HEAT POLYNOMIALS, UMBRAL CORRESPONDENCE
AND BURGERS EQUATIONS.

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Abstract. We show that the umbral correspondence between differential equations can be achieved by means of a suitable transformation preserving the algebraic structure of the problems. We present the general properties of these transformations, derive explicit examples and discuss them in the case of the Appé and Sheffer polynomial families. We apply these transformations to non-linear equations, and discuss how the relevant solutions should be interpreted.

1. Introduction

In this article we show the importance of the umbral calculus, together with the monomiality principle, for solving differential equations.

Umbral calculus [1] provides us with a simple method to solve a large class of linear equations that are the umbral images of well known solvable differential equations. In this framework, we present some new umbral variables and use the umbral correspondence to solve modifications of the heat equation in terms of the solutions of the ordinary heat problem.

We consider the correspondence between the umbral methods and the more recently introduced principle of monomiality [2]. According to the monomiality principle, classes of polynomials can be treated as ordinary monomials in suitable coordinates. These coordinates are obtained from a couple of suitable operators spanning a Weyl algebra and playing the role of derivative and multiplicative operators.

We represent explicitly the umbral correspondence by a transformation given in term of an operator that preserves the algebraic structure of the differential problems under study. We consider the case of non-linear systems and comment on the application of umbral and operational techniques in the linearizable case.

In Section 2 we introduce the umbral correspondence between different coordinate and “differential” operators by a transformation and apply it to the case of the heat equation and its modifications. Section
3 is devoted to the discussion of the nonlinear Burgers equation and its modification when expressed in terms of different coordinates and “differential” operators related by the umbral correspondence, while Section 4 is devoted to present some concluding remarks, including some considerations on a more general view on time ordering problems.

2. Umbral Correspondence and Heat Polynomials.

According to ref. [1] we can define the umbral image of a given differential equation
\[ F(x, u(x), \partial_x u(x), \ldots) = 0, \]
by the operator equation
\[ F(\hat{q}, u(\hat{q}), \hat{p} u(\hat{q}), \ldots) = 0. \]
In eq. (2) \( \hat{q}, \hat{p} \) are some multiplicative and derivative operators such that
\[ [\hat{p}, \hat{q}] = \hat{1}, \]
and \( \hat{1} \) is a “ground state” on which the operators \( \hat{q}, \hat{p} \) are acting\(^1\), usually a constant, which can be set in the whole generality equal to one. From eq. (3) we get that formally we can always write:
\[ \hat{p} = \frac{\partial}{\partial \hat{q}}. \]
The operator \( \hat{q} \) will be referred as a multiplicative operator as, when acting on a basic polynomial of order \( n \), \( P_n(\hat{q}) \), we have:
\[ \hat{q} P_n(\hat{q}) = P_{n+1}(\hat{q}). \]
The derivative operator \( \hat{p} \) is an operator such that, when acting on a basic polynomial of order \( n \), \( P_n(\hat{q}) \), we get:
\[ \hat{p} P_n(\hat{q}) = n P_{n-1}(\hat{q}). \]
From an abstract point of view, the two problems \( (1) \) and \( (2) \) are equivalent under the umbral correspondence, even though the operators, playing the role of multiplication and derivative, can be realized in totally different ways.

We can construct, starting from the operator equation \( (2) \) a scalar equation by projecting it onto the “ground state” \( \hat{1} \):
\[ F(\hat{q}, u(\hat{q}), \hat{p} u(\hat{q}), \ldots) \hat{1} = 0. \]
\(^1\)Our operator algebra acts on this constant to generate its own irreducible representation, as it happens in the Lie algebraic treatment of special functions.
If eq. (1) is an equation for \( u(x) \) and \( u(x) \) is an entire solution, than \( u(\hat{q}) \) will be the corresponding operator solution of eq. (2). If eq. (1) is a linear equation than \( u(\hat{q})\hat{1} \) will be the solution of eq. (7).

We can, in principle, always construct a map \( \hat{T} \) which transforms one set of coordinates and the corresponding “differential” operator into the other. If such a transformation \( \hat{T} \) exists, it should have the following properties:

\[
\hat{T}\hat{T}^{-1} = \hat{1}, \quad \hat{T}x\hat{T}^{-1} = \hat{q}, \quad \hat{T}\partial_x\hat{T}^{-1} = \hat{p}, \quad \hat{T}\hat{1} = \hat{1}.
\]

From eqs. (8) it follows that \([\hat{p}, \hat{q}] = \hat{T}[\partial_x, x]\hat{T}^{-1}\). Thus constant commutation relations are preserved. So if there exists an umbral map \( \hat{T} \), for any choice of coordinates \( \hat{q} \) and “differential” operators \( \hat{p} \), the commutation relation (3) is satisfied. Moreover for any entire function \( f(x) \) we have:

\[
\hat{T} f(x)\hat{T}^{-1} = f(\hat{q}).
\]

We give here an example of the construction of such a transformation. Let us consider the following \( \hat{T} \) operator:

\[
\hat{T}_y = e^{y\partial_x^2}
\]

and with its inverse \( \hat{T}^{-1}_y = e^{-y\partial_x^2} \), where by \( \partial_x^2 \) we mean \( \partial_x \partial_x \). This transformation leaves the derivative operator unchanged, as

\[
[\hat{T}_y, \partial_x] = 0,
\]

but it modifies the \( x \) coordinate. From eq. (8) the new coordinates and derivative operators are:

\[
\hat{q} = x + 2y\partial_x, \quad \hat{p} = \partial_x.
\]

It is immediate to prove that eqs. (5, 6) are satisfied, together with any of their combinations. For example, we have:

\[
x\partial_x (x^n) = nx^n,
\]

and, by the umbral correspondence, we will also have:

\[
\hat{q}\hat{p}(\hat{q}^n) = n\hat{q}^n.
\]

It is interesting to note that, according to the modified Burchnall identity (2), once \( \hat{q}^n \) acts on unity we get:

\[
\hat{q}^n\hat{1} = (x + 2y\partial_x)^n 1 = \sum_{s=0}^{n} \binom{n}{s} H_{n-s}(x, y)(2y\partial_x)^s 1 = n! \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} x^{n-2r} y^r \frac{(n-2s)!}{(n-2s)!} = H_n(x, y).
\]
$H_n(x, y)$ is a two variable Hermite polynomial of the Kampè de Fériet type \cite{[3]}\textsuperscript{2}. Eq. \ref{eq:15} is just an example of the monomiality principle. Eq. \ref{eq:14}, which has as solution the monomials $\hat{q}^n$, can be rewritten as the Ordinary Differential Equation (ODE)

\begin{equation}
2yx\partial_x H_n(x, y) + \partial_{x,x} H_n(x, y) = nH_n(x, y).
\end{equation}

Eq. \ref{eq:15} is the ODE defining the family of basic polynomials \ref{eq:15}.

In conclusion the use of the transformation \ref{eq:10} has shown the umbral equivalence between ordinary monomials $x^n$ and the Hermite polynomials $H_n(x, y)$, which, according to the language of Ref. \cite{[2]} are “quasi monomials”.

Let us now consider the heat equation

\begin{equation}
\partial_t \Psi = \partial_x^2 \Psi,
\end{equation}

for a function $\Psi = \Psi(x, t)$, with the initial condition

\begin{equation}
\Psi|_{t=0} = x^n.
\end{equation}

The solution of eq. \ref{eq:16} \ref{eq:17} can be formally written as:

\begin{equation}
\Psi = (e^{t\partial_x^2} x^n) 1.
\end{equation}

Introducing the operator $\hat{T}_t = e^{t\partial_x^2}$, as in eq. \ref{eq:10}, eqs. \ref{eq:16} \ref{eq:17} can be solved in terms of the two variable Hermite polynomials \ref{eq:15}:

\begin{equation}
\Psi = (\hat{T}_t x^n) 1 = \hat{q}^n 1 = H_n(x, t).
\end{equation}

For this reason, this family of polynomials is some time referred to as the heat polynomials (hp) \cite{[4]}.

In the previous example we have considered a particular initial condition consisting of an ordinary monomial. If we replace such an initial condition by a generic entire function

\begin{equation}
\Psi|_{t=0} = f(x),
\end{equation}

we can write the solution of the heat equation \ref{eq:16} \ref{eq:20} as:

\begin{equation}
\Psi(x, t) = e^{t\partial_x^2} f(x) = f(\hat{q}) 1.
\end{equation}

In other words the umbrae $f(\hat{q}) 1$ of the initial conditions $f(x)$ are solutions of the heat equations \ref{eq:16} \ref{eq:20}.

\textsuperscript{2}The polynomials $H_n(x, y)$ are linked to the ordinary Laguerre polynomials by the identity $H_n(x, y) = i^n y^{\frac{n}{2}} H_n(\frac{ix}{\sqrt{y}})$, where $H_n(x) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r (2x)^{n-2r}}{(n-2r)! r!}$. 

Let us consider now the action of the transformation (10) with $y = \tau$ on the heat equation (16, 17). According to the previous discussion we find

\begin{equation}
\hat{T}_\tau \partial_t \Psi = \hat{T}_\tau \partial_x^2 \Psi \Rightarrow \partial_t \Psi(\hat{q}, t) \hat{1} = \hat{p}^2 \Psi(\hat{q}, t) \hat{1},
\end{equation}

with $\hat{q} = x + 2\tau \partial_x$. The solution of eq. (22) can now be written as:

\begin{equation}
\Psi(\hat{q}, t) \hat{1} = \hat{T}_t \hat{q}^n \hat{1} = \hat{H}_n(x, t + \tau).
\end{equation}

The transformation (10) transforms eq. (16) into the following heat equation

\begin{equation}
\partial_t \Psi(x + 2\tau \partial_x, t) \hat{1} = \partial_x^2 \Psi(x + 2\tau \partial_x, t) \hat{1},
\end{equation}

whose solution can be expressed in terms of the shifted in time two variable Hermite polynomials (23).

Analogous results can be obtained using the transformations induced by the operators

\begin{equation}
\hat{T}_{m,y} = e^{y\partial_x^m}, m > 2.
\end{equation}

In this case the corresponding coordinates and “differential” operators are:

\begin{equation}
\hat{q} = x + my\partial_x^{m-1}, \quad \hat{p} = \partial_x,
\end{equation}

and the corresponding basic polynomials, called higher order Hermite polynomials, are:

\begin{equation}
\hat{q}^n \hat{1} = H_n^{(m)}(x, y) = n! \sum_{r=0}^{[n/m]} \frac{x^{n-mr} y^r}{(n-mr)!r!}.
\end{equation}

So the solution of the higher order spatial derivative heat equation

\begin{equation}
\partial_t \Psi(x, t) = \partial_x^m \Psi(x, t), m > 2,
\end{equation}

\begin{equation}
\Psi(x, 0) = x^n,
\end{equation}

can be expressed in terms of the higher order Hermite polynomials (27).

The similarity transformation (10) transforms ordinary monomials into the two variable Hermite polynomials. This is not the only example. The Appèl polynomials [11] are characterized by an analogous property.

The Appèl polynomials $a_n(x)$ are defined through the generating function [11]

\begin{equation}
\sum_{n=0}^{\infty} \frac{t^n}{n!} a_n(x) = A(t) e^{xt},
\end{equation}

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A(t) = A(t) e^{xt},
\end{equation}

\begin{equation}
P_n(x, y) = H_n^{(m)}(x, y).
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where $A(t)$ is an undetermined function. Let us assume that there exists a domain for the variable $t$ where its Taylor expansion converges. By the obvious identity

$$t e^{xt} = \partial_x e^{xt}$$

and by the assumption that $A(t)$ is an entire function, we can rewrite eq. (29) as:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} a_n(x) = A(\partial_x) e^{tx}.$$  

By expanding in power series the exponential in eq. (31) and by equating the coefficients of the various powers of $t$, we end up with the following definition of the Appel polynomials:

$$a_n(x) = A(\partial_x) x^n.$$  

The operator $A(\partial_x)$ will be referred to as the Appel operator. Let us assume that also its inverse $[A(\partial_x)]^{-1}$ is well defined so that

$$[A(\partial_x)]^{-1} A(\partial_x) = \hat{1}.$$  

When $A(\partial_x) = \hat{T}_t$, we get the Hermite polynomials while, by choosing $A(\partial_x) = \frac{1}{1 - \partial_x}$ we get the Truncated Exponential Polynomials (TEP):

$$\bar{e}_n(x) = \frac{1}{1 - \partial_x} x^n,$$

i.e.

$$\bar{e}_n(x) = n! \sum_{r=0}^{n} x^r r!.$$  

We can introduce the Appel transformation $\hat{T}_A = A(\partial_x)$, as it satisfies all conditions. Then the Appel coordinates and derivatives are:

$$\hat{q} = A(\partial_x)x[A(\partial_x)]^{-1} + \frac{A'(\partial_x)}{A(\partial_x)},$$

$$\hat{p} = A(\partial_x) \partial_x [A(\partial_x)]^{-1} = \partial_x.$$  

The application of $\hat{T}_A$ on eq. (13) yields

$$(x + \frac{A'(\partial_x)}{A(\partial_x)}) \partial_x a_n(x) = na_n(x).$$  

In the case of the TEP, eq. (36) reads:

$$x \partial_x^2 \bar{e}_n(x) - (x + n) \partial_x \bar{e}_n(x) + n \bar{e}_n(x) = 0.$$  

As the derivative is invariant under an Appel transformation $\hat{T}_A$, the heat equation writes:

$$\partial_t \Psi(\hat{q}, t) = \partial_x^2 \Psi(\hat{q}, t).$$

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3 In deriving eq. (35) we have used the identity $[f(\partial_x), x] = f'(\partial_x)$
If the initial condition of our problem is specified by
\( \Psi(\hat{q}, t)|_{t=0} = \hat{q}^n \)
we can derive the correspondent associated heat polynomials as
\[
H_n(\hat{q}, t) = (e^{t\hat{q}^2} \hat{q}^n) = A_H n(x, t) = \frac{n!}{(n-2r)!r!} \sum_{r=0}^{n} a_{n-2r}(x) t^r.
\]
This family of polynomials satisfies the following recurrences
\[
\partial_x A H_n(x, t) = n A H_{n-1}(x, t),
\]
\[
(x + \frac{A'(\partial_x)}{A(\partial_x)} + 2t \partial_x) A H_n(x, t) = A H_{n+1}(x, t).
\]
We have shown here a simple procedure to obtain the \( \hat{T} \) operator which generate the umbral correspondence in the case of linear equations with constant coefficients. On the basis of such a correspondence we can define a wide class of Hermite type polynomials, playing the role of heat polynomials for various differential equations which are in umbral correspondence with the heat equation.

The situation is different if the equation of interest has \( x \)-dependent coefficients like, for example:
\[
\partial_t F(x, t) = x F(x, t) - \partial_x F(x, t), \quad F(x, 0) = f(x).
\]
In this case the umbral counterpart writes:
\[
\partial_t F(\hat{q}, t) = \hat{q} F(\hat{q}, t) - \hat{p} F(\hat{q}, t), \quad F(\hat{q}, 0) = f(\hat{q}).
\]
Taking into account the Weyl decoupling identity:
\[
e^{\hat{A}+\hat{B}} = e^{-\frac{1}{2} t^2} e^{\hat{A}} e^{\hat{B}}, \quad \text{if} \quad [\hat{A}, \hat{B}] = k \in C.
\]
the general solution of eq. (43) can be written, \(^4\)
\[
F(\hat{q}, t) = e^{(\hat{q}-\hat{p})t} f(\hat{q}) = e^{-\frac{1}{2} t^2} e^{\hat{q}t} e^{-\hat{p}t} f(\hat{q}) = e^{-\frac{1}{2} t^2} e^{\hat{q}t} f(\hat{q} - t).
\]
Equation (42) projected
\[
\partial_t F(\hat{q}, t) \hat{1} = \hat{q} F(\hat{q}, t) \hat{1} - \hat{p} F(\hat{q}, t) \hat{1}, \quad F(\hat{q}, 0) \hat{1} = f(\hat{q}) \hat{1},
\]
will have the solution
\[
F(\hat{q}, t) \hat{1} = \frac{e^{-\frac{1}{2} t^2} e^{\hat{q}t} f(\hat{q} - t)}{1 - t},
\]
if \( A(\partial_x) = \frac{1}{1-\partial_x} \) and \( f(x) = 1. \)

\(^4\)Let us note that if \([\hat{p}, \hat{q}] = 1 \rightarrow e^{\hat{pt}} f(\hat{q}) = f(\hat{q} + t).\)
We will now show how the umbral transformation can be exploited to solve in a straightforward way modified heat equations. Let us consider the equation

\[(47) \quad \partial_t(t\partial_t \Psi) = \partial_x^2 \Psi.\]

Eq. (47) can be thought a heat equation, in which the time derivative is replaced by

\[(48) \quad \hat{p} = L \hat{D}_t = \partial_t \partial_t.\]

\(\hat{p}\) is sometimes referred to as the Laguerre derivative \([5,6]\) and it satisfies the identity \(\hat{p}^n = L \hat{D}^n_t = \partial_t^n t^n \partial_t^n\). We can associate to the Laguerre derivative a multiplicative operator

\[(49) \quad \hat{q} = \partial_t^{-1}\]

such that \(5\)

\[(50) \quad [\hat{p}, \hat{q}] = [\partial_t \partial_t, \partial_t^{-1}] = 1.\]

The action of the operator \(\partial_t^{-n}\) on a given function \(f(t)\) can be written as a Cauchy repeated integral, namely

\[(51) \quad \partial_t^{-n} f(t) = \frac{1}{\Gamma(n+1)} \int_0^\infty (t - \xi)^n f(\xi) d\xi.\]

From eq. (51) follows immediately \([2,7]\) that

\[(52) \quad (\partial_t^{-1})^n 1 = \frac{t^n}{n!}.\]

In the transformed variables, eq. (47) can be written as:

\[(53) \quad \hat{p} \Psi(x, \hat{q}) = \partial_x^2 \Psi(x, \hat{q}).\]

The associated \(t\) reads:

\[(54) \quad 2H_n(x, t) = H_n(x, \hat{q}) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2r} t^r}{(n-2r)1 (r!)^2}.\]

The polynomial (54) belongs to the family of hybrid polynomials. These polynomials are situated in between the Hermite and Laguerre polynomials \([2,5]\). By direct calculation one can prove that the polynomials (54) solve eq. (47).

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\(^5\)We define \(\partial_t^{-1}\) as the inverse of the derivative operator so that \(\partial_t^{-1} \partial_t = \partial_t \partial_t^{-1} = 1\) \([2,12]\). This assumption only holds if \(\partial_t^{-1}\) is applied to entire functions.
In this example we have introduced the umbral correspondence between eq. (47) and (53) by defining the new coordinates and derivatives, \((\hat{q}, \hat{p})\), given by eqs. (48, 49). Let us look for the transformation \(\hat{T}_L\) which provide the “transition”

\[
H_n(x, y) \rightarrow 2H_n(x, y).
\]

Formally this transformation is obtained by requiring

\[
\hat{T}_L \hat{T}_L^{-1} = \partial_t^{-1} = \hat{q}.
\]

The explicit form of \(\hat{T}_L\) can be derived by noting that the correspondence between \(t\) and \(\hat{q}\) implies \(t^n \rightarrow \frac{\hat{t}^n}{n!}\).

Given a function \(f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f_n\), taking into account eqs. (52, 53) we can write:

\[
\hat{T}_L^{-1} f(t) = \int_0^\infty e^{-t \xi} f(x \xi) d\xi = \sum_{n=0}^{\infty} f_n t^n.
\]

Thus \(\hat{T}_L\) can be viewed as the inverse of the Laplace transform (57) (see also ref. [8]).

In the previous paragraphs we have considered transformations associated with Appel polynomials which leave the derivative operator invariant. Here we will consider more general transformations, when also the derivative operator is changed. We will call one such transformation a Sheffer transformation.

The Sheffer polynomials \(\sigma_n(x)\) are a natural extension of the Appel polynomials \(a_n(x)\). They are generated by

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} \sigma_n(x) = A(t) \exp(xB(t)),
\]

where \(B(t)\) is, as \(A(t)\), an entire function. Following ref. [9] we can prove the quasi-monomiality of \(\sigma_n(x)\). If we take \(A(t) = 1\) we can easily construct \(\hat{p}\)-operators which do not coincide with the derivative operator \(\partial_x\) and the associated Sheffer polynomials \(s_n(x)\) are given by

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} s_n(x) = \exp(xB(t)).
\]

Multiplying both sides of eq. (59) by the operator \(B^{-1}(\partial_x)\), we find:

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} B^{-1}(\partial_x) s_n(x) = B^{-1}(\partial_x) \exp(xB(t)) = t \exp(xB(t)).
\]
From eq. (60) we deduce that the derivative operator for the Sheffer polynomials \( s_n(x) \) is given by

\[
\hat{p} = B^{-1}(\partial_x).
\]

The multiplicative operator can be obtained by taking the derivative with respect to \( t \) of both sides of eq. (59):

\[
\sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!} s_n(x) = xB'(t) \exp(xB(t)) = xB'(B^{-1}(\partial_x)) \exp(xB(t)).
\]

Thus the multiplicative operator corresponding to \( \hat{p} \) given by eq. (61) takes the form

\[
\hat{q} = xB'(\hat{p}).
\]

A typical example of coordinate and momenta operators associated to a Sheffer transform are

\[
\hat{p} = e^{\partial_x} - 1, \quad \hat{q} = xe^{-\partial_x},
\]

when \( B(\hat{p}) = \ln(\hat{p}+1) \) and thus \( B'(\hat{p}) = \frac{1}{\hat{p}+1} \). As \( e^{\partial_x} f(x) = f(x+1) \), the derivative \( \hat{p} \) acts on a function \( f(x) \) as a shift operator. The associated polynomials are the lower factorial specified by

\[
s_n(x) = \frac{\Gamma(x+1)}{\Gamma(x+1-n)} = (x)_n.
\]

The heat equation, in which the spatial derivatives are substituted by \( \hat{p} \) given by eq. (63), admits a solution given by the corresponding hp, namely

\[
H_n(\hat{q}, t) = n! \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(x)_{n-2r} t^r}{(n-2r)!r!}.
\]

By a proper definition of the function \( B(\hat{p}) \) we can obtain any discrete representation of the derivative operator. For example, by choosing \( B(\hat{p}) = \ln(\hat{p} + \sqrt{\hat{p}^2 + 1}) \) we get \( \hat{p} = \frac{\hat{p}^2 - e^{-\hat{p}}}{2}, \quad B'(\hat{p}) = \frac{1}{\sqrt{\hat{p}^2 + 1}} \) and thus

\[
\hat{q} = 2x(e^{\partial_x} + e^{\partial_x})^{-1} \quad [?].
\]

\[\text{If } A(t) \neq 1 \text{ the Sheffer polynomials can now be defined by the operational rule } \hat{s}_n(x) = A(\hat{p}) s_n(x). \text{ The transformation } T_S \text{ is therefore specified by the following identities, which preserve the Weyl algebra structure:}
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T_S \partial_x T_S^{-1} = \hat{p} = B^{-1}(\partial_x),
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T_S x T_S^{-1} = \hat{q} = xB'(\hat{p}) + \frac{A'(\hat{p})}{A(\hat{p})}.
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when \( B(\hat{p}) = \ln(\hat{p}+1) \) and thus \( B'(\hat{p}) = \frac{1}{\hat{p}+1} \). As \( e^{\partial_x} f(x) = f(x+1) \), the derivative \( \hat{p} \) acts on a function \( f(x) \) as a shift operator. The associated polynomials are the lower factorial specified by

\[
s_n(x) = \frac{\Gamma(x+1)}{\Gamma(x+1-n)} = (x)_n.
\]

The heat equation, in which the spatial derivatives are substituted by \( \hat{p} \) given by eq. (63), admits a solution given by the corresponding hp, namely

\[
H_n(\hat{q}, t) = n! \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(x)_{n-2r} t^r}{(n-2r)!r!}.
\]

By a proper definition of the function \( B(\hat{p}) \) we can obtain any discrete representation of the derivative operator. For example, by choosing \( B(\hat{p}) = \ln(\hat{p} + \sqrt{\hat{p}^2 + 1}) \) we get \( \hat{p} = \frac{\hat{p}^2 - e^{-\hat{p}}}{2}, \quad B'(\hat{p}) = \frac{1}{\sqrt{\hat{p}^2 + 1}} \) and thus

\[
\hat{q} = 2x(e^{\partial_x} + e^{\partial_x})^{-1} \quad [?].
\]

\[\text{If } A(t) \neq 1 \text{ the Sheffer polynomials can now be defined by the operational rule } \hat{s}_n(x) = A(\hat{p}) s_n(x). \text{ The transformation } T_S \text{ is therefore specified by the following identities, which preserve the Weyl algebra structure:}
\]

\[
T_S \partial_x T_S^{-1} = \hat{p} = B^{-1}(\partial_x),
\]

\[
T_S x T_S^{-1} = \hat{q} = xB'(\hat{p}) + \frac{A'(\hat{p})}{A(\hat{p})}.
\]
We have considered so far the case in which the $\hat{T}$ transform has affected the spatial or the time components of the equation but not both. Let us consider the heat equation

\begin{equation}
[(e^{\partial_t} - 1)\Psi]_1 = [(e^{\partial_x} - 1)^2\Psi]_1,
\end{equation}

whose solution will be generated by a double transformation. We thus get:

\begin{equation}
\Psi(x, t) = n! \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(x)^{n-2r}}{(n-2r)!r!} (t)^r.
\end{equation}

The solution of the heat equation

\begin{equation}
\ln(1 + \partial_t)\Psi = (e^{\partial_x} - 1)^2\Psi_1,
\end{equation}

can be obtained using transformations associated with Sheffer Bell-type polynomials in the $t$-variable. In this case we have:

\begin{equation}
\hat{q} = t(1 + \partial_t), \quad \hat{q}^n_1 = b_n(t) = \sum_{k=1}^{n} S_2(n, k)t^k.
\end{equation}

In eq. (68) $S_2(n, k)$ is a Stirling number of second kind. The associated hp can be obtained from eq. (66) by replacing $(t)^r \rightarrow b_r(t)$.

### 3. Burgers Equations and Heat Polynomials

It is well known that the Hopf-Cole transformation \cite{10}

\begin{equation}
u(x, t) = \frac{\partial_x \Psi}{\Psi},
\end{equation}

allows us to recast the heat equation \cite{16} as a non linear equation for the function $u(x, t)$, i.e.

\begin{equation}\partial_t u = \partial_x^2 u + \partial_x(u^2).
\end{equation}

From the discussion of the previous section it follows that the hp are natural candidates for the solution of equation (70) as

\begin{equation}u(x, t) = \frac{n H_{n-1}(x, t)}{H_n(x, t)}.
\end{equation}

An analogous result can be obtained from the higher order heat equations \cite{28} for higher order Burgers equations \cite{10}. For example, in the case of eq. \cite{28} with $m = 3$, the transformation \cite{69} yields

\begin{equation}\partial_t u = \partial_x(u^3) + 3(\partial_x u)^2 + 3u\partial_x^2 u + \partial_x^3 u.
\end{equation}

Eq. (72) is satisfied by the higher order hp (hohp) of order 3, given by eq. \cite{27} with $m=3$. 
It is obvious that any operator function \( f(\hat{q}) \), solution of the umbral heat equation can be used to get, via the corresponding Hopf-Cole transformation,

\[
(73) \quad u = [f(\hat{q})]^{-1}\hat{p}f(\hat{q}).
\]

a solution of the Burgers equation \( \partial_t u = \hat{p}^2 \hat{u} \).\

(74) \quad \partial_t \hat{u} = \hat{p}^2 \hat{u} + \hat{p}(\hat{u}^2).

We can use the case of the Laguerre derivative \( (48, 49) \) to exemplify the problems one encounters when considering the nonlinear Burgers in umbral variables. Taking advantage of the umbral correspondence and using the Hopf-Cole transformation

\[
(75) \quad \omega(x, \partial_t^{-1}) = (\Psi(x, \partial_t^{-1}))^{-1}\Psi_x(x, \partial_t^{-1}),
\]

we can write down the non-linear operator equation

\[
(76) \quad \partial_t \partial_t \omega(x, \partial_t^{-1}) = \omega_{xx}(x, \partial_t^{-1}) + 2[\omega_x(x, \partial_t^{-1})\omega(x, \partial_t^{-1})].
\]

By projection onto a constant, we obtain from eq. \( (76) \) a nonlinear functional equation.

The difficulty is now to interpret properly the umbral operator function \( \omega(x, \partial_t^{-1}) \) and its functional form \( \tilde{\omega}(x, t) = \omega(x, \partial_t^{-1})\hat{1} \) in relation with the equation obtained by projecting eq. \( (76) \). As will be explained in the following, the umbral non-linear equation \( (76) \) should be interpreted as

\[
(77) \quad p_t \omega = \partial_x^2 \omega + 2(\hat{\Omega} \omega), \quad \hat{\Omega} = \partial_x \omega.
\]

In other words eq. \( (77) \) should be considered more like an identity rather than an equation.

The logical steps to check the validity of the above statement are the following:

(1) We use the \( \hat{T} \) transformation to derive the umbral form of the Burgers equation,

(2) We use again \( \hat{T} \) to infer the solution from that of the differential case.

Let us consider a particular solution of the Laguerre heat equation \( (50) \), the hybrid hp \( (51) \) with \( n = 2 \):

\[
(78) \quad H_{2,0}(x, \partial_t^{-1}) = x^2 + 2\partial_t^{-1}.
\]

A solution of the Hopf-Cole transformation \( (75) \) is given by

\[
(79) \quad \omega(x, \partial_t^{-1}) = 2(x^2 + 2\partial_t^{-1})^{-1}x.
\]
The solution (79) in the coordinates $x, t$ is obtained by expanding its right hand side in power series. We find:

\[
\tilde{\omega} = \frac{2}{x} \sum_{r=0}^{\infty} \left( -\frac{2\partial^{-1}}{x^2} \right)^r 1.
\]  

(80)

Taking into account eq. (52), eq. (80) can be summed up and it reads:

\[
\tilde{\omega} = \frac{2}{x} e^{-\frac{2t}{x^2}}.
\]  

(81)

The operator $\hat{\Omega} = -2 \frac{x^2-2\partial^{-1}}{(x^2+2\partial^{-1})^2}$ is clearly different from the derivative with respect to $x$ of the function $\tilde{\omega}$.

The considerations developed so far suggest that, strictly speaking, the projection of eq. (77) cannot be considered a non-linear equation but rather an umbral identity, satisfied by the “umbralized” operator solutions of the original equation. The umbral correspondences we have considered before are only relevant for linear equations. The effect of $\hat{T}$ on a non-linear equation does not present any difficulty provided we define the rules defining the transformation in a clear way and we properly understand the mathematical meaning of the obtained results.

According to eqs. (8) we have

\[
\hat{T}[f(x)^2]1 = f(\hat{q})\hat{T}f(x)1 = [f(\hat{q})]^21 \neq [f(\hat{q})1]^2.
\]  

(82)

In the case of $\hat{q}$ given by eq. (12) we find that

\[
(\hat{q}1)^{2n} = H_n(x, y)^2 \neq \hat{q}^{2n}1 = H_{2n}(x, y).
\]  

(83)

The remarks contained in eqs. (83, 82) complete the considerations presented up above. Therefore, we can state the following result: given a non-linear differential equation for a scalar field $\Phi(x, t)$, its umbral image under the $\hat{T}$ map does not provide, by projection, a nonlinear scalar equation. The nonlinear umbral operator equation admits the solution

\[
\Phi(\hat{q}, t) = \hat{T}\Phi(x, t)\hat{T}^{-1}.
\]  

(84)

Following this prescription, let us consider, as a final example, the umbral counterpart of the sine-Gordon equation

\[
[\partial_t^2 - \hat{p}^2] \Phi(\hat{q}, t)1 = \tilde{F} \Phi(\hat{q}, t)1, \quad \tilde{F} = \sum_{r=1}^{\infty} \frac{(-1)^r (2(\Phi(\hat{q}, t))^2r}{(2r+1)!}.
\]  

(85)

$\Phi(\hat{q}, t)1$ will not be the solution of eq. (85). The equation

\[
[\partial_t^2 - \hat{p}^2] \Phi(\hat{q}, t)1 = [\sin(\Phi(\hat{q}, t))1],
\]  

(86)
is not the umbral counterpart of the sine-Gordon equation.

4. Concluding Remarks

In this paper we have made extensive use of operator methods; in particular we have obtained the solutions of some differential equations using umbral calculus. For the ordinary heat equation the evolution operator coincides with a $\hat{T}$ transformation. In the more general example of eq. (42) we have found that the solution depends on the operator

\[ \hat{T}\hat{U}\hat{T}^{-1} = e^{(\hat{q} - \hat{p})t}, \]

where by $\hat{U} = e^{(x - \partial_x)t}$ we mean the evolution operator of the equation in ordinary space. The two equations (42, 43) are formally equivalent as the $\hat{T}$ transformation leaves the Weyl group invariant.

Problems may arise when we consider transformations which involve the time and in which the derivative operator is modified. In this case the exponential is not an eigenfunction of the corresponding derivative operator. For example, in the case of the Laguerre derivative we have

\[ e^{\alpha t} 1 = e^{\alpha \partial_t^{-1}} 1 = \sum_{r=0}^{\infty} \frac{(\alpha t)^r}{r!} - 1 = C_0(\alpha t) = I_0(2\sqrt{\alpha t}), \]

where $C_0(t)$ is the 0-th order Tricomi function and $I_0(2\sqrt{\alpha t})$ the modified Bessel function of first kind \[12\]. It is evident that the function $C_0(t)$ is an eigenfunction of the Laguerre derivative as

\[ (\partial_t \partial_t)C_0(\alpha t) = \alpha C_0(\alpha t). \]

Accordingly we can write the solution of the equation

\[ LD_t F(x, t) 1 = xF(x, t) 1 - \partial_x F(x, t) 1, \quad F(x, 0) = 1, \]

as

\[ F(x, t) = e^{-\frac{t^2}{2}} e^{xt} = \sum_{r=0}^{\infty} \frac{t^r}{(r!)^2} H_n(x, -\frac{1}{2}). \]

The above series expansion defines a 0-th order Bessel-Hermite function discussed in Refs. [2] [5].

The above example shows that even in the case of linear equations, non trivial problems may arise.

\[ \text{The Laguerre derivative } \partial_t \partial_t = \partial_t + t\partial_t^2 \text{ contains a derivative of second order, therefore we may expect to have two eigenfunctions. However, the second eigenfunctions has a singularity at } t = 0 \text{ and therefore it cannot be used to fulfill the condition } \hat{U}(0) = 1. \]
A further problem arises when discussing the meaning of the transformations from the physical point of view. The use of this transformations to solve a Schrödinger equation in its umbral form may give rise to problems due to the non hermiticity of the variable involved. In fact such a property is not in general preserved by these transformation and this aspect of the problem requires a careful understanding which will be discussed in a forthcoming paper.

In a forthcoming investigation we will discuss more in detail the mathematical properties of the evolution operator under umbral transformation and we will carefully treat the problems arising when time ordered products are involved [13].

Work is also in progress for a better characterization of the solutions of the umbral Burgers equation, using perturbation methods.

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