SURVEY ON A QUANTUM STOCHASTIC EXTENSION
OF STONE’S THEOREM

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Abstract. From Kümmerer’s investigations on stationary Markov processes has emerged an operator algebraic definition of white noises which captures many examples from classical as well as from non-commutative probability. Within non-commutative $L^p$-spaces associated to a white noise, the role of (non-)commutative Lévy processes is played by additive cocycles for the white noise shift, and moreover, the notion for exponentials of classical Lévy processes is generalized by unitary cocycles. As a main result we report a bijective correspondence between additive and unitary cocycles for white noise shifts. If the cocycles are required to be differentiable, the presented correspondence reduces to Stone’s theorem (for norm continuous unitary groups).

The correspondence needs the development of background results for additive cocycles with $L^\infty$-bounded covariance operators: an operator-valued stochastic Itô integration, quadratic variations and non-commutative martingale inequalities as well as stochastic differentiation. Related results and recent progress towards the case of additive cocycles with unbounded variance operators are reported.

1. Introduction

The developments of non-commutative (or quantum) probability led to many examples of non-commutative analogues of Brownian motion or, more generally speaking, Lévy processes. They are realized through creation and annihilation operators on deformed Fock spaces. Motivated by Classical Probability a natural question arises in quantum probability: do non-commutative Markov processes appear as solutions of quantum stochastic differential equations when the increments are given by non-commutative analogues of Lévy processes?

An early answer to this question was given in the 80’s by Hudson and Parthasarathy for the Bosonic Fock space, as well as Barnett, Streater and Wilde for the Fermionic Fock space, and was followed by the work of many others (cf. [M, P] and references therein). They succeeded in extending the stochastic Itô integral, in particular, to creation and annihilation operators as increments. Moreover, quantum Markov processes were constructed as solutions of Bosonic or Fermionic quantum stochastic differential equations. Beginning in the 90’s, a third example appeared on the scene with Voiculesu’s Free Probability [VDN]: free Brownian motion which also
provides a rich stochastic calculus for the construction of Markov processes [KS]. Meanwhile, the stochastic calculi of these examples have been further developed and of particular importance for the present approach are [AF, BSW1, BS, HKR]. Investigations on other examples of non-commutative analogues of Brownian motion on Fock spaces, e.g., the family of q-Brownian motions, become recently a subject of increasing interest (cf. [BS1, BS2, BKS, GM, Kr] and references therein).

At about the same time in the 80’s, Kümmerer followed an operator algebraic approach to stationary Markov processes [Ku1]. His investigations showed that many of these processes can be realized as so-called couplings $\text{Ad} u_t$ to a stationary white noise shift $S_t$ [Ku2, Ku3]. Here the shift $S_t$ is an automorphism group acting on a von Neumann algebra $\mathcal{A}$ and the coupling is given by a unitary cocycle $u_t$ for $S_t$. Consequently, the Markovian evolution appears as $\mathcal{A} \ni x \mapsto \text{Ad} u_t S_t(x) u_t$. From this approach the question arises whether a unitary cocycle $u_t$ for a white noise shift $S_t$ can still be identified as the solution of a certain stochastic differential equation $du_t = db_t u_t$. Here the increments $db_t$ are provided by an additive cocycle $b_t$ (for $S_t$), the operator algebraic generalization of a classical Lévy process. This question was affirmatively answered in [HKK1], in the form of a bijective correspondence between additive and unitary cocycles. In this survey we present this result in an improved form within the framework of non-commutative $L^p$-spaces associated with a trace. We emphasize that the result relies only on the operator algebraic white noise and its cocycles. A priori, it does not refer to any kind of examples on Fock spaces. Nevertheless, (deformed) Fock spaces provide rich sources for examples of white noises and additive cocycles [Ku3, BG].

For the purpose of this survey we restrict our attention to white noises with non-commutative $L^p$-spaces associated with a trace. Appropriate generalizations of the results to white noises on von Neumann algebras of type III are available by passing to Haagerup’s $L^p$-spaces.

After introducing necessary notations and properties of non-commutative probability as well as non-commutative $L^p$-spaces in Section 2, we proceed in Section 3 with the crucial definition of operator-valued white noise. The presented definition of white noise includes a suitable notion of independence, similar to the structure of commuting squares in subfactor theory [GHJ]. Moreover, additive and unitary cocycles for white noise shifts are defined. They will play the part of operator-valued Lévy processes resp. their unitary exponentials. Finally, the introduced framework is justified by a result of Kümmerer [Ku2] which states that many non-commutative Markov processes come from cocycle pertubations of white noises.

The main result is contained in Section 4. In the presence of an operator-valued white noise, it states a bijective correspondence between additive and unitary cocycles for the white noise shift: Briefly formulating, $u_t$ is a unitary cocycle iff $b_t$ is an additive cocycle satisfying the structure equation $db_t^* db_t + db_t + db_t^* = 0$. As a Corollary, Stone’s Theorem appears when the cocycles are further required to be differentiable. Another familiar Corollary is the construction of stationary Markov processes from the perturbation of the white noise shift with a unitary cocycle. Due to the correspondence, the generator of the associated Markovian semigroup can easily be expressed in terms of the corresponding additive cocycle. A first, preliminary version of such a bijective correspondence is contained in [HKK1] using the language of Hilbert modules. The present approach, using non-commutative
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$L^p$-spaces, enables us to prove the existence of all moments of an additive cocycle whenever it comes from the correspondence $[KÖ1]$.

To establish the correspondence, one has to construct a unitary cocycle from an additive one and vice versa. The first direction needs stochastic integration for additive cocycles of white noise shifts $[KÖ1]$ and constructs the unitary cocycle as the solution of a certain stochastic differential equation. This approach combines the structure of an operator-valued white noise with general stochastic integration on non-commutative $L^2$-spaces as can be found in $[BSW2]$. We will only briefly review the related concepts in Section 5 as far as it necessary for the formulation of the main result.

Section 6 is devoted to quadratic variations of additive cocycles $[KÖ2]$, which give rise to the non-commutative analogues of the famous Itô corrections $db_t^*db_t$ in classical stochastic analysis. The stated results provide a rigorous formulation of the structure equation $db_t^*db_t + db_t + db_t^* = 0$. Furthermore, we present estimates on the growth of higher moments of centred additive cocycles $[KÖ2]$. This gives applications to non-commutative martingale inequalities as they have been established in the work of Junge, Pisier and Xu for discretely indexed martingales in $[PX1, JX]$.

In Section 7 we present the construction of additive cocycles from unitary cocycles. We call this procedure stochastic differentiation and its idea is already present in the work of Skorokhod on operator-valued stochastic semigroups in Classical Probability $[Sk]$. Independently, the idea reappeared for scalar-valued white noises (cf. $[KÖ3]$). We will present results on the existence of all moments of the stochastic derivative and the general case which leads to additive cocycles with $\tau$-measurable covariance operators.

Finally, we investigate in Section 8 the structure equation $db_t^*db_t + db_t + db_t^* = 0$ and introduce flows of additive cocycles. Consequently, the structure equation is identified as a specific case of a non-commutative fluctuation-dissipation theorem. This provides sufficient information to control the unbounded generator of the Markovian semigroup.

2. Preliminaries

In the following we introduce non-commutative probability spaces, their morphisms, filtrations and notions of non-commutative processes which we will use throughout this paper. For this survey we will limit our considerations to non-commutative $L^p$-spaces associated with a trace, but we emphasize that the presented setting carries over to Haagerup’s $L^p$-spaces $[H, T]$, after some suitable modifications.

$\mathcal{A}$ will always denote a finite von Neumann algebra with norm separable predual $\mathcal{A}_r$. A faithful normal tracial state on $\mathcal{A}$ will be denoted by $\tau$. For $1 \leq p \leq \infty$ the non-commutative $L^p$-spaces $L^p(\mathcal{A}, \tau)$ are defined by the completion of $\mathcal{A}$ in the norm $\|x\|_p = (\tau(|x|^p))^{1/p}$, $x \in \mathcal{A}$, where $|x| = (x^*x)^{1/2}$. We recall that $L^\infty(\mathcal{A}, \tau)$ is just $\mathcal{A}$ with the usual operator norm and that $L^1(\mathcal{A}, \tau)$ is isometrically isomorphic to $\mathcal{A}_r$. Note that in the presented setting, a densely-defined closed operator affiliated with $\mathcal{A}$ is already $\tau$-measurable. We will denote all $\tau$-measurable operators by $L^0(\mathcal{A}, \tau)$. Moreover, the trace $\tau$ extends to a positive tracial functional on the positive part $|x|$ of all $\tau$-measurable operators $x$ and its extension will still be denoted by $\tau$. Note that $x \in L^p(\mathcal{A}, \tau)$ iff $\tau(|x|^p) < \infty$ $(1 \leq p < \infty)$. For the definitions of $\tau$-measurability and further details on non-commutative $L^p$-spaces we refer the reader to $[PX2]$ and the references therein.
In the sequel the pair \((A, \tau)\) and its associated non-commutative \(L^p\)-spaces will be called (non-commutative) probability spaces. A morphism \(T : (A, \tau) \to (\hat{A}, \hat{\tau})\) between two probability spaces is a completely positive operator such that \(\hat{\tau} \circ T = \tau\) and \(T(1_A) = 1_{\hat{A}}\). Note that a morphism \(T\) extends to a contraction from \(L^p(A, \tau)\) to \(L^p(\hat{A}, \hat{\tau})\) \((1 \leq p < \infty)\), denoted also by \(T\). Endomorphism and automorphism of a probability space are accordingly understood and denoted by \(\text{End}(A, \tau)\) resp. \(\text{Aut}(A, \tau)\). If \(B \subset A\) is a von Neumann subalgebra it is well known that there exists a unique conditional expectation \(E : A \to B\) as endomorphism. It extends to a contractive projection from \(L^p(A, \tau)\) onto \(L^p(B, \tau)\), which is still called a conditional expectation. Moreover, \(L^p(B, \tau)\) is always naturally identified with a subspace of \(L^p(A, \tau)\).

A family of von Neumann subalgebras \((A_I)_I \subset A\), indexed by ‘intervals’ \(I = [s, t]\) with \(-\infty \leq s \leq t \leq \infty\), is called a filtration of \((A, \tau)\) if \(A_I\) and \(A_J\) generates \(A_K\) whenever \(I \cup J = K\). The filtration \((A_I)_I\) is called minimal if \(A_{\mathbb{R}} = A\) and lower (resp. upper) continuous if \(\bigcup_{t \in J} A_I\) generates \(A_J\) (resp. \(\bigcap_{t \in J} A_I = A_J\)) as von Neumann algebra. Here we used the convention \([s, t]^\circ = (s, t)\) for \(s < t\) and \([s, s]^\circ = \emptyset\). Let \(E_I\) denote the conditional expectation from \(A\) onto \(A_I\). The filtration \((A_I)_I\) is called \(A_0\)-Markov filtration if \(E_I(-\infty, 0)E_I[0, \infty) = E_0\). Note that the filtration and its related conditional expectations extend to non-commutative \(L^p\)-spaces for \(1 \leq p < \infty\) and will be called the same.

Let \(A_0 \subset A\) be a distinguished von Neumann subalgebra and let \(T \equiv (T_t)_{t \in \mathbb{R}} \subset \text{Aut}(A, \tau)\) be a pointwise weakly* continuous group. The tuple \((A, \tau, T; A_0)\) is called a \(A_0\)-valued (stationary) stochastic process. The group \(T\) and \(A_0\) induce a continuous filtration \((A_I)_I\) on the probability space \((A, \tau)\) through the von Neumann subalgebras \(A_I\) generated by \(\bigcup_{t \in I} T_t(A_0)\). If \((A_I)_I\) is an \(A_0\)-Markov filtration then \((A, \tau, T; A_0)\) is called a (stationary) \(A_0\)-Markov process. Note that in this case the compressions \(R = E_0T E_0 \in \text{End}(A_0, \tau)\) define a semigroup of contractions on \(A_0\). Again, these notions extend canonically to non-commutative \(L^p\)-spaces.

Finally, following the approach of Kümmerer [Kü2], we formulate the non-commutative counterpart to operator-valued classical stationary generalized processes. We start with a filtration \((A_I)_I \subset A\) (which contains \(A_0\)) and a pointwise weakly* continuous group \(T \subset \text{Aut}(A, \tau)\) such that both are compatible, i.e., \(T_t(A_{r,s}) = A_{r+t,s+t}\) for any \(-\infty \leq r \leq s \leq \infty\) and \(t \in \mathbb{R}\). Then \((A, \tau, T; (A_I)_I)\) is called \(A_0\)-valued generalized (stationary) process. By the continuity of the automorphism group its filtration is automatically lower continuous. Upper continuity follows for the families \((A_{(-\infty, t]}_t \in \mathbb{R})\) and \((A_{[t, \infty)}_t \in \mathbb{R})\). The generalized stationary process \((A, \tau, T; (A_I)_I)\) is called \(A_0\)-valued noise if the filtration \((A_I)_I\) is minimal and \(A_0\) is the fixed point algebra of \(T\). This notions extend as before to non-commutative \(L^p\)-spaces. We will use the same notation for them.

3. White noises and their cocycles

We will present the non-commutative version of an operator-valued classical white noise. Recall that \(A\) denotes a finite von Neumann algebra which is equipped with a faithful normal tracial state \(\tau\) and a minimal filtration \((A_I)_I\). We remind that \(E_I\) denotes the conditional expectation from \(A\) onto \(A_I\). Finally, \(S\) denotes an automorphism group which is compatible with the given filtration and has the fixed point algebra \(A_0\). For the precise definition of a noise we refer to the preliminaries.
Definition 3.1. A noise \((\mathcal{A}, \tau, S; (A_i)_I)\) is called a \((\mathcal{A}_0\text{-valued})\) white noise if \(E_I \circ E_J = E_0\) whenever \(I \cap J = \emptyset\). The von Neumann subalgebras \(A_I\) and \(A_J\) of \(\mathcal{A}\) are called independent (over \(\mathcal{A}_0\)) if \(I \cap J = \emptyset\) implies \(E_I \circ E_J = E_0\).

If the von Neumann algebra \(\mathcal{A}\) is commutative, one gets back classical examples of white noises (cf. [1]: for the example of Gaussian white noise and [5] for its reformulation to a white noise in our sense). Crucial for the noise to be 'white' is the notion of independence which parallels commuting squares in subfactor theory [GHJ]. Moreover, it reduces to the classical notion of stochastic independence whenever the underlying von Neumann algebra is commutative and the noise scalar-valued.

Throughout the paper we will always assume that an \(\mathcal{A}_0\)-valued white noise \((\mathcal{A}, \tau, S; (A_i)_I)\) and its canonical extensions to non-commutative \(L^p\)-spaces \((1 \leq p < \infty)\) are given. For brevity, the automorphism group of the white noise and its extensions to the associated non-commutative \(L^p\)-spaces will just be called shift \(S\).

Remark 3.2. (i) The notion of white noise does not refer to Fock spaces. However, many examples of non-commutative white noises come along with generalized Brownian motions on deformed Fock spaces [2, 3, 5, 6, 7, 8, 9, 10]. Other examples come from classical compound processes which trigger non-commutative Bernoulli shifts [5].

(ii) All examples of white noises constructed from so called white noise functors satisfy \(E_I E_J = E_{I \cap J}\) [6, 7, 8, 9, 10]. This implies independence as stated in definition 3.1. The converse is an open problem.

We define the analogue of an operator-valued class of classical Lévy processes.

Definition 3.3. Let \(1 \leq p \leq \infty\). An additive \(L^p\)-cocycle \(b\) for the shift \(S\) is a strongly continuous family \((b_t)_{t \geq 0}\) in \(L^p(\mathcal{A}, \tau)\) such that

- \(b_t \in L^p(\mathcal{A}_{[0,t]}, \tau)\) (adaptedness);
- \(b_{s+t} = S_t(b_s) + b_t\) for all \(s, t \geq 0\) (cocycle identity).

If \(E_0(b_t) = 0\) for any \(t \geq 0\) then \(b\) is called centred. If \(E_0(b_t) = b_t\) then \(b\) is called drift. \(E_0(b_t^* b_t)\) is the covariance (operator) of \(b\).

Frequently, an additive \(L^p\)-cocycle will also be called a \(L^p\text{-Lévy process.}\) A centred additive \(L^p\)-cocycle \(b\) is automatically continuous since it is a martingale with respect to the continuous filtration \((\mathcal{A}_{(-\infty,t)})_{t \geq 0}\). Moreover, the continuity of a centred additive \(L^p\)-cocycle \(c\) implies the identity

\[ E_0(c_t^* x c_t) = E_0(c_t^* x c_t) t \]

for all \(t > 0\), whenever \(x \in L^2(\mathcal{A}_0, \tau)\) and \(1 \geq 2/p + 1/q\). The equality features that additive cocycles lead to operator-valued Lebesgue measures. This observation is a cornerstone for a non-commutative version of the Itô integral (cf. Section 5).

Note that the continuity assumption is only needed for a drift and insures that a drift is always of the form \((td)_{t \geq 0}\) for some \(d \in L^p(\mathcal{A}_0, \tau)\). We remark that an additive cocycle \(b\) uniquely decomposes into a centred additive cocycle \(c\) and a drift \(d\) such that \(b_t = c_t + td\) for any \(t \geq 0\). Our main result will require an additive \(L^2\)-cocycle \(b\) with a symmetric covariance \(E_0(b_t^* b_t + b_t b_t^*) \in L^\infty(\mathcal{A}_0, \tau)\).

Let us now present the non-commutative analogue of exponentials of Lévy processes on the unit circle.
Definition 3.4. A unitary cocycle $u$ for the shift $S$ is a weakly* continuous family of unitary operators $(u_t)_{t \geq 0} \in L^\infty(A, \tau)$ with the following properties:

(i) $u_t \in A[0,t]$ (adaptedness);
(ii) $u_{s+t} = S_t(u_s)u_t$ for all $s, t \geq 0$ (cocycle identity).

An immediate consequence of (i) and (ii) is that $t \mapsto E_0(u_t) \subset L^\infty(A_0, \tau)$ defines a strongly continuous semigroup of contractions on $L^p(A_0, \tau)$ ($1 \leq p \leq \infty$). For our main result we will assume that $E_0(u_t)$ is uniformly continuous. Note that $E_0(u_t)$ is uniformly continuous iff its generator $d$ is an operator in $L^\infty(A_0, \tau)$. The more general case will be further considered in Section 7.

Stationary Markov processes appear now as follows. Let us extend the unitary cocycle $u$ to negative times by defining $u_t = S_t u^*_t$ for $t < 0$ and let $\text{Ad} u_t(x) = u^*_t xu_t$ for any $x \in L^\infty(A, \tau)$.

Proposition 3.5. Let $(A, \tau, S, (A_I)_I)$ be an $A_0$-valued white noise with unitary cocycle $u$. Then $(A, \tau, \text{Ad} u S; A_0)$ is a stationary Markov process with values in $A_0$.

Such a representation of a Markov process is called a coupling to white noise [Kü2]. Finally, let us state a result from [Kü2] which establishes the converse and justifies the definition of a white noise. $M_n$ will denote the $n \times n$-matrices which are identified with their isomorphic embedding in $A$.

Theorem 3.6. Let $(A, \tau, T; M_n)$ be a stationary Markov process. If there exists a minimal projection $e \in M_n$ which is invariant under $T$, then the Markov process is a coupling to a $M_n$-valued white noise $(A, \tau, S, (A_I)_I)$.

This result indicates how intimately white noises and Markov processes are related and motivated investigations which led to the main result stated below.

4. The main result

We are now in the position to present our main result. From a stochastic point of view it states that many Markov processes are driven by Lévy processes. From the perspective of functional analysis it is an extension of Stone's theorem to cocycles of unitary groups.

Theorem 4.1. For an $A_0$-valued white noise $(A, \tau, S, (A_I)_I)$ there exists a bijective correspondence between:

(a) $u \subset L^\infty(A, \tau)$ is a unitary adapted cocycle for $S$ such that its semigroup $E_0(u)$ is norm continuous in $L^\infty(A_0, \tau)$.
(b) $b \subset L^2(A, \tau)$ is an additive adapted cocycle for $S$ with symmetric covariance $E_0(b^*b + bb^*) \subset L^\infty(A_0, \tau)$ that satisfies the structure equation
$$[b, b] + b^* + b = 0.$$ 

The correspondence from (a) to (b) is given by the stochastic differentiation
$$b_t = \lim_{N \to \infty} \sum_{j=0}^{N-1} S_{t/N}(u_{t/N} - 1)$$
and from (b) to (a) by the stochastic differential equation
$$u_t = 1 + \int_0^t db_s u_s.$$
Here, $[b,b]$ denotes the quadratic variation of the additive cocycle $b$ (cf. Section 6), the definition and background of the stochastic differential equation is presented in Section 6. The main result reduces to Stone’s theorem (for bounded generators) if we require that the cocycles are differentiable.

**Corollary 4.2.** If the unitary cocycle $u$ in (a) or the additive cocycle $b$ in (b) is differentiable then $u_t = \exp(itb)$ and $b_t = ith$ for some selfadjoint operator $h$ in $L^\infty(A_0, \tau)$.

**Proof.** We will show that a differentiable unitary cocycle $u$ lies in $A_0$, the fixed point algebra of the shift $S$. Since $u_t - 1 \in A_{(-\infty,t]} \cap A_{[0,\infty)}$ for any $t > 0$ we conclude $\lim_{t \to 0} \frac{1}{t}(u_t - 1) \in \bigcap_{t > 0} A_{(-\infty,t]}$. The continuity from above of the filtration $(A_{(-\infty,t]})_{t \in \mathbb{R}}$ implies $\lim_{t \to 0} \frac{1}{t}(u_t - 1) \in A_{(-\infty,0]}$. But it holds also $\lim_{t \to 0} \frac{1}{t}(u_t - 1) \in A_{[0,\infty)}$. Thus the derivative $u'_0$ has to be in $A_0$. Furthermore, the cocycle identity implies $u'_t = S_1(u'_0)u_t = u'_0u_t$ and therefore $u_t = 1 + \int_0^t u'_s ds = 1 + \int_0^t u'_0u_s ds$ is in $A_0$ for any $t \geq 0$. Since the solution $u_t = \exp(u'_0t)$ is unitary, we conclude $u'_0 = ith$ for some selfadjoint $h \in A_0$.

On the other hand, the additive cocycle uniquely decomposes as $b_t = c_t + ith$, where $c$ is a centred additive cocycle and $h \in A_0$. From $E_0(|c_t|^2) = E_0(|c_1|^2)t$ and the differentiability of $c_t$ we conclude $c_1 = 0$ and from this $c_t = 0$. The selfadjointness of $h$ follows immediately from the structure equation. □

By the means of our main results stationary non-commutative Markov processes can easily be constructed from additive cocycles for a white noise shift.

**Corollary 4.3.** Let $b$ be an additive cocycle (with centred part $c$ and drift $d$) which satisfies the conditions stated in (a) and let $u$ be the corresponding unitary cocycle. Then $(A, \tau, Ad u S; A_0)$ is a stationary Markov process with values in $A_0$. Moreover, the generator $L$ of the contractive semigroup $R = E_0(Ad u)E_0$ has the form

$$L(x) = E_0(c_1^*x c_1) + dx^*x + xd, \quad x \in A_0.$$  

As already stated, additive cocycles are non-commutative generalizations of classical Lévy processes. In our main result we assumed that second moments of the additive cocycle exist. But the structure equation (in $L^1(A, \tau)$) is very restrictive as the following result shows:

**Theorem 4.4.** If an additive $L^2$-cocycle $b$ satisfies the condition (b) in Theorem 4.1 then all moments of $b$ exist, i.e., $\tau(|b|^p) < \infty$ for any $1 \leq p < \infty$.

The proof uses non-commutative martingale inequalities and was obtained in [Kö1] by approximations starting from the non-commutative Burkholder-Gundy inequalities in [PXi]. Meanwhile an alternative proof is given in [Kö3] which starts from the non-commutative Burkholder/Rosenthal inequalities in [IX].

We conjecture that the assumptions in our main result on the norm continuity of the semigroup $E_0(u)$ and the boundedness of the symmetric variance $E_0(b^*b + bb^*)$ can be tremendously relaxed. This is indicated by results which are used in the proof of the bijective correspondence between additive and unitary cocycles.

5. Stochastic integration

In this section we review non-commutative stochastic integrals and differential equations of Itô type as far as they are needed in theorem 4.1.
Let an \( A_0 \)-valued white noise \((A, \tau, S; (A_t)_t)\), its canonical extensions to \( L^p \)-spaces \((1 \leq p < \infty)\) and an additive \( L^p \)-cocycle for the shift \( S \) be given. The following observation is the key to build Itô integrals on non-commutative \( L^2 \)-spaces of white noises.

**Proposition 5.1.** Let \( x \in L^2(A_t, \tau) \) and \( y \in L^2(A_t, \tau) \) be independent, i.e., \( I^o \cap J^o = \emptyset \). Then \( E_0(x^*x) \in L^\infty(A_0, \tau) \) implies \( xy \in L^2(A, \tau) \). Moreover,

\[
\|xy\|_2 = \|(E_0(x^*x))^{1/2}\|_2 \text{ and } E_0(y^*xy) = E_0(y^*E_0(x^*x)y).
\]

With this result we are prepared to define stochastic integrals for additive cocycles. Let \( \{0 = s_0 < s_1 < \ldots < s_n = t\} \) be a subdivision of the interval \([0, t]\) and let the simple stochastic process \( y = \sum_{j=0}^{n-1} y_{s_j} \chi_{[s_j, s_{j+1})} \subset L^2(A, \tau) \) be adapted to the filtration \((A_{[0,s]}), s \geq 0\), i.e., \( y_s \in L^2(A_{[0,s]}, \tau) \) for any \( s \geq 0 \). Then the (left) Itô integral of \( x \) w.r.t. the centred additive \( L^2 \)-cocycle \( b \) is defined to be

\[
\int_0^t db_s y_s = \sum_{j=0}^{n-1} (b_{s_{j+1}} - b_{s_j}) y_{s_j}.
\]

This leads to the following Itô isometry resp. identity.

**Proposition 5.2.** Let \( y \subset L^2(A, \tau) \) be a simple adapted process, then

\[
\left\| \int_0^t db_s y_s \right\|_2^2 = \int_0^t \left\| (E_0(b_1^*b_1))^{1/2} x_s \right\|_2^2 \, ds,
\]

\[
E_0\left( \int_0^t db_s y_s^2 \right) = \int_0^t E_0(x_s^*E_0(b_1^*b_1)x_s) \, ds.
\]
6. Quadratic variations

Quadratic variations of Lévy processes are fundamental for the development of stochastic calculi in Classical Probability. They give rise to the famous Itô corrections. This section is devoted to quadratic variations of non-commutative Lévy processes as they appear in the white noise approach.

Let $\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ denote a subdivision of the interval $[0, t]$ with mesh $|\mathcal{P}| = \max_j \{t_{j+1} - t_j\}$.

**Theorem 6.1.** Let $b$ be an additive adapted cocycle for $S$ in $L^p(\mathcal{A}, \tau)$ where $2 \leq p < \infty$. Then its quadratic variation 

$$[b, b]_t = L^1 \lim_{|\mathcal{P}| \to 0} \sum_{j=0}^{n_p} |b_{t_{j+1}} - b_{t_j}|^2$$

exists for any $t > 0$ and defines the (non-centred) additive cocycle $[b, b]$ in $L^{p/2}(\mathcal{A}, \tau)$.

If $b$ and $c$ are two additive cocyclics in $L^p(\mathcal{A}, \tau)$ ($2 \leq p < \infty$) then the mutual quadratic variation $[b, c]$ is defined by polarisation. The quadratic variation extends easily to a non-centred additive $L^2$-cocycle $b$ and it holds $[b, b] = [c, c]$, where $c$ denotes the centred part of $b$. As usual, the quadratic variation of a drift as well as its mutual quadratic variation with an additive cocycle vanishes. In addition, the identity

$$[b, c]_t = b^*_t c_t - \int_0^t b^*_s dc_s - \int_0^t db^*_s c_s, \quad t \geq 0,$$

holds for additive $L^4$-cyclics $b$ and $c$.

Centred additive cocycles are $L^p$-continuous martingales w.r.t. the filtration $(\mathcal{A}_{(-\infty, t)})_{t \geq 0}$. Moreover, by the above result, their quadratic variation exists. Starting from the non-commutative Burkholder-Gundy inequalities of Pisier and Xu [PX1] for martingales indexed by discrete time, we established [Ko2]:

**Theorem 6.2.** Let $2 \leq p < \infty$. Then there exists constants $\alpha_p$ and $\beta_p$ only depending on $p$ such that for any centred additive $L^p$-cocycle $b$ and any $t \geq 0$

$$\alpha_p \|b_t\|_{H_p(\mathcal{A})} \leq \|b_t\|_p \leq \beta_p \|b_t\|_{H_p(\mathcal{A})}$$

where $\|b_t\|_{H_p(\mathcal{A})} = \max\{\|\langle b, b \rangle_t^{1/2}\|_p, \|\langle b^*, b^* \rangle_t^{1/2}\|_p\}$.

Note that $\alpha_p$ and $\beta_p$ are the constants which appear in the Burkholder-Gundy inequalities for discretely indexed martingales as presented in [PX1] or [JX]. Explicit bounds of growth for centred additive cocycles are found in [Ko2] by an application of non-commutative Burkholder/Rosenthal inequalities [JX].

**Theorem 6.3.** Let $2 \leq p < \infty$. There exist positive constants $\tilde{\alpha}_p$ and $\tilde{\beta}_p$ such that for every centred Lévy $L^p$-process $b$ and all $t \geq 0$

$$\tilde{\alpha}_p \max\{\mu^+ p t^{1/p}, \mu^- p t^{1/2}\} \leq \|b_t\|_p \leq \tilde{\beta}_p \max\{\mu^+ p t^{1/p}, \mu^- p t^{1/2}\},$$

where the constants are defined to be

$$\lambda_p = \max\{\|(E_0(b^*_t b_1))^{1/2}\|_p, \|(E_0(b_1 b^*_t))^{1/2}\|_p\}$$
and
\[ \mu_p^- = \liminf_{s \to 0} \frac{1}{s^{1/p}} \|b_s\|_{p^*}, \quad \mu_p^+ = \limsup_{s \to 0} \frac{1}{s^{1/p}} \|b_s\|_p < \infty. \]

The constants \( \lambda_p \) describe norms of the covariance operator of the additive cocycle. If the \( p \)-th absolute moment \( M_p(t) := \tau(|b_t|^p) \) is right differentiable at \( t = 0 \) then \( \mu_p^- = \mu_p^+ = (M_p'(0))^{1/p} \). In the case \( p = 2, 4 \) this is a matter of fact, nevertheless we strongly conjecture that this property holds for any \( 2 \leq p < \infty \).

7. Stochastic differentiation

This section presents results which are related to the construction of additive cocycles from unitary cocycles for the shift \( S \) of a given white noise. We call this procedure stochastic differentiation because it reduces to the usual differentiation as soon as we require the unitary cocycle to be differentiable.

In the following we always assume that an \( \mathcal{A}_0 \)-valued white noise \( (\mathcal{A}, \tau, S; (\mathcal{A}_t)_t) \) and its extensions to (non-)commutative \( L^p \)-spaces \( (1 \leq p < \infty) \) are given. We first present the result which is relevant for Theorem 4.1, i.e., the construction of additive cocycles with a covariance operator in \( L^{\infty}(\mathcal{A}_0, \tau) \).

**Theorem 7.1.** Let \( u \) be a unitary cocycle for \( S \) such that the semigroup \( E_0(u) \) has the generator \( d \in \mathcal{A}_0 \). Let \( 1 \leq p < \infty \). Then
\[ b_t = L^p \lim_{N \to \infty} \sum_{j=0}^{N-1} S_{jt/N}(u_{t/N} - 1) \]
exists for any \( t > 0 \) and defines an additive cocycle \( b \) for \( S \) in \( \bigcap_{p < \infty} L^p(\mathcal{A}, \tau) \).

A first, direct proof of this result is contained in [Kö1] and motivated further investigations on quadratic variations of additive cocycles. The proof is quite technical and its strategy is as follows. In a first step one establishes the convergence of the ansatz in the norm topology of \( L^2(\mathcal{A}, \tau) \). Next one considers the case \( 2 < p < \infty \). Since the ansatz is already convergent in \( L^2(\mathcal{A}, \tau) \), it is sufficient to find a uniform bound of the approximating sequence in \( L^p(\mathcal{A}, \tau) \). This already guarantees that the limit is in \( L^p(\mathcal{A}, \tau) \) and that the ansatz is convergent in the norm topology on \( L^q(\mathcal{A}, \tau) \) for \( q < p \). Thus one is left with the problem to find a uniform bound of the approximating sequence in \( L^p(\mathcal{A}, \tau) \). In [Kö1] this bound was established by tedious approximations which are based on the non-commutative Burkholder-Gundy inequalities of Pisier and Xu [PX1]. This concrete example anticipated in parts more general work of Junge and Xu on non-commutative Burkholder-Rosenthal inequalities [JX]. Meanwhile, a simpler proof of the theorem is available in [Kö3] which is based on the results in [JX].

In the following we investigate the case that the semigroup \( E_0(u) \) of the unitary cocycle is no longer continuous in the uniform norm topology on \( L^\infty(\mathcal{A}_0, \tau) \). This situation leads to the construction of additive cocycles with unbounded covariance operators. Before stating the results let us apply to our case some general theory of strongly continuous contractive semigroups on Banach spaces [D].

From \( L^\infty(\mathcal{A}_0, \tau) \subset \mathcal{B}(L^p(\mathcal{A}_0, \tau)) \) for any \( 1 \leq p \leq \infty \) we conclude that the generator of \( E_0(u) \) is a \( \tau \)-measurable operator \( d \in L^0(\mathcal{A}_0, \tau) \) which acts by left-multiplication on the domain \( \mathcal{D}^p(d) \subset L^p(\mathcal{A}_0, \tau) \). Note that \( \mathcal{D}^p(d) \supset \mathcal{D}^q(d) \supset \mathcal{D}^\infty(d) \) for \( 1 \leq p < q \leq \infty \) and, in addition, that \( \mathcal{D}^\infty(d) \) is dense in \( L^\infty(\mathcal{A}_0, \tau) \).
Consequently $D^\infty(d)$ is dense in $L^2(\mathcal{A}_0, \tau)$. Let us remind that $E_0(u)$ is uniformly continuous on $L^p(\mathcal{A}_0, \tau)$ iff $d \in L^\infty(\mathcal{A}_0, \tau)$. Finally, recall that the vector $a \in L^p(\mathcal{A}_0, \tau)$ is said to be an analytic vector for the generator $d$ of the semigroup $E_0(u)$ on $L^p(\mathcal{A}, \tau)$ if $\sum_{n=0}^{\infty} \frac{1}{n!} \|d^n a\|_p < \infty$.

The following result is already contained in [Kô1]:

**Theorem 7.2.** Let $u$ be a unitary cocycle for $S$ such that $1 \in L^\infty(\mathcal{A}_0, \tau)$ is an analytic vector for the generator $d$ of the semigroup $E_0(u)$ acting by left-multiplication on $L^p(\mathcal{A}_0, \tau)$. Then

$$b_t = L^p - \lim_{N \to \infty} \sum_{j=0}^{N-1} S_{jt/N}(u_{t/N} - 1)$$

exists for any $1 \leq p < \infty$ and defines an additive cocycle $b$ for $S$ in $\bigcap_{p<\infty} L^p(\mathcal{A}, \tau)$.

Recently we could establish the most general case of stochastic differentiation:

**Theorem 7.3.** Let $u$ be a unitary cocycle for the shift $S$ and let $\eta \in D(\infty)(d) \subset L^2(\mathcal{A}_0, \tau)$. Then

$$B_t(\eta) = L^2 - \lim_{N \to \infty} \sum_{j=0}^{N-1} S_{jt/N}(u_{t/N} - 1)\eta$$

exists for any $t > 0$ and defines an additive $L^2$-cocycle $B(\eta)$ for any $\eta \in D(\infty)(d)$.

From this result we conjecture that the conditions in our presented main result can be relaxed such that it will include Stone’s theorem for strongly continuous unitary groups on $L^2(\mathcal{A}_0, \tau)$ whenever their generator is a $\tau$-measurable operator.

8. Structure Equations

This section will shed some light on the background of the structure equation $[b, b] + b + b^* = 0$ which appears in Theorem 7.1. As always in this survey, we assume the presence of an $\mathcal{A}_0$-valued white noise $(\mathcal{A}, \tau, S, (A_j)_j)$. Let us first focus on the situation that occurs in our extension of Stone’s Theorem.

**Theorem 8.1.** Let $u$ be a unitary cocycle for $S$ such that its associated semigroup $E_0(u)$ has the generator $d \in L^\infty(\mathcal{A}_0, \tau)$. Then the stochastic derivative $b$ of the unitary cocycle $u$ is an additive cocycle in $L^2(\mathcal{A}, \tau)$ which enjoys the structure equation

$$[b, b] + b + b^* = 0.$$ 

Investigations on Theorem 8.1 turned out that the structure equation comes from a non-commutative fluctuation-dissipation theorem for additive cocycles. Let us define for a given additive $L^p$-cocycle $b$ the flow of additive cocycles

$$\beta: \mathcal{A}_0 \times \mathbb{R}_0^+ \to L^2(\mathcal{A}, \tau)$$

by $\beta_t(x) = [b, xb]_t + xb_t + b_t^* x$.

Now the structure equation can be interpreted as follows. The fluctuation part $[b, xb]_t$ is balanced by the dissipation part $xb_t + b_t^* x$ for $x = 1$. This interpretation is supported by the following result. We remind that $d$ is the generator of the semigroup $E_0(u)$ and $L$ denotes the generator of the semigroup $E_0(Ad u)E_0$. 
Theorem 8.2. Let $u$ be a unitary cocycle for $S$ and $b$ its stochastic derivative as stated in Theorem 8.1. Then it holds for any $x \in A_0$ and $t \geq 0$

$$
\beta_t(x) = L^1 - \lim_{N \to \infty} \sum_{j=0}^{N-1} S_{jt/N}(u_{jt/N}xu_{jt/N} - x) \quad \text{and}
$$

$$
L(x) = E_0(\beta_1(x)) = E_0(b^*_1xb_1) - d^*x + d^*x + xd.
$$

From this theorem one easily concludes the following fixed point properties.

corollary 8.3. For a projection $e \in A_0$ the following are equivalent:

(a) $e$ is a fixed point of the Markov process $(A, \tau, (\text{Ad } u)S, A_0)$

(b) $e$ is a fixed point of the semigroup $E_0(\text{Ad } u)E_0$.

(a') The flow $\beta(e)$ vanishes.

(b') $L(e) = 0$

Let us turn our attention to the case when the semigroup $E_0(u)$ has an unbounded generator $d$ as stated in Theorem 7.3. Looking only at the semigroups $E_0(u)$ and $E_0(\text{Ad } u)E_0$, it is not clear how the domains of the generators $d$ and $L$ are related. But by passing to the level of flows and stochastic derivatives, the domain of $L$ can be explicitly controlled by the domain of $d$. Recall that $D^{(\infty)}(d) \subset L^\infty(A_0, \tau)$.

Theorem 8.4. Let $u$ be a unitary cocycle for $S$ and let $\eta, \xi \in D^{(\infty)}(d)$. Then

$$
\beta_t(\eta^*x\xi) = L^1 - \lim_{N \to \infty} \sum_{j=0}^{N-1} S_{jt/N}((u_{jt/N}\eta)^*xu_{jt/N}\xi - \eta^*x\xi)
$$

exists for any $x \in A_0$ and $t \geq 0$. Moreover,

$$
\beta_t(\eta^*x\xi) = (B_t(\eta)^*x\xi + \eta^*xB_t(\xi) + [B(\eta), xB(\xi)]_t) \quad \text{and}
$$

$$
L(\eta^*x\xi) = E_0((B_1(\eta)^*xB_1(\xi)) - (d\eta)^*xd\xi + (d\eta)^*x\xi + \eta^*xd\xi.
$$

This theorem can be refined to give explicit information on the domain of $L$ on the level of $L^p$-spaces for $1 \leq p < \infty$. Let us illustrate this:

corollary 8.5. If $1 \in D^{(\infty)}(d)$ is an analytic vector for the generator $d$ of the semigroup $E_0(u)$ then

$$
L(x) = E_0(b^*_1xb_1) - d^*x + d^*x + xd
$$

holds for any $x \in A_0$. Moreover, the generator $L$ of the semigroup $E_0(\text{Ad } u)E_0$ maps $\bigcap_{p<\infty} L^p(A_0, \tau)$ into $\bigcap_{p<\infty} L^p(A_0, \tau)$.

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