MODULI OF POLARISED ABELIAN SURFACES

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Abelian surfaces over \( \mathbb{C} \) with a polarisation of type \((1,t)\), \( t \) a positive integer, are parametrised by a coarse moduli space \( \mathcal{A}_t \) which is a quasiprojective variety. In this paper we shall concentrate on the case where \( t \) is a prime \( p \geq 5 \), and show that for \( p \) sufficiently large (in fact \( p \geq 173 \)) any algebraic compactification of \( \mathcal{A}_p \) is of general type.

A few results similar to this are already known. O’Grady, in [O’G], considers the case \( t = p^2 \) and shows that a compactification of \( \mathcal{A}_{p^2} \) is of general type for \( p \geq 17 \) (improved to \( p \geq 11 \) in [GS]). The special feature here is the existence of a finite morphism from \( \mathcal{A}_{p^2} \) to the moduli space of principally polarised abelian surfaces (the case \( t = 1 \)). This is also the case in [Bor], where it is shown that a compactification of a Siegel modular threefold coming from a subgroup \( \Gamma < \text{Sp}(4, \mathbb{Z}) \) of finite index is of general type except for finitely many \( \Gamma \). Another moduli space, referred to in this paper as \( \mathcal{A}_p^{(\text{lev})} \) and parametrising abelian surfaces with a polarisation of type \((1,p)\) and a level structure, is studied in depth by Hulek, Kahn and Weintraub in the book [HKW2]. Its singularities are described in [HKW1] and it has been shown, by Hulek, Gritsenko and me, that it is of general type if the prime \( p \) is at least 37: see [HS] and [GH].

There is a finite morphism \( \mathcal{A}_p^{(\text{lev})} \to \mathcal{A}_p \). This morphism, the singularities of \( \mathcal{A}_p \) and its toroidal compactifications have been studied in [Br] by Brasch, who gives an analysis in the spirit of [HKW2]. Our main tools are Brasch’s results, the calculations relating to \( \mathcal{A}_p^{(\text{lev})} \) found in [HKW1], [HKW2] and [HS], and some special cusp forms constructed by Gritsenko (see [G]). In principle we do not need to know about \( \mathcal{A}_p^{(\text{lev})} \) but, like Brasch, we
are able to save a considerable amount of effort by making use of the existing knowledge of it.

The paper is organised as follows. In Section 1 we give a more detailed outline of the proof, collecting necessary facts from elsewhere and establishing notation. Section 2 is devoted to estimating the dimension of the space of cusp forms of high weight for the relevant subgroup of $\text{Sp}(4, \mathbb{Q})$. In Section 3 we describe a method, due to Gritsenko, that enables us to handle the smooth points at infinity in a suitable compactification of $\mathcal{A}_p$. Section 4 deals with the obstructions arising inside $\mathcal{A}_p$. Section 5 deals with the singularities at infinity, and in Section 6 we assemble all the parts of the proof and add some other remarks and corollaries.

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1 Preliminaries.

Suppose $\Lambda \subseteq \mathbb{C}^2$ is a lattice of rank 4. The complex surface $S = \mathbb{C}^2/\Lambda$ is called an abelian surface if it admits an ample line bundle. A polarisation on an abelian surface $S$ is a class $\lambda \in \text{NS}(X)$ such that $\lambda = c_1(\mathcal{L})$ for some ample line bundle $\mathcal{L}$ on $S$: see [LB] for details. As is well known (see [LB] or [Mum]) a polarisation corresponds to an alternating nondegenerate integral bilinear form on $\Lambda$, which can be expressed, by choosing a suitable $\mathbb{Z}$-basis of $\Lambda$, by the matrix

$$
\begin{pmatrix}
0 & T \\
-T & 0
\end{pmatrix}
$$
where $T = \text{diag} \,(t_1, t_2)$, for some positive integers $t_1$ and $t_2$ with $t_1 | t_2$. The integers $t_1$ and $t_2$ are uniquely determined by $\lambda$, which is said to be a polarisation of type $(t_1, t_2)$, or a $(t_1, t_2)$-polarisation. We may as well suppose that $\lambda$ is not divisible in $\text{NS}(S)$; then $t_1 = 1$ and $\lambda$ is a polarisation of type $(1, t)$ for some positive integer $t$.

Denote by $\mathbb{H}_2$ the Siegel upper half-plane of degree 2,

$$\mathbb{H}_2 = \left\{ Z = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mid Z \in M_{2 \times 2}(\mathbb{C}), Z = \tau Z, \text{Im } Z > 0 \right\},$$

and define the paramodular group $\Gamma_t$ to be the subgroup of the symplectic group $\text{Sp}(4, \mathbb{Q})$ given by

$$\Gamma_t = \left\{ \gamma \in \text{Sp}(4, \mathbb{Q}) \mid \gamma \in \begin{pmatrix} Z & Z & Z & tZ \\ tZ & Z & Z & tZ \\ Z & Z & Z & tZ \\ Z & Z & Z & tZ \end{pmatrix} \right\}.$$

$\Gamma_t$ acts on $\mathbb{H}_2$ by fractional linear transformations:

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto (AZ + B)(CZ + D)^{-1}.$$

The action is properly discontinuous and we denote the quotient $\Gamma_t \backslash \mathbb{H}_2$ by $\mathcal{A}_t$.

**Proposition 1.1.** $\mathcal{A}_t$ is a coarse moduli space for pairs $(S, \lambda)$, where $S$ is an abelian surface over $\mathbb{C}$ and $\lambda$ is a polarisation of type $(1, t)$.

**Proof:** See [HKW2], Part I, Chapter 1, where the prime $p$ may be replaced by an arbitrary positive integer with no other consequential changes. ■

Similarly we introduce the group

$$\Gamma_t^{(\text{lev})} = \left\{ \gamma \in \text{Sp}(4, \mathbb{Q}) \mid \gamma - 1_4 \in \begin{pmatrix} Z & Z & Z & tZ \\ tZ & tZ & tZ & t^2Z \\ Z & Z & Z & tZ \\ Z & Z & Z & tZ \end{pmatrix} \right\}$$

(in general we use $1_n$, or just 1, to denote the $n \times n$ unit matrix) and the moduli space $\mathcal{A}_t^{(\text{lev})} = \Gamma_t^{(\text{lev})} \backslash \mathbb{H}_2$ of $(1, t)$-polarised abelian surfaces with a level structure.
Proposition 1.2. $\Gamma_t$ and $\Gamma_t^{(lev)}$ are related as follows:

a) $\Gamma_t^{(lev)}$ is a normal subgroup of $\Gamma_t$;

b) $\Gamma_t/\Gamma_t^{(lev)} \cong \text{SL}(2,\mathbb{Z}/t)$;

c) $\Gamma_t$ is conjugate in $\text{Sp}(4,\mathbb{Q})$ to a subgroup of $\text{Sp}(4,\mathbb{Z}) = \Gamma_1$ if and only if $t$ is a square;

d) for an odd prime $p$ there is a morphism $\mathcal{A}_p^{(lev)} \to \mathcal{A}_p$ of degree $p(p^2 - 1)/2$ exhibiting $\mathcal{A}_p$ as a quotient of $\mathcal{A}_p^{(lev)}$ by an action of $\text{PSL}(2,\mathbb{Z}/p)$.

Proof: (a), (b) and (d) are in [HKW2] and (c) is easily checked. ■

From now on we shall concentrate on the case where $t$ is an odd prime $p \geq 5$. Denote by $\mathcal{A}_p^*$ and $\mathcal{A}_p^{(lev)*}$ the toroidal compactifications of $\mathcal{A}_p$ and $\mathcal{A}_p^{(lev)}$ constructed in [Br] and [HKW2] respectively (also called Igusa compactifications). Note that these constructions are compatible with each other in the sense that the group action and morphism of Proposition 1.2(d) extend to $\mathcal{A}_p^{(lev)*}$ and to $\mathcal{A}_p^{(lev)*} \to \mathcal{A}_p^*$.

For any arithmetic subgroup $\Gamma$ of $\text{Sp}(4,\mathbb{Q})$ we write $\mathcal{M}_k^*(\Gamma)$ for the space of cusp forms of weight $k$ for $\Gamma$. Following Tai ([T]), we consider modular forms of weight $3n$ and look at $F\omega^{\otimes n}$, where $F \in \mathcal{M}_3^*(\Gamma_p)$ and $\omega = d\tau_1 \wedge d\tau_2 \wedge d\tau_3$ is a differential 3-form on $\mathbb{H}_2$. It is $\Gamma_p$-invariant so it gives an $n$-canonical form where the map $\mathbb{H}_2 \to \mathcal{A}_p$ is unbranched.

To obtain pluricanonical forms on a resolution of singularities of $\mathcal{A}_p^*$ we have to be able to extend over the smooth points of the boundary, over the branch locus (including the singularities of $\mathcal{A}_p$) and over the singularities in the boundary. The first and third of these may be dealt with by choosing special cusp forms, partly in the manner of [GH] and [GS]: for the second we have to estimate the growth with $p$ of the space of obstructions. These
three types of obstacle will be tackled in sections 3–5, but first we need to know how many
cusp forms there are.

2 Cusp forms for $\Gamma_p$.

In order to proceed as we intend we need a nontrivial cusp form of weight 2 and a plentiful
supply of cusp forms of large weight for $\Gamma_p$.

**Proposition 2.1.** There exists a nontrivial cusp form of weight 2 for $\Gamma_p$ if $p > 71$.

**Proof:** By [G], Theorem 3, a Jacobi cusp form of weight 2 and index $p$ can be lifted to a
weight 2 cusp form for $\Gamma_p$, i.e.

$$\dim M^*_2(\Gamma_p) \geq \dim \mathcal{S}_{2,p}^J$$

where $\mathcal{S}_{k,t}^J$ is the space of Jacobi cusp forms of weight $k$ and index $t$. But there is a formula
for the latter dimension (see [EZ] and [SZ]):

$$\dim \mathcal{S}_{2,p}^J = \sum_{j=1}^{p} (1 + j)_6 - \left\lfloor \frac{j^2}{4p} \right\rfloor$$

where

$$\{m\}_6 = \begin{cases} \left\lfloor \frac{m}{6} \right\rfloor & \text{if } m \not\equiv 1 \mod 6 \\ \left\lfloor \frac{m}{6} \right\rfloor - 1 & \text{if } m \equiv 1 \mod 6. \end{cases}$$

It is easy to check that this number is positive if $p > 71$. ■

**Remark:** In fact this shows that $\dim \mathcal{S}_{2,t}^J > 0$ for all $t > 180$ and many smaller $t$. 


Proposition 2.2. The space of cusp forms of weight $k$ for $\Gamma_p$ satisfies
\[
\dim M^*_k(\Gamma_p) = \frac{p^2 + 1}{8640} k^3 + O(k^2)
\]
for any odd prime $p$.

Proof: The corresponding result for $\Gamma_p^{(\text{lev})}$ is given in [HS], Proposition 2.1, where it is shown that
\[
\dim M^*_k(\Gamma_p^{(\text{lev})}) = \frac{p(p^4 - 1)}{17280} k^3 + O(k^2)
\]

We proceed in the same way: writing $\bar{\Gamma}(1) = \text{Sp}(4, \mathbb{Z})/\pm 1$ and $\Gamma(l)$ for the principal congruence subgroup of level $l$, we have
\[
\dim M^*_k(\Gamma(l)) = \frac{k^3}{8640} \left[ \bar{\Gamma}(1) : \Gamma(l) \right]
\]
and if $p^2 | l$ then $\Gamma(l) \subseteq \Gamma_p^{(\text{lev})} \subseteq \Gamma_p$. Furthermore, $\Gamma(l)$ is a normal subgroup of $\Gamma_p$: denote the quotient by $\Gamma_p(l)$. Then
\[
M^*_k(\Gamma_p) = M^*_k(\Gamma(l))^{\Gamma_p(l)}
\]
and we can estimate the dimension, as in [T], by using the method of Hirzebruch in [Hir].

We have (cf. [T], [GS])
\[
\dim M^*_k(\Gamma_p) = \dim M^*_k(\Gamma(l))^{\Gamma_p(l)}
\]
\[
= \frac{1}{|\Gamma_p(l)|} \sum_{\gamma \in \Gamma_p(l)} \text{Trace} \left( \gamma^* | M^*_k(\Gamma(l)) \right)
\]
and by the Atiyah-Bott fixed point theorem
\[
\text{Trace} \left( \gamma^* | M^*_k(\Gamma(l)) \right) = O(k^{\dim \text{Fix}(\gamma)})
\]
so we need to consider only \( \gamma = \pm 1 \), as otherwise \( \dim \text{Fix}(\gamma) < 3 \).

But \(-1\) acts trivially, so we get

\[
\dim \mathcal{M}_k^*(\Gamma_p) = \frac{2}{|\Gamma_p(l)|} \dim \mathcal{M}_k^*(\Gamma(l)) = \frac{2}{[\Gamma_p : \Gamma(l)]} \frac{k^3}{8640} [\bar{\Gamma}(1) : \Gamma(l)] + O(k^2)
\]

\[
= \frac{[\bar{\Gamma}(1) : \Gamma_p^{(\text{lev})}]}{[\Gamma_p : \Gamma_p^{(\text{lev})}]} \frac{k^3}{4320} + O(k^2)
\]

\[
= \frac{p(p^4 - 1)/2}{|SL(2, \mathbb{Z}/p)|} \frac{k^3}{4320} + O(k^2)
\]

\[
= \frac{p^2 + 1}{8640} k^3 + O(k^2)
\]

since \([\bar{\Gamma}(1) : \Gamma_p^{(\text{lev})}] = p(p^4 - 1)/2\) by [HW], p.413, and \(\Gamma_p/\Gamma_p^{(\text{lev})} \cong SL(2, \mathbb{Z}/p)\) (Proposition 1.2), which has order \(p(p^2 - 1)\).

Remarks. a) The degree of the covering \(A_p^{(\text{lev})} \rightarrow A_p\) is \(p(p^2 - 1)/2\) because \(-1 \in \Gamma_p\) but \(-1 \notin \Gamma_p^{(\text{lev})}\). Thus the Galois group is \(PSL(2, \mathbb{Z}/p)\), not \(SL(2, \mathbb{Z}/p)\).

b) This is the first place where we have assumed that \(p\) is prime. Compare the corresponding calculation for \(t = p^2\) in [GS].

3 Extension of differential forms over the smooth part of the boundary

Suppose \(F_2\) is a nontrivial weight 2 cusp form for \(\Gamma_p\) and \(F_n\) is a cusp form of weight \(n\). Then \(F = F_2^n F_n\) is a cusp form of weight \(3n\). The differential form \(F \omega^{\otimes n}\) on \(\mathbb{H}_2\) descends under the action of \(\Gamma_p\) to give an \(n\)-canonical form on \(A_p\) away from the branch locus of \(\mathbb{H}_2 \rightarrow A_p\). Because \(F\) has been chosen to vanish to high order at infinity we get even more.
Proposition 3.1. The differential form on $A_p$ coming from $F_\omega^{\otimes n}$ extends holomorphically over the generic point of each codimension 1 boundary component of $A_p^*$.

Proof: This is a straightforward application of [SC], Chapter IV, Theorem 1. The details are in [GS] (Proposition 3.2) for the case $t = p^2$, and as the case $t = p$ is no different we omit them here. □

Note that we do not need to know anything about the codimension 1 boundary components, not even how many there are (two, in fact). Of course the forms also extend over smooth points in boundary components of codimension greater than 1, so the remaining obstructions to extension come from the branch locus of $\mathbb{H}_2 \to A_p$ and from the singularities of $A_p^*$.

4 Branch locus and singularities of $A_p$.

The singularities of $A_p^*$ and the branch locus of $\mathbb{H}_2 \to A_p$ are fully described in [Br]. In this section we shall not deal with the boundary $A_p^* \setminus A_p$, except when we deal with the points that lie in the closure of the branch locus. Components of the singular locus of $A_p^*$ which are entirely contained in the boundary will be dealt with in the next section.

We want to make use of the information about $A_p^{(\text{lev})^*}$ which is already available in [HKW1], [HKW2] and [HS] (and also [Z], though we shall not need that information directly). For this reason we need a morphism $A_p^{(\text{lev})^*} \to A_p^*$ which we can control, and which extends the quotient morphism $A_p^{(\text{lev})} \to A_p$. Perhaps there is an Igusa compactification functor sending an arithmetic subgroup of $\text{Sp}(4, \mathbb{Q})$ to the Igusa compactification of the
corresponding Siegel modular 3-fold and respecting inclusions among such groups. In de-
default of a result to that effect we prove the following, which is sufficient for our immediate purposes.

**Proposition 4.1.** The group \( G = \Gamma_p/\Gamma_p^{(\text{lev})} \cong \text{SL}(2, \mathbb{Z}/p) \) acts on the Igusa compactification \( \mathcal{A}_p^{(\text{lev})*} \). The action agrees with the natural action of \( G \) on \( \mathcal{A}_p^{(\text{lev})} \) and the quotient is isomorphic to the Igusa compactification \( \mathcal{A}_p^* \) of \( \mathcal{A}_p \).

**Proof:** The Igusa compactifications \( \mathcal{A}_p^{(\text{lev})*} \) and \( \mathcal{A}_p^* \) are by definition the compactifications constructed in [HKW2] (Part I, Chapter 3) and in [Br] respectively. The existence of a \( G \)-action on \( \mathcal{A}_p^{(\text{lev})*} \) extending the action on \( \mathcal{A}_p^{(\text{lev})} \) is an immediate consequence of the construction of \( \mathcal{A}_p^{(\text{lev})*} \): see Remark 3.95 on page 89 of [HKW2]. So we must check that the quotient agrees with Brasch’s construction. By [HKW2], Part I, Proposition 3.154, the quotient \( \mathcal{A}_p^{(\text{lev})*}/G \) is the compactification of \( \mathcal{A}_p \) determined by the Legendre decomposition. But this is precisely what is constructed in Chapter 3, §2 of [Br].

Both \( \Gamma_p \) and \( \Gamma_p^{(\text{lev})} \) contain elements of order 2, among others

\[
I_1 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad I_2 = \begin{pmatrix}
-1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

which are in \( \Gamma_p^{(\text{lev})} \) and hence in \( \Gamma_p \), and \(-1_4\) which is in \( \Gamma_p \) only. (Here we are using the notation and conventions of [HKW1], which differ slightly from those of [Br].)

We put

\[
\mathcal{H}_1 = \text{Fix}(I_1) = \left\{ \begin{pmatrix}
\tau_1 & 0 \\
0 & \tau_3
\end{pmatrix} \in \mathbb{H}_2 \mid \tau_1, \tau_3 \in \mathbb{H}_1 \right\},
\]

\[
\mathcal{H}_2 = \text{Fix}(I_2) = \left\{ \begin{pmatrix}
\tau_1 & -\frac{1}{2} \tau_3 \\
-\frac{1}{2} \tau_3 & \tau_3
\end{pmatrix} \in \mathbb{H}_2 \mid \tau_1, \tau_3 \in \mathbb{H}_1 \right\}.
\]
Denote by $H_i$ and $H_i^{(\text{lev})}$ for $i = 1, 2$ the closures of the images in $\mathcal{H}_i$ in $\mathcal{A}_p^*$ and $\mathcal{A}_p^{(\text{lev})^*}$ respectively. These are the Humbert surfaces referred to in [HKW1], [HKW2] and [Br]. $\mathcal{H}_1$ parametrises products of elliptic curves and $\mathcal{H}_2$ parametrises bielliptic abelian surfaces.

**Proposition 4.2.** Every involution (element of order 2 different from $-1_4$) in $\Gamma_p$ is conjugate in $\Gamma_p$ to $\pm I_1$ or $\pm I_2$. Every involution in $\Gamma_p^{(\text{lev})}$ is conjugate to $I_1$ or $I_2$.

**Proof:** See [Br] and [HKW1]. ■

Recall that $-1_4$ acts trivially on $\mathbb{H}_2$, so that $\Gamma_p$ acts on $\mathbb{H}_2$ through the effective action of the quotient $\bar{\Gamma}_p = \Gamma_p/\pm 1_4$. The elements $I_1$ and $I_2$ act by reflection near $\mathcal{H}_1$, $\mathcal{H}_2$, so in view of Proposition 4.2 the maps $\mathbb{H}_2 \to \mathcal{A}_p$ and $\mathbb{H}_2 \to \mathcal{A}_p^{(\text{lev})}$ are branched over $H_i$ (respectively $H_i^{(\text{lev})}$) and the singular locus, but nowhere else.

$\mathcal{A}_p^*$ and $\mathcal{A}_p^{(\text{lev})^*}$ are normal projective varieties with only finite quotient singularities. In particular they are smooth in codimension 1, so $H_i$ and $H_i^{(\text{lev})}$ are the only branch divisors, and are $\mathbb{Q}$-Cartier Weil divisors.

**Corollary 4.3.** If $n \in \mathbb{N}$ is sufficiently divisible, $F_2 \in \mathcal{M}_2^*(\Gamma_p)$ and $F_n \in \mathcal{M}_n^*(\Gamma_p)$, then $n(K_{\mathcal{A}_p^*} + \frac{1}{2}H_1 + \frac{1}{2}H_2)$ is Cartier and $F_2^n F_n \omega^{\otimes n}$ determines an element of $H^0\left(\mathcal{A}_p^*, n\left(K_{\mathcal{A}_p^*} + \frac{1}{2}H_1 + \frac{1}{2}H_2\right)\right)$.

**Proof:** Clear: cf. [HS], Proposition 4.2. ■

**Corollary 4.4.** We may also also consider $F_2$ and $F_n$ as cusp forms for $\Gamma_p^{(\text{lev})}$ and $F_2^n F_n \omega^{\otimes n}$ as a $G$-invariant form with poles on $\mathcal{A}_p^{(\text{lev})}$, namely

$$F_2^n F_n \omega^{\otimes n} \in H^0\left(\mathcal{A}_p^{(\text{lev})}, n\left(K_{\mathcal{A}_p^{(\text{lev})}} + \frac{1}{2}H_1^{(\text{lev})} + \frac{1}{2}H_2^{(\text{lev})}\right)\right)^G.$$
Proof: Obvious.

As well as the Humbert surfaces, we must consider the non-canonical singularities of $A_p^*$, which also provide an obstruction to extending our pluricanonical forms to the whole of a smooth model.

**Proposition 4.5.** The non-canonical singularities of $A_p^*$ lie either in the boundary of $A_p^* \setminus A_p$ or in $H_1$.

Proof: [Br], Hauptsatz.

For the remainder of this section we shall be concerned with what happens in $H_1$.

**Proposition 4.6.** The non-canonical singularities of $A_p$ lying in $H_1$ are precisely the points of of the two curves $C_{3,1}$ and $C_{5,1}$, which are the images in $A_p$ of

$$C_{3,1} = \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \rho \end{pmatrix} \mid \tau_1 \in \mathbb{H} \right\}$$

$$C_{5,1} = \left\{ \begin{pmatrix} \rho & 0 \\ 0 & \tau_3 \end{pmatrix} \mid \tau_3 \in \mathbb{H} \right\}$$

where $\rho = e^{2\pi i/3}$. The transverse singularity at the generic point of each of these curves is the cone on the twisted cubic curve.

Proof: According to [Br], Hilfsätze 2.24 and 2.25, $H_1 \cap \text{Sing} A_p$ consists of four curves, $C_{2,1}, C_{3,1}, C_{4,1}$ and $C_{5,1}$, forming a square (see Figure 1).

The transverse singularity at a general point of $C_{4,1}$ or $C_{6,1}$ (specifically, at a point of either of these curves which is not also in another $C_{i,1}$ nor in the boundary of $A_p^*$) is an ordinary double point. The isotropy group $Z(x)$ of such a point is of order 8 but includes $-1_4$ and $I_1$, which generate a normal subgroup of index 2. At $x$, $-1_4$ acts trivially on the tangent space and $I_1$ acts by reflection, so the singularity is isomorphic to a quotient
singularity by an action of $Z(x)/\langle -\mathbf{1}_4, I_1 \rangle$, i.e., a threefold ordinary double point. Such a singularity is, of course, canonical.

At a general point of $C_{3,1}$ or $C_{5,1}$ the isotropy group has order 12 and again $-\mathbf{1}_4$ and $I_1$ generate a normal subgroup, of index 3. We could calculate the transverse singularity directly, but it is better to argue as follows. Suppose $x \in C_{5,1}$ but $x \neq P_{3,1}$ or $P_{1,1}$ (i.e. $x$ is in no other component of Sing $A_p$). Let $\tilde{x} \in A_p^{(lev)}$ be such that $\tilde{x} \mapsto x$ under $A_p^{(lev)} \to A_p$: then $\tilde{x} \in C_2$, in the notation of [HKW2]. The isotropy group $Z(\tilde{x})$ of $\tilde{x}$ in $\Gamma_p^{(lev)}$ has order 6 and does not contain $-\mathbf{1}_4$, so $Z(x) = \langle Z(\tilde{x}), -\mathbf{1}_4 \rangle$. But $-\mathbf{1}_4$ acts trivially so the singularity at $x \in A_p$ is the same as the singularity at $\tilde{x} \in A_p^{(lev)}$, and this is of transverse type the cone on the twisted cubic by [HKW2], Theorem 1.8. If instead $x \in C_{3,1}$, we use the existence of an extra automorphism $\Theta$ on $A_p$ which sends an abelian variety to its dual: it is represented by the element

$$
\Theta = \begin{pmatrix}
0 & \frac{1}{\sqrt{p}} & 0 & 0 \\
\sqrt{p} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{p} \\
0 & 0 & \frac{1}{\sqrt{p}} & 0 \\
\end{pmatrix} \in \text{Sp}(4, \mathbb{R})
$$
acting on $\mathbb{H}_2$ (one verifies that $\Theta$ normalises $\Gamma_p$ in $\text{Sp}(4, \mathbb{R})$). This automorphism interchanges $C_{3,1}$ and $C_{5,1}$, which therefore have the same singularities.

Finally, the singularity at the corner $P_{2,1} = C_{4,1} \cap C_{6,1}$ is canonical and the singularities at the other three corners are not, by the criterion of Reid, Shepherd-Barron and Tai: see [YPG] §4.11.

Let $\phi : A_p^{(\text{lev})*} \to A_p^*$ be the quotient morphism of Proposition 4.1. Let $\beta_1^{(\text{lev})} : A^{(\text{lev})} = \tilde{A}_p^{(\text{lev})*} \to A_p^{(\text{lev})*}$ be the resolution of singularities defined in [HS], p.18: that is, blow up along $C_1$ and $C_2$ in $A_p^{(\text{lev})*}$ and take a $G$-invariant resolution of the other singularities of $A_p^{(\text{lev})*}$. Then there is a diagram

$$
\begin{array}{ccc}
A^{(\text{lev})} & \xrightarrow{\phi} & \tilde{A}_p^{(\text{lev})*} \\
\downarrow & & \downarrow \\
A_p^{(\text{lev})*} & \xrightarrow{\beta_1} & A_p^*
\end{array}
$$

where $\phi$ and $\tilde{\phi}$ are quotient maps under the action of $G$, and $\beta_1$ factors through the blow-up of $A_p^*$ along $\phi(C_1) = C_{6,1}$ and $\phi(C_2) = C_{5,1}$. ($\beta_1$ also makes some other modifications in the boundary, which do not concern us.) $A^{(\text{lev})}$ is the same as the smooth variety $A$ studied in §4 of [HS]. The singularities at the general points of $C_{5,1}$ and $C_{6,1}$ are resolved by $\beta_1$.

We have used $C_{5,1}$ (etc.) to denote both a curve in $A_p$ and its closure in $A_p^*$. To this abuse of notation we add another: we will denote the strict transforms of $H_1$ and $H_i^{(\text{lev})}$ under $\beta_1$ and $\beta_1^{(\text{lev})}$ by $H_i$ and $H_i^{(\text{lev})}$ again. We denote the exceptional divisor of $\beta_1$ over $C_{5,1}$ by $E_{5,1}$, and so on, and similarly the exceptional divisor of $\beta_1^{(\text{lev})}$ over $C_i$ will be $E_i$; thus $E_1$ and $E_2$ are what are called $E$ and $E'$ in [HS], and $\tilde{\phi}(E_1) = E_{6,1}$, $\tilde{\phi}(E_2) = E_{5,1}$. 

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Proposition 4.7. The obstructions to extending pluricanonical forms over the general points of $E_{6,1}, H_1$ and $H_2$ are zero.

Proof: The singularities along $C_{6,1}$ are canonical at the general point. The obstructions coming from $E_1, H_1^{(\text{lev})}$ and $H_2^{(\text{lev})}$ are in any case shown to be zero in [HS], Corollary 4.7, Theorem 4.19 and Theorem 4.25. The obstructions we are interested in are just the $G$-invariant parts of those obstructions, hence also zero. ■

If $F_2$ and $F_n$ are cusp forms for $\Gamma_p$ of weights 2 and $n$ ($n$ sufficiently divisible) then $F_2^n F_n \omega^{\otimes n}$ gives, exactly as in [HS], a $G$-invariant element

$$\eta \in H^0(\mathcal{A}^{(\text{lev})}; n(K_{\mathcal{A}}^{(\text{lev})} + \frac{1}{2}H_1^{(\text{lev})} + \frac{1}{2}H_2^{(\text{lev})} + \frac{1}{4}E_1 + \frac{1}{2}E_2))^{G}$$

and by Proposition 4.7, above, and Proposition 4.8 of [HS], we have at once the following.

Proposition 4.8. The form $\eta$ is in fact a $G$-invariant element

$$\eta \in H^0(\mathcal{A}^{(\text{lev})}; nK_{\mathcal{A}}^{(\text{lev})})^{G}$$

provided it lies outside a subspace of dimension at most

$$\sum_{j=1}^{n/2} \dim H^0([n(K_{\mathcal{A}}^{(\text{lev})} + \frac{1}{2}H_1^{(\text{lev})} + \frac{1}{2}H_2^{(\text{lev})} + \frac{1}{4}E_1 + \frac{1}{2}E_2) - (\frac{n}{2} - j)E_2]^{G}).$$

Proof: As for [HS], Proposition 4.3, but taking $G$-invariant sections. ■

Put $n = 12n'$ and $L_j = [n(K_{\mathcal{A}}^{(\text{lev})} + \frac{1}{2}H_1^{(\text{lev})} + \frac{1}{2}H_2^{(\text{lev})} + \frac{1}{4}E_1 + \frac{1}{2}E_2) - (\frac{n}{2} - j)E_2]^{G}$. Let $\Sigma'$ and $\Phi'$ be a section and a fibre of the ruled surface $E_2 = E'$, as in [HS].
Theorem 4.9. The obstruction coming from \( E_2 \) to extending \( F^n_2 F_n \omega \otimes n \), where \( n \) is sufficiently divisible, is

\[
\left( \frac{7}{108} - \frac{1}{6p} \right) n^3 + O(n^2).
\]

Proof: We want to calculate \( \sum_{j=1}^{6n'} \dim H^0(L_j)^G \). By [HS], Proposition 4.11,

\[
L_j \equiv (12n' - 3j)\Sigma' + [6n'(\mu - 2\nu_{\infty}) - j\mu/2] \Phi',
\]

where \( \nu_{\infty} = (p^2 - 1)/12 \) and \( \mu = p\nu_{\infty} \): as in the proof of [HS], Theorem 4.12, \( H^0(L_j) = 0 \) for \( j > 4n' \) and \( L_j - K_{E_2} \) is ample for \( j \leq 4n' \).

To estimate \( \dim H^0(L_j)^G \) for \( j \leq 4n' \) we use the Atiyah-Bott fixed point theorem, much as we did in §2 above. By an elementary result about finite group representations (e.g. [Se], I.2.3)

\[
\dim H^0(L_j)^G = \frac{1}{|G|} \sum_{g \in G} \text{Trace} \left( g^* \big|_{H^0(L_j)} \right)
\]

\[
= \frac{1}{|G|} \sum_{g \in G} \sum_i (-1)^i \text{Trace} \left( g^* \big|_{H^i(L_j)} \right)
\]

since the higher cohomology vanishes. But by the fixed point theorem ([AS], [Hir]; cf. [T], Appendix to §2)

\[
\sum_{g \in G} \sum_i (-1)^i \text{Trace} \left( g^* \big|_{H^i(L_j)} \right) = \left\{ \text{ch}(L_j|_{E_2^g})(g) \cdot \mu_g \right\}[E_2^g]
\]

where \( \mu_g \) is a class depending only on \( g \), not \( L_j \). This is a polynomial whose total degree in \( n' \) and \( j \) is the degree of \( \text{ch}(L_j|_{E_2^g})(g) \) in \( n' \) and \( j \), which is at most \( \dim E_2^g \). Therefore
the only \( g \in G \) which make a contribution involving \( n'^2, n'j \) or \( j^2 \) are \( \pm 1 \). So

\[
\sum_{j=1}^{6n'} \dim H^0(L_j)^G = \sum_{j=1}^{4n'} \frac{1}{|G|} \sum_{g=\pm 1} \sum_i (-1)^i \text{Trace}(g^*|_{H^0(L_j)}) + O(n^2)
\]

\[
= \sum_{j=1}^{4n'} \frac{1}{|G|} \left( \text{Trace}(1^*|_{H^0(L_j)}) + \text{Trace}((-1)^*|_{H^0(L_j)}) \right) + O(n^2)
\]

\[
= \frac{2}{|G|} \sum_{j=1}^{4n'} \dim H^0(L_j) + O(n^2)
\]

\[
= \frac{2}{|G|} \left( \frac{7}{108} \mu - \frac{1}{6} \nu_\infty \right) n^3 + O(n^2)
\]

\[
= \left( \frac{7}{108} - \frac{1}{6p} \right) n^3 + O(n^2)
\]

using the calculation of \( \dim H^0(L_j) \) in [HS].

The obstruction coming from \( C_{3,1} \) could be calculated directly in a similar way but there is no need to do this. Instead we make use of the extra symmetry \( \Theta \) that corresponds to interchanging an abelian surface and its dual.

**Theorem 4.10.** The obstruction to extending a form \( \eta \) to a pluricanonical form on a resolution of the singularities of \( \mathcal{A}_p^* \) that lie in \( \mathcal{A}_p \) is contained in a space of dimension at most

\[
\left( \frac{7}{54} - \frac{1}{3p} \right) n^3 + O(n^2).
\]

**Proof:** Consider the blow-up \( \beta_2 : \tilde{\mathcal{A}}_p^* \to \tilde{\mathcal{A}}_p^* \) along the curves \( C_{3,1}, C_{4,1} \subseteq \tilde{\mathcal{A}}_p^* \). Clearly this resolves the singularities at the general points of \( C_{3,1} \) and \( C_{4,1} \). It is easy to see that the singularity of \( \tilde{\mathcal{A}}_p^* \) above \( P_{1,1} \) is an ordinary double point and thus canonical: hence all the singularities of \( \tilde{\mathcal{A}}_p^* \) away from the boundary are canonical. But we could also blow up \( C_{3,1} \)
and $C_{4,1}$ first and then $C_{5,1}$ and $C_{6,1}$. Let $\beta'_2 : \tilde{A}_p^* \to A_p^*$ and $\beta'_1 : \tilde{A}_p^* \to \tilde{A}_p'^*$ be these blow-ups. The left and right halves of the diagram

$$
\begin{array}{ccc}
\tilde{A}_p^* & \xrightarrow{\beta_1 \beta_2} & \tilde{A}_p'^*\\
\downarrow & & \uparrow \\
\bar{A}_p^* & & \bar{A}_p'^*
\end{array}
$$

are interchanged by $\Theta$, so if $\mathcal{V}$ is the space of forms $\eta$ extending over a resolution of $C_{5,1}$ then $\Theta^* \mathcal{V}$ is the space of forms extending over a resolution of $C_{3,1}$. Furthermore, if $\eta \in \mathcal{V} \cap \Theta^* \mathcal{V}$ then $\eta$ extends over the smooth part of the open subsets of $\tilde{A}_p^*$ and $\tilde{A}_p'^*$ where the birational map $\tilde{A}_p^* \to \tilde{A}_p'^*$ is an isomorphism. But this birational map is an isomorphism except over the corners $P_{1,1}$, $P_{1,2}$, $P_{1,3}$ and $P_{1,4}$ in $A_p^*$, and there the singularities of $\tilde{A}_p^*$ are canonical, so such an $\eta$ extends as in the statement of the theorem.

The codimension in $\mathcal{M}_{\nu}^n(\Gamma_p)$ of $\mathcal{V} \cap \Theta^* \mathcal{V}$ is at most twice the codimension of $\mathcal{V}$, which is what we need.

Remarks. a) Although we have not considered singularities in the boundary, the blow-ups we have made do affect them. We shall see later that they have changed them for the better, from our point of view.

b) In fact we do not expect the conditions coming from $C_{3,1}$ and $C_{5,1}$ to be independent, because the curves meet. But the space of common obstructions will be small, that is to say $O(n^2)$, so we cannot gain anything by calculating it.
5 Singularities in the boundary.

Unlike $A_p^{(lev)*}$, the Igusa compactification $A_p^*$ has non-canonical singularities in the boundary, which present a further obstacle to extending forms.

The boundary of $A_p$ consists of two disjoint open surfaces, $D^o(\ell_0)$ and $D^o(\ell_1)$ (equivalent under $\Theta$), whose closures $D(\ell_0)$ and $D(\ell_1)$ meet in a corank 2 boundary component, a curve called $E(h)$. This corresponds to the structure of the Tits building of $\Gamma_p$, shown in Figure 2. We shall also use $D(\ell_0)$, $D(\ell_1)$ and $E(h)$ to denote the strict transforms of these subvarieties in $\bar{A}_p^*$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Figure 2}
\end{figure}

**Theorem 5.1.** All the non-canonical singularities of $\bar{A}_p^*$ lie in the corank 2 boundary component $E(h)$.

**Proof:** The singularities of $A_p^*$ at the boundary are calculated in [Br]. In $D^o(\ell_0)$ (respectively $D^o(\ell_1)$) there are exactly four singular points, $Q_{1,0}$, $Q_{2,0}$, $Q_{3,0}$ and $Q_{4,0}$ (respectively $Q_{1,1}$, $Q_{2,1}$, $Q_{3,1}$ and $Q_{4,1}$). The singularities at $Q_{3,0}$, $Q_{4,0}$, $Q_{3,1}$ and $Q_{4,1}$ are isolated, but the others are not: in fact $Q_{1,0} = C_{4,1} \cap D^o(\ell_0)$, $Q_{2,0} = C_{3,1} \cap D^o(\ell_0)$, $Q_{1,1} = C_{6,1} \cap D^o(\ell_1)$ and $Q_{2,1} = C_{5,1} \cap D^o(\ell_1)$. All this is shown in [Br], Kapitel 3, Sätze 4.6, 4.7.
In the proofs of those theorems the isotropy groups are calculated. A neighbourhood of $D^0(\ell_0)$ in $A_p^*$ is isomorphic to a neighbourhood of $t_1 = 0$ in $\mathbb{C} \times \mathbb{C} \times \mathbb{H}/P''(\ell_0)$, where

$$P''(\ell_0) = \left\{ g'' = \begin{pmatrix} \varepsilon & \varepsilon m & \varepsilon n \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \mid \varepsilon = \pm 1, (m, n) \in \mathbb{Z}^2 E, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in E^{-1} \text{SL}(2, \mathbb{Z})E \right\}$$

for $E = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \in \text{GL}(2, \mathbb{Q})$. Here the action of $P''(h)$ on $(t_1, \tau_2, \tau_3) \in \mathbb{C} \times \mathbb{C} \times \mathbb{H}$ is given by

$$g'' : \begin{pmatrix} t_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} t_1 \exp\left\{2\pi i [m\tau_2 + (dm - c\tau_2 - cn)\varepsilon\tau_3] \right\} \\ \varepsilon(\tau_2 + m\tau_3 + n)(c\tau_3 + d)^{-1} \\ (a\tau_3 + b)(c\tau_3 + d)^{-1} \end{pmatrix}.$$

In particular $-1_3$ acts trivially and we are really concerned with the action of $P''(\ell_0)/\pm 1_3$.

$Q_{1,0}$ is represented by $(t_1, \tau_2, \tau_3) = (0, 0, p\iota)$ and the effective isotropy group is generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & p \\ 0 & -\frac{1}{p} & 0 \end{pmatrix}.$$  
This element has order 2 modulo $-1_3$ and $I_1 = \text{diag}(-1, 1, 1)$ (which acts as a reflection and therefore introduces no singularity): thus $Q_{1,0}$ is a quotient singularity of order 2, actually of type $\frac{1}{2}(0, 1, 1)$ in the usual notation for cyclic quotient singularities (see [YPG]). In any case it is resolved by $\beta_2$.

$Q_{2,0}$ is represented by $(t_1, \tau_2, \tau_3) = (0, 0, \rho)$, where $\rho = e^{2\pi i/3}$ and the effective isotropy group is generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -p \\ 0 & \frac{1}{p} & 1 \end{pmatrix}.$$  
This element has order 3 modulo $(-1_3, I_1)$ and the singularity is of type $\frac{1}{3}(0, 1, 1)$, neither isolated nor canonical. However, $Q_{2,0}$ is thus a point of $C_{3,1}$ with the same singularity as the general point and is therefore resolved by $\beta_2$. 

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\(Q_{3,0}\) is represented by \((t_1, \tau_2, \tau_3) = (0, \frac{p}{2}(i - 1), p\hat{i})\), and the isotropy comes from
\[
\begin{pmatrix}
1 & p & 0 \\
0 & 0 & p \\
0 & -\frac{1}{p} & 1
\end{pmatrix}.
\]
The singularity here is of type \(\frac{1}{2}(1, 1, 1)\) (the cone on the Veronese). One can see this by carrying out the same argument as in \([\text{HKW1}], \text{Proposition 2.8}\) (for the point \(Q'_1\)), or by the following direct algebraic argument. The point \(Q_{3,1}\), which is of the same type as \(Q_{3,0}\), is the image of \(Q'_1\) under \(\phi: \mathcal{A}_p^{(\text{lev})^*} \to \mathcal{A}_p^*\). The orbit of \(Q'_1\) under \(G\) consists of \(p^2 - 1\) points, one on each \(D_{\ell(a,b)}^0\) (each peripheral boundary component: see \([\text{HKW1}]\)), so \(|\text{Stab}_G Q'_1| = 2p\). Since \(-1 \in \text{Stab}_G Q'_1\) and acts trivially, the singularity at \(Q_{3,1}\) must either be the same as at \(Q'_1\) or a quotient of it by a cyclic group of order \(p\), depending on whether an element of order \(p\) in \(\text{Stab}_G Q'_1\) has trivial image in \(P''(\ell_0)\) or not (i.e. whether it acts trivially on the Zariski tangent space or not). But \(P''(\ell_0)\) has no \(p\)-torsion.

Exactly the same argument shows that the singularity at \(Q_{4,1}\), and hence the one at \(Q_{4,0}\), is the same as the singularity of \(\mathcal{A}_p^{(\text{lev})^*}\) at \(Q'_2\), which by \([\text{HKW1}]\) is a cyclic quotient singularity of type \(\frac{1}{3}(1, 2, 1)\). Both \(\frac{1}{2}(1, 1, 1)\) and \(\frac{1}{3}(1, 2, 1)\) are canonical singularities, so we have finished.

The singularities of \(\mathcal{A}_p^*\) lying on \(E(h)\) are described in \([\text{Br}]\). The picture is a little complicated, depending among other things on the residue class of \(p\) modulo 12. There are many (about \(p/6\)) isolated cyclic quotient singularities present, including all the ones of type \(\frac{1}{p}(r + 1, -r, r(r + 1))\) where the residue class of \(r\) mod \(p\) is not 0, 1 or a primitive cube or fourth root of unity in \(\mathbb{F}_p\). Such singularities are in general not canonical, as follows from \([\text{Mor}]\) and \([\text{MorS}]\). Moreover, the plurigenera of a non-canonical 3-fold cyclic quotient singularity \(P \in X\) of index \(p\) (that is, the dimension of the obstruction to extending
sections of $nK_X$ to a resolution) can be expected to be close to $p^2n^3$, and thus almost as big as $\dim \mathcal{M}_n^*(\Gamma_p)$. So a straightforward dimension count is unlikely to succeed and we shall have to find a special property of the pluricanonical forms we have chosen that allows them to extend.

**Remark.** To see why one expects the plurigenera to grow in this way it is easiest to use toric methods, as described in [YPG]. Put $N = \mathbb{Z}^3$, $M = \text{Hom}(N, \mathbb{Z})$, $N' = \mathbb{Z}^3 + \mathbb{Z} \frac{1}{p}(\nu_1, \nu_2, \nu_3) = \mathbb{Z}^3 + \mathbb{Z} \cdot n$ and $M' = \text{Hom}(N', \mathbb{Z})$, with $0 < \nu_i < p$. Then the plurigenus $P_n(n)$ associated with the toric morphism corresponding to the ray $\mathbb{R} \cdot n$ is given by the number of points of $M' \cap n \Delta_n^\circ$, where

$$\Delta_n = \{ x = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 = M \otimes \mathbb{R} \mid x_i > 1, \langle n, x \rangle < 1 \}$$

and $\Delta_n^\circ$ denotes the Fine interior (see [YPG], appendix to §4). If we are only interested in the growth of $P_n(n)$ with $n$ we may as well take the topological interior instead. We have $|M : M'| = p$ and

$$\text{Vol}_M(\Delta_n) = \frac{1}{6} \left( \frac{p}{\nu_1} - 1 \right) \left( \frac{p}{\nu_2} - 1 \right) \left( \frac{p}{\nu_3} - 1 \right)$$

so $P_n(n) \sim \frac{p^2n^3}{6\nu_1\nu_2\nu_3}$. If the obstruction for arbitrary forms is not to dominate $\dim \mathcal{M}_n^*(\Gamma_p)$, therefore, we should need $\nu_1\nu_2\nu_3 > 1440$; and this is the obstruction arising from just one blow-up on the way to resolving just one of the singularities. In general it is clear that the obstruction will be far too big.

We need a little more information about the singularities of $\mathcal{A}_p^*$ first.
**Lemma 5.2.** The non-canonical singularities of $A^*_p$ lying in the corank 2 boundary component $E(h)$ are all isolated and not in the closure of the branch locus of the quotient map $\mathbb{H}_2 \to A_p$.

**Proof:** According to [Br], the singularities in $E(h)$ that lie in the closure of the branch locus are either cyclic quotient singularities of order 2, of type $\frac{1}{2}(1, 2, 1)$, or quotients by $\mathbb{Z}_p \rtimes \mathbb{Z}_3$ in which generators $\zeta_3$ of $\mathbb{Z}_3$ and $\zeta_p$ of $\mathbb{Z}_p$ act by

\[
\zeta_3 : (z_1, z_2, z_3) \mapsto (z_3, z_1, z_2)
\]

\[
\zeta_p : (z_1, z_2, z_3) \mapsto (e^{2\pi i(1+r)/p}z_1, e^{-2\pi ir/p}z_2, e^{-2\pi i/p}z_3)
\]

for some $r \in \mathbb{Z}$. The first of these types occurs if $p \equiv 1$ or $p \equiv 5 \mod 12$, the second if $p \equiv 1$ or $p \equiv 7 \mod 12$. By the criterion of Reid, Shepherd-Barron and Tai these singularities are canonical. $\blacksquare$

We are going to use the weight 2 form a second time. We assume, for the moment, that we have a form that extends over the part we have already covered and produce from it a form (of higher weight) that extends everywhere.

**Theorem 5.3.** Suppose that $n = 3n'$ and that the form $F_n \omega^{\otimes n'}$ extends to an $n'$-canonical form on a resolution of singularities of $A^*_p$ except perhaps over the exceptional set coming from the singularities in the corank 2 boundary component $E(h)$ of $A^*_p$. Then the form $F^{2n}_n \omega^{\otimes n}$ extends to an $n$-canonical form on the resolution.

**Proof:** Let $\psi : \hat{A}^*_p \to A^*_p$ be a resolution of the non-canonical singularities in $E(h)$. We may assume that the exceptional locus of $\psi$ is a normal crossings divisor. It is enough to show that $F^{2n}_n \omega^{\otimes n}$ extends to a pluricanonical section near a general point of any exceptional component $E$. Suppose, then, that $P \in E$ and that $P$ is not in any other
exceptional component. Let \( \alpha \) be the discrepancy at \( E \): that is, \( K_{\hat{A}_p} - \psi^* K_A^* = \alpha E + E' \), where \( E' \) is supported on the other exceptional components. If \( \alpha \geq 0 \) there is nothing to prove, so assume \( \alpha < 0 \). Then we can choose an analytic open neighbourhood \( U \) of \( P \) in \( \hat{A}_p \) such that \( U \) does not meet the branch locus of \( \pi : \mathbb{H}_2 \to A_p \), by Lemma 5.2. Put \( \tilde{U} = \pi^{-1}(U) \cap F_p \), where \( F_p \) is a fundamental domain for the action of \( \Gamma_p \) on \( \mathbb{H}_2 \) (there is an explicit description of \( F_p \) in [Br], following [Fr]). Then \( \pi|_{\tilde{U}} : \tilde{U} \to U \setminus E \) is an isomorphism and we can identify \( \tilde{U} \) with \( U \setminus E \).

We have \( F_2^n \omega \otimes \pi \circ \pi = (F_3^2 \omega \otimes ^2 \otimes (n') \otimes (F_n \omega \otimes ^n') \). Choose local coordinates \((z_1, z_2, z_3)\) near \( P \) so that \( E \) is given by \( z_1 = 0 \) and take \( U \) to be the polycylinder \( \{|z_1| < \epsilon, |z_2| < \delta, |z_3| < \delta\} \), so \( \tilde{U} \cong \Delta_\epsilon^* \times \Delta_\delta^2 \). If we take a triple cover \( \hat{U} \to U \) branched along \( E \), by setting \((w_1^3, w_2, w_3) = (z_1, z_2, z_3)\), we can write, formally

\[
F_2 \omega^{2/3} = f(w)(dz_1 \wedge dz_2 \wedge dz_3)
\]

(for \( \hat{U} \) sufficiently small), and there is a Laurent series

\[
f(w) = \sum_{r=-\infty}^{\infty} a_r w_1^r
\]

where the \( a_r \) are analytic functions of \( w_2 \) and \( w_3 \). This is because on \( U \) we have \( f = F_2 J^{2/3} \), where \( J = \det(\partial \tau_i / \partial z_j) \) is the Jacobian determinant. \( J(z) \) is finite and nonzero on \( \hat{U} \) (since \( \tau_i \) and \( z_j \) are local coordinates at each point) so we can take a cube root of \( J \) on the triple cover \( \hat{U} \).

I claim that \( a_r = 0 \) for \( r \leq 0 \), that is, that the order of vanishing \( v_E(F_2^3 J^2) \) of \( F_2^3 J^2 \) along \( E \) is positive. Certainly \( v_E(F_2) > 0 \) as \( F_2 \) is a cusp form, so \( v_E(F_2^3) \geq 3 \). We need
to understand $J$, which means having a good description of the geometry near $E$. So we need to look a little more closely at the construction of the boundary.

Near $E(h)$ the structure of $A_p^*$ is toroidal: it is the quotient by a certain discrete group $P''(h)$ of an open subset of a torus embedding $T_N \text{emb}(\Sigma_h)$, where $\Sigma_h$ is a suitable fan. The singularities that we are concerned with arise because the fan $\Sigma_h$ that one naturally chooses is not basic (see [D] or [Od] for the terminology of torus embeddings). There are also some singularities arising from the fixed points of torsion elements in $P''(h)$ but they are canonical. So by taking an equivariant subdivision $\Sigma^+_h$ of $\Sigma_h$ we can obtain a resolution of singularities of the type we want. Our exceptional component $E$ then comes from orb$(\sigma)$ for some 1-dimensional cone $\sigma \in \Sigma^+_h \setminus \Sigma_h$. A local equation for $E$ at $P$ is thus $t_1^{b_1} t_2^{b_2} t_3^{b_3} = 0$, for suitable $b_i \in \mathbb{Z}$, where the $t_i$ are coordinate functions on the torus $T_N \cong (\mathbb{C}^*)^3$. This will be valid on some affine open toric variety, one of the pieces of $T_N \text{emb}(\Sigma_h)$.

So we may take $z_1 = t_1^{b_1} t_2^{b_2} t_3^{b_3}$. However, the torus embedding is defined by setting $t_1 = e^{2\pi i \tau_1}$, $t_2 = e^{2\pi i \tau_2/p}$ and $t_3 = e^{2\pi i \tau_3/p}$, so we have

$$z_1 = e^{2\pi i \{b_1 \tau_1 + b_2 \tau_2/p + b_3 \tau_3/p\}}$$

and $\partial z_1/\partial \tau_j = 2\pi i b_j z_i$. Hence $J^{-2}$ is of degree at most 2 in $z_1$, and thus $v_E(J^2) \geq -2$. Therefore $v_E(F_2^3 J^2) \geq 1$.

What this means is that the form $F_2^3 \omega \otimes^2$ has a zero along $E$, so that (over $U$, i.e., near $P$) $F_2^3 \omega \otimes^2 \in H^0(2K - E)$. By assumption, $F_n \omega \otimes n'$ is a section of $\psi^*(n'K_{A^*_p}) = n'(K - \alpha E)$. Therefore $F_2^n F_n \omega \otimes = (F_2^3 \omega \otimes^2 \otimes (F_n \omega \otimes n')$ gives a section of $n'(2K - E) + n'(K - \alpha E) = nK - n'(1 + \alpha)E$. But $E$ came from resolving a cyclic quotient.
singularity (see [Br]) and cyclic quotient singularities are log terminal (see, for instance, [CKM], Proposition 6.9): that is, $\alpha > -1$. So we have a section of $nK$, as required.

**Remark.** It is possible to show that $f(w)$ has a removable singularity along $E$ by $L_2$ methods, following Freitag ([Fr]) and Sakai ([Sak]). $F_2$ is $L_3$-integrable and this implies that $F_2\omega^{2/3}$ is $L_{2/3}$, that is, $\eta_2 = F_2^3\omega^2$ satisfies $\{\int_U (\eta_2 \wedge \bar{\eta}_2)\} < \infty$. The argument of [Sak], Lemma 1.1, shows that $a_r = 0$ if $r \leq -k$ for an $L_k$ form. In this case all we get is $a_r = 0$ for $r < 0$, which is not quite good enough. I do not know whether a better bound can be obtained by a refinement of the method of [Sak] in the case of non-integral $k$.

**6 Conclusions.**

We can now establish the main result by assembling the results we have proved so far.

**Theorem 6.1.** Any algebraic compactification of the moduli space $A_p$ of abelian surfaces with a polarisation of type $(1, p)$ for $p$ a prime is of general type if $p \geq 173$.

By Proposition 2.1 there is a weight 2 cusp form if $p > 71$. Suppose $n$ is sufficiently divisible. The obstructions to extending $\eta = F_2^n F_n \omega \otimes n$ to the whole of a resolution are the numerical obstruction given in Theorem 4.10 and the condition in Theorem 5.3 that $F_n \omega \otimes n'$ (where $n = 3n'$) should extend to the boundary near an isolated non-canonical singular point in $E(h)$. This second condition can be met, according to Proposition 3.1, by assuming that $F_n$ itself expressible as $F_2^{n'} F_{n'}$, where $F_{n'}$ is a cusp form of weight $n'$. So we need a form $\eta = F_2^{4n'} F_{n'} \omega \otimes 3n'$. The dimension of the space of such forms is $\frac{p^2 + 1}{8640} n'^3 + O(n'^2)$, by
Proposition 2.2. So, comparing this with the obstruction in Theorem 4.10, we see that the moduli space is of general type as long as $p > 71$ and

$$\frac{p^2 + 1}{8640} > \frac{7}{2} - \frac{9}{p}$$

which gives $p \geq 173$. ■

Remarks. a) This bound is not at all likely to be sharp. There will in general be weight $n$ forms $F_n$ not expressible as $F_2^{n'}F_{n'}$ such that the corresponding differential form $F_n\omega^{\otimes n'}$ nevertheless extends to the general point of the boundary. And for large $p$ there will be many independent weight 2 forms and therefore one expects there to be many more forms expressible as $F_2^{n'}F_{n'}$ than we have actually written down. However, it is not clear that such an expression should be unique if it exists, so we cannot easily determine the dimension of the relevant space.

b) In [GH] and [GS] the use of the weight 2 form was purely to get a better bound, but here we have made essential use of the existence of such a form (in Theorem 5.3) in order to get any bound at all.

It is probably true that the result of [Bor] still holds if $\text{Sp}(4, \mathbb{Z})$ is replaced by $\Gamma_p$. If this could be shown (it would be enough to show it for the primes not covered by Theorem 6.1) then all but finitely many moduli spaces of prime-polarised abelian surfaces would be of general type. Practically, it would be best to improve Theorem 6.1 first, so as to have only a few cases to deal with, if possible. Ideally one would like to drop the restriction that $p$ be a prime, but the singularities and the boundary become more complicated.

There ought to be a similar result even for $t$ composite, but the singularities become complicated (there is a partial description in [Br]): instead of four singular curves in $H_1$.
we have to handle two such curves for each ordered pair \((t', t'') \in \mathbb{N} \times \mathbb{N}\) such that \(t't'' = t\).

In general one expects \(\mathcal{A}_t^*\) to become closer to general type as \(t\) gets larger, but if \(t\) has many prime factors that tends to make \(\mathcal{A}_t^*\) have lower Kodaira dimension. For instance, it is shown in \([G]\) that \(p_g(\mathcal{A}_{13}^*) \geq 1\) and therefore \(\kappa(\mathcal{A}_{13}^*) \geq 0\), but Gross and Popescu ([GP]) have recently shown that \(\mathcal{A}_{14}^*\) is unirational. In some special cases, though, we can say a bit more.

**Corollary 6.2.** If \(q \in \mathbb{N}\) and \(p \geq 173\) is a prime, then any compactification of \(\mathcal{A}_{pq^2}^*\) is of general type.

**Proof:** If we put \(Q = \text{diag}(1, q^{-2}, 1, q^2) \in \text{Sp}(4, \mathbb{Q})\) then \(Q\Gamma_{pq^2}Q^{-1} \subseteq \Gamma_p\). This means that there is a surjective morphism \(\mathcal{A}_{pq^2}^* \to \mathcal{A}_p\) and this can be extended to suitable smooth compactifications. Hence \(\mathcal{A}_{pq^2}^*\) is of general type as long as \(\mathcal{A}_p^*\) is. \(\blacksquare\)

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