Quantum Corrections to Non-Abelian SUSY Theories on Orbifolds

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Abstract

We consider supersymmetric non-Abelian gauge theories coupled to hyper multiplets on five and six dimensional orbifolds, $S^1/\mathbb{Z}_2$ and $T^2/\mathbb{Z}_N$, respectively. We compute the bulk and local fixed point renormalizations of the gauge couplings. To this end we extend supergraph techniques to these orbifolds by defining orbifold compatible delta functions. We develop their properties in detail. To cancel the bulk one-loop divergences the bulk gauge kinetic terms and dimension six higher derivative operators are required. The gauge couplings renormalize at the $\mathbb{Z}_N$ fixed points due to vector multiplet self interactions; the hyper multiplet renormalizes only non-$\mathbb{Z}_2$ fixed points. In 6D the Wess-Zumino-Witten term and a higher derivative analogue have to renormalize in the bulk as well to preserve 6D gauge invariance.
1 Introduction

Theories of extra dimensions have been investigated for a long time after the pioneering work by Kaluza and Klein. More recently, string theory has been very important to stimulate research into this direction, because the natural number of dimensions for string theory seems to be ten. Not only in the string theory community the topic of extra dimensions has attracted a lot of attention, also phenomenologists looked at this possibility. This was initiated by the papers \[1, 2\]. Most of the phenomenological activity has focused on five dimensional (5D) models, in particular models on simple 1D orbifolds like \(S^1/\mathbb{Z}_2\) or \(S^1/\mathbb{Z}_2 \times \mathbb{Z}_2\) \[3, 4, 5\]. One of the main reasons to turn to orbifolds is that they naturally lead to chiral fermions. And if the extra dimensional theory is supersymmetric then only by orbifolding 4D \(\mathcal{N} = 1\) supersymmetry can be recovered. Also 2D orbifolds like \(T^2/\mathbb{Z}_N\) have been considered in phenomenological applications in 6D. To obtain phenomenological models from (heterotic) string theory one often uses 6D orbifolds. In this paper we will focus primarily on orbifolds in 5D and 6D, but these results can be easily extended to the 10D string theory and 11D M-theory settings.

There have been many investigations of quantum corrections to field theories on orbifolds. An issue that received particular attention is the generation of the Fayet-Iliopoulos terms at the fixed points of 1D orbifolds \[6, 7, 8, 9, 10, 11\]. Another issue of investigations in extra dimensions is the question of the running of the gauge coupling and possible gauge coupling unification. Initial investigations like \[12, 13\] took a naive approach to this problem, but it was soon widely accepted that this running is in principle sensitive to the ultra–violet (UV) completion of the theory \[14\].

There are various issues that one has to be aware of when discussing 5D and 6D theories. In 6D the constraints of anomalies are very severe \[15, 16, 17\]. This means that one has to be very careful when one tries to obtain a consistent theory in a bottom-up approach. However, if one considers (heterotic) string compactifications it is essentially guaranteed that no dangerous anomalies can ever arise. In the present work the issue of anomalies is not so important, because we are simply interested in the corrections to bulk and brane gauge operators due to various supermultiplets. These results can then be in particular be applied to anomaly free models in 6D.

In a previous publication \[18\] we studied the gauge coupling running by calculating the photon self-energy in extra dimensions. We focused on the renormalization of the gauge operators in Abelian supersymmetric field theories on 5D and 6D orbifolds. In general one would expect that both bulk and fixed point gauge couplings would renormalize \[19, 20, 21\], but we found that for a charged bulk hyper multiplet the contributions cancel out at the fixed points of \(S^1/\mathbb{Z}_2\). However, for a 6D orbifold \(T^2/\mathbb{Z}_N\) this cancellation does not persist, except for the fixed points that are invariant under a \(\mathbb{Z}_2\) symmetry. The other feature we found is that in the 6D case also a dimension six higher derivative term for the gauge multiplet is required to cancel all divergences. The observation that such higher derivative operators are generated is no surprise, it is simply one of the consequences that we consider 6D theories which are non-renormalizable.

Such higher derivative operators have also been found recently for other quantities in theories of extra dimensions: In a 6D supersymmetric model compactified on \(T^2/\mathbb{Z}_2\) such operators were obtained in the context of Scherk-Schwarz supersymmetry breaking \[22\]. And even in 5D orbifold models they can arise if brane localized interactions for bulk fields are considered \[23\]. This can be understood in the Kaluza-Klein mode picture by realizing that for brane localized interactions, the Kaluza-Klein number is not conserved, so that double sums can arise at one loop, raising the degree of divergence of the corresponding sum/integral. Such higher derivative theories may have remarkable UV properties,
and might actually be renormalizable, as speculated in \[24, 25, 26\].

In this work we extend and complete the work that was initiated in our previous paper \[18\] to include non-Abelian gauge interactions. We employ again the method of representing 5D and 6D SUSY theories by $\mathcal{N} = 1$ 4D superfields \[27, 28, 29, 30\] and give a detailed account of how to apply supergraph techniques to 5D and 6D orbifolds. (For applications of supergraph techniques in the context of supergravity see \[31\].) While the renormalization of the gauge couplings due to the hyper multiplets is a straightforward extension of the previous work, the new issue presented here is the inclusion of the self interactions of the non-Abelian gauge multiplet. This is interesting in particular because in order to obtain the vector multiplet propagator a proper gauge fixing is required. When a generic gauge is applied both the higher dimensional Lorentz invariance is lost and there one observes a mixing between the various $\mathcal{N} = 1$ superfields. However, there is a convenient gauge choice available in which these problems can be avoided \[32, 33\]. Using these ingredients we perform our calculation of brane and bulk gauge operators on 5D and 6D orbifolds.

Even though our investigation is restricted to one–loop corrections only, we expect that the results in fact hold to all orders in perturbation theory up to infra–red (IR) effects. Both at the fixed points and in the bulk holomorphicity arguments \[34, 35, 36, 37, 38\] of $\mathcal{N} = 1$ SUSY field theories in 4D apply. Using such arguments the behavior of uncompactified supersymmetric gauge theories in 5D were investigated by \[39, 40, 41\] starting from an anomaly argument by Witten \[42\]. In the direct perturbative calculation that we will be performing, we should of course be able to reproduce those results, and so they can serve as important cross checks.

This paper is organized as follows: In section 2 we give the classical action for a hyper multiplet coupled to a gauge multiplet in 5D and motivate the gauge fixing we employ. We modify the action such that it is formulated consistently on the orbifold $S^1/Z_2$. We introduce orbifold compatible delta functions which are necessary for functional differentiation in order to calculate Feynman graphs on the orbifold. We generalize this concept to 6D and the orbifold $T^2/Z_N$ in section 3. Section 4 describes the quantum calculation of the vector multiplet self-energy in the 5D case. We present the relevant vertices and calculate the Feynman graphs. In a detailed example we demonstrate how to compute an amplitude directly on the orbifold. We take the sum of the graphs to obtain the vector multiplet self-energy. We regularize the result and calculate the bulk and $S^1/Z_2$ fixed points counterterms. Section 5 follows the same logic for the 6D case and $T^2/Z_N$. Evaluating the vector multiplet self-energy here shows that one has special cancellations at those fixed points of $T^2/Z_N$ that are invariant under a $Z_2$ subgroup of $Z_N$. In the final section 6 we give some illustrating examples in which we relate our results to 4D on the zero mode level before we conclude. Appendix A has all explicit results for the calculated Feynman graphs, while appendices B and C show our conventions for Fourier transformation and theta functions, respectively. We regularize the relevant momentum integrals in appendix D.

2 Hyper and non-Abelian gauge multiplets in five dimensions

In this section we consider a hyper multiplet charged under a (non-)Abelian vector multiplet on the orbifold $S^1/Z_2$ in 5D. We begin our discussion with a review of these 5D multiplets using a 4D superfield language. Next we determine the propagators for these superfields. For the vector multiplet this requires gauge fixing and the introduction of ghost multiplets. In the final part of this section we explain how this theory can be extended to the 5D orbifold $S^1/Z_2$ and introduce orbifold compatible delta functions that arise from functional differentiation.
2.1 Classical hyper and gauge multiplet actions

We consider the classical theory of a supersymmetric 5D theory containing a hyper multiplet that is
coupled to a gauge multiplet. We describe these multiplets in terms of 4D superfields [27, 28, 29, 13].
In this language the degrees of freedom of the 5D hyper multiplet are described by two 4D chiral
multiplets $\Phi_+$ and $\Phi_-$. These fields transform in a given representation (for example the fundamental
or adjoint representation) of the gauge group. The degrees of freedom of the 5D gauge multiplet are
contained in one 4D vector multiplet $V = V^i T_i$ and one 4D chiral multiplet $S = S^i T_i$ which both
transform in the adjoint representation of the gauge group. Here $T_i$ are the Hermitian generators of
the gauge group. The algebra of these generators $[T_i, T_j] = f_{ijk} T_k$ defines the purely imaginary structure
coefficients. The Killing metric, denoted by $\eta_{i,j}$, will be used to raise and lower adjoint indices, for example $f_{ijk} = f_{ij}^k \eta_{kl}$. We denote the trace in the representation of the chiral multiplets by $\text{tr}$ and the
trace in the adjoint representation by $\text{tr}_{\text{Ad}}$. The latter is given by $\text{tr}_{\text{Ad}}(XY) = -f_{ijk} f_{lmn} \eta^{lm} \eta^{kn} X^i Y^j$, where the matrix $X$ and $Y$ are defined in the adjoint: $(X)_{jk} = X^i (T_i)_{jk} = X^i f_{ijk}$, etc.

The kinetic action of the hyper multiplet with its coupling to the gauge multiplet is described by

$$S_H = \int d^5x \left[ \int d^4 \theta \ (\bar{\Phi}_+ e^{2V} \Phi_+ + \Phi_- e^{-2V} \bar{\Phi}_-) + \right.$$

$$\left. + \int d^2 \theta \ (\bar{\Phi}_-(\partial_5 + \sqrt{2} S) \Phi_+) + \int d^2 \bar{\theta} \ (\bar{\Phi}_+ (-\partial_5 + \sqrt{2} S) \Phi_-) \right].$$

Here we have indicated the derivative in the fifth direction by $\partial_5$. This action is invariant under the
supergauge transformations

$$\Phi_+ \rightarrow e^{-2\Lambda} \Phi_+, \quad \Phi_- \rightarrow e^{2\Lambda} \Phi_-, \quad S \rightarrow e^{-2\Lambda} \left( S + \frac{1}{\sqrt{2}} \partial_5 \right) e^{2\Lambda}, \quad e^{2V} \rightarrow e^{2\Lambda} e^{2V} e^{2\Lambda},$$

where $\Lambda$ is a chiral superfield and $\bar{\Lambda}$ its conjugate. These conventions (that lead to various factors of
2) ensure that the scalar and fermionic components have charges normalized to unity.

The kinetic action for the 5D gauge multiplet in a 4D superfield language comprises the standard
terms for the 4D gauge field $V$ and one extra term for the 4D chiral multiplet $S$

$$S_V = \frac{1}{g^2} \int d^5x \left[ \frac{1}{4} \int d^2 \theta \ W^\alpha W_\alpha + \frac{1}{4} \int d^2 \bar{\theta} \ W_\dot{\alpha} \bar{W}^{\dot{\alpha}} + \frac{1}{4} \int d^4 \theta \ e^{2V_5} e^{-2V} e^{2V_5} e^{-2V} \right],$$

where we have defined

$$W_\alpha = -\frac{1}{8} \bar{D}^2 (e^{-2V} D_\alpha e^{2V}) \quad \text{and} \quad e^{2V_5} = \partial_5 e^{2V} - \sqrt{2} e^{2V} S - \sqrt{2} \bar{S} e^{2V}. \quad (4)$$

Application of the gauge transformations [2] shows that $W_\alpha$ and $e^{2V_5}$ transform covariantly

$$W_\alpha \rightarrow e^{-2\Lambda} W_\alpha e^{2\Lambda} \quad \text{and} \quad e^{2V_5} \rightarrow e^{2\Lambda} e^{2V_5} e^{2\Lambda} \quad (5)$$
such that the vector multiplet action is gauge invariant. The reduction to the Abelian case is trivial,
where one finds in particular that the super field strengths $W_\alpha$ and $V_5$ are gauge invariant. When we compute the renormalization of the vector multiplet at one loop, we perform a direct computation
This expression is obtained from (3) after some partial integrations and the absorption of a $-\frac{1}{4}D^2$ in the Grassmannian integration measure. There is a mixing between the 4D vector multiplet $V$ and the chiral multiplet $S$. The presence of this mixing is not surprising, because $S$ behaves like a Goldstone superfield since it transforms with a shift under gauge transformations, see (2). From a computational standpoint this mixing is a nuisance, but luckily, it can be removed by a suitable choice of gauge fixing, as we discuss below.

This description is clearly not manifestly 5D Lorentz invariant. Lorentz invariance is recovered after eliminating the auxiliary fields by their equations of motion. Therefore, this description is not an off-shell formulation of the 5D supersymmetric theories. However, for us the main advantage of this approach is that perturbation theory is greatly simplified over a component approach and all kinds of cancellations due to $N=1$ supersymmetry are built in.

### 2.2 Propagators, gauge fixing and ghosts

After this strictly classical discussion of the 5D hyper and vector multiplets we now turn towards the quantization of the theory using path integral methods. To this end we need to determine the propagators of the 4D superfields $\Phi_+, \Phi_-, V$ and $S$ by coupling them to the sources $J_+, J_-, J_V$ and $J_S$, respectively. As usual the interactions can be obtained by functional differentiation with respect to these sources, after the original superfields are integrated out using their corresponding quadratic actions.

By considering the quadratic part of the hyper multiplet action (1) and using some standard superspace identities, we thus obtain

$$S_{H2} = \frac{1}{g^2} \int d^5x \, d^4\theta \, \text{tr} \left[ \frac{1}{8} V D^\alpha \bar{D}^2 D_\alpha V + (\partial_5 V)^2 - \sqrt{2} \partial_5 V (S + \bar{S}) + \bar{S} S \right].$$

(7)

Hence as for massive chiral multiplets in 4D we have both non-chiral propagators between $\bar{J}_\pm$ and $J_\pm$, as well as chiral propagators between $J_+$ and $J_-$ and their conjugates. In figure I we depict our drawing conventions of these chiral propagators: The first propagator in this picture gives the correlation between the sources $J_+$ and $J_-$, and the second one between $J_+$ and $J_-$. Obviously, there is also the conjugate propagator between $\bar{J}_+$ and $\bar{J}_-$. 

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**Figure I:** This picture gives our drawing conventions for the propagators which we employ throughout this paper. In particular, there are two chiral multiplet propagators: The first one corresponds to the diagonal terms in (7), while the second refers to the off-diagonal parts.

Rather than a background field method. Therefore, we will be able to recover only the quadratic part of the vector multiplet action

$$S_{V2} = \frac{1}{g^2} \int d^5x \, d^4\theta \, \text{tr} \left[ \frac{1}{8} V D^\alpha \bar{D}^2 D_\alpha V + (\partial_5 V)^2 - \sqrt{2} \partial_5 V (S + \bar{S}) + \bar{S} S \right].$$

(6)
For the 5D vector multiplet we need to do more work because of gauge invariance. The problem of resulting zero modes can be made manifest by representing the quadratic action (6) in the following matrix form

\[ S_{V^2} = \frac{1}{2} \int d^5x d^4\theta \text{ tr } \begin{pmatrix} V \\ S \\ \bar{S} \end{pmatrix} A \begin{pmatrix} V \\ S \\ \bar{S} \end{pmatrix}, \quad A = \begin{pmatrix} -\Box P_0 - \partial_5^2 & \frac{1}{2}\sqrt{2} P_+ \partial_5 & \frac{1}{2}\sqrt{2} P_- \partial_5 \\ -\frac{1}{2}\sqrt{2} P_- \partial_5 & 0 & \frac{1}{2} P_- \\ -\frac{1}{2}\sqrt{2} P_+ \partial_5 & \frac{1}{2} P_+ & 0 \end{pmatrix}, \]  

(8)

using the transversal projector \( P_0 = \frac{D_\alpha \bar{D}_2 D_\alpha}{8} \) and its chiral counterparts \( P_+ = \frac{\bar{D}_2 D_2}{16} \) and \( P_- = \frac{D_2 \bar{D}_2}{16} \).

The operator \( A \) has chiral zero modes corresponding to the gauge directions \( X \). Indeed, we see that

\[ X = \delta \Lambda \begin{pmatrix} V \\ S \\ \bar{S} \end{pmatrix} = \begin{pmatrix} \Lambda + \bar{\Lambda} \\ \sqrt{2} \partial_5 \Lambda \\ \sqrt{2} \partial_5 \bar{\Lambda} \end{pmatrix} : AX = 0. \]  

(9)

This shows explicitly that also in five dimensions in order to define the propagator of the vector multiplet, we need to perform a gauge fixing to modify the quadratic form \( A \) so that it becomes invertible.

The procedure to determine the gauge fixed action follows the conventional 4D superfield methods for gauge multiplets, see the textbooks [44, 45] for example. As usual we start by choosing a gauge fixing functional

\[ \Theta = \frac{\bar{D}^2}{4} \left( \sqrt{2} V + \frac{1}{\Box} \partial_5 \bar{S} \right). \]  

(10)

This gauge fixing functional has been previously considered in refs. [32, 33]. To motivate this choice we observe, that taking the imaginary part of the restriction

\[ \frac{D^2}{4} \Theta = \frac{1}{\sqrt{2}} \left( \Box C + D + \partial_5 \varphi - i \partial_M A^M \right) \]  

(11)

reveals that with the gauge fixing functional \( \Theta \) 5D Lorentz invariant gauge fixing like \( \partial_M A^M = 0 \) is incorporated. The gauge fixing condition \( \Theta = F \), with \( F \) an arbitrary chiral superfield, is implemented into the path integral via the standard procedure as the argument of a delta function together with a compensating Fadeev-Popov determinant \( \Delta(\Theta) \). One is free to include a Gaussian weighting factor \( \exp i \int d^5x d^4\theta \text{ tr } \bar{F} F \) and to perform functional integration over \( F \). Because of the delta functions, that implement the gauge fixing, this Gaussian integration is trivial and results in the gauge fixing action

\[ S_{gf} = -\int d^5x d^4\theta \text{ tr } \left[ \Theta \Theta \right] . \]  

(12)

Combining this gauge fixing action with (8) gives rise to invertible quadratic operators

\[ S_{V^2} + S_{gf} = \int d^5x d^4\theta \text{ tr } \left[ -V (\Box + \partial_5^2) V + \bar{S} \left( 1 + \frac{\partial_5^2}{\Box} \right) S \right] . \]  

(13)

Here we see a further motivation for the gauge fixing functional (10): The mixing between the \( V \) and the \( S \) and \( \bar{S} \) fields, which was present in (8), has been removed. Consequently, the propagators for \( V \) and \( S \) are decoupled

\[ S_{V^2} = \int d^5x d^4\theta \text{ tr } \left[ \frac{1}{4} J_V \bar{J}_V + \frac{1}{2} J_{\bar{S}} \bar{J}_S \right] . \]  

(14)
This decoupling amounts to a major simplification of the supergraph computations performed in later sections. Notice that the nonlocal term in \( \delta \) has given rise to a perfectly regular propagator for the superfield. The propagators are depicted in figure I. As observed above, the superfield \( S \) can be thought of as Goldstone boson superfield, therefore in this sense this gauge fixing is an application of the supersymmetric 't Hooft \( R \xi \) gauge \[46\].

To finish the description of the gauge fixing procedure, we rewrite the Faddeev-Popov determinant \( \Delta(\Theta) \) using ghosts as usual. In the supersymmetric setting the ghosts \( C \) and \( C' \) are anti-commuting chiral superfields. To determine their action we consider the infinitesimal supergauge variations of the fields \( V \) and \( S \)

\[
\delta \Lambda V = L_V (\Lambda - \overline{\Lambda}) + \coth (L_V) L_V (\Lambda + \overline{\Lambda}), \quad \delta \Lambda S = \sqrt{2} \partial_5 \Lambda + 2 [S, \Lambda],
\]

that are present in the gauge fixing functional \[10\]. Here \( L_V(X) = [V, X] \) denotes the Lie derivative. These variations determine the Fadeev-Popov determinant

\[
\Delta(\Theta) = \int D\Lambda D\Lambda' \exp \left( -\frac{i}{\sqrt{2}} \int d^5 x \left[ \int d^2 \theta \Lambda' \delta \Lambda \Theta + \int d^2 \bar{\theta} \bar{\Lambda}' \delta \bar{\Lambda} \bar{\Theta} \right] \right).
\]

The inverse of the Fadeev Popov determinant is obtained by replacing \( \Lambda \) and \( \Lambda' \) by the ghosts \( C \) and \( C' \), respectively. In this way \( \Delta(\Theta)^{-1} \) can be written as exponential of the ghost action

\[
S_{gh} = \frac{1}{\sqrt{2}} \int d^5 x d^4 \theta \text{ tr} \left[ \sqrt{2} (C' + \overline{C'}) (L_V (C - \overline{C}) + \coth (L_V) L_V (C + \overline{C})) \right.
\]

\[
\left. + C' \frac{\partial_5}{\Box} \left( \sqrt{2} \partial_5 \overline{C} - 2 [S, \overline{C}] \right) + C' \frac{\partial_5}{\Box} \left( \sqrt{2} \partial_5 C + 2 [S, C] \right) \right].
\]

From this action the ghost propagators can be read off easily

\[
S_{gh2} = \int d^5 x d^4 \theta \text{ tr} \left[ -\overline{J}_C \frac{1}{\Box + \partial_5^2} J_C - J'_C \frac{1}{\Box + \partial_5^2} \overline{J}_C \right].
\]

Notice that even though the (quadratic) action \[17\] appears to include non-local terms, the ghosts have perfectly normal 5D propagators. These propagators are given in figure I. Even though there are two types of propagators, we use only one notation for both of them, because the two propagators are the same.

This completes our description of the quantum field theory of hyper and vector multiplets in 5D. The vertices can be obtained straightforwardly by expanding the various actions and will not be given here. In section 4 we will only give those interaction terms that will be relevant for the computations performed there.

\[2.3\] The five dimensional orbifold \( S^1/\mathbb{Z}_2 \)

In the discussion so far we have only considered vector and hyper multiplets in 5D Minkowski space. We now turn to the situation where the fifth dimension is compactified on the orbifold \( S^1/\mathbb{Z}_2 \). As far as the perturbation theory is concerned we only need to reconsider the functional differentiation w.r.t. the sources \( J_\pm, J_V \) and \( J_S \). This naturally leads to the definition of orbifold compatible delta functions.
To describe the orbifold $S^1/Z_2$, we begin by defining the circle $S^1$ by the identifications
\[ y \sim y + \Lambda_W, \quad \Lambda_W = 2\pi R \mathbb{Z}, \quad (19) \]
where $\Lambda_W$ is the winding mode lattice. The length of the circle (the “volume” of a fundamental region of the lattice $\Lambda_W$) is equal to $\text{Vol}_W = 2\pi R$. We denote the delta function on the torus by $\delta(y) = \delta_R(y + \Lambda_W)$. The momentum in the fifth direction $p^5$ is quantized and takes values in the Kaluza-Klein lattice such that the 5D integral is defined as
\[ \int \frac{d^5p}{(2\pi)^5} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\pi R} \sum_{p^5 \in \Lambda_K}, \quad \Lambda_K = \mathbb{Z}/R. \quad (20) \]
The volume of the Kaluza-Klein lattice is given by $\text{Vol}_K = \frac{1}{R}$.

To construct the orbifold $S^1/Z_2$, we need to divide out a $Z_2$ point group. We implement the $Z_2$ action as a reflection $y \rightarrow -y$. This implies that the derivative in the extra dimension transforms as $\partial_5 \rightarrow -\partial_5$. The fundamental domain of the $S^1/Z_2$ orbifold is the interval $[0, \pi R]$. It has two fixed points located at $y = 0$ and $y = \pi R$. The delta function that peaks at these two fixed points is given by $\delta(2y)$ and can be expanded into
\[ \delta(2y) = \frac{1}{2} (\delta(y) + \delta(y - \pi R)). \quad (21) \]
The normalization using the number of fixed points, 2 for $S^1/Z_2$, ensures that the integral of this delta function over the circle is unity.

To describe the five dimensional hyper multiplet coupled to the gauge multiplet on this orbifold, the fields have to be orbifold compatible such that their action is invariant under the orbifold symmetry. This means that they must transform covariantly under the orbifold action
\[ \Phi_+ \rightarrow Z\Phi_+, \quad \Phi_- \rightarrow -\Phi_-Z, \quad V \rightarrow ZVZ, \quad S \rightarrow -ZSZ. \quad (22) \]
Such orbifold compatible (super)fields and sources can always be constructed by taking suitable linear combinations of the fields defined on the covering space and their $Z_2$ reflections. Invariance of the action implies that the transformation of the hyper and vector multiplets are encoded in a single unitary matrix $Z$. Because this is a $Z_2$ action, the matrix $Z$ fulfills $Z^2 = \mathbb{1}$. Hence $Z$ is a real symmetric matrix with the eigenvalues $\pm 1$. As it is often convenient to make the adjoint indices on $V$ and $S$ explicit, we introduce the matrix $Q^i_j$ to write the transformation rules for the $V$ and $S$ superfields as
\[ V^i \rightarrow Q^i_j V^j, \quad S_i \rightarrow -Q^i_j S^j, \quad Q^i_j = \text{tr}[T^i T^j Z] \quad (23) \]
The invariance of the action requires that the matrix $Q$ fulfills
\[ Q^i_v Q^j_{v'} \eta_{ij} = \eta_{v'v}, \quad f_{ijk} Q^i_{v} Q^j_{v'} Q^k_{v''} = f_{v'v''v}, \quad (24) \]
such that it is orthogonal with respect to the Killing metric $\eta_{ij}$. We infer that all matrix elements $Q^i_j$ are real. And due to the $Z_2$ symmetry we know that $Q^2 = \mathbb{1}$ and hence $Q$ is a real symmetric matrix. In the computation of the one loop self-energies, see section 4, we will be making frequent use of the properties of the matrices $Z$ and $Q$. 

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For such Feynman super graph computations using the path integral formalism it is important to know the orbifold compatible delta functions obtained by functional differentiation w.r.t. orbifold compatible sources:

\[
\frac{\delta J_{\pm 2h}}{\delta J_{\pm 1a}} = -\frac{1}{4} \bar{D}^2 \tilde{\delta}_{21}^{(\pm)a}_b, \quad \frac{\delta J_{-2}^b}{\delta J_{-1}^a} = -\frac{1}{4} \bar{D}^2 \tilde{\delta}_{21}^{(-)ab}, \quad \frac{\delta J_{V2}^i}{\delta J_{V1}^j} = \tilde{\delta}_{21}^{(V)i}_j, \quad \frac{\delta J_{S2}^i}{\delta J_{S1}^j} = -\frac{1}{4} \bar{D}^2 \tilde{\delta}_{21}^{(S)i}_j. \quad (25)
\]

Because (except for \(J_Y\)) all these sources are chiral, the functional differentiation w.r.t. them leads to chiral delta functions in superspace: \(-\bar{D}^2 \tilde{\delta}(\theta_2 - \theta_1)\). For later convenience we have defined the superspace orbifold compatible delta functions, indicated as \(\tilde{\delta}\), containing full Grassmann delta functions \(\delta(\theta_2 - \theta_1)\). As a consequence, the factor \(-\frac{1}{4} \bar{D}^2\) appears explicitly for the chiral sources in (25). The \(\mathbb{Z}_2\) properties of orbifold compatible fields imply that

\[
J_+ \to J_+ Z, \quad J_- \to -Z J_-, \quad J_{iV} \to Q^i_j J_{Vj}, \quad J_{Si} \to -Q^i_j J_{Sj}. \quad (26)
\]

where we have used the orthogonality of \(Q\) in (24). From the transformation properties of the sources we infer that the orbifold compatible delta functions are given by

\[
\begin{align*}
\tilde{\delta}_{21}^{(\pm)a}_b &= \frac{1}{2} \left( \delta^a_b \delta(y_2 - y_1) + Z^a_b \delta(y_2 + y_1) \right) \delta^4(x_2 - x_1) \delta^4(\theta_2 - \theta_1), \\
\tilde{\delta}_{21}^{(-)ab} &= \frac{1}{2} \left( \delta^b_a \delta(y_2 - y_1) - Z^b_a \delta(y_2 + y_1) \right) \delta^4(x_2 - x_1) \delta^4(\theta_2 - \theta_1), \\
\tilde{\delta}_{21}^{(V)i}_j &= \frac{1}{2} \left( \delta^i_j \delta(y_2 - y_1) + Q^i_j \delta(y_2 + y_1) \right) \delta^4(x_2 - x_1) \delta^4(\theta_2 - \theta_1), \\
\tilde{\delta}_{21}^{(S)i}_j &= \frac{1}{2} \left( \delta^i_j \delta(y_2 - y_1) - Q^i_j \delta(y_2 + y_1) \right) \delta^4(x_2 - x_1) \delta^4(\theta_2 - \theta_1).
\end{align*} \quad (27)
\]

These delta functions are the key elements of our formalism for calculating Feynman graphs directly on the orbifold, since they contain all the geometric information about the orbifold compatible superfields. Therefore, it is important to develop some of their properties: All delta functions are symmetric in their spacetime and gauge indices, while under a reflection of either \(y_1\) or \(y_2\) the delta functions transform as

\[
\begin{align*}
\tilde{\delta}_{21}^{(\pm)a}_b &\to \pm Z^a_a \tilde{\delta}_{21}^{(\pm)a}_b, & \tilde{\delta}_{21}^{(V)i}_j &\to Q^i_j \tilde{\delta}_{21}^{(V)i}_j, & \tilde{\delta}_{21}^{(S)i}_j &\to -Q^i_j \tilde{\delta}_{21}^{(S)i}_j.
\end{align*} \quad (28)
\]

In calculating amplitudes one often makes use of partial integration. But as the delta function is a function of two coordinates \((x_2, y_2)\) and \((x_1, y_1)\), one sometimes needs to change the coordinate w.r.t. which a derivative \(\partial_5\) acts before one can perform the partial integration. When this \(\partial_5\) acts on the delta function, the change of the coordinate may not only bring in a minus sign as one expects, but may also switch between the types of delta functions:

\[
(\partial_5)_2 \tilde{\delta}_{21}^{(\pm)a}_b = -(\partial_5)_1 \tilde{\delta}_{21}^{(\pm)a}_b, \quad (\partial_5)_2 \tilde{\delta}_{21}^{(V)i}_j = -(\partial_5)_1 \tilde{\delta}_{21}^{(V)i}_j, \quad (\partial_5)_2 \tilde{\delta}_{21}^{(S)i}_j = -(\partial_5)_1 \tilde{\delta}_{21}^{(S)i}_j. \quad (29)
\]

With this technology we are ready to perform supergraph computations on the 5D orbifold \(S^1/\mathbb{Z}_2\) in section 4. But before that we extend this discussion on the classical level to 6D and the orbifold \(T^2/\mathbb{Z}_2\).
3 Hyper and non-Abelian gauge multiplets in six dimensions

In this section we extend our 5D analysis of the previous section to 6D supersymmetric theories on $T^2/\mathbb{Z}_N$. As this is in principle straightforward and in order to avoid many repetitions, we only indicate where we encounter modifications. Most of these modifications have to do with the question, whether the 5D derivative $\partial_5$ has to be replaced by $\partial = \partial_5 + i\partial_6$ or $\bar{\partial} = \partial_5 - i\partial_6$. Here we employ complex coordinates $z = \frac{1}{2}(x_5 - ix_6)$ and $\bar{z} = \frac{1}{2}(x_5 + ix_6)$. To make these modifications easy to trace we use the same structure for this section as was employed in section 2. Since the properties of the orbifold $T^2/\mathbb{Z}_N$ are more complicated than those of $S^1/\mathbb{Z}_2$, we describe them more explicitly.

3.1 Classical hyper and gauge multiplet actions

The only terms in the hyper multiplet action (1) that are changed contain the 5D derivative operator $\partial_5$ and take the form:

$$S_H \supset \int d^6x \left[ \int d^2\theta \Phi_- \partial \Phi_+ - \int d^2\bar{\theta} \bar{\Phi}_+ \bar{\partial} \bar{\Phi}_- \right].$$

(30)

The supergauge transformations are the same as the 5D transformations (2) except for the superfield $S$, which transforms as

$$S \rightarrow e^{-2\Lambda} \left( S + \frac{1}{\sqrt{2}} \partial \right) e^{2\Lambda}.$$

(31)

Notice that in both these expressions the holomorphic derivative $\partial$ appears only in those places where chiral superfields are present.

For the vector multiplet the derivative term, i.e. the last term in (3), has to be modified to

$$S_V \supset \int d^6x \int d^4\theta \text{tr} \left[ \left( -\frac{1}{\sqrt{2}} \bar{\partial} + \hat{S} \right) e^{2V} \left( \frac{1}{\sqrt{2}} \partial + S \right) e^{-2V} + \frac{1}{4} \partial e^{-2V} \bar{\partial} e^{2V} \right].$$

(32)

Notice that in the 6D case it is not possible to represent this result in terms of a single gauge covariant superfield like the superfield $V_5$ defined in (4). In addition to this obvious modification a Wess-Zumino-Witten term has to be added in order to preserve the supergauge invariance [43].

3.2 Propagators, gauge fixing and ghosts

The propagators for the hyper multiplet in 6D,

$$S_{H2} = \int d^6x d^4\theta \left( J_+ J_- \right) \left( -\frac{1}{\Box + \partial} \frac{1}{\partial} \frac{\partial D^2}{4\Box} \frac{\partial D^2}{4\Box} \left( \frac{1}{\partial} \frac{\partial D^2}{4\Box} \right) \left( \frac{1}{\partial} \frac{\partial D^2}{4\Box} \right) \right),$$

(33)

are the direct generalization of the expressions given in (7). Only in those places where a single $\partial_5$ derivative appears, it is not automatically obvious if it has to be replaced by $\partial$ or $\bar{\partial}$.

For the vector multiplet in 6D the gauge fixing functional (10) is generalized to

$$\Theta = \frac{\bar{D}^2}{4} \left( \sqrt{2}V + \frac{1}{\Box} \partial S \right),$$

(34)
and the restriction to the highest component now yields

\[
\frac{D^2}{-4} \Theta = \frac{1}{\sqrt{2}} \left( \Box C + D + \partial_6 A_5 - \partial_5 A_6 + i \partial_M A^M \right).
\] 

(35)

Hence the imaginary part gives rise to a 6D Lorentz invariant gauge fixing for the vector field \(A_M\). Following the same computation for the gauge fixed propagators then gives rise to

\[
S_{V'} = \int d^6x d^4\theta \text{tr} \left[ \frac{1}{4} J_V - \frac{1}{\Box + \partial \partial} J_V + J_S \right].
\] 

(36)

Since in the 5D propagators (14) only \(\partial^2_5\) are present, this 6D results is precisely as expected.

Finally, in order to determine the ghost propagators in 6D we have to take into account the following modifications: The infinitesimal version of the 6D transformation law (31) for the superfield \(S\) reads

\[
\delta_{\Lambda} S = \sqrt{2} \partial \Lambda + 2 [S, \Lambda],
\] 

(37)

and requires the last two terms of the ghost action (17) to be modified to

\[
S_{\text{gh}} \supset \frac{1}{\sqrt{2}} \int d^6x d^4\theta \text{tr} \left[ C' \frac{\Box}{\partial} \left( \sqrt{2} \partial \partial C - 2 [S, C] \right) + C' \frac{\Box}{\partial} \left( \sqrt{2} \partial \partial C + 2 [S, C] \right) \right].
\] 

(38)

As for the vector multiplet, this leads to the obvious generalization of the ghost 5D propagators (18):

\[
S_{\text{gh}2} = \int d^6x d^4\theta \text{tr} \left[ - J_C \frac{1}{\Box + \partial \partial} J_C - J_C \frac{1}{\Box + \partial \partial} J_C \right].
\] 

(39)

Thus, we see that the propagators in 6D are to a large extent simple generalizations of the 5D propagators given in section 2. Therefore, we use the same conventions to draw the propagators in 6D as given in 10

### 3.3 The six dimensional orbifold \(T^2/\mathbb{Z}_N\)

Next we consider the compactification of the 6D multiplets on the orbifold \(T^2/\mathbb{Z}_N\). Because the torus \(T^2\) is compact, the only possible values for the orbifold order \(N\) are 2, 3, 4, 6, but we will keep our discussion general here. The torus \(T^2\) is defined by the identifications

\[
z \sim z + \Lambda_W, \quad \Lambda_W = \pi \left( R_1 \mathbb{Z} + R_2 e^{i\theta} \mathbb{Z} \right).
\] 

(40)

Here \(\Lambda_W\) denotes the winding mode lattice of the torus with the volume \(\text{Vol}_W = (2\pi)^2 R_1 R_2 \sin \theta\), where \(R_1\) and \(R_2\) are the radii of the torus and \(\theta\) defines its angle, i.e. \(\theta = \pi/2\) gives the square torus. Inspired by the string literature, we can introduce the complex structure modulus \(U\) and the Kähler modulus \(T\) of the torus

\[
\Lambda_W = \pi \sqrt{\frac{\text{Im} (T)}{\text{Im} (U)}} (\mathbb{Z} + U \mathbb{Z}), \quad U = \frac{R_2}{R_1} e^{i\theta}, \quad T = i R_1 R_2 \sin \theta.
\] 

(41)
In terms of these variables the volume of the torus reads $\text{Vol}_W = (2\pi)^2 \text{Im} (T)$. The momenta $p$ and $\bar{p}$ of the torus mode functions $\psi_p(z, \bar{z}) = e^{i(pz + \bar{p}\bar{z})}$ are quantized: $p$ lies on the Kaluza-Klein lattice $\Lambda_K$ (and $\bar{p}$ on the complex conjugate lattice). The 6D momentum integral is defined as

$$
\int \frac{d^6 p}{(2\pi)^6} = \int \frac{d^4 p}{(2\pi)^4} \left(\frac{1}{(2\pi)^2 \text{Im} (T)}\right) \sum_{p \in \Lambda_K} i \frac{i}{\sqrt{\text{Im} (T) \text{Im} (U)}} (2\pi)^2 \text{Im} (T) \sum_{p \in \Lambda_K} e^{i(pz + \bar{p}\bar{z})}.
$$

The volume of the Kaluza-Klein lattice is given by $\text{Vol}_K = \frac{1}{\text{Im} (T)}$.

To define the orbifold $T^2/\mathbb{Z}_N$, we implement the $\mathbb{Z}_N$ action of the orbifold group as $z \to e^{-i\varphi}z$, with $\varphi = \frac{2\pi}{N}$. Consequently, the holomorphic derivative $\partial$ transforms as $\partial \to e^{i\varphi} \partial$. The delta function, that peaks at the orbifold fixed points $z_f$, is given by

$$
\delta^2 ((1 - e^{i\varphi})z) = \frac{1}{4 \sin \frac{\varphi}{2}} \sum_f \delta^2 (z - z_f),
$$

in terms of the torus delta function $\delta^2(z)$. The factor $4 \sin \frac{\varphi}{2}$ equals the number of fixed points of the $T^2/\mathbb{Z}_N$ orbifold.

The hyper and gauge multiplets on the orbifold need to be covariant w.r.t. the $\mathbb{Z}_N$ orbifold action. Hence, their transformation behaviour under $z \to e^{-i\varphi}z$ is found to be

$$
\Phi_+ \to Z_+ \Phi_+, \quad \Phi_- \to \Phi_- Z_-, \quad V \to Z_+ V Z_+, \quad S \to Z_- S Z_+,
$$

with the properties $Z_+^N = Z_-^N = 1$, because the transformations are $\mathbb{Z}_N$ actions. Invariance of the action requires in addition that the matrices $Z_+$ and $Z_-$ be unitary are related to each other via: $Z_+ Z_- e^{i\varphi} = \mathbb{1}$. Therefore, we only need the matrix $Z_+$ in principle, however, it turns out to be convenient to keep using the notation $Z_{\pm}$. The transformation rules for the $V$ and $S$ superfields with the adjoint indices made explicit are given by

$$
V^i \to Q^i_j V^j, \quad S^i \to e^{+i\varphi} Q^i_j S^j, \quad Q^i_j = \text{tr}[T^i Z_+ T^j Z_+].
$$

This implies that all matrix elements $Q^i_j$ are real. Invariance of the action requires $Q$ to have the properties (24) and $Q^N = \mathbb{1}$ as it defines a $\mathbb{Z}_N$ action. The reduction to the $\mathbb{Z}_2$ orbifold group with $\varphi = \pi$ and $Z_+ = -Z_-$ is $Z$ is interesting, because then many of the properties of the 5D case, discussed in subsection 2.3, are recovered.

To obtain the orbifold compatible delta functions for the various superfields, we write down the transformation behaviour of orbifold compatible sources under $z \to e^{-i\varphi}z$

$$
J_+ \to J_+ Z_+^{-1}, \quad J_- \to Z_-^{-1} J_-, \quad J^i_+ \to Q^i_j J^j_+, \quad J^i_- \to e^{-i\varphi} Q^i_j J^j_-,
$$

where the orthogonality property of $Q$ in (24) has been used. This is also reflected in the orbifold
compatible delta functions for the $T^2/\mathbb{Z}_N$ orbifold

$$\tilde{\delta}_{21}^{(\pm)}_{ab} = \frac{1}{N} \sum_{b=0}^{N-1} \left[ Z^b \right] a_b \delta^2(z_2 - e^{i \phi} z_1) \delta^4(x_2 - x_1) \delta^4(\theta_2 - \theta_1),$$

$$\tilde{\delta}_{21}^{(-)}_{ab} = \frac{1}{N} \sum_{b=0}^{N-1} \left[ Z^b \right] b_a \delta^2(z_2 - e^{i \phi} z_1) \delta^4(x_2 - x_1) \delta^4(\theta_2 - \theta_1),$$

$$\tilde{\delta}_{21}^{(V)i}_{j} = \frac{1}{N} \sum_{b=0}^{N-1} \left[ Q^{-b} \right] i_j \delta^2(z_2 - e^{i \phi} z_1) \delta^4(x_2 - x_1) \delta^4(\theta_2 - \theta_1),$$

$$\tilde{\delta}_{21}^{(S)i}_{j} = \frac{1}{N} \sum_{b=0}^{N-1} e^{i \phi} \left[ Q^{-b} \right] i_j \delta^2(z_2 - e^{i \phi} z_1) \delta^4(x_2 - x_1) \delta^4(\theta_2 - \theta_1).$$

Under $z_2 \to e^{-i \phi} z_2$ these delta functions transform in the same way as the corresponding sources

$$\tilde{\delta}_{21}^{(\pm)}_{ab} \to \left[ Z_{\pm}^{-1} \right] a^a \tilde{\delta}_{21}^{(\pm)}_{ab}, \quad \tilde{\delta}_{21}^{(V)i}_{j} \to Q^{i'} \tilde{\delta}_{21}^{(V)j}_{i'}, \quad \tilde{\delta}_{21}^{(S)i}_{j} \to e^{-i \phi} Q^{i'} \tilde{\delta}_{21}^{(S)j}_{i'},$$

and under $z_1 \to e^{-i \phi} z_1$ inversely

$$\tilde{\delta}_{21}^{(\pm)}_{ab} \to \left[ Z_{\pm} \right] a^a \tilde{\delta}_{21}^{(\pm)}_{ab}, \quad \tilde{\delta}_{21}^{(V)i}_{j} \to \left[ Q^{-1} \right] i_j \tilde{\delta}_{21}^{(V)i}_{j'}, \quad \tilde{\delta}_{21}^{(S)i}_{j} \to e^{i \phi} \left[ Q^{-1} \right] i_j \tilde{\delta}_{21}^{(S)i}_{j}.$$

In contrast to the orbifold compatible delta functions [27] in 5D, these delta functions are no longer symmetric in their indices: The exchange of the spacetime labels results in

$$\tilde{\delta}_{12}^{(\pm)}_{ab} = \tilde{\delta}_{21}^{(\pm)}_{ab}, \quad \tilde{\delta}_{12}^{(V)i}_{j} = \tilde{\delta}_{21}^{(V)i}_{j}, \quad \tilde{\delta}_{12}^{(S)i}_{j} = \tilde{\delta}_{21}^{(S)i}_{j}.$$

because $Z_+$ and $Z_-$ are unitary and $Q$ is orthonormal. Derivatives with respect to the compactified coordinates always act on the $\delta^2(z_2 - e^{i \phi} z_1)$ factor. Therefore, changing a spacetime index of such a derivative also changes the type of delta function as

$$\partial_Z \tilde{\delta}_{21}^{(\pm)}_{ab} = -\partial_1 \tilde{\delta}_{21}^{(\pm)}_{ab}, \quad \partial_Z \tilde{\delta}_{21}^{(V)i}_{j} = -\partial_1 \tilde{\delta}_{21}^{(V)i}_{j}, \quad \partial_Z \tilde{\delta}_{21}^{(S)i}_{j} = -\partial_1 \tilde{\delta}_{21}^{(S)i}_{j}.$$

Notice that for the hyper multiplet delta functions also a complex conjugation is performed.

This completes the discussion of the supersymmetric field theory on the 6D orbifold $T^2/\mathbb{Z}_N$. We have seen, that even though many properties are very similar to the ones encountered for the $S^1/\mathbb{Z}_2$ orbifold discussed in subsection [23], there are also some important additional complications in the 6D case.

4 Quantum corrections in the 5D theory

This section is concerned with the calculation of the running of the gauge coupling of the 5D gauge multiplet due to vector and hyper multiplets on the 5D orbifold $S^1/\mathbb{Z}_2$. The classical action and the propagators were given in section [2]. Here we first write down the vertices, after that we evaluate the Feynman graphs that lead to a correction of the gauge coupling at one loop. The relevant vertices are obtained by expanding the action: To construct genuine self energy supergraphs we need three point interactions, and to generate tadpole (seagull) graphs four point vertices are required. Hence it is sufficient for us to expand the action to fourth order in the fields.
Figure II: These vertices encode the self interactions of the gauge multiplet involving the vector superfield $V$ and the chiral superfield $S$.

We perform these Feynman graph calculations directly on the orbifold with the help of our orbifold compatible delta functions (27) obtained in section 2. The combination of all these graphs can be divided into two types: One part of these amplitudes corresponds to bulk effects that are also present when the 5D theory is compactified on the circle $S^1$ rather than on the orbifold $S^1/Z_2$. The other part of the amplitudes is sourced by the orbifold fixed points. The divergent piece of the bulk amplitudes is proportional to the quadratic vector multiplet action (6) and therefore leads to the renormalization of the bulk gauge coupling. The divergent piece of the amplitude sourced by the fixed gives rise to renormalization of the gauge coupling at the 4D fixed points. In addition to this, the part of the bulk superfield $S$ that is not projected away at these fixed points receives wave function renormalization.

We calculate the divergences and determine the counter terms.

4.1 Gauge multiplet contributions to the vector multiplet self energy

In this subsection we compute the one loop vector multiplet self energy due to the vector multiplet self interactions. Because of the gauge fixing described in section 2.2 we encounter the superfields $V$, $S$ and the ghosts $C, C'$ in the loops. After describing the vertices we list the resulting diagrams.

Performing the expansion to fourth order in the gauge sector (3) leads to the following interactions

$$
\Delta S_V \supset \int d^5x\,d^4\theta\, \text{tr} \left[ \frac{1}{4} [V, D^\alpha V] \bar{D}^2 D_\alpha V - \frac{1}{8} [V, D^\alpha V] \bar{D}^2 [V, D_\alpha V] - \frac{1}{6} [V, [V, D^\alpha V]] \bar{D}^2 D_\alpha V + \\
+ \sqrt{2} \partial_5 V [V, \bar{S} - S] + 2S [V, \bar{S}] + \frac{1}{3} \partial_5 V [V, [V, \partial_5 V]] - \frac{2}{3} \sqrt{2} \partial_5 V [V, [V, S + \bar{S}]] + 2S [V, [V, \bar{S}]] \right].
$$

(52)

To indicate that we display only the terms of the expansion up to fourth order we use the notation “$\supset$” instead of “$=$”. In deriving (52) from (3) we have rewritten the (anti-)chiral superspace integrals into full superspace integration in the standard way. We use the convention that the derivative operator $\partial_5$ only acts on the field it is immediately adjacent to. The interaction vertices have been collected in figure II.

In the ghost sector we obtain the following interactions from the expansion of (17)

$$
\Delta S_{gh} \supset \int d^5x\,d^4\theta\, \text{tr} \left[ (C' + \bar{C}') [V, C - \bar{C}] + \sqrt{2} \partial_5 C' [\bar{S}, \bar{C}] - \sqrt{2} \partial_5 C' [S, C] + \\
+ \frac{1}{3} (C' + \bar{C}') [V, [V, C + \bar{C}]] \right].
$$

(53)
These vertices are depicted in figure III. One might worry about a possible non-locality of the interaction of a $V$ field with two ghosts $C'$ and $\bar{C}$ in (53), because the term contains a four dimensional d’Alembertian operator $\Box$ in the denominator. But such terms do not necessarily pose a problem, because physical amplitudes may also contain a bunch of supercovariant derivatives, which give rise to additional $\Box$ operators in the numerator so that cancellations can take place. In our calculation this issue does not arise at all, because it is impossible to construct one loop corrections to the $SS$ self energy with ghosts in the loop. The only graphs that could be constructed would be one loop contributions that are purely chiral, such that they vanish upon superspace integration.

The supergraphs for the gauge corrections due to gauge interactions consist of the $VV$, $SS$ and $VS$ self-energies, depicted in figures IV and V. In the first line of figure IV the genuine self energy graphs are labeled IV.A to IV.D. Because there are two ghost propagator diagrams, IV.D gives rise to four contributions. We will use this notation to refer to these supergraphs throughout the remainder of this paper. Similarly, we use the notation IV.E to IV.G to indicate the tadpole supergraphs in the second line. The contributions from these tadpole graphs are necessary to cancel non gauge invariant terms from the total amplitude. The first three graphs in figures V are the $S\bar{S}$ self energy diagrams. Finally, figure V.D gives the self energy due to the mixing between $S$ and $V$.

We have calculated all graphs directly on the orbifold. The results are given in appendix A as they appear in the more general calculation in 6D which we discuss in section 5; the reduction to the 5D case is straightforward. Before we turn to discuss the result of the full amplitude evaluated in the bulk and at the fixed points, we would like to illustrate the main steps that are required for the calculation of such supergraphs on orbifolds by considering one such graph in particular.

4.2 Example of supergraph computation on an orbifold: $VV$ self energy graph due to $S$ superfield

To illustrate the self energy computations on orbifolds, we have chosen the $VV$ self energy contribution due to the chiral superfield $S$ depicted in figure IV.A. As a supergraph it is quite simple and therefore we can focus on the special issues of computing diagrams on $S^1/\mathbb{Z}_2$. These techniques can easily be extended to 6D orbifolds like $T^2/\mathbb{Z}_N$.

The relevant $SV\bar{S}$ interaction term, given in (52), is used twice in diagram IV.A. To calculate this self energy graph the $S$ and $\bar{S}$ superfields are replaced by the corresponding sources that act on the exponential of the propagators (14). After functional derivations we obtain orbifold compatible delta functions (27) (indicated by the twiddles), so that the expression for the supergraph IV.A on
Figure IV: The gauge contributions to the $VV$ part of the gauge multiplet self energy are due to the $V$self coupling, the interactions with the chiral superfield $S$ and the ghost superfields $C$ and $C'$. In the first line the genuine self energy graphs are labeled IV.A to IV.D. The tadpole graphs on the second line are referred to as IV.E to IV.G.

the orbifold reads

$$
\text{IV A} = 2 f_{ijk} f_{\ell mn} \int (d^5 x d^4 \theta)_{1234} \ V_i V_j^\ell \ S_{31} \ j \ p \ D_2 D_2 \ \delta^{(S)} \ p' \ n \ \times
\delta^{(S)} \ q \ m \ \eta \ \eta' \ \delta^{(S)} \ q' \ k. \ (54)
$$

Here we have used that in the 5D case there is no distinction between the orbifold delta functions for $S$ and $\bar{S}$: $\tilde{\delta}(\bar{S}) = \tilde{\delta}(S)$. First we try to replace as many orbifold compatible delta functions by ordinary delta functions as possible. This is always possible for all but one delta function. The strategy to replace an orbifold delta function by an ordinary one is always the same: One expands the orbifold delta function into a sum and performs a substitution such that all the summands are equal.

For example, we can replace the first orbifold delta function in the final factor in the expression (54) for diagram IV.A. We begin by expanding the first delta function

$$
\text{IV A} = 2 f_{ijk} f_{\ell mn} \int (d^5 x d^4 \theta)_{1234} \ V_i V_j^\ell \ S_{31} \ j \ p \ D_2 D_2 \ \delta^{(S)} \ p' \ n \ \times
\delta^{(S)} \ q \ m \ \eta \ \eta' \ \delta^{(S)} \ q' \ k. \ (55)
$$

We perform the reflection $y_4 \rightarrow -y_4$ to show that

$$
-Q_q \ m \ \int dy_4 \ \delta(y_4 + y_2) \ \eta \ \eta' \ \delta^{(S)} \ q' \ k = \ \delta_q \ m \ \int dy_4 \ \delta(y_4 - y_2) \ \eta \ \eta' \ \delta^{(S)} \ q' \ k, \ (56)
$$

where we have used the transformation properties of $\tilde{\delta}^{(S)} \ q' \ k$ and the orthogonality of $Q$ in (24). Here we have not copied the propagators because they contain $\partial_5^2$ which is invariant under this reflec-
Here we used that both the transformation \( V \) and the \( \eta \)′ substitution this back into the original expression, we obtain

\[
\nabla A = 2 f_{ijk} f_{\ell mn} \int (d^5 x d^4 \theta)_{12345} V_i^j V_2^{\tilde{\eta}_{21}^{(S)}} \frac{\eta^m}{(\Box + \partial_5^2)^2} \frac{\tilde{D}_2^2 \tilde{D}_2^2}{16} \tilde{\eta}_{21}^{(S)} \delta_{21} \delta_{41}^{(S) m k}.
\]

Hence we have removed the orbifold projection on the first delta function.

In the same fashion we can remove one of the orbifold delta functions in the first factor. We choose to make the replacement

\[
\tilde{\eta}_{21}^{(S)} j \rightarrow \delta^i_p \delta (y_3 - y_1) \delta^4 (x_3 - x_1) \delta^4 (\theta_3 - \theta_1).
\]

Now we can integrate over \((x, \theta)_3\) and \((x, \theta)_4\) and are left with

\[
\nabla A = 2 f_{ijk} f_{\ell mn} \int (d^5 x d^4 \theta)_{12} V_i^j V_2^{\tilde{\eta}_{21}^{(S)}} \frac{1}{(\Box + \partial_5^2)^2} \frac{\tilde{D}_2^2 \tilde{D}_2^2}{16} \delta_{21}^{(S) m n} \delta_{21}^{(S) m k}.
\]

We can replace one more orbifold delta function. We choose to expand the second delta function

\[
\nabla A = 2 f_{ijk} f_{\ell mn} \int (d^5 x d^4 \theta)_{12} V_i^j V_2^{\tilde{\eta}_{21}^{(S)}} \frac{1}{(\Box + \partial_5^2)^2} \frac{\tilde{D}_2^2 \tilde{D}_2^2}{16} \delta_{21}^{(S) m n} \delta_{21}^{(S) m k} \times
\]

\[
\times \frac{1}{(\Box + \partial_5^2)^2} \frac{\tilde{D}_2^2 \tilde{D}_2^2}{16} \frac{1}{2} \left( \eta^{m k} \delta (y_2 - y_1) - Q^{m k} \delta (y_2 + y_1) \right) \delta^4 (x_2 - x_1) \delta^4 (\theta_2 - \theta_1).
\]

Performing the transformation \( y_1 \rightarrow -y_1 \) one shows that

\[
-f_{ijk} Q^{m k} \int dy_1 V_i^j \delta (y_2 + y_1) \tilde{\eta}_{21}^{(S) m n} = f_{ijk} \eta^{m k} \int dy_1 V_i^j \delta (y_2 - y_1) \tilde{\eta}_{21}^{(S) m n}.
\]

Here we used that both the transformation of \( V \) in (28) and of the orbifold compatible delta function in (29) bring in a matrix \( Q \). Then we applied the orthogonality property of \( Q \) in (24) in order to place the indices of all three \( Q \)’s alike. Subsequently, we took advantage of the fact that three \( Q \)'s contracted with the structure constants leave the structure constants invariant as found in (62). Thus, we find

\[
\nabla A = 2 f_{ijk} f_{\ell mn} \eta^{m k} \int (d^5 x d^4 \theta)_{12} V_i^j V_2^{\tilde{\eta}_{21}^{(S)}} \frac{1}{(\Box + \partial_5^2)^2} \frac{\tilde{D}_2^2 \tilde{D}_2^2}{16} \delta_{21}^{(S) m n} \delta_{21}^{(S) m k}.
\]

Hence we see that in this diagram we have been able to replace all but one orbifold compatible delta functions by ordinary delta functions. The final step in the evaluation of this diagram in the coordinate space representation is to make the expression local in the Grassmann variables. Making use of standard identities for the covariant supersymmetric derivatives, we perform the integration over \( \theta_2 \)

\[
\nabla A = f_{ijk} f_{\ell mn} \eta^{m k} \int (d^5 x)_{12} d^4 \theta \left[ - V_i^j \Box P_0 V_2^{\tilde{\eta}_{21}^{(S)}} \frac{1}{(\Box + \partial_5^2)^2} \frac{\tilde{D}_2^2 \tilde{D}_2^2}{16} \delta_{21}^{(S) m n} \frac{1}{(\Box + \partial_5^2)^2} \delta_{21}^{(S) m k} + 2 V_i^j V_2^{\tilde{\eta}_{21}^{(S)}} \frac{\Box^2}{(\Box + \partial_5^2)^2} \delta_{21}^{(S) m n} \frac{1}{(\Box + \partial_5^2)^2} \delta_{21}^{(S) m k} \right].
\]
Figure V: The $SS\bar{S}$ self energy graph is given in figure V.A. The mixing between the 4D superfields $V$ and $S$ corresponding to the third term of (6) is renormalized by the diagrams V.B and V.C. The last diagram has ghosts in the loop.

Since the expression only contains $\theta_1$, it is local in $\theta_1$ and we simply dropped the subscript “1” on $\theta$.

The structure of the calculation is the same in 5D and in 6D except for the fact that the orbifold compatible delta functions involve $N$ summands instead of two. The result for the 6D counterpart of the example calculation can be found in appendix A. One observes that the reduction of the 6D result to 5D is straightforward by making use of the fact that $\tilde{\delta}(\bar{S}) = \delta(S)$. Hence we refer to appendix A for the expressions for the other diagrams in figure IV.

We note that in this example it did not matter which of the two last orbifold delta functions we replaced, the result is the same. For some other diagrams that we have computed, however, the final result depends on which of the last two orbifold delta functions one replaces. As both possible forms are correct, one can use a linear combination of the two final expressions to make some cancellations explicit. This happens in particular if we encounter a $\tilde{\delta}(V)$ and a $\delta(S)$. For example we will see in section 4.3 that due to such a cancellation only a bulk contribution is left over in the supergraphs V. For this reason we have given the expressions for the other diagrams in appendix A and in the remainder of the chapter at the level of two orbifold compatible delta functions.

4.3 Vector multiplet renormalization due to self interactions

Now we turn to discuss the result of the combined amplitude of gauge multiplet self energy due to self interaction in the bulk and at the fixed points. This amplitude consists of four parts $\Sigma_{VV}$, $\Sigma_{VS}$, $\Sigma_{V\bar{S}}$ and $\Sigma_{\bar{S}S}$, because the vector multiplet is described by the 4D superfields $V$ and $S$.

The $VV$ self energy arises from the supergraphs IV. Using the results for these graphs given in appendix A, this self energy is found to be

$$\Sigma_{VV} = f_{ijk}f_{lmn} \int (d^5x)_{12} d^4\theta \left[ -3 V_1^i \Box_2 P_1 V_2^j \frac{1}{(\Box + \partial_5^2)_{21}} \tilde{\delta}_{21}^{5(V) mj} \frac{1}{(\Box + \partial_5^2)_{21}} \tilde{\delta}_{21}^{5(V) nk} + \frac{1}{(\Box + \partial_5^2)_{21}} \tilde{\delta}_{21}^{5(S) mj} \frac{1}{(\Box + \partial_5^2)_{21}} \tilde{\delta}_{21}^{5(S) nk} \right].$$

This expression still contains two orbifold compatible delta functions. In the first term in (64) it does not make a difference which orbifold delta function we replace by an ordinary delta function, because they are both of the same type: $\tilde{\delta}(V)$. As the remaining delta function $\tilde{\delta}(S)$ contains a sum, see (27), both bulk and fixed point contributions have the same sign. For the second term in (64), we conclude that the fixed point contributions have the opposite sign as compared to the bulk contribution. Only for the last term it makes a difference which delta function we reduce. To take
possible cancellations into account, we write the amplitude as half of the sum of both possibilities to reduce one delta function. Then the fixed point contribution of the last term in \(66\) vanishes, leaving only a bulk contribution.

For the same reason also the \(VS, V\bar{S}\) and \(S\bar{S}\) self energies, given in figure \(V\), only have a bulk contribution, because their two orbifold compatible delta functions expressions are given by

\[
\Sigma_{VS} = -2\sqrt{2}f_{ijk}f_{\ell mn} \int (d^5x)_{12} d^4\theta \partial_5 V_1^i S_2^\ell \frac{1}{(\Box + \partial_5^2)_{21}} \delta_{21}^{5(V) mj} \frac{1}{(\Box + \partial_5^2)_{21}} \delta_{21}^{5(S) nk}
\]

for \(V\bar{S}\), the complex conjugate for \(VS\), and

\[
\Sigma_{S\bar{S}} = 2f_{ijk}f_{\ell mn} \int (d^5x)_{12} d^4\theta S_1^i S_2^\ell \frac{1}{(\Box + \partial_5^2)_{21}} \delta_{21}^{5(V) mj} \frac{1}{(\Box + \partial_5^2)_{21}} \delta_{21}^{5(S) nk}
\]

for the \(S\bar{S}\) self energy.

By combining these results and expanding the final orbifold compatible delta functions according to their definitions in \(27\), we can identify the bulk and fixed point contributions. The bulk amplitude is obtained by taking their summation index \(b = 0\), and it can be expressed as

\[
\Sigma_{\text{bulk}}^\text{gauge} = f_{ijk}f_{\ell mn} \eta^{mj} \eta^{nk} \int (d^5x)_{12} d^4\theta \left[ -V_1^i \Box_2 P_0 V_2^\ell + \partial_5 V_1^i \partial_5 V_2^\ell - \sqrt{2}\partial_5 V_1^i (S_2^\ell + \bar{S}_2^\ell) + S_1^i S_2^\ell \right] \times \\
\times \frac{1}{(\Box + \partial_5^2)_{21}} \delta(y_2 - y_1) \frac{1}{(\Box + \partial_5^2)_{21}} \delta(y_2 - y_1).
\]

The \(\mathbb{Z}_2\) fixed point contributions of \(64\)-\(66\) are simply the \(b = 1\) terms in the expansion of the last orbifold delta functions. As we have explained above, the only non-vanishing self energy contribution at the fixed points is given by

\[
\Sigma_{\text{fp}}^{\text{gauge}} = 2f_{ijk}f_{\ell mn} \eta^{mj} Q^{nk} \int (d^5x)_{12} d^4\theta \left[ -V_1^i \Box_2 P_0 V_2^\ell \right] \frac{1}{(\Box + \partial_5^2)_{21}} \delta(y_2 - y_1) \frac{1}{(\Box + \partial_5^2)_{21}} \delta(y_2 + y_1).
\]

Both the bulk and the fixed point contributions are divergent and therefore need to be regularized and renormalized. In the next subsections we perform this task.

4.3.1 Bulk renormalization

We now compute the divergent bulk scalar integral corresponding to \(67\). Because we need to perform the same analysis in the 6D situation in section 4, we already employ a suitable notation which has a straightforward reduction to 5D.

The 5D bulk contribution \(67\) has the structure

\[
\mathcal{I}_D = \int (d^5x)_{12} A(x_1) B(x_2) \frac{1}{(\Box + \partial \partial^5 - m^2)_{21}} \delta_{21} \frac{1}{(\Box + \partial \partial - m^2)_{21}} \delta_{21},
\]

with \(m\) an infrared regulator mass. Here \(\delta_{21}\) denotes the delta function on the circle or the torus in the 5D and 6D case, respectively. (For our application here in five dimensions one replaces \(\partial \partial \rightarrow \partial_5^2\) and
uses $z = y, \bar{z} = 0$.) We insert a Fourier transformation \[^{B.1}\] to represent this integral in momentum space as

$$I_D = \frac{1}{2^2} \int \frac{d^d p}{(2\pi)^d} \sum_{n \in \Lambda_K} \int \frac{1}{Vol_W} p^2 + |m|^2 + m^2 \left( p - k \right)^2 + |n - l|^2 + m^2.$$  \quad (70)

where Vol$_W$ is the volume of the circle or the torus in the 5D and 6D cases, respectively. The $\mu$ dependence is a result of our Fourier transformation conventions \[^{B}\]. Here $k$ is the continuous external momentum in 4D and $n$ the discrete Kaluza-Klein momentum in the extra dimensions. In order to find the counter terms, we need to calculate the divergent part of

$$I_D |_{\text{div}} = i\alpha_1 + i\alpha_2 (k^2 + |l|^2).$$  \quad (72)

In 5D the second term is not present, i.e.

$$\alpha_1 = -\frac{1}{(4\pi)^2} |m|, \quad \alpha_2 = 0,$$  \quad (73)

while in 6D we find

$$\alpha_1 = \frac{1}{(4\pi)^3} \left[ \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right) m^2 + m^2 \right], \quad \alpha_2 = \frac{1}{6 (4\pi)^3} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right),$$  \quad (74)

where $\frac{1}{\epsilon} = \frac{1}{\epsilon} - \gamma + \ln 4\pi$. In 6D the second term in \[^{72}\] is present because $\alpha_2 \neq 0$, and it requires the introduction of a higher dimensional operator in the action. Transforming back into position space we obtain the local terms

$$I_D |_{\text{div}} = i \int d^D x \left[ \alpha_1 A(x) B(x) - \alpha_2 A(x) \left( \Box + \partial\bar{\partial} \right) B(x) \right].$$  \quad (75)

Turning back to the 5D case, we find that the local one loop counterterms which cancel these divergencies read

$$S_{\text{bulk}}^{\text{gauge}} = -\frac{1}{(4\pi)^2} |m| \int d^5 x \, d^4 \theta \, \text{tr}_{\mathbf{Ad}} \left[ -V \Box P_0 V + \partial_5 V \partial_5 V - \sqrt{2} \partial_5 V (S + \bar{S}) + S \bar{S} \right].$$  \quad (76)

### 4.3.2 Fixed points renormalization

Next we discuss the renormalization at the fixed points, starting from \[^{68}\]. As in the previous section \[^{4.3.1}\] we perform the discussion such that it can be applied in both 5D and 6D. The structure of \[^{68}\] is

$$J_D = \int (d^D x)_{12} A(x_1) B(x_2) \frac{1}{(\Box + \partial\bar{\partial} - m^2)^2} \delta(z_2 - e^{ik\varphi} z_1) \frac{1}{(\Box + \partial\bar{\partial} - m^2)^2} \delta(z_2 - z_1),$$  \quad (77)
Figure VI: These interaction vertices involve the coupling of the gauge superfields $V$ and $S$ to the hyper multiplet chiral superfields $\Phi_+$ and $\Phi_-$. 

with obvious reduction to five dimensions, and where in the delta function only the compact dimensions have been indicated for notational simplicity. In momentum space

$$J_D = \frac{1}{2^2} \int \frac{d^4k}{(2\pi\mu)^2d} \sum_{\ell_1,\ell_2} (2\pi)^d A(k, e^{ik\ell_1} + \ell_2) B(-k, -\ell_1 - \ell_2) J_0.$$  

(78)

The divergence is due to the 4D integral

$$J_0 = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + |\ell_1|^2 + m^2} \frac{1}{(p-k)^2 + |\ell_2|^2 + m^2},$$  

(79)

which is calculated in (D.9). One obtains after the transformation into position space

$$J_D|_{\text{div}} = \frac{i}{(4\pi)^2} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right) \int d^5 x (d^2 z)_{12} A(x, z_1) B(x, z_2) \delta^2(z_2 - e^{ik\varphi} z_1) \delta^2(z_2 - z_1).$$  

(80)

This expression is local in the uncompactified 4D directions. In the compactified dimensions, it is localized on the fixed point, because of the two delta functions with the two different arguments. We apply the result to (68) in order to find the counter terms that cancel the divergencies on the fixed points

$$S_{\text{gauge}}^{\text{fp}} = \frac{-2}{(4\pi)^2} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right) \int d^5 x d^4 \theta \text{tr}_{\text{Ad}} \left[ QV(x,y) \Box P_0 V(x,y) \right] \delta(2y).$$  

(81)

4.4 Vector multiplet renormalization due to a hyper multiplet

The calculation of the hyper multiplet contributions to the vector multiplet self energy is similar to the computation of the corrections from the gauge sector. For the hyper multiplet action (I) the expansion to fourth order gives rise to the interactions

$$\Delta S_H \supset \int d^5 x \left[ \int d^4 \theta \tilde{\Phi}_\pm (\pm 2V + 2V^2) \Phi_\pm + \int d^2 \theta \sqrt{2} \Phi_+ S \tilde{\Phi}_+ + \int d^2 \theta \sqrt{2} \tilde{\Phi}_+ S \tilde{\Phi}_- \right].$$  

(82)

We have depicted the corresponding vertices in figure VII.

The supergraphs with the hyper multiplet in the loop are depicted in figure VII. The graph VII.A contains the propagators that connect the chiral sources $J_\pm$ with the anti-chiral sources $\tilde{J}_\pm$, while diagram VII.B involves the chiral sources $J_+$ and $J_-$. This diagram also has a Hermitian conjugate partner, which we refer to as VII.B. The tadpole graphs are the final diagrams VII.C.
The VV self-energy takes the form
\[ \Sigma_{VV} = \int (d^5x)_{12} d^4\theta \text{tr} \left[ -V_1 \frac{1}{(\Box + \partial^2_5)} \delta^{5(+)}_{21} \Box_2 P_0 V_2 \frac{1}{(\Box + \partial^2_5)} \delta^{5(+)}_{21} + \right. \\
\left. - V_1 \frac{1}{(\Box + \partial^2_5)} \delta^{5(-)}_{21} \Box_2 P_0 V_2 \frac{1}{(\Box + \partial^2_5)} \delta^{5(-)}_{21} + 2 \partial_5 V_1 \frac{1}{(\Box + \partial^2_5)} \delta^{5(+)}_{21} \partial_5 V_2 \frac{1}{(\Box + \partial^2_5)} \delta^{5(-)}_{21} \right]. \quad (83) \]

The VS and SS self-energies are given by
\[ \Sigma_{VS} = -2\sqrt{2} \int (d^5x)_{12} d^4\theta \text{tr} \left[ \partial_5 V_1 \frac{1}{(\Box + \partial^2_5)} \delta^{5(+)}_{21} \bar{S}_2 \frac{1}{(\Box + \partial^2_5)} \delta^{5(-)}_{21} \right], \quad (84) \]
\[ \Sigma_{SS} = 2 \int (d^5x)_{12} d^4\theta \text{tr} \left[ S_1 \frac{1}{(\Box + \partial^2_5)} \delta^{5(+)}_{21} S_2 \frac{1}{(\Box + \partial^2_5)} \delta^{5(-)}_{21} \right], \quad (85) \]

where the VS term is just the complex conjugate of the result for VS. The corresponding diagrams are given in figures VII. Here it is interesting that it does not make a difference which last delta function is removed. This is in contrast to the self-energy results from the gauge sector where one had to be careful not to miss important cancellations.

The bulk amplitude is found by replacing one more orbifold delta function, expanding the remainder and taking the b = 0 contribution
\[ \Sigma_{\text{hyper}}^{\text{bulk}} = \int (d^5x)_{12} d^4\theta \text{tr} \left[ -V_1 \Box_2 P_0 V_2 + \partial_5 V_1 \partial_5 V_2 - \sqrt{2} \partial_5 V_1 (S_2 + \bar{S}_2) + S_1 \bar{S}_2 \right] \times \frac{1}{(\Box + \partial^2_5)} \delta(y_2 - y_1) \frac{1}{(\Box + \partial^2_5)} \delta(y_2 - y_1). \quad (86) \]

By adding the contributions with one of the two orbifold delta functions removed we find that at the fixed points
\[ \Sigma_{\text{hyper}}^{\text{fp}} = \frac{1}{2} \int (d^5x)_{12} d^4\theta \text{tr} \left\{ \left[ \partial_5 V_1, Z \right] \partial_5 V_2 - \sqrt{2} \left[ \partial_5 V_1, Z \right] (S_2 - \bar{S}_2) + [S_1, Z] \bar{S}_2 \right\} \times \frac{1}{(\Box + \partial^2_5)} \delta(y_2 - y_1) \frac{1}{(\Box + \partial^2_5)} \delta(y_2 + y_1). \quad (87) \]

This shows that in the case when Z is proportional to the identity and in the Abelian case the amplitude vanishes at the fixed points.
Figure VIII: The gauge multiplet receives $\bar{S}S$ self energy corrections from the hyper multiplet as is depicted in figure VIII A±. In addition the hyper multiplet gives rise to mixing between the 4D superfields $V$ and $S$, see VIII B±.

4.4.1 Bulk renormalization

As in subsection 4.3.1 we can extract the divergent parts and determine the counter terms, which leads to

$$S_{\text{bulk}}^{\text{hyper}} = \frac{1}{(4\pi)^2} |m| \int d^5 x d^4 \theta \text{tr} \left[ -V \Box P_0 V + \partial_5 V \partial_5 V - \sqrt{2} \partial_5 V (S + \bar{S}) + S \bar{S} \right]$$

(88)

for the correction due to the hyper multiplet.

4.4.2 Fixed Points renormalization

At the fixed points we can write the counter terms as

$$S_{\text{fp}}^{\text{hyper}} = -\frac{1}{2(4\pi)^2} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right) \int d^5 x d^4 \theta \text{tr} Z \left[ \frac{\bar{S} - \sqrt{2} \partial_5 V}{\delta(2y)} \right]$$

(89)

after we have extracted the 4D divergent parts. As we saw at the level of the amplitude, in the Abelian case the hyper multiplet does not induce a correction at the fixed points. The $(\partial_5 V)^2$ parts of this expression have been obtained before, see [47].

Moreover, note that this expression is not gauge invariant. The non-linear extension

$$S_{\text{fp n.l. ext}}^{\text{hyper}} = -\frac{1}{2(4\pi)^2} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right) \int d^5 x d^4 \theta \text{tr} Z \left[ \left( \frac{\bar{S} - \frac{1}{\sqrt{2}} \partial_5}{\delta(2y)} \right) e^{2V}, \left( S + \frac{1}{\sqrt{2}} \partial_5 \right) e^{-2V} \right]$$

(90)

is gauge invariant w.r.t. the zero mode supergauge group, which is defined by $\partial_5 \Lambda = \partial_5 \bar{\Lambda} = 0$ and $[Z, \Lambda] = [Z, \bar{\Lambda}] = 0$. The second condition is a consequence of the orbifold projection at the $\mathbb{Z}_2$ fixed points. However, for the full supergauge group at the fixed points $\partial_5 \Lambda$ and $\partial_5 \bar{\Lambda}$ do not necessarily vanish and do not commute with $Z$. Consequently this expression, as it stands, is not gauge invariant under the full bulk gauge transformations. As we will speculate below this might be cured by a Wess-Zumino-Witten-like term.

5 Quantum corrections in the 6D theory

The approach to calculate the gauge coupling running on the 6D orbifold $T^2/\mathbb{Z}_N$ parallels the procedure described in 5D. Therefore, it will suffice for us to indicate the points that deviate from our treatment in the preceding chapter.
5.1 Gauge multiplet contributions to the vector multiplet self energy

The following gauge field self interactions (52) change, because they contain the 5D derivative operator \( \partial_5 \)

\[
\Delta S_V \supset \int d^6x \ d^4\theta \frac{1}{2} \left[ \sqrt{2} \partial V[V, \bar{S}] - \sqrt{2} \bar{\partial} V[V, S] + \frac{1}{3} \right. \\
+ \left. \frac{2}{3} \sqrt{2} \partial V[V, \partial V] - \frac{2}{3} \sqrt{2} \bar{\partial} V[V, [V, S]]\right] 
\]

and the interaction in the ghost sector (53) changes as

\[
\Delta S_{gh} \supset \int d^6x \ d^4\theta \frac{1}{2} \left[ \sqrt{2} \nabla C'[\bar{S}, \bar{C}] - \sqrt{2} \bar{\nabla} C'[S, C]\right].
\]

We already mentioned above that in 6D gauge invariance requires the presence of an additional WZW term for the gauge multiplet \( V \). This term leads in principle to a three point gauge field self interaction. However, it turns out that all graphs that can be constructed with this additional interaction add up to zero because of the symmetry of the structure constants. Thus, for our calculation, in 6D we are left with the same set of relevant graphs as in the 5D situation.

5.2 Vector multiplet renormalization due to self interactions

The resulting expressions for the amplitudes are consequently also very similar to the ones given in the 5D case on the orbifold \( S^1/\mathbb{Z}_2 \) which we discussed in section [44]. There are of course the obvious modifications of the dimensionality of the integration measure and \( \partial^2 \rightarrow \bar{\partial} \partial \). In particular, the effects of the vector multiplet self interaction given in section [34] are modified as follows: The \( \Sigma_{VV} \) self energy is the same as in (64) except for the term that involves \( \partial_5 \) derivatives w.r.t. the fifth dimension. That term is modified to

\[
\Sigma_{VV} \supset f_{ijk} f_{\ell mn} \int (d^6x)_{12} d^4\theta \left[ \frac{2}{3} \frac{1}{(\partial + \bar{\partial})_2} \delta_{21}^{6(V) mj} \frac{1}{(\partial + \bar{\partial})_2} \delta_{21}^{6(S) nk} \right].
\]

Also the mixing between \( V \) and \( S \) given by the amplitude (65) involves a derivative. In the 6D case it reads

\[
\Sigma_{V\bar{S}} \supset -2\sqrt{2} f_{ijk} f_{\ell mn} \int (d^6x)_{12} d^4\theta \partial V_i \bar{S}_j \delta_{21}^{5(V) mj} \frac{1}{(\partial + \bar{\partial})_2} \delta_{21}^{5(S) nk}.
\]

The amplitude \( \Sigma_{\bar{S}\bar{S}} \) does not involve any single \( \partial_5 \), so that its generalization to 6D is obvious.
5.2.1 Bulk renormalization

Taking these modifications into account we find the following expression for the counter terms in the bulk

\[
S_{\text{gauge}}^{\text{bulk}} = \frac{2m^2}{(4\pi)^3 N} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} + 1 \right) \int d^6 x d^4 \theta \text{tr}_A \left[ -V \Box P_0 V + \partial V \bar{\partial} V - \sqrt{2} (\partial V \bar{S} + \bar{\partial} V S ) + S \bar{S} \right] + \\
- \frac{1}{3} \frac{1}{(4\pi)^3 N} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right) \int d^6 x d^4 \theta \text{tr}_A \left[ -V \Box P_0 (\Box + \partial \bar{\partial}) V + \partial V (\Box + \partial \bar{\partial}) \bar{\partial} V + \\
- \sqrt{2} \partial V (\Box + \partial \bar{\partial}) \bar{S} - \sqrt{2} \bar{\partial} V (\Box + \partial \bar{\partial}) S + S (\Box + \partial \bar{\partial}) \bar{S} \right] . \tag{95}
\]

We note that the expression in the second line is the quadratic approximation of the kinetic action of the vector multiplet, see (6). The remaining part of this expression corresponds to the renormalization of the quadratic approximation of the higher derivative term.

We have also encountered these effects in the Abelian case in 5D and 6D, which we studied in [18]. By gauge invariance we can infer some additional effects. As we reminded the reader below (32), the action is not gauge invariant unless also a Wess-Zumino-Witten term is added [43]. Therefore, to preserve gauge invariance, also this Wess-Zumino-Witten term has to be renormalized. Moreover, because also a higher derivative operator is generated, also a higher derivative analogue of the Wess-Zumino-Witten term must exist and renormalize. We have not performed an explicit calculation to confirm the renormalization. However, we can say that the Wess-Zumino-Witten term and its higher derivative counterpart will have to renormalize with the same multiplicative coefficients as the corresponding terms in the quadratic part of the action in order for the theory to be gauge invariant at the one-loop level.

5.2.2 $Z_2$ fixed point renormalization

Even ordered orbifolds have fixed points which are invariant under a $Z_2$ symmetry, where special cancellations take place that are not present at the other fixed points. For these $Z_2$ fixed points we find the counter term

\[
S_{Z_2}^{\text{gauge}} = \frac{-4}{(4\pi)^2 N} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right) \int d^6 x d^4 \theta \text{tr}_A \left[ -Q^{N/2} V \Box P_0 V \right] \delta^2(2z) . \tag{96}
\]
5.2.3 Non-$Z_2$ fixed point renormalization

At the non-$Z_2$ fixed points we obtain instead the result

$$S_{\text{gauge non-$Z_2$}} = \frac{1}{(4\pi)^2 N} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right)^{[N/2]_s} \sum_{b=1}^{[N/2]_s} \int d^6 x \, d^4 \theta \text{tr} A_d \left\{ \begin{array}{l}
(6 \cos bH - \cos b(H + \varphi) - \cos b(H - \varphi)) (-V \Box P_0 V) + \\
+ 2 \cos b(H + \varphi) \left( \partial V \bar{\partial} V - \sqrt{2} \partial V S - \sqrt{2} S \bar{\partial} V + SS \right) + \\
+ 2 \cos bH \left( \partial V \bar{\partial} V - \sqrt{2} \partial V \bar{S} - \sqrt{2} S \bar{\partial} V + SS \right) \right\} \delta^2 \left( 1 - e^{ib\varphi} \right) z, \end{array} \right.$$  

(97)

where we introduced the hermitean matrix $H$ via $Q = e^{iH}$. To arrive at this expression we have used that the matrices $\cos bH$, etc. are symmetric, which is a consequence of the fact that $Q$ is orthogonal.

The symbol $[N/2]_s$ is defined as $[N/2]_s = \frac{N}{2}$ for $N$ even and $[N/2]_s = \frac{N-1}{2}$ for $N$ odd. Because we have only computed a two point function, this expression for the one loop counterterm is clearly not gauge invariant. Inspired by the expression (32), we expect that the non-linear form of (97) is given by

$$S_{\text{gauge non-$Z_2$ n.l. ext}} = \frac{1}{(4\pi)^2 N} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right)^{[N/2]_s} \sum_{b=1}^{[N/2]_s} \int d^6 x \, \text{tr} A_d \left\{ \begin{array}{l}
(6 \cos bH - \cos b(H + \varphi) - \cos b(H - \varphi)) \left( \frac{1}{4} \int d^2 \theta W^a W_a + \frac{1}{4} \int d^2 \bar{\theta} \bar{W}^\dot{a} \bar{W}^{\dot{a}} \right) + \\
+ 2 \cos b(H + \varphi) \left[ \left( \frac{1}{\sqrt{2}} \partial + S \right) e^{-2V} \left( - \frac{1}{\sqrt{2}} \bar{\partial} + \bar{S} \right) e^{2V} + \frac{1}{4} \partial e^{-2V} \bar{\partial} e^{2V} \right] + \\
+ 2 \cos bH \left[ \left( - \frac{1}{\sqrt{2}} \bar{\partial} + \bar{S} \right) e^{2V} \left( \frac{1}{\sqrt{2}} \partial + S \right) e^{-2V} + \frac{1}{4} \bar{\partial} e^{2V} \partial e^{-2V} \right] \right\} \delta^2 \left( 1 - e^{ib\varphi} \right) z. \end{array} \right.$$  

(98)

This is expression is gauge invariant under the zero mode supergauge group, as defined below (30). However, as was discussed there also this term is not gauge invariant under the full supergauge transformations. This is not surprising when one takes into account that (32) is also not gauge invariant by itself: One needs to add a Wess-Zumino-Witten term to make the theory gauge invariant. Therefore we expect that also the expression above can be made gauge invariant by adding a suitable extension of a Wess-Zumino-Witten interaction.

5.3 Vector multiplet renormalization due to a hyper multiplet

The expansion of the hyper multiplet action to fourth order in the gauge coupling (82) remains valid in 6D. The self energies $\Sigma_{VV}$, $\Sigma_{VS}$, $\Sigma_{VS}$ and $\Sigma_{SS}$ are the same as in 5D except for the changes of the derivative operator in the quadratic part of the vector multiplet action which leads to the following
replacements in (83) and (84)

\[
\Sigma_{VV} \supset \int (d^6x) d^4 \theta \text{tr} \left[ 2 \partial V_1 \frac{1}{(\square + \partial \bar{\partial})_2} \delta_{21}^{(+)} \partial V_2 \frac{1}{(\square + \partial \bar{\partial})_2} \delta_{21}^{(-)} \right],
\]

(99)

\[
\Sigma_{VS} = -2\sqrt{2} \int (d^6x) d^4 \theta \text{tr} \left[ \partial V_1 \frac{1}{(\square + \partial \bar{\partial})_2} \tilde{\delta}_{21}^{(+)} \bar{S}_2 \frac{1}{(\square + \partial \bar{\partial})_2} \tilde{\delta}_{21}^{(-)} \right],
\]

(100)

while (85) stays the same in 6D. After the reduction of one more orbifold projection the bulk amplitude for \( b = 0 \) and the amplitude at the \( \mathbb{Z}_2 \) fixed points of an even ordered orbifold for \( b = N/2 \) are calculated straightforwardly as in 5D.

### 5.3.1 Bulk renormalization

We extract the divergence and determine the local bulk counter term in 6D

\[
S_{\text{bulk}}^{\text{hyper}} = \frac{-2m^2}{(4\pi)^3 N} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} + 1 \right) \int d^6x d^4 \theta \text{tr} \left[ -V \square P_0 V + \partial V \partial V - \sqrt{2} (\partial V \bar{S} + \bar{V} S) + S \bar{S} \right] +
\]

\[
+ \frac{1}{3} \frac{1}{(4\pi)^3 N} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right) \int d^6x d^4 \theta \text{tr} \left[ -V \square P_0 (\square + \partial \bar{\partial}) V + \partial V (\square + \partial \bar{\partial}) \partial V +
\]

\[- \sqrt{2} \partial V (\square + \partial \bar{\partial}) \bar{S} - \sqrt{2} \partial V (\square + \partial \bar{\partial}) S + S (\square + \partial \bar{\partial}) \bar{S} \right].
\]

(101)

In the second and third lines we find the higher dimensional operator which we already alluded to above equation (75).

### 5.3.2 \( \mathbb{Z}_2 \) fixed point renormalization

The following counter term is located at the \( \mathbb{Z}_2 \) fixed points of an even ordered orbifold

\[
S_{\mathbb{Z}_2}^{\text{hyper}} = \frac{-1}{(4\pi)^3 N} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right) \int d^6x d^4 \theta \text{tr} Z_+^{N/2} \left[ [\bar{S} - \sqrt{2} \partial V, S - \sqrt{2} \partial V] - [\partial V, \partial V] \right] \delta^2(2z).
\]

(102)

We note that we recover the factors of the quadratic contribution of (32) enclosed in commutators. As discussed below (76) it is possible to construct a non-linear extension of this term which is invariant under the zero mode gauge group, but such an expression is not gauge invariant under the full bulk gauge transformations.

### 5.3.3 Non-\( \mathbb{Z}_2 \) fixed point renormalization

The counter term at the non-\( \mathbb{Z}_2 \) fixed points involves the delta function \( \delta^2((1 - e^{ib\phi})z) \) which is symmetric under a reflection of \( b \). By summing the contributions to \( b \) and \(-b\) explicitly and introducing the algebra element \( A_+ \) that corresponds to the unitary matrix \( Z_+ \equiv e^{iA_+} \) the local counter
term can be written as

\[ S_{\text{non-Z}_2}^{\text{hyper}} = \frac{-2}{(4\pi)^2 N} \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right) \sum_{b=1}^{[N/2]} \int d^6x \ d^4\theta \ \text{tr} \left[ - (\cos b(A_+ + \varphi) + \cos bA_+) V \Box P_o V + \right. \\
+ \cos b(A_+ + \varphi)(\bar{\partial}V \bar{\partial}V - \sqrt{2} \partial \bar{\partial}V S - \sqrt{2} \bar{\partial}V \bar{S} + S \bar{S}) \\
+ \cos (bA_+)(\bar{\partial}V \bar{\partial}V - \sqrt{2} \bar{\partial}V \bar{S} - \sqrt{2} \bar{\partial}V S + S \bar{S}) \right] \delta^2((1 - e^{ib\varphi}) z). \] (103)

By formally replacing the matrix \( A_+ \) by a scalar \( a_+ \) and the trace \( \text{tr} \) by the square of the charge \( q^2 \) one obtains the abelian result found previously in [13]. Here we can make the same comments about non-linear extensions and gauge invariance as below (97).

6 Some examples as cross checks

In this section we would like to give a few illustrative applications of the general formulae for the gauge couplings discussed in this paper. In addition, we use these examples to perform some simple cross checks of our general results. These checks are inspired by the discussions in [14] to determine the fixed point beta functions using zero mode results on orbifolds and their covering spaces.

6.1 Hyper multiplet on \( S^1/\mathbb{Z}_2 \)

We consider an SU(N) supersymmetric gauge theory in 5D on \( S^1/\mathbb{Z}_2 \). Besides the vector multiplet we take a single hyper multiplet in the fundamental representation of SU(N). The matrix \( Z \) that defines the orbifold action (22) can be chosen to be diagonal

\[ Z = \begin{pmatrix} \mathbb{I}_{N_0} & 0 \\ 0 & -\mathbb{I}_{N_1} \end{pmatrix}, \] (104)

where \( N_0 + N_1 = N \). At the fixed points and in the effective 4D theory the gauge symmetry is then broken to

\[ \text{SU}(N) \rightarrow \text{SU}(N_0) \times \text{SU}(N_1) \times \text{U}(1). \] (105)

To compute the bulk and fixed point gauge coupling renormalization we can use the results given in section 1.4. In particular we found in subsection 1.4.2 that there is no gauge coupling renormalization at the fixed points.

We now present a way to check this result by comparing it with the results for the zero modes on the circle \( S^1 \) and the orbifold \( S^1/\mathbb{Z}_2 \). First of all the renormalization of the gauge couplings can be directly computed by considering the zero mode spectrum. For the hyper multiplet we can write

\[ \Phi_+ = \begin{pmatrix} \Phi_{+0} \\ \Phi_{+1} \end{pmatrix}, \quad \Phi_- = \begin{pmatrix} \Phi_{-0} \\ \Phi_{-1} \end{pmatrix}. \] (106)

These components transform under the orbifold action as

\[ \Phi_{+I} \rightarrow (-)^I \Phi_{+I}, \quad \Phi_{-I} \rightarrow (-)^{I+1} \Phi_{-I}, \] (107)
with $I = 0, 1$, hence the zero modes $\Phi_{+0}$ and $\Phi_{-1}$ form $N_0$ and $N_1$ representations of the zero mode group $\text{SU}(N_0) \times \text{SU}(N_1)$, respectively. Since these representations are the fundamental and anti-fundamental of these gauge groups, we obtain by using the standard beta-functions for super Yang-Mills theories

\[
\frac{1}{g_0^2(\mu)} = \frac{1}{g_0^2} - \frac{1}{16\pi^2} \ln \mu^2, \quad \frac{1}{g_1^2(\mu)} = \frac{1}{g_1^2} - \frac{1}{16\pi^2} \ln \mu^2.
\]  

(108)

Here we denote with $g_{0/1}^2$ the coupling at scale $\mu = 1$, and $g_{0/1}^2(\mu)$ the coupling at scale $\mu$. In our discussion we disregard the gauge coupling running of the $U(1)$ factor, as it does not play a significant role in the cross check we consider below.

We can obtain these results from our local results: By integrating over the orbifold we obtain a relation between the 4D zero mode gauge couplings $g_{0/1}^2$, the 5D bulk gauge coupling $g_5^2$ and the local fixed point gauge couplings $g_{f,p}^2$:

\[
\frac{1}{g_0^2(\mu)} = \frac{1}{2} \frac{2\pi R}{g_5^2(\mu)} + \frac{1}{4} \sum_{f,p} \frac{1}{g_{f,p,0}^2(\mu)} , \quad \frac{1}{g_1^2(\mu)} = \frac{1}{2} \frac{2\pi R}{g_5^2(\mu)} + \frac{1}{4} \sum_{f,p} \frac{1}{g_{f,p,1}^2(\mu)}.
\]  

(109)

Here we have included the appropriate factors of $1/2$, because our results are stated on the covering circle of size $2\pi R$ rather than on the fundamental domain of the orbifold. In addition, the definition of the orbifold delta function (21) contains a factor of $1/2$. Since we have not included Wilson-lines in our analysis, the quantum corrections at the two fixed points are the same, so that the sum simply gives a factor of two. In fact, we conclude from (89) that there is no renormalization at the orbifold fixed points, i.e. $g_{f,p,1}^2(\mu) = g_{f,p,1}^2$ is constant. As our notation is indicating $2\pi R/g_5^2$ can be interpreted as a 4D coupling that runs with $\mu$. To find this dependence, we note that in the bulk there is no difference between the theory on the orbifold $S^1/\mathbb{Z}_2$ and on the circle $S^1$. On the circle we find at the zero mode level one full hyper multiplet in the fundamental of $\text{SU}(N)$, hence

\[
\frac{2\pi R}{g_5^2(\mu)} = \frac{2\pi R}{g_5^2} - \frac{2}{16\pi^2} \ln \mu^2.
\]  

(110)

Inserting this and the fact that the fixed point gauge couplings do not run into (109), we see that we exactly reproduce (108). This means that the local fixed point computation is consistent with the 4D zero mode result.

Of course in this example the cross check is rather easy precisely because at the fixed points the couplings do not renormalize. In the subsequent subsections we consider examples where the fixed point contributions do not vanish anymore.

### 6.2 Hyper multiplet on $T^2/\mathbb{Z}_3$

Now we consider a slightly less simple example of the hyper multiplet contributions to the gauge couplings on $T^2/\mathbb{Z}_3$. The basic logic is the same as in the previous section, so we will be brief and only emphasize the new features here. The matrix $Z_+$ in this case induces a symmetry breaking of the form

\[
Z_+ = \text{diag}(\mathbb{1}_{N_0}, e^{i\phi} \mathbb{1}_{N_1}, e^{2i\phi} \mathbb{1}_{N_2}), \quad \text{SU}(N) \rightarrow \text{SU}(N_0) \times \text{SU}(N_1) \times \text{SU}(N_2),
\]  

(111)
with $\phi = 2\pi/3$. And using the corresponding notations for the chiral superfields that form the hyper multiplet we find the transformations

$$
\Phi_+ I \rightarrow e^{i\phi I} \Phi_+ I, \quad \Phi_- I \rightarrow e^{i\phi(2-I)} \Phi_- I,
$$

(112)

with $I = 0, 1, 2$. From this we infer that only $\Phi_+ 0$ and $\Phi_- 2$ have zero modes. As these chiral superfields live in the $N_0$ and $\bar{N}_2$ representation, respectively, the zero mode gauge couplings renormalize as

$$
\frac{1}{g_0^2(\mu)} = \frac{1}{g_0^2(\mu)} - \frac{1}{16\pi^2} \ln \mu^2, \quad \frac{1}{g_1^2(\mu)} = \frac{1}{g_1^2(\mu)}, \quad \frac{1}{g_2^2(\mu)} = \frac{1}{g_2^2(\mu)} - \frac{1}{16\pi^2} \ln \mu^2.
$$

(113)

In the bulk the contribution to the 6D gauge coupling is the same as on the torus. In terms of the 4D renormalization scale $\mu$ we have

$$
\frac{\text{Vol}_W}{g_6^2(\mu)} = \frac{\text{Vol}_W}{g_6^2(\mu)} - \frac{2}{16\pi^2} \ln \mu^2.
$$

(114)

On the fixed points, however, the results are now more complicated than in the $Z_2$ case, as they are given by (103). The matrix $A_+$ can be read off from $Z_+$, hence we infer that the matrix combination in (103) is given by

$$
A_+ = \text{diag}(0, 1, 2)\phi, \quad \cos A_+ + \cos(A_+ + \phi) = \text{diag}\left(\frac{1}{2}, -1, \frac{1}{2}\right).
$$

(115)

This results in the following expressions for the renormalization of the fixed point gauge couplings

$$
\frac{1}{g_{f,p,0}^2(\mu)} = \frac{1}{g_{f,p,0}^2(\mu)} - \frac{1}{16\pi^2} \ln \mu^2, \quad \frac{1}{g_{f,p,1}^2(\mu)} = \frac{1}{g_{f,p,1}^2(\mu)} + \frac{2}{16\pi^2} \ln \mu^2, \quad \frac{1}{g_{f,p,2}^2(\mu)} = \frac{1}{g_{f,p,2}^2(\mu)} - \frac{1}{16\pi^2} \ln \mu^2.
$$

(116)

Notice that the beta coefficient of the fixed point gauge coupling $g_{f,p,1}$ has the opposite sign as compared to the other two fixed point couplings.

The relation between the 4D zero mode gauge couplings, the 6D bulk gauge coupling and the fixed point gauge couplings on a $Z_3$ orbifold read

$$
\frac{1}{g_1^2(\mu)} = \frac{1}{3} \frac{\text{Vol}_W}{g_6^2(\mu)} + \frac{1}{3} \frac{1}{g_{f,p,1}^2(\mu)}.
$$

(117)

Here we have summed over the fixed points of the $Z_3$ orbifold. Inserting the above results we find that the 4D zero mode gauge expressions given in (113) are indeed reproduced. In particular, the 4D zero mode gauge coupling $g_1(\mu)$ does not renormalize at all. Hence we see that also in this case our bulk and fixed point results are consistent with an analysis using the zero modes on the covering space ($T^2$) and the orbifold ($T^2/Z_3$).

### 6.3 Vector multiplet self interactions on $S^1/Z_2$

Both illustrations discussed above involved a single hyper multiplet in the fundamental representation. The final two examples consider the effects of the self interactions of the non-Abelian vector multiplet on $Z_2$ and $Z_3$ orbifolds. We follow the same methodology as for the hyper multiplet examples: First
compute the zero mode running and then see if it can be reproduced by combining the bulk and fixed point couplings in the appropriate way.

We again consider the $\mathbb{Z}_2$ case on the circle for simplicity, and take the same matrix $Z$ defined in \cite{104}. Writing the vector multiplet superfields in corresponding block matrices

\[
V = \begin{pmatrix} V_0 & V_{01} \\ V_{10} & V_1 \end{pmatrix}, \quad S = \begin{pmatrix} S_0 & S_{01} \\ S_{10} & S_1 \end{pmatrix},
\]

we infer that only $V_0, V_1$ and $S_{01}, S_{10}$ have zero modes, because of the orbifold action \cite{222}. Using some trace identities to express all traces of the gauge group generators in the fundamental representation we obtain the following zero mode beta functions

\[
\frac{1}{g_0^2(\mu)} = \frac{1}{g_0^2} + \frac{3 \cdot 2 N_0}{16\pi^2} \ln \mu^2, \quad \frac{1}{g_1^2(\mu)} = \frac{1}{g_1^2} + \frac{3 \cdot 2 N_1}{16\pi^2} \ln \mu^2 - \frac{2 N_0}{16\pi^2} \ln \mu^2.
\]

The factor of 3 and $-1$ result, because $V_0$ and $V_1$ are 4D vector multiplets while $S_{01}$ and $S_{10}$ are chiral multiplets.

The relation between the bulk, fixed point and zero mode gauge couplings are as stated in \cite{109}. For the 5D bulk gauge coupling we find

\[
\frac{2\pi R}{g_5^2}(\mu) = \frac{2\pi R}{g_5^2} + \frac{2 \cdot 2(N_0 + N_1)}{16\pi^2} \ln \mu^2,
\]

because the zero modes on $S^1$ are the full vector and chiral multiplets $V$ and $S$. To compute the fixed point contributions using \cite{51}, we notice that by standard representation theory

\[
\text{tr}_{\text{Ad}}(V_0 + V_1)^2 = 2N_0 \text{tr}_{N_0} V_0^2 + 2N_1 \text{tr}_{N_1} V_1^2 + 2N_1 \text{tr}_{N_0} V_0^2 + 2N_0 \text{tr}_{N_1} V_1^2.
\]

In analogy to the definition of the trace $\text{tr}_{\text{Ad}}$ we define $\text{tr}_{Q,\text{Ad}}(XY) = -f_{ijk} f_{\ell mn} \eta^{mij} Q^{nk} X^i Y^\ell$. Using this definition and the fact that $Q$ is equal to $-1$ when it corresponds to off-diagonal entries, we infer that

\[
\text{tr}_{Q,\text{Ad}}(V_0 + V_1)^2 = 2N_0 \text{tr}_{N_0} V_0^2 + 2N_1 \text{tr}_{N_1} V_1^2 - 2N_1 \text{tr}_{N_0} V_0^2 - 2N_0 \text{tr}_{N_1} V_1^2.
\]

Hence we find for the 4D fixed point gauge couplings

\[
\frac{1}{g_{f,0}^2(\mu)} = \frac{1}{g_{f,0}^2} + \frac{4 \cdot 2(N_0 - N_1)}{16\pi^2} \ln \mu^2, \quad \frac{1}{g_{f,1}^2(\mu)} = \frac{1}{g_{f,1}^2} + \frac{4 \cdot 2(N_1 - N_0)}{16\pi^2} \ln \mu^2.
\]

Combining these fixed point results with the bulk gauge coupling, we see that we precisely reproduce the 4D zero mode gauge couplings.

### 6.4 Vector multiplet self interactions on $T^2/\mathbb{Z}_3$

Our final example discusses the non-Abelian vector multiplet self interactions on the orbifold $T^2/\mathbb{Z}_3$. Using a similar analysis as presented above, we find that the zero modes are the vector multiplets $V_0, V_1, V_2$ and chiral multiplets $S_{20}, S_{12}, S_{01}$. Consequently, the zero mode gauge couplings read

\[
\frac{1}{g_0^2(\mu)} = \frac{1}{g_0^2} + \frac{1}{16\pi^2} \left\{ 3 \cdot 2N_0 - N_1 - N_2 \right\} \ln \mu^2,
\]
and cyclic permutations of the labels 0, 1, 2. The bulk contribution to the 4D zero mode couplings is of course the same, for the fixed point contributions we find from (95) that we get

\[ \frac{1}{g_{T,0}^2(\mu)} = \frac{1}{g_{T,0}^2} + \frac{1}{16\pi^2} \left\{ -14N_0 + 7(N_1 + N_2) \right\} \ln \mu^2, \]  
(125)

and cyclic permutations. When these results are combined we see again that the 4D zero mode gauge couplings can be obtained from the 6D bulk and the fixed point gauge couplings according to (117).

7 Conclusions

In this paper we considered the renormalization of gauge kinetic operators on orbifolds. With possible applications in string phenomenology in mind, we focused on supersymmetric theories in 5D and 6D, as our results can be straightforwardly extended to 10D string models. The \( Z_N \) orbifolds under investigation preserved 4D Lorentz invariance and \( N = 1 \) supersymmetry, which motivated us to use the language of 4D \( N = 1 \) superfields to describe these theories.

We presented in detail one loop computations on orbifolds using orbifold compatible delta functions. Using this technique we computed the gauge coupling renormalization for non-Abelian supersymmetric gauge theories. This extended our previous work [18] which was only concerned with Abelian theories. For the hyper multiplet in the non-Abelian case we have established that the renormalization of the gauge couplings at the \( Z_2 \) fixed points is absent, but for the other \( Z_N \) fixed points this is not the case. This result is similar to what we had obtained in the Abelian case before. In the non-Abelian theory there are also vector multiplet self interactions. We computed the self energy due to these interactions on the orbifold and found that they always give rise to renormalization both in the bulk and at all fixed points, including the \( Z_2 \) fixed points.

In this work we performed a direct computation of the required counter terms and therefore the renormalization at the fixed points. However, some of our results can also be obtained indirectly by carefully considering what happens at the zero mode level when the theory is compactified on the orbifold or its covering space. This technique has been advocated for example in [14]. For us this provided an important cross check of our results for both hyper and vector multiplet on both \( Z_2 \) and \( Z_3 \) orbifolds.

Aside from the gauge coupling renormalization, we have also encountered some other findings. First of all in the non-Abelian case both in 5D and 6D the local fixed point counter terms appear not to be gauge invariant under the full bulk gauge group. It is possible to give a non-linear completion of these terms which is invariant under the zero mode part of the supergauge transformation. This does not necessarily mean that gauge invariance is broken by quantum effects: Our one loop computation only focused on two point functions. This means that the full gauge invariant renormalized theory can only be guessed by using arguments of supergauge invariance. In the 6D bulk case the more or less obvious kinetic terms for the vector multiplet have to be completed by a Wess-Zumino-Witten term, see section 3 and also [43]. Therefore it might not be so surprising that this could also be the case for the fixed point contributions. What is more surprising is that this also seems to be the case in the 5D setting. To determine these generalizations of Wess-Zumino-Witten terms is very interesting but lies somewhat outside the scope of the present paper. However, this question might be interesting for future research.
The other important quantum effect is that, as in the Abelian case, also higher derivative operators of the vector multiplet are required in order to cancel all divergences in 6D. However, in the non-Abelian case in 6D we also concluded that the Wess-Zumino-Witten term must also renormalize. And in addition there must exist a higher derivative analogue of this term, that ensures that the kinetic higher derivative operator is gauge invariant. To determine the precise form of this is again beyond the scope of the paper, but an interesting question for future investigations.

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A Results for the Feynman graphs

Here we give the results for the Feynman graphs that have been calculated. All results are formulated in 6D notation on the orbifold \( T^2/\mathbb{Z}_N \). These results can also be applied to the orbifold \( S^1/\mathbb{Z}_2 \): The 5D situation is obtained when one replaces \( z = y \) and neglects all dependence on \( \bar{z} \), the derivatives change as \( \partial = \bar{\partial} = \partial_5 \), and instead of the six dimensional orbifold compatible delta functions (47) one uses their five dimensional counterparts (27).

A.1 Gauge multiplet corrections to the vector multiplet self energy

In this appendix we give the superspace and the resulting scalar integral for the gauge multiplet contributions to the vector multiplet corresponding to the diagrams in figure IV. We use the labels for these diagrams suggested by that figure to refer to the contributions of the various topologies and superfields.

\[
\begin{align*}
\mathbf{A} &= f_{ijk} f_{\ell mn} \int (d^6x)_{12} d^4\theta \left[ V^i_1 \Box P_0 V^\ell_2 \frac{1}{(\Box + \bar{\partial}\partial)_2} \tilde{\delta}^{6(s) mj}_{21} \frac{1}{(\Box + \bar{\partial}\partial)_2} \tilde{\delta}^{6(s) nk}_{21} + \\
&\quad - 2 V^i_1 V^\ell_2 \frac{1}{(\Box + \bar{\partial}\partial)_2} \tilde{\delta}^{6(S) mj}_{21} \frac{\Box_2}{(\Box + \bar{\partial}\partial)_2} \tilde{\delta}^{6(S) nk}_{21} \right].
\end{align*}
\]

\[
\begin{align*}
\mathbf{B} &= f_{ijk} f_{\ell mn} \int (d^6x)_{12} d^4\theta V^i_1 \Box \left( \frac{1}{2} P_+ + \frac{1}{2} P_- - \frac{5}{2} P_0 \right) V^\ell_2 \times \\
&\quad \times \frac{1}{(\Box + \bar{\partial}\partial)_2} \tilde{\delta}^{6(V) mj}_{21} \frac{1}{(\Box + \bar{\partial}\partial)_2} \tilde{\delta}^{6(V) nk}_{21}.
\end{align*}
\]
\[ \mathbf{V} C = f_{ijk} f_{\ell mn} \int (d^6x)_{12} \, d^4\theta [\partial V_1^i \partial V_2^\ell \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)mj}_{21} \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(S)nk}_{21} + \\
- \partial V_1^i V_2^\ell \frac{\partial_2}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)mj}_{21} \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(S)nk}_{21} - V_1^i \partial V_2^\ell \frac{\partial_1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)mj}_{21} \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(S)nk}_{21} + \\
+ V_1^i V_2^\ell \frac{\partial_1 \partial_2}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)mj}_{21} \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(S)nk}_{21}] . \quad (A.3) \]

\[ \mathbf{V} D = f_{ijk} f_{\ell mn} \int (d^6x)_{12} \, d^4\theta [2 V_1^i V_2^\ell \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)mj}_{21} \frac{\Box_2}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)nk}_{21} + \\
- \frac{1}{2} V_1^i \Box (P_+ + P_- + P_0) V_2^\ell \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)mj}_{21} \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)nk}_{21}] . \quad (A.4) \]

\[ \mathbf{V} E = -\frac{1}{3} f_{ijk} f_{\ell mn} \eta^{nk} \int (d^6x)_{12} \, d^4\theta V_1^i V_2^\ell \tilde{\delta}^{6(V)mj}_{21} a^j \frac{\eta^{ab}}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)nk}_{21} b^m . \quad (A.5) \]

\[ \mathbf{V} F = 2 f_{ijk} f_{\ell mn} \eta^{nk} \int (d^6x)_{12} \, d^4\theta V_1^i V_2^\ell \tilde{\delta}^{6(S)mj}_{21} a^m \frac{\eta^{ab}}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(S)nk}_{21} b^j . \quad (A.6) \]

\[ \mathbf{V} G = -\frac{2}{3} f_{ijk} f_{\ell mn} \eta^{nk} \int (d^6x)_{12} \, d^4\theta V_1^i V_2^\ell \tilde{\delta}^{6(V)mj}_{21} a^j \frac{\eta^{ab}}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)nk}_{21} b^m . \quad (A.7) \]

The diagrams in figure \[ \mathbf{V} \] give rise to the $SS$ and $VS$ self energies. Written in terms of two orbifold compatible delta functions we have:

\[ \mathbf{V} A = 2 f_{ijk} f_{\ell mn} \int (d^6x)_{12} \, d^4\theta S_1^i S_2^\ell \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)mj}_{21} \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(S)nk}_{21} . \quad (A.8) \]

\[ \mathbf{V} B = -\sqrt{2} f_{ijk} f_{\ell mn} \int (d^6x)_{12} \, d^4\theta \left( 2 \partial V_1^i S_2^\ell \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)mj}_{21} \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(S)nk}_{21} + \\
+ V_1^i \tilde{S}_2^\ell \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)mj}_{21} \frac{\partial_1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(S)nk}_{21} \right) . \quad (A.9) \]

\[ \mathbf{V} C = +\sqrt{2} f_{ijk} f_{\ell mn} \int (d^6x)_{12} \, d^4\theta V_1^i \tilde{S}_2^\ell \frac{1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(V)mj}_{21} \frac{\partial_1}{(\Box + \partial \partial)_{21}} \tilde{\delta}^{6(S)nk}_{21} . \quad (A.10) \]
A.2 Hyper multiplet corrections to the vector multiplet self energy

In this appendix we give the scalar integral expression for the supergraphs given in figures VII and VIII at the level of two remaining orbifold compatible delta functions.

\[ \text{VII.A} \quad A_{\pm} = 2 \int (d^6x)_{12} d^4 \theta \text{ tr} \left[ V_1 \frac{1}{(\Box + \partial \bar{\partial})_2} \delta_{21}^{6(+)} \left( \frac{1}{2} \Box p_0 V_2 + V_2 \Box_2 \right) \frac{1}{(\Box + \partial \bar{\partial})_2} \delta_{21}^{6(+)} \right. \\
+ \left. V_1 \frac{1}{(\Box + \partial \bar{\partial})_2} \delta_{21}^{6(-)} \left( \frac{1}{2} \Box p_0 V_2 + V_2 \Box_2 \right) \frac{1}{(\Box + \partial \bar{\partial})_2} \delta_{21}^{6(-)} \right]. \quad (A.11) \]

\[ \text{VII.B} = 4 \int (d^6x)_{12} d^4 \theta \text{ tr} \left[ V_1 \frac{\partial_1}{(\Box + \partial \bar{\partial})_2} \delta_{21}^{6(+)} V_2 \frac{\bar{\partial}_2}{(\Box + \partial \bar{\partial})_2} \delta_{21}^{6(-)} \right]. \quad (A.12) \]

\[ \text{VII.C} = -2 \int (d^6x)_{12} d^4 \theta \text{ tr} \left[ V_1 V_2 \delta_{21}^{6(+)} \frac{1}{(\Box + \partial \bar{\partial})_2} \delta_{21}^{6(+)} - 2V_1 V_2 \frac{1}{(\Box + \partial \bar{\partial})_2} \delta_{21}^{6(-)} \right]. \quad (A.13) \]

For the \( S\bar{S} \) and \( SV \) self energies we find from figure VII

\[ \text{VII.A} \quad A_{\pm} = 2 \int (d^6x)_{12} d^4 \theta \text{ tr} \left[ S_1 \frac{1}{(\Box + \partial \bar{\partial})_2} \delta_{21}^{6(+)} \bar{S}_2 \frac{1}{(\Box + \partial \bar{\partial})_2} \delta_{21}^{6(-)} \right]. \quad (A.14) \]

\[ \text{VII.B} = -2\sqrt{2} \int (d^6x)_{12} d^4 \theta \text{ tr} \left[ \partial V_1 \frac{1}{(\Box + \partial \bar{\partial})_2} \delta_{21}^{6(+)} \bar{S}_2 \frac{1}{(\Box + \partial \bar{\partial})_2} \delta_{21}^{6(-)} \right]. \quad (A.15) \]

B Fourier transformation conventions

In the main text we need to perform Fourier transformations between coordinate and momentum space. We describe our conventions for the 6D situation compactified on the torus \( T^2 \). Because of the notation in 6D and 5D introduced in sections 2.3 and 3.3 respectively, the reduction to the 5D integrals on the circle \( S^1 \) is straightforward. We define the Fourier transformations as

\[ A(x, z) = \int \frac{d^4p}{(2\pi \mu)^d} \sum_{n\in \Lambda_K} A(p, n)e^{i(px+nz+\bar{n}\bar{z})}, \quad (B.1) \]

and

\[ A(p, n) = \frac{2\mu^d}{\text{Vol}_W} \int d^dxd^2z A(x, z)e^{-i(px+nz+\bar{n}\bar{z})}. \quad (B.2) \]

We have introduced the regularization scale \( \mu \) such that the coordinate and momentum Fourier transforms have the same mass dimension. The coordinate space delta function is given by

\[ \delta^d(x_2 - x_1) \delta^2(z_2 - z_1) = \int \frac{d^4p}{(2\pi)^d} \frac{1}{\text{Vol}_W} \sum_{n\in \Lambda_K} e^{i(p(x_2-x_1)+n(z_2-z_1)+\bar{n}(\bar{z}_2-\bar{z}_1))}. \quad (B.3) \]

The delta function in momentum space can be expanded as

\[ \delta^d(p_2 - p_1) \delta^2(n_2 - n_1) = 2 \int \frac{d^dxd^2z}{(2\pi)^d\text{Vol}_W} e^{i(p_2-p_1)x+(n_2-n_1)z+(\bar{n}_2-\bar{n}_1)\bar{z}). \quad (B.4) \]
C Theta functions

C.1 Genus one theta functions

The genus one theta function on the Kaluza Klein lattice is defined as

\[
\theta^K[\alpha, \beta](\sigma|\tau) = \sum_{n \in \mathbb{Z}/R} e^{i\tau(n-\alpha)^2 + i[(\sigma-\beta)(n-\alpha)]},
\]

(C.1)

The theta function is translation invariant under a shift of \(\alpha\) by an element of the Kaluza Klein lattice or a shift of \(\beta\) by an element of the winding mode lattice

\[
\theta^K[\alpha + n, \beta](\sigma|\tau) = \theta^K[\alpha, \beta](\sigma|\tau), \quad n \in \Lambda_K,
\]

(C.2)

\[
\theta^K[\alpha, \beta + w](\sigma|\tau) = \theta^K[\alpha, \beta](\sigma - w|\tau), \quad w \in \Lambda_W.
\]

(C.3)

The relation between \(\theta^K\) and \(\theta^W\) is

\[
\theta^K[\alpha, \beta](\sigma|\tau) = R \sqrt{2\pi} e^{-\frac{1}{4\tau} r^2 + i\alpha \beta} \theta^W[\beta, -\alpha](\frac{-\sigma}{\tau} |\frac{-1}{\tau}).
\]

(C.5)

This can be obtained by using Poisson resummation, which allows us to rewrite a complex exponential function that is summed over the Kaluza-Klein lattice \(\Lambda_K\) into a delta function that is summed over the winding mode lattice and vice versa. Concretely, we have

\[
\frac{1}{2\pi R} \sum_{n \in \Lambda_K} e^{iny} = \sum_{w \in \Lambda_W} \delta(y - w), \quad y \in \mathbb{R}
\]

(C.6)

and

\[
R \sum_{w \in \Lambda_W} e^{iwp} = \sum_{n \in \Lambda_K} \delta(p - n), \quad p \in \mathbb{R}.
\]

(C.7)

C.2 Genus two theta functions

The genus two theta function on the Kaluza-Klein lattice \(\Lambda_K\) is defined by

\[
\theta^K[\alpha, \beta](\sigma|\tau) = \sum_{n \in \Lambda_K} e^{i\tau(n-\alpha)^2 + i[(\sigma-\beta)(n-\alpha)]}.
\]

(C.8)

Also the genus two theta function fulfills the translation invariance properties (C.2) and (C.3). The genus two theta function on the winding mode lattice is defined by

\[
\theta^W[\beta, \alpha](\sigma|\tau) = \sum_{w \in \Lambda_W} e^{2i\tau|w|\alpha^2 + i[(\sigma-\beta)(w-\alpha)]}.
\]

(C.9)
The relation between $\theta_K$ and $\theta_W$ is given by
\[
\theta_K \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (\sigma | \tau) = \text{Vol} \left( \frac{2\pi}{i \tau} e^{i \left( -\frac{\tau}{2} | \sigma^2 + \alpha \beta + \bar{\alpha} \bar{\beta} \right)} \right) \theta_W \left[ \begin{array}{c} \beta \\ -\alpha \end{array} \right] \left( \frac{\sigma}{\tau} \right) \left( \frac{1}{\tau} \right).
\]
(C.10)

This is obtained by Poisson resummation on the torus
\[
\frac{1}{\text{Vol}_W} \sum_{n \in \Lambda_K} e^{i (n \zeta + \bar{n} \bar{\zeta})} = \sum_{w \in \Lambda_W} \frac{1}{2} \delta^2 (z - w), \quad z \in \mathbb{C},
\]
(C.11)
\[
\frac{1}{\text{Vol}_K} \sum_{w \in \Lambda_W} e^{i (w \zeta + \bar{w} \bar{\zeta})} = \sum_{n \in \Lambda_K} 2 \delta^2 (p - n), \quad p \in \mathbb{C},
\]
(C.12)

where
\[
\delta^2 (p) = \frac{1}{2} \delta (p_5) \delta (p_6), \quad \delta^2 (z) = 2 \delta (x_5) \delta (x_6).
\]
(C.13)

Therefore, in the case $\tau = \frac{2i \mu}{\mu^*}$, $\alpha = sl$, and $\sigma = \beta = 0$
\[
\theta_K \left[ \begin{array}{c} sl \\ 0 \end{array} \right] \left( 0 \left| \frac{2i \mu}{\mu^*} \right) = \text{Vol} \left( \frac{\pi \mu^2}{l} \right)^{D-2} \theta_W \left[ \begin{array}{c} 0 \\ -sl \end{array} \right] \left( 0 \left| \frac{2i \mu}{\mu^*} \right) \right.
\]
(C.14)

holds both for the theta functions on the circle and on the torus.

D Regularization of the momentum integral

In order to determine the counterterms, we need to calculate the divergent part of the integral
\[
I_D = i \int \frac{d^d p_E}{(2\pi)^d} \frac{1}{\text{Vol}_W} \sum_n \frac{1}{p^2 + |n|^2 + m^2 (p - k)^2 + |n - l|^2 + m^2},
\]
(D.1)

which is obtained after a Wick rotation. We can replace the integration over the volume of the continuous momenta by the integration over the radius
\[
\int \frac{d^d p_E}{(2\pi)^d} = \frac{2(\mu)^{1-d}}{(4\pi)^{d/2} \Gamma \left( \frac{d}{2} \right)} \int_0^\infty dp \, p^{d-1}.
\]
(D.2)

The non-compact 4D integral is extended to $d = 4 - 2\epsilon$ dimensions using the standard procedure of dimensional regularization of scalar integrals. Furthermore, we use the identity
\[
\frac{1}{M_i^2} = \frac{1}{\mu^2} \int_0^\infty dt \, e^{-tM_i^2/\mu^2}
\]
(D.3)

where $M_i$ are the momentum-dependent denominators of (D.1). With the help of a Feynman parameter $s$ the integral (D.1) can be written as
\[
I_D = i \frac{1}{(4\pi)^{d/2}} \frac{1}{(2\pi)^{D-d}} \text{Vol} \int_0^1 ds \int_0^\infty \frac{dt}{t^{d/2-1}} e^{-t [s(1-s)(k^2 + |l|^2) + m^2]/\mu^2} \theta_K \left[ \begin{array}{c} sl \\ 0 \end{array} \right] \left( 0 \left| \frac{2i \mu}{\mu^*} \right) \right. \]
(D.4)
where for $D - d = 1, 2$ we take for $\theta_K$ the $\theta$ function of the circle or the torus, defined in (C.1) and (C.8), respectively. After application of the Poisson resummation formula (C.14) which is valid both on the circle and on the torus, we obtain

$$I_D = i \frac{\mu^{D-d}}{(4\pi)^{D/2}} \int_0^1 ds \int_0^\infty dt t^{D/2-1} e^{-t[s(1-s)(k^2+l^2)+m^2]/\mu^2} \left( \frac{0}{-sl} \right) \left( \frac{\mu^2}{2l} \right)$$

(D.5)

Because $\theta_W - 1$ cannot lead to UV divergencies, we can put $\theta_W = 1$ in order to determine the divergent part. We find

$$I_D \Big|_{\text{div}} = i \frac{1}{\mu^d} \left( \frac{\mu^2}{m^2} \right)^2 \left( \frac{m^2}{4\pi} \right)^{D/2} \sum_{n \geq 0} (-)^n \frac{\Gamma(n+2-D/2)\Gamma(n+1)}{\Gamma(2n+2)} \left( \frac{K^2}{m^2} \right)^n.$$  

(D.6)

In the 6D case we obtain

$$I_{6-2\epsilon} \Big|_{\text{div}} = i \frac{1}{(4\pi)^3} \left[ \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{m^2} \right) \left( m^2 + \frac{1}{6}(k^2 + |l|^2) + m^2 \right) \right],$$

(D.7)

where we have defined $\frac{1}{\epsilon} = \frac{1}{\epsilon} - \gamma + \ln 4\pi$. Here only the terms with $n \in \{0, 1\}$ contribute to the divergent part and we have neglected terms with higher $n$. In the 5D case the expression reads

$$I_{5-2\epsilon} \Big|_{\text{div}} = -i \frac{1}{(4\pi)^2} |m|,$$

(D.8)

where only the $n = 0$ term has been taken into account. The four dimensional case can also be traced back when one neglects the summation $\frac{1}{\text{Vol}_W} \sum_n$. This results in

$$I_{4-2\epsilon} \Big|_{\text{div}} = -i \frac{1}{(4\pi)^2} \left( \frac{1}{\epsilon} - \gamma + \ln 4\pi + \ln \frac{\mu^2}{m^2} \right).$$

(D.9)

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