Vector autoregression process.  
Stationarity and simulation

T M Tovstik  
Saint Petersburg State University, Universitetskaya emb., 7/9, St Petersburg, 199034, Russian Federation  
E-mail: peter.tovstik@mail.ru

Abstract. For vector discrete-parameter random autoregressive processes and for a mixed autoregression / moving-average model, we obtain conditions which should be satisfied by the correlation functions or the model coefficients in order that the process be weakly stationary. Fairly simple tests are used. Algorithms for modeling such vector stationary processes are given. Examples are presented clarifying testing criteria for stationarity of models defined in terms of the coefficients or the correlation functions of the process.

1. Introduction  
Our first aim in the present paper is to present a test for stationarity of random processes under consideration. For vector discrete-parameter random autoregression processes $VAR(n)$ and the mixed autoregression / moving-average model $VARMA(n, m)$, we give conditions which should be satisfied by the correlation functions or the model coefficients in order that the process be weakly stationary. Examples are presented clarifying testing criteria for stationarity of models. A fairly simple test criteria will be given, which are based on general stationarity conditions of vector autoregressive processes described in the classical monographs [1, 2].

The second purpose of the present paper is to model vector stationary gaussian autoregressive processes and an autoregression with moving-average residuals. The modeling algorithms of the corresponding one-dimensional processes are described in [3].

In recent years, many studies have appeared [4–9] dedicated to forecasts of real data, which are looked upon as vector stationary autoregressive processes. These practical problems involve the selection of stationary models of autoregression. The present paper is devoted to stationarity tests of models.

2. Vector autoregressive process  
Consider a real weakly stationary discrete-parameter vector process

$$\xi(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_k(t))^*, \quad t = 0, \pm 1, \pm 2, \ldots,$$

which is a vector autoregressive process $VAR(n)$ of order $n$

$$A_0 \xi(t) + A_1 \xi(t - 1) + \ldots + A_n \xi(t - n) = \varepsilon(t), \quad A_0 = I.$$
Here, $\ast$ is the transpose sign, and $A_j$, $J$, $I$ are square matrices of order $k$, $I$ is the identity matrix. Without loss of generality it can be assumed that $J$ is a lower-triangular matrix. The vectors $\varepsilon(t) = (\varepsilon_1(t), \varepsilon_2(t), \ldots, \varepsilon_k(t))^\ast$ are independent and consist of independent random variables satisfying

$$E\varepsilon_q(t) = 0, \quad E(|\varepsilon_q(t)|^2) = 1, \quad E\varepsilon_q(t)\varepsilon_r(s) = \delta_{qr}\delta_{ts}, \quad 1 \leq q, r \leq k,$$

(3)

where $\delta_{ij}$ is the Kronecker delta.

From (2)-(3) it follows that $E\xi(t) = 0$.

The covariation function of the vector stationary processes $\xi(t)$ is the matrix $R(t)$ with entries

$$R(t) = ||R_{ij}(t)||_{(i,j)^\ast} = E\xi(t+s)\xi_j(s)^\ast = ||E\xi_i(t+s)\xi_j(s)||_{((i,j)^\ast)}^\ast, \quad R(-t) = R^T(t).$$

(4)

A relation between covariations (4) and the autoregression coefficients (2) is given by the Yule–Walker equations [2], which we write as follows:

$$\sum_{j=0}^n A_j R(l-j) = 0, \quad l = 1, 2, \ldots, n.$$  (5)

$$\sum_{j=0}^n A_j R(-j) = G, \quad G = J^T R.$$  (6)

3. Modeling a gaussian stationary vector autoregressive process

To model the process $\xi(t)$ one should get the vector

$$\hat{\xi}_n(t)^\ast = (\xi(t-1), \ldots, \xi(t-n)) = (\xi_1(t-1), \ldots, \xi_k(t-1), \ldots, \xi_1(t-n), \ldots, \xi_k(t-n))$$

of dimension $k \cdot n$, model the vector $\varepsilon(t)$, evaluate the vector $J\varepsilon(t)$, and determine $\xi(t)$ by (2).

In particular, if the modeling starts at $t = 0$, then we find $\hat{\xi}_n(0)$, next we get $\xi(0)$, and at next time instants $t = 1, 2, \cdots$ we model $\varepsilon(t)$, and then evaluate $\xi(t)$ by (2).

The components $\hat{\xi}_n(t)$ are stationary related. The correlation matrix $\hat{R}_n$ of the vector $\hat{\xi}_n(t)$ has the form

$$\hat{R}_n = \begin{pmatrix}
R(0) & R(1) & \cdots & R(n-1) \\
R^*(1) & R(0) & \cdots & R(n-2) \\
\vdots & \vdots & \ddots & \vdots \\
R^*(n-1) & R^*(n-2) & \cdots & R(0)
\end{pmatrix}.$$  (7)

Note that in (7) $R(s)$ are square matrices of order $k$. If $\xi(t)$ is a stationary processes, then so is the process $\hat{\xi}_n(t)$, since the conditions of positive definiteness of the correlation function $R(t)$ and $\hat{R}_n$ are the same (see Rozanov [1]). The matrix $\hat{R}_n$ can be written as

$$\hat{R}_n = H_{kn}^T H_{kn},$$  (8)

where $H_{kn}$ is a lower-triangular matrix. The possibility of representation (8) can be looked upon as a test for positive definiteness of the corresponding matrix $\hat{R}_n$.

If we model a Gaussian process $\xi(t)$, for example, at the point $t = 0$, then

$$\hat{\xi}_n(0) = H_{kn}W_{kn}, \quad W_{kn} = (w_1, w_2, \cdots, w_{kn})^\ast, \quad w_j \in N(0,1),$$  (9)

here, $w_j$ are independent standard normal random variables. From $\hat{\xi}_n(0)$, we get the Gaussian vector

$$(\xi(-1), \xi(-2), \cdots, \xi(-n))^\ast$$  (10)

and then model the process by the above algorithm.
4. Modeling a stationary vector autoregressive process with moving-average residuals

A vector autoregressive process $VARMA(n, m)$ with moving-average residuals satisfies the equation

$$\sum_{j=0}^{n} A_j \xi(t - j) = \sum_{j=0}^{m} B_j \varepsilon(t - j), \quad A_0 = I, \quad (11)$$

here $A_j$ and $B_l$ are square matrices of order $k$, and the process $\varepsilon(t)$ is the same as in Section 1.

We will model a Gaussian vector process $VARMA(n, m)$ starting from the point $t = 0$. First, as in the previous section, we model vector (10). Next, we model independent vectors $\varepsilon(0), \varepsilon(-1), \cdots, \varepsilon(-m)$ with independent normally distributed standard components, and then use formula (11) to find $\xi(0)$.

For each successive $t = 1, 2, \cdots$, we model vector $\varepsilon(t)$, which is independent of the previous ones, and then evaluate $\xi(t)$.

5. Stationarity of the autoregressive process

The autoregressive process $VAR(n)$ is completely defined if the correlation matrices

$$R(0), R(1), \cdots, R(n), \quad (12)$$

or the autoregression coefficients

$$A_1, A_2, \cdots, A_n, G \quad (13)$$

are given. A vector process $\xi(t)$ is stationary if the matrix function $R(t)$ is positive definite (see [1]).

If correlations (12) are given, then for a stationary check of the process $VAR(n)$ it suffices to verify that the correlation matrix

$$\hat{R}_{n+1} = \begin{pmatrix} R(0) & \cdots & R(n) \\ \cdots & \cdots & \cdots \\ R^*(n) & \cdots & R(0) \end{pmatrix} \quad (14)$$

of the vector $(\xi(t), \xi(t-1), \cdots, \xi(t-n))^*$ is positive definite. In the stationary case, from equations (5) we find the matrices $A_1, A_2, \cdots, A_n$, verify that $G = \sum_{j=0}^{n} A_j R(-j)$ is positive definite, and by factoring it we evaluate the matrix $J$ such that $G = J J^*$. This completes the process of parameter selection.

In another variant, as original data we have matrices (13), where the matrix $G$ is positive definite. Hannan [2] proves that if a process (2) is stationary, then all the roots $z_j$ of the characteristic polynomial $\varphi(z) = \det(\sum_{j=0}^{n} A_j z^{-j})$ of order $p = nk$ lie inside the unit disk, i.e.,

$$\varphi(z) = \det\left(\sum_{j=0}^{n} A_j z^{-j}\right) = \prod_{j=1}^{p} (z - z_j) = 0, \quad |z_j| < 1, \quad 1 \leq j \leq p. \quad (15)$$

**Remark.** Let $\zeta(t)$ be an autoregressive process in the one-dimensional case

$$\sum_{j=0}^{n} a_j \zeta(t - j) = b \eta(t), \quad a_0 = 1, \quad \mathbb{E} \eta(t) \bar{\eta}(s) = \delta_{ts}. \quad (16)$$

The process $\zeta(t)$ will be stationary (see Rozanov [1]) if the polynomial

$$a(y) = \sum_{j=0}^{n} a_j y^{-j} = \prod_{j=1}^{n} (y - y_j) = 0, \quad |y_j| < 1, \quad 1 \leq j \leq n. \quad (16)$$

has roots smaller than 1 in absolute value.

Even though there is an analogy between conditions (15) and (16), however, the mere positive definiteness of the matrix $G$ and condition (15) are not sufficient for the vector process to be stationary. This conclusion will be supported by Examples 3 and 4 that follow.
Let us find conditions under which the vector process $\xi(t)$ of the form (2) will be stationary.

Assume that the determinant of the matrix $R(0)$ is nonzero. This is the case, for example, if the process $\xi(t)$ is stationary. One may formally introduce the “correlation coefficients”

$$\rho(l) = R(l)R^{-1}(0), \quad l = 1, \ldots, n, \quad \rho(0) = I.$$ 

Let us assume that the autoregression coefficients (13) satisfy condition (15). From equations (5) we get the system of linear equations

$$\sum_{j=0}^{n} A_j \rho(l-j) = 0, \quad l = 1, 2, \ldots, n, \quad A_0 = I, \quad (17)$$

for the matrices

$$\rho(l), \quad l = 1, \ldots, n. \quad (18)$$

The tuples $(A_j, \ j = 1, \ldots, n)$ and (18) can be uniquely determined from each other.

The matrix $G$ is given, and hence the correlation matrix $R(0)$ of the vector $\xi(t)$ can be found from equation (6):

$$R(0) = \left(\sum_{j=0}^{n} A_j \rho(-j)\right)^{-1} G. \quad (19)$$

If the determinant $\det(R(0)) \leq 0$, then the process $\xi(t)$ will be nonstationary.

So, the vector process $\xi(t)$ will be a stationary autoregressive process $AR(n)$ if the following conditions are met.

1°. The matrix $G = J(t)J(t)^*$ is positive definite.

2°. Condition (15) is met.

3°. The matrix $R(0)$, which is obtained from equation (19), is positive definite.

4°. The matrix $R_{n+1}$, which is defined by (14), is positive definite.

Condition 4° is a necessary and sufficient condition for the process $AR(n)$ to be stationary.

6. Stationarity of the process $VARMA(n, m)$

The Yule–Walker equations (5) of the process $VARMA(n, m)$ assume the form

$$\sum_{j=0}^{n} A_j R(l-j) = 0, \quad l = m+1, m+2, \ldots, m+n, \quad (20)$$

and equation (6) is replaced by

$$\sum_{j=0}^{n} \sum_{l=0}^{m} A_j R(-j+l)A_l^* = \sum_{j=0}^{m} B_j B_j^*, \quad (21)$$

inasmuch as

$$\mathbb{E}\left|\sum_{j=0}^{n} A_j \xi(t-j)\right|^2 = \mathbb{E}\left|\sum_{j=0}^{m} B_j \xi(t-j)^2\right| = \sum_{j=0}^{m} B_j B_j^*. \quad (22)$$

The stationarity conditions (11) of the process $VARMA(n, m)$ are nearly the same as for process (2). In condition 1°, the matrix $G$ should be replaced by the matrix $\sum_{j=0}^{m} B_j B_j^*$ (this follows from (21) and (22)). Condition 2° was proved in [2], and condition 4° reduces to the following form. Equations (20) and (21) involve the correlations $R(0), R(1), \ldots, R(m+n)$. Hence the stationarity of the process $VARMA(n, m)$ follows from the positive definiteness of the matrix $R_{m+n+1}$, which is the correlation matrix of the vector $(\xi(t), \xi(t-1), \ldots, \xi(t-m-n))^*$.

Note that for the processes $VAR(n)$ the coefficients (12) and the correlations (13) can be uniquely determined from each other. The model coefficients for the processes $VARMA(n, m)$ can be found from the known correlations by special methods. In the one-dimensional case, one of the algorithms, which from sample correlations estimates the parameters of the correlation function and delivers the density, can be found in [10].
7. Examples

In the first example, we are given correlations (12), for which the matrix (14) is positive definite, and, therefore, the process is stationary. We give the autoregression coefficients (13) and verify the stationarity conditions.

The matrix (14), which is composed of correlations (12) from the second example, is not positive definite, and, therefore, the process is not stationary. Condition 2° is satisfied, but condition 1° is not met.

In the third example, the parameters (13) are taken from the second example, and hence condition 2° is satisfied, condition 1° is also met, but condition 3° is not met. Hence the process is not stationary.

In the fourth example, conditions 1°–3° are met, but the process is not stationary, even though it rapidly settles down to stationary behavior in the model.

Example 1. Consider the process \( \xi_t \) with given correlation matrices \( R(0), R(1), R(2) \). From these correlations, the second-order autoregression coefficients \( A_1, A_2, G, J \) are found:

\[
R(0) = \begin{pmatrix} 4 & 0.8 \\ 0.8 & 1 \end{pmatrix}, \quad R(1) = \begin{pmatrix} -2 & 0.6 \\ 0.6 & 0.6 \end{pmatrix}, \quad R(2) = \begin{pmatrix} 1.2 & 0.4 \\ 0.4 & 0.4 \end{pmatrix},
\]

\[
A_1 = \begin{pmatrix} 1.360 & -1.648 \\ -0.089 & -0.488 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.801 & -0.868 \\ -0.072 & -0.004 \end{pmatrix}, \quad G = \begin{pmatrix} 0.904 & 0.600 \\ 0.600 & 0.626 \end{pmatrix}, \quad J = \begin{pmatrix} 0.951 & 0 \\ 0 & 0.631 \end{pmatrix}.
\]

The roots of the polynomial (15)

\[
\det \left( \sum_{j=0}^{2} A_j z^{n-j} \right) = z^4 + 0.873z^3 - 0.005z^2 - 0.582z - 0.059 = 0
\]

are as follows: \( z_1 = 0.663, z_2 = -0.104, z_{3,4} = -0.716 \pm i0.593 \). They are smaller than 1 in absolute value, since \( |z_{3,4}| = 0.929 \).

All the parameters are known, the process is stationary.

Example 2. Consider the process \( \xi_t \) with given correlation matrices \( R(0), R(1), R(2) \), which coincide with the correlations from the first example, except \( R_{2,2}(2) = 0.2 \) (in place of 0.4). From these correlations, the second-order autoregression coefficients \( A_1, A_2, G \) are found:

\[
A_1 = \begin{pmatrix} 1.360 & -1.648 \\ -0.193 & -0.581 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.801 & -0.868 \\ -0.193 & 0.419 \end{pmatrix}, \quad G = \begin{pmatrix} 0.904 & 0.774 \\ 0.774 & 0.542 \end{pmatrix}.
\]

The roots of the polynomial (15)

\[
\det \left( \sum_{j=0}^{2} A_j z^{n-j} \right) = z^4 + 0.7796z^3 + 0.1115z^2 - 0.3816z + 0.1678 = 0
\]

are as follows: \( z_{1,2} = 0.343 \pm i0.247, z_{3,4} = -0.733 \pm i0.633 \), their absolute values \( |z_{1,2}| = 0.423, |z_{3,4}| = 0.969 \) are smaller than 1. It is easily checked that the matrix \( G \) is not positive definite, and, therefore, one cannot find a real matrix \( J_\eta \) such that \( J_\eta R_\eta^{-1} = G \). As a corollary, the process is not stationary.

Example 3. Consider the process \( \xi_t \) for which \( G = I \) is the identity matrix, and the autoregression coefficients \( A_1, A_2 \) are the same as in Example 2. Therefore, the roots of the characteristic polynomial are smaller than 1 in absolute value.

In this case, the stationarity conditions 1° and 2° of the process AP(2) are satisfied. The matrix \( R(0) \), which is obtained from (19), has the entries \( R_{1,1}(0) = -14.183, R_{1,2}(0) = 21.737, R_{2,1}(0) = 3.120, R_{2,2}(0) = -2.612 \). This matrix is not positive definite, and, therefore, the process is not stationary.

If in Example 3 in place of the identity matrix one considers any positive definite matrix \( \tilde{G} \), then the new correlation matrices will be given by \( \tilde{R}(i) = R(i)\tilde{G} \), and the autoregression coefficients will remain the same, because they follow from equalities (17), which involve the correlation coefficients. The stationary property of the process \( \xi(t) \) will remain the same, the process will be nonstationary.
Example 4. Let \( \hat{\xi}(t) \) be a second-order vector autoregressive process with the parameters
\[
A_1 = \hat{A}_1, \quad A_2 = \hat{A}_2, \quad \hat{G} = G(R(0))^{-1},
\]
where \( A_j, R(j), j = 0, 1, 2, G, \) and \( \rho(j), j = 0, 1, 2 \) are parameters of the process \( \xi(t) \) from Example 1.

We have
\[
\hat{G} = \hat{J} \hat{J}^* = \begin{pmatrix} 0.126 & 0.499 \\ 0.029 & 0.603 \end{pmatrix}, \quad \hat{J} = \begin{pmatrix} 0.355 & 0 \\ 0.083 & 0.772 \end{pmatrix},
\]
which implies condition 1°. Condition 2° follows from (23), since \( \xi(t) \) is stationary. For the processes \( \hat{\xi}(t) \), the Yule–Walker equations with correlations
\[
\hat{R}(j) = R(j)R^{-1}(0) = \rho(j), \quad 0 \leq j \leq 2
\]
are satisfied. Hence
\[
\hat{R}(0) = \rho(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{R}(1) = \rho(1) = \begin{pmatrix} -0.738 & 1.190 \\ 0.036 & 0.571 \end{pmatrix}, \quad \hat{R}(2) = \rho(2) = \begin{pmatrix} 0.261 & 0.190 \\ 0.024 & 0.381 \end{pmatrix},
\]
and therefore, condition 3° is met. However the matrix
\[
\hat{R}_{n+1} = \begin{pmatrix} \hat{R}(0) & \hat{R}(1) & \hat{R}(2) \\ \hat{R}^*(1) & \hat{R}(0) & \hat{R}(1) \\ \hat{R}^*(2) & \hat{R}^*(1) & \hat{R}(0) \end{pmatrix} = \begin{pmatrix} I & \rho(1) & \rho(2) \\ \rho^*(1) & I & \rho(1) \\ \rho^*(2) & \rho^*(1) & I \end{pmatrix}
\]
is not positive definite. Therefore, the process \( \hat{\xi}(t) \) is nonstationary.

Examples show that if a process is stationary, then the roots of the polynomial (15) are smaller than 1 in absolute value, while the converse result is not true.

The general form of a process \( \xi(t) \) satisfying (2) is given in Hannan’s book [2]. If the roots of the polynomial (15) are smaller than 1 in absolute value and if the matrix \( G \) is positive definite, then
\[
\xi(t) = c(t) + \sum_{j=0}^{\infty} \Lambda_j \varepsilon(t-j), \quad (n \sum_{i=0}^{n} A_i z^i)^{-1} = \sum_{j=0}^{\infty} L(j) z^j, \quad \Lambda_j = L(j) J.
\]  
(24)

where the entries of the matrices \( \Lambda_j \) tend exponentially to zero as \( j \to \infty \). We write the expression for \( c(t) \) in the case where the roots of the polynomial (15) are different
\[
c(t) = \sum_{j=1}^{n} z_j^t b(j), \quad \sum_{j=0}^{n} A_j c(t-j) = 0,
\]  
(25)

and the vector \( b(j) \) satisfies the relation
\[
\sum_{i=0}^{n} A_i z_j^{n-i} b(j) = 0.
\]

From formulas (24) and (25) it follows that if relations (15) are satisfied, then as \( t \to \infty \) the process converges to the stationary processes
\[
\hat{\xi}(t) = \sum_{j=0}^{\infty} \Lambda_j \varepsilon(t-j), \quad \mathbf{E} \hat{\xi}(t) \xi(s)^* = \sum_{j=0}^{\infty} L(j) G L^*(t-s+j).
\]  
(26)

Formula (26) for \( \hat{\xi}(t) \) is a Wold decomposition.
8. Modeling a nonstationary autoregressive process for which the stationarity conditions 1°–3° are satisfied, while condition 4° is not satisfied

Let us now discuss the model of a nonstationary autoregressive processes AR(n) of the form (2), which satisfies conditions 1°–3°, but condition 4° is not met. The difficulty here is in modeling the vector $\tilde{\xi}_n(t) = (\xi(t - 1), \cdots, \xi(t - n))$, because this vector is nonstationary.

Consider the stationary vector $\tilde{x}_n(t) = (x(t - 1), \cdots, x(t - n))^*$, $x(t) = (x_1(t), x_2(t), \cdots, x_k(t))^*$, whose distribution is close to that of $\tilde{\xi}_n(t)$ in the following sense. The components $x_j(t), j = 1, \cdots, n$, of the process $x(t)$ are independent, and each of which is a one-dimensional stationary autoregressive process of order $\leq n$. The correlations, and therefore, the variances of the components are the same as for the components of $\xi(t)$.

Once $\tilde{x}_n(t)$ is modeled, it will be used in place of the vector $\tilde{\xi}_n(t)$, and then we model the process $\xi(t)$ following the approach of Section 2. The realization thus obtained will be eventually close to that of the original process.

Realizations of Gaussian processes are shown in figure 1 for the stationary processes $\xi(t)$ from Example 1 and in figure 2 for the nonstationary processes $\tilde{\xi}(t)$ from Example 4. The processes differ only by matrices $J$ and $\tilde{J}$. The same random matrices were used for process modeling.

Realizations of the stationary processes $\xi(t)$ and of the nonstationary processes $\tilde{\xi}(t)$ feature considerable fluctuations. For $\sigma(\xi_1(i)) = 2, \sigma(\xi_2(i)) = 1$, and for a sample of size $N = 300$, the sample estimates are as follows: $\hat{\sigma}(\xi_1(i)) = 2.33, \hat{\sigma}(\xi_2(i)) = 0.97, -6.64 \leq \hat{\xi}_1(i) \leq 7.32, -2.57 \leq \hat{\xi}_2(i) \leq 2.61$.

For the nonstationary processes $\tilde{\xi}(t)$ the following results are obtained: $-6.47 \leq \tilde{\xi}_1(i) \leq 6.89, -2.53 \leq \tilde{\xi}_2(i) \leq 2.78$, and the sample mean-square deviations are as follows: $\hat{\sigma}(\tilde{\xi}_1(i)) = 2.29, \hat{\sigma}(\tilde{\xi}_2(i)) = 1.01$. The realization shows that the process $\tilde{\xi}(t)$ settles down to stationary behavior.

Note that if the process $\tilde{\xi}(t)$ with $\tilde{R}(0) = I$ would be stationary, then both its components of the sample variance would be close to unit.
9. Conclusions

In the present paper, we propose a method of testing the stationarity of vector processes $VARMA(n, m)$ and $VAR(n) = VARMA(n, 0)$. To this end, it suffices to check that the vector composed of successive $(n + m + 1)$ vectors of the process has positive definite correlation matrix. A process failing to satisfy at least one of conditions 1°–3° is nonstationary, and if only condition 4° is met, then the process is not stationary, but settles down to stationary behavior. We also describe methods of modeling the above gaussian vector stationary processes.

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