Steiner Distance in Graphs—A Survey∗

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Abstract

For a connected graph $G$ of order at least 2 and $S \subseteq V(G)$, the Steiner distance $d_G(S)$ among the vertices of $S$ is the minimum size among all connected subgraphs whose vertex sets contain $S$. In this paper, we summarize the known results on the Steiner distance parameters, including Steiner distance, Steiner diameter, Steiner center, Steiner median, Steiner interval, Steiner distance hereditary graph, Steiner distance stable graph, average Steiner distance, and Steiner Wiener index. It also contains some conjectures and open problems for further studies.

Keywords: Distance, Steiner tree, Steiner distance, Steiner diameter, Steiner radius, Steiner eccentricity, Steiner center, Steiner median, Steiner interval, Steiner distance hereditary graph, Steiner distance stable graph, average Steiner distance, Nordhaus-Gaddum-type result, graph product, line graph, extremal graph, algorithm and complexity.

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1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to [19] for graph theoretical notation and terminology not described here. For a graph $G$, let $V(G)$, $E(G)$, $e(G)$, $L(G)$ and $\overline{G}$ denote the set of vertices, the set of edges, the size of $G$, the line graph, and the complement, respectively. The degree, $\deg_G(v)$, of a vertex $v$ of $G$ is the number of edges incident with it. The minimum degree of $G$, $\delta(G)$, is the smallest of the degrees of vertices in $G$ and the maximum degree, $\Delta(G)$, of $G$ is the largest of the degrees of the vertices in $G$. For any subset $X$ of $V(G)$, let $G[X]$ denote the subgraph induced

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pairwise internally disjoint paths. The number of vertices whose deletion renders
is not planar. A subdivision is a threshold graph resulting graph is no longer

In this paper, we let \( K_n \), \( P_n \) and \( C_n \) be the complete graph of order \( n \), the path of order \( n \), and the cycle of order \( n \), respectively. The closed neighborhood of a vertex \( v \) in a graph \( G \) is the set \( N_G[v] = \{ u \in V(G) | d(u, v) \leq 1 \} \), and the open neighborhood is \( N_G(v) = \{ u \in V(G) | d(u, v) = 1 \} \).

A graph is planar if it can be drawn in the plane with no crossing edges. A graph is maximal planar if it is planar, but after the addition of any edge the resulting graph is not planar. A subdivision of \( G \) is a graph obtained from \( G \) by replacing edges with pairwise internally disjoint paths. The connectivity, \( \kappa(G) \), of \( G \) is defined as the minimum number of vertices whose deletion renders \( G \) disconnected or a trivial graph. A graph \( G \) is a threshold graph, if there exists a weight function \( w : V(G) \rightarrow R \) and a real constant \( t \) such that two vertices \( g, g' \in V(G) \) are adjacent if and only if \( w(g) + w(g') \geq t \). A graph is said to be minimally \( k \)-connected if it is \( k \)-connected but omitting any of the edges the resulting graph is no longer \( k \)-connected.

Let \( f(G) \) be a graph invariant and \( n \) a positive integer, \( n \geq n \). The Nordhaus–Gaddum Problem is to determine sharp bounds for \( f(G) + f(\overline{G}) \) and \( f(G) \cdot f(\overline{G}) \), as \( G \) ranges over the class of all graphs of order \( n \), and to characterize the extremal graphs, i.e., graphs that achieve the bounds. Nordhaus–Gaddum type relations have received wide attention; see the recent survey [5].

The join, Cartesian product, lexicographic product, corona, and cluster are defined as follows.

The join or complete product of two disjoint graphs \( G \) and \( H \), denoted by \( G \vee H \), is the graph with vertex set \( V(G) \cup V(H) \) and edge set \( E(G) \cup E(H) \cup \{ uv | u \in V(G), v \in V(H) \} \).

The Cartesian product of two graphs \( G \) and \( H \), written as \( G \Box H \), is the graph with vertex set \( V(G) \times V(H) \), in which two vertices \( (u, v) \) and \( (u', v') \) are adjacent if and only if \( u = u' \) and \( (v, v') \in E(H) \), or \( v = v' \) and \( (u, u') \in E(G) \).

The lexicographic product of two graphs \( G \) and \( H \), written as \( G \circ H \), is defined as follows: \( V(G \circ H) = V(G) \times V(H) \), and two distinct vertices \( (u, v) \) and \( (u', v') \) of \( G \circ H \) are adjacent if and only if either \( (u, u') \in E(G) \) or \( u = u' \) and \( (v, v') \in E(H) \).

The corona \( G \ast H \) is obtained by taking one copy of \( G \) and \( |V(G)| \) copies of \( H \), and by joining each vertex of the \( i \)-th copy of \( H \) with the \( i \)-th vertex of \( G \), where

by \( X \); similarly, for any subset \( F \) of \( E(G) \), let \( G[F] \) denote the subgraph induced by \( F \). We use \( G \setminus X \) to denote the subgraph of \( G \) obtained by removing all the vertices of \( X \) together with the edges incident with them from \( G \); similarly, we use \( G \setminus F \) to denote the subgraph of \( G \) obtained by removing all the edges of \( F \) from \( G \). If \( X = \{ v \} \) and \( F = \{ e \} \), we simply write \( G - v \) and \( G \setminus e \) for \( G - \{ v \} \) and \( G \setminus \{ e \} \), respectively. For two subsets \( X \) and \( Y \) of \( V(G) \) we denote by \( E_G[X, Y] \) the set of edges of \( G \) with one end in \( X \) and the other end in \( Y \). If \( X = \{ x \} \), we simply write \( E_G[x, Y] \) for \( E_G[\{ x \}, Y] \).

In this paper, we let \( K_n \), \( P_n \) and \( C_n \) be the complete graph of order \( n \), the path of order \( n \), and the cycle of order \( n \), respectively. The closed neighborhood of a vertex \( v \) in a graph \( G \) is the set \( N_G[v] = \{ u \in V(G) | d(u, v) \leq 1 \} \), and the open neighborhood is \( N_G(v) = \{ u \in V(G) | d(u, v) = 1 \} \).
\[ i = 1, 2, \ldots, |V(G)|. \]

The *cluster* \( G \odot H \) is obtained by taking one copy of \( G \) and \( |V(G)| \) copies of a rooted graph \( H \), and by identifying the root of the \( i \)-th copy of \( H \) with the \( i \)-th vertex of \( G \), where \( i = 1, 2, \ldots, |V(G)| \).

We divide our introduction into the following subsections to state the motivations and our results of this paper.

### 1.1 Distance parameters and their generalizations

Distance is one of the most basic concepts of graph-theoretic subjects. If \( G \) is a connected graph and \( u, v \in V(G) \), then the *distance* \( d_G(u, v) \) between \( u \) and \( v \) is the length of a shortest path connecting \( u \) and \( v \). If \( v \) is a vertex of a connected graph \( G \), then the *eccentricity* \( e(v) \) of \( v \) is defined by \( e(v) = \max\{d_G(u, v) \mid u \in V(G)\} \). Furthermore, the *radius* \( rad(G) \) and *diameter* \( diam(G) \) of \( G \) are defined by \( rad(G) = \min\{e(v) \mid v \in V(G)\} \) and \( diam(G) = \max\{e(v) \mid v \in V(G)\} \). These last two concepts are related by the inequalities \( rad(G) \leq diam(G) \leq 2rad(G) \). Goddard and Oellermann gave a survey paper on this subject; see [57].

#### 1.1.1 Steiner distance

The distance between two vertices \( u \) and \( v \) in a connected graph \( G \) also equals the minimum size of a connected subgraph of \( G \) containing both \( u \) and \( v \). This observation suggests a generalization of distance. The Steiner distance of a graph, introduced by Chartrand, Oellermann, Tian and Zou in 1989, is a natural and nice generalization of the concept of classical graph distance. For a graph \( G(V, E) \) and a set \( S \subseteq V(G) \) of at least two vertices, an *S-Steiner tree* or a *Steiner tree connecting* \( S \) (or simply, an *S-tree*) is a subgraph \( T(V', E') \) of \( G \) that is a tree with \( S \subseteq V' \). Let \( G \) be a connected graph of order at least 2 and let \( S \) be a nonempty set of vertices of \( G \). Then the *Steiner distance* \( d_G(S) \) among the vertices of \( S \) (or simply the distance of \( S \)) is the minimum size among all connected subgraphs whose vertex sets contain \( S \). Note that if \( H \) is a connected subgraph of \( G \) such that \( S \subseteq V(H) \) and \( |E(H)| = d_G(S) \), then \( H \) is a tree. Observe that \( d_G(S) = \min\{e(T) \mid S \subseteq V(T)\} \), where \( T \) is subtree of \( G \). Furthermore, if \( S = \{u, v\} \), then \( d_G(S) = d(u, v) \) is the classical distance between \( u \) and \( v \). Set \( d_G(S) = \infty \) when there is no S-Steiner tree in \( G \).

**Observation 1.1** Let \( G \) be a graph of order \( n \) and \( k \) be an integer with \( 2 \leq k \leq n \). If \( S \subseteq V(G) \) and \( |S| = k \), then \( d_G(S) \geq k - 1 \).

The problem of finding the Steiner distance of a set of vertices is called the *Steiner Problem* and is NP-complete (see [55]). Steiner trees are well known for their combinatorial optimization aspects and applications to network design and transportation.
1.1.2 Steiner eccentricity, Steiner diameter, and Steiner radius

Let \( n \) and \( k \) be two integers with \( 2 \leq k \leq n \). The Steiner \( k \)-eccentricity \( e_k(v) \) of a vertex \( v \) of \( G \) is defined by \( e_k(v) = \max\{d(S) \mid S \subseteq V(G), |S| = k, \text{ and } v \in S\} \). The Steiner \( k \)-radius of \( G \) is \( \text{srad}_k(G) = \min\{e_k(v) \mid v \in V(G)\} \), while the Steiner \( k \)-diameter of \( G \) is \( \text{sdiam}_k(G) = \max\{e_k(v) \mid v \in V(G)\} \). Note for every connected graph \( G \) that \( e_2(v) = e(v) \) for all vertices \( v \) of \( G \) and that \( \text{srad}_2(G) = \text{rad}(G) \) and \( \text{sdiam}_2(G) = \text{diam}(G) \).

\[
\begin{array}{|c|c|}
\hline
\text{Distance} & \text{Steiner distance} \\
\hline
\{ d_G(u,v) = \min\{e(P) \mid S \subseteq V(P)\}, \\
\text{where } P \text{ is subpath of } G. \} & \{ d_G(S) = \min\{e(T) \mid S \subseteq V(T)\}, \\
\text{where } T \text{ is subtree of } G. \} \\
\hline
\text{Eccentricity} & \text{Steiner } k\text{-eccentricity} \\
\hline
e(v) = \max\{d_G(u,v) \mid u \in V(G)\} & e_k(v) = \max\{d(S) \mid S \subseteq V(G), \\
\text{where } S \text{ is connected subgraph of } G. \} \} \\
\hline
\text{Diameter} & \text{Steiner } k\text{-diameter} \\
\hline
\text{diam}(G) = \max\{e(v) \mid v \in V(G)\} & \text{sdiam}_k(G) = \max\{e_k(v) \mid v \in V(G)\} \\
\hline
\text{Radius} & \text{Steiner } k\text{-radius} \\
\hline
\text{rad}(G) = \min\{e(v) \mid v \in V(G)\} & \text{srad}_k(G) = \min\{e_k(v) \mid v \in V(G)\} \\
\hline
\end{array}
\]

Table 1.1. Classical distance parameters and Steiner distance parameters

The following observation is immediate.

**Observation 1.2** Let \( k, n \) be two integers with \( 2 \leq k \leq n \).

1. If \( H \) is a spanning subgraph of \( G \), then \( \text{sdiam}_k(G) \leq \text{sdiam}_k(H) \).

2. For a connected graph \( G \), \( \text{sdiam}_k(G) \leq \text{sdiam}_{k+1}(G) \).

1.1.3 \( k \)-diameter

Let \( G \) be a \( k \)-connected graph, and let \( u, v \) be any pair of vertices of \( G \). Let \( P_k(u,v) \) be a family of \( k \) inner vertex-disjoint paths between \( u \) and \( v \), i.e., \( P_k(u,v) = \{P_1, P_2, \ldots, P_k\} \), where \( p_1 \leq p_2 \leq \cdots \leq p_k \) and \( p_i \) denotes the number of edges of path \( P_i \). The \( k \)-distance \( d_k(u,v) \) between vertices \( u \) and \( v \) is the minimum \( p_k \) among all \( P_k(u,v) \) and the \( k \)-diameter \( d_k(G) \) of \( G \) is defined as the maximum \( k \)-distance \( d_k(u,v) \) over all pairs \( u, v \) of vertices of \( G \). The concept of \( k \)-diameter emerges rather naturally when one looks at the performance of routing algorithms. Its applications to network routing in distributed and parallel processing are studied and discussed by various authors including Chung [32], Du, Lyuu and Hsu [42], Hsu [79, 80], Meyer and Pradhan [107].
1.1.4 Steiner center and Steiner median

The center $C(G)$ of a connected graph $G$ is the subgraph induced by the vertices $v$ of $G$ with $e(v) = \text{rad}(G)$. As a generalization of the center of a graph, the Steiner $k$-center $C_k(G)$ ($k \geq 2$) of a connected graph $G$ is the subgraph induced by the vertices $v$ of $G$ with $e_k(v) = \text{srad}_k(G)$. Hence the Steiner 2-center of a graph is simply its center. The Steiner $k$-median of $G$ is the subgraph of $G$ induced by the vertices of minimum Steiner $k$-distance in $G$. Similarly, Steiner 2-median of a graph is simply its median. For Steiner centers and Steiner medians, we refer to [113, 114, 117].

| Center | Steiner $k$-center |
|--------|-------------------|
| The subgraph induced by the vertices in $\{v \in V(G) \mid e(v) = \text{rad}(G)\}$ | The subgraph induced by the vertices in $\{v \in V(G) \mid e_k(v) = \text{srad}_k(G)\}$ |
| Median | Steiner $k$-median |
| The subgraph induced by the vertices of minimum distance in $G$. | The subgraph induced by the vertices of minimum Steiner $k$-distance in $G$. |

Table 1.2. Center, median, and their generalizations.

1.1.5 Steiner intervals in graphs

Let $G$ be a graph and $u, v$ two vertices of $G$. Then the interval from $u$ to $v$, $I_G(u, v)$ or $I(u, v)$, is defined by

$$I_G(u, v) = \{w \in V(G) \mid w \text{ lies on a shortest } u-v \text{ path in } G\}.$$  

Thus if $x \in I_G(u, v)$, then $d_G(u, u) = d_G(u, x) + d_G(x, v)$. In the case of a tree $T$ the interval between two vertices $u$ and $v$ in $T$ consists of the vertices on the unique $u$-$v$ path in $T$. However, in general the interval between two vertices $u$ and $v$ in a (connected) graph may contain vertices from more than just one shortest $u$-$v$ path. Mulder [109] devoted an entire monograph to several topics related to intervals in graphs.

Motivated by the notion of the interval between two vertices and the Steiner distance of a set of vertices, Kubicka, Kubicki, and Oellermann [87] defined the Steiner interval, $I_G(S)$ or $I(S)$, of a set $S$ by

$$I_G(S) = \{w \in V(G) \mid w \text{ lies on a Steiner tree for } S \text{ in } G\}.$$  

Thus if $|S| = 2$, then the Steiner interval of $S$ is the interval between the two vertices of $S$.

Let $S$ be an $k$-set and $r \leq k$. Then the $r$-intersection interval of $S$, denoted by $I_r(S)$ is the intersection of all Steiner intervals of $r$-subsets of $S$. Graphs for which the 2-intersection intervals of every 3-set consist of a unique vertex are called median graphs. Median graphs were introduced independently by Avann [10] and Nebesky [110].
1.1.6 Steiner distance hereditary graphs

Howorka \cite{78} in 1977 defined a graph $G$ to be *distance hereditary* if each connected induced subgraph $F$ of $G$ has the property that $d_F(u,v) = d_G(u,v)$ for each pair $u,v \in V(F)$. As a generalization of distance hereditary graphs, Day, Oellermann, and Swart \cite{38} introduced the concept of $k$-Steiner distance hereditary. A connected graph $G$ is defined to be *$k$-Steiner distance hereditary*, $k \geq 2$, if for every connected induced subgraph $H$ of $G$ of order at least $k$ and a set $S$ of $k$ vertices of $H$, $d_H(S) = d_G(S)$. Thus, 2-Steiner distance hereditary graphs are distance hereditary.

1.1.7 Steiner distance stable graphs

In \cite{3} a connected graph is defined to be *vertex (edge) distance stable* if the distance between nonadjacent vertices is unchanged after the deletion of a vertex (edge) of $G$. A more general definition of an equivalent concept was introduced and studied in \cite{3}. It was shown in \cite{3} that a graph is vertex distance stable if and only if it is edge distance stable. We will thus refer to vertex or edge distance stable graphs as *distance stable graphs*. From a more general result established in \cite{3}, it can be deduced that a graph is distance stable if and only if for every pair $u,v$ of nonadjacent vertices, $|N_G(u) \cap N_G(v)| = 0$ or $|N_G(u) \cap N_G(v)| \geq 2$. Thus a graph $G$ is *distance stable* if and only if distances between pairs of vertices in $G$ at distance 2 apart remain unchanged after the deletion of a vertex or an edge.

Further generalizations of distance stable graphs are studied in \cite{60}. In particular, for nonnegative integers $k$ and $\ell$ not both zero and $D \subseteq N - \{\ell\}$ a connected graph $G$ is defined to be $(k,\ell,D)$-stable if for every pair $u,v$ of vertices of $G$ that are at distance $d_G(u,v) \in D$ apart and every set $A$ consisting of at most $k$ vertices of $G - \{u,v\}$ and at most $\ell$ edges of $G$, the distance between $u$ and $v$ in $G - A$ equals $d_G(u,v)$. For a positive integer $m$, let $N_{\geq m} = \{x \in N \mid x \geq m\}$. In \cite{60} it is established that a graph is $(k,\ell,\{m\})$-stable if and only if it is $(k,\ell,N_{\geq m})$-stable. It is further shown that for a positive integer $x$ a graph is $(k+x,\ell,\{2\})$-stable if and only if it is $(k,\ell+x,\{2\})$-stable, but that $(k,\ell+x,\{m\})$-stable graphs need not be $(k+x,\ell,\{m\})$-stable for $m \geq 4$.

The concepts of Steiner distance in graphs and distance stable graphs suggest another generalization of distance stable graphs. Goddard, Oellermann, and Swart \cite{59} introduced the concept of Steiner distance stable graphs. We assume that $k$, $\ell$, $s$ and $m$ are nonnegative integers with $m \geq s \geq 2$ and $k$ and $\ell$ not both zero. If $S$ is a set of $s$ vertices in a connected graph $G$ such that $d_G(S) = m$, then $S$ is called an $(s,m)$-set. A connected graph $G$ is said to be *$k$-vertex $\ell$-edge $(s,m)$-Steiner distance stable* if, for every $(s,m)$-set $S$ of $G$ and every set $A$ consisting of at most $k$ vertices of $G - S$ and at most $\ell$ edges of $G$, $d_{G-A}(S) = d_G(S)$. Thus $k$-vertex $\ell$-edge $(2,m)$-Steiner distance stable graphs are the $(k,\ell,\{m\})$-stable graphs. Note that if $S$ is a set of $s$ vertices such that $d_G(S) = s - 1$ then $d_{G-A}(S) = d_G(S)$ for any set $A$ of at most $k$ vertices of $G - S$ and at most $\ell$ edges of $G$. 

For this reason we require that \( m \geq s \).

For any integers \( k, \ell, m \) and \( s \) with \( m \geq s \geq 2 \) and \( k \) and \( \ell \) not both 0 there exists a \( k \)-vertex \( \ell \)-edge \((s, m)\)-Steiner distance stable graph. To see this let \( G \) be obtained from \( m - 1 \) disjoint copies of \( K_{k+\ell+1} \), say \( H_1, \ldots, H_{m-1} \), by joining every vertex of \( H_i \) to every vertex of \( H_{i+1} \) for \( 1 \leq i < m - 1 \) and then adding a vertex \( v_0 \), and joining it to every vertex of \( H_1 \) and a vertex \( v_m \), and joining it to every vertex of \( H_{m-1} \). It is not difficult to see that \( G \) is \( k \)-vertex \( \ell \)-edge \((s, m)\)-Steiner distance stable.

1.1.8 Average Steiner distance

Let \( G = (V, E) \) be a connected graph of order \( n \). The average distance of \( G \), \( \mu(G) \), is defined to be the average of all distances between pairs of vertices in \( G \), i.e.

\[
\mu(G) = \left( \frac{n}{2} \right)^{-1} \sum_{\{u,v\} \subseteq V(G)} d_G(u, v).
\]

where \( d_G(u, v) \) denotes the length of a shortest \( u \)-\( v \) path in \( G \). This parameter, introduced in architecture as well as by the chemist Wiener, turned out to be a good measure for analysing transportation networks. It indicates the average time required to transport a commodity between two destinations rather than the maximum time required, as is indicated by the diameter; see \[35\].

The average Steiner distance \( \mu_k(G) \) of a graph \( G \), introduced by Dankelmann, Oellermann, and Swart in \[34\], is defined as the average of the Steiner distances of all \( k \)-subsets of \( V(G) \), i.e.

\[
\mu_k(G) = \binom{n}{k}^{-1} \sum_{S \subseteq V(G), |S|=k} d_G(S).
\]

If \( G \) represents a network, then the average Steiner \( k \)-distance indicates the expected number of communication links needed to connect \( k \) processors.

In \[15\], the Steiner \( k \)-distance \( (k \geq 2) \) of a vertex \( v \in V(G) \) in a connected graph \( G \) on \( n \geq k \) vertices, denoted by \( d_k(v, G) \), is defined by

\[
d_k(v) = \sum_{S \subseteq V(G), |S|=k, v \in S} d_G(S).
\]

1.1.9 Steiner distance parameters in chemical graph theory

In \[90\], Li, Mao, and Gutman proposed a generalization of the Wiener index concept, using Steiner distance. Thus, the \( k \)-th Steiner Wiener index \( SW_k(G) \) of a connected graph \( G \) is defined by

\[
SW_k(G) = \sum_{S \subseteq V(G), |S|=k} d_G(S).
\]
For \( k = 2 \), the Steiner Wiener index coincides with the ordinary Wiener index. It is usual to consider \( SW_k \) for \( 2 \leq k \leq n - 1 \), but the above definition implies \( SW_1(G) = 0 \) and \( SW_n(G) = n - 1 \) for a connected graph \( G \) of order \( n \). It should be noted that the average Steiner distance is related to the Steiner Wiener index via 

\[
SW_k(G) = \sum_{S \subseteq V(G), |S| = k} d_G(S) = k^{-1} \sum_{v \in V(G)} d_k(v, G) = \binom{n}{k} \mu_k(G).
\]

For more details on Steiner Wiener index, we refer to [63, 85, 90, 91, 103, 105, 106].

Gutman [62] offered an analogous generalization of the concept of degree distance, Eq. (1.1). Thus, the \( k \)-center Steiner degree distance \( SDD_k(G) \) of \( G \) is defined as

\[
SDD_k(G) = \sum_{S \subseteq V(G), |S| = k} \left[ \sum_{v \in S} \deg_G(v) \right] d_G(S),
\]

see [62, 104, 132] for more details.

Furtula, Gutman, and Katanić [54] introduced the concept of Steiner Harary index. The \( k \)-center Steiner Harary index \( SH_k(G) \) of \( G \) is defined as

\[
SH_k(G) = \sum_{S \subseteq V(G), |S| = k} \frac{1}{d_G(S)},
\]

see [54, 96] for more details.

Mao and Das [98] generalized the concept of Gutman index by Steiner distance. The \( k \)-center Steiner Gutman index \( SGut_k(G) \) of \( G \) is defined by

\[
SGut_k(G) = \sum_{S \subseteq V(G), |S| = k} \left( \prod_{v \in S} \deg_G(v) \right) d_G(S),
\]

see [98, 133] for more details.

The Steiner Wiener index, Steiner Harary index, the Steiner degree distance, and Steiner Gutman index are shown in the following Table 1.3.

| Wiener index | Steiner Wiener index |
|--------------|----------------------|
| \( W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) \) | \( SW_k(G) = \sum_{S \subseteq V(G), |S| = k} d_G(S) \) |
| Harary index | Steiner Harary index |
| \( H(G) = \sum_{\{u,v\} \subseteq V(G)} \frac{1}{d_G(u,v)} \) | \( SW_k(G) = \sum_{S \subseteq V(G), |S| = k} \frac{1}{d_G(S)} \) |
| Degree distance | \( k \)-center Steiner degree distance |
| \( DD(G) = \sum_{\{u,v\} \subseteq V(G)} [\deg_G(u) + \deg_G(v)] d_G(u,v) \) | \( SDD_k(G) = \sum_{S \subseteq V(G), |S| = k} \left[ \sum_{v \in S} \deg_G(v) \right] d_G(S) \) |
| Gutman index | \( k \)-center Steiner Gutman index |
| \( SGut(G) = \sum_{\{u,v\} \subseteq V(G)} [\deg_G(u)\deg_G(v)] d_G(u,v) \) | \( SGut_k(G) = \sum_{S \subseteq V(G), |S| = k} \left[ \prod_{v \in S} \deg_G(v) \right] d_G(S) \) |
1.2 Some other Steiner structural parameters

In this subsection, we introduce some other structural parameters related to Steiner distance.

1.2.1 Steiner geodetic numbers

Let $G$ be a connected graph and $u, v$ two vertices of $G$. Then the interval between $u$ and $v$, denoted by $I[u, v]$ is the union of all vertices that belong to some shortest $u-v$ path. The closure of a set $S$ of vertices in a connected graph $G$ is $\bigcup_{u,v \in S} I[u, v]$. If the closure of a set $S$ of vertices in $G$ is $V(G)$, then $S$ is called a geodetic set. The geodetic number of $G$, denoted by $g(G)$, is the smallest cardinality of a geodetic set in $G$. It was shown in [9] that the problem of finding the geodetic number of a graph is $NP$-hard.

The Steiner interval of a set $S$ of vertices in a connected graph $G$, denoted by $I(S)$, is the union of all vertices of $G$ that lie on some Steiner tree for $S$. A set $S$ of vertices in a connected graph $G$ is a Steiner geodetic set if $I(S) = V(G)$. The Steiner geodetic number of $G$, denoted by $sg(G)$, is the smallest cardinality of a Steiner geodetic set. As noted in [119], the result of [38] that stated that $g(G) \leq sg(G)$ for all connected graphs $G$ it is not true.

1.2.2 Steiner distance and convexity

Abstract convexity started to develop in the early sixties, with the searching of an axiom system to define a set to be convex, in order to generalize, in some way, the classical concept of Euclidean convex set. These concepts can be found in [128]. Among the wide variety of structures that has been studied under this point of view, such as metric spaces, ordered sets or lattices, we are particularly interested in graphs, where several convexities associated to the vertex set are well-known.

Distance optimization properties of Steiner trees have given a way to define the Steiner distance as a generalization of the usual distance in graphs; see [28]. Following that, Cáreres, Márquez, and Puertas [25] define an abstract convexity in the context of graphs by means of the Steiner distance.

There are several well-known definitions of convex vertex sets in graphs, and these convexities are usually defined by means of certain paths. In this fashion, a subset $S$ of vertices of a graph $G$ is monophonically (geodesically) convex (see [54]) if $S$ contains
every vertex of any chordless (shortest) path between vertices in $S$. These sets are called $m$-convex sets ($g$-convex sets).

The definitions above follow the general scheme of abstract convexities. A family $\mathcal{C}$ of subsets of a set $X$ is called a convexity (see [15]) on $X$ if contains the empty set and universal set $X$, is closed under intersections, and is closed under nested unions; that is, if $\mathcal{D} \subseteq \mathcal{C}$ is non-empty and totally ordered by inclusion, then $\bigcup \mathcal{D}$ is in $\mathcal{C}$. Note that last property is trivial if $X$ is a finite set. The elements of $\mathcal{C}$ are called convex sets. It is clear that any subset $A$ of a convex structure is included in a smallest convex set, $CH_{\mathcal{C}}(A) = \bigcap \{C \in \mathcal{C} : A \subseteq C\}$, called the convex hull of $A$. A point $p$ in a convex set $S$ is said to be an extreme point if $S \setminus \{p\}$ is convex. The preceding definitions correspond to $m$-convexity and $g$-convexity in graphs and their convex hulls are denoted by $CH_m$ and $CH_g$ respectively.

Let $G$ be a connected graph. A subset $S \subseteq V(G)$ is said to be St-convex if, for any $A \subseteq S$, all vertices in every Steiner tree of $A$ belong to $S$. The family of all St-convex sets of $V(G)$ defines a convexity called St-convexity.

1.3 Application backgrounds

Steiner distance parameters have their application background in both network science and mathematical chemistry.

1.3.1 Application background of Steiner distance

The Steiner tree problem in networks, and particularly in graphs, was formulated in 1971-by Hakimi (see [67]) and Levi (see [89]). In the case of an unweighted, undirected graph, this problem consists of finding, for a subset of vertices $S$, a minimal-size connected subgraph that contains the vertices in $S$. The computational side of this problem has been widely studied, and it is known that it is an NP-hard problem for general graphs (see [81]). The determination of a Steiner tree in a graph is a discrete analogue of the well-known geometric Steiner problem: In a Euclidean space (usually a Euclidean plane) find the shortest possible network of line segments interconnecting a set of given points. Steiner trees have application to multiprocessor computer networks. For example, it may be desired to connect a certain set of processors with a subnetwork that uses the least number of communication links. A Steiner tree for the vertices, corresponding to the processors that need to be connected, corresponds to such a desired subnetwork.

Steiner distance has application to multiprocessor communication. For example, suppose the primary requirement when communicating a message from a processor $P$ to a collection $S$ of other processors is to minimize the number of communication links that are used. Then a Steiner tree for $S \cup \{P\}$ is an optimal way of connecting these vertices. There are efficient algorithms that find the distance between two vertices or between the vertices
of the entire vertex set. However, if $S$ is a set of $k$ vertices, where $2 < n < |V(G)|$, the only known algorithms that compute the Steiner distance of $S$ have complexity which is a polynomial exponential in $k$. Indeed, it is known that the general problem of finding the Steiner distance of a set is $NP$-hard (see [55]), and several heuristics for finding approximations to it have been developed (see Winter [138] for an extensive survey). However, if the graph $G$ is a tree, then the Steiner distance of any set $S$ and a Steiner tree for $S$ (it is unique) can be found efficiently.

1.3.2 Application backgrounds of Steiner center, Steiner median, and average Steiner distance

Graphs lend themselves as natural models of transportation networks as well as communication and computer networks. Consequently, it is natural to study network problems such as optimal facility location problems for graphs. In almost all such problems, an optimal location is a point that is in some sense central to the network. For example, the center of a connected graph is the subgraph induced by those vertices for which the distance to the most remote vertex is least, and the median is the subgraph induced by those vertices for which the sum of the distances to all of the other vertices is least.

Any vertex in the center of a graph would be a suitable location for an emergency facility, since the distance from the vertex to the furthest vertex from it is minimized, whereas a vertex in the median is a good location for a service facility since the average distance from that vertex to all other vertices is minimized. Slater [7] has given an overview of a variety of other ways of determining centrality, and he introduced and studied some new measures of centrality.

If $G$ represents a network, then the average Steiner $k$-distance indicates the expected number of communication links needed to connect $k$ processors. In contrast the Steiner $k$-diameter of $G$, $sdiam_k(G)$, defined as the maximum distance of the $k$-subsets of $V(G)$, indicates the number of communication links needed in the worst case.

1.3.3 Application background in chemical graph theory

Recall that Wiener index $W(G)$ of the graph $G$ is $W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v)$. Details on this oldest distance–based topological index can be found in numerous surveys, e.g., in [41, 121, 122, 139]. Li, Mao, and Gutman [90] put forward a Steiner–distance–based generalization of the Wiener index concept. A chemical application of $SW_k$ was recently reported in [63], where it is shown that the term $W(G) + \lambda SW_k(G)$ provides a better approximation for the boiling points of alkanes than $W(G)$ itself, and that the best such approximation is obtained for $k = 7$. Furtula, Gutman, and Katanić [54] introduced the concept of Steiner Harary index and gave its chemical applications.
2 Steiner Distance

The classical distance defined on a connected graph $G$ is a metric on its vertex set. As such, certain properties are satisfied. Among these are:

(i) $d(u, v) = 0$ for vertices $u, v$ of $G$ and $d(u, v) = 0$ if and only if $u = v$;

(ii) $d(u, w) \leq d(u, v) + d(v, w)$ for vertices $u, v, w$ of $G$.

The extensions of these properties to the Steiner distance are given by Chartrand, Oellermann, Tian, and Zou [28].

(1) Let $G$ be a connected graph and let $S \subseteq V(G)$, where $S \neq \emptyset$. Then $d_G(S) \geq 0$. Furthermore, $d_G(S) = 0$ if and only if $|S| = 1$. This is an extension of (i).

(2) To provide an extension of (ii), let $S, S_1$ and $S_2$ be subsets of $V(G)$ such that $\emptyset \neq S \subseteq S_1 \cup S_2$ and $S_1 \cap S_2 \neq \emptyset$. Then $d_G(S) \leq d_G(S_1) + d_G(S_2)$. To see this, let $T_i \ (i = 1, 2)$ be a tree of size $d_G(S_i)$ such that $S_i \subseteq V(T_i)$. Let $H$ be the graph with vertex set $V(T_1) \cup V(T_2)$ and edge set $E(T_1) \cup E(T_2)$. Since $T_1$ and $T_2$ are connected and $V(T_1) \cap V(T_2) \neq \emptyset$, the graph $H$ is connected. Since $S \subseteq V(H)$, $d_G(S) \leq e(H) \leq d_G(S_1) + d_G(S_2)$.

It is useful to observe that if $T$ is a nontrivial tree and $S \subseteq V(T)$, where $|S| \geq 2$, then there is a unique subtree $T_S$ of size $d_T(S)$ containing the vertices of $S$. We refer to such a tree as the tree generated by $S$.

Observation 2.1 [28] If $S$ is a set of vertices of a tree $T$ and $v$ is a vertex in $V(T) - S$, then the tree generated by $S \cup \{v\}$ contains the tree generated by $S$.

Observation 2.2 [28] Let $w$ be the (necessarily unique) vertex of $T_S$ whose distance from $v$ is a minimum. Then $T_{S \cup \{v\}}$ contains the unique $v$-$w$ path and

$$d_T(S \cup \{v\}) = d_T(S) + d_T(v, w).$$

Observation 2.3 [28] If $H$ is a subgraph of a graph $G$ and $v$ is a vertex of $G$, then

$$d_G(S \cup \{v\}) = d_G(S) + d_G(v, T_S),$$

$d(v, H)$ denotes the minimum distance from $v$ to a vertex of $H$.

For a tree $T$, we denote by $V_1(T)$ the set of end-vertices of $T$ and $n_1 = |V_1(T)|$. If $S = V_1(T)$, then $T_S = T$ so that $d_T(S) = e(T)$ and $d_T(S \cup \{v\}) = e(T)$ for all $v \in V(T)$. Hence if $T$ is a tree and $k \geq 2$ and integer with $n_1 < k$, then $e_k(v) = e(T)$ for all $v \in V(T)$.

Chartrand, Oellermann, Tian, and Zou [28] derived the following results for Steiner distance.
Observation 2.4 Let $k \geq 2$ be an integer and suppose that $T$ is a tree of order $n$ with $n_1 \geq k$. Let $v \in V(T)$. If $S \subseteq V(T)$ such that $v \notin S$, $|S| = k - 1$ and $d_T(S \cup \{v\}) = e_k(v)$, $S \subseteq V_1(T)$.

Corollary 2.1 [28] Let $k \geq 2$ be an integer and $T$ a tree with $n_1 \geq k$. Then $srad_k(T) = d_T(S)$, where $S$ is a set of $k$ end-vertices of $T$.

Proposition 2.2 [28] Let $k \geq 3$ be an integer and suppose that $T$ is a tree with $n_1 \geq k$ end-vertices. If $v$ is a vertex of $T$ with $e_k(v) = srad_k(T)$, then there exists a set $S$ of $k - 1$ end-vertices of $T$ such that $d_T(S \cup \{v\}) = e_k(v)$ and $v \in V(T_S)$.

Corollary 2.2 [28] Let $k \geq 3$ be an integer and suppose that $T$ is a tree with at least $k$ end-vertices. If $v$ is a vertex of $T$ with $e_k(v) = srad_k(T)$, then $v$ is not an end-vertex of $T$.

Let $k$ and $n$ be integers with $2 \leq k \leq n$. A graph $G$ of order $n$ is called an $(k; n)$-graph if it is of minimum size with the property that $d_G(S) = k - 1$ for all sets $S$ of vertices of $G$ with $|S| = k$.

Chartrand, Oellermann, Tian, and Zou [28] determined the size of an $(k; n)$ graph for each pair $k, n$ of integers with $2 \leq k \leq n$.

Theorem 2.1 [28] Let $k$ and $n$ be integers with $2 \leq k \leq n$. The size of an $(k; n)$-graph is $k - 1$ if $k = n$ and $\lceil (n - k + 1)n/2 \rceil$ if $n > k$.

2.1 Steiner distance in some graph classes

The following observation is easy to make from the definition of a threshold graph.

Observation 2.4 Let $G([n], E)$ be a threshold graph with a weight function $w : V(G) \to \mathbb{R}$. Let the vertices be labelled so that $w(1) \geq w(2) \geq \cdots \geq w(n)$. Then

1. $d_1 \geq d_2 \geq \cdots \geq d_n$, where $d_i$ is the degree of vertex $i$.
2. $I = \{i \in V(G) : d_i \leq i - 1\}$ is a maximum independent set of $G$ and $G \setminus I$ is a clique in $G$.
3. $N(i) = \{1, 2, \cdots , d_i\}$ for every $i \in I$. Thus, the neighborhoods of vertices in $I$ form a linear order under set inclusion. Furthermore, if $G$ is connected, then every vertex in $G$ is adjacent to 1.

Let $C_r$ and $I_{n-r}$ denote the clique and the maximum independent set of $G$, respectively, with $V(C_r) = \{u_1, u_2, \ldots , u_r\}$ and $V(I_{n-r}) = \{u'_1, u'_2, \ldots , u'_{n-r}\}$ such that $deg_G(u_1) \geq deg_G(u_2) \geq \cdots \geq deg_G(u_r)$ and $deg_G(u'_1) \geq deg_G(u'_2) \geq \cdots \geq deg_G(u'_{n-r})$.
Wang, Mao, Cheng, and Melekian [135] derived the following results for the Steiner distance of threshold graphs.

**Proposition 2.3** [135] Let \( k, n \) be two integers with \( 3 \leq k \leq n \), and let \( G \) be a threshold graph of order \( n \). Let \( S \) be a set of distinct vertices of \( G \) such that \( |S| = k \). Let \( u_i \) be the vertex in \( S \cap V(C_r) \) with the minimum subscript, and \( u'_j \) be the vertex in \( S \cap V(I_{n-r}) \) with the maximum subscript.

1. If \( S \subseteq V(C_r) \), then \( d_G(S) = k - 1 \).
2. If \( S \subseteq V(I_{n-r}) \), then \( d_G(S) = k \).
3. If \( S \cap V(C_r) \neq \emptyset \), \( S \cap V(I_{n-r}) \neq \emptyset \), and \( u_iu'_j \in E(G) \), then \( d_G(S) = k - 1 \).
4. If \( S \cap V(C_r) \neq \emptyset \), \( S \cap V(I_{n-r}) \neq \emptyset \), and \( u_iu'_j \notin E(G) \), then \( d_G(S) = k \).

For two graphs \( G \) and \( H \) with \( V(G) = \{u_1, u_2, \ldots, u_n\} \) and \( V(H) = \{v_1, v_2, \ldots, v_m\} \), from the definition of corona graphs, \( V(G * H) = V(G) \cup \{(u_i, v_j) \mid 1 \leq i \leq n, \ 1 \leq j \leq m\} \), where \(*\) denotes the corona product operation. For \( u \in V(G) \), we use \( H(u) \) to denote the subgraph of \( G * H \) induced by the vertex set \( \{(u, v_j) \mid 1 \leq j \leq m\} \). For fixed \( i \ (1 \leq i \leq n) \), we have \( u_i(u_i, v_j) \in E(G * H) \) for each \( j \ (1 \leq j \leq m) \). Then \( V(G * H) = V(G) \cup V(H(u_1)) \cup V(H(u_2)) \cup \ldots \cup V(H(u_n)) \).

Wang, Mao, Cheng, and Melekian [135] also derived the following results on the Steiner distance of corona graphs.

**Theorem 2.2** [135] Let \( k, m, n \) be three integers with \( 3 \leq k \leq n(m + 1) \), and let \( G, H \) be two connected graphs with \( V(G) = \{u_1, u_2, \ldots, u_n\} \) and \( V(H) = \{v_1, v_2, \ldots, v_m\} \). Let \( S \) be a set of distinct vertices of \( G * H \) such that \( |S| = k \).

\[
d_{G*H}(S) = d_G(S'_G) + k - t,
\]

where \( |S \cap V(G)| = t \), and \( S'_G \) is the maximum subset of \( V(G) \) such that \( S \cap (V(H(u)) \cup \{u\}) \neq \emptyset \) for each \( u \in S'_G \).

From the definition of cluster, \( V(G \odot H) = \{(u_i, v_j) \mid 1 \leq i \leq n, \ 1 \leq j \leq m\} \), where \( \odot \) denotes the cluster product operation. For \( u \in V(G) \), we use \( H(u) \) to denote the subgraph of \( G \odot H \) induced by the vertex set \( \{(u, v_j) \mid 1 \leq j \leq m\} \). Without loss of generality, we assume \( u_i(u_i, v_1) \) is the root of \( H(u_i) \) for each \( u_i \in V(G) \). Let \( G(h_1) \) be the graph induced by the vertices in \( \{(u_i, v_1) \mid 1 \leq i \leq n\} \). Clearly, \( G(h_1) \simeq G \), and \( V(G \odot H) = V(H(u_1)) \cup V(H(u_2)) \cup \ldots \cup V(H(u_n)) \).

Wang, Mao, Cheng, and Melekian [135] obtained the following results on the Steiner distance of cluster graphs.

**Theorem 2.3** [135] Let \( k, m, n \) be three integers with \( 3 \leq k \leq n(m + 1) \), and let \( G, H \) be two connected graphs with \( V(G) = \{u_1, u_2, \ldots, u_n\} \) and \( V(H) = \{v_1, v_2, \ldots, v_m\} \). Let
S = {(u_{i1}, v_{j1}), (u_{i2}, v_{j2}), \ldots, (u_{i_k}, v_{jk})} be a set of distinct vertices of G ∪ H. Let S_G = \{u_{i1}, u_{i2}, \ldots, u_{i_k}\} and S_H = \{v_{j1}, v_{j2}, \ldots, v_{jk}\}.

1. If \( S \subseteq V(G(v_1)) \), then \( d_{G \odot H}(S) = d_G(S_G) \).

2. If there exists some \( H(u_i) \) (1 ≤ i ≤ n) such that \( S \subseteq V(H(u_i)) \), then \( d_{G \odot H}(S) = d_H(S_H) \).

3. If there is no \( H(u_i) \) (1 ≤ i ≤ n) such that \( S \subseteq V(H(u_i)) \), then

\[
d_G(S'_G) + k - t \leq d_{G \odot H}(S) \leq \begin{cases} rd_H(S_H) + d_G(S'_G) & v_1 \in S_H, \\ rd_H(S_H \cup \{h_1\}) + d_G(S'_G) & v_1 \not\in S_H, \end{cases}
\]

where \( |S \cap V(G(v_1))| = t \), \( |S'_G| = r \), and \( S'_G \) is the maximum subset of \( V(G) \) such that \( S \cap V(H(u)) \neq \emptyset \) for each \( u \in S'_G \).

Moreover, the upper and lower bounds are sharp.

To show the sharpness of the above lower and upper bounds, Wang, Mao, Cheng, and Melekan consider the following examples.

**Example 2.1.**\(^{\text{[135]}}\) Let \( G = P_n = u_1u_2\ldots u_n \) and \( H = P_n = v_1 v_2 \ldots v_m \) with 3 ≤ k ≤ mn. Note that \( H(u_i) \cong P_m \) for each \( u_i \) (1 ≤ i ≤ n), and \( G(v_1) \cong P_n \). For \( v_i \not\in S_H \), if k ≤ n, then we choose \( S = \{(u_1, v_m), (u_2, v_m), \ldots, (u_{k-1}, v_m)\} \cup \{(u_n, v_m)\} \). Then \( r = k \), \( d_H(S_H \cup \{v_1\}) = m - 1 \), \( d_G(S'_G) = n - 1 \). Since the tree induced by the edges in \( E(G(v_1)) \cup E(H(u_1)) \cup E(H(u_2)) \cup \ldots \cup E(H(u_{k-1})) \cup E(H(u_n)) \) is the unique \( S \)-Steiner tree, it follows that \( d_{G \odot H}(S) \geq k(m - 1) + (n - 1) \). From Theorem\(^{\text{[23]}}\),

\( d_{G \odot H}(S) \leq rd_H(S_H \cup \{v_1\}) + d_G(S'_G) = k(m - 1) + (n - 1) \). So, the upper bound for \( v_1 \not\in S_H \) is sharp. For \( v_1 \in S_H \), if \( k \leq n \), then we choose \( S = \{(u_1, v_1), (u_2, v_1), \ldots, (u_k, v_1)\} \). Then \( r = k \), \( d_H(S_H) = 0 \), \( d_G(S'_G) = k - 1 \). Then \( d_{G \odot H}(S) \geq k - 1 \). From Theorem\(^{\text{[23]}}\),

\( d_{G \odot H}(S) \leq rd_H(S_H) + d_G(S'_G) = k - 1 \). So, the upper bound for \( v_1 \in S_H \) is sharp.

**Example 2.2.**\(^{\text{[135]}}\) Let \( G = P_n = u_1u_2\ldots u_n \) and \( H = K_m \) with 3 ≤ k ≤ mn, where \( V(H) = \{v_1, v_2, \ldots, v_m\} \). Note that \( H(u_i) \cong K_m \) for each \( u_i \) (1 ≤ i ≤ n), and \( G(v_1) \cong P_n \). Choose \( S = \{(u_1, v_m), (u_2, v_m), \ldots, (u_{k-1}, v_m)\} \cup \{(u_n, v_m)\} \) (m ≥ 2). Then \( d_G(S'_G) = n - 1 \) and \( t = 0 \), and hence \( d_{G \odot H}(S) \geq n - 1 + k \). Clearly, the tree induced by the edges in \( E(G(v_1)) \cup E_{G \odot H}[V(G(v_1))], S \) is an \( S \)-Steiner tree in \( G \odot H \), and hence \( d_{G \odot H}(S) \leq n - 1 + k \). So, we have \( d_{G \odot H}(S) = n - 1 + k \), which implies that the lower bound is sharp.

**Corollary 2.3**\(^{\text{[135]}}\) Let \( k, m, n \) be three integers with 3 ≤ k ≤ n(m + 1), and let \( G, H \) be two connected graphs with \( V(G) = \{u_1, u_2, \ldots, u_n\} \) and \( V(H) = \{v_1, v_2, \ldots, v_m\} \). Let \( (u_i, v_1) \) be the root of \( H(u_i) \) for each \( u_i \) (1 ≤ i ≤ n). Let \( S \) be a set of distinct vertices of \( G \odot H \) such that \( |S| = k \). Then

\[
d_{G \odot H}(S) \leq rd_H(S_H \cup \{v_1\}) + d_G(S'_G),
\]
where $|S'_G| = r$, and $S'_G$ is the maximum subset of $V(G)$ such that $S \cap V(H(u)) \neq \emptyset$ for each $u \in S'_G$.

### 2.2 Steiner distance of graph products

Product networks were proposed based upon the idea of using the product as a tool for "combining" two known graphs with established properties to obtain a new one that inherits properties from both [7].

Wang, Mao, Cheng, and Melekian [135] gave the exact value for Steiner distance of joined graphs.

**Proposition 2.4** [135] Let $k, m, n$ be three integers with $3 \leq k \leq m + n$, and let $G, H$ be two connected graphs with $n, m$ vertices, respectively. Let $S$ be a set of distinct vertices of $G \sqcap H$ such that $|S| = k$.

1. If $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$, then $d_{G \sqcap H}(S) = k - 1$.

2. If $S \cap V(H) = \emptyset$ and $G[S]$ is connected, then $d_{G \sqcap H}(S) = k - 1$; if $S \cap V(H) = \emptyset$ and $G[S]$ is not connected, then $d_{G \sqcap H}(S) = k$.

3. If $S \cap V(G) = \emptyset$ and $H[S]$ is connected, then $d_{G \sqcap H}(S) = k - 1$; if $S \cap V(G) = \emptyset$ and $H[S]$ is not connected, then $d_{G \sqcap H}(S) = k$.

In [61], Gologranc obtained the following lower bound for Steiner distance.

**Theorem 2.4** [61] Let $k \geq 2$ be an integer, and let $G, H$ be two connected graphs. Let $S = \{(u_{i_1}, v_{j_1}), (u_{i_2}, v_{j_2}), \ldots, (u_{i_k}, v_{j_k})\}$ be a set of distinct vertices of $G \square H$. Let $S_G = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ and $S_H = \{v_{j_1}, v_{j_2}, \ldots, v_{j_k}\}$. Then

$$d_{G \square H}(S) \geq d_G(S_G) + d_H(S_H).$$

Mao, Cheng, and Wang [100] showed that the inequality in Theorem 2.4 can be equality if $k = 3$; shown in following Corollary 2.5. But, for general $k$ ($k \geq 4$), from Theorem 2.4 and Corollary 2.5 one may conjecture that for two connected graphs $G, H$, 

![Figure 2.1: Graphs for Remark 2.1.](image-url)
Theorem 2.5 \[ d_G \square H(S) = d_G(S_G) + d_H(S_H), \] where \( S = \{(u_{i_1}, v_{j_1}), (u_{i_2}, v_{j_2}), \ldots, (u_{i_k}, v_{j_k})\} \subseteq V(G \square H), \) \( S_G = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\} \subseteq V(G) \) and \( S_H = \{v_{j_1}, v_{j_2}, \ldots, v_{j_k}\} \subseteq V(H). \)

Remark 2.1 Actually, the equality \( d_G \square H(S) = d_G(S_G) + d_H(S_H) \) is not true for \( |S| \geq 4. \) For example, let \( G \) be a tree with degree sequence \( (3,2,1,1,1) \) and \( H \) be a path of order 5. Let \( S = \{(u_1,v_1),(u_2,v_2),(u_3,v_3),(u_4,v_4)\} \) be a vertex set of \( G \square H \) shown in Figure 2.1. Then \( d_G(S_G) = 4 \) for \( S_G = \{u_1,u_2,u_3,u_4\}, \) and \( d_H(S_H) = 4 \) for \( S_H = \{v_1,v_2,v_3,v_4\}. \) One can check that there is no \( S\)-Steiner tree of size 8 in \( G \square H, \) which implies \( d_G \square H(S) \geq 9. \)

Although the conjecture of such an ideal formula is not correct, it is possible to give a strong upper bound for general \( k (k \geq 3). \) Remark 2.1 also indicates that obtaining a nice formula for the general case may be difficult.

Mao, Cheng, and Wang \[100\] got such an upper bound of \( d_G \square H(S) \) for \( S \subseteq V(G \square H) \) and \( |S| = k. \)

Theorem 2.5 \[100\] Let \( k,m,n \) be three integers with \( 3 \leq k \leq mn, \) and let \( G,H \) be two connected graphs with \( V(G) = \{u_1,u_2,\ldots,u_n\} \) and \( V(H) = \{v_1,v_2,\ldots,v_m\}. \) Let \( S = \{(u_{i_1},v_{j_1}),(u_{i_2},v_{j_2}),\ldots,(u_{i_k},v_{j_k})\} \) be a set of distinct vertices of \( G \square H, \) \( S_G = \{u_{i_1},u_{i_2},\ldots,u_{i_k}\}, \) and \( S_H = \{v_{j_1},v_{j_2},\ldots,v_{j_k}\}, \) where \( S_G \subseteq V(G), S_H \subseteq V(H) (S_G,S_H \text{ are both multi-sets}). \) Then

\[
d_G(S_G) + d_H(S_H) \leq d_{G \square H}(S) \leq \min\{d_G(S_G) + (r+1)d_H(S_H), d_H(S_H) + (t+1)d_G(S_G)\},
\]

where \( r,t \) \((0 \leq r,t \leq k-3)\) are defined as follows.

- Let \( X_G^i \) \((1 \leq i \leq \binom{k}{3})\) be all the \((k-3)\)-multi-subsets of \( \{u_{i_1},u_{i_2},\ldots,u_{i_k}\} \) in \( G, \) and let \( r_i \) be the numbers of distinct vertices in \( X_G^i \) \((1 \leq i \leq \binom{k}{3})\), and let \( r = \max\{r_i| 1 \leq i \leq \binom{k}{3}\} \).

- Let \( Y_H^j \) \((1 \leq j \leq \binom{k}{3})\) be all the \((k-3)\)-multi-subsets of \( \{v_{j_1},v_{j_2},\ldots,v_{j_k}\} \) in \( H, \) and let \( t_j \) be the numbers of distinct vertices in \( Y_H^j \) \((1 \leq j \leq \binom{k}{3})\), and let \( t = \max\{t_j| 1 \leq j \leq \binom{k}{3}\} \).

The following corollaries are immediate from Theorem 2.5.

Corollary 2.4 \[100\] Let \( G,H \) be two connected graphs of order \( n,m, \) respectively. Let \( k \) be an integer with \( 3 \leq k \leq mn. \) Let \( S = \{(u_{i_1},v_{j_1}),(u_{i_2},v_{j_2}),\ldots,(u_{i_k},v_{j_k})\} \) be a set of distinct vertices of \( G \square H. \) Let \( S_G = \{u_{i_1},u_{i_2},\ldots,u_{i_k}\} \) and \( S_H = \{v_{j_1},v_{j_2},\ldots,v_{j_k}\}. \) Then

\[
d_G(S_G) + d_H(S_H) \leq d_{G \square H}(S) \leq \min\{d_G(S_G) + (k-2)d_H(S_H), d_H(S_H) + (k-2)d_G(S_G)\} \]
\[
= d_G(S_G) + d_H(S_H) + (k-3)\min\{d_H(S_H),d_G(S_G)\}.
\]
Corollary 2.5 \[100\] Let $G, H$ be two connected graphs, and let $(u, v), (u', v')$ and $(u'', v'')$ be three vertices of $G\square H$. Let $S_G = \{u, u', u''\}$, $S_H = \{v, v', v''\}$, and $S = \{(u, v), (u', v'), (u'', v'')\}$. Then

$$d_{G\square H}(S) = d_G(S_G) + d_H(S_H)$$

To show the sharpness of the above upper and lower bound, Mao, Cheng, and Wang \[100\] considered the following example.

Example 2.3. \[100\] (1) For $k = 3$, from Corollary 2.4, we have $d_{G\square H}(S) = d_G(S_G) + d_H(S_H)$, which implies that the upper and lower bounds in Corollary 2.4 and Theorem 2.5 are sharp.

(2) Let $G, H$ be two connected graphs. Suppose $4 \leq k \leq |V(H)|$. Choose $k$ vertices in $H(u)$ for fixed $u \in V(G)$, say $(u, v_1), (u, v_2), \ldots, (u, v_k)$. Clearly, $d_G(S_G) = 0$. From Corollary 2.4, $d_{G\square H}(S) \geq d_G(S_G) + d_H(S_H) = d_H(S_H)$ and $d_{G\square H}(S) \leq d_G(S_G) + d_H(S_H) + (k - 3) \min\{d_H(S_H), d_G(S_G)\} = d_H(S_H)$, and hence $d_{G\square H}(S) = d_H(S_H)$. This implies that the upper and lower bounds in Corollary 2.4 are sharp. From the definition of $r$, we have $r = 1$ and $d_{G\square H}(S) \leq d_G(S_G) + d_H(S_H) + r \min\{d_H(S_H), d_G(S_G)\} = d_H(S_H)$, and hence $d_{G\square H}(S) = d_H(S_H)$. This also implies that the upper and lower bounds in Theorem 2.5 are sharp.

(3) Let $G = P_n$ and $H = P_m$ with $n \leq m$, where $P_n = u_1 u_2 \ldots u_n$ and $P_m = v_1 v_2 \ldots v_m$. Choose $S = \{(u_1, v_1), (u_1, v_m), (u_n, v_1), (u_n, v_m)\}$. Then $d_G(S_G) = n - 1$, $d_H(S_H) = m - 1$ and $d_{G\square H}(S) = 2(n - 1) + (m - 1) = d_G(S_G) + d_H(S_H) + (4 - 3) \min\{d_H(S_H), d_G(S_G)\}$, which implies that the upper bound in Corollary 2.4 are sharp.

From the definition, the lexicographic product graph $G \circ H$ is a graph obtained by replacing each vertex of $G$ by a copy of $H$ and replacing each edge of $G$ by a complete bipartite graph $K_{m,m}$, where $m = |V(H)|$.

Theorem 2.6 \[68\] Let $(u, v)$ and $(u', v')$ be two vertices of $G \circ H$. Then

$$d_{G \circ H}((u, v), (u', v')) = \begin{cases} 
  d_G(u, u'), & \text{if } u \neq u'; \\
  d_H(v, v'), & \text{if } u = u' \text{ and } \deg_G(u) = 0; \\
  \min\{d_H(v, v'), 2\}, & \text{if } u = u' \text{ and } \deg_G(u) \neq 0.
\end{cases}$$

Anand, Changat, Klavžar, and Peterin \[6\] obtained the following formula.

Theorem 2.7 \[6\] Let $k \geq 2$. Let $S = \{(u_{i_1}, v_{j_1}), (u_{i_2}, v_{j_2}), \ldots, (u_{i_k}, v_{j_k})\}$ be a set of distinct vertices of $G \circ H$ such that $u_{i_p} \neq u_{i_q}$ $(1 \leq p, q \leq k)$. Let $S_G = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$. Then

$$d_{G \circ H}(S) = d_G(S_G).$$

For general case, Mao, Cheng, and Wang \[100\] had the following formula for Steiner distance of lexicographic product graphs.
Theorem 2.8 \cite{100} Let \( k, n, m \) be three integers with \( 2 \leq k \leq mn \). Let \( G \) be a connected graph of order \( n \), and \( H \) be a graph of order \( m \). Let \( S = \{(u_{i1}, v_{i1}), (u_{i2}, v_{i2}), \ldots, (u_{ik}, v_{ik})\} \) be a set of distinct vertices of \( G \circ H \). Let \( S_G = \{u_{i1}, u_{i2}, \ldots, u_{ik}\} \) and \( S_H = \{v_{j1}, v_{j2}, \ldots, v_{jk}\} \) (note that \( S_G, S_H \) are both multi-set). Let \( r \) be the number of distinct vertices in \( S_G \), where \( 1 \leq r \leq k \).

1. If \( r = 1 \) and \( H[S_H] \) is connected in \( H \), then \( d_{G \circ H}(S) = k - 1 \).
2. If \( r = 1 \) and \( H[S_H] \) is not connected in \( H \), then \( d_{G \circ H}(S) = k \).
3. If \( r \geq 2 \), then \( d_{G \circ H}(S) = d_G(S_G) + k - r \).

In Theorem \ref{2.8} they assumed that \( G \) is a connected graph. For \( k = 3 \), they had the following by assuming that \( G \) is not connected.

Proposition 2.5 \cite{100} Let \( G \) and \( H \) be two graphs, and let \((u, v), (u', v')\) and \((u'', v'')\) be three vertices of \( G \circ H \). Let \( S = \{(u, v), (u', v'), (u'', v'')\} \), \( S_G = \{u, u', u''\} \) and \( S_H = \{v, v', v''\} \). Then

\[
d_{G \circ H}(S) = \begin{cases} 
  d_H(S_H), & \text{if } u = u' = u'' \text{ and } deg_G(u) = 0; \\
  \min\{d_H(S_H), 3\}, & \text{if } u = u' = u'' \text{ and } deg_G(u) \neq 0; \\
  \infty, & \text{if } u \neq u', u' = u'' \text{ and } d_G(u, u') = \infty; \\
  d_G(u, u') + 1, & \text{if } u \neq u', u' = u'' \text{ and } d_G(u, u') \neq \infty; \\
  d_G(S_G), & \text{if } u \neq u', u \neq u'' \text{ and } u' \neq u''.
\end{cases}
\]

2.3 Steiner distance and convexity

In \cite{25}, Cáceres, Márquez, and Puertas formulated the following relation between three convex hulls in any graph.

Proposition 2.6 \cite{25} Let \( G \) be a connected graph and let \( S \subseteq V(G) \). Then the following chain of inclusions holds: \( CH_g(S) \subseteq CH_{St}(S) \subseteq CH_m(S) \).

A graph is called house-hole-domino free (HHD-free) (see \cite{12}) if it contains no induced house, domino, or induced cycle \( C_k, k \geq 5 \).

Theorem 2.9 \cite{25} For any set of vertices \( S \) of a connected HHD-free graph \( G \), any Steiner tree \( T \) of \( S \) is contained in the geodesical convex hull of \( S \).

Theorem 2.10 \cite{25} Let \( G \) be a connected HHD-free graph and let \( S \subseteq V(G) \). Then \( S \) is a \( g \)-convex set if and only if it is a \( St \)-convex set.

A convexity is said to be a convex geometry (see \cite{128}) if every convex set is the convex hull of its extreme points. This property gives good behavior to a convexity in
graphs, because in this case we can keep all information about a convex vertex sets just in its extreme points. To find conditions under which St-convexity shares this property, we firstly need to characterize extreme points of a St-convex set. Recall that a vertex is called simplicial if its neighborhood is a complete subgraph. The next lemma shows that St-extreme points are simplicial vertices, the same condition as in case of \textit{g}-convex sets.

Note that, in any case, the extreme points of the convex hull of a set of vertices \( A \) belong to \( A \), because if \( p \in CH(A) \setminus A \) is an extreme point; then \( CH(A) \setminus \{p\} \) is a convex set containing \( A \), which contradicts the minimality of convex hull.

Cáceres, Márquez, and Puertas \cite{25} characterized the class of graphs in which St-convexity becomes a convex geometry.

**Theorem 2.11** \cite{25} The St-convexity in a connected graph \( G \) is a convex geometry if and only if \( G \) is chordal and contains no induced 3-fan.

Let \( G \) be a connected graph of order \( n \geq 2 \) and \( k \) be an integer with \( 2 \leq k \leq n \). Let \( S \subseteq V(G) \) and \( v \in S \), the \( k \)-eccentricity \( e_{k,S}(v) \) de \( v \) en \( S \) is defined by \( e_{k,S}(v) = \max\{d_S(K) : K \subseteq S, |K| = k, v \in K\} \). In case \( S = V(G) \), we denote \( e_{k,S} \) simply by \( e_k \).

**Proposition 2.7** \cite{25} Let \( G \) be a connected graph and let \( uv \) be an edge of \( G \), then \( e_k(u) - 1 \leq e_k(v) \leq e_k(u) + 1 \).

Let \( G \) be a connected graph and \( S \subseteq V(G) \). A vertex \( v \in S \) is called \( k \)-contour of \( S \) if it satisfies \( e_{k,S}(v) \geq e_{k,S}(u) \), for any \( u \in N_S[v] \). The set \( Ct_{k,S}(G) \) of \( k \)-contour vertices of \( S \) is called the \( k \)-contour set of \( S \).

The \( k \)-contour set is an enlargement of St-extreme point set.

**Proposition 2.8** \cite{25} Let \( G \) be a connected graph and \( S \subseteq V(G) \). Then \( Ct_{k,S}(G) \) contains all St-extreme points of \( S \).

Cáceres, Márquez, and Puertas \cite{25} proved the main result for \( k \)-contour vertices: these vertices can rebuild any St-convex set by means of a Steiner convex hull operation. This result provides, in some sense, a generalization of Theorem 2.11 using a vertex set bigger than St-extreme points, that works in any connected graph.

**Theorem 2.12** \cite{25} Let \( G \) be a connected graph and \( S \subseteq V(G) \) a St-convex set. Then \( CH_{St}(Ct_{k,S}(S)) = S \).

### 3 Steiner Diameter

For a graph \( G \) of order \( n \geq 2 \), the Steiner diameter sequence of \( G \) is defined as the sequence

\[ sdiam_2(G), sdiam_3(G), \ldots, sdiam_n(G), \]
while the Steiner radius sequence is the sequence
\[ \text{srad}_2(G), \text{srad}_3(G), \ldots, \text{srad}_n(G). \]

### 3.1 Steiner diameter of some graph classes

The following results are immediate.

**Proposition 3.1** [95, 96] Let \( k, n \) be two integers with \( 2 \leq k \leq n \).

1. For a complete graph \( K_n \), \( \text{sdiam}_k(K_n) = k - 1 \);
2. For a path \( P_n \), \( \text{sdiam}_k(P_n) = n - 1 \);
3. For a cycle \( C_n \), \( \text{sdiam}_k(C_n) = \left\lfloor \frac{n(k-1)}{k} \right\rfloor \);
4. For complete \( r \)-partite graph \( K_{n_1,n_2,\ldots,n_r} \) (\( n_1 \leq n_2 \leq \cdots \leq n_r \)),
   \[ \text{sdiam}_k(K_{n_1,n_2,\ldots,n_r}) = \begin{cases} 
   k - 1, & \text{if } k > n_r; \\
   k, & \text{if } k \leq n_r.
   \end{cases} \]

For Steiner diameter of threshold graphs, Wang, Mao, Cheng, and Melekian [135] derived the following results.

**Proposition 3.2** [135] Let \( k, n \) be two integers with \( 3 \leq k \leq n \), and let \( G \) be a threshold graph of order \( n \). Let \( i \) be the subscript of vertices in \( V(C_r) \) such that \( g_i g'_n \notin E(G) \) but \( g_i+1 g'_n \notin E(G) \).

(i) If \( 3 \leq k \leq n - i \), then \( \text{siam}_k(G) = k \).

(ii) If \( n - i + 1 \leq k \leq n \), then \( \text{siam}_k(G) = k - 1 \).

Chartrand, Oellermann, Tian, and Zou [28] established a relation between the Steiner \( k \)-diameter and Steiner \( (k - 1) \)-diameter of a tree, where \( k \geq 3 \) is an integer.

**Theorem 3.1** [28] Let \( k \geq 3 \) be an integer and \( T \) a tree of order \( k \leq n \). Then
\[ \text{sdiam}_{k-1}(T) \leq \text{sdiam}_k(T) \leq \frac{k}{k-1} \text{sdiam}_{k-1}(T). \]

The following proposition will aid us in deriving a relation between the Steiner \( k \)-diameter and Steiner \( k \)-radius of a tree.

**Proposition 3.3** [28] Let \( k \geq 3 \) be an integer and \( T \) a tree of order \( k \leq n \). Then
\[ \text{sdiam}_{k-1}(T) = \text{srad}_k(T). \]

**Corollary 3.1** [28] Let \( k \geq 2 \) be an integer and \( T \) a tree of order \( k \leq n \). Then
\[ \text{srad}_k(T) \leq \text{sdiam}_k(T) \leq \frac{k}{k-1} \text{srad}_k(T). \]
Chartrand, Oellermann, Tian, and Zou [28] conjectured that Corollary 3.1 can be extended to any connected graph.

**Conjecture 3.1** [28] Let \( \ell \geq 2 \) be an integer and \( G \) is a connected graph of order \( \ell n \leq n \). Then

\[
srad_\ell(G) \leq \text{sdiam}_\ell(G) \leq \frac{k}{k-1}srad_\ell(G).
\]

Chartrand, Oellermann, Tian, and Zou [28] presented the desired characterization of diameter sequences of trees.

**Theorem 3.2** [28] A sequence \( D_2, D_3, \ldots, D_n \) of positive integers is the diameter sequence of a tree of order \( n \) having \( r \) end-vertices if and only if

1. \( D_{k-1} < D_k < \frac{k}{k-1}D_{k-1} \) for \( 3 \leq k \leq r \);
2. \( D_k = k \) for \( r \leq k \leq n \);
3. \( D_{k+1} - D_k \leq D_k - D_{k-1} \) for \( 3 \leq k \leq n-1 \).

**Corollary 3.2** [28] A sequence \( R_2, R_3, \ldots, R_n \) of positive integers is the radius sequence of a tree of order \( n \geq 2 \) having \( r \) end-vertices if and only if

1. \( R_{k-1} < R_k < \frac{k}{k-1}R_{k-1} \) for \( 3 \leq k \leq r + 1 \);
2. \( R_k = n - 1 \) for \( r + 1 \leq k \leq n \);
3. \( R_{k+1} - R_k \leq R_k - R_{k-1} \) for \( 4 \leq k \leq n \).

The maximum number of vertices of maximal planar graphs of given diameter and maximum degree has been determined. Hell and Seyffarth [72] have shown that the maximum number of vertices in a planar graph with diameter 2 and maximum degree \( \Delta \geq 8 \) is \( \lfloor \frac{3}{2}\Delta + 1 \rfloor \). It was shown in [123] that maximal planar graphs of diameter 2 and maximum degree \( \Delta \geq 8 \) have no more than \( \frac{3}{2}\Delta + 1 \) vertices. It was also shown that there exist maximal planar graphs with diameter two and exactly \( \lfloor \frac{3}{2}\Delta + 1 \rfloor \) vertices. Yang, Lin, and Dai [?] have computed the exact maximum number of vertices in planar graphs and maximal planar graphs with diameter two and maximum degree \( \Delta \), for \( \Delta < 8 \).

Fulek, Morigi and Pritchard [53] derived the following upper bound for diameter.

**Theorem 3.3** [53] For every connected planar graph \( G \) of order \( n \) and size \( m \),

\[
diam(G) \leq \frac{4(n-1) - m}{3}.
\]

Since for 3, 4 and 5-connected maximal planar graphs \( m = 3n - 6 \), the bound in Theorem 3.3 becomes \( \text{diam}(G) \leq \frac{n+2}{3} \). It is well known that for the ordinary diameter, i.e., for the case \( k = 2 \), if \( G \) is a \( \ell \)-connected graph of order \( n \), then

\[
diam(G) \leq \frac{n + \ell - 2}{\ell},
\]
which yields
\[ \text{diam}(G) \leq \frac{n + 1}{3} \text{ for 3-connected graphs } G; \]
\[ \text{diam}(G) \leq \frac{n + 2}{4} \text{ for 4-connected graphs } G; \]
\[ \text{diam}(G) \leq \frac{n + 3}{5} \text{ for 5-connected graphs } G. \]

So the ordinary diameters of 3, 4 and 5-connected maximal planar graphs do not exceed \( \frac{n + 1}{3}, \frac{n + 2}{4} \) and \( \frac{n + 3}{5} \), respectively.

Ali, Mukwembi, and Dankelmann [5] derived the following upper bounds for Steiner \( k \)-diameter of maximal planar graphs.

**Theorem 3.4** [5] (1) Let \( G \) be a 3-connected maximal planar graph of order \( n \). If \( k \) is an integer with \( 2 \leq k \leq n \), then
\[
\text{sdiam}_k(G) \leq \frac{n}{3} + \frac{8k}{3} - 5.
\]

(2) Let \( G \) be a 4-connected maximal planar graph of order \( n \). If \( k \) is an integer with \( 2 \leq k \leq n \), then
\[
\text{sdiam}_k(G) \leq \frac{n}{4} + \frac{19k}{4} - 9.
\]

(3) Let \( G \) be a 5-connected maximal planar graph of order \( n \). If \( k \) is an integer with \( 2 \leq k \leq n \), then
\[
\text{sdiam}_k(G) \leq \frac{n}{5} + \frac{24k}{5} - 9.
\]

The following example (see Figure 3.1 (a) for an illustration) shows that, for constant \( k \), the bound in (1) of Theorem 2.4 is best possible, apart from the value of the additive constant.

**Example 3.1.** [5] For an integer \( \ell \geq \lceil \frac{n}{3} \rceil \) let \( G_1, G_2, \ldots, G_\ell \) be disjoint copies of the cycle \( C_3 \), and let \( a_i, b_i, c_i \in V(G_i) \). Let \( G_\ell' \) be the graph obtained from the union of \( G_1, G_2, \ldots, G_\ell \) by adding the edges \( a_{i+1}a_i, b_{i+1}b_i, c_{i+1}c_i, a_{i+1}b_i, c_{i+1}b_i, a_{i+1}c_i \) for \( i = 1, 2, \ldots, \ell - 1 \). Clearly, \( |V(G_\ell')| = 3\ell \) so that \( \ell = \left\lceil \frac{|V(G_\ell')|}{3} \right\rceil. \) Clearly, \( \text{diam}(G_\ell') = \ell - 1 = \frac{|V(G_\ell')|}{3} - 1 \) and \( \text{sdiam}_3(G_\ell') = d(\{a_1, b_1, c_1\}) = \ell = \frac{|V(G_\ell')|}{3}. \) For \( k \geq 4 \), let \( k = 3q + r \) with \( r \in \{1, 2, 3\} \) and define \( S \) to be the set of \( k \) vertices \( \{a_1, b_1, c_1, a_2, b_2, c_2, \ldots, a_q, b_q, c_q\} \cup R \), where \( R \subseteq \{u_\ell, v_\ell, w_\ell\} \) is a set with \( |R| = r \). It is easy to see that \( d(S) = \ell - 1 + 2q + r - 1 = \frac{|V(G_\ell')|}{3} + 2\left\lceil \frac{n}{3} \right\rceil + r - 4 \geq \frac{|V(G_\ell')|}{3} + 2\left\lceil \frac{n}{3} \right\rceil - 3. \) Hence \( \text{sdiam}_k(G_\ell') \geq \frac{|V(G_\ell')|}{3} + 2\left\lceil \frac{n}{3} \right\rceil - 3 \) for \( k \geq 2 \).

The following example (see Figure 3.1 (b) for an illustration) shows that, for constant \( k \), the bound in (2) of Theorem 2.4 is best possible, apart from the value of the additive constant.
Example 3.2. [5] For an integer $\ell \geq k$, let $G_1, G_2, \ldots, G_\ell$ be disjoint copies of the 4-cycle $C_4$, and let $a_i, b_i, c_i, d_i \in V(G_i)$. Let $G''_\ell$ be the graph obtained from the union of $G_1, G_2, \ldots, G_\ell$ by adding the edges $a_{i+1}a_i, b_{i+1}b_i, c_{i+1}c_i, d_{i+1}d_i, a_{i+1}d_i, b_{i+1}a_i, c_{i+1}b_i, d_{i+1}c_i$, for $i = 1, 2, \ldots, k-1$, $a_kc_\ell$ and $a_1c_1$. Clearly, $|V(G''_\ell)| = 4\ell$ so that $\ell = \frac{|V(G''_\ell)|}{4}$. Clearly, $\text{diam}(G''_\ell) = \frac{|V(G''_\ell)|}{4} - 1$. For $k \geq 3$, let $k = 2q + r$ with $r \in \{1, 2\}$ and define the set $S$ of $k$ vertices as $\{b_1, d_1, b_3, d_3, b_5, d_5, \ldots, b_{2q-1}, d_{2q-1}\} \cup R$, where $R \subseteq \{b_\ell, d_\ell\}$ is a set with $|R| = r$. It is easy to verify that $d(S) = \ell - 1 + 2q + 2(r - 1) = \ell - 3 + k + r \geq \ell - 2 + k$. Hence we have $\text{sdiam}_k(G''_\ell) \geq \frac{|V(G''_\ell)|}{4} + k - 2$ for $k \geq 3$.

Figure 3.1: (a) The graph $G'_\ell$; (b) The graph $G''_\ell$; (c) A 4-connected planar graph for which (1), (2) of Theorem 3.3 do not hold.

It is essential in (1), (2) of Theorem 3.3 that $G$ is maximal planar and not just planar, as the following example demonstrates. (For an illustration see Figure 3.1 (c))

Example 3.3. [5] Let $H$ be the Cartesian product of $K_2$ and a cycle $C_{\frac{n}{2}}$, where $n$ is even, i.e. let $V(H) = \{a_0, a_1, \ldots, a_{\frac{n}{2}-1}, b_0, b_1, \ldots, b_{\frac{n}{2}-1}\}$ and $E(H) = \{a_i a_{i+1} | i = 0, 1, \ldots, \frac{n}{2} - 1\} \cup \{b_i b_{i+1} | i = 0, 1, \ldots, \frac{n}{2} - 1\} \cup \{a_i b_i | i = 0, 1, \ldots, \frac{n}{2} - 1\}$ where subscripts are taken modulo $\frac{n}{2}$. Now let $H'$ be the planar graph obtained from $H$ by adding the edges $a_{i+1}b_i$ for $i = 0, 1, \ldots, \frac{n}{2} - 1$. If $k$ divides $\frac{n}{2}$, then the set $S_k = \{a_i | i \in \{0, \frac{2}{k}, 2\frac{2}{k}, 3\frac{2}{k}, (k-1)\frac{2}{k}\}\}$ has $k$ vertices and $d(S_k) = \frac{k-1}{k} \frac{n}{2}$. Hence $\text{diam}_k(H') \geq \frac{k-1}{k} \frac{n}{2}$ which for constant $k \geq 4$ and large $n$ is greater than both $\frac{k}{k} + \frac{k}{k} - 5$ and $\frac{k}{k} + \frac{k}{k} - 9$. Since $H'$ is planar and 4-connected, this shows that (1), (2) of Theorem 3.3 do not hold for 4-connected planar graphs.

The following graphs (see Figure 3.2 (a) for an illustration) show that, for constant $k$, the bound in (3) of Theorem 3.3 is best possible, apart from the value of the additive constant.

Example 3.4. [5] For an integer $\ell \geq k$, let $G_1, G_2, \ldots, G_\ell$ be disjoint copies of the 5-cycle, $C_5$, and let $a_i, b_i, c_i, d_i, w_i \in V(G_i)$. Let $G'''_\ell$ be the graph obtained from the union of $G_1, G_2, \ldots, G_\ell$ by adding the edges $a_{i+1}a_i, b_{i+1}b_i, c_{i+1}c_i, d_{i+1}d_i, w_{i+1}w_i, a_{i+1}w_i$, $b_{i+1}a_i, c_{i+1}b_i, d_{i+1}c_i, w_{i+1}d_i$, for $i = 1, 2, \ldots, \ell - 1$ and new vertices $v_\ell$ adjacent to $a_\ell, b_\ell, c_\ell, d_\ell, w_\ell$ and $v_1$ adjacent to $a_1, b_1, c_1, d_1, w_1$. Clearly, $|V(G'''_\ell)| = 5\ell + 2$ so that $\ell = \frac{|V(G'''_\ell)|}{5} - 2$. 

![Diagram](image-url)
Now $diam(G''_{\ell}) = \ell + 1 = \frac{|V(G''_{\ell})| - 2}{5} + 1$. For $k \geq 3$ let $k = 2q + r$ with $r \in \{0,1\}$ and define the set $S$ of $k$ vertices as $\{v_1, v_{r}\} \cup \{a_2, a_4, a_6, \ldots, a_{2(k-1)}, c_{2(k-1)}\} \cup R$, where $R \subseteq \{a_{2q}\}$ is a set with $|R| = r$. It is easy to verify that $d(S) = \ell - 1 + 2(q - 1) \geq \ell - 1 + 2q + r - 3 = k + \ell - 4$. Hence we have $sdiam_k(G''_{\ell}) \geq \frac{|V(G''_{\ell})| - 2}{5} + \ell - 4$ for $k \geq 3$.

The following example shows that in (3) of Theorem 3.4 it is essential that $G$ is maximal planar, and not just planar. (For an illustration see Figure 3.2 (b))

**Example 3.5.** [5] For $n$ a multiple of $4k$, let $H''$ be the graph of order $n$ obtained from the disjoint union of two cycles of length $n/4$ with vertices $a_0, a_1, \ldots, a_{n/4-1}$ and $c_0, c_1, \ldots, c_{n/4-1}$, respectively, and a cycle of length $n/2$ with vertices $b_0, b_1, \ldots, b_{n/2-1}$, by adding edges $a_i b_{2i-1}, a_i b_{2i}, a_i b_{2i+1}, c_i b_{2i}, c_i b_{2i+1}, c_i b_{2i+2}$ for $i = 0, 1, \ldots, n/4$, where the subscripts are taken modulo $n/4$ for $a_j$ and $c_j$, and modulo $n/2$ for $b_j$. Then the set $S_k = \{a_i | i \in \{0, \frac{n}{4}, 2\frac{n}{4}, 3\frac{n}{4}, (k - 1)\frac{n}{4}\}\}$ has $k$ vertices and $d(S_k) = \frac{k-1}{2}$, $n$. Hence $sdiam_k(H'') \geq \frac{k-1}{4k}n$ which for constant $k \geq 5$ and large $n$ is greater than $\frac{n}{5} + \frac{2k}{5} - 9$. Since $H''$ is planar and 5-connected, this shows that (3) of Theorem 3.4 does not hold for 5-connected planar graphs.

![Figure 3.2: (a) The graph $G''_{\ell}$; (b) A 5-connected planar graph for which (3) of Theorem 3.4 does not hold.](image)

**Remark 3.1** [5] All bounds on the Steiner $k$-diameter given in this paper are sharp except for an additive constant provided that $k$ is constant. It would be interesting to know if for infinitely many values of $n$ and $k$ there are graphs that come within an additive constant, neither dependent on $n$ nor on $k$, of our bounds, or if our bounds can be improved by a term that is linear in $k$. 

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3.2 Bounds for Steiner diameter

In [29], Chartrand, Okamoto, and Zhang derived the upper and lower bounds for $sdiam_k(G)$.

**Theorem 3.5** [29] Let $k, n$ be two integers with $2 \leq k \leq n$, and let $G$ be a connected graph of order $n$. Then

$$k - 1 \leq sdiam_k(G) \leq n - 1.$$ 

Moreover, the bounds are sharp.

3.2.1 In terms of order and minimum degree

For the ordinary diameter, Erdös, Pach, Pollack, and Tuza [45] give the following bounds in terms of order and minimum degree.

**Theorem 3.6** [45] (1) For all connected graphs $G$,

$$diam(G) \leq \frac{3n}{\delta + 1} - 1.$$ 

(2) If $G$ is a triangle-free graph, then

$$diam(G) \leq 4 \left\lceil \frac{n - \delta - 1}{2\delta} \right\rceil.$$ 

(3) If $G$ is a $C_4$-free graph, then

$$diam(G) \leq 4 \left\lceil \frac{5n}{\delta^2 - 2\lceil\delta/2\rceil + 1} \right\rceil.$$ 

The result for all connected graphs was extended by Dankelmann, Swart, and Oellermann in [36].

**Theorem 3.7** [36] Let $G$ be a connected graph of order $n$ and minimum degree $\delta \geq 2$. If $2 \leq k \leq n$, then

$$sdiam_k(G) \leq \frac{3n}{\delta + 1} + 3k.$$ 

Ali, Mukwembi, and Dankelmann [4] improved this upper bound and generalized the corresponding result by Erdös, Pach, Pollack, and Tuza [45].

**Theorem 3.8** [4] (1) Let $G$ be a connected graph of order $n$ and minimum degree $\delta \geq 2$. If $k$ is an integer with $2 \leq k \leq n$, then

$$sdiam_k(G) \leq \frac{3n}{\delta + 1} + 2k - 5.$$
(2) Let $G$ be a connected triangle-free graph of order $n$ and minimum degree $\delta \geq 2$. If $k$ is an integer with $2 \leq k \leq n$, then
\[
\text{sdiamp}_k(G) \leq \frac{2n}{\delta} + 3k - 6.
\]

(3) Let $G$ be a connected $C_4$-free graph of order $n$ and minimum degree $\delta \geq 2$. If $k$ is an integer with $2 \leq k \leq n$, then
\[
\text{sdiamp}_k(G) \leq \frac{5n}{\delta^2 - 2\lceil\delta/2\rceil + 1} + 4k - 9.
\]

The following corollary is derived in the same paper.

**Corollary 3.3** [4]  
(1) Let $G$ be a connected graph of order $n$ and minimum degree $\delta \geq 2$. If $k$ is an integer with $2 \leq k \leq n$, then $G$ has a spanning tree $T$ with
\[
\text{sdiamp}_k(T) \leq \frac{3n}{\delta + 1} + 2k - 3.
\]

(2) Let $G$ be a connected triangle-free graph of order $n$ and minimum degree $\delta \geq 2$. If $k$ is an integer with $2 \leq k \leq n$, then $G$ has a spanning tree $T$ with
\[
\text{sdiamp}_k(T) \leq \frac{2n}{\delta} + 3k - 3.
\]

(3) Let $G$ be a connected $C_4$-free graph of order $n$ and minimum degree $\delta \geq 2$. If $k$ is an integer with $2 \leq k \leq n$, then $G$ has a spanning tree $T$ with
\[
\text{sdiamp}_k(T) \leq \frac{5n}{\delta^2 - 2\lceil\delta/2\rceil + 1} + 4k - 5.
\]

### 3.2.2 In terms of the girth and minimum degree

Using methods initiated by Dankelmann and Entringer [33] and methods used in [4], Ali [2] proved in their paper that

**Theorem 3.9** [2]  
Let $G$ be a connected graph of order $n$, minimum degree $\delta \geq 3$, and girth $g$.

(1) If $k$ is an integer with $2 \leq k \leq n$ and $g$ is odd, then
\[
\text{sdiamp}_k(G) \leq g\frac{n}{K} + (g - 1)k - 2g + 1,
\]
where $K = 1 + \delta \frac{(\delta - 1)(g - 1)^{\frac{1}{2}} - 1}{\delta - 2}$.

(2) If $k$ is an integer with $2 \leq k \leq n$ and $g$ is even, then
\[
\text{sdiamp}_k(G) \leq g\frac{n}{L} + (g - 1)k - 2g + 2,
\]
where $L = 2\delta \frac{(\delta - 1)^{\frac{1}{2}} - 1}{\delta - 2}$.
Ali [2] showed that the bounds in Theorem 3.9 are asymptotically sharp, apart from an additive constant.

**Example 3.6.** If \( \delta \) and \( g \) are such that there exists a Moore graph of minimum degree \( \delta \) and girth \( g \), i.e., a graph with minimum degree \( \delta \), girth \( g \), and order \( 1 + \delta + \delta(\delta - 1) + \delta(\delta - 1)^2 + \cdots + (\delta - 1)(g - 3)/2 \) (if \( g \) is odd), or \( 2(1 + (\delta - 1) + (\delta - 1)^2 + \cdots + (\delta - 1)(g - 2)/2) \) (if \( g \) is even). For a given integer \( \ell > 0 \), let \( G_1, G_2, \ldots, G_\ell \) be disjoint copies of the \((\delta, g)\)-Moore graph, and let \( u_iv_i \in E(G_i) \). Let \( G_{\ell, \delta, g} \) be the graph obtained from the union of \( G_1, G_2, \ldots, G_\ell \) by deleting the edges \( u_iv_i \) for \( i = 2, 3, \ldots, \ell - 1 \) and adding the edges \( u_{i+1}v_i \) for \( i = 1, 2, \ldots, \ell - 1 \).

1. If \( g \) is odd, then \( |V(G_{\ell, \delta, g})| = \ell K \), so \( \ell = \frac{|V(G_{\ell, \delta, g})|}{K} \). If \( 2 \leq n \leq 2\delta \) then, by a simple calculation, \( \text{diam}_k(G_{\ell, \delta, g}) = g \ell + k - 5 \), and so \( \text{sdiam}_k(G_{\ell, \delta, g}) = g \frac{|V(G_{\ell, \delta, g})|}{K} + k - 5 \). Note that the set \( S \) of \( k \) vertices of \( G_{\ell, \delta, g} \) contains \( \delta \) vertices of \( G_1 \) and \( \delta \) vertices of \( G_\ell \). In this case, the difference between \( \text{diam}_k(G_{\ell, \delta, g}) \) and the bound in Theorem 3.9 is \( (g - 2)k - 2g + 6 \). For \( k > 2\delta \), we use the estimate \( \text{diam}_k(G_{\ell, \delta, g}) \geq \text{diam}(G_{\ell, \delta, g}) = g \sqrt{\frac{|V(G_{\ell, \delta, g})|}{K}} - 3 \). In this case, the difference between the Steiner \( k \)-diameter of \( G_{\ell, \delta, g} \) and the bound in Theorem 3.9 is bounded by the additive constant \( (g - 1)k - 2g + 4 \).

2. If \( g \) is even, then \( |V(G_{\ell, \delta, g})| = L \ell \), and thus \( \ell = \frac{|V(G_{\ell, \delta, g})|}{L} \). If \( k \leq 2\delta - 2 \) then, by a simple calculation, \( \text{sdiam}_k(G_{\ell, \delta, g}) = g \ell + k - 7 \), and thus \( \text{sdiam}_k(G_{\ell, \delta, g}) = g \frac{|V(G_{\ell, \delta, g})|}{L} + k - 7 \). Note that the set \( S \) of \( k \) vertices of \( G_{\ell, \delta, g} \) contains \( \delta - 1 \) vertices of \( G_1 \) and \( \delta - 1 \) vertices of \( G_\ell \). In this case, the difference between \( \text{sdiam}_k(G_{\ell, \delta, g}) \) and the bound in Theorem 3.9 is \( (g - 2)k - 2g + 9 \). For \( k > 2\delta - 2 \), we use the estimate \( \text{sdiam}_k(G_{\ell, \delta, g}) \geq \text{diam}(G_{\ell, \delta, g}) = g \sqrt{\frac{|V(G_{\ell, \delta, g})|}{L}} - 5 \). In this case, the difference between the Steiner \( k \)-diameter of \( G_{\ell, \delta, g} \) and the bound in Theorem 3.9 is bounded by the additive constant \( (g - 1)k - 2g + 7 \).

A slight modification of the proof of Theorem 3.9 yields the following result.

**Corollary 3.4** [2] Let \( G \) be a connected graph of order \( n \), minimum degree \( \delta \geq 3 \), and girth \( g \).

1. If \( k \) is an integer with \( 2 \leq k \leq n \) and \( g \) is odd, then \( G \) has a spanning tree \( T \) with

\[
\text{sdiam}_k(G) \leq g \frac{n^2}{K} + (g - 1)k - g,
\]

where \( K = 1 + \delta \frac{(\delta - 1)(g - 1)^2/2 - 1}{\delta - 2} \).

2. If \( k \) is an integer with \( 2 \leq k \leq n \) and \( g \) is even, then \( G \) has a spanning tree \( T \) with

\[
\text{sdiam}_k(G) \leq g \frac{n^2}{L} + (g - 1)k - g + 1,
\]

where \( L = \delta \frac{(\delta - 1)^2/2 - 1}{\delta - 2} \).
3.2.3 In terms of Steiner radius

It was conjectured in [73] that for all integers \( k \geq 2 \) and every connected graph \( G \) of order \( n \geq k \),

\[
\text{sdiam}_k(G) \leq \frac{2(k + 1)}{2k - 1} \text{srad}_k(G). \tag{3.2}
\]

In [73], it was shown that for each integer \( k \geq 2 \) a graph \( G \) exists for which equality is attached in (1) and that (3.2) is valid if \( k \in \{2, 3, 4\} \), but further progress has been reported.

Ore [118] showed that, for a connected graph \( G \) of order \( n \), size \( m \) and diameter \( d \),

\[
e(G) \leq d + \frac{1}{2}(n - d - 1)(n - d + 4).
\]

Extensions of Ore’s result were given be Caccetta and Smyth [22, 23, 24].

Ore’s result is generalized by Dankelmann, Swart, and Oellermann [36] in the next theorem, which can be used to obtain an upper bound on the Steiner \( k \)-diameter of \( G \) in terms of the order and size of \( G \).

**Theorem 3.10** [36] If \( G \) is a connected graph of order \( n \), size \( m \) and \( \text{sdiam}_k(G) = d_k \), where \( 2 \leq k \leq n - 1 \), then

\[
e(G) \leq d_k + \left(\frac{k - 1}{2}\right) + \left(\frac{k - d_k - 1}{2}\right) + (k - d_k - 1)(k + 1).
\]

An analysis of the above proof shows that, for \( k \geq 4 \) and \( n \geq 2k + 1 \) the graph \( G \) obtained from the union of three disjoint graphs \( G_1, G_2, G_3 \) with \( G_1 \cong K_{k+1}, G_2 \cong K_k \) and \( G_3 \cong K_{n-2k-1} \) by joining each vertex in \( G_2 \) to the same end vertex of \( G_3 \) and to each vertex of \( G_1 \) is such that, for \( G \), equality is attained in (1) of Theorem 3.10.

Since the Steiner \( k \)-diameter of a tree is easy to compute, bounds for the Steiner \( k \)-diameter of graphs in classes that do not contain trees are of interest. In the following theorem, they obtained an upper bound on the Steiner \( k \)-diameter of a 2-connected graph of order \( n \).

**Theorem 3.11** [36] Let \( G \) be a 2-connected graph of order \( n \), and let \( 2 \leq k \leq n \). Then

\[
\text{sdiam}_k(G) \leq \left\lfloor \frac{n(k - 1)}{k} \right\rfloor
\]

and equality is attained for the cycle \( C_n \).

Theorem 3.11 shows that among all 2-connected graphs of given order \( n \), the cycle \( C_n \) has the largest possible Steiner \( k \)-diameter. Dankelmann, Swart, and Oellermann [36] conjectured that more generally

**Conjecture 3.2** [36] For a 2\( k \)-connected graph \( G \) of order \( n \),

\[
\text{sdiam}_k(G) \leq \text{sdiam}_k(C_n^{2k}),
\]

where \( C_n^{2k} \) denotes the 2\( k \)-th power of the cycle \( C_n \).
3.3 Graphs with given Steiner diameter

The following observation is immediate.

**Observation 3.1** Let $G$ be a connected graph of order $n$. Then

1. $\text{diam}(G) = 1$ if and only if $G$ is a complete graph;
2. $\text{diam}(G) = n - 1$ if and only if $G$ is a path of order $n$.

Let $uv$ be an edge in $G$. A **double-star** on $uv$ is a maximal tree in $G$ which is the union of stars centered at $u$ or $v$ such that each star contains the edge $uv$. Bloom [16] characterized the graphs with $\text{diam}(G) = 2$.

**Theorem 3.12** [16] Let $G$ be a connected graph of order $n$. Then $\text{diam}(G) = 2$ if and only if $G$ is non-empty and $G$ does not contain a double star of order $n$ as its subgraph.

In [99], Mao, Melekian, and Cheng derived the following result.

**Theorem 3.13** Let $\ell, n$ be two integers with $1 \leq \ell \leq n - 2$, and let $G$ be a graph of order $n$. Then $\kappa(G) \geq \ell$ if and only if $\text{sdiam}_{n-\ell}(G) = n - \ell$.

In [134], Wang, Mao, Li, and Ye obtained the structural properties of graphs with $\text{sdiam}_k(G) = n - 1$.

**Theorem 3.14** [134] Let $k, n$ be two integers with $3 \leq k \leq n - 1$. Let $G$ be a connected graph of order $n$. Then $\text{sdiam}_k(G) = n - 1$ if and only if the number of non-cut vertices in $G$ is at most $k$.

### 3.3.1 Results for small $k$

In [94], Mao characterized the graphs with $\text{sdiam}_3(G) = 2$, which can be seen as an extension of (1) of Observation 3.1.

**Theorem 3.15** [94] Let $G$ be a connected graph of order $n$. Then $\text{sdiam}_3(G) = 2$ if and only if $0 \leq \Delta(G) \leq 1$ if and only if $n - 2 \leq \delta(G) \leq n - 1$.

A **triple-star** $H_3$ is defined as a connected graph of order $n$ obtained from a triangle and three stars $K_{1,a}, K_{1,b}, K_{1,c}$ by identifying the center of a star and one vertex of the triangle, where $0 \leq a \leq b \leq c$, $c \geq 1$ and $a + b + c = n - 3$; see Figure 3.3 (a). Let $H_2$ be a connected graph of order $n$ obtained from a path $P = uvw$ and $n - 3$ vertices such that for each $x \in V(H_2) - \{u, v, w\}$, $xu, xv, xw \in E(H_2)$, or $xu, xv \in E(H_2)$ but $xw \notin E(H_2)$, or $xv, xw \in E(H_2)$ but $xu \notin E(H_2)$, or $xu, xw \in E(H_2)$ but $xv \notin E(H_2)$, or $xw \notin E(H_2)$; see Figure 3.3 (b).
Figure 3.3: The graphs $H_1$ and $H_2$.

**Theorem 3.16** [94] Let $G$ be a connected graph of order $n$. Then $\text{sdiam}_3(G) = 3$ if and only if $G$ satisfies the following conditions.

- $\Delta(G) \geq 2$;
- $G$ does not contain a triple-star $H_1$ as its subgraph;
- $G$ does not contain $H_2$ as its subgraph.

Denote by $T_{a,b,c}$ a tree with a vertex $v$ of degree 3 such that $T_{a,b,c} - v = P_a \cup P_b \cup P_c$, where $0 \leq a \leq b \leq c$ and $1 \leq b \leq c$ and $a + b + c = n - 1$; see Figure 3.4 (a). Observe that $T_{0,b,c}$ where $b + c = n - 1$ is a path of order $n$. Denote by $\triangle_{p,q,r}$ a unicyclic graph containing a triangle $K_3$ and satisfying $\triangle_{p,q,r} - V(K_3) = P_p \cup P_q \cup P_r$, where $0 \leq p \leq q \leq r$ and $p + q + r = n - 3$; see Figure 3.4 (b).

Figure 3.4: The graphs $T_{a,b,c}$ and $\triangle_{p,q,r}$.

**Theorem 3.17** [94] Let $G$ be a connected graph of order $n$ ($n \geq 3$). Then $\text{sdiam}_3(G) = n - 1$ if and only if $G = T_{a,b,c}$ where $a \geq 0$ and $1 \leq b \leq c$ and $a + b + c = n - 1$, or $G = \triangle_{p,q,r}$ where $0 \leq p \leq q \leq r$ and $p + q + r = n - 3$.

In [134], Wang, Mao, Li, and Ye characterized the graphs with $\text{sdiam}_4(G) = 3, 4, n - 1$, respectively.
Theorem 3.18 \[131\] Let $G$ be a connected graph of order $n$ ($n \geq 4$).

(i) If $n = 4$, then $sdiam_4(G) = 3$;

(ii) If $n \geq 5$, then $sdiam_4(G) = 3$ if and only if $n - 3 \leq \delta(G) \leq n - 1$ and $C_4$ is not a subgraph of $\overline{G}$.

A graph $A_1$ is defined as a connected graph of order $n$ ($n \geq 5$) obtained from a $K_4$ with vertex set $\{u_1, u_2, u_3, u_4\}$ and four stars $K_1,a, K_1,b, K_1,c, K_1,d$ by identifying the center of one star and one vertex in $\{u_1, u_2, u_3\}$, where $0 \leq a \leq b \leq c \leq d$, $d \geq 1$, and $a + b + c + d = n - 4$; see Figure 3.3.

A graph $A_2$ is defined as a connected graph of order $n$ ($n \geq 5$) obtained from $K_4 - e$ with vertex set $\{u_1, u_2, u_3, u_4\}$, $e = u_1u_4$ and two stars $K_1,a, K_1,b$ by identifying the center of a star and one vertex in $\{u_2, u_3\}$, and then adding the paths $u_1z_iu_4$ ($1 \leq i \leq c$), where $0 \leq a \leq b$, $b \geq 0$, $c \geq 0$ and $a + b + c = n - 4$; see Figure 3.5.

A graph $A_3$ is defined as a connected graph of order $n$ ($n \geq 5$) obtained from a cycle $C_4 = u_1u_2u_3u_4u_1$ by adding the paths $u_1z_iu_2$ ($1 \leq i \leq a$) and the paths $u_3y_ju_4$ ($1 \leq j \leq b$), where $0 \leq a \leq b$, $b \geq 1$ and $a + b = n - 4$; see Figure 3.5.

A graph $A_4$ is defined as a connected graph of order $n$ ($n \geq 5$) obtained from a star $K_1,3$ with vertex set $\{u_1, u_2, u_3, u_4\}$ and a star $K_1,a$ by identifying $u_3$ and the center of $K_1,a$, where $u_3$ is the center of $K_1,3$, and then adding the vertices $y_i$ and the edges.
$y_i u_j$ ($1 \leq i \leq b$, $j = 1, 2, 4$), where $0 \leq a \leq b$, $b \geq 1$ and $a + b = n - 4$; see Figure 3.5.

**Theorem 3.19** [134] Let $G$ be a connected graph of order $n$ ($n \geq 5$). Then $sdiam_4(G) = 4$ if and only if $G$ satisfies one of the following conditions.

(i) $\delta(G) = n - 3$ and $C_4$ is a subgraph of $\overline{G}$;

(ii) $\delta(G) \leq n - 4$ and each $A_i$ ($1 \leq i \leq 4$) is not a spanning subgraph of $\overline{G}$ (see Figure 3.3).

In [134], Wang, Mao, Li, and Ye also defined the following graph classes.

- Let $T_{a,b,c,d}$ ($0 \leq a, b, c, d \leq n - 1$, $a + b + c + d \leq n - 1$) be a tree of order $n$ ($n \geq 5$) obtained from three paths $P_1, P_2, P_3$ of length $n - b - c - 1, b, c$ respectively by identifying the $(a+1)$-th vertex of $P_1$ and one endvertex of $P_2$, and then identifying the $(n - b - c - d)$-th vertex of $P_1$ and one endvertex of $P_3$ (Note that $u$ and $v$ can be the same vertex);

- Let $\triangle_{a,b,c,d}$ ($0 \leq a, b, c, d \leq n - 2$, $a + b + c + d \leq n - 2$) be an unicyclic graph of order $n$ ($n \geq 5$) obtained from three paths $P_1, P_2, P_3$ of length $n - b - c - 1, b + 1, c$ respectively by identifying the $(a+1)$-th vertex of $P_1$ and one endvertex of $P_2$, and then identifying the $(n - b - c - d)$-th vertex of $P_1$ and one endvertex of $P_3$ (Note that $u$ and $v$ can be the same vertex);

- Let $B_i$ ($1 \leq i \leq 3$) be a graph obtained from three paths $P_1, P_2, P_3$ of length $n - b - c - 1, b + 1, c$ respectively by identifying the $(a+1)$-th vertex of $P_1$ and one endvertex of $P_2$, and then identifying the $(n - b - c - d)$-th vertex of $P_1$ and one endvertex of $P_3$ (Note that $u$ and $v$ can be the same vertex).

Figure 3.6: The graphs $T_{a,b,c,d}, \triangle_{a,b,c,d}, \triangle'_{a,b,c,d}$ and $B_i$ ($1 \leq i \leq 3$).
then identifying the \((n - b - c - d)\)-th vertex of \(P_1\) and one endvertex of \(P_3\), and then adding an edge \(u_{b+1}v_{a+2}\) (Note that \(v_{a+2}\) and \(v\) can be the same vertex).

- Let \(\Delta'_{a,b,c,d}\) \((0 \leq a, b, c, d \leq n - 3, a + b + c + d \leq n - 3)\) be an bicyclic graph of order \(n (n \geq 5)\) obtained from three paths \(P_1, P_2, P_3\) of length \(n - b - c - 1, b + 1, c + 1\) respectively by identifying the \((a + 1)\)-th vertex of \(P_1\) and one endvertex of \(P_2\), and then identifying the \((n - b - c - d)\)-th vertex of \(P_1\) and one endvertex of \(P_3\), and then adding two edges \(u_{b+1}v_{a+2}\) and \(w_{c+1}x_{d+2}\) (Note that \(v_{a+2}\) and \(v\) can be the same vertex).

- Let \(B_1\) be a graph of order \(n (n \geq 5)\) obtained from a clique of order 4 and four paths \(P_1, P_2, P_3, P_4\) of length \(a, b, c, d \leq n - 4, a + b + c + d = n - 4\) respectively by identifying each vertex of this clique with an endvertex of one of the four paths.

- Let \(B_2\) be a graph of order \(n (n \geq 5)\) obtained from a cycle of order 4 and four paths \(P_1, P_2, P_3, P_4\) of length \(a, b, c, d \leq n - 4, a + b + c + d = n - 4\) respectively by identifying each vertex of this cycle with an endvertex of one of the four paths.

- Let \(B_3\) be a graph of order \(n (n \geq 5)\) obtained from \(K^4\) and four paths \(P_1, P_2, P_3, P_4\) of length \(a, b, c, d \leq n - 4, a + b + c + d = n - 4\) respectively by identifying each vertex of \(K^4\) with an endvertex of one of the four paths, where \(K^4\) denotes the graph obtained from a clique of order 4 by deleting one edge.

**Theorem 3.20** \([134]\) Let \(G\) be a connected graph of order \(n (n \geq 5)\). Then \(sdiam_4(G) = n - 1\) if and only if \(G = T_{a,b,c,d}\) or \(G = \Delta_{a,b,c,d}\) or \(G = \Delta'_{a,b,c,d}\) or \(G = B_1\) or \(G = B_2\) or \(G = B_3\).

### 3.3.2 Results for large \(k\)

The following result is immediate.

**Observation 3.2** \([99]\) Let \(G\) be a graph of order \(n\). Then \(sdiam_n(G) = n - 1\) if and only if \(G\) is connected.

Mao, Melekian, and Cheng \([99]\) characterized the graphs with \(sdiam_{n-1}(G) = d (n - 2 \leq d \leq n - 1)\) and \(sdiam_{n-2}(G) = d (n - 3 \leq d \leq n - 1)\).

**Proposition 3.4** \([99]\) Let \(G\) be a connected graph of order \(n\). Then

1. \(sdiam_{n-1}(G) = n - 2\) if and only if \(G\) is 2-connected;
2. \(sdiam_{n-1}(G) = n - 1\) if and only if \(G\) contains at least one cut vertex.
Theorem 3.21 Let $G$ be a connected graph of order $n$ ($n \geq 5$). Then

1. $sdiam_{n-2}(G) = n - 3$ if and only if $\kappa(G) \geq 3$.
2. $sdiam_{n-2}(G) = n - 2$ if and only if $\kappa(G) = 2$ or $G$ contains only one cut vertex.
3. $sdiam_{n-2}(G) = n - 1$ if and only if there are at least two cut vertices in $G$.

3.4 Nordhaus-Gaddum-type results

It is well known that if $diam(G) \geq 3$, then $diam(G) \leq 3$, a result first proved by Harary and Robinson [71]. A similar result, namely that if $diam(G) \geq 4$, then $diam(G) \leq 2$, is due to Straffin [124]. A common generalization of both results is given in the next theorem.

Theorem 3.22 (1) Let $G$ be a connected graph. If $sdiam_k(G) \geq k + r$, $1 \leq r \leq k - 1$, then $sdiam_k(G) \leq 2k - r$.

(2) Let $G$ be a connected graph. If $sdiam_k(G) \geq 2k$, then $sdiam_k(G) \leq k$.

Xu [139] obtained the Nordhaus-Gaddum results for the diameter of graphs. In [94], Mao got the Nordhaus-Gaddum results for the Steiner $k$-diameter of graphs.

Theorem 3.23 Let $G \in G(n)$ be a connected graph with a connected complement. Let $k$ be an integer with $3 \leq k \leq n$. Then

(i) $2k - 1 - x \leq sdiam_k(G) + sdiam_k(\overline{G}) \leq \max\{n + k - 1, 4k - 2\}$;

(ii) $(k - 1)(k - x) \leq sdiam_k(G) \cdot sdiam_k(\overline{G}) \leq \max\{k(n - 1), (2k - 1)^2\}$,

where $x = 0$ if $n \geq 2k - 2$ and $x = 1$ if $n < 2k - 2$.

For $k = n, n - 1, n - 2, 3$, Mao [94] improved the above Nordhaus-Gaddum results of Steiner $k$-diameter and derived the following results.

Observation 3.3 Let $G \in G(n)$ ($n \geq 3$) be a connected graph with a connected complement. Then

(i) $sdiam_n(G) + sdiam_n(\overline{G}) = 2n - 2$;

(ii) $sdiam_n(G) \cdot sdiam_n(\overline{G}) = (n - 1)^2$.

Proposition 3.5 Let $G \in G(n)$ ($n \geq 5$) be a connected graph with a connected complement. Then

(i) $2n - 4 \leq sdiam_{n-1}(G) + sdiam_{n-1}(\overline{G}) \leq 2n - 2$;

(ii) $(n - 2)^2 \leq sdiam_{n-1}(G) \cdot sdiam_{n-1}(\overline{G}) \leq (n - 1)^2$.

Moreover,
(a) $\text{sdi}_n(G) + \text{sdi}_n(\overline{G}) = 2n - 4$ or $\text{sdi}_n(G) \cdot \text{sdi}_n(\overline{G}) = (n - 2)^2$
if and only if both $G$ and $\overline{G}$ are 2-connected;

(b) $\text{sdi}_n(G) + \text{sdi}_n(\overline{G}) = 2n - 3$ or $\text{sdi}_n(G) \cdot \text{sdi}_n(\overline{G}) = (n - 1)(n - 2)$
if and only if $\lambda(G) = 1$ and $\overline{G}$ are 2-connected, or $\lambda(\overline{G}) = 1$ and $G$ are 2-connected.

(c) $\text{sdi}_n(G) + \text{sdi}_n(\overline{G}) = 2n - 2$ or $\text{sdi}_n(G) \cdot \text{sdi}_n(\overline{G}) = (n - 1)^2$
if and only if $G$ satisfies one of the following conditions.

- $\kappa(G) = 1$, $\Delta(G) = n - 2$;
- $\kappa(G) = 1$, $\Delta(G) \leq n - 3$ and $G$ has a cut vertex $v$ with pendant edge $e$ and pendant vertex $u$ such that $G - u$ contains a spanning complete bipartite subgraph.

**Proposition 3.6** \cite{135} Let $G \in \mathcal{G}(n)$ $(n \geq 5)$ be a connected graph with a connected complement. If both $G$ and $\overline{G}$ contains at least two cut vertices, then

(i) $2n - 6 \leq \text{sdi}_n(G) + \text{sdi}_n(\overline{G}) \leq 2n - 2$;

(ii) $(n - 3)^2 \leq \text{sdi}_n(G) \cdot \text{sdi}_n(\overline{G}) \leq (n - 1)^2$.

Otherwise,

(iii) $2n - 6 \leq \text{sdi}_n(G) + \text{sdi}_n(\overline{G}) \leq 2n - 3$;

(iv) $(n - 3)^2 \leq \text{sdi}_n(G) \cdot \text{sdi}_n(\overline{G}) \leq (n - 1)(n - 2)$.

Moreover, the upper and lower bounds are sharp.

**Proposition 3.7** \cite{135} Let $G \in \mathcal{G}(n)$ $(n \geq 10)$ be a connected graph with a connected complement. Then

(i) $6 \leq \text{sdi}_3(G) + \text{sdi}_3(\overline{G}) \leq n + 2$;

(ii) $9 \leq \text{sdi}_3(G) \cdot \text{sdi}_3(\overline{G}) \leq 3(n - 1)$.

Moreover, the bounds are sharp.

### 3.5 Steiner diameter of graph products

For Steiner diameter of joined, corona, cluster graphs, Wang, Mao, Cheng, and Melekian \cite{135} had the following.

**Proposition 3.8** \cite{135} Let $k, n, m$ be two integers with $3 \leq k \leq n(m + 1)$, and let $G, H$
be two connected graphs with $n, m$ vertices, respectively.

(i) If $3 \leq k \leq n$, then $\text{siam}_k(G \ast H) = \text{siam}_k(G) + k$.

(ii) If $n + 1 \leq k \leq mn$, then $\text{siam}_k(G \ast H) = n - 1 + k$.

(iii) If $mn + 1 \leq k \leq (m + 1)n$, then $\text{siam}_k(G \ast H) = n - 1 + mn$. 
Proposition 3.9 Let $G$ be a connected graph with $n$ vertices, and let $H$ be a connected graph with $m$ ($n \leq m$) vertices. Let $k$ be an integer with $3 \leq k \leq n + m$.

(i) If $k > m$, then $\text{siam}_k(G \lor H) = k - 1$.

(ii) If $n < k \leq m$ and $\text{siam}_k(H) = k - 1$, then $\text{siam}_k(G \lor H) = k - 1$; if $n < k \leq m$ and $\text{siam}_k(H) \geq k$, then $\text{siam}_k(G \lor H) = k$.

(iii) If $3 \leq k \leq n$, and $\text{siam}_k(G) \geq k$ or $\text{siam}_k(H) \geq k$, then $\text{siam}_k(G \lor H) = k$; If $3 \leq k \leq n$ and $\text{siam}_k(G) = \text{siam}_k(H) = k - 1$, then $\text{siam}_k(G \lor H) = k - 1$.

Proposition 3.10 Let $k, n, m$ be two integers with $3 \leq k \leq nm$, and let $G, H$ be two connected graphs with $n, m$ vertices, respectively.

(i) If $m > n$ and $3 \leq k \leq n$, then

$$\text{siam}_k(G) + k \leq \text{siam}_k(G \odot H) \leq k \cdot \text{siam}_{k+1}(H) + \text{siam}_k(G).$$

(ii) If $m > n$ and $n + 1 \leq k \leq m - 1$, then

$$n - 1 + k \leq \text{siam}_k(G \odot H) \leq n \cdot \text{siam}_{k+1}(H) + n - 1.$$

(iii) If $m > n$ and $m \leq k \leq nm - n$, then

$$n - 1 + k \leq \text{siam}_k(G \odot H) \leq mn - 1.$$

(iv) If $m > n$ and $nm - n \leq k \leq nm$, then

$$\text{siam}_k(G \odot H) = mn - 1.$$

(v) If $m \leq n$ and $3 \leq k < m$, then

$$\text{siam}_k(G) + k \leq \text{siam}_k(G \odot H) \leq k \cdot \text{siam}_{k+1}(H) + \text{siam}_k(G).$$

(vi) If $m \leq n$ and $m \leq k \leq n$, then

$$\text{siam}_k(G) + k \leq \text{siam}_k(G \odot H) \leq k(m - 1) + \text{siam}_k(G).$$

(vii) If $m \leq n$ and $n < k \leq mn - n$, then

$$n - 1 + k \leq \text{siam}_k(G \odot H) \leq mn - 1.$$

(viii) If $m \leq n$ and $mn - n < k \leq mn$, then

$$\text{siam}_k(G \odot H) = mn - 1.$$

For Steiner $k$-diameter of Cartesian product graphs, Mao, Cheng, and Wang [100] had the following.
Theorem 3.24 [100] Let $k, m, n$ be an integer with $3 \leq k \leq mn$ and $n \leq m$. Let $G, H$ be two connected graphs of order $n, m$, respectively.

(1) If $k \leq n$, then
\[
\text{sdiamp}_k(G) + \text{sdiamp}_k(H) \leq \text{sdiamp}_k(G \Box H) \leq \text{sdiamp}_k(G) + \text{sdiamp}_k(H) + (k - 3) \min\{\text{sdiamp}_k(G), \text{sdiamp}_k(H)\}.
\]

(2) If $n < k \leq m$, then
\[
n - 1 + \text{sdiamp}_k(H) \leq \text{sdiamp}_k(G \Box H) \leq n - 1 + \text{sdiamp}_k(H) + (k - 3) \min\{n - 1, \text{sdiamp}_k(H)\}.
\]

(3) If $m < k \leq mn$, then
\[
n + m - 2 \leq \text{sdiamp}_k(G \Box H) \leq m - 1 + (k - 2)(n - 1).
\]

(4) If $mn - \kappa(G \Box H) + 1 \leq k \leq mn$, then $\text{sdiamp}_k(G \Box H) = k - 1$.

The following corollary is immediate from Theorem 3.24.

Corollary 3.5 [100] Let $G, H$ be two connected graphs of order at least 3. Then
\[
\text{sdiamp}_3(G \Box H) = \text{sdiamp}_3(G) + \text{sdiamp}_3(H).
\]

To show the sharpness of the above upper and lower bound, we consider the following example.

Example 3.7. [100] (1) For $k = 3$, from Corollary 3.5, we have $\text{sdiamp}_k(G \Box H) = \text{sdiamp}_k(G) + \text{sdiamp}_k(H)$, which implies that the upper and lower bounds in Theorem 3.24 are sharp.

(2) Let $G = P_n$ and $H = P_m$ with $5 \leq n \leq m$. Then $\text{sdiamp}_4(G) = n - 1$, $\text{sdiamp}_4(H) = m - 1$ and $\text{sdiamp}_4(G \Box H) = 2(n - 1) + (m - 1)$, which implies that all the upper bounds in Theorem 3.24 are sharp.

By Theorem 2.8, Mao, Cheng, and Wang [100] derived the following results for Steiner diameter of lexicographic product graphs.

Theorem 3.25 [100] Let $k, m, n$ be three integers with $2 \leq k \leq mn$. Let $G$ be a connected graph of order $n$, and $H$ be a graph of order $m$. Then
\[
\text{sdiamp}_k(G \circ H) \leq \begin{cases} 
\text{sdiamp}_k(G) + k - 2, & \text{if } 2 \leq k \leq n; \\
\max\{n + k - 3, k\}, & \text{if } n < k \leq mn.
\end{cases}
\]
and
\[
\text{sdiam}_k(G \circ H) \geq \begin{cases} 
\text{sdiam}_k(G), & \text{if } m + 1 \leq k \leq n; \\
n - 1, & \text{if } \max\{n, m + 1\} \leq k \leq mn; \\
k - 1, & \text{if } 2 \leq k \leq m.
\end{cases}
\]

To show the sharpness of the upper and lower bound in Theorem 3.25, they considered the following example.

**Example 3.8.** [100] Let \( G = P_n \), and \( H \) be a graph of order \( m \). If \( k \leq \min\{2m, n\} \), then \( \text{sdiam}_k(G \circ H) = n + k - 3 = \text{sdiam}_k(G) + k - 2 \). If \( \max\{n, m + 1\} \leq k \leq 2m \), then \( \text{sdiam}_k(G \circ H) = n + k - 3 = \max\{n + k - 3, k\} \). These implies that the upper bounds in Theorem 3.25 are sharp. Let \( G = K_n \) and \( H = K_m \). Then \( G \circ H \) is a complete graph of order \( mn \). If \( 2 \leq k \leq m \), then \( \text{sdiam}_k(G \circ H) = k - 1 \). If \( m + 1 \leq k \leq n \), then \( \text{sdiam}_k(G \circ H) = k - 1 = \text{sdiam}_k(G) \). These implies that the lower bounds in Theorem 3.25 are sharp.

The following result is immediate from Proposition 2.5.

**Proposition 3.11** [100] Let \( G, H \) be two connected graphs. Then
\[
\text{sdiam}_3(G \circ H) = \begin{cases} 
\text{diam}(G) + 1, & \text{if } G = P_n, \text{ diam}(G) \geq 2; \\
\text{sdiam}_3(G), & \text{if } G \neq P_n, \text{ diam}(G) \geq 2; \\
\min\{\text{sdiam}_3(H), 3\}, & \text{if } G = K_n.
\end{cases}
\]

## 4 Average Steiner Distance and Steiner Wiener Index

Average Steiner distance is related to the Steiner Wiener index via \( SW_k(G)/(\binom{n}{k}) \).

The following results are due to Dankelmann, Oellermann, Swart [34], and Li, Mao, Gutman [90, 91].

**Proposition 4.1** [34, 90, 91] Let \( k, n \) be two integers with \( 2 \leq k \leq n \).

1. For a complete graph \( K_n \), \( SW_k(K_n) = \binom{n}{k}(k - 1) \).
2. For a path \( P_n \), \( \mu_k(P_n) = \frac{(k-1)(n+1)}{k+1}; SW_k(P_n) = (k-1)\binom{n+1}{k+1} \).
3. For a star \( S_n \), \( SW_k(S_n) = \binom{n-1}{k-1}(n-1) \).
4. For a complete bipartite graph \( K_{a,b} \) (\( 2 \leq k \leq a + b \)),
\[
SW_k(K_{a,b}) = \begin{cases} 
(k-1)\binom{a+b}{k} + \binom{a}{k} + \binom{b}{k}, & \text{if } 1 \leq k \leq a; \\
(k-1)\binom{a+b}{k} + \binom{b}{k}, & \text{if } a < k \leq b; \\
(k-1)\binom{a+b}{k}, & \text{if } b < k \leq a + b.
\end{cases}
\]
(5) Let $T$ be a graph obtained from a path $P_t$ and a star $S_{n-t+1}$ by identifying a pendant vertex of $P_t$ and the center $v$ of $S_{n-t+1}$, where $1 \leq t \leq n-1$ and $k \leq n$. Then

$$SW_k(T) = t \left(\frac{n-1}{k}\right) - \left(\frac{t}{k+1}\right) - \left(\frac{n}{k+1}\right) + \left(\frac{n-t+1}{k+1}\right) + (k-1) \left(\frac{n}{k}\right).$$

(5) Let $G$ be a graph obtained from a clique $K_{n-r}$ and a star $S_{r+1}$ by identifying a vertex of $K_{n-r}$ and the center $v$ of $S_{r+1}$. For $k \leq r \leq n-1-k$,

$$SW_k(G) = (n-1) \left(\frac{n-1}{k-1}\right) - \left(\frac{n-r-1}{k}\right).$$

For $2 \leq r < k$, Dankelmann, Oellermann, and Swart [34] established a relation between $\mu_r(G)$, $\mu_{k+1-r}(G)$, and $\mu_k(G)$.

**Theorem 4.1** [34] Let $G$ be connected weighted graph and $2 \leq r \leq k-1$. Then

$$\mu_k(G) \leq \mu_r(G) + \mu_{k+1-r}(G).$$

**Corollary 4.1** [34] For $k \geq 3$, $\mu_k(G) \leq (k-1)\mu(G)$.

The bounds in Theorem 4.1 and Corollary 4.1 are sharp for the complete graph.

**Remark 4.1** [34] With essentially the same methods as those used in [75] it can be shown that for each connected graph $G$ of order $n$ and $3 \leq k \leq n$,

$$\mu_k(G) \leq \frac{k+1}{k-1} \mu_{k-1}(G).$$

It remains an open problem to determine a lower bound for $\mu_k(G)$ in terms of $\mu(G)$, but they conjectured that the smallest ratio $\mu_k(G)/\mu(G)$ taken over all connected graphs $G$ of order $n$ where $n \geq k$, is attained if $G$ is the path. More formally:

**Conjecture 4.1** [34] If $G$ is a connected graph of order $n$ and $3 \leq k \leq n$, then

$$\mu_k(G) \geq \frac{3}{k+1} \mu(G).$$

In [34], Dankelmann, Oellermann, and Swart proved that the conjecture is true for $k = 3$ and $k = n$.

### 4.1 Results for trees

It is usual to consider $SW_k$ for $2 \leq k \leq n-1$, but the definition implies $SW_1(G) = 0$ and $SW_n(G) = n-1$ for a connected graph $G$ of order $n$. 

Theorem 4.2 [90] Let $T$ be a tree of order $n$, possessing $p$ pendant vertices. Then

$$SW_{n-1}(T) = n(n - 1) - p,$$

irrespective of any other structural detail of $T$.

Theorem 4.3 [91] Let $T$ be a tree of order $n$, possessing $p$ pendant vertices. Let $q$ be the number of vertices of degree 2 in $T$ that are adjacent to a pendant vertex. Then

$$SW_{n-2}(T) = \frac{1}{2} \left( n^3 - 2n^2 + n - 2np + 2p - 2q \right).$$

Let $G$ be any graph (not necessarily connected) with vertex set $V(G)$. Let $e$ be an edge of $G$, connecting the vertices $x$ and $y$. Define the sets

$$\mathcal{N}_1(e) = \{ u \mid u \in V(G), d(u, x) < d(u, y) \}$$
$$\mathcal{N}_2(e) = \{ u \mid u \in V(G), d(u, x) > d(u, y) \}$$

and let their cardinalities be $n_1(e) = |\mathcal{N}_1(e)|$ and $n_2(e) = |\mathcal{N}_2(e)|$, respectively. In other words, $n_1(e)$ counts the vertices of $G$, lying closer to one end of the edge $e$ than to its other end, and the meaning of $n_2(e)$ is analogous.

In his seminal paper [137], Wiener discovered the following result:

Proposition 4.2 [137] If $T$ is a tree, then for its Wiener index holds:

$$W(T) = \sum_{e \in E(T)} n_1(e) n_2(e).$$

Li, Mao, and Gutman [90] stated the generalization of Proposition 4.2 to Steiner Wiener indices:

Theorem 4.4 [90] Let $k$ be an integer such that $2 \leq k \leq n$. If $T$ is a tree, then for its Steiner $k$-Wiener index holds:

$$SW_k(T) = \sum_{e \in E(T)} \sum_{i=1}^{k-1} \binom{n_1(e)}{i} \binom{n_2(e)}{k-i}. \quad (4.3)$$

Corollary 4.2 [90] (1) Proposition 4.2 is obtained from Eq. (4.3) by setting $k = 2$.

(2) If $k = 3$, then the Steiner $k$-Wiener index of a tree of order $n$ is directly related to the ordinary Wiener index as

$$SW_3(T) = \frac{n - 2}{2} W(T).$$

Dankelmann, Oellermann, and Swart [34] investigated the average Steiner $k$-distance of trees by establishing sharp upper and lower bounds for this parameter.
Theorem 4.5 Let $T$ be a tree of order $n \geq k$ and $2 \leq r \leq k - 1$. Then

$$\mu_k(T) \leq \frac{n}{r} \mu_r(T).$$

Furthermore, equality holds if and only if $T$ is a star.

Corollary 4.3 (1) If $T$ is a tree of order $n \geq k$, then

$$\mu_k(T) \leq \frac{k}{2} \mu(T).$$

(2) If $T$ is a tree of order $n \geq k$ and $2 \leq r \leq k - 2$, then

$$\mu_k(T) \leq \mu_r(T) + \mu_{k-r}(T).$$

The upper and lower bounds for average Steiner distance is also obtained in the same paper.

Proposition 4.3 If $T$ is a tree of order $n$ ($2 \leq k \leq n$), then

$$k \left(1 - \frac{1}{n}\right) \leq \mu_k(T) \leq \frac{k-1}{k+1} (n+1)$$

equality holds if and only if $T$ is a star or path, respectively, or in either case if $k = n$.

Remark 4.2 For $k = 2$, Proposition 4.3 was already observed in [44] and [93].

4.2 Algorithmic aspect for average Steiner distance

An $O(kn^2)$ procedure is developed in [15] for calculating the $k$-distances of all vertices of a tree $T$ of order $n$. Since the Steiner distance of any set $S$ of $k$ vertices contributes the same amount to the $k$-distance of each vertex in $S$, it follows that

$$\mu_k(T) = \sum_{v \in V(T)} \frac{d_k(v)}{n} \binom{n}{k}.$$

Hence, the procedure developed in [15] gives a $O(kn^2)$ procedure for finding $\mu_k(T)$ for a tree $T$ which is considerably more efficient than the brute force method of calculating the Steiner distance of all $\binom{n}{k}$ sets of vertices if $n > k$.

Dankelmann, Oellermann, and Swart outlined an even more efficient algorithm that computes the average $k$-distance of a tree without first computing the $k$-distance of each vertex. For a graph $G$ let $m(G)$ denote the maximum order among all components of $G$. The algorithm we now describe is based on the proof of Theorem 4.5 where the average $k$-distance of a tree is expressed implicitly in terms of the values $m(T - e)$ for $e \in E(T)$. It computes the average $k$-distance of a tree of order $n$ using $O(n)$ graph operations and $O(kn)$ arithmetic operations. In [34], the edge weight $\omega_k(e)$ for $e \in E(T)$ is defined by

$$\omega_k(e) = \binom{n}{k} - \binom{m(T - e)}{k} - \binom{n - m(T - e)}{k}.$$
Then $\omega_k(e)$ counts the number of $k$-sets $S \subseteq V(T)$ that have at least one vertex in each component of $T - e$. Thus $\omega_k(e)$ equals the number of Steiner trees containing $e$ and we have

$$\mu_k(T) = \sum_{e \in E(T)} \omega_k(e) \left( \binom{n}{k} \right)^{-1} = n - 1 - \sum_{e \in E(T)} \left[ \binom{m(T-e)}{k} + \binom{n-m(T-e)}{k} \right] \left( \binom{n}{k} \right)^{-1}.$$ 

Therefore it suffices to compute the values $m(T-e)$ for all $e \in E(T)$, which is possible in $O(n)$ time and to apply the above equality which requires at most $O(kn)$ multiplications and divisions.

### 4.3 Upper and lower bounds

In [34], the range for the average $k$-distance of a connected graph of given order was determined, generalizing a result for $k = 2$ obtained by Entringer, Jackson, and Snyder [44], Doyle and Graver [40], and Lovász [93].

**Theorem 4.6** [34] Let $G$ be a connected graph of order $n$ and let $2 \leq k \leq n - 1$. Then

$$k - 1 \leq \mu_k(G) \leq \frac{k - 1}{k + 1}(n + 1).$$

Equality holds on the left (or right) if and only if $G$ is $(n + 1 - k)$-connected (or if $G$ is a path, respectively).

The upper bound for the average $k$-distance given in Theorem 4.6 can be improved for 2-connected graphs. Plesník [120] showed that the cycle of order $n$, $C_n$, is the unique 2-connected graph with given order $n$ and maximum average distance. This result was generalized for the average $k$-distance in [34]. It is remarkable that Plesník’s result can easily be generalized for $k = 2$ and $2\ell$-connected graphs (see [49]), which seems not to be the case for $k \geq 3$.

**Theorem 4.7** [35] Let $G$ be a 2-connected graph of order $n$ and let $2 \leq k \leq n$. Then

$$\mu_k(G) \leq \mu_k(C_n).$$

Equality holds if and only if $G = C_n$, or $k \geq n - 1$.

In [120], Plesník proved that, apart from the obvious restriction $1 \leq \mu(G) \leq \text{diam}(G)$, the average distance is independent of the diameter and the radius.

**Theorem 4.8** [120] Let $r, d$ be positive integers with $d \leq 2r$ and let $t \in \mathbb{R}$ with $1 \leq t \leq d$. For every $\varepsilon > 0$ there exists a graph $G$ with diameter $d$, radius $r$, and

$$|\mu(G) - t| < \varepsilon.$$
It is natural to ask if there is a similar statement for the $k$-diameter and the average $k$-distance. An answer in the affirmative is stated by Dankelmann, Swart, and Oellermann in [35].

**Theorem 4.9** [16] Let $k, d$ be positive integers, $k \geq 2$, and let $t \in \mathbb{R}$ with $k - 1 \leq t \leq d$. For every $\varepsilon > 0$ there exists a graph $G$ with $k$-diameter $d$ and

$$|\mu_k(G) - t| < \varepsilon.$$ 

Dankelmann, Swart, and Oellermann [35] remarked that Theorem 4.9 is not a generalization of Plesnık’s Theorem 4.8, since it does not allow us to prescribe also the Steiner $k$-radius. The problem of finding such a generalization requires the determination of the possible values for the Steiner $k$-radius of a graph of given Steiner $k$-diameter. This problem is still unsolved.

In [28] it was conjectured that the Steiner $k$-diameter of a graph $G$ never exceeds $\frac{k}{k-1}\text{srad}_k(G)$. This conjecture was disproved by Henning, Oellermann, and Swart [73], where the bound $\text{sdim}_k(G) \leq \frac{2(k+1)}{(2k-1)}\text{srad}_k(G)$ was conjectured.

The problem of determining a sharp lower bound for the average $k$-distance of a connected graph with $n$ vertices and $m$ edges, where $k \geq 3$, is considerably more difficult than the corresponding problem for $k = 2$. The latter one was solved in [44]. The following bound shows that the complete $r$-partite Turán graphs are optimal in this regard. It remains an open problem to determine the graphs of given order and size that minimize the average $k$-distance.

**Theorem 4.10** [35] Let $G$ be a graph of order $n$ and size $m$. Then

$$\mu_k(G) \geq k - 1 + n \left( n - \frac{2m}{n} - 1 \right) \left( \frac{n}{k} \right)^{-1},$$

where for a real number $a$ and a positive integer $b$ the binomial coefficient $\binom{a}{b}$ is defined as $a(a - 1) \ldots (a - b + 1)/b!$.

The bound given is sharp if $k$ is a multiple of $r$. It is attained by the complete $r$-partite Turán graph.

Tomescu and Melter [127] determined the range for the average distance of a graph of given order and chromatic number and also the extremal graphs. Dankelmann, Swart, and Oellermann in [35] showed in the following generalization, that the same graphs are also extremal for $k \geq 3$, though there are other ones as well.

For $r < n$, let $H_{n,r}$ be the graph obtained from a complete graph $K_r$ and a path of order $n - r$ with end vertices $v_1$ and $v'_1$ by joining $v'_1$ to one vertex of $K_r$. For $r = n$, let $H_{n,r}$ be the complete graph $K_n$ and let $v_1$ be a vertex of $K_n$. 
Theorem 4.11 [35] Let $G$ be a connected graph of order $n$ ($2 \leq k \leq n$) and chromatic number $r$ and let $v$ be a vertex of $G$. Then

\begin{align*}
(1) \quad d_k(v, G) & \leq d_k(v_1, H_{n,r}) \\
(2) \quad \mu_k(G) & \leq \mu_k(H_{n,r}),
\end{align*}

with equality if and only if $v = v_1$ and $G = H_{n,r}$, respectively.

Dankelmann, Swart, and Oellermann in [35] remarked that Theorem 4.10 yields a sharp lower bound for the $k$-distance of a connected graph $G$ of given order $n$ and chromatic number $r$. From
\[ e(G) \leq n^2 \frac{r - 1}{2r} \]
and Theorem 4.10 we have immediately
\[ \mu_k(G) \geq k - 1 + n \left( \frac{n/r - 1}{k - 1} \right) \left( \frac{n}{k} \right)^{-1}. \]

This bound is sharp if $n$ is a multiple of $r$. Examples for equality in the above equation are the $r$-partite Turán graph $T_{n,r}$ and, for $k > n/r$, the graph $T_{n,r} - e$.

4.4 Inverse problem

The seemingly elementary question: “which natural numbers are Wiener indices of graphs?” was much investigated in the past; see [51, 64, 65, 129, 131].

Li, Mao, and Gutman [91] considered the analogous question for Steiner Wiener indices:

Problem 4.1 [91] Fixed a positive integer $k$, for what kind of positive integer $w$ does there exist a connected graph $G$ (or a tree $T$) of order $n \geq k$ such that $SW_k(G) = w$ (or $SW_k(T) = w$)?

For $k = 2$, the authors have nice results in [64, 130], completely solved a conjecture by Lepović and Gutman [88] for trees, which states that for all but 49 positive integers $w$ one can find a tree with Wiener index $w$. This is different from Problem 4.1 for trees, since Li, Mao, and Gutman [91] considered graphs or trees with order $n$.

If $G$ is a connected graph or a tree of order $n$, then for $k = n$, $SW_k(G) = n - 1$. Thus the following result is immediate.

Proposition 4.4 [91] For a positive integer $w$, there exists a connected graph $G$ or a tree $T$ of order $n$ such that $SW_n(G) = w$ or $SW_n(T) = w$ if and only if $w = n - 1$.

For $k = n - 1$, Li, Mao, and Gutman [91] had the following results.
Proposition 4.5 \[91\] For a positive integer \(w\), there exists a connected graph \(G\) of order \(n\) such that \(SW_{n-1}(G) = w\), if and only if \(n^2 - 2n \leq w \leq n^2 - n - 2\).

Proposition 4.6 \[91\] For a positive integer \(w\), there exists a tree \(T\) of order \(n\) such that \(SW_{n-1}(T) = w\) if and only if \(n^2 - 2n + 1 \leq w \leq n^2 - n - 2\).

For \(k = n - 2\), Li, Mao, and Gutman \[91\] derived the following result for trees.

Theorem 4.12 \[91\] For a positive integer \(w\), there exists a tree \(T\) of order \(n\) (\(n \geq 5\)) possessing \(p\) pendant vertices, such that \(SW_{n-2}(T) = w\) if and only if \(w = \frac{1}{2}(n^3 - 2n^2 + n - 2np + 2p - 2q)\), where \(q\) is the number of vertices of degree 2 in \(T\) that are adjacent to a pendant vertex, and one of the following holds:

1. \(2 \leq q \leq \frac{n - 4}{2}\) and \(q \leq p \leq n - q - 1\);
2. \(q = 1\) and \(3 \leq p \leq n - 2\);
3. \(q = 0\) and \(4 \leq p \leq n - 1\).

Proposition 4.7 \[91\] For a positive integer \(w\), there exists a tree \(T\) of order \(n\) such that \(SW_k(T) = w\) if

\[
w = t \binom{n-1}{k} - \binom{t}{k+1} + \binom{n}{k+1} + \binom{n-t+1}{k+1} + (k-1)\binom{n}{k},
\]

where \(1 \leq t \leq n - 1\) and \(k \leq n\).

Proposition 4.8 \[91\] For a positive integer \(w\), there exists a connected graph \(G\) of order \(n\) such that \(SW_k(G) = w\) if \(w\) satisfies one of the following conditions:

1. \(w = t \binom{n-1}{k} - \binom{1}{k+1} - \binom{n}{k+1} + \binom{n-t+1}{k+1} + (k-1)\binom{n}{k},\) where \(1 \leq t \leq n - 1\) and \(k \leq n\).
2. \(w = (n-1)\binom{n-1}{k-1} - \binom{n-r-1}{k},\) where \(k \leq r \leq n - 1 - k\) and \(k \leq n\).

4.5 Graph products

Yeh and Gutman \[141\] investigated the Wiener index of graph products and obtained the following results.

Theorem 4.13 \[141\] Let \(G\) be a connected graph with \(n\) vertices, and let \(H\) be a connected graph with \(m\) vertices. Then

\[
W(G \vee H) = e(G) + e(H) + mn + 2 \left[ \binom{n}{2} - e(G) + \binom{m}{2} - e(H) \right].
\]
(2) \( W(G \circ H) = m^2 \left[ (W(G) + n) - n(e(H) + m) \right] \).

(3) \( W(G \Box H) = m^2 W(G) + n^2 W(H) \).

(4) \( W(G \odot H) = m^2 W(G) + n W(H) + m(n^2 - n) d(v|H) \), where \( v \) is the root-vertex of \( H \) and
\[
d(v|H) = \sum_{u \in V(H)} d(u,v) .
\]

(5) \( W(G \otimes H) = (m + 1)^2 W(G) + n \left[ m^2 - e(H) \right] + mn(m + 1)(n - 1) \).

In [103], Mao, Wang, Gutman studied the \( k \)-th Steiner Wiener index of the above specified graph products.

**Theorem 4.14** [103] Let \( G \) be a connected graph with \( n \) vertices, and let \( H \) be a connected graph with \( m \) \((n \geq m)\) vertices. Let \( k \) be an integer, \( 3 \leq k \leq n + m \).

1. If \( k > n \), then
\[
SW_k(G \lor H) = (k - 1) \binom{n + m}{k} .
\]

2. If \( k \leq m \), then
\[
SW_k(G \lor H) = (k - 1) \binom{n + m}{k} + \binom{n}{k} + \binom{m}{k} - x - y,
\]
where \( x \) is the number of the \( k \)-subsets of \( V(G) \) such that the subgraph induced by each \( k \)-subset is connected, and \( y \) is the number of the \( k \)-subsets of \( V(H) \) such that the subgraph induced by each \( k \)-subset is connected.

3. If \( m < k \leq n \), then
\[
SW_k(G \lor H) = (k - 1) \binom{n + m}{k} + \binom{n}{k} + (k - 1) \binom{m}{k} - x .
\]

**Theorem 4.15** [103] Let \( G \) be a connected graph with \( n \) vertices, and let \( H \) be a connected graph with \( m \) vertices. Let \( k \) be an integer, \( 2 \leq k \leq nm \). Then
\[
SW_k(G \circ H) = nk \binom{m}{k} - nx + \sum_{\ell=2}^{k} \binom{m}{r_1} \binom{m}{r_2} \cdots \binom{m}{r_{\ell}} SW_{\ell}(G)
\]
\[
+ \sum_{\ell=2}^{k} (k - \ell) \binom{n}{\ell} \binom{m\ell - \ell}{k - \ell}
\]
where \( \sum_{i=1}^{\ell} r_i = k, r_i \geq 1 \) and \( x \) is the number of the \( k \)-subsets of \( V(H) \) such that the subgraph induced by each \( k \)-subset is connected in \( H \).
Theorem 4.16 \[103\] Let $G$ be a connected graph with $n$ vertices, and let $H$ be a connected graph with $m$ vertices. Let $k$ be an integer with $2 \leq k \leq nm$. Then

$$\sum_{x=2}^{k} \left( m \right)^{x} \left( r_{1} \right)^{x} \cdots \left( r_{x} \right)^{x} SW_{x}(G) + \sum_{y=2}^{k} \left( n \right)^{y} \left( s_{1} \right)^{y} \cdots \left( s_{y} \right)^{y} SW_{y}(G)$$

$$\leq SW_{k}(G \square H) \leq \frac{k}{2} \left[ \sum_{x=2}^{k} \left( m \right)^{x} \left( r_{1} \right)^{x} \cdots \left( r_{x} \right)^{x} SW_{x}(G) + \sum_{y=2}^{k} \left( n \right)^{y} \left( s_{1} \right)^{y} \cdots \left( s_{y} \right)^{y} SW_{y}(G) \right]$$

where $\sum_{i=1}^{x} r_{i} = k$ and $r_{1} \geq 1$, and $\sum_{i=1}^{y} s_{i} = k$ and $s_{1} \geq 1$.

Remark 4.3 \[103\] Suppose that $k = 2$. Then $x = y = 2$, $r_{1} = r_{2} = \ldots = r_{x} = 1$, $\sum_{i=1}^{x} r_{i} = 2$, $s_{1} = s_{2} = \ldots = s_{y} = 1$, $\sum_{i=1}^{y} s_{i} = 2$. Therefore,

$$SW_{2}(G \square H) = m^{2}W(G) + n^{2}SW(H).$$

Thus, the upper and lower bounds in Theorem 4.16 are sharp.

Let $v$ is the root vertex of $H$ and

$$d(v, k|H) = \sum_{v \in V(H), S \subseteq V(H) \atop |S| = k} d(S).$$

Theorem 4.17 \[103\] Let $G$ be a connected graph with $n$ vertices, and let $H$ be a connected graph with $m$ vertices. Let $k$ be an integer, $2 \leq k \leq nm$. Then

$$SW_{k}(G \circ H) = n SW_{k}(H) + \sum_{\ell=2}^{k} \left( \begin{array}{c} m \nonumber \\ r_{1} \end{array} \right) \left( \begin{array}{c} m \nonumber \\ r_{2} \end{array} \right) \cdots \left( \begin{array}{c} m \nonumber \\ r_{\ell} \end{array} \right) SW_{\ell}(G)$$

$$+ \sum_{\ell=2}^{k} \left( \begin{array}{c} n \nonumber \\ \ell \end{array} \right) \left[ \sum_{j=1}^{\ell} \prod_{x=1}^{\ell} \left( \begin{array}{c} m \nonumber \\ r_{x} \end{array} \right) d(v, k|H) \right]$$

where $\sum_{x=1}^{\ell} r_{x} = k$, $r_{x} \geq 1$ and $v$ is the root-vertex of $H$.

Theorem 4.18 \[103\] Let $G$ be a connected graph with $n$ vertices, and let $H$ be a connected graph with $m$ vertices. Let $k$ be an integer, $2 \leq k \leq nm$. Then

$$SW_{k}(G \oplus H) = \sum_{\ell=2}^{k} \left( \begin{array}{c} m+1 \nonumber \\ r_{1} \end{array} \right) \left( \begin{array}{c} m+1 \nonumber \\ r_{2} \end{array} \right) \cdots \left( \begin{array}{c} m+1 \nonumber \\ r_{\ell} \end{array} \right) SW_{\ell}(G) + \left( \begin{array}{c} m \nonumber \\ k-1 \end{array} \right) (k-1)n + kn \left( \begin{array}{c} m \nonumber \\ k \end{array} \right) - xn + \sum_{\ell=2}^{k} \left( \begin{array}{c} n \nonumber \\ \ell \end{array} \right) \left[ \sum_{j=1}^{\ell} \prod_{x=1}^{\ell} \left( \begin{array}{c} m+1 \nonumber \\ r_{x} \end{array} \right) \left( \begin{array}{c} m \nonumber \\ r_{j-1} \end{array} \right) + r_{j} \left( \begin{array}{c} m \nonumber \\ r_{j} \end{array} \right) - x_{j} \right]$$

where $\sum_{x=1}^{\ell} r_{x} = k$, $r_{x} \geq 1$, $x$ is the number of the $k$-subsets of $V(H)$ such that the subgraph induced by each $k$-subset is connected in $H$, and $x_{j}$ is the number of the $r_{j}$-subsets of $V(H)$ such that the subgraph induced by each $r_{j}$-subset is connected in $H$. 

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Remark 4.4 One can see that Theorems Theorem 4.14, 4.15, 4.16, 4.17, and 4.18 are extensions (1), (2), (3), (4), (5) of Theorem 4.13 respectively. In all considered case for $k = 2$ the new results can be reduced to already known ones.

4.6 Nordhaus-Gaddum-type results

In [142], Zhang and Wu studied the Nordhaus–Gaddum problem for the Wiener index and proved that for $G \in \mathcal{G}(n)$,

$$3 \left( \binom{n}{2} \right) \leq W(G) + W(\overline{G}) \leq \frac{1}{6} (n^3 + 3n^2 + 2n - 6).$$

Mao, Wang, Gutman and Li [105] investigated the analogous problem for the Steiner Wiener index.

**Theorem 4.19** [105] Let $G \in \mathcal{G}(n)$ be a connected graph with a connected complement $\overline{G}$. Let $k$ be an integer such that $3 \leq k \leq n$. Then:

(1) $\left( \binom{n}{k} \right)(2k-1-x) \leq SW_k(G) + SW_k(\overline{G}) \leq \max\{n + k - 1, 4k - 2\} \left( \binom{n}{k} \right)$

where if $n \geq 2k - 2$, then $x = 0$; $x = 1$ for positive integer $n$.

(2) $(k-1)^2 \left( \binom{n}{k} \right)^2 \leq SW_k(G) \cdot SW_k(\overline{G}) \leq \max\{k(n-1), (2k-1)^2\} \left( \binom{n}{k} \right)^2$.

Moreover, the lower bounds are sharp.

For $k = n$, the following bounds result is immediate.

**Observation 4.1** [105] Let $G \in \mathcal{G}(n)$ be a connected graph with a connected complement $\overline{G}$. Then

(1) $SW_n(G) + SW_n(\overline{G}) = 2n - 2$;

(2) $SW_n(G) \cdot SW_n(\overline{G}) = (n-1)^2$.

For $k = n-1$, they derived [105] the following result.

**Proposition 4.9** [105] Let $G \in \mathcal{G}(n)$ ($n \geq 5$) be a connected graph with a connected complement $\overline{G}$.

(1) If $G$ and $\overline{G}$ are both 2-connected, then $SW_{n-1}(G) + SW_{n-1}(\overline{G}) = 2n(n-2)$ and $SW_{n-1}(G) \cdot SW_{n-1}(\overline{G}) = n^2(n-2)^2$.

(2) If $\kappa(G) = 1$ and $\overline{G}$ is 2-connected, then $SW_{n-1}(G) + SW_{n-1}(\overline{G}) = 2n(n-2) + p$ and $SW_{n-1}(G) \cdot SW_{n-1}(\overline{G}) = n(n-2)(n^2 - 2n + p)$, where $p$ is the number of cut vertices in $G$.

(3) If $\kappa(G) = \kappa(\overline{G}) = 1$, $\Delta(G) \leq n-3$, and $G$ has a cut vertex $v$ with pendant edge $uv$ such that $G - u$ contains a spanning complete bipartite subgraph, and $\Delta(\overline{G}) \leq n-3$ and
$G$ has a cut vertex $q$ with pendent edge $pq$ such that $G - p$ contains a spanning complete bipartite subgraph, then $SW_{n-1}(G) + SW_{n-1}(\overline{G}) = 2(n-1)^2$ and $SW_{n-1}(G) \cdot SW_{n-1}(\overline{G}) = (n-1)^4$.

(4) If $\kappa(G) = \kappa(\overline{G}) = 1$, $\Delta(G) = n - 2$, $\Delta(\overline{G}) \leq n - 3$ and $G$ has a cut vertex $v$ with pendent edge $uv$ such that $G - u$ contains a spanning complete bipartite subgraph, then

$$SW_{n-1}(G) + SW_{n-1}(\overline{G}) = 2(n-1)^2 \text{ or } SW_{n-1}(G) + SW_{n-1}(\overline{G}) = 2(n-1)^2 + 1$$

and

$$SW_{n-1}(G) \cdot SW_{n-1}(\overline{G}) = (n-1)^4 \text{ or } SW_{n-1}(G) \cdot SW_{n-1}(\overline{G}) = (n-1)^2(n^2 - 2n + 2).$$

(5) If $\kappa(G) = \kappa(\overline{G}) = 1$, $\Delta(G) = \Delta(\overline{G}) = n - 2$, then

$$2(n-1)^2 \leq SW_{n-1}(G) + SW_{n-1}(\overline{G}) \leq 2(n-1)^2 + 2$$

and

$$(n-1)^4 \leq SW_{n-1}(G) \cdot SW_{n-1}(\overline{G}) \leq (n^2 - 2n + 2)^2.$$ For $k = 3$ and $n \geq 10$, from Theorem 4.19 one can see that

$$5\left(\begin{array}{c} n \\ 3 \end{array}\right) \leq SW_{3}(G) + SW_{3}(\overline{G}) \leq (n + 2)\left(\begin{array}{c} n \\ 3 \end{array}\right)$$

and

$$4\left(\begin{array}{c} n \\ 3 \end{array}\right)^2 \leq SW_{3}(G) \cdot SW_{3}(\overline{G}) \leq 3(n - 1)\left(\begin{array}{c} n \\ 3 \end{array}\right)^2.$$ Mao, Wang, Gutman and Li [105] improved these bounds and proved the following result.

**Theorem 4.20** [105] Let $G \in \mathcal{G}(n)$ ($n \geq 4$) be a connected graph with a connected complement $\overline{G}$. Then

(1)

$$5\left(\begin{array}{c} n \\ 3 \end{array}\right) \leq SW_{3}(G) + SW_{3}(\overline{G})$$

and

$$4\left(\begin{array}{c} n \\ 3 \end{array}\right)^2 \leq SW_{3}(G) \cdot SW_{3}(\overline{G}) \leq 3(n - 1)\left(\begin{array}{c} n \\ 3 \end{array}\right)^2.$$
\[
6 \left( \frac{n}{3} \right)^2 + (n - 2) \left( \frac{n}{3} \right) - (n - 2)^2 \leq SW_3(G) \cdot SW_3(\overline{G}) \leq \begin{cases} 
\frac{1}{4} \left[ 7 \left( \frac{n}{3} \right) - 3n + 8 \right]^2, & \text{if } n = 6, 7, \text{ and } sdiam_3(G) = 5; \\
\frac{1}{4} \left[ (\frac{n+1}{4}) + (\frac{n-3}{3}) + \frac{1}{4} (7n^2 - 35n + 48) \right]^2, & \text{otherwise.}
\end{cases}
\]

Moreover, the bounds are sharp.

### 4.7 Steiner Wiener index and Steiner betweenness centrality

The *betweenness centrality* \( B(v) \) of a vertex \( v \in V(G) \) is defined as the sum of the fraction of all pairs of shortest paths that pass through \( v \) across all pairs of vertices in a graph:

\[
B(v) = \sum_{x,y \in V(G) \setminus \{v\}, x \neq y} \frac{\sigma_{x,y}(v)}{\sigma_{x,y}},
\]

where \( \sigma_{x,y} \) denotes the number of all shortest paths between vertices \( x \) and \( y \) in a graph \( G \) and \( \sigma_{x,y}(v) \) denotes the number of all shortest paths between vertices \( x \) and \( y \) in graph \( G \) passing through the vertex \( v \).

In a case when a graph models a social or communication network, as the name suggests, it measures the centrality of a vertex in a graph, by the influence of a vertex in the dissemination of information over a network. It has been independently introduced by Anthonisse in [7] and by Freeman in [52], and among other applications has been applied to detect communities in networks [56, 111].

For a graph \( G \), let \( n(G) \) denote the number of its vertices. For a forest (acyclic graph) \( F \) with \( p \) \((p > 1)\) connected components \( T_1, T_2, \ldots, T_p \) denote by \( N_k(F) \) the sum over all partitions of \( k \) into at least two nonzero parts of products of combinations distributed among the \( p \) components of \( F \):

\[
N_k(F) = \sum_{\ell_1 + \ell_2 + \ldots + \ell_p = k, \ 0 \leq \ell_1, \ell_2, \ldots, \ell_p < k} \binom{n(T_1)}{\ell_1} \binom{n(T_2)}{\ell_2} \cdots \binom{n(T_p)}{\ell_p}.
\]

For a tree \( T \), we define \( N_k(T) = 0 \). Note that by the definition \( \binom{n}{0} = 1 \) and \( \binom{n}{k} = 0 \) whenever \( n < k \).

Kovš [85] derived the following result for Steiner Wiener index.

**Theorem 4.21** [85] (1) Let \( T \) be a tree on \( n \) vertices. Then

\[
SW_k(T) = \sum_{e \in E(T)} N_k(T - e).
\]
(2) Let $T$ be a tree on $n$ vertices. Then

$$SW_k(T) = \sum_{v \in V(T)} N_k(T - v) + (k - 1) \binom{n}{k}.$$ 

The Steiner $k$-betweenness centrality $B_k(v)$ of a vertex $v \in V(G)$ is defined as the sum of the fraction of all $k$-Steiner trees that include $v$ as its non-terminal vertex across all combinations of $k$ vertices of $G$:

$$B_k(v) = \sum_{S \in V(G) \setminus \{v\}, |S| = k} \frac{\sigma_S(v)}{\sigma_S},$$

where $\sigma_S$ denotes the number of all Steiner trees between vertices of $S$ in a graph $G$ and $\sigma_S(v)$ denotes the number of all Steiner trees between vertices of $S$ in a graph $G$ that include also the vertex $v$ as a non-terminal vertex.

**Theorem 4.22** [85] (1) Let $G$ be a connected graph on $n$ vertices. Then

$$SW_k(G) = \sum_{v \in V(G)} B_k(v) + (k - 1) \binom{n}{k}.$$ 

For a graph $G$ on $n$ vertices the average $k$-Steiner betweenness $\overline{B}_k(G)$ is defined as

$$\overline{B}_k(G) = \frac{1}{n} \sum_{v \in V(G)} B_k(v).$$

**Corollary 4.4** [85] Let $G$ be a connected graph on $n$ vertices. Then

$$\overline{B}_k(G) = \frac{1}{n} \binom{n}{k} (\mu_k(G) - k + 1).$$

## 5 Steiner Center and Steiner Median

In a graph $G$, a vertex $x$ is a cut-vertex if deleting $x$ and all edges incident to it increases the number of connected components. A block of a graph is a maximal connected vertex-induced subgraph that has no cut vertices. A block graph is a connected graph whose blocks are complete graphs. Note that trees are block graphs.

### 5.1 Results for Steiner center

Hedetniemi [21] verified that every graph $H$ is the center of some graph $G$. As an extension of this result, Oellermann and Tian [117] derived the following result for Steiner center.
Theorem 5.1 \[117\] Let $k \geq 2$ be an integer and $H$ be a graph. Then $H$ is the Steiner $k$-center of some graph $G$.

The construction given in \[117\] was described in \[113\]. Let $H$ be a given graph and $k \geq 2$. Let $G$ be obtained from $H$ by first adding $2k$ new vertices $\{v_1, v_2, \ldots, v_k\} \cup \{u_1, u_2, \ldots, u_k\}$ and then joining $v_i$ to every vertex of $H$ for $1 \leq i \leq k$ and next adding the edges $u_iv_i$ for $1 \leq i \leq k$. Then, $e_k(u_i) = 2k$, $e_k(v_i) = 2k - 1$ for $1 \leq i \leq k$, and $e_k(u) = 2k - 2$ for all $u \in V(H)$. Hence, $H$ is the Steiner $k$-center of $G$.

Even though every graph is the Steiner $k$-center of some graph, the problem of finding the Steiner $k$-center of any given graph appears to be quite difficult.

However, for trees, an efficient solution to this problem was developed in \[117\] and extends the work done by Jordan \[84\] in his 1869 paper on centers and centroids of trees.

The key result that leads to a recursive procedure for finding the Steiner $k$-center of a tree states: For any tree of order $n \geq 3$ and integer $k$ with $3 \leq k \leq n$, if $T$ has at least $k$ end-vertices and $T'$ is obtained by deleting the end-vertices from $T$, then $C_k(T) \subseteq C_k(T')$. Moreover, if $T$ has at most $k - 1$ end-vertices, then $C_k(T) = T$.

Thus, Steiner $k$-centers of trees of order $3 \leq k \leq n$ was characterized as follows.

Theorem 5.2 \[117\] A tree $H$ is the $k$-center of some tree if and only if

1. $k \geq 3$ and $H$ has at most $k - 1$ end-vertices, or
2. $k = 2$ and $H$ is isomorphic to $K_1$ or $K_2$.

Based on these results, the following procedure for finding the $k$-center of a tree was developed in \[117\].

![Figure 5.1: Graphs for Algorithm 5.1.](image)

**Algorithm 5.1.** Finding the Steiner $k$-center of a tree $T$ of order $n \geq k \geq 2$. 

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(i) $H \leftarrow T$

(ii) If $H$ has at most $k - 1$ end-vertices, or if $H \cong K_1$ or $K_2$ and $k = 2$, output $H$ since $H$ is the Steiner $k$-center of $T$ and stop; otherwise, continue.

(iii) Delete the end-vertices from $H$ and let $H$ be the resulting tree. Return to (ii).

Figure 5.1 illustrates Algorithm 5.1 with $n = 3, 4, 5, 6$.

This procedure gives the following result.

**Corollary 5.1** [117] Let $k$ ($k \geq 3$) be an integer and $T$ a tree of order $n$ ($k \leq n$). Then $C_{k-1}(T) \subseteq C_k(T)$.

Oellermann [113] asked whether these containment relationships hold for general graphs. However, Yeh, Chiang, and Peng [140] had a tree-like graph $F_1$ which has $C_3(F_1) \not\subseteq C_4(F_1)$, where $F_1$ is a graph obtained from $K_4^-$ by adding a pendant edge. Notice that $H$ is a partial 2-tree, an interval graph, and is also a distance-hereditary graph.

The following proposition even shows that graphs $G$ with $C_{n-2}(G) = V(G)$ and $C_{n-1}(G) = \{z\}$ can be constructed systematically, where $n = |V(G)|$.

**Proposition 5.1** [140] Let $G$ be a connected graph having exactly one cut-vertex $z$. Let $n = |V(G)|$. If $G$ has a 2-vertex cut $\{a, b\}$ such that $z \notin \{a, b\}$, then $C_{n-2}(G) = V(G)$ and $C_{n-1}(G) = \{z\}$.

![Diagram](image)

Figure 5.2: The graphs $F_2, F_3, B_{6,4}$.

Let $F_r$ be a graph such that $V(F_r) = \bigcup_{i=1}^{r} \{y, z, x_i, z_i, w_i\}$ and $E(F_r) = \bigcup_{i=1}^{r} \{w_iw_{i+1}, x_iw_{i+1}, y_iy_{i+1}, z_iz_{i+1}, yxy, yy, zzy, zzy, zz, zw\}$. 

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For positive integers $a > b$, let $B_{a,b}$ be a block graph with $V(B_{a,b}) = \{x_1, x_2, \ldots, x_a\} \cup \{y_1, y_2, \ldots, y_b\}$ and $E(B_{a,b}) = \{x_i x_j \mid 1 \leq i < j \leq a\} \cup \{x_i y_s \mid 1 \leq s \leq b\}$. As examples, graphs $F_2, F_3, B_{6,4}$ are depicted in Figure 5.2.

**Proposition 5.2** Let $T$ be a tree of order $n \geq k$ and let $e = uv$ be an edge of $T$. If $T_u$ and $T_v$ have orders $r$ and $s$, respectively, then

\[ d_k(v) - d_k(u) = \binom{r-1}{k-1} - \binom{s-1}{k-1}. \]

Observe that the difference $d_k(v) - d_k(u) = \binom{r-1}{k-1} - \binom{s-1}{k-1}$ is 0 if and only if $r = s$ or both $r < n$ and $s < n$. Therefore, they had the following corollary.

**Corollary 5.2** Let $T$ be a tree of order $n \geq k$ and let $e = uv$ be an edge of $T$. Let $r$ and $s$ be the orders of $T_u$ and $T_v$, respectively. Then $d_k(v) < d_k(u)$ if and only if $r > s$ and $r \geq k$.

(2) Let $T$ be a tree of order $n \geq k$. If $v_0 v_1 \ldots v_r$ is a path in $T$ and $d_k(v_0) < d_k(v_1)$, then $d_k(v_1) < d_k(v_2) < \ldots < d_k(v_r)$.

(3) The $k$-median of any tree is connected.

(4) Let $T$ be a tree of order $n > k$. If $T$ has no edge $e = uv$ for which the orders of $T_u$ and $T_v$, either (i) are equal or (ii) are both less than $k$, then $M_k(T)$ is a single vertex.

(5) If $T$ is a tree of order at least $2k - 1$, then its $k$-median is either $K_1$ or $K_2$.

In [15], the Steiner $k$-medians of trees are characterized and a linear algorithm for finding the Steiner $k$-median of a tree is developed.
One of the main results in Corollary 5.2 leading to a characterization of the Steiner $k$-medians of trees states that if $T$ is a tree of order $n \geq k$ and if $v_0, v_1, \ldots, v_r$ is a path in $T$ with $d_k(v_0) < d_k(v_1)$, then $d_k(v_1) < d_k(v_2) < \cdots < d_k(v_r)$.

From this result, it follows that the Steiner $k$-median of a tree is connected and thus a tree. The Steiner $k$-medians of trees are characterized in [15] as follows:

**Theorem 5.4** [15] (1) A tree $H$ is the Steiner $k$-median, $k \geq 3$, of some tree of order at least $2k - 1$ if and only if $H \cong K_1$ or $K_2$.

(2) A tree $H$ is a Steiner $k$-median of some tree of order at most $2k - 2$ if and only if $H$ is $K_1$ or has exactly $k$ vertices or if $k_1$ is the number of end-vertices of $H$ and $m$ is the number of internal vertices of $H$; then, $m + 2k_1 - 1 \leq k$.

Beineke, Oellermann, and Pippert [15] obtained the following algorithm for finding the Steiner $k$-median of a tree.

**Algorithm 5.2.** Finding the Steiner $k$-median of a tree $T$ of order $n \geq k \geq 2$.

1. Construct a digraph $D$ having $T$ as underlying graph by replacing each edge $e = uv$ by

   (a) The arc $(u, v)$ if $T_v$ has at least $k$ vertices and the number of vertices in $T_v$ exceeds the number of vertices in $T_u$.

   (b) The arc $(v, u)$ if $T_u$ has at least $k$ vertices and the number of vertices in $T_u$ exceeds the number of vertices in $T_v$.

   (c) The symmetric pair of arcs $(u, v)$ and $(v, u)$ if $T_u$ and $T_v$ have the same number of vertices or if $T_u$ and $T_v$ both have fewer than $k$ vertices.

2. If $D$ has symmetric pairs of arcs, then the subgraph $H$ of $T$ induced by those edges corresponding to the symmetric pairs of arcs in $D$ are output since this is the Steiner $k$-median of $T$; otherwise, let $H$ be the vertex with outdegree 0. Output $H$ since it is the Steiner $k$-median of $T$ and stop.

Figure 5.3 illustrates Algorithm 5.2 with $k = 4$ and 5. The Steiner $k$-median is induced by the gray vertices.

**Theorem 5.5** [15] If $T$ is a tree of order $n \geq k + 1 \geq 3$, then $M_k(T) \subseteq M_{k+1}(T)$.

Yeh, Chiang, and Peng [140] showed that the containment relation between the Steiner $k$-median and Steiner $(k + 1)$-median is also true for block graphs.

**Theorem 5.6** [140] If $G$ is a block graph with $|V(G)| \geq k + 1 \geq 3$, then

$$M_k(T) \subseteq M_{k+1}(T).$$
It would be interesting to ask if Theorem 5.6 holds for distance-hereditary graphs, particularly since the distance-hereditary graphs are also Steiner distance hereditary. Yeh, Chiang, and Peng [140] showed that for each $k \geq 2$ there is a distance-hereditary graph $G$ such that $M_k(G) \not\subseteq M_{k+1}(G)$.

Figure 5.4: The graphs $G_i$ ($1 \leq i \leq 3$) and $D_i$ ($i = 3, 4$).

Let $G_r$ be a graph with $V(G_r) = \bigcup_{i=1}^{r} \{x, y, u_i, d_i, f_i\}$ and

$$E(G_r) = \bigcup_{i=1}^{r} \{xy, xu_i, xd_i, yd_i, yf_i, uf_i\}.$$
For $k \geq 3$, let $D_r$ be a graph with $V(D_r) = \{w, x, y_1, y_2\} \cup \{z_1, z_2, \ldots, z_{2r-4}\}$ and $E(D_r) = \{wx, xy_1, xy_2, y_1y_2\} \cup \{yz_i | 1 \leq i \leq 2 \text{ and } 1 \leq j \leq 2r - 4\} \cup \{z_i z_j | 1 \leq i < j \leq 2r - 4\}$.

It is easy to see that $G_1$ and $D_r$, $r \geq 3$, are distance-hereditary graphs [78]. As examples, graphs $G_i$ ($1 \leq i \leq 3$) and $D_i$ ($i = 3, 4$) are depicted in Figure 5.4.

**Theorem 5.7** [15] If $T$ is a tree of order $n \geq k + 1 \geq 3$, then $M_k(T) \subseteq M_{k+1}(T)$.

**Proposition 5.3** [140] (1) For each positive integer $r \geq 2, M_2(G_r) \not\subseteq M_3(G_r)$.

(2) For each distance-hereditary graph $D_r$, $r \geq 3$, $M_r(D_r) \not\subseteq M_{r+1}(D_r)$.

Yeh, Chiang, and Peng [140] derived the following result, and presented a linear time algorithm for finding the Steiner $k$-median of a block graph.

**Theorem 5.8** [140] Let $G$ be a block graph with $|V(G)| \geq k \geq 2$ such that $G$ is not a complete graph. If vertex $x$ is not a cut-vertex of $G$, then either $x$ is not a vertex in the Steiner $k$-median of $G$, or $|V(G)| = k$.

In the following algorithm we use the following notation: $\overrightarrow{G_c}$ has two kinds of arcs $u \leftarrow v$ and $u \leftrightarrow v$. For an arc of the form $u \leftarrow v$ (resp. $u \rightarrow v$), we say that it is an out-edge of vertex $v$ (resp. $u$). For the purpose of this algorithm we assume that an arc of the type $u \leftrightarrow v$ is not an out-edge of $u$ or $v$.

**Algorithm 5.3.** Finding the Steiner median $(G, k)$.

**Input:** A block graph $G$ with $|V(G)| \geq k \geq 2$.

**Output:** The Steiner $k$-median $M_k(G)$ of $G$.

**Begin**

1. if $n = |V(G)|$ or $G$ is a complete graph, then $M_k(G) \leftarrow G$ and STOP.

2. Let $G_c$ be the subgraph of $G$ induced by the cut-vertices of $G$.

3. Construct a graph $\overrightarrow{G_c}$ with two kinds of arcs from $G_c$ as follows:

4. for each edge $uv$ in $G_c$,
   - if $|V_{uv}| > |V_{vu}|$ and $|V_{uv}| + |V_{uv}| \geq k$,
     then replace the edge $uv$ by an arc of the form $u \leftarrow v$,
   - else if $|V_{uv}| < |V_{vu}|$ and $|V_{vu}| + |V_{uv}| \geq k$,
     then replace the edge $uv$ by an arc of the form $u \rightarrow v$,
   - else replace the edge $uv$ by an arc of the form $u \leftrightarrow v$.

5. Let $M_k(G)$ be those vertices in $\overrightarrow{G_c}$ with no out-edges.
End.

**Theorem 5.9** [140] Let \( G = (V, E) \) be a block graph with \( |V| \geq k \geq 2 \). Algorithm 5.3 correctly finds the Steiner \( k \)-median of \( G \) in time \( O(|V| + |E|) \).

A block of a block graph \( G \) that contains exactly one cut-vertex is called an end-block of \( G \).

Yeh, Chiang, and Peng [140] showed that one can easily find the Steiner \( k \)-distance of all vertices in a block graph in polynomial time.

**Theorem 5.10** [140] Suppose \( x \) is a vertex in an end-block of a block graph \( G \) with \( |V(G)| \geq k \geq 2 \), and \( x \) is not a cut-vertex of \( G \). Let \( y \) be a vertex of \( G \) adjacent to \( x \). Let \( d'_k(y) \) (resp. \( d_k(y) \)) denote the Steiner \( k \)-distance of \( y \) in \( G - x \) (resp. \( G \)). Then

\[
d_k(y) = d'_k(y) + d'_{k-1}(y) + \left( \frac{|V(G)| - 2}{k - 2} \right).
\]

Beineke, Oellermann, and Pippert [15] turned their attention to the minimum value of the Steiner \( k \)-distances of the vertices of a graph. In particular, for a connected graph \( G \) of order \( n \geq k \), let the Steiner \( k \)-median value \( m_k(G) \) be defined as \( \min \{ d_k(v) \mid v \in V(G) \} \).

Sharp bounds on the Steiner \( k \)-median value of trees of order \( n \geq 2k - 1 \) are also established in [15].

**Theorem 5.11** [15] If \( T_n \) is a tree of order \( n \geq 2k - 1 \), then

\[
(k - 1) \binom{n - 1}{k - 1} \leq m_k(T_n)
\]

\[
\leq \begin{cases} 
(n - 1) \binom{n - 1}{k - 1} - 2 \sum_{j=1}^{(n-1)/2} \binom{n-j-1}{k-1}, & \text{if } n \text{ is odd}; \\
(n - 1) \binom{n - 1}{k - 1} - 2 \sum_{j=1}^{(n-2)/2} \binom{n-j-1}{k-1} \left( \frac{2}{k-1} \right)^j, & \text{if } n \text{ is even}.
\end{cases}
\]

Furthermore, these bounds are sharp.

### 5.3 From Steiner centers to Steiner medians

In the preceding two subsections, the focus has been on finding the Steiner \( k \)-centers and Steiner \( k \)-medians of trees. In [113], it was shown that, except for trees of small order, these two types of “centers” can be arbitrarily far apart.

Let \( T \) be a tree of order \( n \) \((2 \leq k \leq n)\). Suppose that \( T \) has at most \( 2k - 2 \) vertices. If \( T \) had at most \( k - 1 \) end-vertices, then by Algorithm 5.1, \( C_k(T) = T \). So, in this case, \( M_k(T) \subset C_k(T) \). Suppose that \( T \) has at least \( k \) end-vertices. Then, by Algorithm 5.1, \( C_k(T) \) is obtained by deleting the end-vertices of \( T \). By Algorithm 5.2, \( M_k(T) \) does not contain any end-vertex of \( T \). Hence, once again, \( M_k(T) \subset C_k(T) \).
Oellermann [113] turned her attention to trees having at least $2k - 1$ vertices. For a given graph $G$ and subgraphs $F$ and $H$ of $G$, the distance $d_G(F, H)$ between $F$ and $H$ is defined as $\min\{d_G(u, v) \mid u \in V(F) \text{ and } v \in V(H)\}$.

Hendry [74] showed that if $F$ and $H$ are any two graphs then there exists a connected graph $G$ such that $C(G) \cong F$ and $M(G) \cong H$. The graph constructed by Hendry [74] had the property that $d_G(F, H) = 1$. Holbert [76] showed that the distance between $C(G)$ and $M(G)$ can be arbitrarily large. It was shown in [113] that the distance between the Steiner $k$-center and Steiner $k$-median of a tree of sufficiently large order can be arbitrarily large. Moreover, it is shown that the structure of the Steiner $k$-center and Steiner $k$-median can be prescribed provided that they satisfy the conditions of Theorem 5.2 and (1) of Theorem 5.4.

**Theorem 5.12** [113] Let $T_1$ be any tree with at most $k - 1$ end-vertices and let $T_2$ be isomorphic to $K_1$ or $K_2$. Let $d \geq 1$ be an integer. Then, there exists a tree $T$ with $C_k(T) = T_1$, $M_k(T) = T_2$ and $d(T_1, T_2) = d$.

Since the Steiner $k$-center and Steiner $k$-median of a graph are both measures of centrality, it seems reasonable to ask whether there are measures of centrality that allow each vertex on a shortest path between the Steiner $k$-center and Steiner $k$-median to belong to the “center” with respect to at least one of these measures.

For $1 \leq k \leq |V(G)|$, and $u \in V(G)$, Slater [125] defined

$$r_k(u) = \max \left\{ \sum_{s \in S} d(u, s) \mid S \subseteq V(G), |S| = k \right\}.$$  

The $t$-centrum of $G$, denoted by $C_G(t)$, is defined to be the subset of vertices $u$ in $G$ for which $r_1(u)$ is a minimum. Thus, the vertices of $C_G(1)$ induce the center and the vertices of $C_G(|V(G)|)$ induce the median of $G$.

Slater [125] showed that for a tree $T$, if $u$ belongs to the center and $v$ to the median of $T$, then the subgraph induced by the vertices in $\bigcup_{t=1}^{||V(T)||} C(T; t)$ is a subtree of $T$ containing the $u-v$ path.

Oellermann [114] turned her attention to general $k$. Let $G$ be a connected graph of order $n$ ($2 \leq k \leq n$) and let $\mathcal{P}_k$ be the collection of all $k$-element subsets of $V(G)$. For $1 \leq t \leq \binom{n-1}{k-1}$ and $u \in V(G)$, let the Steiner $(k, t)$-eccentricity of $u$ be defined by

$$e^{(k)}_G(u; t) = \max \left\{ \sum_{s \in S} d(S) \mid S \subseteq \mathcal{P}_n, |S| = k \text{ and } u \in S \text{ for all } s \in S \right\}.$$  

The Steiner $(k, t)$-center $C_k(G; t)$ is the subgraph induced by the vertices of $G$ with minimum Steiner $(k, t)$-eccentricity. Thus, if $t = 1$, then $C_k(G; t) = C_k(G)$, and if $t = \binom{n-1}{k-1}$, then $C_k(G; t) = M_k(G)$. The following result was established in [113]:

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Theorem 5.13: If \( T \) is a tree of order \( n \geq k \) and \( w \) is a vertex on a shortest path from \( C_k(T) \) to \( M_k(T) \), then \( w \) is a vertex of \( C_k(T; t) \) for some \( t, 1 \leq t \leq \binom{n-1}{k-1} \).

Whether or not Theorem 5.12 can be extended to general graphs or even to other classes of graphs for which the Steiner problem can be solved efficiently is still an open problem.

6 Steiner Distance Hereditary Graphs

To be able to state the characterizations of distance hereditary graphs given by Howorka [78], we need the following terminology: An induced path of \( G \) is a path that is an induced subgraph of \( G \). Let \( u, v \in V(G) \). Then, a \( u\)-\( v \) geodesic is a shortest \( u\)-\( v \) path. Let \( C \) be a cycle of \( G \). A path \( P \) is an essential part of \( C \) if \( P \) is a subgraph of \( C \) and \( \frac{1}{2} e(C) < e(P) < e(C) \). An edge of \( G \) that joins two vertices of \( C \) that are not adjacent in \( C \) is called a diagonal of \( C \). We say that two diagonals \( e_1, e_2 \) of \( C \) are skew diagonals if \( C + e_1 + e_2 \) is homeomorphic with \( K_4 \).

Theorem 6.1: The following are equivalent:

1. \( G \) is distance hereditary;
2. Every induced path is a geodesic;
3. No essential part of a cycle is induced;
4. Each cycle of length at least 5 has at least two diagonals and each 5-cycle has a pair of skew diagonals;
5. Each cycle of \( G \) of length at least 5 has a pair of skew diagonals.

In [38], it was pointed out that, in general, it appears to be a difficult problem to determine the Steiner distance of a set of vertices in a graph. However, it was shown in [38] that if \( G \) is \( k \)-Steiner distance hereditary, then the Steiner distance of any set of \( k \) vertices of \( G \) can be determined efficiently. Furthermore, the following result is established in [38]:

Theorem 6.2: If \( G \) is 2-Steiner distance hereditary, then \( G \) is \( k \)-Steiner distance hereditary for all \( k \geq 3 \).

A vertex \( v \) in a graph \( G \) is a true (false) twin of a vertex \( v' \) if \( v \) and \( v' \) have the same closed (respectively, open) neighborhood in \( G \). A twin of a vertex \( v \) is a vertex that is either a true or false twin of \( v \).

Bandelt and Mulder [70], Hammer and Maffray [69] derived the following result for distance hereditary graphs.
Theorem 6.3 A graph $G$ of order $n$ is distance hereditary if and only if there is a sequence of subgraphs $G_1, G_2, \ldots, G_{n-1}$ such that $G_1 \cong K_2$ and for $2 \leq i \leq n-1$, $G_i$ is obtained by adding a new vertex $v$ as pendant, or twin of some vertex $v'$ of $G_{i-1}$.

6.1 Steiner geodetic sets of distance-hereditary graphs

In general there is no relation between $g(G)$ and $sg(G)$. To see that $sg(G)$ can be much larger than $g(G)$ observe that $g(K_{m,n}) = 4$ and $sg(K_{m,n}) = m$ for $5 \leq m \leq n$. The convex hull of a set $S$ of vertices (that is not necessarily convex) is the smallest convex set of vertices that contains $S$. A vertex $v$ of a convex set $S$ is an extreme point of $S$ if $S \setminus \{v\}$ is still convex.

Oellermann and Puertas showed that if $G$ is a distance-hereditary graph, then the geodetic number never exceeds the Steiner geodetic number.

Theorem 6.4 If $G$ is a distance-hereditary graph, then $g(G) \leq sg(G)$.

A vertex is a contour vertex if its eccentricity is at least as large as the eccentricity of each of its neighbors. The collection of all contour vertices of $G$ is called its contour and is denoted by $Ct(G)$.

The next result is used to find minimum Steiner geodetic sets in distance-hereditary graphs.

Lemma 6.1 Let $G$ be a distance-hereditary graph and $T$ a Steiner tree for $Ct(G)$. Then every vertex of $G$ not in $T$ is adjacent to some vertex of $T$.

In [115], Oellermann and Puertas stated a result that describes a procedure for finding minimal Steiner geodetic sets in all distance-hereditary graphs.

Theorem 6.5 Let $G$ be a distance-hereditary graph and suppose that $\overline{T} = V(G) \setminus I(Ct(G))$. Let $S = Ct(G) \cup \overline{T}$. Then $S$ is a Steiner geodetic set for $G$ that is minimal in the sense that for all $v \in S$, $v \notin I(S \setminus v)$.

Except for some special cases, the Steiner geodetic set described in Theorem 6.5 is a minimum Steiner geodetic set as we now see.

Theorem 6.6 Suppose $G$ is a distance-hereditary graph and $S$ a minimum Steiner geodetic set for $G$. If $\text{diam}(G) \geq 3$, then $S$ contains all contour vertices of $G$.

Let $G$ be a distance-hereditary graph. If $G$ has diameter at least 3 it follows, from Theorem 6.6, that a minimum Steiner geodetic set $S$ contains $Ct(G)$. Let $\overline{T} = V(G) \setminus I(Ct(G))$. If $\overline{T} \notin S$, let $u$ be a vertex in $\overline{T}$ that does not belong to $S$. Then there is a
Steiner tree $T$ for $S$ that contains $u$. Since $G$ is Steiner distance hereditary, the induced subgraph $< V(T) >$ must contain a Steiner tree $H$ for $Ct(G)$ that does not contain $u$. Let $W = S \cap \overline{T}$. Then, by Lemma 6.1, every vertex of $W$ is adjacent with some vertex in $H$. So the tree obtained from $H$ by adding an edge from every vertex in $W$ to some vertex in $H$ is a tree that contains $S$ but not $u$ and thus has size less than the size of $T$. This is not possible. So every vertex in $Ct(G) \cup \overline{T}$ belongs to every minimum Steiner geodetic set. By Theorem 6.5, $I(Ct(G) \cup \overline{T}) = V(G)$. Thus, $Ct(G) \cup \overline{T}$ is a minimum Steiner geodetic set. If $G$ has diameter 1, then $\overline{T} = \emptyset$ and a Steiner geodetic set for $G$ must contain all (contour) vertices. So in this case $Ct(G) \cup \overline{T}$ is a minimum Steiner geodetic set. Similarly if $diam(G) = 2$ and $rad(G) = 1$, $Ct(G) \cup \overline{T}$ is a minimum Steiner geodetic set. The next theorem summarizes these results.

**Theorem 6.7** [115] If $G$ is a distance-hereditary graph with

(i) $diam(G) = 1$; or

(ii) $diam(G) = 2$ and $rad(G) = 1$; or

(iii) $diam(G) \geq 3$;

then $sg(G) = |Ct(G)| + |V(G) \setminus I(Ct(G))|$ and $Ct(G) \cup (V(G) \setminus I(Ct(G)))$ is the unique minimum Steiner geodetic set.

### 6.2 3-Steiner distance hereditary graphs

Several new characterizations of 2-Steiner distance hereditary graphs which lead to efficient algorithms that test whether a graph is 2-Steiner distance hereditary have been established; see [70, 37, 69].

A structural characterization of 3-SDH graphs is given in [39]. Suppose $C : v_1, v_2, \ldots, v_\ell, v_1$ is a cycle in a graph $G$. An edge of $G$ that joins two vertices of $C$ that are not adjacent on $C$ is called a diagonal or a chord of $C$. Two chords $e_1$ and $e_2$ of $C$ are skew or crossing, if $C + e_1 + e_2$ is homeomorphic to $K_4$.

**Theorem 6.8** [39] A graph $G$ is 3-Steiner distance hereditary if and only if it is 2-Steiner distance hereditary or if the following conditions hold:

1. Every cycle $C : v_1, v_2, \ldots, v_\ell, v_1$ of length $\ell \geq 6$
   (a) has at least two skew diagonals, or, if $\ell = 6$, then $v_1v_3v_5v_1$ or $v_2v_4v_6v_2$ is a cycle in $< V(C) >$;
   (b) has no two adjacent vertices neither of which is on a diagonal of $C$.

2. $G$ does not contain an induced subgraph isomorphic to any of the following (any subset of dotted edges may be included in the graph).
A *hole* is an induced cycle of length at least 5; a *house* is a 5-cycle with exactly one chord and a *domino* is a 6-cycle with exactly one chord that joins two vertices distance 3 apart on the cycle. The $HHD$-free graphs are characterized as those graphs for which every cycle of length at least 5 contains at least two chords; see [12].

**Theorem 6.9** [39] (1) If $G$ is a 3-$SDH$ graph, then the contour of $G$ is a geodetic set.

(2) If $G$ is a $HHD$-free graph, then the contour of $G$ is a geodetic set.

We show here that the Steiner geodetic number is an upper bound for the geodetic number for 3-$SDH$ graphs.

**Theorem 6.10** [39] If $G$ is a 3-$SDH$ graph and $S \subseteq V(G)$, then $I(S) \subseteq I[S]$.

**Corollary 6.1** [39] If $G$ is a 3-$SDH$ graph, then $g(G) \leq sg(G)$.

We say that a 6-cycle $v_1, v_2, v_3, v_4, v_5, v_6, v_1$ has a triangle of chords if $v_1, v_3, v_5, v_1$ or $v_2, v_4, v_6, v_2$ is a cycle in $G$.

Oellermann and Spinrad [116] obtained the following polynomial algorithm to test whether a graph is 3-$SDH$.

**Algorithm 6.1.** [116] To test whether a given graph $G$ is 3-$SDH$

(1) If $G$ is distance hereditary, then output “$G$ is 3-$SDH$” and halt;

(2) If $G$ has a bad edge, then output “$G$ is not 3-$SDH$” and halt;

(3) If $G$ has a 7-cycle without crossing chords or if $G$ has a 6-cycle without crossing chords and without a triangle of chords, then output “$G$ is not 3-$SDH$” and halt;

(4) If $G$ has any of the forbidden subgraphs of Fig. 1 as an induced subgraph, then output “$G$ is not 3-$SDH$” else output “$G$ is 3-$SDH$”.

Goddard [57] showed that if a graph is $k$-$SDH$, then it is $t$-$SDH$ for all $t \geq k$. The following algorithm, see [38], finds the Steiner distance of a set $S$ of vertices in a $k$-$SDH$ graph. We use this algorithm to find the Steiner interval for $S$. We say that a set $S$ of vertices of a graph $G$ is separated in an induced subgraph $H$ of $G$ that contains $S$ if the vertices of $S$ do not belong to the same component of $H$.

**Algorithm 6.2.** [38] Algorithm to find the Steiner distance of a set $S$, of at least three vertices, in a 3-$SDH$ graph $G$.

- label $V(G) \setminus S$ in arbitrary order $v_1, v_2, \ldots, v_m$
- $G_1 = G$
• for $i = 1$ to $m$
  if $S$ is separated in $G_i - v_i$
    then $G_{i+1} \leftarrow G_i$
  else $G_{i+1} \leftarrow G_i - v_i$
• $d_G(S) = |V(G_{m+1})| - 1$

Eroh and Oellermann [48] derived the following result.

**Theorem 6.11** [48] If $G$ is a 3-SDH graph and if $v_m$, in the final step of the algorithm above, does not separate $S$ in $G_m$, then $v_m \notin I(S)$.

Thus, in order to find $I(S)$ in a 3-SDH graph, we may perform the given algorithm $m$ times, with each of the vertices of $V(G) \setminus S$ in turn as the last vertex $v_m$ in the sequence of vertices input in the algorithm. If $v_m$ does not separate $S$ in $G_m$, then it is not in $I(S)$; otherwise, it is.

### 6.3 $k$-Steiner-distance-hereditary

For $k \geq 2$ and $d \geq k - 1$, Goddard [57] defined the property $S(k,d)$ as meaning that for all sets $S$ of $k$ vertices with Steiner distance $d$, the distance of $S$ is preserved in any connection for $S$. The property $S(k)$ as meaning $S(k,d)$ for all $d$ is also defined in [57]. Day, Oellermann, and Swart [38] introduced the property and called such graphs $k$-Steiner-distance-hereditary. Thus distance-hereditary graphs are the ones obeying $S(2)$.

Day, Oellermann, and Swart [38] conjectured being $k$-Steiner-distance-hereditary implies being $(k+1)$-Steiner-distance-hereditary. Goddard [57] showed that there is a partial converse:

**Theorem 6.12** [57] (1) For all $k \geq 2$ it holds that $S(k,k)$ is equivalent to $S(k)$.

(2) For all $k \geq 2$ it holds that $S(k)$ implies $S(k+1)$.

(3) For all $k \geq 3$ it holds that $S(k)$ implies $S(k-1,d)$ for all $d \geq k$.

### 7 Steiner Intervals in Graphs

The following result in [87] shows how the Steiner distance of an $k$-set can be found if its 2-intersection interval is nonempty.

**Theorem 7.1** [87] Let $S = \{u_1, u_2, \ldots, u_k\}$ be a set of $k \geq 2$ vertices of a graph $G$. If the 2-intersection interval of $S$ is nonempty and $x \in I_2(S)$, then $d(S) = \sum_{i=1}^{r} d(u_i, x)$. 
Since it can be determined in polynomial time whether an $k$-set satisfies the hypothesis of Theorem 7.1, the Steiner distance for such an $k$-set can be found in polynomial time. The result of Theorem 7.1 is best possible in the sense that we will now discuss. Let $G$ be the graph shown in (1) of Figure 7.1 and let $S = \{u_1, u_2, \ldots, u_k\}$. Then $x$ lies on a shortest $u_i$-$u_j$ path for all $1 \leq i < j \leq n$ except on a shortest $u_{k-1}$-$u_k$ path, but $x$ does not belong to a Steiner tree for $S$ and $d(S) \neq \sum_{i=1}^{k} d(u_i, x)$. In fact, $d(S) = 2k - 1$ but a tree with the least number of edges and containing $S$ and $x$ has $2k$ edges and $\sum_{i=1}^{k} d(u_i, x) = 2k$. This graph also serves to illustrate that there may be $k$-sets for which the 2-intersection intervals are empty.

Kubicka, Kubicki, and Oellermann [87] observed also that the $r$-intersection interval of $S$ for all $r$, $3 \leq r \leq k$, contains $z$ and is thus nonempty. Indeed, it is our belief that if the $r$-intersection interval of some $k$-set $S$ is nonempty, then the $\ell$-intersection interval of $S$ is nonempty for all $\ell > r$ and if $x \in I_r(S)$, then $x \in I_\ell(S)$. To prove this statement, it would suffice to show that if $S$ is an $k$-set and $x \in I_{k-1}(S)$, then there exists a Steiner tree for $S$ that contains $x$.

![Graphs for Theorems 7.1 and 7.3](image)

**Figure 7.1:** Graphs for Theorems 7.1 and 7.3

### 7.1 Results for Steiner intervals

Graphs for which the 2-intersection interval of every 3-set is nonempty have been studied; see [109]. The only graphs which have the property that the 2-intersection interval of every $k$-set is nonempty, where $k > 3$, are the stars. To see this, note that if $G$ is a connected graph but not a star, then there exist two independent edges in $G$. If we place the four end vertices of these two edges in an $k$-set, then the 3-intersection interval of this $k$-set is empty.

Kubicka, Kubicki, and Oellermann [87] investigated graphs with the property that all $n$-sets have nonempty 3-intersection intervals for some fixed $n \geq 4$.

**Theorem 7.2** [87] A graph $G$ has the property that the 3-intersection interval of every 4-set is nonempty if and only if $G$ has no cycles of length other than 3 or 5.
In order to characterize graphs with the property that the 3-intersection intervals of every 5-set is nonempty, they established the following result for trees which is of interest in its own right.

**Theorem 7.3** [87] For a tree of order at least $2r - 1$ the $r$-intersection interval of every $(2r - 1)$-set consists of exactly one vertex.

**Theorem 7.4** [87] A graph $G$ has the property that the 3-intersection interval of every 5-set is nonempty if and only if every block of $G$ is isomorphic to $K_2$ or $K_3$ and those blocks isomorphic to $K_3$ are end-blocks.

**Theorem 7.5** [87] For $k = 6$, a graph $G$ has the property that the 3-intersection interval of every $k$-set is nonempty if and only if its order is at least $k$ and, for some vertex $v$ every component of $G - v$ is $K_1$ or $K_2$ (see (2) of Figure 7.1).

Kubicka, Kubicki, and Oellermann [87] characterized those graphs $G$ for which $I_r(S)$ is nonempty for every $k$-set $S$, provided $k$ is sufficiently large in comparison to $r$. In order to present our characterization we need the following characterization of 2-connected graphs that appears in [34] and is a stronger characterization of 2-connected graphs given independently by Györi [66] and Lovász [92].

**Theorem 7.6** [87] A graph $G$ of order $n$ is 2-connected if and only if for every two distinct vertices $a, b$ in $V(G)$ there exists an ordering of the vertices of $G$, $a = x_1, x_2, \ldots, x_n = b$ such that for each $\ell$ with $1 \leq \ell \leq n$ the subgraphs induced by $\{x_1, x_2, \ldots, x_\ell\}$ and $\{x_\ell, x_{\ell+1}, \ldots, x_n\}$ are connected.

**Theorem 7.7** [87] Let $G$ be a graph of order $n \geq k$ and suppose $k \geq 2r$. Then $I_r(S)$ is nonempty for every $k$-set $S$ of vertices of $G$ if and only if $G$ has a cut vertex $v$ such that every component of $G - v$ has at most $r - 1$ vertices.

### 7.2 Finding Steiner intervals in distance-hereditary graphs

Let $G$ be a distance-hereditary graph and $S \subseteq V(G)$ with $|S| \geq 2$. Oellermann and Puertas [115] developed an algorithm for finding the Steiner interval $I(S)$ of $S$ in $G$. Let $I_2(S) = \bigcup_{a,b \in S} I[a,b]$. If $G$ is a graph and $\mathcal{S} \subseteq V(G)$, then $\langle \mathcal{S} \rangle$ denotes the subgraph induced by $\mathcal{S}$.

**Proposition 7.1** [115] If $G$ is a distance-hereditary graph and $S$ a set of vertices of $G$, then $I(S) \subseteq I_2(S)$.

**Algorithm 7.1** [115] (Finding $I(S)$ for a set $S \subseteq V(G)$ where $G$ is a distance-hereditary graph and $|S| \geq 2$).
Let \( H = < I_2(S) >. \) Since \( H \) is connected and induced, \( H \) is distance hereditary.

Mark every vertex of \( H \) that belong to \( S \) with \( T \) and all other vertices with \( F \). (We associate with each vertex \( v \) of \( H \) a set \( s(v) \) of vertices of the original graph \( H \). These sets will be used to construct the Steiner interval for \( S \).) For each \( v \in V(H) \) initialize \( s(v) \leftarrow \{v\} \) and let \( I(S) \leftarrow \emptyset \).

If \( |S| = 2 \), go to Step 7.

While \( |S| \geq 3 \) and \( H \) contains a pendant \( u \), proceed as follows: if \( u \) is marked \( T \), let \( I(S) \leftarrow I(S) \cup s(u) \), mark the neighbor \( u' \) of \( u \) with \( T \), let \( H \leftarrow H - u \) and \( S \leftarrow (S - u) \cup \{u'\} \); otherwise, if \( u \) is marked \( F \) let \( H \leftarrow H - u \). Now if \( |S| \geq 3 \) (but \( H \) contains no pendants), go to Step 5; otherwise, go to Step 7 (since \( |S| = 2 \)).

While \( |S| \geq 3 \) and \( H \) has a pair \( u, v \) of twins which have both been marked \( T \) (or \( F \), respectively) delete the vertex with the larger label (say \( v \)) from \( H \) and \( S \) and modify \( s(u) \), i.e., \( H \leftarrow H - v \), \( S \leftarrow S - \{v\} \) and let \( s(u) \leftarrow s(u) \cup s(v) \). Now if \( |S| \geq 3 \) (but no such twins remain), proceed to Step 6; otherwise, go to Step 7 (since \( |S| = 2 \)).

While \( |S| \geq 3 \) and \( H \) contains a pair of twins, one marked \( T \) (say \( u \)) and the other \( F \) (say \( v \)), then delete the vertex marked \( F \), i.e., define \( H \leftarrow H - v \) and return to Step 4.

If \( |S| = 2 \), say \( S = \{x, y\} \), let \( I(S) \leftarrow I(S) \cup s(u)_{u \in I_H[x,y]} \). Output \( I(S) \) and stop.

**Theorem 7.8** \([115]\) If \( G \) is a distance-hereditary graph and \( S \subseteq V(G), |S| \geq 2 \), then the set \( I(S) \) output by the algorithm is the Steiner interval for \( S \).

**Remark 7.1** \([115]\) Step 1 of the Algorithm is not essential but is useful and allows \( I(S) \) to be found more efficiently if \( I(S) \) represents only a small proportion of the vertices of the graph.

## 8 Steiner Distance Stable Graphs

In this section, we summarize the known results on Steiner distance stable graphs and independent Steiner distance stable graphs.

### 8.1 Steiner distance stable graphs

The next result, due to Goddard, Oellermann, and Swart \([59]\), shows that if distances of \((s,m)\)-sets in a connected graph are preserved after the deletion of certain numbers of vertices and edges, then so are distances preserved for \((s,d)\)-sets where \( d > m \).
Theorem 8.1 If a connected graph $G$ is $k$-vertex $\ell$-edge $(s,m)$-Steiner distance stable, then it is $k$-vertex $\ell$-edge $(s,m+1)$-Steiner distance stable.

Corollary 8.1 If a connected graph is $k$-vertex $\ell$-edge $(s,m)$-Steiner distance stable, then it is $k$-vertex $\ell$-edge $(s,n)$-Steiner distance stable for all $n \geq m$.

The next theorem implies another result of this type.

Theorem 8.2 If a connected graph $G$ is $k$-vertex $\ell$-edge $(s,m)$-Steiner distance stable, then $G$ is $k$-vertex $\ell$-edge $(s,m+1)$-Steiner distance stable.

Corollary 8.2 If a connected graph is $k$-vertex $\ell$-edge $(s,m)$-Steiner distance stable, then it is $k$-vertex $\ell$-edge $(s',m)$-Steiner distance stable for all $s'$ $(2 \leq s' \leq s)$.

In Theorem 8.1 one can see that the condition that a connected graph is $k$-vertex $\ell$-edge $(s,m)$-Steiner distance stable is sufficient for the graph to be $k$-vertex $\ell$-edge $(s,m+1)$-Steiner distance stable. The next result shows that this condition is not necessary.

Theorem 8.3 For any integers $k$, $s$ and $m$ such that $s \geq 2$, $2s - 2 \geq m \geq s$ and $k \geq 1$, there exists a graph $G$ which is $k$-vertex 0-edge $(s,m+1)$-Steiner distance stable, but not $k$-vertex 0-edge $(s,m)$-Steiner distance stable.

If we let $m = s - 1$ in the construction of the proof of Theorem 8.3, we obtain a graph that is $k$-vertex 0-edge $(s,s)$-Steiner distance stable and not $k$-vertex 0-edge $(s-1,s-1)$-Steiner distance stable.

They showed that the converse of Theorem 8.2 does not hold.

Theorem 8.4 For $s \geq 3$ there is a graph which is 1-vertex 0-edge $(s-1,s-1)$-Steiner distance stable but not 1-vertex 0-edge $(s,s)$-Steiner distance stable.

Since the graph $G$ of the proof of Theorem 8.4 is 1-vertex 0-edge $(s-1,s-1)$-Steiner distance stable, it follows, by Theorem 8.1, that $G$ is 1-vertex 0-edge $(s-1,s)$-Steiner distance stable. Since $G$ is not 1-vertex 0-edge $(s,s)$-Steiner distance stable, it follows that the converse of Theorem 8.2 does not hold in general.

Recall it was shown in [13], for a positive integer $k$, that a graph is $(k,0,\{2\})$-stable if and only if it is $(0,k,\{2\})$-stable. So a graph is $k$-vertex 0-edge $(2,2)$-Steiner distance stable if and only if it is 0-vertex $k$-edge $(2,2)$-Steiner distance stable. The next result shows that the necessity of this condition has an extension to $(3,3)$-sets.

Theorem 8.5 For a positive integer $k$, a graph $G$ is $k$-vertex 0-edge $(3,3)$-Steiner distance stable if it is 0-vertex $k$-edge $(3,3)$-Steiner distance stable.
The converse of Theorem 8.5 does not hold; see [59]. Let $H_1 \cong K_s - uv (s \geq 3)$ for some pair $u, v$ of vertices and $H_2 \cong K_2$ where $V(H_2) = \{x, y\}$. Let $G$ be obtained from $H_1 \cup H_2$ by adding the edges $ux$ and $vy$. Then $G$ is 1-vertex 0-edge $(s, s)$-Steiner distance stable, but $G$ is not 0-vertex 1-edge $(s, s)$-Steiner distance stable. To see this let $z \in V(H_1) - \{u, v\}$. Then $d_G(\{x, y, z\}) = 3$ but $d_{G-xy}(\{x, y, z\}) = 4$.

Theorem 8.5 cannot be extended to $(s, s)$-sets for $s \geq 4$; see [59]. To see this, let $G \cong (K_2 \cup K_{s-2} + K_1$, i.e., $G$ is obtained by joining a new vertex to every vertex in disjoint copies of $K_2$ and $K_{s-2}$. Then $G$ is 0-vertex 1-edge $(s, s)$-Steiner distance stable but $G$ is not 1-vertex 0-edge $(s, s)$-Steiner distance stable.

### 8.2 Independent Steiner distance stable graphs

Goddard, Oellermann, and Swart [59] also focused their attention on independent sets of vertices of a graph. Their first result shows that in a certain sense the problem of finding Steiner trees for sets of independent vertices is equivalent to the problem of finding the Steiner trees of sets of vertices that are not necessarily independent.

Let $\Pi_1$ be the problem of finding a Steiner tree for a nonempty set of vertices of a connected graph and $\Pi_2$ the problem of finding a Steiner tree for a nonempty independent set of vertices of a connected graph. Let $G$ be a connected graph and $S$ a nonempty set of vertices of $G$. Suppose $G_1, G_2, \ldots, G_n$ are the components of $< S >_G$. Let $R(G; S)$ be the graph with vertex set $(V(G) - S) \cup \{v_1, v_2, \ldots, v_n\}$ (where $v_i$ corresponds to $G_i$, $1 \leq i \leq n$) and edge set $\{uv | uv \in E(G-S)\} \cup \{v_i | u \in V(G-S)\}$ and $u$ is adjacent in $G$ to some vertex of $G_i$. Thus $R(G; S)$ is the contraction of $G$ that results from the partition $\bigcup_{i=1}^{n} V(G_i) \cup \{u | u \in V(G-S)\}$.

**Theorem 8.6** [59] There is an (efficient) algorithm that solves $\Pi_1$, is and only if there is an (efficient) algorithm that solves $\Pi_2$.

Theorem 8.6 also follows directly from the nearest vertex reduction test described by Beasley [14].

The concepts presented in Subsection 9.2 and Theorem 8.6 suggest the next topic. If $G$ is a connected graph and $S$ an independent set of $s$ vertices of $G$ such that $d_G(S) = m$, then $S$ is called an $I(s, m)$-set. A connected graph is defined to be $k$-vertex $\ell$-edge $I(s, m)$-Steiner distance stable if, for every $I(s, m)$-set $S$ and every set $A$ of at most $k$ vertices of $G - S$ and at most $\ell$ edges of $G$, $d_{G - A}(S) = m$.

The following result in [59] establishes an analogue of Theorem 8.1 with respect to $I(3, m)$-sets.

**Theorem 8.7** [59] If $G$ is a $k$-vertex $\ell$-edge $I(3, m)$-Steiner distance stable graph $m \geq 4$, then $G$ is a $k$-vertex $\ell$-edge $I(3, m + 1)$-Steiner distance stable graph.
It remains an open problem to determine if a $k$-vertex $\ell$-edge $I(s, m)$-Steiner distance stable graph $m \geq 4$, is a $k$-vertex $\ell$-edge $I(s, m + 1)$-Steiner distance stable graph, where $s > 4$.

9 Extremal problems on Steiner diameter

What is the minimal size of a graph of order $n$ and diameter $d$? What is the maximal size of a graph of order $n$ and diameter $d$? It is not surprising that these questions can be answered without the slightest effort (see [17]) just as the similar questions concerning the connectivity or the chromatic number of a graph. The class of maximal graphs of order $n$ and diameter $d$ is easy to describe and reduce every question concerning maximal graphs to a not necessarily easy question about binomial coefficient, as in [77, 78, 118, 136]. Therefore, the authors studied the minimal size of a graph of order $n$ and under various additional conditions.

Erdős and Rényi [46] introduced the following problem. Let $d$, $\ell$ and $n$ be natural numbers, $d < n$ and $\ell < n$. Denote by $\mathcal{H}(n, \ell, d)$ the set of all graphs of order $n$ with maximum degree $\ell$ and diameter at most $d$. Put

$$e(n, \ell, d) = \min\{e(G) : G \in \mathcal{H}(n, \ell, d)\}.$$ 

If $\mathcal{H}(n, \ell, d)$ is empty, then, following the usual convention, we shall write $e(n, \ell, d) = \infty$. For more details on this problem, we refer to [17, 18, 46, 47].

Mao [96] considered the generalization of the above problem. Let $d$, $\ell$ and $n$ be natural numbers, $d < n$ and $\ell < n$. Denote by $\mathcal{H}_k(n, \ell, d)$ the set of all graphs of order $n$ with maximum degree $\ell$ and $sdiam_k(G) \leq d$. Put

$$e_k(n, \ell, d) = \min\{e(G) : G \in \mathcal{H}_k(n, \ell, d)\}.$$ 

If $\mathcal{H}_k(n, \ell, d)$ is empty, then, following the usual convention, we shall write $e_k(n, \ell, d) = \infty$. From Theorem 3.5 we have $k - 1 \leq d \leq n - 1$.

The following results can be easily proved.

Proposition 9.1 [96, 101] (1) For $2 \leq \ell \leq n - 1$ and $3 \leq k \leq n$, $e_k(n, \ell, n - 1) = n - 1$.

(2) For three integers $n, d, \ell$ with $2 \leq d \leq n - 2$ and $n - d + 2 \leq \ell \leq n - 2$, $e_3(n, \ell, d) = n - 1$.

9.1 Results for small $k$

If $sdiam_3(G) = 2$, then $n - 2 \leq \delta(G) \leq n - 1$, and hence $n - 2 \leq \Delta(G) \leq n - 1$. So one can assume that $n - 2 \leq \ell \leq n - 1$ for $d = 2$. 

Theorem 9.1 \[96\]

(1) For $\ell = n - 1$, $e_3(n, \ell, 2) = \binom{n}{2} - \frac{n-1}{2}$ for $n$ odd; $e_3(n, \ell, 2) = \binom{n}{2} - \frac{n-2}{2}$ for $n$ even.

(2) For $\ell = n - 2$, $e_3(n, \ell, 2) = \binom{n}{2} - \frac{n}{2}$ for $n$ even; $e_3(n, \ell, 2) = \infty$ for $n$ odd.

In \[96\], Mao got the following results for $d = 3$.

Theorem 9.2 \[96\]

(1) For $\ell = n - 1$, $e_3(n, n - 1, 3) = n - 1$;

(2) For $\ell = n - 2$, $e_3(n, n - 2, 3) = 2n - 5$;

(3) For $\ell = n - 3$, $e_3(n, n - 3, 3) = 2n - 5$;

(4) For $\ell = 2$, $e_3(n, 2, 3) = 3$ for $n = 4$; $e_3(n, 2, 3) = 5$ for $n = 5$; $e_3(n, 2, 3) = \infty$ for $n \geq 6$.

(5) For $\frac{n}{2} \leq \ell \leq n - 4$, $n \leq e_3(n, \ell, 3) \leq \ell(n - \ell)$.

For $d = n - 2, n - 3, n - 4$, Mao obtained the following results.

Theorem 9.3 \[96\]

(1) For $n \geq 4$, $e_3(n, 2, n - 2) = n$.

(2) For $n \geq 4$,

$$e_3(n, 3, n - 2) = \begin{cases} 
  n + 1 & \text{if } n = 4, \\
  n & \text{if } n = 5, \\
  n - 1 & \text{if } n \geq 6.
\end{cases}$$

(3) For $n \geq 5$ and $4 \leq \ell \leq n - 1$, $e_3(n, \ell, n - 2) = n - 1$.

Theorem 9.4 \[96\]

(1) For $n \geq 5$,

$$e_3(n, 2, n - 3) = \begin{cases} 
  \infty & \text{if } n = 5, 6, \\
  n & \text{if } n \geq 7.
\end{cases}$$

(2) For $n \geq 5$,

$$e_3(n, 3, n - 3) = \begin{cases} 
  \infty, & \text{if } n = 5, \\
  n + 1 & \text{if } n = 6, \\
  n & \text{if } n = 7, \\
  n - 1 & \text{if } n \geq 8.
\end{cases}$$

(3) For $n \geq 5$,

$$e_3(n, 4, n - 3) = \begin{cases} 
  \binom{n}{2} - 2 & \text{if } n = 5, \\
  n + 1 & \text{if } n = 6, \\
  n - 1 & \text{if } n \geq 7.
\end{cases}$$

(4) For $n \geq 6$ and $5 \leq \ell \leq n - 1$, $e_3(n, \ell, n - 3) = n - 1$. 

\[72\]
Theorem 9.5\textsuperscript{96} (1) For $n \geq 5$, 
\[
e_3(n, 2, n - 4) = \begin{cases} 
\infty & \text{if } 5 \leq n \leq 9, \\
n & \text{if } n \geq 10.
\end{cases}
\]
\[(2)\] For $n \geq 6$, 
\[
e_3(n, 3, n - 4) = \begin{cases} 
\infty & \text{if } n = 6, \\
n + 3 & \text{if } n = 7, \\
n + 2 & \text{if } n = 8, \\
n + 1 & \text{if } n = 9, \\
n - 1 & \text{if } n \geq 10.
\end{cases}
\]
\[(3)\] For $n \geq 6$, 
\[
e_3(n, 4, n - 4) = \begin{cases} 
2n & \text{if } n = 6, \\
n + 2 & \text{if } n = 7, \\
n - 1 & \text{if } n \geq 8.
\end{cases}
\]
\[(4)\] For $n \geq 6$, 
\[
e_3(n, 5, n - 4) = \begin{cases} 
2n + 1 & \text{if } n = 6, \\
n + 2 & \text{if } n = 7, \\
n - 1 & \text{if } n \geq 8.
\end{cases}
\]
\[(5)\] For $n \geq 7$ and $6 \leq \ell \leq n - 1$, $e_3(n, \ell, n - 4) = n - 1$.

Mao\textsuperscript{96} also constructed a graph and gave an upper bound of $e_3(n, \ell, d)$ for general $\ell$ and $d$.

Proposition 9.2 For $4 \leq d \leq n - 1$ and $2 \leq \ell \leq n - 1$, 
\[
e_3(n, \ell, d) \leq \frac{(n - d + 1)(n - d + 2)}{2} + d - 3.
\]

9.2 Results for large $k$

The following result is immediate.

Proposition 9.3\textsuperscript{101} For $2 \leq \ell \leq n$, $e_n(n, \ell, n - 1) = n - 1$.

For $k = n - 1$ and $k = n - 2$, Mao and Wang\textsuperscript{101} derived the following results.

Proposition 9.4\textsuperscript{101} (1) For $2 \leq \ell \leq n - 1$, $e_{n-1}(n, \ell, n - 1) = n - 1$.

(2) For $2 \leq \ell \leq n - 1$, $e_{n-1}(n, \ell, n - 2) = n + \ell - 2$.

Proposition 9.5\textsuperscript{101} For $2 \leq \ell \leq n - 1$ and $n \geq 5$, 
\[
e_{n-2}(n, \ell, n - 2) = \begin{cases} 
n & \text{if } 2 \leq \ell \leq n - 2; \\
n - 1 & \text{if } \ell = n - 1.
\end{cases}
\]
Let $P^i_j$ be a path of order $j$, where $1 \leq i \leq r + 2$. We call the graph $K_1 \lor (K_1 \cup P^i_j)$ as a $(u_i, v_i, P^i_j)$-Fan; see Figure 9.1 (a). For $1 \leq i \leq r$, we choose $(u_i, v_i, P^i_j)$-Fan, and choose $(u_{r+1}, v_{r+1}, P^{r+1}_s)$-Fan and $(u_{r+2}, v_{r+2}, P^{r+2}_s)$-Fan. Let $H_n$ be a graph obtained from the above $(r + 2)$ Fans by adding the edges in

$$\{w^{r+1}_1w^{r+2}_s, w^{r+2}_s w^i_1\} \cup \{w^i_1w^{i+1}_1 | 1 \leq i \leq r - 1\} \cup \{w^r_s w^{r+1}_1\}$$

$$\cup \{v_iv_{i+1} | 1 \leq i \leq r + 1\} \cup \{v_{r+2}v_1\};$$

see Figure 9.1 (b), where $4r + \ell + s + 3 = n$, $2 \leq s \leq 5$ and $1 \leq r \leq \frac{n-4}{4}$.

Figure 9.1: Graphs for (3) of Theorem 9.6

By the above graph class, Mao and Wang [101] derived the result in (3) of Theorem 9.6 for $6 \leq \ell \leq n - 9$.

**Theorem 9.6** [101] (1) For $2 \leq \ell \leq n - 1$,

$$e_{n-2}(n, \ell, n-1) = n - 1.$$

(2) For $2 \leq \ell \leq n - 1$ and $n \geq 5$,

$$e_{n-2}(n, \ell, n-2) = \begin{cases} n, & \text{if } 2 \leq \ell \leq n - 2 \\ n - 1, & \text{if } \ell = n - 1. \end{cases}$$

(3) For $n - 8 \leq \ell \leq n - 1$, $e_{n-2}(n, n - 1 - i, n - 3) = 2n - 2$ for $n \geq 5 + i$ and $i = 0, 1$; $e_{n-2}(n, n - 3 - i, n - 3) = 2n - 3$ for $n \geq 7 + 2i$ and $i = 0, 1$; $e_{n-2}(n, n - 5 - i, n - 3) = 2n - 4$ for $n \geq 11 + 2i$ and $i = 0, 1$; $e_{n-2}(n, n - 7 - i, n - 3) = 2n - 5$ for $n \geq 15 + 2i$ and $i = 0, 1$. For $6 \leq \ell \leq n - 9$,

$$\frac{1}{2}(3n + \ell - 3) \leq e_{n-2}(n, \ell, n-3) \leq \frac{1}{2}(3n + \ell + s - 5),$$

where $2 \leq s \leq 5$. Furthermore, if $s = 2$, then $e_{n-2}(n, \ell, n-3) = \frac{1}{2}(3n + \ell - 3)$.

Let $A_{32}$ be a minimally 4-connected graph shown in Figure 9.2 (a) (see [17], Page 18). We now give a graph $H_n$ of order $n$ ($n \geq 96$) such that $\Delta(H_n) = \ell$ and $sdiam_{n-3}(H_n) = n - 4$ constructed by the following steps.
Step 1: For each $i$ ($1 \leq i \leq x$), we let $A_{32}^i$ be the copy of $A_{32}$, where $n = 32x + y$, $x = \lfloor n/32 \rfloor$, and $0 \leq y \leq 31$. Let $V(A_{32}^i) = \{u_j^i | 1 \leq j \leq 12\} \cup \{v_j^i | 1 \leq j \leq 20\}$ such that $d_G(u_j^i) = 5$ for $1 \leq j \leq 12$, and $d_G(v_j^i) = 4$ for $1 \leq j \leq 20$; see Figure 9.2 (a). Let $B_{32x}$ be a graph obtained from $A_{32}^i$ ($1 \leq i \leq x$) by adding the edges in $\{v_j^i v_j^{i+1} | 1 \leq i \leq x\} \cup \{v_j^i v_j^{i+1} | 1 \leq i \leq x-1\} \cup \{v_j^i v_j^{i+1} | 1 \leq i \leq x-1\} \cup \{v_j^i v_j^{i+1} | 1 \leq i \leq x-1\}$; see Figure 9.2 (b).

Step 2: Let $y = 4z + a$, where $z = \lfloor y/4 \rfloor$, $0 \leq a \leq 3$. For each $j$ ($1 \leq j \leq z$), we let $K_4^j$ be the complete graph of order 4. Furthermore, let $K_4^j$ be the graph obtained from $K_4^j$ by adding four pendant vertices $w_1^j, w_2^j, w_3^j, w_4^j$ with four pendant edges such that another end vertex of each pendant edge is attached on only one vertex in $K_4^j$; see Figure 9.2 (c). For each $j$ ($1 \leq j \leq a$), we let $S_5^j$ be the star of order 5 with its leaves $p_1^j, p_2^j, p_3^j, p_4^j$. Since $n \geq 96$, it follows that $A_{32}^1, A_{32}^2, A_{32}^3$ all exist. Set $S_1 = \{v_j^1 | 1 \leq j \leq 4\}$, $S_2 = \{v_j^2 | 9 \leq j \leq 20\}$, and $S_3 = \{v_j^3 | 9 \leq j \leq 20\}$, and $S_4 = \{v_j^4 | 9 \leq j \leq 20\}$, and $S_5 = \{v_j^5 | 9 \leq j \leq 20\}$. Then $|S_1 \cup S_2 \cup S_3| = 40$. If $n \equiv 0 \pmod{32}$, then $D_n = B_{32x}$. If $n \not\equiv 0 \pmod{32}$ and $n - 32x \equiv 0 \pmod{4}$, then $D_n$ is a graph obtained from $B_{32x}$ and $K_4^1, K_4^2, \ldots, K_4^z$ by identifying each vertex in $S' = \{w_j^i | 1 \leq i \leq 4, 1 \leq j \leq z\}$ and only one vertex in $S_1 \cup S_2 \cup S_3$. Since $|S'| = 4z < 40 = |S_1 \cup S_2 \cup S_3|$, for any vertex in $S'$, we can find a vertex in $S_1 \cup S_2 \cup S_3$ and then identify the two vertices. If $n \not\equiv 0 \pmod{32}$ and $n - 32x \not\equiv 0 \pmod{4}$, then $D_n$ is a graph obtained from $B_{32x}$, $K_4^1, K_4^2, \ldots, K_4^z$, and $S_5^1, S_5^2, \ldots, S_5^z$ by identifying
each vertex in $S' = \{w_j^i | 1 \leq i \leq 4, 1 \leq j \leq z\} \cup \{p_j^i | 1 \leq i \leq 4, 1 \leq j \leq a\}$ and only one vertex in $S_1 \cup S_2 \cup S_3$. Since $|S'| = 4z + 4a \leq 28 + 12 = 40 = |S_1 \cup S_2 \cup S_3|$, for any vertex in $S'$, we can find a vertex in $S_1 \cup S_2 \cup S_3$ and then identify the two vertices.

**Step 3:** Let $H_n$ be the graph $D_n$ by adding $\ell - 5$ edges between $u_{12}^1$ and $V(G) - u_{12}^1$.

By the above graph class, they derived the result in (3) of Theorem 9.7.

**Theorem 9.7** \([101]\) (1) For $2 \leq \ell \leq n - 1$, $e_{n-1}(n, \ell, n - 1) = n - 1$.

(2) For $2 \leq \ell \leq n - 1$ and $n \geq 4$,

$$e_{n-3}(n, \ell, n - 2) = \begin{cases} n, & \text{if } 2 \leq \ell \leq \left\lfloor \frac{n}{2} \right\rfloor - 1; \\
\text{or } \ell = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is odd}; \\
n - 1, & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq \ell \leq n - 1; \\
\text{or } \ell = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is even}. \end{cases}$$

(3) For $2 \leq \ell \leq n - 1$ and $n \geq 96$,

$$2n - 2 - \left\lceil \ell/2 \right\rceil \leq e_{n-3}(n, \ell, n - 4) \leq 74 \left\lfloor \frac{n}{32} \right\rfloor + 2i + \ell - 9,$$

where $n \equiv i \pmod{32}$, $1 \leq i \leq 31$.

(4) If $2 \leq \ell \leq n - 1$, then $e_{n-3}(n, \ell, n - 3) \geq \max\{n - 1 + \left\lfloor \ell/2 \right\rceil, \frac{3n - \ell - 5}{2}\}$. If $\left\lceil n/2 \right\rceil + 1 \leq \ell \leq n - 1$, then $e_{n-3}(n, \ell, n - 3) \leq 2n - \ell + 1$. If $5 \leq \ell \leq \left\lceil n/2 \right\rceil$, then

$$e_{n-3}(n, \ell, n - 3) \leq (2\ell + 3) \left\lfloor \frac{n}{\ell + 1} \right\rfloor + \ell + \begin{cases} -8, & \text{if } n \equiv 0 \pmod{\ell + 1}; \\
-5, & \text{if } n \equiv 1 \pmod{\ell + 1}; \\
-2, & \text{if } n \equiv 2 \pmod{\ell + 1}; \\
\text{or } n \equiv 3 \pmod{\ell + 1}; \\
2i - 7, & \text{if } n \equiv i \pmod{\ell + 1}, \end{cases}$$

where $n = (\ell + 1)x + i$ and $0 \leq i \leq \ell$.

Mao and Wang \([101]\) constructed a graph and gave an upper bound of $e_k(n, \ell, d)$ for general $k, \ell, d$.

**Theorem 9.8** \([101]\) Let $k, \ell, d$ be three integers with $2 \leq k \leq n$, $2 \leq \ell \leq n - 1$, and $k - 1 \leq d \leq n - 1$.

(1) If $d = k - 1$, $\left\lceil \frac{n+1}{2} \right\rceil \leq k \leq n$, and $\max\{n - k + 1, 1\} \leq \ell \leq n - 1$, then

$$\left\lceil \frac{\ell + (n - 1)(n - k + 1)}{2} \right\rceil \leq e_k(n, \ell, d) \leq \frac{(n - 1)^2}{4} + \ell.$$

(2) If $2 \leq k \leq d$, $k \leq d \leq n - 1$, and $2 + \left\lceil \frac{n - d + k - 3}{d-k+1} \right\rceil \leq \ell \leq n - 1$, then

$$e_k(n, \ell, d) = n - 1.$$
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