DISJOINTNESS OF THE MÖBIUS TRANSFORMATION AND MÖBIUS FUNCTION

EL HOUCEIN EL ABDALAOUI AND IGOR E. SHPARLINSKI

Abstract. We study the distribution of the sequence of elements of the discrete dynamical system generated by the Möbius transformation $x \mapsto (ax + b)/(cx + d)$ over a finite field of $p$ elements. Motivated by a recent conjecture of P. Sarnak, we obtain nontrivial estimates of exponential sums with such sequences that imply that trajectories of this dynamical system are disjoined with the Möbius function.

1. Introduction

Let, as usual $\mu(n)$ denote the Möbius function, that is, $\mu(n) = 0$ if $n$ is not squarefree and $\mu(n) = (-1)^s$ if $n$ is a product of $s$ distinct primes. Furthermore, given a compact topological space $X$ and a homeomorphism $T : X \to X$, we consider the flow $\mathcal{X} = (T, X)$. The Möbius disjointness conjecture of Sarnak [50] asserts that for any flow $\mathcal{X} = (T, X)$ of topological entropy zero, we have

\begin{equation}
\sum_{n \leq N} \mu(n) f(T^n x) = o(N), \quad N \to \infty,
\end{equation}

for any $x \in X$ and a continuous complex-valued function $f$ on $X$. This conjecture has recently attracted very active interest and has actually been established for several classes of flows, see [2,15,16,18,21,24,25,29,33,41,49] and references therein. Moreover, for the connection between the Sarnak and Chowla conjectures, we refer to very recent works of el Abdalaoui [1], Gomilko, Kwietniak and Lemańczyk [28], Tao [52] and Tao and Teräväinen [53].

As usual, we use $\mathbb{F}_q$ to denote the finite field of $q$ elements.
Here we consider a discrete analogue of this conjecture for the flow $\mathcal{M} = (A, \mathbb{F}_p)$ formed by the Möbius map
\begin{equation}
A : x \mapsto \frac{ax + b}{cx + d}
\end{equation}
over $\mathbb{F}_p$, where $p$ is a sufficiently large prime.

This transformation, over the complex numbers and also in finite fields and rings, has been extensively studied because of their relevance to the ergodic theory, and dynamical systems, see [4, 14, 32, 34–37, 48] and references therein. We also recall that this transformation on the 2-dimensional torus $T^2 = (\mathbb{R}/\mathbb{Z})^2$ is also sometimes called the cat map. It also has important links with theoretical physics, see, for example, [8, Section 4.3] or [42, Appendix A].

Here we aim to establish an appropriate version of the Sarnak conjecture for the Möbius map over $\mathbb{F}_p$. This leads us to investigating of exponential sums along the trajectories of (1.2) twisted with the Möbius function.

In turn, exponential sums with the Möbius function are closely related to sums over primes, which is associated with the behaviour of dynamical systems at “prime” times, see [51] for a general point of view and also specific results for dynamical systems on $\text{SL}_2(\mathbb{R})$.

We note that the study of ergodic dynamical system along sequences of arithmetic interest, initiated by Bourgain [9–12], see also the surveys by Rosenblatt and Wierdl [47] and by Thouvenot [54]. In particular, the case of primes takes its origin in the works of Bourgain [10] and Wierdl [56]; we refer also to the results of Nair [43, 44]. For several more results on the Prime Ergodic Theorem and Ergodic Theorem with Arithmetical Weights, we refer to [3, 17, 20–22, 43, 44], see also the references therein and a very recent survey by Eisner and Lin [22].

For the orbits of the one-dimensional dynamical system such as $x \mapsto gx$ over $\mathbb{F}_p$ such results are given in [5–7, 13, 27, 45].

2. Formal set-up

For $c \neq 0$ also extend the definition (1.2) by setting
\begin{equation}
A(-d/c) = a/c.
\end{equation}
It is now easy to check that this extended map $x \mapsto A(x)$ induces a permutation of $\mathbb{F}_p$. 

In fact, we always identify the map (1.2) with a nonsingular matrix
\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(F_p), \]
and we also always assume that \( c \neq 0 \) (so \( A \) is not a linear map).

Moreover, after an appropriate scaling of the coefficient of \( A \) we can always assume that
\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(F_p). \]

Furthermore, for \( \xi_0 \in F_p \) we consider the trajectory
\[ \xi_n = A(\xi_{n-1}) = A^n(\xi_0), \quad n = 1, 2, \ldots, \]
generated by iterations of \( A \).

It is easy to see that each sequence of the form (2.3) either terminates after finitely many steps (if \( c\xi_{n-1} + d = 0 \)) of is eventually periodic, and then, as \( A \) is a permutation it is purely periodic.

It is known that showing (1.1) can be reduced to estimating exponentials sums along trajectories of \( X \) twisted by the Möbius function. In our case, we are interested in the sums
\[ S_\psi(N) = \sum_{n \leq N} \mu(n)\psi(\xi_n) \]
twisted by the Möbius function along the trajectory (2.3) with a non-trivial additive character \( \psi \) of \( F_p \).

We remark, that similar sums, however associated with a linear map \( x \mapsto gx \) over \( F_p \), that is, of the sequence \( \xi_0g^n \), have been estimated in [5, Theorem 5.1]. In fact, using the ideas of [7] it is possible to improve [5, Theorem 5.1], see also [13]. Furthermore, exponential sums over primes, associated with similar dynamical systems on elliptic curves over \( F_p \) have been estimated in [6] (see also [45, Section 4]), and can easily be extended to sums with the Möbius function.

3. Our approach and main result

One of the ingredients of our approach is an explicit formula for the elements of the sequence (2.3), which is essentially based on the spectral decomposition, see Lemma 4.2.

We then combine it with a new general result which holds for arbitrary multiplicative functions and which is based on the ideas of Bourgain,
Sarnak and Ziegler [16, Theorem 2] and Kátaí [31], see Lemma 6.1, which we believe is of independent interest.

This is combined with some bounds of exponential sums which in turn are based on the Weil bound, see Lemmas 5.1 and 5.2 and allows us to obtain a non-trivial estimate for the sums (2.4).

Throughout the paper, the implied constants in the symbols ‘$O$’, ‘$\ll$’ and ‘$\gg$’ may occasionally, where obvious, depend on the real positive parameter $\varepsilon$, and are absolute otherwise (we recall that $U \ll V$ and $V \gg U$ are both equivalent to $U = O(V)$).

In all our bounds we have to assume that

\begin{equation}
\tag{3.1}
t \geq p^{1/2 + \varepsilon},
\end{equation}

which is not a severe restriction as it is satisfied by the majority of the sequences, see, for example, [19].

Our main result is the following bound:

**Theorem 3.1.** Let $\varepsilon > 0$ be a fixed sufficiently small real number. If the characteristic polynomial of the matrix $A$ of the form (2.2) has two distinct roots in $\mathbb{F}_{p^2}$ and the period length $t$ of the sequence (2.3) satisfies (3.1) then, for any real $\alpha$ with

\begin{equation}
\tag{3.2}
\alpha \geq \frac{3(\log \log p)^6}{\varepsilon \log p}
\end{equation}

and integer $N$ with

\begin{equation}
\tag{3.3}
N \geq p^{1/2} \exp \left(5\alpha^{-1}(\log(1/\alpha))^6\right) \log p.
\end{equation}

uniformly over all nontrivial additive characters $\psi$ of $\mathbb{F}_p$, we have

\[ |S_\psi(N)| \ll \alpha N. \]

We remark that Theorem 3.1 is nontrivial starting from the values of $N$ slightly larger than $p^{1/2}$ which is certainly the best possible range until the the condition (3.1) is relaxed (which is presently necessary for nontrivial estimates of exponential sums along consecutive integers).

### 4. Möbius Transformation and Binary Recurrences

**Lemma 4.1.** Let $f(Z) = Z^2 - eZ + 1 \in \mathbb{F}_p[Z]$, where $e = a + d$, be the characteristic polynomial of the matrix $A$ of the form (2.2). Then there are two binary recurrence sequences $u_n$ and $v_n$ satisfying

\[ u_{n+2} = eu_{n+1} + u_n \quad \text{and} \quad v_{n+2} = ev_{n+1} + v_n \]
with the initial values

\[(u_0, u_1) = (\xi_0, a\xi_0 + b) \quad \text{and} \quad (v_0, v_1) = (1, c\xi_0 + d)\]

such that

\[\xi_n = u_n / v_n\]

for \(n = 0, 1, \ldots\).

**Proof.** It is easy to check that the recursive definition of \(u_n\) and \(v_n\) can be rewritten as

\[
\begin{pmatrix}
  u_{n+1} \\
  v_{n+1}
\end{pmatrix}
= A \begin{pmatrix}
  u_n \\
  v_n
\end{pmatrix},
\quad n = 0, 1, \ldots
\]

with the initial values

\[(u_0, u_1) = (\xi_0, a\xi_0 + b) \quad \text{and} \quad (v_0, v_1) = (1, c\xi_0 + d)\]

Then one verifies that the desired statement by induction on \(n\). \(\square\)

Using the well known expression of linear recurrence sequences via the roots of characteristic polynomials, see, for example, [23], we immediately derive from Lemma 4.1, in a straightforward fashion, the following explicit formula:

**Lemma 4.2.** Let \(f(Z) = Z^2 - eZ + 1 \in \mathbb{F}_p[Z]\), where \(e = a + d\), be the characteristic polynomial of the matrix \(A\) of the form (2.2), which has two distinct roots \(\vartheta\) and \(\vartheta^{-1}\) in \(\mathbb{F}_{p^2}\). Then there exist elements \(\alpha, \beta, \gamma \in \mathbb{F}_{p^2}\) such that

\[\xi_n = \alpha + \frac{\beta}{\vartheta^{2n} + \gamma}, \quad n = 0, 1, \ldots\]

**5. Bounds on single character sums**

Let \(p\) be the characteristic of \(\mathbb{F}_p\) and let \(\overline{\mathbb{F}}_p\) denote the algebraic closure of \(\mathbb{F}_p\). For an exhaustive account on the character sums over finite fields we refer to [30, Chapter 11].

One of our main tools is the bound on hybrid sums of multiplicative and additive characters, which in its classical form is given by Weil [55, Example 12 of Appendix 5]; see also [38, Theorem 3 of Chapter 6].

**Lemma 5.1.** For any polynomials \(g(X), h(X) \in \mathbb{F}_p[X]\) and any non-trivial additive character \(\psi\) and arbitrary multiplicative character \(\chi\) of \(\mathbb{F}_p\) we have

\[
\sum_{x \in \mathbb{F}_p, \ g(x) \neq 0} \psi \left( \frac{h(X)}{g(X)} \right) \chi(x) \ll \max \{\deg g, \deg h\} p^{1/2}.
\]
We also need a modification of a bound of Li [40, Theorem 2] which also applies to rational functions rather than to polynomials. We recall that the trace map and norm map from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$ are given by

$$\text{Tr}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(z) = \sum_{i=0}^{n-1} z^i \text{ and } \text{Nm}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(z) = \prod_{i=0}^{n-1} z^{p^i}.$$ 

To simplify the notation, we use $\text{Tr}(z)$ and $\text{Nm}(z)$ to denote the trace and the norm, respectively, of an element $z$ of the quadratic extension $\mathbb{F}_{p^2}$ in $\mathbb{F}_p$. Combining the arguments of the proof of [40, Theorem 2] with those in [39, Section 4], one easily derives:

**Lemma 5.2.** For any polynomials $g(X), h(X) \in \mathbb{F}_{p^2}[X]$ and any non-trivial additive character $\psi$ and arbitrary multiplicative character $\chi$ of $\mathbb{F}_p$ we have

$$\sum_{x \in \mathbb{F}_{p^2}, \psi(x) \neq 0} \psi \left( \text{Tr} \left( h(X)/g(X) \right) \right) \chi(x) \ll \max\{\deg g, \deg h\} p^{1/2}.$$ 

Note that Lemma 5.2 can be further extended in several directions.

We now need a bound on the character sums

$$Q_\psi(u, v; k, m, N) = \sum_{n \leq N} \psi(u \xi_{kn} + v \xi_{mn})$$

with $u, v \in \mathbb{F}_p$ and non-negative integers $k$ and $m$, along consecutive values of the trajectory (2.3).

**Lemma 5.3.** Assume that the characteristic polynomial of the matrix $A$ of the form (2.2) has two distinct roots in $\mathbb{F}_{p^2}$. If $t$ is the period length of the sequence (2.3), then, for any $u, v \in \mathbb{F}_p$ with $(u, v) \neq (0, 0)$ and integers $0 \leq k < m$, for any $N \leq t$ we have

$$Q_\psi(u, v; k, m, N) \ll mp^{1/2} \log p.$$ 

**Proof.** Let $\vartheta$ be as in Lemma 4.2. It is clear that $t$ is the multiplicative order of $\vartheta^2$. For every integer $h$, We now define the sums

$$Q_{h,\psi}(u, v; k, m) = \sum_{n=1}^{t} \psi \left( u \xi_{kn} + v \xi_{mn} \right) e(hn/t),$$

where

$$e(z) = \exp(2\pi i z).$$
We first consider the case \( \vartheta \in \mathbb{F}_p \). Since \( \vartheta^2 \) is of order \( t \) it can be written as \( \vartheta^2 = g^s \) for \( s = (p - 1)/t \) and some primitive root \( g \) of \( \mathbb{F}_p^* \).

For \( x \in \mathbb{F}_p^* \), we define \( \text{ind} \, x \) by the conditions

\[
g^{\text{ind} \, x} = x \quad \text{and} \quad 0 \leq \text{ind} \, x \leq p - 2.
\]

Hence, using Lemma 4.2 and the additivity of \( \psi \), we write

\[
Q_{h,\psi}(u, v; k, m) = \psi \left( \alpha \left( u + v \right) \right) \sum_{n=1}^{t} \psi \left( \frac{\beta u}{g^{kns} + \gamma} + \frac{\beta v}{g^{mns} + \gamma} \right) e\left( hn/t \right)
\]

\[
= \psi \left( \alpha \left( u + v \right) \right) \sum_{n=1}^{t} \psi \left( \frac{\beta u}{g^{kns} + \gamma} + \frac{\beta v}{g^{mns} + \gamma} \right) e\left( hsn/(p - 1) \right)
\]

\[
= \frac{1}{s} \psi \left( \alpha \left( u + v \right) \right) \sum_{n=1}^{p-1} \psi \left( \frac{\beta u}{g^{kns} + \gamma} + \frac{\beta v}{g^{mns} + \gamma} \right) e\left( hsn/(p - 1) \right).
\]

Now, denote \( x = g^n \) and using that \( g \) is a primitive root, we obtain

\[
Q_{\psi}(h, u, v; k, m) = \frac{1}{s} \psi \left( \alpha \left( u + v \right) \right) \sum_{x \in \mathbb{F}_p^*} \psi \left( \frac{\beta u}{x^{k} + \gamma} + \frac{\beta v}{x^{m} + \gamma} \right) e\left( hsn/(p - 1) \right).
\]

Since the function \( x \mapsto e\left( hsn/(p - 1) \right) \) is a multiplicative character of \( \mathbb{F}_p^* \), recalling Lemma 5.1, we obtain

\[
Q_{h,\psi}(u, v; k, m) \ll \frac{1}{s} \sum_{m} p^{1/2} = mp^{1/2}.
\]

Using the standard reduction between complete and incomplete sums, see [30, Section 12.2], we conclude the proof in this case.

If \( \vartheta \in \mathbb{F}_p^2 \setminus \mathbb{F}_p \), then obviously \( \text{Nm}(\vartheta) = 1 \). We now choose \( g \) to get a generator of the norm group, consisting of the elements \( z \in \mathbb{F}_p^2 \) with \( \text{Nm}(z) = 1 \) and we proceed as in he above, however using Lemma 5.2 instead of Lemma 5.1 in the appropriate place.

For sums with one term

\[
R_{\psi}(u; m, N) = \sum_{n \leq N} \psi \left( u \xi_{mn} \right)
\]

with \( u \in \mathbb{F}_p \) and a non-negative integer \( m \), we have a slightly more precise statement.
Lemma 5.4. Assume that the characteristic polynomial of the matrix $A$ of the form (2.2) has two distinct roots in $\mathbb{F}_p$. If $t$ is the period length of the sequence (2.3), then, for any $u \in \mathbb{F}_p^*$ and an integers $m > 0$, for any $N \leq t$ we have

$$R_u(u; m, N) \ll \gcd(m, t) p^{1/2} \log p.$$  

Proof. Let $d = \gcd(m, t)$. We set $k = m/d$ and $s = t/d$. Let $B = A^k$. We also consider the sequences $\zeta_n = \xi_{kn}$ then instead of (2.3), we can write

$$\zeta_n = B (\zeta_{n-1}) = B^n (\xi_0), \quad n = 1, 2, \ldots.$$  

Hence, by the standard arguments as before, we see that the period of the sequence $\zeta_n, n = 1, 2, \ldots$, is $t$. Using a special case (with only one term) applied to $\zeta_{dn} = \xi_{mn}$ instead of $\xi_n$, we obtain the result. \qed

6. Double sums and correlations with multiplicative functions

Now, by applying the machinery in the proof of the criterion of Bourgain, Sarnak and Ziegler [16, Theorem 2] (which in turn improves the result of Kátai [31]), we obtain our main technical result.

We present it in a form which is more general and flexible than we need here, since we believe it may find other applications.

Lemma 6.1. Let $\nu$ be a multiplicative function and $F$ an arbitrary periodic arithmetic function with period $t$. Assume

$$|\nu(n)| \leq 1 \quad \text{and} \quad |F(n)| \leq 1, \quad n \in \mathbb{N}.$$  

We further assume that for any primes $r \neq s$, and for any positive integer $h \leq t$ we have

$$\left| \sum_{n \leq h} F(nr) F(ns) \right| \ll \max\{r, s\} t \rho$$  

for some real $\rho < 1$. Then for any real $\alpha$ with

$$\alpha^{-2} \exp \left(2\alpha^{-1} \left(\log(1/\alpha)\right)^6\right) \ll \rho^{-1}$$  

and integer $N$ with

$$N \gg t \rho \exp \left(4\alpha^{-1} \left(\log(1/\alpha)\right)^6\right),$$

(6.1)

(6.2)
we have
\[
\left| \sum_{n \leq N} \nu(n) F(n) \right| \ll \alpha N.
\]

Proof. We follow the proof of [16, Theorem 2]. In particular, let \( \alpha > 0 \) be some sufficiently small (as otherwise there is nothing to prove). As in [16, Equation (2.1)] we define
\[
(6.3) \quad j_0 = \left( \frac{\log(1/\alpha)}{\alpha} \right)^3 \quad \text{and} \quad j_1 = j_0^2.
\]
For every integer \( j \in [j_0, j_1 + 1] \) we also define
\[
R_j = (1 + \alpha)^j \quad \text{and} \quad M_j = N/R_{j+1}.
\]
We note that we do not assume that these quantities are integer numbers.

Furthermore, for every integer \( j \in [j_0, j_1] \) we define \( \mathcal{P}_j \) as the set of primes in the interval \([R_j, R_{j+1})\) and then we also define the set
\[
\mathcal{Q}_j = \left\{ m \in [1, M_j] : m \text{ has no prime factors in } \bigcup_{i \leq j} \mathcal{P}_j \right\}.
\]
We note that, by the prime number theorem (with an explicit bound on the error term, we do not however need the full power of the current knowledge such as [30, Corollary 8.30]), we have the following bound on the cardinality of \( \mathcal{P}_j \), for every \( j \in [j_0, j_1] \):
\[
(6.4) \quad \# \mathcal{P}_j \leq R_j \left( \frac{1}{j} + \frac{1}{\alpha j^2} + O \left( \exp \left( -\sqrt{\alpha j} \right) \right) \right) \ll \frac{1}{j} R_j,
\]
see [16, Equation (2.8)], where we have also used that \( \alpha j \geq \alpha j_0 \gg 1 \).

As in the proof of [16, Theorem 2] we notice that the products of the \( mr \) with \( r \in \mathcal{P}_j, m \in \mathcal{Q}_j \) for some \( j \in [j_0, j_1] \) are pairwise distinct and obviously belong the interval \([1, N]\), so we conclude
\[
(6.5) \quad \sum_{j_0 \leq j \leq j_1} \# \mathcal{P}_j \# \mathcal{Q}_j \leq N,
\]
which is also used in the derivation of [16, Equations (2.20) and (2.21)].

Furthermore, using (6.4) and recalling choice of the parameters (6.3), we obtain
\[
\sum_{j_0 \leq j \leq j_1} \# \mathcal{P}_j \ll \sum_{j_0 \leq j \leq j_1} \frac{1}{j} (1 + \alpha)^j \leq (1 + \alpha)^{j_1} \sum_{j_0 \leq j \leq j_1} \frac{1}{j} \leq (1 + \alpha)^{j_1} \log(j_1/j_0) \leq \exp(\alpha j_1) \log j_0.
\]
Hence
\begin{equation}
\sum_{j_0 \leq j \leq j_1} \# \mathcal{P}_j \ll \exp \left( 1.5 \alpha^{-1} (\log(1/\alpha))^6 \right),
\end{equation}
provided that \( \alpha \) is sufficiently small.

Now to establish the desired result, we recall that by \[16, \text{Equation (2.16)}\]
\begin{equation}
\sum_{n \leq N} \nu(n) F(n) \ll \sum_{j_0 \leq j \leq j_1} W_j + \alpha N,
\end{equation}
where
\[ W_j = \sum_{m \in \mathcal{Q}_j} \left| \sum_{r \in \mathcal{P}_j} \nu(r) F(mr) \right|, \quad j_0 \leq j \leq j_1. \]

Using the Cauchy–Schwarz inequality, extending the range of summation over \( m \) to all positive integers up to \( M_j \), changing the order of summation and recalling that \( |\nu(n)| \leq 1 \), we obtain
\begin{equation}
W_j^2 \leq \# \mathcal{Q}_j \sum_{r,s \in \mathcal{P}_j} \left| \sum_{m \leq M_j} F(mr) \overline{F(ms)} \right|, \quad j_0 \leq j \leq j_1,
\end{equation}
see \[16, \text{Equation (2.17)}\].

The contribution \( T_{1,j} \) to the right hand side of \(6.8\) from the diagonal terms can estimated as in \[16, \text{Equation (2.20)}\] by
\begin{equation}
T_{1,j} \ll M_j \# \mathcal{P}_j.
\end{equation}

To estimate the remaining contribution \( T_{2,j} \) from the off-diagonal terms we recall our assumption on bilinear sums with the function \( F \). More precisely, splitting the interval of summation into at most \( M_j/t \) intervals of length \( t \) and at most 1 interval of length \( h \leq t \), we obtain
\[ \left| \sum_{n \leq M_j} F(nr) \overline{F(ns)} \right| \leq \max\{r,s\} (M_j/t + 1)t\rho. \]

Hence, using (for simplicity) that \( r, s \leq R_{j+1} \leq 2R_j \) and \( M_jR_j \leq N \), we derive
\begin{equation}
T_{2,j} \leq 2 \left( \# \mathcal{P}_j \right)^2 R_j (M_j + t) \rho \ll \left( \# \mathcal{P}_j \right)^2 (N + R_j t) \rho.
\end{equation}
Substituting the bound (6.9) and (6.10) in (6.8), we obtain
\[
W_j^2 \ll M_j \# P_j \# Q_j + \left( \# P_j \right)^2 \left( N + R_j \right) t \rho \# Q_j
\]
\[
\leq M_j \# P_j \# Q_j + N \left( \# P_j \right)^2 \# Q_j \rho + \left( \# P_j \right)^2 \# Q_j R_j t \rho,
\]
which after the substitution in (6.7) implies
\[
(6.11) \quad \sum_{n \leq N} \nu(n) F(n) \ll S_1 + S_2 \sqrt{N \rho} + S_3 \sqrt{t \rho} + \alpha N,
\]
where
\[
S_1 = \sum_{j_0 \leq j \leq j_1} \left( M_j \# P_j \# Q_j \right)^{1/2},
\]
\[
S_2 = \sum_{j_0 \leq j \leq j_1} \left( \# P_j \right)^{1/2} \# Q_j, \quad S_3 = \sum_{j_0 \leq j \leq j_1} \left( \# P_j \right)^{1/2} \# Q_j R_j.
\]

To bound the sum $S_1$, we use the Cauchy–Schwarz inequality and write
\[
S_1 \ll \left( \sum_{j_0 \leq j \leq j_1} \left( \# P_j \# Q_j \right) \right)^{1/2} \left( \sum_{j_0 \leq j \leq j_1} M_j \right)^{1/2}.
\]
We estimate the first sum using (6.5), while the second sums is easily estimated as
\[
\sum_{j_0 \leq j \leq j_1} M_j = N \sum_{j_0 \leq j \leq j_1} (1 + \alpha)^{-j} \leq N (1 + \alpha)^{-j_0} \sum_{j=0}^{\infty} (1 + \alpha)^{-j}
\]
\[
= N \frac{1 + \alpha}{\alpha} (1 + \alpha)^{-j_0} \ll N \frac{1}{\alpha} \exp(-j_0 \log(1 + \alpha)).
\]
Therefore, by the definition of $j_0$ in (6.3) combined with the inequality $\log(1 + x) \geq x/2$ for $x \in [0, 1]$, we obtain
\[
(6.12) \quad S_1 \ll N \exp \left( -0.25 (\log(1/\alpha))^3 \right).
\]
Note that (6.12) is stronger than the bound recorded in [16, Equation (2.20)], however this does not affect the final result as it is dominated by the term $\alpha N$, which is already present in (6.11). In particular, we rewrite as
\[
(6.13) \quad S_1 \ll \alpha N.
\]

For the sum $S_2$, writing
\[
\# P_j \left( \# Q_j \right)^{1/2} = \left( \# P_j \# Q_j \right)^{1/2} \left( \# P_j \right)^{1/2},
\]
and applying again the Cauchy–Schwarz inequality, we obtain
\[
S_2 \leq \left( \sum_{j_0 \leq j \leq j_1} \#P_j \#Q_j \right)^{1/2} \left( \sum_{j_0 \leq j \leq j_1} \#P_j \right)^{1/2}.
\]

Now, we see from (6.5) and (6.6) that
\[
S_2 \ll N^{1/2} \exp \left( \alpha^{-1} \left( \log(1/\alpha) \right)^6 \right),
\]
We now see that (6.14), under the condition (6.1) implies
\[
S_2 \sqrt{N \rho} \ll N \sqrt{\rho} \exp \left( \alpha^{-1} \left( \log(1/\alpha) \right)^6 \right) \ll \alpha N.
\]

Therefore, it remains to estimate the sum $S_3$. We notice that the trivial inequality $\#Q_j R_j \leq N$ yields
\[
S_3 \leq N^{1/2} \sum_{j_0 \leq j \leq j_1} \#P_j,
\]
which together with (6.6) implies
\[
S_3 \ll N^{1/2} \exp \left( 1.5 \alpha^{-1} \left( \log(1/\alpha) \right)^6 \right).
\]
We now see that (6.16), under the condition (6.2) implies
\[
S_3 \sqrt{t \rho} \leq \sqrt{N t \rho} \exp \left( 1.5 \alpha^{-1} \left( \log(1/\alpha) \right)^6 \right) \ll \alpha N.
\]
Substituting the bounds (6.13), (6.15) and (6.17) in (6.11), we derive the desired result. \(\square\)

7. Proof of Theorem 3.1

We see from Lemma 5.3 that in Lemma 6.1 we can take some $\rho$ with
\[
t^{-1} p^{1/2} \log p \ll \rho \ll t^{-1} p^{1/2} \log p.
\]

It is now easy to see that (3.2) and (3.3) ensure the validity of (6.1) and (6.2), respectively. Indeed, using (3.2) we see that for a sufficiently large $p$ we have $\alpha^{-1} \leq \log p$ and thus
\[
\alpha^{-2} \leq (\log p)^2 \quad \text{and} \quad (\log(1/\alpha))^6 \leq (\log \log p)^6
\]
we derive
\[
\alpha^{-2} \exp \left( 2\alpha^{-1} \left( \log(1/\alpha) \right)^6 \right) \leq (\log p)^2 \exp \left( \frac{2\varepsilon \log p}{3 \left( \log \log p \right)^6} \left( \log \log p \right)^6 \right) = p^{2\varepsilon/3} (\log p)^2 \ll \rho^{-1}.
\]

Thus (6.1) holds.
Finally, since $p^{1/2} \log p \gg t \rho$ then (3.3) implies (6.2) provided that $\alpha$ is small enough and the result follows.

8. Further perspectives

We also note that our results behind the estimates of Theorem 3.1, in particular Lemma 6.1 below can be applied to estimating exponential sum along sequences with other arithmetic constraints such as square-freeness (in which case one can expect stronger results) or smoothness.

Furthermore, one can also obtain essentially the same results for analogues of the sums (2.4) with sequences of the form $r(\xi_n)$ with a rational function $r(X) \in \mathbb{Q}(X)$. The same approach also works, without any charges, for sums of multiplicative characters with $r(\xi_n)$.

One can also apply our approach to other dynamical systems such as polynomial dynamical systems $x \mapsto f(x)$ for a polynomial $f \in \mathbb{F}_p[X]$ of a fixed degree $d \geq 2$ or to monomial dynamical systems $x \mapsto x^e$ for an integer $e \geq 1$ with $\gcd(e, p - 1) = 1$, which however can be rather large in terms of $p$.

We also hope that a similar approach may be applied to estimating exponential and character sums along squares, that is, with $\xi_n^2$. This kind of questions has also been introduced by Bourgain [9,10,12]. Note that for the dynamical system $x \mapsto gx$ over $\mathbb{F}_p$ such a bound is given in [26] (which one can probably improve using some arguments from [46]).

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References

[1] E. H. el Abdalaoui, ‘On Veech’s proof of Sarnak’s theorem on the Möbius flow’, Preprint, 2017, (available from https://arxiv.org/abs/1711.06326). (p. 1)
[2] E. H. el Abdalaoui, M. Lemańczyk and T. de la Rue, ‘On spectral disjointness of powers for rank-one transformations and Möbius orthogonality’, J. Funct. Analysis, 266 (2014), 284–317. (p. 1)
[3] E. H. el Abdalaoui, J. Kulaga-Przymus, M. Lemańczyk and T. de la Rue, ‘The Chowla and the Sarnak conjectures from ergodic theory point of view’, Discr. Cont. Dyn. Syst., Ser. A, 37 (2017), 2899–2944. (p. 2)
[4] M. Baake, N. Neumärker and J. A. G. Roberts, ‘Orbit structure and (reversing) symmetries of toral endomorphisms on rational lattices’, Discr. Cont. Dyn. Syst., Ser.A, 33 (2013), 527–553. (p. 2)
[5] W. Banks, A. Conflitti, J. B. Friedlander and I. E. Shparlinski, ‘Exponential sums over Mersenne numbers’, Compos. Math., 140 (2004), 15–30. (pp. 2 and 3)
[6] W. D. Banks, J. B. Friedlander, M. Z. Garaev and I. E. Shparlinski, ‘Double character sums over elliptic curves and finite fields’, Pure and Appl. Math. Quart., 2 (2006), 179–197. (pp. 2 and 3)
[7] W. D. Banks, J. B. Friedlander, M. Z. Garaev and I. E. Shparlinski, ‘Exponential and character sums Mersenne numbers’, J. Aust. Math. Soc., 92 (2012), 1–13. (pp. 2 and 3)
[8] R. Blümel, and W. P. Reinhardt, Chaos in atomic physics, Cambridge Univ. Press, Cambridge, 1997. (p. 2)
[9] J. Bourgain, ‘On the maximal ergodic theorem for certain subsets of the integers’, Israel J. Math., 61 (1988), 39–72. (pp. 2 and 13)
[10] J. Bourgain, ‘On the pointwise ergodic theorem on $L^p$ for arithmetic sets’, Israel J. Math., 61 (1988), 73–84. (pp. 2 and 13)
[11] J. Bourgain, ‘An approach to pointwise ergodic theorems’, Geometric aspects of functional analysis (1986/87), Lecture Notes in Math., 1317 Springer, Berlin, 1988, 204–223. (p. 2)
[12] J. Bourgain, ‘Pointwise ergodic theorems for arithmetic sets (with an appendix by J. Bourgain, H. Furstenberg, Y Katznelson and D. S. Ornstein), Inst. Hautes Études Sci. Publ. Math., 69 (1989), 5–45. (pp. 2 and 13)
[13] J. Bourgain, ‘Estimates on exponential sums related to Diffie-Hellman distributions’, Geom. and Funct. Anal., 15 (2005), 1–34. (pp. 2 and 3)
[14] J. Bourgain, ‘A remark on quantum ergodicity for CAT maps. Geometric aspects of functional analysis’, Lecture Notes in Math., 1910, Springer, Berlin, 2007, 89–98. (p. 2)
[15] J. Bourgain, ‘On the correlation of the Möbius function with rank-one systems’, J. Anal. Math., 120 (2013), 105–130. (p. 1)
[16] J. Bourgain, P. Sarnak and T. Ziegler, ‘Disjointness of Möbius from horocycle flow’, From Fourier Analysis and Number Theory to Radon Transforms and Geometry, Devel. Math., 28, Springer, New York, 2013, 67–83. (pp. 1, 4, 8, 9, 10, and 11)
[17] Z. Buczolich, ‘Ergodic averages with prime divisor weights in $L^1$, Ergodic Theory and Dynam. Syst., (to appear). (p. 2)
[18] D. Carmon and Z. Rudnick, ‘The autocorrelation of the Möbius function and Chowla’s conjecture for the rational function field’, Quart. J. Math., 65 (2014), 53–61. (p. 1)
[19] W.-S. Chou, ‘On inversive maximal period polynomials over finite fields’, Appl. Algebra Engrg. Comm. Comput., 6 (1995), 245–250. (p. 4)
[20] C. Cuny and M. Weber, ‘Ergodic theorems with arithmetical weights’, Israel J. Math., 217 (2017), 139–180. (p. 2)
[21] T. Eisner, ‘A polynomial version of Sarnak’s conjecture’, C. R. Math. Acad. Sci., Paris, 353 (2015), 569–572. (pp. 1 and 2)
[22] T. Eisner and M. Lin, ‘On modulated ergodic theorems’, Preprint, 2017, (available from https://arxiv.org/abs/1709.05322). (p. 2)
[23] G. Everest, A. van der Poorten, I. E. Shparlinski and T. Ward, Recurrence sequences, Math. Surveys and Monogr., 104, Amer. Math. Soc., Providence, RI, 2003. (p. 5)
[24] S. Ferenczi, J. Kulaga-Przymus and M. Lemańczyk, ‘Sarnak’s conjecture – What’s new’, Preprint, 2017, (available from https://arxiv.org/abs/1710.04039) (p. 1)
[25] É. Fouvry and S. Ganguly, ‘Strong orthogonality between the Möbius function, additive characters and Fourier coefficients of cusp forms’, Compos. Math., 150 (2014), 763–797. (p. 1)
[26] J. B. Friedlander, J. Hansen and I. E. Shparlinski, ‘On character sums with exponential functions’, Mathematika, 47 (2000), 75–85. (p. 13)
[27] M. Z. Garaev and I. E. Shparlinski, ‘The large sieve inequality with exponential functions and the distribution of Mersenne numbers modulo primes’, Intern. Math. Research Notices, 39 (2005), 2391–2408. (p. 2)
[28] A. Gomilko, D. Kwietniak and M. Lemańczyk, ‘Sarnak’s conjecture implies the Chowla conjecture along a subsequence’, Preprint, 2017, (available from https://arxiv.org/abs/1710.07049). (p. 1)
[29] B. J. Green and T. Tao, ‘The Möbius function is strongly orthogonal to nilsequences’, Ann. Math., 175 (2012), 541–566. (p. 1)
[30] H. Iwaniec and E. Kowalski, Analytic number theory, Amer. Math. Soc., Providence, RI, 2004. (pp. 5, 7, and 9)
[31] I. Kátai, ‘A remark on a theorem of H. Daboussi’, Acta Math. Hungar., 47 (1986), 223–225. (pp. 4 and 8)
[32] D. Kelmer, ‘On matrix elements for the quantized cat cap modulo prime powers’, Ann. Henri Poincaré, 9 (2008), 1479–1501. (p. 2)
[33] J. Kulaga-Przymus and M. Lemańczyk, ‘The Möbius function and continuous extensions of rotations’, Monat. Math., 178 (2015), 553–582. (p. 1)
[34] P. Kurlberg, ‘Bounds on supremum norms for Hecke eigenfunctions’, Ann. Henri Poincaré, 8 (2007), 75–89. (p. 2)
[35] P. Kurlberg, L. Rosenzweig and Z. Rudnick, ‘Matrix elements for the quantum cat map: Fluctuations in short windows’, Nonlinearity, 20 (2007), 2289–2304. (p. 2)
[36] P. Kurlberg and Z. Rudnick, ‘Hecke theory and equidistribution for the quantization of linear maps of the torus’, Duke Math. J., 103 (2000), 47–77. (p. 2)
[37] P. Kurlberg and Z. Rudnick, ‘On the distribution of matrix elements for the quantum cat map’, Ann. Math., 161 (2005), 489–507. (p. 2)
[38] W.-C. W. Li, *Number theory with applications*, World Scientific, Singapore, 1996. (p. 5)

[39] W.-C. W. Li, ‘Character sums over $p$-adic fields,’ *J. Number Theory*, **174** (1999), 181—229. (pp. 6 and 13)

[40] W.-C. W. Li, ‘Character sums over norm groups’, *Finite Fields Appl.*, **12** (2006), 1–15. (pp. 6 and 13)

[41] J. Liu and P. Sarnak, ‘The M"obius disjointness conjecture for distal flows’, *Duke Math. J.*, **164** (2015), 1353–1399. (p. 1)

[42] D. H. U. Marchetti and W. F. Wreszinski, *Asymptotic time decay in quantum physics*, World Sci., 2013. (p. 2)

[43] R. Nair, ‘On polynomials in primes and J. Bourgain’s circle method approach to ergodic theorems’, *Ergodic Th. Dyn. Syst.*, **11** (1991), 485–499. (p. 2)

[44] R. Nair, ‘On polynomials in primes and J. Bourgain’s circle method approach to ergodic theorems II’, *Studia Math.*, **105** (1993), 207–233. (p. 2)

[45] A. Ostafe and I. E. Shparlinski, ‘Exponential sums over points of elliptic curves with reciprocals of primes’, *Mathematika*, **58** (2012), 21–33. (pp. 2 and 3)

[46] A. Ostafe and I. E. Shparlinski, ‘On the power generator of pseudorandom numbers and its multivariate analogue’, *J. Complexity*, **28** (2012), 238–249. (p. 13)

[47] J. M. Rosenblatt and M. Wierdl, ‘Pointwise ergodic theorems via harmonic analysis’, *Ergodic Theory and its Connections with Harmonic Analysis (Alexandria, 1993)*, London Math. Soc. Lecture Note Ser., **205**, Cambridge Univ. Press, Cambridge, 1995, 3–151. (p. 2)

[48] L. Rosenzweig, ‘Fluctuations of matrix elements of the quantum cat map’, *Intern. Math. Research Notices*, **2011** (2011), 4884–4933. (p. 2)

[49] V. V. Ryzhikov, ‘Bounded ergodic constructions, disjointness, and weak limits of powers’, *Trans. Moscow Math. Soc.*, **74** (2013), 165–171. (p. 1)

[50] P. Sarnak, ‘M"obius randomness and dynamics’, *Not. South Afr. Math. Soc.*, **43** (2012), 89–97. (p. 1)

[51] P. Sarnak and A. Ubis, ‘The horocycle flow at prime times’, *J. Math. Pures Appl.*, **103** (2015), 575–618. (p. 2)

[52] T. Tao, ‘Equivalence of the logarithmically averaged Chowla and Sarnak conjectures’, *Number Theory – Diophantine problems, Uniform Distribution and Applications; Festschrift in Honour of Robert F. Tichy’s 60th Birthday (C. Elsholtz and P. Grabner, eds.)*, Springer, 2017, 391–421. (p. 1)

[53] T. Tao and J. Teräväinen, ‘The structure of logarithmically averaged correlations of multiplicative functions, with applications to the Chowla and Elliott conjectures’, *Preprint*, 2017, (available from https://arxiv.org/abs/1708.02610). (p. 1)

[54] J.-P. Thouvenot, ‘La convergence presque sûre des moyennes ergodiques suivant certaines sous-suites d’entiers (d’après Jean Bourgain)’, *Séminaire Bourbaki, Vol. 1989/90. Astérisque* **189–190** (1990), Exp. No. 719, 133–153. (p. 2)

[55] A. Weil, *Basic number theory*, Springer-Verlag, New York, 1974. (p. 5)

[56] M. Wierdl, ‘Pointwise ergodic theorem along the prime numbers’, *Israel J. Math.*, **64** (1988), 315–336. (p. 2)
Laboratoire de Mathématiques Raphaël Salem, Université de Rouen Normandie, F76801 Saint-Étienne-du-Rouvray, France

E-mail address: elhoucein.elabdalaoui@univ-rouen.fr

School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia

E-mail address: igor.shparlinski@unsw.edu.au