The probabilistic approach to limited packings in graphs

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Abstract
We consider (closed neighbourhood) packings and their generalization in graphs. A vertex set $X$ in a graph $G$ is a $k$-limited packing if for every vertex $v \in V(G)$, $|N[v] \cap X| \leq k$, where $N[v]$ is the closed neighbourhood of $v$. The $k$-limited packing number $L_k(G)$ of a graph $G$ is the largest size of a $k$-limited packing in $G$. Limited packing problems can be considered as secure facility location problems in networks.

In this paper, we develop a new application of the probabilistic method to limited packings in graphs, resulting in lower bounds for the $k$-limited packing number and a randomized algorithm to find $k$-limited packings satisfying the bounds. In particular, we prove that for any graph $G$ of order $n$ with maximum vertex degree $\Delta$,

$$L_k(G) \geq \frac{kn}{(k+1)^k \binom{\Delta}{k} (\Delta + 1)}.$$

Also, some other upper and lower bounds for $L_k(G)$ are given.

Keywords: $k$-Limited packings, The probabilistic method, Lower and upper bounds, Randomized algorithm

1. Introduction

We consider simple undirected graphs. If not specified otherwise, standard graph-theoretic terminology and notations are used (e.g., see [1, 2]). We are interested in the classical packings and packing numbers of graphs as introduced in [9], and their generalization, called limited packings and limited packing numbers, respectively, as presented in [6]. In the literature, the classical packings are often referred to under different names: for example, as (distance) 2-packings [9, 13], closed neighborhood packings [11] or strong stable sets [8]. They can also be considered as generalizations of independent (stable) sets which, following the terminology of [9], would be (distance) 1-packings.

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Formally, a vertex set $X$ in a graph $G$ is a $k$-limited packing if for every vertex $v \in V(G)$,

$$|N[v] \cap X| \leq k,$$

where $N[v]$ is the closed neighbourhood of $v$. The $k$-limited packing number $L_k(G)$ of a graph $G$ is the maximum size of a $k$-limited packing in $G$. In these terms, the classical (distance) 2-packings are 1-limited packings, and hence $\rho(G) = L_1(G)$, where $\rho(G)$ is the 2-packing number.

The problem of finding a 2-packing (1-limited packing) of maximum size is shown to be $NP$-complete by Hochbaum and Schmoys [8]. In [4], it is shown that the problem of finding a maximum size $k$-limited packing is $NP$-complete even for the classes of split and bipartite graphs.

Graphs usually serve as underlying models for networks. A number of interesting application scenarios of limited packings are described in [6], including network security, market saturation, and codes. These and others can be summarized as secure location or distribution of facilities in a network. In a more general sense, these problems can be viewed as (maximization) facility location problems to place/distribute in a given network as many resources as possible subject to some (security) constraints.

2-Packings (1-limited packings) are well-studied in the literature from the structural and algorithmic point of view (e.g., see [8, 9, 11, 12]) and in connection with other graph parameters (e.g., see [3, 7, 9, 11, 13]). In particular, several papers discuss connections between packings and dominating sets in graphs (e.g., see [3, 4, 6, 7, 11]). Although the formal definitions for packings and dominating sets may appear to be similar, the problems have a very different nature: one of the problems is a maximization problem not to break some (security) constraints, and the other is a minimization problem to satisfy some reliability requirements. For example, given a graph $G$, the definitions imply a simple inequality $\rho(G) \leq \gamma(G)$, where $\gamma(G)$ is the domination number of $G$ (e.g., see [11]). However, the difference between $\rho(G)$ and $\gamma(G)$ can be arbitrarily large as illustrated in [3]: $\rho(K_n \times K_n) = 1$ for the Cartesian product of complete graphs, but $\gamma(K_n \times K_n) = n$.

In this paper, we develop an application of the probabilistic method to $k$-limited packings in general and to 2-packings (1-limited packings) in particular. In Section 2 we present the probabilistic construction and use it to derive two lower bounds for the $k$-limited packing number $L_k(G)$. Also, using a greedy algorithm approach, we provide an improved lower bound for the 2-packing (1-limited packing) number $\rho(G) = L_1(G)$. The probabilistic construction implies a randomized algorithm to find $k$-limited packings satisfying the lower bounds. The algorithm and its analysis are presented in Section 3. Section 4 shows that the main lower bound is asymptotically sharp, and discusses the improvement for 1-limited packings from the greedy algorithm approach. Finally, Section 5 provides upper bounds for $L_k(G)$, e.g. in terms of the $k$-tuple domination number $\gamma_{\times k}(G)$.

Notice that the probabilistic construction and approach are different from the well-known probabilistic constructions used for independent sets (e.g., see [1], p.27–28). In terms of packings, an independent set in a graph $G$ is a distance 1-packing: for any two vertices in an independent set, the distance between them in $G$ is greater
than 1. To the best of our knowledge, the proposed application of the probabilistic method is a new approach to work with packings and related maximization problems.

2. The probabilistic construction and lower bounds

Let $\Delta = \Delta(G)$ denote the maximum vertex degree in a graph $G$. Notice that $L_k(G) = n$ when $k \geq \Delta + 1$. We define

$$c_t = c_t(G) = \binom{\Delta}{t} \quad \text{and} \quad \tilde{c}_t = \tilde{c}_t(G) = \binom{\Delta + 1}{t}.$$ 

In what follows, we put $\binom{a}{b} = 0$ if $b > a$.

The following theorem gives a new lower bound for the $k$-limited packing number. It may be pointed out that the probabilistic construction used in the proof of Theorem 1 implies a randomized algorithm for finding a $k$-limited packing set, whose size satisfies the bound of Theorem 1 with a positive probability (see Algorithm 1 in Section 3).

**Theorem 1.** For any graph $G$ of order $n$ with $\Delta \geq k \geq 1$,

$$L_k(G) \geq \frac{kn}{\tilde{c}_{k+1}^{1/k} (1 + k)^{1+1/k}}.$$ 

**Proof.** Let $A$ be a set formed by an independent choice of vertices of $G$, where each vertex is selected with the probability

$$p = \left( \frac{1}{\tilde{c}_{k+1} (1 + k)} \right)^{1/k}.$$ 

For $m = k, \ldots, \Delta$, we denote

$$A_m = \{ v \in A : |N(v) \cap A| = m \}.$$ 

For each set $A_m$, we form a set $A'_m$ in the following way. For every vertex $v \in A_m$, we take $m - (k-1)$ neighbours from $N(v) \cap A$ and add them to $A'_m$. Such neighbours always exist because $m \geq k$. It is obvious that

$$|A'_m| \leq (m - k + 1)|A_m|.$$ 

For $m = k + 1, \ldots, \Delta$, let us denote

$$B_m = \{ v \in V(G) - A : |N(v) \cap A| = m \}.$$ 

For each set $B_m$, we form a set $B'_m$ by taking $m - k$ neighbours from $N(v) \cap A$ for every vertex $v \in B_m$. We have

$$|B'_m| \leq (m - k)|B_m|.$$ 

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Let us construct the set $X$ as follows:

$$X = A - \left( \bigcup_{m=k}^{\Delta} A'_m \right) - \left( \bigcup_{m=k+1}^{\Delta} B'_m \right).$$

It is easy to see that $X$ is a $k$-limited packing in $G$. The expectation of $|X|$ is

$$E[|X|] \geq E\left[ |A| - \sum_{m=k}^{\Delta} |A'_m| - \sum_{m=k+1}^{\Delta} |B'_m| \right]$$

$$\geq E\left[ |A| - \sum_{m=k}^{\Delta} (m - k + 1)|A_m| - \sum_{m=k+1}^{\Delta} (m - k)|B_m| \right]$$

$$= pn - \sum_{m=k}^{\Delta} (m - k + 1)E[|A_m|] - \sum_{m=k+1}^{\Delta} (m - k)E[|B_m|].$$

Let us denote the vertices of $G$ by $v_1, v_2, ..., v_n$ and the corresponding vertex degrees by $d_1, d_2, ..., d_n$. We will need the following lemma:

**Lemma 2.** If $p = \left( \frac{1}{\tilde{c}_{k+1}(1+k)} \right)^{1/k}$, then, for any vertex $v_i \in V(G)$,

$$\binom{d_i}{m} (1 - p)^{d_i - m} \leq \binom{\Delta}{m} (1 - p)^{\Delta - m}, \quad m \geq k. \quad (3)$$

**Proof.** The inequality (3) holds if $d_i = \Delta$. It is also true if $d_i < m$ because in this case $\binom{d_i}{m} = 0$. Thus, we may assume that

$$m \leq d_i < \Delta.$$

Now, it is easy to see that inequality (3) is equivalent to the following:

$$(1 - p)^{\Delta - d_i} \geq \binom{d_i}{m} / \binom{\Delta}{m} = \frac{(\Delta - m)!/(d_i - m)!}{\Delta!/d_i!} = \prod_{i=0}^{\Delta - d_i - 1} \frac{\Delta - m - i}{\Delta - i}. \quad (4)$$

Further, $\Delta \geq k$ implies $\frac{\Delta}{k} \leq \frac{\Delta - i}{k - i}$, where $0 \leq i \leq k - 1$. Taking into account that $\Delta > 0$, we obtain

$$\binom{\Delta}{k}^{k} \leq \prod_{i=0}^{k-1} \frac{\Delta - i}{k - i} = c_k < \tilde{c}_{k+1}(1 + k)$$

or

$$\frac{1}{\tilde{c}_{k+1}(1 + k)} < \left( \frac{k}{\Delta} \right)^{k}.$$

Thus,

$$p^k < \left( \frac{k}{\Delta} \right)^{k} \quad \text{or} \quad p < \frac{k}{\Delta} \leq \frac{m}{\Delta}.$$
We have $p < \frac{m}{\Delta}$, which is equivalent to $1 - p > \frac{\Delta - m}{\Delta}$. Therefore,

$$(1 - p)^{\Delta - d_i} > \left(\frac{\Delta - m}{\Delta}\right)^{\Delta - d_i} \geq \prod_{i=0}^{\Delta - d_i - 1} \frac{\Delta - m - i}{\Delta - i},$$

as required in (4). \hfill \Box

Now we go on with the proof of Theorem 1. By Lemma 2,

$$E[|A_m|] = \sum_{i=1}^{n} P[v_i \in A_m]$$

$$= \sum_{i=1}^{n} p \left(\frac{d_i}{m}\right) p^{m}(1 - p)^{d_i - m}$$

$$\leq p^{m+1} \sum_{i=1}^{n} \left(\frac{\Delta}{m}\right) (1 - p)^{\Delta - m}$$

$$= p^{m+1} (1 - p)^{\Delta - m} c_m n,$$

where $p \left(\frac{d_i}{m}\right) p^{m}(1 - p)^{d_i - m}$ is the probability of having vertex $v_i, i = 1, \ldots, n,$ in the set $A_m, m = k, \ldots, \Delta$. Again, by Lemma 2,

$$E[|B_m|] = \sum_{i=1}^{n} P[v_i \in B_m]$$

$$= \sum_{i=1}^{n} (1 - p) \left(\frac{d_i}{m}\right) p^{m}(1 - p)^{d_i - m}$$

$$\leq p^{m} \sum_{i=1}^{n} \left(\frac{\Delta}{m}\right) (1 - p)^{\Delta - m + 1}$$

$$= p^{m} (1 - p)^{\Delta - m + 1} c_m n,$$

where $(1 - p) \left(\frac{d_i}{m}\right) p^{m}(1 - p)^{d_i - m}$ is the probability of having vertex $v_i, i = 1, \ldots, n,$ in the set $B_m, m = k + 1, \ldots, \Delta$.

Taking into account that $c_{\Delta+1} = \left(\frac{\Delta}{\Delta + 1}\right) = 0$, we obtain

$$E[|X|] \geq pm - \sum_{m=k}^{\Delta} (m - k + 1)p^{m+1}(1 - p)^{\Delta - m} c_m n - \sum_{m=k+1}^{\Delta+1} (m - k)p^{m}(1 - p)^{\Delta - m + 1} c_m n$$

$$= pm - \sum_{m=0}^{\Delta-k} (m+1)p^{m+k+1}(1 - p)^{\Delta - m - k} c_{m+k} n$$

$$- \sum_{m=0}^{\Delta-k} (m+1)p^{m+k+1}(1 - p)^{\Delta - m - k} c_{m+k+1} n$$
\[\begin{align*}
\ &= pn - \sum_{m=0}^{\Delta-k} (m+1)p^{m+k+1}(1-p)^{\Delta-m-k} n (c_{m+k} + c_{m+k+1}) \\
\ &= pn - p^{k+1}n \sum_{m=0}^{\Delta-k} (m+1)\tilde{c}_{m+k+1} p^m (1-p)^{\Delta-k-m}.
\end{align*}\]

Furthermore,
\[
(m+1)\tilde{c}_{m+k+1} = \binom{\Delta - k}{m} \frac{(m+1)!((\Delta + 1)!}{(m+k+1)!((\Delta - k)!} 
\leq \binom{\Delta - k}{m} \frac{(\Delta + 1)!}{(k+1)!((\Delta - k)!} = \binom{\Delta - k}{m} \tilde{c}_{k+1}.
\]

We obtain, by the binomial theorem,
\[
E[|X|] \geq pn - p^{k+1}n \sum_{m=0}^{\Delta-k} \binom{\Delta - k}{m} \tilde{c}_{k+1} p^m (1-p)^{\Delta-k-m} \\
= pn - p^{k+1}n\tilde{c}_{k+1} \\
= pn(1-p\tilde{c}_{k+1}) \\
= \frac{kn}{\tilde{c}_{k+1}^{1/k} (1+k)^{1+1/k}}.
\]

Since the expectation is an average value, there exists a particular \(k\)-limited packing of size at least \(\frac{kn}{\tilde{c}_{k+1}^{1/k} (1+k)^{1+1/k}}\), as required. The proof of the theorem is complete. \(\square\)

The lower bound of Theorem 1 can be written in a simpler but weaker form as follows:

**Corollary 3.** For any graph \(G\) of order \(n\),
\[L_k(G) \geq \frac{kn}{e(1+\Delta)^{1+1/k}}.\]

**Proof.** It is not difficult to see that
\[\tilde{c}_{k+1} \leq \frac{(\Delta + 1)^{k+1}}{(k+1)!}\]
and, using Stirling’s formula,
\[(k!)^{1/k} > \left( \frac{\sqrt{2\pi k}}{e} \right)^{1/k} = \frac{2^{1/k} \sqrt{2\pi k}}{e} \frac{k}{e}.
\]

By Theorem 1,
\[L_k(G) \geq \frac{kn ((k+1)!)^{1/k}}{(\Delta + 1)^{1+1/k} (1+k)^{1+1/k}} > \frac{kn}{e(1+\Delta)^{1+1/k}} \times \frac{2^{1/k} \sqrt{2\pi k}}{1+k} > \frac{kn}{e(1+\Delta)^{1+1/k}}.\]
Note that \(\frac{2^{k/2}\pi k}{1+k} > 1\). The last inequality is obviously true for \(k = 1\), while for \(k \geq 2\) it can be rewritten in the equivalent form: \(2\pi k > (1 + 1/k)^2k = e^2 - o(1)\).

In the case \(k = 1\), Theorem 1 gives the following lower bound for the 2-packing (1-limited packing) number:

**Corollary 4.** For any graph \(G\) of order \(n\) with \(\Delta \geq 1\),

\[
\rho(G) = L_1(G) \geq \frac{n}{2\Delta(\Delta + 1)}. \tag{5}
\]

Let \(\delta = \delta(G)\) denote the minimum vertex degree in a graph \(G\). The lower bound of Corollary 4 can be improved as follows:

**Theorem 5.** For any graph \(G\) of order \(n\),

\[
\rho(G) = L_1(G) \geq \frac{n + \Delta(\Delta - \delta)}{\Delta^2 + 1} \geq \frac{n}{\Delta^2 + 1}. \tag{6}
\]

**Proof.** Choose any vertex \(v \in V(G)\) of the minimum degree \(\delta\) in \(G\). Then add \(v\) to a set \(X\) and remove vertices of \(N[N[v]]\) from the graph to obtain \(G' = G - N[N[v]]\), where \(N[N[v]] = \{w : w \in N[u] \text{ for some } u \in N[v]\}\) is the so-called second closed neighbourhood of \(v\) in \(G\). Recursively apply the same procedure to the remaining graph \(G'\) until it is empty. It is not difficult to see that \(X\) is a 1-limited packing (distance 2-packing) of size at least \(\left\lceil \frac{n + \Delta(\Delta - \delta)}{\Delta^2 + 1} \right\rceil\): we remove at most \(1 + \Delta + \Delta(\Delta - 1) = 1 + \Delta^2\) vertices at each iteration, but at most \(1 + \delta + \delta(\Delta - 1) = 1 + \delta\Delta\) vertices at the first iteration, and \((1 + \Delta^2) - (1 + \delta\Delta) = \Delta(\Delta - \delta)\).

The proof of Theorem 5 provides a greedy algorithm to find a distance 2-packing (1-limited packing) satisfying bound (6). We explain later in Section 4 why the lower bound of Theorem 5 is as good as lower bound (5) of Corollary 4 for almost all graphs.

### 3. Randomized algorithm

A pseudocode presented in Algorithm 1 explicitly describes a randomized algorithm to find a \(k\)-limited packing set, whose size satisfies bound (1) with a positive probability. Notice that Algorithm 1 constructs a (preliminary) \(k\)-limited packing \(X'\) by recursively removing unwanted vertices from a randomly generated set \(A\). This is different from the probabilistic construction used in the proof of Theorem 1. The recursive removal of vertices from the set \(A\) may be more effective and efficient, especially if one tries to remove overall as few vertices as possible from \(A\) by maximizing intersections of the sets \(A'_m\) \((m = k, \ldots, \Delta)\) and \(B'_m\) \((m = k+1, \ldots, \Delta)\).

At the final stage, Algorithm 1 does a (greedy) extension of the preliminary \(k\)-limited packing \(X'\) derived from the randomly generated set \(A\). Our experimental tests with randomly generated problem instances show the following: although
the randomized part of Algorithm 1 may eventually return a preliminary $k$-limited packing set slightly smaller than lower bound (1), the extension of this set to a maximal $k$-limited packing always satisfies (1). This is of no surprise, because the expectation of the size of randomly formed set $A$ in Algorithm 1 is $E[|A|] = pn$, where $p = \left( \frac{\Delta}{k} \right) (\Delta + 1)^{-1/k}$, while the expression for lower bound in (1) yields a smaller value:

$$\frac{kn}{(1 + k)^{1+1/k}} = \frac{k}{k+1}pn = \frac{k}{k+1}E[|A|] < E[|A|].$$

From the experimental tests, an initially formed set $A$ may contain only few redundant vertices to be removed to obtain the preliminary $k$-limited packing $X'$. As a result, the preliminary $k$-limited packing $X'$ in many cases satisfies lower bound (1), and the extension of $X'$ to a maximal $k$-limited packing $X$ seems to always satisfy (1). In our view, since the problem is $NP$-hard, Algorithm 1 constitutes a simple efficient approach to tackle the problem in practice and, hopefully, can be useful to solve some hard instances of the problem.

**Algorithm 1**: Randomized $k$-limited packing

**Input**: Graph $G$ and integer $k$, $1 \leq k \leq \Delta$.

**Output**: $k$-Limited packing $X$ in $G$.

begin
  Compute $p = \left( \frac{\Delta}{k} \right) (\Delta + 1)^{-1/k}$;
  Initialize $A = \emptyset$; /* Form a set $A \subseteq V(G)$ */
  foreach vertex $v \in V(G)$ do
    with the probability $p$, decide whether $v \in A$ or $v \notin A$;
  end /* Recursively remove redundant vertices from $A$ */
  foreach vertex $v \in V(G)$ do
    Compute $r = |N(v) \cap A|$;
    if $v \in A$ and $r \geq k$ then
      remove any $r - k + 1$ vertices of $N(v) \cap A$ from $A$;
    end
    if $v \notin A$ and $r > k$ then
      remove any $r - k$ vertices of $N(v) \cap A$ from $A$;
    end
  end
  Put $X' = A$; /* $X'$ is a $k$-limited packing */
  Extend $X'$ to a maximal $k$-limited packing $X$;
  return $X$;
end

Algorithm 1 can be implemented to run in $O(n^2)$ time. To compute the probability $p = \left( \frac{\Delta}{k} \right) (\Delta + 1)^{-1/k}$, the binomial coefficient $\binom{\Delta}{k}$ can be computed
by using the dynamic programming and Pascal’s triangle in $O(k\Delta) = O(\Delta^2)$ time using $O(k) = O(\Delta)$ memory. The maximum vertex degree $\Delta$ of $G$ can be computed in $O(m)$ time, where $m$ is the number of edges in $G$. Then $p$ can be computed in $O(m + \Delta^2) = O(n^2)$ steps. It takes $O(n)$ time to find the initial set $A$. Computing the intersection numbers $r = |N(v) \cap A|$ and removing unwanted vertices of $N(v) \cap A$’s from $A$ can be done in $O(n + m)$ steps. Finally, checking whether $X'$ is maximal or extending $X'$ to a maximal $k$-limited packing $X$ can be done in $O(n + m)$ time: try to add vertices of $V(G) - X'$ to $X'$ recursively one by one, and check whether the addition of a new vertex $v \in V(G) - X'$ to $X'$ violates the conditions of a $k$-limited packing for $v$ or at least one of its neighbours in $G$ with respect to $X' \cup \{v\}$. Thus, overall Algorithm 1 takes $O(n^2)$ time, and, since $m = O(n^2)$ in general, it is linear in the graph size $(m + n)$ when $m = \Theta(n^2)$.

Also, this randomized algorithm for finding $k$-limited packings in a graph $G$ can be implemented in parallel or as a local distributed algorithm. As explained in [5], this kind of algorithms are especially important, e.g. in the context of ad hoc and wireless sensor networks. We hope that this approach can be also extended to design self-stabilizing or on-line algorithms for $k$-limited packings. For example, a self-stabilizing algorithm searching for maximal 2-packings in a distributed network system is presented in [12]. Notice that self-stabilizing algorithms are distributed and fault-tolerant, and use the fact that each node has only a local view/knowledge of the distributed network system. This provides another motivation for efficient distributed search and algorithms to find $k$-limited packings in graphs and networks.

4. Sharpness of the lower bounds

We now show that the lower bound of Theorem 1 is asymptotically best possible for some values of $k$. The bound of Theorem 1 can be rewritten in the following form for $\Delta \geq k$:

$$L_k(G) \geq \frac{kn}{(k+1)^k \left(\frac{\Delta}{k}\right)(\Delta + 1)}.$$

Combining this bound with the upper bound of Lemma 8 from [6], we obtain that for any connected graph $G$ of order $n$ with minimum degree $\delta(G) \geq k$,

$$\frac{1}{k^{k/k}} \times \frac{k}{k+1} n \leq L_k(G) \leq \frac{k}{k+1} n. \quad (7)$$

Notice that the upper bound in the inequality (7) is sharp (see [6]), so these bounds provide an interval of values for $L_k(G)$ in terms of $k$ and $\Delta$ when $k \leq \delta$. For regular graphs, $\delta = \Delta$, and, when $k = \Delta$, we have

$$\frac{1}{k^{k/k}} \left(\frac{\Delta}{k}\right)(\Delta + 1) = \frac{1}{(k+1)^{1/k}} \to 1 \quad \text{as} \quad k \to \infty.$$
Therefore, the bound of Theorem 1 is asymptotically sharp for regular connected graphs in the case $k = \Delta$. In other words, there are graphs whose $k$-limited packing number is arbitrarily close to the bound of Theorem 1. Thus, the following result holds:

**Theorem 6.** When $n$ is large, there exist graphs $G$ such that

$$L_k(G) \leq \frac{kn}{c_{k+1}^{1/k}(1 + k)^{1 + 1/k}(1 + o(1))}. \quad (8)$$

As shown above, the graphs satisfying Theorem 6 contain regular connected ones for $k = \Delta$. This class of graphs can be extended, because it is possible to prove that the bound of Theorem 1 is asymptotically sharp for connected graphs with $k = \Delta(1 - o(1))$, $\delta(G) \geq k$.

Notice that, for regular graphs, the condition $k = \Delta$ and Lemma 5 from [6] imply $L_k(G) = n - \gamma(G)$. Then the classical upper bound (9) for $\gamma(G)$ gives a weaker lower bound for $L_k(G)$ than Theorem 1.

As shown in Theorem 5, in contrast to the situation for relatively ‘large’ values of $k$, bound (1) of Theorem 1 (see Corollary 4) can be improved for distance 2-packings (1-limited packings), i.e. when $k = 1$. However, this improvement is irrelevant for almost all graphs. A 1-limited packing set $X$ in $G$ has a very strong property that any two vertices in $X$ are at distance at least 3 in $G$. It is well known that almost every graph has diameter equal to 2 (e.g., see [10]). Therefore, $\rho(G) = L_1(G) = 1$ for almost all graphs. Thus, in the case $k = 1$, Theorem 1 yields a lower bound of 1 for almost all graphs and is as good as Theorem 5. Notice that the bound of Theorem 5 is sharp, for example, for any number of disjoint copies of the Petersen graph. In the other cases, when $G$ has a diameter larger than 2, one is encouraged to use the greedy algorithm and lower bound (6) provided by Theorem 5, because it improves bound (5) of Corollary 4 by a factor of $2 + o(1)$.

5. Upper bounds

As mentioned earlier, $\rho(G) = L_1(G) \leq \gamma(G)$. In [6], the authors provide several upper bounds for $L_k(G)$, e.g. $L_k(G) \leq k\gamma(G)$ for any graph $G$. Using the well-known bound (see e.g. [1])

$$\gamma(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} n, \quad (9)$$

we obtain

$$L_k(G) \leq \frac{\ln(\delta + 1) + 1}{\delta + 1} kn. \quad (10)$$

Even though this bound does not work well when $k$ is ‘close’ to $\delta$, it is very reasonable for small values of $k$.

We now prove an upper bound for the $k$-limited packing number in terms of the $k$-tuple domination number. A set $X$ is called a $k$-tuple dominating set of $G$ if for every vertex $v \in V(G)$, $|N[v] \cap X| \geq k$. The minimum cardinality of a $k$-tuple dominating set of $G$ is the $k$-tuple domination number $\gamma_{\times k}(G)$. The $k$-tuple domination number is only defined for graphs with $\delta \geq k - 1$. 
**Theorem 7.** For any graph $G$ of order $n$ with $\delta \geq k - 1$, 

$$L_k(G) \leq \gamma_{xk}(G). \quad (11)$$

**Proof.** We prove inequality (11) by contradiction. Let $X$ be a maximum $k$-limited packing in $G$ of size $L_k(G)$, and let $Y$ be a minimum $k$-tuple dominating set in $G$ of size $\gamma_{xk}(G)$. We denote $B = X \cap Y$, i.e. $X = A \cup B$ and $Y = B \cup C$, where $A$ and $C$ are disjoint. Assume to the contrary that $L_k(G) > \gamma_{xk}(G)$, thus $|A| > |C|$.

Since $Y$ is $k$-tuple dominating set, each vertex of $A$ is adjacent to at least $k$ vertices of $Y$. Hence the number of edges between $A$ and $B \cup C$ is as follows:

$$e(A, B \cup C) \geq k|A|.$$ 

Now, every vertex of $C$ is adjacent to at most $k$ vertices of $X$, because $X$ is a $k$-limited packing set. Therefore, the number of edges between $C$ and $A \cup B$ satisfies

$$e(C, A \cup B) \leq k|C|.$$ 

We obtain

$$e(C, A \cup B) \leq k|C| < k|A| \leq e(A, B \cup C),$$

i.e. $e(C, A \cup B) < e(A, B \cup C)$. By eliminating the edges between $A$ and $C$, we conclude that

$$e(C, B) < e(A, B).$$

Now, let us consider an arbitrary vertex $b \in B$ and denote $s = |N(b) \cap A|$. Since $X = A \cup B$ is a $k$-limited packing set, we obtain $|N(b) \cap X| \leq k - 1$, and hence $|N(b) \cap B| \leq k - s - 1$. On the other hand, $Y = B \cup C$ is $k$-tuple dominating set, so $|N(b) \cap Y| \geq k - 1$. Therefore, $|N(b) \cap C| \geq s$. Thus, $|N(b) \cap C| \geq |N(b) \cap A|$ for any vertex $b \in B$. We obtain

$$e(C, B) > e(A, B),$$

a contradiction. We conclude that $L_k(G) \leq \gamma_{xk}(G).$ \hfill \(\square\)

Notice that it is possible to have $k = \Delta + 1$ in the statement of Theorem 7, which is not covered by Theorem 1. Then $\delta = \Delta$, which implies the graph is regular. However, $L_k(G) = \gamma_{xk}(G) = n$ for $k = \delta + 1 = \Delta + 1$. In non-regular graphs, $\delta + 1 \leq \Delta$, and $k \leq \Delta$ to satisfy the conditions of Theorem 1 as well.

For $t \leq \delta$, we define

$$\delta' = \delta - k + 1 \quad \text{and} \quad \bar{b}_t = \bar{b}_t(G) = \left(\frac{\delta + 1}{t}\right).$$

Using the upper bound for the $k$-tuple domination number from [5], we obtain:

**Corollary 8.** For any graph $G$ with $\delta \geq k$,

$$L_k(G) \leq \left(1 - \frac{\delta'}{\bar{b}_k^{-1/\delta'}(1 + \delta'^{1+1/\delta'})}\right) n. \quad (12)$$

In some cases, Theorem 1 and Corollary 8 simultaneously provide good bounds for the $k$-limited packing number. For example, for a 40-regular graph $G$:

$$0.312n < L_{25}(G) < 0.843n.$$
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