Lower ground state due to counter-rotating wave interaction in a trapped ion system

T Liu¹, K L Wang¹ ² and M Feng³

¹ The School of Science, Southwest University of Science and Technology, Mianyang 621010, People’s Republic of China
² The Department of Modern Physics, University of Science and Technology of China, Hefei 230026, People’s Republic of China
³ State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan, 430071, People’s Republic of China

E-mail: mangfeng@wipm.ac.cn

Received 16 November 2006
Published 18 May 2007
Online at stacks.iop.org/JPhysB/40/1967

Abstract
We consider a single ion confined in a trap under the radiation of two travelling waves of lasers. In the strong-excitation regime and without the restriction of the Lamb–Dicke limit, the Hamiltonian of the system is similar to a driving Jaynes–Cummings model without the rotating wave approximation (RWA). The approach we developed enables us to present complete eigensolutions, which makes it possible to compare with the solutions under the RWA. We find that the ground state in our non-RWA solution is energetically lower than the counterpart under the RWA. If we have the ion in the ground state, it is equivalent to a spin dependent force on the trapped ion. Discussion is given for the difference between the solutions with and without the RWA, and for the relevant experimental test, as well as for the possible application to quantum information processing.

1. introduction

Ultracold ions trapped as a line are considered to be a promising system for quantum information processing [1]. Since the first quantum gate performed in an ion trap [2], there have been a series of experiments with trapped ions to achieve nonclassical states [3], simple quantum algorithms [4] and quantum communication [5].

There have been also a number of proposals to employ trapped ions for quantum computing, most of which work only in the weak excitation regime (WER), i.e., the Rabi frequency smaller than the trap frequency. While a bigger Rabi frequency would lead to faster quantum gating, some proposals [6–8] have aimed to achieve operations in the case of the Rabi frequency larger than the trap frequency, i.e., the so-called strong excitation regime (SER).
The difference of the WER from the SER is mathematically reflected in the employment of the rotating wave approximation (RWA), which averages out the fast oscillating terms in the interaction Hamiltonian. As the RWA is less valid with the larger Rabi frequency, the treatment for the SER is complicated, incomplete [9] and sometimes resorts to numerics [10].

In addition, the Lamb–Dicke limit strongly restricts the application of the trapped ions due to technical challenge and the slow quantum gating. We have noticed some ideas [11, 12] to remove the Lamb–Dicke limit in designing quantum gates, which are achieved by using some complicated laser pulse sequences.

In the present work, we investigate, from another research angle, the system mentioned above in SER and in the absence of the Lamb–Dicke limit. The main idea, based on an analytical approach we have developed, is to check the eigenvectors and the eigenenergies of such a system, with which we hope to obtain new insight into the system for more application.

The main result in our work is a newly found ground state, energetically lower than the ground state calculated by a standard Jaynes–Cummings model. We will also present the analytical forms of the eigenvectors and the variance of the eigenenergies with respect to the parameters of the system, which might be used in understanding the time evolution of the system.

The paper is organized as follows. In section 2 we will solve the system in the absence of the RW A. Then some numerical results will be presented in comparison with the RW A solutions in section 3. We will discuss the new results for their possible application. More extensive discussion and a conclusion are made in section 4. Some detailed analytical deduction can be found in the appendix.

2. The analytical solution of the system

As shown in figure 1, we consider a Raman A-type configuration, which corresponds to the actual process in NIST experiments. As in [13], we will employ some unitary transformations to get rid of the assumption of Lamb–Dicke limit and the WER. So our solution is more general than most of the previous work. For a single trapped ion experiencing two off-resonant counter-propagating travelling wave lasers with frequencies $\omega_1$ and $\omega_2$, respectively, and in the case of a large detuning $\delta$, we have an effective two-level system with the lasers driving the electric-dipole forbidden transition $|g\rangle \leftrightarrow |e\rangle$ by the effective laser frequency

---

Most of the previous work in this respect was carried out by cutting off the expansion of the exponentials regarding the quantized phonon operators, which is only reasonable in the WER and within the Lamb–Dicke limit. In contrast, our treatment can be used in both the SER and the WER cases.
\( \omega_L = \omega_1 - \omega_2 \). So we have the dimensionless Hamiltonian
\[
H = \frac{\Delta}{2} \sigma_z + a^\dagger a + \frac{\Omega}{2} (\sigma_x e^{i\eta \hat{x}} + \sigma_- e^{-i\eta \hat{x}}),
\]
in the frame rotating with \( \omega_L \), where \( \Delta = (\omega_0 - \omega_L)/\nu \), \( \omega_0 \) and \( \nu \) are the resonant frequency of the two levels of the ion and the trap frequency, respectively. \( \Omega \) is the dimensionless Rabi frequency in units of \( \nu \) and \( \eta \) the Lamb–Dicke parameter. \( \sigma_{\pm, z} \) are usual Pauli operators, and we have \( \hat{x} = a^\dagger + a \) for the dimensionless position operator of the ion with \( a^\dagger \) and \( a \) being operators of creation and annihilation of the phonon field, respectively. We suppose that both \( \Omega \) and \( \nu \) are much larger than the atomic decay rate and the phonon dissipative rate so that no dissipation is considered below.

As in [13], we first carry out some unitary transformations on equation (1) to avoid the expansion of the exponentials. So we have
\[
H' = U H U^\dagger = \frac{\Omega}{2} \sigma_z + a^\dagger a + g(a^\dagger + a)\sigma_x + \epsilon \sigma_x + g^2,
\]
where
\[
U = \frac{1}{\sqrt{2}} e^{i\sigma_y a^\dagger /2} \begin{pmatrix} F(\eta) & F(\eta) \\ -F(\eta) & F(\eta)\end{pmatrix},
\]
with \( F(\eta) = \exp[i(\eta(a^\dagger + a)/2)] \), \( g = \eta/2 \) and \( \epsilon = -\Delta/2 \). Equation (2) is a typical driving Jaynes–Cummings model including the counter-rotating wave terms. In contrast to the usual treatments that consider the Lamb–Dicke limit by using the RWA in a frame rotation, we retain the counter-rotating wave interaction in the third term of the right-hand side of equation (2) in our case. To continue our treatment, we make a further rotation with \( V = \exp(i\pi \sigma_y/4) \), yielding
\[
H' = VH'V^\dagger = -\frac{\Omega}{2} \sigma_z + a^\dagger a + g(a^\dagger + a)\sigma_x + \epsilon \sigma_x + g^2,
\]
where we have used \( \exp(i\sigma_x)\sigma_z\exp(-i\sigma_x) = \cos(2\theta)\sigma_z + \sin(2\theta)\sigma_x \), and \( \exp(-i\sigma_x)\sigma_z\exp(i\sigma_x) = \cos(2\theta)\sigma_z - \sin(2\theta)\sigma_x \). For the convenience of our following treatment, we rewrite equation (3) to be
\[
H' = (|\psi_1\rangle\langle e| - |g\rangle\langle g|) - \frac{\Omega}{2} (|e\rangle\langle g| + |g\rangle\langle e|) + a^\dagger a + g(a^\dagger + a)(|e\rangle\langle e| - |g\rangle\langle g|) + g^2.
\]

Using the Schrödinger equation, and the orthogonality between \( |e\rangle \) and \( |g\rangle \), we suppose
\[
|\rangle = |\psi_1\rangle|e\rangle + |\psi_2\rangle|g\rangle,
\]
which yields
\[
\epsilon |\psi_1\rangle + a^\dagger a |\psi_1\rangle + g(a^\dagger + a)|\psi_1\rangle - \frac{\Omega}{2} |\psi_2\rangle + g^2 |\psi_1\rangle = E |\psi_1\rangle,
\]
\[
-\epsilon |\psi_2\rangle + a^\dagger a |\psi_2\rangle - g(a^\dagger + a)|\psi_2\rangle - \frac{\Omega}{2} |\psi_1\rangle + g^2 |\psi_2\rangle = E |\psi_2\rangle.
\]
To make the above equations concise, we apply the displacement operator \( \hat{D}(g) = \exp[g(a^\dagger - a)] \) on \( a^\dagger \) and \( a \), which gives \( A = \hat{D}(g)^\dagger a \hat{D}(g) = a + G \), \( A^\dagger = \hat{D}(g)^\dagger a^\dagger \hat{D}(g) = a^\dagger + g \), \( B = \hat{D}(-g)^\dagger a \hat{D}(-g) = a - g \) and \( B^\dagger = \hat{D}(-g)^\dagger a^\dagger \hat{D}(-g) = a^\dagger - g \). So we have
\[
(A^\dagger A + \epsilon)|\psi_1\rangle - \frac{\Omega}{2} |\psi_2\rangle = E |\psi_1\rangle,
\]
\[
(B^\dagger B - \epsilon)|\psi_2\rangle - \frac{\Omega}{2} |\psi_1\rangle = E |\psi_2\rangle.
\]
Obviously, the new operators work in different subspaces, which lead to different evolutions regarding different internal levels \( |g\rangle \) and \( |e\rangle \). We will later regard this feature to be relevant.
to spin-dependent force. The solution of the two equations above can simply be set as

\[ |\psi_1\rangle = \sum_{n=0}^{N} c_n |n\rangle_A, \tag{10} \]

\[ |\psi_2\rangle = \sum_{n=0}^{N} d_n |n\rangle_B, \tag{11} \]

with \( N \) a large integer to be determined later. \( |n\rangle_A = \frac{1}{\sqrt{n!}} (a^\dagger + g)^n |0\rangle_A = \frac{1}{\sqrt{n!}} (a^\dagger + g)^n \exp(-ga^\dagger - g^2/2)|0\rangle \), \( |n\rangle_B = \frac{1}{\sqrt{n!}} (a^\dagger - g)^n |0\rangle_B = \frac{1}{\sqrt{n!}} (a^\dagger - g)^n \exp(ga^\dagger - g^2/2)|0\rangle \). Taking equations (10) and (11) into equations (8) and (9), respectively, and multiplying by \( A|m\rangle \) and \( B|m\rangle \), respectively, we have

\[ (m+\epsilon)c_m - \frac{\Omega}{2} \sum_{n=0}^{N} (-1)^n D_{mn} d_n = E c_m, \tag{12} \]

\[ (m-\epsilon)d_m - \frac{\Omega}{2} \sum_{n=0}^{N} (-1)^n D_{mn} c_n = E d_m, \tag{13} \]

where we have set \(-1)^n D_{mn} = A(m|n\rangle_B \) and \((-1)^m D_{mn} = B(m|n\rangle_A \), whose deduction can be found in the appendix. Diagonizing the relevant determinants, we may have the eigenenergies \( E_i \) and the eigenvectors regarding \( c_n^i \) and \( d_n^i \) \((n = 0, \ldots, N, i = 0, \ldots, N) \). Therefore, as long as we could find a closed subspace with \( c_{N+1} \) and \( d_{N+1} \) approaching zero for a certain large integer \( N \), we may have a complete eigensolution of the system.

3. Discussion based on numerics

Before doing numerics, we first consider a treatment involving the RWA. As the RWA solution could easily present complete eigenenergy spectra, it is interesting to make a comparison between the RWA solution and our non-RWA one. We consider a rotation in equation (2) with respect to \( \exp(-i((\Omega/2)\sigma_z + a^\dagger a)t) \), which results in

\[ H_A = \frac{\Omega}{2} \sigma_z + a^\dagger a + g(a\sigma_+ + a^\dagger \sigma_-) + g^2, \tag{14} \]

where the RWA has been made by setting \( \Omega = 1 \), and we have the corresponding eigenenergies

\[ E_n^\pm = (n + g^2 + 1/2) \pm g \sqrt{n+1}. \tag{15} \]

So the system is degenerate in the case of \( n = 0 \) and there are two eigenenergy spectra corresponding to \( E_n^\pm \) as long as \( n \neq 0 \).

Figures 2(a) and (b) demonstrate two spectra, respectively, and in each figure we compare the differences between the RWA and non-RWA solutions. In contrast to the two spectra in the RWA solution, the non-RWA solution includes only one spectrum. Comparing the two eigensolutions, we find that the even-number and odd-number excited levels in the non-RWA case correspond to \( E_n^+ \) and \( E_n^- \) of the RWA case, respectively, and the difference becomes bigger and bigger with the increase of \( n \). It is physically understandable for these differences because the RWA solution, valid only for small \( \eta \), does not work beyond the Lamb–Dicke regime. The above comparison also demonstrates the change of the ion trap system from

5 We take throughout this paper \( N = 40 \) in which the coefficients \( c_{N+1}^i \) and \( d_{N+1}^i \), with \( i = 0, 1, \ldots, 40 \), are negligible in the case of \( \Omega = 1 \) and 2. Although with the increase of values of \( \Omega \) the diagonalization space has to be enlarged, our analytical method generally works well in a wide range of parameters.
Lower ground state due to counter-rotating wave interaction in a trapped ion system

Figure 2. The eigenenergy spectra with $\Omega = 1$, where (a) and (b) correspond to two different sets of eigenenergies with respect to the Lamb–Dicke parameter. In (a) a comparison is made between $E^+_{n}$ in the RWA case (dashed-dotted curves) and $E_{n}$ with $n = \text{even numbers}$ in the non-RWA case (star curves for $n = 0$ and solid curves for others). In (b) the comparison is for $E^-_{n}$ in the RWA case (dashed-dotted curves) to $E_{n}$ with $n = \text{odd numbers}$ in the non-RWA case (solid curves).

an integrable case (i.e., with RWA validity) to the non-integrable case (i.e., without RWA validity). But besides these differences, we find an unusual result in this comparison, i.e., a new level without the counterpart in the RWA solution appearing in our solution, which is lower than the ground state in the RWA solution by $\nu + x \eta$, with $x$ being an $\eta$-dependent coefficient. From the viewpoint of physics, due to additional counter-rotating wave interaction involved, it is reasonable to have something more in our solution than the RWA case, although this does not surely lead to a new level lower than the previous ground state. Anyway, this is a good news for quantum information processing with trapped ions. As the situation in SER and beyond the Lamb–Dicke limit involves more instability, a stable confinement of the ion requires a stronger trapping condition. In this sense, our solution, with the possibility of having the ion stay in an energetically lower state, gives hope in this respect. We will return to this point later.

Since no report of the new ground state had been found either theoretically or experimentally in previous publications, we suggest to check it experimentally by a resonant absorption spectrum. As shown above, in the case of non-zero Lamb–Dicke parameter, the degeneracy of the neighbouring level spacing is released, and the bigger the $\eta$, the larger the spacing difference between the neighbouring levels. Therefore, an experimental test of the newly found ground state should be available by resonant transition between the ground and the first excited states in figure 2, once the SER is reached. We have noticed that the SER could be achieved by first cooling the ions within the Lamb–Dicke limit and under the WER, and then by decreasing the trap frequency by opening the trap adiabatically [6]. Since it is lower in energy than the previously recognized ground states, the new ground state we found is more stable, and thereby more suitable for storing quantum information. Once the trapped ion is cooled down to the ground state in the SER, it is, as shown in equation (5) with $n = 0$, actually equivalent to the effect of a spin-dependent force on the trapped ion [14]. If we make a Hadamard gate on the ion by $\lvert g \rangle \rightarrow (\lvert g \rangle + \lvert e \rangle)/\sqrt{2}$ and $\lvert e \rangle \rightarrow (\lvert g \rangle - \lvert e \rangle)/\sqrt{2}$, we reach a Schrödinger cat state, i.e., $(1/2)[D^\dagger(g)\lvert 0 \rangle + D^\dagger(-g)\lvert 0 \rangle] \lvert g \rangle - [D^\dagger(g)\lvert 0 \rangle - D^\dagger(-g)\lvert 0 \rangle] \lvert e \rangle$. Two ions confined in a trap in the above situation will yield two-qubit gates without really exciting the vibrational mode [11]. It is also the way with this spin-dependent force towards scalable quantum information processing [12]. As in SER,
we may have larger Rabi frequency than in WER, the quantum gate could be in principle carried out faster in the SER.

In addition, as it is convergent throughout the parameter subspace, our complete eigensolution enables us to accurately write down the state of the system at an arbitrary evolution time, provided that we know the initial state. This would be useful for future experiments in preparing non-classical states and in designing any desired quantum gates with trapped ions in the SER and beyond the Lamb–Dicke limit. Moreover, as shown in figures 3(a)–(c), our present solution is helpful for us to understand the particular solutions in previous publications [13]. The comparison in the figures shows that the results in [13] are actually mixtures of different eigensolutions. For example, the lowest level in figure 2 in [13], corresponding to $\Omega_1 = 2$ and $\eta = 0.2$, is actually constituted at least by the third, the fourth and the fifth excited states of the eigensolution.

4. Further discussion and conclusion

The observation of counter-rotating effects is an interesting topic discussed previously. In [15], a standard method is used to study the observable effects regarding the rotating and the counter-rotating terms in the Jaynes–Cummings model, including to observe Bloch–Siegert shift [16] and quantum chaos in a cavity QED by using differently polarized lights. A recent
work [17] for a two-photon Jaynes–Cummings model has also investigated the observability of the counter-rotating terms. By using perturbation theory, the authors claimed that the counter-rotating effects, although very small, can be in principle observed by measuring the energy of the atom going through the cavity. Actually, for the cavity QED system without any external source involved, it is generally thought that the counter-rotating terms only make a contribution in some virtual fluctuations of the energy in the weak coupling regime, while the interference between the rotating and counter-rotating contributions could result in some phase dependent effects [18]. In any case, if there is an external source, for example, the laser radiating a trapped ultracold ion, the counter-rotating terms will show their effects, e.g., related to heating in the case of WER [19]. In this sense, our result is somewhat amazing because the counter-rotating interaction in the SER, making entanglement between the internal and vibrational states of the trapped ion, plays a positive role in the ion trapping.

We argue that our approach is applicable to different physical processes involving the counter-rotating interaction. Since the counter-rotating terms result in energy nonconservation in single quantum processes, usual techniques cannot solve the Hamiltonian with eigenstates spanning in an open form. In this case, the path-integral approach [20] and the perturbation approach [18], assisted by numerical techniques, were employed in the weak coupling regime of the Jaynes–Cummings model. In contrast, our method, based on the diagonalization of the coherent-state subspace, could in principle study the Jaynes–Cummings model without the RWA in any cases. We have also noticed a recent publication [21] treating a strongly coupled two-level system to a quantum oscillator under an adiabatic approximation, in which something is similar to our work in the solution of the Hamiltonian in the absence of the RWA. But due to different features in their system from our atomic case, the two-level splitting term, much smaller compared to other terms, can be taken as a perturbation. So the advantage of that treatment is the possibility of analytically obtaining good approximate solutions. In contrast, no approximation is used in our solution, which should be more efficient to do the relevant job.

In summary, we have investigated the eigensolution of the system with a single trapped ion, experiencing two travelling waves of lasers, in the SER and in the absence of the Lamb–Dicke limit. We have found the ground state in the non-RWA case to be energetically lower than the counterpart of the solution with RWA, which would be useful for quantum information storage and for quantum computing. The analytical forms of the eigenfunction and the complete set of the eigensolutions would be helpful for us to understand a trapped ion in the SER and with a large Lamb–Dicke parameter. We argue that our work could be applied to different systems in dealing with strong coupling problems.

Acknowledgments

This work is supported in part by NNSFC no 10474118, by Hubei Provincial Funding for Distinguished Young Scholars, and by Sichuan Provincial Funding.

Appendix

We give the deduction of \( A(m|n)_B \) and \( B(m|n)_A \) below,

\[
A(m|n)_B = \frac{1}{\sqrt{m!n!}} (0) e^{-g^2/2} (a + g)^m (a - g)^n e^{g a - g^2/2} (0) \\
= \frac{1}{\sqrt{m!n!}} e^{-2g^2} (0) (a + g)^m e^{ga} (a - g)^n (0) \\
= \frac{1}{\sqrt{m!n!}} e^{-2g^2} (0) (a + 2g)^m (a - 2g)^n (0) = (-1)^n D_{mn},
\]
with
\[
D_{mn} = e^{-2g^2} \sum_{i=0}^{\min[m,n]} (-1)^{-i} \frac{\sqrt{m!n!(2g)^m} n-i}{(m-i)!(n-i)!}.
\]

It is easily proven following a similar step to the above that
\[
\langle m|n \rangle_A = \frac{1}{\sqrt{m!n!}} \langle 0| e^{a-a^2/2} (a - g)^m (a^\dagger + g)^n e^{-ga^\dagger - g^2/2} |0 \rangle,
\]
would finally get to \((-1)^m D_{mn}\).

References

[1] Cirac J I and Zoller P 1995 Phys. Rev. Lett. 74 4091
[2] Monroe C, Meekhof D M, King B E, Itano W M and Wineland D J 1995 Phys. Rev. Lett. 75 4714
[3] Turchette Q A, Wood C S, King B E, Myatt C J, Leibfried D, Itano W M, Monroe C and Wineland D J 1998 Phys. Rev. Lett. 81 3631
Sackett C A et al 2000 Nature 404 256
[4] Gulde S, Riebe M, Lancaster G P T, Becher C, Eschner J, Haeffner H, Schmidt-Kaler F, Chuang I L and Blatt R 2003 Nature 421 48
[5] Riebe M et al 2004 Nature 429 734
Barrett M D et al 2004 Nature 429 737
[6] Poyatos J F, Cirac J I, Blatt R and Zoller P 1996 Phys. Rev. A 54 1532
Poyatos J F, Cirac J I and Zoller P 1998 Phys. Rev. Lett. 81 1322
[7] Zheng S, Zhu X W and Feng M 2000 Phys. Rev. A 62 033807
[8] Feng M 2004 Eur. Phys. J. D 29 189
[9] Feng M, Zhu X, Fang X, Yan M and Shi L 1999 J. Phys. B: At. Mol. Opt. Phys. 32 701
Feng M 2002 Eur. Phys. J. D 18 371
[10] Zeng H, Lin F, Wang Y and Segawa Y 1999 Phys. Rev. A 59 4589
[11] Garcia-Ripoll J J, Zoller P and Cirac J I 2003 Phys. Rev. Lett. 91 157901
[12] Duan L-M 2004 Phys. Rev. Lett. 93 100502
[13] Feng M 2001 J. Phys. B: At. Mol. Opt. Phys. 34 451
[14] Haljan P C, Brickman K -A, Deslauriers L, Lee P J and Monroe C 2005 Phys. Rev. Lett. 94 153602
[15] Crisp M D 1991 Phys. Rev. A 43 2430
[16] Bloch F and Siegert A 1940 Phys. Rev. 57 522
[17] Janowicz M and Orlowski A 2004 Rep. Math. Phys. 54 71
[18] Phoenix S J D 1989 J. Mod. Optics 3 127
[19] Leibfried D, Blatt R, Monroe C and Wineland D J 2003 Rev. Mod. Phys. 75 281
[20] Zaheer K and Zubairy M S 1998 Phys. Rev. A 37 1628
[21] Irish E K, Gea-Banacloche J, Martin I and Schwab K C 2005 Phys. Rev. B 72 195410