Decoherence limit of quantum systems obeying generalized uncertainty principle: new paradigm for Tsallis thermostatistics

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The generalized uncertainty principle (GUP) is a phenomenological model whose purpose is to account for a minimal length scale (e.g., Planck scale or characteristic inverse-mass scale in effective quantum description) in quantum systems. In this Letter, we study possible observational effects of GUP systems in their decoherence domain. We first derive coherent states associated to GUP and unveil that in the momentum representation they coincide with Tsallis’ probability amplitudes, whose non-extensivity parameter $q$ monotonically increases with the GUP deformation parameter $\beta$. Secondly, for $\beta < 0$ (i.e., $q < 1$), we show that, due to Bekner–Babenko inequality, the GUP is fully equivalent to information-theoretic uncertainty relations based on Tsallis-entropy-power. Finally, we invoke the Maximal Entropy principle known from estimation theory to reveal connection between the quasi-classical (decoherence) limit of GUP-related quantum theory and non-extensive thermostatistics of Tsallis. This might provide an exciting paradigm in a range of fields from quantum theory to analog gravity. For instance, in some quantum gravity theories, such as conformal gravity, aforementioned quasi-classical regime has relevant observational consequences. We discuss some of the implications.

Introduction. — There are indications from various studies such as String Theory, Loop Quantum Gravity, Quantum Geometry or Doubly Special Relativity (DSR) theories, that the uncertainty relation between positions and momenta acquire corrections due to gravity effects and should be modified accordingly [7]. These modifications implement, in one way or another, the minimal length scale and/or the maximum momentum. The ensuing modified uncertainty relations are known as the generalized uncertainty principles — GUP’s. A paradigmatic form of GUP is the quadratic GUP, namely

$$\delta x \delta p \geq \frac{\hbar}{2} \left( 1 + \frac{\beta \delta p^2}{m_p^2} \right), \quad (1)$$

where $c = 1$, $m_p = \sqrt{\hbar c/G} \approx 2.2 \times 10^{-8}$ kg is the Planck mass and $\beta$ is a dimensionless deformation parameter. The symbol $\delta$ denotes uncertainty of a given observable and it does not need to be a priori related to the standard deviation. More like in the original Heisenberg uncertainty relation $\delta$ can represent Heisenberg’s “unge- nauigkeiten” (i.e., error-disturbance uncertainties caused by the backreaction in simultaneous measurement of $x$ and $p$) or $\delta p = |\langle \psi | p | \psi \rangle| \equiv |\langle \psi | p | \psi \rangle|_\psi$, see, e.g. [8].

The quadratic GUP (11) has served as an incubator for a number of important studies in quantum mechanics [9–11], particle physics [12, 13], finite-temperature quantum field theory [14] or cosmology [15]. In addition, the mass parameter in (11) does not need to be necessarily $m_p$ but it might be identified with a characteristic mass scale in the effective quantum description, e.g., in condensed matter and atomic physics or in non-linear optics [10, 12].

In cases when $\delta$ represents the standard deviation (henceforth denoted as $\Delta$), the GUP inequality (11) can be deduced from the deformed (Jacobi identity satisfying) commutation relations (DCR)

$$[\hat{x}, \hat{p}] = i\hbar \left( 1 + \frac{\beta \delta p^2}{m_p^2} \right), \quad [\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0, \quad (2)$$

by means of the Cauchy–Schwarz or covariance inequality [13, 21], provided we focus on mirror symmetric states where $\langle \hat{x} \rangle_\psi = \langle \hat{p} \rangle_\psi = 0$, e.g., $\psi$ are parity eigenstates [22].

Most of recent discussions of GUP in quantum gravity have focused on heuristic applications in cosmology and astrophysics (for review, see, e.g. [17]). Comaprably less attention has been devoted to study of GUP in a quasi-classical domain. It is, however, the quasi-classical quantum regime that is pertinent in observational cosmology and astrophysics [23, 24]. Important theoretical instruments used in quasi-classical quantum theory are coherent states (CS’s). This is because CS’s are least
susceptible to the loss of quantum coherence. In a sense, CS’s are the privileged states in the transition to classical reality, as they are the only states that remain pure in the decoherence process.

Various classes of CS’s have been studied. Here we will discuss the Schrödinger-type minimum-uncertainty CS’s associated with GUP. We derive precise forms of such GUP CS’s both in the momentum and position representation, though our focus will be on the momentum representation where CS’s coincide with Tsallis probability amplitudes. For $\beta < 0$ we also reformulate the GUP in terms of one-parameter class of Tsallis entropy-power based uncertainty relations (EPUR), which are saturated by the GUP CS’s. Since thermodynamics alongside with its various generalizations crucially hinges on the Maximum entropy principle (MEP) (i.e., thermodynamic entropy is the statistical entropy evaluated at the maximal entropy distribution), we are led to the conclusion that the combination of GUP CS’s with Tsallis entropy provides a natural framework to discuss the quasi-classical regime of GUP in terms of non-extensive thermostatistics of Tsallis (NTT). We apply this insight to find ensuing GUP generalization of Verlinde’s entropic gravity force and resulting cosmological implications in such a regime.

**Coherent states for GUP.** — We first summarize the steps leading to from the DCR. To this end, we quantify the uncertainty of an observable $\hat{O}$ with respect to a density matrix $\rho$ via its standard deviation. In particular, for variance, i.e., square of the standard deviation we have

$$\langle (\Delta \hat{O})^2 \rangle_e = \text{Tr}(\hat{O}^2 \rho) - \text{Tr}(\hat{O} \rho)^2 = \int_\mathbb{R} (\lambda - \langle \hat{O} \rangle_e)^2 d\text{Tr}(E^{(\hat{O})}_\lambda \rho).$$

Here $E^{(\hat{O})}_\lambda$ is the projection–value–measured of $\hat{O}$ corresponding to spectral value $\lambda$. By confining our study to the observables $\hat{x}$ and $\hat{p}$, the passage from the DCR to GUP is as follows: we set $\hat{O}_1 = \hat{x} - \langle \hat{x} \rangle_e$ and $\hat{O}_2 = \hat{p} - \langle \hat{p} \rangle_e$ so that $(\Delta x^2)_e = \langle \hat{O}^2_1 \rangle_e$, $(\Delta p^2)_e = \langle \hat{O}^2_2 \rangle_e$ and $[\hat{x}, \hat{p}]_e = [\hat{O}_1, \hat{O}_2]_e$, then for arbitrary vector $\psi \in \text{Range} \rho$ and any $\gamma \in \mathbb{R}$ we have

$$0 \leq \langle [\hat{O}_2 - i\gamma \hat{O}_1] \psi \rangle^2 = \langle \psi | [\hat{O}_2 - i\gamma \hat{O}_1] | \psi \rangle$$

$$= \langle \psi | \hat{O}^2_2 | \psi \rangle + i\gamma \langle \psi | [\hat{O}_1, \hat{O}_2] | \psi \rangle + \gamma^2 \langle \psi | \hat{O}^2_1 | \psi \rangle,$$  

and therefore

$$\text{Tr}(\hat{O}^2_2 \rho) + i\gamma \text{Tr}([\hat{O}_1, \hat{O}_2] \rho) + \gamma^2 \text{Tr}(\hat{O}^2_1 \rho) \geq 0.$$  

The LHS is smallest for $\gamma = i\text{Tr}([\hat{O}_2, \hat{O}_1] \rho) / (2\text{Tr}(\hat{O}^2_1 \rho))$, which turns to

$$\text{Tr}(\hat{O}^2_2 \rho) / \text{Tr}(\hat{O}^2_1 \rho) = (\Delta x^2)_e (\Delta p^2)_e \geq \frac{1}{4} \text{Tr}((i[x, p])^2).$$

This is nothing but quantum mechanical version of the covariance inequality. Now we can use to obtain

$$\langle (\Delta x)_e (\Delta p)_e \rangle \geq \frac{\hbar}{2} \left( 1 + \frac{\beta (\Delta p^2)_e + (\langle \hat{p} \rangle)^2}{m^2 p^2} \right).$$

For mirror symmetric $\rho$’s satisfying $(\langle \hat{p} \rangle)_e = 0$ inequality clearly coincides with the GUP, with variances in place of generic $\delta$.

To find $\rho$ that saturates the GUP, we observe from that the inequality is saturated if and only if for all $\psi \in \text{Range} \rho$ the equation $(\hat{O}_2 - i\gamma \hat{O}_1) | \psi \rangle = 0$. If this equation has for given $\gamma$, $(\langle \hat{x} \rangle_e)$ and $\langle \hat{p} \rangle_e$ more than one solution, the corresponding minimum–uncertainty $\rho$ is a mixture of CS’s (i.e., pure minimum–uncertainty states).

It is apparent, cf. Eq. that on the class of mirror symmetric $\rho$’s the equation

$$(\hat{p} - i\gamma \hat{x}) | \psi \rangle = 0,$$

has only one solution for $\psi \in L^2(\mathbb{R})$, so that the minimum–uncertainty $\rho$ is a pure state — CS. It is convenient to seek the solution to in the momentum representation, i.e. $| \psi \rangle \mapsto \rho(p) = | \psi(p) \rangle \langle \psi(p) |$. In the momentum space, $\hat{x}$ and $\hat{p}$ satisfying GUP can be represented as in.

However, by doing so, the non-symmetric nature of $\hat{x}$ would provide inconsistent variance for the ensuing CS, cf. Eq. For this reason we resort to another representation of $\hat{x}$ and $\hat{p}$ complying with , namely

$$\hat{p}(p) = p \langle \psi(p) \rangle,$$

$$\hat{x}(p) = i\hbar \left( \frac{p}{\beta m^2 p^2} \right) \langle \psi(p) \rangle,$$

with $[ \cdot, \cdot ]_+$ being an anticommutator. With this, we can cast into an equivalent form

$$\frac{d}{dp} \langle \psi(p) \rangle = -\frac{(1 + \frac{\gamma}{m^2 p^2})}{\gamma \hbar} p \langle \psi(p) \rangle,$$

which admits the generic solution

$$\langle \psi(p) \rangle = N \left[ 1 + (\beta p^2/m^2)^2 \frac{m^2}{\gamma \hbar} + \frac{1}{\gamma \hbar} \right].$$

The coefficient $N$ ensures that $\int |\psi(p)|^2 dp = 1$ and for $\beta > 0$

$$N_\gamma = \sqrt{\frac{\beta}{m^2 \pi} \frac{\Gamma(\frac{\beta p^2}{m^2 \gamma \hbar} + 1)}{\Gamma(\frac{\beta p^2}{m^2 \gamma \hbar} + 2)}}.$$
for $\beta < 0$ Eq. (11) involves noninteger powers of negative reals, which lead to multi-valued CS. Because wave functions must be single-valued, CS has to have bounded support, which in turn means that $\hat{p}$ must be bounded with spectrum $|\sigma(\hat{p})| \leq m_p/\sqrt{\beta}$. The ensuing operator $\hat{x}$ corresponding to the formal differential expression \[ \psi(p) = N_c \left[ 1 - (|\beta| p^2)/m_p^2 \right]^{\frac{m_p^2}{2m_p} - \frac{1}{2}}, \] where $[z]_+ = \max\{z, 0\}$ with \[ N_c = \sqrt{\frac{|\beta|}{m_p^2 \pi}} \left( \frac{1}{\Gamma(\frac{1}{2} + \frac{m_p^2}{2|\beta| \pi})} \right), \] (14)

In passing, we observe that as $\beta \to 0$ both (11) and (13) reduce to the usual minimum uncertainty Gaussian wave-packet (Glauber coherent state) associated with the conventional Heisenberg uncertainty relation.

To find a physical meaning for $\gamma$, we note [see sentence after (3)] that for CS $\psi$ \[ \gamma = -i \langle [\hat{x}, \hat{p}] \rangle / 2(\Delta x)^2 \psi = -2(\Delta p)^2 \Gamma(\frac{1}{2} + \frac{m_p^2}{|\beta| \pi}) \], (15)

where in the second and third identity we utilized the fact that $\gamma$ saturates (7). Note also that CS’s (11) satisfy $\langle \hat{p} \rangle = \langle \hat{x} \rangle = 0$.

Tsallis distribution. — Let us now consider the following substitutions (valid for $\beta \leq 0$) in (11) and (13): \[ q = \frac{\beta \gamma h}{m_p^2 + \beta \gamma h}, \quad b = \frac{2m_p}{\gamma h} + \frac{2\beta}{m_p}. \]

With this, we can rewrite (11) and (13) as \[ \psi(p) = N_c \left[ 1 - b(1 - q) \frac{p^2}{2m_p} \right]^{\frac{1}{2} - q} \], (17)

This is nothing but the probability amplitude for the Tsallis distribution of a free non-relativistic particle \[ q_s(p, q, b) = |\psi(p)|^2 = \frac{1}{Z} \left[ 1 - b(1 - q) \frac{p^2}{2m_p} \right]^{\frac{1}{2} - q}, \] (18)

with $Z = N_c^{-2}$ being the “partition function”.

A few remarks concerning (13) are now in order. Tsallis distribution of this type is also known as $q$-Gaussian distribution and denoted as $\exp_q(-b p^2/2 m_p)$. In the limit $q \to 1$, $\exp_q(-b p^2/2 m_p) \to \exp(-b p^2/2 m_p)$. Note that because of (11) $q \to 1$ is equivalent to $\beta \to 0$. In addition, since for $\beta > 0$ the $\hat{p}$ operator is unbounded, CS (17) is normalizable only for values of $1 \leq q < 3$. For values $q < 1$ (i.e. $\beta < 0$), the distribution (18) has a finite support with $|p| < \sqrt{2m_p/b(1 - q)}$. Moreover, for $q \geq 5/3$ the variance of (13) is undefined (infinite) and thus the GUP cannot even be formulated. When $q < 5/3$, then (see, e.g. Ref. 41)

\[ (\Delta p)^2 = \frac{2m_p}{b(5 - 3q)} \Rightarrow \gamma = \frac{2(\Delta p)^2}{h \left( 1 + \beta (\Delta p)^2/m_p^2 \right)}, \] (19)

which coincides with (15) [this, in turn, justifies our choice of the representation of $\hat{x}$ and $\hat{p}$ operators]. Furthermore, the mean value does not exist for $q > 2$, so such CS cannot be mirror-symmetric. Thus, the only physically relevant domain of $q$ in CS is $q < 5/3$, which ensures that $\beta$ is monotonically increasing function of $q$ and that $\beta > -m_p^2/3(\Delta p)^2$.

Connection with entropic uncertainty relations. — Probability distribution (13) decays asymptotically following power law. If variance and mean are the only observables, power-law type distributions are incompatible with the conventional MEP based on Shannon–Gibbs’ entropy (SGE). Nonetheless, distribution (13) is a maximizer of Tsallis (differential) entropy (TE) $S_q^{(T)}$, where \[ S_q^{(T)}(F) = \frac{k_B}{1 - q} \left( \int dp F^q(p) - 1 \right), \] (20)

($F$ is a probability density function) subject to a constraint $\langle \hat{p}^2 \rangle = 2m_p/b(5 - 3q)$, cf. [33, 41–43]. $k_B$ is the Boltzmann constant. In the limit $q \to 1$, the TE tends to SGE by L’Hopital’s rule.

When dealing with GUP that is saturated by Tsallis CS, it is convenient to employ the concept of Tsallis entropy power (TEP) [44]. TEP $M_q^{(T)}$ of a random vector $\mathcal{X}$ is the unique number that solves the equation \[ S_q^{(T)}(\mathcal{X}) = S_q^{(T)} \left( \sqrt{M_q^{(T)}(\mathcal{X})} \cdot Z^T \right), \] (21)

Here $Z^T$ represents a Tsallis random vector with zero mean and unit covariance matrix. Such a vector is distributed with respect to the Tsallis distribution that extremizes $S_q^{(T)}$. In Supplemental Material [29] we use Beckner–Babenko theorem [44] to prove that for $\beta < 0$ the DCR (2) implies the following one-parameter class of EPURs \[ M_{q/2}^{(T)}(|\psi|^2) M_{1/(2 - 2q)}^{(T)}(|\psi|^2) \geq h^2/4, \quad q \in [1, 2]. \] (22)

Here $\psi$ is the position-space wave function associated with $\hat{p}$. The clear advantage of EPUR (22) over GUP (1) is in that the RHS has an irreducible and state-independent lower bound. Moreover, (22) is also saturated by the GUP CS’s (29). Numerical simulations based the Markovian master equations for reduced density matrix coupled with predictability sieve method [23, 45, 47] indicate that CSs belong among the so-called pointer states, i.e. states that are least affected by the interaction with the environment (external degrees of freedom). Such states
belong to the quasi-classical domain of quantum theory as they are maximally predictable despite of decoherence [47, 48]. Among all pointer states in the would-be GUP driven universe, CS’s have the highest TE. Moreover, EPUR [22] indicates that TE is at the same time, a pertinent entropy functional in the GUP context. So, when we want to discuss a statistical physics of an emergent world. In NTT the heat one-form $\delta S$ is the distance of the interior of the screen — it operates in the emergent graphic surface, respectively and $\delta x$ is the distance of the particle from the screen. It should be stressed that Verlinde’s thermodynamic relation is not directly related to the interior of the screen — it operates in the emergent world. In NTT the heat one-form $\delta S$ must be replaced with $\delta S = [1 + ((1 - q)/k_B)]S_q^T$ (in our context $S_p \rightarrow S_{2-q}$). If $L$ is a (dimensionless) characteristic length scale (e.g. radius $R/\ell_p$) then the Bekenstein–Hawking entropy $S_{BH} = \ln W(L) \propto L^2$, which implies that the total number of internal configurations $W$ behaves for $L > 1$ as $W(L) = \phi(L)\nu^L$, where $\phi$ is any positive function satisfying $\lim_{L \rightarrow \infty} \ln \phi(L)/\nu^L = 0$ and $\nu > 1$ is some constant [51]. So, from the outside the holographic screen has entropy

\[
S_{2-q}^T = k_B \ln_{2-q} W(L) = \frac{k_B}{q - 1} \left[ \left( \phi(L)\nu^L \right)^{q-1} - 1 \right].
\]

Consequently, the entropic force follows from

\[
F_\delta x = \frac{T \delta S_{2-q}}{1 + (q-1)(\omega_1 L^3 + \omega_2 L^2 + \cdots)},
\]

where $\omega_2, \omega_3 > 0$ are intensive coefficients known from Hills’ entropy expansion in (conventional) thermodynamics of small and mesoscopic systems [51]. To comply with Hills’ expansion we have formally included term $\omega_3 L^3$ even if it is not supported by EG prescription. It will be seen that such a term is cosmologically unfeasible in the quasi-classical regime, so that $\omega_3 \approx 0$.

By holographic scaling, the energy residing inside the holographic screen is related with the on-screen degrees of freedom via the equipartition theorem $E = N k_B T/2$, with $E = M$ being the total mass enclosed by the surface and $N = A/(G\hbar)$ the number of bits connected with the area by the holographic principle [49].

EG paradigm posits that the minimum possible increase in the screen entropy (equivalent to one bit of Shannon’s information) happens if a particle of radius of Compton wave length $\lambda_C$ is added to a holographic sphere [49]. This happens when a point-like quantum particle appears at the distance $\lambda_C$ from the screen [49]. By setting $\delta x = \lambda_C = h/\hbar$ and using the non-extensive version of Landauer principle [52, 55], which states that the erasure of information leads to an entropy increase $\delta S_q^T = 2\pi k_B ((3 - 2q)/q)$ per erased bit, we derive the following modified Newton’s law

\[
F(R) = \frac{GMm}{wR^2} \left[ 1 - \frac{\kappa_3\varepsilon_q R^3}{\kappa_2 \varepsilon_q R^2} \right],
\]

with $\varepsilon_q = 1 - q$, $w = 1 + 2\varepsilon_q$ and $\kappa_n = \omega_n/\ell_p^n$, $n = 2, 3$. Since $2\varepsilon_q$ is small (see below), we can set $w = 1$. Ensuing gravitational potential up to the first-order in $\varepsilon_q$ is

\[
V(R) = \frac{\varepsilon_q}{2} \left[ -1 + \frac{1}{R} + \varepsilon_q \kappa_2 R + \frac{\varepsilon_q \kappa_3}{2} R^2 \right],
\]

where $r_s = 2GM$ is the Schwarzschild radius. Eq. [20] formally coincides with Mannheim–Kazanas external gravitational potential of a static, spherically symmetric source of mass $M$ in conformal gravity (CG) [24]. Strictly speaking, in CG a given local gravitational source generates only a gravitational potential

\[
V_{MK}(R) = -\frac{\varepsilon_q}{2R} + \frac{\chi}{2R}.
\]

The would-be term $\propto R^2$ corresponds to a trivial vacuum solution of CG and hence does not couple to matter sources [50, 51]. Fitting with CG thus implies that $\omega_3 \approx 0$. The magnitude of the constant $\chi$ can be associated with the inverse Hubble radius [61], i.e. $\chi \approx 1/R_H$. One should point out that by means of $V_{MK}$ it has been successfully fitted more than two hundred galactic rotation curves (with no need for dark matter or other exotic modification of gravity) [56, 57]. Besides CG, the spherically-symmetric gravitational potential with a linear potential also occurs, e.g., in the dilaton-reduced action of gravity $[61, 62]$ or $f(R)$ gravity [63].

To be more quantitative, let us assume that the GUP particle in question is inflaton. In such a case a quasi-classical (decoherence) description is valid at the late-inflation epoch (after the first Hubble radius crossing) and perhaps even after its end during reheating [64, 65]. In this period the NTT should be a pertinent framework.
for the description of the “inflaton gas”. E.g., by viewing the “inflaton gas” as the ideal gas NTT predicts that the inflaton pressure should satisfy for $0 < q < 1$ a polytropic relation $p \propto \rho^{q/3}$ ($\rho$ is energy density) [66, 67]. Relation of this type frequently appears in phenomenological studies on late inflation [68, 69]. We can fix $\beta$ by matching the linear terms in Eqs. (26) and (27). By using $r_\kappa \approx R_H$, $\kappa_2 = \pi^2/12$ cm$^{-2}$ (Bekenstein–Hawking value) we get $\varepsilon_q = l_p^2/(\pi R_H^3)$. Note that in this setting the effect of the linear potential is comparable to that of the Newtonian potential on length scales $1/R_H^2 \approx \varepsilon_q \kappa_2$, i.e. $R \approx l_p/(1/(\pi \varepsilon_q)) = R_H$. By solving (16) with respect to $\beta$ and employing (15), we obtain $|\beta| \approx m_p^2 l_p^2/(2\pi (\Delta p)^2 R_H^2)$. To estimate $\beta$, we express the Hubble radius as $R_H(t) = H^{-1}(t) = a(t)/\dot{a}(t)$, where $H$ is the Hubble parameter and the scale factor $a(t)$ can be evaluated from the Vilenkin–Ford model [70]: $a(t) = A \sqrt{\sinh (Bt)}$, with $B = 2\sqrt{\Lambda/3}$ ($\Lambda$ is the cosmological constant). On the other hand, from the relativistic equipartition theorem we have $(\Delta p)^2 \psi \approx 12(k_B T)^2$, cf. [20]. A straightforward computation gives

$$|\beta| \equiv |\beta|(t) = \frac{m_p^2 l_p^2 A}{72\pi (k_B T)^2 \tanh^2 (2\beta^2)}.$$ 

(28)

For concreteness sake, let us consider the late-inflation/reheating epoch, i.e. time scale $t \approx 10^{-33}$s. By assuming $T$ of the order of the reheating temperature $T_R \approx 10^7 \div 10^8$GeV, we obtain $|\beta| \sim 10^{-2} \div 1$, which is in agreement with the values predicted by string theory, cf. e.g. [1]–[2]. In passing we stress that the above connection with the CG potential works only for $\beta < 0$, or else in [20] we would have a wrong sign front of the linear potential.

We finally note that the DCR [2] with $\beta < 0$ given by (28) is consistent with Magueijo–Smolin DSR [6, 7]. In a nutshell, DSR is a theory which coherently tries to implement a second invariant (namely $m_p$ or equivalently $l_p$), besides the speed of light, into the transformations among inertial reference frames. The Magueijo–Smolin DSR model predicts that the DCR should vanish at Planck scale (thus physics should be deterministic there) while at low energies it approaches the conventional canonical commutator. In our case we indeed see from (28) that for allowed cosmological times $|\beta|(t)$ monotonically decreases with increasing $t$.

Conclusions. — To conclude, we have derived explicit form of coherent states for generalized uncertainty principle and showed that in the momentum representation they coincide with Tsallis probability amplitudes. Furthermore, for $\beta < 0$, we have reformulated GUP in terms of Tsallis-entropy based entropic uncertainty relations and by invoking the Maximal Entropy principle, we showed that in the semi-classical (decoherence) limit one can establish equivalence between the GUP quantum systems and non-extensive thermostatistics of Tsallis. This provides a novel framework to discuss transition between the GUP quantum substrate and classical reality and opens a viable route for tabletop experiments to explore possible GUP-based quantum gravitational phenomena via analog gravity models.

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A. EIGENSTATES OF THE POSITION OPERATOR

Because the canonical commutator has the form (2), the plane waves are not any more eigenstates of the operator $\hat{x}$ in the momentum representation and $\hat{p}$ in the position representation. This in turn means that wave functions in position and momentum representation are not connected via Fourier transform. To see how they are related, let us first consider the eigenstates of the operator $\hat{x}$ in the momentum representation. This is given by solving the following eigenvalue equation

$$\hat{x}|p\rangle = x\psi_x(p),$$

that can be equivalently rewritten as

$$\frac{d}{dp}\psi_x(p) = \frac{x - i\hbar \beta m_p p}{i\hbar \left(1 + \frac{\beta m_p^2 p^2}{2}ight)} \psi_x(p).$$

This has the solution

$$\psi_x(p) = A_x e^{-ixm_p \arctan \left(p\sqrt{\beta/m_p}/\hbar \sqrt{|\beta|}\right)} \frac{1}{\sqrt{m_p^2 + p^2 \beta}},$$

for positive $\beta$, and

$$\psi_x(p) = B_x e^{-ixm_p \arctanh \left(p\sqrt{\beta}/m_p\right)/\hbar \sqrt{|\beta|}} \frac{1}{\sqrt{m_p^2 - p^2 |\beta|}},$$

for negative $\beta$ (i.e. $\beta = -|\beta|$).

Let us now discuss the two cases ($\beta > 0$ and $\beta < 0$) separately.

Positive $\beta$ case

In this case $\psi_x(p)$ is quadratically integrable and the normalization factor $A_x$ can be chosen so as to ensure the normalizability to 1. It is easy to see that

$$A_x = \sqrt{\frac{m_p}{\pi}} \beta^{1/4}. $$
It should be noted that in the limit \( \beta \to 0 \) the eigenstate (3) with \( \mathcal{A} \) given by (5) converges to zero and not to \( e^{-i xp/\hbar} / \sqrt{2\pi \hbar} \) due to different normalization (plane waves are not quadratically integrable). On the other hand when \( \mathcal{A} \) is not considered then \( \lim_{\beta \to 0} \psi_x(p) \) is proportional to a plane wave, namely \( e^{-i xp/\hbar} / \sqrt{\beta} \).

If \( \hat{x} \) be self-adjoint, the corresponding eigenfunctions \( \psi_x(p) \) must be orthogonal for different \( x \). By requiring that \( x \) is real and the ensuing eigenfunctions \( \psi_x(p) \) are orthogonal, the completeness relation reads

\[
\langle x| x' \rangle = \int dp \psi_x^*(p) \psi_{x'}(p) = \int dp \frac{\sqrt{m_p} \exp\left(-i(x-x')p\right) / \sqrt{\beta}}{m_p^2 + p^2 / \beta} = \int_{\frac{m_p^2}{2\beta}}^{m_p^2} dz \sqrt{\beta / m_p} \exp\left(-i(x-x')z\right) = \frac{2\hbar / \sqrt{\beta}}{m_p \pi (x-x')} \sin\left(\frac{m_p \pi (x-x')}{2\hbar / \sqrt{\beta}}\right),
\]

where \( \psi_x^*(p) = \langle x | p \rangle \equiv \psi_p(x) \). So, \( x \) can acquire only discrete values \( n \hbar \sqrt{\beta} / m_p + x_0 \) with \( n \in \mathbb{Z} \) and \( x_0 \in (0,2\hbar \sqrt{\beta} / m_p) \). In other words, we have a one-parameter family of eigenvectors \( \{ \psi_x = n(2\hbar \sqrt{\beta} / m_p + x_0(p)) \}_{n \in \mathbb{Z}} \) parametrized by \( x_0 \). Note that the difference between nearest eigenvalues is \( 2 \hbar \sqrt{\beta} / m_p = 2(\Delta x)_{\text{min}} = 2(\Delta x)_{\text{max}} \), which goes to zero as \( \beta \to 0 \). Note that this \( \Delta x_{\text{min}} \) coincides with the minimal positional variance implied by (1). In contrast to \( (\Delta x)_{\text{min}} \), the GUP (1) does not provide any universal bound for \( (\Delta p)_{\text{min}} \) or \( (\Delta p)_{\text{max}} \).

It should be noticed that \( \hat{x} \) is not self-adjoint on the natural domain

\[
\mathcal{D}_x(\mathbb{R}) = \{ \psi \in L^2(\mathbb{R}) : \psi \in \mathcal{A}(\mathbb{R}), \hat{x} \psi \in L^2(\mathbb{R}) \}, \tag{7}
\]

\((AC(\mathbb{R}) \) stands for \textit{absolutely continuous functions on } \( \mathbb{R} \). This is because \( \hat{x} \) has eigenvalues \( \pm i \) with eigenstates belonging to \( \mathcal{D}_x(\mathbb{R}) \). Since the dimensionality of the respective Hilbert subspace is 1, the \textit{deficiency index} of such operator \( \hat{x} \) is \((1, 1) \). It is possible to appropriately restricting the definition region of \( \hat{x} \) so that the deficiency index is \((0, 0) \). For instance, we can require that \( \psi(p = \infty) = \theta \psi(p = -\infty) \) where \( |\theta| = 1 \). So, if we define \( \mathcal{D}_x^{\theta}(\mathbb{R}) = \{ \psi \in \mathcal{D}_x(\mathbb{R}) : \psi(\infty) = \theta \psi(-\infty), |\theta| = 1 \} \) we obtain a bijective correspondence between symmetric extension of the operator \( \hat{x} \) on \( L^2(\mathbb{R}) \) and complex numbers with \( |\theta| = 1 \); each of these operators \( \hat{x}_\theta = \hat{x} \) \( \mathcal{D}_x^{\theta}(\mathbb{R}) \) is self-adjoint, with a discrete spectrum. In addition, by writing \( \theta = e^{i\alpha} \) a setting \( \alpha = x_0 m_p \pi / (\hbar \sqrt{\beta}) \) the spectrum of \( \hat{x}_\theta = n \hbar \sqrt{\beta} / m_p + x_0 \), with \( n \in \mathbb{Z} \).

It should be stressed that the discreteness of the spectrum for the operator \( \hat{x} \) is not compatible with \( \beta > 0 \). Indeed, the expectation value of the canonical commutation relation (2) with respect to any eigenstate of \( \hat{x} \) gives a zero left-hand side of (2), while the right-hand side is always non-zero for \( \beta > 0 \). Such a situation would not happen should both \( \hat{p} \) and \( \hat{x} \) have a continuous spectrum because then the corresponding eigenstates do not belong to the domain of the commutator (see next subsection). On the other hand, the expectation value of the right-hand side of (2) can be zero but only if \( \beta < 0 \) but this would contradict our original assumption that \( \beta > 0 \).

### Negative \( \beta \) case

In this case \( \psi_x(p) \) is not quadratically integrable. Indeed, the corresponding integral

\[
|\psi_x|^2 = \langle x| x \rangle = |B_x|^2 \int_{-m_p / \sqrt{\beta}}^{m_p / \sqrt{\beta}} dp \frac{e^{-i(x-x') m_p \arctan(p \sqrt{\beta} / m_p) / \sqrt{\beta}}}{m_p^2 - p^2 / \beta} = \{ z = m_p \arctan(p \sqrt{\beta} / m_p) / \sqrt{\beta} \}
\]

\[
= \left( \frac{|B_x|}{m_p} \right)^2 \int_{-\infty}^{\infty} dz \sqrt{\beta / m_p} \exp\left(-i(x-x') z / \hbar \right) = \left( \frac{|B_x|}{m_p} \right)^2 \int_{-\infty}^{\infty} dz \exp\left(-i(x-x') z / \hbar \right) = 2\pi \hbar \delta(x-x').
\]

(8)

Ensuing scalar product for two eigenstates is

\[
\langle x'| x \rangle = |B_x|^2 \int_{-m_p / \sqrt{\beta}}^{m_p / \sqrt{\beta}} dp \frac{e^{-i(x-x') m_p \arctan(p \sqrt{\beta} / m_p) / \sqrt{\beta}}}{m_p^2 - p^2 / \beta} = \left( \frac{|B_x|}{m_p} \right)^2 \int_{-\infty}^{\infty} dz \exp\left(-i(x-x') z / \hbar \right)
\]

\[
= \left( \frac{|B_x|}{m_p} \right)^2 2\pi \hbar \delta(x-x').
\]

(9)

So, we can set \( B_x = \sqrt{m_p^2 / 2\pi \hbar} \). Such eigenstates do not belong to the usual \( L^2((-m_p / \sqrt{\beta}, m_p / \sqrt{\beta})) \) space. Instead they belong to the space \( S'((-m_p / \sqrt{\beta}, m_p / \sqrt{\beta})) \) of complex valued tempered distributions. Hence, there
is a spectral transition from discrete to continuous spectrum when $\beta$ becomes negative. In the following subsection we show that the $\hat{x}$ operator is self-adjoint.

In passing we should note that although continuous observables such as position $x$ or $p$ are routinely employed in quantum theory, they are really unphysical idealizations: the set of possible outcomes in any realistic measurement is always countable, since the state space of any apparatus with finite spatial extent has a countable basis. So, our reasonings related to $\beta < 0$ should be thus understood in this mathematically idealized sense — as done with conventional Heisenberg $p$-$x$ uncertainty relations.

The case with $\beta < 0$ is interesting from yet another point. While $\beta > 0$ predicts existence of $(\Delta x)_{\text{min}} = \sqrt{\beta} \hbar / m_p = \sqrt{\beta} \ell_p$ ($\ell_p = \hbar G/c^3 \approx 10^{-3} \text{cm}$ is the Planck length) but does not provide any universal bound for $(\Delta p)_{\text{min}}$ or $(\Delta p)_{\text{max}}$ the case with $\beta < 0$ allows only for momenta from the interval $(-m_p/\sqrt{|\beta|}, m_p/\sqrt{|\beta|})$ which implies that there exists $(\Delta p)_{\text{max}} = m_p / \sqrt{|\beta|}$.

**Self-adjointness of the position operator — negative $\beta$ case**

First we note that the operator $\hat{p}$ defined in (9) is the conventional operator of multiplication (by $p$), which is defined on the domain

$$D(\hat{p}) = \{ \psi \in L^2(I) : p \psi \in L^2(I) \} ,$$

with $I$ being an open interval with the endpoints $-a, a$ ($a = m_p / \sqrt{|\beta|}$). Operators of multiplication are known to be dense (i.e., closure $D(\hat{p}) = L^2(I)$) and self-adjoint [1].

Let us now show that the operator $\hat{x}$ corresponding to the formal differential expression (9) is self-adjoint. The reason why we are interested in a self-adjointness of the position operator is that only self-adjoint operators (and not symmetric) have guaranteed real spectrum [2]. To this end we define the subspace

$$D_x(I) = \{ \psi \in L^2(I) : \psi \in AC(I), \hat{x} \psi \in L^2(I) \} ,$$

($AC(I)$ stands for absolutely continuous functions on $I$, see [3]) which is dense in $L^2(I)$ as it contains, e.g., set of $C^\infty(I)$ functions that is dense in $L^2(I)$. $D_x(I)$ is chosen so that it is the largest reasonable domain on which the operator $\hat{x}$ could be expected to act.

Let us now denote with a symbol $\hat{x}$ the operator [4]

$$\hat{x} \psi \equiv i\hbar \left( \frac{d}{dp} + \frac{\beta}{m_p^2} p^2 \frac{d}{dp} + \frac{\beta}{m_p^2} p \right) \psi, \quad D(\hat{x}) \equiv D_x(I) .$$

We claim that this operator is self-adjoint when $\beta < 0$.

Suppose first that $\psi$ and $\phi$ are arbitrary representatives from $D(\hat{x})$. Then integration by parts gives

$$\langle \psi | \hat{x} \phi \rangle = i\hbar \int_{-a}^{a} dp \left[ \psi^*(p) \phi'(p) + \frac{\beta}{m_p^2} p^2 \psi^*(p) \phi'(p) + \frac{\beta}{m_p^2} p \psi^*(p) \phi(p) \right]$$

$$= i\hbar [\psi, \phi]_a - i\hbar \int_{-a}^{a} dp \left[ \psi^*(p) \phi'(p) + \frac{\beta}{m_p^2} p^2 \psi^*(p) \phi'(p) + \frac{\beta}{m_p^2} p \psi^*(p) \phi(p) \right]$$

$$= i\hbar [\psi, \phi]_a + \langle \hat{x} \psi | \phi \rangle ,$$

where

$$[\psi, \phi]_a = \lim_{p \to a^-} \left( 1 - \frac{|\beta|}{m_p^2} p^2 \right) \psi^*(p) \phi(p) - \lim_{p \to -a^+} \left( 1 - \frac{|\beta|}{m_p^2} p^2 \right) \psi^*(p) \phi(p) .$$

Note that both limits exist for any $\psi, \phi \in D(\hat{x})$, even in the cases when the endpoints are singular or the one–sided limits of the functions $\psi, \phi$ in them make no sense. This is because for any $\psi, \phi \in D(\hat{x})$ we have that $\psi^*(\hat{x} \phi) - (\hat{x} \psi)^* \phi$ belongs to $L^1(I)$, and so for any $d < a$

$$\lim_{c \to -a^+} \int_{c}^{d} dp \left[ \psi^* (\hat{x} \phi) - (\hat{x} \psi)^* \phi \right] = \int_{-a}^{d} dp \left[ \psi^* (\hat{x} \phi) - (\hat{x} \psi)^* \phi \right] = \text{finite} .$$


Thus,
\[
\lim_{p \to - a_+} \left( 1 - \frac{|\beta|}{m_p^2 p^2} \right) \psi^*(p) \phi(p),
\]
has in \(-a\) a proper limit. Similarly for the \(p \to a_-\) limit. So, both limits approach a finite constant, which by the convergence criterion for (improper) integrals must be zero \[5\]. This shows that \(\hat{x}\) is symmetric on the domain \(D(\hat{x})\). From \((13)\) we can thus conclude \[6\] that every element \(\varphi\) from \(D(\hat{x})\) is also in \(D(\hat{x}^\dagger)\) (so that \(D(\hat{x}) \subset D(\hat{x}^\dagger)\)), and then \(\hat{x}\varphi = \hat{x}^\dagger \varphi\). That is, the adjoint of \(\hat{x}\) is an extension of \(\hat{x}\). To prove that \(\hat{x}\) is self-adjoint, we must show that every element in \(D(\hat{x}^\dagger)\) is also in \(D(\hat{x})\). Thus, let \(\varphi, \eta \in L^2(I)\), with \(\langle \varphi | \chi \psi \rangle = \langle \eta | \psi \rangle\) for every \(\psi \in D(\hat{x})\). We must show that any such \(\varphi\) is in \(AC(I)\), with \(\eta = \hat{x}\varphi\). To this end we consider an arbitrary compact interval \(I' = [a, b] \subset I\) and define on it
\[
\tilde{\eta}_{I'}(p) = \frac{c}{\sqrt{1 + \frac{2}{m_p^2}p^2}} + \frac{1}{\sqrt{1 + \frac{2}{m_p^2}p^2}} \int_a^p dt \frac{\eta(t)}{\sqrt{1 + \frac{2}{m_p^2}t^2}},\]
where the constant \(c\) will be determined shortly. From \((17)\) it follows that \(\tilde{\eta}_{I'}(p) \sqrt{1 + \frac{2}{m_p^2}p^2} \in AC(I')\). Since also \(\sqrt{1 + \frac{2}{m_p^2}p^2}\) belongs to \(AC(I')\) and \(\sqrt{1 + \frac{2}{m_p^2}p^2} \neq 0\) on \(I'\), we have that \(\tilde{\eta}_{I'} \in AC(I')\) and
\[
\left( \frac{d}{dp} + \frac{\beta}{m_p^2} p^2 \frac{d}{dp} + \frac{\beta}{m_p^2} p \right) \tilde{\eta}_{I'}(p) = \eta(p),
\]
holds in \(I'\) in the \(L^2\) sense. Let now \(\psi_{I'}\) be an arbitrary function from \(D(\hat{x})\) whose support is in \(I'\), then
\[
\langle \varphi | \hat{x} \psi_{I'} \rangle = \langle \eta | \psi_{I'} \rangle = \int_{I'} dp \left[ \left( \frac{d}{dp} + \frac{\beta}{m_p^2} p^2 \frac{d}{dp} + \frac{\beta}{m_p^2} p \right) \tilde{\eta}_{I'} \right]^* \psi_{I'},
\]
which can be written as
\[
\langle \varphi_{I'} + (i/\hbar) \tilde{\eta}_{I'} | \hat{x} \psi_{I'} \rangle = 0,
\]
where \(\varphi_{I'}\) is the corresponding restriction of \(\varphi\) from \(I\) to \(I'\) (i.e. in terms of characteristic function \(\chi_{I'}\) of a set \(I'\) we have \(\varphi_{I'} = \chi_{I'} \varphi\)). At this stage we define the function \(\zeta_{I'}\) on \(I\), that is zero outside of \(I'\) and for \(p \in I'\) it is defined by the expression
\[
\zeta_{I'}(p) = \frac{1}{\sqrt{1 + \frac{2}{m_p^2}p^2}} \int_a^p dt \frac{\varphi_{I'}(t) + (i/\hbar) \tilde{\eta}_{I'}(t)}{\sqrt{1 + \frac{2}{m_p^2}t^2}}.
\]
Now we chose \(c\) from \((17)\) so that \(\zeta_{I'}(b) = 0\). This ensures that \(\zeta_{I'} \in AC(I')\). Besides
\[
\left( \frac{d}{dp} + \frac{\beta}{m_p^2} p^2 \frac{d}{dp} + \frac{\beta}{m_p^2} p \right) \zeta_{I'}(p) = \varphi_{I'}(p) + (i/\hbar) \tilde{\eta}_{I'}(p).
\]
Since \(\varphi_{I'}, \tilde{\eta}_{I'}\) and \(\zeta_{I'}\) are absolutely continuous on a compact interval \(I'\) they belong to \(L^2(I)\) and due to \((22)\) \(\zeta_{I'}\) belongs to a set of function from \(D(\hat{x})\) with support in \(I'\). Because \((20)\) is valid for any \(\psi_{I'} \in D(\hat{x})\) with support in \(I'\) we can chose \(\psi_{I'} = \zeta_{I'}\) in which case \((20)\) implies
\[
\varphi_{I'} + (i/\hbar) \tilde{\eta}_{I'} = 0.
\]
Hence \(\varphi_{I'} \in AC(I')\) and \(\hat{x} \varphi_{I'}(p) = \eta_{I'}(p)\) by \((18)\) for almost all \(p \in I'\). Since \(I'\) is an arbitrary compact interval in \(I\) it follows that \(\hat{x} \varphi = \eta\) almost everywhere in \(I\) and because \(\eta \in L^2(I)\) we finally arrive at the desired result that \(\varphi \in AC(I)\) [by \((23)\)] and \(\hat{x} \varphi \in L^2(I)\).

As an independent check we compute the deficiency indices of the symmetric operator \(\hat{x}\). For symmetric operator \(\hat{A}\) there are two deficiency indices defined as \(n_\pm(\hat{A}) = \text{dim Ker}(\hat{A}^\dagger \pm i)\) and usually written as the ordered pair \((n_+(A), n_-(A))\). A symmetric operator \(\hat{A}\) is self-adjoint if and only if it is closed and \((n_+(A), n_-(A)) = (0, 0),\) see,
e.g. [1]. Since \( D(\hat{x}) = AC(I) \) and \( \hat{x} \varphi = i\hbar [\varphi' + (\beta/m_p^2)p^2\varphi' + (\beta/m_p^2)p\varphi] \), the problem reduces to finding solutions of the equation

\[
i\hbar \left( \frac{d}{dp} + \frac{\beta}{m_p^2} p^2 \frac{d}{dp} + \frac{\beta}{m_p^2} p \right) \varphi_{\pm} = \mp i \varphi_{\pm},
\]

(24)

belonging to \( AC(I) \). From (4) we see that the general solutions are

\[
\varphi_{\pm}(p) = B_{\pm} e^{\mp m_p \beta \sqrt{\beta}} \left. \frac{\psi_{\pm}(x)}{\sqrt{m_p^2 - p^2}} \right|_{x = n \pi \beta / m_p},
\]

(25)

which are not quadratically integrable on \( I \) (and hence they cannot belong to \( AC(I) \)) so that the deficiency indices are \((0, 0)\). This reconfirms our finding that the \( \hat{x} \) operator is self-adjoint for \( \beta < 0 \).

**B. CONNECTION BETWEEN WAVE FUNCTIONS IN MOMENTUM AND POSITION REPRESENTATION**

**Positive \( \beta \) case**

Any one-dimensional wave function \( |\psi\rangle \) can be written in the momentum basis as

\[
|\psi\rangle = \int dp \langle p | \psi \rangle |p\rangle,
\]

(26)

which implies that

\[
\psi(x) = \int dp \psi_p(x) \tilde{\psi}(p) = \int dp \sqrt{\frac{m_p}{\pi}} e^{i \frac{m_p}{\pi} \beta^{1/4}} \frac{\psi_{\pm}(x)}{\sqrt{m_p^2 + p^2 \beta}} \tilde{\psi}(p),
\]

(27)

where \( \tilde{\psi}(p) = \langle p | \psi \rangle \). Eq. (27) provides a dictionary between position and momentum representation. Similarly, the inverse transformation reads

\[
\tilde{\psi}(p) = \sum_{n \in \mathbb{Z}} \psi_{x = n \pi \beta / m_p}(p) \psi(x = n \pi \beta / m_p)
\]

\[
= \sum_{n \in \mathbb{Z}} \sqrt{\frac{m_p}{\pi}} e^{-i \frac{m_p}{\pi} \beta^{1/4}} \frac{\psi_{\pm}(x = n \pi \beta / m_p)}{\sqrt{m_p^2 + p^2 \beta}} \psi(x = n \pi \beta / m_p).
\]

(28)

As a consistency check one can multiply both sides of Eq. (27) by \( \psi_p(x') \) and then sum over \( n \in \mathbb{Z} \). This yields

\[
\tilde{\psi}(p') = \sum_{n \in \mathbb{Z}} \int dp \sqrt{\frac{m_p}{\pi}} e^{i \beta \left[ \arctan(p \sqrt{\beta} / m_p) - \arctan(p' \sqrt{\beta} / m_p) \right]} \tilde{\psi}(p).
\]

(29)

By employing the identity

\[
\frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{inz} = \sum_{k \in \mathbb{Z}} \delta(z - 2\pi k),
\]

(30)

we can rewrite (29) as

\[
\tilde{\psi}(p') = \sum_{n = \{-1, 0, 1\}} \int dp \sqrt{\frac{m_p}{\pi}} \beta \frac{\delta \left( \arctan \left( \frac{p \sqrt{\beta}}{m_p} \right) - \arctan \left( \frac{p' \sqrt{\beta}}{m_p} \right) - \pi n \right)}{\sqrt{m_p^2 + p^2 \beta} \sqrt{m_p^2 + p^2 \beta}} \tilde{\psi}(p)
\]

\[
= \int dp \delta(p - p') \tilde{\psi}(p) = \tilde{\psi}(p'),
\]

(31)
where on the first line we have used the fact that in $\sum_{k \in \mathbb{Z}}$ survive only terms with $n = \{-1, 0, 1\}$ because $\max_{z} [\arctan(z)] = \pi/2$ (reached for $z \to \infty$) and $\min_{z} [\arctan(z)] = -\pi/2$ (reached for $z \to -\infty$). On the second line we have realized that contributions from $n = \pm 1$ do not need to be considered as they contribute only for $p' \to \infty$ and $p \to -\infty$ or $p' \to -\infty$ and $p \to \infty$, and in these cases $\tilde{\psi} \to 0$ (as it is an element of the $L^2$ Hilbert space).

Similarly, by multiplying Eq. (28) by $\psi_p(x')$ and integrating over $p$ we get

\[
\psi(x') = n' 2\hbar \sqrt{\beta/m_p}
= \int_{\mathbb{R}} dp \sqrt{\frac{m_p}{\pi}} \beta^{1/4} e^{ix_p \arctan(p\sqrt{\beta/m_p})/\hbar \sqrt{\beta}} \sum_{n \in \mathbb{Z}} \sqrt{\frac{m_p}{\pi}} \beta^{1/4} e^{-2i \arctan(p\sqrt{\beta/m_p})} \psi(x = n2\hbar \sqrt{\beta/m_p})
= \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} dp \frac{m_p}{\pi} \sqrt{\beta} e^{-2i(n-n') \arctan(p\sqrt{\beta/m_p})} \psi(x = n2\hbar \sqrt{\beta/m_p})
= \sum_{n \in \mathbb{Z}} \int_{-\pi}^{\pi} dz \frac{1}{2\pi} e^{-i(n-n')z} \psi(x = n2\hbar \sqrt{\beta/m_p})
= \sum_{n \in \mathbb{Z}} \sin [(n-n')\pi] \psi(x = n2\hbar \sqrt{\beta/m_p}) = \psi(x' = n' 2\hbar \sqrt{\beta/m_p}).
\]

Unfortunately, as we have already mentioned, positive value of $\beta$ is not compatible with canonical commutation relation (2).

**Negative $\beta$ case**

In this case the position and momentum representations of a wave function are related via relation

\[
\psi(x) = \int_{-m_p/\sqrt{\beta}}^{m_p/\sqrt{\beta}} dp \frac{e^{ix_p \arctan(p\sqrt{\beta}/m_p)}}{\sqrt{2\pi \hbar}} \psi(p)
= \{ z = m_p \arctanh \left( p \sqrt{\beta/m_p} / \sqrt{\beta} \right) \}
= \int_{\mathbb{R}} dz \frac{1}{\sqrt{2\pi \hbar}} e^{ixz/\hbar} \tilde{\psi}(m_p \tanh(z \sqrt{\beta/m_p}) / \sqrt{\beta})
= \int_{\mathbb{R}} dz \frac{1}{\sqrt{2\pi \hbar}} e^{ixz/\hbar} \tilde{\psi}(z),
\]

where $\tilde{\psi}(z) = \tilde{\psi}(m_p \tanh(z \sqrt{\beta/m_p}) / \sqrt{\beta}) / \cosh(z \sqrt{\beta/m_p})$. Note that this formula holds only in $D = 1$ dimensions. In passing we can easily check that the analogue of Parseval–Plancherel theorem holds, namely

\[
\int_{\mathbb{R}} dx |\psi(x)|^2 = \int_{-m_p/\sqrt{\beta}}^{m_p/\sqrt{\beta}} dp |\tilde{\psi}(p)|^2 = \int_{\mathbb{R}} dz |\tilde{\psi}(z)|^2 \iff ||\psi||_2 = ||\tilde{\psi}||_2 = ||\tilde{\psi}||_2.
\]

From the last line in (33) one can also easily deduce that the momentum operator in the position representation has the form

\[
\hat{p}^{(x)} = m_p \frac{d}{dx} \left( -i\hbar \sqrt{\beta/m_p} \frac{d}{dx} \right) / \sqrt{\beta}.
\]

It can be easily checked that this operator indeed satisfies the canonical commutation relation (2).
There is yet another interesting consequence of Eq. (33), namely one can directly compute from it the corresponding position-space coherent state. In particular, by using the Tsallis probability amplitude (11) (i.e., momentum-space coherent state) we can write for the corresponding position-space coherent state $\psi_{CS}(x)$ that

$$
\psi_{CS}(x) = N \int_{\mathbb{R}} \frac{dz}{\sqrt{2\pi\hbar}} e^{izx/\hbar} \left[ \frac{m_p^2 - m_p^2 (\tanh(z\sqrt{|\beta|/m_p}))^2}{\cosh(z\sqrt{|\beta|/m_p})} \right]^{m_p^2/(2|\beta|\hbar)-1/2} 
= N \frac{m_p^2/(|\beta|\hbar)-1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} \frac{dz}{\sqrt{2\pi\hbar}} e^{izx/\hbar} \cosh(z\sqrt{|\beta|/m_p})^{-m_p^2/(|\beta|\hbar)} 
= \sqrt{\frac{m_p}{4\pi\sqrt{|\beta|\hbar}}} \Gamma\left(\frac{m_p^2/(|\beta|\hbar)}{4}\right) \left| \Gamma \left( \frac{m_p^2/(|\beta|\hbar)}{2} + i \frac{xm_p}{2\sqrt{|\beta|\hbar}} \right) \right|^2 
= \sqrt{\frac{m_p}{4\pi\sqrt{|\beta|\hbar}}} \Gamma\left(\frac{2m_p^2/(|\beta|\hbar)}{4}\right) \left| \Gamma \left( \frac{m_p^2/(|\beta|\hbar)}{2} + i \frac{xm_p}{2\sqrt{|\beta|\hbar}} \right) \right|^2.
$$

(36)

In passage from 2nd to 3rd line we used the fact that the 2nd line represents a Fourier transform (characteristic function) of generalized logistic density of type III [16], cf. also Ref. [17]. Alternatively one can obtain the result (36) from the Ramanujan formula [18]

$$
\int_{-\infty}^{\infty} e^{-ixs} \left| \Gamma(a+is) \right|^2 ds = \sqrt{\pi} \Gamma(a) \Gamma(a+1/2) [\cosh(\xi/2)]^{-2a} = \frac{2\pi}{2a} \gamma(2a) [\cosh(\xi/2)]^{-2a},
$$

(37)

that is valid for $a \in (-1, 0) \cup (0, \infty)$. The normalization factor on the last line of (36) was obtained by using the Mellin–Barnes beta integral (cf. Ref. [17])

$$
\int_{-\infty}^{\infty} |\Gamma(a+ibs)|^4 ds = \frac{2\pi}{b} \Gamma^4(2a),
$$

(38)

In passing we note that the state $\psi_{CS}(x)$ is even-parity state (as required) and, in addition, it belongs to the Schwartz class, i.e., it decays rapidly at infinity along with all derivatives.

For consistency we can now check that $\psi_{CS}(x)$ from Eq. (36) provides a correct positional variance, which together with the momentum variance deduced from $\psi_{CS}(p)$ (cf. Eq. (19)) saturates the GUP (1). To this end we use the formula for the Fourier transform of $|\Gamma(a+ibs)|^4$ (see, [17], Eq. (274), p. 46) to show that

$$
\frac{b\Gamma(4a)}{2\pi\Gamma^3(2a)} \int_{-\infty}^{\infty} s^2 |\Gamma(a+ibs)|^4 ds = \frac{a^2}{b^2(1+4a)\Gamma(2a)}.
$$

(39)

If we now use from (38) that $a = m_p^2/(2|\beta|\hbar)$ and $b = m_p/(2\sqrt{|\beta|\hbar})$ we get

$$
(\Delta x)^2_{CS} = \int_{-\infty}^{\infty} x^2 \psi_{CS}(x) dx = \frac{\hbar m_p^2}{2m_p^2\gamma + \hbar|\beta|\gamma^2} = \frac{(\Delta p)^2_{CS}}{\gamma^2},
$$

(40)

where the last identity results from (15). This is equivalent to the saturated GUP.

C. GENERALIZATION OF BECKNER–BABENKO’S THEOREM

Beckner–Babenko’s theorem for Pontryagin class of transformations

Important for Fourier-transform based entropic uncertainty relations is the following theorem [13, 14]:

Theorem 1 (Beckner–Babenko’s theorem) Let

$$
f^{(2)}(x) \equiv \hat{f}^{(1)}(x) = \int_{\mathbb{R}^D} e^{2\pi i x \cdot y} f^{(1)}(y) dy,
$$
then for \( p \in [1, 2] \) we have

\[
\| \hat{f} \|_{p'} \leq \frac{|p|^{D/2}|1/p'}{|(p')^{D/2}|1/p'} \| f \|_p, \tag{41}
\]

or, equivalently

\[
|p|^{D/2}|1/p'} \| f \|_2 \leq |p|^{D/2}|1/p'} \| f \|_p. \tag{42}
\]

Here, \( p \) and \( p' \) are the usual Hölder conjugates (i.e., \( p' \in [2, \infty) \)). For any \( X \in \ell^p(\mathbb{R}^D) \) the \( p \)-norm \( \| X \|_p \) is defined as

\[
\| X \|_p = \left( \int_{\mathbb{R}^D} |X(y)|^p \, dy \right)^{1/p}.
\]

Due to symmetricity of the Fourier transform also the reverse inequality holds:

\[
\| f \|_{p'} \leq \frac{|p|^{D/2}|1/p'}{|(p')^{D/2}|1/p'} \| \hat{f} \|_p.
\]

Lieb [15] proved that the inequality (41) is saturated (jointly for all \( p \)) only for Gaussian functions. This relation directly implies entropy-power uncertainty relation for Pontryagin class of transformations (cf. Section ).

B-B theorem can be directly generalized to our situation (33). This will be discussed in Section .

### D. ENTROPY POWERS

#### Entropy powers based on Gaussian distribution

**Rényi entropy power**  Rényi entropy power \( N^R_p(X) \) is defined as the solution of the equation

\[
S^R_q(X) = S^R_q\left(\sqrt{N^R_q(X) \cdot Z^q}\right), \tag{43}
\]

where \( \{Z^q\} \) represents a Gaussian random vector with zero mean and unit covariance matrix. So, \( N^R_p(X) \) denotes the variance of a would be Gaussian distribution that has the same Rényi information content as the random vector \( \{X\} \) described by the PDF \( F(x) \). Expression (43) was studied in [8–10] where it was shown that the only class of solutions of (43) is

\[
N^R_q(X) = \frac{1}{2\pi} p^{-q/\nu} \exp\left(\frac{2}{D} S^R_q(X)\right), \tag{44}
\]

with \( 1/p + 1/p' = 1 \) and \( p \in \mathbb{R}^+ \). In addition, when \( p \to 1_+ \) one has \( N^R_p(X) \to N(X) \), where \( N(X) \) is the conventional Shannon entropy power [11].

**Tsallis entropy power**  Tsallis entropy power \( N^T_p(X) \) is defined as the solution of the equation

\[
S^T_q(X) = S^T_q\left(\sqrt{N^T_q(X) \cdot Z^q}\right). \tag{45}
\]

Corresponding entropy power has not been studied in the literature yet but it can easily be derived by observing that the following scaling property for differential Tsallis entropy holds, namely

\[
S^T_q(aX) = S^T_q(X) \oplus q \ln_q |a|^D, \tag{46}
\]

where \( a \in \mathbb{R} \) and the \( q \)-deformed “sum” and logarithm are defined as [12]: \( x \oplus_q y = x + y + (1 - q)xy \) and \( \ln_q x = (x^{1-q} - 1)/(1 - q) \), respectively. Relation (46) follows directly from the following chain of identities

\[
S^T_q(aX) = \frac{1}{1-q} \left[ \int d^Dy \left( \int d^Dx \delta(y - ax) F(x) \right)^q - 1 \right] = \frac{1}{1-q} \left[ |a|^{D(1-q)} \int d^Dy F^q(y) - 1 \right]

= |a|^{D(1-q)} \left( S^T_q(X) + \frac{1}{1-q} \right) - \frac{1}{1-q} = |a|^{D(1-q)} S^T_q(X) + \ln_q |a|^D

= \left[ (1 - q) \ln_q |a|^D + 1 \right] S^T_q(X) + \ln_q |a|^D = S^T_q(X) \oplus q \ln_q |a|^D. \tag{47}
\]
We can further use the simple fact that
\[
S_q^T(Z_G) = \ln_q(2\pi q^{q'}/q)^{D/2}.
\] (48)

Here \( p \) and \( p' \) are Hölder’s double, i.e., \( 1/q + 1/q' = 1 \). Combining (45), (46) and (48) we get that
\[
S_q^T(X) = \ln_q(2\pi q^{q'}/q)^{D/2} \oplus_q \ln_q(N_q^T)^{D/2} = \ln_q(2\pi q^{q'}/q N_q^T)^{D/2},
\] (49)

where we have used the sum rule from the \( q \)-deformed calculus: \( \ln_q x \oplus_q \ln_q y = \ln_q xy \). Equation (49) can be resolved for \( N_q^T \) when we employ the \( q \)-exponential, i.e., \( e_q^x = [1 + (1 - q)x]^{1/(1-q)} \), which among others satisfies the relation \( \ln_q e_q^x = \ln_q(e_q^x) = x \). With this we have that
\[
N_q^T(X) = \frac{1}{2\pi} q^{-q'/q} \left[ \exp_q \left( S_q^T(X) \right) \right]^{2/D} = \frac{1}{2\pi} q^{-q'/q} \exp_{1-(1-q)D/2} \left( \frac{2}{D} S_q^T(X) \right).
\] (50)

In addition, when \( p \to 1_+ \) one has
\[
\lim_{q \to 1} N_q^T(X) = \frac{1}{2\pi e} \exp \left( \frac{2}{D} H(X) \right) = N(X),
\] (51)

where \( N(X) \) is the conventional Shannon entropy power and \( H(X) \) is Shannon entropy [11].

In connection with Tsallis entropy power we might notice one interesting fact, namely by starting with Rényi entropy power we have
\[
N_q^R(X) = \frac{1}{2\pi} q^{-q'/q} \exp \left( \frac{2}{D} S_q^R(X) \right) = \frac{1}{2\pi} q^{-q'/q} \left( \int d^D x \mathcal{F}^q(x) \right)^{2/(D(1-q))}
\]
\[
= \frac{1}{2\pi} q^{-q'/q} \left[ e_q^{S_q^T(X)} \right]^{2/D} = N_q^T(X).
\] (52)

Here we have used the obvious identity
\[
\left( \int d^D x \mathcal{F}^q(x) \right)^{1/(1-q)} = [(1-q)S_q^T(X) + 1]^{1/(1-q)} = e_q^{S_q^T(X)}.
\] (53)

So, we have obtained that Rényi and Tsallis entropy powers match each other.

**Entropy powers based on Tsallis entropy**

When dealing with GUP that is saturated by Tsallis probability amplitude states, it is more convenient to work with entropy powers based on Tsallis distribution (17). In this connection we can again formally define the ensuing entropy powers for both Rényi and Tsallis entropies as solutions of the equations
\[
S_q^R(X) = S_q^R \left( \sqrt{M_q^R(X) \cdot Z^T} \right),
\]
\[
S_q^T(X) = S_q^T \left( \sqrt{M_q^T(X) \cdot Z^T} \right),
\] (54)

where \( \{ Z^T \} \) represents a Tsallis random vector with zero mean and unit covariance matrix. Such a vector is distributed with respect to the \( q \)-Gaussian probability density function that extremizes \( S_q^T \) (and hence also \( S_q^R \)). To solve Eqs. (54) we use the scaling relations for Rényi and Tsallis entropies [see, Eq. (47)], namely
\[
S_q^R(aX) = S_q^R(X) + D \ln |a|,
\]
\[
S_q^T(aX) = S_q^T(X) \oplus_q \ln_q |a|^D.
\] (55)

In the next step we need to know \( S_q^R \left( Z^T \right) \) and \( S_q^T \left( Z^T \right) \). To this end we first realise that the \( q \)-Gaussian distribution extremizing \( S_{2-q} \) with zero mean and unit covariance matrix has the form [12]
\[
\mathcal{F}(q, p) = N_q \left[ 1 - b(1-q)p^x \right]^{1/(1-q)} \\
\text{with } b = \frac{1}{2(2-q) - D(q-1)},
\] (56)
Note that $b > 0$ for all $q < (4 + D)/(2 + D)$. For $q \geq (4 + D)/(2 + D)$ the variance is not defined. The normalization factor $N_S(q)$ is such that for $q < 1$

$$N_S(q) = \left(1 + \frac{D}{2}(1-q)\right) \frac{1}{\pi^{D/2}} \frac{1}{(b(1-q))^{D/2}} \frac{\Gamma\left(\frac{1}{1-q} + \frac{D}{2}\right)}{\Gamma\left(\frac{1}{1-q}\right)}$$

$$= \frac{1}{\pi^{D/2}} \frac{1}{(b(1-q))^{D/2}} \frac{\Gamma\left(\frac{2-q}{1-q} + \frac{D}{2}\right)}{\Gamma\left(\frac{2-q}{1-q}\right)}, \quad (57)$$

and for $1 < q < (4 + D)/(2 + D)$

$$N_S(q) = \frac{1}{\pi^{D/2}} \frac{1}{(b(q-1))^{D/2}} \frac{\Gamma\left(\frac{1}{1-q}\right)}{\Gamma\left(\frac{1}{1-q} - \frac{D}{2}\right)}.$$  \quad (58)

Since $\mathcal{F}(2-q,p)$ extremises $S_q^T$ (or equivalently $\mathcal{F}(q,p)$ extremises $S_{2-q}^T$), we can write

$$\int d^Dp \ [\mathcal{F}(2-q,p)]^q = N_S^q(2-q) \int d^Dp \ [1 - b(q-1)p^2]^q \Omega_D \int_0^\infty dz \ z^{D-1} [1 - b(q-1)z^2]^q \Omega_D.$$  \quad (59)

Let us now discuss a bit more the situations with $b(q-1) < 0$ that is relevant in the main text. When $b(q-1) > 0$ [i.e. when $q < 1$] we work with $N_S(2-q)$ and Eq. (59) turns to

$$N_S^q(2-q) \frac{\pi^{D/2}}{\Gamma(D/2)} \left(\frac{1}{b(1-q)}\right)^{D/2} \int_0^\infty dx \ x^{D/2-1} [1 + x^q \Omega_D]^{q/(q-1)}$$

$$= N_S^q(2-q) \frac{\pi^{D/2}}{\Gamma(D/2)} \left(\frac{1}{b(1-q)}\right)^{D/2} \frac{\Gamma\left(\frac{1}{1-q} + \frac{D}{2}\right)}{\Gamma\left(\frac{1}{1-q}\right)}$$

$$= N_S^q(2-q) \left(\frac{\pi}{b(1-q)}\right)^{D/2} \frac{\Gamma\left(\frac{1}{1-q} + \frac{D}{2}\right)}{\Gamma\left(\frac{1}{1-q}\right)}$$

$$= \left(\frac{b(1-q)}{\pi}\right)^{D/2(1-q)} \frac{\Gamma\left(\frac{1}{1-q} + \frac{D}{2}\right)}{\Gamma\left(\frac{1}{1-q}\right)} \left[ \frac{\Gamma\left(\frac{1}{1-q} + \frac{D}{2}\right)}{\Gamma\left(\frac{1}{1-q}\right)} \right]^q$$

$$= \left[ \frac{b(1-q)}{\pi}\right]^{D/2} \frac{\Gamma\left(\frac{1}{1-q}\right)}{\Gamma\left(\frac{1}{1-q} + \frac{D}{2}\right)} \left(1 - \frac{D}{2q}(1-q)\right)^{-1}. \quad (60)$$

With (60) we get that

$$S_q^R(ZT) = \log \left[ \left(\frac{\pi}{b(1-q)}\right)^{D/2} \frac{\Gamma\left(\frac{1}{1-q} + \frac{D}{2}\right)}{\Gamma\left(\frac{1}{1-q}\right)} \left(1 - \frac{D}{2q}(1-q)\right)^{1/(q-1)} \right],$$

$$S_q^T(ZT) = \ln_q \left[ \left(\frac{\pi}{b(1-q)}\right)^{D/2} \frac{\Gamma\left(\frac{1}{1-q} + \frac{D}{2}\right)}{\Gamma\left(\frac{1}{1-q}\right)} \left(1 - \frac{D}{2q}(1-q)\right)^{1/(q-1)} \right], \quad (61)$$
and consequently the ensuing entropy powers are

\[ M_q^R(\mathcal{X}) = \exp \left[ \frac{2}{D} \left( S_q^R(\mathcal{X}) - S_q^R(\mathcal{Z}^T) \right) \right] = A \exp \left( \frac{2}{D} S_q^R(\mathcal{X}) \right) \]

\[ M_q^T(\mathcal{X}) = A \left[ \exp \left( S_q^T(\mathcal{X}) \right) \right]^{2/D} = A \exp_{1-(1-q)D/2} \left( \frac{2}{D} S_q^T(\mathcal{X}) \right) , \quad (62) \]

where

\[ A = \left[ \left( \frac{\pi}{b(1-q)} \right)^{D/2} \Gamma \left( \frac{1}{1-q} - \frac{b}{D} \right) \frac{1}{\Gamma \left( \frac{1}{1-q} \right)} \right]^{1/(q-1)} \]. \quad (63) \]

As a consistency check we can take limit \( q \to 1 \). Indeed, by realizing that in the present case \( \sigma^2 = 1 \) implies \( b = [2q - D(1 - q)]^{-1} \), we get

\[ \lim_{q \to 1} M_q^R(\mathcal{X}) = \lim_{q \to 1} M_q^T(\mathcal{X}) = \frac{1}{2\pi e} \exp \left( \frac{2}{D} R(\mathcal{X}) \right) = N(\mathcal{X}) , \quad (64) \]

where \( N(\mathcal{X}) \) is the conventional Shannon entropy power. Finally we can again check that

\[ M_q^R(\mathcal{X}) = M_q^T(\mathcal{X}) . \quad (65) \]

**F. ENTROPY POWER UNCERTAINTY RELATIONS**

Entropy power uncertainty relations for Pontryagin class of transformations

In conventional QM the \( x \) and \( p \) representation wave functions \( \psi(x) \) and \( \hat{\psi}(p) \), respectively are related via Fourier transform relations

\[ \psi(x) = \int_{\mathbb{R}^D} e^{ip \cdot x / h} \hat{\psi}(p) \frac{dp}{(2\pi\hbar)^{D/2}} , \]

\[ \hat{\psi}(p) = \int_{\mathbb{R}^D} e^{-ip \cdot x / h} \psi(x) \frac{dx}{(2\pi\hbar)^{D/2}} . \quad (66) \]

Plancherel (or Riesz–Fischer) equality then implies that \( \|\psi\|_2 = \|\hat{\psi}\|_2 = 1 \). Let us define new functions, namely

\[ f^{(2)}(x) = (2\pi\hbar)^{D/4} \psi(\sqrt{2\pi\hbar} x) , \]

\[ f^{(1)}(p) = (2\pi\hbar)^{D/4} \hat{\psi}(\sqrt{2\pi\hbar} p) . \quad (67) \]

The factor \( (2\pi\hbar)^{D/4} \) ensures that also the new functions are normalized (in sense of \( \| \ldots \|_2 \) ) to unity. With these we will have the same structure of the Fourier transform as in the Beckner–Babenko inequality. Beckner–Babenko inequality (42) can be then rewritten as

\[ \left[ \left( \frac{q'}{2\pi\hbar} \right)^D \right]^{1/q'} \|\psi\|_2^2 \leq \left[ \left( \frac{q}{2\pi\hbar} \right)^D \right]^{1/q} \|\hat{\psi}\|_2^2 . \quad (68) \]

This is equivalent to

\[ \left[ \left( \frac{q'}{2\pi\hbar} \right)^D \right]^{1/q'} \exp \left[ \frac{2(1 - q/2)}{q'} S_q^{R_{q/2}} (|\psi|)^2 \right] \leq \left[ \left( \frac{q}{2\pi\hbar} \right)^D \right]^{1/q} \exp \left[ \frac{2(1 - q/2)}{q} S_q^{R_{q/2}} (|\hat{\psi}|)^2 \right] . \quad (69) \]

Now we take power \( q/(D(1 - q/2)) \) of both left and right side and use the fact that \( 2/q - 1 = 1 - 2/q' \), whis gives

\[ \left( \frac{q}{2\pi\hbar} \right)^{(1/q)(1-q/2)} \exp \left[ \frac{2}{D} S_{q/2}^{R_{q/2}} (|\psi|^2) \right] \left( \frac{q'}{2\pi\hbar} \right)^{(1/q')(1-q/2)} \exp \left[ \frac{2}{D} S_{q/2}^{R_{q/2}} (|\hat{\psi}|^2) \right] \geq 1 . \quad (70) \]
This is identical to (use that for Hölder double one has $1/(1 - q/2) + 1/(1 - q'/2) = 2$)
\[
\frac{1}{2\pi} \left( \frac{q}{2} \right)^{1/(1 - q/2)} \exp \left[ \frac{2}{D} S_{q/2}^R (|\psi|^2) \right] \frac{1}{2\pi} \left( \frac{q'}{2} \right)^{1/(1 - q'/2)} \exp \left[ \frac{2}{D} S_{q'/2}^R (|\psi|^2) \right] \geq \frac{\hbar^2}{4},
\]
(71)

By using (52) this is equivalent to
\[
N_{q/2}^T(|\psi|^2)N_{q'/2}^T(|\psi|^2) \geq \frac{\hbar^2}{4}.
\]
(72)

This result can be generalized to class of Pontryagin dual wave functions.

### Entropy power inequalities for GUP transformations

Using the relation (33) together with (67) we can write
\[
\left[ \left( \frac{q'}{2\pi \hbar} \right)^D \right]^{1/q} |||\psi||^2|_{q/2} \leq \left[ \left( \frac{q}{2\pi \hbar} \right)^D \right]^{1/q'} |||\psi||^2|_{q'/2},
\]
(73)

where $q' \in [2, \infty)$ while $q \in [1, 2]$. It can be checked numerically that this is indeed saturated for coherent states $|\psi|^2_{CS}(p) = q_T(p)^2 - q'/2, b)$ [given by (17)] and associated $\psi_{CS}(x)$ [given by (36)] with the non-extensivity index $2 - q/2$. Analytical proof can be readily done, e.g., for cases $q = 1$ and $q = 2$. We can further rewrite (73) as
\[
\left\{ \frac{1}{2\pi} \left( \frac{q}{2} \right)^{1/(1 - q/2)} \exp \left[ \frac{2}{D} S_{q/2}^R (|\psi|^2) \right] \right\} \left\{ \frac{1}{2\pi} \left( \frac{q'}{2} \right)^{1/(1 - q'/2)} \exp \left[ \frac{2}{D} S_{q'/2}^R (|\psi|^2) \right] \right\} \geq \frac{\hbar^2}{4}.
\]
(74)

Since the relevant wave function is not $\tilde{\psi}$ but $\tilde{\psi}$, cf. Eq. (33), we need to reformulate (74) in terms of $\tilde{\psi}$. This can be done by realizing that
\[
\left\{ \frac{1}{2\pi} \left( \frac{q'}{2} \right)^{1/(1 - q'/2)} \exp \left[ \frac{2}{D} S_{q'/2}^R (|\tilde{\psi}|^2) \right] \right\}
= \frac{1}{2\pi} \left( \frac{q'}{2} \right)^{1/(1 - q'/2)} \left[ \int_{m_p/\sqrt{\pi}}^{m_p/\sqrt{\pi}} dp \left| \tilde{\psi}(p) \right|^2 q'/2 \left( 1 - \frac{p^2 \beta}{m_p^2} \right)^{2/(1 - q'/2)} \right]
= \left\{ \frac{1}{2\pi} \left( \frac{q'}{2} \right)^{1/(1 - q'/2)} \exp \left[ \frac{2}{D} S_{q'/2}^R (|\tilde{\psi}|^2) \right] \right\} \left\{ \left( 1 - \frac{p^2 \beta}{m_p^2} \right)^{q'/2 - 1} \right\}^{2/(1 - q'/2)}_{q'/2},
\]
(75)

where $\langle \ldots \rangle_x$ denotes the average value with respect to escort distribution of order $z$ associated with $f \equiv |\tilde{\psi}|^2$, namely
\[
f(p) \mapsto F_z(p) = \frac{f(x)}{\int dp f(x)}.
\]
(76)

This allows to write (74) as
\[
N_{q/2}^T(|\psi|^2)N_{q'/2}^T(|\tilde{\psi}|^2) \left\{ \left( 1 - \frac{p^2 \beta}{m_p^2} \right)^{q'/2 - 1} \right\}^{2/(1 - q'/2)}_{q'/2} \geq \frac{\hbar^2}{4}.
\]
(77)

Now, without loss of generality, we may assume that the same value of $\langle \ldots \rangle_{q'/2}$ in (77) could be computed also with distribution $|\tilde{\psi}|^2_{CS}(p) = q_T(p)^2 - q'/2, b)$ for some unknown parameter $b$, so that also $\beta = b m_p(q'/2 - 1)/2 \neq \tilde{\beta} = \tilde{b} m_p(q'/2 - 1)/2$. For such $|\tilde{\psi}|^2_{CS}(p)$ and ensuing $|\tilde{\psi}|^2_{CS}(x)$ one would have
\[
N_{q/2}^T(|\tilde{\psi}|^2_{CS})N_{q'/2}^T(|\tilde{\psi}|^2_{CS}) \left\{ \left( 1 - \frac{p^2 \beta}{m_p^2} \right)^{q'/2 - 1} \right\}^{2/(1 - q'/2)}_{q'/2} = \frac{\hbar^2}{4}.
\]
(78)
By dividing (77) by (78) we obtain

\[ M_{q'/2}^T(|\psi|^2)M_{q'/2}^T(|\tilde{\psi}|^2) \geq \sigma_p^2(\tilde{b}, 2 - q'/2) \sigma_x^2(\tilde{b}, 2 - q'/2) \]

where \( \sigma_p^2(\tilde{b}, 2 - q'/2) \) denotes the variance of \( |\tilde{\psi}|^2_{CS}(p) = q_T(p) 2 - q'/2, \tilde{b} \) and similarly \( \sigma_x^2(\tilde{b}, 2 - q'/2) \) is the variance of the associated \( |\tilde{\psi}|^2_{CS}(x) \). In deriving (79) we used the fact that

\[ M_{q'/2}^T(|\tilde{\psi}|^2) = \exp \left[ \frac{2}{D} \left( S_{q'/2}^R(|\tilde{\psi}|^2) - S_{q'/2}^R(|\tilde{\psi}|^2_{CS,1}) \right) \right] \]

\[ = \exp \left[ \frac{2}{D} \left( S_{q'/2}^R(|\tilde{\psi}|^2) - S_{q'/2}^R(|\tilde{\psi}|^2_{CS}) \right) \right] \left[ \sigma_p^2(\tilde{b}, 2 - q'/2) \right]. \]

Here \( S_{q'/2}^R(|\tilde{\psi}|^2_{CS,1}) \) denotes the Rényi entropy of the coherent state distribution with unit variance. Similar relation holds also for \( |\tilde{\psi}|^2_{CS}(x) \). At this stage we use the formula (40) and (19) to write (79) as

\[ M_{q'/2}^T(|\psi|^2)M_{q'/2}^T(|\tilde{\psi}|^2) \geq \sigma_p^2(\tilde{b}, 2 - q'/2) \frac{\sigma_x^2(\tilde{b}, 2 - q'/2)}{\gamma^2} \]

\[ = \frac{\hbar^2}{4} \frac{q^2}{(3q/2 - 1)(3q'/2 - 1)} = \frac{\hbar^2}{4} f(q). \]

Note, that the unknown \( \tilde{b} \) parameter completely factored out from the RHS of (81). The function \( f(q) \) is positive and monotonically increasing for \( q \in [1, 2] \) with max \( f(q) = 1 \). It is important to stress that \( f(q) \) is an universal function of \( q \). So, by using the fact that

\[ f(q) = \frac{2}{([2/q - 1] + 1)(3q/2 - 1)} \frac{2}{([2/q' - 1] + 1)(3q'/2 - 1)} = \phi(q/2)\phi(q'/2), \]

we may define the rescaled entropy power \( \tilde{M}_p^T = \phi^{-1}(x)M_{q'/2}^T \) and rewrite the EPUR in the form

\[ \tilde{M}_p^T(|\psi|^2) \tilde{M}_q^T(|\tilde{\psi}|^2) \geq \frac{\hbar^2}{4}. \]

It is this form of the EPUR that is used in the main text (with tilde omitted).

Since Tsallis distribution maximizing Tsallis entropy \( S_{q'/2}^R \) has the non-extensivity parameter \( 2 - q'/2 \), and because for \( \beta < 0 \) the non-extensivity parameter \( 2 - q'/2 < 1 \) for \( q' > 2 \) we have (as expected) that \( q < 2 \). The lower bound for \( q \) is fixed by the Beckner–Babenko inequality to be \( \geq 1 \).

G. \((\Delta p)^2_\beta\) AND ULTRA-RELATIVISTIC EQUIPARTITION THEOREM

Here we provide a simple evaluation of \((\Delta p)^2_\beta\) the ultra-relativistic limit by employing equipartition theorem. Let us first use the general equipartition relation [20]

\[ \langle p_i \frac{\partial H}{\partial p_i} \rangle = k_B T. \]

For a relativistic particle (no modification of the dispersion relation at high energies is assumed) this implies

\[ \langle p \cdot \frac{\partial H}{\partial p} \rangle = \sum_{i=1}^{3} \langle p_i c p_i \sqrt{p + m^2 c^2} \rangle = \langle \frac{p^2 c^2}{p_0} \rangle, \]

which in the ultra-relativistic limit gives \( \langle |p| \rangle c = 3k_B T \).

In order to find \( \langle p^2 \rangle \) we can employ the fact that the usual derivation of the equipartition theorem also directly gives the identity

\[ \langle p_i \frac{\partial H^2}{\partial p_i} \rangle = 2k_B T \langle H \rangle + 2k_B T \langle p_i \frac{\partial H}{\partial p_i} \rangle. \]
While the LHS gives
\[
\sum_{i=1}^{3} \left( p_i \frac{\partial H^2}{\partial p_i} \right) = 2c^2 \langle p^2 \rangle = 2c^2(\Delta p)^2,
\] (87)
the RHS implies in the ultra-relativistic limit
\[
6k_B T \langle H \rangle + 6(k_B T)^2 = 6k_B T \langle |p| \rangle c + 6(k_B T)^2 = 24(k_B T)^2.
\] (88)
Consequently, we get in the ultra-relativistic limit that
\[
c^2 \langle p^2 \rangle = c^2(\Delta p)^2 = 12(k_B T)^2.
\] (89)
Since inflaton’s rest mass \( \simeq 10^{12} - 10^{13}\) GeV, cf. e.g. Ref. [19], the inflaton is during the late inflation/reheating epoch (i.e., \( \simeq 10^{15} - 10^{16}\) GeV) in ultra-relativistic regime. If we now employ the fact that in semi-classical regime thermal and quantum fluctuations should be of the same order we get for inflaton \( (\Delta p)^2 \simeq 12(k_B T)^2 \).

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[2] In fact, symmetric operator \( \hat{A} \) has real spectrum only if its both deficiency indices \( n_\pm(\hat{A}) = \dim \text{Ker}(\hat{A}^\dagger \pm i) \) are zero, which is however happening only when \( \hat{A} \) is self-adjoint operator [1].
[3] A complex valued function \( \psi \) on \( \mathbb{R} \) is absolutely continuous if it is an indefinite integral, that is if there is a measurable function \( \tilde{\psi} \), integrable on closed interval \( I \subset \mathbb{R} \) with endpoints \( a < b \), such that \( \psi(x) = c + \int_a^x \tilde{\psi}(p) \, dp \) for any \( a \leq x \leq b \) and some constant \( c \). If \( I \subset \mathbb{R} \) is open, \( \psi \) is absolutely continuous on \( I \) if it is absolutely continuous on each compact interval \( I' \subset I \). Absolute continuity ensures that \( \psi' \) is final almost everywhere in \( I \) and that rules of integration by parts are valid.
[4] Recall, that operator is defined by its differential expression and by its domain. In particular, without specifying a domain one cannot decide whether a given differential expression corresponds, e.g., to symmetric or self-self adjoint operator.
[5] Convergence criterion for improper integrals \( \int_a^b \, dp \, f(p) \) states [7] that the integral of a bounded function \( f(p) \) on \( (a,b) \) converges if both \( \lim_{p \to a+} (p - a) f(p) = 0 \) and \( \lim_{p \to b-} (b - p) f(p) = 0 \). Since in our case \( \psi, \phi \) are from \( D(\hat{x}) \), the product \( \psi^*(p) \phi(p) \) is bounded and integrable in \( I \). So, potential singular points can only be at boundaries and thus \( \psi^*(p) \phi(p) \) must obey the above convergence criterion in order to ensure that \( \int_a^b \, dp \, \psi^*(p) \phi(p) \) is finite.
[6] Let us recall that \( D(\hat{x}^\dagger) \) is the collection of all vectors \( \varphi \) in \( L^2(\mathcal{D}) \) for which there exists a \( \tilde{\varphi} \in L^2(\mathcal{D}) \) such that \( \langle \varphi | \tilde{\varphi} \rangle = \langle \tilde{\varphi} | \psi \rangle \) for every \( \psi \in D(\hat{x}) \). This defines a linear mapping (adjoint operation) \( \hat{x}^\dagger \varphi = \tilde{\varphi} \) for any \( \varphi \in D(\hat{x}^\dagger) \).
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