REDUCTION AND BIFURCATION OF TRAVELING WAVES OF THE KDV-BURGERS-KURAMOTO EQUATION

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1. Introduction. This paper considers the KdV-Burgers-Kuramoto (KBK) equation [14]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^3 u}{\partial x^3} + \gamma \frac{\partial^4 u}{\partial x^4} = 0,
\]

which is an appropriate model to describe phenomena that are simultaneously involved in nonlinearity, dissipation, dispersion and instability, where $\alpha, \beta, \gamma$ are nonzero real constants. Equation (1) plays important roles in describing physical processes in motion of turbulence and unstable systems [30, 17, 18] and is also known as the generalized Kuramoto-Sivashinsky equation [17] or Benney equation [25]. The KBK equation has been investigated widely and various direct methods have been proposed to obtain exact traveling wave solutions of it, such as the Weiss-Tabor-Carnevale transformation method [17], trial-function method [25], tanh-function method and extended tanh-function method [19, 28, 4]. In recent decade, more methods are applied to obtain new exact solutions of it, including the trigonometric function expansion method [6], generalized F-expansion method [34], unified ansätze approach [15], a combination method [32] and Exp-function...
method \cite{16, 24}. In addition, some numerical methods for solving the fractional KBK equation can also be found in \cite{29, 31, 10}.

Though there have been so many profound results about traveling wave solutions of the KBK equation which contributed to our understanding of nonlinear physical phenomena and wave propagation, some traveling wave solutions could be still lost because of defects of the direct methods caused by auxiliary equations and hypotheses about the forms of solutions. Not only that, it is well known that not even all traveling wave solutions can be expressed analytically. So, qualitative analysis of traveling waves of the KBK equation is still necessary and significant. In addition, we also want to explore whether there exist other types of solutions of the KBK equation except the traveling wave solutions, for example, the series solutions. These problems arouse our great interest in surveying the KBK equation again.

In order to obtain new solutions of the KBK equation, we need to reduce it as simple as possible. It is known that the Lie group method is a powerful approach to reduce nonlinear partial differential equations and construct their group-invariant solutions. The fundamental basis of the technique is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. Most of the required theory and description of the techniques of this method can be found in \cite{26, 27} as well as in \cite{11, 1, 2}. By using the Lie symmetry analysis method \cite{26}, we obtain finite-dimensional symmetries of the KBK equation, which form a three-dimensional Lie algebra. Subsequently, a one-dimensional optimal system for the KBK equation is given according to the adjoint representation of vector field. Based on the optimal system, we get four classes of reduced equations of the KBK equation and corresponding group-invariant solutions which are not reported before.

Particularly, as an important reduced equation, the traveling wave equation of the KBK equation is investigated qualitatively in detail. Our strategy is to transform it to a three-dimensional differential dynamical system with singular perturbation. This idea is firstly introduced by Fu and Liu \cite{7} in 2010. From geometric singular perturbation point of view, they give the existence condition of kink waves by proving that a strictly increasing traveling front of the KdV-Burgers equation can persist in the KBK equation for sufficiently small dissipation parameter $\bar{\gamma}$. But, interactions and combination of nonlinearity, dissipation, dispersion and instability could lead to more complicated behaviors of traveling waves, such as the periodic motion, oscillation. It means that other types of bounded traveling waves could occur to the KBK equation besides the traveling fronts. In fact, three basic types of bounded traveling waves could occur for a PDE, which are periodic waves, kink waves (or shock waves) and solitary waves. Recall that heteroclinic orbits are trajectories which have two distinct equilibria as their $\alpha$ and $\omega$-limit sets and homoclinic orbits are trajectories whose $\alpha$ and $\omega$-limit sets consist of the same equilibrium. So, the three basic types of bounded traveling waves mentioned above correspond to periodic, heteroclinic and homoclinic orbits of the traveling wave system of a PDE respectively \cite{9, 13, 20, 21, 22}. It is just the relation that offers an effective way to study traveling waves of a PDE from the point of view of differential dynamical system. So, the dynamical system methods play important roles in our paper, which is the most remarkable difference from \cite{7}. It will be seen that this method allows detailed analysis on phase space geometry of traveling wave system of the KBK
equation so that all possible bounded traveling waves and corresponding existence conditions can be identified clearly.

We first prove that it has a two-dimensional invariant submanifold $\mathcal{M}_c$ in $\mathbb{R}^3$ by the geometric singular perturbation theory [5, 12]. Restricted on $\mathcal{M}_c$, the singular perturbation system is reduced to a dynamical system with perturbation in $\mathbb{R}^2$. By using the dynamical system methods and some techniques such as tracking unstable manifold of the saddle and studying equilibria at infinity, we investigate the phase space geometry of the corresponding unperturbed system in detail. The result clearly shows that under appropriate wavespeed conditions there exist the saddle-spiral shock waves besides the traveling fronts mentioned in [7] for the unperturbed system. Subsequently, by using the Fredholm theorem, we prove that the two types of traveling waves can persist in the KBK equation. Furthermore, Mel’nikov type computation is carried out to study the homoclinic bifurcation and Poincaré bifurcation of the reduced system. Various wavespeed conditions are determined to guarantee the existence of solitary waves and periodic waves of the KBK equation. Not only that, based on the bifurcation results, we also discuss the existence of two types of oscillatory bounded traveling waves for the KBK equation.

2. Reduction and series solutions of the KBK equation. To seek a symmetry $\sigma(x, t, u)$ of the KBK equation, we set

$$\sigma = a(x, t)u_t + b(x, t)u_x + d(x, t)u + e(x, t),$$

where $u = u(x, t)$ satisfies equation (1) and $a(x, t), b(x, t), d(x, t), e(x, t)$ are functions to be determined later. Based on Lie group theory [26], $\sigma$ satisfies the following equation

$$\sigma_t + u_x \sigma + u \sigma_x + \tilde{\alpha} \sigma_{xx} + \tilde{\beta} \sigma_{xxx} + \tilde{\gamma} \sigma_{xxxx} = 0. \quad (2)$$

Noting that $u_t = -uu_x - \tilde{\alpha} u_{xx} - \tilde{\beta} u_{xxx} - \tilde{\gamma} u_{xxxx}$, we can calculate the expressions of $\sigma_t, \sigma_x, \sigma_{xx}, \sigma_{xxx}$ and $\sigma_{xxxx}$ (see Appendix 1).

Substituting them into (2), we get the differential equations with respect to $a(x, t), b(x, t), d(x, t)$ and $e(x, t)$. Solving them yields

$$a(x, t) = c_1, b(x, t) = c_2 t + c_3, d(x, t) = 0, e(x, t) = -c_2,$$

where $c_1, c_2, c_3$ are arbitrary real constants. Thus, the symmetry of the KBK equation can be written as

$$\sigma = c_1 u_t + c_2 (tu_x - 1) + c_3 u_x.$$

It follows that the vector field of the KBK equation is spanned by the vectors

$$V_1 = \frac{\partial}{\partial t}, V_2 = \frac{\partial}{\partial x}, V_3 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

Next, we construct the one-dimensional optimal system for the KBK equation. Firstly, one can check that the following commutation relations hold

$$[V_1, V_2] = 0, [V_1, V_3] = V_2, [V_2, V_3] = 0. \quad (3)$$

It means that the vector field is closed under the Lie bracket. From the commutation relations (3), we calculate adjoin representations of the vector field as follows

$$\Ad(\exp(\varepsilon V_i)) V_i = V_i, \quad i = 1, 2, 3,$$
\[
\text{Ad}(\exp(\varepsilon V_1))V_2 = V_2, \quad \text{Ad}(\exp(\varepsilon V_1))V_3 = V_3 - \varepsilon V_2,
\]
\[
\text{Ad}(\exp(\varepsilon V_2))V_1 = V_1, \quad \text{Ad}(\exp(\varepsilon V_2))V_3 = V_3,
\]
\[
\text{Ad}(\exp(\varepsilon V_3))V_1 = V_1 + \varepsilon V_2, \quad \text{Ad}(\exp(\varepsilon V_3))V_2 = V_2
\]
for any \(\varepsilon \in \mathbb{R}\).

Given a nonzero vector
\[
V = a_1 V_1 + a_2 V_2 + a_3 V_3,
\]
we expect to simplify as many of the coefficients \(a_i\) as possible through suitable applications of adjoint maps to \(V\).

Firstly, suppose that \(a_3 \neq 0\). Without loss of generality, we can set \(a_3 = 1\). If we act on the vector \(V\) by \(\text{Ad}(\exp(a_2 V_1))\), the coefficient of the \(V_2\) can be eliminated:
\[
\tilde{V} = \text{Ad}(\exp(a_2 V_1))V = a_1 V_1 + V_3.
\]
It is easy to see that \(\tilde{V}\) can not be reduced further by above adjoint maps. So, every one-dimensional subalgebra generated by \(V\) with \(a_3 \neq 0\) is equivalent to the subalgebra spanned by \(a_1 V_1 + V_3(\varepsilon_1 \neq 0)\) or \(V_3(\varepsilon_1 = 0)\).

The remaining one-dimensional subalgebras are spanned by vectors \(V\) with \(a_3 = 0\) i.e., \(V = a_1 V_1 + a_2 V_2\). Similarly, suppose that \(a_1 = 1\). If we act on the vector \(V\) by \(\text{Ad}(\exp(-a_2 V_3))\), the coefficient of the \(V_2\) can be eliminated:
\[
\tilde{V} = \text{Ad}(\exp(-a_2 V_3))V = V_1.
\]
So, every one-dimensional subalgebra generated by \(V\) with \(a_3 = 0\) is equivalent to the subalgebra spanned by \(V_1(\varepsilon_1 \neq 0)\) or \(V_2(\varepsilon_1 = 0)\). Thus, we obtain an optimal system of one-dimensional subalgebras of the KBK equation as follows:
\[
\{V_1, V_2, V_3, a_1 V_1 + V_3\}
\]
where \(a_1 \neq 0\) is an arbitrary constant.

According to the optimal system, we will reduce the KBK equation and construct its group-invariant solutions.

Case 1. For the generator \(V_2\), we have
\[
u(x, t) = f_1(t).
\]
Substituting (4) into (1), we reduce the KBK equation to the form:
\[
f_1' = 0,
\]
where \(f_1' = \frac{df_1}{dt}\). Solving it, we get the trivial solution of the KBK equation \(u(x, t) = C\), where \(C\) is an arbitrary constant.

Case 2. For the generator \(V_3\), we have
\[
u(x, t) = f_2(t) + xt^{-1}.
\]
Substituting (5) into (1), we reduce the KBK equation to the form:
\[
t f_2' + f_2 = 0,
\]
where \(f_2' = \frac{df_2}{dt}\). Solving it, we get the trivial solution of the KBK equation
\[
u(x, t) = C t^{-1} + xt^{-1},
\]
where \(C\) is an arbitrary constant.

Case 3. For the generator \(V_1\), we have
\[
u(x, t) = f_3(x).
\]
Substituting (6) into (1), we reduce the KBK equation to the form:

\[ \gamma f''' + \beta f'' + \alpha f' + f^3 f' = 0, \]  

(7)

where \( f'_3 = \frac{df}{dx} \).

Integrating (7) once, we have

\[ \gamma f'' + \beta f' + \alpha f + \frac{1}{2} f^2 + e = 0, \]  

(8)

where \( e \) is an integral constant.

We assume a solution of Eq.(8) in a power series of the form

\[ f_3(x) = \sum_{n=0}^{\infty} c_n x^n. \]  

(9)

Substituting (9) into (8) leads to

\[ 6\gamma c_3 + \gamma \sum_{n=1}^{\infty} (n+3)(n+2)(n+1)c_{n+3}x^n + 2\beta c_2 \]

\[ +\beta \sum_{n=1}^{\infty} (n+2)(n+1)c_{n+2}x^n + \alpha c_1 + \alpha \sum_{n=1}^{\infty} (n+1)c_{n+1}x^n \]

\[ +\frac{1}{2} c_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (\sum_{k=0}^{n} c_k c_{n-k})x^n + e \equiv 0. \]  

(10)

From (10), we have

\[ c_3 = \frac{-1}{6\gamma}(2\beta c_2 + \alpha c_1 + \frac{1}{2} c_0^2 + e) \]  

(11)

and

\[ c_{n+3} = \frac{-\beta(n+2)(n+1)c_{n+2} + \alpha(n+1)c_{n+1} + \frac{1}{2} \sum_{k=0}^{n} c_k c_{n-k}}{\gamma(n+3)(n+2)(n+1)}, \]  

(12)

for \( n \geq 1 \).

Thus, the power series solution of the KBK equation can be written as follows

\[ u(x,t) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \sum_{n=1}^{\infty} c_{n+3} x^{n+3} \]

\[ = c_0 + c_1 x + c_2 x^2 - \frac{1}{\gamma}(2\beta c_2 + \alpha c_1 + \frac{1}{2} c_0^2 + e)x^3 \]

\[ -\sum_{n=1}^{\infty} \frac{\beta(n+2)(n+1)c_{n+2} + \alpha(n+1)c_{n+1} + \frac{1}{2} \sum_{k=0}^{n} c_k c_{n-k}}{\gamma(n+3)(n+2)(n+1)} x^{n+3}, \]  

(13)

where \( c_i (i = 0, 1, 2) \) are arbitrary constants, the other coefficients \( c_n (n \geq 3) \) are determined by (11) and (12). By a simple application of the Implicit Function Theorem, it is easy to check that power series (13) is convergent.

Case 4. For the generator \( a_1 V_1 + V_3 \), we have

\[ u(x,t) = f_4(\xi) + \frac{t}{a_1}, \]  

(14)
where \( \xi = x - \frac{t^2}{2a_1} \). Substituting (14) into (1), we reduce the KBK equation to the form:

\[
\dddot{f} + \ddot{f} + \dot{f} + f = 0,
\]

(15)

where \( f' = \frac{df}{d\xi} \).

Similarly, we can obtain the corresponding series solution of Eq. (15) as follows

\[
u(x,t) = \frac{t}{a_1} + c_0 + c_1(x - \frac{t^2}{2a_1}) + c_2(x - \frac{t^2}{2a_1})^2 + c_3(x - \frac{t^2}{2a_1})^3 + c_4(x - \frac{t^2}{2a_1})^4 - \sum_{n=2}^{\infty} c_{n+3}(x - \frac{t^2}{2a_1})^{n+3},
\]

(16)

where \( c_i (i = 0, 1, 2) \) are arbitrary constants, \( c_3 = -\frac{1}{6\gamma} (2\beta c_2 + \alpha c_1 + \frac{1}{2} c_0^2 + e) \), \( c_4 = -\frac{1}{24\gamma} (6\beta c_3 + 2\alpha c_2 + c_1 c_0 + \frac{1}{6} c_0^2) \) and other coefficients \( c_n \) are determined by

\[
c_{n+3} = (\gamma(n+3)(n+2)(n+1) + \frac{1}{2} \sum_{k=0}^{n+3} c_k c_{n-k}) / (\gamma(n+3)(n+2)(n+1)), \quad (n \geq 2).
\]

Especially, the traveling wave solutions correspond to the symmetry group generated by

\[
\text{Ad}(\exp(-\bar{c}V_2))V_1 = V_1 + \bar{c}V_2.
\]

For the generator \( V_1 + \bar{c}V_2 \), we have

\[
u(x,t) = f(\xi) = f(x - \bar{c}t),
\]

which convert the KBK equation into its traveling wave system

\[
\dddot{f} + \ddot{f} + \dot{f} + f = 0,
\]

(17)

where \( ' \) denotes \( d/d\xi \) and constant \( \bar{c} > 0 \) is wavespeed.

3. Existence of invariant submanifold \( \mathcal{M}_\epsilon \) in \( \mathbb{R}^3 \) and the flow on it. Integrating (17) once will yield

\[
\dddot{f} + \ddot{f} + \dot{f} + \frac{1}{2} f^2 - \bar{c} f = 0,
\]

(18)

where we set the integration constant \( e = 0 \) (Otherwise, we can always make a suitable homeomorphic transformation to eliminate it and convert the equation into the same form as equation (18) ). Firstly, we rescale the parameter \( \bar{\gamma} = \epsilon \gamma \) for the small \( \epsilon > 0 \). Then, the homeomorphic transformation \( f(\xi) = \gamma U(\xi) \) convert (18) into the form

\[
\epsilon U''' + \beta U'' + \alpha U' + \frac{1}{2} U^2 - \bar{c} U = 0,
\]

where \( \beta = \frac{\bar{\beta}}{\gamma}, \alpha = \frac{\bar{\alpha}}{\gamma}, \epsilon = \frac{\bar{\epsilon}}{\gamma} \). The equivalent system of it is

\[
\begin{cases}
U' = V, \\
V' = Y, \\
\epsilon Y' = \epsilon U - \frac{1}{2} U^2 - \alpha V - \beta Y.
\end{cases}
\]

(19)

With \( \xi = \epsilon \tau \), the ‘fast system’ associated with (19) has the form

\[
\begin{cases}
U' = \epsilon V, \\
V' = \epsilon Y, \\
\epsilon Y' = \epsilon U - \frac{1}{2} U^2 - \alpha V - \beta Y,
\end{cases}
\]

(20)
where \( \dot{\cdot} \) denotes \( d/d\tau \). (20) has two equilibria \( E_0 : (0, 0, 0) \) and \( E_1 : (2c, 0, 0) \) at which the Jacobian matrices are

\[
J(E_0) = \begin{bmatrix} 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \\ c & -\alpha & -\beta \end{bmatrix} \quad \text{and} \quad J(E_1) = \begin{bmatrix} 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \\ -c & -\alpha & -\beta \end{bmatrix}
\]

with corresponding characteristic equations

\[
\Delta(x) - c\epsilon^2 = 0 \quad \text{and} \quad \Delta(x) + c\epsilon^2 = 0
\]

respectively, where \( \Delta(x) = x^3 + \beta x^2 + \alpha x \). Function \( \Delta(x) \) has three zeros \( 0, -\frac{1}{2}\beta \pm \frac{1}{2}\sqrt{\beta^2 - 4\alpha c} \) in \( \mathbb{C} \). Noting that \( \left. \frac{d\Delta(x)}{dx} \right|_{x=0} \neq 0 \), we can see that \( J(E_0) \) and \( J(E_1) \) have at least a pair nonzero real eigenvalues with opposite signs. We want to prove that system (19) has two manifolds intersecting along a one-dimensional curve in \( \mathbb{R}^3 \). This curve just corresponds to a bounded traveling wave solution of the KBK equation.

If \( \epsilon \) is set to zero in (19), then \( U \) and \( V \) are governed by

\[
\begin{cases}
U' = V \\
V' = Y
\end{cases}
\]  (21)

where \( Y \) lies on the set

\[
\mathcal{M}_0 = \{(U, V, Y) \in \mathbb{R}^3 : cU - \frac{1}{2}U^2 - \alpha V - \beta Y = 0\}
\]

which is a two-dimensional submanifold in \( \mathbb{R}^3 \).

From [5], the manifold \( \mathcal{M}_0 \) is said to be normally hyperbolic if the linearization of the fast system, restricted to \( \mathcal{M}_0 \), has exactly \( \dim \mathcal{M}_0 \) eigenvalues on the imaginary axis, with the remainder of the system hyperbolic. The ‘fast system’ (20), restricted to the manifold \( \mathcal{M}_0 \), has the Jacobian matrix

\[
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c - U & -\alpha & -\beta \end{bmatrix}
\]

which has the eigenvalues 0, 0, \( -\beta \). It means that the manifold \( \mathcal{M}_0 \) is normally hyperbolic. So, Fenichels invariant manifold theory [5] guarantees that there exists a two-dimensional submanifold \( \mathcal{M}_\epsilon \) diffeomorphic to \( \mathcal{M}_0 \) in \( \mathbb{R}^3 \), which is within the distance \( \epsilon \) of \( \mathcal{M}_0 \) and is invariant for the flow (19).

Next, we assume that the manifold \( \mathcal{M}_\epsilon \) can be written as

\[
\mathcal{M}_\epsilon = \{(U, V, Y) \in \mathbb{R}^3 : Y = h(U, V, \epsilon)\},
\]

where function \( h(U, V, \epsilon) \) satisfies \( h(U, V, 0) = \frac{U^2}{\beta^3} - \frac{\alpha V}{\beta^3} - \frac{U^2}{2\beta^3} \). In order to obtain the approximation of manifold \( \mathcal{M}_\epsilon \), we expand function \( h(U, V, \epsilon) \) in Taylor series in the variable \( \epsilon \)

\[
h(U, V, \epsilon) = \frac{cU}{\beta} - \frac{\alpha V}{\beta} - \frac{U^2}{2\beta^3} + \epsilon h_1(U, V, 0) + O(\epsilon^2).
\]  (22)

Substituting (22) into (19), we get

\[
h_1(U, V, 0) = \frac{\alpha cU}{\beta^3} - \frac{(\alpha^2 + \beta c) V}{\beta^3} - \frac{\alpha U^2}{2\beta^3} + \frac{UV}{\beta^2}
\]
by power of \( \epsilon \). This allows one to write (19) as the following system

\[
\begin{align*}
U' &= V, \\
V' &= \frac{dU}{\partial U} - \frac{aV}{\partial V} - \frac{u^2}{2\beta} + \epsilon h(U,V,0) + O(\epsilon^2),
\end{align*}
\]

(23)

which determines the dynamics on the ‘slow’ manifold \( \mathcal{M}_\epsilon \).

Letting \( \epsilon = 0 \) in (23), we have

\[
\begin{align*}
U' &= V = P(U,V), \\
V' &= \frac{dU}{\partial U} - \frac{aV}{\partial V} - \frac{u^2}{2\beta} = Q(U,V),
\end{align*}
\]

(24)

which has two equilibria \( E_0' : (0,0) \) and \( E_1' : (2c,0) \), at which the Jacobian matrices are

\[
J(E_0') = \begin{bmatrix} 0 & 1 \\ \frac{\alpha}{\beta} & -\frac{\alpha}{\beta} \end{bmatrix} \quad \text{and} \quad J(E_1') = \begin{bmatrix} 0 & 1 \\ -\frac{\alpha}{\beta} & -\frac{\alpha}{\beta} \end{bmatrix}
\]

with the eigenvalues \( \lambda_{1,2}(E_0') = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta \epsilon}}{2\beta} \) and \( \lambda_{1,2}(E_1') = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta \epsilon}}{2\beta} \) respectively.

Without loss of generality, we only consider the case \( \alpha > 0, \beta > 0 \) for (24). In fact, all other cases can be converted to the case by suitable homeomorphic transformations. For instance, if \( \alpha < 0, \beta > 0 \), we can make the homeomorphic transformation \( V = -V, \xi = -\tau \). If \( \alpha > 0, \beta < 0 \), we can make the homeomorphic transformation \( U = -U + 2c, \xi = -\tau \). Similarly, if \( \alpha < 0, \beta < 0 \), we can make the homeomorphic transformation \( U = -U + 2c, V = -V \). For the case \( \alpha > 0, \beta > 0 \), \( E_0' \) is a saddle and \( E_1' \) is a stable node when \( \alpha^2 \geq 4\beta \epsilon \) or a stable focus when \( \alpha^2 < 4\beta \epsilon \). We will prove that there exists a heteroclinic orbit connecting \( E_0' \) and \( E_1' \).

**Theorem 3.1.** Suppose that \( \alpha > 0, \beta > 0 \). Then system (24) has a saddle-node heteroclinic orbit when \( c \leq \frac{\alpha^2}{4\beta} \) or a saddle-focus heteroclinic orbit when \( c > \frac{\alpha^2}{4\beta} \).

**Proof.** Note that the expression

\[
\frac{\partial P(U,V) / \partial U + \partial Q(U,V) / \partial V}{\partial V} = -\frac{\alpha}{\beta}
\]

has a fixed sign. By the Bendixon Theorem, system (24) has no closed orbit in phase plane (U-V plane).

When \( \alpha > 0, \beta > 0 \) and \( c \leq \frac{\alpha^2}{4\beta} \), we consider the problem in the triangle region

\[
D := \{(U,V) \in \mathbb{R}^2 : 0 < U < 2c, 0 < V < k(U - 2c)\}
\]

which is enclosed by three lines \( U = 0, V = 0 \) and \( V = k(U - 2c) \) in phase plane of (24), where \( k < 0 \) is a constant to be determined. In region \( D \), there is no equilibrium of (24). Firstly, by [3], there exists an unstable manifold \( \Gamma \) of the saddle \( E_0' \) in the first quadrant, which intersects neither the \( U \)-axis nor the \( V \)-axis in an enough small neighborhood of the origin \( E_0' \).

The vector field defined by (24) guarantees that orbits confined to the first quadrant move to the right as \( \xi \) increases. It means that \( \Gamma \) can not intersect the boundary \( \partial D_1 := \{(U,V) \in \mathbb{R}^2 : U = 0, 0 \leq V \leq -2kc\} \). On the boundary \( \partial D_2 := \{(U,V) \in \mathbb{R}^2 : 0 < U < 2c, V = 0\} \), \( \frac{dV}{\partial \xi} |_{\partial D_2} = \frac{dV}{\partial U} - \frac{u^2}{2\beta} > 0 \), which means
that Γ can not intersect the boundary ∂D₂. On the boundary ∂D₃ := \{(U,V) ∈ \mathbb{R}^2 : 0 < U < 2c, V = k(U - 2c)\}, we have
\[
\frac{dV}{dU} \big|_{∂D₃} = \frac{F(U)}{βk(U - 2c)} - \frac{α}{β},
\]
where F(U) = cU - \frac{1}{2}U^2. From the fact F(2c) = 0,
\[
\frac{F(U)}{U - 2c} = \frac{F(U) - F(2c)}{U - 2c} > F'(2c) = -c,
\]
which leads to \(\frac{dV}{dU} \big|_{∂D₃} = \frac{F(U)}{βk(U - 2c)} - \frac{α}{β} < -\frac{α}{β} - \frac{α}{β}\). So, there exists negative constant k satisfying\(\frac{c}{α} - \frac{α}{β} ≤ k\) since \(α^2 ≥ 4βc\). To be more specific, we can choose the constant k in the interval \((-\frac{α - \sqrt{α^2 - 4βc}}{2β}, -\frac{α + \sqrt{α^2 - 4βc}}{2β})\) freely to guarantee Γ not to intersect the boundary ∂D₃.

From the facts above, it concludes that Γ can not go out of the region D and \(E'₁\) is exactly the ω-limit set of it. Thus, we prove the existence of saddle-node heteroclinic orbit connecting \(E₀\) and \(E'₁\). Moreover, the fact \(dU/dξ = V > 0\) means that the bounded kink wave solution corresponding to Γ is monotone increasing with respect to ξ.

When \(α > 0, β > 0\) and \(c > \frac{α^2}{17}\), we need to consider the problem globally. With the Poincaré transformation \(U = 1/y, V = x/y\) and \(dτ = dξ/y\), (24) can be changed into
\[
\begin{cases}
  x' = -\frac{1}{2β} + \frac{μ}{β} - \frac{αxy}{β} - x^2y, \\
  y' = -xy^2,
\end{cases}
\]
which has no equilibrium in \((x, y)\)-plane when \(y = 0\).

Then by another Poincaré transformation \(U = x/y, V = 1/y\) and \(dτ = dξ/y\), (24) can be changed into
\[
\begin{cases}
  x' = y + P₂(x, y), \\
  y' = Q₂(x, y),
\end{cases}
\]
which has an equilibrium \((0, 0)\) corresponding to the equilibrium at infinity in V-axis, where \(P₂(x, y) = \frac{αxy}{β} - \frac{αx^2y}{β} + \frac{x^3}{2β}, Q₂(x, y) = \frac{αxy}{β} - \frac{αx^2y}{β} + \frac{x^2y}{2β}\). \((0, 0)\) is a degenerate equilibrium with nilpotent Jacobian matrix. So, we need more precise analysis for it.

By implicit function theorem, we can solve the equation \(y + p₂(x, y) = 0\) in an enough small neighborhood of the origin \((0, 0)\) and obtain
\[
y = φ(x) = -\frac{x^3}{2β} + \frac{αx^4}{2β^2} + O(x^5).
\]
Let
\[
Ψ(x) := Q₂(x, φ(x)) = -\frac{x^5}{4β^2} + \frac{αx^6}{2β^3} + O(x^7),
\]
\[
δ(x) := \frac{∂P₂(x, φ(x))}{∂x} + \frac{∂Q₂(x, φ(x))}{∂y} = \frac{2x^2}{β} - \frac{3αx^3}{2β^2} + O(x^4).
\]

By Theorem 7.2 and its corollary in [35], the degenerate equilibrium \((0, 0)\) is an unstable degenerate node. So, we can give the global phase portrait of system (24) in figure 1. It shows that there exists a saddle-focus heteroclinic orbit connecting \(E₀\) and \(E'₁\), which corresponds to the saddle-spiral shock wave. □
Assume that the heteroclinic connection of (24) can be expressed as \((U_0, V_0)\). The one thing left is to prove there exists analogous heteroclinic connection for (23) when \(\epsilon\) is sufficiently small.

In order to seek such connection in (23), we set
\[
U = U_0 + \epsilon \tilde{U}, \quad V = V_0 + \epsilon \tilde{V}.
\]
(25)
Substituting (25) into (23) and taking the lowest order in \(\epsilon\), we obtain the approximate system
\[
L \left[ \begin{array}{c} \tilde{U} \\ \tilde{V} \end{array} \right] = \left[ \begin{array}{c} 0 \\ h_1(U_0, V_0, 0) \end{array} \right]
\]
which governs \(\tilde{U}\) and \(\tilde{V}\), where the linear differential operator is defined by
\[
L \left[ \begin{array}{c} \tilde{U} \\ \tilde{V} \end{array} \right] = \frac{d}{d\xi} \left[ \begin{array}{c} \tilde{U} \\ \tilde{V} \end{array} \right] - \left[ \begin{array}{cc} 0 & 1 \\ -\frac{\alpha - U_0}{\beta} & -\frac{\alpha}{\beta} \end{array} \right] \left[ \begin{array}{c} \tilde{U} \\ \tilde{V} \end{array} \right].
\]

Next, we want to prove system (26) has a solution satisfying
\[
\tilde{U}, \quad \tilde{V} \to 0 \text{ as } \xi \to \pm\infty.
\]

By Fredholm theory [33], system (26) has a square-integrable solution if and only if the following compatibility condition holds
\[
\int_{-\infty}^{+\infty} \left\langle X(\xi), \left[ \begin{array}{c} 0 \\ h_1(U_0(\xi), V_0(\xi), 0) \end{array} \right] \right\rangle d\xi = 0 \quad (27)
\]
for all functions \(X(\xi)\) in the kernel of the adjoint operator \(L^*\), where \(\left\langle \cdot, \cdot \right\rangle\) is the inner product on \(\mathbb{R}^2\).

The adjoint system for (26) can be expressed as
\[
\frac{dX}{d\xi} = \left[ \begin{array}{cc} 0 & -\frac{\epsilon - U_0}{\beta} \\ -1 & \frac{\alpha}{\beta} \end{array} \right] X. \quad (28)
\]
Noting that \(\xi \to +\infty, U_0 \to 2c\), we can see that (28) has a constant matrix with two eigenvalues \(\lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4 \beta \epsilon}}{2\beta}\). Obviously, both eigenvalues \(\lambda_1\) and \(\lambda_2\) have positive real parts. Any solutions of (28), other than the zero solution, must grow.

Figure 1. Global phase portrait of (24) for \(\alpha > 0, \beta > 0\) and \(\alpha^2 < 4\beta c\).
exponentially. The only solution in $L^2$ is therefore a zero solution $X(\xi) = 0$, and consequently the Fredholm orthogonality condition (27) trivially holds. Thus, we prove the existence of analogous heteroclinic orbits for (23) when $\epsilon$ is sufficiently small, which implies the desired existence of kink wave and saddle-spiral shock wave for the KBK equation.

4. **Solitary wave and periodic wave solutions.** In this section, under the condition that $\alpha > 0$ and $\beta > 0$, we will prove the existence of homoclinic and periodic orbits of (23), which corresponds to the existence of solitary wave and periodic wave solutions of the KBK equation respectively.

Rescale the parameter $\alpha = \epsilon \tilde{\alpha}$. System (23) becomes

$$
\begin{align*}
U' &= V, \\
V' &= \frac{cU}{\beta} - \frac{U^2}{2\beta} + \epsilon G(U, V, \epsilon),
\end{align*}
$$

where $G(U, V, \epsilon) = -\frac{(\tilde{\alpha}\beta + c)V}{\beta^2} + \frac{U^2}{\beta^2} + O(\epsilon)$. When $\epsilon = 0$, (29) has a saddle $\tilde{E}_0 : (0, 0)$ and a center $\tilde{E}_1 : (2c, 0)$. In fact, in this case, (29) is a Hamiltonian system with the first integral

$$
H(U, V) := \frac{1}{2}V^2 - \frac{cU^2}{2\beta} + \frac{U^3}{6\beta}.
$$

By the properties of Hamiltonian system, there exists a homoclinic orbit $\Upsilon_0$ determined by the level curve $\frac{1}{2}V^2 - \frac{cU^2}{2\beta} + \frac{U^3}{6\beta} = 0$. In the compact region enclosed by homoclinic loop $\Upsilon_0 \cup \{\tilde{E}_0\}$, there is a family of periodic orbits $\Gamma_{\tilde{E}_1}(h)$ surrounding the center $\tilde{E}_1$ (see figure 2), where

$$
\Gamma_{\tilde{E}_1}(h) := \{(U, V) \in \mathbb{R}^2 : H(U, V) = h, h \in (-\frac{2c^3}{3\beta}, 0)\}.
$$

![Figure 2. Global phase portrait of (29) for $c = 1, \beta = 1$ and $\epsilon = 0$.](image)

**Theorem 4.1.** Suppose that $\alpha = \epsilon \tilde{\alpha} > 0$, $\beta > 0$ and $0 < \epsilon \ll 1$.

1. For arbitrary $\epsilon$, there exists a $c = c(\epsilon)$ ($c(0) = \frac{7}{5}\tilde{\alpha}\beta$) to guarantee that system (29) has a homoclinic orbit.

2. System (29) has a periodic orbit when $\tilde{\alpha}\beta < c < \frac{7}{5}\tilde{\alpha}\beta$. 
Proof. Consider the Mel’nikov function

\[ M(h, \alpha, \beta, c) = \oint_{\gamma_1} G(U, V, 0) dU = \oint_{\gamma_1} \left( -\frac{1}{\beta} - c + R(h) \right) dU = \oint_{\gamma_2} (-\beta c - R(h)), \]

where \( I_k := \oint_{\gamma_1} U^k dU, \) \( k = 0, 1, \) \( R(h) := I_1/I_0 \) and \( \Gamma \) corresponds to homoclinic loop \( \gamma_0 \cup \{ \tilde{\gamma}_0 \} \) or periodic orbits \( \Gamma_{E_1}(h). \)

First, we claim that

\[ \lim_{h \to -\frac{2c}{3\beta}} R(h) = 2c, \]

so that

\[ \lim_{h \to 0} R(h) = \frac{12}{7} c. \]

In fact, assume that \( (a(h), 0) \) and \( (b(h), 0) \) \( (a(h) < b(h)) \) are the points where periodic orbit \( \Gamma_{E_1}(h) \) intersects the \( U \)-axis. Noting that \( b(h) - a(h) \to 0 \) when \( h \to -\frac{2c}{3\beta}, \) we have

\[ \lim_{h \to -\frac{2c}{3\beta}} R(h) = \lim_{h \to -\frac{2c}{3\beta}} \frac{I_1(h)}{I_0(h)} = \lim_{h \to -\frac{2c}{3\beta}} \frac{\int_{a(h)}^{b(h)} UVdU}{\int_{a(h)}^{b(h)} VdU} = 2c, \]

implying (32).

Note the facts that homoclinic orbit \( \gamma_0 \) intersects \( U \)-axis at the point \((3c, 0)\) and periodic orbits \( \Gamma_{E_1}(h) \) approach the homoclinic orbit \( \gamma_0 \) as \( h \to 0. \) Thus

\[ \lim_{h \to 0} R(h) = \lim_{h \to 0} \frac{I_1(h)}{I_0(h)} = \frac{I_1^*}{I_0^*}, \]

where

\[ I_0^* := I_0(0) = \oint_{\gamma_0 \cup \{ E_1 \}} VdU = 2\int_0^{3c} V(U) dU, \]

\[ I_1^* := I_1(0) = \oint_{\gamma_0 \cup \{ E_1 \}} UVdU = 2\int_0^{3c} UV(U) dU, \]

and \( V(U) = \sqrt{-\frac{3}{\beta} - \frac{3}{\beta} + U}, U \in (0, 3c) \) is the explicit expression of the curve \( \gamma_0. \) Direct computation gives the result \( I_1^*/I_0^* = \frac{12}{7} c \) and thus proves (33). From the fact that \( I_0 > 0 \) is the area of the region enclosed by \( \Gamma, \) the Mel’nikov function has a simple zero \( c = \frac{7}{2} \beta. \) By [8], we know that for every sufficiently small \( \epsilon, \) there exists a corresponding \( c = c(\epsilon) \) \( (c(0) = \frac{7}{2} \beta) \) to guarantee that the homoclinic orbit survive from homoclinic bifurcation and therefore prove the existence of homoclinic orbit.

In order to prove the result of periodic orbit in theorem 4.1, we need to investigate Poincaré bifurcation for system (29). Firstly, we claim that function \( R(h) \) is monotone for \( h \in (-\frac{2c}{3\beta}, 0). \) Our strategy is to use Theorem 2 in [23] to prove it. Note that the energy function (30) has the form of variable separation. By Theorem 2 in [23], in order to prove the monotonicity of \( R(h), \) it suffices to verify the following conditions

(LZ1): \( \tilde{G}(U)(U - 2c) > 0(0 < 0) \) for \( U \in (0, 3c) \setminus \{2c\}, \)

(LZ2): \( f_1(U)f_2(\tilde{U}) > 0 \) for \( U \in (0, 2c), \)

(LZ3): \( \zeta(U) < 0(0 > 0) \) for \( U \in (0, 2c), \)

where \( \tilde{G}(U) := -\frac{U^2}{2\beta} + \frac{U^3}{6\beta}, \) \( \tilde{U} = \tilde{U}(U) \) is defined by \( \tilde{G}(U) = \tilde{G}(\tilde{U}), \) \( f_i(U) = U^{i-1}, i = 1, 2, \) and

\[ \zeta(U) := \frac{f_2(U)\tilde{G}''(\tilde{U}) - f_2(\tilde{U})\tilde{G}''(U)}{f_1(U)\tilde{G}'(U) - f_1(U)\tilde{G}'(U)}. \]

(34)
Condition (LZ1) can be verified since $\tilde{G}'(U)(U - 2c) = \frac{U(U - 2c)^2}{2h} > 0$ for $U \in (0, 3c) \setminus \{2c\}$. Condition (LZ2) holds naturally. Furthermore, from (34) one can calculate

$$\zeta'(U) = \frac{\tilde{M}(\tilde{U}, U) + \tilde{M}(U, \tilde{U})}{(U - 2c + U)^2},$$

where $\tilde{M}(U, \tilde{U}) := U(U - 2c)$. Note that $\tilde{M}(\tilde{U}, U) > 0$, $\tilde{M}(U, \tilde{U}) < 0$ and

$$\frac{d\tilde{U}}{dU} = \frac{G'(U)}{G'(\tilde{U})} \frac{U(U - 2c)}{U(U - 2c)} < 0$$

for $0 < U < 2c < \tilde{U} < 3c$. It follows from (35) that $\zeta'(U) > 0$ for $U \in (0, 2c)$. This concludes that $R'(h) < 0$ for $h \in (-\frac{2\gamma}{3\beta}, 0)$ by Theorem 2 in [23]. Thus, the monotonicity of $R(h)$ is proved.

The monotonicity of $R(h)$ implies that if $\tilde{\alpha}\beta < c < \frac{7}{5}\tilde{\alpha}\beta$, there exists a $h^* \in (-\frac{2\gamma}{3\beta}, 0)$ satisfies $M(h^*, \tilde{\alpha}, \beta, c) = 0$. Furthermore, $h^*$ is a simple zero of the Mel'nikov function $M(h, \tilde{\alpha}, \beta, c)$, since $M'(h^*, \tilde{\alpha}, \beta, c) = \frac{\tilde{M}(h^*)}{\beta}(-\tilde{\alpha}\beta - c + R(h^*)) + \frac{\tilde{M}(h^*)}{\beta}R'(h^*) = \frac{\tilde{M}(h^*)}{\beta}R'(h^*) < 0$. Therefore, by the Poincaré bifurcation theory [3], when $\tilde{\alpha}\beta < c < \frac{7}{5}\tilde{\alpha}\beta$ and $\epsilon$ is small enough, there exists a limit cycle for system (29), which is close to the periodic orbit $\Gamma_{E_1}(h^*)$.

5. Existence of other bounded traveling waves. In section 3 and 4, we obtain different wavespeed conditions to guarantee the existence of three basic types of bounded traveling waves of the KBK equation. From these bifurcation results, existence of more bounded traveling waves of the KBK equation can be identified. In fact, one can note that the region enclosed by a periodic orbit or a homoclinic loop is compact. It means that orbits in the compact region are bounded, and therefore correspond to the bounded traveling waves of the KBK equation. In addition, from properties of the dynamical system, it is well known that the limit cycle and the homoclinic loop are limit sets. So, when a limit cycle appears from the Poincare bifurcation, there will exist some connections between the equilibrium and the limit cycle, which correspond to a kind of oscillatory bounded traveling wave. Similarly, if a homoclinic orbit persists from the homoclinic bifurcation, there will exist some connections between the equilibrium and the homoclinic loop, which corresponds to another kind of oscillatory bounded traveling wave.

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Appendix 1.

$$\sigma_t = -a_t\tilde{\alpha}u_{x,x} - a_t u_{x} - a_t\tilde{\beta}u_{x,x,x} - a_t\tilde{\gamma}u_{x,x,x,x} + 3 a_t 2 u_{x,x}^2$$
$$-b_t u_{x,x} + a_t 2 u_{x,x} - \tilde{b}_t u_{x,x} + a_t\tilde{\beta}u_{x,x,x} + 2 a_t u_{x}^2$$
$$-b_t u_{x,x,x} + a_t\tilde{\gamma}u_{x,x,x,x} - d_t u_{x,x,x,x} + a_t\tilde{\gamma}u_{x,x,x,x,x,x}$$
$$-b_t u_{x,x,x,x} - d_t u_{x,x} + a_t\tilde{\beta}u_{x,x,x} + b_t u_{x,x} + d_t u + e_t$$
$$+5 a_t u_{x,x} + a_t\tilde{\gamma}u_{x,x,x} + 2 a u_{x,x} + 2 a_t \tilde{\alpha}u_{x,x}$$
$$+2 a\tilde{\beta}u_{x,x,x} + 2 a\tilde{\gamma}u_{x,x,x,x} + 2 a\tilde{\gamma}u_{x,x,x,x,x,x}$$
\[2 a \beta \gamma u_{x,x,x,x,x} + 10 a \gamma u_{x,x} u_{x,x} + 4 a u_x \bar{a} u_{x,x} - b u_x^2 - d u_{x,x} + 2 a u \bar{a} u_{x,x,x,x} \]

\[\sigma_x = \begin{align*}
- a_x u u_x & - a_x \bar{a} u_{x,x} - a_x \beta u_{x,x,x} - a_x \gamma u_{x,x,x,x} - a u_x^2 \\
- a u u_x & - a \bar{a} a u_{x,x} - a \beta u_{x,x,x} - a \gamma u_{x,x,x,x} + b u_x & \\
+ b u_{x,x} + d u_x & + d u_x + e_x \end{align*} \]

\[\sigma_{xx} = \begin{align*}
- a_x x u_x & - a_x \bar{a} u_{x,x} - a_x \beta u_{x,x,x} - a_x \gamma u_{x,x,x,x} \\
- 2 a_x u_x^2 & - 2 a_x u u_x - 2 a_x \bar{a} u_{x,x} - 2 a_x \beta u_{x,x,x} \\
- 2 a_x \gamma u_{x,x,x,x} - 3 a u x u_{x,x} - a u u_x & - a \bar{a} a u_{x,x} \\
- a \beta u_{x,x,x,x} & - a \gamma u_{x,x,x,x,x} + b u_{x,x} + b u_x & \\
+ b u_{x,x,x} + d u_{x,x} & + 2 d u_x + d u_{x,x} + e_{x,x} \end{align*} \]

\[\sigma_{xxx} = \begin{align*}
- 3 a_x \bar{a} u_{x,x} & - a_x \gamma u_{x,x,x} - a_x \beta u_{x,x,x} \\
- 3 a_x x u_x & - a_x \bar{a} u_{x,x} - a_x \beta u_{x,x,x} - a_x \gamma u_{x,x,x,x} \\
- 3 a_x \gamma u_{x,x,x,x} & - 3 a_x \beta u_{x,x,x,x} - 3 a_x \gamma u_{x,x,x,x,x} \\
- 4 a u_{x,x} & - a u u_{x,x} - a \bar{a} u_{x,x,x} - a \beta u_{x,x,x,x} \\
- a \gamma u_{x,x,x,x,x} & - 3 a_x \beta u_{x,x,x,x,x} - 3 a_x \gamma u_{x,x,x,x,x} \\
- 3 a_x \beta u_{x,x,x,x,x} & - a \bar{a} u_{x,x,x,x,x} \end{align*} \]

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