A FINITE FIELD ANALOGUE FOR APPELL SERIES $F_3$

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Abstract. In this paper we introduce a finite field analogue for the Appell series $F_3$ and give some reduction formulae and certain generating functions for this function over finite fields.

1. Introduction

Let $q$ be a power of a prime. $\mathbb{F}_q$ and $\hat{\mathbb{F}}_q^*$ are denoted as the finite field of $q$ elements and the group of multiplicative characters of $\mathbb{F}_q^*$ respectively. Then the domain of all characters $\chi$ of $\mathbb{F}_q^*$ can be extended to $\mathbb{F}_q$ by setting $\chi(0) = 0$ for all characters. Let $\overline{\chi}$ and $\varepsilon$ denote the inverse of $\chi$ and the trivial character respectively. For more details about characters, please see $\mathbb{F}_q$ and $\hat{\mathbb{F}}_q$, Chapter 8.

Greene in 1987 developed the theory of hypergeometric functions over finite fields and proved a number of transformation and summation identities for hypergeometric functions over finite fields which are analogues to those in the classical case (see $\mathbb{F}_q$ for the definition of the hypergeometric functions). Greene in $\mathbb{F}_q$ introduced the notation

$$2F_1\left(\begin{array}{c} A, B \\ C \end{array} \right| x\right)^G = \varepsilon(x) \frac{BC(-1)}{q} \sum_y B(y)BC(1-y)\overline{A}(1-xy)$$

for $A, B, C \in \hat{\mathbb{F}}_q$ and $x \in \mathbb{F}_q$ and defined the finite field analogue of the binomial coefficient as

$$\left(\begin{array}{c} A \\ B \end{array} \right)^G = \frac{B(-1)}{q} J(A, \overline{B}),$$

where $J(\chi, \lambda)$ is the Jacobi sum given by

$$J(\chi, \lambda) = \sum_u \chi(u)\lambda(1-u).$$

See $\mathbb{F}_q$, $\hat{\mathbb{F}}_q$, $\mathbb{F}_q$, $\mathbb{F}_q$, $\mathbb{F}_q$, $\mathbb{F}_q$, $\mathbb{F}_q$ for more information about the finite field analogue of the hypergeometric functions.

In this paper, for the sake of simplicity, we use the notation

$$\left(\begin{array}{c} A \\ B \end{array} \right) = q \left(\begin{array}{c} A \\ B \end{array} \right)^G = B(-1)J(A, \overline{B})$$

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and define the finite field analogue of the classic Gauss hypergeometric function as

\[ \genfrac{[}{]}{}{2}{1}{A, B}{C}{x} = q \cdot \genfrac{[}{]}{}{2}{1}{A, B}{C}{x}^G = \varepsilon(x)BC(-1) \sum_y B(y)BC(1 - y)\overline{A}(1 - xy). \]

Then

\[ \genfrac{[}{]}{}{2}{1}{A, B}{C}{x} = \frac{1}{q - 1} \sum_{\chi} \left( A\chi \overline{A}\chi \right) \left( B\chi \overline{C}\chi \right) \chi(x) \]

for any \( A, B, C \in \widehat{\mathbb{F}}_q \) and \( x \in \mathbb{F}_q \). Similarly, the finite field analogue of the generalized hypergeometric function for any \( A_0, A_1, \ldots, A_n, B_1, \ldots, B_n \in \widehat{\mathbb{F}}_q \) and \( x \in \mathbb{F}_q \) is defined by

\[ \genfrac{[}{]}{}{n+1}{F_n}{A_0, A_1, \ldots, A_n}{B_1, \ldots, B_n}{x} = \frac{1}{q - 1} \sum_{\chi} \left( A_0\chi \overline{A}\chi \right) \left( A_1\chi \overline{A}\chi \right) \cdots \left( A_n\chi \overline{A}\chi \right) \chi(x). \]

In our notations one of Greene’s theorems is as follows.

**Theorem 1.1.** (See [7, Theorem 4.9]) For any characters \( A, B, C \in \widehat{\mathbb{F}}_q \), we have

\[ \genfrac{[}{]}{}{2}{1}{A, B}{C}{1} = A(-1) \left( \frac{B}{AC} \right). \]

The results in the following proposition follows readily from some properties of Jacobi sums.

**Proposition 1.1.** (See [7, (2.6), (2.7), (2.8) and (2.13)]) If \( A, B \in \widehat{\mathbb{F}}_q \), then

\[ \left( \begin{array}{c} A \\ B \end{array} \right) = \left( \begin{array}{c} A \\ AB \end{array} \right), \]

\[ \left( \begin{array}{c} A \\ B \end{array} \right) = \left( \begin{array}{c} BA \\ B \end{array} \right) B(-1), \]

\[ \left( \begin{array}{c} A \\ B \end{array} \right) = \left( \begin{array}{c} B \\ A \end{array} \right) AB(-1), \]

\[ \left( \begin{array}{c} \varepsilon \\ A \end{array} \right) = -A(-1) + (q - 1)\delta(A). \]

where \( \delta(\chi) \) is a function on characters given by

\[ \delta(\chi) = \begin{cases} 1 & \text{if } \chi = \varepsilon \\ 0 & \text{otherwise} \end{cases}. \]

Among these interesting double hypergeometric functions in the field of hypergeometric functions, Appell’s four functions may be the most important functions. Three of them are
as follows.

\[
F_1(a; b, b'; c; x, y) = \sum_{m, n \geq 0} \frac{(a)_{m+n}(b)_{m}(b')_{n}}{m!n!(c)_{m+n}} x^m y^n, \ |x| < 1, \ |y| < 1,
\]

\[
F_2(a; b, b'; c, c'; x, y) = \sum_{m, n \geq 0} \frac{(a)_{m+n}(b)_{m}(b')_{n}}{m!n!(c)_{m}(c')_{n}} x^m y^n, \ |x| + |y| < 1,
\]

\[
F_3(a, a'; b, b'; c; x, y) = \sum_{m, n \geq 0} \frac{(a)_{m}(a')_{n}(b)_{m}(b')_{n}}{m!n!(c)_{m+n}} x^m y^n, \ |x| < 1, \ |y| < 1.
\]

See [1, 2, 4, 12] for more detailed material about Appell's functions.

Inspired by Greene’s work, Li et al in [10] gave a finite field analogue of the Appell series \(F_1\) and obtained some transformation and reduction formulas and the generating functions for the function over finite fields. In that paper, the finite field analogue of the Appell series \(F_1\) was given by

\[
F_1(A; B, B'; C; x, y) = \varepsilon(xy)AC(-1) \sum_u A(u) A(1-u)B(1-ux)B'(1-uy).
\]

Motivated by the work of Greene [7] and Li et al [10], the author et al in [8] introduced a finite field analogue of the Appell series \(F_2\) which is

\[
F_2(A; B, B'; C, C'; x, y) = \varepsilon(xy) BB' C C'(-1) \sum_{u, v} B(u) B'(v) BC(1-u)B'C'(1-v)A(1-ux-vy)
\]

and deduced certain transformation and reduction formulas and the generating functions for this function over finite fields.

In this paper we will give a finite field analogue for the Appell series \(F_3\). Since the Appell series \(F_3\) has the following double integral representation [2, Chapter IX]:

\[
F_3(a, a'; b, b'; c; x, y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}
\]

\[
\cdot \int \int u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}(1-ux-a(1-vy)-a'(1-vy)) du dv,
\]

where the double integral is taken over the triangle region \{\((u, v)| u \geq 0, v \geq 0, u + v \leq 1\}\,

we now give the finite field analogue of \(F_3\) in the form:

\[
F_3(A, A'; B, B'; C; x, y) = \varepsilon(xy) BB'(-1) \sum_{u, v} B(u) B'(v) C B B'(1-u-v)A(1-ux)A'(1-vy),
\]

where \(A, A', B, B', C \in \mathbb{F}_q\), \(x, y \in \mathbb{F}_q\) and each sum ranges over all the elements of \(\mathbb{F}_q\). In the above definition, the factor \(\frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}\) is dropped to obtain simpler results. We choose the factor \(\varepsilon(xy) \cdot BB'(-1)\) to get a better expression in terms of binomial coefficients.

From the definition of \(F_3(A, A'; B, B'; C; x, y)\) we know that

\[
F_3(A, A'; B, B'; C; x, y) = F_3(A', A; B', B; C; y, x)
\]

for any \(A, B, B', C, C' \in \mathbb{F}_q\) and \(x, y \in \mathbb{F}_q\).
The aim of this paper is to give several reduction formulas and certain generating functions for the Appell series $F_3$ over finite fields. The fact that the Appell series $F_3$ does not have a single integral representation but has a double one leads us to giving a finite field analogue for the Appell series $F_3$ which is more complicated than that for $F_1$. Consequently, the results on the reduction formulas and the generating functions for the Appell series $F_3$ over finite fields are also quite complicated.

We give two other expressions for $F_3(A, A'; B, B'; C; x, y)$ in the next section. Several reduction formulae for $F_3(A, A'; B, B'; C; x, y)$ is given in Section 3. The last section is devoted to deducing certain generating functions for $F_3(A, A'; B, B'; C; x, y)$.

2. Other expressions for $F_3(A, A'; B, B'; C; x, y)$

In this section we give two other expressions for $F_3(A, A'; B, B'; C; x, y)$.

**Theorem 2.1.** For any $A, A', B, B', C, \in \hat{\mathbb{F}_q}$ and $x, y \in \mathbb{F}_q$, we have

$$F_3(A, A'; B, B'; C; x, y) = \frac{1}{(q-1)^2} \sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( \frac{CBB'}{C{\chi}\lambda} \right) \chi(x)\lambda(y)$$

$$+ \varepsilon(y) B'(-1) \left( \frac{A}{B'C} \right) B'C(x)A'(1-y),$$

where each sum ranges over all multiplicative characters of $\mathbb{F}_q$.

In order to prove Theorem 2.1 we need an auxiliary result.

**Proposition 2.1.** (Binomial theorem, see [7, (2.5)]) For any character $A \in \hat{\mathbb{F}_q}$ and $x \in \mathbb{F}_q$, we have

$$A(1 + x) = \delta(x) + \frac{1}{q-1} \sum_{\chi} \left( \frac{A}{\chi} \right) \chi(x),$$

where the sum ranges over all multiplicative characters of $\mathbb{F}_q$ and $\delta(x)$ is a function on $\mathbb{F}_q$ given by

$$\delta(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x \neq 0 
\end{cases}.$$
It is easy to see from the binomial theorem that
\begin{align}
(2.2) & \quad \overline{A}(1 - ux) = \delta(ux) + \frac{1}{q-1} \sum_{\chi} \left( \overline{A} \right) \chi(-ux), \\
(2.3) & \quad \overline{A}(1 - vy) = \delta(vy) + \frac{1}{q-1} \sum_{\lambda} \left( \overline{A}' \right) \lambda(-vy).
\end{align}

Applying (2.1)–(2.3) and using that fact that \(\delta(u)B(u) = \delta(ux)B(u)\varepsilon(xy) = \delta(vy)B'(v)\varepsilon(xy) = 0\), [7, (1.15)] and (1.2) yield
\begin{align*}
\varepsilon(xy) & \sum_{u \in \mathbb{F}_q, v \neq 1} B(u)B'(v)\overline{CBB'}(1 - u - v)\overline{A}(1 - ux)\overline{A}(1 - vy) \\
& = \frac{1}{(q-1)^2} \sum_{u \in \mathbb{F}_q, v \neq 1} B(u)B'(v) \sum_{\mu} \left( \frac{CBB'}{\mu} \right) \mu(-u)CBB'\overline{\mu}(1 - v) \\
& \quad \cdot \sum_{\chi} \left( \overline{A} \right) \chi(-ux) \sum_{\lambda} \left( \overline{A}' \right) \lambda(-vy) \\
& = \frac{1}{(q-1)^2} \sum_{u \in \mathbb{F}_q, v \in \mathbb{F}_q} B(u)B'(v) \sum_{\mu} \left( \frac{CBB'}{\mu} \right) \mu(-u)CBB'\overline{\mu}(1 - v) \\
& \quad \cdot \sum_{\chi} \left( \overline{A} \right) \chi(-ux) \sum_{\lambda} \left( \overline{A}' \right) \lambda(-vy) \\
& = \frac{1}{(q-1)^2} \sum_{\chi} \left( \overline{A} \right) \chi(-x) \sum_{\lambda} \left( \overline{A}' \right) \lambda(-y) \sum_{\mu} \left( \frac{CBB'}{\mu} \right) \mu(-1) \\
& \quad \cdot \sum_{u \in \mathbb{F}_q} B\chi\mu(u) \sum_{v \in \mathbb{F}_q} B'(v)\lambda(v)CBB'\overline{\mu}(1 - v) \\
& = \frac{B(-1)}{(q-1)^2} \sum_{\chi} \left( \overline{A} \right) \chi(-x) \sum_{\lambda} \left( \overline{A}' \right) \lambda(-y) \sum_{v} B'(v)\lambda(v)CBB'\overline{\chi}(1 - v) \\
& = \frac{BB'(-1)}{(q-1)^2} \sum_{\chi, \lambda} \left( \overline{A} \right) \left( \overline{A}' \right) \left( \frac{CBB'}{C\chi} \right) \left( \frac{CBB'}{CB'\chi} \right) \chi(x)\lambda(y).
\end{align*}

On the other hand, by (2.2), the fact that \(\varepsilon(xy)\overline{CBB'}(u)\delta(ux) = 0\) and [7, (1.15)]
\begin{align*}
\varepsilon(xy) & \sum_{u \in \mathbb{F}_q, v = 1} B(u)B'(v)\overline{CBB'}(1 - u - v)\overline{A}(1 - ux)\overline{A}(1 - vy) \\
& = \varepsilon(xy)\overline{CBB'}(-1)\overline{A}(1 - y) \sum_{u \in \mathbb{F}_q} CBB'(u)\overline{A}(1 - ux) \\
& = \frac{\varepsilon(y)CBB'(-1)\overline{A}(1 - y)}{q-1} \sum_{\chi} \left( \overline{A} \right) \chi(-x) \sum_{u \in \mathbb{F}_q} CBB\chi(u),
\end{align*}
\[
F_3(A, A'; B, B'; C; x, y) = \varepsilon(xy) B'(-1) \left( \sum_{u \in \mathbb{F}_q, v \neq 1} \frac{A}{\chi} \frac{A'}{\lambda} \left( CBB'C \right) \left( C B' \chi \right) \chi(x) \lambda(y) \right) + \varepsilon(y) B'(-1) \left( \frac{\overline{A}}{B'C} B'(x) \overline{A'}(1 - y) \right).
\]

This completes the proof of Theorem 2.1.

Corollary 2.2. For any \( A, A', B, B', C, \in \hat{\mathbb{F}}_q \) and \( x, y \in \mathbb{F}_q \), we have

\begin{align*}
F_3(A, A'; B, B'; C, x, 1) &= B'C(-1)_{3} F_2 \left( A, B, \overline{A'B'C} \big| x \right), \\
F_3(A, A'; B, B'; C; 1, y) &= BC(-1)_{3} F_2 \left( A', B', \overline{A'B'C} \big| y \right).
\end{align*}

Proof. We first show (2.4). It follows from Theorem 2.1, (1.3) and (1.1) that

\begin{align*}
F_3(A, A'; B, B'; C; x, 1) &= \frac{1}{(q - 1)^2} \sum_{\chi, \lambda} \frac{A}{\chi} \frac{A'}{\lambda} \left( CBB'C \right) \left( C B' \chi \right) \chi(x) \lambda(1) \\
&= \frac{C(-1)}{(q - 1)^2} \sum_{\chi} \left( \frac{\overline{A}}{\chi} \right) \left( CBB'C \right) \chi(-x) \sum_{\lambda} \left( \frac{A'}{\lambda} \right) \left( B'C \right) \chi(1) \\
&= \frac{A'C(-1)}{q - 1} \sum_{\chi} \left( \frac{\overline{A}}{\chi} \right) \left( CBB'C \right) \chi(-x) \left( B'C \right) \chi(1) \\
&= \frac{B'C(-1)}{q - 1} \sum_{\chi} \left( \frac{A}{\chi} \right) \left( CBB'C \right) \chi(x) \left( B'C \right) \chi(1) \\
&= B'(-1)_{3} F_2 \left( A, B, \overline{A'B'C} \big| x \right),
\end{align*}

which proves (2.4). (2.5) follows readily from (2.4) and (1.6).
Theorem 2.3. For any $A, A', B, B', C, \in \hat{F}_q$ and $x, y \in F_q$, we have

$$F_3(A, A'; B, B'; C; x, y) = \frac{1}{(q - 1)^2} \sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( C_{\chi \lambda} \right) \left( \frac{BB'\chi\lambda}{B'\lambda} \right) \chi(x)\lambda(y)$$

$$+ \frac{B'(-1)}{q - 1} \sum_{\chi\lambda=BB'} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \chi(x)\lambda(-y),$$

where the sum in the second term of the right side ranges over the region $\chi, \lambda \in \widehat{F}_q, \chi\lambda = B'B'$.

Proof. It follows from the binomial theorem that

$$(2.6) \sum_{\lambda} \left( \frac{A'}{\lambda} \right) \lambda(-y) = (q - 1)\overline{A'}(1 - y) - \delta(y) = (q - 1)\varepsilon(y)\overline{A'}(1 - y).$$

According to [7, (2.15)], we have

$$\left( \frac{A}{B} \right) \left( \frac{C}{A} \right) = \left( \frac{C}{B} \right) \left( \frac{BB'}{AB} \right) - (q - 1)B(-1)\delta(A) + (q - 1)AB(-1)\delta(BC)$$

for any $A, B, C \in \hat{F}_q$. Then

$$\left( \frac{BB'}{C_{\chi \lambda}} \right) \left( \frac{BB'\chi\lambda}{C_{\chi \lambda}} \right) = \left( \frac{BB'}{C_{\chi \lambda}} \right) \left( \frac{B'B'\chi\lambda}{B'\lambda} \right) - (q - 1)C\chi\lambda(-1)\delta(CB')\chi$$

$$+ (q - 1)B'\chi(-1)\delta(BB'\chi\lambda).$$

Using the above identity in Theorem 2.1 and by (2.6), we get

$$F_3(A, A'; B, B'; C; x, y) = \frac{1}{(q - 1)^2} \sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( C_{\chi \lambda} \right) \left( \frac{BB'\chi\lambda}{B'\lambda} \right) \chi(x)\lambda(y)$$

$$- \frac{C(-1)B'\overline{C}(-x)}{q - 1} \sum_{\lambda} \left( \frac{A'}{\lambda} \right) \chi(-y)$$

$$+ \frac{B'(-1)}{q - 1} \sum_{\chi\lambda=BB'} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \chi(x)\lambda(-y)$$

$$+ \varepsilon(y)B'(-1) \left( \frac{A}{B'\overline{C}} \right) B'\overline{C}(x)\overline{A'}(1 - y)$$

$$= \frac{1}{(q - 1)^2} \sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( C_{\chi \lambda} \right) \left( \frac{BB'\chi\lambda}{B'\lambda} \right) \chi(x)\lambda(y)$$

$$+ \frac{B'(-1)}{q - 1} \sum_{\chi\lambda=BB'} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \chi(x)\lambda(-y),$$

which ends the proof of Theorem 2.3.

Actually, from Theorem 2.3 we can also deduce (1.6), (2.4) and (2.5).
3. Reduction Formulae

In this section we give some reduction formulae for \( F_3(A, A'; B, B'; C; x, y) \). In order to derive these formulae we need some auxiliary results.

**Proposition 3.1.** (See [7, Corollary 3.16 and Theorem 3.15]) For any \( A, B, C, D \in \mathbb{F}_q \) and \( x \in \mathbb{F}_q \), we have

\[
\begin{align*}
(3.1) & \quad _2F_1 \left( \frac{\varepsilon, B}{C} \left| x \right. \right) = \left( \frac{B}{C} \right) \varepsilon(x) - \overline{C}(x)B\overline{C}(1-x), \\
(3.2) & \quad _2F_1 \left( \frac{A, \varepsilon}{C} \left| x \right. \right) = \left( \frac{C}{A} \right) A(-1)\overline{C}(x)A\overline{C}(1-x) - C(-1)\varepsilon(x) \\
& \quad \quad \quad + (q - 1)A(-1)\delta(1-x)\delta(\overline{C}C), \\
(3.3) & \quad _2F_1 \left( \frac{A, B}{A} \left| x \right. \right) = \left( \frac{B}{A} \right) \varepsilon(x)\overline{B}(1-x) - \overline{A}(-x) \\
& \quad \quad \quad + (q - 1)A(-1)\delta(1-x)\delta(B), \\
(3.4) & \quad _3F_2 \left( \frac{A, B, C, D}{A, B} \left| x \right. \right) = \left( \frac{CD}{BD} \right) \_2F_1 \left( \frac{A, C}{D} \left| x \right. \right) - BD(-1)\overline{B}(x)\left( \frac{AB}{B} \right) \\
& \quad \quad \quad + (q - 1)BD(-1)\delta(\overline{CD})\varepsilon(x)\overline{A}(1-x).
\end{align*}
\]

From the definition of \( F_3(a, a'; b, b'; c; x, y) \) we know that

\[
F_3(a, 0; b, b'; c; x, y) = F_3(a, a'; b, 0; c; x, y) = \_2F_1 \left( \frac{a, b}{c} \left| x \right. \right), \\
F_3(0, a'; b, b'; c; x, y) = F_3(a, a'; 0, b'; c; x, y) = \_2F_1 \left( \frac{a, b'}{c'} \left| y \right. \right).
\]

We now give finite field analogues of the above identities.

**Theorem 3.1.** Let \( A, B, B', C, C' \in \mathbb{F}_q \) and \( x, y \in \mathbb{F}_q \). If \( y \neq 0 \), then

\[
\begin{align*}
(3.5) & \quad F_3(A, \varepsilon; B, B'; C; x, y) = CB'(-1)\left( \frac{\overline{B}}{B'} \right) \_2F_1 \left( \frac{A, B}{C} \left| x \right. \right) - \overline{C}B'(x)\delta(y - 1)B'(-1)\left( \frac{A}{\overline{C}B'} \right) \\
& \quad \quad \quad - B'(-1)\overline{C}(y)\overline{B}C(1-y)\_2F_1 \left( \frac{A, B}{\overline{C}B} \right) - \frac{x(1-y)}{y} \\
& \quad \quad \quad + (q - 1)\varepsilon(x)C(-1)\delta(\overline{B'C})\overline{A}(1-x); \\
\end{align*}
\]

if \( x \neq 0 \), then

\[
\begin{align*}
(3.6) & \quad F_3(\varepsilon, A'; B, B'; C; x, y) = CB(-1)\left( \frac{\overline{B}}{C} \right) \_2F_1 \left( \frac{A', B'}{C} \left| y \right. \right) - \overline{C}B(y)\delta(x - 1)B(-1)\left( \frac{A'}{\overline{C}B} \right) \\
& \quad \quad \quad - B(-1)\overline{C}(x)\overline{B}C(1-x)\_2F_1 \left( \frac{A', B'}{\overline{C}B} \right) - \frac{y(1-x)}{x} \\
& \quad \quad \quad + (q - 1)\varepsilon(y)C(-1)\delta(\overline{B'C})\overline{A'}(1-y).
\end{align*}
\]
Proof. We first prove (3.5). It is easily known from (3.1) and (3.4) that
\[
2F_1 \left( \frac{\varepsilon, B'}{C \chi} \middle| y \right) = \left( \frac{B'}{C \chi} \right) - \overline{C}(y) \overline{B} C \chi (1 - y),
\]
\[
3F_2 \left( \frac{A, C B'; B}{C, C B'} \middle| x \right) = \frac{B C}{B'} 2F_1 \left( \frac{A, B}{C \chi} \middle| x \right) - B'(-1) \overline{C} B'(x) \left( \frac{A C B'}{C B'} \right)
\]
\[
+ (q - 1) B'(-1) \delta(BC) \varepsilon(x) \overline{A}(1 - x).
\]
From Theorem 2.1, the above two identities and (1.3) we deduce that
\[
F_3(A, \varepsilon; B, B'; C; x, y) = \frac{C(-1)}{(q - 1)^2} \sum_x \chi \left( A \chi \right) \left( \frac{C B B'}{C B' \chi} \right) \chi(-x) \sum_x \chi \left( \lambda \right) \left( \frac{B' \lambda}{C \chi} \right) \lambda(y)
\]
\[
+ B'(-1) \left( A \overline{B} C \right) B' \overline{C}(x) \varepsilon(1 - y)
\]
\[
= \frac{C(-1)}{q - 1} \sum_x \chi \left( A \chi \right) \left( \frac{C B B'}{C B' \chi} \right) \chi(-x) 2F_1 \left( \varepsilon, B' \middle| C \chi \right) y
\]
\[
+ B'(-1) \left( A \overline{B} C \right) B' \overline{C}(x) \varepsilon(1 - y)
\]
\[
= \frac{C B'(-1)}{q - 1} \sum_x \chi \left( A \chi \right) \left( \frac{B \chi}{C B' \chi} \right) \left( \frac{C B' \chi}{C \chi} \right) \chi(x)
\]
\[
- \frac{B'(-1) \overline{C}(y) \overline{B} C (1 - y)}{q - 1} \sum_x \chi \left( A \chi \right) \left( \frac{B \chi}{C B' \chi} \right) \chi \left( -\frac{x(1 - y)}{y} \right)
\]
\[
+ B'(-1) \left( A \overline{B} C \right) B' \overline{C}(x) \varepsilon(1 - y)
\]
\[
= C B'(-1) 3F_2 \left( \frac{A, C B'; B}{C, C B'} \middle| x \right)
\]
\[
- B'(-1) \overline{C}(y) \overline{B} C (1 - y) 2F_1 \left( \frac{A, B}{C B'} \middle| -\frac{x(1 - y)}{y} \right)
\]
\[
+ B'(-1) \left( A \overline{B} C \right) B' \overline{C}(x) \varepsilon(1 - y)
\]
\[
= C B'(-1) \left( \frac{B \overline{C}}{B'} \right) 2F_1 \left( \frac{A, B}{C \chi} \middle| x \right) - C(-1) \overline{C} B'(x) \left( \frac{A C B'}{C B'} \right)
\]
\[
+ (q - 1) \varepsilon(x) C(-1) \delta(BC) \overline{A}(1 - x) + B'(-1) \left( A \overline{B} C \right) B' \overline{C}(x) \varepsilon(1 - y)
\]
\[
- B'(-1) \overline{C}(y) \overline{B} C (1 - y) 2F_1 \left( \frac{A, B}{C B'} \middle| -\frac{x(1 - y)}{y} \right),
\]
which is equivalent to (3.5). (3.6) follows from (3.5) and (1.6). This completes the proof of Theorem 3.1. \(\square\)
Theorem 3.2. Let $A, B, B', C, C' \in \mathbb{F}_q$ and $x, y \in \mathbb{F}_q$. If $y \neq 0$, then
\[
F_3(A, A'; B, \varepsilon; C; x, y) = -C(-1)_{2}^{1} F_1 \left( \begin{array}{c} A', B' \\ C \\ x \end{array} \right) + \left( \frac{A}{A'C} \right) \left( \frac{B'C}{A'} \right) A'C(x)\delta(1-y)
\]
\[
+ A'(-1)C(y)A'C(1-y) \frac{A'B'C}{A'} 2 \left( \begin{array}{c} A, B \\ C'A' \\ y \end{array} \right) - \frac{x(1-y)}{y}
\]
\[
+ (q-1)\varepsilon(x)C(y)A'C(1-y)\delta(A'B'C)A \left( 1 + \frac{x(1-y)}{y} \right) ;
\]
if $x \neq 0$, then
\[
F_3(A, A'; \varepsilon, B'; C; x, y) = -C(-1)_{2}^{1} F_1 \left( \begin{array}{c} A', B' \\ C \\ y \end{array} \right) + \left( \frac{A}{A'C} \right) \left( \frac{B'C}{A} \right) A'C(y)\delta(1-x)
\]
\[
+ A(-1)C(x)A'C(1-x) \frac{A'B'C}{A} 2 \left( \begin{array}{c} A', B' \\ C'A \\ x \end{array} \right) - \frac{y(1-x)}{x}
\]
\[
+ (q-1)\varepsilon(y)C(x)A'C(1-x)\delta(A'B'C)A \left( 1 + \frac{y(1-x)}{x} \right) .
\]
Proof. It follows from (3.2) and (3.3) that
\[
2 \left( \begin{array}{c} A', \varepsilon \\ C \chi \end{array} \right) \left. \begin{array}{c} y \\ \chi \end{array} \right) = \left( \frac{C\chi}{A'} \right) A'(-1)C(x)\delta(1-y) - C\chi(-1)
\]
\[
+ (q-1)A'(-1)\delta(1-y)\delta(A'C\chi),
\]
\[
3 \left( \begin{array}{c} A, C, B \\ C'A', C \\ x \end{array} \right) - \frac{x(1-y)}{y} = \left( \frac{A'B'C}{A'} \right) 2 \left( \begin{array}{c} A, B \\ C'A \\ x \end{array} \right) - \frac{x(1-y)}{y}
\]
\[
- A'(-1)C(y)C(x(1-y)) \left( \frac{A'C}{C} \right)
\]
\[
+ (q-1)A'(1-y)\delta(A'B'C)\varepsilon(x(1-y))A \left( 1 + \frac{x(1-y)}{y} \right) .
\]
We deduce from Theorem 2.1 the above identities and (1.3) that
\[
F_3(A, A'; B, \varepsilon; C; x, y) = \frac{1}{(q-1)^2} \sum_x \left( \frac{A_{\chi}}{\chi} \right) \left( \frac{B_{\chi}}{C_{\chi}} \right) \chi(-x) \sum_{\lambda} \left( \frac{A'_{\lambda}}{\lambda} \right) \left( \frac{\chi_{\lambda}}{C_{\lambda}} \right) \lambda(y)
\]
\[
+ \left( \frac{A}{C} \right) C(x)A'(1-y)
\]
\[
= \frac{1}{q-1} \sum_x \left( \frac{A_{\chi}}{\chi} \right) \left( \frac{B_{\chi}}{C_{\chi}} \right) \chi(-x) \left( \frac{A', \varepsilon}{C_{\chi}} \right) y
\]
\[
+ \left( \frac{A}{C} \right) C(x)A'(1-y)
\]
over finite fields.
For any which proves the first identity. The second identity follows from the first identity and (1.6). This concludes the proof of Theorem 3.2.

4. Generating functions

In this section, we establish some generating functions for $F_3(A, A'; B, B'; C; x, y)$. 

**Theorem 4.1.** For any $A, A', B, B', C \in \mathbb{F}_q$ and $x, t \in \mathbb{F}_q^* \setminus \{1\}$, $y \in \mathbb{F}_q$, we have

$$
\frac{1}{q-1} \sum_\theta \binom{A \theta}{C} F_3(A \theta, A'; B, B'; C; x, y) \theta(t)
= \overline{A}(1-t) F_3 \left( A, A'; B, B'; C; \frac{x}{1-t}, y \right)
- \overline{A}(-t) B(-1) \overline{C}(x) \overline{B} C(1-x) \, _2F_1 \left( A', B'; BC \mid -\frac{y(1-x)}{x} \right).
$$

**Proof.** It follows from (3.3) and (1.4) that

$$
_2F_1 \left( \frac{B' \lambda, BC \lambda}{B' \lambda} \mid x \right) = \left( \frac{BC \lambda}{B' \lambda} \right) \overline{B} C \lambda(1-x) - B' \lambda(-x)
= \left( \frac{B' \lambda}{BC \lambda} \right) \overline{B} B' C(-1) \overline{B} C \lambda(1-x) - B' \lambda(-x).
$$
Then, by \([1.3]\) and \([7, (2.11)\]}

\[
\sum_{\chi, \lambda} \left( \frac{A'}{\lambda} \right) \left( \frac{CBB'}{C\lambda} \right) \left( \frac{CB'\chi}{C\chi\lambda} \right) \chi(-x)\lambda(y) \\
= CB'(-1) \sum_{\lambda} \left( \frac{A'\lambda}{\lambda} \right) \lambda(-y) \sum_{\chi} \left( \frac{B\chi}{CB'\chi} \right) \left( \frac{CB'\chi}{C\chi\lambda} \right) \chi(x) \\
= CB'(-1)C(x) \sum_{\lambda} \left( \frac{A'\lambda}{\lambda} \right) \lambda \left( \frac{-y}{x} \right) \sum_{\chi} \left( \frac{BC\lambda\chi}{BC\chi\lambda} \right) \left( \frac{B'\lambda\chi}{B\lambda\chi} \right) \chi(x) \\
= (q - 1)CB'(-1)C(x) \sum_{\lambda} \left( \frac{A'\lambda}{\lambda} \right) \left( \frac{B'\lambda}{BC\lambda} \right) \lambda \left( \frac{-y(1 - x)}{x} \right) \\
- (q - 1)C(-x)B'(x) \sum_{\lambda} \left( \frac{A'\lambda}{\lambda} \right) \lambda(y) \\
= (q - 1)^2B(-1)C(x)BC(1 - x) \sum_{\lambda} \left( \frac{A'\lambda}{\lambda} \right) \left( \frac{B'\lambda}{BC\lambda} \right) \lambda \left( \frac{-y(1 - x)}{x} \right) \\
- (q - 1)^2C(-x)B'(x)\varepsilon(y)\overline{A}(1 - y),
\]

where in the second step we have used the substitution \(\chi \rightarrow \overline{C}\lambda\chi\). It is easily known from Theorem 2.1 that

\[
\sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A}{\lambda} \right) \left( \frac{CBB'}{C\lambda} \right) \left( \frac{CB'\chi}{C\chi\lambda} \right) \chi \left( \frac{x}{1 - t} \right) \lambda(y) \\
= (q - 1)^2 \left( F_3 \left( A, A'; B, B'; C; \frac{x}{1 - t}, y \right) - B'(-1)\varepsilon(y)B'BC(1 - t)\overline{A}(1 - y) \left( \frac{A}{B'C} \right) \right).
\]

It follows from \([3.3]\) that

\[
\sum_{\theta} \left( \frac{A\theta}{A} \right) \left( \frac{A\chi\theta}{A} \right) \theta(t) = (q - 1)_{2F1} \left( \frac{A, A\chi}{A} \right) t \\
= (q - 1)_{2F1} \left( \frac{A\chi}{A} \right) \overline{A}(1 - t) - (q - 1)_{2F1} \overline{A}(-t),
\]

\[
\sum_{\theta} \left( \frac{A\theta}{A} \right) \left( \frac{AB'C\theta}{A} \right) \theta(t) = (q - 1)_{2F1} \left( \frac{A, AB'C}{A} \right) t \\
= (q - 1)_{2F1} \left( \frac{AB'C}{A} \right) \overline{A}(1 - t) - (q - 1)_{2F1} \overline{A}(-t).
\]
By (1.2), (1.4) and (4.3), we have

\[
\sum_{\chi, \lambda} \left( \begin{array}{c} A' \\ \lambda \end{array} \right)_C \left( \begin{array}{c} CBB' \\ C\lambda \end{array} \right)x(-x)\lambda(y) \sum_{\theta} \left( \begin{array}{c} A\theta \\ \lambda \end{array} \right)_A \left( \begin{array}{c} A\chi\theta \\ C\lambda \end{array} \right)_A \theta(t)
\]

\[
= (q - 1)A(1 - t) \sum_{\chi, \lambda} \left( \begin{array}{c} A' \\ \lambda \end{array} \right)_C \left( \begin{array}{c} CBB' \\ C\lambda \end{array} \right)x(-x)\lambda(y) \sum_{\theta} \left( \begin{array}{c} A\theta \\ \lambda \end{array} \right)_A \left( \begin{array}{c} A\chi\theta \\ C\lambda \end{array} \right)_A \theta(t)
\]

\[
- (q - 1)A(-t) \sum_{\chi, \lambda} \left( \begin{array}{c} A' \\ \lambda \end{array} \right)_C \left( \begin{array}{c} CBB' \\ C\lambda \end{array} \right)x(-x)\lambda(y).
\]

Substituting (4.1) and (4.2) into (4.5), then applying Theorem 2.1, (1.2), (1.3), (4.4) and (4.5) in the left side and cancelling some terms gives

\[
\frac{1}{q - 1} \sum_{\theta} \left( \begin{array}{c} A\theta \\ \theta \end{array} \right)_F 3(A\theta, A'; B, B'; C; x, y)\theta(t)
\]

\[
= \frac{1}{(q - 1)^2} \sum_{\chi, \lambda} \left( \begin{array}{c} A' \\ \chi \end{array} \right)_C \left( \begin{array}{c} CBB' \\ C\chi \end{array} \right)x(-x)\lambda(y) \sum_{\theta} \left( \begin{array}{c} A\theta \\ \theta \end{array} \right)_A \left( \begin{array}{c} A\chi\theta \\ A\theta \end{array} \right)\theta(t)
\]

\[
+ \frac{1}{q - 1}e(y)C(-1)B'\bar{C}(x)A'(1 - y) \sum_{\theta} \left( \begin{array}{c} A\theta \\ \theta \end{array} \right)_A \left( \begin{array}{c} AB'C\theta \\ A\theta \end{array} \right)\theta(t)
\]

\[
= \overline{A}(1 - t)F_3 \left( A, A'; B, B'; C; \frac{x}{1 - t}, y \right)
\]

\[
- \overline{A}(-t)B(-1)\bar{C}(x)\overline{BC}(1 - x)F_1 \left( \frac{A'B'}{B'C} \mid \frac{x(1 - y)}{y} \right).
\]

which is exactly the right side. This finishes the proof of Theorem 4.1.

From Theorem 4.1 and (1.6) we can easily deduce another generating function for $F_3(A, A'; B, B'; C; x, y)$.

**Theorem 4.2.** For any $A, A', B, B', C \in \overline{F}_q$ and $y, t \in F_q \setminus \{1\}, \ x \in F_q$, we have

\[
\frac{1}{q - 1} \sum_{\theta} \left( \begin{array}{c} A\theta \\ \theta \end{array} \right)_F 3(A, A\theta; B, B'; C; x, y)\theta(t)
\]

\[
= \overline{A'}(1 - t)F_3 \left( A, A'; B, B'; C; x, \frac{y}{1 - t} \right)
\]

\[
- \overline{A'}(-t)B'(-1)\bar{C}(y)\overline{B'C}(1 - y)F_1 \left( \frac{A, B}{B'C} \mid \frac{-x(1 - y)}{y} \right).
\]

We also establish two other generating functions for $F_3(A, A'; B, B'; C; x, y)$.
Theorem 4.3. For any $A, A', B, B', C \in \mathbb{F}_q$, and $x, t \in \mathbb{F}_q \setminus \{1\}$, $y \in \mathbb{F}_q$, we have

$$\frac{1}{q-1} \sum_{\theta} \left( \frac{BB'C\theta}{\theta} \right) F_3(A, A'; B\theta, B', C; x, y) \theta(t)$$

$$= (1-t)F_3 \left( A, A'; B, B'; C; \frac{x}{1-t}, y \right) + \frac{B'B'C(-t)}{BB'} F_3(A, A'; CBB', B', C; x, y)$$

$$- \frac{B'(y)B(-t)C'B'(-\frac{x}{t})}{BB'} \left( \frac{A}{BB'} \right) \left( \frac{A'}{B'B} \right).$$

Proof. It is easily known from Theorem 2.1 that

$$F_3(A, A'; CBB', B'; C; x, y) = \frac{1}{(q-1)^2} \sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( \frac{\varepsilon}{CBB} \right) \left( \frac{\chi}{C\lambda} \right) \chi(x)\lambda(y)$$

$$+ \varepsilon(y)B'(-1) \left( \frac{A}{BB'} \right) B'C'(x)\overline{A}(1-y).$$

Then

$$\sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{\varepsilon}{B'B} \right) \lambda(y) = -B'(-1) \sum_{\lambda} \left( \frac{A'}{\lambda} \right) \lambda(y) + (q-1) \left( \frac{A}{B'} \right) BB'$$

$$= -(q-1)B'(-1)\varepsilon(y)\overline{A}(1-y) + (q-1) \left( \frac{A}{B'} \right) BB'.$$

Then, by (1.5),

$$\sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( \frac{\varepsilon}{CBB} \right) \left( \frac{\chi}{C\lambda} \right) \chi(x)\lambda(y)$$

$$= -CB'(-1) \sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( \frac{CBB}{C\lambda} \right) \chi(-x)\lambda(y) + (q-1) \overline{C}B'(x) \left( \frac{A}{BB'} \right) \sum_{\lambda} \left( \frac{A}{\lambda} \right) \left( \frac{\varepsilon}{B'} \right) \lambda(y)$$

$$= -CB'(-1) \sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( \frac{CBB}{C\lambda} \right) \chi(-x)\lambda(y) - (q-1)^2 B'(-1) \overline{C}B'(x) \left( \frac{A}{CBB} \right) \varepsilon(y)\overline{A}(1-y)$$

$$+ (q-1)^2 \overline{C}B'(x) \left( \frac{A}{CBB} \right) \left( \frac{A}{B'} \right) BB'.$$
From (4.6) and (4.7) we deduce that
\[
(4.8) \quad \sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( \frac{C B' \chi}{C \chi \lambda} \right) \chi(-x) \lambda(y) = (q - 1)^2 \left( C B'(-x) \left( \frac{A}{C B'} \right) \left( \frac{A'}{B'} \right) \overline{B}(y) - C B'(-1) F_3(A, A'; C B', B'; C; x, y) \right).
\]

It follows from (3.3) that
\[
\sum_{\theta} \left( \frac{B B' C \theta}{\theta} \right) \left( \frac{B x \theta}{B B' C \theta} \right) \theta(t) = (q - 1) F_1 \left( \frac{B B' C, B x}{B B' C} \right) t \quad \Rightarrow \quad (q - 1) \left( \frac{B x}{B B' C} \right) \overline{B}(1 - t) = (q - 1) B B' C(-t).
\]

Then
\[
(4.9) \quad \sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( \frac{C B' \chi}{C \chi \lambda} \right) \chi(-x) \lambda(y) \sum_{\theta} \left( \frac{B B' C \theta}{\theta} \right) \left( \frac{B x \theta}{B B' C \theta} \right) \theta(t) = (q - 1) C B'(-1) B(1 - t) \sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( \frac{C B' \chi}{C \chi \lambda} \right) \chi \left( \frac{x}{1 - t} \right) \lambda(y)
\]
\[
- (q - 1) B B' C(-t) \sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( \frac{C B' \chi}{C \chi \lambda} \right) \chi(-x) \lambda(y).
\]

Substituting (1.6) and (4.8) into (4.9), applying (1.2), (1.3), (4.9) and [7, (2.10)] in the left side and cancelling some terms yields
\[
\frac{1}{q - 1} \sum_{\theta} \left( \frac{B B' C \theta}{\theta} \right) F_3(A, A'; B \theta, B'; C; x, y) \theta(t)
\]
\[
= \frac{C B'(-1)}{(q - 1)^3} \sum_{\chi, \lambda} \left( \frac{A}{\chi} \right) \left( \frac{A'}{\lambda} \right) \left( \frac{C B' \chi}{C \chi \lambda} \right) \chi(-x) \lambda(y) \sum_{\theta} \left( \frac{B B' C \theta}{\theta} \right) \left( \frac{B x \theta}{B B' C \theta} \right) \theta(t)
\]
\[
+ \frac{1}{q - 1} \varepsilon(y) B'(-1) \left( \frac{A}{B' C} \right) B' C(x) \overline{A}(1 - y) \sum_{\theta} \left( \frac{B B' C \theta}{\theta} \right) \theta(t)
\]
\[
= \overline{B}(1 - t) F_3 \left( A, A'; B, B'; C; \frac{x}{1 - t}, y \right)
\]
\[
- \overline{B}(-t) B' C(t) \left( C B'(-x) \left( \frac{A}{C B'} \right) \left( \frac{A'}{B'} \right) \overline{B}(y) - C B'(-1) F_3(A, A'; C B', B'; C; x, y) \right),
\]
which equals the right side. The proof of Theorem 4.3 is completed.

From Theorem 4.3 and (1.6) we can also deduce another generating function for $F_3(A, A'; B, B'; C; x, y)$.
Theorem 4.4. For any \( A, A', B, B', C \in \widehat{\mathbb{F}}_q \) and \( y, t \in \mathbb{F}_q^* \setminus \{1\} \), \( x \in \mathbb{F}_q \), we have

\[
\frac{1}{q-1} \sum_{\theta} \begin{pmatrix} BB'C \theta \\ \theta \end{pmatrix} F_3(A, A'; B, B'; \theta; C; x, y) \theta(t)
= B'(1-t) F_3 \left( A, A'; B, B'; C; x, \frac{y}{1-t} \right) + \overline{BB'C}(-t) F_3(A, A'; B, \overline{CB}; C; x, y)
- \overline{B(y)B'(-t)CB} \left( -\frac{y}{t} \right) \begin{pmatrix} A' \\ \overline{CB} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.
\]

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