Dynamical origin of the
\( \star_\theta \)-noncommutativity in field theory from quantum mechanics

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Abstract

We show that introducing an extended Heisenberg algebra in the context of the Weyl-Wigner-Groenewold-Moyal formalism leads to a deformed product of the classical dynamical variables that is inherited to the level of quantum field theory, and that allows us to relate the operator space noncommutativity in quantum mechanics to the quantum group inspired algebra deformation noncommutativity in field theory.

Key words: Noncommutativity, star-products

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1 Introduction

Theoretical physics has provided us a fairly deep understanding of the microscopic structure of matter, but very little is known regarding the microscopic structure of space-time.

From a methodological point of view, the use of a noncommutative struc-
ture for space-time coordinates had already been proposed in the early days of field theory as a failed hope at finding an effective and Lorentz invariant cutoff needed to control the ultraviolet divergences plaguing the theory. From a conceptual and theoretical point of view there is a simple heuristic argument - based on Heisenberg’s Uncertainty Principle, the Einstein Equivalence Principle and the Schwarzschild metric - which shows that the Planck length seems to be a lower limit to the possible precision measurement of position, and that shorter distances do not appear to have an operational meaning [1]. Thus Quantum Mechanics and Field Theory, at dimensions of the order of the Planck length, ought to incorporate in their very structure the noncommutativity of space-time by replacing the concept of a space-time point by a cell of a dimension given by the Planck scale area. Under these premises the very concept of manifold as an underlying mathematical structure of physical theories becomes questionable and some people are convinced that a new paradigm of geometrical space is needed. The noncommutative geometry of Connes [2], which by resorting to arbitrary and noncommutative $C^*$-algebras dualizes geometry and replaces its usual notions of manifolds and points by a new calculus based on operators in Hilbert space and the use of spectral analysis, epitomizes this line of thought. More recently there has been further evidence of space-time noncommutativity [3] coming from certain models of string theory which, although with a geometry quite different from that of noncommutative geometry is not incompatible with it, and has led to the same issue of noncommutativity of space-time at short distances. In the noncommutative quantum field theory rooted on the phenomenology of the low energy approximation of string theory in the presence of a strong magnetic background, the fields on a target space of space-time canonical coordinates are replaced by a $C^*$-algebra of functions with a deformed product
given by the so called Groenewold-Moyal star-product:

\[ f(x) \star_\theta g(x) = f(x)e^{\left(\frac{i}{2}[\theta^{ij}, x^i - \partial_j]_\theta\right)}g(x), \]

(1)

where the constant real and invertible anti-symmetric tensor \( \theta^{ij} \) has dimensions of length squared. One interpretation (see e.g. [4]) for the origin of this noncommutativity is based on postulating the replacement of the space-time argument of canonical coordinates \( x^i \) of field operators by a “space-time” of Hermitian operators obeying the Heisenberg algebra

\[ [\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \quad i, j = 1, \ldots, 2d \]

(2)

where \( I \) is an identity operator. Operators \( \mathcal{O}(\hat{x}) \), acting on a Hilbert space of delta-function normalizable functions in \( d \)-dimensions, are then defined in terms of the basic operators (2) by means of the Weyl basis \( g(\alpha, \hat{x}) = e^{i\alpha \cdot \hat{x}} \). Using now the Weyl-Moyal correspondence

\[ \mathcal{O}(\hat{x}) = \int d^{2d} \alpha g(\alpha, \hat{x}) \tilde{\mathcal{O}}_W(\alpha), \]

(3)

where \( \tilde{\mathcal{O}}_W(\alpha) \) is the Fourier transform of the Weyl function corresponding to \( \mathcal{O} \), it follows, in complete analogy to the results derived from the Weyl-Wigner-Groenewold-Moyal (WWGM) formalism of quantum mechanics (see the following section), that the Weyl function corresponding to the operator product \( \mathcal{O}_1 \mathcal{O}_2 \) is given by

\[ (O_1)_W \star_\theta (O_2)_W. \]

(4)

For a review of noncommutative quantum field theory based on these criteria see, e.g., [5].

An alternative and Lorentz invariant (in the twisted symmetry sense) interpretation of the origin of the star-product (1) comes from considering the
twisted coproduct of the Hopf algebra $\mathcal{H}$ of the universal enveloping $\mathcal{U}(\mathcal{P})$ of the Poincaré algebra $\mathcal{P}$. It can be shown (see e.g. [9]) that for a certain Drinfeld twisting of the coproduct with an invertible $\mathcal{F} \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$ such that

$$\mathcal{F}_{12}(\Delta \otimes \text{id})\mathcal{F} = \mathcal{F}_{23}(\text{id} \otimes \Delta), \quad (\epsilon \otimes \text{id})\mathcal{F} = 1 = (\text{id} \otimes \epsilon)\mathcal{F},$$

(5)

this coproduct induces a deformation in the product, $m \rightarrow m_F$, of the module algebra $\mathcal{A} = C^\infty(M)$ over $\mathcal{H}$, such that the action of $\mathcal{H}$ on $\mathcal{A}$ preserves covariance, i.e.

$$h \triangleright m_F(a \otimes b) = m \circ [(\mathcal{F}_{-1}^{-1} \triangleright a) \otimes (\mathcal{F}_{-1}^{-1} \triangleright b)] = a \star_{\theta} b,$$

(6)

where $a, b \in \mathcal{A}$ and $h \in \mathcal{H}$, and we have used the Sweedler notation throughout. In particular, considering the coordinates $x^i$ as elements of $\mathcal{A}$, equation (6) implies that

$$[x^i, x^j]_{\star_{\theta}} \equiv x^i \star_{\theta} x^j - x^j \star_{\theta} x^i = i\theta^{ij}.$$  

(7)

Note, however, that although both of the above described representative lines of thought lead to the same algebra of operators for noncommutative quantum field theory, the origins of this noncommutativity appear to be quite different. In the later case, as has been stressed by Chaichian et al., the product (7) is inherited from the twist of the operator product of quantum fields and no noncommutativity of the coordinates was used in the derivation of (6); while in the line of thought described in [4] the assumed noncommutativity of the space-time operators forms an essential part of the ensuing arguments. However, the inference that the multiplication in the algebra of fields is given by the star-product (6) is an external ingredient imported from the phenomenology of string theory.
Since quantum mechanics is strongly interwoven into noncommutative geometry, and since single particle quantum mechanics can be seen, in the free field or weak coupling limit, as a mini-superspace sector of quantum field theory where most degrees of freedom have been frozen (*i.e.*, as a one-particle sector of field theory), it is suggestive that a further study of quantum mechanics in this noncommutative context, and in particular in the WWGM formalism based on a Heisenberg algebra extended to incorporate space noncommutativity, may help to shed some additional light on the origins of the product (7) in the algebra of noncommutative field theory.

Observe, however, that in the strict sense of quantum mechanics only expectation values have a physical meaning. This, in the WWGM quantum formalism, translates to the fact that the c-equivalent of a quantum operator, or to that effect of a product of operators, appears together with the Wigner quasi-distribution function inside of a phase-space integral. In the case of the standard Heisenberg algebra of usual quantum mechanics, the Wigner function is the same as the Weyl equivalent of the von Neumann density matrix and the Weyl equivalent of a product of operators (given by the Groenewold-Moyal product of their respective Weyl equivalents) is indeed the c-function that would appear in the integrand multiplying the Wigner function. On the other hand, as it is shown in the next section, this is not true for the case of a quantum mechanics with an extended Heisenberg algebra. In fact, as shown in equations (27) or (28) there, either of which can be used to evaluate the expectation value of a product of operators, the Weyl equivalent of a product of operators (given by (30) with a composite ★-product defined by (25), (31) and (32) ) is not the one required in the integrands in order to arrive at the correct expectation values. Hence this ★-product does not appear as a nat-
ural ingredient of the quantum mechanical formalism when considering only Schrödinger operators.

The purpose of this work is to show nonetheless that when considering in addition Weyl equivalents of Heisenberg operators, the $\star_\theta$ product for the algebra of what can then be identified as canonical dynamical variables, emerges naturally within the theory and thus allows for a further link between the points of view of quantum operator space noncommutativity, as presented in [4], and the quantum group inspired algebra deformation noncommutativity, discussed in [9]. Lastly we could expect as well that a detailed study of exactly solvable models in the frame of this extended Heisenberg algebra WWGM formalism may also be helpful to achieve a further understanding of the possible phenomenological consequences in space of the noncommutativity in field theory. In this context, the above observations as well as some additional ones contained below are also pertinent to some works that have appeared recently in the literature on what has been called noncommutative quantum mechanics.

2 Quantum Mechanics on Extended Heisenberg Algebras in the WWGM Formalism

By an extended Heisenberg algebra we understand the algebra of position and momentum operators satisfying the commutation relations

\begin{align}
[\hat{R}_i, \hat{R}_j] &= i\theta_{ij}, \\
[\hat{P}_i, \hat{P}_j] &= i\hbar\delta_{ij}, \\
[\hat{R}_i, \hat{P}_j] &= i\hbar\delta_{ij},
\end{align}

(8) (9) (10)
where $\hat{R}_i, \hat{P}_i \ i = 1, \ldots, d$ are the components of the position and momentum quantum operators, respectively, with component eigenvalues on $\mathbb{R}^d$, and $\theta_{ij}$ and $\bar{\theta}_{ij}$ are evidently antisymmetric matrices, which in the most general case can be functions of the generators of the above algebra. For our present purposes and algebraic simplicity, in what follows we shall set $\bar{\theta}_{ij} = 0$ and $d = 2$, and consider only the zeroth order constant term of the Taylor expansion of $\theta_{12} \equiv \theta$. (For $\theta$ constant, the formalism described below can be generalized to include more spatial dimensions in a fairly straightforward way, and it also can be extended to incorporate space-time noncommutativity by parameterizing the time and considering it as an extra variable. See for example [10]). From an intrinsically noncommutative operator point of view, the development of a formulation for the quantum mechanics based on the above extended Heisenberg algebra of operators requires first a specification of a representation for the generators of the algebra, second a specification of the Hamiltonian which governs the time evolution of the system and last a specification of the Hilbert space on which these operators and the other observables of the theory act. As for the choice of the Hilbert space, a reasonable assumption is that it can be taken to be the same as that for the corresponding system in the usual quantum mechanics, but for a realization of the extended Heisenberg algebra, because of the noncommutativity (8), we cannot use configuration space as a basis. We can use, however, for a basis either of the eigenkets $|p_1, p_2\rangle$, $|q_1, p_2\rangle$, $|q_2, p_1\rangle$, of the commuting pairs of observables $(\hat{P}_1, \hat{P}_2)$, $(\hat{R}_1, \hat{P}_2)$, or $(\hat{R}_2, \hat{P}_1)$, respectively, or any combination of the $(R, P)$ such that they form a complete set of commuting observables. Having in mind generalizations to include the noncommutativity (9), we choose as the realization of our extended Heisenberg algebra the one based on $|q_1, p_2\rangle$. 
The construction follows standard procedures (cf. [6]): Consider the unitary operator \( \hat{S}(\gamma) = e^{\gamma \hat{R}_2} \) (\( \gamma \) is an arbitrary parameter) and evaluate its commutators with \( \hat{R}_1 \) and \( \hat{P}_2 \). It is easy to show that

\[
\hat{S}(\gamma)|q_1, p_2\rangle = |q_1 - \theta \gamma, p_2 + \hbar \gamma\rangle. \tag{11}
\]

Assuming now that \( \gamma \) is an infinitesimal and evaluating \( \langle q_1, p_2 | \hat{S}(\gamma) | q'_1, p'_2 \rangle \) to first order in \( \gamma \) results in

\[
\langle q_1, p_2 | \hat{R}_2 | q'_1, p'_2 \rangle = (-i \theta \partial_{q_1} + i \hbar \partial_{p_2}) \langle q_1, p_2 | q'_1, p'_2 \rangle,
\]

so the realization of \( \hat{R}_2 \) in this basis is

\[
\hat{R}_2 = -i \theta \partial_{q_1} + i \hbar \partial_{p_2}. \tag{12}
\]

Considering next the unitary operator \( \hat{S}(\lambda) = e^{\lambda \hat{P}_1} \) and following a similar procedure we get

\[
\hat{P}_1 = -i \hbar \partial_{q_1}. \tag{13}
\]

The representations for the remainder of the generators \( \hat{R}_1 \) and \( \hat{P}_2 \) of the algebra are obviously simply multiplicative. (Note that by making use of (11) we can readily make the change of basis \( |q_1, p_2\rangle \rightarrow |p_1, p_2\rangle \) and derive the representations \( \hat{R}_1 = i \hbar \partial_{p_1} \) and \( \hat{R}_2 = i \hbar \partial_{p_2} + \frac{\theta}{\hbar} p_1 \) for the extended Heisenberg algebra generators in the momentum representation. In this case \( \hat{P}_1 \) and \( \hat{P}_2 \) are obviously just multiplicative. All our calculations could then be related to that basis.)

For later calculations we shall be needing to evaluate the transition function \( \langle q_1, p_2 | q_2, p_1 \rangle \). This can be derived [7] by noting that

\[
\langle q_1, p_2 | \hat{R}_2 | q_2, p_1 \rangle = q_2 \langle q_1, p_2 | q_2, p_1 \rangle = i(\hbar \partial_{p_2} - \theta \partial_{q_1}) \langle q_1, p_2 | q_2, p_1 \rangle, \tag{14}
\]

8
\[ \langle q_1, p_2 | \hat{P}_1 | q_2, p_1 \rangle = p_1 \langle q_1, p_2 | q_2, p_1 \rangle = -i \hbar \partial_{q_1} \langle q_1, p_2 | q_2, p_1 \rangle. \]  

(15)

Combining these two expressions yields

\[ (\hbar q_2 - \theta p_1) \langle q_1, p_2 | q_2, p_1 \rangle = i \hbar \partial_{p_2} \langle q_1, p_2 | q_2, p_1 \rangle, \]

(16)

which can be readily solved to give, after normalization,

\[ \langle q_1, p_2 | q_2, p_1 \rangle = \frac{1}{2\pi \hbar} \exp\left[ -\frac{i}{\hbar} (q_2 p_2 - \theta p_1 p_2 - q_1 p_1) \right]. \]

(17)

Making use of (17) and the Baker-Campbell-Hausdorff (BCH) theorem, it is fairly direct to show that

\[ \frac{1}{(2\pi \hbar)^2} \text{Tr}\{ \exp[i \hbar ((y - y') \cdot \hat{R} + (x - x') \cdot \hat{P})] \} = \delta(x - x') \delta(y - y'), \]

(18)

where \( x = (x_1, x_2) \quad y = (y_1, y_2). \)

Thus for our extended Heisenberg algebra also the \( \{(2\pi \hbar)^{-1} \exp[i \hbar (y \cdot \hat{R} + x \cdot \hat{P})]\} \) form a complete set of orthonormal operators, and any Schrödinger operator (which may depend explicitly on time) \( A(\hat{P}, \hat{R}, t) \) can be written as

\[ A(\hat{P}, \hat{R}, t) = \int \int dx \ dy \alpha(x, y, t) \exp\left[ \frac{i}{\hbar} (x \cdot \hat{P} + y \cdot \hat{R}) \right], \]

(19)

where, by (18), the \( c \)-function \( \alpha(x, y, t) \) is determined by

\[ \alpha(x, y, t) = (2\pi \hbar)^{-2} \text{Tr}\{ A(\hat{P}, \hat{R}, t) \exp[-i \hbar (x \cdot \hat{P} + y \cdot \hat{R})] \}. \]

(20)

The Weyl function corresponding to the quantum operator \( A(\hat{P}, \hat{R}, t) \) is then given by

\[ A_W(p, q, t) = \int \int dx \ dy \ \alpha(x, y, t) \exp\left[ \frac{i}{\hbar} (x \cdot p + y \cdot q) \right] = \]

\[ \int \int dx_1 dx_2 e^{i(x_1 p_1 + y_2 q_2)} \langle q_1 - \frac{x_1}{2} - \frac{\theta y_2}{2\hbar}, p_2 + \frac{y_2}{2} | \hat{A} | q_1 + \frac{x_1}{2} + \frac{\theta y_2}{2\hbar}, p_2 - \frac{y_2}{2} \rangle. \]

(21)
To derive the expectation value of a product of two Schrödinger operators, one writes the expectation value of the product in terms of the von Neumann density matrix \( \rho \) as

\[
\langle \hat{A}_1 \hat{A}_2 \rangle = \text{Tr}[\rho \hat{A}_1 \hat{A}_2],
\]

and evaluates the trace in the above chosen basis. After a rather lengthy but fairly straightforward calculation the result obtained is

\[
\langle \hat{A}_1 \hat{A}_2 \rangle = \int \ldots \int dp_1 dp_2 dq_1 dq_2 \frac{1}{(2\pi\hbar)^2} \int d\xi d\eta e^{-i(\eta q_2 - \xi p_1)} \\
\langle q_1 - \frac{\xi}{2}, p_2 - \frac{\eta}{2} | \rho | q_1 + \frac{\xi}{2}, p_2 + \frac{\eta}{2} \rangle e^{i\theta p_1 \partial q_2} ((A_1)_W \ast_h (A_2)_W),
\]

where \( \ast_h := \exp[i\hbar \Lambda] := \exp \left[ i\hbar (\vec{\nabla}_q \cdot \vec{\nabla}_p - \vec{\nabla}_p \cdot \vec{\nabla}_q) \right] \),

is the Gronewold-Moyal star-product bidifferential of the usual WWGM quantum mechanics formalism. If we now let

\[
\rho_{(\text{Wigner})} := \frac{1}{(2\pi\hbar)^2} \int d\xi d\eta e^{-i(\eta q_2 - \xi p_1)} \langle q_1 - \frac{\xi}{2}, p_2 - \frac{\eta}{2} | \rho | q_1 + \frac{\xi}{2}, p_2 + \frac{\eta}{2} \rangle
\]

denote the standard Wigner quasi-probability distribution in our chosen basis, then (23) reads as

\[
\langle \hat{A}_1 \hat{A}_2 \rangle = \int \int dp dq \ \rho_{(\text{Wigner})} e^{i\theta p_1 \partial q_2} ((A_1)_W \ast_h (A_2)_W).
\]

Note that we could equally well have integrated the above equation by parts to get

\[
\langle \hat{A}_1 \hat{A}_2 \rangle = \int \int dp dq \ \rho_W ((A_1)_W \ast_h (A_2)_W).
\]

where the Weyl function \( \rho_W \) corresponding to \( \rho \) is related to \( \rho_{(\text{Wigner})} \) by

\[
\rho_W = e^{-i\theta p_1 \partial q_2} (\rho_{(\text{Wigner})}),
\]

in contradistinction to what happens in the usual quantum mechanics where they are the same. So in the calculation of the expectation value of the product
of two Schrödinger operators, the quantities that enter in the quantum mechanics based on the extended Heisenberg algebra are either \((A_1)_W \star_h (A_2)_W\), when averaging with \(\rho_W\), or \(e^{\hat{\theta} p_1 \partial_{q_2}} ((A_1)_W \star_h (A_2)_W)\) when averaging with the usual Wigner function. However, also contrary to what happens in ordinary quantum mechanics, these quantities are not equal to the Weyl equivalent \((\hat{A}_1 \hat{A}_2)_W\) of the product \(\hat{A}_1 \hat{A}_2\).

To evaluate \((\hat{A}_1 \hat{A}_2)_W\) we use (20) and (21), and following steps entirely analogous to the ones treated in more detail in the following section when considering Heisenberg operators, it can be shown that

\[
(\hat{A}_1 \hat{A}_2)_W = (\hat{A}_1)_W \star (\hat{A}_2)_W, \tag{29}
\]

where \(\star\) is defined by the composition of operator bi-differentials:

\[
\star := \star_\theta \circ \star_h, \tag{30}
\]

with \(\star_h\) as defined in (24) and

\[
\star_\theta := e^{\frac{i}{2} \theta (\overrightarrow{\partial}_{q_1} \overleftarrow{\partial}_{q_2} - \overrightarrow{\partial}_{q_2} \overleftarrow{\partial}_{q_1})}. \tag{31}
\]

Furthermore and similarly to what occurs in ordinary quantum mechanics, there is a stronger star-value equation related to (27). There are again however important differences. Thus, given a Hamiltonian operator \(\hat{H}\) and a pure energy state satisfying the eigenvalue equation \(\hat{H}|\psi\rangle = E|\psi\rangle\), it can be shown that the star-value equation for the quantum mechanics with our extended Heisenberg algebra is

\[
\hat{H}_W \star_h \rho_{(Wigner)} = E \rho_{(Wigner)}, \tag{32}
\]

where

\[
\hat{H}_W(p, q) = e^{\frac{i}{\hbar} \theta p_1 \partial_{q_2}} H_W(p, q). \tag{33}
\]
Because of space limitations we omit here the details of the proof of this theorem. These, together with other more detailed aspects of our previous discussion as well examples where specific implications of the quantum mechanics here summarized are displayed and compared with other approaches, will be dealt with in a forthcoming paper to appear elsewhere.

3 Weyl Equivalent of Heisenberg Operators

Let

$$\Omega^H := \Omega(\hat{\mathbf{P}}, \hat{\mathbf{R}}, t) := e^{i\frac{\hat{H}}{2\hbar}} \Omega(\hat{\mathbf{P}}, \hat{\mathbf{R}}, 0)e^{-i\frac{\hat{H}}{2\hbar}},$$

(34)

be the Heisenberg operator corresponding to the Schrödinger operator $\Omega(\hat{\mathbf{P}}, \hat{\mathbf{R}}, 0)$. As for Schrödinger operators the $c$-function $\alpha_\Omega(x, t)$, associated with the Weyl function $(\Omega^H)_W$ defined as in (21), is given by (see (20))

$$\alpha_\Omega(x, y, t) = (2\pi\hbar)^{-2} \text{Tr}\{e^{i\frac{\hat{H}}{2\hbar}} \Omega(\hat{\mathbf{P}}, \hat{\mathbf{R}}, 0)e^{-i\frac{\hat{H}}{2\hbar}} e^{-i(x\cdot \hat{\mathbf{P}} + y\cdot \hat{\mathbf{R}})}\}. \quad (35)$$

Differentiating (21) with respect to $t$ and taking the Fourier transform gives immediately

$$\frac{\partial \alpha_\Omega}{\partial t} = \frac{i(2\pi\hbar)^{-2}}{\hbar} \int dq_1 dp_2 \langle q_1 - \frac{x_1}{2} - \frac{y_1}{2\hbar}, p_2 + \frac{y_2}{2\hbar} | [H, \Omega^H] | q_1 + \frac{x_1}{2} + \frac{y_2}{2\hbar}, p_2 - \frac{y_2}{2} \rangle \exp[-\frac{i}{\hbar}(y_1 q_1 + x_2 p_2)]. \quad (36)$$

Consider now the quantity

$$\int dq_1 dp_2 \exp[-\frac{i}{\hbar}(y_1 q_1 + x_2 p_2)] \langle q_1 - \frac{x_1}{2} - \frac{y_1}{2\hbar}, p_2 + \frac{y_2}{2\hbar} | H \Omega^H | q_1 + \frac{x_1}{2} + \frac{y_2}{2\hbar}, p_2 - \frac{y_2}{2} \rangle,$$
which, after making use of (19), (17), the BCH theorem and performing several fairly direct integrations, yields

\[
(2\pi\hbar)^{-2} \int dq_1 dp_2 \exp[-\frac{i}{\hbar}(y_1 q_1 + x_2 p_2)] \langle q_1 - \frac{x_1}{2} - \frac{y_2}{2\hbar}, p_2 + \frac{y_2}{2\hbar} | H \Omega^H | q_1 + \frac{x_1}{2} + \frac{y_2}{2\hbar}, p_2 - \frac{y_2}{2} \rangle = \\
\int dx' dy' \alpha_H (x'y') \alpha_H (x - x', y - y', t) \times \exp\left[-\frac{i}{\hbar}(y'_1 y_2 + y_1 y'_2) + x'_2 y_2 - x_2 y'_2 + y_1 x'_1 - y'_1 x_1 \right].
\]

(37)

Rewriting (37) in terms of \(H_W\) and \((\Omega^H)_W\), by making use of the Fourier inverse of the first equality in (21), and substituting the result into (36) it readily follows that

\[
\frac{\partial \alpha}{\partial t} = i \frac{2\pi \hbar}{\hbar} \int \ldots \int d\mathbf{p}' d\mathbf{q}' d\mathbf{p}'' d\mathbf{q}'' dx' dy' e^{-\frac{i}{\hbar}(x' \cdot \mathbf{p}' + y' \cdot \mathbf{q}')} \times [H_W (\mathbf{p}', \mathbf{q}') \Omega^H_W (\mathbf{p}'', \mathbf{q}'', t) - \Omega^H_W (\mathbf{p}', \mathbf{q}') H_W (\mathbf{p}'', \mathbf{q}'', t)] \times \exp[-\frac{i}{\hbar}((x - x') \cdot \mathbf{p}'' + (y - y') \cdot \mathbf{q}'')]
\times \exp\left[\frac{i}{2\hbar}(y'_1 y_2 + y_1 y'_2) + x'_2 y_2 - x_2 y'_2 + y_1 x'_1 - y'_1 x_1 \right].
\]

(38)

Finally, double Fourier transforming both sides of (38), rearranging terms and performing the integrals, yields

\[
\frac{\partial \Omega^H_W}{\partial t} = i \frac{\hbar}{\hbar} [H_W (\mathbf{p}, \mathbf{q}) \star \Omega^H_W (\mathbf{p}, \mathbf{q}) - \Omega^H_W (\mathbf{p}, \mathbf{q}) \star H_W (\mathbf{p}, \mathbf{q})].
\]

(39)

Note that by interchanging the ordering of the Weyl functions in the second term inside the square brackets in (39), we alternatively have

\[
\frac{\partial \Omega^H_W}{\partial t} = i \hbar H_W [e^{\frac{i}{\hbar}(h \Lambda + \theta \Lambda')} - e^{-\frac{i}{\hbar}(h \Lambda + \theta \Lambda')}] \Omega^H_W = -\frac{2}{\hbar} H_W \sin\left[\frac{1}{2}(h \Lambda + \theta \Lambda')\right] \Omega^H_W,
\]

(40)

where

\[
\Lambda := \nabla_\mathbf{q} \cdot \nabla_\mathbf{p} - \nabla_\mathbf{p} \cdot \nabla_\mathbf{q}, \quad \Lambda' := \nabla_{q_1} \cdot \nabla_{q_2} - \nabla_{q_2} \cdot \nabla_{q_1}.
\]

(41)
Equation (40) can be formally integrated to give

\[
\Omega^H_W(p, q, t) = \exp\left\{-\frac{2t}{\hbar} H_W \sin\left[\frac{1}{2}(\hbar \Lambda + \theta \Lambda^\prime)\right]\right\}\Omega_W(p, q, 0). \tag{42}
\]

Note that (39) is in agreement with the derivation in [8] for the time evolution of the Wigner function, although the calculation there is somewhat circular from our point of view as it assumes the $\star_{\theta}$-product to be valid \textit{ab initio}.

4 Noncommutative Field Theory from extended Heisenberg algebra Quantum Mechanics

Up to this point in the WWGM formalism the $q$’s and $p$’s (the continuum of eigenvalues of $\hat{R}$ and $\hat{P}$) are only variables of integration. In order to be able to interpret them as canonical dynamical variables, as it is the case for ordinary WWGM quantum mechanics, let us consider the specific cases when the Heisenberg operator $\Omega^H$ in Section 3 is $\hat{P}(t)$ or $\hat{R}(t)$. Making use of (21) and (42), and recalling that $P_W(p, q, 0) = p$ and $R_W(p, q, 0) = q$, we get for this particular cases, and a mechanical Hamiltonian of the form $\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{R})$,

\[
\begin{align*}
\frac{dP^H_W}{dt}|_{t=0} &= -\frac{1}{\hbar} H(h\Lambda + \theta \Lambda')p = -\nabla_q V, \\
\frac{d(R^H_1)_W}{dt}|_{t=0} &= -\frac{1}{\hbar} H(h\Lambda + \theta \Lambda')q_1 = \frac{p_1}{2m} + \frac{\theta}{\hbar} \partial_{q_2} V, \\
\frac{d(R^H_2)_W}{dt}|_{t=0} &= -\frac{1}{\hbar} H(h\Lambda + \theta \Lambda')q_2 = \frac{p_2}{2m} - \frac{\theta}{\hbar} \partial_{q_1} V. \tag{43}
\end{align*}
\]
Introducing now the following fundamental Poisson brackets as part of the algebra structure of the $q$’s and $p$’s:

\[
\{ p_i, p_j \} = 0, \quad \{ q_i, q_j \} = \frac{\delta_{ij}}{\hbar}, \quad \{ q_i, p_j \} = \delta_{ij},
\] (44)

we have that (43) read

\[
\frac{d(P^H_i)}{dt}|_{t=0} = \{ p_i, H \} = \dot{p}_i, \quad \frac{d(R^H_i)\hbar}{dt}|_{t=0} = \{ q_i, H \} = \dot{q}_i,
\] (45)

and therefore with this additional Poisson structure the $q$’s and $p$’s satisfy the Hamilton equations and can be considered formally as canonical dynamical variables in the theory.

A representation for the above Poisson brackets can be constructed by defining the twisted product

\[
q_i \star_\theta q_j := q_i e^{\frac{i}{\hbar} \sum_{lm} \theta_{qlm} \partial_{q_{lm}} q_j},
\] (46)

where we have generalized our arguments to $\mathbb{R}^d$ (with $d \geq 2$), and letting

\[
\{ q_i, q_j \} := -\frac{i}{\hbar}[q_i, q_j] \star_\theta := -\frac{i}{\hbar}[q_i \star_\theta q_j - q_j \star_\theta q_i].
\] (47)

We can consequently argue that the noncommutativity of the extended Heisenberg algebra in Quantum Mechanics manifests itself as a twisting in the product of the algebra of the corresponding classical canonical dynamical variables which, in accordance with [9], may be interpreted in turn as an Abelian Drinfeld twisting of the coproduct in the Hopf algebra $\mathcal{H}$ of the universal envelope $\mathcal{U}(\mathcal{G})$ of the Galileo symmetry algebra. If we now view the module algebra $\mathcal{A}_\theta$ (the so called Groenewold-Moyal plane), described in the Introduction, as a certain completion of the algebra generated by the $q_i$ and describe fields as elements of $\mathcal{A}_\theta$, then fields will clearly inherit the $\star_\theta$-product.
As a final parenthetical remark, note from Sec 2 that in all the expressions based on the WWGM formalism containing the $\theta$, it always appears in the form of the quotient $\frac{\theta}{\hbar}$. If we claim that the noncommutativity (8) in the extended Heisenberg algebra is originated from quantum gravity, then it is reasonable to assume (as already mentioned in the Introduction) that $\theta \sim l_p^2 = \frac{k \hbar}{c^3}$, where $l_p$ is the Planck length and $k$ is the gravitational coupling constant. Thus $\frac{\theta_{ij}}{\hbar} \sim \frac{k}{c^3}$. This shows that corrections, due to this noncommutativity, to calculations such as energy spectra and equations of motion such as (43), are indifferent to the value of $\hbar$, and that even in the limit $\hbar \to 0$ there is what may appear as a remanent of quantum gravity.

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