POSITIVE SOLUTIONS FOR A NONLINEAR SCHRÖDINGER-POISSON SYSTEM

CHUNHUA WANG
School of Mathematics and Statistics & Hubei Key Laboratory of Mathematical Sciences
Central China Normal University
Wuhan 430079, China

JING YANG
School of Science, Jiangsu University of Science and Technology
Zhenjiang 212003, China

(Communicated by Yanyan Li)

ABSTRACT. In this paper, we study the following nonlinear Schrödinger-Poisson system
\[
\begin{align*}
-\Delta u + u + \epsilon K(x)\Phi(x)u &= f(u), \\ -\Delta \Phi &= K(x)u^2,
\end{align*}
\]
where $K(x)$ is a positive and continuous potential and $f(u)$ is a nonlinearity satisfying some decay condition and some non-degeneracy condition, respectively. Under some suitable conditions, which are given in section 1, we prove that there exists some $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the above problem has infinitely many positive solutions by applying localized energy method. Our main result can be viewed as an extension to a recent result Theorem 1.1 of Ao and Wei in [3] and a result of Li, Peng and Wang in [26].

1. Introduction and the main result. In this paper, we are mainly motivated by the following problem
\[
\begin{align*}
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial t} &= -\frac{\hbar^2}{2m}\Delta \psi + (V(x) + E)\psi + \epsilon K(x)\Phi(x)\psi - f(x, \psi), \\ -\Delta \Phi &= K(x)|\psi|^2,
\end{align*}
\]
where $\hbar$ is the Planck constant, $i$ is the imaginary unit, $m$ is a positive constant, $E$ is a real number, $\epsilon > 0$, $\psi : \mathbb{R}^3 \times [0, T] \mapsto \mathbb{C}$. It is well known that this type of equations has a strong physical meaning because it appears in quantum mechanics models (see e.g. [10, 12, 28]) and in semiconductor theory (see [8, 29, 30]).
We want to investigate the existence of standing wave \( \psi(x,t) = e^{-iEt} u(x) \), where \( u(x) \) is a real function. If \( f(x,|\psi|) = f(x,\psi) \), then \( u \) solves
\[
\begin{cases}
-\frac{\hbar^2}{2m} \Delta u + V(x)u + \epsilon K(x)\Phi(x)u = f(x,u), & x \in \mathbb{R}^3, \\
-\Delta \Phi = K(x)u^2, & x \in \mathbb{R}^3.
\end{cases}
\tag{1.1}
\]

Since in (1.1), the first equation is a nonlinear stationary equation (where the non-linear term simulates the interaction between many particles) that is coupled with a Poisson equation to be satisfied by \( \Phi \), meaning that the potential is determined by the charge of the wave function, we refer (1.1) as a nonlinear Schrödinger-Poisson system. When \( \hbar \to 0 \) in (1.1), the existence of the so-called semi-classical states in nonlinear Schrödinger-Poisson system, we refer the readers to [22-24] and the references therein.

When \( \frac{\hbar^2}{2m} = 1 \), (1.1) becomes
\[
\begin{cases}
-\Delta u + V(x)u + \epsilon K(x)\Phi(x)u = f(x,u), & x \in \mathbb{R}^3, \\
-\Delta \Phi = K(x)u^2, & x \in \mathbb{R}^3,
\end{cases}
\tag{1.2}
\]
which was first introduced in [9] as a model describing solitary waves for the nonlinear stationary Schrödinger equations integrating with the electrostatic field.

In recent years, when \( \epsilon = 1 \) and \( K(x) \equiv 1 \), problem (1.2) with \( V(x) \equiv 1 \) or being radially symmetric, has been widely studied under various conditions on \( f \) (see [2,17,35]). The case of positive and bounded nonradial potential \( V \) has been considered in [37], when \( f \) is asymptotically linear, in [5,6], when \( f(x,u) = |u|^{p-1}u \), with \( 3 < p < 5 \). Moreover, in [5], the existence of ground state solutions for problem (1.2) has been proved in several situations, including the positive constant potential case. In [6], Azzollini and Pomponio considered problem (1.2) with a class of more general potential which may be unbounded from below, and the existence of ground state solutions was proved. When \( 3 < p < 5 \) and \( V \) was not a constant, the existence of ground state solutions was established in [5,6,14]. For the autonomous case, namely \( K(x) \equiv 1 \) and when \( f(u) = |u|^{p-2}u \), it has been investigated in [9,29].

In [2,35], it was proved that (1.2) has infinitely many pairs radial solutions for \( 3 < p < 6 \), and has multiple solutions (but not infinitely many) for small \( \epsilon \) and \( 2 < p < 3 \). In [36], when \( \epsilon = 1 \), Ruiz proved that (1.2) admits no nontrivial solutions for \( 2 < p \leq 3 \) and possesses a positive radial solution for \( 3 < p < 6 \). In [11,17], they used the mountain pass theorem and obtained a radial positive solution for \( 4 \leq p < 6 \). In [14], when \( \epsilon = 1 \) and \( V(x) \equiv 1 \), Cerami and Vaira studied (1.2) with \( f(x,u) = Q(x)|u|^{p-1}u, 3 < p < 5 \). They have proved the existence of positive solutions to (1.2) when \( Q \) and \( K \) were nonsymmetric and nonnegative functions satisfying \( \lim_{|x| \to \infty} Q(x) = Q_\infty \) and \( \lim_{|x| \to \infty} K(x) = 0 \). When \( V(x) \equiv 1 \) and \( f(x,u) = Q(x)|u|^{p-1}u, 1 < p < 5 \), in [25], Li, Peng and Yan proved that (1.2) has infinitely many non-radial positive solutions under the assumptions that \( K(x), Q(x) \) were positive bounded radial functions in \( \mathbb{R}^3 \) satisfying some decaying conditions. In [26], applying the finite reduction method, Li, Peng and Wang proved that there exists \( \epsilon(m) \) such that, for \( 0 < \epsilon < \epsilon(m) \), (1.2) with \( f(x,u) = |u|^{p-2}u, 2 < p < 6 \), has an \( m \)-bump solution under the assumption that
\[
(K) \quad K(x) \in C(\mathbb{R}^3, \mathbb{R}^+), \quad \lim_{|x| \to \infty} K(x) = 0 \quad \text{and} \quad \lim_{|x| \to \infty} \frac{\ln(K(x))}{|x|} = 0.
\]
For more results about (1.1), one can refer to [1, 8, 15, 18, 20, 31, 36] and the reference therein.

Recently in [13], Cerami, Passasseeo and Solimini developed a localized Nahari’s manifold argument and localized variational method to prove the existence of infinitely many positive solutions of the following equation

\[
\begin{cases}
-\Delta u + (1 + \epsilon K(x))u = u^p, & x \in \mathbb{R}^N, \\
u > 0 \text{ in } \mathbb{R}^N, & u \in H^1(\mathbb{R}^N),
\end{cases}
\] (1.3)

where the potential \( K \) satisfies suitable decay assumptions (see below \((K_1) - (K_2)\)). Unlike [13], Ao and Wei in [3] used localized energy method to prove that there existed some \( \epsilon \) such that \( 0 < \epsilon < \epsilon_0 \), (1.3) with more general nonlinearity \( f(u) \) had infinitely many solutions. This generalized and gave a new proof of the results by Cerami, Passasseeo and Solimini in [13]. They also used the new techniques to establish the existence of infinitely many positive bound states for elliptic systems.

Motivated by [3, 26], in this paper we study the case \( \epsilon^2 = V(x) = 1 \) and \( f(x, u) = f(u) \), i.e., the following system of Schrödinger-Poisson equations

\[
\begin{cases}
-\Delta u + u + \epsilon K(x)\Phi(x)u = f(u), & x \in \mathbb{R}^3, \\
-\Delta \Phi = K(x)u^2, & x \in \mathbb{R}^3.
\end{cases}
\] (1.4)

In order to state our main result, we give the conditions imposed on \( K(x) \) (similar to [3, 13]) and \( f \):

\begin{align*}
(K_1) & \quad K(x) \in C(\mathbb{R}^3, \mathbb{R}^+) , \quad \lim_{|x| \to \infty} K(x) = 0; \\
(K_2) & \quad \exists 0 < \alpha < 1, \quad \lim_{|x| \to \infty} K(x)e^{\alpha|x|} = +\infty; \\
(f_{1}) & \quad f : \mathbb{R} \to \mathbb{R} \text{ is of class } C^{1+\delta} \text{ for some } 0 < \delta \leq 1 \text{ and } f(u) = 0 \text{ for } u \leq 0, f'(0) = 0; \\
(f_{2}) & \quad \text{The equation}
\begin{align*}
-\Delta w + w &= f(w), & w > 0 \text{ in } \mathbb{R}^3, \\
\lim_{|x| \to \infty} w(x) &= 0, & w(0) = \max_{x \in \mathbb{R}^3} w(x),
\end{align*}
(1.5)
\end{align*}

has a nondegenerate solution \( w \), i.e.,

\[
\ker(\Delta - 1 + f'(w)) \cap L^\infty(\mathbb{R}^3) = \text{span} \left\{ \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \frac{\partial w}{\partial x_3} \right\}.
\] (1.6)

From the well-known results of [21], we know that \( w \) is radially symmetric with exponential decay. Moreover, we have the following asymptotic behavior of \( w \)

\[
w(r) = A_N r^{-\frac{N-1}{2}} e^{-r} \left( 1 + O \left( \frac{1}{r} \right) \right), \quad w'(r) = -A_N r^{-\frac{N-1}{2}} e^{-r} \left( 1 + O \left( \frac{1}{r} \right) \right)
\] (1.7)

for \( r \) large, where \( A_N > 0 \) is a generic constant.

Note that the function

\[
f(w) = w^p - aw^q, \quad \text{for } t \geq 0
\]

with a constant \( a \geq 0 \) satisfies the above conditions \((f_1) - (f_2)\) if \( 1 < q < p < 5 \) (see Appendix C of [33], also [3]).

Nondegeneracy is a generic condition. We want to point out that there do exist nonlinearities with degenerate ground states; the first example seems to be given by Dancer [16]. See also Polacik [34].
Under the nondegeneracy assumption \((f_2)\), the spectrum of the linearized operator
\[
\Delta \varphi - \varphi + f'(u)\varphi = \lambda \varphi, \quad \varphi \in H^1(\mathbb{R}^3)
\] (1.8)
admits the following decompositions
\[
\lambda_1 > \lambda_2 > \cdots > \lambda_n > \lambda_{n+1} = 0 > \lambda_{n+2},
\] (1.9)
where each of the eigenfunction corresponding to the positive eigenvalue \(\lambda_j\) decays exponentially. These eigenfunctions will play important role in our secondray Liapunov-Schmidt reduction (see Section 3 below).

In the sequel, the Sobolev space \(H^1(\mathbb{R}^3)\) is endowed with the standard norm
\[
\|u\| = \left( \int |\nabla u|^2 + |u|^2 \right)^{\frac{1}{2}},
\]
which is induced by the inner product
\[
\langle \nabla u, \nabla v \rangle = \int (\nabla u \cdot \nabla v + uv).
\]
Let \(|\cdot|_p\) be the usual norm of \(L^p(\mathbb{R}^3)\). Define \(D^{1,2}(\mathbb{R}^3)\) to be the completion of \(C^\infty_0(\mathbb{R}^3)\) with respect to the Dirichlet norm \(\|u\|_D = \left( \int |\nabla u|^2 \right)^{\frac{1}{2}}\).

We will look for the solutions \((u, \Phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\). Now we reduce (1.4) to a single equation with a non-local term. By the assumption of \((K_1)\) and \(u \in L^q(\mathbb{R}^3)\) for all \(q \in [2, 6]\), then \(K u^2 \in L^{\frac{6}{5}}(\mathbb{R}^3)\) for all \(u \in H^1(\mathbb{R}^3)\), and there holds
\[
\int |K u^2 v| \leq \left( \int |K u^2|^{\frac{6}{5}} \right)^{\frac{5}{6}} \left( \int |v|^{\frac{6}{5}} \right)^{\frac{5}{6}} \leq C \left( \int |K u^2|^{\frac{6}{5}} \right)^{\frac{5}{6}} \|v\|_D, \quad \forall v \in D^{1,2}(\mathbb{R}^3).
\]

By Riesz theorem, there exists a unique \(\Phi_u \in D^{1,2}(\mathbb{R}^3)\) such that
\[
\int \nabla \Phi_u \nabla v = \int K u^2 v, \quad \forall v \in D^{1,2}(\mathbb{R}^3).
\] (1.10)
It follows that \(\Phi_u\) satisfies the Poisson equation
\[
-\Delta \Phi_u = K(x)u^2
\]
and there holds
\[
\Phi_u(x) = \int \frac{K(z)u^2(z)}{|x - z|} \, dz.
\]
Moreover, taking \(v = \Phi_u\) in (1.10), we have
\[
\|\Phi_u\|_D \leq C\|u\|^2. \quad (1.11)
\]

Substituting \(\Phi_u\) in (1.4), we obtain
\[
-\Delta u + u + \epsilon K(x)\Phi_u u = f(u), \quad u \in H^1(\mathbb{R}^3).
\] (1.12)

Now we consider the functional \(I : H^1(\mathbb{R}^3) \to \mathbb{R}\) given by
\[
I(u) = \frac{1}{2} \int (|\nabla u|^2 + u^2) + \frac{\epsilon}{4} \int K(x)\Phi_u(x)u^2 - \int F(u), \quad \forall u \in H^1(\mathbb{R}^3). \quad (1.13)
\]
Noting that
\[
\int K(x)\Phi_u(x)u^2 \leq \max_{x \in \mathbb{R}^3} |K(x)| \left( \int |\Phi_u|^6 \right)^{\frac{1}{6}} \left( \int |u|^2 \right)^{\frac{5}{6}} \leq C\|\Phi_u\|_D\|u\|^2 \leq C\|u\|^4,
\]
so \(I\) is a well defined \(C^1\) functional. If \(u \in H^1(\mathbb{R}^3)\) is a critical point of it, then the pair \((u, \Phi_u)\) is a classical solution of (1.4).

Our main result of this paper is as follows:
Theorem 1.1. Assume that \((K_1) - (K_2)\) and \((f_1) - (f_2)\) hold. Then there exists \(\epsilon_0 > 0\) such that \(0 < \epsilon < \epsilon_0\), problem \((1.4)\) has infinitely many positive solutions.

Before we close this introduction, let us outline the main idea in the proof of Theorem 1.1.

First we introduce some notations. Let \(\mu > 0\) be a real number such that \(w(x) \leq c e^{-|x|}\) for \(|x| > \mu\) and some constant \(c\) independent of \(\mu\) large. Now we define the configuration space
\[
\Omega_1 = \mathbb{R}^3, \quad \Omega_m := \{P_m = (P_1, P_2, \ldots, P_m) \in \mathbb{R}^{3m} | \min_{k \neq j} |P_k - P_j| \geq \mu\}, \forall m > 1.
\]

Let \(w\) be the nondegenerate solution of \((1.5)\) and \(m \geq 1\) be an integer. Define the sum of \(m\) spikes as
\[
w_{P_j} = w(x - P_j) \quad \text{and} \quad w_{P_m} = \sum_{j=1}^{m} w_{P_j}.
\]

Let the operator be
\[
S(u) = \Delta u - u - \epsilon K \Phi_u u + f(u).
\]

Fixing \(P_m = (P_1, \ldots, P_m)\) in \(\Omega_m\), we define the following functions as the approximate kernels:
\[
Q_{j,k} = \frac{\partial w_{P_j}}{\partial x_k} \eta_j(x), \quad \text{for} \quad j = 1, \ldots, m, k = 1, 2, 3,
\]
where \(\eta_j(x) = \eta(2|x-P_j|)\) and \(\eta(t)\) is a cut off function, such that \(\eta(t) = 1\) for \(|t| \leq 1\) and \(\eta(t) = 0\) for \(|t| \geq \frac{\mu^2}{\mu - 1}\). Note that the support of \(Q_{j,k}\) belongs to \(B_{\frac{\mu^2}{\mu - 1}}(P_j)\).

Applying \(w_{P_m}\) as the approximate solution and performing the Liapunov-Schmidt reduction, we can show that there exists a constant \(\mu_0\), such that for \(\mu \geq \mu_0\), and \(\epsilon < c_\mu\), for some constant \(c_\mu\) depending on \(\mu\) but independent of \(m\) and \(P_m\), we can find a \(\varphi_{P_m}\) such that
\[
S(w_{P_m} + \varphi_{P_m}) = \sum_{j=1}^{m} \sum_{k=1,2,3} c_{jk} Q_{j,k},
\]
and we can show that \(\varphi_{P_m}\) is \(C^1\) in \(P_m\). This is done in Section 2.

After that, for any \(m\), we define a new function
\[
\mathcal{M}(P_m) = I(w_{P_m} + \varphi_{P_m}),
\]
and we maximize \(\mathcal{M}(P_m)\) over \(\Omega_m\).

At the maximum point of \(\mathcal{M}(P_m)\), we show that \(c_{jk} = 0\) for all \(j, k\). Therefore we prove that the corresponding \(w_{P_m} + \varphi_{P_m}\) is a solution of \((1.12)\). And \((w_{P_m} + \varphi_{P_m}, \Phi_{w_{P_m} + \varphi_{P_m}})\) is a pair solution of \((1.4)\). By the arguments before, we know that there exists \(\mu_0\) large such that \(\mu \geq \mu_0\) and \(\epsilon < c_\mu\) and for any \(m\), there exists a spike solution to \((1.12)\) with \(m\) spikes in \(\Omega_m\). Considering that \(m\) is arbitrary, there exists infinitely many spikes solutions for \(\epsilon < c_\mu_0\) independent of \(m\).

There are three main difficulties in the maximization process. Firstly, we need to show that the maximum points will not go to infinity. This is guaranteed by the slow decay assumption on the potential \(K(x)\). Secondly, we have to detect the difference in the energy when the spikes move to the boundary of the configuration space. Here we use the induction method and detect the difference of the \(m\)-spikes energy and the \(m + 1\)-spikes energy. A crucial estimate is Lemma 3.2, where we
prove that the accumulated error can be controlled from step $m$ to step $m+1$. To this end, we make a secondary Liapunov-Schmidt reduction. This is done in Section 3. Thirdly, the Poisson potential brings some new difficulties which involves many complex and technical estimates.

Motivated by [3,4,27,32], our main idea is to use the Liapunov-Schmidt reduction method. We want to point out that the only assumption we need is the nondegeneracy of the bump. We have no requirements on the structure of the nonlinearity. Unlike in [3], in order to deal with Poisson potential, we have to introduce two different norms (see (2.1) and (2.2) in section 2 below). Moreover, the Poisson potential makes our problem is more difficult than the single Schrodinger equation in [3].

Our paper is organized as follows. In section 2, we carry out Lyapunov-Schmidt reduction. Then we perform a second Liapunov-Schmidt reduction in section 3. Finally, we prove our main result in section 4. We put some technical estimates in Appendix A.

2. Finite-dimensional reduction. In this section, we perform a finite-dimensional reduction.

Let $\gamma \in (0, 1)$. We denote

$$E_1(\cdot) := \sum_{j=1}^{m} e^{-\gamma |\cdot - P_j|} := \sum_{j=1}^{m} E_{1,j}(\cdot)$$

and

$$E_2(\cdot) := \left\{ \begin{array}{ll}
\sum_{j=1}^{m} e^{-\gamma |\cdot - P_j|} := \sum_{j=1}^{m} E_{2,j}(\cdot), & \cdot \in \mathbb{R}^3 \setminus \{P_1, P_2, \cdots, P_m\}, \\
+\infty, & \cdot \in \{P_1, P_2, \cdots, P_m\},
\end{array} \right.$$ (2.2)

where $P_m = (P_1, \cdots, P_m) \in \Omega_m$ and $\Omega_m$ is defined in (1.14).

Now we introduce the following two norms

$$\|u\|_* = \sup_{x \in \mathbb{R}^3} |E_1(x)^{-1} u(x)|$$

and

$$\|f\|_{**} = \sup_{x \in \mathbb{R}^3} |E_2(x)^{-1} f(x)|.$$ (2.4)

Then from [20], we have the following relations between the norms $\|\cdot\|_{L^\infty(\mathbb{R}^3)}$, $\|\cdot\|_*$ and $\|\cdot\|_{**}$:

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq C \|u\|_*$$ (2.5)

and

$$\|f\|_{L^\infty(\mathbb{R}^3)} \leq C \|f\|_{**},$$ (2.6)

where $C > 0$ independent of $\mu, m$ and $P_m \in \Omega_m$.

Given $P_m = (P_1, P_2, \cdots, P_m) \in \mathbb{R}^{3m}$ with $\min_{k \neq j} |P_k - P_j| \geq \mu$, we divide $\mathbb{R}^3$ into $m+1$ parts

$$\Lambda_j := \left\{ x \mid |x - P_j| \leq \frac{\mu}{2} \right\}, \quad \Lambda = \bigcup_{j=1}^{m} \Lambda_j \quad \text{and} \quad \Lambda^c = \mathbb{R}^3 \setminus \Lambda.$$
Now we consider
\[
L(\varphi_{P_m}) := \Delta \varphi_{P_m} - \varphi_{P_m} - \epsilon K(\Phi_{w_{P_m}} w_{P_m})' \varphi_{P_m} + f'(w_{P_m}) \varphi_{P_m}
\]
\[
= h + \sum_{j=1,2,\ldots,m,k=1,2,3} c_{j,k} Q_{j,k}, \quad \text{in } \mathbb{R}^3,
\]
\[
\int \varphi_{P_m} Q_{j,k} = 0 \quad \text{for } j = 1, 2, \ldots, m, k = 1, 2, 3,
\]
where
\[
\epsilon K(\Phi_{w_{P_m}} w_{P_m})' \varphi_{P_m} = \epsilon K \Phi_{w_{P_m}} \varphi_{P_m} + 2 \epsilon K \left( \int \frac{K(z) w_{P_m}(z) \varphi_{P_m}(z)}{|x-z|} \, dz \right) w_{P_m}(x).
\]

Firstly, we give a result which will be used later.

**Lemma 2.1.** ([20], Lemma 3.4) There exists a constant $C_3 = 6^3$ such that for any $m \in \mathbb{N}^+$ and any $P_m = (P_1, P_2, \ldots, P_m) \in \mathbb{R}^{3m}$,
\[
\sum_{l=1}^{l} \left\{ P_j \mid \frac{1}{2} \mu \leq |x-P_j| < \left( \frac{l+1}{2} \mu \right) \right\} \leq C_3(l+1)^2
\]
for all $x \in \mathbb{R}^3$ and all $l \in \mathbb{N}$. Particularly, we have
\[
\sum_{l=1}^{l} \left\{ P_j \mid 0 \leq |x-P_j| < \frac{\mu}{2} \right\} \leq C_3.
\]

In the following, $\beta$ will denote a positive constant depending on $\epsilon$ and $\gamma$ but independent of $\mu, m, P_m$ and may vary from line to line.

**Lemma 2.2.** Let $h$ with $\|h\|_{**}$ bounded and assume that $(\varphi_{P_m}, \{c_{j,k}\})$ is a solution to (2.7). Then there exist positive numbers $\mu_0$ and $C$, such that for all $0 < \epsilon < e^{-2\mu}, \mu > \mu_0$ and $P_m \in \Omega_m$, one has
\[
\|\varphi_{P_m}\|_* \leq C \|h\|_{**},
\]
where $C$ is a positive constant independent of $\mu, m$ and $P_m \in \Omega_m$.

**Proof.** We prove it by contradiction. Assume that there exists a solution $\varphi_{P_m}$ to (2.7) and $\|h\|_{**} \to 0, \|\varphi_{P_m}\|_* = 1$.

Multiplying the equation in (2.7) by $Q_{j,k}$ and integrating in $\mathbb{R}^3$, we get
\[
\int L(\varphi_{P_m}) Q_{j,k} = \int h Q_{j,k} + c_{j,k} \int Q_{j,k}^2.
\]

Considering the exponential decay at infinity of $\partial_x w$ and the definition of $Q_{j,k}$, we have
\[
\int Q_{j,k}^2 = \int \left( \frac{\partial w_{P_j}}{\partial x_k} \eta \left( \frac{2|x-P_j|}{\mu-1} \right) \right)^2 = \int \left( \frac{\partial w}{\partial x_k} \right)^2 + \int \left[ \eta^2 \left( \frac{2|x|}{\mu-1} - 1 \right) \left( \frac{\partial w}{\partial x_k} \right)^2 \right]
\]
\[
= \int \left( \frac{\partial w}{\partial x_k} \right)^2 + \int_{B_{\frac{\mu-1}{2}}} \left[ \eta^2 \left( \frac{2|x|}{\mu-1} - 1 \right) \left( \frac{\partial w}{\partial x_k} \right)^2 \right]
\]
\[
\leq \int \left( \frac{\partial w}{\partial x_k} \right)^2 + C \int_{\mathbb{R}^3} e^{-2r} \, dr = \int \left( \frac{\partial w}{\partial x_k} \right)^2 + O(e^{-\mu}), \quad \text{as } \mu \to \infty.
\]
On the other hand, applying (2.6), we can check that

\[
\left| \int hQ_{j,k} \right| \leq C\|h\|_{\ast\ast} \int \left| \frac{\partial w_{P_j}}{\partial x_k} \right| \eta \left( \frac{2|x - P_j|}{\mu - 1} \right)
\]

\[
\leq C\|h\|_{\ast\ast} \int_{B_{\frac{\mu}{2}}(P_j)} e^{-|x - P_j|} \leq C\|h\|_{\ast\ast} \int_0^{\frac{\mu}{2}} s^2 e^{-s} ds \leq C\|h\|_{\ast\ast}.
\]

(2.13)

Here and in what follows, \(C\) stands for a positive constant independent of \(\epsilon\) and \(\mu\), as \(\epsilon \to 0\).

Now if we write \(\tilde{Q}_{j,k} = \frac{\partial w_{P_j}}{\partial x_k}\), then we have

\[
\int L(\varphi_{P_m}) Q_{j,k} = \int L(Q_{j,k}) \varphi_{P_m}
\]

\[
\leq \int_{B_{\frac{\mu}{2}}(P_j)} (\Delta \tilde{Q}_{j,k} - \tilde{Q}_{j,k} + f'(w_{P_j}) \tilde{Q}_{j,k}) \eta_j \varphi_{P_m}
\]

\[
+ \int_{B_{\frac{\mu}{2}}(P_j) \setminus B_{\frac{\mu}{2} - 1}(P_j)} \varphi_{P_m} (\tilde{Q}_{j,k} \Delta \eta_j + 2\nabla \eta_j \nabla \tilde{Q}_{j,k})
\]

\[
+ \int_{B_{\frac{\mu}{2}}(P_j) \setminus B_{\frac{\mu}{2} - 1}(P_j)} (f'(w_{P_m}) - f'(w_{P_j})) \varphi_{P_m} \tilde{Q}_{j,k} \eta_j
\]

\[
- \epsilon \int_{B_{\frac{\mu}{2}}(P_j) \setminus B_{\frac{\mu}{2} - 1}(P_j)} \mathcal{K}(\Phi_{w_{P_m}} w_{P_m})' \varphi_{P_m} \tilde{Q}_{j,k} \eta_j.
\]

(2.14)

Since

\[
\Delta \tilde{Q}_{j,k} - \tilde{Q}_{j,k} + f'(w_{P_j}) \tilde{Q}_{j,k} = 0,
\]

we have

\[
\int_{B_{\frac{\mu}{2}}(P_j) \setminus B_{\frac{\mu}{2} - 1}(P_j)} (\Delta \tilde{Q}_{j,k} - \tilde{Q}_{j,k} + f'(w_{P_j}) \tilde{Q}_{j,k}) \eta_j \varphi_{P_m} = 0.
\]

(2.15)

Moreover, by Lemma A.1 we have

\[
\left| \int_{B_{\frac{\mu}{2}}(P_j) \setminus B_{\frac{\mu}{2} - 1}(P_j)} \varphi_{P_m} (\tilde{Q}_{j,k} \Delta \eta_j + 2\nabla \eta_j \nabla \tilde{Q}_{j,k}) \right|
\]

\[
\leq C\|\varphi_{P_m}\|_{\ast} \int_{B_{\frac{\mu}{2}}(P_j) \setminus B_{\frac{\mu}{2} - 1}(P_j)} e^{-(1+\gamma)s} ds \leq C e^{-(1+\beta)\frac{\mu}{2}} \|\varphi_{P_m}\|_{\ast},
\]

(2.16)

for some \(\beta > 0\).

Observing that in \(B_{\frac{\mu}{2}}(P_j) \setminus B_{\frac{\mu}{2} - 1}(P_j)\) the following holds

\[
|f'(w_{P_m}) - f'(w_{P_j})| \leq C \sum_{k \neq j} w_{P_k} \delta,
\]

where \(\delta\) is a positive constant independent of \(\epsilon\) and \(\mu\), as \(\epsilon \to 0\).
by Lemma A.1 we have
\[
\left| \int_{B_{\rho/2}(P_j)} (f'(w_{P_m}) - f'(w_{P_j})) \varphi_{P_m, \tilde{Q}_j, \kappa_j}) \right|
\leq C\|\varphi_{P_m}\| \int_{B_{\rho/2}(P_j)} \left| \sum_{k \neq j} w_{P_k} \sum_{l=1}^m e^{-\gamma|x-P_l|} \right|
\leq C\|\varphi_{P_m}\| \int_{B_{\rho/2}(P_j)} e^{-\frac{\mu}{2} \left( e^{-\gamma|x-P_j|} + \sum_{l \neq j} e^{-\gamma|x-P_l|} \right)}
\leq C\|\varphi_{P_m}\| \epsilon^{-\frac{\mu}{2}} \left( \int_0^{\frac{\mu}{2}} s^2 e^{-\gamma s} ds + \epsilon^{-\frac{\mu}{2}} \int_0^{\frac{\mu}{2}} s^2 ds \right)
\leq Ce^{-\beta \frac{\mu}{2}} \|\varphi_{P_m}\| \epsilon
\]
for some $\beta > 0$. Finally, by the assumption of $\epsilon$ we have
\[
\left| -\epsilon \int_{B_{\rho/2}(P_j)} K(\Phi_{\rho/2}(P_j)) \varphi_{P_m, \tilde{Q}_j, \kappa_j}) \right|
\leq Ce^{-2\mu} \int_{B_{\rho/2}(P_j)} K(\Phi_{\rho/2}(P_j)) \varphi_{P_m, \tilde{Q}_j, \kappa_j}) + Ce^{-2\mu} \int_{B_{\rho/2}(P_j)} K \left( \int \frac{Kw_{P_m} \varphi_{P_m} dz}{|x-z|} \right) w_{P_m}
\leq Ce^{-\beta \frac{\mu}{2}} \|\varphi_{P_m}\| \epsilon
\]
for some $\beta > 0$, where we use the following estimates whose proofs we put in Appendix A
\[
\left| \int_{B_{\rho/2}(P_j)} K \Phi_{\rho/2}(P_j) \varphi_{P_m} \right| \leq C\|\varphi_{P_m}\| \epsilon
\]
\[
\left| \int_{B_{\rho/2}(P_j)} K \left( \int \frac{Kw_{P_m} \varphi_{P_m} dz}{|x-z|} \right) w_{P_m} \right| \leq C\|\varphi_{P_m}\| \epsilon
\]
It follows from (2.11) to (2.18) that
\[
|c_{jk}| \leq C(e^{-\beta \frac{\mu}{2}} \|\varphi_{P_m}\| \epsilon + \|h\|_{\ast \ast}).
\]
Now let $\vartheta \in (0, 1)$. It is easy to check that the function $E_2$ (defined in (2.2)) satisfies
\[
LE_2 \leq \frac{1}{2}(\vartheta^2 - 1)E_2, \ \text{in} \ \mathbb{R}^3 \setminus \bigcup_{j=1}^m B_{\tilde{\mu}}(P_j)
\]
provided $\tilde{\mu}$ is large enough but independent of $\mu$.
Indeed, by Lemma 2.1 we have
\[
w_{P_m} \leq \sum_{|x-P_j| < \frac{\mu}{2}} w(x-P_j) + \sum_{l=1}^\infty \sum_{\frac{\mu}{2} \leq |x-P_j| < \frac{\mu}{2} + \frac{1}{l}} w(x-P_j)
\leq Cw(\tilde{\mu}) + C \sum_{l=1}^\infty l^2 e^{-\frac{\mu}{2}} \leq Cw(\tilde{\mu}).
\]
Then
\[
|f'(w_{P_m})| \leq C(w_{P_m})^\delta \leq Cw(\tilde{\mu}) \leq \frac{1-\vartheta^2}{4}, \ \text{in} \ \mathbb{R}^3 \setminus \bigcup_{j=1}^m B_{\tilde{\mu}}(P_j).
\]
Noting that here \( x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{m} B_{\bar{\mu}}(P_j) \), then
\[
E_2(x) = \sum_{j=1}^{m} \frac{e^{-\gamma|x-P_j|}}{|x-P_j|} := \sum_{j=1}^{m} E_{2,j}(x).
\]
Moreover, it follows from Lemmas A.1 to A.5 that
\[
|eK(x)\Phi_{w_{m}}| \leq Ce^{-2\mu} \sum_{l=1}^{m} \left[ \sum_{i=1}^{2} \frac{|x-P_l|^i}{|x-P_l|^i} (e^{-\gamma|x-P_l|} + 1) + \frac{1}{|x-P_l|} \right] \leq Ce^{-2\mu} \left[ \frac{1}{\mu} + 1 + \bar{\mu} \right] \sum_{l=1}^{m} e^{-\gamma|x-P_l|}
\]
and
\[
|eK(x) \int \frac{K(z) w_{m}(z) E_2(z)}{|x-z|} dz w_{m}| \leq Ce \int \frac{(\sum_{j=1}^{m} e^{-\gamma|z-P_j|}) (\sum_{l=1}^{m} e^{-\gamma|x-P_l|}) |z-P_k|^{-1}}{|x-z|} dz \sum_{k=1}^{m} e^{-\gamma|x-P_k|}
\]
where
\[
|eK(x)(\Phi_{w_{m}} w_{m})' E_2| \leq \frac{1 - \vartheta^2}{8} E_2, \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{j=1}^{m} B_{\bar{\mu}}(P_j),
\]
which implies that
\[
|eK(x)(\Phi_{w_{m}} w_{m})' E_2| \leq \frac{1 - \vartheta^2}{4} E_2, \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{j=1}^{m} B_{\bar{\mu}}(P_j).
\]

From (2.23) and (2.24), by simple computation we have
\[
-LE_2 = -[\Delta E_2 - E_2 - eK(x)(\Phi_{w_{m}} w_{m})' E_2 + f'(w_{m}) E_2]
= \sum_{j=1}^{m} \left[ -\gamma^2 + \gamma \frac{N-3}{|x-P_j|} + \frac{N-3}{|x-P_j|^2} + \epsilon K(x)(\Phi_{w_{m}} w_{m})' - f'(w_{m}) \right] E_2
\geq \frac{1}{2} (1 - \vartheta^2) E_2, \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{j=1}^{m} B_{\bar{\mu}}(P_j),
\]
which yields that (2.22) is true.

Hence the function \( E_2 \) can be used as a barrier to prove \( \varphi_{m}(x) \in C(\mathbb{R}^3 \setminus \bigcup_{j=1}^{m} B_{\bar{\mu}}(P_j)) \). Then applying the maximum principle for the linear operator \( L \) in \( \mathbb{R}^3 \setminus \bigcup_{j=1}^{m} B_{\bar{\mu}}(P_j) \), we have
\[
|\varphi_{m}(x)| \leq C \left( ||L \varphi_{m}||_{**} + \sup_{j} \| \varphi_{m} \|_{L^\infty(\partial B_{\bar{\mu}}(P_j))} E_2(x) \right)
\leq C \left( ||L \varphi_{m}||_{**} + \sup_{j} \| \varphi_{m} \|_{L^\infty(\partial B_{\bar{\mu}}(P_j))} E_1(x) \right),
\]
which implies that
\[
\| \varphi_{P_m}(x) \|_* \leq (\| L \varphi_{P_m} \|_{**} + \sup_j \| \varphi_{P_m} \|_{L^\infty(\partial B_\mu(P_j))}).
\] (2.25)
for all \( x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^m B_\mu(P_j) \).

Now we prove it by contradiction. We assume that there exist a sequence of \( \epsilon \) tending to 0, \( \mu \) tending to \( \infty \) and a sequence of solutions of (2.7) for which (2.10) is not true. The problem being linear, we can reduce to the case where we have a sequence \( \epsilon^{(n)} \) tending to 0, \( \mu^{(n)} \) tending to \( \infty \) and sequences \( h^{(n)}, \varphi^{(n)}, \{ c^{(n)}_{j,k} \} \) such that
\[
\| h^{(n)} \|_{**} \to 0, \quad \text{and} \quad \| \varphi^{(n)}_{P_m} \|_* = 1.
\]
By (2.21), we have
\[
\left\| \sum_{j,k} c^{(n)}_{j,k} Q_{j,k} \right\|_{**} \to 0.
\]
Then (2.25) implies that there exists \( P_j^{(n)} \in \Omega_m \) such that
\[
\| \varphi^{(n)}_{P_m} \|_{L^\infty(\partial B_{\mu^{(n)}}(P_j^{(n)}))} \geq C
\]
(2.26)
for some fixed constant \( C > 0 \). Applying elliptic estimates together with Ascoli-Arzela’s theorem, we can find a sequence \( P_j^{(n)} \) and we can extract, from the sequence \( \varphi^{(n)}(\cdot - P_j^{(n)}) \) a subsequence which will converge to \( \varphi_\infty \) a solution of
\[(\Delta - 1 + f'(w)) \varphi_\infty = 0, \quad \text{in} \quad \mathbb{R}^3,
\]
which is bounded by a constant times \( e^{-\gamma|x|} \), with \( \gamma > 0 \). Moreover, recall that \( \varphi^{(n)}_{P_m} \) satisfies the orthogonality conditions in (2.7). Hence the limit function \( \varphi_\infty \) also satisfies
\[
\int \varphi_\infty \frac{\partial w}{\partial x_j} = 0, \quad j = 1, 2, 3.
\]
Since \( w \) is non-degenerate, we have that \( \varphi_\infty \equiv 0 \) which contradicts to (2.26).

From Lemma 2.2, we can obtain the following result.

**Proposition 2.3.** Then there exist positive numbers \( \gamma \in (0, 1), \mu_0 > 0 \) and \( C > 0 \), such that for all \( 0 < \epsilon < e^{-2\mu}, \mu > \mu_0 \) and for any given \( h \) with \( \| h \|_{**} \) norm bounded, there is a unique solution \( (\varphi_{P_m}, \{ c_{j,k} \}) \) to problem (2.7). Moreover,
\[
\| \varphi_{P_m} \|_* \leq C \| h \|_{**}.
\] (2.27)

**Proof.** Here we consider the space
\[
\mathcal{H} = \left\{ u \in H^1(\mathbb{R}^3) : \int u Q_{j,k} = 0, P_m \in \Omega_m \right\}.
\]
Observe that problem (2.7) in \( \varphi_{P_m} \) is rewritten as
\[
\varphi_{P_m} + K(\varphi_{P_m}) = \tilde{h}, \quad \text{in} \quad \mathcal{H},
\]
where \( \tilde{h} \) is defined by duality and \( K : \mathcal{H} \to \mathcal{H} \) is a linear compact operator. By Fredholm’s alternative, we know that the equation (2.28) has a unique solution for each \( \tilde{h} \) is equivalent to showing that the equation has a unique solution for \( \tilde{h} = 0 \), which in turn follows from Lemma 2.2. The estimate (2.27) follows from directly from (2.10) in Lemma 2.2. The proof is completed. □
In the sequel, if \( \varphi_{P_m} \) is the unique solution given by Proposition 2.3, we denote
\[
\varphi_{P_m} = A(h).
\]
(2.29)

By (2.27), we have
\[
\|A(h)\|_* \leq C\|h\|_{**}.
\]
(2.30)

Now, we consider
\[
\begin{cases}
\Delta(w_{P_m} + \varphi_{P_m}) - (w_{P_m} + \varphi_{P_m}) - \epsilon K \Phi(w_{P_m} + \varphi_{P_m})(w_{P_m} + \varphi_{P_m}) \\
+ f(w_{P_m} + \varphi_{P_m}) = \sum_{j=1,2,\ldots,m,k=1,2,3} c_{j,k} Q_{j,k}, \quad \text{in } \mathbb{R}^3,
\end{cases}
\]
(2.31)

Furthermore, Proposition 2.4. Given \( 0 < \gamma < 1 \). There exist positive numbers \( \mu_0, C \) and \( \beta > 0 \)(independent of \( \mu, m \) and \( P_m \in \Omega_m \)) such that for all \( \mu \geq \mu_0 \), and for any \( P_m \in \Omega_m, \epsilon < e^{-2\mu}, \) there is a unique solution \( (\varphi_{P_m}, \{c_{j,k}\}) \) to problem (2.31). Furthermore, \( \varphi_{P_m} \) is \( C^1 \) in \( P_m \) and we have
\[
\|\varphi_{P_m}\|_* \leq Ce^{-\beta \mu}, \quad |c_{j,k}| \leq Ce^{-\beta \mu}.
\]
(2.32)

Note that the first equation in (2.31) can be rewritten as
\[
L(\varphi_{P_m}) = -\bar{S}(w_{P_m}) + N(\varphi_{P_m}) + \sum_{j=1,2,\ldots,m,k=1,2,3} c_{j,k} Q_{j,k},
\]
(2.33)

where \( \bar{S}(\cdot) \) is defined as (1.19),

\[
L(\varphi_{P_m}) = \Delta \varphi_{P_m} - \varphi_{P_m} - \epsilon K (\Phi w_{P_m} w_{P_m})' \varphi_{P_m} + f'(w_{P_m}) \varphi_{P_m}
\]
(2.34)

and
\[
N(\varphi_{P_m}) = -[f(w_{P_m} + \varphi_{P_m}) - f(w_{P_m}) - f'(w_{P_m}) \varphi_{P_m}]
\]
\[
+ \epsilon K [\Phi(w_{P_m} + \varphi_{P_m})(w_{P_m} + \varphi_{P_m}) - \Phi w_{P_m} w_{P_m} - (\Phi w_{P_m} w_{P_m})' \varphi_{P_m}].
\]
(2.35)

In order to use the contraction mapping theorem to prove that (2.33) is uniquely solvable in the set that \( \|\varphi_{P_m}\|_* \) is small, we need to estimate \( \|\bar{S}(w_{P_m})\|_{**} \) and \( \|\bar{N}(\varphi_{P_m})\|_{**} \) respectively.

Lemma 2.5. Given \( 0 < \gamma < 1 \). For \( \mu \) large enough, and any \( P_m \in \Omega_m, \epsilon < e^{-2\mu}, \) we have
\[
\|\bar{S}(w_{P_m})\|_{**} \leq Ce^{-\beta \mu}
\]
(2.36)

for some constants \( \beta > 0 \) and \( C \) independent of \( \mu, m \) and \( P_m \).

Proof. Note that
\[
\bar{S}(w_{P_m}) = -\epsilon K \Phi w_{P_m} w_{P_m} + f(w_{P_m}) - \sum_{j=1}^{m} f(w_j).
\]
(2.37)

Similar to (2.5) and (2.6) of section 2.1 in [3], we can prove
\[
\left|f(w_{P_m}) - \sum_{j=1}^{m} f(w_j)\right| \leq Ce^{-\beta \mu} E_2(x),
\]
(2.38)

for a proper choice of \( \beta > 0 \).
Moreover, by the assumption of ε, Lemmas A.1 to A.4, we can prove that

\[ |eK\Phi_{w_{p_m}} w_{p_m}| \leq C e^{-\beta \mu} E_2(x), \tag{2.39} \]

for some \( \beta > 0 \). In fact, on one hand, fix \( j \in \{1, 2, \ldots, m\} \) and consider the region \( \Lambda_j \). In this region, if \( x \in \mathbb{R}^3 \setminus \{P_1, P_2, \ldots, P_m\} \), we have

\[
\left| eK\Phi_{w_{p_m}} w_{p_m} \right| \\
\leq C e^{-2\mu} \left| K\Phi_{w_{p_m}} w_{p_m} \right| \\
\leq C e^{-2\mu} \left\{ \sum_{i=1}^{m} \left[ \sum_{i=1}^{2} \frac{|x-P_i|^i}{|x-P_i|^j} \left( e^{-\frac{|x-P_i|^j}{2}} + 1 \right) + \frac{1}{|x-P_i|^j} \right] \sum_{k=1}^{m} e^{-|x-P_k|} \right\} \\
\leq C e^{-2\mu} \left[ \sum_{i=1}^{2} \frac{|x-P_i|^i}{|x-P_i|^j} \left( e^{-\frac{|x-P_i|^j}{2}} + 1 \right) + \frac{1}{|x-P_i|^j} \right] \\
+ \sum_{i=1}^{2} \left( \frac{1}{\mu} + 1 + \mu + \mu^2 \right) e^{-\frac{|x-P_i|^j}{2}} \left( e^{-\frac{|x-P_i|^j}{2}} + 1 \right) + \frac{1}{\mu} e^{-|x-P_i|} \\
\leq C e^{-\beta \mu} E_{2,j}(x),
\]

where we use the fact that

\[ |x-P_i| \geq |P_j - P_i| - |x-P_j| \geq \frac{\mu}{2}, \text{ for any } l \neq j. \]

And if \( x = P_{j_0}, j_0 \in \{1, 2, \ldots, m\} \), then we also have

\[
\left| eK\Phi_{w_{p_m}} w_{p_m} \right| \\
\leq C e^{-2\mu} \sum_{l \neq j_0} \left[ \sum_{i=1}^{2} \frac{|P_{j_0} - P_i|^i}{|P_{j_0} - P_i|^j} \left( e^{-\frac{|P_{j_0} - P_i|^j}{2}} + 1 \right) + \frac{1}{|P_{j_0} - P_i|^j} \right] \sum_{k \neq j_0} e^{-|P_{j_0} - P_k|} \\
\leq C e^{-2\mu} \left[ \sum_{i=1}^{2} \frac{|P_{j_0} - P_i|^i}{|P_{j_0} - P_i|^j} \left( e^{-\frac{|P_{j_0} - P_i|^j}{2}} + 1 \right) + \frac{1}{|P_{j_0} - P_i|^j} \right] \\
+ \sum_{l \neq j_0} \left( \frac{1}{\mu} + 1 + \mu + \mu^2 \right) e^{-\frac{|P_{j_0} - P_i|^j}{2}} \left( e^{-\frac{|P_{j_0} - P_i|^j}{2}} + 1 \right) + \frac{1}{\mu} e^{-|P_{j_0} - P_i|} \right] \\
\times \left( e^{-|P_{j_0} - P_i|} + \sum_{k \neq j_0} e^{-|P_{j_0} - P_k|} \right) \]
\[ C^{-2\mu} \left[ \sum_{i=1}^{2} \left| P_{j_i} - P_{j} \right|^i e^{-\frac{|P_{j_0} - P_j|}{2}} + 1 \right] e^{-|P_{j_0} - P_j|} \]
\[ + C e^{-2\mu} \left( \sum_{i=1}^{2} \left| P_{j_0} - P_{j} \right|^i e^{-\frac{|P_{j_0} - P_j|}{2}} + 1 \right) e^{-\mu} \]
\[ + C e^{-2\mu} \left( \frac{1}{\mu} + 1 + \mu^2 \right) e^{-\frac{\mu}{2}} (e^{-\frac{\mu}{2}} + 1) + \frac{1}{\mu} \right] e^{-|P_{j_0} - P_j|} \]
\[ + C e^{-2\mu} \left( \frac{1}{\mu} + 1 + \mu^2 \right) e^{-\frac{\mu}{2}} (e^{-\frac{\mu}{2}} + 1) + \frac{1}{\mu} \right] e^{-\mu} \]
\[ \leq C e^{-\beta \mu} E_{2,j}(P_{j_0}). \]

On the other hand, considering the region \( \Lambda^C \), if \( x \in \mathbb{R}^3 \setminus \{ P_1, P_2, \ldots, P_m \} \), from Lemmas A.1 to A.4 we have
\[
\left| eK_{\Phi_{wP_m}} P_{m} \right| \leq C e^{-2\mu} \sum_{i=1}^{m} \left[ \sum_{i=1}^{2} \left| P_{j_i} - P_{j} \right|^i e^{-\frac{|P_{j_0} - P_j|}{2}} + 1 \right] e^{-|P_{j_0} - P_j|} \]
\[ \leq C e^{-\mu} \sum_{i=1}^{m} \left[ \sum_{i=1}^{2} \left| P_{j_i} - P_{j_0} \right|^i e^{-\frac{|P_{j_0} - P_j|}{2}} + 1 \right] \sum_{k=1}^{m} e^{-|P_{j_0} - P_k|} \]
\[ \leq C e^{-\beta \mu} E_2(x). \]

If \( x = P_{j_0}, j_0 \in \{1, 2, \ldots, m\} \), then we also have
\[
\left| eK_{\Phi_{wP_m}} P_{m} \right| \leq C e^{-2\mu} \sum_{i \neq j_0}^{m} \left[ \sum_{i=1}^{2} \left| P_{j_i} - P_{j_0} \right|^i e^{-\frac{|P_{j_0} - P_j|}{2}} + 1 \right] \sum_{k \neq j_0} e^{-|P_{j_0} - P_k|} \]
\[ \leq C e^{-\mu} \sum_{i \neq j_0} \left[ \sum_{i=1}^{2} \left| P_{j_i} - P_{j_0} \right|^i e^{-\frac{|P_{j_0} - P_j|}{2}} + 1 \right] \sum_{k \neq j_0} e^{-|P_{j_0} - P_k|} \]
\[ \leq C e^{-\beta \mu} E_2(P_{j_0}). \]

It follows from (2.38) and (2.39) that
\[
\| S(w_{P_m}) \|_{\ast \ast} \leq C e^{-\beta \mu} \] (2.40)
for some \( \beta > 0 \) independent of \( \mu, m \) and \( P_m \).

**Lemma 2.6.** For any \( P_m \in \Omega_m \) satisfying \( \| \varphi_{P_m} \|_{\ast} + \| \varphi^1_{P_m} \|_{\ast} + \| \varphi^2_{P_m} \|_{\ast} \leq 1 \), we have
\[
\| \mathcal{N}(\varphi_{P_m}) \|_{\ast \ast} \leq C \| \varphi_{P_m} \|^{1+\delta} \] (2.41)
and
\[
\| \mathcal{N}(\varphi^1_{P_m}) - \mathcal{N}(\varphi^2_{P_m}) \|_{\ast \ast} \leq C (\| \varphi^1_{P_m} \|^{\delta} + \| \varphi^2_{P_m} \|^{\delta}) \| \varphi_{P_m} - \varphi^2_{P_m} \|_{\ast}. \] (2.42)
Proof. By direct computation and applying the mean-value theorem, we have

\[ N(\varphi_{P_m}) = -(f'(w_{P_m} + \varphi_{P_m})\varphi_{P_m} - f'(w_{P_m})\varphi_{P_m}) + \epsilon K \left( \varphi_{P_m} - 2 \int \frac{Kw_{P_m} \varphi_{P_m} d\gamma}{|x-z|} \right) \varphi_{P_m} \]  

(2.43)

\[ =: N_1(\varphi_{P_m}) + N_2(\varphi_{P_m}). \]

Since \( f' \) is Hölder continuous with the exponent \( \delta \), we deduce

\[ |N_1(\varphi_{P_m})| \leq C|\varphi_{P_m}|^{1+\delta} \leq C\|\varphi_{P_m}\|_{1+\delta} \left( \sum_{j=1}^{m} e^{-\gamma|x-x_j|} \right)^{1+\delta} \]  

(2.44)

\[ \leq C\|\varphi_{P_m}\|_{1+\delta} E_2(x). \]

Similar to (2.39), if \( x \in \mathbb{R}^3\setminus\{P_1, P_2, \cdots, P_m\} \), applying Lemmas A.1 to A.4 we have

\[ |\epsilon K \varphi_{P_m} \varphi_{P_m}| \leq C\|\varphi_{P_m}\|^2 \left( \sum_{j=1}^{m} \frac{e^{-\gamma|x-x_j|}}{|x-z|} \right) \sum_{k=1}^{m} e^{-|x-x_k|} \]

\[ \leq C\|\varphi_{P_m}\|^2 \left( \sum_{j=1}^{m} \frac{e^{-\gamma|x-x_j|}}{|x-z|} \right) \sum_{k=1}^{m} e^{-|x-x_k|} \]

\[ \leq C\|\varphi_{P_m}\|^2 E_2(x), \]

and

\[ |2 \int \frac{Kw_{P_m} \varphi_{P_m} d\gamma}{|x-z|} \varphi_{P_m}| \leq C\|\varphi_{P_m}\|^2 \left( \sum_{j=1}^{m} \frac{e^{-\gamma|x-x_j|}}{|x-z|} \right) \sum_{k=1}^{m} e^{-|x-x_k|} \]

\[ \leq C\|\varphi_{P_m}\|^2 \left( \sum_{j=1}^{m} \frac{e^{-\gamma|x-x_j|}}{|x-z|} \right) \sum_{k=1}^{m} e^{-|x-x_k|} \]

\[ \leq C\|\varphi_{P_m}\|^2 E_2(x), \]

which implies that

\[ |N_2(\varphi_{P_m})| \leq C E_2(x)(\|\varphi_{P_m}\|^2 + \|\varphi_{P_m}\|_{1+\delta}^3) \leq C\|\varphi_{P_m}\|^2 E_2(x). \]  

(2.45)

It follows from (2.43), (2.44) and (2.45) that

\[ \|N(\varphi_{P_m})\|_{**} \leq \|N_1(\varphi_{P_m})\|_{**} + \|N_2(\varphi_{P_m})\|_{**} \]
which implies that

\[ \|\varphi_{p_m}\|_1^{1+\delta} + C\|\varphi_{p_m}\|_2^2 \leq C\|\varphi_{p_m}\|_1^{1+\delta}. \]

Also, similar to (2.39), \(x = P_{j_0}, j_0 \in \{1, 2, \cdots, m\}\), applying Lemmas A.1 to A.4 we have

\[
|eK\Phi_{\varphi_{p_m}w_{p_m}}| \\
\leq C\|\varphi_{p_m}\|_2^2 \left( \int \frac{ \sum_{j=1}^{m} e^{-\gamma|z-P_j|} (\sum_{k=1}^{m} e^{-\gamma|z-P_k|}) }{|P_{j_0} - z|} dz \right) \sum_{k \neq j_0} e^{-\gamma|P_{j_0} - P_k|} \\
\leq C\|\varphi_{p_m}\|_2^2 \sum_{k \neq j_0} \left[ \sum_{i=1}^{2} \frac{|P_{j_0} - P_i|^{|}\sum_{k=1}^{m} e^{-\gamma|z-P_k|} + 1}{|P_{j_0} - P_i|} \right] \sum_{k \neq j_0} e^{-\gamma|P_{j_0} - P_k|} \\
\leq C\|\varphi_{p_m}\|_2^2 E_2(P_{j_0}),
\]

and

\[
|2\left( \int \frac{Kw_{p_m}\varphi_{p_m}w_{p_m}}{|P_{j_0} - z|} dz \right) \varphi_{p_m}| \\
\leq C\|\varphi_{p_m}\|_2^2 \left( \int \frac{ \sum_{j=1}^{m} e^{-\gamma|z-P_j|} (\sum_{k=1}^{m} e^{-\gamma|z-P_k|}) }{|P_{j_0} - z|} dz \right) \sum_{k \neq j_0} e^{-\gamma|P_{j_0} - P_k|} \\
\leq C\|\varphi_{p_m}\|_2^2 \sum_{k \neq j_0} \left[ \sum_{i=1}^{2} \frac{|P_{j_0} - P_i|^{|}\sum_{k=1}^{m} e^{-\gamma|z-P_k|} + 1}{|P_{j_0} - P_i|} \right] \sum_{k \neq j_0} e^{-\gamma|P_{j_0} - P_k|} \\
\leq C\|\varphi_{p_m}\|_2^2 E_2(P_{j_0}),
\]

which implies that

\[
|N_2(\varphi_{p_m})| \leq C E_2(P_{j_0})\|\varphi_{p_m}\|_2^2 + \|\varphi_{p_m}\|_1^3 \leq C\|\varphi_{p_m}\|_2^2 E_2(P_{j_0}). \tag{2.46}
\]

It follows from (2.43), (2.45) and (2.46) that

\[
\|N(\varphi_{p_m})\|_{**} \leq \|N_1(\varphi_{p_m})\|_{**} + \|N_2(\varphi_{p_m})\|_{**} \leq C\|\varphi_{p_m}\|_1^{1+\delta} + C\|\varphi_{p_m}\|_2^2 \leq C\|\varphi_{p_m}\|_1^{1+\delta}.
\]

By direct computation, we have

\[
\|N(\varphi_{p_m}^1) - N(\varphi_{p_m}^2)\|_{**} \\
= \|f'(w_{p_m} + \varphi_{p_m}^1) - f'(w_{p_m} + \varphi_{p_m}^2)\|_{**} + [f'(w_{p_m} + \varphi_{p_m}^1) - f'(w_{p_m} + \varphi_{p_m}^2)]\varphi_{p_m}^2.
\]
\[-\epsilon K (\Phi \varphi^1_m - \Phi \varphi^1_m) w \varphi^1_m - \epsilon K (\Phi \varphi^2_m - \Phi \varphi^1_m) \varphi^1_m - \epsilon K (\Phi \varphi^2_m - \Phi \varphi^1_m) \varphi^1_m \]
\[-2 \epsilon K \left( \int \frac{K w \varphi^1_m}{|x-z|} \varphi^1_m \right) dx dz \]
\[-2 \epsilon K \left( \int \frac{K w \varphi^1_m}{|x-z|} \varphi^1_m \right) \parallel \varphi^1_m - \varphi^1_m \parallel_{*} \]
\[\leq C (\parallel \varphi^1_m \parallel_{*}^2 + \parallel \varphi^2_m \parallel_{*}^3) \parallel \varphi^1_m - \varphi^1_m \parallel_{*}, \quad (2.47)\]

since just by the same argument as (2.45) it follows from direct computations and Lemmas A.1 to A.4 that
\[\left| [f'(w \varphi^1_m + \varphi^1_m)](\varphi^2_m - \varphi^1_m) \right| \]
\[\leq C (\parallel \varphi^1_m \parallel_{*}^4 + \parallel \varphi^2_m \parallel_{*}^3) \parallel \varphi^2_m - \varphi^1_m \parallel_{*}, \]
\[\leq C \epsilon K (\Phi \varphi^2_m - \Phi \varphi^1_m) w \varphi^1_m \]
\[\leq C e^{-2\mu} \int \frac{(|\varphi^1_m| + |\varphi^2_m|)|\varphi^2_m - \varphi^1_m|}{|x-z|} dz dx \]
\[\leq C e^{-2\mu} (\parallel \varphi^1_m \parallel_{*} + \parallel \varphi^2_m \parallel_{*}) \parallel \varphi^2_m - \varphi^1_m \parallel_{*} \]
\[\times \left( \int \frac{\left( \sum_{j=1}^{m} e^{-\gamma|z-P_j|} \right) \left( \sum_{k=1}^{m} e^{-\gamma|z-P_k|} \right)}{|x-z|} \right) \sum_{l=1}^{m} e^{-|x-P_l|} \]
\[\leq C E_2 (x) (\parallel \varphi^1_m \parallel_{*} + \parallel \varphi^2_m \parallel_{*}) \parallel \varphi^2_m - \varphi^1_m \parallel_{*}, \]
\[\left| -\epsilon K (\Phi \varphi^2_m - \Phi \varphi^1_m) \right| \varphi^1_m \]
\[\leq C e^{-2\mu} \left( \parallel \varphi^1_m \parallel_{*} + \parallel \varphi^2_m \parallel_{*} \right) \parallel \varphi^2_m - \varphi^1_m \parallel_{*} \]
\[\times \left( \int \frac{\left( \sum_{j=1}^{m} e^{-\gamma|z-P_j|} \right) \left( \sum_{k=1}^{m} e^{-\gamma|z-P_k|} \right)}{|x-z|} \right) \sum_{l=1}^{m} e^{-|x-P_l|} \]
\[\leq C E_2 (x) (\parallel \varphi^1_m \parallel_{*} + \parallel \varphi^2_m \parallel_{*}) \parallel \varphi^2_m - \varphi^1_m \parallel_{*}, \]
\[\left| -\epsilon K (\Phi \varphi^2_m - \Phi \varphi^1_m) \right| \varphi^1_m \]
\[\leq C e^{-2\mu} \left( \parallel \varphi^1_m \parallel_{*} + \parallel \varphi^2_m \parallel_{*} \right) \parallel \varphi^2_m - \varphi^1_m \parallel_{*} \]
\[\times \left( \int \frac{\left( \sum_{j=1}^{m} e^{-\gamma|z-P_j|} \right) \left( \sum_{k=1}^{m} e^{-\gamma|z-P_k|} \right)}{|x-z|} \right) \sum_{l=1}^{m} e^{-|x-P_l|} \]
Proof of Proposition 2.4. We will use the contraction theorem to prove it. Observe
that
Proof of Proposition 2.4. We will use the contraction theorem to prove it. Observe
that
that
Proof of Proposition 2.4. We will use the contraction theorem to prove it. Observe
that
Proof of Proposition 2.4. We will use the contraction theorem to prove it. Observe
that
Proof of Proposition 2.4. We will use the contraction theorem to prove it. Observe
that
Proof of Proposition 2.4. We will use the contraction theorem to prove it. Observe
that
Hence by the contraction theorem there exists a unique $\varphi_{P_m} \in B$ such that (2.48) holds. So

$$\|\varphi_{P_m}\|_* = \|T(\varphi_{P_m})\|_* \leq Ce^{-\beta \mu}.$$ 

Combining (2.21), (2.36) and (2.41) we have

$$|c_{j,k}| \leq C(e^{-\beta \mu}) \|\varphi_{P_m}\|_* + \|S(w_{P_m})\|_* + \|N(\varphi_{P_m})\|_* \leq Ce^{-\beta \mu}.$$  

Now we leave to prove that $\varphi_{P_m}$ is $C^1$ in $P_m$. Consider the following mapping $H : \Omega_m \times (H \cap H^1(\mathbb{R}^3)) \times \mathbb{R}^{3m} \rightarrow (H \cap H^1(\mathbb{R}^3)) \times \mathbb{R}^{3m}$ of class $C^1$

$$H(P_m, \varphi_{P_m}, c) = \begin{pmatrix} (\Delta - 1)^{-1}S(w_{P_m} + \varphi_{P_m}) - \sum_{j=1,\ldots,m,k=1,2,3} c_{j,k}(\Delta - 1)^{-1}Q_{j,k} \\ \langle \varphi_{P_m}, (\Delta - 1)^{-1}Q_{1,1} \rangle \\ \vdots \\ \langle \varphi_{P_m}, (\Delta - 1)^{-1}Q_{m,3} \rangle \end{pmatrix}.$$ 

Then problem (2.31) is equivalent to $H(P_m, \varphi_{P_m}, c) = 0$. We know that, given $P_m \in \Omega_m$, there is a unique solution $(\varphi_{P_m}, \{c_{j,k}\})$. We prove that the operator $\frac{\partial H(P_m, \varphi_{P_m}, c)}{\partial (\varphi_{P_m}, c)}|_{(P_m, \varphi_{P_m}, \{c_{j,k}\})} : (H \cap H^1(\mathbb{R}^3)) \times \mathbb{R}^{3m} \rightarrow (H \cap H^1(\mathbb{R}^3)) \times \mathbb{R}^{3m}$ is invertible for $\mu$ large. Then $\varphi_{P_m} \in C^1$ follows from the implicit function theorem. Indeed, observe that

$$\frac{\partial H(P_m, \varphi_{P_m}, c)}{\partial (\varphi_{P_m}, c)}|_{(P_m, \varphi_{P_m}, \{c_{j,k}\})} \langle \psi, d \rangle = \begin{pmatrix} (\Delta - 1)^{-1}(S'(w_{P_m} + \varphi_{P_m})\psi - \sum_{j=1,\ldots,m,k=1,2,3} d_{j,k}(\Delta - 1)^{-1}Q_{j,k} \\ \langle \psi, (\Delta - 1)^{-1}Q_{1,1} \rangle \\ \vdots \\ \langle \psi, (\Delta - 1)^{-1}Q_{m,3} \rangle \end{pmatrix}.$$ 

Since $\|\varphi_{P_m}\|_*$ is small, the same proof as in that of Lemma 2.2 shows that $\frac{\partial H(P_m, \varphi_{P_m}, c)}{\partial (\varphi_{P_m}, c)}|_{(P_m, \varphi_{P_m}, \{c_{j,k}\})}$ is invertible for $\mu$ large. This completes the proof of Proposition 2.4. \qed

3. A secondary Liapunov-Schmidt reduction. In this section, as in [3, 4, 27], applying secondary Lyapunov-Schmidt reduction we give a key estimate on the difference between the solutions in the $m$-th step and $(m + 1)$-th step. For $P_m \in \Omega_m$, we denote $w_{P_m}$ as $w_{P_m} + \varphi_{P_m}$, where $\varphi_{P_m}$ is the unique solution given by Proposition 2.4. The main result below shows that the difference between $w_{P_m}$ and $w_{P_m} + \phi_{m+1}$ is small globally in $H^1(\mathbb{R}^3)$ norm. For this purpose, we now write

$$w_{P_{m+1}} = w_{P_m} + \phi_{m+1} = \bar{u} + \phi_{m+1}, \quad (3.1)$$

where

$$\bar{u} = u_{P_m} + w_{P_{m+1}}.$$
By the definition of the $\| \cdot \|_*$ norm in (2.3) that it depends on the spikes $P$, we now denote the norm by $\| \cdot \|_{*,P_{m+1}}$ to indicate the dependence. It follows from the definition of $\phi_{m+1}$ that
\[
\| \phi_{m+1} \|_{*,P_{m+1}} = \| \phi_{P_{m+1}} - \phi_{P_m} \|_{*,P_{m+1}}.
\] (3.2)

Applying Proposition 2.4, we get
\[
\| \phi_{P_{m+1}} \|_{*,P_{m+1}} \leq Ce^{-\beta \mu}. \tag{3.3}
\]
and
\[
\| \phi_{P_m} \|_{*,P_{m+1}} \leq \| \phi_{P_m} \|_{*,P_{m+1}} \leq Ce^{-\beta \mu}. \tag{3.4}
\]
Then it follows from (3.2) to (3.4) that
\[
\| \phi_{m+1} \|_{*,P_{m+1}} \leq Ce^{-\beta \mu}. \tag{3.5}
\]
But the estimate (3.5) is not sufficient. We need a crucial estimate for $\phi_{m+1}$ which will be given later. (In the following we will always assume that $\gamma > \frac{1}{2}$.) In order to obtain the crucial estimate, we will need the following lemma.

Lemma 3.1. (Lemma 2.4, [3]) For $|P_j - P_k| \geq \mu$ large, it holds that
\[
\int f(w(x - P_j))w(x - P_k)dx = (\vartheta + o(1))w(|P_j - P_k|) \tag{3.6}
\]
as $\mu \to \infty$ and
\[
\vartheta = \int f(w)e^{-x_1}dx > 0. \tag{3.7}
\]

Lemma 3.2. Let $\mu, \epsilon, \gamma$ and $P_m$ be as in Proposition 2.4. Then it holds
\[
\int (|\nabla \phi_{m+1}|^2 + \phi_{m+1}^2) \leq Ce^{-\beta \mu} \sum_{j=1}^{m} w(|P_j - P_{m+1}|) + C\epsilon^2 \left[ \int K^{2} \Phi_{w_{P_{m+1}}}^2 w_{P_{m+1}}^2 + \left( \int K \Phi_{w_{P_{m+1}}} w_{P_{m+1}} \right)^2 \right] \tag{3.8}
\]
for some constant $C > 0, \beta > 0$ independent of $\mu, m$ and $P_{m+1} \in \Omega_{m+1}$ (the constant may depend on the choice of $\gamma$).

Proof. In order to prove (3.8), we have to make a further decomposition. As Lemma 2.2 in [3], the basic idea is as follows: around each spike, we project $\phi_{m+1}$ into the orthogonal space of the unstable eigenfunctions and kernels. By this way, we get a linear operator which is possibly definite. Therefore we have to estimate three components of $\phi_{m+1}$: the coefficients of projections to the unstable eigenfunctions and kernels, and the orthogonal part. Now we perform this procedure in details.

By the non-degeneracy $(f_2)$, the following eigenvalue problem
\[
\Delta \varphi - \varphi + f'(w)\varphi = \lambda \varphi, \quad \varphi \in H^1(\mathbb{R}^3) \tag{3.9}
\]
adopts the following set of eigenvalues
\[
\lambda_1 > \lambda_2 > \cdots > \lambda_{n+1} = 0 > \lambda_{n+2} > \cdots. \tag{3.10}
\]
We denote the eigenfunctions corresponding to the positive eigenvalues $\lambda_j$ as $\varphi_j(x), j = 1, \cdots, n$. By $(f_2)$, we conclude that there is a positive generic constant $c_0$ such that
\[
\int [\| \nabla \varphi \|^2 + \varphi^2 - f'(w)\varphi^2] \geq c_0 \| \varphi \|^2_{H^1(\mathbb{R}^3)} \tag{3.11}
\]
for all $H^1$ functions satisfying $\int_{\mathbb{R}^3} \varphi_i^j = \int_{\mathbb{R}^3} \varphi \frac{\partial w}{\partial x_i} = 0$, $i = 1, 2, 3$, $j = 1, \cdots, n$. We fix $\varphi_i^j$ such that $\max_{x \in \mathbb{R}^3} \varphi_i^j = 1$. Denote by $\varphi_{ij} = \eta_i \varphi_i^j(x - P_j)$, where $\eta_i$ is the cut-off function introduced in section 1.

By the equation satisfied by $\phi_{m+1}$, we have

$$\tilde{L}\phi_{m+1} = -\tilde{S} + \sum_{j=1, \cdots, m+1, k=1,2,3} c_{jk}Q_{j,k}$$

(3.12)

for some constants $\{c_{jk}\}$, where

$$\tilde{L} = \Delta - 1 - \epsilon K(\Phi \tilde{u})' + f'(\tilde{u}),$$

$$(\Phi \tilde{u})' = \begin{cases} \Phi(\tilde{u} + \phi_{m+1}) - \Phi \tilde{u}, & \text{if } \phi_{m+1} \neq 0, \\ (\Phi \tilde{u})', & \text{if } \phi_{m+1} = 0, \end{cases}$$

$$f'(\tilde{u}) = \begin{cases} f(\tilde{u} + \phi_{m+1}) - f(\tilde{u}), & \text{if } \phi_{m+1} \neq 0, \\ f'(\tilde{u}), & \text{if } \phi_{m+1} = 0, \end{cases}$$

and

$$\tilde{S} = -[f(uP_m + w_{P_{m+1}}) - f(uP_m) - f(w_{P_{m+1}})] + \epsilon K(\Phi \tilde{u} - \Phi uP_m uP_m).$$

Here we may write $\tilde{u}$ as $\tilde{u} = \tilde{u} + \tau \phi_{m+1}$, where $\tau \in [0,1]$.

Now we proceed the proof into a few steps.

First we estimate the $L^2$-norm of $\tilde{S}$.

By the estimate in Proposition 2.4, we have the following estimate

$$\int |f(uP_m + w_{P_{m+1}}) - f(uP_m) - f(w_{P_{m+1}})|^2 dx \leq C e^{-\beta \mu} \sum_{j=1}^m w(|P_{m+1} - P_j|).$$

(3.13)

By direct computation and Lemma A.4, we have

$$\epsilon K(\Phi \tilde{u} - \Phi uP_m uP_m) = \epsilon \Phi uP_m w_{P_{m+1}} + \epsilon \Phi w_{P_{m+1}} uP_m + \epsilon \Phi w_{P_{m+1}} w_{P_{m+1}}$$

$$+ 2\epsilon K \left( \int \frac{K uP_m w_{P_{m+1}}}{|x - z|} dz \right) uP_m + 2\epsilon K \left( \int \frac{K uP_m w_{P_{m+1}}}{|x - z|} dz \right)^2 w_{P_{m+1}}$$

and

$$\int |\epsilon K(\Phi \tilde{u} - \Phi uP_m uP_m)|^2 dx$$

$$\leq C \left[ \int (\epsilon^2 K^2 \Phi^2 uP_m w_{P_{m+1}} + \epsilon^2 K^2 \Phi^2 uP_m uP_m + \epsilon^2 K^2 \Phi^2 w_{P_{m+1}} w_{P_{m+1}}$$

$$+ 4\epsilon^2 K^2 \left( \int \frac{K uP_m w_{P_{m+1}}}{|x - z|} dz \right)^2 uP_m + 4\epsilon^2 K^2 \left( \int \frac{K uP_m w_{P_{m+1}}}{|x - z|} dz \right)^2 w_{P_{m+1}}$$

$$\leq C e^{-\beta \mu} \sum_{j=1}^m w(|P_{m+1} - P_j|) + C \int \epsilon^2 K^2 \Phi^2 w_{P_{m+1}} w_{P_{m+1}}. \right]$$

(3.14)
It follows from (3.13) and (3.14) that
\[ |\bar{S}|^2 \leq C e^{-\beta \mu} \sum_{j=1}^{m} w(|P_{m+1} - P_j|) + C \int \epsilon^2 K^2 \Phi^2_{w_{m+1}} w_{m+1}^2. \] (3.15)
By the estimate (3.5), we have the following estimate
\[ \tilde{u} = \sum_{j=1}^{m+1} w(x - P_j) + O(e^{-\beta \mu}). \] (3.16)
Decompose \( \phi_{m+1} \) as
\[ \phi_{m+1} = \psi + \sum_{j=1, \ldots, m+1, l=1, \ldots, n} t_{j,l} \varphi_{j,l} + \sum_{j=1, \ldots, m+1, k=1, 2, 3} d_{j,k} Q_{j,k} \] (3.17)
for some \( t_{j,l}, d_{jk} \) such that
\[ \int \psi \varphi_{j,l} = \int \psi Q_{j,k} = 0, \quad j = 1, \ldots, m, \quad k = 1, 2, 3, \quad l = 1, \ldots, n. \] (3.18)
Since
\[ \phi_{m+1} = \varphi_{P_{m+1}} - \varphi_{P_m}, \] (3.19)
we have for \( j = 1, \ldots, m, \)
\[ d_{j,k} = \int \phi_{m+1} Q_{j,k} + \sum_{j=1}^{m} (t_{j,l} Q_{j,k}) = \int (\varphi_{P_{m+1}} - \varphi_{P_m}) Q_{j,k} + \sum_{j=1}^{m} (t_{j,l} Q_{j,k}) \]
\[ = e^{-\beta \mu} \sum_{j=1}^{m} t_{j,l}, \] (3.20)
and
\[ d_{m+1,k} = \int \phi_{m+1} Q_{m+1,k} + \sum_{j=1}^{m} (t_{m+1,l} Q_{m+1,k}) \]
\[ = \int (\varphi_{P_{m+1}} - \varphi_{P_m}) Q_{m+1,k} + \sum_{j=1}^{m} (t_{m+1,l} Q_{m+1,k}) \] (3.21)
\[ = - \int \varphi_{P_m} Q_{m+1,k} + e^{-\beta \mu} \sum_{j=1}^{m} t_{m+1,l}, \]
where we use the orthogonality conditions satisfied by \( \varphi_{P_m} \) and \( \varphi_{P_{m+1}} \). Hence by Proposition 2.4, we have
\[ \begin{cases} |d_{j,k}| \leq ce^{-\beta \mu} \sum_{j=1}^{m} t_{j,l}, \text{ for } j = 1, 2, \ldots, m, \\ |d_{m+1,k}| \leq Ce^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma |P_j - P_{m+1}|} + e^{-\beta \mu} \sum_{j=1}^{m} t_{m+1,l}. \end{cases} \] (3.22)
By (3.17), we can rewrite (3.12) as
\[ \bar{L}(\psi) + \sum_{j=1, \ldots, m+1, l=1, \ldots, n} t_{j,l} \bar{L}(\varphi_{j,l}) + \sum_{j=1, \ldots, m+1, k=1, 2, 3} d_{j,k} \bar{L}(Q_{j,k}) \]
\[ = -\bar{S} + \sum_{j=1, \ldots, m+1, k=1, 2, 3} c_{jk} Q_{j,k}. \] (3.23)
In order to estimate the coefficients $t_{j,l}$, we use the equation (3.23). First, multiplying (3.23) by $\varphi_{jl}$ and integrating over $\mathbb{R}^3$, we have

\[ t_{j,l} \int \bar{L}(\varphi_{jl}) \varphi_{jl} = - \sum_{k=1}^{3} d_{j,k} \int \bar{L}(Q_{j,k}) \varphi_{jl} - \int S \varphi_{jl} \\
- \sum_{k \neq l} t_{l,j} \int \bar{L}(\varphi_{jk}) \varphi_{jl} - \int \bar{L}(\varphi_{jl}) \varphi_{jl}, \tag{3.24} \]

where

\[
\begin{aligned}
\left| \int S \varphi_{jl} \right| &\leq Ce^{-\beta \mu} e^{-\gamma|P_j - P_{m+1}|} + \epsilon \left| \int K \Phi_{wp_{m+1}} w P_{m+1} \varphi_{jl} \right|, \\
\text{for } j = 1, 2, \ldots, m, \\
\left| \int S \varphi_{m+1,l} \right| &\leq Ce^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma|P_j - P_{m+1}|} + \epsilon \left| \int K \Phi_{wp_{m+1}} w P_{m+1} \varphi_{m+1,l} \right|.
\end{aligned} \tag{3.25} \]

By (3.16), we see that

\[ \int \bar{L}(\varphi_{jk}) \varphi_{jl} = - \lambda_{k} \delta_{k,l} \int \varphi_{j,k} \varphi_{0} + O(e^{-\beta \mu}). \tag{3.26} \]

Combining (3.22), (3.24), (3.25) and (3.26), and the orthogonal conditions satisfied by $\psi$, we have

\[
\begin{aligned}
|t_{j,l}| &\leq Ce^{-\beta \mu} e^{-\gamma|P_j - P_{m+1}|} + \epsilon \left| \int K \Phi_{wp_{m+1}} w P_{m+1} \varphi_{jl} \right| + e^{-\beta \mu} \left\| \psi \right\|_{H^1(B_{\frac{1}{2}}(P_j))}, \\
j = 1, \ldots, m, \\
|t_{m+1,l}| &\leq Ce^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma|P_j - P_{m+1}|} + \epsilon \left| \int K \Phi_{wp_{m+1}} w P_{m+1} \varphi_{m+1,l} \right| + e^{-\beta \mu} \left\| \psi \right\|_{H^1(B_{\frac{1}{2}}(P_{m+1}))},
\end{aligned} \tag{3.27} \]

and

\[
\begin{aligned}
|d_{j,l}| &\leq Ce^{-\beta \mu} e^{-\gamma|P_j - P_{m+1}|} + \epsilon \left| \int K \Phi_{wp_{m+1}} w P_{m+1} \varphi_{jl} \right| + e^{-\beta \mu} \left\| \psi \right\|_{H^1(B_{\frac{1}{2}}(P_j))}, \\
j = 1, \ldots, m, \\
|d_{m+1,l}| &\leq Ce^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma|P_j - P_{m+1}|} + \epsilon \left| \int K \Phi_{wp_{m+1}} w P_{m+1} \varphi_{m+1,l} \right| + e^{-\beta \mu} \left\| \psi \right\|_{H^1(B_{\frac{1}{2}}(P_{m+1}))}.
\end{aligned} \tag{3.28} \]

Next, we estimate $\psi$. Multiplying (3.23) by $\psi$ and integrating over $\mathbb{R}^3$, we find

\[ \int \bar{L}(\psi) \psi = - \int S \psi - \sum_{j=1, \ldots, m+1, k=1,2,3} d_{j,k} \int \bar{L}(Q_{j,k}) \psi \tag{3.29} \]

\[- \sum_{j=1, \ldots, m+1, k=1,2,3} t_{j,l} \int \bar{L}(\varphi_{jl}) \psi. \]
We claim that
\[ \int [\bar{L}(\psi)\psi] \geq c_0 \|\psi\|^2_{H^1} \]  
(3.30)
for some constant \( c_0 > 0 \) (independent of \( m \) and \( P_{m+1} \)).

Since the approximate solution is exponentially decay away from the points \( P_j \), we have
\[ \int_{\mathbb{R}^3 \setminus \bigcup_j B_{\mu_j}(P_j)} [\bar{L}(\psi)\psi] \geq \frac{1}{2} \int_{\mathbb{R}^3 \setminus \bigcup_j B_{\mu_j}(P_j)} (|\nabla \psi|^2 + |\psi|^2). \]  
(3.31)

Now we only need to prove the above estimate in the domain \( \bigcup_j B_{\mu_j}(P_j) \). We prove it by contradiction. Otherwise, there exists a sequence \( \mu_n \to \infty \), and \( P_j(P_n) \) such that
\[ \int_{B_{\mu_n}(P_j(P_n))} (|\nabla \psi|^2 + |\psi|^2) = 1, \quad \int_{B_{\mu_n}(P_j(P_n))} \bar{L}(\psi_n)\psi_n \to 0, \text{ as } n \to \infty. \]

Then we can extract from the sequence \( \psi_n \cdot P_j(P_n) \) a subsequence which will converge weakly in \( H^1(\mathbb{R}^3) \) to \( \psi_\infty \) such that
\[ \int |\nabla \psi_\infty|^2 + |\psi_\infty|^2 - f'(w)\psi_\infty^2 = 0 \]  
(3.32)
and
\[ \int \psi_\infty \varphi_0' = \int \psi_\infty \frac{\partial w}{\partial x_j} = 0, \quad \text{for } j = 1, 2, 3, l = 1, 2, \ldots, n. \]  
(3.33)

It follows from (3.32) and (3.33) that \( \psi_\infty = 0 \). Therefore
\[ \psi_n \to 0 \text{ weakly in } H^1(\mathbb{R}^3). \]  
(3.34)

Hence, we have
\[ \int_{B_{\mu_n}(P_j(P_n))} f'(\tilde{u})\psi_n^2 \to 0, \text{ as } n \to \infty. \]  
(3.35)

Then
\[ \|\psi_n\|_{H^1(B_{\mu_n}(P_j(P_n)))} \to 0 \text{ as } n \to \infty, \]
which contradicts to the assumption \( \|\psi_n\|_{H^1(B_{\mu_n}(P_j(P_n)))} = 1 \). Therefore (3.30) holds.

It follows from (3.29) and (3.30) that
\[ \|\psi\|^2 \leq C \left( \sum_{j,k} |d_{j,k}| \int L(Q_{j,k})\psi + \sum_{j,l} |\epsilon_{j,l}| \int L(\varphi_{ji})\psi + \int |\Sigma\psi| \right) 
\leq C \left( \sum_{j,k} |d_{j,k}|\|\psi\| + \sum_{j,l} |\epsilon_{j,l}|\|\psi\|_{H^1(B_{\mu_j}(P_j))} + \|\Sigma\|_{L^2}\|\psi\| \right). \]  
(3.36)

By (3.28) and (3.36), we have
\[ \|\psi\| \leq C \left( \sum_{j,k} |d_{j,k}| + e^{-\beta \mu} \sum_{j=1}^m e^{-\gamma(P_{m+1}-P_j)} + \epsilon \int K \Phi_{w_{P_{m+1}}} w_{P_{m+1}} + \|\Sigma\|_{L^2} \right). \]  
(3.37)
From (3.15) and (3.37), recalling that $\gamma > \frac{1}{2}$, we get
\[
\|\phi_{m+1}\| \leq C \left( e^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma |P_{m+1} - P_j|} + \epsilon \int K \Phi w_{P_{m+1}} w_{P_{m+1}} + |\mathcal{S}| \right) 
\leq C \left[ e^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma |P_{m+1} - P_j|} + e^{-\beta \mu} \left( \sum_{j=1}^{m} w(|P_{m+1} - P_j|) \right)^{\frac{1}{2}} \right] + \epsilon \int K \Phi w_{P_{m+1}} w_{P_{m+1}} + \epsilon \left( \int K^2 \Phi^2 w_{P_{m+1}}^2 w_{P_{m+1}}^2 \right)^{\frac{1}{2}}.
\]
(3.38)

Since we choose $\gamma > \frac{1}{2}$, by the definition of the configuration space, we have
\[
\left( \sum_{j=1}^{m} e^{-\gamma |P_{m+1} - P_j|} \right)^{2} \leq C \sum_{j=1}^{m} w(|P_{m+1} - P_j|).
\]
(3.39)

It follows from (3.38) and (3.39) that
\[
\|\phi_{m+1}\| \leq C \left[ e^{-\beta \mu} \left( \sum_{j=1}^{m} w(|P_{m+1} - P_j|) \right)^{\frac{1}{2}} \right] + \epsilon \int K \Phi w_{P_{m+1}} w_{P_{m+1}} + \epsilon \left( \int K^2 \Phi^2 w_{P_{m+1}}^2 w_{P_{m+1}}^2 \right)^{\frac{1}{2}}.
\]
(3.40)

Hence (3.8) holds.

Moreover, from the estimates (3.22) and (3.28), and taking into consideration that $\eta_j$ is supposed in $B_{\frac{\mu}{2}}(P_j)$, using Hölder inequality, we can get a more accurate estimate on $\phi_{m+1}$,
\[
\|\phi_{m+1}\| \leq C \left[ e^{-\beta \mu} \left( \sum_{j=1}^{m} w(|P_{m+1} - P_j|) \right)^{\frac{1}{2}} \right] + \epsilon \sum_{j=1,\ldots,m+1} \left( \int_{B_{\frac{\mu}{2}}(P_j)} K^2 \Phi^2 w_{P_{m+1}}^2 w_{P_{m+1}}^2 \right)^{\frac{1}{2}} + \epsilon \left( \int K^2 \Phi^2 w_{P_{m+1}}^2 w_{P_{m+1}}^2 \right)^{\frac{1}{2}}.
\]
(3.41)

4. Proof of the main result. In this section, first we investigate a maximization problem. Then we prove our main result.

Fixing $P_m \in \Omega_m$, we define
\[
\mathcal{M}(P_m) = J(w_{P_m}) = J(w_{P_m} + \varphi_{P_m}) : \Omega_m \to \mathbb{R}
\]
and
\[
\mathcal{C}_m = \sup_{P_m \in \Omega_m} \mathcal{M}(P_m).
\]
(4.1)

Observe that $\mathcal{M}(P_m)$ is continuous in $P_m$. We will prove below that the maximization problem has a solution. Denote $\mathcal{M}(P_m)$ as the maximum where $P_m = (\bar{P}_1, \ldots, \bar{P}_m) \in \Omega_m$, that is
\[
\mathcal{M}(P_m) = \max_{P_m \in \Omega_m} \mathcal{M}(P_m),
\]
(4.3)

and we denote the solution by $u_{\bar{P}_m}$.

First we show that the maximum can be attained at finite points for each $\mathcal{C}_m$. 

Lemma 4.1. Assume that \((K_1), (K_2)\) hold. If the assumptions in Proposition 2.4 are satisfied, then for all \(m\):

(i) There exists \(P_m \in \Omega_m\) such that

\[
C_m = \mathcal{M}(P_m); \tag{4.4}
\]

(ii) There holds

\[
C_{m+1} > C_m + I(w), \tag{4.5}
\]

where \(I(w)\) is the energy of \(w\),

\[
I(w) = \frac{1}{2} \int (|\nabla w|^2 + w^2) - \int F(w). \tag{4.6}
\]

Proof. Here we follow the proofs in [3, 13] and we need to use the estimate (3.8). We divide the proof into the following two steps.

**Step 1.** \(C_1 > I(w)\), and \(C_1\) can be attained at a finite point.

First applying standard Liapunov-Schmidt reduction, we have

\[
\|\varphi_P\| \leq C\|K\Phi_{wp}w_P\|_{L^2}. \tag{4.7}
\]

Supposing that \(|P| \to \infty\), then by (4.7) we have

\[
J(u_P) = \frac{1}{2} \int (|\nabla u_P|^2 + u_P^2) + \frac{\epsilon}{4} \int K(x)\Phi_{u_P}u_P^2 - \int F(u_P)
\]

\[
= \left[ \frac{1}{2} \int (|\nabla w_P|^2 + w_P^2) - \int F(w_P) \right] + \frac{1}{2} \int (|\nabla \varphi_P|^2 + \varphi_P^2)
\]

\[
+ \int (\nabla w_P \nabla \varphi_P + w_P \varphi_P) - \int (F(u_P) - F(w_P)) + \frac{\epsilon}{4} \int K(x)\Phi_{u_P}u_P^2
\]

\[
= I(w) + \frac{1}{2} \|\varphi_P\| - \int (F(u_P) - F(w_P) - f(w_P)\varphi_P) + \frac{\epsilon}{4} \int K(x)\Phi_{u_P}u_P^2
\]

\[
\geq I(w) + \frac{\epsilon}{8} \int K(x)\Phi_{wp}w_P^2 - C\|\varphi_P\|^2
\]

\[
\geq I(w) + \frac{\epsilon}{8} \int K(x)\Phi_{wp}w_P^2 - \int \epsilon^2 K^2\Phi_{wp}^2 w_P^2
\]

\[
\geq I(w) + \frac{\epsilon}{16} \int K(x)\Phi_{wp}w_P^2
\]

\[
\geq I(w) + \frac{1}{16} \int_{B_\frac{16}{2}(\varphi_P)} \epsilon K(x)\Phi_{wp}w_P^2 - \sup_{B_{\frac{w_P}{2}}(0)} \frac{w_P^2}{w_P^2} \int_{\text{supp} K^-} \epsilon K\Phi_{wp}w_P^2
\]

\[
\geq I(w) + \frac{\epsilon}{16} \int_{B_\frac{1}{2}(\varphi_P)} K(x)\Phi_{wp}w_P^2 - O(\epsilon \epsilon^{-\frac{2|P|}{\pi}}),
\]

where we use the following estimates

\[
\left| - \int (F(u_P) - F(w_P) - f(w_P)\varphi_P) \right| = \left| \int (f(w_P + \theta_1 \varphi_P) - f(w_P)\varphi_P) \right|
\]

\[
= \left| \int f'(w_P + \theta_2 \varphi_P)\varphi_P^2 \right| \leq C\|\varphi_P\|^2 \tag{4.8}
\]

and

\[
\frac{\epsilon}{4} \int K(x)\Phi_{u_P}u_P^2 \geq \frac{\epsilon}{8} \int K(x)\Phi_{wp}w_P^2 - C\|\varphi_P\|^2, \tag{4.9}
\]

whose proof we put in the Appendix A.
By the assumption \((K_2)\), we have
\[
\frac{1}{16} \int_{B_{\frac{1}{2}}(P)} \epsilon K \Phi_{w_P} w_P^2 - O(\epsilon e^{-\frac{\epsilon}{2}|P|}) > 0 \quad \text{for } |P| \text{ large.}
\]
So we have
\[
C_1 \geq J(u_P) > I(w). \quad (4.10)
\]
Now we will prove that \(C_1\) can be attained at a finite point. Let \(\{P_j\}\) be a sequence such that \(\lim_{j \to \infty} M(P_j) = C_1\), and assume that \(|P_j| \to +\infty,\)
\[
J(u_{P_j}) = \frac{1}{2} \int (|\nabla u_{P_j}|^2 + u_{P_j}^2) + \frac{\epsilon}{4} \int K(x) \Phi_{u_{P_j}} u_{P_j}^2 - \int F(u_{P_j})
\]
\[
= \left[ \frac{1}{2} \int (|\nabla w_{P_j}|^2 + w_{P_j}^2) - \int F(w_{P_j}) \right] + \frac{1}{2} \int (|\nabla \varphi_{P_j}|^2 + \varphi_{P_j}^2)
\]
\[
+ \int \nabla w_{P_j} \nabla \varphi_{P_j} + w_{P_j} \varphi_{P_j} - f(w_{P_j}) \varphi_{P_j}
\]
\[
- \int [F(u_{P_j}) - F(w_{P_j}) - f(w_{P_j}) \varphi_{P_j}]
\]
\[
+ \frac{\epsilon}{4} \int K(x) \Phi_{(w_{P_j} + \varphi_{P_j})} (w_{P_j} + \varphi_{P_j})^2
\]
\[
\leq I(w) + \epsilon C \|\varphi_{P_j}\|^2 + \frac{\epsilon}{4} \int K(x) \Phi_{(w_{P_j} + \varphi_{P_j})} (w_{P_j} + \varphi_{P_j})^2
\]
\[
\leq I(w) + O \left( \int \epsilon^2 K^2 \Phi^2_{w_{P_j}} w_{P_j}^2 \right) + \frac{\epsilon}{4} \int K(x) \Phi_{(w_{P_j} + \varphi_{P_j})} (w_{P_j} + \varphi_{P_j})^2.
\]
By the assumption \((K_1)\), we have
\[
O \left( \int \epsilon^2 K^2 \Phi^2_{w_{P_j}} w_{P_j}^2 \right) + \frac{\epsilon}{4} \int K(x) \Phi_{(w_{P_j} + \varphi_{P_j})} (w_{P_j} + \varphi_{P_j})^2 \to 0
\]
as \(j \to \infty\). Then we obtain
\[
C_1 = \lim_{j \to \infty} J(u_{P_j}) \leq I(w),
\]
which contradicts to \((4.10)\). Thus \(C_1\) can be attained at a finite point.

**Step 2.** Assume that there exists \(P_m = (P_1, \ldots, P_m) \in \Omega_m\) such that \(C_m = M(P_m)\) and we denote the solution by \(u_{P_m}\). Next we prove that there exists \(P_{m+1} = (P_1, \ldots, P_{m+1}) \in \Omega_{m+1}\) such that \(C_{m+1}\) can be attained. Let \(P_{m+1}^{(n)}\) be a sequence such that
\[
C_{m+1} = \lim_{n \to \infty} M(P_{m+1}^{(n)}). \quad (4.11)
\]
We claim that \(P_{m+1}^{(n)}\) is bounded. We prove it by contradiction. Without loss of generality, we assume that \(|P_{m+1}^{(n)}| \to \infty\) as \(n \to \infty\). In the following we omit the index \(n\) for simplicity. By direct computation, we have
\[
J(u_{P_{m+1}}) = J(u_{P_m} + w_{P_{m+1}} + \phi_{m+1})
\]
\[
= J(u_{P_m} + w_{P_{m+1}}) - \int (f(u_{P_m} + w_{P_{m+1}}) - f(u_{P_m}) - f(w_{P_{m+1}})) \phi_{m+1}
\]
\[
+ \epsilon \int K(\Phi u_{P_m} + \phi_{u_{P_m}} u_{P_m}) \phi_{m+1}
\]
\[ + \frac{\epsilon}{4} \int K[\Phi (u_{p_m} + w_{p_{m+1}} + \phi_{m+1}) (u_{p_m} + w_{p_{m+1}} + \phi_{m+1})^2 - \Phi (u_{p_m} + w_{p_{m+1}}) (u_{p_m} + w_{p_{m+1}})^2 - 4\Phi \bar{u}\phi_{m+1}] \\
- \int \sum_{j=1,\ldots,n; k=1,2,3} c_{jk} Q_{j,k} \phi_{m+1} \\
- \int f'(u_{p_m} + w_{p_{m+1}} + \partial \phi_{m+1}) \phi_{m+1}^2 + \frac{1}{2} \int (|\nabla \phi_{m+1}|^2 + \phi_{m+1}^2) \\
= J(u_{p_m} + w_{p_{m+1}}) + O(\|\phi_{m+1}\|^2 + \|S(u_{p_m} + w_{p_{m+1}})\|\phi_{m+1}\|) \\
- \int \sum_{j=1,\ldots,n; k=1,2,3} c_{jk} Q_{j,k} \phi_{m+1} \\
= J(u_{p_m} + w_{p_{m+1}}) + O \left( e^{-\beta \mu} \sum_{j=1}^{m} w(|P_{m+1} - P_j|) \right) + \int e^2 K^2 \Psi_{p_{m+1}}^2 w_{p_{m+1}}^2 \\
+ \left( \int K(x) \Phi w_{p_{m+1}} w_{p_{m+1}}^2 \right), \\
\text{(4.12)} \]
\[
+ \frac{\epsilon}{4} \int K [\Phi(ue_{m} + \varphi_{m+1})](ue_{m} + \varphi_{m+1})^2 - \Phi(ue_{m})u_{m}^2 - 4\Phi(ue_{m})ue_{m}w_{m+1} \]
\leq C_{m} + I(w) + \int \left( f(ue_{m}) - \sum_{j=1,\ldots,m; k=1,2,3} c_{jk}Q_{j,k} \right) w_{m+1}
+ \frac{\epsilon}{4} \int K \Phi w_{m+1}^2 w_{m+1} \]
\leq \int [f(ue_{m})w_{m+1} + f(w_{m+1})ue_{m}] + O \left( e^{-\beta \mu} \sum_{j=1}^{m} w(|P_{m+1} - P_{j}|) \right)
\leq C_{m} + I(w) + \frac{\epsilon}{4} \int K \Phi w_{m+1}^2 w_{m+1} \]
\leq \int f(w_{m+1})ue_{m} + O \left( e^{-\beta \mu} \sum_{j=1}^{m} w(|P_{m+1} - P_{j}|) \right),
\]
(4.13)

since
\[
\frac{\epsilon}{4} \int K [\Phi(ue_{m} + \varphi_{m+1})](ue_{m} + \varphi_{m+1})^2 - \Phi(ue_{m})u_{m}^2 - 4\Phi(ue_{m})ue_{m}w_{m+1} \]
\leq \frac{\epsilon}{4} \int K \Phi w_{m+1}^2 w_{m+1} \]
\leq \int K \left( \int \frac{Kue_{m}w_{m+1}dz}{|x - z|} \right) (w_{m+1}^2 + 2ue_{m}w_{m+1})
\leq \frac{\epsilon}{4} \int K \Phi w_{m+1}^2 w_{m+1} + O \left( e^{-\beta \mu} \sum_{j=1}^{m} w(|P_{m+1} - P_{j}|) \right),
\]

By (2.71) in [3], we have
\[
\left| \sum_{j=1,\ldots,m; k=1,2,3} c_{jk}Q_{j,k} w_{m+1} \right| \leq C e^{-\beta \mu} \sum_{j=1}^{m} w(|P_{m+1} - P_{j}|)
\]
(4.14)

for some proper \( \beta > 0 \). By the equation satisfied by \( \varphi_{m} \),
\[
\Delta \varphi_{m} - \varphi_{m} + f'(w_{m})\varphi_{m} = -S(w_{m}) + \bar{N}(\varphi_{m})
+ \epsilon K(\Phi w_{m} \varphi_{m} + \Phi \varphi_{m} u_{m})
+ 2\epsilon K \left( \int \frac{Kue_{m}w_{m}dz}{|x - z|} \right) (w_{m} + \varphi_{m}) + \sum_{j=1,\ldots,m; k=1,2,3} c_{jk}Q_{j,k},
\]
(4.15)

where
\[
\bar{N}(\varphi_{m}) = \left[ f(w_{m} + \varphi_{m}) - f(w_{m}) - f'(w_{m})\varphi_{m} \right],
\]
we have
\[
\int f(w_{m+1})\varphi_{m} = \int (\Delta - 1)w_{m+1}\varphi_{m} = \int (\Delta - 1)\varphi_{m}w_{m+1}
= \int [-S(w_{m}) + \bar{N}(\varphi_{m}) + \epsilon K(\Phi w_{m} \varphi_{m} + \Phi \varphi_{m} u_{m})]
\]
and

\[
\left| \int [\tilde{N}(\varphi_{P_m}) - f'(w_{P_m})\varphi_{P_m}]w_{P_{m+1}} \right| \leq C e^{-\beta \mu} \sum_{j=1}^{m} w(|P_{m+1} - P_j|),
\]

Moreover, we can choose \( \gamma \) such that \( \gamma + \delta > 1, (1 + \delta) \gamma > 1 \). Then it follows from the first two estimates of page-22 in [3] that

\[
\int \sum_{j=1,\ldots,m;k=1,2,3} c_{jk} Q_{j;k} w_{P_{m+1}} \leq C e^{-\beta \mu} \sum_{j=1}^{m} w(|P_{m+1} - P_j|)
\]

and

\[
\int \left[ -S(w_{P_m}) + \epsilon K(\Phi_{w_{P_m}} \varphi_{P_m} + \Phi_{\varphi_{P_m}} w_{P_m}) 
+ 2\epsilon K \left( \int \frac{K w_{P_m} \varphi_{P_m} + \varphi_{P_m}}{|x - z|} w_{P_{m+1}} \right) \right] \leq C \epsilon \int K \Phi_{w_{P_m}} w_{P_{m+1}} w_{P_{m+1}}
\]

+ \epsilon \int K(\Phi_{w_{P_m}} \varphi_{P_m} + \Phi_{\varphi_{P_m}} w_{P_m}) w_{P_{m+1}}

+ 2\epsilon K \left( \int \frac{K w_{P_m} \varphi_{P_m}}{|x - z|} \right) (w_{P_m} + \varphi_{P_m}) w_{P_{m+1}}

\leq C \epsilon \int K \Phi_{w_{P_m}} w_{P_{m+1}} w_{P_{m+1}} + \epsilon e^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma |x - P_j|} \int K \Phi_{w_{P_m}} w_{P_{m+1}}

+ \epsilon e^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma |x - P_j|} \int K \frac{\sum_{j=1}^{m} e^{-\gamma |x - P_j|}}{|x - z|} w_{P_m} w_{P_{m+1}}

+ \epsilon e^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma |x - P_j|} \int K \frac{\sum_{j=1}^{m} e^{-\gamma |x - P_j|}}{|x - z|} d z (w_{P_m} + \sum_{l=1}^{m} e^{-\gamma |x - P_l|}) w_{P_{m+1}}

So, from (4.16) to (4.19), we obtain

\[
\int f(w_{P_{m+1}})\varphi_{P_m}
\leq C \epsilon \int K \Phi_{w_{P_m}} w_{P_{m+1}} w_{P_{m+1}} + \epsilon e^{-\beta \mu} \sum_{j=1}^{m} K e^{-\gamma |x - P_j|} \Phi_{w_{P_m}} w_{P_{m+1}}

+ \epsilon e^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma |x - P_j|} \int K \frac{\sum_{j=1}^{m} e^{-\gamma |x - P_j|}}{|x - z|} d z w_{P_m} w_{P_{m+1}}
\]
Combing (4.12), (4.13), (4.14) and (4.21), we obtain

\[ +\epsilon e^{-\beta \mu} \int K \left( \int \frac{K w_{P_m} \sum_{j=1}^{m} e^{-\gamma|x-P_j|}}{|x-z|} \, dz \right) (w_{P_m} + \sum_{l=1}^{m} e^{-\gamma|x-P_l|}) w_{P_{m+1}} \\
+ \epsilon e^{-\beta \mu} \sum_{j=1}^{m} w(|P_{m+1} - P_j|) \right]. \]

Hence by Lemma 3.1, we have

\[
\int f(w_{P_{m+1}})u_{P_m} = \int f(w_{P_{m+1}})(w_{P_m} + \varphi_{P_m}) \\
= \int f(w_{P_{m+1}})w_{P_m} + O\left( \epsilon \int K \Phi_{w_{P_m}} w_{P_m} w_{P_{m+1}} \right) \\
+ \epsilon e^{-\beta \mu} \int \sum_{j=1}^{m} Ke^{-\gamma|x-P_j|} \Phi_{w_{P_m}} w_{P_{m+1}} \\
+ \epsilon e^{-\beta \mu} \int K \left( \int \frac{K w_{P_m} \sum_{j=1}^{m} e^{-\gamma|x-P_j|}}{|x-z|} \, dz \right) w_{P_m} w_{P_{m+1}} \\
+ \epsilon e^{-\beta \mu} \int K \left( \int \frac{K w_{P_m} \sum_{j=1}^{m} e^{-\gamma|x-P_j|}}{|x-z|} \, dz \right) (w_{P_m} + \sum_{l=1}^{m} e^{-\gamma|x-P_l|}) w_{P_{m+1}} \\
+ e^{-\beta \mu} \sum_{j=1}^{m} w(|P_{m+1} - P_j|) \geq \frac{1}{4} \vartheta \sum_{j=1}^{m} w(|P_{m+1} - P_j|) \\
+ O\left( \epsilon \int K \Phi_{w_{P_m}} w_{P_m} w_{P_{m+1}} + \epsilon e^{-\beta \mu} \int \sum_{j=1}^{m} e^{-\gamma|x-P_j|} \Phi_{w_{P_m}} w_{P_{m+1}} \right) \\
+ \epsilon e^{-\beta \mu} \int K \left( \int \frac{K w_{P_m} \sum_{j=1}^{m} e^{-\gamma|x-P_j|}}{|x-z|} \, dz \right) w_{P_m} w_{P_{m+1}} \\
+ \epsilon e^{-\beta \mu} \int K \left( \int \frac{K w_{P_m} \sum_{j=1}^{m} e^{-\gamma|x-P_j|}}{|x-z|} \, dz \right) (w_{P_m} + \sum_{l=1}^{m} e^{-\gamma|x-P_l|}) w_{P_{m+1}} \\
+ e^{-\beta \mu} \sum_{j=1}^{m} w(|P_{m+1} - P_j|) \right]. \]

Combing (4.12), (4.13), (4.14) and (4.21), we obtain

\[ J(u_{P_{m+1}}) = J(u_{P_m} + w_{P_{m+1}} + \phi_{m+1}) \\
\leq C_m + \frac{C}{4} \int K \Phi_{w_{P_{m+1}}} w_{P_{m+1}}^2 - \frac{1}{4} \vartheta \sum_{j=1}^{m} w(|P_{m+1} - P_j|) \\
+ O\left( \epsilon \int K \Phi_{w_{P_m}} w_{P_m} w_{P_{m+1}} + \epsilon e^{-\beta \mu} \int \sum_{j=1}^{m} Ke^{-\gamma|x-P_j|} \Phi_{w_{P_m}} w_{P_{m+1}} \right) \\
+ \epsilon e^{-\beta \mu} \int K \left( \int \frac{K \sum_{j=1}^{m} e^{-\gamma|x-P_j|}}{|x-z|} \, dz \right) w_{P_m} w_{P_{m+1}} \]
Combining (4.11), (4.22), (4.23) and (4.24), we have

\[ +\epsilon e^{-\beta \mu} \int K \left( \frac{\sum_{j=1}^{m} e^{-\gamma |x-P_j|^2}}{|x-z|} dz \right) (w_{\mathbf{P}_m}^2 + \sum_{i=1}^{m} e^{-\gamma |x-P_i|^2} w_{P_{i+1}}^2) \\
+\epsilon e^{-\beta \mu} \sum_{j=1}^{m} w(|P_{m+1} - P_j|) + \epsilon^2 \left[ \int K \Phi^2_{\mathbf{P}_m} w_{P_{m+1}}^2 + \epsilon^2 \left( \int K \Phi_{\mathbf{P}_m} w_{P_{m+1}}^2 \right)^2 \right]. \]

(4.22)

By the assumption that \( |P_{m+1}^{(n)}| \to \infty \), we have

\[ \epsilon \int \int K \Phi_{\mathbf{P}_m^{(n)}}^2 w_{P_{m+1}^{(n)}}^2 + \epsilon \int K \Phi_{\mathbf{P}_m} w_{P_{m+1}}^2 + \epsilon e^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma |x-P_j|^2} \Phi_{\mathbf{P}_m} w_{P_{m+1}}^2 \\
+\epsilon e^{-\beta \mu} \int K \left( \frac{\sum_{j=1}^{m} e^{-\gamma |x-P_j|^2}}{|x-z|} dz \right) (w_{\mathbf{P}_m}^2 + \sum_{i=1}^{m} e^{-\gamma |x-P_i|^2} w_{P_{i+1}}^2) \\
+\epsilon e^{-\beta \mu} \int K \left( \frac{\sum_{j=1}^{m} e^{-\gamma |x-P_j|^2}}{|x-z|} dz \right) \sum_{j=1}^{m} e^{-\gamma |x-P_j|^2} w_{P_{m+1}}^2 \\
+\epsilon^2 \int K \Phi_{\mathbf{P}_m} w_{P_{m+1}}^2 + \epsilon^2 \left( \int K \Phi_{\mathbf{P}_m} w_{P_{m+1}}^2 \right)^2 \to 0, \quad \text{as} \quad n \to +\infty, \]

(4.23)

and

\[ -\frac{1}{4} \theta \sum_{j=1}^{m} w(|P_{m+1} - P_j|) + O \left( \frac{\epsilon e^{-\beta \mu}}{\sum_{j=1}^{m} w(|P_{m+1} - P_j|)} \right) < 0. \]

(4.24)

Combining (4.11), (4.22), (4.23) and (4.24), we have

\[ C_{m+1} \leq C_m + I(w). \]

(4.25)

On the other hand, since by the assumption, \( C_m \) can be attained at \((\bar{P}_1, \cdots, \bar{P}_m)\), so there exists other point \( P_{m+1} \) which is far away from the \( m \) points which be determined later. Next let’s consider the solution concentrated at the points \((\bar{P}_1, \cdots, \bar{P}_m, P_{m+1})\), and we denote the solution by \( u_{\mathbf{P}_m, P_{m+1}} \), then similar with the above argument, applying the estimate (3.41) of \( \phi_{m+1} \) instead of (3.20), we have the following estimate:

\[ J(u_{\mathbf{P}_m, P_{m+1}}) \]

\[ \leq J(u_{\mathbf{P}_m}) + I(w) + \left\{ \epsilon \int K \Phi_{\mathbf{P}_m}^2 w_{P_{m+1}}^2 \\
+O \left( \sum_{j=1}^{m} \left( \int B_{\frac{1}{2}}(P_j) \right)^2 K^2 \Phi_{\mathbf{P}_m}^2 w_{P_{m+1}}^2 \right) \right\}^2 \\
+O \left( \epsilon^2 K^2 \Phi_{\mathbf{P}_m}^2 w_{P_{m+1}}^2 \right) - O \left( \sum_{j=1}^{m} w(|P_{m+1} - P_j|) \right) \]

\[ +O \left[ \epsilon e^{-\beta \mu} \sum_{j=1}^{m} K e^{-\gamma |x-P_j|} \Phi_{\mathbf{P}_m} w_{P_{m+1}} \right]. \]
Proof. So then we can get Proposition 4.2. The maximization problem

\[ C = \text{impossible.} \]

Hence we prove that it follows from (4.25) and (4.28) that

\[ \Omega \]

we assume \((j, k)\), for some \(\alpha < 1\), then we can choose \(P_{m+1}\) such that

\[ |P_{m+1}| > \frac{(\max \sum_{j=1}^{m} |P_j| - \ln \epsilon)}{\gamma - \alpha}. \]

Then we can get

\[ \left\{ \frac{\epsilon}{4} \int K \Phi w_{P_{m+1}} w_{P_{m+1}}^2 + O \left( \sum_{j=1}^{m} \left( \int_{B_{2}^{\epsilon}(P_j)} \epsilon^2 K^2 \Phi^2 w_{P_{m+1}}^2 w_{P_{m+1}}^2 \right)^{\frac{1}{2}} \right) \right\} \]

\[ + O \left( \int \epsilon^2 K^2 \Phi^2 w_{P_{m+1}} w_{P_{m+1}}^2 \right) - O \left( \sum_{j=1}^{m} w(||P_{m+1} - P_j||) \right) \]

\[ + O \left[ \epsilon e^{-\beta \mu} \int \sum_{j=1}^{m} K e^{-\gamma |x - P_j|} \Phi w_{P_{m+1}} w_{P_{m+1}} \right] \]

\[ + \epsilon e^{-\beta \mu} \int K \left( \sum_{j=1}^{m} e^{-\gamma |x - P_j|} \sum_{k=1}^{m} e^{-\gamma |x - P_k|} \right) dz \right) w_{P_{m+1}} \]

\[ + \epsilon \int K \Phi w_{P_{m+1}} w_{P_{m+1}} \] \]

\[ \geq C \epsilon e^{-\alpha P_{m+1}} - O \left( \sum_{j=1}^{m} e^{-\gamma |P_j - P_{m+1}|} \right) > 0. \]

So

\[ C_{m+1} \geq J(u_{\bar{P}_{m+1}}) > C_m + I(w). \]

It follows from (4.25) and (4.28) that

\[ C_m + I(w) < C_{m+1} \leq C_m + I(w), \]

which is impossible. Hence we prove that \(C_{m+1}\) can be attained at finite points in \(\Omega_{m+1}\). \(\square\)

**Proposition 4.2.** The maximization problem

\[ \max_{P_{m} \in \Omega_{m}} \mathcal{M}(P_{m}) \]

has a solution \(P_{m} \in \Omega_{m}'\), i.e., the interior of \(\Omega_{m}\).

**Proof.** We prove it by an indirect method. Assume that \(P_{m} = (\bar{P}_{1}, \bar{P}_{2}, \cdots, \bar{P}_{m}) \in \partial \Omega_{m}\). Then there exists \((j, k)\) such that \(|\bar{P}_{j} - \bar{P}_{k}| = \mu\). Without loss of generality, we assume \((j, k) = (j, m)\). Then following the estimates (4.12), (4.13), (4.14) and
(4.21), we have
\[ C_m = J(u_{P_m}) \]
\[ \leq C_{m-1} + I(w) + \frac{\vartheta}{4} \int K \Phi_{w_{P_m}} w_{P_m}^2 - \frac{\vartheta}{4} \sum_{j=1}^{m-1} e^{-|P_m - \bar{P}_j|} \]
\[ + O(e^{-\beta\mu} \sum_{j=1}^{m-1} e^{-|P_m - \bar{P}_j|}) + O(\varepsilon) \]
\[ \leq C_{m-1} + I(w) + O(\varepsilon) + \frac{\vartheta}{4} \sum_{j=1}^{m-1} w(|P_m - \bar{P}_j|) + O\left(e^{-\beta\mu} \sum_{j=1}^{m-1} w(|P_m - \bar{P}_j|)\right). \] (4.30)

By the definition of the configuration set, we observe that given a ball of size \( \mu \), there are at most \( C_3 := 6^3 \) number of non-overlapping ball of size \( \mu \) surrounding this ball. Since \(|P_j - \bar{P}_k| = \mu\), we have
\[ \sum_{j=1}^{m-1} w(|P_m - \bar{P}_j|) = w(|P_m - \bar{P}_j|) + \sum_{k \neq j} w(|P_m - \bar{P}_k|) \]
and
\[ \sum_{k \neq j} w(|P_m - \bar{P}_k|) \leq Ce^{-\mu} + C_3 e^{-\mu - \frac{\mu}{4}} + \ldots + C_3^k e^{-\mu - \frac{\mu}{4}} \]
\[ \leq Ce^{-\mu} \sum_{j=0}^{\infty} e^{k(\ln C_3 - \frac{\mu}{4})} \leq Ce^{-\mu}, \]
if \( C_3 < \frac{\mu}{4} \), which is true for \( \mu \) large enough.

Hence, we have
\[ C_m \leq C_{m-1} + I(w) + Ce - \frac{\vartheta}{4} w(\mu) + O(e^{-(1+\beta)\mu}) \leq C_{m-1} + I(w), \] (4.31)
which contradicts to (4.5) in Lemma 4.1.

Now we are in position to prove our main result.

Proof of Theorem 1.1. By Proposition 2.4 in Section 2, there exists \( \mu_0 \) such that for \( \mu > \mu_0 \), we have a \( C^1 \) map which, to any \( P^0 \in \Omega_m \), associates \( \varphi_{P^0} \) such that
\[ S(w_{P^0} + \varphi_{P^0}) = \sum_{j=1,2,\ldots,m,k=1,2,3} c_{jk} Q_{j,k}, \int \varphi_{P^0} Q_{j,k} dx = 0 \] (4.32)
for some constants \( c_{jk} \in \mathbb{R}^{3m} \).

By Proposition 4.2, there is a \( P \in \Omega_{m}^* \) that achieves the maximum for the maximization problem in Proposition 4.2. Let \( u_{P^0} = w_{P^0} + \varphi_{P^0} \). Then we have
\[ D_{P_j} \mathcal{M}(P^0)|_{P_j = P^0} = 0, j = 1, 2, \ldots, m, k = 1, 2, 3. \] (4.33)

Therefore, we have
\[ \int (\nabla w_{P} \nabla \frac{\partial (w_{P} + \varphi_{P})}{\partial P_{jk}}|_{P_j = P^0} + w_{P} \frac{\partial (w_{P} + \varphi_{P})}{\partial P_{jk}}|_{P_j = P^0}) \]
\[ + \epsilon \int K \Phi_{w_{P}} w_{P} \frac{\partial (w_{P} + \varphi_{P})}{\partial P_{jk}}|_{P_j = P^0} - \int f(w_{P}) \frac{\partial (w_{P} + \varphi_{P})}{\partial P_{jk}}|_{P_j = P^0} = 0, \]
which implies
\[ \sum_{j=1, \cdots, m, k=1,2,3} c_{jk} \int Q_{j,k} \frac{\partial (w_P + \varphi_P)}{\partial P_{sl}} \bigg|_{P_s = P_{j,k}^0} = 0 \] (4.34)
for \( s = 1, \cdots, m, l = 1, 2, 3 \).

We can prove that (4.34) is a diagonally dominant system. Indeed, noting that
\[ \int \varphi_P Q_{s,l} = 0, \]
we have
\[ \int Q_{s,l} \frac{\partial \varphi_P}{\partial P_{j,k}} \bigg|_{P_j = P_{j,k}^0} = - \int \varphi_P \frac{\partial Q_{j,l}}{\partial P_{j,k}} \bigg|_{P_j = P_{j,k}^0} = O(e^{-\beta \mu}). \]

We can prove that (4.34) is a diagonally dominant system. Indeed, noting that
\[ \int \varphi_P Q_{s,l} = 0, \]
we have
\[ \int Q_{s,l} \frac{\partial \varphi_P}{\partial P_{j,k}} \bigg|_{P_j = P_{j,k}^0} = - \int \varphi_P \frac{\partial Q_{j,l}}{\partial P_{j,k}} \bigg|_{P_j = P_{j,k}^0} = O(e^{-\beta \mu}). \]

It follows from (4.35) and (4.36) that equation (4.34) becomes a system of homogeneous equations for \( c_{sl} \), and the matrix of the system is nonsingular. Hence \( c_{sl} = 0 \) for \( s = 1, \cdots, m, l = 1, 2, 3 \). Therefore \( u_{P0} + \varphi_{P0} \) is a solution of (1.12). Similar to the argument in Section 6 of [27], we can get that \( u_{P0} + \varphi_{P0} \) is a positive pair solution of (1.4). \( \square \)

Appendix A. Some technical estimates. In this section, we give some technical estimates which are used before. Recall that
\[ \Lambda_j := \{ x \mid \| x - P_j \| \leq \frac{\mu}{2} \}, \quad \Lambda = \bigcup_{j=1}^{m} \Lambda_j \]
and
\[ \Lambda^C = \mathbb{R}^3 \setminus \Lambda. \]

First we analyzes \( w_{P_m} \) in \( \mathbb{R}^3 \).

Lemma A.1. For any \( x \in \Lambda_j (j = 1, \cdots, m) \) and any \( \vartheta > 0 \), we have
\[ \sum_{i=1}^{m} e^{-\vartheta \| x - P_i \|} = e^{-\vartheta \| x - P_j \|} + O(e^{\frac{-\vartheta \mu}{2}}). \] (A.1)

For any \( x \in \Lambda^C \), we have
\[ \sum_{i=1}^{m} e^{-\vartheta \| x - P_i \|} = O(e^{-\vartheta \frac{\mu}{2}}). \] (A.2)
Proof. Note that given a ball of size $\mu$, there are at most $C_3 := 6^3$ number of non-overlapping ball of size $\mu$ surrounding this ball. For any $x \in \Lambda_j$, i.e. $|x - P_j| \leq \frac{\mu}{2}$, we have

$$|x - P_i| \geq |P_i - P_j| - |x - P_j| \geq \frac{\mu}{2} \text{ for all } i \neq j.$$ 

Then we have

$$\sum_{i=1}^{m} e^{-\vartheta|x-P_i|} = e^{-\vartheta|x-P_j|} + e^{-\vartheta|x-P_i|} \leq e^{-\vartheta|x-P_j|} + \sum_{k=1}^{\infty} C_3^k e^{-\frac{k\vartheta}{2}} = e^{-\vartheta|x-P_j|} + \sum_{k=1}^{\infty} e^{k(\ln C_3 - \frac{\vartheta}{2})} = e^{-\vartheta|x-P_j|} + O(e^{-\frac{\vartheta}{2}},$$

if $\ln C_3 < \frac{\vartheta}{2}$, which is true for $\mu$ large enough.

The proof of (A.4) is similar. \qed

By the same arguments as above, we also have

**Lemma A.2.** For any $x \in \Lambda_j (j = 1, \cdots, m)$, we have

$$\sum_{i=1}^{m} \frac{1}{|x - P_i|} = \frac{1}{|x - P_j|} + O\left(\frac{1}{\mu}\right). \quad (A.3)$$

For any $x \in \Lambda^c$, we have

$$\sum_{i=1}^{m} \frac{1}{|x - P_i|} = O\left(\frac{1}{\mu}\right). \quad (A.4)$$

**Lemma A.3.** For any $x \in \Lambda_j (j = 1, \cdots, m)$ and any $\vartheta_1, \vartheta_2 > 0$, we have

$$\sum_{i=1}^{m} \frac{|x - P_i|^\vartheta_1}{e^{\vartheta_2|x-P_i|}} = \frac{|x - P_j|^\vartheta_1}{e^{\vartheta_2|x-P_j|}} + O\left(\frac{\mu^{\vartheta_1}}{e^{\frac{\vartheta_1}{2} \mu}}\right). \quad (A.5)$$

For any $x \in \Lambda^c$, we have

$$\sum_{i=1}^{m} \frac{|x - P_i|^\vartheta_1}{e^{\vartheta_2|x-P_i|}} = O\left(\frac{\mu^{\vartheta_1}}{e^{\frac{\vartheta_1}{2} \mu}}\right). \quad (A.6)$$

The following lemma is very crucial.

**Lemma A.4.** For any $P_m \in \Omega_m$ and $\vartheta_1, \vartheta_2 > 0$, we have

$$\int \frac{(\sum_{i=1}^{m} e^{-\vartheta_1|z-P_i|})(\sum_{k=1}^{m} e^{-\vartheta_2|z-P_k|})}{|x - z|} dz$$
Firstly, we consider \( x \in \mathbb{R}^3 \setminus \{ P_1, \ldots, P_m \} \),

\[
C \sum_{i=1}^{m} \left[ \sum_{i=1}^{2} \frac{|x - P_i|^i}{e^{\frac{\theta_i|x - P_i|}{\theta_i}} + 1} + \frac{1}{|x - P_i|} \right],
\]

\[
C \sum_{i \neq j} \left[ \sum_{i=1}^{2} \frac{|P_{j0} - P_i|^i}{e^{\frac{\theta_i|P_{j0} - P_i|}{\theta_i}} + 1} + \frac{1}{|P_{j0} - P_i|} \right],
\]

where \( x = P_{j0}, j_0 \in \{1, \ldots, m\} \).

Proof. Note that

\[
\int \frac{(\sum_{j=1}^{m} e^{-\theta_1|x - P_j|})(\sum_{k=1}^{m} e^{-\theta_2|x - P_k|})}{|x - z|} \, dz
\]

\[
= \sum_{j=1}^{m} \int \frac{e^{-(\theta_1 + \theta_2)|z - P_j|}}{|x - z|} \, dz + \sum_{j \neq k} \int \frac{e^{-\theta_1|x - P_j|}e^{-\theta_2|x - P_k|}}{|x - z|} \, dz
\]

\[
:= I_1 + I_2.
\]

Firstly, we consider \( x \in \mathbb{R}^3 \setminus \{ P_1, P_2, \ldots, P_m \} \) and estimate \( I_1 \). Let \( r_j = \frac{1}{2} |x - P_j| \). Then we have

\[
\int_{B_{r_j}(x)} \frac{e^{-(\theta_1 + \theta_2)|z - P_j|}}{|x - z|} \, dz \leq \int_{B_{r_j}(x)} \frac{e^{-(\theta_1 + \theta_2)|z - P_j|}}{|x - z|} \, dz
\]

\[
= Ce^{-(\theta_1 + \theta_2)r_j} \int_{r_j}^{r_j} t \, dt = Ce^{-(\theta_1 + \theta_2)r_j}r_j^2
\]

\[
= C|x - P_j|^2e^{-\frac{(\theta_1 + \theta_2)}{2} |x - P_j|}
\]

and

\[
\int_{B_{r_j}(P_j)} \frac{e^{-(\theta_1 + \theta_2)|z - P_j|}}{|x - z|} \, dz \leq \frac{1}{r_j} \int_{B_{r_j}(P_j)} \frac{e^{-(\theta_1 + \theta_2)|z - P_j|}}{|x - z|} \, dz
\]

\[
= \frac{C}{r_j} \int_{0}^{r_j} e^{-(\theta_1 + \theta_2)t^2} \, dt
\]

\[
= \frac{C}{|x - P_j|} + C \sum_{i=-1}^{1} |x - P_j|^i e^{-\frac{(\theta_1 + \theta_2)}{2} |x - P_j|}.
\]

Suppose that \( z \in \mathbb{R}^3 \setminus (B_{r_j}(P_j) \cup B_{r_j}(x)) \). Then

\[
|z - P_j| \geq \frac{|x - P_j|}{2}, \quad |z - x| \geq \frac{|x - P_j|}{2}.
\]

If \( |z - P_j| \geq 2|x - P_j| \), then \( |z - x| \geq |z - P_j| - |x - P_j| \geq \frac{|x - P_j|}{2} \). As a result,

\[
\frac{e^{-(\theta_1 + \theta_2)|z - P_j|}}{|x - z|} \leq \frac{2e^{-(\theta_1 + \theta_2)|z - P_j|}}{|z - P_j|}.
\]

If \( |z - P_j| \leq 2|x - P_j| \), then \( |z - x| \geq \frac{|x - P_j|}{4} \geq \frac{|x - P_j|}{4} \). Thus we have proved that

\[
\frac{e^{-(\theta_1 + \theta_2)|z - P_j|}}{|x - z|} \leq 4e^{-(\theta_1 + \theta_2)|z - P_j|}.
\]
It follows from (A.10) and (A.11) that
\[
\int_{\mathbb{R}^3 \setminus (B_{r_j}(P_j) \cup B_{r_j}(x))} e^{-\theta_1 |z - P_j|} \frac{dz}{|x - z|} \leq C \int_{\mathbb{R}^3 \setminus (B_{r_j}(P_j) \cup B_{r_j}(x))} e^{-\theta_1 |z - P_j|} \frac{dz}{|z - P_j|}
\]
\[
\leq C \int_{\mathbb{R}^3 \setminus B_{r_j}(P_j)} e^{-\theta_1 |z - P_j|} \frac{dz}{|z - P_j|}
\]
\[
\leq C \int_{r_j}^{+\infty} e^{-\theta_1 t} t \, dt
\]
\[
\leq C (1 + |x - P_j|) e^{-\theta_2 |x - P_j|}.
\]
Combining (A.8), (A.9) and (A.12), we know that
\[
I_1 \leq C \sum_{j=1}^{m} \left( \sum_{i=-1}^{2} |x - P_j|^i e^{-\frac{(\theta_1 + \theta_2)|x - P_j|}{2}} + \frac{1}{|x - P_j|} \right). \tag{A.13}
\]
Now we estimate $I_2$. Since
\[
I_2 = \sum_{j \neq k} \int_{\mathbb{R}^3 \setminus (B_{r_j}(P_j) \cup B_{r_k}(x))} e^{-\theta_1 |z - P_j|} e^{-\theta_2 |z - P_k|} \frac{dz}{|x - z|}
\]
\[
= \sum_{j \neq k} \int_{\bigcup_{i=1}^{m} \Lambda_i} e^{-\theta_1 |z - P_j|} e^{-\theta_2 |z - P_k|} \frac{dz}{|x - z|} + \sum_{j \neq k} \int_{\Lambda^c} e^{-\theta_1 |z - P_j|} e^{-\theta_2 |z - P_k|} \frac{dz}{|x - z|}
\]
\[
= \sum_{l=1}^{m} \sum_{j \neq k} \int_{\Lambda_l} e^{-\theta_1 |z - P_j|} e^{-\theta_2 |z - P_k|} \frac{dz}{|x - z|} + \sum_{j \neq k} \int_{\Lambda^c} e^{-\theta_1 |z - P_j|} e^{-\theta_2 |z - P_k|} \frac{dz}{|x - z|}
\]
\[
= \sum_{l=1}^{m} \sum_{j \neq k} \int_{\Lambda_l} e^{-\theta_1 |z - P_j|} e^{-\theta_2 |z - P_k|} \frac{dz}{|x - z|} + \sum_{j \neq k} \int_{\Lambda^c} e^{-\theta_1 |z - P_j|} e^{-\theta_2 |z - P_k|} \frac{dz}{|x - z|}
\]
\[
+ \sum_{l=1}^{m} \sum_{k \neq j} \int_{\Lambda_l} e^{-\theta_1 |z - P_j|} e^{-\theta_2 |z - P_k|} \frac{dz}{|x - z|} + \sum_{j \neq k} \int_{\Lambda^c} e^{-\theta_1 |z - P_j|} e^{-\theta_2 |z - P_k|} \frac{dz}{|x - z|}
\]
\[
:= I_{21} + I_{22} + I_{23} + I_{24}, \tag{A.14}
\]
where
\[
\Lambda_l = \left\{ x \in \mathbb{R}^3 \mid |z - P_l| \leq \frac{\mu}{2} \right\}, \quad \Lambda = \bigcup_{l=1}^{m} \Lambda_l \text{ and } \Lambda^c = \mathbb{R}^3 \setminus \Lambda.
\]
We will estimate $I_{21}, I_{22}, I_{23}$ and $I_{24}$ respectively. Similar to $I_1$, we have
\[
I_{21} \leq C \sum_{l=1}^{m} \int_{\Lambda_l} e^{-\theta_1 |z - P_l|} e^{-\theta_2 |z - P_k|} \frac{dz}{|x - z|}
\]
\[
\leq C \sum_{l=1}^{m} \int_{\Lambda_l} e^{-\theta_1 |z - P_l|} \sum_{h=1}^{\infty} C_h e^{-\theta_2 h \frac{|z - P_k|}{2}} \frac{dz}{|x - z|}
\]
\[
\begin{align*}
&\leq C \sum_{l=1}^{m} \int_{\Lambda_l} e^{-\theta_1 |z-P_l|} e^{-\theta_2 \frac{z}{2}} \frac{dz}{|x-z|} \\
&\leq C \sum_{l=1}^{m} \int_{\Lambda_l} e^{-(\theta_1 + \theta_2) |z-P_l|} \frac{dz}{|x-z|} \tag{A.15} \\
&\leq C \sum_{l=1}^{m} \left( \sum_{i=-1}^{2} |x-P_l|^i e^{-(\theta_1 + \theta_2) |z-P_l|} \right) + \frac{1}{|x-P_l|},
\end{align*}
\]
where we use the fact that
\[
\sum_{h=1}^{\infty} C_3^h e^{-\theta_2 h \frac{z}{2}} = \lim_{h \to \infty} \frac{C_3^h}{e^{\theta_2 h \frac{z}{2}}} \left[ 1 - \left( \frac{C_3}{e^{\theta_2 \frac{z}{2}}} \right)^h \right] = C e^{-\theta_2 \frac{z}{2}} \leq C e^{-\theta_2 |z-P_l|}, \quad z \in \Lambda_l.
\]

Just by the same argument as (A.15), we can also have
\[
I_{22} \leq C \sum_{l=1}^{m} \left( \sum_{i=-1}^{2} |x-P_l|^i e^{-(\theta_1 + \theta_2) |z-P_l|} \right) + \frac{1}{|x-P_l|}. \tag{A.16}
\]

Next, we estimate
\[
\begin{align*}
I_{23} \leq C & \sum_{l=1}^{m} \left( \sum_{j \neq k \neq l} e^{-\theta_1 |z-P_j|} e^{-\theta_2 |z-P_k|} \right) \\
& \leq C \sum_{l=1}^{m} \int_{\Lambda_l} e^{-(\theta_1 + \theta_2) |z-P_l|} \frac{dz}{|x-z|} \tag{A.17} \\
& \leq C \sum_{l=1}^{m} \left( \sum_{i=-1}^{2} |x-P_l|^i e^{-(\theta_1 + \theta_2) |z-P_l|} \right) + \frac{1}{|x-P_l|},
\end{align*}
\]
where we use the fact that
\[
\begin{align*}
&\sum_{j \neq k \neq l} e^{-\theta_1 |z-P_j|} e^{-\theta_2 |z-P_k|} \leq \left( \sum_{j \neq l} e^{-\theta_1 |z-P_j|} \right) \left( \sum_{k \neq l} e^{-\theta_2 |z-P_k|} \right) \\
&\quad \leq \left( \sum_{k \neq l} e^{-\theta_2 |z-P_k|} \right) \left( \sum_{h=1}^{\infty} C_3^h e^{-\theta_1 h \frac{z}{2}} \right) \\
&\quad \leq C \left( \sum_{k \neq l} e^{-\theta_2 |z-P_k|} \right) e^{-\theta_1 \frac{z}{2}} \\
&\quad \leq \left( \sum_{h=1}^{\infty} C_3^h e^{-\theta_2 h \frac{z}{2}} e^{-\theta_1 \frac{z}{2}} \right) \\
&\quad \leq C e^{-\theta_2 \frac{z}{2}} \leq C e^{-(\theta_1 + \theta_2) |z-P_l|}.
\end{align*}
\]

Finally, we estimate \(I_{24}\). Then we have
\[
\begin{align*}
I_{24} &= \sum_{j \neq k} \int_{\Lambda^c} e^{-\theta_1 |z-P_j|} e^{-\theta_2 |z-P_k|} \frac{dz}{|x-z|} \\
&\leq \int_{\Lambda^c} \left( \sum_{j=1}^{m} e^{-\theta_1 |z-P_j|} \right) \left( \sum_{k=1}^{m} e^{-\theta_2 |z-P_k|} \right) \frac{dz}{|x-z|}
\end{align*}
\]
Observe that Lemma A.5. For any $\phi \in C^{\infty}_{c}(\mathbb{R}^{d})$ and $x, y \in \mathbb{R}^{d}$, we have

$$\int \frac{e^{\theta_{j} |x-y|} \sum_{k=1}^{m} e^{-\theta_{j} |x-y|} e^{-\theta_{j} |x-y|}}{|x-y|} \,dz \leq C \sum_{j=1}^{m} \int_{A^{j}} e^{-\theta_{j} |x-y|} |x-y|^{\frac{2}{d-2}} \,dz \leq C \sum_{j=1}^{m} \int_{A^{j}} e^{-\theta_{j} |x-y|} \,dz \leq C \sum_{j=1}^{m} \left( \sum_{i=1}^{m} |x-y|^{i} e^{-\theta_{j} |x-y|} + \frac{1}{|x-y|} \right).$$

(A.18)

From (A.7) to (A.18), the first case holds.

When $x = P_{j}, j \in \{1, 2, \cdots, m\}$, just by the same argument as above, we can show that this case is also true.

Just by the same argument as Lemma A.4, we also have the following estimate.

Lemma A.5. For any $P_{m} \in \Omega_{m}$, we have

$$\int \frac{\left( \sum_{j=1}^{m} e^{-|x-P_{j}|} \right) \left( \sum_{k=1}^{m} e^{-|y-P_{k}|} \right)}{|x-y|} \,dz \leq C \sum_{l=1}^{m} \left[ \sum_{i=1}^{m} \frac{|x-P_{l}|^{i}}{e^{|x-P_{l}|}} \left( e^{-|x-P_{l}|} + 1 \right) + \frac{1}{|x-P_{l}|} \right],$$

where $x \in \mathbb{R}^{3} \setminus \{P_{1}, \cdots, P_{m}\}$,

$$\leq \left\{ \begin{array}{ll}
C \sum_{l \neq j_{0}} \left[ \sum_{i=1}^{m} \frac{|P_{j_{0}} - P_{l}|^{i}}{e^{|P_{j_{0}} - P_{l}|}} \left( e^{-|P_{j_{0}} - P_{l}|} + 1 \right) + \frac{1}{|P_{j_{0}} - P_{l}|} \right], & x = P_{j_{0}}, j_{0} \in \{1, \cdots, m\}.
\end{array} \right.$$
It follows from Lemmas A.1 to A.4 that

\[ (2.20) \]

\[
\frac{1}{|x - P_i|} \left( e^{-\gamma|x-P_i|} + 1 \right) \]

\[
\times \left( e^{-\gamma|x-P_j|} + \sum_{k \neq j} e^{-\gamma|x-P_k|} \right) \]

\[
\leq C \int_0^\infty \left[ \sum_{i=1}^4 e^{\frac{t^i}{(n+2)l}} \left( e^{-\frac{1}{4}t} + 1 \right) + te^{-\gamma t} \right]
\]

\[
+ C \int_{B_{\tilde{R}}(P_j)} \left[ \sum_{i=1}^2 \frac{\mu^i}{2} e^{-\gamma|x-P_j|} \left( e^{-\frac{2}{4}t} + 1 \right) + \frac{1}{\mu} e^{-\frac{2}{4}t} \right] \leq C. \]

So, from (A.19), we have if \( x \in \mathbb{R}^3 \setminus \{P_1, P_2, \ldots, P_m\} \),

\[
\left| \int_{B_{\tilde{R}}(P_j)} K \Phi w_{P_m} \varphi_{P_m} \right| \leq C \| \varphi_{P_m} \|. \tag{A.20}
\]

On the other hand, when \( x = P_j, j \in \{1, 2, \ldots, m\} \), using Lemma A.4 and proceeding as done in (A.20), we also have

\[
\left| \int_{B_{\tilde{R}}(P_j)} K (P_j) \Phi w_{P_m} (P_j) \varphi_{P_m} (P_j) \right| \leq C \| \varphi_{P_m} \|. \tag{A.21}
\]

Then (2.19) has been proved. \( \square \)

Proof of (2.20). Without loss of generality, we assume that \( x \in \mathbb{R}^3 \setminus \{P_1, \ldots, P_m\} \). It follows from Lemmas A.1 to A.4 that

\[
\left| \int_{B_{\tilde{R}}(P_j)} K \left( \int_{\mathbb{R}^3} \frac{K w_{P_m}}{|x-z|} \, dz \right) w_{P_m} \right|
\]

\[
\leq C \| \varphi_{P_m} \| \int_{B_{\tilde{R}}(P_j)} \left( \int \frac{\sum_{j=1}^m e^{-|z-P_j|}}{|x-z|} \left( \sum_{k=1}^m e^{-\gamma|z-P_k|} \right) \, dz \right) \sum_{l=1}^m e^{-|x-P_l|}
\]

\[
\leq C \| \varphi_{P_m} \| \| \sum_{l=1}^m \left( \sum_{i=1}^2 \frac{|x - P_l|^i}{e^{\frac{2}{2} |x-P_l|}} (e^{-\frac{2}{4}t} + 1) + \frac{1}{|x - P_l|} \right) \sum_{i=1}^m e^{-|x-P_l|}
\]

\[
=: C \| \varphi_{P_m} \| \tilde{I}. \tag{A.22}
\]

Note that

\[
\tilde{I} = \int_{B_{\tilde{R}}(P_j)} \left[ \left( \sum_{i=1}^2 \frac{|x - P_j|^i}{e^{\frac{2}{4} |x-P_j|}} (e^{-\frac{2}{4}t} + 1) + \frac{1}{|x - P_j|} \right) \right]
\]

\[
+ \sum_{l \neq j} \left( \sum_{i=1}^2 \frac{|x - P_l|^i}{e^{\frac{2}{4} |x-P_l|}} (e^{-\frac{2}{4}t} + 1) + \frac{1}{|x - P_l|} \right) \right] (e^{-|x-P_j|} + \sum_{l \neq j} e^{-|x-P_l|})
\]

\[
\leq C \int_{B_{\tilde{R}}(P_j)} \left[ \left( \sum_{i=1}^2 \frac{|x - P_j|^i}{e^{\frac{2}{4} |x-P_j|}} (e^{-\frac{2}{4}t} + 1) + \frac{1}{|x - P_j|} \right) \right]
\]

\[
+ \left( \sum_{i=1}^2 \frac{\mu^i}{e^{\frac{2}{4} |x-P_j|}} (e^{-\frac{2}{4}t} + 1) + \frac{1}{\mu} \right) \right] \left( e^{-|x-P_j|} + e^{-\frac{2}{4}t} \right]
\]
It follows all the inequalities above that (4.9) holds.

Proof of (4.9). By direct computations, we have

\[
\frac{\epsilon}{4} \int K(x) \Phi_{w_P} w_P^2 \\
= \frac{\epsilon}{4} \int K(x) \Phi_{w_P} w_P^2 + \frac{\epsilon}{4} \int K(x) \Phi_{w_P} \varphi_P^2 + \epsilon \int K(x) \Phi_{w_P} w_P \varphi_P \\
+ \frac{\epsilon}{4} \int K(x) \Phi_{\varphi_P} w_P^2 + \frac{\epsilon}{4} \int K(x) \Phi_{\varphi_P} \varphi_P^2 + \epsilon \int K(x) \Phi_{\varphi_P} w_P \varphi_P \\
+ \epsilon \int K(x) \left( \int \frac{K(z) \varphi_P(z) w_P(z)}{|x-z|} dz \right) w_P \varphi_P ,
\]

\[
\left| \int K(x) \Phi_{w_P} \varphi_P^2 \right| \leq \max_{x \in \mathbb{R}^3} |K(x)| \left( \int |\Phi_{w_P}|^6 \right)^{\frac{1}{6}} \left( \int |\varphi_P|^{12} \right)^{\frac{1}{12}} \\
\leq C \|\Phi_{w_P}\|_{L^6} \|\varphi_P\|^2 \leq C \|\varphi_P\|^2 ,
\]

\[
\left| \int K(x) \Phi_{w_P} w_P \varphi_P \right| \leq \sigma \int K \Phi_{w_P} w_P^2 + \frac{1}{4\sigma} \int K \Phi_{w_P} \varphi_P^2 \\
\leq \sigma \int K \Phi_{w_P} w_P^2 + C \|\varphi_P\|^2 , \text{ where } \sigma > 0 \text{ small enough},
\]

\[
\left| \int K(x) \Phi_{\varphi_P} w_P^2 \right| \leq \max_{x \in \mathbb{R}^3} |K(x)| \left( \int |\Phi_{\varphi_P}|^6 \right)^{\frac{1}{6}} \left( \int |w_P|^{12} \right)^{\frac{1}{12}} \\
\leq C \|\Phi_{\varphi_P}\|_{L^6} \|w_P\|^2 \leq C \|\varphi_P\|^2 ,
\]

\[
\left| \int K(x) \Phi_{\varphi_P} \varphi_P^2 \right| \leq \max_{x \in \mathbb{R}^3} |K(x)| \left( \int |\Phi_{\varphi_P}|^6 \right)^{\frac{1}{6}} \left( \int |\varphi_P|^{12} \right)^{\frac{1}{12}} \\
\leq C \|\Phi_{\varphi_P}\|_{L^6} \|\varphi_P\|^2 \leq C \|\varphi_P\|^4 \leq C \|\varphi_P\|^2 ,
\]

\[
\left| \int K(x) \Phi_{\varphi_P} w_P \varphi_P \right| \leq \max_{x \in \mathbb{R}^3} |K(x)| \left( \int |\Phi_{\varphi_P}|^6 \right)^{\frac{1}{6}} \left( \int |w_P|^{12} \right)^{\frac{1}{12}} \left( \int |\varphi_P|^{12} \right)^{\frac{1}{12}} \\
\leq C \|\Phi_{\varphi_P}\|_{L^6} \|w_P\| \|\varphi_P\| \leq C \|\varphi_P\|^3 \|w_P\| \leq C \|\varphi_P\|^2 ,
\]

and

\[
\left| \int K(x) \left( \int \frac{K(z) \varphi_P(z) w_P(z)}{|x-z|} dz \right) w_P \varphi_P \right| \\
\leq \max_{x \in \mathbb{R}^3} |K(x)| \int |\Phi_{\varphi_P}|^{\frac{1}{2}} |\Phi_{w_P}|^{\frac{1}{2}} |w_P| |\varphi_P| \\
\leq C \left( \int |\Phi_{\varphi_P}|^6 \right)^{\frac{1}{6}} \left( \int |\Phi_{w_P}|^6 \right)^{\frac{1}{6}} \left( \int |w_P|^2 \right)^{\frac{1}{2}} \left( \int |\varphi_P|^3 \right)^{\frac{1}{3}} \\
\leq C \|\Phi_{\varphi_P}\|_{L^6}^{\frac{1}{3}} \|\Phi_{w_P}\|_{L^6}^{\frac{1}{3}} \|w_P\| \|\varphi_P\| \leq C \|w_P\|^2 \|\varphi_P\|^2 \leq C \|\varphi_P\|^2 .
\]

It follows all the inequalities above that (4.9) holds.
REFERENCES

[1] A. Ambrosetti, On Schrödinger-Poisson systems, *Milan J. Math.*, **76** (2008), 257–274.

[2] A. Ambrosetti and D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem, *Commun. Contemp. Math.*, **10** (2008), 391–404.

[3] W. Ao and J. Wei, Infinitely many positive solutions for nonlinear equations with non-symmetric potential, *Calc. Var. Partial Differential Equ.*, **51** (2014), 761–798.

[4] W. Ao, J. Wei and J. Zeng, An optimal bound on the number of interior spike solutions for the Lin-Ni-Takagi problem, *J. Funct. Anal.*, **265** (2013), 1324–1356.

[5] A. Azzollini and A. Pomponio, Ground state solutions for the non-linear Schrödinger-Maxwell equations, *J. Math. Anal. Appl.*, **345** (2008), 90–108.

[6] A. Azzollini and A. Pomponio, Ground state solutions for the non-linear Schrödinger-Maxwell equations with a singular potential, arXiv:0706.1679 [math.AP].

[7] A. Bahri and Y. Y. Li, On a min-max procedure for the existence of a positive solution for certain scalar field equations in $\mathbb{R}^N$, *Rev. Mat. Iberoamericana*, **6** (1990), 1–15.

[8] V. Benci and D. Fortunato, Solitary waves of the nonlinear Klein-Gordon equation coupled with Maxwell’s equations, *Rev. Math. Phys.*, **14** (2002), 409–420.

[9] V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, *Topol. Methods Nonlinear Anal.*, **11** (1998), 283–293.

[10] R. Benguria, H. Brézis and E. Lieb, The Thomas-Fermi-Von Weizsäcker theory of atoms and molecules, *Comm. Math. Phys.*, **79** (1981), 167–180.

[11] D. Cassani, Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell’s equations, *Nonlinear Anal.*, **58** (2004), 733–747.

[12] I. Catto and P. Loins, Binding of atoms and stability of molecules in Hartree and Thomas-Fermi type theories. I. A necessary and sufficient condition for the stability of general molecular systems, *Comm. Partial differential equations*, **17** (1992), 1051–1110.

[13] G. Cerami, D. Passaseo and S. Solimini, Infinitely many positive solutions to some scalar field equations with non symmetric coefficients, *Comm. Pure Appl. Math.*, **66** (2013), 372–413.

[14] G. Cerami and G. Vaira, Positive solutions for some non-autonomous Schrödinger-Poisson systems, *J. Differential Equations*, **248** (2010), 521–543.

[15] G. Coclite, A multiplicity result for the nonlinear Schrödinger-Maxwell equations, *Commun. Appl.*, **7** (2003), 417–423.

[16] E. N. Dancer, On the uniqueness of the positive solution of a singularly perturbed problem, *Rocky Mountain J. Math.*, **25** (1995), 957–975.

[17] T. D’Aprile and D. Mugnai, Non-existence results for the coupled Klein-Gordon-Maxwell equations, *Adv. Nonlinear Stud.*, **4** (2004), 307–322.

[18] T. D’Aprile and D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, *Proc. Roy. Soc. Edinburg Sect.*, **134** (2004), 893–906.

[19] T. D’Aprile and J. Wei, On bound states concentration on spheres for the Maxwell-Schrödinger equations, *SIAM J. Math. Anal.*, **37** (2005), 321–342.

[20] M. Delpino, J. Wei and W. Yao, Intermediate reduction method and infinitely many positive solutions of nonlinear Schrödinger equations with non-symmetric potentials, *Calc. Var. Partial Differential Equations*, **53** (2015), 473–523.

[21] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$. In: *Mathematical Analysis and Applications*, part A, 369–402. Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London, 1981.

[22] I. Ianni, Solutions of the Schrödinger-Poisson system concentrating on spheres, part II: Existence, *Math. Models Methods Appl. Sci.*, **19** (2009), 877–910.

[23] I. Ianni and G. Vaira, Solutions of the Schrödinger-Poisson system concentrating on spheres, part I: Necessary conditions, *Math. Models Methods Appl. Sci.*, **19** (2009), 707–720.

[24] I. Ianni and G. Vaira, Solutions of the Schrödinger-Poisson system concentrating on spheres, part with potential, *Adv. Nonlinear Stud.*, **8** (2008), 573–595.

[25] G. Li, S. Peng and S. Yan, Infinitely many positive solutions for the nonlinear Schrödinger-Poisson system, *Commun. Contemp. Math.*, **12** (2010), 1069–1092.

[26] G. Li, S. Peng and C. Wang, Multi-bump solutions for the nonlinear Schrödinger-Poisson system, *J. Math. Phys.*, **52** (2011), 053505, 19 pp.

[27] F. Lin, W. Ni and J. Wei, On the number of interior peak solutions for a singularly perturbed Neumann problem, *Comm. Pure Appl. Math.*, **60** (2007), 252–281.
[28] E. Lions and B. Simon, The Thomas-Fermi theory of atoms, molecules and solids, *Adv. Math.*, 23 (977), 22–116.
[29] P. Lions, Solutions of Hartree-Fock equations for Coulomb systems, *Comm. Math. Phys.*, 109 (1987), 33–97.
[30] P. Markowich, C. Ringhofer and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag, Vienna, 1990.
[31] C. Mercuri, Positive solutions of nonlinear Schrödinger-Poisson systems with radial potentials vanishing at infinity, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 19 (2008), 211–227.
[32] M. Monica, F. Pacard and J. Wei, Finite-energy sign-changing solutions with dihedral symmetry for the stationary nonlinear Schrödinger equation, *J. Eur. Math. Soc. (JEMS)*, 14 (2012), 1923–1953.
[33] W. M. Ni and I. Takagi, Locating the peaks of least energy solutions to a semilinear Neumann problem, *Duke Math. J.*, 70 (1993), 247–281.
[34] P. Poláčik, Morse indices and bifurcations of positive solutions of $\Delta u + f(u) = 0$ on $\mathbb{R}^N$, *Indiana Univ. Math. J.*, 50 (2001), 1407–1432.
[35] D. Ruiz, Semiclassical states for coupled Schrödinger-Maxwell equations: concentration around a sphere, *Math. Models Methods Appl. Sci.*, 15 (2005), 141–164.
[36] D. Ruiz, The nonlinear Schrödinger-Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.*, 237 (2006), 655–674.
[37] Z. Wang and H. Zhou, Positive solution for a nonlinear stationary Schrödinger-Poisson system in $\mathbb{R}^3$, *Discrete Contin. Dyn. Syst.*, 18 (2007), 809–816.

Received for publication September 2017.

E-mail address: chunhuawang@mail.ccnu.edu.cn
E-mail address: yyangecho@163.com