1. Introduction

Forty years have elapsed since Freund and Rubin [1] opened the way for the study of the seven-dimensional coset compact manifold that occurs in the descent of the eleven-dimensional supergravity to four dimensions.

Can one find traces of this coset manifold in the standard model? To any group-inclined theorist, a compact seven-dimensional manifold is the domain of the continuous group $G_2$, the smallest exceptional Lie group well known as the automorphism group of Cayley numbers.

Spurred by eleven-dimensional physics, manifolds of $G_2$ holonomy have been studied ever since the Freund–Rubin paper. These works concern themselves with continuous $G_2$. While (continuous) $SU_3$ is a subgroup of $G_2$, its discrete subgroups have not received the same attention.

The search for an organizing pattern that explains the tripling of standard model’s chiral families, their mixings and masses initially suggested the continuous family symmetry $SU_3$. The discovery of neutrino oscillations with two large lepton mixing angles pointed to an underlying Yukawa crystallography described by a discrete family symmetry—ergo a plethora of models [2, 3] some with triplet representations of discrete subgroups of $SU_3$.

This author’s research has concentrated on the discrete family symmetries generated by $T_7$, the Frobenius group with 21 elements, and more recently [4] on $T_{13}$, the 39-element Frobenius group. The original physics motivation: both are $SU_3$ subgroups with triplet representations, can unify quark and lepton flavour models.

The mathematical literature, offers an alternate raison d'être for $T_7$ as the maximal subgroup of the simple group $PSL_2(7)$ of order 168.

Similarly, $T_{13} \times Z_2$ is the maximal subgroup of another simple group, $PSL_2(13)$ of order 1092.

* In Memoriam Peter G O Freund.
Interestingly both $PSL_2(7)$ and $PSL_2(13)$ are discrete subgroups of continuous $G_2$; both have seven-dimensional representations embedded in the septet of the Lie algebra $[5] G_2$.

This paper summarizes well-known mathematical facts which may offer a discrete path from the standard model to the Freund–Rubin $G_2$ manifold.

2. Some Frobenius groups and progenitors

We begin with a description of the essentials of the Frobenius groups or order 21 and 39 and 78.

- $T_7 = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$
  This 21-element Frobenius group has two generators which satisfy the presentation
    $\langle S, T | S^3 = T^7 = e, \ ST^2 = S^2 \rangle$
  It contains five conjugacy classes,
    $C_1^{(1)} = \{ e \}$, $C_7^{(2)} = \{ T \}$, $C_7^{(3)} = \{ S^2 TS^2 \}$, $C_7^{(4)} = \{ TS^2 \}$, $C_7^{(5)} = \{ S \}$,
  with (1, 3, 3, 7, 7) elements of order (1, 3, 3, 13, 13), respectively.
  Its five representations break into three singlets representations:
    $1$: $T = S = 1$, $1'$: $T = 1, S = \omega$, $1''$: $T = 1, S = \omega^2$, $\omega^3 = 1$,
  and one complex triplet representation with conjugate:
    $S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $3$: $T = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta^2 & 0 \\ 0 & 0 & \eta^4 \end{pmatrix}$, $\bar{3}$: $T = \begin{pmatrix} \eta^6 & 0 & 0 \\ 0 & \eta^5 & 0 \\ 0 & 0 & \eta^3 \end{pmatrix}$,
  where $\eta^7 = 1$.

- $T_{13} = \mathbb{Z}_{13} \rtimes \mathbb{Z}_3$
  The 39-element Frobenius group has two generators with a similar presentation
    $\langle S, T | S^3 = T^{13} = 1, \ ST^2 = S^3 \rangle$
  It contains seven conjugacy classes and irreps ($n = 0, 1, \ldots, 12$):
    $C_1^{(1)} = \{ e \}$, $C_7^{(2)} = \{ ST^n \}$, $C_7^{(3)} = \{ S^2 T^n \}$, $C_7^{(4)} = \{ T, T^3, T^6 \}$,
    $C_7^{(5)} = \{ T^2, T^5, T^6 \}$, $C_7^{(6)} = \{ T^4, T^{10}, T^{12} \}$, $C_7^{(7)} = \{ T^7, T^8, T^{11} \}$
  with (1, 13, 13, 3, 3, 3, 3) elements of order (1, 3, 3, 13, 13, 13, 13), respectively.
  Its seven representations break into three singlets representations:
    $1$: $T = S = 1$, $1'$: $T = 1, S = \omega$, $1''$: $T = 1, S = \omega^2$, $\omega^3 = 1$,
  and two inequivalent triplet representations and their conjugates ($\rho^{13} = 1$):
    $S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $3_1$: $T = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho^3 & 0 \\ 0 & 0 & \rho^9 \end{pmatrix}$, $\bar{3}_1$: $T = \begin{pmatrix} \rho^{12} & 0 & 0 \\ 0 & \rho^{10} & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}$,
\[ T_2 := \begin{pmatrix} \rho^2 & 0 & 0 \\ 0 & \rho^6 & 0 \\ 0 & 0 & \rho^8 \end{pmatrix}, \quad T_3 := \begin{pmatrix} \rho^{11} & 0 & 0 \\ 0 & \rho^7 & 0 \\ 0 & 0 & \rho^8 \end{pmatrix}. \]

Their embeddings into continuous \( SU_3 \) are straightforward

\[ SU_3 \subset T_7 : \quad \mathbf{3} = 3, \quad \mathbf{8} = 1' + 1' + \bar{3} + \bar{3} \]

and two equivalent embeddings for \( T_{13} \),

\[ SU_3 \supset T_{13} : \begin{cases} \mathbf{3} = 3_1, \quad \mathbf{8} = 1' + 1' + \bar{3}_2 + \bar{3}_2, \\ \mathbf{3} = 3_2, \quad \mathbf{8} = 1' + 1' + 3_1 + 3_1. \end{cases} \]

### 3. Frobenius groups and spin lattices

Textures based on Frobenius groups such as \( T_7 \) and \( T_{13} \) can be viewed as mappings between the world of data (standard model parameters) and a potential \( V(\varphi) \), depending on local familons fields \( \varphi(x, \alpha) \), with only family charges (no gauged quantum numbers). The familon potential is invariant the family texture group.

They are labelled by \( x \) the space-time coordinate, and \( \alpha \) the family symmetry index. From the Freund–Rubin perspective, these two labels should unify into those of an eleven-dimensional manifold.

Familons develop vacuum values which can be mapped backwards to the standard model and explain/predict its experimental consequences. This is the familon’s ‘standard model portal’. One may ask if there exist other portals into different physical systems, with the same family symmetry expressed in terms of different physical variables, without the gauge accoutrement of the Standard Model.

In 1981, it was noted [6] that non-abelian discrete symmetries expressed as semi-direct products of cyclic abelian groups, emerge from spin lattice models with specialized couplings.

Consider a square lattice with each lattice site labelled by \( n \). There sits a ‘spin’ which can assume one of \( p \) values where \( p \) is prime (for the groups discussed above \( p = 7, 13 \)), described by the abelian \( \mathbb{Z}_p \) symmetry, a cyclic group with \( p \) one-dimensional representations with characters,

\[ \chi_r = \exp \left[ \frac{2\pi i}{p} r \right], \quad r = 0, 1, \ldots, p - 1. \]

The state of the spin at each lattice point \( n \) is labelled by the integers \( \sigma_n = \{0, 1, 2, \ldots, p - 1\} \).

Their interactions between two lattice points \( n \) and \( m \), are determined by the Lagrangian,

\[ L(\sigma_n, \sigma_m) = L(\sigma_n - \sigma_m), \]

which preserve global translation invariance over the whole lattice. For simplicity interaction occurs only between nearest neighbours, with the action

\[ S = \sum_{(nm)} L(\sigma_n - \sigma_m). \]

Consider the transformations

\[ \sigma_n \rightarrow \sigma'_n = g(\sigma_n). \]
generated by the automorphism group of $\mathbb{Z}_p$, which maps the spins into themselves at each site. Global invariance is achieved whenever

$$L(\sigma_n - \sigma_m) \rightarrow L(g(\sigma_n) - g(\sigma_m)) = L(\sigma_n - \sigma_m).$$

When $p$ is an odd prime, the automorphism group of $\mathbb{Z}_p$ is $\mathbb{Z}_{p-1}$; Its elements are powers of generator(s) so that

$$g(\sigma_n) = v^m \sigma_n \pmod{p}$$

The other symmetry is the global translation (affine) symmetry under which

$$\sigma_n \rightarrow \sigma_n + \tau \pmod{p}$$

at all lattice sites, where $\tau$ is a fixed element of $\mathbb{Z}_p$. When combined, these two symmetries generate a non-abelian discrete symmetry, the semi-direct product of the two cyclic symmetries.

The generic action is then

$$S(\{y_0, y_1, \ldots, y_{p-1}\}) = \sum_{\sigma_n} \sum_{r=0}^{p-1} y_r \exp \left[ \frac{2\pi i}{p} r(\sigma_n - \sigma_m) \right].$$

The global symmetry acts on the coupling constants $y_0, y_1, \ldots, y_{p-1}$ which values determined by the desired symmetry.

- Consider the case $p = 7$ where the automorphism group is $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$. Our construction yields three non-abelian groups, $\mathbb{T}_7$, $\mathbb{Z}_7 \times \mathbb{Z}_2$, and $\mathbb{Z}_7 \times \mathbb{Z}_3$.

  For $\mathbb{T}_7$, let $v$ generate the homomorphic subgroup $\mathbb{Z}_3$. The general action on the spins at each site is then

  $$\sigma_n \rightarrow v^m \sigma_n + \tau \pmod{7}.$$  

  The integer $v$ is determined by the requirement

  $$v^3 = 1 \pmod{7} \rightarrow \begin{cases} v = 2 : 2^3 = 8 = 1 \pmod{7} \cr v = 4 : 4^3 = 64 = 1 \pmod{7}. \end{cases}$$

  This symmetry is realized only for special values of the coupling constants, determined by the $\mathbb{T}_7$ representations, that is $y_1 = y_2 = y_4$ and/or $y_3 = y_5 = y_6$.

  The same construction with the full automorphism group yields $\mathbb{Z}_7 \times \mathbb{Z}_6 = \mathbb{T}_7 \times \mathbb{Z}_2$ which is realized only if all six coupling constants are equal.

- If we take $p = 13$ the $\mathbb{X}_{13}$ automorphism group is $\mathbb{Z}_{12} = \mathbb{Z}_6 \times \mathbb{Z}_2 = \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The construction of the non-abelian symmetries proceeds in the same way. The action depends on thirteen coupling constants $y_0, y_1, \ldots, y_{12}$.

  $\mathbb{T}_{13}$ is constructed by considering $\mathbb{Z}_3$ as the automorphism group. Its generator $w$ is determined by requiring that

  $$w^3 = 1 \pmod{13} \rightarrow \begin{cases} w = 3 : 3^3 = 27 = 1 \pmod{13} \cr w = 4 : 9^3 = 729 = 1 \pmod{13}. \end{cases}$$

  Global invariance under $\mathbb{T}_{13}$, is obtained whenever the twelve couplings assemble themselves in four groups of triplets corresponding to $\mathbb{3}_1$, $\mathbb{3}_2$ and their conjugates.
By choosing $\mathbb{Z}_6$ as the automorphism group, the same construction realizes the $T_{13} \times \mathbb{Z}_2$ symmetry by grouping the couplings into groups of six with equal values, reflecting its two sextet representations.

4. The road to $G_2$

Searching for a theoretical origin of these Frobenius symmetries is tantamount to finding their progenitor simple groups [7]. We begin with some mathematical factoids:

- $T_7$ is the largest maximal subgroup of $\mathbb{PSL}_2(7)$:

$$\mathbb{PSL}_2(7) \supset T_7.$$ 

- $T_{13}$ is naturally embedded into the 78-element Frobenius group $\mathbb{Z}_{13} \rtimes \mathbb{Z}_6$, itself embedded into $\mathbb{PSL}_2(13)$, the simple group of order 1092:

$$\mathbb{PSL}_2(13) \supset \mathbb{Z}_{13} \rtimes \mathbb{Z}_6 \supset T_{13}.$$ 

These two simple groups have a common feature: real seven-dimensional representations, which suggests that they are embeddable into continuous $G_2$.

This is indeed the case, as shown by various authors [8–10].

A list of $G_2$’s seven irreducible discrete subgroup can be found in [11, 12]; as expected, all have real septet representations. To begin, note that the $G_2$ Kronecker product of two fundamentals

$$7 \times 7 = [7 + 14]_a + [1 + 27]_s$$

contains the septet in its antisymmetric product. Therefore a necessary condition for a good embedding is that the Kronecker product of the subgroup representations must satisfy this requirement, thus limiting the possible embeddings. The rest of the decomposition in the antisymmetric product then expresses the $G_2$ adjoint in terms of the subgroup’s representations.

We now summarize from Evans and Pughs [11] the specific embeddings for each of the seven cases.

- $G_2 \supset \mathbb{PSL}_2(7)$
  Order 168. Irreps: $1, 3, \bar{3}, 6, 7, 8$.
  Kronecker products:

$$7 \times 7 = (7 + 3 + \bar{3} + 8)_a + (1 + 6 + 6 + 7 + 8)_s (1 + 3 + \bar{3})(1 + 3 + \bar{3})$$

$$= (1 + 3 + \bar{3} + 3 + 8)_a + (1 + 6 + 6 + 7 + 8)_s$$

Two embedding:

$$7 = 7, \quad 14 = 3 + 3 + 8 \ 7 = 1 + 3 + \bar{3}, \quad 14 = 3 + 3 + 8$$

- $G_2 \supset \mathbb{PGL}_2(7)$
  Order 336. Irreps: $1, 1_1, 6_1, 6_2, 6_3, 7_1, 7_2, 8_1, 8_2$.
  Kronecker product:

$$7_2 \times 7_2 = (7_2 + \ldots)_a + (1 + \ldots)_s (1_1 + 6_1) \times (1_1 + 6_1)$$

$$= (1_1 + 6_1) + 1_1^t + 6_1 + 8_1)_a + (1 + 6_1 + 6_2 + 6_3 + 8_2)_s$$
Two embeddings:

\[ 7 = 7_2, \quad 14 = 6_1 + 8_1, \quad 7 = 1_1 + 6_1, \quad 14 = \ldots. \]

- \( G_2 \supset PSL_2(7) \times (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \)
  Order 1344. Irreps: \( 1, 3, 6, 7_1, 7_2, 7_3, 8, 14, 21_1, 21_2 \).
  Kronecker product: \( 7_1 \times 7_1 = (7_1 + 14)_o + (1 + 6 + 21)_o \).
  Two embeddings: \( 7 = 7_1 \) and \( 7 = 7_2 \).

- \( G_2 \supset PSL_2(8) \)
  Order 504. Irreps: \( 1, 7_1, 7_2, 7_3, 7_4, 8, 9_1, 9_2, 9_3 \).
  Kronecker product: \( 7_1 \times 7_1 = 7_2 \times 7_2 = (7_1 + 7_2 + 7_3)_o + (1 + 9_1 + 9_2 + 9_3)_o \).
  Two embeddings: \( 7 = 7_1, 14 = 14_1 \) or \( 7 = 7_2, 14 = 14_1 \).

- \( G_2 \supset PSL_2(13) \)
  Order 1092. Irreps: \( 1, 7_1, 7_2, 12_1, 12_2, 12_3, 13, 14_1, 14_2 \).
  Kronecker product:
  \[ 7_1 \times 7_1 = (7_1 + 14)_o + (1 + 13 + 14)_o \],
  \[ 7_2 \times 7_2 = (7_2 + 14)_o + (1 + 13 + 14)_o \],
  Two embeddings: \( 7 = 7_1, 14 = 14_1 \) or \( 7 = 7_2, 14 = 14_1 \).

- \( G_2 \supset PU(3, 3) \)
  Order 6048. Irreps: \( 1, 6, 7_1, 7_2, 7, 14, 21_1, 21_2, 27, 28, 28, 32, 32 \).
  Kronecker product: \( 7 \times 7 = (7 + 14)_o + (1 + 27)_o \).
  One embedding: \( 7 = 7 \).

- \( G_2 \supset G_2(\mathbb{Z}_2) \)
  Order 12096. Irreps: \( 1, 1_1, 1_2, 6, 6, 7_1, 7_2, 14_1, 14_2, 14_1, 21_1, 21_2, 27_1, 27_2, 42, 56, 64 \).
  Kronecker product: \( 7 \times 7 = (7 + 14)_o + (1 + 27)_o \).
  One embedding: \( 7 = 7, 14 = 14 \).

The interested reader can find more information in [11], especially on the McKay graphs which provide a graphical rendition of the Kronecker products.

We mention for completeness the embeddings of \( \mathbb{Z}_{13} \times \mathbb{Z}_6 \), with two real sextuplet representations into \( PSL_2(13) \):

\[ 7_1 = 1 + 6_1 = 1 + 3_1 + 3_1; \quad 7_2 = 1 + 6_2 = 1 + 3_2 + 3_2. \]

It is not clear if any of these mathematical connections will prove relevant to an understanding of the roots of the standard model. It is nevertheless helpful to gather some of the information in on place which perhaps will bear fruit.

5. Englert fluxes and \( PSL_2(7) \)

I have to mention the work of Pietro Fré and collaborators [13] who compactify eleven-dimension supergravity to a four-dimensional space-time with M2 branes. Their Freund–Rubin manifold is generalized to a kind of squashed sphere of the type introduced by Englert [14], where the three form satisfies a complicated equations.

They construct solutions for the three-form on a seven-dimensional manifold modded out by a ‘crystallographic lattice’ which they take to be \( PSL_2(7) \). They find a supersymmetric solution with the Frobenius symmetry \( \widetilde{T} \).

Their work provides yet another example where these non-abelian discrete symmetries arise. In fact they believe that it can shed light on M-theory.
It would be very interesting if their construction generalizes to $T_{13}$, or to any of the finite subgroups of $G_2$ described above.

6. A personal note

I met Peter in person soon after arriving at FermiLab (then NAL). My early recollection was at a seminar by Nambu–Sensei: received in silence by an attentive audience desperately trying to understand the wisdom of the master, until a booming voice arose ‘Yoichiro Yoichiro, no, no, no . . .’, and rushing to the blackboard to explain; I do not remember the rest. My friend Lou Clavelli who had been a student at Chicago told me, this is Peter Freund.

This stentorian voice belonged to an extremely friendly and inquisitive person, who knew an amazing amount of physics, mathematics, and not least stories! A font of knowledge on many fronts.

The last time I saw Peter was at the Nambu memorial in Osaka, where we shared a few days which I very much enjoyed in his civilized company.

I am honoured by writing this paper on one of his fundamental contributions to physics.

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References

[1] Freund P G O and Rubin M A 1980 Phys. Lett. B 97 233–35
[2] Feruglio F and Romanino A 2019 Neutrino flavour symmetries (arXiv:1912.06028 [hep-ph])
[3] Altarelli G and Feruglio F 2010 Rev. Mod. Phys. 82 2701
[4] Jay Pérez M, Rahat M H, Ramond P, Stuart A J and Xu B 2020 Tribimaximal mixing in the $SU_5 \times T_{13}$ texture (arXiv:2001.04019 [hep-ph])
[5] King S F and Luhn C 2013 Rep. Prog. Phys. 76 056201
[6] Altarelli G and Feruglio F 2010 Rev. Mod. Phys. 82 2701
[7] Mohapatra R N et al 2007 Rep. Prog. Phys. 70 1757
[8] King S F and Luhn C 2013 Rep. Prog. Phys. 76 056201
[9] Wilson R et al 2017 ATLAS of finite group representations http://brauer.maths.qmul.ac.uk/Atlas/v3/
[10] Wales D B 1970 Finite linear groups of degree seven II Pac. J. Math. 34 207–35
[11] Cohen A M and Wales D B 1982 Finite subgroup of G2(C) Commun. Algebra 11 441–59
[12] Meurman A (1982) An embedding of PSL(2,13) in $G_2(C)$ Lie Algebras and Related Topics (Lecture Notes in Mathematics) ed D Winter vol 933 (Berlin: Springer)
[13] He Y-H 2003 J. High Energy Phys. JHEP 02(2003)023
[14] Cerchiai B L, Fré P and Trigiante M 2019 The role of PSL(2, 7) in M-Theory: M2 Branes, Englert equations and the septuples (arXiv:1812.11049v2 [hep-th])
[15] Fré P, Grassi P A, Ravera L and Trigiante M 2016 Fortschr. Phys. 64 425–62
[16] Englert F 1982 Phys. Lett. B 119 339–42