Half-quantum vortices in strongly correlated Bose liquids.

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Abstract

We discuss the structure of a vortex in a superfluid Bose liquid with a suppressed Bose-Einstein condensate and an intensive pair correlated condensate. The vortex represents the pair of half-quantum vortices topologically confined by the soliton.

In strongly interacting and strongly correlated Bose liquid – superfluid $^4$He – the density of the Bose-condensate is small compared to the total mass density of the liquid, $\rho_0 \ll \rho$. This fact leads to speculations that the rest part of the liquid can be described in terms of the Cooper-like pair-correlated condensate (the most recent discussion of this idea is in Ref. [1]; references to the older papers can be found in [2]). We would like to point out that if this idea is correct, it must have a pronounced effect on the core structure of the elementary vortex, which must be non-axisymmetric.

Let us accept that the Cooper-like pair condensate is dominating and gives the dominating contribution to the superfluid density. Then the phenomenological free energy describing interacting pair-
and Bose-condensates, which contains all the relevant physics, can be written as follows

\[ F = \frac{1}{2} \rho \nabla_s^2 + \epsilon(\rho) + F\{\Psi\}. \]  

(1)

Here \( \nabla_s \) is the superfluid velocity expressed in terms of the of pair condensate phase \( \phi \); and \( \rho \) is the total mass density of the liquid (we consider \( T \rightarrow 0 \)). The single-particle Bose-condensate produces the small correction which as we assume can be described in terms of the Ginzburg-Landau functional, the last term in Eq.(1). The Galilean invariant Ginzburg-Landau functional for the wave function of Bose condensate \( \Psi = |\Psi|e^{i\Phi} \) has the form

\[ F\{\Psi\} = \frac{\beta}{2\rho} \left(|\Psi|^2 - \rho_0\right)^2 + \frac{\hbar^2}{2m^2} \left(-i\nabla - m\nabla_s\right) |\Psi|^2 + \bar{\alpha}|\Psi|^2 \sin^2 \left(\Phi - \frac{\phi}{2}\right). \]  

(3)

Here \( \rho_0 \ll \rho \) is the density of the Bose condensate in equilibrium, which is much smaller that the total density. The last term in Eq.(3) is the Josephson interaction coupling the two condensates; it provides the phase coherence of the two condensates in equilibrium, when \( \Phi = \frac{\phi}{2} \) and thus the two condensates have the common superfluid velocity.

We are interested in the structure of the \( N = 1 \) vortex in this mixture of two condensates. Around the \( N = 1 \) vortex the phase of the Bose-condensate \( \Phi \) changes by \( 2\pi \), while the phase \( \phi \) of the pair-condensate wave-function changes by \( 4\pi \). In other words, from the point of view of the pair condensate, the elementary vortices of pair condensate have \( 2\pi \) winding of \( \phi \), and thus they have twice smaller elementary circulation \( \kappa_2 = \frac{1}{2}\kappa_0 \), where \( \kappa_0 = 2\pi\hbar/m \).

From the point of view of the Bose condensate they represent vortices with \( N = 1/2 \) (discussion of the half-quantum vortices – Alice strings – can be found in the book [3]). For nonzero but small \( \rho_0 \ll \rho \), the half-quantum vortices are combined into pairs forming \( N = 1 \) vortices (Fig. 1). Let us consider the structure of such a vortex molecule.

Introducing

\[ \tilde{\Phi} = \Phi - \frac{\phi}{2}, \quad \tilde{\Psi} = |\Psi|e^{i\tilde{\Phi}} \]  

(4)
Figure 1: Asymmetric vortex as pair of half-quantum vortices. (a) Vortex structure in case of strong pinning of the Bose-condensate phase $\Phi$ by the phase $\phi$ of the pair condensate. Half-quantum vortices of the pair condensate are confined by $\pi$-soliton of the Bose-condensate. Within the soliton the Bose-condensate order parameter crosses zero. (b) Weak-pinning case. The amplitude of the Bose-condensate order parameter has equilibrium value $\sqrt{\rho_0}$ everywhere except for the core of the $N = 1$ vortex in the Bose-condensate. In both Figures thin dashed lines terminating on half-quantum vortices are lines where the phase $\tilde{\Phi}$ has a $\pi$-jump required by the Aharonov-Bohm effect experienced by the Bose condensate in the presence of the half-quantum vortices of pair condensate.
one obtains
\[
F\{\tilde{\Psi}\} = \frac{\beta}{2\rho} \left( |\tilde{\Psi}|^2 - \rho_0 \right)^2 + \frac{1}{2m^2} \left| \nabla \tilde{\Psi} \right|^2 + \tilde{\alpha} |\tilde{\Psi}|^2 \sin^2 \tilde{\Phi}.
\] (5)

This equation is completely uncoupled from equation for $v_s$. However, in the presence of half-quantum vortices there is a topological connection due to the Aharonov-Bohm effect: for the Bose condensate, the half-quantum vortex in the pair condensate is viewed as the Aharonov-Bohm tube with the half-quantum magnetic flux. Since $\phi$ has $2\pi$ winding around each half-quantum vortex, the phase $\tilde{\Phi}$ must have a $\pi$-jump across some line terminating on a half-quantum vortex.

The structure of the whole system can be easily found in two extreme cases determined by the Josephson coupling $\tilde{\alpha}$.

If the Josephson coupling is big, the phase of the Bose-condensate $\Phi$ is strongly pinned by the phase $\phi$ of the pair condensate, and one has either $\tilde{\Phi} = 0$ or $\tilde{\Phi} = \pi$. In this case the $\pi$-jump is realized due to the $\pi$-soliton in Fig. 1(a). The Bose-condensate wave function can be represented as
\[
\tilde{\Psi}(x, y) = \sqrt{\rho_0} = (\Theta(-x - x_0) + \Theta(x - x_0)) \text{sign } y + \Theta(x + x_0) \Theta(x_0 - x) \tanh \frac{y}{\xi},
\] (6)

where $\Theta$ is the step function; $(x, y) = (x_0, 0)$ and $(x, y) = (-x_0, 0)$ are positions of half-quantum vortices, which are topologically confined by the $\pi$-soliton; and $\xi$ is the coherence length of the Bose-condensate. The phase $\tilde{\Phi}$ has $\pi$-jumps on the dashed lines terminating on half-quantum vortices in Fig. 1(a).

If the tension $\sigma$ of the $\pi$-soliton is known, the distance $R = 2x_0$ between the Alice strings in such a non-axisymmetric vortex can be easily found from the consideration of the $R$-dependent part of the energy of the vortex per unit length:
\[
U_{\text{vortex}}(R) = \sigma R - \frac{\pi \hbar^2}{2m^2} \rho \ln R.
\] (7)

Here the first term is the energy of confinement of half-quantum vortices due to the tension of the soliton, and the second term is the hydrodynamic repulsion of half-quantum vortices. Minimization gives
the equilibrium distance:

\[ R = \frac{\pi \hbar^2 \rho}{2 \sigma m^2}. \]  

(8)

For dilute Bose condensate, the coherence length of the Bose condensate is large compared with the coherence length of the pair condensate, \( \xi \gg a \); the coherence length of the pair condensate \( a \), which determines the core of the Alice string, is on the order of interatomic distance. Taking into account that the relative density of the Bose condensate \( \rho_0/\rho \sim a^2/\xi^2 \); the surface tension \( \sigma \sim \hbar^2/(ma\xi^3) \); and \( \rho \sim m/a^2 \), one obtains \( R/a \sim \xi^3/a^3 \sim (\rho/\rho_0)^{3/2} \gg 1 \).

In the other extreme case, when the pinning of the Bose-condensate phase is weak, it is more advantageous to fix the order parameter magnitude \( |\Psi| = \sqrt{\rho_0} \), and vary the phase \( \Phi \). In this configuration instead of \( \pi \)-soliton, the Bose condensate contains the fat soliton with the \( N = 1 \) vortex in the center (Fig. 1(b)). Within the fat soliton the phase of the condensate changes from \( \pm \pi/2 \) to 0 in the region of thickness \( \tilde{\xi} = \hbar/(m\tilde{\alpha}^{1/2}) \). The tension of the fat soliton, which now enters the confinement term in Eq. (7), is \( \sigma \sim \tilde{\xi}\tilde{\alpha}\rho_0 \sim \tilde{\alpha}^{1/2}\rho_0(h/m) \sim \hbar^2/(ma\xi^2\tilde{\xi}) \).

The weak-pinning regime occurs when \( \tilde{\xi} \gg \xi \), while the strong-pinning regime occurs when \( \tilde{\xi} \ll \xi \). In both regimes, and also in the intermediate regime when \( \tilde{\xi} \sim \xi \), the structure of the vortex core is highly anisotropic, and the core size considerably exceeds the interatomic distance \( a \).

In conclusion, if the idea that the dominating non-condensate particles form the pair-correlated state is valid for superfluid \(^4\)He, vortices in superfluid \(^4\)He must be highly anisotropic in the limit \( \rho_0/\rho \to 0 \).

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**References**

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