Weak KAM theory for action minimizing random walks

Kohei Soga

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Abstract
We introduce a class of controlled random walks on a grid in $\mathbb{T}^d$ and investigate global properties of action minimizing random walks for a certain action functional together with Hamilton–Jacobi equations on the grid. This yields an analogue of weak KAM theory, which recovers a part of original weak KAM theory through the hyperbolic scaling limit.

Mathematics Subject Classification 37J50 · 49L25 · 60G50 · 65M06

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1 Introduction

We introduce a class of controlled random walks and investigate action minimizing problems in the class based on the framework of weak KAM theory and Aubry–Mather theory.

1.1 Action minimizing random walk

Let $G_h$ be the grid with a small unit length $0 < h \ll 1$ in $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$, i.e., $G_h = h\mathbb{Z}^d \cap [0, 1]^d$, $h^{-1} \in \mathbb{N}$ (even number) with identification of 0 and 1. Let $\{e_i\}$ be the standard basis of $\mathbb{R}^d$ and set $B = \{\pm e_i\}_{i=1,\ldots,d}$. Consider the jump process from $x \in G_h$ to one of the points $(x + h\omega)_{\omega \in B}$ with a transition probability $\rho(\omega)$. Iteration of the jump process yields a random walk whose paths are given as

$$
\gamma^0 = x \in G_h, \quad \gamma^{k+1} = \gamma^k + h\omega,
$$

where $k$ indicates the number of iteration. We associate $k$ with time in such a way that each jump process takes place within a small unit time $0 < \tau \ll 1$, $\tau^{-1} \in \mathbb{N}$ (even number).

We introduce time-1-periodic inhomogeneous transition probabilities: for a given function $\xi : G_h \times \{\tau k \mid k \in \mathbb{Z}\} \to \mathbb{R}^d$ with $\xi(\cdot, t + 1) = \xi(\cdot, t)$, define

$$
\rho^\pm(x, t; \omega) := \frac{1}{2d} \pm \frac{\tau}{2h} \xi(x, t) \cdot \omega,
$$

where $\rho^+(x, t; \omega)$ (resp. $\rho^-(x, t; \omega)$) stands for the transition probability of the jump from $(x, t)$ to $(x + h\omega, t + \tau)$ (resp. from $(x, t)$ to $(x + h\omega, t - \tau)$). The paths $\gamma^k$ for $\rho^-$ are re-indexed as $\gamma^{-k}$ with $k \geq 0$, since it is a time-backward process. We refer to an example of such a random walk derived from diffusive discretization of an ODE

$$
x'(s) = f(x(s), s), \quad x(0) = x \quad \text{with} \quad f : \mathbb{T}^d \times \mathbb{T} \to \mathbb{T}^d, \quad \text{Lipschitz}.
$$

In fact, one can prove that, if $\xi = f|_{G_h \times \{\tau k \mid k \in \mathbb{Z}\}}$ and $\rho = \rho^+$ (resp. $\rho = \rho^-$), the random walk tends to a solution $x(t) : [0, T] \to \mathbb{T}^d$ (resp. $x(t) : [-T, 0] \to \mathbb{T}^d$) of the ODE as a consequence of the hyperbolic scaling limit, i.e., $h, \tau \to 0$ with $0 < \lambda_0 \leq \tau/h \leq \lambda_1$ (the law of large numbers: see Soga [44]).

Let us relate the random walks with optimal control. We regard each $\xi$ as a control and look for controls that maximize/minimize the cost functionals

$$
L^\rho_+(\xi; v^+) := E \left[ \sum_{0 \leq k < l} -L^{(c)}(\gamma^k, \tau k + \tau, \xi(\gamma^k, \tau k))\tau + v^+(\gamma', \tau l) \right] \quad \text{for } \rho = \rho^+, \tag{1.1}
$$

$$
L^\rho_-(\xi; v^-) := E \left[ \sum_{-l < k \leq 0} L^{(c)}(\gamma^k, \tau k - \tau, \xi(\gamma^k, \tau k))\tau + v^-(\gamma^{-l}, -\tau l) \right] \quad \text{for } \rho = \rho^-, \tag{1.2}
$$

where $E$ stands for the average with respect to the probability measure of the random walk generated by $\xi$ in the above manner; $L^{(c)} := L(x, t, \xi) - c \cdot \xi$ with a constant $c \in \mathbb{R}^d$. 

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and a Tonelli Lagrangian $L : \mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}$; $v^\pm : G_h \times \{\tau k \mid k \in \mathbb{Z}\} \to \mathbb{R}$ with $v^\pm(\cdot, t+1) = v^\pm(\cdot, t)$ are given functions. Later, we will set up the problem more precisely.

In this paper, we find appropriate families of functions $v^\pm$ via Hamilton–Jacobi type equations on the grid and investigate properties of the maximizers $\xi^{**}$ of $\sup_{\xi} L^i_+(\xi; v^+) \text{ and the minimizers } \xi^{*-} \text{ of } \inf_{\xi} L^i_-(\xi; v^-)$ including the asymptotics of the maximizing/minimizing random walks as $l \to \infty$ in the framework of weak KAM theory and Aubry–Mather theory; we also investigate the hyperbolic scaling limit of the issue to recover (a part of) exact weak KAM theory and Aubry–Mather theory. The essential tool of our investigation is the value function of a time-depending Hamilton–Jacobi equation on the grid, which was investigated in Soga [48] from a viewpoint of numerical analysis of viscosity solutions to initial value problems, i.e., we associate $L^i_+(\xi; v^+)$ with the value function of a discrete equation corresponding to

$$v_t + H(x, t, c + v_x) = 0, \quad x \in \mathbb{T}^d, \quad t < 0 \tag{1.3}$$

and $L^i_-(\xi; v^-)$ with that of the discrete equation corresponding to

$$v_t + H(x, t, c + v_x) = 0, \quad x \in \mathbb{T}^d, \quad t > 0, \tag{1.4}$$

where $H$ is the Legendre transform of $L$ in the sense of

$$H(x, t, p) = \sup_{\zeta \in \mathbb{R}^d} (p \cdot \zeta - L(x, t, \zeta));$$

then, we investigate our action maximizing/minimizing problem with $v^\pm$ being time-1-periodic solutions of the discrete Hamilton–Jacobi equations. Hence, the first half of this paper is devoted to prove existence of time-1-periodic discrete solutions. We will see that our investigation of $L^i_+(\xi; v^\pm)$ corresponds to that of the Lax–Oleinik semigroup $T^i_\pm$ and weak KAM solutions $u^\pm$ (see the notation below). Unlike the standard weak KAM theory, our “dynamical system” has a diffusion effect that makes time-forward evolution and time-backward evolution distinct. Therefore, we treat the problem with $L^i_+(\xi; v^+)$ and that with $L^i_-(\xi; v^-)$ independently. Since the investigation with $L^i_+(\xi; v^+)$ is quite parallel to that of $L^i_-(\xi; v^-)$, we omit detailed discussion in this paper.

In Sect. 2, we set up a class of controlled random walks and the corresponding Hamilton–Jacobi equation on a grid. Then, we investigate initial value problems of the Hamilton–Jacobi equation within the time interval $[0, 1]$ through stochastic calculus of variations, to obtain necessary a priori estimates and convergence properties. In Sect. 3 and 4, we find time-1-periodic solutions of the Hamilton–Jacobi equation and investigate action minimizing random walks, to obtain an analogue of weak KAM theory and Aubry–Mather theory.

### 1.2 Weak KAM theory and related literature

Before going into details, we give a short summary of weak KAM theory established by Albert Fathi [17–21] and its analogues recently developed by many researchers, so that readers can get a clear view of the current paper as one of such analogous theories. Briefly speaking, weak KAM theory investigates global properties of the action maximizing/minimizing curves for

$$L^i_+(\gamma; v^+) := -\int_0^t L(c)(\gamma(s), \gamma'(s))ds + v^+(\gamma(t)), \quad \gamma \in AC([0, t]; \mathbb{T}^d), \quad \gamma(0) = x,$$

$$L^i_-(\gamma; v^-) := \int_{-t}^0 L(c)(\gamma(s), \gamma'(s))ds + v^-(\gamma(-t)), \quad \gamma \in AC([-t, 0]; \mathbb{T}^d), \quad \gamma(0) = x.$$
with $v^\pm$ being weak solutions of the stationary Hamilton–Jacobi equation
\[
H(x, c + v_x(x)) = \bar{H} \quad \text{in } \mathbb{T}^d,
\]
where $AC(I; \mathbb{T}^d)$ stands for the family of all absolutely continuous curves $\gamma : I \to \mathbb{T}^d$ and $\bar{H} \in \mathbb{R}$ is some constant. This problem is closely related to one of the central issues of Hamiltonian dynamics, i.e., the issue to investigate structures of the phase space and to understand time global properties of motions. Kolmogorov–Arnold–Moser (KAM) theory provides an excellent framework to show existence of invariant manifolds called KAM tori for nearly integrable Hamiltonian systems, i.e., Hamiltonian systems
\[
x'(s) = H_p(x(s), p(s)), \quad p'(s) = -H_x(x(s), p(s))
\]
generated by Hamiltonians $H(x, p) = H_0(p) + H_1(x, p) : \mathbb{T}^d \times G \to \mathbb{R}$ with $G \subset \mathbb{R}^d$ and $\|H_1\|_{C^0} \ll 1$ (KAM theory is available also for nearly integrable twist maps). After KAM theory was established, lots of efforts have been made to understand the situation where $H_1$ gets larger or Hamiltonian systems far from being integrable. A rigorous answer to this kind of questions was given by Aubry and Le Daeron [2] and Mather [35] for twist maps on the annulus (Aubry–Mather theory), where they found invariant sets called Aubry–Mather sets which can be seen as a generalization of smooth invariant circles obtained in KAM theory. Then, Moser [39] showed that for each smooth twist map there exists a certain Mather sets which can be seen as a generalization of smooth invariant circles obtained in KAM theory.

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\[
L(x, \zeta) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}, \quad \text{strictly convex and superlinear with respect to } \zeta,
\]
which is nowadays called Tonelli Lagrangians. Here, the Euler–Lagrange system is given as
\[
d \overline{d} s L_\zeta(x(s), x'(s)) = L_x(x(s), x'(s)),
\]
which is equivalent to the Hamiltonian system (1.6) generated by the Legendre transform $H$ of $L$. The main tool of Mather’s investigation is the family of action minimizing invariant measures for

**Mather’s minimizing problem:** For each constant $c \in \mathbb{R}^d$, consider
\[
\inf_{\mu} \int_{\mathbb{T}^d \times \mathbb{R}^d} L^{(c)}(x, \zeta) \, d\mu,
\]
where the infimum is taken over all probability measures on $\mathbb{T}^d \times \mathbb{R}^d$ that are invariant under the Euler–Lagrange flow generated by $L$.

A minimizing measure of (1.8) is called a Mather measure, and the closure of the union of the supports of all Mather measures for each fixed $c$ is called the Mather set. Each Mather set provides an invariant set analogous to a KAM torus in the corresponding Hamiltonian system.

Fathi related KAM theory and Aubry–Mather theory to weak solutions of Hamilton–Jacobi equations (1.5). This seems to be quite natural, since existence of a KAM torus implies the existence of $c \in \mathbb{R}^n$, $\bar{H} \in \mathbb{R}$ and $v \in C^2(\mathbb{T}^d; \mathbb{R})$ that satisfy (1.5) (each KAM torus is a Lagrangian submanifold and it is given by the graph of the derivative of $v$). More generally with an arbitrary $C^2$-Hamiltonian $H$, if $c \in \mathbb{R}^d$ and $v \in C^2(\mathbb{T}^d; \mathbb{R})$ have the property that
\[
\text{graph}(c + v_x) := \{(x, c + v_x(x)) \mid x \in \mathbb{T}^d \}
\]
is invariant under the Hamiltonian flow $\phi^t_H$ of (1.6), then, $v$ satisfies (1.5) with some constant $\bar{H} \in \mathbb{R}$; conversely, if there exists a $C^2$-solution $v$ of (1.5), then, graph$(c + v,x)$ is invariant under $\phi^t_H$. Unfortunately, one cannot always expect a smooth solution of (1.5). Lions et al. [32] analyzed (1.5) in the class of viscosity solutions, where they proved that for each $c \in \mathbb{R}^d$ there exists a unique number $\bar{H}(c) \in \mathbb{R}$ such that (1.5) admits at least one viscosity solution only when $\bar{H} = \bar{H}(c)$. This result is based on methods of viscosity solutions without any motivation from dynamical systems. Fathi investigated (1.5) by means of the Lax–Oleinik semigroup $\{T^t_\pm\}_{t \geq 0}$ defined in $C^0(\mathbb{T}^n; \mathbb{R})$ to itself as

$$T^t_+ u(x) := \sup_{\gamma \in AC([0,t]; \mathbb{T}^d), \gamma(0) = x} \mathcal{L}^t_+ (\gamma; u), \quad \text{ (1.9)}$$

$$T^t_- u(x) := \inf_{\gamma \in AC([-t,0]; \mathbb{T}^d), \gamma(0) = x} \mathcal{L}^t_- (\gamma; u). \quad \text{ (1.10)}$$

Tonelli's calculus of variations guarantees that maximizing/minimizing curves for the right hand side of (1.9) and (1.10) exist, and both of which are $C^2$-solutions of the Euler–Lagrange system (1.7) for any $c \in \mathbb{R}^d$. Hence, the results from $T^t_\pm$ are closely related to each other. The key fact of weak KAM theory is the existence of a constant $\bar{H}(c) \in \mathbb{R}$ and $u_\pm \in C^0(\mathbb{T}^n; \mathbb{R})$ such that

$$T^t_+ u_+ - \bar{H}(c)t = u_+, \quad T^t_- u_- + \bar{H}(c)t = u_- \quad \text{for all } t \geq 0,$$

where $u_\pm$ are called weak KAM solutions and $\bar{H}(c)$ the effective Hamiltonian (there are other names of $\bar{H}(c)$). Here are some important consequences [21]:

- $u_+$ is semiconvex, and an a.e. solution of (1.5) with $\bar{H} = \bar{H}(c)$. 
- $u_-$ is semiconcave, and a viscosity solution of (1.5) with $\bar{H} = \bar{H}(c)$. 
- graph$(c + \frac{\partial}{\partial x} u_+)$ is invariant under $\phi^t_H$ for $s \geq 0$. 
- graph$(c + \frac{\partial}{\partial x} u_-)$ is invariant under $\phi^t_H$ for $s \leq 0$. 
- There exists $\mathcal{M}^*(u_\pm) \subset$ graph$(c + \frac{\partial}{\partial x} u_\pm)$ that is invariant under $\phi^t_H$ for $s \in \mathbb{R}$ and contains the Mather set for $c$. 
- $-\bar{H}(c)$ coincides with the infimum (1.8). 
- The “recurrence rate” of a minimizing curve $\gamma^*$ of $T^t_+ u_-$ extended to $- (\infty, 0]$ yields a Mather measure, i.e., roughly speaking, the probability measure of $\mathbb{T}^d \times \mathbb{R}^d$ defined as

$$\text{meas}[A] := \lim_{k \to \infty} \frac{\text{length}\{s \in [-t_k, 0] \mid (\gamma^*(s), \gamma^*'(s)) \in A\}}{t_k}, \quad A \subset \mathbb{T}^d \times \mathbb{R}^d,$$

where $(t_k)_{k \in \mathbb{N}}$ is a sequence tending to $+ \infty$, is a Mather measure (the same holds for an extended maximizing curve of $T^t_+ u_+$).

Weak KAM theory for twist maps was given by E [12]. We refer also to the works by Evans and Gomes [15,16] and Fathi and Siconolfi [23] for further ideas and methods in weak KAM theory.

As an analogous problem, consider the action maximizing/minimizing curves for

$$\mathcal{L}^t_+ (\gamma; v^+) := - \int_0^t e^{-\varepsilon s} L^{(c)}(\gamma(s), \gamma'(s))ds + e^{-\varepsilon t} v^+(\gamma(t)), \quad \gamma \in AC([0, t]; \mathbb{T}^d), \quad \gamma(0) = x,$$

$$\mathcal{L}^t_- (\gamma; v^-) := \int_{-t}^0 e^{\varepsilon s} L^{(c)}(\gamma(s), \gamma'(s))ds + e^{-\varepsilon t} v^-(\gamma(-t)), \quad \gamma \in AC([-t, 0]; \mathbb{T}^d), \quad \gamma(0) = x,$$
where $\varepsilon > 0$, the discount factor, gives a dissipation effect. Then, we have an analogue of weak KAM theory with $v^\pm$ being weak solutions of the discounted Hamilton–Jacobi equation
\[
\varepsilon v^\varepsilon(x) + H(x, c + v^\varepsilon_x(x)) = 0 \quad \text{in } \mathbb{T}^d.
\] (1.11)
The corresponding dynamical system is the discounted Euler–Lagrange system
\[
\frac{d}{ds} \left( L_\varepsilon(x(s), x'(s)) \right) = L_\varepsilon(x(s), x'(s)) - \varepsilon L_\varepsilon(x(s), x'(s)) + \varepsilon c,
\] (1.12)
which is equivalent to the discounted Hamiltonian system
\[
\begin{cases}
    x'(s) = H_p(x(s), p(s)), \\
    p'(s) = -H_\varepsilon(x(s), p(s)) + \varepsilon c - \varepsilon p(s),
\end{cases}
\] (1.13)
where both of maximizing/minimizing curves for $\mathcal{L}_\varepsilon^i$ are $C^2$-solutions of (1.12). We refer to Marò and Sorrentino [34] and Mitake and Soga [37] for Aubry–Mather theory and weak KAM theory for the discounted problems; Gomes [26], Iturriaga and Sanchez Morgado [30], Davini et al. [10, 11] for analysis of the selection problem in the vanishing discount process of (1.11) based on weak KAM theory, i.e., the problem whether or not the whole sequence $\{v^\varepsilon\}_{\varepsilon>0}$ converges to some weak KAM solution as $\varepsilon \to 0+$ (convergence up to subsequence is well-known and is already used in [32]); Mitake and Tran [38] for a result similar to [10] on the selection problem for degenerate viscous Hamilton–Jacobi equations based on a PDE approach called the nonlinear adjoint method introduced by Evans [14]. Recently, Wang et al. [50–52] developed weak KAM theory for contact Lagrangian/Hamiltonian dynamics and contact Hamilton–Jacobi equations, which include discounted problems as a particular case. The contact problem arises from the action maximizing/minimizing curves for some implicitly given action functionals. The selection problem in the vanishing contact process was studied in Chen et al. [9].

Related to smooth approximation, an analogue of weak KAM theory has been developed for viscous Hamilton–Jacobi equations. The first result in such a direction was provided by Moser [40] (though it is not about viscous Hamilton–Jacobi equations), where he showed smooth approximation of Aubry–Mather sets by a regularization technique. After weak KAM theory was announced, Jauslin et al. [31] demonstrated smooth approximation of $\text{graph}(c + \frac{\partial}{\partial s} u_-)$ in the context of weak KAM theory for twist maps through the vanishing viscosity method for the forced Burgers equations (they are equivalent to Hamilton–Jacobi equations in 1-dimensional space), via a PDE approach. Furthermore, they made the first attempt to solve the selection problem in the vanishing viscosity process. The regularized problems with artificial viscosities can be treated also in terms of stochastic optimal control, based on the pioneering work by Fleming [24]: consider the action maximizing/minimizing controls for $\mathcal{L}_\varepsilon^i(\xi; v^\pm) := E \left[ -\int_0^t L^{(c)}(\gamma(s), s, \xi(\gamma(s), s))ds + v^+(\gamma(t), t) \right]$

\[
d\gamma(s) = \xi(\gamma(s), s)ds + \sqrt{2\varepsilon}dW(s), \quad \gamma(0) = x, \quad 0 \leq s \leq t
\]
and
\[
\mathcal{L}_\varepsilon^i(\xi; v^-) := E \left[ \int_{-t}^0 L^{(c)}(\gamma(s), s, \xi(\gamma(s), s))ds + v^-(\gamma(-t), -t) \right],
\]

\[
d\gamma(s) = \xi(\gamma(s), s)ds + \sqrt{2\varepsilon}dW(|s|), \quad \gamma(0) = x, \quad -t \leq s \leq 0\text{(backward sense)},
\]
where $\xi \in C^1(\mathbb{T}^d \times \mathbb{T}; \mathbb{R}^d)$ are controls, $W$ is the standard Brownian motion and $E[\cdot]$ stands for the expectation with respect to the Wiener measure; $\mathcal{L}_\varepsilon^i, \mathcal{L}_\varepsilon^i$ involve the viscous...
Hamilton–Jacobi equations
\[
\begin{align*}
v^v_t + H(x, t, c + v^v_x) &= -v\Delta v^v, \quad x \in \mathbb{T}^d, \ t < 0, \\
v^v_t + H(x, t, c + v^v_x) &= v\Delta v^v, \quad x \in \mathbb{T}^d, \ t > 0,
\end{align*}
\]
respectively, with \( v > 0 \). There are many results based on the above \( L^\pm \) and ideas of weak KAM theory: we refer to Gomes [25] and Iturriaga and Sanchez Morgado [29] for analysis of \( H(x, c + v^v_x) = \bar{H}^v(c) + v\Delta v^v \) in \( \mathbb{T}^d \) and a stochastic analogue of Mather measures; Bessi [4] and Anantharaman et al. [1] for partial answers to the selection problem in the vanishing viscosity process as a generalization of Jauslin et al. [31].

The problem of action maximizing/minimizing random walks for (1.1) and (1.2) with \( d = 1 \) was partially studied in Soga [45,46]. Then, Soga [47] applied the results in [45,46] to an investigation of the selection problem in the limit process of finite difference approximation, where the result is apparently similar to that of Bessi [4], but there is a crucial difference due to the finite propagation speed of random walks under the hyperbolic scaling.

In addition to the above mentioned analogues of weak KAM theory, we refer to Bernard and Buffoni [3] and Zavidovique [53] for weak KAM like formulation of abstract functional equations; Evans [13] for a quantum analogue of weak KAM theory; Bessi [5] for an Aubry–Mather theory approach to the Vlasov equation.

The results of this paper can be seen also as numerical methods of weak KAM theory and Hamilton–Jacobi equations. In fact, we construct analogous objects of exact weak KAM solutions, calibrated curves, effective Hamiltonians, Mather measures, etc., which tend to the exact ones at the hyperbolic scaling limit. Let us relate such results to the literature of numerical analysis of Hamilton–Jacobi equations and weak KAM theory. Crandall and Lions [8] showed an abstract result that monotone finite difference schemes for Hamilton–Jacobi equations yield viscosity solutions of initial value problems, which was generalized by Souganidis [49]. Verification of the monotonicity of a scheme under consideration is highly non-trivial, also for Tonelli Hamiltonians. Soga [45] developed mathematical analysis of the Lax–Friedrichs finite difference scheme applied to hyperbolic scalar conservation laws and the corresponding Hamilton–Jacobi equations in the 1-dimensional setting through techniques of optimal control theory, and extended the classical results by Oleinik [42] to obtain monotonicity and convergence within an arbitrary time interval. Then, Soga [46] showed existence of time-1-periodic discrete solutions, which corresponds to weak KAM solutions \( u_- \). Soga [48] generalized the work [45] to problems with a multi space dimension.

The current paper discusses existence of time-1-periodic discrete solutions based on [48]. We mention that there is a big literature on numerical analysis or computational observation of weak KAM theory based on other techniques to approximate a specific object such as viscosity solutions, effective Hamiltonians, e.g., Gomes and Oberman [27], Rorro [43], Nishida and Soga [41], Bouillard et al. [6]. In contrast to these works, the main feature of our current investigation is that we give a framework that produces analogues of viscosity solutions, their first-order derivatives, their characteristic curves, etc., all at once; there are explicit equations for these objects, for which one can find a structure similar to exact weak KAM theory; these objects and the structure are rigorously convergent.

## 2 Random walk and Hamilton–Jacobi equation on grid

We set up a class of controlled random walks and Hamilton–Jacobi equations on a grid. Then, based on the stochastic and variational approach [48], we analyze the initial value problems
of the equations within the time interval $[0, 1]$ to obtain the solution maps, i.e., the time-1 maps with good a priori estimates and convergence properties.

The function $L$ is assumed to satisfy the following (L1)–(L4):

\begin{enumerate}[(L1)]
  \item \(L(x, t, \zeta) : \mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}, C^2\),
  \item \(L_{\zeta \zeta}(x, t, \zeta)\) is positive definite in \(\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d\),
  \item \(L\) is uniformly superlinear with respect to \(\zeta\), i.e., for each \(a \geq 0\) there exists \(b_1(a) \in \mathbb{R}\) such that \(L(x, t, \zeta) \geq a|\zeta| + b_1(a)\) in \(\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d\),
  \item \(\exists \alpha > 0\) such that \(|L_{\zeta \zeta}(x, t, \zeta)| \leq \alpha (1 + |L|)\) in \(\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d\) for \(j = 1, \ldots, d\),
  \item \(\exists L_{\zeta \zeta} \), i.e., for each \(a \geq 0\) there exists \(b_1(a) \in \mathbb{R}\) such that \(L(x, t, \zeta) \geq a|\zeta| + b_1(a)\) in \(\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d\).
\end{enumerate}

The function \(H\) defined by

\[H(x, t, p) = \sup_{\zeta \in \mathbb{R}^d} \{ p \cdot \zeta - L(x, t, \zeta) \}\]

with the properties:

\begin{enumerate}[(H1)]
  \item \(H(x, t, p) : \mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}, C^2\),
  \item \(H_{pp}(x, t, p)\) is positive definite in \(\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d\),
  \item \(H\) is uniformly superlinear with respect to \(p\), i.e., for each \(a \geq 0\) there exists \(b_2(a) \in \mathbb{R}\) such that \(H(x, t, p) \geq a|p| + b_2(a)\) in \(\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d\),
\end{enumerate}

For each \(c \in \mathbb{R}^d\), let \(L^{(c)}(x, t, c + p)\) be the Legendre transform of \(H(x, t, c + p)\) with respect to \(p\), i.e.,

\[L^{(c)}(x, t, \zeta) = H(x, t, \zeta) - c \cdot \zeta.\]

Throughout this paper, the dependence on a variable in \(\mathbb{T}\) is regarded as the dependence on a variable in \(\mathbb{R}\) with 1-periodicity.

### 2.1 Controlled random walk

Let \(\delta = (h, \tau)\) be a pair of a unit small length of space and a unit small time. We choose 

\[h = (2N)^{-1}, \tau = (2K)^{-1}\]

with \(N, K \in \mathbb{N}\), so that our grid can always contain 1. Introduce the following notation:

\[x_m := (hm^1, \ldots, hm^d), \ t_k := k \tau \text{ for } m = (m^1, \ldots, m^d) \in \mathbb{Z}^d, k \in \mathbb{Z},\]

\[
G_{\text{even}} := \{x_m \mid m \in \mathbb{Z}^d, \ m^1 + \cdots + m^d = \text{even}\},
\]

\[
G_{\text{odd}} := \{x_m \mid m \in \mathbb{Z}^d, \ m^1 + \cdots + m^d = \text{odd}\},
\]

\[
\tilde{G}_{\delta} := \bigcup_{k \in \mathbb{Z}} \left\{(G_{\text{even}} \times \{t_{2k}\}) \cup (G_{\text{odd}} \times \{t_{2k+1}\})\right\}
\]

(summation of the indexes of each point is even),

\[
\hat{G}_{\delta} := \bigcup_{k \in \mathbb{Z}} \left\{(G_{\text{odd}} \times \{t_{2k}\}) \cup (G_{\text{even}} \times \{t_{2k+1}\})\right\}
\]

(summation of the indexes of each point is odd),

\[
\{e_1, \ldots, e_d\} \text{ is the standard basis of } \mathbb{R}^d,
\]

\[
B := \{\pm e_1, \ldots, \pm e_d\}.
\]

Note that \(x_{m \pm 2Ne_i} = x_m \pm e_i, \ t_{k \pm 2K} = t_k \pm 1\) and

\[
\text{pr.} \ G_{\text{even}} := \{x_m \in G_{\text{even}} \mid 0 \leq m_i \leq 2N - 1\}.
\]
We use the following notation to describe such random walks: $G_{\text{even}} := \{x_m \in G_{\text{odd}} \mid 0 \leq m_i \leq 2N - 1\}$, $G_{\delta} := \{(x_m, t_k) \in G_{\delta} \mid 0 \leq m_i \leq 2N - 1, \ 0 \leq k \leq 2K - 1\}$, $\tilde{G}_{\delta} := \{(x_m, t_k) \in \tilde{G}_{\delta} \mid 0 \leq m_i \leq 2N - 1, \ 0 \leq k \leq 2K - 1\}$, where $G_{\delta}$, $\tilde{G}_{\delta}$ are seen as discretization of $\mathbb{T}^d \times \mathbb{T}$ with the mesh size $\delta = (h, \tau)$.

Figure 1 shows the two-dimensional $G_{\text{even}}$ by the symbol $\circ$ and $G_{\text{odd}}$ by $\bullet$. We sometimes use the notation $(x_m, t_k), (x_{m+1}, t_{k+1})$ to indicate points of $G_{\delta}$ and $(x_{m+1}, t_k), (x_m, t_{k+1})$ to indicate points of $\tilde{G}_{\delta}$ with $1 := (1, 0, \ldots, 0) \in \mathbb{Z}^d$. For $(x, t) \in G_{\delta} \cup \tilde{G}_{\delta}$, the notation $m(x), k(t)$ denotes the index of $x, t$, respectively.

For each point $(x_n, t_{l+1}) \in \tilde{G}_{\delta}$, we consider the backward random walks $\gamma$ within $[t_{l'}, t_{l+1}]$ which start from $x_n$ at $t_{l+1}$ and move by $\omega h, \omega \in B$ in each backward time step $\tau$:

$$\gamma = \{\gamma^k\}_{k=l', \ldots, l+1}, \quad \gamma^{l+1} = x_n, \quad \gamma^k = \gamma^{k+1} + \omega h.$$  

We use the following notation to describe such random walks:

$$X_n^{l+1,k} := \{x_{m+1} \mid (x_{m+1}, t_k) \in \tilde{G}_{\delta}, \max_{1 \leq j \leq d} |x_j^m - x_j^l| \leq (l+1-k)h\}$$

(the set of all reachable points of the random walk at time $k$),

$$G_n^{l+1,l'} := \bigcup_{l' \leq k \leq l+1} (X_n^{l+1,k} \times \{t_k\}) \subset \tilde{G}_{\delta}$$

(the set of all reachable space-time points of the random walk within $l' \leq k \leq l+1$),

$$\xi : G_n^{l+1,l'+1} \ni (x_m, t_{k+1}) \mapsto \xi_{m+1}^{k+1} \in \left[-(d\lambda)^{-1}, (d\lambda)^{-1}\right] \quad \lambda := \tau/h,$$

$$\rho : G_n^{l+1,l'+1} \times B \ni (x_m, t_{k+1}, \omega) \mapsto \rho_{m+1}^{k+1}(\omega) := \frac{1}{2d} - \frac{\lambda}{2} \left(\omega \cdot \xi_{m+1}^{k+1}\right) \in [0, 1],$$

$$\gamma : [l', l'+1, \ldots, l+1] \ni k \mapsto \gamma^k \in X_n^{l+1,k}, \gamma^{l+1} = x_n, \gamma^k = \gamma^{k+1} + \omega h, \omega \in B.$$
\( \Omega_{n}^{l+1,l'} : \) the family of the above \( \gamma \),

where \( \xi \) and \( \rho \) are not defined at \( l' \). We see that \( \rho_{m+1}^{k+1}(\omega) \), \( \omega \in B \) are the transition probability from \( (x_{m}, t_{k+1}) \) to \( (x_{m} + \omega h, t_{k}) \), because

\[
\sum_{\omega \in B} \rho_{m+1}^{k+1}(\omega) = \sum_{i=1}^{d} (\rho_{m+1}^{k+1}(e_{i}) + \rho_{m+1}^{k+1}(-e_{i})) = 1.
\]

Transition of random walks is controlled by \( \xi \). We define the probability density of each path \( \gamma \in \Omega_{n}^{l+1,l'} \) as

\[
\mu_{n}^{l+1,l'}(\gamma) := \prod_{l' \leq k \leq l} \rho_{m(\gamma^{k+1})}^{k+1}(\omega^{k+1}), \quad \omega^{k+1} := \frac{\gamma^{k} - \gamma^{k+1}}{h}.
\]

For each control \( \xi \), the probability density \( \mu_{n}^{l+1,l'}(\cdot) = \mu_{n}^{l+1,l'}(\cdot; \xi) \) yields a probability measure of \( \Omega_{n}^{l+1,l'} \), i.e.,

\[
\text{Prob}(A) = \sum_{\gamma \in A} \mu_{n}^{l+1,l'}(\gamma; \xi) \quad \text{for } A \subset \Omega_{n}^{l+1,l'}.
\]

The expectation with respect to this probability measure is denoted by \( E_{\mu_{n}^{l+1,l'}(\cdot; \xi)} [...], \) i.e., for a function \( f : \Omega_{n}^{l+1,l'} \to \mathbb{R} \),

\[
E_{\mu_{n}^{l+1,l'}(\cdot; \xi)}[f(\gamma)] := \sum_{\gamma \in \Omega_{n}^{l+1,l'}} \mu_{n}^{l+1,l'}(\gamma; \xi) f(\gamma).
\]

In particular, the average of sample paths \( \gamma \in \Omega_{n}^{l+1,l'} \) with a control \( \xi \) is denoted by \( \bar{\gamma}^{k} \), i.e.,

\[
\bar{\gamma}^{k} := \sum_{\gamma \in \Omega_{n}^{l+1,l'}} \mu_{n}^{l+1,l'}(\gamma; \xi) \gamma^{k}, \quad l' \leq k \leq l + 1.
\]

As shown in [48], we have

\[
\bar{\gamma}^{l+1} = x_{n}, \quad \bar{\gamma}^{k} = \bar{\gamma}^{k+1} - \xi^{k+1} \tau \quad \text{with} \quad \bar{\xi}^{k} := \sum_{\gamma \in \Omega_{n}^{l+1,l'}} \mu_{n}^{l+1,l'}(\gamma; \xi) \xi_{m(\gamma)}^{k}, \quad (2.1)
\]

There is another formulation of the probability measure of the random walk in terms of the configuration space, not the path space \( \Omega_{n}^{l+1,l'} \), i.e., the distribution on \( X_{n}^{l+1,k} \) for each \( l' \leq k \leq l + 1 \). Define \( p(\xi) : G_{n}^{l+1,l'} \ni (x_{m+1}, t_{k}) \mapsto p_{m+1}^{k}(\xi) \in [0, 1] \) as

\[
p_{m+1}^{k}(\xi) := \text{Prob}( \{ \gamma \in \Omega_{n}^{l+1,l'} \mid \gamma^{k} = x_{m+1} \} ) \quad \text{(2.2)}
\]

It follows from the definition of random walks that \( p_{m+1}^{k} \) is independent from the choice of \( l' \) and

\[
\sum_{\{m \mid x_{m+1} \in X_{n}^{l+1,k}\}} p_{m+1}^{k}(\xi) = 1 \quad \text{for each } k.
\]

Furthermore, it holds that

\[
p_{m+1}^{k}(\xi) = \sum_{\omega \in B} p_{m+1+\omega}^{k+1}(\xi) \rho_{m+1+\omega}^{k+1}(-\omega), \quad (2.3)
\]
where \( p_{m+1+\omega}^{k+1}(\xi) = \rho_{m+1+\omega}^{k+1}(\omega) = 0 \) if \( m_{m+1+\omega} \neq x_{m+1+1+k} \). We will see in Sect. 3 that \( p(\xi) \) plays an important role to derive an analogue of Mather’s minimizing problem and the construction of Mather measures.

Since our transition probabilities are space-time inhomogeneous, the well-known law of large numbers does not always hold in the hyperbolic scaling limit, i.e., \( \delta = (h, \tau) \to 0 \) under \( 0 < \lambda_0 \leq \lambda := \tau/h \) with a constant \( \lambda_0 \). The author investigated the asymptotics of the probability measure of \( \Omega_n^{l+1,l'} \) as \( \delta \to 0 \) in [44,48] as follows: Let \( \eta(\gamma) : \{l', l'+1, \ldots, l+1\} \to \mathbb{R}^d \) be a function defined for each \( \gamma \in \Omega_n^{l+1,l'} \) as

\[
\eta^k(\gamma) = \eta^{k+1}(\gamma) - \xi^{k+1}(m_{\gamma^{k+1}}), \quad \eta^{l+1}(\gamma) = x_n.
\]

Define \( \tilde{\sigma}_i^{l+1,k} \) and \( \hat{\sigma}_i^{l+1,k} \) for \( i = 1, \ldots, d \) as

\[
\tilde{\sigma}_i^{l+1,k} := E_{\mu_{l+1,l'}(\cdot; \xi)}[(\eta^k(\gamma) - \gamma^k)^i], \quad \hat{\sigma}_i^{l+1,k} := E_{\mu_{l+1,l'}(\cdot; \xi)}[(\eta^k(\gamma) - \gamma^k)^i].
\]

where \( (\eta^k(\gamma) - \gamma^k)^i \) denotes the \( i \)-th component of \( \eta^k(\gamma) - \gamma^k \).

Lemma 2.1 [44,48] For any control \( \xi \), we have

\[
(\tilde{\sigma}_{i}^{l+1,k})^2 \leq \hat{\sigma}_{i}^{l+1,k} \leq (t_{l+1} - t_k) \frac{h}{\lambda} \quad \text{for} \quad 0 \leq k \leq l + 1.
\]

Note that \( \tilde{\sigma}_i^{l+1,k} \) can be seen as a generalization of the standard variance; the standard variance is of \( O(1) \) as \( \delta \to 0 \) under hyperbolic scaling in general for space-time inhomogeneous random walks; however, \( \tilde{\sigma}_i^{l+1,k} \) and \( \hat{\sigma}_i^{l+1,k} \) always tend to 0 for any control \( \xi \); in the space-homogeneous case, i.e., \( \xi \) is constant for each \( k \), \( \tilde{\sigma}_i^{l+1,k} \) is equal to the standard variance. The above hyperbolic scaling limit of the random walks plays an important role to investigate convergence of our theory.

Forward random walks are defined in the same manner with \( \gamma^{l+1} = x_n, \gamma^{k+1} = y^k + \omega h \) for \( k = l + 1, \ldots, l' - 1 \) and \( \rho_{m+1}(\omega) := \frac{1}{2d} + \frac{\sigma_k^k}{2}(\omega \cdot \xi_m^{k+1}) \).

2.2 Hamilton–Jacobi equation on grid

Let \( v \) denote a function: \( \tilde{G}_\delta \ni (x_{m+1}, t_k) \mapsto v(x_{m+1}, t_k) = v_{m+1}^k \in \mathbb{R} \). Introduce the spatial discrete derivatives of \( v \) that are defined at each point \( (x_m, t_k) \in \tilde{G}_\delta \) as

\[
(D_{x}^j v)(x_m, t_k) = (D_{x}^j v)^k_m := \frac{v_m^k + \omega_j - v_{m-\omega_j}^k}{2h},
\]

\[
(D_{x} v)(x_m, t_k) = (D_{x} v)^k_m := ((D_{x_1} v)^k_m, \ldots, (D_{x_d} v)^k_m).
\]

Introduce the temporal discrete derivative of \( v \) that is defined at each point \( (x_m, t_{k+1}) \in \tilde{G}_\delta \) as

\[
(D_{t} v)(x_m, t_{k+1}) = (D_{t} v)^k_{m+1} := \left( v_{m+1}^k - \frac{1}{2d} \sum_{\omega \in B} v_{m+\omega}^k \right) \frac{1}{\tau}.
\]
Let $P \subset \mathbb{R}^d$ be an arbitrary convex and compact set. Fix any $r > 0$. For each fixed $c \in P$, consider the initial value problems of the Hamilton–Jacobi equation on the grid

$$\begin{cases} v : \tilde{G}_{\delta} | 0 \leq k \leq 2K \ni (x_{m+1}, t_k) \mapsto v^k_{m+1} \in \mathbb{R}, \\ v^k_{m+1 \pm 2Nei} = v^k_{m+1} \quad (i = 1, \ldots, d), \\ v(\cdot, 0) = v^0 : G_{\text{odd}} \ni x_{m+1} \mapsto v^0_{m+1} \in \mathbb{R} \text{ is given so that} \\ v^0_{m+1 \pm 2Nei} = v^0_{m+1} \quad \text{and} \quad |D_xv^0| \leq r \quad (i = 1, \ldots, d), \\ (\tilde{D}_tv)^{k+1}_m + H(x_m, t_k, c + (D_xv)^k_m) = 0, \end{cases}$$

which corresponds to the initial value problems of the exact Hamilton–Jacobi equations (1.4) with initial data from $\text{Lip}_r(T^d; \mathbb{R})$ (the family of Lipschitz functions with a Lipschitz constant bounded by $r$). In (2.4), the quantity $v^{k+1}_m$ is unknown to be determined by $\{v^k_{m+\omega}\}_{\omega \in B}$ as a recursion. As explained in [48], evolution of (2.4) can be intuitively seen as Fig. 2: The value $v^{k+1}_m$ is determined by the values of the grid points $\cdot$ contained in the pyramid in the figure, where the pyramid grows up to $k = 0$ keeping the aspect ratio determined by $\lambda := \tau/h$ (a finite speed of propagation). We call the pyramid “a pyramid of dependence”. The key point is that the contribution of the value at each grid point $\cdot$ within the pyramid of dependence to the value $v^{k+1}_m$ can be characterized by the probability measure of a controlled backward random walk starting at $(x_m, t_{k+1})$.

The Hamilton–Jacobi equation on the grid corresponding to (1.3) is given as

$$\begin{cases} v : \tilde{G}_{\delta} | -2K \leq k \leq 0 \ni (x_{m+1}, t_k) \mapsto v^k_{m+1} \in \mathbb{R}, \\ v^k_{m+1 \pm 2Nei} = v^k_{m+1} \quad (i = 1, \ldots, d), \\ v(\cdot, 0) = v^0 : G_{\text{odd}} \ni x_{m+1} \mapsto v^0_{m+1} \in \mathbb{R} \text{ is given so that} \\ v^0_{m+1 \pm 2Nei} = v^0_{m+1} \quad |D_xv^0| \leq r \quad (i = 1, \ldots, d), \\ (\tilde{D}_tv)^{k-1}_m + H(x_m, t_k, c + (D_xv)^k_m) = 0 \end{cases}$$

with

$$(\tilde{D}_tv)^{k-1}_m := \left( v^{k-1}_m - \frac{1}{2d} \sum_{\omega \in B} v^{k}_{m+\omega} \right) \frac{1}{-\tau}.$$
2.3 Lax–Oleinik type solution map

We recall the results \[48\] on solvability of (2.4) in terms of the Lax–Oleinik type representation formula with the action functional (1.2). To be more precise than \(L^l\_\_\), we use the following notation: define the action functional for each \(v^0 : G_{\text{odd}} \rightarrow \mathbb{R}\) as

\[
E_n^{l+1}(\xi; v^0, c) := E_{\xi, n}^{l+1, 0}(\xi, \xi_0) \left[ \sum_{0<k \leq l+1} \left( L(c) v_k^{\xi} t_{k-1}, \xi_k^{m(\psi)} \right) + v_{m(\psi)}^0 \right].
\]

**Theorem 2.2** \[48\] For each \(r > 0\) and the set \(P\) (the set of \(c\)), there exists \(\lambda_1 > 0\) for which the following statements hold for any small \(\delta = (h, \tau)\) with \(\lambda := \tau/h < \lambda_1\), any \(c \in P\) and any initial data \(v(\cdot, 0) = v^0\) of (2.4):

1. For each \(n\) and \(l\) with \(0 < l + 1 \leq 2K\) such that \((x_n, t_{l+1}) \in \tilde{G}_0\), the action functional \(E_n^{l+1}(\xi; v^0, c)\) has the infimum within all controls \(\xi : G^{l+1, 1}_n \rightarrow \left( (d\lambda)^{-1}, (d\lambda)^{-1} \right)^d\). There exists the unique minimizing control \(\xi^*\) to attain the infimum, which satisfies

\[
|\xi^*|_{\infty} \leq (d\lambda)^{-1} < (d\lambda)^{-1} on G^{l+1, 1}_n for all 1 \leq j \leq d.
\]

2. Define the function \(v : \tilde{G}_0 | 0 \leq k \leq 2K \rightarrow \mathbb{R}\) as

\[
v(x_m, t_{k+1}) := \inf_\xi E_{m}^{k+1}(\xi; v^0, c), \quad v(x_{m+1}, 0) := v_{m+1}^0. \tag{2.6}
\]

Then, the minimizing control \(\xi^*\) for \(\inf_\xi E_n^{l+1}(\xi; v^0, c)\) satisfies

\[
\xi_{m,k}^* = H_\rho(x_m, t_k, c + (D_x v_k)^m) \quad (\Leftrightarrow (D_x v_k)^m = L_\xi(x_m, t_k, \xi_{m,k}^*)) - c).
\]

In particular, \((D_x v_k)^m\) is uniformly bounded on \(\tilde{G}_0 | 0 \leq k \leq 2K\) independently from \(\delta\) (this is a CFL-type condition).

3. The function \(v\) defined in the claim 2 is the unique solution of (2.4).

Throughout this paper, \(\lambda_1\) stands for the constant mentioned in Theorem 2.2.

The following families of the maps are well-defined as the Lax–Oleinik type solution maps for (2.4): for any \(v^0\) given in (2.4),

\[
\{\varphi^k_{\delta}(\cdot; c)\}_{k \in \mathbb{N}} | \{0\}, \quad \varphi^k_{\delta}(\cdot; c) : v^0 \mapsto v(\cdot, t_k) \quad (v \text{ is given as (2.6)}).
\]

In addition, we set

\[
\{\psi^k_{\delta}(\cdot; c)\}_{k \in \mathbb{N}} | \{0\}, \quad \psi^k_{\delta}(\cdot; c) : u^0 = D_x v^0 \mapsto u^k = D_x v(\cdot, t_k) \quad (v \text{ is given as (2.6)}),
\]

where \(\varphi^k_{\delta}\) is indeed the solution map of the system of discrete conservation laws with restricted initial data derived from (2.4): \(u^k_m = (u_{1m}^k, u_{2m}^k, \ldots, u_{dm}^k) := D_x v(\cdot, t_k) : G \rightarrow \mathbb{R}^d\) satisfies for \(i = 1, 2, \ldots, d\),

\[
\frac{1}{\tau} \left( u_{m+1}^{k+1} - \frac{1}{2d} \sum_{\omega \in B} u_{m+e_\omega}^{k+1} \right) + \frac{1}{2h} \left\{ H(x_{m+2\epsilon_i}, t_k, c + u_{m+2\epsilon_i}^k) - H(x_m, t_k, c + u_m^k) \right\} = 0.
\]

In Sect. 3, we will seek for a pair of constant \(\widetilde{H}_\delta(c)\) and a function \(\bar{v}^0\) such that \(\varphi^{2K}_{\delta}(\bar{v}^0; c) + \widetilde{H}_\delta(c) = \bar{v}^0\). The pair yields a time-1-periodic solution \(\bar{v}\) of

\[
(D_x v_m)^{k+1} + H(x_m, t_k, c + (D_x v_m)^k) = \widetilde{H}_\delta(c).
\]

Then, with 1-periodic extension of \(\bar{v}\) to the whole \(\tilde{G}_0\), we complete the set up of the action minimizing problem for \(L^l\_\_ (\xi; \bar{v})\). For this purpose, we need more preliminary investigations.
As for (2.5), we have
\[
E_{n-1}(\xi; c) := E_{\mu_{n-1,0}}(\xi) \left[ \sum_{-l-1 \leq k < 0} -L^{(c)}(y^k, I_{k+1}, \xi^k_{m(y^k)}) \tau + v_0(y^0) \right],
\]
\[
v(x_m, t_{-k-1}) = \sup_{\xi} E_m^{k-1}(\xi; c),
\]
where the solution map is denoted by \( \varphi^k_\delta(\cdot; c) \) and \( \tilde{\psi}^k_\delta(\cdot; c) \).

### 2.4 Semiconcavity of Lax–Oleinik type solution map

Due to the variational structure of the solution \( v_{m+1}^k = \varphi^k_\delta(v^0, c)(x_{m+1}) \) to (2.4), we have a kind of semiconcavity property, i.e.,
\[
(D_{ij}^2 v)^k_{m+1} := \left( v_{m+1}^{k+2e_j} + v_{m+1}^{k-2e_j} - 2v_{m+1}^k \right) \frac{1}{4h^2} = \left( D_{ij} v \right)_{m+1}^{k+e_j} - \left( D_{ij} v \right)_{m+1}^{k-e_j}
\]
is bounded from the above. An instant observation shows that the upper bound is given by \( \sup_m (D_{ij}^2 v^0)_{m+1} \). Our aim is to obtain a sharper estimate independent from the initial data \( v^0 \). Note that our “semiconcavity” estimate is restricted to the directions of \( e_1, \ldots, e_d \), not every direction of \( \mathbb{R}^d \). The sharper semiconcavity estimate implies a lot in regards to the behaviors of the derivative of discrete solutions.

Introduce the following notation:
\[
M^k_\delta := \sup_{m,j} (D_{ij}^2 v)^k_{m+1},
\]
\[
u^* := \max_{x \in \mathbb{T}^d, t \in \mathbb{T}, \| \gamma \| \leq (d\lambda_1)^{-1}, c \in P} \left| L^{(c)}(x, t, \gamma) \right|, \quad \text{(note that } |(D_{ij} v)^k_{m+1}|_\infty \leq u^* \text{ for any solution of (2.4))},
\]
\[
H^*_p := \max_{x \in \mathbb{T}^d, t \in \mathbb{T}, \| u \| \leq u^*} \| H_p(x, t, u) \|_\infty,
\]
\[
H^*_{xx} := \max_{x \in \mathbb{T}^d, t \in \mathbb{T}, \| u \| \leq u^*} \| H_{xx}(x, t, u) \|,
\]
\[
H^*_{xp} := \max_{x \in \mathbb{T}^d, t \in \mathbb{T}, \| u \| \leq u^*} \| H_{xp}(x, t, u) \|,
\]
\[
H^*_pp := \inf_{x \in \mathbb{T}^d, t \in \mathbb{T}, \| u \| \leq u^*} \frac{H_{pp}(x, t, u) y \cdot y}{|y|^2}, \quad \text{where } H_{pp}^* > 0 \text{ due to (H2)},
\]
\[
M^* := \frac{H^*_pp}{\sqrt{(1 + d)(H^*_{xp})^2 + H^*_{pp} H^*_{xx}}},
\]
\[
\eta^* := M^*_+ - M^*_-,
\]
\[
M(t) := M^*_+ + \frac{\eta^* e^{-\eta^* t H_{pp}^*}}{1 - e^{-\eta^* H_{pp}^*}}, \quad t > 0, \quad \text{where } M(t) \to M^*_+ \text{ as } t \to \infty.
\]
Theorem 2.3 Suppose that $\delta = (h, \tau)$ with $\lambda := \tau / h < \lambda_1$ is such that

$$
\lambda \leq \min \left\{ \frac{1 - 2d H^*_x \tau}{2d H^*_p + d H^*_x}, \frac{1}{10r H^*_p} \right\},
$$

$$
\tau < \min \left\{ \frac{1}{2d H^*_x}, \frac{1 - d \lambda H^*_p}{2d (H^*_p M^*_+ + H^*_x)}, \frac{1}{M^*_p (M^*_+ - M^-)}, \frac{\log 2}{\eta^* H^*_p} \right\},
$$

Then, we have $M^k_\delta \leq M(t_k)$ for all $k = 1, \ldots, 2K$. In particular, if $M^0_\delta \leq M^*_+$, we have $M^k_\delta \leq M^*_+$ for all $0 \leq k \leq 2K$; if $v^k$ is extended to $k \to \infty$ keeping the boundedness $|D_x v^k|_\infty \leq u^*$, we have $M^k_\delta \leq M(t_k)$ for all $k > 0$.

Proof We insert

$$
v^{k+1}_m = \frac{1}{2d} \sum_{\omega \in B} v^k_{m+\omega} - H(x_m, t_k, c + (Dv)^k_m) \tau
$$

into $(D^2 v^k_m)^{k+1}$ and apply Taylor’s formula with short notation $H_{pp}, H_{xx}, H_{xp},$ etc., for the remainder terms, to get

$$
(D^2 v^k_m)^{k+1} \cdot 4h^2 = \frac{1}{2d} \sum_{\omega \in B} \left( v^k_{m+2e_j+\omega} + v^k_{m-2e_j+\omega} - 2v^k_{m+\omega} \right)
$$

$$
- \left\{ H(x_m+2e_j, t_k, c + (D_x v)^k_{m+2e_j}) + H(x_m-2e_j, t_k, c + (D_x v)^k_{m-2e_j}) \right\}
$$

$$
- 2H(x_m, t_k, c + (D_x v)^k_m) \tau
$$

$$
= \frac{1}{2d} \sum_{\omega \in B} \left( \frac{D^2 v^k_{m+\omega}}{2d} \cdot 4h^2 \right)
$$

$$
- \left\{ H(x_m+2e_j, t_k, c + (D_x v)^k_{m+2e_j}) - H(x_m, t_k, c + (D_x v)^k_{m+2e_j}) \right\}
$$

$$
+ H(x_m, t_k, c + (D_x v)^k_{m+2e_j}) - H(x_m, t_k, c + (D_x v)^k_m)
$$

$$
+ H(x_m-2e_j, t_k, c + (D_x v)^k_{m-2e_j}) - H(x_m, t_k, c + (D_x v)^k_{m-2e_j})
$$

$$
+ H(x_m, t_k, c + (D_x v)^k_{m-2e_j}) - H(x_m, t_k, c + (D_x v)^k_m) \right\} \tau
$$

$$
= \frac{1}{2d} \sum_{i=1}^d \left( (D^2 v^k_{m+e_i}) + (D^2 v^k_{m-e_i}) \right) \cdot 4h^2
$$

$$
- \left\{ H_x(x_m, t_k, c + (D_x v)^k_{m+2e_j}) \cdot (2he_j) + \frac{1}{2} H_{xx} \times (2he_j) \cdot (2he_j) \right\}
$$

$$
+ H_p(x_m, t_k, c + (D_x v)^k_m) \cdot ((D_x v)^k_{m+2e_j} - (D_x v)^k_m)
$$

$$
+ \frac{1}{2} H_{pp} \times ((D_x v)^k_{m+2e_j} - (D_x v)^k_m) \cdot ((D_x v)^k_{m+2e_j} - (D_x v)^k_m)
$$

$$
+ H_x(x_m, t_k, c + (D_x v)^k_{m-2e_j}) \cdot (-2he_j) + \frac{1}{2} H_{xx} \times (-2he_j) \cdot (-2he_j)
$$

$$
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\begin{align*}
+ H_p(x_m, t_k, c + (D_x v)^k_m) & \cdot ((D_x v)^k_{m-2e_j} - (D_x v)^k_m) \\
+ \frac{1}{2} H_{pp} & \times ((D_x v)^k_{m-2e_j} - (D_x v)^k_m) \cdot ((D_x v)^k_{m-2e_j} - (D_x v)^k_m) \bigg\} \tau \\
& = \frac{2h^2}{d} \sum_{i=1}^d \left( (D_x^2 v)^k_{m+e_i} + (D_x^2 v)^k_{m-e_i} \right) \\
& \quad - \left\{ \sum_{i=1}^d H_{p^j} (x_m, t_k, c + (D_x v)^k_m) \left( \frac{v^k_{m+2e_j+e_i} - v^k_{m+2e_j-e_i}}{2h} \right) \\
& \quad + \frac{v^k_{m-2e_j+e_i} - v^k_{m-2e_j-e_i}}{2h} \right\} 2h \\
& \quad + H_{x^j} \cdot \left( (D_x v)^k_{m+2e_j} - (D_x v)^k_m + (D_x v)^k_m - (D_x v)^k_{m-2e_j} \right) \tau \\
& \quad + \frac{1}{2} H_{pp} \times ((D_x v)^k_{m+2e_j} - (D_x v)^k_m) \cdot ((D_x v)^k_{m+2e_j} - (D_x v)^k_m) \\
& \quad + \frac{1}{2} H_{pp} \times ((D_x v)^k_{m-2e_j} - (D_x v)^k_m) \cdot ((D_x v)^k_{m-2e_j} - (D_x v)^k_m) \\
& \quad + \frac{1}{2} H_{x^j} \times 4h^2 + \frac{1}{2} H_{x^j} \times 4h^2 \bigg\} \tau.
\end{align*}

Since

\begin{align*}
\frac{v^k_{m+2e_j+e_i} - v^k_{m+2e_j-e_i}}{2h} + \frac{v^k_{m-2e_j+e_i} - v^k_{m-2e_j-e_i}}{2h} - 2 \frac{v^k_{m+e_i} - v^k_{m-e_i}}{2h} \\
= \frac{2h^2}{d} \left( (D_x^2 v)^k_{m+e_i} \cdot \frac{d}{h} - (D_x^2 v)^k_{m-e_i} \cdot \frac{d}{h} \right),
\end{align*}

we have with \( \lambda = \tau/h \),

\begin{align*}
(D_x^2 v)^{k+1}_m \cdot 4h^2 & = \frac{2h^2}{d} \sum_{i=1}^d \left\{ (D_x^2 v)^k_{m+e_i} \left( 1 - d \lambda H_{p^j} (x_m, t_k, c + (D_x v)^k_m) \right) \\
& \quad + (D_x^2 v)^k_{m-e_i} \left( 1 + d \lambda H_{p^j} (x_m, t_k, c + (D_x v)^k_m) \right) \right\} \\
& \quad - \left\{ H_{x^j} \cdot \left( (D_x v)^k_{m+2e_j} - (D_x v)^k_m + (D_x v)^k_m - (D_x v)^k_{m-2e_j} \right) \right\} 2h \\
& \quad + \frac{1}{2} H_{pp} \times ((D_x v)^k_{m+2e_j} - (D_x v)^k_m) \cdot ((D_x v)^k_{m+2e_j} - (D_x v)^k_m) \\
& \quad + \frac{1}{2} H_{pp} \times ((D_x v)^k_{m-2e_j} - (D_x v)^k_m) \cdot ((D_x v)^k_{m-2e_j} - (D_x v)^k_m) \\
& \quad + \frac{1}{2} H_{x^j} \times 4h^2 + \frac{1}{2} H_{x^j} \times 4h^2 \bigg\} \tau \\
& \leq \frac{2h^2}{d} \sum_{i=1}^d \left\{ (D_x^2 v)^k_{m+e_i} \left( 1 - d \lambda H_{p^j} (x_m, t_k, c + (D_x v)^k_m) \right) \\
& \quad + (D_x^2 v)^k_{m-e_i} \left( 1 + d \lambda H_{p^j} (x_m, t_k, c + (D_x v)^k_m) \right) \right\} + 4h^2 \tau H_{x^j}^* \\
& \quad - \left\{ H_{x^j} \cdot \left( (D_x v)^k_{m+2e_j} - (D_x v)^k_m + (D_x v)^k_m - (D_x v)^k_{m-2e_j} \right) \right\} 2h
\end{align*}
\[
+ \frac{H^*_{pp}}{2} \left( (D_x v)^{k}_{m+2e_j} - (D_x v)^{k}_{m} \right) \cdot \left( (D_x v)^{k}_{m+2e_j} - (D_x v)^{k}_{m} \right)
+ \frac{H^*_{pp}}{2} \left( (D_x v)^{k}_{m-2e_j} - (D_x v)^{k}_{m} \right) \cdot \left( (D_x v)^{k}_{m-2e_j} - (D_x v)^{k}_{m} \right) \tau
\]
\[
= 2 \frac{h^2}{d} \sum_{i=1}^{d} \left\{ \frac{d^2}{d} \left( (D_x v)^{k}_{m+e_i} \left( 1 - d^2 H_{p,l} \left( x_m, t_k, c + (D_x v)^{k}_{m} \right) \right) \right) \right\} + 4h^2 \tau H^*_{xx}
\]
\[
- \left\{ \frac{H^*_{pp}}{2} \left| \left( (D_x v)^{k}_{m+2e_j} - (D_x v)^{k}_{m} \right) + \frac{2h}{H^*_{pp}} H_{x_l p} \right|^2 - \frac{2h^2}{H^*_{pp}} \right| H_{x_l p} \right|^2 \right\} \tau
\]
\[
= \left\{ ((D_x v)^{k}_{m+2e_j} - (D_x v)^{k}_{m}) + \frac{2h}{H^*_{pp}} H_{x_l p} \right\}^2
\]
\[
\leq 2 \frac{h^2}{d} \sum_{i=1}^{d} \left\{ (D_j^2 v)^{k}_{m+e_i} \left( 1 - d^2 H_{p,l} \left( x_m, t_k, c + (D_x v)^{k}_{m} \right) \right) \right\} + 4h^2 \tau H^*_{xx} + \frac{4h^2 \tau d H^*_{xp}^2}{H^*_{pp}}
\]
\[
- \frac{\tau H^*_{pp}}{2} \left\{ ((D_x v)^{k}_{m+2e_j} - (D_x v)^{k}_{m}) + \frac{2h}{H^*_{pp}} H_{x_l p} \right\}^2
\]
\[
= 4 \frac{h^2}{2d} \sum_{i=1}^{d} \left\{ (D_j^2 v)^{k}_{m+e_i} \left( 1 - d^2 H_{p,l} \left( x_m, t_k, c + (D_x v)^{k}_{m} \right) \right) \right\} + 4h^2 \tau H^*_{xx} + \frac{4h^2 \tau d H^*_{xp}^2}{H^*_{pp}}
\]
\[
- \frac{4h^2 \tau}{2} H^*_{pp} \left\{ ((D_x v)^{k}_{m+e_j} + \frac{H_{x_l p}^*}{H^*_{pp}} \right)^2 + ((D_x v)^{k}_{m-e_j} + \frac{H_{x_l p}^*}{H^*_{pp}} \right)^2 \right\}.
\]

Set \( g_\pm(y) : \mathbb{R} \rightarrow \mathbb{R} \) as
\[
g_\pm(y) := \frac{1}{2d} \left( 1 \pm d^2 H_{p,l} \left( x_m, t_k, c + (D_x v)^{k}_{m} \right) \right) y - \frac{\tau H^*_{pp}}{2} \left( y + \frac{H_{x_l p}^*}{H^*_{pp}} \right)^2.
\]
We see that \( g(y) \geq 0 \), if
\[
y \leq \frac{1 - d\lambda H_p^*}{2d\tau H_{pp}^*} - \frac{H_{kp}^*}{H_{pp}^*} \left( \leq \frac{1 - d\lambda H_{pi}(x_m, t_k, c + (D_x v)_m)}{2d\tau H_{pp}^*} - \frac{H_{xi(pi)}}{H_{pp}^*} \right).
\]

Since \( \lambda \leq (1 - 2dH_{xp}^*)/(2dH_{pp}^* + dH_p^*) \), we have for all initial data \( v^0 \) in (2.4),
\[
M_0^0 = \sup_{m,j} (D_j^2 v(0))_{m+1} \leq \frac{r}{h} \leq \frac{r\lambda}{\tau} \leq \frac{1 - 3dH_{xp}^*}{2d\tau H_{pp}^*} = \frac{1 - d\lambda H_p^*}{2d\tau H_{pp}^*} - \frac{H_{xp}^*}{H_{pp}^*}.
\]

Suppose that for some \( k \geq 0 \),
\[
M_k^k = \sup_{m,j} (D_j^2 v(k))_{m+1} \leq \frac{1 - d\lambda H_p^*}{2d\tau H_{pp}^*} - \frac{H_{xp}^*}{H_{pp}^*}.
\]

Then, we have
\[
(D_j^2 v(k+1))_m \leq \frac{1}{2d} \sum_{i \in \{1, 2, \ldots, d\} \setminus \{j\}} \left\{ (D_j^2 v(k))_{m+\epsilon_i} \left(1 - d\lambda H_{pi}(x_m, t_k, c + (D_x v)_m) \right) \right. \\
+ (D_j^2 v(k))_{m-\epsilon_i} \left(1 + d\lambda H_{pi}(x_m, t_k, c + (D_x v)_m) \right) \right\} + \tau H_{xx}^* + \frac{\tau dH_{xp}^*}{H_{pp}^*} + g(M_k^k).
\]

Since \( 1 \pm d\lambda H_{pi}(x_m, t_k, c + (D_x v)_m) \geq 0 \) due to the CFL-type condition given in Theorem 2.2, we have
\[
(D_j^2 v(k+1))_m \leq M_k^k + \tau H_{xx}^* + \frac{\tau dH_{xp}^*}{H_{pp}^*} - \tau H_{pp}^*(M_k^k)^2 + 2\tau H_{xp}^* M_k^k,
\]

and hence,
\[
M_{k+1}^k \leq M_k^k + \tau G(M_k^k), \quad G(y) := \left(-H_{pp}^*(y - M_+^k)(y - M_-^k), \quad G(y) > 0 \text{ for } 0 \leq y < M_+^k \text{ and } G(y) < 0 \text{ for } y > M_+^k. \right)
\]

Since \( \tau \leq (1 - d\lambda H_p^*)/(2d(H_{pp}^* M_+^k + H_{xp}^*)) \) and \( \tau \leq 1/|H_{pp}^*(M_+^k - M_-^k)| \), we have
\[
M_+^* < \frac{1 - d\lambda H_p^*}{2d\tau H_{pp}^*} - \frac{H_{xp}^*}{H_{pp}^*}, \quad \tau G(y) \leq M_+^* - y \text{ for all } 0 \leq y \leq M_+^*.
\]

Therefore, the following two cases happen:

(i) If \( M_k^k \leq M_+^* \), we may have \( M_{k+1}^k = M_k^k \), but we certainly have \( M_{k+1}^k \leq M_+^* \).

(ii) If \( M_k^k > M_+^* \), we have \( M_{k+1}^k < M_k^k \).

In both cases, we have \( M_{k+1}^k \leq \frac{1 - d\lambda H_p^*}{2d\tau H_{pp}^*} - \frac{H_{xp}^*}{H_{pp}^*} \). By induction, we see that (2.7) holds for all \( 0 \leq k < 2K \), and thus, (2.8) holds for all \( 0 \leq k < 2K \). Now, it is clear that, if \( M_0^0 \leq M_+^* \), we have \( M_k^k \leq M_+^* \) for all \( 0 \leq k < 2K \). Note that these statements are true beyond \( k = 2K \) as long as \( |(D_x v)_m|_\infty \leq u^* \) holds.

We estimate the decay in the case (ii). Consider the initial value problem
\[
w'(t) = G(w(t)), \quad w(0) = M_+^* + \alpha,
\]
where \( G(y) = -H_{pp}^*(y - M_+^k)(y - M_-^k) \).
The solution satisfies
\[
    w(t) = M_+^* + \frac{\eta^* e^{-\eta^* H_{pp}^* \tau}}{1 - e^{-\eta^* H_{pp}^* \tau} + \frac{\eta^*}{\alpha}} = M_+^* + \frac{\eta^*}{1 - \frac{\eta^*}{\alpha} e^{\eta^* H_{pp}^* \tau}} \leq M(t).
\]

Since \( \tau \leq \log 2/(H_{pp}^* \eta^*) \implies e^{\eta^* H_{pp}^* \tau} - 1 \leq 2\eta^* H_{pp}^* \tau \), \( \tau \leq [4(1 + d)(H_{xx}^* \eta^* + H_{pp}^* H_{xx}^*)^{-1} - 1]^{-1} \Rightarrow \alpha^{-1} \leq 10H_{pp}^* \tau \) and \( \lambda \leq 1/(10rH_{pp}^*) \), we have
\[
    w(\tau) \geq M_+^* + \frac{2\eta^* H_{pp}^* \tau + 2\eta^*}{2\eta^* H_{pp}^* \tau + 2\eta^*} \geq M_+^* + \frac{1}{10H_{pp}^* \tau},
\]
\[
    M_\delta^k < M_\delta^0 < \frac{r\lambda}{\tau} \leq \frac{1}{10H_{pp}^* \tau} \leq M_\delta^* + \frac{1}{10H_{pp}^* \tau} \leq w(t_k) = w(\tau).
\]

Suppose that \( M_\delta^k \leq w(t_k) \) for some \( k \geq 1 \). Note that \( y + \tau G(y) \) is increasing for \( y \leq (1 + 2H_{xp}^* \tau)/(2H_{pp}^* \tau) \) and that
\[
    w(t_k) \leq w(0) = M_+^* + \frac{1 + 2H_{xp}^* \tau}{2H_{pp}^* \tau}.
\]

Therefore, we see that
\[
    M_\delta^{k+1} \leq M_\delta^k + \frac{\tau}{2} w''(t_k + \theta \tau) \quad (\exists \theta \in (0, 1))
\]
\[
    \leq w(t_k),
\]
where we note that \( w''(t) > 0 \). By induction, we obtain our assertion. \( \square \)

Throughout this paper, we take \( \tau, h > 0 \) small enough to satisfy the condition of Theorem 2.3.

As for (2.5), we have the semiconvex estimate
\[
    (D_j^2 v)_{m+1}^k = (v_{m+1+2e_j}^k + v_{m+1-2e_j}^k - 2v_{m+1}^k) \frac{1}{4h^2} \geq -M(|t_k|), \quad k < 0.
\]

### 2.5 Hyperbolic scaling limit of Lax–Oleinik type solution map

We state convergence of the solution map \( \varphi_\delta^k \) of (2.4) as \( \delta = (h, \tau) \to 0 \).

We set the Lax–Oleinik type operator \( \varphi^t(\cdot; c) : \text{Lip}(\mathbb{T}^d; \mathbb{R}) \to \text{Lip}(\mathbb{T}^d; \mathbb{R}), \eta \geq 0 \) as
\[
    \varphi^0(w; c) = w, \quad \varphi^t(w; c)(x) = \inf_{\gamma \in AC([0, t]; \mathbb{T}^d)} \left\{ \int_0^t L(\gamma(s), s, \gamma'(s))ds + w(\gamma(0)) \right\},
\]
where we sometimes treat \( \gamma : [0, t] \to \mathbb{T}^d \) as \( \gamma : [0, t] \to \mathbb{R}^d \). The viscosity solution \( v \) of
\[
\begin{align*}
    v_t(x, t) + H(x, t, c + v_x(x, t)) &= 0 & \text{in} \quad \mathbb{T}^d \times (0, 1), \\
    v(x, 0) &= w(x) & \text{on} \quad \mathbb{T}^d
\end{align*}
\]
(2.9)
is given as \( v(\cdot, t) = \varphi^t(w; c) \) (see, e.g., [7]). By Tonelli’s theory, we have a minimizing curve \( \gamma^* \) for each value \( v(x, t) = \varphi^t(v^0; c)(x) \).

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Before discussing the hyperbolic scaling limit of $\varphi^k_0$, we state Lipschitz interpolation of a function on $G_{\text{od}}$ or $G_{\text{ev}}$.

**Lemma 2.4** For each function $u : G_{\text{od}} \ni x_{m+1} \mapsto u_{m+1} \in \mathbb{R}$ with $|D_\nu u|_{\infty} \leq r$ on $G_{\text{od}}$, we have a Lipschitz continuous function $w : \mathbb{R}^d \to \mathbb{R}$ such that

$$w|_{G_{\text{od}}} = u, \quad |w(x)|_{\infty} \leq \beta r \quad \text{a.e. } x \in \mathbb{R}^d \quad \text{as } h \to 0+,$$

where $\beta > 0$ is a constant depending only on $d$.

**Proof** Consider the $d$-dimensional cube

$$C^d_{m+1} := \{x_{m+1} + a_1 e_1 + \cdots + a_d e_d \mid a_1, \ldots, a_d \in [0, 1]\}.$$

We inductively construct $w$: For $d = 1$, it is clear that

$$f^1_0(x^1) := v_{m+1} + \frac{v_{m+1+2e_1} - v_{m+1}}{2h}(x^1 - x^1_{m+1}), \quad 0 \leq x^1 - x^1_{m+1} \leq 1$$

is defined on $C^d_{m+1}$ and can be connected with respect to $m$ to be a desired Lipschitz function. For $d = 2$, in addition to the above $f^1_0(x^1)$, we set

$$f^2_{0,1}(x^1) := f^1_0(x^1) + \frac{v_{m+1+2e_2} - v_{m+1}}{2h}(x^2 - x^2_{m+1}), \quad 0 \leq x^2 - x^2_{m+1} \leq 1$$

and define

$$f^2_{0,2}(x^1, x^2) := f^1_0(x^1) + \frac{f^2_{0,1}(x^1) - f^1_0(x^1)}{2h}(x^2 - x^2_{m+1}), \quad 0 \leq x^i - x^i_{m+1} \leq 1 \quad (i = 1, 2).$$

We see that $f^2_{0,2}(x^1, x^2)$ is defined on $C^d_{m+1}$ and can be connected with respect to $m$ to be a desired Lipschitz function. For $d = 3$, in addition to $f^2_{0,2}(x^1, x^2)$, we set $f^3_{0,2}(x^1, x^2)$ as

$$f^3_{0,1}(x^1) := v_{m+1+2e_3} + \frac{v_{m+1+2e_3+2e_1} - v_{m+1+2e_1}}{2h}(x^1 - x^1_{m+1}),$$

$$f^3_{0,2}(x^1) := v_{m+1+2e_2+2e_1} + \frac{v_{m+1+2e_2+2e_1} - v_{m+1+2e_1}}{2h}(x^1 - x^1_{m+1}),$$

$$f^3_{0,2}(x^1, x^2) := f^3_{0,1}(x^1) + \frac{f^3_{0,2}(x^1) - f^3_{0,1}(x^1)}{2h}(x^2 - x^2_{m+1})$$

and define

$$f^3_{0,2,3}(x^1, x^2, x^3) := f^2_{0,2}(x^1, x^2) + \frac{f^3_{0,2}(x^1, x^2) - f^2_{0,2}(x^1, x^2)}{2h}(x^3 - x^3_{m+1}),$$

$$0 \leq x^i - x^i_{m+1} \leq 1 \quad (i = 1, 2, 3).$$

We see that $f^3_{0,2,3}(x^1, x^2, x^3)$ is defined on $C^d_{m+1}$ and can be connected with respect to $m$ to be a desired Lipschitz function. For $d = 4$, in addition to $f^3_{0,2,3}(x^1, x^2, x^3)$, we set $f^4_{0,2,3}(x^1, x^2, x^3)$ as

$$f^4_{0,1}(x^1) := v_{m+1+2e_4} + \frac{v_{m+1+2e_4+2e_1} - v_{m+1+2e_1}}{2h}(x^1 - x^1_{m+1}),$$

$$f^4_{0,2}(x^1) := v_{m+1+2e_2+2e_4} + \frac{v_{m+1+2e_2+2e_4} - v_{m+1+2e_4}}{2h}(x^1 - x^1_{m+1}),$$

$$f^4_{0,2}(x^1, x^2) := f^4_{0,1}(x^1) + \frac{f^4_{0,2}(x^1) - f^4_{0,1}(x^1)}{2h}(x^2 - x^2_{m+1}),$$

$$f^4_{0,2,3}(x^1, x^2, x^3) := f^3_{0,2,3}(x^1, x^2, x^3) + \frac{f^4_{0,2,3}(x^1, x^2, x^3) - f^3_{0,2,3}(x^1, x^2, x^3)}{2h}(x^3 - x^3_{m+1}),$$

$$0 \leq x^i - x^i_{m+1} \leq 1 \quad (i = 1, 2, 3, 4).$$
\[
\begin{align*}
    f_{4,3}^1(x^1) &:= v_{m+1+2\varepsilon_3+2\varepsilon_4} + \frac{v_{m+1+2\varepsilon_3+2\varepsilon_4+2\varepsilon_1} - v_{m+1+2\varepsilon_3+2\varepsilon_4}}{2h}(x^1 - x_{m+1}^1), \\
    f_{4,3,2}^1(x^1) &:= v_{m+1+2\varepsilon_2+2\varepsilon_3+2\varepsilon_4} + \frac{v_{m+1+2\varepsilon_2+2\varepsilon_3+2\varepsilon_4+2\varepsilon_1} - v_{m+1+2\varepsilon_2+2\varepsilon_3+2\varepsilon_4}}{2h}(x^1 - x_{m+1}^1), \\
    f_{4,3}^{1,2}(x^1, x^2) &:= f_{4,3}^1(x^1) + \frac{f_{4,3,2}^1(x^1) - f_{4,3}^1(x^1)}{2h}(x^2 - x_{m+1}^2), \\
    f_{4}^{1,2,3}(x^1, x^2, x^3) &:= f_{4,3}^{1,2}(x^1, x^2) + \frac{f_{4,3}^{1,2}(x^1, x^2) - f_{4,3}^{1,2}(x^1, x^2)}{2h}(x^3 - x_{m+1}^3)
\end{align*}
\]

and define
\[
\begin{align*}
    f_{4}^{1,2,3,4}(x^1, x^2, x^3, x^4) &:= f_{4}^{1,2,3}(x^1, x^2, x^3) \\
    &+ \frac{f_{4}^{1,2,3}(x^1, x^2, x^3) - f_{4}^{1,2,3}(x^1, x^2, x^3)}{2h}(x^4 - x_{m+1}^4).
\end{align*}
\]

We see that \( f_{4}^{1,2,3,4}(x^1, x^2, x^3, x^4) \) is defined on \( C_{m+1}^d \) and can be connected with respect to \( m \) to be a desired Lipschitz function. We may repeat the same argument for \( d = 5, 6, \ldots \). \( \square \)

**Theorem 2.5** Suppose that a sequence \( \delta = (h, \tau) \to 0 \) is such that \( 0 < \lambda_0 \leq \lambda = \tau / h < \lambda_1 \) with any constant \( \lambda_0 \in (0, \lambda_1) \). Let \( \psi_\delta^k(v^0_\delta; c) \) be the solution of (2.4) with initial data \( v^0 = v^0_\delta \) for each element \( \delta \) of the sequence, where \( v^0_\delta \) may depend on \( \delta \).

1. If \( |v^0_\delta| \leq R \) for all \( \delta \) with some constant \( R \geq 0 \), there exist a subsequence of \( \{\psi_\delta^k(v^0_\delta; c)\}_\delta \), denoted by \( \{\psi_\delta^k(v^0_\delta; c)\}_k \), and a function \( w \in \text{Lip}(\mathbb{T}^d; \mathbb{R}) \) such that
   \[
   \sup_{0 \leq k \leq 2K} \sup_m \left| \psi_\delta^k(v^0_\delta; c)(x_{m+1}) - \psi^k(w; c)(x_{m+1}) \right| \to 0 \quad \text{as} \quad \delta' \to 0,
   \]
   where \( \sup_m \) stands for the supremum with respect to \( m \) such that \( (x_{m+1}, t_\delta) \in \tilde{G}_\delta \).

2. If \( v^0_\delta \) is such that its Lipschitz interpolation converges uniformly to a function \( w \in \text{Lip}(\mathbb{T}^d; \mathbb{R}) \) as \( \delta \to 0 \), the whole sequence \( \{\psi_\delta^k(v^0_\delta; c)\}_\delta \) satisfies
   \[
   \sup_{0 \leq k \leq 2K} \sup_m \left| \psi_\delta^k(v^0_\delta; c)(x_{m+1}) - \psi^k(w; c)(x_{m+1}) \right| \to 0 \quad \text{as} \quad \delta \to 0.
   \]

3. If \( v^0_\delta \) is such that \( v^0_\delta(x_{m+1}) = w(x_{m+1}) \) with a fixed \( w \in \text{Lip}_r(\mathbb{T}^d) \), it holds that
   \[
   \sup_{0 \leq k \leq 2K} \sup_m \left| \psi_\delta^k(v^0_\delta; c)(x_{m+1}) - \psi^k(w; c)(x_{m+1}) \right| \leq \beta_0 \sqrt{h},
   \]
   where \( \beta_0 > 0 \) is a constant independent of \( \delta \), \( w \) and \( c \).

4. Let \( w_\delta \) be the Lipschitz interpolation of \( v^0_\delta \). Then, it holds that
   \[
   \sup_{0 \leq k \leq 2K} \sup_m \left| \psi_\delta^k(v^0_\delta; c)(x_{m+1}) - \psi^k(w_\delta; c)(x_{m+1}) \right| \leq \tilde{\beta}_0 \sqrt{h},
   \]
   where \( \tilde{\beta}_0 > 0 \) is a constant independent of \( \delta \), \( v^0_\delta \) and \( c \).

**Proof** By Lemma 2.4, the sequence of the Lipschitz interpolation of \( v^0_\delta \) is uniformly bounded and equi-Lipschitz (\( r \) is fixed in (2.4) for all \( \delta \)). Hence, we find a convergent subsequence with the limit \( w \in \text{Lip}(\mathbb{T}^d; \mathbb{R}) \). Then, the claim 1 is reduced to the claim 2. We can prove the claims 2, 3 and 4 in the same way as the proof of Theorem 2.2 in [48], by means of Lemma 2.1. \( \square \)
As for (2.5), $\bar{\varphi}^k(\cdot; c)$ tends to $\bar{\varphi}^l(\cdot; c)$, where
\[
\bar{\varphi}^l(w; c)(x) := \sup_{y \in AC([t,0];\mathbb{T}^d)} \left\{ -\int_0^t L(c)(y(s), s, \gamma(s)) ds + w(\gamma(0)) \right\}, \quad t \in [-1, 0].
\]

Note that $\tilde{v}(\cdot, t) = \bar{\varphi}^l(w; c)$ is the semiconvex a.e. solution of
\[
\begin{cases}
v_t(x, t) + H(x, t, c + v_x(x, t)) = 0 & \text{in } \mathbb{T}^d \times [-1, 0),
v(x, 0) = w(x)
\end{cases}
\]
(2.10)

### 2.6 Hyperbolic scaling limit of derivative of solution

We prove that, when $u^k = \varphi^k_\delta(v^0; c)$ converges to a viscosity solution $u(\cdot, t) = \varphi^l(w; c)$ of (2.9) (we use the symbol $u$ instead of $v$) for some sequence $\delta = (h, \tau) \to 0$ in the sense of the claim 2 of Theorem 2.5, each partial derivative $(D_x^i v)^k$ also converges to $u_{x^i}$ pointwise a.e. A similar statement was proved in [48] under the assumption that $v^0\}_{m+1} = w(x_{m+1})$ with a fixed semiconcave function $w \in \text{Lip}_c(\mathbb{T}^d, \mathbb{R})$. Here, we remove this assumption by means of a different approach based on Theorem 2.3.

**Theorem 2.6** Consider the situation of the claim 2 of Theorem 2.5. Let $(x, t) \in \mathbb{T}^d \times (0, 1]$ be such that $u_{x^i}(x, t)$ exists, where $u(\cdot, t) := \varphi^l(w; c)$. Note that a.e. points of $\mathbb{T}^d \times [0, 1]$ have such a property. Let $(x_n, t_l) \in \mathcal{G}_\delta$ be such that $(x_n, t_l) \to (x, t)$ as $\delta \to 0$. Then, $u_{m+1}^k = \varphi^k_\delta(v^0; c)(x_{m+1})$ satisfies
\[
| (D_x^i v)^l_n - u_{x^i}(x, t) | \to 0 \quad \text{as } \delta \to 0.
\]

**Proof** The strategy is the following: if the convergence fails, $(D_x^i v)^l_m$ must keep away from $u_{x^i}(\cdot, t)$ within a certain interval in the $e_i$-direction or $(-e_i)$-direction because of semiconcavity, which violates Theorem 2.5.

We proceed by contradiction. Suppose that we have an $\varepsilon_0 > 0$ and a subsequence of $\delta \to 0$, still denoted by the same symbol, such that $| (D_x^i v)^l_n - u_{x^i}(x, t) | \geq \varepsilon_0$ for $\delta \to 0$. Then, there are two cases

(i) $(D_x^i v)^l_n \geq u_{x^i}(x, t) + \varepsilon_0$, \quad (ii) $(D_x^i v)^l_n \leq u_{x^i}(x, t) - \varepsilon_0$.

Since $s(y) := u(x^1, \ldots, x^{i-1}, y, x^{i+1}, \ldots, x^d, t)$ is a semiconcave function of $y$ due to the semiconcave feature of $\varphi^l(w; c)$, we have
\[
\text{ess sup}_{\{y : |y - x^i| \leq v\}} |s'(y) - u_{x^i}(x, t)| \to 0 \quad \text{as } v \to 0.
\]

Hence, there exists $v_0 > 0$ such that $|s'(y) - u_{x^i}(x, t)| \leq \varepsilon_0/3$ for a.e. $y$ with $|y - x^i| \leq v_0$ ($y$ must be a point of differentiability of $s(\cdot)$). Let $r_0 \in \mathbb{N}$ be a natural number satisfying
\[
\frac{1}{2} \min \left\{ v_0, \frac{\varepsilon_0}{3M(t_l)} \right\} \leq 2hr_0 + 2h \leq \min \left\{ v_0, \frac{\varepsilon_0}{3M(t_l)} \right\},
\]
where $M(t_l)$ is mentioned in Theorem 2.3.

**Case (i):** Due to Theorem 2.3, we have for any $0 < r \leq r_0$,
\[
(D_x^i v)^l_n - (D_x^i v)^l_{n-2re_i} = \sum_{r'=0}^{r-1} \left( (D_x^i v)^l_{n-2re_i+2r'+1e_i} - (D_x^i v)^l_{n-2re_i+2r'e_i} \right)
\]
\[
\begin{align*}
(D_x v)^l_{n-2r e_i} - s'(y) & \geq \left( (D_x v)^l_n - \frac{\varepsilon_0}{3} \right) - \left( u_x(x, t) + \frac{\varepsilon_0}{3} \right) \\
& \geq \frac{\varepsilon_0}{3} \text{ for a.e. } y \in [x^i - v_0, x^i + v_0].
\end{align*}
\]

Therefore, we see that for \( \delta \to 0 \),
\[
\begin{align*}
\left( v^l_{n+h} - u(x + he_i, t) \right) - \left( v^l_{n-2r_0 e_i - e_i} - u(x - 2hr_0 e_i - he_i, t) \right) \\
= \sum_{r=0}^{r_0} \int_{x^i - 2hr - h}^{x^i - 2hr + h} \left( (D_x v)^l_{n-2r e_i} - s'(y) \right) dy \\
\geq \frac{\varepsilon_0}{3} (2hr_0 + 2h) \geq \frac{\varepsilon_0}{3} \cdot \frac{1}{2} \min \left\{ v_0, \frac{\varepsilon_0}{3M(\eta_t)} \right\}.
\end{align*}
\]

Since Theorem 2.5 implies that the first line tends to 0 as \( \delta \to 0 \), we reach a contradiction.

Case (ii): Similarly, we have for any \( 0 < r \leq r_0 \),
\[
\begin{align*}
(D_x v)^l_{n+2r e_i} - (D_x v)^l_{n-2r e_i} & \leq 2hr M(\eta_t) \leq 2hr_0 M(\eta_t) \leq \frac{\varepsilon_0}{3}, \\
s'(y) - (D_x v)^l_{n+2r e_i} & \geq \left( u_x(x, t) - \frac{\varepsilon_0}{3} \right) - \left( (D_x v)^l_n + \frac{\varepsilon_0}{3} \right) \\
& \geq \frac{\varepsilon_0}{3} \text{ for a.e. } y \in [x^i - v_0, x^i + v_0].
\end{align*}
\]

Therefore, we see that for \( \delta \to 0 \),
\[
\begin{align*}
(u(x + 2hr_0 e_i + he_i, t) - v^l_{n+2r_0 e_i + e_i}) - (u(x - he_i, t) - v^l_{n-e_i}) \\
= \sum_{r=0}^{r_0} \int_{x^i + 2hr - h}^{x^i + 2hr + h} \left( s'(y) - (D_x v)^l_{n+2r e_i} \right) dy \\
\geq \frac{\varepsilon_0}{3} (2hr_0 + 2h) \geq \frac{\varepsilon_0}{3} \cdot \frac{1}{2} \min \left\{ v_0, \frac{\varepsilon_0}{3M(\eta_t)} \right\},
\end{align*}
\]

which is a contradiction. \( \square \)

As for (2.5), we obtain a similar result for \( \tilde{\psi}^k_\delta (\cdot ; c) \).

### 2.7 Hyperbolic scaling limit of minimizing random walk

It was proved in [48] that a minimizing random walk of \( \varphi^\delta \) converges to a minimizing curve with an end point \((x, t)\) of \( \varphi^l \) as \( \delta \to 0 \) under the hyperbolic scaling, provided a minimizing curve with the end point \((x, t)\) is unique (a.e. points \((x, t)\) have such a property). In this subsection, we extend the result without any assumption. Note that there is a case where we have uncountably many minimizing curves with a common end point.

Let \( \gamma_\delta (\cdot) : [0, t] \to \mathbb{R}^d, \eta_\delta (\gamma)(\cdot) : [0, t] \to \mathbb{R}^d, t \leq 1 \) be the linear interpolations of each sample path \( \gamma \in \Omega_{n+1,0} \), \( \eta(\gamma) \), respectively, where \( t \in [t_{l+1}, t_{l+2}] \):

\[
\gamma_\delta(s) := \begin{cases} 
  x_n & \text{for } s \in [t_{l+1}, t], \\
  y^k + \frac{y^{k+1} - y^k}{\tau} (s - t_k) & \text{for } s \in [t_k, t_{k+1}].
\end{cases}
\]

\( \delta \) Springer
Lemma 2.7 Let $f : [0, t] \to \mathbb{R}^d$ be a Lipschitz function with a Lipschitz constant $\theta$ satisfying $f(t) = 0$. Then, it holds that $\| f \|_{C^0([0,t])} \leq \theta$ $\| f \|_{L^2([0,t])} + \sqrt{\int f^2} \| L^2([0,t])}$.

See the proof of Lemma 3.5 in [45].

Theorem 2.8 Consider the situation of the claim 2 of Theorem 2.5. Let $(x, t) \in \mathbb{R}^d \times (0, 1]$ be an arbitrary point. Let $\Gamma^*(x, t)$ be the set of all minimizing curves $\gamma^* : [0, t] \to \mathbb{R}^d$ for $u(x, t) := \phi'(w; c)(x)$. Let $(x_n, t_{i+1}) \in \tilde{G}_\delta$ be such that $(x_n, t_{i+1}) \to (x, t)$ as $\delta \to 0$. For each $\delta$, let $\gamma \in \Omega^{l+1,0}_n$ be the random walk generated by the minimizing control $\xi^*$ for $v_n^{l+1}$.

1. Fix an arbitrary $\varepsilon_1 > 0$ and define the set

$$\bar{\Omega}^{\varepsilon_1}_\delta := \{ \gamma \in \Omega^{l+1,0}_n \mid \text{there exists } \gamma^* = \gamma^*(\gamma) \in \Gamma^*(x, t) \text{ such that } \| \eta_\delta'(\gamma) - \gamma^* \|_{L^2([0,t])} \leq \varepsilon_1 \}.$$

Then, we have $\text{Prob}(\bar{\Omega}^{\varepsilon_1}_\delta) \to 1$ as $\delta \to 0$.

2. Fix an arbitrary $\varepsilon_2 > 0$ and define the set

$$\bar{\Omega}^{\varepsilon_2}_\delta := \{ \gamma \in \Omega^{l+1,0}_n \mid \text{there exists } \gamma^* = \gamma^*(\gamma) \in \Gamma^* \text{ such that } \| \eta_\delta - \gamma^* \|_{C^0([0,t])} \leq \varepsilon_2 \}.$$

Then, we have $\text{Prob}(\bar{\Omega}^{\varepsilon_2}_\delta) \to 1$ as $\delta \to 0$.

3. Fix an arbitrary $\varepsilon_3 > 0$ and define the set

$$\bar{\Omega}^{\varepsilon_3}_\delta := \{ \gamma \in \Omega^{l+1,0}_n \mid \text{there exists } \gamma^* = \gamma^*(\gamma) \in \Gamma^* \text{ such that } \| \gamma^* - \gamma^* \|_{C^0([0,t])} \leq \varepsilon_3 \}.$$

Then, we have $\text{Prob}(\bar{\Omega}^{\varepsilon_3}_\delta) \to 1$ as $\delta \to 0$.

Proof For each $\gamma \in \bar{\Omega}^{\varepsilon_1}_\delta$, we have $\gamma^* \in \Gamma^*(x, t)$ such that

$$\| \eta_\delta'(\gamma) - \gamma^* \|_{L^1([0,t])} \leq \sqrt{t} \| \eta_\delta'(\gamma) - \gamma^* \|_{L^2([0,t])} \leq \varepsilon_1 \sqrt{t} \leq \varepsilon_1,$$

which implies

$$|\eta_\delta'(\gamma)(s) + x - x_n - \gamma^*(s)| = \left| \int_s^t \left( \eta_\delta'(\gamma)'(\tilde{s}) - \gamma^*(\tilde{s}) \right) d\tilde{s} \right| \leq \varepsilon_1 \text{ for all } s \in [0, t],$$

where $\eta_\delta'(\gamma)(s) = x_n$ for $s \in [t_{i+1}, t]$. Hence, for sufficiently small $\delta$ such that $|x - x_n| \leq \varepsilon_1/2$, we have $\| \eta_\delta'(\gamma) - \gamma^* \|_{C^0([0,t])} \leq 3\varepsilon_1/2$. Therefore, $\bar{\Omega}^{\varepsilon_1}_\delta \subset \bar{\Omega}^{\varepsilon_3}_\delta$ holds and the claim 2 follows from the claim 1.

For any $\varepsilon > 0$, define the set

$$\Gamma^\varepsilon := \{ r : [0, t] \to \mathbb{R}^d \mid r \in \text{Lip}, \| r \|_{\infty} \leq (dx_1)^{-1}, \quad r(t) = x, \| r' - \gamma^* \|_{L^2([0,t])} \leq \varepsilon \text{ for all } \gamma^* \in \Gamma^*(x, t) \}.$$
Suppose that there is no such \( v(\varepsilon) > 0 \). Then, we have a sequence \( \{r_i\} \in \mathbb{N} \subset \Gamma^\varepsilon \) such that

\[
\int_0^t L^{(c)}(r_i(s), s, r'_i(s))ds + w(r_i(0)) \rightarrow u(x, t) = \inf_{\gamma \in AC, \gamma(t) = x} \left\{ \int_0^t L^{(c)}(\gamma(s), s, \gamma'(s))ds + w(\gamma(0)) \right\}.
\]

By Tonelli’s theory (see [21]), we find a subsequence of \( \{r_i\} \), still denoted by the same symbol, which converges uniformly to a curve \( \gamma^* \in \Gamma^\varepsilon(x, t) \). Observe that

\[
\kappa_i := \int_0^t L^{(c)}(r_i(s), s, r'_i(s))ds + w(r_i(0)) - \left\{ \int_0^t L^{(c)}(\gamma^*(s), s, \gamma'^*(s))ds + w(\gamma^*(0)) \right\}
\]

\[
= \int_0^t \left[ L^{(c)}(r_i(s), s, r'_i(s)) - L^{(c)}(\gamma^*(s), s, r'_i(s)) \right]ds + \int_0^t \left[ L^{(c)}(\gamma^*(s), s, r'_i(s)) - L^{(c)}(\gamma^*(s), s, \gamma'^*(s)) \right]ds + (w(r_i(0)) - w(\gamma^*(0)))
\]

\[
= \int_0^t L^{(c)}_x(\gamma^*(s) + \theta_i(s), s, r'_i(s)) \cdot (r_i(s) - \gamma^*(s))ds
\]

\[
+ \int_0^t L^{(c)}_\xi(\gamma^*(s), s, \gamma'^*(s)) \cdot (r'_i(s) - \gamma'^*(s))ds
\]

\[
+ \int_0^t \frac{1}{2} L^{(c)}_{\xi\xi}(\gamma^*(s), s, \gamma'^*(s) + \tilde{\theta}_i(s))(r'_i(s) - \gamma'^*(s)) \cdot (r'_i(s) - \gamma'^*(s))ds
\]

\[
+(w(r_i(0)) - w(\gamma^*(0))) \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty,
\]

where \( \theta_i, \tilde{\theta}_i \) come from the Taylor’s formula. In the right hand side, the first term and the last term are bounded by \( O(||r_i - \gamma^*||c_0([0, t])) \); since every element of \( \Gamma^\varepsilon(x, t) \) is \( C^2 \)-solution of the Euler–Lagrange equation, the second term is treated as

\[
\int_0^t L^{(c)}_x(\gamma^*(s), s, \gamma'^*(s)) \cdot (r'_i(s) - \gamma'^*(s))ds
\]

\[
= L^{(c)}_x(\gamma^*(s), s, \gamma'^*(s)) \cdot (r_i(s) - \gamma^*(s))|_{s=0}^{s=t} - \int_0^t \frac{d}{ds} L^{(c)}_x(\gamma^*(s), s, \gamma'^*(s)) \cdot (r_i(s) - \gamma^*(s))ds
\]

\[
= L^{(c)}_x(\gamma^*(s), s, \gamma'^*(s)) \cdot (r_i(s) - \gamma^*(s))|_{s=0}^{s=t} - \int_0^t L^{(c)}_x(\gamma^*(s), s, \gamma'^*(s)) \cdot (r_i(s) - \gamma^*(s))ds
\]

\[
\leq O(||r_i - \gamma^*||c_0([0, t]));
\]

by convexity of \( L^{(c)} \), the third term is bounded from the below by \( \alpha' ||r'_i - \gamma'^*||^2_{L^2([0, t])} \) with a positive constant \( \alpha' \) independent from \( i \). Hence, we obtain

\[
0 < \alpha' \varepsilon^2 \leq \alpha' ||r'_i - \gamma'^*||^2_{L^2([0, t])} \leq \kappa_i + O( ||r_i - \gamma^*||c_0([0, t]) ) \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty,
\]

which is a contradiction.

If the claim 1 does not hold, we have \( b > 0 \) and a sequence \( \delta_i = (h_i, \tau_i) \rightarrow 0 \) as \( i \rightarrow \infty \) for which \( \text{Prob}(\Omega^{l+1, 0}_n \backslash \hat{\Omega}^{l+1, 0}_b) = 1 - \text{Prob}(\hat{\Omega}^{l+1, 0}_b) \geq b \) for all \( i \). For any \( \gamma \in \Omega^{l+1, 0}_n \backslash \hat{\Omega}^{l+1, 0}_b \), we
have $\eta_\delta(\gamma)(\cdot) + x - x_n \in \Gamma^{\varepsilon_1}$. Hence, by Lemma 2.1 and (2.11), we have

$$v_n^{l+1} = E_{\mu_n^{l+1,0}(\cdot; \xi_n^*)} \left[ \int_0^t L^{(c)}(\eta_\delta(\gamma)(s), s, \eta_\delta(\gamma)'(s))ds + w(\eta_\delta(\gamma')(0)) \right]$$

$$+ O(\sqrt{h_i}) + O(|t - t_{i+1}|) + O(\max |v_{h_i}^0 - w|)$$

$$= \sum_{\gamma \in \Omega_n^{l+1,0} \setminus \Omega_{\delta_1}^{l+1,0}} \mu_n^{l+1,0}(\gamma; \xi_n^*) \left\{ \int_0^t L^{(c)}(\eta_\delta(\gamma)(s) + x - x_n, s, \eta_\delta(\gamma)'(s))ds ight. \\

$$+ w(\eta_\delta(\gamma)(0) + x - x_n) \bigg\}$$

$$+ \sum_{\gamma \in \Omega_n^{l+1,0}} \mu_n^{l+1,0}(\gamma; \xi_n^*) \left\{ \int_0^t L^{(c)}(\eta_\delta(\gamma)(s) + x - x_n, s, \eta_\delta(\gamma)'(s))ds ight. \\

$$+ w(\eta_\delta(\gamma)(0) + x - x_n) \bigg\}$$

$$+ O(|x - x_n|_{\infty}) + O(\sqrt{h_i}) + O(|t - t_{i+1}|) + O(\max |v_{h_i}^0 - w|)$$

$$\geq \sum_{\gamma \in \Omega_n^{l+1,0} \setminus \Omega_{\delta_1}^{l+1,0}} \sum_{\gamma \in \Omega_n^{l+1,0}} \mu_n^{l+1,0}(\gamma; \xi_n^*) \left\{ \int_0^t \|\eta_\delta(\gamma)(s) - \gamma_\delta\|_L^2 ds \right. \\

$$+ O(\sqrt{h_i}) + O(|t - t_{i+1}|) + O(\max |v_{h_i}^0 - w|)$$

$$\geq b v(\varepsilon_1) + u(x, t)$$

$$+ O(|x - x_n|_{\infty}) + O(\sqrt{h_i}) + O(|t - t_{i+1}|) + O(\max |v_{h_i}^0 - w|)$$

$$> u(x, t) + \frac{1}{2} b v(\varepsilon_1) \quad \text{as } i \to \infty.$$ 

Since we have the convergence $v_n^{l+1} \to u(x, t)$ as $\delta \to 0$, we reach a contradiction.

We prove the claim 3. For any $\varepsilon > 0$, define the set

$$B_\delta^\varepsilon := \{ \gamma \in \Omega^{l+1,0}_n \mid \|\eta_\delta(\gamma) - \gamma_\delta\|_{L^2([0,t])} \leq \varepsilon \}.$$ 

Then, by Lemma 2.1, we have

$$\sum_{\gamma \in \Omega_n^{l+1,0} \setminus B_\delta^\varepsilon} \mu_n^{l+1,0}(\gamma; \xi_n^*) \varepsilon^2 \leq \sum_{\gamma \in \Omega_n^{l+1,0}} \mu_n^{l+1,0}(\gamma; \xi_n^*) \int_0^t |\eta_\delta(\gamma)(s) - \gamma_\delta(s)|^2 ds$$

$$= \sum_{\gamma \in \Omega_n^{l+1,0}} \mu_n^{l+1,0}(\gamma) \left\{ \sum_{0 \leq k \leq l} |\eta_\delta^k(\gamma) - \gamma_\delta^k|^2 \tau \right\} + O(h) = O(h).$$

Hence, for any fixed $\varepsilon_3 \gg \varepsilon > 0$, we obtain

$$\text{Prob}(\Omega_n^{l+1,0} \setminus B_\delta^\varepsilon) = 1 - \text{Prob}(B_\delta^\varepsilon) \leq \frac{O(h)}{\varepsilon^2} \to 0 \quad \text{as } \delta \to 0. \quad (2.12)$$

Below, $\beta_1, \beta_2, \ldots$ are some positive constants independent of $\delta$. By Lemma 2.7, we have a constant $\beta_1 > 0$ such that for any $\gamma \in B_\delta^\varepsilon$,

$$\|\eta_\delta(\gamma) - \gamma_\delta\|_{C^0([0,t])} \leq \beta_1 \sqrt{\varepsilon}.$$
With $\sum_{\gamma \in \Omega^{l+1.0}_N} = \sum_{\gamma \in B^\epsilon_3} + \sum_{\gamma \in \Omega^{l+1.0}_N \setminus B^\epsilon_3}$, we have
\[
\sum_{\gamma \in \Omega^{l+1.0}_N} \mu^{l+1.0}_N(\gamma) \parallel \eta_\delta(\gamma) - \gamma_0 \parallel C^0((0,1)) \leq \beta_1 \sqrt{\epsilon} + \beta_2 \left(1 - \text{Prob}(B^\epsilon_3)\right). \tag{2.13}
\]
For each $\gamma \in B^\epsilon_3 \setminus \Omega^{\epsilon^3}_\delta$, it holds that
\[
\parallel \eta_\delta(\gamma) - \gamma_0 \parallel C^0((0,1)) \leq \beta_1 \sqrt{\epsilon}, \quad \parallel \gamma_0 - \gamma^* \parallel C^0((0,1)) > \epsilon_3 \quad \text{for all } \gamma^* \in \Gamma^*,
\]
which implies
\[
\parallel \eta_\delta(\gamma) - \gamma^* \parallel C^0((0,1)) \geq \parallel \gamma_0 - \gamma^* \parallel C^0((0,1)) - \parallel \eta_\delta(\gamma) - \gamma_0 \parallel C^0((0,1)) > \epsilon_3 - \beta_1 \sqrt{\epsilon} \quad \text{for all } \gamma^* \in \Gamma^*.
\]
Therefore, we see that for $0 < \epsilon \ll \epsilon_3$,
\[
\gamma \in B^\epsilon_3 \setminus \Omega^{\epsilon^3}_\delta \Rightarrow \gamma \in \Omega^{l+1.0}_N \setminus \tilde{\Omega}^{\epsilon^3 - \beta_1 \sqrt{\epsilon}}. \tag{2.14}
\]
For each $\gamma \in \Omega^{l+1.0}_N$, let $\gamma^* = \gamma^*(\gamma)$ denote the element of $\Gamma^*$ nearest to $\gamma_0$ in $\parallel \cdot \parallel C^0$ (if there are several, it is one of them). Then, by (2.13) and (2.14), we have
\[
\beta_1 \sqrt{\epsilon} + \beta_2 \left(1 - \text{Prob}(B^\epsilon_3)\right) \geq \sum_{\gamma \in \Omega^{l+1.0}_N} \mu^{l+1.0}_N(\gamma) \parallel \gamma_0 - \eta_\delta(\gamma) \parallel C^0((0,1))
\geq \sum_{\gamma \in B^\epsilon_3 \setminus \Omega^{\epsilon^3}_\delta} \mu^{l+1.0}_N(\gamma) \parallel \gamma_0 - \eta_\delta(\gamma) \parallel C^0((0,1))
\geq \sum_{\gamma \in B^\epsilon_3 \setminus \Omega^{\epsilon^3}_\delta} \mu^{l+1.0}_N(\gamma) \parallel \gamma_0 - \gamma^*(\gamma) \parallel C^0((0,1))
- \sum_{\gamma \in \Omega^{l+1.0}_N \setminus \tilde{\Omega}^{\epsilon^3 - \beta_1 \sqrt{\epsilon}}} \mu^{l+1.0}_N(\gamma) \parallel \gamma^*(\gamma) - \eta_\delta(\gamma) \parallel C^0((0,1))
\geq \text{Prob}(B^\epsilon_3 \setminus \Omega^{\epsilon^3}_\delta) \epsilon_3 - \beta_3 \text{Prob}\left(\Omega^{l+1.0}_N \setminus \tilde{\Omega}^{\epsilon^3 - \beta_1 \sqrt{\epsilon}}\right),
\]
which implies with $\text{Prob}(\Omega^{l+1.0}_N \setminus \tilde{\Omega}^{\epsilon^3 - \beta_1 \sqrt{\epsilon}}) = 1 - \text{Prob}(\tilde{\Omega}^{\epsilon^3 - \beta_1 \sqrt{\epsilon}})$,
\[
\text{Prob}(B^\epsilon_3) - \text{Prob}(\Omega^{\epsilon^3}_\delta) \leq \text{Prob}(B^\epsilon_3 \setminus \Omega^{\epsilon^3}_\delta)
\leq \beta_1 \sqrt{\epsilon} + \beta_2 \left(1 - \text{Prob}(B^\epsilon_3)\right) + \beta_3 \left(1 - \text{Prob}(\Omega^{\epsilon^3}_\delta)\right) + \epsilon_3.
\]
Sending $\delta \to 0$, we obtain with the claim 2 and (2.12),
\[
1 - \frac{\beta_1 \sqrt{\epsilon}}{\epsilon_3} \leq \liminf_{\delta \to 0} \text{Prob}(\Omega^{\epsilon^3}_\delta) \leq 1.
\]
Since we may take $\epsilon > 0$ arbitrarily small so that $\sqrt{\epsilon}/\epsilon_3 > 0$ can be arbitrarily small, we conclude
\[
\lim_{\delta \to 0} \text{Prob}(\Omega^{\epsilon^3}_\delta) = 1.
\]
\[\square\]
As for (2.5), we obtain a similar result for maximizing random walks of $\tilde{\vartheta}_d^k(\cdot; c)$. 
2.8 Time-global extension of Lax–Oleinik type solution map

We extend the solution map \( \varphi_{\delta}^k(\cdot; c) \) of (2.4) to \( k \to \infty \) with a fixed \( \delta = (h, \tau) \). This is a non-trivial issue, because the constant \( \lambda_1 \) found in Theorem 2.2 that guarantees the solvability of (2.4) within \([0, 1]\) depends on the terminal time, i.e. an instant observation only shows that \( v^0 \) with \( |(D_x v^0)|_\infty \leq r \) yields \( v^{2K} \) such that \( |(D_x v^0)|^{2K}_\infty \leq u^* \) with \( u^* > r \) in general. Hence, we need an additional argument to prove \( |(D_x v^0)|^{2K}_\infty \leq r \) for some \( r > 0 \) so that we can solve (2.4) within \([0, 1]\) with initial data \( v^{2K} \) with the same \( \delta \).

Our strategy is the following: we first observe a priori boundedness of minimizing curves of \( u(\cdot, t) = \varphi^t(w; c) \) independently from \( w \in \text{Lip}(\mathbb{T}^d; \mathbb{R}) \), which implies a priori boundedness of \( u_t(\cdot, 1) \) independently from \( w \in \text{Lip}(\mathbb{T}^d; \mathbb{R}) \). Theorems 2.3 and 2.5 imply a similar boundedness for \( v^{2K} = \varphi_{\delta}^{2K}(v^0; c) \).

**Lemma 2.9** For each \( t > 0 \), there exist a constant \( \beta(t) > 0 \) independent of \( c \in P \) and \( w \in \text{Lip}(\mathbb{T}^d; \mathbb{R}) \) such that every minimizing curve \( \gamma^* \) for \( u(\cdot, t) = \varphi^t(w; c) \) satisfies \( |\gamma^*(s)|_\infty \leq \beta(t) \) for all \( s \in [0, t] \).

**Proof** This is proved in [21] for time-independent Lagrangians. For readers’ convenience, we give a proof. Let \( \gamma^* \) be a minimizing curve for \( u(x, t) = \varphi^t(w; c)(x) \) regarded as \( \gamma^*: [0, t] \to \mathbb{R}^d \) with \( x \in [0, 1]^d \). Set \( y := \gamma^*(0) + z \) with \( z \in \mathbb{Z}^d \) such that \( \gamma^*(0) + z \in [0, 1]^d \). We see that

\[
\int_0^t L^{(c)}(\gamma^*(s), s, \gamma^*(s))ds \leq \inf_{\gamma \in AC([0, t]; \mathbb{R}^d)} \gamma(0) = \gamma(0) = y
\]

In fact, if not, there exists \( \gamma \in AC([0, t]; \mathbb{R}^d) \), \( \gamma(t) = x, \gamma(0) = y \) such that

\[
\int_0^t L^{(c)}(\gamma^*(s), s, \gamma^*(s))ds \geq \int_0^t L^{(c)}(\gamma(s), s, \gamma(s))ds,
\]

\[
\Rightarrow \int_0^t L^{(c)}(\gamma^*(s), s, \gamma^*(s))ds + w(\gamma^*(0)) > \int_0^t L^{(c)}(\gamma(s), s, \gamma(s))ds + w(\gamma(0)),
\]

where we note that \( w(\gamma^*(0)) = w(y) = w(\gamma(0)) \) due to the periodicity of \( w \); this violates the assumption that \( \gamma^* \) is a minimizing curve for \( u(x, t) = \varphi^t(w; c)(x) \). Set \( \gamma(s) := x + \frac{x-y}{t}(s-t) \). Since \( |x-y|_\infty \leq 1 \), we have \( |\gamma'(s)|_\infty \leq t^{-1} \) for all \( s \in [0, t] \). Hence, we have

\[
\int_0^t L^{(c)}(\gamma^*(s), s, \gamma^*(s))ds \leq \int_0^t L^{(c)}(\gamma(s), s, \gamma(s))ds
\]

\[
\leq \sup_{x \in \mathbb{Z}^d, t \in \mathbb{T}, |\xi|_\infty \leq t^{-1}, c \in P} |L^{(c)}(x, t, \xi)|t =: \beta_1(t) t.
\]

Since \( L^{(c)}(\gamma^*(s), s, \gamma^*(s)) \) is continuous with respect to \( s \), we have \( s^* \in [0, t] \) such that

\[
L^{(c)}(\gamma^*(s^*), s^*, \gamma^*(s^*)) = \int_0^t L^{(c)}(\gamma^*(s), s, \gamma^*(s))ds \leq \beta_1(t) t.
\]

Due to (L3), there exists a constant \( \beta_2(t) > 0 \) depending only on \( t \) and \( \beta_1(t) \) such that \( |\gamma^*(s^*)|_\infty \leq \beta_2(t) \). Therefore, with the Euler–Lagrange flow (this is independent of \( c \)) \( \Phi_L^{s_0} = (\Phi_L^{s_0}, \Phi_L^{s_0}): \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d \) and the periodicity of \( L \) in \((x, t)\), we see that

\[
\gamma^*(s) \in \bigcup_{0 \leq s_0 \leq t} \bigcup_{0 \leq s \leq t} \Phi^{s_0, s}_L([-\beta_2(t), \beta_2(t)]^d) \quad \text{for all } s \in [0, t],
\]

where the right hand side is a compact set independent of \( c \in P \) and \( w \in \text{Lip}(\mathbb{T}^d; \mathbb{R}) \). \(\square\)
Lemma 2.10  Let \( \beta(1) \) be the number mentioned in Lemma 2.9 with \( t = 1 \). Set
\[
\tilde{\beta} := \sup_{x \in \mathbb{T}^d, t \in T, |\xi|_\infty \leq \beta(1)}.j
\]
Then, \( u(x, t) := \varphi^t(w; c)(x) \) satisfies for any \( w \in \text{Lip}(\mathbb{T}^d; \mathbb{R}) \) and any \( x \in \mathbb{T}^d \),
\[
-\tilde{\beta} \leq \liminf_{\varepsilon \to 0} \frac{u(x+\varepsilon e_i, 1) - u(x, 1)}{\varepsilon} \leq \limsup_{\varepsilon \to 0} \frac{u(x+\varepsilon e_i, 1) - u(x, 1)}{\varepsilon} \leq \tilde{\beta} \quad (i = 1, \ldots, d).
\]

Proof  Let \( \gamma^* : [0, 1] \to \mathbb{T}^d \) be a minimizing curve for \( u(x, t) \). It is well-known that we have for any \( 0 < \tau < 1 \),
\[
u(x, 1) = \int_\tau^1 L^{(c)}(\gamma^*(s), s, \gamma^*(s))ds + u(\gamma^*(\tau), \tau)
\]
and \( \mu_x(\gamma^*(\tau), \tau) = L^{(c)}(\gamma^*(\tau), \tau, \gamma^*(\tau)) \). By the variational structure and Lemma 2.9, we obtain
\[
\limsup_{\varepsilon \to 0} \frac{u(x+\varepsilon e_i, 1) - u(x, 1)}{\varepsilon} \leq \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \int_\tau^1 L^{(c)}(\gamma^*(s) + \varepsilon e_i, s, \gamma^*(s)) - L^{(c)}(\gamma^*(s), s, \gamma^*(s))ds + u(\gamma^*(\tau) + \varepsilon e_i, \tau) - u(\gamma^*(\tau), \tau) \right\} = \int_\tau^1 L^{(c)}(\gamma^*(s), s, \gamma^*(s))ds + L^{(c)}(\gamma^*(\tau), \tau, \gamma^*(\tau)) \leq \tilde{\beta}.
\]
Let \( \gamma^*_\varepsilon : [0, 1] \to \mathbb{T}^d \) be a minimizing curve for \( u(x + \varepsilon e_i, t) \). By the variational structure and Lemma 2.9, we obtain
\[
\liminf_{\varepsilon \to 0} \frac{u(x+\varepsilon e_i, 1) - u(x, 1)}{\varepsilon} \geq \liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \int_\tau^1 L^{(c)}(\gamma^*_\varepsilon(s), s, \gamma^*_\varepsilon(s)) - L^{(c)}(\gamma^*_\varepsilon(s) - \varepsilon e_i, s, \gamma^*_\varepsilon(s))ds + u(\gamma^*_\varepsilon(\tau) - \varepsilon e_i, \tau) \right\} = \liminf_{\varepsilon \to 0} \left\{ - \int_\tau^1 L^{(c)}(\gamma^*_\varepsilon(s), s, \gamma^*_\varepsilon(s))ds - L^{(c)}(\gamma^*_\varepsilon(\tau), \tau, \gamma^*_\varepsilon(\tau)) \right\} \geq -\tilde{\beta}.
\]

Theorem 2.11  Set \( h_0 = (12\tilde{\beta}_0 M(1))^{-2} \), where \( \tilde{\beta}_0 \) is mentioned in the claim 4 of Theorem 2.5 and \( M(1) \) in Theorem 2.3. If \( h \in (0, h_0) \) with \( 0 \leq \lambda_0 \leq \lambda := \tau/h < \lambda_1 \), the solution \( v_{m+1} \)
of (2.4) satisfies with the constant \( \tilde{\beta} \) in Lemma 2.10,
\[
|D_{x^i} v|_{m}^{2K} \leq \tilde{\beta} + 1 \quad \text{for all } m(i = 1, \ldots, d),
\]
where \( \tilde{\beta}_0, M(1) \) and \( \lambda_1 \) depend on \( r \) but \( \tilde{\beta} \) does not.

Proof  We proceed by contradiction: Suppose that there exists \( m \) such that \( |(D_{x^i} v)|_{m}^{2K} > \tilde{\beta} + 1 \). We deal with the case \( (D_{x^i} v)|^{2K}_{m} > \tilde{\beta} + 1 \). Take \( r_0 \in \mathbb{N} \) such that \( \frac{1}{5M(1)} \leq 2hr_0 \leq \frac{1}{2M(1)} \). By Theorem 2.3, we have for all \( r = 0, \ldots, r_0 \),
\[
(D_{x^i} v)|^{2K}_{m} - (D_{x^i} v)|^{2K}_{m-2r e_i} \leq M(1) \cdot 2hr \leq \frac{1}{2}, \quad (D_{x^i} v)|^{2K}_{m-2r e_i} > \tilde{\beta} + \frac{1}{2}.
\]

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Let \( w_\delta \) be the Lipschitz interpolation of \( v^0 \), whose Lipschitz constant is bounded by \( \beta r \) due to Lemma 2.4. Set \( u(\cdot, t) := \varphi^t(w_\delta; c) \). Then, we have with the claim 4 of Theorem 2.5 and Lemma 2.10,
\[
2\tilde{\beta}_0\sqrt{h} \geq (v_{2K}^{2K})_{m+1} - u(x_m + he_i, 1) - (v_{2K}^{2K})_{m-2re_i} - u(x_m - 2hr_0e_i - he_i, t)) \geq (r_0 + 1) \left( \tilde{\beta} + \frac{1}{2} - \bar{\beta} \right) \cdot 2h \geq \frac{1}{6M(1)},
\]
which is a contradiction for \( h \in (0, h_0) \). The other case \( (D_x v_m^{2K})_{m+1} < -\tilde{\beta} - 1 \) is treated in the same way. \( \square \)

Theorem 2.11 implies that the solution \( v_{m+1}^{k+1} \) of (2.4) with \( r \geq \tilde{\beta} + 1 \) satisfies
\[
| (D_x v_m^{2K})_{m+1} |_{\infty} \leq \tilde{\beta} + 1 \leq r.
\]

Hence, we may solve (2.4) with initial data \( v^{2K}_{m+1} \) up to \( k = 2K \), which means that we have \( v_{m+1}^{k+1} \) up to \( k = 4K \) with \( |(D_x v)^{4K}_{m+1}|_{\infty} \leq \tilde{\beta} + 1 \leq r \) and \( |H_p(x_m, t_k, c + (D_x v)^{k}_{m+1})|_{\infty} \leq (d\lambda_1)^{-1} \) for all \( 0 \leq k \leq 4K \) and \( m \). In this way, the solution \( v_{m+1}^{k+1} \) of (2.4) can be extended to \( k \to \infty \) with
\[
| (D_x v_m^{2K+l})_{m+1} |_{\infty} \leq \tilde{\beta} + 1 \leq r \quad \text{for all } l \in \mathbb{N} \text{ and } m,
\]
\[
|H_p(x_m, t_k, c + (D_x v)^{k}_{m+1})|_{\infty} \leq (d\lambda_1)^{-1} \quad \text{for all } k \geq 0 \text{ and } m.
\]

Furthermore, the bound (2.15) and the proof of Theorem 2.1 of [48] imply that
\[
v_{n+1}^{l+1} = \inf_{\xi} E_{n+1}^{l+1}(\xi; v_0, c) \quad \text{for all } l \geq 0 \text{ and } n
\]
with the unique minimizing control \( \xi^* \) given as
\[
\xi^*_m = H_p(x_m, t_k, c + (D_x v)^{k}_{m+1}), \quad |\xi^*_m|_{\infty} \leq (d\lambda_1)^{-1}.
\]

**Theorem 2.12** Under the condition of Theorem 2.11 with \( r \geq \tilde{\beta} + 1 \), the solution map \( \varphi^k_\delta(\cdot; c) \) of (2.4) is extended to all \( k \in \mathbb{N} \cup \{ 0 \} \), satisfying \( |D_x \varphi^k_\delta(\cdot; c)|_{\infty} \leq \tilde{\beta} + 1 \) for all \( l \in \mathbb{N} \). Furthermore, every minimizing control of \( \varphi^k_\delta(\cdot; c) \) is bounded by \( (d\lambda_1)^{-1} \).

As for (2.5), we obtain the extension of \( \varphi^k_\delta(\cdot; c) \) to \( k \to -\infty \).

### 3 Weak KAM theory: non-autonomous case

We formulate an analogue of weak KAM theory and Aubry–Mather theory for the problem of action minimizing random walks. We construct analogues of the weak KAM objects for each \( \delta = (h, \tau) \) and investigate their hyperbolic scaling limit.

Suppose that \( \delta = (h, \tau) \) and \( r \) are taken so that the assertions in Sect. 2 hold. Introduce the following notation:
\[
W_\delta := \{ w : G_{\text{odd}} \to \mathbb{R} | w(x_{m+1 \pm 2Ne_i}) = w(x_{m+1}) \text{ for } i = 1, \ldots, d, \quad |(D_x w)|_{\infty} \leq r \},
\]
\[
X_\delta := \{ w|_{x \in \mathbb{R}^d} : w \in W_\delta \}.
\]
\[ \bar{X}_\delta := \{ u : \text{pr}.G_{\text{even}} \rightarrow \mathbb{R}^d \mid u = (D_x w), \ w \in W_\delta \}, \]
\[ \bar{X}_\delta := \{ (u(y_1), u(y_2), \ldots, u(y_a)) \in \mathbb{R}^{da} \mid u \in \bar{X}_\delta \}, \]
where the grid points of pr.\(G_{\text{even}}\) are re-labeled as \(y_1, \ldots, y_a\) with \(a := \text{pr}.G_{\text{even}}\).
\[ \mathcal{P}_\delta : \bar{X}_\delta \ni u \mapsto (u(y_1), \ldots, u(y_a)) \in \bar{X}_\delta \] (clearly one to one and onto).

We may reduce the time-1 maps of (2.4) as
\[
\varphi_\delta (\cdot ; c) := \varphi_{2K}^\delta (\cdot ; c) : X_\delta \rightarrow X_\delta, \ \psi_\delta (\cdot ; c) := \psi_{2K}^\delta (\cdot ; c) : \bar{X}_\delta \rightarrow \bar{X}_\delta.
\]
We may also reduce \(\hat{\psi}_\delta (\cdot ; c) := \mathcal{P}_\delta \circ \psi_\delta (\cdot ; c) \circ \mathcal{P}_\delta^{-1}\) to be
\[
\hat{\psi}_\delta (\cdot ; c) : \bar{X}_\delta \rightarrow \bar{X}_\delta.
\]
As for (2.5), we set \(\bar{\varphi}_\delta (\cdot ; c) := \bar{\varphi}_{2K}^\delta (\cdot ; c) : X_\delta \rightarrow X_\delta, \ \bar{\psi}_\delta (\cdot ; c) := \bar{\psi}_{2K}^\delta (\cdot ; c) : \bar{X}_\delta \rightarrow \bar{X}_\delta\).

### 3.1 Weak KAM solution on grid

We look for a fixed point of \(\psi_\delta (\cdot ; c)\), which yields an analogue of a weak KAM solution. We see that \(X_\delta\) is a convex compact set. Since the images of \(\varphi_\delta\) and \(\psi_\delta\) are obtained through a finite number of the four arithmetic operations, they are continuous in the topology of \(\sup_{\text{pr}.\text{odd}} | \cdot |\) or \(\sup_{\text{pr}.\text{even}} | \cdot | \infty\). Hence, we see that \(\hat{\psi}_\delta (\cdot ; c)\) is continuous in the topology of \(| \cdot | \infty\) of \(\mathbb{R}^{da}\). Brouwer’s fixed-point theorem yields a fixed point of \(\hat{\psi}_\delta (\cdot ; c)\) in \(\bar{X}_\delta\), from which we obtain \(\bar{v}^0 = \bar{v}^0 (c) \in X_\delta\) for each \(c \in \bar{P}\) such that
\[
D_x \bar{v}^0 = \psi_\delta (D_x \bar{v}^0 ; c) = D_x \varphi_\delta (\bar{v}^0 ; c).
\]
Therefore, we have a constant \(\bar{H}_\delta (c) \in \mathbb{R}\) such that
\[
\varphi_\delta (\bar{v}^0 ; c) + \bar{H}_\delta (c) = \bar{v}^0. \tag{3.1}
\]
This leads to a version of weak KAM theorem:

**Theorem 3.1** For each \(c \in \bar{P}\), there exists the unique number \(\bar{H}_\delta (c) \in \mathbb{R}\) for which we have \(\bar{v}^0 \in X_\delta\) such that \(\varphi_\delta (\bar{v}^0 ; c) + \bar{H}_\delta (c) = \bar{v}^0\).

**Proof** Existence of a pair of \(\bar{H}_\delta (c)\) and \(\bar{v}^0\) is shown in (3.1). We prove the uniqueness of \(\bar{H}_\delta (c)\). Suppose that for some \(c\), we have \(A \in \mathbb{R}\) and \(\bar{v}^0 \in X_\delta\) such that \(\varphi_\delta (\bar{v}^0 ; c) + A = \bar{v}^0\). Since we have \((\varphi_\delta)^l (\bar{v}^0 ; c) = \bar{v}^0 - l \bar{H}_\delta (c)\) and \((\varphi_\delta)^l (\bar{v}^0 ; c) = \bar{v}^0 - l A\) for any \(l \in \mathbb{N}\), where \((\varphi_\delta)^l\) stands for \(l\)-iteration of \(\varphi_\delta (\cdot ; c)\), it holds that
\[
(\varphi_\delta)^l (\bar{v}^0 ; c)(x_{n+1}) = E_{\mu_{n+1}^{\delta \xi \bar{v}^0}} \left[ \sum_{1 \leq k \leq 2lK} L^{(c)} (y^k, t_{k-1}^{\delta \xi \bar{v}^0}, \bar{v}^0) \right] = \bar{v}^0_{n+1} - l \bar{H}_\delta (c),
\]
\[
(\varphi_\delta)^l (\bar{v}^0 ; c)(x_{n+1}) = E_{\mu_{n+1}^{\delta \xi \bar{v}^0}} \left[ \sum_{1 \leq k \leq 2lK} L^{(c)} (y^k, t_{k-1}^{\delta \xi \bar{v}^0}, \bar{v}^0) \right] = \bar{v}^0_{n+1} - l A,
\]
where $\xi^*, \tilde{\xi}^*$ are minimizing controls. Then, we have

\[
v_{n+1}^0 - lA \leq E_{v_{n+1}^0}^{2K,0}(\cdot; \xi^*) \left[ \sum_{1 \leq k \leq 2K} L(c) (\gamma^k, t_k - 1, \xi^*_{m(\gamma^k)}) \tau + v_{m(\gamma^0)}^0 \right],
\]

\[
\tilde{H}_\delta(c) - A \leq \frac{1}{l} E_{v_{n+1}^0}^{2K,0}(\cdot; \xi^*) \left[ \frac{v_{m(\gamma^0)}^0}{l} - \tilde{v}_{m(\gamma^0)}^0 \right] - \frac{1}{l} \left( v_{n+1}^0 - \tilde{v}_{n+1}^0 \right).
\]

Letting $l \to \infty$, we have $\tilde{H}_\delta(c) - A \leq 0$. Similar reasoning yields $\tilde{H}_\delta(c) - A \geq 0$. \qed

As for (2.5), we have a pair of the unique constant $\tilde{H}_\delta(c)$ and a function $\tilde{v}^0$ such that $\tilde{v}^0(\tilde{v}^0; c) - \tilde{H}_\delta(c) = \tilde{v}^0$ [we set “−” to have the statements following (3.3)].

In Sect. 4, we treat the autonomous case.

### 3.2 Property of effective Hamiltonian

We call $\tilde{H}_\delta(\cdot)$ the effective Hamiltonian of (2.4). Here are the properties of $\tilde{H}_\delta(\cdot)$:

**Theorem 3.2** 1. The problem with a constant $A$

\[
\begin{align*}
    v : \mathcal{G}_\delta |_{k \geq 0} &\to \mathbb{R}, \\
    v^k_{m+1 \pm 2N} &= v^k_{m+1} \quad (i = 1, \ldots, d), \\
    v^0 &\in X_\delta, \\
    (D_t v)^{k+1} + H(x_m, t_k, c + (D_x v)^k_m) &= A
\end{align*}
\]

admits a time-1-periodic solution (i.e., $v^{k+2K} = v^k$ for all $k \geq 0$), if and only if $A = \tilde{H}_\delta(c)$ and $v^0$ satisfies $\varphi_\delta(v^0; c) + \tilde{H}_\delta(c) = v^0$.

2. Let $\tilde{v}$ be a time-1-periodic solution of (3.2) with $A = \tilde{H}_\delta(c)$. Then, we have

\[
\tilde{H}_\delta(c) = \sum_{\{(m,k) | (x_m, t_k) \in \text{pr.} \mathcal{G}_\delta\}} H(x_m, t_k, c + (D_x \tilde{v})^k_m) (2h)^d \tau.
\]

3. Let $v^k_{m+1}$ be the solution of (2.4) with arbitrary initial date $v^0 \in X_\delta$. Then, we have

\[
\lim_{l \to \infty} \frac{v^l_{n+1}}{l_{n+1}} = -\tilde{H}_\delta(c) \quad \text{for any } n.
\]

4. $\tilde{H}_\delta(\cdot) : P \to \mathbb{R}$ is convex (hence, Lipschitz continuous).

**Proof** 1. The time-1 map of (3.2), denoted by $\tilde{v}_\delta(\cdot; c)$, is given as $\tilde{v}_\delta(\cdot; c) = \varphi_\delta(\cdot; c) + A$. Note that a solution of (3.2) is time-periodic, if and only if $v^0$ is a fixed point of $\varphi_\delta(\cdot; c)$. If $A = \tilde{H}_\delta(c)$ and $\varphi_\delta(v^0; c) + \tilde{H}_\delta(c) = v^0$, we have $\tilde{v}_\delta(v^0; c) = \varphi_\delta(v^0; c) + \tilde{H}_\delta(c) = v^0$. If $\varphi_\delta(v^0; c) = v^0$, we have $\varphi(v^0; c) + A = v^0$; Theorem 3.1 implies that $A$ must be $\tilde{H}_\delta(c)$.

2. Since $\tilde{v}$ is time-1-periodic, we have

\[
\tilde{H}_\delta(c) = \sum_{\{(m,k) | (x_m, t_k) \in \text{pr.} \mathcal{G}_\delta\}} H(x_m, t_k, c + (D_x \tilde{v})^k_m) (2h)^d \tau
\]

\[
+ \sum_{\{(m,k) | (x_m, t_k) \in \text{pr.} \mathcal{G}_\delta\}} \left( \tilde{v}^{k+1}_m - \frac{1}{2d} \sum_{\omega \in B} (\tilde{v}^k_{m+\omega}) (2h)^d, \right)
\]

where the second summation is 0 due to periodicity of $\tilde{v}$.
3. Let $\xi^*$ be the minimizing control of $v_{n+1}^l$, i.e.,
\[
v_{n+1}^l = E_{\mu_n^{l+1,0}(:\xi^*)} \left[ \sum_{1 \leq k \leq l+1} L(c) \left( y^k, t_k-1, \xi^*k \right) \tau + v_0^0(m(y^k)) \right].
\]

Let $\tilde{v}$ be a time-1-periodic solution of (3.2) with $A = \tilde{H}_\delta(c)$. We have the minimizing control $\tilde{\xi}^*$ of $\tilde{v}_{n+1}^l$, i.e.,
\[
v_{n+1}^l = E_{\mu_n^{l+1,0}(:\tilde{\xi}^*)} \left[ \sum_{1 \leq k \leq l+1} L(c) \left( y^k, t_k-1, \tilde{\xi}^*k \right) \tau + \tilde{v}_0^0(m(y^k)) \right] + \tilde{H}_\delta(c)t_{l+1}.
\]

Due to the variational property, we have
\[
v_{n+1}^l - \tilde{v}_{n+1}^l \leq E_{\mu_n^{l+1,0}(:\tilde{\xi}^*)} \left[ v_0^0(m(y^k)) - \tilde{v}_0^0(m(y^k)) \right] - \tilde{H}_\delta(c)t_{l+1},
\]
\[
v_{n+1}^l - \tilde{v}_{n+1}^l \geq E_{\mu_n^{l+1,0}(:\tilde{\xi}^*)} \left[ v_0^0(m(y^k)) - \tilde{v}_0^0(m(y^k)) \right] - \tilde{H}_\delta(c)t_{l+1}.
\]

We divide the inequalities by $t_{l+1}$ and sending $l \to \infty$. Due to boundedness of $v^0, \tilde{v}_0$ and $\tilde{v}$, we obtain the assertion.

4. Consider the solution $v_{m+1}^k = v_{m+1}^k(c)$ of (2.4) for each $c$ with common initial date $v^0 \in X_\delta$. It is enough to show that the function $\theta \mapsto v_{n+1}^1(c)$ is concave for fixed $l, n$. In fact, if so, $c \mapsto v_{n+1}^1(c)/t_{l+1}$ is also concave for any $l$; due to the claim 3, $\lim_{l \to \infty} v_{n+1}^1(c)/t_{l+1} = -\tilde{H}_\delta(c)$ is also concave. Let $\xi^*$ be the minimizing control for $v_{n+1}^1(c^*)$ with $c^* := \theta c + (1 - \theta)\tilde{c}$, $\theta \in [0, 1], c, \tilde{c} \in P$. Then, the variational property yields
\[
v_{n+1}^1(c^*) - \left\{ \theta v_{n+1}^1(c) + (1 - \theta)v_{n+1}^1(\tilde{c}) \right\}
\geq \theta E_{\mu,:\xi^*} \left[ \sum_{1 \leq k \leq l+1} -(c^* - c) \cdot \xi^*k \right] \tau
\]
\[+(1 - \theta)E_{\mu,:\xi^*} \left[ \sum_{1 \leq k \leq l+1} -(c^* - \tilde{c}) \cdot \xi^*k \right] \tau
\]
\[= 0.
\]

We remark that, if $d = 1$, $\tilde{H}_\delta(\cdot)$ has $C^1$-regularity (see [41,46]).

As for the pair of $\tilde{H}_\delta(c)$ and $\tilde{v}_0$, we have a similar result. In particular, it holds that the problem with a constant $\tilde{A}$
\[
\begin{cases}
v : \tilde{G}_\delta|_{k \leq 0} \to \mathbb{R}, \\
v_k^{m+1} = v_k^{m+1} + 2N_{e_i} (i = 1, \ldots, d), \\
v_0 \in X_\delta, \\
(\tilde{D}_t v)^{k+1}_m + H(x_m, t_k, c + (D_x v)^{k}_m) = \tilde{A}
\end{cases}
\]

admits a time-1-periodic solution $\tilde{v}_3(c)$, if and only if $\tilde{A} = \tilde{H}_\delta(c)$ and $v^0$ satisfies $\tilde{\varphi}_\delta(\tilde{v}_0; \tilde{c}) - \tilde{H}_\delta(c) = \tilde{v}_0$. We remark that $\tilde{H}_\delta(c) = \tilde{H}_\delta(c)$ would not be true in general. In fact, the reasoning to compare $\tilde{H}_\delta(c)$ and $A$ demonstrated in the proof of Theorem 3.1 does not work in the comparison of $\tilde{H}_\delta(c)$ and $\tilde{H}_\delta(c)$, because switching a minimizer and maximizer.
between the two variational problems for \( \varphi^k \) and \( \bar{\varphi}^k \) does not make any sense. Nevertheless, as we will see later, \( \bar{H}_\delta(c) \) and \( \tilde{H}_\delta(c) \) converge to the same quantity at the hyperbolic scaling limit.

In the rest of the paper, we consider the problem

\[
\begin{cases}
(D_t v)^{k+1}_m + H(x_m, t_k, c + (D_x v)_m^k) = \bar{H}_\delta(c), \\
\bar{v}^k_{m+1} = \bar{v}^k_{m+1} + 2N_r (j = 1, \ldots, d).
\end{cases}
\]

(3.4)

### 3.3 Uniqueness of weak KAM solution

We discuss (up to constants) uniqueness of a solution \( \bar{v}^0 \) of (3.1) with respect to \( c \). When \( d = 1 \), it is unique up to constants \([41,46]\). The same holds in the higher dimensional case:

**Theorem 3.3** Let \( \bar{v}^0 \in X_\delta \) satisfy \( \varphi_\delta(\bar{v}^0; c) + \bar{H}_\delta(c) = \bar{v}^0 \). Then, any other function \( v^0 \in X_\delta \) satisfying \( \varphi_\delta(v^0; c) + \bar{H}_\delta(c) = v^0 \) differs from \( \bar{v}^0 \) up to a constant. Furthermore, a time-1-periodic solutions of (3.4) is unique up to constants.

**Proof** Let \( \bar{v}^0, v^0 \in X_\delta \) satisfy (3.1) with the same \( c \). Set \( \bar{v}^k := \varphi_\delta(\bar{v}^0; c) + t_k \bar{H}_\delta(c), v^k := \varphi_\delta(v^0; c) + t_k \bar{H}_\delta(c), S_{k+1}^m := v^k_{m+1} - \bar{v}^k_{m+1} \) and \( S_k := \max_m \bar{S}_m^k \). Note that \( \bar{v}^k \) and \( v^k \) are time-1-periodic solutions of (3.4). It follows from the discrete Hamilton–Jacobi equation, Taylor’s formula and (H2) that

\[
S_{k+1}^m = \sum_{i=1}^d \left( \frac{1}{2d} - \frac{\lambda}{2} H_{p_i}(x_m, t_k, (D_x \bar{v})^k_m) \right) \bar{S}_{m+e_i}^k + \left( \frac{1}{2d} + \frac{\lambda}{2} H_{p_i}(x_m, t_k, (D_x \bar{v})^k_m) \right) \bar{S}_{m-e_i}^k - \frac{\tau}{2} k_{m+1}^k \left| (D_x v)^k_m - (D_x \bar{v})^k_m \right|^2,
\]

where \( k_{m+1}^k > \kappa > 0 \) for all \( k, m \). The CFL-type condition stated in Theorem 2.2 implies that \( \frac{1}{2d} \pm \frac{\lambda}{2} H_{p_i}(x_m, t_k, (D_x \bar{v})^k_m) > 0 \). Hence, we have \( S_{k+1}^k \leq S_k \) for all \( k \). On the other hand, since \( \bar{v}^{2K} = \varphi_\delta(\bar{v}^0; c) + \bar{H}_\delta(c) = \bar{v}^0 \) and \( v^{2K} = \varphi_\delta(v^0; c) + \bar{H}_\delta(c) = v^0 \), we have \( S_{k+1}^m = \bar{S}_m^k \) and \( S_{k+1}^m + S_{k+2K}^m = \bar{S}_m^k \) for all \( k \geq 0 \). Therefore, it holds that \( S_k = S_{2K} \leq S_k \leq S_0 \) for all \( 0 \leq k \leq 2K \), i.e., \( S_k = S^0 \) for all \( k \geq 0 \).

Let \( m \) be such that \( S_{k+1}^m = \bar{S}_{m+1}^k (= S^0) \). Then, the recurrence equation of \( S_{m+1}^k \) implies that we must have \( S_{m+\omega}^k = S^0 \) for all \( \omega \in B \), because otherwise \( S_{k+1}^k < S^0 \). Let \( \Lambda^k := \{ x_{m+\omega} \in \text{pr.h}\mathbb{Z}^d | x_m, t_k \in \text{pr}, \bar{S}_m^k < S^0 \} \). If \( x_{m+\omega} \in \Lambda^k \), we must have \( x_{m+1+\omega} \in \Lambda^{k+1} \) for all \( \omega \in B \). Hence, if \( \Lambda^0 = \emptyset \), it holds that \( \# \Lambda^k < \# \Lambda^{k+1} \) for all \( k \geq 0 \). Until we have \( \Lambda^{k+1} = \text{pr}.G_{\text{even}} \) or \( \text{pr}.G_{\text{odd}} \), which means that \( S_{m+1}^k < S^0 \) for all \( m \). Therefore, we must have \( \Lambda^0 = \emptyset \). Furthermore, the equality \( S_k = S^0 \) for all \( k \geq 0 \) requires \( (D_x v)^k_m = (D_x \bar{v})^k_m \) for all \( m, k \). This implies that there exists a constant \( \beta \in \mathbb{R} \) such that \( v^0 = \bar{v}^0 + \beta \).

Since \( v^k = \varphi_\delta(\bar{v}^0; \beta; c) + t_k \bar{H}_\delta(c) = \varphi_\delta(v^0; c) + t_k \bar{H}_\delta(c) + \beta = \bar{v}^k + \beta \) for all \( k \geq 0 \), a time-1-periodic solutions of (3.4) is unique up to constants. □

### 3.4 Long-time behavior of Lax–Oleinik type solution map

In autonomous weak KAM theory, Fathi [20] elegantly made clear the long-time behavior of the Lax–Oleinik semigroup: \( T^t u + t \bar{H}(c) \) converges to a weak KAM solution (the limit
depends on \( u \) as \( t \to \infty \) for all \( u \in C^0(\mathbb{T}^d; \mathbb{R}) \). In other words, the viscosity solution \( v \) of

\[
v_t + H(x, c + v_x) = \tilde{H}(c), \quad v(x, 0) = u(x), \quad x \in \mathbb{T}^d, \quad t > 0
\]

converges to a stationary solution as \( t \to \infty \). We remark that this is not true in a time-dependent case in general [22]. In our version of weak KAM theory, however, we have convergence of any solution to a time-1-periodic solution as \( k \to \infty \), even in the non-autonomous case.

**Theorem 3.4** For each \( v^0 \in X_\delta \), the solution \( u^k = \phi^k_\delta (v^0; c) + t_k \tilde{H}(c) \) of (3.4) converges to some time-1-periodic solution of (3.4) as \( k \to \infty \), i.e., there exist \( \bar{v}^0 \) such that \( \phi^k_\delta (\bar{v}^0; c) + \bar{H}(c) = \bar{v}^0 \) for which \( v^k \) and \( \bar{v}^k := \phi^k_\delta (\bar{v}^0; c) + t_k \tilde{H}(c) \) satisfy

\[
\max_m |v^{2Kl+k} - \bar{v}^k| \to 0 \quad \text{as } l \to \infty, \quad \text{for all } k \geq 0,
\]

where the constant level of \( \bar{v}^0 \) depends on \( v^0 \).

**Proof** Fix \( v^0 \in X_\delta \) arbitrarily. Let \( \bar{v}^0 \in X_\delta \) satisfy (3.1). If \( \bar{v}^0 \leq v^0 \) does not hold, we add a constant to \( \bar{v}^0 \) and rewrite it as \( \bar{w}^0 \), to have \( \bar{w}^0 \leq v^0 \), where \( \bar{w}^0 \) still satisfies (3.1). We find a constant \( A \in \mathbb{R} \) such that

\[
\bar{w}^0 \leq v^0 \leq \bar{w}^0 + A.
\]

For simpler presentation, set \( \tilde{\phi}^k_\delta \cdot := \phi^k_\delta (\cdot; c) + t_k \tilde{H}(c) \). Since \( \tilde{\phi}^k_\delta \) preserves the order, \( v^k := \phi^k_\delta v^0 \) and \( \bar{v}^k := \phi^k_\delta \bar{w}^0 \) satisfy

\[
\bar{w}^0 \leq v^k \leq \bar{v}^k + A \quad \text{for all } k \in \mathbb{N}.
\]

Set \( S_{m+1}^k := v^k_{m+1} - \bar{w}^k_{m+1} \) and \( S^k := \max_m S_{m+1}^k \). As we already saw in the proof of Theorem 3.3, it holds that

\[
S_{m+1}^{k+1} = \sum_{i=1}^d \left\{ \left( \frac{1}{2d} - \frac{\lambda}{2} H_p (x, t_k, (D_x \bar{w})^k_m) \right) S^k_{m+1} \right\} + \left( \frac{1}{2d} + \frac{\lambda}{2} H_p (x, t_k, (D_x \bar{w})^k_m) \right) S^{k+1}_m - \frac{\tau}{2} k_{m+1} |(D_x v)_m^k - (D_x \bar{w})^k_m|^2,
\]

where \( k_{m+1} > \kappa > 0 \) for all \( k, m \) and \( S^{k+1} \leq S^k \) for all \( k \). Since \( |S^k| \) is uniformly bounded, we have \( S^k \to S^* \) as \( k \to \infty \).

Since \( \{v^{2Kl}_k\}_{l \in \mathbb{N}} \) is bounded (in \( \mathbb{R}^{2G_{\text{odd}}} \)) due to (3.5), we have a convergent subsequence \( \{v^{2Kl}_{k, a_l}\}_{l \in \mathbb{N}} \), such that \( v^{2K} = \bar{v}^0 \) as \( l \to \infty \). Since \( v^{2Kl_{a_l}+k} = \phi^k_\delta v^{2Kl_{a_l}} \) and \( \tilde{\phi}^k_\delta \) is continuous, we have \( v^{2Kl_{a_l}+k} \to \bar{v}^k := \phi^k_\delta \bar{v}^0 \) as \( l \to \infty \) for all \( k \in \mathbb{N} \).

Furthermore, for \( \tilde{S}_{m+1} \) := \( v^k_{m+1} - \bar{w}^k_{m+1} \) and \( \tilde{S}^k := \max_m \tilde{S}_{m+1}^k \), we have \( \tilde{S}^k \to S^* \) as \( k \to \infty \). Hence, due to the same reasoning as the proof of Theorem 3.3, we must have \( (D_x v)_m^k = (D_x \bar{w})_m^k \) for all \( m, k \), which imply that there exists some constant \( \beta \in \mathbb{R} \) such that \( \bar{v}^0 = \bar{w}^0 + \beta \). Therefore, \( \bar{v}^0 \) is a time-1-periodic solution of (3.4).

Note that \( \bar{\phi}^k_\delta \) is non-expanding, i.e., \( \max_x |\bar{\phi}^k_\delta (\bar{v}^0 - \bar{\phi}^k_\delta \bar{w}^0)| \leq \max_x |\bar{v}^0 - \bar{w}^0| \) for all \( k \geq 0 \) and \( \bar{w}, \bar{v} \in X_\delta \). For any \( \epsilon > 0 \), we have \( l = l(\epsilon) \in \mathbb{N} \) such that \( \max_x |\phi^{2Kl_k}_{\delta} - \bar{v}^0| < \epsilon \). If \( l \geq a(\epsilon) \), we have for all \( k \geq 0 \),

\[
\max_x |\bar{\phi}^k_\delta (v^0 - \bar{v}^k)| = \max_x |\phi^{2Kl_k}_{\delta} - \bar{v}^k| + \max_x |\phi^{2Kl_k}_{\delta} - \bar{v}^0| \leq \max_x |\phi^{2Kl_k}_{\delta} - \bar{v}^0| + \epsilon < \epsilon.
\]
3.5 Asymptotics of minimizing random walk and Mather measure

From now on, we consider time-1-periodic solutions \( \tilde{v} \) of (3.4), where \( \tilde{v} \) is also denoted by \( \tilde{v}(c) \) when we specify the value of \( c \). Note that the solution map of (3.4) is given as \( \phi_{\delta}^k(\cdot; c) + \tilde{H}_\delta(c) t_k \). If \( \tilde{v}(c) \) is a time-1-periodic solution of (3.4), we may periodically extend \( \tilde{v}(c) \) to the whole \( \tilde{\mathcal{G}}_\delta \); controlled random walks and the representation formula stated in Theorems 2.2 and 2.12 are well-defined with an arbitrary (negative) terminal time, i.e., we have for any \( l, l' \in \mathbb{Z} \) with \( l' \leq l \) and \( n \),

\[
\tilde{v}(c)|_{n+1}^{l+1} = \inf_{\xi} E_{\mu_{n+1}^{l+1} (\cdot; \xi)} \left[ \sum_{l' < k \leq l+1} L(c) \left( \gamma^k, t_k \xi_{m(\gamma^k)} \right) \tau + \tilde{v}(c)|_{m(l')} \right] + \tilde{H}_\delta(c)(t_{l+1} - t_l),
\]

where its minimizing control is given as

\[
\xi_{m_{l+1}}^{k+1} = H_p(x_m, t_k, c + (D_x \tilde{v})_m^k), \ |\xi_{m_{l+1}}^{k+1}|_\infty \leq (d \lambda_1)^{-1};
\]

the minimizing control \( \xi^* \) and minimizing random walk can be extended to \( l' \rightarrow - \infty \) with

\[
\tilde{v}(c)|_{n+1}^{l+1} = H_p(x_m, t_k, c + (D_x \tilde{v})_m^k) \text{ on } \tilde{\mathcal{G}}_\delta. \]

Then, we are ready to consider the action minimizing problem

\[
\inf_{\tilde{v}} \mathcal{L}(\tilde{v})
\]

stated in Introduction. We will see that the asymptotic behavior of minimizing random walks is very similar to that of calibrated curves in weak KAM theory. Furthermore, if we look at the asymptotic behavior on \( \text{pr}_1 \mathcal{G}_\delta \), we find an object very similar to Mather measures. Roughly speaking, a Mather measure shows “recurrence rate” of trajectories of Euler–Lagrange system in standard autonomous weak KAM theory (see construction of a Mather measure by calibrated curves [21]). We will observe that, in our problem, the quantity corresponding to “recurrence rate” is given by the long-time average of the (configuration-space-based) probability measure of a minimizing random walk projected on \( \text{pr}_1 \mathcal{G}_\delta \).

The next theorem is reminiscent of the backward rotation vector of calibrated curves.

**Theorem 3.5** Let \( c \in P \) be such that \( \tilde{H}_\delta(c) \) is differentiable (a.e. points have such a property due to convexity). Let \( \tilde{v}(c) \) be a time-1-periodic solution of (3.4). Then, every minimizing random walk of \( \tilde{v}(c) \) has the backward rotation vector \( \frac{\partial}{\partial \gamma} \tilde{H}_\delta(c) \), i.e., the average \( \tilde{\gamma}^k \) for \( l \leq k \leq l+1 \) of the minimizing random walk (see (2.1)) for \( \tilde{v}(c) |_{n+1}^{l+1} \) satisfies

\[
\lim_{l' \to -\infty} \tilde{\gamma}^k |_{l'} = \frac{\partial}{\partial \gamma} \tilde{H}_\delta(c),
\]

where we note that the averaged path defined for \( l' \leq k \leq l+1 \) and that defined for \( l'' \leq k \leq l+1 \) are identical for max\{l', l''\} \( \leq k \leq l + 1 \).

**Proof** Let \( \xi^* \) be the minimizing control for \( \tilde{v}(c) |_{n+1}^{l+1} \) up to \( l' \leq l \). Recall (2.1): \( \tilde{\gamma}^k = \tilde{\gamma}^k + \tilde{\xi}^k + \tau \). For \( \tilde{v}(c) |_{n+1}^{l+1} \) with \( \tilde{c} \neq c \), we have

\[
\tilde{v}(c) |_{n+1}^{l+1} (\tilde{c}) \leq E_{\mu_{n+1}^{l+1} (\cdot; \xi^*)} \left[ \sum_{l' < k \leq l+1} L(c) \left( \gamma^k, t_k \xi_{m(\gamma^k)} \right) \tau + \tilde{v}(c) |_{m(l')} \right] + \tilde{H}_\delta(c)(t_{l+1} - t_l),
\]
\[ v^{l+1}_n(c) - v^{l+1}_n(c) \leq E_{\mu_n^{l+1}(:\xi^*}) \left[ \sum_{l'<k \leq l+1} -\left( \tilde{c} - c \right) \cdot \xi^{*k}_{m(y_k)} \tau + \tilde{v}^{l'}_{m(y_k)}(\tilde{c}) - \tilde{v}^{l'}_{m(y'_k)}(c) \right] \]

Hence, we have
\[
\begin{align*}
  v^{l+1}_n(c) - v^{l+1}_n(c) &= E_{\mu_n^{l+1,l'}(:\xi^*)} \left[ \tilde{v}^{l'}_{m(y_k)}(\tilde{c}) - \tilde{v}^{l'}_{m(y'_k)}(c) \right] \\
  &\leq -\frac{\left( \tilde{c} - c \right) \cdot \sum_{l'<k \leq l+1} \tilde{\xi}^{k} \tau + \tilde{H}_{\delta}(\tilde{c}) - \tilde{H}_{\delta}(c)}{t_{l+1} - t_l} \\
  &= -(\tilde{c} - c) \cdot \frac{x_n - \tilde{y}^{l'}}{t_{l+1} - t_l} + \tilde{H}_{\delta}(\tilde{c}) - \tilde{H}_{\delta}(c).
\end{align*}
\]

Since \( \tilde{v}^k_{m+1}(c), \tilde{v}^k_{m+1}(\tilde{c}) \) are uniformly bounded, we obtain
\[
0 \leq -\left( \tilde{c} - c \right) \cdot \left( \liminf_{l' \to -\infty} \frac{\tilde{y}^{l'}}{t_l} \right) + \tilde{H}_{\delta}(\tilde{c}) - \tilde{H}_{\delta}(c)
\]
\[
\leq -(\tilde{c} - c) \cdot \left( \limsup_{l' \to -\infty} \frac{\tilde{y}^{l'}}{t_l} \right) + \tilde{H}_{\delta}(\tilde{c}) - \tilde{H}_{\delta}(c).
\]

For \( \tilde{c} = c + \varepsilon e_i \) and \( \varepsilon \to 0^+ \), we have
\[
e_i \cdot \left( \limsup_{l' \to -\infty} \frac{\tilde{y}^{l'}}{t_l} \right) \leq \frac{\tilde{H}_{\delta}(\tilde{c}) - \tilde{H}_{\delta}(c)}{\varepsilon} \to e_i \cdot \frac{\partial}{\partial \tilde{c}} \tilde{H}_{\delta}(c).
\]

For \( \tilde{c} = c + \varepsilon e_i \) and \( \varepsilon \to 0^- \), we have
\[
e_i \cdot \left( \liminf_{l' \to -\infty} \frac{\tilde{y}^{l'}}{t_l} \right) \geq \frac{\tilde{H}_{\delta}(\tilde{c}) - \tilde{H}_{\delta}(c)}{\varepsilon} \to e_i \cdot \frac{\partial}{\partial \tilde{c}} \tilde{H}_{\delta}(c).
\]

We observe asymptotics of the probability measures of controlled random walks on pr.\( \tilde{G}_\delta \) and derive an analogue of Mather’s minimizing problem as well as Mather measures. We call \( \xi : \text{pr.} \tilde{G}_\delta \to \left( -(d\lambda)^{-1}, (d\lambda)^{-1} \right)^d, \lambda := \tau / h < \lambda_1 \) an admissible control, where we do not distinguish \( \xi \) and its 1-periodic extension to \( \tilde{G}_\delta \). Let \( \mu_{n+1}^{0,-l}(:\xi)(G_{n+1}^{0,-l}) \), re-denoted simply by \( \mu_{n+1}^{0,-l}(:\xi) \), be the probability density of the backward random walk controlled by \( \xi \), where we may take any \( l \in \mathbb{N} \). For each admissible control \( \xi \), define the linear functional
\[
\mathcal{F}_{\delta}^l(:\xi) : C_c(T^d \times \mathbb{T} \times \mathbb{R}^d; \mathbb{R}) \to \mathbb{R},
\]
\[
\mathcal{F}_{\delta}^l(f; \xi) := E_{\mu_{n+1}^{0,-l}(:\xi)} \left[ \frac{1}{l} \sum_{-l < k \leq 0} f \left( \gamma^k, t_k, \xi^k_{m(y_k)} \right) \tau \right],
\]
where \( C_c(T^d \times \mathbb{T} \times \mathbb{R}^d; \mathbb{R}) \) is the family of compactly supported continuous functions defined in \( T^d \times \mathbb{T} \times \mathbb{R}^d \). The Riesz representation theorem yields the unique probability measure \( \mu_{\delta}^l(\xi) \) of \( T^d \times \mathbb{T} \times \mathbb{R}^d \) such that
\[
\mathcal{F}_{\delta}^l(f; \xi) = \int_{T^d \times \mathbb{T} \times \mathbb{R}^d} f \, d\mu_{\delta}^l(\xi) \quad \text{for any } f \in C_c(T^d \times \mathbb{T} \times \mathbb{R}^d; \mathbb{R}).
\]
We see that, for all \( l \in \mathbb{N} \), \( \text{supp}(\mu^l(\xi)) \) is contained in the compact set
\[
K_\delta := \left\{ (x_{m+1}, t_k, \xi_{m+1}^k) \mid (x_{m+1}, t_k) \in \text{pr.} \tilde{G}_\delta \right\} \subset \mathbb{T}^d \times \mathbb{T} \times [- \frac{(d \lambda)^{-1}}{2}, \frac{(d \lambda)^{-1}}{2}]^d.
\]
Hence, we have a sequence \( l \to \infty \) for which \( \mu^l(\xi) \) converges weakly to a probability measure \( \mu(\xi) \) on \( \mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d \). Let \( \mathcal{P}_\delta \) be the family of all such probability measures for all admissible controls.

We demonstrate more concrete construction of each \( \mu^1(\xi) \) and \( \mathcal{P}_\delta \) by means of the configuration-space-based distribution \( p^1_{m+1}(\xi) \) (mentioned in Sect. 2.2) of the random walk starting at \( (x_{m+1}, 0) \) and controlled by \( \xi \). Re-define \( p^1_{m+1}(\xi) \) as \( p(\cdot; \xi) : \tilde{G}_\delta \to [0, 1] \); \( p(\cdot; \xi) = 0 \) outside \( G_{n+1}^{0, -\infty} \); otherwise
\[
p(x_{n+1}, 0)(\xi) := 1, \quad p(x_{m+1}, t_k; \xi) := \sum_{\gamma \in \Omega_{n+1}^{0, -l}} \mu^0_{n+1}(\gamma; \xi) \quad \text{with} \quad -l \leq k,
\]
where we note that \( p(x_{m+1}, t_k; \xi) \) is independent of the choice of \( l \). We have for each \( k \leq 0 \),
\[
\sum_{\{x \mid (x, t_k) \in \tilde{G}_\delta\}} p(x, t_k; \xi) = 1. \tag{3.6}
\]
Observe that
\[
\mathcal{F}^1_\delta(f; \xi) = \frac{1}{l_t} \sum_{\gamma \in \Omega_{n+1}^{0, -l}} \mu^0_{n+1}(\gamma; \xi) \left( \sum_{-l \leq k \leq 0} f \left( \gamma^k, t_k, \xi_{m}^k \right) \tau \right)
\]
\[
= \frac{1}{l_t} \sum_{-l \leq k \leq 0} \left( \sum_{\gamma \in \Omega_{n+1}^{0, -l}} \mu^0_{n+1}(\gamma; \xi) f \left( \gamma^k, t_k, \xi_{m}^k \right) \right) \tau
\]
\[
= \frac{1}{l_t} \sum_{-l \leq k \leq 0} \left( \sum_{\{m \mid (x_{m+1}, t_k) \in \tilde{G}_\delta\}} p(x_{m+1}, t_k; \xi) f \left( x_{m+1}, t_k, \xi_{m+1}^k \right) \right) \tau.
\]
Define \( p^1_\delta(\cdot; \xi) : \text{pr.} \tilde{G} \to [0, 1] \) as
\[
p^1_\delta(x_{m+1}, t_k; \xi) := \frac{1}{l_t} \sum_{\{m', k' \mid \text{pr.} (x_{m'+1}, t_{k'}) = (x_{m+1}, t_k), -l \leq k' \leq 0\}} p(x_{m'+1}, t_{k'}; \xi) \tau,
\]
where, for \( y \in \mathbb{R}^q \), \( \text{pr.} y := y + z \) with \( z \in \mathbb{Z}^q \) such that \( y + z \in [0, 1]^q \). We note that (3.6) implies
\[
\sum_{(x, t) \in \text{pr.} \tilde{G}} p^1_\delta(x, t; \xi) = 1 \quad \text{for each} \quad l \in \mathbb{N}.
\]
Due to the periodicity of \( f \in C_c(\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d; \mathbb{R}) \) with respect to \( (x, t) \), we have
\[
\mathcal{F}^1_\delta(f; \xi) = \sum_{(x, t) \in \text{pr.} \tilde{G}} p^1_\delta(x, t; \xi) f \left( x, t, \xi_{m(x)}^k \right)
\]
Now, we see that the above \( \mu^1_\delta(\xi) \) is represented as
\[
\mu^1_\delta(\xi) = \sum_{(x, t) \in \text{pr.} \tilde{G}} p^1_\delta(x, t; \xi) \tilde{\omega}_{x, t, \xi_{m(x)}^k},
\]
where \( \mathcal{F}_{x,t} \) is the Dirac measure of \( \mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d \) supported by the point \((x, t, \zeta)\). Since the function \( p^l_{\delta}(\cdot; \xi) \) is determined by a finite number of values in \([0, 1]\), the sequence \(\{p^l_{\delta}(\cdot; \xi)\}_{l \in \mathbb{N}}\) can be regarded as a sequence of \([0, 1]^a\) with \(a = \sharp(\text{pr} \hat{\mathcal{G}})\). Hence, we find a convergent subsequence with the limit \( p_{\delta}(\cdot; \xi) \). Let \( Q_{\delta}(\xi) \) be the set of all limits of convergent subsequences of \(\{p^l_{\delta}(\cdot; \xi)\}_{l \in \mathbb{N}}\). Then, we see that

\[
\mathcal{P}_{\delta} = \bigcup_{\xi} \left\{ \mu_{\delta}(\xi) = \sum_{(x, t) \in \text{pr} \hat{\mathcal{G}}} p_{\delta}(x, t; \xi) \delta_{x, t, \xi_{m(\xi)}} \left| p_{\delta}(\cdot; \xi) \in Q_{\delta}(\xi) \right. \right\},
\]

where the union is taken over all admissible controls.

We investigate an a priori constraint of \(\mathcal{P}_{\delta}\), which corresponds to the holonomic constraint in Mather’s minimizing problem [33]. Fix an arbitrary admissible control \(\xi\). For each function \(g : \text{pr} \hat{\mathcal{G}} \to \mathbb{R}\), consider a continuous function \(f : \mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}\) whose restriction to \(\text{pr} \hat{\mathcal{G}} \times \mathbb{R}^d\) is

\[
f(x_m, t_{k+1}, \xi) := \left( D_x g \right)_{m+1} + \left( D_t g \right)_{m} \cdot \zeta, \quad (x_m, t_{k+1}, \xi) \in \text{pr} \hat{\mathcal{G}} \times \mathbb{R}^d. \quad (3.7)
\]

**Proposition 3.6** Each \(\mu_{\delta}(\xi) \in \mathcal{P}_{\delta}\) satisfies the constraint

\[
\int_{\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d} f \, d\mu_{\delta}(\xi) = 0
\]

for any continuous function \(f : \mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}\) whose restriction to \(\text{pr} \hat{\mathcal{G}} \times \mathbb{R}^d\) is of the form (3.7).

**Proof** We cut off \(f\) with respect to \(\xi\) without changing the values for \(\xi \in [-(d\lambda)^{-1}, (d\lambda)^{-1}]^d\) so that \(f\) belongs to \(C_c(\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d; \mathbb{R})\) and periodically extend it to \(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d\). Then, we have

\[
\mathcal{F}_{\delta}(f; \xi) = \frac{1}{l_t} \sum_{-l < k \leq 0} \left( \sum_{m \mid (x_{m+1}, t_k) \in \hat{\mathcal{G}}} p(x_{m+1}, t_k; \xi) f \left( x_{m+1}, t_k, \xi_{m+1} \right) \right) \tau
\]

\[
= \frac{1}{l_t} \sum_{-l < k \leq 0} \left( \sum_{m \mid (x_{m+1}, t_k) \in G_{n+1}^{-l}} p^{l+1}_{m+1}(\xi) f \left( s_{m+1}, t_k, \xi_{m+1} \right) \right) \tau
\]

\[
= \frac{1}{l_t} \sum_{-l < k \leq 0} \left( \sum_{m \mid (x_{m+1}, t_k) \in G_{n+1}^{-l}} p^{l+1}_{m+1}(\xi) \left[ s_{m+1} - \frac{1}{2 \lambda} \sum_{i=1}^d \left( g_{m+1+e_i} - g_{m+1-e_i} \right) s_{m+1}^{1-\lambda} \right] \right)
\]

\[
= \frac{1}{l_t} \sum_{-l < k \leq 0} \left( \sum_{m \mid (x_{m+1}, t_k) \in G_{n+1}^{-l}} \left( p^{l+1}_{m+1}(\xi) s_{m+1} - \sum_{\omega \in B} p^{l+1}_{m+1}(\xi) p^{l+1}_{m+1}(\omega) s_{m+1+\omega} \right) \right).
\]

It follows from (2.3) that

\[
\sum_{m \mid (x_{m+1}, t_k) \in G_{n+1}^{-l}} \sum_{\omega \in B} p^{l+1}_{m+1}(\xi) p^{l+1}_{m+1}(\omega) g_{m+1+\omega} = \sum_{m \mid (x_{m+1}, t_k) \in G_{n+1}^{-l}} p^{l+1}_{m+1}(\xi) g_{m+1}.
\]
Hence, we obtain
\[
\mathcal{F}_b^l(f; \xi) = \frac{1}{l!} \sum_{-l < k \leq 0} \left( \sum_{\{m \mid (x_m, t_k) \in G_{n+1}^0\}} p_{m+1}^k(\xi) g_{m+1}^k - \sum_{\{m \mid (x_m, t_{k-1}) \in G_{n+1}^0\}} p_m^{k-1}(\xi) g_m^{k-1} \right)
\]
\[
= \frac{1}{l!} \left( g_0^{n+1} - \sum_{\{m \mid (x_{m+1}, t_l) \in G_{n+1}^0\}} p_{m+1}(\xi) g_{m+1} - \sum_{\{m \mid (x_m, t_{l-1}) \in G_{n+1}^0\}} p_m(\xi) g_m \right)
\]
\[
= \int_{T^d \times \mathbb{R}^d} f \, d\mu_b^l(\xi).
\]

Since \(\mu_b(\xi)\) is the weak limit of a subsequence of \(\{\mu_b^l\}_{l \in \mathbb{N}}\), the assertion is proved. \(\square\)

The next theorem is an analogue of Mather’s minimizing problem.

**Theorem 3.7** For each \(c \in P\), we have
\[
\inf_{\mu_b^l \in \mathcal{P}_b} \int_{T^d \times T \times \mathbb{R}^d} L^{(c)}(x, t - \tau, \xi) \, d\mu_b = -\tilde{H}_b(c),
\]
where there exists at least one minimizing probability measure \(\mu_b^* \in \mathcal{P}_b\) that attains the infimum.

**Proof** Let \(\bar{v} = \bar{v}(c)\) be a time-1-periodic solution of (3.4) and consider the admissible control
\[
\xi^* : \text{pr}. \tilde{G} \ni (x_m, t_{k+1}) \mapsto H_p(x_m, t_k, c + (D_x \bar{v})^\xi_m) \in \left[ - (d\lambda_1)^{-1}, (d\lambda_1)^{-1} \right]^d
\]
Then, for any \(x_{n+1}\), we have
\[
\int_{T^d \times T \times \mathbb{R}^d} L^{(c)}(x, t - \tau, \xi) + \tilde{H}_b(c) \, d\mu_b^l(\xi^*) = \mathcal{F}_b^l(L^{(c)}(\cdot, \cdot - \tau, \cdot) + \tilde{H}_b(c); \xi^*)
\]
\[
= \frac{1}{l!} \left( \bar{v}_n^{0} - E_{\mu_{n+1}^l(\cdot; \xi^*)} \left( \bar{v}^{-l}_{m(\gamma^{-1})} \right) \right).
\]
Since \(\bar{v}\) is bounded, the weak limit \(\mu_b^* (\xi^*) \in \mathcal{P}_b\) of a subsequence of \(\{\mu_b^l (\xi^*)\}_{l \in \mathbb{N}}\) yields
\[
\int_{T^d \times T \times \mathbb{R}^d} L^{(c)}(x, t - \tau, \xi) + \tilde{H}_b(c) \, d\mu_b^*(\xi^*) = 0.
\]
Let \(\mu_b\) be an arbitrary element of \(\mathcal{P}_b\), where \(\mu_b\) is the weak limit of a subsequence of \(\{\mu_b^l (\xi)\}_{l \in \mathbb{N}}\) with some admissible control \(\xi\). Due to variational structure of (3.4), we have
\[
\frac{1}{l!} \bar{v}_n^{0} = \mathcal{F}_b^l(L^{(c)}(\cdot, \cdot - \tau, \cdot) + \tilde{H}_b(c); \xi^*) + \frac{1}{l!} E_{\mu_{n+1}^l(\cdot; \xi^*)} \left( \bar{v}^{-l}_{m(\gamma^{-1})} \right)
\]
\[
\leq \mathcal{F}_b^l(L^{(c)}(\cdot, \cdot - \tau, \cdot) + \tilde{H}_b(c); \xi) + \frac{1}{l!} E_{\mu_{n+1}^l(\cdot; \xi)} \left( \bar{v}^{-l}_{m(\gamma^{-1})} \right),
\]
which leads to
\[
\int_{T^d \times T \times \mathbb{R}^d} L^{(c)}(x, t - \tau, \xi) + \tilde{H}_b(c) \, d\mu_b^*(\xi^*) \leq \int_{T^d \times T \times \mathbb{R}^d} L^{(c)}(x, t - \tau, \xi) + \tilde{H}_b(c) \, d\mu_b(\xi).
\]
\(\square\)
3.6 Mather set and Aubry set

We define an analogue of the Mather set $\mathcal{M}_\delta(c)$ for each $c$ as

$$\mathcal{M}_\delta(c) := \bigcup_{\mu^*_\delta} \text{supp}(\mu^*_\delta) \subset \text{pr.}\tilde{\mathcal{G}}_\delta \times [-(d\lambda)^{-1}, (d\lambda)^{-1}]^d,$$

where the union is taken over all minimizing probability measures $\mu^*_\delta$ of (3.8). We state properties of $\mathcal{M}_\delta(c)$.

**Theorem 3.8** For each $c$, the support of any minimizing probability measure for (3.8) is contained in the set

$$\mathcal{A}_\delta(c) := \{ (x_m, t_{k+1}, H_{p}(x_m, t_k, c + (D_x\tilde{v})^k_m)) \mid (x_m, t_{k+1}) \in \text{pr.}\tilde{\mathcal{G}}_\delta \},$$

where $\tilde{v} = \tilde{v}(c)$ is the (up to constant) unique time-$1$-periodic solutions of (3.4); namely we have

$$\mathcal{M}_\delta(c) \subset \mathcal{A}_\delta(c) \subset \text{pr.}\tilde{\mathcal{G}}_\delta \times [-(d\lambda)^{-1}, (d\lambda)^{-1}]^d.$$

**Proof** Suppose that there exists a minimizing probability measure $\mu^*_\delta$ whose support is not contained in the set $\mathcal{A}_\delta(c)$. Then, we have a point $(x_n, t_{l+1}, \xi^*) \in \text{supp}(\mu^*_\delta)$ such that $\xi^* \neq H_{p}(x_n, t_l, c + (D_x\tilde{v})^k_m)$ for some time-$1$-periodic solution $\tilde{v} = \tilde{v}(c)$ of (3.4). It holds that

$$-\tilde{H}_\delta(c) = -(D_t\tilde{v})^k_{m+1} - H((x_m, t_k, c + (D_x\tilde{v})^k_m))$$

$$\leq -(D_t\tilde{v})^k_{m+1} - (D_x\tilde{v})^k_{m} \cdot \xi + L(c)(x_m, t_k, \xi) \quad \text{for any } \xi \in \mathbb{R}^d,$$

which is a strict inequality for any $\xi \neq H_{p}(x_m, t_k, c + (D_x\tilde{v})^k_m)$ due to the Legendre transform. Consider a continuous function $f : \mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d \rightarrow \mathbb{R}$ whose restriction to $\text{pr.}\tilde{\mathcal{G}}_\delta$ is given as $f(x_m, t_{k+1}, \xi) = -((D_t\tilde{v})^k_{m+1} + (D_x\tilde{v})^k_{m} \cdot \xi) + L(c)(x_m, t_{k+1} - \tau, \xi)$. Then, Proposition 3.6 implies that

$$\int_{\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d} f(x, t, \xi) d\mu^*_\delta = \int_{\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d} L(c)(x, t - \tau, \xi) d\mu^*_\delta > -\tilde{H}_\delta(c).$$

This is a contradiction.  

One can see the set $\mathcal{A}_\delta(c)$ as an analogue of the Aubry set.

3.7 Hyperbolic scaling limit

We observe the hyperbolic scaling limit of what we obtained in Sect. 3.1–3.4, where we mean by “$\delta = (h, r) \rightarrow 0$” that $\delta \rightarrow 0$ with the condition $0 < \lambda_0 \leq \lambda = \tau/h \leq \lambda_1$ with the constant $\lambda_1$ found in Sect. 2.

First, we state convergence of the effective Hamiltonian $\tilde{H}_\delta$ to that of the exact problem

$$v_t + H(x, t, c + v_x) = \tilde{H}(c) \quad \text{in } \mathbb{T}^d \times \mathbb{T}. \quad (3.9)$$

Hereinafter $b_1, b_2, \ldots$ are positive constants independent of $\delta$ and $c$.

**Theorem 3.9** There exists a constant $b_1 > 0$ for which we have

$$\sup_{c \in P} |\tilde{H}_\delta(c) - \tilde{H}(c)| \leq b_1 \sqrt{h} \quad \text{for } \delta \rightarrow 0.$$
Hence, we have with Lemma 2.1,

\[
|\eta^k(\gamma) - \gamma^*(t_k)|_\infty \leq b_3 h
\]

Hence, we have with Lemma 2.1,

\[
u_\delta(x_{n+1}) = \tilde{u}_{n+1}^{2K} \\
\leq E_{\tilde{\nu}_{n+1}^{2K,0}}(\xi) \left[ \sum_{0 < k \leq 2K} L^c \left( \gamma^k, t_{k-1}, \xi^{k}_{m(\gamma^k)} \right) \tau + u_\delta(\eta^0) \right] + \tilde{H}_\delta(c) \\
\leq E_{\tilde{\nu}_{n+1}^{2K,0}}(\xi) \left[ \sum_{0 < k \leq 2K} L^c \left( \eta^k(\gamma), t_{k-1}, \xi^{k}_{m(\gamma^k)} \right) \tau + u_\delta(\eta^0) \right] + \tilde{H}_\delta(c) + b_4 \sqrt{h} \\
\leq E_{\tilde{\nu}_{n+1}^{2K,0}}(\xi) \left[ \sum_{0 < k \leq 2K} L^c \left( \gamma^*(t_k), t_{k-1}, \gamma^*(t_k) \right) \tau + u_\delta(\gamma^*(0)) \right] + \tilde{H}_\delta(c) + b_5 \sqrt{h} \\
\leq \int_0^1 L^c(\gamma^*(s), s, \gamma^*(s))ds + u_\delta(\gamma^*(0)) + \tilde{H}_\delta(c) + b_6 \sqrt{h}.
\]

Since we have

\[
\tilde{\nu}(x_{n+1}, 1) = \int_0^1 L^c(\gamma^*(s), s, \gamma^*(s))ds + \tilde{\nu}(\gamma^*(0), 0) + \tilde{H}(c),
\]

with \(\tilde{\nu}(\gamma^*(0), 0) = \tilde{\nu}(\gamma^*(0), 1)\), we obtain

\[
u_\delta(x) - \tilde{\nu}(x, 1) \leq \nu_\delta(x_{n+1}) - \tilde{\nu}(x_{n+1}, 1) + b_7 h \\
\leq \nu_\delta(\gamma^*(0)) - \tilde{\nu}(\gamma^*(0), 1) + \tilde{H}(c) - \tilde{H}(c) + b_6 \sqrt{h} + b_7 h.
\]

By the choice of \(x\), we see that

\[
\tilde{H}_\delta(c) - \tilde{H}(c) \geq [\nu_\delta(x) - \tilde{\nu}(x, 1)] - [\nu_\delta(\gamma^*(0)) - \tilde{\nu}(\gamma^*(0), 1)] - b_8 \sqrt{h} \\
\geq -b_8 \sqrt{h}.
\]

We prove the converse inequality. We switch the above \(x\) to the one attaining \(\min_{x \in \mathbb{T}^d}(\nu_\delta(x) - \tilde{\nu}(x, 1))\). Let \(\xi^*\) be the minimizing control for \(\nu_\delta(x_{n+1}) = \tilde{u}_{n+1}^{2K}\). Then, we have

\[
u_\delta(x_{n+1}) = E_{\tilde{\nu}_{n+1}^{2K,0}}(\xi) \left[ \sum_{0 < k \leq 2K} L^c \left( \gamma^k, t_{k-1}, \xi^{k}_{m(\gamma^k)} \right) \tau + u_\delta(\eta^0) \right] + \tilde{H}_\delta(c) \\
\geq E_{\tilde{\nu}_{n+1}^{2K,0}}(\xi) \left[ \sum_{0 < k \leq 2K} L^c \left( \eta^k(\gamma), t_{k-1}, \xi^{k}_{m(\gamma^k)} \right) \tau + u_\delta(\eta^0) \right] + \tilde{H}_\delta(c) - b_9 \sqrt{h}.
\]
Let $\eta_\delta(\gamma)$ be the linear interpolation of $\eta^k(\gamma)$, where $\eta_\delta(\gamma)(t) = \xi^k_{m(\gamma^k)}$ for $t \in (t_{k-1}, t_k)$. Then, for each $\gamma$, we have

$$
\bar{w}(x_{n+1}, 1) \leq \int_0^1 L^c(\eta_\delta(\gamma)(s), s, \eta_\delta(\gamma)'(s))ds + \bar{w}(\eta_\delta(\gamma)(0), 0) + \bar{H}(c)
$$

$$
\leq \sum_{0 < k \leq 2K} L^c \left( \eta^k(\gamma), t_{k-1}, \xi^k_{m(\gamma^k)} \right) t + \bar{w}(\eta^0(\gamma), 0) + \bar{H}(c) + b_9 h.
$$

Therefore, we see that

$$
\bar{u}_\delta(x) - \bar{w}(x, 1) \geq \bar{u}_\delta(x_{n+1}) - \bar{w}(x_{n+1}, 1) - b_{10h}
$$

$$
\geq E_{\mu(\xi^k)} \left[ \bar{u}_\delta(\eta^0(\gamma)) - \bar{w}(\eta^0(\gamma), 1) \right] + \bar{H}(c) - \bar{H}(c) - b_{11}\sqrt{h}.
$$

Due to the choice of $x$, we conclude $\bar{H}(c) - \bar{H}(c) \leq b_{11}\sqrt{h}$. □

As for $\bar{H}(c)$, we have the same convergence result.

**Theorem 3.10** Let $\{\bar{v}_\delta(c)\}_\delta$ be a uniformly bounded sequence of time-1-periodic solutions of (3.4) with $\delta \to 0$. Then, there exists a subsequence of $\{\bar{v}_\delta(c)\}_\delta$, still denoted by the same symbol, such that $\{\bar{v}_\delta(c)\}_\delta$ tends to a viscosity solution $\bar{v}$ of (3.9) as $\delta \to 0$ in the sense that

$$
\max_{\{(m,k+1)\} \in \mathbb{G} \mid (x_m, t_{k+1}) \in \text{pr.}\mathbb{G}} |\bar{v}_\delta(c)^k_{m+1} - \bar{v}(x_m, t_{k+1})| \to 0 \quad \text{as } \delta \to 0. \quad (3.10)
$$

**Proof** Let $u_\delta: \mathbb{T}^d \to \mathbb{R}$ be the Lipschitz interpolation of $\bar{v}_\delta(c)^0$. Then, $\{u_\delta\}_\delta$ is uniformly bounded and equi-Lipschitz to have convergent subsequence with the limit $w$. Let $\bar{v}$ be the viscosity solution of

$$
v_t + H(x, t, c + v_x) = \bar{H}(c) \text{ in } \mathbb{T}^d \times (0, 1], \quad v(\cdot, 0) = w(\cdot) \text{ on } \mathbb{T}^d.
$$

Since $\bar{H}(c) \to \bar{H}(c)$ as $\delta \to 0$, we see that (3.10) follows from the same reasoning as the proof of Theorem 2.5, where the time periodicity of $\bar{v}_\delta$ implies that $\bar{v}$ is time-1-periodic. □

As for a sequence of time-1-periodic solutions of (3.3), it tends to a semiconvex a.e. solution of (3.9) as $\delta \to 0$ (up to a subsequence).

The same reasonings as the proofs of Theorems 2.6 and 2.8 yield the following theorems:

**Theorem 3.11** Consider the subsequence $\{\bar{v}_\delta(c)\}_\delta$ and $\bar{v}$ in Theorem 3.10. Let $(x, t) \in \mathbb{T}^d \times (0, 1]$ be such that $\bar{v}_{\delta'}(x, t)$ exists. Note that a.e. points of $\mathbb{T}^d \times [0, 1]$ have such a property. Let $(x_n, t_n) \in \mathbb{G} \delta$ be such that $(x_n, t_n) \to (x, t)$ as $\delta \to 0$. Then, we have

$$
|(D_{x, t} \bar{v}_\delta(c))^\bar{v}_\delta(x, t)| \to 0 \quad \text{as } \delta \to 0.
$$

**Theorem 3.12** Consider the subsequence $\{\bar{v}_\delta(c)\}_\delta$ and $\bar{v}$ in Theorem 3.10. Let $(x, t) \in \mathbb{R}^d \times (0, 1]$, $\bar{v}$ be arbitrary. Let $\Gamma^*(x, t)$ be the set of all minimizing curves $\gamma^*: [-\bar{t}, \bar{t}] \to \mathbb{R}^d$ for $\bar{v}(x, t)$. Let $(x_n, t_{n+1}) \in \mathbb{G} \delta$, $\bar{v}$ be such that $(x_n, t_{n+1}) \to (x, t)$, $t_{n+1} \to -\bar{t} - 0$ as $\delta \to 0$.

For each $\delta$, let $\gamma \in \Omega_{n+1}^{\bar{t}} - \bar{t}$ be the random walk generated by the minimizing control $\xi^*$ for $\bar{v}_{\delta}^{n+1}$. 1. Fix an arbitrary $\varepsilon_1 > 0$ and define the set

$$
\mathbb{G}^{\varepsilon_1} := \{\gamma \in \Omega_{n+1}^{\bar{t}} - \bar{t} \mid \text{there exists } \gamma^* = \gamma^*(\gamma) \in \Gamma^*(x, t) \text{ such that } \| \eta_\delta(\gamma)' - \gamma^{*\prime} \|_{L^2([-\bar{t}, \bar{t}])} \leq \varepsilon_1 \}.
$$

Then, we have $\text{Prob}(\mathbb{G}^{\varepsilon_1}) \to 1$ as $\delta \to 0$.
2. Fix an arbitrary \( \varepsilon_2 > 0 \) and define the set

\[
\tilde{\Omega}_\delta^{\varepsilon_2} := \{ \gamma \in \Omega_{\delta,n}^{l+1,-1} \mid \text{there exists } \gamma^* = \gamma^*(\gamma) \in \Gamma^* \text{ such that } \| \eta_{\delta}(\gamma) - \gamma^* \|_{C^0([-\tilde{t}, t])} \leq \varepsilon_2 \}.
\]

Then, we have \( \text{Prob}(\tilde{\Omega}_\delta^{\varepsilon_2}) \to 1 \) as \( \delta \to 0 \).

3. Fix an arbitrary \( \varepsilon_3 > 0 \) and define the set

\[
\Omega_\delta^{\varepsilon_3} := \{ \gamma \in \Omega_{\delta,n}^{l+1,-1} \mid \text{there exists } \gamma^* = \gamma^*(\gamma) \in \Gamma^*(x, t) \text{ such that } \| \gamma_{\delta} - \gamma^* \|_{C^0([-\tilde{t}, t])} \leq \varepsilon_3 \}.
\]

Then, we have \( \text{Prob}(\Omega_\delta^{\varepsilon_3}) \to 1 \) as \( \delta \to 0 \).

Theorem 3.13 Let \( \{\mu_\delta^*\} \) with \( \delta \to 0 \) be a sequence of minimizing measures for (3.8). Then, there exist a subsequence of \( \{\mu_\delta^*\} \), still denoted by the same symbol and an exact Mather measure \( \mu^* \) for \( L^{(c)} \) such that \( \mu_\delta^* \) converges weakly to \( \mu^* \).

Proof Since \( \{\mu_\delta^*\} \) is supported by a compact set independently from \( \delta \), we find a weakly convergent subsequence with the limit \( \mu^* \). Proposition 3.6 implies the holonomic constraint of \( \mu^* \). Theorems 3.7 and 3.9 imply that \( \mu^* \) satisfies

\[
\int_{\mathbb{T}^d \times \mathbb{T} \times \mathbb{R}^d} L^{(c)}(x, t, \xi) d\mu^* = -\tilde{H}(c),
\]

which shows that \( \mu^* \) is a Mather measure for \( L^{(c)} \) (see [28,33]). \( \square \)

4 Weak KAM theory: autonomous case

We investigate the case with a time-independent Lagrangian \( L(x, \xi) / \text{Hamiltonian } H(x, p) \). The presentation of Sect. 3 becomes simpler in the autonomous case. We state several things which differ from Sect. 3.

4.1 Weak KAM solution on grid

Since the configuration grid switches between \( G_{\text{even}} \) and \( G_{\text{odd}} \) depending on the time index, the discrete Hamilton–Jacobi equation always needs at least two unit-time-evolutions, i.e., with \( \phi_\delta := \varphi_\delta^2 \), we have

\[
\varphi_\delta := \varphi_\delta^{2K} = \phi_\delta \circ \phi_\delta \circ \cdots \circ \phi_\delta = (\phi_\delta)^K \text{ (i.e., } K\text{-iteration of } \phi_\delta).\]

We say that \( v \) is a stationary solution of (3.4), if it satisfies \( v(\cdot, t_{2k}) = v(\cdot, 0) \), \( v(\cdot, t_{2k+1}) = v(\cdot, t_1) \) for all \( k \) (one cannot have \( v^0 = v(\cdot, t_k) = v(\cdot, t_{k+1}) \)). Note that \( v^0 \in X_\delta \) yields a stationary solution, if and only if \( v^0 \) admits \( \phi_\delta v^0 + 2\tau \tilde{H}_\delta(c) = v^0 \). The reasoning in Sect. 3.1 yields a pair of \( \tilde{H}_\delta(c) \in \mathbb{R} \) and \( v^0 \in X_\delta \) such that \( \varphi_\delta(v^0; c) + \tilde{H}_\delta(c) = v^0 \).

Proposition 4.1 If \( \tilde{v}^0 \in X_\delta \) satisfies \( \phi_\delta(\tilde{v}^0; c) + \tilde{H}_\delta(c) = \tilde{v}^0 \), it satisfies also \( \phi_\delta \tilde{v}^0 + 2\tau \tilde{H}_\delta(c) = \tilde{v}^0 \), i.e., \( \tilde{v}^k := \varphi_\delta^k(\tilde{v}^0; c) + t_k \tilde{H}_\delta(c) \) is a stationary solution of (3.4).
Proof For \( \bar{v}^k := \phi_\delta^k(\bar{v}^0; c) + t_k \bar{H}_\delta(c) \) (\( \bar{v}^k \) solves (3.4)), set \( Z^{k+1}_{m+1} := \bar{v}^{k+2}_{m+1} - \bar{v}^k_{m+1} \) and \( S^k := \max_m Z^{k}_{m+1} \). It follows from (3.4), Taylor’s formula and (H2) that

\[
Z^{k+1}_m = \sum_{i=1}^d \left\{ \left( \frac{\lambda}{2d} H_p(x_m, (D_x \bar{v})^k_m) \right) Z^{k+1}_{m+1} + \left( \frac{1}{2d} + \frac{\lambda}{2} H_p(x_m, (D_x \bar{v})^k_m) \right) Z^{k+1}_{m+1} \right\} - \frac{\tau}{2k^m} \bar{v}^{k+2}_{m+1} - \bar{v}^k_{m+1},
\]

where \( k^m > \kappa > 0 \) for all \( k, m \). The CFL-type condition stated in Theorem 2.2 implies that \( \frac{\lambda}{2d} + \frac{\lambda}{2} H_p(x_m, (D_x \bar{v})^k_m) \geq 0 \). Hence, we have \( S^{k+1} \leq S^k \) for all \( k \). On the other hand, since \( \bar{v}^{2K} = \phi_\delta(\bar{v}^0; c) + \bar{H}_\delta(c) = \bar{v}^0 \), we have \( Z^{k+2}_{m+1} = Z^{k+1}_{m+1} \) and \( S^{k+2} = S^k \) for all \( k \geq 0 \). Therefore, it holds that \( S^0 \leq S^2K \leq S^k \leq S^0 \) for all \( 0 \leq k \leq 2K \), i.e., \( S^k = S^0 \) for all \( k \geq 0 \).

Let \( m \) be such that \( S^{k+1}_{m+1} = S^0 \). Then, the recurrence equation of \( Z^{k+1}_{m+1} \) implies that we must have \( Z^{k+1}_{m+1} = S^0 \) for all \( \omega \in B \), because otherwise \( S^{k+1} < S^0 \). Let \( \Lambda^k := \{ x_m+1 \in pr.h\mathbb{Z}^d \mid x_{m+1}, t_k \in pr.\bar{\mathcal{G}}, \ Z^{k+1}_{m+1} < S^0 \} \). If \( x_m+1 \in \Lambda^k \), we must have \( x_{m+1}+\omega \in \Lambda^{k+1} \) for all \( \omega \in B \). Hence, if \( \Lambda^0 \neq \emptyset \), it holds that \( \mathbb{Z}\Lambda^k < \mathbb{Z}\Lambda^{k+1} \) for all \( k \geq 0 \) until we have \( \Lambda^{k+1} \) = pr.\( G_{even} \) or pr.\( G_{odd} \), which means that \( Z^{k}_{m+1} < S^0 \) for all \( m \) at some \( k \). Therefore, we must have \( \Lambda^0 = \emptyset \). Furthermore, the equality \( S^k = S^0 \) for all \( k \geq 0 \) requires \( (D_x \bar{v})^{k+2} = (D_x \bar{v})_m^k \) for all \( m, k \). This implies that there exists a constant \( \beta \in \mathbb{R} \) such that \( \bar{v}^0 + \beta = \bar{v}^2 \). Since \( \bar{v}^2 + \beta = \phi_\delta(\bar{v}^0 + \beta) + 2\tau \bar{H}_\delta(c) = \phi_\delta \bar{v}^2 + 2\tau \bar{H}_\delta(c) = \bar{v}^4 \), \( \bar{v}^4 + \beta = \bar{v}^6 \), \ldots, \( \bar{v}^{2K-2} + \beta = \bar{v}^{2K} \), we have \( \bar{v}^0 + K\beta = \bar{v}^{2K} = \bar{v}^0 \). Thus, we conclude that \( \beta = 0 \) and \( \bar{v}^0 = \bar{v}^2 = \phi_\delta \bar{v}^0 + 2\tau \bar{H}_\delta(c) \).

\[ \square \]

4.2 Mather measure

We re-formulate the analogue of Mather’s minimizing problem in the autonomous case. We call \( \xi : pr.h\mathbb{Z}^d = pr.(G_{even} \cup G_{odd}) \to \{-(d\lambda)^{-1}, (d\lambda)^{-1}\}^d \) an admissible control, where we identify \( \xi \) as the function \( \xi : pr.\bar{\mathcal{G}} \to \{-(d\lambda)^{-1}, (d\lambda)^{-1}\}^d \). Define the functional for each \( l \in \mathbb{N} \) as

\[
\mathcal{F}^l_\delta(\xi; \xi) : C_c(\mathbb{T}^d \times \mathbb{R}^d; \mathbb{R}) \to \mathbb{R},
\]

\[
\mathcal{F}^l_\delta(f; \xi) := E_{\mu_{k+1}^l; \xi} \left[ \frac{1}{l!} \sum_{-l < k \leq 0} f(\gamma^k; \xi_{m(l^k)}) \right].
\]

The Riesz representation theorem yields the unique probability measure \( \mu_\delta^l(\xi) \) of \( \mathbb{T}^d \times \mathbb{R}^d \) such that

\[
\mathcal{F}^l_\delta(f; \xi) = \int_{\mathbb{T}^d \times \mathbb{R}^d} f \, d\mu_\delta^l(\xi) \quad \text{for any } f \in C_c(\mathbb{T}^d \times \mathbb{R}^d; \mathbb{R}).
\]
We give the representation of $\mu^l_\delta(\xi)$ through $p(\cdot; \xi) : \tilde{G}_\delta \to [0, 1]$. Define $p^l_\delta(\cdot; \xi) : \pr.h\mathbb{Z}^d \to [0, 1]$ as

$$p^l_\delta(x; \xi) := \begin{cases} 1 & \sum \{ (m, k) \mid \pr.x_{m+1} = x, \ k = \text{even}, \ -l < k \leq 0 \} p(x_{m+1}, t_k; \xi) \tau \quad \text{for } x \in \pr.G_{\text{odd}}, \\ 1 & \sum \{ (m, k) \mid \pr.x_m = x, \ k+1 = \text{odd}, \ -l < k+1 \leq 0 \} p(x_m, t_{k+1}; \xi) \tau \quad \text{for } x \in \pr.G_{\text{even}} \end{cases}$$

where we note that

$$\sum_{x \in \pr.h\mathbb{Z}^d} p^l_\delta(x; \xi) = 1 \quad \text{for each } l \in \mathbb{N}.$$  

Due to the periodicity of $f \in C_c(T^d \times \mathbb{R}^d; \mathbb{R})$ and $\xi$, we have

$$\mathcal{F}^l_\delta(f; \xi) = \sum_{x \in \pr.h\mathbb{Z}^d} p^l_\delta(x; \xi) f(x, \xi(x)),$$

$$\mu^l_\delta(\xi) = \sum_{x \in \pr.h\mathbb{Z}^d} p^l_\delta(x; \xi) \delta_{x, \xi(x)},$$

where $\delta_{x, \xi}$ is the Dirac measure of $T^d \times \mathbb{R}^d$ supported by the point $(x, \xi)$. Let $\mathcal{Q}_\delta(\xi)$ be the set of all limits of convergent subsequences of $\{p^l_\delta(\cdot; \xi)\}_{l \in \mathbb{N}}$ to have

$$\mathcal{P}_\delta = \bigcup_{\xi} \left\{ \mu_\delta(\xi) = \sum_{x \in \pr.h\mathbb{Z}^d} p_\delta(x; \xi) \delta_{x, \xi(x)} \mid p_\delta(\cdot; \xi) \in \mathcal{Q}_\delta(\xi) \right\},$$

where the union is taken over all admissible controls.

We re-state the holonomic-like constraint of $\mathcal{P}_\delta$ in the autonomous case. For each function $g : \pr.h\mathbb{Z}^d \to \mathbb{R}$, consider a continuous function $f : T^d \times \mathbb{R}^d \to \mathbb{R}$ whose restriction to $\pr.h\mathbb{Z}^d \times \mathbb{R}^d$ is

$$f(x_m, \xi) := \frac{1}{\tau} \left( g(x_m) - \frac{1}{2d} \sum_{\omega \in B} g(x_{m+\omega}) \right) + \sum_{i=1}^d \frac{g(x_{m+1_i}) - g(x_{m-1_i})}{2h} \xi^i. \quad (4.1)$$

**Proposition 4.2** Each $\mu_\delta(\xi) \in \mathcal{P}_\delta$ satisfies the constraint

$$\int_{T^d \times \mathbb{R}^d} f \ d\mu_\delta(\xi) = 0$$

for any continuous function $f : T^d \times \mathbb{R}^d \to \mathbb{R}$ whose restriction to $\pr.h\mathbb{Z}^d \times \mathbb{R}^d$ is of the form (4.1).

**Proof** We identify each $g : \pr.h\mathbb{Z}^d \to \mathbb{R}$ with the function $g : \tilde{G} \to \mathbb{R}$ such that $g_{1_{x_m+1}} = g(x_{m+1})$ and $g_{2_{x_m+1}} = g(x_m)$ for all $k$, where $g$ is also 1-periodically extended to $\tilde{G}$. Then, a function $f$ with (4.1) can be seen as the function $f : T^d \times T \times \mathbb{R}^d \to \mathbb{R}$ whose restriction to $\pr.\tilde{G} \times \mathbb{R}^d$ is

$$f(x_m, t_{k+1}, \xi) := (D_1 g)^{k+1}_m + (D_1 g)^k_m \cdot \xi, \quad (x_m, t_{k+1}, \xi) \in \pr.\tilde{G} \times \mathbb{R}^d.$$

The rest is the same as the proof of Proposition 3.6. □
If $g$ is the restriction of $g \in C^1(\mathbb{T}^d; \mathbb{R})$ to $pr.h\mathbb{Z}^d$, we have $\frac{1}{\delta}(g(x_m) - \frac{1}{2d} \sum_{\omega \in B} g(x_{m+\omega})) \to 0$ as $\delta \to 0$ with the hyperbolic scaling. Hence, Proposition 4.2 implies the exact holonomic condition at the hyperbolic scaling limit.

The analogue of Mather’s minimizing problem is the following:

**Theorem 4.3** For each $c$, we have

$$\inf_{\mu_\delta \in \mathcal{P}_\delta} \int_{\mathbb{T}^d \times \mathbb{R}^d} L^{(c)}(x, \xi) \, d\mu_\delta = -\bar{H}_\delta(c),$$

where there exists at least one minimizing probability measure $\mu_\delta^* \in \mathcal{P}_\delta$ that attains the infimum.

We define an analogue of the Mather set $\mathcal{M}_\delta(c)$ for each $c$ as

$$\mathcal{M}_\delta(c) := \bigcup_{\mu_\delta^*} \text{supp}(\mu_\delta^*) \subset pr.h\mathbb{Z}^d \times [-d\lambda, (d\lambda)]^d,$$

where the union is taken over all minimizing probability measures $\mu_\delta^*$ of (4.2).

**Theorem 4.4** Let $\bar{v}^0 \in X_\delta$ be such that $\varphi_\delta(\bar{v}^0; c) + \bar{H}_\delta(c) = \bar{v}^0$ and $\bar{v}^1 := \varphi_\delta^1(\bar{v}^0; c) + \tau \bar{H}_\delta(c)$. Define $\bar{v} = \bar{v}(c) : h\mathbb{Z}^d \to \mathbb{R}$ as $\bar{v}(x_{m+1}) = \bar{v}^0_{m+1}$ on $pr.G_{\text{odd}}$ and $\bar{v}(x_m) = \bar{v}^1_m$ on $pr.G_{\text{even}}$. For each $c$, the support of any minimizing probability measure for (4.2) is contained in the set

$$\mathcal{A}_\delta(c) := \{ (x_m, H_p(x_m, c + (D_x \bar{v})_m)) \mid x_m \in pr.h\mathbb{Z}^d \}.$$
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