ROBUSTNESS TO MODELING ERRORS IN RISK-SENSITIVE
MARKOV DECISION PROBLEMS WITH MARKOV RISK
MEASURES

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Abstract. We consider risk-sensitive Markov decision processes (MDPs), where the MDP model is influenced by a parameter which takes values in a compact metric space. We identify sufficient conditions under which small perturbations in the model parameters lead to small changes in the optimal value function and optimal policy. We further establish the robustness of the risk-sensitive optimal policies to modeling errors. Implications of the results for data-driven decision-making, decision-making with preference uncertainty, and systems with changing noise distributions are discussed.

1. Introduction. Risk-sensitive Markov decision processes (MDPs) are an essential paradigm in applications where reliability is a key decision factor. Risk-sensitivity is often relevant to financial optimization and portfolio planning where the risk is due to extreme market events, and the decision maker (DM) is concerned with more than just expected performance. Risk-sensitive policies are also frequently deployed in critical infrastructure systems. For example, the electric grid needs to reliably meet random demand in the face of uncertainties due to weather, input prices, and renewable power. Similarly, industrial equipment, vehicles, supply chains, etc. all have to meet functional operating and safety requirements under a wide range of environmental conditions. In healthcare, planning for patient quality of life outcomes is fundamentally a risk-sensitive problem as well.

To solve any MDP in practice, we need to estimate or construct the model from data. In addition, in the risk-sensitive paradigm, we need to elicit and input the DM’s risk preferences to identify a specific risk-sensitive objective. In this paper, we capture both of these components via a single key ‘model parameter’ that completely characterizes the risk-sensitive MDP model. Specifically, it determines the state transition kernel, admissible action set, cost function, and risk-sensitive objective.

There is always fundamentally some modeling error in the choice of this parameter. First, when estimating the transition kernel and cost function, some statistical uncertainty is introduced. Second, there is modeling error in the risk-sensitive objective, due to the complexity of dynamic risk models and the difficulty of precisely eliciting the DM’s preferences. Third, the underlying physical system may change and degrade over time due to fatigue and equipment failures, etc. All of these effects may then be expressed as perturbations of the model parameter.

In this paper, we ask the following question: (Q) Under what conditions are the value functions and optimal policies in risk-sensitive MDPs robust to parameter perturbations? Suppose that an optimal risk-sensitive policy has been computed for a nominal parameter value and implemented. While the system is in operation, these parameters can drift leading to a perturbed risk-sensitive MDP. Question (Q) can then be rephrased as: (Q’) Under what conditions on the risk-sensitive MDP, do the value functions and optimal policies under the perturbed parameters approximate those for the original nominal one? Indeed, question (Q’) is equivalent to identifying sufficient conditions under which the value functions and optimal policies are continuous in the parameter. Next we present some examples illustrating the practical importance of this question.
1.1. Applications with Parametric Uncertainty. We overview three specific applications that help motivate the problem of sequential optimization under parametric uncertainty, and the issue of sensitivity to parameter perturbations.

1.1.1. EV Charging Systems. Consider a grid aggregator providing EV charging services to the customers in a city. The goal of the aggregator is to maximize operational profits by using renewable energy and scheduling the charging processes. Due to the uncertainties in the renewable energy generation and the charging requests from customers, this problem can be formulated as a risk-sensitive MDP, since the aggregator needs to take the risk of failing to serve demand into account.

These uncertainties are represented as functions or random variables in the system, which are usually parametric. For instance, due to the increase in renewable production over time, the statistics of renewable generation will drift. Due to spreading adoption of EVs, the charging time statistics of the EVs will also drift over time. On the other side, customer demand is affected by the price of charging, traffic, and time, all of which change dynamically. Therefore, in seeking robust strategies for using renewable energy and scheduling the charging processes, it is necessary to consider the impact of changes in these parameters on the risk-sensitive optimal profits and policies.

1.1.2. Reinforcement Learning. The DM in reinforcement learning (RL) sequentially evaluates the cost of taking certain actions in certain states. If the system is unknown to the DM, or if it is difficult to formulate an explicit model, then the DM will approximate the system with some parametrized one. For instance, the linear-quadratic-regulator (LQR) problem, which is widely applied in the field of robotics, uses a linear model to approximate the state transition function, and a quadratic model to approximate the cost function. The safe-RL problem is another example, where the DM cannot safely explore the entire state space because some exploration policies may lead to system instability. Thus, safe-RL algorithms only deploy conservative policies, which ensure that the reachable states are within a “safe set”. Usually, the safe set is represented by a parametric model that can be updated during the training process for policy exploration.

Such RL methods are successful because of the inherent connection between MDPs, RL, and perturbation analysis [10]. By the robustness property of MDPs, if the approximate model is close enough to the system model, then the DM will arrive at a near optimal policy.

1.1.3. Preference Uncertainty. There is an extensive literature on the problem of preference ambiguity in optimization and the difficulty of eliciting the DM’s risk preferences. In [2, 12], the authors develop robust models for risk-aware optimization where the DM’s risk preferences are expressed as an uncertainty set of utility/risk functions. The related stochastic dominance constrained optimization approach is developed in [13], where the dominance constraints express a requirement for an entire class of risk-sensitive DMs. The problem of preference uncertainty has not yet been studied extensively in the dynamic setting.

1.2. Related Works. The theory of risk-sensitive MDP is well-established. Howard and Matheson in [20] first incorporated risk sensitivity into an MDP by optimizing the expected exponential utility function of rewards/costs. Jaquette et. al. [22, 23] investigated MDPs with exponential utility functions and moment optimality, which lexicographically maximize the sequence of signed moments of the total discounted reward. Moreover, Porteus [30] identified certain conditions where
risk-sensitive MDPs can be solved with Bellman equations.

Other risk criteria have also been applied to the total cost of an MDP, for instance, mean-variance [25], average value-at-risk [4, 5], target value [39, 8] that measures the probability of the cost exceeding a target, and general monotone functions [11, 6]. In addition, Ruszczyński [32] proposed a dynamic risk measure that sequentially measures the risk of costs in the future with a nested decomposition, and proved that the risk-sensitive MDP can be solved with Bellman equations. We will further introduce the details of these risk-sensitive MDP models in subsection 2.2.

Our analysis relies on the theory of continuous parametric MDPs. This theory was first investigated in [27, 17] for classical risk-neutral MDPs, and conditions were identified such that the value function is continuous in the state. Stigum [37] used the continuity of a finite-horizon parametric dynamic programming (DP) problem to prove the existence of a competitive equilibrium in the context of the economy. This result was extended and refined later by Jordan [24] to establish continuity of the value function with respect to the parameter in an infinite-horizon parametric MDP. Dutta et. al. [14] also studied continuity of the value function, and relaxed the joint continuity assumption required by [24] to separate continuity in a parametric MDP with monotone value functions. All of these results are for a risk-neutral DM, whereas in this paper, we study the continuity of the value function and optimal policy for a risk-sensitive DM.

1.3. Contributions and Outline of this Paper. Our present work generalizes the continuity results for risk-neutral MDPs in [14] to risk-sensitive MDPs. Our key contributions are as follows:

1. We show that if a parametric risk measure is jointly continuous on its domain and parameter space, then the risk envelop in its biconjugate representation is hemicontinuous with respect to the parameter. This allows us to employ Berge’s Maximum Theorem to establish the continuity of the value function of the MDP with Markov risk measures.

2. We prove that if the cost function, the transition kernel, and the admissible action set are jointly continuous in the state, action, and parameter, then the value function of the risk-sensitive MDP is jointly continuous in the state and the parameter and the optimal policy is lower semicontinuous in the state and parameter.

3. We relax the above joint continuity conditions. We assume separate continuity of the cost function, the transition kernel, and the admissible action set in the state-action pair and the action-parameter pair. We further make some monotonicity assumptions on the MDP, so that the value function is a monotone non-decreasing function of the state. Under these conditions, we establish that the value function remains continuous in the state and parameter.

4. Finally, we propose sufficient conditions for the value function to be Lipschitz continuous with respect to the state and parameter. The corresponding Lipschitz coefficients of the value functions are also provided for both infinite and finite-horizon risk-sensitive MDPs. These coefficients explicitly bound the change of the value function in terms of the perturbation in the parameters. We further demonstrate that the policy remains lower semicontinuous in the state and the parameter in this setting.

This paper is organized as follows: in section 2, we formulate the risk-sensitive MDP and provide some preliminaries. We also pose our main questions about para-
metric risk-sensitive MDPs here. In section 3, we review the parametric continuity results for risk-neutral MDPs. In section 4, we present our main results: the sufficient conditions for the value function of the risk-sensitive MDPs to be continuous (we provide the proofs separately in section 5 for easier readability). In subsection 4.3, we determine sufficient conditions for the value function to be Lipschitz continuous, and identify the Lipschitz coefficients. We then provide some examples to illustrate the joint continuity of the risk measure in section 6. We conclude the paper in section 7.

1.4. Notation and Definitions.

1.4.1. Spaces. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, where \(\Omega\) is the set of scenarios, \(\mathcal{F}\) is the \(\sigma\)-algebra of events, and \(\mathbb{P}\) is the probability measure on \(\mathcal{F}\). We let \(L_p(\Omega, \mathcal{F}, \mathbb{P})\) denote the space of \(p\)-integrable random variables, i.e., \(\|X\|_p := \left(\int |X(\omega)|^p \mathbb{P}(d\omega)\right)^{\frac{1}{p}} < \infty\) for all \(X \in L_p(\Omega, \mathcal{F}, \mathbb{P})\). For a topological space \(\mathcal{X}\), let \(\mathcal{B}(\mathcal{X})\) denote the collection of all Borel measurable subsets, and \(\mathcal{M}(\mathcal{X})\) denote the collection of all probability measures on \(\mathcal{X}\).

We let \(\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}\) denote the extended real line. For any function \(f : \mathcal{X} \to \mathbb{R}\), we let \(\mathcal{X}\) be a normed space, we say \(\mu\) is \(L\)-Lipschitz if \(|f(x) - f(y)| \leq L\|x - y\|\) for all \(x, y \in \text{dom}(f)\).

1.4.2. Ordering. When \(\mathcal{X} = \mathbb{R}^d\), we endow it with the usual component-wise order: if \(x, x' \in \mathcal{X}\) with \(x \leq x'\), then \(x_i' \leq x_i'\) for all \(i = 1, \ldots, d\). For any \(X \in \mathcal{X}\), let \(X_+ := \max\{0, X\}\).

Given a random variable \(X : \Omega \to \mathcal{X}\), we say \(X \sim \mu\) for a probability distribution \(\mu : B(\mathcal{X}) \to [0, 1]\) if \(\mu(B) = \mathbb{P}(X \in B)\) for all \(B \in \mathcal{B}(\mathcal{X})\). The cumulative distribution function (CDF) of \(X\) is denoted by \(F_X(x) := \mathbb{P}(X \leq x)\) for all \(x \in \mathcal{X}\).

We endow the space of random variables on \(\mathcal{X}\) with the first stochastic order \(\preceq\). For two random variables \(X, X' : \Omega \to \mathcal{X}\) with \(X \sim \mu\) and \(X' \sim \mu'\) for \(\mu, \mu' \in \mathcal{M}(\mathcal{X})\), we have \(X \preceq X'\) if

\[\mathbb{P}(X \geq x) \leq \mathbb{P}(X' \geq x), \text{ for all } x \in \mathcal{X}.\]

In this case, we write \(\mu \preceq \mu'\).

1.4.3. Convergence. Let \(\mathcal{X} = L_p(\Omega, \mathcal{F}, \mathbb{P})\) for \(p \in (1, \infty)\) and \(\mathcal{X}^* = L_p^*(\Omega, \mathcal{F}, \mathbb{P})\) be the dual space of \(\mathcal{X}\). In this paper, the space \(\mathcal{X}^*\) is endowed with the weak* topology: We say a sequence of functions \(\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{X}^*\) converges in the weak* sense to \(g \in \mathcal{X}^*\), denoted by \(g_n \overset{w^*}{\to} g\), if for all \(f \in \mathcal{X}\),

\[\int f(\omega)g_n(\omega)\mathbb{P}(d\omega) \to \int f(\omega)g(\omega)\mathbb{P}(d\omega), \text{ as } n \to \infty.\]

A sequence of probability measures \(\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}(\Omega)\) converges in the weak* sense to \(\mu \in \mathcal{M}(\Omega)\), denoted by \(\mu_n \overset{w^*}{\to} \mu\), if for all \(f \in \mathcal{C}_b(\Omega)\),

\[\int f(\omega)\mu_n(d\omega) \to \int f(\omega)\mu(d\omega), \text{ as } n \to \infty.\]

We say that \(\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{M}(\Omega)\) converges to \(\mu \in \mathcal{M}(\Omega)\) setwise, denoted by \(\mu_n \overset{s}{\to} \mu\), if the above convergence holds for all measurable and bounded functions \(f \in L_\infty(\Omega, \mathcal{F}, \mathbb{P})\).
Let \( \mathcal{Y} \) be a metric space. A transition kernel \( q : \mathcal{B}(\mathcal{X}) \times \mathcal{Y} \to [0, 1] \) is weakly continuous if \( q(\cdot, y_n) \xrightarrow{w_N} q(\cdot, y) \) for all sequences \( \{y_n\}_{n \in \mathbb{N}} \) with \( y_n \to y \). Further, we say \( q \) is setwise continuous if for all measurable and bounded functions \( f : \mathcal{X} \to \mathbb{R} \),

\[
\int f(x)q(dx, y_n) \to \int f(x)q(dx, y), \text{ as } n \to \infty.
\]

In this case, we denote \( q(\cdot, y_n) \xrightarrow{w} q(\cdot, y) \).

A sequence of random variables \( \{X_n\}_{n \in \mathbb{N}} \) converges to \( X \) in \( \mathcal{L}_p \), denoted by \( X_n \xrightarrow{\mathcal{L}_p} X \), if \( \|X_n - X\|_p \to 0 \).

### 1.4.4. Correspondences

Let \( \mathcal{A} \) be a metric space and \( \mathcal{B} \) be a Hausdorff topological space. A correspondence \( \Psi : \mathcal{A} \rightrightarrows \mathcal{B} \) is a set-valued map such that \( \Psi(a) \subset \mathcal{B} \) for all \( a \in \mathcal{A} \). A correspondence is closed-valued (or compact-valued) if \( \Psi(\cdot) \) is closed (or compact) in \( \mathcal{B} \) for every \( a \in \mathcal{A} \).

We next recall the definition of upper/lower hemicontinuity of \( \Psi \) from [1]. A closed-valued correspondence \( \Psi : \mathcal{A} \rightrightarrows \mathcal{B} \) is upper hemicontinuous at \( a \in \mathcal{A} \) if and only if for any sequence \( \{a_n\}_{n \in \mathbb{N}} \subset \text{dom}(\Psi) \), and any sequence \( \{b_n\}_{n \in \mathbb{N}} \) with \( b_n \in \Psi(a_n) \), we have that \( a_n \to a \in \mathcal{A} \) and \( b_n \to b \in \mathcal{B} \) implies \( b \in \Psi(a) \). A correspondence \( \Psi : \mathcal{A} \rightrightarrows \mathcal{B} \) is lower hemicontinuous at \( a \in \mathcal{A} \) if and only if for any \( b \in \Psi(a) \) and for any sequence \( \{a_n\}_{n \in \mathbb{N}} \subset \text{dom}(\Psi) \) with \( a_n \to a \), there exists a sequence \( \{b_n\}_{n \in \mathbb{N}} \) and \( b_n \in \Psi(a_n) \) for all \( n \in \mathbb{N} \) such that \( b_n \to b \). A correspondence is continuous if it is both upper and lower hemicontinuous at all points \( a \in \mathcal{A} \).

#### 2. Problem Formulation

In this section, we define parametric MDPs, where all of the model information is expressed by a model parameter. The model parameter, denoted \( \theta \in \Theta \) where \( \Theta \) is a compact metric space, describes the cost function, transition kernel, admissible action set, and the DM’s risk preferences.

The underlying probability space is \( (\Omega, \mathcal{F}, \mathbb{P}) \), the state space is \( \mathcal{S} \), and the action space is \( \mathcal{A} \). The state and action spaces are assumed to be Borel subsets of Euclidean spaces. The MDP can be either finite-horizon with time index \( \{1, \ldots, T\} \) for \( T < \infty \) or infinite-horizon. The initial state \( S_0 = s_0 \) is fixed. We have a filtration \( \mathcal{F}_t \subset \mathcal{F} \subset \cdots \subset \mathcal{F}_T \), where \( \mathcal{F}_t \) is the \( \sigma \)-algebra generated by the random variables \( \{S_0, A_0, \ldots, S_t\} \) (where \( S_t \) and \( A_t \) are the random state and action at time \( t \)).

We write the dynamics of the system at time \( t \) as a Borel measurable function \( q_t : \mathcal{B}(\mathcal{S}) \times \mathcal{S} \times \mathcal{A} \times \Theta \to [0, 1] \). That is, under parameter \( \theta \) and state-action pair \((s_t, a_t) \in \mathcal{S} \times \mathcal{A} \), \( q_t(\cdot|s_t, a_t, \theta) \) is a probability measure on \( \mathcal{S} \), i.e.,

\begin{equation}
\mathbb{P}(S_{t+1} \in B|s_t, a_t, \theta) = q_t(B|s_t, a_t, \theta), \quad \text{for all } B \in \mathcal{B}(\mathcal{S}).
\end{equation}

We denote this measure succinctly as \( q_t(s_t, a_t, \theta) \), so (2.1) yields \( S_{t+1} \sim q_t(s_t, a_t, \theta) \).

The cost function at time \( t \) is \( c_t : \mathcal{S} \times \mathcal{A} \times \Theta \to [0, \infty) \), and \( c_T : \mathcal{S} \times \Theta \to [0, \infty) \) is the terminal cost function (which does not depend on the action) for the finite-horizon case.

An MDP is said to be stationary if \( q_t \equiv q \) and \( c_t \equiv c \) for all time \( t \in \mathbb{N} \). If an MDP is infinite-horizon, we assume that it is stationary. Let \( \gamma : \Theta \to [0, \bar{\gamma}] \) be the discount factor of future costs in the infinite-horizon MDP, where we assume that \( \bar{\gamma} < 1 \).

The set of admissible actions in state \( s \in \mathcal{S} \) with parameter \( \theta \) is given by a correspondence \( \Gamma : \mathcal{S} \times \Theta \to \mathcal{B}(\mathcal{A}) \). We let \( \mathcal{D}(\theta) := \{(s, a) \in \mathcal{S} \times \mathcal{A} : a \in \Gamma(s, \theta)\} \) denote the set of all feasible state-action pairs for parameter \( \theta \). For each time \( t \), the DM picks a map \( \pi_t : \mathcal{S} \times \Theta \to \mathcal{A} \) with \( \pi_t(s, \theta) \in \Gamma(s, \theta) \) for all \((s, \theta) \in \mathcal{S} \times \Theta \). Then
π := (π₀, π₁, . . .) ∈ Π denotes a policy for the MDP, where Π is the space of all feasible policies. A policy is said to be stationary if πᵣ = πᵣ' for all t ≠ t' ∈ N. Under the parameter θ ∈ Θ, the DM selects a policy π ∈ Π and faces the sequence of costs:

\[ Z_T(s₀, θ; π) := c₀(s₀, π₀(s₀, θ), \ldots, c_T(s_T, θ)) \]

for a finite-horizon;

\[ Z_∞(s₀, θ; π) := (c₀(s₀, π₀(s₀, θ), c₁(s₁, π₁(s₁, θ), \ldots) \]

for an infinite-horizon,

where we suppress the dependence on the underlying scenario for simplicity (i.e., \( Z_T(\cdot|s₀, θ; π) : Ω → ℝ^{T+1} \) and \( Z_∞(\cdot|s₀, θ; π) : Ω → ℝ^∞ \) are mappings from the underlying probability space to cost sequences).

2.1. Risk-Neutral Problem. The risk-neutral finite-horizon performance criteria is the expected total cost

\[ J_{RΚ,T}(s₀, θ; π) := \mathbb{E}\left[\sum_{t=0}^{T-1} cₜ(Sₜ, πₜ(Sₜ, θ), θ) + c_T(S_T, θ)\right], \]

and the risk-neutral finite-horizon MDP is

\[ \Psi_{RΚ,T} : \min_{π ∈ Π} J_{RΚ,T}(s₀, θ; π), \]

with optimal policy \( π^*_{RΚ,T} = \arg\min_{π ∈ Π} J_{RΚ,T}(s₀, θ; π) \). The (stationary) infinite-horizon performance criteria is the expected discounted total cost

\[ J_{RΚ,∞}(s₀, θ; π) := \mathbb{E}\left[\sum_{t=0}^{∞} \gamma^t c(Sₜ, π(Sₜ, θ), θ)\right], \]

and the risk-neutral infinite-horizon MDP is

\[ \Psi_{RΚ,∞} : \min_{π ∈ Π} J_{RΚ,∞}(s₀, θ; π), \]

with optimal policy \( π^*_{RΚ,∞} = \arg\min_{π ∈ Π} J_{RΚ,∞}(s₀, θ; π) \).

2.2. Risk-Sensitive Problem. We now consider risk-sensitive MDPs. To begin, we formalize the notion of a risk measure. Let \( X = \mathcal{L}_p(Ω, F, P) \) for \( 1 < p < ∞ \) be an admissible space of random variables. We have a risk measure \( ρ_θ : X → ℝ \) for each value of the parameter θ. A risk measure \( ρ_θ \) is coherent if it satisfies the following conditions, which were first introduced in [3].

DEFINITION 2.1 (Coherent Risk Measures). A risk measure \( ρ_θ \) is coherent if it satisfies:

(i) Monotonicity: If \( X, X' ∈ X \) and \( X(ω) ≤ X'(ω) \) for all \( ω ∈ Ω \), then \( ρ_θ(X) ≤ ρ_θ(X') \).

(ii) Convexity: If \( X, X' ∈ X \) and \( α ∈ [0, 1] \), then \( ρ_θ(αX + (1 - α)X') ≤ αρ_θ(X) + (1 - α)ρ_θ(X') \).

(iii) Translation equivalence: If \( α ∈ ℝ \) and \( X ∈ X \), then \( ρ_θ(X + α) = ρ_θ + α \).

(iv) Positive homogeneity: If \( a > 0 \) and \( X ∈ X \), then \( ρ_θ(aX) = aρ_θ(X) \).

Let \( L_t := \mathcal{L}_p(Ω, F_t, P) \) for all \( t ∈ N \), and let \( \{ρ_{θ,t}\}_{t∈ℕ} \) be a sequence of one-step conditional risk measures [32] where each \( ρ_{θ,t} : L_{t+1} → L_t \). We also suppose all \( \{ρ_{θ,t}\}_{t∈ℕ} \) are coherent as in Definition 2.1. For the finite-horizon case, the risk-sensitive objective is:

\[ J_T(s₀, θ; π) := c₀(s₀, π₀(s₀, θ), θ) + ρ_{θ,1}(c₁(S₁, π₁(S₁, θ), θ) + \cdots \]
This objective is a risk measure on finite sequences \( \rho_\theta : \prod_{t=0}^T \mathcal{L}_t \to \mathbb{R} \) constructed by composing the one-step risk measures. The corresponding risk-sensitive MDP is:

\[
\mathfrak{P}_T : \min_{\pi \in \Pi} J_T(s_0, \theta; \pi),
\]

with optimal policy \( \pi^* = \arg\min_{\pi \in \Pi} J_T(s_0, \theta; \pi) \). Let \( \{v_t\}_{t=0}^T \), where \( v_t : \mathcal{S} \times \mathcal{G} \to \mathbb{R}, \) be the value functions for \( \mathfrak{P}_T \). Similarly, for the infinite-horizon case, the risk-sensitive objective is:

\[
J_\infty(s_0, \theta; \pi) := c_0(s_0, \pi(s_0, \theta), \theta) + \rho_{\theta,1}\left(\gamma(\theta)c(S_1, \pi(S_1, \theta), \theta)
\right.
+ \rho_{\theta,2}\left(\gamma(\theta)^2c(S_2, \pi(S_2, \theta), \theta) + \cdots\right).
\]

This objective is a risk measure on infinite sequences \( \rho_\theta : \prod_{t=0}^\infty \mathcal{L}_t \to \mathbb{R} \), which is well-defined by [32, Theorem 3] under mild assumptions. The corresponding risk-sensitive MDP is:

\[
\mathfrak{P}_\infty : \min_{\pi \in \Pi} J_\infty(s_0, \theta; \pi),
\]

with optimal stationary policy denoted by \( \pi^*_\infty = \arg\min_{\pi \in \Pi} J_\infty(s_0, \theta; \pi) \). Let \( v : \mathcal{S} \times \mathcal{G} \to \mathbb{R} \) be the value function for \( \mathfrak{P}_\infty \) (in the stationary case).

### 2.3. Perturbation of Risk-Sensitive MDPs

The goal of the risk-sensitive MDP is to obtain the optimal value function and optimal policy \((\{v_t\}_{t=0}^T \text{ and } \pi^* \text{ for } \mathfrak{P}_T \text{ or } v \text{ and } \pi^*_\infty \text{ for } \mathfrak{P}_\infty)\). The main objective of this paper is to establish the continuity properties of the value function and policy as a function of \( \theta \). In particular, suppose \( \{\theta_n\}_{n \in \mathbb{N}} \) converges to \( \theta \), then we ask under what conditions:

Q1. Does \( \min_{\pi \in \Pi} J(s_0, \theta_n; \pi) \) converge to \( \min_{\pi \in \Pi} J(s_0, \theta; \pi) \) as \( n \to \infty \)?

Q2. Does the optimal policy \( \pi^*(s_0, \theta_n) \) converge to \( \pi^*(s_0, \theta) \) as \( n \to \infty \)?

As stated in the Introduction, the above questions are frequently encountered in market design, control of safety-critical systems, and distributional reinforcement learning. In particular, if the answer to Q1 is affirmative, then the value function of the risk-sensitive MDP under the nominal parameter \( \theta \) is “close” to the value function under the perturbed parameter \( \theta' \), when \( \theta \) and \( \theta' \) are close to each other. In addition, if the answer to Q2 is affirmative, then the respective optimal policies are also close. Thus, the DM can ignore minor perturbations of the model parameter and not recompute the value functions and optimal policies every time the parameter drifts. Indeed, in practice initial control policies are often designed at the time of manufacturing/installation but then not tuned for the rest of the system lifetime, even though the system components degrade and the underlying distributions of the operating conditions change over the lifetime.

### 3. Results for the Risk-Neutral Case

We briefly review the existing continuity results for \( \mathfrak{P}_{RN,T} \) and \( \mathfrak{P}_{RN,\infty} \) from [24, 14]. Our goal is to derive analogous results for the risk-sensitive MDPs \( \mathfrak{P}_T \) and \( \mathfrak{P}_\infty \). Under mild assumptions on the risk-neutral MDP, the value functions \( \{v_t(\cdot, \theta)\}_{t=0}^T \) and \( v(\cdot, \theta) \) exist for all \( \theta \in \mathcal{G} \); see, for example, [18, 38, 19]. We now recall conditions under which the value functions of an MDP are continuous, see [14].

**Assumption 3.1 (Jointly Continuous MDP).** For all \( t \in \mathbb{N} \):

\[
+ \rho_{\theta,1}(T, s_0, \theta) + \rho_{\theta,1}(T, \pi(s_0, \theta), \theta) + \rho_{\theta,2}(T, \gamma(\theta)c(S_1, \pi(S_1, \theta), \theta) + \cdots).
\]
(i) \( q_t \) is weak* continuous on \( S \times A \times \Theta \).
(ii) \( c_t \) is continuous on \( S \times A \times \Theta \) and bounded.
(iii) \( \Gamma : S \times \Theta \rightrightarrows A \) is continuous and is a compact-valued correspondence.
(iv) \( \gamma : \Theta \to (0, \bar{\gamma}] \) is continuous and \( \bar{\gamma} < 1 \).

Assumption 3.1 requires joint continuity with respect to the state, action, and parameter for all system components: cost functions, transition kernels, admissible action sets, and discount factor.

**Theorem 3.2** ([14], Theorem 1). Suppose Assumption 3.1 holds.

(i) The value functions \( \{v_t\}_{t=0}^T \) are continuous on \( S \times \Theta \), and \( \pi_{R^T}^*(\cdot, \cdot) \) is lower semi-continuous on \( S \times \Theta \).

(ii) The value function \( v \) is continuous on \( S \times \Theta \), and \( \pi_{R^\infty}^*(\cdot, \cdot) \) is lower semi-continuous on \( S \times \Theta \).

[14, Theorem 1] does not establish continuity for the finite-horizon case. However, one can readily adopt the proof technique of [14, Theorem 1] to arrive at the continuity result for finite-horizon MDPs by essentially the same argument.

Next we identify regularity assumptions for the class of monotone MDPs. Recall \( S \) is equipped with the element-wise order (see subsection 1.4.2).

**Assumption 3.3** (Monotone MDP). For every \( t \in \mathbb{N} \) and every \( s, s' \in S \) such that \( s \leq s' \), we have

(i) \( q_t(s, a, \theta) \leq q_t(s', a, \theta) \) for all \( (a, \theta) \in A \times \Theta \).
(ii) \( c_t(s, a, \theta) \leq c_t(s', a, \theta) \) for all \( (a, \theta) \in A \times \Theta \).
(iii) \( \Gamma(s, \theta) \geq \Gamma(s', \theta) \) for all \( \theta \in \Theta \).

Under Assumption 3.3, [38, Chapter 9] shows that the value function of a risk-neutral MDP is monotonically increasing (a simpler proof is presented in [26, Theorem 5]). In the following, we appeal to weaker separate continuity assumptions for monotone MDPs compared to Assumption 3.1.

**Assumption 3.4** (Separately Continuous MDP). For every \( t \in \mathbb{N} \):

(i) \( q_t(\cdot, \cdot, \theta) \) is weak* continuous on \( S \times A \) for every \( \theta \in \Theta \), and \( q_t(s, \cdot, \cdot) \) is weak* continuous on \( A \times \Theta \) for every \( s \in S \).
(ii) \( c_t(\cdot, \cdot, \theta) \) is continuous on \( S \times A \) for every \( \theta \in \Theta \), and \( c_t(s, \cdot, \cdot) \) is continuous on \( A \times \Theta \) for every \( s \in S \).
(iii) \( \Gamma(\cdot, \theta) \) is continuous on \( S \) for every \( \theta \in \Theta \), and \( \Gamma(s, \cdot) \) is continuous on \( \Theta \) for every \( s \in S \).
(iv) \( \gamma : \Theta \to (0, \bar{\gamma}] \) is continuous and \( \bar{\gamma} < 1 \).

**Theorem 3.5** ([14], Theorem 3). Suppose Assumption 3.3 and Assumption 3.4 hold. Then:

(i) \( \min_{\pi \in \Pi} J_{R^T}(\cdot, \cdot, \pi) \) is continuous on \( S \times \Theta \), and \( \pi_{R^T}^*(\cdot, \cdot) \) is lower semi-continuous on \( S \times \Theta \).
(ii) \( \min_{\pi \in \Pi} J_{R^\infty}(\cdot, \cdot, \pi) \) is continuous on \( S \times \Theta \), and \( \pi_{R^\infty}^*(\cdot, \cdot) \) is lower semi-continuous on \( S \times \Theta \).

4. **Main Results.** Although the desired parametric continuity results have been established for risk-neutral MDPs in [14], the risk-sensitive extension remains challenging. In the risk-neutral case, continuity of the value function follows directly from weak continuity of the transition kernel (since it is based on an expectation). In the risk-sensitive case, \( \rho_\theta \) can be any coherent risk measure, and these have widely differing forms.

We first need a universal representation of coherent risk measures to discuss their
continuity properties. The Fenchel-Moreau Theorem establishes that every law invariant\(^1\), proper\(^2\), and coherent risk measure \(\rho\) can be represented as its biconjugate as follows. Let \(\mathcal{X}^* = \mathcal{L}^*_p(\Omega, \mathcal{F}, \mathbb{P})\) be the dual space of \(\mathcal{X}\) endowed with the weak* topology. Then let

\[
P_\Omega := \left\{ \phi \in \mathcal{X}^* : \int \phi(\omega) \mathbb{P}(d\omega) = 1, \ \phi \geq 0 \right\}
\]

be the collection of probability density functions with respect to \(\mathbb{P}\) on \(\Omega\). Every element in \(P_\Omega\) can be identified with a probability measure on \((\Omega, \mathcal{F})\), which features \(\phi\) as its Radon-Nikodym derivative (or density) with respect to \(\mathbb{P}\).

The robust representation of \(\rho\) is then:

\[
\rho_\theta(X) := \sup_{\phi \in \Phi(\theta)} \langle X, \phi \rangle = \sup_{\phi \in \Phi(\theta)} \int X(\omega) \phi(\omega) \mathbb{P}(d\omega),
\]

where \(\Phi(\theta) \subset P_\Omega\) is the risk envelope, and we write \(\Phi : \Theta \ni \rho \mapsto P_\Omega\) as a correspondence to emphasize the dependence on \(\theta\). According to [31], the risk envelope is explicitly:

\[
\Phi(\theta) = \left\{ \phi \in P_\Omega : \langle \phi, X \rangle \leq \rho_\theta(X) \text{ for all } X \in \mathcal{X} \right\} = \bigcap_{X \in \mathcal{X}} \{ \phi \in P_\Omega : \langle \phi, X \rangle \leq \rho_\theta(X) \} \subset P_\Omega.
\]

Representation (4.1) essentially amounts to taking the supremum of expectations of the value function over a set of “tilted” distributions. We must establish the relationship between the continuity of the risk-sensitive value function and the continuity of the risk envelope.

For risk-sensitive MDPs, solving the DP decomposition requires some form of continuity (indeed, lower hemicontinuity) of the risk envelope. However, this assumption is not an obvious condition even if the one-step risk measures are continuous. In addition, the risk envelope \(\Phi\) is parameterized by the state, action, transition kernel, and parameter. Continuity of the supremum of the integrals in (4.1) will follow from the Berge Maximum Theorem. This proof technique requires us to establish the continuity of the integral and continuity of the risk envelope \(\Phi\) in (4.1). To show that the integral in (4.1) is continuous, we need to appeal to Lebesgue’s dominated convergence theorem with varying measures.

### 4.1. Jointly Continuous MDPs.

Continuity of sequential risk measures requires additional conditions on the one-step risk measures \(\rho_{\theta,t}\). Our main result leverages the class of Markov risk measures, which was first studied by Ruszczynski et. al. [32, Definition 6].

**Definition 4.1 (Markov Conditional Risk Measure and Risk Transition Mapping).** Let \(V_S := \mathcal{L}_p(S,\mathcal{B}(S),\mathbb{P})\). A sequence of risk measures \(\{\rho_{\theta,t}\}_{t \in \mathbb{N}}\), where \(\rho_{\theta,t} : V_S \to \mathbb{R}\), is Markov with respect to \(\{s_t\}_{t \in \mathbb{N}}\) under the following conditions. For any \(v(\cdot, \theta) \in V_S\) and \(\theta \in \Theta\), there exists a mapping \(\sigma_t : V_S \times S \times \mathcal{P}_S \times \Theta \to \mathbb{R}\) such that

\[
\rho_{\theta,t}(v(\cdot, \theta)) = \sigma_t(v(\cdot, \theta), s_t, q_t(s_t, a_t, \theta), \theta),
\]

where \(\{\sigma_t\}_{t \in \mathbb{N}}\) satisfies for every \(t \in \mathbb{N}\):

\(^1\)\(\rho\) is law invariant if two random variables \(X, X' \in \mathcal{X}\), \(\rho_\theta(X) = \rho_\theta(X')\) if \(F_X(u) = F_{X'}(u)\) for all \(u \in \mathbb{R}\). A detailed discussion of law invariant risk measures is given in [35].

\(^2\)\(\rho\) is proper if \(\rho_\theta(X) > -\infty\) for all \(x \in \mathcal{X}\) and its domain \(\text{dom}(\rho_\theta) := \{x \in \mathcal{X} : \rho_\theta(X) < \infty\} \neq \emptyset\).
1. For all \((s,a) \in D(\theta)\), the mapping

\[ v(\cdot, \theta) \mapsto \sigma_t(v(\cdot, \theta), s, q_t(s,a,\theta), \theta) \]

is a coherent risk measure on \(\mathcal{V}_S\).

2. For all \(v(\cdot, \theta) \in \mathcal{V}_S\) and every policy \(\pi(\cdot, \theta)\) measurable on \(\mathcal{S}\), the mapping

\[ s \mapsto \sigma_t(v(\cdot, \theta), s, q_t(s,\pi(s,\theta),\theta), \theta) \]

is an element of \(\mathcal{V}_S\).

Under these conditions, \(\{\sigma_t\}_{t \in \mathbb{N}}\) are referred to as risk transition mappings. For infinite-horizon risk-sensitive MDP, the risk mappings \(\sigma_t\) are stationary (i.e., \(\sigma_t \equiv \sigma\) for all \(t \in \mathbb{N}\)).

By [34, Theorem 2.2] (see also, [33]), each \(\sigma_t\) has the form

\[ \sigma_t(v(\cdot, \theta), s, q_t(s,a,\theta), \theta) = \sup_{\phi \in \Phi_t(s,q_t(s,a,\theta),\theta)} \langle v(\cdot, \theta), \phi \rangle, \]

where \(\Phi_t(s,q_t(s,a,\theta),\theta) \subset \mathcal{L}^*_p(\mathcal{S}, \mathcal{B}(\mathcal{S}), \mathbb{Q}_t)\). This aligns with the representation (4.1) and captures the dependence on the state \(s\), transition kernel \(q_t(s,a,\theta)\), and parameter \(\theta\). For the (stationary) infinite-horizon case, we have \(\mathbb{Q}_t \equiv \mathbb{Q}\) and \(\Phi_t \equiv \Phi\) for some \(\Phi\) such that

\[ \sigma_t(v(\cdot, \theta), s, q(s,a,\theta), \theta) = \sup_{\phi \in \Phi_t(s,q(s,a,\theta),\theta)} \langle v(\cdot, \theta), \phi \rangle. \]

We now present the key assumption on the parametric risk sensitive MDP to have continuous value functions.

**Assumption 4.2 (Jointly Continuous MDP).** For all \(t \in \mathbb{N}\):

(i) \(q_t\) is setwise continuous on \(\mathcal{S} \times \mathcal{A} \times \Theta\) and there exists a measure \(\mathbb{Q}_t \in \mathcal{M}(\mathcal{S})\) and a measurable function \(m_t : \mathcal{S} \times \mathcal{S} \times \mathcal{A} \times \Theta \rightarrow [0, \infty)\) such that \(q_t(ds'|s,a,\theta) = m_t(s';s,a,\theta)\mathbb{Q}_t(ds')\) and \(m_t(\cdot;s,a,\theta) \in \mathcal{L}^*_p(\mathcal{S}, \mathcal{B}(\mathcal{S}), \mathbb{Q}_t)\).

(ii) \(\Gamma: \mathcal{S} \times \Theta \rightarrow \mathcal{A}\) is continuous and is a compact-valued correspondence.

(iii) \(\gamma: \Theta \rightarrow (0, \gamma]\) is continuous and \(\gamma < 1\).

Next we present our main result for MDPs satisfying the joint continuity conditions given in Assumption 4.2.

**Theorem 4.3.** Suppose Assumption 4.2 holds. In addition, suppose the conditional risk measures \(\{\rho_{\theta,t}\}_{t \in \mathbb{N}}\) are Markov, coherent, and the risk envelopes \(\Phi_t\) are jointly continuous on \(\mathcal{S} \times \mathcal{M}(\mathcal{S}) \times \Theta\).

(i) Then, \(\min_{\pi \in \Pi} J_T(\cdot, \pi)\) is continuous on \(\mathcal{S} \times \Theta\) and \(\pi^*_T(\cdot, \cdot)\) is lower semi-continuous on \(\mathcal{S} \times \Theta\).

(ii) Suppose in addition that the conditional risk measures \(\{\rho_{\theta,t}\}_{t \in \mathbb{N}}\) are stationary. Then, \(\min_{\pi \in \Pi} J_\infty(\cdot, \pi)\) is continuous on \(\mathcal{S} \times \Theta\) and \(\pi^*_\infty(\cdot, \cdot)\) are lower semi-continuous on \(\mathcal{S} \times \Theta\).

### 4.2. Separately Continuous Monotone MDPs.

Next we weaken the joint continuity condition in Assumption 4.2 to separate continuity for monotone MDPs (see Assumption 3.3 for the requirements on monotone MDPs and Assumption 4.4 for the separate continuity requirements). We recall that if \(s_t, s'_t \in \mathcal{S}\) with \(S_{t+1} \sim q(s_t,a,\theta)\) and \(S'_{t+1} \sim q(s'_t,a,\theta)\), then \(q(s_t,a,\theta) \preceq q(s'_t,a,\theta)\) implies \(S_{t+1} \preceq S'_{t+1}\).
**Assumption 4.4 (Separately Continuous MDP).** For every \( t \in \mathbb{N} \):

(i) \( q_t(\cdot, \cdot, \theta) \) is setwise continuous on \( S \times A \) for any \( \theta \in \Theta \), and \( q_t(s, \cdot, \cdot) \) is setwise continuous on \( A \times \Theta \) for any \( s \in S \).

(ii) \( c_t(\cdot, \cdot, \theta) \) is continuous on \( S \times A \) for any \( \theta \in \Theta \), and \( c_t(s, \cdot, \cdot) \) is continuous on \( A \times \Theta \) for any \( s \in S \).

(iii) \( \Gamma(\cdot, \theta) \) is continuous on \( S \) for any \( \theta \in \Theta \), and \( \Gamma(s, \cdot) \) is continuous on \( \Theta \) for any \( s \in S \).

(iv) \( \gamma : \Theta \to (0, \bar{\gamma}] \) is continuous and \( \bar{\gamma} < 1 \).

**Theorem 4.5.** Suppose Assumption 3.3 and Assumption 4.4 hold. Also suppose the conditional risk measures \( \{\rho_{\theta,t}\}_{t \in \mathbb{N}} \) are Markov, coherent, and such that \( \Phi_t \) is jointly continuous on \( S \times M(S) \times \Theta \).

(i) Then, \( \min_{\pi \in \Pi} J_T(\cdot, \cdot; \pi) \) is continuous on \( S \times \Theta \) and \( \pi_T^* \) is lower semi-continuous on \( S \times \Theta \).

(ii) Suppose in addition that the conditional risk measures \( \{\rho_{\theta,t}\}_{t \in \mathbb{N}} \) are stationary. Then, \( \min_{\pi \in \Pi} J_\infty(\cdot, \cdot; \pi) \) is continuous on \( S \times \Theta \) and \( \pi_\infty^* \) is lower semi-continuous on \( S \times \Theta \).

In Section 6, we identify the conditions on the \( \Phi_t \) such that \( \Phi_t \) is a continuous correspondence. This yields continuity of \( \sigma_t \) in the value functions, state, action, and the parameter by an application of Berge’s maximum theorem [1].

**4.3. Lipschitz MDPs.** Coherent risk measures are subdifferentiable, see, e.g., [34, Section 3]. Since bounded subgradients imply Lipschitz continuity, this motivates us to demonstrate that the parametric value function is Lipschitz continuous in the state and parameter. This result allows us to establish explicit perturbation bounds for the value functions of risk-sensitive MDPs.

For any two metric spaces \( Y \times U \) with metrics \( d_Y \) and \( d_U \), we define the metric on \( Y \times U \) to be

\[
d_{Y \times U}((y, u), (y', u')) := d_Y(y, y') + d_U(u, u').
\]

Let \( 2^U \) denote the set of all compact subsets of \( U \). We endow \( 2^U \) with the Hausdorff metric

\[
d_H(U, U') := \max \left\{ \sup_{u \in U} d_U(u, U'), \sup_{u' \in U'} d_U(u', U) \right\},
\]

for all \( U, U' \in 2^U \), where

\[
d_U(u', U) := \inf_{u \in U} d_U(u', u).
\]

We recall from [19, Definition (ii), p.5]: a mapping \( f : X \times Y \to Z \) is uniformly Lipschitz continuous on \( Y \) if

\[
\sup_{y \in Y} \sup_{x \neq x'} \frac{d_Z(f(x, y), f(x', y))}{d_X(x, x')} < \infty.
\]

By [34, Corollary 3.1], a coherent risk measure \( \rho_{\theta,t} \) is continuous and subdifferentiable on the interior of its domain. Thus, by making additional boundedness assumptions on \( \rho_{\theta,t} \), we can preserve Lipschitz continuity over the entirety of \( \text{dom}(\rho_{\theta,t}) \) by [21, Lemma 2.1]. In this case, we let \( L_{\rho_{\theta,t}} < \infty \) be the Lipschitz coefficient of \( \rho_{\theta,t} \) for every \( \theta \in \Theta \), and we assume \( L_{\Theta,t} := \sup_{\theta \in \Theta} \{L_{\rho_{\theta,t}}\} < \infty. \)
Let us define $\mathcal{M}_{W_1}(\mathcal{S})$ as the set of measures over $\mathcal{S}$ endowed with the Wasserstein metric, denoted by $W_1$, (which makes it a complete separable metric space). For a risk measure $\rho_{\theta,t}$, define $\Psi_t : \mathcal{S} \times \mathcal{A} \times \Theta \Rightarrow \mathcal{M}_{W_1}(\mathcal{S})$ as

\[
\Psi_t(s,a,\theta) = \{ \psi \in \mathcal{M}_{W_1}(\mathcal{S}) : \psi(ds') = \phi(s')q(ds'|s,a,\theta), \phi \in \Phi_t(s,q(s,a,\theta),\theta) \}.
\]

We can define the usual Hausdorff metric on the compact subsets of $\mathcal{M}_{W_1}(\mathcal{S})$.

**Assumption 4.6 (Lipschitz MDP).** The following statements hold:

(i) $\Gamma$ is compact-valued and there exists $L_D \geq 0$, such that, for all $s,s' \in \mathcal{S}$, we have

\[
d_H(\Gamma(s,\theta), \Gamma(s',\theta')) \leq L_D \left( d_S(s, s') + d_\Theta(\theta, \theta') \right).
\]

(ii) The correspondence $\Psi$ is compact-valued and Lipschitz continuous with Lipschitz coefficient $L_{\rho,t}$:

\[
d_W(\Psi(s,a,\theta), \Psi(s',a',\theta')) \leq L_{\rho,t} \left( d_S(s, s') + d_A(a, a') + d_\Theta(\theta, \theta') \right).
\]

(iii) The cost function $c_t$ is $L_{c_t}$-Lipschitz continuous on $\mathcal{S} \times \mathcal{A} \times \Theta$.

(iv) For infinite-horizon risk-sensitive MDP, $L_{c_t} = L_c$ and $L_{\rho_t} = L_{\rho}$. Further, $L_D$ in (i) satisfy $(1 + L_{\rho})(1 + L_D) < 1/\bar{\gamma}$.

**Theorem 4.7.** Suppose Assumption 4.6 holds. Also suppose the conditional risk measures $\{\rho_{\theta,t}\}_{t \in \mathbb{N}}$ are Markov, coherent, and such that $\Phi_t$ is jointly continuous on $\mathcal{S} \times \mathcal{M}(\mathcal{S}) \times \Theta$. Then

(i) $\min_{\pi \in \Pi} J_T(\cdot, \cdot; \pi)$ is $L_{v_T}$-Lipschitz, where $L_{v_T}$ is recursively defined as

\[
L_{v_T} = L_{c_T},
\]

\[
L_{v_t} = L_{c_t}(1 + L_D) + L_{v_{t+1}}(1 + L_{\rho_{t+1}})(1 + L_D), \text{ for } t = 0, \ldots, T - 1.
\]

(ii) Suppose in addition $\{\rho_{\theta,t}\}_{t \in \mathbb{N}}$ are stationary, then $\min_{\pi \in \Pi} J_\infty(\cdot, \cdot; \pi)$ is $L_{\infty}$-Lipschitz where

\[
L_{\infty} = \frac{L_c(1 + L_D)}{1 - \bar{\gamma}(1 + L_{\rho})(1 + L_D)}.
\]

**4.4. Discussion.** Theorem 4.3 and Theorem 4.5 resolve the questions Q1 and Q2 under different hypotheses on the risk sensitive MDPs. Indeed, the continuity of $\min_{\pi \in \Pi} J_T(\cdot, \cdot)$ and $\min_{\pi \in \Pi} J_\infty(\cdot, \cdot)$ follows by assuming that: (i) the one-step risk measures are Markov and coherent (and also stationary for the infinite-horizon case) with continuous risk envelopes; and (ii) the cost function, the transition kernel, the admissible action set, and the discount factor are all continuous (as given in Assumption 4.2 and Assumption 3.3 with Assumption 4.4).

Under further assumptions on the MDP and the risk measure, Theorem 4.7 gives explicit bounds on the difference in value functions obtained from the nominal parameter and perturbed parameter, since the value functions are Lipschitz in the parameter.

**5. Proofs of Main Results.** We prove Theorem 4.3, Theorem 4.5, and Theorem 4.7 in this section.
5.1. Proof of Theorem 4.3. These claims are proven with the help of [14, Theorem 1], [32, Theorem 2], and [32, Theorem 4]. We first give the proof for $\mathfrak{P}_\infty$, and then elaborate on the variations needed for $\mathfrak{P}_T$.

**Case of $\mathfrak{P}_\infty$:** The proof for $\mathfrak{P}_\infty$ consists of the following steps:

1. We apply [32, Theorem 4] to show that $\mathfrak{P}_\infty$ has a DP decomposition.
2. We next show that the risk-sensitive Bellman operator is a contraction in $C_b(S \times \Theta)$.
3. Then, the conclusion follows by Berge’s Maximum Theorem.

We start by verifying conditions (i)-(v)3 required by [32, Theorem 4] for $\mathfrak{P}_\infty$. For every $\theta \in \Theta$:

1. By Assumption 4.2 (i), $q(\cdot,\cdot,\theta)$ is setwise continuous, which yields condition (i).
2. The $\{p_t,\gamma_t\}_{t \in \mathbb{N}}$ in Definition 4.1 are stationary Markov risk measures. Then, the fact that $\Phi$ is continuous yields condition (ii).
3. By Assumption 4.2 (ii), boundedness and continuity of $c$ yields conditions (iii) and (iv).
4. Assumption 4.2 (iii) yields condition (v).

Then, by [32, Theorem 4], $\mathfrak{P}_\infty$ can be solved by computing the optimal risk-sensitive value function $v(\cdot,\theta)$ which satisfies:

\begin{equation}
(5.1) \quad v(s,\theta) = \min_{a \in \Gamma(s,\theta)} c(s,a,\theta) + \gamma(\theta)\sigma(v(\cdot,\theta), s, q(s,a,\theta), \theta), \text{ for all } s \in S,
\end{equation}

where $\sigma$ is defined in (4.3). Now define the mapping $\hat{T} : C_b(S \times \Theta) \to C_b(S \times \Theta)$ by

\begin{equation}
(5.2) \quad \hat{T}(\hat{v})(s,\theta) := \min_{a \in \Gamma(s,\theta)} c(s,a,\theta) + \gamma(\theta)\sigma(\hat{v}(\cdot,\theta), s, q(s,a,\theta), \theta), \text{ for all } (s,\theta) \in S \times \Theta.
\end{equation}

We show that $v$ in (5.1) is the fixed point of $\hat{T}$ (i.e., $v = \hat{T}(v)$), which requires the following auxiliary result.

**Lemma 5.1.** For any $\hat{v} \in C_b(S \times \Theta)$, the mapping:

\begin{equation}
(5.3) \quad \sigma_{q,\hat{v}} : (s,a,\theta) \mapsto \sigma(\hat{v}(\cdot,\theta), s, q(s,a,\theta), \theta),
\end{equation}

is jointly continuous and bounded on $S \times A \times \Theta$.

**Proof.** See Appendix A.

By Lemma 5.1, together with the continuity of $c, A$, and $\gamma$ from Assumption 4.2 (ii), (iii), and (iv), we establish that $\hat{T}(\hat{v}) \in C_b(S \times \Theta)$ by applying Berge’s Maximum Theorem to the RHS of (5.2). Then, $\hat{T}$ is a contraction mapping by the following result.

**Lemma 5.2.** The mapping $\hat{T} : C_b(S \times \Theta) \to C_b(S \times \Theta)$ is a $\overline{\gamma}$-contraction in the supremum norm.

**Proof.** See Appendix B.

Now, by Lemma 5.2 we see that $v$ defined in (5.1) is the unique fixed point of $\hat{T}$. Thus, the fixed point satisfies $v \in C_b(S \times \Theta)$, as $C_b(S \times \Theta)$ is complete under the supremum norm, which proves the first part of Theorem 4.3 since $v(\cdot,\cdot) = \min_{\pi \in \Pi} J_\infty(\cdot,\cdot,\pi)$. 

---

3The joint continuity of $q, \Phi, c$ and $A$ are further required by [33, p.604].
For the second part, by [32, Theorem 4] the optimal policy \( \pi^* \) exists, and for each \( \theta \in \Theta \) it satisfies:

\[
(5.4) \quad \pi^*(s, \theta) \in \arg \min_{a \in \Gamma(s, \theta)} c(s, a, \theta) + \gamma(\theta)\sigma(v(\cdot, \theta), s, q(s, a, \theta), \theta), \text{ for all } s \in S.
\]

Again, Lemma 5.1 along with Assumption 4.2 (ii) and (iii) imply that \( \pi^*(\cdot, \cdot) \) is lower semicontinuous on \( S \times \Theta \) by Berge’s Maximum Theorem. This completes the proof of continuity for \( \mathcal{P}_T \).

**Case of \( \mathcal{P}_\infty \):** The proof for \( \mathcal{P}_T \) is by mathematical induction. Note that Assumption 4.2 and Definition 4.1 give the conditions required by [32, Theorem 2], and thus \( \min_{\pi \in \Pi} J(s_0, \theta; \pi) \) can be solved by the DP decomposition:

\[
\begin{align*}
v_T(s, \theta) &= c_T(s, \theta), \\
v_1(s, \theta) &= \min_{a \in \Gamma(s, \theta)} c_T(s, a, \theta) + \sigma_T(v_{t+1}(\cdot, \theta), s, q_T(s, a, \theta), \theta), \quad t = 0, \ldots, T - 1,
\end{align*}
\]

which gives the initial value function \( v_0 \). Starting from \( t = T \), the terminal cost \( v_T = c_T \) is jointly continuous and bounded by Assumption 4.2 (ii). For the induction step for time \( t = 0, \ldots, T - 1 \), suppose that \( v_{t+1} \in C_b(S \times \Theta) \). Then, by Lemma 5.1, the mapping \( \sigma_{q_T, v_{t+1}} \) is continuous and bounded. Thus, Assumption 4.2 implies that \( v_t \in C_b(S \times \Theta) \) and so \( \pi^*_t \) is lower semicontinuous by Berge’s Maximum Theorem, which completes the induction step. Taking \( v_0(s_0, \theta) = \min_{\pi \in \Pi} J_T(s_0, \theta; \pi) \), we establish continuity for \( \mathcal{P}_\infty \).

**5.2. Proof of Theorem 4.5.** This proof is similar to [14, Theorem 3], except we work with the risk transition mapping \( \sigma \) instead of the usual expectation of the value function.

**Case of \( \mathcal{P}_\infty \):** For every \( \theta \in \Theta \), Assumption 4.4 and Definition 4.1 yield the conditions required by [32, Theorem 4], which then implies that the optimal value function \( v \) solves the DP decomposition (5.1).

Let \( C_b^1(S \times \Theta) \) be the set of bounded and continuous functions on \( S \times \Theta \) that are monotonically increasing on \( S \). We will show that for any \( \tilde{v} \in C_b^1(S \times \Theta) \), the mapping \( \tilde{T} : C_b^1(S \times \Theta) \to C_b^1(S \times \Theta) \), defined by

\[
(5.5) \quad \tilde{T}(\tilde{v})(s, \theta) := \min_{a \in \Gamma(s, \theta)} c(s, a, \theta) + \gamma(\theta)\sigma(\tilde{v}(\cdot, \theta), s, q(s, a, \theta), \theta), \text{ for all } (s, \theta) \in S \times \Theta,
\]

has the value function \( v \) defined in (5.1) as its unique fixed point.

**Lemma 5.3.** The mapping:

\[
\sigma_{q, \tilde{v}} : (s, a, \theta) \mapsto \sigma(\tilde{v}(\cdot, \theta), s, q(s, a, \theta), \theta),
\]

is continuous on \( S \times A \) for every \( \theta \in \Theta \) and continuous on \( A \times \Theta \) for every \( s \in S \).

**Proof.** The proof is similar to the proof of Lemma 5.1. By fixing \( \theta \in \Theta \), we can prove that \( \sigma_{q, \tilde{v}} \) is jointly continuous on \( S \times A \), and by fixing \( s \in S \), we can prove that \( \sigma_{q, \tilde{v}} \) is jointly continuous on \( A \times \Theta \). We omit the details here for brevity.

Now, for any \( t \in \mathbb{N} \), \( (a, \theta) \in A \times \Theta \), and \( s_t, s'_t \in S \), let \( S_{t+1} \sim q(s_t, a, \theta) \) and \( S'_{t+1} \sim q(s'_t, a, \theta) \). Then \( s_t \leq s'_t \) implies \( S_{t+1} \preceq S'_{t+1} \) according to Assumption 3.2 (i). Thus, (4.3) yields

\[
\sigma(\tilde{v}(\cdot, \theta), s_t, q(s_t, a, \theta), \theta) = \rho_{a, t}(\tilde{v}(S_{t+1}, \theta))
\]
where inequality (a) holds because \( \tilde{v} \in C_b^1(\mathcal{S} \times \Theta) \) is monotone in \( s \), and the coherent risk measure \( \rho_{\theta,t} \) preserves the first stochastic order by [34, Lemma 5.1]. This directly implies that the mapping \( \sigma_{q,\bar{v}} \) is also monotone on \( \mathcal{S} \). Combined with Lemma 5.3 we have that \( \sigma_{q,\bar{v}} \) is jointly continuous on \( \mathcal{S} \times \mathcal{A} \times \Theta \) as a result of [14, Lemma 2]. Then, applying Berge’s Maximum Theorem to (5.5) we obtain \( \bar{T}(\bar{v}) \in C_b(\mathcal{S} \times \Theta) \).

Furthermore, for any \( s, s' \in \mathcal{S} \) with \( s \leq s' \), we have

\[
\min_{a \in \Gamma(s,\theta)} c(s, a, \theta) + \gamma(\theta)\sigma(\bar{v}(. , \theta), s, q(s, a, \theta), \theta) \\
\leq \min_{a \in \Gamma(s',\theta)} c(s, a, \theta) + \gamma(\theta)\sigma(\bar{v}(. , \theta), s, q(s, a, \theta), \theta) \\
\leq \min_{a \in \Gamma(s',\theta)} c(s', a, \theta) + \gamma(\theta)\sigma(\bar{v}(. , \theta), s', q(s', a, \theta), \theta),
\]

by Assumption 3.3 and the monotonicity of \( \sigma_{q,\bar{v}} \). It follows that \( \bar{T}(\bar{v}) \) is also monotone on \( \mathcal{S} \), and so \( \bar{T}(\bar{v}) \in C_b^1(\mathcal{S} \times \Theta) \). Therefore, \( \bar{T} \) is a contraction on \( C_b(S \times \Theta) \) with \( v \) as its fixed point by reasoning similarly to Lemma 5.2. Consequently, using Berge’s Maximum Theorem again we conclude that \( \pi^* \) is lower semicontinuous.

**Case of \( \mathcal{P}_\infty \):** The finite-horizon MDP can be proven analogously with mathematical induction, we omit the details here for brevity.

### 5.3. Proof of Theorem 4.7.

We apply the results of [19] to establish Lipschitz continuity of the value functions for both \( \mathcal{P}_\infty \) and \( \mathcal{P}_T \).

**Case of \( \mathcal{P}_\infty \):** Assumption 4.6 implies Assumption 4.2, which in turn implies \( v(s_0, \theta) = \min_{\pi \in \Pi} J_{\infty}(s_0, \theta; \pi) \) can be computed by (5.1). We recall the mapping \( \bar{T} \) defined in (5.2), and we restrict its domain to \( C_{L_0}(\mathcal{S} \times \Theta) \) for some \( L_0 < \infty \). We can then apply the results of [19] to show that its fixed point \( v = \bar{T}(v) \) is Lipschitz continuous. This argument requires the following auxiliary lemmas.

**Lemma 5.4.** The mapping \( (s, \theta) \mapsto \rho_{\theta,t}(\bar{v}(s, \theta)) \) is \( L_0 L_\rho \)-Lipschitz continuous on \( \mathcal{S} \times \Theta \) for any \( \bar{v} \in C_{L_0}(\mathcal{S} \times \Theta) \).

*Proof. See Appendix C.*

**Lemma 5.5.** The mapping \( \sigma_{q,\bar{v}} \) defined in (5.3) is \( L_0(1 + L_\rho) \)-Lipschitz continuous for any \( \bar{v} \in C_{L_0}(\mathcal{S} \times \Theta) \).

*Proof. See Appendix D.*

As a consequence of Lemma 5.5 with Assumption 4.6 parts (i), (iii), and (iv), we establish that the fixed point \( v = \bar{T}(v) \) is \( L_\infty \)-Lipschitz continuous by [19, Theorem 4.1], where

\[
L_\infty = \frac{L_c(1 + L_D)}{1 - \gamma(1 + L_\rho)(1 + L_D)}.
\]

This completes the first part of the proof.

**Case of \( \mathcal{P}_T \):** Lipschitz continuity for the value functions of the finite-horizon problem is again proven by mathematical induction. First note that the terminal cost \( v_T = c_T \) is \( L_{c_T} \)-Lipschitz continuous. For the induction step for time \( t = 0, \ldots, T - 1 \), assume \( v_{t+1} \) is \( L_{v_{t+1}} \)-Lipschitz continuous, then Lemma 5.5 yields that the mapping \( \sigma_{q,v_{t+1}} \) is \( L_{v_{t+1}}(1 + L_{\rho_T}) \)-Lipschitz continuous. Next, by [19, Lemma 3.2], \( v_t \) is \( L_{v_t} \)-Lipschitz continuous where \( L_{v_t} = L_{v_t}(1 + L_D) + L_{v_{t+1}}(1 + L_{\rho_T})(1 + L_D) < \infty \), which
completes the induction step. By picking \( v_0(\cdot, \cdot) = \min_{\pi \in \Pi} J_T(\cdot, \cdot ; \pi) \), we prove the result for \( \mathcal{P}_t \).

6. Continuity of Risk Measures and Risk Transition Mappings. Our main results assume the joint continuity of the risk envelope \( \Phi \) on \( S \times M(S) \times \Theta \). Under this condition, through an application of Berge’s Maximum Theorem, we are able to demonstrate that the risk measure is continuous in its arguments. This technique compels us to ask two questions:

1. Are there general parametric risk measures that are continuous over closed subsets of \( X \times \Theta \)?
2. Are there sufficient conditions under which risk transition mappings are continuous?

We devote this section to answering these two questions. First, we focus on deriving classes of parametric risk measures that are continuous under certain assumptions. Then, we identify a sufficient condition on the risk envelope under which it is both upper and lower hemicontinuous (and therefore, a continuous correspondence). A simple application of Berge’s Maximum Theorem then yields the desired continuity of the risk transition mapping.

6.1. Continuity of Risk Measures. In this section, we identify some examples of risk measures that are continuous on \( X \times \Theta \). We first recall the following useful result regarding convergence in \( L^1 \).

**Theorem 6.1** ([9], Theorem 4.5.4). Let \( \mathbb{P} \) be a probability measure. Suppose that \( f \) is a \( \mathbb{P} \)-measurable function and \( \{f_n\}_{n \in \mathbb{N}} \) is a sequence of \( \mathbb{P} \)-integrable functions. Then the following assertions are equivalent:

1. The sequence \( \{f_n\}_{n \in \mathbb{N}} \) converges to \( f \) in measure and is uniformly integrable.
2. The function \( f \) is integrable and the sequence \( \{f_n\}_{n \in \mathbb{N}} \) converges to \( f \) in \( L^1 \).

Note that if \( 1 \leq q < p < \infty \), then \( X_n \overset{\mathcal{L}_q}{\rightarrow} X \) implies \( X_n \overset{\mathcal{L}_p}{\rightarrow} X \) since

\[
\mathbb{E}[|X_n - X|^q]^{\frac{1}{q}} = \left( \mathbb{E}[|X_n - X|^p]^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq \mathbb{E}[|X_n - X|^p]^{\frac{1}{p}},
\]

where inequality (a) holds by Jensen’s inequality since \( f(x) = x^\frac{q}{p} \) is convex in \( x \). Furthermore, \( \{X_n\}_{n \in \mathbb{N}} \subset \mathcal{X} \) implies \( |X_n|^p \) is \( \mathbb{P} \)-integrable, and so \( X_n \) is \( \mathbb{P} \)-integrable. That is, the conditions required by Theorem 6.1 are fulfilled if we assume \( X_n \overset{\mathcal{L}_p}{\rightarrow} X \).

In all of the examples below, \( X \) is a real-valued random variable. In addition, we introduce a function \( \lambda : \Theta \rightarrow \mathbb{R}_+ \) that expresses the DM’s degree of risk-sensitivity (higher values of \( \lambda(\theta) \) mean the DM is more risk-sensitive).

**Example 6.2** (Worst-Loss Risk Measure). Consider a risk measure:

\[
\rho_\theta(X) := \mathbb{E}[X] + \lambda(\theta) \text{ess inf } X,
\]

where \( \text{ess inf} \) is the essential infimum of \( X \), i.e.,

\[
\text{ess inf } X = \sup \{ x \in \mathbb{R} : \mathbb{P}(X < x) = 0 \}.
\]

This \( \rho_\theta \) satisfies Definition 2.1.

Now suppose \( \Omega \) is compact and \( X \) and the sequence \( \{X_n\}_{n \in \mathbb{N}} \) are continuous on \( \Omega \), so \( \|X\|_\infty < \infty \) and we can replace \( \text{ess inf} \) with \( \inf \). It follows that

\[
\rho_\theta(X) = \mathbb{E}[X] + \lambda(\theta) \inf_{\omega \in \Omega} X(\omega),
\]
is continuous on $C_0(\Omega) \times \Theta$. This claim is based on the following observations. First, $X_n \xrightarrow{\mathcal{L}} X$ for $p \geq 1$ implies $X_n \to X$ in $\mathbb{P}$ and that $\{X_n\}_{n \in \mathbb{N}}$ is $\mathbb{P}$-uniformly integrable by Theorem 6.1, which directly implies $\mathbb{E}[X_n] \to \mathbb{E}[X]$. Next, since $\Omega$ is compact and $X$ is continuous, by Berge’s Maximum Theorem we have $\inf_{\omega \in \Omega} X_n(\omega) \to \inf_{\omega \in \Omega} X(\omega)$. Then, $\lambda(\theta_n) \inf X_n(\omega) \to \lambda(\theta) \inf X(\omega)$ since $\lambda$ is also continuous. It follows that $\rho_\theta(X)$ is jointly continuous.

**Example 6.3 (Mean-Deviation of Order $p$).** For $p \in [1, \infty)$, define

$$\rho_\theta(X) := \mathbb{E}[X] + \lambda(\theta) \|X - \mathbb{E}[X]\|_p.$$  

For $p = 2$, this is the standard mean-deviation model introduced in [28]. In this case, $\rho_\theta$ meets all but the monotonicity requirement of Definition 2.1 for $p > 1$. However, for $p = 1$, if $\mathbb{P}$ is non-atomic, then $\rho_\theta$ is coherent if and only if $\lambda(\theta) \in [0, 1/2]$ by [36, Example 6.19].

We establish continuity for $p = 1$ and $\lambda(\theta_n) \in [0, 1/2]$. Indeed, by the triangle inequality:

$$0 \leq \lim_{n \to \infty} (\|X_n - \mathbb{E}[X_n]\|_1 - \|X - \mathbb{E}[X]\|_1) \leq \lim_{n \to \infty} (\|X_n - X\|_1 + |\mathbb{E}[X_n] - \mathbb{E}[X]|) \overset{(a)}{=} 0,$$

where equality (a) holds since: (i) $\lim_{n \to \infty} \|X_n - X\|_1 = 0$ by $X_n \xrightarrow{\mathcal{L}} X$, and (ii) $\lim_{n \to \infty} |\mathbb{E}[X_n] - \mathbb{E}[X]| = 0$ as we have shown in Example 6.2. Then it follows that

$$\lambda(\theta_n) \|X_n - \mathbb{E}[X_n]\|_1 \to \lambda(\theta) \|X - \mathbb{E}[X]\|_1,$$

which implies $\rho_\theta(X)$ is jointly continuous.

**Example 6.4 (Mean-Upper-Semideviation of Order $p$).** Recall

$$\rho_\theta(X) := \mathbb{E}[X] + \lambda(\theta) \|(X - \mathbb{E}[X])_+\|_p,$$

is the mean-upper-semideviation, which is a coherent risk measure. In contrast to the mean-deviation, $\rho_\theta$ is monotonic if $\lambda(\theta) \in [0, 1]$ and $\mathbb{P}$ is non-atomic [29].

We show that if $\lambda(\theta) \in [0, 1]$, then

$$\rho_\theta(X) := \mathbb{E}[X] + \lambda(\theta) \|(X - \mathbb{E}[X])_+\|_p,$$

is continuous. Indeed, since $|(X_n - \mathbb{E}[X_n])_+|^p \leq |X_n - \mathbb{E}[X_n]|^p$ for all $n \in \mathbb{N}$ and $p \in [1, \infty)$, the Dominated Convergence Theorem implies

$$\|(X_n - \mathbb{E}[X_n])_+\|_p \to \|(X - \mathbb{E}[X])_+\|_p,$$

as $n \to \infty$,

where the convergence $\|(X_n - \mathbb{E}[X_n])_+\|_p \to \|(X - \mathbb{E}[X])_+\|_p$ is based on a similar argument as (6.1). This yields the joint continuity of $\rho_\theta(X)$.

**Example 6.5 (Certainty Equivalent).** Let $\mathfrak{U}_\theta : \mathbb{R} \to \mathbb{R}$ be a utility function which is continuous and monotonically increasing (and thus $\mathfrak{U}_\theta^{-1}$ exists), for all $\theta \in \Theta$. The corresponding certainty equivalent is $\rho_\theta(X) = \mathfrak{U}_\theta^{-1}(\mathbb{E}[\mathfrak{U}_\theta(X)])$. For general $\mathfrak{U}_\theta$, the certainty equivalent is not coherent as it may fail to satisfy positive homogeneity and convexity. However, for the exponential utility function $\mathfrak{U}_\theta(x) = \exp(\lambda(\theta)x)/\lambda(\theta)$, the homogenization procedure [36] produces the following coherent risk measure:

$$\rho_\theta(X) = \inf_{\tau > 0} \tau \mathfrak{U}_\theta^{-1}\left(\mathbb{E}\left[\mathfrak{U}_\theta\left(\frac{X}{\tau}\right)\right]\right) \text{ is coherent.}$$
Suppose $\Upsilon_\theta(X)$ is jointly continuous on $\mathcal{V}_{S \times \Theta} \times \Theta$. If we further assume that the homogenization given in (6.2) satisfies
\[
\lim_{\tau \to 0} \tau \Upsilon^{-1}_\phi \mathbb{E}[\Upsilon_\theta(X/\tau)] = \infty,
\]
then $X \to \tau \Upsilon^{-1}_\phi \mathbb{E}[\Upsilon_\theta(X/\tau)]$ is KN-inf-compact on the graph $\{(\tau, X, \theta) \in \mathbb{R} \times \mathcal{V}_{S \times \Theta} \times \Theta : \tau > 0\}$ by [15, Definition 1.3]. Then, the generalization of Berge’s Maximum Theorem [15, Theorem 1.4] yields the joint continuity of $\rho_\theta(X) = \inf_{\tau > 0} \tau \Upsilon^{-1}_\phi \mathbb{E}[\Upsilon_\theta(X/\tau)]$ on $\mathcal{V}_{S \times \Theta} \times \Theta$.

**Example 6.6 (Conditional Value-at-Risk).** The Value-at-Risk (VaR) at level $u \in [0, 1]$ is defined as
\[
\text{VaR}_u(X) := F^{-1}_X(u) = \inf \{x : \mathbb{P}(X \leq x) \geq u\},
\]
where $F^{-1}_X$ is also referred to as the inverse CDF or quantile function. It is well known that VaR does not satisfy the sub-additivity property (which is implied by convexity and positive homogeneity). A common alternative to VaR is the CVaR (at level $\lambda(\theta) \in [0, 1]$), which is defined as:
\[
\text{CVaR}_{\lambda(\theta)}(X) := \frac{1}{1-\lambda(\theta)} \int_{\lambda(\theta)}^1 \text{VaR}_u(X)du,
\]
and we let $\rho_\theta(X) = \text{CVaR}_{\lambda(\theta)}(X)$.

Suppose the CDF $F_{X_n}$ of $X_n$ is continuous for all $n \in \mathbb{N}$, and all $X_n : \Omega \to [0, \infty)$. We will show that $\text{CVaR}_{\lambda(\theta)}(X)$ is jointly continuous on $\mathcal{V}_{S \times \Theta} \times \Theta$ by the Dominated Convergence Theorem in Appendix F.

**6.2. Continuity of Risk Transition Mapping.** We now address the question of establishing the continuity of the risk transition mapping defined in (4.4). Towards this end, define $\mathcal{H}$ to be the set of sequentially continuous functions:
\[
\mathcal{H} = \left\{ h : S \times \mathcal{L}^*_p(S, \mathcal{B}(S), \mathbb{Q}) \times \Theta \to \mathcal{L}^*_p(S, \mathcal{B}(S), \mathbb{Q}) : h \text{ is sequentially continuous} \right\}.
\]
For a transition kernel $q(s, a, \theta) \in \mathcal{M}(S)$, we identify it by its Radon-Nikodym derivative $m \in \mathcal{L}^*_p(S, \mathcal{B}(S), \mathbb{Q})$ with respect to $\mathbb{Q}$. We use $M$ to denote this mapping, that is, $m(s') = M(q(s, a, \theta))(s') \in \mathcal{L}^*_p(S, \mathcal{B}(S), \mathbb{Q})$. We make the following assumption to proceed.

**Assumption 6.7.** Let $\mathcal{H}' \subset \mathcal{H}$ be any subset. There exists $\tilde{\phi} \in \mathcal{L}^*_p(S, \mathcal{B}(S), \mathbb{Q})$ satisfying $\int \tilde{\phi}(s')\mathbb{Q}(ds') < \infty$ such that $\Phi(s, q(s, a, \theta), \theta)$ is given by
\[
\Phi(s, q(s, a, \theta), \theta) = \left\{ \phi \in \mathcal{L}^*_p(S, \mathcal{B}(S), \mathbb{Q}) : \phi = h(s, m, \theta) m, h \in \mathcal{H}' \subset \mathcal{H}, m = M(q(s, a, \theta)), 0 \leq \phi(s') \leq \tilde{\phi}(s') \right\}.
\]

**Lemma 6.8.** Suppose that the Assumption 6.7 holds. Then, the correspondence $\Phi : S \times \mathcal{L}^*_p(S, \mathcal{B}(S), \mathbb{Q}) \times \Theta \Rightarrow \mathcal{L}^*_p(S, \mathcal{B}(S), \mathbb{Q})$ defined in (4.4) is a continuous correspondence.

*Proof.* See Appendix E for the proof.\[
\square
\]
One can now simply apply Berge’s Maximum Theorem to show that the risk transition mapping $\sigma_t(\cdot, s, q_t(s, a, \theta), \theta)$ is continuous in $v(\cdot, \theta)$. We note here that in [33], the author provides an example of a continuous risk-transition mapping in Example 1. Lemma 6.8 generalizes that result and [33, Example 1] is a special case of Lemma 6.8.
7. Conclusion. In this paper, we consider risk-sensitive MDPs based on nested Markov risk measures, as elucidated in [32]. In our framework, both the system parameters and the DM’s risk preferences are encoded through a model parameter that is subject to perturbation. We examine sufficient conditions for the value functions to be continuous on the parameter space for both finite-horizon and infinite-horizon MDPs. Our first result requires the system model and the risk measure to be jointly continuous over the state, action, and parameter spaces. Then, we relax this assumption to only require separate continuity for monotone MDPs. In this way, our results generalize the parametric continuity results for risk-neutral MDPs in [14] to the class of risk-sensitive MDPs.

Appendix A. Proof of Lemma 5.1. We prove the claim by applying Berge’s Maximum Theorem [1, Theorem 17.31] to \( \hat{\sigma} \) defined in (4.5). For \( \hat{\sigma} \in C_b(S \times \Theta) \), we define:

\[
\sigma(\hat{\sigma}(.), s, q(s, a, \theta), \theta) = \sup_{\phi \in \Phi(s, q(s, a, \theta), \theta)} \langle \hat{\sigma}(.), \phi \rangle.
\]

(A.1)

Consider the map

\[
(s, a, \theta, \phi) \mapsto \int \hat{v}(s', \theta)\phi(s')q(ds'|s, a, \theta),
\]

where \( \phi \in \mathcal{P}_S \) and

\[
\mathcal{P}_S := \left\{ \phi \in \mathcal{X}^* : \int \phi(s)Q(ds) = 1, \phi \geq 0 \right\}
\]

is endowed with the weak* topology.

For any sequence \( \{\theta_n\}_{n \in \mathbb{N}} \subset \Theta \) with \( \theta_n \to \theta \), since \( \hat{v} \) is jointly continuous on \( S \times \Theta \), we have

\[
\left\{ s' \in S : \lim_{n \to \infty} \hat{v}(s'_n, \theta_n) \neq \hat{v}(s', \theta), \text{ for all } s'_n \to s' \right\} = \emptyset.
\]

Moreover, since \( \phi_n \xrightarrow{w^*} \phi \) and \( q(s_n, a_n, \theta_n) \xrightarrow{w} q(s, a, \theta) \), we conclude that

\[
\phi_n q(s_n, a_n, \theta_n) \xrightarrow{w^*} \phi q(s, a, \theta).
\]

Thus, for any \( \hat{v} \in C_b(S \times \Theta) \),

\[
\int \hat{v}(s', \theta)\phi_n(s')q(ds'|s_n, a_n, \theta_n) \to \int \hat{v}(s', \theta)\phi(s')q(ds'|s, a, \theta),
\]

which establishes that the map (A.2) is continuous by [7, Theorem 5.5].

Next, by [34, Proposition 6.2] we have that \( \Phi \) is weakly compact for every \( (s, a, \theta) \in S \times A \times \Theta \). Then Assumption 4.2 (i) and the joint continuity of \( \Phi \) imply that the mapping \( (s, a, \theta) \mapsto \Phi(s, q(s, a, \theta), \theta) \) is continuous. Thus, an application of Berge’s Maximum Theorem [1] yields the continuity of \( \sigma_{\hat{\sigma}, \hat{\theta}} \) on \( S \times A \times \Theta \).

Appendix B. Proof of Lemma 5.2. Fix a policy \( \pi \in \Pi \) and define the mapping \( \hat{T} \) by

\[
\hat{T}_x(\hat{v})(s, \theta) = c(s, \pi(s, \theta), \theta) + \gamma(\theta)\sigma(\hat{\sigma}(., \theta), s, q(s, \pi(s, \theta), \theta, \theta), \theta).
\]
For any \( \hat{v}_1, \hat{v}_2 \in C_b(S \times \Theta) \) (endowed with the supremum norm) we have:

\[
\left\| \hat{T}_\pi(\hat{v}_1) - \hat{T}_\pi(\hat{v}_2) \right\|_\infty = \left\| c(s, \pi(s, \theta), \theta) + \gamma(\theta)\sigma(\hat{v}_1(\cdot, \theta), s, q(s, \pi(s, \theta), \theta)) - c(s, \pi(s, \theta), \theta) - \gamma(\theta)\sigma(\hat{v}_2(\cdot, \theta), s, q(s, \pi(s, \theta), \theta)) \right\|_\infty \\
= \sup_{(s, \theta) \in S \times \Theta} \left| \gamma(\theta) \left( \sigma(\hat{v}_1(\cdot, \theta), s, q(s, \pi(s, \theta), \theta), \theta) - \sigma(\hat{v}_2(\cdot, \theta), s, q(s, \pi(s, \theta), \theta), \theta) \right) \right| \\
\leq \gamma \sup_{(s, \theta) \in S \times \Theta} \left| \hat{v}_1(\cdot, \theta) - \hat{v}_2(\cdot, \theta) \right| \\
\leq \gamma \left\| \hat{v}_1 - \hat{v}_2 \right\|_\infty
\]

where (a) holds due to the bound on the discount factor \( \gamma(\theta) \leq \bar{\gamma} \) for every \( \theta \in \Theta \), and (b) holds since \( \Phi \) is a set of probability density functions. Then, \( \hat{T}_\pi \) is a contraction for any \( \pi \in \Pi \). Taking \( \pi = \pi^* \) to be the optimal policy defined in (5.4), we conclude that \( \hat{T} \) is also a contraction.

**Appendix C. Proof of Lemma 5.4.** We first use [19, Remark 2] to show that \( \rho_{\theta,t}(X) \) is jointly Lipschitz on \( \mathcal{X} \times \Theta \), then the result will hold by composition of Lipschitz continuous functions.

By [21, Lemma 2.1] and [34, Corollary 3.1], we have that \( \rho_{\theta,t} \) is \( L_\rho \)-Lipschitz continuous on the interior of \( \text{dom}(\rho_{\theta,t}) \). Then

\[
\sup_{\theta \in \Theta} \sup_{X \neq X'} \frac{\left\| \rho_{\theta,t}(X) - \rho_{\theta,t}(X') \right\|_\infty}{\left\| X - X' \right\|_\infty} = \sup_{\theta \in \Theta} L_{\rho_{\theta,t}} \leq L_\rho < \infty,
\]

which establishes that \( \rho_{\theta,t}(X) \) is uniformly Lipschitz continuous on \( \Theta \). Then, Assumption 4.6 (ii) implies that \( \rho_{\theta,t}(X) \) is jointly \( L_\rho \)-Lipschitz continuous on \( \mathcal{V}_{S \times \Theta} \times \Theta \) according to [19, Remark 2].

Finally, since \( \hat{v} \in C_{L_0}(S \times \Theta) \), we have that \( \rho_{\theta,t}(\hat{v}(s, \theta)) \) is \( L_0 \)-\( L_\rho \)-Lipschitz continuous on \( S \times \Theta \) by [19, Lemma 2.1 (b)], and the proof is complete.

**Appendix D. Proof of Lemma 5.5.** Let \( \psi, \psi' \in \mathcal{M}(S) \) and pick any \( \theta, \theta' \in \Theta \). Since \( \hat{v} \in C_{L_0}(S \times \Theta) \), we have

\[
\left| \int \hat{v}(\hat{s}, \theta)\psi(d\hat{s}) - \int \hat{v}(\hat{s}, \theta')\psi'(d\hat{s}) \right| \\
\leq \left| \int \hat{v}(\hat{s}, \theta)\psi(d\hat{s}) + \int \hat{v}(\hat{s}, \theta')\psi'(d\hat{s}) \right| + \left| \int \hat{v}(\hat{s}, \theta) - \hat{v}(\hat{s}, \theta') \psi'(d\hat{s}) \right| \\
\leq L_0 \left( W_1(\psi, \psi') + d_{\Phi}(\theta, \theta') \right),
\]

which shows that \( (\psi, \theta) \mapsto \int \hat{v}(\hat{s}, \theta)\psi(d\hat{s}) \) is \( L_0 \)-Lipschitz continuous. Due to Assumption (ii) and an application of [19, Lemma 3.2], we have

\[
\sup_{\psi \in \Psi_{(s, a, \theta)}} \int \hat{v}(\hat{s}, \theta)\psi(d\hat{s}) - \sup_{\psi' \in \Psi_{(s', a', \theta')}} \int \hat{v}(\hat{s}, \theta')\psi'(d\hat{s})
\]
\[ \leq (1 + L_\rho) L_0 \left( d_S(s, s') + d_A(a, a') + d_\Theta(\theta, \theta') \right). \]

This directly yields that \( \sigma_{q, \bar{v}} \) is \( L_0(1 + L_\rho) \)-Lipschitz continuous for any \( \bar{v} \in C_{L_0}(S \times \Theta) \), which completes the proof.

**Appendix E. Proof of Lemma 6.8.** First, we prove the upper hemicontinuity of the correspondence \( \Phi \). Consider a sequence of triples \( \{(s_n, m_n, \theta_n)\}_{n \in \mathbb{N}} \subset S \times \mathcal{L}_p^*(S, \mathcal{B}(S), \mathbb{Q}) \times \Theta \) such that \( s_n \to s \in S \), \( m_n \to m \in \mathcal{L}_p^*(S, \mathcal{B}(S), \mathbb{Q}) \) in the weak* sense, and \( \theta_n \to \theta \in \Theta \). For any \( \phi_n \in \Phi(s_n, m_n, \theta_n) \), we have \( \phi_n = h(s_n, m_n, \theta_n)m_n \) by (6.4). In this case, for any bounded measurable function \( f : S \to \mathbb{R} \), we have

(E.1) \[ |h(s_n, m_n, \theta_n)m_n| \leq \|f\|_\infty \phi_n \text{ and } \int f(s') \phi(s')(ds') \leq \int \|f\|_\infty \phi(s')(ds') < \infty. \]

Therefore, if \( \phi_n \to \phi \) in the weak* sense (which yields \( \phi_n \mathbb{Q} \xrightarrow{\ast} \phi \mathbb{Q} \)), then

\[
\int f(s')h(s, m, \theta)m(s')Q(ds') \xrightarrow{(a)} \lim_{n \to \infty} \int f(s')h(s_n, m_n, \theta)m_n(s')Q(ds') \\
= \lim_{n \to \infty} \int f(s')\phi_n(s')Q(ds') \xrightarrow{(b)} \int f(s')\phi(s')Q(ds'),
\]

where (a) holds due to the Dominated Convergence Theorem, and (b) holds by the definition of setwise convergence. In this case, \( \phi = h(s, m, \theta)m \), implying \( \phi \in \Phi(s, m, \theta) \) by (6.4), which shows that \( \Phi \) is upper hemicontinuous at \((s, m, \theta)\).

We next show that \( \Phi \) is lower hemicontinuous. Again consider a sequence of triples \( \{(s_n, m_n, \theta_n)\}_{n \in \mathbb{N}} \subset S \times \mathcal{L}_p^*(S, \mathcal{B}(S), \mathbb{Q}) \times \Theta \) such that: \( s_n \to s \in S \), \( m_n \to m \in \mathcal{L}_p^*(S, \mathcal{B}(S), \mathbb{Q}) \) pointwise, and \( \theta_n \to \theta \in \Theta \). For any \( \phi \in \Phi(s, m, \theta) \), there exists \( h \in \mathcal{H} \) such that \( \phi = h(s, m, \theta)m \). We need to prove that there exists \( \phi_n \in \Phi(s_n, m_n, \theta_n) \) such that \( \phi_n \to \phi \) in the weak* sense. Towards this end, pick \( \phi_n = h(s_n, m_n, \theta_n)m_n \), then by (6.4), \( \phi_n \in \Phi(s_n, m_n, \theta_n) \). Thus, by the Dominated Convergence Theorem with (E.1) again, we have

\[
\int f(s')\phi_n(s')Q(ds') = \int f(s')h(s_n, m_n, \theta_n)m_n(s')Q(ds') \\
\quad \to \int f(s')h(s, m, \theta)m(s')Q(ds') = \int f(s')\phi(s')Q(ds'),
\]

which yields \( \phi_n \mathbb{Q} \xrightarrow{\ast} \phi \mathbb{Q} \), and thus, \( \phi_n \to \phi \) in the weak* sense. Then \( \Phi \) is lower hemicontinuous at \((s, m, \theta)\). Since \( \Phi \) is both upper/lower hemicontinuous, and \((s, m, \theta)\) is arbitrary, we can conclude that \( \Phi \) is a continuous correspondence.

**Appendix F. Proof of Claims in Example 6.6.** By (6.3), we have \( \text{VaR}_u(X_n) = F_{X_n}^{-1}(u) = \inf \{ x : \mathbb{P}(X_n \geq x) < 1 - u \} \). Then by the Markov inequality, if \( X_n \geq 0 \) then

\[
\mathbb{P}(X_n \geq x) = \mathbb{P}(X_n^p \geq x^p) \leq \frac{\mathbb{E}[X_n^p]}{x^p}, \text{ for all } x > 0.
\]

For any \( u \in [0, 1) \) and \( p > 1 \), we see

\[
\{ x : \mathbb{P}(X_n \geq x) < 1 - u \} \supset \{ x : \mathbb{E}[X_n^p] < (1 - u)x^p \}.
\]
Taking the infimum over $x$ on both sides, we have

$$F_{X_n}^{-1}(u) \leq \inf \{ x : \mathbb{E}[X_n^p] < (1 - u)x^p \} = \left( \frac{\mathbb{E}[X_n^p]}{1 - u} \right)^{\frac{1}{p}}.$$ 

Pick

$$(F.1) \quad \tilde{f}_n(u) = \frac{\mathbb{E}[X_n^p]^{\frac{1}{p}}}{(1 - u)^{\frac{1}{p}}} \mathbb{1}_{\{u \in [\lambda(\theta_n), 1]\}},$$

then $F_{X_n}^{-1}(u) \leq \tilde{f}_n(u)$ for all $u \in [0, 1)$ and $n \in \mathbb{N}$. Moreover,

$$\lim_{n \to \infty} \int_0^1 \tilde{f}_n(u)du = \lim_{n \to \infty} \mathbb{E}[X_n^p]^{\frac{1}{p}} \int_0^1 \mathbb{1}_{\{u \in [\lambda(\theta_n), 1]\}} \frac{p(1 - \lambda(\theta_n))^{1 - \frac{1}{p}}}{p - 1} du$$

$$= \mathbb{E}[X^p]^{\frac{1}{p}} \frac{p(1 - \lambda(\theta))^{1 - \frac{1}{p}}}{p - 1} = \int_0^1 \bar{f}(u)du < \infty,$$

since $X_n \overset{d}{\to} X$, and $\lambda : \Theta \to [0, 1)$ is continuous and uniformly bounded. Furthermore, as we stated in Example 6.2, $X_n \overset{c}{\to} X$ implies $F_{X_n} \to F_X$ and thus [16, Proposition 5, p.250] implies that $F_{X_n}^{-1} \to F_X^{-1}$ pointwise$^4$. In this case, we have $\left| F_{X_n}^{-1}(u) \right| \leq \tilde{f}_n(u) + M$ for all $n \in \mathbb{N}$, which yields

$$\int_{[\lambda(\theta_n)]} \text{VaR}_u(X_n)du = \int_0^1 \mathbb{1}_{\{u \in [\lambda(\theta_n), 1]\}} \text{VaR}_u(X_n)du$$

$$(a) \quad \int_0^1 \mathbb{1}_{\{u \in [\lambda(\theta), 1]\}} \text{VaR}_u(X)du = \int_{[\lambda(\theta)]} \text{VaR}_u(X)du,$$

where (a) holds by the generalized Dominated Convergence Theorem. This establishes that $\text{CVaR}_{\lambda(\theta)}(X)$ is jointly continuous on $\mathcal{V} \times \Theta$.

**Acknowledgments.** Shiping Shao and Abhishek Gupta would like to acknowledge Ford Motor Company for supporting this research through a University Alliance Project.

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$^4$In [16, Proposition 5, p.250], the assumption of the convergence of probability measures is indeed the convergence of distribution functions. See [16, Definition 1, p.244] for details.

22
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