Novel $|V_{us}|$ Determination Using Inclusive Strange $\tau$ Decay and Lattice HVPs

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We propose and apply a new approach to determining $|V_{us}|$ using dispersion relations with weight functions having poles at Euclidean (space-like) momentum which relate strange hadronic $\tau$ decay distributions to hadronic vacuum polarization functions (HVPs) obtained from lattice QCD. We show examples where spectral integral contributions from the region where experimental data have large errors or do not exist are strongly suppressed but accurate determinations of the relevant lattice HVP combinations remain possible. The resulting $|V_{us}|$ agrees well with determinations from $K$ physics and 3-family CKM unitarity. Advantages of this new approach over the conventional hadronic $\tau$ decay determination employing flavor-breaking sum rules are also discussed.

INTRODUCTION

Precise determinations of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{us}|$ are important in the context of 3-family unitarity tests and searches for physics beyond the Standard Model (SM). The most precise such determination, $|V_{us}| = 0.2253(7)$, is from $\Gamma[K_{\mu2}]/\Gamma[\pi_{\mu2}]$, using the lattice input $f_K/f_\pi = 1.193(3)$ [3]. Three-family unitarity and $|V_{ud}| = 0.97417(21)$ [3], similarly, imply $|V_{us}| = 0.2258(9)$, while $\Gamma_{K_{\mu2}}$ with lattice $f_\pi(0)$ input, yields $|V_{us}| = 0.2237(10)$ [3]. It is a long-standing puzzle that conventional flavor-breaking (FB) finite-energy sum rules (FESRs) employing hadronic $\tau$ decay data yield much lower $|V_{us}|$, most recently $0.2186(21)$ [2] (0.2207(27) when $\Gamma[K_{\mu2}]$ and dispersive $K\pi$ form factor constraints are incorporated [2]).

The conventional FB FESR implementation employs assumptions for unknown dimension $D = 6$ and 8 OPE condensates which turn out to fail self-consistency tests [2]. An alternate implementation, fitting $D > 4$ condensates to data, yields results passing these tests and compatible with determinations from other sources [2]. The resulting error is dominated by uncertainties on the relevant weighted inclusive flavor $us$ spectral integrals and a factor $> 2$ larger than that of $K$-decay-based approaches. Improved branching fractions (BFs) used in normalizing low-multiplicity $us$ exclusive-mode Belle and BaBar distributions would help, but $\sim 25\%$ errors on higher-multiplicity $us$ “residual mode” contributions [10], involving modes not re-measured at the B-factories, preclude a factor of 2 improvement [2, 11].

This paper presents a novel dispersive approach to determining $|V_{us}|$ using inclusive strange hadronic $\tau$ decay data, hadronic vacuum polarization functions (HVPs) computed on the lattice, and weight functions, $\omega_N(s) = \Pi_N^{K_{\mu2}}(s + Q^2_{K_{\mu2}})^{-1}$, $Q^2_{K_{\mu2}} > 0$, having poles at Euclidean $Q^2 = Q^2_{K_{\mu2}} > 0$. We show examples of such $\omega_N$ which strongly suppress spectral contributions from the high-multiplicity $us$ “residual” region without blowing up errors on the related lattice HVP combinations. The approach yields $|V_{us}|$ in good agreement with $K$-decay analysis results and 3-family CKM unitarity expectations. The lattice error is comparable to the experimental one, and the total error is less than that of the inclusive FB FESR $\tau$ decay determination.

NEW INCLUSIVE DETERMINATION

The conventional inclusive $FB\tau$ decay determination is based on the FESR relation [12, 13]

$$
\int_0^{s_0} \omega(s)\Delta\rho(s)ds = \frac{1}{2\pi i} \int_{|s| = s_0} \omega(s)\Delta\Pi(-s)ds, \quad (1)
$$

connecting, for any $s_0$ and analytic $\omega(s)$, the relevant FB combination, $\Delta\Pi(-s) = \Pi_{us}(-s) - \Pi_{ud}(-s)$, of spin $J = 0, 1$ HVPs and associated spectral function $\Delta\rho(s) = \frac{1}{2i}\text{Im}\Delta\Pi(-s)$. Experimental data is used on the LHS and, for large enough $s_0$, the OPE on the RHS.

In the SM, the differential distribution, $dR_{V/A;ij}/ds$, associated with the flavor $ij = ud, us$ vector ($V$) or axial vector ($A$) current-induced decay ratio $R_{V/A;ij} = \Gamma[\tau^- \to \nu_\tau\text{hadrons}_{V/A;ij}]/\Gamma[\tau^- \to e^-\nu_e\nu_\tau]$, is related to
the $J = 0, 1$ spectral functions $\rho_{ij,V/A}^{(J)}(s)$, by [14]

$$\frac{dR_{ij,V/A}}{ds} = \frac{12\pi^2|V_{ij}|^2 S_{EW}}{m_T^2} \times \left[ \omega_r(s) \rho_{ij,V/A}^{(0+1)}(s) - \omega_L(y_r) \rho_{ij,V/A}^{(0)}(s) \right],$$

where $y_r = s/m_T^2$, $\omega_r(y) = (1-y)^2(1+2y)$, $\omega_L(y) = 2y(1-y)^2$, and $S_{EW}$ is a known short-distance electroweak correction [15, 16]. Experimental $dR_{ij,V/A}/ds$ distributions thus determine, up to factors of $\prod_{N=1}^3$, combinations of the $\rho_{ij,V/A}$. The low $|V_{us}|$ noted above results from a conventional implementation [17] of Eq. (1) which employs fixed $s_0 = m_T^2$ and $\omega = \omega_r$ and assumptions for experimentally unknown $D = 6$ and 8 condensates. With $s_0 = m_T^2$ and $\omega = \omega_r$, inclusive non-strange and strange BFs determine the $ud$ and $us$ spectral integrals. Testing $D = 6$ and 8 assumptions by varying $s_0$ and/or $\omega$, however, yields $|V_{us}|$ with significant unphysical $s_0$- and $\omega$-dependence, motivating an alternate implementation employing variable $s_0$ and $\omega$ which allows a simultaneous fit of $|V_{us}|$ and the $D > 4$ condensates. Significantly larger (now stable) $|V_{us}|$ are found, the conventional implementation results $|V_{us}| = 0.2186(21)$ [14] and $0.2207(27)$ [3], shifting up to $0.2208(23)$ and $0.2231(27)$ [14], respectively, with the new implementation. us spectral integral uncertainties dominate the error, with current $\sim 25\%$ residual mode contribution errors precluding a competitive determination [3].

Motivated by this limitation, we switch to generalized dispersion relations involving the experimental $us$ V+A inclusive distribution and weights, $\omega_N(s) \equiv \prod_{\tau=1}^N (s + Q_{\tau}^2)^{-1}$, $0 < Q_{\tau}^2 < Q_{\tau+1}^2$, having poles at $s = -Q_{\tau}^2$. From Eq. (2), $dR_{us,V+A}/ds$ directly determines $|V_{us}|\rho_{us}(s)$, with

$$\rho_{us}(s) \equiv \left( 1 + 2 \frac{s}{m_T^2} \right) \rho_{us,V+A}^{(1)}(s) + \rho_{us,V+A}^{(0)}(s).$$

For $N \geq 3$, the associated HVP combination

$$\hat{\Pi}_{us} \equiv \left( 1 - 2 \frac{Q^2}{m_T^2} \right) \Pi_{us,V+A}^{(1)}(Q^2) + \Pi_{us,V+A}^{(0)}(Q^2)$$

satisfies the convergent dispersion relation

$$\int_0^\infty \rho_{us}(s)\omega_N(s)ds = \sum_{k=1}^N \text{Res}_{s=-Q_k^2} \left[ \hat{\Pi}_{us}(-s)\omega_N(s) \right] = \sum_{k=1}^N \hat{\Pi}_{us,V+A}(Q_k^2) \equiv \tilde{F}_{\omega_N}. \quad (5)$$

With $\hat{\Pi}_{us}(Q_k^2)$ measured on the lattice, $dR_{us,V+A}/ds$ used to fix $s < m_T^2$ spectral integral contributions, and $s > m_T^2$ contributions approximated using pQCD, one has,

$$|V_{us}| = \sqrt{\frac{\tilde{R}_{us;\omega_N}}{\tilde{F}_{\omega_N}} - \int_{m_T^2}^{\infty} \rho_{us}^{(pQCD)}(s)\omega_N(s)ds} \quad (6)$$

where $\tilde{R}_{us;\omega_N} = \frac{m_T^2}{12\pi^2 S_{EW}} \int_0^{m_T^2} \frac{1}{(1-y_{\tau})^2} \frac{dR_{us,V+A}(s)}{ds} \omega_N(s)ds$.

Choosing uniform pole spacing $\Delta$, $\omega_N$ can be characterized by $\Delta, N$, and the pole-interval midpoint, $C = (Q_1^2 + Q_N^2)/2$. With large enough $N$, and all $Q_k^2$ below $\sim 1 \text{ GeV}^2$, spectral integral contributions from $s > m_T^2$ and the higher-$s$, larger-error part of the experimental distribution can be strongly suppressed. Increasing $N$ lowers the error of the LHS in Eq. (3) but increases the relative RHS error. With results insensitive to modest changes of $\Delta$, we fix $\Delta = 0.2/(N-1) \text{ GeV}^2$, ensuring $\omega_N$ with the same $C$ but different $N$ have poles spanning the same $Q^2$ range. $C$ and $N$ are varied to minimize the error on $|V_{us}|$.

We employ the following $us$ spectral input: $K_{IJ2}$ or $\tau \rightarrow K\nu_e$ [7] for $K$ pole contributions, unit-normalized Belle or BaBar distributions for $K\pi$ [18, 19], $K^-\pi^+\pi^-\pi^-[20]$, $K^0\pi^-\pi^0$ [21], and $K\bar{K} \rightarrow \pi^+\pi^-$ [22, 23], the most recent HFLAV BFs [3], and 1999 ALEPH results [10], modified for current, the residual mode distribution. Multiplication of a unit-normalized distribution by the ratio of corresponding exclusive-mode to electron BFs, converts that distribution to the corresponding contribution to $dR_{us,V+A}(s)/ds$. The dispersively constrained $K\pi$ BFs of Ref. [3] (ACLP) provide an alternate $K\pi$ normalization. In what follows, we illustrate the lattice approach using the HFLAV $K\pi$ normalization. Alternate results using the ACLP normalization are given in Ref. [24].

**LATTICE CALCULATION METHOD**

We compute the two-point functions of the flavor us V and A currents, $J_{\mu,V/A}(x,t) = J_{\mu,V/A}(x)$, $\mu = x,y,z,t$, via

$$C_{\mu V/A}^{(\mu)}(t) = \sum_{x} (J_{\mu,V/A}(x,t)(J_{\mu,V/A}(0,0))^\dagger). \quad (7)$$

The continuum spin $J = 0, 1$, us HVPs, $\Pi_{us,V/A}^{(J)}$, are related to the two-point functions by

$$\sum_{x} e^{iHx} (J_{\mu,V/A}(x)(J_{\mu,V/A}(0))^\dagger) =$$

$$(q^2g_{\mu\nu} - q^\mu q^\nu)\Pi_{us,V/A}^{(1)}(q^2) + q^\mu q^\nu \Pi_{us,V/A}^{(0)}(q^2),$$

up to finite volume (FV) and discretization corrections. With $C_{\mu V/A}(t) = \frac{1}{t} \sum_{k=x,y,z} C_{\mu V/A}^{(k)}(t), C_{\mu V/A}^{(0)}(t) = C_{\mu V/A}^{(0)}(t)$, the analogous $J = 0, 1$ parts of $C_{\mu V/A}$ are [25, 26]

$$\Pi_{\mu V/A}^{(J)}(Q^2) - \Pi_{\mu V/A}^{(J)}(0) = \sum_{t} K(q,t) C_{\mu V/A}^{(J)}(t). \quad (9)$$
with \( K(q,t) = \frac{\cos q - 1}{q^2 + \hat{q}^2} \) and \( \hat{q} \) the lattice momentum, \( \hat{q}_\mu = 2 \sin q_\mu / 2 \). FV corrections to this infinite volume result are discussed below. We use lattice HVPs measured on the near-physical-quark-mass, 2 + 1 flavor 48\(^3 \times 96\) and 64\(^3 \times 128\) Möbius domain wall fermion ensembles of the RBC and UKQCD collaborations [27], employing all-mode-averaging (AMA) [28, 29] to reduce costs. Slight \( u, d, s \) mass mistunings are corrected by measuring the HVPs with partially quenched (PQ) physical valence quark masses [27], also using AMA.

\( \bar{F}_{\omega_N} \) in Eq. (6) can be decomposed into four contributions, \( \bar{F}_{V/A: \omega_N} \), labelled by the spin \( J = 0 \) or 1, and current type, \( V \) or \( A \), \( \bar{F}_{V/A: \omega_N} = \text{lim}_{t \to \infty} \bar{L}^{(j)}_{V/A: \omega_N}(t) \), where \( \bar{L}^{(j)}_{V/A: \omega_N}(t) = \sum_{l=-t}^{t} \omega_N(l) \gamma_l^{(j)}(t) \). From Eqs. (9), (11),

\[
\omega^{(1)}_N = \sum_{k=1}^{N} K \left( \sqrt{Q_k^2}, t \right) \left( 1 - \frac{2Q_k^2}{m_c^2} \right) \text{Res}_{\omega_N} \left( \frac{\omega_N(s)}{s - Q_k^2} \right),
\]

\[
\omega^{(0)}_N = \sum_{k=1}^{N} K \left( \sqrt{Q_k^2}, t \right) \text{Res}_{\omega_N} \left( \frac{\omega_N(s)}{s - Q_k^2} \right).
\]

With finite lattice temporal extent, finite-time effects may exist. Increasing \( N \) increases the level of cancelation and relative weight of large-\( t \) contributions on the RHS of Eq. (5). The restrictions 0.1 GeV\(^2 \leq C < 1 \) GeV\(^2 \) and \( N \leq 5 \), chosen to strongly suppress large-\( t \) contributions, allows us to avoid modelling the large-\( t \) behavior. Fig. 1 shows, as an example, the large-\( t \) plateaus of the partial sums \( \bar{L}^{(j)}_{V/A: \omega_N}(t) \), obtained in all four channels, for \( N = 4, C = 0.5 \) GeV\(^2 \) on the 48\(^3 \times 96\) ensemble.

The upper panel of Fig. 2 shows the relative sizes of the four \( C \)-dependent lattice contributions, \( \gamma^{(j)}, A^{(j)} \), for \( N = 4 \). The lower panel, similarly, shows the relative sizes of different contributions to the weighted \( u \) spectral integrals. \( Kt \) denotes the sum of \( K^{-\pi} \) and \( K^\pi \) contributions, pQCD the contribution from \( s > m_c^2 \), evaluated using the 5-loop-truncated pQCD form [30, 31]. Varying \( C \) (and \( N \)) varies the level of suppression of the pQCD and higher-multiplicity contributions, the relative size of \( K \) and \( K^\pi \) contributions, and hence the level of “inclusiveness” of the analysis. The stability of \( |V_{us}| \) under such variations provides additional systematic cross-checks.

**ANALYSIS AND RESULTS**

The \( A^{(0)} \) channel produces the largest RHS contribution to Eq. (5). On the LHS, the \( K \) pole dominates \( \rho_{us; A}^{(0)}(s) \), with continuum contributions doubly chirally suppressed. Estimated LHS continuum \( A^{(0)} \) contributions, obtained using sum-rule \( K(1460) \) and \( K(1830) \) decay constant results [32], are numerically negligible for the \( \omega_N \) we employ. An “exclusive” \( A^{(0)} \) analysis relating \( \tilde{F}_{\omega_N}^{(0)} \) to the \( K \)-pole contribution, \( \tilde{R}_{\omega_N}^{K} = \gamma_K \omega_N(m_K^2) \), is, therefore, possible, with \( \gamma_K = 2 |V_{us}|^2 \int F_{\omega_N}^{K} \) obtained from either \( K_{\mu 2} \) or \( \Gamma[\tau \to K\nu_\tau] \). Since the simulations underlying \( \tilde{F}_{\omega_N}^{(0)} \) are isospin symmetric, we correct \( \gamma_K \) for leading-order electromagnetic (EM) and strong isospin-breaking (IB) effects [8]. With PDG \( \tau \) lifetime [8] and HFLAV \( \tau \to K\nu_\tau \) BF [4] input, \( \gamma_K[\tau_K] = 0.0012061(167)_{\text{exp}}(13)_{IB} \) GeV\(^2 \). \( \gamma_K[\tau_K] \) is employed in our main, fully \( \tau \)-based analysis. The more precise result \( \gamma_K[K_{\mu 2}] = 0.0012347(29)_{\text{exp}}(22)_{IB} \) [4]...
from $\Gamma[K_{\mu2}]$ can also be used if SM dominance is assumed. Exclusive analysis $|V_{us}|$ results are independent of $C$ for $C < 1$ GeV$^2$ (confirming tiny continuum $A^{(0)}$ contributions) and agree with the results, $|V_{us}| = 0.2233(15)_{\exp}(12)_{th}$ and $0.2260(3)_{\exp}(12)_{th}$, obtained using $|V_{us}| = \sqrt{\gamma_K/(2f_K^2)}$, the isospin-symmetric lattice result $F_K \equiv \sqrt{2f_K} = 0.15551(83)$ GeV $^{27}$ and $\gamma_K = \gamma_K[\gamma_K]$ and $\gamma_K[K_{\mu2}]$, respectively. See $^{24}$ for further details.

For the fully inclusive analysis, statistical and systematic uncertainties are reduced by employing $2f_K^2\omega_N(m_K^2)$, with measured $f_K$, for the $K$ pole $A^{(0)}$ channel contribution. The residual, continuum $A^{(0)}$ contributions are compatible with zero within errors, as anticipated above. IB corrections, beyond those applied to $\gamma_K$, are numerically relevant only for $K\pi$. We account for (i) $\pi^0$-$\eta$ mixing, (ii) EM effects, and (iii) IB in the phase space factor, with $\pi^0$-$\eta$ mixing numerically dominant, evaluating these corrections, and their uncertainties, from the results presented in Ref. $^{8}$. A 2% uncertainty, estimated using results from a study of duality violations in the $SU(3)_F$-related flavor $ud$ channels $^{33}$, is assigned to pQCD contributions. Since our analysis is optimized for $\omega_N$ strongly suppressing higher-multiplicity and $s > m_s^2$ contributions, such an uncertainty plays a negligible role in our final error.

Several systematic uncertainties enter the lattice computation. With an assumed continuum extrapolation linear in $a^2$ but only two lattice spacings, $O(a^2)$ discretization uncertainties must be estimated. For the $\omega_N$ we employ, the two ensembles yield $\tilde{F}_{\omega_N}$ differing by less than (typically significantly less than) 10%, compatible with $\sim C a^2$ or smaller $O(a^2)$ errors. Anticipating a further $\sim C a^2$ reduction of $O(a^2)$ relative to $O(a^4)$ corrections, we estimate residual $O(a^4)$ continuum extrapolation uncertainties to be $\sim 0.1 C a_0^2$, with $a_0^{-1} = 2.36$ GeV $^{27}$ the smaller of the two lattice spacings. We also take into account the lattice scale setting uncertainty. The dominant FV effect is expected to come from $K\pi$ loop contributions in the $V^{(1)}$ channel, which we estimate using a lattice regularized version of finite-volume Chiral Perturbation Theory (ChPT). It is known, from Ref. $^{33}$, that one-loop ChPT for HVPs involving the light $u,d$ quarks yields a good semi-quantitative representation of observed FV effects $^{35}$; we thus expect it to also work well for the flavor $us$ case considered here, where FV effects involving the heavier $s$ quark should be suppressed relative to those in the purely light $u,d$ quark sector. The result of our one-loop ChPT estimate is a 1% FV correction. We thus assign a 1% FV uncertainty to our $V^{(1)}$ channel contributions $^{37}$. Regarding the impact of the slight $u, d, s$ sea-quark mass mismistunings on the PQ results, the shift from slightly mis-tuned unitary to PQ shifted-valence-mass results for $\tilde{F}(\omega_N)$ corresponds to shifts in $|V_{us}|$ of $< 0.4\%$ for both ensembles. With masses and decay constants typically much less sensitive to sea-quark mass shifts than to the same valence-quark mass shifts, we expect sea-mass PQ effects to be at the sub-$\sim 0.1\%$ level, and hence negligible on the scale of the other errors in the analysis.

Fig. 3 shows the $C$ dependence of relative, non-data, inclusive analysis error contributions. $K$ labels the $f_K$-induced $A^{(0)}$ uncertainty, other that induced by the statistical error on the sum of $V^{(1)}, V^{(0)}, A^{(1)}$, and tiny continuum $A^{(0)}$ channel contributions. The statistical error dominates for low $C$, the discretization error for large $C$.

Fig. 4 shows our $|V_{us}|$ results. These agree well for different $N$, and $C < 1$ GeV$^2$. The slight trend toward lower central values for weights less strongly suppressing high-$s$ spectral contributions ($N = 3$ and higher $C$) suggests the residual mode distribution may be somewhat underestimated due to missing higher-multiplicity contributions. Such missing high-$s$ strength would also lower the $|V_{us}|$ obtained from FB FESR analyses. Table II lists relative spectral integral contributions for selected $\omega_N$. Note the significantly larger (6.8% and 21%) residual mode and pQCD contributions for $N = 3$ and $C = 1$ GeV$^2$. Restricting $C$ to $< 1$ GeV$^2$ keeps these from growing further and helps control higher-order discretization errors. The error budget for various sample weight choices is summarized in Table II.

Our optimal inclusive determination is obtained for $N = 4$, $C = 0.7$ GeV$^2$, where residual mode and pQCD contributions are highly suppressed, and yields results

$$|V_{us}| = \begin{cases} 0.2228(15)_{\exp}(13)_{th}, & \text{for } \gamma_K[\gamma_K] \\ 0.2245(11)_{\exp}(13)_{th}, & \text{for } \gamma_K[K_{\mu2}] \end{cases}$$

consistent with determinations from $K$ physics and 3-family unitarity. Theoretical (lattice) errors are comparable to experimental ones and combined errors improve on those of the corresponding inclusive FB FESR determinations. A comparison to the results of other determinations is given in Fig. 5.
that more effectively suppress spectral contributions to the OPE, and (ii) the existence of weight functions analytically improvable precision lattice data in place of systematic errors added in quadrature. The determination using $\gamma[\tau_K]$ and lattice $f_k$ is shown for comparison.

![Graph](image)

**FIG. 4.** $|V_{us}|$ vs. $C$ for $N = 3, 4, 5$. $N = 3, 5$ results are shifted horizontally for presentational clarity and statistical and systematic errors added in quadrature. The determination using $\gamma[\tau_K]$ and lattice $f_k$ is shown for comparison.

| contribution | [N, C](GeV^2) | [3, 0.3] | [3, 1] | [4, 0.7] | [5, 0.9] |
|-------------|---------------|---------|--------|---------|---------|
| $K$         | 65.5          | 30.9    | 61.7   | 66.9    |
| $K\pi$      | 21.4          | 28.6    | 26.4   | 25.2    |
| $K^0\pi^+\pi^-$ | 2.4    | 5.6     | 2.8    | 2.1     |
| $\bar{K}\pi^0$ | 3.1    | 7.3     | 3.6    | 2.7     |
| residual    | 2.7           | 6.8     | 2.9    | 2.1     |
| pQCD        | 4.9           | 20.8    | 2.7    | 1.1     |

**TABLE I.** Sample relative spectral integral contributions.

**CONCLUSION AND DISCUSSION**

We have presented a novel method for determining $|V_{us}|$ using inclusive strange hadronic $\tau$ decay data. Key advantages over the related FB FESR approach employing the same $us$ data are (i) the use of systematically improvable precision lattice data in place of the OPE, and (ii) the existence of weight functions that more effectively suppress spectral contributions from the larger-error, high-$s$ region without blowing up theory errors. The results provide not only the most accurate inclusive $\tau$ decay sum rule determination of $|V_{us}|$ but also evidence that high-$s$ region systematic errors may be underestimated in the alternate FB FESR approach. The combined experimental uncertainty can be further reduced through improvements to the experimental $\tau \rightarrow K, K\pi$ BFVs, while the largest of the current theoretical errors, that due to lattice statistics, is improvable by straightforward lattice computational effort. Such future improvements will help constrain the flavor dependence of any new physics contributions present in hadronic tau decays, contributions expected to be present at some level if the apparent violation of lepton flavor universality seen recently in semileptonic $b \rightarrow c$ decays involving $\tau$ persists.

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![Graph](image)

**FIG. 5.** Our $|V_{us}|$ determinations (inclusive, $\gamma_K[\tau_K]$-based exclusive (filled square) and $\gamma_K[K_{\mu2}]$-based exclusive (empty square)) c.f. results from other sources.

| contribution | relative error (%) |
|-------------|--------------------|
| [N, C](GeV^2) | [3, 0.3] | [3, 1] | [4, 0.7] | [5, 0.9] |
| theory      | $f_K$           | 0.37   | 0.20   | 0.34   | 0.36   |
|             | others, stat.   | 0.41   | 0.19   | 0.34   | 0.41   |
|             | discretization  | 0.10   | 0.80   | 0.25   | 0.27   |
|             | scale setting   | 0.09   | 0.08   | 0.11   | 0.11   |
|             | IB              | 0.10   | 0.21   | 0.11   | 0.10   |
|             | FV              | 0.10   | 0.04   | 0.13   | 0.18   |
|             | pQCD            | 0.05   | 0.26   | 0.03   | 0.01   |
| total       | 0.59            | 0.91   | 0.38   | 0.58   |

| experiment  | $K$             | 0.48   | 0.27   | 0.44   | 0.47   |
|             | $K\pi$          | 0.20   | 0.32   | 0.23   | 0.22   |
|             | $K^-\pi^+\pi^-$ | 0.06   | 0.16   | 0.06   | 0.05   |
|             | $K^0\pi^0$      | 0.03   | 0.09   | 0.03   | 0.03   |
|             | residual        | 0.41   | 1.35   | 0.41   | 0.28   |
| total       | 0.66            | 1.43   | 0.65   | 0.59   |

**TABLE II.** Error budget for the inclusive $|V_{us}|$ determination.
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We show additional, supplementary plots and tables containing detailed information on the experimental distributions, expanding on the exclusive determination of $|V_{us}|$ using lattice $A^{(0)}$ contributions and $f_K$ input, and providing the ACLP $K\pi$ normalization analogues of the HFLAV $K\pi$ normalization results shown in the main text.

Fig. 6 shows the experimental exclusive mode distributions for both the HFLAV and ACLP $K\pi$ normalization choices. Fig. 7 shows an example of the weighted versions thereof, for the $N = 4$ and $C = 0.5$ GeV$^2$ weight choice. The figures include, for reference, the pQCD contribution, evaluated using the 5-loop-truncated $D = 0$ OPE form and, for illustrative purposes, the sample value $|V_{us}| = 0.2253$.

The experimental results for $|V_{us}|^2\tilde{\rho}_{us}(s)$ are obtained as a sum over exclusive and semi-exclusive mode contributions. B-factory unit-normalized, $\frac{dN}{ds}$ distributions are used for the $K\pi$ $^{19}, K^-\pi^+\pi^-^{20}, \bar{K}^0\pi^-\pi^0$ $^{21}$ and $K^-\pi^-K^-\pi^-$ $^{22}$ modes. The corresponding exclusive-mode $dR_{K\pi}(s)/ds$ contributions are obtained by multiplying the unit-normalized $\tau \to X\nu\tau$ distributions by $B_X/B_{\tau}$, with $B_{\tau}$ the $\tau \to e\nu_e\nu_\tau$ branching fraction (BF) and $B_X$ that of $\tau \to X\nu$. In the case of the two $K\pi$ modes, the preliminary BaBar results for the $K^-\pi^0$ distribution, reported in Ref. 22, were never finalized. With the unit-normalized $K^-\pi^0$ and $K_S\pi^-$ distributions identical in the isospin limit, this problem can be dealt with, up to small isospin-breaking corrections, by using the unit-normalized Belle $K_S\pi^-$ distribution $^{19}$ for both $K\pi$ modes. This leads to

$$\frac{dR_{K\pi}(s)}{ds} = \left(\frac{B_{K^-\pi^0} + B_{\bar{K}^0\pi^-}}{B_{\tau}}\right) \left[\frac{1}{N} \frac{dN}{ds}\right]_{K_S\pi^-}.$$

As noted in the main text, two choices exist for the 2-mode $K\pi$ BF sum, that given by HFLAV $^7$ and that obtained from the extended analysis of ACLP $^8$, in which additional dispersive constraints were imposed on the timelike $K\pi$ form factors (at the cost of somewhat larger final errors on the two $K\pi$ BFs). HFLAV BFs $^7$ are used for all other modes. For $K^-\pi^-\pi^-$, we employ the full covariances of the unit-normalized distribution, provided by BaBar $^{20}$. The errors on the corresponding weighted spectral integrals turn out to be dominated by the BF uncertainty. For example, for the weight choice $N = 4$, $C = 0.5$ GeV$^2$, 2.29% of the total 2.42% relative error comes from the uncertainty in the 2-mode $K\pi$ BF sum. It is also worth noting that correlations play a relatively minor role in determining the small remaining contribution to the total error; including the correlations in the unit-normalized distribution results shifts this (already small) contribution by only 25%. For the $K^0\pi^-\pi^0$ mode, at present only the errors on the unit-normalized distribution are known. Ignoring the (presently unknown) correlations, one finds, for the weight choice $N = 4$, $C = 0.5$ GeV$^2$, 3.38% of the total 3.41% relative error generated by the HFLAV BF uncertainty. In view of the minor
role played by correlations in determining the error on the exclusive \( K^- \pi^+ \pi^- \) contribution, we conclude that our lack of knowledge of the correlations in the unit-normalized \( K^0 \pi^- \pi^0 \) distribution will have a negligible impact on final error on contributions for this exclusive mode. The residual mode \( dR_{\text{us},V+A}/ds \) distribution is obtained by summing 1999 ALEPH results \( ^{10} \) for those modes \( (K^- \pi^0 \pi^0, K 3\pi, K \eta, K 4\pi \text{ and } K 5\pi) \) included in the ALEPH distribution but not remeasured by either Belle or BaBar. Each such contribution is first rescaled by the ratio of the current to the 1999 ALEPH BF for the mode (or sum of modes), and the rescaled contributions then re-summed to produce the updated version of the residual mode distribution.

Exclusive and semi-exclusive mode contributions to the product \( |V_{\text{us}}|^2 \rho_{\text{us}}(s) \) are obtained from the corresponding contributions to \( dR_{\text{us},V+A}/ds \) by division by \( 12\pi^2 S_{\text{EW}} \left(1 - \frac{s}{m^2}\right)^2 / m^2_r \).

Fig. 8 shows the result of the exclusive \( A^{(0)} \) determination of \( |V_{\text{us}}^A| = \sqrt{R_{\omega_N}^{K \pi} / F_{\omega_N}^{A^{(0)}}} \) for \( N = 4 \) as an example.

Table S I shows the relative spectral integral contributions, as a function of \( C \), for \( N = 4 \) and the ACLP \( K \pi \) normalization. Table S II shows the experimental error budget obtained using the ACLP \( K \pi \) normalization for various sample weight choices. (Results for the same weight choices and the HFLAV \( K \pi \) normalization are shown in the bottom half of Table II in the main text.) The last line of the table gives the quadrature sums of the total experimental error and corresponding total theory error, the latter shown in the upper half of Table II in the main text.

Finally, in Fig. 11 we show the comparison of our

![Relative spectral contributions (N=4)](image)

![Vus vs. C for N = 3, 4, 5, for the ACLP Kπ normalization. N = 3, 5 results are shifted horizontally for presentation clarity and statistical and systematic errors added in quadrature. The determination using γ|τK| and lattice fK is shown for comparison.](image)

![|Vus| vs. C for N = 3, 4, 5, for the ACLP Kπ normalization.](image)

| contribution     | value [%] |
|------------------|------------|
| \( N, C[\text{GeV}^2] \) | [3,0.3] [3,1] [4,0.7] [5,0.9] |
| \( K \)          | 64.9 30.4 61.0 66.2 |
| \( K \pi \)      | 22.1 29.3 27.2 26.0 |
| \( K^- \pi^+ \pi^- \) | 2.4 5.5 2.7 2.1 |
| \( K^0 \pi^- \pi^0 \) | 3.1 7.2 3.5 2.7 |
| residual         | 2.6 6.7 2.8 2.0 |
| pQCD             | 4.9 20.8 2.7 1.1 |

**TABLE S1.** Sample relative spectral integral contributions, ACLP \( K \pi \) normalization.
| contribution | relative error (%) |
|--------------|--------------------|
| \(|N, C[GeV^2]|\) | [3, 0.3] [3, 1] [4, 0.7] [5, 0.9] |
| experiment \(K\) | 0.47 0.27 0.43 0.46 |
| \(K\pi\) | 0.43 0.69 0.51 0.48 |
| \(K^-\pi^+\pi^-\) | 0.06 0.15 0.06 0.05 |
| \(K^0\pi^-\pi^0\) | 0.03 0.09 0.03 0.03 |
| residual | 0.40 1.33 0.41 0.27 |
| total | 0.76 1.33 0.79 0.72 |
| Combined total | 0.96 1.79 0.98 0.98 |

**TABLE SII.** Error budget for experimental contributions to the inclusive |\(V_{us}\)| determination with ACLP \(K\pi\) normalization.

![Graph](image)

**FIG. 11.** Our |\(V_{us}\)| determinations (inclusive, \(\gamma_K[\tau_K]\)-based exclusive (filled square) and \(\gamma_K[K_{\mu2}]\)-based exclusive (empty square)) c.f. other determination results for the ACLP \(K\pi\) normalization case.

ACLP-\(K\pi\)-normalization-based results for |\(V_{us}\)| to those obtained by other methods, analogous to the HFLAV-\(K\pi\)-normalization-based comparison shown in Fig. 5 of the main text.