QUASI-STABILITY PROPERTY AND ATTRACTORS FOR A SEMILINEAR TIMOSHENKO SYSTEM

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Abstract. This paper is concerned with the classical Timoshenko system for vibrations of thin rods. It has been studied by many authors and most of known results are concerned with decay rates of the energy, controllability and numerical approximations. There are just a few references on the long-time dynamics of such systems. Motivated by this scenario we establish the existence of global and exponential attractors for a class of semilinear Timoshenko systems with linear frictional damping acting on the whole system and without assuming the well-known equal wave speeds condition.

1. Introduction. The classical Timoshenko system

\begin{align}
\rho_1 \varphi_{tt} - k (\varphi_x + \psi)_x &= 0, \quad x \in (0, L), \quad t > 0, \\
\rho_2 \psi_{tt} - b \psi_{xx} + k (\varphi_x + \psi) &= 0, \quad x \in (0, L), \quad t > 0,
\end{align}

models the transverse vibrations of a thin rod of length $L$ by taking into account shear stress and bending moment components. Here, $\varphi = \varphi(x, t)$ and $\psi = \psi(x, t)$ describe, respectively, the transverse displacement and the rotation angle of the center of mass of a beam element. The constants $\rho_1, \rho_2, k$ and $b$ are taken positive and stand for, respectively, mass density, moment of mass inertia, shear coefficient and flexural rigidity. The original derivation of the model was presented in Timoshenko [32]. See also Timoshenko [33, Sec. 55].

Timoshenko systems are nowadays a major research subject in second order evolution problems. With respect to their asymptotic behavior the nature of the damping term is very important since problem (1)-(2) is conservative. In this direction,

2010 Mathematics Subject Classification. Primary: 35B40, 35B41, 35L53; Secondary: 74K10, 93D20.

Key words and phrases. Timoshenko system, global attractor, exponential attractor, long-time dynamics, quasi-stability.

The first author was supported by Fundação Araucária grant 308/2012,
The second author was supported by CNPq grant 441414/2014-1,
The third author was supported by CAPES grant 20132268.

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it seems there is a kind of dichotomy based on a “equal speeds” assumption
\[
\frac{k}{\rho_1} = \frac{b}{\rho_2},
\]
proposed by Soufyane [29]. The main result in [29] asserts that the Timoshenko system with a damping \(\psi_t\) acting only in the second equation (2) is exponentially stable if and only if (3) holds. This means that condition (3) is sufficient to stabilize the system exponentially with only one damping term, yet in the equation for rotation angle.

By taking condition (3) as a starting assumption several generalizations and extensions were established, including elastic, viscoelastic and thermoelastic models, see e.g. [1, 2, 4, 5, 10, 14, 15, 17, 21, 23, 30] and the references therein. In addition, recently, Almeida Júnior et al. [3] proved a complementary result which asserts that the Timoshenko system under assumption (3) is also exponentially stable with a damping \(\varphi_t\) acting only in the equation for the displacement (1).

The stability of Timoshenko systems can also be studied without assuming condition (3). In this case, as a consequence of [29] and [3], the exponential stability is only achieved by adding damping term in both equations (1)-(2). Early works include [12, 20, 27]. More recently Cavalcanti et al. [7] proved that exponential stability of system (1)-(2), without assumption (3), can be obtained adding two locally distributed nonlinear damping \(\alpha_1(x)g_1(\varphi_t)\) and \(\alpha_2(x)g_2(\psi_t)\) on the equations (1) and (2), respectively.

Now, with respect to the dynamical systems generated by Timoshenko systems there are just a few published works. The only one we found is due to Grasselli, Pata and Prouse [16]. It establishes the existence of a uniform attractor for a non-autonomous viscoelastic Timoshenko system where the dissipation is given by two memory terms acting on both equations, for displacement and rotation angle.

The purpose of the present work is to complement and extend some early works on Timoshenko systems by establishing new results on the existence of attractors and their properties. We shall consider the following damped semilinear Timoshenko system
\[
\begin{align*}
\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + f_1(\varphi, \psi) + \varphi_t &= h_1 & &\text{in } (0, L) \times (0, \infty), \\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + f_2(\varphi, \psi) + \psi_t &= h_2 & &\text{in } (0, L) \times (0, \infty),
\end{align*}
\]
where \(f_1, f_2\) are nonlinear source terms representing the elastic foundation and \(h_1, h_2\) are external forces. The hypotheses on these functions will be given later. To this system we consider initial conditions
\[
\begin{align*}
\varphi(\cdot, 0) &= \varphi_0(\cdot), & \varphi_1(\cdot, 0) &= \varphi_1(\cdot), & \psi(\cdot, 0) &= \psi_0(\cdot), & \psi_1(\cdot, 0) &= \psi_1(\cdot) & &\text{in } (0, L), \\
\end{align*}
\]
and Dirichlet boundary condition
\[
\begin{align*}
\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = 0, & & t \geq 0.
\end{align*}
\]
We observe that our problem has damping terms in both equations (4) and (5). Therefore we shall not assume the equal wave speeds assumption (3). As a matter of fact, there is a criticism on this assumption since it never occurs physically. See for instance [24, 7].

The main features of our work are summarized as follows. (a) It is the first work addressing the existence of a finite-dimensional attractor with optimal regularity
and fractal exponential attractor for these kind of systems. This is achieved by using the recent quasi-stability theory of Chueshov and Lasiecka [8, 9]. (b) It is also the first work that explores the gradient structure of Timoshenko systems in order to establish geometrical properties of their attractors. Our main result is Theorem 3.1.

The rest of the paper is organized as follows. In Section 2 we present our assumptions and state the results on existence and global well-posedness to the system (4)-(7). In Section 3 we consider the corresponding dynamical system and state our results on existence and global well-posedness to the system (4)-(7). In Section 4 we consider the Hilbert space $H^1_0(0, L) \times L^2(0, L) \times H^1_0(0, L) \times L^2(0, L)$, equipped with the following inner product and norm (equivalent to the usual one)

\begin{equation}
\langle U, \tilde{U} \rangle_{\mathcal{H}} = \rho_1 \langle \Phi, \tilde{\Phi} \rangle + \rho_2 \langle \Psi, \tilde{\Psi} \rangle + b(\psi_x, \tilde{\psi}_x) + k(\varphi_x + \psi, \tilde{\varphi}_x + \tilde{\psi})
\end{equation}

\begin{equation}
\| U \|_{\mathcal{H}} = \rho_1 \| \Phi \|_2^2 + \rho_2 \| \Psi \|_2^2 + b \| \psi_x \|_2^2 + k \| \varphi_x + \psi \|_2^2,
\end{equation}

for any $U = (\varphi, \Phi, \psi, \Psi), \tilde{U} = (\varphi, \tilde{\Phi}, \psi, \tilde{\Psi}) \in \mathcal{H}$, where $(\cdot, \cdot)$ and $\| \cdot \|_2$ stand for usual inner product and norm in $L^2(0, L)$, respectively. More generally, throughout this paper the notation $\| \cdot \|_p$ will stand for usual norm in $L^p(0, L)$, $1 \leq p \leq \infty$. Here, we use the simplified Poincaré inequality

\[ \| u \|_2^2 \leq L^2 \| u_x \|_2^2, \quad \forall u \in H^1_0(0, L). \]

Besides, it is easy to check that

\[ D(A) = (H^2(0, L) \cap H^1_0(0, L)) \times H^1_0(0, L) \times (H^2(0, L) \cap H^1_0(0, L)) \times H^1_0(0, L). \]

Therefore the well-posedness of (4)-(7) is given through equivalent problem (8) with respect to mild and strong solutions. Let us now recall these concepts:
A function \( U \in C([0,T), \mathcal{H}) \), \( T > 0 \), satisfying the following integral equation

\[
U(t) = e^{At}U_0 + \int_0^t e^{A(t-s)}F(U(s)) \, ds, \quad t \in [0,T),
\]

is called a mild solution of the initial value problem (8) on \([0,T)\).

A function \( U : [0,T) \rightarrow \mathcal{H} \) is called strong solution of (8) on \([0,T), \, T > 0\), if \( U \) is continuous on \([0,T), \) continuously differentiable on \((0,T), \) with \( U(t) \in D(A) \) for \( t \in (0,T), \) and (8) is satisfied on \([0,T)\) almost everywhere.

### 2.2. Well-posedness result

In order to consider local existence and uniqueness of mild and strong solutions to Cauchy problem (8) we initially consider the following assumptions on \( f_i, \) and \( h_i, \) for each \( i = 1,2. \)

(A1) \( h_i \in L^2(0,L). \)

(A2) \( f_i : \mathbb{R}^2 \rightarrow \mathbb{R} \) is locally Lipschitz continuous on each of its arguments, namely, there exist a constant \( \gamma_i \geq 1 \) and a continuous function \( \sigma_i : \mathbb{R} \rightarrow \mathbb{R}^+ \) such that

\[
\begin{align*}
|f_i(s_1, r) - f_i(s_2, r)| &\leq \sigma_i(|r|)(1 + |s_1|^{\gamma_i} + |s_2|^{\gamma_i})|s_1 - s_2|, \\
|f_i(s, r_1) - f_i(s, r_2)| &\leq \sigma_i(|s|)(1 + |r_1|^{\gamma_i} + |r_2|^{\gamma_i})|r_1 - r_2|,
\end{align*}
\]

for every \( (s, r), (s, r_j) \in \mathbb{R}^2, \) \( j = 1,2. \)

**Theorem 2.1** (Local Existence). Under assumptions (A1)-(A2) we have:

(i) If \( U_0 \in \mathcal{H}, \) then there exists \( T_{\text{max}} > 0 \) such that (8) has a unique mild solution \( U \in C([0,T_{\text{max}}), \mathcal{H}). \)

(ii) If \( U_0 \in D(A), \) then the corresponding mild solution \( U(t) \) is strong one.

(iii) If \( U(t) \) and \( V(t) \) are two mild or strong solutions corresponding to initial data \( U_0 \) and \( V_0, \) respectively, then

\[
\|U(t) - V(t)\|_\mathcal{H} \leq e^{Ct}\|U_0 - V_0\|_\mathcal{H}, \quad \forall \ t \in [0,T_{\text{max}}),
\]

for some constant \( C = C(||U_0||_\mathcal{H}, ||V_0||_\mathcal{H}) > 0. \)

**Remark 1.** The proof of Theorem 2.1 is made in Section 4. Now, in order to show that both mild and strong solutions are globally defined (i.e. \( T_{\text{max}} = +\infty \)) we additionally consider dissipative hypotheses on the source as follows.

(A3) There is a function \( F : \mathbb{R}^2 \rightarrow \mathbb{R} \) satisfying

\[
\frac{\partial F}{\partial s}(s, \cdot) = f_1(s, \cdot) \quad \text{and} \quad \frac{\partial F}{\partial r}(\cdot, r) = f_2(\cdot, r),
\]

and

\[
F(s, r) \geq -\theta_2 - \mu_1|r|^2 - \theta_1|s|^2, \quad \forall \ (s, r) \in \mathbb{R}^2,
\]

\[
F(s, r) \leq f_1(s, r)s + f_2(s, r)r + \theta_1|s|^2 + \mu_1|r|^2 + \theta_2, \quad \forall \ (s, r) \in \mathbb{R}^2,
\]

for some constants

\[
0 \leq \theta_1 \leq \min \left\{ \frac{k}{8L^2}, \frac{b}{16L^2} \right\}, \quad 0 \leq \mu_1 \leq \frac{b}{8L^2}, \quad \text{and} \quad \theta_2, \mu_2 \geq 0.
\]

**Theorem 2.2** (Global Well-Posedness). Under assumptions (A1)-(A3), then problem (8), and consequently system (4)-(7), is global well posed with respect to strong and mild solutions.

**Remark 2.** The proof of Theorem 2.2 follows as an immediate consequence of Theorem 2.1 and Lemma 4.3 also proved afterward in Section 4.
Remark 3. Given $U_0 \in \mathcal{H}$ and the (corresponding) unique mild solution $U$ of (8) lying in $C([0, \infty); \mathcal{H})$, then using standard density arguments it is possible to find a sequence of regular (strong) solutions $U^n$ such that

$$U^n \to U \quad \text{in} \quad C([0, T], \mathcal{H}), \quad \forall \, T > 0.$$ 

Therefore, the above regularity is sufficient to justify any procedure on the calculations performed in this paper.

3. Long-time dynamics.

3.1. Generation of a dynamical system. From Theorem 2.2 we can define the dynamical system $(\mathcal{H}, S(t))$, where $\mathcal{H}$ is introduced in (11), and the evolution operator $S(t) : \mathcal{H} \to \mathcal{H}$ is given by relation

$$S(t)U_0 = U(t), \quad U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1) \in \mathcal{H},$$

where $U(t) = (\varphi(t), \varphi_t(t), \psi(t), \psi_t(t)), \ t \geq 0$, is the unique mild solution of (8).

Essential concepts within the theory of infinite-dimensional dynamical systems can be found e.g. in [6, 8, 9, 11, 18, 31]. For the sake of the reader we present some of these preliminary concepts applied to the dynamical system $(\mathcal{H}, S(t))$ given in (20), see e.g. Chueshov and Lasiecka [8, 9]. More precisely, all properties stated in the next main result (see Theorem 3.1) follow the theoretical-lines introduced below.

- The dynamical system $(\mathcal{H}, S(t))$ given in (20) is called quasi-stable on a set $B \subset \mathcal{H}$ (in accordance with [9, Definition 7.9.2]) if there exist a compact seminorm $n_X(\cdot, \cdot)$ on $X := H^1_0(\Omega) \times H^1_0(\Omega)$ and nonnegative scalar functions $a(t)$ and $c(t)$ locally bounded in $[0, \infty)$, and $b(t) \in L^1(\mathbb{R}^+) \setminus L^1(\mathbb{R}^+)$ with $\lim_{t \to \infty} b(t) = 0$, such that

$$\|S(t)U_1 - S(t)U_2\|_H^2 \leq a(t)\|U_1 - U_2\|_H^2, \quad (21)$$

and

$$\|S(t)U_1 - S(t)U_2\|_H^2 \leq b(t)\|U_1 - U_2\|_H^2 + c(t) \sup_{s \in [0, t]} [n_X(u(s), v(s))]^2, \quad (22)$$

for any $U_1, U_2 \in B$, where we use the notation $S(t)U_i = (\varphi^i(t), \varphi^*_i(t), \psi^i(t), \psi^*_i(t)), \ i = 1, 2, \ and \ (u, v) = (\varphi^1 - \varphi^2, \psi^1 - \psi^2)$.

- A global attractor for $(\mathcal{H}, S(t))$ is a bounded closed set $\mathfrak{A} \subset \mathcal{H}$ which is fully invariant and uniformly attracting, namely, $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$, and

$$\lim_{t \to +\infty} \text{dist}_H(S(t)B, \mathfrak{A}) = 0, \quad \text{for every bounded subset} \ B \subset \mathcal{H}.$$

- A global minimal attractor for $(\mathcal{H}, S(t))$ is a bounded closed set $\mathfrak{A}_{\text{min}} \subset \mathcal{H}$ which is positively invariant $(S(t)\mathfrak{A}_{\text{min}} \subset \mathfrak{A}_{\text{min}})$ and attracts uniformly every point, that is,

$$\lim_{t \to +\infty} \text{dist}_H(S(t)U_0, \mathfrak{A}_{\text{min}}) = 0, \quad \text{for any} \ U_0 \in \mathcal{H},$$

and $\mathfrak{A}_{\text{min}}$ has no proper subsets possessing these two properties.

- The unstable manifold emanating from a set $\mathcal{N}$, denoted by $\mathcal{M}_+(\mathcal{N})$, is a set of $\mathcal{H}$ such that for each $U_0 \in \mathcal{M}_+(\mathcal{N})$ there exists a full trajectory $\Gamma = \{U(t) \mid t \in \mathbb{R}\}$ satisfying

$$U(0) = U_0 \quad \text{and} \quad \lim_{t \to -\infty} \text{dist}_H(U(t), \mathcal{N}) = 0.$$
• The fractal dimension of a compact set $\mathcal{A} \subset \mathcal{H}$ is defined by

$$\dim_{\mathcal{H}} \mathcal{A} = \limsup_{\varepsilon \to 0} \frac{\ln n(\mathcal{A}, \varepsilon)}{\ln(1/\varepsilon)},$$

where $n(\mathcal{A}, \varepsilon)$ is the minimal number of closed balls in $\mathcal{H}$ of radius $\varepsilon$ which covers $\mathcal{A}$. Since the Hausdorff dimension does not exceed the fractal one (see e.g. [18, Chapter 2]) it is enough to prove finiteness of the fractal dimension.

• A compact set $\mathcal{A}_{\text{exp}} \subset \mathcal{H}$ is said to be a fractal exponential attractor of the dynamical system $(\mathcal{H}, S(t))$ if $\mathcal{A}_{\text{exp}}$ is a positively invariant set of finite fractal dimension in $\mathcal{H}$ and for every bounded set $B \subset \mathcal{H}$ there exist positive constants $t_B, C_B$ and $\sigma_B$ such that

$$\sup_{t \in t_B} \text{dist}_{\mathcal{H}}(S(t)U_0, \mathcal{A}_{\text{exp}}) \leq C_B e^{-\sigma_B(t-t_B)}, \quad t \geq t_B.$$ 

If there exists an exponential attractor only having finite dimension in some extended space $\mathcal{H} \supset \mathcal{H}$, then this exponentially attracting set is called generalized fractal exponential attractor.

Remark 4. In order to construct a compact seminorm $n_X(\cdot, \cdot)$ on $X := H^1_0(\Omega) \times H^1_0(\Omega)$, we need to rewrite the phase space $\mathcal{H}$ given (11) as $\mathcal{H} := X \times Y$ with compact embedding $X \hookrightarrow Y$. By isomorphism, we consider from now on that $\mathcal{H}$ can be expressed as

$$\mathcal{H} \cong (H^1_0(\Omega) \times H^1_0(\Omega)) \times (L^2(\Omega) \times L^2(\Omega)),$$

whose solution trajectories $(\varphi(t), \psi(t), \varphi_t(t), \psi_t(t))$ are taken with inner product and norm (12). Such isomorphic phase space shall allow us to achieve the quasi-stability property according to Definition 7.9.2 in [9].

3.2. Main result. Our main result is concerned with long-time behavior for the dynamical system $(\mathcal{H}, S(t))$ defined in (20). It reads as follows.

Theorem 3.1 (Main Result). Under assumptions (A1)-(A3) we have:

(i) The dynamical system $(\mathcal{H}, S(t))$ is quasi-stable on any bounded positively invariant set $B \subset \mathcal{H}$.

(ii) The dynamical system $(\mathcal{H}, S(t))$ possesses a unique compact global attractor $\mathcal{A} \subset \mathcal{H}$, which is characterized by the unstable manifold $\mathcal{A} = \mathcal{M}^u(N)$, emanating from the set $N = \{U = (\varphi, 0, \psi, 0) \in \mathcal{H} ; \quad AU + F(U) = 0\}$ of stationary solutions.

(iii) Every trajectory stabilizes to the set $N$, namely, for any $U \in \mathcal{H}$ one has

$$\lim_{t \to +\infty} \text{dist}_{\mathcal{H}}(S(t)U, N) = 0.$$ 

In particular, there exists a global minimal attractor $\mathcal{A}_{\text{min}}$ given by $\mathcal{A}_{\text{min}} = N$.

(iv) The attractor $\mathcal{A}$ has finite fractal and Hausdorff dimension $\dim_{\mathcal{H}} \mathcal{A}$.

(v) Every trajectory $\Gamma = \{ (\varphi(t), \varphi_t(t), \psi(t), \psi_t(t)) ; \ t \in \mathbb{R}\}$ from the attractor $\mathcal{A}$ has the smoothness property

$$(\varphi_t, \varphi_{tt}, \psi_t, \psi_{tt}) \in L^\infty(\mathbb{R}; \mathcal{H}).$$

Moreover, there exists a constant $R > 0$ such that

$$\sup_{t \in \mathbb{R}} \sup_{\Gamma \subset \mathcal{A}} \left( \|\varphi_t(t)\|^2_2 + \|\varphi_{tt}(t)\|^2_2 + \|\psi(t)\|^2_2 + \|\psi_{tt}(t)\|^2_2 \right) \leq R^2. \quad (25)$$
Lemma 4.2. Under assumptions of Theorem 3.1, let us consider $H$ is a locally Lipschitz continuous operator.

Proof. Now let us estimate the right side terms in (26). Firstly we note that $\sigma$ is dissipative in $H_{-1}$, which is seen as isomorphic to $L^2(0, L) \times L^2(0, L) \times H^{-1}(0, L)$. In addition, from interpolation theorem, there exists a generalized exponential attractor whose fractal dimension is finite in a smaller extended space $H_{-\delta}$, where

$$H \subset H_{-\delta} \subseteq H_{-1}, \quad 0 < \delta \leq 1.$$ 

Remark 5. The proof is made in second part of Section 4. Actually, we first prove several lemmas which provide sufficient tools to conclude all items of Theorem 3.1 as an application of abstract results from the general theory in dynamical systems.

4. Proofs.

4.1. Proof of local existence. The proof of Theorem 2.1 will follow as consequence of the next two lemmas.

Lemma 4.1. Under assumptions of Theorem 2.1, then $A : D(A) \subset H \rightarrow H$ defined in (9) is the infinitesimal generator of a $C_0$-semigroup of contractions $T(t) = e^{At}$ on $H$.

Proof. Following e.g. [22, 27, 28] it is not so difficult to prove that $R(I - A) = H$, where $R(I - A)$ stands for range of the operator $I - A$, and that $A$ is dissipative in $H$, namely, for all $U = (\varphi, \Phi, \psi, \Psi) \in D(A)$,

$$\text{Re}(AU, U)_H = -(\|\Phi\|^2_2 + \|\Psi\|^2_2) \leq 0.$$ 

Therefore, from Lumer-Phillips Theorem (see e.g. [25, Chapter 1]), $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions $T(t) = e^{At}$ on $H$. \hfill $\square$

Lemma 4.2. Under assumptions of Theorem 2.1, then $F : H \rightarrow H$ defined in (10) is a locally Lipschitz continuous operator.

Proof. Let us consider $R > 0$ and initial data $U = (\varphi, \Phi, \psi, \Psi), \tilde{U} = (\tilde{\varphi}, \tilde{\Phi}, \tilde{\psi}, \tilde{\Psi}) \in H$ such that $\|U\|_H, \|\tilde{U}\|_H \leq R$. From definition (10) and norm $H$ we have

$$\|F(U) - F(\tilde{U})\|_H^2 \leq \frac{2}{\rho_1} \int_0^L |f_1(\varphi, \psi) - f_1(\tilde{\varphi}, \tilde{\psi})|^2 dx + \frac{2}{\rho_2} \int_0^L |f_2(\varphi, \psi) - f_2(\tilde{\varphi}, \tilde{\psi})|^2 dx.$$ 

(26)

Now let us estimate the right side terms in (26). Firstly we note that

$$\Delta f_i := f_i(\varphi, \psi) - f_i(\tilde{\varphi}, \tilde{\psi}) = [f_i(\varphi, \psi) - f_i(\varphi, \tilde{\psi})] + [f_i(\varphi, \tilde{\psi}) - f_i(\tilde{\varphi}, \tilde{\psi})], \quad i = 1, 2.$$ 

Also, since $\sigma_1$ is continuous and $H^1_0(0, L) \hookrightarrow L^\infty(0, L)$, there exists a constant $\tilde{c}_\infty > 0$ such that $\|\varphi\|_\infty, \|\tilde{\varphi}\|_\infty, \|\psi\|_\infty, \|\psi\|_\infty \leq \tilde{c}_\infty R$ and so

$$\sigma_1^2(|\varphi|), \sigma_1^2(|\tilde{\varphi}|) \leq \max_{\tau \in [0, \tilde{c}_\infty R]} \sigma_1^2(|\tau|) \leq K_1^1,$$ 

where

$$(vi) \quad \text{The dynamical system } (H, S(t)) \text{ possesses a generalized fractal exponential attractor } \mathfrak{A}_{ex} \text{ with finite dimension in the extended space }$$

$$H_{-1} := L^2(0, L) \times H^{-1}(0, L),$$

which is seen as isomorphic to $L^2(0, L) \times L^2(0, L) \times H^{-1}(0, L)$. In addition, from interpolation theorem, there exists a generalized exponential attractor whose fractal dimension is finite in a smaller extended space $H_{-\delta}$, where

$$H \subset H_{-\delta} \subseteq H_{-1}, \quad 0 < \delta \leq 1.$$
for some constant $K^4_R > 0$. From assumptions (14)-(15), we have

\[
\frac{2}{\rho_1} \int_0^L |\Delta f_1|^2 \, dx \leq \frac{4}{\rho_1} \int_0^L \sigma_1^2(|\varphi|) \left( 1 + |\psi|^{\gamma_1} + |\tilde{\psi}|^{\gamma_1} \right)^2 |\psi - \tilde{\psi}|^2 \, dx \\
+ \frac{4}{\rho_1} \int_0^L \sigma_1^2(|\tilde{\varphi}|) \left( 1 + |\varphi|^{\gamma_1} + |\tilde{\varphi}|^{\gamma_1} \right)^2 |\varphi - \tilde{\varphi}|^2 \, dx \\
\leq \frac{16K^4_R}{\rho_1} \int_0^L \left( 1 + |\psi|^{2\gamma_1} + |\tilde{\psi}|^{2\gamma_1} \right) |\psi - \tilde{\psi}|^2 \, dx \\
+ \frac{16K^4_R}{\rho_1} \int_0^L \left( 1 + |\varphi|^{2\gamma_1} + |\tilde{\varphi}|^{2\gamma_1} \right) |\varphi - \tilde{\varphi}|^2 \, dx \\
\leq K^2_R \left( \|\psi - \tilde{\psi}\|_2^2 + \|\varphi - \tilde{\varphi}\|_2^2 \right),
\]

for some constant $K^2_R > 0$. We also note that

\[
\|u_x\|_2^2 = \|u_x + v - v\|_2^2 \\
\leq 2 \left( \|u_x + v\|_2^2 + L^2 \|v_x\|_2^2 \right) \\
\leq 2 \max \left\{ \frac{1}{k}, \frac{L^2}{b} \right\} \left( k \|u_x + v\|_2^2 + b \|v_x\|_2^2 \right), \quad \forall u \in H_0^1(0, L).
\]

Thus, there exists a constant $K^3_R > 0$ depending on $R > 0$ such that

\[
\frac{2}{\rho_1} \int_0^L |\Delta f_1|^2 \, dx \leq K^3_R \left( k \|(\varphi - \tilde{\varphi})_x + (\psi - \tilde{\psi})\|_2^2 + b \|(\psi - \tilde{\psi})_x\|_2^2 \right). \quad (27)
\]

Analogously, there exists a constant $K^4_R > 0$ such that

\[
\frac{2}{\rho_2} \int_0^L |\Delta f_2|^2 \, dx \leq K^4_R \left( k \|(\varphi - \tilde{\varphi})_x + (\psi - \tilde{\psi})\|_2^2 + b \|(\psi - \tilde{\psi})_x\|_2^2 \right). \quad (28)
\]

Finally, inserting (27)-(28) in (26), we conclude

\[
\|\mathcal{F}(U) - \mathcal{F}(\tilde{U})\|_H^2 \leq K_R \|U - \tilde{U}\|_H^2,
\]

for some constant $K_R > 0$ depending on $R > 0$. \hfill \Box

4.1.1. Proof of Theorem 2.1: (i) and (ii). It is immediate consequence of Lemmas 4.1 and 4.2 and [25, Thms 1.4 and 1.6]. \hfill \Box

4.1.2. Proof of Theorem 2.1: (iii). Let us consider two mild or strong solutions $U(t)$ and $V(t)$ on the interval $[0, T_{\text{max}})$ corresponding to initial data $U_0$ and $V_0$, respectively, and set $W = U - V$, $W_0 = U_0 - V_0$. From (13) we have

\[
W(t) = e^{At}W_0 + \int_0^t e^{A(t-s)} \left[ \mathcal{F}(U(s)) - \mathcal{F}(V(s)) \right] \, ds, \quad t \in [0, T_{\text{max}}).
\]

Also, from Lemma 4.2 there exists a constant $C = C(\|U_0\|_H, \|V_0\|_H) > 0$ such that

\[
\|\mathcal{F}(U(t)) - \mathcal{F}(V(t))\|_H \leq C\|W(t)\|_H, \quad t \in [0, T_{\text{max}}),
\]
from where it follows that
\[ \| W(t) \|_{\mathcal{H}} \leq \| W_0 \|_{\mathcal{H}} + C \int_0^t \| W(s) \|_{\mathcal{H}} \, ds, \quad t \in [0, T_{\text{max}}). \]

Therefore, (16) is obtained after applying Gronwall inequality. This completes the proof of Theorem 2.1.

\[ \square \]

4.2. **Proof of global well-posedness.** The proof of Theorem 2.2 is obtained from Theorem 2.1 and Lemma 4.3 below.

Let us first consider the associated energy \( E(t) := E(\varphi(t), \varphi_t(t), \psi(t), \psi_t(t)), \ t \geq 0, \) to problem (4)-(7), namely,
\[
E(t) = \frac{\rho_1}{2} \| \varphi_t(t) \|_2^2 + \frac{\rho_2}{2} \| \psi_t(t) \|_2^2 + \frac{K}{2} \| \varphi_x(t) + \psi(t) \|_2^2 + \frac{b}{2} \| \psi_x(t) \|_2^2 
\]
\[ + \left( \int_0^L F(\varphi(t), \psi(t)) \, dx - \int_0^L h_1 \varphi(t) \, dx - \int_0^L h_2 \psi(t) \, dx. \right) \tag{29} \]

In what follows, whenever convenient, we omit the parameter \( t > 0 \) inside norms and integrals just to simplify the notation.

**Lemma 4.3.** Let \( U = (\varphi, \varphi_t, \psi, \psi_t) \) be a strong or mild solution of (8). Then
\[
\frac{d}{dt} E(t) = - \int_0^L \left[ |\varphi_t|^2 + |\psi_t|^2 \right] \, dx, \quad t > 0. \tag{30} \]

Also, there exists a constant \( K = K(\| h_1 \|_2, \| h_2 \|_2) > 0 \) such that
\[
E(t) \geq \frac{1}{4} \| U(t) \|_{\mathcal{H}}^2 - K, \quad t \geq 0. \tag{31} \]

In addition, \( U(t) \) is globally bounded in \( \mathcal{H} \) and thus \( T_{\text{max}} = +\infty \).

**Proof.** The proof is first made for strong solutions. Then, it holds for mild solutions by taking standard density arguments. Equality (30) follows by taking the multipliers \( \varphi_t \) in (4) and \( \psi_t \) in (5), and adding the resulting expression. Now we define
\[
\tilde{E}(t) = E(t) + K, \tag{32} \]
where
\[
K := (\theta_2 + \mu_2) L + \left( \frac{2 L^2}{k} + \frac{4 L^4}{b} \right) \| h_1 \|_2^2 + \frac{4 L^2}{b} \| h_2 \|_2^2 > 0. \]

From assumption (17) one has
\[
\int_0^L F(\varphi, \psi) \, dx \geq -\theta_2 L - \mu_1 \| \psi_x \|_2^2 - \theta_1 \| \varphi \|_2^2 
\geq -\theta_2 L - \mu_1 L^2 \| \psi_x \|_2^2 - \theta_1 L^2 \| \varphi_x \|_2^2 
\geq -\theta_2 L - \left( \mu_1 L^2 + 2 \theta_1 L^4 \right) \| \psi_x \|_2^2 - 2 \theta_1 L^2 \| \varphi_x \|_2^2. \tag{33} \]

Besides, from Cauchy-Schwarz and Poincaré inequalities we get
\[
\int_0^L h_1 \varphi \, dx \leq L \| h_1 \|_2 \| \varphi_x + \psi \|_2 
\leq L \| h_1 \|_2 \| \varphi_x + \psi \|_2 + L^2 \| h_1 \|_2 \| \psi_x \|_2 
\leq \frac{2 L^2}{k} \| h_1 \|_2^2 + \frac{k}{8} \| \varphi_x + \psi \|_2^2 + \frac{4 L^2}{b} \| h_1 \|_2^2 + \frac{b}{16} \| \psi_x \|_2^2, \tag{34} \]
\]
and
\[ \int_0^L h_2 \psi \, dx \leq L \|h_2\|_2 \|\psi\|_2 \leq \frac{4L^2}{b} \|h_2\|_2^2 + \frac{b}{16} \|\psi\|_2^2. \] (35)

Replacing (33)-(35) in (32) we deduce
\[ \bar{E}(t) \geq \frac{1}{4} \|(\varphi(t), \varphi(t), \psi(t), \psi(t))\|_H^2 = \frac{1}{4} \|U(t)\|_H^2, \quad t \geq 0, \]
from where it follows inequality (31). Finally, we note that (30)-(31) are enough to conclude that solutions are globally bounded and from [25, Thm 1.4] one gets \( T_{\text{max}} = +\infty \). This completes the proof of Lemma 4.3. \( \square \)

4.3. Proof of the main result. The proof of Theorem 3.1 will be given through next three lemmas.

We first recall that dynamical system \((\mathcal{H}, S(t))\) given in (20) is said to be gradient if there exists a strict Lyapunov functional on \( \mathcal{H} \), namely, there exists a continuous functional \( \Phi \) such that \( t \mapsto \Phi(S(t)U_0) \) is non-increasing for any \( U_0 \in \mathcal{H} \), and the condition \( \Phi(S(t)U_0) = \Phi(U_0) \) for all \( t > 0 \) and some \( U_0 \in \mathcal{H} \) implies that \( S(t)U_0 = U_0 \) for all \( t > 0 \).

**Lemma 4.4.** There exists a strict Lyapunov functional \( \Phi \) for the dynamical system \((\mathcal{H}, S(t))\) defined in (20). In other words, \((\mathcal{H}, S(t))\) is gradient. Besides,
- the Lyapunov functional \( \Phi \) is bounded from above on any bounded subset of \( \mathcal{H} \);
- the set \( \Phi_R = \{ U_0 \in \mathcal{H} ; \Phi(U_0) \leq R \} \) is bounded in \( \mathcal{H} \) for every \( R > 0 \).

**Proof.** Let us consider the functional \( \Phi := E \). From equality (30) we see that \( t \mapsto \Phi(S(t)U_0) \) is non-increasing for every \( U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1) \in \mathcal{H} \) with
\[ \Phi(S(t)U_0) + \int_0^L \left( \|\varphi(t)\|_2^2 + \|\psi(t)\|_2^2 \right) \, dx \leq \Phi(U_0), \quad U_0 \in \mathcal{H}. \] (36)

Let us suppose now that \( \Phi(S(t)U_0) = \Phi(U_0) \) for all \( t > 0 \). Thus,
\[ \|\varphi(t)\|_2^2 + \|\psi(t)\|_2^2 = 0, \quad t > 0, \]
and so \( \varphi(t) \equiv \varphi_0 \) and \( \psi(t) \equiv \psi_0 \) for all \( t \geq 0 \). This implies that \( U(t) = S(t)U_0 = (\varphi_0, 0, \psi_0, 0) \) is a stationary solution, that is, \( S(t)U_0 = U_0 \) for all \( t > 0 \). Therefore \( \Phi \) is a strict Lyapunov functional for the system \((\mathcal{H}, S(t))\).

In addition, from (36) it is obvious that \( \Phi \) is bounded from above on bounded subsets of \( \mathcal{H} \). Also, from (31) and (36) we have
\[ \|U(t)\|_H^2 \leq 4\Phi(S(t)U_0) + 4K \leq 4\Phi(U_0) + 4K, \]
for any mild solution \( U(t) \) corresponding to \( U_0 \in \mathcal{H} \). This is enough to show that \( \Phi_R \) is a bounded set of \( \mathcal{H} \). The proof of Lemma 4.4 is complete. \( \square \)

**Lemma 4.5.** The set \( \mathcal{N} \) of stationary solutions of (8) is bounded in \( \mathcal{H} \).

**Proof.** From Lemma 4.4 the set of stationary solutions can be expressed by
\[ \mathcal{N} = \{ U = (\varphi, 0, \psi, 0) \in \mathcal{H} ; \mathcal{A}U + \mathcal{F}(U) = 0 \}. \]
Given \( U = (\varphi, 0, \psi, 0) \in \mathcal{N} \) and using (9)-(10) we have the stationary system
\begin{align*}
-k(\varphi_x + \psi)_x + f_1(\varphi, \psi) & = h_1, \quad (37) \\
-b\psi_{xx} + k(\varphi_x + \psi) + f_2(\varphi, \psi) & = h_2. \quad (38)
\end{align*}
Multiplying (37) by \( \varphi \) and (38) by \( \psi \), integrating over \((0, L)\), and adding the resulting expression results
\[
k\|\varphi_x + \psi_x\|^2 + b\|\psi_x\|^2 + \int_0^L \left[ f_1(\varphi, \psi)\varphi + f_2(\varphi, \psi)\psi \right] dx = \int_0^L \left[ h_1 \varphi + h_2 \psi \right] dx. \tag{39}
\]
Applying assumptions (17)-(18) it follows that
\[
\int_0^L \left[ f_1(\varphi, \psi)\varphi + f_2(\varphi, \psi)\psi \right] dx \geq -\theta_1 \int_0^L |\varphi|^2 dx - \mu_1 \int_0^L |\psi|^2 dx - \theta_2 L
\geq -2\theta_1 L^2 \|\varphi_x + \psi_x\|^2 - L^2(2\theta_1 L^2 + \mu_1) \|\psi_x\|^2 - \theta_2 L.
\]
Also, Hölder and Poincaré inequalities imply
\[
\int_0^L \left[ h_1 \varphi + h_2 \psi \right] dx \leq \|h_1\|_2 \|\varphi\|_2 + \|h_2\|_2 \|\psi\|_2
\leq \frac{k}{2} \|\varphi_x + \psi_x\|^2 + \frac{b}{2} \|\psi_x\|^2 + \left( \frac{L^2}{2k} + \frac{L^4}{b} \right) \|h_1\|^2_2 + \frac{L}{b} \|h_2\|^2_2.
\]
Replacing these two last estimates in (39) and using (19) we obtain
\[
\frac{1}{4} \|U\|^2_{\mathcal{H}} = \frac{k}{4} \|\varphi_x + \psi_x\|^2 + \frac{b}{4} \|\psi_x\|^2 \leq \theta_2 L + \left( \frac{L^2}{2k} + \frac{L^4}{b} \right) \|h_1\|^2_2 + \frac{L}{b} \|h_2\|^2_2, \tag{40}
\]
from where it follows that \( \mathcal{N} \) is bounded in \( \mathcal{H} \). This concludes the proof of Lemma 4.5.

Now we consider a key result to conclude the existence of a global attractor as well as its properties for the dynamical system \((\mathcal{H}, S(t))\). It brings up a stability inequality which allows us to conclude the quasi-stability property for \((\mathcal{H}, S(t))\).

**Lemma 4.6.** Let \( B \) be a bounded set of \( \mathcal{H} \). Given initial data \( U_i = (\varphi^i_0, \varphi^i_1, \psi^i_0, \psi^i_1) \in B \), \( i = 1, 2 \), let us consider the mild solutions \( S(t)U_i = (\varphi^i(t), \varphi^i_1(t), \psi^i(t), \psi^i_1(t)) \) of (8), \( t \geq 0 \), respectively. Then there exist constants \( \gamma > 0 \), and \( K_B > 0 \) depending on the size of \( B \), such that
\[
\|S(t)U_1 - S(t)U_2\|^2_{\mathcal{H}} \leq 3e^{-\gamma t} \|U_1 - U_2\|^2_{\mathcal{H}} + K_B \int_0^t e^{-\gamma(t-\tau)} \left[ \|u(\tau)\|^2_{\mathcal{H}} + \|v(\tau)\|^2_{\mathcal{H}} \right] d\tau, \tag{41}
\]
for any \( t \geq 0 \), where we denote \( u = \varphi^1 - \varphi^2 \) and \( v = \psi^1 - \psi^2 \).

**Proof.** Let us start by denoting \( U(t) := S(t)U_1 - S(t)U_2 = (u(t), u_1(t), v(t), v_1(t)) \), \( t \geq 0 \). Thus \( U \) solves the following problem in the mild sense
\[
\rho_1 u_{tt} - k(u_x + v_x) + f_1(\varphi^1, \psi^1) - f_1(\varphi^2, \psi^2) + u_t = 0, \tag{42}
\]
\[
\rho_2 v_{tt} - b v_{xx} + k(u_x + v) + f_2(\varphi^1, \psi^1) - f_2(\varphi^2, \psi^2) + v_t = 0, \tag{43}
\]
with initial condition
\[
U(0) = (u(0), u_1(0), v(0), v_1(0)) = U_1 - U_2.
\]
Taking the multipliers \( u_t \) in (42) and \( v_t \) in (43), and adding the resulting expression, we have
\[
\frac{1}{2} \frac{d}{dt} \|U(t)\|^2\|_2^2 + \|v_t(t)\|^2_2 \leq -\int_0^L (\Delta f_1) u_t(t) dx - \int_0^L (\Delta f_2) v_t(t) dx, \tag{44}
\]
where we denote
\[
\mathcal{L}(t) = \rho_1 \|u_t(t)\|^2_2 + \rho_2 \|v_t(t)\|^2_2 + b \|v_{xx}(t)\|^2_2 + k \|u_x(t) + v(t)\|^2_2 \equiv \|U(t)\|^2_{\mathcal{H}}, \tag{45}
\]
and, for each \(i = 1, 2\),
\[
\Delta f_i = f_i(\varphi^1, \psi^1) - f_i(\varphi^2, \psi^2) = [f_i(\varphi^1, \psi^1) - f_i(\varphi^1, \psi^2)] + [f_i(\varphi^1, \psi^2) - f_i(\varphi^2, \psi^2)].
\]
Let us estimate the right hand side of (44). From (30)-(31) there exists a constant \(K^1_B > 0\) depending on \(B\) such that
\[
\|S(t)U_1\|_{H}^2, \|S(t)U_2\|_{H}^2 \leq K^1_B, \quad \forall \ t \geq 0.
\]
Besides, since \(\sigma_i\) is continuous and \(H_0^1(0, L) \hookrightarrow L^\infty(0, L)\), there exists a constant \(K^2_B > 0\) so that
\[
\sigma_i(\|\varphi^1\|), \sigma_i(\|\psi^2\|) \leq K^2_B \ a.e. \ in \ (0, L) \times (0, \infty), \ i, j = 1, 2.
\]
From assumptions (14)-(15) and H"older inequality we get
\[
\left| \int_0^L (\Delta f_1) u(t) \, dx \right| \leq \int_0^L \sigma_1(\|\varphi^1(t)\|) \left(1 + |\psi^1(t)|^{\gamma_1} + |\psi^2(t)|^{\gamma_2}\right) |v(t)| u(t) \, dx
\]
\[
\quad + \int_0^L \sigma_1(\|\varphi^2(t)\|) \left(1 + |\varphi^1(t)|^{\gamma_1} + |\varphi^2(t)|^{\gamma_2}\right) |u(t)| u(t) \, dx
\]
\[
\leq K^2_B (1 + \|\psi^1(t)\|_{L^{\gamma_1}} + \|\psi^2(t)\|_{L^{\gamma_2}}) \int_0^L |v(t)| u(t) \, dx
\]
\[
\quad + K^2_B (1 + \|\varphi^1(t)\|_{L^{\gamma_1}} + \|\varphi^2(t)\|_{L^{\gamma_2}}) \int_0^L |u(t)| u(t) \, dx
\]
for some constant \(K^3_B > 0\) depending on \(B\). Applying Young inequality with \(\epsilon = \alpha_1/4\), there exists a constant \(K^4_B > 0\) such that
\[
\left| \int_0^L (\Delta f_1) u(t) \, dx \right| \leq K^4_B \left(\|u(t)\|_{H}^2 + \|v(t)\|_{H}^2\right) + \frac{\alpha_1}{2} \|u(t)\|_{H}^2.
\]  \hspace{1cm} (46)
Similarly there exists a constant \(K^5_B > 0\) such that
\[
\left| \int_0^L (\Delta f_2) v(t) \, dx \right| \leq K^5_B \left(\|u(t)\|_{H}^2 + \|v(t)\|_{H}^2\right) + \frac{\alpha_2}{2} \|v(t)\|_{H}^2.
\]  \hspace{1cm} (47)
Replacing (46)-(47) in (44) and denoting \(K^6_B := 2(K^4_B + K^5_B)\) we obtain
\[
\frac{d}{dt} \mathcal{L}(t) \leq -\|u(t)\|_{H}^2 - \|v(t)\|_{H}^2 + K^6_B \left(\|u(t)\|_{H}^2 + \|v(t)\|_{H}^2\right).
\]  \hspace{1cm} (48)
Now let us define the perturbed functional
\[
\mathcal{L}_\varepsilon(t) = \mathcal{L}(t) + \varepsilon \mathcal{I}(t) + \varepsilon \mathcal{J}(t),
\]
where \(\varepsilon > 0\) will be fixed later and
\[
\mathcal{I}(t) = \rho_1 \int_\Omega u(t)u(t) \, dx \quad \text{and} \quad \mathcal{J}(t) = \rho_2 \int_\Omega v(t)v(t) \, dx.
\]
First, it is easy to check that there exists a constant \(C > 0\) such that
\[
|\mathcal{L}_\varepsilon(t) - \mathcal{L}(t)| \leq \varepsilon C \mathcal{L}(t), \quad \forall \ t \geq 0, \ \forall \ \varepsilon > 0.
\]  \hspace{1cm} (49)
Next, we show that there exist constants \(K^7_B > 0\) and \(\varepsilon_1 > 0\) so that
\[
\frac{d}{dt} \mathcal{L}_\varepsilon(t) + \frac{\varepsilon}{2} \mathcal{L}(t) \leq K^7_B \left(\|u(t)\|_{H}^2 + \|v(t)\|_{H}^2\), \quad \forall \ t > 0, \ \forall \ \varepsilon \in (0, \varepsilon_1].
\]  \hspace{1cm} (50)
Indeed, deriving \( \mathcal{I} \) and \( \mathcal{J} \), and using equations (42)-(43), it follows
\[
\frac{d}{dt} \left[ \mathcal{I}(t) + \mathcal{J}(t) \right] = \rho_1 \| u_t(t) \|_2^2 + \rho_2 \| v_t(t) \|_2^2 - \varepsilon b \| u_x(t) + v(t) \|_2^2 + I_1 + I_2 \tag{51}
\]
where
\[
I_1 = -\int_0^L \left[ (\Delta f_1)u(t) + (\Delta f_2)v(t) \right] dx,
\]
\[
I_2 = -\int_0^L \left[ u_t(t)u(t) + v_t(t)v(t) \right] dx.
\]
Now we estimate the terms \( I_1 \) and \( I_2 \). Using analogous arguments to conclude (46) and (47), but replacing functions \( u_t \) and \( v_t \) by \( u \) and \( v \), respectively, we infer
\[
|I_1| \leq C_B^1 (\| u(t) \|_2^2 + \| v(t) \|_2^2),
\]
for some constant \( C_B^1 > 0 \) depending on \( B \). Also, Hölder and Young inequalities lead us to
\[
|I_2| \leq \int_0^L |u(t)||u_t(t)| dx + \int_0^L |v(t)||v_t(t)| dx
\]
\[
\leq L \| u_x(t) + v(t) \|_2 \| u_t(t) \|_2 + L^2 \| v_x(t) \|_2 \| u_t(t) \|_2 + L \| v_x(t) \|_2 \| v_t(t) \|_2
\]
\[
\leq \frac{b}{2} \| v_x(t) \|_2^2 + \frac{k}{2} \| u_x(t) + v(t) \|_2^2 + \left( \frac{L^2}{2 \rho_1 k} + \frac{L^4}{\rho_1 b} \right) \rho_1 \| u_t(t) \|_2^2 + \frac{L^2}{2 b \rho_2} \rho_2 \| v_t(t) \|_2^2.
\]
Inserting these two last estimates in (51) and neglecting nonnegative terms we arrive at
\[
\frac{d}{dt} \left[ \mathcal{I}(t) + \mathcal{J}(t) \right] \leq -\frac{b}{2} \| v_x(t) \|_2^2 - \frac{k}{2} \| u_x(t) + v(t) \|_2^2 + K_1 \left( \rho_1 \| u_t(t) \|_2^2 + \rho_2 \| v_t(t) \|_2^2 \right) + C_B^1 (\| u(t) \|_2^2 + \| v(t) \|_2^2),
\]
with \( K_1 = 1 + \frac{L^2}{2 \rho_1 k} + \frac{L^4}{\rho_1 b} + \frac{L^2}{2 b \rho_2} \). Deriving \( \mathcal{L}_c \), utilizing (48), (52), and adding \( \frac{\varepsilon}{2} \mathcal{L}(t) \) in both sides of the resulting expression, we get
\[
\frac{d}{dt} \mathcal{L}_c(t) + \frac{\varepsilon}{2} \mathcal{L}(t) \leq -\left[ 1 - \varepsilon \left( K_1 + \frac{1}{2} \right) \rho_1 \right] \| u_t(t) \|_2^2 - \left[ 1 - \varepsilon \left( K_1 + \frac{1}{2} \right) \rho_2 \right] \| v_t(t) \|_2^2
\]
\[
+ \left( \varepsilon C_B^1 + K_B^6 \right) \left( \| u(t) \|_2^2 + \| v(t) \|_2^2 \right).
\]
Taking \( \varepsilon \leq \varepsilon_1 := \frac{2}{K_{\alpha+1}} \min \left\{ \frac{1}{\rho_1}, \frac{1}{\rho_2} \right\} > 0 \), and \( K_B^7 = \varepsilon C_B^1 + K_B^6 \), then (50) holds true.

Now we fix \( \varepsilon_2 = \min \left\{ \varepsilon_1, \frac{1}{2 \alpha} \right\} > 0 \). So choosing \( \varepsilon \leq \varepsilon_2 \) we derive from (49) that
\[
\frac{1}{2} \mathcal{L}(t) \leq \mathcal{L}_c(t) \leq \frac{3}{2} \mathcal{L}(t), \quad t \geq 0.
\tag{53}
\]
Further, from (50) we have
\[
\frac{d}{dt} \mathcal{L}_c(t) + \frac{\varepsilon}{3} \mathcal{L}_c(t) \leq K_B^7 \left( \| u(t) \|_2^2 + \| v(t) \|_2^2 \right), \quad t > 0.
\tag{54}
\]
Finally, combining (53) and (54), we conclude
\[
\mathcal{L}(t) \leq 3e^{-\alpha t} \mathcal{L}(0) + K_B \int_0^t e^{-\gamma(t-\tau)} \left( \| u(\tau) \|_2^2 + \| v(\tau) \|_2^2 \right) d\tau, \quad t > 0,
\]
where we take $\gamma = \frac{\alpha}{2} > 0$ and $K_B = K_B^1 > 0$. Regarding that $L(t) = \|S(t) U_1 - S(t) U_2\|^2_{\mathcal{H}}, \ t \geq 0$, then estimate (41) follows. The proof of Lemma 4.6 is complete.

Now we are in conditions to conclude the proof of Theorem 3.1.

4.3.1. Proof of Theorem 3.1: (i). We need to show that $(\mathcal{H}, S(t))$ satisfies conditions (21)-(22). Let $B \subset \mathcal{H}$ be a bounded positively invariant set of $S(t)$ and take $U_1, U_2 \in B$. As before, we denote

$$S(t) U_i = (\varphi^i(t), \phi^i(t), \psi^i(t), \psi^i(t)), \ i = 1, 2, \text{ and } (u, v) = (\varphi^1 - \varphi^2, \psi^1 - \psi^2).$$

Firstly, from (16) in Theorem 2.1-(iii) we see that (21) follows promptly with $a(t) = e^{2C_B t} > 0$, for some $C_B > 0$, which is locally bounded in $[0, \infty)$. Next, we consider the seminorm

$$n_X(u, v) = \|u\|_2 + \|v\|_2, \ X = H^1_0(\Omega) \times H^1_0(\Omega).$$

Keeping in mind that $\mathcal{H}$ can be represented as in (23) and since embedding $X \hookrightarrow L^2(\Omega) \times L^2(\Omega)$ is compact, then it follows that $n_X(\cdot, \cdot)$ is compact seminorm on $X$. Besides, from Lemma 4.6 results

$$\|S(t) U_1 - S(t) U_2\|^2_{\mathcal{H}} \leq b(t)\|U_1 - U_2\|^2_{\mathcal{H}} + c(t) \sup_{\tau \in [0, t]} [n_X(u(\tau), v(\tau))]^2,$$

where

$$b(t) = 3e^{-\gamma t} \text{ and } c(t) = K_B \int_0^t e^{-\gamma(t - \tau)} d\tau, \ t \geq 0.$$ 

Finally, we note that $b \in L^1(\mathbb{R}^+) \text{ with } \lim_{t \to \infty} b(t) = 0, \text{ and } c_\infty = \sup_{t \in \mathbb{R}^+} c(t) \leq \frac{K_B}{\gamma} < \infty.$ Thus, condition (22) also holds true. This shows that $(\mathcal{H}, S(t))$ is quasi-stable on any bounded positively invariant set in $\mathcal{H}$.

4.3.2. Proof of Theorem 3.1: (ii). From Theorem 3.1-(i) and [9, Prop. 7.9.4], $(\mathcal{H}, S(t))$ is asymptotically smooth. Then applying Lemmas 4.4 and 4.5 from above, and also [9, Cor. 7.5.7], $(\mathcal{H}, S(t))$ possesses a compact global attractor given by $\mathfrak{A} = M_+ (\mathcal{N})$.

4.3.3. Proof of Theorem 3.1: (iii). It follows promptly from Theorem 3.1-(ii) and [9, Thm. 7.5.10].

4.3.4. Proof of Theorem 3.1: (iv) and (v). From items (i)-(ii) of Theorem 3.1, $(\mathcal{H}, S(t))$ is quasi-stable on the attractor $\mathfrak{A}$. Applying [9, Thm. 7.9.6] it follows that $\mathfrak{A} \text{ has finite fractal dimension } \dim^H_1 \mathfrak{A}$. Moreover, since condition (22) holds with property $c_\infty = \sup_{t \in \mathbb{R}^+} c(t) < \infty$, then the smoothness properties (24)-(25) follow as consequence of [9, Thm. 7.9.8].

4.3.5. Proof of Theorem 3.1: (vi). Let us consider the bounded set $\mathcal{B} = \{U_0 \in \mathcal{H} : \Phi(U_0) \leq R\}$ where $\Phi$ is the strict Lyapunov functional obtained in Lemma 4.4 and every $R > 0$. Thus, for $R > 0$ large enough, it is easy to check that $\mathcal{B}$ is a positively invariant bounded absorbing set. This means that $(\mathcal{H}, S(t))$ is dissipative and from Theorem 3.1-(i) it is also quasi-stable on $\mathcal{B}$. Besides, given $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1) \in \mathcal{B}$ we can see from (4)-(5) and (31) that

$$(\varphi_t, \varphi_t, \psi_1, \psi_1) \in L^\infty(0, T; \mathcal{H}_{-1}), \ \forall \ T > 0.$$
Keeping in mind that $\mathcal{H}_{-1}$ is isomorphic to $L^2(0,L) \times L^2(0,L) \times H^{-1}(0,L)$, then following similar arguments as presented in [19, 26] we can prove that mapping $t \mapsto S(t)U_0 \equiv U(t)$, for any $U_0 \in \mathcal{B}$, is Hölder continuous in $\mathcal{H}_{-1}$ with exponent $\delta = 1$. Hence, from [9, Thm. 7.9.9] the dynamical system $(\mathcal{H}, S(t))$ has a generalized fractal exponential attractor $\mathfrak{A}_{\exp}$ with finite fractal dimension in $\mathcal{H}_{-1}$. The remaining conclusion follows after applying interpolation theorem analogously to [19, 26]. The proof of Theorem 3.1 is now complete.

**Acknowledgments.** We would like to thank the anonymous referee for his/her helpful comments and remarks that have helped us to improve and clarify this work.

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Received December 2015; revised February 2016.

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