THE SUPPORT THEOREM FOR THE SINGLE RADIUS SPHERICAL MEAN TRANSFORM

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ABSTRACT. Let $f \in L^p(\mathbb{R}^n)$ and $R > 0$. The transform is considered that integrates the function $f$ over (almost) all spheres of radius $R$ in $\mathbb{R}^n$. This operator is known to be non-injective (as one can see by taking Fourier transform). However, the counterexamples that can be easily constructed using Bessel functions of the 1st kind, only belong to $L^p$ if $p > 2n/(n-1)$. It has been shown previously by S. Thangavelu that for $p$ not exceeding the critical number $2n/(n-1)$, the transform is indeed injective.

In this article, the support theorem is proven that strengthens this injectivity result. Namely, if $K$ is a convex bounded domain in $\mathbb{R}^n$, the index $p$ is not above $2n/(n-1)$, and (almost) all the integrals of $f$ over spheres of radius $R$ not intersecting $K$ are equal to zero, then $f$ is supported in the closure of the domain $K$.

In fact, convexity in this case is too strong a condition, and the result holds for any what we call an $R$-convex domain.

1. Introduction

We consider the transform acting on functions defined on $\mathbb{R}^n$ by integrating them over all spheres of a fixed radius $R > 0$. Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then such integrals exist for almost every center. One can easily construct examples of non-injectivity of this transform at least for some values of $p$ (see the proof of Theorem 1 below for details). However, such constructions, which use Bessel functions of the 1st kind, work only when $p > 2n/(n-1)$. And indeed, it was shown by S. Thangavelu [21] that for $p \leq 2n/(n-1)$ the transform is injective. In this article, we prove a stronger statement (comparable to S. Helgason’s “hole” support theorem [10, Theorem 2.6 and Corollary 2.8] for the Radon transform):

Theorem 1. Let $K$ be the closure of a bounded convex domain in $\mathbb{R}^n$ with $n > 1$, a function $f$ belong to $L^p(\mathbb{R}^n)$ with $p \leq 2n/(n-1)$,
and $R > 0$. If the integrals of $f$ over almost all spheres of radius $R$ contained in $\mathbb{R}^n \setminus K$ are equal to zero, then $f$ is compactly supported and its support is contained in $K$.

This conclusion does not hold for $p > 2n/(n - 1)$.

It is interesting to notice the appearance of the same critical power $2n/(n - 1)$ in a similar situation, where however the set of spheres of integration is defined differently: one allows arbitrary radii of the spheres, but restricts the set of their centers to the points of a closed hypersurface $S \subset \mathbb{R}^n$ only. It is shown in [1] that this transform is injective when $p \leq 2n/(n - 1)$ and injectivity fails otherwise, for instance when $S$ itself is a sphere.

Convexity is too strong condition in this case. The statement holds for a larger class of domains that is natural for the problem under the consideration.

**Definition 2.** Let $R$ be a positive number. A bounded closed domain $K \subset \mathbb{R}^n$ is said to be $R$-convex, if

1. Its complement $CK := \mathbb{R}^n \setminus K$ is the union of all closed balls $B \in CK$ of radius $R$.
2. The set of centers of all such balls is connected.

**Theorem 3.** The statement of Theorem 1 holds for $R$-convex bounded domains $K$.

Theorem 1 is proven in the next section. In the following section, an auxiliary local result is established in Theorem 10. In the next section, Theorem 3 is derived from Theorems 1 and 10. The paper ends with the remarks and acknowledgments sections.

## 2. Proof of Theorem 1

We start with the hardest part of the proof, when $p \leq 2n/(n - 1)$. Let $f \in L^p(\mathbb{R}^n)$, $p \in [1, 2n/(n - 1)]$ be such that

\[
\int_{|\omega|=1} f(y + R\omega)d\sigma(\omega) = 0
\]

for almost all $y \in \mathbb{R}^n$ such that $\text{dist}(y, K) > R$, where $d\sigma(\omega)$ is the standard surface area measure on the unit sphere in $\mathbb{R}^n$. We need to show that then $f(x) = 0$ for almost all $x \notin K$.

Since $K$, being a closed bounded convex domain, is the intersection of all balls it is contained within, it is sufficient to prove the statement when $K$ is a ball. Rescaling and shifting, we can assume without loss of generality that $K$ is the unit ball $B(0,1)$ centered at the origin.
Convolving with small support smooth radial functions, one reduces the problem to the case when \( f \) is infinitely differentiable and, moreover, all its derivatives belong to the same space \( L^p \) as \( f \) itself.

Consider for each \( m \in \mathbb{Z}^+ \) an orthonormal basis \( Y_l^m, 1 \leq l \leq d(m) \) of the space of all spherical harmonics of degree \( m \) in \( \mathbb{R}^n \) (the natural representation of the group \( O(n) \) in this space is irreducible). Then function \( f \) can be expanded into the Fourier series with respect to spherical harmonics as follows:

\[
(2) \quad f(x) = \sum_{m,l} f_{m,l}(|x|) Y_l^m(\theta),
\]

where \( \theta = \frac{x}{|x|} \) and

\[
(3) \quad f_{m,l}(|x|) = \int_{\theta \in S} f(|x|\theta) Y_l^m(\theta) d\theta.
\]

Due to the obvious rotational invariance of the problem, each term \( f_{m,l}(|x|) Y_l^m(\theta) \) of the series also has the corresponding spherical integrals (1) vanishing. Since clearly \( f_m \) belongs to the same \( L^p \)-space that \( f \) does, it is sufficient to prove the statement of the theorem for the functions of the form

\[
(4) \quad f(|x|) Y_l^m(\theta)
\]

only, where, as before, \( \theta = \frac{x}{|x|} \). Hence, we will assume (4) from now on.

Let \( \delta_R(x) \) be the delta function supported on the sphere of radius \( R \) centered at the origin. Then condition (1) can be rewritten as follows:

\[
(5) \quad h(x) := (f \ast \delta_R)(x) = 0 \text{ for } |x| > R + 1,
\]

where the star \( \ast \) denotes the \( n \)-dimensional convolution. Considering \( f(x) \) as a tempered distribution, one can pass to Fourier images in the left hand side of (5) to get

\[
(6) \quad \widehat{h}(\xi) = \widehat{f(\xi)} \widehat{\delta_R(\xi)}, \xi \in \mathbb{R}^n.
\]

Notice that due to (5), function \( h := f \ast \delta_R \) is compactly supported (with the support in the ball of radius \( R+1 \)) and smooth, and thus the standard Paley-Wiener theorem applies [20]. Therefore, the Fourier transform \( \widehat{h}(\xi) \) of \( h \) is an entire function satisfying for any \( N > 0 \) the estimate:

\[
(7) \quad |\widehat{h}(\xi)| \leq C_N (1 + |\xi|)^{-N} e^{-(R+1)|\Im \xi|}.
\]

We also recall that \( \widehat{\delta_R(\xi)} \) coincides, up to a constant factor, with \( j_{(n-2)/2}(R|\xi|) \), where \( j_p \) is the so called normalized or spherical Bessel
function [18]:\[ j_p(\lambda) = \frac{2^p \Gamma(p + 1) J_p(\lambda)}{\lambda^p}. \]

Here we use the standard notation $J_p(\lambda)$ for Bessel functions of the first kind.

Due to (6), we have
\[ \hat{h}(\xi) = \text{const} j_{(n-2)/2}(R|\xi|) \hat{f}(\xi). \]

We can now explain the strategy of the proof. The claim we are proving is equivalent to $\hat{f}(\xi)$ being an entire function of the following Paley-Wiener class:
\[ |\hat{f}(\xi)| \leq C_N(1 + |\xi|)^{-N} e^{R|3\xi|} \]
(notice the exponent $R$ in (10) instead of $R+1$ present in (7)). Taking into account (9), this task will be achieved, if we could show that:

1. The distribution $\hat{f}$ does not have any delta-type terms supported at zeros of $j_{(n-2)/2}(R|\xi|)$, and thus $\hat{f}$ can be obtained by dividing $\hat{h}$ by $j_{(n-2)/2}(R|\xi|)$.
2. This ratio is entire, i.e. $\hat{h}$ in fact vanishes at zeros of $j_{(n-2)/2}(R|\xi|)$.
3. The estimate (10) holds, which due to (7) requires one to get an estimate from below for $j_{(n-2)/2}(R|\xi|)$ that would eliminate the unnecessary $+1$ in $R+1$ in (7).

We will deal with these steps in the reverse order. The last one is achieved by the following simple statement:

**Lemma 4.** (e.g., [4, Lemma 6] or [2, Lemma 4]) On the entire complex plane, except for a disk $S_0$ centered at the origin and a countable number of disks $S_k$ of radii $\pi/6$ centered at points $\pi(k + \frac{2^{\nu+3}}{4})$, one has
\[ |J_\nu(z)| \geq C e^{|\text{Im} z|/\sqrt{|z|}}, \quad C > 0. \]

In order to handle the other two issues, we need to do some preparations.

The following lemma allows one to represent spherical means as volume integrals.

**Lemma 5.** Let $\lambda_0 > 0$ satisfy
\[ j_{(n-2)/2}(R\lambda_0) = 0. \]
Then the spherical mean $h = \delta_R * f$ can be represented as
\[ h = \text{const}(\Delta + \lambda_0^2)(f * \Psi_R) \]
where
\[ \Psi(x) = j_{(n-2)/2}(\lambda_0|x|) \chi_R(x) \]
and \( \chi_R \) is the characteristic function of the ball of radius \( R \) centered at the origin.

Proof. Indeed, this follows easily from Stokes formula. Denoting by \( B(x, R) \) and \( S(x, R) \) the ball and sphere centered at \( x \) and of radius \( R \), one gets
\[ \int_{B(x, R)} \{[\Delta + \lambda_0^2]f(v)j_{(n-2)/2}(\lambda_0|x-v|) - f(v)[(\Delta + \lambda_0^2)j_{(n-2)/2}(\lambda_0|x-v|)]dv \] \[ = \int_{S(x, R)} \{f(v)\frac{dj_{(n-2)/2}}{dr}(R\lambda_0) - \frac{df}{dr}(v)j_{(n-2)/2}(R\lambda_0)\}dA(v). \] (13)

Here \( r = |x| \) and \( \frac{\partial}{\partial r} \) is the external normal derivative on the sphere \( |x| = t \).

We now take into account that, according to our choice of \( \lambda_0 \), the Bessel function \( j_{(n-2)/2}(\lambda_0u) \) satisfies the following two equalities:
\[ j_{(n-2)/2}(R\lambda_0) = 0 \]
and
\[ (\Delta + \lambda_0^2)j_{(n-2)/2}(\lambda_0|y|) = 0. \]

Also, due to the simplicity of zeros of \( j_{(n-2)/2} \),
\[ j'_{(n-2)/2}(R\lambda_0) \neq 0. \]

These features, combined with (13), prove the statement of the lemma. \( \square \)

Lemma 6. Let \( f(x) = \sum_{l=1}^{d(m)} f_l(r)Y_i^m(\theta), \ x = r\theta, |\theta| = 1. \) Then for any radial compactly supported continuous function \( \psi \) the convolution \( F = \psi * f \) has the similar representation \( F(x) = \sum_{l=1}^{d(m)} F_l(r)Y_i^m(\theta). \)

Proof. The convolution operator \( f \rightarrow \psi * f \) is rotationally invariant. Indeed:
\[ (\psi * f)(x) = \int f(y)\psi(|x-y|)dy, \]
and thus for any rotation \( T \) and the rotated function \( f_T(x) = f(Tx) \) one has:
\[ (\psi * f_T)(x) = \int f(Ty)\psi(|x-y|)dy = \int f(Ty)\psi(|Tx - Ty|)dy \]
\[ = \int f(y)\psi(|Tx - y|)dy = (\psi * f)_T(x). \]

This implies that the convolution preserves the subspaces of harmonics of a fixed degree, which proves the lemma. \( \square \)
End of the proof of Theorem 1

According to our strategy, the next step is to prove that \( \hat{f} \) is an entire function.

Due to (9), outside of zeros of \( j_{(n-2)/2}(R|\xi|) \), one has

\[
\hat{f}(\xi) = \text{const} \frac{\hat{h}(\xi)}{j_{(n-2)/2}(R|\xi|)}.
\]

Notice that the denominator is an entire function of the variable \( \xi \in \mathbb{C}^n \), since \( j_{\nu}(u) \) is an even entire function of the real argument \( u \) and hence is an entire function of \( u^2 \). The next lemma shows that the numerator in (14) vanishes at (simple) zeros of the denominator. Therefore, the zeros cancel, and the ratio in the right hand side of (14) is an entire function, as needed.

**Lemma 7.** For any \( \lambda_0 \) such that \( j_{(n-2)/2}(R\lambda_0) = 0 \), function \( \hat{h}(\xi) \) vanishes on the complex quadric

\[ Q = \{ \xi \in \mathbb{C}^n \mid \xi_1^2 + \cdots + \xi_n^2 = \lambda_0^2 \} \]

*Proof. Since \( \lambda_0 \neq 0 \), the quadric \( Q \) is irreducible and has a maximal dimension intersection with the real subspace. Thus, due to analytic continuation, it suffices to check vanishing of the entire function \( \hat{h}(\xi) \) on the intersection \( Q \cap \mathbb{R}^n \), i.e. on the sphere \( |\xi| = \lambda_0 \) in \( \mathbb{R}^n \).

Since, by assumption, \( h \) vanishes outside of the unit ball, we can write

\[
\hat{h}(\xi) = \int_{|x| \leq t} h(x)e^{-i\xi \cdot x}dx,
\]

for arbitrary \( t > 1 \).

Let us substitute for \( h \) the representation (12). Then, by Stokes’ formula

\[
\hat{h}(\xi) = \text{const} \int_{|x| \leq t} (\Delta + \lambda_0^2)(f \ast \Psi_R)e^{-i\xi \cdot x}dx
\]

\[
= \text{const} \int_{|x| \leq t} (f \ast \Psi_R)(\Delta + \lambda_0^2)e^{-i\xi \cdot x}dx
\]

\[
+ \text{const} \int_{|x| = t} \left( \frac{\partial}{\partial r}(f \ast \Psi_R)e^{-i\xi \cdot x} - (f \ast \Psi_R)\frac{\partial}{\partial r}e^{-i\xi \cdot x} \right)dA(x).
\]

Since \( |\xi| = \lambda_0 \), the exponential function \( e^{-i\xi \cdot x} \) is annihilated by the operator \( \Delta + \lambda_0^2 \). Therefore, \( \hat{h}(\xi) \) is expressed by the surface term alone:

\[
\hat{h}(\xi) = \text{const} \int_{|x| = t} \left( \frac{\partial}{\partial r}(f \ast \Psi_R)e^{-i\xi \cdot x} - (f \ast \Psi_R)\frac{\partial}{\partial r}e^{-i\xi \cdot x} \right)dA(x).
\]
Here, as before, \( r = |x| \) and \( \frac{\partial}{\partial r} \) is the external normal derivative on the sphere \( |x| = t \).

The function \( \Psi_f \) is radial and thus, due to Lemma 6, the convolution \( F := f * \Psi \) has the form \( F(x) = \sum_{l=1}^{d(m)} F_l(r)Y_l(\theta) \). Projection of the exponential function \( e^{-i\xi \cdot x} \) on the space of spherical harmonics of degree \( m \) can be given in terms of Bessel functions (see [20, Theorem 3.10]), which leads to the following formula:

\[
(16) \quad \hat{h}(\xi) = c_m\lambda_0^m t^{n+m-1} \sum_{l=1}^{d(m)} \left( F'_l(t) j_{n+2m-2} (\lambda_0 t) - F_l(t) j'_{n+2m-2} (\lambda_0 t) \right).
\]

In what follows, the estimate is done the same way for any \( l \) between 1 and \( d(m) \), so we will drop the sum over \( l \) and work with a single term.

In order to prove that \( h(\xi) = 0 \), it suffices to check that the expression in the right hand side tends to 0 as \( t \to \infty \). This can now be easily shown using the \( L^p \) condition on \( F \) and the known estimate for Bessel functions:

\[
(17) \quad j_{n/2m-1}(t), j'_{n/2m-1}(t) = 0(t^{-\frac{n+2m-1}{2}}), \quad t \to \infty.
\]

Indeed, let us pick \( t_0 > t \) and average both sides of (16) for \( t \) from \( t_0 \) to \( 2t_0 \):

\[
(18) \quad \frac{1}{2t_0 - t_0} \int_{t_0}^{2t_0} |F'_l(t) j_{n/2m-1}(\lambda_0 t) - F_l(t) j'_{n/2m-1}(\lambda_0 t)|^{n+m-1} dt.
\]

Let \( A(t) := |F'_l(t)| + |F_l(t)| \). From (17) and (18) one obtains:

\[
(19) \quad |\hat{h}(\xi)| \leq \frac{c_m}{t_0} \int_{t_0}^{2t_0} A(t) t^{n-1} \frac{1}{2} dt = \frac{c_m}{t_0} \int_{t_0}^{2t_0} A(t) t^{\frac{n-1}{p} - t^{\frac{n-1}{q} - \frac{2n-2}{p}} dt.
\]

Functions \( F_l(r) \) and \( F'_l(r) \) are the radial parts of functions in \( L^p(\mathbb{R}^n) \) and therefore belong to \( L^p((0, \infty), r^{n-1} dr) \). So is the function \( A(r) \).

We now apply Hölder inequality to (19) to get

\[
(20) \quad |\hat{h}(\xi)| \leq \frac{c_m}{t_0} \left( \int_{t_0}^{2t_0} A^p(t) t^{n-1} \frac{1}{p} dt \right)^{\frac{1}{p}} \left( \int_{t_0}^{2t_0} t^{\frac{n-1}{2} - t^{\frac{n-1}{q} - \frac{2n-2}{p}}} \frac{1}{q} dt \right)^{\frac{1}{q}},
\]

where the index \( q \) dual to \( p \) is introduced in the standard manner: \( p^{-1} + q^{-1} = 1 \), or \( q = p/(p-1) \). The second factor in (20) can be easily
computed:
\[
\left( \int_{t_0}^{2t_0} t^{\frac{n-1}{2} - \frac{n-1}{p}} dt \right)^\frac{1}{q} = C t_0^{\frac{n-1}{2} - \frac{n}{p} + 1},
\]
and hence (20) leads to the estimate:
\[
(21) \quad |\hat{h}(\xi)| \leq c_m \|A\|_{L^p((t_0, 2t_0), t^{n-1} dt)} t_0^{\frac{n-1}{2} - \frac{n}{p} + 1}.
\]
The condition \( p \leq 2n/(n-1) \) shows that \((n-1)/2 - n/p \leq 0\), and hence the last factor in (21) is bounded. Since the condition that \( F \in L^p(\mathbb{R}^n) \) implies
\[
\|A\|_{L^p((t_0, 2t_0), t^{n-1} dt)} \to 0 \text{ when } t_0 \to \infty,
\]
this shows the required equality \( \hat{h}(\xi) = 0 \).

**Corollary 8.** The function
\[
\Phi(\xi) := \frac{\hat{h}(\xi)}{j_{(n-2)/2}(R|\xi|)}
\]
is entire of the Paley-Wiener class (10).

The only remaining step is to show that the same statement as in Corollary 8 applies to the function \( \hat{f}(\xi) \):

**Lemma 9.** The Fourier transform \( \hat{f}(\xi) \) is an entire function of the Paley-Wiener class (10).

**Proof.** Corollary 8 says that the right hand side in (14) is an entire function of the Paley-Wiener class (10). The Lemma (and thus the Theorem 1) will be proven if we show that in fact \( \hat{f} = \Phi \).

The tempered distribution \( \hat{f}(\xi), \xi \in \mathbb{R}^n \) coincides with \( \Phi(\xi) \) outside of the union of the discrete set of spheres \( S_k \) defined by (simple) zeros of Bessel function:
\[
S_k = \{ \xi \in \mathbb{R}^n : \xi_1^2 + \ldots + \xi_n^2 = \lambda_k^2 \},
\]
where
\[
j_{(n-2)/2}(\lambda_k R) = 0.
\]
This means that \( \hat{f} \) can differ from \( \Phi \) only by terms supported on these spheres:
\[
\hat{f}(\xi) = \Phi(\xi) + \sum_k c_k(\xi) \delta(|\xi| - |\lambda_k|).
\]
Observe that since \( f(x) = \sum_{l=1}^{d(m)} f_l(r)Y_l^m(\theta) \), the coefficients \( c_k(\xi) \) must have the similar form

\[
c_k(\xi) = \sum_{l=1}^{d(m)} a_{k,l} Y_l^m(\eta), a_k = \text{const}, \xi = |\xi|, |\eta| = 1.
\]

Our aim is to show that there are no such distributional terms in \( \hat{f} \), i.e. all coefficients \( a_{k,l} \) must vanish.

Fix \( k \) and choose a positive number \( \varepsilon \) so small that the spherical layer \( L := \{ \lambda_k - \varepsilon \leq |\xi| \leq \lambda_k + \varepsilon \} \) containing \( S_k \), does not contain other spheres \( S_m \) with \( m \neq k \).

Let now \( \psi \) be a radial function from the Schwartz class, whose Fourier transform vanishes outside the spherical layer \( L \) and such that \( \hat{\psi}(\xi) = 1, \xi \in S_k \). We can now localize the sphere \( S_k \) in the spectrum of \( f \) by considering the convolution \( g = \psi * f \). By construction,

\[
\hat{g}(\xi) = \Phi(\xi)\hat{\psi}(\xi) + c_k Y_l(\eta)(\delta(|\xi| - \lambda_k)).
\]

The first term is in the Schwartz class, while the second one is, up to a constant factor, Fourier transform of Bessel function \( j_{n/2+1-\ell}|x| \) and therefore after convolving with \( \psi \) we have

\[
g(x) = \psi * \varphi + \text{const} a_k j_{n/2+1-\ell}(|x|),
\]

where \( \psi \) is inverse Fourier transform of \( \Psi \) and hence is also a Schwartz function. By the condition for \( \hat{f} \) and by the construction, the functions \( g \) and \( \psi * \varphi \) belong to \( L^p(\mathbb{R}^n) \) with \( p < 2n/n - 1 \) while Bessel function \( j_{n/2+1-\ell} \) is not in this class. Therefore the coefficient \( a_k = 0 \).

Thus, there are no \( \delta \)-function terms in \( \hat{f} \) and \( \hat{f} = \Psi \) is an entire function in \( \mathbb{C}^n \) satisfying, as it was explained above, Paley-Wiener estimate which implies \( \text{supp} f \subset \overline{B}(0, R) \).

Let now \( p > 2n/(n-1) \). Then one can find a counterexample, where even compactness of support of \( f \) cannot be guaranteed, using Bessel functions. The function

\[
f(x) = |x|^{1-n/2} J_{n/2-1}(\lambda|x|)
\]

provides such a counterexample (due to L. Zalcman). Indeed, consider the following spherical mean mapping \( M \):

\[
M g(x, t) = \frac{1}{\omega_n} \int_{S(0,1)} g(x + t\theta)d\theta,
\]
that averages any continuous function \( g \) over spheres. It is well known that \( f \) defined in (22) satisfies the following functional identity:

\[
Mf(x, t) = \text{const} f(x)f(t).
\]  

Thus, if \( \lambda \) is chosen as a zero of \( J_{n/2-1} \), (23) implies that the spherical means of \( f(x) \) over all spheres of radius 1 are equal to zero. Also, asymptotic behavior of the Bessel functions shows that \( f \in L^q(\mathbb{R}^n) \) for any \( q > 2n/(n-1) \). This completes the proof of Theorem 1. □

3. A LOCAL RESULT

In order to extend the statement of Theorem 1 to all \( R \)-convex domains, we need to establish first the following local theorem, which in some particular cases as well as in different related versions has been established previously [12, 23].

**Theorem 10.** Let \( f(x) \) be an infinitely differentiable function in the ball \( B(0, R + \varepsilon) \subset \mathbb{R}^n \) and its spherical averages over all spheres of radius \( R \) contained in this ball are equal to zero. If \( f \) vanishes in the ball \( B(0, R) \), then it vanishes in the whole ball \( B(0, R + \varepsilon) \).

**Proof.** Without loss of generality, we can assume that \( R = 1 \). As in [12, 23], we will exploit relations between spherical and plane waves [12, Ch. 1 and 4].

For a function \( u(x) \) on \( \mathbb{R}^n \) we will denote by \( u^\#(x) \) its radialization

\[
u^\#(x) := \int_{k \in SO(n)} u(kx)dk,
\]

where \( dk \) is the normalized Haar measure on \( SO(n) \). Function \( u^\#(x) \) is clearly radial and thus is a function of a single variable \( |x| \). Abusing notations, we will write \( u^\#(x) = u^\#(|x|) \).

The following simple statement (which we will prove for completeness) will be useful:

**Lemma 11.** Let \( u(x), v(x) \) be continuous functions on \( \mathbb{R}^n \) and \( v(x) \) be radial and compactly supported. Then

\[
(u * v)^\# = u^\# * v.
\]

**Proof.** Indeed,

\[
(u * v)^\#(x) = \int_{SO(n)} \int_{\mathbb{R}^n} u(kx - y)v(y)dydk.
\]
Changing the variables in the $y$-integral from $y$ to $ky$, using the rotational invariance of $v$, and changing the order of integration, one gets

$$(u * v)^\#(x) = \int_{\mathbb{R}^n} \left( \int_{SO(n)} u(kx - ky)dk \right) v(y)dy = (u^\# * v)(x).$$

This proves the lemma. □

In particular, the convolution of two radial functions is radial.

The relation between plane waves and radial functions that we need is contained in the following result of [12, Ch.4, formulas (4.13) and (4.16)]:

**Lemma 12.** [12] Let $e \in \mathbb{R}^n$ and $g(p)$ be a function of a scalar variable $p \in \mathbb{R}$. We consider the ridge function $g(\langle x, e \rangle)$ and its radialization $g(\langle \cdot, e \rangle)^\#$, which we will identify with a function $f(r)$ of scalar variable $r$. Then the relations between the functions $f(r)$ and $g(p)$ are provided by the following Abel type transforms:

\begin{align*}
(24) \quad f(r) &= (\mathcal{A}g)(r) := 2\frac{(n-1)}{\omega_n} r^{2-n} \int_0^r (r^2 - s^2)^{\frac{n-3}{2}} g(p)dp \\
\text{and} \quad (25) \quad g(p) &= (\mathcal{A}^{-1}f)(p) := \frac{2^{n-1}p}{(n-2)!} \left( \frac{d}{dp^2} \right)^{n-1} \int_0^p r^{n-1}(p^2 - r^2)^{\frac{n-4}{2}} f(r)dr.
\end{align*}

We can now derive the following useful relation:

**Lemma 13.** Let $\delta_S$ denote the normalized measure supported by the unit sphere. Let also $g(p)$ be a continuous function on $\mathbb{R}$. Then

$$\mathcal{A}(g * \delta_S)(p) = \text{const} \mathcal{A}(g *_1 (1 - |p|^2)^{\frac{n-3}{2}}),$$

where $*_1$ denotes one-dimensional convolution and $\mathcal{A}g$ in the left hand side is considered as a radial function on $\mathbb{R}^n$, i.e. $\mathcal{A}g(|x|)$ for $x \in \mathbb{R}^n$.

**Proof.** Since $\mathcal{A}g = g(\langle \cdot, e \rangle)^\#$, the left hand side, according to Lemma 11 can be rewritten as

$$(g(\langle \cdot, e \rangle) * \delta_S)^\#.$$

It is straightforward to check that

$$(g(\langle \cdot, e \rangle) * \delta_S)(x)$$
is equal to the ridge function
\[
(g(p) *_1 (1 - |p|^2)^{\frac{n-3}{2}})|_{p=\langle x,e \rangle}.
\]
Now radialization of this ridge function gives the right hand side expression in (26).

We can complete now the proof of our theorem. We start with the case of a radial function, which we write as \( f(|x|) \) for some function \( f(r) \) of a single variable. By the assumption, \( (f * \delta_S)(x) = 0 \) for \( |x| < \varepsilon \). Then (26) implies that
\[
(g *_{\mathbb{R}^1} (1 - |p|^2)^{\frac{n-3}{2}})(s) = 0
\]
for \( s \leq \varepsilon \), where \( g(p) := (A^{-1}f)(p) \). It follows from (25) that the condition \( f(x) = 0 \) for \( |x| \leq 1 \) implies \( g(p) = 0 \) for \( |p| \leq 1 \), and therefore (27) can be rewritten as
\[
\int_1^{1+\varepsilon} g(p)(1 - |p-s|^2)^{\frac{n-3}{2}} dp = 0, \quad s \leq \varepsilon.
\]

Thus the Titchmarsh convolution theorem [22] (see also [11, Theorem 4.3.3], [16, Lecture 16], or [27, Ch. VI]) implies that \( g(p) = 0 \) for \( 1 \leq p \leq 1 + \varepsilon \). Since \( f = Ag \), the relation (24) leads to the conclusion that \( f(x) = 0 \) for \( |x| \leq 1 + \varepsilon \). This proves the statement of the theorem in the radial case.

It remains now to pass from radial to non-radial functions. To this end, we observe that the \( C^\infty \) function \( f \) has zero integrals over all spheres of radius 1 centered in the open ball \( B(0, \varepsilon) \). Thus, all its partial derivatives \( D^\alpha f \) have the same property. Since this vanishing condition is invariant under rotations, according to Lemma 11, it also holds for radializations \( (D^\alpha f)^\# \). Since the theorem is already proven for radial functions, all these radializations vanish, i.e.
\[
(28) \quad \int_{|x|=r} D^\alpha f(x) dA(x) = 0
\]
for all \( 0 < r < 1 + \varepsilon \).

Let us prove now that on each sphere \( |x| = t \) for \( t \in [0, 1 + \varepsilon) \) the function \( f \), along with all its derivatives, is orthogonal to all monomials. This, due to the Weierstrass Theorem will imply the needed property that \( f = 0 \) in \( B(0, 1 + \varepsilon) \).

We prove this claim by induction with respect to the degree of the monomial. For a zero degree monomial, the claim is true, due to (28).
Suppose that
\begin{equation}
\int_{|x|=t} p(x)D^\alpha f(x)dA(x) = 0,
\end{equation}
for all monomials \( p(x) \) of degree not exceeding \( N \) and all multiindices \( \alpha \). Integrating both sides of this identity with respect to \( t \) from 0 to any \( r < 1 + \varepsilon \) yields
\[ \int_{|x|\leq r} p(x)D^\alpha f(x)dx = 0. \]
We now replace the multiindex \( \alpha \) with \( \beta = \alpha + \delta_j \), where the multiindex \( \delta_j \) has 1 in \( j \)th place and 0s otherwise and write
\begin{equation}
(30) \quad p(x)D^\beta f(x) = \frac{\partial}{\partial x_j} (p(x)D^\alpha f(x)) - \frac{\partial p}{\partial x_j} D^\alpha f(x).
\end{equation}
The second term on the right does not contribute to the integral over the ball \( |x| \leq r \), due to the induction assumption, and thus identity (29), where \( \alpha \) is replaced by \( \beta \), reduces to
\[ \int_{|x|\leq r} \frac{\partial}{\partial x_j} (p(x)D^\alpha f(x))dx = 0. \]
Using Stokes’ formula, we obtain
\[ \int_{|x|=r} x_j p(x)D^\alpha f(x)dA(x) = 0. \]
Since \( j = 1, \ldots, n \) is arbitrary, we conclude that identity (29) holds for all monomials of degree \( N + 1 \). This completes the proof of theorem.

\[ \square \]

4. Proof of Theorem 3

We can now prove Theorem 3 that extends Theorem 1 to the case of \( R \)-convex domains. So, we assume that \( K \subset \mathbb{R}^n \) is a closed \( R \)-convex domain and a function \( f \in L^p(\mathbb{R}^n) \) with \( p \leq 2n/(n-1) \) is such that its spherical means over almost every sphere of radius \( R \) not intersecting \( K \) is zero. As it has been shown before, one can assume, without restriction of generality, that the function is smooth. Consider the set \( C \) of centers of all balls of radius \( K \) not intersecting \( K \). Due to \( R \)-convexity of \( K \), this set is connected, and the union of the corresponding balls covers the whole complement of \( K \). Consider also the subset \( C_f \subset C \) of such centers \( x \) that \( f \) vanishes in the ball \( B(x, R) \). If we establish that in fact \( C_f = C \), this will prove the theorem.
Theorem 1 implies that $f = 0$ outside the convex hull of $K$. Thus, in particular, the set $C_f$ is non-empty, since it contains all points $x$ with a sufficiently large norm. It is also obvious that, due to continuity of $f$, the set $C_f$ is relatively closed in $C$. Let us now prove that it is also relatively open. Due to connectedness of $C$, this will imply that $C_f = C$ and thus $f = 0$ in the whole complement of $K$, which is the statement of the theorem.

Indeed, let $x \in C_f$. This means that $f = 0$ in $B(x, R)$. There exists a positive $\varepsilon$ such that the ball $B(x, R + \varepsilon)$ is inside the complement of $K$. Then the function $f$ satisfies the conditions of Theorem 10 in $B(x, R + \varepsilon)$, and thus $f = 0$ in $B(x, R + \varepsilon)$. In particular, $f$ vanishes in the ball $B(y, R)$ for any $y \in \mathbb{R}^n$ such that $|y - x| < \varepsilon$. This means that all such points $y$ belong to $C_f$, and hence $C_f$ is open. This finishes the proof of the theorem. □

5. Remarks

(1) In V. Volchkov’s book [23] a result similar to Theorem 1 was proven when integration is done over balls rather than spheres. The proof does not seem to translate to the case of spheres. In particular, a comparable result for the case of integrals over spheres was proven in [23] only when two values of radii are allowed.

(2) The statement of Theorem 10 holds also for functions of finite smoothness, if one knows that spherical averages of $f$ vanish for all spheres of radius $r < R$ (rather than $r = R$ as in Theorem 10).

(3) The local Theorem 10 has been established previously in some particular cases, as well as in different related versions in [12, 23]. For instance, one can check that the consideration in the second section of [12, Ch. VI] provides such a result in $3D$, although the local formulation is not stated there explicitly. In [23], a theorem similar to Theorem 10 is proven for the case of integrals over balls rather than spheres.

(4) The use of Lemma 4 could have been avoided. Indeed, it is known (e.g., [16, Lecture 16]) that estimates from below of the kind provided in the lemma hold for any function of Paley-Wiener class, not just for Bessel function of the lemma. In fact, this claim is equivalent to Titchmarsh convolution theorem quoted before [22] (see also [11, Theorem 4.3.3], [16, Lecture 16], or [27, Ch. VI]). We, however, decided to use the statement easiest to verify.
Moreover, one could do with a much cruder statement than the one of Lemma 4 or Titchmarsh theorem, which would not give in Theorem 1 that the support is inside $K$, but just its compactness. Then the local Theorem 10 would still bootstrap this to the correct statements of Theorems 1 and 3.

ACKNOWLEDGMENTS

The work of the first author was performed when he was visiting Texas A&M University and was partially supported by the ISF (Israel Science Foundation) grant 688/08. The second author was partially supported by the NSF grant DMS 0604778 and by the KAUST grant KUS-CI-016-04. The authors express their gratitude to ISF, NSF, Texas A&M University, and KAUST for the support. They also are grateful to Y. Lyubarskii for useful information.

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