A dense neutrino medium could experience self-induced flavor conversions on relatively small scales in the presence of the so-called fast flavor conversion modes. Owing to the fact that fast conversion scales could be much smaller than the ones of the traditional collective neutrino oscillations, it has been speculated that fast modes could lead to some sort of flavor equilibrium. We study the evolution of fast modes in the nonlinear regime and we show that not only fast modes are not guaranteed to lead to flavor equilibrium, but also they could lead to usual collective neutrino oscillations on scales determined by neutrino number density. For the $\nu_e$ dominated case, we observe large amplitude collective oscillations, whereas a sort of flavor stabilization is reached for the $\bar{\nu}_e$ dominated case.

1. INTRODUCTION

The most extreme astrophysical sites such as core-collapse supernovae and neutron star mergers include dense neutrino media. The nature of the neutrino evolution in such a dense neutrino gas could be very different from the one in vacuum and matter. Indeed, due to the presence of neutrino-neutrino interactions, the evolution of neutrinos in dense neutrino media is a nonlinear correlated problem [1–3].

The first studies on this problem were carried out in simplified symmetric models such as the stationary spherically symmetric neutrino bulb model [3–5]. The most prominent feature observed in these studies was the presence of collective neutrino oscillations where neutrinos evolve collectively due to the correlation induced by neutrino-neutrino interaction. The most remarkable observational consequence of this collective evolution is the well-known spectral swapping in which $\nu_e$ ($\bar{\nu}_e$) exchange its spectrum with $\nu_x$ ($\bar{\nu}_x$) for a range of neutrino energies [4–9].

It was just more recently that it was realised the spatial/time symmetries could be broken spontaneously in a dense neutrino gas [10–18]. This, in principle, could lead to new neutrino flavor conversion phenomena. On the one hand, the breaking of the spatial symmetry allows for neutrino flavor conversion at very large neutrino number densities. On the other hand, the breaking of time symmetry could remove the matter suppression of neutrino oscillations and might lead to flavor conversion at very large matter densities.

Another important development was the discovery of fast flavor conversion modes that could occur on very small scales [19–31]. Unlike the case of the traditional collective (slow) modes which do occur on scales determined by neutrino vacuum frequency $\omega = \Delta m^2_{21}/2E$ (which is $\sim O(1)$ km for a 10 MeV neutrino), fast modes occur on scales $\sim G_F^{-1}n_\nu^{-1}$ with $n_\nu$ being the neutrino number density. Obviously, the conversion scales for fast modes could be much smaller than the ones for slow modes at large enough neutrino number densities. It has also been argued that the presence of crossing in the angular distribution of electron lepton number carried by neutrinos ($\nu$ELN) is a necessary condition for the occurrence of fast modes [23, 26–28]. In particular, it has been shown that the presence of crossing(s) in $\nu$ELN could allow $G_F n_\nu$ to play the role of $\omega$ [28].

Not only is fast modes an amazing phenomenon by itself, but also it could have important implications for the physics of supernovae. Firstly, it could help removing matter suppression by activating unstable neutrino modes (with large frequencies in time) on small enough scales. Secondly, it could cause neutrino flavor conversion within SN regions that have long been thought to be the realm of scattering processes. In fact, this is expected since fast modes do occur on scales $\sim G_F^{-1}n_\nu^{-1}$, whereas scattering processes occur on scales $\sim G_F^{-2}E^{-2}n_B^{-1}$ with $n_B$ being the baryon number density [32, 33]. Within the neutrino decoupling region, the former could have values $\lesssim O(1)$ cm, whereas the latter could be much larger $\gtrsim O(1)$ km.

Since fast conversion modes could occur on scales much smaller than the ones of the traditional collective modes, it has been widely speculated that fast modes could lead to some sort of flavor decoherence/equilibrium. Needless to say, the occurrence of flavor equilibrium could significantly simplify the physics of supernova neutrinos from both theoretical and observational points of view. On the theoretical side, it could remarkably reduce the computational difficulties. On the observational side, it could notably improve the analysis of the supernova neutrino signals.

Although the total flavor equilibrium might be inaccessible due to the lepton number conservation laws, some sort of partial flavor equilibrium (decoherence) has been proposed to occur. This could be very similar to the evolution of decoherence for small $\nu_x$-$\bar{\nu}_e$ asymmetries ($\alpha = n_{\nu_x}/n_{\nu_e}$ close to one) within slow modes, where $\bar{\nu}_e$ could experience flavor equilibrium and full decoherence\(^1\) with survival probability $P_{\nu_e,\bar{\nu}_e} \approx 1/2$ [34–36]. Likewise, $\nu_e$ could experience decoherence up to the extents allowed

\(^1\) This is the case for $\alpha < 1$. For the $\nu_e$ dominated case, it might be $\nu_e$ that experiences equilibrium.
by the conservation laws.

In this work, we study fast neutrino flavor conversion modes in the nonlinear regime by using a one dimensional schematic model. We show that not only fast modes are not guaranteed to lead to flavor equilibrium, but also they could lead to large amplitude collective neutrino oscillations on very small scales for $\alpha < 1$. Moreover, for $\alpha > 1$, an out of flavor equilibrium stabilization could be reached.

2. THE NEUTRINO LINE MODEL

To study the evolution of fast modes in the simplest multiangle configuration, we consider a stationary one dimensional schematic model in a two-flavor scenario in which electron neutrinos and antineutrinos are emitted from an infinite line with emission angles in the range $[-\vartheta_{\text{max}}, \vartheta_{\text{max}}]$ [12, 13]. We also assume that the symmetry is preserved in the transverse direction (along the line).

We assume that neutrinos and antineutrinos are emitted monochromatically \(^2\) and with the normalised angular distributions $f_{\nu_e}(\vartheta)$ and $f_{\bar{\nu}_e}(\vartheta)$. In addition, to observe fast modes we allow for different $f_{\nu_e}(\vartheta)$ and $f_{\bar{\nu}_e}(\vartheta)$ so that there can exist crossing in the $\nu$ELN.

At each point $z$, the state of a neutrino which is traveling in direction $\vartheta$ could be specified by its density matrix $\rho_\vartheta(z)$, and, in the absence of collision, its flavor evolution is given by the equation of motion (EOM) [2, 37–42]

$$i \cos \vartheta \partial_z \rho_\vartheta = [H_\vartheta, \rho_\vartheta],$$

with $H_\vartheta = H_{\text{vac}} + H_{\text{mat}} + H_{\nu\nu, \vartheta}$ being the total Hamiltonian where

$$H_{\text{vac}} = \frac{1}{2} \begin{bmatrix} -\omega \cos 2\theta_e & \omega \sin 2\theta_e \\ \omega \sin 2\theta_e & \omega \cos 2\theta_e \end{bmatrix},$$

$$H_{\text{mat}} = \frac{\lambda}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

are the contributions from vacuum and matter, $\theta_e$ is the neutrino vacuum mixing angle, $\lambda = \sqrt{2}G_F n_e$ with $n_e$ being the electron number density and

$$H_{\nu\nu, \vartheta} = \mu \int_{-\vartheta_{\text{max}}}^{\vartheta_{\text{max}}} [f_{\nu_e}(\vartheta')\rho_{\vartheta'}(z) - \alpha f_{\bar{\nu}_e}(\vartheta')\bar{\rho}_{\vartheta'}(z)]$$

$$\times [(1 - \cos(\vartheta - \vartheta'))d\vartheta',$$

is the contribution from $\nu - \nu$ interaction where $\mu = \sqrt{2}G_F n_{\bar{\nu}_e}$. The EOM for antineutrinos is the same except that $\omega \rightarrow -\omega$. Moreover, it is assumed that $\Delta m^2_{\text{atm}} > 0$ ($< 0$) in the normal (inverted) mass hierarchy.

Due to the neutron richness of the supernova medium and the neutron star merger environment, $\bar{\nu}_e$'s are decoupled at smaller radii than $\nu_e$'s. This means that one should naively expect $\bar{\nu}_e$ to be more peaked in the forward direction than $\nu_e$. In this study, we set $f_{\nu_e}(\vartheta)$ to be constant within the range $[-\vartheta_{\text{max}}, \vartheta_{\text{max}}]$ and

$$f_{\bar{\nu}_e}(\vartheta) \propto \exp(-\vartheta^2/2\sigma^2)$$

with $\sigma^2 = \pi/60$. It should be noted that to observe fast modes, one could also take uniform $f_{\nu_e}(\vartheta)$ and $f_{\bar{\nu}_e}(\vartheta)$ with different opening angles for neutrinos and antineutrinos. However, the choice we made here allows for a smoother transition from $\bar{\nu}_e$ to $\nu_e$ dominated angular range. Nevertheless, as we will discuss in the next section, the qualitative features of our results do not depend on the choice of angular distributions.

3. RESULTS AND DISCUSSION

We have studied fast neutrino flavor conversion modes in the nonlinear regime. We assumed $\omega = -1$ (inverted hierarchy), $\vartheta_{\text{max}} = \pi/3$ and very small $\theta_e$. \(^3\) Nevertheless, we have confirmed that the results we are presenting here do not seem to depend qualitatively on the choice of $\omega$, $\theta_e$, $\vartheta_{\text{max}}$ or even on the details of the shape of $f_{\nu_e}(\vartheta)$ and $f_{\bar{\nu}_e}(\vartheta)$ as long as fast modes exist. We investigated the evolution of fast modes for a number of $\alpha$'s for both $\nu_e$ dominated ($\alpha < 1$) and $\bar{\nu}_e$ ($\alpha > 1$) dominated cases of which we show the results for $\alpha = .5$ and 2 in Figs. 2

\(^2\) This assumption is made for the sake of definiteness. Otherwise, the EOM should be approximately blind to the neutrino frequency in the presence of fast modes.

\(^3\) Note that one can make this choice if the matter density is very large. However, one should bear in mind that if the growth rate $\kappa \gtrsim \lambda$, then it is not physically allowed to assume small effective $\theta_e$ since one may not find such a rotating frame where the vacuum term oscillates so quickly that the off-diagonal term averages to zero.
FIG. 2: The angle-averaged survival probabilities of neutrinos (blue line) and antineutrinos (red line) for $\alpha = 0.5$, $\mu = 10^3$, $10^4$ and $10^5$ km$^{-1}$ with $\lambda = 0.3\mu$ (upper panels) and $\lambda = \mu$ (lower panels) as a function of the unitless distance defined in the text. The actual distance is then $z = 10^4 r/\mu$ km for each case. On the lower left panel, we provide a zoomed up subplot of survival probabilities in the range $r = 1$ to 1.1.

FIG. 3: The same information as in Fig. 2 for $\alpha = 2$ except that three matter densities $\lambda = 0$, $\lambda = 0.3\mu$ and $\lambda = \mu$ are shown.
and 3. In general, for the $\nu_e$ dominated cases we observed large amplitude oscillations, whereas a sort of flavor stabilization was reached for the $\bar{\nu}_e$ dominated case.

In Fig. 2, we show the angle-averaged neutrino and antineutrino survival probabilities for $\mu = 10^3, 10^4$ and $10^5$ km$^{-1}$ and $\lambda = 0.3\mu$ and $\mu$ as a function of the unitless distance defined by $r = \mu z/10^4$. Since the relevant scale for fast modes is $\sim 1/\mu$, the unitless quantity $r$ provides a physical measure of length in the problem. For these values of $\lambda$, the value of the growth rate is $\kappa \approx 0.05\mu$. The most prominent feature of this plot is the presence of large amplitude collective oscillations. On the lower left panel, we provide a zoomed up subplot of the survival probabilities in the range $\theta/\pi \approx 0.09$ to 0.1.

FIG. 4: Angular distributions of the survival probabilities of neutrinos (blue line) and antineutrinos (red line) at different radii. On the middle panel, we provide a zoomed up subplot of survival probabilities in the range $\theta/\pi = 0.09$ to 0.1.

and it does not seem that any sort of flavor equilibrium is generally reached since neither $P_{\nu_e,\nu_e} = 1/2$ nor $P_{\bar{\nu}_e,\bar{\nu}_e} = 1/2$ (except for the lowest left panel in which $P_{\nu_e,\nu_e} \approx 0.52$). Note that the oscillation amplitude gets very tiny for the maximum matter density.

In spite of the ongoing speculation, fast modes do not necessarily lead to flavor equilibrium. This speculation stems from the assumption that large amplitude fast flavor conversions could occur on scales (determined by $\mu$) much smaller than the scales of collective modes (which were assumed to be determined by $\omega$). However, as it has been analytically shown in Ref. [28], in the presence of fast modes in neutrino gas, $\mu$ (or even $\lambda$) could play the role of $\omega$. Thus, the collective scales are set by a combination of $\mu$ and $\omega$ (rather than only $\omega$) in the presence of fast modes. This means that the logic of comparing scales mentioned above could totally fail. Then one might be tempted to expect that the nature of the evolution of fast modes (in the nonlinear regime) may not be much different from the one of the usual collective slow modes.

Since the values of $\mu$ for which fast conversion modes occur are very large, one might have to use a relatively large number of angle bins (at least several thousands in our case) to reach convergence in the simulations. This simply comes from the fact that for such large values of $\mu$, the neighbouring neutrino beams could experience quite different potentials during their evolution. This implies that the angular distribution of the neutrino quantities could be completely uneven. In Fig. 4, we present the angular distributions of the neutrino and antineutrino

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Note that for $\alpha < 1$, the presence of matter is necessary to observe fast mode instabilities [22, 28].
FIG. 5: Artificial flavor equilibrium due to insufficient number of angle bins. The angle-averaged survival probabilities of neutrinos (blue line) and antineutrinos (red line) are plotted as a function of the unitless distance $r = \mu z/10^4$ for $\alpha = 0.5$, $\mu = 10^5$, $\lambda = \mu$ and using 100 angle bins. To guide the eye, we have plotted the value 0.5 (brown dashed line) which corresponds to flavor equilibrium.

Survival probabilities

![Graph showing survival probabilities vs. radius.]

Although flavor equilibrium could be hard to reach in a valid treatment of the problem accounting well for numerical convergence, a too small number of angle bins may lead to an approximate artificial flavor equilibrium. This is shown explicitly in Fig. 5 where only 100 angle bins are used for $\alpha = 0.5$, $\mu = 10^5$ and $\lambda = \mu$ (to be compared with the lower right panel in Fig. 2).

In this study, we have assumed that neutrino beams are only propagating in the forward direction. However, if the backward propagating modes exist, one has to consider both spatial and temporal evolution of neutrinos [26]. Here we do not want to consider a multi-dimensional problem which could be very tough in multiangle configuration. However, to have an idea of the temporal evolution of fast modes in the nonlinear regime, we considered a homogenous non-stationary neutrino gas so that the time is enough (and the relevant parameter) to describe the evolution of the system. The neutrinos were assumed to be emitted in a multiangle scenario with emission angles $\theta$ within the range $[-\theta_{\text{max}}, \theta_{\text{max}}]$. One then has to solve an EOM which is similar to Eq. (1) except that $\cos \theta \partial_z$ must be replaced by $\partial_t$. We repeated our simulations for the temporal evolution of fast modes and we did not observe any qualitative difference for $\alpha > 1$ for which we observed fast modes in the neutrino gas.

Another remark concerns the neutrino lepton number conservation law (for $\theta_{\nu} \approx 0$) within the comoving frame [43–45]. Although if one considers only the temporal evolution of the patterns in the angular distributions.

FIG. 6: The evolution of the net lepton number $N$ (blue solid line) and the lepton number flux $F$ (red dashed line). The values are rescaled by the initial ones at $z = 0$.

Survival probabilities

![Graph showing survival probabilities vs. distance.]

The evolution of the net lepton number

$$N = n_{\nu_e} \int [f_{\nu_e}(\vartheta') (\rho_{\nu_e, \nu_e} - \rho_{\nu_e, \nu_x}) - \alpha f_{\nu_e}(\vartheta') (\rho_{\nu_e, \nu_x} - \rho_{\bar{\nu}_e, \bar{\nu}_x})] \, d\vartheta',$$

(6)

is conserved, it is the flux of the lepton number

$$F = n_{\nu_e} \int [f_{\nu_e}(\vartheta') (\rho_{\nu_e, \nu_e} - \rho_{\nu_e, \nu_x}) - \alpha f_{\nu_e}(\vartheta') (\rho_{\nu_e, \nu_x} - \rho_{\bar{\nu}_e, \bar{\nu}_x})] \cos \vartheta' \, d\vartheta',$$

(7)

which is conserved if only the spatial evolution of the neutrino gas is considered. One should note that the net lepton number is not conserved any more. In particular, a nontrivial evolution of the angular distribution of the νELN could allow for large variations of $\mathcal{N}$. We observed that this variations could be larger for smaller $\nu_e$–$\bar{\nu}_e$ asymmetries. This is explicitly shown in Fig. 6 where (rescaled) $F$ and (rescaled) $N$ are plotted as a function of $z$ for $\alpha = 0.8$ and $\lambda = 0.3 \mu$. Although (rescaled) $F$ is expectedly conserved, the (rescaled) net lepton number $\mathcal{N}$ could vary by more than 50%. Though the extent to which the net lepton number could be violated depends on the details of the physical quantities, it should be noted that one might not be, in principle, seriously limited by the conservation of the neutrino net lepton number.

By studying a one dimensional schematic model, we have shown that fast modes do not necessarily lead to flavor equilibrium or even flavor stabilization. The non-occurrence of flavor equilibrium makes the problem of flavor evolution in dense neutrino media more involved. Obviously, it remains to be investigated if multi-dimensional models can induce flavor equilibrium and a degeneracy of
the neutrino spectra which would remarkably simplify the issue of neutrino evolution in core-collapse supernovae or accretion disks around compact objects such as black holes or binary neutron star merger remnants. In conclusion, unless future studies demonstrate that such an equilibration can occur, the present work shows that the study of the evolution of a dense neutrino gas requires a seven dimensional non-linear description, implementing symmetry breaking and possible small scale instabilities due to fast modes.

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