The field theoretic renormalization group (RG) is applied to a near-equilibrium fluid model associated with a scalar field (like temperature or density of an impurity) that is active, that is, affects the dynamics of the fluid itself. It is shown that the only possible nontrivial infrared asymptotic regimes are governed by “passive” fixed points of the RG equations, where the back reaction is irrelevant. This result resembles the result obtained by Nandy and Bhattacharjee (1998) in a model describing active convection by fully developed turbulence. Furthermore, the existence of “exotic” fixed points with negative and complex effective couplings and transport coefficients that may suggest possible directions for future studies is established. Bibliography: 56 titles.

1. Introduction

Over forty years ago, Forster, Nelson, and Stephen published their seminal paper in which the renormalization group (RG) method was applied to statistical theory of fluid turbulence [1]; see also the preliminary short publication [2]. They studied the stochastic Navier–Stokes (NS) equation subjected to various kinds of external stirring force, the stochastic Burgers equation, 1 passively advected scalar field, and mentioned briefly an active scalar field. The paper was mostly devoted to the problem of “long tails”, that arises in the derivation of hydrodynamic equation in two dimensions, but it gave a decisive impact to the RG approach in fully developed turbulence. Pioneered in [4–11], the RG theory of turbulence still remains a perspective and developing area of theoretical physics [12–16]. The most powerful field theoretic RG approach is reviewed in the monographs [17,18] and papers [19–21]; see also references therein.

It should be recognized that so far constructive analytical approaches based on the underlying NS dynamics have had limited success in describing real turbulent flows. Most of such approaches (like DIA or EDQMN approximation) can be viewed as one-loop approximations to certain self-consistency equations that involve infinite series of perturbative terms [22].2 The first successful attempt to work with the whole perturbative series was undertaken in the seminal paper by Wyld [24] and was continued, e.g., in a series of works by L’vov et. al. [25,26].

An alternative approach to work with the whole perturbative series is provided by the field theoretic RG. Its important advantage is that it is based on a regular expansion in a formal small parameter and, as such, it respects the Galilean symmetry of the problem in any finite-order approximation. Furthermore, it can be naturally combined with exact functional relations (like the Ward identities and the Dyson or Schwinger equations) and the short-distance operator-product expansion; see [17–21] for the reviews and references.

One of the marker problems is to establish the existence of so-called anomalous (multi-) scaling on the base of a dynamical model and calculation of the corresponding anomalous exponents within a certain perturbation expansion [27]. Despite great efforts, this problem

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Later, it was reintroduced in a scalar form by Kardar, Parisi, and Zhang [3], and became a paradigmatic model in non-equilibrium critical phenomena.

In this respect, they are similar to the Hartree-Fock and Gor’kov equations in many-body problems, which can be interpreted as the leading-order approximations of certain general schemes; see, e.g., monograph [23].
remains essentially open for real turbulence, or, to be more specific, for the stochastic NS equation [26].

However, this problem was solved analytically in exhaustive way for the celebrated Kraichnan’s rapid-change model: a scalar field advected passively by the velocity field with given statistics [28]; see also, e.g., [29] for a review and references. This problem can be accessed by a few approaches: the so-called zero-mode approach [30,31], numerical simulations [32], and the RG [33].

The latter allows one to construct a regular perturbative expansion for the anomalous exponents, similar to famous ε-expansion of the critical exponents [18,34,35], and to calculate the anomalous exponents to the order ε^3 for the rapid-change model [36] and to the order ε^2 for its generalization with finite correlation time [37] and for advection by the stirred NS equation [38].

The next step to access the real turbulence is to take into account the back influence of the scalar field on the dynamics of the fluid (“active scalar”). It was studied in [39–41] in connection to the phase transitions in binary fluids.

In turbulence itself, the comparison of the passive and scalar advection was performed, and a promising resemblance between their spectra was established [42–46]. In particular, the so-called statistically conservation laws (on the microscopic level related to some preserved geometric configurations of convected particles) play an important role in the both cases [44–46]. But the situation is not yet entirely clear [42,43].

The RG approach was applied to the active scalar advection by a turbulent flow by Nandy and Bhattacharjee in [41]. In that situation, it is necessary to describe the velocity statistics by a full-scale dynamical model, like the stochastic NS equation (and not by a synthetic Gaussian ensemble, like the rapid-change model). They found out that the only infrared (IR) attractive fixed point of the RG equations (and, therefore, the only possible type of IR asymptotic behavior) corresponds to a situation where the active term is IR irrelevant (in the sense of Wilson). This observation gives an explanation to previous numerical or phenomenological findings about the close resemblance between the passive and active cases.

It is generally recognized that the RG analysis is reliable and self-contained at small values of the expansion parameter, the exponent entering the stirring force correlation function in the NS equation (called y or 2ε different studies). Extrapolation to real finite value is a separate problem. And it needs to apply additional methods like the operator expansion at short distances; see [17] and the references. Strong IR divergences arise in the model at finite values of y = 2ε, related to most divergent contributions from the “dangerous” composite operators built solely by velocity fields and their time derivatives [17,19,48].

In our paper, we present the results of the RG analysis of the active scalar field advected by the velocity statistics introduced and studied in [1,2].

It corresponds to the fluid in thermal equilibrium, and, therefore, does not describe the fully developed turbulence. The advantage is that here, the problem of extrapolation to finite values of the RG expansion parameter does not exist, and perturbative results of the RG parameter can be trusted (no “dangerous” composite operators are expected to exist). The active term in the NS equation is taken in the standard form, applied in [39–41]. It can be derived on the base of microscopic hydrodynamic theory and arises in the so-called model H of equilibrium critical behavior; see surveys [49,50] and Chap. 5 of monograph [18].

From a more phenomenological point of view, it is the simplest term that can be constructed of minimal number of fields and derivatives, and cannot be removed by a redefinition of the pressure term.

\(^3\)The relation between the RG, statistical conservation laws, and operator-product expansion is discussed in [47].
Our main result is qualitatively the same as that of [41]: the only nontrivial IR attractive fixed point corresponds to the passive scalar, where the active term in the NS equation is irrelevant.

The plan of the paper is the following. In Sec. 2, we introduce the model: the NS equation with the “active” term and the diffusion-advection equation for scalar field, with the respective random stirred noises. Section 3 is devoted to the field theoretic formulation of the problem: the corresponding action functional and the elements of the Feynman diagrammatic techniques. In Sec. 4, one can find the analysis of the ultraviolet (UV) divergences: the canonical dimensions are derived for all the fields and parameters, and possible counterterms are presented. As a result, the multiplicative renormalizability of the model is established, and the renormalized action functional is written down with all needed renormalization constants. In the following Sec. 5, the one-loop expressions for the relevant diagrams and all the renormalization constants are presented. In Sec. 6, the RG equations are derived, and the RG functions (the $\beta$ functions and the anomalous dimensions) are given, with the explicit expressions for their one-loop approximations. Section 7 lists all the fixed points of the RG equations along with eigenvalues of the stability matrices, that determine their character as attractors of the RG equations. The last Sec. 8 is reserved for conclusions.

2. The model

In the present paper, we consider advection of an active scalar field by the incompressible fluid. To begin with, let us define the equation of the scalar field advection. It has a form of usual diffusion-advection equation subjected to a random force:

$$\nabla_t \theta(x) = \kappa_0 \partial^2 \theta(x) + f(x),$$

where $x = \{t, x\}$, $\theta(x)$ is the scalar field, $\kappa_0$ is the molecular diffusivity coefficient, $v = \{v_i(x)\}$ is the fluid velocity field,

$$\nabla_t = \partial_t + v_k(x) \partial_k$$

is the Lagrangian (Galilean covariant) derivative, $\partial^2 = \partial_k \partial_k$ is the Laplace operator (with implied summation over the repeated index), and $f(x)$ is a Gaussian noise with zero mean and given correlation function:

$$\langle f(t, x) f(t', x') \rangle = B_0 \delta(t - t') \delta(x - x') \quad (3)$$

The amplitude $B_0 > 0$ is set equal to one as follows: this can be achieved by a proper rescaling of the field $\theta$, noise, and parameters.

It is worth noting that there are two types of scalar fields: one is the density of a conserved quantity (for example, density of a pollutant), while the second one is a “tracer” (concentration of a pollutant, temperature, or enstrophy). Expression (6) corresponds to the tracer field (no conservation is implied). For the density field, the noise correlation function necessarily contains a derivative and therefore has the form

$$\langle f(t, x) f(t', x') \rangle = -B_0 \delta(t - t') \partial^2 \delta(x - x'), \quad B_0 > 0. \quad (4)$$

In what follows, we mostly consider the case of a tracer. The density field is discussed later: it turns out that the results for this case are derived in a more simple way.

The dynamics of the fluid is determined by the forced NS equation for a viscous incompressible fluid. This equation has the form

$$\nabla_t v_i = \nu_0 \partial^2 v_i + \partial_i \varphi - \alpha_0 (\partial_i \theta)(\partial^2 \theta) + f_i,$$

where $\varphi(x)$ is the pressure, $f_i(x)$ is the external random stirring force, $\partial^2$ is the Laplacian, and $\nabla_t$ is the Lagrangian derivative (2).
The “active” term $(\partial_\theta)(\partial^2\theta)$ in the NS equation contains two fields and three derivatives, and as such, is the simplest nontrivial construction built of the minimal number of derivatives and scalar fields. (As already said, the tracer field enters the equations only in the form of spatial derivative.) The simplest contribution of the form $\partial_\theta$ can be absorbed by a redefinition of the pressure $\varphi$, and, therefore, does not affect the fluid dynamics. On the other hand, this term can be derived in a microscopic approach, within the framework of the model II of equilibrium critical hydrodynamics, see [18, 49, 50]. This derivation also shows that $\alpha_0 > 0$.

The statistics of the random force in (5) is chosen in the following way [1, 2]:

$$
\langle f_i(x)f_j(x') \rangle = D_0 \delta(t-t') \int \frac{dk}{(2\pi)^d} k^2 e^{ik(x-x')} P^\perp_{ij}(k).
$$

Here, the symbol $P^\perp_{ij}(k) = \delta_{ij} - k_i k_j/k^2$ denotes the transverse projector. The correlation function is interpreted as a thermal noise. The constant amplitude $D_0 > 0$ is positive, we take a closer look on it below.

The factor of $k^2$ in (6) respects the conservation of the fluid momentum and, simultaneously, ensures the fluctuation-dissipation relation. As a result, the equal-time correlation functions of the velocity field are described by the simple Maxwell distribution [1]; for the detailed proof see also [17].

3. Field theoretic formulation

According to the general theorem of De Dominicis–Janssen [17,18,35], the stochastic model (1), (5) can be reformulated as the field-theoretic one with the doubled set of fields. The reformulated theory has the action functional

$$
S(\Phi) = \frac{1}{2} v'_i D_{i k}^f v'_k + v'_i \left\{ -\nabla_i v_i + \nu_0 \partial^2 v_i - \alpha_0 (\partial_\theta)(\partial^2 \theta) \right\} + \frac{1}{2} \theta' \theta' + \theta' \left\{ -\nabla_i \theta + \kappa_0 \partial^2 \theta \right\}.
$$

Here, $D_{i k}^f$ is the correlation function (6), $\Phi = \{v'_i, \theta', v_i, \theta\}$ is the full set of fields which includes the usual velocity and scalar fields $\{v_i, \theta\}$ and the additional auxiliary “response” fields $\{v', \theta'\}$. Here and below, the summations over the repeated indices and the integrations over $x = \{t, \mathbf{x}\}$ are implied, for example:

$$
v'_i \partial^2 v_i = \sum_{i=1}^{d} dt dx v'_i \partial^2 v_i.
$$

The first term in (7) comes from the correlation function (6) and can be written as

$$
\frac{1}{2} v'_i D_{i k}^f v'_k = \frac{1}{2} D_0 (\partial_k v'_i)(\partial_k v'_i) = -\frac{1}{2} D_0 v'_i \partial^2 v'_i.
$$

This field-theoretic model can be interpreted the following way: various correlation and response functions of the original stochastic problem can be represented as functional averages over all the fields of the full set with weight exp $S(\Phi)$. So they can be considered as the Green functions of the field-theoretic model with action functional (7). This model corresponds to the standard Feynman diagrammatic technique with the propagators

$$
\langle v_i v'_j \rangle = \langle v'_j v_i \rangle^* = \frac{P^{\perp}_{ij}(k)}{-i\omega + \nu_0 k^2},
$$

$$
\langle v_i v_j \rangle = \frac{P^{\perp}_{ij}(k)}{|-i\omega + \nu_0 k^2|^2},
$$

$$
\langle \theta \theta' \rangle = \frac{1}{| -i\omega + \kappa_0 k^2 |^2}.
$$

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\[ \langle \theta \theta \rangle = \frac{1}{| - i \omega + \kappa_0 k^2 |^2}, \]  

and the three vertices

\[
V_{ijl} = i(k_l \delta_{ij} + k_j \delta_{il}),
\]

\[
V_i = -i k_i,
\]

\[
V_j = i \alpha_0 (k_i q^2 + q_i k^2) = -i \alpha_0 (k_i (pq) + q_i (pk)).
\]

The role of the coupling constants is played by the parameters \( g_0, w_0, \) and \( u_0 \) defined by the relations

\[ D_0 = g_0 \nu_0^3, \quad \alpha_0 = w_0 \nu_0^3, \quad \kappa_0 = u_0 \nu_0. \]  

The dimensional analysis (see the next section) shows that the first two coupling constants scale as \( g_0, w_0 \sim \Lambda^{-\varepsilon} \), where \( \Lambda \) is a characteristic UV momentum, while \( u_0 \) is dimensionless. Thus, the model as a whole becomes logarithmic (all the couplings become dimensionless) at \( \varepsilon \equiv 2 - d = 0 \), and the UV divergences have the form of poles in \( \varepsilon \) in the Green functions. Although \( u_0 \) is not an expansion parameter, it should be treated on the same footing as \( g_0, w_0, \) because the renormalization constants and the RG functions depend on it.

4. UV DIVERGENCES AND RENORMALIZATION

It is well known that the analysis of UV divergences is based on the analysis of canonical dimensions; see, e.g., [18,34,35]. Dynamical models like (7) have two independent scales: the time scale \( T \) and the length scale \( L \). Thus the canonical dimension of any quantity \( F \) (a field or a parameter) is described by two numbers, the frequency dimension \( d_\omega F \) and the momentum dimension \( d_k F \), defined so that \( [F] \sim [T]^{-d_\omega F} [L]^{-d_k F} \). The obvious consequences of the definition are the relations

\[ d_k = -d_\omega = 1, \quad d_\omega = 0, \quad d_{\omega_0} = 0, \quad d_{\omega_0} = 1. \]  

The other dimensions are found from the requirement that each term of the action functional is dimensionless (with respect to the momentum and the frequency dimensions separately). Then one introduces the total canonical dimension [18]

\[ d_F = d_k^F + 2d_\omega^F, \]
which plays in the theory of renormalization of dynamical models the same role as the conventional canonical dimension plays in static problems. The canonical dimensions for the model (7) are given in Table 1, including renormalized parameters (without the subscript “o”), which are introduced later.

| Field | Symbol | Dimension | Operator | Terms |
|-------|--------|-----------|----------|-------|
| F     | v'     | d + 1     | -1       | 0     |
| F     | v      | d/2       | 1/2      | 0     |
| F     | θ'     | d/2       | 1/2      | 0     |
| θ'    |       | -d - 4    |           | 0     |
| θ'    |       | 2 - d     |           | 0     |
| θ'    |       | w, u      |           | 0     |

Table 1. Canonical dimensions of the fields and parameters in the models.

From Table 1, it follows that, as already mentioned, the model becomes logarithmic (the coupling constants become dimensionless) at \( \varepsilon = 2 - d = 0 \), and the UV divergences have the form of poles in \( \varepsilon \) in the Green functions.

The total canonical dimension of any 1-irreducible Green function \( \Gamma \) (the formal index of UV divergence) is given by the expression

\[
\delta_\Gamma = d + 2 - \sum_{\Phi} N_\Phi d_\Phi
\]

in the logarithmic theory (that is, at \( \varepsilon = 0 \)). Here, \( N_\Phi \) is the number of the fields entering into the function \( \Gamma \), \( d_\Phi \) is its total canonical dimension, and the summation over all types of the fields \( \Phi \) is implied. Superficial UV divergences, whose removal requires counterterms, can be present only in the functions \( \Gamma \) with nonnegative integer \( \delta_\Gamma \). The counterterm is a polynomial in frequencies and momenta of degree \( \delta_\Gamma \) provided that \( \omega \sim k^2 \).

Dimensional analysis should be augmented by the following considerations.

If, for some reason, a number of external momenta occurs as an overall factor in all diagrams of a certain 1-irreducible Green function, then the real index of divergence should be properly reduced.

In the present case, the fields \( \theta' \) and \( v' \) enter all the vertices in the form of spatial derivatives, see (7). Thus, any appearance of \( \theta' \) or \( v' \) in a certain 1-irreducible function gives an external momentum, and the real index of divergence takes the form

\[
\delta_\Gamma' = \delta_\Gamma - N_{\theta'} - N_{v'}. \tag{16}
\]

As a manifestation of causality, all the 1-irreducible diagrams without external “tails” of the response fields \( v', \theta' \) contain self-contracted circuits of retarded propagators and therefore vanish. Thus, it suffices to consider the functions with \( N_{\theta'} + N_{v'} \geq 1 \) only.

The action functional (7) is invariant with respect to the simultaneous reflection of the scalar fields, \( \theta \rightarrow -\theta, \theta' \rightarrow -\theta' \). Therefore all the Green functions with odd total number of the scalar fields vanish (no diagrams can be constructed). In particular, this excludes the 1-irreducible function \( (\theta' \theta') \) and the corresponding counterterm \( \theta'(\partial \theta)^2 \).

The counterterms having the form of total derivatives (or reduced to such form using the integration by parts) vanish after the integration over \( x \) and should be ignored; consequently, the counterterms that differ by a total derivative should be identified.

Of course, one should not forget the transversality conditions \( \partial_i v_i = \partial_i v'_i = 0 \) for the vector fields.

The analysis shows that our model is multiplicatively renormalizable: all the counterterms necessary to eliminate of UV divergences can be reproduced by rescaling the terms already present in action (7); moreover, some terms do not need to be rescaled (some counterterms
allowed by dimensional analysis are forbidden by additional considerations discussed above). The needed counterterms are \( \partial v' \partial v', v' \partial^2 v, v' (\partial \theta) \partial^2 \theta, \) and \( \theta' \partial^2 \theta. \) The counterterms \( v' \nabla_t v, \theta' \nabla_t \theta, \) and \( \theta' \theta' \) are forbidden by Galilean symmetry and/or by the real index.

The resulting renormalized action functional has the form

\[
S_R(\Phi) = \frac{1}{2} Z^D D \partial_k v'_i \partial_k v'_i + v'_i \left\{ -\nabla_t v_i + Z_\nu \nu \partial^2 v_i \right\} \\
- v'_i Z_\alpha (\partial_i \theta)(\partial^2 \theta) + \frac{1}{2} \theta' \theta' + \theta' \left\{ -\nabla_t \theta + Z_\kappa \kappa \partial^2 \theta \right\},
\]

(17)

where the dimensionless renormalization constants \( Z_i \) are naturally reproduced as multiplicative renormalization of the parameters

\[
D_0 = D Z_D, \quad \nu_0 = \nu Z_\nu, \quad \alpha_0 = \alpha Z_\alpha, \quad \kappa_0 = \kappa Z_\kappa,
\]

(18)

where \( D, \nu, \alpha, \kappa \) are renormalized parameters and the reference mass \( \mu \) is an additional parameter of the renormalized theory. From now on, we assume that the minimal subtraction (MS) scheme of renormalization is used.

To pass to dimensionless couplings \( g, w, u, \) we make the substitutions, cf. Eq. (12),

\[
D = g \mu^3 \nu^3, \quad \alpha = w \mu^3 \nu^3, \quad \kappa = u \nu,
\]

(19)

which leads to the following relations for \( Z_i \):

\[
Z_g = Z_D Z_\nu^{-3}, \quad Z_w = Z_\alpha Z_\nu^{-3}, \quad Z_u = Z_\kappa Z_\nu^{-1}.
\]

(20)

No renormalization of the fields is required.

5. ONE-LOOP RESULTS

In order to calculate the renormalization constants \( Z_i \) in the one-loop approximation, one has to calculate UV divergent parts of the diagrams shown below. We do not give details of the calculation, which is rather standard for dynamical models; see, e.g., Appendix in [51] and Sec. C in [52]. Below we give the coefficients in front of the pole \( 1/\varepsilon \) in the diagrams.

For \( Z_D - 1 \)-irreducible function \( \langle v'_i v'_j \rangle \):

\[
\begin{align*}
\text{Diagram 1} & \quad = \frac{g^2 \nu^3}{8} p^2 P_{ij}^\perp(p) \ast \frac{1}{2}, \\
\text{Diagram 2} & \quad = \frac{w^2 \nu^3}{8 u^3} p^2 P_{ij}^\perp(p) \ast \frac{1}{2}.
\end{align*}
\]

For \( Z_\nu - 1 \)-irreducible function \( \langle v'_i v_j \rangle \):

For \( Z_\kappa - 1 \)-irreducible function \( \langle \theta' \theta \rangle \):

\[
\begin{align*}
& = \frac{-g \nu}{4(u+1)} p^2.
\end{align*}
\]
For $Z_\alpha - 1$-irreducible function $\langle \nu_j \theta \theta \rangle$:

\[
= -iw \nu^3 [p_j (p + q, q) + q_j (p + q, p)] \frac{g}{16(u + 1)},
\]

\[
= iw \nu^3 [p_j (p + q, q) + q_j (p + q, p)] \frac{w}{16u(u + 1)},
\]

\[
= -iw \nu^3 [p_j (p + q, q) + q_j (p + q, p)] \frac{g}{16u(u + 1)},
\]

\[
= iw \nu^3 [p_j (p + q, q) + q_j (p + q, p)] \frac{w}{16(u + 1)u^2}.
\]

In the first two diagrams, the additional factor $1/2$ is the symmetry coefficient. Note that the contribution of the second diagram for $Z_\kappa$ is positive, in contrast to the three preceding diagrams. This means that the contribution of the active term reduces the effective diffusivity.

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coefficient, which is not typical for models of advection. The same effect takes place for the model studied in [41].

After calculating the one-loop diagrams, one can find the expressions for $Z's$ which are necessary to eliminate the divergences: the Green functions of the renormalized model should be UV finite in the limit $\varepsilon \to 0$. Using the MS scheme, one obtains the following results:

$$
Z_D = 1 - \frac{g}{32\pi\varepsilon} - \frac{w^2}{32\pi g u^2\varepsilon},
$$
$$
Z_\nu = 1 - \frac{g}{32\pi\varepsilon} - \frac{w}{32\pi u^2\varepsilon},
$$
$$
Z_\kappa = 1 - \frac{g}{8\pi u(u + 1)\varepsilon} + \frac{w}{8\pi u^2(u + 1)\varepsilon},
$$
$$
Z_\alpha = 1 + \frac{1}{32\pi(u + 1)\varepsilon} \left\{ g + \frac{g}{u} - \frac{w}{u^2} - \frac{w}{u} \right\},
$$

(21)

with corrections of higher orders in $g$ and $w$. The other constants are found from relations (20).

6. RG EQUATIONS

We proceed to the derivation of the RG equations; a detailed discussion can be found in [18,34,35].

Consider the renormalized correlation function $G^R = \langle \Phi \cdots \Phi \rangle_R$. Owing to the absence of the fields renormalization, it differs from the original (unrenormalized) function $G = \langle \Phi \cdots \Phi \rangle$ only by the choice of parameters. Indeed, the relation $S_R(\Phi, e, \mu) = S(\Phi, e_0)$ between the functionals (7) and (17) results in the relations

$$
G(e_0, \ldots) = G^R(e, \mu, \ldots)
$$

(22)

between the correlation functions. Here, $N_\phi$ and $N_{\phi'}$ are the numbers of the corresponding fields entering into $G$, $e_0 = \{\nu_0, g_0, u_0, v_0\}$ is the full set of bare parameters, and $e = \{\nu, g, u, v\}$ are their renormalized counterparts; the ellipsis stands for the other arguments (times, coordinates, momenta, etc.).

We use $\tilde{D}_\mu$ to denote the differential operation $\mu \partial_\mu$ for fixed $e_0$ and operate on both sides of equation (22) with it. This gives the basic RG differential equation

$$
\mathcal{D}_{RG} G^R(e, \mu, \ldots) = 0.
$$

(23)

Here, $\mathcal{D}_{RG}$ is the operation $\tilde{D}_\mu$ expressed in terms of the renormalized variables,

$$
\mathcal{D}_{RG} = \mathcal{D}_\mu + \beta_g \partial_g + \beta_u \partial_u + \beta_w \partial_w - \gamma_\nu \mathcal{D}_\nu.
$$

(24)

Here, we write $\mathcal{D}_s \equiv s \partial_s$ for any variable $s$. The anomalous dimension $\gamma_e$ of a certain parameter $e$ is defined as

$$
\gamma_e = Z_e^{-1} \tilde{D}_\mu Z_e = \tilde{D}_\mu \ln Z_e,
$$

(25)

and the $\beta$ functions for the three dimensionless coupling constants $g$, $u$, and $w$ are

$$
\beta_g = \tilde{D}_\mu g = g[-\varepsilon - \gamma_g],
$$
$$
\beta_w = \tilde{D}_\mu w = w(-\varepsilon - \gamma_w),
$$
$$
\beta_u = \tilde{D}_\mu u = -u \gamma_u,
$$

(26)

where the second equalities result from the definitions and relations (20).

All the renormalization constants in the MS scheme have the form

$$
Z_e = 1 + \sum_{n=1}^{\infty} z^{(n)} \varepsilon^{-n},
$$

(27)
where the coefficients $z^{(n)}$ do not depend on $\varepsilon$. Then from the definition and expressions (26), it follows that the corresponding anomalous dimension is determined solely by the first-order coefficient,

$$\gamma_\varepsilon = -(D_g + D_w)z^{(1)},$$

(28)

see, e.g., monograph [18]. Thus, from expressions (21) we obtain the following one-loop answers for the anomalous dimensions:

$$\gamma_D = \gamma_g + 3\gamma_\nu = \frac{g}{32\pi} + \frac{w^2}{g32\pi u^2},$$

$$\gamma_\nu = \frac{g}{32\pi} + \frac{w}{32\pi u^2},$$

$$\gamma_\kappa = \gamma_u + \gamma_\nu = \frac{g}{8\pi u(u+1)} - \frac{w}{8\pi u^2(u+1)},$$

$$\gamma_\alpha = \gamma_w + 3\gamma_\nu = -\frac{1}{32\pi(u+1)} \left\{ g + \frac{g}{u} - \frac{w}{u^2} - \frac{w}{u} \right\},$$

(29)

with higher-order corrections in $g$ and $w$.

Then it is easy to find all the $\beta$ functions. In what follows, to simplify the expressions we introduce the variables $\hat{g} = g/(2\pi)$ and $\hat{w} = w/(2\pi)$. Then the one-loop answers for $\beta$ functions are:

$$\beta_u = -w\gamma_u = u(\gamma_\nu - \gamma_\kappa) = -\frac{1}{4(u+1)} \left( \hat{g} - \frac{\hat{w}}{u} \right) + \frac{u}{16} \left( \hat{g} + \frac{\hat{w}}{u^2} \right),$$

(30)

$$\beta_w = -\varepsilon w - w\gamma_w = -\varepsilon w + w(3\gamma_\nu - \gamma_\alpha)$$

$$= -\varepsilon w + \frac{w}{16(u+1)} \left( \hat{g} + \frac{\hat{g}}{u} - \frac{\hat{w}}{u^2} - \frac{\hat{w}}{u} \right) + \frac{3w}{16} \left( \hat{g} + \frac{\hat{w}}{u^2} \right),$$

(31)

$$\beta_g = -\varepsilon g - g\gamma_g = -\varepsilon g + g(3\gamma_\nu - \gamma_1)$$

$$= -\varepsilon g - g \left( \frac{\hat{g}}{16} + \frac{\hat{w}}{16u^2} \right) + 3g \left( \frac{\hat{g}}{16} + \frac{\hat{w}}{16u^2} \right).$$

(32)

7. Fixed points

Possible IR asymptotic regimes of a renormalizable field-theoretic model are defined by IR attractive fixed points of the corresponding RG equations. The coordinates $g_*$ of the fixed points are found from the equations

$$\beta_i(g*) = 0,$$

(33)

where $g = \{g_i\}$ is the full set of coupling constants and $\beta_i$ is the full set of the $\beta$ functions. The type of a fixed point is determined by the matrix

$$\Omega_{ij} = \partial \beta_i / \partial g_j|_{g=g_*}.$$  

(34)

For the IR attractive fixed points, the matrix $\Omega$ is positive, that is, the real parts of their eigenvalues are positive.

In our model, $g = \{g, u, w\}$ and the $\beta$ functions are given by relations (30)–(32). Analysis of these expressions reveals the following fixed points:

1) the line of Gaussian (free) fixed points

$$g_* = w_* = 0, \quad u_* \text{ arbitrary},$$

(35)

with eigenvalues

$$-\varepsilon, \quad -\varepsilon, \quad 0.$$  

(36)
This line is IR attractive for $\varepsilon < 0$ ($d > 2$). All the nonlinearities are unimportant in the IR region, as can be anticipated from the analysis of canonical dimensions. The vanishing of one of the eigenvalues reflects the fact that this line is indifferent to the change of the ratio $u = \kappa/\nu$: the equations for the velocity and the scalar field are not related.

2a) and 2b) passive fixed points

$$w_* = 0, \quad u_* = \frac{-1 \pm \sqrt{17}}{2}, \quad \hat{g}_* = 8\varepsilon. \quad (37)$$

The eigenvalues

$$\varepsilon, \quad \frac{9}{16} \varepsilon + \frac{\sqrt{17}}{16} \varepsilon, \quad \frac{17}{16} \varepsilon - \frac{\sqrt{17}}{16} \varepsilon, \quad (38)$$

and

$$\varepsilon, \quad \frac{9}{16} \varepsilon - \frac{\sqrt{17}}{16} \varepsilon, \quad \frac{17}{16} \varepsilon + \frac{\sqrt{17}}{16} \varepsilon, \quad (39)$$

are IR attractive for $\varepsilon > 0$ ($d < 2$). The active term is irrelevant, and one returns to the ordinary passive advection.\(^4\)

3) the “active” fixed point

$$u_* = -3, \quad \hat{w}_* = \frac{72\varepsilon}{5}, \quad \hat{g}_* = \frac{24\varepsilon}{5} \quad (40)$$

with eigenvalues

$$\varepsilon, \quad \frac{7}{20} \varepsilon + \frac{\sqrt{145}}{20} \varepsilon, \quad \frac{7}{20} \varepsilon - \frac{\sqrt{145}}{20} \varepsilon, \quad (41)$$

which is always unstable (the eigenvalues have different signs).

4a) and 4b) complex “active” fixed points

$$u_* = i\sqrt{7},$$

$$\hat{w}_* = \frac{56(-11+i\sqrt{7})}{3+7i} \varepsilon = \frac{28\varepsilon}{11} + \frac{140i\sqrt{7}\varepsilon}{11}, \quad (42)$$

$$\hat{g}_* = \frac{8(5+i\sqrt{7})\sqrt{7}}{3+7\sqrt{7}i} \varepsilon = \frac{56\varepsilon}{11} + \frac{16i\sqrt{7}\varepsilon}{11}$$

with eigenvalues

$$\varepsilon, \quad -\frac{i\varepsilon}{352}(-11\sqrt{17} + 33i + \sqrt{-38770 - 15958i\sqrt{7}}),$$

$$-\frac{i\varepsilon}{352}(-11\sqrt{17} + 33i - \sqrt{-38770 - 15958i\sqrt{7}}), \quad (43)$$

or, approximately

$$\varepsilon, \quad (-0.5289574958 - 0.1909278136i)\varepsilon, \quad (0.7164574958 + 0.3562872705i)\varepsilon. \quad (44)$$

And, finally, the complex conjugate point

$$u_* = -i\sqrt{7},$$

$$\hat{w}_* = \frac{56(11+i\sqrt{7})}{-3+7i} \varepsilon = \frac{28\varepsilon}{11} - \frac{140i\sqrt{7}\varepsilon}{11}, \quad (45)$$

$$\hat{g}_* = \frac{8(5-i\sqrt{7})\sqrt{7}}{-3+7\sqrt{7}i} \varepsilon = \frac{56\varepsilon}{11} - \frac{16i\sqrt{7}\varepsilon}{11}$$

\(^4\)There is a misprint in the expression (3.69) in [1], that corresponds to ours $u_*$ in (37). It is also worth mentioning that the authors call this expression “intriguing”.

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with eigenvalues

$$\varepsilon, \quad -\frac{i\varepsilon}{352}(11\sqrt{17} + 33i + \sqrt{-38770 + 15958i\sqrt{7}}),$$

$$-\frac{i\varepsilon}{352}(11\sqrt{17} + 33i - \sqrt{-38770 + 15958i\sqrt{7}}),$$

(46)

or, approximately

$$\varepsilon, \quad (-0.5289574958 + 0.1909278136i)\varepsilon, \quad (0.7164574958 - 0.3562872705i)\varepsilon.$$

(47)

These two points are always unstable (the real parts of the eigenvalues have different signs for all \(\varepsilon\)).

For \(\varepsilon = 0\) (\(d = 2\)), there is a line of fixed points \(g_* = w_* = 0\), \(u_*\) is arbitrary. A direct numerical integration of the differential equations for the running coupling constants shows\(^\text{5}\) that this point is IR attractive for the physical initial data \(g > 0, w > 0\), at least for some interval of values for \(u\). The sample RG flow for the special case \(u = 1\) is depicted in Fig. 1.

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**Fig. 1.** The RG flow in the \(g-w\) plane for \(u = 1\) and \(d = 2\).

We conclude with brief discussion of the density model with correlation function (4). Again, we can set \(B_0 = 1\). Then the analysis of the canonical dimensions gives

$$d^\theta_\theta = d/2 + 1, \quad d^\theta_\psi = d/2 - 1, \quad d^\alpha_\alpha = -d - 6, \quad d^\theta_\psi = -d$$

for the momentum dimensions, and

$$d_\theta = d_\psi = d/2, \quad d_\alpha = d_\psi = -d$$

for the total dimensions. All the other dimensions remain the same as in Table 1.

\(^5\)We thank N. M. Lebedev for his help.
From these results, it follows that the total dimension of the coupling constant $w_0$ is negative for all $d > 0$, so that the active term appears IR irrelevant already at the level of simple dimensional analysis: no refined RG analysis is needed; the fact also mentioned in [1].

8. Conclusion

At first glance, our results look rather disappointing. The IR attractive fixed point is trivial (Gaussian) for $d > 2$ and passive for $d < 2$. The back reaction of the scalar field on the velocity dynamics is always negligible in the leading term of the IR asymptotic scaling behavior. But this result appears robust in the sense that it agrees with the result derived earlier by Nandy and Bhattacharjee who considered another type of the random noise, corresponding to advection by a turbulent fluid [41].

In agreement with [41], the nonlinearity gives contributions of different signs into effective viscosity and diffusion coefficients. This means that the back reaction of the scalar field would produce the mechanical energy from the thermodynamic one. This is probably the physical reason why the active IR regime is not realized as an attractive RG fixed point.

To avoid possible misunderstanding, it should be stressed that the “active” term remains to be present in the model: it can strongly affect nonuniversal quantities (e.g., amplitudes in scaling laws, like the Kolmogorov or Batchelor constants) and correction terms to the leading IR behavior (which can quantitatively be essential). Probably, these results give an explanation to the similarity between the spectra of passive and active fields, in spite of the serious differences between the underlying motion of the impurity particles [42–46].

Furthermore, the apparently “nonphysical” fixed points 3 and 4 may have a sound physical interpretation.

The negative value $u_* < 0$ means that one of the effective viscosity or, more likely, diffusivity coefficients tend to negative values at some intermediate scales. A similar effect was discussed a long time ago in a number of studies of non-equilibrium stochastic models [53]. Along with the instability of the corresponding RG fixed points, this suggests that the original model itself might be incomplete and can be modified by adding the higher-order terms.

The complex value of the viscosity coefficient (or better to say, of its effective analog) was encountered for stochastic models of the Langmuir plasma turbulence [54, 55] and for the stochastic version of the nonlinear Schrödinger equation [56].

These considerations suggest that our results may justify the future efforts in studying generalized versions of the active advection models: inclusion of another new terms (like the KPZ interaction and its numerous anisotropic versions), complex coupling constants and transport parameters, non-local in-space and/or in-time random forces, etc. This work remains for the future and partly is already in progress.

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6The relevant dimensionless parameter $w_0 k^d$ vanishes in the IR asymptotic region $k \to 0$. 
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