Quantum Fuzzy Sets: Blending Fuzzy Set Theory and Quantum Computation

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Abstract
In this article we investigate a way in which quantum computing can be used to extend the class of fuzzy sets. The core idea is to see states of a quantum register as characteristic functions of quantum fuzzy subsets of a given set. As the real unit interval is embedded in the Bloch sphere, every fuzzy set is automatically a quantum fuzzy set. However, a generic quantum fuzzy set can be seen as a (possibly entangled) superposition of many fuzzy sets at once, offering new opportunities for modeling uncertainty. After introducing the main framework of quantum fuzzy set theory, we analyze the standard operations of fuzzification and defuzzification from our viewpoint. We conclude this preliminary paper with a list of possible applications of quantum fuzzy sets to pattern recognition, as well as future directions of pure research in quantum fuzzy set theory.

1 Introduction
Fuzzy set theory (FST) has been around for four decades, producing a remarkable bulk of theoretical and practical knowledge. Its relevance to the real-world is beyond doubt, as many expert systems based on fuzzy set theory and its companion, fuzzy logic (FL), have been implemented and are currently running throughout the world. The chief reason behind FST's remarkable success lies in its ability to build decision systems in presence of vagueness: it provides a suitable mathematical tool to translate properties that are essentially “fuzzy” in common language usage, such as “being quite hot” when talking about temperature level for a thermal device.

It should be noted that, although for a given fuzzy property there are a number of different fuzzy characteristic functions that could express it (indeed, at least in principle, an infinite amount of them), the choice of one or the other is left to the fuzzy expert system designer, based on his/her understanding of the problem at hand. It would be desirable, at least in certain cases, to be able
to represent vague predicates in more than one way at once, especially when our knowledge of the real-life system to be modeled is insufficient to lead one way or the other.

It is the contention of this paper that quantum computation can provide the proper arena for introducing quantum fuzzy sets, i.e. "sets" that are in a sense superpositions of various fuzzy sets at once. These quantum fuzzy sets can be seen as the extension of "quantum properties", or quantum predicates.

This paper is broken down as follows: Section 1 starts with investigating the Logic of Quantum Computing and its relation to Fuzzy Logic. This relation is the core "philosophical" motivation behind our approach. Section 2 shows how standard fuzzy sets and their operations can be represented within the quantum computation framework. Section 3 introduces the main topic of this paper, quantum fuzzy sets, and sketches the way in which they add more "room" to FST. Section 4 describes future directions, both theoretical and applied, of the Quantum Fuzzy Sets program.

One final word. This article is not self-contained: it assumes the reader to be familiar with the basic ideas of Quantum Computation (QC) and Fuzzy Set Theory. References abound: for QC the standard one is [17], and for fuzzy set theory (FST), two excellent recent books covering much ground are [12] and [10].

2 From Fuzzy Logic to the Logic of Quantum Computation

Almost since its inception, quantum mechanics has inspired new approaches to logic. The first seminal work in this area was John Von Neumann’s proposal of a quantum logic. Von Neumann’s key idea was that quantum mechanics may entail a radical change of view not only of the way we understand and do physics, but of logic itself. Quantum logic (QL) is, from an algebraic standpoint, the orthocomplemented complete lattice of projections on a given Hilbert space. QL has been developed for almost fifty years: a comprehensive and well-written survey is [9]. A set theory based on quantum logic has been investigated by several authors; for instance, in the seventies and eighties Gaisi Takeuti (see [21]) proposed a quantum equivalent of boolean valued models of set theory, thereby creating a “universe of discourse” where the internal logic (in the sense of categorical logic) is QL. Later, other approaches based on more general structures, like quantales, have been proposed (see [15]).

It has been pointed out in [4] that QL, with all its undisputable merits, may not be the proper framework to model quantum computation. The reason being, that QL is the logic of quantum measurements, i.e. the logic of a physical system as it appears to an external observer.
In quantum computing measurement happens, as it were, at the very end of the computational process. Most of the computation is the successive application of basic unitary (and therefore reversible) operations, known as quantum gates. To reason about this process, it would be desirable to introduce a new type of quantum logic, the “internal logic” of the processor. Indeed, something along these lines has been proposed very recently, in [4] and [5]. In the two mentioned articles, a sketch for a symmetric, para-consistent logic of internal computation is offered, for a single-qubit and a two-qubit quantum processor. Such a logic is quite similar in spirit to the minimalist version of linear logic, Sambin’s Basic Logic.

In the author’s opinion, this internal quantum logic (IQL for short) is a very promising starting point, albeit incomplete. An enhanced version that takes into account “phase rotation” modalities, and their geometrical content, will be presented in a forthcoming work [13]. Meanwhile, for the purpose of the present article, we will spend a few words on the connection between IQL and FL.

Consider a one-qubit processor. Its state space is a two-dimensional Hilbert space, and it can be mapped onto the so-called Bloch sphere. The north pole and the south pole are the states 0 and 1, respectively. The various meridians of the Bloch sphere differ by a phase. The zero meridian is in one-to-one correspondence with the unit interval, in an obvious way. But the unit interval is the set of truth values of Zadeh’s Fuzzy Logic. We are thus lead to think of the Bloch sphere as a generalized set of “quantum truth values”; once one specifies a phase, the Bloch sphere collapses, as it were, onto a copy of the unit interval. Indeed, the Bloch sphere is comprised of uncountable many copies of the unit interval, just like the earth is sliced by its meridians.

We wish to stress right away that the Bloch sphere is, at least from traditional perspectives on logic, a rather odd set of truth values:

- The Bloch sphere is not a lattice, as opposed to most supports for standard and not-standard logics (boolean algebras, Heyting algebras, distributive lattices, etc.).
- Truth values, i.e. points on the Bloch sphere, can differ by a phase, something that has no classical counterpart.
- The status of the pair true/false is relative. Any couple of antipodal points on the Bloch sphere is a candidate for playing the true/false role.

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1 We use here the standard mapping from the qubit in normal form \( \cos(\theta) |0\rangle + e^{i\phi} \sin(\theta) |1\rangle \) with \( 0 \leq \theta \leq \frac{\pi}{2} \) and \( 0 \leq \phi \leq \pi \), to the point on the sphere of coordinates \( (\sin(2\theta) \cos(\phi), \sin(\phi) \sin(2\theta), \cos(2\theta)) \).

2 The author is indebted with Noson Yanofsky for this insight: originally, the unit interval was realized as the vertical axis of the “thick” Bloch sphere, i.e. the one containing all states, pure and impure alike.

3 An immediate consequence is that, upon choosing an ordered pair of antipodal points on
As we already mentioned, an in-depth investigation on the logic of the Bloch sphere and its higher dimensional analogues will be tackled in another paper. For now, though, from the naive considerations above, it should be clear that, in a sense, its logic extends traditional FL.

Now, the step to quantum fuzzy set theory is not a big one: once one sees the unit interval as a set of truth values, one can use it to build “characteristic functions”. So, why not use the entire Bloch sphere to do just the same?

3 Standard Fuzzy Sets from the Quantum Computation standpoint

We recall that a fuzzy set is just a map:

\[ f : X \rightarrow [0, 1] \]

from a set \( X \) into the unit interval. The idea behind the definition is simply that \( f \) is a generalized characteristic function of a subset of \( X \) with unclear (fuzzy) boundaries.

We are now going to describe fuzzy sets from the viewpoint of quantum computation. In the following we are going to assume that the set \( X \) is finite, although everything we shall say would carry through to infinite sets as well, provided that we have a quantum machine with an infinite quantum register at our disposal.

Let us assume that \( X \) has cardinality \( N \): \( |X| = N \). Without loss of generality, we can think of \( X \) as the set of the first \( N \) natural numbers: \( X = \{1, 2, \ldots, N\} \). We can associate to any fuzzy characteristic function \( f \) on \( X \) the following quantum state:

\[ |s_f\rangle = \bigotimes_{1 \leq i \leq N} \left[ f(i)\frac{1}{2}|1\rangle + (1 - f(i))\frac{1}{2}|0\rangle \right] \]

in other words, we set each qubit in the register to the appropriate mix given by \( f \).

**Definition 1** A state in the \( N \) quantum register that can be written as \( |s_f\rangle \) for some \( f : X \rightarrow [0, 1] \) will be referred to as a Classical Fuzzy State (CFS).

Classical Fuzzy States are exactly the ones whose components live on the phase zero meridian of the Bloch sphere. Notice that the standard basis of \( C^{2^{|X|}} \) becomes nothing else but the set of characteristic functions of crisp subsets of \( X \). For instance, the state \( |1100\ldots0\rangle \) corresponds to the subset \( \{1, 2\} \) of \( X \). This harmless observation leads to the main insight behind this paper: states the sphere (i.e. an observable), we have another embedding of the unit interval. This trivial observation is nevertheless rich in consequences, as it will be shown in [13].
in which our quantum register can be found encode “characteristic functions”
crisp or not, classical or quantum), of a set \( X \).
The state \( |s_f\rangle \) can be expanded in the standard basis as
\[
|s_f\rangle = f(1)^{\frac{1}{\sqrt{2}}} f(2)^{\frac{1}{\sqrt{2}}} \cdots f(n)^{\frac{1}{\sqrt{2}}} |1\ldots1\rangle + \\
\quad\quad\quad\quad\quad f(1)^{\frac{1}{\sqrt{2}}} (1 - f(2))^{\frac{1}{\sqrt{2}}} \cdots f(n)^{\frac{1}{\sqrt{2}}} |10\ldots1\rangle + \cdots \\
\quad\quad\quad\quad\quad (1 - f(1))^{\frac{1}{\sqrt{2}}} (1 - f(2))^{\frac{1}{\sqrt{2}}} \cdots (1 - f(n))^{\frac{1}{\sqrt{2}}} |00\ldots0\rangle.
\]
In other words, from the quantum computation perspective, even a standard fuzzy set is a quantum superposition of crisp sets.

Expansion of (♣) above tells us a bit more, namely that an observation of \( s_f \) in the standard basis collapses it to (the characteristic function of) the crisp subset \( S \subseteq X \) with probability
\[
p(S) = |\alpha_1\alpha_2\ldots\alpha_n|^2
\]
where
\[
\alpha_i = \begin{cases} 
  f(i)^{\frac{1}{\sqrt{2}}} & \text{if } i \in S; \\
  (1 - f(i))^{\frac{1}{\sqrt{2}}} & \text{if } i \notin S.
\end{cases}
\]

One last important fact: by their very definition classical fuzzy states are not entangled. If we want quantum magic to set in, we must be prepared to go a bit farther.

Summing up, we can say that, from our standpoint, classical fuzzy sets are unentangled, zero-phase superpositions of crisp sets. Observation collapses them into one or the other of their crisp constituents.

If we start our register in the default, “ground state” \( |00\ldots0\rangle \), we can think of \( f \) as the unitary map
\[
U_f = \bigotimes U_{f(i)}
\]
where
\[
U_{f(i)} = \begin{pmatrix} 
  f(i)^{\frac{1}{\sqrt{2}}} & -(1 - f(i))^{\frac{1}{\sqrt{2}}} \\
  (1 - f(i))^{\frac{1}{\sqrt{2}}} & f(i)^{\frac{1}{\sqrt{2}}}
\end{pmatrix}
\]
And hence,
\[
U_f|00\ldots0\rangle = |s_f\rangle.
\]
Each \( U_{f(i)} \) does what is supposed to do, namely sending the \( i \)-th qubit from the ground state \( |0\rangle \) to the proper mixing given by \( f(i) \). The geometrical interpretation is straightforward: \( U_{f(i)} \) is the rotation of the Bloch sphere around its \( y \) axis by the angle \( \theta_i = 2 \arcsin(f(i)^{\frac{1}{2}}) \).

To show that we can indeed do fuzzy set theory from within quantum computation, we need to be able to carry out fuzzy logical operations on fuzzy sets. Before we embark on this step, let us remind ourselves that FST has a
large variety of alternatives families of logical connectives (for instance, there are several complements, several AND, etc.). We shall choose one possible set of fuzzy operations, including logical connectives, and simply point out that similar methods could equally well implement different choices. Our pick is the “probabilistic set”:

| complement | $1 - f(i)$ |
|------------|-----------|
| intersection | $f(i)g(i)$ |

We begin with unitary connectives. The standard well-known NOT gate, $egin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the standard basis, works as desired on the zero-phase meridian:

$$\text{NOT}(\cos(\theta)|0\rangle + \sin(\theta)|1\rangle) = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle = \cos(\frac{\pi}{2} - \theta)|0\rangle + \sin(\frac{\pi}{2} - \theta)|1\rangle$$

thus its action on classical fuzzy states is precisely the one given by the complement:

$$\text{NOT}(s_f = \bigotimes_i f(i)\frac{\pi}{2}|1\rangle + (1 - f(i))\frac{\pi}{2}|0\rangle) = \bigotimes_i (1 - f(i))\frac{\pi}{2}|1\rangle + f(i)\frac{\pi}{2}|0\rangle.$$

As far as binary connectives, one encounters the same issues as with standard logic gates: unitary transformations are reversible, and thus represent only reversible gates. However, just as in the classical logic case, there is a way around, namely adding control qubits.

As it is well known, the classical AND can be implemented using the three-qubits Toffoli’s Gate $T$. For the pair $|\psi\rangle$ and $|\phi\rangle$, where $\psi, \phi \in \{0, 1\}$, it can be computed as follows:

$$\text{AND}(|\psi\rangle, |\phi\rangle) = T(|\psi\rangle \otimes |\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\phi\rangle \otimes |\psi\phi \oplus 0\rangle$$

Here $\oplus$ denotes as usual multiplication modulo 2. Let us now see the action of AND on a pair of CFS, $|s_f\rangle$ and $|s_g\rangle$. We shall compute it at the $i$-th component:

$$\text{AND}(f(i)\frac{\pi}{2}|1\rangle + (1 - f(i))\frac{\pi}{2}|0\rangle, g(i)\frac{\pi}{2}|1\rangle + (1 - g(i))\frac{\pi}{2}|0\rangle) =$$

$$f(i)\frac{\pi}{2}g(i)\frac{\pi}{2}T(|1\rangle, |1\rangle, |0\rangle) + f(i)\frac{\pi}{2}(1 - g(i))\frac{\pi}{2}T(|1\rangle, |0\rangle, |0\rangle) +$$

$$(1 - f(i))\frac{\pi}{2}g(i)\frac{\pi}{2}T(|0\rangle, |1\rangle, |0\rangle) + (1 - f(i))\frac{\pi}{2}(1 - g(i))\frac{\pi}{2}T(|0\rangle, |0\rangle, |0\rangle) =$$

$$f(i)\frac{\pi}{2}g(i)\frac{\pi}{2}|11\rangle |1\rangle + (f(i)\frac{\pi}{2}(1 - g(i))\frac{\pi}{2}|10\rangle + (1 - f(i))\frac{\pi}{2}g(i)\frac{\pi}{2}|01\rangle +$$

The contradiction is only apparent: probabilities only occur after measurements have been taken; before, there are only “quantum truth values”.

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4Our choice is suggested by QC itself. As we have already seen, fuzzy characteristic functions look like “complex-valued probability distributions”, from the standpoint of QC. This fact goes, at first sight, against the general consensus that FL is not probability logic. The contradiction is only apparent: probabilities only occur after measurements have been taken; before, there are only “quantum truth values”.
\[(1 - f(i))^{\frac{1}{2}} (1 - g(i))^{\frac{1}{2}} |000\rangle |0\rangle\]

An observation on the third qubit of the output produces \(|1\rangle\) with probability \(f(i)g(i)\): AND has successfully multiplied the amplitudes, at every \(i\).

Having the set-theoretical complement and intersection at our disposal, we can as usual obtain all the fuzzy connectives by sheer logic.

We now turn our attention to fuzzification and defuzzification. Let us tackle fuzzification first: what is needed here, is an operator that takes as input a crisp value (i.e. the register in the state where \(n - 1\) qubits are set to \(|0\rangle\), and one is set to \(|1\rangle\), and fuzzifies it around the sharp value. The exact shape of this operator depends on the desired fuzzifier. Here, we choose a naive square-shaped fuzzifier of diameter \(k\): it creates a fuzzy value centered at the crisp one, square-shaped, such that the base has size \(2k + 1\). The value at points surrounding the crisp value will be uniformly \(\frac{1}{2}\). Let us denote such operator as \(FUZ\).

If \(|\alpha_1 \ldots \alpha_n\rangle\), where \(\alpha \in \{0, 1\}\), is an element of the standard basis, here is how \(FUZ\) operates on it: \(FUZ(|\alpha_1 \ldots \alpha_n\rangle) = |\gamma_1, \ldots \gamma_n\rangle\) where

\[
\gamma_i = \begin{cases} 
\frac{1}{\sqrt{2}} (|1\rangle + |0\rangle) & \text{if } \exists j \ |j - i| \leq k \text{ and } \alpha_j = 1 \\
0 & \text{otherwise}
\end{cases}
\]

\(FUZ\) is linear but not unitary (fuzzification loses information). However, we can manufacture a unitary map with the standard trick of remembering the input vector:

\[|s_f\rangle |000 \ldots 0\rangle \longrightarrow |s_f\rangle |U_{FUZ}(s_f)\rangle\]

Now, defuzzification. Again, as before with the AND connective, we rely on the indirect representation of boolean maps, given by the following recipe:

if \(f : 2^n \rightarrow 2^m\) is a boolean function, we can associate to it the unitary transformation \(U_f : 2^{n+m} \rightarrow 2^{n+m}\) given by

\[(\blacklozenge) \ \forall u \in 2^n, \forall v \in 2^m: U_f((u,v)) = (u, v \oplus f(u)).\]

Before using formula \((\blacklozenge)\) above, one must choose a classical defuzzification operator, and discretize it (indeed, turn it into a boolean map). Just like in the case of logical connectives, there are many choices in FST. We shall choose as an example the center-of-mass defuzzification \(COM : 2^n \rightarrow 2^n\). If \(|\alpha_1 \ldots \alpha_n\rangle\), where \(\alpha \in \{0, 1\}\), is a binary string, here is how it operates:

\[COM(\alpha_1 \ldots \alpha_n) = \gamma_1, \ldots \gamma_n\]

where

\[
\gamma_i = \begin{cases} 
1 & \text{if } i = \lfloor \frac{\sum_j g(ij)}{n} \rfloor \\
0 & \text{otherwise}
\end{cases}
\]

Now, if we want to defuzzify a state \(|s_f\rangle\), we first apply \(U_{COM}\) to the state of length \(2n\) \(|s_f\rangle |00 \ldots 0\rangle\) (i.e. \(|s_f\rangle\) padded with \(n\) 0s), and then measure the last
n qubits. The outcome will be a sharp center-of-mass value, as desired (to be sure, our defuzzifier is only a probabilistic one. It will return the center-of-mass after repeated trials).

Our list is now complete. Armed with fuzzification, logical connectives, and defuzzification, we can simulate any fuzzy inference engine on a quantum machine, provided, of course, that a sufficient amount of quantum memory is available.

4 Quantum Fuzzy Sets

We have just seen how classical fuzzy set theory takes a rather natural place inside Quantum Computation. It is now time to move a step forward and introduce the main topic of the paper. Note: this Section aims to barely sketch quantum fuzzy sets. A full treatment of them will be tackled in [14].

The way to quantum fuzzy sets is straightforward: we have just seen that fuzzy characteristic functions can be represented by suitable states of a quantum register, so why not postulate that all states are characteristic functions of some new subsets?

**Definition 2** A quantum fuzzy subset of a set $X$ is a point in the state space $\mathbb{C}^{2^{|X|}}$.

The reader may wonder what we have actually gained. The answer is twofold:

- we can consider quantum fuzzy sets that are superpositions of several standard fuzzy subsets, thereby creating fuzzy sets that have different “shapes” at once.
- we can create quantum fuzzy sets that are entangled superpositions of crisp (or fuzzy) subsets. In fact, from our perspective, every entangled state is an entangled superposition of crisp subsets.

If

$$f_i : X \rightarrow [0, 1]$$

where

$$i = 1, \ldots, k$$

is a collection of fuzzy subsets of $X$, and

$$|s_{f_1}, \ldots, s_{f_k}\rangle$$

is the corresponding set of classical fuzzy states, one can combine all the previous shapes into a quantum fuzzy set via superposition:

$$|s\rangle = c_1 |s_{f_1}\rangle + \ldots + c_k |s_{f_k}\rangle$$

The generic quantum fuzzy state above can be entangled or not, depending on the family of fuzzy states and the particular amplitudes chosen.

Notice that in general classical fuzzy states are not orthogonal to one another. Indeed, when are two CFS orthogonal to one another?
Theorem 1 Two states $|s_f\rangle$, $|s_g\rangle$ are orthogonal to one another, if and only if there is an $i \in X$ such that $f(i) = 0, g(i) = 1$, or the other way round.

The proof is immediate from expansion $\clubsuit$. An important consequence of this fact is that the mingling of different non-orthogonal fuzzy shapes prevent us from retrieving them sharply at a later stage: no observable contains those shapes as its eigenvectors.

As far as entanglement goes, quantum fuzzy sets that are not entangled can be transformed into classical fuzzy states by applying suitable rotations of the Bloch sphere on each component (one just rotates the meridian till one reaches zero phase). Thus, the most interesting (and enigmatic) quantum fuzzy sets are the entangled ones.  

Fuzzy connectives and logical operators defined as unitary maps, as mentioned in the previous sections, can be applied to any quantum fuzzy set. The result is a natural parallelization of fuzzy operators, out of linearity. Here is a simple example using $\text{NOT}$:

$$\text{NOT}\, |s\rangle = \text{NOT}(c_1|s_{f_1}\rangle + \ldots + c_k|s_{f_k}\rangle) = c_1\text{NOT}|s_{f_1}\rangle + \ldots + c_k\text{NOT}|s_{f_k}\rangle.$$  

Just the same applies to all fuzzy connectives, as well as any pair of fuzzification and defuzzification operators expressed by unitary transformations.

We left the previous Section knowing that standard fuzzy engines can be simulated in a quantum machine. We know more now, namely that such inference engine accept variables that are quantum fuzzy sets. Such an engine can be aptly called a quantum fuzzy inference engine (QFE). Unlike its classical counterpart, running on a classical machine, it can process in parallel superpositions of classical fuzzy sets. It is thus reasonable to think of potential applications of QFE, the topic of the last Section.

5 Applications of Quantum Fuzzy Sets. Future Directions

Fuzzy sets are an eminently practical tool, familiar to AI engineers and software designers alike. Fuzzy Set Theory is generally employed in the design of expert systems, where variable and rules allow for vagueness. An important subclass of applications of FST is in Pattern Recognition (PR)(see for instance [6] and [18]).

The type of possible PR applications of Quantum Fuzzy Sets that we envision exploit the features of QFS described in the last Section. Here is a partial list:

\(^{5}\)\footnote{However, this observation does not mean that unentangled quantum fuzzy states are trivial; far from it: their phase may entail interesting interference phenomena when they are combined by fuzzy connectives.}
• PR applications where the training data set is very large (for instance, large databases of blurred images). The idea here is to use the inner parallelism of Quantum Computing to implement a parallel fuzzy inference engine.

• PR application where different features are assumed to have a high degree of correlation, but the exact nature of this correlation is unknown. Here, we would create entangled superposition of standard fuzzy sets, to account for new statistical patterns. The level of entanglement (and therefore of correlation among the variables) could be controlled by suitable entanglement unitary operators.

• The design of entirely new PR algorithms based on Quantum Computing. At this initial stage, we envision algorithms based on amplitude amplification, like the well-known Grover’s algorithm on database search. An initial quantum fuzzy set would be initialized, representing a coarse model of the data. The algorithm would update the fuzzy set by amplifying some of its amplitudes that best fit training data. We expect that applications of this scheme will be relevant to fuzzy mathematical morphology (16), as well as in the design of fuzzy filters for image processing (16).

On the theoretical side, we see at least three independent, though closely interrelated, lines of research:

• There are several categories of Fuzzy Sets, such as Goguen’s category (11), and Barr’s category (3). In a sequel to this work (14), we will introduce the category of Quantum Fuzzy Sets, present its categorical logic, and investigate how the cited fuzzy sets categories are related to it.

• We will try to describe a general notion of quantum set, including both quantale sets and quantum fuzzy sets into a unified framework.

• Lastly, there is another thread we intend to pursue, going back from Quantum Fuzzy Sets to Quantum Computation itself, namely the attempt to recast quantum algorithms in the language of QFS. From this perspective, a quantum algorithm is a series of controlled quantum set-theoretical operations on some underlying set. The hope is that certain known quantum algorithms will appear a bit more intuitive when regarded from this angle, and that new heuristic may emerge to help creating entirely ones.

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