Abstract

In this paper we investigate trigonometric vertex models associated with solutions of the Yang-Baxter equation which are invariant relative to $q$-deformed superalgebras $\mathfrak{sl}(r|2m)^{(2)}$, $\mathfrak{osp}(r|2m)^{(1)}$ and $\mathfrak{osp}(r = 2n|2m)^{(2)}$. The associated $R$-matrices are presented in terms of the standard Weyl basis making possible the formulation of the quantum inverse scattering method for these lattice models. This allowed us to derive the eigenvectors and the eigenvalues of the corresponding transfer matrices as well as explicit expressions for the Bethe ansatz equations.

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1 Introduction

In the course of years it has become clear that classical vertex models of statistical mechanics are ideal paradigm of the theory of two-dimensional integrable systems [1]. It turns out that a \( R \)-matrix satisfying the Yang-Baxter relation generates in a natural manner the Boltzmann weights of an exactly solved vertex model. An important family of such models are given by the trigonometric solutions of the Yang-Baxter equation connected with the fundamental representation of generic \( q \)-deformed Lie algebras [2, 3] and Lie superalgebras [4, 5].

The physical understanding of vertex models includes necessarily the exact diagonalization of their transfer matrices, which can provide us information about the free energy behaviour and on the nature of the elementary excitations. This step has been successfully achieved for standard Lie algebras either by the analytical Bethe ansatz [6], a phenomenological technique yielding us solely the transfer matrix eigenvalues, or through the quantum inverse scattering method [7, 8] which gives us also the eigenvectors. The latter is a more powerful mathematical approach, offering us the foundation for studying two-dimensional vertex models from first principles which culminated in an algebraic formulation of the Bethe ansatz [9, 10]. The majority of the algebraic Bethe ansatz results for superalgebras, however, have been concentrated on the rational \( q \to 1 \) limit of deformed universal enveloping algebras associated with the \( sl(r|m) \) [11, 12] and \( osp(r|2m) \) [13] symmetries. A similar algebraic program for \( q \)-deformed superalgebras is still very far from being completed though some progress already appeared in the literature. Most of it is associated with the \( U_q[sl(r|m)\] supersymmetry, whose \( R \)-matrix elements are the statistical weights of the Perk-Schultz vertex model [15], and the corresponding transfer matrix can be diagonalized by a graded version of the nested algebraic Bethe ansatz developed originally by Kulish [11]. By way of contrast, the other superalgebras have been studied on a rather case by case basis and representative examples are the vertex models associated with certain \( q \)-deformations of the \( osp(1|2) \) [16] and \( osp(2|2) \) [17, 18, 19] symmetries.

A unified algebraic Bethe ansatz solution of the fundamental vertex models invariant rela-
tive to affine $q$-deformed Lie superalgebras is indeed a long-standing open problem in the field of integrable systems. In order to establish this formulation it is indispensable to have at hand explicit expressions for the $R$-matrices much like that presented by Jimbo for nonexceptional Lie algebras [3]. Despite of recent advances [20, 21, 22, 23] in developing methods for constructing solutions of the Yang-Baxter equation, the results for the $R$-matrices are usually written in terms of projection operators that still need to be calculated in a convenient orthonormal coordinates. This unsatisfactory situation is probably related to the fact that Lie superalgebras possess a more involved representation theory as compared with ungraded algebras [24]. This is particularly complicated for twisted superalgebras, making it difficult even to carry on an intuitive analysis such as the analytical Bethe ansatz [25].

The purpose of this paper is to start to bridge this gap, by presenting the quantum inverse scattering formulation for the $U_q[sl(r|2m)^{(2)}]$, $U_q[osp(r|2m)^{(1)}]$ and $U_q[osp(r = 2n|2m)^{(2)}]$ vertex models. We recall that the symbol $\sigma$ in $U_q[G^{(\sigma)}]$ refers to the type of automorphism admitted by the superalgebra $G$ and their explicit forms have been summarized in the work pioneered by Bazhanov and Shadrikov [4]. We have organized this paper as follows. In next section we present the $R$-matrices of these lattice models in terms of the elementary Weyl basis, paving the way for a Bethe ansatz analysis. In section 3 we describe the essential tools to solve the eigenvalue problem for the corresponding transfer matrices by the algebraic Bethe ansatz approach. We use this knowledge in section 4 to present explicit expressions for the eigenvalues and the Bethe ansatz equations. Our conclusions are discussed in section 5. In four appendices we summarize the crossing matrices, extra commutation rules and technical details concerning the nested Bethe ansatz analysis.

## 2 The quantum $R$-matrices

In this section we shall present some solutions of the Yang-Baxter equation

$$R_{12}(\lambda)R_{13}(\lambda + \mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda + \mu)R_{12}(\lambda),$$

(1)
where the $R$-matrix $R_{ab}(\lambda)$ act on the tensor product of two $Z_2$ graded vector spaces $V_a$ and $V_b$.

In general a graded vector space $V$ is defined by the direct sum $V^{(0)} \oplus V^{(1)}$ where $V^{(0)}$ and $V^{(1)}$ represents its even and odd subspaces. The $\alpha$-th degree of freedom of these subspaces are distinguished by their Grassmann parity which is a function $p_\alpha$ with values in the group $Z_2$,

$$p_\alpha = \begin{cases} 
0 & \text{for } \alpha \text{ even} \\
1 & \text{for } \alpha \text{ odd}
\end{cases} \quad (2)$$

We recall that the tensor products in Eq.(1) take into account the gradation of the respective subspaces. This means that the matrix elements of Eq.(1) will depend crucially on the parities of the coordinates, see refs. [11, 12] for detailed discussion. It is possible, however, to define a new matrix $\tilde{R}_{ab}(\lambda)$ satisfying a different relation that is insensitive to grading, namely

$$\tilde{R}_{12}(\lambda) \tilde{R}_{23}(\lambda + \mu) \tilde{R}_{12}(\mu) = \tilde{R}_{23}(\mu) \tilde{R}_{12}(\lambda + \mu) \tilde{R}_{23}(\lambda). \quad (3)$$

This matrix plays a direct role in the quantum inverse scattering formulation and it is simply related to $R_{ab}(\lambda)$ by the following expression

$$\tilde{R}_{ab}(\lambda) = P_{ab} R_{ab}(\lambda), \quad (4)$$

where $P_{ab} = \sum_{\alpha, \beta = 1}^{N} (-1)^{p_\alpha p_\beta} \hat{e}_{\alpha \beta}^{(a)} \otimes \hat{e}_{\beta \alpha}^{(b)}$ is the graded permutation operator and the integer $N$ represents the dimension of the spaces $V_a$. As usual $\hat{e}_{\alpha \beta}^{(a)} \in V_a$ denotes $N \times N$ matrices having only one non-vanishing element with value 1 at row $\alpha$ and column $\beta$.

In what follows we will exhibit explicit $\tilde{R}$-matrices expressions associated with $U_q[sl(r|2m)^{(2)}]$, $U_q[osp(r|2m)^{(1)}]$ and $U_q[osp(r = 2n|2m)^{(2)}]$ Lie superalgebras. We remark that the needed informations about these algebras, including the forms of the possible Coxeter automorphisms and the corresponding $R$-matrices in terms of projectors have been described in ref. [4]. Therefore, here we restrict ourselves in presenting only the main results for the $\tilde{R}$-matrices in suitable basis for an algebraic Bethe ansatz analysis. In order to obtain explicit formulas it is convenient to work with a specific grading and we have chosen the following one

$$p_\alpha^{(b)} = \begin{cases} 
1 & \text{for } \alpha = 1, \ldots, m \text{ and } \alpha = r + m + 1, \ldots, r + 2m \\
0 & \text{for } \alpha = m + 1, \ldots, r + m
\end{cases} \quad (5)$$

3
where we have introduced the label $l_0 \equiv (r|2m)$ to emphasize the numbers of even (r) and odd (2m) elements of the graded vector space we are considering.

It turns out that the above mentioned quantum $\tilde{R}$-matrices in the Weyl basis are

$$
\tilde{R}_{ab}^{(l_0)}(\lambda) = \sum_{\alpha=1}^{N_0} a_\alpha^{(l_0)}(\lambda) \hat{e}_\alpha^{(a)} \otimes \hat{e}_\alpha^{(b)} + b^{(l_0)}(\lambda) \sum_{\alpha,\beta=1}^{N_0} \hat{e}_\alpha^{(a)} \otimes \hat{e}_\beta^{(b)} + c^{(l_0)}(\lambda) \sum_{\alpha,\beta=1}^{N_0} \hat{e}_\alpha^{(a)} \otimes \hat{e}_\beta^{(b)} + N_0 \sum_{\alpha,\beta=1}^{N_0} d_{\alpha,\beta}^{(l_0)}(\lambda) \hat{e}_\alpha^{(a)} \otimes \hat{e}_\beta^{(b)}
$$

where each index $\alpha$ has its conjugated $\alpha' = N_0 + 1 - \alpha$ with $N_0$ being the dimension of the graded vector space with $r$ even and $2m$ odd elements. The Boltzmann weights $a_\alpha^{(l_0)}(\lambda)$, $b^{(l_0)}(\lambda)$, $c^{(l_0)}(\lambda)$ and $\bar{c}^{(l_0)}(\lambda)$ are determined by

$$
a_\alpha^{(l_0)}(\lambda) = (e^{2\lambda} - \zeta^{(l_0)})(e^{2\lambda(1-p_\alpha^{(l_0)})} - q^2 e^{2\lambda p_\alpha^{(l_0)}})
$$

$$
b^{(l_0)}(\lambda) = q(e^{2\lambda} - 1)(e^{2\lambda} - \zeta^{(l_0)})
$$

$$
c^{(l_0)}(\lambda) = (1 - q^2)(e^{2\lambda} - \zeta^{(l_0)})
$$

$$
\bar{c}^{(l_0)}(\lambda) = e^{2\lambda} \zeta^{(l_0)}(\lambda),
$$

while $d_{\alpha,\beta}^{(l_0)}(\lambda)$ has the form

$$
d_{\alpha,\beta}^{(l_0)}(\lambda) = 
\begin{cases}
q(e^{2\lambda} - 1)(e^{2\lambda} - \zeta^{(l_0)}) + e^{2\lambda}(q^2 - 1)(\zeta^{(l_0)} - 1) & \text{for } \alpha = \beta = \beta' \\
(e^{2\lambda} - 1)(e^{2\lambda} - \zeta^{(l_0)})(-1)^{p_\alpha^{(l_0)}-1}q^{2p_\alpha^{(l_0)}} + e^{2\lambda}(q^2 - 1) & \text{for } \alpha = \beta \neq \beta' \\
(q^2 - 1)\zeta^{(l_0)}(e^{2\lambda} - 1)\frac{\epsilon_{\alpha}}{\epsilon_{\beta}}q^{t_{\alpha} - t_{\beta}} - \delta_{\alpha,\beta'}(e^{2\lambda} - \zeta^{(l_0)}) & \text{for } \alpha < \beta \\
(q^2 - 1)e^{2\lambda}\left[(e^{2\lambda} - 1)\frac{\epsilon_{\alpha}}{\epsilon_{\beta}}q^{t_{\alpha} - t_{\beta}} - \delta_{\alpha,\beta'}(e^{2\lambda} - \zeta^{(l_0)})\right] & \text{for } \alpha > \beta
\end{cases}
$$

In table 1 we have collected the values of the dimension $N_0$ and the dependence of $\zeta^{(l_0)}$ with the parameter $q$ for each Lie superalgebra. The other variables $\epsilon_{\alpha}$ and $t_{\alpha}$ for the
$U_q[osp(2n|2m)^{(2)}]$ are related to the grading by

\[
\epsilon_\alpha = \begin{cases} 
(-1)^{\frac{l_0}{2}} & \text{for } 1 \leq \alpha \leq \frac{N_0}{2}, \\
1 & \text{for } \frac{N_0}{2} + 1 \leq \alpha \leq N_0,
\end{cases}
\]

(12)

\[
t_\alpha = \begin{cases} 
\alpha - \frac{1}{2} + p_\alpha^{(l_0)} - 2 \sum_{\beta=\alpha}^{N_0} p_\beta^{(l_0)} & \text{for } 1 \leq \alpha \leq \frac{N_0}{2}, \\
\alpha + \frac{1}{2} + p_\alpha^{(l_0)} - 2 \sum_{\beta=\frac{N_0}{2}+1}^{N_0} p_\beta^{(l_0)} & \text{for } \frac{N_0}{2} + 1 \leq \alpha \leq N_0,
\end{cases}
\]

(13)

and for the remaining superalgebras we have

\[
\epsilon_\alpha = \begin{cases} 
(-1)^{\frac{l_0}{2}} & \text{for } 1 \leq \alpha < \frac{N_0 + 1}{2}, \\
1 & \text{for } \alpha = \frac{N_0 + 1}{2}, \\
(-1)^{\frac{l_0}{2}} & \text{for } \frac{N_0 + 1}{2} < \alpha \leq N_0
\end{cases}
\]

(14)

\[
t_\alpha = \begin{cases} 
\alpha + \frac{1}{2} - p_\alpha^{(l_0)} + 2 \sum_{\alpha \leq \beta < \frac{N_0 + 1}{2}}^{N_0} p_\beta^{(l_0)} & \text{for } 1 \leq \alpha < \frac{N_0 + 1}{2}, \\
\frac{N_0 + 1}{2} - \sum_{\frac{N_0 + 1}{2} < \beta \leq \alpha}^{N_0+1} p_\beta^{(l_0)} & \text{for } \alpha = \frac{N_0 + 1}{2}, \\
\frac{N_0 + 1}{2} - \sum_{\frac{N_0 + 1}{2} < \beta \leq \alpha}^{N_0 + 1} p_\beta^{(l_0)} & \text{for } \frac{N_0 + 1}{2} < \alpha \leq N_0.
\end{cases}
\]

(15)

We note that the $R$-matrix $R_{12}(\lambda)$ defined by Eqs.(4,6) satisfies important relations besides the standard properties of regularity and unitarity. One of them is the so-called $PT$ symmetry given by

\[
P_{12}R_{12}(\lambda)P_{12} = R_{12}^{st_1st_2}(\lambda),
\]

(16)

where the symbol $st_k$ denotes the supertransposition in the space with index $k$. The other is the crossing symmetry, namely

\[
R_{12}(\lambda) = \frac{\rho(\lambda)}{\rho(-\lambda - \eta)} V_{1} R_{12}^{st_2}(-\lambda - \eta) V_{1}^{-1},
\]

(17)
where $\rho(\lambda)$ is a convenient normalization, $\eta$ is the crossing parameter and $V$ is an anti-diagonal matrix. Since the expressions of some of these quantities are sufficiently cumbersome we have collected them in appendix A.

We would like to point out that our findings for the $\hat{R}$-matrices of the $U_q[sl(2n+1|2m)]^{(2)}$ and $U_q[osp(2n+1|2m)]^{(1)}$ vertex models remains valid even for $n = 0$. In this case, however, such $\hat{R}$-matrices can be related, by transformations that are compatible with the Yang-Baxter equation, to that of the $U_{\tilde{q}}[B_m]$ and $U_{\tilde{q}}[A_{2m}]^{(2)}$ vertex models [3] with $\tilde{q} = -q^{-1}$, respectively.

3 The algebraic Bethe ansatz

Each Yang-Baxter solution presented in last section may be interpreted as the local Boltzmann weights of an integrable vertex models on a square lattice of size $L \times L$ [1]. Its corresponding row-to-row transfer matrix $T^{(l_0)}(\lambda)$ can be conveniently written as the supertrace, over an auxiliary space $A \equiv C_{N_0}$, of an operator denominated monodromy matrix $T^{(l_0)}(\lambda)$ [11, 12]

$$T^{(l_0)}(\lambda) = \text{Str}[T^{(l_0)}(\lambda)] = \sum_{\alpha=1}^{N_0} (-1)^{p^{(l_0)}_{a_\alpha}} T^{(l_0)}_{\alpha\alpha}(\lambda)$$

where $T^{(l_0)}_{\alpha\beta}(\lambda)$ denotes the elements of the monodromy matrix which is given by the following ordered product of $R$-matrices

$$T^{(l_0)}(\lambda) = R^{(l_0)}_{AL}(\lambda) R^{(l_0)}_{AL-1}(\lambda) \ldots R^{(l_0)}_{A1}(\lambda).$$

The local weights of the above expression are obtained from Eq.(6) by the relation $R^{(l_0)}_{A_{\beta j}}(\lambda) = P_{A_{\beta j}} R^{(l_0)}_{A_{\beta j}}(\lambda)$. They are viewed as $N_0 \times N_0$ matrices on the auxiliary space $A$ whose elements are operators acting nontrivially in the $j$-th quantum space $\prod_{j=1}^{L} \otimes C_{N_0}$. The monodromy operator (19) is a basic object in the quantum inverse scattering method and with help of the Yang-Baxter equation (3) one can show that it satisfy the following quadratic algebra

$$\hat{R}^{(l_0)}_{12}(\lambda - \mu) T^{(l_0)}(\lambda) \otimes_{s^0} T^{(l_0)}(\mu) = T^{(l_0)}(\mu) \otimes_{s^0} T^{(l_0)}(\lambda) \hat{R}^{(l_0)}_{12}(\lambda - \mu),$$

where the matrix elements of $\hat{R}^{(l_0)}_{12}(\lambda - \mu)$ are the weights (7-15) defined on the tensor product $A \otimes A$. The symbol $\otimes_{s^0}$ stands for the supertensor product [11] with respect to the auxiliary
space $\mathcal{A}$. We recall that such product between two matrices with elements $A_{ab}$ and $B_{cd}$ should be understood by $A \otimes B = \sum_{abcd} (-1)^{p_{a}^{(l_{0})} + p_{d}^{(l_{0})}} A_{ac} B_{bd} \hat{e}_{ac} \otimes \hat{e}_{bd}$.

The Yang-Baxter algebra (20) plays a fundamental role in the solution of the transfer matrix eigenvalue problem,

$$T^{(l_{0})}(\lambda) |\Phi\rangle = \Lambda^{(l_{0})}(\lambda) |\Phi\rangle,$$

by means of an exact operator formalism. Other important ingredient is the existence of a pseudovacuum state $|\Phi_{0}\rangle$ in which the monodromy matrix acts triangularly. This state help us to identify the off-diagonal elements of the monodromy matrix as potential creation and annihilation fields. For the vertex models considered in this paper we can choose $|\Phi_{0}\rangle$ as the highest weight state vector

$$|\Phi_{0}\rangle = \prod_{j=1}^{L} \otimes |0\rangle_{j}, \quad |0\rangle_{j} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{N_{0}},$$

where $|0\rangle_{j}$ is the local reference state at the $j$-th lattice site with $N_{0}$ components. The action of each operator $R^{(l_{0})}_{A_{j}}(\lambda)$ in this state gives

$$R^{(l_{0})}_{A_{j}}(\lambda) |0\rangle_{j} = \begin{pmatrix} \omega_{1}^{(l_{0})}(\lambda) |0\rangle_{j} & \dagger & \dagger & \cdots & \dagger & \dagger \\ 0 & \omega_{2}^{(l_{0})}(\lambda) |0\rangle_{j} & 0 & \cdots & 0 & \dagger \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_{N_{0}-1}^{(l_{0})}(\lambda) |0\rangle_{j} & \dagger \\ 0 & 0 & 0 & \cdots & 0 & \omega_{N_{0}}^{(l_{0})}(\lambda) |0\rangle_{j} \end{pmatrix}_{N_{0} \times N_{0}}$$

where the symbol $\dagger$ stands for non-null values and the functions $\omega_{\alpha}^{(l_{0})}(\lambda)$ are given by

$$\omega_{\alpha}^{(l_{0})}(\lambda) = \begin{cases} (-1)^{p_{1}^{(l_{0})}} a_{1}^{(l_{0})}(\lambda) & \text{for } \alpha = 1 \\ (-1)^{p_{\alpha}^{(l_{0})}} b_{\alpha}^{(l_{0})}(\lambda) & \text{for } \alpha = 2, \ldots, N_{0} - 1 \\ (-1)^{p_{N_{0}}^{(l_{0})}} d_{N_{0},N_{0}}^{(l_{0})}(\lambda) & \text{for } \alpha = N_{0} \end{cases}$$

To make further progress one needs to seek for an appropriate representation of the monodromy matrix that is able to distinguish possible creation and annihilation fields.

Previous
experience with the vertex models [13] whose weights have similar triangular property such as exhibited in Eq.(23) suggests us that a promising ansatz should be

\[
\mathcal{T}^{(l_0)}(\lambda) = \begin{pmatrix}
B(\lambda) & \tilde{B}(\lambda) & F(\lambda) \\
\tilde{C}(\lambda) & \hat{A}(\lambda) & \tilde{B}^*(\lambda) \\
C(\lambda) & \tilde{C}^*(\lambda) & D(\lambda)
\end{pmatrix}_{N_0 \times N_0}, \tag{25}
\]

where \(\tilde{B}(\lambda)\) (\(\tilde{B}^*(\lambda)\)) and \(\tilde{C}^*(\lambda)\) (\(\tilde{C}(\lambda)\)) are \((N_0 - 2)\)-component row (column) vectors, \(\hat{A}(\lambda)\) is a \((N_0 - 2) \times (N_0 - 2)\) matrix whose elements will be denoted by \(A_{ab}(\lambda)\) and the remaining operators play the role of scalars. Taking into account this representation and the grading choice (5), the diagonalization of the transfer matrix becomes equivalent to the problem

\[
\begin{pmatrix}
(-1)^{p_1^{(l_0)}} B(\lambda) + \sum_{a=1}^{N_0-2} (-1)^{p_a^{(l_0)}} \hat{A}_{aa}(\lambda) + (-1)^{p_{N_0}^{(l_0)}} D(\lambda)
\end{pmatrix} \phi = \Lambda^{(l_0)}(\lambda) \phi. \tag{26}
\]

Direct comparison between Eq.(19) and Eq.(23) reveals that \(\tilde{B}(\lambda)\), \(\tilde{B}^*(\lambda)\) and \(F(\lambda)\) are creation fields with respect to the reference state \(|\Phi_0\rangle\). Furthermore, the diagonal elements of \(\mathcal{T}^{(l_0)}(\lambda)\) satisfy the relations

\[
B(\lambda) |\Phi_0\rangle = [\omega_1(\lambda)]^L |\Phi_0\rangle \quad \quad \quad D(\lambda) |\Phi_0\rangle = [\omega_{N_0}(\lambda)]^L |\Phi_0\rangle,
\]

\[
A_{aa}(\lambda) |\Phi_0\rangle = [\omega_{a+1}(\lambda)]^L |\Phi_0\rangle \quad \text{for } a = 1, \ldots, N_0 - 2, \tag{27}
\]

as well as the annihilation properties

\[
\tilde{C}(\lambda) |\Phi_0\rangle = 0 \quad \tilde{C}^*(\lambda) |\Phi_0\rangle = 0 \quad C(\lambda) |\Phi_0\rangle = 0
\]

\[
A_{ab}(\lambda) |\Phi_0\rangle = 0 \quad \text{for } a, b = 1, \ldots, N_0 - 2 \quad a \neq b, \tag{28}
\]

implying that the reference state \(|\Phi_0\rangle\) is one of the transfer matrix eigenstates whose respective eigenvalue is

\[
\Lambda^{(l_0)}(\lambda) = (-1)^{p_1^{(l_0)}} [\omega_1(\lambda)]^L + \sum_{a=1}^{N_0-2} (-1)^{p_{a+1}^{(l_0)}} [\omega_{a+1}(\lambda)]^L + (-1)^{p_{N_0}^{(l_0)}} [\omega_{N_0}(\lambda)]^L. \tag{29}
\]

Within the algebraic Bethe ansatz approach we now seek for other transfer matrix eigenvectors as linear combinations of products of creations fields acting on \(|\Phi_0\rangle\). In order to do that we need to find the appropriate set of commutation rules between the diagonal and creation fields which in principle are encoded in the Yang-Baxter algebra (20). The procedure of deriving
commutation rules in a convenient form is similar to that describe in ref.[13], requiring in some cases the substitution of the exchange rules between the scalar operator $B(\lambda)$ and $F(\mu)$ or the vector field $\vec{B}^*(\lambda)$ back on the original commutation relations coming from the algebra (20). A considerable amount of additional work is however necessary to include some adaptations that take into account the grading structure. For example, the commutation relations between the diagonal fields and the creation operator $\vec{B}(\lambda)$ are

$$B(\lambda) \otimes \vec{B}(\mu) = (-1)^{p_{12}^{(lo)} b_{12}^{(lo)} (\mu - \lambda)} \vec{B}(\mu) \otimes B(\lambda) + (-1)^{p_{12}^{(lo)} c_{12}^{(lo)} (\mu - \lambda)} \vec{B}(\lambda) \otimes B(\mu)$$

(30)

$$D(\lambda) \otimes \vec{B}(\mu) = (-1)^{p_{12}^{(lo)} b_{12}^{(lo)} (\mu - \lambda)} \vec{B}(\mu) \otimes D(\lambda) - \frac{d_{N_0,1}^{(lo)} (\lambda - \mu)}{d_{N_0,N_0}^{(lo)} (\lambda - \mu)} F(\lambda) \otimes \vec{C}^*(\mu)$$

$$+ \frac{c_{12}^{(lo)} (\lambda - \mu)}{d_{N_0,N_0}^{(lo)} (\lambda - \mu)} F(\mu) \otimes \vec{C}^*(\lambda) - \frac{\vec{\xi}^{(lo)}_{1} (\lambda - \mu)}{d_{N_0,N_0}^{(lo)} (\lambda - \mu)} \cdot [\vec{B}^*(\lambda) \otimes \hat{A}(\mu)]$$

(31)

$$\hat{A}(\lambda) \otimes \vec{B}(\mu) = \frac{1}{b_{12}^{(lo)} (\lambda - \mu)} \vec{B}(\mu) \otimes \hat{A}(\lambda) \cdot \vec{\xi}_{12}^{(lo)} (\lambda - \mu) - \frac{\vec{\xi}^{(lo)}_{1} (\lambda - \mu)}{b_{12}^{(lo)} (\lambda - \mu)} \vec{B}(\lambda) \otimes \hat{A}(\mu)$$

$$+ \frac{1}{d_{N_0,N_0}^{(lo)} (\lambda - \mu)} \left[ (-1)^{p_{12}^{(lo)} b_{12}^{(lo)} (\mu - \lambda)} \vec{B}^*(\lambda) \otimes B(\mu) + \frac{\vec{\xi}^{(lo)}_{1} (\lambda - \mu)}{b_{12}^{(lo)} (\lambda - \mu)} F(\lambda) \otimes \vec{C}^*(\mu) \right] \otimes \vec{\xi}_{12}^{(lo)} (\lambda - \mu)$$

$$+ \frac{1}{b_{12}^{(lo)} (\lambda - \mu)} \left[ F(\mu) \otimes \vec{C}(\lambda) \right] \otimes \vec{\xi}_{12}^{(lo)} (\lambda - \mu)$$

(32)

where $p_{ab}^{(lo)} = p_{a}^{(lo)} + p_{b}^{(lo)}$ and the symbol $\otimes$ denotes the supertensor product with new Grassmann parities $p_{a}^{(1)}$ related to the previous ones by $p_{a}^{(1)} = p_{a+1}^{(lo)}$, $\alpha = 1, \ldots, N_0 - 2$. Furthermore, the vectors $\vec{\xi}_{1}^{(lo)}$ and $\vec{\xi}_{2}^{(lo)}$ are given by

$$\vec{\xi}_{1}^{(lo)} (\lambda) = \sum_{a=1}^{N_0 - 2} d^{(lo)}_{N_0,N_0} + 1 (\lambda) \hat{e}_a \otimes \hat{e}_{N_0 - 1 - a}$$

(33)

and

$$\vec{\xi}_{2}^{(lo)} (\lambda) = \sum_{a=1}^{N_0 - 2} \left[ d^{(lo)}_{1,1+a} (\lambda) - d^{(lo)}_{N_0,a+1} (\lambda) \right] \frac{d^{(lo)}_{1,N_0} (\lambda)}{d^{(lo)}_{N_0,N_0} (\lambda)} \hat{e}_a \otimes \hat{e}_{N_0 - 1 - a},$$

(34)

such that $\hat{e}_i$ is a vector of length $N_0 - 2$ with only one non-null unitary element at $i$-th position. The label $l_\alpha$ generalizes previous definition, characterizing the graded vector space
with $N_\alpha = N_0 - 2\alpha$ degrees of freedom whose number of even and odd elements is determined by the following rule

$$l_\alpha \equiv \begin{cases} 
(r|2m - 2\alpha) & \text{for } m \geq \alpha \\
(r + 2m - 2\alpha|0) & \text{for } 0 \leq m < \alpha 
\end{cases} \quad (35)$$

The final definition entering the commutation rules is concerned with the auxiliary $\tilde{R}$-matrix $\tilde{r}_{ab}^{(l_1)}(\lambda)$. It is obtained from Eq.(6) by the expression

$$\tilde{r}_{ab}^{(l_\alpha)}(\lambda) = \kappa^{(l_{a-1})}(\lambda) \tilde{R}_{ab}^{(l_\alpha)}(\lambda) \quad \kappa^{(l_\alpha)}(\lambda) = q^{(l_\alpha)} \frac{\tilde{b}^{(l_\alpha)}(\lambda)}{d_{N_\alpha,N_\alpha}^{(l_\alpha)}(\lambda)} \quad (36)$$

where $q^{(l_\alpha)} = (-1)^{l_1^{(l_\alpha)}} q^{1 - 2p_1^{(l_\alpha)}}$ and $\tilde{R}_{ab}^{(l_\alpha)}(\lambda)$ is the $\tilde{R}$-matrix (6) defined in the graded space labeled by $l_\alpha$.

The other sets of commutation rules necessary in the solutions of the eigenvalue problem are those between the diagonal fields and the scalar creation operator $F(\lambda)$ as well as among all the creation fields. In order to avoid overcrowding this section with extra heavier formulae we have summarized them in Appendix B. It turns out that the role analysis of the transfer matrix eigenvalue problem described in ref.[13] can be adjusted to cover the analogous problem (21) for the trigonometric vertex models described in section 2. Considering that this procedure has been well explained in the above mentioned reference, there is no need to repeat it here again, and in what follows we will present only the essential points concerning the properties of the eigenvectors and eigenvalues. As usual the eigenvectors of the transfer matrix are built up in terms of a linear combination of products of the many creations operators acting on the pseudo vacuum state $|\Phi_0\rangle$. They form a multiparticle state structure characterized by a set of rapidities $\{\lambda_j^{(l_1)}\}$, that parameterize the creation fields and can be written in terms of the following scalar product,

$$|\Phi_{m_1}\rangle = \Phi_{m_1}^{(l_1)}(\lambda_1^{(l_1)}, \ldots, \lambda_{m_1}^{(l_1)}) \cdot \vec{F} |\Phi_0\rangle, \quad (37)$$

where the vector $\vec{F} \in \prod_{j=1}^{m_1} \otimes C_{N_0-2}^{N_0}$ whose coefficients are going to be denoted by $F^{a_{m_1} \ldots a_1}$ and the indices $a_j$ run over $N_0 - 2$ possible values. The structure of the vector $\Phi_{m_1}^{(l_1)}(\lambda_1^{(l_1)}, \ldots, \lambda_{m_1}^{(l_1)})$
obeys the following second order recursion relation

\[
\Phi_{m_1}(\lambda_1^{(l_1)}, \ldots, \lambda_{m_1}^{(l_1)}) = B(\lambda_1^{(l_1)}) \otimes \Phi_{m_1-1}(\lambda_2^{(l_1)}, \ldots, \lambda_{m_1}^{(l_1)}) - \sum_{j=2}^{m_1} (-1)^{p_j^{(l_0)}} \xi^{(l_0)}_j(\lambda_1^{(l_1)} - \lambda_j^{(l_1)}) \prod_{k=2, k \neq j}^{m_1} \frac{a_1^{(l_0)}(\lambda_k^{(l_1)} - \lambda_j^{(l_1)})}{b^{(l_0)}(\lambda_k^{(l_1)} - \lambda_j^{(l_1)})} \\
+ F(\lambda_1^{(l_1)}) \otimes \Phi_{m_1-2}(\lambda_2^{(l_1)}, \ldots, \lambda_{j-1}^{(l_1)}, \lambda_{j+1}^{(l_1)}, \ldots, \lambda_{m_1}^{(l_1)}) \\
\times B(\lambda_j^{(l_1)}) \prod_{k=2}^{j-1} \frac{\tilde{x}_k^{(l_1)}(\lambda_k^{(l_1)} - \lambda_j^{(l_1)})}{a_1^{(l_0)}(\lambda_k^{(l_1)} - \lambda_j^{(l_1)})} \tag{38}
\]

where we see that the vector $\xi_1^{(l_0)}(\lambda)$ projects out from the linear combination (38) certain states that describe pair of excitations with the same bare momenta $\lambda_j^{(l_1)}$. It therefore plays the role of a generalized exclusion rule, forbidding certain states at the same site of lattice.

In order to make the eigenkets defined by Eqs.(37-38) true eigenvectors of the transfer matrix $T^{(l_0)}(\lambda)$ it is required that the vector $\tilde{F}$ be an eigenstate of a inhomogeneous transfer matrix $\tilde{T}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\})$ whose Boltzmann weights $r_{(l_1)}^{(l_1)}(\lambda)$ are directly related to the auxiliary matrix $r_{ab}^{(l_1)}(\lambda)$ by

\[
r_{(l_1)}^{(l_1)}(\lambda) = P_{(l_1)} r_{(l_1)}^{(l_1)}(\lambda) \tag{39}
\]

where now $A^{(l_1)} \in C^{N_0-2}$, i.e a space with two less degrees of freedom as compared with $A$. As before $\tilde{T}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\})$ is given in terms of the supertrace of a monodromy matrix over the space $A^{(l_1)}$ by

\[
\tilde{T}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\}) = \text{Str}_{A^{(l_1)}} r_{A^{(l_1)} m_1}^{(l_1)} (\lambda - \lambda_{m_1}^{(l_1)}) r_{A^{(l_1)} m_{1-1}}^{(l_1)} (\lambda - \lambda_{m_{1-1}}^{(l_1)}) \ldots r_{A^{(l_1)} 1}^{(l_1)} (\lambda - \lambda_{1}^{(l_1)}) \tag{40}
\]

Following the same kind of arguments explained in ref.[13] and considering the form of our commutations rules we find that the corresponding eigenvalues are given by the expression,

\[
\Lambda^{(l_0)}(\lambda) = (-1)^{p_i^{(l_0)}} [\omega_i(\lambda)]^{L} \prod_{j=1}^{m_1} (-1)^{p_j^{(l_0)}} \frac{a_i^{(l_0)}(\lambda_i^{(l_1)} - \lambda)}{b^{(l_0)}(\lambda_i^{(l_1)} - \lambda)} \\
+ (-1)^{p_j^{(l_0)}} [\omega_{N_0}(\lambda)]^{L} \prod_{j=1}^{m_1} (-1)^{p_j^{(l_0)}} \frac{b^{(l_0)}(\lambda - \lambda_j^{(l_1)})}{d^{(l_0)}_{N_0,N_0}(\lambda - \lambda_j^{(l_1)})} \\
+ [b^{(l_0)}(\lambda)]^{L} \tilde{A}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\}) \prod_{i=1}^{m_1} \frac{1}{b^{(l_0)}(\lambda - \lambda_i^{(l_1)})} \tag{41}
\]
where $\tilde{\Lambda}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\})$ is the eigenvalue of the inhomogeneous transfer matrix $\tilde{T}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\})$, and provided that the rapidities $\{\lambda_j^{(l_1)}\}$ satisfy the Bethe ansatz equations

$$
\left[ (-1)^{p_1^{(l_0)}} \frac{a_1^{(l_0)}(\lambda_j^{(l_1)})}{b^{(l_0)}(\lambda_j^{(l_1)})} \right]^{L} a_1^{(l_0)}(0)
\prod_{j=1, j \neq i}^{m_{l_1}} (-1)^{p_1^{(l_0)}} b^{(l_0)}(\lambda_j^{(l_1)} - \lambda_i^{(l_1)}) \frac{a_1^{(l_0)}(\lambda_j^{(l_1)} - \lambda_i^{(l_1)})}{b^{(l_0)}(\lambda_j^{(l_1)} - \lambda_i^{(l_1)})} = \tilde{\Lambda}^{(l_1)}(\lambda = \lambda_i^{(l_2)}, \{\lambda_j^{(l_1)}\})
$$

(42)

This completes only the first step of the Bethe ansatz analysis because we still need to determine the eigenvalues $\tilde{\Lambda}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\})$. We have reached a point which is typical of nested Bethe ansatz problems that are going to be discussed in the next section.

4 Eigenvalues and Bethe Ansatz Equations

This section is concerned with the diagonalization of the auxiliary transfer matrix $\tilde{T}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\})$ which will be carried out by another algebraic Bethe ansatz analysis. The corresponding monodromy matrix can be read off from Eq.(40) and it is

$$
\mathcal{T}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\}) = r_{A^{(l_1)}}^{(l_1)}(\lambda - \lambda^{(l_1)}_i) r_{A^{(l_2)}}^{(l_2)}(\lambda - \lambda^{(l_2)}_{m_{l_1}} - 1) \ldots r_{A^{(l_2)}}^{(l_2)}(\lambda - \lambda^{(l_2)}_{m_{l_1-1}})\ldots r_{A^{(l_1)}}^{(l_1)}(\lambda - \lambda^{(l_1)}_1),
$$

(43)

that satisfies the following intertwining relation

$$
r_{12}^{(l_1)}(\lambda - \mu)\mathcal{T}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\}) \otimes \mathcal{T}^{(l_1)}(\mu, \{\lambda_j^{(l_1)}\}) = \mathcal{T}^{(l_1)}(\mu, \{\lambda_j^{(l_1)}\}) \otimes \mathcal{T}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\}) r_{12}^{(l_1)}(\lambda - \mu).
$$

(44)

As long as $N_1 \geq 3$ the structure of the Boltzmann weights $r_{A^{(l_1)}}^{(l_1)}(\lambda)$ resembles much that of the original vertex operator $R_{A_j}^{(l_0)}$ we have begun with. In this situation, we can proceed by adjusting the main results of previous section but now with $N_0 - 2$ degrees of freedom as well as by taking into account the presence of the inhomogeneities $\{\lambda_j^{(l_1)}\}$. The new pseudovacuum state $|\Phi_0^{(l_1)}\rangle$ in which the monodromy matrix (43) acts triangularly is given by the following
ferromagnetic state

\[ |\Phi_0^{(1)}\rangle = \prod_{j=1}^{m_1} |0^{(1)}\rangle_j, \quad |0^{(1)}\rangle_j = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \]

(45)

and the action of the vertex operator \( r_{A^{(1)}j}^{(l_1)}(\lambda) \) on it satisfies the relation

\[
\begin{pmatrix}
\omega_1^{(l_1)}(\lambda) |0^{(1)}\rangle_j \\
\omega_2^{(l_1)}(\lambda) |0^{(1)}\rangle_j \\
\vdots \\
\omega_{N_1-1}^{(l_1)}(\lambda) |0^{(1)}\rangle_j \\
\omega_{N_1}^{(l_1)}(\lambda) |0^{(1)}\rangle_j
\end{pmatrix} =
\begin{pmatrix}
\omega_1^{(l_1)}(\lambda) \\
\omega_2^{(l_1)}(\lambda) \\
\vdots \\
\omega_{N_1-1}^{(l_1)}(\lambda) \\
\omega_{N_1}^{(l_1)}(\lambda)
\end{pmatrix}_{N_1 \times N_1}
\]

(46)

where the non-null values \( \omega_{\alpha}^{(l_1)}(\lambda) \) are now given by

\[
\omega_{\alpha}^{(l_1)}(\lambda) = \begin{cases} 
\kappa^{(l_0)}(\lambda) (-1)^{p_1^{(l_1)} \cdot a_1^{(l_1)}(\lambda)} & \text{for } \alpha = 1 \\
\kappa^{(l_0)}(\lambda) (-1)^{p_\alpha^{(l_1)} \cdot b^{(l_1)}(\lambda)} & \text{for } \alpha = 2, \ldots, N_1 - 1 \\
\kappa^{(l_0)}(\lambda) (-1)^{p_{N_1}^{(l_1)} \cdot d_{N_1,N_1}^{(l_1)}(\lambda)} & \text{for } \alpha = N_1
\end{cases}
\]

(47)

As before, the triangularity property of the operator \( r_{A^{(1)}j}^{(l_1)}(\lambda) \) suggest us to take the following representation for the corresponding monodromy matrix

\[
\mathcal{T}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\}) = \begin{pmatrix} B^{(1)}(\lambda, \{\lambda_j^{(l_1)}\}) & \tilde{B}^{(1)}(\lambda, \{\lambda_j^{(l_1)}\}) & F^{(1)}(\lambda, \{\lambda_j^{(l_1)}\}) \\
\tilde{C}^{(1)}(\lambda, \{\lambda_j^{(l_1)}\}) & \tilde{A}^{(1)}(\lambda, \{\lambda_j^{(l_1)}\}) & \tilde{B}^{\ast(1)}(\lambda, \{\lambda_j^{(l_1)}\}) \\
C^{(1)}(\lambda, \{\lambda_j^{(l_1)}\}) & C^{\ast(1)}(\lambda, \{\lambda_j^{(l_1)}\}) & D^{(1)}(\lambda, \{\lambda_j^{(l_1)}\}) \end{pmatrix}_{N_1 \times N_1}
\]

(48)

As a consequence of this assumption and property (46) we find that the elements of this monodromy matrix satisfy the following relations

\[
B^{(1)}(\lambda, \{\lambda_j^{(l_1)}\}) |\Phi_0^{(1)}\rangle = \prod_{i=1}^{m_1} \omega_{i}^{(l_1)}(\lambda - \lambda_i^{(l_1)}) |\Phi_0^{(1)}\rangle,
\]

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as well as the annihilation properties

\begin{align}
\tilde{C}^{(1)}(\lambda, \{\lambda_j^{(l_1)}\}) |\Phi_0^{(1)}\rangle &= 0, \\
\tilde{C}^*^{(1)}(\lambda, \{\lambda_j^{(l_1)}\}) |\Phi_0^{(1)}\rangle &= 0, \\
C^{(1)}(\lambda, \{\lambda_j^{(l_1)}\}) |\Phi_0^{(1)}\rangle &= 0.
\end{align}

(50)

To implement the diagonalization of the transfer matrix \(\tilde{T}^{(l_1)}(\lambda, \{\lambda_j^{(l_1)}\})\) we need to introduce a second Bethe ansatz whose multiparticle eigenstates are going to be parameterized by a new set of inhomogeneities \(\{\lambda_1^{(l_2)}, \ldots, \lambda_{m_2}^{(l_2)}\}\). Clearly, the structure of the commutation relations for the elements of the monodromy matrix (48) as well as the eigenvalue construction is similar to that presented in section 3 and appendix B. We basically have to change the original Boltzmann weights by the corresponding ones related to the \(\tilde{R}\)-matrix \(\tilde{r}^{(l_1)}(\lambda)\), each operator \(\hat{O}(\lambda)\) by its corresponding \(\hat{O}^{(1)}(\lambda, \{\lambda_j^{(l_1)}\})\), to replace the parameters \(\{\lambda_j^{(l_1)}\}\) by \(\{\lambda_j^{(l_2)}\}\) and to substitute the parities \(p_\alpha^{(l_1)}\) by \(p_\alpha^{(l_2)} = p_\alpha^{(l_2)} + 1\) for \(\alpha = 1, \ldots, N_1 - 2\) in the tensor products. As a consequence of that, the role analysis of the previous section can be repeated for each step \(l_\alpha\) whose corresponding Boltzmann weights \(r^{(l_\alpha)}(\lambda)\) share the essential features presented by the original \(R\)-matrix we started with. To make our notation clear we stress that we are using \(\tilde{T}^{(l_\alpha)}(\lambda)\) for the transfer matrix associated to \(r^{(l_\alpha)}(\lambda)\) with eigenvalues \(\tilde{\Lambda}^{(l_\alpha)}(\lambda)\), and \(T^{(l_\alpha)}(\lambda)\) for the one associated with \(R^{(l_\alpha)}(\lambda)\) with eigenvalues \(\Lambda^{(l_\alpha)}(\lambda)\). Note that these \(R\)-matrices differs only by a multiplicative factor as explained in Eq.(36). In particular, the eigenvalues at nearest neighbor steps \(l_\alpha\) and \(l_{\alpha+1}\) are going to satisfy a recurrence relation similar to that exhibited by expression (41). By taking into account our results so far it is not difficult to derive that such relation is given by

\[
\Lambda^{(l_\alpha)}(\lambda, \{\lambda_1^{(l_\alpha)}, \ldots, \lambda_{m_\alpha}^{(l_\alpha)}\}) = (-1)^{p^{(l_\alpha)}} \prod_{i=1}^{m_\alpha} (-1)^{p^{(l_\alpha)}} a_i^{(l_\alpha)}(\lambda - \lambda_i^{(l_\alpha)}) \prod_{i=1}^{m_{\alpha+1}} (-1)^{p^{(l_\alpha)}} \frac{a_i^{(l_\alpha)}(\lambda_i^{(l_{\alpha+1})} - \lambda)}{b^{(l_\alpha)}(\lambda_i^{(l_{\alpha+1})} - \lambda)}
\]

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By following this recipe, it is straightforward to find the final expressions for the eigenvalues of diagonalizing the transfer matrix \( T \). These expressions can be iterate, beginning on \( \alpha = 0 \), until we reach the final step \( l_f \) and therefore up to \( \alpha = f - 1 \). The number of steps necessary in such nested Bethe ansatz construction as well as the underlying \( R \)-matrix \( R_{\mathcal{A}(l_f),j}^{(l_f)}(\lambda) \) will depend much on the family of the vertex model we are diagonalizing. In Table 2 we present the index \( l_f \) and describe the type of vertex models appearing in the lowest Bethe ansatz analysis for each superalgebra \( sl(r|2m)^{(2)} \), \( osp(r|2m)^{(1)} \) and \( osp(2n|2m)^{(2)} \).

For sake of completeness in appendix C we show the explicit expressions for the matrices \( R_{\mathcal{A}(l_f),j}^{(l_f)}(\lambda) \) and the corresponding eigenvalues of the inhomogeneous transfer matrix \( T^{(l_f)}(\lambda, \{\lambda_1^{(l_f)}, \ldots, \lambda_{m_{l_f}}^{(l_f)}\}) \). We have now reached a point which all the results can be gathered together in order to find the eigenvalues expression and the Bethe ansatz equations for the vertex models presented in section 3. For instance, we start with the eigenvalue formula (41) and use the recurrence relation (51) until we reach the problem of diagonalizing the transfer matrix \( T^{(l_f)}(\lambda, \{\lambda_1^{(l_f)}, \ldots, \lambda_{m_{l_f}}^{(l_f)}\}) \). We then take into account the results for the eigenvalues of these vertex systems, which have been collected in appendix C.

By following this recipe, it is straightforward to find the final expressions for the eigenvalues.

\[
+(-1)^{p_{l_{a+1}}^{(l_a)}} \prod_{i=1}^{m_{l_a}} (-1)^{p_{l_a}} d_{N_{l_a},N_{l_a}}^{(l_a)} (\lambda - \lambda_i^{(l_a)}) \prod_{i=1}^{m_{l_a+1}} (-1)^{p_{l_{a+1}}^{(l_a)}} d_{N_{l_{a+1}},N_{l_{a+1}}}^{(l_{a+1})} (\lambda - \lambda_i^{(l_{a+1})})
\]

where \( p_{l_{a+1}}^{(l_a)} = p_{l_{a+1}}^{(l_a)} \) for \( \beta = 1, \ldots, N_a - 2 \).

In order to be consistent, we need to set \( \lambda_j^{(l_a)} = 0 \) for \( j = 1, \ldots, m_{l_0} \) and also to make the identification \( m_{l_0} \equiv L \). By the same token, the variables \( \{\lambda_j^{(l_{a+1})}\} \) that parameterize the eigenvectors of the inhomogeneous transfer matrix \( T^{(l_a)}(\lambda, \{\lambda_1^{(l_a)}, \ldots, \lambda_{m_{l_a}}^{(l_a)}\}) \) are required to satisfy the following Bethe ansatz equation

\[
\prod_{i=1}^{m_{l_a+1}} (-1)^{p_{l_{a+1}}^{(l_a)}} a_i^{(l_{a+1})} (\lambda_i^{(l_{a+1})} - \lambda_{i-1}^{(l_{a-1})}) = \prod_{i \neq j}^{} (-1)^{p_{l_2}^{(l_1)}} a_j^{(l_1)} (\lambda_j^{(l_1)} - \lambda_i^{(l_1)}) b_i^{(l_1)} (\lambda_i^{(l_1)} - \lambda_j^{(l_1)}) d_{N_{l_{a+1}},N_{l_{a+1}}}^{(l_{a+1})} (\lambda_i^{(l_{a+1})} - \lambda_j^{(l_{a+1})})
\]

\[
\times \prod_{i=1}^{m_{l_{a+1}}} (-1)^{p_{l_{a+1}}^{(l_{a+1})}} a_i^{(l_{a+1})} (\lambda_i^{(l_{a+1})} - \lambda_{i+1}^{(l_{a+1})}) b_i^{(l_{a+1})} (\lambda_i^{(l_{a+1})} - \lambda_{i+1}^{(l_{a+1})}) d_{N_{l_{a+1}},N_{l_{a+1}}}^{(l_{a+1})} (\lambda_i^{(l_{a+1})} - \lambda_{i+1}^{(l_{a+1})}),
\]

(51)
of the original vertex models which we shall begin to list below. In order to do that it is convenient to define the function \( Q_\alpha(\lambda) = \prod_{i=1}^{m_{\lambda_0}} \sinh(\lambda - \lambda_i^{(t)}) \) and recalling that we have set \( q = e^{i\gamma} \). The final results are:

\[ U_q[sl(2n + 1|2m)^{(2)}]: \]

\[
\Lambda^{(l_0)}(\lambda) = (-1)^{p_{l_0}^{(l_0)}} \left[ (-1)^{p_{l_0}^{(l_0)}} a_{1}^{(l_0)}(\lambda) \right]^{L} \left[ \frac{Q_1(\lambda - i\frac{\gamma}{2})}{Q_1(\lambda + i\frac{\gamma}{2})} \right]^{2p_{l_0}^{(l_0)} - 1} + (-1)^{p_{l_0}^{(l_0)}} \left[ (-1)^{p_{l_0}^{(l_0)}} d_{1}^{(l_0)}(\lambda) \right]^{L} \left[ \frac{Q_1(\lambda + i\frac{\gamma}{2})}{Q_1(\lambda - i\frac{\gamma}{2})} \right]^{2p_{l_0}^{(l_0)} - 1} + \left[ b^{(l_0)}(\lambda) \right]^{L} \sum_{\alpha=1}^{N_{0} - 2} G_\alpha(\lambda|\{\lambda_j^{(l_0)}\}) \]

(53)

\[
G_\alpha(\lambda|\{\lambda_j^{(l_0)}\}) = \begin{cases} 
Q_\alpha(\lambda+i\frac{\alpha\gamma}{2}) Q_{\alpha+1}(\lambda+i\frac{\alpha\gamma}{2}) 
& \text{for } \alpha = 1, \ldots, m - 1 \\
Q_\alpha(\lambda+i\frac{\alpha\gamma}{2}) Q_{\alpha+1}(\lambda+i\frac{\alpha\gamma}{2}) 
& \text{for } \alpha = m \\
Q_\alpha(\lambda+i\frac{\alpha\gamma}{2}) Q_{\alpha+1}(\lambda+i\frac{\alpha\gamma}{2}) 
& \text{for } \alpha = m + 1, \ldots, m + n - 1 \neq m + n \\
Q_\alpha(\lambda+i\frac{\alpha\gamma}{2}) Q_{\alpha+1}(\lambda+i\frac{\alpha\gamma}{2}) 
& \text{for } \alpha = m + n \\
G_{\alpha-(m+n)}(-i\frac{\gamma}{2} - i(m - n - \frac{1}{2})\gamma - \lambda - \{\lambda_j^{(l_0)}\}) 
& \text{for } \alpha = m + n + 1, \ldots, 2m + 2n - 1
\end{cases}
\]

\[ U_q[sl(2n|2m)^{(2)}]: \]

\[
\Lambda^{(l_0)}(\lambda) = (-1)^{p_{l_0}^{(l_0)}} \left[ (-1)^{p_{l_0}^{(l_0)}} a_{1}^{(l_0)}(\lambda) \right]^{L} \left[ \frac{Q_1(\lambda - i\frac{\gamma}{2})}{Q_1(\lambda + i\frac{\gamma}{2})} \right]^{2p_{l_0}^{(l_0)} - 1} + (-1)^{p_{l_0}^{(l_0)}} \left[ (-1)^{p_{l_0}^{(l_0)}} d_{1}^{(l_0)}(\lambda) \right]^{L} \left[ \frac{Q_1(\lambda + i\frac{\gamma}{2})}{Q_1(\lambda - i\frac{\gamma}{2})} \right]^{2p_{l_0}^{(l_0)} - 1} + \left[ b^{(l_0)}(\lambda) \right]^{L} \sum_{\alpha=1}^{N_{0} - 2} G_\alpha(\lambda|\{\lambda_j^{(l_0)}\}) 
\]

(54)
\[ G_\alpha(\lambda|\{\lambda_{j}^{(l_\alpha)}\}) = \begin{cases} 
\frac{Q_\alpha(\lambda+i(\frac{\alpha_2}{2})\gamma) \ Q_{\alpha+1}(\lambda+1(\frac{\alpha_2}{2})\gamma)}{Q_\alpha(\lambda+i(\frac{\alpha_1}{2})\gamma) \ Q_{\alpha+1}(\lambda+1(\frac{\alpha_1}{2})\gamma)} \text{ for } \alpha = 1, \ldots, m - 1 \\
\frac{Q_\alpha(\lambda+i(\frac{\alpha_2}{2})\gamma) \ Q_{\alpha+1}(\lambda+1(\frac{\alpha_2}{2})\gamma)}{Q_\alpha(\lambda+i(\frac{\alpha_1}{2})\gamma) \ Q_{\alpha+1}(\lambda+1(\frac{\alpha_1}{2})\gamma)} \text{ for } \alpha = m \neq m + n - 1 \\
\frac{Q_\alpha(\lambda+i(\frac{\alpha_2}{2})\gamma) \ Q_{\alpha+1}(\lambda+1(\frac{\alpha_2}{2})\gamma)}{Q_\alpha(\lambda+i(\frac{\alpha_1}{2})\gamma) \ Q_{\alpha+1}(\lambda+1(\frac{\alpha_1}{2})\gamma)} \text{ for } \alpha = m + 1, \ldots, m + n - 2 \neq m + n - 1 \\
\frac{Q_\alpha(\lambda+i(\frac{\alpha_2}{2})\gamma) \ Q_{\alpha+1}(\lambda+1(\frac{\alpha_2}{2})\gamma)}{Q_\alpha(\lambda+i(\frac{\alpha_1}{2})\gamma) \ Q_{\alpha+1}(\lambda+1(\frac{\alpha_1}{2})\gamma)} \text{ for } \alpha = m + n - 1 = m \\
\frac{Q_\alpha(\lambda+i(\frac{\alpha_2}{2})\gamma) \ Q_{\alpha+1}(\lambda+1(\frac{\alpha_2}{2})\gamma)}{Q_\alpha(\lambda+i(\frac{\alpha_1}{2})\gamma) \ Q_{\alpha+1}(\lambda+1(\frac{\alpha_1}{2})\gamma)} \text{ for } \alpha = m + n - 1 \neq m \\
\frac{Q_\alpha(\lambda+i(\frac{\alpha_2}{2})\gamma) \ Q_{\alpha+1}(\lambda+1(\frac{\alpha_2}{2})\gamma)}{Q_\alpha(\lambda+i(\frac{\alpha_1}{2})\gamma) \ Q_{\alpha+1}(\lambda+1(\frac{\alpha_1}{2})\gamma)} \text{ for } \alpha = m + n, \ldots, 2m + 2n - 2 
\end{cases} \]

- \( U_q[osp(2n|2m)^{(1)}] \):

\[ \Lambda^{(l_\alpha)}(\lambda) = (-1)^{l_\alpha} \left[ (-1)^{l_\alpha} a_1^{(l_\alpha)}(\lambda) \right]^{L} \left[ \frac{Q_1 \left( \lambda - i \frac{\alpha_2}{2} \right)^{2p_1^{(l_\alpha)} - 1}}{Q_1 \left( \lambda + i \frac{\alpha_2}{2} \right)} \right]^{L} \]

\[ + \left( -1 \right)^{p_{l_\alpha}^{(l_\alpha)}} \left[ (-1)^{p_{l_\alpha}^{(l_\alpha)}} d_{N_\alpha}^{(l_\alpha)}(\lambda) \right]^{L} \left[ \frac{Q_1 \left( \lambda + i \left( m - n + \frac{3}{2} \right) \right)^{2p_1^{(l_\alpha)} - 1}}{Q_1 \left( \lambda + i \left( m - n + \frac{1}{2} \right) \right)} \right]^{L} \]

\[ + \left[ b^{(l_\alpha)}(\lambda) \right]^{L} \sum_{\alpha=1}^{N_{l_\alpha}} G_\alpha(\lambda|\{\lambda_{j}^{(l_\alpha)}\}) \]

(55)
• \( U_q[osp(2n + 1|2m)^{(1)}] \):

\[
\Lambda^{(l_0)}(\lambda) = (-1)^{p_1(l_0)} \left[ (-1)^{p_1(l_0)} a_1^{(l_0)}(\lambda) \right]^L \left[ \frac{Q_1 \left( \lambda - i \frac{\gamma}{2} \right)}{Q_1 \left( \lambda + i \frac{\gamma}{2} \right)} \right]^{2p_1(l_0) - 1} \\
+ (-1)^{p_N_0} \left[ (-1)^{p_N_0} d_{N_0,N_0}^{(l_0)}(\lambda) \right]^L \left[ \frac{Q_1 \left( \lambda + i (m - n + 1) \gamma \right)}{Q_1 \left( \lambda + i (m - n) \gamma \right)} \right]^{2p_N_0 - 1} \\
+ \left[ b^{(l_0)}(\lambda) \right]^L \sum_{\alpha=1}^{N_0-2} G_\alpha(\lambda|\{\lambda_j^{(l_0)}\}) 
\]

\( (56) \)

\[
G_\alpha(\lambda|\{\lambda_j^{(l_0)}\}) = \\
\begin{cases} 
 Q_\alpha(\lambda+i(\frac{\alpha+1}{2})\gamma) & \text{for } \alpha = 1, \ldots, m-1 \\
 Q_\alpha(\lambda+i(\frac{\alpha+1}{2})\gamma) & \text{for } \alpha = m \\
 Q_\alpha(\lambda+i(\frac{m-n+1}{2})\gamma) & \text{for } \alpha = m+1, \ldots, m+n-1 \neq m+n \\
 Q_\alpha(\lambda+i(\frac{m-n}{2})\gamma) & \text{for } \alpha = m+n \\
 Q_\alpha(\lambda-i(\frac{1}{2})\gamma - \lambda - \{\lambda_j^{(l_0)}\}) & \text{for } \alpha = m+n+1, \ldots, 2m+2n-1 
\end{cases}
\]

• \( U_q[osp(2n|2m)^{(2)}] \):

\[
\Lambda^{(l_0)}(\lambda) = (-1)^{p_1(l_0)} \left[ (-1)^{p_1(l_0)} a_1^{(l_0)}(\lambda) \right]^L \left[ \frac{Q_1 \left( \lambda - i \frac{\gamma}{2} \right)}{Q_1 \left( \lambda + i \frac{\gamma}{2} \right)} \right]^{2p_1(l_0) - 1} \\
+ (-1)^{p_N_0} \left[ (-1)^{p_N_0} d_{N_0,N_0}^{(l_0)}(\lambda) \right]^L \left[ \frac{Q_1 \left( \lambda + i \left( m - n - \frac{1}{2} \right) \gamma \right)}{Q_1 \left( \lambda + i \left( m - n - \frac{3}{2} \right) \gamma \right)} \right]^{2p_N_0 - 1} \\
+ \left[ b^{(l_0)}(\lambda) \right]^L \sum_{\alpha=1}^{N_0-2} G_\alpha(\lambda|\{\lambda_j^{(l_0)}\}) 
\]

\( (57) \)
\[ G_\alpha(\lambda|\{\lambda_j^{(l_\alpha)}\}) = \begin{cases} 
\frac{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\right)\gamma)}{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\right)\gamma)} \frac{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\right)\gamma)}{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\right)\gamma)} & \text{for } \alpha = 1, \ldots, m - 1 \\
\frac{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\left(m+n-\frac{\pi}{2}\right)\right)\gamma)}{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\left(m+n-\frac{\pi}{2}\right)\right)\gamma)} \frac{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\left(m+n-\frac{\pi}{2}\right)\right)\gamma)}{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\left(m+n-\frac{\pi}{2}\right)\right)\gamma)} & \text{for } \alpha = m \neq m + n - 1 \\
\frac{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\left(m+n-\frac{\pi}{2}\right)\right)\gamma)}{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\left(m+n-\frac{\pi}{2}\right)\right)\gamma)} \frac{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\left(m+n-\frac{\pi}{2}\right)\right)\gamma)}{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\left(m+n-\frac{\pi}{2}\right)\right)\gamma)} & \text{for } \alpha = m + 1, \ldots, m + n - 2 \neq m + n - 1 \\
\frac{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\left(m+n-\frac{\pi}{2}\right)\right)\gamma)}{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\left(m+n-\frac{\pi}{2}\right)\right)\gamma)} \frac{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\left(m+n-\frac{\pi}{2}\right)\right)\gamma)}{Q_\alpha(\lambda+i\left(\frac{\pi}{2}\left(m+n-\frac{\pi}{2}\right)\right)\gamma)} & \text{for } \alpha = m + n - 1 \neq m \\
G_{\alpha-(m+n-1)}(-i(m-n-1)\gamma - \lambda|\{-\lambda_j^{(l_\beta)}\}) & \text{for } \alpha = m + n, \ldots, 2m + 2n - 2 
\end{cases} \]

Before proceeding with the Bethe ansatz equations, we note that in the expressions (53-57) we have performed the shifts \(\{\lambda_j^{(l_\alpha)}\} \rightarrow \{\lambda_j^{(l_\alpha)}\} - \delta^{(l_\alpha)}\) in order to bring the final results in a more symmetrical way. In Table 3 we show the values for the displacements \(\delta^{(l_\alpha)}\). The same procedure described above for the eigenvalues also works for determining the Bethe ansatz equations for the shifted rigidities. We begin with Eq.(42), each step of the nesting is disentangled with the help of the relation (52) and when we reach the last step \(l_f\) we use the Bethe ansatz results exhibited in appendix C. It turns out that the Bethe ansatz equations for these vertex models are given by

- \(U_q[sl(2n+1|2m)^{(l_\beta)}]:(\)

\[ \prod_{i=1}^{m_{l_\alpha}} \frac{\sinh \left(\lambda_i^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} + i\gamma \right)}{\sinh \left(\lambda_i^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} - i\gamma \right)} = \prod_{i \neq j} \frac{\sinh \left(\lambda_j^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} + i\gamma \right)}{\sinh \left(\lambda_j^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} - i\gamma \right)} \quad \alpha = 1, \ldots, m - 1 \]

\[ \prod_{i=1}^{m_{l_\alpha}} \frac{\sinh \left(\lambda_i^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} - i\gamma \right)}{\sinh \left(\lambda_i^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} + i\gamma \right)} = \prod_{i \neq j} \frac{\sinh \left(\lambda_j^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} - i\gamma \right)}{\sinh \left(\lambda_j^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} + i\gamma \right)} \quad \alpha = m \]

\[ \prod_{i=1}^{m_{l_\alpha}} \frac{\sinh \left(\lambda_i^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} - i\gamma \right)}{\sinh \left(\lambda_i^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} + i\gamma \right)} = \prod_{i \neq j} \frac{\sinh \left(\lambda_j^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} - i\gamma \right)}{\sinh \left(\lambda_j^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} + i\gamma \right)} \quad \alpha = m + 1, \ldots, m + n - 1 \neq m + n - 1 \]

\[ \prod_{i=1}^{m_{l_\alpha}} \frac{\sinh \left(\lambda_i^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} - i\gamma \right)}{\sinh \left(\lambda_i^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} + i\gamma \right)} = \prod_{i \neq j} \frac{\sinh \left(\lambda_j^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} - i\gamma \right)}{\sinh \left(\lambda_j^{(l_\alpha)} - \lambda_i^{(l_{\alpha-1})} + i\gamma \right)} \quad \alpha = m + n, \ldots, 2m + 2n - 2 \]
\[ U_q[\text{sl}(2n|2m)^{(2)}] : \]

\[
\prod_{i=1}^{m-1} \frac{\sinh \left( \lambda_{i+1} - \lambda_i + i\frac{\gamma}{2} \right)}{\sinh \left( \lambda_{i+1} - \lambda_i - i\frac{\gamma}{2} \right)} = \prod_{i=1}^{m} \frac{\sinh \left( \lambda_{i+1} - \lambda_i + i\gamma \right)}{\sinh \left( \lambda_{i+1} - \lambda_i - i\gamma \right)} \prod_{i=1}^{m+1} \frac{\sinh \left( \lambda_{i+1} - \lambda_i + i\gamma \right)}{\sinh \left( \lambda_{i+1} - \lambda_i - i\gamma \right)}
\]

\[ \alpha = m + n \]

\[ (58) \]

\[ U_q[\text{osp}(2n|2m)^{(1)}] : \]

\[
\prod_{i=1}^{m-1} \frac{\sinh \left( \lambda_{i+1} - \lambda_i + i\frac{\gamma}{2} \right)}{\sinh \left( \lambda_{i+1} - \lambda_i - i\frac{\gamma}{2} \right)} = \prod_{i=1}^{m} \frac{\sinh \left( \lambda_{i+1} - \lambda_i + i\gamma \right)}{\sinh \left( \lambda_{i+1} - \lambda_i - i\gamma \right)} \prod_{i=1}^{m+1} \frac{\sinh \left( \lambda_{i+1} - \lambda_i + i\gamma \right)}{\sinh \left( \lambda_{i+1} - \lambda_i - i\gamma \right)}
\]

\[ \alpha = m + n \]

\[ (59) \]
\[
\prod_{i=1}^{m_{α-1}} \frac{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} + i\frac{γ}{2})}{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} - i\frac{γ}{2})} = \prod_{i=1}^{m_{α}} \frac{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} + i\frac{γ}{2})}{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} - i\frac{γ}{2})} \prod_{i=1}^{m_{α+1}} \frac{\sinh (\lambda_i^{(l_{α+1})} - \lambda_i^{(l_α)} + i\frac{γ}{2})}{\sinh (\lambda_i^{(l_{α+1})} - \lambda_i^{(l_α)} - i\frac{γ}{2})}
\]

\[\prod_{i=1}^{m_{α-1}} \frac{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} + i\frac{γ}{2})}{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} - i\frac{γ}{2})} = \prod_{i=1}^{m_{α}} \frac{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} + i\frac{γ}{2})}{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} - i\frac{γ}{2})} \prod_{i=1}^{m_{α+1}} \frac{\sinh (\lambda_i^{(l_{α+1})} - \lambda_i^{(l_α)} + i\frac{γ}{2})}{\sinh (\lambda_i^{(l_{α+1})} - \lambda_i^{(l_α)} - i\frac{γ}{2})}
\]

\[\prod_{i=1}^{m_{α-1}} \frac{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} + i\frac{γ}{2})}{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} - i\frac{γ}{2})} = \prod_{i=1}^{m_{α}} \frac{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} + i\frac{γ}{2})}{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} - i\frac{γ}{2})} \prod_{i=1}^{m_{α+1}} \frac{\sinh (\lambda_i^{(l_{α+1})} - \lambda_i^{(l_α)} + i\frac{γ}{2})}{\sinh (\lambda_i^{(l_{α+1})} - \lambda_i^{(l_α)} - i\frac{γ}{2})}
\]

\[\prod_{i=1}^{m_{α-1}} \frac{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} + i\frac{γ}{2})}{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} - i\frac{γ}{2})} = \prod_{i=1}^{m_{α}} \frac{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} + i\frac{γ}{2})}{\sinh (\lambda_i^{(l_α)} - \lambda_i^{(l_{α-1})} - i\frac{γ}{2})} \prod_{i=1}^{m_{α+1}} \frac{\sinh (\lambda_i^{(l_{α+1})} - \lambda_i^{(l_α)} + i\frac{γ}{2})}{\sinh (\lambda_i^{(l_{α+1})} - \lambda_i^{(l_α)} - i\frac{γ}{2})}
\]

(61)
\[ \frac{m_{\alpha-1}}{\prod_{i=1}^{m_{\alpha-1}} \sinh \left( \lambda_i^{(\ell)} - \lambda_i^{(\ell-1)} + i\frac{\gamma}{2} \right)} \sinh \left( \lambda_i^{(\ell)} - \lambda_i^{(\ell-1)} - i\frac{\gamma}{2} \right) = \prod_{i=1}^{m_{\alpha+1}} \sinh \left( \lambda_i^{(\ell+1)} - \lambda_i^{(\ell)} - i\alpha \frac{\gamma}{2} \right) \sinh \left( \lambda_i^{(\ell+1)} - \lambda_i^{(\ell)} + i\alpha \frac{\gamma}{2} \right) \alpha = m \]

\[ \frac{m_{\alpha-1}}{\prod_{i=1}^{m_{\alpha-1}} \sinh \left( \lambda_j^{(\ell)} - \lambda_j^{(\ell-1)} - i\frac{\gamma}{2} \right)} \sinh \left( \lambda_j^{(\ell)} - \lambda_j^{(\ell-1)} + i\frac{\gamma}{2} \right) = \prod_{i \neq j}^{m_{\alpha+1}} \sinh \left( \lambda_j^{(\ell)} - \lambda_j^{(\ell)} - i\gamma \right) \sinh \left( \lambda_j^{(\ell)} - \lambda_j^{(\ell)} + i\gamma \right) \alpha = m + n \]

where \( g_{\alpha} \) has two possible values defined by

\[ g_{\alpha} = \begin{cases} 2 & \alpha = m + n - 1 \\ 1 & \text{otherwise} \end{cases} \]

For the sake of completeness we have presented the Bethe ansatz results concerning the vertex models \( U_q[sl(1|2m)^{(2)}] \) and \( U_q[osp(1|2m)^{(1)}] \) in appendix D. We close this section with the following remark. It is possible to verify that the systems of Bethe ansatz equations exhibited above are the conditions of analyticity of \( \Lambda(\lambda) \) as a function of the rapidities \( \{\lambda_j^{(\ell_1)}\}, \ldots, \{\lambda_j^{(\ell_{f+1})}\} \). This is indeed an extra check of the validity of our Bethe ansatz results since the eigenvalue does not know a priori about the existence of such poles.

5 Conclusion

In this paper we have presented the \( R \)-matrices of the fundamental trigonometric vertex models based on the superalgebras \( sl(r|2m)^{(2)} \), \( osp(r|2m)^{(1)} \) and \( osp(2n|2m)^{(2)} \) in terms of the Weyl basis. The structure of the corresponding Boltzmann weights is therefore explicitly unveiled, opening up an opportunity to investigate the physical properties of such vertex models from the statistical mechanics viewpoint. In fact, the transfer matrix eigenvalue problem was formulated and solved by a first principle algebraic framework called quantum inverse scattering method. From our results for the transfer matrix eigenvalues and Bethe ansatz equations one can in principle derive the free-energy thermodynamics, the quasi-particle excitation behaviour as
well as the classes of universality governing the criticality of gapless regimes. Furthermore, the rather universal formula we obtained for the eigenvectors could be useful in future computations of off-shell properties such as form factors [28] and correlation functions [10, 29] of relevant operators.

This work also paves the way to undertake formal study of these vertex models with open boundary conditions [30]. We remark that the \( \tilde{R}_{ab}^{(i_0)}(\lambda) \) commutes for different values of the rapidity \( \lambda \). As a consequence of that the trivial diagonal solution of the reflection equation \( K_-(\lambda) = Id \) and \( K_+(\lambda) = V^{st}V \) [31] does hold for all these vertex models. It seems an interesting problem to classify the solutions of the reflexion equation for such models, extending the recent efforts made in the case of super-Yangian \( R \)-matrices [32] by employing for instance the technique developed in ref.[33].

Finally, we observe that the vertex models discussed in this paper share a common algebraic structure denominated braid-monoid algebra [34]. We hope that this property will help us to improve our understanding of such systems and to provide new insights into other related problems. One of them would be the explicit formulae for the \( \tilde{R} \)-matrices in an arbitrary grading structure, which we plan to study in a future publication.

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Appendix A : The crossing symmetry

The purpose of this appendix is to present the explicit expressions for the crossing parameter \( \eta \), the normalization function \( \rho(\lambda) \) and the crossing matrix \( V \). The crossing parameter \( \eta \) is better written in terms of the anisotropy \( \gamma \) such that \( q = e^{i\gamma} \) and it turns out that \( \eta = i(m - n + 1)\gamma, i\frac{\pi}{2} + i(m - n - \frac{1}{2})\gamma, i\frac{\pi}{2} + i(m - n)\gamma, i(m - n + \frac{1}{2})\gamma, i(m - n - 1)\gamma \) for the
superalgebras $U_q[osp(2n|2m)_{(1)}]$, $U_q[sl(2n+1|2m)_{(2)}]$, $U_q[sl(2n|2m)_{(2)}]$, $U_q[osp(2n+1|2m)_{(1)}]$ and $U_q[osp(2n|2m)_{(2)}]$, respectively. The normalization function is given by

$$\rho(\lambda) = q(e^{2\lambda} - 1)(e^{2\lambda} - \zeta^{l_0})$$  \hspace{1cm} (A.1)

The only non-null entries of the matrix $V$ are the anti-diagonal elements $V_{aa'}$. Up to an arbitrary normalization and for each superalgebra discussed here they are

- $U_q[osp(2n|2m)_{(1)}]$ and $U_q[sl(2n|2m)_{(2)}]$:

$$V_{aa'} = \begin{cases} 
(1)^{1-p_{l_0}} & \text{for } \alpha = 1 \\
(1)^{1/p_{l_0}} q \left( \alpha - p_{l_0} - p_{l_0} - 2 \sum_{\beta=2}^{\alpha-1} p_{l_0} \right) & \text{for } 1 < \alpha < \frac{N_0+1}{2} \\
(1)^{1/p_{l_0}} q \left( \alpha - 2p_{l_0} - 2 \sum_{\beta=2, \beta \neq \frac{N_0+1}{2}+1}^{\alpha-1} p_{l_0} \right) & \text{for } \frac{N_0+1}{2} < \alpha \leq N_0
\end{cases}$$  \hspace{1cm} (A.2)

- $U_q[osp(2n|2m)_{(2)}]$:

$$V_{aa'} = \begin{cases} 
(1)^{1-p_{l_0}} & \text{for } \alpha = 1 \\
(1)^{1/p_{l_0}} q \left( \alpha - p_{l_0} - 2 \sum_{\beta=2}^{\alpha-1} p_{l_0} \right) & \text{for } 1 < \alpha < \frac{N_0+1}{2} \\
(1)^{1/p_{l_0}} q \left( \alpha - p_{l_0} - 2 \sum_{\beta=2, \beta \neq \frac{N_0+1}{2}+1}^{\alpha-1} p_{l_0} \right) & \text{for } \frac{N_0+1}{2} < \alpha \leq N_0
\end{cases}$$  \hspace{1cm} (A.3)
\* \( U_q[osp(2n + 1|2m)^{(1)}] \) and \( U_q[sl(2n + 1|2m)^{(2)}] \):

\[
V_{\alpha\alpha'} = \begin{cases} 
- \frac{1}{2} \frac{1-p_{(0)}^{(l_0)}}{p_{(0)}^{(l_0)}} q & \text{for } \alpha = 1 \\
- \frac{1}{2} \frac{1-p_{(0)}^{(l_0)}}{p_{(0)}^{(l_0)}} \left( \alpha-1-p_{(0)}^{(l_0)}-p_{(0)}^{(l_0)} \right) & \text{for } 1 < \alpha < \frac{N_0+1}{2} \\
- \frac{1}{2} \frac{N_0-1-p_{(0)}^{(l_0)}-p_{(0)}^{(l_0)}}{p_{(0)}^{(l_0)}} & \text{for } \alpha = \frac{N_0+1}{2} \\
- \frac{1}{2} \frac{1-p_{(0)}^{(l_0)}}{p_{(0)}^{(l_0)}} \left( \alpha-2-p_{(0)}^{(l_0)}-p_{(0)}^{(l_0)} \right) & \text{for } \frac{N_0+1}{2} < \alpha \leq N_0
\end{cases}
\]

Appendix B : Commutations rules

This appendix is devoted to complement the commutation relations presented in the main text that are needed in the solution of the transfer matrix eigenvalue problem. The first set is between the diagonal fields and the scalar creation field \( F(\mu) \). The relations among \( B(\lambda) \) and \( D(\lambda) \) with \( F(\mu) \) comes directly from the Yang-Baxter algebra but that between \( \hat{A}(\lambda) \) and \( F(\mu) \) requires also the knowledge of the commutation rule between \( \bar{B}(\lambda) \) and \( \bar{B}^*(\mu) \). They have the following form

\[
B(\mu)^{s_1} \otimes F(\lambda) = \frac{a_1^{(l_0)}(\lambda - \mu)}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} F(\lambda)^{s_1} \otimes B(\mu) - \frac{d_{l_0,1}^{(l_0)}(\lambda - \mu)}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} F(\mu)^{s_1} \otimes B(\lambda) \\
- \frac{1}{2} \frac{p_{12}^{(l_0)}}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} \left[ \bar{B}(\mu)^{s_1} \otimes \bar{B}(\lambda) \right] \cdot [\xi_3^{(l_0)}(\lambda - \mu)]^t,
\]

\[
D(\lambda)^{s_1} \otimes F(\mu) = \frac{a_1^{(l_0)}(\lambda - \mu)}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} F(\mu)^{s_1} \otimes D(\lambda) - \frac{d_{l_0,1}^{(l_0)}(\lambda - \mu)}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} F(\lambda)^{s_1} \otimes D(\mu) \\
- \frac{\xi_4^{(l_0)}(\lambda - \mu)}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} \cdot \left[ \bar{B}^*(\lambda)^{s_1} \otimes \bar{B}^*(\mu) \right],
\]

\[
\hat{A}(\lambda)^{s_1} \otimes F(\mu) = \left[ 1 - \frac{c^{(l_0)}(\lambda - \mu)}{b^{(l_0)}(\lambda - \mu)^2} \right] F(\mu)^{s_1} \otimes \hat{A}(\lambda) + \left[ \frac{c^{(l_0)}(\lambda - \mu)}{b^{(l_0)}(\lambda - \mu)} \right]^2 F(\lambda)^{s_1} \otimes \hat{A}(\mu) \\
- \frac{c^{(l_0)}(\lambda - \mu)}{b^{(l_0)}(\lambda - \mu)} \left[ \bar{B}(\lambda)^{s_1} \otimes \bar{B}^*(\mu) - (-1)^{p_{12}^{(l_0)}} \bar{B}^*(\lambda)^{s_1} \otimes \bar{B}(\mu) \right],
\]

25
where \( \xi^{(l_0)}_3(\lambda) = \sum_{a=1}^{\mathcal{N}_0-2} d_{a+1,N_0}^{(l_0)}(\lambda) \hat{c}_{N_0-1-a} \otimes \hat{e}_a \).

On the other hand the commutation relations between the creation fields are given by

\[
\bar{B}(\lambda) \otimes \bar{B}(\mu) = \hat{B}(\mu) \otimes \hat{B}(\lambda) \cdot \tilde{f}_{12}^{(l_0)}(\lambda - \mu) + \frac{(-1)^{p_{12}^{(l_0)}}}{a_1^{(l_0)}(\lambda - \mu)} \xi_1^{(l_0)}(\lambda - \mu) F(\lambda) \bar{B}(\mu) + \frac{(-1)^{p_{12}^{(l_0)}}}{a_1^{(l_0)}(\lambda - \mu)} \xi_2^{(l_0)}(\lambda - \mu) F(\mu) \bar{B}(\lambda),
\]

\[ [F(\lambda), F(\mu)] = 0, \]

\[
F(\mu) \otimes \bar{B}(\lambda) = (-1)^{p_{12}^{(l_0)}} \frac{b_1^{(l_0)}(\lambda - \mu)}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} \bar{B}(\lambda) \otimes F(\mu) - (-1)^{p_{12}^{(l_0)}} \frac{c^{(l_0)}(\lambda - \mu)}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} \bar{B}(\mu) \otimes F(\lambda),
\]

\[
\bar{B}(\mu) \otimes F(\lambda) = (-1)^{p_{12}^{(l_0)}} \frac{b_1^{(l_0)}(\lambda - \mu)}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} F(\lambda) \otimes \bar{B}(\mu) - (-1)^{p_{12}^{(l_0)}} \frac{c^{(l_0)}(\lambda - \mu)}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} F(\mu) \otimes \bar{B}(\lambda).
\]

There are other commutation rules that either important to write appropriate relations between the diagonal and the creation fields or to disentangle the eigenvalue problem. They are listed below

\[
B(\lambda) \otimes \bar{B}^*(\mu) = (-1)^{p_{12}^{(l_0)}} \frac{b_1^{(l_0)}(\lambda - \mu)}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} \bar{B}^*(\mu) \otimes B(\lambda) - \bar{B}^*(\lambda) \otimes \hat{A}(\mu) \cdot \frac{[\xi_3^{(l_0)}(\mu - \lambda)]^i}{d_{N_0,N_0}^{(l_0)}(\mu - \lambda)} + \frac{c^{(l_0)}(\mu - \lambda)}{d_{N_0,N_0}^{(l_0)}(\mu - \lambda)} F(\mu) \otimes \bar{C}(\lambda) - \frac{d_{1,N_0}^{(l_0)}(\mu - \lambda)}{d_{N_0,N_0}^{(l_0)}(\mu - \lambda)} F(\lambda) \otimes \bar{C}(\mu)
\]

\[
\bar{B}(\lambda) \otimes \bar{B}^*(\mu) = (-1)^{p_{12}^{(l_0)}} \bar{B}^*(\mu) \otimes \bar{B}(\lambda) + \frac{c^{(l_0)}(\mu - \lambda)}{b_1^{(l_0)}(\mu - \lambda)} F(\mu) \otimes \hat{A}(\lambda) - \frac{c^{(l_0)}(\mu - \lambda)}{b_1^{(l_0)}(\mu - \lambda)} F(\lambda) \otimes \hat{A}(\mu)
\]

\[
(-1)^{p_{12}^{(l_0)}} \bar{C}(\lambda) \otimes \bar{B}(\mu) = \bar{B}(\mu) \otimes \bar{C}(\lambda) + \frac{c^{(l_0)}(\mu - \lambda)}{b_1^{(l_0)}(\mu - \lambda)} \left[ B(\mu) \otimes \hat{A}(\lambda) - B(\lambda) \otimes \hat{A}(\mu) \right]
\]
\[ C^*(\lambda) \otimes B(\mu) = B(\mu) \otimes C^*(\lambda) \cdot \frac{\mathcal{R}_{12}(\lambda - \mu)}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} \cdot \frac{d_{N_0,0}^{(l_0)}(\lambda - \mu)}{d_{N_0,N_0}^{(l_0)}(\lambda - \mu)} B(\lambda) \otimes C^*(\mu) \]

\[ - (-1)^{l_1^{(l_0)}} \xi_4^{(l_0)}(\lambda - \mu) \cdot \hat{A}(\lambda) \otimes \hat{A}(\mu) + (-1)^{l_2^{(l_0)}} \xi_5^{(l_0)}(\lambda - \mu) B(\mu) D(\lambda) \]

\[ + (-1)^{l_2^{(l_0)}} \xi_5^{(l_0)}(\lambda - \mu) F(\mu) C(\lambda) \]

where the vectors \( \xi_4^{(l_0)}(\lambda) \) and \( \xi_5^{(l_0)}(\lambda) \) are

\[ \xi_4^{(l_0)}(\lambda) = \sum_{a=1}^{N_0-2} d_{N_0,N_0-a}^{(l_0)}(\lambda) \delta_{N_0-1-a} \otimes \delta_a \]

\[ \xi_5^{(l_0)}(\lambda) = \sum_{a=1}^{N_0-2} d_{1,N_0-a}^{(l_0)}(\lambda) \delta_{N_0-1-a} \otimes \delta_a \]

Finally, the matrix \( \mathcal{R}_{12}(\lambda) \) can be represented by

\[ \mathcal{R}_{12}(\lambda) = \sum_{abcd} R_{a,c}^{+1}\cdot c^{+1}(\lambda) \delta_{ab}^{(1)} \otimes \delta_{cd}^{(2)} \]

where \( R_{a,c}^{+1} \) are the matrix elements of \( \hat{R}_{12}^{(l_0)}(\lambda) \). Here we recall that we have used the convention

\[ \hat{R}_{12}^{(l_0)}(\lambda) = \sum_{abcd} R_{a,c}^{+1}\cdot c^{+1}(\lambda) \delta_{ab}^{(1)} \otimes \delta_{cd}^{(2)} \].

**Appendix C : Auxiliary Bethe Ansatz**

In this appendix we present the Bethe ansatz results concerning the last step \( l_f \) of the nested construction presented in section 4. In general, we need to diagonalize the following inhomogeneous transfer matrix

\[ T^{(l_f)}(\lambda, \{\lambda_1^{(l_f)}, \ldots, \lambda_{m_f}^{(l_f)}\}) \]

\[ = \text{Str}_{\mathcal{A}^{(l_f)}} \left[ R_{\mathcal{A}^{(l_f)},m_f}^{(l_f)}(\lambda - \lambda_{m_f}) R_{\mathcal{A}^{(l_f)},m_f-1}^{(l_f)}(\lambda - \lambda_{m_f-1}) \ldots R_{\mathcal{A}^{(l_f)},1}^{(l_f)}(\lambda - \lambda_{1}) \right] \]

We now begin to list the respective \( R \)-matrix \( R_{\mathcal{A}^{(l_f)},j}^{(l_f)}(\lambda) \), the eigenvalue expression \( \Lambda^{(l_f)}(\lambda, \{\lambda_1^{(l_f)}, \ldots, \lambda_{m_f}^{(l_f)}\}) \) and the corresponding Bethe ansatz equations of the vertex models.
mentioned in Table 2. The $U_q[sl(2n + 1|2m)^{(2)}]$ and $U_q[osp(2n + 1|2m)^{(1)}]$ vertex models have $l_f = (3|0)$ and the corresponding $R^{(l_f)}(\lambda)$ is given by

$$
R^{(l_f)}(\lambda) = \begin{pmatrix}
a_{1}^{(l_f)}(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b^{(l_f)}(\lambda) & 0 & c^{(l_f)}(\lambda) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{1,1}^{(l_f)}(\lambda) & 0 & d_{1,2}^{(l_f)}(\lambda) & 0 & d_{1,3}^{(l_f)}(\lambda) & 0 & 0 \\
0 & c^{(l_f)}(\lambda) & 0 & b^{(l_f)}(\lambda) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{2,1}^{(l_f)}(\lambda) & 0 & d_{2,2}^{(l_f)}(\lambda) & 0 & d_{2,3}^{(l_f)}(\lambda) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b^{(l_f)}(\lambda) & 0 & c^{(l_f)}(\lambda) \\
0 & 0 & d_{3,1}^{(l_f)}(\lambda) & 0 & d_{3,2}^{(l_f)}(\lambda) & 0 & d_{3,3}^{(l_f)}(\lambda) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c^{(l_f)}(\lambda) & 0 & b^{(l_f)}(\lambda) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{1}^{(l_f)}(\lambda)
\end{pmatrix}
$$

(C.2)

where the Boltzmann weights for each superalgebra are given by the set of relations (7 - 15).

For the $U_q[sl(2n + 1|2m)^{(2)}]$ vertex model the underlying $R^{(l_f)}(\lambda)$ operator is related to that of the Izergin-Korepin [26] model and has the following expressions for the eigenvalues and Bethe Ansatz equations

$$
\Lambda^{(l_f)}(\lambda, \{\lambda_{1}^{(l_f)}, \ldots, \lambda_{m_f}^{(l_f)}\}) = \prod_{i=1}^{m_f} a_{i}^{(l_f)}(\lambda - \lambda_{i}^{(l_f)}) \frac{Q_{f+1}(\lambda + i\gamma)}{Q_{f+1}(\lambda)} + \prod_{i=1}^{m_f} d_{1,1}^{(l_f)}(\lambda - \lambda_{i}^{(l_f)}) \frac{Q_{f+1}(\lambda - i\gamma)}{Q_{f+1}(\lambda)} + \prod_{i=1}^{m_f} b^{(l_f)}(\lambda - \lambda_{i}^{(l_f)}) \frac{Q_{f+1}(\lambda - i\gamma)}{Q_{f+1}(\lambda)} \frac{Q_{f+1}(\lambda + i\gamma)}{Q_{f+1}(\lambda)}.
$$

(C.3)

$$
\prod_{i=1}^{m_f} \frac{\sinh(\lambda_{j}^{(l_f+1)} - \lambda_{i}^{(l_f)} - i\gamma)}{\sinh(\lambda_{j}^{(l_f+1)} - \lambda_{i}^{(l_f)})} = \prod_{i \neq j} \frac{\sinh(\lambda_{j}^{(l_f+1)} - \lambda_{i}^{(l_f+1)} - i\gamma) \cosh(\lambda_{j}^{(l_f+1)} - \lambda_{i}^{(l_f+1)} + i\gamma)}{\sinh(\lambda_{j}^{(l_f+1)} - \lambda_{i}^{(l_f+1)} + i\gamma) \cosh(\lambda_{j}^{(l_f+1)} - \lambda_{i}^{(l_f+1)} - i\gamma)}
$$

(C.4)
For the $U_q[osp(2n+1|2m)^{(1)}]$ vertex model, however, the underlying $R^{(l_f)}(\lambda)$ operator is similar to that of the Fateev-Zamolodchikov [27] model and expressions for the eigenvalues and Bethe Ansatz equations are given by

$$\Lambda^{(l_f)}(\lambda\mid \{\lambda_1^{(l_f)}, \ldots, \lambda_{m_f}^{(l_f)}\}) = \prod_{i=1}^{m_f} a_1^{(l_f)}(\lambda - \lambda_i^{(l_f)}) \frac{Q_{f+1}(\lambda + i\gamma)}{Q_{f+1}(\lambda)} + \prod_{i=1}^{m_f} d_{N_f,N_f}^{(l_f)}(\lambda - \lambda_i^{(l_f)}) \frac{Q_{f+1}(\lambda - i\frac{\pi}{2})}{Q_{f+1}(\lambda + i\frac{\pi}{2})} + \prod_{i=1}^{m_f} b^{(l_f)}(\lambda - \lambda_i^{(l_f)}) \frac{Q_{f+1}(\lambda + i\gamma)}{Q_{f+1}(\lambda)} \frac{Q_{f+1}(\lambda - i\frac{\pi}{2})}{Q_{f+1}(\lambda + i\frac{\pi}{2})}, \quad (C.5)$$

$$\prod_{i=1}^{m_f} \frac{\sinh \left(\lambda_{i+1}^{(l_f)} - \lambda_i^{(l_f)} - i\gamma\right)}{\sinh \left(\lambda_{i+1}^{(l_f)} - \lambda_i^{(l_f)}\right)} = \prod_{i\neq j}^{m_f+1} \frac{\sinh \left(\lambda_{j+1}^{(l_f)} - \lambda_{i+1}^{(l_f)} - i\frac{\pi}{2}\right)}{\sinh \left(\lambda_{j+1}^{(l_f)} - \lambda_i^{(l_f)} + i\frac{\pi}{2}\right)}. \quad (C.6)$$

The last step for the $U_q[sl(2n|2m)^{(2)}]$ and $U_q[osp(2n|2m)^{(2)}]$ is $l_f = (2|0)$ and the form of the underlying R-matrix is that of the six-vertex model, namely

$$R^{(l_f)}(\lambda) = \begin{pmatrix}
a_1^{(l_f)}(\lambda) & 0 & 0 & 0 \\
0 & d_{1,1}^{(l_f)}(\lambda) & d_{1,2}^{(l_f)}(\lambda) & 0 \\
0 & d_{2,1}^{(l_f)}(\lambda) & d_{2,2}^{(l_f)}(\lambda) & 0 \\
0 & 0 & 0 & a_1^{(l_f)}(\lambda)
\end{pmatrix} \quad (C.7)$$

where each superalgebra has its own Boltzmann weights defined in (7 - 15).

It turns out that for the $U_q[sl(2n|2m)^{(2)}]$ vertex models the last step has the following eigenvalues and Bethe Ansatz equations,

$$\Lambda^{(l_f)}(\lambda, \{\lambda_1^{(l_f)}, \ldots, \lambda_{m_f}^{(l_f)}\}) = \prod_{i=1}^{m_f} a_1^{(l_f)}(\lambda - \lambda_i^{(l_f)}) \frac{Q_{f+1}(\lambda + i\gamma)}{Q_{f+1}(\lambda)} \frac{Q_{f+1}(\lambda + i\gamma + i\frac{\pi}{2})}{Q_{f+1}(\lambda + i\frac{\pi}{2})} + \prod_{i=1}^{m_f} d_{N_f,N_f}^{(l_f)}(\lambda - \lambda_i^{(l_f)}) \frac{Q_{f+1}(\lambda - i\gamma)}{Q_{f+1}(\lambda)} \frac{Q_{f+1}(\lambda - i\gamma + i\frac{\pi}{2})}{Q_{f+1}(\lambda + i\frac{\pi}{2})}, \quad (C.8)$$

29
\[
\prod_{i=1}^{m_f} \frac{\sinh \left[ 2 \left( \lambda_j^{(l_f+1)} - \lambda^{(l_f)}_i - i\gamma \right) \right]}{\sinh \left[ 2 \left( \lambda_j^{(l_f+1)} - \lambda^{(l_f)}_i \right) \right]} = \prod_{i \neq j} \frac{\sinh \left[ 2 \left( \lambda_j^{(l_f+1)} - \lambda^{(l_f+1)}_i - i\gamma \right) \right]}{\sinh \left[ 2 \left( \lambda_j^{(l_f+1)} - \lambda^{(l_f+1)}_i + i\gamma \right) \right]}
\]  
(C.9)

while for the \( U_q[osp(2n|2m)^{(2)}] \) vertex models the last step results are

\[
\Lambda^{(l_f)}(\lambda, \{\lambda_1^{(l_f)}, \ldots, \lambda_{m_f}^{(l_f)}\}) = \prod_{i=1}^{m_f} a_1^{(l_f)}(\lambda - \lambda_i^{(l_f)}) \frac{Q_{f+1}(\lambda + 2i\gamma)}{Q_{f+1}(\lambda)} + \prod_{i=1}^{m_f} \frac{d^{(l_f)}_{N_f, N_f}(\lambda - \lambda_i^{(l_f)}) Q_{f+1}(\lambda - 2i\gamma)}{Q_{f+1}(\lambda)}.
\]  
(C.10)

\[
\prod_{i=1}^{m_f} \frac{\sinh \left( \lambda_j^{(l_f+1)} - \lambda_i^{(l_f)} - 2i\gamma \right)}{\sinh \left( \lambda_j^{(l_f+1)} - \lambda_i^{(l_f)} \right)} = \prod_{i \neq j} \frac{\sinh \left( \lambda_j^{(l_f+1)} - \lambda_i^{(l_f+1)} - 2i\gamma \right)}{\sinh \left( \lambda_j^{(l_f+1)} - \lambda_i^{(l_f+1)} + 2i\gamma \right)}.
\]  
(C.11)

For the \( U_q[osp(2n|2m)^{(1)}] \) vertex models the last step occurs at \( l_f = (4|0) \) and a more careful analysis is required. First we perform the transformation \( R^{(l_f)}_{A^{(l_f)}_j} \rightarrow M_j^{-1} R^{(l_f)}_{A^{(l_f)}_j} M_j \), that preserves the spectrum of the transfer matrix associated. For each \( j \)-th site of the lattice the matrix \( M_j \) is

\[
M_j = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]  
(C.12)

Now it is not difficult to show that the transformed \( R^{(l_f)}_{A^{(l_f)}_j} \) matrix can be decomposed in terms of the tensor product of two 6-vertex models. More precisely, the new \( R^{(l_f)} \) can be written as

\[
R^{(l_f)}(\lambda) = R^{6v}_\sigma(\lambda) R^{6v}_\tau(\lambda)
\]  
(C.13)

where \( \sigma \) and \( \tau \) represent two commuting basis. These basis can be easily constructed in terms of Pauli matrices by the following relations

\[
\sigma^\alpha = \sigma^\alpha_P \otimes I_{2 \times 2},
\]  
(C.14)

\[
\tau^\alpha = I_{2 \times 2} \otimes \sigma^\alpha_P,
\]
where $\sigma^\alpha$ and $\tau^\alpha$ are elements of $\sigma$ and $\tau$ basis respectively, $\sigma_p^\alpha$ is a Pauli matrix (with the identity included) and $I_{2\times2}$ is the identity matrix with dimensions $2 \times 2$.

Performing the above procedure we have the following expressions for the $R$-matrix of these two 6-vertex models,

$$R_{\sigma}^{6v}(\lambda) = \begin{pmatrix} a_\sigma(\lambda) & 0 & 0 & 0 \\ 0 & b_\sigma(\lambda) & \bar{c}_\sigma(\lambda) & 0 \\ 0 & c_\sigma(\lambda) & b_\sigma(\lambda) & 0 \\ 0 & 0 & 0 & a_\sigma(\lambda) \end{pmatrix}, \quad R_{\tau}^{6v}(\lambda) = \begin{pmatrix} a_\tau(\lambda) & 0 & 0 & 0 \\ 0 & b_\tau(\lambda) & \bar{c}_\tau(\lambda) & 0 \\ 0 & c_\tau(\lambda) & b_\tau(\lambda) & 0 \\ 0 & 0 & 0 & a_\tau(\lambda) \end{pmatrix},$$

(C.15)

whose Boltzmann weights are given by

$$a_\sigma(\lambda) = 1, \quad a_\tau(\lambda) = (e^{2\lambda} - q^2)^2,$$

$$b_\sigma(\lambda) = q \frac{(e^{2\lambda} - 1)}{(e^{2\lambda} - q^2)}, \quad b_\tau(\lambda) = q(e^{2\lambda} - 1)(e^{2\lambda} - q^2),$$

$$c_\sigma(\lambda) = \frac{(1 - q^2)}{(e^{2\lambda} - q^2)}, \quad c_\tau(\lambda) = e^{2\lambda}(1 - q^2)(e^{2\lambda} - q^2),$$

$$\bar{c}_\sigma(\lambda) = e^{2\lambda} \frac{(1 - q^2)}{(e^{2\lambda} - q^2)}, \quad \bar{c}_\tau(\lambda) = (1 - q^2)(e^{2\lambda} - q^2).$$

(C.16)

Consequently, the transfer matrix eigenvalues associated to $R^{(l_f)}$ vertex model can be decomposed as a product of the eigenvalues of two 6-vertex models defined in Eq.(C.15-C.16). In the presence of inhomogeneities $\{\lambda_{1}^{(l_f)},\ldots,\lambda_{m_l}^{(l_f)}\}$ we find that these eigenvalues are given by

$$\Lambda^{(l_f)}(\lambda, \{\lambda_{1}^{(l_f)},\ldots,\lambda_{m_l}^{(l_f)}\}) = \left[ \frac{Q_+ (\lambda + i\gamma)}{Q_+ (\lambda)} + \prod_{i=1}^{m_l} \frac{b_\sigma(\lambda - \lambda_{i}^{(l_f)})Q_+ (\lambda - i\gamma)}{Q_+ (\lambda)} \right] \times \left[ \prod_{i=1}^{m_l} \frac{a_\tau(\lambda - \lambda_{i}^{(l_f)})Q_- (\lambda + i\gamma)}{Q_- (\lambda)} + \prod_{i=1}^{m_l} \frac{b_\tau(\lambda - \lambda_{i}^{(l_f)})Q_- (\lambda - i\gamma)}{Q_- (\lambda)} \right],$$

(C.17)

and the Bethe Ansatz equations are
and the respective Bethe Ansatz equations are
\[
\prod_{i=1}^{m_{l_j}} \frac{\sinh \left( \lambda_j^{(l_j)} - \lambda_i^{(l_j)} - i \gamma \right)}{\sinh \left( \lambda_j^{(l_j)} - \lambda_i^{(l_j)} + i \gamma \right)} = \prod_{i \neq j} \frac{\sinh \left( \lambda_j^{(l_j)} - \lambda_i^{(l_j)} + i \gamma \right)}{\sinh \left( \lambda_j^{(l_j)} - \lambda_i^{(l_j)} - i \gamma \right)}. 
\tag{C.18}
\]

Appendix D: The $U_q[sl(1|2m)^{(2)}]$ and $U_q[osp(1|2m)^{(1)}]$ results

In this appendix we list the Bethe ansatz results for the $U_q[sl(1|2m)^{(2)}]$ and $U_q[osp(1|2m)^{(1)}]$ vertex models. In both cases the last step occurs at $l_f = (1|2)$ and the corresponding $R^{(l)}$ can be related to that of the Fateev-Zamolodchikov [27] or Izergin-Korepin [26] vertex models, respectively. Since the problem of diagonalizing these systems have already been discussed in the previous appendix we restrict ourselves here in presenting only the main results. The eigenvalue expression for the $U_q[sl(1|2m)^{(2)}]$ vertex model is
\[
\Lambda^{(l_0)}(\lambda) = - \left[ -a_1^{(l_0)}(\lambda) \right]^L \frac{Q_1 \left( \lambda - i \frac{\pi}{2} \right)}{Q_1 \left( \lambda + i \frac{\pi}{2} \right)} - \left[ -d_{N_0,N_0}^{(l_0)}(\lambda) \right]^L \frac{Q_1 \left( \lambda + im \gamma + i \frac{\pi}{2} \right)}{Q_1 \left( \lambda + i (m-1) \gamma + i \frac{\pi}{2} \right)}
+ \left[ b^{(l_0)}(\lambda) \right]^L \sum_{\alpha=1}^{N_0-2} G_\alpha(\lambda|\{\lambda_j^{(l_j)}\}) \tag{D.1}
\]

\[
G_\alpha(\lambda|\{\lambda_j^{(l_j)}\}) = \begin{cases} 
Q_\alpha(\lambda + (\frac{\pi}{2} - 1) \gamma) Q_{\alpha+1}(\lambda + (\frac{\pi}{2} - 1) \gamma) & \alpha = 1, \ldots, m - 1 \\
Q_\alpha(\lambda + (\frac{\pi}{2} - 1) \gamma) Q_{\alpha+1}(\lambda + (\frac{\pi}{2} - 1) \gamma) & \alpha = m \\
G_{\alpha-m}(\lambda^{(l_j)} - i \frac{\pi}{2} - i (m-\frac{1}{2}) \gamma - \lambda - \{\lambda_j^{(l_j)}\}) & \alpha = m + 1, \ldots, 2m - 1
\end{cases}
\]

and the respective Bethe Ansatz equations are
\[
\prod_{i=1}^{m_{l_j}} \frac{\sinh \left( \lambda_j^{(l_j)} - \lambda_i^{(l_j)} + i \frac{\pi}{2} \right)}{\sinh \left( \lambda_j^{(l_j)} - \lambda_i^{(l_j)} - i \frac{\pi}{2} \right)} = \prod_{i \neq j} \frac{\sinh \left( \lambda_j^{(l_j)} - \lambda_i^{(l_j)} - i \gamma \right)}{\sinh \left( \lambda_j^{(l_j)} - \lambda_i^{(l_j)} + i \gamma \right)} \prod_{i=1}^{m_{l_j+1}} \frac{\sinh \left( \lambda_j^{(l_j+1)} - \lambda_i^{(l_j+1)} + i \frac{\pi}{2} \right)}{\sinh \left( \lambda_j^{(l_j+1)} - \lambda_i^{(l_j+1)} - i \frac{\pi}{2} \right)} \tag{D.2}
\]
On the other hand, for the $U_q[osp(1|2m)^{(1)}]$ we have

$$
\begin{align*}
\Lambda^{(l_0)}(\lambda) &= -\left[-a_1^{(l_0)}(\lambda)\right]^L \frac{Q_1(\lambda - i \frac{\gamma}{2})}{Q_1(\lambda + i \frac{\gamma}{2})} - \left[-d_{N_0,N_0}^{(l_0)}(\lambda)\right]^L \frac{Q_1(\lambda + i (m + 1) \gamma)}{Q_1(\lambda + im \gamma)} \\
&+ \left[b^{(l_0)}(\lambda)\right]^L \sum_{\alpha=1}^{N_0-2} G_\alpha(\lambda|\{\lambda_j^{(l_0)}\})
\end{align*}
$$

(D.3)

$$
G_\alpha(\lambda|\{\lambda_j^{(l_0)}\})
= \begin{cases}
\frac{Q_\alpha(\lambda+i(\frac{\alpha+1}{2})\gamma)}{Q_\alpha(\lambda+i\frac{\gamma}{2})} \frac{Q_{\alpha+1}(\lambda+i(\frac{\alpha+1}{2})\gamma)}{Q_{\alpha+1}(\lambda+i\frac{\gamma}{2})} & \alpha = 1, \ldots, m-1 \\
\frac{Q_\alpha(\lambda+i(m-\frac{1}{2})\gamma)}{Q_\alpha(\lambda+i\frac{\gamma}{2})} \frac{Q_{\alpha}(\lambda+i(m+\frac{1}{2})\gamma)}{Q_{\alpha}(\lambda+i\frac{\gamma}{2})} & \alpha = m \\
G_{\alpha-m}(-i(m+\frac{1}{2})\gamma - \lambda| - \{\lambda_j^{(l_0)}\}) & \alpha = m+1, \ldots, 2m-1
\end{cases}
$$

while the Bethe Ansatz equations are given by

$$
\begin{align*}
\prod_{i=1}^{m_{\alpha-1}} \frac{\sinh(\lambda_j^{(l_0)} - \lambda_i^{(l_0-1)} + i \frac{\gamma}{2})}{\sinh(\lambda_j^{(l_0)} - \lambda_i^{(l_0-1)} - i \frac{\gamma}{2})} &= \prod_{i \neq j}^{m_{\alpha-1}} \frac{\sinh(\lambda_j^{(l_0)} - \lambda_i^{(l_0)} + i \gamma)}{\sinh(\lambda_j^{(l_0)} - \lambda_i^{(l_0)} - i \gamma)} \prod_{i=1}^{m_{\alpha+1}} \frac{\sinh(\lambda_j^{(l_0+1)} - \lambda_i^{(l_0)} + i \frac{\gamma}{2})}{\sinh(\lambda_j^{(l_0+1)} - \lambda_i^{(l_0)} - i \frac{\gamma}{2})} \\
&\alpha = 1, \ldots, m-1 \\
\prod_{i=1}^{m_{\alpha-1}} \frac{\sinh(\lambda_j^{(l_0)} - \lambda_i^{(l_0-1)} + i \frac{\gamma}{2})}{\sinh(\lambda_j^{(l_0)} - \lambda_i^{(l_0-1)} - i \frac{\gamma}{2})} &= \prod_{i \neq j}^{m_{\alpha-1}} \frac{\sinh(\lambda_j^{(l_0)} - \lambda_i^{(l_0)} + i \gamma)}{\sinh(\lambda_j^{(l_0)} - \lambda_i^{(l_0)} - i \gamma)} \prod_{i=1}^{m_{\alpha+1}} \frac{\sinh(\lambda_j^{(l_0+1)} - \lambda_i^{(l_0)} + i \gamma)}{\sinh(\lambda_j^{(l_0+1)} - \lambda_i^{(l_0)} - i \gamma)} \\
&\alpha = m
\end{align*}
$$

(D.4)
Tables

Table 1: The values of the dimension $N_0$ and the parameter $\zeta^{(l_0)}$. In general the integers $n, m \geq 1$ except for the $U_q[osp(2n|2m)^{(1)}]$ where $n \geq 2$.

| $U_q[\mathcal{G}]$ | $N_0$ | $\zeta^{(l_0)}$ |
|------------------|-------|-----------------|
| $U_q[sl(2n+1|2m)^{(2)}]$ | $2n + 2m + 1$ | $-q^{2n-2m+1}$ |
| $U_q[sl(2n|2m)^{(2)}]$ | $2n + 2m$ | $-q^{2n-2m}$ |
| $U_q[osp(2n|2m)^{(1)}]$ | $2n + 2m$ | $q^{2n-2m-2}$ |
| $U_q[osp(2n+1|2m)^{(1)}]$ | $2n + 2m + 1$ | $q^{2n-2m-1}$ |
| $U_q[osp(2n|2m)^{(2)}]$ | $2n + 2m$ | $q^{2n-2m+2}$ |

Table 2: Parameters of the vertex models associated with the last step Bethe ansatz analysis for the $q$-deformed Lie superalgebras. The symbols IK and FZ stand for Izergin-Korepin [26] and Fateev-Zamolodchikov models [27], respectively.

| Superalgebra | $l_f$ | $R^{(l_f)}$ matrix |
|--------------|-------|-------------------|
| $U_q[sl(2n+1|2m)^{(2)}]$ | (3|0) | nineteen-vertex IK model |
| $U_q[sl(2n|2m)^{(2)}]$ and $U_q[osp(2n|2m)^{(2)}]$ | (2|0) | six-vertex model |
| $U_q[osp(2n+1|2m)^{(1)}]$ | (3|0) | nineteen-vertex FZ model |
| $U_q[osp(2n|2m)^{(1)}]$ | (4|0) | two decoupled six-vertex models |
Table 3: Table with the shifts performed in the rapidities.

| Lie Superalgebra | $\delta^{(\alpha)}$ |
|------------------|----------------------|
| $sl(2n+1|2m)^{(2)}$, $osp(2n+1|2m)^{(1)}$ and $sl(2n|2m)^{(2)}$ | $\begin{cases} 
\frac{i\alpha}{2} \gamma & 1 \leq \alpha \leq m \\
 i \left(m - \frac{\alpha}{2}\right) \gamma & m < \alpha \leq m + n 
\end{cases}$ |
| $osp(2n|2m)^{(2)}$ | $\begin{cases} 
\frac{i\alpha}{2} \gamma & 1 \leq \alpha \leq m \\
 i \left(m - \frac{\alpha}{2}\right) \gamma & m < \alpha < m + n \\
 i \left(\frac{m-n-1}{2}\right) \gamma & \alpha = m + n 
\end{cases}$ |
| $osp(2n|2m)^{(1)}$ | $\begin{cases} 
\frac{i\alpha}{2} \gamma & 1 \leq \alpha \leq m \\
 i \left(m - \frac{\alpha}{2}\right) \gamma & m < \alpha \leq m + n - 2 \\
 i \left(\frac{m-n+1}{2}\right) \gamma & \alpha = \pm n 
\end{cases}$ |

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