Constraints on Rindler Hydrodynamics

Adiel Meyer and Yaron Oz

School of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel

(Dated: May 11, 2014)

Abstract

We study uncharged Rindler hydrodynamics at second order in the derivative expansion. The equation of state of the theory is given by a vanishing equilibrium energy density. We derive relations among the transport coefficients by employing two frameworks. First, by the requirement of having an entropy current with a non-negative divergence, second by studying the thermal partition function on stationary backgrounds. The relations derived by these two methods are equivalent. We verify the results by studying explicit examples in flat and curved space-time geometries.

PACS numbers: 04.70.-s, 11.25.Tq, 47.75.+f

*Electronic address: adielmey@post.tau.ac.il
†Electronic address: yaronoz@post.tau.ac.il
## Contents

I. Introduction and Summary \hspace{1cm} 3

II. The Fluid Data \hspace{1cm} 4

III. The Entropy current \hspace{1cm} 7

A. The entropy current at second order \hspace{1cm} 7
B. Constraints on the entropy current and the transport coefficients \hspace{1cm} 8
C. Flat space-time background \hspace{1cm} 10
D. Curved space-time background \hspace{1cm} 12

IV. Thermal Partition Function \hspace{1cm} 13

A. The stress-energy tensor \hspace{1cm} 14
B. The partition function \hspace{1cm} 15

Acknowledgements \hspace{1cm} 16

A. The entropy current divergence \hspace{1cm} 17
B. Curved background solution \hspace{1cm} 17

References \hspace{1cm} 20
I. INTRODUCTION AND SUMMARY

There has been much interest in recent years in the holographic relation between the hydrodynamics of quantum field theories defined on \((d + 1)\)-dimensional space-times and deformations of black hole geometry in one higher space dimension. Aspects of this relation have been studied for a large class of field theories in diverse dimensions including conformal and non-conformal theories, relativistic as well as non-relativistic ones, anomalous theories and theories with spontaneously broken symmetries.

One intriguing example, which will be the subject of this paper, is the Rindler hydrodynamics, that is the hydrodynamics induced on a codimension one hypersurface in Rindler geometry \([1, 5]\). The Rindler geometry is a solution to the vacuum Einstein equations, and correspondingly the equilibrium energy density in the hypersurface hydrodynamics vanishes. Despite this unconventional equation of state, the hydrodynamic description is well defined. Viewing hydrodynamics as an effective description of quantum field theory at large length scales and at local thermal equilibrium, one may hope that understanding Rindler hydrodynamics can shed light on how Holography works in asymptotically flat space-times.

In the hydrodynamic description one upgrades the local charge densities to local currents. In this framework, the entropy current in hydrodynamics is an upgrade of the local entropy density to a local current \([6]\). The entropy current has not been given yet a microscopic origin and it is not a unique object. It is constructed phenomenologically order by order in the hydrodynamic derivative expansion. One can generalize the second law of thermodynamics, by requiring that the entropy current has a non-negative divergence order by order in this expansion. The requirement imposes constraints on the transport coefficients of the hydrodynamic description, which in turn impose constraints on the underlying microscopic quantum field theory. This has been applied, for instance, in the study of anomalous hydrodynamics \([7, 8]\), uncharged hydrodynamics \([9]\), and in the analysis of holographic hydrodynamics \([10, 11]\).

There are two types of such constraints. One type gives equality constraints that relate different transport coefficients. The second type gives inequalities, such as the requirement that the shear and bulk viscosities at the first viscous order are non-negative. In order to fully exploit these constraints one studies them in the presence background fields. In the case of uncharged hydrodynamics the background field is a curved metric.
A second method to derive the first type of constraints employs the thermal partition function on a curved background with a timelike Killing vector $\text{[12, 13]}$. In this framework the constraints are related to the underlying gauge and diffeomorphism symmetries.

The aim of this paper is to study the first type of these constraints in the case of Rindler hydrodynamics. We will employ the two methods discussed above and compare the results. The paper is organized as follows. In section 2 we will classify the independent fluid data, i.e. modulo the conservation laws of the stress-energy tensor in ideal hydrodynamics. We will use as variables the fluid velocity and pressure as well as the curved background data. In section 3 we will construct the entropy current up to second order in the derivative expansion and its divergence. By requiring the latter to be non-negative we will derive constraints on the transport coefficients. We will check the constraints in two cases: flat hypersurface (with and without a bulk Gauss-Bonnet term) and curved hypersurface in Einstein gravity. In section 4 we will constrict the thermal partition function and derive the constraints on the transport coefficients. The constraints obtained by the two methods are equivalent. Some details of the calculation are outlined in the appendices.

II. THE FLUID DATA

We would like to study the (d+1)-dimensional Rindler hydrodynamics in a general curved background up to second order in the derivative expansion. The fluid variables that we will use are the fluid velocity $u^\mu$ normalized as $u_\mu u^\mu = -1$, and the fluid pressure $p$. The curvature data will be given by the Riemann tensor $R^\mu_{\nu\rho\sigma}$. In Rindler hydrodynamics we have a vanishing equilibrium energy density $\epsilon_0 = 0$. We will start by classifying the independent fluid and curvature data, that is, modulo the ideal fluid hydrodynamic equations and the curvature identities. The classification of fluid and curvature data when $\epsilon_0 \neq 0$ has been carried out in $\text{[9]}$. One cannot simply use this classification by setting $\epsilon_0 = 0$, since the ideal fluid hydrodynamic equations are different. We will use, however, the same classification scheme according to the transformation properties under the local $SO(d)$ symmetry group that leaves the (d+1)-velocity $u^\mu(x)$ invariant, as employed in $\text{[9]}$.

The Rindler hydrodynamics stress-energy tensor at the ideal level is

$$T_{\mu\nu}^{(0)} = p P_{\mu\nu}$$

(1)
and the conservation equations $\nabla_{\nu} T^{\sigma\nu} = 0$ projected on $u_\sigma$ and on $P^\mu_\sigma$ are:

$$\Theta \equiv \nabla_{\mu} u^{\mu} = 0 , \quad (2)$$
$$u^{\sigma} \nabla_{\sigma} u^{\mu} + P^{\mu\sigma} \nabla_{\sigma} \ln p = 0 . \quad (3)$$

$P^{\mu\nu} = g^{\mu\nu} + u^{\mu} u^{\nu}$ is the projector on the vector space perpendicular to $u^\mu$. The fact that the fluid expansion $\Theta$ vanishes will be of importance.

In the following tables we classify all the possible terms after imposing the ideal fluid conservation equations. The independent fluid data at first order is given in Table I.

| Independent data                        |
|----------------------------------------|
| Scalars (1)                            |
| $D \ln p$                              |
| Vectors (1)                            |
| $a^\mu \equiv (u. \nabla) u^\mu$       |
| Tensors (2)                            |
| $\sigma^{\mu\nu} \equiv \nabla_{(\mu} u_{\nu)}$ |
| $\omega^{\mu\nu} \equiv \nabla_{[\mu} u_{\nu]}$ |

**TABLE I: Fluid data at first order in derivatives**

$D = u^\mu \nabla_\mu$, $a_\mu$ is the fluid acceleration, $\sigma^{\mu\nu}$ and $\omega^{\mu\nu}$ are the fluid shear and rotation tensors, respectively. We define $A_{(\mu} B_{\nu)} = \frac{1}{2} (A_{\mu} B_{\nu} + A_{\nu} B_{\mu})$ and $A_{[\mu} B_{\nu]} = \frac{1}{2} [A_{\mu} B_{\nu} - A_{\nu} B_{\mu}]$.

At this order there is no curvature data. The independent fluid data at second order is derivatives in given in Table II. The composite expressions of fluid data that we will need for the analysis are given in Table III. We define

$$A_{(\mu\nu)} \equiv P^\alpha_\mu P^\beta_\nu \left( \frac{A_{\alpha\beta} + A_{\beta\alpha}}{2} - g_{\alpha\beta} \left( \frac{P^{\rho\sigma} A_{\rho\sigma}}{d} \right) \right) . \quad (4)$$

| Independent data                        |
|----------------------------------------|
| Scalars (1)                            |
| $D(D \ln p)$                           |
| Vectors (2)                            |
| $P^{\mu\alpha} \nabla_{\alpha} (D \ln p)$ |
| $P^\alpha_\alpha \nabla_\beta \sigma^{\alpha\beta}$ |
| Tensors (1)                            |
| $P^{\mu\alpha} P^{\nu\beta} \nabla_{\alpha} \nabla_{\beta} \ln p$ |

**TABLE II: Fluid data at second order in derivatives**
Scalars (4) \n\( (D\ln p)^2, \ a^2, \ \omega^2, \ \sigma^2 \)

Vectors (3) \n\( a^\mu D\ln p, \ a_\mu \omega^{\mu\nu}, \ a_\nu \sigma^{\mu\nu} \)

Tensors (5) \n\( D\ln p \sigma_{\mu\nu}, \ \sigma^{(\mu}_\alpha \sigma^{\alpha)\nu}, \ \omega^{(\mu}_\alpha \sigma^{\alpha)\nu}, \ \omega^{(\mu}_\alpha \omega^{\alpha)\nu}, \ a_{(\mu} a_{\nu)} \)

**TABLE III:** Composite expressions in fluid data at second order

where \( \omega^2 = \omega_{\mu\nu} \omega^{\mu\nu} \). The classification of the independent curvature data is the same as in [9] and for completeness we give it in table IV.

| Scalars (2) | Independent data |
|-------------|------------------|
| \( R \equiv R^{\mu\nu} \_\mu\nu \) |
| \( R_{00} \equiv u^{\mu} u^{\nu} R_{\mu\nu} \equiv u^{\mu} u^{\nu} R^{\alpha}_{\_\mu\alpha\nu} \) |

| Vectors(1) | \( P^{\mu\alpha} R_{\alpha\beta} u^{\beta} \) |

| Tensors(2) | \( R_{(\mu\nu)} \) |
|-------------|------------------|
| \( K_{(\mu\nu)} \) |
| where \( K_{\mu\nu} \equiv u^{\alpha} u^{\beta} R_{\mu\alpha\nu\beta} \) |

**TABLE IV:** Independent type curvature data at second order

Consider next the most general form of the stress-energy tensor of Rindler hydrodynamics in curved space. We fix the ambiguities at the derivative order by requiring that [3–5]

\[
P^{\mu\sigma}_\tau T^{(n)}_{\mu\nu} u^\nu = 0 ,
\]

and the pressure receives no derivative corrections. With this choice, the stress-energy tensor up to second order in derivatives takes the form.

6
\[ T_{\mu\nu}^{(0)} + \Pi_{\mu\nu} = pP_{\mu\nu} - 2\eta\sigma_{\mu\nu} - \zeta u_\mu u_\nu D\ln p \]
\[ + p^{-1} \left[ \tau \nabla_\mu \nabla_\nu \ln p + \kappa_1 R_{\mu\nu} + \kappa_2 K_{\mu\nu} + \lambda_0 D\ln p \sigma_{\mu\nu} \right. \]
\[ + \left. \lambda_1 \sigma_\mu^\alpha \sigma_{\alpha\nu} + \lambda_2 \sigma_{(\mu}^\alpha \omega_{\nu)} + \lambda_3 \omega^\alpha_{\mu \alpha\nu} + \lambda_4 a_\mu a_\nu \right] \]
\[ + u_\mu u_\nu \left[ d_1 \sigma^2 - d_2 \omega^2 + d_3 (D\ln p)^2 + d_4 D(D\ln p) + d_5 a^2 + e_1 R + e_2 R_{00} \right] \]

where
\[ K_{\mu\nu} = R^{\mu\nu\alpha\beta} u_\alpha u_\beta, \quad R_{\mu\nu} = R^{\alpha\nu\beta\nu} g_{\alpha\beta}. \]

There are two transport coefficients at first order: \( \eta \) is the shear viscosity and \( \zeta \) is a contribution to the energy density. There are fifteen transport coefficients at second order. This is the same number as for a generic non-conformal hydrodynamics with non-vanishing equilibrium energy density. However, some of the transport coefficients multiply different expressions in the fluid variables. In particular, seven of these terms can be viewed as corrections to the energy density. As we saw in the previous section, the fluid expansion \( \Theta = \nabla_\mu u^\mu \) vanishes and the scalar \( D\ln p \) appears instead in the stress-energy tensor. Note, however, that unlike the case where the equilibrium energy density is nonzero, here this scalar is not related to \( \Theta \).

### III. THE ENTROPY CURRENT

In this section we will construct the most general entropy current that exhibits vanishing equilibrium energy density. We will construct it up to second order in derivatives and derive the constraints on its form and on the transport coefficients in the stress-energy tensor. These follow from the requirement that its divergence is non-negative.

#### A. The entropy current at second order

In \cite{9}, the most general entropy current when the equilibrium energy density is nonzero has been constructed. In the following we will use a similar procedure and outline the differences. As above, we will use as fluid variables the normalized fluid velocity \( u^\mu \) and the pressure \( p \), and replace the fluid expansion \( \Theta \) by the scalar \( D\ln p \).
The entropy current $J^\mu$ has a derivative expansion

$$J^\mu = \sum_{l \geq 0} J^\mu_{(l)} ,$$

where $J^\mu_{(0)} = su^\mu$ and $J^\mu_{(1)} = a_1(p)u^\mu D\ln p + a_2(p)a^\mu$. At second order in derivatives we get that the most general entropy current depends on thirteen parameters, which correspond to the six independent vectors (three of them are composite), and seven independent scalars (multiplied by $u^\mu$) at second order, as outlined in the tables. It is convenient to parametrize the entropy current at the second order as follows:

$$J^\mu_{(2)} = \nabla_\nu [A_1(u^\mu \nabla^\nu \ln p - u^\nu \nabla^\mu \ln p)] + \nabla_\nu (A_2 \ln p \omega^{\mu\nu})$$

$$+ A_3 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) u_\nu + [A_4 D(D\ln p) + A_5 R + A_6 R_{00}] u^\mu$$

$$+ (B_1 \omega^2 + B_2 (D\ln p)^2 + B_3 \sigma^2) u^\mu + B_4 \left[ a^2 u^\mu - 2(D\ln p) \nabla_\perp \ln p \right]$$

$$+ D\ln p \nabla^\mu B_5 - P^{\alpha\beta}(\nabla_\beta u^\mu)(\nabla_\alpha B_5)] + B_6 D\ln p a^\mu + B_7 a_\nu \sigma^{\mu\nu} .$$

The divergence of the entropy current up to second order reads:

$$\nabla_\mu J^\mu = \nabla_\mu (su^\mu) + \nabla_\mu J^\mu_{(1)} + \nabla_\mu J^\mu_{(2)}$$

$$= -\frac{s}{p} \sigma_{\mu\nu} \Pi^{\mu\nu} - \frac{s}{p} D\epsilon + \nabla_\mu J^\mu_{(1)} + \nabla_\mu J^\mu_{(2)} \geq 0 ,$$

where the energy density up to second order $\epsilon = T^{(0+1+2)}_{\mu\nu} u^\mu u^\nu$ reads

$$\epsilon = -\zeta D\ln p + d_1 \sigma^2 - d_2 \omega^2 + d_3 (D\ln p)^2 + d_4 D^2 \ln p + d_5 a^2 + e_1 R + e_2 R_{00} .$$

Note, that the entropy density $s$ is constant at zeroth order since it is related to the energy density by the thermodynamic relation $d\epsilon = T ds$.

B. Constraints on the entropy current and the transport coefficients

In order to constrain the entropy current and to find relations among the transport coefficients in the stress-energy tensor, we have to analyze the terms that can appear in the divergence of the entropy current. We omit in the entropy divergence terms that cannot be completed to a positive expression, that is, to a complete square. We will encounter two cases. In the first case, we find terms that cannot appear in the second order divergence of the entropy current, i.e. $a^2$. We will therefore set to zero the coefficients of all the third
order terms in the entropy current divergence that take the form of second order vectors times $a^\mu$. In the second case, we identify terms that cannot appear at the fourth order divergence. This analysis was carried out in [9] for non-vanishing energy density. We can use this analysis with the following modifications:

- In [9] change $\Theta$ to $D\ln p$.
- In order to use the basis used in [9], we compute the entropy divergence (A1) and change the basis from $P^{\mu\alpha} P^{\nu\beta} \nabla_\alpha \nabla_\beta \ln p$ to $P^{\mu\alpha} P^{\nu\beta} D\sigma_{\alpha\beta}$ by,

$$P^{\mu\alpha} P^{\nu\beta} \nabla_\alpha \nabla_\beta \ln p = a^\mu a^\nu + \sigma^{\mu\nu} D\ln p - \sigma^{\mu\rho} \sigma_{\rho\nu} - \omega^{\mu\rho} \omega_{\rho\nu} - P^{\mu\alpha} P^{\nu\beta} D\sigma_{\alpha\beta} - K^{\mu\nu}. \quad (12)$$

Therefore, the following terms cannot appear at the fourth order divergence: $(\omega)^4$, $(\omega.a)^2$, $(a)^4$, $R^2$, $R^2_{\mu\nu}$, $R_{\mu\nu} R^{\mu\nu}$, $K_{\mu\nu} K^{\mu\nu}$. We will, thus, set to zero the coefficients of all the third order terms in the entropy current divergence that cannot be organized in a complete square, i.e.

$D\ln p R_{\mu\nu}, R D\ln p, R^{\mu\nu} \sigma_{\mu\nu}, R^{\mu\nu} a_{\mu} u_{\mu}, a^2 D\ln p, D^3 \ln p, a^2 \nabla_\beta (D\ln p), \sigma^{\beta\mu} a_{\beta} a_{\mu}, D R, D R_{\mu\nu}, \omega^2 D\ln p, \sigma^{\mu\lambda} K_{\mu\lambda}, \sigma^{\mu\lambda} \omega_{\lambda\sigma} \omega_{\sigma\nu}, a_{\mu} \nabla_\mu \sigma^{\mu\nu}$.

For this we will have to construct the most general entropy current to third order in derivatives and take its divergence.

Carrying out this analysis, we find the most general entropy current with positive divergence:

$$J^\mu_{(1)} = -\frac{s}{p} \zeta D\ln p$$
$$J^\mu_{(2)} = \nabla_\nu [A_1 (u^{\mu} \nabla^\nu \ln p - u^\nu \nabla^\mu \ln p)] + \nabla_\nu (A_2 \ln p \omega^{\mu\nu})$$
$$+ A_3 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) u_\nu + \left[ A_4 D(D\ln p) - \frac{p}{2} \frac{dA_3}{dp} R + A_6 R_{\mu\nu} \right] u^\mu$$
$$+ (B_1 \omega^2 + B_2 (D\ln p)^2 + B_3 \sigma^2) u^\mu$$
$$- \left( \frac{p}{2} \frac{dA_3}{dp} + \frac{p^2}{2} \frac{d^2 A_3}{dp^2} + \frac{1}{2} A_6 + \frac{p}{2} \frac{dA_6}{dp} \right) [a^2 u^\mu - 2(D\ln p)\nabla^\mu \ln p]$$
$$+ \left( A_3 + \frac{p}{2} \frac{dA_3}{dp} \right) [D\ln p \nabla^\mu \ln p - P^{\alpha\beta} (\nabla_\beta u^\alpha)(\nabla_\alpha \ln p)]$$
$$+ \left( A_3 + 4p \frac{dA_3}{dp} + p^2 \frac{d^2 A_3}{dp^2} + 2A_6 + p \frac{dA_6}{dp} \right) D\ln p a^\mu$$

(13)
with,
\[ A_3 = p^{-2}\kappa_1, \quad A_4 = p^{-1}d, \quad A_6 = p^{-1}e, \quad B_1 = \frac{1}{4} \left( A_3 + p\frac{dA_3}{dp} + p^{-2}(\tau - \lambda - 4d) \right) \] (14)
which leave four free parameters, \( A_1, A_2, B_2, B_3 \). For a detailed computation of its divergence see Appendix [A].

The relations among the stress-energy tensor transport coefficients that we get are:

\[ \kappa_2 = \tau - \kappa_1 + p\frac{d\kappa_1}{dp} \] (15)
\[ pe_1 = \kappa_1 - \frac{p}{2} \frac{d\kappa_1}{dp} \] (16)
\[ \lambda_4 = p(2e_1 - e_2) + p\frac{d}{dp}(\tau + \kappa_1 - \kappa_2) - 2\tau - \kappa_1 + \kappa_2 \] (17)
\[ 0 = pd_2 + pe_1 + pe_2 - 2\kappa_1 - \kappa_2 + \lambda_3 + \frac{p}{4} \frac{d}{dp}(2\kappa_1 + \kappa_2 - \lambda_3) \] (18)
\[ pd_5 = pe_1 - pe_2 + p^2 \frac{d}{dp}(2e_1 - e_2) . \] (19)

C. Flat space-time background

In the particular case of hydrodynamics in flat space-time, we set to zero the curvature terms in the entropy current and in the stress-energy tensor. This restrict the coefficients of the entropy current and we get the relations:

\[ A_1 + A_3 = A_6, \quad A_2 + A_3 = 0, \quad A_3 = 2A_5. \] (20)

The relations among the coefficients of the entropy current that follow from the requirement of non-negative divergence of the entropy current are,

\[ 4B_1 + \frac{4d_2}{p} + \frac{\lambda_3}{p^2} - p\frac{dB_5}{dp} - \frac{\tau}{p^2} = 0 \] (21)
\[ -2B_4 - p^2 \frac{d^2B_5}{dp^2} + \frac{2d_5}{p} - \frac{\lambda_4}{p^2} - p\frac{dB_5}{dp} - \frac{\tau}{p^2} = 0 \] (22)
\[ -p\frac{dB_4}{dp} - 2B_4 + p^2 \frac{d^2B_5}{dp^2} + p\frac{dB_5}{dp} - p\frac{dB_6}{dp} + \frac{2d_5}{p} - \frac{dd_5}{dp} = 0 \] (23)
\[ A_4 = \frac{d_4}{p} \]  \hspace{1cm} (24)
\[ p \frac{dB_5}{dp} = B_6 + \frac{2d_5}{p} \]  \hspace{1cm} (25)
\[ p \frac{dB_1}{dp} - 2B_1 + 2B_4 - 2p \frac{dB_5}{dp} + B_6 + \frac{dd_2}{dp} - \frac{2d_2}{p} = 0 \]  \hspace{1cm} (26)
\[ B_7 = 0 , \]  \hspace{1cm} (27)

We can eliminate the coefficients of the entropy current from (21-27) to get a relation between the transport coefficients of the stress-energy tensor,
\[ \frac{4d_2}{p} - 4 \frac{dd_2}{dp} + \frac{8\lambda_3}{p^2} - \frac{5d\lambda_3}{p dp} + \frac{d^2\lambda_3}{dp^2} - \frac{8\lambda_4}{p^2} + \frac{1d\lambda_4}{p dp} - \frac{16\tau}{p^2} + \frac{6 d\tau}{p dp} - \frac{d^2\tau}{dp^2} + \frac{6d_5}{p} = 0 . \]  \hspace{1cm} (28)

The relations (21-27) can be checked against two examples of Rindler hydrodynamics derived via the gravitational solutions in one higher space dimension. The first example is the fluid hydrodynamics obtained in the framework of Einstein’s gravity, i.e. from a gravitational background obeying the Einstein equations for an empty universe \( R_{AB} = 0 \). This example was studied in [3, 4], and its results were:
\[ A_1 = 2sp^{-2}, \quad A_2 = -A_3 = sp^{-2}, \quad A_4 = 0, \quad A_5 = -\frac{1}{2}sp^{-2}, \quad A_6 = sp^{-2}, \]
\[ B_1 = \frac{1}{2}sp^{-2}, \quad B_2 = B_4 = B_6 = 2sp^{-2}, \quad B_3 = 3sp^{-2}, \quad p \frac{dB_5}{dp} = 2sp^{-2}, \quad B_7 = 0 \]
\[ d_1 = -2p^{-1}, \quad d_2 = d_3 = d_4 = d_5 = e_1 = e_2 = 0, \]
\[ \tau = \lambda_0 = \lambda_2 = \lambda_3 = -\lambda_4 = -4, \quad \lambda_1 = -2 \]
\[ \eta = 1, \quad \zeta = 0 . \]  \hspace{1cm} (29)

\( s = 4\pi \) is the entropy density in equilibrium in this case.

The second example is an empty universe in Gauss-Bonnet gravity. This case has been worked out in [5], its results are the same as in the previous example with the following modifications:
\[ A_2 = -A_3 = sp^{-2}(1 + 4\alpha p^2), \quad A_5 = -\frac{1}{2}sp^{-2}(1 + 4\alpha p^2), \quad A_6 = sp^{-2}(1 - 4\alpha p^2), \]
\[ B_1 = \frac{1}{2}sp^{-2}(1 + 4\alpha p^2), \quad B_2 = B_6 = 2sp^{-2}(1 - 2\alpha p^2), \quad B_3 = sp^{-2}(3 + 2\alpha p^2), \]
\[ p \frac{dB_5}{dp} = 2sp^{-2}(1 - 2\alpha p^2), \]
\[ \lambda_3 = -4(1 + 3\alpha p^2), \quad \lambda_1 = -2(1 + 2\alpha p^2) , \]  \hspace{1cm} (30)
where $\alpha$ is the coefficient of the Gauss-Bonnet term. In both cases the relations (20-27) are satisfied. Therefore, the relation (28) is satisfied as well.

Consider now the special case $\zeta = 0$. We get two additional constraints:

$$
\frac{A_4}{p} + 2B_2 - 2p \frac{dB_5}{dp} - \frac{2d_3}{p} - \frac{dd_4}{dp} = 0
$$

(31)

$$
p^2 \frac{d^2 B_5}{dp^2} + p \frac{dB_5}{dp} - p \frac{dB_2}{dp} + \frac{dd_3}{dp} = 0.
$$

(32)

The case $\zeta = 0$ in flat space has been studied in the holographic framework in [3], where it was found that

$$
d_2 = d_3 = d_4 = d_5 = 0, \quad \tau = \lambda_3 = -\lambda_4, \quad B_i = \tilde{B}_i p^{-2},
$$

(33)

with $i = 1...7$ and $\tilde{B}_i$ constants. Repeating the analysis for this case, and evaluating the constraints for (33) we get the linearly independent equations

$$
\tilde{B}_2 + 2 \tilde{B}_5 = 0
$$

(34)

$$
A_4 = 0
$$

(35)

$$
2 \tilde{B}_5 + \tilde{B}_6 = 0
$$

(36)

$$
2 \tilde{B}_1 - \tilde{B}_4 - \tilde{B}_5 = 0
$$

(37)

$$
\tilde{B}_7 = 0
$$

(38)

$$
\tilde{B}_2 - \tilde{B}_4 = 0.
$$

(39)

We find six constraints instead of the five that were found in [3]. Indeed, the higher order entropy current yields one more constraint [39].

**D. Curved space-time background**

In this subsection we will construct a solution in Einstein gravity that describes holographically Rindler hydrodynamics in a curved background. The additional space coordinates will be denoted by $r$ and the hydrodynamics is defined on the co-dimension one hypersurface $r = r_c$. We will follow the procedure in [2-4], where a flat metric $\eta_{\mu\nu}$ was chosen on $r = r_c$, and we will replace it by a curved metric $g_{\mu\nu}$. The details of the construction are presented in appendix [B]. We outline them in the following. We need to construct the bulk metric up to second order in derivative expansion. We do that by solving the bulk equations $R_{AB} = 0,$
requiring regularity of the solution at \( r = 0 \) and fixing the metric on the hypersurface \( r = r_c \).

The difference compared to [3, 4] appears at second order in the derivative expansion, because curvature effects are of second order or higher. The second order correction to the bulk metric reads

\[
\delta_{\text{curv}}G^{(2)}_{\mu\nu} = 2u_{\mu}u_{\nu}(r - r_c)R - 2p^2(r - r_c)^2u_{\mu}P_{\nu}R^\lambda_{\mu} + 2(r - r_c)(R_{\mu\nu} + K_{\mu\nu}) ,
\]  

where \( R, R_{\mu\nu} \) and \( K_{\mu\nu} \) refer to the curvature of the hyper surface metric. The entropy (area) current \( J^\mu = \frac{1}{4G_N} \sqrt{G} \ell^\mu \), where \( G_N \) is the Newton constant, is affected by the corrections of the metric determinant and the normal to the event horizon:

\[
\delta_{\text{curv}}G^{(2)} = -\frac{2}{p}(R + R_{00}) ,
\]

\[
\delta_{\text{curv}}\ell^\mu(2) = -\frac{1}{p^3}P_{\mu\rho}R^\lambda_{\rho}u_\lambda .
\]

The final result is the same as (29) with the following modifications:

\[
A_3 = A_5 = -2sp^{-2}, \quad e_1 = -2p^{-1}, \quad \kappa_1 = \kappa_2 = -2 .
\]

It is easy to see that this result satisfies the constraint equations: (15-19).

**IV. THERMAL PARTITION FUNCTION**

In this section we will consider the Rindler thermal partition function on curved backgrounds with a time-like killing vector [12, 13]. We will use the framework employed in [12] to find relations among the transport coefficients for uncharged non-conformal hydrodynamics. As above, since the equilibrium energy density vanishes in our case, we cannot simply borrow the results of [12] that assume its non-vanishing. In the case of non-vanishing energy density, the relations among the transport coefficients derived in [12] are the same as those obtained from the non-negativity of the entropy current divergence [9]. We will see this in here as well.

The equilibrium backgrounds posses a time-like killing vector and can be written in the Kaluze-Klein form

\[
 ds_{KK}^2 = G_{\mu\nu}dx^\mu dx^\nu = -e^{2\sigma(\vec{x})} \left( dt + a_i(\vec{x})dx^i \right)^2 + g_{ij}(\vec{x})dx^i dx^j ,
\]

(44)
where \( i = 1 \ldots d \).

The fluid variables take the form

\[
\begin{align*}
\mu_0 & = e^{-\sigma(\vec{x})}(1, 0, \ldots, 0), \\
p & = p_0 e^{-\sigma(\vec{x})}, \\
\epsilon_0 & = 0.
\end{align*}
\]

(45)

A. The stress-energy tensor

Evaluating the derivative part of the stress-energy tensor (6) on the curved equilibrium background we get

\[
\Pi^{ij}_{\text{eq}} = a_1 \left( R^{ij}_g - \frac{R}{2} g^{ij} \right) + a_2 \left( \nabla^i \nabla^j \sigma - \nabla^2 \sigma g^{ij} \right) + a_3 \left( \nabla^i \sigma \nabla^j \sigma - \frac{\nabla^2 \sigma}{2} g^{ij} \right) \\
+ a_4 \left( f^{ik} f^j_k + \frac{f^2}{4} g^{ij} \right) e^{2\sigma} + g^{ij} \left( b_1 R_g + b_2 \nabla^2 \sigma + b_3 (\nabla \sigma)^2 + b_4 \frac{1}{4} f^2 e^{2\sigma} \right)
\]

(46)

where,

\[
\begin{align*}
 f_{ij} & = \partial_i a_j - \partial_j a_i, \\
b_1 p & = \frac{1}{2} \kappa_1, \\
b_2 p & = \kappa_2 - \kappa_1 - \tau \\
b_3 p & = \frac{1}{2} (\kappa_2 - \kappa_1 + \lambda_4) \\
b_4 p & = \frac{1}{4} (2 \kappa_1 + \kappa_2 - \lambda_3), \\
a_1 p & = \kappa_1 \\
a_2 p & = \kappa_2 - \kappa_1 - \tau, \\
a_3 p & = \kappa_2 - \kappa_1 + \lambda_4, \\
a_4 p & = -\frac{1}{4} (2 \kappa_1 + \kappa_2) + \frac{1}{4} \lambda_3.
\end{align*}
\]

(47)

\( R^{ij}_g, R_g \) are Ricci tensor and Ricci scalar respectively, calculated from the spatial metric \( g_{ij}(\vec{x}) \) in (43). The corrections of the fluid variables to second order are:

\[
\begin{align*}
\mu^\mu & = b_0 \mu_0^\mu + v_1 e^{\sigma} \nabla_j \sigma f^{ji} + v_2 e^{\sigma} \nabla_j f^{ji}, \\
b_0 & = 1 - \left( v_1 e^{2\sigma} a_i \nabla_j \sigma f^{ji} + v_2 e^{2\sigma} a_i \nabla_j f^{ji} \right) \\
p & = p_0 e^{-\sigma} + t_1 R_g + t_2 \nabla^2 \sigma + t_3 (\nabla \sigma)^2 + t_4 \frac{1}{4} f^2 e^{2\sigma} \\
\epsilon & = r_1 R_g + r_2 \nabla^2 \sigma + r_3 (\nabla \sigma)^2 + r_4 \frac{1}{4} f^2 e^{2\sigma},
\end{align*}
\]

where

\[
\begin{align*}
r_1 & = e_1, \\
r_2 & = e_2 - e_1, \\
r_3 & = e_2 - e_1 + d_5, \\
r_4 & = -(2 e_1 + e_2 + d_2).
\end{align*}
\]

(48)
The second source of corrections arises from inserting the velocity and pressure corrections into the zeroth order stress-energy tensor. We find that the modification of the stress-energy tensor due to these corrections is given by

\[
T_{ij} = g_{ij} \left( t_1 R_g + t_2 \nabla^2 \sigma + t_3 (\nabla \sigma)^2 + t_4 \frac{1}{4} f^2 e^{2\sigma} \right)
\]

\[
T_{00} = e^{2\sigma} \left( r_1 R_g + r_2 \nabla^2 \sigma + r_3 (\nabla \sigma)^2 + r_4 \frac{1}{4} f^2 e^{2\sigma} \right)
\]

\[
T_0^i = -p_0 \left( v_1 e^\sigma \nabla_j \sigma f^{ji} + v_2 e^\sigma \nabla_j f^{ji} \right)
\]

Adding (46) to (49) we get

\[
T_{ij} = g_{ij} \left( t_1 R_g + t_2 \nabla^2 \sigma + t_3 (\nabla \sigma)^2 + t_4 \frac{1}{4} f^2 e^{2\sigma} \right) + \Pi_{ij}^{eq}
\]

\[
T_{00} = e^{2\sigma} \left( r_1 R_g + r_2 \nabla^2 \sigma + r_3 (\nabla \sigma)^2 + r_4 \frac{1}{4} f^2 e^{2\sigma} \right)
\]

\[
T_0^i = -p_0 \left( v_1 e^\sigma \nabla_j \sigma f^{ji} + v_2 e^\sigma \nabla_j f^{ji} \right)
\]

where \( \Pi_{ij}^{eq} \) was listed in (46).

**B. The partition function**

The thermal partition function takes the form

\[
W = \log Z = -\frac{1}{2} \int d^d x \sqrt{-G} \left[ S_1 R_g + p_0^2 S_2 f_{ij} f^{ij} + S_3 (\nabla \sigma)^2 \right],
\]

(51)

where \( S_1, S_2, S_3 \) are three arbitrary function of \( \sigma \).

In order to evaluate the stress-energy tensor from the partition function, we vary the partition function

\[
\delta W = \int d^{d+1} x \sqrt{-G_{d+1}} \left( -\frac{1}{2} T_{\mu\nu} \delta g^{\mu\nu} \right) = \frac{s}{p_0} \int d^d x \sqrt{-G_{d+1}} \left( -\frac{1}{2} T_{\mu\nu} \delta g^{\mu\nu} \right),
\]

(52)

where we integrated over the time coordinate \( t \) which is compactified on a circle with radius \( \frac{p_0}{s} \). One obtains the stress-energy tensor by varying with respect to the metric,

\[
T_{00} = -\frac{p_0 e^{2\sigma}}{s \sqrt{-G_{d+1}}} \frac{\delta W}{\delta \sigma}, \quad T_0^i = \frac{p_0}{s \sqrt{-G_{d+1}}} \frac{\delta W}{\delta a_i},
\]

\[
T^{ij} = -\frac{2p_0}{s \sqrt{-G_{d+1}}} g^{ij} \frac{\delta W}{\delta g^{im}}.
\]

(53)
The stress-energy tensor derived from the thermal partition functions (51) reads

\[ sT_{ij} = pS_1 (R^i_j - \frac{1}{2} R g^{ij}) + 2p^2 \rho S_2 (f^i_k f_{jk} - \frac{1}{4} f^2 g^{ij}) + p (S_3 - S'_1) (\nabla^i \sigma \nabla^j \sigma - \frac{1}{2} (\nabla \sigma)^2 g^{ij}) \]

\[ - \frac{1}{2} (\nabla \sigma)^2 g^{ij} - p S'_1 (\nabla^i \nabla^i \sigma - g^{ij} \nabla^2 \sigma) + \frac{1}{2} p S''_1 (\nabla \sigma)^2 g^{ij} \]

\[ sT_{00} = \frac{p^2}{2 \rho} (S'_1 R_g + p^2 S'_2 f^2 - S'_3 (\nabla \sigma)^2 - 2S_3 \nabla^2 \sigma) \]

\[ sT^i_0 = 2p^2 \rho (S'_2 \nabla^i \sigma f^{ji} + S_2 \nabla^i f^{ji}) \]

(54)

where ' denotes derivative with respect to \( \sigma \).

Comparing the equations (50) and (54), one can express the transport coefficients in terms of the three coefficients \( S \) appearing in (51). We find

\[ sa_1 = pS_1, \quad sa_2 = -pS'_1, \quad sa_4 = 2p^2 \rho S_2, \quad sa_3 = p (S_3 - S''_1) \]

\[ sb_1 = -pt_1, \quad sb_2 = -pt_2, \quad sb_3 = -pt_3 + \frac{1}{2} p S''_1, \quad sb_4 = -pt_4. \]

(55)

One can eliminate the coefficients \( S' \)'s from above set of relations which gives a relation among transport coefficients. These relations are the same as those we obtained in the previous section (15-19), where we imposed nonnegativity for the divergence of the entropy current.

**Acknowledgements**

We would like to thank Synati Bhattacharya for valuable discussions. The work is supported in part by the Israeli Science Foundation center of excellence.
Appendix A: The entropy current divergence

Here we present the full computation of the entropy current divergence \( \nabla J^\mu = \frac{2ns}{p} \sigma_{\alpha\beta} \sigma^{\alpha\beta} + (a_1 + \frac{s\zeta}{p}) D^2 \ln p + \frac{da_1}{dp} p (D \ln p)^2 + \frac{da_2}{dp} p a^\mu \nabla_\mu \ln p + a_2 (\sigma^2 - \omega^2 + R_{00}) \)

\[
+ \left( \frac{dA_6}{dp} - p \frac{dA_3}{dp} + 2B_4 - p \frac{dB_5}{dp} + B_6 - \frac{dc_2}{dp} \right) D \ln p R_{00} \\
+ \left( \frac{dA_5}{dp} - p \frac{dA_3}{dp} - \frac{dc_1}{dp} \right) RD \ln p \\
+ \left( A_3 - p^{-2} \kappa_1 \right) R_{\mu \nu} \sigma_{\mu \nu} - \left( A_3 + p \frac{dA_3}{dp} - p \frac{dB_5}{dp} \right) R_{\mu \nu} a_\nu u_\mu \\
+ \left( p \frac{dB_4}{dp} - 2B_2 - 2p \frac{dB_5}{dp} - 2 \frac{dA_5}{dp} - p \frac{dB_6}{dp} + 2 \frac{dA_3}{dp} - 2 \frac{dc_4}{dp} \right) D^3 \ln p D \ln p \\
+ \left( -p \frac{dB_4}{dp} - 2B_4 + p^2 \frac{d^2 B_5}{dp^2} + p \frac{dB_5}{dp} - p \frac{dB_6}{dp} + 2 \frac{dA_3}{dp} - 2 \frac{dc_4}{dp} \right) a^2 D \ln p \\
+ \left( A_4 - \frac{dA_4}{p} \right) D^3 \ln p + \left( p \frac{dB_4}{dp} + B_6 + 2 \frac{dA_3}{dp} - p \frac{dB_6}{dp} + 2 \frac{dA_3}{dp} - 2 \frac{dc_4}{dp} \right) a^2 D \ln p \\
+ \left( +2B_3 - 2B_4 - p^2 \frac{d^2 B_5}{dp^2} - p \frac{dB_5}{dp} - 2 \frac{dA_3}{dp} - p \frac{dB_6}{dp} + 2 \frac{dA_3}{dp} - 2 \frac{dc_4}{dp} \right) \sigma_{\mu \nu} \nabla_\mu \ln p \nabla_\beta \ln p \\
+ \left( A_5 - \frac{dA_5}{p} \right) DR + \left( A_6 - \frac{dA_5}{p} \right) DR_{00} \\
+ \left( p \frac{dB_4}{dp} - 2B_1 + 2B_3 - 2p \frac{dB_5}{dp} + B_6 + \frac{dd_2}{dp} - \frac{2dd_2}{p} \right) \omega^2 D \ln p \\
+ \left( p \frac{dB_4}{dp} - 2B_3 + 2B_1 - 2p \frac{dB_5}{dp} + B_6 - B_7 - \frac{dd_1}{dp} + 2 \frac{dA_3}{dp} - p^{-2} \lambda_0 \right) \sigma^2 D \ln p \\
+ \left( -p^{-2} \lambda_1 - 2B_3 + 2 \frac{dA_3}{dp} \right) \sigma^\nu \lambda \sigma_\nu \sigma_\lambda \\
+ \left( -p^{-2} \kappa_2 - 2B_3 + 2 \frac{dA_3}{dp} \right) \sigma^\nu \lambda K_\nu \lambda \\
- \left( 2B_3 + 4B_1 + \frac{4dd_2}{p} + p^{-2} \lambda_3 - 2 \frac{dA_3}{dp} \right) \sigma^\nu \lambda \omega_\lambda \omega_\sigma_\nu \\
- \left( p^2 \frac{d^2 B_5}{dp^2} + p \frac{dB_5}{dp} - p \frac{dB_6}{dp} + \frac{dd_3}{dp} \right) (D \ln p)^3 \\
- \left( 2B_3 + p \frac{dB_5}{dp} + B_7 - \frac{2dd_1}{dp} - p^{-2} \tau \right) \sigma^\mu \nabla_\mu \nabla_\nu \ln p - B_7 a_\nu \nabla_\mu \sigma^{\mu \nu} \quad (A1)
\]

Appendix B: Curved background solution

In this appendix, we explain briefly how to get the results of the stress tensor and entropy current for a curved hypersurface. The procedure is similar to that used in [2–4] for a flat
hypothesis. We solve the Einstein equations: $R_{AB} = 0$, and boost the solution with the velocity vector $u^\mu$. Then, we promote the parameters of the solution to be depended on space-time coordinates of the cut off hyper surface located on $r = r_c$. Thus, we obtain the solution:

$$ds^2 = -(1 + p^2(x^\lambda)(r - r_c))u_\mu(x^\lambda)u_\nu(x^\lambda)dx^\mu dx^\nu - 2p(x^\lambda)u_\mu(x^\lambda)dx^\mu dr + P_{\mu\nu}(x^\lambda)dx^\mu dx^\nu.$$

where we choose the gauge choice to be:

$$g_{rr} = 0; \quad g_{r\mu} = -pu_\mu.$$

Since we promote the parameters to be depended on the coordinates of space-time, the metric does not satisfy the Einstein equations. We will solve the Einstein equations order by order in the derivative expansion. The Einstein equations at the $n^{th}$ order are:

$$\delta R_{AB}^{(n)} + \hat{R}_{AB}^{(n)} = 0,$$

where $\hat{R}_{AB}^{(n)}$ is the $n^{th}$ order Ricci tensor calculated from the $(n-1)^{th}$ order metric, and $\delta R_{AB}^{(n)}$ is calculated from the correction of the metric to the $n^{th}$ order. The final result is

$$\delta R_{rr}^{(n)} = -\frac{1}{2}\partial^2_r (P^{\lambda\sigma} g_{\lambda\sigma}^{(n)}),$$

$$\delta R_{r\mu}^{(n)} = \frac{1}{4}pu_\mu \partial_r (P^{\lambda\sigma} g_{\lambda\sigma}^{(n)}) + \frac{1}{2}p^{-1}\partial^2_r (u^\lambda g_{\mu\lambda}^{(n)}),$$

$$\delta R_{\mu\nu}^{(n)} = -\frac{1}{2} \left( u_\mu \partial_r (u^\lambda g_{\nu\lambda}^{(n)}) + u_\nu \partial_r (u^\lambda g_{\mu\lambda}^{(n)}) \right) - \frac{1}{2} \partial_r (\delta g_{\mu\nu}^{(n)}) - \frac{1}{2} p^{-2} \Phi \partial_r^2 (\delta g_{\mu\nu}^{(n)}) - \frac{1}{2} u_\mu u_\nu \partial^2_r (u^\lambda g_{\lambda\sigma}^{(n)}) + \frac{1}{4} \Phi u_\mu u_\nu \partial_r (P^{\lambda\sigma} g_{\lambda\sigma}^{(n)}).$$

where $\Phi = 1 + p^2(r - r_c)$.

This solution needs to be consistent with the following boundary conditions: (i) no singularity at $r = 0$ and (ii) a given curved induced metric on $r = r_c$. The second implies that all the $n \geq 1$ order corrections vanish at $r = r_c$. Projecting into components normal and transverse to $u^\mu$ we find,

$$P^{\lambda\nu} P^\rho_{\mu} g_{\lambda\sigma}^{(n)} = 2p^2 \int_{r_c}^{r'} \frac{1}{\Phi} dr' \int_{r_c}^{r'} P^{\lambda\nu} P^\rho_{\mu} \hat{R}_{\lambda\sigma}^{(n)} dr'' ,$$

$$u^\lambda P^\rho_{\mu} g_{\lambda\sigma}^{(n)} = (1/2)(1 - r/r_c)V_\mu^{(n)}(x) - 2p \int_r^{r_c} dr' \int_{r_c}^{r'} dr'' P^{\lambda\nu} \hat{R}_{\lambda\sigma}^{(n)} ,$$

$$u^\lambda u^\sigma g_{\lambda\sigma}^{(n)} = (1 - r/r_c)A^{(n)}(x) + p \int_r^{r_c} dr' \int_{r_c}^{r'} dr'' \left( pP^{\lambda\sigma} \hat{R}_{\lambda\sigma}^{(n)} - p^{-1} \Phi \hat{R}_{rr}^{(n)} - 2\hat{R}_{r\lambda}^{(n)}u^\lambda \right).$$
where \(V^{(n)}_\mu (V^{(n)}_\mu u^\mu = 0)\) and \(A^{(n)}\) are free, undetermined functions at this stage. To determine these functions we calculate the Brown-York stress-energy tensor on the hyper surface located at \(r = r_c\). We impose a frame in which the pressure does not receive any derivative corrections. This condition eliminates the derivative parts of the term proportional to \(g_{\mu\nu}\) in the Brown-York stress-energy tensor and fixes \(A^{(2)}\) to a non-zero value. Note, that the stress tensor in this frame has a term proportional to \(u^\mu u^\nu\). Hence, at this point we see the traditional Landau frame \(T_{\mu\nu}u^\mu = 0\) will be inconsistent. To fix \(V^{(2)}_\mu\) we instead require \(T^{\mu\nu}_{(n)}P^\lambda_\mu u_\nu = 0\).

With these values fixed, the stress-energy tensor can be put in a more conventional form:

\[
T_{\mu\nu} = pP_{\mu\nu} - 2\sigma_{\mu\nu} - 2p^{-1}u_\mu u_\nu(\sigma_{\alpha\beta}\sigma^{\alpha\beta} + R) - 2p^{-1}\sigma_{\mu\rho}\sigma^\rho_\nu - 4p^{-1}\sigma^\rho_{(\mu}\omega_{\nu)} - 4p^{-1}\omega_{\mu\nu}\omega^\rho_\rho \\
- 4p^{-1}P^\alpha_\mu P^\beta_\nu \nabla_\alpha \nabla_\beta \ln p - 4p^{-1}\sigma_{\mu\nu} D\ln p + 4p^{-1}D^\alpha_\mu \ln p D^\beta_\nu \ln p - 2p^{-1}P^\alpha_\mu P^\beta_\nu R_{\alpha\beta} - 2p^{-1}K_{\mu\nu}
\]

Due to the \(u_\mu u_\nu\) term, one can see that the energy density is no longer zero. It is corrected at the second order:

\[
\epsilon = T_{\mu\nu}u^\mu u^\nu = -\frac{2}{p}\sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{2}{p}R.
\]

The corrections due to the curved hypersurface metric alone are presented in (10).

Since metric solution is no longer stationary, the event horizon is dynamical and its location varies in time and space. In order to find \(r_h(x^\mu)\) we will need to solve the following equation in the derivative expansion

\[
g^{AB}\partial_A(r - r_h(x^\mu))\partial_B(r - r_h(x^\mu)) = 0.
\]

Using our previous results for the metric, it is straightforward to show at second order

\[
r_h = r_c - \frac{1}{p^2} + 2\frac{p^2}{p^2}u^\mu \nabla_\mu \ln p - \frac{3}{2p^4}\sigma_{\alpha \beta}\sigma^{\alpha \beta} - \frac{1}{2p^4}\omega_{\alpha \beta}\omega^{\alpha \beta} - \frac{8}{p^4}D\ln p D\ln p \\
+ \frac{1}{p^4}D^\alpha_\mu \ln p D^\beta_\nu \ln p + \frac{4}{p^4}D(D\ln p) - \frac{3}{2p^4}(R - R_{00})
\]

Now, it is easy to construct the normal to the event horizon by \(\ell^A = g^{AB}\partial_B(r - r_h)\) and the metric determinant. We find the entropy current

\[
J^\mu = \frac{\ell^\mu}{4G} \left(1 - \frac{1}{p^2}\left(\sigma_{\alpha \beta}\sigma^{\alpha \beta} - \frac{5}{2}\omega_{\alpha \beta}\omega^{\alpha \beta} + 2P^{\alpha \beta} \nabla_\alpha \nabla_\beta \ln p - 2P^{\alpha \beta} \nabla_\alpha \ln p \nabla_\beta \ln p + R + 2R_{00}\right)\right).
\]
[1] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, “From Navier-Stokes To Einstein,” JHEP 1207, 146 (2012) [arXiv:1101.2451 [hep-th]].

[2] G. Compere, P. McFadden, K. Skenderis and M. Taylor, “The Holographic fluid dual to vacuum Einstein gravity,” JHEP 1107, 050 (2011) [arXiv:1103.3022 [hep-th]].

[3] G. Compere, P. McFadden, K. Skenderis and M. Taylor, “The relativistic fluid dual to vacuum Einstein gravity,” JHEP 1203, 076 (2012) [arXiv:1201.2678 [hep-th]].

[4] C. Eling, A. Meyer and Y. Oz, “The Relativistic Rindler Hydrodynamics,” JHEP 1205, 116 (2012) [arXiv:1201.2705 [hep-th]].

[5] C. Eling, A. Meyer and Y. Oz, “Local Entropy Current in Higher Curvature Gravity and Rindler Hydrodynamics,” JHEP 1208, 088 (2012) [arXiv:1205.4249 [hep-th]].

[6] L. D. Landau and E. M. Lifshitz and Fluid Mechanics, Pergamon, New York (1959).

[7] D. T. Son and P. Surowka, “Hydrodynamics with Triangle Anomalies,” Phys. Rev. Lett. 103, 191601 (2009) [arXiv:0906.5044 [hep-th]].

[8] Y. Neiman and Y. Oz, “Relativistic Hydrodynamics with General Anomalous Charges,” JHEP 1103, 023 (2011) [arXiv:1011.5107 [hep-th]].

[9] S. Bhattacharyya, “Constraints on the second order transport coefficients of an uncharged fluid,” JHEP 1207, 104 (2012) [arXiv:1201.4654 [hep-th]].

[10] S. Bhattacharyya, V. E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, T. Morita, M. Rangamani and H. S. Reall, “Local Fluid Dynamical Entropy from Gravity,” JHEP 0806, 055 (2008) [arXiv:0803.2526 [hep-th]].

[11] S. Chapman, Y. Neiman and Y. Oz, “Fluid/Gravity Correspondence, Local Wald Entropy Current and Gravitational Anomaly,” JHEP 1207, 128 (2012) [arXiv:1202.2469 [hep-th]].

[12] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla and T. Sharma, “Constraints on Fluid Dynamics from Equilibrium Partition Functions,” JHEP 1209, 046 (2012) [arXiv:1203.3544 [hep-th]].

[13] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz and A. Yarom, “Towards hydrodynamics without an entropy current,” Phys. Rev. Lett. 109, 101601 (2012) [arXiv:1203.3556 [hep-th]].