The Schrödinger-equation presentation of any oscillatory classical linear system that is homogeneous and conservative

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Abstract

The time-dependent Schrödinger equation with time-independent Hamiltonian matrix is a homogeneous linear oscillatory system in canonical form. We investigate whether any classical system that itself is linear, homogeneous, oscillatory and conservative is guaranteed to linearly map into a Schrödinger equation. Such oscillatory classical systems can be analyzed into their normal modes, which are mutually independent, uncoupled simple harmonic oscillators, and the equation of motion of such a system linearly maps into a Schrödinger equation whose Hamiltonian matrix is diagonal, with $\hbar$ times the individual simple harmonic oscillator frequencies as its diagonal entries. Therefore if the coupling-strength matrix of such an oscillatory system is presented in symmetric, positive-definite form, the Hamiltonian matrix of the Schrödinger equation it maps into is $\hbar$ times the square root of that coupling-strength matrix. We obtain a general expression for mapping this type of oscillatory classical equation of motion into a Schrödinger equation, and apply it to the real-valued classical Klein-Gordon equation and the source-free Maxwell equations, which results in relativistic Hamiltonian operators that are strictly compatible with the correspondence principle. Once such an oscillatory classical system has been mapped into a Schrödinger equation, it is automatically in canonical form, making second quantization of that Schrödinger equation a technically simple as well as a physically very interpretable way to quantize the original classical system.

Introduction

A time-dependent Schrödinger equation, viewed as a real-valued equation of motion that couples the real and imaginary parts of its wave vector, is a homogeneous linear oscillatory canonical classical system (its classical Hamiltonian function is the presentation in the appropriate real canonical variables of the quantum expectation value of its Hamiltonian matrix). Here we shall see that a general homogeneous linear oscillatory conservative classical system’s equation of motion can always be linearly mapped into a Schrödinger equation, and that this mapping is invertible if the classical system has no zero-frequency normal modes. When the oscillatory classical system’s coupling-strength matrix is presented in symmetric form and is positive definite (i.e., no zero-frequency normal modes), the corresponding Schrödinger equation’s quantum Hamiltonian matrix comes out to be $\hbar$ times the positive-definite square root of that classical coupling-strength matrix. The Schrödinger equation’s complex-valued mapped wave vector can be sensibly normalized such that the quantum system’s energy expectation value equals the oscillatory classical system’s energy function. Moreover, that classical system can then be immediately quantized by means of the very straightforward second quantization of the Schrödinger equation that it maps into, which by its nature is in canonical form. This is not only technically simple, it is as well automatically accompanied by a detailed physical interpretation—e.g., one has a mathematical depiction of classical-wave/quantum-particle complementarity via the linear mapping of the original classical oscillatory degrees of freedom (which have Hermitian representation) into the second-quantized Schrödinger-equation wave vector’s annihilation and creation components (which have non-Hermitian representation). Mapping into a Schrödinger equation of the real-valued classical scalar-field Klein-Gordon equation with mass parameter $m$ yields a complex-valued scalar wave function and the Hamiltonian operator $\left(\frac{\left|\hat{p}\right|^2}{2} + m^2 c^4\right)^{1/2}$, which is in accord with the correspondence-principle prescription for a
relativistic free particle of mass $m$ [1]. Such mapping of the classical source-free Maxwell equations yields a complex-valued transverse-vector wave function and the Hamiltonian operator $c\hat{p}$, which is relativistically appropriate to the massless free photon [2].

Oscillatory classical linear systems which are homogeneous and conservative are described by second-order equations of motion that have the form,

$$\ddot{q} + Kq = 0,$$

where $K$ is a nonvanishing real-valued matrix, all of whose eigenvalues are real and nonnegative, and whose real-valued eigenvectors completely span the real-valued vector space on which $K$ naturally operates. Note that the use of the terms “vector” and “matrix” in this article is not intended to exclude vectors that have a continuum of components (e.g., functions) or matrices that have a continuum of entries (e.g., operators on function spaces). However, in the interest of cutting down on notational clutter, all of the didactic generic formulas that are presented in this article which involve vector components or matrix entries display only the case that these are discrete—that is notwithstanding the fact that the interesting examples which are discussed in the last part of this article all have continuum character.

We note that first-order classical equations of motion which have the simple form,

$$\dot{s} = Ws,$$

also describe homogeneous linear oscillatory conservative classical systems when $W$ is a nonvanishing real-valued matrix that has exclusively imaginary eigenvalues whose associated complex-valued eigenvectors completely span the extended complex-valued vector space on which the real-valued $W$ can operate. This is so because Eq. (1b) implies that,

$$\ddot{s} - W^2s = 0,$$

and the matrix $-W^2$ can be shown to conform to all the requirements stipulated for the matrix $K$ below Eq. (1a). To see this, note that if $s_\omega$ is any complex-valued eigenvector of $W$, with $i\omega$ its corresponding imaginary eigenvalue, where $\omega$ is a real number, then because $W$ is real-valued, the particular complex-conjugated vector $s_\omega^*$ is as well an eigenvector of $W$, but with eigenvalue $-i\omega$. Therefore the real-valued vector $s_\omega + s_\omega^*$ is an eigenvector of the real-valued matrix $-W^2$ with the real, nonnegative eigenvalue $\omega^2$.

In addition, since the complex-valued eigenvectors of $W$ of the form $s_\omega$ are assumed to completely span the extended complex-valued vector space on which $W$ can operate, it is apparent that the real-valued eigenvectors of $-W^2$ that have the form $s_\omega + s_\omega^*$ completely span the real-valued vector space on which the real-valued matrix $-W^2$ naturally operates—and of course the eigenvalues $\omega^2$ of $-W^2$ associated to each member of this complete set of its real-valued eigenvectors are themselves real-valued and nonnegative. Therefore the nonvanishing real-valued matrix $-W^2$ of Eq. (1c) possesses all of the properties that are required of the real-valued matrix $K$ of Eq. (1a).

It is further to be noted at this point that if the nonvanishing real-valued matrix $W$ is antisymmetric, then it automatically fulfills the remaining requirements that are stipulated below Eq. (1b), and, in addition, a linear mapping of Eq. (1b) into a Schrödinger equation is immediately manifest. This is so because if $W$ is real-valued and antisymmetric, then $iW$ is Hermitian on the extended complex-valued vector space on which $W$ can operate. By virtue of its Hermitian property, $iW$ necessarily possesses a complete set of complex-valued eigenvectors, for each of which it has a corresponding real eigenvalue. Those real eigenvalues of $iW$ correspond, of course, to imaginary eigenvalues of $W$ with the same corresponding eigenvectors, and that set of eigenvectors of course completely spans the extended complex-valued vector space on which $W$ can operate. In addition, if we multiply both sides of Eq. (1b) by the factor $i\hbar$, it becomes a Schrödinger equation with the Hermitian Hamiltonian matrix $i\hbar W$.

In the next section we shall show that classical equations of motion given by Eq. (1a), with the restrictions on the matrix $K$ that are stipulated below Eq. (1a), can always be linearly mapped into Schrödinger equations—consequently the same is true for classical equations of motion given by Eq. (1b) with the restrictions on the matrix $W$ that are stipulated below Eq. (1b)). This task will be greatly facilitated by the fact that the oscillatory classical Eq. (1a) can be analyzed into its normal modes, which, of course, behave as mutually independent simple harmonic oscillators. It turns out that a classical simple harmonic oscillator equation of motion which has the natural angular frequency $\omega$ can be linearly mapped into a Schrödinger
equation for an ultra-basic single-state quantum system whose one-by-one Hamiltonian “matrix” is either the real number \( h\omega \) or the real number \( -h\omega \). The classical equation of motion for a collection of such mutually independent simple harmonic oscillators (i.e., an oscillatory classical system that has been analyzed into its normal modes) correspondingly linearly maps into a Schrödinger equation whose Hamiltonian matrix is diagonal, with its diagonal entries corresponding in one-to-one fashion to the angular frequencies of the independent simple harmonic oscillators which comprise that particular collection: each such Hamiltonian-matrix diagonal entry is a unique one of those angular frequencies times one of the two allowed factors \( \pm h \).

We now turn to the technical details of the analysis of Eq. (1a) into its normal modes, and the subsequent linear mapping of such collections of independent simple harmonic oscillators into Schrödinger equations.

**Analysis into normal modes and their mapping into Schrödinger equations**

The real-valued eigenvectors \( q_j \) of Eq. (1a) completely span the real-valued vector space on which \( K \) naturally operates, and each \( q_j \) corresponds to a nonnegative eigenvalue \( \omega_j^2 \), where we take \( \omega_j \) to be real and nonnegative. Therefore the \( q_j \) satisfy eigenvalue equations of the form,

\[
K q_j = \omega_j^2 q_j
\]  

(2a)

It turns out that we can use these eigenvectors \( q_j \) to construct a matrix \( S \) which is invertible, and for which the composite matrix \( S^{-1}KS \) is in diagonal form, with all of its nondiagonal entries being equal to zero, while its diagonal entries embrace all the eigenvalues \( \omega_j^2 \) of \( K \). Because of this diagonal form of the matrix \( S^{-1}KS \), it will be the case that each of the components \((S^{-1}q)_j \) of the transformation \( S^{-1}q \) of the dynamical vector \( q \) of Eq. (1a) satisfies an independent simple harmonic oscillator equation whose natural angular frequency \( \omega_j \) is the nonnegative square root of one of the eigenvalues \( \omega_j^2 \) of the matrix \( K \). In short, the components of the transformed vector \( S^{-1}q \) are the normal modes of Eq. (1a).

We shall now construct the matrix \( S \) by filling its columns with the components of a set of linearly independent \( q_j \), where that set is sufficiently large to completely span the real-valued vector space on which \( K \) naturally operates,

\[
S_{ij} \equiv (q_j)_i \quad (2b)
\]

Because the columns of the matrix \( S \) are linearly independent and completely span the real-valued vector space on which \( S \) (and \( K \)) naturally operate, the matrix \( S \) will have an inverse \( S^{-1} \). In addition, because of the eigenvalue equations given by Eq. (2a) and the definition of \( S \) given by Eq. (2b), it is readily verified that,

\[
(KS)_{kj} = K_{kl}(q_j)_l = (Kq_j)_k = \omega_j^2 S_{kj}
\]  

(2c)

This result permits us to verify that \( S^{-1}KS \) is precisely the diagonal form of the matrix \( K \) mentioned below Eq. (2a),

\[
(S^{-1}KS)_{mj} = (S^{-1})_{mk}(KS)_{kj} = \omega_j^2(S^{-1})_{mk}S_{kj} = \omega_j^2(S^{-1}S)_{mj} = \omega_j^2 \delta_{mj}
\]  

(2d)

It is convenient to denote this diagonal form of \( K \) as \( K_S \),

\[
K_S \equiv S^{-1}KS
\]  

(3a)

If we now multiply Eq. (1a) through by the matrix \( S^{-1} \) and further define,

\[
q_S \equiv S^{-1}q
\]  

(3b)

we obtain from Eq. (1a) that,

\[
\ddot{q}_S + K_S q_S = 0
\]  

(3c)

which, from Eqs. (3a) and (2d), reads when written in component form,

\[
\ddot{q}_S(j)/dt^2 + \omega_j^2 (q_S)_j = 0
\]  

(3d)
which is a set of mutually independent simple harmonic oscillator equations whose natural angular frequencies \(\omega_j\) are given by the nonnegative square roots of the nonnegative eigenvalues \(\omega^2_j\) of the matrix \(K\). From Eq. (3d) we see that the normal-mode simple harmonic oscillator variables are the components of the vector \(q_s\).

We wish at this point to further linearly map the set of mutually independent simple harmonic oscillator equations encompassed by Eq. (3c) into a Schrödinger equation. Notwithstanding that they have the same form, Eqs. (3c) and (1a) crucially differ in that \(K_S\) in Eq. (3c) is known to be diagonal (with real nonnegative entries on the principal diagonal and uniformly zero entries elsewhere), whereas \(K\) in Eq. (1a) is not guaranteed to be diagonal. When we now attempt to pass to a Schrödinger equation, we obviously do not wish to undo the simplicity that having only diagonal matrices present confers on an equation of motion. Therefore we now make it a rigid rule that any attempted further linear mapping of Eq. (3c) into (hopefully) a Schrödinger equation may only be attempted with diagonal matrices. This affords an immediate benefit: diagonal matrices all mutually commute.

Now a Schrödinger equation has the form,

\[
i\hbar \dot{\psi} = H\psi, \tag{4a}\]

and, of course, our cardinal rule stated above requires that the Hermitian matrix \(H\) be diagonal.

Eq. (3c) is second-order in time, whereas the Schrödinger Eq. (4a) is first-order in time. To reconcile this difference in order, it is necessary to take \(\psi\) to be a linear mapping of \(\dot{q}_s\), and possibly of \(qs\) itself as well. Therefore we now make the ansatz,

\[
\psi = iN(Wqs + \dot{q}_s), \tag{4b}\]

where the matrices \(N\) and \(W\) are of course both assumed to be diagonal. We make the further assumption that the matrix \(N\) is invertible (i.e., has no vanishing entries on its principal diagonal), which implies that it simply factors out of the linear, homogeneous Schrödinger Eq. (4a), and therefore is not determined by it. From Eq. (3c), we know that \(\ddot{q}_s = -K_qqs\). Therefore putting the ansatz of Eq. (4b) into the Schrödinger Eq. (4a) results in,

\[
i\hbar \ddot{q}_s - i\hbar K_qqs = HWqs + H\dot{q}_s, \tag{4c}\]

which yields the two equations,

\[
i\hbar W = H, \quad -i\hbar K_q = HW, \tag{4d}\]

that have the solutions,

\[
H = \hbar(K_q)^\dagger, \quad W = -i(K_q)^\dagger, \tag{4e}\]

which are consistent with our assumption that \(H\) and \(W\) are diagonal matrices, and also imply that \(H\) is Hermitian. Putting the results of Eq. (4e) into Eq. (4b) together with the definition of \(K_S\) given by Eq. (3a) and that of \(q_s\) given by Eq. (3b) yields the desired linear mapping of \(q\) and \(\dot{q}\) of Eq. (1a) into the Schrödinger equation wave vector \(\psi\), and also yields the associated Hamiltonian matrix \(H\) of that Schrödinger equation,

\[
\psi = N((S^{-1}K_S)^\dagger S^{-1}q + iS^{-1}\dot{q}), \quad H = \hbar(S^{-1}K_S)^\dagger. \tag{4f}\]

From Eq. (4f), bearing in mind that both \(N\) and \((S^{-1}K_S)^\dagger\) are mutually commuting diagonal matrices and \(N\) is invertible, it can readily be shown that the Schrödinger Eq. (4a) for \(\psi\) follows from the underlying classical Eq. (1a) for \(q\).

We as well note from Eq. (4f) that if all the eigenvalues of \(K\) are positive, i.e., the classical system is purely oscillatory, then the diagonalized matrix \(S^{-1}K_S\) is invertible, as is the diagonal matrix \((S^{-1}K_S)^\dagger\), and therefore the linear mapping between \(q\) and \(\psi\) is also invertible.

An interesting mathematical point is that since the diagonal entries of \(S^{-1}K_S\) are all real and nonnegative (they are the the eigenvalues of \(K\), \((S^{-1}K_S)^\dagger\) is certainly defined as a diagonal matrix, but multiply so, i.e., the signs of the nonzero diagonal entries of \((S^{-1}K_S)^\dagger\) can be chosen at will. So from a strictly mathematical point of view, Eq. (4f) specifies a whole set of distinct linear mappings of the classical \(q\) into Schrödinger wave vectors \(\psi\), with equally distinct Hamiltonian matrices \(H = \hbar(S^{-1}K_S)^\dagger\) to accompany each distinct linear mapping.
Although the Schrödinger Eq. (4a) does not determine the invertible diagonal “normalization” matrix $N$ of our Schrödinger wave vector $\psi$ of Eq. (4f), we can ask if there is an additional physically sensible requirement which impinges on the value of that “normalization” diagonal matrix $N$.

Now the behavior of quantum expectation values frequently closely parallels that of their classical counterparts, as Ehrenfest’s Theorem attests, and that is particularly the case for simple linear systems. Specifically, the expectation value of the Hamiltonian matrix $H$, namely $\psi^* H \psi$, is a real-valued function of $\psi$ and $\psi^*$ with the dimension of energy which is conserved because the time evolution of $\psi$ is governed by the Schrödinger Eq. (4a) and the Hamiltonian matrix is Hermitian—this conservation of $\psi^* H \psi$ can be explicitly verified. The clear classical analog of $\psi^* H \psi$ is therefore, of course, the classical conserved energy that is associated with the Eq. (3c) classical equation of motion. That classical conserved energy is the nonnegative entity,

$$\mathcal{E}_{K_S}(qs, \dot{qs}) \triangleq (\dot{qs}qs + qsK_qs)/\gamma,$$

where the dimension and magnitude of the real positive number $\gamma$ depends on the dimension and normalization of $qs$—note that $\mathcal{E}_{K_S}(qs, \dot{qs})$ is required to have the dimension of energy. That $\mathcal{E}_{K_S}(qs, \dot{qs})$ is conserved, i.e., that its time derivative vanishes, follows directly from Eq. (3c) itself and the fact that $K_S$ is diagonal.

Therefore it is completely sensible physically to attempt to determine $N$ by additionally imposing the utterly natural requirement that,

$$\psi^* H \psi = \mathcal{E}_{K_S}(qs, \dot{qs}) = (\dot{qs}qs + qsK_qs)/\gamma,$$

whenever this is possible—we shall see that Eq. (5b) requires the real nonnegative diagonal matrix $K_S$ to be positive definite, i.e., the classical system must be purely oscillatory. Furthermore, the strictly nonnegative character of the classical energy $\mathcal{E}_{K_S}(qs, \dot{qs})$ now precludes the possibility that the diagonal Hamiltonian matrix $H = \hbar(S^{-1}KS)^\dagger$ can have anything other than nonnegative entries. Unlike the Schrödinger Eq. (4a), Eq. (5b) is, of course, neither linear nor homogeneous in $\psi$. We now reexpress Eq. (4f) in the more compact form,

$$\psi = N((K_S)^{\dagger}qs + i\dot{qs}), \quad H = \hbar(K_S)^{\dagger},$$

and substitute the right-hand sides of both the first and second equalities of Eq. (5c) into the left hand side of Eq. (5b). For the left and right hand sides of Eq. (5b) to then be able to be equal, the following equation involving the diagonal matrices $N^*$, $N$ and $(K_S)^{\dagger}$ must be satisfied,

$$N^* N (K_S)^{\dagger} = I/(2\hbar \gamma^2),$$

where I is the identity matrix. Of course this is not possible if $(K_S)^{\dagger}$ has any vanishing or negative diagonal entries. If $(K_S)^{\dagger}$ indeed has only positive entries, which can only be the case if the classical system is purely oscillatory, then the simplest solution for the diagonal matrix $N$ is one with only real-valued diagonal entries, namely,

$$N = (K_S)^{-\frac{1}{2}}/(2^\gamma \hbar)^\frac{1}{2}.$$

Putting this determination of $N$ into Eq. (5c) results in the properly normalized Schrödinger wave vector,

$$\psi = ((S^{-1}KS)^{\dagger}S^{-1}q + i(S^{-1}KS)^{-\frac{1}{2}}\dot{S}^{-1}q)/(2^\gamma \hbar)^\frac{1}{2}, \quad H = \hbar(S^{-1}KS)^{\dagger},$$

which in the more explicit notation used in Eq. (4f) reads,

$$\psi = ((S^{-1}KS)^{\dagger}S^{-1}q + i(S^{-1}KS)^{-\frac{1}{2}}\dot{S}^{-1}q)/(2^\gamma \hbar)^\frac{1}{2}, \quad H = \hbar(S^{-1}KS)^{\dagger},$$

where the diagonal matrices $S^{-1}KS$ and $(S^{-1}KS)^{\dagger}$ now both need to be positive definite, and the real positive constant $\gamma$ comes from the classical energy function $\mathcal{E}_{S^{-1}KS}(S^{-1}q, S^{-1}\dot{q})$ of Eq. (5a) that is appropriate to the purely oscillatory classical equation of motion system of Eq. (3c),

$$\mathcal{E}_{S^{-1}KS}(S^{-1}q, S^{-1}\dot{q}) = ((S^{-1}\dot{q})(S^{-1}q) + (S^{-1}q)(S^{-1}KS)(S^{-1}q))/\gamma^2.$$
Because the diagonal matrix \((S^{-1}KS)^\dagger\) is positive definite, the linear mapping of \(q\) and \(\dot{q}\) into \(\psi\) given in Eq. (5g) is invertible,

\[
q = ((\gamma^2 h)/2)^{\dagger} S (S^{-1} KS)^{-\dagger} (\psi + \psi^*), \quad \dot{q} = -i((\gamma^2 h)/2) S (S^{-1} KS)^{\dagger} (\psi - \psi^*).
\]  

(5i)

While the Eq. (5g) route to the desired invertible linear mapping of Eq. (1a) into Schrödinger Eq. (4a) is of great generality in principle, in practice it suffers from the need to explicitly know all the eigenvectors of \(K\) in order to be able to construct \(S\), and, in addition, from the need to explicitly invert \(S\).

We see from Eq. (5g) that one of the consequences of having the matrix \(S\) and its inverse \(S^{-1}\) in hand is that the Schrödinger equation’s Hamiltonian matrix \(H = h(S^{-1} KS)^\dagger\) is presented to us in already diagonal form. It is certainly not essential that that be the case. In the next section we therefore simply expunge \(S\) and its inverse from Eq. (5g), which of course will work if \(K\) is diagonal. However it quickly becomes clear that the resulting expression still works when \(K\) is merely symmetric.

### Schrödinger-equation presentation of symmetrically coupled oscillatory systems

The result of expunging \(S\) and \(S^{-1}\) from Eq. (5g) is,

\[
\psi = (K\delta q + iK^{-\dagger}\delta q)/(2\gamma^2 h)^{\dagger}, \quad H = hK\delta^{\dagger},
\]

(6a)

and if \(K\) is a real-valued symmetric positive-definite matrix, all the expressions in it still make sense: in those circumstances \(H = hK\delta^{\dagger}\) is well defined as a real-valued symmetric positive-definite matrix itself. Therefore \(H\) is Hermitian, as required, and \(K\delta\) and \(K^{-\dagger}\) are well-defined as real-valued symmetric invertible matrices. Furthermore, it is straightforwardly verified that in consequence of the basic oscillatory classical equation of motion of Eq. (1a), the wave vector \(\psi\) of Eq. (6a) satisfies the Schrödinger Eq. (4a) with the Hamiltonian matrix \(H = hK\delta^{\dagger}\) given by Eq. (6a). In addition, when \(K\) is a real-valued symmetric positive-definite matrix, Eq. (6a) yields,

\[
\psi^* H \psi = (\dot{q}\dot{q} + qKq)/(2\gamma^2),
\]

(6b)

and if \(\gamma\) has been appropriately selected such that,

\[
\mathcal{E}_K(q, \dot{q}) \overset{\text{def}}{=} (\dot{q}\dot{q} + qKq)/(2\gamma^2),
\]

(6c)

has the dimension of energy, then it is clear that,

\[
L_K(q, \dot{q}) \overset{\text{def}}{=} (\dot{q}\dot{q} - qKq)/(2\gamma^2),
\]

(6d)

also has the dimension of energy. Moreover, it is easily verified that the Euler-Lagrange equation which follows from the Lagrangian \(L_K(q, \dot{q})\) of Eq. (6d) is precisely the Eq. (1a) classical equation of motion. Now the conserved energy of any classical system that has a Lagrangian \(L\) is well-known to be uniquely given by \((q \nabla_q L - L)\), which, for the particular Eq. (1a) case that \(L\) is given by \(L_K(q, \dot{q})\) of Eq. (6d), is straightforwardly verified to be \(\mathcal{E}_K(q, \dot{q})\), as defined by Eq. (6c). Therefore, Eqs. (6b) and (6c) show that when \(K\) is real-valued, symmetric and positive definite, then the expectation value of the Hamiltonian matrix which follows from Eq. (6a) is equal to the conserved energy of the classical system of Eq. (1a), as required.

Finally, when \(K\) is real-valued, symmetric and positive definite, the inverse of the Eq. (6a) linear mapping of \(q\) and \(\dot{q}\) into \(\psi\) is readily calculated to be,

\[
q = ((\gamma^2 h)/2)^{\dagger} K^{-\dagger} (\psi + \psi^*), \quad \dot{q} = -i((\gamma^2 h)/2)^{\dagger} K^\dagger (\psi - \psi^*),
\]

(6e)

which is, as expected, the result of expunging \(S\) and \(S^{-1}\) from Eq. (5i).

What if the matrix \(K\) of the classical Eq. (1a) is nonsymmetric? We then first need to find a real-valued invertible matrix \(S\) such that the similarity-transformed \(K_S \overset{\text{def}}{=} S^{-1} KS\) is symmetric. Eq. (6a) is extended to cover this situation by replacing \(K\) by \(K_S\) and \(q\) by \(q_S = S^{-1} q\), precisely as in Eq. (5f), except that now \(K_S\) is merely symmetric and positive definite, not necessarily diagonal.
In the next section, we use the machinery of Eq. (5f) and its associated Eq. (3c) similarity-transformed version of the Eq. (1a) classical equation of motion (albeit always bearing in mind that $K_S$ is merely symmetric and positive definite, not diagonal) to show that the real and imaginary parts of $\psi$ times the factor $\sqrt{2\hbar}$ obey a simple first-order coupled equation of motion which can immediately be Hamiltonized and then quantized. This second quantization of an oscillatory classical system’s linear mapping into a Schrödinger equation is a very easy route to that underlying system’s quantization, and one which as well automatically yields considerable physical insight.

**Hamiltonization and quantization of the Schrödinger-equation presentation**

Taking the real-valued similarity-transformed $K_S$ in both Eqs. (3c) and (5f) to now be, as discussed above, merely symmetric and positive definite rather than necessarily diagonal, we note that the real and imaginary parts of the wave vector $\psi$ of Eq. (5f), each multiplied (for later convenience) by the factor $\sqrt{2\hbar}$, are given by,

$$q_c \equiv \sqrt{\hbar} (\psi + \psi^*) = (K_S)^\dagger q_S / \gamma, \quad p_c \equiv -i(\hbar / 2)(\psi - \psi^*) = (K_S)^{-\dagger} \dot{q}_S / \gamma,$$

which are readily seen, as a consequence of Eq. (3c), which is a similarity-transformed version of the underlying Eq. (1a) classical equation of motion, to satisfy the simple first-order coupled antisymmetrical equation of motion,

$$\dot{q}_c = (K_S)^\dagger p_c, \quad \dot{p}_c = -(K_S)^{-\dagger} q_c.$$  \hspace{1cm} (7b)

With a little effort, it can also be verified that the Eq. (7b) system implies the Schrödinger Eq. (4a) with $H = \hbar (K_S)^\dagger$. Moreover, for $H = \hbar (K_S)^\dagger$, where $(K_S)^\dagger$ is real-valued and symmetric, the two equalities of the Eq. (7b) system follow from simply the real and imaginary parts of the Schrödinger Eq. (4a). In other words, for the situation that we are concerned with here, namely that $H = \hbar (K_S)^\dagger$, where $(K_S)^\dagger$ is real-valued, symmetric and positive definite, the real-valued coupled antisymmetrical system of Eq. (7b) is completely equivalent to the complex valued Schrödinger Eq. (4a).

In addition, Eq. (7b) also follows from a simple bilinear classical Hamiltonian, namely,

$$H_{K_S}(q_c, p_c) = (q_c^\dagger K_S q_c + p_c^\dagger K_S p_c)/2,$$  \hspace{1cm} (7c)

via the classical canonical Hamiltonian equations of motion, i.e.,

$$\dot{q}_c = \nabla_{p_c} H_{K_S}(q_c, p_c), \quad \dot{p}_{c} = -\nabla_{q_c} H_{K_S}(q_c, p_c),$$  \hspace{1cm} (7d)

and the fact that $(K_S)^\dagger$ is a real symmetric matrix.

By putting the definition of $(q_c, p_c)$ given in Eq. (7a) into Eq. (7c) we can reexpress our system’s classical Hamiltonian in terms of its Schrödinger-equation presentation wave vector $\hat{\psi}$ and complex conjugate $\hat{\psi}^*$,

$$H_{K_S}(q_c, p_c) = (\psi^* H \psi + \psi H \psi^*)/2 = \psi^* H \psi,$$  \hspace{1cm} (7e)

where the last equality in Eq. (7e) follows from the fact that $H = \hbar (K_S)^\dagger$ is a real, symmetric matrix. It is pleasing to once again see the quantum expectation value of the Hamiltonian matrix $H$ come out to be equal to the Schrödinger-equation presentation’s classical energy, i.e., to its classical Hamiltonian.

Since Eqs. (7c) and (7d) assure us that the classical equations of motion of Eq. (7b) obeyed by $(q_c, p_c)$ are indeed presented in canonical Hamiltonian form, we can now safely quantize this classical system by imposing first Dirac’s canonical commutation rules on the components of $(q_c, p_c)$, and next Heisenberg’s equations of motion on the now quantized $(\hat{q}_c, \hat{p}_c)$. Dirac’s canonical commutation rules promote the components of $(q_c, p_c)$ into noncommuting Hermitian operators which obey the commutation relations,

$$\left[[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0, \quad [[\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}. \right.$$  \hspace{1cm} (8a)

The Eq. (8a) commutation relations, in turn, imply that the components of the non-Hermitian quantized wave vector $\hat{\psi} = (\hat{q}_c + i\hat{p}_c)/(2\hbar)^{\dagger}$ satisfy, in conjunction with the components of this quantized wave vector’s Hermitian conjugate $\hat{\psi}^\dagger = (\hat{q}_c - i\hat{p}_c)/(2\hbar)^{\dagger}$, the following commutation relations,

$$[\hat{\psi}_i, \hat{\psi}_j] = [\hat{\psi}_i^\dagger, \hat{\psi}_j^\dagger] = 0, \quad [\hat{\psi}_i, \hat{\psi}_j^\dagger] = \delta_{ij}. \hspace{1cm} (8b)$$

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These Eq. (8b) commutation relations are the \textit{fundamental} ones for the components of our Schrödinger-equation presentation \textit{quantized} wave vector, and they give those quantized wave-vector components and their Hermitian conjugates, respectively, their well-known interpretation as \textit{annihilation} and \textit{creation} \textit{operators}, which is so often crucial to physical understanding. They are \textit{as well} the key to constructing \textit{physically useful} orthogonal basis sets for the \textit{second-quantized} Hilbert space that is the result of the imposition of Dirac’s canonical commutation rules on the components of the dynamical-variable vector \((\mathbf{q}_c, \mathbf{p}_c)\).

The Hamiltonian \textit{operator} for this quantized (i.e., \textit{second} quantized) Schrödinger-equation presented system is obtained by substituting the \textit{quantized} dynamical-variable vector \((\mathbf{q}_c, \mathbf{p}_c)\) into the system’s \textit{classical} Hamiltonian of Eq. (7c), namely, by writing down,

\[
\mathcal{H}_K(\mathbf{q}_c, \mathbf{p}_c) = (\mathbf{q}_c(K_S)^* \hat{\mathbf{q}}_c + \mathbf{p}_c(K_S)^* \hat{\mathbf{p}}_c)/2,
\]

which could have ambiguities due to \textit{operator-ordering} issues, but it is apparent that \textit{those do not arise} in this case. Noting that the quantized dynamical-variable vector \((\mathbf{q}_c, \mathbf{p}_c)\) is given in terms of the quantized wave vector \(\hat{\psi}\) and its Hermitian conjugate \(\hat{\psi}^\dagger\) by the quantized analog of the two definitions in Eq. (7a), namely \(\hat{\mathbf{q}}_c = (\hbar/2)(\hat{\psi} + \hat{\psi}^\dagger)\) and \(\hat{\mathbf{p}}_c = -i(\hbar/2)(\hat{\psi} - \hat{\psi}^\dagger)\), we reexpress the \textit{uniquely defined} second-quantized Hamiltonian operator \(\mathcal{H}_K(\hat{\psi}_c, \hat{\psi}_c)\) of Eq. (8c) in terms of the \textit{quantized wave vector} \(\hat{\psi}\) and its Hermitian conjugate \(\hat{\psi}^\dagger\),

\[
\mathcal{H}_K(\hat{\psi}_c, \hat{\psi}_c) = (\hat{\psi}^\dagger\mathbf{H}\hat{\psi} + \hat{\psi}\mathbf{H}\hat{\psi}^\dagger)/2,
\]

where \(\mathbf{H} = \hbar(K_S)^\dagger\), a real symmetric positive definite matrix.

If we now apply Heisenberg’s equation of motion and the commutation rules for the components of the quantized \(\hat{\psi}\) and \(\hat{\psi}^\dagger\) that are given by Eq. (8b) to the second-quantized Hamiltonian operator written in the form given by Eq. (8d), we can calculate the time derivative of any component of the Schrödinger-equation presentation \textit{quantized} wave vector \(\hat{\psi}\),

\[
d\hat{\psi}_i/dt = (-i/\hbar)[\hat{\psi}_i, (\hat{\psi}^\dagger\mathbf{H}\hat{\psi} + \hat{\psi}\mathbf{H}\hat{\psi}^\dagger)/2] = (-i/\hbar)((\mathbf{H}\hat{\psi})_i + (\hat{\psi}\mathbf{H})_i)/2 = (-i/\hbar)(\mathbf{H}\hat{\psi})_i,
\]

where the last step reflects the real \textit{symmetric} character of the Hamiltonian matrix \(\mathbf{H} = \hbar(K_S)^\dagger\). Thus we have shown that,

\[
i\hbar d\hat{\psi}/dt = \mathbf{H}\hat{\psi},
\]

i.e., the Schrödinger Eq. (4a) which the Schrödinger-equation presentation wave vector \(\psi\) satisfies is \textit{also} satisfied by that wave vector’s operator \textit{quantization} \(\hat{\psi}\), which itself is, of course, a vector of the \textit{annihilation operators} of the complete set of quantum states which the \textit{components} of the wave vector \(\psi\) individually describe.

We next turn to the Schrödinger-equation presentations of specifically the classical Klein-Gordon equation and the source-free Maxwell equations.

The \textit{spinless} quantum free particle from the classical Klein-Gordon equation

The classical Klein-Gordon equation for the real-valued scalar field \(\phi\) differs from the classical wave equation by a simple mass term \([3, 1],\)

\[
\ddot{\phi} + (c^2 \nabla^2 + \omega^2)\phi = 0,
\]

where \(\omega = (mc^2)/\hbar\). Eq. (9a) has the form of Eq. (1a) with,

\[
K = -c^2 \nabla^2 + \omega^2,
\]

which, on the space of real-valued scalar fields, is a real-valued, symmetric, positive-definite operator with the dimension of frequency squared. Therefore, starting with Eq. (6a) above and going right through to Eq. (8f), we have results that can all be transcribed for the real-valued classical Klein-Gordon equation. We need to bear in mind that during this exercise \(K\) is specifically defined by Eq. (9b) and that the real-valued classical dynamical vector \(q\) is defined by the real-valued \(\phi\), which, as a real-valued vector, of course has a three-dimensional \textit{continuous} index instead of a discrete one. In such a case the \textit{summation} that
defines index contraction is willy-nilly supplanted by three-dimensional integration, which compels some systematic technical changes in the formalism, for example in the dimension of the variables that one deals with (summation is over dimensionless indices, integration here involves the three space dimensions) and in the fact that Kronecker deltas give way to three-dimensional delta functions. That notwithstanding, most of the results properly transcribed to the case of the real-valued classical Klein-Gordon equation remain very similar in appearance to the formulas that run from Eq. (6a) through Eq. (8f).

In particular, Eq. (6a) needs essentially no modification; one simply bears in mind that the operator \( K \) is given by Eq. (9b), and one replaces the occurrences of \( q \) and \( \dot{q} \) by \( \phi \) and \( \dot{\phi} \). The only remaining issue is one of a global reconciliation of dimension, which requires the determination of the parameter \( \gamma \) that appears Eq. (6a) so as to accord with the conventions one intends to adopt for the classical Klein-Gordon theory. Now one conventional choice of dimension for \( \phi \) is the same as that of the electromagnetic vector potential \( \mathbf{A} \) \cite{3, 1}, which implies that \( \int |\nabla \phi|^2 d^3 \mathbf{r} \) has the dimension of energy. A glance at the classical conserved energy given by Eq. (6c) reveals that \( \gamma \) must have the dimension of \( c \), so we choose the value \( c \) for \( \gamma \). With that, Eq. (6a) yields the mapping into the wave function and Hamiltonian operator of the Schrödinger equation that corresponds to the classical Klein-Gordon theory,

\[
\psi = (K^+ \phi + iK^- \dot{\phi})/(2c^2 \hbar)^{1/2}, \quad H = \hbar K^+,
\]

where the operator \( K \) is, of course, given by Eq. (9b). The inverse of this mapping from \( \phi \) and \( \dot{\phi} \) into the complex-valued Schrödinger wave function \( \psi \) is easily calculated, or may be transcribed from Eq. (6e),

\[
\phi = c(\hbar/2)^{1/2} K^{-1/2} (\psi + \psi^*), \quad \dot{\phi} = -i c(\hbar/2)^{1/2} K^{-1/2} (\psi - \psi^*).
\]

Now let’s take a closer look at the Schrödinger equation’s Hamiltonian operator,

\[
H = \hbar K^+ = \hbar (-c^2 \nabla^2 + ((mc^2)/\hbar)^2)^{1/2}
\]

In configuration space the quantum momentum operator \( \hat{\mathbf{p}} \) is well-known to be given by,

\[
\hat{\mathbf{p}} = -i\hbar \nabla,
\]

so that,

\[
-\nabla^2 = |\hat{\mathbf{p}}|^2/\hbar^2,
\]

which, when substituted into the expression for \( H \) in Eq. (9e), yields,

\[
H = (|\hat{\mathbf{p}}|^2 + m^2 c^4)^{1/2},
\]

which is precisely the quantization of the standard relativistic energy of a free particle of mass \( m \). Thus we have the fascinating state of affairs that the classical Klein-Gordon equation (i.e., with real-valued \( \phi \)) is linearly isomorphic to the very Schrödinger equation with the correspondence-principle mandated square-root Hamiltonian for the free particle of mass \( m \) that Klein and Gordon were in fact trying to sideline. If Klein and Gordon had but been aware of the Eq. (6a) theorem with its square-root character of the Hamiltonian matrix \( H = \hbar K^+ \), the history of relativistic quantum mechanics and its second quantization might have taken a different route, one in closer harmony with the correspondence principle.

Second quantization of the Schrödinger wave function \( \psi \) for the classical Klein-Gordon theory can be transcribed from Eqs. (8). Here the different dimension of \( \psi \) that is imposed by its continuum character results in its basic commutation relations coming out in terms of a three-dimensional delta function instead of in terms of the Kronecker delta of Eq. (8b).

\[
[\tilde{\psi}(\mathbf{r}), \tilde{\psi}^\dagger(\mathbf{r}')] = \delta^{(3)}(\mathbf{r} - \mathbf{r}'), \quad [\tilde{\psi}(\mathbf{r}), \tilde{\psi}(\mathbf{r}')] = 0, \quad [\tilde{\psi}^\dagger(\mathbf{r}), \tilde{\psi}^\dagger(\mathbf{r}')] = 0.
\]

This promotion of the Schrödinger wave function \( \psi(\mathbf{r}) \) to operator field is the most straightforward and physically transparent route to the quantization of the classical Klein-Gordon field \( \phi(\mathbf{r}) \), which, of course, is explicitly given by Eq. (9d) in terms of the Schrödinger wave function and its complex conjugate. The
familiar physical interpretation attached to the commutation relations of Eq. (9i) is that the operator field \( \hat{\psi}(\mathbf{r}) \) creates a relativistic spinless particle of mass \( m \) at location \( \mathbf{r} \), while the operator field \( \hat{\psi}^\dagger(\mathbf{r}) \) destroys such a particle. Such particle creation and destruction operator fields are non-Hermitian. However, from the first equality of Eq. (9d) we note that the quantized classical Klein-Gordon field \( \hat{\phi}(\mathbf{r}) \) itself will, on the contrary, turn out to be Hermitian, and will be ambiguously capable of both particle creation and annihilation. In light of the second equality in Eq. (9d), the same comments apply to the quantization of the time derivative that by themselves they only obey the original second-order real-valued classical Klein-Gordon equation. Eqs. (9c), (9d) and (9i) thus mathematically depict the complementarity of the quantized particle outlook (oriented toward Hermitian second-quantized wave-functions that unambiguously either annihilate or create particles, and obey a first-order complex-valued quantum Schrödinger equation) to the classical wave outlook (oriented toward Hermitian fields that by themselves only obey a real-valued second-order classical wave equation).

Finally, we wish to exhibit, in terms of these quantized Schrödinger wave functions that create or destroy particles, the Hamiltonian operator functional that oversees free relativistic spinless particles in the second quantized world (we already met this operator in schematic form in Eq. (8d)),

\[
\hat{H}[\hat{\psi}, \hat{\psi}^\dagger] = \frac{i}{\hbar} \int \left[ \hat{\psi}^\dagger(\mathbf{r})(-c^2\hbar^2\nabla^2 + m^2c^4)\hat{\psi}(\mathbf{r}) + \hat{\psi}(\mathbf{r})(-c^2\hbar^2\nabla^2 + m^2c^4)\hat{\psi}^\dagger(\mathbf{r}) \right] d^3\mathbf{r}.
\]  

(9j)

We now turn to the similar Schrödinger equation that corresponds to the real-valued homogeneous linear source-free Maxwell equations. The differences to the Schrödinger-equation results for the classical Klein-Gordon equation are that the resulting relativistic particle is massless, and that its wave function is a vector field which is strictly transverse.

Free-photon quantum mechanics from the source-free Maxwell equations

In the source-free case, the Coulomb and Gauss laws tell us that both the electric and magnetic fields are purely transverse, i.e., \( \nabla \cdot \mathbf{E} = 0 \) and \( \nabla \cdot \mathbf{B} = 0 \). The results of the Maxwell law and Faraday’s law in the source-free case are,

\[ \mathbf{E} = c\nabla \times \mathbf{B}, \quad \mathbf{B} = -c\nabla \times \mathbf{E}. \]  

(10a)

This first-order equation system has the simple antisymmetrical character of Eq. (7b), which readily produces a Schrödinger equation. For example, the extremely simple transverse-vector wave function ansatz \( \Psi = \mathbf{E} + \imath\mathbf{B} \) will in consequence of Eq. (10a) satisfy the Schrödinger equation which has the Hamiltonian operator \( \hbar c \text{curl} \). Unfortunately this operator has odd parity, and therefore is not a physically appropriate Hamiltonian for electromagnetism. The reason that a Hamiltonian of odd parity has manifested itself here is that the transverse vector fields on either side of each of the two equations of Eq. (10a) are of opposite intrinsic parity: namely \( \mathbf{E} \) is a polar vector field, while \( \mathbf{B} \) is an axial vector field. So it should be feasible to extract a physically appropriate even-parity Schrödinger-equation Hamiltonian operator from source-free electromagnetic theory by first recasting its linear homogeneous equations of motion such that they involve only transverse vector fields which all have the same intrinsic parity. We shall do this here by mapping the transverse axial-vector magnetic field \( \mathbf{B} \) into a transverse polar-vector field that is already well-known to electromagnetic theory, namely the vector potential in radiation gauge [4]. Specifically, we define,

\[
\mathbf{A} \overset{\text{def}}{=} (-\nabla^2)^{-1}(\nabla \times \mathbf{B}),
\]  

(10b)

where by \((-\nabla^2)^{-1}\) we mean the standard real-valued symmetric integral operator with the Coulomb kernel. Eq. (10b) implies that,

\[ \nabla \cdot \mathbf{A} = 0, \]  

(10c)
i.e., \( \mathbf{A} \) is a transverse vector field. Furthermore, since \( \mathbf{B} \) itself is a transverse vector field, Eq. (10b) implies that,

\[ \nabla \times \mathbf{A} = \mathbf{B}, \]  

(10d)
which is, of course, the basic property of a vector potential \( A \). We can further delineate the properties of \( A \) in source-free electromagnetism by using its definition together with Faraday’s law (i.e., the second equality in Eq. (10a)) to calculate its time derivative,

\[
\dot{A} = (-\nabla^2)^{-1}(\nabla \times \dot{B}) = -c(-\nabla^2)^{-1}(\nabla \times (\nabla \times E)) = -cE, \tag{10e}
\]

where the last equality holds when \( E \) is transverse, which is, of course the case for source-free electromagnetism. So in that case,

\[
E = -\dot{A}/c. \tag{10f}
\]

Eqs. (10d) and (10f) together imply that for source-free electromagnetism, we can obtain both of \( B \) and \( E \) from \( A \), so we only need to concern ourselves with calculating the polar transverse vector field \( A \). Therefore we now substitute Eqs. (10d) and (10f) into the Maxwell law, which in the case of source-free electromagnetism is the first equality of Eq. (10a), to obtain a linear homogeneous second-order equation which involves the polar transverse vector field \( A \) alone,

\[
\dot{A} - c^2\nabla^2 A = 0. \tag{10g}
\]

This is, of course, the \textit{classical wave equation}, and it bears a \textit{marked resemblance} to the classical Klein-Gordon equation of Eq. (9a). The \textit{only} differences are that in Eq. (10g) the parameter \( \omega \) that appears in Eq. (9a) \textit{vanishes identically}, and, of course, in Eq. (10g) the transverse vector field \( A \) replaces the scalar field \( \phi \) of Eq. (9a). Even the \textit{dimension} of the transverse vector field \( A \) is the \textit{same} as the dimension that we \textit{chase} for \( \phi \) by adhering to a common convention [3, 1]. Therefore, for the linear mapping, and its inverse, of the real-valued transverse-vector fields \( A \) and \( \dot{A} \) into a complex-valued transverse-vector \( \text{Schrödinger-equation wave function} \Psi \), we can simply transcribe Eqs. (9b), (9c) and (9d) for the real-valued classical scalar Klein-Gordon theory, taking \( \omega \) (and \( m \)) to be zero identically, and replacing \( \phi, \dot{\phi}, \text{and} \psi \) by, respectively, \( A, \dot{A}, \) and \( \Psi \). Thus our basic real, symmetric operator is,

\[
K = -c^2\nabla^2, \tag{10h}
\]

which, to be sure, is not positive-definite in the broadest sense. However, Fourier transformation methodology indicates that on a sufficiently restricted function space, \(-\nabla^2\) can indeed be regarded as positive definite. The operators we \textit{actually require} in the following mapping formulas are \((-\nabla^2)^{\frac{1}{2}}, (-\nabla^2)^{-\frac{1}{2}}, \text{and} (-\nabla^2)^{\frac{3}{2}}\), and they \textit{themselves} have the tractable-looking positive-definite Fourier representations \(|k|, |k|^{-\frac{1}{2}}\) and \(|k|^{\frac{1}{2}}\) respectively.

Transcribing Eq. (9c) as described above, the linear mapping of the real-valued transverse-vector fields \( A \) and \( \dot{A} \) into the complex-valued transverse-vector \( \text{Schrödinger-equation wave function} \Psi \), together with the associated \( \text{Schrödinger-equation Hamiltonian operator} \), is given by,

\[
\Psi = (K^{\frac{1}{2}} A + iK^{-\frac{1}{2}} \dot{A})/(2c^2\hbar)^{\frac{1}{2}}, \quad H = \hbar K^{\frac{1}{2}}. \tag{10i}
\]

The \textit{inverse} of this linear mapping from \( A \) and \( \dot{A} \) into the complex-valued \( \text{Schrödinger-equation wave function} \Psi \) is,

\[
A = c(h/2)^{\frac{1}{2}} K^{-\frac{1}{2}}(\Psi + \Psi^*), \quad \dot{A} = -ic(h/2)^{\frac{1}{2}} K^{\frac{1}{2}}(\Psi - \Psi^*). \tag{10j}
\]

In light of Eq. (10h) and the fact that in configuration representation \( \hat{p} = -i\hbar \nabla \), we have from the second equality in Eq. (10i) that the \( \text{Schrödinger-equation Hamiltonian operator} \) can be written,

\[
H = \hbar K^{\frac{1}{2}} = \hbar(-c^2\nabla^2)^{\frac{1}{2}} = (c^2|\hat{p}|^2)^{\frac{1}{2}} = \hbar |\hat{p}|. \tag{10k}
\]

This Hamiltonian operator is clearly the quantized version of the relativistic energy of a \textit{massless} free particle, which is appropriate to the free photon, and it as well has even parity.

By using Eqs. (10b) and (10e), the vector potential can be removed from the \( \text{Schrödinger-equation linear mapping of Eq. (10i)} \) in favor of the \( E \) and \( B \) fields,

\[
\Psi = (cK^{-\frac{1}{2}}(\nabla \times B) - iK^{-\frac{1}{2}}E)/(2\hbar)^{\frac{1}{2}}, \quad H = \hbar K^{\frac{1}{2}}. \tag{11a}
\]
The mapping of $E$ and $B$ into $\Psi$ given in Eq. (11a) has the inverse,

$$B = c(\hbar/2)^{\frac{1}{2}}K^{-\frac{1}{2}}(\nabla \times (\Psi + \Psi^*)),$$
$$E = i(\hbar/2)^{\frac{1}{2}}K^{\frac{1}{2}}(\Psi - \Psi^*).$$

(11b)

We invite the reader to verify that the complex-valued linear mapping of the classical $E$ and $B$ fields into the wave function $\Psi$ which Eq. (11a) specifies, along with its specified Hamiltonian operator $H = \hbar K^{\frac{1}{2}}$ (where $K = -c^2\nabla^2$), actually satisfies the Schrödinger equation. (Hint: use the source-free Maxwell and Faraday laws of Eq. (10a) and the transverse nature of the source-free $E$ field.) One should also verify that the quantum expectation value of the Hamiltonian operator agrees with the classical energy of the $E$ and $B$ field system, i.e., that,

$$\int \Psi^*(\mathbf{r}) \cdot (H \Psi(\mathbf{r})) \, d^3\mathbf{r} = \frac{i}{2} \int (|E(\mathbf{r})|^2 + |B(\mathbf{r})|^2) \, d^3\mathbf{r}.$$  

(11c)

Upon their second quantization, Eqs. (11a) and (11b) manifest the expected tantalizing complementary interplay of the potential for photon creation and annihilation with the familiar, workaday transverse electric and magnetic fields.

In addition to its zero mass parameter, the second special feature of electromagnetic theory vis-à-vis classical Klein-Gordon theory is, of course, the free photon’s always transverse polarization (spin) states. This signature free-photon characteristic does not cause much in the way of complications, but there is one formula concerning second quantization which it notationally impacts, albeit no substantive physical effect is involved. The canonical commutation rule for second quantization of the free photon’s transverse vector wave function might naively be expected to read,

$$[(\tilde{\Psi}(\mathbf{r})), (\tilde{\Psi}^\dagger(\mathbf{r}'))] = \delta_{ij} \delta^{(3)}(\mathbf{r} - \mathbf{r'}),$$

(12a)

but this is not mathematically consistent with the transverse character of the second-quantized photon wave-functions, i.e., it is mathematically inconsistent with the fact that $\nabla \cdot \Psi = 0$. The nature of the right-hand side of Eq. (12a) is one of completeness, but the transverse wave function creation and annihilation operators are incomplete in that they do not pertain to vector fields which are the gradients of scalar fields, i.e., they do not pertain to vector fields which fail to be transverse. Now the $ij$ components of the projection operator onto the subspace of such purely gradient vector fields is given by,

$$P_{ij} = -\partial_i (\nabla^2)^{-1} \partial_j.$$  

(12b)

We note that $P_{ij}$ is Hermitian, and that its contraction with itself yields itself, which are the two essential properties of the $ij$ components of projection operators. Of course its contraction with the components of any transverse vector field vanishes. Thus $(\delta_{ij} - P_{ij})$ are the $ij$ components of the projection operator onto the subspace of transverse vector fields, and therefore,

$$[(\tilde{\Psi}(\mathbf{r})), (\tilde{\Psi}^\dagger(\mathbf{r}'))] = \langle r | (\delta_{ij} - P_{ij}) | r' \rangle = (2\pi)^{-3} \int e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r'})} (\delta_{ij} - \mathbf{k}_i \mathbf{k}_j | \mathbf{k} |^{-2}) \, d^3\mathbf{k}.$$  

(12c)

Notwithstanding these fancy maneuvers with projection operators, the only issue which is involved here is the simple fact that free-photon creation and annihilation operators (and as well free photon wave functions in the first quantized regime) are purely transverse, and therefore any expression involving these operators, e.g., the expression which describes their canonical commutation relation, must, of course, correctly reflect this fact. There is obviously no physics implication which flows from this requirement of mere notational correctness.

**Conclusion**

It is a remarkable fact that any classical system whose equation of motion is linear, homogeneous, purely oscillatory and conservative is effectively already first-quantized: once its Eq. (1a) coupling-strength matrix $K$ has been similarity-transformed to a symmetric, positive-definite presentation, Eq. (6a) invertibly linearly maps that equation of motion into explicit time-dependent Schrödinger-equation form with Hamiltonian matrix $\hbar K^{\frac{1}{2}}$. Thus we see that Michael Faraday and James Clerk Maxwell were actually the first to
effectively elucidate a quantized particle, namely the very important and not exactly simple ultra-relativistic massless transverse-vector free photon.

Any complex-valued solution wave function of a time-dependent Schrödinger-equation has the familiar characteristic expansion in terms of the complete set of mutually orthogonal eigenfunctions of that equation’s Hamiltonian operator. The one-to-one linear mapping of any purely oscillatory linear classical system that is homogeneous and conservative into a Schrödinger equation thus implies a characteristic two-component eigenfunction expansion of such a classical system’s solutions. For the case of certain wave equations that fall into the class of Eq. (1a), precisely such a solution expansion has been described in detail by Leung, Tong and Young [5].

The natural correspondence-principle version of the relativistic free-particle Schrödinger equation was iterated by Klein, Gordon and Schrödinger for no physically motivated reason, but merely in an effort to rid it of its calculationally unpalatable square-root Hamiltonian operator [6, 1, 7]. If this iterated equation is still regarded as a complex-valued quantum-mechanical entity, a large class of completely extraneous, highly unphysical unbounded-below negative-energy solutions are injected by that iteration. These also destroy its probability interpretation, and the fact that it depends on only the square of a Hamiltonian cuts it adrift from the Heisenberg picture and Ehrenfest theorem. However, if this iterated equation is regarded as the description of a classical, real-valued field, it thereupon becomes strongly analogous to the classical wave equation, and has an eminently sensible nonnegative conserved energy [3, 1]. This classical Klein-Gordon equation is as well one of those classical equation systems which is linearly equivalent to a Schrödinger equation: it quite marvelously chooses to be equivalent to precisely the Schrödinger equation with the natural correspondence-principle square-root Hamiltonian operator which Klein, Gordon and Schrödinger had tried to sideline by concocting it.

It is a pity that Klein, Gordon and Schrödinger had no idea of the theorem presented by this paper, and thus were not equipped to unearth this astonishing fact themselves. If they had but grasped the full consequences of the real-valued classical Klein-Gordon equation, they might well have abandoned their physically unmotivated rejection of the correspondence-principle mandated relativistic free-particle square-root Hamiltonian operator $\left(\left|\mathbf{p}\right|^2 + m^2 c^4\right)^{\frac{1}{2}}$ [7, 1].

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