ON BERGMAN COMPLETENESS OF
NON-HYPERCONVEX DOMAINS

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ABSTRACT. In the paper we study the problems of the boundary behaviour of the
Bergman kernel and the Bergman completeness in some classes of bounded pseudo-
convex domains, which contain also non-hyperconvex domains. Among the classes
for which we prove the Bergman completeness and the convergence of the Bergman
kernel to infinity while tending to the boundary are all bounded pseudonvex balanced
domains, all bounded Hartogs domains with balanced fibers over regular domains and
some bounded Laurent-Hartogs domains.

0. Introduction. The aim of the paper is to present some new results concern-
ing Bergman completeness and the boundary behaviour of the Bergman kernel in
bounded pseudoconvex but not necessarily hyperconvex domains. We are interested
in the following exhausting property of the Bergman kernel:

\[(\ast) \quad K_D(z) \to \infty \quad \text{as} \quad z \to \partial D.\]

The starting point for our considerations may be the following two recent results:
- any bounded hyperconvex domain satisfies \((\ast)\) (see [Ohs 2]),
- any bounded hyperconvex domain is Bergman complete (see [Blo-Pfl] and
  [Her]).

Both properties mentioned above are closely related. In particular, the Bergman
completeness is often proved after proving the property \((\ast)\). To the best of our
knowledge there are no known examples of bounded Bergman complete domains
not satisfying \((\ast)\).

The existence of non-hyperconvex bounded domains with \((\ast)\) is very well-known
and easy (take the Hartogs triangle). On the other hand the existence of bounded
pseudoconvex but non-hyperconvex Bergman complete domains is not so trivial but
also known (see [Chen], [Her], [Zwo]).

In our paper we shall present a class of domains satisfying the above prop-
ties. The classes of domains which we consider are the following: bounded pseudo-
convex balanced domains, Hartogs domains with \(m\)-dimensional balanced fibers,
Hartogs-Laurent domains and Zalcman type domains (domains in the unit disc with
complements composing of infinitely many closed discs).

Among others we prove the following results.

All bounded pseudoconvex balanced domains satisfy \((\ast)\) and are Bergman com-
plete. The latter result gives the positive answer to the question posed in [Jar-Pfl 1]

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and [Jar-Pfl 2]. Note that if the Minkowski functional of the considered domain is continuous then the domain is hyperconvex and the result follows from the above mentioned theorems.

Any bounded pseudoconvex Hartogs domain with \( m \)-dimensional balanced fibers over a domain with the property \((*)\) satisfies \((*)\). Any bounded pseudoconvex Hartogs domain over a \( c^1 \)-complete domain (which implies automatically Bergman completeness) is Bergman complete. In particular, there are bounded and pseudoconvex non-fat domains that are Bergman complete and satisfy \((*)\).

On the other hand we show that there are bounded fat domains in \( \mathbb{C} \) (some Zalcman type domains) not satisfying \((*)\) – this gives an answer to a question posed in [Jar-Pfl 2].

1. Definitions and known results. Let us denote by \( E \) the unit disc in \( \mathbb{C} \).

Let \( D \) be a bounded domain in \( \mathbb{C}^n \). Let us denote by \( L^2_h(D) \) square integrable holomorphic functions on \( D \). \( L^2_h(D) \) is a Hilbert space with the scalar product induced from \( L^2(D) \). Let us define the Bergman kernel of \( D \)

\[
K_D(z) = \sup \{ \frac{|f(z)|^2}{||f||^2_{L^2(D)}} : f \in L^2_h(D), f \not\equiv 0 \}.
\]

Among other well-known properties let us recall only two of them (see e.g [Jar-Pfl 2]).

If \( D_1 \subset D_2 \) are bounded domains in \( \mathbb{C}^n \) then \( K_{D_2}(z) \leq K_{D_1}(z), z \in D_1 \).

If \( \{D_j\}_{j=1}^{\infty} \) is an increasing sequence of domains in \( \mathbb{C}^n \) whose union is a bounded domain \( D \), then \( K_{D_j} \) tends decreasingly and locally uniformly to \( K_D \).

It is well-known that \( \log K_D \) is a smooth plurisubharmonic function. Therefore, we may define

\[
\beta_D(z; X) := \left( \sum_{j,k=1}^n \frac{\partial^2 \log K_D(z)}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k \right)^{1/2}, \quad z \in D, \quad X \in \mathbb{C}^n.
\]

\( \beta_D \) is a pseudometric called the Bergman pseudometric.

For \( w, z \in D \) we put

\[
b_D(w, z) := \inf \{ L_{\beta_D}(\alpha) \},
\]

where the infimum is taken over piecewise \( C^1 \)-curves \( \alpha : [0, 1] \to D \) joining \( w \) and \( z \) and \( L_{\beta_D}(\alpha) := \int_0^1 \beta_D(\alpha(t); \alpha'(t)) \, dt \).

We call \( b_D \) the Bergman distance of \( D \).

The Bergman distance (as well as the Bergman metric) is invariant with respect to biholomorphic mappings. In other words, for any biholomorphic mapping \( F : D \mapsto G (D, G \subset \subset \mathbb{C}^n) \) we have

\[
b_G(F(w), F(z)) = b_D(w, z), \quad \beta_G(F(w); F'(w)X) = \beta_D(w; X), \quad w, z \in D, X \in \mathbb{C}^n.
\]

A bounded domain \( D \) is called Bergman complete if any \( b_D \)-Cauchy sequence is convergent to some point in \( D \) with respect to the standard topology of \( D \).

Any bounded Bergman complete domain is pseudoconvex (see [Bre]). Let us recall that a bounded domain \( D \) is called hyperconvex if it admits a continuous negative plurisubharmonic exhaustion function. Now we may formulate the following very general result:
Theorem 1.1 (see [Blo-Pfl], [Her], [Ohs 2]). Let $D$ be a bounded hyperconvex domain in $\mathbb{C}^n$. Then $D$ satisfies $(\ast)$ and $D$ is Bergman complete.

Our aim is to study the boundary behaviour of the Bergman kernel and the problem of Bergman completeness. We shall make use of following powerful tools; namely, the extension theorem of $L^2_h$-functions, localization principle of the Bergman kernel and the Bergman metric and criteria for a domain to be Bergman complete and for the Bergman kernel to tend to infinity near the fixed point from the boundary. Let us recall below these results.

Theorem 1.2 (see [Ohs-Tak]). Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$. Let $H$ be any affine subspace of $\mathbb{C}^n$. Then there is a constant $C \in \mathbb{R}$ dependent only on diameter of $D$ such that for any $f \in L^2_h(D \cap H)$ there is $F \in L^2_h(D)$ such that $\|F\|_{L^2_h(D \cap H)} \leq C \|f\|_{L^2_h(D)}$.

In particular, we get from Theorem 1.2 that

$$K_{D \cap H}(z) \leq \tilde{C} K_D(z), \quad z \in D \cap H,$$

where $\tilde{C} \in \mathbb{R}$ is a constant dependent only on diameter of $D$.

Theorem 1.3 (see [Die-For-Her], [Ohs 1]). Let $D$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, $z^0 \in \partial D$. Then for any neighbourhoods $U_1 = U_1(z^0) \subset U_2(z^0) = U_2$ there is a positive constant $C$ such that for any connected component $V$ of $D \cap U_2$ and for any $z \in U_1 \cap V$, $X \in \mathbb{C}^n$ we have:

$$\frac{1}{C} K_V(z) \leq K_D(z) \leq K_V(z),$$

$$\frac{1}{C} \beta_V(z; X) \leq \beta_D(z; X) \leq \beta_V(z; X).$$

Theorem 1.4 (see [Kob], [Pfl 1]). Let $D$ be a bounded domain such that $(\ast)$ is satisfied and $H^\infty(D)$ is dense in $L^2_h(D)$. Then $D$ is Bergman complete.

Theorem 1.5 (see [Pfl 1]). Let $D$ be a bounded pseudoconvex domain. Let $z^0 \in \partial D$ be such that there are $r \in (0, 1]$, $\varepsilon \geq 1$ and a sequence $\{z^\nu\}_{\nu=1}^\infty$ of points from $\mathbb{C}^n \setminus \bar{D}$ tending to $z^0$ such that $B(z, r||z^\nu - z^0||\varepsilon) \cap D = \emptyset$ (so called 'outer cone condition'). Then $\lim_{z \to z^0} K_D(z) = \infty$.

2. Balanced domains. Recall that a set $D$ is balanced if $z \in D$ and $\lambda \in \bar{E}$ implies $\lambda z \in D$.

In this section we deal with bounded pseudoconvex balanced domains. We prove the following result.

Theorem 2.1. Let $D$ be a bounded pseudoconvex balanced domain. Then $D$ satisfies $(\ast)$ and $D$ is Bergman complete.

Note that if the Minkowski functional of $D$ is continuous then $D$ is hyperconvex and the result follows from Theorem 1.1. Additionally, in this case the theorem has already been known for a long time (see [Jar-Pfl 1]). Using only a little more refined methods than the ones used in the last paper we prove the theorem in general case. Let us mention here that the problem whether bounded pseudoconvex balanced domains are Bergman complete has been stated in [Jar-Pfl 1] and [Jar-Pfl 2].
Proof of Theorem 2.1. First we prove the property (⋆). Take any point \( z^0 \in \partial D \).

Fix any \( M > 0 \). In view of (1.1) (applied to \( H = \mathbb{C}z^0 \) — remember that \( \mathbb{C}z^0 \) is a disc) there is some \( z^1 = sz^0 \), \( 0 < s < 1 \) such that \( z^1 \in D \) and \( K_D(z^1) > M \). It follows from continuity of \( K_D \) that there is some open neighbourhood \( U \subset D \) of \( z^1 \) such that \( K_D(z) > M \) for \( z \in U \). Note that for any \( z \in U \) the function

\[
u_z : \frac{1}{h(z)} E \ni \lambda \mapsto K_D(\lambda z)
\]

is subharmonic and radial. Therefore, \( u_z(t), 0 \leq t < \frac{1}{h(z)} \) is increasing (see e.g. [Jak-Jar]). Consequently, \( K_D(z) > M \) for any \( z \in ([1, \infty) \cup D \). Since \([1, \infty) \cup D \) is a neighbourhood of \( z^0 \), we finish the proof.

To finish the proof it is sufficient to show that \( H^\infty(D) \) is dense in \( L_h^2(D) \) (and then use Theorem 1.4).

It is well-known that any holomorphic function \( F \) on \( D \) is a local uniform limit of a series \( \sum_{k=0}^\infty Q_k(z) \), where \( Q_k \) is a homogeneous polynomial of degree \( k \) (see e.g. [Jak-Jar]). Since all \( Q_k \) are orthogonal (in \( L_h^2(D) \)) and there is an exhausting family of compact balanced sets of the domain \( D \) (on each of them the functions \( Q_k \) are orthogonal), the standard approximation process leads to the convergence of \( (F_N := \sum_{k=0}^N Q_k) \) to \( F \) in \( L^2(D) \) norm (under the assumption that \( F \in L_h^2(D) \)).

Since \( D \) is bounded, all \( F_N \)’s are bounded, which finishes the proof. \( \square \)

3. Hartogs domains. In the present section we consider bounded pseudoconvex Hartogs domains with \( m \)-dimensional balanced fibers. Let \( G_D \subset \mathbb{C}^{n+m} \) denote a bounded pseudoconvex Hartogs domain over \( D \subset \mathbb{C}^n \) with \( m \)-dimensional balanced fibers, i.e.

\[
G_D = \{(z, w) \in D \times \mathbb{C}^m : H(z, w) < 1\},
\]

where \( D \) is bounded and pseudoconvex, \( \log H \) is plurisubharmonic on \( D \times \mathbb{C}^m \), \( H(z, \lambda w) = |\lambda|H(z, w) \), \( (z, w) \in D \times \mathbb{C}^m, \lambda \in \mathbb{C} \), and \( G_D \) is bounded (i.e. \( H(z, w) \geq C |w| \) for some \( C > 0, z \in D, w \in \mathbb{C}^m \)).

Let \( G_D \) be as above. For any \( f \in L_h^2(D) \) we define a function \( F(z, w) := f(z), (z, w) \in G_D \). Since \( G_D \subset D \times (RE)^m \) for some \( R > 0 \), we easily get that \( \|F\|_{L_h^2(G_D)} \leq C_1 \|f\|_{L_h^2(D)} \) for some \( C_1 > 0 \) independent of the choice of \( f \) (therefore, \( F \in L_h^2(G_D) \)). In particular, we get that

\[
(3.1) \quad K_D(z) \leq C_2 K_{G_D}(z, 0), \; z \in D.
\]

Theorem 3.1. Let \( G_D \) be a bounded pseudoconvex Hartogs domain over \( D \) with \( m \)-dimensional balanced fibers. Fix a point \( (z^0, w^0) \in \partial G_D \). Assume that one of the following three conditions is satisfied:

(i) \( z^0 \in D \),

(ii) \( z^0 \in \partial D \) and \( K_D(z) \to \infty \) as \( z \to z^0 \),

(iii) there is some neighbourhood \( U \) of \( (z^0, w^0) \) such that

\[
U \cap G_D \subset \{(z, w) \in \mathbb{C}^{n+m} : |w| < ||z - z^0||^\delta \} \text{ for some } \delta > 0 \text{ (in particular, } w^0 = 0 \).
\]

Then \( K_{G_D}(z, w) \to \infty \) as \( (z, w) \to (z^0, w^0) \).
In particular, if $D$ satisfies $(*),$ then $G_D$ satisfies $(*).$

Proof. Consider the case $z^0 \in D$ (that is we consider the case (i)). Then $H(z^0, w^0) \geq 1.$ In view of the $L^2$-extension theorem for any $M \in \mathbb{R}$ there is $w^1 = tw^0, 0 < t < 1,$ such that $(z^0, w^1) \in G_D$ and $K_{G_D}(z^0, w^1) > M.$ Continuity of $K_{G_D}$ gives us the existence of an open neighbourhood $U := U_1 \times U_2$ of $(z^0, w^1)$ in $G_D$ (with $0 \notin U_2)$ such that $K_{G_D}(z, w) > M$ for $(z, w) \in U.$ Similarly, as earlier considering the function

$$u(z, w): \frac{1}{H(z, w)} E \ni \lambda \mapsto K_{G_D}(z, \lambda w)$$

we get a radial subharmonic function such that $u(z, w)(1) > M,$ which gives us that $K_{G_D}(z, w) > M$ for $(z, w) \in (U_1 \times [1, \infty)U_2) \cap G_D.$

Consider now the case (ii). Then $z^0 \in \partial D.$ It follows from (3.1) and (ii) that for any $M \in \mathbb{R}$ there is some open neighbourhood $U$ of $z^0$ such that $K_{G_D}(z, 0) > M,$ $z \in U \cap D.$

Fix $z \in U \cap D.$ Fix additionally for a while $w$ such that $0 < H(z, w) < 1.$ Then the function $\frac{1}{H(z, w)} E \ni \lambda \mapsto K_{G_D}(z, \lambda w)$ is larger than $M$ at $0$ and is radial and subharmonic; therefore, increasing. Consequently, $K_{G_D}(z, w) > M$ for any $w$ with $H(z, w) < 1.$ Since $z \in U$ was chosen arbitrarily we get that $K_{G_D}(z, w) > M$ for any $(z, w) \in G_D$ with $z \in U.$

We are left with the case (iii). Without loss of generality we may assume that $z^0 = 0.$ Consider points $(0, w_0) \notin \bar{D}$ (i.e. $w_0 \neq 0$). Let us consider the balls $B((0, w_0), r||w_0||^\varepsilon),$ where $\varepsilon > 0, 0 < r < 1$ will be chosen later (independently of $w_0$). Our aim is to verify that the outer cone condition form Theorem 1.5 is satisfied for a suitable $0 < r \leq 1$ and $\varepsilon \geq 1.$

Fix $r = \frac{1}{2}.$ Consider only $||w_0|| < \frac{1}{2}.$ Take a point $(z, w) \in B((0, w_0), \frac{1}{2}||w_0||^\varepsilon) \cap \bar{G}_D.$ Then $||z|| < \frac{1}{2}||w_0||^\varepsilon$ and $||w_0|| - ||w|| \leq ||w - w_0|| \leq \frac{1}{2}||w_0||^\varepsilon.$ Consequently, $||w_0|| - \frac{1}{2}||w|| \leq ||w|| \leq ||z|| < \left(\frac{1}{2}\right)\delta||w_0||^\delta\varepsilon.$ So assuming that $\varepsilon$ is large enough ($\varepsilon - 1 > 0, \varepsilon\delta - 1 > 0, \delta + \varepsilon\delta - 2 > 0$) we get:

$$\frac{1}{2} < 1 - \frac{1}{2}||w_0||^{\varepsilon - 1} < \left(\frac{1}{2}\right)^\delta||w_0||^{\varepsilon\delta - 1} < \frac{1}{2}$$

- contradiction. Therefore, in view of Theorem 1.5 we finish the proof. $\square$

The idea of the condition (iii) comes from generalizing the phenomenon, which appears in the Hartogs triangle and the point $(z^0, w^0) = (0, 0).$

It turns out that there are bounded pseudoconvex Hartogs domains and points from the boundary, which do not satisfy any from the conditions (i)–(iii) but such that the limes as in Theorem 3.1 exists.

**Example 3.2.** Let $\{a_j\}_{j=1}^\infty \subset (0, 1)$ be a sequence tending to $0.$ Let us define $u_k(\lambda) = \log \left(\sum_{j=1}^k \frac{a_j}{2^j|\lambda - a_j|^{n_j}}\right), \lambda \in E \setminus \{a_1, \ldots, a_k\},$ where $n_j \geq j.$ Note that $u_k(0) < 0, k = 1, 2, \ldots.$ Define $u := \lim_{k \to \infty} u_k = \log \left(\sum_{j=1}^\infty \frac{a_j}{2^j|\lambda - a_j|^{n_j}}\right)$ on $E_\infty := E \setminus \{a_j\}_{j=1}^\infty \cup \{0\}.$ The construction ensures us that the sequence $\{u_k\}_{k=1}^\infty$ locally bounded from above and globally bounded from below increasing sequence on $E_\infty$ and, therefore, $u$ is a subharmonic function on $E_\infty$ bounded from below. Moreover, $\lim_{x < 0, x \to 0} u(x) \leq 0.$ Define $G_{E_\infty} := \{(z, w) \in E_\infty \times \mathbb{C} : |w| < \exp(-u(z))\}.$ Then $G_{E_\infty}$ is a bounded pseudoconvex Hartogs domain with one dimensional fibers. Note
that the point \((0,0)\) does not satisfy any from the conditions (i)–(iii) but one may easily verify that, choosing if necessary \(n_j\) larger, the outer cone condition from Theorem 3.1 is satisfied (for instance for points \((a_j, a_j)\)). Therefore, the claim of Theorem 3.1 is also satisfied. Note that \(\{(0,0)\} \subseteq \partial G_{E\infty} \cap \{(0) \times \mathbb{C}^m\} \).

We may prove even more. Namely, the domain \(G_{E\infty}\) satisfies \((\ast)\). In fact, the points \((z, w) \in \partial G_{E\infty}, z \in \partial E\), satisfy (ii). The points \((a_k, w) \in \partial G_{E\infty}\) (and then automatically \(w = 0\)) satisfy (iii). The points \((z, w) \in \partial G_{E\infty}, z \in E_{\infty}\), satisfy (i). Finally, one may easily verify (proceeding similarly as in the case \((0,0)\)) that the points \((0, w) \in \partial G_{E\infty}\) satisfy the outer cone condition from Theorem 1.5.

**Lemma 3.3.** Let \(G_D\) be a bounded pseudoconvex Hartogs domain over \(D\) with \(m\)-dimensional balanced fibers such that \(H^\infty(D)\) is dense in \(L^2_h(D)\) and, additionally, assume that there is some \(\varepsilon > 0\) such that \(D \times P(0, \varepsilon) \subset G_D\). Then \(H^\infty(G_D)\) is dense in \(L^2_h(G_D)\).

**Proof.** Take \(F \in L^2_h(G_D)\). We know that

\[
F(z, w) = \sum_{\nu=0}^{\infty} F_{\nu}(z, w) := \sum_{\nu=0}^{\infty} \sum_{\beta\in \mathbb{Z}^m_+: |\beta|=\nu} f_\beta(z)w^{\beta}
\]

where the convergence of \(G_N := \sum_{\nu=0}^{N} F_{\nu}\) to \(F\) is locally uniform (see e.g. [Jak-Jar]). Consequently, because of orthogonality of \(w^{\beta}\), similarly as in the proof of Theorem 2.1 the functions \(G_N\) converge in \(L^2(G_D)\) to \(F\). It is therefore sufficient to approximate \(f_\beta(z)w^{\beta}\) with bounded functions. But because of the assumption of the lemma one may easily conclude from the Fubini theorem that \(f_\beta \in L^2_h(D)\) so \(h_N(z)w^{\beta}\), where \(h_N \in H^\infty(D)\) and \(h_N \to f_\beta\) in \(L^2_h(D)\), tends to \(f_\beta(z)w^{\beta}\) in \(L^2_h(G_D)\). \(\square\)

**Remark 3.4.** Note that the assumption \(D \times P(0, \varepsilon) \subset G_D\) is essential. For instance, \(H^\infty(E_\varepsilon) = H^\infty(E)|_{E_\varepsilon}\) is dense in \(L^2_h(E_\varepsilon) = L^2_h(E)|_{E_\varepsilon}\) and \(H^\infty(G_{E_\varepsilon})\) is not dense in \(L^2_h(G_{E_\varepsilon})\), where \(G_{E_\varepsilon}\) is the Hartogs triangle, \(G_{E_\varepsilon} := \{(z, w) \in E_\varepsilon \times \mathbb{C} : |w| < |z|\}\).

**Theorem 3.5.** Let \(G_D\) be a bounded pseudoconvex Hartogs domain over \(D\) with \(m\)-dimensional balanced fibers. Assume that \(D\) satisfies \((\ast)\), \(H^\infty(D)\) is dense in \(L^2_h(D)\) and there is \(\varepsilon > 0\) such that \(D \times P(0, \varepsilon) \subset G_D\). Then \(G_D\) is Bergman complete.

**Proof.** Combine Theorem 3.1, Lemma 3.3 and Theorem 1.4. \(\square\)

Note that Theorem 3.5 cannot be even applied to arbitrary pseudoconvex bounded Hartogs domain with one dimensional fibers. However, small change in assumptions on the domain \(D\) in Theorem 3.5 will make it possible to prove Bergman completeness of \(G_D\) without additional assumptions on the shape of \(G_D\). But before formulating the result we have to introduce the notion of the inner Carathéodory pseudodistance.

For a domain \(D \subset \mathbb{C}^n\) we define the Carathéodory-Reiffen pseudometric

\[
\gamma_D(z; X) := \sup \{|f'(z)X| : f \in \mathcal{O}(D, E), f(z) = 0\}, \ z \in D, \ X \in \mathbb{C}^n.
\]

The inner Carathéodory pseudodistance is the integrated form of \(\gamma_D\), i.e.

\[
d^\infty (w, z) := \inf \{L_w(z) \circ \gamma_D(z; X) : D \text{ is a piecewise } C^1 \text{ curve joining } w \text{ and } z\}.
\]
where $L_{\gamma_D}(\alpha) := \int_0^1 \gamma_D(\alpha(t); \alpha'(t))dt$. It is well-known that holomorphic mappings are contractions with respect to $c^i$ (i.e. $c^i_G(F(w), F(z)) \leq c^i_D(w, z)$ for any $F \in \mathcal{O}(D, G)$, $w, z \in D$). The last property is not shared by the Bergman distance (in the class of bounded domains – see e.g. [Jar-Pfl 2]). We have additionally that (see e.g. [Jar-Pfl 2])

$$c^i_D \leq b_D. \tag{3.2}$$

Exactly as in the case of the Bergman distance we introduce for bounded domains the notion of $c^i$-completeness.

**Theorem 3.6.** Let $G_D$ be a bounded pseudoconvex Hartogs domain over $D$ with $m$-dimensional balanced fibers. Assume that $D$ is $c^i$-complete. Then $G_D$ is Bergman complete.

**Proof.** Take any point $(z_0, w_0) \in \partial G_D$. Suppose that there is a $b_{G_D}$-Cauchy sequence $\{(z_\nu, w_\nu)\}$ tending (in the natural topology of $D$) to $(z_0, w_0)$. Because of (3.2) and the contractivity of $c^i_D$ with respect to the projection we exclude the case $z^0 \in \partial D$.

So assume that $z^0 \in D$. Let $U_1, U_2$ be small open balls with the centre at $z^0$ such that $U_1 \subset U_2 \subset D$. There is a sequence of $C^1$-piecewise curves $\gamma_{\nu, \mu} : [0, 1] \mapsto G_D$ such that $\gamma_{\nu, \mu}(0) = (z_\nu, w_\nu)$, $\gamma_{\nu, \mu}(1) = (z_\mu, w_\mu)$ and $L_{\beta_{G_D}}(\gamma_{\nu, \mu}) < b_{G_D}((z_\nu, w_\nu), (z_\mu, w_\mu)) + \frac{1}{\nu}, 1 \leq \nu < \mu$. We claim that there is some $\nu_0$ such that $\gamma_{\nu, \mu}([0, 1]) \subset G_{U_1}$, ($G_{U_j} := (U_j \times \mathbb{C}^m) \cap G_D, j = 1, 2$) for $\mu > \nu > \nu_0$. Actually, if it were not the case, then there would be a sequence of $t_k \in (0, 1)$ such that $(u_k, v_k) := \gamma_{\nu_k, \mu_k}(t_k) \notin G_{U_1}$, $u_k \notin U_1$ and $b_{G_D}((z_{v_k}, w_{v_k}), (u_k, v_k)) \to 0$ as $k$ tends to infinity. But then also

$$0 \leq c^i_D(z_{n_k}, u_k) \leq c^i_D((z_{n_k}, w_{n_k}), (u_k, v_k)) \to 0$$

– contradiction (to the boundedness of $D$).

Note that $G_{U_j}$ satisfies the assumptions of Theorem 3.5, so $G_{U_j}$ is Bergman complete, $j = 1, 2$.

Applying the localization principle of the Bergman metric (Theorem 1.3) we get that

$$b_{G_{U_2}}((z_\nu, w_\nu), (z_\mu, w_\mu)) \leq L_{\beta_{G_{U_2}}}(\gamma_{\nu, \mu}) \leq \frac{CL_{\beta_{G_D}}(\gamma_{\nu, \mu})}{L_{\beta_{G_D}}} \leq C(b_{G_D}((z_\nu, w_\nu), (z_\mu, w_\mu)) + \frac{1}{\nu}), \mu > \nu > \nu_0$$

so $\{(z_\nu, w_\nu)\}_{\nu \gg 1}$ is a $b_{G_{U_2}}$-Cauchy sequence tending to the boundary of $G_{U_2}$ (in the natural topology of $G_{U_2}$), which, however, contradicts the Bergman completeness of $G_{U_2}$. \( \square \)

**Remark 3.7.** Since any Kobayashi complete bounded domain is taut (a bounded domain $D$ in $\mathbb{C}^n$ is taut if for any convergent sequence of mappings $\varphi_\nu \in \mathcal{O}(E, D)$ its limit $\varphi$ satisfies that $\varphi(E) \subset D$ or $\varphi(E) \subset \partial D$), there are bounded pseudoconvex balanced domains (in fact any such that the Minkowski functional is not continuous) in $\mathbb{C}^2$ such that no estimate of the type $b_D \leq C k_D$ holds (compare [Jar-Pfl 2]).

Note that there are bounded balanced pseudoconvex domains which are not fat (i.e. $\text{int}(D) \neq D$, see [Sic]), so there are Bergman complete domains satisfying (3).
which are not fat (use Theorem 2.1). Other domains having the same property
(but in the class of Hartogs domains) are given below.

Theorem 3.1, Theorem 3.5 and Theorem 3.6 apply among others to the following
domain

\[ G_E := \{ (z, w) \in E \times \mathbb{C} : |w| < \exp(-\exp(\sum_{j=1}^{\infty} \alpha_j \log |z - a_j|)) \}, \]

where \( \alpha_j > 0, \{a_j\}_{j=1}^{\infty} \) is dense in \( E_0 \) and \( \sum_{j=1}^{\infty} \alpha_j \log |a_j| > -\infty \). Note that \( G_E \subset E^2, G_E \neq E^2 \) but \( \text{int}(G_E) = E^2 \).

It follows from Theorem 3.6 that any bounded pseudoconvex Hartogs domain
over a complete bounded pseudoconvex Reinhardt domain (e.g. the unit disc) is Bergman complete (see [Pfl 2]).

It seems to be natural to ask the question whether Theorem 3.6 remains true
under the assumption that \( H \) is continuous. Proposition 3.8.

\[ \text{Proposition 3.8. Let } G_D \text{ be a bounded pseudoconvex Hartogs domain over } D \text{ with } m \text{-dimensional balanced fibers.} \]

\[ G_D \text{ is taut iff } D \text{ is taut and } H \text{ is continuous;} \]

\[ G_D \text{ is hyperconvex iff } D \text{ is hyperconvex and } H \text{ is continuous.} \]

\[ \text{Proof. Note that non-continuity of } H \text{ gives us the existence of the sequence } \{(z_\nu, w_\nu)\} \subset G_D \text{ convergent to } (z, w) \in D \times \mathbb{C}^m \text{ such that } \lim_{\nu \to \infty} H(z_\nu, w_\nu) = \delta < H(z, w) = 1. \]

Then the sequence \( \varphi_\nu(\lambda) = (z_\nu, \frac{w_\nu \lambda}{H(z_\nu, w_\nu)}) \), \( \lambda \in E \), satisfies \( \varphi_\nu(E) \subset G_D \) and \( \varphi_\nu \)
converges locally uniformly to \( \varphi \), where \( \varphi(\lambda) = (z, \frac{\lambda}{H(z, w)}) \), \( \varphi(0) \in G_D \) but \( \varphi(E) \not\subset G_D \), so \( G_D \) cannot be taut.

It is trivial to see that tautness (respectively, hyperconvexity) of \( G_D \) implies tautness (respectively, hyperconvexity) of \( D \).

Hyperconvexity of \( D \) delivers us the existence of negative continuous plurisubharmonic
exhaustion function \( u \) of \( D \). Note that if \( H \) is continuous, then the function \( \max\{u(z), \log H(z, w)\} \) is a continuous negative exhaustion function of \( G_D \).

Assume now tautness of \( D \) and continuity of \( H \). Consider a sequence \( \varphi^\nu := (\varphi_1^\nu, \varphi_2^\nu) \in \mathcal{O}(E, G_D) \), which converges locally uniformly to \( \varphi^0 \). Because of tautness of \( D \) either \( \varphi_1^0 \in \mathcal{O}(E, D) \) or \( \varphi_2^0 \in \partial D \), in the second case \( \varphi_2^0(E) \subset \partial G_D \). So consider the first case. It easily follows from the maximum principle for subharmonic functions that either \( H(\varphi^0(\lambda)) = 1 \) or \( H(\varphi^0(\lambda)) < 1, \lambda \in E \), which finishes the proof. \( \square \)

4. Hartogs-Laurent domains. In this section we consider Hartogs-Laurent
domains. More precisely, let \( D \) be a bounded pseudoconvex domain in \( \mathbb{C}^n \) and let \( u, v \)
be plurisubharmonic functions on \( D \), \( u + v < 0 \) on \( D \). Then we define the Hartogs-Laurent domain \( G := \{ (z, z_{n+1}) \in D \times \mathbb{C} : \exp(u(z)) < |z_{n+1}| < \exp(-v(z)) \} \) over \( D \). We assume additionally that there is some constant \( C \in \mathbb{R} \) such that \( u(z) > C \)
(i.e. \( G \) is bounded) and \( v \neq -\infty \).
Proposition 4.1. Let \( G \) be as above (with some \( D, u \) and \( v \)). Assume additionally that \( D \) satisfies \((\ast)\). Then \( G \) satisfies \((\ast)\).

Proof. Since \( G \subset \{(z, z_{n+1}) \in D \times \mathbb{C} : |z_{n+1}| < \exp(-v(z))\} \), we get in view of Theorem 3.1 and because of the contraction property of the Bergman kernel under inclusion of domains that \( K_G(z, z_{n+1}) \to \infty \) whenever \((z, z_{n+1}) \to (w, w_{n+1}) \in \partial G\), where \( w \in \partial D \) or \(|w_{n+1}| \geq \exp(-v(w))\).

Now we consider the case when \((z, z_{n+1}) \to (w, w_{n+1}) \in \partial G\), where \( w \in D \) and \(|w_{n+1}| \leq \exp(u(w))\).

First we prove that \( K_G(z, z_{n+1}) \to \infty \) as \((z, z_{n+1}) \to (w, 0) \in \partial G \) with \( w \in D \). Take a small ball \( U \subset \subset D \) with the centre at \( w \). Put \( G_U := G \cap (U \times \mathbb{C}) \). We claim that the function \( \frac{1}{z_{n+1}} \) is from \( L^2_h(G_U) \). In fact,

\[
\int_{G_U} \frac{1}{|z_{n+1}|^2} d\mathcal{L}^{2n+2}(z, z_{n+1}) = \int_U \left( \int_{\exp(u(z))}^{\min\{u, \log \epsilon\}} |z_{n+1}|^{-2} d\mathcal{L}^2(z_{n+1}) \right) d\mathcal{L}^{2n}(z) = 2\pi \int_U (-v(z) - u(z)) d\mathcal{L}^{2n}(z).
\]

Therefore, in view of the local summability of pluriharmonic functions (not identical to \(-\infty\)) the last expression is finite. Consequently, \( K_{G_U}(z, z_{n+1}) \to \infty \) as \((z, z_{n+1}) \to (w, 0) \). And now the localization property of the Bergman kernel (Theorem 1.3) implies that

\[
K_G(z, z_{n+1}) \to \infty \quad \text{as} \quad (z, z_{n+1}) \to (w, 0).
\]

We are left with the case \((z, z_{n+1}) \to (w, w_{n+1}) \in \partial G\), \( w \in D \), \( 0 < \epsilon < |w_{n+1}| \leq \exp(u(w)) \). Consider now the new Laurent-Hartogs domain \( \tilde{G}_1 \) defined over \( D \) with \( u \) replaced by \( \tilde{u} := \max\{u, \log \epsilon\} \) (and the same \( v \)). Taking now \( \tilde{G}_1 \) to be \( \{(z, 1/z_{n+1}) : (z, z_{n+1}) \in G_1\} \) we get that the convergence of \( K_{G_1}(z, z_{n+1}) \to \infty \) as \((z, z_{n+1}) \to (w, w_{n+1}) \) is equivalent to the convergence of \( K_{G_1}(z, z_{n+1}) \to \infty \) as \((z, z_{n+1}) \to (w, 1/w_{n+1}) \) (use the invariance of the Bergman kernel with respect to biholomorphic mappings). Since \( \tilde{G}_1 \subset \{(z, z_{n+1}) : z \in D, \ |z_{n+1}| < \exp(-\tilde{u}(z))\} \) and \( 1/|w_{n+1}| \geq \exp(-u(w)) \) we get (using the contractivity of the Bergman kernel under inclusion and Theorem 3.1) that \( K_{G_1}(z, z_{n+1}) \to \infty \) as \((z, z_{n+1}) \to (w, w_{n+1}) \). And now the localization of the Bergman kernel (Theorem 1.3) implies that \( K_G(z, z_{n+1}) \to \infty \) as \((z, z_{n+1}) \to (w, w_{n+1}) \). \( \Box \)

Theorem 4.2. Let \( D \) be a bounded domain pseudoconvex domain in \( \mathbb{C}^n \), which is \( c^2 \)-complete. Let \( G \) be as above with the additional property that there is some constant \( C \) such that \( u(z) > C > -\infty \) for any \( z \in D \). Then \( G \) is Bergman complete.

Proof. We proceed similarly as in the proofs of results in Section 3. Take any \( b_G \)-Cauchy sequence \( \{(z^w, z^0_{n+1})\} \) converging to \((z^0, z^0_{n+1}) \in \partial G\). One easily excludes (because of \( c^2 \)-completeness of \( D \)) the case \( z^0 \in \partial D \). In case \( z^0 \in D \) we may exactly as in the proof of Theorem 3.6 reduce the problem to the problem of completeness of \( \tilde{G} := G \cap (U \times \mathbb{C}) \), where \( U \) is some small ball around \( z^0 \), \( U \subset \subset D \), such that \( U \times A \subset G \), where \( A \) is some annulus. Similarly, as in the proof of Lemma 3.3, expanding any \( L^2 \) function in the series \( F(z, z_{n+1}) := \sum_{\nu=0}^{\infty} b_{\nu}(z)z^{\nu} w_{n+1}(z, z_{n+1}) \in \tilde{G} \), we
easily get that \( F_N(z, \z_n+1) := \sum_{\nu=-N}^N h_\nu(z)z_n^{\nu+1} \) tends in \( L^2_h(\tilde{G}) \) to \( F \). Moreover, \( h_\nu(z) \in L^2_h(U) \). Since \( H^{\infty}(U) \) is dense in \( L^2_h(U) \), we get consequently, that \( H^{\infty}(\tilde{G}) \) is dense in \( L^2_h(\tilde{G}) \) (approximate \( h_\nu(z)z_n^{\nu+1} \) with \( h_{\nu,j}(z)z_n^{\nu+1} \), where \( h_{\nu,j} \in H^{\infty}(U) \) tends to \( h_\nu \) in \( L^2(U) \), and then use inequality \( u > C \), which in connection with Proposition 4.1 and Theorem 1.4 finishes the proof. □

5. Zalcman type domains. In Section 3 we saw that there are non-fat domains satisfying (*) and being Bergman complete. In this chapter we go into the opposite direction and we find domains, which are bounded pseudoconvex and fat but which do not satisfy (*) (which gives the answer about the existence of such domains in [Jar-Pfl 2]). This counterexample is found in the class of Zalcman type domains, which were considered in the context of (*) and Bergman completeness in [Ohs 2] and [Chen]. It follows from the papers above that there are Zalcman type domains satisfying (*) and being Bergman complete and non-hyperconvex. We show that there are Zalcman type domains, which neither satisfy (*) nor are Bergman complete.

Let us fix a sequence (of pairwise different points) \( \{a_j\}_{j=1}^\infty \subset E \) and a closed disc \( B \subset E \) such that \( a_j \notin B \), \( a_j \to 0 \) and \( 0 \in B \) (automatically \( 0 \in \partial B \)).

Below we shall consider only sequence of positive numbers \( r_j \) such that \( \bar{\Delta}(a_j, r_j) \cap \bar{\Delta}(a_k, r_k) = \emptyset \) for any \( j \neq k \) and \( B \cap \bar{\Delta}(a_j, r_j) = \emptyset \).

**Lemma 5.1.** We may choose \( r_j \) so that there is a constant \( M < \infty \) such that

\[
K_{D_N}(z) < M \text{ for any } z \in B, \ N = 1, 2, \ldots,
\]

where \( D_N := E \setminus (\bigcup_{j=1}^N \bar{\Delta}(a_j, r_j)) \).

**Proof.** We define \( r_j \) inductively. Since \( E \setminus \bar{\Delta}(a_1, r_1) \) increases to \( E \setminus \{a_1\} \) as \( r_1 \) decreases to 0 and \( K_{E \setminus \{a_1\}} \) coincides with \( K_E \) on \( E \setminus \{a_1\} \) we get that there is a constant \( M \in \mathbb{R} \) such that \( K_{E \setminus \bar{\Delta}(a_1, r_1)} < M \) on \( B \) for sufficiently small \( r_1 > 0 \).

Assume that we have already chosen \( r_1, \ldots, r_N \) such that

\[
K_{D_N} < M \text{ on } B
\]

(\( D_N \) is defined as in the lemma). Since \( D_N \setminus \bar{\Delta}(a_{N+1}, r_{N+1}) \) increases to \( D_N \setminus \{a_{N+1}\} \) and \( K_{D_N \setminus \{a_{N+1}\}} \) coincides with \( K_{D_N} \) on \( D_N \setminus \{a_{N+1}\} \), we get as previously (use (5.2)) that \( K_{D_N \setminus \bar{\Delta}(a_{N+1}, r_{N+1})} < M \) on \( B \) for sufficiently small \( r_{N+1} > 0 \), which completes the proof. □

**Proposition 5.2.** There is a sequence \( s_j \to 0, \ 0 < s_j \leq r_j \) and a domain \( G := E \setminus (\bigcup_{j=1}^\infty \bar{\Delta}(a_j, s_j) \cup \{0\}) \) satisfying the property

\[
K_G(z) < M, \quad z \in B \cap G.
\]

**Proof.** Let us fix an increasing sequence of compact sets \( L_N \) such that \( \bigcup_{N=1}^\infty L_N = \text{int } B \).

We claim that we may choose a family of positive numbers \( \{s_N^j\}_{1 \leq N < \infty, N \leq j} \) such that \( s_{N+1}^j \leq s_N^j \leq r_j \) for \( j \geq N + 1 \) and \( s_N^N \leq r_N \) such that for the domain

\[
G_N := E \setminus (\bigcup_{N=1}^\infty \bar{\Delta}(a_j, s_N^j) \cup \bigcup_{j=1}^N \bar{\Delta}(a_j, s_j^j) \cup \{0\})
\]
we have $K_{G_N} < M$ on $L_N$.

Assume for a while that such a choice can be done. Then define $s_j := s_j^j$. Since $G_N \subset G$, we have $K_{G_N} \geq K_G$ for any $N$, in particular, for any $N$ $K_G < M$ on $L_N$, which completes the proof.

We define the desired family inductively with respect to $N$. Let $0 < t < 1$. Since $E \setminus (\bigcup_{j=2}^\infty \Delta(a_j, tr_j) \cup \Delta(a_1, r_1) \cup \{0\})$ increases to $E \setminus (\bigcup_{j=2}^\infty \{a_j\} \cup \Delta(a_1, r_1) \cup \{0\})$ as $t$ decreases to 0 and the Bergman kernel of the last domain is the restriction to this domain of $K_{D_1}$, we get that for $t$ sufficiently small $K_{G_1} < M$ on $L_1$, where $s_1^1 := r_1, s_1^j = tr_j, j \geq 2$.

Assume that the construction has been successful for $N$ (i.e. we have defined already all $s_k^j, j \geq k, k \leq N$). Let $0 < t < 1$. Since $E \setminus (\bigcup_{j=N+2}^\infty \Delta(a_j, ts_j^N) \cup \Delta(a_{N+1}, s_{N+1}^N) \cup \bigcup_{j=1}^N \Delta(a_j, t s_j^N) \cup \{0\})$ increases to $E \setminus (\bigcup_{j=N+2}^\infty \{a_j\} \cup \Delta(a_{N+1}, s_{N+1}^N) \cup \bigcup_{j=1}^N \Delta(a_j, t s_j^N) \cup \{0\})$ as $t$ decreases to 0 and the Bergman kernel of the last domain is the restriction to this domain of $K_{E \setminus (\bigcup_{j=1}^{N+1} \Delta(a_j, t s_j^{N+1}))}$ (which is smaller than or equal to $K_{D_{N+1}}$ because $D_{N+1}$ is a subset of the considered domain), we get that defining for $t$ sufficiently small $s_{N+1}^{N+1} := s_{N+1}^N, s_j^{N+1} := ts_j^N, j \geq N + 2$ the inequality $K_{G_{N+1}} < M$ holds on $L_{N+1}$.

Let us remark that because of the property $K_{D_j}(z) \to K_D(z)$ locally uniformly for any sequence $\{D_j\}_{j=1}^\infty$ of domains such that $D_j \subset D_{j+1}$ and $\bigcup_{j=1}^\infty D_j = D$ (D is a bounded domain) we get easily that $\beta_{D_j} \to \beta_D$ locally uniformly on $D \times \mathbb{C}^n$ (although the convergence in contrast to the convergence of Bergman kernels need not be monotonic).

Based on the above property of the Bergman kernel we present below a similar construction (to that from Proposition 5.2) leading to a domain having the assumptions as in Proposition 5.2 and, additionally, not Bergman complete. We denote $\beta_D(z) := \beta_D(z; 1)$.

**Lemma 5.3.** There are a constant $M_1 \in \mathbb{R}$ and a family of tuples $\Lambda = \bigcup_{N=0}^\infty \Lambda_N,$ where:

\[ \Lambda_0 = \emptyset, \quad \Lambda_N \subset (0, s_1] \times \ldots \times (0, s_N]; \]

for any $N$ if $\lambda \in \Lambda_N$ then there is some $\lambda_{N+1}$ such that for any $0 < s \leq \lambda_{N+1}$, $(\lambda, s) \in \Lambda_{N+1};$

for any $\lambda = (\lambda_1, \ldots, \lambda_N) \in \Lambda$ we have that $\beta_{D_\lambda} < M_1$ on $B$, where

\[ D_\lambda := E \setminus \bigcup_{j=1}^N \Delta(a_j, \lambda_j). \]

**Proof.** The proof goes similarly as that of Lemma 5.1. We proceed using induction. Since $E \setminus \Delta(a_1, t)$ increases to $E \setminus \{a_1\}$ as $t$ decreases to 0 and $\beta_{E \setminus \{a_1\}}$ coincides with $\beta_E$ on $E \setminus \{a_1\}$ we get that there is a constant $M_1 \in \mathbb{R}$ such that $\beta_{E \setminus \Delta(a_1, t)} < M_1$ on $B$ for any $0 < t \leq \lambda_1 \leq s_1$. We define $\Lambda_1 := (0, \lambda_1]$.

Assume that we have already defined $\Lambda_1, \ldots, \Lambda_N$ such that Lemma is satisfied, in particular,

\[ \beta_{D_\lambda} < M_1 \text{ on } B \]

for any $\lambda \in \Lambda_N$.
Fix any \( \lambda \in \Lambda_N \). Since \( D_\lambda \setminus \bar{\Delta}(a_{N+1}, t) \) increases to \( D_\lambda \setminus \{a_{N+1}\} \) and \( \beta_{D_\lambda \setminus \{a_{N+1}\}} \) coincides with \( \beta_{D_\lambda} \) on \( D_\lambda \setminus \{a_{N+1}\} \), we get as previously that \( \beta_{D_\lambda \setminus \bar{\Delta}(a_{N+1}, t)} < M_1 \) on \( B \) for sufficiently small \( t > 0 \), which completes the proof. \( \square \)

**Proposition 5.4.** There is a sequence \( \lambda_j \to 0 \), \( 0 < \lambda_j \leq s_j \) and a domain \( G := E \setminus (\bigcup_{j=1}^\infty \bar{\Delta}(a_j, \lambda_j) \cup \{0\}) \) satisfying the property

\[
\beta_G(z) \leq M_1, \quad z \in B \cap G.
\]

**Proof.** Let us fix an increasing sequence of compact sets \( L_N \) such that \( \bigcup_{N=1}^\infty L_N = \text{int} B \).

Without loss of generality we may assume that \( s_1 = \lambda^1 \) (\( s_1 \) is from Proposition 5.2 and \( \lambda^1 \) from Lemma 5.3).

It is sufficient to find sequences \( \{\lambda^j\}_{j=1}^\infty \) and \( \{t_j\}_{j=1}^\infty \subset (0, 1)^N \) such that \( \lambda^N \in \Lambda_N \), \( \lambda_N = t_1 \cdots t_N s_N \), \( \lambda^{N+1} = (\lambda^N, \lambda_{N+1}) \), and \( \beta_D < M_1 \) on \( L_N \), where

\[
D_N := D_{\lambda_N} \setminus \bigcup_{j=N+1}^\infty \bar{\Delta}(a_j, t_1 \cdots t_N s_j).
\]

Put \( \lambda_1 := \lambda^1(= s_1) \), \( t_1 := 1 \). Then for \( 1 > t > 0 \) small enough the Bergman metric on \( D_{\lambda^1} \setminus (\bigcup_{j=2}^\infty \bar{\Delta}(a_j, t s_j) \cup \{0\}) \) is less than \( M_1 \) on \( L^1 \) for \( 0 < t \leq t_1 < 1 \), we may also assume that \( \lambda^2 := (\lambda^1, t_2 s_2) \in \Lambda^2 \).

Assume that the construction has been succesful for \( N \) (i.e. we have already all \( t_j, j = 1, \ldots, N \) and \( \lambda^j, j = 1, \ldots, N \)). Let \( 0 < t < 1 \). Since \( D_{\lambda_N} \setminus (\bigcup_{j=N+1}^\infty \bar{\Delta}(a_j, t t_1 \cdots t_N s_j) \cup \{0\}) \) increases to \( D_{\lambda_N} \setminus (\bigcup_{j=N+1}^\infty \{a_j\} \cup \{0\}) \) as \( t \) decreases to 0 and the Bergman metric of the last domain is the restriction to this domain of \( \beta_{D_{\lambda_N}} \) we may choose \( t_{N+1} \) and then define \( \lambda^{N+1} := (\lambda^N, t_1 \cdots t_{N+1} s_{N+1}) \in \Lambda^{N+1} \) having the desired properties. \( \square \)

**Remark 5.5.** Note that the above mentioned results may be put in some more general context. Two principal properties that were used were the following both: \( K_D \) and \( \beta_D \) do not change after deleting a discrete subset and both are continuous with respect to the increasing family of domains. Applying the same procedure we may prove for instance that there are Zalcman type domains, which are not Carathéodory complete. Consequently, there are Zalcman type domains, without peak functions in 0 (see [Rud]).

**References**

[Blo-Pfl] Z. Blocki & P. Pflug, *Hyperconvexity and Bergman completeness*, Nagoya Math. J. **151** (1998), 221–225.

[Bre] J. Bremermann, *Holomorphic continuation of the kernel and the Bergman metric*, in 'Lectures on functions of a complex variable', Univ. of Mich. Press (1955), 349–383.

[Chen] B.-Y. Chen, *Completeness of the Bergman metric on non-smooth pseudoconvex domains*, Ann. Pol. Math. (to appear).

[Die-For-Her] K. Diederich, J. E. Fornaess & G. Herbort, *Boundary behaviour of the Bergman metric*, Proc. Symp. in Pure Math. **41** (1984), 59–67.

[Her] G. Herbort, *The Bergman metric on hyperconvex domains*, Math. Z. (to appear).

[Jak-Jar] P. Jakóbczak & M. Jarnicki, *Lectures on Holomorphic Functions of Several Complex Variables* (to appear).

[Jar-Pfl 1] M. Jarnicki & P. Pflug, *Bergman completeness of complete circular domains*, Ann. Pol. Math. **50** (1989), 219-222.

[Jar-Pfl 2] M. Jarnicki & P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, Walter de Gruyter, Berlin, 1993.
[Kob] S. Kobayashi, *On complete Bergman metrics*, Proc. Amer. Math. Soc. **13** (1962), 511–513.

[Ohs 1] T. Ohsawa, *Boundary behaviour of the Bergman kernel function on pseudoconvex domains*, Publ. RIMS Kyoto Univ. **20** (1984), 897–902.

[Ohs 2] T. Ohsawa, *On the Bergman kernel of hyperconvex domains*, Nagoya Math. J. **129** (1993), 43–52.

[Ohs-Tak] T. Ohsawa & K. Takegoshi, *On the extension of $L^2$-holomorphic functions*, Math. Z. **195** (1987), 197–204.

[Pfl 1] P. Pflug, *Quadratintegrable holomorphe Funktionen und die Serre Vermutung*, Math. Ann. **216** (1975), 285–288.

[Pfl 2] P. Pflug, *About the Carathéodory completeness of all Reinhardt domains*, In: Functional Analysis, Holomorphy and Approximation Theory II, (1984), North-Holland, Amsterdam, 331-337, G. Zapata ed.

[Rud] W. Rudin, *Some theorems on bounded analytic functions*, Trans. Amer. Math. Soc. **78** (1955), 333–342.

[Sic] J. Siciak, *Extremal plurisubharmonic functions and capacities in $\mathbb{C}^n$*, Sophia Kokyuroku in Mathematics **14** (1982).

[Zwo] W. Zwonek, *On Bergman completeness of pseudoconvex Reinhardt domains*, (preprint).

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