ON SERIES OF FREE $R$-DIAGONAL OPERATORS

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ABSTRACT. For a series of free $R$-diagonal operators, we prove an analogue of the three series theorem. We show that a series of free $R$-diagonal operators converges almost uniformly if and if two numerical series converge.

1. Introduction

Free probability theory, introduced by Voiculescu, has many similarities with the classic probability theory. In this theory, free independence replaces the usual one. The free central limit theorem was proved by Voiculescu in [13]. Later, other limit laws were studied by different authors. Pata and the first-named author proved that there is one-to-one correspondence of the weak limits between the free case and the classic case in [3] [4]. One might be interested in other types of limit theorems in free probability. In [2], the first-named author proved an analogue of the three series theorem, which characterizes the almost uniform convergence of a series of random variables by the convergence of three numerical series. In this article, we consider series of $R$-diagonal operators. We prove that the uniform convergence of a series of $R$-diagonal operators is equivalent to the convergence of two numerical series. To prove this result, we use the tools introduced recently by Nica and Noyes and the first-named author.

We review some results of $⊞$-convolution in the Section 2. We recall the definition of the $⊞_{RD}$-convolution, the properties of this convolution and prove some relevant theorems in the Section 3. We prove our main theorem in the Section 4.

2. Free independence and the $⊞$-convolution

We use the notation $(\mathcal{M}, \tau)$ to denote a tracial non-commutative $W^*$-probability space, where $\mathcal{M}$ is a von Neumann algebra, and $\tau$ is a faithful normal tracial state. We assume that $\mathcal{M}$ acts on a Hilbert space $H$. The collection of closed, densely defined operators affiliated with $\mathcal{M}$ is a *-algebra, which we denote by $\widetilde{\mathcal{M}}$. An operator in $\widetilde{\mathcal{M}}$ is also called a random variable. If $X$ is a closed densely defined operator on $H$ and $X$ has the polar decomposition $X = u|X|$, then $X$ is affiliated with $\mathcal{M}$ if and only if $u$ and the spectral projections $e_\lambda$ of $|X|$ are in $\mathcal{M}$. If $X$ is a self-adjoint operator, the distribution $\mu_X$ of $X$ is the probability measure on $\mathbb{R}$ so that $\mu_X(\sigma) = \tau(e_X(\sigma))$.

In Voiculescu’s free probability theory, a family subalgebras $(\mathcal{M}_i)_{i \in I}$ of $\mathcal{M}$ is said to be free if, given a natural number $n$ and elements $X_1 \in \mathcal{M}_{i_1}, X_2 \in \mathcal{M}_{i_2}, \cdots,
X_n ∈ M_{n}, such that \( \tau(X_1) = \tau(X_2) = \cdots = \tau(X_n) = 0 \) and \( i_j \neq i_{j+1} \) for \( 1 \leq j < n \), we have \( \tau(X_1X_2 \cdots X_n) = 0 \). Consider operators \( X_i = U_i|X_i| ∈ M \), \( i ∈ I \), the family \( (X_i)_{i ∈ I} \) is said to be \*\)-free if the family \( (A_i)_{i ∈ I} \) is a free family of algebra, where \( A_i \) is the smallest unital von Neumann subalgebra containing \( U_i \) and the spectral projections of \( |X_i| \). The free additive convolution \( ⊞ \) is an operation on probability measures on the real line such that, if \( X_1 \) and \( X_2 \) are free self-adjoint random variables, we have \( μ_{X_1 ⊞ X_2} = μ_{X_1} ⊸ μ_{X_2} \).

Now we review the calculation of the free additive convolution. Denote by \( C \) the complex plane, and \( C^+ = \{ z ∈ C : \Re z > 0 \} \). Given a probability measure \( μ \) on \( R \), the Cauchy transform of \( μ \) is defined as

\[
G_μ(z) = \int_{-∞}^{∞} \frac{dμ(t)}{z - t}, \quad z ∈ C^+.
\]

Given two positive numbers \( α \) and \( β \), let us denote \( Γ_{α,β} = \{ z = x + iy ∈ C^+ : y > α|x|, |z| > β \} \). Set \( F_μ(z) = 1/G_μ(z) \), then \( F_μ^{-1}(z) \) exists in some domain of the form \( Γ_{α,β} \), where \( F_μ^{-1}(z) \) is the right inverse of \( F_μ(z) \) with respect to composition. The Voiculescu transform of \( μ \) is defined as \( ϕ_μ(z) := F_μ^{-1}(z) - z \). Let \( μ_1, μ_2 \) be two probability measures on \( R \), and \( μ_1 ⊸ μ_2 \) be their free additive convolution. A remarkable property of the Voiculescu transform is

\[
ϕ_μ_1 ⊸ μ_2(z) = ϕ_μ_1(z) + ϕ_μ_2(z).
\]

The above equation is valid in some domain of the form \( Γ_{α,β} \). The \( R \)-transform of \( μ \) is an analytic function defined as \( R_μ(z) = ϕ_μ(1/z) \).

Weak convergence of probability measure can be translated in terms of the corresponding \( ϕ \)-functions. The following theorem is from [6].

**Theorem 2.1.** Let \( \{ μ_n \}_{n=1}^{∞} \) be a sequence of probability measures on \( R \). The following assertions are equivalent.

1. \( μ_n → μ \) weakly,
2. There exist \( α, β > 0 \) and a function \( ϕ \) such that \( ϕ_μ_n → ϕ \) uniformly on the compact subsets of \( Γ_{α,β} \).

Moreover, if (1) and (2) are satisfied, we have \( ϕ = ϕ_μ \) in \( Γ_{α,β} \).

Almost uniform convergence is an appropriate replacement of almost sure convergence in classical probability theory. We say that a sequence \( X_n \) in \( M \) converges almost uniformly to zero, if for every \( ε > 0 \), there is a projection \( p ∈ M \) with \( τ(1 - p) < ε \) such that \( ∥X_np∥ → 0 \) as \( n → ∞ \). The measure topology extends the classical concept of convergence in probability. This topology was introduced by Nelson in [9], one may also find an excellent exposition in [12].

**Definition 2.1.** The measure topology of \( M \) (with respect to \( τ \)) is the uniform topology given a neighborhood system \( \{ x + N(ε, δ) : ε, δ > 0 \} \), \( x ∈ M \) where \( N(ε, δ) = \{ a ∈ \}

\( \mathcal{M} : \|ap\| < \varepsilon \) and \( \tau(p^+) \leq \delta \) for some projection \( p \) in \( \mathcal{M} \). We say \( X_n \) converges in measure to \( X \) if \( X_n \) converges to \( X \) in this topology.

It is clear that convergence in measure is weaker than almost uniform convergence. When \( X_n, X \) are self-adjoint random variables, then \( X_n \to X \) in measure implies \( \mu_{X_n} \to \mu_X \) weakly.

3. R-DIAGONAL OPERATORS AND THE \( \boxplus_{RD} \)-CONVOLUTION

The class of \( R \)-diagonal elements in free probability was introduced by Nica and Speicher in [10]. This notion has a natural extension to the unbounded case. Let us recall the following definitions in [7].

**Definition 3.1.**

1. Let \( S, T \in \widetilde{\mathcal{M}} \), we say \( S \) and \( T \) have the same \( * \)-distribution if there exists a trace-preserving \( * \)-isomorphism \( \phi \) from \( W^*(S) \) onto \( W^*(T) \) with \( \phi(S) = T \), where \( \widetilde{\phi} \) is the natural extension of \( \phi \) to unbounded operators.

2. \( T \in \widetilde{\mathcal{M}} \) is said to be \( R \)-diagonal if there exist a von Neumann algebra \( \mathcal{N} \), with a faithful, normal, tracial state and \( * \)-free elements \( U \) and \( H \) in \( \mathcal{N} \), such that \( U \) is Haar unitary, \( H \geq 0 \), and such that \( T \) has the same \( * \)-distribution as \( UH \).

It is well-known that if \( T \in \widetilde{\mathcal{M}} \) is \( R \)-diagonal with \( \ker(T) = 0 \), and the partial decomposition of \( T \) is \( T = u|T| \), then \( u \) is a Haar unitary which is \( * \)-free from \( |T| \). The distribution of an \( R \)-diagonal operator \( T \) is uniquely determined by \( |T| \). Given two \( R \)-diagonal operators \( X, Y \in \widetilde{\mathcal{M}} \), such that \( X \) and \( Y \) are \( * \)-free, then the operator \( X + Y \in \widetilde{\mathcal{M}} \) is also \( R \)-diagonal. In [5], Nica and Noyes and the first-named author introduced a new kind of convolution of probability measures on the positive real line \( \mathbb{R}^+ \), which they denoted by \( \boxplus_{RD} \). The defining property of this new convolution is \( \mu_{Z \boxplus Z} = \mu_{X \boxplus RD \mu_Y} \), if \( X \) and \( Y \) are \( * \)-free \( R \)-diagonal operators and \( Z = X + Y \). We will follow the paper [5] to describe the functional which linearizes \( \boxplus_{RD} \)-convolution.

For convenience, given two positive numbers \( \alpha \) and \( \beta \), we denote \( \Lambda_\alpha = \{ z = x + iy \in \mathbb{C} : x > 0, |y| < \alpha|x| \} \) and \( \Lambda_{\alpha, \beta} = \{ z \in \mathbb{C} : |z| > \beta \}\setminus \Lambda_\alpha \). Let \( \mu \) be a probability measure \( \mu \) on \([0, +\infty)\), then its Cauchy transform \( C_\mu \) is defined in \( \mathbb{C}\setminus[0, +\infty) \) and its Voiculescu transform \( \varphi_\mu \) is defined in some domain of the form \( \Lambda_{\alpha, \beta} \). It is known that for \( z \) in a domain \( \Lambda_{\alpha, \beta} \), we have \( \lim_{|z| \to \infty} \varphi_\mu(z)/z = 0 \). We set

\[
V_\mu(z) = \frac{1}{z} \left( 1 + \frac{1}{z} \varphi_\mu(z) \right),
\]

and let \( W_\mu(z) = 1/V_\mu(z) \), then \( W^{-1}_\mu(z) \) exists in some domain of the form \( \Lambda_{\alpha, \beta} \). The \( \varphi \varphi \)-transform of \( \mu \) is defined by \( \varphi\varphi_\mu(z) = W^{-1}_\mu(z) - z \). This transform linearizes the \( \boxplus_{RD} \)-convolution, namely if \( \mu_1 \) and \( \mu_2 \) are two probability measures on \([0, +\infty)\), then \( \varphi\varphi_{\mu_1 \boxplus RD \mu_2}(z) = \varphi\varphi_{\mu_1}(z) + \varphi\varphi_{\mu_2}(z) \) in some domain of the form \( \Lambda_{\alpha, \beta} \). One might find detailed description of this definition in [5].
The weak convergence of probability measures is equivalent to convergence properties of the corresponding $ϕϕ$-transforms.

**Theorem 3.1.** Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of probability measures on $\mathbb{R}^+$. The following assertions are equivalent.

1. $\mu_n \to \mu$ weakly,
2. There exist two positive numbers $\alpha$, $\beta$, a domain of the form $\Lambda_{\alpha,\beta}$ and a function $ϕϕ$ such that $ϕϕ\mu_n \to ϕϕ$ uniformly on the compact subsets of $\Lambda_{\alpha,\beta}$.

Moreover, if (1) and (2) are satisfied, we have $ϕϕ = ϕϕ\mu$ in $\Lambda_{\alpha,\beta}$ and $\sup_n |ϕϕ\mu_n(z)| = o(z)$ as $z \to \infty$ for $z \in \Lambda_{\alpha,\beta}$.

We would also like to recall some basic combinatorial theory of bounded $R$-diagonal operators. If $X$ is a bounded $R$-diagonal operator, $X = a + bi$, where $a, b$ are self-adjoint operators, then $X$ is a $R$-diagonal operator if and only if the joint $R$-transform of $(X, X^*)$ is a series $R_{X,X^*}$ in two non-commuting indeterminates $z$ and $z^*$ of the form

$$R_{X,X^*}(z, z^*) = \sum_{n=1}^\infty \alpha_n(z z^*)^n + \sum_{n=1}^\infty \alpha_n(z^* z)^n.$$  

In other words, the only non-vanishing cumulants of $(X, X^*)$ are

$$k_{2n}(X, X^*, \cdots, X, X^*) = k_{2n}(X^*, X, \cdots, X^*, X) = \alpha_n, \quad n \in \mathbb{N}.$$  

We denote $f_X(z) := \sum_{n=1}^\infty \alpha_n z^n$. As indicated in [3], we have

$$f_X(z) = \frac{1}{z} ϕϕ_{\nu_{X^*X}} \left( \frac{1}{z} \right).$$

Given a bounded self-adjoint operator $a \in M$, we set $R_a(z) := \sum_{n=1}^\infty k_n(a, \cdots, a) z^n$, where $k_n$ is the $n$-th free cumulant of $a$. When $a$ is the real part of $X$, then the odd moments of $\mu_a$ vanish, therefore $\mu_a$ is a symmetric measure on $\mathbb{R}$. By the linearity property of the free cumulant functions and the property of $R$-diagonal operators mentioned above, we have

$$k_{2n}(a, \cdots, a) = k_{2n} \left( \frac{X + X^*}{2}, \cdots, \frac{X + X^*}{2} \right)$$

$$= \frac{1}{2^{2n}} [k_{2n}(X, X^*, \cdots, X, X^*) + k_{2n}(X^*, X, \cdots, X^*, X)]$$

and $k_{2n+1}(a, \cdots, a) = 0$. Therefore, we have

$$R_a(z) = \sum_{n=1}^\infty k_n(a, \cdots, a) z^n$$

(3.3)  

$$= \sum_{n=1}^\infty 2\alpha_n \left( \frac{z}{2} \right)^{2n} = 2f_X \left( \frac{z}{2} \right)^2$$

$$+ 2f_X \left( \frac{z^*}{2} \right)^2.$$
We recall that the above function is a little different from the \( R \)-transform defined in the Section 2, see [11, 13] for details. In fact, we have \( R_a(z) = z R_{\mu_a}(z) \). From the equations (3.2) and (3.3), we obtain that
\[
\varphi_{\mu_a}(z) = R_{\mu_a} \left( \frac{1}{z} \right) = z R_a \left( \frac{1}{z} \right)
\]
(3.4)
\[
= 2 z f_X \left( \frac{1}{2z} \right)^2
\]
\[
= (2z)^3 \varphi_{\mu_{X^*X}} ((2z)^2)
\]
The above properties of the real part of \( R \)-diagonal operators is also true for unbounded operators.

**Proposition 3.1.** Let \( X \) be a \( R \)-diagonal operator, \( a \) be its real part, then \( \mu_a \) is a symmetric measure on \( \mathbb{R} \). Moreover, we have \( \varphi_{\mu_a}(z) = (2z)^3 \varphi_{\mu_{X^*X}} ((2z)^2) \) in some domain of the form \( \Gamma_{\alpha,\beta} \).

**Proof.** If \( X \) is bounded, the proposition follows from our previous discussion. For the unbounded case, we use the truncated operators \( X_n := X e_{[0,n]} \) to approximate \( X \). Let \( a_n \) be the real part of \( X_n \), then the argument in [6] tells us that the sequence \( \mu_{a_n} \) converges to \( \mu_a \) weakly and the sequence \( \mu_{X^*X_n} \) converges to \( \mu_{X^*X} \) weakly. The result follows from the results of Theorems 2.1 and 3.1. □

Roughly, the function \( \varphi_{\varphi_{\mu}} \) is very close to the function \( \varphi_{\mu} \) for a probability measure \( \mu \) on the positive real line. The following results are from [5].

**Lemma 3.1.** Let \( \{\mu_n\}_{n=1}^{\infty} \) be a sequence of probability measures on \( \mathbb{R}^+ \). If \( \varphi_{\varphi_{\mu_n}}(z) = o(z) \) as \( z \to \infty \) in some domain \( \Lambda_{\alpha,\beta} \), then \( \{\mu_n\}_{n=1}^{\infty} \) is a tight family.

**Theorem 3.2.** For any probability measure \( \mu \) on \( \mathbb{R}^+ \) we have
\[
\varphi_{\varphi_{\mu}}(z) = z^2 [G_{\mu}(z) - 1/z] (1 + o(z))
\]
as \( z \to \infty \) within some domain \( \Lambda_{\alpha,\beta} \). Moreover, this estimate is uniform if \( \mu \) varies in a tight family of measures.

4. **Convergence of series of \( R \)-diagonal operators**

Now we prove our main theorem which is an analogue of the three series theorem of Kolmogorov and Lévy. Note that the condition (4) in Theorem 4.1 below involves only two numerical series, this is due to the circular symmetry of the \( R \)-diagonal operators.

**Theorem 4.1.** Let \( \{X_n\}_{n=1}^{\infty} \) be a free sequence of \( R \)-diagonal operators. The following conditions are equivalent.

1. \( \sum_{n=1}^{\infty} X_n \) converges almost uniformly,
2. \( \sum_{n=1}^{\infty} X_n \) converges in measure,
Let $S_n := \sum_{k=1}^{n} X_k$, then the sequence of probability measures $\{\mu_{S_n^* S_n}\}_{n=1}^{\infty}$ converges weakly.

(4) the following two series converge:

\[ \sum_{n=1}^{\infty} \tau(|X_n|^2 : |X_n| \leq 1), \sum_{n=1}^{\infty} \tau(e_{|X_n|((1, +\infty))}). \]

Proof. Suppose (4) is true, for $R$-diagonal operators $X_n$, we have $\tau(X_n : |X_n| \leq 1) = 0$. (4) ⇒ (1) follows from a result of Batty [1, 8].

(1) ⇒ (2) follows directly from definitions. Assume we have (2), by the properties of convergence in measure given in [9, Theorem 1], the adjoint operation and the joint multiplication are uniformly continuous on bounded sets in the measure topology, thus (2) implies $S_n^* S_n$ converges in measure. Therefore, (3) is true.

Now we concentrate on the proof of (3) ⇒ (4). We have $S_n := \sum_{k=1}^{n} X_k$, and that the sequence of probability measures $\{\mu_{S_n^* S_n}\}_{n=1}^{\infty}$ converges weakly. Let us denote by $\mu_n$ the distribution of $X_n^* X_n$, and denote by $\nu_n$ the distribution of $S_n^* S_n$. We also let $\nu$ be the limit distribution of $\nu_n$. Then by the result in Section 3 we have

\[ \varphi_{\nu_n} = \varphi_{\mu_1} \cdots + \varphi_{\mu_n} \]

in some domain. By Theorem 3.1 there is a domain of the form $\Lambda_{\alpha, \beta}$ where all the functions $\varphi_{\nu_n}, \varphi_{\mu}$ are defined and moreover $\lim_{n \to \infty} \varphi_{\nu_n}(z) = \varphi_{\nu}(z)$ uniformly on the compact subsets of $\Lambda_{\alpha, \beta}$ and $\varphi_{\nu_n}(z) = o(z)$ uniformly as $z \to \infty$ in the domain of the form $\Lambda_{\alpha, \beta}$.

Let $a_n$ be the real part of $X_n$ and $b_n$ be the imaginary part of $X_n$, and set $A_n = \sum_{k=1}^{n} a_k$. The convergence of the functions $\varphi_{\nu_n}, \varphi_{\mu}$ to the function $\varphi_{\nu}$ and Proposition 3.1 imply that the functions $\varphi_{A_n}$ converges to the function $\varphi_{R(S)}$ uniformly on compact subsets of a domain of the form $\Gamma_{\alpha', \beta'}$ for some operator $S$. The main result of [2] implies that $A_n$ converges almost uniformly. This implies that $a_n$ converges to 0 in the measure topology and likewise $b_n$ converges to 0 in the measure topology. Therefore, $X_n^* X_n$ converges to 0 in the measure topology and the probability measures $\{\mu_n\}_{n=1}^{\infty}$ form a tight family. By the estimate in Theorem 3.2 we have

\[ \left| z^2 \left[ G_{\mu_n}(z) - \frac{1}{z} \right] \right| \leq 2|\varphi_{\mu_n}(z)| \]

for $z$ in the domain $\Lambda_{\alpha, \beta}$ which is large enough.

For $z = -y$ with $y > 0$, we have

\[
 z^2 \left[ G_{\mu_n}(z) - \frac{1}{z} \right] = y^2 \int_{0}^{\infty} \left[ \frac{1}{-y - t} - \frac{1}{-y} \right] d\mu_n(t) = \int_{0}^{\infty} \frac{yt}{y+t} d\mu_n(t),
\]

which is always positive. On the other hand, by Proposition 3.1 we have $\varphi_{\mu_n}(z) = (2z)^3 \varphi_{\mu_n}((2z)^2)$ where $\mu_{a_n}$ is the distribution of $a_n$. By the same proposition, $\mu_{a_n}$ is
symmetric, we have $G_{\mu_n}(z) = -\overline{G_{\mu_n}(-z)}$, therefore we obtain

$$\varphi_{\mu_n}(z) = -\overline{G_{\mu_n}(-z)}$$

in the domain where $\varphi_{\mu_n}$ is defined. In particular, $\varphi_{\mu_n}(i\sqrt{y}/2)$ is purely imaginary for $y > 0$. This implies $\varphi(-y)$ is always negative.

We choose $y$ large enough such that when $z = -y$ we have estimate as in (4.2). Moreover, the terms in the right hand side of the equation (4.1) are all negative, and their summation is convergent. Therefore, the series

$$\sum_{1}^{\infty} \int_{0}^{\infty} \frac{yt}{y+t} d\mu_n(t)$$

is also convergent. Now we obtain the inequality,

$$\sum_{n=1}^{\infty} \tau(|X_n|^2 : |X_n| \leq 1) + \sum_{n=1}^{\infty} \tau(e_{|X_n|((1, +\infty))}) \leq (y + 1) \int_{0}^{\infty} \frac{t}{y+t} d\mu_n(t),$$

which implies (4.1). The proof is complete.

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