Characterizing \([h,2,1]\) graphs by minimal forbidden induced subgraphs

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Abstract

An undirected graph \(G\) is called a VPT graph if it is the vertex intersection graph of a family of paths in a tree. The class of graphs which admit a VPT representation in a host tree with maximum degree at most \(h\) is denoted by \([h, 2, 1]\). The classes \([h, 2, 1]\) are closed by taking induced subgraphs, therefore each one can be characterized by a family of minimal forbidden induced subgraphs. In this paper we associate the minimal forbidden induced subgraphs for \([h, 2, 1]\) which are VPT with (color) \(h\)-critical graphs. We describe how to obtain minimal forbidden induced subgraphs from critical graphs, even more, we show that the family of graphs obtained using our procedure is exactly the family of VPT minimal forbidden induced subgraphs for \([h, 2, 1]\). The members of this family together with the minimal forbidden induced subgraphs for VPT \([12,15]\), are the minimal forbidden induced subgraphs for \([h, 2, 1]\), with \(h \geq 3\). Notice that by taking \(h = 3\) we obtain a characterization by minimal forbidden induced subgraphs of the class VPT \(\cap\) EPT = Chordal = \([3, 2, 2] = [3, 2, 1]\) (see \([7]\)).

Keywords: intersection graphs, representations on trees, VPT graphs, critical graphs, forbidden subgraphs.

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1 Introduction

The intersection graph of a set family is a graph whose vertices are the members of the family, and the adjacency between them is defined by a non-empty intersection of the corresponding sets. Classic examples are interval graphs and chordal graphs.

An interval graph is the intersection graph of a family of intervals of the real line, or, equivalently, the vertex intersection graph of a family of subpaths of a path. A chordal graph is a graph without chordless cycles of length at least four. Gavril [5] proved that a graph is chordal if and only if it is the vertex intersection graph of a family of subtrees of a tree. Both classes have been widely studied [2].

In order to allow larger families of graphs to be represented by subtrees, several graph classes are defined imposing conditions on trees, subtrees and intersection sizes [9,10]. Let $h$, $s$ and $t$ be positive integers; an $(h,s,t)$-representation of a graph $G$ consists in a host tree $T$ and a collection $(T_v)_{v \in V(G)}$ of subtrees of $T$, such that (i) the maximum degree of $T$ is at most $h$, (ii) every subtree $T_v$ has maximum degree at most $s$, (iii) two vertices $v$ and $v'$ are adjacent in $G$ if and only if the corresponding subtrees $T_v$ and $T_{v'}$ have at least $t$ vertices in common in $T$. The class of graphs that have an $(h,s,t)$-representation is denoted by $[h,s,t]$. When there is no restriction on the maximum degree of $T$ or on the maximum degree of the subtrees, we use $h = \infty$ and $s = \infty$ respectively. Therefore, $[\infty, \infty, 1]$ is the class of chordal graphs and $[2, 2, 1]$ is the class of interval graphs. The classes $[\infty, 2, 1]$ and $[\infty, 2, 2]$ are called VPT and EPT respectively in [7]; and UV and UE, respectively in [13].

In [6,14], it is shown that the problem of recognizing VPT graphs is polynomial time solvable. Recently, in [11], generalizing a result given in [7], we have proved that the problem of deciding whether a given VPT graph belongs to $[h, 2, 1]$ is NP-complete even when restricted to the class VPT \( \cap \) Split without dominated stable vertices. The classes $[h, 2, 1]$, $h \geq 2$, are closed by taking induced subgraphs, therefore each one can be characterized by a family of minimal forbidden induced subgraphs. Such a family is known only for $h = 2$ [11] and there are some partial results for $h = 3$ [4]. In this paper we associate the VPT minimal forbidden induced subgraphs for $[h, 2, 1]$ with (color) $h$-critical graphs. We describe how to obtain minimal forbidden induced subgraphs from critical graphs, even more, we show that the family of graphs obtained using our procedure is exactly the family of VPT minimal forbidden induced subgraphs for $[h, 2, 1]$. The members of this family together with the minimal forbidden induced subgraphs for VPT (see Figure 2) [12,15], are the minimal forbidden induced subgraphs for $[h, 2, 1]$, with $h \geq 3$. Notice that by taking $h = 3$ we obtain a characterization by minimal forbidden induced
The paper is organized as follows: in Section 2, we provide basic definitions and basic results. In Section 3, we give necessary conditions for VPT minimal non \([h, 2, 1]\) graphs. In Section 4, we show a procedure to construct minimal non \([h, 2, 1]\) graphs. In Section 5, we characterize minimal non \([h, 2, 1]\) graphs.

2 Preliminaries

Throughout this paper, graphs are connected, finite and simple. The vertex set and the edge set of a graph \(G\) are denoted by \(V(G)\) and \(E(G)\) respectively. The open neighborhood of a vertex \(v\), represented by \(N_G(v)\), is the set of vertices adjacent to \(v\). The closed neighborhood \(N_G[v]\) is \(N_G(v) \cup \{v\}\). The degree of \(v\), denoted by \(d_G(v)\), is the cardinality of \(N_G(v)\). For simplicity, when no confusion can arise, we omit the subindex \(G\) and write \(N(v)\), \(N[v]\) or \(d(v)\).

Two vertices \(x, y \in V(G)\) are called true twins if \(xy \in E(G)\) and \(N(x) = N(y)\).

A complete set is a subset of mutually adjacent vertices. A clique is a maximal complete set. The family of cliques of \(G\) is denoted by \(C(G)\). A stable set is a subset of pairwise non-adjacent vertices.

A graph \(G\) is \(k\)-colorable if its vertices can be colored with at most \(k\) colors in such a way that no two adjacent vertices share the same color. The chromatic number of \(G\), denoted by \(\chi(G)\), is the smallest \(k\) such that \(G\) is \(k\)-colorable.

A vertex \(v \in V(G)\) or an edge \(e \in E(G)\) is a critical element of \(G\) if \(\chi(G - v) < \chi(G)\) or \(\chi(G - e) < \chi(G)\). A graph \(G\) with chromatic number \(h\) is \(h\)-vertex critical (resp. \(h\)-edge critical) if each of its vertices (resp. edges) is a critical element and it is \(h\)-critical if both hold.

A VPT representation of \(G\) is a pair \((\mathcal{P}, T)\) where \(\mathcal{P}\) is a family \((P_v)_{v \in V(G)}\) of subpaths of a host tree \(T\) satisfying that two vertices \(v\) and \(v'\) of \(G\) are adjacent if and only if \(P_v\) and \(P_{v'}\) have at least one vertex in common, in such case we say that \(P_v\) intersects \(P_{v'}\). When the maximum degree of the host tree is \(h\) the VPT representation of \(G\) is called an \((h, 2, 1)\)-representation of \(G\). The class of graphs which admit an \((h, 2, 1)\)-representation is denoted by \([h, 2, 1]\).

Since a family of vertex paths in a tree satisfies the Helly property \([3]\), if \(C\) is a clique of \(G\) then there exists a vertex \(q\) of \(T\) such that \(C = \{v \in V(G) : q \in V(P_v)\}\). On the other hand, if \(q\) is any vertex of the host tree \(T\), the set \(\{v \in V(G) : q \in V(P_v)\}\), denoted by \(C_q\), is a complete set of \(G\), but not necessarily a clique. In order to avoid this drawback we introduce the notion of full representation at \(q\).
Let \(\langle P, T \rangle\) be a VPT representation of \(G\) and let \(q\) be a vertex of degree \(h\) of \(T\). The connected components of \(T - q\) are called the branches of \(T\) at \(q\). A path is contained in a branch if all its vertices are vertices of the branch. Notice that if \(N_T(q) = \{q_1, q_2, \ldots, q_h\}\) then \(T\) has exactly \(h\) branches at \(q\). The branch containing \(q\) is denoted by \(T_q\). Two branches \(T_i\) and \(T_j\) are linked by a path \(P_v \in P\) if both vertices \(q_i\) and \(q_j\) belong to \(V(P_v)\).

**Definition 1** A VPT representation \(\langle P, T \rangle\) is full at a vertex \(q\) of \(T\) if, for every two branches \(T_i\) and \(T_j\) of \(T\) at \(q\), there exist paths \(P_v, P_w, P_u \in P\) such that: (i) the branches \(T_i\) and \(T_j\) are linked by \(P_v\); (ii) \(P_w\) is contained in \(T_i\) and intersects \(P_v\) in at least one vertex; and (iii) \(P_u\) is contained in \(T_j\) and intersects \(P_v\) in at least one vertex.

A clear consequence of the previous definition is that if \(\langle P, T \rangle\) is full at a vertex \(q\) of \(T\), with \(d_T(q) = h \geq 3\), then \(C_q\) is a clique of \(G\).

The following theorem shows that a VPT representation which is not full at a vertex \(q\) of \(T\), with \(d_T(q) = h \geq 4\), can be modified to obtain a VPT representation without increasing the maximum degree of the host tree; and, even more, decreasing the degree of the vertex \(q\).

**Theorem 2** [7] Let \(\langle P, T \rangle\) be a VPT representation of \(G\). Assume there exists a vertex \(q \in V(T)\) with \(d_T(q) = h \geq 4\) and two branches of \(T\) at \(q\) which are linked by no path of \(P\). Then there exists a VPT representation \(\langle P', T' \rangle\) of \(G\) with \(V(T') = V(T) \cup \{q'\}\), \(q' \notin V(T)\), and

\[
d_T'(x) = \begin{cases} 3, & \text{if } x = q' \\ h - 1, & \text{if } x = q \\ d_T(x), & \text{if } x \in V(T') \setminus \{q, q'\}. \end{cases}
\]

In what follows we give the definition of the branch graph which can be used to describe intrinsic properties of representations of VPT graphs.

**Definition 3** [7] Let \(C \in \mathcal{C}(G)\). The branch graph of \(G\) for the clique \(C\), denoted by \(B(G/C)\), is defined as follows: its vertices are the vertices of \(V(G) \setminus C\) which are adjacent to some vertex of \(C\). Two vertices \(v\) and \(w\) are adjacent in \(B(G/C)\) if and only if

1. \(vw \notin E(G)\);
2. there exists a vertex \(x \in C\) such that \(xv \in E(G)\) and \(xw \in E(G)\);
3. there exists a vertex \(y \in C\) such that \(yv \in E(G)\) and \(yw \notin E(G)\);
4. there exists a vertex \(z \in C\) such that \(zv \notin E(G)\) and \(zw \in E(G)\).

It is clear that if \(C \in \mathcal{C}(G)\) and \(v \in V(G) - C\) then \(C \in \mathcal{C}(G - v)\). The
following claim says what happens with the branch graphs when we remove such vertices. Its proof is trivial.

**Claim 4** Let $C \in \mathcal{C}(G)$ and let $v \in V(G) - C$: (i) If $v \notin V(B(G/C))$ then $B(G-v/C) = B(G/C)$; (ii) if $v \in V(B(G/C))$ then $B(G-v/C) = B(G/C) - v$.

As will be seen in what follows, branch graphs of VPT graphs can be used to describe intrinsic properties of representations.

**Lemma 5** [1] Let $C$ be a clique of a VPT graph $G$, $\langle P, T \rangle$ be a VPT representation of $G$ and $q$ be a vertex of $T$ such that $C = C_q$. If $v$ is a vertex of $B(G/C)$ then $P_v$ is contained in some branch of $T$ at $q$. If two vertices $v$ and $w$ are adjacent in $B(G/C)$ then $P_v$ and $P_w$ are not contained in a same branch of $T$ at $q$.

In [1] we proved the following two results which show that there is a relation between the VPT graphs that can be represented in a tree with maximum degree at most $h$ and the chromatic number of their branch graphs.

**Lemma 6** [1] Let $\langle P, T \rangle$ be a VPT representation of $G$. Let $C \in \mathcal{C}(G)$ and $q \in V(T)$ such that $C = C_q$. If $d_T(q) = h$, then $B(G/C)$ is $h$-colorable.

**Theorem 7** [1] Let $G \in \text{VPT}$ and $h \geq 4$. The graph $G$ belongs to $[h,2,1] - [h-1,2,1]$ if and only if $\text{Max}_{C \in \mathcal{C}(G)}(\chi(B(G/C))) = h$. The reciprocal implication is also true for $h = 3$.

**Definition 8** A clique $K$ of a graph $G$ is called **principal** if $\text{Max}_{C \in \mathcal{C}(G)}(\chi(B(G/C))) = \chi(B(G/K))$.

A graph $G$ is **split** if $V(G)$ can be partitioned into a stable set $S$ and a clique $K$. The pair $(S,K)$ is the **split partition** of $G$ and this partition is unique up to isomorphisms. The vertices in $S$ are called **stable vertices**, and $K$ is called the **central clique** of $G$. We say that a vertex $s$ is a **dominated stable vertex** if $s \in S$ and there exists $s' \in S$ such that $N(s) \subseteq N(s')$. Notice that if $G$ is split then $\mathcal{C}(G) = \{K, N[s] \text{ for } s \in S\}$. We will call **Split** to the class of split graphs.

**Lemma 9** Let $G \in \text{VPT} \cap \text{Split}$ with split partition $(S,K)$. Then, $K$ is a principal clique of $G$.

**Proof.** Let $s \in S$, we know that $N[s] \in \mathcal{C}(G)$. Observe that $V(B(G/N[s])) = (K - N[s]) \cup S'$, with $S' = \{x \in S : N(x) \cap N(s) \neq \emptyset\}$. We claim that the vertices of $K - N[s]$ are isolated in $B(G/N[s])$. Indeed, let $x \in K - N[s]$, if $y \in K -$
Theorem 12
Let \( N \mid \chi \) then, by Theorem 7, that is, \( K \) is a principal clique of \( G \).

\[ \text{PROOF.} \]

\[ \chi(B(G/N[s])) \leq \chi(B(G/K)) - 1 \]

Thus, \( \chi(B(G/N[s])) \leq \chi(B(G/K)) \). Hence, \( \text{Max}_{C \in \mathcal{C}}(\chi(B(G/C))) = \chi(B(G/K)) \).

3 Necessary conditions for VPT minimal non \([h,2,1]\) graphs

In this Section we give some necessary conditions for VPT minimal non \([h,2,1]\) graphs, with \( h \geq 3 \); recall that:

Definition 10 A minimal non \([h,2,1]\) graph is a minimal forbidden induced subgraph for the class \([h,2,1]\), this means any graph \( G \) such that \( G \not\in [h,2,1] \) and \( G - v \in [h,2,1] \) for every vertex \( v \in V(G) \).

Theorem 11 Let \( G \in \text{VPT} \) and let \( h \geq 3 \). If \( G \) is a minimal non \([h,2,1]\) graph then \( G \in [h+1,2,1] \).

\[ \text{PROOF.} \]

Let \( C \in \mathcal{C}(G) \) and let \( v \not\in C \). We know that \( G - v \in [h,2,1] \) then, by Theorem \[7\], \( \chi(B(G - v/C)) \leq h \). By Claim \[4\], \( \chi(B(G - v/C)) = \chi(B(G/C) - v) \geq \chi(B(G/C)) - 1 \). Thus, \( \chi(B(G/C)) - 1 \leq h \) and hence \( \chi(B(G/C)) \leq h + 1 \). Then, by Theorem \[7\], \( G \in [h + 1,2,1] \). \( \square \)

Theorem 12 Let \( K \) be a principal clique of a VPT minimal non \([h,2,1]\) graph \( G \), with \( h \geq 3 \). Then: (i) \( V(B(G/K)) = V(G) - K \); (ii) if \( v \in V(G) - K \) then \( |N(v) \cap K| > 1 \); (iii) \( B(G/K) \) is \((h+1)\)-vertex critical; (iv) if \( s_1, s_2 \in V(G) - K \) then \( N(s_1) \cap K \neq N(s_2) \cap K \).

\[ \text{PROOF.} \]

By Theorem \[11\], \( G \in [h + 1,2,1] \). Then, by Theorem \[7\] and since \( K \) is a principal clique of \( G \), \( \chi(B(G/K)) = h + 1 \).

(i) It is clear that \( V(B(G/K)) \subseteq V(G) - K \). Suppose there exists \( v \in V(G) - K \) such that \( v \not\in V(B(G/K)) \). Thus, by Claim \[4\], \( B(G - v/K) = B(G/K) \). Since \( G \) is a minimal non \([h,2,1]\) graph, \( G - v \in [h,2,1] \) and, by Theorem \[7\], \( B(G - v/K) \) is \( h \)-colorable. Thus, \( B(G/K) \) is \( h \)-colorable which contradicts the fact that \( K \) is a principal clique of \( G \).

(ii) By item (i) we know that \( v \in V(B(G/K)) \), then \( |N(v) \cap K| \geq 1 \). If \( |N(v) \cap K| = 1 \), \( v \) will be an isolated vertex of \( B(G/K) \) and \( \chi(B(G/K)) = \chi(B(G/K) - v) \). But, by Claim \[4\] and Theorem \[7\], \( \chi(B(G/K)) = \chi(B(G/K) - v) = \chi(B(G - v/K)) = h \), which also contradicts the fact that \( K \) is a principal clique of \( G \).
(iii) We know that $\chi(B(G/K)) = h + 1$. Suppose that $B(G/K)$ is not $(h + 1)$-vertex critical, that is, there is $v \in V(B(G/K))$ such that $\chi(B(G/K) - v) = h + 1$. Then, since $v \in V(B(G/K))$, by Claim \[ \chi(B(G - v/K)) = \chi(B(G/K) - v) = h + 1, \] which contradicts the fact that $G$ is a minimal non $[h, 2, 1]$ graph.

(iv) We will see that if $N(s_1) \cap K = N(s_2) \cap K$ then $s_1s_2 \notin E(B(G/K))$ and $N_{B(G/K)}(s_1) = N_{B(G/K)}(s_2)$, which contradicts the fact that $B(G/K)$ is $(h+1)$-vertex critical. Indeed, if $N(s_1) \cap K = N(s_2) \cap K$ then $s_1s_2 \notin E(B(G/K))$ by definition of branch graph. Moreover, if $s_3 \in N_{B(G/K)}(s_1)$ then there exist $k_1, k_2, k_3 \in K$ such that: (i) $k_1s_1 \in E(G)$, $k_1s_3 \in E(G)$; (ii) $k_2s_1 \in E(G)$, $k_2s_3 \notin E(G)$; (iii) $k_3s_1 \notin E(G)$, $k_3s_3 \in E(G)$. And, since $N(s_1) \cap K = N(s_2) \cap K$, $k_1s_2 \in E(G)$, $k_2s_2 \in E(G)$, $k_3s_2 \notin E(G)$. In addition, $s_3s_2 \notin E(G)$ because in other case there would be an induced 4-cycle $\{s_2, k_2, k_3, s_3\}$ in $G$, which contradicts the fact that $G$ is in VPT (see Figure 2). Hence, $s_3 \in N_{B(G/K)}(s_2)$; we have proven that $N_{B(G/K)}(s_1) \subseteq N_{B(G/K)}(s_2)$. In a similar way, it is easy to see that $N_{B(G/K)}(s_2) \subseteq N_{B(G/K)}(s_1)$.

Theorem [15] shows that all VPT minimal non $[h, 2, 1]$ graphs are split without dominated stable vertices.

To prove this theorem we give the following lemma.

**Lemma 13** Let $h \geq 3$, let $G$ be a VPT minimal non $[h, 2, 1]$ and let $K$ be a principal clique of $G$. Then, $K - \{k\} \in C(G - k)$, for all $k \in K$.

**PROOF.** Let $\langle P, T \rangle$ be an $(h + 1, 2, 1)$-representation of $G$ and let $q \in V(T)$ such that $K = C_q$. We claim that $\langle P, T \rangle$ is full at $q$. Indeed, suppose, for a contradiction, that $\langle P, T \rangle$ is not full at $q$. We can assume, without loss of generality, that if $x$ is an end vertex of a path $P_v \in P$ then there exists a path $P_u \in P$ intersecting $P_v$ only in $x$, in other case the vertex $x$ can be removed from $P_v$. This implies that any path of $P$ linking two branches intersects paths contained in those branches. Hence, since $\langle P, T \rangle$ is not full at $q$, there exist branches $T_i$ and $T_j$ of $T$ at $q$ which are linked by no path of $P$. Then, by Theorem [2] we can obtain a new VPT representation $\langle P', T' \rangle$ of $G$ with $d_{T'}(q) \leq h$. Thus, by Lemma [3] $B(G/C_q)$ is $h$-colorable which contradicts the fact that $C_q$ is a principal clique of $G$.

Hence, since $\langle P, T \rangle$ is full at $q$, every pair of branches of $T$ at $q$ are linked by a path of $P$. If there exists $k \in C_q$ such that $C_q - \{k\}$ is not a clique of $G - k$, there must exists $v \in V(G) - C_q$ such that $v$ is adjacent to all the vertices of $C_q - \{k\}$. Let $T_1, T_2, \ldots, T_{h+1}$ be the branches of $T$ at $q$. Assume, without loss of generality, that $P_k$ links the branches $T_1$ and $T_2$. Since $v \in V(G) - C_q$, there exists $i$, such that $P_v$ is contained in $T_i$. And, since $h \geq 3$, there exists a branch
where

Definition 14

A canonical VPT representation of $G$ is a pair $\langle P, T \rangle$ where $T$ is a tree whose vertices are the members of $C(G)$, $P$ is the family $(P_v)_{v \in V(G)}$ with $P_v = \{ C \in C(G) : v \in C \}$ and $P_v$ is a subpath of $T$ for all $v \in V(G)$.

The following definition will be used in the proof of Theorem 15.

Definition 14 A canonical VPT representation of $G$ is a pair $\langle P, T \rangle$ where $T$ is a tree whose vertices are the members of $C(G)$, $P$ is the family $(P_v)_{v \in V(G)}$ with $P_v = \{ C \in C(G) : v \in C \}$ and $P_v$ is a subpath of $T$ for all $v \in V(G)$.

In [13] it was proved that every VPT graph admits a canonical VPT representation.

Theorem 15 Let $G$ be a VPT graph and let $h \geq 3$. If $G$ is a minimal non $[h,2,1]$ graph, then $G \in \text{Split}$ without dominated stable vertices.

PROOF. Case (1): Suppose that $G \in \text{Split}$ with split partition $(S,K)$, and $G$ has dominated stable vertices. Let $\langle P, T \rangle$ be a canonical VPT representation of $G$, and let $q \in V(T)$ such that $K = C_q$. Assume that $N_T(q) = \{ q_1,q_2,\ldots,q_k \}$, with $k > h$, and call $T_1,T_2,\ldots,T_k$ to the branches of $T$ at $q$ containing the vertices $q_1,q_2,\ldots,q_k$ respectively. It is clear that for each $q_i$, with $1 \leq i \leq k$, there exists $P_{w_i} \in P$ such that $q_i \in V(P_{w_i})$ and $q \notin V(P_{w_i})$. Notice that every $w_i \in S$.

Suppose that $S = \{ w_1, w_2, \ldots, w_k \}$. Since $G$ has dominated stable vertices, by item (iv) of Theorem 12 we can assume, without loss of generality, that $N(w_1) \subsetneq N(w_2)$. This means that $w_1$ and $w_2$ are not adjacent in $B(G/C_q)$; thus, by item (iii) of Theorem 12, $N_{B(G/C_q)}(w_1) \not\subset N_{B(G/C_q)}(w_2)$. Hence, there exists $l \in V(B(G/C_q)) - \{ w_1, w_2 \}$, such that $l \in N_{B(G/C_q)}(w_1) - N_{B(G/C_q)}(w_2)$. Since $V(B(G/C_q)) = S$ we can assume that $l = w_3$. Then, by definition of branch graph, there exists $z \in C_q$ such that $zw_1 \in E(G)$, $zw_3 \in E(G)$ and, since $N(w_1) \subsetneq N(w_2)$, $zw_2 \in E(G)$, which implies that $P_z$ contains the vertices $q_1, q_2$ and $q_3$. Then $P_z$ is not a path. This contradicts the fact that $\langle P, T \rangle$ is a VPT representation of $G$.

We conclude that $S' = S - \{ w_1, w_2, \ldots, w_k \} \neq \emptyset$. Let $G' = G - S'$. Notice that $C_q \in C(G')$ and $V(B(G'/C_q)) = \{ w_1, w_2, \ldots, w_k \}$. Since $G$ is a minimal non $[h,2,1]$ graph, then $G' \in [h,2,1]$ and $\chi(B(G'/C_q)) \leq h$.

We claim that there exists an $h$-coloration of $B(G'/C_q)$ such that if there exists $x \in C_q$ and $w_i, w_j \in \{ w_1, w_2, \ldots, w_k \}$ with $xw_i \in E(G)$, $xw_j \in E(G)$ then $w_i$ and $w_j$ have different colors in $B(G'/C_q)$. (*)
Indeed, if $w_i$ and $w_j$ have the same color in $B(G'/C_q)$ then $w_i w_j \notin E(B(G'/C_q))$. Then we can assume that $N(w_i) \subseteq N(w_j)$, since, by hypothesis, there exists $x \in C_q$ such that $x w_i \in E(G)$ and $x w_j \in E(G)$. Which implies that $w_i$ is an isolated vertex of $B(G'/C_q)$. Therefore, we can change the color of $w_i$ to either of the $h-1$ remaining colors. This process can be done as often as necessary until we have the desired $h$-coloration of $B(G'/C_q)$.

Hence, we consider an $h$-coloration, say $c'$, of $B(G'/C_q)$ satisfying condition $(\ast)$.

Now, we give an $h$-coloration, say $c$, of $B(G/C_q)$ as follows: given $w \in V(B(G/C_q))$, by Lemma 5, there exists $1 \leq i \leq k$ such that $P_w$ is contained in $T_i$, we define $c(w) = c'(w_i)$. Notice that, in particular, $c(w_i) = c'(w_i)$.

We will see that $c$ is a proper coloration of $B(G/C_q)$. That is, we have to see that if $uw \in E(B(G/C_q))$ then $c(u) \neq c(v)$. Since $uw \in E(B(G/C_q))$, by Lemma 5, $P_u$ and $P_v$ are in different branches of $T$ at $q$ say $T_i$ and $T_j$. Moreover, there exists $x \in C_q$ such that $x u \in E(G)$ and $x v \in E(G)$, but this implies that $x w_i \in E(G)$ and $x w_j \in E(G)$. Hence, since our coloration satisfies condition $(\ast)$, $c'(w_i) \neq c'(w_j)$. Thus, $c(u) \neq c(v)$. Therefore, our coloration is proper.

Thus, we have an $h$-coloration of $B(G/C_q)$ which contradicts the fact that $C_q$ is a principal clique of $G$. We conclude that, if $G \in \text{Split}$ then $G$ has no dominated stable vertices.

Case (2): Suppose that $G \notin \text{Split}$. Since $G$ is a minimal non $[h,2,1]$ graph, by Theorem 11, $G \in [h+1,2,1]$. Let $(P,T)$ be an $(h+1,2,1)$-representation of $G$ and let $q \in V(T)$ such that $C_q$ is a principal clique of $G$. We know, by item (i) of Theorem 12, that $V(B(G/C_q)) = V(G) - C_q$. Since $G \notin \text{Split}$ there exist $x, y \in V(B(G/C_q))$ such that $x y \in E(G)$.

Let $\tilde{G}$ be the graph which has an $(h+1,2,1)$-representation $(P', T)$, where $P' = (P'_v)_{v \in V(G)}$ such that:

$$
P'_v = \begin{cases} 
P_v, & \text{if } v \in C_q \\
q_v, & \text{if } v \in V(G) - C_q, \text{ where } q_v \text{ is the vertex of } P_v \text{ closest to } q. \end{cases}
$$

Notice that $V(\tilde{G}) = V(G)$. We claim that $\tilde{G}$ is a split graph, with split partition $(V(G) - C_q, C_q)$. Indeed, if $x, y \in V(G) - C_q$ and $x y \in E(\tilde{G})$ then $q_x = q_y$. Thus, $N_{\tilde{G}}(x) \cap C_q = N_{\tilde{G}}(y) \cap C_q$ which contradicts item (iv) of Theorem 12. Hence, $\tilde{G} \in \text{Split}$ and, by Lemma 9, $C_q$ is a principal clique of $\tilde{G}$.
On the other hand, we can assume that \( N_G(x) \cap C_q \subsetneq N_G(y) \cap C_q \), because in other case it would be an induced 4-cycle in \( G \) which contradicts the fact that \( G \in \text{VPT} \) (see Figure 2). Then, there exists \( w \in C_q \) such that \( wx \in E(G) \), \( wy \in E(G) \). And, since \( xy \in E(G) \) then \( P_x \) and \( P_y \) are in a same branch of \( T \) at \( q \). Hence, by the existence of \( w \), \( q \) lies on the path of \( T \) between \( q \) and \( q \). Which implies that \( \tilde{G} \) has dominated stable vertices. Now it is easy to see that \( B(G/C_q) = B(\tilde{G}/C_q) \), therefore \( \tilde{G} \in [h + 1, 2, 1] - [h, 2, 1] \).

Then, by Case (1), \( \tilde{G} \) is not a minimal non \( [h, 2, 1] \) graph. Thus, there exists \( v \in V(\tilde{G}) \) such that \( (\tilde{G} - v) \in [h + 1, 2, 1] \).

If \( v \in V(B(\tilde{G}/C_q)) \), then \( \chi(B(\tilde{G} - v/C_q)) = h + 1 \). Moreover, by Claim 4 and since \( B(\tilde{G}/C_q) = B(G/C_q) \), we have that \( B(\tilde{G} - v/C_q) = B(G/C_q) - v = B(G/C_q) - v = B(G - v/C_q) \). Hence, \( \chi(B(G - v/C_q)) = h + 1 \) which contradicts the fact that \( G \) is a minimal non \( [h, 2, 1] \) graph.

If \( v \in C_q \), then, by Lemma 13, \( C_q - v \in C(G - v) \); therefore \( C_q - v \in C(\tilde{G} - v) \). Thus, \( \tilde{G} - v \in \text{Split with split partition} (V(G) - C_q, C_q - v) \). Then, by Lemma 9, \( C_q - v \) is a principal clique of \( \tilde{G} - v \). Hence, \( \chi(B(\tilde{G} - v/C_q - v)) = h + 1 \). Moreover, it is easy to see that \( B(\tilde{G} - v/C_q - v) = B(G - v/C_q - v) \); thus \( \chi(B(\tilde{G} - v/C_q - v)) = \chi(B(G - v/C_q - v)) = h + 1 \) which contradicts the fact that \( G \) is a minimal non \( [h, 2, 1] \) graph.

We conclude that \( G \in \text{Split} \). \( \square \)

In Theorem 12 we give some necessary conditions on the branch graph with respect to a principal clique of a minimal non \( [h, 2, 1] \) graph. In Theorem 16 using the fact that all minimal non \( [h, 2, 1] \) graphs are split without dominated stable vertices and the fact that the central clique of a split graph is principal, we will give more necessary conditions for minimal non \( [h, 2, 1] \) graphs.

**Theorem 16** Let \( G \) be a VPT graph and let \( h \geq 3 \). If \( G \) is a minimal non \( [h, 2, 1] \) graph with split partition \((S, K)\) then: (i) for all \( k \in K \), \( |N(k) \cap S| = 2 \); (ii) \( |E(B(G/K))| = |K| \); (iii) \( B(G/K) \) is \((h + 1)\)-critical.

**Proof.** By Theorem 15, \( G \in \text{Split without dominated stable vertices} \). Let \((S, K)\) be a split partition of \( G \). By Lemma 9, \( K \) is a principal clique of \( G \).

(i) Since \( G \in \text{VPT} \cap \text{Split without dominated stable vertices} \), \( |N(k) \cap S| \leq 2 \), for all \( k \in K \). Suppose there exists \( k \in K \) such that \( |N(k) \cap S| < 2 \).

By Theorem 11, \( G \in [h + 1, 2, 1] \). Let \((P, T)\) be an \((h + 1, 2, 1)\)-representation of \( G \) and let \( q \in V(T) \) such that \( K = C_q \). By Lemma 13, \( C_q - \{k\} \in C(G - k) \)
1. If $|N(k) \cap S| = 0$: Then $B(G - k/C_q - \{k\}) = B(G/C_q)$. Thus, $\chi(B(G - k/C_q - \{k\})) = \chi(B(G/C_q)) = h + 1$, which contradicts the fact that $G$ is a minimal non-[$h, 2, 1$] graph.

2. If $|N(k) \cap S| = 1$: We will see that $B(G - k/C_q - \{k\}) = B(G/C_q)$. It is clear, by item (ii) of Theorem 12, that $V(B(G - k/C_q - \{k\})) = V(B(G/C_q))$ and $E(B(G - k/C_q - \{k\})) \subseteq E(B(G/C_q))$. Let $uv \in E(B(G/C_q))$ such that $uv \notin E(B(G - k/C_q - \{k\}))$. Since $|N(k) \cap S| = 1$ we can assume, without loss of generality, that $\{N(v) \cap C_q\} - \{N(u) \cap C_q\} = \{k\}$. Therefore, since, for all $k \in C_q$, $|N(k) \cap S| \leq 2$ we have that $N_{B(G/C_q)}(v) = \{u\}$ then $d_{B(G/C_q)}(v) = 1$, which contradicts the fact that $H$ is $(h + 1)$-vertex critical.

(ii) First we will prove that $|E(B(G/K))| \leq |K|$. Let $e = s_is_j \in E(B(G/K))$. By definition of branch graph, there exists $k \in K$ such that $ks_i \in E(G)$, $ks_j \in E(G)$. Thus, for each $e \in E(B(G/K))$ there exists $k \in K$. Hence, by item (i), $|E(B(G/K))| \leq |K|$. Now we will see that $|K| \leq |E(B(G/K))|$. Let $k \in K$. By item (i), $|N(k) \cap S| = 2$. Suppose that $N(k) \cap S = \{s_i, s_j\}$, hence $N(s_i) \cap N(s_j) \neq \emptyset$. Since there are not dominated stable vertices, then $N(s_i) \notin N(s_j), N(s_j) \notin N(s_i)$. Thus, $s_is_j \in E(B(G/K))$. Hence, for each $k \in K$ there exist $s_is_j \in S$ such that $s_is_j \in E(B(G/K))$. Observe that if $\hat{k} \neq k$, then $N(\hat{k}) \cap S \neq N(k) \cap S$. Because if $N(\hat{k}) \cap S = N(k) \cap S$, then $\hat{k}$ and $k$ are true twins in $G$ which contradicts the fact that $G$ is a minimal non-[h, 2, 1] graph. Therefore, $|K| \leq |E(B(G/K))|$.

(iii) By item (iii) of Theorem 12, $B(G/K)$ is $(h + 1)$-vertex critical. Then, $\chi(B(G/K)) = h + 1$. We want to see that $B(G/K)$ is $(h + 1)$-edge critical, that is, $\chi(B(G/K) - e) = h$, for all $e \in E(B(G/K))$. By item (i), for all $k \in K$, $|N(k) \cap S| = 2$ then there are not vertices of $K$ of degree 1. Moreover, $V(B(G/K)) = \{s_1, s_2, ..., s_n\}$ with $\{s_1, s_2, ..., s_n\} = S$. Let $e = s_is_j$ and let $k \in K$ such that $ks_i \in E(G)$, $ks_j \in E(G)$. Since there are not dominated stable vertices, $B(G - k/K - \{k\}) = B(G/K) - e$. Then, $\chi(B(G - k/K - \{k\})) = \chi(B(G/K) - e) = h$, because $G$ is a minimal non-[h, 2, 1] graph. Hence, $B(G/K)$ is $(h + 1)$-edge critical. Thus, $B(G/K)$ is $(h + 1)$-critical. \qed

4 Building minimal non-[h, 2, 1] graphs

The construction presented here is similar to that done in [1], and a generalization of that used in [2]. Given a graph $H$ with $V(H) = \{v_1, ..., v_n\}$, let $G_H$ be the graph with vertices:
The usefulness of Lemma 17, PROOF. Assume that $H$ and only if

Theorem 18

Let $h \geq 3$. The graph $G_H$ is a minimal non $[h,2,1]$ graph if and only if $H$ is $(h+1)$-critical.

PROOF. Assume that $G_H$ is a minimal non $[h,2,1]$ graph. By item (ii) of Lemma 17, $B(G_H/K_H) = H$. Hence, by item (iii) of Theorem 16, $H$ is $(h+1)$-critical.

Let $H$ be an $(h+1)$-critical graph with $V(H) = \{v_1, v_2, ..., v_n\}$. By Lemma 9 and Lemma 17, $\max_{C \subseteq C(G_H)}(\chi(B(G_H/C))) = \chi(B(G_H/K_H)) = \chi(H) = h+1$. Hence, by Theorem 7, $G_H \in [h+1,2,1] - [h,2,1]$. Let us see that $G_H - v \in [h,2,1]$, for all $v \in V(G_H)$. First, if $v = v_i \in V(H)$, using Lemma 4 and item (ii) of Lemma 17, $B(G_H - v_i/K_H) = B(G_H/K_H) - v_i = H - v_i$. Thus, since $H$ is $(h+1)$-vertex critical, $\chi(B(G_H - v_i/K_H)) = h$. Hence, $G_H - v_i \in [h,2,1]$. Secondly, if $v = v_{ij}$ being $e = v_iv_j \in E(H)$, since $B(G_H - v_{ij}/K_H - \{v_{ij}\}) = H - e$, then $\chi(B(G_H - v_{ij}/K_H - \{v_{ij}\})) = \chi(H - e)$. And, $\chi(H - e) = h$ because $H$ is $(h+1)$-edge critical. Hence, $G_H - v_{ij} \in [h,2,1]$. Since $H$ has no degree 1 vertices, $G_H$ has no more vertices. \qed
5 Characterization of minimal non \([h,2,1]\) graphs

In this Section, we give a characterization of VPT minimal non \([h, 2, 1]\) graphs, with \(h \geq 3\). The main result of this Section is Theorem 19 which states that the only VPT minimal non \([h, 2, 1]\) graphs are the constructed from \((h+1)\)-critical graphs.

Moreover, in Theorem 20 we show that the family of graphs constructed from \((h+1)\)-critical graphs together with the family of minimal forbidden induced subgraphs for VPT 12,15, is the family of minimal forbidden induced subgraphs for \([h, 2, 1]\), with \(h \geq 3\).

**Theorem 19** Let \(h \geq 3\) and let \(G\) be a VPT graph. \(G\) is a minimal non \([h, 2, 1]\) graph if and only if there exists an \((h+1)\)-critical graph \(H\) such that \(G \simeq G_H\).

**PROOF.** The reciprocal implication follows directly applying Theorem 18.

Let \(G\) be a minimal non \([h, 2, 1]\) graph. By Theorem 15 we know that \(G \in \text{Split without dominated stable vertices}\). Let \((S, K)\) be a split partition of \(G\). By Theorem 11 \(G \in [h + 1, 2, 1]\). Let \(H = B(G/K)\). By item (iii) of Theorem 16 \(H\) is an \((h+1)\)-critical graph. Let us see that \(G \simeq G_H\). Let \(G_H = (S_H, K_H)\). By item (i) of Lemma 17, \(B(G_H/K_H) = H\). Then, since \(V(B(G_H/K_H)) = V(B(G/K))\), \(S_H = S\). Moreover, since \(E(B(G_H/K_H)) = E(B(G/K))\), by item (ii) of Theorem 16 \(|K_H| = |K|\) and, by item (i) of Theorem 16 \(|N(k) \cap S| = 2\) for all \(k \in K\). Suppose that \(N(k) \cap S = \{v_i, v_j\}\) we will see that \(v_iv_j \in E(H)\). It is clear that \(v_ik \in E(G)\) and \(v_jk \in E(G)\). Moreover, by item (ii) of Theorem 12 there exist \(k', k'' \in K\) such that \(k'v_i \in E(G), k''v_j \in E(G)\). If \(k' = k''\) then, since \(|N(k) \cap S| = 2\) for all \(k \in K\), we have that \(k'\) and \(k\) are true twins in \(G\), which contradicts the fact that \(G\) is minimal non \([h, 2, 1]\) graph. Hence, \(k' \neq k''\). Thus, \(k'v_j \notin E(G)\) and \(k''v_i \notin E(G)\). Therefore, \(v_iv_j \in E(H)\).

Hence, we can define a function that assigns to each vertex \(k \in K\) an edge \(v_iv_j \in E(H)\), that is, an element of \(K_H\). Note that in \(G_H\) the vertex \(v_{ij} \in K_H\) is adjacent exactly to \(v_i\) and \(v_j\). Hence, the function \(f\) can be extended to a new function \(\hat{f}\) from \(K \cup S\) to \(K_H \cup S_H\), being the identity function from \(S\) to \(S_H\). Moreover, \(\hat{f}\) is an isomorphism between \(G\) and \(G_H\). \(\Box\)

**Theorem 20** Let \(h \geq 3\). A graph \(G\) is a minimal non \([h, 2, 1]\) if and only if \(G\) is one of the members of \(F_0, F_1, \ldots, F_6\) or \(G \simeq G_H\), being \(H\) an \((h+1)\)-critical graph.
PROOF. By Theorem 19, if \( G \cong G_H \) being \( H \) an \((h+1)\)-critical graph, then \( G \) is a minimal non \([h,2,1]\) graph.

If \( G \) is any of the members of \( F_0, \ldots, F_{16} \) then \( G \notin \text{VPT} \) and \( G - v \in \text{VPT} \), for all \( v \in V(G) \). Moreover, in [4] it was proved that \( G - v \in \text{EPT} \), for all \( v \in V(G) \). Thus, \( G - v \in \text{VPT} \cap \text{EPT} = [3,2,1] \) \( \Box \), which implies that \( G - v \in [h,2,1] \). Hence, \( G \) is a minimal non \([h,2,1]\) graph.

Let \( h \geq 3 \) and let \( G \) be a minimal non \([h,2,1]\) graph.

Case (1): \( G \notin \text{VPT} \). Since \( G \) is a minimal non \([h,2,1]\) graph, then \( G - v \in [h,2,1] \) for all \( v \in V(G) \). Thus, \( G - v \in \text{VPT} \) for all \( v \in V(G) \). Then, \( G \) is a minimal forbidden induced subgraph for \( \text{VPT} \). Hence, \( G \) is one of the members of \( F_0, F_1, \ldots, F_{16} \).

Case (2): \( G \in \text{VPT} \). Then, by Theorem 19, \( G \cong G_H \), being \( H \) an \((h+1)\)-critical graph.

Notice that, since every \( G_H \) is \( \text{VPT} \) no member of \( F_0, F_1, \ldots, F_{16} \) is an induced subgraph of \( G_H \). On the other hand, suppose that there exists \( G \) a member of \( F_0, F_1, \ldots, F_{16} \) that has a \( G_H \) as induced subgraph. Then, there exists \( \{v_1, \ldots, v_n\} \subseteq V(G) \), with \( n \geq 1 \), such that \( G - \{v_1, \ldots, v_n\} = G_H \). Hence, since \( G_H \) is a minimal non \([h,2,1]\) graph, \( G - \{v_1, \ldots, v_n\} \notin [h,2,1] \) which contradicts the fact that \( G \) is a minimal non \([h,2,1]\) graph. \( \Box \)

Fig. 2. Minimal forbidden induced subgraphs for \( \text{VPT} \) graphs (the vertices in the cycle marked by bold edges form a clique).
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