ON PARTIAL SUMS OF NORMALIZED MITTAG-LEFFLER FUNCTIONS

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Abstract. This article deals with the ratio of normalized Mittag-Leffler function \( E_{\alpha,\beta}(z) \) and its sequence of partial sums \((E_{\alpha,\beta})_m(z)\). Several examples which illustrate the validity of our results are also given.

1. Introduction

Let \( \mathcal{A} \) be the class of functions \( f \) normalized by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disk \( \mathcal{U} = \{ z \in \mathbb{C} : |z| < 1 \} \).

Denote by \( \mathcal{S} \) the subclass of \( \mathcal{A} \) which consists of univalent functions in \( \mathcal{U} \).

Consider the function \( E_{\alpha}(z) \) defined by

\[
E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \quad z \in \mathcal{U},
\]

where \( \Gamma(s) \) denotes the familiar Gamma function. This function was introduced by Mittag-Leffler in 1903 [9] and is therefore known as the Mittag-Leffler function.

Another function \( E_{\alpha,\beta}(z) \), having similar properties to those of Mittag-Leffler function, was introduced by Wiman [19], [20] and is defined by

\[
E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \quad \beta > 0, \quad z \in \mathcal{U}.
\]

During the last years the interest in Mittag-Leffler type functions has considerably increased due to their vast potential of applications in applied problems such as fluid flow, electric networks, probability, statistical distribution theory etc. For a detailed account of properties, generalizations and applications of functions \( \text{(1.2)} - \text{(1.3)} \) one may refer to [6], [7], [12], [16].

Geometric properties including starlikeness, convexity and close-to-convexity for the Mittag-Leffler function \( E_{\alpha,\beta}(z) \) were recently investigated by Bansal and Prajapati in [11]. Differential subordination results associated with generalized Mittag-Leffler function were also obtained in [14].

The function defined by \( \text{(1.3)} \) does not belong to the class \( \mathcal{A} \). Therefore, we consider the following normalization of the Mittag-Leffler function \( E_{\alpha,\beta}(z) \):

\[
E_{\alpha,\beta}(z) = \Gamma(\beta) z E_{\alpha,\beta}(z) = z + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)} \frac{1}{\beta} z^{n+1}, \quad \alpha > 0, \quad \beta > 0, \quad z \in \mathcal{U}.
\]

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Note that some special cases of $E_{\alpha,\beta}(z)$ are:

$$
\begin{align*}
E_{2,1}(z) &= z \cosh \sqrt{z} \\
E_{2,2}(z) &= \sqrt{z} \sinh(\sqrt{z}) \\
E_{2,3}(z) &= 2[\cosh(\sqrt{z}) - 1] \\
E_{2,4}(z) &= 6[\sinh(\sqrt{z}) - \sqrt{z}] / \sqrt{z}.
\end{align*}
$$

(1.5)

Recently, several results related to partial sums of special functions, such as Bessel \[10\], Struve \[21\], Lommel \[2\] and Wright functions \[3\] were obtained.

Motivated by the work of Bansal and Prajapat \[1\] and also by the above mentioned results, in this paper we investigate the ratio of normalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ defined by (1.4) to its sequence of partial sums

$$
\begin{align*}
(\mathbb{E}_{\alpha,\beta})_0(z) &= z \\
(\mathbb{E}_{\alpha,\beta})_m(z) &= z + \sum_{n=1}^{m} A_n z^{n+1}, \ m \in \{1, 2, \ldots\},
\end{align*}
$$

(1.6)

where

$$
A_n = \frac{\Gamma(\beta)}{\Gamma((\alpha n + \beta)\beta)}, \quad \alpha > 0, \ \beta > 0, \ n \in \mathbb{N}.
$$

We obtain lower bounds on ratios like

$$
\Re\left\{ \frac{E_{\alpha,\beta}(z)}{(E_{\alpha,\beta})_m(z)} \right\}, \quad \Re\left\{ \frac{(E_{\alpha,\beta})_{m}(z)}{E_{\alpha,\beta}(z)} \right\}, \quad \Re\left\{ \frac{E'_{\alpha,\beta}(z)}{(E_{\alpha,\beta})'_{m}(z)} \right\}, \quad \Re\left\{ \frac{(E_{\alpha,\beta})'_m(z)}{E'_{\alpha,\beta}(z)} \right\}.
$$

Several examples will be also given.

Results concerning partial sums of analytic functions may be found in \[4\], \[8\], \[11\], \[13\], \[17\], \[18\] etc.

2. Main results

In order to obtain our results we need the following lemma.

**Lemma 2.1.** Let $\alpha \geq 1$ and $\beta \geq 1$. Then the function $E_{\alpha,\beta}(z)$ satisfies the next two inequalities:

$$
|E_{\alpha,\beta}(z)| \leq \frac{\beta^2 + \beta + 1}{\beta^2}, \ z \in \mathbb{U}
$$

(2.1)

$$
|E'_{\alpha,\beta}(z)| \leq \frac{\beta^2 + 3\beta + 2}{\beta^2}, \ z \in \mathbb{U}.
$$

(2.2)

**Proof.** Under the hypothesis we have $\Gamma(n + \beta) \leq \Gamma(\alpha n + \beta)$ and thus

$$
\frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)} \leq \frac{\Gamma(\beta)}{\Gamma(n + \beta)} = \frac{1}{(\beta)_n}, \ n \in \mathbb{N},
$$

(2.3)

where

$$
(x)_n = \begin{cases} 
1 & n = 0 \\
1 & x(x+1)\ldots(x+n-1) & n \in \mathbb{N}
\end{cases}
$$

is the well-known Pochhammer symbol.

Note that

$$
(x)_n = x(x+1)_{n-1}, \ n \in \mathbb{N}
$$

(2.4)

and

$$
(x)_n \geq x^n, \ n \in \mathbb{N}.
$$

(2.5)
Making use of (2.3) - (2.5) and also of the well-known triangle inequality, for \( z \in \mathcal{U} \), we obtain

\[
|E_{\alpha, \beta}(z)| = \left| z + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)} z^{n+1} \right| \leq 1 + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)} \leq 1 + \sum_{n=1}^{\infty} \frac{1}{(\beta)_n}
\]

and thus, inequality (2.1) is proved.

Using once more the triangle inequality, for \( z \in \mathcal{U} \), we obtain

\[
(2.6) \quad |E'_{\alpha, \beta}(z)| = \left| 1 + \sum_{n=1}^{\infty} \frac{(n+1)\Gamma(\beta)}{\Gamma(\alpha n + \beta)} z^n \right| \leq 1 + \sum_{n=1}^{\infty} \frac{n\Gamma(\beta)}{\Gamma(\alpha n + \beta)} + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)}
\]

For \( \beta \geq 1 \) we have

\[
(2.7) \quad \frac{n}{(\beta)_n} = \frac{n}{\beta(\beta+1)_{n-1}} = \frac{n}{\beta(\beta+1)_{n-2}(\beta+n-1)} \leq \frac{1}{\beta(\beta+1)_{n-2}}.
\]

Taking into account inequalities (2.3) - (2.5) and (2.7), from (2.6), we obtain

\[
|E'_{\alpha, \beta}(z)| \leq 1 + \frac{\sum_{n=1}^{\infty} \frac{n}{(\beta)_n}}{\sum_{n=1}^{\infty} \frac{1}{(\beta)_n}} \leq 1 + \frac{1}{\beta} + \frac{1}{\beta} \sum_{n=2}^{\infty} \frac{1}{(\beta+1)_{n-2}} + \frac{1}{\beta} \sum_{n=1}^{\infty} \frac{1}{(\beta+1)_{n-1}}
\]

and thus, inequality (2.2) is also proved.

Let \( w(z) \) be an analytic function in \( \mathcal{U} \). In the sequel, we will frequently use the following well-known result:

\[
\Re \left\{ \frac{1+w(z)}{1-w(z)} \right\} > 0, \quad z \in \mathcal{U} \quad \text{if and only if} \quad |w(z)| < 1, \quad z \in \mathcal{U}.
\]

**Theorem 2.1.** Let \( \alpha \geq 1 \) and \( \beta \geq \frac{1+\sqrt{5}}{2} \). Then

\[
(2.8) \quad \Re \left\{ \frac{E_{\alpha, \beta}(z)}{(E_{\alpha, \beta})_m(z)} \right\} \geq \frac{\beta^2 - \beta - 1}{\beta^2}, \quad z \in \mathcal{U}
\]

and

\[
(2.9) \quad \Re \left\{ \frac{(E_{\alpha, \beta})_m(z)}{E_{\alpha, \beta}(z)} \right\} \geq \frac{\beta^2}{\beta^2 + \beta + 1}, \quad z \in \mathcal{U}.
\]

**Proof.** From inequality (2.1) we get

\[
1 + \sum_{n=1}^{\infty} A_n \leq \frac{\beta^2 + \beta + 1}{\beta^2}, \quad \text{where} \quad A_n = \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)}, \quad n \in \mathbb{N}.
\]

The last inequality is equivalent to

\[
\frac{\beta^2}{\beta + 1} \sum_{n=1}^{\infty} A_n \leq 1.
\]

In order to prove the inequality (2.8), we consider the function \( w(z) \) defined by

\[
\frac{1+w(z)}{1-w(z)} = \frac{\beta^2}{\beta + 1} \frac{E_{\alpha, \beta}(z)}{(E_{\alpha, \beta})_m(z)} - \frac{\beta^2 - \beta - 1}{\beta + 1}
\]
or

\begin{equation}
(2.10) \quad \frac{1 + w(z)}{1 - w(z)} = \frac{1 + \sum_{n=1}^{m} A_n z^n + \frac{\beta^2}{\beta + 1} \sum_{n=m+1}^{\infty} A_n z^n}{1 + \sum_{n=1}^{m} A_n z^n}.
\end{equation}

From (2.10), we obtain

\begin{equation}
\begin{aligned}
\frac{1}{1 + w(z)} &= \frac{\beta^2}{\beta + 1} \sum_{n=m+1}^{\infty} A_n z^n \\
\frac{1}{1 - w(z)} &= 2 + 2 \sum_{n=1}^{m} A_n z^n + \frac{\beta^2}{\beta + 1} \sum_{n=m+1}^{\infty} A_n z^n
\end{aligned}
\end{equation}

and

\begin{equation}
|w(z)| < \frac{\beta^2}{\beta + 1} \sum_{n=m+1}^{\infty} A_n.
\end{equation}

The inequality $|w(z)| < 1$ holds true if and only if

\begin{equation}
\frac{2\beta^2}{\beta + 1} \sum_{n=m+1}^{\infty} A_n \leq 2 - 2 \sum_{n=1}^{m} A_n
\end{equation}

which is equivalent to

\begin{equation}
(2.11) \quad \sum_{n=1}^{m} A_n + \frac{\beta^2}{\beta + 1} \sum_{n=m+1}^{\infty} A_n \leq 1.
\end{equation}

To prove (2.11), it suffices to show that its left-hand side is bounded above by

\begin{equation}
\frac{\beta^2}{\beta + 1} \sum_{n=1}^{\infty} A_n
\end{equation}

which is equivalent to

\begin{equation}
\frac{\beta^2 - \beta - 1}{\beta + 1} \sum_{n=1}^{m} A_n \geq 0.
\end{equation}

The last inequality holds true for $\beta \geq \frac{1 + \sqrt{5}}{2}$.

We use the same method to prove inequality (2.9). Consider the function $w(z)$ given by

\begin{equation}
1 + w(z) = \frac{\beta^2 + \beta + 1 \left( E_{\alpha,\beta} \right)_{m}(z) - \frac{\beta^2}{\beta + 1}}{\beta + 1}.
\end{equation}

From the last equality we obtain

\begin{equation}
\begin{aligned}
w(z) &= \frac{-\frac{\beta^2 + \beta + 1}{\beta + 1} \sum_{n=m+1}^{\infty} A_n z^n}{2 + 2 \sum_{n=1}^{m} A_n z^n - \frac{\beta^2 - \beta - 1}{\beta + 1} \sum_{n=m+1}^{\infty} A_n z^n}
\end{aligned}
\end{equation}
and
\[ |w(z)| < \frac{\beta^2 + \beta + 1}{\beta + 1} \frac{\infty}{n=m+1} A_n \]
\[ \frac{2 - 2 \sum_{n=1}^{m} A_n - \frac{\beta^2 - \beta - 1}{\beta + 1} \infty}{n=m+1} A_n. \]

Then, \(|w(z)| < 1\) if and only if
\[ (2.12) \quad \frac{\beta^2}{\beta + 1} \frac{\infty}{n=m+1} A_n + \frac{m}{n=1} A_n \leq 1. \]

Since the left-hand side of (2.12) is bounded above by
\[ \frac{\beta^2}{\beta + 1} \frac{\infty}{n=1} A_n \]
we have that the inequality (2.9) holds true. Now, the proof of our theorem is completed.

In the next theorem we consider ratios involving derivatives.

**Theorem 2.2.** Let \( \alpha \geq 1 \) and let \( \beta \geq \frac{3 + \sqrt{17}}{2} \). Then
\[ (2.13) \quad \Re \left\{ \frac{E_{\alpha,\beta}'(z)}{(E_{\alpha,\beta})_m(z)} \right\} \geq \frac{\beta^2 - 3\beta - 2}{\beta^2}, \quad z \in \mathcal{U} \]
and
\[ (2.14) \quad \Re \left\{ \frac{(E_{\alpha,\beta})'_m(z)}{E_{\alpha,\beta}'(z)} \right\} \geq \frac{\beta^2}{\beta^2 + 3\beta + 2}, \quad z \in \mathcal{U}. \]

**Proof.** From (2.2) we have
\[ 1 + \sum_{n=1}^{\infty} (n + 1)A_n \leq \frac{\beta^2 + 3\beta + 2}{\beta^2}, \quad \text{where} \quad A_n = \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)}, \quad n \in \mathbb{N}. \]
The above inequality is equivalent to
\[ \frac{\beta^2}{3\beta + 2} \sum_{n=1}^{\infty} (n + 1)A_n \leq 1. \]
To prove (2.13), define the function \( w(z) \) by
\[ \frac{1 + w(z)}{1 - w(z)} = \frac{\beta^2}{3\beta + 2} \frac{E_{\alpha,\beta}'(z)}{(E_{\alpha,\beta})'_m(z)} - \frac{\beta^2 - 3\beta - 2}{3\beta + 2} \]
which gives
\[ w(z) = \frac{\frac{\beta^2}{3\beta + 2} \sum_{n=m+1}^{\infty} (n + 1)A_n z^n}{2 + 2 \sum_{n=1}^{m} (n + 1)A_n z^n + \frac{\beta^2}{3\beta + 2} \sum_{n=m+1}^{\infty} (n + 1)A_n z^n}. \]
and

\[ |w(z)| < \frac{\beta^2}{\beta^2 + 3\beta + 2} \sum_{n=m+1}^{\infty} (n + 1)A_n \]

The condition \(|w(z)| < 1\) holds true if and only if

\[ \sum_{n=1}^{m} (n + 1)A_n + \beta^2 \sum_{n=m+1}^{\infty} (n + 1)A_n \leq 1. \quad (2.15) \]

The left-hand side of \((2.15)\) is bounded above by

\[ \frac{\beta^2}{\beta^2 + 3\beta + 2} \sum_{n=1}^{m} (n + 1)A_n \]

if \(\frac{\beta^2 - 3\beta - 2}{\beta^2 + 3\beta + 2} \sum_{n=1}^{m} (n + 1)A_n \geq 0\)

which holds true for \(\beta \geq \frac{3 + \sqrt{17}}{2}\).

The proof of \((2.14)\) follows the same pattern. Consider the function \(w(z)\) given by

\[ 1 + w(z) = \frac{\beta^2 + 3\beta + 2}{\beta^2 + 3\beta + 2} \left( \frac{E_{\alpha,\beta}''(z)}{E_{\alpha,\beta}'(z)} \right) - \frac{\beta^2}{3\beta + 2} \]

\[ = \frac{1 + \sum_{n=1}^{m} (n + 1)A_n z^n - \beta^2 \sum_{n=m+1}^{\infty} (n + 1)A_n z^n}{1 + \sum_{n=1}^{\infty} (n + 1)A_n z^n}. \quad (2.16) \]

From \((2.16)\), we can write

\[ w(z) = \frac{-\beta^2 + 3\beta + 2}{3\beta + 2} \sum_{n=m+1}^{\infty} (n + 1)A_n z^n \]

\[ 2 + 2 \sum_{n=1}^{m} (n + 1)A_n z^n - \beta^2 \sum_{n=m+1}^{\infty} (n + 1)A_n z^n \]

and

\[ |w(z)| < \frac{\beta^2}{\beta^2 + 3\beta + 2} \sum_{n=m+1}^{\infty} (n + 1)A_n \]

\[ 2 - 2 \sum_{n=1}^{m} (n + 1)A_n - \beta^2 \sum_{n=m+1}^{\infty} (n + 1)A_n \]

The last inequality implies that \(|w(z)| < 1\) if and only if

\[ \frac{2\beta^2}{3\beta + 2} \sum_{n=m+1}^{\infty} (n + 1)A_n \leq 2 - 2 \sum_{n=1}^{m} (n + 1)A_n \]

or equivalently

\[ \sum_{n=1}^{m} (n + 1)A_n + \frac{\beta^2}{3\beta + 2} \sum_{n=m+1}^{\infty} (n + 1)A_n \leq 1. \quad (2.17) \]
It remains to show that the left-hand side of (2.17) is bounded above by
\[
\frac{\beta^2}{3\beta + 2} \sum_{n=1}^{\infty} (n + 1)A_n.
\]
This is equivalent to
\[
\frac{\beta^2 - 3\beta - 2}{3\beta + 2} \sum_{n=1}^{m} (n + 1)A_n \geq 0 \text{ which holds true for } \beta \geq \frac{3 + \sqrt{17}}{2}.
\]
Now, the proof of our theorem is completed. \(\square\)

3. Examples

In this section we give several examples which illustrate our theorems.

A result involving the functions \(E_{2,2}(z)\) and \(E_{2,3}(z)\), defined by (1.5), can be obtained from Theorem 2.1 by taking \(m = 0, \alpha = 2, \beta = 2\) and \(m = 0, \alpha = 2, \beta = 3\), respectively.

Corollary 3.1. The following inequalities hold true:
\[
\Re \left\{ \frac{\sinh(\sqrt{z})}{\sqrt{z}} \right\} \geq \frac{1}{4} = 0.25 \quad \Re \left\{ \frac{\sqrt{z}}{\sinh(\sqrt{z})} \right\} \geq \frac{4}{7} \approx 0.57
\]
and
\[
\Re \left\{ \frac{\cosh(\sqrt{z}) - 1}{z} \right\} \geq \frac{5}{18} \approx 0.28 \quad \Re \left\{ \frac{z}{\cosh(\sqrt{z}) - 1} \right\} \geq \frac{18}{13} \approx 1.38.
\]

Setting \(m = 0, \alpha = 2\) and \(\beta = 4\) in Theorem 2.1 and Theorem 2.2 respectively, we obtain the next result involving the function \(E_{2,4}(z)\), defined by (1.5), and its derivative.

Corollary 3.2. The following inequalities hold true:
\[
\Re \left\{ \frac{\sinh(\sqrt{z}) - \sqrt{z}}{z \sqrt{z}} \right\} \geq \frac{11}{96} \approx 0.11 \quad \Re \left\{ \frac{z \sqrt{z}}{\sinh(\sqrt{z}) - \sqrt{z}} \right\} \geq \frac{32}{7} \approx 4.57
\]
and
\[
\Re \left\{ \frac{\sqrt{z} \cosh(\sqrt{z}) - \sinh(\sqrt{z})}{z \sqrt{z}} \right\} \geq \frac{1}{24} \approx 0.04 \quad \Re \left\{ \frac{z \sqrt{z}}{\sqrt{z} \cosh(\sqrt{z}) - \sinh(\sqrt{z})} \right\} \geq \frac{8}{5} = 1.6.
\]

Remark 3.1. If we consider \(m = 0\) in inequality (2.13), we obtain \(\Re \left\{ E'_{\alpha,\beta}(z) \right\} > 0\). In view of Noshiro-Warschawski Theorem (see [5]), we have that the normalized Mittag-Leffler function is univalent in \(U\) for \(\alpha \geq 1\) and \(\beta \geq \frac{3 + \sqrt{17}}{2}\).

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