EVALUATION OF BIOT-SAVART INTEGRALS ON TETRAHEDRAL MESHES

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Abstract. An arithmetically simple method has been developed for the evaluation of Biot–Savart integrals on tetrahedralized distributions of vorticity. In place of the usual approach of analytical formulae for the velocity induced by a linear distribution of vorticity on a tetrahedron, the integration is performed using Gaussian quadrature and a ray tracing technique from computer graphics. This eliminates completely the need for the evaluation of square roots, logarithms and arc tangents, and almost completely eliminates the requirement for trigonometric functions, with no operation more costly than a division required during the main calculation loop. An assessment of the algorithm’s performance is presented, demonstrating its accuracy, second order convergence and near-linear speedup on parallel systems.

Key words. Biot-Savart integral, tetrahedral mesh, vortex method, numerical integration, ray tracing

AMS subject classifications. 76B47, 65D30

1. Introduction. An important part of many calculations in fluid dynamics and electromagnetism is the evaluation of a Biot-Savart integral, for velocity in fluid dynamics and for magnetic field in electromagnetism. In fluid dynamics, the source term is vorticity while in electromagnetism it is current. The Biot-Savart integral for velocity \( \mathbf{v} \) due to a distribution of vorticity \( \omega \) over a volume \( V \) is:

\[
\mathbf{v}(\mathbf{x}) = -\frac{1}{4\pi} \int_V \frac{\mathbf{r} \times \omega(\mathbf{x}_1)}{R^3} \, dV,
\]

where \( \mathbf{r} = \mathbf{x} - \mathbf{x}_1 \), \( R = |\mathbf{r}| \) and subscript 1 indicates a variable of integration. For electromagnetic calculations, \( \mathbf{v} \) is the magnetic field and \( \omega \) the current in the region \( V \). For the purposes of this paper, it is assumed that the volume \( V \) is discretized into tetrahedral elements within which the source varies linearly. The question then is how to compute the resulting field \( \mathbf{v} \). This paper is motivated by the requirement to evaluate velocities in Lagrangian vortex methods, where a moving distribution of control points is tetrahedralized at each time step as an aid to velocity calculation. Such a method has been implemented by Marshall et al. [4], using a mixture of analytical formulae and Gaussian quadrature to carry out the integration of equation 1.1. The aim of this paper is to develop a method which is arithmetically simpler than that used hereto.

A number of analytical formulae and numerical procedures have been developed for the evaluation of equation 1.1. A recent general paper is that of Suh [11] who applies Stokes’ theorem to reduce the volume integral over an element to a number of surface integrals which are in turn reduced to line integrals which can be evaluated analytically. This work is similar to that of Newman [6] who derived equivalent formulae, focussing on applications in fluid dynamics. In electromagnetism, there is an extensive literature on the evaluation of equation 1.1, including analytical [2, 7, 12–14] and numerical [3] approaches. As this paper is motivated by the fluid dynamical

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problem, it will use the terms ‘velocity’ and ‘vorticity’ but it should be noted that there are also many useful related results in the electromagnetism literature.

In evaluating the velocity at the nodes of a vorticity distribution which is discretized into tetrahedra, a Gaussian quadrature can be used for tetrahedra which are far from the evaluation point \( \mathbf{x} \), but (integrable) singularities in the integrand make this awkward for nearby elements. In this case, the standard approach is to use analytical formulae which have the benefit of being exact and non-singular. The disadvantage, as noted by Marshall et al. [4], is that such formulae [6] are computationally expensive, requiring in this case 12 logarithms and 24 arc tangents per tetrahedron. Even if the analytical formulae are only used for ‘nearby’ elements, they still represent a large part of the velocity computation. The aim of this paper is to present a velocity computation method which is arithmetically simple, requiring no operation more onerous than a division during the main calculation, by using a ray-tracing method borrowed from computer graphics. The resulting method can then be used in codes as a ‘plug-in’ replacement for previous techniques.

2. Velocity evaluation. The integral of equation 1.1 is to be evaluated at a number of points \( \mathbf{x} \), which also form the nodes of a distribution of vorticity \( \omega \). In this case, as in the work of Marshall et al. [4], the nodes are tetrahedralized so that they form the vertices of a collection of tetrahedra. It is further assumed that vorticity varies linearly over the elements. The aim is now to avoid the computational effort involved in the standard analytical approach to quadrature and develop a method which is arithmetically as simple as possible. In the words of Richardson, this is a problem where it may be quicker to arrive at a destination at the “footpace of arithmetic”, rather than “on the swift steed of symbolic analysis” [8].

First, equation 1.1 is rewritten in spherical polar coordinates centred on the evaluation point \( \mathbf{x} \):

\[
v = \frac{1}{4\pi} \int_{0}^{\pi} \int_{0}^{\pi} \int_{-\infty}^{\infty} \hat{s} \times \omega \, dR \sin \phi \, d\phi \, d\theta, \tag{2.1}
\]

where \( \mathbf{r} = \mathbf{x} - \mathbf{x}_1 = -R\hat{s} \) with the unit vector \( \hat{s} = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \). This transformation also eliminates the integrable singularity in the integrand, making the integration rather easier from a numerical point of view. The azimuthal and polar integrals are evaluated using Gaussian quadratures so that:

\[
v \approx \sum_{n=1}^{N} \sum_{m=1}^{M} \sin \phi_n \frac{w_{n}^{(N)} w_{m}^{(M)}}{16\pi^2} \int_{-\infty}^{\infty} \hat{s}_{nm} \times \omega \, dR, \tag{2.2}
\]

where

\[
\phi_n = (1 + t_n^{(N)})\pi/2,
\]
\[
\theta_m = (1 + t_m^{(M)})\pi/2,
\]
\[
\hat{s}_{nm} = (\sin \phi_n \cos \theta_m, \sin \phi_n \sin \theta_m, \cos \phi_n),
\]

and \((t_i^{(N)}, w_i^{(N)}), i = 1, \ldots, N\), are the nodes and weights of an \( N \) point Gaussian quadrature with \(-1 \leq t \leq 1\). The main computation is now reduced to integrals over \( R \) along rays in the direction \( \hat{s} \).
2.1. Integration over tetrahedra. The radial integral of equation 2.2 must now be evaluated, to include the contribution of the mesh tetrahedra. The approach is summarized in figure 2.1: if the ray from $x$ in the direction $\hat{s}$ intersects the tetrahedron, i.e. passes through two of its faces, the tetrahedron makes a finite contribution to the integral. The overall integral is:

$$I(x, \theta, \phi) = \int_{-\infty}^{\infty} \hat{s} \times \omega \, dR.$$  \hfill (2.3)$$

If the ray enters a tetrahedron at $R = R_0$ and exits at $R = R_1$,

$$I(x, \theta, \phi) = \sum \int_{R_0}^{R_1} \hat{s} \times \omega \, dR = \frac{1}{2} \sum (R_1 - R_0)\hat{s} \times (\omega_0 + \omega_1),$$  \hfill (2.4)$$

summing over tetrahedra which are intersected by the ray. The vorticities $\omega_0$ and $\omega_1$ are those at the entry and exit points $R_0$ and $R_1$ and, as noted previously, $\omega$ varies linearly between them, so that equation 2.4 is exact. If the ray enters or exits a face of the tetrahedron at area coordinates $(u,v)$, the vorticity $\omega$ is:

$$\omega = (1-u-v)\omega_0 + u\omega_1 + v\omega_2,$$

where $\omega_i, \ i = 0,1,2$ are the vorticities at the triangle vertices. This integral is arithmetically very simple, with no operation more complex than a division, not even requiring a square root for the calculation of a distance.

The remaining part of the algorithm is how the intersection, if any, of a ray with a tetrahedron can be determined. This is found using a method from computer graphics. Möller and Trumbore [5] give a method and C source code for determining the area coordinates $(u,v)$ and directed distance $R$ of the intersection point of a ray with a triangle. Their method is very efficient requiring at most one division and no operation more complex than this. It is used in the algorithm of this paper to find the intersections of a ray with a tetrahedron, by checking the faces in turn. In order to accelerate the procedure, an initial check is carried out on the tetrahedron as a whole.
Figure 2.2 shows the check for the case of a triangle. The vector $\mathbf{r}$ is the vector from $\mathbf{x}$ to an arbitrary vertex of the tetrahedron. The scalar product of $\hat{s}$ and $\mathbf{r}$ is:

$$\mathbf{r}.\hat{s} = |\mathbf{r}| \cos \theta,$$

where $\theta$ is the angle between $\hat{s}$ and $\mathbf{r}$. If the tetrahedron subtends less than this angle, none of its faces are intersected and no further check is needed. If it has a maximum edge length $h$, the maximum angle it can subtend is $\psi \approx h/|\mathbf{r}|$ and $\cos \theta < \cos \psi$ is a necessary condition for the tetrahedron to be intersected. Using the small angle approximation of $\cos \psi$ and neglecting fourth order terms:

$$\cos \theta < 1 - h^2/|\mathbf{r}|^2,$$
$$\cos^2 \theta < 1 - 2h^4/|\mathbf{r}|^4,$$

so that the condition can be written:

$$(\mathbf{r}.\hat{s})^2 < |\mathbf{r}|^2 - 2h^2.$$  

The test is implemented by setting $h^2$ to the square of the longest edge length on the element and by applying it only for $|\mathbf{r}|$ greater than some minimum value.

**2.2. Summary of algorithm.** In summary, given a set of points $\mathbf{x}_i$ each with an associated vorticity $\mathbf{\omega}_i$, the Biot-Savart integral at each point can be evaluated as follows:

1. Generate a tetrahedralization of the points $\mathbf{x}_i$ using, for example, the TETGEN code of Si [10].
2. Select Gaussian quadrature rules of order $N$ and $M$ for integration in $\phi$ and $\theta$ respectively.
3. Precompute the factors $k_{nm} = \sin \phi_n w_n^{(N)} w_m^{(M)}/16\pi^2$ and the ray directions $\hat{s}_{nm} = (\sin \phi_n \cos \theta_m, \sin \phi_n \sin \theta_m, \cos \phi_n)$, equation 2.2, for $n = 1, \ldots, N$ and $m = 1, \ldots, M$.
4. For each tetrahedron, loop over the nodes $\mathbf{x}_i$ and check for intersection of each ray $\hat{s}$ in turn. If the ray intersects the tetrahedron, add the contribution of equation 2.4 to the velocity $\mathbf{v}_i$. 

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**FIG. 2.2. Pre-check to decide if tetrahedron is likely to be intersected by a ray**
If the calculation is being carried out on a parallel system, the tetrahedra can be divided amongst the processors for the velocity calculation with the total velocity being found by a summation at the end of the computation. This only requires one communication per velocity calculation.

3. Numerical tests. A number of tests have been carried out to evaluate the performance of the method of §2.2 with respect to speed, accuracy and parallelization. The method has been implemented in a Lagrangian vortex code currently under development, similar to that of Marshall et al. [4]. The control points are tetrahedralized using the TETGEN code of Si [10] and the program runs on serial and parallel systems using the MPI message passing interface.

3.1. Speed and accuracy. The first test case considered is Hill’s spherical vortex [9, pp23–25]. This is a distribution of vorticity with a linear variation of azimuthal vorticity $\omega_\theta = Ar$ inside a sphere of radius $a$. Within the sphere, radial and axial velocities are:

$$ u = \frac{Az}{5}, \quad v = \frac{A}{5} \left( 2r^2 + z^2 + \frac{5}{3} a^2 \right). $$

This is an especially useful test case because, with the exception of errors in discretizing the boundary of the sphere, the vorticity distribution is exactly duplicated by linear interpolation over elements. The test was carried out by evaluating the velocity at $N$ randomly placed points with a sphere of radius $a = 1$ with $N$ varying from 1000 to 16000. Calculations were carried out on a 500MHz Intel Pentium III laptop with 4 point Gaussian quadrature in $\phi$ and $\theta$ and with 16 point quadratures on an 8 node 3.2GHz Intel Xeon cluster. Table 3.1 shows the r.m.s error in velocity and the computation time for these two cases and, for comparison, the corresponding data for Marshall et al.’s calculations on a Cray C-90 [4]. From the data presented, it appears that for small numbers of nodes, the method is more accurate than the use of analytical formulae and that it remains better up to distributions, in this case, of about 4000 points. The second point to note is that the accuracy appears to be controlled, up to the limit of the resolution of the vorticity, by the number of quadrature points. The $4 \times 4$ quadrature error stops reducing at about 4000 nodes while the $16 \times 16$ is still improving at 16000 nodes.

3.2. Convergence. As a second test on the accuracy of the calculation method, and to assess the convergence properties, the velocity due to a Gaussian core ring was computed. This has exponentially small vorticity on the boundary of the vorticity distribution so that errors due to discretization of the boundary should be reduced,
Fig. 3.1. Convergence of velocity for a Gaussian vortex ring, r.m.s. error against number of quadrature points: solid line 16×17 points; dashed line 64×65 points; dotted line 128×257 points.

Fig. 3.2. Error in velocity against discretization length $h$, for 64×64 point quadrature: circles: errors; solid line: linear fit $\epsilon = 8.03h^{2.15}$.

compared to the case of Hill’s spherical vortex. For comparison, the velocity was computed using the stream function for an axisymmetric vortex ring [9, pages 192–194], differentiating it numerically on a dense grid to give the velocity. This velocity was interpolated to find the velocity at the control points of the test distribution. Three different distributions of points were considered, each made up of a number of stations equally spaced in azimuth, with a square grid of points at each station. The low resolution distribution had 16×17 points, the intermediate 64×65 and the high resolution 128×257. The velocity was evaluated using equal numbers of quadrature points in $\phi$ and $\theta$. Figure 3.1 shows the r.m.s. error in velocity as a function of the number of quadrature points for all three test cases.

It is clear that, as proposed in §3.1, the accuracy of the integration is controlled by the order of Gaussian quadrature, up to a limit fixed by the resolution of the vorticity distribution. This is investigated in figure 3.2 which shows the r.m.s. error $\epsilon$ of figure 3.1 plotted against a typical discretization length scale $h$. The velocities
were computed using 64 quadrature points in $\phi$ and $\theta$. At this stage the two lower resolution test cases have reached their minimum error. The error scales as $h^{2.15}$ showing that the method is approximately second order.

3.3. Parallelization. The final requirement of a velocity calculation algorithm is that it be possible to implement it efficiently on a parallel system. The method of §2 has been coded for an MPI system by sharing the tetrahedra out amongst the processors and combining the velocity contributions at the end of the calculation. Only one communication is needed, to combine the velocities computed on each processor so that the network overhead is low. Figure 3.3 shows the computation time for velocity on Hill’s spherical vortex discretized with 16000 points, with 4–64 3.2GHz processors. The fitted line shows the calculation time scaling as $N^{-0.98}$, a near linear speedup.

4. Conclusions. A method for the evaluation of Biot–Savart integrals has been presented which eliminates the need for complicated mathematical operations over most of the calculation. With the exception of a negligible number of trigonometric factors which must be pre-computed, the method requires no operation more costly than a division. Calculations performed using the technique demonstrate that it is approximately second order accurate, that it converges correctly and that it achieves near-linear speedup on a parallel system. Future work will consider how to use the approach in methods similar to fast multipole and how to improve the ray tracing algorithm used, perhaps by taking advantage of adjacency relationships between tetrahedra.

In conclusion, we note that the method is based on a ray-tracing technique from computer graphics and that such methods are now being implemented at hardware level on mass market systems. This opens the possibility of using the calculation method on relatively cheap hardware, giving further acceleration at little additional cost.

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