This paper introduces a discrete limit order book model where new orders are placed with a fixed displacement from the mid-price. Further, the trade event occurs whenever the mid-price hits the price level on which there is some volume. Therefore, the dynamics of the limit order book model leads to two trading mechanisms, namely Type I trade and Type II trade. A Type I trade takes place whenever the price maximum increases, while a Type II trade occurs if the price drops by $\mu$ or more and then increases by $\mu$ again. Our focus is mainly on the distribution of order avalanches length, and by an avalanche length we consider a series of order executions where the length of periods with no trade event cannot exceed $\epsilon$.

1 Introduction

The emergence of automated high frequency trading in the absence of a Tobin tax, i.e. a microscopic but nevertheless non-zero tax on financial transactions, has put financial markets recently at times into high distress. This is highlighted by the large scale occurrences of flash crashes, which are fast and deep falls in security prices, together with an as rapid upward movement to previous levels. Among the most notable of those events, it is worth to mention the one occurring on 6th of May 2010 when the Dow Jones incurred a loss of 9.5% just to recover within 15 minutes (see [KKST17]); the April 2013 flash crash which removed about 136 billions of dollars from the S&P 500, only to be regained within minutes; as well as the Singapore 2013 flash crash which momentarily wiped out about 6.9 billion dollars of capitalization.
There is an ongoing debate about the causes and responsibilities of players involved. However, we are not entering this discussion which is intensely debated and where there are plenty of sources easily to be found out, but rather study distributional properties of this phenomenon.

The interest into order avalanches is motivated by the theory of self-organized criticality (SOC), see [SC06].

Our study is targeting an avalanche object in the specific context of financial markets.

In recent years, there has been increasing interest in studies of the dynamics of the limit order book, both from the theoretical and practical point of view, see [AAC∗16]. Modeling the dynamics of the order book as an interacting Markovian queueing system is exhibited in [CdL13], and the analytical tractability of the model provides the relation between order flow and price dynamics.

The stochastic auction model in which buy and sell orders came at the auction by following independent renewal processes is studied in [Kru03], and the number of present orders are modeled by stochastic processes and further limiting distributions are investigated. Delattre et al. [DRR13] investigate the efficient price that can be employed in practice, and also they developed price statistical estimation using the order flow.

The authors in [AJ13] investigated the order book model as a multi-dimensional continuous-time Markov chain, and further using the functional central limit theorem authors showed that the rescaled price process converges to Brownian motion. In [BHQ14] the authors investigate a stochastic LOB model that converges to a continuous-time limit, and the limits of the buy and sell volume densities are modeled by stochastic partial differential equations coupled with a two-dimensional Brownian motion. A model in which limit order book dynamics depend on the available prices and volume that corresponds to the current prices is introduced and studied in [HK17]. Especially, in [HK17] the authors concluded that when the order size and tick size is converging to zero, by a weak law of large numbers, the volume densities tend to non-linear PDEs, that are coupled with non-linear ODEs, which correspond to the best ask and best bid price.

The structure of the paper is as follows. Section 2 introduces a discrete limit order book model and a key quantity of our research, namely avalanche length. Further, in Section 3 the probability generating function for the simplified avalanche length is established. Section 4 contains limit results for the full avalanche length and the generating function for the full avalanche length. The generating function of the time to the first trade, if we start with the initially empty book, is presented in Section 5.
2 A simple limit order book model with discrete time and space

In [dW13, Ric13, Spo14] a simple order book model driven by arithmetic Brownian motion has been studied. These studies lead to interesting known and new Brownian path functionals involving Brownian local time and the theory of Brownian excursions. In the past it has been shown that such results can be illustrated, found, and often even proven by limit arguments with or from corresponding results for random walks.

For example, let us refer to [Ver79, Tak95, Tak99, Csá94, CR92, CH04, LH07, CMS86, BCP03] and in particular [SD88, CH03, PW14].

Further (possibly) related references are [Ald98, Föl94, DT96, Csá96, LM07].

In this section we follow this approach and define a simple order book process driven by a simple symmetric random walk.

2.1 The basic setting

We start with a probability space $(\Omega,\mathcal{F},P)$ that carries a simple symmetric random walk $\{S_n : n \geq 0\}$. This process is interpreted as the mid price, i.e. the arithmetic average between best bid and ask price. See also [DRR13] for a general discussion of various price concepts, or rather price substitutes in the context of order book modelling.

For better readability we use mixed index and function notation freely, i.e. is $S_k$ and $S(k)$ means the same.

For simplicity we focus on the the ask-side of the order book and ignore order cancellations. We fix an integer spread parameter $\mu \geq 1$. Further, let us introduce the ask order volume process $\{V(n,u) : n \geq 0, u \in \mathbb{Z}\}$, which is a two-parameter process and $V(n,u)$ denotes the volume of order at time $n$ and price $u$. As a further simplification, we ignore the size of the order volume, and distinguish only the cases $V(n,u) > 0$, if there is an order, or $V(n,u) = 0$, when there is no order in the book.

We assume, that we start from an order book that is initially full, which is convenient for our analysis. So this means

$$V(0,u) = I_{u \geq 0}, \quad u \in \mathbb{Z}.\quad (1)$$

Later we will consider an order book that is initially empty.

The dynamic development of the order book is described as follows. If there is an order at time $n$ at the level $u = S_n$, indicated by $V(n,S_n) > 0$, it is executed and the corresponding entry is removed from the order book. Thus at the next time step $V(n+1,S_n) = 0$. This is also true, if there was no order, indicated by $V(n,S_n) = 0$. Furthermore we assume a new order will be placed.

\[\text{Some exchanges actually clear the order book before trading starts on a new day, e.g. the Istanbul Stock Exchange, see [VZPR14] for a statistical analysis.}\]
Figure 1: Type I trade (depicted with the blue rhombus) example.

at distance $\mu$ above the price $S_n$, thus $V(n, S_n + \mu) = V(n - 1, S_n + \mu) + 1$. To summarize, we have

$$V(n, u) = \begin{cases} 0 & \text{if } u = S_{n-1} \\ V(n-1, u) + 1 & \text{if } u = S_{n-1} + \mu \\ V(n-1, u) & \text{otherwise}, \end{cases}$$

$n \geq 1, u \in \mathbb{Z}$. (2)

2.2 Type I and Type II trades, trading times and inter-trading times

A trade event occurs at time $n$ if the mid-price reaches the price level $U$, i.e. $U = S_n$, and there is some volume at the price level $U$, i.e. if $V(n, S_n) > 0$.

**Definition 1.** We define trading times $\{\tau_i : i \geq 0\}$ and intertrading times $(T_i)_{i \geq 1}$ by

$$\tau_0 = 0, \quad \tau_i = \inf\{n > \tau_{i-1} : V(n, S_n) > 0\}, \quad T_i = \tau_i - \tau_{i-1}, \quad i \geq 1. \quad (3)$$

It is quite intuitive that typically a trade on the ask-side occurs, if the midprice moves up. We formalize this as follows. For $i \geq 1$ we say the $i$-th trade is a Type I trade, if $S(\tau_i) > S(\tau_{i-1})$. Otherwise we call the trade a Type II trade. Figure 1 displays an example for a typical Type I trade. A Type II trade occurs, if the price drops by $\mu$ or more, and then increases by $\mu$ or more again. If this happens in a very short time interval, say of
Figure 2: Example of a Type II trade (depicted with the red squares), that is followed by two consecutive Type I trades (depicted with the blue rhombus).

length less than \( \varepsilon > 0 \), we say, we have a flash-crash trade. Figure 2 displays an example for a typical Type II trade (blue rhombs depict Type I trades, while red rectangles depict Type II trades). Intuitively, Type II trades, and in particular, flash-crash trades are less frequent than Type I trades. Note that the first trade that occurs at the initial position (at the level \( u = 0 \)) at the starting time (\( n = 0 \)), we define as a Type II trade.

**Remark 1.** Note that all pictures included in this paper are generated by the program that simulate the dynamics summarized in equation (2).

A sufficient condition for a Type I trade to occur is given by the following easy lemma.

**Lemma 1.** If \( n \geq 1 \) is a strict ascending ladder time of the random walk, then there is a Type I trade at \( n \).

**Proof.** Suppose \( n \geq 1 \) is a strict ascending ladder time. Then, we know that \( S_n > \max(S_0, \ldots, S_{n-1}) \geq 0 \) and there was no trade at level \( S_n \) or higher before \( n \). Since we assume \( V(0, u) > 0 \) for \( u \geq 0 \) it follows \( V(n, S_n) > 0 \), thus there is a trade, and at a price level which is higher than the level of the last trade. \( \square \)
Motivated by this lemma, and also in agreement with [3W13], we introduce the simplified trading times and the simplified intratradings times, that are obtained by ignoring Type II trades, by following definition.

**Definition 2.** Define the simplified trading times \(\{\rho_i : i \geq 0\}\) and the simplified intratradings times \(\{R_i\}_{i \geq 1}\) as

\[
\rho_0 = 0, \quad \rho_i = \inf\{n > \rho_{i-1} : S_n > S(\rho_{i-1})\}, \quad R_i = \rho_i - \rho_{i-1}, \quad i \geq 1.
\]

**Remark 2.** Note that \(\{\rho_j : j \geq 0\}\) \(\subset\) \(\{\tau_i : i \geq 0\}\) and that with Definition 2 we are ignoring Type II trades, but we may also ignore some Type I trades that occur after a Type II trade (see Figure 3 for an illustration, in which simplified trading times are depicted by circles). In Figure 3 we can see Type II trade which is followed with two consecutive Type I trades, of which first Type I trade is ignored by Definition 2.

### 2.3 On the best ask price

Our next goal is to show that the best ask price can be described by the price process without reference to the full order volume process. For that purpose we define the best ask price process \(\{\alpha_n\}_{n \geq 0}\) by

\[
\alpha_0 = 0, \quad \alpha_{n+1} = \alpha_n + I_{\{\alpha_n = S_n\}} - I_{\{\alpha_n = S_{n+1}\}}, \quad n \geq 0.
\]
Equation (5) can be interpreted as follows: if \( \alpha_n = S_n \) the midprice hits the best ask, and therefore the next best ask is higher than the previous one; if \( \alpha_n = S_n + \mu + 1 \) the order is placed is the previous best ask and the next best ask is lower than the previous one.

**Lemma 2.** We have

\[
S_n \leq \alpha_n \leq S_n + \mu + 1, \quad n \geq 0. \tag{6}
\]

**Proof.** By using induction on \( n \) and distinguishing for the step \( n \mapsto n + 1 \) the six cases corresponding to: \( \alpha_n = S_n \) and \( S_{n+1} = S_n \pm 1 \), \( \alpha_n = S_n + \mu + 1 \) and \( S_{n+1} = S_n \pm 1 \), \( S_n < \alpha_n < S_n + \mu + 1 \) and \( S_{n+1} = S_n + 1 \).

**Lemma 3.** We have for all \( n \geq 0 \) that \( V(n, u) > 0 \) iff \( u \geq \alpha_n \).

**Proof.** We do again induction on \( n \geq 0 \). For \( n = 0 \) the initial conditions show the claim is true. For the induction step \( n \mapsto n + 1 \) we distinguish again three cases. Case 1: Suppose \( S_n = \alpha_n \). Then \( \alpha_{n+1} = \alpha_n + 1 \). By the induction hypothesis \( V(n, u) > 0 \) iff \( u \geq \alpha_n \). There is a trade at time \( n \) and thus \( V(n+1, \alpha_n) = V(n+1, \alpha_n) = 0 \). A new order is placed above \( V(n+1, S_n + \mu) = V(n, S_n + \mu) + 1 \geq 0 \), but we had already \( V(n, S_n + \mu) > 0 \).

All other positions are unchanged. Case 2: Suppose \( \alpha_n = S_n + \mu + 1 \). Then \( S_n < \alpha_n \) and \( V(n, S_n) = 0 \), thus no trade takes place. A new order is placed at \( S_n + \mu = \alpha_n - 1 = \alpha_{n+1} \), thus \( V(n+1, \alpha_{n+1}) > 0 \). All other positions are unchanged. Case 3: Suppose \( S_n < \alpha_n < S_n + \mu + 1 \). Then \( \alpha_{n+1} = \alpha_n \). No trade takes place also in this case. The new order is placed at \( S_n + \mu \geq \alpha_n \). All other positions are unchanged.

**Corollary 1.** For \( i \geq 1 \) we have

\[
\tau_i = \inf\{n > \tau_{i-1} : S_n = \alpha_n\}. \tag{7}
\]

**Remark 3.** The trade happens exactly when the mid-price hits the best ask price, i.e. when \( \alpha_n = S_n, n \geq 1 \).

**Remark 4.** In Figure 4 red points represent the best ask price process \( \{\alpha_n\}_{n \geq 0} \). Figure 4 is generated by program simulation and it confirms the aforementioned results for the best ask price process \( \{\alpha_n\}_{n \geq 0} \).

We can now define the trading excursion process. We define it as a process that takes values in the set of simple random paths of finite length. More precisely, for \( n \geq 0 \) let

\[
U^{(n)} = \{ (s_0, \ldots, s_n) \in \mathbb{Z}^{n+1} : s_0 = 0, |s_j - s_{j-1}| = 1 \text{ for } j = 1, \ldots, n \}, \tag{8}
\]

and \( U^{(\infty)} = \bigcup_{n \geq 0} U^{(n)} \). Let \( \mathcal{U}^{(\infty)} \) denote the power set of \( U^{(\infty)} \) and define a discrete measure \( \nu \) by \( \nu((s_0, \ldots, s_n)) = 2^{-n} \). Then \( (U^{(\infty)}, \mathcal{U}^{(\infty)}, \nu) \)
becomes a $\sigma$-finite measure space. Next we define the \emph{trading excursion process} $(e_i)_{i \geq 1}$ by

$$e_{in} = S(\tau_{i-1} + n) - S(\tau_{i-1}), \quad 0 \leq n \leq T_i, \quad i \geq 1,$$

which takes values in $U(\infty)$.

\textbf{Remark 5.} For each $i \geq 1$ if $e_{iT_i} > 0$ the Type I trade occurs at time $\tau_i$, otherwise if $e_{iT_i} \leq 0$ the Type II trade occurs at time $\tau_i$.

\textbf{Lemma 4.} The trading excursions $e_i, i \geq 1$ are iid.

\textit{Proof.} Since $\tau_i, i \geq 0$ are stopping times for the random walk, the claim follows from the Markov property of the random walk. To give some details, fix $i \geq 1$ and consider the processes

$$S'(n) = S(\tau_{i-1} + n) - S(\tau_{i-1}), \quad \alpha'(n) = \alpha(\tau_{i-1} + n) - \alpha(\tau_{i-1}), \quad n \geq 0. \quad (10)$$

The trading excursion process is uniquely determined by the price process and conversely, the price process can be reconstructed from the trading excursion process by glueing the excursions together, similar as in the classical excursion theory, cf. [RY99, Prop.XII.2.5, P.482].
2.4 Avalanche length

Instead of studying the dynamics of the two-parameter order-book process, following [Ric13] and other earlier work, and motivated by [SC06], we focus on a scalar key quantity, the avalanche length. When there is a longer up-movement of the asset price, then a substantial portion of the order book is "eaten up" until the price drops again. This can perhaps be described as an "order avalanche".

But long up movements are rare for a random walk and when we think of the corresponding model for continuous time, the probability that Brownian motion increases on an interval of positive length is zero. Therefore we consider as an avalanche a period of trade executions, but allow a small window, of size $\varepsilon$ at most, without trading. Only when there is no trade for a period of length larger than $\varepsilon$ the avalanche terminates. Figure 5 shows an avalanche consisting of a sequence of typical Type I trades. For the simplified avalanche length we consider only Type I trades, and completely ignore (do not take into account) Type II trades and even some Type I trades which follow Type II trade (as depicted in the Figure 3).

But an avalanche may continue, even if the price drops, but recovers quickly, that is, when we have a Type II flash-crash trade. This situation is illustrated in Figure 6 for $\varepsilon = 8, \mu = 2$. Formal definitions will be given below.

Figure 5: Avalanche example without flash-crash trades when $\varepsilon = 7$
3 Simplified avalanche length

3.1 Discrete results, generating function

The simplified avalanche length \( L_\varepsilon \) is defined as follows: if \( R_1 \leq \varepsilon, \ldots, R_k \leq \varepsilon, R_{k+1} > \varepsilon, k \geq 1 \), then

\[
L_\varepsilon = R_1 + \ldots + R_k. \tag{11}
\]

This is illustrated in Figure 5 above particularly for \( k = 3 \) and \( \varepsilon = 7 \).

**Proposition 1.** The probability generating function for the simplified avalanche length is given by

\[
E[z^{L_\varepsilon}] = \frac{P[R_1 > \varepsilon]}{E[1 - z^{R_1}; R_1 \leq \varepsilon] + P[R_1 > \varepsilon]}, \tag{12}
\]

**Proof.** From (11) we have:

\[
E[z^{L_\varepsilon}] = \sum_{k \geq 0} E[z^{R_1 + \ldots + R_k} : R_1 \leq \varepsilon, R_2 \leq \varepsilon, \ldots, R_k \leq \varepsilon, R_{k+1} > \varepsilon]
\]

\[
= \sum_{k \geq 0} E[z^{R_1} z^{R_2} \ldots z^{R_k} ; R_1 \leq \varepsilon, R_2 \leq \varepsilon, \ldots, R_k \leq \varepsilon, R_{k+1} > \varepsilon] \tag{13}
\]

Figure 6: Full versus simplified avalanche length example with flash-crash trade
Since $\rho_j$ are the strict ascending ladder times, by following [Fe68, XIII.1d, P.305] it is clear that $R_j$ are independent and identically distributed (iid). Thus, we obtain

$$E[z^{R_1}] = \sum_{k \geq 0} E[z^{R_1} : R_1 \leq \varepsilon]^k E[1 : R_{k+1} > \varepsilon] = \frac{P[R_{k+1} > \varepsilon]}{1 - E[z^{R_1} : R_1 \leq \varepsilon]}$$

$$= \frac{P[R_{k+1} > \varepsilon]}{E[1 : R_1 \leq \varepsilon] + E[1 : R_1 > \varepsilon] - E[z^{R_1} : R_1 < \varepsilon]} = \frac{P[R_1 > \varepsilon]}{E[1 - z^{R_1} : R_1 < \varepsilon] + P[R_1 > \varepsilon]}$$

$$E[1 - z^{R_1}] = P[R_1 > \varepsilon]$$

(14)

**Remark 6.** The random variable $R_1$ is the first passage time of the random walk through 1. Its distribution can be computed by the reflection principle, and can be found in [Fe68, Theorem 2 of III.7, P.89]. Feller defines the probability that the first passage through $r$ occurs at $n$ by $\varphi_{r,n}$, and by reflection principle:

$$\varphi_{r,n} = r n \left( \frac{n}{n + r} \right) 2^{-n}. \quad (15)$$

Using this notation we have $P[R_1 = 2n + 1] = \varphi_{1,2n+1}$.

The corresponding probability generating function can be found in [Fe68, XIII.(4.10), P.315], it is

$$E[z^{R_1}] = 1 - \sqrt{1 - z^2} \quad . \quad (16)$$

**Corollary 2.** Denoting by $a_k$ the number of the paths of length $k$ representing the first simplified Type I paths, we obtain:

$$a_k = \frac{1}{k} \left( \frac{k}{(k+1)/2} \right) \quad (17)$$

when $k$ is odd, and $a_k = 0$ otherwise.

Since $(R_j)_{j \in \{1, \ldots, n+1\}}$ are independent and identically distributed (iid), instead of $R_1$ from now on we write $R$. The probability generating function in (12) can be rewritten in various different and more explicit ways. We have

$$P[R > \varepsilon] = \frac{3 + \varepsilon'}{2 + \varepsilon} \left( \frac{2 + \varepsilon'}{3 + \varepsilon'} \right)^{2 + \varepsilon'}, \quad (18)$$

where we write

$$\varepsilon' = 2 \left[ \frac{\varepsilon - 1}{2} \right] + 1,$$
for an easy notation to distinguish odd and even $\varepsilon$.

We have also

$$E[z^L_{\varepsilon}] = \frac{1 - \Phi_{\varepsilon}(1)}{1 - \Phi_{\varepsilon}(z)},$$  \hfill(19)

where $\Phi_{\varepsilon}(z)$ is a polynomial, namely

$$\Phi_{\varepsilon}(z) = \sum_{k=0}^{(\varepsilon'-1)/2} \varphi_{1,2k+1} z^{2k+1}. \hfill(20)$$

It can be expressed in terms of the Gaussian hypergeometric function $2F_1(a, b, c, z)$,

$$\Phi_{\varepsilon}(z) = \frac{1 - \sqrt{1 - z^2}}{z} - \frac{\left(2 + \varepsilon'\right)}{3 + \varepsilon'} \frac{z^{2+\varepsilon'}}{(2 + \varepsilon')^{2+\varepsilon'}} 2F_1(1, 1 + \frac{\varepsilon'}{2}, \frac{5 + \varepsilon'}{2}, z^2). \hfill(21)$$

**Remark 7.** The hypergeometric function with parameter 1 can be expressed also in terms of (generalizations) of the associated Legendre Polynomials, see [AS64, 15.4.13, P.562].

**Corollary 3.**

$$E[L_{\varepsilon}] = 2 + \varepsilon' - \frac{2 + \varepsilon'}{3 + \varepsilon'} \cdot 2^{2+\varepsilon'} \left(\frac{2 + \varepsilon'}{3 + \varepsilon'}\right) \hfill(22)$$

$$\text{Var}[L_{\varepsilon}] = \frac{4}{3} \cdot (2 + 3\varepsilon' + \varepsilon'^2) - \frac{6 + 7\varepsilon' + 2\varepsilon'^2}{3 + \varepsilon'} \cdot \frac{2^{2+\varepsilon'}}{(3 + \varepsilon')^2} \hfill(23)$$

**Proof.** We obtain the moments from the derivatives of the probability generating function at $z = 1$. 

**3.2 Brownian excursion limit for the simplified avalanche length**

For a first limit result we consider $\frac{1}{n}R$ for $n \geq 1$ and introduce the Laplace-Stieltjes transform for its distribution, namely $E[e^{-\frac{\lambda}{n}R}]$.

**Lemma 5.** We have for fixed $\Re(\lambda) > 0$ the asymptotics

$$E[e^{-\frac{\lambda}{n}R}] = 1 - \sqrt{2\lambda} \cdot n^{-\frac{1}{2}} + O(n^{-1}), \quad n \to \infty. \hfill(24)$$

**Proof.** This lemma is proved by combining [16] with the elementary asymptotics, namely the exponential series $e^{-\lambda/n}$ as $n \to \infty$, and the power series of $\sqrt{1 - z^2}$ as $z \to 0$. 

\[12\]
Thus $\frac{1}{n} R$ converges in distribution under $P$ to zero. This is not surprising, as it is essentially the length of the first excursion of the random walk. For the Brownian limit this degenerates to zero, as arbitrary small excursions accumulate near time zero. But the second term in the asymptotic expansion is relevant for the simplified avalanche length. We can get another point of view by connecting the limit to Itô’s excursion measure.

Let $G$ denote the distribution of Brownian excursion length under the upper Ito measure $n_+$, see Appendix 6.1. Then it is well-known, see [RY99, XII], that $G$ has a density, namely

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x > 0.$$  \hfill (25)

Note that

$$\int_0^\infty (1 - e^{-\lambda x}) g(x) dx = \sqrt{2\lambda}, \quad \Re(\lambda) > 0.$$  \hfill (26)

**Proposition 2.** For $n \geq 1$ let $\mu_n = \text{Law}(\frac{1}{n} R)$. Then we have

$$\lim_{n \to \infty} \int_0^\infty (1 - e^{-\lambda x}) \frac{1}{n} \mu_n(dx) = \sqrt{2\lambda}, \quad \Re(\lambda) > 0,$$  \hfill (27)

and thus $\lim_{n \to \infty} \frac{1}{n} \mu_n = G$ vaguely on $(0, \infty)$.

**Proof.** This follows from (24). Let $\mu_n = \text{Law}(\frac{1}{n} R)$ and $\nu$ denote the measure with density $g$. Then we can apply Lemma 13 and furthermore Proposition 8. \hfill \square

In the following limit theorem $\varepsilon$ denotes a strictly positive real number. We write for brevity $L_{n\varepsilon}$ instead of the more precise $L_{\lfloor n\varepsilon \rfloor}$. Let erf be the error function (see [OLBC10]), which is defined as:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$  \hfill (28)

**Theorem 1.** The scaled simplified avalanche length for the simple symmetric random walk converges in distribution to the simplified Brownian avalanche length. Analytically we have

$$\lim_{n \to \infty} E[e^{-\frac{1}{n} L_{n\varepsilon}}] = \frac{1}{\sqrt{\lambda \varepsilon \pi}} \text{erf}(\sqrt{\lambda \varepsilon}) + e^{-\lambda \varepsilon},$$  \hfill (29)

and

$$\lim_{n \to \infty} E[\frac{1}{n} L_{n\varepsilon}] = \varepsilon, \quad \lim_{n \to \infty} \text{Var}[\frac{1}{n} L_{n\varepsilon}] = \frac{4}{3} \varepsilon^2.$$  \hfill (30)

\footnote{The distribution under the lower Ito measure $n_-$ is the same.}
Proof. The Laplace transform of the avalanche length $L_\varepsilon$ in the context of Parisian options is presented in [DW15]. Also, that formula can be derived from the Lévy measure of the subordinator consisting of Brownian passage times, see (Dudok de Wit [dW13], Theorem 16).

In [dW13] avalanche length $L$ was defined as a random variable modeling the first time after which no orders get executed in an $\varepsilon$-time interval, where the orders are executed due to the price increase. Therefore, it corresponds to the simplified avalanche length in our model. Theorem 16 in [dW13] states that the Laplace transform of the avalanche length is given by

$$E\left[e^{-\lambda L}\right] = \frac{1}{\sqrt{\lambda \varepsilon \pi \text{erf} \left(\sqrt{\lambda \varepsilon}\right) + e^{-\lambda \varepsilon}}}. \quad (31)$$

In order to derive the formula (31) in [dW13] the author first considered the random variable $L_y$ given by conditioning $L$ on $H = y$ for some $y \geq 0$, where $H$ is the avalanche height defined as the highest level on which the order get executes throughout the period of avalanche.

Theorem 11 in [dW13] states that the avalanche height $H$ is exponentially distributed with parameter $\sqrt{\frac{2}{\pi \varepsilon}}$, so we have:

$$E\left[e^{-\lambda L}\right] = E\left[E\left[e^{-\lambda L_y} \mid H\right]\right] = \int_0^\infty E\left[e^{-\lambda L_y}\right] \sqrt{\frac{2}{\pi \varepsilon}} e^{-y \sqrt{\frac{2}{\pi \varepsilon}}} dy. \quad (32)$$

Theorem 14 in [dW13] established the Laplace transform of the $L_y$, and it is

$$E\left[e^{-\lambda L_y}\right] = \exp \left( -y \sqrt{\frac{2}{\pi \varepsilon}} \left(\sqrt{\lambda \varepsilon \pi \text{erf} \left(\sqrt{\lambda \varepsilon}\right)} + e^{-\lambda \varepsilon} - 1\right) \right). \quad (33)$$

Therefore, the Laplace transform of the avalanche length (31) is obtained by combining (32) and (33), i.e.:

$$E\left[e^{-\lambda L}\right] = \int_0^\infty \exp \left( -y \sqrt{\frac{2}{\pi \varepsilon}} \left(\sqrt{\lambda \varepsilon \pi \text{erf} \left(\sqrt{\lambda \varepsilon}\right)} + e^{-\lambda \varepsilon} - 1\right) \right) \sqrt{\frac{2}{\pi \varepsilon}} e^{-y \sqrt{\frac{2}{\pi \varepsilon}}} dy$$

$$= \sqrt{\frac{2}{\pi \varepsilon}} \int_0^\infty \exp \left( -y \sqrt{\frac{2}{\pi \varepsilon}} \left(\sqrt{\lambda \varepsilon \pi \text{erf} \left(\sqrt{\lambda \varepsilon}\right)} + e^{-\lambda \varepsilon} \right) \right) dy$$

$$= \frac{1}{\sqrt{\lambda \varepsilon \pi \text{erf} \left(\sqrt{\lambda \varepsilon}\right) + e^{-\lambda \varepsilon}}}. \quad (34)$$

In the following, the approach is to derive this formula by excursion theory. Note that $G(x)$ is defined as

$$G(x) = \int_x^\infty g(y) \, dy, \quad \text{where} \quad g(y) = \frac{1}{\sqrt{2\pi}} y^{-3/2}. \quad (35)$$
We fix \( n \in \mathbb{N} \) such that \( n > 1/\varepsilon \) and consider only excursions with length \( R_i \in \left[ \frac{1}{n}, \varepsilon \right] \). Note that there exist only finitely many of those, i.e. \( R_1^{(n)}, ..., R_j^{(n)} \), and then there is a first excursion \( R_{j+1} \) (this does not depend on \( n \)), whose length is more than \( \varepsilon \). Define the modified avalanche length \( L_{\varepsilon,n} \) by

\[
L_{\varepsilon,n} = R_1^{(n)} + ... + R_j^{(n)} \leq L_{\varepsilon}
\]

(36)

where \( L_{\varepsilon} \) is the combined length of all excursions until one whose length exceeds \( \varepsilon \), see (11). Since there are a \( \sigma \)-discrete number of excursions we translate into the continuous time case, following the geometric series. Since the \( R_i^{(n)} \) are iid, we sum the geometric series:

\[
E \left[ e^{-\lambda L_{\varepsilon,n}} \right] = \frac{P (R > \varepsilon)}{1 - E [e^{-\lambda R_i^{(n)} \mathbb{1}_{R_i^{(n)} \leq \varepsilon}}]}.
\]

(37)

Letting \( n \to \infty \), we obtain by dominated convergence that

\[
E \left[ e^{-\lambda L_{\varepsilon}} \right] = \frac{P (R > \varepsilon)}{1 - E [e^{-\lambda R \mathbb{1}_{R \leq \varepsilon}}]}.
\]

(38)

In order to estimate the denominator

\[
E [\mathbb{1}_{R \leq \varepsilon}] + E [\mathbb{1}_{R > \varepsilon}] - E \left[ e^{-\lambda R \mathbb{1}_{R \leq \varepsilon}} \right] = E \left[ (1 - e^{-\lambda R}) \mathbb{1}_{R \leq \varepsilon} \right] + P (R > \varepsilon),
\]

(39)

we define

\[
\phi (x) = \left( 1 - e^{-\lambda x} \right) \mathbb{1}_{x \leq \varepsilon}.
\]

(40)

Thus:

\[
\phi' (x) = \lambda e^{-\lambda x} \mathbb{1}_{x \leq \varepsilon} + \left( e^{-\lambda x} - 1 \right) \delta_{\varepsilon} (x).
\]

(41)

Following the distribution of \( R \) under the Ito measure \( n \) the first equality in (42) is obtained by the Master Formula (92), see [RY99, XII, Proposition1.10]. Further, we progress by Fubini theorem, so we obtain:

\[
E \left[ (1 - e^{-\lambda R}) \mathbb{1}_{R \leq \varepsilon} \right] + E [\mathbb{1}_{R > \varepsilon}] = \int \phi (R (u)) n (du) + \int \mathbb{1}_{R(u) > \varepsilon} n (du)
\]

\[
= \int_0^\infty \phi' (x) G (x) \, dx + G (\varepsilon)
\]

\[
= \lambda \int_0^\varepsilon e^{-\lambda x} G(x) \, dx + e^{-\lambda \varepsilon} G (\varepsilon) - G (\varepsilon) + G (\varepsilon)
\]

\[
= \left( e^{-\lambda \varepsilon} - 1 \right) G (\varepsilon) + \lambda \int_0^\varepsilon e^{-\lambda x} G(x) \, dx + G (\varepsilon)
\]

\[
= \left( e^{-\lambda \varepsilon} - 1 \right) G (\varepsilon) + \lambda \int_0^\varepsilon e^{-\lambda x} G(x) \, dx + \int_\varepsilon^\infty g(x) \, dx
\]

\[
= \int_0^\varepsilon \left( 1 - e^{-\lambda x} g(x) \right) \, dx + \int_\varepsilon^\infty g(x) \, dx,
\]

(42)
where last equality is obtained by the partial integration. Furthermore, (30) is obtained from differentiating the generating function.

We note that the moments from (30) are consistent with the asymptotics of the moments from Corollary [3].

**Remark 8.** The distribution of the simplified Brownian avalanche length was obtained by [dW13]. We see that the limit of the random walk avalanche length agrees with the Brownian avalanche length. To prove that the limit distribution actually is the Brownian avalanche length distribution, without referring to the result in [dW13], would require further justification, perhaps a continuity argument for the avalanche length as a functional on the Skorohod (or Wiener) space. This is postponed for now.

The analytical result above allows to study the distribution of $L_\varepsilon$ in more detail. We conjecture it has an exponentially decaying tail, and thus positive integer moments of all orders, which can be obtained from differentiating the probability generating function [HR14].

### 4 More excursions: The full avalanche length

In this section we are going to analyze the full avalanche length with parameter $\mu \geq 1$ and windows parameter $\varepsilon > 0$, which we denote by $L^*_\mu,\varepsilon$. The order execution avalanche continues, until there is a time interval of length greater than $\varepsilon$ that contains neither Type I nor Type II trades, and thus the full avalanche length is obtained from a sequence of trades with intratrading time less than $\varepsilon$ followed by a trade with intratrading time bigger than $\varepsilon$.

Formally $L^*_\mu,\varepsilon$ is defined as follows: If $k \geq 1$ and $T_1 \leq \varepsilon, \ldots, T_k \leq \varepsilon, T_{k+1} > \varepsilon$, then

$$L^*_\mu,\varepsilon = T_1 + \ldots + T_k. \quad (43)$$

**4.1 Generating functions for the full avalanche length**

Given the parameter $\mu \geq 1$ we define three sets of random walk paths of length $n \geq 1$ by

$$\mathcal{A}_{n,\mu} = \{ s \in U_n : s_0 = 0, -\mu < s_k \leq 0 \text{ for } 0 < k \leq n-1 \text{ and } s_n = +1 \}, \quad (44)$$

$$\mathcal{B}_{n,\mu} = \{ s \in U_n : s_0 = 0, -\mu < s_k \leq 0 \text{ for } 0 < k \leq n-1, s_{n-1} = 0 \text{ and } s_n = -1 \}, \quad (45)$$

$$\mathcal{C}_{n,\mu} = \{ s \in \mathcal{A}_{n,\mu} : \min(s_0, \ldots, s_{n-1}) = -\mu + 1 \}. \quad (46)$$
Next we define the corresponding sets with arbitrary, finite path length, that is,

$$\mathcal{A}_\mu = \bigcup_{n \geq 1} \mathcal{A}_{n,\mu}, \quad \mathcal{B}_\mu = \bigcup_{n \geq 1} \mathcal{B}_{n,\mu}, \quad \mathcal{C}_\mu = \bigcup_{n \geq 1} \mathcal{C}_{n,\mu}. \quad (47)$$

Examples are given in Figure 7, Figure 8 and Figure 9.

![Figure 7: Some elements of $\mathcal{A}_\mu$](image)

![Figure 8: Some elements of $\mathcal{B}_\mu$](image)

![Figure 9: Some elements of $\mathcal{C}_\mu$](image)

We will count all trading paths of length $n$ and of a certain type by combinatorial enumeration. If $n \geq 0$ and $(s_0, \ldots, s_n)$ is a path we assign the size $n$ to it. We associate with Cartesian products of sets of trading paths the set of all concatenations of the paths from the components, and obviously their sizes are added. As we have the symmetric Bernoulli model, the corresponding probability generating functions are obtained from the ordinary generating functions by the substitution $z \mapsto z/2$.

As in [Fel68, XIV.4] let us introduce

$$\lambda_1(s) = \frac{1 + \sqrt{1 - s^2}}{s}, \quad \lambda_2(s) = \frac{1 - \sqrt{1 - s^2}}{s}. \quad (48)$$
Then we have the following lemma:

**Lemma 6.** The probability generating functions for the combinatorial classes \( \mathcal{A}_\mu, \mathcal{B}_\mu, \mathcal{C}_\mu \) are given by

\[
A_\mu(s) = \frac{\lambda_1(s)^\mu - \lambda_2(s)^\mu}{\lambda_1(s)^{\mu+1} - \lambda_2(s)^{\mu+1}}, \quad (49)
\]

\[
B_\mu(s) = A_\mu(s), \quad (50)
\]

\[
C_\mu(s) = A_\mu(s) - A_{\mu-1}(s). \quad (51)
\]

**Proof.** Shifting a path from \( \mathcal{A}_\mu \) by \( \mu \) we obtain a path corresponding to absorption at zero at the \( n \)-th trial in a symmetric game of gamblers ruin with initial position \( \mu \) and absorbing barriers at 0 and \( \mu + 1 \). This generating function is derived in [Fel68, XIV.4, (4.12), P.351]. Using that formula for \( p = 1/2, q = 1/2, z = \mu, a = \mu + 1 \) yields (49). The last step for a path in \( \mathcal{B}_{n,\mu} \) is down. If we replace it by an up-step, we obtain a path in \( \mathcal{A}_{n,\mu} \). Thus we have a bijection from \( \mathcal{B}_{n,\mu} \) to \( \mathcal{A}_{n,\mu} \) yielding (50). The set \( \mathcal{C}_{n,\mu} \) contains paths of length \( n \), that hit \( -\mu + 1 \), but not \( -\mu \), thus

\[
\mathcal{C}_{n,\mu} = \mathcal{A}_{n,\mu} \setminus \mathcal{A}_{n,\mu-1}. \quad (52)
\]

Since \( \mathcal{A}_{n,\mu-1} \subseteq \mathcal{A}_{n,\mu} \) equation (51) follows.

Next we state the key result for the description of Type II trades, with is a path decomposition for trading excursions that lead to a Type II trade. which is, in fact, related to the decomposition illustrated in [RY99, Sec.VI.3, Fig.5,P.256]. We state first the path decomposition and the proof, which refers to a few lemmas, which we provide for the clarity of exposition afterwards, at the end of this subsection.

**Proposition 3.** Suppose that the first trade is a Type II. Then there exists an integer \( K \geq 1 \), such that \((S(0),\ldots,S(T_1))\) is the concatenation of \( K \) elements from \( \mathcal{B}_\mu \) and one element from \( \mathcal{C}_\mu \).

**Proof.** Assume that the first trade is Type II. Then \((S(0),\ldots,S(T_1))\) contains no Type I trade, thus \(S(0),\ldots,S(T_1)\) are non positive by Lemma 9. A Type II trade occurs when the price goes down by \( \mu \) or more steps and then goes up by \( \mu \) steps. The first time this behavior is completed, the excursion ends.

Let \( K = -S(T_1 - 1) \) and \( \ell^* = \max\{0 \leq n < T_1 - 1 : S(n) = -K\} \), see Figure 10 for an illustration particularly for \( \mu = 5, K = 4 \) and \( \ell^* = \ell_4 \). Since the Type II trade is completed, the path \((S(\ell^*),\ldots,S(T_1))\) must go down to the level \(-K - \mu + 1\), and at that moment the new order is placed at the level \(-K + 1\). Then, the Type II trade occurs at time \( T_1 \), i.e. \( S(T_1) = -K + 1 \). Note that if the path \((S(\ell^*),\ldots,S(T_1))\) did go deeper, a Type II trade would
be followed by Type I trade which would complete before $T_1$. Thus this part of the path, namely the path $(S(\ell^*),...,S(T_1))$, belongs to the class $\mathcal{C}_\mu$.

Further, note that every level from 0 to $-K$ is visited by $(S(0),...,S(\ell^*))$. Let $\ell_0 = 0$ and for $k = 1,...,K$ define $\ell_k$ as the last time level $-k$ is visited before $\ell^*$ plus one, i.e. $\ell_k = \max\{0 < n < \ell^* + 1 : S(n) = -k\}$. Since there is no trade at all, in particular no Type I trade during this initial period, by Lemma 8, the paths $(S(\ell_{j-1}),...,S(\ell_j))$, $j = 1,...,K$, have depth less than $\mu$. Therefore, there are $K$ paths, precisely $(S(\ell_{j-1}),...,S(\ell_j))$ $j = 1,...,K$, that belong to class $\mathcal{B}_\mu$.

**Remark 9.** The $\ell_j$ are not stopping times, but honest times.

**Remark 10.** The final down segments in the $\mathcal{B}_\mu$ paths seem to vanish in the Brownian limit, but they correspond to the support of the local time at the minimum.

**Lemma 7.** If the first trade is a Type II trade, then

\[
\max\{S(0),...,S(T_1)\} \leq 0, \quad \min\{S(0),...,S(T_1)\} \leq -\mu.
\]

**Proof.** Assume by contradiction there is a time $n \in [0,T_1]$ with $S_n > 0$. We can assume that this $n$ is the minimal number with this property. Then we had $V(n,S_n) > 0$ and $S_n > S_0$, thus the first trade would be a Type I trade. Assume next by contradiction that $-\mu < S_n \leq 0$ for all $n \in [0,T_1]$. Then the recursion for $\alpha_n$ shows $\alpha_n = 1$ for all $n \in [1,T_1]$ and no trade could take place.

**Lemma 8.** Suppose $0 \leq a \leq b$ and $S(a) = S(b)$. If

\[
\min\{S(a),...,S(b)\} < S(a) - \mu,
\]

Figure 10: Path decomposition of a Type II trading excursion, $\mu = 5$, $K = 4$
there is a Type II trade followed by a Type I trade in \([a, b]\). Moreover, if
\[
\min\{S(a), \ldots, S(b)\} = S(a) - \mu,
\]
there is a Type II trade at time \(b\).

**Proof.** Let
\[
\gamma = \min(S(a), \ldots, S(b)), \quad \beta = \min\{n \in [a, b] : S(n) = \gamma\},
\]
and
\[
\delta = \min\{n \in [\beta, b] : S(n) = \gamma + \mu\}.
\]
At time \(\beta\) a new order is placed at the price level \(S_n + \mu = \gamma + \mu\), and therefore we have \(V(\beta + 1, \gamma + \mu) > 0\). This order at price level \(\gamma + \mu\) gets executed at time \(\delta\).

The second part of the lemma follows directly from the definition of the Type II trade.

\[\square\]

**Lemma 9.** The first trade after zero is a Type I trade iff
\[
\max(S(0), \ldots, S(T_1 - 1)) = 0, \quad S(T_1) = 1
\]
and
\[
\min(S(0), \ldots, S(T_1 - 1)) > -\mu.
\]

**Proof.** Suppose that the first trade is Type I trade. Then there is no Type II trade during the time interval \([0, T_1]\), and thus by Lemma 8 inequality (59) holds. Note that the price level 1 is the strict ladder time, and therefore, according to Lemma 1, a Type I trade occurs at the price level 1. Since the first trade occurs at time \(T_1\), it directly implies that (58) holds. Conversely, assume that (59) and (58) hold. Then, by Lemma 1, a Type I trade occurs at time \(T_1\), and there is no earlier Type I trade. Further, Lemma 7 shows there is no Type II trade earlier.

\[\square\]

### 4.2 Generating function for the full avalanche length

**Proposition 4.** The generating function of the time to the next trade is given by
\[
E[z^{T_1}] = A_\mu(z) + \frac{B_\mu(z)}{1 - B_\mu(z)} \cdot C_\mu(z).
\]

**Proof.** Let us define the set \(\mathcal{T}_{\mu,n}\) of all paths where the first trade occurs at step \(n\). and \(\mathcal{T}_\mu = \bigcup_{n \geq 1} \mathcal{T}_{\mu,n}\). This set contains elements that are either Type I trades, and thus correspond by Lemma 9 to the class \(\mathcal{A}_\mu\) or Type II trades, which correspond by the path decomposition in Proposition 3 bijectively to the Cartesian product of the set of finite sequences of length 1 or...
more of elements in \( \mathcal{B}_\mu \) times the class \( \mathcal{C}_\mu \). Using the symbolic notation from [FS09] Section I] this is written more clearly as
\[
\mathcal{F}_\mu = \mathcal{A}_\mu \cup \text{SEQ}_{\geq 1}(\mathcal{B}_\mu) \times \mathcal{C}_\mu. \tag{61}
\]
From [FS09] Theorem I.1, P.27 and the section on restricted constructions, in particular SEQ\(_{\geq k}\), in [FS09] P.30, we obtain (60).

Once the law of \( T_1 \) is found, we can compute the avalanche length distribution.

**Theorem 2.** The full avalanche length for the symmetric random walk has probability generating function
\[
E[z^{T_1}] = P[T_1 > \varepsilon] \frac{1}{E[1 - z^{T_1}; T_1 \leq \varepsilon] + P[T_1 > \varepsilon]}, \tag{62}
\]
where \( T_1 \) has the probability generating function given in (60) above.

**Proof.** This follows exactly from the same arguments as in the proof of Proposition I with \( R \) replaced by \( T \).

Let us now consider also the following quantities: The time to the first trade, assuming it is a Type I trade, that is, \( T_1 \) on \( \{S(T_1) > 0\} \); the time to the first trade, assuming it is a Type II trade, that is, \( T_1 \) on \( \{S(T_1) \leq 0\} \); and the time to the first Type II trade.

Furthermore, denote by \( D \) the index of the first Type II trade in the sequence of all trades, i.e.:
\[
D = \inf\{i \geq 1 : S(\tau_i) < S(\tau_{i-1})\}. \tag{63}
\]

Consequently the time to the first Type II trade is \( \tau_D \).

**Lemma 10.** We have [3]
\[
E[z^{T_1}; S(T_1) > 0] = A_\mu(z), \quad E[z^{T_1}; S(T_1) \leq 0] = \frac{B_\mu(z)}{1 - B_\mu(z)} \cdot C_\mu(z) \tag{64}
\]
and
\[
E[z^D] = \frac{1}{1 - E[z^{T_1}; S(T_1) \leq 0]} \cdot E[z^{T_1}; S(T_1) \leq 0]. \tag{65}
\]

**Proof.** By following Lemma 9 a Type I trading excursion corresponds to an element in \( \mathcal{A}_\mu \), and therefore by [FS09] Theorem I.1, P.27 the left equation in (64) holds. By Proposition 3 path of a Type II trading excursion corresponds to the second term on the right hand side of (61), and thus by [FS09] Theorem I.1, P.27 the right equation in (64) holds. The path to the next Type II trades consists of a (possible empty) sequence of Type I trading excursions followed by a Type II trading excursions, thus to SEQ(\( \mathcal{B}_\mu \)) × \( \mathcal{C}_\mu \). We can apply again [FS09] Theorem I.1, P.27 to obtain the corresponding generating functions.

[3]We use semicolon notation, \( E[X; A] = E[X I_A] \) for an integrable random variable \( X \) and an event \( A \).
Corollary 4.

\[ P[S(T_1) > 0] = \frac{\mu}{\mu + 1} \quad (66) \]

Proof. We set \( z = 1 \) in the previous result.

### 4.3 The particular probabilities

By combining (60) with (49), (50) and (51) we have

\[ T_\mu(z) = A_\mu(z) + \frac{A_\mu(z)}{1 - A_\mu(z)}(A_\mu(z) - A_{\mu-1}(z)) = \frac{A_\mu - A_\mu(z)A_{\mu-1}(z)}{1 - A_\mu(z)}. \quad (67) \]

For \( \mu = 1, 2, 3, 4, 5, 6, 7 \), we use series expansion and expand the corresponding formula (67). Thus, we obtain probabilities \( p_n = P[T_1 = n] \) for particular values see Table 1 when \( n = 1, \ldots, 10 \) and \( \mu = 1, \ldots, 7 \). Furthermore, the following holds:

\[ T_\mu(z) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} p_k \right) z^n (1 - z). \quad (68) \]

Let \( q_n = P[T_1 > n] \), i.e. \( q_n = \sum_{k \geq n+1} P[T_1 = k] \), and let \( Q_\mu(z) = \sum_{n \geq 0} q_n z^n \). Then, we have

\[ Q_\mu(z) = \sum_{n \geq 0} \left( \sum_{k \geq n+1} p_k \right) z^n = \frac{1 - T_\mu(z)}{1 - z}. \]

For particular values of probabilities \( q_\varepsilon = P[T_1 > \varepsilon] \) when \( \varepsilon = 1, \ldots, 10, 11 \) and \( \mu = 1, \ldots, 7 \) see Table 2.

Recall (62), i.e. we have:

\[ E[z^{L_{\mu,\varepsilon}}] = \frac{P[T_1 > \varepsilon]}{1 - E[z^{T_1} : T_1 \leq \varepsilon]} = \frac{q_\varepsilon}{1 - \sum_{k=1}^{\varepsilon} z^k p_k}. \quad (69) \]
Table 2: The probabilities $q_\varepsilon = P[T_1 > \varepsilon]$ for $\varepsilon = 1, \ldots, 9$ and $\mu = 1, \ldots, 7$

Thus, by plugging particular values from Table 1 and Table 2, when $\mu = 1, 2, 3, 4$ and $\varepsilon = 1, 2, 3, 4, 5$, in equation (69), the probability generating functions of the full avalanche length $L_{\mu, \varepsilon}^*$ are derived, see Table 3.

Table 3: The probability generating function of the full avalanche length $L_{\mu, \varepsilon}^*$ when $\mu = 1, 2, 3, 4$ and $\varepsilon = 1, 2, 3, 4, 5$

Furthermore, we expand the probability generating function of the full avalanche length presented in Table 3 and calculate the individual probabilities see Table 4 (for $\mu = 1$), Table 5 (for $\mu = 2$) and Table 6 (for $\mu = 3$ and $\mu = 4$).

4.4 Limit results for the full avalanche length

We consider the Brownian limits

$$\left\{ \frac{1}{\sqrt{n}} S_{[nt]} : t \geq 0 \right\} \rightarrow \{W(t) : t \geq 0\}$$

in distribution on the Skorohod space, see [Bil99, Theorem 14.1. of XIV,P.146]. Linear interpolation on the Wiener space works similarly see [Bil99, Theorem 8.6. of VIII,P.90].
Table 4: For $\mu = 1$ and for $\varepsilon = 1, 2, 3, 4, 5$, the probabilities $P[L_{1,\varepsilon}^* = k]$ that the full avalanche length takes values $k = 1, \ldots, 8$.

| $\varepsilon$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 1             | 1/4 | 1/8 | 1/16| 1/32| 1/64| 1/128| 1/256| 1/512|
| 2             | 1/8 | 1/8 | 3/32| 5/64| 1/16| 13/256| 21/512| 17/512|
| 3             | 1/16| 1/16| 1/16| 7/128| 13/256| 3/64| 11/256| 81/2048|
| 4             | 1/32| 1/32| 1/32| 1/32| 15/512| 29/1024| 7/256| 27/1024|
| 5             | 1/64| 1/64| 1/64| 1/64| 1/64| 31/2048| 61/4096| 15/1024|

Table 5: For $\mu = 2$ and for $\varepsilon = 1, 2, 3, 4, 5$, the probabilities $P[L_{2,\varepsilon}^* = k]$ that the full avalanche length takes values $k = 1, \ldots, 8$.

| $\varepsilon$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 1             | 1/4 | 1/8 | 1/16| 1/32| 1/64| 1/128| 1/256| 1/512|
| 2             | 1/4 | 1/8 | 1/16| 1/32| 1/64| 1/128| 1/256| 1/512|
| 3             | 3/16| 3/32| 3/32| 9/128| 3/64| 9/256| 27/1024| 39/2048|
| 4             | 5/32| 5/64| 5/64| 5/64| 15/256| 45/1024| 75/2048| 125/4096|
| 5             | 1/8 | 1/16| 1/16| 1/16| 1/16| 13/256| 21/512| 37/1024|

We note that time must be scaled linearly, $t \mapsto nt$, and space must be scaled with the square-root $\mu \mapsto \sqrt{n}\mu$. Excursion length refers to time, excursion depth to space.

Up to here we have omitted in the notation for the intratrading times the dependence on the parameter $\mu$. Now we must use $\mu$ as a variable in the limiting procedure, thus we write from now on $T_{1}^{(\mu)}$ instead of $T_{1}$, and for brevity $T_{1}^{(n\mu)}$ for the more precise $T_{1}^{(\lfloor n\mu \rfloor)}$.

**Proposition 5.** We have for the Laplace-Stieltjes transform

$$E[e^{-sT_{1}^{(\sqrt{n}\mu)}}] = 1 - \sqrt{2}\tanh\left(\mu\sqrt{2}s\right) \cdot n^{-\frac{1}{2}} + \mathcal{O}(n^{-1}), \quad n \to \infty.$$  \hspace{1cm} (71)

*Proof.* Slightly lengthy, but still elementary asymptotic expansions\(^4\) using the expansion of $\sqrt{1 - z^2}$ for $z \to 1$ obtain the expansion of $\lambda_{1}(z)$ namely:

$$\lambda_{1}(z) = 1 + \sqrt{2}(1 - z)^{1/2} + (1 - z) + \frac{3}{2\sqrt{2}}(1 - z)^{3/2} + \mathcal{O}(1 - z)^{2}.$$  

---

\(^4\)The detailed calculations can be performed by hand in a few pages but can be also performed (and checked) automatically with a computer algebra system.
Table 6: For \( \mu = 3, 4 \) and for \( \varepsilon = 1, 2, 3, 4, 5 \), the probabilities \( P[L^*_3,\varepsilon = k] \) that the full avalanche length takes values \( k = 1, \ldots, 8 \).

| \( \varepsilon \) | 1/4 | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 | 1/256 | 1/512 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1/4 | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 | 1/256 | 1/512 |
| 2 | 1/4 | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 | 1/256 | 1/512 |
| 3 | 3/16 | 3/32 | 3/32 | 9/128 | 3/64 | 9/256 | 27/1024 | 39/2048 |
| 4 | 3/16 | 3/32 | 3/32 | 9/128 | 3/64 | 9/256 | 27/1024 | 39/2048 |
| 5 | 5/32 | 5/64 | 5/64 | 15/256 | 15/256 | 25/512 | 75/2048 | 125/4096 |

Inserting this into the power series for \( \log(1 + x) \) at \( x = 0 \) yields

\[
\log \lambda_1(z) = \sqrt{2}(1 - z)^{1/2} + \frac{5}{6\sqrt{2}}(1 - z)^{3/2} + O(1 - z)^2 ,
\]
and a similar calculation yields

\[
\log \lambda_2(z) = -\sqrt{2}(1 - z)^{1/2} - \frac{5}{6\sqrt{2}}(1 - z)^{3/2} + O(1 - z)^2 .
\]

Now we expand \( e^{-s/n} \) for \( n \to \infty \) using the exponential series, and the elementary formula

\[
\lambda_1(e^{-s/n})^{\mu\sqrt{n}} = \exp(\mu\sqrt{n} \log \lambda_1(e^{-s/n}))
\]
and combine the asymptotic expansions, similarly for \( \lambda_2 \). If we consider enough terms to account for cancellations in the differences in [19] and [51] we obtain expansion for \( A_{\mu\sqrt{n}}(e^{-\frac{\pi}{2}}) \) which coincides with \( B_{\mu\sqrt{n}}(e^{-\frac{\pi}{2}}) \), and we find also the expansion of \( C_{\mu\sqrt{n}}(e^{-\frac{\pi}{2}}) \) as \( n \to \infty \). Finally we get a quotient of exponentials, which can be expressed as hyperbolic tangent.

The following limit results involve the hyperbolic tangent \( \tanh \), the hyperbolic cosecant \( \text{csch} \), the hyperbolic secant \( \text{sech} \) and the hyperbolic cotangent \( \text{coth} \).

**Proposition 6** (Hyperbolic function table). We have pointwise for \( n \to \infty \) the asymptotic relations

\[
\begin{align*}
E[e^{-\frac{\pi}{2} T_{1}^{\mu\sqrt{n}}}] &= 1 - \sqrt{2s} \tanh(\mu\sqrt{2s}) \cdot n^{-\frac{1}{2}} + O(n^{-1}), \\
E[e^{-\frac{\pi}{2} T_{1}^{\mu\sqrt{n}}}; S(T_{1}^{\mu\sqrt{n}}) > 0] &= 1 - \sqrt{2s} \coth(\mu\sqrt{2s}) \cdot n^{-\frac{1}{2}} + O(n^{-1}), \\
E[e^{-\frac{\pi}{2} T_{1}^{\mu\sqrt{n}}}; S(T_{1}^{\mu\sqrt{n}}) \leq 0] &= 2\sqrt{2s} \text{csch}(2\mu\sqrt{2s}) + O(n^{-1/2}), \\
E[e^{-\frac{\pi}{2} T_{D}^{\mu\sqrt{n}}}] &= \text{sech}(2\mu\sqrt{2s})^2 + O(n^{-1/2}).
\end{align*}
\]
Proof. Elementary asymptotic calculations are performed, similarly as in the proof of Proposition 5. Clearly the entries \((73)\) and \((74)\), add up to \((72)\) by the law of total probability. This corresponds in the limit to the elementary identity
\[
\tanh \left( \frac{z}{2} \right) = \coth(z) - \text{csch}(z),
\]
see [AS64, 4.5.30, P.84].

Lemma 11. Let
\[
h(x) = \frac{1}{\sqrt{2\pi}} x^{-3/2} + 2 \sum_{k \geq 1} \left[ \frac{1}{\sqrt{2\pi}} \cdot x^{-3/2} - 2 \frac{\sqrt{2}}{\pi} k^2 \mu^2 \cdot x^{-5/2} \right] \exp \left( -\frac{2k^2 \mu^2}{x} \right),
\]
then we have
\[
\int_0^\infty (1 - e^{-\lambda x}) h(x) \, dx = \sqrt{2\lambda} \tanh \left( \mu \sqrt{2\lambda} \right).
\]
Proof. We justify termwise integration and sum the resulting series.

Theorem 3. The Laplace transform of the scaled full avalanche length for the simple symmetric random walk satisfies:
\[
\lim_{n \to \infty} E\left[ e^{-\frac{\lambda}{n} T^*_{\mu,\sqrt{n},\epsilon}} \right] = \frac{\int_0^\infty h(x) \, dx}{\int_0^\epsilon (1 - e^{-\lambda x}) h(x) \, dx + \int_\epsilon^\infty h(x) \, dx}.
\]
Proof. We combine the limit results for the first trade in Proposition 5, Lemma 11 and Lemma 13.

Remark 11. Integrating the series for \(h\) term by term, we find series representations for the integrals in the numerator and denominator in \((79)\). The terms for \(\int_0^\epsilon (1 - e^{-\lambda x}) h(x) \, dx\) can be expressed in terms of the error and complementary error functions. Termwise integration of \(\int_\epsilon^\infty h(x) \, dx\) yields there exists the series representation which can be recognize as a series representation of a Jacobi theta function. We have
\[
\lim_{n \to \infty} P \left[ \frac{1}{n} T^*_{\mu,\sqrt{n}} > \epsilon \right] = \vartheta_4(0, \epsilon).
\]
5 An initially empty order book

Let us assume that we start with an order book that is initially empty, i.e.

\[ V(0, u) = 0, \quad u \in \mathbb{Z}. \]  

(81)

Furthermore, as in the case when the initially full limit order book is considered, the dynamics involves with respect to the equation (2). In order to be precise, specifically trading times \( \{ \tilde{\tau}_i : i \geq 0 \} \) and intertrading times \( (\tilde{T}_i)_{i \geq 1} \) corresponding to the model when the limit order book is initially empty are defined by

\[ \tilde{\tau}_0 = 0, \quad \tilde{\tau}_i = \inf \{ n > \tilde{\tau}_{i-1} : V(n, S_n) > 0 \}, \quad \tilde{T}_i = \tilde{\tau}_i - \tilde{\tau}_{i-1}, \quad i \geq 1. \]  

(82)

The first trade (either Type I or Type II trade) will occur at smallest time \( \tilde{T}_1 \) for which it is satisfied:

\[ |\min(S(0), S(1), \ldots, S(\tilde{T}_1 - 1)) - S(\tilde{T}_1)| = \mu. \]

Remark 12. Due to the fact that the limit order book model dynamics follow \( (2) \), note that if we start with an initially empty order book after occurring the first trade (either Type I or Type II trade) the analysis of the avalanche length is equivalent as we have started with initially full order book.

![Figure 11: Example of the path that depicts the trivial case for the first Type I trade.](image)
5.1 The first Type I trade in an initially empty book

If the first Type I trade in an initially empty book occurs at the price level $k$, for $k \in \{1, 2, ..., \mu\}$, then the path $(S(0), S(1), ..., S(T_1))$ goes down by $\mu - k$ steps, but not below, and then goes up to the level $k$, as illustrated in Figure 12. The trivial case when $k = \mu$ is depicted at Figure 11, the new order is placed at the start at distance $S_0 + \mu = \mu$ and the first trade is occued at level $\mu$, after exactly $\mu$ steps up.

For $n \geq 1$, $\mu \geq 1$ and $k \in \{1, 2, ..., \mu\}$ we introduce the paths:

$$G_{n, \mu, k} = \{ s \in U_n : s_0 = 0, k - \mu < s_j < \mu \text{ for } 1 \leq j \leq n - 1 \text{ and } s_n = k - \mu \},$$  \quad (83)$$

$$F_{n, \mu, k} = \{ s \in U_n : s_0 = k - \mu, k - \mu - 1 < s_j < k \text{ for } 1 \leq j \leq n - 1 \text{ and } s_n = k \}. $$ \quad (84)

![Figure 12](image)

Figure 12: The path of the first Type I trade in the initially empty order book can be seen as a concatenation of two paths: first one which is starting at 0, forbidden level is $\mu$ and absorbing at level $k - \mu$; second path which is starting at $k - \mu$, forbidden level is $k - \mu - 1$ and absorbing at level $k$, where $k \in \{1, 2, ..., \mu\}$

Furthermore, define the corresponding sets with arbitrary, finite path length by:

$$G_{\mu, k} = \bigcup_{n \geq 1} G_{n, \mu, k}, \quad F_{\mu, k} = \bigcup_{n \geq 1} F_{n, \mu, k} \quad (85)$$
Lemma 12. For \( k \in \{1, 2, \ldots, \mu\} \) denote by \( G_{\mu,k}, F_{\mu,k} \) the probability generating functions for the combinatorial classes \( \mathcal{G}_{\mu,k}, \mathcal{F}_{\mu,k} \) respectively. We have

\[
G_{\mu,k}(s) = \frac{\lambda_1(s)^\mu - \lambda_2(s)^{\mu-k}}{\lambda_1(s)^{2\mu-k} - \lambda_2(s)^{2\mu-k}}, \tag{86}
\]

\[
F_{\mu,k}(s) = \frac{\lambda_1(s) - \lambda_2(s)}{\lambda_1(s)^{\mu+1} - \lambda_2(s)^{\mu+1}}. \tag{87}
\]

Proof. The generating function \( G_{\mu,k} \) is derived following the formula [Fel68, XIV.4, (4.11), P.351], by shifting a path from \( G_{n,\mu,k} \) by \( \mu - k \). Thus, we obtain a path corresponding to absorption at zero at the \( n \)-th trial in a symmetric game of gamblers ruin with initial position \( \mu - k \) and absorbing barriers at 0 and \( 2\mu - k \). The formula [Fel68, XIV.4, (4.11), P.351] for \( p = 1/2, q = 1/2, z = \mu - k, a = 2\mu - k \) yields (86).

The generating function \( F_{\mu} \) is derived following the formula [Fel68, XIV.4, (4.12), P.351] by shifting a path from \( F_{n,\mu,k} \) by \( \mu - k + 1 \). Therefore, the formula is obtained for \( p = 1/2, q = 1/2, z = 1, a = \mu + 1 \).

Since the generating function \( F_{\mu,k} \) does not depend on \( k \), from now on we can omit subscript \( k \) and write \( F_{\mu} \).

5.2 The first Type II trade in initially empty book

The path of the first Type II trade in the initially empty order book needs to down by \( \mu \) or more, and then to go up by \( \mu \) steps. Therefore, if the first trade in initially empty order book is a Type II trade, then the path \( (S(0), \ldots, S(\tilde{T}_1)) \) is a concatenation of \( K \geq 1 \) elements from \( \mathcal{B}_\mu \) and one element from \( \mathcal{C}_\mu \). The proof is same as in the Proposition 3.

5.3 The first trade in initially empty book

Proposition 7. If the order book is initially empty, the generating function of the time to the first trade is given by

\[
E[z^{\tilde{T}_1}] = \sum_{k=0}^{\mu} G_{\mu-k}(z) F_{\mu}(z) + B_{\mu}(z) C_{\mu}(z). \tag{88}
\]

Proof. Let us define the set \( \tilde{\mathcal{T}}_{\mu,n} \) of all paths where the first trade in an initially empty order book occurs at step \( n \). and \( \tilde{\mathcal{T}}_{\mu} = \bigcup_{n \geq 1} \tilde{\mathcal{T}}_{\mu,n} \). This set contains elements that are either Type I trades or Type II trades. By Lemma 12 the path of the Type I trade is the concatenation of an element from the class \( \mathcal{G}_{\mu,k} \) and an element from the class \( \mathcal{F}_{\mu,k} \). The path of the Type II trade corresponds bijectively to the Cartesian product of the set of
finite sequences of length 1 or more of elements in $B$ times the class $C$.

Using the symbolic notation from [FS09, Section I] this is written more clearly as

$$\tilde{T}_\mu = \sum_{k=0}^{\mu} T_{\mu-k}(z) \cup \text{SEQ}_{\geq 1}(B_{\mu} \times C_{\mu}). \quad (89)$$

From [FS09, Theorem I.1, P.27] and the section on restricted constructions, in particular $\text{SEQ}_{\geq k}$, in [FS09, P.30], the probability generating function (88) is obtained.

6 Auxiliary material

6.1 Some results of excursion theory

We recall some results of excursion theory, in particular we refer to Revuz and Yor [RY99, XII.2, P.480]. Let $(U, \mathcal{U})$ be the measurable space of Brownian excursions, and let $(\epsilon_t, t > 0)$ be the excursion process. Since the Define by $U_\delta = U \cup \{\delta\}$ the set enhanced by the zero-excursion (which is set equal to $\delta$) on the set where the local time at zero is strictly increasing. Further, this set is equipped with the $\sigma$-algebra $\mathcal{U}_\delta = \sigma(\mathcal{U}, \{\delta\})$.

For a measurable subset $\Gamma$ of $\mathcal{U}_\delta$, the function

$$N^\Gamma_t(\omega) = \sum_{0 < u \leq t} 1_\Gamma(\epsilon_u(\omega)) \quad (90)$$

is measurable.

The Itô measure $n$ is the $\sigma$-finite measure defined on $\mathcal{U}$ by

$$n(\Gamma) := E\left[N^\Gamma_1\right] \quad (91)$$

and extended to $\mathcal{U}_\delta$ by $n(\delta) = 0$.

Since these functions graphs are either entirely above or below the $t$-axis, we denote by $U^+$ and $U^-$ the subsets of the set $U$. Further, $n^+$ and $n^-$ are upper and lower Itô measures, i.e. restrictions of $n$ to $U^+$ and $U^-$ respectively.

It turns out that the excursion process is a Poisson Point Process, and hence the Itô measure is its characteristic measure. An important consequence of this is the Master Formula, see [RY99 XII, Proposition 1.10] for a general version, which states that for a positive $\mathcal{U}_\delta$-measurable function $H$ defined on $U_\delta$ we have

$$E[H(e(\omega))] = \int_U H(u) n(du). \quad (92)$$
6.2 On convergence in distribution and vague convergence

Lemma 13. Suppose we are given a real number \( \varepsilon > 0 \), a sequence of probability measures \((\mu_n)_{n \geq 1}\) on \( \mathcal{B}((0, \infty)) \) and a non-negative measure \( \nu \) on \( \mathcal{B}((0, \infty)) \), not the zero measure, such that

\[
\int_0^\infty (1 \wedge x) \nu(dx) < \infty. \tag{93}
\]

If \( \varepsilon \) is a continuity point for \( \nu \) and

\[
\int_0^\infty e^{-sx} \mu_n(dx) = 1 - \hat{\nu}(s) \cdot n^{-1/2} + \mathcal{O}(n^{-1}) \quad n \to \infty \quad (94)
\]

pointwise for all \( s \geq 0 \), where

\[
\hat{\nu}(s) = \int_0^\infty (1 - e^{-sx}) \nu(dx), \tag{95}
\]

then we have

\[
\lim_{n \to \infty} n^{1/2} \int_0^\varepsilon (1 - e^{-sx}) \mu_n(dx) = \int_0^\varepsilon (1 - e^{-sx}) \nu(dx) \tag{96}
\]

and

\[
\lim_{n \to \infty} n^{1/2} \int_\varepsilon^{\infty} \mu_n(dx) = \int_\varepsilon^{\infty} \nu(dx) \tag{97}
\]

for all \( \lambda \geq 0 \).

Proof. Assumption (93) implies that the integral in (95) is finite for all \( s \geq 0 \).

Let us denote the integral on the left hand side of (96), which is simply the Laplace transform of \( \mu_n \) by \( \tilde{\gamma}_n(s) \).

Since we assumed that \( \mu_n \) lives on \((0, \infty)\) we have \( 0 \leq \tilde{\gamma}_n(\lambda) < 1 \) for \( \lambda > 0 \).

Since we assumed that \( \nu \) is not the zero measure, we have also \( \hat{\nu}(\lambda) > 0 \). Let us define another new measure \( \nu_\lambda \) by

\[
\nu_\lambda(dx) = \frac{1 - e^{-\lambda x}}{\hat{\nu}(\lambda)} \nu_n(dx). \tag{98}
\]

Note that \( \nu_\lambda \) is a probability measure. Denote by \( \hat{\nu}_\lambda(s) \) its Laplace transform. The asymptotics (94) imply

\[
\lim_{n \to \infty} \tilde{\gamma}_n(s) = \hat{\nu}_\lambda(s) \tag{99}
\]

pointwise for all \( s \geq 0 \). By the continuity theorem for Laplace transforms, e.g. [Fel71, Theorem XIII.1.2a, P.433], it follows that \( \gamma_n \to \nu_\lambda \) weakly as \( n \to \infty \). As \( \varepsilon \) is also a continuity point for the limit distribution \( \nu_\lambda \), it follows

\[
\lim_{n \to \infty} \int_0^\varepsilon \gamma_n(dx) = \int_0^\varepsilon \nu_\lambda(dx). \tag{100}
\]
Relation (94) implies also
\[ \lim_{n \to \infty} n^{1/2}(1 - \tilde{\mu}_n(\lambda)) = \tilde{\nu}(\lambda). \]  
(101)
Combining (100) and (101) yields (97). Now consider the function
\[ f(x) = \frac{\tilde{\nu}(\lambda)}{1 - e^{-\lambda x}} I_{x > \varepsilon}, \quad x > 0. \]  
(102)
It is bounded and continuous except for the point \( x = \varepsilon \). By the continuous mapping theorem, as formulated for example in [Dur10] Theorem 3.2.4, P.101] we get
\[ \lim_{n \to \infty} \int_{\varepsilon}^{\infty} f(x) \gamma_n(dx) = \int_{\varepsilon}^{\infty} f(x) \nu(\lambda)(dx). \]  
(103)
Using the definitions of \( \gamma_n \) and \( \nu(\lambda) \) and (101) we obtain (96).

**Proposition 8.** In the setting of Lemma 15 we have
\[ \lim_{n \to \infty} n^{1/2} \mu_n = \nu \]  
(104)
vaguely on \((0, \infty)\).

**Proof.** Suppose \( f \) is a continuous function with compact support in \((0, \infty)\). Define \( g(x) = \tilde{\nu}(\lambda)f(x)/(1 - e^{-\lambda x}) \) for \( x > 0 \). Since \( f \) vanishes is some neighbourhood of \( x = 0 \), it follows that \( g \) is a bounded continuous function. Thus we have from weak convergence
\[ \lim_{n \to \infty} \int_{0}^{\infty} g(x) \gamma_n(dx) = \int_{0}^{\infty} g(x) \nu(dx), \]  
(105)
which can be rewritten with (101) as
\[ \lim_{n \to \infty} \int_{0}^{\infty} f(x) n^{1/2} \mu_n(dx) = \int_{0}^{\infty} f(x) \nu(dx), \]  
(106)
thus showing vague convergence.

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