Not conformally Einstein metrics in conformal gravity

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Abstract
The equations of motion of four-dimensional conformal gravity, whose Lagrangian is the square of the Weyl tensor, require that the Bach tensor $E_{\mu\nu} = (\nabla^\rho \nabla^\sigma + \frac{1}{4} R^{\rho\sigma}) C_{\mu\rho\nu\sigma}$ vanishes. Since $E_{\mu\nu}$ is zero for any Einstein metric, and any conformal scaling of such a metric, it follows that large classes of solutions in four-dimensional conformal gravity are simply given by metrics that are conformal to Einstein metrics (including Ricci flat). In fact it becomes more intriguing to find solutions that are not conformally Einstein. We obtain five new such vacua, which are homogeneous and have asymptotic generalized Lifshitz anisotropic scaling symmetry. Four of these solutions can be further generalized to metrics that are conformal to classes of pp-waves, with a covariantly-constant null vector. We also obtain large classes of generalized Lifshitz vacua in Einstein–Weyl gravity.

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1. Introduction

Einstein’s theory of gravity can be further extended by adding higher-order curvature terms, without violating the underlying principles of general relativity. Such terms also arise naturally as perturbations to the low-energy effective actions of string theories. Because the equations of motion now involve higher derivatives and a higher degree of nonlinearity, much less is known about the solutions in higher-derivative gravities. In four dimensions, since the Gauss–Bonnet quadratic curvature invariant is a total derivative, the possible independent quadratic curvature modifications are equivalent just to $R^2$ and $R^{\rho\nu} R_{\mu\nu}$. This implies that Einstein metrics, including Schwarzschild-AdS or Kerr-AdS metrics, are automatically solutions in
cosmological gravities extended with quadratic curvature terms. It is an interesting, although generally rather hard, problem to find new solutions that are not merely Einstein metrics.

In this paper, we shall first consider solutions of purely conformal gravity in four dimensions, for which the Lagrangian can be written as

\[
\mathcal{L} = \frac{1}{2} \sqrt{-g} C_{\mu
u\rho\sigma} C^{\mu
u\rho\sigma},
\]

where \(C_{\mu
u\rho\sigma}\) is the Weyl tensor. The equations of motion following from this Lagrangian imply that the Bach tensor

\[
E_{\mu\nu} = (\nabla^\rho \nabla_\sigma + \frac{1}{2} R^\rho_{\sigma\rho}) C_{\mu\rho\nu\sigma}
\]

must vanish.

The most general spherically-symmetric black hole solution in conformal gravity, up to an overall conformal factor, was obtained in [1]. It turns out that it is locally conformal to the Schwarzschild-AdS black hole [2]; however, since the conformal factor can be singular at infinity these black holes are globally distinct from Schwarzschild-AdS. (Charged static solutions with toroidal and hyperbolic topologies were given in [3, 4], where the AdS/CFT applications to Fermi and non-Fermi liquids were studied.)

The most general metrics in conformal gravity within the Plebanski class have also been constructed, in [5]. It was shown in [6] that these again include new (rotating) black holes that are globally distinct from the usual Kerr-AdS solutions. The general metric obtained in [5] is conformal to the Plebanski–Demianski metric, which is the most general type D Einstein metric.

This leads one to wonder about the existence of solutions in conformal gravity that are not conformal to Einstein metrics. One such solution was indeed constructed in [7]6. One goal of the present paper is to find more metrics in conformal gravity that are not conformal to Ricci-flat or Einstein metrics. (In this paper, when we refer to Einstein metrics, we shall include Ricci-flat metrics as well.) It is not necessarily simple to test whether two ostensibly different solutions are actually conformal to each other. For this reason, we focus first on classes of metric ansätze of the form given by (9) that are homogeneous with a generalized Lifshitz scaling symmetry given by (10). Because of the homogeneity and the symmetries, it is evident in these cases that such solutions are not conformally related to each other.

Although conformal gravity in itself does not seem to be a physically-acceptable theory owing to the presence of ghostlike massive spin-2 modes, it has been argued that these non-unitary modes can be truncated out by imposing appropriate boundary conditions [9]. This process selects the Einstein solutions, which could be taken as an indication that solutions that are not conformally Einstein capture the unphysical elements of conformal gravity. Since conformal gravity is in any case of some considerable theoretical interest, it would seem to be worthwhile to study the full space of solutions in greater detail, and, especially, to explore the properties of those that are not Einstein or conformally Einstein.

In section 2, we review a condition derived in [10] for testing whether a solution is conformally Einstein. It provides a necessary, but not sufficient, criterion for a metric to be conformal to an Einstein metric. Thus, if this condition is not obeyed for a given metric, then the metric cannot be conformally transformed to be Einstein. We use the gyrating vacua [11] in conformal gravity as examples to demonstrate that for appropriate choices of the parameters, the metrics can be either (I) Einstein, or (II) conformally Ricci flat, or (III) not conformal to any Einstein metric.

In section 3, we consider a large class of homogeneous metrics with generalized Lifshitz scaling symmetry. By demanding that these metrics satisfy the equations of motion of

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6 It is interesting to note that, as was shown recently in [8], a partially massless spin-2 field does not propagate in a solution to conformal gravity unless it is conformally Einstein.
conformal gravity, we obtain six solutions that are not conformally Einstein, including one that is conformal to the previously-known metric [7]. We argue that these solutions are distinct, in the sense that they are also not conformal to each other. We find that four solutions can be viewed as conformal to certain pp-wave metrics (whose defining property is that they admit a covariantly-constant null vector).

In section 4, we show that the four solutions mentioned above can be generalized to much broader classes of inhomogeneous pp-waves in conformal gravity. In section 5, we consider cohomogeneity-one Bianchi IX metrics in conformal gravity. Whilst there exist triaxial solutions that are not conformally Einstein, all the biaxial solutions are conformally Einstein. We argue that this suggests that it is unlikely that there will exist any rotating black holes in conformal gravity beyond those that are already known, which are conformally Einstein.

In section 6, we consider the more general Einstein–Weyl theory of gravity, for which the Lagrangian is Einstein–Hilbert with a cosmological constant in addition to the Weyl-squared term. We obtain some solutions that are homogeneous metrics with generalized Lifshitz scaling symmetries. This paper ends with conclusions in section 7.

2. Testing of conformally Einstein metrics

It was shown in [10] that if a metric $d\bar{s}^2$ is conformal to an Einstein metric $d\tilde{s}^2$ with $d\bar{s}^2 = \Omega^{-2}d\tilde{s}^2$, then one must have

$$\nabla_{\mu}C_{\mu\nu\rho\sigma} - (D - 3)V_{\mu}C_{\mu\nu\rho\sigma} = 0,$$

for some appropriate $V_{\mu}$. The conformal factor $\Omega$ can then be obtained by $V_{\mu} = \partial_{\mu}\log \Omega$. This was used recently to distinguish the spherically-symmetric solutions in some six-dimensional conformal gravity theories that are conformally Einstein from those that are not [12].

The existence of a solution for the vector $V_{\mu}$ is a necessary condition for the metric to be Einstein. Thus, if we have a metric for which there is no such solution, then the metric is not conformally Einstein. On the other hand, if there exists a solution for $V_{\mu}$ then it is not guaranteed that the metric is conformally Einstein, and it is still necessary to find the conformal factor to check whether the metric is indeed conformally Einstein or not.

Let us consider the gyrating homogeneous vacua ansatz proposed in [11]:

$$d\bar{s}^2 = \ell^2 \left( \frac{dr^2 - 2 du dv + dx^2}{r^2} + \frac{2c_1 du dx}{r^{c_1}} + \frac{c_2 du^2}{r^{c_2}} \right),$$

where $c_1$ and $c_2$ are constants. The metric is AdS when $z = 1$ and/or $c_1 = 0 = c_2$. For non-vanishing $c_1$, aside from the AdS metric, there are three solutions in conformal gravity:

I: $z = -2$, $c_2 = -\frac{1}{2}c_1^2$;
II: $z = 0$, $c_1$ arbitrary;
III: $z = -1$, $c_2 = -\frac{1}{2}c_1^2$.

If we substitute the metric I into (3), we find that $V_{\mu} = 0$. Indeed, it is easy to verify that the metric I is already Einstein, with $R_{\mu\nu} = -(3/\ell^2)g_{\mu\nu}$.

If we substitute metric II into (3), we find that

$$V_r = \frac{1}{r}, \quad V_v = 0 = V_x.$$

7 A cohomogeneity-one metric admits an isometric action by a compact Lie group and has a one-dimensional orbit space. In other words, any constant radius slice of such a metric describes a homogeneous spacetime.
and that there is no constraint on $V_u$. This implies that the conformal factor is $\Omega^2 = r^2 f(u)^2$. Since condition (3) is necessary but not sufficient, it is still necessary to find a solution for the function $f(u)$ that makes the conformally transformed metric Einstein. In fact, the metric becomes Ricci flat, if $f$ is taken to be

$$f(u) = \frac{1}{\cos \left( \frac{1}{2}\sqrt{c_1^2 + 2c_2 \, u} \right)}.$$  

(7)

The Ricci-flat metric can be cast in the form

$$\mathrm{d}s^2 = \cos^2 \theta \left( \mathrm{d}r^2 + \left( 2c_2 - c_1^2 \right) \mathrm{d}x^2 \right) + \frac{4c_2 r^2}{2c_2 - c_1^2} \, \mathrm{d}\theta^2 + 4 \cos \theta \, \mathrm{d}\theta \left( \mathrm{d}v + c_1 r \, \mathrm{d}x \right).$$  

(8)

The metric is of the pp-wave type, which we define to be the one with a covariantly-constant null vector $\ell = \partial_v$, which also happens to be a Killing vector in this example. Note that all the curvature polynomials vanish identically.

For the metric III, there is no solution of $V_u$, and hence the metric cannot be conformally transformed to any Einstein metric. The construction and the classification of these solutions go beyond Einstein metrics. One focus of this paper is to construct further examples of such solutions.

So far we have considered the gyratons with $c_1$ non-vanishing. The solution (4) with $c_1 = 0$ belongs to the pp-wave type of Schrödinger metrics [13, 14]. In conformal gravity, we have $z = \pm \frac{1}{2}$. The $z = -\frac{1}{2}$ solution is the Kaigorodov metric [15] which is Einstein, whilst the $z = +\frac{1}{2}$ solution is conformal to a Ricci-flat pp-wave.

3. Homogeneous and not conformally Einstein vacua

3.1. The metrics and their properties

In this section, we consider the metric ansatz

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{r^2} + \sum_{i=k}^3 \mathrm{d}r z_i \, \mathrm{d}x_i^2 + \frac{1}{2} \sum_{i=p, p \neq k} \sqrt{c_k r^2 + z_i} \, \mathrm{d}x_i \, \mathrm{d}x_j.$$  

(9)

Here $z_k$, $d_k$ and $c_k$ are constant parameters. Note that if all $z_k$’s are equal, then the metric becomes the usual local patch of AdS, which we shall not consider further. The metrics are all homogeneous, with the generalized Lifshitz scaling symmetry

$$r \to \lambda r, \quad x_i \to \lambda^{-z_i} x_i, \quad i = 1, 2, 3.$$  

(10)

The components of the curvature tensor in the vielbein base

$$\epsilon^i = \frac{\mathrm{d}r}{r}, \quad \epsilon^i = r^2 \mathrm{d}x_i$$  

are all constants, independent of $r$. It is clear that the metric ansatz contains the Lifshitz [16, 17], Schrödinger [13, 14] and gyraton [11] solutions as special cases. In Einstein gravity, the Lifshitz and Schrödinger solutions require some appropriate matter energy–momentum tensor, whilst they can arise naturally in higher-derivative gravity theories without additional matter [2].

Three classes of solutions can arise. (I) The metric (9) is Einstein, with non-zero cosmological constant. (II) It is conformal to an Einstein metric (which is in fact Ricci flat in this case). or (III) It is not conformal to any Einstein metric. In this paper, we shall be concerned with only the third class of solutions, since the study of solutions in the first and second classes reduces to a study of solutions in Einstein gravity.
For the diagonal case, all $c_k$ vanish and the $d_k$ can be set to unity by rescaling the $x_k$. After a Wick rotation, the metric becomes
\[\text{d}x^2 = -r^{2z_1} \text{d}t^2 + \frac{r^{2z_1}}{r^2} \text{d}r^2 + r^{2z_1} \text{d}x_1^2 + r^{2z_1} \text{d}x_2^2.\] (12)
Metrics of this kind, with multiple anisotropic scaling symmetries, were constructed in Einstein gravity coupled to multiple massive vectors [18]. The metric (12) is a solution in conformal gravity if
\[z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3 = 0.\] (13)
It turns out that these metrics can be rendered Ricci flat by means of a scaling with the conformal factor
\[\Omega^2 = r^{-z_1-z_2-z_3}.\] (14)
Thus, any solution of the form (9) that is not conformally Einstein necessarily involves at least one off-diagonal term. We find six such solutions. In presenting these solutions, we have performed appropriate analytical continuation so that the metrics are all real and of Lorentzian signature.

**Solution 1.**
\[\text{d}x^2 = \frac{\text{d}r^2}{r^2} + r^2(2 \text{d}u \text{d}v + \text{d}x^2) + r^2 \text{d}u^2 - \frac{5 \text{d}u^2}{r}.\] (15)
The metric has the Ricci tensor $R_{ij} = \lambda_{ij} g_{ij}$ (no sum) where $\lambda_{ij}$ are:
\[\lambda_{rr} = -\frac{27}{4}, \quad \lambda_{uu} = \frac{3}{4}, \quad \lambda_{xx} = -\frac{33}{4}, \quad \lambda_{uv} = -\frac{3}{4}.\] (16)
Using the criteria for determining the Petrov type that we summarize in the appendix, we find that this metric is of Petrov type I.

**Solution 2.**
\[\text{d}x^2 = \frac{\text{d}r^2}{r^2} + \frac{2 \text{d}u \text{d}v}{r} + \frac{\text{d}x^2}{r^2} + r^2 \text{d}u^2.\] (17)
with
\[\lambda_{rr} = -\frac{9}{2}, \quad \lambda_{uu} = -\frac{3}{2}, \quad \lambda_{xx} = -6, \quad \lambda_{uv} = -\frac{3}{2}.\] (18)
It is of Petrov type II.

**Solution 3.**
\[\text{d}x^2 = \frac{\text{d}r^2}{r^2} + \frac{2 \text{d}u \text{d}v}{r} + \text{d}x^2 + 6r \text{d}u \text{d}x + 7r^2 \text{d}u^2,\] (19)
with
\[\lambda_{rr} = \lambda_{xx} = \lambda_{uv} = -\frac{1}{2}, \quad \lambda_{uu} = -\frac{13}{14}, \quad \lambda_{xx} = 0.\] (20)
It is of Petrov type III.

**Solution 4.**
\[\text{d}x^2 = \frac{\text{d}r^2}{r^2} + \frac{2 \text{d}u \text{d}v}{r} + \frac{\text{d}x^2}{r^2} + 6r^2 \text{d}u \text{d}x + 2r^8 \text{d}u^2,\] (21)
with
\[\lambda_{rr} = -\frac{9}{2}, \quad \lambda_{uu} = -\frac{33}{4}, \quad \lambda_{xx} = \lambda_{uv} = -6, \quad \lambda_{uv} = -\frac{3}{2}.\] (22)
It is of Petrov type II.

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8 Non-conformally Einstein metrics with vanishing Bach tensor were discussed in the context of diagonal Bianchi I models in [19].
Table 1. The curvatures and Petrov types of the six homogeneous and not conformally Einstein vacua. Here we have defined $\text{Riem}^{(1)}_{\mu
u} = R_{\mu\nu\lambda\rho} R^{\lambda\rho} R_{\alpha\beta}$ and $\text{Riem}^{(2)}_{\mu
u} = R_{\mu\nu\lambda\rho} R^{\lambda\rho} R_{\alpha\beta}$.

| Solution | $R_{\mu\nu} R^{\mu\nu}/R^3$ | $\text{Riem}^{(1)}/R^3$ | $\text{Riem}^{(2)}_{(1)}/R^3$ | $\text{Riem}^{(2)}_{(2)}/R^3$ | Petrov class |
|----------|-----------------|----------------|----------------|----------------|---------------|
| 1        | 179            | 713         | 2429         | 5571         | Type I        |
| 2        | 3              | 203         | 6501         | 145          | Type II       |
| 3        | 1/3            | 4           | 4/3          | 4/3          | Type III      |
| 4        | 1/3            | 243         | 6561         | 145          | Type II       |
| 5        | 1/4            | 1           | 1/4          | 1/4          | Type III      |
| 6        | 1/13           | 17          | 13           | 17           | Type N        |

Solution 5. The fifth solution is a gyrating Schrödinger geometry discussed in section 2 (with $z = -1, c_1 = \sqrt{2}$ and $c_2 = -1$ in the gyrating Schrödinger ansatz):
\[
ds^2 = \frac{dr^2 + 2 du dv + dx^2}{r^2} + 4 du dx + 2r^2 du^2,
\] (23)
with
\[
\lambda_{rr} = \lambda_{xx} = -3, \quad \lambda_{uu} = -1, \quad \lambda_{uv} = -2.
\] (24)
It is of Petrov type III.

Solution 6:
\[
ds^2 = \frac{dr^2}{r^2} + 10 du \left( \frac{dv}{r^2} - \frac{2 dx}{r} \right) + 2r dx dv + \frac{10 du^2}{r^4} + r^2 dx^2.
\] (25)
It is of Petrov type N. This solution is conformal to the one obtained in [7], with the conformal factor $r^2$.

Having obtained these six solutions, and determined that they are not conformally Einstein, it is natural to ask whether these solutions are conformal to each other. Since the Petrov classification is conformally invariant, the only possible candidates for being conformally related are solutions 2 and 4, or else solutions 3 and 5. It is clear that if the putative conformal factor in either of these cases is a function of $r$ only, then it will break the scaling symmetry. On the other hand, if the conformal factor involves the coordinates of $u, v, x$ as well, then constant shift symmetries along these directions will be broken. Thus, we expect that these solutions are not conformal to each other.

The properties of the six solutions are summarized by the following table.

3.2. PP-waves in conformal gravity

The solutions 2–6, unlike solution 1, all admit a global null vector $\ell = \partial_v$. However, it is not covariantly constant for any of these solutions. In fact we find no covariantly-constant null vector for any of our metrics, and hence none of them are of the pp-wave type. However, solutions 2, 3, 4 and 5 can be transformed into pp-waves by multiplying the metric by a particular conformal factor that depends on the $r$ coordinate. With the appropriate conformal factors, the metrics 2, 3, 4 and 5 all then admit a global null vector $V = \partial_v$ that is also covariantly constant. This null vector also happens to be a Killing vector. Of course, as we have established, these are pp-waves of conformal gravity which are neither Ricci flat nor Einstein. Here we list the corresponding pp-waves and present some of their properties. Their Petrov classification is the same as the corresponding homogeneous vacuum solution.
**PP-wave 1.** The solution is conformal to the homogeneous solution 2, with a conformal factor \( r \), namely

\[
ds^2 = \frac{dr^2}{r} + 2\, du \, dv + \frac{dx^2}{r^3} + r^3 \, du^2.
\]
(26)

The Ricci-tensor becomes diagonal in the coordinate base with the following components:

\[
R_{rr} = -\frac{3}{r} g_{rr}, \quad R_{uu} = -\frac{3r}{2r^3} g_{uu}, \quad R_{xx} = -\frac{3}{r} g_{xx}.
\]
(27)

We find that the Ricci scalar is \( R = -6/r \) and that the other non-vanishing curvature invariants are

\[
R_{uu} R^{uu} = \frac{1}{2} R^2, \quad \text{Riem}^2 = R^2, \quad \text{Riem}^3_{(1)} = -R^3, \quad \text{Riem}^3_{(2)} = 0.
\]
(28)

It is clear that the metric has a curvature singularity at \( r = 0 \).

**PP-wave 2.** This is conformal to the homogeneous solution 3, with a conformal factor \( r \), namely

\[
ds^2 = \frac{dr^2}{r} + 2\, du \, dv + \frac{dx^2}{r^3} + 6r^2 \, du \, dx + 7r^3 \, du^2.
\]
(29)

All the curvature polynomials vanish, but with the following non-vanishing Ricci-tensor components

\[
R_{uu} = -\frac{27}{14r^3} g_{uu}, \quad R_{ax} = -\frac{1}{r} g_{ax}.
\]
(30)

Thus the metric has a curvature singularity at \( r = 0 \).

**PP-wave 3.** This is conformal to the homogeneous solution 4, with a conformal factor \( r \), i.e.

\[
ds^2 = \frac{dr^2}{r} + 2\, du \, dv + \frac{dx^2}{r^3} + 6r^2 \, du \, dx + 7r^3 \, du^2.
\]
(31)

The non-vanishing Ricci-tensor components are

\[
R_{rr} = -\frac{3}{r} g_{rr}, \quad R_{uu} = -\frac{45}{4r^3} g_{uu}, \quad R_{xx} = -\frac{3}{r} g_{xx}, \quad R_{ax} = -\frac{6}{r} g_{ax}.
\]
(32)

Thus we have \( R = -6/r \), together with

\[
R_{uu} R^{uu} = \frac{1}{2} R^2, \quad \text{Riem}^2 = R^2, \quad \text{Riem}^3_{(1)} = -R^3, \quad \text{Riem}^3_{(2)} = 0,
\]
(33)

and there is a curvature singularity at \( r = 0 \).

**PP-wave 4.** This is conformal to the homogeneous solution 5, but now with a conformal factor \( r^2 \). In other words, we have

\[
ds^2 = dr^2 + 2\, du \, dv + dx^2 + 4r^2 \, du \, dx + 2r^3 \, du^2.
\]
(34)

The non-vanishing Ricci-tensor components are given by

\[
R_{uu} = -\frac{2}{r^2} g_{uu}, \quad R_{ax} = -\frac{1}{r^2} g_{ax}.
\]
(35)

However, all the curvature polynomials vanish identically. The metric is nevertheless singular at \( r = 0 \).

It is worth pointing out that the Schrödinger solution also belongs to the pp-wave class of metrics. However, it is homogeneous whilst the pp-waves we obtained above are of cohomogeneity-one.
4. More general classes of not conformally Einstein metrics

As in the case of the Schrödinger solution which is a special case of a more general pp-wave solution, we show that the homogeneous vacuum solutions 2–5 can be further generalized to become much more general non-homogeneous metrics. These metrics can be viewed as flowing from one vacuum in \( r = 0 \) to another in \( r = \infty \). We consider the conditions such that the solutions are not conformally Einstein. It turns out that all of these solutions are conformal to pp-waves with a covariantly-constant null vector.

**Generalising solution 2.**

\[
\frac{ds^2}{r^2} + \frac{2 du dv}{r} + \frac{dx^2}{r^4} + r^2 h du^2 = 
\]

\( h = c_0 \log r + c_1 + \frac{c_2 \log r}{r^3} + \frac{c_3}{r^3} \). \hspace{1cm} (37)

Note that the \( c_i \)'s can be arbitrary functions of \( u \). It is of interest to note that logarithmic modes emerge in this solution.

We now examine the condition for which the metric is not conformally Einstein. Substituting the solution into (3), we find that all of the components of \( V_\mu \) vanish except for \( V_r \). We can solve for \( V_r \), and substituting it into the remaining equations yields the constraint

\( c_0 + c_1 r = 0 \). \hspace{1cm} (38)

Thus, as long as \( c_0 \) and \( c_1 \) are not simultaneously zero, the metric is not conformally Einstein. On the other hand, if \( c_0 = c_1 = 0 \), then we find that the metric \( ds^2 = r^2 dx^2 \) indeed becomes Ricci flat. Note that, as in the case of the homogeneous solution, the metric \( ds^2 = r^2 dx^2 \) describes a pp-wave in conformal gravity with the covariantly-constant null vector \( \ell = \partial_v \).

**Generalising solution 3.**

\[
\frac{ds^2}{r^2} + \frac{2 du dv}{r} + dx^2 + rf du dx + r^2 h du^2 = 
\]

where

\[ f = c_1 + \frac{c_2 \log r}{r^3} + \frac{c_3}{r^3} + \frac{c_4}{r^3} \]

\[ h = \frac{7}{36} c_1^2 + \frac{c_1 (2 c_3 - c_2)}{4 r} + \frac{c_6}{r^2} + \frac{c_8}{r^3} + \left( \frac{c_1 c_2}{2 r} + \frac{c_5}{r^2} + \frac{c_7}{r^3} \right) \log r + \frac{3 c_1^3 (\log r)^2}{8 r^2}. \hspace{1cm} (40)\]

The \( c_i \)'s are functions of \( u \). The metric is not conformal to an Einstein (Ricci-flat) metric provided that \( c_1 c_2 \neq 0 \). As in the case of the homogeneous solution, the metric \( ds^2 = r d L^2 \) describes a pp-wave in conformal gravity.

**Generalising solution 4.**

\[
\frac{ds^2}{r^2} + \frac{2 du dv}{r} + \frac{dx^2}{r^4} + r^2 f du dx + r^2 h du^2 = 
\]

where

\[ f = c_1 + \frac{c_2 \log r}{r^3} + \frac{c_3}{r^3} + \frac{c_4}{r^6} \]

\[ h = \frac{1}{18} c_1^2 + \frac{c_1 c_2}{12 r^3} - \frac{c_2^2 (\log r)^2}{8 r^6} + \frac{c_2}{r^6} \log r + \frac{c_6}{r^6} + \frac{c_7 \log r}{r^9} + \frac{c_8}{r^9} + \frac{c_2^2}{4 r^{12}}. \hspace{1cm} (42)\]

The \( c_i \)'s are functions of \( u \). Note that if \( c_1 = c_2 = c_3 = c_4 = 0 \), then the solution reduces to the generalized solution 2. The metric is not conformally Einstein provided that one of the...
\((c_1, c_2, c_5, c_6)\) is non-vanishing. As in the previous examples, the metric \(ds^2 = r ds^2\) describes a pp-wave in conformal gravity.

**Generalising solution 5.**

\[
d s^2 = \frac{dr^2 + 2 du \, dv + dr^2}{r^2} + f \, du \, dx + r^2 h \, du^2,
\]

where

\[
f = c_1 r + c_2 + \frac{c_3}{r} + \frac{c_4}{r^2},
\]

\[
h = \frac{1}{8} c_1^2 r^2 + \frac{1}{4} c_1 c_2 r + \frac{1}{8} (c_2^2 + 2 c_1 c_3) + \frac{c_5}{r} + \frac{c_6}{r^2} + \frac{c_7}{r^3} + \frac{c_8}{r^4}.
\]

The \(c_i\)'s are functions of \(u\). In this case, it is easier to describe the situations when the metric is conformally Einstein: (i) \(c_1 = c_2 = 0\); (ii) \(c_1 \neq 0\) and \(c_2 = c_3 = c_6 = 0\); (iii) \(c_1 c_2 \neq 0\) and \(c_3 = c_2^2/(3c_1)\) and \(c_5 = c_1 c_6/c_2 + c_1^2/(72c_1)\). The metric \(ds^2 = r^2 \, ds^2\) describes a pp-wave in conformal gravity.

Apparently none of these generalized solutions are characterized by their curvature invariants, which do not depend on the functions \(f\) and \(h\). Therefore, according to theorem 2.3 in [20], these metrics are all contained in the Kundt class. This implies that the global null vector admitted by each of these metrics is geodesic, expansion-free, shear-free and twist-free. In fact, these metrics are all written in the canonical Kundt form. Also, as previously mentioned, these metrics are all conformal to pp-waves, which are known to be of Kundt class.

### 5. Bianchi IX metrics

In this section, we consider a different class of metric ansatz: the cohomogeneity-one triaxial Bianchi IX metrics

\[
d s^2 = dr^2 + a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2,
\]

where \(a\), \(b\) and \(c\) are functions of \(r\), and the \(SU(2)\) left-invariant 1-forms \(\sigma_i\) satisfy \(d \sigma_i = -\frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k\). Substituting this into the equations of motion of conformal gravity, we obtain a (rather complicated) system of fourth-order differential equations for the functions \(a\), \(b\) and \(c\). It can be verified that, in general, the necessary condition (3) for the metric to be conformally Einstein cannot be satisfied for any vector \(V_{\mu}\) if one only imposes the aforementioned equations of motion. Thus, the vanishing of the Bach tensor does not in general imply that the Bianchi IX metrics are conformally Einstein.

If we specialize to the case of biaxial Bianchi IX metrics by setting, for example, \(a(r) = b(r)\), then it turns out that the equations following from the vanishing of the Bach tensor do now imply that a solution to (3) for a vector \(V_{\mu}\) exists. Furthermore, it can then be explicitly verified that the equations following from requiring that \(\Omega^2 ds^2\) be an Einstein metric imply that \(\partial_{\mu} \log \Omega\) is equal to the solution for \(V_{\mu}\). In other words, for biaxial Bianchi IX metrics, the vanishing of the Bach tensor is sufficient to imply that the metric is conformally Einstein, but this is no longer true for the case of triaxial Bianchi IX metrics.

Let us now present the biaxial case in some detail. Owing to the conformal symmetry, the most general ansatz can be expressed as

\[
d s^2 = \frac{dr^2}{r^2 h} + h a_3^2 + a_1^2 + a_2^2.
\]

The equations of motion reduce to

\[
2 r^4 h' h''' - r^4 h'^2 + 4 r^3 h'' h'' - 4 r^2 h^2 + 4 h^2 - 8 h + 4 = 0,
\]
which can be solved analytically to give
\[ h = 1 + \frac{c_{-2}}{r^2} + \frac{c_{-1}}{r} + c_1 r + c_2 r^2 \]  
(48)
with
\[ 4c_{-2} c_2 = c_{-1} c_1 . \]  
(49)
It is then straightforward to verify that the metric is conformally Einstein.

For rotating black holes in the extremal limit, the near-horizon geometry is described by a cohomogeneity-one metric whose components are functions of the latitude coordinate \( \theta \), with the level surfaces being \( U(1) \) bundles over the AdS_2. With an appropriate conformal transformation, the metric takes the form
\[ ds^2 = \frac{d\theta^2}{h} + h (d\phi + r \, dt)^2 + \frac{dr^2}{r^2} - r^2 dt^2 , \]  
(50)
where
\[ h = 1 + p_1 \cos \theta + p_2 \cos 2\theta + q_1 \sin \theta + q_2 \sin 2\theta , \]  
(51)
and
\[ p_1^2 + q_1^2 = 4(p_2^2 + q_2^2) . \]  
(52)
Note that the metric is simply a Lorentzian continuation of the biaxial metric considered earlier. It is conformally Einstein, with the vector \( V_\mu \) in (3) having the non-vanishing component
\[ 2V_\theta = -\frac{p_1 \cos \theta - 2p_2 \cos 2\theta + q_1 \sin \theta - 2q_2 \sin 2\theta}{q_1 \cos \theta + 2q_2 \cos 2\theta - p_1 \sin \theta - 2p_2 \sin 2\theta} . \]  
(53)
This demonstrates that the near-horizon geometry of the most general extremal black hole in conformal gravity is Einstein. This is a strong indication that the rotating black hole obtained in [6] is the most general black hole solution in conformal gravity, and it is conformally Einstein.

6. Homogeneous vacua in Einstein–Weyl gravity

In this section, we consider solutions of Einstein–Weyl gravity with a cosmological constant:
\[ e^{-\Lambda} \mathcal{L} = R - 2\Lambda + \frac{1}{2} \alpha R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} . \]  
(54)
The linearized theory was studied in [21]. It was shown that the trace scalar mode decouples and hence the theory contains one massless graviton and one ghost-like massive spin-2 mode. There exists a special point in the parameter space, known as critical gravity, for which the massive spin-2 mode becomes a logarithmic mode [21]. New black holes, Lifshitz and Schrödinger solutions can arise in Einstein–Weyl gravity [2]. In this section, we show that more general classes of homogeneous vacua (9) can also occur in Einstein–Weyl gravity. Note that Einstein–Weyl gravity, unlike pure Weyl-squared gravity, is not itself conformally invariant, and so in this case the issue of whether solutions might be conformally Einstein does not arise. (If a solution were conformally Einstein then the Bach tensor contribution to the field equation would vanish, and so such a solution would in fact necessarily then have to be Einstein.)
6.1. Diagonal metrics

There are Lorentzian solutions given by

\[ ds^2 = \ell^2 \left( -r^{2z_1} \frac{dr^2}{r^2} + r^{2z_2} \, dx_1^2 + r^{2z_3} \, dx_2^2 \right), \quad (55) \]

for

\[ \Lambda = -\frac{1}{2\ell^2} \left( \sum_i z_i^2 + \sum_{i<j} z_i z_j \right), \quad \alpha^{-1} = \frac{2}{3\ell^2} \left( \sum_i z_i^2 - 2 \sum_i z_i \right), \quad (56) \]

where \( i \) and \( j \) are summed from 1 to 3. Thus, we see that general anisotropic Lifshitz solutions can arise in higher-derivative gravity without the need for any matter energy–momentum tensor.

6.2. Metrics with one off-diagonal term

The first class of solutions with a single off-diagonal term in the metric are all Lorentzian signature and take the form

\[ ds^2 = \ell^2 \left( \frac{dr^2}{r^2} + r^{2z_2} \, dx_1^2 + r^{2z_3} \, dx_2^2 + r^{2z_4} \, dx_1 \, dx_2 \right), \quad (57) \]

for

\[ (1): z_2 = 2z_3 - z_1, \quad \Lambda = -\frac{3z_2^2}{\ell^2}, \quad \alpha = \frac{\ell^2}{2z_1(z_3 - 2z_1)}, \]

\[ (2): z_1 = -\frac{z_3}{2}, \quad \Lambda = -\frac{12z_3^2 + 4z_3z_3 + 11z_3^4}{32\ell^2}, \quad \alpha = \frac{3\ell^2}{4z_3(z_3 - z_2)}, \]

\[ (3): z_1 = \frac{z_3}{4}, \quad z_2 = \frac{z_3}{4}, \quad \Lambda = -\frac{3z_3^2}{\ell^2}, \quad \alpha = \frac{4\ell^2}{\frac{3}{c_3}}, \]

\[ (4): z_1 = -\frac{z_3}{2}, \quad z_2 = \frac{5z_3}{2}, \quad \Lambda = -\frac{3z_3^2}{\ell^2}, \quad (58) \]

The second class of solutions can be of either Lorentzian or Euclidean signature, depending on the parameters. The metric for this class is given by

\[ ds^2 = \ell^2 \left( \frac{dr^2}{r^2} + r^{2z_2} \, dx_1^2 + r^{2z_3} \, dx_2^2 + r^{2z_4} \, dx_1 \, dx_2 \right), \quad (59) \]

with the constraints

\[ z_1 = -\frac{z_3}{2}, \quad z_2 = \frac{5z_3}{2}, \quad \Lambda = \frac{3(11 - 2c_3)z_3^4}{2(c_3 - 4)\ell^2}, \quad \alpha = \frac{(4 - c_3)\ell^2}{4(4 + 3c_3)z_3^2}. \quad (60) \]

For \( c_3 > 4 \), the metric has Lorentzian signature, while for \( c_3 < 4 \) the metric has Euclidean signature.

In addition, we find a Lorentzian solution

\[ ds^2 = \ell^2 \left( \frac{dr^2}{r^2} - r^{2z_1} \, dx_1^2 + r^{2z_2} \, dx_2^2 + r^{2z_3} \, dx_1 \, dx_2 + \sqrt{c_3} r^{2z_4} \, dx_1 \, dx_2 \right), \quad (61) \]

for

\[ z_1 = \frac{1}{4} (-2z_3 + z_3 \sqrt{3(c_3 - 4)}), \quad z_2 = \frac{1}{4} (-2z_3 - z_3 \sqrt{3(c_3 - 4)}), \]

\[ \Lambda = -\frac{3c_3z_3^2}{4(4 + c_3)\ell^2}, \quad \alpha = \frac{4(c_3 + 4)\ell^2}{(12 - c_3)(3c_3 - 4)z_3^2}, \quad (62) \]

where the reality of the \( z_i \) requires that \( c_3 > 4 \).
We find two classes of solutions with two off-diagonal terms. The first takes the form

\[ ds^2 = \ell^2 \left( \frac{dr^2}{r^2} + r^{z_2} d\chi^2 + r^{z_3} dx_1 dx_2 + r^{z_1+z_2} dx_1 dx_3 \right), \]

with

(1) \( z_1 = -2z_2, \quad z_3 = 4z_2, \quad \Lambda = -\frac{3z_2^2}{\ell^2}, \quad \alpha = -\frac{\ell^2}{20z_2^2}, \)

(2) \( z_1 = -\frac{z_2}{2}, \quad z_3 = \frac{5z_2}{2}, \quad \Lambda = -\frac{3z_2^2}{\ell^2}, \quad \alpha = \frac{4\ell^2}{z_2^2}, \)

and

(3) \( z_1 = \frac{1}{7}(\pm 3\sqrt{2} - 2)z_2, \quad z_3 = \frac{2}{7}(2 \mp 3\sqrt{2})z_2, \quad \Lambda = \frac{3(\pm 20\sqrt{2} - 53)z_2^2}{196\ell^2}, \quad \alpha = \frac{(\pm 2\sqrt{2} - 1)\ell^2}{2z_2^2}. \)

In all three of these examples, the metric is Lorentzian.

The second class of solutions take the form

\[ ds^2 = \ell^2 \left( \frac{dr^2}{r^2} + r^{z_1} d\chi^2 + r^{z_2} dx_1 dx_2 + \sqrt{c_3^2} r^{z_3} dx_1 dx_2 + r^{z_1+z_2} dx_1 dx_3 \right), \]

with

(4) \( z_1 = -2z_2, \quad z_3 = \frac{(4 \pm 3\sqrt{2})c_3 + c_1)z_2}{c_3 - 2}, \quad \alpha = \frac{(c_3 - 2)\ell^2}{2(c_3 \mp \sqrt{8c_3 - 6})z_2^2}, \quad \Lambda = -\frac{3(48 \pm 40\sqrt{2}/c_3 + 10c_3 \mp \sqrt{8c_3^2 + c_1^2})z_2^2}{8(c_3 - 2)^2\ell^2}. \)

and

(5) \( z_1 = \frac{\sqrt{c_3}(2\sqrt{c_3} \pm 3\sqrt{2}\sqrt{2(c_3 - 4)})z_2}{36 - 7c_3}, \quad z_3 = \frac{2\sqrt{c_3}(2\sqrt{c_3} \pm 3\sqrt{2}(c_3 - 4))z_2}{36 - 7c_3}, \quad \alpha = \frac{2z_2}{7(c_3 - 12) \mp \sqrt{8(c_3 - 4)c_3}} \right)^2, \quad \Lambda = \frac{-3z_2^2([864 - 420c_3 + 53c_3^2] \pm 4(5c_3 - 18)\sqrt{2(c_3 - 4)c_3})}{4(36 - 7c_3)^2\ell^2}. \)

The metrics are Lorentzian also, provided that \( c_3 \) is chosen appropriately so that all the parameters are real.

### 7. Conclusions

In this paper, we have constructed a variety of new solutions in conformal gravity in four dimensions. Quadratic gravities in four dimensions are special in that Einstein metrics are automatically solutions. In conformal gravity, Einstein metrics with an arbitrary conformal factor are also solutions. Indeed, the most general spherically-symmetric black holes in conformal gravity can be shown to be locally conformal to the Schwarzschild-AdS black hole. The most general metric in conformal gravity in the Plebanski class is also locally conformal to the (neutral) Plebanski–Demianski metric, which is Einstein. It is of interest to
look for solutions that are not merely conformal scalings of Einstein metrics. Such solutions are known to exist, although there are few explicit examples.

We have constructed five new examples of homogeneous vacua solutions that are not conformally Einstein. These metrics exhibit generalized Lifshitz anisotropic scaling. We also studied their Petrov classification. It turns out that four of the solutions are conformal to pp-waves (defined to be metrics admitting a covariantly-constant null vector). We were able to generalize these solutions further, to obtain inhomogeneous metrics with multiple independent free functions.

We have also obtained new homogeneous vacua with generalized multiple anisotropic Lifshitz scaling symmetry in Einstein–Weyl gravity. Our results show that there exist wide classes of homogeneous vacua in higher-derivative gravities. Their classification and application to non-relativistic holography require further investigation.

It could be of interest to generalize the investigation of metrics that are not conformally Einstein to the case of higher-dimensional conformal gravities. For instance, in six dimensions there can be three independent conformally invariant terms, but only one specific combination admits Einstein metrics as solutions [22–25]. Thus the general solutions in conformal gravities in six dimensions are not conformally Einstein. Furthermore, even for the theory that does admit Einstein metrics, the study of its spherically-symmetric solutions shows that there exist classes of solution that are not conformally Einstein [12].

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Appendix. Petrov classification

Here, we summarize some key aspects of the Petrov classification of the Weyl tensor. This amounts to a classification of the structure of the eigenvalues $\lambda$ of the equation

$$\frac{1}{2} C_{\mu\nu\rho\sigma} X^{\rho\sigma} = \lambda X_{\mu\nu},$$

It can be restated as follows. We begin by defining the tensor

$$Q_{\mu\nu\rho\sigma} = \frac{1}{2} (C_{\mu\nu\rho\sigma} - i^* C_{\mu\nu\rho\sigma} - i^* C_{\mu\nu\rho\sigma} - i^* C_{\mu\nu\rho\sigma}),$$

(A.1)

where the left and right duals of the Weyl tensor are defined by

$$i^* C_{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} C_{\alpha\beta\rho\sigma}, \quad C_{\mu\nu\rho\sigma}^* = \frac{1}{2} \epsilon_{\rho\sigma\alpha\beta} C_{\mu\nu}^{\alpha\beta}.$$  \hspace{1cm} (A.2)

Quadratic and cubic curvature invariants $I$ and $J$ are then defined by

$$I = \frac{1}{16} Q_{\mu\nu\rho\sigma} Q^{\mu\nu\rho\sigma}, \quad J = \frac{1}{192} Q_{\mu\nu\rho\sigma} Q_{\mu\sigma\rho\sigma} Q_{\sigma\rho\sigma}.$$  \hspace{1cm} (A.3)

If the spacetime is conformally flat, $C_{\mu\nu\rho\sigma} = 0$, then it is of Petrov type O. For all other spacetimes, we have the possibilities:

$I^3 \neq 27 J^2 :$ \quad type I,

$$I^3 = 27 J^2 \neq 0 :$$ \quad type II or type D,

$I = J = 0 :$ \quad type N or type III \hspace{1cm} (A.4)
The six distinct Petrov types arise as follows. At the top, with \( I^3 \neq 27J^2 \), is type I. In
the second level, where \( I^3 = 27J^2 \neq 0 \), lie type II and type D. At the lowest level, where
\( I = J = 0 \), lie type III, type N and type O:

```
I ------  \( I^3 \neq 27J^2 \) ------ II -------  \( I^3 = 27J^2 \neq 0 \) ------ D
    |    \( \cdots \cdots \)          |    \( \cdots \cdots \)          |    \( \cdots \cdots \)
III ---- N ------       O  \( \cdots \cdots \) I = J = 0
```

Type II can be distinguished from type D by the fact that, in a type II spacetime, there
exists a non-vanishing vector \( K^\mu \) such that

\[
C_{\mu\nu\rho\sigma}K^\mu K^\rho = aK_\nu K_\sigma, \quad C_{\mu\nu\rho\sigma}K^\mu K^\rho = bK_\nu K_\sigma, \quad \text{(A.5)}
\]

for some \( a \) and \( b \).

Type O is conformally flat, with \( C_{\mu\nu\rho\sigma} = 0 \). Type N can be distinguished from type III by
the fact that in a type N spacetime there exists a non-vanishing vector such that

\[
C_{\mu\nu\rho\sigma}K^\sigma = 0. \quad \text{(A.6)}
\]

The orders of specialization from one Petrov type to another are indicated by the arrows in the
diagram.

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