A tighter bound on the number of relevant variables in a bounded degree Boolean function

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March 22, 2019

Abstract

A classical theorem of Nisan and Szegedy says that a boolean function with degree $d$ as a real polynomial depends on at most $d^2 \cdot 2^{d-1}$ of its variables. In recent work by Chiarelli, Hatami and Saks, this upper bound was improved to $C \cdot 2^d$, where $C = 6.614$. Here we refine their argument to show that one may take $C = 4.416$.

1 Introduction

Given a Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$, there is a unique multilinear polynomial in $\mathbb{R}[x_1, \ldots, x_n]$ which agrees with $f$ on every input in $\{0,1\}^n$. One important feature of this polynomial is its degree, denoted $\deg(f)$, which is known to be polynomially related to many other complexity measures, such as block sensitivity $\text{bs}(f)$, certificate complexity $C(f)$, decision tree depth $D(f)$, and approximate degree $\widetilde{\deg}(f)$ (see [4] and [1]). One can also bound the number of relevant variables of $f$ (i.e. the variables which actually show up in a term with non-zero coefficient in the polynomial for $f$, also called the \textit{junta size of $f$}) entirely in terms of the degree:

**Theorem 1** (Nisan-Szegedy [4]). A function $f : \{0,1\}^n \rightarrow \{0,1\}$ with degree $d$ has at most $\frac{d}{2} \cdot 2^d$ relevant variables.

The idea of Nisan and Szegedy’s original proof is to lower bound the influence of a relevant variable: a polynomial of degree $d$ has total influence at most $d$, and yet the derivative in the direction of a relevant coordinate is a degree $d - 1$ polynomial which is not identically zero, so it is non-zero on a random input with probability at least $1/2^{d-1}$. In other words, each relevant coordinate has influence at least $1/2^{d-1}$, so there can be at most $d \cdot 2^d$ of them.

Theorem 1 is has the correct exponential dependence on $d$ – indeed, consider the function $f$ given by the complete binary decision tree of depth $d$ which queries a distinct coordinate at each vertex. This function has degree $d$ and $2^d - 1$ relevant variables. However, it remained open whether the multiplicative factor of $\Theta(d)$ was necessary until a recent paper by Chiarelli, Hatami and Saks showed that $O(2^d)$ suffices.\(^1\)

\(^1\)In the same paper, the authors also give an improved lower bound construction, namely, for each $d$, a degree $d$ function with $\frac{d}{2} \cdot 2^d - 2$ relevant variables.
Theorem 2 (Chiarelli, Hatami, Saks, [2]). A function $f : \{0,1\}^n \rightarrow \{0,1\}$ with degree $d$ has at most $(6.614) \cdot 2^d$ relevant variables.

The main idea in [2] is to replace influence by a different measure – one which behaves more stably under restrictions of variables. Specifically, they define

$$W(f) = \sum_{i \in R(f)} 2^{-\deg_i(f)}$$

where $R(f)$ is the set of relevant variables for $f$, and $\deg_i(f)$ is the degree of $f$ in $x_i$. It is straightforward to check that $W(f)$ does not decrease by more than $|H| \cdot 2^{-d}$ in expectation when randomly restricting a set $H$ of coordinates with $\deg_i(f) = \deg(f) = d$. If $H$ is chosen well, these contributions are summable, and hence $W(f)$ is bounded above by some universal constant. Since $|R(f)| \leq 2\deg(f)$, this implies Theorem 2.

The heart of the proof is therefore in choosing the set $H$ which is both small enough so that $W(f)$ does not incur a heavy loss, and yet significant enough that the restricted functions are of reduced complexity. The idea used in [2] (originating in unpublished work of Nisan and Smolensky, see [1]) is to build $H$ from a maximal collection of disjoint monomials of full degree. The number of such disjoint monomials is limited by the block sensitivity $bs(f)$, which is always at most $d^2$, and by maximality, all of the resulting restricted functions have degree $\leq d - 1$.

1.1 Our improvements

The above idea certainly does the trick, but there are two somewhat substantial sources of slack in the analysis: one is the global use of the worst-case bound $bs(f) \leq d^2$, which can be improved for any fixed $d$ with a finite computation. The other is that restricting a large disjoint collection of degree $d$ monomials actually causes a large drop in block sensitivity, which can be exploited. By leveraging both of these ideas, we are able to improve the constant 6.614 by about 33%.

Theorem 3 (Main result). A function $f : \{0,1\}^n \rightarrow \{0,1\}$ with degree $d$ has at most $(4.416) \cdot 2^d$ relevant variables.

2 Preliminaries

Restrictions: For a function $f : \{0,1\}^n \rightarrow \{0,1\}$, a set $H \subset [n]$, and an assignment $\alpha : H \rightarrow \{0,1\}$, we denote by $f_\alpha$ the restricted function obtained by setting the variables $x_h$ to $\alpha(h)$ for $h \in H$. We will sometimes use $f(\alpha_H, x)$ for $f_\alpha(x)$ if we want to be explicit about the set of coordinates which have received the assignment by $\alpha$.

Influence: The influence of coordinate $i$ on $f$, or $\Inf_i[f]$, is the probability that, for a uniformly random input $x$, flipping the $i$th bit of $x$ causes the value of $f(x)$ to flip. The total influence $\Inf[f] = \sum_{i \in [n]} \Inf_i[f]$ can also be expressed in terms of the Fourier coefficients of $f$,
namely
\[
\text{Inf}[f] = \sum_{S \subseteq [n]} |S|\hat{f}(S)^2.
\]
Since the degree of \( f \) remains unchanged when \( f \) is expressed as a multilinear polynomial over \( \{0,1\}^n \) (as we consider in this paper) or \( \{1,-1\}^n \) (as in the Fourier expansion), the above formula makes it clear that a Boolean function of degree \( d \) has \( \text{Inf}[f] \leq d \). As mentioned in the introduction, the following useful fact is from \([4]\), and can be proved by induction:
\[
\text{Inf}_i[f] \geq 2^{1-\deg_i(f)}.
\] (1)

**Block sensitivity:** For a set \( B \subseteq [n] \) and a string \( x \in \{0,1\}^n \), we denote by \( x^B \) the string obtained from \( x \) by flipping all the bits \( x_b \) for \( b \in B \). Recall that the block sensitivity of \( f \) at an input \( x \) (denoted \( \text{bs}_x(f) \)) is the maximum number \( b \) of disjoint blocks \( B_1, \ldots, B_b \subseteq [n] \) such that \( f(x) \neq f(x^{B_i}) \) for all \( 1 \leq i \leq b \), and the block sensitivity of \( f \) (denoted \( \text{bs}(f) \)) is the maximum of \( \text{bs}_x(f) \) over all inputs \( x \). It is well-known that block sensitivity and degree are polynomially related:
\[
\deg(f)^{1/3} \leq \text{bs}(f) \leq \deg(f)^2
\] (2)
although neither bound is known to be sharp. The best known constructions have \( \text{bs}(f) = \Theta(\deg(f)^{1/2}) \) and \( \text{bs}(f) = \Theta(\deg(f)\log_3(6)) = \Theta(\deg(f)^{1.6309\ldots}) \) respectively. See \([1]\) and \([3]\) for details and for relationships to many other complexity measures.

**The measure \( W(f) \):** Recall that
\[
W(f) := \sum_{i \in R(f)} 2^{-\deg_i(f)},
\]
where \( R(f) \) is the set of relevant coordinates (i.e. coordinates \( i \) for which \( \text{Inf}_i[f] > 0 \)) and \( \deg_i(f) \) is the degree of largest degree monomial appearing in \( f \) (with non-zero coefficient) that contains \( x_i \). The behavior of \( W \) under restrictions boils down to the following inequality, whose simple proof we reproduce below for completeness.

**Fact 4 \([2]\).** For any relevant coordinates \( i \neq j \), let \( f_0 \) and \( f_1 \) be the restrictions obtained from \( f \) by setting \( x_j = 0 \) and 1 respectively. Then
\[
2^{-\deg_i(f)} \leq 2^{-\deg_i(f_0) - 1} + 2^{-\deg_i(f_1) - 1}
\] (3)

*Proof.* Write \( f = x_jf_1 + (1-x_j)f_0 = x_j(f_1 - f_0) + f_0 \), from which it is clear that \( \deg_i(f) \geq \deg_i(f_0) \). If \( \deg_i(f) \geq 1 + \deg_i(f_0) \) then the inequality is true independently of \( \deg_i(f_1) \). Otherwise, it must be that \( \deg_i(f) = \deg_i(f_0) \), in which case the leading degree monomials for \( x_i \) must cancel in \( f_1 - f_0 \). But this implies \( \deg_i(f) = \deg_i(f_1) = \deg_i(f_0) \), and so the inequality becomes an equality in this case. \( \square \)

By summing (3) over \( i \in R(f) \) and iterating over restrictions of more variables, one obtains
\[
W(f) \leq |H| \cdot 2^{-d} + \frac{1}{2|H|} \sum_{\alpha:H \rightarrow \{0,1\}} W(f_{\alpha})
\] (4)
for any set $H \subseteq [n]$ with $\deg_i(f) = d$ for all $i \in H$. As in [2], we define
\[ W_d := \max_{\deg(f) = d} W(f). \]

If $H$ is chosen as a maximal collection of degree $d$ monomials in $f$, then each $f_{\alpha}$ has degree at most $d - 1$. An unpublished argument of Nisan and Smolensky (which we essentially use in the proof of Lemma 6 below) implies that $|H| \leq \deg(f) \cdot \bs(f) \leq d^3$, and so (4) yields the recursive inequality
\[ W_d \leq d^3 \cdot 2^{-d} + W_{d-1}. \]

This is already summable, but the bound $W(f) \leq \frac{1}{2} \Inf[f] \leq d/2$ is preferable for small $d$, and optimizing over the choice of the two bounds yields $W_d \leq 6.614$ for all $d$.

3 Improving the constant

3.1 Don’t spend it all in one place

Our first new idea is simply to keep track of block sensitivity through the restriction process: the main observation is Proposition 5 below, which says that if $f$ has $\ell$ disjoint monomials of maximum degree, then by assigning any values to the variables in these monomials, the block sensitivity of the restricted function decreases by $\ell$. So, if we have to restrict many variables in order to drop the degree of $f$ (i.e. to hit all the maximum degree monomials), then we must “spend” our limited supply of block sensitivity, and in the future it will become much easier to lower the degree again.

Proposition 5. If $f : \{0, 1\}^n \to \{0, 1\}$ has $\ell$ disjoint monomials $M_1, \ldots, M_\ell$, each of degree $d = \deg(f)$, then for any assignment $\alpha : \bigcup M_i \to \{0, 1\}$, the restricted function $f_{\alpha}$ has
\[ \bs(f_{\alpha}) \leq \bs(f) - \ell. \]

In particular, $\ell \leq \bs(f)$.

Proof. Let $M = \bigcup_{i=1}^{\ell} M_i$ and $b = \bs(f_{\alpha}) = \bs_y(f_{\alpha})$, for some $y \in \{0, 1\}^{[n]} \setminus M$. Then there are $b$ disjoint blocks $B_1, \ldots, B_b \subseteq [n] \setminus M$ with $f(\alpha_M, y) \neq f(\alpha_M, y^{B_j})$ for each $j$. Since $M_i$ is a maximum degree monomial in $f$, each of the functions $\{0, 1\}^{M_i} \ni x \mapsto f(x, z)$ is non-constant for any $z$. Therefore, for each $i$, there is a block $C_i \subseteq M_i$ with $f(\alpha_M^C, y) \neq f(\alpha_M, y)$. Therefore $\{C_1, \ldots, C_\ell, B_1, \ldots, B_b\}$ is a collection of disjoint sensitive blocks for $f$ at the input $(\alpha_M, y) \in \{0, 1\}^n$, and so $\bs(f) \geq b + \ell$. \qed

To keep track of $W$, degree and block sensitivity simultaneously, we define
\[ W(b, d) := \max_{f \text{ with } \bs(f) \leq b \text{ and } \deg(f) = d} W(f). \]

Note that $W(0, d) = 0$, $W(b, 0) = 0$, and $W(b, d) \leq W_d$ for any $b$. By (2), we have $W(d^2, d) = W_d$, and we make the convention that $W(b, d) = 0$ for $b > d^2$. 

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Lemma 6. For each $b, d$ with $b \leq d^2$, we have

$$W(b, d) \leq \max_{(\ell, k) \in \{1, \ldots, b\} \times \{1, \ldots, d\}} \left( \ell \cdot d \cdot 2^{-d} + W(b - \ell, d - k) \right)$$

Proof. Suppose $f$ has degree $d$ and $\text{bs}(f) \leq b$. Let $M_1, \ldots, M_\ell$ be a maximal collection of disjoint degree $d$ monomials in $f$, and let $H = \cup_i M_i$. By inequality (4),

$$W(f) \leq |H| \cdot 2^{-d} + \sum_{\alpha: H \to \{0, 1\}} \mathbb{E}[W(f_{\alpha})]$$

Because the collection $\{M_1, \ldots, M_\ell\}$ is maximal, $H$ hits every degree $d$ monomial and hence each $f_{\alpha}$ has degree $d_{\alpha} \leq d - 1$. By Proposition 5, each $f_{\alpha}$ has $\text{bs}(f_{\alpha}) \leq b - \ell$. Since $W(\cdot, d)$ is monotone (for feasible inputs), it follows that for each $\alpha$, $W(f_{\alpha}) \leq W(b - \ell, d - k')$, where $k' = \arg \max_{k \in \{1, \ldots, d\}} W(b - \ell, d - k)$. Taking the maximum over all possible values of $\ell \in \{1, \ldots, b\}$ yields the desired bound. \qed

Since $W_d$ is bounded and increasing\(^2\), so $W^* := \lim_{d \to \infty} W_d$ exists. Since $W_d = W(d^2, d)$, the following corollary comes easily from Lemma 6.

Corollary 7. For any $d$,

$$W^* \leq W(d^2, d) + \sum_{r=d+1}^{\infty} r^3 2^{-r}$$

Lemma 6 yields explicit bounds on $W(b, d)$ for any finite $(b, d)$, which in turn yields an explicit bound on $W^*$ via Corollary 7. For small values ($d \leq 9$), the bound

$$W(b, d) \leq W_d \leq \max_{\deg(f) = d} \sum_{i \in R(f)} \frac{\text{Inf}_i[f]}{2} = \frac{d}{2}$$

is better than the one from Lemma 6. Extracting numerical bounds recursively yields

$$W(50^2, 50) \leq 5.07812...$$

which implies the same bound (to around 10 decimal digits) on $W^*$.

3.2 Tighter bounds on block sensitivity for low degree functions

To further reduce our estimate of $W^*$, we focus on functions of low degree, which clearly have the most influence on the bounds. Specifically, we produce sharper upper bounds on the block sensitivity of such functions, by solving a small set of linear programs. We begin with a simple reduction to linear program feasibility, using ideas from the original proof of $\text{bs}(f) \leq 2 \deg(f)$ from [4].\(^3\)

\(^2\)It is shown in [2] that $W_d \geq 2^{-d} + W_{d-1}$, and in fact their lower bound construction can be turned into a proof that $W_d \geq 2 \cdot 2^{-d} + W_{d-1}$.

\(^3\)The “2” in this bound can be removed by using repeated function composition (or tensorization), as shown in [6].

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and obtain bounds from Lemma 8.

**Proof.** If \( f(x) \) attains maximal block sensitivity at \( z \), then \( f(x \oplus z) \) attains maximal block sensitivity at 0, so without loss of generality we may assume \( z = 0 \), and possibly replacing \( f \) by \( 1 - f \) we may also assume that \( f(0) = 0 \). If \( B_1, \ldots, B_b \) are sensitive blocks for \( f \) at 0, then define

\[
g(y_1, \ldots, y_b) = f(y_{B_1}, \ldots, y_{B_i}, \ldots, y_{B_b})
\]

so that for each coordinate vector \( e_i \), \( g(e_i) = f(1_{B_i}) = f(0_{B_i}) = 1 \).

For any \( d \geq 1 \), define the moment map \( m_d: \mathbb{R} \to \mathbb{R}^d \) by \( m(t) = (t, t^2, \ldots, t^d) \).

**Proposition 9.** If there exists a degree \( d \) function \( f: \{0, 1\}^n \to \{0, 1\} \) with block sensitivity \( b \), then there exists \( \tau \in \{0, 1\} \) such that the following set of linear inequalities has a solution \( p \in \mathbb{R}^d \):

\[
\begin{align*}
\langle p, m_d(1) \rangle & = 1 \\
0 & \leq \langle p, m_d(k) \rangle \leq 1 \quad \text{for each } k \in \{2, \ldots, b - 1\} \\
\langle p, m_d(b) \rangle & = \tau
\end{align*}
\]  

**Proof.** If such an \( f \) exists, then let \( g(x_1, \ldots, x_b) = \frac{1}{n} \sum_{\sigma \in S_b} g(x_{\sigma(1)}, \ldots, x_{\sigma(b)}) \), where \( g \) comes from Fact 8, and set \( \tau = g(1, 1, \ldots, 1) \). It is well known (see [1]) that there is a univariate polynomial \( p: \mathbb{R} \to \mathbb{R} \) of degree at most \( d \) such that for any \( x \in \{0, 1\}^b \), \( g(x_1, \ldots, x_b) = p(x_1 + \cdots + x_b) \). For each \( k \in \{1, \ldots, b\} \), \( p(k) \) is therefore the average value of \( g \) on boolean vectors with hamming weight \( k \), so in particular \( p(k) \in [0, 1] \). We also know \( p(0) = g(0) = 0 \), \( p(b) = g(1, \ldots, 1) = \tau \), and \( p(1) = \frac{1}{n} \sum_i g(e_i) = 1 \), and hence the coefficients of \( p \) provide a solution to the set of linear inequalities.

Using the simplex method with exact (rational) arithmetic in Maple, we compute the largest \( b = b(d) \) for which the LP (5) is feasible for \( 1 \leq d \leq 14 \), which yields upper bounds on block sensitivity for low degree boolean functions. These bounds are summarized in Table 1.

| \( \text{deg}(f) \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|-------------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| \( \text{bs}(f) \leq \) | 1 | 3 | 6 | 10 | 15 | 21 | 29 | 38 | 47 | 58 | 71 | 84 | 99 | 114 |

**Remark:** The largest known separation between block sensitivity and degree is exhibited by a function on 6 variables with degree 3. By a tensorization lemma in [6], any degree \( d \) function \( f \)
Table 2: Bounds on $W(f)$ for low degrees, obtained using Lemma 6 and Table 1.

with block sensitivity $b$ yields an infinite family of boolean functions $f_k$ with $\deg(f_k) = d^k$ and $\bs(f_k) \geq b^k$. Hence, if an entry $(d, b(d))$ in Table 1 is tight for some $d \geq 4$, then by Fact 8 there is a function on $b(d)$ variables exhibiting a larger-than-currently-known separation between degree and block sensitivity. If $f = \deg(f)^{\log_3(d)}$ is in fact the optimal separation, then our techniques would show $W^* < 3.96$.

4 Discussion and concluding remarks

While our methods are unlikely to produce the optimal $W^*$, they do suggest a few interesting questions.

- The proof seems to suggest that block sensitivity limits junta size, and for small $d$, the values of $W(b, d)$ are much lower when $b \ll d^2$ than when $b \sim d^2$. A classical result of Simon [5] says that $|R(f)| \leq s(f)^4s(f)$. Interestingly, we can obtain a proof of a weaker version of Simon’s theorem using only the techniques in this paper and in [2]. The idea is to define an analogue of $W$ for sensitivity instead of degree:

$$S(f) := \sum_{i \in R(f)} 2^{-s_i(f)}, \quad \text{where } s_i(f) := \max_{\{x: f(x') \neq f(x)\}} (s_x(f) + s_{x_i}(f)).$$

Claim: Fact 4 holds with $\deg_i(f)$ replaced by $s_i(f)$.

Proof: Since $s_i(f) \geq \max\{s_i(f_0), s_i(f_1)\}$ is clear from the definitions, we can assume that $f_0$ does not depend on $x_i$. Without loss of generality suppose $j = 1$ and that $y$ has $f(1, y) = 1 \neq f(1, y')$ and $s_i(f_1) = s_y(f_1) + s_{y'}(f_1)$. First suppose $f_0(y) = 0$. Since $f(1, y) = 1$, this means $f$ is also sensitive to $i$ at input $(1, y)$, and so $s_i(f) \geq 1 + s_i(f_1)$. If $f_0(y) = 1$, then $f_0(y')$ is also 1 because $f_0$ does not depend on $x_i$. But then $f$ is sensitive to $j$ at input $(1, y')$, and so either way $s_i(f) \geq 1 + s_i(f_1)$, which implies the claim. □

From here we arrive at the analogue of (4), and we can proceed in a number of ways. As in the proof of Lemma 6, we can restrict maximum degree monomials until we run out of block sensitivity, yielding $|R(f)| \leq \deg(f) \cdot \bs(f) \cdot 4^{s(f)}$.

In any case, it seems reasonable to conjecture a Nisan-Szegedy theorem for block sensitivity, namely that any boolean function $f$ is a poly$(\bs(f)) \cdot 2^{\bs(f).}$-junta.

- More generally, it would be interesting to characterize certain ternary relationships between complexity measures. Many of the examples we know which achieve optimal or best-known separations between two measures tend to have the property that a third measure is equal or very close to one of the other two. (For example, the best known gap of the form $\bs(f) \ll \deg(f)$ is attained by a Tribes function on $n$ variables with $\deg(f) = n$ and $\bs(f) = s(f) = C(f) = \sqrt{n}$.) Moreover, these examples almost always
have $|R(f)| = \text{poly}(\deg(f))$. Meanwhile, the known examples of functions with nearly-optimal junta size do not exhibit any super-constant separation between the measures $\deg(f), \bs(f), s(f)$ and $D(f)$. (It is possible to hybridize small, well-separated functions with large juntas, but the separations and the junta size both suffer some loss.)

- Finally, Table 1 suggests that $\bs(f) \leq c_0 \deg(f)^2$, for $c_0 \approx 0.59$. If you enjoyed reading this paper, perhaps you would enjoy trying to compute the optimal value of $c_0$. One consequence of showing that $\bs < \deg^2$ is that any separation between $\bs$ and degree proven by simply tensorizing a single example would necessarily (in the absence of more sophisticated arguments) look like $\bs(f) \geq \deg(f)^2 - \epsilon$, for some $\epsilon > 0$.

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