Emergent Geometry and Gravity from Matrix Models

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Motivation

- expect quantum structure of space-time at Planck scale due to Gravity ↔ Quantum Mechanics
- fine-tuning problems (cosm. const. etc.)
- "dark matter, dark energy" ... ??

⇒ perhaps gravity is modified?

pre-geometric theory of gravity:
Matrix Models → noncommutative space-time & gravity
Introduction

Outline:

- geometry from matrix models:
  - NC branes
  - effective geometry
  - dynamics
  - examples

- gauge theory point of view

- quantization

- curvature, etc.

review: H.S., arXiv:1003.4134
D. Blaschke, H. S. arXiv:1003.4132, arXiv:1005.0499
Matrix Models

candidate for quantum theory of fundamental interactions

\[ S = - Tr \left( [X^a, X^b][X^{a'}, X^{b'}] \eta_{aa'} \eta_{bb'} + \bar{\Psi} \Gamma^a [X_a, \Psi] \right) \]

\[ X^a \in \text{Mat}(\mathbb{C}), \quad a = 1, \ldots, 10 \]

IKKT Model 1996

- no geometrical pre-requisites, extremely simple
- \{ NC space-time metric (=gravity) \} emerge
- \{ nonabelian gauge fields gravitons \} ... fluctuations of NC space
- well-behaved under quantization
- new perspectives for dark energy / dark matter

Ishibashi, Kawai, Kitazawa and Tsuchiya 1996, ff
Rivelles 2002, Yang 2006, H.S. 2007 ff, ...
Space-time & geometry from matrix models:

\[ \text{e.o.m.: } \delta S = 0 \Rightarrow [X^a, [X^{a'}, X^{b'}]] \eta_{aa'} = 0 \]

solutions:

1. \[ [X^a, X^b] = i \theta^{ab} 1, \quad \text{rank } \theta^{ab} = 2n \]
   separate \( X^a = (X^\mu, \Phi^i), \quad \mu = 1, \ldots, 2n \)
   \[ [X^\mu, X^\nu] = i \theta^{\mu\nu} 1 \quad \Phi^i = 0 \]
   ..."quantum plane" \( \mathbb{R}^{2n}_\theta \)

2. \[ [X^a, X^b] = i \theta^{ab}(X) \sim i \theta^{ab}(x) \quad \text{...generic quantum space} \]
   \( \rightarrow \) space-time as 3+1-dimensional brane solution \( M^4 \subset \mathbb{R}^{10} \)
   \[ X^a = (X^\mu, \Phi^i), \quad \mu = 1, \ldots, 4; \]
   \[ \Phi^i = \Phi^i(X^\mu) \]

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Emergent Geometry and Gravity, from Matrix Models
Space-time & geometry from matrix models:

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   \[ [X^\mu, X^\nu] = i \theta^{\mu\nu} \mathbf{1} \quad \Phi^i = 0 \]
   ..."quantum plane" \( \mathbb{R}^{2n}_\theta \)
2. \[ [X^a, X^b] = i \theta^{ab}(X) \sim i \theta^{ab}(x) \]
   ...generic quantum space
   \[ \rightarrow \text{space-time as 3+1-dimensional brane solution} \quad \mathcal{M}^4 \subset \mathbb{R}^{10} \]
   \[ X^a = (X^\mu, \Phi^i), \quad \mu = 1, ..., 4; \]
   \[ \Phi^i = \Phi^i(X^\mu) \]
1) Moyal-Weyl quantum plane $\mathbb{R}^4_\theta$

\[
[X^\mu, X^\nu] = i \theta^{\mu \nu} 1, \quad \mu, \nu = 1, ..., 4
\]

\[
\phi^i = 0
\]

... Heisenberg algebra, interpreted as space of functions on $\mathbb{R}^4_\theta$

uncertainty relations $\Delta x^\mu \Delta x^\nu \geq |\theta^{\mu \nu}|$

relation with classical $\mathbb{R}^4$:

\[
\mathcal{C}(\mathbb{R}^4) \ni \phi(x) = \int d^4 k \ e^{i k_\mu x^\mu} \iff \int d^4 k \ e^{i k_\mu X^\mu} =: \Phi(X) \in \text{Mat}(\infty, \mathbb{C})
\]

note:

\[
X^\mu \in \text{Mat}(\infty, \mathbb{C}) \quad \ldots \text{quantized coordinate functions on } \mathbb{R}^4_\theta
\]

\[
\Phi(X^\mu) \in \text{Mat}(\infty, \mathbb{C}) \quad \ldots \text{general function on } \mathbb{R}^4_\theta
\]

\[
[X^\mu, \Phi] =: i \theta^{\mu \nu} \partial_\nu \Phi \sim i \theta^{\mu \nu} \partial_\nu \phi(x) \rightarrow \text{NC field theory}
\]
2) Noncommutative spaces and Poisson structure

$\left(\mathcal{M}, \theta^{\mu\nu}(x)\right)$ ... $2n$-dimensional manifold with Poisson structure

Its quantization $\mathcal{M}_\theta$ is NC algebra such that

$$I : C(\mathcal{M}) \rightarrow \mathcal{A} \cong \text{Mat}(\infty, \mathbb{C})$$

$$f(x) \mapsto \hat{f}(X)$$

$$x^i \mapsto X^i, \quad e^{ikx} \mapsto e^{ikX}$$

such that

$$[\hat{f}(X), \hat{g}(X)] = I(i\{f(x), g(x)\}) + O(\theta^2)$$

("nice") $\Phi \in \text{Mat}(\infty, \mathbb{C}) \leftrightarrow$ quantized function on $\mathcal{M}$

Furthermore:

$$(2\pi)^2 \text{Tr}(\phi(X)) \sim \int d^4x \rho(x) \phi(x)$$

$$\rho(x) = \text{Pfaff}(\theta^{-1}_{\mu\nu}) \ldots \text{symplectic volume}$$

(cf. Bohr-Sommerfeld quantization)
Interpretation of $X^a$ in matrix model:

$$X^a = (X^\mu, \phi^i(X^\mu)) : \mathcal{M}^4 \hookrightarrow \mathbb{R}^D \quad \text{...(quantized) embedding function}$$
Effective geometry of NC brane:

consider scalar field coupled to Matrix Model ("test particle")

use \([X^\mu, \phi] \sim i \theta^{\mu\nu}(x) \partial_\nu \phi \quad \Rightarrow \quad \theta^{\mu\nu} = \{x^\mu, x^\nu\}\]

\[
S[\phi] = Tr \left[ X^a, \phi \right] \left[ X^b, \phi \right] \eta_{ab} \quad (U(\mathcal{H}) \text{ gauge inv.})
\]

\[
\sim \int d^4x \sqrt{|G_{\mu\nu}|} \ G^{\mu\nu}(x) \partial_\mu \phi \partial_\nu \phi
\]

\[
G^{\mu\nu}(x) = e^{-\sigma} \theta^{\mu\nu'}(x) \theta^{\nu\nu'}(x) \ g^{\mu\nu'}(x) \quad \text{effective metric (cf. open string m.)}
\]

\[
g_{\mu\nu}(x) = \partial_\mu x^a \partial_\nu x^b \eta_{ab} \quad \text{induced metric on } \mathcal{M}^4_\theta \quad (\text{cf. closed string m.})
\]

\[
e^{-2\sigma} = \frac{|\theta^{-1}_{\mu\nu}|}{|g_{\mu\nu}|}, \quad |G_{\mu\nu}| = |g_{\mu\nu}| \quad \text{for } \dim(\mathcal{M}) = 4
\]

\(\phi\) couples to metric \(G^{\mu\nu}(x)\), determined by \(\theta^{\mu\nu}(x)\) & embedding \(\phi^i(x)\)

same for gauge fields, fermions

... quantized Poisson manifold with metric \((\mathcal{M}, \theta^{\mu\nu}(x), G_{\mu\nu}(x))\)
so: all matter couples to dynamical metric $G_{\mu\nu} \Rightarrow$ effective gravity

however: metric is not fundamental d.o.f.

rather: matrices $X^a$ resp. $(\phi^i, \theta^{\mu\nu})$ resp. $(\phi^i, F_{\mu\nu})$

$\Rightarrow$ dynamics of gravity NOT given by Einstein equations

turns out to be different from GR (long distances!)
may be close enough to observation (?)

note: $D = 10$ just enough to describe most general $g_{\mu\nu}(x)$ (locally)
result:

\((\mathcal{M}, \omega)\) symplectic manifold, \(\omega = \frac{1}{2} \theta^{-1}_{\mu\nu} dx^\mu \wedge dx^\nu\)

\(x^a : \mathcal{M} \hookrightarrow \mathbb{R}^D\) ... embedding in \(\mathbb{R}^D\)

induced metric \(g_{\mu\nu}\) and \(G^{\mu\nu}\) as above. Then:

\[
\{x^a, \{x^b, \varphi}\}\eta_{ab} = e^\sigma \Box_G \varphi \\
\nabla^\mu_G (e^\sigma \theta^{-1}_{\mu\nu}) = G_{\nu\rho} \theta^{\rho\mu} (e^{-\sigma} \partial_\mu \eta + \partial_\mu x^a \Box_G x^b \eta_{ab})
\]

for \(\varphi \in C^\infty(\mathcal{M})\), \(\nabla_G\) ... Levi-Civita, \(\Box_G\) ... Laplace-Op. w.r.t. \(G_{\mu\nu}\), and

\[\eta(x) := \frac{1}{4} e^\sigma G^{\mu\nu} g_{\mu\nu}.\]

(H.S., 2008)
in particular:
matrix e.o.m: \[ [X^a, [X^b, X^{a'}]]\eta_{aa'} = 0 \iff \Delta_G \Phi^i = 0, \Delta_G x^\mu = 0 \]
\[
\nabla^\mu (e^\sigma \theta^{-1}_{\mu \nu}) = e^{-\sigma} G_{\rho \nu} \theta^{\rho \mu} \partial_\mu \eta \\
\eta = \frac{1}{4} e^\sigma G^{\mu \nu} g_{\mu \nu}
\]

... covariant formulation in semi-classical limit

in particular:

\[ \mathcal{M}^4 \hookrightarrow \mathbb{R}^D \] is harmonic embedding (w.r.t. \( G_{\mu \nu} \))
minimal surface
dynamics of NC structure $\theta^{\mu\nu}$:

$$S_{YM} = - \text{Tr}[X^a, X^b][X^a, X^b] \sim \int d^4x \sqrt{g} e^{-\sigma} \eta$$

Euclidean case: at $p \in \mathcal{M}$, diagonalize $g_{\mu\nu} = (1, 1, 1, 1)$ using $SO(4) \rightarrow$ standard form

$$\theta^{\mu\nu} = \theta \begin{pmatrix} 0 & -\alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & \mp \alpha^{-1} & 0 \\ 0 & 0 & \pm \alpha^{-1} & 0 \end{pmatrix}.$$ 

effective metric $G^{\mu\nu} = (\alpha^2, \alpha^2, \alpha^{-2}, \alpha^{-2})$.

Note

$$\frac{1}{4} G^{\mu\nu} g_{\mu\nu} = e^{-\sigma} \eta = \frac{1}{2}(\alpha^2 + \alpha^{-2}) \geq 1$$

$$\star \omega = \pm \omega \iff e^{-\sigma} \eta = 1 \iff G_{\mu\nu} = g_{\mu\nu} \iff S_{YM} \text{ minimal}$$

minimum of $S_{YM} \iff \theta^{\mu\nu}$ (A)SD $\iff G_{\mu\nu} = g_{\mu\nu}$. 

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special class of solutions:

\[
g_{\mu\nu} = G_{\mu\nu}, \\
\Delta G_{\phi^i} = 0, \\
\nabla^\mu \theta^{-1}_{\mu\nu} = 0
\]

holds for (anti)self-dual symplectic structure \( \theta^{-1}_{\mu\nu} \),

\[
\star(\theta^{-1}) = \pm \theta^{-1} \quad \text{Euclidean} \\
\star(\theta^{-1}) = \pm i\theta^{-1} \quad \text{Minkowski (Wick rotation } X^0 \to it \text{ )}
\]

then

\[
S_{MM} \sim Tr[X^a, X^b][X^{\alpha'}, X^{\beta'}] = \int d^4x \sqrt{|g_{\mu\nu}|}
\]

... same structure as vacuum energy, “brane tension”.

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Dynamics of emergent NC gravity

**Effective action**

\[ S = \int d^4 x \sqrt{|g|} \left(-2\Lambda^4 + \Lambda^2_4 R\right) + S_{\text{matter}} \]

leads to

\[
\delta S = \int d^4 x \sqrt{|g|} \delta g_{\mu\nu} \left(-\Lambda^4 g^{\mu\nu} + 8\pi T^{\mu\nu} - \Lambda^2_4 G^{\mu\nu}\right)
\]

\[
= -2 \int \delta \phi^i \partial_\mu (\sqrt{|g|} \left(-\Lambda^4 g^{\mu\nu} + 8\pi T^{\mu\nu} - \Lambda^2_4 G^{\mu\nu}\right)) \partial_\nu \phi^i
\]

since \( g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \phi^i \partial_\nu \phi^i \)

1. "Einstein branch"

\[
\Lambda^4 g^{\mu\nu} + \Lambda^2_4 G^{\mu\nu} = 8\pi T^{\mu\nu}
\]

2. "harmonic branch"

\[
\Lambda^4 \Box g\phi = (8\pi T^{\mu\nu} - \Lambda^2_4 G^{\mu\nu}) \nabla_\mu \partial_\nu \phi
\]

prototype: flat space \( \mathbb{R}^4_\theta \subset \mathbb{R}^{10} \), even for \( \Lambda \gg 0 \)!
1) harmonic branch

... (NC) minimal surfaces $\mathcal{M} \subset \mathbb{R}^D$, deformed by matter

- prototype: flat space $\mathbb{R}^4_\theta \subset \mathbb{R}^{10}$
  insensitive to vacuum energy (minimal surface)!

- interesting “near-realistic “ cosmological solution
  (FRW, big bounce) D. Klammer, H.S. arXiv:0903.0986, PRL 102
  compatible with type Ia supernovae without fine-tuning

- matter $\rightarrow$ deformed ”gravity bags“, Newtonian gravity
  H.S.: arXiv:0909.4621
  (but post-newtonian corrections probably not acceptable)
2) Einstein branch

example: Schwarzschild geometry

embedding $\mathcal{M} \subset \mathbb{R}^7$, asymptotically flat (harmonic), $e^\sigma \to \text{const}$

$$x^a = \begin{pmatrix} t \\ r \cos \sin \\ r \sin \sin \\ r \cos \\ \omega \sqrt{\frac{r_c}{r}} \cos (\omega (t + r)) \\ \omega \sqrt{\frac{r_c}{r}} \sin (\omega (t + r)) \\ \omega \sqrt{\frac{r_c}{r}} \end{pmatrix},$$

with $\eta_{ab} = (-, +, +, +, +, +, -)$.

central singularity: embedding $\hookrightarrow \infty$

(presumably modified e.g. via fuzzy sphere ...)

with complexified SD symplectic form

$\star \theta^{-1} = i \theta^{-1}$, \quad $\theta^{-1} \to \text{const}$ for $r \to \infty$
alternative interpretation of M.M: NC gauge theory

parametrize matrices as fluctuations around $\mathbb{R}^4_\theta$:

\[
X^\mu = \bar{X}^\mu + \bar{\theta}^{\mu\nu} A_\nu, \quad \bar{X}^\mu \text{...Moyal-Weyl}
\]

\[
[X^\mu, X^\nu] = i\bar{\theta}^{\mu\mu'}\bar{\theta}^{\nu\nu'} (\partial_{\nu'} A_{\mu'} - \partial_{\mu'} A_{\nu'} + [A_{\mu'}, A_{\nu'}]) + i\bar{\theta}^{\mu\nu}
\]

\[
F_{\mu\nu} (x) \text{... u(1) field strength}
\]

action:

\[
S_{YM} \sim \int d^4 x (F_{\mu\nu} + i\bar{\theta}^{-1}_{\mu\nu})(F_{\mu'\nu'} + i\bar{\theta}^{-1}_{\mu'\nu'}) \bar{G}^{\mu\mu'} G^{\nu\nu'}
\]

... NC $U(1)$ gauge theory on $\mathbb{R}^4_\theta$

however:

- $U(1)$ sector does not decouple from $SU(n)$ sector, ...
- one-loop: UV/IR mixing, except in $N = 4$ SUSY case: finite (!?)

.... understood in interpretation in terms of emergent gravity.
Quantization

Quantization of matrix model:

\[ Z = \int dX^a d\psi \ e^{-S[X]-S[\psi]} = e^{-S_{\text{eff}}} \]

2 interpretations:

1. **NC SYM on** \( \mathbb{R}^4_\theta \): UV/IR mixing \((U(1) \text{ sector only!})\)
   except for IKKT model \((\mathcal{N} = 4 \text{ SUSY, } D = 10)\): perturb. finite !(?)

2. **U(1)** absorbed in \( \theta^{\mu
u}(x) \rightarrow \text{gravity}, \text{ induced E-H. action} \)

\[ S_{\text{eff}} \sim \int d^4x \sqrt{|G|} \left( \Lambda^4 + c\Lambda^2_4 R[G] + \cdots \right) \]

\((R[G] \text{ due to UV/IR mixing in NC gauge theory})\)

- explanation for UV/IR mixing & **U(1)** entanglement
- \( D = 10 \) required for quantization (maximal SUSY)
Quantization

Quantization of matrix model:

\[ Z = \int dX^a d\psi e^{-S[X] - S[\psi]} = e^{-S_{\text{eff}}}. \]

2 interpretations:

1. **NC SYM on \( \mathbb{R}^4_{\theta} \):** UV/IR mixing \((U(1) \text{ sector only!})\)
   except for IKKT model \((N = 4 \text{ SUSY, } D = 10)\): perturb. finite !(?)

2. **U(1) absorbed in \( \theta^{\mu\nu}(x) \rightarrow \text{gravity} \), induced E-H. action**

\[ S_{\text{eff}} \sim \int d^4x \sqrt{|G|} \left( \Lambda^4 + c\Lambda_4^2 R[G] + \ldots \right) \]

\((R[G] \text{ due to UV/IR mixing in NC gauge theory})\)

- explanation for UV/IR mixing & \(U(1)\) entanglement
- \(D = 10\) required for quantization (maximal SUSY)
**su(n) gauge fields:** same model, new vacuum

\[ \mathbf{Y}^a = \begin{pmatrix} Y^\mu \\ Y^i \end{pmatrix} = \begin{pmatrix} X^\mu \otimes 1_n \\ \phi^i \otimes 1_n \end{pmatrix} \]

include fluctuations:

\[ \mathbf{Y}^a = (1 + A^\rho \partial_\rho) \begin{pmatrix} X^\mu \otimes 1_n \\ \phi^i \otimes 1_n + \Phi^i \end{pmatrix} \]

where

\[ A^\mu = -\theta^{\mu\nu} A_{\nu,\alpha} \otimes \lambda^\alpha, \quad \lambda^\alpha \in \text{su}(n) \]

\[ \Phi^i = \Phi^i_{\alpha} \otimes \lambda^\alpha \]

\[ \Rightarrow \text{effective action:} \]

\[ S_{\text{YM}} = \int d^4x \sqrt{G} e^\sigma G^{\mu\nu} G^{\nu\nu'} \text{tr} F_{\mu\nu} F_{\mu'\nu'} + 2 \int \eta(x) \text{tr} F \wedge F \]

(H.S., JHEP 0712:049 (2007), JHEP 0902:044,(2009) )

... **su(n) Yang-Mills coupled to metric** \( G^{\mu\nu}(x) \)
higher-order terms, curvature

\[ H^{ab} := \frac{1}{2} [[X^a, X^c], [X^b, X_c]] + \]
\[ T^{ab} := H^{ab} - \frac{1}{4} \eta^{ab} H, \quad H := H^{ab} \eta_{ab} = [X^c, X^d][X_c, X_d], \]
\[ \Box X := [X^b, [X_b, X]] \]

result:

for 4-dim. \( M \subset \mathbb{R}^D \) with \( g_{\mu\nu} = G_{\mu\nu} \):

\[ Tr \left( 2 T^{ab} \Box X_a \Box X_b - T^{ab} \Box H_{ab} \right) \sim \frac{2}{(2\pi)^2} \int d^4 x \sqrt{g} e^{2\sigma} R \]
\[ Tr(\{[[X^a, X^c], [X_c, X^b]][X_a, X_b] - 2 \Box X^a \Box X^a \}) \]
\[ \sim \frac{1}{(2\pi)^2} \int d^4 x \sqrt{g} e^{\sigma} \left( \frac{1}{2} e^{-\sigma} \theta^{\mu\eta} \theta^{\rho\alpha} R_{\mu\eta\rho\alpha} - 2R + \partial^\mu \sigma \partial_\mu \sigma \right) \]

(Blaschke, H.S. arXiv:1003.4132)

(cf. Arnlind, Hoppe, Huisken arXiv:1001.2223)

⇒ Einstein-Hilbert-type action for gravity as matrix model
pre-geometric version of (quantum?) gravity, background indep.!
proof: (assume $g = G$)

\[
H^{ab} = \frac{1}{2} [[X^a, X^c], [X^b, X_c]] + \sim -e^\sigma G^{\mu \nu} \partial_\mu x^a \partial_\nu x^b \overset{g=G}{=} e^\sigma P^{ab}_T,
\]

\[
T^{ab} = H^{ab} - \frac{1}{4} \eta^{ab} H \sim e^\sigma P^{ab}_N
\]

$P_N, P_T$ ... projector on normal / tangential bundle vof $M \subset \mathbb{R}^D$. note

\[
R_{\nu \mu \lambda \kappa} = P^{ab}_N (-\partial_\kappa \partial_\nu x_a \partial_\lambda \partial_\mu x_b + \partial_\kappa \partial_\mu x_a \partial_\nu \partial_\lambda x_b)
\]

\[
= -\nabla_\kappa \nabla_\nu x^a \nabla_\lambda \nabla_\mu x_a + \nabla_\kappa \nabla_\mu x^a \nabla_\nu \nabla_\lambda x_a
\]

(i.e. Gauss-Codazzi theorem) and

\[
T^{bc}[X^a, [X_a, T_{bc}]] \sim e^{2\sigma} P^{bc}_N \nabla_\mu \nabla_\nu (e^\sigma \eta_{bc} - e^\sigma \partial^\nu x_b \partial_\nu x_c))
\]

\[
= e^{2\sigma} \left( (D - 4) \Box e^\sigma - 2 P^{bc}_N (e^\sigma \nabla^\mu \partial^\nu x_b \nabla_\mu \partial_\nu x_c) \right)
\]

\[
= e^{2\sigma} \left( (D - 4) \Box e^\sigma - 2 e^\sigma \nabla^\mu \partial^\nu x^a \nabla_\mu \partial_\nu x_a \right)
\]
further comments:

- **generalization for** $g \neq G$:
  still find M.M. terms with purely tensorial meaning $(\mathcal{M}, G_{\mu \nu}, \theta^{\mu \nu})$

  obtain

  $$S_{10} = \int \sqrt{g} e^{2 \sigma} R[G] + \nabla G g \nabla G g + ...$$

  $$\approx \int \sqrt{g} e^{2 \sigma} (R[g] + 3 R^{\mu \nu} h_{\mu \nu}),$$

  $$h_{\mu \nu} = - e^{-1} (\bar{\theta} g F)_{\mu \nu} - e^{-1} (F g \bar{\theta}^{-1})_{\mu \nu} - \frac{1}{2} g_{\mu \nu} (\bar{\theta} F)$$

  $$S_4 = S_{YM} = \int (\sqrt{g} + (F - \ast g F) \ast g (F - \ast g F))$$

  $$\theta^{-1}_{\mu \nu} = \bar{\theta}^{-1} + F_{\mu \nu}, \quad \ast \bar{\theta}^{-1} = \pm \bar{\theta}^{-1}$$

  ... work in progress (with D. Blaschke)

- probably (!?!) $F_{\mu \nu}$ resp. $\theta^{-1}_{\mu \nu}$ should be integrated out

- $\exists$ “extrinsic“ terms $Tr \Box x^a \Box x^a$, depend on embedding $\mathcal{M} \subset \mathbb{R}^D$

  (minimal surfaces preferred ... )
natural (only?) action

\[ S[\Psi] = \text{Tr} \bar{\Psi} \gamma_a [X^a, \Psi] \]
\[ \sim \int d^4 x \rho(x) \bar{\Psi} i \gamma_{\mu} \theta^{\mu\nu}(x) \partial_\nu \Psi, \quad \{ \gamma_{\mu}, \gamma_{\nu} \} = 2G_{\mu\nu} \]

**note:**
- naturally SUSY \(\rightarrow\) IKKT model
- couple to \(G_{\mu\nu}\), but non-standard spin connection (submanifold!)
- quantization induces E-H action plus additional terms

\[ \Gamma_\Psi = \frac{1}{4\pi^2} \int d^4 x \sqrt{|g|} \left( 2\Lambda^4 + \Lambda^2 \left( -\frac{1}{3} R[g] + \frac{1}{4} \partial_\mu \sigma \partial^\mu \sigma \right. \right. \]
\[ + \frac{1}{8} e^{-\sigma} R[g]_{\mu\nu\rho\sigma} \theta^{\mu\nu} \theta^{\rho\sigma} + \frac{1}{4} (\Box_g x^a)(\Box_g x^b) \eta_{ab} \left. \right) + \mathcal{O}(\log \Lambda) \right). \]

- precise matching with UV/IR mixing (checked in \(D = 4\))

(D. Klammer, H.S., arXiv:0901.2322, arXiv:0909.5298)
matrix-model \( Tr[X^a, X^b][X^{a'}, X^{b'}] \eta_{aa'} \eta_{bb'} \)

dynamical NC spaces \(\leftrightarrow\) emergent gravity & gauge thy

not same as G.R., long-distance corrections (extrinsic geometry)

intriguing cosmological solutions,
physics of vacuum energy different from GR

suitable for quantizing gravity!

(IKKT model, \(N = 4\) SUSY in \(D = 4\))

... more work is needed!
Deformations of Moyal-Weyl plane: gravitons

Dynamical \( X^\mu \Rightarrow \) Dynamical \( (\theta^{\mu\nu}(x), G^{\mu\nu}(x)) \)

Parametrize fluctuations

\[
X^\mu = \bar{X}^\mu + \bar{\theta}^{\mu\nu} A^\nu
\]

\[
i \theta^{\mu\nu}(x) \sim [X^\mu, X^\nu] = i \bar{\theta}^{\mu\nu'} (\partial_{\mu'} A_{\nu'} - \partial_{\nu'} A_{\mu'} + [A_{\mu'}, A_{\nu'}]) + i \bar{\theta}^{\mu\nu}
\]

\[
G^{\mu\nu}(x) = \bar{\eta}^{\mu\nu} - h^{\mu\nu} (+O(F^2))
\]

Therefore

\[
h_{\mu\nu} = \bar{\eta}_{\nu\nu'} \bar{\theta}^{\nu'}_{\nu} F_{\rho\mu} + \bar{\eta}_{\mu\mu'} \bar{\theta}^{\mu'}_{\mu} F_{\eta\nu} - \frac{1}{2} \bar{\eta}_{\mu\nu} (\bar{\theta}^{\rho\eta} F_{\rho\eta})
\]

... Linearized metric fluctuation (cf. Rivelles [hep-th/0212262])
e.o.m for fluctuations of Moyal-Weyl plane (linearized):

\[
[X^\mu, [X^\nu, X^{\mu'}]]\eta_{\mu\mu'} = 0
\]
\[
\Rightarrow \partial^\mu F_{\mu\nu} = 0
\]
\[
\Rightarrow R_{\mu\nu}[G] = 0 \quad (\partial^\mu h_{\mu\nu} = 0 \ldots \text{harm. gauge})
\]

cf. Rivelles [hep-th/0212262]

while \( R_{\mu\nu\rho\eta} \neq 0 \) ... nonvanishing curvature

\[
\Rightarrow \text{on-shell d.o.f. of gravitational waves on Minkowski space}
\]

i.e.: trace-\( U(1) \) photons on \( \mathbb{R}^4_\theta \) are actually gravitons

NC \( U(1) \) does not decouple, couples like graviton
Relation with string theory: branes in background $B$ field

brane $\mathcal{M} \subset \mathbb{R}^{10}$ in $B$-field background:

DBI action: $S_{DBI} \sim \int d^4x \sqrt{\det(g_{\mu\nu} + (B_{\mu\nu} + F_{\mu\nu}))}$

where $g_{\mu\nu}$ ... closed-string metric (pull-back from bulk)

$G^{\mu\nu} \sim B^{\mu\mu'} B^{\nu\nu'} g_{\mu'\nu'}$ ... open-string metric on $\mathcal{M}$

($\partial F$ neglected...)

here:

- NO 10-D bulk! fields only live on brane
- $U(1)$ field strength $F$ absorbed in

$$\theta^{-1}_{\mu\nu}(x) = B_{\mu\nu} + F_{\mu\nu}$$

(splitting is unphysical)

eff. metric for nonabelian gauge fields etc. on $\mathcal{M}$:

$$G^{\mu\nu} \sim \theta^{\mu\mu'}(x)\theta^{\nu\nu'}(x)g_{\mu'\nu'}$$
assume: vacuum energy $\Lambda^4 \gg$ energy density $\rho$

$\Rightarrow$ look for harmonic embedding $\Delta x^a = 0$ of FRW metric

$$ds^2 = -dt^2 + a(t)^2(d\chi^2 + \sinh^2(\chi)d\Omega^2),$$

Ansatz

$$x^a(t, \chi, \theta, \varphi) = \begin{pmatrix}
  a(t) \begin{pmatrix}
    \cos \psi(t) \\
    \sin \psi(t)
  \end{pmatrix} \\
  \begin{pmatrix}
    \sinh(\chi) \sin \theta \cos \varphi \\
    \sinh(\chi) \sin \theta \sin \varphi \\
    \sinh(\chi) \cos \theta \\
    \cosh(\chi)
  \end{pmatrix} \\
  0 \\
  x_c(t)
\end{pmatrix} \in \mathbb{R}^{10}$$

(cf. B. Nielsen, JGP 4, (1987))

Evolution $a(t), \psi(t), x_c(t)$ determined by $\Delta x^a = 0$

solution of M.M + leading term $\int d^4x \sqrt{G} \Lambda^4$ in $\Gamma_{1-loop}$
harmonic embedding $\Delta_g x^a = 0$ leads to

analog of Friedmann equations

$$H^2 = \frac{\dot{a}^2}{a^2} = -b^2 a^{-10} + d^2 a^{-8} - \frac{k}{a^2}.$$  
$$\frac{\ddot{a}}{a} = -3d^2 a^{-8} + 4b^2 a^{-10}.$$  

largely independent of detailed matter/energy content as long as $\Lambda^4 \gg \rho$

$k = -1$ (negative spatial curvature) most interesting
Implications:

1) early universe:

- big bounce: $\dot{a} = 0$ for $a = a_{\text{min}} \sim b^{1/4}$
  ($\exists$ bound for energy density $\rho$ vs. vacuum energy $\Lambda^4$)

- inflation-like phase $a(t) \sim t^2$, ends at $a(t_{\text{exit}}) = \sqrt{\frac{4}{3}} \frac{b}{d}$

geometric mechanism (no scalar field required), no fine-tuning
2) late evolution (now): \( \dot{a} \to 1 \)

approaches Milne-like universe \( (k = -1, \text{spatial curvature}) \),

in remarkably good agreement with observation
(age \( 13.8 \cdot 10^9 \text{ yr} \), type Ia supernovae)
different physics for early universe (recombination etc.)

A. Benoit-Levy and G. Chardin, [arXiv:0903.2446]

CMB acoustic peak argued to be at correct scale (?)

no fine-tuning of cosm. const., no need for dark energy!