QCD as topologically ordered system.

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We argue that QCD belongs to a topologically ordered phase similar to many well-known condensed matter systems with a gap such as topological insulators or superconductors. Our arguments are based on analysis of the so-called “deformed QCD” which is a weakly coupled gauge theory, but nevertheless preserves all crucial elements of strongly interacting QCD, including confinement, nontrivial $\theta$ dependence, degeneracy of the topological sectors, etc. Specifically, we construct the so-called topological “BF” action which reproduces the well known infrared features of the theory such as non-dispersive contribution to the topological susceptibility which can not be associated with any propagating degrees of freedom. Furthermore, we interpret the well known resolution of the celebrated $U(1)_A$ problem when would be $\eta'$ Goldstone boson generates its mass as a result of mixing of the Goldstone field with a topological auxiliary field characterizing the system. We identify the non-propagating auxiliary topological field in BF formulation in deformed QCD with the Veneziano ghost (which plays the crucial role in resolution of the $U(1)_A$ problem). Finally, we elaborate on relation between “string-net” condensation in topologically ordered condensed matter systems and long range coherent configurations, the “skeletons”, studied in QCD lattice simulations.

I. INTRODUCTION AND MOTIVATION

The main motivation for the present studies is some recent lattice QCD results which can not be easily interpreted in terms of the conventional quantum field theory with a gap. To be more specific, the gauge configurations studied in [1–4] display a laminar structure in the vacuum consisting of extended, thin, coherent, locally low-dimensional sheets of topological charge embedded in 4d space, with opposite sign sheets interleaved. A similar structure has been also observed in QCD by different groups [5–10] and also in two dimensional $CP^{N-1}$ model [11]. A correlation length of the percolating objects is order of size of the system $\sim L$ while the width of these objects apparently vanish in the continuum limit. This is in drastic contrast with conventional expectation that in a gapped QCD in hadronic phase the fluctuations should have a typical scale $\sim \Lambda_{QCD}$ while the gauge invariant correlation functions must display a conventional exponentially weak sensitivity to the size of the system $\sim \exp(-\Lambda_{QCD} L)$. Furthermore, the studies of localization properties of Dirac eigenmodes have also shown evidence for the delocalization of low-lying modes on effectively low-dimensional surfaces.

It is important to emphasize that the observed long range configurations can not be identified with any propagating degrees of freedom such as physical gluons with transverse polarizations. In particular, these long range objects contribute to the contact term in topological susceptibility which has an on opposite sign in comparison with conventional contributions related to propagating physical states, see details below. In different words, lattice studies are consistent with non-dispersive nature of these long range objects.

It is not a goal of the present paper to cover this subject [1–11] with a number of subtle points which accompany it. Instead, we attempt to understand the observed long range structure on a deeper theoretical level using “deformed QCD” as a toy model [12, 13]. This is a simplified version of QCD which, on one hand, is a weakly coupled gauge theory wherein computations can be performed in theoretically controllable manner. On other hand, the corresponding deformation preserves all the relevant elements of strongly coupled QCD such as confinement, degeneracy of topological sectors, nontrivial $\theta$ dependence, and many other important aspects which allow us to test some fascinating features of strongly interacting QCD, including the long range order observed in lattice QCD simulations. It is claimed [12, 13] that there is no phase transition in passage from weakly coupled “deformed QCD” to strongly coupled real QCD. The ground state in “deformed QCD” is saturated by the fractionally charged weakly interacting pseudo-particles (monopoles) which live in 3 dimensions.

Precisely 3d feature of this model offers a new perspective on possible deep relation between “deformed QCD” and previously studied $(2 + 1)$ condensed matter (CM) systems with a gap which are known to lie in topologically ordered phases [14]. These CM systems include, but not limited to such systems as: quantum hall states (monopoles) which live in 3 dimensions.

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large gauge transformation, and therefore must be identified as they correspond to the same physical state. It is very different from conventional degeneracy in topologically ordered CM systems when distinct degenerate states are present in the system. Nevertheless, the systems in both cases demonstrate a huge sensitivity to the large distance physics formulated in terms of the boundary conditions, see few additional comments in [20] on analogies between topologically ordered CM systems and long range features in QCD. In what follows we discuss some manifestations of this long range order in QCD which may be considered as a complementary tool, along with well-established characteristics such as topological entanglement entropy [21, 22].

We start our presentation in section II by reviewing the relevant parts of the model [12, 13]. In section III we argue that the infrared description can be formulated in terms of Chern-Simons effective Lagrangian in BF form. We reproduce the expression for the topological susceptibility using this BF-type action. In section IV we present the physical interpretation of the obtained results by using analogy with CM systems. In section V we consider some profound consequences of our findings for high energy particle physics if topological order persists in a passage from weakly coupled “deformed QCD” to strongly coupled physical QCD. Section VI is our Conclusion where we propose to test very fascinating topological properties of the QCD quantum vacuum in a CM laboratory.

II. DEFORMED QCD

Here we overview the “center-stablized” deformed Yang-Mills developed in [12, 13]. In the deformed theory an extra term is put into the Lagrangian in order to prevent the center symmetry breaking that characterizes the QCD phase transition between “confined” hadronic matter and “deconfined” quark-gluon plasma. The nature of the gap in this model is reviewed in section II A, while in section II B we review the computation of the non-dispersive contact term in topological susceptibility [23]. This term will be our starting point in construction of the corresponding Chern Simons Lagrangian in section III.

A. The model

We start with pure Yang-Mills (gluodynamics) with gauge group SU(N) on the manifold \( R^3 \times S^1 \) with the standard action

\[
S^{YM} = \int_{R^3 \times S^1} d^4x \frac{1}{2g^2} \text{tr} \left[ F^2_{\mu\nu}(x) \right],
\]

and add to it a deformation action,

\[
\Delta S \equiv \int_{R^3} d^3x \frac{1}{L^3} P[\Omega(x)],
\]

built out of the Wilson loop (Polyakov loop) wrapping the compact dimension

\[
\Omega(x) \equiv \mathcal{P} \left[ e^{i \int \theta(x) A_4(x)} \right].
\]

Parameter \( L \) here is the length of the compactified dimension which is assumed to be small. The coefficients of the polynomial \( P[\Omega(x)] \) can be suitably chosen such that the deformation potential (2) forces unbroken symmetry at any compactification scales. At small compactification \( L \) the gauge coupling is small so that the semiclassical computations are under complete theoretical control [12, 13].

As described in [12, 13], the proper infrared description of the theory is a dilute gas of \( N \) types of monopoles, characterized by their magnetic charges, which are proportional to the simple roots and affine root \( \alpha_a \in \Delta_{aff} \) of the Lie algebra for the gauge group \( U(1)^N \). For a fundamental monopole with magnetic charge \( \alpha_a \in \Delta_{aff} \), the topological charge is given by

\[
Q = \int_{R^3 \times S^1} d^4x \frac{1}{16\pi^2} \text{tr} \left[ F_{\mu\nu} \tilde{F}^{\mu\nu} \right] = \pm \frac{1}{N},
\]

and the Yang-Mills action is given by

\[
S_{YM} = \int_{R^3 \times S^1} d^4x \frac{1}{2g^2} \text{tr} \left[ F^2_{\mu\nu} \right] = \frac{8\pi^2}{g^2} |Q|.
\]

The \( \theta \)-parameter in the Yang-Mills action can be included in conventional way,

\[
S_{YM} \rightarrow S_{YM} + i\theta \int_{R^3 \times S^1} d^4x \frac{1}{16\pi^2} \text{tr} \left[ F_{\mu\nu} \tilde{F}^{\mu\nu} \right],
\]

with \( \tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \).

The system of interacting monopoles, including \( \theta \) parameter, can be represented in the dual sine-Gordon form as follows [12, 13, 23],

\[
S_{dual} = \int_{R^3} d^3x \left( \frac{\theta}{2\pi} \right)^2 (\nabla \sigma)^2 - \zeta \int_{R^3} d^3x \sum_{a=1}^{N} \cos \left( \alpha_a \cdot \sigma + \frac{\theta}{N} \right),
\]

where \( \zeta \) is magnetic monopole fugacity which can be explicitly computed in this model using conventional semiclassical approximation. The \( \theta \)-parameter enters the effective Lagrangian (7) as \( \theta/N \) which is the direct consequence of the fractional topological charges of the monopoles (4). Nevertheless, the theory is still \( 2\pi \) periodic. This \( 2\pi \) periodicity of the theory is restored not due to the \( 2\pi \) periodicity of Lagrangian (7). Rather, it is restored as a result of summation over all branches of the theory when the levels cross at \( \theta = \pi (mod \ 2\pi) \) and one branch replaces another and becomes the lowest energy state as discussed in [23].

Finally, the dimensional parameter which governs the dynamics of the problem is the Debye correlation length of the monopole’s gas,

\[
\eta_\sigma^2 \equiv L\zeta \left( \frac{4\pi}{g} \right)^2.
\]
The average number of monopoles in a “Debye volume” is given by

\[ N \equiv m_π^3 ζ = \left( \frac{g}{4\pi} \right)^3 \frac{1}{L^3 ζ} \gg 1, \]  

(9)

The last inequality holds since the monopole fugacity is exponentially suppressed, \( ζ \sim e^{-1/g^2} \), and in fact we can view (9) as a constraint on the validity of the approximation where semiclassical approximation is justified.

**B. Topological susceptibility**

The topological susceptibility \( χ \) which plays a crucial role in resolution of the \( U(1)_A \) problem [24-29] and is defined as follows \(^1\)

\[ χ(θ = 0) = \left. \frac{∂^2 E_{vac}(θ)}{∂θ^2} \right|_{θ = 0} \]

\[ = \lim_{k → 0} \int d^4 x e^{ikx} \langle T\{q(x), q(0)\} \rangle, \]

where \( θ \) is the parameter which enters the Lagrangian (6) along with topological density operator \( q(x) \) and \( E_{vac}(θ) \) is the vacuum energy density determined by (7).

It is important that the topological susceptibility \( χ \) does not vanish in spite of the fact that \( q = \partial_μ K^μ \) is total divergence. Furthermore, any physical state gives a negative contribution to this diagonal correlation function

\[ χ_{\text{dispersive}} \sim \lim_{k → 0} \int d^4 x e^{ikx} \langle T\{q(x), q(0)\} \rangle \]

\[ \sim \lim_{k → 0} \sum_n \frac{⟨0|q|n⟩⟨n|q|0⟩}{−k^2 − m_n^2} \sim − \sum_n \frac{|c_n|^2}{m_n^2} < 0, \]

where \( m_n \) is the mass of a physical state, \( k → 0 \) is its momentum, and \( ⟨0|q|n⟩ = c_n \) is its coupling to topological density operator \( q(x) \). At the same time the resolution of the \( U(1)_A \) problem requires a positive sign for the topological susceptibility (12), see the original reference [26] for a thorough discussion,

\[ χ_{\text{non-dispersive}} = \lim_{k → 0} \int d^4 x e^{ikx} \langle T\{q(x), q(0)\} \rangle > 0. \]  

(12)

Therefore, there must be a contact contribution to \( χ \), which is not related to any propagating physical degrees of freedom, and it must have the “wrong” sign. The “wrong” sign in this paper implies a sign which is opposite to any contributions related to the physical propagating degrees of freedom (11). In the framework [24] the contact term with “wrong” sign has been simply postulated, while in refs. [25, 26] the Veneziano ghost had been introduced into the theory to saturate the required property (12). Furthermore, as we discuss below the contact term has in fact the structure \( χ \sim \int d^4 x δ^4(x) \). The significance of this structure is that the gauge variant correlation function in momentum space

\[ \lim_{k → 0} \int d^4 x e^{ikx} \langle K_μ(x), K_ν(0) \rangle \sim \frac{k_μ k_ν}{k^4} \]  

(13)

develops a topologically protected “unphysical” pole. Furthermore, the residue of this pole has a “wrong sign”, which precisely corresponds to the Veneziano ghost contribution saturating the non-dispersive term in gauge invariant correlation function (12),

\[ ⟨q(x)q(0)⟩ \sim ⟨∂_μ K^μ(x), ∂_ν K^ν(0)⟩ \sim δ^4(x) \]  

(14)

The singular behaviour \( ⟨q(x)q(0)⟩ \) with “wrong sign” has been well confirmed by the lattice simulations [3, 5-7, 30].

The topological susceptibility in “deformed QCD" model can be explicitly computed as this model is a weakly coupled gauge theory. In this model it is saturated by fractionally charged weakly interacting monopoles, and it is given by [23]

\[ χ_{YM} = \int d^4 x ⟨q(x), q(0)⟩ = \frac{ζ}{N_L} \int d^4 x [δ(x)] \]  

(15)

It has the required “wrong sign” as this this contribution is not related to any physical propagating degrees of freedom, and it has a \( δ(x) \) function structure which implies the presence of the pole (13). However, there are no any physical massless states in the system, and computations [23] leading to (15) are accomplished without any ghosts or any other unphysical degrees of freedom. Instead, this term is described in terms of the tunnelling events between different (but physically equivalent) topological sectors in the system.

One should emphasize that the \( δ(x) \) function which appears in the expression for topological susceptibility (15) is not an artifact of small size monopole-approximation used in [23]. Instead, this singular behaviour is a generic feature which is shared by many other models, including exactly solvable two dimensional Schwinger model and QCD when the contact term is saturated by the Veneziano ghost. In fact, this singular behaviour is measured in the QCD lattice simulations at strong coupling [3, 5-7, 30] as we already mentioned. One should emphasize again that the significance of this structure is that the gauge variant correlation function in momentum space develops a topologically protected “unphysical” pole similar to eq. (13) which will play an important role in our discussions in section III when this pole will be identified with unphysical topological fields in BF-formulation of the theory.

The \( δ(x) \) function in (15) should be understood as total divergence related to the infrared (IR) physics, rather
than to ultraviolet (UV) behaviour as explained in Ref. [23]

\[
\chi_{YM} \sim \int \delta(x) d^3x
\]

\[
= \int d^3x \partial_\mu \left( \frac{x^\mu}{4\pi x^2} \right) = \oint_{S_2} d\Sigma_\mu \left( \frac{x^\mu}{4\pi x^2} \right). \quad (16)
\]

In different words, the non-dispersive contact term with “wrong” sign (15) is highly sensitive to the boundary conditions and behaviour of the system at arbitrary large distances. Therefore, it is natural to expect that a variation of the boundary conditions would change the topological susceptibility (15) despite of the fact that the system has a gap (8). We will reproduce the \( \delta(x) \) function in (15) in terms of the topological quantum field theory for deformed QCD constructed in the next section III B. This will further illuminate the IR nature of the contact term.

The light quarks can be easily inserted into the system. The corresponding generalization of eq. (15) reads [23]

\[
\chi_{QCD} = \int d^4x(q(x)q(0)) \quad (17)
\]

\[
= \frac{\zeta}{NL} \int d^3x \left[ \delta(x) - m_q \frac{e^{-m_q x^2}}{4\pi^2} \right].
\]

The first term in this equation has non-dispersive nature and has the positive sign. As explained above this contact term is not related to any physical propagating degrees of freedom. Instead, it emerges as a result of the tunnelling transitions between the degenerate topological sectors ². The positive sign of this term is the crucial element for the resolution of the \( U(1)_A \) problem. The second term in eq. (17) is a standard dispersive contribution, can be restored through the absorptive part using conventional dispersion relations, and has a negative sign in accordance with general principles (11). This conventional physical contribution is saturated in this model by the lightest \( \eta' \) field. It enters \( \chi_{QCD} \) precisely in such a way that the Ward Identity (WI) expressed as \( \chi_{QCD}(m_q = 0) = 0 \) is automatically satisfied as a result of cancellation between the two terms. If the contact non-dispersive term with “wrong sign” was not present in the system, the WI could not be satisfied as physical states always contribute with negative sign in eqs. (11,17).

### III. BF THEORY FOR DEFORMED QCD

In many respects the deformed QCD which is an effective 3d theory defined by the action (7) is very similar to the abelian Higgs model describing superconductivity. The Higgs theory is known to belong to the topologically ordered phase [17]. Therefore, our goal here is to provide some arguments suggesting that these two models are in fact behave very similarly in the infrared. This similarity suggests that the deformed QCD also lies in a topological phase. Separately, it has been claimed [12, 13] that the passage from the deformed QCD to strongly coupled QCD is smooth, without any phase transitions. If true, it would imply that QCD also lies in a topologically ordered phase.

#### A. BF theory for the deformed QCD

**Construction.**

The action (7) describes the theory with a gap, similar to the conventional Landau-Ginsburg effective action describing the abelian Higgs model. In both cases the crucial part related to the topological order is missing in these effective actions. We want to reconstruct this missing topological term. We follow [17] in this derivation. As our 3d model (7) has the Euclidean metric, we proceed with the Euclidean path integral computations. It is different from physical case of the Minkowski 2+1 BF theory discussed in [17] when one can discuss the braiding phases of quasiparticles, the Hamiltonian formulation, etc. Nevertheless, the crucial points can be explained using the Euclidean path integral approach.

We wish to derive the topological action for deformed QCD by using the same technique exploited in ref. [17] for the Higgs model. We start with construction of the source term for a configuration of \( M^{(a)} \) point-like magnetic monopoles coupled to \( \sigma \) field entering the low energy action (7)

\[
S_f = -\beta \int d^3x J(x)\sigma
\]

\[
J(x) = \frac{i}{\beta} \sum_{a=1}^{N} \sum_{k=1}^{M^{(a)}} Q_k^{(a)} \delta(x(x_k^{(a)} - x) \alpha_a,
\]

where \( Q_k^{(a)} = \pm 1 \), and \( \alpha_a \in \Delta_{aff} \) is the affine root of the Lie algebra for the gauge group \( U(1)^N \) and beta is defined as

\[
\beta \equiv \frac{1}{L} \left( \frac{g}{2\pi} \right)^2.
\]

It has been demonstrated in [12, 13] that summation over all possible positions and orientations of the monopoles leads to the low energy action (7), see also Ref. [23] with some technical details.

The \( \sigma \) field plays the role of the scalar magnetic potential as one can see from expression for the \( U(1)^N \) mag-

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² In the context of this paper the “degeneracy” implies the existence of winding states \( |n\rangle \) constructed as follows: \( \mathcal{T}|n\rangle = |n+1\rangle \). In this formula the operator \( \mathcal{T} \) is the large gauge transformation operator which commutes with the Hamiltonian \( [\mathcal{T}, H] = 0 \). The physical vacuum state is unique and constructed as a superposition of \( |n\rangle \) states. In QFT approach the presence of \( n \) different sectors in the system is reflected by summation over \( n \in \mathbb{Z} \) in definition of the path integral in QCD. It should not be confused with conventional term “degeneracy” when two or more physically distinct states are present in the system.
netic field $B_i = \epsilon^{ijk} F_{jk}/2g$

$$F^{(a)}_{ij} = \frac{g^2}{2\pi L} \epsilon^{ijk} K^{(a)} \sigma^{(a)} = \frac{g}{2\pi L} \nabla \sigma^{(a)}. \ (20)$$

Essentially, the $\sigma^{(a)}$ fields are the physical photons in effectively three dimensional space. They become massive as a result of the Debye screening which eventually determines their masses (8). These fields are dynamical fields, and the corresponding Maxwell term $\frac{1}{2g} (B^{(a)} \cdot B^{(a)})$ is expressed in eq. (7) in terms of the $\sigma^{(a)}$ fields.

We now turn to analysis of the topological density operator $\rho(x)$ computed for background monopole’s configurations. It can be represented in terms of the effective scalar $\sigma$ field as follows

$$\rho(x) = \frac{1}{16\pi^2} \text{tr} \left[ F_{\mu\nu} \tilde{F}^{\mu\nu} \right] = \frac{1}{8\pi^2} \epsilon^{ijk} N \int_{a=1} F^{(a)}_{jk} (x)$$

$$= \frac{1}{8\pi^2} \sum_{a=1}^N \left\langle A^{(a)}_4 \right\rangle \left[ \epsilon^{ijk} \partial_i F^{(a)}_{jk} (x) \right]$$

$$= \left( \frac{g}{2\pi} \right)^2 \left( \frac{1}{2\pi L} \right) \sum_{a=1}^N \left\langle A^{(a)}_4 \right\rangle \nabla^2 \sigma^{(a)}$$

$$= \frac{1}{LN} \sum_{a=1}^N \sum_{k=1}^{M^{(a)}} Q^{(a)}_k \delta (t^{(a)}_k - x). \ (21)$$

The vacuum expectation value of $A^{(a)}_4$ equals $2\mu^a \pi/NL$, where $\mu^a \alpha_b = \delta^a_b$ and it plays the role of the Higgs field in this model as explained in [12, 13]. One can explicitly see that the topological charge in formula (21) for a single monopole or antimonopole is properly normalized $\int d^4 x = \pm 1/N$. Now we introduce truly singlet abelian field $f_{jk}(x) \equiv \sum_{a=1}^N \left\langle A^{(a)}_4 \right\rangle \left[ \epsilon^{ijk} \right] (x)$ such that the topological density operator $\rho(x)$ for background monopoles is expressed in terms of this new field as follows,

$$\rho(x) = \frac{1}{16\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} = \frac{1}{4\pi NL} \left[ \epsilon^{ijk} \partial_i f_{jk}(x) \right]. \ (22)$$

One should emphasize that the expression on the right hand side of eq. (22) does not represent all properties of the topological density operator, e.g. it does not include all non-abelian fluctuations which are present in the system. Rather it should be treated as a classical background long distance behaviour of $\rho(x)$. Further to this point: this new abelian field $f_{jk}(x)$ is not an abelian projection describing the abelian $U(1)^N$ magnetic monopoles in this model. Rather, the combination $\epsilon^{ijk} \partial_i f_{jk}(x)$ describes the gauge invariant topological density distribution (22). Field $f_{jk}(x)$ itself is not a gauge invariant object under generic gauge transformations. Indeed, from relation $q(x) = \partial_a K^a(x)$ one can infer that $\epsilon^{ijk} f_{jk}(x)$ transforms like $K^i(x)$. Transformation properties of $K^a(x)$ are well known: though this object does not carry the colour indices, it is not a gauge invariant object. In fact, $K^a(x)$ transforms in a nontrivial way under the large gauge transformations. In particular $\int d^4x K^a(x)$ determines the winding number of a “degenerate” vacuum state. For our specific case of deformed QCD we need to know the transformation properties for the following operators,

$$\left[ \epsilon^{ijk} \partial_i f_{jk}(x) \right] \text{transforms like } q(x) \ \text{(i.e.invariant)}$$

$$\left[ \epsilon^{ijk} f_{jk}(x) \right] \text{transforms like } K^i(x). \ (23)$$

Furthermore, $f_{jk}(x)$ field does not discriminate different types of monopoles and anti-monopoles which are classified by the affine root of the Lie algebra $\alpha_a \in \Delta_{aff}$. Instead, this new field is sensitive exclusively to the topological charge density of these objects $Q = \int d^4 x q(x) = \pm 1/N$, not to their abelian magnetic charges $\sim \alpha_a$. As a final comment: there is no Maxwell dynamical term for this field. This is in drastic contrast with dynamical Maxwell term $(\nabla \sigma)^2$ describing the abelian magnetic components in the effective action (7). There is no mystery here: as we shall discuss below the dynamics of $f_{jk}(x)$ is governed by pure topological field theory with no Maxwell counterpart.

One can define the gauge potential $a_i(x)$ in association with tensor field $f_{jk}(x)$ introduced above and new scalar potential $a(x)$ describing the divergent portion of this tensor as follows:

$$f_{jk}(x) = \left( \partial_j a_i(x) - \partial_i a_j(x) \right) - \frac{1}{2} \epsilon^{ijk} \partial^i a(x). \ (24)$$

The $a_i(x), a(x)$ potentials in eq.(24) are not directly related to the abelian magnetic potentials and effective $\sigma$ fields discussed above. The fractional topological charge of the monopoles can be expressed in terms of $a(x)$ potential as follows,

$$Q = \int_{\mathbb{R}^3 \times S^1} \frac{d^4 x q(x)}{4\pi N} = \frac{1}{4\pi N} \int_{\mathbb{R}^3} d^4 x \left[ \epsilon^{ijk} \partial_i f_{jk}(x) \right]$$

$$= \frac{1}{4\pi N} \int_{\mathbb{R}^3} d^3 x \nabla^2 a(x) = \frac{1}{4\pi N} \int_{\Sigma} d^3 \Sigma \cdot \nabla a(x), \ (25)$$

\footnote{We believe a short historical detour on fractionalization of the topological charge in QFT is warranted here. In given context the fractional topological objects appear in 2 dimensional $CP^{N-1}$ model [31] which were coined as instanton quarks (instanton partons). These quantum objects carry fractional topological charge $Q = \pm 1/N$, and they are very similar to our monopoles in deformed QCD discussed in section II A. These objects do not appear individually in path integral; instead, they appear as configurations consisting $N$ different $1/N$ objects such that the total topological charge of each configuration is integer. Nevertheless, these objects are highly delocalized: they may emerge on opposite sides of the space time or be close to each other with similar probabilities. Later on, similar objects have been discussed in a number of papers in different context [32], [33], [34], [35], [36], [37]. In particular, it has been argued that the well-established $\theta/N$ dependence in strongly coupled QCD unambiguously implies that the relevant configurations in QCD must carry the fractional topological charges, see review preprint [34] and references therein. The weakly coupled deformed QCD model [12, 13] considered in this paper is a precise dynamical realization of this idea.}
where surface $\Sigma$ defines the boundaries of our system. We have to take this surface to infinity if we define our system on $\mathbb{R}^3$. Our normalization is chosen in such a way that a single monopole classified by $\alpha_0$ and fractional topological charge $Q = 1/N$ is described by an effective long distance field $a(x)$ which satisfies $\nabla^2 a(x) = -4\pi \delta^3(x)$ with asymptotic behaviour $a(x) = 1/|x|$.

Our next step is to insert the delta function into the path integral with field $b(x)$ as a Lagrange multiplier

$$
\delta \left( q(x) - \frac{1}{4\pi NL} \left[ \epsilon^{ijk} \partial_i f_{jk}(x) \right] \right) \sim \int \mathcal{D}[b] e^{i \int d^4x \, b(x) \cdot (q(x) - \frac{1}{4\pi} \epsilon^{ijk} \partial_i f_{jk}(x))},
$$

where $q(x) \approx \sqrt{F_{\mu\nu} F^{\mu\nu}}$ in this formula is treated as the original expression (4) for the topological density operator including the fast non-abelian gluon degrees of freedom, while $f_{jk}(x)$ is treated as a slow varying external source describing the large distance physics for a given monopole’s configuration, similar to the treatment in ref. [17] of external currents for quasiparticles.

Our task now is to integrate out the original non-abelian fast degrees of freedom and describe the large distance physics in terms of new slow varying fields in form of the effective action $S[\sigma, f_{jk}, b]$. We use the same semiclassical approximation as we did before which is expressed in terms of the low energy effective action (6). The only new element in comparison with previous computations is that the fast degrees of freedom must be integrated out in the presence of new slow varying background fields $f_{jk}, b$ which appear in eq. (26). Fortunately, the computations can be easily performed if one notices that the background field $b(x)$ enters eq. (26) exactly in the same manner as $\theta$ parameter enters (6). Therefore, assuming that $b(x)$ is slow varying background field we arrive to the following effective action

$$
\mathcal{Z} \sim \int \mathcal{D}[b] \mathcal{D}[\sigma] \mathcal{D}[f] e^{-S_{\text{top}}[b,f] - S_{\text{dual}}[\sigma,b]} \quad (27)
$$

$$
S_{\text{top}}[b,f] = \frac{i}{4\pi N} \int_{\mathbb{R}^3} d^3 x b(x) \epsilon^{ijk} \partial_i f_{jk}(x)
$$

$$
= -\frac{i}{4\pi N} \int_{\mathbb{R}^3} d^3 x b(x) \nabla^2 a(x);
$$

$$
S_{\text{dual}}[\sigma,b] = \int_{\mathbb{R}^3} d^3 x \frac{1}{2L} \left( \frac{\theta}{2\pi} \right)^2 \left( \nabla \sigma \right)^2 - \zeta \int_{\mathbb{R}^3} d^3 x \sum_{a=1}^N \cos \left( \alpha_a \cdot \sigma + \frac{\theta + b(x)}{N} \right).
$$

There are two new elements in comparison with our previous expression (7). First, the topological term $S_{\text{top}}$ emerges. This term can be also written as

$$
S_{\text{top}} = -\frac{i}{4\pi N} \int_{\mathbb{R}^3} d^3 x \frac{b(x) \epsilon^{ijk} f_{jk}(x)}{4\pi N}, \quad b_i(x) \equiv \partial_i b(x),
$$

which brings it into the line with conventional expression employed in the Higgs model [17]. The second new element which appears in (27) is that $S_{\text{dual}}[\sigma,b]$ now depends on pure topological field $b(x)$ which has no Maxwell counterpart.

One should comment here that we neglected the surface terms in expression for $S_{\text{top}}[b,f]$ such that only scalar potential $a(x)$ from eq. (24) enters the final expression for $S_{\text{top}}[b,f]$. These surface terms very often are crucial in similar studies in condensed matter systems defined on a finite manifold with the boundaries which may have nontrivial topologies such as torus. These surface terms are known to be responsible for the dynamics of the so-called “edge states” in topological field theories. We come back to this point later in the text. However, for the purpose of this work the surface terms can be neglected as we mostly discuss a trivial topology $S^2$ in this work. There is no physical degeneracy in this case and the system itself is characterized by a single unique vacuum state. Nevertheless, the topological feature of the theory such as the topological long range order described by topological action (27), (28) does not go away when topologically trivial manifold is considered.

In fact, these features are manifested in a different way. To be more specific, the main goal of the rest of the section is to argue that the well-known resolution of the celebrated $U(1)_A$ problem is a direct consequence of the topological order described by topological action (27), (28). In different words, we want to argue that the (would be) Goldstone boson receives its mass in this system (in apparent contradiction with conventional symmetry arguments) as a result of topological features of the Chern-Simons action (28).

## B. Topological susceptibility in BF theory.

We now want to compute the correlation function $\langle q(x), q(0) \rangle$ entering the expression for the topological susceptibility (15) by integrating out $b, a$ fields

$$
\langle q(x), q(0) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[b] \mathcal{D}[\sigma] \mathcal{D}[a] e^{-S}[\sigma, a(x), \nabla^2 a(0)} (4\pi NL)^2.
$$

To carry out the computations we limit ourself by considering $\theta = 0$ vacuum state where $\langle \sigma \rangle = 0$ for the massive $\sigma$ fields. We expand thecos term in (27) by keeping the quadratic term for long range field $b(x)$,

$$
\zeta \sum_{a=1}^N \cos \left( \frac{b(x)}{N} \right) \approx \zeta N \left[ 1 - \frac{1}{2} \left( \frac{b(x)}{N} \right)^2 \right].
$$

4 In fact, the constraints on field $b_i(x)$ from [17] require that $b_i(x) \sim \partial_i \Lambda$ when the boundary of $\mathbb{R}^3$ is a topologically trivial $S^2$, such that we are not losing much information with identification $b_i(x) \equiv \partial_i b(x)$ in eq. (28).
Now, the obtained Gaussian integral over $D[b]$ can be explicitly executed, and we are left with the following Gaussian integral over $D[a]$

$$
(q(x), q(0)) = \frac{1}{Z} \int D[a] e^{-S[a] - \frac{1}{2} \int_{\mathbb{R}^3} d^3 x \left[ a(x) \overline{\nabla}^2 a(x) \right]} (4\pi NL)^2 (30)
$$

$$
S[a] = \frac{1}{2} \overline{a} \overline{a} \overline{\nabla}^2 a \overline{\nabla}^2 a
$$

As the next step we rescale $a(x)$ field

$$
a'(x) \equiv \frac{a(x)}{4\pi\sqrt{N_\zeta}} (31)
$$

to bring $S[a]$ to more conventional form

$$
S[a'] = \frac{1}{2} \int_{\mathbb{R}^3} d^3 x \left[ a'(x) \overline{\nabla}^2 a'(x) \right] (32)
$$

With this normalization, the corresponding Gaussian integral over $\int D[a']$ can be easily computed

$$
\frac{\int D[a'] e^{-S[a']} - \frac{1}{2} \int_{\mathbb{R}^3} d^3 x \left[ a'(x) \overline{\nabla}^2 a'(x) \right]}{\int D[a'] e^{-S[a']}} = \delta^3(x), (33)
$$

where $S[a']$ is defined by eq. (32). Now we are ready to complete the computations of the topological susceptibility using the topological BF action (27). We express the original topological density operator (25) in terms of $a(x)$ and take into account the expression for Gaussian integral (33). Final expression for the gauge invariant correlation function

$$
(q(x), q(0)) = \frac{\zeta}{NL^2} \delta^3(x) (34)
$$

precisely reproduces the original formula (15) which was derived without any mentioning of any auxiliary fields [23]. One can also compute a gauge variant correlation function

$$
\lim_{k \to 0} \int d^3 x e^{ikx} \langle \nabla_i a(x), \nabla_j a(0) \rangle \sim \frac{k_i k_j}{k^4}. (35)
$$

This object is very similar to the computations (13) using the Veneziano ghost. The unphysical pole (35) has precisely the same nature as the pole (13) in the Veneziano construction (13). In fact, the transformation properties of our field $\nabla_i a(x)$ are the same as $K_i(x)$ field (23) in the Veneziano construction (13).

Based on this observation and comparing (13) with (35) we identify our topological fields constructed for the deformed QCD with the effective Veneziano ghost. This identification uncovers the nature of the Veneziano ghost as an effective topological non-propagating field. In both cases this pole is not related to any physical massless degrees of freedom which are not present in the system as the theory is gapped. Rather, it contributes to the non-dispersive portion of the gauge invariant correlation function (12), (14), (34). Still, this unphysical topological field does contribute to the $\theta$ dependent portion of the ground state energy.

In weakly coupled deformed QCD one can carry out all the computations without even mentioning the topological fields or the Veneziano ghost as formula (15) shows. However, formulation of this phenomenon in terms of the topological QFT reveals its deep nature which is otherwise hard to understand. As explained above this contact term (34) is not related to any physical propagating degrees of freedom. In computations (15) it emerges as a result of the tunnelling transitions between the degenerate topological sectors. The non-dispersive nature of this term in present computations (based on topological effective Lagrangian (27)) is manifested itself in saturation of the $\chi_{YM}$ by non-propagating, non-dynamical long range $b(x), a(x)$ fields. These fields are not dynamical fields as they do not have the Maxwell counterpart. Nevertheless, these fields are crucial as they saturate the non-dispersive contact term in topological susceptibility, as we have demonstrated above.

Entire framework advocated in this paper is in fact a matter of convenience rather than necessity. The same comment also applies to CM systems: the BF formulation [14–19] using the topological QFT is simply a matter of convenience to represent the known and previously established results (such as braiding phases, the ground state degeneracy, etc) using a beauty of topological quantum field theory. As we shall discuss in section IV the manifestations of this long range order in QCD and in CM systems are somewhat different, but the beauty of the topological BF formulation remains the same.

C. The mass generation for (would be) Goldstone boson in topologically ordered system

We want to reproduce the behaviour (17) in deformed QCD using an appropriate generalization of the topological BF action (27) when the massless quarks are introduced into the system. This study will further illuminate the relation between the auxiliary $a(x), b(x)$ fields and the unphysical Veneziano ghost. To proceed with this task we have to introduce the light matter field which is represented in this model by the $\eta'$ -field [23]. If $\chi_{YM}$ were vanished the $\eta'$ would be conventional massless Goldstone boson which is nothing but the phase of the chiral condensate. However, $\chi_{YM} \neq 0$ in 4d QCD (12) as well as in deformed QCD (15). In different words, the $\eta'$ field receives its mass exclusively as a result of generating of the nonzero contribution to $\chi_{YM}$ with “wrong sign” (12), which is the key element in resolution of the celebrated $U(1)_A$ problem [24–29]. In context of the present work we want to see how the $\eta'$ physical contribution exactly cancels (as Ward Identities require) the non-dispersive term in the topological density (17). We want to see how it happens in BF formulation using the topological action (27). It will shed a new light on a very deep relation, already mentioned above, between the Veneziano ghost and the topological $a(x), b(x)$ fields. Essentially we want to see how the (would be) Goldstone
The Goldstone boson becomes a massive field in topological QFT in apparent contradiction with conventional symmetry arguments.

In the dual sine-Gordon theory the \( \eta' \) meson field appears exclusively in combination with the \( \phi \) parameter as \( \theta \rightarrow \theta - \phi(x) \), where \( \phi \) is the phase of the chiral condensate which, up to dimensional normalization parameter, is identified with physical \( \eta' \) meson in QCD. As it is well known, this property is the direct result of the transformation properties of the path integral measure under the chiral transformations \( \psi \rightarrow \exp(i\gamma_5 \frac{\phi}{\sqrt{c}})\psi \). Therefore, \( \phi(x) \) enters the effective action exactly in the same way as the \( b(x) \) field does as it couples to the topological density operator exactly in the same way (26). Therefore, one can integrate out the fast degrees of freedom exactly in the same way as we did before to arrive at the following effective action which now includes the (would be) Goldstone boson field

\[
Z \sim \int D[b] D[\sigma] D[a] D[\phi] e^{-(S_{\text{top}} + S_{\text{dual}}[\sigma, b, \phi] + S_{a})} \tag{36}
\]

\[
S_{\phi} = \int_{\mathbb{R}^3} d^3 x \frac{c}{2} (\nabla \phi)^2
\]

\[
S_{\text{top}}[b, a] = -\frac{i}{4\pi N} \int_{\mathbb{R}^3} d^3 x b(x) \nabla^2 a(x);
\]

\[
S_{\text{dual}}[\sigma, b, \phi] = \int_{\mathbb{R}^3} d^3 x \frac{1}{2}\left( \frac{g}{2\pi} \right)^2 (\nabla \sigma)^2
\]

\[-\zeta \int_{\mathbb{R}^3} d^3 x \sum_{\alpha = 1}^N \cos \left( \alpha_\sigma \cdot \sigma + \frac{\theta + b(x) - \phi(x)}{N} \right),
\]

where coefficient \( c \) determines the normalization of the \( \phi \) field and has dimension one. This coefficient, in principle, can be computed in this model, but the results do not depend on its numerical value. In terms of these parameters the \( \phi \) mass is given by \( 23 \),

\[
m_{\eta'}^2 = \frac{\zeta}{cN}. \tag{37}
\]

There are two new elements which appear in (36) in comparison with (27). First, kinetic term for \( \phi \) field is generated and parametrized by \( S_{\phi} \). Secondly, \( \phi(x) \) field enters the \( S_{\text{dual}}[\sigma, b, \phi] \) in a very precise and specific way consistent with Ward Identities.

Our task now is to compute the topological susceptibility in the presence of massless quark field by integrating out the auxiliary \( b(x), a(x) \) fields. We repeat the same steps which led us to (30). The only difference is that \( a(x) \) and \( \phi(x) \) fields are mixing such that effective action \( S_{QCD}[a, \phi] \) now takes the form

\[
\langle q(x), q(0) \rangle_{QCD} = \frac{1}{Z} \int D[a] e^{-S_{QCD}} \left[ \frac{\nabla^2 a(x), \nabla^2 a(0)}{(4\pi NL)^2} \right]
\]

\[
S_{QCD}[a, \phi] = \frac{1}{2N\zeta(4\pi N)^2} \int_{\mathbb{R}^3} d^3 x \left[ a(x) \nabla^2 \nabla^2 a(x) \right]
\]

\[+ \int_{\mathbb{R}^3} d^3 x \left[ \frac{c}{2} (\nabla \phi(x))^2 + \frac{i}{4\pi N} \nabla \phi(x) \cdot \nabla a(x) \right], \tag{38}
\]

where we performed the integration by parts of the mixing term \( \int d^3 x \phi \nabla^2 a = -\int d^3 x \nabla \phi \cdot \nabla a \). Our next step is to eliminate the non-diagonal term \( \int d^3 x \nabla \phi \cdot \nabla a \) in (38) by making a shift

\[
\phi_2(x) = \phi(x) + \frac{i}{4\pi cN} a(x), \tag{39}
\]

where \( 1/\sqrt{c} \) is inserted to the left hand side of eq. (39) in order to bring the kinetic term for \( \phi_2(x) \) field to its canonical form. The problem is reduced to the computations of the Gaussian integral similar to (30) with the only difference that the effective action now, after the rescaling (31), takes the form

\[
S_{QCD}[a', \phi_2] = \frac{1}{2} \int_{\mathbb{R}^3} d^3 x \left( \nabla \phi_2(x) \right)^2 \tag{40}
\]

\[+ \frac{1}{2} \int_{\mathbb{R}^3} d^3 x a'(x) \left[ \nabla^2 \nabla^2 - m_{\eta'}^2 \nabla^2 \right] a'(x)
\]

which replaces previous expression (32). In formula (40) parameter \( m_{\eta'}^2 \) is the \( \eta' \) mass in this model and it is given by eq (37). In terms of this rescaled field \( a'(x) \) the Gaussian integral which enters (38) can be easily computed and it is given by

\[
\frac{\int D[a'] e^{-S_{QCD}} \nabla^2 a'(x), \nabla^2 a'(0)}{\int D[a'] e^{-S_{QCD}[a']}} = [\delta(x) - m_{\eta'}^2 G(x)] \tag{41}
\]

where \( S_{QCD} \) is defined by (40) and the massive Green’s function \( G_m(x) = \frac{e^{-m_{\eta'}^2 r}}{4\pi r} \) is normalized in conventional way \( m^2 \int d^3 x G_m(x) = 1 \). Collecting all numerical coefficients from (31), (38) and (41) the final expression for the topological susceptibility in the presence of massless quark takes the form

\[
\langle q(x), q(0) \rangle_{QCD} = \frac{\zeta}{NL^2} \left[ \delta(x) - m_{\eta'}^2 e^{-m_{\eta'}^2 r} \right]. \tag{42}
\]

It precisely reproduces our formula (17) which was derived by explicit integration over all possible monopole’s configurations without even mentioning the topological auxiliary fields. It now has two terms: the first term with “wrong sign” which has non-dispersive nature, and which was computed previously (34) using the same auxiliary topological fields. It also has a new dispersive term related to the massive \( \eta' \) propagating degrees of freedom. It has a negative sign in accordance with general principles (11). The Ward Identities are satisfied \( \chi_{QCD} = \int d^3 x \langle q(x), q(0) \rangle_{QCD} = 0 \) as a result of exact cancellation between the two terms. The celebrated \( U(1)_A \) problem is resolved in this framework exclusively as a result of dynamics of the topological \( a(x), b(x) \) fields. These fields are not propagating degrees of freedom, but nevertheless they generate a crucial non-dispersive contribution with “wrong sign” which is the key element of formulation and resolution of the \( U(1)_A \) problem and the generation of the \( \eta' \) mass.
D. Topological fields and the Veneziano ghost.

The expression for the correlation function (41) with action (40) can be represented in a complementary way which makes the connection between the Veneziano ghost and topological fields much more explicit and precise.

To proceed with our task, we use a standard trick to represent the 4-th order operator $\hat{S}^4 \hat{2}^2 - m^2 \hat{2}^2$ which enters the effective action (40) as a combination of two terms with the opposite signs: the ghost $\phi_1$ and a massive physical $\phi$ field. To be more specific, we write

$$\frac{1}{\hat{2}^2 \hat{2}^2 - m^2 \hat{2}^2} = \frac{1}{m^2} \left( \frac{1}{\hat{2}^2 - m^2} - \frac{1}{\hat{2}^2} \right), \quad (43)$$

such that the Green’s function for $a(x)$ field enters the expression for the topological susceptibility (41) can be represented as a combination of two Green’s functions: for physical massive field with conventional kinetic term and the ghost field with “wrong” sign for the kinetic term. As usual, the presence of 4-th order operator in eq. (40) is a signal that the ghost is present in the system. This signal is explicitly manifested in eq. (43). The contact term in this framework is precisely represented by the ghost contribution. Indeed, the relevant correlation function which enters the expression for the topological susceptibility (41) can be explicitly computed using expression (43) for inverse operator as follows

$$\left\langle D[a]e^{-S_{QCD}[a] } \hat{2}^2 a(x), \hat{2}^2 a'(0) \right\rangle = \frac{1}{(2\pi)^4} \frac{d^3p}{m_q^2} \left[ -\frac{1}{p^2 + m_q^2} + \frac{1}{p^2} \right]$$

$$= \frac{1}{(2\pi)^4} \frac{d^3p}{m_q^2} \left[ \frac{1}{p^2 + m_q^2} \right] = \left\langle \delta(x) - m_q^2 \frac{e^{-m_q r}}{4\pi r} \right\rangle,$$

which, of course, is the same final expression we had before (41) with the only difference is that it is now explicitly expressed as a combination of two terms: physical massive $q'$ contribution and unphysical contribution which saturates the contact term with “wrong” sign.

A new twist in this computation is to represent the correlation function (44) by introducing two fields $\phi_1(x)$ and $\hat{\phi}(x)$ replacing the $a'(x)$ which enters the effective action (40) as the 4-th order operator. To be more precise, we rewrite our action (40) in terms of new fields $\phi_1(x)$ and $\hat{\phi}(x)$ as follows

$$S_{QCD}[\hat{\phi}, \phi_1, \phi_2] = \frac{1}{2} \int d^3x \left[ \left( \hat{\nabla} \phi_1(x) \right)^2 - \left( \hat{\nabla} \phi_2(x) \right)^2 \right]$$

$$+ \frac{1}{2} \int d^3x \left[ \left( \nabla \phi_1(x) \right)^2 + m^2 \phi_1(x)^2 \right] \quad (45)$$

while $a'(x)$ field is expressed in terms of new fields $\phi_1(x)$ and $\hat{\phi}(x)$ as

$$a'(x) = \frac{1}{m_q} \left( \hat{\phi}(x) - \phi_1(x) \right), \quad (46)$$

while the topological density $q(x)$ operator is expressed in terms of these fields as follows

$$q = \sqrt{\frac{\zeta}{NL^2}} \hat{2} a' = \sqrt{\frac{\zeta}{NL^2 m_q^2}} \hat{2} \left( \hat{\phi} - \phi_1 \right). \quad (47)$$

This redefinition obviously leads to our previous result (42), (44) when we use the Green’s functions determined by the Lagrangian (45) for the physical massive field $\hat{\phi}$ and the ghost $\phi_1$, i.e.,

$$\left\langle q(x), q(0) \right\rangle_{QCD} = \frac{\zeta}{NL^2} \left[ \delta(x) - m_q^2 \frac{e^{-m_q r}}{4\pi r} \right]. \quad (48)$$

It is quite amazing that precisely this structure (45) had emerged previously in study of of the $U(1)_A$ problem in 2d Schwinger model in Kogut-Susskind (KS) formulation [38], see also [39] with related discussions. Topological density operator in 2d Schwinger model $\epsilon_{\mu\nu} F^{\mu\nu}$ is also expressed as $\epsilon_{\mu\nu} F^{\mu\nu} \sim \partial_\mu \hat{\phi} \hat{\phi}$ similar to (47). Furthermore, one can show that this structure still holds even with non-vanishing quark mass $m_q$, in which case an additional term $\sim m_q \cos(\phi + \phi_2 - \phi_1)$ appears in effective action (45), similar to analogous expression in KS description [38]. In fact, our notations for $\phi_1, \phi_2, \phi_3$ fields entering (45) are precisely the same notations which had been used in ref. [38] to emphasize the similarity. Furthermore, analogous structure also emerges in 4d QCD when the topological density operator is expressed in terms of the Veneziano ghost where $q \sim \Box (\hat{\phi} - \phi_1)$ has precisely the same structure [40].

An important point here is that the contact term in this framework is explicitly saturated by the topological non-propagating auxiliary fields expressed in terms of the ghost field $\phi_1$, similar to 2d KS ghost or 4d Veneziano ghost 5. From our original formulation without any auxiliary fields reviewed in section IIIB it is quite obvious that our theory is unitary and causal. When we introduce the auxiliary fields (which are extremely useful when one attempts to study the long range order) the unitarity, of course, still holds. Formally, the unitary holds in this formulation because the ghost field $\phi_1$ is always paired up with $\phi_2$ in each and every gauge invariant matrix element as explained in [38] (with the only exception being the topological density operator (47) which requires a special

5 It is important to emphasize that KS and Veneziano ghosts should not be confused with the conventional Faddeev-Popov ghost which is normally introduced into the theory to cancel out unphysical polarizations of the gauge fields. Instead, the KS, Veneziano ghost is introduced to account for the existence of topological sectors in the theory, see [39] for references and details. In four dimensional case the Veneziano ghost can not be confused with Faddeev-Popov ghost as the Veneziano ghost being a singlet does not carry a colour index, in contrast with Faddeev-Popov ghost. The sole purpose of the Veneziano ghost is to saturate the contact term with “wrong sign” in topological susceptibility, similar to eq. (48) in deformed QCD model.
treatment presented in this section). The condition that enforces this statement is the Gupta-Bleuler-like condition on the physical Hilbert space $\mathcal{H}_{\text{phys}}$ which reads like

$$(\phi_2 - \phi_1)^{(+) | \mathcal{H}_{\text{phys}}} = 0 ,$$

where the (+) stands for the positive frequency Fourier components of the quantized fields. The crucial point here is that the formulation of the theory using the topological fields has an enormous advantage as the long range order is explicitly present in formulation (27), (36) and therefore, in equivalent formulation in terms of the ghost field as eq. (45) states.

- Our arguments, based on analysis of a simplified version of QCD essentially suggest that this key element of the $U(1)_A$ problem represented by eq.(15) is a direct manifestation of the long distance topological action (27), (28), (36). Similar structure in CM systems is known to describe topologically ordered phases. It is naturally to assume that the deformed QCD also belongs to a topologically ordered phase. Furthermore, one can explicitly see from our computations above that the $\gamma^\mu$ generates its mass as a result of mixture of (would be) Goldstone field with topological auxiliary field. Therefore, we interpret the well known resolution of the $U(1)_A$ problem in deformed QCD as a result of dynamics of a topological field describing the long range order of the system.

IV. PHYSICAL INTERPRETATION. ANALOGIES WITH CONDENSED MATTER SYSTEMS.

The central line of all our previous discussions is that the resolution of the celebrated $U(1)_A$ problem in deformed QCD is a direct manifestation of the topological order of the system. The main argument is based on analysis of the topological action (27), (28), (36) which exactly reproduces the crucial element of the $U(1)_A$ problem, the topological susceptibility as eqs. (34), (42), (48) demonstrate. A similar topological action in condensed matter (CM) systems is known to lead to a variety of very non-trivial properties as a manifestation of topologically ordered phases realized in these systems, see [14–19] and many references therein.

A. Differences and Similarities between CM systems and deformed QCD

As we already mentioned, there is a fundamental difference between CM systems defined in Minkowski space time $d = (2 + 1)$ and Euclidean 3d “deformed QCD” which has been studied in the present work. In particular, instead of propagating quasiparticles in CM systems we have pseudo-particles (monopoles) which saturate the path integral. As a result of this difference we can not use many standard tools which normally would detect the topological order. For example, we can not compute the braiding phases of charges and vortices which are normally used in CM systems simply because our system does not support such kind of excitations. Further to this point, the topological $b(x), a(x)$ fields entering the abelian BF action (27), (28), (36) are not related in any way to the physical $E&M$ field in contrast with CM systems where topological $a_\mu$ field couples to the physical $E&M$ charges. It prevents us from forming a real vortex which carries magnetic field. Also, these topological $b(x), a(x)$ fields do not have canonical Maxwell counterparts. Instead, the abelian topological field in our case decouples from $E&M$ charges, and behaves like $K_\mu$ field as discussed in section III A, see eq.(23).

Furthermore, the “degeneracy” in deformed QCD model is related to degenerate of winding states $|n\rangle$ which are connected to each other by large gauge transformation, and therefore must be identified as they correspond to the same physical state. It is very different from conventional term “degeneracy” in topologically ordered CM systems when distinct degenerate states are present in the system as a result of formulation of a theory on a topologically non-trivial manifold such as torus.

In case of deformed QCD one should anticipate a similar behaviour when the system is defined on topologically nontrivial manifold. To be more specific, if the boundary of the Euclidean space $\mathbb{R}^3$ in eq.(1) is a topologically trivial $S^2$ than one should expect a unique vacuum state. At the same time, if the Euclidean space $\mathbb{R}^3$ in eq.(1) is additionally compactified on a large torus, i.e. $\mathbb{R}^3 \rightarrow \mathbb{T}^2 \times \mathbb{R}^1$, than one should expect an additional topological $Z_2$ symmetry for $SU(2)$, in close analogy with behaviour of CM system [17]. In fact, the emergence of this additional topological $Z_2$ symmetry has been explicitly demonstrated in deformed QCD in [41]. This symmetry is realized as physical degeneracy in the limit of large $\mathbb{T}^2$. Such a behaviour of the system further supports our claim that the deformed QCD belongs to a topologically ordered phase. Indeed, a conventional gapped theory can not be sensitive to any variation of the boundary conditions at arbitrary large distances.

The moral is: the systems in both cases demonstrate a huge sensitivity to the large distance physics formulated in terms of the boundary conditions. For example, even if we formulate a CM system on a topologically trivial manifold such as sphere, there would be a unique ground state in this “trivial” case. However, it is quite obvious that the long range order in this system governed by topological BF action, does not go away. One can not use the conventional tools, such as degeneracy of the ground state, to detect a topological order. Instead, one should use some different observables to detect the long range order which obviously remains in the system even if it is formulated on a topologically trivial manifold.

In fact we discussed previously some similarities between long range order in CM systems and in QCD in [20]. In particular, the analysis of the Aharonov -Casher effect as formulated in [42] in many respects is very similar to our discussions in QCD. The relevant part of that
work can be stated as follows. If one inserts an external charge into superconductor when the electric field is exponentially suppressed $\sim \exp(-r/\lambda)$ with $\lambda$ being the penetration depth, a neutral magnetic fluxon will be still sensitive to an inserted external charge at arbitrary large distance. The effect is pure topological and non-local in nature. The crucial element why this phenomenon occurs in spite of the fact that the system is gapped is very similar to the key point in deformed QCD discussed in this section. Indeed, the system has different topological states $u_n$ (number of Cooper pairs) are analogous to the topological sectors $|n\rangle$ in our work. As a result of the “tunnelling”, an appropriate ground state $U(\theta)$ must be constructed as discussed in [42], analogous to the $|\theta\rangle$ vacuum construction in gauge theories. This state $U(\theta)$ is an eigenstate of the so-called “modular operator” which commutes with the Hamiltonian. In our work an analogous role plays the large gauge transformation operator $T$ such that $T|\theta\rangle = \exp(-i\theta)|\theta\rangle$ which also commutes with the Hamiltonian $[T, H] = 0$, such that our system must be transparent to topologically nontrivial pure gauge configurations, similar to transparency of the superconductor to the “modular electric field” from ref. [42].

As we already mentioned at the beginning of this section: there is a fundamental difference between our system and CM analogues because our topological fields, while abelian, do not have a Maxwell counterpart, do not couple to the physical charges, and do not propagate, in contrast with physical E&M field discussed in ref. [42]. Nevertheless, both systems behave very similarly as a result of “degeneracy” mentioned above. The dynamics of these degenerate states can be studied in terms of the long range auxiliary topological field (or, what is the same, in terms of à la Veneziano ghost) as we discussed in section III.

B. “String-net condensation” in CM [43] vs long range “Skeleton” in the lattice QCD [1, 2]

While we demonstrated a number of supporting arguments that QCD indeed lies in a topologically ordered phase, its microscopical nature remains a mystery. In this section we attempt to uncover (at least partly) this mystery by presenting an analogy with CM systems where it has been suggested [43] that the microscopical mechanism for topological phases is the so-called “String-net condensation”. We shall argue below that a similar structure has been observed in the lattice QCD [1, 2] which was coined as “Skeleton”. In fact, the lattice results [1, 2] was our original motivation for the present studies in search for a long range order in QCD, see Introduction. By providing this analogy with “String-net condensation” in CM we speculate that the emergence of the “Skeleton” is the microscopical mechanism leading to the long range topological order in QCD.

The “Skeleton” represents the extended, thin, coherent, locally low-dimensional configurations embedded in 4d space, with opposite sign sheets interleaved. We present few arguments below to argue that “string -net” condensate in CM systems as constructed in [43] is a direct analog of the “Skeleton” structure observed in lattice simulations in QCD [1–10].

Indeed, in both cases the configurations themselves have lower dimensionality than the space itself. However, these low-dimensional configurations are so dense, and they fluctuate so strongly that they almost fill all of entire space. In both cases an effective tension of the configurations vanishes as a result of large entropies of the objects which overcome the internal tension. It leads to the condensation of the “string-nets” and percolation of the “Skeleton” correspondingly. If the effective tensions of these configurations were not vanished, we would observe a finite number of fluctuating objects with finite size in the system instead of observed percolation of the “Skeleton” and condensation of the “string-nets”. Furthermore, typically a “Skeleton” is spreading over maximal possible distances percolating through the entire volume of the system similar to “string-nets” which condense. Finally, in both cases the $P$ and $T$ invariance holds as a result of presence of two coherent networks. In String-net condensation this is achieved by considering two topological QFT’s with opposite chiralities. In “Skeleton” studies there are two oppositely-charged sign-coherent connected structures (sheets). The $P$ and $T$ invariance holds in QCD as a result of delicate cancellation between the opposite sign sheets. This delicate cancelation may be locally violated as a result of an external impact such as heavy ion collision. Apparently, such kind of local $P$ and $CP$ violation indeed has been observed in heavy ion collision experiments at RHIC, Brookhaven, and LHC, Geneva, see section VB for the details and references.

The crucial difference between the two cases is of course the nature of the objects: CM systems live in real Minkowski space-time while lattice QCD measurements are done in Euclidean space-time where corresponding configurations saturate the path integral. In the present context it implies that while in CM systems the corresponding “string -nets” are made of real particles/quasiparticles organized into extended objects which may condense, in QCD the corresponding extended “skeleton” configurations live in 4d Euclidean space. Therefore, they should be interpreted as the objects describing the tunnelling events in Minkowski space time, similar to instantons. The term “condensation” normally used in CM literature is not quite appropriate for such 4d objects. Therefore, it is more appropriate to describe the relevant physics by term “percolation” of extended configurations.

What could be an appropriate analytical tools to study this structure in QCD? In deformed weakly coupled QCD we argued [44, 45] that the required structure may indeed emerge in form of the domain walls corresponding to interpolation between topologically different but physically equivalent winding vacuum states. When one
slowly moves from weak coupling regime to strong coupling regime these extended domain walls become very crumpled and wrinkled objects with large number of foldings and effectively vanishing tension as a result of large internal entropy. In this case an arbitrary number of such objects can be formed and they can percolate through the entire volume of the system, forming a kind of a vacuum condensate consisting these crumpled objects. Nevertheless, the topological charge distribution with such striking feature as extended coherence should still persist as the transition from weak to strong coupling regime should be smooth [12, 13]. It is quite likely that an appropriate description for this physics in strong coupling regime should be formulated in terms of holographic dual model rather than in terms of quantum field theory, as argued in [20], however we leave this subject for future studies.

To conclude this section: we speculate that the “Skeleton” structure observed in the QCD lattice simulations [1–10] may represent a basic microscopical mechanism supporting the topological long range order in QCD, similar to “string-net” condensation in CM systems [43]. It is clear that much work needs to be done before this speculation becomes a workable framework. Nevertheless, in next section V we discuss some possible manifestations of the long range order in QCD if our speculations and analogies presented above in this section IV turn out to be correct.

V. POSSIBLE MANIFESTATIONS OF THE LONG RANGE ORDER IN QCD

In section III we argued that the resolution of the celebrated $U(1)_A$ problem is a direct manifestation of the topological order of the system. The arguments were based on study of the weakly coupled deformed QCD model. In section IV we argued that the long range structure observed in the lattice simulations [1–10] may be considered as an evidence that the transition from weakly coupled deformed QCD to the strong coupled QCD is indeed smooth as similar “strings-net” structure in CM is normally attributed to topologically ordered phase. If the topological order indeed persists in this transition to the strong coupling regime than we should expect a number of other profound consequences (along with resolution of the celebrated $U(1)_A$ problem which has been already discussed) of this long range topological order in QCD.

The main goal of this section is to argue that we might be indeed witnessing some of the manifestations of this long range order. We elaborate below on two phenomena which are well established experimentally. At the same time, a theoretical understanding of these effects is still lacking. We present some arguments suggesting that these two phenomena can be naturally understood if QCD indeed belongs to a topologically ordered phase.

A. Universal hadronization temperature

$T_H \simeq (160 – 170) \text{ MeV}$

Over the years, hadron production studies in a variety of high energy collision experiments from $e^+e^-$ annihilation to $pp$ and $p\bar{p}$ interactions with energies from a few GeV up to the TeV range, the production pattern always shows striking thermal aspects with apparently quite universal temperature around $T_H \simeq (160 – 170) \text{ MeV}$. It is very difficult to understand the nature of this “apparent thermalization” as one can not even speak about kinetic equilibration. In different words, the thermal spectrum in all high energy collisions emerges in spite of the fact that the statistical thermalization could never be reached in those systems. Hagedorn concluded that the hadrons are simply born in equilibrium [46], in apparent contradiction with causality.

We argue below that the source for such striking thermal features could be related to the coherent structure of sheets making the “skeleton”, see section IV B. In this case the “apparent thermalization” emerges as a result of tunnelling events (described by these coherent 4d configurations) accompanied by particle production, rather than a result of interaction of the produced particles.

As we discussed in section IV B the “skeleton” configurations should be interpreted as tunnelling processes which are happening all the time in Minkowski vacuum without interruptions. These tunnelling processes in Minkowski vacuum (when no any external sources are present in the system) do not lead to any emission or absorption of real particles, similar to the persistent tunnelling events in Bloch’s case. These tunnelling events simply select an appropriate ground state of the system which is a specific superposition of winding $|n\rangle$ states.

However, when an external source emerges in the system (e.g. as a result of high energy collision) a delicate cancellation mentioned in section IV B between the opposite sign sheets will lead to a production of real particles. We expect that the high energy collisions including $e^+e^-$, $pp$ and $p\bar{p}$ interactions do not completely destroy the coherent vacuum structure discussed in section IV B as the vacuum remains in the same confined topologically ordered phase. However, the collisions lead to some distortions of these low-dimensional sheets rather than breaking them. It is crucial that these Euclidean long range vacuum configurations with vanishing width (as observed in lattice simulations) actually describe instantaneous tunnelling events (with no violation of causality) in Minkowski space. Relaxation of these long range configurations to their conventional form (which existed in vacuum before the collision occurred) leads to emission of real physical particles.

What is expected spectrum of the emitted particles as a result of this mechanism? With a number of additional assumptions one can explicitly see emergence of the thermal spectrum as a result of the Unruh-Hawking radiation [47]. We refer to papers [40, 48, 49] and references therein where all relevant formulae in the present con-
text with emerging Planck spectrum have been discussed. One should emphasize that the Planck spectrum in this approach is not resulted from the kinetics. Rather, the Planck spectrum in high energy collisions in this framework is resulted from the stochastic tunnelling processes when the event horizon emerges as a result of strong interactions. In such circumstances the spectrum must be thermal with $T$ given by

$$T = \frac{a}{2\pi} \sim \Lambda_{QCD}. \quad (50)$$

The acceleration $a$ in this framework describes the dynamics of distorted low-dimensional sheets which organize the “skeleton” as discussed in [49]. This parameter in case of a small distortion (such as peripheral heavy ion collisions, see section V B) can be, in principle, expressed in terms of the auxiliary axion field, similar to $a(x), b(x)$ fields from section III, see footnote 7 to avoid confusion with term “axion”. In general, computation of $a$ is a hard problem of strongly coupled QCD.

An approximate universality of the temperature with no dependence on energy of colliding particles nor their nature (including $e^+e^−, pp$ and $\bar{p}p$ collisions) in this framework is due to the fact that the emission occurs from the distorted long range vacuum configurations rather than from the colliding particles themselves. Furthermore, it is expected that the acceleration $a$ (as a dynamical field in this framework) is not fluctuating on $\Lambda_{QCD}$ scales, but sensitive to larger scales such as size of the system. This is because its dynamics is governed by long range topological fields entering the topological action (similar to (27), (28) in deformed QCD) rather than by conventional fast gluon fluctuations. In deformed QCD this property can be easily seen from the action written in form (45) when topological fields are represented by the ghost $\phi_{\Lambda}(x)$ which does not “know” about $\Lambda_{QCD}$ scale. It should be contrasted with physical massive field $\phi(x)$ which of course “knows” about $\Lambda_{QCD}$ as $m_{\phi} \sim \Lambda_{QCD}$. The initial energy of colliding particles determines the normalization of the fluctuating topological field, but not its spectral properties.

What could be an appropriate technique to study this question in strongly coupled regime? As suggested in [20, 48], it is very likely that the dual holographic description might be an appropriate technical tool to analyze these questions. However, we leave this subject for future studies. Much work needs to be done before this proposal initiated in [48] and further elaborated in [40, 49] becomes a workable framework in strongly coupled regime.

However, we believe that the crucial element of this idea (which is formulated in terms of the tunnelling events of the long range coherent “skeleton” described by the auxiliary axion field, similar to $a(x), b(x)$ fields from section III) has a potential to eventually explain Hagedorn’s notion that the states were prepared before the collision occurs. This explanation would not violate causality as topological auxiliary fields are not propagating degrees of freedom in this framework. In different words, these fields can not send or receive signals as they describe entanglements rather than correlations.

B. Local $P$ and $CP$ violation in QCD. Chiral Magnetic Effect (CME)

The violation of local $P$ and $CP$ invariance in QCD has been a subject of intense studies for the last couple of years as a result of very interesting ongoing experiments at RHIC (Relativistic Heavy Ion Collider) [50, 51] and, more recently, at the LHC (Large Hadron Collider) [52–55].

The main idea to explain the observed asymmetries is to assume that an effective $\theta(\vec{x}, t)_{ind} \neq 0$ is induced as a result of collision. As the $\theta(\vec{x}, t)_{ind}$ parameter enters the Lagrangian (6) with the topological density operator $q(q)$, the local $P$ and $CP$ invariance of QCD is broken on the same scales $\Lambda$ where $\theta(\vec{x}, t)_{ind} \neq 0$ is correlated. As a result of this violation, one should expect a number of $P$ and $CP$ violating effects (such as Chiral Magnetic Effect (CME), Charge Separation Effect (CSE), Chiral Vortical Effect (CVE)) taking place in large region $L$ as suggested in [56].

It is important to emphasize that $\theta(\vec{x}, t)_{ind}$ must be correlated on large scales $L \gg \Lambda_{QCD}$; otherwise, all $P$ and $CP$ odd effects will be washed out. Furthermore, the effective Lagrangian approach advocated in [56] is justified if and only if $\theta(\vec{x}, t)_{ind}$ is correlated on large scales $L \gg \Lambda_{QCD}^{-1}$. These ideas with specific applications to heavy ion collisions were further developed in [57, 58], see recent review papers [59, 60] with a complete list of references and latest developments in this active field, including some recent experimental works.

One of the crucial questions for the applications of the CME to heavy ion collisions is a correlation length of the induced $\langle \theta(\vec{x}, t)_{ind} \rangle \neq 0$. Why are these $P$ odd domains large? In applications to heavy ion collisions the size of the system $L \approx 10$ fm is much larger than conventional $\Lambda_{QCD}^{-1} \approx 1$ fm. Apparently, a relatively large correlation length is a required feature for interpretation of the observed asymmetry [50–55] in terms of CME as argued in [40, 45, 49].

We think that the crucial element in understanding this key question is deeply rooted to the properties of topological order as we advocate in this work. To be more concrete: as we discussed above, the emission of real particles in this framework is resulted from relaxation of the distorted long range sheets of the “skeleton” to its vacuum configuration. These distortions (resulted from the collision) spoil the delicate exact cancellations between oppositely charged long range vacuum configurations which existed in vacuum before the collision occurred. The size of these $P$ odd distortions is order of the size of the system, i.e. $\sim L$. This long range order explains why CME, CSE and CVE are operational in this system and how the asymmetry can be coherently accumulated from entire system. Exactly this long range
order explains how $\langle \theta(\vec{x}, t)_{\text{ind}} \rangle \neq 0$ could be correlated on large scales. It also justifies the assumption made in [56] where $\theta(\vec{x}, t)_{\text{ind}}$ was treated as an external background field in effective Lagrangian approach with correlation length much larger than any conventional QCD fluctuations, $L \gg \Lambda_{\text{QCD}}^{-1}$.

The key element in estimation of measured asymmetries is algebraic sensitivity to acceleration parameter $a$ introduced above. This parameter for peripheral collisions with large impact parameter can be estimated as $a \sim L^{-1}$ where $L^{-1}$ is the size of the system. The algebraic sensitivity to large distances leads to sufficiently large intensity for asymmetries as estimated in [49]:

$$\text{asymmetries from [50-55]} \sim \frac{e \cdot \kappa}{L \Lambda_{\text{QCD}}} \sim 10^{-3}, \quad (51)$$

where electric charge $e \sim \sqrt{\sigma} \sim 10^{-1}$ enters the estimate (51) because CME, CSE and CVE effects are proportional to anomalous coupling of the topological fluctuations with E&M field. Also, the factor $\kappa \sim N^{-2} \sim 0.1$ in eq. (51) describes conventional suppression $\sim N^{-2}$ related to any $\theta$ dependent effects. Finally, for numerical estimates we assume $\Lambda_{\text{QCD}}/a \sim \Lambda_{\text{QCD}} \sim 10$. If a long range order were not present in the system, we would get a strong suppression $\exp(-\Lambda_{\text{QCD}}L)$ instead of power like suppression $(\Lambda_{\text{QCD}})^{-1}$ in eq. (51).

Numerical estimation (51) is consistent with intensities of the observed asymmetries presented in refs. [50–55]. Furthermore, this entire framework is consistent with all known experimental results. In particular, the correlations due to the local $P$ violation should demonstrate the universal behaviour similar to the “universal apparent thermalization” discussed in section V.A as the source for the both effects is the same and related to long range “skeletons” as argued above. For example, the effect should not depend on energy of colliding ions. Such independence on energy is indeed supported by observations where correlations measured in Au+Au and Cu+Cu collisions at $\sqrt{s_{NN}} = 62$ GeV and $\sqrt{s_{NN}} = 200$ GeV are almost identical and independent on energy. The same tendency also continues for the LHC energies: the ALICE Pb-Pb results for 3-particle correlator at $\sqrt{s_{NN}} = 2.76$ TeV are almost identically coincide with RHIC Au+Au results at $\sqrt{s_{NN}} = 200$ GeV, which is precisely what was anticipated as the sizes for $\text{Au}^{79}$ and $\text{Pb}^{208}$ are almost the same, and therefore, parameter $L$ entering (51) is also the same, see [49] and references to the experimental results therein.

The algebraic decay $(a/\Lambda_{\text{QCD}})^p$ with $p \sim 1$ which appears in estimate (51) is a direct manifestation of the topological long range order in QCD. This algebraic scaling can be, in principle, experimentally tested as the size of the system entering (51) is related to the size of the colliding heavy ions. Some suppression of the measured correlations with increasing the size of the system indeed has been observed. For example, the effect for Au+Au collisions with $A \sim 197$ is suppressed in comparison with Cu+Cu collisions with $A \sim 64$, see [49] and references to the experimental results therein.

We conclude this section with the following proposal based on analogy discussed in section IV. We argued that the long range order in CM systems and in QCD is resulted from very similar structures: “strings-net” condensation in CM and “skeleton” in QCD correspondingly. We also argued that in both cases the $P$ and $T$ invariance holds on average as a result of presence of two coherent networks with delicate cancellations between them. In “string-net” condensation the delicate cancellation is achieved by considering two topological QFT’s with opposite chiralities. In “skeleton” studies there are two oppositely-charged sign-coherent connected structures such that all $P$ and $T$ odd effects vanish on average. As we discussed above in heavy ion collisions (in event by event studies) this cancellation is not perfect and it is not absolute, and in principle may lead to some $P$ odd effects as a result of formation of large $P$ odd domains which can be described in terms of the auxiliary axion field. This field is similar to our topological $b(x)$ field in our studies of the deformed QCD in section III, see eqs. (27), (28).

It would be interesting to find a similar effect in CM systems where a local violation of $P$ and $T$ invariance may also occur as a result of non-cancellation between two topological QFT’s with opposite chiralities as a result of some external impact which locally may destroy its delicate cancellation. The point is that this non-cancellation may happen on a macroscopic scale of a sample, rather than on a scale of the correlation length determined by a gap of the system. In fact, the effect similar to CME and CSE[56] (mentioned above in regards to heavy ion collisions) has been discussed in CM literature where it was coined as topological magneto-electric effect [64]. Furthermore, the topological effects considered in [64] are described in terms of the domain walls which are very similar (in spirit) to the domain walls studied in weakly coupled deformed QCD in [44, 45], and which were interpreted there as a direct manifestation of the long range

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6 The algebraic sensitivity with respect to $a$ can be tested using the Veneziano ghost model[40, 61, 62]. As we discussed in section III the topological fields $a(x), b(x)$ in deformed QCD essentially represent the ghost field governed by the effective action (45). Separately: the structure (45) exactly coincides with corresponding expression in 4d QCD when $U(1)A$ problem is treated using the Veneziano ghost. Therefore, the testing of the algebraic sensitivity using the Veneziano ghost in [40, 61, 62] essentially implies that the same algebraic sensitivity to a will emerge in deformed QCD when the topological fields $a(x), b(x)$ are expressed in terms of the ghost field.

7 We use term “axion” which is a jargon here. There is no real new dynamical degree of freedom such as axion in QCD, see reviews [63] about physical axion as real degree of freedom and dark matter candidate. However, the $P$ odd domain described in terms of $b_{\text{ind}}(x)$ justifies our terminology as the dynamical $\theta(x)$ is, by definition, the axion.
VI. CONCLUSION AND FUTURE DIRECTIONS

The main “technical” result of the present work can be formulated as follows. The analysis of a simplified version of QCD suggests that the non-dispersive contribution to the topological susceptibility (12) (which is a key element of formulation and resolution of celebrated $U(1)_A$ problem) emerges because the deformed QCD can be described in terms of the auxiliary topological $a(x), b(x)$ fields governed by BF-like action (27), (28), (36). Such an effective action in CM systems is normally considered as a signal for emergence of a topologically ordered phase with a number of known striking features. Therefore, it is naturally to assume that the deformed QCD also lies in a topologically ordered phase. While we can not use many standard tools (such as the braiding phases, physical degeneracy etc as explained in section IV) which would confirm the topological order, we still can feel the long range order in the system through different phenomena discussed in this paper. One could argue that this long range order persists in strongly coupled QCD as well, as there should be no any phase transitions in passage from “deformed QCD model” to real QCD.

However, the computations in “deformed QCD model” do not say much about the source, the nature, the “mechanism” of this long range order. At this point we return to our first line in introductory section I on very puzzling recent lattice results [1–9] which actually motivated this study. What are the observable manifestations of the long range order observed in these simulations? One can inverse the question: what is the microscopical realization of the topological order in QCD if it is confirmed by future studies? In CM systems it has been shown that a physical mechanism for topological phases can be formulated in terms of the string-net condensation [43]. We argued in section IV B that the structure observed on the lattices and coined as “skeleton” is analogous to “String-net condensation” in CM systems. In different words, we argued that the microscopical mechanism for the long range order in QCD if it is confirmed by future numerical and analytical computations could be precisely the long range structure which already had been studied [1–9].

However, the observable manifestations of this topological ordered phase is much more difficult to detect as explained in IV B because $E\&M$ field as a probe does not couple directly to the topological structure observed on the lattice simulations in contrast with CM systems where external $E\&M$ field is a perfect probe of the long range order. Nevertheless, there are different manifestations of the long range order such as generating the mass of would be Goldstone boson (the so-called $U(1)_A$ problem in QCD) as well as some other manifestations which were shortly reviewed in section V. We think that all these phenomena are already signalling us that the long range order is present in the system. Much work needs to be done before the speculations reviewed in section V become a working framework.

A first step along this line of study could be testing the ideas formulated in section V B using some CM systems, which are known to belong to a topologically ordered phase. Specifically, a local violation of $P$ and $T$ invariance (similar to observed at RHIC and the LHC) may occur in a CM system formulated in terms of “string-net condensation” of two topological QFT’s with opposite chiralities. Such a local violation could be a result of non-cancellation between two topological QFT’s due to some external impact which may locally destroy the delicate cancellation between the two “string-networks”. It would be interesting to see if this non-cancellation may occur on a macroscopic scale of a sample, rather than on a scale of the correlation length determined by a gap of the system. It would be also interesting to test in CM systems the idea that “would be Goldstone” boson may generate its mass through mixing with a topological field, similar to what happens in deformed QCD as discussed in section III C.

Finally, one should note that the crucial physics which is responsible for a number of striking features discussed in this paper and which are due to the tunnelling events between “degenerate” winding states\(^8\) can be in principle tested in the Maxwell theory defined on a compact manifold [65]. To be more specific, when the Maxwell system is defined on a compact manifold there will be tunnelling events which are very similar in nature to the effects discussed in the present work. Corresponding topological effects are also not related to the physical propagating degrees of freedom, similar to our discussions of the topological susceptibility in section III. Furthermore, the resulting topological contribution to the Casimir pressure, while numerically very small has the sign which is opposite to the well known expression for the Casimir effect, similar to the properties of the non-dispersive contribution to the topological susceptibility discussed in section III. Essentially, this extra topological contribution is precisely the source of a tiny violation of a commonly accepted (but generally wrong) receipt that the Casimir effect due to Maxwell photons could be obtained by multiplying the corresponding scalar expressions by a factor of two. This extra contribution is exactly analogous to the topological terms discussed in this work, which can not be expressed in terms of conventional propagating degrees of freedom.

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\(^8\) see a comment on term “degeneracy” in the given context in footnote 2.
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