THE CHARACTER OF NON-MANIPULABLE COLLECTIVE CHOICES BETWEEN TWO ALTERNATIVES

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February 28, 2024

Abstract

We consider classes of non-manipulable social choice functions with range of cardinality at most two within a set of at least two alternatives. We provide the functional form for each of the classes we consider. This functional form is a characterization that explicitly describes how a social choice function of that particular class selects the collective choice corresponding to a profile. We provide a unified formulation of these characterizations using the new concept of “character”. The choice of the character, depending on the class of social choice functions, gives the functional form of all social choice functions of the class.

JEL Code: D71
Keywords: social choice functions, weak group strategy-proofness, anonymity, functional form, character function, preferences, restricted domain.

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1 Introduction

The characterization of classes of social choice functions is a central theme in Social Choice Theory. There are characterizations that describe in an explicit way the rule that determines the collective choice once a profile of preferences is chosen.

Such kind of characterizations are referred to as functional form characterizations by Barberà et al. [2], whereas Hagiwara and Yamamura [7] call them closed characterizations.

In the theory of non-manipulable social choice functions there are classical results which really are functional form characterizations. The celebrated Gibbard-Satterthwaite theorem is an example. When the collective choice is taken between two alternatives and anonymity of agents is required, the quota majority method is another example. A classical reference is Moulin [10].

This paper focuses on the functional form of non-manipulable collective choice between two alternatives. We consider a model for social choice in which the realizable alternatives are two, indifference is permitted, and the collective decision is based upon a preference profile of agents who also consider other alternatives.

As compared to Barberà et al. [2] and Hagiwara and Yamamura [7], we do not restrict the set of agents to be finite. As such non-manipulability is based on groups of agents rather than on a single agent.

We deal mainly with weakly group strategy-proof social choice functions, according to Definition 2 in Barberà et al. [2]. It is well known that this class of functions does not coincide in general with the class of individually strategy-proof social choice functions. However, since we only deal with social choice functions with range of cardinality two, weak group strategy-proofness is the appropriate non-manipulability notion for the case of infinite agents (see [1, Corollary 1] and [6, Theorem 3.3. (ii)]). Moreover, our results extend to individual strategy-proofness for finitely many agents.

Contributions of the paper. Although Barberà et al. [2] characterize non-manipulability, they do not provide the functional form of non-manipulable social choice functions. Hagiwara and Yamamura [7] present as a closed characterization their main result Theorem 2. Neither papers considers anonymity.

The two of the three significant contributions of our paper are: we provide the functional form of

(1) all non-manipulable social choice functions;

(2) all non-manipulable, anonymous social choice functions.

The third notable contribution is the notion of character, which we use to introduce the canonical functional form of social choice functions with range of cardinality two within a set of at least two alternatives. The functional forms of the classes (1) and (2) can be unified, being both canonical under an appropriate choice of a character. The unifying nature of the canonical functional form is demonstrated by discussing several classes of social choice functions.

1Even if one assumes that there are only finitely many agents, some assumptions are necessary in order to ensure that individual strategy-proofness is equivalent to weak group strategy-proofness.
With these three results, our paper completes the work initiated by Barberà et al. [2], and further developed by Hagiwara and Yamamura [7]. In addition, it significantly generalizes results of Basile et al. [5], for the model with only two alternatives.

**Relevance of the setting.** Analyzing the non-manipulable collective choice between two alternatives $a$ and $b$ (candidates, public projects, the introduction of a new law versus the status quo, ...) is different if agents are asked to report their preference just between $a$ and $b$, or if they are asked, as we assume in our more general setting, for preferences over a larger set of alternatives that includes $a$ and $b$. The fact that the collectivity “implements” either $a$ or $b$, may induce to think that the social choice is independent of the remaining alternatives (principle of independence of irrelevant alternatives). This is not always the case, even if strategic reporting is ruled out by the social choice functions adopted. The following example demonstrates this.

**Example 1.1** There are only two agents $v_1$, and $v_2$ and three alternatives $a, b, c$. The social choice function we consider for the collective decision of implementing either $a$ or $b$ is as described next. Corresponding to a profile of preferences over the three alternatives, the collective choice is $b$ if and only if either one of the following circumstances is true:

– agent $v_1$ prefers $b$ to $a$;

– agent $v_1$ is indifferent between $a$ and $b$ but prefers $c$ to both $a, b$;

In all the other cases the rule selects $a$.

Neither agent may gain by lying, i.e. this scf is non-manipulable.

Now, consider the following two profiles $P$ and $Q$. In profile $P$: agent $v_1$ is indifferent between $a$ and $b$ that are both worse than $c$. In profile $Q$: agent $v_1$ is indifferent between $a$ and $b$, but both alternatives are strictly preferred to $c$; agent $v_2$ reports the same preference as in profile $P$.

Even if both profiles are indistinguishable between $a$ and $b$, the collective choice described above selects $b$ for $P$ and $a$ for $Q$. $\square$

Hence a non-manipulable social choice function does not guarantee the independence of the collective choice from the irrelevant alternatives. In particular, this tells us that investigating the setting of the present paper is not obvious from Basile et al. [5].

On the other hand, considering our model, is not merely an intellectual exercise. Some arguments supporting its relevance are presented by Barberà et al. [2] and by Hagiwara and Yamamura [7]. We like to add that it can be also seen as a model for balancing simplicity and full attention to agents’ necessities. An intuition that supports this idea is the following, simple, real-life situation. Suppose that a community has to decide about a public project like building an educational institution. “Educational Institutions” differ by many dimensions. They differ by students they target (primary, secondary, a college, ...), or by learning environments provided for teaching e.g; online, in person, blended, residential, vocational or other format. Of course they differ by location, and so on.

So, many potential alternatives are on the table. Naturally, every citizen has his/her opinion about the various options. With respect to some, they may be indifferent. Even if the potential alternatives are many, for several reasons, the designer of the public choice mechanism may limit the outcome of the collective decision only to either build solution $a$ or build solution $b$. 

Nonetheless, the implementation of a social choice function that asks for preferences on the entire spectrum of possibilities rather than only on \( \{a, b\} \), allows us to take into account the preferences of the agents also regarding the other alternatives. By doing so, there is a kind of balance between simplicity (either \( a \) or \( b \)) and respect for the needs of citizens (the input for the social choice is a profile of preferences over all potential alternatives). There is only one possible drawback of this idea: if agents are asked preferences over the entire set of potential alternatives, but only either \( a \) or \( b \) is realized, then it looks like agents have rights that the social planner don’t care. If we think carefully, we see that this is not the case, as shown in Example 1.1 according to which the collective choice is not independent of the full spectrum of alternatives. We believe that here clearly emerges “the choice to restrict the range (of the social choice function) as a possible tool for the mechanism designer”, as emphasized by Barberà et al. [2] in their final remarks.

Together with the previous argument (of philosophical flavor, say) in favor of the study of our setting, there are also theoretical arguments. For example, Barberà et al. in a different paper, [3], show that a non-manipulable social choice function, over the domain made of all profiles consisting of single-dipped preferences relative to some linear order, necessarily has range of cardinality at most two.

The plan of the paper is as follows. After presenting in Section 2 the social choice model we adopt, Section 3 is devoted to obtain the representation of general non-manipulable social choice functions with range of cardinality at most two. We also introduce the notions of character and canonical functional forms. In Section 4 we analyze the anonymous case. Section 5, by considering several classes of social choice functions, demonstrates how the character function approach can be systematically applied to them. This also shows the role of the character in unifying the functional form characterizations of these classes.

The paper ends with Conclusions and an Appendix with some proofs.

## 2 The social choice model

Let \( V \) be a set representing a collectivity whose members we refer to as agents. Let \( \mathcal{W}(A) \) be the set of all complete, transitive binary relations over an arbitrary set \( A \) of alternatives. We refer to such relations as preferences. With reference to a preference \( W \in \mathcal{W}(A) \), we adopt standard notations: \( x \succ W y \) stands for \((x, y) \in W \) and \((y, x) \notin W \), and the notation \( x \sim W y \) stands for \([ (x, y) \in W \) and \((y, x) \in W \].

Functions \( P \) from \( V \) to \( \mathcal{W}(A) \) are named preference profiles. The class of all preference profiles is denoted by \( \mathcal{W}(A)^V \). A partial profile of preferences is a function \( \pi \) from a subset of \( V \) to \( \mathcal{W}(A) \). A partial profile is therefore \( \pi = (\pi_v)_{v \in T} \) where \( T \) is a subset of \( V \) and \( \pi_v \in \mathcal{W}(A) \) for every \( v \in T \). A particular partial profile is that which involves the empty set as domain; we speak of this as the empty profile.

For a profile \( P = (P_v)_{v \in V} \) we shall also use the notation \( P = [P_T, P^c_T] \) if \( T \) is a subset of \( V \), the set \( T^c \) is its complement, and \( P_T, P^c_T \) are the obvious restrictions \( P_T = (P_v)_{v \in T} \), \( P^c_T = (P_v)_{v \notin T} \) of \( P \). Extending this notation to arbitrary partitions of \( V \) or to partial profiles is straightforward.

Let \( \mathcal{P} \) be a subset of \( \mathcal{W}(A)^V \). We refer to \( \mathcal{P} \) as the class of feasible profiles. Possible assumptions
on \( \mathcal{P} \) are:

- universal domain: \( \mathcal{P} = \mathcal{W}(A)^V \),
- cartesian restricted domain: \( \mathcal{P} = \times_{v \in V} \mathcal{W}_v \) with \( \emptyset \neq \mathcal{W}_v \subseteq \mathcal{W}(A) \),
- quasi-cartesian restricted domain: the set \( \mathcal{P} \) has the property
  \[ P, Q \in \mathcal{P}, T \subseteq V \Rightarrow [P_T, Q_T] \in \mathcal{P}. \]

Trivially, cartesian implies quasi-cartesian. For a finite set of agents, the converse is also true. In the case of infinitely many agents, quasi-cartesian is a weaker assumption than cartesian.\(^2\)

Note that the possibility of dealing with strict preferences (i.e. with the elements of the subset \( S(A) \) of \( \mathcal{W}(A) \) made of preferences that are also antisymmetric) is not excluded in the case of cartesian restricted domain (or quasi-cartesian). The case \( \mathcal{P} = S(A)^V \), will be referred to as strict universal domain.

Throughout the sequel, \( \phi : \mathcal{P} \to A \) stands for a social choice function (scf, for short). If the range of \( \phi \) has cardinality at most two, we say that \( \phi \) is a two-valued scf.

### 2.1 Non-manipulability.

Our setting coincides with that of Barberà et al.\(^2\) except for the fact that the sets \( V \), of agents, and \( A \), of alternatives, need not to be finite.

Dealing with an arbitrary set \( V \), the notion of non-manipulability we adopt is based on coalitions of agents rather than on a single agent. A “coalition of agents” stands for a “non-empty group of agents”. A desirable property of a scf is that there are no incentives for the agents to coalesce forming a group that, with false reporting, can manipulate the social outcome for the advantage of its own members. Groups can organize manipulation in several ways. Because of this, Barberà et al.\(^2\) present different notions of group manipulations. We adopt the following notion of group manipulation.

**Definition 2.1** Let \( \phi \) be scf. We say that a coalition \( D \) can strongly manipulate a profile \( P \in \mathcal{P} \) under \( \phi \) if there is another profile \( Q \in \mathcal{P} \) such that

- every agent \( v \) in \( D^c \) has the same preference in both \( P \) and \( Q \), i.e. \( P_v = Q_v \);
- every agent \( v \) in \( D \) prefers \( \phi(Q) \) to \( \phi(P) \) according to \( P_v \), i.e. \( \phi(Q) \succ_{P_v} \phi(P) \).

The impossibility of the above form of manipulation leads to the notion of weak group strategy-proofness.

\(^2\)For example, let \( V \) be the set of natural numbers and \( W_0, W_1 \) two distinct preferences. The domain \( \mathcal{P} \) consisting of all profiles \( P \) that can be written as \( P_v = \begin{cases} W_1, & \text{if } v \in F \\ W_0, & \text{otherwise} \end{cases} \) for some finite subset \( F \) of \( V \), is quasi-cartesian but not cartesian.
Definition 2.2 We say that a scf $\phi$ is weakly group strategy-proof if no coalition of agents can strongly manipulate any feasible profile under $\phi$. Moreover, we shorten the expression “weakly group strategy-proof” as wGSP.

When $\phi$ is wGSP, we also say sometimes that $\phi$ is non-manipulable. Evidently, the above weak group strategy-proofness of a scf $\phi$ is the same as the validity of the following implication for $\phi$:

$$[D \text{ is a coalition, } P, Q \in \mathcal{P} \text{ are identical on } D^c] \Rightarrow \exists v \in D : \phi(P) \succeq_{P_v} \phi(Q).$$

We have adopted the terminology of Barberà et al. [2, Definition 2]. However, note that the term “coalitional strategy-proofness” is sometime used for the same concept (see [9] and [1], for example).

A further property equivalent to wGSP is the following

$$P, Q \in \mathcal{P}, \phi(P) \neq \phi(Q) \Rightarrow \exists v \in V \text{ such that } \begin{cases} \phi(P) \succeq_{P_v} \phi(Q) \\ P_v \neq Q_v \end{cases}$$

introduced in [6] under the name almost preference reversal (APR, for short).

In the previous Definition 2.2, if we replace coalitions with singletons, the corresponding notion is known as individual strategy-proofness.

Definition 2.3 A scf $\phi$ is individually strategy-proof (ISP, for short), if, according to $P_v$, the alternative $\phi(P_v, P_{-v})$ is at least as good as $\phi(Q_v, P_{-v})$, for every agent $v$, for every profile $P \in \mathcal{P}$, and for every preference $Q_v$ such that $(Q_v, P_{-v}) \in \mathcal{P}$.

It is well known that wGSP and ISP are not equivalent notions in general. Even if we restrict the setting by assuming that $V$ is finite, further assumptions are necessary in order to ensure that ISP implies wGSP. Le Breton and Zaporozhets [9] and Barberà et al. [1] investigates this in depth. However, some plain cases of equivalence of the two notions can be recalled without resorting to technicalities.

Suppose $V$ is finite:

– ISP implies wGSP when $\mathcal{P}$ is universal (as it can be verified with a direct proof);
– ISP implies wGSP when $\mathcal{P}$ is a cartesian restricted domain but the scf has range of cardinality at most three ([1, Corollary 1] and [6, Theorem 3.3 (ii)]).

Remark 2.4 We concentrate our attention on weak group strategy-proofness. This property is more stringent than individual strategy-proofness but it allows us to skip the assumption of the finiteness of $V$ in our results. The statements we have just recalled above tell us that when $V$ is finite our results extend to the case of individual strategy-proofness. In a sense this is a point of view opposite to Hagiwara and Yamamura [7]. They first obtain the closed characterization of ISP scfs, and, on the basis of [1, Corollary 1], present it as the closed characterization of wGSP scfs (which they name group strategy-proof scf).
3 The functional form of all two-valued weakly group strategy-proof social choice functions

In this section we obtain the canonical representation of a two-valued wGSP scf. Without loss of generality, we adopt the following approach. For every set \( \{a, b\} \subseteq A \) consisting of two distinct alternatives we determine the functional form of the wGSP social choice functions with range contained in \( \{a, b\} \).

With this in mind, we adopt the following notation: by \( D(x, P) \) we denote the subset of \( V \) consisting of agents that, under the profile \( P \), prefer the alternative \( x \in \{a, b\} \) to the other remaining alternative in \( \{a, b\} \); by \( I(P) \) we denote the subset of \( V \) consisting of agents that, under \( P \) are indifferent between \( a \) and \( b \). We also set \( \tilde{D}(x, P) = D(x, P) \cup I(P) \).

3.1 Monotonicity

The identification of the functional form of two-valued wGSP scfs goes through a characterization of non-manipulability by means of a monotonicity à la Maskin: suppose that the society selects \( a \) when profile \( Q \) prevails, then it is natural to ask that \( a \) is again selected under a new profile \( P \) that shows no less support for the alternative \( a \). A relevant issue is the precise meaning that needs to be given to the verbal expression \( \text{(✠)} \) “the profile \( P \) supports the alternative \( a \) at least as the profile \( Q \) does”.

In the following we discuss this, demonstrating the value this paper add to Basile et al. [5].

**Example 1.1 (continuation).** Comparing the model adopted in [5] to the one adopted here, we note that here we do not assume that \( A = \{a, b\} \).

In [5] by the sentence \( \text{(*)} \) we exactly meant that, changing the profile of preferences of the society from \( Q \) to \( P \), agents favoring \( a \) against \( b \) remain so, and agents that were indifferent between \( a \) and \( b \), and have not moved to prefer \( a \), remain indifferent between \( a \) and \( b \).

On the basis of this, the monotonicity (à la Maskin) characterizes non-manipulability [5, Theorem 2.5]. However, in Example 1.1 we see that even if the support for the alternative \( a \) shown by \( P \) is not less than the one shown by \( Q \), the change of profile from \( Q \) to \( P \) changes the collective choice from \( a \) to \( b \). So the scf of that example is not monotone. In other words, in the setting of the present paper Theorem 2.5 of [5] verbatim does not hold true. We need to enlarge the class of monotone scfs in order to keep the equivalence between monotonicity and non-manipulability. To these goal the present section is devoted.

We propose a more stringent interpretation of the sentence \( \text{(*)} \). In contrast to \( \text{(*)}_{\text{ba}} \) in our more general setting we change the meaning as follows:

\( \text{(*)}_{\text{two}} \) changing the profile of preferences of the society from \( Q \) to \( P \), agents in favor of \( a \) against \( b \) remain such, and agents that were indifferent between \( a \) and \( b \), and have not moved to prefer \( a \), **do not change their opinion about the entire set of alternatives**.
Based on this, a weaker, yet straightforward, notion of monotonicity can be introduced that characterizes the non-manipulability (Theorem 3.3, infra).

The following definition formalizes the meaning of (✠ two).

**Definition 3.1** Given two profiles \( P \) and \( Q \), we say that \( P \) supports the alternative \( a \) at least as \( Q \) does, and write \( P \geq^a Q \), if

\[
D(a, Q) \subseteq D(a, P), \quad \text{and} \quad v \in I(Q) \setminus D(a, P) \text{ implies that } P_v = Q_v.
\]

Replacing \( b \) to \( a \) in the two conditions above, we define the relation \( P \geq^b Q \).

The relation \( P \geq Q \) can be interpreted similarly: “the profile \( P \) supports the alternative \( b \) at least as the profile \( Q \) does”. The fact that in our exposition we privilege the reference to \( a \) in the definition of monotonicity, does not affect the generality of our results.

It is obvious that:

\[
P \geq^a Q \iff Q \geq^b P.
\]

The next definition introduces the mentioned extension of the monotonicity property used in [5]. To emphasize that the new definition is weaker than the monotonicity property in [5], we refer to it as *almost monotonicity*.

**Definition 3.2** Let \( V, \mathcal{P}, A \) be arbitrary and \( a, b \) two distinct elements of \( A \). We say that a scf \( \phi : \mathcal{P} \to \{a, b\} \) is almost monotone if

\[
[P, Q \in \mathcal{P}, P \geq Q, \quad \phi(Q) = a] \Rightarrow \phi(P) = a.
\]

The relation of almost monotonicity with non-manipulability is the focus of the next theorem.

**Theorem 3.3** Let \( a, b \) be two distinct elements of \( A \). Consider a scf \( \phi : \mathcal{P} \to \{a, b\} \). Then, \( \phi \) is wGSP if it is almost monotone. Conversely, on a quasi–cartesian restricted domain, \( \phi \) is almost monotone if it is wGSP.

**Proof:** We show that almost monotonicity implies almost preference reversal. Suppose \( \phi \) is not APR. By definition this means that we find feasible profiles \( P, Q \) such that \( \phi(P) \neq \phi(Q) \) and moreover with, for every \( v \in V \), either \( \phi(Q) \succ_{P_v} \phi(P) \) is true or \( P_v = Q_v \) is true.

If \( \phi(P) = a \) we get \( P \geq^b Q \). If \( \phi(P) = b \) we get \( P \geq^a Q \). This is immediate. In both cases almost monotonicity is violated.

Now, under the assumption that the scf is defined on a quasi–cartesian restricted domain, we prove the converse. So, let us assume that \( \phi \) is wGSP and show, by contradiction, that it must be almost monotone.

Suppose almost monotonicity is violated. Then we have two profiles \( P, Q \in \mathcal{P} \) such that \( P \geq^a Q, \quad \phi(Q) = a, \) and \( \phi(P) = b \). Let us consider the following partition \( \{V_0, V_1, V_2\} \) of \( V \):

\[
V_0 = \{v \in V : P_v = Q_v\},
\]

\[
V_1 = \{v \in V : P_v \neq Q_v\} \cap D(b, Q),
\]

\[
V_2 = \{v \in V : P_v \neq Q_v\} \cap \overline{D}(a, Q).
\]
Since we have the hypothesis that $\phi$ is wGSP and this is the same as APR, we have an agent $v$ (at least one exists because of APR) for which one has both $P_v \neq Q_v$ and $b \succ a$. In other words we have $\emptyset \neq \tilde{D}(b, P) \cap \{v \in V : P_v \neq Q_v\}$. Observe that the agents belonging to $\tilde{D}(b, P) \cap \{v \in V : P_v \neq Q_v\}$ necessarily prefer $b$ to $a$ under $Q$. This is a consequence of $P \geq Q$, i.e. of the validity of the two conditions $D(a, Q) \subseteq D(a, P)$ and $v \in I(Q) \setminus D(a, P) \Rightarrow P_v = Q_v$. So, we can write

$$\emptyset \neq \tilde{D}(b, P) \cap \{v \in V : P_v \neq Q_v\} \subseteq V_1.$$ 

Let us consider the profile $P' = [P_{v_1}, Q_{v_1}, P_{v_2}] \in \mathcal{P}$. Since $\phi$ is wGSP, then $\phi(P') = b$. Otherwise the coalition $V_1$ manipulates $P'$ by means of $P$.

If the set $V_2$ is empty, then $P' = Q$ and we contradict $\phi(Q) = a$. So, it must be the case that $V_2$ is nonempty. However, we again contradict $\phi(Q) = a$. Otherwise, having $\phi(Q) = a$, the coalition $V_2$ manipulates $P'$ by means of $Q$ (observe that, again by the two conditions $D(a, Q) \subseteq D(a, P)$ and $v \in I(Q) \setminus D(a, P) \Rightarrow P_v = Q_v$, every agent in the coalition $V_2$ prefers $a$ to $b$ under $P$, i.e. $V_2 \subseteq D(a, P)$).

Example 3.4 In order to prove the implication wGSP $\Rightarrow$ almost monotonicity, we cannot dispense with the assumption on $\mathcal{P}$ of being quasi–cartesian. Indeed, let us consider $V = \{v_1, v_2\}$, $A = \{a, b, c, d\}$, and the domain $\mathcal{P}$ consisting of only two feasible profiles $P, Q$. Assume $P$ is given by: $P_{v_1} = a \succ b \succ c \succ d$, and $P_{v_2} = b \succ a \succ d \succ c$. Assume $Q$ is: $Q_{v_1} = c \succ d \succ a \succ b$, and $Q_{v_2} = d \succ c \succ b \succ a$. Define $\phi(P) = a$, $\phi(Q) = b$. The scf $\phi$ is APR hence wGSP but it is not almost monotone.

Remark 3.5 With reference to scfs whose range consists of alternatives named $x, y$, Barberà et al (2012) introduce the notion of essentially xy-monotonic scf [4, Definition 5] and the notion of essentially xy-based scf [4, Definition 7]. Considering that in the present paper we refer to $\{a, b\}$-valued functions, we observe, see [4, Remark 2.12], that our notion of almost monotonicity is equivalent to that of being simultaneously essentially ab-based and essentially ab-monotonic.

3.2 The character function

Before introducing all the “ingredients” needed for establishing the canonical representation of a two-valued non-manipulable scf, we recall some notions about general partially ordered sets.

We recall that a partially ordered set (poset, briefly) is a set in which a binary relation is defined which is reflexive, transitive, and antisymmetric. The notation we adopt for a poset is $L$. When we want to emphasize that on the set $L$, the underlying set that can be made partially ordered with the adoption of different binary relations, we are considering a specific relation denoted by $\geq$, then we write $(L, \geq)$ to denote a poset. As is customary, for $x, y \in L$, the validity of $x \geq y$ is also denoted by $y \leq x$. 

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Definition 3.6 Let $(L, \geq)$ be a partially ordered set. A subset $C$ of $L$ is super order closed if

$$x \in L, f \in C, x \geq f \Rightarrow x \in C.$$ 

A subset $M$ of $L$ is an antichain when

$$x, x' \in M, x \neq x' \Rightarrow x' \geq x$$

is false.

When $C$ is a super order closed subset of a poset we write $C$ is a SOC subset or $C$ is SOC, in short.

Of course the empty set is an antichain and it is SOC.

If $X$ is a subset of $L$, its subset $m(X)$, defined as $m(X) = \{ x \in X : x \geq y, y \in X \Rightarrow y = x \}$, denotes the set of the minimal elements of $X$. Of course $m(X)$ is an antichain.

Note that for a finite poset $L$ it is possible to prove the following (see the Appendix for the proof).

Proposition 3.7 The antichains of a finite poset $L$ are in one–to–one correspondence with the SOC subsets of $L$.

Definition 3.8 A character on the set $P$ of the feasible profiles is a function $\chi$ from $P$ to the underlying set $L$ of a poset $(L, \geq)$.

By using a fixed character $\chi$, one can introduce two-valued scfs in a straightforward way. For each subset $C$ of the image set $\chi(P)$, we define a scf.

Definition 3.9 Let $a$ and $b$ be two distinct elements of $A$, and $\chi$ be a character. For every $C \subseteq \chi(P)$, by setting

$$\phi_{(\chi,C)}(P) = a \iff \chi(P) \in C,$$

we define a function $\phi_{(\chi,C)} : P \to \{a, b\}$. The scf $\phi_{(\chi,C)}$ will be called the canonical scf.

When no ambiguity occurs about the character, we shorten $\phi_{(\chi,C)}$ as $\phi_C$.

Now we introduce a notion of monotonicity of $\{a, b\}$-valued scfs with respect to a character. As in the previous notions of monotonicity we can equivalently choose to refer to $a$ or to $b$.

Definition 3.10 Let $\chi$ be a character on the set $P$ of the feasible profiles. A scf $\phi : P \to \{a, b\} \subseteq A$ is said to be $(\chi, a)$-monotone when

$$[P, Q \in P, \chi(P) \geq \chi(Q), \phi(Q) = a] \Rightarrow \phi(P) = a.$$ 

Next proposition identifies the scfs $\phi : P \to \{a, b\} \subseteq A$ that are $(\chi, a)$-monotone. Such scfs are all and only the canonical scfs corresponding to subsets $C$ of the image set $\chi(P) (\subseteq L)$ which are SOC with respect to the poset consisting of $\chi(P)$ endowed with the restriction of the ordering of the poset $L$. 

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Proposition 3.11 Let $\chi : \mathcal{P} \to L$ be a character, $L$ being a poset with respect to a partial order $\geq$. For a scf $\phi : \mathcal{P} \to \{a, b\} \subseteq A$, $\phi$ is $(\chi, a)$-monotone if and only if $\phi = \phi_{\chi}^C$ for some SOC subset $C$ of the poset $(\chi(\mathcal{P}), \geq)$. When such $C$ exists, it is unique.

PROOF:
⇒
Assume $\phi$ is $(\chi, a)$-monotone, and consider the set

$$C = C_\phi := \{ x \in L : x = \chi(P) \text{ for some } P \in \mathcal{P} \text{ such that } \phi(P) = a \}.$$ 

This is the desired SOC subset of $(\chi(\mathcal{P}), \geq)$.

Indeed:
– by $(\chi, a)$-monotonicity $C$ is SOC in the poset $(\chi(\mathcal{P}), \geq)$;
– by $(\chi, a)$-monotonicity we have $\phi(P) = a \Leftrightarrow \phi_C(P) = a$;
– $C$ and $C'$ subsets of $(\chi(\mathcal{P}), \geq)$ with $\phi_C = \phi_C' \Rightarrow C = C'$.
⇐
Suppose $C$ is a SOC subset of $(\chi(\mathcal{P}), \geq)$, and $\phi = \phi_C$. Assume $[P, Q \in \mathcal{P}, \chi(P) \geq \chi(Q), \phi(Q) = a]$. Since $\phi = \phi_C$, the condition $\phi(Q) = a$ means, by Definition 3.9, that $\chi(Q) \in C$. By super order closedness of $C$, also $\chi(P) \in C$, hence $\phi(P) = \phi_C(P) = a$. □

3.3 The character of two-valued non-manipulable social choice functions

The canonical scf in Definition 3.9 becomes the functional form of the two-valued non-manipulable scfs if we suitably define the character function. To do this we preliminarily have to introduce a poset.

Definition 3.12 We denote by $\mathcal{D}$ the set consisting of triples $(S, T, \pi)$ where $S, T$ are disjoint subsets of $V$, and $\pi$ is a partial profile whose domain is $T$. We endow the set $\mathcal{D}$ with the partial order $\leq_\mathcal{D}$ defined as follows:

$$(S, T, \pi) \leq_\mathcal{D} (S', T', \pi')$$ means that $S \subseteq S'$, $S \cup T \subseteq S' \cup T'$, and $v \in T \cap T' \Rightarrow \pi_v = \pi'_v$.

The proof that the relation $\leq_\mathcal{D}$ is a partial order is given in the Appendix. We now present the character of the two-valued wGSP scfs.

Definition 3.13 The character function for the two-valued weakly group strategy-proof scfs is the function $\chi : \mathcal{P} \to \mathcal{D}$ defined as

$$\chi(P) = (D(a, P), I(P), (P_v)_{v \in I(P)}).$$

A direct verification shows that $P \geq a Q \Leftrightarrow \chi(P) \geq_\mathcal{D} \chi(Q)$. Hence, we can reformulate Theorem 3.3 as follows.

$^3$Equivalently one can define the relation $(S, T, \pi) \leq_\mathcal{D} (S', T', \pi')$ by requiring the validity of the conditions $S \subseteq S'$, $T \setminus S' \subseteq T'$, and $v \in T \setminus S' \Rightarrow \pi_v = \pi'_v$. 

10
Theorem 3.14 Let $a, b$ be two distinct elements of $A$. Consider a scf $\phi : \mathcal{P} \to \{a, b\}$ defined over a quasi-cartesian restricted domain. Then, $\phi$ is wGSP if and only if $\phi$ is $(\chi, a)$-monotone.

We present below the canonical functional form of the non-manipulable two-valued scfs.

Theorem 3.15 Let $a, b$ be two distinct elements of $A$. Assume $\mathcal{P}$ is a quasi-cartesian restricted domain. Let $\chi$ be the character introduced in Definition 3.13. Then, the map $C \mapsto \phi(\chi, C)$ is a bijection from the set of all SOC subsets of $\chi(\mathcal{P})$, $\leq_D$ to the set of all wGSP scfs from $\mathcal{P}$ to $\{a, b\}$.

Proof: Apply Proposition 3.11 and Theorem 3.14.

Observe that to determine the collective choice corresponding to a profile $P$ we do not have to know the actual profile $P$, but only need to know the components of $\chi(P)$. Therefore, these constitute the primitive elements for the formation of the collective decision.

An element of $\chi(\mathcal{P})$ is a triple $(S, T, \pi)$ consisting of disjoint subsets $S, T$ of agents and a partial $\{a, b\}$-indifference profile $\pi$. The latter is a partial profile $\pi$ such that for every agent $v$ in its domain $T$, one has that $a$ and $b$ are indifferent according to $\pi_v$. So, we are dealing with a generalized version of the veto pairs considered in [5]. With $W = S \cup T$, the elements of $\chi(\mathcal{P})$ can be seen as pairs consisting of a veto pair and a $\pi$.

Intuitively: the triple $(S, T, \pi)$ acts similarly to a veto unit, determining a non-manipulable rule for the collective choice and expressing a veto against $b$. Indeed, to block the alternative $b$ (i.e. $a$ is the collective choice) it suffices that $(S, T, \pi)$ supports $a$, i.e. the agents of $S$ vote for $a$, and those $v \in T$ that don’t move in favor of $a$, are adamant at the $\{a, b\}$-indifferent preference $\pi_v$.

Remark 3.16 In [4] Basile et al. also identify a functional form of the wGSP scfs. They are all and only the scfs of $\psi$-type. The definition of $\psi$-type functions relies on an extension of the concept of committee, and one needs a well ordered collection of extended committees in order to define a $\psi$-type scf $\phi$. The social choice $\phi(P)$ will then be the choice of the first extended committee which is not indifferent, if there is one; otherwise it will be the the alternative associated to a profile of unanimous indifference between the alternatives $a, b$. As we see this kind of representation is quite different from the canonical functional form. Moreover does not guarantee uniqueness of the representation. Finally, in [4] the domain is universal. Theorem 3.15 is a notable improvement of the representation theorem proved in [4].

When $V$ and $A$ are finite, $\chi(\mathcal{P})$ is finite too, and since SOC subsets of $\chi(\mathcal{P})$ are in a one-to-one correspondence with antichains of $\chi(\mathcal{P})$ (due to Proposition 3.7), the canonical functional form can be expressed in terms of sets of incomparable elements of $\chi(\mathcal{P}), \leq_D$.

According to the following representation theorem, rules for a non-manipulable collective choice between two alternatives $a$ and $b$ can be formed uniquely in the following way: with reference to some incomparable generalized vetoers $(S_1, T_1, \pi^{(1)}), (S_2, T_2, \pi^{(2)}), (S_3, T_3, \pi^{(3)}), \ldots$ the rule selects $a$ if and only if at least one of the $(S_i, T_i, \pi^{(i)})$ supports $a$.

4We recall that a committee is a nonempty family of coalitions of agents which is closed under supersets. See subsection 5.1, infra.

5Note that incomparability in the poset $(\chi(\mathcal{P}), \leq_D)$ may be thought of as a form of independence in vetoing.
Theorem 3.17  Suppose $V$ and $A$ are finite, $\mathcal{P} = \times_{v \in V} \mathcal{W}_v$, and $\chi$ is the character of Definition 3.13. Then, there is a bijection $\Delta \mapsto \phi_\Delta$ from the set of all antichains $\Delta$ of $(\chi(\mathcal{P}), \leq_D)$ to the set of all $wGSP$ scfs from $\mathcal{P}$ to $\{a, b\}$. For an antichain $\Delta$ the scf $\phi_\Delta$ is defined as follows
\[
\phi_\Delta(P) = \begin{cases} 
a, & \text{if } \exists (S, T, \pi) \in \Delta: (D(a, P), I(P), (P_v)_{v \in I(P)}) \geq_D (S, T, \pi) 
b, & \text{otherwise}
\end{cases}
\]

Remark 3.18  Hagiwara and Yamamura (2020) prove the following result, which they call closed characterization, for $wGSP$ scfs with range of cardinality two within a larger set of alternatives.

Theorem 3.19  Let $\phi$ be a scf with range $\{a, b\}$. Then, $\phi$ is $wGSP$ if and only if it is an upper set rule $g^Q$. To define an upper set rule one needs a set $Q \subseteq \mathcal{P}$ of profiles with the property of being an “upper set with respect to $\prec_{(b,a)}$” in the sense defined in [2, page 662]. Then the definition of the rule is the following:
\[
g^Q(P) = \begin{cases} 
a, & \text{if } P \in Q 
b, & \text{if } P \notin Q
\end{cases}
\]
There are substantial differences between Theorem 3.19 and our results. Since Hagiwara and Yamamura assume that $V$ is finite, we can compare Theorem 3.19 and Theorem 3.17. We observe that
– an upper set rule identifies the collective choice corresponding to a profile $P$ by verifying if $P$ belongs to the upper set $Q$.
– the knowledge of $Q$ is required to implement this upper set rule.

Our characterization does not need a large set of profiles exhausting the set of profiles that will select $a$. First of all, we just refer to the primitives which are necessary to the formation of the collective choice. These are: the sets $D(x, P)$ of agents in favor of $x \in \{a, b\}$, the sets $I(P)$ of agents which are indifferent and, for each agent $v \in I(P)$, the particularly chosen $\{a, b\}$-indifference preference $P_v$. Second, a scf is identified by means of a small number of incomparable generalized vetoers\footnote{The number of incomparable elements within a SOC set $C$ is much smaller than the the number of elements of $C$.} each leading the collectivity to choose $a$. Then the value of $\phi(P)$ comes from the comparison of the primitives of the profile $P$ with the vetoers.

To conclude this Remark, we observe that the notion of upper set with respect to $\prec_{(b,a)}$ can be framed within our more general notion of SOC subset of a poset. Due to this, Theorem 3.19 can also be seen as a corollary of our Theorem 3.15. Moreover, Hagiwara and Yamamura assume, more restrictively, that $\mathcal{P}$ is a cartesian restricted domain with identical factors.
4 Anonymous social choice functions

In this section we suppose that the set $V$ of agents is finite, and $\mathcal{P} = \mathcal{W}^V$, with $\mathcal{W} \subseteq \mathcal{W}(A)$. Let $\Phi_{AN}$ consists of all scfs $\phi : \mathcal{P} \to \{a, b\} \subseteq A$ which are non-manipulable (in this case one can refer to ISP or wGSP equivalently, see Remark [2.4]) and anonymous.

**Definition 4.1** A scf $\phi$ is anonymous if, for every profile $P$ and for every permutation $\sigma$ of $V$, it results $\phi(P) = \phi(P \circ \sigma)$, where $P \circ \sigma$ is the profile in which the preference of each agent $v$ is $P_{\sigma(v)}$.

We also assume that there are finitely many preferences in $\mathcal{W}$ for which the alternatives $a$ and $b$ are indifferent. Hence, setting $\mathcal{I} = \{ W \in \mathcal{W} : a \sim_W b \}$ we assume this set is finite, say of cardinality $\tau$ ($A$ need not to be finite; $\tau$ may be zero). Let us enumerate the elements of $\mathcal{I}$ as follows $\mathcal{I} = \{ W_1, W_2, \ldots, W_\tau \}$, and introduce the sets $I_i(P) = \{ v \in V : P_v = W_i \}$, for $i = 1, \ldots, \tau$. Of course the sets $I_i(P)$ partition $I(P)$.

Now we describe the character function we associate to the class $\Phi_{AN}$. Let us fix the number of agents to be $n$, and define $L$ to be the set $L = \{ 0, 1, \ldots, n \}^{\tau+1}$. Let us use the notation $t^+$ for the positive part of a number $t$, namely $t^+ = \max\{ t, 0 \}$. We make $L$ a poset by introducing the following partial order.

For $k, \ell \in L$, define:

$$k \leq_L \ell \iff \sum_{i=1}^\tau (k_i - \ell_i)^+ \leq l_0 - k_0.$$

It is straightforward to see that $k \leq_L \ell$ is equivalent to $k_0 + \sum_{i \in J} k_i \leq \ell_0 + \sum_{i \in J} \ell_i$ for every subset $J$ of $\{ 1, \ldots, \tau \}$.

Indeed, if $J$ is a subset of $\{ 1, \ldots, \tau \}$ and we assume $k \leq_L \ell$, then we have

$$\sum_{i \in J} (k_i - \ell_i)^+ \leq \sum_{i=1}^\tau (k_i - \ell_i)^+ \leq l_0 - k_0,$$

proving one implication. For the converse observe that we can write

$$\sum_{i=1}^\tau (k_i - \ell_i)^+ = \sum_{i \in J} (k_i - \ell_i)^+ + \sum_{i \notin J} (k_i - \ell_i)^+$$

if we set $J$ to be the subset of $\{ 1, \ldots, \tau \}$ consisting of all indices for which $k_i \geq \ell_i$. Consequently, $\sum_{i=1}^\tau (k_i - \ell_i)^+ = \sum_{i \in J} (k_i - \ell_i)^+$, and this gives immediately $k \leq_L \ell$, in virtue of the present assumption.

Having introduced the reference poset, we present the definition of character for the anonymous, weakly group strategy-proof scfs.

**Definition 4.2** The character function for the two-valued, anonymous, weakly group strategy-proof scfs is the function $\chi : \mathcal{P} \to \{ 0, 1, \ldots, n \}^{\tau+1} = L$, defined by

$$\chi = (\chi_0, \chi_1, \ldots, \chi_\tau) \text{ with } \chi_0(P) = |D(a, P)|, \chi_i(P) = |I_i(P)| \text{ for } i = 1, \ldots, \tau.$$

We now present the main result of this section.

**Theorem 4.3** Let $\phi : \mathcal{P} \to \{ a, b \} \subseteq A$ be a scf. Then,

$$\phi \in \Phi_{AN} \iff \phi \text{ is } (\chi, a)-\text{monotone}.$$
proof: Let us suppose that \( \phi \) is \((\chi, a)\)-monotone. Because of Proposition 3.11, we know that \( \phi = \phi_C \) for some SOC subset \( C \) of the poset \((\chi(P), \leq_L)\). Since for every profile \( P \) and every permutation \( \sigma \) of agents the cardinalities \( |D(a, P \circ \sigma)|, |I_i(P \circ \sigma)| \) are respectively the same as \( |D(a, P)|, |I_i(P)| \), the character values \( \chi(P) \) and \( \chi(P \circ \sigma) \) are identical. Hence, by definition of \( \phi_C \), we obtain that \( \phi_C \) is anonymous. It remains to prove that it is also weakly group strategy-proof.

By appealing to Theorem 3.3, we can prove that it is almost monotone:

\[
P \geq_a Q, \phi_C(Q) = a \Rightarrow \phi_C(P) = a,
\]

or, equivalently, that

\[
P \geq Q, \chi(Q) \in C \Rightarrow \chi(P) \in C.
\]

The assertion will follow from showing that

CLAIM 1: \( P \geq Q \Rightarrow \chi(P) \geq_L \chi(Q) \),

and from the super order closedness of \( C \). CLAIM 1 is proved in Appendix 7.3.

For the converse, let us assume that \( \phi \in \Phi_{AN} \). In Appendix 7.4 the following claim is proved.

CLAIM 2: The set \( C = \{ \chi(P) : \phi(P) = a \} \) is SOC in \((\chi(P), \leq_L)\).

As a consequence, we have that \( \phi = \phi_C \), i.e. \( \phi(P) = a \leftrightarrow \phi_C(P) = a \). Indeed the implication \( \phi(P) = a \Rightarrow \phi_C(P) = a \) is true by definition of \( C \), whereas the converse \( \phi(P) = a \Leftarrow \phi_C(P) = a \) derives from the anonymity of \( \phi \) together with the definition of the character function. Finally, Proposition 3.11 applies.

Because of Proposition 3.7 and Proposition 3.11 we have the following corollary of the above Theorem 4.3.

Corollary 4.4 Suppose the set \( \mathcal{P} \) of the feasible profiles is \( \mathcal{W}^V \). Let \( a, b \) be two alternatives belonging to the set \( A \). Assume that \( V \), as well as the preferences in \( \mathcal{W} \) for which the alternatives \( a \) and \( b \) are indifferent, are finite. Then, there is a bijection \( \Delta \mapsto \phi_\Delta \) from the set of all antichains \( \Delta \) of \((\chi(P), \leq_L)\) to the set \( \Phi_{AN} \) of all anonymous, weakly group strategy-proof scfs with values in \( \{a, b\} \). For an antichain \( \Delta \) the scf \( \phi_\Delta \) is defined as follows

\[
\phi_\Delta(P) = \begin{cases} 
a, & \text{if } \exists (x_0, x_1, \ldots, x_\tau) \in \Delta : (|D(a, P)|, |I_1(P)|, \ldots, |I_\tau(P)|) \geq_L (x_0, x_1, \ldots, x_\tau) \\
b, & \text{otherwise}
\end{cases}
\]

Hence in the anonymous case for the formation of the collective decision the primitives are the components of a numerical vector. Precisely, the number of agents that vote for \( a \), and, for \( i = 1, \ldots, \tau \), the number of agents that express the \{a, b\}-indifference preference \( W_i \).

Example 4.5 We illustrate with an example the role of the character function.

To fix ideas let us suppose that in \( \mathcal{W} \) there are three preferences \( W_1, W_2, W_3 \) under which \( a \) and \( b \) are indifferent. The agents are eleven: \( V = \{v_1, \ldots, v_{11}\} \).
In this case it turns out that the appropriate poset $L$ to use consists of the 4-tuples of elements of $\{0,1,\ldots,11\}$, where for $x = (x_0, x_1, x_2, x_3)$, $y = (y_0, y_1, y_2, y_3) \in L$ we define

$$x \leq y \iff x_0 + \sum_{i=1}^{3} (x_i - y_i)^+ \leq y_0.$$ 

The character function is $P \in \mathcal{P} \mapsto \chi(P) \in L$ by letting $(\chi(P))_0$ be the number of agents $v \in V$ that prefer $a$ to $b$ under $P_v$, and $(\chi(P))_i$ be the number of agents for which $P_v = W_i$ ($i = 1, 2, 3$). Hence,

$$\chi(P) = \left\{ x \in L : \sum_{i=0}^{3} x_i \leq 11 \right\}.$$ 

The elements of $\Phi_{AN}$ are all and only the $\{a,b\}$-valued scfs $\phi$ defined on $\mathcal{P}$ which are monotone in the following sense

$$P, Q \in \mathcal{P}, \phi(Q) = a, \chi(P) \geq \chi(Q) \Rightarrow \phi(P) = a,$$

or, equivalently, all and only the scfs $\phi(\chi,C)$ where $C$ runs over the SOC subsets of $(\chi(\mathcal{P}), \geq)$. Since the latter is finite, we can equivalently parametrize $\Phi_{AN}$ with the minimal elements of $C$ rather than with $C$ itself. This identification tells us that the character function permits the description of all scfs in simple understandable terms. For example, if the minimal elements of $C$ are the following three incomparable quadruples $x = (2,4,0,3)$, $y = (4,1,3,1)$, $z = (4,2,4,0)$ of $\chi(\mathcal{P})$, the correspondent anonymous non-manipulable two-valued scf $\phi$ is

$$\phi(P) = \begin{cases} a, & \text{if at least one of the following } \chi(P) \geq x, \chi(P) \geq y, \chi(P) \geq z, \text{ holds true,} \\ b, & \text{otherwise.} \end{cases}$$

In words: the corresponding scf is the one that selects $a$ in each profile that shows at least the same support for the alternative $a$ than either $x$, or $y$, or $z$. Otherwise, the collective choice is $b$. The profile $x = (2,4,0,3)$, for example, simply means that two votes are for $a$, four agents choose $W_1$, three choose $W_3$, and no one chooses $W_2$. Whereas to support alternative $a$ no less than $x$ means that a profile $P$ forms under which:

- at least two votes are for $a$,
- the number of agents that are either in favor of $a$ or express the preference $W_j$ is at least $2 + x_j$,
- the number of agents that are either in favor of $a$ or express one of the two preferences $W_i, W_j (i \neq j)$, is at least $2 + x_i + x_j$,
- the number of agents that are either in favor of $a$ or choose one of the three preferences $W_i$ is at least nine. \hfill \Box

\footnote{Like the profile $P$ whose primitives are $(3,3,1,4)$.}
\footnote{Like for the profile $Q$ whose primitives are $(4,1,1,2)$.}
5 The unifying role of character and canonical functional form

Let us denote by $\Phi$ the class of all wGSP scfs $\phi: \mathcal{P} \to \{a, b\}$, under the basic assumptions that we have adopted, i.e.

- $\mathcal{P}$ is a quasi-cartesian restricted domain,
- the distinct alternatives $a, b$ belong to a larger set $A$ of alternatives.

In Section 3, for the class $\Phi$, we have defined a character with the help of which in Theorem 3.15 we have provided the canonical functional form for all the scfs in $\Phi$.

In the same way, in Section 4, we have defined a character for the class $\Phi_{AN}$ and provided the canonical functional form for all the scfs in this class.

In each of the respective cases above, if $\chi$ is the character corresponding to the respective class, the scfs in that class are all and only the functions

$$(\star) \quad \{\phi_{(X,C)}: C \text{ is a SOC subset of } \chi(\mathcal{P})\}.$$ 

In other words these two classes admit the same functional representation introduced in Definition 3.9. The character acts as a parameter that distinguishes the classes, and the SOC set $C$ acts as a parameter that distinguishes the scfs within a class.

The purpose of this section is to consider other classes of scfs, provide the corresponding character functions and describe the corresponding class in the fashion described by $(\star)$ above.

We will first consider the classes:

- $\Phi^S$, the class of all wGSP scfs $\phi: \mathcal{P} \to \{a, b\}$, under the assumption that $\mathcal{P} = S(A)^V$ is the strict universal domain;
- $\Phi^{bi}$, the class of all wGSP scfs $\phi: \mathcal{P} \to \{a, b\}$, under the assumptions that $A = \{a, b\}$;
- $\Phi^{bi}_{AN}$, the subclass of $\Phi_{AN}$ under the assumptions that $A = \{a, b\}$.

As a further exemplification of the unifying possibilities offered by the structure of the function $\phi_{(X,C)}$, we also consider the subclass $\Phi_s$ of $\Phi$ consisting of all strongly group strategy-proof scfs (see Definition 4 in Barberà et al. [2]).

The possibility of unifying by means of $(\star)$ the description of so many relevant classes, is the reason why we have proposed the name canonical for such scfs.

5.1 Strict preferences.

We recall that a committee (also named simple game in the Game Theory literature) is (see [11]), a nonempty family of coalitions of agents which is closed under supersets. We also recall that a “voting by committee” is a scf $P \mapsto \phi_C(P)$, where $C$ is a committee of agents, and $\phi_C(P) = a \iff D(a, P) \in C$. 

16
When $V$ is finite, and $A = \{a, b\}$ Larsson and Svensson proved the following [8, Theorem 2]: “a voting rule defined for all strict preference profiles, is strategy-proof and onto if and only if it is voting by committees.”

Let us observe that this functional form characterization is canonical. Indeed, it is such by using the character function $\chi$ introduced in Definition 3.13 with reference to the strict universal domain. Let us suppose that $\mathcal{P}$ is the strict universal domain. The second and the third components of the character are, respectively, the empty set and the empty profile. Consequently we can identify $(\chi(\mathcal{P}), \leq_D)$ with the power set of $V$ ordered by the set inclusion and the concept of super order closed subset of $\chi(\mathcal{P})$ gives back either stricto sensu a committee or the empty set or the entire power set. Hence Theorem 3.15 gives

**Corollary 5.1** Let $V, A$ be arbitrary and $a, b$ two distinct elements of $A$. Suppose that $\mathcal{P}$ is the strict universal domain. Let $\chi : \mathcal{P} \to 2^V$ defined by $\chi(P) = D(a, P)$. Then, $\Phi^S$ consists of all and only the functions $P \mapsto \phi(\chi, C)(P)$, where $C$ is a subset of agents closed under supersets.

The above corollary generalizes [8, Theorem 2].

### 5.2 Two alternatives only

In this subsection we assume $A = \{a, b\}$, i.e. agents express preferences only about the realizable alternatives. This is the framework of Basile et al. [5] in which finiteness of $V$ and universality of the domain $\mathcal{P}$ is assumed. We do not assume these hypothesis in the following.

In this framework Theorems 3.15 and 3.17 can be simplified on the basis of the following observations. Since, given a profile $P$, the partial profile $(P_v)_{v \in I(P)}$ is uniquely determined, we identify the triple $(\chi(P), I(P), (P_v)_{v \in I(P)})$ with the pair $(D(a, P), \sim_D(a, P))$. The role of the poset $L$, instead of $D$, can be played by the simpler poset $\mathcal{V}$ consisting of the set $\{(S, W) \in 2^V \times 2^V : S \subseteq W\}$ endowed with the componentwise set inclusion as partial order $\leq \mathcal{V}$. Likewise the role of the character of Definition 3.13 can be played by a simpler one. We mean the character $\chi_{bi} : \mathcal{P} \to \mathcal{V}$ defined as $\chi_{bi}(P) = (D(a, P), \sim_D(a, P))$.

We recall that $\mathcal{V}$ has been introduced in Basile et al. in [5]. Its elements were named veto pairs. Now, Theorems 3.15 and 3.17 specialize as follows in case $A = \{a, b\}$.

**Theorem 5.2** (Theorem 3.15 in case of two alternatives only). Assume $A = \{a, b\}$. On the quasi–cartesian restricted domain $\mathcal{P}$, the class $\Phi_{bi}$ consists of all and only the functions $\phi(\chi_{bi}, C)$, with $C$ SOC set of veto pairs.

The above theorem is new with respect to [5], since $V$ is arbitrary and $\mathcal{P}$ need not to be the universal domain.

The Corollary below is the veto pairs representation of non-manipulable scfs provided in [5, Theorem 3.3], limitedly to the universal domain. Again we have a functional form characterization of canonical type.

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9 They mean individually, but this is the same as wGSP, as we saw previously.

10 The poset $\mathcal{V}$ is clearly embeddable into $D$. 17
Corollary 5.3 (Theorem 3.17 in case of two alternatives only). Suppose $V$ is finite, $A = \{a, b\}$, and $\mathcal{P} = \times_{v \in V} \mathcal{W}_v$. Then, there is a bijection $\Sigma \mapsto \phi_\Sigma$ from the set of all antichains $\Sigma$ of $(\chi_b(\mathcal{P}), \leq_V)$ to $\Phi^{bi}$. For an antichain $\Sigma$ the scf $\phi_\Sigma$ is defined as follows

$$\phi_\Sigma(P) = \begin{cases} a, & \text{if } \exists (S, T) \in \Sigma : (D(a, P), \tilde{D}(a, P)) \geq_V (S, T) \\ b, & \text{otherwise} \end{cases}.$$

Example 1.1 marks once more the distinction between assuming or not that $A = \{a, b\}$. Indeed if we consider, with $A \supset \{a, b\}$, the $\{a, b\}$-valued scfs $\phi_\Sigma$ defined above are wGSP but do not exhaust all $\{a, b\}$-valued wGSP scfs. The function of Example 1.1 cannot be represented as a $\phi_\Sigma$. Its functional form comes from Theorem 3.17 and cannot come from Corollary 5.3. This is so because $\phi_\Sigma$, differently from $\phi_\Delta$ in Theorem 3.17, takes only care of which agents $v$ are indifferent and not even of their preference $P_v$.

This tells us that investigating the setting with $A \supset \{a, b\}$ is not obvious from [5].

Similar considerations to the previous ones can be carried out in the anonymous case with exactly two alternatives. The poset $L$ of Section 4 becomes \{0,1,\ldots,n\} \times \{0,1,\ldots,n\}, endowed with the order relation: $x \leq_L y \iff x_0 \leq y_0$, and $x_0 + x_1 \leq y_0 + y_1$. Assume $\mathcal{P}$ is the universal domain, so that we are in the framework of [5]. Let $N_0$ be the set \{0,1,2,\ldots\}. It is straightforward to verify that the set $\chi(P) = \{(x_0,x_1) \in N_0 \times N_0 : x_0 + x_1 \leq n\}$ with the order $\geq_L$ is order isomorphic to the poset (introduced in [5] Subsection 4.1) $(G, \geq_G)$ where

$$G = \{(x_0,x_1) \in \{0,1,\ldots,n\}^2 : x_0 \leq x_1\},$$

and the order is $y_{\geq_G} x \iff y_i \geq x_i$, $i = 0,1$.

Hence, Corollary 4.4 says that the veto cardinality representation [5, Theorem 4.3] by of Basile et al. is a functional form characterization of $\Phi_{AN}^{bi}$ of canonical type.

We conclude this section by remarking that even the quota majority method (see [10, Corollary p. 63]) is a functional form characterization of canonical type. Indeed, in the case of strict preferences only, the poset $L$ of Section 4 becomes \{0,1,\ldots,n\}, endowed with the usual order. The character is $\chi(P) = |D(a, P)|$. The fact that all and only the non-manipulable scfs are the functions $\phi_C$ with $C$ SOC subset of \{0,1,\ldots,n\}, gives back the quota majority method.

### 5.3 Strong group strategy-proofness

In this section we exemplify how the character function approach applies to the result by Barberà et al. [2] concerning the functional form of strongly group strategy-proof scfs. Notice that their results require the assumption about $\mathcal{P}$ they refer to as “minimal assumption”:

$\mathcal{P} = \times_{v \in V} \mathcal{W}_v$ with $\mathcal{W}_v \subseteq \mathcal{W}(A)$, and every $\mathcal{W}_v$ contains at least three preferences $W^i \in \mathcal{W}(A)$ for $i = \sim, a, b$ (depending on $v$) such that in $W^\sim a, b$ are indifferent, in $W^a a$ is preferred to $b$, in $W^b b$ is preferred to $a$.

Let us recall that a scf is strongly group strategy-proof when it is immune to weak manipulation. In contrast to Definition 2.1 of strong manipulation the weak manipulation is defined as follows.
Definition 5.4 Let $\phi$ be scf. We say that a coalition $D$ can weakly manipulate a profile $P \in \mathcal{P}$ under $\phi$ if there is another profile $Q \in \mathcal{P}$ such that

- every agent $v$ in $D^c$ has the same preference in both $P$ and $Q$, i.e. $P_v = Q_v$;
- for every agent $v$ in $D$, $\phi(Q) \succsim_{P_v} \phi(P)$, and at least for one agent $v_o$ of $D$, $\phi(Q) \succ P_{v_o} \phi(P)$.

i.e. all members of the manipulating coalition are not worse off and at least one is better off.

Let $\widehat{\mathcal{U}}$ be the set $\{P \in \mathcal{P} : I(P) = V\}$, i.e. the set of all feasible profiles under which all agents, unanimously, are indifferent between alternative $a$ and alternative $b$.

To fix ideas we consider the case that there are at least three agents in $V$.

It is known from [2, Theorem 3] that a scf $\phi$ with range of cardinality two is strongly group strategy-proof if and only if it is, with a suitable choice of a subset $\mathcal{U}$ of $\widehat{\mathcal{U}}$, either a **veto for $b$**: 

$$\phi(P) = a \overset{\text{def}}{\iff} \text{either } D(a, P) \neq \emptyset \text{ or } P \in \mathcal{U},$$

or a **veto for $a$**: 

$$\phi(P) = a \overset{\text{def}}{\iff} \text{either } [D(a, P) \neq \emptyset \text{ and } D(b, P) = \emptyset] \text{ or } P \in \mathcal{U}.$$ 

Let $(L, \geq)$ be the poset (see the figure below) with $L = \{-1, 0, 1\} \cup \widehat{\mathcal{U}}$ and where 1 and $-1$ are respectively the maximum and the minimum, and the remaining elements being pairwise incomparable.

![Fig 1](image)

Let us define the character function $\chi : \mathcal{P} \rightarrow L$ as follows:

$$\chi(P) = \begin{cases} 
1, & \text{if } D(a, P) \neq \emptyset \text{ and } D(b, P) = \emptyset, \\
0, & \text{if } D(a, P) \neq \emptyset \text{ and } D(b, P) \neq \emptyset, \\
-1, & \text{if } D(a, P) = \emptyset \text{ and } D(b, P) \neq \emptyset, \\
P, & \text{if } D(a, P) = \emptyset \text{ and } D(b, P) = \emptyset.
\end{cases}$$

The SOC subsets $C$ of $(L, \geq)$ are: the empty set; $L$; the sets $\{0, 1\} \cup \mathcal{U}$ with $\emptyset \subseteq \mathcal{U} \subseteq \widehat{\mathcal{U}}$; the sets $\{1\} \cup \mathcal{U}$ with $\emptyset \subseteq \mathcal{U} \subseteq \widehat{\mathcal{U}}$. When $C$ is empty $\phi_{(\chi, C)}$ is constantly $b$. When $C = L$ then $\phi_{(\chi, C)}$

\footnote{There is a difference in case $|V| = 2$, but for our illustrative purposes the case $|V| > 2$ is sufficient.}
is constantly $a$. For $C = \{0, 1\} \cup \mathcal{U}$, the scf $\phi_{(\chi,C)}$ is the veto for $b$ and in the latter case we get the veto for $a$, in this way exhausting the class $\Phi_s$ of all strongly group strategy-proof scfs with range of cardinality at most two. In other words the canonical functional characterization applies to $\Phi_s$ too.

6 Conclusions

In this study we have considered a social choice model with an arbitrary set of agents $V$, and a set $A$ of at least two alternatives. The collective choice is limited to either $a$ or $b$ ($a, b \in A$), and is based upon a preference profile of the agents who also consider the other alternatives in $A$. Indifference is permitted. The non-manipulability notion adopted is the weak group strategy-proofness.

We have provided the functional form for each of the following classes of $\{a, b\}$-valued scfs $\phi$:
- all non-manipulable scfs,
- all non-manipulable and anonymous scfs (here $V$ is a finite set of agents),
- all non-manipulable scfs defined on the strict universal domain,
- all non-manipulable scfs, under the assumptions that $A = \{a, b\}$,
- all non-manipulable anonymous, under the assumptions that $A = \{a, b\}$,
- all strongly group strategy-proof scfs.

In each of these cases, for an appropriate poset $(L, \geq_L)$, an appropriate function $\chi$ from the set $\mathcal{P}$ of feasible profiles to $L$, and appropriate subset $C$ of $L$, we have shown that

$$\phi(P) = a \text{ if and only if } \chi(P) \in C,$$

The set $C$ is required to have the following properties
(1) $C$ is contained into the $\chi$-image of $\mathcal{P}$,
(2) $\chi(Q) \in C$, and $\chi(P) \geq_L \chi(Q) \Rightarrow \chi(P) \in C$.

The distinction of the different cases is marked by a specific pair $L, \chi$ for each case. So, the different character of the above types of social choice functions is captured by $\chi : \mathcal{P} \rightarrow L$, and for this reason we named it character function.

The character function with the help of the set $C$, describes how a given scf selects the alternative, i.e. gives functional forms characterizations of a particular class. Such characterizations share the same mathematical structure. For this unifying effect we have spoken of canonical functional forms.

When the number of agents and the number of alternatives are finite, in the parametrization of $\phi$ we can replace $C$ with the set of its $\geq_L$-minimal elements. This forms a set (of incomparable elements in $L$) whose cardinality is much smaller than the cardinality of $C$. This way of functional form expression of a social choice function is not only a characterization per se, but also an implementable algorithm. Indeed, the identification of $\phi(P)$ is based on the comparison of $\chi(P)$ with the minimal elements parametrizing $\phi$. It is not necessary to know entirely the profile $P$. For the formation of the collective choice, $\chi(P)$ gives the primitives one needs.
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7 Appendix

In this section we present the proof of Proposition 3.7, verify that the relation $\leq_{D}$ of Definition 3.12 is a partial order, and prove the two claims contained in the proof Theorem 4.3.
7.1 The proof of Proposition 3.7

A lemma precedes the proof of Proposition 3.7. Assume here that \((L, \geq)\) is a given poset.

Lemma 7.1 Let \(F\) be a nonempty finite subset of \(L\). Then, every element of \(F\) is greater than a minimal element of \(F\). In other words: \(m(F) \neq \emptyset\).

**Proof:** We show that for every \(f_0 \in F\) there is \(x \in m(F)\) such that \(f_0 \geq x\). If \(f_0\) is not minimal itself, the set \(F_0 = \{x \in F \setminus \{f_0\} : f_0 \geq x\}\) \(\neq \emptyset\). Given an element \(f_1 \in F_0\), if it belongs to \(m(F)\), we are done. Otherwise the set \(F_1 = \{x \in F \setminus \{f_0, f_1\} : f_1 \geq x\} \neq \emptyset\).

Given an element \(f_2 \in F_1\), if it belongs to \(m(F)\), we are done. Otherwise the set \(F_2 = \{x \in F \setminus \{f_0, f_1, f_2\} : f_2 \geq x\} \neq \emptyset\). Since \(F\) is finite, this procedure stops, finding a minimal element of \(F\) smaller than the starting point \(f_0\). \(\square\)

Proposition 3.7 The antichains of a finite poset \(L\) are in a one–to–one correspondence with the SOC subsets of \(L\).

**Proof:** We show that the map \(C \mapsto m(C)\) which associates to every SOC subset \(C\) of \(L\) the antichain consisting of its minimal elements is both

- injective: \([m(C_1) = m(C_2), \text{ where } C_1, C_2 \text{ are SOC subsets of } L] \Rightarrow C_1 = C_2\),

and

- surjective: every antichain \(X\) of \(L\) is the image \(m(C)\) of some SOC subset \(C\) of \(L\).

Assume \(m(C_1) = m(C_2)\), where \(C_1, C_2\) are SOC subsets of \(L\), by symmetry it is enough proving that \(C_1 \subseteq C_2\) to achieve injectivity. So, take \(x \in C_1\). Since \(L\) is finite, we can appeal to the previous lemma to find a minimal element \(m\) of \(C_1\) such that \(x \geq m\). Since \(m(C_1) = m(C_2)\), the element \(m\) belongs to \(C_2\) also. Because \(C_2\) is SOC, then it contains \(x\) too, as desired.

To achieve surjectivity, take a antichain \(X\) and set \(C\) to be the SOC subset of \(L\) defined as follows

\[C = \{x \in L : \exists m \in X \text{ such that } x \geq m\}.\]

We show that \(X \subseteq m(C) \subseteq X\).

For the first inclusion, take \(x \in X\). The element \(x\) also belongs to \(C\), more precisely we have to show that it is also a minimal element of \(C\), i.e. from \([x \geq y, y \in C]\) we must be able to derive that \(y = x\). By definition of the set \(C\), we get the existence of \(m \in X\) such that \(y \geq m\). So by transitivity, for the two elements \(x, m \in X\) we have \(x \geq m\). Since \(X\) is an antichain, we get \(x = m\). Summarizing \(m = x \geq y = x = m\) and by antisymmetry we get \(y = x\), as desired.

For the second inclusion, let us take \(x \in m(C)\). Being \(x\) an element of \(C\) by definition, we can take \(m \in X\) with \(x \geq m\). Being \(C\) a superset of \(X\), we have \([x \geq m, \text{ and } x, m \in C]\). Being \(x\) a minimal element of \(C\), the elements \(x\) and \(m\) are identical so \(x \in X\). \(\square\)
7.2 The relation $\leq_D$ introduced in Definition 3.12 is a partial order

Transitivity. Let us suppose that $(S, T, \pi) \leq_D (S', T', \pi')$, and $(S', T', \pi') \leq_D (S'', T'', \pi'')$. This two conditions mean respectively

$- S \subseteq S'$, $S \cup T \subseteq S' \cup T'$, and $v \in T \cap T' \Rightarrow \pi_v = \pi_v'$,

$- S' \subseteq S''$, $S' \cup T' \subseteq S'' \cup T''$, and $v \in T' \cap T'' \Rightarrow \pi_v' = \pi_v''$,

Hence the two conditions $S \subseteq S''$, $S \cup T \subseteq S'' \cup T''$ that are part of the relation $(S, T, \pi) \leq_D (S'', T'', \pi'')$, that we want to prove are immediate. So we only have to verify that $\pi_v = \pi_v''$. Observe that if $v \in T \cap T''$, then either $v \in S'$ or $v \in T'$. But the first possibility would give $v \in S''$ which is impossible since $S''$ and $T''$ are disjoint. So the only possibility is $v \in T'$. Consequently we have $v \in T \cap T'$, and $v \in T' \cap T''$. Hence, $\pi_v = \pi_v' = \pi_v''$.

Antisymmetry. Let us suppose: $(S, T, \pi) \leq_D (S', T', \pi')$, and $(S', T', \pi') \leq_D (S, T, \pi)$. Namely that

$- S \subseteq S'$, $S \cup T \subseteq S' \cup T'$, and $v \in T \cap T' \Rightarrow \pi_v = \pi_v'$,

$- S' \subseteq S$, $S' \cup T' \subseteq S \cup T$, and $v \in T' \cap T \Rightarrow \pi_v' = \pi_v$.

So $S = S'$ and $S \cup T = S' \cup T'$ are immediate. Since $S$ and $T$ are disjoint (and $S'$ and $T'$ as well), one gets that $T$ and $T'$ are identical and the partial profiles $\pi$ and $\pi'$ too.

7.3 CLAIM 1 of Theorem 4.3

We have to prove the following implication: $P \geq Q \Rightarrow L(\chi(P)) \geq L(\chi(Q))$.

PROOF: Let us assume that $P \geq Q$. This is equivalent to:

\[(1) \quad D(a, Q) \subseteq D(a, P), \quad D(b, P) \subseteq D(b, Q), \quad \text{and} \quad v \in I(P) \cap I(Q) \Rightarrow P_v = Q_v.\]

From the two partitions $\{D(a, Q), I(Q), D(b, Q)\}$, and $\{D(a, P), I(P), D(b, P)\}$, of $V$, due to (1), we have the partitions

\[(2) \quad V = D(a, Q) \cup [D(a, P) \setminus D(a, Q)] \cup [I(P) \cap I(Q)] \cup [D(b, Q) \cap I(P)] \cup D(b, P), \]
and

\[(3) \quad V = D(a, Q) \cup [I(Q) \cap D(a, P)] \cup [I(Q) \cap I(P)] \cup [D(b, Q) \setminus D(b, P)] \cup D(b, P).\]

Let us set: $D = I(Q) \cap I(P)$, $E = [D(a, P) \setminus D(a, Q)]$, $F = [D(b, Q) \setminus D(b, P)]$. Then, for every $i \in \{1, \ldots, \tau\}$, using (1) again, we have:

$I_i(Q) = [I_i(Q) \cap E] \cup [I_i(Q) \cap D]$, from (2),

and

$I_i(P) = [I_i(P) \cap F] \cup [I_i(P) \cap D]$, from (3).
Also notice that \( I_i(Q) \cap D = I_i(P) \cap D \) \[12\]

Now, because of the latter equation and since the set \( D \) is disjoint from both \( E \) and \( F \), by setting \( J = \{ i : |I_i(Q)| > |I_i(P)| \} \), we have

\[
\sum_{i=1}^\tau [ |I_i(Q)| - |I_i(P)| ]^+ = \sum_{i \in J} |I_i(Q)| - |I_i(P)| = \sum_{i \in J} |I_i(Q) \cap E| - |I_i(P) \cap F| \leq \]

\[
\leq \sum_{i \in J} |I_i(Q) \cap E| \leq \sum_{i=1}^\tau |I_i(Q) \cap E| = |I(Q) \cap E| \leq |E| = |D(a, P)| - |D(a, Q)|
\]

(again because of (1)) which is the same as the desired relation \( \chi(P) \geq \chi(Q) \).

\[\square\]

### 7.4 CLAIM 2 of Theorem 4.3

The claim is: the set \( C = \{ \chi(P) : \phi(P) = a \} \) is SOC in \( (\chi(P), \leq_L) \).

**proof:** Let us consider two profiles \( P', Q' \) with \( \phi(P') = a \), and \( \chi(P') \leq_L \chi(Q') \). We show that \( \phi(Q') = a \).

For brevity, we set \( \chi(P') = (k_0, k_1, \ldots, k_\tau) \), and \( \chi(Q') = (\ell_0, \ell_1, \ldots, \ell_\tau) \). Moreover, we partition the set \( \{ 1, 2, \ldots, \tau \} \) as \( J \cup J^c \) where \( j \in J \Leftrightarrow k_j > \ell_j \). Without loss of generality we assume that \( J \neq \emptyset \) (the case \( J = \emptyset \) is similar and in a sense easier). Let us enumerate the elements of \( J \) and \( J^c \) as follows: \( J = \{ j_1, j_2, \ldots, j_\beta \} \) \( J^c = \{ \ell_1, \ell_2, \ldots, \ell_\alpha \} \).

We shall introduce two new profiles \( P \) and \( Q \), respectively permutations of \( P' \), and \( Q' \), and show that from \( \phi(P) = a \) (due to anonymity) we get \( \phi(Q) = a \). Again anonymity will produce the asserted \( \phi(Q') = a \).

For convenience we denote the agents as \( v_1, v_2, \ldots, v_n \), and use the intuitive interval notations: \( V = [v_1, v_n] \), \( \{ v_{\gamma+1}, v_{\gamma+2}, \ldots, v_\beta \} = [v_\gamma, v_\beta] \), and so forth.

Let us consider the sequence of integers

\[
0 \leq k_0 \leq \gamma_1 \leq \cdots \leq \gamma_\alpha \leq \delta_1 \leq \cdots \leq \delta_\beta \leq n,
\]

where:

\[
\gamma_r = k_0 + k_{i_1} + \cdots + k_{i_r} \text{ for } r = 1, \ldots, \alpha, \text{ and }
\delta_r = \gamma_1 + \ell_{j_1} + \cdots + \ell_{j_r} \text{ for } r = 1, \ldots, \beta.
\]

Obviously,

\[
\gamma_\alpha = k_0 + \sum_{i \in J^c} k_i, \quad \delta_\beta = k_0 + \sum_{i \in J^c} k_i + \sum_{j \in J} \ell_j,
\]

and since we have assumed \( J \neq \emptyset \) we have

\[
\delta_\beta < w(P') = \sum_{i=0}^\tau k_i.
\]

\[\footnote{We show the inclusion \( \subseteq \). From } v \in I_i(Q) \cap D = I_i(Q) \cap I(P), \text{ due to (1), we have } P_v = Q_v = W_i. \text{ Hence } v \in I_i(P).\]

\[24\]
On the other hand, from \((k_0, k_1, \ldots, k_\tau) \leq L(\ell_0, \ell_1, \ldots, \ell_\tau)\) we get \(w(P') \leq \delta_\beta + (\ell_0 - k_0)\). Hence, summing up, we have the following sequence of integers:

\[
0 \leq k_0 \leq \gamma_1 \leq \cdots \leq \gamma_\alpha \leq \delta_1 \leq \cdots \leq \delta_\beta < w(P') \leq \delta_\beta + (\ell_0 - k_0) \leq w(Q') \leq n
\]

by means of which we define the mentioned profiles \(P\) and \(Q\).

The next figure illustrates these definitions.

**Definition of \(P\):**
- on the set \([v_1, v_{k_0}]\) of agents we assign (arbitrarily) the preferences of the agents in \(D(a, P)\);
- on the set \([v_{k_0}, v_{\gamma_1}]\) we assign the preference \(W_{i_1}\) to all agents;
- \(\ldots\)
- on the set \([v_{\gamma_{\alpha-1}}, v_{\gamma_\alpha}]\) we assign the preference \(W_{i_\alpha}\) to all agents;
- on the set \([v_{\gamma_\alpha}, v_{\delta_1}]\) we assign the preference \(W_{j_1}\) to all agents;
- \(\ldots\)
- on the set \([v_{\delta_{\beta-1}}, v_{\delta_\beta}]\) we assign the preference \(W_{j_\beta}\) to all agents;
- on the set of agents \([v_{\delta_\beta}, v_{w(P')}\)] we distribute the remaining preferences \(W_{j}\), each being present \((k_j - \ell_j)\) more times in the original profile \(P'\);
- to the remaining agents in \([v_{w(P')}, v_n]\) we assign (arbitrarily) the preferences of the agents in \(D(b, P')\).

**Definition of \(Q\):**
- on the set \([v_1, v_{k_0}]\) of agents we assign (arbitrarily) preferences of \(k_0\) agents in \(D(a, Q')\); since preferences of remaining \((\ell_0 - k_0)\) agents in \(D(a, Q')\) have not been selected, we distribute them (arbitrarily) to the agents of the set \([v_{\delta_\beta}, v_{\delta_\beta} + \ell_0 - k_0]\);
- to the agents of the sets \([v_{k_0}, v_{\gamma_1}], \ldots, [v_{\gamma_{\alpha-1}}, v_{\gamma_\alpha}], [v_{\gamma_\alpha}, v_{\delta_1}], \ldots, [v_{\delta_{\beta-1}}, v_{\delta_\beta}]\) we assign respectively the preference \(W_{i_1}, \ldots, W_{i_\alpha}, W_{j_1}, \ldots, W_{j_\beta}\);
- on the set of agents \([v_{\delta_\beta} + \ell_0 - k_0, v_{w(Q')}\)] we distribute the remaining preferences \(W_{i}\) each being present \((\ell_i - k_i)\) more times in the original profile \(Q'\);
– to the remaining agents in \([v_{w(Q')}, v_n]\) we assign (arbitrarily) the preferences of the agents in
\(D(b, Q')\).

Let us also introduce the profile \(Q^1\) identical to \(P\) from agent \(v_1\) till agent \(v_{\delta_\beta + \ell_0 - k_0}\), and identical to \(Q\) on the remaining agents, i.e. \(Q^1 = [P[v_1, v_{\delta_\beta + \ell_0 - k_0}], Q[v_{\delta_\beta + \ell_0 - k_0}, v_n]]\).

Now suppose that on the contrary \(\phi(Q) = b\). Considering the profile \(Q^1\), we have that \(\phi(Q^1)\) must be \(b\), otherwise the coalition \([v_1, v_{k_0}] \cup [v_{\delta_\beta}, v_{\delta_\beta + \ell_0 - k_0}]\) manipulates \(Q\) by presenting \(Q^1\). If \(\delta_\beta + (\ell_0 - k_0) = n\), then \(Q^1\) coincides with \(P\) and this is not possible since \(\phi(P) = a\). Therefore it must be the case that \(\delta_\beta + (\ell_0 - k_0) < n\), hence the coalition \([v_{\delta_\beta + (\ell_0 - k_0)}, v_n]\) manipulates \(P\) by presenting \(Q^1\). \(\square\)