Contribution of twist-3 fragmentation function to single transverse-spin asymmetry in semi-inclusive deep inelastic scattering

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Abstract

We study the contribution of the twist-3 fragmentation function to the single-transverse spin asymmetry in semi-inclusive deep inelastic scattering within the framework of the collinear factorization. Making use of the Ward-Takahashi identity in QCD, we establish the formalism in the Feynman gauge to calculate the non-pole contribution of the twist-3 fragmentation function to the asymmetry and derive the complete cross-section formula in the leading order QCD perturbation theory. The obtained twist-3 hadronic tensor is shown to satisfy the electromagnetic gauge-invariance. The behavior in the small transverse-momentum region for the five structure functions is also given.
1 Introduction

Clarifying the origin of the single spin-asymmetries (SSA) in hard inclusive processes has been a big challenge in high energy spin physics. Since the asymmetries reflect the effect of parton’s intrinsic transverse momentum and correlations in the hadrons, proper description of SSAs requires an extension of the theoretical frameworks for high-energy processes beyond the conventional twist-2 level. Within the framework of the collinear factorization, which is suited to describe the hadron production with a large transverse momentum $p_T$, SSAs appear as a twist-3 observable representing the multi-parton (quark-gluon or purely gluonic) correlations in the nucleon and in the fragmentation process. So far the formalism for calculating the contribution from the twist-3 distributions to SSA has been well developed [1, 2, 3, 4], and there have been many works on them. Those works for the quark-gluon correlation functions include SSAs in semi-inclusive deep-inelastic scattering (SIDIS, $ep^+ \rightarrow ehX$) [3, 5, 6, 7, 8, 9], hadron production in $pp$ collision ($p^+p \rightarrow hX$) [10, 11, 12, 13, 14, 15, 16, 17], the Drell-Yan and direct-photon production ($p^+p \rightarrow \gamma^{(s)}X$) [2, 18, 19], dijet production [20] and $W$-production [21] in $pp$ collision, and $ep \rightarrow jetX$ [22], etc. The works on the three-gluon correlation functions include those for the $D$-meson production in SIDIS ($ep^+ \rightarrow eDX$) [4, 23, 24, 25] and $pp$-collision ($p^+p \rightarrow DX$) [26, 27] and the Drell-Yan and direct-photon production ($p^+p \rightarrow \gamma^{(s)}X$) [28], etc. It has been also shown that the partonic hard cross section for the soft-gluon-pole contribution from the twist-3 distributions can be obtained from a certain twist-2 partonic hard cross section [7, 14, 24, 27, 28]. This “master formula” greatly simplifies the actual calculation, making the structure of the cross section transparent, and is expected to be a useful tool to include higher order corrections. Despite these remarkable developments in our handling of the twist-3 distributions, study on the twist-3 fragmentation functions is still limited [29, 30, 31].

In this paper, we shall derive the contribution from the twist-3 fragmentation function to the single-spin-dependent cross section for SIDIS, $ep^+ \rightarrow ehX$. In the case of the twist-3 distribution functions, a pole part of an internal propagator in the partonic hard scattering amplitude gives rise to the real cross sections, being combined with the real twist-3 multiparton distribution functions. On the other hand, the twist-3 fragmentation functions corresponding to the quark-gluon correlation are, in general, complex because the time-reversal invariance does not give any constraint owing to the final state interaction [32], and thus the imaginary part of the fragmentation matrix elements contributes to the cross section combined with a nonpole partonic hard-scattering part. One should also recall that the soft-gluon-pole matrix elements of the twist-3 quark-gluon fragmentation function has shown to be identically zero, which is connected to the universality property (process-independence) of the Collins fragmentation function [33, 34]. Therefore we shall focus on the nonpole contribution of the twist-3 fragmentation in this work. As in our previous study for the twist-3 quark-gluon and three-gluon correlation functions [3, 4], we will employ the Feynman-gauge to calculate the cross section, since this formulation makes clear how the color-gauge-invariant fragmentation matrix elements are constructed and factorized in the cross section. The electromagnetic gauge invariance is also manifest in our formulation.
Here we mention previous works on the nonpole contributions from the twist-3 fragmentation functions. Yuan and Zhou [29] focused on the small-$p_T$ behavior of the twist-3 fragmentation contribution to the Collins asymmetry in SIDIS in comparison with the transverse-momentum-dependent (TMD) factorization approach, but did not present the twist-3 cross section. We will present the cross section formula for all five structure functions in $ep^\uparrow \rightarrow ehX$. We will also see that our leading small-$p_T$ behavior of the cross section for the Collins asymmetry disagrees with the result in [29]. Kang et al. [30] calculated the so-called “derivative term” of the contribution to $A_N$ of $p^\uparrow p \rightarrow hX$. Metz and Pitonyak [31] derived the complete leading-order formula for the twist-3 fragmentation contribution to this process, using the lightcone-gauge formulation which was developed for the nonpole contribution from the twist-3 distribution to other $p_T$-dependent processes [35, 36, 37]. Our formulation in Feynman gauge as well as its application to SIDIS will shed new light for the study on the twist-3 fragmentation functions. We also mention that one of the present authors (KK) applied the present Feynman-gauge formulation for the nonpole contribution of the three-gluon correlation functions to $A_{LT}$ in $pp$-collision [38]. Together with the contribution from the twist-3 quark-gluon and the three-gluon correlation functions derived in [3, 9, 4], the present study will complete the twist-3 cross section for $ep^\uparrow \rightarrow ehX$ which is relevant for the future eRHIC experiment [39].

The remainder of this paper is organized as follows: In section 2, we define the twist-3 fragmentation functions for a spin-0 hadron which are relevant in our analysis. Both 2-parton and 3-parton correlation functions contribute to $ep^\uparrow \rightarrow ehX$, and all of them are chiral-odd, appearing in a pair with the transversity distribution from the polarized nucleon. In section 3, we will describe our Feynman-gauge formalism to calculate the nonpole contribution from the twist-3 fragmentation function. To this end we apply the collinear expansion to the partonic hard scattering part. Use of the Ward-Takahashi (WT) identities allows us to reorganize all the contribution into a color-gauge-invariant factorized form. We also show the obtained hadronic tensor satisfies the electro-magnetic gauge invariance, using the relation among twist-3 fragmentation functions. In section 4, we will show the result for the five independent structure functions corresponding to different azimuthal structures for $ep^\uparrow \rightarrow ehX$. In order to make connection with the TMD factorization approach, we also show the small $p_T$-limit of each structure function, in comparison with those from the twist-3 distributions. Section 5 will be devoted to a brief summary.

2 Twist-3 fragmentation functions

In this section, we introduce twist-3 fragmentation functions for a spin-0 hadron relevant to the present study. In the twist-3 accuracy, the four momentum of the hadron $P_h$ can be regarded as lightlike. The 2-quark correlator defines two real twist-3 fragmentation functions $\hat{e}_1(z)$ and $\hat{e}_\bar{1}(z)$ as [32]

$$
\Delta_{ij}(z) = \frac{1}{N} \sum_X \int \frac{d\lambda}{2\pi} e^{-i\lambda z} \langle 0|\psi_i(0)|h(P_h)X\rangle \langle h(P_h)X|\bar{\psi}_j(\lambda w)|0\rangle
$$
\[ \Delta^\alpha_{Fij}(z_1, z_2) = \frac{1}{N} \sum_X \int \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} e^{-i\tilde{\lambda}\frac{\delta}{2}} e^{-i\mu\frac{1}{2}} \langle 0|\psi_i(0)|hX\rangle \langle hX|\bar{\psi}_j(\lambda w)gF^{\alpha\beta}(\mu w)w_\beta|0 \rangle \\
= \frac{M_N}{2z}(\gamma_5 p_h\gamma_\lambda)_{ij} \epsilon^{\lambda\alpha\mu} \tilde{E}_F(z_1, z_2) + \cdots, \quad (2) \]

\[ \tilde{\Delta}^\alpha_{Fij}(z_1, z_2) = \frac{1}{N} \sum_X \int \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} e^{-i\tilde{\lambda}\frac{\delta}{2}} e^{-i\mu\frac{1}{2}} \langle 0|\bar{\psi}_j(\lambda w)\psi_i(0)|hX\rangle \langle hX|gF^{\alpha\beta}(\mu w)w_\beta|0 \rangle \\
= \frac{M_N}{2z}(\gamma_5 p_h\gamma_\lambda)_{ij} \epsilon^{\lambda\alpha\mu} \tilde{E}_F(z_1, z_2) + \cdots, \quad (3) \]

where \( F^{\alpha\beta} \equiv F^{\alpha\beta}T^a \) is the gluon’s field strength with \( T^a \) being the color matrices. \( \tilde{E}_F \) and \( \tilde{E}_D \) are, in general, complex functions. \( \tilde{E}_F(z_1, z_2) \) has a support on \( 1 > z_2 > 0 \) and \( z_1 > z_2 \), while \( \tilde{E}_F(z_1, z_2) \) has a support on \( \frac{1}{z_2} - \frac{1}{z_1} > 1, \frac{1}{z_1} < 0 \) and \( \frac{1}{z_2} > 0 \). Replacing the field strength tensor \( gF^{\alpha\beta}(\mu w)w_\beta \) by the covariant derivative \( D^a(\mu w) = \partial^a - igA^{\alpha}(\mu w) \) in \( (2) \), one can define (complex) \( D \)-type function \( \tilde{E}_D(z_1, z_2) \) by the same decomposition in the right-hand-side. However, \( \tilde{E}_D(z_1, z_2) \) is related to \( \tilde{E}_F(z_1, z_2) \) as \( [6] \)

\[ \tilde{E}_D(z_1, z_2) = P \left( \frac{1}{1/z_1 - 1/z_2} \right) \tilde{E}_F(z_1, z_2) + \delta \left( \frac{1}{z_1} - \frac{1}{z_2} \right) \bar{e}(z_2), \quad (4) \]

where \( \bar{e}(z) \) is given by

\[ \Delta^\alpha_{\partial ij}(z) = \frac{1}{N} \sum_X \int \frac{d\lambda}{2\pi} e^{-i\tilde{\lambda}\frac{\delta}{2}} \langle 0|(-\infty w, 0)|\psi_i(0)|hX\rangle \langle hX|\bar{\psi}_j(\lambda w)|\lambda w, \infty w|0 \rangle \tilde{e}_\alpha \]

\[ = \frac{M_N}{2z}(\gamma_5 p_h\gamma_\lambda)_{ij} \epsilon^{\lambda\alpha\mu} \tilde{e}(z) + \cdots. \quad (5) \]

\[^{1}\text{This convention is convenient, since all contributions from the twist-3 distribution (quark-gluon and three-gluon) and fragmentation functions have a common factor in the cross section.}\]
Here we have restored the gauge-link operators to emphasize that the derivative $\partial^\alpha$ hits both $\bar{\psi}(\lambda w)$ and $[\lambda w, \infty w]$, and in writing down the relation (4) we have used the identity $\bar{E}_F(z, z) = 0 \ [33, 34]$. The function $\tilde{e}(z)$ can be shown to be purely imaginary, i.e., $\text{Im} \tilde{e}(z) = -i \tilde{e}(z)$. From QCD equation of motion, one has the relation [31],

$$ z \int \frac{dz'}{z'^2} \bar{E}_D(z', z) = \tilde{e}_1(z) + i \tilde{e}_1(z), \quad (6) $$

which, combined with the relation (4), leads to an important relation in our analysis

$$ z \int \frac{dz'}{z'^2} P \left( \frac{1}{1/z' - 1/z} \right) \text{Im} \bar{E}_F(z_1, z_2) + z \text{Im} \tilde{e}(z) = \tilde{e}_1(z). \quad (7) $$

In what follows, we shall obtain the contribution of the twist-3 fragmentation function to the single-spin-dependent cross section in terms of $\text{Im} \bar{E}_F(z_1, z_2)$, $\text{Im} \tilde{e}(z)$ and $\tilde{e}_1(z)$. But the relation (7) allows us to eliminate $\tilde{e}_1(z)$ (or $\text{Im} \tilde{e}(z)$) in favor of the other two functions.

3 Collinear twist-3 formalism in Feynman gauge

3.1 Kinematics

Following [24, 40], we summarize the kinematics for the SIDIS process,

$$ e(l) + p^+(p, S) \rightarrow e(l') + h(P_h) + X, \quad (8) $$

where $l$, $l'$ and $P_h$ are 4-momenta of the initial lepton, scattered lepton, and the produced hadron, respectively. This process is described by the five independent Lorentz invariants:

$$ S_{ep} = (p + l)^2, \quad x_{bj} = \frac{Q^2}{2p \cdot q}, \quad Q^2 = -q^2 = -(l - l')^2, \quad (9) $$

$$ z_f = \frac{p \cdot P_h}{p \cdot q}, \quad q_T = \sqrt{-q_t^2}, $$

where the space-like momentum $q_t$ is the “transverse” component of $q$ defined as

$$ q_t^\mu = q^\mu - \frac{P_h \cdot q}{p \cdot P_h} p^\mu - \frac{p \cdot q}{p \cdot P_h} P_h^\mu, \quad (10) $$

satisfying $q_t \cdot p = q_t \cdot P_h = 0$. In the hadron frame where the virtual photon and the initial nucleon are collinear, i.e., both move along the $z$-axis, the momenta $q$ and $p$ are given by

$$ q^\mu = (q^0, \vec{q}) = (0, 0, 0, -Q), \quad p^\mu = \left( \frac{Q}{2x_{bj}}, 0, 0, \frac{Q}{2x_{bj}} \right). $$
The azimuthal angle of the hadron plane as measured from the \(xz\) plane is taken to be \(\chi\) and thus the momentum of the final hadron is parameterized as

\[
P_h^\mu = \frac{z_f Q}{2} \left(1 + \frac{q_T^2}{Q^2}, \frac{2q_T}{Q} \cos \chi, \frac{2q_T}{Q} \sin \chi, -1 + \frac{q_T^2}{Q^2}\right).
\]  

(11)

Correspondingly \(w^\mu\) becomes

\[
w^\mu = \frac{1}{z_f Q \left(1 + \frac{q_T^2}{Q^2}\right)^2} \left(1 + \frac{q_T^2}{Q^2}, \frac{2q_T}{Q} \cos \chi, -\frac{2q_T}{Q} \sin \chi, 1 - \frac{q_T^2}{Q^2}\right).
\]  

(12)

The transverse momentum of the final hadron in this frame is given by \(P_h T = z_f q_T\), which is true in any frame where the 3-momenta \(\vec{q}\) and \(\vec{p}\) are collinear. The azimuthal angle of the lepton plane measured from the \(xz\) plane is taken to be \(\phi\) and thus the lepton momentum can be parameterized as

\[
l^\mu = \frac{Q}{2} \left(\cosh \psi, \sinh \psi \cos \phi, \sinh \psi \sin \phi, -1\right),
\]

\[
l'^\mu = \frac{Q}{2} \left(\cosh \psi, \sinh \psi \cos \phi, \sinh \psi \sin \phi, 1\right),
\]

(13)

with \(\cosh \psi = \frac{2z_f S_{ep} S_T}{Q^2} - 1\). The transverse spin vector of the initial nucleon \(S^\mu\) is parametrized as

\[S^\mu = (0, \cos \Phi_S, \sin \Phi_S, 0),\]

(14)

with the azimuthal angle \(\Phi_S\) of \(\vec{S}\). Although three azimuthal angles \(\phi\), \(\chi\) and \(\Phi_S\) are defined above, it is obvious that the cross section for \(ep^+ \to ehX\) depends on them only through the relative angles \(\phi - \chi\) and \(\Phi_S - \chi\). Thus, it can be expressed in terms of \(S_{ep}\), \(x_{bj}\), \(Q^2\), \(z_f\), \(q_T^2\), \(\phi - \chi\) and \(\Phi_S - \chi\) in the above hadron frame. Note that \(\phi\), \(\chi\) and \(\Phi_S\) are invariant under boosts in the \(\vec{q}\)-direction, so that the cross section presented below is the same in any frame where \(\vec{q}\) and \(\vec{p}\) are collinear.

With the kinematical variables defined above, the differential cross section for the SIDIS process with the unpolarized lepton is expressed as

\[
\frac{d^6 \Delta \sigma}{d x_{bj} dQ^2 dz_f dq_T^2 d\phi d\chi} = \frac{\alpha^2_{em}}{128 \pi^4 x_{bj} S^2_{ep} Q^2} z_f L^{\mu \nu}(l, l') W_{\mu \nu}(p, q, P_h),
\]

(15)

where \(L^{\mu \nu}(l, l') = 2(l^\mu l'^\nu + l'^\mu l^\nu) - Q^2 g^{\mu \nu}\) is the leptonic tensor, \(W_{\mu \nu}(p, q, P_h)\) is the hadronic tensor in the same normalization as in [1], and \(\alpha_{em} = e^2/(4\pi)\) is the QED coupling constant. The single-spin-dependent part in the cross section (15) describes the SSA in the process (8).
3.2 Analysis of hadronic tensor

We now consider the contribution from the twist-3 fragmentation function to the hadronic tensor $W_{\mu\nu}(p, q, P_h)$. Figure 1 shows the generic diagrams which give rise to the twist-3 effect originating from the final-state hadron. In these contributions, the quark transversity distribution $h(x)$ can be immediately factorized from the hadronic tensor as

$$W_{\mu\nu}(p, q, P_h) = \int \frac{dx}{x} h(x) w_{\mu\nu}(xp, q, P_h),$$

where $x$ denotes the momentum fraction of the quark in the nucleon, and the summation over quark flavors is implicit. $w_{\mu\nu}$ receives contribution from each of the five diagrams depicted in Fig. 1 as

$$w_{\mu\nu} \equiv w^{(a)}_{\mu\nu} + w^{(b)}_{\mu\nu} + w^{(c)}_{\mu\nu} + w^{(d)}_{\mu\nu} + w^{(e)}_{\mu\nu}$$

$$\equiv \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \Delta^{(0)}(k) S_{\mu\nu}(k) \right]$$

$$+ \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \left\{ \text{Tr} \left[ \Delta_A^{(1)}(k_1, k_2) S_{\mu\nu,\alpha}(k_1, k_2) \right] + \text{Tr} \left[ \Delta_A^{(1)}(k_1, k_2) S_{\mu\nu,\alpha}(k_1, k_2) \right] \right\}$$

Figure 1: Generic diagrams giving rise to the twist-3 fragmentation function contribution to the polarized cross section $\Delta \sigma$. The upper blob represents the fragmentation matrix elements for the final hadron and the lower blob is the quark-transversity distribution. The middle blob denotes the partonic hard scattering part.
\[ + \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \left\{ \text{Tr} \left[ \Delta^{(1)\alpha}_A(k_1, k_2) S_{\mu\nu,\alpha}(k_1, k_2) \right] + + \text{Tr} \left[ \tilde{\Delta}^{(1)\alpha}_A(k_1, k_2) \tilde{S}^R_{\mu\nu,\alpha}(k_1, k_2) \right] \right\}, \]

(17)

where \( k \) and \( k_{1,2} \) are the 4-momenta of the quarks fragmenting into the final hadron, and we have suppressed the other momentum arguments \( P_h, q \) and \( p \) for simplicity. \( S_{\mu\nu}(k), S^{L,R}_{\mu\nu,\alpha}(k_1, k_2) \) and \( \tilde{S}^{L,R}_{\mu\nu,\alpha}(k_1, k_2) \) are the partonic hard matrix elements from \( \Delta^{(1)}(k_1, k_2) \) to Fig. 1(a), (b) and (d), respectively. \( \text{Tr}[\cdots] \) indicates the trace over spinor indices while the color trace is implicit. The hadronic matrix elements \( \Delta^{(0)}(k), \Delta^{(1)\alpha}_A(k_1, k_2) \) and \( \tilde{\Delta}^{(1)\alpha}_A(k_1, k_2) \) in Eq. (17) corresponding to Fig. 1(a), (b) and (d), respectively, are defined as

\[
\Delta^{(0)}_\gamma(k) = \frac{1}{N} \sum_X \int d^4\xi e^{-ik\cdot\xi} \langle 0|\psi_i(0)|hX\rangle \langle hX|\bar{\psi}_j(\xi)|0\rangle, \tag{18}
\]

\[
\Delta^{(1)\alpha}_A(k_1, k_2) = \frac{1}{N} \sum_X \int d^4\xi \int d^4\eta e^{-ik\cdot\xi} e^{-i(k_2-k_1)\cdot\eta} \langle 0|\psi_i(0)|hX\rangle \langle hX|\bar{\psi}_j(\xi)gA^\alpha(\eta)|0\rangle, \tag{19}
\]

\[
\tilde{\Delta}^{(1)\alpha}_A(k_1, k_2) = \frac{1}{N} \sum_X \int d^4\xi \int d^4\eta e^{-ik\cdot\xi} e^{-i(k_2-k_1)\cdot\eta} \langle 0|\psi_i(0)|hX\rangle \langle hX|gA^\alpha(\eta)|0\rangle. \tag{20}
\]

Here the numbers in the superscripts denote the number of the coherent gluon lines coming out of the fragmentation matrix elements. \( \Delta^{(1)\alpha}_A(k_1, k_2) \) for the diagram Fig. 1(c) is obtained from \( \Delta^{(1)\alpha}_A(k_1, k_2) \) by shifting the gluon field \( gA^\alpha(\eta) \) in (20) to the right of the cut. Likewise, \( \tilde{\Delta}^{(1)\alpha}_A(k_1, k_2) \) for the diagram Fig. 1(e) is obtained from \( \tilde{\Delta}_A^{(1)\alpha}(k_1, k_2) \) by the interchange of the gluon field \( \bar{\psi}_j(\xi) \) and \( gA^\alpha(\eta) \) in (20). We note the relations among these matrix elements and the partonic hard parts, \( \Delta^{(1)\alpha}_A(k_1, k_2) = \gamma^0 \Delta^{(1)\alpha}_A(k_2, k_1)^\dagger \gamma^0, \tilde{\Delta}^{(1)\alpha}_A(k_1, k_2) = \gamma^0 \tilde{\Delta}^{(1)\alpha}_A(k_2, k_1)^\dagger \gamma^0, S_{\mu\nu,\alpha}(k_1, k_2) = \gamma^0 S^{L}_{\mu\nu,\alpha}(k_2, k_1)^\dagger \gamma^0 \) and \( \tilde{S}^R_{\mu\nu,\alpha}(k_1, k_2) = \gamma^0 \tilde{S}^{L}_{\mu\nu,\alpha}(k_2, k_1)^\dagger \gamma^0 \). These relations guarantees the cross section is real. Since the leptonic tensor is symmetric (and real) under \( \mu \leftrightarrow \nu \) for the unpolarized electron, only the symmetric parts of \( S_{\mu\nu} \) and \( S^{L,R}_{\mu\nu,\alpha} \) contribute to the cross section. Therefore, in what follows, we shall omit the Lorentz indices \( \mu \) and \( \nu \) from \( S_{\mu\nu}(k) \) and \( S^{L,R}_{\mu\nu,\alpha}(k_1, k_2) \) and represent their symmetric parts as \( S(k) \) and \( S^{L,R}_{\alpha}(k_1, k_2) \), respectively, for simplicity. Correspondingly, we simply write \( w^{(a,b,c,d,e)} \) for \( u^{(\alpha,\beta)}_{\mu\nu} \).

In order to extract twist-3 effects we perform the collinear expansion of the hard part with respect to \( k \) and \( k_{1,2} \) around the momentum \( P_h \). In the present study it is useful to define a projection tensor

\[
\Omega^\alpha\beta = g^\alpha\beta - P_h^\alpha w_\beta. \tag{21}
\]

With this tensor the momentum of a quark is expressed as \( k^\alpha = (k \cdot w)P_h^\alpha + \Omega^\alpha_\beta k^\beta \). Accordingly the collinear expansion of \( S(k) \) may be performed as

\[
S(k) = S(z) + \left. \frac{\partial S(k)}{\partial k^\alpha} \right|_{k \to 0} \Omega^\alpha_\beta k^\beta + \cdots \tag{22}
\]

8
where \( k \cdot w = 1/z \) and we have used the short-hand notation as \( S(z) \equiv S(k = P_h/z) \). Here and below “c.l.” indicates the collinear limit \( k \to P_h/z \). Having this expansion, the first term in (17) becomes up to twist-3

\[
w^{(a)} = \int \frac{dz}{z^2} \text{Tr} \left[ \Delta^{(0)}(z) S(z) \right] - i \Omega^\alpha \int \frac{dz}{z^2} \text{Tr} \left[ \Delta^{(0)\beta}(z) \frac{\partial S(k)}{\partial k^\alpha} \bigg|_{\text{c.l.}} \right],
\]

with the light-cone matrix elements

\[
\Delta^{(0)}_{ij}(z) = \frac{1}{N} \sum_X \int \frac{d\lambda}{2\pi} e^{-i\lambda z} \langle 0| \psi_i(0)| hX \rangle \langle hX| \bar{\psi}_j(\lambda w)|0 \rangle,
\]

\[
\Delta^{(0)\alpha}_{ij}(z) = \frac{1}{N} \sum_X \int \frac{d\lambda}{2\pi} e^{-i\lambda z} \langle 0| \psi_i(0)| hX \rangle \langle hX| \bar{\psi}_j(\lambda w) \partial^\alpha|0 \rangle.
\]

Next, we consider the twist-3 effects arising from the quark-gluon correlation matrix elements in \( w^{(b,c)} \). As was the case for the contribution from the twist-3 distribution [3], we need to reorganize the twist-3 terms appearing in the collinear expansion to construct gauge-invariant \( F \)-type matrix elements. A main difference for the present case is that the nonperturbative matrix elements differ between \( w^{(b)} \) and \( w^{(c)} \). Since the coupling of the transversely polarized gluon \( A_{\perp}^\sigma \) onto \( S^\sigma_L(k_1, k_2) \) gives rise to a power suppressed contribution compared with the coupling of the longitudinally polarized gluon \( (A \cdot w) P_h^\sigma \), one needs collinear expansion only for the latter. For convenience we introduce the notation

\[
S^L(k_1, k_2) \equiv P_h^\sigma S^\sigma_L(k_1, k_2)
\]

and perform the collinear expansion for this quantity as

\[
S^L(k_1, k_2) = S^L(z_1, z_2) + \frac{\partial S^L(k_1, k_2)}{\partial k_1^\alpha} \bigg|_{\text{c.l.}} \Omega^\alpha \beta k_1^\beta
\]

\[
+ \frac{\partial S^L(k_1, k_2)}{\partial k_2^\alpha} \bigg|_{\text{c.l.}} \Omega^\alpha \beta k_2^\beta + \cdots.
\]

Substituting this expression into \( w^{(b)} \), one finds in the twist-3 accuracy

\[
w^{(b)} = \int \frac{dz_1}{z_1^2} \frac{dz_2}{z_2^2} \text{Tr} \left[ \Delta^{(1)}_{\sigma}(z_1, z_2) w_\sigma S^L(z_1, z_2) \right]
\]

\[
+ \Omega^\alpha \beta \int \frac{dz_1}{z_1^2} \frac{dz_2}{z_2^2} \text{Tr} \left[ \Delta^{(1)\beta}_{\sigma}(z_1, z_2) S^L_\sigma(z_1, z_2) \right]
\]

\[\text{Here } \perp \text{ indicates the transverse component with respect to } P_h.\]
\[-i\Omega^\alpha \beta \int \frac{dz_1 dz_2}{z_1^2 z_2^2} \text{Tr} \left[ \Delta^{(1)\beta}_{\delta_1} (z_1, z_2) \frac{\partial S^L(k_1, k_2)}{\partial k_1^\alpha} \right] \text{c.l.} \]

\[-i\Omega^\alpha \beta \int \frac{dz_1 dz_2}{z_1^2 z_2^2} \text{Tr} \left[ \Delta^{(1)\beta}_{\delta_2} (z_1, z_2) \frac{\partial S^L(k_1, k_2)}{\partial k_2^\alpha} \right] \text{c.l.}, \tag{29} \]

where \( S^L_\sigma(z_1, z_2) \equiv S^L_\sigma(P_\mu / z_1, P_\mu / z_2) \), c.l. indicates the collinear limit \( k_i \to P_{\mu i} (i = 1, 2) \), and the light-cone matrix elements are given by

\[ \Delta^{(1)\beta}_{\delta_{ij}} (z_1, z_2) = \frac{1}{N} \sum_X \int \frac{d\lambda d\mu}{2\pi 2\pi} e^{-i \frac{\lambda}{z_1} \mu} e^{-i \left( \frac{z_1}{z_2} - \frac{1}{z_1} \right) \mu} \langle 0 | \psi_i(0) | hX \rangle \]

\[ \times \langle hX | \bar{\psi}_j(\lambda \mu) g A^\beta (\mu \nu) | 0 \rangle, \tag{30} \]

\[ \Delta^{(1)\beta}_{\delta_{1ij}} (z_1, z_2) = \frac{1}{N} \sum_X \int \frac{d\lambda d\mu}{2\pi 2\pi} e^{-i \frac{\lambda}{z_1} \mu} e^{-i \left( \frac{z_1}{z_2} - \frac{1}{z_1} \right) \mu} \langle 0 | \psi_i(0) | hX \rangle \]

\[ \times \langle hX | \bar{\psi}_j(\lambda \mu) \partial^\beta g A^w (\mu \nu) | 0 \rangle, \tag{31} \]

\[ \Delta^{(1)\beta}_{\delta_{2ij}} (z_1, z_2) = \frac{1}{N} \sum_X \int \frac{d\lambda d\mu}{2\pi 2\pi} e^{-i \frac{\lambda}{z_1} \mu} e^{-i \left( \frac{z_1}{z_2} - \frac{1}{z_1} \right) \mu} \langle 0 | \psi_i(0) | hX \rangle \]

\[ \times \langle hX | \bar{\psi}_j(\lambda \mu) \partial^\beta g A^w (\mu \nu) + \tilde{\bar{\psi}}_j(\lambda \mu) g(\partial^\beta A^w (\mu \nu)) | 0 \rangle. \tag{32} \]

Note the first term in (29) may be rewritten by the use of the tree-level Ward identity

\[ P_\mu S^L_\sigma(z_1, z_2) = \frac{S(z_2)}{1/z_2 - 1/z_1 + i\epsilon}. \tag{33} \]

By performing the integration over \( 1/z_1 \), it is straightforward to see this term eventually becomes \( O(g) \) contribution of the gauge-link operator \( [\lambda w, \infty w] \) for \( \Delta(z) \) in (11). The remaining terms in (29) can be reorganized as

\[ w^{(b)} = -i \Omega^\alpha \beta \int \frac{dz_1 dz_2}{z_1^2 z_2^2} \text{Tr} \left[ \Delta^{(1)\beta}_{\delta_1} (z_1, z_2) \left( \frac{\partial S^L(k_1, k_2)}{\partial k_1^\alpha} \right) + \right. \]

\[-i \Omega^\alpha \beta \int \frac{dz_1 dz_2}{z_1^2 z_2^2} \text{Tr} \left[ \left( \Delta^{(1)\beta}_{\delta_2} (z_1, z_2) - \Delta^{(1)\beta}_{\delta_1} (z_1, z_2) \right) \frac{\partial S^L(k_1, k_2)}{\partial k_2^\alpha} \right] \text{c.l.} \]

\[ + \Omega^\alpha \beta \int \frac{dz_1 dz_2}{z_1^2 z_2^2} \text{Tr} \left[ \Delta^{(1)\beta}_{\delta_{ij}} (z_1, z_2) S^L_\sigma(z_1, z_2) \right]. \tag{34} \]

By writing \( \Delta^{(1)\beta}_{\delta_2} - \Delta^{(1)\beta}_{\delta_1} = \Delta^{(1)}_{F} + \widehat{\Delta}^{(1)}_{A} \) with

\[ \Delta^{(1)\beta}_{Fij} = \frac{1}{N} \sum_X \int \frac{d\lambda d\mu}{2\pi 2\pi} e^{-i \frac{\lambda}{z_1} \mu} e^{-i \left( \frac{z_1}{z_2} - \frac{1}{z_1} \right) \mu} \langle 0 | \bar{\psi}_i(0) | hX \rangle \]
\[
\times \langle hX | \bar{\psi}_j(\lambda w) g \left( \partial^3 A^w(\mu w) - w \cdot \partial A^\beta(\mu w) \right) | 0 \rangle, \quad (35)
\]

\[
\tilde{\Delta}^{(1)\beta}_{ij} = \frac{1}{N} \sum_X \int \frac{d\lambda d\mu}{2\pi^2} e^{-i\xi e^{-i\frac{\lambda}{\mu}} \mu} \langle 0 | \bar{\psi}_i(0) | hX \rangle \langle hX | \bar{\psi}_j(\lambda w) g \partial^\sigma A^\beta(\mu w) w_\sigma | 0 \rangle
\]
\[
= i \left( \frac{1}{z_2} - \frac{1}{z_1} \right) \Delta^{(1)\beta}_{ii}, \quad (36)
\]

we find for the diagram (b) in Fig. 1
\[
w^{(b)} = -i\Omega^\alpha_\beta \int \frac{dz_1 dz_2}{z_1^2 z_2^2} Tr \left[ \Delta^{(1)\beta}_{FR}(z_1, z_2) \frac{\partial S^L(k_1, k_2)}{\partial k_2^\alpha} \right] \]
\[
+ \Omega^\alpha_\beta \int \frac{dz_1 dz_2}{z_1^2 z_2^2} Tr \left[ \Delta^{(1)\beta}_{AR}(z_1, z_2) \left( \frac{1}{z_2} - \frac{1}{z_1} \right) \frac{\partial S^L(k_1, k_2)}{\partial k_2^\alpha} \right] \]
\[
- i\Omega^\alpha_\beta \int \frac{dz_1 dz_2}{z_1^2 z_2^2} Tr \left[ \Delta^{(1)\beta}_{Ri}(z_1, z_2) \left( \frac{1}{z_2} - \frac{1}{z_1} \right) \frac{\partial S^L(k_1, k_2)}{\partial k_2^\alpha} \right] \]. \quad (37)

Carrying out a similar analysis for the diagram (c) in Fig. 1 one obtains
\[
w^{(c)} = -i\Omega^\alpha_\beta \int \frac{dz_1 dz_2}{z_1^2 z_2^2} Tr \left[ \Delta^{(1)\beta}_{FR}(z_1, z_2) \frac{\partial S^R(k_1, k_2)}{\partial k_2^\alpha} \right] \]
\[
+ \Omega^\alpha_\beta \int \frac{dz_1 dz_2}{z_1^2 z_2^2} Tr \left[ \Delta^{(1)\beta}_{AR}(z_1, z_2) \left( \frac{1}{z_2} - \frac{1}{z_1} \right) \frac{\partial S^R(k_1, k_2)}{\partial k_2^\alpha} \right] \]
\[
- i\Omega^\alpha_\beta \int \frac{dz_1 dz_2}{z_1^2 z_2^2} Tr \left[ \Delta^{(1)\beta}_{Ri}(z_1, z_2) \left( \frac{1}{z_2} - \frac{1}{z_1} \right) \frac{\partial S^R(k_1, k_2)}{\partial k_2^\alpha} \right] \], \quad (38)

where \( \Delta^{(1)\beta}_{ij} \) and \( \Delta^{(1)\beta}_{FR} \) are the correlation functions obtained by shifting the gluon fields from the left of the cut to the right of the cut, respectively, in (31) and (35), and are connected to \( \Delta^{(1)\beta}_{ij} \) and \( \Delta^{(1)\beta}_{FR} \) by the relation \( \Delta^{(1)\beta}_{ij}(z_1, z_2) = \gamma^0 \Delta^{(1)\beta}_{ij}(z_2, z_1)^\dagger \gamma^0 \) and \( \Delta^{(1)\beta}_{FR}(z_1, z_2) = \gamma^0 \Delta^{(1)\beta}_{FR}(z_2, z_1)^\dagger \gamma^0 \). In our lowest order calculation with respect to the number of the coherent gluon lines, \( \Delta^{(1)\beta}_{FR} \) can be regarded as the F-type correlation function defined in (2).

Therefore, except for the first terms in \( w^{(a,b,c)} \), each term in \( w^{(a,b,c)} \) is not gauge invariant.

For the case of the pole contributions for the twist-3 distribution functions, sum of the diagrams corresponding to the same type of poles satisfies a particular WT identity due to the on-shell condition for an internal line. This leads to a cancellation among all gauge-noninvariant terms, leaving the gauge-invariant factorized expression for the corresponding twist-3 cross section [3][4]. In the present case, however, such special on-shell condition is lacking, so that one has to reconsider the factorization property and prove the color gauge-invariance of the contribution of the twist-3 fragmentation functions. In the next subsection, we present such argument to establish the collinear twist-3 formalism for the non-pole contribution.
3.3 Ward-Takahashi identities and relation among hard parts

In order to show the factorization property for the non-pole contribution, one has to prove that all these gauge-dependent terms are combined into the matrix elements for the gauge-invariant correlation functions. To show this, we first note the amplitude with an off-shell quark external line obeys a WT identity as (see Fig. 2)

\[(k_2 - k_1)^\sigma S_{\sigma}^{La}(k_1, k_2) = T^a S(k_2) + G^a(k_1, k_2),\]  

where we have supplied the color index $a$ explicitly. The first term $T^a S(k_2)$ results from the sum of the first two terms in the right-hand-side of Fig. 2 and the ghost-like term $G^a(k_1, k_2)$ appears due to the off-shellness of the fragmenting quark. This ghost-like term, however, can be neglected in our analysis as we will see below. The ghost-like term has the following structure,

\[G^a(k_1, k_2) = M_{\mu\lambda}(xp, q, k_1)(k_1 - xp - q)_\lambda (k_1 - xp - q)_\rho P^{\rho\sigma}(xp + q - k_2) \times \cdots,\]

where $M^{\mu\lambda}$ denotes $\gamma^\nu q \rightarrow gq$ scattering amplitude and the polarization tensor $P^{\rho\sigma}$ for the final gluon can be taken as

\[P^{\rho\sigma}(k) = -g^{\rho\sigma} + \frac{k_\rho P_{h\sigma} + k_\sigma P_{h\rho}}{k \cdot P_h}.\]

Then $G^a(k_1, k_2)$ contains two factors which become zero in the collinear limit, i.e.,

\[M^{\mu\lambda}(xp, q, \frac{P_h}{z_1})(\frac{P_h}{z_1} - xp - q)_\lambda = 0,\]  

Figure 2: WT identity for a coupling of the scalar-polarized gluon with the momentum $(k_1 - k_2)^\sigma$ onto the amplitude in the left of the final-state cut of $S_{\sigma}^{La}(k_1, k_2)$. For the quark line with the on-shell momentum $xp$, the spinor factor $u(xp)$ is understood to be multiplied. The gluon-line goes into the final-state cut, and thus has the on-shell momentum and the physical polarization. The quark line with the momentum $k_1$ enters the fragmentation matrix element and thus the factor for this external line is amputated.
\[
\left( \frac{P_h}{z_1} - xp - q \right) \rho \mathcal{P}^{\rho \sigma}(xp + q - \frac{P_h}{z_2}) = 0. \tag{43}
\]

Accordingly, the ghost-like term and its first derivatives with respect to the momenta \(k_{1,2\perp}^\alpha\) vanish in the collinear limit:

\[
G^\alpha(z_1, z_2) = 0, \quad \frac{\partial G^\alpha(k_1, k_2)}{\partial k_{1,2\perp}^\alpha}|_{c.l.} = 0. \tag{44}
\]

We now rewrite the WT identity (39) as

\[
P_\sigma^\alpha S_{\sigma}^{L\alpha}(k_1, k_2) = \frac{1}{1/z_2 - 1/z_1 + i\epsilon}\left[ T^\alpha S(k_2) + G^\alpha(k_1, k_2) - (k_{2\perp} - k_{1\perp})^\sigma S_{\sigma}^{L\alpha}(z_1, z_2) - (k_2 - k_1) \cdot P_\sigma^\alpha S_{\sigma}^{L\alpha}(z_1, z_2) w^\sigma \right], \tag{45}
\]

where the sign of the \(i\epsilon\)-prescription is chosen so that this identity reproduces (33) in the collinear limit. Noting the relation (44), we take the derivative of (45) with respect to \(k_{1,2\perp}^\alpha\) and then take the collinear limit to obtain

\[
\frac{\partial S_{\sigma}^{L\alpha}(k_1, k_2)}{\partial k_{1\perp}^\alpha}|_{c.l.} = \frac{1}{1/z_2 - 1/z_1 + i\epsilon}S_{\sigma}^{L\alpha}(z_1, z_2), \tag{46}
\]

\[
\frac{\partial S_{\sigma}^{L\alpha}(k_1, k_2)}{\partial k_{2\perp}^\alpha}|_{c.l.} = \frac{1}{1/z_2 - 1/z_1 + i\epsilon}\left[ T^\alpha \frac{\partial S(k_2)}{\partial k_{2\perp}^\alpha}|_{c.l.} - S_{\sigma}^{L\alpha}(z_1, z_2) \right]. \tag{47}
\]

Substituting these relations into the expression of \(w^{(b)}\) in (37) and integrating over \(1/z_1\), one obtains

\[
w^{(b)} = i\Omega^\alpha \beta \int \frac{d z_1}{z_1^2} \frac{d z_2}{z_2^2} P \left( \frac{1}{1/z_2 - 1/z_1} \right) \text{Tr} \left[ \Delta_{F}^{(1)\beta}(z_1, z_2) S_{\alpha}^{L\alpha}(z_1, z_2) \right]
\]

\[
- i\Omega^\alpha \beta \int \frac{d z}{z^2} \text{Tr} \left[ \left\{ \Delta_{F}^{(1)\beta}(z) + i\Delta_{A}^{(1)\beta}(z) + \Delta_{A}^{(1)\beta}(z) \right\} \frac{\partial S(k)}{\partial k^\alpha}|_{c.l.} \right], \tag{48}
\]

where the single-variable light-cone matrix elements are given by

\[
\Delta_{F}^{(1)\beta}(z) = \frac{1}{N} \sum_X \int \frac{d \lambda}{2\pi} e^{-i\lambda} \langle 0 | \psi \rangle \langle hX | \bar{\psi} \rangle
\]

\[
\times \langle hX | \bar{\psi} \rangle i g \int_{\lambda}^{\infty} d \mu F^{\beta\lambda}(\mu \lambda) \left\{ i g \int_{\lambda}^{\infty} d \mu F^{\beta\lambda}(\mu \lambda) \right\} \langle 0 | \psi \rangle \langle hX | \bar{\psi} \rangle, \tag{49}
\]

\[
\Delta_{A}^{(1)\beta}(z) = \frac{1}{N} \sum_X \int \frac{d \lambda}{2\pi} e^{-i\lambda} \langle 0 | \psi \rangle \langle hX | \bar{\psi} \rangle \langle hX | \bar{\psi} \rangle g A^{\beta}(\lambda \lambda) \langle 0 | \psi \rangle \langle hX | \bar{\psi} \rangle, \tag{50}
\]

\[
\Delta_{A}^{(1)\beta}(z) = \frac{1}{N} \sum_X \int \frac{d \lambda}{2\pi} e^{-i\lambda} \langle 0 | \psi \rangle \langle hX | \bar{\psi} \rangle
\]
with the light-cone matrix element defined by
\[ \psi_j(\lambda w) T^\beta \{ ig \int_\infty^\lambda d\mu A^{w}(\mu w) \} |0\rangle. \] (51)

Here we observe that the operators in the matrix elements appearing in the second term of (48) can be regarded as the \( O(g) \) part of those in the expansion of \( [\bar{\psi}(\lambda w) [\lambda w, \infty w]] T^\beta \). This term may be combined with the matrix element (25) which can be regarded as \( O(1) \) part of the one for \( [\bar{\psi}(\lambda w) [\lambda w, \infty w]] T^\beta \).

By a similar analysis for the diagram (c), it is straightforward to derive
\[ w^{(c)} = -i\Omega^\alpha_\beta \int \frac{dz_1}{z_1^2} \frac{dz_2}{z_2^2} P \left( \frac{1}{1/z_1 - 1/z_2} \right) \text{Tr} \left[ \Delta^{(1)\beta}_{FR}(z_1, z_2) S^R_{\alpha}(z_1, z_2) \right] \]
\[ -i\Omega^\alpha_\beta \int \frac{dz}{z^2} \text{Tr} \left[ \Delta^{(1)\beta}_{\alpha\beta}(z) \frac{\partial S(k)}{\partial k^\alpha} \right]_{\text{c.l.}}, \] (52)

with the light-cone matrix element defined by
\[ \Delta^{(1)\beta}_{\alpha\beta}(z) = \frac{1}{N} \sum X \int \frac{d\lambda}{2\pi} e^{-i\frac{\lambda}{2}} \left\{ ig \int_0^\infty d\mu A^w(\mu w) \right\} \psi_1(0) \langle hX | \bar{\psi}_j(\lambda w) T^\beta |0\rangle. \] (53)

The operator appearing in this matrix element can be identified as one of \( O(g) \) part of the gauge-link operator \([\infty w, 0]\) in \( \Delta^\beta_\alpha(\lambda w) \) defined in (5). In this way, one can identify the sum of (25), (50), (51) and (53) as \( \Delta^\beta_\alpha(\lambda w) \). Therefore, taking the sum of (23), (48) and (52), we find that the sum of \( w^{(a)} + w^{(b)} + w^{(c)} \) can be written in a color-gauge-invariant form as
\[ w^{(a)} + w^{(b)} + w^{(c)} = \int \frac{dz}{z^2} \text{Tr} [\Delta(z) S_{\mu\nu}(z)] + \Omega^\alpha_\beta \int \frac{dz}{z^2} \text{ImTr} \left[ \Delta^\beta_\alpha(z) \frac{\partial S_{\mu\nu}(k)}{\partial k^\alpha} \right]_{\text{c.l.}} \]
\[ - \Omega^\alpha_\beta \int \frac{dz_1}{z_1^2} \frac{dz_2}{z_2^2} P \left( \frac{1}{1/z_2 - 1/z_1} \right) \]
\[ \times \left\{ \text{ImTr} \left[ \Delta^\beta_\alpha(z_1, z_2) S^L_{\mu\nu}(z_1, z_2) \right] + (\mu \leftrightarrow \nu) \right\}, \] (54)

where we have dropped the superscripts (0) and (1), and have restored the Lorentz indices \( \mu \) and \( \nu \) to make the symmetrization for \( S^L_{\mu\nu}(z_1, z_2) \) explicit.

Finally, we consider the twist-3 contributions arising from the diagrams (d,e) in Fig. In this case, the corresponding hard part \( \tilde{S}^L_{\mu\nu}(k_1, k_2) \) obey a simple Ward identity, i.e.,
\[ (k_2 - k_1)\sigma \tilde{S}^L_{\mu\nu}(k_1, k_2) = 0, \] (55)

and the resulting factorization formula is expressed solely in terms of the gauge-invariant \( F \)-type matrix element (3) as
\[ w^{(d)} + w^{(e)} = -\Omega^\alpha_\beta \int \frac{dz_1}{z_1^2} \frac{dz_2}{z_2^2} P \left( \frac{1}{1/z_2 - 1/z_1} \right) \]
\[ \times \left\{ \text{ImTr} \left[ \Delta^\beta_\alpha(z_1, z_2) \tilde{S}^L_{\mu\nu}(z_1, z_2) \right] + (\mu \leftrightarrow \nu) \right\}. \] (56)

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Equations (54) and (56) show that all the contributions arising from the twist-3 fragmentation functions have definite factorization property with manifest color gauge-invariance. We remark that the technique developed in this section can also be used to calculate other twist-3 observables which receive non-pole contribution, such as the double-spin asymmetry $A_{LT}$, c.f. [38].

3.4 Electromagnetic gauge-invariance of hadronic tensor

Here we show that the hadronic tensor obtained in the previous subsection satisfies the electromagnetic gauge-invariance. We first note that each contribution in (54) from different twist-3 fragmentation function does not separately satisfy electromagnetic current conservation. However, one can show the sum of them, $w^{(a)} + w^{(b)} + w^{(c)}$, satisfies this property. To show this, we note the twist-3 relation (7) allows us to rewrite the 2-quark correlator (5) as

\[
\Delta_{ij}(z) = \frac{M_N}{2}(\gamma_5\gamma_\alpha\gamma_\lambda)_{ij}\epsilon^{\lambda\alpha\nu P_h}\left[\int \frac{dz'}{z'^2} P\left(\frac{1}{1/z' - 1/z}\right)\text{Im}\hat{E}_F(z', z) + \text{Im}\bar{c}(z)\right].
\] (57)

Inserting this expression and other matrix elements into (54), we find

\[
q^\mu\left[w^{(a)} + w^{(b)} + w^{(c)}\right]_{\mu\nu} = -\frac{M_N}{2}\epsilon^{\lambda\alpha\nu P_h}\int \frac{dz}{z^2}\text{Im}\bar{c}(z)
\]
\[
\times \text{Tr}\left[\gamma_5\gamma_\lambda\left(\gamma_\alpha q^\mu S_{\mu\nu}(z) + \frac{P_h}{z} q^\mu \frac{\partial S_{\mu\nu}(k)}{\partial k^\alpha} \right)\right]
\]
\[
- \frac{M_N}{2}\epsilon^{\lambda\alpha\nu P_h}\int \frac{dz}{z^2} \int \frac{dz'}{z'^2} P\left(\frac{1}{1/z' - 1/z}\right)\text{Im}\hat{E}_F(z', z)
\]
\[
\times \text{Tr}\left[\gamma_5\gamma_\lambda\left(\gamma_\alpha q^\mu S_{\mu\nu}(z) + \frac{P_h}{z} q^\mu S_{\mu\nu,\alpha}(z', z)\right)\right].
\] (58)

In deriving this relation, we have used the fact that when $q^\mu$ (or $q'^\mu$) is contracted with the right virtual-photon-quark vertex of the diagram in Fig. 1 (b), the result vanishes due to the WT identity in QED. From this expression, one finds that (58) vanishes if the hadronic tensor satisfies the following relation:

\[
\epsilon^{\lambda\alpha\nu P_h}\left[\gamma_\alpha q^\mu S_{\mu\nu}(z) + \frac{P_h}{z} q^\mu \frac{\partial S_{\mu\nu}(k)}{\partial k^\alpha} \right] = 0,
\] (59)

\[
\epsilon^{\lambda\alpha\nu P_h}\left[\gamma_\alpha T^a q^\mu S_{\mu\nu}(z) + \frac{P_h}{z} q^\mu S_{\mu\nu,\alpha}(z', z)\right] = 0.
\] (60)

To prove that (59) is indeed satisfied, we use the WT identity in QED,

\[
q^\mu k S_{\mu\nu}(k) = 0,
\] (61)
which holds for the on-shell quark momentum \( k^\mu = (k^+, k^-) = \frac{k^2}{2k^+}, \vec{k}_\perp \). By applying \( \epsilon^{\alpha \omega P_h} \partial/\partial k^\alpha \) to this relation and then taking the collinear limit \( k \to \frac{P}{z} \), one obtains the relation (59). For the proof of the second relation (60), we rewrite the left-hand-side of (60) as

\[
\epsilon^{\alpha \omega P_h} q^\mu \left[ \gamma_\alpha T^a S_{\mu \nu}(z) + \frac{P_h}{z} S^{La}_{\mu \nu \alpha}(z', z) \right] = \epsilon^{\alpha \omega P_h} q^\mu \left[ \frac{i}{P_h} \frac{\gamma_\alpha T^a}{z} + i \epsilon \right] S_{\mu \nu}(z) + S^{La}_{\mu \nu \alpha}(z', z) \right]. \tag{62}
\]

One finds that this expression vanishes due to the WT identity in QED if one recalls that \( S^{La}_{\mu \nu \alpha}(z', z) \) does not contain diagrams in which the collinear quark and gluon lines merge into a single quark line and it is exactly the first term in \( [ \ ] \), and that \( \epsilon^{\alpha \omega P_h} \) guarantees the physical polarization for the coherent gluon. Therefore (60) is also satisfied. We thus find (54) meets the requirements for the current conservation.

Regarding the contribution of \( \tilde{E}_F \) to the hadronic tensor (56), it is easy to check

\[
q^\mu \left[ w^{(d)} + w^{(e)} \right]_{\mu \nu} = q^\nu \left[ w^{(d)} + w^{(e)} \right]_{\mu \nu} = 0, \tag{63}
\]

by using the WT identity in QED. All in all, the non-pole contribution of the twist-3 fragmentation function to the hadronic tensor satisfies the electromagnetic gauge-invariance.

4 Calculation of \( L^{\mu \nu} W_{\mu \nu} \)

4.1 Single-spin dependent cross section

With the gauge-invariant factorized expression at hand, we now turn to derive the Twist-3 fragmentation contribution to the SSA in SIDIS. To calculate the contraction \( L^{\mu \nu}(l, l')W_{\mu \nu}(p, q, P_h) \) in (15), we introduce the following 4-vectors which are orthogonal to each other [40, 24]:

\[
T^\mu = \frac{1}{Q} (q^\mu + 2 x Bj p^\mu),
\]

\[
X^\mu = \frac{1}{q_T} \left\{ \frac{P_h^\mu}{z_f} - q^\mu - \left( 1 + \frac{q_T^2}{Q^2} \right) x Bj p^\mu \right\},
\]

\[
Y^\mu = \epsilon^{\mu \nu \rho \sigma} Z^\nu X^\rho T^\sigma,
\]

\[
Z^\mu = -\frac{q^\mu}{Q}. \tag{64}
\]

The hadronic tensor \( W^{\mu \nu} \) can be expanded in terms of the six independent tensors \( V_k^{\mu \nu} \) \((k = 1, \ldots, 4, 8, 9)\) [7]:

\[
V_1^{\mu \nu} = X^\mu X^\nu + Y^\mu Y^\nu, \quad V_2^{\mu \nu} = g^{\mu \nu} + Z^\mu Z^\nu,
\]

\[
V_3^{\mu \nu} = g^{\mu \nu}, \quad V_4^{\mu \nu} = X^\mu Y^\nu + Y^\mu X^\nu, \\
V_5^{\mu \nu} = Y^\mu X^\nu, \quad V_6^{\mu \nu} = X^\mu Y^\nu,
\]

\[
V_7^{\mu \nu} = X^\mu Z^\nu + Z^\mu X^\nu, \quad V_8^{\mu \nu} = Y^\mu Z^\nu + Z^\mu Y^\nu,
\]

\[
V_9^{\mu \nu} = Z^\mu Z^\nu.
\]
\[ \mathcal{V}_{3}^{\mu\nu} = T^{\mu}X^{\nu} + X^{\mu}T^{\nu}, \quad \mathcal{V}_{4}^{\mu\nu} = X^{\mu}X^{\nu} - Y^{\mu}Y^{\nu}, \]

\[ \mathcal{V}_{8}^{\mu\nu} = T^{\mu}Y^{\nu} + Y^{\mu}T^{\nu}, \quad \mathcal{V}_{9}^{\mu\nu} = X^{\mu}Y^{\nu} + Y^{\mu}X^{\nu}. \]  

(65)

Using their inverse tensors \( \tilde{V}_{k}^{\mu\nu} \) given by

\[ \tilde{V}_{1}^{\mu\nu} = \frac{1}{2}(2T^{\mu}T^{\nu} + X^{\mu}X^{\nu} + Y^{\mu}Y^{\nu}), \quad \tilde{V}_{2}^{\mu\nu} = T^{\mu}T^{\nu}, \]

\[ \tilde{V}_{3}^{\mu\nu} = -\frac{1}{2}(T^{\mu}X^{\nu} + X^{\mu}T^{\nu}), \quad \tilde{V}_{4}^{\mu\nu} = \frac{1}{2}(X^{\mu}X^{\nu} - Y^{\mu}Y^{\nu}), \]

\[ \tilde{V}_{8}^{\mu\nu} = -\frac{1}{2}(T^{\mu}Y^{\nu} + Y^{\mu}T^{\nu}), \quad \tilde{V}_{9}^{\mu\nu} = \frac{1}{2}(X^{\mu}Y^{\nu} + Y^{\mu}X^{\nu}), \]  

(66)

one obtains

\[ L_{\mu\nu}W^{\mu\nu} = \sum_{k=1, 2, 4, 8, 9} [L_{\mu\nu}V_{k}^{\mu\nu}] [W_{\rho\sigma}\tilde{V}_{k}^{\rho\sigma}] = Q^{2} \sum_{k=1, 2, 4, 8, 9} A_{k}(\phi - \chi) [W_{\rho\sigma}\tilde{V}_{k}^{\rho\sigma}], \]  

(67)

where \( A_{k}(\phi - \chi) \equiv L_{\mu\nu}V_{k}^{\mu\nu}/Q^{2} \) is given by

\[ A_{1}(\phi) = 1 + \cosh^{2} \psi, \quad A_{2}(\phi) = -2, \quad A_{3}(\phi) = -\cos \phi \sinh 2\psi, \]

\[ A_{4}(\phi) = \cos 2\phi \sinh^{2} \psi, \quad A_{8}(\phi) = -\sin \phi \sin 2\psi, \quad A_{9}(\phi) = \sin 2\phi \sinh^{2} \psi. \]  

(68)

By the expansion (67), the cross section for the SIDIS process consists of the five structure functions associated with \( A_{1,2}, A_{3}, A_{4}, A_{8} \) and \( A_{9} \), respectively, which have different dependencies on the azimuthal angle \( \phi - \chi \). Calculating the Feynman diagrams shown in Fig.3 the single-spin-dependent cross section associated with the transversity distribution and the twist-3 fragmentation functions can be obtained to be

\[
\begin{align*}
\frac{d^{6}\Delta\sigma}{dx_{b}dQ^{2}dz_{f}dz_{T}d\phi d\chi} & = \frac{\alpha_{em}^{2} \alpha_{s} M_{N}}{16\pi^{2}x_{b}^{2}S_{e}^{2}Q^{2}} \sum_{k=1, 2, 4, 8, 9} A_{k}S_{k} \int_{x_{min}}^{1} \frac{dx}{x} \int_{z_{min}}^{1} \frac{dz}{z} \delta \left( \frac{q_{T}^{2}}{Q^{2}} - \left( 1 - \frac{1}{x} \right) \left( 1 - \frac{1}{z} \right) \right) \\
& \times \sum_{a} \epsilon_{a}^{2} h_{a}^{q}(x) \left[ \frac{\hat{e}_{a}^{q}(z)}{z} \Delta\hat{\sigma}_{k}^{1} + \frac{d}{d(1/z)} \left\{ \frac{\text{Im}\tilde{e}_{a}^{q}(z)}{z} \right\} \Delta\hat{\sigma}_{k}^{2} + \text{Im}\tilde{e}_{a}^{q}(z)\Delta\hat{\sigma}_{k}^{3} \right. \\
& \quad - 2 \int_{z}^{\infty} \frac{dz'}{z'^{2}} \left\{ P \left( \frac{1}{1/z - 1/z'} \right) \text{Im}\tilde{E}_{F}^{q}(z', z)\Delta\hat{\sigma}_{k}^{4} \\
& \quad \left. + z\text{Im}\tilde{E}_{F}^{q}(-z', (1/z - 1/z')^{-1})\Delta\hat{\sigma}_{k}^{5} \right\} \right].
\end{align*}
\]  

(69)
Figure 3: Feynman diagrams giving rise to the contribution of the twist-3 fragmentation functions (a) \( \hat{e}_1, \bar{e} \), (b) \( \hat{E}_F \) and (c) \( \hat{E}_F \). For (b) and (c), an extra coherent gluon line coming out of the fragmentation insertion \( \otimes \) attaches to one of the dots in the diagrams.

where \( A_k \equiv A_k(\phi - \chi), S_k \equiv \sin(\Phi_S - \chi) \) for \( k = 1, \ldots, 4 \) and \( S_k \equiv \cos(\Phi_S - \chi) \) for \( k = 8, 9 \). We have introduced new variables \( \hat{x} \equiv \frac{x_{bj}}{x} \) and \( \hat{z} \equiv \frac{z_f}{z} \). The symbol \( e_a \) denotes the electric charge for quark-flavor \( a \) and the superscript \( a \) is supplied to each distribution and fragmentation function. The lower limits of the integrations are given by

\[
x_{\text{min}} = x_{bj} \left( 1 + \frac{z_f}{1 - z_f} \frac{q_T^2}{Q^2} \right),
\]

\[
z_{\text{min}} = z_f \left( 1 + \frac{x_{bj}}{1 - x_{bj}} \frac{q_T^2}{Q^2} \right).
\]

Partonic hard cross sections \( \Delta \hat{\sigma}_k^i \) \( (i = 1, \ldots, 5, k = 1, \ldots, 4, 8, 9) \) are the functions of \( \hat{x}, \hat{z} \) and \( \hat{z}' \equiv \frac{z_f}{z} \) and are given as follows:

\[
\Delta \hat{\sigma}_1^1 = -\frac{4C_F(-2 + 5\hat{z} - 3\hat{z}^2 + \hat{x}(4 - 7\hat{z} + 6\hat{z}^2))}{q_T(1 - \hat{z} + \hat{x}(-1 + 2\hat{z}))},
\]

\[
\Delta \hat{\sigma}_2^1 = -\frac{8C_F\hat{z}}{q_T},
\]

\[
\Delta \hat{\sigma}_3^1 = -\frac{2C_FQ((-1 + \hat{z})^3 - 2\hat{x}\hat{z}(4 - 7\hat{z} + 3\hat{z}^2) + \hat{x}^2(1 + 7\hat{z} - 14\hat{z}^2 + 8\hat{z}^3))}{q_T^2\hat{z}\hat{x}(1 - \hat{z} + \hat{x}(-1 + 2\hat{z}))},
\]
\[
\Delta \hat{\sigma}^1_4 = -\frac{4C_F Q^2 (-1 + \hat{z})(1 - \hat{z} - \hat{x}(-4 + \hat{z})\hat{z} + \hat{x}^2(-1 - 3\hat{z} + 2\hat{z}^2))}{q_T^2 \hat{x} \hat{z}(1 - \hat{z} + \hat{x}(-1 + 2\hat{z}))},
\]

\[
\Delta \hat{\sigma}^1_8 = \frac{2C_F Q(4\hat{x}(-1 + \hat{z})\hat{z} - (-1 + \hat{z})^2(1 + \hat{z}) + \hat{x}^2(1 + 5\hat{z} - 8\hat{z}^2 + 4\hat{z}^3))}{q_T^2 \hat{x} \hat{z}(1 - \hat{z} + \hat{x}(-1 + 2\hat{z}))},
\]

\[
\Delta \hat{\sigma}^1_9 = -\Delta \hat{\sigma}^1_4,
\]

\[
\Delta \hat{\sigma}^2_1 = -\frac{4C_F (-1 + 2\hat{x})(-1 + \hat{z})}{q_T [1 - \hat{z} + \hat{x}(-1 + 2\hat{z})]},
\]

\[
\Delta \hat{\sigma}^2_2 = 0,
\]

\[
\Delta \hat{\sigma}^2_3 = -\frac{4C_F Q(-1 + 2\hat{x})(-1 + \hat{z})}{q_T^2 [1 - \hat{z} + \hat{x}(-1 + 2\hat{z})]},
\]

\[
\Delta \hat{\sigma}^2_4 = -\frac{4C_F Q^2(-1 + 2\hat{x})(-1 + \hat{z})}{q_T^2 [1 - \hat{z} + \hat{x}(-1 + 2\hat{z})]},
\]

\[
\Delta \hat{\sigma}^2_5 = -\Delta \hat{\sigma}^2_3,
\]

\[
\Delta \hat{\sigma}^2_9 = -\Delta \hat{\sigma}^2_4,
\]

\[
\Delta \hat{\sigma}^3_1 = \frac{4C_F (-1 + 3\hat{z})}{q_T},
\]

\[
\Delta \hat{\sigma}^3_2 = \frac{8C_F \hat{z}}{q_T},
\]

\[
\Delta \hat{\sigma}^3_3 = \frac{2C_F Q [-(1 + \hat{z})^2 + \hat{x}(1 - 3\hat{z} + 4\hat{z}^2)]}{q_T^2 \hat{x} \hat{z}},
\]

\[
\Delta \hat{\sigma}^3_4 = \frac{4C_F Q^2 [-1 + \hat{z} + \hat{x}(1 - \hat{z} + \hat{z}^2)]}{q_T^3 \hat{x} \hat{z}},
\]

\[
\Delta \hat{\sigma}^3_8 = \frac{2C_F Q [1 - \hat{z}^2 - \hat{x}(1 - \hat{z} + 2\hat{z}^2)]}{q_T^2 \hat{x} \hat{z}},
\]

\[
\Delta \hat{\sigma}^3_9 = -\Delta \hat{\sigma}^3_4,
\]

\[
\Delta \hat{\sigma}^4_1 = \frac{1}{Nq_T} \left[ -1 + \hat{x} + 4\hat{z} - 3\hat{x} \hat{z} + 3(-1 + 2\hat{x})\hat{z}^2 \right. - \left. \frac{2(-1 + \hat{x} + \hat{z} - \hat{x} \hat{z})}{(-1 + \hat{x} + \hat{x} - 2\hat{x} \hat{z})(\hat{z} - \hat{z}')} \right]
\]
\[
\Delta \sigma_2^4 = \frac{2 \hat{\xi}(2\hat{x}^2(-1 + \hat{\xi})(\hat{\xi} - \hat{\xi}')) - (1 + \hat{\xi})(1 - 2\hat{\xi} + 4\hat{\xi}^2)}{Nq_T(\hat{x}z - (1 + \hat{x} + \hat{\xi})\hat{\xi}')}
+ \frac{2N\hat{x}(-\hat{x}(1 + 2\hat{\xi})(\hat{\xi} - \hat{\xi}')) + \hat{x}'(-2 - \hat{x} + \hat{\xi}(3 - 2\hat{\xi} + 2\hat{\xi}'))}{Nq_T(\hat{x}z + (1 + \hat{x} + \hat{\xi})\hat{\xi}')}, (90)
\]

\[
\Delta \sigma_3^4 = \frac{Q}{2Nq_T^2} \left[ \frac{(-1 + \hat{\xi})^3 - 2\hat{x}(-1 + \hat{\xi})(1 - 2\hat{\xi} + 4\hat{\xi}^2)}{\hat{x}(1 + \hat{x} + \hat{\xi})} \right.
+ \frac{4(-1 + \hat{x} + \hat{\xi} - \hat{x}\hat{\xi})}{(-1 + \hat{x} + \hat{\xi} - 2\hat{x}\hat{\xi})}(\hat{\xi}' - \hat{\xi}'')
- \frac{4(-1 + \hat{x})(-1 + \hat{\xi})\hat{\xi}(1 - \hat{\xi} + \hat{x}(1 + 2\hat{\xi}))}{(-1 + \hat{x} + \hat{\xi})(\hat{x}\hat{\xi} + \hat{x}' - (\hat{x} + \hat{\xi})\hat{\xi}')}
\left. + \frac{4(-1 + \hat{\xi})\hat{\xi}^2}{(-1 + 2\hat{\xi})(-1 + \hat{x} + \hat{\xi} - 2\hat{x}\hat{\xi})(\hat{\xi}' - \hat{\xi}'')} + \frac{4(-1 + \hat{\xi})\hat{\xi}^3(-1 + 2\hat{\xi}')}{\hat{x}\hat{\xi} + (1 + \hat{x} + \hat{\xi})\hat{\xi}'} \right], (91)
\]

\[
\Delta \sigma_4^4 = \frac{1}{Nq_T} \left[ \frac{(-1 + \hat{\xi})^2 + \hat{x}(1 + \hat{\xi}(-1 + 2\hat{\xi}))}{-1 + \hat{x} + \hat{\xi}} - \frac{(1 + 2(-1 + \hat{\xi}))(-1 + \hat{\xi})\hat{x}}{(-1 + \hat{x} + \hat{\xi})\hat{x}'} \right.
- \frac{2\hat{x}\hat{\xi}}{(-1 + \hat{x} + \hat{\xi} - 2\hat{x}\hat{\xi})(\hat{\xi}' + \hat{\xi}'')} + \frac{\hat{\xi}(-1 + \hat{\xi})^2 + 2\hat{x}(-1 + \hat{\xi})^2 + \hat{x}^2(-1 - 2(-1 + \hat{\xi}))}{(-1 + \hat{x} + \hat{\xi})(\hat{x}\hat{\xi} + \hat{x}' - (\hat{x} + \hat{\xi})\hat{\xi}')}
\left. + \frac{N}{q_T} \left[ \frac{(-1 + \hat{\xi})^2 + \hat{x}(-1 + \hat{\xi} - 2\hat{\xi}^2)}{-1 + \hat{x} + \hat{\xi}} - \frac{2\hat{x}(-1 + 2\hat{\xi})\hat{x}^2}{(-1 + \hat{x})(-1 + \hat{x} + \hat{\xi} - 2\hat{x}\hat{\xi})(\hat{\xi}' - \hat{\xi}'')} \right. \right.
\left. + \frac{\hat{x}^2(-1 + \hat{\xi})^2 + 2\hat{x}(-1 + \hat{\xi})^2 + \hat{x}^2(-1 - 2(-1 + \hat{\xi}))}{(-1 + \hat{x})(-1 + \hat{x} + \hat{\xi})(\hat{x}\hat{\xi} + \hat{x}' - (\hat{x} + \hat{\xi})\hat{\xi}')}, (93) \right.
\]

\[\text{20}\]
\[
\Delta \sigma_8^4 = \frac{Q}{2Nq_T^2} \left[ -\frac{(1 + \hat{\varepsilon})(1 - \hat{\varepsilon} + \hat{x}(1 + 2\hat{\varepsilon}))}{\hat{x}\hat{\varepsilon}} - \frac{4(-1 + \hat{x})(1 + \hat{\varepsilon})}{(-1 + \hat{x} + \hat{\varepsilon} - 2\hat{x}\hat{\varepsilon})(\hat{\varepsilon} - \hat{\varepsilon}')} + \frac{2 - 6\hat{\varepsilon} + 4\hat{\varepsilon}^2}{\hat{\varepsilon}'} + \frac{4(-1 + \hat{x})(1 + \hat{\varepsilon})\hat{\varepsilon}}{\hat{x}\hat{\varepsilon} + \hat{\varepsilon}' - (\hat{x} + \hat{\varepsilon})\hat{\varepsilon}'} \right] \\
+ \frac{NQ}{2q_T^2\hat{\varepsilon}} \left[ -1 + \hat{\varepsilon} + 2\hat{\varepsilon}^2 \right] + \frac{1 - \hat{\varepsilon}^2}{\hat{x}} + \frac{4(-1 + 2\hat{x})(1 + \hat{\varepsilon})\hat{\varepsilon}^2}{(-1 + \hat{x} + \hat{\varepsilon} - 2\hat{x}\hat{\varepsilon})(\hat{\varepsilon} - \hat{\varepsilon}')} + \frac{4(-1 + \hat{\varepsilon})\hat{\varepsilon}^3}{\hat{x}\hat{\varepsilon} + \hat{\varepsilon}' - (\hat{x} + \hat{\varepsilon})\hat{\varepsilon}'} \right], \tag{94}
\]
\[
\Delta \sigma_9^4 = \frac{1}{Nq_r} \left[ -1 + \hat{\varepsilon} \left\{ -1 - \frac{2\hat{x}}{(-1 + \hat{x} + \hat{\varepsilon} - 2\hat{x}\hat{\varepsilon})(\hat{\varepsilon} - \hat{\varepsilon}')} \right. \right. \\
+ \frac{(-1 + 2\hat{x})(1 + \hat{\varepsilon})}{(-1 + \hat{x})\hat{\varepsilon}'} + \frac{-1 + \hat{x} + \hat{\varepsilon} - 2\hat{x}\hat{\varepsilon}}{-\hat{x}\hat{\varepsilon} + (-1 + \hat{x} + \hat{\varepsilon})\hat{\varepsilon}'} \left\} \right] \\
+ \frac{N}{q_r} \left[ 1 + \hat{\varepsilon} \left\{ 1 + \frac{\hat{\varepsilon}}{(-1 + \hat{x})(1 - \hat{x} + \hat{x}(1 + 2\hat{\varepsilon}))} \left( \frac{2(1 - 2\hat{x})\hat{x}}{\hat{\varepsilon} - \hat{\varepsilon}'} + \frac{(-1 + \hat{x} + \hat{\varepsilon} - 2\hat{x}\hat{\varepsilon})^2}{\hat{x}\hat{\varepsilon} + \hat{\varepsilon}' - (\hat{x} + \hat{\varepsilon})\hat{\varepsilon}'} \right) \right\} \right]. \tag{95}
\]
\[
\Delta \sigma_1^5 = \frac{\hat{x}(1 - \hat{\varepsilon})[1 + 6(-1 + \hat{\varepsilon})\hat{\varepsilon}]}{Nq_r\hat{\varepsilon}} F, \tag{96}
\]
\[
\Delta \sigma_2^5 = \frac{4\hat{x}(-1 + \hat{\varepsilon})^2}{Nq_r} F, \tag{97}
\]
\[
\Delta \sigma_3^5 = \frac{Q(-1 + 2\hat{x})(1 + \hat{\varepsilon})^2}{Nq_T^2\hat{\varepsilon}} F, \tag{98}
\]
\[
\Delta \sigma_4^5 = \frac{Q^2(1 + 2(-1 + \hat{x})\hat{x})(1 + \hat{\varepsilon})^3}{Nq_T^3\hat{x}\hat{\varepsilon}} F, \tag{99}
\]
\[
\Delta \sigma_5^5 = \frac{-Q(-1 + \hat{\varepsilon})^2(-1 + 2\hat{\varepsilon})}{Nq_T^2\hat{\varepsilon}} F, \tag{100}
\]
\[
\Delta \sigma_6^5 = \frac{-Q^2(-1 + 2\hat{x})(1 + \hat{\varepsilon})^3}{Nq_T^3\hat{x}\hat{\varepsilon}} F, \tag{101}
\]

where \( F \) is a dimensionless factor given by
\[
F = \frac{\hat{x}\hat{\varepsilon}^3 + \hat{\varepsilon}(\hat{x} - (-1 + \hat{\varepsilon})^2 - 3\hat{x}\hat{\varepsilon})\hat{\varepsilon}' + (-1 + \hat{x} + \hat{\varepsilon})^2\hat{\varepsilon}'^2}{\hat{\varepsilon}'((-1 + \hat{\varepsilon})\hat{\varepsilon} - (-1 + \hat{x} + \hat{\varepsilon})\hat{\varepsilon}'\hat{\varepsilon}'\hat{\varepsilon}' - (\hat{x}\hat{\varepsilon} + (-1 + \hat{x} + \hat{\varepsilon})\hat{\varepsilon}')}. \tag{102}
\]
4.2 Azimuthal asymmetries and their asymptotic behavior at small-\( q_T \)

Using the obtained cross-section formula (69), we investigate the small-\( q_T \) behavior of the azimuthal asymmetries, which becomes important to check the consistency with the description based on the TMD factorization approach.

We now define new angle variables as,

\[
\phi - \chi = \phi_h, \quad \Phi_S - \chi = \phi_h - \phi_S,
\]

where \( \phi_h \) and \( \phi_S \) are, respectively, the azimuthal angles of the hadron plane and the nucleon’s spin vector measured from the lepton plane. With these variables, the polarized cross-section is now recast into

\[
\frac{d^6\Delta\sigma}{dx_tq_T^2dz_fdz_d\phi_d\chi} = \sin(\phi_h - \phi_S) [F_1 + F_2 \cos \phi_h + F_3 \cos 2\phi_h]
\]

\[
+ \cos(\phi_h - \phi_S) [F_4 \sin \phi_h + F_5 \sin 2\phi_h]
\]

\[
= F^{\sin(\phi_h - \phi_S)}(\phi_h - \phi_S) + F^{\sin(2\phi_h - \phi_S)}(2\phi_h - \phi_S) + F^{\sin \phi_S} \sin \phi_S
\]

\[
+ F^{\sin(3\phi_h - \phi_S)}(3\phi_h - \phi_S) + F^{\sin(\phi_h + \phi_S)}(\phi_h + \phi_S),
\]

where the structure functions with different azimuthal dependencies are given by

\[
F^{\sin(\phi_h - \phi_S)} = F_1, \quad F^{\sin(2\phi_h - \phi_S)} = \frac{F_2 + F_4}{2}, \quad F^{\sin \phi_S} = -\frac{F_2 + F_4}{2},
\]

\[
F^{\sin(3\phi_h - \phi_S)} = \frac{F_3 + F_5}{2}, \quad F^{\sin(\phi_h + \phi_S)} = -\frac{F_3 + F_5}{2}.
\]

In order to see the asymptotic behavior of the structure functions at small-\( q_T \), we need the expression for the \( \delta \)-function in the \( q_T \to 0 \) limit;

\[
\delta\left(\frac{q_T^2}{Q^2} - \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{\tilde{z}}\right)\right) \to \hat{x}\hat{z}\left[\frac{\delta(\hat{x} - 1)}{(1 - \hat{x})_+} + \frac{\delta(\hat{z} - 1)}{(1 - \hat{x})_+} + \delta(\hat{x} - 1)\delta(\hat{z} - 1)\ln\frac{Q^2}{q_T^2}\right]
\]

Using this form in (69), we find \( F^{\sin(\phi_h - \phi_S)} \sim 1/q_T, \ F^{\sin(2\phi_h - \phi_S)} \sim 0, \) and \( F^{\sin(3\phi_h - \phi_S)} \sim 0 \) for the contributions from the twist-3 fragmentation functions, and thus they are suppressed at small-\( q_T \) compared with the contribution from the quark-gluon correlation functions inside the transversely polarized nucleon [3] [9] [11]. On the other hand twist-3 fragmentation function gives a leading contribution to the other two structure functions as

\[
F^{\sin \phi_S} \sim \frac{F_0}{Qq_T^2} \sin 2\psi \sum_a e_a^2 \int \frac{dx}{x} h^a(x) \int \frac{dz}{z} A_1 \delta(\hat{x} - 1),
\]

\[
F^{\sin(\phi_h + \phi_S)} \sim -\frac{F_0}{q_T^2} \sinh^2 \psi \sum_a e_a^2 \int \frac{dx}{x} h^a(x) \int \frac{dz}{z} \left[A_2 \delta(\hat{x} - 1) + B_2 \delta(\hat{z} - 1)\right],
\]
with \( F_0 = \frac{\alpha_s^2 m_{\phi_3}^2}{16\pi^2 \alpha_s^2 M_N^2} \). Here \( A_1, A_2 \) and \( B_2 \) are, respectively, given by

\[
A_1 = 2C_F \left[ \frac{\hat{c}_1^a(z)}{z} \left( 3\hat{z} - \frac{2}{(1-\hat{z})_+} \right) + 2\frac{d}{d(1/z)} \left\{ \text{Im} \hat{c}_a(z) \frac{\hat{z}}{z} \right\} + \text{Im} \hat{c}_a(z) \frac{\hat{z}(-1 + 3\hat{z})}{(1-\hat{z})_+} \right]
- 2 \int_z^\infty \frac{dz'}{z'^2} \left[ P \left( \frac{1}{1/z - 1/z'} \right) \text{Im} \hat{E}_F(z', z) \hat{z} \right.
\left. \times \frac{1}{N} \left( \frac{2(2\hat{z} - 1)}{\hat{z}'} + \frac{1 + \hat{z}}{(1-\hat{z})_+} \right) + N \left( \frac{4\hat{z}}{1 - \hat{z}' - \hat{z} - \hat{z}', \hat{z} + \hat{z}'} \right) \right],
\]

\[
A_2 = 4C_F \left[ \frac{\hat{c}_1^a(z)}{z} \hat{z} + \frac{d}{d(1/z)} \left\{ \text{Im} \hat{c}_a(z) \frac{\hat{z}}{z} \right\} + \text{Im} \hat{c}_a(z) \frac{\hat{z}^2}{(1-\hat{z})_+} \right]
- 2 \int_z^\infty \frac{dz'}{z'^2} \left[ P \left( \frac{1}{1/z - 1/z'} \right) \text{Im} \hat{E}_F(z', z) \hat{z} \right. \hat{z} \left. \times N \left( \frac{\hat{z}}{1 - \hat{z}' - \hat{z} - \hat{z}', \hat{z} + \hat{z}'} \right) \right],
\]

\[
B_2 = 4C_F \text{Im} \hat{c}_a(z_f) \left[ \frac{\hat{x}}{(1 - \hat{x})_+} + \delta(\hat{x} - 1) \ln \frac{Q^2}{q_T^2} \right].
\]

Note the terms proportional to \( \delta(\hat{x} - 1) \) in \((107)\) cancel due to the twist-3 relation \((7)\). Twist-3 quark-gluon correlation function in the nucleon gives rise to these structure functions as \( F_{\text{distribution}}^{s\sin(\phi_S)} \sim 1/q_T^2 \) and \( F_{\text{distribution}}^{s\sin(\phi_3 + \phi_S)} \sim 1/q_T \), which may be compared with the above result \((107)\) and \((108)\). Thus, for the \( \sin(\phi_S) \)-asymmetry, two contributions are equally important. The obtained asymptotic part \((108)\) for the Collins azimuthal asymmetry does not agree with that derived in the previous study \(29\) in which the authors did not include the contribution from \( \hat{c}_1 \) and \( \hat{E}_F \). We emphasize that the inclusion of \( \hat{c}_1 \)-contribution is crucial to guarantee the electromagnetic gauge-invariance, as we saw in Sec. 3.4.

For a confirmation of the matching/mismatching between the collinear twist-3 approach and the TMD factorization approach for all structure functions, we need to know the asymptotic behavior of the relevant TMD functions in its high transverse-momentum region. We will investigate this issue in the future publication.

## 5 Summary

In this paper, we have calculated the contribution of the twist-3 fragmentation function to the SSA in SIDIS. We have established the collinear twist-3 formalism in the Feyn-
man gauge to derive the single-spin dependent cross section formula. There, the relations among hard parts based on the WT identities play a crucial role in reorganizing the matrix elements into the color gauge-invariant ones. We have also shown the obtained twist-3 hadronic tensor satisfies the electromagnetic gauge-invariance. Together with the formalism for the pole contributions [3, 4], present work completes the theoretical formalism to calculate the twist-3 cross sections for $p_T$-dependent processes within the framework of the collinear factorization in QCD. Using the complete cross-section formula, we have derived the asymptotic behavior of the azimuthal asymmetries in the small-$q_T$ region.

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