DICHOTOMY OF POINCARE MAP AND BOUNDEDNESS OF SOLUTIONS OF CERTAIN NON-AUTONOMOUS PERIODIC CAUCHY PROBLEMS

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Abstract. In this paper we study the dichotomy of the Poincaré map and give the relations between the dichotomy of the Poincaré map and boundedness of solutions of the following periodic Cauchy problems
\[
\begin{align*}
\dot{X}(t) &= A(t)X(t) + e^{i\mu t}Pb, \quad t \geq 0 \\
X(0) &= 0,
\end{align*}
\]
and
\[
\begin{align*}
\dot{X}(t) &= -X(t)A(t) + e^{i\mu t}(I - P)b, \quad t \geq 0 \\
X(0) &= 0,
\end{align*}
\]
where \(A(t)\) is a square size matrix of order \(m\), \(\mu\) is any real number, \(b\) is a non-zero vector in \(\mathbb{C}^m\) and \(P\) is an orthogonal projection.

1. Introduction

The aim of this paper is to study the relation between dichotomy of Poincaré map and boundedness of the solutions of the \(q\)-periodic \((q > 0)\) Cauchy problems in the continuous case. For a well-posed non-autonomous Cauchy problem
\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + e^{i\mu t}I, \quad t \geq 0 \\
x(0) &= 0,
\end{align*}
\]
where \(A(t)\) an \(m \times m\) matrix, the solution leads to an evolution family \(U = \{U(t, s), t \geq s \geq 0\}\), i.e. \(U(t, s)U(s, r) = U(t, r)\) and \(U(t, t) = I\) for all \(t \geq s \geq r \geq 0\). When the Cauchy problem \((A(t), \mu, Pb, 0)\) is \(q\)-periodic, i.e. \(A(t + q) = A(t)\) for all \(t \geq 0\), then the family \(U\) is \(q\)-periodic as well, i.e. \(U(t + q, s + q) = U(t, s)\) for all \(t \geq s \geq 0\). It is given in [1] that the evolution family \(U\) is uniformly exponentially stable if and only if the spectral radius of \(U(q, 0)\) is less than one, i.e.
\[
r(U(q, 0)) := \sup\{|\lambda|, \lambda \in \sigma(U(q, 0))\} = \inf_{n \geq 1} \|U(q, 0)^n\|^\frac{1}{n} < 1.
\]
We show that \(U(q, 0)\) is dichotomous if for each \(\mu \in \mathbb{R}\) the matrices
\[
\Phi_\mu(q) = \int_0^q U(q, s)e^{i\mu s}ds \quad \text{and} \quad \Psi_\mu(q) = \int_0^q U^{-1}(q, s)e^{i\mu s}ds
\]
are invertible and there exits a projection \(P\) which commutes with \(U(q, 0)\), \(\Phi_\mu(q)\) and \(\Psi_\mu(q)\) such that for each real \(\mu \in \mathbb{R}\) and each vector \(b \in \mathbb{C}^m\), the solutions of the Cauchy Problems \((A(t), \mu, Pb, 0)\) and \((-A(t), \mu, (I - P)b, 0)\) are bounded on \(\mathbb{R}_+\). We give an example that invertibility of the matrices \(\Phi_\mu(q)\) and \(\Psi_\mu(q)\) is
necessary condition and boundedness of the Cauchy problems \((A(t), \mu, Pb, 0)\) and \((-A(t), \mu, (I - P)b, 0)\) is not sufficient for the dichotomy of \(U(q, 0)\).

In [1] and [3] stability of the poincaré map have been studied in the discrete and continuous case respectively. These papers give a connection between stability of the Poincaré map and boundedness of the solutions of Cauchy problems. Results regarding the dichotomy of a matrix have been discussed in [2] and [6]. For connection between stability and periodic systems see the papers [1], [3], [5] and [7]. General theory of dichotomy of infinite dimensional systems has given in the monograph [4].

The paper is organized as follows: In section 2 we recall basic well known properties of the evolution family. In section 3 we established the results regarding the connection between dichotomy of the Poincaré map \(U(q, 0)\) and boundedness of solutions for some periodic Cauchy problems.

2. Preliminary Results

Let \(X\) be a Banach space and let \(\mathcal{L}(X)\) be the space of all bounded linear operators acting on \(X\). The norm in \(X\) and in \(\mathcal{L}(X)\) is denoted by the same symbol \(||\cdot||\).

A family \(\mathcal{U} = \{U(t, s) : t \geq s \geq 0\} \subseteq \mathcal{L}(X)\) is called evolution family if the following properties are satisfied

(i) \(U(t, t) = I\), for all \(t \in \mathbb{R}_+\),

(ii) \(U(t, s)U(s, r) = U(t, r)\) for all \(t \geq s \geq r \geq 0\),

where \(I\) denote the identity operator on \(\mathcal{L}(X)\). If the later condition is satisfied for all \(t, s, r \in \mathbb{R}_+\) then we say that \(\mathcal{U}\) is reversible evolution family on \(X\). In this case \(U(t, s)\) is invertible for all \(t, s \in \mathbb{R}_+\).

An evolution family \(\mathcal{U}\) is called strongly continuous if for each \(x \in X\) the map

\[(t, s) \mapsto U(t, s)x : (t, s) \in \mathbb{R}^2 \to X\]

is continuous for all \(t \geq s \geq 0\). Such a family is called \(q\)-periodic (with some \(q > 0\)) if

\[U(t + q, s + q) = U(t, s), \text{ for all } t \geq s \geq 0.\]

Clearly, a \(q\)-periodic evolution family also satisfies

(i) \(U(pq + v, pq + u) = U(v, u)\), for all \(p \in \mathbb{N}\), for all \(v \geq u \geq 0\),

(ii) \(U(pq, rq) = U((p - r)q, 0) = U(q, 0)^{p-r}\), for all \(p, r \in \mathbb{N}, p \geq r\).

Let \(\{U(t, s) : t \geq s \geq 0\}\) be \(q\)-periodic evolution family then the operator \(U(q, 0)\) is called Poincaré map or monodromy operator.

The family \(\mathcal{U}\) is called uniformly exponentially stable if there exist two positive constants \(N\) and \(\omega\) such that

\[||U(t, s)|| \leq Ne^{-\omega(t-s)}, \text{ for all } t \geq s \geq 0.\]

The set of all \(m \times m\) matrices having complex entries would be denoted by \(\mathcal{M}(m, \mathbb{C})\). Assume that the map \(t \mapsto A(t) : \mathbb{R} \mapsto \mathcal{M}(m, \mathbb{C})\) is continuous. Then the Cauchy Problem

\[
\begin{cases}
\dot{X}(t) = A(t)X(t), & t \in \mathbb{R} \\
X(0) = I,
\end{cases}
\]
has a unique solution denoted by $\Phi(t)$. It is well known that $\Phi(t)$ is an invertible matrix and that its inverse is the unique solution of the Cauchy Problem
\begin{equation}
\begin{cases}
\dot{X}(t) = -X(t)A(t), & t \in \mathbb{R} \\
X(0) = I.
\end{cases}
\tag{2}
\end{equation}
Set $U(t, s) := \Phi(t)\Phi^{-1}(s)$ for all $t, s \in \mathbb{R}$. Obviously, the family $U = \{U(t, s), t, s \in \mathbb{R}\}$, has the following properties:

(i) $U(t, t) = I$, for all $t \in \mathbb{R}$;

(ii) $U(t, s) = U(t, r)U(r, s)$ for all $t, s, r \in \mathbb{R}$;

(iii) $U(t, s)$ is invertible for all $t, s \in \mathbb{R}$;

(iv) $\frac{\partial}{\partial t}U(t, s) = A(t)U(t, s)$ for all $t, s \in \mathbb{R}$;

(v) $\frac{\partial}{\partial s}U(t, s) = -U(t, s)A(s)$ for all $t, s \in \mathbb{R}$;

(vi) The map $(t, s) \mapsto U(t, s) : \mathbb{R}^2 \to \mathcal{M}(m, \mathbb{C})$ is continuous.

If, in addition, the map $A(\cdot)$ is $q$-periodic, for some positive number $q$, then:

(vii) $U(t + q, s + q) = U(t, s)$ for all $t, s \in \mathbb{R}$;

(viii) There exists $\omega \in \mathbb{R}$ and $M \geq 1$ such that
\[ \|U(t, s)\| \leq Me^{\omega(t-s)}, \quad \text{for all} \quad t \geq s, \]
i.e. the family $U$ has an exponential growth.

For a given real number $\mu$ and a given family $(A(t))$ we consider the Cauchy Problem
\begin{equation}
\begin{cases}
\dot{X}(t) = A(t)X(t) + e^{i\mu t}I, & t \geq 0 \\
X(0) = 0,
\end{cases}
(A(t), \mu, I, 0)
\end{equation}
and the differential matrix system
\[ \dot{X}(t) = A(t)X(t), \quad t \in \mathbb{R}. \quad (A(t)) \]
Obviously, the solution of $(A(t), \mu, I, 0)$ is given by
\[ \Phi_{\mu}(t) = \int_0^t U(t, s)e^{i\mu s}ds. \]
Now we define
\[ V(t, s) := U^{-1}(t, s) = \Phi(s)\Phi^{-1}(t), \quad t, s \in \mathbb{R} \]
then the family $V = \{V(t, s), t, s \in \mathbb{R}\}$ is an evolution family if
\[ \Phi(t)\Phi^{-1}(s) = \Phi^{-1}(s)\Phi(t) \quad \text{for all} \quad t, s \in \mathbb{R}. \quad (2.1) \]
Throughout the paper we assume that equation (2.1) is satisfied for all $t, s \in \mathbb{R}$. Consider the Cauchy problem
\begin{equation}
\begin{cases}
\dot{Y}(t) = -Y(t)A(t) + e^{i\mu t}I, & t \geq 0 \\
Y(0) = 0.
\end{cases}
(-A(t), \mu, I, 0)
\end{equation}
The solution of \((-A(t), \mu, I, 0)\) is given by
\[
\Psi_{\mu}(t) = \int_0^t V(t, s) e^{\mu s} ds.
\]

Let \(p_L\) be the characteristic polynomial associated to the matrix \(L \in \mathcal{M}(m, \mathbb{C})\) and let \(\sigma(L) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}, k \leq m\) be its spectrum. There exist integer numbers \(m_1, m_2, \ldots, m_k \geq 1\) such that
\[
p_L(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}, \quad m_1 + m_2 + \cdots + m_k = m.
\]
Let \(j \in \{1, 2, \ldots, k\}\) and \(Y_j := \ker(L - \lambda_j I)^{m_j}\) then in [2] we have the following important theorem which is useful latter on.

**Theorem 2.1.** For each \(z \in \mathbb{C}^m\) there exists \(y_j \in Y_j, j \in \{1, 2, \ldots, k\}\) such that
\[
L^n z = L^n y_1 + L^n y_2 + \cdots + L^n y_k.
\]
Moreover, if \(y_j(n) := L^n y_j\) then \(y_j(n) \in Y_j\) for all \(n \in \mathbb{Z}_+\) and there exist a \(\mathbb{C}^m\)-valued polynomials \(p_j(n)\) with \(\deg(p_j) \leq m_j - 1\) such that
\[
y_j(n) = \lambda_j^n p_j(n), \quad n \in \mathbb{Z}_+, \quad j \in \{1, 2, \ldots, k\}.
\]
Indeed from the Hamilton-Cayley theorem and using the well known fact that
\[
\ker[pr(L)] = \ker[p(L)] \oplus \ker[r(L)]
\]
whenever the complex valued polynomials \(p\) and \(r\) are relatively prime, follows
\[
\mathbb{C}^m = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_k.
\]  
(3.1)

Let \(z \in \mathbb{C}^m\). For each \(j \in \{1, 2, \ldots, k\}\) there exists a unique \(y_j \in Y_j\) such that
\[
z = y_1 + y_2 + \cdots + y_k
\]
and then
\[
L^n z = L^n y_1 + L^n y_2 + \cdots + L^n y_k, \quad n \in \mathbb{Z}_+.
\]

3. Dichotomy and Boundedness

Let us denote \(\Gamma_1 = \{z \in \mathbb{C} : |z| = 1\}, \Gamma_1^+ := \{z \in \mathbb{C} : |z| > 1\}\) and \(\Gamma_1^- := \{z \in \mathbb{C} : |z| < 1\}\).

A matrix \(L\) is called:

(i) *stable* if \(\sigma(L)\) is the subset of \(\Gamma_1^-\) or, equivalently, if there exist two positive constants \(N\) and \(T\) such that \(\|L^n\| \leq Ne^{-Tn}\) for all \(n = 0, 1, 2, \ldots,\)

(ii) *expansive* if \(\sigma(L)\) is the subset of \(\Gamma_1^+\) and

(iii) *dichotomic* if \(\sigma(L)\) does not intersect the set \(\Gamma_1\).

**Remark 3.1.** If \(L\) is a dichotomic matrix then there exists \(\eta \in \{1, 2, \ldots, \xi\}\) such that
\[
|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_\eta| < 1 < |\lambda_{\eta+1}| \leq \cdots \leq |\lambda_\xi|.
\]

Having in mind the decomposition of \(\mathbb{C}^m\) given by (3.1) let us consider
\[
X_1 = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_\eta \quad \text{and} \quad X_2 = Y_{\eta+1} \oplus Y_{\eta+2} \oplus \cdots \oplus Y_\xi.
\]
Then \(\mathbb{C}^m = X_1 \oplus X_2\).

Recall that a linear map \(P : \mathbb{C}^m \to \mathbb{C}^m\) is called projection if \(P^2 = P\). In the following theorem we give our first result.
Theorem 3.2. Let \( q > 0 \). If the matrix \( L := U(q,0) \) is dichotomic and there exists a projection \( P \) commuting with \( L \), \( \Phi_\mu(q) \) and \( \Psi_\mu(q) \) then for each \( \mu \in \mathbb{R} \) and each non-zero vector \( b \in \mathbb{C}^m \) the solutions of the following Cauchy problems

\[
\begin{cases}
  \dot{X}(t) = A(t)X(t) + e^{i\mu t}Pb, & t \geq 0 \\
  X(0) = 0,
\end{cases}
\quad (A(t), \mu, Pb, 0)
\]

and

\[
\begin{cases}
  \dot{X}(t) = -X(t)A(t) + e^{i\mu t}(I - P)b, & t \geq 0 \\
  X(0) = 0,
\end{cases}
\quad (-A(t), \mu, (I - P)b, 0)
\]

are bounded.

Proof. Assume that \( L \) is dichotomic, then by Remark 3.1 we have a decomposition of \( \mathbb{C}^m \), i.e. \( \mathbb{C}^m = X_1 \oplus X_2 \).

We define \( P : \mathbb{C}^m \rightarrow \mathbb{C}^m \) by \( Px = x_1 \), where \( x = x_1 + x_2 \), such that \( x_1 \in X_1 \) and \( x_2 \in X_2 \). It is clear that \( P \) is a projection.

Moreover for all \( x \in \mathbb{C}^m \) and all \( k \in \mathbb{Z}_+ \), this yields

\[
PL^k x = P(L^k(x_1 + x_2)) = P(L^k(x_1) + L^k(x_2)) = L^k(x_1) = L^k P x.
\]

Hence \( PL^k = L^k P \) for all \( k \in \mathbb{Z}_+ \). Also we have \( P\Phi_\mu(q)x = P(\Phi_\mu(q)(x_1 + x_2)) = P(\Phi_\mu(q)(x_1) + \Phi_\mu(q)(x_2)) = \Phi_\mu(q)(x_1) = \Phi_\mu(q)x \) and similarly we conclude that \( P\Psi_P(q) = \Psi_\mu(q)P \).

Now the solution of the Cauchy problem \( (A(t), \mu, Pb, 0) \) is given by

\[
\Phi_{(\mu,P,b)}(t) = \int_0^t U(t,s) e^{i\mu s} P bd s.
\]

Let \( n \) be the integer part of \( \frac{t}{q} \) and let \( r := (t - qn) \in [0, q) \). Then

\[
\begin{align*}
\int_0^t U(t,s) e^{i\mu s} P bd s &= \int_0^{qn+r} U(t,s) e^{i\mu s} P bd s \\
&= \int_0^{qn} U(t,s) e^{i\mu s} P bd s + \int_{qn}^{qn+r} U(t,s) e^{i\mu s} P bd s \\
&= \int_{qn}^{qn+r} U(t,s) e^{i\mu s} P bd s + \sum_{k=0}^{n-1} \int_{qk}^{q(k+1)} U(qn+r,s) e^{i\mu s} P bd s \\
&= \int_{qn}^{qn+r} U(t,s) e^{i\mu s} P bd s + U(r,0) \sum_{k=0}^{n-1} \int_{qk}^{q(k+1)} U(qn,s) e^{i\mu s} P bd s \\
&= \int_{qn}^{qn+r} U(t,s) e^{i\mu s} P bd s + U(r,0) \sum_{k=0}^{n-1} \int_{kq}^{qk} U(qn,qk + \tau)e^{i\mu(qk+\tau)} P bd \tau \\
&= \int_{qn}^{qn+r} U(t,s) e^{i\mu s} P bd s + U(r,0) \sum_{k=0}^{n-1} e^{i\mu k} \int_{0}^{q} U(q(n-k),\tau)e^{i\mu \tau} P bd \tau \\
&= \int_{qn}^{qn+r} U(t,s) e^{i\mu s} P bd s + U(r,0) \sum_{k=0}^{n-1} e^{i\mu qk} U(q(0)^{n-k-1} \int_{0}^{q} U(q,\tau)e^{i\mu \tau} P bd \tau \\
&= I_1 + I_2.
\end{align*}
\]
where $I_1 = \int_{qn}^{qn+r} U(t, s)e^{i\mu s} P b ds$, and $I_2 = U(r, 0) \sum_{k=0}^{n-1} e^{i\mu q k} L^{n-k-1} \Phi_{\mu}(q) P b$.

Now the family $\mathcal{U}$ has a growth bound and $0 \leq t - s \leq r < q$, so we have

$$||I_1|| = \left\| \int_{qn}^{qn+r} U(t, s)e^{i\mu s} P b \, ds \right\| \leq M \int_{qn}^{qn+r} e^{\omega(t-s)} ||P b|| \leq q M e^{\omega} ||P b||,$$

where $\omega$ is a real number and $M \geq 1$. Hence $I_1$ is bounded.

Next let $z_\mu = e^{i\mu q}$, and $\Phi_{\mu}(q)b = l \in \mathbb{C}^m$ then

$$I_2 = U(r, 0)(L^{n-1} z_\mu^0 + L^{n-2} z_\mu^1 + \cdots + L^0 z_\mu^{n-1}) P l$$

By our assumption we know that $L$ is dichotomic and $|z_\mu| = 1$ thus $z_\mu$ is contained in the resolvent set of $L$ therefore the matrix $(z_\mu I - L)$ is an invertible matrix. Hence

$$I_2 = U(r, 0)(z_\mu I - L)^{-1}(z_\mu^n I - L^n) P l.$$

Taking norm of both sides

$$||I_2|| \leq ||U(r, 0)(z_\mu I - L)^{-1} z_\mu^n P l|| + ||U(r, 0)(z_\mu I - L)^{-1} P L^n t|| = ||U(r, 0)|| ||(z_\mu I - L)^{-1}|| ||P l|| + ||U(r, 0)|| ||(z_\mu I - L)^{-1}|| ||P L^n t||.$$

Using Theorem 2.1 we have

$$L^n l = \lambda_1^n p_1(n) + \lambda_2^n p_2(n) + \cdots + \lambda_\xi^n p_\xi(n),$$

thus

$$P L^n l = \lambda_1^n p_1(n) + \lambda_2^n p_2(n) + \cdots + \lambda_\xi^n p_\xi(n),$$

where each $p_i(n)$ are $\mathbb{C}^m$-valued polynomials with degree at most $(m_i - 1)$ for any $i \in \{1, 2, \ldots, \xi\}$. From hypothesis we know that $|\lambda_i| < 1$ for each $i \in \{1, 2, \ldots, \eta\}$. So $||P L^n t|| \rightarrow 0$ when $n \rightarrow \infty$. Thus $I_2$ is bounded, hence the solution of $(A(t), \mu, P b, 0)$ is bounded.

Next, since the solution of the Cauchy problem $(-A(t), \mu, (I - P)b, 0)$ is given by

$$\Psi_{(\mu, I - P)b}(t) = \int_0^t V(t, s)e^{i\mu s}(I - P)b \, ds.$$ 

By similar method we obtain that

$$\Psi_{(\mu, I - P)b}(t) = J_1 + J_2$$

where $J_1 = \int_{qn}^{qn+r} V(t, s)e^{i\mu s}(I - P)b ds$ and

$$J_2 = V(r, 0)(z_\mu^0 - (n-1) + z_\mu^1 L^{(n-2)} + \cdots + z_\mu^{n-1} L^0) \Psi_{\mu}(q)(I - P)b.$$ 

Proceeding as before we can show that $J_1$ is bounded. Now for $J_2$ we have since $PL = LP$, therefore $(I - P)L = L(I - P)$. By our assumption we know that $L$
is invertible and since $L^{-1}$ is also dichotomic hence using the same arguments as above we have

$$J_2 = V(r, 0)(z_\mu I - L^{-1})^{-1}(z_\mu^n I - L^{-n})\Psi_{\mu}(q)(I - P)b$$

$$= V(r, 0)(z_\mu I - L^{-1})^{-1}(z_\mu^n I - L^{-n})(I - P)\Psi_{\mu}(q)b.$$

Taking norm of both sides we get

$$\|J_2\| \leq \|V(r, 0)\|\|(z_\mu I - L^{-1})^{-1}\|(I - P)\|\|\Psi_{\mu}(q)b\| + \|V(r, 0)\|\|(z_\mu I - L^{-1})^{-1}\|\|L^{-n}(I - P)\|\|\Psi_{\mu}(q)b\|.$$  

First we prove that $L^{-n}x \to 0$ as $n \to \infty$ for any $x \in X_2$. Since $(I - P)\Psi_{\mu}(q)b \in X_2$ the assertion would follows. Now since $X_2 = Y_{\eta+1} \oplus Y_{\eta+2} \oplus \cdots \oplus Y_\xi$. So any $x \in X_2$ can be written as a sum of $\xi - \eta$ vectors $y_{\eta+1}, y_{\eta+2}, \ldots, y_\xi$. It would be sufficient to prove that $L^{-n}y_i \to 0$ as $n \to \infty$ for any $i \in \{\eta + 1, \eta + 2, \ldots, \xi\}$. Let $Y \in \{Y_{\eta+1}, Y_{\eta+2}, \ldots, Y_\xi\}$ say $Y = ker(L - \lambda I)^\rho$, where $\rho \geq 1$ is an integer number and $|\lambda| > 1$. Consider $d_1 \in Y \setminus \{0\}$ such that $(L - \lambda I)d_1 = 0$ and let $d_2, d_3, \ldots, d_\rho$ given by $(L - \lambda I)d_i = d_{i-1}$. Then $A := \{d_1, d_2, \ldots, d_\rho\}$ is a basis in $Y$. So it is sufficient to prove that $L^{-n}d_i \to 0$ as $n \to \infty$ for any $i \in \{1, 2, \ldots, \rho\}$. For $i = 1$, we have that $L^{-n}d_1 = \frac{1}{\lambda}d_1 \to 0$ as $n \to \infty$. For $i = 2, 3, \ldots, \rho$, denote $B_n = L^{-n}d_i$. Then $(L - \lambda I)^\rho B_n = 0$, i.e.

$$B_n - C_\rho^1 B_{n-1} + C_\rho^2 B_{n-2} \alpha^2 + \cdots + C_\rho^\rho B_{n-\rho} \alpha^\rho = 0, \text{ for all } n \geq \rho$$  \hspace{1cm} (3.2)

where $\alpha = \frac{1}{\lambda}$. 

Passing for instance at the components, it follows that there exists a $C^m$-valued polynomial $P_\rho$ having degree at most $\rho - 1$ and verifying (3.2) such that $B_n = \alpha^n P_\rho(n)$. Thus $B_n \to 0$, when $n \to \infty$ i.e. $L^{-n}d_i \to 0$ for any $i \in \{1, 2, \ldots, \rho\}$. Thus $J_2$ is bounded.

The converse statement of the above theorem is not straight forward and we need to put an extra condition i.e. the matrices $\Phi_{\mu}(q)$ and $\Psi_{\mu}(q)$ are invertible, at the end of the paper we have given an example which shows that the invertibility conditions on matrices $\Phi_{\mu}(q)$ and $\Psi_{\mu}(q)$ can not be removed. Due to this reason we put the converse statement of the above theorem as a new theorem which is stated as.

**Theorem 3.3.** Let there exists a projection $P$ commuting with $L$, $\Phi_{\mu}(q)$ and $\Psi_{\mu}(q)$ and let for each $\mu \in \mathbb{R}$ the matrices $\Phi_{\mu}(q)$ and $\Psi_{\mu}(q)$ are invertible then if for each real number $\mu$ and each non-zero vector $b \in \mathbb{C}^m$, the solutions of the Cauchy Problems $(A(t), \mu, Pb, 0)$ and $(-A(t), \mu, (I - P)b, 0)$ bounded then the Poincare map $L$ is dichotomic.

**Proof.** Suppose on contrary that the matrix $L$ is not dichotomic then $\sigma(L) \cap \Gamma_1 \neq \phi$. Let $\omega \in \sigma(L) \cap \Gamma_1$ then there exists a non zero $y \in \mathbb{C}^m$ such that $Ly = \omega y$, it is easy to see that $L^ky = u^k y$. Here we have two cases:

**Case 1:** If $Py \neq 0$. Choose $\mu_1 \in \mathbb{R}$ such that $\omega = e^{i\mu_1 q}$, then $L^k y = e^{i\mu_1 q k} y$. Since
\(\Phi_{\mu_1}(q)\) is invertible so there exists \(b_1 \in \mathbb{C}^m\) such that \(\Phi_{\mu_1}(q)b_1 = y\). Then

\[
\Phi_{(\mu_1, P, b_1)}(t) = \int_{qn}^{qn+r} U(t, s)e^{i\mu_1 sP}b_1 ds + U(r, 0) \sum_{k=0}^{n-1} e^{i\mu_1 qk}PL^{n-k-1}y \\
= \int_{qn}^{qn+r} U(t, s)e^{i\mu_1 sP}b_1 ds + U(r, 0) \sum_{k=0}^{n-1} e^{i\mu_1 qk}Pe^{i\mu_1 q(n-k-1)}y \\
= \int_{qn}^{qn+r} U(t, s)e^{i\mu_1 sP}b_1 ds + U(r, 0) \sum_{k=0}^{n-1} e^{i\mu_1 q(n-1)}Py \\
= \int_{qn}^{qn+r} U(t, s)e^{i\mu_1 sP}b_1 ds + U(r, 0)nqPbds.
\]

Now clearly \(U(r, 0)nqPb \to \infty\) as \(n \to \infty\). Hence there exist \(\mu_1 \in \mathbb{R}\) and \(b_1 \in \mathbb{C}^m\) such that \(\Phi_{(\mu_1, P, b_1)}\) is unbounded. Therefore contradiction arises.

Case 2: If \(Py = 0\) then surely \((I - P)y \neq 0\). Since \(PL = LP\) therefore \((I - P)L = L(I - P)\). Choose \(\mu_2 \in \mathbb{R}\) such that \(\omega = e^{-iq\mu_2}\). In this case we note that \(L^{-k}y = e^{i\mu_2 qk}y\). Also \(\Psi_{\mu_2}(q)\) is invertible so there exists \(b_2 \in \mathbb{C}^m\) such that \(\Psi_{\mu_2}(q)b_2 = y\). Now consider the solution of \((-A(t), \mu_2, b_2, 0)\) we have

\[
\Psi_{(\mu_2, I - P, b_2)}(t) = J_{1, \mu_2} + J_{2, \mu_2},
\]

where

\[
J_{1, \mu_2} = \int_{qn}^{qn+r} V(t, s)e^{i\mu_2 s(I - P)b_2} ds,
\]

and

\[
J_{2, \mu_2} = V(r, 0) \sum_{k=0}^{n-1} e^{i\mu_2 qk}L^{-(n-k-1)}\Psi_{\mu_2}(q)(I - P)b_2 \\
= V(r, 0) \sum_{k=0}^{n-1} e^{i\mu_2 qk}(I - P)L^{-(n-k-1)}y \\
= V(r, 0) \sum_{k=0}^{n-1} e^{i\mu_2 qk}(I - P)e^{i\mu_2 q(n-k)}y \\
= V(r, 0) \sum_{k=0}^{n-1} e^{i\mu_2 q(n-1)(I - P)y} \\
= V(r, 0)ne^{i\mu_2 q(n-1)(I - P)y}.
\]

Clearly we see that \(J_{2, \mu_2} = V(r, 0)ne^{i\mu_2 q(n-1)(I - P)y} \to \infty\) as \(n \to \infty\). Hence there exist \(\mu_2 \in \mathbb{R}\) and \(b_2 \in \mathbb{C}^m\) such that \(\Psi_{(\mu_2, I - P, b_2)}(t)\) is unbounded. Which is again an absurd. This completes the proof. \(\square\)

The following theorem is taken from [1] which we used to obtained theorem 3.5.

**Theorem 3.4.** The matrix \(L\) is stable if and only if for each \(b \in \mathbb{C}^m\), the solution of \((A(t), \mu, Pb, 0)\) is bounded on \(\mathbb{R}_+\) uniformly with respect to the parameter \(\mu \in \mathbb{R}\), i.e.

\[
\sup_{\mu \in \mathbb{R}} \sup_{t \geq 0} \left\| \int_0^t U(t, s)e^{i\mu bs} ds \right\| := K(b) < \infty.
\]
Theorem 3.5. The matrix $L$ is dichotomic if and only if there exists a projection $P$ such that for each vector $b \in \mathbb{C}^m$, the solutions of the Cauchy Problems $(A(t), \mu, Pb, 0)$ and $(-A(t), \mu, (I - P)b, 0)$ are uniformly bounded on $\mathbb{R}_+$ with respect to the parameter $\mu \in \mathbb{R}$, i.e.

$$
\sup_{\mu \in \mathbb{R}} \sup_{t \geq 0} \left\| \int_0^t U(t, s)e^{is\mu}Pbds \right\| := K_P(b) < \infty, \quad (3.3)
$$

and

$$
\sup_{\mu \in \mathbb{R}} \sup_{t \geq 0} \left\| \int_0^t V(t, s)e^{is\mu}(I - P)bds \right\| := K_{1-P}(b) < \infty. \quad (3.4)
$$

Proof. Suppose the matrix $L$ is dichotomic and let $L_1$ and $L_2$ be the restrictions of $L$ on $X_1$ and $X_2$ respectively. Consider the spectral decomposition of $\mathbb{C}^m$ as given in Remark 3.1, that is we can write

$$
\mathbb{C}^m = X_1 \oplus X_2.
$$

Then $L_1$ is stable on $X_1$ and $L_2^{-1}$ is stable on $X_2$. Define the projection $P : \mathbb{C}^m \to \mathbb{C}^m$ as $Px = x_1$ where $x = x_1 + x_2$ such that $x_1 \in X_1$ and $x_2 \in X_2$. Then clearly $PC^m = X_1$ and $(I - P)C^m = X_2$.

Since $Pb \in X_1$ for each $b \in \mathbb{C}^m$, therefore Theorem 3.4 implies that

$$
\sup_{\mu \in \mathbb{R}} \sup_{t \geq 0} \left\| \int_0^t U(t, s)e^{is\mu}Pbds \right\| := K_P(b) < \infty.
$$

Also $(I - P)b \in X_2$ for each $b \in \mathbb{C}^m$ then again Theorem 3.4 implies that

$$
\sup_{\mu \in \mathbb{R}} \sup_{t \geq 0} \left\| \int_0^t V(t, s)e^{is\mu}(I - P)bds \right\| := K_{1-P}(b) < \infty.
$$

Conversely let $P$ be the projection for which (3.3) and (3.4) are satisfied. Assume that $PC^m = W_1$ and $(I - P)C^m = W_2$. Then clearly $C^m = W_1 \oplus W_2$. So by (3.3) and using Theorem 3.4 we have $L$ is stable on $W_1$. Similarly by (3.4) and again using Theorem 3.4 we obtain that $L^{-1}$ is stable on $W_2$. Hence $L$ is dichotomic on $\mathbb{C}^m$. \hfill \Box

Now we will present an example which shows that in Theorem 3.3 the invertibility condition on the matrices $\Phi_\mu(q)$ and $\Psi_\mu(q)$ can not be removed.

Example 3.6. Let

$$
\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},
$$

then

$$
\Phi^{-1}(t) = \Phi(-t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.
$$

So in this case the evolution family $\mathcal{U} = \{U(t, s), t, s \in \mathbb{R}_+\}$, is given by

$$
U(t, s) = \Phi(t)\Phi^{-1}(s) = \begin{pmatrix} \cos(t - s) & \sin(t - s) \\ -\sin(t - s) & \cos(t - s) \end{pmatrix}.
$$

Since $\sin t$ and $\cos t$ are $2\pi$-periodic functions so this evolution family is $2\pi$-periodic. Now

$$
U(2\pi, s) = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix},
$$
and
\[ \Phi_\mu(2\pi) = \int_0^{2\pi} U(2\pi, s)e^{i\mu s} ds. \]

Choose \( \mu = 0 \), we have
\[ \Phi_0(2\pi) = \int_0^{2\pi} U(2\pi, s) ds = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

which is not invertible. The solution of the Cauchy problem \((A(t), \mu, Pb, 0)\) is given by
\[ \Phi_\mu(t)Pb = \int_{2\pi n}^{2\pi n + r} U(t, s)e^{i\mu s} Pb ds + U(r, 0) \sum_{k=0}^{n-1} z^k_\mu U(2\pi, 0)^{n-k-1} \Phi_\mu(2\pi)Pb, \]

where \( r \in [0, 2\pi) \). Now the family has growth bound and \( 0 \leq t - s < 2\pi \), so we have
\[ \left\| \int_{2\pi n}^{2\pi n + r} U(t, s)e^{i\mu s} Pb ds \right\| \leq rMe^{2\pi\omega}\|Pb\| \leq 2\pi Me^{2\pi\omega}\|Pb\| < \infty, \]

where \( \omega \in \mathbb{R} \) and \( M \geq 1 \).

Next we have since
\[ U(2\pi, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

therefore
\[ \sum_{k=0}^{n-1} z^k_\mu U(2\pi, 0)^{n-k-1} = \begin{pmatrix} \sum_{k=0}^{n-1} z^k_\mu & 0 \\ 0 & \sum_{k=0}^{n-1} z^k_\mu \end{pmatrix}. \]

For \( z_\mu \neq 1 \), we obtain
\[ \left| \sum_{k=0}^{n-1} z^k_\mu \right| = \left| \frac{z^k_\mu - 1}{1 - z_\mu} \right| \leq \frac{2}{|1 - z_\mu|}. \]

So for the corresponding values of \( \mu \in \mathbb{R} \) and each \( b \in \mathbb{C}^2 \) the solution is bounded. If \( z_\mu = 1 \) i.e. \( \mu = 0 \), then
\[ \sum_{k=0}^{n-1} z^k_\mu U(2\pi, 0)^{n-k-1}\Phi_0(2\pi) = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \]

Thus the solution is bounded.

Also we have
\[ V(t, s) = U^{-1}(t, s) = \begin{pmatrix} \cos(t - s) & -\sin(t - s) \\ \sin(t - s) & \cos(t - s) \end{pmatrix}. \]

Similarly as above we can see that \( \Psi_0(2\pi) \) is not invertible and the solution \( \Psi_{(\mu, \lambda - P, b)}(t) \) is bounded for each \( \mu \in \mathbb{R} \) and \( b \in \mathbb{C}^2 \). But \( 1 \in \sigma(U(2\pi, 0)) \), i.e. the matrix \( U(2\pi, 0) \) is not dichotomic.
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