SPINNING PARTICLE AS A NON-TRIVIAL ROTATING SUPER BLACK HOLE WITH BROKEN N=2 SUPERSYMMETRY

A.Ya. Burinskii

Gravity Research Group, NSI Russian Academy of Sciences, B.Tulskaya 52, 113191 Moscow, Russia

Abstract

A non-trivial supergeneralization of the Kerr-Newman solution is considered as representing a combined model of the Kerr-Newman spinning particle and superparticle.

We show that the old problem of obtaining non-trivial super black hole solutions can be resolved in supergravity broken by Goldstone fermion. Non-linear realization of broken N=2 supersymmetry specific for the Kerr geometry is considered and some examples of the super-Kerr geometries generated by Goldstone fermion are analyzed. The resulting geometries acquire torsion, Rarita-Schwinger field and extra wave contributions to metric and electromagnetic field caused by Grassmann variables.

One family of the self-consistent super–Kerr–Newman solutions to broken N=2 supergravity is selected, and peculiarities of these solutions are discussed. In particular, the appearance of extra ‘axial’ singular line and traveling waves concentrated near ‘axial’ and ring-like singularities.

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1 e-mail: grg@ibrae.ac.ru
1 Introduction

Since 1968 it has been mentioned that the Kerr-Newman solution possesses some remarkable properties which allow to consider it as a model of spinning particle [1, 2, 3, 4]. Some string-like structures were obtained in Kerr geometry. The first one is connected with a singular ring of the Kerr solution [2, 5], two others are linked with a complex representation of Kerr geometry [6] in which the Kerr-Newman solution is considered as a retarded-time field generated by a complex source propagating along a complex world line [4]. The point of view that some of black holes can be treated as elementary particles was also suggested by superstring theory [7, 8, 9, 4, 5]. In particular, Sen [8] has obtained a generalization of the Kerr solution to low energy string theory, and it was shown [9] that near the Kerr singular ring the Kerr-Sen solution acquires a metric similar to the field around a heterotic string.

However, description of spinning particle based only on the bosonic fields cannot be complete, and involving fermionic degrees of freedom is required. The most natural way to involve fermions is to treat corresponding super black holes in supergravity. In previous papers [10, 11] we considered a method of supergeneralization of the Kerr-Newman geometry leading to a combined model of the Kerr spinning particle and superparticle. It was mentioned that this method leads to the broken or non-linear realization of supersymmetry.

In the present paper we show that the breaking of supersymmetry caused by Goldstone fermion allows to resolve the old problem of non-triviality of the super black holes solutions [14, 15, 16, 17]. Using Deser-Zumino model of broken N=1 supergravity [12] generalized to the N=2 supergravity described by Ferrara and Nieuwenhuizen [13], we analyze some non-trivial supergeneralizations of the Kerr-Newman geometry and show the appearance of torsion and traveling waves.

We select one self-consistent class of the super-Kerr-Newman solutions to N=2 supergravity with supersymmetry broken by Goldstone fermion and discuss peculiarities of these solutions. In particular, the appearance of extra ‘axial’ singular line and traveling waves concentrated near the ‘axial’ and ring-like singularities.
Non-trivial super-solutions from trivial ones

The problem of non-trivial supergravity solutions is connected with the fact that any solution of Einstein gravity is indeed a trivial solution of supergravity field equations with a zero spin-3/2 field [14, 15, 16, 17]. By using a supergauge freedom of supergravity (supertranslations) one can turn the gravity solutions into a form containing spin-3/2 field. However, since this field can be gauged away by the reverse transformation, such supersolutions are indeed trivial. There existed even an opinion that all the super black hole solutions are trivial. However, some examples of the non-trivial super black hole solutions were given by Aichelburg and Güven [18, 19], and also in two dimensional dilaton supergravity by Knutt-Wehau and Mann [20].

In previous papers [10, 11] we showed that non-trivial examples of the super-Kerr geometry can be obtained by a trivial supershift of the Kerr solution taking into account some non-linear B-slice constraints. Indeed, the complex structure of the Kerr geometry prompts how to avoid this triviality problem.

The Kerr-Schild form of the Kerr geometry [21]

\[ g_{ik} = \eta_{ik} + 2hk_i k_k \]  

allows to give a complex representation of the Kerr solution as a geometry generated by a complex source propagating along a complex world line \( x^0(\tau) \) in auxiliary Minkowski space \( \eta = \text{diag}(-, +, +, +) \). This representation shows that from complex point of view the Schwarzschild and Kerr geometries are equivalent and connected by a trivial complex shift.

The non-trivial twisting structure of the Kerr geometry arises as a result of the shifted real slice of the complex retarded-time construction [6, 4]. If the real slice is passing via ‘center’ of the solution \( x_0 \) there appears a usual spherical symmetry of the Schwarzschild geometry. The specific twisting structure results from the complex shift of the real slice regarding the source.

Similarly, it is possible to turn a trivial super black hole solution into a non-trivial if one finds an analogue to the real slice in superspace.

The trivial supershift can be represented as a replacement of the complex world line by a superworldline

\[ X^i_0(\tau) = x^i_0(\tau) - i\theta^i\bar{\zeta} + i\zeta^i\bar{\theta}, \]  

allows to give a complex representation of the Kerr solution as a geometry generated by a complex source propagating along a
parametrized by Grassmann coordinates $\zeta, \bar{\zeta}$, or as a corresponding coordinate replacement in the Kerr solution

$$x'^i = x^i + i\theta\sigma^i \bar{\zeta} - i\zeta \sigma^i \bar{\theta}; \quad \theta' = \theta + \zeta, \quad \bar{\theta}' = \bar{\theta} + \bar{\zeta},$$ (3)

Assuming that coordinates $x^i$ before the supershift are the usual c-number coordinates one sees that coordinates acquire nilpotent Grassmann contributions after supertranslations. Therefore, there appears a natural splitting of the space-time coordinates on the c-number ‘body’-part and a nilpotent part - the so called ‘soul’. The ‘body’ subspace of superspace, or B-slice, is a submanifold where the nilpotent part is equal to zero, and it is a natural analogue to the real slice in complex case.

It has been shown in [10, 11] that reproducing the real slice procedure of the Kerr geometry in superspace we have to consider super light cone constraints

$$s^2 = [x^i - X^i_0(\tau)][x^i - X^i_0(\tau)] = 0,$$ (4)

and B-slice, where coordinates $x^i$ do not contain nilpotent contributions. Selecting the body and nilpotent parts of this equation we obtain three equations. The first one is the real slice condition of the complex Kerr geometry claiming that complex light cones, described by set

$$x = x_0(\tau) + \Psi \sigma \bar{\Psi},$$ (5)

can reach the real slice. Here $\Psi$ and $\bar{\Psi}$ are the commuting two-component spinors. On the real slice field $\Psi(x)$ determines the principal null congruence of the Kerr geometry (1)

$$k_i(x) = P^{-1} \Psi \sigma_i \bar{\Psi}.$$ (6)

The nilpotent part of (1) yields two B-slice conditions

$$[x^i - x^i_0(\tau)](\theta \sigma_i \bar{\zeta} - \zeta \sigma_i \bar{\theta}) = 0;$$ (7)

$$(\theta \sigma_i \bar{\zeta} - \zeta \sigma_i \bar{\theta})^2 = 0.$$ (8)

2These constraints are similar to the complex light cone constraints of the standard Kerr geometry connected with a retarded-time construction. The physical sense of these constraints is existence of the real slice for the light cones placed at the points of complex world line $x^i_0(\tau)$. Similarly, super light cone constraints demand existence of the body-slice for the super light cones placed at the points of the super world line $x^i_0(\tau)$. 

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Equation (7) may be rewritten using (5) in the form
\[ (\theta^\alpha \sigma_{\alpha\alpha} \tilde{\zeta}^\alpha - \zeta^\alpha \sigma_{\alpha\alpha} \bar{\theta}^\alpha) \Psi^\beta \sigma^i_{\beta\delta} \bar{\Psi}^\delta = 0 \] (9)
which yields
\[ \bar{\Psi} \bar{\theta} = 0, \quad \bar{\Psi} \bar{\zeta} = 0, \] (10)
which in turn is a condition of proportionality of the commuting spinors \( \bar{\Psi}(x) \) and anticommuting spinors \( \bar{\theta} \) and \( \bar{\zeta} \), this condition providing the left null superplanes of the supercones to reach B-slice.

Finally, by introducing the Kerr projective spinor coordinate \( Y(x) \) we have \( \bar{\Psi}^2 = Y(x) \), \( \bar{\Psi}^1 = 1 \), and we obtain
\[ \bar{\theta}^\alpha = \begin{pmatrix} \bar{\theta}^1 \\ Y(x) \bar{\theta}^1 \end{pmatrix}, \] (11)
\[ \bar{\zeta}^\alpha = \begin{pmatrix} \bar{\zeta}^1 \\ Y(x) \bar{\zeta}^1 \end{pmatrix}. \] (12)

It also leads to \( \bar{\theta} \bar{\theta} = \bar{\zeta} \bar{\zeta} = 0 \), and equation (8) is satisfied automatically.

Thus, as a consequence of the B-slice and superlightcone constraints we obtain a non-linear submanifold of superspace \( \bar{\theta} = \bar{\theta}(x) \), \( \bar{\theta} = \bar{\theta}(x) \). The original four-dimensional supersymmetry is broken, and the initial supergauge freedom which allowed to turn the super geometry into trivial one is lost. Nevertheless, there is a residual supersymmetry based on free Grassmann parameters \( \theta^1, \bar{\theta}^1 \).

It was mentioned that the above B-slice constraints yield in fact the non-linear realization of broken supersymmetry introduced by Volkov and Akulov \[24, 25\] and considered in N=1 supergravity by Deser and Zumino \[12\]. In terminology of broken supersymmetry the Grassmann parameters \( \zeta^\alpha(x) \), \( \bar{\zeta}^\alpha(x) \) represent some fermion Goldstone field on space-time which has to be eaten by spin-3/2 Rarita-Schwinger field. As a result supergauge will be fixed.

In this paper we consider the application of this approach to obtaining the superversions of the Kerr-Newman solution to N=2 supergravity formulated in \[13\].
3 Broken Supersymmetry in N=1 Supergravity

Here we use spinor notations of the book [23]. The indices $i, j, k, l...$ are related to coordinate, and $a, b, c, d...$ are reserved for tetrad.

The Volkov-Akulov model of non-linear realization of supersymmetry [22, 23] is based on selecting a submanifold of superspace setting the correspondence of the Grassmann coordinates

$$\Theta = \left( \frac{\theta^\alpha}{\bar{\theta}^\dot{\alpha}} \right)$$

(13)

to a Goldstone field $\lambda(x)$ which is a Majorana fermion. This yields the submanifold $\Theta(x) = b\lambda(x)$ which is non-linear in general case. The non-linear residual supertransformations are

$$\delta_\epsilon \lambda = b^{-1} \epsilon + ib(\bar{\epsilon}\gamma^i\lambda)\partial_i\lambda(x),$$

(14)

and contain inhomogeneous term $b^{-1} \epsilon = \left( \frac{\zeta^\alpha}{\bar{\zeta}^\dot{\alpha}} \right)$.

Considered by Deser and Zumino case of broken N=1 supergravity is based on this model [12], and it is proposed that $\epsilon$ admits local transformations $\epsilon(x)$.

The Lagrangian is given by

$$\mathcal{L} = -(i/2)\bar{\lambda}\gamma^i\mathcal{D}\lambda - (i/2b)\bar{\lambda}\gamma^i\psi_i + \mathcal{L}_{sg},$$

(15)

where the supergravity Lagrangian is

$$\mathcal{L}_{sg} = -eR/2k^2 - i/2\epsilon^{ijkl}\bar{\psi}_i\gamma_5\gamma_j\mathcal{D}_k\psi_l,$$

(16)

and

$$\mathcal{D}_i = \partial_i - \frac{1}{2}\omega_{iab}\Sigma^{ab}; \quad \Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b].$$

(17)

3 Further we use the Dirac four-component spinor notations.

4 We have omitted here cosmological term since here we shall consider only the region of massless fields.
The Lagrangian is invariant under the above non-linear supertransformations, and tetrad $e^a$ and Rarita-Schwinger fields $\psi_i$ are transformed as follows
\begin{align*}
\delta_\epsilon \lambda &= b^{-1} \epsilon + ib(\bar{\epsilon} \gamma^i \lambda) \partial_i \lambda, \\
\delta_\epsilon e_i^a &= -ik\bar{\epsilon} \gamma^a \psi_i, \\
\delta_\epsilon \psi_i &= -2/k \mathcal{D}_i \epsilon.
\end{align*}

It is assumed that this construction is similar to the Higgs mechanism of the usual gauge theories. The Goldstone fermion $\lambda(x)$ can be eaten by appropriate local transformation $\epsilon(x)$ with a corresponding redefinition of the tetrad and spin-3/2 field. It means that starting from the gravity solution with zero spin-3/2 field and some Goldstone fermion field $\lambda$ one can obtain in such a way a non-trivial supergravity solution with non-linear realization of broken supersymmetry.

There are two obstacles for indirect application of this scheme to the Kerr-Newman case. First one is the electromagnetic charge which demands to change the expression for supercovariant derivative that leads to non-Majorana values for spin-3/2 field. The second one is the complex character of supertranslations in the Kerr case that also yields the non-Majorana supershifts. Thus, this scheme has to be extended to $N=2$ supergravity.

4 Broken supersymmetry in $N=2$ supergravity

The consistent $N=2$ supergravity described by Ferrara and Nieuwenhuizen \cite{Ref13} is based on the complex (non-Majorana) supergauge field $\epsilon = (\epsilon_1 + i\epsilon_2)/\sqrt{2}$ and contains a complex spin-3/2 field $\chi = (\psi + i\phi)/\sqrt{2}$ and the vector potential of electromagnetic field $A_i$.

The expression for supercovariant derivative (see Appendix B) is extended by electromagnetic contribution $F_{ab}$ and torsion terms $\mathcal{D}_{Ni} = (k^2/2)\Sigma^{ab} k_{iab}$,

\begin{equation}
\tilde{\mathcal{D}}_i = \mathcal{D}_i + \mathcal{D}_{Ni} - (ik/2\sqrt{2})F_{ab}\Sigma^{ab}\gamma_i.
\end{equation}

The action is invariant under the following complex local supersymmetry transformations
\begin{equation}
\delta_\epsilon e_i^a = k(\bar{\epsilon}\gamma^a \chi_i - \bar{\chi}_i \gamma^a \epsilon),
\end{equation}
\[ \delta \epsilon \chi_i = \frac{2}{k} \tilde{D}_i \epsilon, \]  
\[ \delta \epsilon A_i = -i \sqrt{2}(\epsilon \chi_i - \bar{\chi}_i \epsilon). \] 

The expression (18) can be extended on the complex non-Majorana spinors

\[ \delta \epsilon \lambda = b^{-1} \epsilon + i b(\bar{\epsilon} \gamma^i \lambda) \partial_i \lambda. \] 

Now we have to use the above considered complex supershift and superlightcone constraints of the Kerr geometry (11, 12) in \( N=2 \) supergravity. One should note, that the above superlightcone constraints restrict values of supershift parameters \( \bar{\zeta} \), leaving the values of \( \zeta \) free, which leads in general case to a non-Majorana spinor

\[ \epsilon = \begin{pmatrix} \zeta_\alpha \\ \bar{\zeta}_\dot{\alpha} \end{pmatrix} \] 

corresponding to a complex supershift. Further, we should note that supercovariant derivative contains the nonlinear in \( \epsilon \) terms from torsion \( D_N \), which hinders the indirect use of finite supertranslations. At the other hand the Kerr constraints on \( \bar{\zeta} \), (12), select a submanifold displaying a remarkable nilpotency \( \bar{\zeta}^2 = 0 \). It means, that the values of \( \bar{\zeta} \) on this submanifold lie in a degenerate subalgebra of the Grassmann algebra \[ 15, 24 \].

To avoid the non-linear terms from torsion we will extend this property on the four-component spinor \( \epsilon \), and will restrict supershift by the form

\[ \epsilon = 2^{-1/2} \begin{pmatrix} s\eta_\alpha \\ s\bar{\zeta}_\dot{\alpha} \end{pmatrix}, \] 

where

\[ s^2 = \bar{s}^2 = 0; \quad \bar{s}s + s\bar{s} = 0. \]

This form possesses the nilpotency \( \epsilon^2 = 0 \) that leads to cancelling the nonlinear term \( D_N \), and gives rise to validity (22), (23), (24) under finite supertranslations. The formulated in \( N=1 \) case problem of the absorption of the

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5It means that the complex expansion \( \zeta_\alpha = \sum_{\alpha\beta}^{(1)} s_q + \sum_{\alpha\beta\gamma\delta}^{(2)} s_p s_q s_r \ldots \), where \( s_p s_q + s_q s_p = 0 \), contains there only the first non-zero term \( s^2 = \bar{s}^2 = 0; \quad \bar{s}s + s\bar{s} = 0 \). This degeneracy take place only on the considered subspace of supershifts, however, even in a very narrow neighborhood of this subspace the full algebra has to be taken into account. In particular, by application spinor supercovariant derivatives when the Grassmann variables have to be considered as independent.
Goldstone fermion \( \lambda \) is reduced to a choice of supergauge field

\[
\epsilon(x) = -b\lambda(x),
\]

that leads to satisfying \((23)\) and yields \( \lambda' = \lambda + \delta\lambda \to 0 \), if we assume that \( \lambda \) has corresponding nilpotency.

Below we present few examples of supershift functions and will see that the most interesting case is when the Goldstone fermion \( \lambda \) satisfies the massless Dirac equation.

5 Examples of N=2 super-geometries

In this section we consider different variants of the Goldstone fermion satisfying the constraints \((12)\) and describe the resulting super-geometries. It is convenient to perform all calculations in the local Lorentz frame of the Kerr geometry where \( \sigma\)-matrices take the form \( [16, 19, 26] \)

\[
\sigma_i = 2^{1/2} \left( \begin{array}{cc} e_i^3 & e_i^2 \\ -e_i^4 & -e_i^1 \end{array} \right); \quad \bar{\sigma}_i = 2^{1/2} \left( \begin{array}{cc} -e_i^4 & -e_i^2 \\ e_i^1 & e_i^3 \end{array} \right). \quad (29)
\]

The corresponding matrices with tetrad indices \( a, b, c, d... \) will be given by \( \sigma^a = \sigma_i e^{ai} \) and take the form

\[
\sigma^a = 2^{1/2} \left( \begin{array}{cc} \delta^a_4 & \delta^a_1 \\ \delta^a_2 & -\delta^a_3 \end{array} \right). \quad (30)
\]

The four component spinor

\[
\phi = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}
\]

will be called as aligned to \( e^3 \) if \( e^3_i \gamma^j \phi^{(3)} = 0 \) that has the consequence \( A = D = 0 \). Similarly, spinor \( \phi^{(4)} \) is aligned to \( e^4 \) if \( e^4_i \gamma^j \phi^{(4)} = 0 \), which yields \( B = C = 0 \). Therefore, any four component spinor can be split on the sum of two aligned spinors \( \phi = \phi^{(3)} + \phi^{(4)} \). Physically, the \( e^3 \) (\( e^4 \)) aligned solution describes the waves in the \( e^3 \) (\( e^4 \)) null direction. In many cases (especially massless) solutions of these aligned parts are independent. In the
algebraically special Kerr-Newman geometry all the tensor fields are aligned to the principal null direction $e^3$. The supershift constraints (12) represent in fact a condition for the two-component spinor $\tilde{\zeta}$ to be aligned to $e^3$. In the local Lorentz frame it takes the form $\tilde{\zeta}^2 = 0$, the second component $\tilde{\zeta}^1 = C$ is free. Therefore, the general form of supershift satisfying the Kerr’s superconstraints and nilpotency condition will be

$$\epsilon = s2^{-1/2} \begin{pmatrix} \eta_\alpha \\ \zeta_\dot{\alpha} \end{pmatrix},$$  

where

$$\eta_\alpha = \begin{pmatrix} A \\ B \end{pmatrix},$$  

and

$$\zeta_\dot{\alpha} = \begin{pmatrix} C \\ 0 \end{pmatrix}.$$  

For the sake of convenience we also give contravariant components of $\eta$

$$\eta^\alpha = \begin{pmatrix} B \\ -A \end{pmatrix}.$$  

The resulting complex Rarita-Schwinger field has contributions from metric

$$\chi_g = \chi_{g\alpha}^\alpha = (s\sqrt{2}/k) \begin{pmatrix} dA + AH - Bd\bar{Y} \\ dB - AG - BH \\ dC - CH \\ CdY \end{pmatrix},$$  

and from electromagnetic field

$$\chi_F = -i(s/\sqrt{2}) \begin{pmatrix} C\bar{N}e^3 \\ C(\bar{S}e^3 - \bar{N}e^1) \\ A(Ne^4 + Se^1) + B(Ne^2 - Se^3) \\ -ANe^1 + BN e^3 \end{pmatrix},$$  

where we have introduced the notations for the geometry parameters

$$H = [(\bar{Z} - Z)h - h_{,4}]e^3/2; \quad G = h\bar{Z}e^1 - (h_{,2} - hY_{,3})e^3;$$  

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and for the combinations of the tetrad components of electromagnetic field

\[ N = F_{12} + F_{34}, \quad S = 2F_{31}. \]  

The function \( Y(x) \) is the main function determining the principal null congruence \( \mathbb{N} \) and the Kerr tetrad. It can be expressed as a projective spinor coordinate \( Y = \psi^0 / \bar{\psi}^1 \), and satisfies the equation \( dY = Z[e^1 - (P \psi / P)e^3] \), where \( P = 2^{-1/2}(1 + YY) \) for the stationary Kerr-Newman background. The corresponding Kerr tetrad \( e^a \) is given in the Appendix C. The resulting nilpotent contribution to tetrad contains two terms, contributions from metric and from electromagnetic field \( \delta e^a = \delta_\gamma e^a + \delta_F e^a \). The metric part will be

\[
\begin{align*}
\delta_\gamma e^1 &= \bar{s}s\sqrt{2}[(\bar{C}C - \bar{B}B)dY - \bar{A}d\bar{B} + Bd\bar{A} + \bar{A}AG + \bar{A}B(H + \bar{H})], \\
\delta_\gamma e^2 &= -\delta e^3, \\
\delta_\gamma e^3 &= \bar{s}s\sqrt{2}[(\bar{A}\bar{B})d\bar{Y} - \bar{A}\bar{B}dY + Ad\bar{A} - \bar{A}dA + A\bar{A}(\bar{H} - H)], \\
\delta_\gamma e^4 &= \bar{s}s\sqrt{2}[(\bar{C}dC - C\bar{d}\bar{C} + \bar{B}d\bar{B} - Bd\bar{B} + \bar{A}\bar{B}\bar{G} - \bar{A}\bar{B}G + (BB - C\bar{C})(H - \bar{H})].
\end{align*}
\]

Electromagnetic contribution to tetrad is

\[
\begin{align*}
\delta_F e^1 &= -ik\bar{s}s/2[e^1(C\bar{A}\bar{N} - \bar{C}AN) - C\bar{A}\bar{S}\bar{e}^3], \\
\delta_F e^2 &= -\delta_F e^3, \\
\delta_F e^3 &= ik\bar{s}s\bar{e}^3(C\bar{A}\bar{N} + \bar{C}AN)/2, \\
\delta_F e^4 &= ik\bar{s}s(Le^4 + Se^1)(C\bar{A} + \bar{C}A)/2.
\end{align*}
\]  

Nilpotent contribution to vector potential \( \bar{A} \) also contains two terms \( \delta \bar{A} = \delta_\gamma A_\alpha e^\alpha + \delta_F A_\alpha e^\alpha \),

\[
\begin{align*}
\delta_\gamma A &= -i(\bar{s}s\sqrt{2}/k)(\bar{C}dA + \bar{A}d\bar{C} - C\bar{d}\bar{A} + 2A\bar{C}H - 2\bar{A}CH + 2B\bar{C}dY - 2BCd\bar{Y}), \\
\end{align*}
\]

and

\[
\begin{align*}
\delta_F A &= \bar{s}s/2\{e^1[S\bar{A}\bar{A} + (\bar{N} - N)A\bar{B}] + e^2[S\bar{A}\bar{A} - (\bar{N} - N)\bar{A}\bar{B}] + e^3[(BB + C\bar{C})(N + \bar{N}) - A\bar{B}\bar{S} - \bar{A}\bar{B}\bar{S}] + e^4A\bar{A}(N + \bar{N})\}. 
\end{align*}
\]

\(^6^\text{This function is determined by the Kerr theorem [6, 21]. A fixation of } Y \text{ selects the null planes in } CM^4 \text{ and null rays of the principal null congruence.}\)
In these expressions we have the free spinor components

\[ \eta_\alpha = \begin{pmatrix} A(x) \\ B(x) \end{pmatrix}, \tag{44} \]

determining the supershift. Now we consider some particular cases leading to simplification of general expressions.

### 5.1 Case I

The simplest spinor shift \( A(x) \neq 0, \quad B = 0 \), considered in the basis having the \( \sigma \)-matrices adapted to the auxiliary Minkowski space \( \eta_{ik} \). The peculiarity of this shift is that while transformed to the aligned basis this spinor takes the same simple form that allows to simplify expressions

\[ \eta_\alpha = \begin{pmatrix} A(x) \\ 0 \end{pmatrix}, \tag{45} \]

We will also assume that \( C = 1 \), and obtain for \( \delta_g \) contribution to tetrad

\[ \begin{align*}
\delta_g e^1 &= \bar{s}s \sqrt{2}[dY + \bar{A}AG], \\
\delta_g e^2 &= -\delta e^1, \\
\delta_g e^3 &= \bar{s}s \sqrt{2}[A\bar{d}\bar{A} - \bar{A}d\bar{A} + \bar{A}\bar{A}(\bar{H} - H)], \\
\delta_g e^4 &= \bar{s}s \sqrt{2}(\bar{H} - H),
\end{align*} \tag{46} \]

and \( \delta_F \) contribution

\[ \begin{align*}
\delta_F e^1 &= -ik\bar{s}s/2[e^1(\bar{A}\bar{N} - AN) - \bar{A}S e^3], \\
\delta_F e^2 &= -\delta_F e^1, \\
\delta_F e^3 &= ik\bar{s}s e^3(\bar{A}\bar{N} + AN)/2, \\
\delta_F e^4 &= ik\bar{s}s(N e^4 + Se^1)(\bar{A} + A)/2.
\end{align*} \tag{47} \]

Contributions to vector potential \( A \) will be

\[ \delta_g A = -i(\bar{s}s \sqrt{2}/k)(dA - d\bar{A} + 2AH - 2\bar{A}H), \tag{48} \]

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7: Transformations of the Dirac spinors from the \( e^3 \)-aligned basis to the basis of auxiliary Minkowski space are given in Appendix D.
\[ \delta_F A = \bar{s}s/2[(e^1 + e^2)A\bar{A} + (e^3 + e^4)A\bar{A})(N + \bar{N})]. \]  

One can see that the wave oscillations of the shift, \( A = e^{ipx^i} \), can lead to the wave oscillations of the terms \( \delta_F e^a \) in tetrad and the term \( \delta_g A \) in the vector potential. Since in the Kerr-Newman geometry function \( N \) has the behavior \( N \sim Z^2 \), the tetrad oscillations are growing near the Kerr singular ring and take the form of waves travelling along the singularity. However, in this case we do not have clear proposals concerning mechanisms or the possible origin of such oscillations in supershift. Some suggestion to this case follows from observation that such oscillating spinor \( \eta \) resembles oscillating solutions of the Dirac equation.

### 5.2 Case II

The spinor shift has parameters \( A = 0, \ B \neq 0 \), considered in the basis where \( \sigma \)-matrices are adapted to the auxiliary Minkowski space \( \eta_{ik} \). Being transformed to the aligned basis this spinor takes the form \( A = B\bar{Y}; \ B \neq 0 \). This case does not lead to essential simplifications in respect to the general case, and we will not write down the expressions for this case, but we should note that appearance of traveling waves for oscillating \( B \) can be observed in this case, too.

### 5.3 Case III

The case of supershifts aligned to \( e^3 \): \( A = 0; \ B \neq 0 \), considered in the aligned \( e^3 \) basis. \(^8\) This case yields maximal simplifications of the expressions and is motivated by super-QED model of broken supersymmetry for the region of matter fields. Besides, as we shall see it leads to a family of self-consistent super-Kerr-Newman solutions.

There is no electromagnetic contributions to tetrad in this case and we have

\[
\begin{align*}
\delta e^1 &= \bar{s}s\sqrt{2}(C\bar{C} - B\bar{B})dY, \\
\delta e^2 &= -\delta e^1, \\
\delta e^3 &= 0, 
\end{align*}
\]

\(^8\)If \( \sigma \)-matrices were adapted to auxiliary Minkowski frame this shift is given by parameters \( A = -\bar{Y}; \ B = 1 \). See matrices of transformation in Appendix D.
$$\delta e^4 = \bar{s}s\sqrt{2}[(CC - BB)h(Z - Z)e^3 + CdC - CdC + BdB - BdB].$$

The contribution to the Kerr-Newman vector potential will be

$$\delta_g A = -i(\bar{s}s2\sqrt{2}/k)[\bar{B}C dY - B\bar{C}d\bar{Y}], \quad (51)$$

$$\delta_F A = -\bar{s}s/2e^3[(B\bar{B} + C\bar{C})(N + \bar{N})]. \quad (52)$$

The case $C = \bar{B}$ could be called pseudo-Majorana. In this case there are no nilpotent contributions to tetrad at all, $\delta e^a = 0$, and metric has pure bosonic form similar to the case of extreme black holes without an angular momentum. However, the Kerr-Newman vector potential has nilpotent contributions, and moreover, it can have traveling waves from the term

$$\delta_g A = -i(\bar{s}s2\sqrt{2}/k)(\bar{B}^2 dY - B^2 d\bar{Y}). \quad (53)$$

In the case of independent $B$ and $C$ there also appears tetrad traveling waves.

In general case there is a non-zero torsion in the super-Kerr geometry $Q^a = \bar{\psi}_b e^b \wedge \gamma^a \psi_e e^e$ which gives rise to corresponding traveling waves of torsion. [9]

The torsion $Q^a$ contains contributions from metric $Q^a_g$, electromagnetic field $Q^a_F$ and terms of interplay $Q^a_I$.

We write down here the torsion terms only for the simplest and the most interesting case III, when $A = 0$. Contributions from metric are

$$Q^1_g = i\bar{s}s\sqrt{2}/k^2(CdC - C\bar{C}H - \bar{B}dB + \bar{B}B\bar{H}) \wedge dY,$$

$$Q^2_g = \bar{Q}^1_g,$$

$$Q^3_g = i\bar{s}s\sqrt{2}/k^2(CC - BB)dY \wedge d\bar{Y}, \quad (54)$$

$$Q^4_g = i\bar{s}s\sqrt{2}/k^2[(d\bar{C} - CH) \wedge (dC - C\bar{H}) + (d\bar{B} - \bar{B}H) \wedge (dB - BH)].$$

Contributions from electromagnetic field are

$$Q^1_F = i\bar{s}s/(2\sqrt{2})N\bar{N}(B\bar{B} - C\bar{C})e^1 \wedge e^3,$$

$$Q^2_F = \bar{Q}^1_F,$$

$$Q^3_F = 0,$$

$$Q^4_F = i\bar{s}s/(2\sqrt{2})(B\bar{B} - C\bar{C})(\bar{N}e^1 - \bar{S}e^3) \wedge (Ne^2 - Se^3). \quad (55)$$

[9] The Kerr-Newman solution with torsion based on the Poincaré gauge theory has been considered in [27].
Terms of interplay are

\[
\begin{align*}
Q^1_I &= \tilde{s}s/(k\sqrt{2})N(Bd\vec{C} - \vec{C}dB) \wedge e^3, \\
Q^2_I &= Q^1_I, \\
Q^3_I &= 0, \\
Q^4_I &= \tilde{s}s/(k\sqrt{2})[B(dC - CH) \wedge (Ne^2 - Se^3) - C(dB - BH) \wedge (Ne^1 - Se^3) + \text{c.c. term}].
\end{align*}
\]

(56)

In the case $\vec{B} = \vec{C}$ all the torsion terms disappear.

6 Self-consistent super-Kerr-Newman solutions to broken N=2 supergravity

The generalized to N=2 supergravity Deser-Zumino Lagrangian (15), (16) takes the form

\[
\mathcal{L} = -\left(\frac{i}{2}\right)[\tilde{\lambda}_\gamma \tilde{D} \lambda - \tilde{D} \lambda \gamma \lambda] - \left(\frac{i}{2b}\right)[\bar{\lambda}_\gamma^i \chi_i - \bar{\chi}_i \gamma^i \lambda] + \mathcal{L}_{2-sg},
\]

(57)

where the N=2 supergravity Lagrangian is

\[
\mathcal{L}_{2-sg} = -eR/2k^2 - 1/4F_{ij}F^{ij} - ie^{ijkl} \bar{\chi}_i \gamma_5 \gamma_j \tilde{D}_k \chi_l.
\]

(58)

It follows from (57) that the self-consistent solutions to broken N=2 supergravity has to take into account the energy-momentum tensor of the Grassmann fields. In particular, when considering the initiate trivial solutions in the super-gauge with zero Rarita–Schwinger field, one can use this Lagrangian with $\chi = 0$ that yields the Einstein–Maxwell–Dirac system of equations. We note that the energy-momentum tensor of the Goldstone field $\lambda$ acts here as fermionic matter. However, when using the exact Kerr-Newman solution as trivial one to perform the super-gauge with absorption of the Goldstone fermion, we do not take into account the energy-momentum tensor of the Goldstone field. Therefore, in general case, the considered above super-geometries cannot be treated as self-consistent. However, one exclusive case can be selected when the self-consistency is guaranteed. It takes place for the ghost Goldstone field possessing the zero energy-momentum tensor.

In this case, starting from the Lagrangian with $\chi = 0$, we have in fact the Einstein-Maxwell system of equation leading to the exact Kerr-Newman
solution and the Dirac equation (on the Kerr-Newman background) for the Goldstone fermion $\lambda$.

This solution can be considered as an exact super-solution to $N=2$ supergravity coupled to Goldstone field. Then, absorption of the Goldstone field by the complex Rarita-Schwinger field $\chi$ turns this solution into self-consistent solution with broken $N=2$ supersymmetry.

In the Appendix B we have given the solutions of the massless Dirac equation on the Kerr-Newman background in the aligned to $e^3$ case (case III). The corresponding functions $B$ and $C$ have the form

$$B = \tilde{Z} f_B(\bar{Y}, \bar{\tau})/P,$$  
$$C = Z f_C(Y, \tau)/P,$$  

(59)

where $f_B$ and $f_C$ are arbitrary analytic functions of the complex angular variable

$$Y = e^{i\phi} \tan(\theta/2),$$  

(60)

and the retarded-time

$$\tau = t - r - ia \cos \theta$$  

(61)

satisfies the relations $\tau_{2,4} = 0$, and $Y_{2,4} = 0$.

The energy-momentum tensor of the Goldstone field $\lambda$ can be expressed via the Grassmann contributions to tetrad (51) as follows

$$T_{ik} = i/2 e_{(ia} \delta e_{k)}.$$

(62)

One sees that the energy-momentum tensor of the Goldstone field $\lambda$ with $C = \tilde{B}$ cancels, $T_{ik} = 0$, and the field takes the ghost character. As it was mentioned earlier, in this case torsion and Grassmann contributions to tetrad are absent, and metric takes the exact Kerr-Newman form.

The main features of the resulting super-Kerr-Newman solutions are the extra wave fields on the bosonic Kerr-Newman background: the complex Rarita-Schwinger field $\chi_i$ and the nilpotent contributions to electromagnetic field given by the expressions (52) and (53).

One should note that the expressions for $B$ and $C$ are singular on the Kerr singular ring, $Z^{-1} \equiv P^{-1}(r + ia \cos \theta) = 0$, and contain traveling waves if there is an oscillating dependence on complex time parameter $\tau$. Indeed, near the Kerr singular ring $\tan \theta \simeq 1$, and angular dependence of these solutions on $\phi$ is determined by the degree of function $Y = e^{i\phi} \tan(\theta/2)$. One sees that any non-trivial analytic dependence on $Y$ will lead to a singularity
Thus, besides the Kerr singular ring the solutions contain an extra axial singularity which is coupled topologically with singular ring threading it. The singular Rarita-Schwinger ‘hair’ was mentioned earlier by Aichelburg and Güven [19, 25] as an obstacle to form a super black hole in the case of the Kerr geometry. However, when considering this solution as a model of spinning particle, we have to treat a region of parameters $a \gg m$ leading to a naked disk-like super-source [1, 3, 10] without horizons that allows to avoid the objections related to rotating black holes. However, even in the case of black hole, one can assume that this ‘axial’ singularity will not hinder to form horizon since it is build of the ghost fields.

One should also note that this ‘axial’ singularity in some solutions can change the position in time scanning the space-time. One simple example of such function is

$$C = \frac{Z}{P}[f_1(\tau) - Yf_2(\tau)]^{-1},$$

where the position of the ‘axial’ singularity is determined by the root of equation $Y = f_1/f_2$ depending on the retarded-time parameter $\tau$. \[11\]

Let us consider elementary fermionic wave excitations in the form

$$C = \bar{B} = \frac{Z}{PY^n}e^{i\omega\tau} = 1/(r + ia \cos \theta) \tan^n(\theta/2). \quad (63)$$

Taking into account the coordinate relation of the Kerr geometry

$$x + iy = (r + ia)e^{i\phi} \sin \theta, \quad z = r \cos \theta, \quad (64)$$

one sees that near the Kerr singular ring (where $r \simeq 0, \sin \theta \simeq 1$) at the point $\phi = 0$, singularity is directed along the $x$-axis. One can introduce the local coordinates $\tilde{z} = z$ and $\tilde{y} = y - a$ on the plane orthogonal to direction of singularity. Then, near the singularity $\tilde{r} \simeq \sqrt{2a(\tilde{y} + i\tilde{z})}$, and (63) takes the form

$$C \simeq \frac{1}{\sqrt{2a(\tilde{y} + i\tilde{z})}}e^{i(n\phi + \omega t)}, \quad (65)$$

that describes a traveling wave along the ring-like singularity parametrized by $\phi$. The first factor shows that this singularity is a branch line corresponding

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\[10\] Corresponding fermionic singular solutions were also considered in [26].

\[11\] Topological coupling of the axial singularity with the Kerr singular ring provides its guiding role for the ring that allows to consider a speculation that this couple of singularities could play the role of a topological wave-pilot construction in the spirit of the old de Broglie ideas.
to the known twofoldedness of the Kerr geometry. Because of that parameter $n$ can take both integer and half-integer values. The potential (53) takes near singular ring the form
\[
\delta g A \simeq (s/s/k)^{3/2} e^{i(2n\phi+2\omega t)}(d\bar{y} + id\bar{z}) + c.c.\text{term}.
\] (66)

The obtained from potential (53) electromagnetic field is
\[
F = -i(s s^{5/2}/k)\{C^2[r^{-1}dY \wedge dt + \bar{r}^{-1}(\bar{r}^{-1} - i\omega)e^1 \wedge e^3] - c.c.\text{term}\}. (67)
\]

The proportional to $e^1 \wedge e^3$ term represents the null electromagnetic field propagating along principal null direction $e^3$. Near the Kerr singular ring this congruence is tangent to ring leading to traveling waves propagating along the ring.

The ‘axial’ singularity coincides with z-axis and can be placed either at $\theta = 0 \ (Y = 0)$ or at $\theta = \pi \ (Y = \infty)$. It is a half-infinite line threading the Kerr singular ring and passing to ‘negative’ sheet of the Kerr geometry. Its position and character depend on the values of $n$. By introducing the distance from ‘axial’ singularity $\rho = \sqrt{x^2 + y^2}$, one can describe its behavior in the asymptotic region of large $r$ by the following expressions:
- if $\theta \simeq 0$ then
  \[
  \delta g A \sim \rho^{2n}r^{-3-2n}(dx + i dy),
  \]
  \[
  \delta F A \sim \rho^{2n}r^{-4-2n}(dz + dt),
  \]
- if $\theta \simeq \pi$ then
  \[
  \delta g A \sim \rho^{-2n}r^{2n-1}(dx - idy),
  \]
  \[
  \delta F A \sim \rho^{-2n}r^{2n-2}(dz - dt).
  \]

One sees that this singularity can be increasing or decreasing function of distance $r$. For some $n$ (for example $n=1/2, -3/2$) dependence on $r$ can disappear. The solutions with ‘increasing’ and ‘even’ singularities cannot be stable. In the cases $n = 0$ and $n = -1$ singularity represents a ‘decreasing’ half-infinite line like the string of the Dirac monopole. The case $n = -1/2$ is exclusive: there are two ‘decreasing’ singularities which are situated symmetrically at $\theta = 0$ and $\theta = \pi$. The space part of the null vector $e^3$ is tangent to axial singularity, and electromagnetic field (67) grows near this singularity and contains in asymptotic region the leading term in the form of the null traveling wave
\[
F \simeq -(s s^{5/2}/k)\{C^2\bar{r}^{-1}\omega e^1 \wedge e^3 + c.c.\text{term}\}. (68)
\]
7 Conclusion

We have shown that the problem of obtaining non-trivial super black hole solutions can be resolved in supergravity broken by Goldstone fermion. In case of the Kerr geometry it leads to a specific non-linear realization of supersymmetry which has to be adjusted with complex structure of the Kerr geometry.

We considered three families of the rotating and charged super black hole geometries representing supergeneralizations of the Kerr-Newman geometry generated by Goldstone fermions of various form. When supersymmetry is broken, the Goldstone fermion disappears and these super-geometries can acquire torsion and traveling waves of the Rarita-Schwinger and electromagnetic fields.

Among these geometries we have selected one exclusive family of the exact super-Kerr-Newman solutions to N=2 supergravity broken by Goldstone fermion aligned to principal null congruence of the Kerr geometry.

Peculiarities of these solutions were analyzed, in particular, the ghost character of Goldstone field, absence of torsion, the appearance of an extra ‘axial’ singular line which is coupled topologically to the Kerr singular ring, and traveling waves of the null electromagnetic field concentrated near the ‘axial’ and ring-like singularities.

The obtained exact solutions are based on the massless Goldstone field. At present stage of investigation our knowledge regarding the origin of the Goldstone fermion is very incomplete. Analyzing the Wess-Zumino model of super-QED [23] with spontaneously broken supersymmetry one can see that it leads to massless Goldstone fermions, at least in the region that is out of core. It takes the place if we are interested mainly in the stationary black hole solutions describing the region of massless fields.

However, as it was mentioned in [10, 11], for the known parameters of spinning particles the angular momentum is very high, regarding the mass parameter. In this case the black hole horizons disappear and a ‘hard core’ region, representing a superconducting disk-like source, has to be taken into account. Treatment of this region is extremely important and complicated problem, however, it is beyond of the frame of this paper. We should only note that investigations of such super-sources should be connected apparently with Seiberg-Witten theory [28] and with some other interesting models of broken supersymmetry together with gauge symmetry breaking [29, 30, 31].
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Appendix A. Some relations of DKS-formalism

The rotation coefficients can be obtained from the following independent forms

\[ \Gamma_{42} = -Ze^1 - Y_{,3} e^3; \]
\[ \Gamma_{12} + \Gamma_{34} = [h_{,4} + (\bar{Z} - Z)h]e^3; \]
\[ \Gamma_{31} = hZe^2 + (h_{,1} - h\bar{Y}_{,3})e^3; \] (69)

Taking into account the property of skew-symmetry, \( \Gamma_{ab} = -\Gamma_{ba} \), and relations to the complex conjugate forms, for example: \( \Gamma_{12} = \bar{\Gamma}_{21} \); \( \Gamma_{42} = \bar{\Gamma}_{41} \); \( \Gamma_{32} = \bar{\Gamma}_{31} \).

Basic relations:

\[ P = 2^{-1/2}(1 + Y\bar{Y}); \quad Y = e^{i\phi}\tan\theta; \] (70)
\[ Z_{,4} = -Z^2; \quad Z_{,4} = -\bar{Z}^2; \quad Z_{,2} = (\bar{Z} - Z)Y_{,3}; \] (71)
\[ Y_{,1} = Z; \quad Y_{,2} = Y_{,4} = 0; \quad \bar{Y}_{,2} = \bar{Z}; \quad \bar{Y}_{,1} = \bar{Y}_{,4} = 0; \] (72)
\[ Y_{,3} = -ZP\bar{Y}/P = -Z\bar{Z}^{-1}ZP_{,2}/P. \] (73)

Some other useful relations:

\[ P_Y P = \frac{1}{4}\sin^2\theta; \] (74)
\[ P_Y - r_1 = -ia(\cos\theta)_{,1} = -iaZ2^{1/2}P_Y/P^2; \] (75)
\[ (r_1 - P_Y)e^1 + (r_2 - P_Y)e^2 = ia2^{1/2}/P^2[P_Y d\bar{Y} - P_Y dY] = \]
\[ 4aP_Y P_Y/P^2 d\phi = a\sin^2\theta d\phi; \] (76)
\[
(r_1 - P_Y)(r_2 - P_{\bar{Y}}) = 2a^2 P_Y P_{\bar{Y}} ZZ/P^4;
= a^2 \sin^2 \theta/(r^2 + a^2 \cos^2 \theta); \tag{77}
\]
\[
dY \wedge d\tau = \tilde{r}^{-1} e^1 \wedge e^3. \tag{78}
\]

The complex radial coordinate \( \tilde{r} = r + ia \cos \theta = PZ^{-1} \).

Some tetrad derivatives
\[
Z_{,1} = (Z^3/P) F_{YY}'' + 2Z^2 P_Y / P, \tag{79}
\]
\[
\bar{Z}_{,2} = (\bar{Z}^3/P) \bar{F}_{YY}'' + 2\bar{Z}^2 \bar{P}_\bar{Y} / P, \tag{80}
\]
\[
(\ln \bar{Z}/P)_{,2} = \bar{Z}^2 A/P - Y_{,3} \bar{Z}/Z, \tag{81}
\]
\[
Z_{,3}/Z = -(Z_{,1}/Z) P_Y / P + hZ - ZP_{YY} / P + (Z - \bar{Z}) P_Y P_{\bar{Y}} / P^2 , \tag{82}
\]
\[
\tilde{r}_{,2} = P_Y, \quad \tilde{r}_{,4} = P. \tag{83}
\]

We should note that in definitions of DKS-paper the "in-going" congruence is used leading to the "advanced" time coordinate \( \tau = \tau_{\text{adv}} = t + \tilde{r}_{\text{adv}} = t + r + ia \cos \theta \). In this case the "out"-congruence takes place on the "negative" sheet of metric where \( r \leq 0 \). Using the redefinition \( \tilde{r} \to -\tilde{r} \) one can interchange the "positive" and "negative" sheets of the Kerr geometry that yields the retarded time \( \tau_{\text{ret}} = t - \tilde{r}_{\text{ret}} = t - r - ia \cos \theta \) for positive values of \( r \). Since this redefinition can be performed in the final expressions we retain the DKS-notations in this paper for the sake of convenience with the exceptions of the expressions for \( \tau \) when we would like to underline that it is the retarded-time coordinate.

### Appendix B. Aligned solutions of the Dirac equation

The Dirac spinor is represented in general form
\[
\Psi_D = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}. \tag{84}
\]

\footnote{The sign of \( \rho \) in the expression (7.9c) of DKS is apparently wrong.}
The supercovariant derivative in complex N=2 case has the form
\[
\tilde{\mathcal{D}}_i = \mathcal{D}_i + \mathcal{D}_{Ni} - (ik/2\sqrt{2})F_{ab}\Sigma^{ab}\gamma_i. \tag{85}
\]

The metric part of the supercovariant derivative is
\[
\mathcal{D}_i = \partial_i - \frac{1}{2}\Gamma_{ab}\Sigma^{ab}. \tag{86}
\]
For the Ricci rotation coefficients we use notations of DKS-paper \[21\]
\[
\omega_{iab} = -\Gamma_{abi}, \tag{89}
\]
and see Appendix A.

The Dirac equation
\[
\gamma^a\mathcal{D}_a\Psi_D = 0.
\]
A special class of $e^3$-aligned solutions satisfying the constraint $\gamma^i e^3_i \Psi_D = 0$, has the form
\[
\Psi_D = \begin{pmatrix} 0 \\ B \\ C \\ 0 \end{pmatrix}. \tag{87}
\]
For the $e^3$-aligned solutions of the Dirac equation the Kerr-Newman electromagnetic field drops out. In the nilpotent case \[32\] the nonlinear term drops out too. The spinor-valued 1-form expression $\mathcal{D}\Psi_D$ takes the form
\[
\mathcal{D}\Psi_D = \begin{pmatrix} -B(\bar{Z}e^2 + Y,3e^3) \\ dB + B[h,4 + (Z - \bar{Z})h/2]e^3 \\ dC + C[h,4 - (Z - \bar{Z})h/2]e^3 \\ C(Ze^1 + Y,3e^3) \end{pmatrix}. \tag{88}
\]
As a result, the Dirac equation $\gamma^a\mathcal{D}_a\Psi_D = 0$ yields the four equations:
\[
C,4 + ZC = 0; \quad C,2 - CY,3 = 0, \tag{89}
\]
and
\[
B,4 + \bar{Z}B = 0; \quad B,1 - B\bar{Y},3 = 0. \tag{90}
\]
These equations can be easily solved by using the known basis relations of the Kerr-Schild formalism (See Appendix A).

First equation \[89\] gives $C = ZC_0$ where $C_{0,4} = 0$. Then, substituting $C$ and into second equation we obtain $C = fZ/P$, where $f$ must satisfy the
conditions \( f_{,2} = f_{,4} = 0 \). Therefore, \( f \) must be a function taking constant values on the null planes spanned by vectors \( e^1 \) and \( e^3 \). These null planes are "left" null planes of a foliation of space-time into complex null cones. In another terminology they represent a geometrical image of twistors and can be parametrized by twistor coordinates. All the twistor coordinates \( \chi \) and \( \mu = x^i \sigma_i \chi \) satisfy the relations \((...)_{,2} = (...)_{,4} = 0\), see [6]. Coordinate \( Y \) is in fact one of the (projective) twistor coordinates. For our problem it will be convenient to use a retarded time coordinate \( \tau \). The retarded time coordinate to a point \( x \) is defined by a point of intersection of the light cone emanated from \( x \) with the world-line of source.

The light cone is split on the "left" and "right" null planes, therefore, the retarded time parameter takes constant values on the "left" null planes of the cone that leads to the relations \( \tau_{,2} = \tau_{,4} = 0 \). \(^{13}\) For the Kerr geometry one can use a known complex interpretation containing a complex world line for the source. \(^{14}\) In the rest frame the complex light cone equation \((t - \tau)^2 = \tilde{r}^2\) can be split with selection of the retarded fold \( \tilde{r} = t - \tau \). Here \( t \) is a real time coordinate, and \( \tilde{r} = r + ia \cos \theta = P/Z \) is a complex radial distance from the real point \( x \) to a point of source at the complex world line. It yields \( \tau = t - \tilde{r} \).

Therefore, the function \( f \) can be an arbitrary analytic function of complex coordinates \( Y \) and \( \tau \), and we have solution \( C = f(Y, \tau)Z/P \). For function \( B \) one can obtain similarly \( B = \tilde{f}(\tilde{Y}, \tilde{\tau})\tilde{Z}/P \), where coordinates \( \tilde{Y} \) and \( \tilde{\tau} \) satisfy the relations \((...)_{,1} = (...)_{,4} = 0\), and are constant on the "right" null planes of the light cone foliation.

**Appendix C. The Kerr tetrad and some useful matrix expressions**

The Kerr tetrad \( e^a \) is determined by function \( Y(x) \):

\[
\begin{align*}
e^1 &= d\zeta - Ydv, \\
e^2 &= d\bar{\zeta} - \bar{Y}dv, \\
e^3 &= du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv, \\
e^4 &= dv + he^3
\end{align*}
\]

\(^{13}\)Details of this construction can be found in [6, 4].

\(^{14}\)One should note that in the case of the complex world line the "left" and "right" roots for the retarded time \( \tau \) are different, and "right" root satisfies the relations \( \tau_{,1} = \tau_{,4} = 0 \).
where the null Cartesian coordinates are used $\sqrt{2}u = z + t$, $\sqrt{2}v = z - t$, $\sqrt{2}u = x + iy$, $\sqrt{2}v = x - iy$.

$\sigma$-matrices in the local Kerr geometry are

$$
\sigma_i = 2^{1/2} \begin{pmatrix} e_i^3 & e_i^2 \\ e_i^1 & -e_i^4 \end{pmatrix}; \quad \bar{\sigma}_i = 2^{1/2} \begin{pmatrix} -e_i^4 & -e_i^2 \\ -e_i^1 & e_i^3 \end{pmatrix}.
$$

(92)

The corresponding matrices with tetrad indices $a, b, c, d...$ will be given by

$$
\sigma^a = \sigma_i e^{ai} \text{ and take the form }
$$

$$
\sigma^a = 2^{1/2} \begin{pmatrix} \delta_i^a & \delta_i^a \\ \delta_i^2 & -\delta_i^3 \end{pmatrix}.
$$

(93)

$$
\Sigma^{ab} = \frac{1}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a) = \frac{1}{4} \begin{pmatrix} \sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a & 0 \\ 0 & \bar{\sigma}^a \sigma^b - \bar{\sigma}^b \sigma^a \end{pmatrix}.
$$

(94)

In particular

$$
\Sigma^{12} = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \quad \Sigma^{14} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ;
$$

(95)

$$
\Sigma^{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} ; \quad \Sigma^{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} ;
$$

$$
\Sigma^{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} ; \quad \Sigma^{34} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .
$$

Appendix D. Transformations of the Dirac spinors from the $e^3$-aligned basis to the basis of auxiliary Minkowski space

Transformations of the Dirac spinors from the $e^3$-aligned basis to the basis of auxiliary Minkowski space are given by matrices

$$
\begin{pmatrix} M^{-1} & 0 \\ 0 & M^\dagger \end{pmatrix} \begin{pmatrix} \phi_{al} \\ \bar{\chi}_{al} \end{pmatrix} = \begin{pmatrix} \phi_{Mink} \\ \bar{\chi}_{Mink} \end{pmatrix}.
$$

(96)
where

\[ M = \begin{pmatrix} 1 & \bar{Y} \\ 0 & 1 \end{pmatrix}, \]  
(97)

\[ M^{-1} = \begin{pmatrix} 1 & \bar{Y} \\ 0 & 1 \end{pmatrix}, \]  
(98)

\[ M^* = \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}, \]  
(99)

\[ M^\dagger = \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}. \]  
(100)
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