On D-Wave Meson Spectroscopy and the $K^*(1410) - K^*(1680)$ Problem

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Abstract

The mass spectrum of D-wave mesons is considered in a nonrelativistic constituent quark model. The results show a common mass degeneracy of the isovector and isodoublet states of the $1^3D_1$ and $1^3D_3$ nonets, and suggest therefore that the $K^*(1680)$ cannot be the $I = 1/2$ member of the $1^3D_1$ nonet. They also suggest that the $\eta_2(1870)$, presently omitted from the Meson Summary Table, should be interpreted as the $I = 0$ $s\bar{s}$ state of the $1^1D_2$ nonet.

Key words: quark model, potential model, D-wave mesons
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1 Introduction

The existence of a gluon self-coupling in QCD suggests that, in addition to the conventional $q\bar{q}$ states, there may be non-$q\bar{q}$ mesons: bound states including gluons (gluonia and glueballs, and $q\bar{q}g$ hybrids) and multiquark states [1]. Since the theoretical guidance on the properties of unusual states is often contradictory, models that agree in the $q\bar{q}$ sector differ in their predictions about new states. Among the naively expected signatures for gluonium are

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i) no place in $q\bar{q}$ nonet,
ii) flavor-singlet coupling,
iii) enhanced production in gluon-rich channels such as $J/\Psi(1S)$ decay,
iv) reduced $\gamma\gamma$ coupling,
v) exotic quantum numbers not allowed for $q\bar{q}$ (in some cases).

Points iii) and iv) can be summarized by the Chanowitz $S$ parameter $^2$

$$S = \frac{\Gamma(J/\Psi(1S) \to \gamma X)}{\text{PS}(J/\Psi(1S) \to \gamma X)} \times \frac{\text{PS}(X \to \gamma\gamma)}{\Gamma(X \to \gamma\gamma)},$$

where PS stands for phase space. $S$ is expected to be larger for gluonium than for $q\bar{q}$ states. Of course, mixing effects and other dynamical effects such as form-factors can obscure these simple signatures. Even if the mixing is large, however, simply counting the number of observed states remains a clear signal for non-exotic non-$q\bar{q}$ states. Exotic quantum number states ($0^{--}, 0^{+-}, 1^{--}, 2^{+-}, \ldots$) would be the best signatures for non-$q\bar{q}$ states. It should be also emphasized that no state has yet unambiguously been identified as gluonium, or as a multiquark state, or as a hybrid.

In this paper we shall discuss $D$-wave meson states, the interpretation of which as members of conventional quark model $q\bar{q}$ nonets encounters difficulties $^3$. We shall be concerned with the four meson nonets which have the following $q\bar{q}$ quark model assignments, according to the most recent Review of Particle Physics $^4$:  
1) $1^1D_2 J^{PC} = 2^{--}, \pi_2(1670), \eta_2(\ ? \ ), \eta_2(\ ? \ ), K_2(1770)$
2) $1^3D_1 J^{PC} = 1^{--}, \rho(1700), \omega(1600), \phi(\ ? \ ), K^*(1680)$
3) $1^3D_2 J^{PC} = 2^{--}, \rho_2(\ ? \ ), \omega_2(\ ? \ ), \phi_2(\ ? \ ), K'_2(1820)$
4) $1^3D_3 J^{PC} = 3^{--}, \rho_3(1690), \omega_3(1670), \phi_3(1850), K^*_3(1780)$

and start with a discussion of the corresponding two problems associated with the isodoublet channel of these nonets. One of them is related to the $K^*(1410) - K^*(1680)$ problem, the other to possible $1^1D_2 - 3^1D_2$ mixing in the $I = 1/2$ channel.

The two mesons, $K^*(1680)$ (with mass $1714 \pm 20$ MeV and width $323 \pm 110$ MeV) and $K^*(1410)$ (1412 $\pm 12$ MeV, 227 $\pm 22$ MeV) are currently assigned to the $1^3D_1$ and $2^3S_1$ nonets, respectively (the latter, $2^3S_1 J^{PC} = 1^{--}, \rho(1450), \omega(1420), \phi(1680), K^*(1410)$, has the same flavor quantum numbers as the former), although, as the Particle Data Group (PDG) states, “the $K^*(1410)$ could be replaced by the $K^*(1680)$ as the $2^3S_1$ state” $^4$. The problem with these mesons is that the $K^*(1410)$ seems too light to be the $2^3S_1$ state, even if one takes into account possible $2^3S_1 - 1^3D_1$ mixing. Similarly, the $K^*(1680)$ seems too light to be the $1^3D_1$. One may doubt even the existence of the $K^*(1410)$, as suggested first by Törnqvist $^5$, since it (as well as the $K^*(1680)$) has been observed by only one group, LASS $^6$, although with superior statistics, in partial wave analyses under the much stronger $K^*_0(1430)$ and $K^*_0(1430)$. Two older experiments $^6, ^7$ quote a considerably higher mass, $\simeq 1500$ MeV. In addition, its $K\pi$ branching ratio is suspiciously small, only $(6.6 \pm 1.3)\%$. On the other hand, the $K^*(1680)$ has a suspiciously large total width ($\sim 400$ MeV), much larger than typical hadron widths, and a natural suspicion would be that it is really composed of two states of normal width ($\sim 150 - 200$ MeV) $^6$, quite analogously to what has been suggested to be the case for the $\rho(1600)$ and $\omega(1600)$ which have been
resolved into \( \rho(1450) \) plus \( \rho(1700) \) and \( \omega(1420) \) plus \( \omega(1600) \) \[10\]. The masses of the two states contained in the \( K^*(1680) \) were determined in ref. \[6\] to be \( 2 \, ^3S_1(\approx 1608) \) and \( 1 \, ^3D_1(\approx 1784) \). This is in agreement with the values obtained by Godfrey and Isgur in a relativized quark model \[11\], \( 2 \, ^3S_1(1580) \), \( 1 \, ^3D_1(1780) \). An older experiment on the \( K^*(1680) \) quotes a mass of the same order, \( \sim 1800 \text{ MeV} \) \[8\].

Theoretically, for the four \((n, L)\)-wave meson nonets, the isoscalar and isovector members of the \( n \, ^3L_L \) and \( n \, ^1L_L \) nonets with the same charge cannot mix, since they have opposite \( C\)- and \( G\)-parity, as long as one neglects \( SU(2) \) breaking. However, their isodoublet counterparts (strange, charmed, ... mesons) do not possess definite \( C\)-parity and, therefore, can in principle mix when only \( SU(3) \) flavor symmetry is broken. This type of mixing can take place for all \( L \geq 1 \) mesons, as follows,

\[
\begin{pmatrix}
Q_{\text{high}} \\
Q_{\text{low}}
\end{pmatrix}
= \begin{pmatrix}
\cos \theta_{nL} & \sin \theta_{nL} \\
-\sin \theta_{nL} & \cos \theta_{nL}
\end{pmatrix}
\begin{pmatrix}
\rho \, ^1L_L \\
\rho \, ^3L_L
\end{pmatrix},
\]

where \( Q \) stands for the \( K, D, D_s, \ldots \). It is known that this mixing actually takes place for the \( P\)-wave mesons where the \( I = 1/2 \) \( K_{1A} \) and \( K_{1B} \) states of the \( 1 \, ^3P_1 \) and \( 1 \, ^1P_1 \) nonets, respectively, mix, leading to the physical \( K(1270) \) and \( K(1400) \) states \[12, 13\]. If such a mixing is also the case for the \( D\)-wave mesons, a question suggests itself regarding the physical masses of the \( I = 1/2 \) states of the \( 3 \, ^3D_2 \) and \( 1 \, ^3D_2 \) nonets, which we call \( K_{2A} \) and \( K_{2B} \), respectively, in the following.

If the assumption of Törnqvist about the \( K^*(1680) \) \[6\] is correct, one would have simultaneous mass near-degeneracy of the \( 1 \, ^3D_1 \) and \( 1 \, ^3D_3 \) meson nonets in the isovector and isodoublet channels, since in this case \( M(\rho(1700)) \approx M(\rho_3(1690)) \), \( M(K^*(1780)) \approx M(K_3^*(1780)) \). As shown in our previous paper \[14\], similar degeneracy of the \( 1 \, ^3P_0 \) and \( 1 \, ^3P_2 \) nonets is an intrinsic property of \( P\)-wave meson spectroscopy and may be straightforwardly understood in a nonrelativistic constituent quark model. We now wish to apply this model to the \( D\)-wave mesons in order to show that near-degeneracy of the \( 3 \, ^3D_3 \) and \( 1 \, ^3D_1 \) nonets mentioned above also takes place. We note that this result is a direct consequence of the nonrelativistic constituent quark model which we discuss below; this mass near-degeneracy of the two nonets does not depend on the values of the input parameters, and cannot be considered as a numerical coincidence, as the results of, e.g., Godfrey and Isgur \[11\], may be viewed (their model finds the values \( M(K^*) = 1780 \text{ MeV} \), \( M(K_3^*) = 1790 \text{ MeV} \) for the \( I = 1/2 \) \( 1 \, ^3D_1 \) and \( 1 \, ^3D_3 \) meson masses). We also expect our model to provide relevant information on possible \( K_{2A} - K_{2B} \) mixing.

2 Nonrelativistic constituent quark model

In the constituent quark model, conventional mesons are bound states of a spin \( 1/2 \) quark and spin \( 1/2 \) antiquark bound by a phenomenological potential which has some basis in QCD \[13\]. The quark and antiquark spins combine to give a total spin
0 or 1 which is coupled to the orbital angular momentum \( L \). This leads to meson parity and charge conjugation given by \( P = (-1)^{L+1} \) and \( C = (-1)^{L+S} \), respectively. One typically assumes that the \( q\bar{q} \) wave function is a solution of a nonrelativistic Schrödinger equation with the generalized Breit-Fermi Hamiltonian\(^1\)

\[
H_{BF} \psi_n(\mathbf{r}) \equiv (H_{kin} + V(\mathbf{p}, \mathbf{r})) \psi_n(\mathbf{r}) = E_n \psi_n(\mathbf{r}),
\]

where \( H_{kin} = m_1 + m_2 + \mathbf{p}^2/2\mu - (1/m_1^2 + 1/m_2^2) \mathbf{p}^4/8, \mu = m_1 m_2/(m_1 + m_2), m_1 \) and \( m_2 \) are the constituent quark masses, and to first order in \((v/c)^2 = \mathbf{p}^2 c^2/E^2 \simeq \mathbf{p}^2/m^2 c^2\), \( V(\mathbf{p}, \mathbf{r}) \) reduces to the standard nonrelativistic result,

\[
V(\mathbf{p}, \mathbf{r}) \simeq V(r) + V_{SS} + V_{LS} + V_T, \tag{3}
\]

with \( V(r) = V_V(r) + V_S(r) \) being the confining potential which consists of a vector and a scalar contribution, and \( V_{SS}, V_{LS} \) and \( V_T \) the spin-spin, spin-orbit and tensor terms, respectively, given by \([3]\)

\[
\begin{align*}
V_{SS} &= \frac{2}{3m_1 m_2} \mathbf{s}_1 \cdot \mathbf{s}_2 \Delta V_V(r), \\
V_{LS} &= \frac{1}{4m_1^2 m_2^2} \left( \left[ (m_1 + m_2)^2 + 2m_1 m_2 \right] \mathbf{L} \cdot \mathbf{S}_+ + (m_2^2 - m_1^2) \mathbf{L} \cdot \mathbf{S}_- \right) \frac{dV_V(r)}{dr} \\
&\quad - \left[ (m_1^2 + m_2^2) \mathbf{L} \cdot \mathbf{S}_+ + (m_2^2 - m_1^2) \mathbf{L} \cdot \mathbf{S}_- \right] \left( \frac{dV_S(r)}{dr} \right), \\
V_T &= \frac{1}{12m_1 m_2} \left( \frac{1}{r} \frac{dV_V(r)}{dr} - \frac{d^2V_V(r)}{dr^2} \right) S_{12}. \tag{5}
\end{align*}
\]

Here \( \mathbf{s}_+ \equiv \mathbf{s}_1 + \mathbf{s}_2, \mathbf{s}_- \equiv \mathbf{s}_1 - \mathbf{s}_2, \) and

\[
S_{12} \equiv \frac{3}{r^2} \left( \frac{(\mathbf{s}_1 \cdot \mathbf{r})(\mathbf{s}_2 \cdot \mathbf{r})}{r^2} - \frac{1}{3} \mathbf{s}_1 \cdot \mathbf{s}_2 \right). \tag{7}
\]

For constituents with spin \( s_1 = s_2 = 1/2, S_{12} \) may be rewritten in the form

\[
S_{12} = 2 \left( 3\left( \frac{\mathbf{S} \cdot \mathbf{r}}{r^2} \right)^2 - \mathbf{S}^2 \right), \quad \mathbf{S} = \mathbf{S}_+ \equiv \mathbf{s}_1 + \mathbf{s}_2. \tag{8}
\]

Since \((m_1 + m_2)^2 + 2m_1 m_2 = 6m_1 m_2 + (m_2 - m_1)^2, m_1^2 + m_2^2 = 2m_1 m_2 + (m_2 - m_1)^2,\) the expression for \( V_{LS} \), Eq. (5), may be rewritten as follows,

\[
V_{LS} = \frac{1}{2m_1 m_2} \frac{1}{r} \left[ \left( \frac{3}{r} \frac{dV_V(r)}{dr} - \frac{dV_S(r)}{dr} \right) + \frac{(m_2 - m_1)^2}{2m_1 m_2} \left( \frac{dV_V(r)}{dr} - \frac{dV_S(r)}{dr} \right) \right] \mathbf{L} \cdot \mathbf{S}_+. \]

\(^1\)The most widely used potential models are the relativized model of Godfrey and Isgur \([1]\) for the \( q\bar{q} \) mesons, and Capstick and Isgur \([2]\) for the \( qqq \) baryons. These models differ from the nonrelativistic quark potential model only in relatively minor ways, such as the use of \( H_{kin} = \sqrt{m_1^2 + \mathbf{p}_1^2} + \sqrt{m_2^2 + \mathbf{p}_2^2} \) in place of that given in (2), the retention of the \( m/E \) factors in the matrix elements, and the introduction of coordinate smearing in the singular terms such as \( \delta(r) \).
\[
+ m_2^2 - m_1^2 \frac{1}{4m_1^2m_2^2} \left( \frac{dV_V(r)}{dr} - \frac{dV_S(r)}{dr} \right) \mathbf{L} \cdot \mathbf{S}_- \equiv V_{LS}^+ + V_{LS}^-.
\]

Since two terms corresponding to the derivatives of the potentials with respect to \( r \) are of the same order of magnitude, the above expression for \( V_{LS}^+ \) may be rewritten as

\[
V_{LS}^+ = \frac{1}{2m_1m_2} \frac{1}{r} \left( 3\frac{dV_V(r)}{dr} - \frac{dV_S(r)}{dr} \right) \mathbf{L} \cdot \mathbf{S} \left[ 1 + \frac{(m_2 - m_1)^2}{2m_1m_2} O(1) \right].
\]

### 3 D-wave spectroscopy

We now wish to apply the Breit-Fermi Hamiltonian to the D-wave mesons. By calculating the expectation values of different terms of the Hamiltonian defined in Eqs. (4),(8),(9), taking into account the corresponding matrix elements \( \langle s_1 \cdot s_2 \rangle \), \( \langle \mathbf{L} \cdot \mathbf{S} \rangle \) and \( S_{12} \), one obtains relations similar to those for the P-wave mesons \( 14, 17 \).

\[
M(3D_1) = M_0 + \frac{1}{4} \langle V_{SS} \rangle - 3\langle V_{LS}^+ \rangle - \frac{1}{2}\langle V_T \rangle,
\]

\[
M(3D_3) = M_0 + \frac{1}{4} \langle V_{SS} \rangle + 2\langle V_{LS}^+ \rangle - \frac{1}{7}\langle V_T \rangle,
\]

\[
M(\rho_2) = M_0 + \frac{1}{4} \langle V_{SS} \rangle - \langle V_{LS}^+ \rangle + \frac{1}{2}\langle V_T \rangle,
\]

\[
M(\pi_2) = M_0 - \frac{3}{4} \langle V_{SS} \rangle.
\]

\[
\begin{pmatrix}
M(K'_2) \\
M(K_2)
\end{pmatrix} = \begin{pmatrix}
M_0 + \frac{1}{4} \langle V_{SS} \rangle - \langle V_{LS}^+ \rangle + \frac{1}{2}\langle V_T \rangle & \sqrt{2}\langle V_{LS}^- \rangle \\
\sqrt{2}\langle V_{LS}^- \rangle & M_0 - \frac{3}{4} \langle V_{SS} \rangle
\end{pmatrix} \begin{pmatrix}
K_{2A} \\
K_{2B}
\end{pmatrix},
\]

where \( M_0 \) stands for the sum of the constituent quark masses in either case. The \( V_{LS}^- \) term acts only on the \( I = 1/2 \) singlet and triplet states giving rise to the spin-orbit mixing between these states\(^2\), and is responsible for the physical masses of the \( K_2 \) and \( K'_2 \). Let us assume, for simplicity, that

\[
\sqrt{2}\langle V_{LS}^- \rangle(K_{2B}) \simeq -\sqrt{2}\langle V_{LS}^- \rangle(K_{2A}) \equiv \Delta.
\]

The masses of the \( K_{2A}, K_{2B} \) are then determined by relations similar to those for the \( \pi_2, \rho_2 \) above, and

\[
M(K'_2) \simeq M(K_{2A}) + \Delta, \quad M(K_2) \simeq M(K_{2B}) - \Delta, \quad \text{or} \]

\[
\Delta \simeq M(K'_2) - M(K_{2A}) \simeq M(K_{2B}) - M(K_2). \tag{11}
\]

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\(^2\)The spin-orbit \( 3D_2 - 1D_2 \) mixing is a property of the model we are considering; the possibility that another mechanism contributes to this mixing, such as mixing via common decay channels \( 13 \), should not be ruled out, but is not included here.

\(^3\)Actually, as follows from Eq. (28) below,

\[
\frac{M(K'_2) - M(K_{2A})}{M(K_{2B}) - M(K_2)} = \frac{M(K_2) + M(K_{2B})}{M(K'_2) + M(K_{2A})} \approx \frac{2M(K_{2B})}{2M(K_{2A})} \approx 1,
\]

when both the deviations \( M(K_{2B}) - M(K_2), M(K'_2) - M(K_{2A}) \) and the mass difference \( M(K_{2A}) - M(K_{2B}) \) are small compared to \( M(K_{2A}), M(K_{2B}) \).
We thus obtain the following formulas for the masses of all eight $I = 1, 1/2$ $D$-wave mesons, $\pi_2, \rho, \rho_1, \rho_2, K_{2B}, K^*, K_{2A}, K^*_3$:

\[
M(1D_2) = m_1 + m_2 - \frac{3}{4\,m_1m_2}a, \quad (12)
\]
\[
M(3D_1) = m_1 + m_2 + \frac{1}{4\,m_1m_2} - \frac{3b}{m_1m_2} - \frac{c}{2m_1m_2}, \quad (13)
\]
\[
M(3D_2) = m_1 + m_2 + \frac{1}{4\,m_1m_2} - \frac{b}{m_1m_2} + \frac{c}{2m_1m_2}, \quad (14)
\]
\[
M(3D_3) = m_1 + m_2 + \frac{1}{4\,m_1m_2} + \frac{2b}{m_1m_2} - \frac{c}{7m_1m_2}, \quad (15)
\]

where $a, b$ and $c$ are related to the matrix elements of $V_{SS}, V_{LS}$ and $V_T$ (see Eqs. (4), (6), (10)) and assumed to be the same for all of the $D$-wave states, and we have ignored the correction to $V_{LS}^+$ in the formula (10) that is due to the difference in the masses of the $n$ and $s$ quarks. These masses, as calculated from (12)-(15), are (in the following, $\pi_2$ stands for the mass of the $\pi_2$, etc., and we assume $SU(2)$ flavor symmetry, $n = m_u = m_d$, $s = m_s$)

\[
n = \frac{5\pi_2 + 3\rho + 5\rho_2 + 7\rho_3}{40},
\]

\[
s = \frac{10K_{2A} + 6K^* + 10K_{2B} + 14K^*_3 - 5\pi_2 - 3\rho - 5\rho_2 - 7\rho_3}{40}.
\]

With the physical values of the meson masses (in GeV), $\pi_2 \cong 1.67, \rho \cong \rho_2 \cong \rho_3 \cong 1.70, \ K_{2A} \cong K_{2B} \cong 1.80, \ K^* \simeq K^*_3 \cong 1.77$, the above relations give

\[
n \cong 850 \text{ MeV}, \quad s \cong 940 \text{ MeV},
\]

so that the abovementioned correction, according to (10), is $\sim 90^2/(2 \cdot 850 \cdot 940) \cong 0.5\%$, i.e., completely negligible. It follows from (12)-(15) that

\[
\frac{15a}{m_1m_2} = 3M(3D_1) + 5M(3D_2) + 7M(3D_3) - 15M(1D_2), \quad (18)
\]
\[
\frac{60b}{m_1m_2} = 14M(3D_3) - 5M(3D_2) - 9M(3D_1), \quad (19)
\]
\[
\frac{30c}{7m_1m_2} = 5M(3D_2) - 2M(3D_3) - 3M(3D_1). \quad (20)
\]

By expressing the ratio $n/s$ in four different ways, viz., directly from (16),(17) and dividing the expressions (18)-(20) for the $I = 1/2$ and $I = 1$ mesons by each other, one obtains the three relations,

\[
\frac{5\pi_2 + 3\rho + 5\rho_2 + 7\rho_3}{10K_{2A} + 6K^* + 10K_{2B} + 14K^*_3 - 5\pi_2 - 3\rho - 5\rho_2 - 7\rho_3} = \frac{3K^* + 5K_{2A} + 7K^*_3 - 15K_{2B}}{3\rho + 5\rho_2 + 7\rho_3 - 15\pi_2}.
\]
\[
\frac{3K^* + 5K_{2A} + 7K_3^* - 15K_{2B}}{3\rho + 5\rho_2 + 7\rho_3 - 15\pi_2} = \frac{14K_3^* - 5K_{2A} - 9K^*}{14\rho_3 - 5\rho_2 - 9\rho},
\]
\[
\frac{14K_3^* - 5K_{2A} - 9K^*}{14\rho_3 - 5\rho_2 - 9\rho} = \frac{5K_{2A} - 2K_3^* - 3K^*}{5\rho_2 - 2\rho_3 - 3\rho}.
\]

First consider Eq. (23) which may algebraically be rewritten as
\[
(K_3^* - K^*)(\rho_3 - \rho_2) = (K_3^* - K_{2A})(\rho_3 - \rho).
\]

Since the \(\rho\) and \(\rho_3\) states are mass near-degenerate, \(\rho \approx \rho_3\) (their masses are 1700\pm20 MeV and 1691 \pm 5 MeV, respectively [4]), it then follows from (24) that either \(\rho_2 \approx \rho \approx \rho_3\), or \(K^* \approx K_3^*\). The first possibility leads, through the relations (19),(20) applied to the \(I = 1\) mesons, to \(b \approx c \approx 0\), which would in turn, from the same relations for the \(I = 1/2\) mesons, imply \(K^* \approx K_{2A} \approx K_3^*\). Although this case may not be excluded on the basis of current experimental data on the meson masses, we consider simultaneous disappearance of both the spin-orbit and tensor terms as dubious. We believe, therefore, that the physical case corresponds to
\[
K^* \approx K_3^*,
\]
so that, the mass near-degeneracy of the \(1\,^3D_1\) and \(1\,^3D_3\) meson nonets in the \(I = 1\) channel, \(\rho \approx \rho_3\), implies similar near-degeneracy also in the \(I = 1/2\) channel. This result is a direct consequence of the model we are considering; the equality \(K^* = K_3^*\) follows from Eq. (24), independent of the values of the input parameters \(a, b, c, n, s\), with the proviso that the result \(\rho = \rho_3\) is borne out experimentally.

With \(K^* = K_3^*\) and \(\rho = \rho_3\), Eqs. (21) and (22) may be rewritten as
\[
(\rho - \rho_2 + K^* - K_{2A})(\pi_2 + \rho_2 + 2\rho) = 2(K^* - K_{2A})(K_{2A} + K_{2B} + 2K^*),
\]
\[
(K_{2A} - K_{2B})(\rho - \rho_2) = (K^* - K_{2A})(\rho_2 - \pi_2).
\]

One now has to determine the values of \(\rho_2\), \(K_{2A}\) and \(K_{2B}\). The remaining equation is obtained from the mixing of the \(K_{2A}\) and \(K_{2B}\) states which results in the physical \(K_2\) and \(K'_2\) mesons. Independent of the mixing angle,
\[
K_{2A}^2 + K_{2B}^2 = K_2^2 + K_2^{'2}.
\]

With (in MeV) \(\pi_2 = 1670 \pm 20\), \(\rho = \rho_3 \cong 1690\), \(K^* = K_3^* \cong 1780\), \(K_2 = 1773\), \(K'_2 = 1816\), the solution to (26)-(28) is
\[
\rho_2 = 1741 \pm 19\text{ MeV}, \quad K_{2A} = 1827 \pm 17\text{ MeV}, \quad K_{2B} = 1762 \pm 18\text{ MeV}.
\]

For this solution, we observe the sum rule
\[
K_{2A}^2 - \rho_2^2 = 0.307\text{ GeV}^2 \simeq K_{2B}^2 - \pi_2^2 = 0.316\text{ GeV}^2,
\]
which may be further generalized to include the near-degenerate \(\rho \approx \rho_3 \cong 1690\text{ MeV}\) and \(K^* \approx K_3^* \cong 1780\text{ MeV}\):
\[
K^{*2} - \rho^2 \approx K_3^{*2} - \rho_3^2 \cong 0.312\text{ GeV}^2.
\]
Relations of the type (30),(31) could have been expected by analogy with the formulas

\[ K^{*2} - \rho^2 = K^2 - \pi^2, \quad K_2^{*2} - a_2^2 = K^2 - \pi^2, \quad \text{etc.}, \]

provided by either the algebraic approach to QCD \[18\] or phenomenological formulas

\[ m_{1/2}^2 = 2Bn + C, \quad m_{1/2}^2 = B(n + s) + C \]

(where \( B \) is related to the quark condensate, and \( C \) is a constant within a given meson nonet) motivated by the linear mass spectrum of a nonet and the collinearity of Regge trajectories of the corresponding \( I = 1 \) and \( I = 1/2 \) states, as discussed in ref. \[19\].

Note from (29) that both the \( K_{2A} \) and \( K_{2B} \) lie in the mass intervals provided by current experimental data on the \( K_2^* \) and \( K_2 \) states, respectively. This simply means that the mixing between these states is negligible (within uncertainties provided by data), or \( \sqrt{2}\langle V_{LS} \rangle << K_{2A} - K_{2B} \). As we will see in Eqs. (32)-(34) below, this is entirely consistent with reasonable expectation based on the decrease of such matrix elements with increasing partial wave (see the corresponding \( P \)-wave results \[14\]).

Thus, the nonrelativistic constituent quark model we are considering suggests the following \( q\bar{q} \) assignments for the isovector and isodoublet states of the \( D \)-wave meson nonets:

\[
\begin{align*}
\pi_2 &\simeq 1680 \text{ MeV}, \quad K_{2B} \simeq 1770 \text{ MeV}, \\
\rho &\simeq 1690 \text{ MeV}, \quad K^* \simeq 1780 \text{ MeV}, \\
\rho_2 &\simeq 1730 \text{ MeV}, \quad K_{2A} \simeq 1820 \text{ MeV}, \\
\rho_3 &\simeq 1690 \text{ MeV}, \quad K^* \simeq 1780 \text{ MeV}.
\end{align*}
\]

Let us now extract the matrix elements of the spin-spin, spin-orbit, and tensor interaction in our model. As follows from (18)-(20) and the above relations for the masses of the \( I = 1, 1/2 \) mesons,

\[
\begin{align*}
\langle V_{SS} \rangle &\simeq \frac{a}{n^2} \simeq \frac{a}{ns} \simeq 23.3 \text{ MeV}, \quad (32) \\
\langle V_{LS}^+ \rangle &\simeq \frac{b}{n^2} \simeq \frac{b}{ns} \simeq -3.3 \text{ MeV}, \quad (33) \\
\langle V_T \rangle &\simeq \frac{c}{n^2} \simeq \frac{c}{ns} \simeq 46.7 \text{ MeV}. \quad (34)
\end{align*}
\]

Also, \( \langle V_{LS}^- \rangle \simeq 0 \), since the \( K_{2A} - K_{2B} \) mixing angle is close to zero. Therefore, the spin-spin and tensor terms of the Hamiltonian (2) are of the same order of magnitude, and the spin-orbit terms are negligibly small.

One may now estimate the masses of the isoscalar mesons of the four nonets assuming that they are pure \( s\bar{s} \) states. Applying (12)-(15) with \( m_1 = m_2 = s \), we find

\[
\eta_2 \simeq 1860 \text{ MeV}, \quad \phi \simeq \phi_3 \simeq 1870 \text{ MeV}, \quad \phi_2 \simeq 1910 \text{ MeV}. \quad (35)
\]

The value 1870 is within 1% of the physical value of the \( \phi_3 \) mass, \( 1854 \pm 7 \text{ MeV} \) \[1\]. There exists an experimental candidate for the \( \eta_2(1860) \) but it was omitted from the
recent Meson Summary Table as “needs confirmation”. This state indicated in PDG as the $\eta_2(1870)$\cite{PDG} has been seen by the Crystal Ball collaboration in the final state $\eta \pi^0 \pi^0$ of a $\gamma \gamma$ reaction as a resonant structure having mass and width $1881 \pm 32 \pm 40$ MeV, $221 \pm 92 \pm 44$ MeV, respectively\cite{CrystalBall}, and as a similar structure in $\gamma \gamma \to \eta \pi^+ \pi^-$ by the CELLO collaboration, with mass and width $1850 \pm 50$ MeV, $\sim 360$ MeV, respectively\cite{CELLO}.

The masses of the remaining isoscalar $n\bar{n}$ states of the four nonets may be calculated by assuming that all four nonets are ideally mixed and using the Sakurai mass formula for an ideally mixed nonet\cite{Sakurai}.

$$M^2(I = 1) + M^2(I = 0, n\bar{n}) + 2M^2(I = 0, s\bar{s}) = 4M^2(I = 1/2).$$

(36)

In this way, one obtains

$$\eta_2' \simeq 1670 \text{ MeV}, \quad \omega \simeq \omega_3 \simeq 1680 \text{ MeV}, \quad \omega_2 \simeq 1720 \text{ MeV}.$$  

(37)

The value 1680 is within 1\% of the physical value of the $\omega_3$ mass, 1667 \pm 4 MeV, and 2\% of that of the $\omega$, 1649 \pm 24 MeV\cite{PDG}.

4 Concluding remarks

We have shown that a nonrelativistic constituent quark model displays a common mass near-degeneracy of the $1^3D_1$ and $1^3D_3$ meson nonets in the isovector and isodoublet channels, and suggests therefore that the $K^*(1680)$ cannot be the $I = 1/2$ member of the $1^3D_1$ nonet. The mass of the true member of the latter is estimated to be $\simeq 1780$ MeV. This may support the assumption of Tornqvist that the $K^*(1680)$ should resolve into two separate resonances which are the $I = 1/2$ members of the $1^3D_1$ and $2^3S_1$ nonets. The analysis of the LASS data on the reaction $K^-p \to \bar{K}^0\pi^-p$ done by Bird\cite{Bird} reveals a resonant structure with mass $1678 \pm 64$ MeV and a huge width of $454 \pm 270$ MeV; the two abovementioned states may be associated with its upper- and lower-mass parts, respectively.

The conclusion that the $K^*(1410)$ does not belong to the $2^3S_1$ nonet agrees with the results obtained by one of the authors in ref.\cite{Ko} on the basis of the linear spectrum of a meson nonet discussed in\cite{Carlstroem}, which does not support the $K^*(1410)$ meson being the member of the $2^3S_1$ nonet. (In\cite{Ko}, out of the two, $K^*(1410)$ and $K^*(1680)$, the preference being the $2^3S_1 I = 1/2$ state was given to the latter). If this is actually the case, and the true member of the $2^3S_1$ nonet is, e.g., the low-mass part of the broad $K^*(1680)$, in agreement with Tornqvist, the question immediately arises as to what the real nature of this state is, if it does exist. A possible answer to this question may be the subject of subsequent investigation.

We close with briefly summarizing our findings:

1. A nonrelativistic constituent quark model displays a common mass near-degeneracy of the $1^3D_1$ and $1^3D_3$ meson nonets in the $I = 1$ and $1/2$ channels, and suggests therefore that the $K^*(1680)$ cannot be the $I = 1/2$ member of the $1^3D_1$ nonet.
2. When matched to current experimental data on the meson masses, this model shows no mixing between the $I = 1/2$ states of the $1^3D_2$ and $1^1D_2$ nonets. The spin-orbit terms of the Hamiltonian appear to be negligibly small.

3. The results suggest a sum rule

$$M^2(I = 1/2) - M^2(I = 1) \approx \text{const} \simeq 0.31 \text{ GeV}^2,$$

which holds for all four $D$-wave meson nonets.

4. The results also suggest that the $\eta_2(1870)$ which is at present omitted from the Meson Summary Table, is the $I = 1 s\bar{s}$ state of the $1^1D_2$ nonet.

5. The $q\bar{q}$ assignments for the $P$-wave nonets obtained on the basis of the results of the work, are

- $1^1D_2 \; J^{PC} = 2^{--}$, $\pi_2(1680), \eta_2'(1670), \eta_2(1860), K_{2B}(1770)$
- $1^3D_1 \; J^{PC} = 1^{--}$, $\rho(1690), \omega(1680), \phi(1870), K^*(1780)$
- $1^3D_2 \; J^{PC} = 2^{--}$, $\rho_2(1730), \omega_2(1720), \phi_2(1910), K_{2A}(1820)$
- $1^3D_3 \; J^{PC} = 3^{--}$, $\rho_3(1690), \omega_3(1680), \phi_3(1870), K^*_3(1780)$

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