Theory of broken symmetry quantum Hall states in the $N = 1$ Landau level of Graphene

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We study many-body ground states for the partial integer fillings of the $N = 1$ Landau level in graphene, by constructing a model that accounts for the lattice scale corrections to the Coulomb interactions. Interestingly, in contrast to the $N = 0$ Landau level, this model contains not only pure delta function interactions but also some of its derivatives. Due to this we find several important differences with respect to the $N = 0$ Landau level. For example at quarter filling when only a single component is filled, there is a degeneracy lifting of the quantum hall ferromagnets and ground states with entangled spin and valley degrees of freedom can become favourable. Moreover at half-filling of the $N = 1$ Landau level, we have found a new phase that is absent in the $N = 0$ Landau level, that combines characteristics of the Kekulé state and an antiferromagnet. We also find that according to the parameters extracted in a recent experiment, at half-filling of the $N = 1$ Landau level graphene is expected to be in a delicate competition between an AF and a CDW state, but we also discuss why the models for these recent experiments might be missing some important terms.

Introduction. The quantum Hall regime in graphene realizes a rich landscape of broken symmetry and topological states, stemming in part from the near four-fold degeneracy of its Landau levels (LLs) associated with its valley and spin degrees of freedom [1]. Most studies to date have focused on the states in the $N = 0$ LL, with transport and magnon transmission experiments favoring an anti-ferromagnetic (AF) state at neutrality [2-6], while STM experiments reporting evidence for Kekulé-type valence-bond-solids and charge density wave states (CDW) [7-9].

While the projected Coulomb interaction is typically the dominant term in the Hamiltonian, it possesses a large symmetry that leaves the quantum Hall ground states undetermined. Therefore, it is crucial to account for the corrections that reflect the lower symmetry of the underlying graphene lattice to select the ground states [1,10-15]. A convenient model to capture these interactions allowed by symmetries is the continuum model of short-range symmetry breaking interactions allowed by symmetries, although a related model containing some of these terms was introduced in Ref. [15]. This model can be viewed as a projection into the $N = 0$ LL of a more general model introduced by Aleiner, Kharzeev and Tsvelik [16] that includes all possible delta-function interactions allowed by symmetries. There is no study to this date that has constructed an analogous model in the $N = 1$ LL that includes all possible short-distance interactions allowed by symmetry, although a related model containing some of these terms was introduced in Ref. [17].

The purpose of our study is therefore to construct this model of symmetry breaking interactions in the $N = 1$ LL and to determine its ground states at partial integer fillings. Interestingly, we will see that in contrast to the $N = 0$ LL [15], the $N = 1$ LL model contains interactions that are not pure delta functions [17]. Therefore, in contrast to the $N = 0$ LL, a unique ground state is selected even when a single component is filled (to be denoted by $\tilde{\nu} = 1$), and some of the possible ground states are spin-valley entangled, in the sense discussed in Ref. [18]. Moreover, when two components are filled (to be denoted by $\tilde{\nu} = 2$), we find a new type of Kekulé-Antiferromagnetic state in addition to those found in the $N = 0$ LL. Based on the parameters estimated in Ref. [17], graphene is expected to be in a delicate competition between an AF and a CDW state. However, as we will discuss, these parameters are possibly missing some important terms.

Model and Symmetries. We begin by reviewing the continuum model of short-range symmetry breaking interactions of Aleiner, Kharzeev and Tsvelik [16] in the absence of a magnetic field. This is described by the following Hamiltonian:

$$\mathcal{H} = \mathcal{H}_D + \mathcal{H}_C + \mathcal{H}_A,$$  \hspace{1cm} (1)

with:

$$\mathcal{H}_D = v_F \sum_i \left( \tau_i^x p_i^x \sigma_i^z + p_i^y \sigma_i^y \right),$$  \hspace{1cm} (2)

being the linearized single particle hamiltonian around the Dirac points,

$$\mathcal{H}_C = \sum_{i<j} \frac{e^2}{\epsilon |\mathbf{r}_i - \mathbf{r}_j|},$$

the Coulomb interaction, and

$$\mathcal{H}_A = \sum_{i<j} \left\{ \sum_{\alpha,\beta} V_{i,\beta} T_{i,\alpha}^z T_{i,\alpha}^\dagger \delta(\mathbf{r}_i - \mathbf{r}_j) \right\},$$  \hspace{1cm} (3)

the sublattice-valley dependent interactions. We have defined $T_{i,\alpha}^z = \tau_{\alpha}^x \otimes \sigma_{\beta}^z \otimes s_{0}^{(i)}$, and $\tau_{\alpha}^x$, $\sigma_{\beta}^z$, $s_{0}^{(i)}$ $\alpha, \beta = 0, x, y, z$ to be the Pauli matrices acting on valley, sublattice and spin respectively.

By denoting the valley (sublattice) states as $|\tau\rangle$, $|\sigma\rangle$, with $\tau(\sigma) = \pm 1$ corresponding to the $K, K'$ ($A, B$) valleys (sublattices), then the action of lattice symmetry on...
these states is given by:

\[ C_{\theta}|\tau,\sigma\rangle = Z^{-\tau\sigma}|-\tau,-\sigma\rangle, \]
\[ M_{x}|\tau,\sigma\rangle = |\tau,\sigma\rangle, \]
\[ M_{y}|\tau,\sigma\rangle = |-\tau,\sigma\rangle, \]
\[ T_{R_{1,2}}|\tau,\sigma\rangle = Z^{\pm\tau}|\tau,\sigma\rangle, \]

with \( Z = e^{i\frac{\pi}{3}} \). \( C_{\theta} \) is the rotation by \( \pi/3 \), \( M_{x}, M_{y} \) the two mirrors and \( T_{R_{1,2}} \) the translations by the two basis vectors of graphene (see Fig. 1(a) for the illustration of these symmetries). These symmetries reduce the couplings of Eq. (3) to nine independent couplings satisfying the following relations [16]:

\[ F_{xz} = V_{xx} = V_{xy}, \]
\[ F_{zy} = V_{yz} = V_{zy}, \]
\[ F_{zz} = V_{y0} = V_{xz} = V_{z0}, \]
\[ F_{01} = V_{zz} = V_{y0}, \]
\[ F_{10} = V_{y0} = V_{zy}, \]
\[ F_{2z} = V_{z0}, \]
\[ F_{02} = V_{z2}, \]
\[ F_{0z} = V_{z0}. \]

Projected Model in the \( N = 1 \) Landau level. By projecting \( \mathcal{H}_{A} \) from Eq. (1), with the constraints in Eq. (5), one obtains the following Hamiltonian of symmetry breaking interactions in the \( N \)th LL:

\[ \mathcal{H}_{A}^{N} = \sum_{i<j}(V_{2}^{N}(r_{ij})\tau_{i}^{0}\tau_{j}^{0} + V_{2}^{N}(r_{ij})\tau_{i}^{0}\tau_{j}^{0}), \]

with \( \tau_{i}^{0} = \tau_{x}^{0} + \tau_{y}^{0} + \tau_{z}^{0} \) (see S-II for further details). As we see there is an effective \( U(1) \) valley conservation arising from the underlying lattice symmetries. Specifically for the \( N = 1 \) LL we have:

\[ V_{\cdot,\cdot}(r_{ij}) = \sum_{n=0}^{2} g_{n}^{\cdot,\cdot} e^{2n\delta(r_{ij})}. \]

Therefore, the main difference between the model of Eq. (7) for the \( N = 1 \) LL and the model of Ref. [14] for the \( N = 0 \) LL is the existence of interactions which are not pure delta functions. As we will show, this leads to several important differences in the physics of these two Landau levels.

Mean-field ground states. We will now derive the Hartree-Fock (HF) functional for the Hamiltonian of Eqs. (9), (7) and obtain the phase diagram in the integer fillings of the \( N = 1 \) LL, \( \tilde{\nu} = 1 (\tilde{\nu} = 2) \) when one (two) out of the four valley-spin degenerate LL are filled [7]. We consider the competition of translational invariant integer quantum Hall ferromagnets that can be described by a particle-hole condensate order parameter of the form, \( \langle c_{X_{1},\pi_{1}}c_{X_{2},\pi_{2}} \rangle = F_{\pi_{1}\pi_{2}}\delta_{X_{1},X_{2}}, \) with \( X_{j} \) labeling intra-LL guiding center coordinates. Here \( c_{X_{\pi}} \) denotes the electron creation operator with valley \( \tau \) and spin \( s \), and \( P \) is the projector in spin-valley space into either one-dimensional subspace (for \( \tilde{\nu} = 1 \) ) or a two-dimensional subspace (for \( \tilde{\nu} = 2 \). The general form of the Hartree-Fock functional is then (\( \mathcal{E}_{HF}[P] = \sum_{\nu=x,y,z} \left( u_{1}^{H}(T_{V}(T_{P})^{2})-u_{1}^{X}(T_{V}(T_{P}^{2}) \right) \)):

\[ \mathcal{E}_{HF}[P] = \sum_{\nu=x,y,z} \left( u_{1}^{H}(T_{V}(T_{P})^{2})-u_{1}^{X}(T_{V}(T_{P}^{2}) \right), \]

with \( u_{1}^{H,X} = u_{1}^{H,X} = u_{1}^{H,X} \). Therefore the possible ground states depend only on 4 effective Hartree and exchange constants, \( u_{2}^{H}, u_{2}^{X}, u_{1}^{H}, u_{1}^{X} \), which are linear combinations of the constants \( g_{n}^{\cdot,\cdot} \) that appear in Eq. (7) (see Eq.(S-35) for explicit relations). Moreover, while in the \( N = 0 \) LL (see Eq.(S-33)) the Hartree and the exchange constants are forced to be equal, \( u_{1}^{H,X} = u_{1}^{X} \) [15], in the \( N = 1 \) LL they are independent due to the appearance of non-delta interactions (see S-III for further details). Similar functionals have been proposed, however phenomenologically, for the \( N = 0 \) LL to capture the physics beyond the delta-functionals in Refs. [13, 19].

We will consider general spin-valley entangled [18, 19] variational states. The following two orthonormal spinors can be used to uniquely parametrize the state characterized by \( P \) in Eq. (9),

\[ |F_{1}\rangle = \cos \frac{a_{1}}{2}|\eta\rangle|\bar{s}\rangle + e^{i\beta_{1}}\sin \frac{a_{1}}{2}|\bar{\eta}\rangle|-\bar{s}\rangle, \]
\[ |F_{2}\rangle = \cos \frac{a_{2}}{2}|\eta\rangle|-\bar{s}\rangle + e^{i\beta_{1}}\sin \frac{a_{2}}{2}|\bar{\eta}\rangle|\bar{s}\rangle. \]

Here \( |\eta\rangle \) and \( |\bar{s}\rangle \) are states parametrized by unit vectors \( \eta \) and \( \bar{s} \) in the spin and valley Bloch spheres respectively and \( a_{1,2} \) and \( \beta_{1,2} \) are real constants. Notice that in general these states might not be separable into a tensor product of spin and valley components and
therefore can account for spin-valley entanglement. \cite{18}
For \( \tilde{\nu} = 1 \), we take \( P = |F_1 > < F_1| \), and for \( \tilde{\nu} = 2 \)
\( P = |F_1 > < F_2| + |F_2 > < F_2| \).

**Ground states for \( \tilde{\nu} = 1 \).** As discussed in Ref. \cite{18},
the energy functional in this case reduces to:

\[
E_{HF}^{\tilde{\nu}=1} = \cos^2 \alpha_1 (\Delta_x \eta_2^2 + \Delta_y \eta_2^2),
\]

with \( \Delta_x = u_{2}^H - u_{X}^X \), \( \Delta_y = u_{1}^H - u_{X}^X \) and \( \eta_2^2 = \eta_2^2 + \eta_2^2 \) (see S-V-A for further details). The resulting phase diagram is shown in Fig. 1(b) and contains four phases. These
are a charge density wave (CDW) with \( \eta = \hat{z}, s = \hat{z} \) and \( \alpha_1 = 0 \), and a Kekulé distortion (KD) state with
\( \eta = \eta_1, s = \hat{z} \) and \( \alpha_1 = 0 \). Interestingly, we see that also
spin-valley entangled phases with \( \eta \) \( \sim \) \( \eta_1 \), \( \sim \hat{z} \) \( \sim \hat{z} \) \( \sim \hat{z} \) and \( \alpha_1 = 0 \). These entangled phases are degenerate in the absence of Zeeman fields, but in their presence they
split antiferromagnetic phase (AFI) with \( \eta = \hat{z}, s = \hat{z} \) and \( \alpha_1 = \frac{\pi}{2} \) and the canted antiferromagnet (CAF) with
\( \eta = \eta_1, s = \hat{z} \) and \( \alpha_1 = \frac{\pi}{2} \), as discussed in Ref. \cite{18}.

Notice that in the \( N = 0 \) LL, \( \Delta_z = \Delta_1 = 0 \), and therefore
all of the above states would be degenerate and with a
vanishing HF energy.

**Ground states for \( \tilde{\nu} = 2 \).** The HF functional for \( \nu = 2 \) is
more difficult to minimize analytically. To make progress,
we first consider the subset of states from Eq. 10) without
spin-valley entanglement. These can be classified into
the valley active states \cite{20}:

\[
|F_1 \rangle = |\eta_1 \rangle |s\rangle, \quad |F_2 \rangle = |\eta_2 \rangle |-s\rangle,
\]

in which the valley degree of freedom varies, and the spin
active states:

\[
|F_1 \rangle = |\eta \rangle |s_1\rangle, \quad |F_2 \rangle = |-\eta \rangle |s_2\rangle,
\]

in which the spin degree of freedom varies. We first
minimize the energy functional within this subspace and
then perform a quadratic expansion of all possible deviations of parameters that account for spin-valley entangled states (see S-V-B, VI, VII for further details). For simplicity we will also neglect the Zeeman term that is typically weak compared to the interaction terms \cite{2} \cite{21} \cite{22}.

In contrast to \( \tilde{\nu} = 1 \), for \( \tilde{\nu} = 2 \) we find that whenever a spin-valley disentangled state is energetically favorable it is also an exact local minima of the energy with respect to all possible quadratic deviations that include spin-valley entanglement. This indicates that these spin-valley disentangled states are also possibly exact global minima of the energy.

Following this procedure, we find a total of five possible ground states for \( \tilde{\nu} = 2 \) that are realized as a function of the four Hartree and exchange parameters \( u_{2}^H, u_{X}^X, u_{1}^H, u_{1}^X \). These possible five states are listed in Table I (see S-V-B) for more details on these states). To visualize the energetic competition among these five phases, we have chosen to draw two-dimensional phase diagrams as functions of the two Hartree parameters \( u_{2}^H, u_{1}^H \) for fixed values of \( \Delta_z = u_{2}^H - u_{X}^X \) \( \Delta_1 = u_{1}^H - u_{1}^X \). We find that there are a total of six different kinds of phase diagrams depending on the values and signs of \( \Delta_z, \Delta_1 \). Two of these representative phase diagrams are depicted in Fig. 2 and the remainder are presented in S-V-B).

Interestingly, according to the model and the estimates of Ref. \cite{17}, \( u_{2}^H = u_{X}^H = 0 \) and \( u_{1}^X > 0, u_{1}^X < 0 \) (see S-IV for further details). This means that graphene in the \( N = 1 \) LL would have a phase diagram like the one in Fig. 2(a), and it would be located exactly at the origin of this phase diagram, which we indicate by a black dot in Fig. 2(a). Therefore, we see that the model and the parameter estimates of Ref. \cite{17} place graphene right at the boundary between the CDW and the AF states. We note that at even this boundary, these phases remain stable against spin-valley entangled rotations (see S-VII for further details).

One of the interesting qualitative differences that we have found in the \( N = 1 \) LL is the existence of a new phase that features a combination of Kekulé state and antiferromagnet, that we term the Kekulé-antiferromagnet (KD-AF). In this phase one set of electrons has an XY vector in the valley sphere with spin up while the others occupy the opposite valley vector with spin down, as

Table I. Competing states at \( \tilde{\nu} = 2 \) and their wavefunctions.
We have studied the ground states of spontaneous symmetry broken integer quantum Hall states in the $N = 1$ LL of graphene. We have constructed a general model consistent with the lattice symmetries of graphene that describes the short-range corrections to the Coulomb interaction. Based on this model we studied the ground states at integer fillings. We have found several important qualitative differences with respect to the $N = 0$ LL. First, we showed that when a single component of the $N = 1$ LL is filled ($\tilde{\nu} = 1$), our model can lift the degeneracy to select the ground states, in contrast to the $N = 0$ LL where states remain undetermined. Moreover, interestingly, among the possible competing states at $\tilde{\nu} = 1$, we find that spin-valley entangled phases can appear. On the other hand, when two components are filled ($\tilde{\nu} = 2$), we have found a qualitatively new state that is absent in the $N = 0$ LL, which features a combination of Kekulé and Antiferromagnet character and that we have termed the Kekulé-AF state.

We have shown that the related model for the $N = 1$ LL that appeared in Ref. [17] is missing terms that are allowed by symmetry and is a special case of our model [6]. In particular Ref. [17] is missing the inter-sublattice scattering interactions that appear in Eqs. [6]. By taking the parameters from Ref. [17], we find that graphene will be near the phase boundary separating the CDW and AF states. However, this prediction should be taken carefully because of the aforementioned absence of A-B scattering processes in the model of Ref. [17]. These processes are known to be crucial in the $N = 1$ LL, because they give rise to “$g_1$” interaction in Eq. [8] that ultimately is needed to stabilize the AF or Kekulé states that are reported in experiments [7–9, 17, 24]. We see no reason why these inter-sublattice scattering terms would be negligible in the higher Landau levels. We hope our study stimulates future experiments to better narrow down the states and parameters realized in the $N = 1$ LL of graphene.

![Figure 2](image-url)  
Figure 2. a) Phase diagram at $\tilde{\nu} = 2$ when $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < \Delta_4$ for $\Delta_i/\Delta_j = -1$. According to the estimates of [17] (see S-II for further details), graphene is located at the dot at the origin and therefore at the boundary between the CDW and AF phases. b) Phase diagram at $\tilde{\nu} = 2$ when $\Delta_1, \Delta_3 > 0$, $\Delta_2 < \Delta_4$ for $\Delta_i/\Delta_j = 1/2$. This contains a new phase, the Kekulé distortion antiferromagnet (KD-AF), which does not appear in the $N = 0$ LL. The thick black boundaries represent special first order transitions (phases become unstable coincidentally with their energy crossing) while the orange ones indicate ordinary first order transitions (energies cross but phases remain metastable).

**Discussion.** We have studied the ground states of spontaneous symmetry broken integer quantum Hall states in the $N = 1$ LL of graphene. We have found that spin-valley entangled phases can appear. On the other hand, when two components are filled ($\tilde{\nu} = 2$), we have found a qualitatively new state that is absent in the $N = 0$ LL, which features a combination of Kekulé and Antiferromagnet character and that we have termed the Kekulé-AF state.

We have shown that the related model for the $N = 1$ LL that appeared in Ref. [17] is missing terms that are allowed by symmetry and is a special case of our model [6]. In particular Ref. [17] is missing the inter-sublattice scattering interactions that appear in Eqs. [6]. By taking the parameters from Ref. [17], we find that graphene will be near the phase boundary separating the CDW and AF states. However, this prediction should be taken carefully because of the aforementioned absence of A-B scattering processes in the model of Ref. [17]. These processes are known to be crucial in the $N = 0$ LL, because they give rise to “$g_1$” interaction in Eq. [8] that ultimately is needed to stabilize the AF or Kekulé states that are reported in experiments [7–9, 17, 24]. We see no reason why these inter-sublattice scattering terms would be negligible in the higher Landau levels. We hope our study stimulates future experiments to better narrow down the states and parameters realized in the $N = 1$ LL of graphene.

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1. M. Goerbig, Electronic properties of graphene in a strong magnetic field, Reviews of Modern Physics 83, 1193 (2011).
2. A. Young, J. Sanchez-Yamagishi, B. Hunt, S. Choi, K. Watanabe, T. Taniguchi, R. Ashoori, and P. Jarillo-Herrero, Tunable symmetry breaking and helical edge transport in a graphene quantum spin hall state, Nature 505, 528 (2014).
3. D. S. Wei, T. Van Der Sar, S. H. Lee, K. Watanabe, T. Taniguchi, B. I. Halperin, and A. Yacoby, Electrical generation and detection of spin waves in a quantum hall ferromagnet, Science 362, 229 (2018).
4. P. Stepanov, S. Che, D. Shcherbakov, J. Yang, R. Chen, K. Thilahar, G. Voigt, M. W. Bockrath, D. Smirnov, K. Watanabe, et al., Long-distance spin transport through a graphene quantum hall ferromagnet, Nature Physics 14, 907 (2018).
5. H. Zhou, C. Huang, N. Wei, T. Taniguchi, K. Watanabe, M. P. Zaletel, Z. Papić, A. H. MacDonald, and A. F. Young, Strong-magnetic-field magnon transport in monolayer graphene, Physical Review X 12, 021060 (2022).
6. A. K. Paul, M. R. Sahu, K. Watanabe, T. Taniguchi, J. Jain, G. Murthy, and A. Das, Electrically switchable tunneling across a graphene pn junction: evidence for canting antiferromagnetic phase in $\nu = 0$ state, arXiv preprint arXiv:2205.00710 (2022).
7. S.-Y. Li, Y. Zhang, L.-J. Yin, and L. He, Scanning tunneling microscope study of quantum hall isospin ferromagnetic states in the zero landau level in a graphene...
monolayer, Physical Review B **100**, 085437 (2019).

[8] X. Liu, G. Farahi, C.-L. Chiu, Z. Papic, K. Watanabe, T. Taniguchi, M. P. Zaletel, and A. Yazdani, Visualizing broken symmetry and topological defects in a quantum hall ferromagnet, Science **375**, 321 (2022).

[9] A. Coissard, D. Wander, H. Vignaud, A. G. Grushin, C. Repellin, K. Watanabe, T. Taniguchi, F. Gay, C. B. Winkelmann, H. Courtois, *et al.*, Imaging tunable quantum hall broken-symmetry orders in graphene, Nature **605**, 51 (2022).

[10] K. Nomura and A. H. MacDonald, Quantum hall ferromagnetism in graphene, Physical review letters **96**, 256602 (2006).

[11] M. O. Goerbig, R. Moessner, and B. Douçot, Electron interactions in graphene in a strong magnetic field, Physical Review B **74**, 161407 (2006).

[12] J. Alicea and M. P. Fisher, Graphene integer quantum hall effect in the ferromagnetic and paramagnetic regimes, Physical Review B **74**, 075422 (2006).

[13] I. F. Herbut, Theory of integer quantum hall effect in graphene, Physical Review B **75**, 165411 (2007).

[14] J. Jung and A. MacDonald, Theory of the magnetic-field-induced insulator in neutral graphene sheets, Physical Review B **80**, 235417 (2009).

[15] M. Kharitonov, Phase diagram for the $\nu = 0$ quantum hall state in monolayer graphene, Physical Review B **85**, 155439 (2012).

[16] I. Aleiner, D. Kharzeev, and A. Tsvelik, Spontaneous symmetry breaking in graphene subjected to an in-plane magnetic field, Physical Review B **76**, 195415 (2007).

[17] F. Yang, A. A. Zibrov, R. Bai, T. Taniguchi, K. Watanabe, M. P. Zaletel, and A. F. Young, Experimental determination of the energy per particle in partially filled landau levels, Physical review letters **126**, 156802 (2021).

[18] J. Atteia and M. O. Goerbig, $SU(4)$ spin waves in the $\nu=\pm 1$ quantum hall ferromagnet in graphene, Physical Review B **103**, 195413 (2021).

[19] A. Das, R. K. Kaul, and G. Murthy, Coexistence of canted antiferromagnetism and bond order in $\nu= 0$ graphene, Physical Review Letters **128**, 106803 (2022).

[20] S. S. Hegde and I. S. Villadiego, Theory of competing charge density wave, kekulé, and antiferromagnetically ordered fractional quantum hall states in graphene aligned with boron nitride, Physical Review B **105**, 195417 (2022).

[21] D. A. Abanin, B. E. Feldman, A. Yacoby, and B. I. Halperin, Fractional and integer quantum hall effects in the zeroth landau level in graphene, Physical Review B **88**, 115407 (2013).

[22] I. Sodemann and A. H. MacDonald, Broken $SU(4)$ symmetry and the fractional quantum hall effect in graphene, Physical Review Letters **112**, 126804 (2014).

[23] F. Wu, I. Sodemann, Y. Araki, A. H. MacDonald, and T. Jolicoeur, $SO(5)$ symmetry in the quantum hall effect in graphene, Physical Review B **90**, 235432 (2014).

[24] F. Amet, A. Bestwick, J. Williams, L. Balicas, K. Watanabe, T. Taniguchi, and D. Goldhaber-Gordon, Composite fermions and broken symmetries in graphene, Nature communications **6**, 1 (2015).
S-I: REVIEW OF THE MODEL FOR GRAPHENE

The tight binding Hamiltonian of graphene is:

\[ H = \sum_{R, R', \sigma, \sigma'} t(R + \sigma a_1 - R' - \sigma' a_1) |R, \sigma\rangle \langle R', \sigma'| \]

\[ = t \sum_k \begin{pmatrix} 0 & M(k) \\ M^*(k) & 0 \end{pmatrix} \]  

(S-1)

in the ordered basis |kA\rangle, |kB\rangle with |k, \sigma\rangle = \sum_R e^{-ikR} |R, \sigma\rangle and \( M(k) = e^{-ikR_1} + e^{-ikR_2} + e^{-ik(R_1 + R_2)} \). \( R \) labels the unit cell and \( \sigma = \pm \) the sublattice. \( \tau_i \) are the pauli matrices acting in valley space and \( \sigma_i \) in sublattice space. Upon linearizing \( M \), this gives the Dirac Hamiltonian:

\[ \mathcal{H}_D = \frac{\sqrt{3}}{2} R t(\tau_z p_x \sigma_x + p_y \sigma_y) \]  

\[ \mathcal{H}_D = v_F (\tau_z p_x \sigma_x + p_y \sigma_y) \]  

(S-2)

with \( R = |R_1| = |R_2| \).

In a similar fashion, we can construct a model for graphene which describes the short range two-body interaction anisotropies appearing at the lattice scale of graphene. These can be written as :

\[ \mathcal{H}_A = \sum_{i<j} \sum_{\tau_1, \ldots, \tau_4; \sigma_1, \ldots, \sigma_4} V_{\tau_1, \ldots, \tau_4; \sigma_1, \ldots, \sigma_4} |\tau_1 \sigma_1; \tau_2 \sigma_2\rangle \langle \tau_3 \sigma_3; \tau_4 \sigma_4| \delta(r_i - r_j) \]  

(S-3)

Here \( |\tau \sigma; \tau' \sigma'| \) label the two body hardcore eigenstates of Eq. (2) with \( \tau = \pm 1 \) (\( \sigma = \pm 1 \)) corresponding to \( \tau = K, K' \) (\( \sigma = A, B \)). Eq. (S-3) contains \( 2^8 \) strengths which are subject to symmetry contraints.
General symmetries

Because of the $SU_s(2)$ in spin space, the interactions can be decomposed into a singlet and triplet component:

$$\mathcal{H}_A = \mathcal{H}^s_A \bigoplus \mathcal{H}^t_A \quad (\text{S-4})$$

If we label the single particle states with a super-spin $|N_s\rangle = |\tau_1\sigma_1\rangle$ which takes 4 values, we see that there are 6 distinct spin triplet anti-symmetrized two body states consistent with this:

$$|\lambda\rangle^t = \frac{1}{\sqrt{2}} (|\tau_1\sigma_1;\tau_2\sigma_2\rangle - |\tau_2\sigma_2;\tau_1\sigma_1\rangle) \quad (\text{S-5})$$

So we write Eq. (S-3) in the form:

$$\mathcal{H}^t_A = \sum_{i<j} \sum_{\lambda,\lambda'} V_{\lambda,\lambda'}^t |\lambda\rangle^t \langle \lambda'|^t \delta(r_i - r_j) \quad (\text{S-6})$$

For the singlet, 10 spin symmetric states exist:

$$|\lambda\rangle^s = \frac{1}{\sqrt{2}} (|\tau_1\sigma_1;\tau_2\sigma_2\rangle + |\tau_2\sigma_2;\tau_1\sigma_1\rangle) \quad (\text{S-7})$$

and the interactions in this subspace can be written:

$$\mathcal{H}^s_A = \sum_{i<j} \sum_{\lambda,\lambda'} V_{\lambda,\lambda'}^s |\lambda\rangle^s \langle \lambda'|^s \delta(r_i - r_j) \quad (\text{S-8})$$

Because of hermiticity, we always have:

$$V_{\lambda,\lambda'} = V_{\lambda',\lambda}^* \quad (\text{S-9})$$

Lattice symmetries of graphene

Below we derive the action of the symmetry operations on the hardcore states $|\tau,\sigma\rangle$ introduced in Eqs. (S). For the definition of the auxiliary vectors used see Fig. [S-1]

- $C_6$.

$$C_6 |R,\sigma\rangle = |C_6 R + \sigma C_6 a_1\rangle = |C_6 R + \sigma R_2, -\sigma\rangle \to C_6 |k,\sigma\rangle = e^{-i\sigma (C_6 k) R_2}$$

$$\to C_6 |\tau b_1, \sigma\rangle = Z^{-\tau \sigma} |\tau b_1, -\sigma\rangle \quad (\text{S-10})$$

since $-b_3 = -b_1 + Q_1 = -b_1$, with $Z = e^{i2\pi/3}$.

- $M_x$. $M_x |\tau b_1, \sigma\rangle = |\tau b_1, -\sigma\rangle$

- $M_y$. $M_y |\tau b_1, \sigma\rangle = |\tau b_1, \sigma\rangle$

- $T_R$.

$$T_R |R,\sigma\rangle = |R + R_1, \sigma\rangle \to T_{R_i} |k,\sigma\rangle = e^{ikR_i} |k,\sigma\rangle$$

$$\to T_{R_i} |\tau,\sigma\rangle = Z^{\tau \sigma} |\tau,\sigma\rangle \quad (\text{S-11})$$
Symmetry reduced model

By using the aforementioned symmetries, the model in Eq. (S-3) can be block-diagonalized and reduced to a total of 9 independent real parameters. This yields:

\[ \mathcal{H}^A = \sum_{i,j} \left\{ F_{00}(\tau_0^i\tau_0^j)(\sigma_0^i\sigma_0^j) + F_{z1}(\tau_0^i\tau_0^j)(\sigma_z^i\sigma_z^j) + \frac{F_{zz}}{2}(\tau_0^i\tau_0^j)(\sigma_z^i\sigma_z^j) \right\} \]

where \( F_{00} = \frac{F_{10} + F_{11}}{2} \) and \( F_{zz} = \frac{F_{10} - F_{11}}{2} \).

We note that the model introduced in Ref. [16] is written in the basis of:

\[
\begin{pmatrix}
|KA\rangle \\
|KB\rangle \\
|K'B\rangle \\
-|K'A\rangle
\end{pmatrix}
\] (S-13)

whereas ours in Eq. (S-12) in the basis of:

\[
\begin{pmatrix}
|KA\rangle \\
|KB\rangle \\
|K'A\rangle \\
|K'B\rangle
\end{pmatrix}
\] (S-14)

S-II: PROJECTED MODEL INTO THE NTH LL

The Dirac Hamiltonian in a magnetic field can be derived from the substitution \( \mathbf{p} \rightarrow \Pi = \mathbf{p} + e\mathbf{A} \), with [\( \Pi_x, \Pi_y \)] = −i. We have:

\[ H = v_F(\tau_z\Pi_x\sigma_x + \Pi_y\sigma_y) \] (S-15)

We flip the basis in valley \( K' \):

\[ \psi_{\tau=K'} = \begin{pmatrix} |\tau = K', A\rangle \\ |\tau = K', B\rangle \end{pmatrix} \] (S-16)

and the Hamiltonian becomes:

\[ H = \tau_zv_F(\Pi_x\sigma_x + \Pi_y\sigma_y) \] (S-18)

By defining \( \hat{a} = \frac{1}{\sqrt{2}}(\Pi_x - i\Pi_y) \) and \( \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\Pi_x + i\Pi_y) \) with \( [\hat{a}, \hat{a}^\dagger] = 1 \), the Hamiltonian becomes:

\[ H = \tau_z\sqrt{2}v_F \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^\dagger & 0 \end{pmatrix} \] (S-19)
The eigenstates of the Dirac Hamiltonian in the Nth LL have a definite valley number and can be written as:

\[ |K\rangle = \frac{1}{\sqrt{2}} \left( |n = N - 1, A\rangle + |n = N, B\rangle \right) \quad (S-20) \]

\[ |K'\rangle = \frac{1}{\sqrt{2}} \left( |n = N - 1, B\rangle - |n = N, A\rangle \right) \]

The model for the short range corrections to the Coulombs in the \( N = 1 \) LL, \( \mathcal{H}_A^N \) can be found by projecting \( \mathcal{H}_A \) in Eq. (S-12) to the states in Eq. (S-20), \( |\tau\rangle \), with \( \tau = K, K' (\bar{\tau} = K', K) \):

\[ \mathcal{H}_A^N = \sum_{i<j} \sum_{\tau_1, \tau_2, \tau_3, \tau_4} \langle \tau_1, \tau_2 | i,j \rangle \mathcal{H}_A^{ij} |\tau_3, \tau_4 \rangle \langle i,j | \tau_1, \tau_2 \rangle + \mathcal{H}_A^{ij} |\tau_1, \tau_2 \rangle \langle i,j | \tau_3, \tau_4 \rangle (S-21) \]

with \( \mathcal{H}_A^{ij} \) the interaction between particles \( i,j \) obtained from Eq. (S-12). Due to the \( U_0(1) \) of \( \mathcal{H}_A \), Eq. (S-21) reduces to:

\[ \mathcal{H}_A^N = \sum_{i<j} \sum_{\tau_1, \tau_2} \{ \mathcal{H}_A^{ij} |\tau_1, \tau_2 \rangle \langle i,j | \tau_1, \tau_2 \rangle + \mathcal{H}_A^{ij} |\tau_1, \tau_2 \rangle \langle i,j | \tau_1, \tau_2 \rangle \}
= \sum_{i<j} \{ V_0^N (r_{ij}) \tau_0^i \tau_0^j + V_2^N (r_{ij}) \tau_2^i \tau_2^j + V_4^N (r_{ij}) \tau_4^i \tau_4^j \} \quad (S-22) \]

Due to \( C_6 \) symmetry, \( \mathcal{H}_A^{KKKK} = \mathcal{H}_A^{K'K'K'K'} \), \( \mathcal{H}_A^{KKK'K} = \mathcal{H}_A^{K'KK'K} \), we have:

\[ V_0 = \frac{1}{2} (\mathcal{H}_A^{KKKK} + \mathcal{H}_A^{K'K'K'}) \]
\[ V_2 = \frac{1}{2} (\mathcal{H}_A^{KKKK} - \mathcal{H}_A^{K'K'K'}) \]
\[ V_4 = \mathcal{H}_A^{KK'K'K} = \mathcal{H}_A^{K'KK'K} \quad (S-23) \]

The \( V_0 \) term is negligible compared to the \( SU(4) \) symmetric Coulombs and will be omitted. Therefore, we reach to the general form for the \( N \)th LL in the main text.

More concretely, by writing the position operators in terms of the guiding center operators \( R_i = r_i + \epsilon_{ij} \Pi_j \), the matrix elements \( \mathcal{H}_A^{ij} \) can be written as:

\[ \mathcal{H}_A^{ij} |\tau_1, \ldots, \tau_4 \rangle = \sum_{\alpha, \beta} e^{i q (R_i - R_j)} F_{a\beta} \langle \tau_1 | \vec{F}_j (\mathbf{q}) \tau_\alpha \sigma_\beta |\tau_3 \rangle \langle \tau_2 | (-\mathbf{q}) \tau_\alpha \sigma_\beta |\tau_4 \rangle \quad (S-24) \]

with \( \vec{F}_j (\mathbf{q}) = e^{i \mathbf{q} \cdot (\hat{z} \times \Pi)} \). By evaluating these matrix elements, we reach to:

\[ V_2^N (\mathbf{q}) = \frac{1}{4} (|F_{N-1,N-1} (\mathbf{q})|^2 + |F_{N,N} (\mathbf{q})|^2) (F_{z0} + F_{zz}) \]
\[ + (F_{N-1,N-1} (\mathbf{q}) F_{N,N} (\mathbf{q}) + F_{N,N} (\mathbf{q}) F_{N-1,N-1} (\mathbf{q})) (F_{z0} - F_{zz}) + |F_{N-1,N} (\mathbf{q})|^2 F_{z1} \] \( (S-25) \)
\[ V_4^N (\mathbf{q}) = \frac{1}{8} (F_{zz} + F_{z0}) (|F_{N-1,N-1} (\mathbf{q})|^2 + F_{N,N} (\mathbf{q})^2) \]

with \( F_{n,n'} = \langle n | e^{i \mathbf{q} \cdot (\hat{z} \times \Pi)} | n' \rangle \) being the form factors in the LL \( n, n' \).

In the \( N = 1 \) LL, the strengths \( V_{z,i} \) take the form:

\[ V_{z,i} (r_{ij}) = \sum_{n=0}^{2} g_n r_{ij}^{2n} \delta (r_{ij}) \quad (S-26) \]
where we have used the form factors:

\[
\begin{align*}
F_{0,0}(q) &= e^{-\frac{|q|^2}{4}}, \\
F_{1,0}(q) &= -\frac{i q^*}{\sqrt{2}} e^{-\frac{|q|^2}{4}}, \\
F_{0,1}(q) &= -\frac{i q}{\sqrt{2}} e^{-\frac{|q|^2}{4}}, \\
F_{1,1}(q) &= (1 - \frac{|q|^2}{2}) e^{-\frac{|q|^2}{4}}.
\end{align*}
\]  

(S-27)

with \(q = q_x - iq_y\). The relation of the parameters of the projected model to the original one is:

\[
\begin{align*}
g_0^z &= F_{z0} \\
g_1^z &= \frac{F_{z1} + F_{z0}}{2} \\
g_2^z &= \frac{F_{z2} + F_{z0}}{2} \\
g_0^\perp &= \frac{F_{z0}}{8} \\
g_1^\perp &= \frac{F_{z1} + F_{z0}}{4} \\
g_2^\perp &= \frac{1}{128} (F_{z2} + F_{z0})
\end{align*}
\]  

(S-28)

For the \(N = 0\) LL the parameters of the projected model are:

\[
\begin{align*}
g_1^z &= F_{z1} + F_{z0} \\
g_2^z &= F_{z2} + F_{z0}
\end{align*}
\]  

(S-29)

**S-III: HARTREE-FOCK THEORY**

In first quantization the projected Hamiltonian both in the \(N = 0\) and \(N = 1\) LL have the following form:

\[
V^P = \sum_a \sum_{i<j} \sum_q e^{i q \cdot (R_i - R_j)} V_a(q) \tau_a^i \tau_a^j
\]  

(S-30)

In second quantization this becomes:

\[
V^P = \frac{1}{2A} \sum_a \sum_{X_1, \ldots, X_4} \sum_{s_1, s_2, \tau_1, \ldots, \tau_4} V_a(q) \phi_{X_1,X_4}(q) \rho_{X_2,X_3}(-q) \tau_a^{\tau_1} \tau_a^{\tau_4} \tau_a^{\tau_2} \tau_a^{\tau_3} c_{X_1}^{s_1} c_{X_2}^{s_2} c_{X_3}^{s_1} c_{X_4}^{s_4}
\]  

(S-31)

with \(\phi_{X_1,X_2}(q) = \delta_{X_1,X_2} e^{i q_x (X_1 + X_2)}\).

We search for the mean field energy functional for translational invariant states, parametrized by:

\[
\left\{ c_{X_1}^{s_1} c_{X_2}^{s_2} \right\} = P_{\tau_1 \tau_2} \delta_{X_1,X_2}.
\]

We find:

\[
E_{HF}[P] = \frac{A}{8\pi^2} \sum_a \left\{ V_a(0) Tr\{ T_a P \} Tr\{ T_a P \} - \frac{1}{2\pi} \int dq_x dq_y V_a(q_x, q_y) Tr\{ T_a P T_a P \} \right\}
\]  

(S-32)

For the \(N = 0\) LL we get:

\[
E_{HF}[P] = \frac{N^2}{2A} \sum_a g_a \left\{ Tr\{ T_a P \} Tr\{ T_a P \} - Tr\{ T_a P T_a P \} \right\}
\]  

(S-33)

where \(g_a\) the interaction strengths in the \(N = 0\) LL in Eq. (S-29) for \(a = \perp, z\).
For the $N = 1$ LL we get:

$$E_{HF}[P] = \frac{N^2}{2A} \left\{ u_z^H Tr\{T_z P \} Tr\{T_z P \} - u_z^X Tr\{T_z P T_z P \} \right.
\left. + u_z^H (Tr\{T_z P \} Tr\{T_z P \} + Tr\{T_y P \} Tr\{T_y P \})
- u_z^X (Tr\{T_z P T_z P \} + Tr\{T_y P T_y P \}) \right\}$$

(S-34)

with

$$u_z^H = F_{z0}$$
$$u_z^X = -\frac{F_{zz}}{4} + F_{zz} + F_{z0}$$
$$u_{\perp}^H = \frac{F_{10}}{8}$$
$$u_{\perp}^X = -\frac{F_{1\perp}}{2} + \frac{F_{1\perp}}{16} + \frac{F_{10}}{16}$$

(S-35)

S-IV: COMPARISON WITH THE MODEL OF REF. [17]

The model proposed in Eq.(S21) of Ref. [17] can be recasted into the following more convenient form before projection,

$$\mathcal{V} = \sum_{i<j} \left\{ V_1 \tau_i^0 \tau_j^0 (\sigma_0^i \sigma_0^j + \sigma_z^i \sigma_z^j) + V_2 \tau_i^1 \tau_j^1 (\sigma_0^i \sigma_0^j + \sigma_z^i \sigma_z^j) \right\} \delta(\mathbf{r}_i - \mathbf{r}_j)$$

(S-36)

Notice that while the above model contains inter-valley scattering terms, it does not contain inter-sublattice scattering terms. This model is a special case of the Aleiner, Kharzeev and Tsvelik, in which the only non-vanishing parameters are $F_{zz} \neq 0$ in Eq. (S-12). Upon projection of Eq. (S-36) we find that,

$$g_{0,1}^z = g_{0,2}^1 = 0$$
$$g_z^2 = \frac{F_{zz}}{16}, \quad g_1^z = \frac{F_{1\perp}}{4}$$

(S-37)

leading to Eq.(1) of Ref. [17]. Moreover, Ref. [17] also estimated that the above constants $g_z^2, g_1^z$ are positive. Therefore this leads to the following values and signs of the parameters of the HF functional:

$$u_{z,\perp}^H = 0$$
$$\Delta_z < 0, \quad \Delta_\perp > 0, \quad \Delta_\perp < \Delta_\perp$$

(S-38)

S-V-A): GROUND STATES AT QUARTER-FILLING ($\nu = 1$)

Let’s label the occupied state by $|\chi_i\rangle$. Then we have $P = |\chi_i\rangle \langle \chi_i|$. So,

$$(Tr\{P T_i\})^2 = Tr\{P T_i P T_i\} = (|\chi_i| T_i |\chi_i\rangle)^2$$

(S-39)

since only one state contributes to the trace. Then the HF energy would be:

$$E_{HF}^{N=0} = 0$$
$$E_{HF}^{N=1} = \frac{N^2}{2A} \left\{ (u_z^H - u_z^X) \langle \chi_i | T_z | \chi_i \rangle^2 +
(u_z^H - u_z^X) \left( \langle \chi_i | T_x | \chi_i \rangle^2 + \langle \chi_i | T_y | \chi_i \rangle^2 \right) \right\}$$

(S-40)

$$= \frac{N^2}{2A} \left\{ \Delta_z \langle \chi_i | T_z | \chi_i \rangle^2 +
\Delta_\perp \langle \chi_i | T_x | \chi_i \rangle^2 + \langle \chi_i | T_y | \chi_i \rangle^2 \right\}$$
S-V-B): GROUND STATES AT HALF-FILLING ($\tilde{\nu} = 2$)

We first consider the disentangled, valley and spin active, states in Eqs. (12), (13). The HF functional for the valley active states is,

$$E_{HF}[P_{\eta}] = \frac{N^2}{2A} \left\{ u_\eta^H (\eta_1^z + \eta_2^z)^2 - u_\eta^X ((\eta_1^z)^2 + (\eta_2^z)^2) + u_\eta^H ((\eta_1^x + \eta_2^x)^2 + (\eta_1^y + \eta_2^y)^2) - u_\eta^X ((\eta_1^x)^2 + (\eta_2^x)^2 + (\eta_1^y)^2 + (\eta_2^y)^2) \right\}$$

\hspace{1cm} (S-41)

and for the spin active,

$$E_{HF}[P_{s}] = \frac{N^2}{2A} \left\{ -(2u_1^X + u_2^X)(1 + s_1 \cdot s_2) - (u_1^X (n_1)^2 + u_2^X (n_2)^2)(1 - s_1 \cdot s_2) \right\}$$

\hspace{1cm} (S-42)

It is easy to recover the $N = 0$ LL functionals by setting $u_{1,2}^H = u_{1,2}^X$. We note that for the spin active states the Hartree energy vanishes, rendering the HF functional for the spin active states the same as in the $N = 0$ LL. In Table S-II, the energies of the different states can be found.

| States appearing in the $\tilde{\nu} = 2$ | Wavefunctions | Energies |
|------------------------------------------|---------------|----------|
| CDW (Charge density wave) | $|\eta_1\rangle |s\rangle$, $|\eta_2\rangle |-s\rangle$ | $E_{HF} = 2\Delta_s + 2u_s^H$ |
| KD (Kekulé distortion) | $|\eta_1\rangle |s\rangle$, $|\eta_2\rangle |-s\rangle$ | $E_{HF} = 2\Delta_s + 4u_s^H$ |
| FM (Ferromagnet) | $|\eta_1\rangle |s\rangle$, $|\eta_2\rangle |-s\rangle$ | $E_{HF} = 4\Delta_s + 4u_s^H - 2u_s^H$ |
| AF (Antiferromagnet) | $|\eta_1\rangle |s\rangle$, $|\eta_2\rangle |-s\rangle$ | $E_{HF} = 2\Delta_s - 2u_s^H$ |
| KD-AF (Kekulé antiferromagnet) | $|\eta_1\rangle |s\rangle$, $|\eta_2\rangle |-s\rangle$ | $E_{HF} = 2\Delta_s - 2u_s^H$ |

Table S-II. Table representing the states appearing in the phase diagram at $\tilde{\nu} = 2$, their wavefunctions and their HF energies.

In addition to the phase diagrams in the main text, we obtain by comparing the HF energies of the states for the other possible cases of the values of $\Delta_{z,1}$ the phase diagrams in Figs. S-2.

S-VI: LINEAR STABILITY ANALYSIS IN VALLEY-SPIN DISENTANGLED SUB-SPACES

By expanding around the states with minimum energy up to quadratic terms, we are able to find the stability lines for spin-valley disentangled fluctuations. For the valley active states, by writing:

$$n_{1,2}^{z,1} \approx \frac{1}{2} (n_{1,2}^{z,1})^2$$

\hspace{1cm} (S-43)

the HF energies up to quadratic fluctuations can be written as:

$$E''[\delta n_{z,1}] = C^i + \frac{1}{2} \delta n_{z,1}^T \mathbb{K}^i \delta n_{z,1}$$

\hspace{1cm} (S-44)

with $i$ representing the state, i.e CDW, KD, KD-AF, AF, $\mathbb{K}^i$ the stability matrix, $C^i$ constants and $\delta n_{z,1} = \left( \begin{array}{c} \delta n_{1,1}^z \\ \delta n_{2,1}^z \end{array} \right)$ the fluctuations around the ground state for the two components.
The stability matrices are:

\[
\begin{align*}
K_{AF} &= \begin{pmatrix} 2u^H_1 \pm 2 & 2u^H_z \\ 2u^H_z & 2u^H_1 \pm 2 \end{pmatrix} \\
K_{CDW} &= \begin{pmatrix} -2u^H_z \pm 2 & 2u^H_1 \\ 2u^H_1 & -2u^H_z \pm 2 \end{pmatrix} \\
K_{KD-AF} &= \begin{pmatrix} 2u^H_1 \pm 2 & 2u^H_z \\ 2u^H_z & 2u^H_1 \pm 2 \end{pmatrix} \\
K_{KD} &= \begin{pmatrix} -2u^H_1 \pm 2 & 2u^H_z \\ 2u^H_z & -2u^H_1 \pm 2 \end{pmatrix}
\end{align*}
\]

(S-45)

, where the upper (the lower) sign corresponds to \( \Delta_z < (>) \Delta_1 \). For the spin active states, an analogous analysis yields,

\[
\begin{align*}
\mathcal{E}_{FM}^{FM}[\delta s_{1,2}] &= C_{FM} - 2u^X_1 \delta s_1 \cdot \delta s_2 \\
\mathcal{E}_{KD,KD-AF}^{KD,KD-AF}[\delta s_{1,2}] &= C_{KD,KD-AF} - 2u^X_1 \delta s_1 \cdot \delta s_2 - 2(u^X_1 - u^X_2)u^2_1
\end{align*}
\]

(S-46)

Our results are presented in Figs. S-3

**S-VII: LINEAR STABILITY ANALYSIS IN VALLEY-SPIN ENTANGLED SPACES FOR THE CDW AND AF**

The general HF functional for the spin-valley entangled states,

\[
\begin{align*}
|F_1\rangle &= \cos \frac{\alpha_1}{2} |\eta\rangle |s\rangle + e^{i\beta_1} \sin \frac{\alpha_1}{2} |-\eta\rangle |-s\rangle \\
|F_2\rangle &= \cos \frac{\alpha_2}{2} |\eta\rangle |-s\rangle + e^{i\beta_2} \sin \frac{\alpha_2}{2} |-\eta\rangle |s\rangle
\end{align*}
\]

(S-47)
Figure S-3. a) Phase diagram (coloured regions in the phase diagram) and stability lines (dotted red lines) for the valley active states for $\Delta_z > \Delta_\perp$. b) Phase diagram (coloured regions in the phase diagram) and stability lines (dotted red lines) for the spin active states. The red dotted line represents the stability line of the FM, while the blue one of the KD-AF and KD. c) Phase diagram (coloured regions in the phase diagram) and stability lines (dotted red lines) for the valley active states for $\Delta_z < \Delta_\perp$.

is:

$$\mathcal{E}_{HF} = 2u^H \{ M^P_1 M^P_2 - |M^P_{1z}|^2 \} + 2u^H \{ M^P_1 M^P_2 + M^P_1 M^P_2 - |M^P_{1'}|^2 - |M^P_{2'}|^2 \} + \Delta_z \{ |M^P_1|^2 + |M^P_2|^2 + 2 |M^P_{1z}|^2 \} + \Delta_\perp \{ |M^P_1|^2 + |M^P_2|^2 + 2 |M^P_{1'}|^2 \}$$

(S-48)

with $M^P_{a(i)} = \langle F_i \tau_a | F_j \rangle$ and we can always write the unit vector $\eta$ as $\eta = \begin{pmatrix} \sin \theta_p \cos \phi_p \\ \sin \theta_p \sin \phi_p \\ \cos \theta_p \end{pmatrix}$.

CDW

By expanding around $a_1 = a_2 = \theta_p = 0$, keeping up to quadratic terms, the energy functional for the CDW is around the minimum is:

$$\mathcal{E}_{CDW} = (a^2_1 + a^2_2)(\Delta_1 - \Delta_2 - u^H_1 - u^H_2) + 2(\Delta_1 - \Delta_2 + u^H_1 - u^H_2) \theta^2_p$$

(S-49)

So we find that the instability lines are:

$$u^H_2 = -u^H_1 + (\Delta_1 - \Delta_2)$$

$$u^H_1 = u^H_1 + (\Delta_1 - \Delta_2)$$

(S-50)
**AF**

By expanding around $a_1 = a_2 = \theta_p = \frac{\pi}{2}$ and $\beta = \beta_1 + \beta_2 = 0$, the energy is:

$$
\mathcal{E}_{AF} = (a_1^2 + a_2^2)(3\Delta_1 - \Delta_z - u^H_1 + u^H_z)
- (\Delta_1 + \Delta_z - 3u^H_1 - u^H_z)a_1a_2
+ (\Delta_1 - \Delta_z - u^H_1 + u^H_z)\beta^2
+ 4(\Delta_1 - \Delta_z - u^H_1 + u^H_z)\theta^2_p
$$

(S-51)

So we find that the instability lines are:

$$
\begin{align*}
    u^H_z &= u^H_1 + (\Delta_z - \Delta_1) \\
    u^H_1 &= \Delta_1 \\
    u^H_z &= -u^H_1 + (\Delta_z - \Delta_1)
\end{align*}
$$

(S-52)

These are the same as the ones which occur from the non spin-valley entangled analysis.