A $\frac{1}{N}$ NASH EQUILIBRIUM FOR NON-LINEAR MARKOV GAMES OF MEAN-FIELD-TYPE ON FINITE STATE SPACE

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ABSTRACT. We investigate mean field games for players, who are weakly coupled via their empirical measure. To this end we investigate time-dependent pure jump type propagators over a finite space in the framework of non-linear Markov processes. We show that the individual optimal strategy results from a consistent coupling of an optimal control problem with a forward non-autonomous dynamics which leads to the well-known McKean-Vlasov dynamics in the limit as the number $N$ of players goes to infinity. The case where one player has an individual preference different to the ones of the remaining players is also covered. The limiting system represents a $\frac{1}{N}$-Nash Equilibrium for the approximating system of $N$ players.

1. Introduction

Mean field game theory is a type of dynamic Game theory where the agents are coupled with each other by their individual dynamics and their empirical mean where the objective for each agent is a function of not only her own preference and decision but also of the decisions of the other players. All in all it is a mathematical tool which aims to describe a control problem with a large number $N$ of agents by letting $N$ tend to infinity where the impact of the individual decisions of the other agents is becoming extremely weak compared to the overall impact as $N$ increases. The limiting model emerges from the fact that each agent constructs her strategy from her own state and from the state of the empirical mean of an infinite number of co-agents of hers which is called the mean field approach.

The mean field approach has been independently developed by J.-M. Lasry and P.-L. Lions in a series of papers see [12] and the references therein using nonlinear PDE’s and by M. Huang, P. Caines, Malhamé, see [14] [15] in the setting of stochastic processes, see also [6].

The investigations in this work are carried out in the framework of non-linear Markovian propagators, respectively time inhomogeneous nonlinear Feller processes, which was developed by Vassili Kolokoltsov [18] [19]. We focus on propagators related to processes of pure jump type with finite intensity measure on a finite set

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\[ \mathbb{X} = \{1, \ldots, k\}, \ k \in \mathbb{N}, \] which can be identified with the possible decisions of the players, respectively with the (financial) positions in the financial instruments of a finite market. Our starting point of the so-called closed-loop construction including an optimal control is the following forward Kolmogorov equation written in the weak form:

\[ \left( \frac{\partial f}{\partial s} - \mathcal{A}[s, \rho, u_s]f_s, \mu_s \right) = 0, \quad 0 \leq t < s \leq T \quad (1.1) \]

\[ f(t, x) = \Phi(x), \quad x \in \hat{\mathbb{X}} \]

where \( f \in C([0, T] \times \hat{\mathbb{X}}) \) is the set of continuous functions on \( \hat{\mathbb{X}} := \cup_{N=1}^{\infty} \mathbb{X}^N, \) \( \rho \in \mathcal{M}([0, T] \times \hat{\mathbb{X}}), \) is the set of functions on \([0, T] \times \hat{\mathbb{X}}\) taking values in the set of finite measures, and the generator \( \mathcal{A} \) is of the form

\[ \mathcal{A}[s, \rho, u_s]f(s, x) = \sum_{i=1}^{\lfloor x \rfloor} A_i^i[s, \rho, u_s]f(s, x) = \sum_{i=1}^{\lfloor x \rfloor} \int_{\mathbb{X}} (f_{i'}(s, y) - f_{i'}(s, x_i)) \nu(s, x_i, \rho, u_s, dy). \quad (1.2) \]

Here we introduced the notation \( f_{i'} \) in order to describe that \( A_i^i \) acts on the component \( x_i \) only, in fact \( f_{i'}(x_i) = f_{x_i'}(x_i) \) where \( x_i' \in \mathbb{X}^{\lfloor x \rfloor - 1} \) is derived by removing the variable corresponding to the \( i^{th} \) agent from \( x \). The length of the vectors describing the number of players is denoted by \( |x| \).

**Hypothesis A**

We assume \( \nu(s, x_i, \rho, u, dy) \) to be a bounded stochastic kernel in \( \mathbb{X} \) vanishing for \( x_i = y \) as for Lévy measures, continuous in \( s \) and Lipschitz continuous in the parameters \( \rho = \rho_s, u = u_s \). The choice of the space \( \mathbb{X} \) means that the integral is a sum.

Moreover, the parameters of the generator \( \mathcal{A} \) are subject to the assumptions that the control law \( u \in U \), later referred to as the set of admissible controls, satisfies \( u_t \in U \) for all \( t \in [0, T] \) and a set \( U \) specified in Section 4 and \( \rho \) is a Lipschitz continuous measure valued function on \([0, T]\) such that for all \( s \in [t, T] \) we have \( \rho_s \in \mathcal{P}_\delta(\hat{\mathbb{X}}), \) the linear hull of Dirac probability measures. The natural domain of the operator \( \mathcal{D}(\mathcal{A}[t, \rho_t, u]) \subset C(\hat{\mathbb{X}}) \) and \( C(\mathbb{X}) \) is the set of continuous functions on the space \( \hat{\mathbb{X}} \) which will be restricted according to technical constraints.

For the sake of completeness we add that \( \mathcal{P}_\delta(\mathbb{X}) \subset \mathcal{M}(\mathbb{X}), \) with \( \mathcal{M}(\mathbb{X}) \) being the set of finite measures on \( \mathbb{X} \). An analogous statement holds when replacing \( \mathbb{X} \) by \( \hat{\mathbb{X}} \). We also introduce the set of continuous measure valued functions \( C([0, T], \mathcal{M}(\mathbb{X})), \) respectively, \( C_\mu = \{ \rho \in C([0, T], \mathcal{M}(\mathbb{X})) \mid \rho_0 = \mu \} \) for later purposes. We mention that \( \mathcal{M}(\mathbb{X}) \) is isomorphic to \( \mathbb{R}^k. \) The sensitivity analysis is carried out on an open neighbourhood \( M \) of the origin.

The construction involves a mean-field type limit consistent with a given optimal control problem. A key role in the theory of measure-valued limits of interacting
particle systems plays the injection from the equivalence class \( S\hat{X} \) of vectors \( x \in \hat{X} \), which are identical up to a permutation of players, into the set of point measures on \( X \) defined by

\[
x = (j_1, \ldots, j_N) \rightarrow \frac{1}{N}(\delta_{j_1} + \cdots + \delta_{j_N}) =: \frac{1}{N}\delta_x
\]

more precisely the bijection between \( S\hat{X}^N \) and the set of \( N \)-point measures \( \mathbf{P}_N(X) \) in \( X \) of the above form for arbitrary \( N \in \mathbb{N} \). Implicitly we identify the set of \( N \)-point measures \( \mathbf{P}_N(X) \) in \( X \) of the above form with the set of Dirac measure in \( S\hat{X}^N \). For each \( N \in \mathbb{N} \) the space \( C^{sym}(\hat{X}^N) \) of \( \mathbb{R} \)-valued continuous functions which are invariant under component-wise permutations of their arguments is equivalent with the space of \( \mathbb{R} \)-valued continuous functions \( C(S\hat{X}^N) \). Moreover, \( C^{sym}(\hat{X}) \) is a core of the operator \( \mathfrak{A}[s, \rho_s, u_s] \). The restriction \( \mathfrak{A}_N[s, \rho_s, u_s] \) of the operator \( \mathfrak{A}[s, \rho_s, u_s] \) to \( C(S\hat{X}^N) \) generates a time inhomogeneous Feller process \( X^N(s) = (X^N_1(s), \ldots, X^N_N(s)) \), \( s \in [t, T] \) in \( \hat{X}^N \), see e.g. [7], [16].

Motivated by similar previous results from mathematical physics and since all agents are assumed to be subject to the same equation, the generator \( \mathfrak{A} \) being of special form (1.2), one investigates the dynamics for each individual agent given by the time inhomogeneous Markov process \( X := X^N_i, 1 \leq i \leq N, \) in \( X \). As the number \( N \) of agents tends to infinity the individual dynamics depend on their own distributions only which is also phrased that "the individual dynamics in the mean field model separate as \( N \rightarrow \infty \)" by physicists.

By assumption the objective for each of them is to find the value function

\[
V(t, x) = \sup_u \mathbb{E}_x \left[ \int_t^T J(s, X(s), \rho^N_t, u_s) \, ds + V^T(X(T), \rho_T) \right] \quad (1.3)
\]

on \([0, T] \times X\), i.e. to maximize her expected payoff, over a suitable class of admissible control processes \( u = \{u(s, X(s)) \mid 0 \leq s \leq T\} \in \mathcal{U} \), where the cost function \( J : [0, T] \times X \times \mathbf{P}_N(X) \times U \) and the terminal cost function \( V^T : X \times \mathbf{P}_N(X) \rightarrow \mathbb{R} \), as well as the final time \( T \) are given.

An explicit expression for the value function can be derived by dynamic programming as solution of the HJB equation (4.2). For admissible control processes the HJB equation is well posed and the resulting optimal control function \( \hat{u} \) is unique for given start value \( x \in X \) and given \( \rho \). The so-called kinetic equation which leads to the nonlinear Markov process in the sense of V. Kolokoltsov with control law \( u^\infty \) given by the limit of the approximating \( N \)-meanfield control problems is derived by making an Ansatz motivated by the weak form of the one player evolution with an intrinsic choice of the parameter \( \rho \) in the generator, namely

\[
\frac{d}{dt}(g, \mu_t) = (A[t, \rho_t, u^\infty]g, \mu_t)|_{\mu = \rho} \quad (1.4)
\]

for arbitrary \( g \in C(X) \) and arbitrary finite measures \( \mu_s \in M(X) \) which are differentiable in \( s \in [0, T] \).

\(^1\)The precise meaning of this equation in the setting of this paper is given in Theorem 3.2
Finally MFG consistency is said to hold if the fixed measure valued function \( \rho \) in the objective function can be replaced by the empirical measures

\[
\mu_s^N = \frac{1}{N}(\delta_{X_1^N(s)} + \ldots + \delta_{X_N^N(s)}), \quad t \leq s \leq T,
\]

of the underlying process while well-posedness of the optimal control problem and uniqueness of the optimal control parameter are conserved - as a result of what could be called a closed loop construction.

A fix point argument reveals that solving the optimal control problem including MFG consistency and performing the mean-field limit are interchangeable which concludes the proof and the article. The construction exhibits the order of convergence to be \( \frac{1}{N} \) which establishes the \( \frac{1}{N} \)-Nash equilibrium.

### 2. Pure Jump Markov Processes

In the entire section let us assume that \( \mathfrak{A} \) decomposes as given in (1.2). Hence we consider a single player \( i \). In order to simplify notations we even drop the index \( i \), i.e. \( \mathbf{A} := \mathbf{A}^i \) whence \( x = x_i \in \mathbb{X} \). At the same time the values of all agents different from \( i \) are kept fix, i.e. \( f(s,i) := f_i(s,i), 1 \leq i \leq N \), with the notation introduced above.

For \( \mathbb{X} = \{1, \ldots, k\} \) real valued functions on \( \mathbb{X} \) can be represented as \( k \)-vectors. We assume that \( \mathbf{A} \) is a time inhomogeneous \( Q \)-matrix on \( \mathbb{X} \), i.e.

- \( -\infty < \nu_j(s,i,\rho_s,u_s) \leq 0 \) for all \( i \in \mathbb{X} \);
- \( \nu_j(s,i,\rho_s,u_s) \geq 0 \) for \( i \neq j, i,j \in \mathbb{X} \);
- \( \sum_j \nu_j(s,i,\rho_s,u_s) = 0 \) for all \( i \).

Thus we have \( \nu_i(s,\rho_s,u_s) := \nu_i(s,i,\rho_s,u_s) = -\sum_{i \neq j} \nu_j(s,i,\rho_s,u_s) \) since the row sum vanishes. We find using matrix form

\[
\mathbf{A}[s,\rho_s,u_s]f(s) = \begin{pmatrix}
\nu_1(s,\rho_s,u_s) & \nu_2(s,1,\rho_s,u_s) & \ldots & \nu_k(s,1,\rho_s,u_s) \\
\nu_1(s,2,\rho_s,u_s) & \nu_2(s,\rho_s,u_s) & \ldots & \nu_k(s,2,\rho_s,u_s) \\
\vdots & \vdots & \ddots & \vdots \\
\nu_1(s,k,\rho_s,u_s) & \nu_2(s,k,\rho_s,u_s) & \ldots & \nu_k(s,\rho_s,u_s)
\end{pmatrix}f(s)
\] (2.1)

where \( f(s) = (f(s,1),\ldots,f(s,k))^t \in C(\mathbb{X}) \).

**Hypothesis B**

For the rest of the paper we complement the assumptions made on the domains of the variables and parameters of operator \( \mathbf{A} \) by regularity conditions, namely: We assume that \( \nu \) is bounded in \( x \in \mathbb{X} \) and \( \rho = \rho_t \in \mathbb{R}^k \) uniformly in \( t \in [0,T] \) and \( u = u_t \in U \). It is assumed to be uniformly Lipschitz continuous in the parameters \( \rho \) and \( u \) and continuous in \( t \). Moreover, the partial derivatives of \( \nabla_{\rho,\mu}(t,x,\rho,u_t) \) exist in \( C_0(\mathbb{R}^k) \) as functions of \( \rho \) and are uniformly Lipschitz continuous uniformly in the other variables. Since \( A \) is a finite dimensional matrix valued function of time we have the following:

\[
|A f| \leq C \| f \|, \quad (2.2)
\]
$f \in \mathbb{R}^k$ which means that the matrix valued function $A$ constitutes a bounded linear operator.

**Proposition 2.1.** Let $M$ be an open subset in $B_1(0) \subset \mathbb{R}^k$ and $U \subset \mathbb{R}^m$ a convex bounded open control set. Assume that the matrix valued function $\nu(s, \rho, u)$ in (2.1) is continuous in $t > t_0$ for some $t_0 \in \mathbb{R}$, respectively of type $C^q$, for possibly different $q \geq 1$ in the parameters $\rho \in M$ and $u \in U$ as well as the initial condition $\Phi \in M$. Then so is the unique linear flow induced by $\nu$.

The proof is a direct consequence of the results on linear ordinary differential equations in [25] CH. XVIII section 4 and [1] CH 2.

The solution to the Kolmogorov equation given by the matrix (2.1) possesses the cocycle property, which replaces the semi group property of autonomous systems see [29], [3]. Intimately related to the cocycle property is the notion of a propagator to be found e.g. in physics publications or in works of Reed and Simon respectively V. Kolokoltsov. A family of mappings $U_{t,s}$, $t \leq s \leq T$, in a set $S$ is called a (forward) propagator (resp. backward propagator) in $S$ if $U_{t,t} = id_S$ and the following iteration equation holds:

$$U_{t,r} = U_{t,s}U_{s,r}$$

for $t \leq s \leq r$. Here $U_{t,s}U_{s,r}$ is to be interpreted as the iteration of mappings. For linear propagators or evolutions it means the application of linear operators, see [2] and [7]. Even more specific, for the linear equation (1.1) with generator $A$ in (2.1) the solution $f$ is of the form $f(s) = \Lambda(t,s)\cdot f(t)$ with $k \times k$-matrix

$$\Lambda(t,s) = \text{Exp} \int_t^s A(\tau, \cdot) \, d\tau, \quad (2.3)$$

where $\text{Exp}$ is the well known replacement for the exponential function in case of non-commutativity, and $\cdot$ stands for all other parameters. The iteration becomes the matrix multiplication:

$$\Lambda(t,s)\Lambda(s,r) = \text{Exp}[\int_t^r A(\tau, \cdot) \, d\tau] = \Lambda(t,r) \quad (2.4)$$

where $t \leq s \leq r$.

**Remark 2.2.** The matrix valued functions $\Lambda(s,r,\cdot)$ constitute bounded linear operators.

A propagator $U_{t,r}$ of bounded linear operators on a Banach space $B(S)$, where $S$ is a Polish space, is called strongly continuous if $U_{t,r}$ depends strongly continuously on $t$ and $r$. The family $\{\Lambda(t,s) \mid 0 \leq t \leq s \leq T\}$ defines a positive, strongly continuous linear propagator or evolution on the set of Euclidean $k$-vectors which trivially coincides with the set of continuous real valued functions on the discrete set $\mathbb{X}$. Strong continuity follows on the diagonal in time since the generator $A$ is bounded by rewriting the definition of strong continuity in terms of the one of the generator where we apply the Euclidean norm in $\mathbb{R}^k$ and Remark 2.2. For arbitrary $i \in \mathbb{X}$ the mapping

$$f \rightarrow \sum_{j \in \mathbb{X}} \Lambda_{i,j}(t,s)\cdot f_j(t)$$
defines a positive continuous linear functional on $\mathbb{R}^k$. Moreover, for each $i \in X$, $\Lambda_{i,j}(t,s,x,\cdot)\delta_j$ constitutes a discrete kernel on $(X,\mathcal{P}(X))$, $\mathcal{P}$ being the power set of $X$, without having to consult the Riesz Markov theorem. Hence (2.4) is a particular type of Chapman Kolmogorov equation. We summarize our findings in the following

**Proposition 2.3.** For each $0 \leq t \leq s \leq T < \infty$ let $A(s,\cdot)$ be a $Q$-matrix in $X$. Set $U_{t,s} = \text{Texp}[\int_t^s A(r,\cdot) \, dr]$ then the family $\{U_{t,s} \mid t \geq 0\}$ has the following properties

1. The cocycle property holds, i.e. $U_{t,r} = U_{t,s}U_{s,r}$ for $0 \leq t \leq s \leq r \leq T$;
2. $U_{t,s}$ is the unique solution to the forward equation
   $$\frac{d}{ds}U_{t,s} = U_{t,s}A(s,\cdot), \quad U_{t,t} = \text{id}_X$$
3. $U_{t,s}$ is the unique solution to the backward equation
   $$\frac{d}{dt}U_{t,s} = -A(t,\cdot)U_{t,s}, \quad U_{s,s} = \text{id}_X$$
4. For $\ell = 0, 1, 2, \ldots$, we have
   $$\left[\frac{d^\ell}{ds^\ell}U_{t,s}\right]_{t=s} = A(t,\cdot)^\ell$$
   and analogously for the backward equation.

Properties i) – iii) have been dealt with above and iv) follows by direct calculation.

We now establish the connection between linear propagators or evolutions and Markov processes which we intend to use for solving the control problem. We adopt the notation in [2] to the time dependent case.

Let $(E, \mathcal{E})$ be a measurable space and $U_{t,r}$ an arbitrary linear propagator. Assume that $x \in E, E \in \mathcal{E}$. We say that $\{p_{t,r}(x,E) := (U_{t,r}(x,E) E = \int_E f(y)p(t,x,r,dy), 0 \leq t \leq r < \infty\}$, is a normal transition family if

1. the maps $x \to p(t,x,r,E)$ are measurable for each $E \in \mathcal{E}$;
2. the Chapman Kolmogorov equation holds;
3. $p(t,x,r,\cdot)$ is a probability measure on $\mathcal{E}$.

For the finite measurable space $(X,\mathcal{P}(X))$ measurability in $x$ is trivially satisfied and the cocycle property together with the existence of a kernel reveal the Chapman Kolmogorov equation. The Markov property follows from the following proposition which is a straightforward adaption from [27].

**Proposition 2.4.** A time inhomogeneous matrix $Q(t), 0 \leq t \leq s \leq T$, on a finite set $I$ is a $Q$-matrix if and only if $p(t,s) = \text{Texp}\int_t^s Q(\tau) \, d\tau$ is a stochastic matrix for all $0 \leq t \leq s \leq T$.

We thus have:

**Lemma 2.5.** The family $\Lambda$ is a normal transition family.

As done in [16] the notion of a projective family for time inhomogeneous transition probabilities see [2] Theorem 3.1.7 can be generalized to general measurable spaces $(E, \mathcal{E})$ and trivially also to finite sets $A_0, \ldots, A_n \subset X, n \in \mathcal{N}$, and arbitrary
probability measures $\mu$ in $\mathcal{X}$. The existence of a process is then guaranteed by the Kolmogorov existence theorem. We only state the existence of a process in the following

**Proposition 2.6.** If $\{ p_{t,r}(x,A) := (A^{t+r}\chi_{A})(x) = \int_A f(y)p(t,x,r,dy), 0 \leq t \leq r < \infty \}$ is a normal transition family and $\mu$ is a fixed probability measure on the finite measurable space $(\mathcal{X}, \mathcal{P}(\mathcal{X}))$, then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P}_{\mu})$, a filtration $(\mathcal{F}_t, t \geq 0)$ and a Markov process $(X_t, t \geq 0)$ on that space such that:

- $\mathbb{P} [ X(r) \in A \mid X(t) = x ] = p_{t,r}(x,A)$ for each $0 \leq t \leq r$, $x \in \mathcal{X}$, $A \in \mathcal{P}(\mathcal{X})$.
- $X(0)$ has law $\mu$.

Since $\mathcal{X}$ is compact the proof follows the line of arguments in Ethier Kurtz Theorem and [2] Theorem 3.1.7. In Section 5 we shall see that the solution of the kinetic equation (3.2) is the limit of the linear $N$-mean field evolutions as $N$ tends to infinity which arises when restricting the generator $\mathfrak{A}$ in (1.1) to $C_{sym}(\mathcal{X}^N)$ and replacing the parameter $\rho$ by the empirical distribution. The corresponding adjoint is denoted by $\mathfrak{A}_N^\ast$. In order to prove the mean field limit we need to unify spaces. This is possible since the factor spaces $S\mathcal{X}$ and the spaces of $N$-point measures $\mathbb{P}_N^N(\mathcal{X})$ as well as the corresponding larger spaces $\mathcal{M}(\mathcal{X})$ and $\mathbb{H}^k$ can be identified. We consider existence and uniqueness and the sensitivity analysis of the solutions of the Kolmogorov equation and the optimal control problems for the $N$-player on the larger space $\mathbb{H}^k$.

Consequently we replace or identify in a first step $\mathfrak{A}_N^\ast$ with the linear operator

$$\hat{\mathfrak{A}}^N[t, \delta_x, u]F(\delta_x) := \mathfrak{A}_N^\ast[t, \delta_x, u]f(x)$$

(2.5)

on $C(\mathbb{P}_N^N(\mathcal{X}))$ which in more detail reads:

$$\hat{\mathfrak{A}}^N[t, \delta_x, u]F(\delta_x) = \sum_{i=1}^{N} \mathfrak{A}^{*i}[t, \delta_x, u]F(\delta_x)$$

$$= \sum_{l=1}^{k} \sum_{\ell=1}^{k} n_{\ell} \nu_{\delta_l}(t, \delta_l, u) \left[ F \left( \sum_{l=1}^{k} n_{\ell} \delta_l + (\delta_l + \delta_{\ell}) - \delta_\ell \right) - F \left( \sum_{l=1}^{k} n_{\ell} \delta_l \right) \right]$$

where $n_{\ell}$ describes how often the value $l$ appears and $n_{\ell}$ is specified by the kernel $\nu$.

In a second step, specific to the case of a finite set $\mathcal{X}$, we identify $\{ \delta_1, \ldots, \delta_k \}$ with the standard basis $\{ e_1, \ldots, e_k \}$ in $\mathbb{H}^k$ to find that $x \in S\mathcal{X}$ with $|x| = N$ corresponds to $x^N = \sum_{\ell=1}^{k} n_{\ell} e_{\ell} \in \mathbb{H}^k$ with $\sum_{\ell=1}^{k} n_{\ell} = N$ and:

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$^2$Where $\nu_{\delta_l}(t, \delta_l, u)$ is the transpose of the matrix $\nu_{\delta_l}(t, \delta_l, u)$. 

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\[ \hat{\mathcal{A}}_i^N[t,x^N,u]F(x^N) := \sum_{i=1}^{N} A^* i[t,x^N,u]F(x^N) \]  
(2.6)

\[ = \sum_{\ell=1}^{k} \sum_{\ell'=1}^{k} n_{\ell,\ell'} F \left( \sum_{\ell=1}^{k} n_{\ell e_{\ell}} + (e_{\ell} + e_{\ell'}) - e_{\ell} - F \left( \sum_{\ell=1}^{k} n_{\ell e_{\ell}} \right) \right) \]

In the sequel we give an alternative representation of the linear operator \( \hat{\mathcal{A}}_i^N \). For practical reasons we introduce a scaling parameter \( h \in \mathbb{R}^+ \). For differentiable functions \( F \) on \( \mathcal{M}(\mathcal{X}) \) a variational derivative \( \frac{dF(Y)}{dY(x)} \) of \( F \) is the Gateaux derivative \( D_{\delta_x} \) of \( F \) in the direction of \( \delta_x, x \in \mathcal{X} \). Since \( \mathcal{M}(\mathcal{X}) \), and \( \mathbb{R}^k \) are isomorphic we are able to work with directional derivatives \( \partial_x \) on \( \mathbb{R}^k \).

**Proposition 2.7.** Assume that \( \nu \) satisfies the Hypothesis A and let \( F \in C^2(\mathbb{R}^2) \) then the operator \( \hat{\mathcal{A}}_i^N[t,hx^N,u] \), where \( t \in [0,T] \), \( x^N = \sum_{i=1}^{N} n_i e_i \in \mathbb{R}^k \) such that \( \sum_{\ell=1}^{k} n_{\ell} = N, u \in U \), and \( h \in \mathbb{R}^+ \), has the representation:

\[ \hat{\mathcal{A}}_i^N[t,hx^N,u]F(hx^N) = h \left( A[t,hx^N,u]D_{h\partial_x}F(hx), e_{\ell} \right) + h^2 \int_0^1 ds(1-s) \]

\[ \times \sum_{\ell,\ell',\ell''} \nu_{\ell,\ell',\ell''} (t, e_{\ell}, e_{\ell'}, e_{\ell''}) \left( \partial_{h\partial_x}F(hx^N + hs(e_{\ell} + e_{\ell'} - e_{\ell''})) \right) . \]

Due to the fact that the agents are indistinguishable the sum with respect to \( \ell \) becomes a factor \( N \)

**Proof.** For \( Y \) and \( Y + \zeta \) such that the whole line \( \{ Y + \theta \zeta \mid 0 \leq \theta \leq 1 \} \) is in \( M \subset \mathbb{R}^k \) and \( F \in C^2(M) \) the Taylor theorem gives the following representation, see [26], [18] Cor 13 of Lemma 12.6.1:

\[ F(Y + \zeta) - F(Y) = \left( \frac{\partial F(Y)}{\partial \zeta}, \zeta \right) + \int_0^1 ds(1-s) \left( \frac{\partial^2 F(Y)}{\partial \zeta^2}, \zeta \otimes \zeta \right) . \]

Inserting the Taylor expansion of order 2 into (2.6) under the integral for the choice \( Y = hx^N \) and \( \zeta = h((e_{\ell} + e_{\ell'}) - e_{\ell}) \) finishes the proof.

We gladly anticipate that the first term coincides with the generator of the Koopman propagator constructed in Proposition (3.4) for the choice \( h = \frac{1}{N} \). \( \square \)

Since the jump type operator \( \hat{\mathcal{A}}_i^N[t,x^N,u] \) is a linear combination of the operator \( A \) all properties previously investigated are conserved, in particular existence and sensitivity results, the fact that \( \hat{\mathcal{A}}_i^N[t,x^N,u] \) generates a strongly continuous contraction propagator \( \psi^k_N \) and that there exists a corresponding Markov process.

### 3. Properties of the Non linear Evolution

In the sequel we investigate the nonlinear Kinetic equation which was motivated by the weak equation [18], namely

\[ \mu_t = A^*[t, \mu_t, u_t] \mu_t, \quad \mu_0 = \mu, \quad t \in [0,T] . \]  
(3.1)
For the finite set $X$ arbitrary spaces of real valued functions coincide with the set of real valued mappings on $X$ which is isomorphic to $\mathbb{R}^k$. Consequently the dual space of $C(X)$, the space of bounded measures $\mathbb{M}(X)$ is isomorphic to $\mathbb{R}^k$. When identifying $\mathbb{M}(X)$ and $\mathbb{R}^k$ the Kinetic equation becomes a nonlinear differential equation in $\mathbb{R}^k$:

$$\dot{x}_t = A^*x_t; u|x_t, \quad x_0 = x, \quad t \in [0, T].$$  \quad (3.2)$$

Under the conditions of Hypothesis A the subsequent theorem gives the existence of a corresponding flow which is continuously differentiable in all variables, parameters and initial conditions.

**Proposition 3.1.** Let $M$ be an open subset in $B_1(0) \subset \mathbb{R}^k$ and $U \subset \mathbb{R}^m$ a convex bounded open control set. Assume that the matrix valued function $\nu(s,x,u)$ in (2.1) is continuous in $t > t_0$ for some $t_0 \in \mathbb{R}$ of type $C^q$, for possibly different $q \geq 1$, in the variable $x \in M$ and the parameter $u \in U$ as well as the initial condition $x_0 \in M$. Then so is the unique nonlinear global flow $\alpha(t_0,t,x_0;u)$, $t_0 \leq t$, induced by $\nu(s,x,u)x$.

Since $\nu$ is bounded the vector valued function $\nu(s,x,u)x$ satisfies a uniform Lipschitz condition on $B_1(0) \subset \mathbb{R}^k$ and hence is bounded in $M$, moreover, the conditions of Theorems 2) and 7) as well Remarks 3) in [25] CH. XVIII and Theorem 2.9 in [1] CH 2 apply which finishes the proof. For the sake of intuition we add that the set $M$ is in fact a subset of the positive part of a simplex within $B_1(0)$. This automatically implies that the solution $x_t$ of (3.2) is Lipschitz continuous in the initial conditions:

**Corollary 3.2.** For all $x \in M \subset \mathbb{R}^k$ the unique solution to equation (3.2) given by Proposition 3.1 is Lipschitz continuous in the initial data i.e:

$$|\alpha(0,t,x,u) - \alpha(0,t,y,u)| \leq C(T) |x - y|$$  \quad (3.3)$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^k$.

We summarize our findings by concluding that the initial value problem (3.2) is well-posed.

**Definition 3.3.** Let $\beta(t,s)$, $0 \leq t \leq s \leq T$, be a nonsingular flow in a set $K$ of a given Banach space, then

$$(\Phi^{t,s}F)(b) := F(\beta(t,s,b)) \quad \beta_t = b \in K$$

defines a linear operator on $C(K)$ which we call the Koopman propagator with respect to $\beta(t,s)$.

The notion coincides with the Koopman operator in [24]. From the general theory, see [24], it follows that the Koopman propagator has the following properties:

- $\Phi^{t,s}$ is a linear propagator.
- $\Phi^{t,s}$ is a contraction on $C(K)$, i.e. $\|\Phi^{t,s}F\| < \|F\|_{C(K)}$ $\forall F \in C(K)$, where $\|\cdot\|$ denotes the norm in the Banach space. Being a contraction the Koopman propagator is bounded.
For $M$ as in Proposition 3.1 we introduce the set $C^1(M)$ of functionals $F = F(x)$ such that the gradient $\nabla_x F$ is continuous. This space becomes a Banach space when equipped with the norm

$$\|F\|_{C^1(M)} := \sup_{x \in K} |(\nabla_x F)(x)|$$

**Proposition 3.4.** Under the conditions given in Hypotheses A and B we have:

i) The time inhomogeneous global flow corresponding to the solution of the kinetic equation defines the time inhomogeneous Koopman propagator:

$$(\phi^{t,s}_F)(x) := F(\alpha(t, s, x, u)) \quad x \in \mathbb{R}^k. \quad (3.4)$$

ii) The generator of the Koopman propagator is defined by

$$A[t, x, u]F(x) = \sum_{i=1}^{k} \frac{\partial F}{\partial x_i} a_i[t, x, u] \quad (3.5)$$

where $a_i[t, x]$ corresponds to line $i$ of the matrix valued function $A[t, x]$.

iii) The Koopman propagator constitutes a strongly continuous family of bounded linear operators on $C^1(M)$.

**Proof.** For the sake of a more comprehensive notation we drop the control parameter $u$. First we shall prove that the global flow induced by the solution to the nonlinear kinetic equation is nonsingular.

i) By differentiating the right hand side of the nonlinear kinetic equation (3.2) we have:

$$\frac{\partial (A^*[t, x] x)}{\partial x}|_{x=x_t} = (A^*[t, x_t] + A^*[t, x]_{x=x_t}) \frac{\partial x_t}{\partial x}$$

If the time is sufficiently small then the above value dose not vanish and the corresponding determinant is positive [29] hence the global flow is a nonsingular transformation.

ii) Under the assumptions at the beginning of Section 2 $(\nabla_x A^*)(x)$ exists and for every $x_0 \in \mathbb{R}^k$ the solution $x_s = \alpha(t, s, x_0)$ exists for all $s \in [0, T]$. By inserting into the definition we find

$$\frac{(\phi^{t,s}_F)(x) - F(x)}{s-t} = \frac{F(\alpha(t, s, x)) - F(x)}{s-t} = \frac{F(x_s) - F(x)}{s-t} \quad (3.6)$$

where $x = x_t$ and $0 \leq t \leq s \leq T$. For $F \in C^1(M)$ with compact support the mean value theorem reveals

$$\frac{(\phi^{t,s}_F)(x) - F(x)}{s-t} = \sum_{i=1}^{k} F_{x_i}(x_0)x_\theta = \sum_{i=1}^{k} F_{x_i}(x_0)a_i^*[\theta, x]x_\theta \quad (3.7)$$

$$= \sum_{i=1}^{k} F_{x_i}(x(t, \theta, x_t))a_i^*[\theta, x]\alpha(t, \theta, x_t) \quad (3.8)$$

where $t \leq \theta \leq s$. Since the derivatives $F_{x_i}$ have compact support by introducing the flow $\alpha(t, s, x_t)$ given by the solution $x_s$ to the kinetic equation (3.2) with $x_t = x$ we obtain for $t \leq \theta \leq s$:

$$\lim_{(s-t) \to 0} F_{x_i}(\alpha(t, \theta, x_t)) \cdot a_i^*[t, x]\alpha(t, \theta, x_t) = F_{x_i}(x)a_i^*[t, x]x$$
uniformly \( \forall x \in M \) thus (3.7) has a strong limit in \( C^1(M) \) and the infinitesimal generator \( A \) is given by

\[
A[t, x]F(x) = \sum_{i=1}^{k} \frac{\partial F}{\partial x_i} a_i[t, x] x .
\]

iii) To show the strong continuity we insert the definition of the Koopman propagator and exploit the properties of the flow given by Proposition 3.1, i.e. we have:

\[
\lim_{(t_0, s_0) \to (t, s)} \| \phi^{t, s} F - \phi^{t_0, s_0} F \| = 0
\]

for every \( F \in C^1(M) \). The fact that the set of continuously differentiable functions with compact support form a dense subset of \( C^1(M) \) concludes the proof. \( \square \)

4. Controlled Jump Markov Process

In this subsection we shall describe the principle of dynamic programming for the finite state Markov Jump Process and the corresponding HJB equation. The optimal payoff is represented by the value function \( V: [0, T] \times X \times \mathbb{R}^k \times U \to \mathbb{R} \) starting at time \( t \) and position \( j \) is defined as:

\[
V(t, j, y) := \sup_{u \in U} \mathbb{E}_j \left[ \int_t^T J(s, X_s, y_s, u_s) \, ds + V_T(X_T, y_T) \right] .
\]

In order to guarantee a unique optimal control law, in the set of Lipschitz continuous functions as a function of the solutions, we confine to quadratic cost functions

\[
J(s, j, y, u) = \sum_{\ell=1}^{k} J_{j, \ell}(s, y) u_{\ell} - |u|^2 ,
\]

for \( J_{j, \ell} \in \mathbb{R} \), and \( s \in [0, T], j \in X, y \in M, u \in U \) see [30]. Moreover, for the set \( U \) of admissible controls we assume either that \( U \) is a bounded convex set with a smooth boundary. We emphasize that the assumptions of Hypotheses A and B hold even for this section.

The methodology in a standard setting reveals the HJB equation i.e. the following system of ordinary differential equations

\[
\frac{\partial V}{\partial t} + \max_u \left[ \sum_{\ell=1}^{k} J_{j, \ell}(t, y) u_{\ell} - |u|^2 + \sum_{\ell=1}^{k} \frac{\partial A_{j, \ell}(t, y)}{\partial y} u_{\ell} \right] = 0 .
\]

\( \textbf{Remark 4.1.} \) Due to Hypothesis B, \( \nu_j(t, i, y, u) \in C_0(\mathbb{R}^k) \) in the variable \( y \), the derivatives \( \frac{\partial A(t, y)}{\partial y} \) are bounded and continuous.

\* For any \( t \in [0, T] \), \( A(t, y) \) is \( C^1 \)-differentiable with respect to the vector \( y \), also there exist a constant \( c_1 \) such that:

\[
\sup_{(t,u)} \left| \frac{\partial A(t, y)}{\partial y} \right|_{\mathbb{R}^k} \leq c_1 |y|
\]

(4.3)
• \( J(t, x) \) is \( C^1 \)-differentiable with respect to the vector \( y \), also there exist a constant \( c_2 \) such that:

\[
\sup_{(t,u)} \left| \frac{\partial J[t,y,u]}{\partial y} \right|_{\mathbb{R}^k} \leq c_2 |y| \tag{4.4}
\]

**Theorem 4.2.** Assume that \( J_j,\ell(s, y) \) above is continuous in \( t \) and \( C^1 \)-differentiable in \( y \). Suppose that \( A[t,y,u] \) is as in (1.2). Then the Cauchy problem for the ordinary differential equation (4.2) is well posed for all terminal data \( V_T \in \mathbb{R}^k \), and the solution \( V_t \) is of class \( C^1 \) for all \( t \in [0,T] \).

The result is a direct consequence of Proposition 3.1. Now we study smooth dependence of the solution of the HJB equation (4.2) above when replacing the parameter \( y \in \mathbb{R}^k \) by a curve \( y(t), t \in [0,T] \), in \( \mathbb{R}^k \). We proceed by studying smooth dependence on the real parameter \( \alpha \).

Under the assumptions of the previous theorem and if we set

\[
H(t, V, \alpha) = \max_u \left[ \sum_{\ell=1}^k J_j,\ell(s, \alpha, y)u_\ell - |u|^2 + A[t, \alpha, y, u]V \right] \tag{4.5}
\]

we can see that \( H \) is continuous in \((t,V,\alpha)\) and \( C^1 \)-differentiable with respect to \((V,\alpha)\).

**Remark 4.3.** Since \( H \) is continuous then the maximum is achieved at a specific measurable \( \hat{u}(t) \) in particular

\[
H(t, V, \alpha) = \sum_{\ell=1}^k J_j,\ell(s, \alpha, y)\hat{u}_\ell - |\hat{u}|^2 + A[t, \alpha, y, \hat{u}]V
\]

and the solution to the ordinary differential equation (4.2) for each \( \alpha \) is:

\[
V(t, \alpha) = e^{\int_0^t A[s, \alpha, y, \hat{u}] ds} V_0 + \int_0^t \sum_{\ell=1}^k \left( J_j,\ell(s, \alpha, y)\hat{u}_\ell - |\hat{u}|^2 \right) e^{\int_s^t A[r, \alpha, y, \hat{u}] dr} ds.
\]

Inserting \( V \) into \( H \) gives that \( H \) is \( C^1 \)-differentiable with respect to \((V,\alpha)\) indeed.

**Theorem 4.4.** Suppose that \( J_j,\ell(s, \alpha, y) \) is \( C^1 \)-differentiable with respect to \( \alpha \), then the solution \( V(t, \alpha) \) is \( C^1 \)-differentiable with respect to \( \alpha \in \mathbb{R} \).

The result is a direct consequence of Proposition 3.1. In order to be able to apply the HJB methodology to the present mean field model we have to replace the parameter \( y \) by a curve \( y \).

\[
\frac{\partial V(t,y)}{\partial t} + \max_u \left[ \sum_{\ell=1}^k J_j,\ell(s, \alpha, y)u_\ell - |u|^2 + A[t, j, \alpha, y, u]V(t,j) \right] = 0, \tag{4.6}
\]

respectively

\[
H(t, j, V, y) = \max_u \left[ \sum_{\ell=1}^k J_j,\ell(s, \alpha, y)u_\ell - |u|^2 + A[t, j, \alpha, y, u]V(t,j) \right]. \tag{4.7}
\]
To insert $y$ in the value function we do the following procedures. For any $(t, x) \in [0, T] \times \mathbb{X}$ and $y^1, y^2 \in C([0, T], M)$ we define:

$$V(t, \alpha, x) := V(t, x, y^1 + \alpha(y^2 - y^1)), \quad \alpha \in [0, 1].$$  \hspace{1cm} (4.8)

Hence the smooth dependence of the solutions of the HJB equation on the functional vector valued parameter $y$ reduces to dependence on the real parameter $\alpha$, i.e. if the directional derivative $\partial_{y^2-y^1} V$ of $V(t, x, y)$ exists and is continuous we have:

$$V(t, x, y^2) - V(t, x, y^1) = \int_0^1 \partial_{y^2-y^1} V(t, x, y^1 + \alpha((y^2 - y^1))) d\alpha.$$  \hspace{1cm} (4.9)

We adopt the assumptions we made when we studied smooth dependence on the real parameter $y$.

**Theorem 4.5.** Under the previous conditions and assumptions A and B we have that for any $y \in C([0, T], \mathbb{R}^k)$ the solution of the equation (4.6) is Lipschitz continuous in $y$ uniformly, i.e. for $y^1, y^2 \in C([0, T], \mathbb{R}^k)$, there exist a constant $K \geq 0$ such that

$$\sup_{(t, x) \in [0, T] \times \mathbb{X}} |V(t, x, y^1) - V(t, x, y^2)| \leq K \sup_{t \in [0, T]} |y^1 - y^2|. \hspace{1cm} (4.9)$$

Now we would like to use the results in the previous sections for building the mean field game structure. We assume here that all agents are from the same class, where each agent wants to maximize her profit by finding the optimal strategy from the HJB equation. Now if the optimal strategy is reached at only one point $\hat{u}$ then one get the unique optimal control from the solution of the HJB equation. Let us denote the resulting unique optimal feedback control by

$$u_t = \Gamma(t, x, y_t).$$

Furthermore, if we assume that the resulting unique optimal control in (4.7) is continuous in $t$ and Lipschitz continuous in $(x, y)$ uniformly with respect to $t$.

**Theorem 4.6.** Under the assumptions of the previous theorems and results, given a trajectory $y \in C([0, T], \mathbb{R}^k)$ and a final payoff $V_T$ the control $\hat{u} = \Gamma(t, x, y_t)$ defined via equations (4.6) and (4.7), is Lipschitz continuous in $y$ i.e., for any $\eta, y \in C([0, T], \mathbb{R}^k)$

$$\Gamma(t, x, \eta_t) - \Gamma(t, x, y_t) \leq k_1 \sup_{s \in [0, T]} |\eta_s - y_s|_{\mathbb{R}^k}, \quad \forall t \in [0, T], x \in \mathbb{X}$$

**Proof.** The proof is done by combing the result from the previous theorem and the fact that the unique optimal control is Lipschitz continuous in $(x, y)$. \hfill \square

5. **Convergence of N-particle Approximations**

In Physics and Biology scaling limits and analyzing scaling limits are well established techniques which allow to focus on particular aspects of the system under consideration. Scaling empirical measures by a small parameter $h$ in such a way that the measure $h(\delta_{x_1} + \ldots + \delta_{x_N})$ remains finite when the number $N$ of particles or species tends to infinity and the individual contribution becomes negligible allows to treat the ensemble as continuously distributed.
Scaling $k^{th}$-order interactions by $h^{k-1}$ reflects the idea that they are more rare than $k-\ell$ order ones for $1 \leq \ell < k$ and makes them neither negligible nor overwhelming. This scaling transforms an arbitrary generator $\Lambda_k$ of a $k^{th}$-order interaction into

$$\Lambda_k^h F(h\delta_x) = h^{k-1} \sum_{I \subset \{1, \ldots, n\}, |I| = k} \int \mathbb{R}^k [F(h\delta_x - h\delta_{x_I} + h\delta_y) - F(h\delta_y)] \times P(x; dy)$$

with positive kernel $P(x; dy)$. The $N$-mean field limit is a law of large numbers for the first order interactions given by the $N$-mean field evolutions. For the special case of pure jump type $N$-mean field evolutions, cf. 1.2, we prove weak convergence to the solution of the kinetic equation (3.2) by exploiting properties of the corresponding propagators. The procedure consists of introducing the scale $h = \frac{1}{N}$ and as explained in Section 3 by unifying space, i.e. it is pursued by substituting $f(x)$ by $F \left( \frac{|x|}{\delta_x} \right)$ where $|x|$ denotes the length of the vector. We admit that one agent has a decision rule different from the one of the others. The property exploited in the construction proving the $N$-mean field limit is:

**Proposition 5.1.** Let $A_i : \mathbb{R}^k \to \mathbb{R}^k, i = 1, 2, t \geq 0$, be two families of arbitrary bounded matrix valued linear functions which are continuous in time. Assume moreover, that $U_{i,t,r}$ are two propagators in $\mathbb{R}^k$ with $\|U_{i,t,r}\| \leq C_1, i = 1, 2$, such that for any $f \in \mathbb{R}^k$ the equation

$$\frac{d}{ds} U_{t,s} f = U_{t,s} A_s f, \quad \frac{d}{ds} U_{s,r} f = -A U_{s,r} f, \quad t \leq s \leq r,$$

holds in $\mathbb{R}^k$ for both pairs $(A_i, U_i)$.

Then we have i)$$U_{2,r}^t - U_{1,r}^t = \int_t^r U_{2,s}^t (A_2^s - A_1^s) U_{1,s}^r ds$$

and ii)$$\|U_{2,r}^t - U_{1,r}^t\|_{B(0,1) \to \mathbb{R}^k} \leq C_1^2 (r - t) \sup_{t \leq s \leq r} \|A_2^s - A_1^s\|_{B(0) \to \mathbb{R}^k}$$

The result is adopted from a well known result on bounded linear operators see e.g. [10], and [18]. The representation is used to derive the subsequent properties. The propagator $\Lambda(t,s, \cdot)$ generated by the operator $A$ in 2.1 is bounded.

As in Section 3 let $\psi_{N,t}^s, t \leq s$, be the $N$-mean field propagator generated by (2.6) and assume that $\phi_t^s$ is the Koopman propagator defined in (3.4). Since the linear combination $\mathfrak{A}^N$ of copies of the bounded operators $A$ in (1.2) is also bounded, Remark 2.2 implies once again that $\psi_{N,t}^s$ is bounded. Exploiting Proposition 5.1i) we derive an estimate for the deviation of the propagator $\psi_{N,t}^s$ from the limiting Koopman propagator $\phi_t^s$. We first study the unrealistic case of a common initial condition $y$.

By construction $\mathfrak{A}^N$ and $\psi_{N,t}^s$ satisfy equation (5.1) then Proposition (5.1) reveals the representation:
\[
[\psi_N^{s,t} - \phi^{s,t}]F(x^N) = \int_s^t \left[ \psi_N^{s,t} (\tilde{\mathcal{A}}^N[r,y] - \mathcal{A}[r,y]) \phi^{s,t} F \right] ds \tag{5.2}
\]
for \( F \in C^1(M) \) and \( x^N = \frac{1}{N} \sum_{\ell=1}^{\ell} n_\ell \delta \) with \( \sum_{\ell=1}^{\ell} n_\ell = N \) and we continue by estimating

\[
\sup_{x^N} \left| \left[ \psi_N^{s,t} - \phi^{s,t} \right] F(x^N) \right| \leq \int_s^t \| \psi_N^{s,t} \| \sup_{x \in M} \left| (\tilde{\mathcal{A}}^N[r,y] - \mathcal{A}[r,y]) \phi^{s,t} F(x^N) \right| ds \tag{5.3}
\]

with \( x \leq \frac{1}{N} \), for \( 0 \leq t \leq s \leq T \), with a constant \( C(T) \) summarizing the three operator norms and integration with respect to time.

This estimate will be a further step be applied to estimate the order of convergence in the mean field limit when the initial conditions suit the operators and hence differ while \( N \) changes. In fact we shall assume that the initial conditions

\[
x_0^N = \frac{1}{N} (\delta_{x_0^N} + \ldots + \delta_{x_0^N}), \tag{5.4}
\]

with \( x_{1,0}, \ldots, x_{N,0} \in \mathbb{X} \), of the Kolmogorov equation for the generators \( \tilde{\mathcal{A}}^N \) converge in \( \mathbb{R}^k \), as \( N \to \infty \), to a vector \( x_0 \in \mathbb{R}^k \) in such a way that

\[
|x_0^N - x_0| \leq \frac{k_1}{N}, \tag{5.5}
\]

with a constant \( k_1 \geq 0 \).

**Theorem 5.2.** Let the assumptions of Proposition 2.7 be satisfied and let the initial conditions \( \frac{1}{N} (\delta_{x_{1,0}} + \ldots + \delta_{x_{N,0}}) \) be subject to (5.5). Then, for \( t \in [0, T] \) with arbitrary \( T \geq 0 \),

\[
\left| (\psi_0^{0,t} F)(x_0^N) - (\phi^{0,t} F)(x_0) \right| \leq \frac{C(T)}{N} \left( T \| F \|_{C^1(M)} + k_1 \right)
\]

with a constant \( C(T) \).

**Proof.** For \( F \in C^1(M) \) we have that

\[
\left| (\psi_0^{0,t} F)(x_0^N) - (\phi^{0,t} F)(x_0) \right| \leq \left| (\psi_0^{0,t} F - \phi^{0,t} F)(x_0^N) \right| + \left| (\phi^{0,t} F)(x_0^N) - (\phi^{0,t} F)(x_0) \right| .
\]

Estimating the first term by (5.3) and the second one by exploiting the Lipschitz continuity of the nonlinear flow guaranteed by Proposition 3.1 finishes the proof. \( \Box \)

We now reintroduce a variable control parameter \( u \). If we assume also that all agents are following a fixed strategy \( \gamma(t,x) \) but one player, for instance the first player, who applies a different control \( u_{1,t} = \tilde{\gamma}(t,x) \), then the controlled process of \( N \) interacting agents will be generated by the following operator

\[
\tilde{\mathcal{A}}^N[t, \gamma, \tilde{\gamma}] F(\frac{1}{N} \delta_x) = \tilde{\mathcal{A}}^N[t, \gamma] F(\frac{1}{N} \delta_x) + \left[ A^1[t, \frac{1}{N} \delta_x, \gamma(t,)] - A^1[t, \frac{1}{N} \delta_x, \gamma(t,)] \right] F(\frac{1}{N} \delta_x) \tag{5.6}
\]
where \( \hat{x}^N \) and \( A^1 = A \) were defined in (2.6) and (1.2), respectively, \( x = \frac{1}{N} \delta_x \), \( F \in C^1_\infty(M) \).

**Remark 5.3.** Let \( \psi^0_{N,t}[\gamma, \tilde{\gamma}] \) the N-mean field propagator on \( C^1(M) \) generated by \( \hat{x}^N \). Since \( \hat{x}^N \) is a linear combination of the linear operator \( A \) it inherits all properties of \( A \). Hence \( \psi^0_{N,t}[\gamma, \tilde{\gamma}] \) possesses the same properties as the propagator \( A \) which is generated by \( A \), i.e. it is linear for \( T \) sufficiently small it is a contraction operator hence bounded, and it is Lipschitz continuous in the initial condition, i.e. it possesses the Feller property.

**Theorem 5.4.** Suppose the assumptions of Theorem (5.2) hold for the family of Markov jump type operators \( A[t, y, \gamma(t, \cdot)] \) with a class of functions \( \gamma : \mathbb{R}^+ \times X \rightarrow U \), which are continuous in the first variable and Lipschitz continuous in the second one, let \( \psi^0_{N}[\gamma, \tilde{\gamma}] \) be as above, and let \( \phi^{0,t}[\gamma] \) the Koopman propagator (3.4). Then the following bounds hold:

i) For \( t \in [0, T] \) with any \( T \geq 0 \):

\[
\left| (\psi^0_{N}[\gamma, \tilde{\gamma}]F)(x^N_0) - [\phi^{0,t}[\gamma]F](x_0) \right| \leq \frac{C(T)}{N} \left( T \| F \|_{C^1(M)} + k_1 \right);
\]

with a constant \( C(T) \) independent of \( \tilde{\gamma} \).

ii) Let \( J(t, x) \in C([0, T], C^1(M)) \). Then the following bounds exist:

\[
\mathbb{E} \left[ \int_t^T J(s, X^N_s[\gamma, \tilde{\gamma}])ds - \int_t^T J(s, X^N_s[\gamma])ds \right] \leq \frac{C(T)}{N} \left( T \| J \|_{C([0, T], C^1(M))} + k_1 \right)
\]

where \( X^N_t[\gamma, \tilde{\gamma}] \) is the Markov process specified by the propagator \( \psi^0_{N}[\gamma, \tilde{\gamma}] \) and \( x_t[\gamma] \) is the solution to kinetic equation (3.2) with initial \( x_0 \).

**Proof.**

i) Having applied the representation (5.6) the proof follows along the same lines as the one of Theorem (5.2).

ii) Let us represent the integral \( \int_t^T J(s, X^N_s[\gamma, \tilde{\gamma}])ds \) as the limit of Riemannian sums. Then the wanted inequality is obtained by applying part i) term by term and passing to the limit.

\[ \square \]

**6. Mean field limits as an \( \epsilon \)-Nash equilibrium**

A strategy portfolio \( \Gamma \) in a game of \( N \) agents with payoffs \( V_i(\Gamma), i = 1, \ldots, N \), is called an \( \epsilon \)-Nash equilibrium if, for each player \( i \) and an acceptable individual strategy \( u_i \)

\[
V_i(\Gamma) \geq V_i(\Gamma_{-i}, u_i) - \epsilon,
\]

where \( (\Gamma_{-i}, u_i) \) denotes the profile obtained from \( \Gamma \) by substituting the strategy of player \( i \) with \( u_i \). Let us extend the estimate given in Proposition 5.4ii) to the case of \( J \) depending on the position of a tagged player and her own strategy. So we have to look at the process of pairs \( (X^N_{i,t}, x^N_t) \), which refers to a chosen tagged agent and an overall mass. The generators of the process of pairs \( (X^N_{i,t}, x^N_t) \) are defined on the space \( C^1_{\infty}(\mathbb{R} \times M) \) and take the form.
with \( \hat{\mathbf{A}}_N^{t, \gamma, \tilde{\gamma}} \) as in (5.6). Moreover, the corresponding propagator will be denoted by, \( \psi_{N; \hat{A}}^{t, \gamma, \tilde{\gamma}} \).

**Remark 6.1.** Since \( \hat{\mathbf{A}}_N^{t, \gamma, \tilde{\gamma}} \) is a finite sum of copies of the operator \( \mathbf{A} \) the corresponding propagator \( \psi_{N; \hat{A}}^{t, \gamma, \tilde{\gamma}} \) possesses the same properties as the propagator \( \Lambda \), generated by \( \mathbf{A} \), and \( \psi_{N}^{0, t} \) i.e. it is linear, for \( T \) sufficiently small it is a contraction operator hence bounded, and it is Lipschitz continuous in the initial condition, i.e. it possesses the Feller property.

**Remark 6.2.** Let \( \phi_{\hat{A}}^{0, t}[\gamma] \) the propagator generated by the family

\[
A^1[t, y, \tilde{\gamma}] I_d + I_d A[t, y, \gamma]
\]

on \( C^1_{\infty}(\mathbb{X} \times M) \). Since the operator \( \mathbf{A} \) is bounded and more regular in the parameters than \( A \), the propagator \( \phi_{\hat{A}}^{0, t}[\gamma, \tilde{\gamma}] \) inherits the properties of the Koopman propagator \( \phi^{0, t} \). Moreover, by inserting (5.6) into the definition and by applying Proposition 2.7 we find:

\[
\hat{\mathbf{A}}_N^{t, \gamma, \tilde{\gamma}} F(x_1, \frac{1}{N} \delta_x) := \left( A^1[t, \frac{1}{N} \delta_x, \tilde{\gamma}] + A[t, y, \gamma] \right) F(x_1, \frac{1}{N} \delta_x) + O(\frac{1}{N}).
\]

For the Kolmogorov equation corresponding to this generator we make the assumptions on the initial conditions that \( x_{1,0}^N \in \mathbb{X} \) converges, as \( N \to \infty \) to a point \( x_{1,0} \in \mathbb{X} \) such that

\[
|x_{1,0}^N - x_{1,0}| \leq \frac{k_2}{N}
\]

with a constant \( k_2 \), which arise in fact additional to (5.5). Then we find the following bounds for the approximation in the mean-field limit of \( N \) players in case one of them has a differing individual preference.

**Theorem 6.3.** Under the assumptions of the Theorem (5.4), let \( \phi_{\hat{A}}^{0, t}, \psi_{N; \hat{A}}^{0, t} \), and the cost function \( J \) be as above. Then the following bounds exist:

i) For \( t \in [0, T] \) with any \( T \geq 0 \),

\[
\left| (\psi_{N; \hat{A}}^{0, t}[\gamma, \tilde{\gamma}] F)(x_{1,0}^N, x_0^N) - (\phi_{\hat{A}}^{0, t}[\gamma] F)(x_{1,0}^N, x_0^N) \right| \leq \frac{C(T)}{N} \left( t |F|_{C^1(M)} + k_1 \right)
\]

with a constant \( C(T) \) not depending on \( \tilde{\gamma} \);

ii) for \( J(t, x, y, u) \in C([0, T] \times U; C^1(M)) \):

\[
\mathbb{E} \left[ \int_t^T J(s, X_{1,s}^N, y_s^N, \tilde{\gamma}, \tilde{\gamma}(X_{1,s}^N)) ds - \int_t^T J(s, X_{1,s}^N, y_s^N, \gamma(s, X_{1,s}^N)) ds \right] \leq \frac{C(T)}{N} \left( (T + k_2) |J|_{C([0, T] \times U; C^1(M))} + k_1 \right)
\]
where the pair \((X_t^N, Y_t^N[\gamma, \tilde{\gamma}])\) is the Markov process specified by the propagator \(\psi^{0,t}_{N,tag}[\gamma, \tilde{\gamma}]\), in particular the Markov process \(X_{1,s}\) is generated by \(A^1[t, y_t[\gamma], \tilde{\gamma}]\) and \(x_{t}[\gamma]\) is the solution to the kinetic equation (3.2) with initial condition \(x_0\), so to say a degenerate Markov process with one sample path having probability one.

**Proof.** The basic idea is to insert definitions and to exploit the properties of the propagators \(\Lambda^{t,s}, \psi^{t,s}_N, \) and \(\psi^{t,s}_{N,tag}\) summarized in Remarks 6.1 and 6.2 and the processes corresponding to the two first ones.

i) For \(F \in C_{\infty}^0(\mathbb{X} \times M)\), we have
\[
\left| \left( \psi^{0,t}_{N,tag}[\gamma, \tilde{\gamma}] F \right) (x_{1,0}^N, x_0^N) - \left( \psi^{0,t}_{tag}[\gamma] F \right) (x_{1,0}, x_0) \right|
\leq \left| \left( \psi^{0,\gamma}_{N,tag}[\gamma, \tilde{\gamma}] - \psi^{0,\gamma}_{tag}[\gamma] \right) F \right| (x_{1,0}^N, x_0^N) + \left| \left( \psi^{0,t}_{tag}[\gamma] F \right) (x_{1,0}, x_0) \right|
- \left( \psi^{0,t}_{tag}[\gamma] F \right) (x_{1,0}, x_0^N) + \left| \left( \psi^{0,t}_{tag}[\gamma] F \right) (x_{1,0}, x_0^N) - \left( \psi^{0,t}_{tag}[\gamma] F \right) (x_{1,0}, x_0) \right|.
\]

We estimate the first term by (5.3) using the operator norm and the second one by using the assumption (5.4) and the fact that, being a solution of a linear ODE, it is linear in the initial conditions. The bound for the third term follows by exploiting the Lipschitz continuity of the solutions to the kinetic equation (3.2) obtained in Corollary 3.3. This finishes the proof of part i).

ii) Let us represent both integrals \(\int_t^T J(s, X_{1,s}^N, y_s[\gamma], \tilde{\gamma}(s, X_{1,s}^N))ds\) and \(\int_t^T J(s, X_{1,s}^N, y_s[\gamma], \tilde{\gamma}(s, X_{1,s}^N))ds\) as the limits of Riemannian sums. Then the result ii) is obtained by applying part i term-by-term and passing to the limit. \(\square\)

The results of this and the previous two sections, and Theorem 6.3 in particular are based on a fixed control parameter, depending on time however, and thus hold independently of the MFG methodology. The main point in the convergence results above is the \(\frac{1}{N}\)-convergence rates for the case of non-smooth (only Lipschitz) drift coefficients.

**Theorem 6.4.** Let \(\{A[t, y, u] \mid t \geq 0, y \in M, u \in \mathcal{U}\}\) be the family of jump type operators given in (2.1) and \(x\) be the solution to equation (3.2). Assume the following

i) The kernel \(\nu(t, x, y, u_t)\) satisfies the Hypothesis A and B;

ii) The time-dependent Hamiltonian \(H_t\) is of the form (4.5):
\(H_t \) is continuous in \(t\), and Lipschitz continuous in \((x, x)\), uniformly with respect to \(t, x, x\).

iii) The gradient \(\nabla_x H_t\) exists and is continuous and uniformly bounded.

iv) The terminal function \(V_T\) is in \(C^1(M)\).

v) The initial conditions \(x_0^N = \frac{1}{N}(\delta_{x_{1,0}^N} + \cdots + \delta_{x_{N,0}^N})\) of an \(N\) players game converge in \((C^1(\mathbb{X}))^N\), as \(N \to \infty\), to a probability law \(x_0 \in \mathbb{P}(\mathbb{X})\) in a way that (5.5) and (6.4) are satisfied.

Then the strategy profile \(u = \hat{\Gamma}(t, x, y_t)\), defined via HJB (4.2) and (4.5) with \(\hat{y} = x\), is a perfect \(\epsilon\)-equilibrium in a \(N\) players game, with
\[
\epsilon = C(T)N^{-1} \left( \|J\|_{C([0,T] \times \mathcal{U}, C^1(\mathbb{X} \times M))} + \|V_T\|_{C^2(\mathbb{X})} + 1 \right).
Proof. Due to Assumption ii) the unique solution to the HJB equation admits a unique optimal control parameter given by (4.5). Let the 1st player choose a different strategy \( \tilde{\gamma} \), other than \( \Gamma \). Denote the state dynamics of the first player by \( X_{1,t}^N \) or \( \tilde{X}_{1,t}^N \) if defined via equation (3.2) with \( \gamma = \Gamma \) or \( \gamma = \tilde{\gamma} \), respectively.

Then,

\[
\left| V^1(0, X_{1,0}^N)[\Gamma] - V^1(0, \tilde{X}_{1,0}^N)[\Gamma - 1, \tilde{\gamma}] \right| \\
\leq \left| \mathbb{E} \left[ \int_0^T J_1(s, X_{1,s}^N, y_s[\Gamma], \Gamma)ds \right] - \mathbb{E} \left[ \int_0^T J_1(s, \tilde{X}_{1,s}^N, y_s[\Gamma - 1, \tilde{\gamma}], \tilde{\gamma})ds \right] \right| \\
+ \left| \mathbb{E} \left[ V^T(X_{1,T}^N) \right] - \mathbb{E} \left[ V^T(\tilde{X}_{1,T}^N) \right] \right|.
\]

By Theorem (6.3),

\[
\left| V^1(0, X_{1,0}^N)[\Gamma] - V^1(0, \tilde{X}_{1,0}^N)[\Gamma - 1, \tilde{\gamma}] \right| \leq \frac{C(T)}{N} \left( \| J \| + \left\| V^T \right\|_{C^1([0,T] \times \mathcal{X})} + 1 \right)
\]

where \( \| J \| = \left\| J \right\|_{C([0,T] \times \mathcal{X})} \). It is clear these estimates hold if we start the game at any time \( t \in [0, T] \). This completes the proof and the construction of the mean-field game in this paper. \( \square \)

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