THE MULTIPLICATIVE STRUCTURE ON HOCHSCHILD
COHOMOLOGY OF A COMPLETE INTERSECTION

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Dedicated to Hans-Bjørn Foxby in admiration

Abstract. We determine the product structure on Hochschild cohomology of commutative algebras in low degrees, obtaining the answer in all degrees for complete intersection algebras. As applications, we consider cyclic extension algebras as well as Hochschild and ordinary cohomology of finite abelian groups.

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1. INTRODUCTION

For any associative algebra\(^1\) \(A\) over a commutative ring \(K\), the Hochschild cohomology ring \(HH(A/K, A)\) with its cup (or Yoneda) product is \textit{graded commutative} by Gerstenhaber’s fundamental result \([20]\).

However, it is not necessarily \textit{strictly} graded commutative, that is, the squaring operation \(HH^{2i+1}(A/K, A) \rightarrow HH^{4i+2}(A/K, A)\), for \(i \geq 0\), from Hochschild cohomology in odd degrees to, necessarily, the 2–torsion of the Hochschild cohomology in twice that degree, might be nontrivial.

Indeed, this is already so in the simplest example. If \(A = K[x]/(x^2)\) with \(2 = 0\) in \(K\), then the Hochschild cohomology ring of \(A\) over \(K\) is isomorphic to a polynomial

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\(^1\)All our rings have a multiplicative identity and ring homomorphisms are unital.
ring over $A$ in one variable, $\text{HH}(A/K, A) \cong A[z]$, where $A \cong \text{HH}^0(A/K, A)$ is of cohomological degree zero, and the variable $z$ is of degree one, representing the class $\partial / \partial x \in \text{Der}_K(A, A) \cong \text{HH}^1(A/K, A)$. As we are in characteristic 2, this ring is indeed graded commutative, but certainly not strictly so, the square of $z^{2i+1} \in \text{HH}^{2i+1}(A/K, A)$ returning the non-zero element $z^{4i+2} \in \text{HH}^{4i+2}(A/K, A).

The purpose here is to clarify the structure of this squaring map in the simplest case, when $i = 0$ and $A$ is commutative. We will then deduce from this the entire ring structure when $A$ is a homological complete intersection over $K$.

The crucial point is that the square of a derivation with respect to cup product, as an element in $\text{HH}^2(A/K, A)$, is determined by the Hessian quadratic forms associated to the defining equations.

The key technical tool we use is the Tate model of the multiplication map $\mu : A^\text{ev} = A \otimes_K A \to A$ for a commutative $K$–algebra $A$. We show that the Postnikov tower of such model carries a family of compatible co-unit al diagonal approximations and we give an explicit description of such approximation in low degrees to obtain the desired results.

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2. Statement of Results

Our treatment requires some simple facts about differential forms and the calculus that goes along with them. While we cover all facts we will use, the reader may gain the full picture by consulting [21].

2.1. If $P = K[x \in X]$ is a polynomial ring over some commutative ring $K$, with $X$ a set of variables, then the module of Kähler differentials of $P$ over $K$ is a free $P$–module, based on the differentials $dx$ for $x \in X$, that is, $\Omega^1_{P/K} \cong \bigoplus_{x \in X} Pdx$.

If now $f \in P$ is any polynomial, then we may consider its Taylor expansion $f(x + dx)$ in $\text{Sym}_P \Omega^1_{P/K} \cong K[x, dx; x \in X] \cong P \otimes_K P$.

In concrete terms, the polynomial will only depend on finitely many variables, say, $x = x_1, \ldots, x_n$, and then

$$f(x + dx) = \sum_{a = (a_1, \ldots, a_n) \in \mathbb{N}^n} \frac{\partial^{|a|} f(x)}{\partial^a x} dx^a,$$

where $|a| = \sum_i a_i$ and the coefficient of $dx^a$ is the corresponding divided partial derivative of $f$, in that the usual partial derivative satisfies

$$\frac{\partial^{|a|} f(x)}{\partial^{a_1} x_1 \cdots \partial^{a_n} x_n} = a_1! \cdots a_n! \frac{\partial^{|a|} f(x)}{\partial^{a_1} x_1 \cdots \partial^{a_n} x_n}.$$

2.2. The linear part with respect to the $dx$ in that expansion can be viewed as the total differential of $f$, to wit, $df = \sum_{i=1}^n \frac{\partial f(x)}{\partial x_i} dx_i \in \Omega^1_{P/K}$. 

It defines the $P$–linear Jacobian of the given polynomial,
\[ \widetilde{\text{jac}}_f : \text{Der}_K(P,P) \to P, \quad D \mapsto D(df) \]
on the $P$–module of $K$–linear derivations from $P$ to $P$.

The quadratic part with respect to the $dx$ in the Taylor expansion,
\[ H_f(dx) = \sum_{1 \leq i \leq j \leq n} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} dx_i dx_j, \]
is the Hessian form defined by $f$. By definition it is an element of $\text{Sym}_2 \Omega^1_{P/K}$ and so defines a $P$–quadratic form
\[ \tilde{h}_f : \text{Der}_K(P,P) \to P \]
that sends $D = \sum_{i=1}^n p_i \frac{\partial}{\partial x_i} + \sum_{x \in X} p_x \frac{\partial}{\partial x}$ to
\[ \tilde{h}_f(D) = (D \otimes D)(H_f(dx)) = \sum_{1 \leq i \leq j \leq n} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} p_ip_j, \]
where we have set $X' = X \setminus \{x\}$. The $P$–quadratic module $\left(\text{Der}_K(P,P), \tilde{h}_f\right)$ depends solely on $f$, not on the chosen coordinate system $x_i$.

### 2.3. If we take a family of polynomials $f = \{f_b\}_{b \in Y}$, then the associated (linear) Jacobian, respectively the (quadratic) Hessian map are given by
\[ \widetilde{\text{jac}}_f = \sum_{b \in Y} \text{jac}_{f_b} \partial_{f_b}, \quad \tilde{h}_f = \sum_{b \in Y} \tilde{h}_{f_b} \partial_{f_b} : \text{Der}_K(P,P) \to \prod_{b \in Y} P\partial_{f_b}, \]
where the symbols $\partial_{f_b}$ simply represent the corresponding $\delta$–functions in the product of the free $P$–modules $P\partial_{f_b}$.

### 2.4. If $A = P/I$, with $I = \{f_b\}_{b \in Y}$, then $A$ is a commutative $K$–algebra and the module of Kähler differentials of $A$ over $K$ has a free $A$–module presentation
\[ \bigoplus_{b \in Y} A[f_b] \xrightarrow{j} A \otimes_P \Omega^1_{P/K} \cong \bigoplus_{x \in X} Adx \to \Omega^1_A \to 0, \]
where the basis element $[f_b]$ is mapped to $j([f_b]) = 1 \otimes df_b$. The $A$–linear map $j$ factors as the surjection $\bigoplus_{b \in Y} A[f_b] \to I/I^2$ onto the conormal module $I/I^2$ that sends $[f_b]$ to $f_b$ mod $I^2$, followed by the $A$–linear map $d: I/I^2 \to A \otimes_P \Omega^1_{P/K}$ that is induced by the universal $K$–derivation, sending $f$ mod $I^2$ to $1 \otimes df$.

It follows that the Jacobian and Hessian descend,
\[ \text{(*)} \quad \text{jac}_f = \sum_{b \in Y} \text{jac}_{f_b} \partial_{f_b}, \quad h_f = \sum_{b \in Y} h_{f_b} \partial_{f_b} : \text{Der}_K(P,A) \to \prod_{b \in Y} A\partial_{f_b}, \]
the $A$–linear map $\text{jac}_f$ taking its values in $N_{A/P} = \text{Hom}_A(I/I^2, A)$, the normal module of $A$ with respect to $P$, $\partial_{f_b}$, viewed as a submodule of $\prod_{b \in Y} A\partial_{f_b}$, the $A$–dual of the free $A$–module $\bigoplus_{b \in Y} A[f_b]$. The kernel of the Jacobian satisfies
\[ \text{Ker} \text{jac}_f \cong \text{Der}_K(A,A) \cong \text{HH}^1(A/K, A) \cong \text{Ext}^1_{A^e}(A,A). \]
Note that the last isomorphism holds for any associative $K$–algebra $A$, when $A^e$ denotes the enveloping algebra of $A$ over $K$, while in higher degrees one only has natural maps $\text{HH}^i(A/K, A) \to \text{Ext}^i_{A^e}(A,A)$ that are isomorphisms when $A$ is projective as $K$–module; see also 2.6 below.
Now we can formulate our first result.

**Theorem 2.5.** Let \( A = P/I \) be a presentation of a commutative \( K \)-algebra as quotient of a polynomial ring \( P = K[x; x \in X] \) over some commutative ring \( K \) modulo an ideal \( I = (f_b)_{b \in Y} \subseteq P \).

1. The Hessian map \( \mathcal{H} f \) induces a well defined \( A \)-quadratic map
   \[
   q : \text{Der}_K(A, A) \to \text{Hom}_A(I/I^2, A)
   \]
   from the module of derivations to the normal module. Explicitly,
   \[
   q \left( \sum_{x \in X} a_x \frac{\partial}{\partial x} \right) (f \mod I^2) = \sum_{1 \leq i \leq j \leq n} \frac{\partial^{(2)} f(x)}{\partial x_i \partial x_j} a_i a_j \mod I,
   \]
   where, as above, \( x_1, \ldots, x_n \in X \) are the variables that actually occur in a presentation of \( f \), linearly ordered in some way.

2. There is a canonical \( A \)-linear map
   \[
   \delta : \text{Hom}_A(I/I^2, A) \to \text{Ext}^2_{A^{op}}(A, A),
   \]
   whose image is \( T^1_{A/K} \subseteq \text{Ext}^2_{A^{op}}(A, A) \), the first tangent or André–Quillen cohomology of \( A \).

3. The map from \( \text{Der}_K(A, A) = \text{Ext}^1_{A^{op}}(A, A) \) to \( \text{Ext}^2_{A^{op}}(A, A) \) that sends a derivation \( D \) to the class \([D \circ D]\) of its square under Yoneda product equals \( \delta \circ q \) and takes its image in the 2–torsion of \( T^1_{A/K} \).

**Remark 2.6.** If \( A \) as in the preceding result is further projective over \( K \), then there is a natural bijection \( \text{HH}^\bullet(A/K, A) \to \text{Ext}^\bullet_{A^{op}}(A, A) \) of graded commutative \( K \)-algebras\(^2\).

Moreover, \( T^1_{A/K} \subseteq \text{HH}^2(A/K, A) \) identifies with the \( A \)-submodule of classes of symmetric Hochschild 2–cocycles.

**2.7.** In our second result, let the \( K \)-algebra \( A = P/I \) be presented as above, but assume moreover that

(a) \( A \) is transversal to itself as \( K \)-module in that
   \[
   \text{Tor}^K_+(A, A) = \oplus_{i>0} \text{Tor}^K_i(A, A) = 0.
   \]

(b) The ideal \( I \subseteq P \) is regular in the sense of Quillen [29, Def.6.10], that is, \( I/I^2 \)
   is a projective \( A \)-module and the canonical morphism of graded commutative algebras
   \[
   \bigwedge_A(I/I^2) \to \text{Tor}^P(A, A)
   \]
   is an isomorphism. In (loc.cit.) Quillen points out that for \( P \) noetherian, \( I \)
   is regular in \( P \) if, and only if, it is generated locally by a \( P \)-regular sequence.

\footnote{Simply using the definition of the algebra structure on source or target, one obtains an isomorphism from the opposite algebra of \( \text{HH}^\bullet(A/K, A) \) to the Yoneda Ext–algebra. However, as already Gerstenhaber pointed out in his original paper, sending an element \( u \) of degree \( m \) to \(-1\)^{\binom{m}{2}}u provides an algebra isomorphism between a graded commutative algebra and its opposite algebra.}
We will say, for short, that \( A \) is a \textit{homological complete intersection algebra} over \( K \) if conditions (a) and (b) are satisfied. Alternative terminology would be to say that the kernel of the multiplication map \( \mu : A^{ev} \to A \) has “free” or “exterior Koszul homology,” a notion introduced and studied by A. Blanco, J. Majadas Soto, and A. Rodicio Garcia in a series of articles following \cite{7}, or to call the multiplication map a “quasi-complete intersection homomorphism” as in \cite{4}.

2.8. The key properties of such homological complete intersection algebras are that the corresponding \textit{Tate model} is small and the associated \textit{cotangent complex} \( \mathbb{L}_{A/K} \) is, by \cite[Cor. 6.14]{29}, concentrated in homological degrees 0 and 1. More precisely,

\[
\mathbb{L}_{A/K} \\equiv \begin{array}{c}
0 \\
I/P^2[1] \\
A \otimes_P \Omega^1_{P/K}[0] \\
0
\end{array}
\]

and its shifted \( A \)-dual, the \textit{shifted tangent complex} \( t_A := \text{Hom}_A(\mathbb{L}_{A/K}[1], A) \) is the complex

\[
\begin{array}{c}
0 \\
\text{Der}_K(P, A)[-1] \\
N_{A/P}[-2] \\
0
\end{array}
\]

concentrated in cohomological degrees one and two.

2.9. The link between tangent and Hochschild cohomology was established in Quillen’s fundamental paper \cite[Sect. 8]{29}. In essence, it says that the complex returning Hochschild homology can be realized as a DG Hopf algebra whose primitive part is represented by the cotangent complex. The qualification here is two-fold: there generally is only a spectral sequence reflecting this interpretation that generally only degenerates in characteristic zero, and, on the dual side, for Hochschild cohomology vis-à-vis the enveloping algebra of the tangent complex, the algebra structures in cohomology do not match up. However, for complete intersection algebras, things are well controlled because the cotangent complex is so short. In detail, we can complement the picture as follows.

2.10. The Hessian \( A \)-quadratic map \( q \) from 2.4(*) above can be interpreted as defining on \( t_A \) the structure of a graded (super) Lie algebra, where we follow the axiomatization of this notion in \cite{2}, and the Jacobian can be viewed as a Lie algebra differential. The enveloping algebra of this DG Lie \( A \)-algebra; see \cite{34}; is readily identified with the \textit{Clifford algebra} \( \text{Cliff}(q) \) on that quadratic map and the Jacobian induces a DG algebra differential \( \partial_{\text{jac}_f} \) on it. In particular, the cohomology of this DG algebra inherits an algebra structure this way.

\textbf{Theorem 2.11.} Let \( A = P/I \) be a homological complete intersection algebra over \( K \), with \( P = K[x_1, \ldots, x_n] \) and \( I = (f_1, \ldots, f_c) \) such that the \( f_j \) induce an \( A \)-basis of the \( A \)-module \( I/P^2 \) that we hence further assume to be free.

The cohomology of the DG algebra \( (\text{Cliff}(q), \partial_{\text{jac}_f}) \) is then the Yoneda \textit{Ext}–algebra \( \text{Ext}^*_{A^{ev}}(A, A) \) of self extensions of \( A \) as bimodule. In detail,

\[
\text{Cliff}(q) \cong A(t_1, \ldots, t_n; s_1, \ldots, s_c),
\]

with the \( t_i \) in degree 1, the \( s_j \) in degree 2, the differential determined by

\[
\partial(t_i) = \sum_{j=1}^c \left( \frac{\partial f_j}{\partial x_i} \mod I \right) s_j, \quad \partial(s_j) = 0.
\]
With respect to the multiplicative structure, the $s_j$ are central and
\[
t_i^2 = \sum_{j=1}^{c} \left( \frac{\partial f_j}{\partial x_i^2} \mod I \right) s_j,
\]
\[
t_it_{i'} + t_i't_i = \sum_{j=1}^{c} \left( \frac{\partial^2 f_j}{\partial x_i \partial x_{i'}} \mod I \right) s_j.
\]
If $A$ is projective over $K$, then the natural homomorphism of graded commutative algebras $\text{HH}^\bullet(A/K) \to \text{Ext}_A^\bullet(A, A)$ is an isomorphism, thus, the Hochschild cohomology is given as cohomology of that DG Clifford algebra too.

Remark 2.12. In concrete terms, expanding on 2.9 above, the Poincaré–Birkhoff–Witt theorem for enveloping algebras of (super) Lie algebras shows that as differential graded coalgebras, in particular, as complexes of graded $A$–modules, one has that $(\text{Cliff}(q), \partial_{jac} f)$ is isomorphic to the Koszul complex $K$ over $S = \text{Sym}_A(N_{A/P}) \cong A[s_1, ..., s_j]$ on the sequence
\[
\sum_{j=1}^{c} \left( \frac{\partial f_j}{\partial x_i} \mod I \right) s_j \in S; \quad i = 1, ..., n.
\]
Bigrading the Koszul complex by placing the variables $t_i = \partial_{x_i} \in \text{Der}_K(P, A)$ into bidegree $(1, 0)$ and $s_j = \partial f_j$ into bidegree $(0, 1)$, one regains the – well-known $[39, 22]$ – $A$–linear Hodge decomposition of the Hochschild cohomology for such a homological complete intersection,
\[
\text{HH}^p(A/K) \cong \bigoplus_{i+2j=p} H^i(K)_j
\]
\[
H^i(K)_j \cong H^{i+2j}(\text{Hom}_A(S_{i+j}(L_{A/K}[1]), A)).
\]
However, the preceding theorem says that, generally, this decomposition is not compatible with the multiplicative structure, in that squaring sends $\text{HH}^1(A/K) = H^1(K)_0$ to
\[
H^0(K)_1 \cong T_{A/K}^1 = H^2(\text{Hom}_A(\mathcal{L}_{A/K}[1], A)) \subseteq \text{HH}^2(A/K),
\]
but not to the other summand
\[
H^2(K)_0 \cong \text{Hom}_A(\Omega^2_{A/K}, A) = H^2(\text{Hom}_A(S_2(\mathcal{L}_{A/K}[1]), A)) \subseteq \text{HH}^2(A/K).
\]
In other words, the “obvious” multiplication on the Hodge decomposition needs to be twisted or “quantized”, here from the exterior multiplicative structure of the Koszul complex to that of the (homogeneous) Clifford algebra.

This phenomenon is analogous; see [13] and the discussion therein; to the situation for non-affine smooth schemes in characteristic zero. Similarly, for arbitrary commutative rings over a field of characteristic zero, the relation between Hodge decomposition and multiplicative structure on Hochschild cohomology is studied from a combinatorial point of view in [6].

We hasten to point out that for the affine (homological) complete intersections considered here this multiplicative twist can only make a difference when 2 is not a unit in $A$. 

3. Tate Models for the Multiplication Map of an Algebra

Let $S \rightarrow R$ be a homomorphism between commutative rings. In his seminal paper [37], John Tate presented in essence the following construction of a free $S$–resolution of $R$; see also [3] for further details, and, in particular, [14] for details on differential graded (DG) algebras with divided powers. The special case of the multiplication map $R \otimes_K R \rightarrow R$ for a commutative $K$–algebra $R$ of finite type that we will be mainly interested in has also been detailed in [5].

3.1. There exists a factorization of the given ring homomorphism as $S \rightarrow (\mathcal{T}, \partial) \xrightarrow{\pi} R$ such that

(i) $\mathcal{T}$ is a strictly graded commutative $S$–algebra of the form

$$\mathcal{T}_* = \text{Sym}_S(F_0) \otimes_S \bigwedge_S \left( \bigoplus_{i \text{ odd}} F_i \right) \otimes_S \Gamma_S \left( \bigoplus_{j > 0, \text{ even}} F_i \right),$$

with divided powers (in even degrees), graded so that $S$ is concentrated in degree 0, each $F_i$, $i \geq 0$, a free $S$–module concentrated in degree $i$.

Here $\text{Sym}$ denotes the symmetric algebra functor, $\bigwedge$ the exterior algebra functor, and $\Gamma$ the divided power algebra functor.

(ii) The $S$–algebra differential $\partial : \mathcal{T}_* \rightarrow \mathcal{T}_{*-1}$ is compatible with divided powers, thus, uniquely determined by its restriction to $\bigoplus_{j \geq 1} F_i \subseteq \mathcal{T}$, vanishing necessarily on $F_0$ for degree reasons.

(iii) The morphism $\pi$ is a quasiisomorphism of DG algebras, where one endows $R$, concentrated in degree zero, with the, only possible, zero differential.

Such factorization of $S \rightarrow R$ is then a Tate model for $R$ over $S$, but, with the usual slight abuse, we also simply call $(\mathcal{T}, \partial)$ a Tate model, the remaining data understood.

For given $i$, the module $\oplus_{j \leq i} F_j$ generates a DG subalgebra $\mathcal{T}^{(i)}$ with divided powers, and $\mathcal{T}^{(0)} \subseteq \cdots \subseteq \mathcal{T}^{(i)} \subseteq \mathcal{T}^{(i+1)} \subseteq \cdots \subseteq \mathcal{T} = \bigcup_i \mathcal{T}^{(i)}$ constitutes the associated Postnikov tower. It follows from the description given that the inclusion $\mathcal{T}^{(i)} \rightarrow \mathcal{T}$ induces isomorphisms $H_j(\mathcal{T}^{(i)}) \xrightarrow{\cong} H_j(\mathcal{T})$ for $j < i$ and that $\partial|_{\mathcal{T}^{(i+1)}} : F_{i+1} \rightarrow \mathcal{T}^{(i)}$ can be interpreted as an $S$–linear attaching map that kills $H_i(\mathcal{T}^{(i)})$.

Remark 3.2. Viewed just as a complex, such a Tate model constitutes in particular a free resolution of $R$ as $S$–module, thus, can be used to calculate $\text{Tor}^S(R, -)$ and $\text{Ext}_S(R, -)$.

If $S$ is noetherian and $S \rightarrow R$ is surjective, one may choose $F_0 = 0$ and each $F_i$, for $i \geq 1$, to be free of finite rank over $S$, and then $\mathcal{T}$ is in turn free of finite rank over $S$ in each degree. If $R$ is only of finite type as an $S$–algebra, then one may choose $F_0$ to be a finite free $S$–module and each $F_i$ a finite free $\mathcal{T}^{(0)} = \text{Sym}_S(F_0)$–module.

3.3. Now consider the particular case, where $A$ is a commutative $K$–algebra over some commutative ring $K$, and $\mu : A^\vee := A \otimes_K A = S \rightarrow A = R$ is the multiplication map, a $K$–algebra homomorphism from the enveloping algebra of $A$ over $K$ to $A$. We denote $\Omega_A^1 = \Omega_{A/K}^1$ the kernel of the multiplication map and
and the proof of the corollary indicate how

\begin{equation}
\tilde{\Omega}_P^1 = \Omega_P^1 / (\Omega_P^1)^2
\end{equation}

represents the functor of taking \(K\)-linear derivations in symmetric \(A\)-bimodules. We use the convention that the universal derivation \(d : A \to \tilde{\Omega}_A^1\) sends \(a \mapsto da = 1 \otimes a - a \otimes 1\).

Assume now that \(A\) is presented as \((\text{Sym}_K F) / I\), for some free \(K\)-module \(F\) and some ideal \(I\) in the polynomial ring \(P = \text{Sym}_K F\) over this free module.

The following result holds for an arbitrary presentation \(A = P / I\), with neither \(P\) nor \(A\) necessarily commutative.

**Lemma 3.4** (cf. [18, Cor.2.11]). There exists an exact sequence of \(A^{ev}\)-modules

\begin{equation}
I/I^2 \xrightarrow{d} A \otimes_P \tilde{\Omega}_P^1 \otimes_P A \xrightarrow{j} A^{ev} \xrightarrow{\mu} A \xrightarrow{0}
\end{equation}

where \(d\) is induced by restricting the universal \(K\)-linear derivation \(d : P \to \tilde{\Omega}_P^1\) to \(I\) and \(j\) is obtained from the inclusion \(i : \tilde{\Omega}_P^1 \subseteq P \otimes_K P\) as \(j \equiv A \otimes_P i \otimes_P A\) taking into account \(A^{ev} \cong A \otimes_P (P \otimes_K P) \otimes_P A\).

Returning to the commutative situation, choose a \(P^{ev}\)-linear surjection \(P \otimes_K G \otimes_K P \to I\) for some free \(K\)-module \(G\). The preceding Lemma has the following immediate consequence.

**Corollary 3.5.** One may construct a Tate model for the multiplication map \(\mu : A \otimes_K A \to A\) with \(F_0 = 0, F_1 = A \otimes_K F \otimes_K A\), and \(F_2 = A \otimes_K G \otimes_K A\).

**Proof.** As the multiplication map is surjective, one may choose \(F_0 = 0\) in the construction of the Tate model.

The image of the restriction of the universal derivation \(d : P \to \tilde{\Omega}_P^1\) to \(F \subseteq P\) generates the ideal \(\tilde{\Omega}_P^1 \subseteq P \otimes_K P\) and this shows that the induced map \(A \otimes_K F \otimes_K A \to A \otimes_P \tilde{\Omega}_P^1 \otimes_P A\) is surjective, thus, the choice of \(F_1\). The resulting DG algebra \(\mathcal{T}^{(1)} = \bigwedge_{A^{ev}} F_1\) is nothing but the Koszul complex over the composition \(A \otimes_K F \otimes_K A \to A \otimes_P \tilde{\Omega}_P^1 \otimes_P A\) whose first homology is isomorphic to \(d(I/I^2)\) — this is exactness of the sequence (1) above at the term \(A \otimes_P \tilde{\Omega}_P^1 \otimes_P A\).

Applying on both sides the tensor product over \(P\) with \(A\) to the \(P^{ev}\)-linear surjection \(P \otimes_K G \otimes_K P \to I\) yields an \(A^{ev}\)-linear surjection first from \(F_2\) onto \(I/I^2\) and then via \(d\) onto that homology. As \(F_2\) is \(A^{ev}\)-projective, one can find a lifting of that surjection into the 1-cycles of \(\mathcal{T}^{(1)}\) and such lifting extends the differential from \(\mathcal{T}^{(1)}\) to

\begin{equation}
\mathcal{T}^{(2)} = \bigwedge_{A^{ev}} (F_1) \otimes_{A^{ev}} \Gamma_{A^{ev}} (F_2),
\end{equation}

whence the claim concerning \(F_2\) follows.

To be even more specific, Lemma 3.4 and the proof of the corollary indicate how to describe the possible differentials on \(\mathcal{T}^{(2)}\).

**3.6.** Let \(x = \{x_a\}_a\) be a basis of the free \(K\)-module \(F\). Then the universal derivation sends these algebra generators to

\begin{equation}
dx_a = x'' - x' \in \tilde{\Omega}_P^1 \subseteq P \otimes_K P \cong \text{Sym}_K (F \oplus F),
\end{equation}
where we abbreviate \( x' = x \otimes 1 \), \( x'' = 1 \otimes x \). Below, we will use the same notation for the respective residue classes in \( A^{ev} \).

Representing a polynomial \( f(x) = \sum_{A=(a_1,\ldots,a_k)} \alpha_A x_{a_1} \cdots x_{a_k} \in P \), with \( \alpha_A \in K \), as a finite sum of (ordered) monomials in (finitely many) basis elements \( x_a \) of \( F \), the definition of \( d \) and then the product rule for this derivation yield

\[
(*) \quad df = 1 \otimes f - f \otimes 1 = f(x') - f(x'')
\]

\[
= \sum_{A=(a_1,\ldots,a_k)} \sum_{\kappa=1}^k \alpha_A x_{a_1}' \cdots x_{a_{\kappa-1}}' \partial f x_{a_{\kappa+1}}'' \cdots x_{a_k}' \]

\[
= \sum_{A=(a_1,\ldots,a_k)} \sum_{\kappa=1}^k \alpha_A x_{a_1}' \cdots x_{a_{\kappa-1}}' (x_{a_{\kappa}}'' - x_{a_{\kappa}}') x_{a_{\kappa+1}}'' \cdots x_{a_k}''
\]

in \( \Omega^1_P = (x'' - x')_A \subset P \otimes_K P \).

3.7. Slightly abusing notation, we may then base the free \( A^{ev} \)-module \( F_1 \) in the construction of \( T \) above on symbols \( dx_a \) and define \( \partial dx_a = x'' - x' \in A^{ev} \). Similarly, if \( \{ f_b \} \) are polynomials in \( I \) whose classes generate \( I/I^2 \), we may base the free \( A^{ev} \)-module \( F_2 \) on symbols \( df_b \) and choose

\[
(**) \quad \partial(df_b) = \sum_{A=(a_1,\ldots,a_k)} \sum_{\kappa=1}^k \alpha_A x_{a_1}' \cdots x_{a_{\kappa-1}}' dx_{a_{\kappa}}'' \cdots x_{a_k}'' \in F_1
\]

for the differential on those basis elements, where the right hand side results from a chosen presentation of \( f = f_b \) as above.

This time, the right hand side will depend on the chosen presentation, but those choices are easily controlled, in that one may add \( \partial(\omega_b) \) to the right hand side for any two-form \( \omega_b \in \bigwedge^2 A^{ev} F_1 \) to change the value of \( \partial(df_b) \). These are also the only choices.

If we collect coefficients of the terms \( dx_a \) in (**) for \( f = f_b \) to write

\[
df_b = \sum_a \Delta_{b,a}(x',x'')dx_a,
\]

with \( \Delta_{b,a}(x',x'') \) in \( P \otimes_K P \), then these coefficients satisfy

\[
\mu \Delta_{b,a}(x',x'') = \Delta_{b,a}(x,x) = \frac{\partial f_b(x)}{\partial x_a}
\]

in \( P \) irrespective of the representation we chose.

4. Diagonal Approximation and Graded Commutativity

With \( T \) a Tate model of the multiplication map \( A^{ev} \to A \) for a commutative \( K \)-algebra as before, note that \( T \) is naturally a complex of \( A \)-bimodules so that we can form the tensor product \( T \otimes_A T \). This tensor product carries again the structure of a differential strictly graded commutative algebra with divided powers.

Denote \( j_{1,2} : T \to T \otimes_A T \) the DG algebra homomorphisms \( j_1(t) = t \otimes_A 1 \), respectively \( j_2(t) = 1 \otimes_A t \), for \( t \in T \).

4.1. The complex underlying \( T \otimes_A T \) is one of free \( A \)-trimodules, that is, free modules over \( A^{ev} \). We make no claim about the homology of this complex, except
to note that the algebra homomorphism \( \epsilon \otimes_A \epsilon : T \otimes_A T \to A \otimes_A A \cong A \) is the natural map onto the zeroth homology.

Note, however, that if \( A \) is flat over \( K \), then \( A \) is \( A \)–flat via both \( j_1 \) or \( j_2 \), and \( T \) constitutes a flat resolution of \( A \) over \( A \) via either \( j_1 \) or \( j_2 \), thus, the homology of \( T \otimes_A T \) is trivial, equal to \( A \) in degree zero.

However, even without flatness we have the following.

**Lemma 4.2.** With notation as introduced above,

1. The algebra homomorphisms
   \[
   \epsilon_1 = \epsilon \otimes_A \epsilon : T \otimes_A T \to A \otimes_A T \cong T \\
   \epsilon_2 = T \otimes_A \epsilon : T \otimes_A T \to T \otimes_A A \cong T
   \]
   are homotopic as morphisms of complexes of \( A \)–trimodules.

2. If \( H_m(T \otimes_A T) = 0 \) for \( 1 \leq m \leq i \) for some \( i \), then Ker \( \epsilon_1 \cap \) Ker \( \epsilon_2 \) has vanishing homology up to and including degree \( i \).

   In particular, if \( A \) is flat over \( K \), then the subcomplex Ker \( \epsilon_1 \cap \) Ker \( \epsilon_2 \) \( \subseteq \) \( T \otimes_A T \) has zero homology.

**Proof.** As \( \epsilon \epsilon_1 = \epsilon \epsilon_2 \), the difference \( \epsilon_2 - \epsilon_1 \) takes its values in the acyclic complex Ker(\( \epsilon : T \to A \)). As its source is a complex of projective, even free \( A \)–trimodules, the claim (1) follows.

Concerning (2), consider the following commutative diagram of complexes with exact rows and columns

\[
\begin{array}{ccccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Ker} \epsilon & \rightarrow & T & \rightarrow & A & \rightarrow & 0 \\
\downarrow & & \downarrow \epsilon_2 & & \downarrow \epsilon & & \downarrow \epsilon & & \downarrow 0 \\
0 & \rightarrow & \text{Ker} \epsilon_1 & \rightarrow & T \otimes_A T & \rightarrow & T & \rightarrow & 0 \\
\downarrow & & \downarrow \epsilon_1 & & \downarrow T & & \downarrow 0 & & \downarrow 0 \\
0 & \rightarrow & \text{Ker} \epsilon_1 \cap \text{Ker} \epsilon_2 & \rightarrow & \text{Ker} \epsilon_2 & \rightarrow & \text{Ker} \epsilon & \rightarrow & 0 \\
\downarrow & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

By construction, \( \epsilon \) is a quasi isomorphism, whence Ker \( \epsilon \) is acyclic. By assumption \( \epsilon_1, \epsilon_2 \) are quasiisomorphisms in degrees up to and including \( i \), where we may take \( i = \infty \) if \( A \) is flat over \( K \). The long exact homology sequence obtained from the short exact sequence, say, at the bottom yields then the claims. \( \square \)

**Definition 4.3** (cf. [32]). Let \( A \) be a complex of \( A \)–modules that comes with an augmentation \( A \xrightarrow{\epsilon} A \). A \emph{co-unital diagonal approximation} on \( A \) is a morphism \( \Phi : A \xrightarrow{\Phi} A \otimes_A A \) of \( A \)–complexes so that the following diagram commutes,

\[
\begin{array}{c}
A \\
\downarrow \Phi \\
A \otimes_A A \end{array}
\xrightarrow{\epsilon \otimes_A \epsilon} A, \]

\[ A \otimes_A A \xrightarrow{\Phi} A, \]

\[ A \otimes_A A \xrightarrow{\Phi} A. \]
where we have identified $A \otimes A \cong A \cong A \otimes A A$.

As is well-known, the existence of a co-unital diagonal approximation on a Tate model of the multiplication map implies graded commutativity of the Yoneda product on $\text{Ext}_{A^{\text{ev}}}^{\bullet}(A, A)$; see, for example, [11, 32, 35].

**Definition 4.4.** Given a co-unital diagonal approximation $\Phi : T \to T \otimes_{A} T$ on the Tate model of $\mu : A^{\text{ev}} \to A$, define a cup product $\cup = \cup_{\Phi}$ on $\text{Hom}_{A^{\text{ev}}}(T, A)$ through

$$f \cup g = \mu(f \otimes g)_{\Phi} : T \xrightarrow{\Phi} T \otimes_{A} T \xrightarrow{f \otimes g} A \otimes_{A} A \xrightarrow{\mu} A,$$

for cochains $f, g : T \to A$.

**Proposition 4.5.** Let $A$ be a commutative $K$–algebra with a Tate model of $\mu : A^{\text{ev}} \to A$ that admits a co-unital diagonal approximation. The corresponding cup product then induces a graded commutative product in $H(\text{Hom}_{A^{\text{ev}}}(T, A)) = \text{Ext}_{A^{\text{ev}}}(A, A)$. This product coincides with the usual Yoneda or composition product on the $\text{Ext}$–algebra.

**Proof.** Given cocycles $f, g \in \text{Hom}_{A^{\text{ev}}}(T, A)$, lift them to morphisms of complexes $\tilde{f}, \tilde{g} \in \text{Hom}_{A^{\text{ev}}}(T, T)$. In the homotopy category of complexes $\epsilon_{1}, \epsilon_{2} : T \otimes_{A} T \to T$ agree because of Lemma 4.2(1), and so the following diagram of complexes of $A^{\text{ev}}$–modules commutes in the homotopy category,

By definition, $f \circ \tilde{g}$ is a cocycle representing the Yoneda product of $f$ with $g$, while the composition across the top and down yields $f \cup g$. Thus, these two products coincide in $\text{Ext}_{A^{\text{ev}}}(A, A)$.

Further, for $\tilde{f} \in \text{End}_{A^{\text{ev}}}(T), \tilde{g} \in \text{End}_{A^{\text{ev}}}(T)$ lifts of homogeneous cocycles as above, one has

$$(\tilde{f} \otimes \text{id})(\text{id} \otimes \tilde{g}) = \tilde{f} \otimes \tilde{g} = (-1)^{|g||\tilde{g}|}(\tilde{f} \otimes \text{id})(\tilde{f} \otimes \text{id})$$

whence combining the above diagram with that corresponding to $(\text{id} \otimes \tilde{g})(\tilde{f} \otimes \text{id})$ yields graded commutativity of the Yoneda or cup product in cohomology, that is, in $\text{Ext}_{A^{\text{ev}}}(A, A)$. \hfill \qed

**Remark 4.6.** The preceding proof shows that a partial co-unital diagonal approximation $\Phi^{(i)} : T^{(i)} \to T^{(i)} \otimes_{A} T^{(i)}$ is enough to express the Yoneda product of two cohomology classes whose degrees add up to at most $i$ through the cup product on $\text{Hom}_{A^{\text{ev}}}(T^{(i)}, A)$. Moreover, these classes will commute in the graded sense with respect to the product.

In the particular case that $A$ is flat over $K$, one can indeed construct a co-unital diagonal approximation that is furthermore a homomorphism of DG algebras with divided powers.
Proposition 4.7. If $A$ is flat over $K$, then there exists a co-unital diagonal approximation $\Phi$ on the Tate model $T$ of $\mu : A^{ev} \to A$ that induces such approximation $\Phi^{(i)}$ on each DG algebra $T^{(i)}$ in the Postnikov tower.

Proof. We construct a desired co-unital diagonal approximation $\Phi$ inductively along the Postnikov tower. At the basis of the Postnikov tower, we have $T^{(0)} = T_0 = A^{ev}$ and $T^{(0)} \otimes_A T^{(0)} = (T \otimes_A T)_0 \cong A^{\otimes 3} \subseteq T \otimes_A T$. Now

$$
\Phi^{(0)} : T^{(0)} \cong A^{\otimes 2} \to T^{(0)} \otimes_A T^{(0)} \cong A^{\otimes 3},
$$

$$
\Phi^{(0)}(a \otimes b) = a \otimes 1 \otimes b
$$

defines a homomorphism of algebras such that $\epsilon_1 \Phi^{(0)} = \text{id}_{A^{\otimes 2}} = \epsilon_2 \Phi^{(0)}$. Via $\Phi^{(0)}$, we can, and will, view $T \otimes_A T$ as a DG $A^{ev}$–algebra.

Now assume a co-unital diagonal approximation $\Phi^{(i)} : T^{(i)} \to T^{(i)} \otimes_A T^{(i)}$ has been defined for some $i \geq 0$. With $T^{(i+1)} = T^{(i)}(F_{i+1})$ for some free $A^{ev}$–module $F_{i+1}$, note that $(j_1 + j_2)|_{F_{i+1}} : F_{i+1} \to T^{(i+1)} \otimes_A T^{(i+1)}$ satisfies

$$
\epsilon_1(j_1 + j_2)|_{F_{i+1}} = \epsilon_2(j_1 + j_2)|_{F_{i+1}}.
$$

For $m = 1, 2$, by the induction hypothesis $\epsilon_m \Phi^{(i)} = \text{id}_{T^{(i)}}$. As $\epsilon_m$ is a morphism of complexes, it follows that

$$
\epsilon_m(\Phi^{(i)} \partial - \partial(j_1 + j_2)) = \partial - \partial \epsilon_m(j_1 + j_2),
$$

vanishes on $F_{i+1}$. Because $\Phi^{(i)}$ is a morphism of complexes,

$$
\partial(\Phi^{(i)} \partial - \partial(j_1 + j_2)) = 0,
$$

whence it follows that $\Phi^{(i)} \partial - \partial(j_1 + j_2)$ maps $F_{i+1}$ into the cycles of $\ker \epsilon_1 \cap \ker \epsilon_2$. As the latter complex is acyclic for $A$ flat over $K$ by Lemma 4.2(2), and as $F_{i+1}$ is projective, even free, we can find $\Phi'_{i+1} : F_{i+1} \to \ker \epsilon_1 \cap \ker \epsilon_2$ such that $\Phi^{(i)} \partial = \partial(\Phi'_{i+1} + j_1 + j_2)$ on $F_{i+1}$.

The algebra homomorphism given by $\Phi^{(i)}$ on $T^{(i)}$ and extended through

$$
\Phi_{i+1} = \Phi'_{i+1} + j_1 + j_2
$$

first to $F_{i+1}$ and, then as algebra homomorphism to $\Phi^{(i+1)}$ on $T^{(i+1)}$ satisfies $\partial \Phi^{(i+1)} = \Phi^{(i+1)} \partial$ and $\epsilon_m \Phi^{(i+1)} = \text{id}$. In other words, we have found an extension of $\Phi^{(i)}$ to a co-unital diagonal approximation on $T^{(i+1)}$, completing the inductive argument. \qed

5. Products in Low Degree

5.1. We can sharpen Proposition 4.7 a bit in low degrees in that we do not need any flatness there.

Proceeding as in the proof of Proposition 4.7, define $\Phi_1 = (j_1 + j_2)|_{F_1} : F_1 \to T^{(1)} \otimes_A T^{(1)}$ and extend $\Phi^{(0)}$, $\Phi_1$ to a homomorphism $\Phi^{(1)} : T^{(1)} \to T^{(1)} \otimes_A T^{(1)}$ of strictly commutative graded algebras with divided powers. It will indeed be a co-unital approximation and homomorphism of DG algebras, as $\Phi^{(0)}$ is such a
homomorphism and
\[ \partial \Phi_1(\text{d}x_a) = \partial(j_1 + j_2)(\text{d}x_a) = (j_1 + j_2)\partial(\text{d}x_a) \]
\[ = j_1(1 \otimes x_a - x_a \otimes 1) + j_2(1 \otimes x_a - x_a \otimes 1) \]
\[ = (1 \otimes x_a \otimes 1 - x_a \otimes 1 \otimes 1 + (1 \otimes 1 \otimes x_a - 1 \otimes x_a \otimes 1) \]
\[ = 1 \otimes 1 \otimes x_a - x_a \otimes 1 \otimes 1 = \Phi^{(0)}(\partial(\text{d}x_a)) \]
for any basis element \( \text{d}x_a \) in \( F_1 \).

Put differently, a diagonal approximation \( \Phi^{(1)} : T^{(1)} \to T^{(1)} \otimes_A T^{(1)} \) can be chosen in such a way that
\[ \Phi^{(1)}(a \otimes b) = a \otimes 1 \otimes b \quad \text{for} \ a \otimes b \in A^{ev}, \]
\[ \Phi^{(1)}(\text{d}x_a) = \text{d}x'_a + \text{d}x''_a \quad \text{for} \ \text{d}x_a \in F_1, \]
where we have shortened notation to \( \text{d}x'_a = \text{d}x_a \otimes 1 = j_1(\text{d}x_a) \) and \( \text{d}x''_a = 1 \otimes \text{d}x_a = j_2(\text{d}x_a) \). We also use the algebra homomorphism \( \Phi^{(0)} : A^{ev} \to A^{ev} \) to identify
\[ x' = \Phi^{(0)}(x') = x \otimes 1 \otimes 1, \quad x = 1 \otimes x \otimes 1, \quad x'' = \Phi^{(0)}(x'') = 1 \otimes 1 \otimes x. \]

5.2. It is noteworthy that the induced cup product on \( \text{Hom}^*(T^{(1)}, A) \) is always strictly graded commutative. Indeed, recall that \( T^{(1)} \) is nothing but the Koszul complex on \((x_a'' - x_a')_a\) in \( A^{ev} \). Dualizing into \( A \), the induced differential becomes zero, and there is a quasi isomorphism
\[ \text{Hom}^*_A(T^{(1)}, A) \cong \bigoplus_{i \geq 0} \text{Hom}^i_A(A \otimes_P \Omega^i_{P/K} \otimes_P A, A) \cong \bigoplus_{i \geq 0} \text{Hom}_P(\Omega^i_{P/K}, A) \]
to the graded module of alternating \( K \)-linear polyvector fields on \( P \) with values in \( A \). If now \( f \in \text{Hom}^{|f|}_A(T^{(1)}, A) \) is a cocycle of odd degree, then \( f \cup f = 0 \). In fact, if \( \omega \in T^{(1)}_{2|f|} \), then \( \Phi^{(1)}(\omega) \), in Sweedler notation, is of the form
\[ \Phi^{(1)}(\omega) = \sum (\omega_1) \otimes_A \omega_2 (1 - (-1)^{|\omega_1||\omega_2|}) \otimes_A \omega_1(1) \]
for suitable homogeneous \( \omega_{(k)} \in T^{(1)}, k = 1, 2 \), whose degrees add up to \( 2|f| \), as the explicit form of \( \Phi^{(1)} \) shows. Now \( f \otimes f \) is zero on any summand with \( |\omega_{(k)}| \neq |f| \), while on summands where \( |\omega_{(k)}| = |f| \) is odd, one has
\[ (f \otimes f)(\omega_{(1)} \otimes_A \omega_{(2)} - \omega_{(2)} \otimes_A \omega_{(1)}) = f(\omega_{(1)})f(\omega_{(2)}) - f(\omega_{(2)})f(\omega_{(1)}) = 0, \]
as \( A \) is commutative.

5.3. As just demonstrated, the construction of \( \Phi^{(1)} \) does not require \( A \) to be flat over \( K \), and we now show by explicit construction that one can find also without any flatness assumption a co-unital diagonal approximation \( \Phi^{(2)} \) on \( T^{(2)} \) that extends the just constructed \( \Phi^{(1)} \).

Recall that \( T^{(2)} = T^{(1)}F_2 \), with \( F_2 \) a free \( A^{ev} \)-module based on elements \( \text{d}f_b \) that correspond to polynomials \( f_b \in I \subseteq P \) whose classes in \( I/I^2 \) generate that \( A^{ev} \)-module. An “attaching map” \( \partial|_{F_2} : F_2 \to T^{(1)} \) that killed the second homology of \( T^{(1)} \) was obtained in 3.7(\text{**}) from some representation of each \( f_b \) as \( K \)-linear combination of monomials in the algebra generators of \( P \) over \( K \).

In those terms, and extending the shorthand notation to \( \text{d}f'_b = \text{d}f_b \otimes 1 = j_1(\text{d}f_b) \) and \( \text{d}f''_b = 1 \otimes \text{d}f_b = j_2(\text{d}f_b) \), we now determine a diagonal approximation in degree 2.
**Proposition 5.4.** The algebra homomorphism $\Phi^{(2)} : \mathcal{T}^{(2)} \to \mathcal{T}^{(2)} \otimes_A \mathcal{T}^{(2)}$ that extends $\Phi^{(1)}$ by means of

$$\Phi_2(df_b) = df'_b + df''_b$$

$$- \sum_{A=(a_1, \ldots, a_k)} \sum_{1 \leq \kappa < \lambda \leq k} \alpha_A x'_{a_1} \cdots x'_{a_{\kappa-1}} dx_{a_\kappa} x_{a_{\kappa+1}} \cdots x_{a_{\lambda-1}} dx_{a_\lambda} x_{a_{\lambda+1}} \cdots x_{a_k}$$

for some finite presentation $f_b(x) = \sum_{A=(a_1, \ldots, a_k)} \alpha_A x_{a_1} \cdots x_{a_k} \in P$ of the given polynomial $f_b$ defines a co-unital diagonal approximation.

**Proof.** Inspection reveals immediately that $\Phi^{(2)}$ is co-unital, that is,

$$\epsilon_1 \Phi^{(2)} = \text{id}_{\mathcal{T}^{(2)}} = \epsilon_2 \Phi^{(2)}.$$ 

It thus remains to verify that $\Phi^{(2)}$ is compatible with the differentials, equivalently, that $\partial \Phi_2(df_b) = \Phi^{(1)}(\partial df_b)$ for each $b$.

This is a straightforward verification. First, we exhibit $\partial(df'_b)$ and $\partial(df''_b)$ for use below,

$$\partial(df'_b) = \sum_{A=(a_1, \ldots, a_k)} \sum_{1 \leq \kappa < \lambda \leq k} \alpha_A x'_{a_1} \cdots x'_{a_{\kappa-1}} dx_{a_\kappa} x_{a_{\kappa+1}} \cdots x_{a_k},$$

$$\partial(df''_b) = \sum_{A=(a_1, \ldots, a_k)} \sum_{1 \leq \kappa < \lambda \leq k} \alpha_A x_{a_1} \cdots x_{a_{\kappa-1}} dx_{a_\kappa} x''_{a_{\kappa+1}} \cdots x''_{a_k}.$$

Next we apply the differential to the double sum in the definition of $\Phi_2(df_b)$ and obtain

$$\partial \left( \sum_{A=(a_1, \ldots, a_k)} \sum_{1 \leq \kappa < \lambda \leq k} \alpha_A x'_{a_1} \cdots x'_{a_{\kappa-1}} dx_{a_\kappa} x_{a_{\kappa+1}} \cdots x_{a_{\lambda-1}} dx_{a_\lambda} x''_{a_{\lambda+1}} \cdots x''_{a_k} \right)$$

$$= \sum_{A=(a_1, \ldots, a_k)} \sum_{1 \leq \kappa < \lambda \leq k} \alpha_A x'_{a_1} \cdots x'_{a_{\kappa-1}} (x_{a_\kappa} - x'_{a_\kappa}) x_{a_{\kappa+1}} \cdots x_{a_{\lambda-1}} dx_{a_\lambda} x''_{a_{\lambda+1}} \cdots x''_{a_k}$$

$$- \sum_{A=(a_1, \ldots, a_k)} \sum_{1 \leq \kappa < \lambda \leq k} \alpha_A x'_{a_1} \cdots x'_{a_{\kappa-1}} dx_{a_\kappa} x_{a_{\kappa+1}} \cdots x_{a_{\lambda-1}} (x''_{a_\lambda} - x_{{a_\lambda}+1}) x''_{a_{\lambda+1}} \cdots x''_{a_k}$$

$$= \sum_{A=(a_1, \ldots, a_k)} \sum_{1 \leq \kappa < \lambda \leq k} \alpha_A (x_{a_1} \cdots x_{a_{\lambda-1}} - x'_{a_1} \cdots x'_{a_{\lambda-1}}) dx_{a_\lambda} x''_{a_{\lambda+1}} \cdots x''_{a_k}$$

$$- \sum_{A=(a_1, \ldots, a_k)} \sum_{1 \leq \kappa < \lambda \leq k} \alpha_A x'_{a_1} \cdots x'_{a_{\kappa-1}} dx_{a_\kappa} (x''_{a_{\kappa+1}} \cdots x''_{a_k} - x_{a_{\kappa+1}} \cdots x_{a_k})$$

$$= \partial(df'_b) + \partial(df''_b) - \Phi^{(1)}(\partial df_b),$$

where the first equality uses that $\partial$ is a skew derivation of degree one, and the second equality applies 3.6 to the monomials $x_{a_1} \cdots x_{a_{\kappa-1}}$, respectively, $x_{a_{\kappa+1}} \cdots x_{a_k}$, while the final equality uses the form of $\partial(df'_b)$ and $\partial(df''_b)$ as recalled above. Reordering the terms, we obtain $\Phi^{(2)}(\partial df_b) = \partial \Phi^{(2)}(df_b)$ as required. 

Note the following special property of the co-unital diagonal approximation just constructed.

**Corollary 5.5.** The co-unital diagonal approximation $\Phi^{(2)}$ is cocommutative in the $df_b$, in that the automorphism of graded algebras $\sigma : \mathcal{T}^{(2)} \otimes_A \mathcal{T}^{(2)} \to \mathcal{T}^{(2)} \otimes_A$
\(\mathcal{T}^{(2)}\), ignoring differentials, that exchanges \(df_b' \leftrightarrow df_b''\), for each \(b\), but leaves the remaining variables unchanged, satisfies \(\sigma \circ \Phi^{(2)} = \Phi^{(2)}\).

Having determined one explicit co-unital diagonal approximation on \(\mathcal{T}^{(2)}\), what choices were involved? Provided \(\mathcal{T} \otimes_A \mathcal{T}\) is exact in low degrees, we can easily describe all co-unital diagonal approximations on \(\mathcal{T}^{(2)}\) that are algebra homomorphisms between algebras with divided powers.

**Theorem 5.6.** Let \(\mathcal{T}\) be a Tate model of \(\mu : A^{ev} \to A\) as before. The just exhibited co-unital diagonal approximation \(\Phi^{(2)}\) can be modified to the DG algebra homomorphism \(\Psi : \mathcal{T}^{(2)} \to \mathcal{T}^{(2)} \otimes_A \mathcal{T}^{(2)}\) that respects divided powers, agrees in degree zero with \(\Phi^{(0)}\) and is given on the algebra generators of \(\mathcal{T}\) in degrees 1 and 2 through

\[
\Psi(dx_a) = dx'_a + dx''_a + \partial\omega_a,
\Psi(df_b) = \Phi^{(2)}(df_b) + \sum_a \Delta_{b,a}(x', x'')\omega_a + \partial\eta_b,
\]

where \(\Delta_{b,a} \in A^{ev}\) are as in 3.6, the \(\omega_a\) are of degree 2 and the \(\eta_b\) are of degree 3 in \(\mathcal{T}^{(2)} \otimes_A \mathcal{T}^{(2)}\), such that \(\partial\omega_a\) and \(\sum_a \Delta_{b,a}(x', x'')\omega_a + \partial\eta_b\) are in \(\text{Ker} \epsilon_1 \cap \text{Ker} \epsilon_2\).

If \(\mathcal{T}^{(2)} \otimes_A \mathcal{T}^{(2)}\) is exact in degrees 1 and 2, then these are the only possible co-unital diagonal approximations that are homomorphisms of DG algebras with divided powers and that agree with \(\Phi^{(2)}\) in degree zero. Moreover, in this case \(\omega_a\) and \(\eta_b\) can already be chosen in \(\text{Ker} \epsilon_1 \cap \text{Ker} \epsilon_2\).

**Proof.** That \(\Psi\) is a co-unital diagonal approximation along with \(\Phi^{(2)}\) is easily verified directly.

Conversely, if \(\Psi\) is some co-unital diagonal approximation on \(\mathcal{T}^{(2)}\), then \((\Psi - \Phi^{(2)})(dx_a)\) is a cycle for each \(a\) and, if \(H_1(\mathcal{T}^{(2)} \otimes_A \mathcal{T}^{(2)}) = 0\), it is a boundary, thus, of the form \(\partial\omega_a\) for suitable \(\omega_a\) in degree two. Moreover, \(\partial\omega_a\) is necessarily a cycle in \(\text{Ker} \epsilon_1 \cap \text{Ker} \epsilon_2\), as both diagonal approximations are co-unital, and Lemma 4.2(2) shows that \(H_1\) of that complex vanishes too. Thus, \(\omega_a\) can be chosen in \(\text{Ker} \epsilon_1 \cap \text{Ker} \epsilon_2\).

If also \(H_2(\mathcal{T}^{(2)} \otimes_A \mathcal{T}^{(2)}) = 0\), then the 2–cycles \(\Psi(df_b) - \Phi^{(2)}(df_b) - \sum_a \Delta_{b,a}\omega_a\) are boundaries, thus, of the form \(\partial\eta_b\) for some form \(\eta_b\) of degree 3. Again, \(\partial\eta_b\) belongs necessarily to \(\text{Ker} \epsilon_1 \cap \text{Ker} \epsilon_2\) and \(H_2\) of that complex vanishes by the same argument as before, whence we may replace \(\eta_b\), if necessary, by an element in \(\text{Ker} \epsilon_1 \cap \text{Ker} \epsilon_2\) that has the same image under the differential.

To illustrate, we determine the cup product of two 1–cochains.

**5.7.** Assume \(\mathcal{T}\) is a Tate model of \(\mu : A^{ev} \to A\) for some commutative \(K\)–algebra \(A\). Let \(f, g : \mathcal{T} \to A\) be \(A^{ev}\)–linear cochains of degree 1. Their cup product is then a cochain of degree 2 with values in \(A\), explicitly given by

\[
(f \cup g)(dx_{a_1} \wedge dx_{a_2}) = \mu(f \otimes_A g)\Phi(dx_{a_1} \wedge dx_{a_2})
= \mu(f \otimes_A g)((dx'_{a_1} + dx''_{a_1}) \wedge (dx'_{a_2} + dx''_{a_2}))
= g(dx_{a_1})f(dx_{a_2}) - f(dx_{a_1})g(dx_{a_2})
\]
and
\[
(f \cup g)(df_b) = \mu(f \otimes_A g)\Phi(df_b)
\]
\[
= \sum_A \sum_{1 \leq \kappa < \lambda \leq k} \alpha_A x_{a_1} \cdots x_{a_{\kappa-1}} f(dx_{a_\kappa}) x_{a_{\kappa+1}} \cdots x_{a_\lambda} \cdots x_{a_k} g(dx_{a_\lambda}) x_{a_{\lambda+1}} \cdots x_{a_k}
\]
\[
= \sum_A \sum_{1 \leq \kappa < \lambda \leq k} \alpha_A x_{a_1} \cdots x_{a_{\kappa-1}} \hat{x}_{a_{\kappa}} x_{a_{\kappa+1}} \cdots x_{a_{\lambda-1}} \hat{x}_{a_\lambda} x_{a_{\lambda+1}} \cdots x_{a_k} f(dx_{a_\kappa}) g(dx_{a_\lambda}) x_{a_{\lambda+1}} \cdots x_{a_k}
\]

It remains to find a palatable form of the coefficients of \(f(dx_{a_\kappa}) g(dx_{a_\lambda})\) in that expression.

**Theorem 5.8.** Given a commutative \(K\)-algebra \(A\), represented as \(A \cong K[x; x \in X]/(f_b)\). If for a given \(b\), the polynomial \(f_b\) depends on the variables \(x_1, \ldots, x_n \in X\), one may choose a co-unit diagonal approximation \(\Phi^{(2)}\) on \(T^{(2)}\) in such a way that
\[
(f \cup g)(df_b) = \sum_{1 \leq i \leq j \leq n} \left( \frac{\partial^{(2)} f_b(x)}{\partial x_i \partial x_j} \right) \mod I \quad f(dx_i) g(dx_j) \in A,
\]
for any \(A^\times\)-linear cochains \(f, g : T \to A\) of degree one.

**Proof.** Let \(f_b = \sum_{e \in \mathbb{N}_0} \alpha_e x_1^e \cdots x_n^e\) with \(\alpha_e \in K\) be the presentation of \(f_b\) as polynomial in the variables \(x_1, \ldots, x_n\). Rewrite the occurring monomials as
\[
\alpha_e x_1^e \cdots x_n^e = \alpha_x a_1 \cdots x_{a|e|},
\]
where
\[
x_{a_1} = \cdots = x_{a_1} = x_1, \\
x_{a_1+1} = \cdots = x_{a_1+a_2} = x_2, \\
\vdots
\]
\[
x_{a_1+\cdots+a_n+1} = \cdots = x_{a_1+\cdots+a_n} = x_n.
\]
In the corresponding choice of \(\Phi^{(2)}(df_b)\) in Proposition 5.4, this term contributes
\[
\sum_{1 \leq \kappa < \lambda \leq |e|} \alpha_x a_1 \cdots x_{a_\kappa} \cdots x_{a_\lambda} \cdots x_{a_\kappa+1} \cdots x_{a_\lambda+1} \cdots x_{a_n} dx'_{a_\kappa} dx'_{a_\lambda} dx''_{a_{\kappa+1}} \cdots dx''_{a_{\lambda+1}}\]
Specializing to \(x_a = x_a' = x_a''\) for each index \(a\), and sorting reduces this sum to
\[
\alpha_e \left( \sum_{i=1}^{n} \left( \binom{e_i}{2} \right) x_1^{e_i} \cdots x_{i-1}^{e_i} x_i^{e_i-2} x_{i+1}^{e_i} \cdots x_n^{e_i} dx_i' dx_i'' + \sum_{1 \leq i < j \leq n} e_i e_j x_1^{e_i} \cdots x_{i-1}^{e_i} x_i^{e_i-1} x_{i+1}^{e_i} \cdots x_{j-1}^{e_j} x_j^{e_j-1} x_{j+1}^{e_j} \cdots x_n^{e_n} dx'_i dx'_j \right)
\]
The coefficient of \(dx'_i dx'_j\) in this expression is indeed the corresponding divided second derivative of the monomial, whence, putting the terms together again, it follows that for this choice of presentation,
\[
\Phi^{(2)}(df_b)(x, x, x) = df'_b + df''_b - \sum_{1 \leq i < j \leq n} \frac{\partial^{(2)} f_b(x)}{\partial x_i \partial x_j} dx'_i dx'_j
\]
and then the cup product takes the form claimed. \(\square\)
6. Proof of Theorem 2.5

Let \( A = P/I \) be as before a presentation of a commutative \( K \)-algebra as quotient of a polynomial ring \( P = K[x; x \in X] \) over some commutative ring \( K \) modulo an ideal \( I = (f_b)_{b \in Y} \subseteq P \). We need to show that the Hessian quadratic map

\[
h_f = \sum_{b \in Y} h_{f_b} \partial_{f_b} : \text{Der}_K(P, A) \to \prod_{b \in Y} A \partial_{f_b}
\]
as defined in 2.4 sends \( \text{Der}_K(A, A) \subseteq \text{Der}_K(P, A) \) to \( N_{A/P} = \text{Hom}_A(I/I^2, A) \subseteq \prod_{b \in Y} A \partial_{f_b} \).

Now an element \((u_b \partial_{f_b})_b \in \prod_{b \in Y} A \partial_{f_b}\), represents an element in the normal module \( N_{A/P} \) if, and only if, for every relation \( \sum_b z_b f_b = 0 \), with \( z_b \in P \) and only finitely many of these elements nonzero, the sum \( \sum_b z_b u_b \) is zero in \( A \).

Let \( X' = \{x_1, \ldots, x_N\} \subset X \) be the finite subset of variables that are involved in the finitely many polynomials \( f_b \) that occur with nonzero coefficient in the given relation.

If \( D = \sum_{x \in X} a_x \partial_{a_x} \in \text{Der}_K(A, A) \) is given, set \( D' = \sum_{x' \in X'} \tilde{a}_x \partial_{\tilde{a}_x} \) where \( \tilde{a}_x \in P \) lifts \( a_x \in A \). One then has \( D'(f_b) \equiv D(f_b) = 0 \) mod \( I \), and

\[
q(D)_b = \sum_{1 \leq i, j \leq N} \frac{\partial^{(2)} f_b(x)}{\partial x_i \partial x_j} \tilde{a}_i \tilde{a}_j \mod I \equiv \tilde{h}_f(D')(f_b) \mod I
\]
for those \( f_b \) that are involved in the relation. Next note that the second divided power \( D^{(2)} = \sum_{1 \leq i, j \leq N} \frac{\partial^{(2)}}{\partial x_i \partial x_j} \tilde{a}_i \tilde{a}_j \) of \( D' \) satisfies

\[
D^{(2)}(fg) = D^{(2)}(f)g + D'(f)D'(g) + fD^{(2)}(g)
\]
for any polynomials \( f, g \in P \).

Apply \( D^{(2)} \) to the relation \( 0 = \sum_b z_b f_b \) to obtain

\[
0 = D^{(2)} \left( \sum_b z_b f_b \right) = \sum_b D^{(2)}(z_b f_b)
\]

\[
= \sum_b \left( D^{(2)}(z_b f_b) + D'(z_b)D'(f_b) + z_b D^{(2)}(f_b) \right)
\]

\[
= \sum_b z_b q(D)_b \mod I
\]
as in the middle equation, each term \( D^{(2)}(z_b f_b) \) is in \( I \) and \( D'(f_b) \equiv D(f_b) = 0 \) mod \( I \) as \( D \) is a derivation on \( A \). Therefore, \( q(D) = (q(D)_b)_b \) is indeed an element of the normal module. This proves part (1) of Theorem 2.5.

Part (2) of Theorem 2.5 just restates a well-known fact, but we give another deduction using the Tate model \( T \) of the multiplication map \( \mu : A^e \to A \) and establish along the way as well Part (3) of Theorem 2.5.

To this end, recall, from 5.2 above, that \( T^{(1)} \) is just the Koszul complex on \( (x''_a - x_a)_a \) in \( A^e \) and that its dual \( \text{Hom}_A^*(T^{(1)}, A) \) carries the cup product induced by \( \Phi^{(1)} \) with respect to which its cohomology \( H^*(\text{Hom}_A^*(T^{(1)}, A)) \cong \text{Hom}_A(\Omega^*_{P/K}, A) \) is a strictly graded commutative algebra.

As the diagonal approximation \( \Phi \) on \( T \) extends the diagonal approximation \( \Phi^{(1)} \) on \( T^{(1)} \), the \( A \)-linear surjective restriction map \( p : T^* = \text{Hom}_A^*(T, A) \to T^{(1)*} = \text{Hom}_A^*(T^{(1)}, A) \) resulting from the inclusion \( T^{(1)} \subset T \) is compatible with the
cup products on source and target, thus, induces a $K$–algebra homomorphism in cohomology.

If $K$ is the kernel of the restriction map on the complexes, then the long exact cohomology sequence contains

$$H^1(T^{(1)*}) \to H^2(K) \to H^2(T^*) \to H^2(T^{(1)*})$$

If now $D \in \text{Der}_K(A, A) = H^1(\text{Hom}^*_\text{ev}(T, A))$ then $H^2(p)(D \cup D) = 0$, as $H^*(p)$ is an algebra homomorphism into a strictly graded commutative algebra. Thus, $D \cup D$ is in the image of the map $\delta : N_{A/P} \to \text{Ext}^2(\text{ev}, A)$, and this image is known to be $T_{A/K}$, the first André–Quillen cohomology of $A$ over $K$. Moreover, the explicit description of the cup product on $\text{Hom}_{A^{\text{ev}}}(T^{(2)}, A)$ shows that $q(D) \in N_{A/P}$ is mapped via $\delta$ to $D \cup D$.

7. Hochschild Cohomology of Homological Complete Intersections

Now we consider the case that $A = P/I$ is a homological complete intersection over $K$ with $I/I^2$ free.

**Proposition 7.1.** (see [7] and [4, 25]) Assume the commutative $K$–algebra $A$ admits a presentation $A = P/I$, with $P = \text{Sym}_K F$ a symmetric $K$–algebra on a free $K$–module $F$, and $I \subseteq P$ an ideal.

If $\text{Tor}^K_+(A, A) = \bigoplus_{i>0} \text{Tor}^K_i(A, A) = 0$, then $T^{(2)}$ as constructed in Corollary 3.5 is already a Tate model of $\mu : A^{\text{ev}} \to A$ if, and only if, $I/I^2$ is a free $A$–module, the natural homomorphism of strictly graded commutative algebras $\Lambda_A I/I^2 \to \text{Tor}^P_+(A, A)$ is an isomorphism, and the surjection $P \otimes_K G \otimes_K P \to I$ was chosen to induce a bijection from a $K$–basis of $G$ to an $A$–basis of $I/I^2$.

**Proof.** In Corollary 3.5 we formed $T^{(1)}$ as the Koszul complex on $(x^a_a - x^c_c)_a$ in $A^{\text{ev}}$, where $x_a$ runs through a $K$–basis of $F$. Its homology can be identified as

$$H(T^{(1)}) \cong \text{Tor}^{P \otimes_K P}_+(P, A \otimes_K A).$$

By [15, Chap. XVI§5(5a)], flatness of $P$ over $K$ together with $\text{Tor}^K_+(A, A) = 0$ yields an isomorphism $\text{Tor}^{P \otimes_K P}_+(P, A \otimes_K A) \cong \text{Tor}^P_+(A, A)$.

If $I/I^2$ is free and the canonical map $\Lambda_A I/I^2 \to \text{Tor}^P_+(A, A) \cong H(T^{(1)})$ is an isomorphism of algebras, then already Tate [37] showed that $T^{(2)} = T$, provided the map $G \to I/I^2$ sends a $K$–basis to an $A$–basis. The main result of [7] yields the converse. □

**Remark 7.2.** This proposition applies in particular when $I = (f_1, ..., f_c) \subseteq P = K[x_1, ..., x_n]$ defines a complete intersection, in the sense that the $f_j$ form a Koszul–regular sequence in $P$. In this case, the condition $\text{Tor}^K_+(A, A) = 0$ is equivalent to exactness of the Koszul complex on $f_1(x'), ..., f_c(x'), f_1(x''), ..., f_c(x'')$ in $P \otimes_K P \cong K[x', x'']$, in that this complex is simply the tensor product of the Koszul complex on $f_1, ..., f_c$ in $P$, a free resolution of $A$ over $K$, with itself over $K$.

The resolution of a complete intersection ring over its enveloping algebra through the Tate model has a long history.
Wolffhardt [39] was the first to exhibit it, but he apparently was unaware of Tate’s result and wrote down the resolution in case \( K \) is a field of characteristic zero — in which case divided powers can be replaced by symmetric ones at the expense of introducing denominators.

In explicit form, this resolution was also described in [10, 12, 22]. As the reviewers of [26] in MathSciNet note: “...the authors recover the calculation of \( HH_*(A/K, A) \) due originally to K. Wolffhardt [Trans. Amer. Math. Soc. 171 (1972), 5166; MR0306192 (46 #5319)], and more recently (since 1988) to a host of other authors, including the reviewers.”

To see that the condition \( \text{Tor}^A_1(A, A) = 0 \) is not void even in this special situation, contrary to what is implied in [19], we know explicitly the form of the cup product on \( \text{Hom}_{A^{ev}}(\mathcal{T}^{(2)}, A) \), thus, on \( \text{Hom}_{A^{ev}}(\mathcal{T}, A) \) in the case at hand.

Set \( F^* = \text{Hom}_{A^{ev}}(F, A) \) for any \( A^{ev} \)-module \( F \). As a complex, \( \text{Hom}_{A^{ev}}^{\bullet}(\mathcal{T}^{(2)}, A) \) is the Koszul complex on \( \left( \sum_{j=1}^c \frac{\partial f_j}{x_i} \partial f_{j^i} \right)_{i=1}^n \) in the polynomial ring \( A[\partial f_1, \ldots, \partial f_c] \).

Here \( \{ \partial f_j \}_{j=1,...,c} \subseteq F_2^* \) form the dual \( A \)-basis to \( \{ df_j \}_{j=1,...,c} \subseteq F_2 \) and we employ the \( A \)-basis \( \partial x_1, \ldots, \partial x_n \in F_1^* \) dual to \( \{ dx_i \}_{i=1,...,n} \subseteq F_1 \). That Koszul complex has then the compact description

\[
\text{Hom}_{A^{ev}}^{\bullet}(\mathcal{T}^{(2)}, A) = \bigwedge^A_A \left( F_1^*[-1] \right) \otimes_A \text{Sym}_A \left( F_2^*[-2] \right)
\]

with differential \( \partial(\partial x_i) = \sum_{j=1}^c \frac{\partial f_j}{x_i} \partial f_{j^i} \) and \( \partial \partial f_j = 0 \). The cohomological grading on \( \text{Hom}_{A^{ev}}^{\bullet}(\mathcal{T}^{(2)}, A) \) is recovered from putting \( \partial x_i \) into degree 1 and \( \partial f_j \) into degree 2 as indicated.

It remains to determine the multiplicative structure.

Identify \( \{ \partial x_{i_1} \wedge \partial x_{i_2} \}_{i_2 < i_1} \subseteq \bigwedge^2_A F_1^* \) with the \( A \)-basis dual to \( \{ dx_{i_2} \wedge dx_{i_1} \}_{i_2 < i_1} \) in \( \bigwedge^2_A F_1 \). The cup product resulting from the diagonal approximation in Theorem 5.8 is then explicitly given by

\[
\partial x_{i_1} \cup \partial x_{i_2} = \begin{cases} 
\partial x_{i_1} \wedge \partial x_{i_2} + \sum_{j=1}^c \frac{\partial f_j}{x_{i_1} x_{i_2}} \partial f_{j^i} & \text{if } i_1 < i_2, \\
\sum_{j=1}^c \frac{\partial f_j}{x_{i_1}} \partial f_{j^i} & \text{if } i_1 = i_2, \\
\partial x_{i_1} \wedge \partial x_{i_2} & \text{if } i_1 > i_2.
\end{cases}
\]

Moreover, the elements \( \partial f_j \) are central, the co-unital diagonal approximation \( \Phi \) being cocommutative in the \( \partial f_j \) as pointed out in Corollary 5.5. These explicit calculations show that \( \text{Hom}_{A^{ev}}^{\bullet}(\mathcal{T}, A) \) with the cup product \( \cup = \cup_\Phi \) from Theorem
5.8 satisfies the relations
\[
\partial_{x_i} \cup \partial_{x_i} = \sum_{j=1}^{c} \frac{\partial^{2j} f_j}{\partial x_i^j} \partial f_j,
\]
\[
\partial_{x_i} \cup \partial_{x_i} + \partial_{x_{i+1}} \cup \partial_{x_{i+1}} = \sum_{j=1}^{c} \frac{\partial^{2j} f_j}{\partial x_{i+1} x_{i+1}} \partial f_j,
\]
\[
\partial_{x_i} \cup \partial_{f_j} = \partial_{f_j} \cup \partial_{x_i},
\]
\[
\partial_{f_{j_1}} \cup \partial_{f_{j_2}} = \partial_{f_{j_2}} \cup \partial_{f_{j_1}}.
\]
for \(i, i_1, i_2 = 1, \ldots, n\) and \(j, j_1, j_2 = 1, \ldots, c\).

Putting it together, this shows \((\text{Hom}_A^\bullet(T, A), \cup) \cong \text{Cliff}(q)\) as graded algebras and the differential on \(\text{Hom}_A^\bullet(T, A)\) readily identifies with \(\partial_{\text{sec}}\) on \(\text{Cliff}(q)\).

This completes the proof of Theorem 2.11. \(\square\)

8. The Case of One Variable

8.1. Here we consider cyclic extensions \(A = K[x]/(f)\), with \(K\) some commutative ring, \(f = a_0 x^d + \cdots + a_d \in P = K[x]\) a polynomial. We set \(I = (f)\) and abbreviate \(\partial_x f = \frac{\partial f}{\partial x}\). Let \(\mathcal{C} = \mathcal{C}_f = (a_0, \ldots, a_d) \subseteq K\) be the content ideal of \(f\). By [27, 5.3 Thm. 7], the polynomial \(f\) is a non-zero-divisor in \(K[x]\) if, and only if, the annihilator of the content ideal in \(K\) is zero, equivalently, \(\text{Hom}_K(K/\mathcal{C}, K) = 0\).

To provide some context, we quote the following results from the literature and refer to the references cited below for unexplained terminology.

**Proposition 8.2.** Let \(f = a_0 x^d + \cdots + a_d \in K[x]\) be a polynomial as above and set \(A = K[x]/(f)\).

1. \(A\) is free as \(K\)-module if, and only if, the ideal \(I\) is generated by a monic polynomial. \(A\) is then necessarily free of finite rank.
2. \(A\) is finitely generated projective as \(K\)-module if, and only if, the ideal \(I\) is generated by a quasi-monic polynomial.
3. \(A\) is projective as \(K\)-module if, and only if, the ideal \(I\) is generated by an almost quasi-monic polynomial.
4. \(A\) is flat as \(K\)-module if, and only if, the content ideal \(\mathcal{C}\) of \(f\) is generated in \(K\) by an idempotent.

In cases (1) and (2), \(f\) is a non-zero-divisor in \(K[x]\), while in cases (3) and (4), the content ideal is generated by an idempotent \(e \in K\), \(e f = (e)\). With \(K_1 = K/(1-e), K_2 = K/(e)\), so that \(K = K_1 \times K_2\), the polynomial \(ef\) is then a non-zero-divisor in \(K_1[x]\), while \((1-e)f = 0\), and \(A \cong K_1[x]/(ef) \times K_2[x]\).

**Proof.** Parts (1), (2) and (3) can be found in [9], where also the terminology “(almost) quasi-monic” is introduced, while part (4) is contained in [28, Thm. 4.3]. \(\square\)

**Examples 8.3.** The polynomial \(f = 15 x^d + 10 x + 6\), for \(d \geq 2\), is quasi-monic over \(K = \mathbb{Z}/30\mathbb{Z}\). The polynomial \(f = 15 x^d + 6\), for \(d \geq 1\), is almost quasi-monic, but not quasi-monic over \(K = \mathbb{Z}/30\mathbb{Z}\). The ring \(A = \mathbb{Z}[x]/(2x - 1) \cong \mathbb{Z}[\frac{1}{2}]\) is flat as module over \(K = \mathbb{Z}\), but not projective.

8.4. In each of the cases considered in Proposition 8.2, we have a decomposition \(K = K_1 \times K_2\) and \(A \cong A_1 \times A_2\), where \(A_1 = K_1[x]/(f)\) with \(\overline{f}\) a non-zero-divisor.
in $K_1[x]$ and $A_2 = K_2[x]$ a polynomial ring. Moreover, as algebras,

$$\text{Ext}^{A_{\text{op}}}_A(A, A) \cong \text{Ext}^{A_{\text{op}}}_{A_1}(A_1, A_1) \times \text{Ext}^{A_{\text{op}}}_{A_2}(A_2, A_2),$$

with the second factor satisfying

$$\text{Ext}^{A_{\text{op}}}_{A_2}(A_2, A_2) \cong K_2[x][0] \oplus K_2[x][\frac{\partial}{\partial x}][1]$$

with obvious multiplicative structure.

We will therefore now concentrate on the case that the given polynomial $f \in K[x]$ is a non-zero-divisor.

**8.5.** If $f$ is a non-zero-divisor, the natural map $A \to I/I^2, 1 \mapsto f$ mod $I$, is an isomorphism, and $0 \to P \xrightarrow{f} P \to A \to 0$ is a free resolution of $A$ both over $P$ and $K$, so that $\text{Tor}^P_i(A, A) = 0$ for $i \geq 2$ and $\bigwedge_A I/I^2 \to \text{Tor}^P_i(A, A)$ is trivially an isomorphism.

In other words, $I = (f) \subseteq P$ is a regular ideal. The only missing piece for $A$ to be a homological complete intersection over $K$ is thus the vanishing of $\text{Tor}^P_i(A, A)$.

That vanishing is equivalent to $f \otimes 1 \in P \otimes_K A$ being a non-zero-divisor, in turn equivalent to $1 \otimes f, f \otimes 1$ forming a regular sequence in $P \otimes_K P$.

In light of [27, 5, Exercise 9], the sequence $1 \otimes f, f \otimes 1$ is regular in $P \otimes_K P$ if, and only if, $\text{Ext}^1_P(K/\epsilon, K) = 0$ for $i = 0, 1$, that is, the depth of the content ideal $\epsilon$ on $K$ is at least 2.

**Example 8.6.** Let $R[s, t, u, v]$ be polynomial ring over a commutative ring $R$, and set $K = R[s, t, u, v]/(sv - tu)$. For $d \geq 1$, the polynomial $f(x) = sx^d + v \in K[x]$ has content ideal $\epsilon_f = (s, v)$ of depth 2, while $g(x) = sx^d + u \in K[x]$ has content ideal $\epsilon_g = (s, u)$ of depth 1.

Both $f, g$ are thus non-zero-divisors in $K[x]$, but only $f$ defines a homological complete intersection $A = K[x]/(f)$ over $K$.

**8.7.** Constructing the beginning of the Tate resolution of the multiplication map $\mu : A^{ev} \cong K[x', x''][/f(x'), f(x'')] \to A, \mu(x') = \mu(x'') = x$, as in Corollary 3.5, one obtains $\mathcal{T}^{(2)}$ as a 2–periodic complex, augmented versus $A$ in degree 0,

$$0 \xrightarrow{\mu} A \xrightarrow{A^{ev} \cdot x'' - x'} A^{ev} \xrightarrow{\Delta} A^{ev} \xrightarrow{x'' - x'} A^{ev} \xrightarrow{\epsilon} A$$

where $\Delta \in A^{ev}$ is the residue class of the unique polynomial $\Delta(x', x'') \in K[x', x'']$ that satisfies $f(x'') - f(x') = (x'' - x')\Delta(x', x'')$.

Summarizing the discussion so far, we have the following characterizations, where we refer to [4, 8] for the terminology and properties of exact zero divisors.

**Theorem 8.8.** If $f = a_0x^d + \cdots + a_d$ is a non-zero-divisor in $K[x]$, then the following are equivalent.

1. $A = K[x]/(f)$ is a homological complete intersection over $K$.
2. $\mathcal{T}^{(2)}$ resolves $A$ as $A^{ev}$–module.
3. $(x'' - x', \Delta)$ is a pair of exact zero-divisors in $A^{ev}$.
4. The content ideal $\epsilon$ of $f$ has grade at least 2, in that $\text{Hom}_K(K/\epsilon, K) = 0 = \text{Ext}^1_K(K/\epsilon, K)$. 

Moreover, $A$ will be flat over $K$ if, and only if the content ideal of $f$ is the unit ideal in $K$.

Proof. The equivalence $(1)\iff (2)$ is a special case of Proposition 7.1, while $(2)\iff (3)$ is a tautology, in that $(x'' - x', \Delta)$ is, by definition, a pair of exact zero-divisors in $A^{ev}$ if, and only if, the ideals $(x'' - x')$ and $(\Delta)$ are each other’s annihilators in $A^{ev}$, if, and only if, the $2$–periodic complex $T^{(2)}$ displayed above is exact. The equivalence $(3)\iff (4)$ has been discussed in 8.5 above.

The final claim follows from Proposition 8.2 in that flatness means that $\tau_f$ is generated by an idempotent. If that idempotent is not $1$, then $f$ is not a non-zero-divisor in $K[x]$. \hfill $\Box$

Assume for the remainder of this section that $f$ and $A$ satisfy the equivalent conditions in Theorem 8.8. We determine the algebra $\text{Ext}_{A^{ev}}^*(A, A)$.

**8.9.** Call $A' = K[x]/(f(x), \partial_x f)$, the ramification algebra of $f$ and introduce as well the annihilator of $\partial_x f$ in $A$, given by

$$\theta = (f : p \partial_x f) / (f) \subseteq A.$$  

Note that $\theta(\partial_x f) = 0$ in $A$, whence $\theta$ is naturally an $A'$–module.

In fact, $\theta \cong \text{Hom}_A(A', A)$, as dualizing the exact sequence

$$0 \to \theta \to A \to \partial_x f \to A' \to 0$$

of $A$–modules shows. The same sequence can be read to identify $\theta \cong \text{Der}_K(A, A)$ and $A' \cong T^1_{A/K}$, the first André–Quillen (tangent) cohomology of $A$ over $K$.

**Theorem 8.10.** Assume $f \in K[x]$ is a polynomial whose content ideal $\mathfrak{c}$ is of depth at least two.

The Yoneda Ext–algebra of $A = K[x]/(f)$ as $A^{ev}$–module is then graded commutative and satisfies

$$\text{Ext}_{A^{ev}}^*(A, A) \cong A \times_{A'} (A' \oplus \theta t)[s] / ((at)(bt) - abf^{(2)} s; a, b \in \theta),$$

where $A, A'$ and $\theta$ are in degree zero, $t$ is of degree $1$, while $s$ is central of degree $2$, and $f^{(2)}$ denotes the residue class of the second divided derivative of $f$ in $A'$.

In particular, the even Yoneda Ext–algebra $\text{Ext}_{A^{ev}}^{2*}(A, A) = \bigoplus_{i \geq 0} \text{Ext}_{A^{ev}}^i(A, A)$ is the fibre product of the ring $A$ over $A'$ with the polynomial ring $A'[s]$ in one variable of degree $2$, 

$$\text{Ext}_{A^{ev}}^{2*}(A, A) \cong A \times_{A'} A'[s].$$

**Proof.** Just note that the algebra displayed is indeed the cohomology algebra of the DG Clifford algebra $A(t, s)$ with relations $t^2 = f^{(2)} s, ts = st$, and differential $\partial(t) = (\partial_x f)s, \partial s = 0$. Now apply Theorem 2.11. \hfill $\Box$

**Remark 8.11.** While we know from graded commutativity of the algebra that the classes $abf^{(2)}$, for $a, b \in \theta$ must be $2$–torsion in $A'$, we can do better, in that these classes are already $2$–torsion in $A$.

To see this, observe that $2f^{(2)} = \partial^2 f / \partial x^2$, the ordinary second derivative of $f$, and use that $a\partial_x f = uf, b\partial_x f = vf$ for suitable polynomials $u, v \in K[x]$, because
\( a, b \in \theta \). One then obtains
\[
abla \frac{\partial^2 f}{\partial x^2} = a (\partial_x (b \partial_x f) - \partial_x b \cdot \partial_x f) \\
= a \partial_x (vf) - a \partial_x b \cdot \partial_x f \\
= a (\partial_x v) f + av \partial_x f - a \partial_x b \cdot \partial_x f \\
= (a \partial_x v + uv - u \partial_x b) f \\
\equiv 0 \mod f.
\]

**Corollary 8.12.** If 2 is a non-zero-divisor in \( A = K[x]/(f) \), for a polynomial in \( K[x] \) with content ideal of depth at least two, then \( \text{Ext}^1_{A^n}(A, A) \) is strictly graded commutative.

**Example 8.13** (Inspired by [31, 19]). The foregoing Corollary fails in more than one variable. Consider \( A = \mathbb{Z}[x, y]/(x^2 - 4xy + y^2 - 1) \) as a \( \mathbb{Z} \)-algebra. As \( f = x^2 - 4xy + y^2 - 1 \) is monic in \( y \) of degree 2, the \( \mathbb{Z}[x] \)-module \( A \) is free of rank 2, in particular, 2 is a non-zero-divisor on \( A \) and \( A \) is projective over \( \mathbb{Z} \).

The Euler identity yields \( x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f + 2 = 2 \in A \), whence the Jacobian ideal of \( A \) over \( \mathbb{Z} \) is \( (2x - 4y, -4x + 2y) = 2A \), and so \( T^1_{A/\mathbb{Z}} \cong A/(2A) \cong \mathbb{Z}_2[x, y]/((x+y+1)^2) \) is a nonzero 2–torsion \( A \)-module.

For \( D = \frac{1}{2} \left( y \frac{\partial f}{\partial y} - x \frac{\partial f}{\partial x} \right) = (2x - y) \frac{\partial f}{\partial x} + (x - 2y) \frac{\partial f}{\partial y} \) one verifies that \( D(f) = 0 \), already in \( \mathbb{Z}[x, y] \). Using Theorem 2.5, one finds
\[
q(D)(f) = \left( 2x - y \right)^2 \frac{\partial f}{\partial x^2} + (2x - y)(x - 2y) \frac{\partial^2 f}{\partial x \partial y} + (x - 2y)^2 \frac{\partial f}{\partial y^2} \\
= (x - 2y)^2 - 4(2x - y)(x - 2y) + (x - 2y)^2 \\
\equiv y^2 + x^2 \equiv 1 \mod (f, 2),
\]
whence \( q(D)(f) \) is not 2–torsion in the normal module \( N_{A^2/\mathbb{Z}} \cong A \).

Further, \( D \cup D \equiv 1 \) (mod 2A) does not vanish in \( T^1_{A/K} \); in fact, the squaring map sends \( \text{Ext}^1_{A^n}(A, A) \cong \text{Der}_2(A, A) \) surjectively onto \( T^1_{A/K} \), and one may verify easily that further \( T^1_{A/K} \cong \text{Ext}^2_{A^n}(A, A) \cong \text{HH}^2(A/K, A) \).

We finish this section by looking at some classical situations.

**The Separable or Unramified Case.** Recall that a polynomial \( f(x) \in K[x] \) is separable over \( K \), if its derivative \( \partial_x f \) is a unit modulo \( f \), equivalently, \( (f, \partial_x f) = K[x] \). In that case, the content ideal \( \mathcal{C}_f \) necessarily equals \( K \), whence \( f \) is a non-zero-divisor, \( A = K[x]/(f) \) is flat over \( K \), and Theorem 8.10 applies. We retrieve that all higher extension groups vanish, \( \text{Ext}^i_{A^n}(A, A) = 0 \), as it should be.

**The Generically Unramified Case.** The polynomial \( f \) is generically unramified, if it is a non-zero-divisor in \( K[x] \) and its derivative \( \partial_x f \) is a non-zero-divisor modulo \( f \). Assuming furthermore that \( A \) is flat over \( K \), equivalently, that the content of \( f \) is the unit ideal, Theorem 8.10 applies and shows that \( f \) is generically unramified if, and only if, \( \text{Ext}^1_{A^n}(A, A) = 0 \), equivalently, all odd extension groups \( \text{Ext}^{2+1}_{A^n}(A, A) \) vanish. In that case, the Yoneda Ext–algebra is thus reduced to its even part, the fibre product of \( A \) over \( A' \) with a polynomial ring \( A'[z] \) generated in cohomological degree 2.
There is no room for a non-zero squaring map and so the Ext–algebra is strictly graded commutative.

**The Totally Ramified Case.** The polynomial \( f \) is totally ramified, if it is a non-zero-divisor in \( K[x] \), but its derivative \( \partial_x f \) is zero modulo \( f \). Equivalently, \( A' = A \).

Assuming that the content of \( f \) is the unit ideal, Theorem 8.10 shows that the Yoneda algebra \( \text{Ext}^*_A(A, A) \) is the Clifford algebra over the \( A \)–quadratic map \( q : A't \rightarrow A's, q(t) = f^{(2)}(x)s. \) If \( f^{(2)}(x) = 0 \) in \( A \), as happens, for example, by Remark 8.11 if \( 2 \) is a non-zero-divisor in \( A \), then the Yoneda algebra is the tensor product of the exterior algebra over \( A \) generated by \( t \) in degree 1 with the symmetric algebra over \( A \) generated by \( s \) in degree 2.

In contrast, if \( f^{(2)}(x) \) is a unit in \( A \), whence \( A \) is necessarily of characteristic 2 by Remark 8.11, then the Yoneda algebra is the tensor algebra over \( A \) on a single generator \( t \) in degree 1. This covers the case \( A = K[x]/(x^2) \) with char \( K = 2 \) mentioned in the Introduction.

**Polynomials over a field.** Last, but not least, let us consider the case when the coefficient ring \( K \) is a field. Then being a non-zero-divisor simply means that \( f \) is not the zero polynomial. Trivially, \( A \) is projective over \( K \), thus the Yoneda Ext–algebra of self-extensions agrees with Hochschild cohomology.

Further, \( K[x] \) being a principal ideal domain, we obtain explicit descriptions of \( A' \) and \( \theta \). Set \( g = \gcd(f, \partial_x f) \) and \( h = f/g \) in \( K[x] \), so that \( A' \cong K[x]/(g) \) and \( \theta = (f(x) : \partial_x f)/(f) \cong (h)/(f) \). Thus, \( \theta \) is the principal ideal generated by \( h \) in \( K[x]/(f) \), isomorphic to \( A' \) as \( A \)–module. Therefore, we obtain the following presentation

\[
\text{HH}^*(A/K) \cong \frac{K[x, y, z]}{(f, gy, gz, y^2 + f^{(2)}h^2z)}
\]

of the Hochschild cohomology ring of \( A = K[x]/(f) \) over \( K \) as a quotient of a graded polynomial ring, where \( x \) is of degree 0, \( y \) is the class of \( h\partial_x \) of degree 1, and \( z \) is of degree 2.

If \( \text{char} \ K \neq 2 \), then necessarily \( f^{(2)}h^2 \equiv 0 \pmod f \) by Remark 8.11 and the description simplifies accordingly.

**Remark 8.14.** This last example encompasses Theorem 3.2, Lemma 4.1, Lemma 5.1, Theorem 5.2 and Theorem 6.2 from [23], completes the characteristic 2 cases that were not handled there, and covers as well Theorem 3.9 from [36].

**9. Cohomology of Finite Abelian Groups**

**9.1.** As our final example, we consider the Hochschild and ordinary cohomology of finite abelian groups. If \( G = \mu_{n_1} \times \cdots \times \mu_{n_r} \) is such a group, with \( \mu_n \) the multiplicatively written cyclic group of order \( n \), we may assume that the orders of the factors satisfy \( 2 \leq n_1 | n_2 | \cdots | n_r \), so that the \( n_i \) are the elementary divisors or invariant factors of the group.

**9.2.** The group algebra on \( G \) over a commutative ring \( K \) is then given by \( A = KG \cong K[x_1,...,x_r]/(x_1^{n_1} - 1,...,x_r^{n_r} - 1). \) It is a homological complete intersection over \( K \), thus, Theorem 2.11 applies. However, because the variables \( x_i \) become units in \( A \), one can simplify that result in this case, and recover as well the fact established in [17] that there is an isomorphism of graded commutative algebras...
\( \text{HH}^*(A/K) \cong A \otimes_K \text{Ext}^*_K(K, K) \), where \( K \) is considered an \( A \)-module via the trivial representation of \( G \).

**Theorem 9.3.** Let \( G = \mu_{n_1} \times \cdots \mu_{n_r} \) be a finite abelian group as above. The Hochschild cohomology of \( KG \) over \( K \) is then the cohomology of the tensor product of differential graded Clifford algebras,

\[
(C(G), \partial) = KG \otimes_K \bigotimes_{j=1}^r \left( \frac{K(\tau_j, \sigma_j)}{[\tau_j^2 - m_j \sigma_j, \tau_j \sigma_j - \sigma_j \tau_j]}, \partial \tau_j = n_j \sigma_j, \partial \sigma_j = 0 \right),
\]

where

\[
m_j = \begin{cases} 
0 & \text{if } n_j \text{ is odd}, \\
n_j/2 & \text{if } n_j \text{ is even},
\end{cases}
\]

and the (cohomological) degree of the \( \tau_j \) is 1, that of the \( \sigma_j \) is 2.

Further, the cohomology of the second factor in the tensor product is the group cohomology \( H^*(G, K) = \text{Ext}^*_K(K, K) \), where \( K \) is the trivial \( G \)-representation, and one retrieves the isomorphism from [17] of graded commutative algebras

\[
\text{HH}^*(KG/K) \cong KG \otimes_K H^*(G, K).
\]

**Proof.** Note that \( A = KG \cong K\mu_{n_1} \otimes_K \cdots \otimes_K K\mu_{n_r} \) and that accordingly the multiplication map \( \mu : A^\text{ev} \to A \) is the tensor product over \( K \) of the multiplication maps \( \mu_j : (K_{\mu_j})^\text{ev} \to K_{\mu_j} \).

It follows that a Tate model for \( A \) over \( K \) is obtainable as tensor product of the corresponding Tate models of the cyclic groups involved. Thus, it remains to discuss the Tate model of a single cyclic group, say \( \mu_n \), for some natural number \( n \). The group algebra \( K\mu_n \cong K[x]/(x^n - 1) \) is a cyclic extension of \( K \), free as \( K \)-module. Therefore, we can apply 8.7 with \( f = x^n - 1 \in K[x] \) to obtain the Tate model \( \mathcal{T}^{(2)} = \mathcal{T} \) as a 2-periodic complex, augmented versus \( K\mu_n \) in degree 0,

\[
\begin{array}{cccccc}
0 & \leftarrow & K\mu_n & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \cdots & \equiv & K\mu_n \\
\mu & & \Delta & & \Delta & & \Delta & & \Delta & & \Delta & & H = \mathcal{T}^{(2)} \\
0 & \leftarrow & K\mu_n^\text{ev} & \leftarrow & K\mu_n^\text{ev} & \leftarrow & K\mu_n^\text{ev} & \leftarrow & K\mu_n^\text{ev} & \cdots & \equiv & \mathcal{T} \end{array}
\]

where \( \Delta \in K\mu_n^\text{ev} \) is the residue class of the unique polynomial

\[
\Delta = \Delta(x', x'') = \frac{(x'')^n - (x')^n}{x'' - x'} = \sum_{i=0}^{n-1} (x'')^i(x')^{n-1-i} \in K[x', x'']
\]

that satisfies \( f(x'') - f(x') = (x'' - x')\Delta(x', x'') \in K[x', x''] \). Written as DG algebra,

\[
\mathcal{T} = K\mu_n^\text{ev}(dx, df) \cong \bigotimes_{K\mu_n^\text{ev}} (dx) \otimes_{K\mu_n^\text{ev}} \Gamma_{K\mu_n^\text{ev}}(df),
\]

with \( dx \) in (homological) degree 1 and \( df \) in degree 2 and differential \( \partial(df) = \Delta(x', x'')dx \) and \( \partial(dx) = x'' - x' \).

Further, the co-unital diagonal approximation \( \Phi : \mathcal{T} \to \mathcal{T} \otimes_{K\mu_n} \mathcal{T} \) from Proposition 5.4 is determined by

\[
\begin{align*}
\Phi_1(dx) & = dx' + dx'' \\
\Phi_2(df) & = dx' + dx'' - \sum_{p+q+r=n-2} (x')^p(dx')(dx'')(x'')^r
\end{align*}
\]
in our earlier notation. Next we adapt this co-unital diagonal approximation using Theorem 5.6. Setting \( x = x' = x'' \), we have

\[
\Phi_2(df) = dx' + dx'' - \left( \frac{n}{2} \right) x^{n-2} dx'dx'' \mod (x'' - x, x - x').
\]

Now the form \( dx'df'' \in \mathcal{T} \otimes_{K\mu_n} \mathcal{T} \) of degree 3 is in the kernel of both augmentation maps \( \epsilon_{1,2} \) and satisfies

\[
\partial(dx'df'') = (x - x')df'' - dx' \Delta(x, x'') dx''
\]

\[
\equiv n x^{n-1} dx'dx'' \mod (x'' - x, x - x').
\]

As \( x \) is a unit in \( K\mu_n \), we can adjust \( \Phi_2 \) by adding \( \partial \eta \), where

\[
\eta = \begin{cases} \frac{n-1}{2} (x')^{-1} dx'df'' & \text{if } n \text{ is odd}, \\ \frac{n}{2} (x')^{-1} dx'df'' & \text{if } n \text{ is even}, \end{cases}
\]

to obtain the adapted co-unital diagonal approximation \( \Psi \) that agrees in degrees 0, 1 with \( \Phi \), but in degree 2 is given by

\[
\Psi(df) = \Phi_2(df) + \partial \eta = dx' + dx'' \mod (x'' - x, x - x')
\]

if \( n \) is odd,

\[
\Psi(df) = \Phi_2(df) + \partial \eta = dx' + dx'' - \frac{n}{2} x^{n-2} dx'dx'' \mod (x'' - x, x - x')
\]

if \( n \) is even. (This adaptation of the multiplicative structure in case of a cyclic group already occurs in [15, Chap. XII §7, p.252].)

At this stage, \( \text{Hom}_{K\mu_n^{ev}}(\mathcal{T}, K\mu_n) \) with the cup product \( \cup_{\Psi} \) and original differential is the DG Clifford algebra

\[
\text{Hom}_{K\mu_n^{ev}}(\mathcal{T}, K\mu_n) \cong \frac{K\mu_n(t, s)}{(t^2 - m x^{n-2}s, st - ts)},
\]

where \( m = 0 \) for \( n \) odd, \( m = n/2 \) for \( n \) even, while the differential satisfies \( \partial(t) = n x^{n-1} s \) and \( \partial s = 0 \).

For the final touch, set \( \tau = xt, \sigma = x^n s \) and note that this coordinate change is indeed invertible, as \( x \) is a unit in \( K\mu_n \). In terms of \( \tau, \sigma \) we obtain

\[
(\text{Hom}_{K\mu_n^{ev}}(\mathcal{T}, K\mu_n), \partial) \cong \left( \frac{K\mu_n(\tau, \sigma)}{(\tau^2 - m x^{n-2} \sigma, \tau \sigma - ts)}, \partial(\tau) = n \sigma, \partial \sigma = 0 \right).
\]

Finally, taking the tensor product over \( K \) of the Tate models for the cyclic groups involved, their adapted diagonal approximations fit together to endow the tensor product with a co-unitial diagonal approximation. Dualizing into \( KG \) over \( KG^{ev} \) results in the DG Clifford algebra \( (C(G), \partial) \) in the statement of the theorem.

To establish the relation to group cohomology, note first that

\[
H(C(G), \partial) \cong KG \otimes_K H \left( \bigotimes_{j=1}^{r} \left( \frac{K}{\langle \tau_j, \sigma_j \rangle}, \partial \tau_j = n_j \sigma_j, \partial \sigma_j = 0 \right) \right)
\]

as \( KG \) is flat over \( K \). Secondly, observe that applying \( K \otimes_{KG} (-) \) to the Tate model \( \mathcal{T} \) of \( \mu : KG^{ev} \to KG \) and then taking cohomology of \( \text{Hom}_{KG}(K \otimes_{KG} \mathcal{T}, K \otimes_{KG} \mathcal{T}) \) yields a homomorphism of graded algebras

\[
\text{Ext}^{\bullet}_{KG^{ev}}(KG, KG) \xrightarrow{K \otimes_{KG} (-)} \text{Ext}^{\bullet}_{KG}(K, K)
\]
that factors through $K \otimes_{KG} \text{Ext}_{KG}^\bullet(KG, KG)$. Thus, one obtains a homomorphism of graded algebras

$$(\varphi) \quad H \left( \bigotimes_{j=1}^r \left( \frac{K(\tau_j, \sigma_j)}{(\tau_j^2 - m_j \sigma_j, \tau_j \sigma_j - \sigma_j \tau_j)} , \partial_j \right) \right) \to \text{Ext}_{KG}^\bullet(K, K),$$

where $\partial_j \tau_j = n_j \sigma_j, \partial_j \sigma_j = 0$.

One may check directly that this is an isomorphism by noting that one may construct a Tate model for the augmentation $KG \to K$ that returns the argument of $H$ in the source of this homomorphism and that this is, at least, a $K$–linear isomorphism, if one initially ignores the multiplicative structure.

Alternatively, one may use [33] to observe that the structure map $K \to KG$ induces an algebra homomorphism

$$\text{Ext}_{KG}(K, K) \to KG \otimes_{KG} \text{Ext}_{KG}(K, K) \cong \text{HH}(KG/K)$$

that provides a splitting to the $K$–algebra homomorphism $(\varphi)$. \qed

**Remark 9.4.** One obstacle to an even more transparent proof that $(\varphi)$ is an isomorphism is that the co-unital diagonal approximation on the Tate model for $\mu : KG^{ev} \to KG$ does not obviously induce such an approximation when applying $K \otimes_{KG} (-)$. This obstacle can be overcome by exhibiting directly such a co-unital diagonal approximation on the Tate model for the augmentation $KG \to K$, as is done in [30], so that it will agree with the one on $K \otimes_{KG} C(G)$ exhibited above.

An explicit co-unital diagonal approximation on the Tate model for the augmentation $KG \to K$ has also been described in [24] and we refer to that reference and [30] for explicit information on the structure of $H(G, K)$, augmenting and reproducing results from [16, 38].

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MULTIPLICATIVE STRUCTURE OF HOCHSCHILD COHOMOLOGY

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