Optimal parameters related with continuity properties of the multilinear fractional integral operator between Lebesgue and Lipschitz spaces

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Abstract

We deal with the boundedness of the multilinear fractional integral operator $I_{\gamma,m}$ from a product of weighted Lebesgue spaces into adequate weighted Lipschitz spaces. Our results generalize some previous estimates not only for the linear case but also for the unweighted problem in the multilinear context. We characterize the classes of weights for which the problem described above holds and show the optimal range of the parameters involved. The optimality is understood in the sense that the parameters defining the corresponding spaces belong to a certain region. We further exhibit examples of weights for the class which cover the mentioned area.

Keywords Multilinear fractional operator · Lipschitz spaces · Weights

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1 Introduction

Given $0 < \gamma < n$, the classical fractional integral operator $I_\gamma$ is defined by

$$I_\gamma f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}} dy,$$

provided the integral is finite. In [5] Muckenhoupt and Wheeden proved that, if $1 < p < n/\gamma$ and $1/q = 1/p - \gamma/n$, this operator maps $L^p(w^p)$ into $L^q(w^q)$ if and only if $w \in A_{p,q}$. Moreover, when $p = n/\gamma$ they showed that $I_\gamma$ maps $L^{n/\gamma}(w^{n/\gamma})$ into a certain weighted version of the bounded mean oscillation spaces BMO if and only if $w^{-n/(n-\gamma)} \in A_1$.

Later on, in [7] the author proved that for $n/\gamma < p < n/(\gamma - 1) + \delta/n = \gamma/n - 1/p$ the operator $I_\gamma$ maps $L^p(w^p)$ into a weighted version of Lipschitz spaces associated to the parameter $\delta$. A two-weighted problem it was also studied, giving the optimal parameters for which the associated classes of weights are nontrivial. Other results related with the continuity properties of $I_\gamma$ in the range of $p$ given above can be found in [2] and for different versions of weighted Lipschitz spaces in [3] and [6].

Given $m \in \mathbb{N}$ and $0 < \gamma < mn$ the multilinear version of order $m$ of the operator above, $I_{\gamma,m}$, is defined as follows

$$I_{\gamma,m} \vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^{m} f_i(y_i)}{\left(\sum_{i=1}^{m} |x-y_i|\right)^{mn-\gamma}} d\vec{y},$$

where $\vec{f} = (f_1, f_2, \ldots, f_m)$ and $\vec{y} = (y_1, y_2, \ldots, y_m)$, provided the integral is finite.

The continuity properties of $I_{\gamma,m}$ were studied for several authors. For example, it was shown in [4] that $I_{\gamma,m} : \prod_{i=1}^{m} L^{p_i} \hookrightarrow L^q$, where $1/p = \sum_{i=1}^{m} 1/p_i$ and $1/q = 1/p - \gamma/n$. The author also considered weighted versions of these estimates, generalizing the result in [5]. On the other hand, in [1] the authors proved unweighted estimates of $I_{\gamma,m}$ between $\prod_{i=1}^{m} L^{p_i}$ and Lipschitz-$\delta$ spaces, with $0 \leq \delta < 1$ and $\delta/n = \gamma/n - 1/p$. For other type of estimates involving $I_{\gamma,m}$ see also [8].

In this paper we study the boundedness of the operator $I_{\gamma,m}$ between a product of weighted Lebesgue spaces and certain weighted Lipschitz spaces, generalizing the linear case proved in [7] and the unweighted problem given in [1]. These spaces, denoted by $L^w(\delta)$ are given by the collection of locally integrable functions for which the inequality

$$\|wX_B\|_\infty \left|\frac{1}{B}|1+\delta/n| \int_B |f(x) - f_B| dx \leq C\right.$$ 

holds for every ball $B$ (see Sect. 2). We do not only consider related weights, which is an adequate extension of the one-weight estimates in the linear case, but also with independent weights exhibiting a generalization of the two-weight problem for $m = 1$. We characterize the classes of weights for which the problem described above holds, by also showing the optimal range of the parameters involved. The optimality is understood in the sense that the parameters defining the corresponding spaces belong
to a certain region, becoming trivial outside of it. Moreover we exhibit examples of weights covering this area giving, in this way, a complete theory. As far as we know, the results in this paper are a first approach to this topic in the weighted multilinear context.

We shall now introduce the classes of weights and the notation required in order to state our main results. Throughout the paper the multilinear parameter will be denoted by $m \in \mathbb{N}$. Let $0 < \gamma < mn$, $\delta \in \mathbb{R}$ and $\vec{p} = (p_1, p_2, \ldots, p_m)$ be an $m$-tuple of exponents where $1 \leq p_i \leq \infty$ for $1 \leq i \leq m$. We define $p$ such that $1/p = \sum_{i=1}^{m} 1/p_i$.

Given the weights $w$, $v_1, \ldots, v_m$, if $\vec{v} = (v_1, v_2, \ldots, v_m)$ we say that the pair $(w, \vec{v})$ belongs to the class $\mathbb{H}_m(\vec{p}, \gamma, \delta)$ if there exists a positive constant $C$ such that the inequality

$$\frac{\|w x_B\|_\infty}{|B|^{(\delta-1)/n}} \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} \frac{v_i^{-p_i}(y)}{|B|^{1/n} + |x_B - y|^{(\gamma - \gamma_i + 1 + \delta)/(p_i)}} \, dy \right)^{1/p_i^\prime} \leq C$$

holds for every ball $B$, where $x_B$ denotes the center of $B$ and $\sum_{i=1}^{m} \gamma_i = \gamma$, with $0 < \gamma_i < n$ for every $i$. The integral above is understood as usual when $p_i = 1$, (see Sect. 2 for further details).

When $m = 1$ the class defined above was first introduced in [7] (see also [5] for the case $\delta = 0$). In that paper the author showed nontrivial weights when $\gamma - n \leq \delta \leq \min\{1, \gamma - n/p\}$, $\gamma$, $n$, $p$, $m$. We shall see that a similar restriction on $\delta$ appears in the multilinear context.

We are in a position to state our first result.

**Theorem 1.1** Let $0 < \gamma < mn$, $\delta \in \mathbb{R}$, and $\vec{p}$ a vector of exponents that verifies $p > n/\gamma$. Let $(w, \vec{v})$ be a pair such that $v_i^{-p_i} \in RH_m$, for $1 \leq i \leq m$. Then the following statements are equivalent:

1. **The operator** $I_{\gamma,m}$ **is bounded from** $\prod_{i=1}^{m} L^{p_i}(v_i^{p_i})$ **to** $L_w(\delta)$;
2. **The pair** $(w, \vec{v})$ **belongs to** $\mathbb{H}_m(\vec{p}, \gamma, \delta)$.

In the linear case the reverse Hölder condition on the weight is trivially satisfied, so our theorem is a well extension of the corresponding result in [7] for $p > n/\gamma$.

We have already observed that, although there is no restrictions on $\delta$ in the previous theorem, they arise as a consequence of the nature of the corresponding weights. The next result gives the range of parameters involved in the class defined above where the weights are trivial, that is, $v_i = \infty$ a.e. for some $i$ or $w = 0$ a.e.

**Theorem 1.2** Let $0 < \gamma < mn$, $\delta \in \mathbb{R}$, and $\vec{p}$ a vector of exponents. The following statements hold:

1. **If** $\delta > 1$ **or** $\delta > \gamma - n/p$ **then condition** $\mathbb{H}_m(\vec{p}, \gamma, \delta)$ **is satisfied if and only if** $v_i = \infty$ **a.e. for some** $1 \leq i \leq m$.
2. **The same conclusion holds if** $\delta = \gamma - n/p = 1$.
3. **If** $\delta < \gamma - mn$ **then condition** $\mathbb{H}_m(\vec{p}, \gamma, \delta)$ **is satisfied if and only if** $v_i = \infty$ **a.e. for some** $1 \leq i \leq m$ **or** $w = 0$ a.e.
We also exhibit non trivial examples of weights showing that the class is non empty (see Sect. 5).

As we shall see if \( w = \prod_{i=1}^{m} v_i \) and \( \delta < \tau = (\gamma - mn)(1 - 1/m) + 1/m \) the corresponding class, denoted by \( \vec{v} \in H_m(\vec{p}, \gamma, \delta) \), is reduced to the \( A_{\vec{p}, \infty} \) condition. This class is defined by the vectors \( \vec{v} = (v_1, \ldots, v_m) \), for which the following inequality holds and it is the endpoint case of the \( A_{\vec{p}, q} \) classes defined in [4]. If \( p_i = 1 \) for some \( i \), the corresponding factor above must be understood as \( \|v_i^{-1} X_B\|_\infty \). When \( m = 1 \) this inequality is equivalent to require \( v^{-p'} \in A_1 \), which is the expected condition (see for example [7]).

The following result summarizes the discussion given above by showing that the parameters \( \delta \) and \( p \) are restricted to a line.

**Theorem 1.3** Let \( 0 < \gamma < mn, \delta \in \mathbb{R} \), and \( \vec{p} \) a vector of exponents. If \( \vec{v} \in H_m(\vec{p}, \gamma, \delta) \), then \( \delta = \gamma - n/p \).

The theorem above proves that if \( \delta = \gamma - n/p \), then \( H_m(\vec{p}, \gamma, \delta) \subset A_{\vec{p}, \infty} \) and both classes coincide for \( \delta < \tau \). When \( m = 1 \) the result above was obtained in [9].

The article is organized as follows. In Sect. 2 we give some previous notation and properties of the classes of weights. In Sect. 3 we study the behaviour of some operators related with \( I_{\gamma, m} \) which will be useful in the proof of the main theorem, given in Sect. 4. Finally, in Sect. 5 we exhibit examples of weights in the optimal range and prove the result dealing with the particular case of related weights.

### 2 Preliminaries

Throughout the paper \( C \) will denote an absolute constant that may change in every occurrence. By \( A \lesssim B \) we mean that there exists a positive constant \( c \) such that \( A \leq cB \). We say that \( A \approx B \) when \( A \lesssim B \) and \( B \lesssim A \).

Let \( m \in \mathbb{N} \). Given a set \( E \), with \( E^n \) we shall denote the cartesian product of \( E \) \( m \) times.

The multilinear fractional integral operator of order \( 0 < \gamma < mn \) is defined by

\[
I_{\gamma, m} \vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^{m} f_i(y_i)}{(\sum_{i=1}^{m} |x - y_i|)^{mn-\gamma}} d\vec{y},
\]

where \( \vec{f} = (f_1, f_2, \ldots, f_m) \) and \( \vec{y} = (y_1, y_2, \ldots, y_m) \). It will be useful for us to consider the operator

\[
J_{\gamma, m} \vec{f}(x) = \int_{(\mathbb{R}^n)^m} \left( \frac{1}{(\sum_{i=1}^{m} |x - y_i|)^{mn-\gamma}} - \frac{1 - X_{B(0,1)^n}(\vec{y})}{(\sum_{i=1}^{m} |y_i|)^{mn-\gamma}} \right) \prod_{i=1}^{m} f_i(y_i) d\vec{y}. \tag{2.1}
\]
which differs from $I_{γ,m}$ only by a constant term. This operator has the same Lipschitz norm as $I_{γ,m}$, so it will be enough to give the results for $J_{γ,m}$.

By a weight we understand any positive and locally integrable function.

Given $δ ∈ ℝ$ and a weight $w$ we say that a locally integrable function $f ∈ L_w(δ)$ if there exists a positive constant $C$ such that

$$\|wX_B\|_∞ \frac{1}{|B|^{1+δ/n}} \int_B |f(x) - f_B| \, dx \leq C \tag{2.2}$$

for every ball $B$, where $f_B = |B|^{-1} \int_B f$. The smallest constant $C$ for which the inequality above holds will be denoted by $\|f\|_L_w(δ)$.

If $δ = 0$ the space $L_w(0)$ coincides with a weighted version of BMO spaces introduced in [5]. Concerning to the unweighted case, when $0 < δ < 1$ these spaces are equivalent to the classical Lipschitz classes $Λ_1(δ)$ given by the collection of functions $f$ satisfying $|f(x) - f(y)| ≤ C|x - y|^δ$ and they are Morrey spaces when $-n < δ < 0$.

These classes of functions were also studied in [7].

As we said in the introduction, the classes $H_{γ,m}(⃗p, γ, δ)$ are given by the pairs $(w, ⃗v)$ for which the inequality

$$\sup_{B ⊂ ℝ^n} \|wX_B\|_∞ \prod_{i=1}^m \left( \int_{ℝ^n} \frac{v_i^{-p'_i}(y)}{|B|^{1/n} + |x_B - y|^{(n-γ_i+1/m)p'_i}} \, dy \right)^{1/p'_i} < \infty \tag{2.3}$$

holds. For those index $i$ such that $p_i = 1$ we understand the corresponding factor on the products above as

$$\left\| \frac{v_i^{-1}}{|B|^{1/n} + |x_B - y|^{(n-γ_i+1/m)}} \right\|_∞ \tag{2.4}.$$ 

Let $I_2 = \{1 ≤ i ≤ m : p_i = 1\}$ and $I_1 = \{1, \ldots, m\} \setminus I_2$. Observe that $(w, ⃗v) ∈ \mathbb{H}_m(⃗p, γ, δ)$ implies that the inequalities

$$\|wX_B\|_∞ \prod_{i ∈ I_1} \|v_i^{-1}X_B\|_∞ \prod_{i ∈ I_2} \left( \frac{1}{|B|} \int_B v_i^{-p'_i}(y) \, dy \right)^{1/p'_i} \leq C \tag{2.5}$$

and

$$\|wX_B\|_∞ \prod_{i ∈ I_1} \left( \int_{ℝ^n \setminus B} \frac{v_i^{-p'_i}(y)}{|x_B - y|^{(n-γ_i+1/m)p'_i}} \, dy \right)^{1/p'_i} \leq C \tag{2.6}$$

holds for every ball $B$. We shall refer to these inequalities as the local and the global conditions, respectively.
On the other hand, under certain properties on \( \vec{v} \), the corresponding local and global conditions imply (2.3). Before state and prove this result, we shall introduce some useful notation.

Given \( m \in \mathbb{N} \) we denote \( S_m = \{0, 1\}^m \). Given a set \( B \) and \( \sigma \in S_m, \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m) \) we define

\[
B^{\sigma_i} = \begin{cases} 
B, & \text{if } \sigma_i = 1 \\
\mathbb{R}^n \setminus B, & \text{if } \sigma_i = 0.
\end{cases}
\]

With the notation \( B^\sigma \) we will understand the cartesian product \( B^{\sigma_1} \times B^{\sigma_2} \times \cdots \times B^{\sigma_m} \). In particular, if we set \( 1 = (1, 1, \ldots, 1) \) and \( 0 = (0, 0, \ldots, 0) \) then we have

\[
B^1 = B \times B \times \cdots \times B = B^m, \quad \text{and} \quad B^0 = (\mathbb{R}^n \setminus B) \times (\mathbb{R}^n \setminus B) \times \cdots \times (\mathbb{R}^n \setminus B) = (\mathbb{R}^n \setminus B)^m.
\]

We recall that a weight \( w \) belongs to the reverse Hölder class \( RH_s, 1 < s < \infty \), if there exists a positive constant \( C \) such that the inequality

\[
\left( \frac{1}{|B|} \int_B w^s \right)^{1/s} \leq C \frac{1}{|B|} \int_B w
\]

holds for every ball \( B \) in \( \mathbb{R}^n \). The smallest constant for which the inequality above holds is denoted by \( [w]_{RH_s} \). It is not difficult to see that \( RH_t \subset RH_s \) whenever \( 1 < s < t \).

We say that \( w \in RH_\infty \) if

\[
\sup_B w \leq \frac{C}{|B|} \int_B w,
\]

for some positive constant \( C \). It is well known that any radial power function \( |\cdot|^\alpha \), with \( \alpha > 0 \) satisfies \( RH_\infty \) condition.

**Lemma 2.1** Let \( 0 < \gamma < mn, \delta \in \mathbb{R}, \vec{p} \) a vector of exponents and \( (w, \vec{v}) \) a pair of weights such that \( v_i^{-1} \in RH_\infty \) for \( i \in I_1 \) and \( v_i^{-p'_i} \) is doubling for \( i \in I_2 \). Then, condition \( \mathbb{H}_m(\vec{p}, \gamma, \delta) \) is equivalent to (2.6).

**Proof** We have already seen that \( \mathbb{H}_m(\vec{p}, \gamma, \delta) \) implies (2.6). Let \( \theta_i = n - \gamma_i + 1/m \), for every \( i \). If \( m_2 = m - m_1 \) where \( m_1 = \#I_1 \), the cardinal of \( I_1 \), after a possible rename of the index \( i \in I_2 \) we have that

\[
\prod_{i \in I_2} \left( \int_{\mathbb{R}^n} \frac{v_i^{-p'_i}}{(|B|^{1/n} + |x_B - \cdot|)^{\theta_i p'_i}} \right)^{1/p'_i} = \sum_{\sigma \in S_{m_2}} \prod_{i=1}^{m_0} \left( \int_{B^{\sigma_i}} \frac{v_i^{-p'_i}}{(|B|^{1/n} + |x_B - \cdot|)^{\theta_i p'_i}} \right)^{1/p'_i}.
\]
Fix $\sigma \in S_{m_2}$. If $\sigma_i = 0$, we have that
\[
\left( \int_{B^{\sigma_i}} \left( |B|^{1/n} + |x_B - \cdot| \right)^{(n-\gamma_i+1/m)p_i'} v_i^{-p_i'} \right)^{1/p_i'} \leq \left( \int_{\mathbb{R}^n \setminus B} \left( |B|^{1/n} + |x_B - \cdot| \right)^{(n-\gamma_i+1/m)p_i'} v_i^{-p_i'} \right)^{1/p_i'}.
\]

For $\sigma_i = 1$, since $v_i^{-p_i'}$ is doubling, we have that
\[
\left( \int_{B^{\sigma_i}} \left( |B|^{1/n} + |x_B - \cdot| \right)^{(n-\gamma_i+1/m)p_i'} (y) \right)^{1/p_i'} \leq \left( \int_{\mathbb{R}^n \setminus B} |x_B - y|^{(n-\gamma_i+1/m)p_i'} dy \right)^{1/p_i'}.
\]

Therefore, for every $\sigma \in S_{m_2}$ we obtain
\[
\prod_{i=1}^{m_2} \left( \int_{B^{\sigma_i}} \left( |B|^{1/n} + |x_B - \cdot| \right)^{(n-\gamma_i+1/m)p_i'} v_i^{-p_i'} \right)^{1/p_i'} \leq \prod_{i \in \mathcal{I}_2} \left( \int_{\mathbb{R}^n \setminus B} |x_B - \cdot|^{\sigma_i p_i'} \right)^{1/p_i'}.
\]

On the other hand, when $i \in \mathcal{I}_1$ we can follow a similar argument with the integral replaced by $\| \cdot \|_\infty$. Indeed, since $v_i^{-1} \in RH_\infty$ observe that
\[
\|v^{-1} \chi_B\|_\infty \leq \frac{[v^{-1}]_{\text{RH}_\infty}}{|B|} \int_B v^{-1} \leq \frac{C}{|B|} \int_{2B \setminus B} v^{-1} \leq C \|v^{-1} \chi_{2B \setminus B}\|_\infty.
\]

This allows us to estimate as follows
\[
\prod_{i \in I_1} \left( \frac{v_i^{-1}}{|B|^{1/n} + |x_B - \cdot|^n \gamma/m + 1/m} \right)_{\infty} \leq C \prod_{i \in I_1} \left( \int_{\mathbb{R}^n} \frac{|v_i^{-p'_i}|}{(|x_B - \cdot|^n \gamma/m + 1/m)^{q_i p'_i}} \right)^{1/p'_i} \leq C,
\]

as desired. \(\square\)

Corollary 2.2 Under the hypotheses of Lemma 2.1 we have that conditions (2.6) implies (2.5).

The following lemma is a local-to-global result for the condition \(\mathbb{H}_m(\tilde{\rho}, \gamma, \delta)\). It will be useful in order to give examples of weights. We shall assume that \(\gamma_i = \gamma/m, \) for every \(i\).

Lemma 2.3 Let \(0 < \gamma < mn, \delta < \tau = (\gamma - mn)(1 - 1/m) + 1/m, \tilde{\rho}\) a vector of exponents and \((w, \tilde{v})\) a pair of weights satisfying condition (2.5). Then \((w, \tilde{v})\) satisfies (2.6).

Proof Let \(\theta = n - \gamma/m + 1/m\). Fix a ball \(B\) and set \(B_k = 2^k B\), for every \(k \in \mathbb{N}\). If \(i \in I_1\) we have that
\[
\left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p'_i}(y)}{|x_B - y|^{\theta p'_i}} \, dy \right)^{1/p'_i} \leq C \sum_{k=1}^\infty \left( \int_{B_k \setminus B_{k+1}} \frac{v_i^{-p'_i}(y)}{|x_B - y|^{\theta p'_i}} \, dy \right)^{1/p'_i} \leq C \sum_{k=1}^\infty |B_k|^{-\theta/n} \left( \int_{B_k} v_i^{-p'_i} \right)^{1/p'_i}.
\]
If we set \( \vec{k} = (k_1, k_2, \ldots, k_m) \), the left-hand side of (2.6) can be bounded by

\[
C \sum_{\vec{k} \in \mathbb{N}^m} \prod_{i \in I_1} |B_{k_i}|^{-\theta/n} \left\| v_i^{-1} \chi_{B_{k_i+1}} \right\|_\infty \prod_{i \in I_2} |B_{k_i}|^{-\theta/n} \left( \int_{B_{k_i+1}} v_i^{-p_i'} \right)^{1/p_i'} = C \sum_{\vec{k} \in \mathbb{N}^m} I(B, \vec{k}).
\]

Observe that \( \mathbb{N}^m \subset \bigcup_{i=1}^m K_i \), where \( K_i = \{ \vec{k} = (k_1, k_2, \ldots, k_m) : k_i \geq k_j \text{ for every } j \} \). Let us estimate the sum over \( K_1 \), being similar for the other sets. Therefore

\[
\sum_{\vec{k} \in K_1} I(B, \vec{k}) \leq \sum_{k_1=1}^\infty |B_{k_1}|^{-\theta/n} \prod_{i \in I_1} \left\| v_i^{-1} \chi_{B_{k_1+1}} \right\|_\infty \prod_{i \in I_2} \left( \int_{B_{k_1+1}} v_i^{-p_i'} \right)^{1/p_i'} \prod_{i \neq 1} \sum_{k_i=1}^{k_1} |B_{k_i}|^{-\theta/n}.
\]

Notice that

\[
\sum_{k_i=1}^{k_1} |B_{k_i}|^{-\theta/n} = |B|^{-\theta/n} \sum_{k_i=1}^{k_1} 2^{-k_i\theta} \lesssim 2^{-k_1\theta} \sum_{k_i=1}^{k_1} 2^{(k_1-k_i)\theta} \lesssim |B_{k_1}|^{-\theta/n} 2^{k_1\theta}.
\]

Thus, from the estimation above and (2.5) we obtain that

\[
\frac{\| \chi_B \chi_{B} \|_\infty}{|B|^{(\delta-1)/n}} \sum_{\vec{k} \in K_1} I(B, \vec{k}) \leq \frac{C}{|B|^{(\delta-1)/n}} \sum_{k_1=1}^\infty 2^{(m-1)k_1\theta} |B_{k_1}|^{-m\theta/n} \| \chi_{B_{k_1+1}} \|_\infty \leq C \sum_{k_1=1}^\infty 2^{(m-1)k_1\theta} 2k_1(\delta-1),
\]

and the last sum is finite provided \( \delta < \tau \). \( \square \)

### 3 Technical results

In this section we introduce some operators involved with \( I_{\gamma, m} \) and useful properties in order to prove our main results.

Let \( B = B(x_B, R) \) and \( \tilde{B} = 2B \). We can formally decompose the operator in (2.1) as

\[
J_{\gamma, m} \tilde{f}(x) = a_B + I \tilde{f}(x),
\]

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where

\[
A_B = \int_{(\mathbb{R}^n)^m} \left( \frac{1 - \chi_{\tilde{B}^m}(\tilde{y})}{(\sum_{i=1}^m |x_B - y_i|)^{mn-\gamma}} - \frac{1 - \chi_{B(0,1)^m}(\tilde{y})}{(\sum_{i=1}^m |y_i|)^{mn-\gamma}} \right) \prod_{i=1}^m f_i(y_i) \, d\tilde{y}
\]

and

\[
I_{\tilde{f}}(x) = \int_{(\mathbb{R}^n)^m} \left( \frac{1}{(\sum_{i=1}^m |x - y_i|)^{mn-\gamma}} - \frac{1 - \chi_{\tilde{B}^m}(\tilde{y})}{(\sum_{i=1}^m |x_B - y_i|)^{mn-\gamma}} \right) \prod_{i=1}^m f_i(y_i) \, d\tilde{y}.
\]

(3.1)

(3.2)

The first step is to show that this operator is well-defined for \( \tilde{f} \) as in Theorem 1.1.

**Lemma 3.1** Let \( 0 < \gamma < mn, \delta \in \mathbb{R}, \) and \( \tilde{p} \) be a vector of exponents that verifies \( p > n/\gamma. \) Let \((w, \tilde{v})\) be a pair of weights in \( H_m(\tilde{p}, \gamma, \delta) \) such that \( v_{i}^{-p_{i}} \in RH_m, \) for \( i \in \mathcal{I}_2. \) If \( \tilde{f} \) satisfies \( f_{i}v_{i} \in L^{p_{i}} \) for every \( 1 \leq i \leq m, \) then \( J_{\gamma,m} \tilde{f} \) is finite in almost every \( x \in \mathbb{R}^n. \)

**Proof** We shall estimate \( A_B \) and \( I_{\tilde{f}} \) separately. Fix \( B = B(x_B, R) \) and consider \( B_{0} = B(0, R_{0}), \) where \( R_{0} = 2(|x_B| + R). \) Then, if \( \tilde{y} \notin B_{0}^m \) then the expression between brackets behaves as

\[
\left( \sum_{i=1}^m |x_B - y_i| \right)^{-mn+\gamma-1}.
\]

We can write

\[
A_B = \int_{(\mathbb{R}^n)^m} \left( \frac{1 - \chi_{\tilde{B}^m}(\tilde{y})}{(\sum_{i=1}^m |x_B - y_i|)^{mn-\gamma}} - \frac{1 - \chi_{B(0,1)^m}(\tilde{y})}{(\sum_{i=1}^m |y_i|)^{mn-\gamma}} \right) \prod_{i=1}^m f_i(y_i) \, d\tilde{y}
\]

\[
= \int_{B_{0}^m} + \int_{(\mathbb{R}^n)^m \setminus B_{0}^m}
\]

\[
= A_{1}^{B} + A_{2}^{B}.
\]

We split the estimate of \( A_{1}^{B} \) into four possible cases.

1. \( \chi_{\tilde{B}^m}(\tilde{y}) = \chi_{B(0,1)^m}(\tilde{y}) = 0. \) In this case we have that \( y_{i_0} \notin \tilde{B}, \) for at least one \( i_0 \in \{1, \ldots, m\}. \) This yields

\[
\sum_{i=1}^m |x_B - y_i| \geq |x_B - y_{i_0}| > C|B|^{1/n},
\]
since \( B \subset B_0 \). On the other hand, \( \vec{y} \notin B(0, 1)^m \) implies that \(|y_{j_0}| \geq 1\) for some \( j_0 \). Thus,

\[
\frac{1 - \mathcal{X}_{B_m}(\vec{y})}{(\sum_{i=1}^m |x_B - y_i|)^{mn-\gamma}} - \frac{1 - \mathcal{X}_{B(0,1)^m}(\vec{y})}{(\sum_{i=1}^m |y_i|)^{mn-\gamma}} \leq 1 + \frac{C}{|B|^{m-\gamma/n}}.
\]

(2) \( \mathcal{X}_{B_m}(\vec{y}) = 0, \mathcal{X}_{B(0,1)^m}(\vec{y}) = 1 \). In this case we have that

\[
\frac{1 - \mathcal{X}_{B_m}(\vec{y})}{(\sum_{i=1}^m |x_B - y_i|)^{mn-\gamma}} - \frac{1 - \mathcal{X}_{B(0,1)^m}(\vec{y})}{(\sum_{i=1}^m |y_i|)^{mn-\gamma}} = \frac{1}{(\sum_{i=1}^m |y_i|)^{mn-\gamma}} \leq 1 + \frac{C}{|B|^{m-\gamma/n}}.
\]

(3) \( \mathcal{X}_{B_m}(\vec{y}) = 1, \mathcal{X}_{B(0,1)^m}(\vec{y}) = 0 \). We obtain that

\[
\frac{1 - \mathcal{X}_{B_m}(\vec{y})}{(\sum_{i=1}^m |x_B - y_i|)^{mn-\gamma}} - \frac{1 - \mathcal{X}_{B(0,1)^m}(\vec{y})}{(\sum_{i=1}^m |y_i|)^{mn-\gamma}} = \frac{1}{(\sum_{i=1}^m |y_i|)^{mn-\gamma}} \leq 1 + \frac{C}{|B|^{m-\gamma/n}}.
\]

(4) \( \mathcal{X}_{B_m}(\vec{y}) = \mathcal{X}_{B(0,1)^m}(\vec{y}) = 1 \). This is the simplest case since the expression is zero and we trivially obtain the desired bound.

Recall that \( I_1 = \{i : p_i = 1\} \) and \( I_2 = \{i : p_i > 1\} \). With the previous estimate we have that

\[
|a_B^1| \leq \left(1 + \frac{C}{|B|^{m-\gamma/n}}\right) \left(\int_{B_0} \prod_{i=1}^m f_i(y_i) \, d\vec{y}\right)
= \left(1 + \frac{C}{|B|^{m-\gamma/n}}\right) \prod_{i=1}^m \left(\int_{B_0} f_i(y_i) \, dy_i\right)
\leq \left(1 + \frac{C}{|B|^{m-\gamma/n}}\right) \prod_{i=1}^m \|f_i\|_{p_i} \prod_{i \in I_1} \|v_i^{-1} X_{B_0}\|_{\infty} \prod_{i \in I_2} \left(\int_{B_0} v_i^{-p'_i}\right)^{1/p'_i} \leq \infty.
\]

We now turn our attention to the estimate of \( a_B^2 \). By noticing that

\[
(B_0^m)^c = \bigcup_{\sigma \in S_m, \sigma \neq 1} B_0^\sigma
\]
we obtain

\[ a_B^2 = \sum_{\sigma \in S_m, \sigma \neq 1} \int_{B_0^\sigma} \frac{\prod_{i=1}^m f_i(y_i)}{\left(\sum_{i=1}^m \left|x_B - y_i\right|\right)^{m-\gamma+1}} d\tilde{y}. \]

Let us estimate a term of this sum for a fixed \( \sigma \in S_m, \sigma \neq 1 \). If we set \( \theta_i = n - \gamma_i + 1/m \), for \( 1 \leq i \leq m \), we have that

\[ \int_{B_0^\sigma} \frac{\prod_{i=1}^m f_i(y_i)}{\left(\sum_{i=1}^m \left|x_B - y_i\right|\right)^{m-\gamma+1}} d\tilde{y} \leq C \left( \prod_{i: \sigma_i = 1} \int_{B_0} \frac{\left|f_i\right|}{\left|B_0\right|^{|\theta_i|/n}} \right) \left( \prod_{i: \sigma_i = 0} \int_{B_0^\sigma} \frac{\left|f_i(y_i)\right|}{\left|x_B - y_i\right|^{|\theta_i|}} d\tilde{y} \right). \]

(3.3)

We shall estimate each factor on the right hand side separately. Observe that, for every \( i \in I_1 \) such that \( \sigma_i = 1 \), we have

\[ \left\| v_i^{-1} X_{B_0} \right\|_\infty \leq C \left| B_0 \right|^{|\theta_i|/n} \left\| v_i^{-1} \right\|_\infty \left( \left| B_0 \right|^{1/n} + \left|x_{B_0} - i\right| \right)^{\theta_i/\theta_i}. \]

On the other hand, if \( \sigma_i = 1 \) and \( i \in I_2 \) we get

\[ \frac{1}{\left| B_0 \right|^{|\theta_i|/n}} \left( \int_{B_0} v_i^{-p_i'} \right)^{1/p_i'} \leq C \left( \int_{B_0} \frac{v_i^{-p_i'}(y_i)}{\left( \left| B_0 \right|^{1/n} + \left|x_{B_0} - y_i\right| \right)^{\theta_i/p_i'}} \right)^{1/p_i'} \leq C \left( \int_{\mathbb{R}^n} \frac{v_i^{-p_i'}(y_i)}{\left( \left| B_0 \right|^{1/n} + \left|x_{B_0} - y_i\right| \right)^{\theta_i/p_i'}} \right)^{1/p_i'}. \]

By combining these two estimates we get

\[ \prod_{i: \sigma_i = 1} \int_{B_0} \frac{\left|f_i(y_i)\right|}{\left|B_0\right|^{|\theta_i|/n}} d\tilde{y} \]

\[ \leq \left( \prod_{i: \sigma_i = 1} \left\| f_i v_i \right\|_p \right) \left\| v_i^{-1} X_{B_0} \right\|_\infty \left( \prod_{i: \sigma_i = 1} \left\| f_i v_i \right\|_p \right) \left( \prod_{i: \sigma_i = 0} \left\| f_i v_i \right\|_p \right) \]

\[ \leq C \left( \prod_{i: \sigma_i = 1} \left\| f_i v_i \right\|_p \right) \left( \prod_{i: \sigma_i = 1} \left\| v_i^{-1} \right\|_\infty \right) \left( \prod_{i: \sigma_i = 0} \left\| v_i^{-p_i'} \right\|_\infty \right) \]

\[ \times \prod_{i: \sigma_i = 1} \left( \int_{\mathbb{R}^n} \frac{v_i^{-p_i'}(y_i)}{\left( \left| B_0 \right|^{1/n} + \left|x_{B_0} - y_i\right| \right)^{\theta_i/p_i'}} d\tilde{y} \right)^{1/p_i'}. \]
We now turn our attention to the second factor on the right hand side of (3.3). If \( z \notin B_0 \) then

\[
|z| \leq |z - x_B| + |x_B| \leq |z - x_B| + \frac{R_0}{2} \leq |z - x_B| + \frac{|z|}{2},
\]

which implies that \( |z - x_B| > |z|/2 \). Therefore, if \( i \in I_1 \) and \( \sigma_i = 0 \) we get

\[
\left\| \frac{1}{|x_B - \cdot|^{\alpha_i}} \right\|_\infty \leq \left\| \frac{v_i^{-1} \mathcal{X}_{B_0}^{\alpha_i}}{(|B_0|^{1/n} + |x_{B_0} - \cdot|)^{\alpha_i}} \right\|_\infty \leq \left\| \frac{v_i^{-1}}{(|B_0|^{1/n} + |x_{B_0} - \cdot|)^{\alpha_i}} \right\|_\infty.
\]

Notice also that, if \( i \in I_2 \) and \( \sigma_i = 0 \), we have

\[
\left( \int_{B_0} |f_i(y_i)| \frac{v_i^{-p_i'}(y_i)}{|x_B - y_i|^{\alpha_i}} \, dy_i \right)^{1/p_i'} \leq C \left( \int_{B_0} |f_i(y_i)| \frac{v_i^{-p_i'}(y_i)}{(|B_0|^{1/n} + |x_{B_0} - y_i|)^{\alpha_i}} \, dy_i \right)^{1/p_i'} \leq C \left( \int_{B_0} |f_i(y_i)| \frac{v_i^{-p_i'}(y_i)}{(|B_0|^{1/n} + |x_{B_0} - y_i|)^{\alpha_i}} \, dy_i \right)^{1/p_i'}.
\]

Thus we can proceed as follows

\[
\prod_{i : \sigma_i = 0} \int_{B_0} \frac{|f_i(y_i)|}{|x_B - y_i|^{\alpha_i}} \, dy_i \leq \prod_{i \in I_1 : \sigma_i = 0} \left\| f_i \right\|_{p_i} \left\| \frac{v_i^{-1} \mathcal{X}_{B_0}^{\alpha_i}}{|x_B - \cdot|^{\alpha_i}} \right\|_\infty \times \prod_{i \in I_2 : \sigma_i = 0} \left\| f_i \right\|_{p_i} \left( \int_{B_0} \frac{v_i^{-p_i'}(y_i)}{|x_B - y_i|^{\alpha_i}} \right)^{1/p_i'} \leq C \left( \prod_{i : \sigma_i = 0} \left\| f_i \right\|_{p_i} \right) \prod_{i \in I_1 : \sigma_i = 0} \left\| \frac{v_i^{-1}}{(|B_0|^{1/n} + |x_{B_0} - \cdot|)^{\alpha_i}} \right\|_\infty \times \prod_{i \in I_2 : \sigma_i = 0} \left( \int_{B_0} \frac{v_i^{-p_i'}(y_i)}{(|B_0|^{1/n} + |x_{B_0} - y_i|)^{\alpha_i}} \, dy_i \right)^{1/p_i'}.
\]

By using these estimates in (3.3) and applying condition (2.3) we obtain that

\[
\int_{B_0^\sigma} \frac{\prod_{i=1}^m f_i(y_i)}{\left( \sum_{i=1}^m |x_B - y_i| \right)^{mn - \gamma + 1}} \, dy \leq C \frac{|B_0|^{m(\delta - 1)/n}}{\| w \mathcal{X}_{B_0} \|_\infty} \prod_{i=1}^m \left\| f_i \right\|_{p_i},
\]

and therefore

\[
a_B^2 \leq C \frac{|B_0|^{m(\delta - 1)/n}}{\| w \mathcal{X}_{B_0} \|_\infty} \prod_{i=1}^m \left\| f_i \right\|_{p_i}.
\]
We proceed with the estimation of \( I \tilde{f}(x) \). We write \( I \tilde{f}(x) = I_1 \tilde{f}(x) + I_2 \tilde{f}(x) \), where

\[
I_1 \tilde{f}(x) = \int_{B_m} \frac{\prod_{i=1}^m f_i(y_i)}{(\sum_{i=1}^m |x - y_i|)^{mn-\gamma}} \, d\tilde{y}
\]

and

\[
I_2 \tilde{f}(x) = \int_{(B_m)^c} \left( \prod_{i=1}^m f_i(y_i) \right) \left( \frac{1}{(\sum_{i=1}^m |x - y_i|)^{mn-\gamma}} - \frac{1}{(\sum_{i=1}^m |x_B - y_i|)^{mn-\gamma}} \right) \, d\tilde{y}.
\]

Let us first estimate \( I_1 \). We shall split the set \( \mathcal{I}_2 \) into \( \mathcal{I}_2^1 \) and \( \mathcal{I}_2^2 \) where

\[
\mathcal{I}_2^1 = \{ i : 1 < p_i < \infty \} \quad \text{and} \quad \mathcal{I}_2^2 = \{ i : p_i = \infty \}.
\]

Let \( m_j^j = \# \mathcal{I}_2^j \), for \( j = 1, 2 \). Then \( m = m_1 + m_2 = m_1 + m_1^1 + m_2^1 \). Observe that

\[
|I_1 \tilde{f}(x)| \leq \int_{B_m} \frac{\prod_{i=1}^m |f_i(y_i)|}{(\sum_{i=1}^m |x - y_i|)^{mn-\gamma}} \, d\tilde{y}
\]

\[
\leq \left( \prod_{i \in \mathcal{I}_2^1} \| f_i \|_\infty \right) \int_{B_m} \frac{\prod_{i \in \mathcal{I}_1,} |f_i(y_i)| \prod_{i \in \mathcal{I}_2^1} v_i^{-1}(y_i)}{(\sum_{i=1}^m |x - y_i|)^{mn-\gamma}} \, d\tilde{y}
\]

\[
\leq \left( \prod_{i \in \mathcal{I}_2^1} \| f_i \|_\infty \right) \left( \prod_{i \in \mathcal{I}_1} \| f_i \|_B^{-1} \right) \int_{B_m^2} \frac{\prod_{i \in \mathcal{I}_1} |f_i(y_i)| \prod_{i \in \mathcal{I}_2^1} v_i^{-1}(y_i)}{(\sum_{i=1}^m |x - y_i|)^{mn-\gamma}} \, d\tilde{y}
\]

\[
= \left( \prod_{i \in \mathcal{I}_2^1} \| f_i \|_\infty \right) \left( \prod_{i \in \mathcal{I}_1} \| f_i \|_B^{-1} \right) \, I(x, B).
\]

Since \( p > n/p \) we have that

\[ \gamma > n/p = n \sum_{i=1}^m \frac{1}{p_i} = m_1 n + \frac{n}{p^*} , \]

where \( 1/p^* = \sum_{i \in \mathcal{I}_2} 1/p_i \). This allows us to split \( \gamma = \gamma^1 + \gamma^2 \), where \( \gamma^1 > m_1 n \) and \( \gamma^2 > n/p^* \). Therefore

\[ mn - \gamma = m_2 n - \gamma^2 + m_1 n - \gamma^1 . \]

Let us sort the sets \( \mathcal{I}_2^1 \) and \( \mathcal{I}_2^2 \) increasingly, so

\[ \mathcal{I}_2^1 = \{ i_1, i_2, \ldots, i_{m_1^1} \} \quad \text{and} \quad \mathcal{I}_2^2 = \{ i_{m_1^1+1}, i_{m_1^1+2}, \ldots, i_{m_2} \} . \]
We now define $\vec{g} = (g_1, \ldots, g_{m_2})$, where

$$g_j = \begin{cases} |f_{ij}| & \text{if } 1 \leq j \leq m_1^1; \\ v_{ij}^{-1} & \text{if } m_1^1 + 1 \leq j \leq m_2. \end{cases}$$

Then we can estimate as follows

$$I(x, B) \leq C \int_{\tilde{B}^{m_2}} \frac{\prod_{i \in I_2} |f_i(y)| \prod_{i \in I_2^2} v_{i}^{-1}(y_i)(\sum_{i \in I_2} |x - y_i|)^{\gamma_1 - nm_1}}{\prod_{i \in I_2} |x - y_i|^{m_2n - \gamma^2}} d\tilde{y}$$

$$\leq C|\tilde{B}|^{\gamma_1/n - m_1} \int_{\tilde{B}^{m_2}} \frac{\prod_{i \in I_2} |f_i(y)| \prod_{i \in I_2^2} v_{i}^{-1}(y_i)}{\prod_{j=1}^{m_2} g_j(y_{ij})} \frac{d\tilde{y}}{(\sum_{j=1}^{m_2} |x - y_{ij}|)^{m_2n - \gamma^2}}$$

$$= C|\tilde{B}|^{\gamma_1/n - m_1} I_{\gamma_2, m_2}(\tilde{g}, X_{\tilde{B}^{m_2}})(x).$$

Next we define the vector of exponents $\vec{r} = (r_1, \ldots, r_{m_2})$ in the following way

$$r_j = \begin{cases} m_2 p_{ij} / (m_2 - 1 + p_{ij}) & \text{if } 1 \leq j \leq m_1^1; \\ m_2 & \text{if } m_1^1 + 1 \leq j \leq m_2. \end{cases}$$

This definition yields

$$\frac{1}{r} = \sum_{j=1}^{m_2} \frac{1}{r_j} = \sum_{j=1}^{m_1^1} \left( \frac{1}{m_2} + \frac{m_2 - 1}{m_2 p_{ij}} \right) + \sum_{j=m_1^1+1}^{m_2} \frac{1}{m_2} = \frac{m_1^1}{m_2} + \frac{m_2 - 1}{m_2 p^*} + \frac{m_2}{m_2} = 1 + \frac{m_2 - 1}{m_2 p^*}.$$

Observe that $1/r > 1/p^*$. We also have $n/p^* < \gamma^2$. Then there exists an auxiliary number $\gamma_0$ such that $n/p^* < \gamma_0 < n/r$. Indeed, if $\gamma^2 < n/r$ we can directly pick $\gamma_0 = \gamma^2$. Otherwise $\gamma_0 < \gamma^2$. Let us first assume that $m_2 \geq 2$. We set

$$\frac{1}{q} = \frac{1}{r} - \frac{\gamma_0}{n}.$$

Then $0 < 1/q < 1$ since

$$\frac{1}{r} - 1 = \left( 1 - \frac{1}{m_2} \right) \frac{1}{p^*} < \frac{1}{p^*} < \frac{\gamma_0}{n}.$$
By using the fact that $I_{\gamma_0, m_2} \colon \prod_{j=1}^{m_2} L^{r_j} \to L^q$ (see [4]) we obtain

\[
\int_B I(x, B) \, dx \leq C |\tilde{B}|^{(1-n)/\alpha} \left( \int_B |I_{\gamma_0, m_2}(\tilde{g}\chi_{\tilde{B}})(x)|^q \, dx \right)^{1/q} |\tilde{B}|^{1/q'}
\leq C |\tilde{B}|^{(1-n)/\alpha} \left( \int_{B^n} |I_{\gamma_0, m_2}(\tilde{g}\chi_{\tilde{B}})(x)|^q \, dx \right)^{1/q}
\leq C |\tilde{B}|^{(1-n)/\alpha} \prod_{j=1}^{m_2} \|g_j \chi_{\tilde{B}}\|_{r_j}.
\]

Observe that $r_j < p_{ij}$ for every $1 \leq j \leq m_1^2$. By applying Hölder inequality we have

\[
\prod_{j=1}^{m_2} \|g_j \chi_{\tilde{B}}\|_{r_j} = \prod_{i \in \mathcal{I}_2^1} \left( \int_{\tilde{B}} |f_{i1}^{-r_i} v_{i1}^{-r_i}) \right)^{1/r_i} \prod_{i \in \mathcal{I}_2^2} \left( \int_{\tilde{B}} v_i^{-m_2 p_i'})^{1/(m_2 p_i')}
\leq \prod_{i \in \mathcal{I}_2^1} \|f_{i1} v_{i1}\|_{p_i} \prod_{i \in \mathcal{I}_2^2} \left( \int_{\tilde{B}} v_i^{-m_2 p_i'})^{1/p_i'}
\leq |\tilde{B}|^{m_2^1/m_2 - 1/(m_2 p^*) + m_2^2/m_2} \prod_{i \in \mathcal{I}_2^1} \|v_i^{-1}\|_{RH_{m_2}} \prod_{i \in \mathcal{I}_2^2} \left( \frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-1} \right)
= C |\tilde{B}|^{1/(m_2 p^*)} \prod_{i \in \mathcal{I}_2^1} \|f_{i1} v_{i1}\|_{p_i} \left( \frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-1} \right)^{1/p_i'} \prod_{i \in \mathcal{I}_2^2} \left( \frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-1} \right).
\]

By combining all these estimates with condition (2.5), we finally get that

\[
\int_B |I_1 \tilde{f}(x)| \, dx \leq \left( \prod_{i \in \mathcal{I}_2^2} \|f_{i1} v_{i1}\|_{\infty} \right) \left( \prod_{i \in \mathcal{I}_1} \|f_{i} \chi_{\tilde{B}}\|_1 \right) \int_B I(x, B) \, dx
\leq C \left( \prod_{i \in \mathcal{I}_2^2} \|f_{i1} v_{i1}\|_{\infty} \right) \left( \prod_{i \in \mathcal{I}_1} \|f_{i} \chi_{\tilde{B}}\|_1 \right) |\tilde{B}|^{(1-n)/\alpha} |\tilde{B}|^{(1-n)/\alpha}
\times \prod_{i \in \mathcal{I}_2^1} \|f_{i1} v_{i1}\|_{p_i} \left( \frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-1} \right)^{1/p_i'} \prod_{i \in \mathcal{I}_2^2} \left( \frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-1} \right)
\leq C \left( \prod_{i=1}^{m} \|f_{i} v_{i}\|_{p_i} \right) \prod_{i \in \mathcal{I}_2} \left( \frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-1} \right)^{1/p_i'} \prod_{i \in \mathcal{I}_1} \|v_i^{-1}\chi_{\tilde{B}}\|_{\infty}
\times |\tilde{B}|^{(1-n)/\alpha} |\tilde{B}|^{(1-n)/\alpha}
\leq C \|w\chi_{\tilde{B}}\|_{\infty}^{-1} |\tilde{B}|^{(1-n)/\alpha} |\tilde{B}|^{(1-n)/\alpha}
\leq C \|w\chi_{\tilde{B}}\|_{\infty}^{-1} |\tilde{B}|^{1+\delta/n}.
\]
Therefore, we can obtain the desired bound for $I_1 \tilde{f}$ provided $m_2 \geq 2$. We now consider the case $0 \leq m_2 < 2$. There are only three possible cases:

1. $m_2 = 0$. In this case we have $m_1^2 = m_2^2 = 0$ and this implies $\bar{p} = (1, 1, \ldots, 1)$. This situation is not possible, because $\bar{p} > n/\gamma$.

2. $m_1^2 = 0$ and $m_2^2 = 1$. In this case $1/p = m - 1$. The condition $p > n/\gamma$ implies $\gamma > (m - 1)n$. Let $i_0$ be the index such that $p_{i_0} = \infty$. By using Fubini’s theorem, we can proceed in the following way

\[
\int_B \int_{\tilde{B}^m} \frac{\prod_{i=1}^{m} |f_i(y_i)|}{(\sum_{i=1}^{m} |x - y_i|)^{mn-\gamma}} d\tilde{y} dx
= \int_{\tilde{B}^m} \prod_{i=1}^{m} |f_i(y_i)| \left( \int_B \left( \sum_{i=1}^{m} |x - y_i| \right)^{\gamma mn} dx \right) d\tilde{y}.
\]

Since

\[
\int_B \left( \sum_{i=1}^{m} |x - y_i| \right)^{\gamma - mn} dx \leq C \int_0^{4R} \rho^{\gamma - mn} \rho^{n-1} d\rho \leq C |B|^\gamma/n - m + 1,
\]

by (2.5), we get

\[
\int_B |I_1 \tilde{f}(x)| dx \leq C |B|^\gamma/n - m + 1 \left( \prod_{i=1}^{m} \|f_i v_i\|_{p_i} \right) \left( \prod_{i \in \mathcal{I}_1} \|v_{i_0}^{-1} X_B\|_{\infty} \right) \left( \frac{1}{|B|} \int_{\tilde{B}} v_{i_0}^{-1} \right)
\]

\[
\leq C \left( \prod_{i=1}^{m} \|f_i v_i\|_{p_i} \right) \frac{|B|^\gamma/n - m + 2 + \delta/n - \gamma/n + 1/p}{\|w\mathcal{X}_{\tilde{B}}\|_{\infty}}
\]

\[
\leq C \left( \prod_{i=1}^{m} \|f_i v_i\|_{p_i} \right) \frac{|B|^{1 + \delta/n}}{\|w\mathcal{X}_{B}\|_{\infty}}.
\]

3. $m_1^2 = 1$ and $m_2^2 = 0$. If $i_0$ denotes the index for which $1 < p_{i_0} < \infty$, the condition $p > n/\gamma$ implies that

\[
\frac{\gamma}{n} > \frac{1}{p} = m - 1 + \frac{1}{p_{i_0}},
\]

and thus $\gamma > (m - 1)n$. We repeat the estimate given in the previous case. Then

\[
\int_B |I_1 \tilde{f}(x)| dx \leq C |B|^\gamma/n - m + 1/p_{i_0} \left( \prod_{i=1}^{m} \|f_i v_i\|_{p_i} \right) \left( \prod_{i \in \mathcal{I}_1} \|v_{i_0}^{-1} X_{\tilde{B}}\|_{\infty} \right) \left( \frac{1}{|B|} \int_{\tilde{B}} v_{i_0}^{-1}\right)^{1/p'_{i_0}}
\]
\[
\leq C \left( \prod_{i=1}^{m} \| f_i \|_{p_i} \right) \frac{\| \tilde{B} \|^{\gamma/n - m + 1 + 1/p' + \delta/n - \gamma/n + 1/p}}{\| w \chi_{\tilde{B}} \|_{\infty}}
\]

\[
\leq C \left( \prod_{i=1}^{m} \| f_i \|_{p_i} \right) \frac{\| B \|^{1 + \delta/n}}{\| w \chi_{B} \|_{\infty}}.
\]

This completes the estimate for \( I_1 \tilde{f} \). For \( I_2 \tilde{f} \), by the mean value theorem, we can write

\[
| I_2 \tilde{f}(x) | \leq | B |^{1/n} \sum_{\sigma \in S_m, \sigma \neq 1} \int_{\tilde{B}^\sigma} \left( \prod_{i=1}^{m} | f_i(y_i) | \right) \left( \sum_{i=1}^{m} | x_B - y_i | \right)^{mn - \gamma + 1} d\tilde{y}.
\]

Notice that this expression is similar to \( a_{B_0}^2 \), with \( B_0 \) replaced with \( \tilde{B} \). Therefore, we can proceed in a similar way to obtain

\[
\int_{B} | I_2 \tilde{f}(x) | \, dx \leq C \frac{| B |^{1 + \delta/n}}{\| w \chi_{B} \|_{\infty}} \prod_{i=1}^{m} \| f_i \|_{p_i}.
\]

This completes the proof of the lemma. \( \square \)

**Remark 3.2** The corresponding estimate obtained for \( I \tilde{f} \) will be used for the proof of Theorem 1.1.

Next we are going to set some geometrical facts that will be useful later. These results were set and proved in \([7]\). For a fixed ball \( B = B(x_B, R) \) we define the sets

\[
A = \{ x_B + h : h = (h_1, h_2, \ldots, h_n) : h_i \geq 0 \text{ for } 1 \leq i \leq n \},
\]

\[
C_1 = B \left( x_B - \frac{R}{12 \sqrt{n}} u, \frac{R}{12 \sqrt{n}} \right) \cap \left\{ x_B - \frac{R}{12 \sqrt{n}} u + h : h_i \leq 0 \text{ for every } i \right\},
\]

and

\[
C_2 = B \left( x_B - \frac{R}{3 \sqrt{n}} u, \frac{2R}{3} \right) \cap \left\{ x_B - \frac{R}{3 \sqrt{n}} u + h : h_i \leq 0 \text{ for every } i \right\},
\]

where \( u = (1, 1, \ldots, 1) \). The following figure shows a sketch of these sets. (Fig. 1)

**Remark 3.3** It is not difficult to see that \( | C_i | \approx | B |, \) for \( i = 1, 2 \).

The next lemma deals with the sets defined above and will be useful in the proof of our main result. The proof is similar to the corresponding result given in \([7]\) for the case \( m = 1 \) and we omit it.

**Lemma 3.4** There exists a positive constant \( C = C(n) \) such that the inequality

\[
\frac{1}{\left( \sum_{j=1}^{m} | x - y_j | \right)^{mn - \gamma}} \leq \frac{1}{\left( \sum_{j=1}^{m} | z - y_j | \right)^{mn - \gamma}}
\]
Fig. 1 The sets $A$, $C_1$ and $C_2$, where $x^1_B = x_B - R/(12\sqrt{n})u$
and $x^2_B = x_B - R/(4\sqrt{n})u$

$$x^1_B \in A \quad x^2_B \in C_2 \quad x_B \in C_1 \quad x^2_B \in C_2$$

The sets $A$, $C_1$ and $C_2$, where $x^1_B = x_B - R/(12\sqrt{n})u$
and $x^2_B = x_B - R/(4\sqrt{n})u$

$$x^1_B \in A \quad x^2_B \in C_2 \quad x_B \in C_1 \quad x^2_B \in C_2$$

$\geq C \frac{|B|^{1/n}}{(|B|^{1/n} + \sum_{j=1}^{m} |x_B - y_j|)^{mn-\gamma + 1}}$

holds for every $x \in C_1$, $z \in C_2$, and $y_j \in A$ for $1 \leq j \leq m$.

4 Proof of the main results

We devote this section to prove the results contained in Sect. 1.

Proof of Theorem 1.1 We shall first prove that (2) implies (1). We shall deal with the operator $J_{\gamma,m}$ since it differs from $I_{\gamma,m}$ by a constant term. We want to prove that for every ball $B$

$$\left\| w\chi_B \right\|_\infty \frac{|B|^{1+\delta/n}}{|B|^{1+\delta/n}} \int_B |J_{\gamma,m} \tilde{f}(x) - (J_{\gamma,m} \tilde{f})_B| \, dx \leq C \prod_{i=1}^{m} \left\| f_i v_i \right\|_{p_i}, \quad (4.1)$$

with $C$ independent of $B$. Fix a ball $B = B(x_B, R)$ and recall that $J_{\gamma,m} \tilde{f}(x) = a_B + I \tilde{f}(x)$. In Lemma 3.1 we proved that

$$\int_B |I \tilde{f}(x)| \, dx \leq C \frac{|B|^{1+\delta/n}}{\left\| w\chi_B \right\|_\infty} \prod_{i=1}^{m} \left\| f_i v_i \right\|_{p_i},$$

$\therefore$ Springer
which implies that

\[
\int_B |J_{\gamma,m} \tilde{f}(x) - a_B| \, dx \leq C |B|^{1+\delta/n} \prod_{i=1}^m \|f_i v_i\|_{p_i},
\]  

(4.2)

On the other hand, observe that

\[
\int_B |J_{\gamma,m} \tilde{f}(x) - (J_{\gamma,m} \tilde{f})_B| \, dx \leq \int_B |J_{\gamma,m} \tilde{f}(x) - a_B| \, dx + \int_B |(J_{\gamma,m} \tilde{f})_B - a_B| \, dx \\
\leq \int_B |J_{\gamma,m} \tilde{f}(x) - a_B| \, dx \\
+ \int_B \frac{1}{|B|} \int_B |J_{\gamma,m} \tilde{f}(y) - a_B| \, dy \, dx \\
\leq 2 \int_B |J_{\gamma,m} \tilde{f}(x) - a_B| \, dx.
\]

By combining this estimate with (4.2) we obtain the desired inequality.

We now prove that (1) implies (2). Assume that the component functions \( f_i \) of \( \tilde{f} \) are nonnegative. We have that (4.1) holds for every ball \( B = B(x_B, R) \). Also observe that

\[
\frac{1}{|B|} \int_B |g(x) - g_B| \, dx \approx \frac{1}{|B|^2} \int_B \int_B |g(x) - g(z)| \, dx \, dz,
\]

and therefore the left hand side of (4.1) is equivalent to

\[
\frac{\|w \chi_B\|_\infty}{|B|^{2+\delta/n}} \int_B \int_B |J_{\gamma,m} \tilde{f}(x) - J_{\gamma,m} \tilde{f}(z)| \, dx \, dz =: I.
\]

Observe that, when \( y_i \in B \) for every \( i \) we have

\[
|B|^{1/n} + |x_B - y_j| \geq \frac{1}{m} \left( |B|^{1/n} + \sum_{i=1}^m |x_B - y_i| \right).
\]

for every \( 1 \leq j \leq m \). By combining Lemma 3.4 and Remark 3.3 with the inequality above we can estimate \( I \) as follows

\[
I \geq \frac{\|w \chi_B\|_\infty}{|B|^{2+\delta/n}} \int_{C_2} \int_{C_1} \int_{A^m} \frac{|B|^{1/n} \prod_{i=1}^m f_i(y_i)}{(|B|^{1/n} + \sum_{i=1}^m |x_B - y_i|)^{m-\gamma+1}} \, d\tilde{y} \, dx \, dz \\
\geq C \frac{\|w \chi_B\|_\infty}{|B|^{(\delta-1)/n}} \prod_{i=1}^m \left( \int_A \frac{f_i(y_i)}{(|B|^{1/n} + |x_B - y_i|)^{\gamma+1/m}} \, dy_i \right).
\]
Since the set \( A \) is a quadrant from \( x_B \), a similar estimation can be obtained for the other quadrants from \( x_B \). Thus, we get

\[
I \geq C \frac{\| w \mathcal{X}_B \|_{\infty}}{|B|^{(\delta-1)/n}} \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} \frac{f_i(y)}{|B|^{1/n} + |x_B - y|^{n-\gamma+1/m}} \, dy \right),
\]

which implies that

\[
\frac{\| w \mathcal{X}_B \|_{\infty}}{|B|^{(\delta-1)/n}} \prod_{i=1}^{m} \left( \int_{\mathbb{R}^n} \frac{f_i(y)}{|B|^{1/n} + |x_B - y|^{n-\gamma+1/m}} \, dy \right) \leq C \prod_{i=1}^{m} \| f_i v_i \|_{p_i}. \tag{4.3}
\]

For every \( i \in I_1 \) and \( k \in \mathbb{N} \) we define \( V^i_k = \{ x : v^{-1}_i(x) \leq k \} \) and the functionals

\[
F^k_i(g) = \int_{\mathbb{R}^n} \frac{g(y) v^{-1}_i(y) \mathcal{X}_V^i(y)}{|B|^{1/n} + |x_B - y|^{n-\gamma+1/m}} \, dy.
\]

Therefore \( F^k_i \) is a functional in \((L^1)^* = L^\infty\). Indeed, if \( g \in L^1 \)

\[
|F^k_i(g)| \leq \| g \|_{L^1} \left\| \frac{v^{-1}_i \mathcal{X}_V^i}{(|B|^{1/n} + |x_B - .|^{n-\gamma+1/m})} \right\|_{\infty} < \infty,
\]

and we also get

\[
\frac{|F^k_i(f_i v_i)|}{\| f_i v_i \|_{L^1}} \leq \left\| \frac{v^{-1}_i \mathcal{X}_V^i}{(|B|^{1/n} + |x_B - .|^{n-\gamma+1/m})} \right\|_{\infty},
\]

for every \( i \in I_1 \).

If \( i \in I_2 \) then we set \( A_k = A \cap B(0, k) \) and consider

\[
f^k_i(y) = \frac{v^{-p'_i}_i(y)}{|B|^{1/n} + |x_B - y|^{(n-\gamma+1/m)/(p_i-1)}} \mathcal{X}_A(y) \mathcal{X}_V^i(y).
\]

Let us choose \( f = (f_1, \ldots, f_m) \), where \( f_i \) is such that \( f_i v_i \in L^1 \) for \( p_i = 1 \) and \( f_i = f^k_i \) for \( p_i > 1 \), for \( k \) fixed. Therefore, the left hand side of (4.3) can be written as follows

\[
\frac{\| w \mathcal{X}_B \|_{\infty}}{|B|^{(\delta-1)/n}} \prod_{i \in I_1} F^k_i(f_i v_i) \prod_{i \in I_2} \left( \int_{A_k \cap V^i_k} \frac{v^{-p'_i}_i(y)}{|B|^{1/n} + |x_B - y|^{(n-\gamma+1/m)p'_i}} \, dy \right)
\]
and it is bounded by

\[ C \prod_{i \in \mathcal{I}_1} \| f_i v_i \|_{L^1} \prod_{i \in \mathcal{I}_2} \left( \int_{A_k \cap V^i_k} \frac{v_i^{-p'_i}(y)}{(|B|^{1/n} + |x_B - y|)^{(n-\gamma_i+1/m)p_i'}} dy \right)^{1/p_i'} . \]

This yields

\[
\frac{\| w \mathcal{X}_B \|_{\infty}}{|B|^{(\delta-1)/n}} \prod_{i \in \mathcal{I}_1} \| f_i v_i \|_{L^1} \prod_{i \in \mathcal{I}_2} \left( \int_{A_k \cap V^i_k} \frac{v_i^{-p'_i}(y)}{(|B|^{1/n} + |x_B - y|)^{(n-\gamma_i+1/m)p_i'}} dy \right)^{1/p_i'} \leq C,
\]

for every nonnegative \( f_i \) such that \( f_i v_i \in L^1, i \in \mathcal{I}_1 \) and for every \( k \in \mathbb{N} \). By taking the supremum over these \( f_i \) we get

\[
\frac{\| w \mathcal{X}_B \|_{\infty}}{|B|^{(\delta-1)/n}} \prod_{i \in \mathcal{I}_1} \| f_i v_i \|_{L^1} \prod_{i \in \mathcal{I}_2} \left( \int_{A_k \cap V^i_k} \frac{v_i^{-p'_i} \mathcal{X}_{A_k \cap V^i_k}}{(|B|^{1/n} + |x_B - y|)^{(n-\gamma_i+1/m)p_i'}} dy \right)_{1/p_i'} \leq C. \]

By taking limit for \( k \to \infty \), the left hand side converges to

\[
\frac{\| w \mathcal{X}_B \|_{\infty}}{|B|^{(\delta-1)/n}} \prod_{i \in \mathcal{I}_1} \| f_i v_i \|_{L^1} \prod_{i \in \mathcal{I}_2} \left( \int_{\mathbb{R}^n} \frac{v_i^{-p'_i} \mathcal{X}}{(|B|^{1/n} + |x_B - y|)^{(n-\gamma_i+1/m)p_i'}} dy \right)_{1/p_i'} ,
\]

which is precisely the condition \( \mathbb{H}_m(\bar{\rho}, \gamma, \delta) \). This completes the proof. \( \square \)

We now proceed to prove Theorem 1.2.

**Proof of Theorem 1.2** Let us begin with item (a). We shall first assume that \( \delta > 1 \). If \((w, \tilde{v}) \in \mathbb{H}_m(\bar{\rho}, \gamma, \delta)\), we choose \( B = B(x_B, R) \) where \( x_B \) is a Lebesgue point of \( w^{-1} \). From (2.3) we obtain

\[
\prod_{i \in \mathcal{I}_1} \left( \frac{v_i^{-1}}{|B|^{1/n} + |x_B - \cdot|^{n-\gamma_i+1/m}} \right)_{\infty} \prod_{i \in \mathcal{I}_2} \left( \int_{\mathbb{R}^n} \frac{v_i^{-p'_i}}{(|B|^{1/n} + |x_B - y|)^{(n-\gamma_i+1/m)p_i'}} dy \right)_{1/p_i'} \leq w^{-1}(B) \frac{|B|^{(\delta-1)/n}}{|B|^{\delta-1}} \leq w^{-1}(B) \frac{|B|^{(\delta-1)/n}}{|B|^{\delta-1}},
\]

\( \square \)
for every $R > 0$. By letting $R \to 0$ and applying the monotone convergence theorem, we conclude that at least one limit factor in the product should be zero. That is, there exists $1 \leq i \leq m$ such that $v_i = \infty$ almost everywhere.

On the other hand, if $\delta > \gamma - n/p$ and $(w, \vec{v})$ belongs to $\mathcal{H}_m(\vec{p}, \gamma, \delta)$, we pick a ball $B = B(x_B, R)$, where $x_B$ is a Lebesgue point of $w^{-1}$ and every $v_i^{-1}$. Then we have

$$m \prod_{i=1}^{m} \frac{1}{|B|} \int_B v_i^{-1} \leq \prod_{i \in I_1} \|v_i^{-1} \chi_B\|_\infty \prod_{i \in I_2} \left( \frac{1}{|B|} \int_B v_i^{-p_i} \right)^{1/p'_i} \leq C \frac{|B|^\delta - \frac{\gamma_i}{p} + \frac{1}{p}}{||w \chi_B||_\infty} \leq C \frac{w^{-1}(B)}{|B|} R^{\delta - \gamma + n/p}$$

for every $R > 0$. By letting $R \to 0$ we get

$$m \prod_{i=1}^{m} v_i^{-1}(x_B) = 0,$$

which yields that $\prod_{i=1}^{m} v_i^{-1}$ is zero almost everywhere. This implies that the set $M = \bigcap_{i=1}^{m} \{v_i^{-1} > 0\}$ has null measure. Since $v_i(y) > 0$ for almost every $y$ and every $i$, there exists $j$ such that $v_j = \infty$ almost everywhere.

We turn now our attention to (b). Suppose $\delta = \gamma - n/p = 1$. We shall prove that if $(w, \vec{v}) \in \mathcal{H}_m(\vec{p}, \gamma, 1)$, there exists $j$ such that $v_j = \infty$ in almost $\mathbb{R}^n$. We define

$$\alpha = \sum_{i=1}^{m} \frac{1}{p'_i} = m - \frac{1}{p}.$$

By applying Hölder inequality we obtain that

$$\left( \int_{\mathbb{R}^n} \left( \prod_{i \in I_2} v_i^{-1} \right)^{1/\alpha} \right)^{\alpha} \leq C \prod_{i \in I_2} \left( \int_{\mathbb{R}^n} \left( \frac{v_i^{1/p'_i}}{|B|^{1/n} + |x_B - y|^{(n-\gamma_i+1/m)p'_i}} \right) \right)^{1/p'_i}$$

and since $(w, \vec{v}) \in \mathcal{H}_m(\vec{p}, \gamma, 1)$ this implies that

$$\prod_{i \in I_1} \left\| \frac{v_i^{-1}}{|B|^{1/n} + |x_B - y|^{n-\gamma_i+1/m}} \right\|_\infty \leq C \left( \int_{\mathbb{R}^n} \left( \prod_{i \in I_2} v_i^{-1} \right)^{1/\alpha} \right)^{\alpha} \frac{w^{-1}(B)}{|B|},$$
and furthermore
\[
\left( \int_{\mathbb{R}^n} \left( \prod_{i=1}^m v_i^{-1} \right)^{1/\alpha} \right)^\alpha \leq \frac{w^{-1}(B)}{|B|}
\]
for every ball \( B = B(x_B, R) \).

We now use an adaptation of an argument of [7]. If the set \( E = \{ x : \prod_{i=1}^m v_i^{-1}(x) > 0 \} \) has positive measure we write \( E = \bigcup_{k \geq 1} E_k \), where \( E_k = \{ x : \prod_{i=1}^m v_i^{-1}(x) > 1/k \} \). Then there exists \( k_0 \) verifying \( |E_{k_0}| > 0 \). Let \( x_B \) be a Lebesgue point of \( w^{-1} \) which also is a density point in \( E_{k_0} \) such that \( w^{-1}(x_B) < \infty \). By letting \( R \to 0 \) and observing that \( (mn - \gamma + 1)/\alpha = n \) we obtain
\[
\left( \int_{\mathbb{R}^n} \left( \prod_{i=1}^m v_i^{-1} \right)^{1/\alpha} \right)^\alpha \leq C w^{-1}(x_B),
\]
so the left-hand side integral is finite for almost every \( x_B \in E_{k_0} \). On the other hand, we have
\[
\left( \int_{\mathbb{R}^n} \left( \prod_{i=1}^m v_i^{-1} \right)^{1/\alpha} \right)^\alpha \geq \left( \int_{B \cap E_{k_0}} \left( \prod_{i=1}^m v_i^{-1} \right)^{1/\alpha} \right)^\alpha \geq \frac{1}{k_0} \left( \int_{B \cap E_{k_0}} \frac{dy}{|x_B - y|^n} \right)^\alpha.
\]
(4.4)

We now define \( \varepsilon(R) := (|B| - |B \cap E_{k_0}|)^{1/n} = |B \setminus E_{k_0}|^{1/n} \). Since \( x_B \) is a density point of \( E_{k_0} \) we have that
\[
\frac{\varepsilon(R)}{R} = c \left( \frac{|B| - |B \setminus E_{k_0}|}{|B|} \right)^{1/n} \to 0
\]
when \( R \) approaches to zero. If we take \( C = \{ y : \varepsilon(R) \leq |x_B - y| \leq R \} \), then
\[
|C| = c(R^n - \varepsilon^n(R)) = c(|B| - (|B| - |B \cap E_{k_0}|)) = |B \cap E_{k_0}|.
\]
We also have
\[
|C| = |B \cap E_{k_0}| = |(B \cap E_{k_0}) \cap C| + |(B \cap E_{k_0}) - C|
\]
and therefore
\[
|C \setminus (B \cap E_{k_0} \cap C)| = |C| - |B \cap E_{k_0} \cap C| = |B \cap E_{k_0} \setminus C|.
\]
Since
\[
\sup_{C \setminus (B \cap E_k_0 \cap C)} |x_0 - y|^{-n} \leq \inf_{B \cap E_k_0 \setminus C} |x_0 - y|^{-n}
\]
we return to (4.4) and write
\[
\left( \int_{\mathbb{R}^n} \left( \prod_{i=1}^m v_i^{-1} \right)^{1/\alpha} \right)^\alpha \geq \frac{1}{k_0} \left( \int_{B \cap E_k_0 \cap C} \frac{dy}{|x_B - y|^n} + \int_{(B \cap E_k_0) \setminus C} \frac{dy}{|x_B - y|^n} \right)^\alpha
\]
\[
\geq \frac{1}{k_0} \left( \int_{B \cap E_k_0 \cap C} \frac{dy}{|x_B - y|^n} + \int_{C \setminus (B \cap E_k_0 \cap C)} \frac{dy}{|x_B - y|^n} \right)^\alpha
\]
\[
\geq \frac{1}{k_0} \left( \int_{C} \frac{dy}{|x_B - y|^n} \right)^\alpha = \frac{1}{k_0} \left( \int_{\mathbb{R}^n} \frac{dy}{r^\alpha} \right)^\alpha
\]
\[
= \frac{1}{k_0} \ln \left( \frac{R}{\varepsilon(R)} \right)^\alpha,
\]
which approaches to $\infty$ when $R \to 0$, a contradiction. This yields
\[|E| = 0,\] that is, $\prod_{i=1}^m v_i^{-1} = 0$ almost everywhere, from where we can deduce that there exists $j$ satisfying $v_j = \infty$ almost everywhere.

We finish with the proof of item (c). If $\delta < \gamma - mn$, given a ball $B = B(x_B, R)$ and $B_0 \subset B$ the condition (2.5) implies that
\[
\|wX_{B_0}\|_{\infty} \prod_{i=1}^m \|v_i^{-1}X_{B_0}\|_{p_i'} \leq \|wX_B\|_{\infty} \prod_{i=1}^m \|v_i^{-1}X_B\|_{p_i'} \leq CR^{\delta - \gamma + mn}.
\]
Observe that the right-hand side of the inequality above tends to zero when $R$ tends to $\infty$. This implies that either $\|wX_{B_0}\|_{\infty} = 0$ or $\|v_i^{-1}X_{B_0}\|_{p_i'} = 0$, for some $i$. By the arbitrariness of $B_0$ we obtain either $w = 0$ or $v_j = \infty$ for some $i$, respectively. \qed

5 The class $\mathbb{H}_m(\vec{p}, \gamma, \delta)$

We begin this section by exhibiting nontrivial pairs of weights satisfying condition $\mathbb{H}_m(\vec{p}, \gamma, \delta)$. Concretely, we shall prove the following theorem.

**Theorem 5.1** Given $0 < \gamma < mn$ there exist pairs of weights $(w, \vec{v})$ satisfying (2.3) for every $\vec{p}$ and $\delta$ such that $\gamma - mn \leq \delta \leq \min\{1, \gamma - n/p\}$, excluding the case $\delta = 1$ when $\gamma - n/p = 1$.

The following figure shows the area in which we can find nontrivial weights, depending on the value of $\gamma$. \hfill \copyright Springer
In order to prove Theorem 5.1, it will be useful the following lemma (see [7]).

**Lemma 5.2** If $R > 0$, $B = B(x_B, R)$ is a ball in $\mathbb{R}^n$ and $\alpha > -n$ then

$$\int_B |x|^\alpha \, dx \approx R^n (\max\{R, |x_B|\})^\alpha .$$

**Proof of Theorem 5.1** Recalling that $\tau = (\gamma - mn)(1 - 1/m) + 1/m$ is the number appearing in Lemma 2.3, we shall split the proof into the following cases:

(a) $\gamma - mn < \delta < \tau \leq \gamma - n/p$;
(b) $\gamma - mn < \delta \leq \gamma - n/p < \tau$;
(c) $\gamma - mn < \delta = \tau < 1 < \gamma - n/p$;
(d) $\gamma - mn < \delta = \tau < \gamma - n/p < 1$;
(e) $\tau < \delta \leq \min\{1, \gamma - n/p\}$;
(f) $\delta = \gamma - mn$.

Let us prove (a). Recall that $I_1 = \{i : p_i = 1\}$, $I_2 = \{i : p_i > 1\}$ and $m_j = \#I_j$, for $j = 1, 2$. Since $m_1 < m$ by the restrictions on the parameters, we can take

$$0 < \varepsilon < \frac{mn - \gamma + \delta}{m - m_1} .$$

For $1 \leq i \leq m$ we define

$$\beta_i = \begin{cases} 0 & \text{if } i \in I_1, \\ \frac{n}{p_i} - \varepsilon & \text{if } i \in I_2. \end{cases}$$

Let $\alpha = \sum_{i=1}^m \beta_i + \delta - \gamma + n/p > 0$. Then we take

$$w(x) = |x|^\alpha \quad \text{and} \quad v_i(x) = |x|^\beta_i .$$
By virtue of Lemma 2.3 it will be enough to show that \((w, \vec{v})\) verifies condition (2.5).
Let \(B = B(x_B, R)\) and \(|x_B| \leq R\), by Lemma 5.2 if \(i \in \mathcal{I}_2\) we get
\[
\left( \frac{1}{|B|} \int_B v_i^{-p'_i} \right)^{1/p'_i} = \left( \frac{1}{|B|} \int_B |x|^{-\beta_i p'_i} \, dx \right)^{1/p'_i} \approx R^{-\beta_i},
\]
and \(\|v_i^{-1} \chi_B\|_{\infty} = 1\) for \(i \in \mathcal{I}_1\). On the other hand, \(\|w \chi_B\|_{\infty} \lesssim R^\alpha\) since \(\alpha > 0\). Therefore,
\[
\|w \chi_B\|_{\infty} \lesssim \|v_i^{-1} \chi_B\|_{\infty} = \left( \frac{1}{|B|} \int_B |x|^{-\beta_i p'_i} \, dx \right)^{1/p'_i} \approx |x_B|^{-\beta_i}.
\]
Thus, the proof of (a).

We now consider the case \(|x_B| > R\). We have that
\[
\|w \chi_B\|_{\infty} \lesssim |x_B|^\alpha
\]
whilst for \(i \in \mathcal{I}_2\)
\[
\left( \frac{1}{|B|} \int_B |x|^{-\beta_i p'_i} \, dx \right)^{1/p'_i} \approx |x_B|^{-\beta_i}.
\]
Consequently, since \(\delta < \gamma - n/p\)
\[
\|w \chi_B\|_{\infty} \lesssim \|v_i^{-1} \chi_B\|_{\infty} = \left( \frac{1}{|B|} \int_B |x|^{-\beta_i p'_i} \, dx \right)^{1/p'_i} \approx |x_B|^{-\beta_i}.
\]
which completes the proof of (a).

We now prove (b). In this case we take \(w = 1\) and \(v_i = |x|^{\beta_i}, \beta_i = (\gamma - \delta)/m - n/p_i\) for every \(1 \leq i \leq m\). By Lemma 2.3 it will be enough to prove that \((w, \vec{v})\) satisfies condition (2.5). Pick a ball \(B = B(x_B, R)\) and assume that \(|x_B| \leq R\). Observe that for every \(i \in \mathcal{I}_1\) we get \(\beta_i < 0\), since we are assuming \(\delta > \gamma - mn\). Then, for \(i \in \mathcal{I}_1\) we get
\[
\|v_i^{-1} \chi_B\|_{\infty} \approx R^{-\beta_i}.
\]
On the other hand, for \(i \in \mathcal{I}_2\) we have \(\beta_i < n/p_i\), so Lemma 5.2 yields
\[
\left( \frac{1}{|B|} \int_B v_i^{-p'_i} \right)^{1/p'_i} \approx R^{-\beta_i}.
\]
These two estimates imply that

$$\frac{\|w_{\mathcal{X}_B}\|_{\infty}}{|B|^{\frac{\alpha}{2n-\gamma}n+\frac{1}{p}}} \prod_{i \in \mathcal{I}_1} \|v_i^{-1} \mathcal{X}_B\|_{\infty} \prod_{i \in \mathcal{I}_2} \left( \frac{1}{|B|} \int_B v_i^{-p'_i} \right)^{1/p'_i} \leq C \frac{R^{-\sum_{i=1}^m \beta_i}}{R^{\delta-\gamma+n/p}} \leq C.$$

If $|x_B| > R$, we have that $\|v_i^{-1} \mathcal{X}_B\|_{\infty} \lesssim |x_B|^{-\beta_i}$ and also

$$\left( \frac{1}{|B|} \int_B v_i^{-p'_i} \right)^{1/p'_i} \approx |x_B|^{-\beta_i}$$

by Lemma 5.2. Thus

$$\frac{\|w_{\mathcal{X}_B}\|_{\infty}}{|B|^{\frac{\alpha}{2n-\gamma}n+\frac{1}{p}}} \prod_{i \in \mathcal{I}_1} \|v_i^{-1} \mathcal{X}_B\|_{\infty} \prod_{i \in \mathcal{I}_2} \left( \frac{1}{|B|} \int_B v_i^{-p'_i} \right)^{1/p'_i} \leq C \frac{|x_B|^{-\sum_{i=1}^m \beta_i}}{R^{\delta-\gamma+n/p}} \leq C,$$

Since $\delta \leq \gamma - n/p$. This concludes the proof of item (b).

In order to prove (c) we pick $(\gamma - \tau)/m - n/p_i < \beta_i < n/p'_i$ for every $i \in \mathcal{I}_2$ and $\beta_i = 0$ for $i \in \mathcal{I}_1$. Notice that this election is possible since $\gamma - \tau < mn$. We also take $\alpha = \sum_{i=1}^m \beta_i + \tau - \gamma + n/p$ and define $w(x) = |x|^\alpha$ and $v_i(x) = |x|^{\beta_i}$, for $1 \leq i \leq m$. By virtue of Lemma 2.1 it will be enough to show that condition (2.6) holds, since every $v_i$ is a doubling weight. We shall prove that

$$\frac{\|w_{\mathcal{X}_B}\|_{\infty}}{|B|^{(\delta-\gamma)n}} \prod_{i \in \mathcal{I}_1} \left( \frac{v_i^{-1} \mathcal{X}_{\mathbb{R}^n \setminus B}}{|x_B - |x|^{n-\gamma/m+1/m}} \right) \prod_{i \in \mathcal{I}_2} \left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p'_i}(y)}{|x_B - y|^{(n-\gamma/m+1/m)p'_i}} dy \right)^{1/p'_i} \leq C$$

for every ball $B = B(x_B, R)$. We first notice that

$$\prod_{i \in \mathcal{I}_1} \left( \frac{v_i^{-1} \mathcal{X}_{\mathbb{R}^n \setminus B}}{|x_B - |x|^{n-\gamma/m+1/m}} \right) \leq R^{-\sum_{i \in \mathcal{I}_1} (n-\gamma/m+1/m)},$$

so there will be enough to show that

$$R^{1-\delta-\sum_{i \in \mathcal{I}_1} (n-\gamma/m+1/m)} \|w_{\mathcal{X}_B}\|_{\infty} \prod_{i \in \mathcal{I}_2} \left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p'_i}(y)}{|x_B - y|^{(n-\gamma/m+1/m)p'_i}} dy \right)^{1/p'_i} \leq C$$

(5.1)

for every ball $B$. We shall first assume that $|x_B| \leq R$. For every $k \in \mathbb{N}$ we take $B_k = B(x_B, 2^k R)$ and for $i \in \mathcal{I}_2$, by Lemma 5.2, we write

$$\left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p'_i}(y)}{|x_B - y|^{(n-\gamma/m+1/m)p'_i}} dy \right)^{1/p'_i} \leq C.$$
Optimal parameters related with continuity...

\[ \lesssim \sum_{k=1}^{\infty} (2^k R)^{-n+\gamma/m-1/m} \left( \int_{B_{k+1} \setminus B_k} |y|^{-\beta_i} |y|^{p'_i} \, dy \right)^{1/p'_i} \]

\[ \lesssim \sum_{k=1}^{\infty} (2^k R)^{-n+\gamma/m-1/m-\beta_i+n/p'_i} \]

\[ \lesssim R^{-n/p_i+\gamma/m-1/m-\beta_i}, \]

Since \(-n/p_i + \gamma/m - 1/m - \beta_i < 0\) by the election of \(\beta_i\). Then the left-hand side of (5.1) is bounded by

\[ CR^{1-\delta - \sum_{i \in I_1} (n-\gamma/m+1/m)+\alpha - \sum_{i \in I_2} (n/p_i-\gamma/m+1/m+\beta_i) = C R^{-n/p+\gamma+\alpha - \sum_{i=1}^{m} \beta_i} = C. \]

We now assume \(|xB| > R\). There exists a number \(N\) such that \(2^N R < |xB| \leq 2^{N+1} R\). If \(i \in I_2\) we have that

\[ \left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p'_i}(y)}{|xB-y|^{(n-\gamma/m+1/m)p'_i}} \, dy \right)^{1/p'_i} \lesssim \sum_{k=1}^{\infty} (2^k R)^{-n+\gamma/m-1/m} \left( \int_{B_k} |y|^{-\beta_i} |y|^{p'_i} \, dy \right)^{1/p'_i} \]

\[ = \sum_{k=1}^{N} + \sum_{k=N+1}^{\infty} \]

\[ = S_1^i + S_2^i. \]

Let \(\theta_i = n/p_i + (1-\gamma)/m\), for \(1 \leq i \leq m\). We shall estimate the sum \(S_1^i + S_2^i\), for \(i \in I_2\), by distinguishing into the cases \(\theta_i < 0, \theta_i = 0\) and \(\theta_i > 0\). We shall first prove that if \(\theta_i < 0\), then

\[ S_j^i \leq C |xB|^{-\beta_i-\theta_i}, \quad (5.2) \]

for \(j = 1, 2\). Indeed, by Lemma 5.2 we obtain

\[ S_1^i \lesssim \sum_{k=1}^{N} (2^k R)^{-n+\gamma/m-1/m+n/p'_i} |xB|^{-\beta_i} \]

\[ \lesssim |xB|^{-\beta_i} R^{-\theta_i} \sum_{k=1}^{N} 2^{-k\theta_i} \]

\[ \lesssim |xB|^{-\beta_i} (2^N R)^{-\theta_i} \]

\[ \lesssim |xB|^{-\beta_i-\theta_i}, \]

since \(\theta_i < 0\). For \(S_2^i\) we apply again Lemma 5.2 in order to get

\[ S_2^i \lesssim \sum_{k=N+1}^{\infty} (2^k R)^{-n+\gamma/m-1/m+n/p'_i} \]
\[
\sum_{k=N+1}^{\infty} (2^k R)^{-\beta_i - \theta_i} = \left(2^{N+1} R\right)^{-\beta_i - \theta_i} \sum_{k=0}^{\infty} 2^{-k(\beta_i + \theta_i)} \lesssim |x_B|^{-\beta_i - \theta_i},
\]

since \(\theta_i + \beta_i = n/p_i + (1 - \gamma)/m + \beta_i > 0\).

Now assume that \(\theta_i = 0\). By proceeding similarly as in the previous case, we have

\[
S_1^i \lesssim |x_B|^{-\beta_i} N \lesssim |x_B|^{-\beta_i} \log_2 \left(\frac{|x_B|}{R}\right),
\]

and

\[
S_2^i \lesssim |x_B|^{-\beta_i}
\]

since \(\beta_i > 0\) when \(\theta_i = 0\). Consequently,

\[
S_1^i + S_2^i \lesssim |x_B|^{-\beta_i} \left(1 + \log_2 \left(\frac{|x_B|}{R}\right)\right) \lesssim |x_B|^{-\beta_i} \log_2 \left(\frac{|x_B|}{R}\right).
\]

We finally consider the case \(\theta_i > 0\). For \(S_2^i\) we can proceed exactly as in the case \(\theta_i < 0\) and get the same bound. On the other hand, for \(S_1^i\) we have that

\[
S_1^i \lesssim \sum_{k=1}^{N} (2^k R)^{-n+\gamma/m-1/m+n/p'_i} |x_B|^{-\beta_i}
\]

\[
\lesssim |x_B|^{-\beta_i} R^{-\theta_i} \sum_{k=1}^{N} 2^{-k\theta_i}
\]

\[
\lesssim |x_B|^{-\beta_i} \left(2^N R\right)^{-\theta_i} 2^{N\theta_i} \frac{1 - 2^{-N\theta_i}}{1 - 2^{-\theta_i}}
\]

\[
\lesssim |x_B|^{-\beta_i - \theta_i} 2^{N\theta_i}.
\]

Therefore, if \(i \in I_2\) and \(\theta_i > 0\) we get

\[
S_1^i + S_2^i \lesssim |x_B|^{-\beta_i - \theta_i} \left(1 + 2^{N\theta_i}\right) \lesssim 2^{N\theta_i} |x_B|^{-\beta_i - \theta_i}.
\]

By combining (5.2), (5.3) and (5.4) we obtain

\[
\prod_{i \in I_2} \left(\int_{\mathbb{R}^n \setminus B} v_{i}^{-p'_i}(y) \frac{|x_B - y|^{(n-\gamma/m+1/m)p'_i}}{|x_B - y|^{(n-\gamma/m+1/m)p'_i}} dy\right)^{1/p'_i}
\]
\[
\lesssim \prod_{i \in I_2, \theta_i < 0} |x_B|^{-\beta_i - \theta_i} \prod_{i \in I_2, \theta_i = 0} |x_B|^{-\beta_i} \log_2 \left( \frac{|x_B|}{R} \right) \\
\times \prod_{i \in I_2, \theta_i > 0} |x_B|^{-\beta_i - \theta_i} 2^{N \theta_i} \\
\lesssim |x_B|^{-\sum_{i \in I_2} (\beta_i + \theta_i)} 2^{N \sum_{i \in I_2, \theta_i > 0} \theta_i} \\
\times \left( \log_2 \left( \frac{|x_B|}{R} \right) \right)^{\# \{i \in I_2, \theta_i = 0\}},
\]
so the left-hand side of (5.1) can be bounded by
\[
CR^{1-\delta-(n-\gamma/m+1/m)m_1}|x_B|^{\alpha-\sum_{i \in I_2} (\beta_i + \theta_i)} 2^{N \sum_{i \in I_2, \theta_i > 0} \theta_i} \left( \log_2 \left( \frac{|x_B|}{R} \right) \right)^{\# \{i \in I_2, \theta_i = 0\}}
\]
which is equal to
\[
C \left( \frac{|x_B|}{R} \right)^{\tau - 1 + (n-\gamma/m+1/m)m_1 + \sum_{i \in I_2, \theta_i > 0} \theta_i} \left( \log_2 \left( \frac{|x_B|}{R} \right) \right)^{\# \{i \in I_2, \theta_i = 0\}}. \tag{5.5}
\]
Since \( \theta_i < n + (1 - \gamma)/m \) for \( i \in I_2 \), there exists \( \varepsilon > 0 \) that verifies
\[
\sum_{i \in I_2, \theta_i > 0} \theta_i + \varepsilon \# \{i \in I_2, \theta_i = 0\} \leq \left( n + \frac{1 - \gamma}{m} \right) \# \{i \in I_2, \theta_i > 0\}.
\]
By using the fact that \( \log_2 t \lesssim \varepsilon^{-1} t^\varepsilon \) for every \( t \geq 1 \), we can majorize (5.5) by a constant factor provided that
\[
\tau - 1 + \left( n + \frac{1 - \gamma}{m} \right) (m_1 + \# \{i \in I_2 : \theta_i > 0\}) \leq \tau - 1 \\
+ \left( n + \frac{1 - \gamma}{m} \right) (m - 1) = 0.
\]
Indeed, if this last inequality did not hold, then we would have that \( \theta_i > 0 \) for every \( i \in I_2 \). We also observe that \( \theta_i > 0 \) for \( i \in I_1 \). This would lead to \( n/p > \gamma - 1 \), a contradiction.

In order to prove (d) we only need to consider two cases. If there exists some \( i \in I_2 \) such that \( \theta_i \leq 0 \), the proof follows exactly as in (c). If not, that is \( \theta_i > 0 \) for every \( i \in I_2 \), observe that
\[
\tau - 1 + \left( n + \frac{1 - \gamma}{m} \right) m_1 + \sum_{i \in I_2, \theta_i > 0} \theta_i = \tau - 1 + \sum_{i=1}^m \left( \frac{n}{p_i} + \frac{1 - \gamma}{m} \right) \\
= \tau + \frac{n}{p} - \gamma \\
< 0,
\]
So we can choose $\varepsilon > 0$ small enough so that the resulting exponent for $|x_B|/R$ in (5.5) is negative. This concludes the proof in this case.

We now proceed with the proof of (e). Let us first suppose that $\delta < \min\{1, \gamma-n/p\}$. We take $\alpha = \delta - \tau > 0$ and $\beta_i = (1 - \tau)/m - \theta_i$, for every $i$. Then we define $w(x) = |x|^\alpha$ and $v_i = |x|^{\beta_i}$, $1 \leq i \leq m$. These functions are locally integrable since $\alpha > 0$ and $\beta_i < n/p_i'$. Furthermore, $\beta_i < 0$ for $i \in I_1$, so $v_i^{-1}$ is RH$\infty$ for these index. Then, by Lemma 2.1, it will be enough to show that condition (2.6) holds. Fix a ball $B = B(x_B, R)$ and assume that $|x_B| < R$. Then we get

$$\|w X_B\|_{\infty} \left| \frac{|x_B|}{R} \right| \lesssim R^{1-\delta+\alpha} = R^{1-\tau}. \quad (5.6)$$

On the other hand, if $i \in I_1$ we have

$$\|v_i^{-1} X_{R^n \setminus B} \|_{|x_B| - |n-\gamma/m+1|} \lesssim \sum_{k=0}^{\infty} \left\| v_i^{-1} X_{B_{k+1} \setminus B_k} \right\|_{|x_B| - |n-\gamma/m+1|} \lesssim \sum_{k=0}^{\infty} (2^k R)^{-\beta_i - n+\gamma/m-1/m} \lesssim R^{(\tau-1)/m}, \quad (5.7)$$

since $\tau < \delta < 1$. This yields

$$\prod_{i \in I_1} \left\| v_i^{-1} X_{R^n \setminus B} \right\|_{|x_B| - |n-\gamma/m+1|} \lesssim R^{m_1 (\tau-1)/m}. \quad (5.7)$$

Finally, since $\beta_i + \theta_i = (1 - \tau)/m > 0$ for $i \in I_2$, we can proceed as in page 21 to obtain

$$\prod_{i \in I_2} \left( \int_{R^n \setminus B} v_i^{-p_i'}(y) \left| x_B - y \right|^{(n-\gamma/m+1)/p_i'} \, dy \right)^{1/p_i'} \lesssim R^{m_2 (\tau-1)/m}. \quad (5.8)$$

By combining (5.6), (5.7) and (5.8), the left-hand side of (2.6) is bounded by a constant $C$. We now consider the case $|x_B| > R$. We have that

$$\|w X_B\|_{\infty} \left| \frac{|x_B|}{R} \right| \lesssim R^{1-\delta} |x_B|^\alpha. \quad (5.9)$$

Since $|x_B| > R$, there exists a number $N \in \mathbb{N}$ such that $2^N R < |x_B| \leq 2^{N+1} R$. For $i \in I_1$ we write

$$\|v_i^{-1} X_{R^n \setminus B} \|_{|x_B| - |n-\gamma/m+1|} \lesssim \sum_{k=0}^{N} \left\| v_i^{-1} X_{B_{k+1} \setminus B_k} \right\|_{|x_B| - |n-\gamma/m+1|} \lesssim \sum_{k=0}^{N} |x_B|^{-|n-\gamma/m+1|}. \quad (5.9)$$
\[ + \sum_{k=N+1}^{\infty} \frac{v_i^{-1} \mathcal{X}_{B_{k+1}} \setminus B_k}{|x_B - \cdot|^{n-\gamma/m+1/m}} \bigg|_{\infty} = S_1^i + S_2^i. \]

By applying Lemma 5.2 and proceeding as in page 23 with \( p_i = 1 \) we have that

\[ \prod_{i \in I_1} \left| \frac{v_i^{-1} \mathcal{X}_{R \setminus B}}{|x_B - \cdot|^{n-\gamma/m+1/m}p_i} \right|^{1/p_i} \lesssim |x_B|^{\sum_{i \in I_1}(\beta_i+\theta_i)} 2^N \sum_{i \in I_1} \theta_i. \tag{5.10} \]

Finally, if \( i \in I_2 \) our choice of \( \beta_i \) allows us to follow the argument given in page 22 to conclude that

\[ \prod_{i \in I_2} \left( \int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^{(n-\gamma/m+1/m)p_i'}} \, dy \right)^{1/p_i'} \lesssim |x_B|^{\sum_{i \in I_2}(\beta_i+\theta_i)} 2^N \sum_{i \in I_2, \theta_i > 0} \theta_i \times \left( \log_2 \left( \frac{|x_B|}{R} \right) \right)^{\# \{i \in I_2, \theta_i = 0\}}. \]

By combining the inequality above with (5.9) and (5.10), the left-hand side of (2.6) can be bounded by

\[ CR^{1-\delta} |x_B|^{\alpha - \sum_{i=1}^{m}(\theta_i + \beta_i) + \sum_{i: \theta_i > 0} \theta_i} \left( \log_2 \left( \frac{|x_B|}{R} \right) \right)^{\# \{i \in I_2, \theta_i = 0\}} \]

or equivalently by

\[ C \left( \frac{R}{|x_B|} \right)^{1-\delta - \sum_{i: \theta_i > 0} \theta_i} \left( \log_2 \left( \frac{|x_B|}{R} \right) \right)^{\# \{i \in I_2, \theta_i = 0\}}. \]

If \( \theta_i < 0 \) for every \( i \) then the exponent of \( R/|x_B| \) is positive. On the other hand, if \( \theta_i \geq 0 \) for every \( i \) then

\[ 1 - \delta - \sum_{i: \theta_i > 0} \theta_i = 1 - \delta - \sum_{i=1}^{m} \theta_i = 1 - \delta - \frac{n}{p} + \gamma - 1 > 0, \]

since \( \delta < \gamma - n/p \). In both cases we can repeat the argument given in page 23 to conclude that \((w, \vec{v})\) belongs to \( \mathbb{H}_m(\vec{p}, \gamma, \delta) \). Let us observe that, for example, if \( \gamma \leq 1 \) then every \( \theta_i \) is nonnegative.

If \( \delta = 1 < \gamma - n/p \) or \( \delta = \gamma - n/p < 1 \) the same estimation as above works when we take \( \theta_i < 0 \) for every \( i \). The second case also works when \( \theta_i > 0 \) for every \( i \).
We finish with the proof of item (f). In this case we fix $\alpha > 0$ and take $w(x) = (1 + |x|^\alpha)^{-m_1}$. If $g_i$ are nonnegative fixed functions in $L^{p_i'}(\mathbb{R}^n)$ for $i \in I_2$, we define

$$v_i(x) = \begin{cases} e^{|x|} & \text{if } i \in I_1, \\ g_i^{-1} & \text{if } i \in I_2. \end{cases}$$

Fix a ball $B = B(x_B, R)$. It is enough to check condition (2.5), since $\delta = \gamma - mn < \tau$. Notice that

$$\|w \chi_B\|_\infty \prod_{i \in I_1} \|v_i^{-1} \chi_B\|_\infty \leq \prod_{i \in I_2} \|1 + |\cdot|^\alpha\|_\infty \|e^{-|\cdot|} \chi_B\|_\infty \leq 1.$$  

Therefore,

$$\frac{\|w \chi_B\|_\infty}{|B|^\delta/n - \gamma/n + 1/p} \prod_{i \in I_1} \|v_i^{-1} \chi_B\|_\infty \prod_{i \in I_2} \left( \frac{1}{|B|} \int_B v_i^{-p_i'}(y) \, dy \right)^{1/p_i'} \leq \prod_{i \in I_2} \|g_i\|_{p_i'},$$

for every ball $B$. This concludes the proof of (f). \hfill \Box

We finish with the proof of Theorem 1.3.

**Proof** Let $\bar{v} \in H_m(\bar{p}, \gamma, \delta)$ and $B$ be a ball. Since condition (2.5) holds, then we have that

$$|B|^\gamma/n - \delta/n - 1/p \|\chi_B\|_\infty \prod_{i=1}^m v_i \prod_{i \in I_1} \|v_i^{-1} \chi_B\|_\infty \prod_{i \in I_2} \left( \frac{1}{|B|} \int_B v_i^{-p_i'}(y) \, dy \right)^{1/p_i'} \leq C,$$

for some positive constant $C$. This implies that

$$|B|^\gamma/n - \delta/n - 1/p \|\chi_B\|_\infty \prod_{i \in I_1} \|v_i^{-1} \chi_B\|_\infty \prod_{i \in I_2} \left( \frac{1}{|B|} \int_B v_i^{-p_i'}(y) \, dy \right)^{1/p_i'} \leq C \inf_B \prod_{i=1}^m v_i^{-1}.$$  

Thus we deduce that

$$|B|^\gamma/n - \delta/n - 1/p \prod_{i=1}^m \inf_B v_i^{-1} \leq C \inf_B \prod_{i=1}^m v_i^{-1}.$$  

Since $\prod_{i=1}^m \inf_B v_i^{-1} \leq \inf_B \prod_{i=1}^m v_i^{-1}$, we get that

$$|B|^\gamma/n - \delta/n - 1/p \leq C$$

holds for every ball $B$, so that we must have $\delta = \gamma - n/p$. \hfill \Box
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