Nonlinear Coupling of Electromagnetic and Electron Acoustic Waves in Multi-Species Degenerate Astrophysical Plasma

N. L. Shatashvili‡
Andronikashvili Institute of Physics, TSU, Tbilisi 0177, Georgia and
Department of Physics, Faculty of Exact and Natural Sciences,
Ivane Javakhishvili Tbilisi State University (TSU), Tbilisi 0179, Georgia

S. M. Mahajan†
Institute for Fusion Studies, The University of Texas at Austin, Austin, Tx 78712

V. I. Berezhiani‡
Andronikashvili Institute of Physics, TSU, Tbilisi 0177, Georgia and
School of Physics, Free University of Tbilisi, Georgia

Nonlinear wave–coupling is studied in a multi-species degenerate astrophysical plasma consisting of two electron species (at different temperatures): a highly degenerate main component plus a smaller classical relativistic flow immersed in a static neutralizing ion background. It is shown that the high frequency electromagnetic (HF EM) waves, through their strong nonlinear interactions with the electron–acoustic waves (sustained by a multi-electron component (degenerate) plasma surrounding a compact astrophysical object) can scatter to lower frequencies so that the radiation observed faraway will be spectrally shifted downwards. It is also shown that, under definite conditions, the EM waves could settle into stationary Solitonic states. It is expected that the effects of such structures may persist as detectable signatures in forms of modulated micro-pulses in the radiation observed far away from the accreting compact object. Both these effects will advance our abilities to interpret the radiation coming out of the compact objects.

I. INTRODUCTION

In compact astrophysical objects (like neutron stars, white dwarfs (WD), active galactic nuclei (AGN)), the matter density is so high that the average inter-particle distance is considerably smaller than the electron De Broglie wave-length. The electron gas at such high densities must obey Fermi-Dirac statistics $\text{[1]}$. Within the framework of an ideal Fermi gas, several authors have, recently, studied the nonlinear dynamics of such highly degenerate systems. One must also note that the Fermi momentum of electrons $p_{F} = m_{e}c\sqrt{n_{d}/n_{cr}}^{1/3}$ ($n_{cr} = 5.9 \times 10^{29} \text{ cm}^{-3}$) is the normalizing critical number-density (see e.g. $\text{[2, 3]}$ and references therein) that acquires highly relativistic values ($\gg m_{e}c$) if $n_{d} \gg n_{cr}$. For such a dense, degenerate system, the Fermi energy, rather than the intrinsic thermal energy (which could be very small) will dominate the system dynamics. In other words the Fermi temperature is much larger than the thermal temperature, $T \ll T_{F} = m_{e}c^{2}(\gamma_{F} - 1)$ where

$$\gamma_{F} = \left(1 + \left(\frac{m_{e}c}{n_{cr}}\right)^{2}\right)^{1/2}$$

Even for the least compact of the compact systems, the white dwarfs (WDs), the densities exceed $n_{cr}$ necessitating a relativistic treatment for dynamics for the degenerate gas.

Most reported work has concentrated on low frequency (LF) longitudinal plasma mode dynamics in quantum/degenerate magneto-plasmas (see e.g. $\text{[4, 5]}$ and references therein). Since the generation of high density plasma is presumably augmented by production of intense pulses of X– and Gamma–rays, several authors have also investigated modulational interactions of such high frequency (HF) waves with variety of plasma modes $\text{[6–13]}$. It was argued that the observed radiation coming from the compact astrophysical objects (harboring multi species plasma) could carry footprints/information from nonlinear interactions like, for instance, the wave self-modulation and soliton formation.

In the present study, we investigate the nonlinear coupling of the Electromagnetic (EM) and Electron Acoustic Waves (EAW) in a Multi-Species Degenerate Astrophysical Plasma consisting of two different temperature electron species: a highly degenerate main component $(d)$ mixed with a smaller classical relativistic flow $(cl)$ immersed in a static neutralizing ion background. One of the principal aims of this work is to explore, possibly, novel nonlinear wave-coupling and modulational interactions induced by the new physics originating in the contamination by the component cl. Such a composite system of a highly degenerate WD plasma co-existing with a classical hot accreting astrophysical flow $(cl)$ and references therein) is an interesting and unusual state of matter. It is expected that this very combination of $d$ and $cl$ will pertain, for example, during the relativistic jet formation from accretion-induced collapsing White Dwarfs to Black Holes $\text{[14, 15]}$.

Two temperature plasmas have been extensively stud-
ied in the past in the context of electron-sound wave generation in both classical and degenerate/quantum plasmas [19, 30]. Normally the cooler component is a smaller fraction with lower density compared to the hotter component [21, 22]. Nonlinear phenomena in multi-component plasmas can be seriously affected by relativistic temperatures. An example of how a two temperature e-p-i plasma can differ from a single temperature system was worked out in [31] where it was shown that the presence of a minority of cold electrons and ions can lead to the scattering of the pump EM wave into the electron-sound and EM waves, and to the instability of relativistically hot e-p plasma against the LF perturbations. EAW could also exist in two temperature plasmas consisting of a degenerate (fermi temperature could also exist in two temperature plasmas consisting of a degenerate (classical temperature $T_F$) and a classical component ($T_{cl} \ll T_F$). The dispersion properties of EAW can be derived from the linear dispersion relation shown, e.g., in [32], modified by the presence of non-degenerate cold electron species (static ions):

$$1 + 3 \frac{\omega_{cl}^2}{k^2 V_F^2} \left[ 1 - \frac{\omega^2}{2 k V_F} \right] + \frac{\omega_{cl}^2}{\omega^2} = 0,$$  

(1)

where $V_F = p_F/m_e \gamma_F$ is the relevant relativistic Fermi velocity, $\Omega_{ed} = \omega_{cl} / \gamma_F$ ($\omega_{cl} = \sqrt{4 \pi e^2 N_{cl} / m_e}$ and $\gamma_F = \sqrt{1 + R_0^2}$, $R_0 = (N_{cl} / n_{cr})^{1/3}$, and $N_{cl}$ $(N_{0cl})$ is equilibrium lab-frame density of the degenerate (classical) plasma. We have assumed $T_{ed} = 0$.

When Landau-damping for degenerate electrons is neglected, Equation (1), in the limit,

$$k V_{ed} \ll \omega \ll k V_F,$$  

(2)

reduces to the so-called Electron-Sound solution with the frequency

$$\omega = \omega_{cl} \frac{k^2 V_F^2}{3 \Omega_{ed}} \left[ 1 + \frac{k^2 V_F^2}{3 \Omega_{ed}^2} \right]^{-1} = k^2 c_s^2 \left[ 1 + \frac{k^2 V_F^2}{3 \Omega_{ed}^2} \right]^{-1},$$  

(3)

where

$$c_s^2 = \frac{\omega_{cl}^2}{3 \Omega_{ed}^2} V_F^2 = \frac{c^2}{3} \left( \frac{N_{0cl}}{N_{0d}} \right) \frac{R_0^2}{\sqrt{1 + R_0^2}},$$  

(4)

which, due to the condition (2), is satisfied for

$$\frac{N_{0cl}}{N_{0d}} \ll \frac{3}{\sqrt{1 + R_0^2}}.$$  

(5)

We note that in above dispersion relation, the quantum effects (like recoil and pair creation) are neglected [30, 32, 33].

II. MODEL

The nonlinear wave dynamics will be studied for a quasi neutral unmagnetized plasma of an immobile classical ion component $(i)$, and two electron species – the bulk relativistic degenerate $(d)$ electron gas with a density $N_{0d}$ and a small contamination of non-degenerate classical $(cl)$ electrons with density $N_{0cl}$ [12]. Quasi neutrality demands

$$N_{0d} + N_{0cl} = N_{0i} = \frac{N_{0i}}{N_{0d}} = 1 + \alpha, \quad \alpha = \frac{N_{0cl}}{N_{0d}},$$  

(6)

where $\alpha (\ll 1)$ is the fraction of the classical to the degenerate electrons.

To study the nonlinear propagation of intense EM waves, the multicomponent plasma dynamics has to be coupled (see [12]) with the Maxwell equations,

$$\vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \phi - \nabla \varphi, \quad \vec{B} = \nabla \times \vec{A},$$  

(7)

take the form (in the Coulomb Gauge $\nabla \cdot \vec{A} = 0)$:

$$\frac{\partial^2 \vec{A}}{\partial t^2} - c^2 \Delta \vec{A} + \frac{e}{m} \frac{\partial}{\partial t} (\nabla \varphi) - 4 \pi c \vec{J} = 0,$$  

(8)

$$\Delta \varphi = -4 \pi \rho,$$  

(9)

where for charge density (sum over all species) and current density (sum over only the two electron species) we have, respectively:

$$\rho = \sum q N = -e N_d - e N_{cl} + e N_i,$$

$$\vec{J} = \sum q \gamma_{d(cl)} n_{d(cl)} \vec{V}_{d(cl)} = -\frac{e}{m} N_d \vec{P}_d - \frac{e}{m} N_{cl} \vec{P}_{cl}.$$  

(10)

Here $\vec{p}_{d(cl)} = \gamma_{d(cl)} m \vec{V}_{d(cl)}$ is the hydrodynamic momentum, $n_{d(cl)} = N_{d(cl)}/\gamma_{d(cl)}$ is the rest-frame particle density ($N_{d(cl)}$ denotes the laboratory frame density) of the degenerate (classical) electron fluid density, $\vec{V}_{d(cl)}$ is the fluid velocity, and $\gamma_{d(cl)} = (1 + \vec{p}_{d(cl)}^2 / m^2 c^2)^{1/2}$. The fully covariant description of the active fluid species (ions are static) is displayed in its familiar vorticity form, for example, in [34, 35]. Since we will be dealing with a particular class of systems for which the generalized (canonical) vorticities

$$\Omega_{d(cl)} = -\frac{(e/c) B + \nabla \times (G_{d(cl)} \vec{p}_{d(cl)})}{0},$$

the remnant dynamics is contained in (see [36] for details)

$$\frac{\partial}{\partial t} \left( G_{d(cl)} \vec{p}_{d(cl)} - \frac{e}{c} \vec{A} \right) + \nabla \left( m e^2 G_{d(cl)} \gamma_{d(cl)} - e \varphi \right) = 0,$$  

(11)

$$\frac{\partial N_{d(cl)}}{\partial t} + \nabla \cdot (N_{d(cl)} \vec{V}_{d(cl)}) = 0,$$  

(12)

where the ”effective mass factors” $G_d$ and $G_{cl}$ are quite different for the two electron species: $G_d = \omega_d / n_d m_e c^2$ , where $\omega_d$ is an enthalpy per unit volume, originates from degeneracy rather than relativistic kinematics. The
The general expression for enthalpy \( w_d \) for arbitrary density and temperature (for a plasma described by local Dirac-Juttner equilibrium distribution function) can be found in Ref. [37]. For a fully (strongly) degenerate electron plasma, however, this very tedious expression smoothly transfers to the one with just density dependence: \( w_d \equiv w_d(n) \). In fact \( w_d/n_d m_e c^2 = (1 + (R_d)^2)^{1/2} \), where \( R_d \equiv (n_d/n_{cr})^{1/3} \). The effective mass factor, then, is simply determined by the plasma rest frame density, \( G_d = [1 + (n_d/n_{cr})^{2/3}]^{1/2} \) for arbitrary \( n_d/n_{cr} \). For relativistically hot classical plasma an expression for effective mass factor \( G_d \) can be found in Refs. [36, 38]. Note, that the degenerate fluid equation \( (11) \) is valid in the long wave-length limit \( (k \ll 1) \); the characteristic frequencies are much greater than the De Broglie frequency: \( \omega \gg \omega_h = h k^3/2m \) justifying the absence of quantum diffraction phenomenon effects [39].

We will, now, investigate the one-dimensional propagation of circularly polarized EM wave \((\partial_z \neq 0, \partial_t \equiv 0, \partial_y \equiv 0)\) with a mean frequency \( \omega_0 \) and a mean wave number \( k_0 \) in the \( z \) direction:

\[
A_\perp = (\hat{x} + i \hat{y}) A(z, t) \exp(ik_0 z - i\omega_0 t) + \text{c.c.} \tag{13}
\]

where \( A(z, t) \) is a slowly varying function of \( z \) and \( t \) and \( \hat{x} \) and \( \hat{y} \) are the standard unit vectors. The vanishing of the generalized vorticities \((\Omega_{d(cl)} = 0)\) guarantees

\[
P_{\perp d(cl)} = -\frac{e}{e G_{d(cl)}} A_\perp , \tag{14}
\]

where the constant of integration is set equal to zero since particle hydrodynamic momenta are assumed to be zero at the infinity where the field vanishes.

The perpendicular component of Eq. (8), using Eqs. (10)-(14), will give the Equation for transverse motion as follows:

\[
\frac{\partial^2 A_\perp}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 A_\perp}{\partial z^2} + \omega_{cd}^2 \left( \frac{N_d}{N_{d0}} \frac{1}{\gamma_d G_d} + \frac{N_{cl}}{N_{d0}} \frac{1}{\gamma_{cl} G_{cl}} \right) A_\perp = 0 , \tag{15}
\]

where

\[
\omega_{cd} = \sqrt{\frac{4\pi e^2 N_{d0}}{m}} , \quad \gamma_{d(cl)} = \sqrt{1 + \frac{e^2 A_\perp^2}{m^2 c^4 G_{d(cl)}^2}} , \tag{16}
\]

\[
G_d = \left[ 1 + R_0^2 \left( \frac{N_d}{\gamma_d N_{d0}} \right)^{2/3} \right]^{1/2} , \quad G_{cl} = G_{cl} (mc^2/T_{cl}) . \tag{17}
\]

We also need to write the equations for longitudinal motion. This motion is driven by the ponderomotive pressure \( \sim \frac{P_{\perp d(cl)}}{2} \) of high frequency (HF) EM wave. Equations (12) and (13), then, give:

\[
\frac{\partial}{\partial t} n_{d(cl)} + \frac{\partial}{\partial z} \left( \frac{P_{\perp d(cl)}}{m c^2 G_{d(cl)}} \right) = 0 \,
\]

\[
\frac{\partial}{\partial t} \left( G_{d(cl)} P_{\perp d(cl)} \right) + \frac{\partial}{\partial z} \left( mc^2 G_{d(cl)} \gamma_{d(cl)} - e \varphi \right) = 0 \tag{19}
\]

and Eq. (9) reads as:

\[
\Delta \varphi = 4\pi e (N_d + N_{cl} - N_i) . \tag{20}
\]

In what follows we assume that the pump wave is weakly relativistic (\(|A|^2 \ll m^2 c^2\)) leading to the expansion of densities so that \( \delta N_{d(cl)} \ll N_{d0(cl)} \). Straightforward algebra gives the following relation for the degenerate fluid:

\[
\frac{1}{G_d \gamma_d} = \frac{1}{G_{0d}} - \frac{R_0^2}{3 (R_0^2 + 1)^{3/2}} \frac{\delta N_d}{N_{0d}} - \frac{1}{2} \frac{1}{(R_0^2 + 1)^{3/2}} \frac{e^2 A_\perp^2}{m^2 c^4} \left( 1 - \frac{1}{3} \frac{R_0^2}{R_0^2 + 1} \right) \tag{21}
\]

while for the classical non-degenerate electron fraction we obtain:

\[
\frac{1}{G_{cl} \gamma_{cl}} = \frac{1}{G_{0cl}} \left( 1 - \frac{1}{2} \frac{e^2 A_\perp^2}{m^2 c^4 G_{0cl}^2} B_1 + \frac{\delta N_{cl}}{N_{0cl}} B_2 \right) , \tag{22}
\]

where

\[
B_1 = \left( 1 + \frac{1}{2} \frac{G_{cl}'(z_0)}{G_{0cl}'} \right) , \quad B_2 = \frac{G_{0cl}'(z_0)}{G_{0cl}'} \tag{23}
\]

In the preceding equations, the prime denotes the \( z \)-derivative, and the effective mass of the classical electrons species is \((K_2(z)\) and \(K_3(z)\) are modified Bessel functions)

\[
G_{cl} = \frac{K_3(z_{cl})}{K_2(z_{cl})} \quad \text{with} \quad z_{cl} = mc^2/T_{cl} \, , \tag{24}
\]

and

\[
\frac{N_{cl}}{\gamma_{cl}} f(z_{cl}) = \text{const} , \quad \frac{N_{d}}{\gamma_{d}} f(z_{cl}) = N_{0cl} f(z_0) , \tag{25}
\]

\[
f(z_{cl}) = \frac{z_{cl}}{K_2(z_{cl})} \exp - z_{cl} G_{cl}(z_{cl}) . \tag{26}
\]

Using the notations

\[
\frac{\delta N_d}{N_{d0}} = \delta \nu_d , \quad \frac{\delta N_{cl}}{N_{0cl}} = \delta \nu_{cl} , \tag{26}
\]
we may express the set of equations for HF and LF motions in a “simplified” form:

\[
\frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} - c^2 \frac{\partial^2 \mathbf{A}_\perp}{\partial z^2} + \omega^2 \frac{A_{cl}}{G_{cl}} (1 + \delta \nu_d) \mathbf{A}_\perp = 0 ,
\]

(27)

\[
\Omega_d^2 \left[ \delta \nu_d - \frac{1}{2} \left( 1 - \frac{R_0^2}{R_0^2 + 1} \right) \frac{e^2 A_{cl}^2}{m^2 c^4 G_{cl}} \right] \mathbf{A}_\perp = 0 ,
\]

(28)

\[
\alpha (1 + \delta \nu_{cl}) \frac{\omega^2}{\gamma_d G_{cl}} = 0 .
\]

(29)

where

\[
\Omega_d^2 = \frac{\omega^2}{R_0^2 + 1} , \quad \Omega_{cl}^2 = \frac{\omega^2}{G_{cl}} .
\]

(30)

Equation for the HF motion, after some simple algebra, can be simplified to obtain

\[
\frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} - c^2 \frac{\partial^2 \mathbf{A}_\perp}{\partial z^2} + \left( \Omega_d^2 + \alpha \Omega_{cl}^2 \right) \mathbf{A}_\perp = 0 .
\]

(31)

The plasma response is contained in Eqs. \(9\) - \(11\):

\[
\frac{\partial^2 \mathbf{A}_\perp}{\partial z^2} \delta \varphi = 4 \pi e N_{ad} (\delta \nu_d + \alpha \delta \nu_{cl}) ,
\]

(32)

\[
\frac{\partial}{\partial t} \delta \nu_d + \frac{\partial}{\partial z} \left( \frac{\delta \rho_{cl}}{m} \right) = 0 .
\]

(33)

After some tedious but straightforward algebra we derive the following equation for, respectively, the degenerate and non-degenerate classical fluids,

\[
\frac{\partial}{\partial t} \left( \frac{\delta \rho_{cl}}{m} \right) = - \frac{e}{m \sqrt{R_0^2 + 1}} \delta \varphi + \frac{c^2 R_0^2}{3 R_0^2 + 1} \frac{\partial}{\partial z} \delta \nu_d
\]

(34)

\[
\frac{\partial}{\partial t} \left( \frac{\delta \rho_{cl}}{m} \right) = - \frac{e}{m G_{cl}} \frac{\partial}{\partial z} \delta \varphi - c^2 B_2 \frac{\partial}{\partial z} \delta \nu_{cl}
\]

(35)

Introducing \(B_2 = - |B_2|\), using \(33\), Eqs. \(34\) and \(35\) can be rewritten in a somewhat more elegant way,

\[
\frac{\partial^2}{\partial z^2} \delta \nu_d = 0 .
\]

(36)

\[
\frac{\partial}{\partial t} \left[ \frac{\delta \rho_{cl}}{m} \right] = - \frac{e}{m G_{cl}} \frac{\partial}{\partial z} \delta \varphi - c^2 B_2 \frac{\partial}{\partial z} \delta \nu_{cl}
\]

(37)

Introducing

\[
\kappa^2 = \left( 1 - \frac{1}{3 \left( R_0^2 + 1 \right)} \right) , \quad \frac{2}{3} < \kappa^2 < 1
\]

(38)

we finally write the set of equations for coupled LF motion:

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{c^2 R_0^2}{3 R_0^2 + 1} \frac{\partial^2}{\partial z^2} + \frac{\omega^2_d}{\sqrt{R_0^2 + 1}} \right) \delta \nu_d
\]

(39)

\[
\left( \frac{\partial^2}{\partial t^2} - c^2 |B_2| \frac{\partial^2}{\partial z^2} + \frac{\alpha \omega^2_{cl}}{G_{cl}} \right) \delta \nu_{cl} - \frac{c^2 B_1}{2 G_{cl}} \frac{\partial^2}{\partial z^2} \left( \frac{e^2 A_{cl}^2}{m^2 c^4} \right)
\]

(40)

From the definition of classical hot electrons effective mass \(G_{cl}\) we may show that generally \(B_2 < 0\).
Our system of equations actually describes the plasma with two different effective masses of electron species – effective mass of degenerate electrons \( (G_{0d}(N_{0d})) \) is determined by their density while the classical hot electrons’ effective mass is determined by their temperature \( (G_{0c}(T_{0c})) \). Although, the derived system is valid for arbitrary temperatures of the classical component, we will limit ourselves (consistent with our earlier stated goal) to the case when the effective mass of degenerate electrons is considerably larger than the classical fraction – the Fermi energy of degenerate electrons is significantly bigger than the thermal energy of the classical electron fraction. For the \( d \) component, then, the factors \( B_1 \approx 1 \), and \( B_2 \approx 0 \). Consequently, after invoking, the equations describing the LF motion, may be written as:

\[
\left[ \Omega_d^2 \frac{\partial^2}{\partial t^2} - \alpha \omega_{ed}^2 \frac{c^2}{2 \left( R_0^2 + 1 \right)} \frac{\partial^2}{\partial z^2} \left( \frac{e^2 A_e^2}{m^2 c^4} \right) \right] \delta \nu_d = \]

\[
= - \alpha \Omega_d^2 \frac{c^2}{2} \left( 1 - \frac{\kappa^2}{\sqrt{R_0^2 + 1}} \right) \frac{\partial^2}{\partial z^2} \left( \frac{e^2 A_e^2}{m^2 c^4} \right) \quad (41)
\]

and

\[
\delta \nu_d = - \frac{1}{\alpha} \delta \nu_d + \frac{c^2}{2 \alpha \omega_{ed}^2} \frac{\partial^2}{\partial z^2} \left( \frac{e^2 A_e^2}{m^2 c^4} \right ) \quad (42)
\]

In the low frequency regime:

\[
\frac{\partial^2}{\partial t^2} \ll \alpha \omega_{ed}^2 \ll \Omega_d^2 \quad (43)
\]

calculating the combination (from the preceding)

\[
\Omega_d^2 \kappa^2 \delta \nu_d + \alpha \omega_{ed}^2 \delta \nu_d =
\]

\[
= - \omega_{ed}^2 \left( 1 - \frac{\kappa^2}{\sqrt{R_0^2 + 1}} \right) \delta \nu_d + \frac{c^2}{2} \frac{\partial^2}{\partial z^2} \left( \frac{e^2 A_e^2}{m^2 c^4} \right) \quad (44)
\]

we obtain the HF equation for the vector potential in its final form:

\[
\frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} - e^2 \frac{\partial^2 \mathbf{A}_\perp}{\partial z^2} + \left( \Omega_d^2 + \alpha \omega_{ed}^2 \right) \mathbf{A}_\perp
\]

\[
= - \omega_{ed}^2 \left( 1 - \frac{\kappa^2}{\sqrt{R_0^2 + 1}} \right) \delta \nu_d \mathbf{A}_\perp + \frac{c^2}{2} \mathbf{A}_\perp \frac{\partial^2}{\partial z^2} \left( \frac{e^2 A_e^2}{m^2 c^4} \right) -
\]

\[
- \frac{\Omega_d^2 \kappa^2}{2 \left( R_0^2 + 1 \right)} \frac{e^2 A_e^2}{m^2 c^4} \mathbf{A}_\perp - \frac{\alpha \omega_{ed}^2}{2} \frac{e^2 A_e^2}{m^2 c^4} \mathbf{A}_\perp = 0 \quad (45)
\]

By imposing the HF dispersion relation: \( \omega_0^2 = k_0^2 c^2 + \Omega_d^2 + \alpha \omega_{ed}^2 \), the final coupled HF/LF dynamics is expressible as (in terms of the dimensionless \( A = eA/mc^2 \) and for arbitrary level of degeneracy):

\[
2i\omega_0 \left( \frac{\partial}{\partial t} + V_g \frac{\partial}{\partial z} \right) A + \omega_0 V_g \frac{\partial^2 A}{\partial z^2}
\]

\[
+ \omega_{ed}^2 \left( 1 - \frac{\kappa^2}{\sqrt{R_0^2 + 1}} \right) \delta \nu_d A -
\]

\[
+ \omega_{ed}^2 \left( \alpha + \frac{\kappa^2}{(R_0^2 + 1)^{3/2}} \right) |A|^2 A = 0 \quad (46)
\]

where \( V_g \) is the group velocity of HF wave and the electron sound velocity \( c_s \) is defined as

\[
c_s^2 = \alpha \frac{c^2}{3} \frac{R_0^2}{R_0^2 + 1} \quad (48)
\]

From (45), we can read the relativistic electron-sound velocity for a 2-temperature degenerate relativistic plasma, velocity \( c_s \sim \sqrt{\frac{\alpha}{3} c R_0/(R_0^2 + 1)^{1/4}} < c/\sqrt{3} \).

We emphasize here, that the presence of the non-degenerate electron fraction is crucial, without a non zero \( \alpha \), the speed would not be finite. In fact, \( \alpha \gg c_s^2 / L^2 \omega_{ed}^2 \) where \( L \) is the characteristic length of LF mode. It must be emphasized that the presence of a small fraction of non-degenerate electrons is the reason that LF longitudinal waves exist together with the HF–EM waves.

### III. NONLINEAR COUPLING OF HF EM WAVES AND ELECTRON-ACOUSTIC WAVES

The system of Equations (46), (47) together with the relation (45), constitutes the system that we now investigate for a possible modulational instability arising from the interaction of HF–EM pump wave with the LF the electron-sound waves.

We begin by rewriting the relevant equations as

\[
2i\omega_0 \left( \frac{\partial}{\partial t} + V_g \frac{\partial}{\partial z} \right) A + \omega_0 V_g \frac{\partial^2 A}{\partial z^2}
\]

\[
+ \omega_{ed}^2 \left( 1 - \frac{\kappa^2}{\sqrt{R_0^2 + 1}} \right) \delta \nu_d A -
\]

\[
+ \omega_{ed}^2 \left( \alpha + \frac{\kappa^2}{(R_0^2 + 1)^{3/2}} \right) |A|^2 A = 0 \quad (49)
\]
\[(\frac{\partial^2}{\partial t^2} - c_s^2 \frac{\partial^2}{\partial z^2})\delta u_d = -3 c_s^2 b_3 \frac{\partial^2 |A|^2}{\partial z^2}\]  

(50)

with

\[b_1 = \left[1 - \frac{\kappa^2}{\sqrt{R_0^2 + 1}}\right], \quad b_2 = \left[\alpha + \frac{\kappa^2}{(R_0^2 + 1)^{3/2}}\right],\]

\[b_3 = \frac{\sqrt{R_0^2 + 1}}{R_0} b_1\]  

(51)

and making the ansatz\]

\[A(z, t) = a(z, t) = e^{i\theta(z, t)};\]

\[\delta u_d(z, t) = \delta u_d(z, t) \exp[i k z - i \Omega t] + \text{c.c.};\]

\[a = a_0 + \delta a \exp[i k z - i \Omega t] + \text{c.c.};\]

\[\theta = \theta_0 + \delta \theta \exp[i k z - i \Omega t] + \text{c.c.},\]  

(52)

where \(a(z, t)\) and \(\theta(z, t)\) are slowly varying functions of space and time, and \(\delta a \ll a_0\), \(\delta \theta \ll \theta_0\). From the resulting linearized Eqs. [40-51], we obtain the dispersion relation:

\[(\Omega^2 - c_s^2 k^2) \left[(\Omega - V_g k)^2 - \frac{1}{4} V_g' k^2 \left(V_g' k^2 - 2 b_2 \frac{\omega_e^2}{\omega_0} a_0^2\right)\right] = \frac{3}{2} b_1 b_3 V_g' c_s^2 k^4 a_0^2 \frac{\omega_e^2}{\omega_0}.\]  

(53)

For coinciding roots and for small amplitudes (with \(\Omega^2 - c_s^2 k^2 \ll 1\)), we get:

\[(\Omega^2 - c_s^2 k^2) \left[(\Omega - V_g k)^2 - \frac{1}{4} V_g' k^2 \right] = \frac{3}{2} b_1 b_3 V_g' c_s^2 k^4 a_0^2 \frac{\omega_e^2}{\omega_0}.\]  

(54)

which with

\[\Omega = k c_s + i \Gamma, \quad \Omega = V_g k - \frac{1}{2} V_g' k^2 + i \Gamma\]

leads to

\[(\Omega^2 - c_s^2 k^2) \left[(\Omega - V_g k)^2 - \frac{1}{4} V_g' k^2 \right] = 2 k c_s V_g' k^2 \Gamma^2\]  

(55)

implying the increment of decay instability:

\[\Gamma^2 = \frac{3}{4} c_s k b_1 b_3 a_0^2 \frac{\omega_e^2}{\omega_0}.\]  

(56)

The principal message of the preceding calculation is that in the multi-species (degenerate in bulk) plasma it is possible to generate lower frequency electromagnetic waves when high transverse HF EM waves scatter on the longitudinal electron-sound: the corresponding relative frequency shift \(\Delta \omega/\omega_0 \sim c_s/c = \sqrt{\alpha/3 R_0}/(1 + R_0^2)^{1/4}\) being defined by the classical electron fraction \(\alpha = N_{od}/N_{0d}\) as well as the degeneracy level \(R_0(N_{0d})\) of bulk electron species. E.g. for intermediate degeneracy level \(R_0 \sim 1\) we get for such shift to be \(\sim 0.5 \sqrt{\alpha}\).

Now let’s look for the modulational instability assuming \(\Omega \simeq V_g k + i \Gamma, V_g k \gg \Gamma\), for the growth rate of instability we obtain:

\[\Gamma^2 = \frac{1}{4} V_g' k^2 \left[2 \frac{\omega_e^2}{\omega_0} a_0^2 B - V_g' k^2\right],\]

with \(B \equiv b_2 - 3 b_1 b_3 \frac{c_s^2}{V_g^2 - c_s^2}.\)  

(57)

For the modulational instability \(B > 0\); this translates into

\[c_s^2 \left(1 + 3 \frac{b_1 b_3}{b_2}\right) < V_g^2 < c_s^2.\]  

(58)

The maximum growth rate occurs at

\[k_m^2 = \frac{\omega_e^2}{V_g^2} a_0^2 B\]

and is

\[\Gamma_m^2 = \frac{1}{4} V_g' k_m^2 \left[2 \frac{\omega_e^2}{\omega_0} a_0^2 B - V_g' k_m^2\right] = \frac{1}{2} V_g' k_m^2,\]  

(59)

with the explicit expression

\[\Gamma_m = \frac{\omega_e^2}{2} \left(\frac{\omega_e^2}{\omega_0}\right) a_0^2 \left(3 b_1 b_3 \frac{c_s^2}{V_g - c_s}\right).\]  

(60)

We must remind the reader that since our final equations were derived under the condition [43], the wave-amplitude can not be very large:

\[\Gamma_m \ll \sqrt{\alpha/\omega_e} \Rightarrow a_0^2 \ll \sqrt{\alpha/\omega_e} B.\]  

(61)

Let us make some estimates:

(i) For super relativistic degenerate bulk electrons \((R_0 \gg 1)\), we find

\[b_1 = \left(1 - \frac{2}{3R_0}\right) = 1, \quad b_2 = \left(\alpha + \frac{2}{3R_0}\right),\]

\[b_3 = \frac{1}{R_0} \Rightarrow B \equiv \alpha + \frac{3}{R_0}\]
and the amplitude is restricted to:

\[ a_2^2 < \sqrt{a} \frac{\omega_0}{\omega_{ed}} \left( \alpha + \frac{3}{R_0} \right)^{-1} . \]  

(ii) For weakly relativistic degeneracy \( (R_0 \ll 1) \), the relevant results are

\[ b_1 = 1 , \quad b_2 = (1 + \alpha) , \quad b_3 = \frac{1}{R_0} \gg 1 \quad \implies \]

\[ B \cong \frac{3}{R_0} \gg 1 \]

and the amplitude is bounded by

\[ a_2^2 \ll \sqrt{a} R_0^2 \frac{\omega_0}{\omega_{ed}} . \]  

Thus, the addition of even very small amount of non-degenerate classical plasma \( (N_e < 0) \), i.e. \( c_s \neq 0 \) leads to the instability of degenerate e-i plasma against the LF perturbations. Note, that the wave modulation/decay instabilities, demonstrated above, do not exist in an e-i plasma that has a single electron (degenerate) component.

For a better understanding of the character of radiation coming from compact objects, let us explore possible stationary solutions to the system of Equations \[ \text{[46, 17]} \]

Let

\[ A_\perp = A_\perp(\xi, \tau), \quad \delta \nu_d = \delta \nu_d(\xi, \tau) , \]

\[ \xi = z - V_g t, \quad t = \tau, \quad \frac{\partial}{\partial \tau} \ll V_g \frac{\partial}{\partial z}. \]  

Solving \[ \text{[20]} \] yields:

\[ \delta \nu_d = -3b_3 \frac{c_s^2}{V_g^2 - c_s^2} |A|^2 , \]

and using it in \[ \text{[39]} \] gives the so-called Nonlinear Schrödinger (NLS) equation:

\[ 2i\omega_0 \frac{\partial}{\partial \tau} A + \omega_0 V_g \frac{\partial^2 A}{\partial z^2} + \omega_{ed}^2 Q |A|^2 A = 0 , \]

\[ Q = \frac{3c_s^2}{c_s^2 - V_g^2} b_1 b_3 + b_2 . \]

As it is well known the NLS allows a stationary solution at \( Q > 0 \) representing: 1) the subsonic \( (V_g < c_s) \) soliton of rarification (the total density variation \( \delta \nu_d + \delta \nu_h \sim \delta \nu_h < 0 \) since \( \alpha \ll 1 - \text{see Eq. [12]} \)). For the supersonic regime \( \sqrt{1 + \frac{2b_1 b_3}{b_2} c_s < V_g \ll V_F} \) we get the soliton of compression.

**IV. CONCLUSIONS**

High density (Fermi degenerate) plasmas associated with compact objects are generated simultaneously with the production of intense pulses of X– and Gamma-rays. Recently it was argued that these strong HF electromagnetic waves could play a fundamental role in shaping/controlling the nature of the final radiation that will emerge from these compact astrophysical objects. Amongst several possibilities, we study here a subclass of phenomena induced by the interaction of the HF waves with the plasma; the relevant plasma consists of two electron components (in a neutral background of static ions) – the bulk component is high density (degenerate with relativistic Fermi energies/relativistic Fermi temperature \( T_F \)) and is “contaminated” with a much lower density classical (non degenerate) component.

All the new and significant results of this paper stem for our inclusion of the classical component that is relatively low density, and has much smaller temperature \( T \) (thermal) as compared to \( T_F \). Because of the strong nonlinear coupling between the HF EM waves and and Electron Acoustic Waves (a normal mode of the plasma), this “multi-electron” system is found to display a modulation/decay instability, and the HF EM waves can be strongly scattered on the acoustic waves. The main contribution of this work is the demonstration that it is the small classical contamination that is the source of this instability; such an instability will not pertain in a system that has a single active electron component.

Since lower frequency electromagnetic waves can be generated by the scattering of HF EM waves on the electron–acoustic waves, one expects that the electromagnetic spectral range can become quite different from the original; in fact the radiation coming out of the compact object will be driven spectrally downwards.

We also found that the system does allow stationary soliton solutions both in the subsonic and supersonic regimes, respectively, the solitons of rarification and compression. One could expect that the effects of such solitonic structures may persist as detectable signatures in forms of modulated micro-pulses \[ \text{[40]} \] in the radiation far away from the accreting compact object.

It is well known that most of the compact astrophysical objects are immersed in strong magnetic fields. It is highly desirable, therefore, to extend the present study to magnetized multi-component plasmas. The importance of the spin and the quantum diffraction effects must be carefully estimated, and if necessary, must be included in the model. These modifications/improvements of the model will be essential before we undertake parametric / numerical studies so that the model could be compared to real observations on compact astrophysical objects.
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