Relativistic effect in galaxy clustering

Jaiyul Yoo

Center for Theoretical Astrophysics and Cosmology, Institute for Computational Science, University of Zürich, Switzerland
Physics Institute, University of Zürich, Winterthurerstrasse 190, CH-8057, Zürich, Switzerland

E-mail: jyoo@physik.uzh.ch

Received 18 June 2014, revised 2 September 2014
Accepted for publication 9 September 2014
Published 12 November 2014

Abstract

The general relativistic description of galaxy clustering provides a complete and unified treatment of all the effects in galaxy clustering such as the redshift-space distortion, gravitational lensing, Sachs–Wolfe effects, and their relativistic effects. In particular, the relativistic description resolves the gauge issues in the standard Newtonian description of galaxy clustering by providing the gauge-invariant expression for the observed galaxy number density. The relativistic effect in galaxy clustering is significant on large scales, in which dark energy models or alternative theories of modified gravity deviate from general relativity. In this paper, we review the relativistic effect in galaxy clustering by providing a pedagogical derivation of the relativistic formula and by computing the observed galaxy two-point statistics. The relativistic description of galaxy clustering is an essential tool for testing general relativity and probing the early Universe on large scales in the era of precision cosmology.

Keywords: large-scale structure of the Universe, galaxy clustering, relativistic effect
PACS numbers: 98.80.-k, 98.62.Py, 98.65.-r, 98.80.Jk

1. Introduction

To understand the nature of dark energy and to probe the early Universe, a large number of galaxy surveys are operational and an even larger number of surveys are planned for the near future. These current and future galaxy surveys will deliver high precision measurements of galaxy clustering, providing enormous statistical power to solve the issues in the standard model of cosmology. However, high precision measurements in the upcoming galaxy surveys simultaneously bring in new challenges, setting a level of accuracy that theoretical predictions are obliged to meet. Given these strict requirements, two critical questions naturally arise in
regard to improving theoretical predictions in galaxy clustering. (1) Various effects contribute to galaxy clustering, such as the redshift-space distortion, the gravitational lensing, and so on. What is the exhaustive list of all the contributions to galaxy clustering? We need a complete description of all the effects in galaxy clustering to control systematics in theoretical modeling. (2) On large scales, perturbations such as the matter density fluctuation are gauge-dependent, as there is no unique choice of hypersurface of simultaneity throughout the entire Universe. The standard description of galaxy clustering is ill-posed to address this issue, i.e. which gauge condition needs to be chosen to describe the observed galaxy clustering and why?

As we note that the standard description of galaxy clustering is Newtonian, the questions in which the speed of light is infinite and the gravity is felt instantaneously across the Universe can be tackled. The light we measure in galaxy surveys, of course, propagates at finite speed through the Universe and is affected by the inhomogeneity and the curvature of the Universe. Therefore, we need proper general relativistic treatments to relate the observables we measure from the light to the physical quantities of source galaxies and the inhomogeneities that affect the photon propagation. This goal can be readily achieved by tracing back the photon path given the observed redshift and the angular position of the source galaxies, and the full relativistic formula of galaxy clustering is constructed from the observable quantities, providing the relation to the inhomogeneities and the source galaxy population [1, 2]. In this way, the relativistic description of galaxy clustering naturally answers the two key questions, since observable quantities are gauge-invariant and receive all the contributions without any theoretical prejudice. The relativistic formula is independently developed [3–5], and many interesting applications are investigated (e.g., [6–19]).

Regarding the detectability of the relativistic effect in galaxy clustering, it was shown [1] that the relativistic effect can be measured in the angular power spectrum and the systematic errors are larger than the cosmic variance on large scales. Furthermore, using the multi-tracer technique [20] to eliminate the cosmic variance limit on large scales, it was shown [14, 16] that the galaxy power spectrum can be used to measure the relativistic effect with great significance in upcoming surveys and can be used to discriminate alternative theories of modified gravity against general relativity on large scales. The relativistic effect in galaxy clustering becomes dominant on large scales, in which modified gravity or dark energy models deviate from general relativity and the information about the inflationary epoch remains intact. Therefore, it is crucial to have a proper relativistic description to avoid misinterpretation of large-scale measurements.

The purpose of this work is to provide a pedagogical derivation of the relativistic description of galaxy clustering. We begin by providing the relation of the photon path to the observed redshift and the angular position of source galaxies (section 2). Using the observable quantities, we construct galaxy clustering observables (section 3) and compute galaxy two-point statistics (section 4). We conclude with a discussion of further applications (section 5). Throughout the paper, we use the Latin indices for the spacetime component and the Greek indices for the spatial component, and we set the speed of light $c \equiv 1$. Symbols used in this paper are summarized in table 1.

2. Observed angular position and redshift of sources

2.1. Metric perturbations and gauge transformation

The background Friedmann–Lemaître–Robertson–Walker (FLRW) universe is described by a spatially homogeneous and isotropic
Table 1. Various symbols used in the paper.

| Symbols | Definition of the symbols | Equation |
|---------|---------------------------|----------|
| $g_{ab}$, $\eta_{ab}$ | FLRW metric and Minkowsky metric | (2.1) |
| $a$, $\widehat{a}$ | comoving scale factor and background three-metric | (2.1) |
| $A$, $B_{ai}$, $C_{ij}$ | components of perturbed metric tensor | (2.2) |
| $a$, $\alpha\beta$, $\bar{g}$ | decomposed metric perturbations | (2.2) |
| $\xi^{a}$, $\alpha$, $\beta$ | components of perturbed metric tensor | (2.2) |
| $\alpha$, $\beta$, $\phi$, $\gamma$ | decomposed metric perturbations | (2.2) |
| $u^{a}$, $U^{a}$, $v$ | four velocity, spatial component of $u^{a}$, scalar velocity | (2.4) |
| $\xi^{a}$, $T$, $L$, $L^\alpha$ | coordinate transformation vector and its decomposition | (2.5) |
| $\alpha\chi$, $\phi\chi$, $\chi^a$, $\delta^a$ | scalar gauge-invariant variables | (2.8) |
| $\Psi_{\alpha}$, $\psi_{\alpha}$ | vector gauge-invariant variables | (2.9) |
| $k_{i}^{a}$, $k^{a}$ | photon wavevector in local and FRW frames | (2.11), (2.16) |
| $\theta$, $\phi$, $z$ | observed angular position $\hat{n}$ and redshift of source galaxy | (2.12), (2.14) |
| $\{e_{i}\}^{a}$, $\{e_{i}\}^{a}$ | tetrad vectors in local frame | (2.15) |
| $v_{i}$, $\lambda$ | physical and conformal affine parameters | (2.18) |
| $\delta_{r}$, $\delta_{\theta}$, $\delta_{\phi}$ | perturbations in photon wavevector | (2.20) |
| $\delta_{r}$, $\delta_{\theta}$, $\delta_{\phi}$ | gauge-invariant variables for $\delta_{r}$, $\delta_{\theta}$, $\delta_{\phi}$ | (2.22) |
| $\hat{x}^{a}$, $x^{a}$ | observationally inferred and true positions of source galaxies | (2.30), (2.32) |
| $\bar{r}$, $\bar{\theta}$ | comoving coordinates in the background | (2.31) |
| $\delta r$, $\delta \theta$, $\delta \phi$ | distortions in the source position between $\hat{x}^{a}$ and $x^{a}$ | (2.32) |
| $\kappa$ | distortion in the observed redshift | (2.38), (2.41) |
| $\delta D_{l}$ | fluctuation in luminosity distance | (3.7), (3.8) |
| $\delta r$, $\kappa$ | gauge-invariant variables for $\delta r$ and $\kappa$ | (3.12) |
| $dV_{phys}$, $dV_{obs}$, $\delta V$ | physical and observationally inferred volumes, and their difference | (3.13), (3.14) |
| $n_{g}$, $n_{g}^{obs}$ | physical and observed galaxy number densities | (3.15) |
| $\bar{n}_{g}$, $\bar{n}_{g}$ | physical and observed mean number densities | (3.16), (3.21) |
| $\delta_{g}^{obs}$, $\delta_{g}^{int}$ | observed and intrinsic galaxy fluctuations | (3.22), (3.16) |
| $\phi$, $\psi_{\alpha}$, $\delta_{m}$ | Newtonian counterparts of $\phi$, $\psi_{\alpha}$, $\delta_{m}$ | (4.3) |
| $T_{l}$, $W_{l}$ | transfer function and its conversion function to $T_{m}$ | (4.6), (4.7) |
| $\Delta_{R}^{2}$ | dimensionless power spectrum of primordial curvature perturbation | (4.9) |
| $n_{g}^{obs}$, $n_{g}^{obs}$, $\delta_{g}^{obs}$, $\delta_{g}^{obs}$ | two dimensional galaxy number density and its fluctuation | (4.11) |
| $P_{\bar{r}}$ | normalized redshift distribution of source galaxies | (4.14) |
| $C_{l}$, $T_{l}$ | angular power spectrum and its multipole function | (4.15) |
| $\nabla^{2}$, $\nabla^{4}$ | gravitational and velocity contributions to $\delta_{g}^{obs}$ | (4.19) |
| $n_{g}(k)$, $a_{in}$ | spherical and angular decompositions | (4.23), (4.12) |
| $S_{l}$, $M_{l}$ | spherical power spectrum and its multipole function | (4.27) |
\[ dx^2 = g_{ab} dx^a dx^b = -a^2(\tau) d\tau^2 + a^2(\tau) \tilde{g}_{ab} dx^a dx^b, \]  
(2.1)

where \( \tau \) is the conformal time, \( a(\tau) \) is the expansion scale factor, and \( \tilde{g}_{ab} \) is the 3-spatial metric tensor with a constant spatial curvature \( K \). The real universe is inhomogeneous, and small deviations from the background metric are represented by

\[
\begin{align*}
\delta g_{00} &\equiv -2a^2A \equiv -2a^2\alpha, \\
\delta g_{0a} &\equiv -a^2B_a \equiv -a^2(\beta_a + B_a), \\
\delta g_{ab} &\equiv 2a^2C_{ab} \equiv 2a^2\left(\rho \tilde{g}_{ab} + \rho_a L_b + \frac{1}{2}C_{a1b} + \frac{1}{2}C_{b1a} + C_{ab}\right),
\end{align*}
\]
(2.2)

where the vertical bar is the covariant derivative with respect to spatial metric \( \tilde{g}_{ab} \).

Perturbations are further decomposed into scalar \((\alpha, \beta, \gamma)\), vector \((B_a, C_a)\) and tensors \((C_{ab})\), which are readily distinguishable by their spatial indices. The fluid quantities are described by the energy-momentum tensor

\[
T_{ab} = \rho u_a u_b + p \left(g_{ab} + u_a u_b\right) + q_a u_b + q_b u_a + \pi_{ab},
\]
(2.3)

where \( \rho \) is the energy density, \( p \) is the isotropic pressure, \( q_a \) is the energy flux, and \( \pi_{ab} \) is the anisotropic pressure. These fluid quantities are measured by the observer moving with timelike four velocity \((u^a u_a = -1)\)

\[
u^a \equiv \frac{1}{a} \left(1 - \mathcal{A}, \ U^a\right), \quad U^a \equiv -U^a + U^a, \]
(2.4)

where we decomposed the four velocity into scalar \((U)\) and vector \((U^a)\). We also define a scalar velocity \( v \equiv U + \beta \).

The general covariance is a symmetry in relativistic theory, and any coordinate system can be used to describe the physical system, providing ample degree of freedom at hand. However, in a cosmological framework, a coordinate transformation accompanies a change in the correspondence of the inhomogeneous universe to the fictitious background universe, known as the gauge transformation. Therefore, it is important to check that the theory under consideration is gauge-invariant. For the most general coordinate transformation

\[
\begin{align*}
\delta x^a &= x^a + \xi^a, \\
\xi^a &= (T, \mathcal{L}^a), \\
\mathcal{L}^a &= L^a + L',
\end{align*}
\]
(2.5)

the scalar quantities transform as

\[
\begin{align*}
\delta \alpha &= \alpha - T' - HT, \\
\delta \beta &= \beta - T + \mathcal{L}', \\
\delta \gamma &= \gamma - L, \\
\delta \mathcal{U} &= U - \mathcal{L}', \\
\delta \mathcal{V} &= \mathcal{V} - T,
\end{align*}
\]
(2.6)

and the vector metric perturbations transform as

\[
\begin{align*}
\delta B_a &= B_a + L'_a, \\
\delta C_a &= C_a - L_a, \\
\delta U_a &= U_a + L'_a.
\end{align*}
\]
(2.7)

where the prime is the derivative with respect to the conformal time and the conformal Hubble parameter is \( \mathcal{H} = a'/a = aH \). Since tensor harmonics are independent of tensors that can be constructed from coordinate transformations, tensor-type perturbations \((C_{ab}, \pi_{ab})\) remain unchanged under the gauge transformation in equation (2.5). Similarly, the gauge-transformation properties of the fluid quantities can be derived, and in particular the matter density fluctuation \( \delta \equiv \delta \rho_g/\rho_g \) transforms as \( \tilde{\delta} = \delta + 3HT \).

Based on the above gauge transformation properties, we can construct linear-order gauge-invariant quantities. The scalar gauge-invariant variables are
\[ \alpha_{\chi} = a - \frac{1}{a} \chi', \quad q_{\chi} = \varphi - H \chi, \quad v_{\chi} = v - \frac{1}{a} \chi, \quad \delta_{\chi} = \delta + 3Hv, \quad (2.8) \]

where \( \chi = a (\beta + \gamma') \) is the scalar shear of the normal observer \((n_a = 0)\) and it is spatially invariant, transforming as \( \dot{\chi} = \chi - aT \). The notation for scalar gauge-invariant variables is set up such that \( \delta_{\chi} \) for example, corresponds to the matter density fluctuation \( \delta \) in the comoving gauge \((v = 0)\) and \( v_{\chi} \) correspond to the scalar velocity \( v \) in the zero-shear gauge \((\chi = 0)\). Similarly, \( v_{\chi} = \delta/3H = v + \delta/3H \) would correspond to the scalar velocity \( v \) in the uniform density gauge \((\delta = 0)\), and many other gauge-invariant variables can be constructed in this way how the gauge correspondence is explicit \([21]\).

The vector gauge-invariant variables are

\[ \Psi_{\alpha} = B_{\alpha} + C'_{\alpha}, \quad v_{\alpha} = U_{\alpha} - B_{\alpha}, \quad (2.9) \]

These gauge-invariant variables \((\alpha_{\chi}, q_{\chi}, v_{\chi}, \Psi_{\alpha}, v_{\alpha})\) correspond to \( \Phi_A, \Phi_H, v^{(0)}, \Psi, \) and \( v_c \) in Bardeen’s notation \([22]\). For future use, we define a gauge-invariant velocity quantity,

\[ V_{\alpha} = -v_{\chi, \alpha} + v_{\alpha}, \quad (2.10) \]

which encompasses the scalar and vector gauge-invariant variables.

Any physical quantities or observable quantities should be gauge-invariant, i.e. the choice of gauge condition for the perturbation should be explicit. One can achieve this goal by choosing a gauge condition before any calculations are performed. However, all the quantities in this case become automatically gauge-invariant, depriving us of a way to verify if the quantities of interest are genuinely gauge-invariant. With fully general metric representation, the verification is explicit in the calculations \([2]\).

### 2.2. Observables in the observer rest frame

Galaxy positions in a redshift survey are identified by measuring photons from the sources by the observer in the rest frame. The photon propagation direction is set orthogonal to the hypersurface defined by constant phase \( \vartheta = k \cdot x - \omega t \). In the observer rest frame, the components of the photon wavevector can be written as

\[ k^a_L = \eta_{ab} \vartheta^b = (\omega, k) = 2\pi \nu \left( 1 - \hat{n} \right), \quad (2.11) \]

where the local metric is Minkowsky \( \eta_{ab} \), the angular frequency is \( \omega = 2\pi \nu \), and the amplitude of a photon wavevector is \(|k| = \omega = 2\pi / \lambda \) \((\lambda = 1)\). The subscript \( L \) is used to emphasize that the components are written in the observer rest frame or the local Lorentz frame. The observed angular position of the source galaxy is then determined by a unit directional vector \( \hat{n} = (\vartheta, \phi) \) for photon propagation in the observer rest frame,

\[ \hat{n} = -\frac{k}{|k|} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (2.12) \]

and the photon frequency measured by the observer is

\[ \omega = 2\pi \nu = -\eta_{ab} u^a_L k^b_L, \quad (2.13) \]

where \( u^a_L = (1, 0, 0, 0) \) in the observer rest frame. The observed redshift of the source galaxy is then determined by the ratio of the observed photon wavelength \( \lambda_L = 1/\nu \) to the wavelength \( \lambda_s \) we would measure in the rest frame of the source galaxy
where we omitted the subscript ‘obs’ for the observed redshift $z$. A prominent Lyα line $\lambda_s = 121.6$ nm, for example, is often used to measure the redshift of source galaxies. In this way, the observed angular position and the redshift of the source are expressed in terms of physical quantities.

In order to compute the photon wavevector in a FRW coordinate (as opposed to the local Lorentz frame), we first construct an orthonormal basis. The observer is moving with a time-like four velocity $u^a$, which defines the proper-time-direction $[e_i]^v \equiv u^i$ in the observer rest frame. Spatial hypersurface orthogonal to $u^a$ can be described by three spacelike four vectors $[e_i]^v$ ($i = [e_x]^v, [e_y]^v, [e_z]^v$). These four vectors, called tetrads, form an orthonormal basis in the observer rest frame. Using the orthonormality condition ($\eta^{ab} = \eta_{ab} = g_{ab} [e_i]^v [e_d]^b$, $c, d = t, x, y, z$), the tetrads in an inhomogeneous universe with the metric tensor in equations (2.1) and (2.2) can be constructed as

$$[e_i]^v = u^a, \quad [e_i]^v = \frac{1}{a} \left[ U_i^a - B_i, \delta_i^a - C_i^a \right],$$

where the tetrad index can be raised or lowered by $(\eta_{ab})$, while the FRW index is raised or lowered by $g_{ab}$. The photon wavevector in a FRW coordinate can be derived by transforming equation (2.11) in the local Lorentz frame as

$$k^a = \frac{2\pi v}{a} \left[ 1 - A - n^i (U_i^a - B_i) \right], \quad n^a = \frac{2\pi v}{a} \left[ -n^a + U^a + n^i C_i^a \right].$$

where $n^i$ is the $i$th spatial component of the unit directional vector $\hat{n}$ in a local Lorentz frame, other perturbation quantities are those in a FRW frame, and the repeated indices indicate the summation over the spatial components. It is noted that the photon wavevectors in equations (2.11) and (2.16) are different, and the components in a FRW frame are affected by the observer velocity and the gravitational potential. Naturally, a unit directional vector $-k^a/k^a$ in a FRW frame cannot be used to describe the observed angular position $\hat{n}$ in the observer rest frame. However, the photon frequency measured by the observer is a Lorentz scalar, i.e. it is the same in both frames

$$-g_{ab} u^a k^b = -\eta_{ab} u^a k^b = \omega = 2\pi v.$$

Likewise, the observed redshift in equation (2.14) is a Lorentz scalar.

2.3. Photon wavevector and conformal transformation

We parametrize the photon path $x^a(v)$ with an affine parameter $v$, and its propagation direction is then $k^a(v) = dx^a/dv$ in equation (2.16), subject to the null condition $k^a k_a = 0$ and the geodesic equation $k^a;_b = 0$. Since null geodesic is conformally invariant, we further simplify the photon propagation equations by using a conformal transformation $g_{ab} \rightarrow a^2 \delta_{ab}$.

The null geodesic $x^a(v)$ described by the conformally transformed wavevector $\tilde{k}^a$ remains unaffected under the conformal transformation, while its affine parameter is transformed to another affine parameter $\lambda$ (see e.g. [23]).

1 We always use $z$ to refer to the observed redshift.
\[
\frac{dv}{dl} = Ca^2, \quad (2.18)
\]

where the proportionality constant \( C \) represents additional degree of freedom from the conformal transformation. The conformally transformed photon wavevector \( \hat{k}^a = Ca^2k^a \) can be explicitly written as
\[
\hat{k}^0 = 2\pi C\alpha \left[ 1 - A - n^i \left( U_i - B_i \right) \right], \quad \hat{k}^a = 2\pi C\alpha \left[ -n^a + U^a + n^i C_i^a \right]. \quad (2.19)
\]

It proves convenient to choose the normalization constant \( 2\pi C\alpha \equiv 1 \) at the observer position \( x^a(\lambda_o) \), and this normalization condition implies that the conformally transformed wavevector can be written as
\[
\hat{k}^a \equiv (1 + \delta v, -n^a - \delta n^a), \quad (2.20)
\]

where the observed angle \( n^a \) measured in the observer rest frame is constant. The product \( \hat{k}^a \hat{k}_a \) is unity at the observer position to all orders in perturbation, and unity everywhere in a homogeneous universe. However, in an inhomogeneous universe it varies along the photon path as fictitious observers measure \( v \) in equation (2.17) and the scale factor \( a \) changes at each point.

For the coordinate transformation in equation (2.5), these perturbations to the wavevector transform as
\[
\delta \tilde{v} = \delta v + 2HT + \frac{d}{dl}T, \quad \delta \tilde{n}^a = \delta n^a + 2HTn^a - \frac{d}{dl}L^a, \quad (2.21)
\]

and we can define two gauge-invariant variables for these perturbations
\[
\delta \nu_x = \delta v + 2H\chi + \frac{d}{dl} \left( \frac{K}{a} \right), \quad \delta \tilde{n}_x^a = \delta n^a + 2H\chi n^a - \frac{d}{dl}G^a, \quad (2.22)
\]

where we used the background photon path \( d/dl = \partial_\tau - n^a\partial_a \) and \( G^a = \gamma^a + C^a \) is a pure gauge term, transforming as \( \tilde{G}^a = G^a - L^a \). The pure gauge term will be absent in observable quantities below.

### 2.4. Gauge-invariant geodesic equation

Having established the gauge-transformation properties of the photon wavevector, we now derive the photon geodesic equation. First, the null condition of the photon wavevector is
\[
0 = \hat{k}^a \hat{k}_a = \left( n^a n_a - 1 \right) + 2 \left( n^a \delta n_a - \delta \nu - A + B_a n^a + C_{ab} n^a n^b \right). \quad (2.23)
\]

and the background relation is trivially satisfied by the construction of the unit directional vector \( n^a \). In terms of the gauge-invariant variables, the null condition implies
\[
n_a \delta n_a^a = \delta \nu + \alpha_\chi - \phi_x - \gamma_a n^a - C_{ab} n^a n^b. \quad (2.24)
\]

Similarly for the geodesic equation \( k^b k^a_{\ b} = 0 \), the background relation is trivially removed, and it yields the propagation equation for the perturbation \( \delta \nu, \delta n^a \). The temporal and spatial components of the geodesic equation are
\[
0 = \hat{k}^a \hat{k}_a ; a = \frac{d}{dl} \delta \nu + \delta \Gamma^0, \quad 0 = \hat{k}^b \hat{k}^a_{\ b} = \frac{d}{dl} \delta n^a + \delta \Gamma^a, \quad (2.25)
\]

where we have defined \( \delta \Gamma^0 \) and \( \delta \Gamma^a \) using the Christoffel symbol \( \hat{k}^b_{\ b} \) based on the conformally transformed metric \( \tilde{g}_{ab} \) as
Rearranging in terms of the gauge-invariant variables, we derive the gauge-invariant geodesic equations for temporal component

\[
\frac{d}{dl}(\delta \nu_x + 2 \alpha_x) = (\alpha_x - \varphi_x)' - (\Psi_{\alpha\beta} + C_{\alpha\beta}) n^\alpha n^\beta, \tag{2.28}
\]

and for spatial component

\[
\frac{d}{dl}(\delta n_x^\alpha + 2 \varphi_x n^\alpha + \Psi^\alpha + 2 C_{\alpha\beta} n^\beta) = (\alpha_x - \varphi_x - \Psi_{\rho\mu} n^\rho - C_{\rho\mu} n^\rho n^\mu)' - \frac{d}{dl}(2 \varphi_x n^\alpha + \Psi^\alpha + 2 C_{\alpha\beta} n^\beta). \tag{2.29}
\]

Fictitious gauge freedoms in \(\delta \nu\) and \(\delta n^\alpha\) are completely removed, and equations (2.24), (2.28), and (2.29) are manifestly gauge-invariant.

### 2.5. Observed source position and redshift

The source galaxy position on the sky is identified by the observed angle \((\theta, \phi)\) in equation (2.12) and the observed redshift \(z\) in equation (2.14). Based on these observables, the observers infer the source position by using the distance-redshift relation in a homogeneous universe, i.e.

\[
\hat{x}_s^a \equiv \left[ \tau_s, \bar{r}_s \right] = \left[ \tau_s, \bar{r}_s \sin \theta \cos \phi, \bar{r}_s \sin \theta \sin \phi, \bar{r}_s \cos \theta \right], \tag{2.30}
\]

where the source position is expressed in a rectangular coordinate, the comoving distance to the source is

\[
\bar{r}_s \equiv \bar{r}(z) = \bar{r}_o - \bar{r}_z = \int_{0}^{z} \frac{dz'}{H(z')}, \tag{2.31}
\]

and a bar is used to indicate that these quantities are computed at the background level. Given a set of cosmological parameters, these quantities are fully determined, and there are no gauge ambiguities associated with coordinate transformations.

However, the real position \(x_o^a\) of the source galaxy is different from the inferred source position \(\hat{x}_s^a\), as the universe is inhomogeneous, affecting the photon propagation, and the components of \(x_o^a\) themselves are gauge-dependent. To represent the source position with respect to the inferred source position, we define the coordinate distortions \((\Delta \tau, \delta \bar{r}, \delta \theta, \delta \phi)\) by expressing the real source galaxy position as

\[
x_o^a \equiv \left[ \tau_z + \Delta \tau, (\bar{r}_z + \delta \bar{r}) \sin (\theta + \delta \theta) \cos (\phi + \delta \phi), (\bar{r}_z + \delta \bar{r}) \sin (\theta + \delta \theta) \sin (\phi + \delta \phi), (\bar{r}_z + \delta \bar{r}) \cos (\theta + \delta \theta) \right]. \tag{2.32}
\]
and it is noted that these coordinate distortions are also gauge-dependent, as we show below. The coordinate distortions can be computed by tracing the photon path backward from the observer and solving for $x^a_s$. First, we consider the source galaxy position $x^a_s$ in a homogeneous universe by integrating the photon wavevector in equation (2.20) over the affine parameter $\lambda$ as

$$x^a_s(\lambda_o) - x^a_o = \left[ \tilde{x}_s - \tilde{x}_o, \tilde{x}_s \right] = \left[ \lambda_s - \lambda_o, (\lambda_o - \lambda_s)n^a \right],$$

(2.33)

and without loss of generality we set $x^a_o = 0$, and $\lambda_o = 0$. The relation in equation (2.33) defines the affine parameter in a given coordinate system as

$$\lambda = \tilde{x} - \tilde{x}_o = -\int_0^\xi \frac{dz'}{H(z')},$$

(2.34)

but in terms of the redshift parameter $1 + \bar{z}(\tau) \equiv 1/a(\tau)$ (or coordinate time $\tau$) of the source position in the background (note that the observed redshift is $z$). Since it depends on the coordinate time of the source position, the redshift parameter transforms $\bar{z} = z - HT$ under the coordinate transformation in equation (2.5), and so does the affine parameter

$$\lambda = \lambda(1 + H_o T_o) - \int_0^\lambda d\lambda' 2HT,$$

(2.35)

where the integration over the affine parameter signifies that the integrand is evaluated along the photon path $x^a_i$.

The coordinate distortions are useful quantities for characterizing the source galaxy position $x^a_i$, as the observer uses the inferred source position $\bar{x}^a_s$ based on the observable quantities. In the same way, it is convenient to define the affine parameter $\lambda_i$, satisfying equation (2.34) in terms of the observed redshift $\bar{z}$. The affine parameter at the source position $x^a_i$ is parametrized as

$$\lambda_i \equiv \lambda_z + \Delta \lambda_i,$$

(2.36)

and the (conformal) time coordinate of the source galaxy position can be rephrased as

$$\tau_i \equiv \tau(\lambda_i) = \tau(\lambda_z + \Delta \lambda_i) + \delta \tau(\lambda_z + \Delta \lambda_i) = \tau_z + \Delta \lambda_i + \delta \tau_z,$$

(2.37)

where the subscript $z$ indicates that quantities are evaluated at the observed redshift (or the affine parameter $\lambda_z$). Since the observed redshift is related to $\tilde{z}$, as

$$1 + z = \frac{1}{a(\tau_z)} = \frac{\lambda_z}{\lambda_i} = \frac{(H_o a_o)}{(H_i a_i)} \equiv \frac{1 + \delta \bar{z}}{a(\tau_i)},$$

(2.38)

substituting equation (2.37) yields that the distortion $\Delta \tau$ in time coordinate is related to the distortion $\delta \bar{z}$ in the observed redshift

$$\Delta \tau = \Delta \lambda_i + \delta \tau_z = \frac{\delta \bar{z}}{H_z},$$

(2.39)

Finally, we evaluate the photon frequency along the photon path $x^a_i$ by using equations (2.13) and (2.17)
\[
2\pi\nu = -k^u u_\alpha = -\frac{\dot{\nu}}{c^2} = \frac{1}{c^2} \left[ 1 + \delta_\nu + \lambda + (V_o - B_0)n^\alpha \right] = \frac{1}{c^2} \left[ 1 + \delta_\nu + \alpha_n + V_o n^\alpha - H \right]. \tag{2.40}
\]

and the distortion in the observed redshift is

\[
\delta z = H_o \delta \tau_o + \left[ \delta_\nu + \alpha_n + V - H \right] \int^\tau_o \dd r, \tag{2.41}
\]

where \( \tilde{r} \) is the comoving line-of-sight distance to the source, \( V = V_o n^\nu \) is the line-of-sight velocity, and \( \tau (\lambda_o) = \tau_o - \delta \tau_o \). We also used equation (2.28) for integration. Spurious spatial gauge freedom in \( \delta z \) is removed and its temporal gauge dependence \( \delta \dot{z} = \delta \nu + H \dot{z} \) leaves the observed redshift \( z \) explicitly gauge-invariant. We define a gauge-invariant variable for the lapse in the observed redshift as \( \delta \dot{\tau}_z = \delta \nu + H \dot{z} \).

3. Relativistic description of galaxy clustering

3.1. Geometric distortions in photon path

Now we derive the geometric distortions \( \delta r, \delta \theta, \delta \phi \) in equation (2.32). Since we clarified the relation between the photon wavevector and the observable quantities, the photon wavevector can be integrated over the affine parameter to obtain the source galaxy position and express it in terms of the observable quantities. The integration over the affine parameter can be subsequently converted into the integration over the mean photon path \( \dd \bar{r} = -\dd \lambda \) in equation (2.34), as we are interested in the linear order effect. Therefore, noting that the affine parameter describing the source galaxy position is \( \lambda_s = \lambda_z + \Delta \lambda_z \), we first integrate the geodesic equation in equation (2.25) to relate the perturbations to the photon wavevector with metric perturbations as

\[
\delta u \bigg|_0 = -2(\Delta z - A_0) - \int_0^{\lambda_s} \dd \bar{r} \left[ A' - \left( B_{a\beta} + C_{a\beta} \right)n^\alpha n^\beta \right],
\]

\[
= -2\alpha_n \left[ -\int_0^{\lambda_s} \dd \bar{r} \left[ (\Delta z - \phi_s)' - \left( \Psi_{a\beta} + C_{a\beta} \right)n^\alpha n^\beta \right] 
-2 H \right] \frac{d}{d\lambda} \left( \dot{z} \right) \bigg|_\lambda \bigg|_0, \tag{3.1}
\]

\[
\delta n^\alpha \bigg|_0 = -\left[ B^a + 2C^a_{\beta n^\beta} \right] \int_0^{\lambda_s} \dd \bar{r} \left( A - B_\beta n^\beta - C_\beta n^\beta n^\gamma \right)^{\alpha^\gamma}
= -\left[ 2 \phi_s n^\alpha + \Psi^a + 2 C^a_{\beta n^\beta} \right] \int_0^{\lambda_s} \dd \bar{r} \left( \Delta z - \phi_s' - \Psi_{\beta n^\beta} - C_\beta n^\beta n^\gamma \right)^\alpha
-2 H \delta n^\alpha \bigg|_0 \frac{d}{d\lambda} \bigg|_0, \tag{3.2}
\]
and then integrate the photon wavevector to obtain the source position

\[
\begin{align*}
\delta r &= x_0^a = \left[ \tilde{r}_z + \delta r_0 + \Delta \lambda_z - \int_0^{\tilde{r}_z} d\tilde{r} \delta u_v, \quad \tilde{r}_v n^a - \Delta \lambda_z n^a + \int_0^{\tilde{r}_v} d\tilde{r} \delta n^a \right] \\
&= \left[ \tilde{r}_z + \delta r_0 + \Delta \lambda_z - \int_0^{\tilde{r}_z} d\tilde{r} \left( \tilde{r}_z - \tilde{r} \right) \delta \Gamma^a, \quad \tilde{r}_z n^a + \tilde{r}_v \delta n^a - \Delta \lambda_z n^a \\
&- \int_0^{\tilde{r}_v} d\tilde{r} \left( \tilde{r}_z - \tilde{r} \right) \delta \Gamma^a \right].
\end{align*}
\]

(3.3)

where quantities \((\delta u_v, \delta n^a)\) are at the observer position. Constructing two unit directional vectors based on the observed angle

\[
\begin{align*}
\vartheta &= \theta (\cos \phi, \cos \phi, -\sin \phi), \\
\varphi &= \frac{1}{\sin \theta} \frac{\partial}{\partial \hat{\mathbf{n}}} \hat{\mathbf{n}} = (-\sin \phi, \cos \phi, 0), 
\end{align*}
\]

(4.3)

the geometric distortions of the source galaxy position can be explicitly computed in terms of metric perturbations as

\[
\begin{align*}
\delta r &= n_a x_0^a = \tilde{r}_z - \Delta \lambda_z + \int_0^{\tilde{r}_v} d\tilde{r} n_a \delta n^a \\
&= \delta r_0 - \frac{\delta \tilde{r}_z}{H_0} + \int_0^{\tilde{r}_v} d\tilde{r} \left( A - B_a n^a - C_{a\beta} n^a n^\beta \right) \\
&= \left( \chi_0 + \delta r_0 \right) - \frac{\delta \tilde{r}_z}{H_0} \\
&+ \int_0^{\tilde{r}_v} d\tilde{r} \left[ \left( \alpha_X - \varphi_X \right) - \left( \Psi_{a\beta} + C_{a\beta} \right) n^a n^\beta \right] - n_a G^a \left[ \right], \\
\end{align*}
\]

(3.5)

\[
\begin{align*}
\delta \theta &= \theta_a x_0^a = \tilde{r}_z \partial_a \delta n^a - \int_0^{\tilde{r}_v} d\tilde{r} \left( \tilde{r}_z - \tilde{r} \right) \partial_a \delta \Gamma^a \\
&= \tilde{r}_z \partial_a \left( \delta n^a + B^a + 2 C_{a\beta} n^\beta \right) - \int_0^{\tilde{r}_v} d\tilde{r} \left[ \partial_a \left( B^a + 2 C_{a\beta} n^\beta \right) \\
&+ \left( \frac{\tilde{r}_z - \tilde{r}}{\tilde{r}} \right) \frac{\partial}{\partial \hat{\mathbf{n}}} \hat{\mathbf{n}} \left( A - B_a n^a - C_{a\beta} n^a n^\beta \right) \right] \\
&= \tilde{r}_z \partial_a \left( \delta n^a + \Psi^a + 2 C_{a\beta} n^\beta \right) - \partial_a G^a \left[ \right] \\
&- \int_0^{\tilde{r}_v} d\tilde{r} \left[ \partial_a \left( \Psi^a + 2 C_{a\beta} n^\beta \right) \\
&+ \left( \frac{\tilde{r}_z - \tilde{r}}{\tilde{r}} \right) \frac{\partial}{\partial \hat{\mathbf{n}}} \hat{\mathbf{n}} \left( \alpha_X - \varphi_X - \Psi_{a\beta} n^a - C_{a\beta} n^a n^\beta \right) \right]. \\
\end{align*}
\]

(3.6)

where we used equation (2.24) for manipulating \(\delta \tau\). The azimuthal distortion \(\tilde{r}_z \sin \theta \delta \phi\) is similar to \(\tilde{r}_z \delta \theta\). It is apparent that these geometric distortions are gauge-dependent quantities. Physically, the radial and angular distortions arise due to the metric perturbations along the photon path and the identification of the source at the observed redshift.

### 3.2. Lensing magnification and luminosity distance

Here we derive two quantities associated with angular distortions of the source galaxy position on the sky. The first quantity, known as the gravitational lensing convergence \(\kappa\), describes the change in the solid angle as part of the distortion in the physical volume. As
such, the convergence itself is not directly associated with observable quantities. The second quantity $D_L(z)$ describes the luminosity distance of a standard candle with known luminosity in the rest frame at the observed redshift. Naturally, the luminosity distance is an observable quantity, and its perturbation $\delta D_L$ is related to the gravitational lensing convergence $\kappa$.

We first compute the gravitational lensing convergence $\kappa$. In galaxy clustering, we only need the change in the solid angle between the observed $\theta \phi$ and the (unobserved) source $\theta \delta \theta \phi \delta \phi$, and the ratio of the solid angles is the Jacobian of the angular transformation or the determinant of the deformation matrix:

$$\frac{\partial (\theta + \delta \theta, \phi + \delta \phi)}{\partial (\theta, \phi)} = \frac{\sin (\theta + \delta \theta)}{\sin \theta} \left[ 1 + \frac{\delta \theta}{\delta \phi} \right] = 1 + \left( \cot \theta + \frac{\delta \theta}{\delta \phi} \right) \delta \phi \equiv 1 - 2\kappa,$$

where we computed the Jacobian only to the linear order in perturbation. At this order, the distortion in the solid angle is completely described by the isotropic expansion in the angle (gravitational lensing convergence), and the angular shear and rotation come at a higher order in the perturbations.

Using equation (3.6), the gravitational lensing convergence can be derived as

$$\kappa = n_a \left( \delta n_a^\alpha + \Psi^a + 2C_\beta^\alpha n^\beta \right)_\theta = \int_0^{\bar{r}} d\bar{r} \frac{n_a \left( \Psi^a + 2C_\beta^\alpha n^\beta \right)}{\bar{r}} + \int_0^{\bar{r}} d\bar{r} \frac{1}{2\bar{r}} \hat{V}_a \left( \Psi^a + 2C_\beta^a e^\beta \right) + \int_0^{\bar{r}} d\bar{r} \left( \frac{\bar{r} - r}{2r_0} \right) \hat{V}^2 \left( \alpha_x - q^a - \Psi^a n^a - C_{ab} n^a n^b \right) - \frac{n_a G^a}{\bar{r}} + \frac{1}{2r_0} \hat{V}_a G^a,$$

where $\hat{V}$ is the angular gradient operator. The gravitational lensing convergence is gauge-dependent, as it is expressed in relation to the observed angular positions to the unobservable source position.

Next, we compute the fluctuation $\delta D_L$ in the luminosity distance. The observed flux $f_{\text{obs}}$ of a source galaxy at the observed redshift $z$ is used to infer its luminosity $L = 4\pi D_L^2(z)f_{\text{obs}}$, where the luminosity distance in a homogeneous universe is $D_L(z) = (1 + z)r$. However, the physical luminosity $L_{\text{phys}} \equiv 4\pi D_L^2(z)f_{\text{obs}}$ of the source is different from the inferred luminosity, as the source galaxy is not at the inferred distance and the photon propagation is affected by fluctuations along the path. We define the dimensionless fluctuation in the luminosity distance $D_L(z) = \tilde{D}_L(z)(1 + \delta D_L)$. As expressed in terms of observable quantities, the fluctuation $\delta D_L$ is an gauge-invariant observable quantity, as we prove below.

Since the luminosity distance is related to the angular diameter distance $D_A(z) = D_L(z)/(1 + z)$, the fluctuation in the angular diameter distance is identical to the fluctuation in the luminosity distance. Thus, we compute the fluctuation in the angular diameter distance by using the geometric distortions we already computed. In the source rest frame, consider a unit area $dA_{\text{phys}}$ that is perpendicular to the observed photon vector $N^a$ parallelly transported along the photon path to the source position. This unit area would appear subtended by a solid angle $d\Omega = \sin \theta d\theta d\phi$ measured by the observer, and it is related to the angular diameter distance as [5]

$$dA_{\text{phys}} = D_A^2(z) d\Omega = \sqrt{-g} \epsilon_{abc} N^a N^b \frac{dx^c}{d\theta} \frac{dx^e}{d\phi} d\theta d\phi,$$
where \( \sqrt{-g} \equiv a^4(1 + \delta g) \) is the metric determinant, \( \epsilon_{abcd} \) is a Levi–Civita symbol, the source position is \( x^a_0 \) in equation (3.3), and the observed photon vector \( N^a = k^a/(k^b u_b) + u^a \) in a FRW frame. The covariant expression in equation (3.9) represents a simple mapping of the solid angle in the observer rest frame to the physical area in the source rest frame defined by the four velocity of the source and perpendicular to the photon wavevector.

Removing the mean angular diameter distance \( D_A = \tilde{r}_z/(1 + z) \), we simplify equation (3.9) to obtain the relation for \( \delta D_L \):

\[
(1 + \delta D_L)^2 = (1 + \delta g)(1 + \delta z) \frac{\epsilon_{abc} \frac{\partial u^a}{\partial x^b} \frac{\partial u^b}{\partial x^c}}{r_z^2 \sin \theta} \frac{d^4 a}{\delta \theta \delta \phi}.
\]

\[
\delta g = \mathcal{A} + C^a_{\alpha}, \quad \hat{u}^a = au^a.
\]

(3.10)

Expanding the equation to the linear order in perturbation, the fluctuation in the luminosity distance is derived as

\[
\delta D_L = \delta z - \kappa + \frac{\delta r}{\tilde{r}_z} + \frac{1}{2} \left( C^a_{\alpha} - C_{\alpha} n^a n^\alpha \right) = \delta z - \kappa + \frac{\delta r}{\tilde{r}_z} + q_{\chi} - \frac{1}{2} C_{\alpha} n^\alpha n^\beta, \tag{3.11}
\]

where we defined two-gauge invariant variables

\[
\delta r = \delta r + n_a G^a \left| \frac{\tilde{r}_z}{r_z} \right|, \quad \kappa = \kappa + \frac{n_a G^a}{2 \tilde{r}_z} \left| - \frac{1}{2\tilde{r}_z} \hat{V}_a G^a \right|.
\]

(3.12)

by removing the gauge-dependent terms in equations (3.5) and (3.8), respectively. Written in terms of gauge-invariant variables, we explicitly verify that the fluctuation \( \delta D_L \) is a gauge-invariant observable, and equation (3.11) recovers the expressions for the luminosity distance computed in the conformal Newtonian gauge [24] and in the synchronous gauge [5]. The observed flux is affected not only by the change \( \kappa \) in the observed solid angle, but also by the change in the radial direction set by the observed redshift.

### 3.3. Physical volume occupied by sources in 4D spacetime

Extending the previous calculation of a unit area in the source rest frame, we now compute the physical 3D volume occupied by the source galaxy at \( x^a_0 \) in 4D spacetime that appears to the observer within the small interval \( dz \) of the observed redshift and the observed solid angle \( d\Omega \) [1, 2, 25]:

\[
dV_{\text{phy}} = \sqrt{-g} \epsilon_{abcd} u^a \frac{\partial x^a}{\partial x^b} \frac{\partial x^b}{\partial x^c} \frac{\partial x^c}{\partial x^d} dz d\theta d\phi \equiv dV_{\text{obs}}(1 + \delta V),
\]

\[
d\tilde{V}_{\text{obs}} = \frac{r_z^2 d\Omega}{H_z(1 + z)^2}.
\]

(3.13)

where the dimensionless volume fluctuation \( \delta V \) is defined with respect to the volume \( d\tilde{V}_{\text{obs}} \) inferred by the observer. Expanding the covariant expression to the linear order in perturbation, we derive
\[
\delta V = 3 \delta z + \delta g + 2 \frac{\delta r}{r} - 2 \kappa + H \frac{\delta}{\delta t} \delta r - A + U^a n_a \\
= 3 \delta z + 3 \delta \phi + 2 \frac{\delta r}{r} - 2 K + H \frac{\delta}{\delta t} \delta r + V + n_a \Psi^a \\
= 3 \delta z + \alpha \chi + 2 \frac{\delta r}{r} - 2 K - H \frac{\delta}{\delta t} \left( \frac{\delta z}{V} \right) + V - C_{\alpha \beta} n^\alpha n^\beta. 
\] (3.14)

This equation is manifestly gauge-invariant, providing the linear-order relativistic effect in the volume distortion. In equation (3.13), the physical volume is mapped by using the three independent variables that are the observable quantities \((z, \theta, \phi)\) in the observer rest frame. Therefore, the derivative terms in equation (3.13) (and hence in equation (3.14)) are the partial derivatives with the other observed quantities held fixed. For example, the derivative with respect to the observed redshift represents the change in response to the variation in the observed redshift, which is the line-of-sight derivative along the past light cone. To the linear order in perturbation, it is the background photon path, involving not only the spatial derivative, but also the time derivative. As the inferred volume is \(dV_{\text{phys}}\), the volume distortion \(\delta V\) in the physical volume \(dV_{\text{phys}}\) is expected to have contributions from each component in \(dV_{\text{obs}}\). Notably, the contribution \(3 \delta z\) from the comoving factor \((1 + z)^3\) in \(dV_{\text{obs}}\), the contribution \(2 \delta r / r\) from \(\delta^2 r / r^2\), the contribution \(z\kappa\) from \(d\Omega\), the contribution \(H \delta \delta r\) from the change of the radial displacement at the observed redshift, and the remaining contribution from defining the source rest frame. As written in terms of geometric distortions, the notation is physically transparent.

3.4. Observed galaxy number density and galaxy clustering

Given the observed redshift and angle in observation, the volume element \(dV_{\text{obs}}\) is used to infer the volume occupied by the source galaxies in the sky, and the observed number density is obtained by counting the number of galaxies within the observed redshift and solid angle:

\[
dN_g^{\text{obs}}(z, \vec{\eta}) = n_g^{\text{obs}} dV_{\text{obs}} = n_g dV_{\text{phys}}, \quad n_g^{\text{obs}} = n_g (1 + \delta V),
\] (3.15)

where \(n_g\) is the physical number density of source galaxies. It is evident that the volume distortion \(\delta V\) in equation (3.14) always contributes to the observed galaxy number density, and its contribution is collectively described as the volume effect [26, 27].

In general, the physical number density \(n_g\) of source galaxies can be written in terms of the mean and the intrinsic fluctuation as

\[
n_g = \bar{n}_g (t_p) \left( 1 + \delta_{g}^{\text{int}} \right), \quad \bar{n}_g (t_p) \equiv \left\{ n_g \right\}_{t_p}, \quad \left\langle \delta_{g}^{\text{int}} \right\rangle_{t_p} = 0,
\] (3.16)

where the mean \(\bar{n}_g\) and the fluctuation \(\delta_{g}^{\text{int}}\) are defined over a hypersurface denoted with \(t_p\). Galaxies are tracers of the underlying matter distribution, and the relation between the galaxy fluctuation \(\delta_{g}^{\text{int}}\) and the matter fluctuation \(\delta_m\) is called galaxy bias. The galaxy bias is known to be linear on large scales \(\delta_{g}^{\text{int}} = b \delta_m\) [28]. However, in the relativistic context, this linear bias relation is ambiguous, as the choice of gauge condition for \(\delta_m\) is unspecified. A physically meaningful choice of \(t_p\) is the proper-time hypersurface, described by the comoving-synchronous gauge in a pressureless medium:

\[
\delta_{g}^{\text{int}} = b \delta_{\mu}^{\text{int}},
\] (3.17)

where \(\delta_{\mu}^{\text{int}}\) is the matter density fluctuation in the comoving-synchronous gauge (\(\delta_{\mu}^{\text{int}} = \delta_\mu\)). It is noted that the matter fluctuation in the comoving-synchronous gauge represents one in the proper-time hypersurface only when the universe is dominated by a pressureless medium, and
it becomes more subtle beyond the linear order [29]. Galaxy formation is a local process, and its dynamics are affected by the long wavelength modes through the change in the local clock that can be measured without any knowledge beyond the local area [7]. This biasing scheme is consistent with other recent studies [3–5, 7, 8, 14].

The other contribution to galaxy clustering, the source effect [27], is associated with the physical quantities of the source galaxies, but expressed in terms of observable quantities such as the observed redshift and flux. The mean galaxy number density is represented in the proper time hypersurface, or the rest frame of galaxies, which is different from the observed redshift. So when the observed number density is expressed at the observed redshift, the physical number density is

\[ n_g = \hat{n}_g(z) \left[ 1 - e \delta \xi_{tp} \right] \left( 1 + \delta_g^{\text{int}} \right), \]

\[ e = -\frac{d \ln \hat{n}_g}{d \ln (1+z)}, \tag{3.18} \]

where the coefficient \( e \) is the evolution bias and the distortion in the observed redshift is evaluated at the proper time \( t_p \). Note that a galaxy sample with constant comoving number density would have \( e = 3 \). Additional contributions arise when we characterize the galaxy sample by using its inferred luminosity as the threshold. As discussed in section 3.2, the inferred luminosity is different from the physical luminosity, and its contribution is

\[ n_g = \hat{n}_g(L) \left[ 1 - t \delta D_L \right] \left( 1 + \delta_g^{\text{int}} \right), \]

\[ t \equiv -2 \frac{d \ln \hat{n}_g}{d \ln L}, \tag{3.19} \]

where the coefficient \( t \) describes the slope of the luminosity function. If the differential luminosity function \( d\hat{n}_g(L) \propto L^{-s} \) is well approximated by a constant slope \( s \), we have \( t = 2(s-1) \). It is often the case that the cumulative luminosity function slope \( p = d \log_{10}\hat{n}_g(M)/dM \) is expressed in terms of magnitude \( M = \text{constant} -2.5 \log_{10}(L/L_0) \), and we have \( p = 0.4(s-1) \) and \( t = 5p \).

The observed galaxy number density is physically well-defined and is expressed in terms of the observed redshift, angle, and the number of galaxies. Collecting all the contributions to the observed galaxy number density, we can relate it to various perturbation contributions as

\[ n_g^{\text{obs}}(z, \hat{n}) = \hat{n}_g(z) \left( 1 + \delta_g^{\text{int}} \right) \left( 1 - e \delta \xi_{tp} \right) \left( 1 - t \delta D_L \right). \tag{3.20} \]

Depending on how the galaxy sample is defined in terms of other observable quantities, additional terms from the source effect may be present in equation (3.20). While all the perturbation contributions can be computed, we have no means to compute the mean number density \( \hat{n}_g(z) \) of the observed galaxy sample, defined in equation (3.16). In observation, the observed mean number density is obtained by averaging the number density over the survey area,

\[ \hat{n}_g(z) \equiv \frac{1}{\Omega} \int_{\Omega} d^2 \hat{n}^{\text{obs}}_g(z, \hat{n}), \tag{3.21} \]

which can introduce additional fluctuations in the estimate of the mean galaxy number density \( \hat{n}_g(z) \), since perturbations of wavelength larger than the survey size will be absorbed in the observed mean \( \hat{n}_g \). The observed galaxy fluctuation is then obtained by using the observed density.

\[ n_g^{\text{obs}}(z, \hat{n}) = \hat{n}_g(z) \left( 1 + \delta_g^{\text{int}} \right) \left( 1 - e \delta \xi_{tp} \right) \left( 1 - t \delta D_L \right). \tag{3.20} \]

In [1, 2], we adopted the simplest approach for biasing \( n_g = F[V], \) i.e. the physical galaxy number density is some unknown function of the matter density at the same spacetime. While it lacks any gauge issues, it is rather physically restrictive, as the galaxy number density evolution is driven only by the matter density evolution: \( n_g = \hat{n}_g(z)(1 + b m) \) and \( e = 3b \), where \( m = \delta_m - 3 \delta \) is a gauge-invariant matter density fluctuation at the observed redshift. Equation (3.17) provides a more physically motivated biasing scheme than the one in [1, 2].
mean number density as
\[
\delta_g^{\text{obs}}(z, \mathbf{n}) \equiv \frac{n_g^{\text{obs}}(z, \mathbf{n})}{\bar{n}_g(z)} - 1, \quad n_g^{\text{obs}}(z, \mathbf{n}) = \bar{n}_g(z) \left( 1 + \delta_g^{\text{obs}} \right).
\] (3.22)

Assuming no residual fluctuation in the mean number density, i.e., \( \bar{n}_g(z) = n_g(z) \), the galaxy fluctuation can be written as
\[
\delta_g^{\text{obs}}(z, \mathbf{n}) = \delta_g^{\text{int}} + \delta V - e\delta \xi_\mu + t\delta D_L
\]
\[
= b\delta m^\mu - e\delta \xi_\mu - t\delta D_L + 3\delta \xi^\mu + \alpha_\mu + 2\varphi_\mu + 2\delta r^\mu\bar{r} - 2K - H^2 \frac{1}{\bar{n}^2} \left( \frac{\delta}{\bar{n}} \right)
\]
\[
+ V - C_{ab} n^a n^b.
\] (3.23)

This equation is the main result and contains all the linear-order relativistic effects that come into play in galaxy clustering. Contributions to galaxy clustering are physically split into the source and the volume effects, and they arise as the photon propagation is affected by subtle relativistic effects between the source and observer positions. It is apparent in equation (3.23) that the observed galaxy fluctuation receives contributions not only from scalar perturbations, but also from vector and tensor perturbations.

4. Cosmological probes: observed galaxy two-point statistics

Here we derive the galaxy two-point statistics that are measurable in galaxy surveys. By definition, the galaxy fluctuation \( \delta_g^{\text{obs}}(z, \mathbf{n}) \) in equation (3.22) vanishes upon averaging over the survey area. The galaxy two-point statistics \( \langle \delta_g^{\text{obs}} \delta_g^{\text{obs}} \rangle \), therefore, provide crucial ways to probe cosmology in galaxy surveys. If the underlying distribution is Gaussian, the two-point statistics (the power spectrum or the correlation function) contain the complete information about the distribution, and in practice the Universe is, to a good approximation, Gaussian on large scales, where the relativistic effect is significant. Higher-order statistics such as the galaxy bispectrum describes the deviation from the Gaussianity, providing crucial clues about the initial condition at very early time. However, it requires the second-order relativistic calculation [30–33], and here we focus on the linear-order relativistic effect in galaxy clustering and the derivation of the observed galaxy two-point statistics. Due to the restriction in length, we refer the reader to papers cited in this section for plots of two-point statistics and their physical explanation.

4.1. Linearized Einstein equations and computation of the observed galaxy fluctuation \( \delta_g^{\text{obs}} \)

Being constructed solely from observable quantities, the observed galaxy fluctuation \( \delta_g^{\text{obs}} \) is gauge-invariant, as verified in equation (3.23). Therefore, it can be evaluated with any choice of gauge conditions. Here we provide a simple way to compute the observed galaxy fluctuation, assuming general relativity\(^4\). The computation of galaxy two-point statistics will follow in the subsequent sections. For simplicity, we assume that there are no vector or tensor modes at the initial condition and ignore their contributions to \( \delta_g^{\text{obs}} \) from now on.

\(^4\) Indeed, \( \delta_g^{\text{obs}} \) in equation (3.23) is derived without using Einstein equations, i.e. the general relativistic description of galaxy clustering is formulated in a general metric representation, and the only assumption was that the photons follow the geodesic, such that it is not restricted to Einstein’s gravity, but applicable to other alternative theories of modified gravity (see, e.g., [16]).
In a flat universe with pressureless medium (CDM and baryons on large scales; \( p = \pi_{\text{ab}} = 0 \) in equation (2.3)), the Einstein equations can be arranged in terms of gauge-invariant variables (e.g. [21, 22, 34]) as

\[
V V q_\chi = - \frac{3H_0^2}{2} \Omega_{\Lambda m} \delta \phi - \frac{\delta \phi}{a}, \quad q_\chi' = H \alpha_\nu, \quad \alpha_\nu = -q_\chi, \quad (4.1)
\]

and the conservation equations yields

\[
v_\nu' + Hv_\nu = \alpha_\nu, \quad \delta \chi_\nu = -3q_\chi - v_\nu, \quad (4.2)
\]

where we defined two additional scalar gauge-invariant variables, two curvature perturbations in the comoving gauge \((v = 0)\), \(\alpha_\nu = \alpha - (av)'/a\) and \(q_\nu = q - Hv\). Since \(\alpha_\nu = \alpha_\nu + (av)_v'/a\), the conservation equation for momentum implies \(\alpha_\nu = 0\), i.e. for a flat universe with pressureless medium, the comoving gauge \((v = 0)\) is identical to the synchronous gauge \((\alpha = 0)\), and from the Einstein equation the comoving-gauge curvature perturbation is conserved \(\phi_\nu' = 0\) [35, 36].

Furthermore, in this circumstance, there exists ‘Newtonian correspondence’ that relates the fully relativistic quantities to the Newtonian quantities. In a flat universe with pressureless medium, the Newtonian matter density \(\delta_\text{m}\) is identical to the comoving gauge matter density \(\delta_v\), and the Newtonian velocity \(v_\nu\) and potential \(\phi_\nu\) are identical to the conformal Newtonian gauge quantities \(\chi_v\) and \(\phi_\chi\) [35, 37]. Therefore, we adopt a simple notation \(\delta_\text{m} \equiv \delta_v\), \(v_\nu \equiv v_\nu\), and \(\phi_\nu \equiv \phi_v = -\alpha_\nu\), but note that it is fully relativistic and no Newtonian approximation is made. The Einstein and the conservation equations can be written in terms of these quantities:

\[
\phi_\nu = \frac{3H_0^2}{2} \Omega_m \delta_\text{m}, \quad v_\nu = -\delta v_{\nu} = -\frac{Hf}{k^2} \delta_\text{m}, \quad V = -\hat{\mathbf{n}} \cdot \nabla v_\nu = i\frac{Hf}{k} \frac{\delta_\text{m}}{k} \mu_k, \quad (4.3)
\]

where the logarithmic growth rate is \(f = d \ln \delta_\text{m}/d \ln a\) (we used \(d/d\tau = H d/d \ln a\), \(\mu_k\) is the cosine angle between the line-of-sight direction and the wavevector \(\mathbf{k} = \mathbf{k} \cdot \hat{\mathbf{n}} / k\), and the equations are in Fourier space.

As the observed galaxy fluctuation in equation (3.23) is composed of many perturbation variables, we first express those perturbation components in terms of \(\delta_\text{m}, v_\nu,\) and \(\phi_\nu\) [14, 17]

\[
\delta_\chi_\nu = V + \phi_\nu + \int_0^\nu d\nu' 2\phi_\nu', \quad \delta_\nu = -\frac{\delta_\chi_\nu}{H}, \quad \delta_\nu' = -V + 1 + z \frac{\partial V}{\partial \nu} = \delta_\chi_\nu + \frac{1 + z}{H} \frac{\partial H}{\partial \nu} \delta_\nu, \quad \delta_\nu'' = \delta_\nu' + Hv_\nu, \quad (4.4)
\]

where we have ignored quantities at the observer position that can be absorbed to the observed mean number density \(\bar{n}_g\) [1, 2]. Furthermore, since the observed galaxies are only those along the past light cone of photons, the partial derivative with respect to the observed redshift in equations (3.23) and (4.4) is

\[
\frac{\partial}{\partial \zeta} = \frac{1}{H} \frac{d}{d\nu} = -\frac{1}{H} \left( \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \nu} \right), \quad (4.5)
\]

The reason for the notation is because we keep the other observable quantities \((\theta, \phi)\) fixed. However, it involves not only the spatial derivative, but also the time derivative, as it is literally the variation of the observed redshift.
To facilitate the computation, we define the transfer functions \( T_f(k, z) \) that relate the amplitude of a perturbation variable \( Y(k, z) \) of Fourier mode \( k \) at \( z \) with the initial conditions at a very early time. In the linear regime, all the perturbations at each wave mode grow only in amplitude, without changing its phase set by the initial conditions. This deterministic growth of perturbation variables is captured by the transfer function, and the phase information in the initial condition is characterized by the curvature perturbation \( \mathcal{R}(k) \) in the comoving gauge during the inflationary period\(^5\). For example, the transfer functions for the matter density fluctuation \( \delta_m \) and the gravitational potential \( \phi_N \) at \( z \) are

\[
\delta_m(k, z) = T_m(k, z) \mathcal{R}(k), \quad \phi_N(k, z) = T_{\phi_N}(k, z) \mathcal{R}(k) = \hat{W}_{\phi_N} T_m(k, z) \mathcal{R}(k),
\]

where we defined a conversion function \( \hat{W}_\mathcal{R} \) that relates the transfer function \( \hat{W}_\mathcal{R} \) for a perturbation variable \( Y \) to the transfer function \( T_m \) for the matter density fluctuation (hence \( \hat{W}_{\phi_N} = 1 \)). Other conversion functions are

\[
\hat{W}_{\phi_N} = \frac{3H_0^2 \Omega_m}{2 \pi^2}, \quad \hat{W}_{\phi} = -\frac{H \partial}{k^2 \varphi}, \quad \hat{W}_\mathcal{R} = (f - 1) \hat{W}_{\phi_N}, \quad \hat{W}_\mathcal{K} = \left( \frac{z}{\pi \Omega} \right) \hat{W}_{\phi_N} \hat{\mathcal{K}}^2.
\]

Transfer functions of other perturbation variables such as \( \delta \chi \), \( \delta \chi_r \), and so on can be computed in terms of the conversion functions \( \hat{W}_\mathcal{R} \) in equation (4.7) by using the relations in equation (4.4).

The power spectrum of these perturbation variables \( Y_j(k, z) \) can be computed as

\[
\left\langle Y_j(k, z_1) Y_j^*(k', z_2) \right\rangle = (2\pi)^3 \delta^3(k + k') T_f(k, z_1) T_f(k, z_2) P_R(k),
\]

where the primordial curvature power spectrum is characterized by the scalar amplitude \( A_s \) and the spectral index \( n_s \) at a pivot scale \( k_0 \)

\[
\Delta^2_R(k) = \frac{k^3}{2\pi^2} P_R(k) = A_s \left( \frac{k}{k_0} \right)^{n_s-1+\frac{1}{2} \frac{dn_s}{d\ln k} \ln \left( \frac{k}{k_0} \right)}.
\]

Since the observed galaxy fluctuation in equation (3.23) is a linear combination of many perturbation variables (some of which are also linear combinations or involve the line-of-sight integration of other perturbation variables), it proves convenient to write the observed galaxy fluctuation as

\[
\delta_{\text{obs}}(x) = \sum_i \int \frac{d^3k}{(2\pi)^3} \int_0^{\tilde{r}_i} d\tilde{r} \mathcal{E}_i(\tilde{r}) T_f(k, \tilde{r}) \mathcal{R}(k) e^{i k \cdot x}, \quad x_s = \tilde{r} \hat{n},
\]

where \( \tilde{r}_i \) is the comoving distance to the source galaxy (i.e. \( \tilde{r}_i = r_i \)), \( x = \tilde{r} \hat{n} \), and the index \( i \) runs for all the components in equation (3.23). \( \mathcal{E}_i \) is the Dirac delta function if the perturbation variable \( Y_i \) is a local function such as \( \phi_N, \chi_N \), and so on, while \( \mathcal{E}_i \) is unity if it involves the line-of-sight integration such as \( \mathcal{K} \).

\(^5\)The comoving-gauge curvature perturbation in our notation (equation (2.2)) is \( \zeta = \mathcal{R} \), i.e. the curvature perturbation \( \varphi \) in the comoving-gauge condition (\( \omega = 0 \)), but \( \mathcal{R} \) is more commonly used in literature in defining the transfer functions. Moreover, it is denoted as \( \zeta \) in some literature, but care must be taken as \( \zeta \) is often used for the curvature perturbation \( \varphi \) in the uniform-matter gauge (\( \delta = 0 \)), i.e. \( \zeta = \varphi_N \) in our notation.
4.2. Galaxy angular power spectrum $C_l$

While galaxy redshift surveys provide information on the radial position of galaxies (based on the observed redshift), two-dimensional angular statistics may be used to probe cosmology, for example, when the radial information is less reliable due to photometric redshift measurements. By counting the number of galaxies $dN^\text{obs}_g(\hat{n})$ within the observed solid angle $d\Omega$, the observed angular galaxy number density $n^\text{obs}_{g,2D}(\hat{n})$ and its fluctuation $\delta^\text{obs}_{g,2D}(\hat{n})$ are defined as

$$dN^\text{obs}_g(\hat{n}) = n^\text{obs}_{g,2D}(\hat{n})d\Omega, \quad \delta^\text{obs}_{g,2D}(\hat{n}) = \frac{n^\text{obs}_{g,2D}(\hat{n})}{\bar{n}^\text{obs}_{g,2D}} - 1, \quad \bar{n}^\text{obs}_{g,2D} = \frac{N^\text{tot}_g}{\Omega},$$

where $N^\text{tot}_g$ is the total number of observed galaxies within the survey area $\Omega$ in angle. The angular fluctuation of the observed galaxy number density is then decomposed in terms of spherical harmonics as

$$\delta^\text{obs}_{g,2D}(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n}), \quad a_{lm} = \int d^2\hat{n} Y_{lm}^*(\hat{n})\delta^\text{obs}_{g,2D}(\hat{n}),$$

$$C_l = \sum_m \frac{\bar{n}^\text{obs}_{g,2D}}{2l+1} \left\langle a_{lm} a_{lm}^* \right\rangle,$$

and the angular power spectrum $C_l$ is the ensemble average of the decomposed angular coefficients. The angular fluctuation and the angular power spectrum are constructed purely based on the observable quantities: the observed angle $\hat{n}$ and the number of galaxies $dN^\text{obs}_g(\hat{n})$.

In order to compare this to the observation, we need to compute the theoretical predictions for $\delta^\text{obs}_{g,2D}(\hat{n})$ and $C_l$. Since the observed angular galaxy number density $n^\text{obs}_{g,2D}(\hat{n})$ is just the volume average of the three-dimensional galaxy number density $n^\text{3D}_g(z, \hat{n})$ in equation (3.15):

$$n^\text{obs}_{g,2D}(\hat{n}) = \int dz \frac{\bar{f}^2(z)}{(1+z)^3 H(z)} n^\text{3D}_g(z, \hat{n}),$$

the angular fluctuation of the observed galaxy number density can be related to the three-dimensional galaxy fluctuation in equation (3.23) as

$$\delta^\text{obs}_{g,2D}(\hat{n}) = \int dz P_z(z) \delta^\text{obs}_{g,3D}(z, \hat{n}), \quad P_z(z) = \frac{\bar{f}^2(z)}{N_g^\text{tot} (1+z)^3 H(z)} \bar{n}^\text{3D}_g(z),$$

where we defined the normalized redshift distribution $P_z(z)$ of the galaxy sample. It is also noted that the dimensions are different for the angular and the three-dimensional galaxy number densities. Therefore, using the decomposition of $\delta^\text{obs}_g(x_s)$ in equation (4.10), the observed galaxy angular power spectrum can be computed as

$$C_l = \left\langle a_{lm} a_{lm}^* \right\rangle = 4\pi \int \frac{dk}{k} \Delta^2(k) T_l^2(k),$$

Here we use $n^\text{3D}_g$ instead of $n^\text{obs}_g$ (and similarly so for its fluctuation), because the three-dimensional quantities may not be available in photometric surveys to be ‘observed.’ In this case, they have to be modeled.
where we defined the angular multipole function

\[ T_\ell(k) = \sum_l \int d\bar{r} \mathcal{P}_\ell (\bar{r}) \int d\bar{z} \bar{E}_\ell(k, \bar{r}) (k\bar{r}). \]  

(4.16)

and used the partial wave expansion

\[ e^{\mathbf{k} \cdot \mathbf{x}} = 4\pi \sum_{lm} \mathcal{P}_l^m(k) Y^m_l(\hat{k}) Y^m_{lm}(\hat{n}), \quad \hat{\mathcal{V}}^2 Y^m_{lm}(\hat{n}) = -l(l+1)Y^m_{lm}(\hat{n}). \]  

(4.17)

Accounting for the relativistic effect in galaxy clustering, the angular power spectrum was computed (see [1, 3, 4, 8, 15] for plots and detailed explanation). Since the angular power spectrum is a projected quantity, the line-of-sight velocity contribution is suppressed in \( \delta_g^{\text{obs}} \), while the gravitational lensing contribution accumulates if the source distribution is located at higher redshift. The gravitational potential contribution is small, but it is dominant over other contributions on large scales. Therefore, the relativistic effect is important on scales \( l \sim k/\bar{r} \sim \mathcal{H}/\bar{r} \), but the cosmic variance accordingly grows larger. The calculation can be readily extended to the angular cross-power spectrum, in which two different galaxy samples or the same galaxy sample but at two different redshift distributions are correlated. In particular, the relativistic effect in galaxy clustering may be isolated with a clever choice of redshift bins (see [3, 38, 39]), while the detection significance needs to be quantified with a realistic covariance matrix, as the radial bins are not independent.

### 4.3. Galaxy power spectrum \( P_g(k) \)

The initial perturbations generated during the inflation epoch are best characterized by its power spectrum such as the comoving-gauge curvature power spectrum \( \mathcal{P}_g(k) \). An understanding of the galaxy power spectrum \( P_g^{\text{obs}}(k) \) measurements is one of the main goals in galaxy redshift surveys, in which galaxy positions are mapped with three-dimensional information. There exist a few complications in predicting the observed galaxy power spectrum. Since the power spectrum is inherently non-local, we have to explicitly account for the survey geometry in consideration. Furthermore, the observed galaxy fluctuation \( \delta_g^{\text{obs}} \) in equation (3.23) receives contributions from the fluctuations along the line-of-sight direction, which are nearly angular quantities and ill-described by the three-dimensional power spectrum. For simplicity, we ignore these complications and proceed to provide a simplified version of the observed galaxy power spectrum that just include the functions of local terms in equation (3.23).

With this simplification and using the Einstein equation, the observed galaxy fluctuation in equation (3.23) can be expressed in terms of its Fourier components as

\[ \delta_g^{\text{obs}}(\mathbf{z}, \hat{n}) \approx \int \frac{d^3k}{(2\pi)^3} e^{\mathbf{k} \cdot \mathbf{x}} \left[ \delta_{\ell}^{\text{Newt}}(\mathbf{k}, \mathbf{z}) + \frac{\mathcal{P}}{(k/\mathcal{H})^2} \delta_m(\mathbf{k}, \mathbf{z}) - i\mu_t \frac{R}{k/\mathcal{H}} \delta_n(\mathbf{k}, \mathbf{z}) \right]. \]  

(4.18)
where the two dimensionless coefficients $P$ and $R$ are defined as \(^7\)

$$
P = ef - \frac{3}{2} \Omega_m(z) \left[ e + f - \frac{1 + z \frac{dH}{dz}}{H} + (t - 2) \left( 2 - \frac{1}{H \tau} \right) \right],$$

$$
R = f \left[ e - \frac{1 + z \frac{dH}{dz}}{H} + (t - 2) \left( 1 - \frac{1}{H \tau} \right) \right].
$$

(4.19)

and the Newtonian description of the observed galaxy fluctuation in the redshift space is \(^{28}\)

$$
\delta^\text{Newt.}_g(k, z) = b \delta_m(k, z) - \mu^2 \frac{k^2 V_V(k, z)}{H} = \left( b + f \mu^2 \right) \delta_m(k, z).
$$

(4.20)

Apparent from the spatial dependence of $R$ and $P$, these two new coefficients in the relativistic description result from the velocity and the gravitational potential contribution to galaxy clustering (see \([5, 14]\) for the time-evolution of $P$ and $R$). It is noted that these coefficients are derived by assuming general relativity, and their values and time evolution differ in other gravity theories \([16]\). The Newtonian description $\delta^\text{Newt.}_g$ is derived from equation (3.23) by ignoring the gravitational potential ($\propto P \frac{\delta m}{H^2}$) and the velocity ($\propto R \frac{\delta m}{k}$) contributions, and the redshift-space distortion term in $\delta^\text{Newt.}_g$ comes from $-H \frac{\partial}{\partial z} (\delta z/\tau)$ in equation (3.23).

The observed galaxy power spectrum can be obtained by taking the ensemble average of the square bracket in equation (4.18)

$$
P^\text{obs}_g(k, z) = P^\text{Newt.}_g(k, z) + \left[ \frac{\mu^2 R^2}{(k/H)^2} + \mu^2 R^2 + 2bP + 2f \mu^2 P \right] P_m(k, z),
$$

(4.21)

where the Newtonian galaxy power spectrum is computed under the distant-observer approximation as

$$
P^\text{Newt.}_g(k, z) = \left( b + f \mu^2 \right)^2 P_m(k, z), \quad P_m(k, z) = T_m^2(k, z) P_R(k).
$$

(4.22)

The relativistic galaxy power spectrum was computed (see \([2, 5, 7, 12, 14, 16]\) for plots and detailed explanation). While the Newtonian contribution in equation (4.20) falls as $k^n$ on large scales, the gravitational potential $P$ and the velocity $R$ contributions becomes larger on large scales, as their power spectra scale with $-b\mu^2$ and $-R$, respectively. Naturally, these relativistic effects are dominant on large scales, and hence it is difficult to measure in low-redshift galaxy surveys. However, with the multi-tracer technique \([20]\), the cosmic variance, i.e. the dominant source of measurement uncertainties on large scales, can be removed, and these relativistic effects in galaxy clustering can be measured with high significance \([14]\).

Last, the effect of the primordial non-Gaussianity is also a relativistic effect in galaxy clustering and can be readily implemented in the relativistic formula \([5, 8, 14]\).

The power spectrum analysis is often performed by embedding the observed sphere in a cubic volume and by taking Fourier transformation of $\delta^\text{obs}_g$. This procedure practically assumes the distant-observer approximation, and the equations in this subsection are valid only in the flat-sky limit, which breaks down on large scales \([18]\). A more careful power spectrum analysis in observation can be performed without the distant-observer approximation, but its application has been so far limited to the Newtonian redshift-space power spectrum \([40]\). Despite these shortcomings, the power spectrum analysis in this section

\(^7\) It is noted that the coefficient $R$ should be distinguished from the comoving curvature $R(k)$.
provides the main relativistic effect in galaxy clustering and its detectability in future surveys [14].

4.4. Spherical galaxy power spectrum $S_l(k)$

To overcome the shortcoming of the power spectrum analysis, an alternative analysis has been developed based on the radial and angular eigenfunctions of the Helmholtz equation [41–43], while its application was limited to the Newtonian expression in equation (4.20).

The angular fluctuation in the observed galaxy fluctuation $\delta_g^{\text{obs}}$ is decomposed in terms of spherical harmonics for all-sky analysis, which naturally implements the angular contributions to $\delta_g^{\text{obs}}$ such as $\mathcal{K}$. The radial fluctuation in $\delta_g^{\text{obs}}$ is decomposed in terms of the spherical Bessel function for spectral Fourier analysis.

Using spherical Fourier analysis, the observed galaxy number density $n_g^{\text{obs}}$ can be written in terms of its spherical Fourier mode $n_{lm}(k)$ as

$$n_g^{\text{obs}}(z, \hat{n}) = \int_0^\infty dk \sum_{lm} \sqrt{\frac{2}{\pi}} j_l(k\hat{n}) Y_{lm}(\hat{n}) n_{lm}(k),$$

$$n_{lm}(k) = \int d^3x \sqrt{\frac{2}{\pi}} j_l(k\hat{r}) Y^*_{lm}(\hat{n}) n_g^{\text{obs}}(x). \quad (4.23)$$

Since the mean galaxy number density evolves in time, we define the survey window function $N(z)$ (or the radial selection function) to separate the radial fluctuation from the mean variation as

$$n_g(z) \equiv n_g N(z), \quad V_r = 4\pi \int dz \frac{r^2}{H} N(z), \quad P_s(z) = 4\pi \frac{r^2}{V_r} N(z), \quad (4.24)$$

where the overall mean number density in the survey is $n_g \equiv N_{\text{tot}}/V_r$. As this radial distribution of the mean number density only affects the spherical monopole $n_{00}(k)$, the spherical Fourier mode of the observed galaxy fluctuation and its spherical power spectrum can be obtained as

$$\delta_{lm}(k) = \frac{n_{lm}(k)}{n_g}, \quad \left\langle \delta_{lm}(k)\delta^*_{lm}(k') \right\rangle = \delta_{ll'}\delta_{mm'} S_l(k, k'). \quad (4.25)$$

If $n_g^{\text{obs}}(z, \hat{n}) \propto \delta_n(x)$, for example, the spherical power spectrum is then $S_l(k) = P_n(k)$, where $S_l(k, k') = \delta^2(k - k')S_l(k)$.

Generalizing the spherical Fourier analysis to the relativistic description in equation (3.23), the spherical Fourier mode of $\delta_g^{\text{obs}}$ is

$$\delta_{lm}(k) = i^l \int d^3\ln \frac{k'k}{2\pi^2} \int d^3\hat{k}' \mathcal{R}(\hat{k}') Y^*_{lm}(\hat{k}') \mathcal{M}_l(k', k), \quad (4.26)$$

and the spherical galaxy power spectrum becomes

$$S_l(k, k') = 4\pi \int d^3\ln \Delta \mathcal{R}(\hat{k}) \mathcal{M}_l(\hat{k}, k) \mathcal{M}_l(\hat{k}', k'), \quad (4.27)$$

where the spherical multipole function $\mathcal{M}_l(k', k)$ is defined as

$$\mathcal{M}_l(k', k) = \sqrt{\frac{2}{\pi}} \int_0^\infty d\hat{r} r^2 N(\hat{r}) k_j(k\hat{r}) \sum \int_0^{\hat{r}} d\hat{r}' \mathcal{P}_l(\hat{r}, \hat{r}', \hat{r}) T_{lj}(k', \hat{r}) j_l(k'\hat{r}). \quad (4.28)$$
The relativistic spherical galaxy power spectrum was computed (see [17] for details). All the relativistic effects in galaxy clustering are naturally implemented in the spherical Fourier analysis, and the spherical power spectrum \( S_l(k) \) reduces to the flat-sky power spectrum \( P(k) \) on small scales. The great advantage in this approach is the spectral Fourier analysis on a sphere, providing the most natural way to describe the relativistic effect in galaxy clustering on large scales. However, a complication is that the observed data \( \delta_k^{\text{obs}}(z, \hat{n}) \) needs to be processed as a function of cosmological parameters, because it requires the conversion of a distance \( r \) to a Fourier mode \( k \), using the observed redshift and angle\(^8\). The current power spectrum analysis produces the observed galaxy power spectrum based on the fiducial cosmology and neglects its model dependence. However, given the measurement uncertainties in the current surveys, these systematic errors are negligible, as long as the fiducial model is close to the best-fit cosmology from the measurements.

5. Conclusion and future prospects

We have provided a pedagogical derivation of the general relativistic description of galaxy clustering and computed the galaxy two-point statistics. The gauge-invariance of individual equations is explicitly verified to show that the final relativistic formula for galaxy clustering is indeed gauge-invariant. Accounting for the relativistic effect in galaxy clustering, various galaxy two-point statistics are derived with particular attention to the relation between various two-point statistics and the observable quantities.

While the relativistic formula provides the most accurate and complete description of galaxy clustering on large scales, the linear-order calculation in this work is limited to the two-point statistics. However, crucial information about the early Universe is encoded in the deviation from the Gaussianity, i.e. higher-order statistics such as the bispectrum, because any deviation from the standard single field inflationary model or any physics beyond the standard model naturally involves multiple fields that played significant roles in the early Universe. Given the numerous upcoming surveys, the second-order relativistic description of galaxy clustering [29, 30] (see also [31–33]) is an essential tool for probing the subtle relativistic effect in galaxy clustering that may decode the dynamics of the early Universe.

Acknowledgments

JY acknowledges useful discussions with Matias Zaldarriaga. JY is supported by the Swiss National Science Foundation and the Tomalla foundation grants.

References

[1] Yoo J, Fitzpatrick A L and Zaldarriaga M 2009 Phys. Rev. D 80 083514
[2] Yoo J 2010 Phys. Rev. D 82 083508
[3] Bonvin C and Durrer R 2011 Phys. Rev. D 84 063505
[4] Challinor A and Lewis A 2011 Phys. Rev. D 84 043516
[5] Jeong D, Schmidt F and Hirata C M 2012 Phys. Rev. D 85 023504
[6] McDonald P 2009 J. Cosmol. Astropart. Phys. 11 26
[7] Baldauf T, Seljak U, Senatore L and Zaldarriaga M 2011 J. Cosmol. Astropart. Phys. 10 31
[8] Brunner M, Crittenden R, Koyama K, Maartens R, Pitrou C and Wands D 2012 Phys. Rev. D 85 043010

\(^8\) This requirement applies both for \( P(k) \) and \( S_l(k) \) analysis, and it can be computationally challenging.
[9] Schmidt F and Jeong D 2012 Phys. Rev. D 86 083527
[10] Jeong D and Schmidt F 2012 Phys. Rev. D 86 083512
[11] Bertacca D, Maartens R, Raccanelli A and Clarkson C 2012 J. Cosmol. Astropart. Phys. 10 25
[12] Lopez-Honorez L, Meno O and Rigolin S 2012 Phys. Rev. D 85 023511
[13] Hall A, Bonvin C and Challinor A 2013 Phys. Rev. D 87 064026
[14] Yoo J, Haman N, Seljak U and Zaldarriaga M 2012 Phys. Rev. D 86 063514
[15] Maartens R, Zhao G-B, Bacon D, Koyama K and Raccanelli A 2013 J. Cosmol. Astropart. Phys. 2 044
[16] Lombriser L, Yoo J and Koyama K 2013 Phys. Rev. D 87 104019
[17] Yoo J and Desjacques V 2013 Phys. Rev. D 88 023502
[18] Yoo J and Seljak U 2013 arXiv:1308.1093
[19] Bonvin C, Hui L and Gaztañaga E 2014 Phys. Rev. D 89 083535
[20] Seljak U 2009 Phys. Rev. Lett. 102 021302
[21] Hwang J-C and Noh H 2001 Phys. Rev. D 65 023512
[22] Bardeen J M 1980 Phys. Rev. D 22 1882
[23] Wald R M 1984 General Relativity (Chicago: The University of Chicago Press)
[24] Bonvin C, Durrer R and Gasparini M A 2006 Phys. Rev. D 73 023523
[25] Weinberg S 1972 Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (New York: Wiley)
[26] Matsubara T 2000 Astrophys. J. Lett. 537 L77
[27] Yoo J 2009 Phys. Rev. D 79 023517
[28] Kaiser N 1984 Astrophys. J. Lett. 284 L9
[29] Yoo J 2014 arXiv:1408.5137
[30] Yoo J and Zaldarriaga M 2014 Phys. Rev. D 90 023513
[31] Bertacca D, Maartens R and Clarkson C 2014 J. Cosmol. Astropart. Phys. 9 037
[32] Bertacca D, Maartens R and Clarkson C 2014 arXiv:1406.0319
[33] di Dio E, Durrer R, Marozzi G and Montanari F 2014 arXiv:1407.0376
[34] Kodama H and Sasaki M 1984 Prog. Theor. Phys. Suppl. 78 1
[35] Hwang J and Noh H 2000 Gen. Rel. Grav. 31 1131
[36] Wands D and Slosar A 2009 Phys. Rev. D 79 123507
[37] Hwang J-C and Noh H 2005 Phys. Rev. D 72 044011
[38] di Dio E, Montanari F, Lesgourgues J and Durrer R 2013 J. Cosmol. Astropart. Phys. 11 044
[39] di Dio E, Montanari F, Durrer R and Lesgourgues J 2014 J. Cosmol. Astropart. Phys. 1 42
[40] Yamamoto K, Nakamichi M, Kamino A, Bassett B A and Nishioka H 2006 Pub. Astron. Soc. Japan 58 93
[41] Binney J and Quinn T 1991 Mon. Not. R. Astron. Soc. 249 678
[42] Fisher K B, Lahav O, Hoffman Y, Lynden-Bell D and Zaroubi S 1995 Mon. Not. R. Astron. Soc. 272 885
[43] Heavens A F and Taylor A N 1995 Mon. Not. R. Astron. Soc. 275 483