0, 1]$, or the logical composition of probability formulas free of quantifiers. In particular, $P(\varphi) \propto p$ means that $\varphi$ is true with probability $\propto p$. For the special case $P(\varphi) = 1$, we write $\varphi$ for short.

In the sequel, we use the standard logical abbreviations, as well as

$$
\varphi \text{ dr } [T_1, T_2] \overset{\text{def}}{=} \forall t. (T_1 \leq t \leq T_2) \Rightarrow \varphi \text{ at } t
$$

$$
\varphi \text{ in } [T_1, T_2] \overset{\text{def}}{=} \exists t. (T_1 \leq t \leq T_2) \wedge \varphi \text{ at } t
$$

**Interpretation** In the following, we will use a random variable $Z : \Omega \rightarrow (\text{Var} \rightarrow \text{Val})$ to describe the current state and a stochastic process $\mathcal{H} : [0, +\infty) \times \Omega \rightarrow (\text{Var} \rightarrow \text{Val})$ to represent the whole evolution. The semantics of a term $\theta$ is a function $[\theta] : (\Omega \rightarrow (\text{Var} \rightarrow \text{Val})) \rightarrow (\Omega \rightarrow \text{Val})$ that maps any random variable $Z$ to a random variable $[\theta]^Z$, defined as follows:

- $[c]^Z = c$
- $[x]^Z = Y$ where $Y(\omega) = Z(\omega)(x)$ for $\omega \in \Omega$
- $[f^k(E_1, ..., E_k)]^Z = f^k([E_1]^Z, ..., [E_k]^Z)$
- $[\theta]^Z = \theta$
- $[\text{now}]^Z = Y$ where $Y(\omega) = Z(\omega)(\text{now})$ for $\omega \in \Omega$
- $[\text{now}]^Z = Y$ where $Y(\omega) = Z(\omega)(\text{now})$ for $\omega \in \Omega$
- $[u^t(T_1, ..., T_i)]^Z = u^t([T_1]^Z, ..., [T_i]^Z)$
- $[\varepsilon]^Z = \varepsilon$
- $[\langle \text{ch.}E, T \rangle]^Z = \langle \text{ch.}E, T \rangle$
- $[\langle \text{ch.}E, T \rangle]^Z = \langle \text{ch.}E, T \rangle$
- $[h_1 : h_2]^Z = [h_1]^Z : [h_2]^Z$
- $[h^*]^Z = ([h]^Z)^*$

The semantics of state formula $S$ is a function $[S] : (\Omega \rightarrow (\text{Var} \rightarrow \text{Val})) \rightarrow (\Omega \rightarrow \{0, 1\})$ that maps any random variable $Z$ describing the current state to a boolean random variable $[S]^Z$, defined as follows:

- $[\bot]^Z = 0$
- $[R^n(\theta_1, ..., \theta_n)]^Z = R^n([\theta_1]^Z, ..., [\theta_n]^Z)$
  where $R^n(\theta_1]^Z, ..., [\theta_n]^Z)$
- $[h.ch?]^Z = \mathcal{I}_{\omega \in \Omega}([h]^Z(\omega), \text{ch?} \in Z(\omega)(\text{ch?}))$
- $[\langle \text{ch.}E, T \rangle]^Z = \mathcal{S}_{\omega \in \Omega}([h]^Z(\omega), \text{ch?} \in Z(\omega)(\text{ch?}))$
- $[\langle \text{ch.}E, T \rangle]^Z = \mathcal{S}_{\omega \in \Omega}([h]^Z(\omega), \text{ch?} \in Z(\omega)(\text{ch?}))$
- $[\text{ch.}]^Z = 1 - [S]^Z$
- $[\langle S_1 \wedge S_2 \rangle]^Z = [S_1]^Z + [S_2]^Z - [S_1]^Z \ast [S_2]^Z$

where given a set $S$, the characteristic function $\mathcal{I}_S$ is defined such that $\mathcal{I}_S(w) = 1$ if $w \in S$ and $\mathcal{I}_S(w) = 0$ otherwise. The semantics of formula $\varphi$ is interpreted over a
stochastic process and an initial random variable. More precisely, it’s a function \( [\varphi] : ([0, +\infty) \times \Omega \rightarrow (\text{Var} \rightarrow \text{Val}) \rightarrow (\Omega \rightarrow (\text{Var} \rightarrow \text{Val})) \rightarrow (\Omega \rightarrow \{0, 1\}) \) that maps a stochastic process \( \mathcal{H} \) with initial state \( Z \) to a boolean random variable \( [\varphi]_{\mathcal{H}, Z} \). The definition is given below:

\[
\begin{align*}
[\bot]_{\mathcal{H}, Z} &= 0 \\
[S \text{ at } T]_{\mathcal{H}, Z} &= [S]_{\mathcal{H}}([T]_{Z}) \\
[-\varphi]_{\mathcal{H}, Z} &= 1 - [\varphi]_{\mathcal{H}, Z} \\
[\varphi_1 \lor \varphi_2]_{\mathcal{H}, Z} &= [\varphi_1]_{\mathcal{H}, Z} + [\varphi_2]_{\mathcal{H}, Z} - [\varphi_1]_{\mathcal{H}, Z} \times [\varphi_2]_{\mathcal{H}, Z} \\
[\forall \varphi \cdot \varphi]_{\mathcal{H}, Z} &= \inf \{ [\varphi[b/v]]_{\mathcal{H}, Z} : b \in \mathbb{R} \} \\
[\forall t \cdot \varphi]_{\mathcal{H}, Z} &= \inf \{ [\varphi[b/t]]_{\mathcal{H}, Z} : b \in \mathbb{R}^+ \}
\end{align*}
\]

The semantics of probability formula \( P \) is defined by function \( [P] : ([0, +\infty) \times \Omega \rightarrow (\text{Var} \rightarrow \text{Val}) \rightarrow (\Omega \rightarrow (\text{Var} \rightarrow \text{Val})) \rightarrow \{0, 1\} \) that maps a stochastic process \( \mathcal{H} \) with initial state \( Z \) to a boolean variable \( [P]_{\mathcal{H}, Z} \). Formally,

\[
[P(\varphi) \propto p]_{\mathcal{H}, Z} = (P([\varphi]_{\mathcal{H}, Z} = 1) = P(\{ \omega \in \Omega : [\varphi]_{\mathcal{H}, Z}(\omega) = 1 \}) \propto p)
\]

The semantics for \( \neg \) and \( \lor \) can be defined as usual.

We have proved that the terms and formulas of the assertion language are measurable, stated by the following theorem:

**Theorem 2 (Measurability).** For any random variable \( Z \) and any stochastic process \( \mathcal{H} \), the semantics of \( [\emptyset]_{\mathcal{H}, Z} \), \( [S]_{\mathcal{H}, Z} \) and \( [\varphi]_{\mathcal{H}, Z} \) are random variables (i.e. measurable).

**Proof.** The proof of this theorem is given in Appendix.

### 5.2 Specifications

Based on the assertion language, the specification for a SHCSP process \( P \) is defined as a Hoare triple of the form \( \{A; E\} P \{R; C\} \), where \( A, E, R, C \) are probability formulas. \( A \) and \( R \) are **precondition** and **postcondition**, which specify the initial state and the terminating state of \( P \) respectively. For both of them, the formulas \( \varphi \) occurring in them have the special form \( S \text{ at now} \), and we will write \( S \) for short. \( E \) is called an **assumption** of \( P \), which expresses the timed occurrence of the dual of communication events provided by the environment. \( C \) is called a **commitment** of \( P \), which expresses the timed occurrence of communication events, and the real-time properties of \( P \).

**Definition 1 (Validity).** We say a Hoare triple \( \{A; E\} P \{R; C\} \) is valid, denoted by \( \models \{A; E\} P \{R; C\} \), iff for any process \( Q \), any initial states \( \rho_1 \) and \( \rho_2 \), if \( P \) terminates, i.e.\( (P|Q, \rho_1 \cup \rho_2) \xrightarrow{\ast} (\epsilon|Q', \rho_1 \cup \rho_2; \mathcal{H}) \) then \( \{A\}^{\rho_1} \) and \( \{E\}^{\rho_2} \) imply \( \{R\}^{\rho_1} \) and \( \{C\}^{\rho_1} \), where \( \mathcal{H} \) is the stochastic process of the evolution.

### 6 Proof System

We present a proof system for reasoning about all valid Hoare triples for SHCSP processes. First we axiomatize SHCSP language by defining the axioms and inference rules...
for all the primitive and compound constructs, and then the general rules and axioms that are applicable to all processes.

**Skip** The rule for skip is very simple. Indicated by $\top$, the skip process requires nothing from the environment for it to execute, and guarantees nothing during its execution.

$$\{A; \top\} \textbf{skip} \{A; \top\}$$

**Assignment** The assignment $x := e$ changes nothing but assigns $x$ to $e$ in the final state, taking no time to complete.

$$\{A[e/x]; \top\} x := e \{A; \top\}$$

**Input** For input $ch ? x$, we use logical variables $o$ to denote the starting time, $h$ the initial trace, and $v$ the initial value of $x$ respectively, in the precondition. The assumption indicates that the compatible output event is not ready during $[o, o_1)$, and at time $o_1$, it becomes ready. As a consequence of the assumption, during the whole interval $[o, o_1)$, the input event keeps waiting and ready, as indicated by the commitment. At time $o_1$, the communication occurs and terminates immediately. As indicated by the postcondition, $x$ is assigned by some value $v'$ received, the trace is augmented by the new pair $\langle ch.v', o_1 \rangle$, and now is increased to $o_1$. Assume $A$ does not contain $tr$ and $o_1$ is finite (and this assumption will be adopted for the rest of the paper). Let $h'$ be $h[v/x, o/now] \cdot \langle ch.v', o_1 \rangle$, the rule is presented as follows:

$$\{A \land now = o \land tr = h \land x = v; \neg h.ch! \text{dr}[o, o_1) \land h.ch! \text{at } o_1 \} ch ? x$$

$$\{A[o/now] \land now = o_1 \land \exists v'.(x = v' \land tr = h'); h.ch! \text{dr}[o, o_1]\}$$

A communication event is equivalent to a sequential composition of a wait statement and an assignment, both of which are deterministic. Thus, as shown above, the formulas related to traces and readiness hold with probability 1.

If such finite $o_1$ does not exist, i.e., the compatible output event will never become available. As a consequence, the input event will keep waiting forever, as shown by the following rule:

$$\{A \land now = o \land tr = h; \neg h.ch! \text{dr}[o, \infty)\} ch ? x$$

$$\{A[o/now] \land now = \infty; h.ch! \text{dr}[o, \infty]\}$$

**Output** Similarly, for output $ch! e$, we have one rule for the case when the compatible input event becomes ready in finite time. Thus the communication occurs successfully.

$$\{A \land now = o \land tr = h; \neg h.ch? \text{dr}[o, o_1) \land h.ch? \text{at } o_1\} ch! e$$

$$\{A[o/now] \land now = o_1 \land tr = h[o/now] \cdot \langle ch.e, o_1\rangle, h.ch! \text{dr}[o, o_1]\}$$

We also have another rule for the case when the compatible input event will never get ready.

$$\{A \land now = o \land tr = h; (\neg h.ch?) \text{dr}[o, \infty)\} ch! e$$

$$\{A[o/now] \land now = \infty; h.ch! \text{dr}[o, \infty]\}$$
Stochastic Differential Equation Let $f$ be a function, and $\lambda > 0, p \geq 0$ are real values. We have the following rule for $\langle ds = bdt + \sigma dW & B \rangle$.

\[
\begin{align*}
&f(s) \in C^2(\mathbb{R}^n, \mathbb{R}) \text{ has compact support on } B, \lambda, p > 0 \text{ and} \\
&A \rightarrow B \rightarrow (f \leq \lambda p) \quad B \rightarrow (f \geq 0) \wedge (Lf \leq 0) \\
\{A \wedge s = s_0 \wedge \text{now} = o; \top\} \langle ds = bdt + \sigma dW & B \rangle \{P(f(s) \geq \lambda) \leq p \wedge A[s_0/s, o/\text{now}] \\
\wedge \text{now} = o + \bar{d} \wedge \text{cl}(B) ; B \wedge P(f(s) \geq \lambda) dr [o, o + \bar{d}] \leq p\}
\end{align*}
\]

where $o, s_0$ are logical variables denoting the starting time and the initial value of $s$ resp., $d$ is the execution time of the SDE, and $\text{cl}(B)$ returns the closure of $B$, e.g. $\text{cl}(x < 2) = x \leq 2$; and the Lie derivative $Lf(s)$ is defined as $\sum b_i(s) \frac{\partial f_i}{\partial s_j}(s) + \frac{1}{2} \sum_{i,j} (\sigma(s)\sigma(s)^T)_{ij} \frac{\partial^2 f_i}{\partial s_j^2}(s)$. The rule states that, if the initial state of the SDE satisfies $f \leq \lambda p$, and in the domain $B$, $f$ is always non-negative and $Lf$ is non-positive, then during the whole evolution of the SDE, the probability of $f(s) \geq \lambda$ is less than or equal to $p$; on the other hand, during the evolution, the domain $B$ holds almost surely, while at the end, the closure of $B$ holds almost surely.

Sequential Composition For $P; Q$, we use $o$ to denote the starting time, and $o_1$ the termination time of $P$, if $P$ terminates, which is also the starting time of $Q$. The first rule is for the case when $P$ terminates.

\[
\{A \wedge \text{now} = o; E\} P \{R_1 \wedge \text{now} = o_1; C_1\} \{R_1 \wedge \text{now} = o_1; C_1\} Q \{R; C\}
\]

\[
\{A; E\} P; Q \{R; C\}
\]

On the other hand, if $P$ does not terminate, the effect of executing $P; Q$ is same to that of executing $P$ itself.

\[
\{A \wedge \text{now} = o; E\} P \{R \wedge \text{now} = \infty; C\} \quad \{A \wedge \text{now} = o; E\} P \{R \wedge \text{now} = \infty; C\}
\]

Conditional There are two rules depending on whether $B$ holds or not initially.

\[
A \Rightarrow B \quad \{A; E\} P \{R; C\} \quad \text{and} \quad \{A; E\} \neg B \Rightarrow P \{A; \top\}
\]

Probabilistic Choice The rule for $P \sqcup_p Q$ is defined as follows:

\[
\{A \wedge \text{now} = o; E\} P \{P(S) \sqsupset_p p_1; P(\varphi) \sqsupset_p q_1\}
\]

\[
\{A \wedge \text{now} = o; E\} Q \{P(S) \sqsupset_p q_1; P(\varphi) \sqsupset_p q_2\}
\]

\[
\{A \wedge \text{now} = o; E\} P \sqcup_p Q \{P(S) \sqsupset_p p_1 + (1 - p)q_1; P(\varphi) \sqsupset_p q_2 + (1 - p)q_2\}
\]

where $\sqsupset_1, \sqsupset_2$ are two relational operators. The final postcondition indicates that, if after $P$ executes $S$ holds with probability $\sqsupset_1 p_1$, and after $Q$ executes $S$ holds with probability $\sqsupset_1 q_1$, then after $P \sqcup_p Q$ executes, $S$ holds with probability $\sqsupset_1 (pp_1 + (1 - p)q_1)$. The historical formula can be understood similarly.

Communication Interrupt We define the rule for the special case $\langle ds = bdt + \sigma dW & B \rangle \sqsupset (ch?x \rightarrow Q)$ for simplicity, which can be generalized to general case without any difficulty. We use $o_P$ to denote the execution time of the SDE. The premise
of the first rule indicates that the compatible event (i.e. $h.ch!$) is not ready after the
continuous terminates. For this case, the effect of executing the whole process is thus
equivalent to that of executing the SDE.

$$\{ A \land now = o; E \} \{ ds = bdt + \sigma dW & B \} \{ R \land now = o + o_F; C \}$$

$$A \land now = o \land E \Rightarrow (tr = h \land \neg h.ch! \lor dr [o, o + o_F])$$

$$\{ A \land now = o; E \} \{ ds = bdt + \sigma dW & B \} \{ R \land now = o + o_F; C \}$$

In contrary, when the compatible event gets ready before the continuous terminates,
the continuous will be interrupted by the communication, which is then followed by
$Q$. Thus, as shown in the following rule, the effect of executing the whole process is
equivalent to that of executing $ch?x; Q$, plus that of executing the SDE before the
communication occurs, i.e. in the first $o_1$ time units.

$$\begin{align*}
\{ A \land now = o; E \} \{ ds = bdt + \sigma dW & B \} \{ R \land now = o + o_F; C \} \\
(A \land now = o \land E) \Rightarrow (tr = h \land h.ch! \land (o + o_1) \land o_1 \leq o_F) \\
\{ A \land B \land now = o; E \} ch?x; Q \{ R_1; C_1 \} \\
\{ A \land now = o; E \} \{ ds = bdt + \sigma dW & B \} \{ R \land now = o + o_F; C \} \\
\{ R_1; R\} |_{[o, o + o_1]} \wedge C_1 \}
\end{align*}$$

where $R\} |_{[o, o + o_1]}$ extracts from $R$ the formulas before $o + o_1$, e.g., $(P(S \text{ at } T) \bowtie p) |_{[o, o + o_1]}$ is equal to $P(S \text{ at } T) \bowtie p$ if $T$ is less or equal to $o + o_1$, and true otherwise.

**Parallel Composition**

For $P \parallel Q$, let $X$ be $X_1 \cap X_2$ where $X_1 = \Sigma(P)$ and $X_2 = \Sigma(Q)$, then

$$\begin{align*}
A \Rightarrow A_1 \land A_2, \quad \{ A_1 \land now = o; E_1 \} P \{ R_1 \land tr = \gamma_1 \land now = o_1; C_1 \} \\
\{ A_2 \land now = o; E_2 \} Q \{ R_2 \land tr = \gamma_2 \land now = o_2; C_2 \} \\
\forall ch \in X. (C_1 |_{o_1/now} \wedge) \Rightarrow E_2 \{ ch \} \land (C_2 |_{o_2/now} \wedge) \Rightarrow E_1 \{ ch \} \\
\forall dh \in X_1 \setminus X. (E_2 |_{dh} \Rightarrow E_1 |_{dh}) \land (E_2 |_{dh} \Rightarrow E_2 |_{dh}) \\
\{ A \land now = o; E \} P \parallel Q \{ R; C_1 \land C_2 \}
\end{align*}$$

where $A_1$ is a property of $P$ (i.e., it only contains variables of $P$), $A_2$ a property of $Q,$
and $o_1$ and $o_2$, $\gamma_1$ and $\gamma_2$ logical variables representing the time and trace at termination
of $P$ and $Q$ respectively. Let $o_m$ be $\max\{o_1, o_2\}$, $R$, $C_1$ and $C_2$ are defined as follows:

$$\begin{align*}
R_1 & \overset{def}{=} R_1 |_{[\gamma_1/tr, o_1/now]} \land R_2 |_{[\gamma_2/tr, o_2/now]} \land now = o_m \land \gamma_1 \land x = \gamma_2 \land tr = \gamma_1 \parallel \gamma_2 \\
C_1 & \overset{def}{=} C_1 |_{[o_1/now]} \land R_1 |_{[o_1/now]} \land dr [o_1, o_m] \text{ for } i = 1, 2
\end{align*}$$

where for $i = 1, 2$, $R_i \Rightarrow R_i'$ but $tr \notin R_i'$. At termination of $P \parallel Q$, the time will be
the maximum of $o_1$ and $o_2$, and the trace will be the alphabetized parallel of the traces of
$P$ and $Q$, i.e. $\gamma_1, \gamma_2$. In $C'_1$ and $C'_2$, we specify that none of variables of $P$ and $Q$ except for
now and $tr$ will change after their termination.

**Repetition** For $P^*$, let $k$ be an arbitrary non-negative integer, then

$$\begin{align*}
\{ A \land now = o + k \ast t \land tr = (h \cdot \alpha^k); E[o/now] \} P \\
\{ A \land now = o + (k + 1) \ast t \land tr = (h \cdot \alpha^{k+1}); C \}
\end{align*}$$

$t$ and $\alpha$ are logical variables representing the time elapsed and trace accumulated respectively
by each execution of $P$, and $o$ and $o'$ denote the starting and termination time of the loop ($o'$
could be infinite).
The general rules that are applicable to all processes, such as Monotonicity, Case Analysis, and so on, are similar to the traditional Hoare Logic. We will not list them here for page limit.

**Theorem 3 (Soundness).** If $\vdash \{ A; E \} P \{ R; C \}$, then $\models \{ A; E \} P \{ R; C \}$, i.e. every theorem of the proof system is valid.

**Proof.** The proof of this theorem can be found in Appendix.

**Example 1.** For the aircraft example, define $f(x, y)$ as $|y|$, assume $f(x_s, y_0) = |y_0| \leq \lambda p$, where $p \in [0, 1]$. Obviously, $B \rightarrow (f \geq 0) \land (Ef \leq 0)$ holds. By applying the inference rule of SDE, we have the following result:

$$\{ \text{now} = o; \text{True} \} P_{Air} \{ \exists d. \text{now} = o + d \land B \land P(f \geq \lambda)s \leq p; \}$$

which shows that, the probability of the aircraft entering the dangerous state is always less than or equal to $p$ during the flight. Thus, to guarantee the safety of the aircraft, $p$ should be as little as possible. For instance, if the safety factor of the aircraft is required to be 99.98%, then $p$ should be less than or equal to 0.0002, and in correspondence, $|y_0| \leq \frac{\lambda}{5000}$ should be satisfied.

### 7 Conclusion

This paper presents stochastic HCSP (SHCSP) for modelling hybrid systems with probability and stochasticity. SHCSP is expressive but complicated with interacting discrete, continuous and stochastic dynamics. We have defined the semantics of stochastic HCSP and proved that it is well-defined with respect to stochasticity. We propose an assertion language for specifying time-related and probability-related properties of SHCSP, and have proved the measurability of it. Based on the assertion language, we define a compositional Hoare Logic for specifying and verifying SHCSP processes. The logic is an extension of traditional Hoare Logic, and can be used to reason about how the probability of a property changes with respect to the execution of a process. To illustrate our approach, we model and verify a case study on a flight planing problem at the end.

### References

1. A. Abate, M. Prandini, J. Lygeros, and S. Sastry. Probabilistic reachability and safety for controlled discrete time stochastic hybrid systems. *Automatica*, 44(11):2724–2734, 2008.
2. E. Altman and V. Gaitsgory. Asymptotic optimization of a nonlinear hybrid system governed by a Markov decision process. *SIAM Journal of Control and Optimization*, 35(6):2070–2085, 1997.
3. M. L. Bujorianu. Extended stochastic hybrid systems and their reachability problem. In *HSCC’04*, volume 2993 of *LNCS*, pages 234–249, 2004.
4. M. L. Bujorianu and J. Lygeros. Toward a general theory of stochastic hybrid systems. *Lecture Notes in Control and Information Sciences (LNCIS)*, 337:3–30, 2006.
5. M. L. Bujorianu, J. Lygeros, and M. C. Bujorianu. Bisimulation for general stochastic hybrid systems. In *HSCC’05*, volume 3414 of LNCS, pages 198–214, 2005.

6. M. Fränzle, E. M. Hahn, H. Hermanns, N. Wolovick, and L. Zhang. Measurability and safety verification for stochastic hybrid systems. In *HSCC’11*, pages 43–52. ACM, 2011.

7. E. M. Hahn, A. Hartmanns, H. Hermanns, and J. Katoen. A compositional modelling and analysis framework for stochastic hybrid systems. *Formal Methods in System Design*, 43(2):191–232, 2013.

8. E. M. Hahn, H. Hermanns, B. Wachter, and L. Zhang. PASS: abstraction refinement for infinite probabilistic models. In *TACAS’10*, volume 6015 of LNCS, pages 353–357, 2010.

9. J. He. From CSP to hybrid systems. In *A Classical Mind, Essays in Honour of C.A.R. Hoare*, pages 171–189. Prentice Hall International (UK) Ltd., 1994.

10. T. A. Henzinger. The theory of hybrid automata. In *LICS’96*, pages 278–292, July 1996.

11. C. A. R. Hoare. An axiomatic basis for computer programming. *Commun. ACM*, 12(10):576–580, 1969.

12. C. A. R. Hoare. *Communicating Sequential Processes*. Prentice-Hall, 1985.

13. J. Hu, J. Lygeros, and S. Sastry. Towards a theory of stochastic hybrid systems. In *HSCC’02*, volume 1790 of LNCS, pages 160–173, 2002.

14. J. Liu, J. Lv, Z. Quan, N. Zhan, H. Zhao, C. Zhou, and L. Zou. A calculus for hybrid CSP. In *APLAS’10*, volume 6461 of LNCS, pages 1–15, 2010.

15. J. Meseguer and R. Sharykin. Specification and analysis of distributed object-based stochastic hybrid systems. In *HSCC’06*, volume 3927 of LNCS, pages 460–475, 2006.

16. C. Morgan, A. McIver, K. Seidel, and J. W. Sanders. Refinement-oriented probability for CSP. *Formal Asp. Comput.*, 8(6):617–647, 1996.

17. A. Platzer. Stochastic differential dynamic logic for stochastic hybrid programs. In *CADE’11*, volume 6803 of LNCS, pages 446–460, 2011.

18. M. Prandini and J. Hu. Application of reachability analysis for stochastic hybrid systems to aircraft conflict prediction. In *47th IEEE Conference on Decision and Control (CDC)*, pages 4036 – 4041. IEEE, 2008.

19. J. Sproston. Decidable model checking of probabilistic hybrid automata. In *Formal Techniques in Real-Time and Fault-Tolerant Systems*, volume 1926 of LNCS, pages 31–45, 2000.

20. L. Zhang, Z. She, S. Ratschan, H. Hermanns, and E. M. Hahn. Safety verification for probabilistic hybrid systems. In *CAV’10*, volume 6174 of LNCS, pages 196–211, 2010.

21. C. Zhou, J. Wang, and A. P. Ravn. A formal description of hybrid systems. In *Hybrid Systems III*, volume 1066 of LNCS, pages 511–530, 1996.

22. P. Zuliani, A. Platzer, and E. M. Clarke. Bayesian statistical model checking with application to stateflow/simulink verification. *Formal Methods in System Design*, 43(2):338–367, 2013.
Appendix

7.1 The Semantics of SHCSP

The semantics of the rest of SHCSP is given in Table 2. The semantics of skip and $x := e$ are defined as usual, except that for each, an internal event occurs. Rule (Idle) says that a terminated configuration can keep idle arbitrarily, and then evolves to itself. For input $ch?x$, the input event has to be put in the ready set if it is enabled (In-1); then it may wait for its environment for any time $d$ during keeping ready (In-2); or it performs a communication and terminates, and accordingly the corresponding event will be removed from the ready set, and $x$ is assigned and $tr$ is extended by the communication (In-3). The semantics of output $ch!e$ is similarly defined by rules (Out-1), (Out-2) and (Out-3).

For $P_1 || P_2$, we always assume that the initial states $P_1$ and $P_2$ are parallelable. There are four rules: both $P_1$ and $P_2$ evolve for $d$ time units in case they can delay $d$ time units respectively; or $P_1$ may progress separately on internal events or external communication events (Par-2), and the symmetric case can be defined similarly (omitted here); or they together perform a synchronized communication (Par-3); or $P_1 || P_2$ terminates when both $P_1$ and $P_2$ terminate (Par-4). At last, the semantics for conditional, sequential, internal choice, and repetition is defined as usual.

7.2 Proof of Theorem

Proof: We will prove the càdlàg, adaptedness and Markov time properties by induction on the structure of SHCSP $P$. To simplify notation, we assume that the process $P$ start at time 0 and $\Delta(P)$ is short for $\Delta(P, P')$ if $P' = e$.

- Cases skip, wait $d$ and $x = e$: Deterministic times $\Delta(\text{skip}) = \Delta(x = e) = 0$ and $\Delta(\text{wait } d) = d$ are trivial Markov times. For skip and wait $d$, $H$ is adapted to the filtration generated by $\rho$. For $x = e$, $H$ is adapted to $\rho$ and $e$. For skip and $x = e$, $H$ is trivially càdlàg as the time domain is $\{0\}$.
- Case In-1: $\Delta(ch?x, ch?x) = 0$ is a trivial Markov time. $H$ is càdlàg and adapted to the filtration generated by $\rho$.
- Case In-2: $\Delta(ch?x, ch?x) = d$ is a trivial Markov time. $H$ is càdlàg and adapted to the filtration generated by $\rho$.
- Case In-3: $\Delta(ch?x) = d$ is a trivial Markov time. $H$ is càdlàg and adapted to the filtration generated by $\rho$ and $e$.

For cases Out-1, Out-2 and Out-3, the fact can be proved similarly.

- Case $(ds = bdt + \sigma dW & B)$: $\Delta(\langle ds = bdt + \sigma dW & B \rangle) = \inf\{t \geq 0 : X_t \notin B\}$ is a Markov time if $B$ is any Borel set. Here, $X_t$ is the solution of SDE $ds = bdt + \sigma dW$. $H$ is adapted to the filtration generated by $(W_s)_{s \leq t}$ and $\rho$.
- Case $B \rightarrow P$: If $B$ is true, executing $B \rightarrow P$ is same as executing $P$. By induction hypothesis, $\Delta(P)$ is a Markov time and $H$ is càdlàg and adapted. If $B$ is false, the fact holds obviously.
- Case $P \cap P Q$: By induction hypothesis, $\Delta(P)$ and $\Delta(Q)$ are both Markov time. So $\Delta(P \cap P Q)$, the sum of two Markov times $p\Delta(P)$ and $(1 - p)\Delta(Q)$, is also a
(skip, ρ) ㄱ→ (ε, ρ[tr → tr · (τ, now)])

(ε, ρ) ㄱ→ (ε, ρ[now → now + d])

(x := e, ρ) ㄱ→ (ε, ρ[x := e, tr → tr · (τ, now)])

ρ(ω)(tr).ch? ǂ ρ(ω)(rdy)

(ch?x, ρ) ㄱ→ (ch?x, ρ[rdy → rdy ∪ {tr.ch?}])

ρ(ω)(tr).ch? ∈ ρ(ω)(rdy)

(ch?x, ρ) ㄱ→ (ch?x, ρ[now → now + d], H^u_d)

ρ(ω)(tr).ch? ∈ ρ(ω)(rdy)

(chl, ρ) ㄱ→ (chl, ρ[rdy → rdy ∪ {tr.ch!}])

ρ(ω)(tr).ch! ∈ ρ(ω)(rdy)

(chl, ρ) ㄱ→ (chl, ρ[now → now + d], H^u_d)

ρ(ω)(tr).ch! ∈ ρ(ω)(rdy)

chl ㄱ→ (chl, ρ[rdy → rdy \ {tr.ch!}], tr → tr · (ch.e, now))

(P_1, ρ_1) ㄱ→ (P^1_1, ρ^1_1), (P_2, ρ_2) ㄱ→ (P^1_2, ρ^1_2)

(P_1 || P_2, ρ_1 ⊕ ρ_2) ㄱ→ (P_1 || P_2, ρ_1 ⊕ ρ_2)

(P_1, ρ_1) ㄱ→ (P_1, ρ_1), Σ(β) ǂ Σ(P_1) \ Σ(P_2)

(P_1 || P_2, ρ_1 ⊕ ρ_2) ㄱ→ (P_1 || P_2, ρ_1 ⊕ ρ_2)

(P_1 || P_2, ρ_1 ⊕ ρ_2) ㄱ→ (P_1 || P_2, ρ_1 ⊕ ρ_2, H_1 \ H_2)

(ε || ρ, ρ_1 ⊕ ρ_2) ㄱ→ (ε, ρ_1 ⊕ ρ_2)

ρ(ω)(B) = T

(B → P, ρ) ㄱ→ (P, ρ[tr → tr · (τ, now)])

ρ(ω)(B) = F

(B → P, ρ) ㄱ→ (ε, ρ[tr → tr · (τ, now)])

P, ρ ㄱ→ (P', ρ', H) P' ǂ ε

P, Q, ρ ㄱ→ (P'; Q, ρ', H)

(P, ρ) ㄱ→ (ε, ρ', H)

(P; Q, ρ) ㄱ→ (Q, ρ', H)

(P, ρ) ㄱ→ (P', ρ', H) P' ǂ ε

(P', ρ) ㄱ→ (P'; P', ρ', H)

(P, ρ) ㄱ→ (ε, ρ', H)

(P', ρ) ㄱ→ (P', ρ', H)

(P', ρ) ㄱ→ (ε, ρ', H)

(P', ρ) ㄱ→ (P', ρ', H)

(P', ρ) ㄱ→ (ε, ρ[tr → tr · (τ, now)])

Table 2. The semantics of the rest of SHCSP
Markov time. By induction hypothesis, $H'$ for $P$ and $H''$ for $Q$ are both càdlàg. Because càdlàg functions form an algebra, $H$ is also càdlàg for every outcome of $\Box$. $H$ is adapted, because $H'$ and $H''$ are adapted and the choice $\Box$ generates the filtration.

- Case $P; Q$: Suppose $(P; Q, \rho) \to (Q, \rho', H')$ and $(Q, \rho', H') \to (\epsilon, \rho'', H'')$. By induction hypothesis, $\Delta(P; Q, Q) = \Delta(P)$ is a Markov time and $H'$ is càdlàg and adapted to $(F'_t)_{t \geq 0}$. $\rho'$ is a random variable. By induction hypothesis, $\Delta(Q)$ is a Markov time and $H''$ is càdlàg and adapted to $(F''_t - \Delta(P))_{t \geq \Delta(P)}$. Obviously, $\Delta(P; Q) = \Delta(P) + \Delta(Q)$ is a Markov time. $H$ is adapted to $(F_t)_{t \geq 0}$, since the two parts $H', H''$ are adapted. By induction hypothesis, $H$ is càdlàg on $[0, \Delta(P; Q, Q))$ and on $(\Delta(P; Q, Q), \infty)$, because the constituent fragments are. At $\Delta(P; Q, Q), H$ is càdlàg, by construction.

- Case $\langle ds = bdt + \sigma dW & B \rangle \geq d Q$: This case can be defined by $t = 0; \langle ds = bdt + \sigma dW & t < d & B \rangle; t \geq d \to Q$. The fact can be proved similarly as the case $P; Q$.

- Case $\langle ds = bdt + \sigma dW & B \rangle \geq \{ \omega_i : \chi_{\, i} \* \to Q_i \}$: If the evolution of SDE terminates before any communication occurs, this case is same as $\langle ds = bdt + \sigma dW & B \rangle$. Otherwise, $H$ is càdlàg and adapted the filtration generated by $\rho, (W_s)_{s \leq t}$ and the weights $\{ \omega_i \}_{i \in I}, \Delta(\langle ds = bdt + \sigma dW & B \rangle \geq \{ \omega_i : \chi_{\, i} \* \to Q_i \})$ is a Markov time, since the communication and $Q_i$ are both Markov times.

- Case $P \parallel Q$: Suppose $(P_1 \parallel P_2, \rho_1 \uplus \rho_2) \to (\epsilon \parallel \epsilon, \rho_1' \uplus \rho_2', H_1 \uplus H_2)$. Because the processes $P$ and $Q$ don’t share variables, by induction hypothesis, $H = H_1 \uplus H_2$ is càdlàg and adapted to the filtration generated by $\rho_1 \uplus \rho_2, (W_s)_{s \leq t}$ and the weights $\{ \omega_i \}_{i \in I}, \Delta(P) | Q) = \max(\Delta(P), \Delta(Q))$ is a Markov time.

\[ \square \]

7.3 Proof of Theorem 2

Proof: We will prove this fact by induction on the structure of $\theta$, $S$ and $\varphi$.

$[\theta]^Z$ is a random variable:

1. $[c]^Z = c$ is a random variable trivially.
2. $[x]^Z = Y$ is a random variable, because $Y(\omega) = Z(\omega)(x)$ for each $\omega \in \Omega$ and $Z$ is measurable. So is $Y$.
3. $[f^k(E_1, ..., E_k)]^Z = f^k([E_1]^Z, ..., [E_k]^Z)$ is a random variable, because $[E_1]^Z, ..., [E_k]^Z$ are measurable and $f^k$ is Borel-measurable. Thus, the composition $f^k([E_1]^Z, ..., [E_k]^Z)$ is measurable (the $\sigma$-algebras in the composition are compatible).

The cases $[0]^Z$, $[\text{now}]^Z$, $[\text{now}^T(T_1, ..., T_k)]^Z$, $[z]^Z$ and $[\langle ch, E, T \rangle]^Z$ can be proved similarly.

$[h_1 \cdot h_2]^Z = [h_1]^Z \cdot [h_2]^Z$ is a product. It is also measurable by induction hypothesis (measurable functions form an algebra).

$[S]^Z$ is a random variable:

1. $[\perp]^Z = 0$ is trivially measurable.
HCSP processes
rule is valid, then the conclusion is also valid. That is, if every premise of the inference rule in the proof system preserves validity. Therefore, if every premise of the proof system preserves validity, then the conclusion is also valid. Hence, the (finite) countable infimum \(\inf\{\|\varphi[b/v]\|^{H,Z} : b \in \mathbb{R}\}\) is measurable for any \(\varphi\). So, the (finite) countable infimum \(\inf\{\|\varphi[b/v]\|^{H,Z} : b \in \pi\}\) is measurable for each \(b\). Consider a rational mesh \(\pi := \{b_1, b_2, \ldots, b_n\} \subset \mathbb{Q}\) with \(b_1 \leq b_2 \leq \cdots \leq b_n\). It's obvious that \(\|\varphi[b/v]\|^{H,Z}\) is measurable for each \(b \in \pi\). So, the (finite) countable infimum \(\inf\{\|\varphi[b/v]\|^{H,Z} : b \in \pi\}\) is measurable. Then, the countable infimum \(\inf\{\|\varphi[b/v]\|^{H,Z} : b \in \pi\}\) is measurable, because the set of rational meshes is countable. Notice that \(\mathcal{H}\) is càdlàg by Theorem 1, so \(\inf\{\|\varphi[b/v]\|^{H,Z} : b \in \mathbb{R}\}\) is measurable. \(\|\varphi_1 \land \varphi_2\|^{H,Z}\) and \(\|\forall v.\varphi\|^{H,Z}\) can be proved similarly.

\[
\Box
\]

### 7.4 Proof of Theorem 3

**Proof:** To prove soundness, we need to show that the axioms are valid, and that every inference rule in the proof system preserves validity. That is, if every premise of the rule is valid, then the conclusion is also valid.

We will prove the soundness theorem by induction on the structure of Stochastic HCSP processes \(S\). In the following proof, we always assume \(S\) executes in parallel with its environment \(\mathcal{E}\), and \(\langle S \|. \mathcal{E}, \rho_1 \cup \rho_2 \rangle \xrightarrow{\alpha} \langle \mathcal{E}', \rho_1' \cup \rho_2' \rangle\); \(\mathcal{H}\) is the stochastic process of the evolution and \(T_0 = \rho_1(\text{now})\) for simplicity. Moreover, for readability, we will write \(\langle A \|| E \rangle^{\mathcal{H},\omega}\) as \(\rho \models A\) and \(\rho, \mathcal{H} \models E\), for any state \(\rho\), any stochastic process \(\mathcal{H}\), any state formula \(A\), and any formula \(E\).

- **Case skip:** The fact holds trivially from the fact \(\rho_1' = \rho_1[tr + \tau]\).

- **Case Assignment** \(x := e\): From the operational semantics, we have \(\rho_1' = \rho_1[x \mapsto e, tr \mapsto tr \cdot (\tau, \text{now})]\). Assume \(\rho_1 \models (A \land tr = h)|e/x\), we need to prove \(\rho_1' \models A \land tr = h + \tau\). Obviously, this holds.

- **Case Input** \(ch\?x\): From the operational semantics, we have \(\rho_1' = \rho_1[\text{now} \mapsto T_0 + d, x \mapsto b, tr \mapsto tr \cdot \langle ch.b, T_0 + d \rangle]\) for some \(d \geq 0\) and \(b\); and for any \(\omega \in \Omega\) and any \(t \in [T_0, T_0 + d]\). Assume \(\rho_1 \models (A \land \text{now} = h \land x = v)\) and \(\rho_2, \mathcal{H} \models -h.ch!\ |	ext{dr}[o, o_1] \land h.ch!\ |	ext{at} o_1\), we need to prove that \(\rho_1' \models A[v/x, o/\text{now}] \land \text{now} = o_1 \land \exists v'.(x = v' \land tr = h')\) and \(\rho_1', \mathcal{H} \models h.ch?\ |	ext{dr}[o, o_1]\), where \(h'\) is \(h[v/x, o/\text{now}]\cdot (ch.v', o_1)\).
First from $\rho_1 \models A \land \text{now} = o \land x = v$ and the assumption that $A$ does not contain $tr$, we have $\rho_1 \models A[v/x, o/\text{now}]$. Compare $\rho'_1$ with $\rho_1$, we can find that only variables $tr$, $now$, and $x$ are changed. Plus that $A$ does not contain $tr$, we obtain $\rho'_1 \models A[v/x, o/\text{now}]$.

From the assumption $\rho_1, \mathcal{H} \models -h.ch! \ d\tau [o, o_1] \land h.ch! \ at \ o_1$, we can get the fact that $\forall t \in [o, o_1), \mathcal{H}(t, \cdot)(h).ch! \ |_{ch} \notin \mathcal{H}(t, \cdot)(\tau(dy)) |_{ch}$, and $\mathcal{H}(t, \cdot)(h).ch! \ |_{ch} \in \mathcal{H}(o_1, \cdot)(\tau(dy)) |_{ch}$. From $\rho_1 \models tr = h$, then $\rho(\cdot)(tr) = \rho(\cdot)(h)$, and obviously $\rho(\cdot)(h) = \mathcal{H}(t, \cdot)(h)$ since the number of $ch$ in $h$ does not change during the waiting time. Plus the fact that $T_0 = o$, we finally obtain $T_0 + d = o_1$. So $\rho'_1, \mathcal{H} \models \text{now} = o_1$ holds.

Denote $\rho'_1(\cdot)(x)$ by $c$, then $\rho'_1 \models \exists v'.x = v'$ holds by assigning $v'$ with $c$. From the semantics of substitution, $\rho'_1(\cdot)(tr) = \rho_1(\cdot)(h) \cdot (ch.c, T_0 + d)$. On the other hand, $\rho'_1(\cdot)(h[v/x, o/\text{now}] \cdot (ch.v', o_1)) = \rho_1(\cdot)(h) \cdot (ch.c, o_1)$. Thus, plus the above fact, we prove that $\rho'_1 \models \exists v'.(x = v' \land tr = h')$.

Finally, from the operational rule, we have $\rho'_1, \mathcal{H} \models \rho_1(\cdot)(tr).ch! \ d\tau [T_0, T_0 + d]$. Based on the facts $T_0 = o$, $T_0 + d = o_1$, and $\rho_1(\cdot)(tr) = \rho_1(\cdot)(h)$, we prove the result.

- **Case Output ch!e**: The fact can be proved similarly to ch?x.

- **Case Continuous $\langle ds = bdt + \sigma dW \& B \rangle$**: First assume the continuous terminates. To prove this, we first introduce two lemmas.

**Lemma 1.** Let $X_t$ an a.s. right continuous strong Markov process (e.g. solution from SDE) and $X_0 = x$. If $f \in C^2(\mathbb{R}^n, \mathbb{R})$ has compact support and $\tau$ is a Markov time with $E^x \tau < \infty$, then

$$E^x f(X_t) = f(x) + E^x \int_0^\tau Af(X_s) ds$$

where $Af(x) := \lim_{t \to 0^+} \frac{E^x f(X_t) - f(x)}{t}$

**Lemma 2.** If $f(X_t)$ is a càdlàg supermartingale with respect to the filtration generated by $(X_t)_{t \geq 0}$ and $f \geq 0$ on the evolution domain of $X_t$, then for all $\lambda > 0$:

$$P(\sup_{t \geq 0} f(X_t) \geq \lambda | \mathcal{F}) \leq \frac{Ef(X_0)}{\lambda}$$

We have $\rho'_1 = \rho_1[\text{now} \mapsto T_0 + d, s \mapsto X(d, \cdot)](tr + \tau)$ for some $d \geq 0$ where $X : [0, +\infty) \times \Omega \to \mathbb{R}^d(s)$ is the solution of the SDE; and for all $t \in [T_0, T_0 + d] \mathcal{H}(t, \cdot)(s) = X(t, \cdot)$. We define another random variable $Y = \sup \{f(X_t) : t \in [0, d]\}$, $f \in C^2(\mathbb{R}^d(s), \mathbb{R})$ has compact support on $B$. Consider any $x \in \mathbb{R}^d(s)$ and any time $r \geq 0$. The deterministic time $r$ is a Markov time with $E^x r = r < \infty$. By Lemma 1, we have

$$E^x f(X_r) = f(x) + E^x \int_0^r Af(X_t) dt$$
where $Af = Lf \leq 0$ by the premise. So $\int_0^T Af(X_t) dt \leq 0$, hence, $E^\rho \int_0^T Af(X_t) dt \leq 0$. This implies $E^\rho f(X_t) \leq f(x)$ for all $x$.

The filtration is right-continuous and $f \in C(\mathbb{R}^d, \mathbb{R})$ is compactly supported, the strong Markov property for $X_t$ implies for all $t \geq r > 0$ that $E^\rho (f(X_t) | F_r) = E^{\rho^r} f(X_{t-r}) \leq f(X_r)$. Thus, $f(X_t)$ is a supermartingale with respect to $X_t$, because it is the filtration of $X_t$ and $E^\rho |f(X_t)| < \infty$ for all $t$ since $f \in C^2(\mathbb{R}^d, \mathbb{R})$ has compact support. Consider any initial state $Y$ for $X$. By Lemma 2 and the premises, we have $P \left( \sup_{t \geq 0} f(X_t) \geq \lambda | F_t \right) \leq \frac{E f(Y)}{\lambda} \leq \frac{2p}{\lambda} = p$.

The fact holds.

The other case is that the continuous does not terminates in finite time. From proof above, for any $d > 0$, we have $P_{\leq \rho}(f(s) \geq \lambda) \mathrm{d}r \ [T_0, T_0 + d]$. So we can get $P_{\leq \rho}(f(s) \geq \lambda) \mathrm{d}r \ [T_0, \infty]$. The result holds.

- Case Sequential Composition $P; Q$: We assume the intermediate state at termination of $P$ is $\rho'_1$ (thus $Q$ will start from $\rho'_1[tr + \tau]$), and the behaviors of $P$ and $Q$ are $H_1$ and $H_2$ respectively, whose concatenation is exactly $H$. Assume we have $\rho_1 \models A \land \text{now} = o$ and $\rho_1, H \models E$, we need to prove that $\rho'_1 \models R$ and $\rho'_1, H \models C_1[0; \text{now} \land C]$, where $\{A \land \text{now} = o; E\} P \{R_1 \land \text{now} = o_1 \land tr = h_1; C_1\}$ and $\{R_1 \land \text{now} = o_1 \land tr = h_1 + \tau; E[o; \text{now}]\} P \{R; C\}$ as in the rule for sequential composition.

According to the inference rules, from $\{A \land \text{now} = o; E\} P \{R_1 \land \text{now} = o_1 \land tr = h_1; C_1\}$, we can get $\{A \land \text{now} = o; E \leq o_1\} P \{R_1 \land \text{now} = o_1 \land tr = h_1; C_1\}$, where $E \leq o_1$ only addresses the behavior of environment before or equal time $o_1$. Then the proof is given as follows: First, from $\rho_1, H \models E$, we have $\rho_1, H \models E \leq o_1$, then by induction hypothesis, for $P$, we have $\rho'_1 \models R_1 \land \text{now} = o_1 \land tr = h_1$ and $\rho''_1, H_1 \models C_1$. Similarly, by induction hypothesis again for $Q$, we have $\rho'_1 \models R$ and $\rho'_2, H_2 \models C$, then $\rho'_1, H \models C$. From $\rho''_1, H_1 \models C_1$, we have $\rho'_1, H \models C_1[0; \text{now}]$. The result is proved.

- Case Probabilistic Choice $P \uparrow_\rho Q$: We may assume $\infty \leq \rho \geq 0$. From operational semantics, we have $\{P \geq \rho_1^P(S); E\} P \{P \geq \rho_1^P(S); C_1\}$ with probability $p$ and $\{P \geq \rho_2^P(S); E\} Q \{P \geq \rho_2^P(S); C_2\}$ with probability $1 - p$. Assume $\rho_1 \models A$, and $\rho_2, H \models E$. By the law of total probability, we can easily get $\rho'_1 \models P \geq \rho_1^p, 1 - p \rho_2^p(S)$ and $H \models C_1 \lor C_2$.

- Case Communication Interrupt: Assume $\rho_1 \models A \land \text{now} = o$, and $\rho_2, H \models E$. For the first case, assume we have $\{A \land \text{now} = o; E\} \{ds = bdt + \sigma dW \land B\} \{R \land \text{now} = o + o_F; C\}$, and $A \land \text{now} = o \land E \models (tr = h \land \text{h.ch}! \mathrm{d}r[o, o + o_F])$, we need to prove $\rho'_1 \models R \land \text{now} = o + o_F$ and $\rho'_1, H \models C$. From the assumption, we have $\rho_1 \models tr = h$ and $\rho_2, H \models \text{h.ch}! \mathrm{d}r[o, o + o_F]$. According to the operational semantics, the final state and the behavior of interrupt are equal to the ones of continuous. The result holds by induction hypothesis.

For the second case, assume we have $\{A \land \text{now} = o; E\} \{ds = bdt + \sigma dW \land B\} \{R \land \text{now} = o + o_F; C\}$, $A \land \text{now} = o \land E \models (tr = h \land \text{h.ch}! \mathrm{at}[o + o_1 \land \text{now} \leq o_F])$, and $A \land \text{now} = o; E \models \text{h.ch}! \mathrm{ch}! [\{R_1; C_1\}$, we need to prove $\rho'_1 \models R_1$ and $\rho'_1, H \models (P \geq \rho(f(s) \geq \lambda) \land \text{B}) \mathrm{d}r[o, o + o_1 \land \text{now}]$. From the assumption, we have $\rho_1 \models tr = h$ and $\rho_2, H \models \text{h.ch}! \mathrm{d}r[o, o + o_1 \land \text{h.ch}! \mathrm{at}[o + o_1 \land \text{now} \leq o_F]$. According to the operational semantics, the final state and the behavior of interrupt
Case Parallel Composition \(P \parallel Q\): From the operational semantics, there must exist \(\rho_{11} \) and \(\rho_{12} \) for initial states and terminating states of \(P \) and \(Q \) respectively, which satisfy: \(\rho_{1} = \rho_{11} \parallel \rho_{12} \) and \(\rho'_{1} = \rho'_{11} \parallel \rho'_{12} \). \(\rho_{11}((t) \triangleright X) = \rho_{12}((t) \triangleright X) \) (assuming \(P \) and \(Q \) terminate at the same time here, which will be generalized in the following proof). Assume we have \(\rho_{1} \models A \land now = o, \) \(\rho_{2} \models E, \) we need to prove \(\rho'_{1} \models R \) and \(\rho'_{1}, H \models C_{1}' \land C_{2}' \), where \(\{A_{1} \land now = o; E_{1}\} \models R_{1} \land tr = \gamma_{1} \land now = o_{1}; C_{1}\} \) and \(\{A_{2} \land now = o; E_{2}\} \models R_{2} \land tr = \gamma_{2} \land now = o_{2}; C_{2}\} \) hold; and compatibility check \(\forall ch \in X. (C_{1}(o_{1}/now) \mid dh \Rightarrow E_{2}(o_{2}/now) \mid dh \Rightarrow E_{1}(ch), \forall dh \in X_{1} \setminus X, E \mid dh \Rightarrow E_{1} \mid dh, \) and \(\forall dh' \in X_{2} \setminus X, E \mid dh' \Rightarrow E_{2} \mid dh' \) hold. Among them, \(R, C_{1}' \) and \(C_{2}' \) are defined as in the rule for parallel composition. The proof is given by the following steps.

First of all, we prove that \(\rho'_{11}, H \models C_{1} \) and \(\rho'_{12}, H \models C_{2} \). If they do not hold, assume \(C_{1} \) fails to hold not later than \(C_{2} \), and the first time for which \(C_{1} \) does not hold is \(t_{1} \) (when it exists), then for all \(t < t_{1}, C_{2} \) holds. There are three kinds of formulas at time \(t_{1} \) in \(C_{1} \): if the formula is for internal variables or internal communication (between \(P \) and \(Q \)) non-readiness, then it will not depend on \(Q \) or \(E \), according to the fact that \(C_{1} \) holds before time \(t_{1} \), it must hold at \(t_{1} \); if the formula is for external communication readiness, first from compatibility check, for any channel \(dh \in X_{1} \setminus X \), it does not occur in \(C_{2} \), then we have \(E \mid dh \Rightarrow E_{1} \mid dh \), where \(E \mid dh \) extracts formulas related to communications along \(dh \) from \(E \). Then from \(\rho_{2}, H \models E, \) we have \(\rho_{2}, H \models E_{1} \mid dh, \) and thus \(\rho_{12} \parallel \rho_{2}, H \models E_{1} \mid dh \). By induction hypothesis, the formula considered must hold at \(t_{1} \); if the formula is for internal communication readiness, then there must exist an open interval \((t_{0}, t_{1}) \) during which it is not satisfied. From the assumption, \(C_{2} \) holds in the interval \((t_{0}, t_{1}) \), thus \(E_{1} \mid X \) holds in the interval \((t_{0}, t_{1}) \). By induction, the internal communication readiness assertions in \(C_{1} \) hold in the interval \((t_{0}, t_{1}) \). We thus get a contradiction. Therefore, we can get the fact that both \(\rho'_{11}, H \models C_{1} \) and \(\rho'_{12}, H \models C_{2} \) hold. On the other hand, if such \(t_{1} \) does not exist, there must exist an open interval \((t_{2}, t_{3}) \) such that for all \(t \leq t_{2}, C_{1} \) and \(C_{2} \) hold, while \(C_{1} \) does not hold in \((t_{2}, t_{3}) \). The proof is very similar to the above case. We omit it here for avoiding repetition.

Based on the above facts, from \(\rho_{1} \models A \) and \(\rho_{1}, H \models E, \) and compatibility check, we have therefore \(\rho_{12} \parallel \rho_{2}, H \models E_{1} \). Similarly, we can get for another process \(Q \) that \(\rho_{12} \models A \land now = o, \) and \(\rho_{11} \parallel \rho_{2}, H \models E_{2} \). Then, by induction on \(P \) and \(Q \), we have \(\rho'_{11} \models R_{1} \land tr = \gamma_{1} \land now = o_{1} \) and \(\rho'_{11}, H \models C_{1} \); \(\rho'_{12} \models R_{2} \land tr = \gamma_{2} \land now = o_{2} \) and \(\rho'_{12}, H \models C_{2} \) respectively.

Notice that \(\rho'_{11} \parallel \rho'_{12} \), i.e. \(\rho'_{1} \), only redefines the values of \(tr \) and \(now \), where the communications are arranged in the order according to their occurring time, and variable \(now \) takes the greater value between \(\rho'_{11}((t) \triangleright X) \) and \(\rho'_{12}((t) \triangleright X) \). Obviously, we have \(\rho'_{1} \models R_{1}[\gamma_{1}/tr, o_{1}/now] \land R_{2}[\gamma_{2}/tr, o_{2}/now] \land now = o_{m} \). And, \(\rho'_{1} \models \gamma_{1} \land X = \gamma_{2} \land X \) holds because of synchronization. From the definition of \(\triangleright, \rho'_{1}(tr)(t) \models \rho'_{11}(tr)(t) \parallel \rho'_{12}(tr)(t) \), we can easily get the fact \(\rho'_{1} \models tr = \gamma_{1} \parallel \gamma_{2} \). Thus \(R \) holds for the final state.
From $\rho_{11}, \mathcal{H} \models C_1$ and $\rho_{12}, \mathcal{H} \models C_2$, considering that only now change and matter, we have $\rho_1, \mathcal{H} \models C_1[o_1/now] \land C_2[o_2/now]$. After $P$ or $Q$ terminates, only rd$y$, tr and now may change, plus the fact that $R_1$ and $R_2$ do not contain readiness, $R_1 \Rightarrow R'_1$, $R_2 \Rightarrow R'_2$, and $R'_1, R'_2$ do not contain tr, we have $\rho'_1, \mathcal{H} \models R'_1[o_1/now] \triangleright [o_1/now]$ and $\rho'_1, \mathcal{H} \models R'_2[o_2/now] \triangleright [o_2/now]$. The whole result is proved.

- **Case Repetition $P^*$:** From the operational semantics, we have there must exist a finite integer $n \geq 0$, and $\rho_{11}, ..., \rho_{1n}$ such that $(P^* \parallel E, \rho_{11} \cup \rho_2) \xrightarrow{\alpha} (E \parallel \rho_{11}, \rho'_2)$ where $\rho_{11} = \rho_1, \rho_{1n} = tr + \tau = \rho'_1$. Assume $\rho_1 \models A \land now = o \land tr = h$ and $\rho_2, \mathcal{H} \models E$, we need to prove that $\rho'_1 \models A \land now = o' \land tr = h \cdot w^k \land \tau$ and $\rho'_1, \mathcal{H} \models C \lor E$ holds as defined in the rule for Repetition for any non-negative integer $k$.

  If $n = 1$, then we have $\rho_1[tr + \tau] = \rho'_1$, let $o = o'$, the fact holds directly. If $n > 1$, from $\rho_1 \models A \land now = o \land tr = h$ and $\rho_2, \mathcal{H} \models E[o/now]$, then let $k$ be 0, by induction hypothesis, we have $\rho_{12} \models A \land now = o' \land tr = h \cdot w$ by assigning $o'$ by $o + t$, and $\rho_{12}, \mathcal{H} \models C$. Recursively repeating the proof, plus the fact for any $k$, $\rho_{1k}, \mathcal{H} \models E[o/now]$, we can prove the result.

□