GLOBAL APPROXIMATION THEOREMS FOR GENERAL GAMMA TYPE OPERATORS

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Dedicated to Prof. P. N. Agrawal

ABSTRACT. In this paper, we obtained some global approximation results for general Gamma type operators.
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1. Introduction

For a measurable complex valued and locally bounded function defined on \([0, \infty)\), Lupas and Müller \[12\] defined and studied some approximation properties of linear positive operators \(\{G_n\}\) defined by

\[G_n(f; x) = \int_0^\infty g_n(x, u)f\left(\frac{u}{n}\right)du,\]

where

\[g_n(x, u) = \frac{x^{n+1}}{n!}e^{-xu}u^n, \quad x > 0.\]

In \[13\], Mazhar gives an important modifications of the Gamma operators using the same \(g_n(x, u)\)

\[F_n(f; x) = \int_0^\infty \int_0^\infty g_n(x, u)g_{n-1}(u, t)f(t)dudt = \frac{(2n)!x^{n+1}}{n!(n-1)!} \int_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}}f(t)dt, \quad n > 1, \quad x > 0.\]

Recently, Karsli \[7\] considered the following Gamma type linear and positive operators

\[L_n(f; x) = \int_0^\infty \int_0^\infty g_{n+2}(x, u)g_n(u, t)f(t)dudt = \frac{(2n+3)!x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}}f(t)dt, \quad x > 0,\]
and obtained some approximation results.

In [11], Karsli and Özarslan obtained some local and global approximation results for the operators $L_n(f;x)$.

In 2007, Mao [14] define the following generalised Gamma type linear and positive operators

$$M_{n,k}(f;x) = \int_0^\infty \int_0^\infty g_n(x,u)g_{n-k}(u,t)f(t)dudt$$

$$= \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!} \int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}}f(t)dt, \quad x > 0,$$

which includes the operators $F_n(f;x)$ for $k = 1$ and $L_{n-2}(f;x)$ for $k = 2$.

Some approximation properties of $M_{n,k}$ were studied in [8] and [9]. Several authors obtain the global approximation results for different operators (see [1], [3] and [4]).

We can rewrite the operators $M_{n,k}(f;x)$ as

$$M_{n,k}(f;x) = \int_0^\infty K_{n,k}(x,t)f(t)dt,$$

where

$$K_{n,k}(x,t) = \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!} \frac{t^{n-k}}{(x+t)^{2n-k+2}}, \quad x, t \in (0, \infty).$$

In this paper, we study some global approximation results of the operators $M_{n,k}$. Let $p \in N_0$(set of non-negative integers), $f \in C_p$, where $C_p$ is a polynomial weighted space with the weight function $w_p$,

$$w_0(x) = 1, \quad w_p(x) = \frac{1}{1+x^p}, \quad p \geq 1,$$

and $C_p$ is the set of all real valued functions $f$ for which $w_p f$ is bounded and uniformly continuous on $[0, \infty)$.

The norm on $C_p$ is defined by

$$||f||_p = \sup_{x \in [0, \infty)} w_p(x)|f(x)|, \quad f \in C_p[0, \infty).$$

We also consider the following Lipschitz classes:

$$\omega^2_p(f;\delta) = \sup_{h \in [0,\delta]} ||\Delta^2 hf||_p,$$

$$\Delta^2_h f(x) = f(x+2h) - 2f(x+h) + f(x),$$

$$\omega^1_p(f;\delta) = \sup \{w_p(x)|f(t) - f(x)| : |t - x| \leq \delta \text{ and } t, x \geq 0\};$$

$$Lip^2_p \alpha = \{f \in C_p[0, \infty) : \omega^2_p(f;\delta) = O(\delta^\alpha) \text{ as } \delta \to 0^+\},$$

where $h > 0$ and $\alpha \in (0, 2]$. 
2. Auxiliary Results

In this section we give some preliminary results which will be used in the proofs of our main theorems.

Let us consider
\[ e_m(t) = t^m, \quad \varphi_{x,m}(t) = (t - x)^m, \quad m \in \mathbb{N}_0. \]

**Lemma 1.** For any \( m \in \mathbb{N}_0 \) (set of non-negative integers), \( m \leq n - k \)
\[
M_{n,k}(t^m; x) = \frac{[n - k + m]_m x^m}{[n]_m} \tag{2.1}
\]
where \( n, k \in \mathbb{N} \) and \([x]_m = x(x - 1)\ldots(x - m + 1), [x]_0 = 1, x \in \mathbb{R}.

In particular for \( m = 0, 1, 2\ldots \) in (2.1) we get

(i) \( M_{n,k}(1; x) = 1 \),

(ii) \( M_{n,k}(t; x) = \frac{n - k + 1}{n} x \),

(iii) \( M_{n,k}(t^2; x) = \frac{(n - k + 2)(n - k + 1)}{n(n - 1)} x^2. \)

**Lemma 2.** Let \( m \in \mathbb{N}_0 \) and fixed \( x \in (0, \infty) \), then
\[
M_{n,k}(\varphi_{x,m}; x) = \left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{(n - m + j)!}{n!(n - k)!} \right) x^m.
\]

**Lemma 3.** For \( m = 0, 1, 2, 3, 4 \), one has

(i) \( M_{n,k}(\varphi_{x,0}; x) = 1 \),

(ii) \( M_{n,k}(\varphi_{x,1}; x) = \frac{1 - k}{n} x \),

(iii) \( M_{n,k}(\varphi_{x,2}; x) = \frac{k^2 - 5k + 2n + 4}{n(n - 1)} x^2 \),

(iv) \( M_{n,k}(\varphi_{x,3}; x) = \frac{-k^3 + 12k^2 - 17k + n(18 - 12k) + 24}{n(n - 1)(n - 2)} x^3 \),

(v) \( M_{n,k}(\varphi_{x,4}; x) = \frac{k^4 - 22k^3 + k^2(143 + 12n) - k(314 + 108n) + 12n^2 + 268n + 192}{n(n - 1)(n - 2)(n - 3)} x^4 \),

(vi) \( M_{n,k}(\varphi_{x,m}; x) = O \left( n^{-\lfloor (m+1)/2 \rfloor} \right). \)

**Proof.** Using Lemma 2 we get Lemma 3 \[ \square \]

**Theorem 1.** For the operators \( M_{n,k} \) and for fixed \( p \in \mathbb{N}_0 \), there exists a positive constant \( N_{p,k} \) such that
\[
w_p(x)M_{n,k} \left( \frac{1}{w_p}; x \right) \leq N_{p,k}. \tag{2.2}
\]

Furthermore, for all \( f \in C_p[0, \infty) \), we have
\[
\|M_{n,k}(f; \cdot)\|_p \leq N_{p,k}\|f\|_p, \tag{2.3}
\]
which guarantees that \( M_{n,k} \) maps \( C_p[0, \infty) \) into \( C_p[0, \infty) \).
Proof. For \( p = 0 \), (2.2) follows immediately. Using Lemma 1, we get

\[
\begin{align*}
  w_p(x)M_{n,k} \left( \frac{1}{w_p}; x \right) &= w_p(x) \left( M_{n,k}(e_0; x) + M_{n,k}(e_p; x) \right) \\
  &= w_p(x) \left( 1 + \frac{(n-p)!(n-k+p)!}{n!(n-k)!} x^p \right) \\
  &\leq N_{p,k}w_p(x)(1 + x^p) = N_{p,k},
\end{align*}
\]

where

\[
N_{p,k} = \max \left\{ \sup_n \frac{(n-p)!(n-k+p)!}{n!(n-k)!}, 1 \right\}.
\]

Observe that for all \( f \in C_p \) and every \( x \in (0, \infty) \), we get

\[
\begin{align*}
  w_p(x) |M_{n,k}(f; x)| &\leq w_p(x) \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!} \int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}} |f(t)| \frac{w_p(t)}{w_p(t)} dt \\
  &\leq \|f\|_p w_p(x) M_{n,k} \left( \frac{1}{w_p}; x \right) \\
  &\leq N_{p,k}\|f\|_p.
\end{align*}
\]

Taking supremum over \( x \in (0, \infty) \), we get (2.3). \( \square \)

**Lemma 4.** For the operators \( M_{n,k} \) and fixed \( p \in N_0 \), there exists a positive constant \( N_{p,k} \) such that

\[
  w_p(x)M_{n,k} \left( \frac{\varphi_{x,2}}{w_p(t)}; x \right) \leq N_{p,k} \frac{x^2}{n}.
\]

**Proof.** Using Lemma (3), we can write

\[
\begin{align*}
  w_0(x)M_{n,k} \left( \frac{\varphi_{x,2}}{w_0(t)}; x \right) &= \frac{k^2 - 5k + 2n + 4}{n(n-1)} x^2 \\
  &\leq N_{p,k} \frac{x^2}{n},
\end{align*}
\]

which gives the result for \( p = 0 \).

Let \( p \geq 1 \). Then using Lemma 1 and Lemma 3, we get

\[
\begin{align*}
  M_{n,k} \left( \frac{\varphi_{x,2}}{w_p(t)}; x \right) &= M_{n,k}(e_{p+2}; x) - 2x M_{n,k}(e_{p+1}; x) + x^2 M_{n,k}(e_p; x) + M_{n,k}(\varphi_{x,2}; x) \\
  &= \frac{(n-p-2)!(n-k+2)!}{n!(n-k)!} x^{p+2} - 2 \frac{(n-p-1)!(n-k+p+1)!}{n!(n-k)!} x^{p+2} \\
  &+ \frac{(n-p)!(n-k+p)!}{n!(n-k)!} x^{p+2} + \frac{k^2 - 5k + 2n + 4}{n(n-1)} x^2 \\
  &\leq N_{p,k} \frac{x^2}{n}(1 + x^p),
\end{align*}
\]

where \( N_{p,k} \) is a positive constant. Hence, the proof is completed. \( \square \)
3. Rate of Convergence

Let \( p \in \mathbb{N}_0 \). By \( C^2_p[0, \infty) \), we denote the space of all functions \( f \in C_p[0, \infty) \) such that \( f', f'' \in C_p[0, \infty) \).

**Theorem 2.** Let \( p \in \mathbb{N}_0, \ n \in \mathbb{N} \) and \( g \in C^1_p[0, \infty) \), there exists a positive constant \( N_{p,k} \) such that

\[
 w_p(x) |M_{n,k}(f; x) - f(x)| \leq N_{p,k} \|f'\|_p \frac{x}{\sqrt{n}}
\]

for all \( x \in (0, \infty) \).

**Proof.** We have

\[
f(t) - f(x) = \int_x^t f'(v)dv.
\]

By using linearity of \( M_{n,k} \) we get

\[
M_{n,k}(f; x) - f(x) = M_{n,k} \left( \int_x^t f'(v)dv; x \right).
\]

(3.1)

Remark that

\[
\left| \int_x^t f'(v)dv \right| \leq \|f'\|_p \left| \int_x^t \frac{dv}{w_p(v)} \right| \leq \|f'\|_p |t - x| \left( \frac{1}{w_p(t)} + \frac{1}{w_p(x)} \right).
\]

From (3.1) we obtain

\[
w_p(x) |M_{n,k}(f; x) - f(x)| \leq \|f'\|_p \left\{ M_{n,k}(\mid \varphi_{x,1} \mid ; x) + w_p(x)M_{n,k} \left( \frac{\mid \varphi_{x,1} \mid }{w_p(t)} ; x \right) \right\}.
\]

Using Cauchy-Schwarz inequality, we can write

\[
M_{n,k}(\mid \varphi_{x,1} \mid ; x) \leq (M_{n,k}(\mid \varphi_{x,2} \mid ; x))^{1/2},
\]

\[
M_{n,k} \left( \frac{\mid \varphi_{x,1} \mid }{w_p(t)} ; x \right) \leq \left( M_{n,k} \left( \frac{1}{w_p(t)} ; x \right) \right)^{1/2} \left( M_{n,k} \left( \frac{\mid \varphi_{x,2} \mid }{w_p(t)} ; x \right) \right)^{1/2}.
\]

Using Lemma 3, Theorem 1 and Lemma 4, we obtain

\[
w_p(x) |M_{n,k}(f; x) - f(x)| \leq N_{p,k} \|f'\|_p \frac{x}{\sqrt{n}}.
\]

\[\square\]

**Lemma 5.** Let \( p \in \mathbb{N}_0, \) If

\[
H_{n,k}(f; x) = M_{n,k}(f; x) - f \left( x + \frac{1 - k}{n} x \right) + f(x),
\]

(3.2)

then there exists a positive constant \( N_{p,k} \) such that for all \( x \in (0, \infty) \) and \( n \in \mathbb{N} \), we have

\[
w_p(x) |H_{n,k}(g; x) - g(x)| \leq N_{p,k} \|g''\|_p \frac{x^2}{n}
\]

for any function \( g \in C^2_p \).
Proof. From Lemma 4 we observe that the operators $H_{n,k}$ are linear and reproduce the linear functions. Hence

$$H_{n,k}(\varphi_{x,1};x) = 0.$$ 

Let $g \in C^2_p$. By the Taylor formula one can write

$$g(t) - g(x) = (t - x) g'(x) + \int_x^t (t - v) g''(v) dv, \quad t \in (0, \infty).$$

Then,

$$|H_{n,k}(g; x) - g(x)|$$

$$= |H_{n,k}(g(t) - g(x)); x| = \left| H_{n,k} \left( \int_x^t (t - v) g''(v) dv; x \right) \right|$$

$$= \left| M_{n,k} \left( \int_x^t (t - v) g''(v) dv; x \right) - \int_x^{x + \frac{1 - k}{n}} \left( x + \frac{1 - k}{n} x - v \right) g''(v) dv \right|.$$ 

Since

$$\left| \int_x^t (t - v) g''(v) dv \right| \leq \frac{\|g''\|_p (t - x)^2}{2} \left( \frac{1}{w_p(x)} + \frac{1}{w_p(t)} \right)$$

and

$$\left| \int_x^{x + \frac{1 - k}{n}} \left( x + \frac{1 - k}{n} x - v \right) g''(v) dv \right| \leq \frac{\|g''\|_p}{2 w_p(x)} \left( \frac{1 - k}{n} x \right)^2,$$

we get

$$w_p(x)|H_{n,k}(g; x) - g(x)| \leq \frac{\|g''\|_p}{2} \left[ M_{n,k}(\varphi_{x,2}; x) + w_p(x) M_{n,k} \left( \frac{\varphi_{x,2}}{w_p(t)}; x \right) \right] + \frac{\|g''\|_p}{2} \left( \frac{1 - k}{n} x \right)^2.$$ 

Hence by Lemma 4 we obtain

$$w_p(x)|H_{n,k}(g; x) - g(x)| \leq N_{p,k} \|g''\|_p \frac{x^2}{n}$$

for any function $g \in C^2_p$. The Lemma is proved. \qed

The next theorem is the main result of this section.

**Theorem 3.** Let $p \in N_0$, $n \in N$ and $f \in C_p[0, \infty)$, then there exists a positive constant $N_{p,k}$ such that

$$w_p(x) |M_{n,k}(f; x) - f(x)| \leq N_{p,k} \omega_p^2 \left( f, \frac{x}{\sqrt{n}} \right) + \omega^1_p \left( f, \frac{1 - k}{n} x \right).$$

Furthermore, if $f \in \text{Lip}_p^\alpha$ for some $\alpha \in (0, 2]$, then

$$w_p(x) |M_{n,k}(f; x) - f(x)| \leq N_{p,k} \left( \frac{x^2}{n} \right)^{\alpha/2} + \omega^1_p \left( f, \frac{1 - k}{n} x \right),$$

holds.
Proof. Let \( p \in \mathbb{N}_0, f \in C_p[0, \infty) \) and \( x \in (0, \infty) \) be fixed. We consider the Steklov means of \( f \) by \( f_h \) and given by the formula
\[
f_h(x) = \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} \{2f(x + s + t) - f(x + 2(s + t))\} dsdt,
\]
for \( h > 0 \) and \( x \geq 0 \). We have
\[
f(x) - f_h(x) = \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} \Delta^2_{s+t} f(x) dsdt,
\]
which gives
\[
\|f - f_h\|_p \leq \omega^2_p(f, h). \tag{3.3}
\]
Furthermore, we have
\[
f''_h(x) = \frac{1}{h^2} \left( 8 \Delta^2_{h/2} f(x) - \Delta^2_h f(x) \right),
\]
and
\[
\|f''_h\|_p \leq \frac{9}{h^2} \omega^2_p(f, h). \tag{3.4}
\]
From (3.3) and (3.4) we conclude that \( f_h \in C^2_p[0, \infty) \) if \( f \in C_p[0, \infty) \).

Moreover
\[
|M_{n,k}(f; x) - f(x)| \leq H_{n,k}(|f(t) - f_h(t); x|) + |f(x) - f_h(x)|
+ |H_{n,k}(f_h; x) - f_h(x)| + \left| f \left( x + \frac{1 - k}{n} x \right) - f(x) \right|,
\]
where \( H_{n,k} \) is defined in (3.2).

Since \( f_h \in C^2_p[0, \infty) \) by the above, it follows from Theorem 1 and Lemma 5 that
\[
w_p(x) |M_{n,k}(f; x) - f(x)| \leq (N + 1) \|f - f_h\|_p + N_{p,k} \|f''_h\|_p \frac{x^2}{n}
+ w_p(x) \left| f \left( x + \frac{1 - k}{n} x \right) - f(x) \right|.
\]

By (3.3) and (3.4), the last inequality yields that
\[
w_p(x) |M_{n,k}(f; x) - f(x)| \leq N_{p,k} \omega^2_p(f; h) \left( 1 + \frac{1}{h^2} \frac{x^2}{n} \right)
+ \omega^1_p \left( f, \frac{1 - k}{n} x \right).
\]

Thus, choosing \( h = \frac{x}{\sqrt{n}} \), the first part of the proof is completed.

The remainder of the proof can be easily obtained from the definition of the space \( \text{Lip}^2_p \).

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