ON A CONSTANT RANK THEOREM FOR NONLINEAR ELLIPTIC PDES

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ABSTRACT. We give a new proof of Bian-Guan’s constant rank theorem for nonlinear elliptic equations. Our approach is to use a linear expression of the eigenvalues of the Hessian instead of quotients of elementary symmetric functions.

1. Introduction. A constant rank theorem asserts that a convex solution \( u \) of an elliptic partial differential equation, satisfying appropriate conditions, must have constant rank. In the 1980’s Caffarelli-Friedman [5] proved such a result for semi-linear elliptic equations, and a similar result was discovered around the same time by Yau (see [22]). These results were extended to more general elliptic and parabolic PDEs by Korevaar-Lewis [19], Caffarelli-Guan-Ma [6] and Bian-Guan [2, 3]. Moreover, the constant rank theorem (also known as the “microscopic convexity principle”) has been shown to hold for a number of geometric differential equations involving the second fundamental form of hypersurfaces [12, 11, 13, 6, 2]. In addition, these ideas have been investigated in the complex setting [20, 10, 15, 14], where there are applications to Kähler geometry. Constant rank theorems are also closely related to the question of convexity of solutions of non-linear PDE on convex domains (the “macroscopic convexity principle”) [4, 7, 18, 17, 16, 1, 21, 27].

A common approach for establishing a constant rank theorem is to consider expressions involving the elementary symmetric polynomials \( \sigma_\ell \) of the eigenvalues \( \lambda_1 \leq \cdots \leq \lambda_n \) of the Hessian \( D^2u \). Indeed Bian-Guan [2] proved a rather general constant rank theorem for nonlinear elliptic equations

\[ F(D^2u, Du, u, x) = 0 \]

subject to a local convexity condition for \( F \) (see (2) below). Their proof relies on a sophisticated computation using the quantity \( \sigma_{\ell+1} + \frac{\sigma_{\ell+2}}{\sigma_{\ell+1}} \). In this paper, we take...
a different approach by computing directly with the eigenvalues of $D^2u$ (compare with the works [28, 24, 25], for example). We consider the simple linear expression
\[ \lambda_\ell + 2\lambda_{\ell-1} + \cdots + \ell \lambda_1, \]
(1)
of the smallest \( \ell \) eigenvalues of \( D^2u \) (more precisely, we perturb \( u \) slightly first). While this expression is not smooth in general, it has the crucial property that it is semi-concave, as long as \( u \) is sufficiently regular.

We now describe our result more precisely. Let \( \Omega \) be a domain in \( \mathbb{R}^n \). Write \( \text{Sym}(n) \) for the space of real symmetric \( n \times n \) matrices, and \( \text{Sym}^+(n) \) for the subset that are strictly positive definite. We consider the real-valued function
\[ F = F(A, p, u, x) \in C^2(\text{Sym}(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega) \]
which satisfies the condition that for each \( p \in \mathbb{R}^n \),
\[ (A, u, x) \in \text{Sym}^+(n) \times \mathbb{R} \times \Omega \mapsto F(A^{-1}, p, u, x) \]
is locally convex. (2)

Now let \( u \in C^3(\Omega) \) be a convex solution of
\[ F(D^2u, Du, u, x) = 0, \]
(3)
subject to the ellipticity condition
\[ F^{ij}(D^2u, Du, u, x) > 0 \quad \text{on} \quad \Omega, \]
(4)
where we write \( F^{ij} \) for the derivative of \( F \) with respect to the \( (i, j) \)th entry \( A_{ij} \) of \( A \). Our main result is a new proof of the following theorem of Bian-Guan [2].

**Theorem 1.1.** With \( u \) and \( F \) as above, satisfying (2), (3) and (4), the Hessian \( D^2u \) has constant rank in \( \Omega \).

We give the proof in Section 2 below. The heart of the proof is to establish a differential inequality (see (9) below) for our expression (1) and then apply a standard Harnack inequality. Once we have perturbed \( u \) so that the eigenvalues of its Hessian are distinct, this is a straightforward computation (simpler than the analogous calculation in [2]). The rest of the proof is concerned with making this formal argument rigorous.

If we replace the condition (2) with a stronger “strict convexity” condition, we can prove the following additional consequence (cf. [2]). Let us write \( n - k \) for the rank of \( D^2u \), which is now constant. Then there exist \( k \) fixed directions \( X_1, \ldots, X_k \) such that \( (D^2u(x))(X_j) = 0 \) for all \( 1 \leq j \leq k \) and all \( x \in \Omega \). We show this in Section 3. Examples (including one of Korevaar-Lewis [19]) show that a stronger condition than (2) is indeed necessary for this conclusion.

We expect that our techniques can also be used to give new proofs of constant rank theorems for parabolic and geometric equations.

2. Proof of Theorem 1.1. As in [2], note that the convexity condition (2) can be written as follows: for every symmetric matrix \( (X_{ab}) \in \text{Sym}(n) \), vector \( (Z_a) \in \mathbb{R}^n \) and \( Y \in \mathbb{R} \), we have
\[ 0 \leq F^{ab,rs} X_{ab} X_{rs} + 2 F^{ar} A^{bs} X_{ab} X_{rs} + F^{x_a x_b} Z_a Z_b \]
\[ - 2 F^{ab, u} X_{ab} Y - 2 F^{ah, x_c} X_{ah} Z_c + 2 F^{u, x_a} Y Z_a + F^{u, u} Y^2, \]
(5)
where we are evaluating the derivatives \( F \) at \( (A, p, u, x) \) for a positive definite matrix \( A \). Here, we are using the usual notation for derivatives of \( F \) (see [2]), we write \( A^{ij} \) for the \( (i, j) \)th entry of \( A^{-1} \), and we use the standard convention of summing repeated indices from 1 to \( n \).
To prove Theorem 1.1, it is sufficient to prove the following. Suppose that at $x_0 \in \Omega$, the Hessian $D^2 u$ has at least $k$ zero eigenvalues. Then there exists $r_0 > 0$ such that the Hessian $D^2 u$ has at least $k$ zero eigenvalues on the ball $B_{r_0}(x_0) \subset \Omega$ of radius $r_0$ centered at $x_0$.

We fix then this point $x_0 \in \Omega$, and write $B = B_{r_0}(x_0)$ for a sufficiently small $r_0 > 0$ (which we may shrink later) so that $B \subset \Omega$.

As pointed out in [2], it follows from our assumptions on $u$ and $F$ and the standard elliptic regularity theory that $u$ is in $W^{4,p}(B)$ for all $p$. We fix, once and for all, $p$ strictly larger than $n$. Let $\varepsilon > 0$. Since the polynomials are dense in $W^{4,p}(B)$, we can find a polynomial $P$ such that

$$\|P - u\|_{W^{4,p}(B)} \leq \varepsilon. \quad (6)$$

By the Sobolev embedding theorem,

$$\|P - u\|_{C^{1,\alpha}(B)} \leq C\varepsilon, \quad (7)$$

for uniform constants $C > 0$ and $\alpha \in (0,1)$.

Now a generic polynomial $P$ will have the property that $D^2 P$ has distinct eigenvalues away from a proper real analytic subset. Note also that by the convexity of $u$, the Hessian $D^2 u$ is nonnegative definite. Hence, by making a small perturbation to $P$, we may assume without loss of generality that the eigenvalues $\Lambda_1 \leq \cdots \leq \Lambda_n$ of $D^2 P$ are positive and distinct away from a proper real analytic subset $V \subset B$.

At $x \in B \setminus V$ we have

$$0 < \Lambda_1 < \cdots < \Lambda_n.$$

We consider, for $\ell = 1, \ldots, k$, the positive quantity

$$Q^{(\ell)} = \Lambda_\ell + 2\Lambda_{\ell-1} + \cdots + \ell\Lambda_1 = \sum_{j=1}^{\ell} (\ell + 1 - j)\Lambda_j.$$

We will prove that on $B \setminus V$ (after possibly shrinking the radius of $B$), and for each $\ell = 1, \ldots, k$,

$$Q^{(\ell)} + |DQ^{(\ell)}| \leq c_\varepsilon, \quad (8)$$

where we write $c_\varepsilon$ to mean a constant satisfying $c_\varepsilon \to 0$ as $\varepsilon \to 0$. Once (8) holds for $\ell = k$ we are done since then $Q^{(k)} \to 0$ on $B$ as $\varepsilon \to 0$, which implies that the first $k$ eigenvalues of $D^2 u$ must vanish everywhere on $B$.

We prove (8) by a finite induction. Assume it holds for $1, 2, \ldots, \ell - 1$ (if $\ell = 1$, we do not assume anything). Write $Q = Q^{(\ell)}$. We first show that $Q$ satisfies the following differential inequality on $B \setminus V$ (on which $Q$ is smooth),

$$F^{ab} |P_{ab}Q_{ab} \leq C|DQ| + CQ + f_\varepsilon, \quad (9)$$

where $f_\varepsilon$ has the property that $\|f_\varepsilon\|_{L^p(B)} \to 0$ as $\varepsilon \to 0$, and for a uniform $C$.

In what follows, we will denote by $c_\varepsilon, f_\varepsilon, C$ any quantities with the same properties as described here, where the uniformity will be clear from the context.

We compute at a fixed point $x \in B \setminus V$, and we assume that $D^2 P$ is diagonal at this point with the eigenvalue $\Lambda_j$ given by $P_{jj}$. Then the first derivative of $\Lambda_j$ is given at $x$ by

$$(\Lambda_j)_a = P_{jja}. \quad (10)$$

The inductive hypothesis tells us that $|D\Lambda_j| \leq c_\varepsilon$ for $j = 1, \ldots, \ell - 1$, and hence at $x$,

$$|P_{jji}| \leq c_\varepsilon, \quad \text{for all } j = 1, \ldots, \ell - 1, \ i = 1, \ldots, n. \quad (11)$$
where

\[ (\Lambda_j)_{ab} = P_{j,ab} + 2 \sum_{m \neq j} \frac{P_{maj} P_{mbj}}{\Lambda_j - \Lambda_m}. \]

Hence

\[ Q_{ab} = \sum_{j=1}^{\ell} (\ell + 1 - j) P_{j,ab} + 2 \sum_{j=1}^{\ell} \sum_{m \neq j} (\ell + 1 - j) \frac{P_{maj} P_{mbj}}{\Lambda_j - \Lambda_m} \]

\[ = \sum_{j=1}^{\ell} (\ell + 1 - j) P_{j,ab} + 2 \sum_{1 \leq j < m \leq \ell} (m - j) \frac{P_{maj} P_{mbj}}{\Lambda_j - \Lambda_m} \]

\[ + 2 \sum_{j=1}^{\ell} \sum_{m > \ell} (\ell + 1 - j) \frac{P_{maj} P_{mbj}}{\Lambda_j - \Lambda_m}, \]

where the second line is obtained after cancelling the positive terms with \( \Lambda_j - \Lambda_m \) for \( j > m \) with the corresponding negative terms.

We now differentiate the equation (3) twice in the \( u \) direction to obtain

\[ 0 = F^{|u|}_{ab} u_{ab,jj} + F^{|P_u|}_{ab} u_{ab,jj} + F^{|F_u|}_{ab} u_{ab,jj} \]

\[ + F^{ab,rs}|a u_{ab,j} u_{rs,j} + F^{|F_{u,p}|}_{ab} u_{ab,j} + F^{u,u}|a u_{ab,j} + F^{x,\lambda}\|u| \]

\[ + 2 F^{ab,pr}|a u_{ab,j} u_{sj} + 2 F^{ab,u}|a u_{ab,j} + 2 F^{ab,x}|a u_{ab,j} \]

\[ + 2 F^{P_u,u}|a u_{ab,j} + 2 F^{P_{u,x}|a u_{ab,j} + 2 F^{u,u,\lambda}|a u_{ab,j}. \]

Replace \( u \) by the polynomial \( P \), at the expense of an error term \( f_\varepsilon \), to get

\[ f_\varepsilon = F^{ab} P_{ab,jj} + F^{P_u} P_{ab,jj} + F^{u} P_{ab,jj} \]

\[ + F^{ab,rs} P_{ab,jj} + F^{P_{u,p}} P_{ab,jj} + F^{u,u} P_{ab,jj} + F^{x,\lambda} P_{ab,jj} \]

\[ + 2 F^{ab,pr} P_{ab,jj} + 2 F^{ab,u} P_{ab,jj} + 2 F^{ab,x} P_{ab,jj} \]

\[ + 2 F^{P_u,u} P_{ab,jj} + 2 F^{P_{u,x}} P_{ab,jj} + 2 F^{u,u,\lambda} P_{ab,jj}, \]

where, here and for the rest of this section, \( F_{ab} \), \( F_{u,p} \) etc are all evaluated at \( P \), and \( \|f_\varepsilon\|_{L^1(B)} \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \). Combining this with (12), we obtain

\[ F^{ab} Q_{ab} \leq 2 \sum_{1 \leq j < m \leq \ell} (m - j) F^{ab} P_{maj} P_{mbj} \]

\[ \Lambda_j - \Lambda_m \]

\[ + 2 \sum_{j=1}^{\ell} \sum_{m > \ell} (\ell + 1 - j) F^{ab} P_{maj} P_{mbj} \]

\[ \Lambda_j - \Lambda_m \]

\[ + C \left( \sum_{i=1}^{n} \sum_{a,b=1}^{\ell} |P_{ab}| \right) + C Q + f_\varepsilon + (\ast), \]

where

\[ (\ast) = - \sum_{j=1}^{\ell} (\ell + 1 - j) \left\{ \sum_{a,b,r,s=\ell+1}^{\ell} F^{ab,rs} P_{ab,jj} + F^{u,u} P_{ab,jj} + F^{x,\lambda} P_{ab,jj} \]

\[ + 2 \sum_{a,b=\ell+1}^{\ell} F^{ab,u} P_{ab,jj} + 2 \sum_{a,b=\ell+1}^{\ell} F^{ab,x} P_{ab,jj} \right\}. \]
Note that we have separated out all terms which involve \( P_{abj} \) with at least two indices between 1 and \( \ell \), as well as all terms involving the eigenvalues \( P_{aa} \) for \( a \leq \ell \).

For each fixed \( j \), we now use (5), with

\[
X_{pq} = \begin{cases} 
- P_{pqj} & \text{if } p, q > \ell \\
0 & \text{otherwise,}
\end{cases}
\]

\[
Z_i = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise,}
\end{cases}
\]

\[ Y = P_j. \tag{16} \]

This implies

\[
0 \leq \sum_{a,b,r,s=\ell+1}^n F_{ab,rs} P_{abj} P_{rsj} + 2 \sum_{a,b,m=\ell+1}^n F_{ab} P_{maj} P_{mbj} \Lambda_m + F_{xj,xj}
\]

\[
+ 2 \sum_{a,b=\ell+1}^n F_{ab,u} P_{abj} P_{j} + 2 \sum_{a,b=\ell+1}^n F_{ab,xj} P_{abj} + 2 F_{u,xj} P_{j} + F_{u,u} P_{j}^2,
\]

and hence, using that \( 0 < \Lambda_{m} - \Lambda_{j} < \Lambda_{m} \) whenever \( j < m \),

\[
\ast \leq 2 \sum_{j=1}^\ell \sum_{a,b,m=\ell+1}^n (\ell + 1 - j) F_{ab} P_{maj} P_{mbj} \Lambda_m
\]

\[
\leq 2 \sum_{j=1}^\ell \sum_{m>\ell}^n (\ell + 1 - j) F_{ab} P_{maj} P_{mbj} \Lambda_m - \Lambda_j. \tag{17}
\]

On the other hand, for a uniform \( c > 0 \) we have

\[
2 \sum_{1 \leq j < m \leq \ell} (m - j) F_{ab} P_{maj} P_{mbj} \Lambda_m - \Lambda_j \geq c Q^{-1} \sum_{i=1}^n \sum_{1 \leq a < b \leq \ell} P_{abi}^2,
\]

using the ellipticity assumption, and that \( Q \geq \Lambda_{m} - \Lambda_{j} \) whenever \( j < m \leq l \). Using this, together with the inequality

\[ C |P_{abi}| \leq c Q^{-1} P_{abi}^2 + C' Q, \]

it follows that

\[
C \sum_{a,b=1}^n \sum_{i=1}^\ell |P_{abi}| \leq C \sum_{a=1}^n \sum_{i=1}^\ell |P_{aai}|
\]

\[
+ 2 \sum_{1 \leq j < m \leq \ell} (m - j) F_{ab} P_{maj} P_{mbj} \Lambda_m - \Lambda_j + C' Q, \tag{18}
\]

for suitable \( C' \). Then combining (15) with (17), (18), and making use of the inductive hypothesis (11), we obtain

\[
F_{ab} Q_{ab} \leq C \sum_{i=1}^n \sum_{a=1}^\ell |P_{aai}| + C Q + f \varepsilon
\]

\[
\leq C \sum_{i=1}^n |Q_i| + C Q + f \varepsilon.
\]

Namely, the differential inequality (9) holds on \( B \setminus V \).

We now wish to apply the Harnack inequality to \( Q \). Note however that \( Q \) may not be smooth on \( V \). On the other hand, \( Q \) is a semi-concave function on the whole of \( B \) (see for example [8, p.40]) which means that \( Q = U + W \) where \( U \) is concave on \( B \) and \( W \in C^{1,1}(B) \). We will use following version of the Harnack inequality (cf. [26, Lemma 2.1.(III)]), where the assumptions are slightly different).
Lemma 2.1. Consider the operator $L$ given by $Lv = a^{ij}D_{ij}v + b^iD_i v + cv$ with bounded coefficients, with $a^{ij}$ satisfying $\lambda|\xi|^2 \leq a^{ij}\xi_i \xi_j \leq \Lambda|\xi|^2$ for all $\xi \in \mathbb{R}^n$, with $\lambda, \Lambda > 0$. Let $v$ be a semi-concave nonnegative function on the ball $B \subset \mathbb{R}^n$ which is smooth on $B \setminus V$, where $V$ is a proper real analytic variety. Suppose that $Lv \leq f$ in $B$ where $f \in L^p(B)$. Then on the half size ball $B'$,

$$\left( \frac{1}{|B'|} \int_{B'} v^q \right)^{1/q} \leq C \left( \inf_{B'} v + \|f\|_{L^p(B)} \right),$$  

(19)

for positive constants $C$ and $q$ depending only on $n, \lambda, \Lambda$, bounds for $b^i$ and $c$ and the radius of the ball $B$.

Proof. We give the proof for the reader’s convenience. If $v \in W^{2,n}(B)$, then this result is standard (see for example [9, Theorem 9.22]). We prove the result we need for $v$ semi-concave using a mollification argument. Let $v_{\varepsilon}$ denote a standard mollification of $v$, with a mollifier whose support has radius $\varepsilon > 0$. Let $\delta > 0$. Outside the $\delta$-neighborhood of $V$ we have a bound $\sum_{|\gamma| \leq 3} |D^\gamma v| \leq C_\delta$. For sufficiently small $\varepsilon$ we will then have

$$\sum_{|\gamma| \leq 2} |D^\gamma(v_{\varepsilon} - v)| \leq \varepsilon C_\delta,$$

away from the $2\delta$-neighborhood of $V$. Applying the differential inequality $Lv \leq f$ outside the $2\delta$-neighborhood of $V$ we will have

$$Lv_{\varepsilon} = Lv + L(v_{\varepsilon} - v) \leq f + C\varepsilon C_\delta$$

where $C$ depends on the bounds for the coefficients of $L$.

On the other hand, since $v$ is semi-concave we have a fixed upper bound for $D^2 v_{\varepsilon}$ everywhere. In particular, near $V$ we have $L(v_{\varepsilon}) \leq C$. Since the $2\delta$-neighborhood of $V$ has measure at most $C\delta$ for some fixed $C$, applying the standard Harnack inequality to the smooth function $v_{\varepsilon}$, we obtain

$$\left( \frac{1}{|B'|} \int_{B'} v_{\varepsilon}^q \right)^{1/q} \leq C(\inf_{B'} v_{\varepsilon} + \|f\|_{L^p(B)} + \varepsilon C_\delta + \delta^{1/n}).$$

We can then first choose $\delta$ very small, and then let $\varepsilon \to 0$, to obtain the required inequality for $v$. \hfill \Box

Applying this lemma to $Q$, we obtain

$$\left( \frac{1}{|B'|} \int_{B'} Q^q \right)^{1/q} \leq C(\inf_{B'} Q + \|f_{\varepsilon}\|_{L^p(B)}) \leq c_\varepsilon,$$

since $\inf_{B'} Q$ tends to zero as $\varepsilon \to 0$. From (10) we have a uniform Lipschitz bound on $Q$, and this implies that $|Q| \leq c_\varepsilon$ on $B'$.

The required bound on $|DQ|$ in (8) then follows from the next lemma, which uses again the semi-concavity of $Q$. Recall from (7) that we have a uniform bound on the $C^{3,\alpha}$ norm of $P$.

Lemma 2.2. There is a constant $C$ depending on the $C^{3,\alpha}$ norm of $P$ with the following property. For $x \in \frac{1}{2}B' \setminus V$ and sufficiently small $\varepsilon$, we have

$$|DQ(x)|^{1+1/\alpha} \leq Cc_\varepsilon,$$

(20)

where $c_\varepsilon = \sup_{B'} Q$. Here $\alpha \in (0,1)$ is the constant from (7).
Proof. Let \( x \in \frac{1}{2} B' \setminus V \) and let \( \xi \) be a unit vector for which \( D_\xi Q(x) < 0 \). We already know that \( D_\xi Q(x) \) is bounded, since \( Q \) is uniformly Lipschitz, and our goal is to obtain the stronger bound (20). For \( r > 0 \) sufficiently small (to be determined later), define \( y = x + rt \xi \in B' \) and \( x_t = (1-t)x + ty = x + rt\xi, \) for \( t \in [0,1] \).

Now define the function \( h : \text{Sym}(n) \to \mathbb{R} \) by
\[
h(A) = \ell \lambda_1(A) + \ldots + \lambda_d(A),
\]
so that \( Q(z) = h(D^2 P(z)) \) for any \( z \). Note that \( h \) is a concave Lipschitz function. Using the concavity of \( h \) we have
\[
(1-t)Q(x) + tQ(y) = (1-t)h(D^2 P(x)) + th(D^2 P(y))
\]
\[
\leq h \left( (1-t)D^2 P(x) + tD^2 P(y) \right).
\]

Next,
\[
\left| (1-t)D^2 P(x) + tD^2 P(y) - D^2 P(x_t) \right|
\]
\[
= \left| (1-t) \left( D^2 P(x) - D^2 P(x_t) \right) + t \left( D^2 P(y) - D^2 P(x_t) \right) \right|
\]
\[
\leq Ct(1-t)|y - x|^{1+\alpha},
\]
by applying the Mean Value Theorem to \( D^2 P(x) - D^2 P(x_t) \) and \( D^2 P(y) - D^2 P(x_t) \) and then using the fact that \( D^3 P \) has bounded \( \alpha \)-Hölder norm. Using this in (21), and writing \( h(D^2 P(x_t)) = Q(x_t) \), we have
\[
(1-t)Q(x) + tQ(y) \leq Q(x_t) + Ct(1-t)|y - x|^{1+\alpha},
\]
where we also used the Lipschitz property of \( h \). This implies
\[
\frac{Q(y) - Q(x)}{|y - x|} \leq \frac{Q(x_t) - Q(x)}{t|y - x|} + C_1 |y - x|^{\alpha}.
\]
(22)

Letting \( t \to 0 \) and recalling that \( c_\varepsilon = \sup_{B'} Q \), we have
\[
-\frac{c_\varepsilon}{r} \leq D_\xi Q(x) + C_1 r^\alpha,
\]
for a uniform \( C_1 \). Choose \( r > 0 \) so that \( C_1 r^\alpha = -\frac{1}{2} D_\xi Q(x) \) (increasing \( C_1 \) if necessary to ensure that \( B_x(x) \subset B' \)). We obtain after rearranging,
\[
|D_\xi Q(x)|^{1+1/\alpha} \leq 2c_\varepsilon(2C_1)^{1/\alpha},
\]
as required.

This completes the proof of (8) and hence Theorem 1.1.

3. Strict convexity. We consider now the case when we replace the condition (2) by a strict convexity type condition. Recall from Section 2 that (2) is equivalent to (5). We now consider the condition: there exists a continuous function \( \eta > 0 \) on \( \Omega \) such that for every symmetric matrix \( (X_{ab}) \in \text{Sym}(n) \), vector \( (Z_a) \in \mathbb{R}^n \) and \( Y \in \mathbb{R} \), we have
\[
\eta |X|^2 \leq F_{ab,rs} X_{ab} X_{rs} + 2 F_{ar} A^{ab} X_{ab} X_{rs} + F_{xa,xb} Z_a Z_b
\]
\[
- 2 F_{ab,rs} X_{ab} Y - 2 F_{ab,rs} X_{ab} Z_r + 2 F_{u,a} Y Z_a + F_{u,a} Y^2, \]
(23)
where we are evaluating the derivatives \( F \) at \((A,p,u,x)\) for a positive definite matrix \( A \).

Namely we replace the 0 on the left hand side of (5) by \( \eta |X|^2 \).

Then we have:
Theorem 3.1. Let $u$ and $F$ be as in Theorem 1.1, except that (2) is replaced by the stronger condition (23). Let the rank of $D^2u$ be $n - k$. Then there exist $k$ fixed directions $X_1, \ldots, X_k$ such that $(D^2u(x))(X_j) = 0$ for all $1 \leq j \leq k$ and all $x \in \Omega$.

Proof. Write $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ for the eigenvalues of $D^2u$. Our assumption is that $\lambda_1 = \cdots = \lambda_k = 0$ and $\lambda_{k+1} > 0$ on $\Omega$. Fix $x_0 \in \Omega$. It suffices to show that there exists a neighborhood $U$ of $x_0$ and fixed directions $X_1, \ldots, X_k$ such that $(D^2u(x))(X_j) = 0$ for $1 \leq j \leq k$ and all $x \in U$. Moreover, by an argument of Korevaar-Lewis [19, p. 29-30] it is enough to show that, for $x$ near $x_0$, we have $D_X D^2u(x) = 0$ for all vectors $X$ in the null space of $D^2u(x)$. Fix $x$ and choose coordinates such that $D^2u(x)$ is diagonal with $u_{ii} = \lambda_i$. We wish to show that $u_{pq}(x) = 0$ for $1 \leq i \leq k$ and all $p, q$.

First, let $Y$ be a vector in the null space of $D^2u(x)$ and extend to a constant vector field in a neighborhood of $x$. Then $D^2u(Y, Y)$ is a nonnegative function which vanishes at $x$. It follows that its first derivative vanishes at $x$, and so $u_{pq} Y^p Y^q = 0$ at $x$. Hence

$$u_{pq}(x) = 0 \quad \text{for } 1 \leq p, q \leq k \text{ and all } i. \quad (24)$$

We now consider the quantity $R = \lambda_1 + \cdots + \lambda_k$, the sum of the $k$ smallest eigenvalues. Since $R = 0$ and $\lambda_{k+1} > 0$ on $\Omega$, it follows that we can differentiate $R$ (more precisely, the “sum of first $k$ eigenvalues” function on the space of symmetric matrices is smooth in a neighborhood of the set of values of $D^2u$), and we obtain

$$0 = R_{ab} = \sum_{j=1}^k u_{jjab} + 2 \sum_{j=1}^k \sum_{m \geq k+1} \frac{u_{maj} u_{mbj}}{-\lambda_m}. \quad (25)$$

Taking the trace of this equation with respect to $F_{ab}$ (evaluating at $u$) and using (13) and (24) gives

$$0 = \sum_{j=1}^k \left\{ \sum_{a,b,s,k \geq k+1} F_{ab,rs} u_{abj} u_{saj} + F_{u,ab} u_{abj} + F_{x,j,aj} + 2 \sum_{a,b \geq k+1} F_{ab,u} u_{abj} u_{aj} + 2 \sum_{a,b \geq k+1} F_{ab} u_{maj} u_{mbj} \frac{1}{\lambda_m} \right\}.$$

We now apply the strict convexity assumption at $u$ using the choices (16) with $P$ replaced by $u$ and $\ell$ replaced by $k$. Note that since $D^2u$ is not strictly positive definite, in (23) we take positive definite matrices $A$ with $A \to D^2u$. This implies

$$0 \geq \eta \sum_{j=1}^k \sum_{p,q \geq k+1} |u_{pqj}|^2,$$

and hence $u_{pqj} = 0$ for $i \leq k$ and $p, q > k$. Together with (24), this completes the proof of the theorem.

To see that the assumptions of Theorem 1.1 are not sufficient for the stronger conclusion of Theorem 3.1, one can consider the example of Korevaar-Lewis [19, p. 31] which corresponds to $F(A, u) = u - 1/\text{tr}(A)$. In our setting, a similar but slightly simpler example is given by

$$F(A, x) = r - \frac{n-1}{\text{tr}(A)}, \quad (25)$$
where $r = \sqrt{x_1^2 + \cdots + x_n^2}$, with $\Omega$ a small ball which does not intersect the origin in $\mathbb{R}^n$. A solution of the equation $F(D^2 u, x) = 0$ is given by $u = r$, which is linear along different lines at different points. One can check that $F$ given by (25) satisfies (2), but not the stronger condition (23).

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