Modelling to find Rank of Matrix when rows are similar / in form of scalar multiple

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Abstract.
Here formulas are formed to calculate rank of matrix. By applying it to the problems we can reduce the steps so it save time for calculations. After reducing the matrix to its upper triangular form, the case may occur that we get, row equivalent matrix / rows in the form of scalar multiple of each other, in such case formula (1) is used to find the rank of matrix. Or it may possible that without reduced the matrix to upper triangular form, we get the matrix whose rows are identical or in form of scalar multiple of each other in such case formula (2) is used. In this paper, the problems are solved for computation of rank by different methods and results were compared.

Key requirements
System of linear equations, row equivalent matrices, rank of a matrix.

1. Introduction
Rank has many applications in different field. It is useful in solving a system of linear equations for number of solutions such as parallelism [1], in control theory to check whether linear system is controllable / observable [2] and many more applications.

Rank of matrix can be found by i) Minor method [3] ii) Counting the number of non-zero rows in a matrix after reducing it in upper triangular form [4]. Minor method is tedious and time consuming for higher order matrix, so the most preferable method is the counting the number of non-zero rows after reducing the matrix to upper triangular form.[5]

2. Methodology
If after reduction to upper triangular form, similar rows are there then some more steps are required to find rank. In such case the below formula is designed to find rank of a matrix of any order.

\[ \text{Rank of Matrix} = \left( \text{Number of non-zero rows in UTM} \right) - \left( \frac{\left( \text{No. of identical rows} \right)}{\text{Number scalar multiple rows in UTM}} \right) - 1 \]
UTM is upper triangular matrix

For example if $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$

After performing certain operations, we get

$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ 0 & 0 & 0 & d_{14} \\ 0 & 0 & 0 & kd_{14} \\ 0 & 0 & 0 & d_{14} \end{bmatrix}$

Where $k$ is scalar

Here, Number of non-zero rows = 4,
No. of identical rows / Number scalar multiple rows = 3

Using formula (1), we get

Then rank = $\{4-(3-1)\} = 2$

**Formula (2)**

$\text{Rank of Matrix} = \text{Number of rows in a matrix} - \{[\text{Number of identical rows OR Rows which in scalar multiple}] - 1\}$

For e.g. if $B= \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ k_1b_{11} & k_1b_{12} & k_1b_{13} & k_1b_{14} \\ k_2b_{11} & k_2b_{12} & k_2b_{13} & k_2b_{14} \\ k_3b_{11} & k_3b_{12} & k_3b_{13} & k_3b_{14} \end{bmatrix}$

By applying formula (2)

Rank = 4-(4-1) = 1

3. Examples

*Example 1* Find the rank of matrix $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 15 \end{bmatrix}$ by Minor method.

Solution: Consider a minor of order three
\[
\begin{bmatrix}
1 & 4 & 5 \\
2 & 6 & 8 \\
3 & 7 & 15
\end{bmatrix}
\]

\[= 1(90-56) - 4(30-24) + 5(14-18) = -10 \neq 0\]

As minor of order three is non-zero therefore the rank of given matrix is 3

**Example 2** Find the rank of matrix B = \[
\begin{bmatrix}
1 & 2 & 1 \\
-1 & 0 & 2 \\
2 & 1 & -3
\end{bmatrix}
\]

Solution: \(R_2 \rightarrow R_2 + R_1, R_3 \rightarrow R_3 - 2R_1\)

We get \[
\begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & 3 \\
0 & -3 & -5
\end{bmatrix}
\]

\(R_3 \rightarrow 2R_3 + 3R_2\)

\[
\begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & 3 \\
0 & 0 & -1
\end{bmatrix}
\]

Therefore, Rank of matrix is the number of non-zero rows in upper triangular matrix = 3

**Example 3** Find the rank of matrix B = \[
\begin{bmatrix}
-1 & -3 & 2 & -4 \\
-2 & -6 & 4 & -10 \\
-5 & -15 & 10 & -16
\end{bmatrix}
\]

Solution:

To reduce the matrix to upper triangular form

\(R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 5R_1\)

\[
\begin{bmatrix}
-1 & -3 & 2 & -4 \\
0 & 0 & 0 & -2 \\
0 & 0 & 0 & 4
\end{bmatrix}
\]

Here, matrix in upper triangular form and the second and third row are scalar multiple of each other i.e. become similar if we multiply by proper scalar to second or third row it become similar.

No. of non-zero rows = 3
Rows in scalar multiple of each other = 2

Using formula (1)

We get, Rank B = 3 - (2-1)

Ans = 2.

(1)

Solution by Minor Method: [5], [6], [7]

Here highest possible minor is of order three, On taking minor of order three

\[
\begin{vmatrix}
-1 & -3 & 2 \\
-2 & -6 & 4 \\
-5 & -15 & 10
\end{vmatrix} = 0,
\begin{vmatrix}
-1 & -3 & -4 \\
-2 & -6 & -10 \\
-5 & -15 & -16
\end{vmatrix} = 0.
\]

As all possible minor of order three are zero therefore rank is less than three.

So consider a minor of order 2

\[
\begin{vmatrix}
2 & -4 \\
4 & -10
\end{vmatrix} = -4 \neq 0
\]

Implies Rank = 2

(2)

Solution by Reducing the matrix in Upper triangular form Method: [5], [6], [7]

\[
\begin{bmatrix}
-1 & -3 & 2 & -4 \\
-2 & -6 & 4 & -10 \\
-5 & -15 & 10 & -16
\end{bmatrix}
\]

\[R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 5R_1\]

\[
\begin{bmatrix}
-1 & -3 & 2 & -4 \\
0 & 0 & 0 & -2 \\
0 & 0 & 0 & 4
\end{bmatrix}
\]

\[R_3 \rightarrow R_3 - 2R_2\]

\[
\begin{bmatrix}
-1 & -3 & 2 & -4 \\
0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Rank is number of non-zero rows in upper triangular matrix

Ans = 2

(3)

On comparing (i), (ii) and (iii),

Result in each case is same.
Example 4 Calculate the rank of \[
\begin{bmatrix}
10 & 20 & 30 \\
-10 & -20 & -30 \\
1 & 2 & 3
\end{bmatrix}
\].

Solution:

i) Using Formula (2):

\[
\text{Rank} = 3 - (3 - 1) = 1
\]  \hspace{1cm} (4)

ii) By Minor method:

\[
\begin{vmatrix}
10 & 20 & 30 \\
-10 & -20 & -30 \\
1 & 2 & 3
\end{vmatrix} = 0
\]

As minor of order three is zero consider a minor of order two.

\[
\begin{vmatrix}
10 & 20 \\
-10 & -20
\end{vmatrix} = 1 
\begin{vmatrix}
10 & 20 \\
1 & 2
\end{vmatrix} = 1 
\begin{vmatrix}
10 & 30 \\
1 & 2
\end{vmatrix} = 1 
\begin{vmatrix}
20 & 30 \\
-10 & -30
\end{vmatrix} = 0 
\begin{vmatrix}
20 & 30 \\
2 & 3
\end{vmatrix} = 2 
\begin{vmatrix}
10 & 30 \\
1 & 3
\end{vmatrix} = 0
\]

All possible minor of order 2 vanishes so consider a minor of order one which is element of matrix so rank is one. \hspace{1cm} (5)

iii) When reduced the given matrix to Echelon form we get

\[
\begin{bmatrix}
10 & 20 & 30 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Rank = no. of non-zero rows in upper triangular matrix

Ans = 1 \hspace{1cm} (6)

Formula (2) verified from (i), (ii) & (iii).

When the rows are not similar/ scalar multiple in a given matrix but while reducing the matrix to Echelon form, we get similar or scalar multiple rows then also Formula (2) is applicable.

Example 5 Find the rank of matrix \[
C = \begin{bmatrix}
-1 & -2 & -3 & -4 \\
-2 & -3 & -4 & -5 \\
-3 & -4 & -5 & -6 \\
-4 & -5 & -6 & -7
\end{bmatrix}
\]

Solution:

By Formula (2)

\[R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 3R_1, R_4 \rightarrow R_4 - 4R_1\]
We get
\[
\begin{bmatrix}
-1 & -2 & -3 & -4 \\
0 & 1 & 2 & 3 \\
0 & 2 & 4 & 6 \\
0 & 3 & 6 & 9
\end{bmatrix}
\]

*Here in this* matrix, Number of rows are 4, last three become identical on multiplying by proper scalar, so to find rank we can apply formula (2).

\[
\text{Rank of } B = 4 - (3 - 1) = 2.
\]  
(7)

**By Minor Method:**

Consider a minor of order four:
\[
\begin{vmatrix}
-1 & -2 & -3 & -4 \\
0 & 1 & 2 & 3 \\
0 & 2 & 4 & 6 \\
0 & 3 & 6 & 9
\end{vmatrix} = 0
\]

As minor of order four vanishes. Take a minor of order three:
\[
\begin{vmatrix}
-1 & -2 & -3 \\
0 & 1 & 2 \\
0 & 2 & 4
\end{vmatrix} = 0 
\begin{vmatrix}
-1 & -2 & -3 \\
0 & 1 & 2 \\
0 & 3 & 6
\end{vmatrix} = 0
\begin{vmatrix}
-2 & -3 & -4 \\
0 & 2 & 4 \\
0 & 3 & 6
\end{vmatrix} = 0 
\begin{vmatrix}
-3 & -5 & -6 \\
-4 & -5 & -6 \\
0 & 2 & 4
\end{vmatrix} = 0
\begin{vmatrix}
-3 & -5 & -6 \\
-4 & -5 & -6 \\
0 & 3 & 6
\end{vmatrix} = 0
\begin{vmatrix}
-2 & -3 & -4 \\
-4 & -6 & -7 \\
0 & 3 & 6
\end{vmatrix} = 0
\begin{vmatrix}
-1 & -2 \\
-2 & -3
\end{vmatrix} = -1 \neq 0
\]

Ans: As minor of order two is non-zero, Hence rank = 2.  
(8)

**Solution by Reduction to the Upper triangular form Method:**

\[ R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 3R_1, R_4 \rightarrow R_4 - 4R_1 \]

We get
\[
\begin{bmatrix}
-1 & -2 & -3 & -4 \\
0 & 1 & 2 & 3 \\
0 & 2 & 4 & 6 \\
0 & 3 & 6 & 9
\end{bmatrix}
\]

\[ R_3 \rightarrow R_3 - 2R_2, R_4 \rightarrow R_4 - 3R_2 \]
\[-1 \quad -2 \quad -3 \quad -4 \\
0 \quad 1 \quad 2 \quad 3 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \]

Here number of non-zero rows = 2

Ans = 2

Result verified.

4. Result
In this paper, Rank of matrix is calculated by three different methods. In example 1 rank is found by Minor method. For example (2) calculation of rank by reducing the matrix to its upper triangular form.

In example number (3), (4) and (5) rank is found by using Minor method, Reduction to upper triangular form and formula (1) / formula (2) method respectively. On comparing answer from all three methods accuracy of 100 percent is obtained.

5. Conclusion
In this paper, formula of rank of matrix is designed and tested with verities of examples and it is observed that this formula gives 100% accurate result. This derived formula will be beneficial to solve industry related problem efficiently such as inventory management where the system is defined in the formed of linear simultaneous equations.

6. References
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