Spectral Telescope: Convergence Rate
Bounds for Random-Scan Gibbs Samplers
Based on a Hierarchical Structure

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Abstract

Random-scan Gibbs samplers possess a natural hierarchical structure. The structure connects Gibbs samplers targeting higher dimensional distributions to those targeting lower dimensional ones. This leads to a quasi-telescoping property of their spectral gaps. Based on this property, we derive three new bounds on the spectral gaps and convergence rates of Gibbs samplers on general domains. The three bounds relate a chain’s spectral gap to, respectively, the correlation structure of the target distribution, a class of random walk chains, and a collection of influence matrices. Notably, one of our results generalizes the technique of spectral independence, which has received considerable attention for its success on finite domains, to general state spaces. We illustrate our methods through a sampler targeting the uniform distribution on a corner of an $n$-cube.
1 Introduction

Gibbs samplers are among the most popular Markov chain Monte Carlo (MCMC) approaches to sample from multivariate probability distributions. They have been applied and studied for sampling, counting, inference, and optimization in a variety of disciplines, including mathematics, statistics, physics, and computer science. This work concerns theoretical properties of random-scan Gibbs samplers, also known as Glauber dynamics. Our key observation is that these types of samplers possess a natural hierarchical, or recursive, structure that facilitates convergence analysis of the underlying Markov chains. Exploiting this structure, we derive a quasi-telescoping property for the spectral gaps of these chains, which leads to several new convergence rate bounds.

Our motivation mainly stems from the spectral independence technique recently developed in the theoretical computer science community, which we now briefly review. Spectral independence was initially introduced in Anari et al. (2021b) to establish a polynomial mixing time of the Gibbs sampler for hardcore models. It has since received a tremendous amount of attention in computer science as it provides a powerful tool for proving fast, and sometimes optimal, mixing time bounds for Gibbs samplers for several important discrete models. It is particularly useful for samplers with many components, and despite being very recently developed, it is already regarded as an attractive alternative to more traditional tools for convergence analysis, such as Dobrushin’s uniqueness condition. See Feng et al. (2021), Chen et al. (2021), Jain et al. (2021), Chen et al. (2022b), Blanca et al. (2022), Chen et al. (2022a), and
the references therein. In the original framework, spectral independence was defined to bound the spectral gaps of samplers on the Boolean domain \( \{0, 1\}^n \), but it has since been improved and extended in various directions. Some notable extensions include entropy factorization (Chen et al., 2021; Blanca et al., 2022), entropic independence (Anari et al., 2021a, 2022), localization schemes (Chen and Eldan, 2022), and spectral independence on general finite domains (Feng et al., 2021).

On continuous domains, convergence analysis of Gibbs samplers with many components remains challenging, despite impressive analyses for some interesting models (Roberts and Sahu, 1997; Smith, 2014; Pillai and Smith, 2017, 2018; Janvresse, 2001; Carlen et al., 2003). Practically speaking, the only existing tools that are designed specifically for convergence analysis of Gibbs samplers on general state spaces are based on the classical Dobrushin’s uniqueness condition (see Wang and Wu, 2014, and references therein). A framework that can be applied to chains outside finite domains thus seems ever so appealing.

The main contribution of this paper is Theorem 2, which describes the aforementioned quasi-telescoping property concerning the spectral gaps of Gibbs samplers on general state spaces. We refer to this property as “the spectral telescope.” Derived from a hierarchical structure of Gibbs samplers, the spectral telescope puts forward a flexible framework for bounding the spectral gap for these samplers on both discrete and continuous state spaces. Based on it, we construct three types of bounds, given in Corollaries 3 to 5. These three corollaries connect the spectral gap to, respectively, the correlation (dependence) structure of the target distribution, a collection of low-dimensional random walk chains, and a set of “influence matrices” which define a spectral-independence-type condition. Corollaries 4 and 5 extend/generalize results in Alev and Lau (2020) and Feng et al. (2021), while Corollary 3 appears to be new even for finite state spaces. In particular, Corollary 5 generalizes Feng et al.’s (2021)
spectral-independence-based bound in two ways. Firstly, Feng et al.’s (2021) result is extended from finite state spaces to general ones. Moreover, whereas Feng et al. (2021) calculate influence matrices based on total variation distances between conditional distributions, Corollary 5 uses influence matrices based on a more general class of Wasserstein divergences. Compared to methods based on the total variation distance, Wasserstein-based methods are often more effective for convergence analysis of Markov chains in high-dimensional settings (see, e.g., Hairer et al., 2011; Qin and Hobert, 2022).

Theorem 2 and its corollaries arm us with techniques to bound the spectral gap beyond those relying on spectral independence. These techniques can further be combined with various tools, such as orthogonal polynomials (see, e.g., Diaconis et al., 2008) and one-shot coupling (see, e.g., Roberts and Rosenthal, 2002), to attain broader applicability. This is illustrated by a concrete example in Section 4. In this example, we study a random-scan Gibbs sampler targeting the uniform distribution on the corner of an $n$-cube. We first invoke Corollary 3 and establish a tight spectral gap bound by analyzing the correlation structure of the target distribution using orthogonal polynomials. In contrast, a straightforward generalization of spectral independence where the influence matrices are calculated from total variation distances would give only trivial bounds. A second non-trivial, but sub-optimal bound is obtained using Corollary 5, where we utilize spectral independence based on suitable Wasserstein divergences. The example also shows that constructing a tight bound via our method requires adequate information on the target distribution. Our method is not a panacea, but rather one of the many steps towards understanding the convergence properties of Gibbs samplers. Applying it to Gibbs samplers in various fields is a direction for future research.

Properties similar to the spectral telescope have been derived for some models
prior to our research. In particular, the spectral telescope is reminiscent of an inductive property of spectral gaps for Kac models, which are commonly used to study the distribution of physical particles (Carlen et al., 2003, Theorem 2.2). The general mathematical setting in Carlen et al. (2003) is quite different from ours, but some of the models they studied can be thought as Gibbs samplers whose target distributions satisfy certain symmetric properties.

The rest of this paper is organized as follows. In the remainder of this section, we briefly explain the hierarchical structure of Gibbs samplers, without getting into technical details. After introducing some preliminaries in Section 2, we formally define the Gibbs algorithm, describe its hierarchical structure, and state our main results in Section 3. Section 4 contains the aforementioned example. The detailed proofs of our main results are provided in Section 5.

1.1 The hierarchical structure: High level ideas

Now we briefly explain the hierarchical (or recursive) structure of the random-scan Gibbs sampler, and defer the formal descriptions to Section 3. Let $X_1, \ldots, X_n$ be random elements whose joint distribution is $\Pi$. For $i \in [n] := \{1, \ldots, n\}$ and $x$ in the range of $X_i$, let $\Pi_{\{i\} \setminus \{i\}}(\cdot \mid x)$ denote the conditional distribution of

$$(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$$

given $X_i = x$. Consider a Gibbs algorithm targeting $\Pi$ that updates one component at a time. Given the current state $(x_1, \ldots, x_n)$, in each iteration, the sampler randomly and uniformly selects one component to update using its full conditional distribution. Of course, selecting one component to update is equivalent to selecting $n - 1$ components to fix. This is, in turn, equivalent to selecting one component, say $x_i$, to fix, and then calling one step of the Gibbs sampler targeting $\Pi_{\{i\} \setminus \{i\}}(\cdot \mid x_i)$,
which randomly selects \( n - 2 \) of the remaining components to fix, and updates the component that was not selected. We can then rewrite the Gibbs sampler as a recursive algorithm, as we can replace \( \Pi \) with \( \Pi_{-\{i\}\{|i\}}(\cdot | x_i) \), and repeat the argument until the target distribution is univariate.

For illustration, suppose that \( n = 4 \), and the current state is \((x_1, \ldots, x_4)\). One step of the Gibbs sampler targeting \( \Pi \) proceeds as follows:

1. Randomly and uniformly select an index \( j \) from \([4] = \{1, 2, 3, 4\}\).
2. Update \( x_j \).

This is equivalent to the following procedure:

1'. Randomly and uniformly select an index \( i_1 \) from \([4]\).
2'. Randomly and uniformly select an index \( i_2 \) from \([4] \setminus \{i_1\}\).
3'. Randomly and uniformly select an index \( i_3 \) from \([4] \setminus \{i_1, i_2\}\).
4'. Update \( x_j \), where \( \{j\} = [4] \setminus \{i_1, i_2, i_3\} \).

Note the hierarchical structure: Steps 2'-4' form one step of the Gibbs sampler targeting \( \Pi_{-\{i_1\}\{|i_1\}}(\cdot | x_{i_1}) \). Step 3'-4' make up one step of the Gibbs sampler targeting the conditional distribution of \( X_{i_3} \) and \( X_j \) given \( X_{i_1} = x_{i_1} \) and \( X_{i_2} = x_{i_2} \). Finally, step 4' alone can be regarded as one step of the Gibbs sampler targeting the conditional distribution of \( X_j \) given all other components.

As we will see, this hierarchical structure not only reformulates the original Gibbs sampler, but also leads to non-trivial bounds on the spectral gap.
2 Preliminaries

Consider a probability space \((E, F, \nu)\). We use \(L^2(\nu)\) to denote the set of measurable functions \(f : E \rightarrow \mathbb{R}\) such that

\[
\int_E f(x)^2 \nu(dx) < \infty.
\]

For \(f, g \in L^2(\nu)\), one can define their inner product

\[
\langle f, g \rangle_\nu = \int_E f(x)g(x) \nu(dx).
\]

In particular, the \(L^2\) norm of a function \(f \in L^2(\nu)\) is \(\|f\|_\nu = \sqrt{\langle f, f \rangle_\nu}\). Two functions in \(L^2(\nu)\) are equal if their difference has a vanishing \(L^2\) norm. \(L^2(\nu)\) forms a Hilbert space. We use \(L^2_0(\nu)\) to denote the subspace of \(L^2(\nu)\) consisting of functions \(f\) such that

\[
\langle f, 1 \rangle_\nu = \int_E f(x) \nu(dx) = 0.
\]

Also, \(L^2_*(\nu)\) is used to denote the set of probability measures \(\omega : F \rightarrow [0, 1]\) such that \(\omega\) is absolutely continuous with respect to \(\nu\), and that \(d\omega/d\nu \in L^2(\nu)\). For \(\omega_1, \omega_2 \in L^2_*(\nu)\), their \(L^2\) distance is

\[
\|\omega_1 - \omega_2\|_\nu = \left\| \frac{d\omega_1}{d\nu} - \frac{d\omega_2}{d\nu} \right\|_\nu.
\]

Let \(K : E \times F \rightarrow [0, 1]\) be a transition kernel that describes the transition law of a Markov chain \((X(t))_{t=0}^\infty\). We say that \(\nu\) is a stationary distribution of \((X(t))\) if

\[
\nu K(\cdot) := \int_E K(x, \cdot) \nu(dx) = \nu(\cdot).
\]

Suppose that \(\nu K = \nu\). For \(f \in L^2(\nu)\) and \(x \in E\), define

\[
K f(x) = \int_E f(x') K(x, dx').
\]
If \( f \in L^2_0(\nu) \), then \( Kf \in L^2_0(\nu) \). Then we can view \( K \) as a linear operator on \( L^2_0(\nu) \), referred to as a Markov operator. Using Cauchy-Schwarz, one can show that

\[
\|K\|_\nu := \sup_{f \in L^2_0(\nu), \ f \neq 0} \frac{\|Kf\|_\nu}{\|f\|_\nu} \leq 1,
\]

where \( \|K\|_\nu \) is called the \( L^2 \) norm of \( K \).

The above framework is particularly useful in the study of reversible chains. A chain associated with \( K \), where \( \nu K = \nu \), is said to be reversible with respect to \( \nu \) if \( K \) is self-adjoint, i.e., for \( f, g \in L^2_0(\nu) \),

\[
\langle Kf, g \rangle_\nu = \langle f, Kg \rangle_\nu.
\]

All chains studied in this paper are reversible with respect to their respective stationary distributions. Suppose that \( K \) is self-adjoint. Then the spectral gap of \( K \) (or that of a chain associated with \( K \)) is \( 1 - \|K\|_\nu \). The magnitude of the spectral gap governs how fast a Markov chain associated with \( K \) converges to its stationary distribution \( \nu \), with a larger gap indicating faster convergence. Indeed, the following well-known result states that \( \|K\|_\nu \) is in fact the \( L^2 \) convergence rate of the chain.

**Lemma 1.** (Roberts and Rosenthal, 1997) Let \( (X(t))_{t=0}^{\infty} \) be a chain reversible with respect to \( \nu \) and let \( K \) be its Markov operator. For \( \omega \in L^2_0(\nu) \) and \( t \geq 0 \), let \( \omega K^t \) be the distribution of \( X(t) \) if \( X(0) \sim \omega \). Then, for \( \rho < 1 \), \( \|K\|_\nu \leq \rho \) if and only if the following holds: For \( \omega \in L^2_0(\nu) \), there exists a constant \( C_\omega < \infty \) such that, for \( t \geq 0 \),

\[
\|\omega K^t - \nu\|_\nu \leq C_\omega \rho^t.
\]

The Markov operator \( K \) is said to be positive semi-definite if it is self-adjoint, and \( \langle f, Kf \rangle_\nu \geq 0 \) for \( f \in L^2_0(\nu) \). In this case, the following formula holds:

\[
\|K\|_\nu = \sup_{f \in L^2_0(\nu), \ f \neq 0} \frac{\langle f, Kf \rangle_\nu}{\|f\|_\nu^2}.
\]
It is well-known (see, e.g., Liu et al., 1995) that operators of random-scan Gibbs algorithms are positive semi-definite.

3 A Hierarchical Structure

3.1 Gibbs samplers and their recursive forms

Let \((X_1, \mathcal{B}_1, \mu_1), \ldots, (X_n, \mathcal{B}_n, \mu_n)\) be \(\sigma\)-finite measure spaces, where \(n\) is a positive integer that is at least 2. Assume that in each space, singletons are measurable. Suppose that, for \(i = 1, \ldots, n\), \(X_i\) is an \(X_i\)-valued random element, and that \(X = (X_1, \ldots, X_n)\) has a joint distribution \(\Pi\).

Assume that \(\Pi\) is absolutely continuous with respect to the base measure \(\mu_1 \times \cdots \times \mu_n\) with Radon-Nikodym derivative (density) \(\pi\), so that \(\pi\) is a measurable function on \(X := X_1 \times \cdots \times X_n\). Although Radon-Nikodym derivatives only need to be defined outside a null set, for concreteness we insist that \(\pi\) is specified everywhere on \(X\). While these assumptions would seem more rigid than necessary, they bring a great deal of technical and notational convenience. For a nonempty set of indices \(\Gamma = \{\gamma_1, \ldots, \gamma_{|\Gamma|}\} \subset [n]\), where \(\gamma_1 < \cdots < \gamma_{|\Gamma|}\), let \(X_\Gamma = X_{\gamma_1} \times \cdots \times X_{\gamma_{|\Gamma|}}\), \(\mu_\Gamma = \mu_{\gamma_1} \times \cdots \times \mu_{\gamma_{|\Gamma|}}\), and \(X_\Gamma = (X_{\gamma_1}, \ldots, X_{\gamma_{|\Gamma|}})\). Also, for \(\Gamma\) given above and \(x = (x_1, \ldots, x_n) \in X\), where \(x_i \in X_i\) for each \(i\), let \(x_\Gamma = (x_{\gamma_1}, \ldots, x_{\gamma_{|\Gamma|}})\). For any \(\Gamma\) such that \(1 \leq |\Gamma| \leq n - 1\), the marginal density of \(X_\Gamma\) evaluated at any \(y \in X_\Gamma\) is

\[
\pi_\Gamma(y) = \int_{X_{-\Gamma}} \pi(x) \, dx_{-\Gamma}, \quad \text{where } x \in X \text{ satisfies } x_\Gamma = y.
\]

In the above equation, \(-\Gamma = [n] \setminus \Gamma\), and \(dx_{-\Gamma}\) is a short-hand notation for \(d\mu_{-\Gamma}\). By convention, \(\pi_{[n]} = \pi\). For nonempty sets \(\Lambda, \Gamma \subset [n]\) such that \(\Lambda \cap \Gamma = \emptyset\), the conditional density of \(X_\Gamma\) given \(X_\Lambda = y \in X_\Lambda\), denoted by \(\pi_{\Gamma|\Lambda}(\cdot \mid y)\), is defined for
$y \in X_\Lambda$ such that $\pi_\Lambda(y) > 0$, and given by

$$
\pi_{\Gamma|\Lambda}(z \mid y) = \frac{\pi_{\Lambda \cup \Gamma}(x_{\Lambda \cup \Gamma})}{\pi_\Lambda(y)}, \quad \text{where } x \in X \text{ satisfies } x_\Lambda = y \text{ and } x_\Gamma = z.
$$

If $\pi_\Lambda(y) = 0$, we let $\pi_{\Gamma|\Lambda}(\cdot \mid y)$ be an arbitrary probability density function on $X_\Gamma$. By convention, if $\Lambda = \emptyset$, $\pi_{\Gamma|\Lambda}(\cdot \mid y)$ means $\pi_\Gamma(\cdot)$, even though $y \in X_\emptyset$ cannot be specified.

A random-scan Gibbs sampler targeting $\pi$ with block size $l$ is described in Algorithm 1. In short, given the current state $x \in X$, the sampler randomly selects a subset $\Gamma$ of indices, and updates $x_\Gamma$ using the conditional distribution of $X_\Gamma$ given $X_{-\Gamma} = x_{-\Gamma}$. In many applications, a block size of 1 is used because it becomes more difficult to draw from the corresponding full conditional distributions when the block size increases. Regardless of the block size, the underlying Markov chain is reversible with respect to $\pi$.

**Algorithm 1** One step of the Gibbs sampler targeting $\pi$, block size $l$, where $l \in \{1, \ldots, n\}$:

- **Input:** Current state $x \in X$.
  
  Randomly and uniformly choose a subset of indices $\Gamma \subset [n]$ under the constraint $|\Gamma| = l$.
  
  Draw $w \in X_\Gamma$ from $\pi_{\Gamma|\Gamma}(\cdot \mid x_{-\Gamma})$.
  
  Let $x' \in X$ be such that $x'_\Gamma = w$ and $x'_{-\Gamma} = x_{-\Gamma}$.

- **Return:** Next state $x'$.

Algorithm 1 is a special case of Algorithm 2, which follows the same procedure, but targets $\pi_{-\Lambda|\Lambda}(\cdot \mid y)$ for some $\Lambda \subset [n]$ such that $|\Lambda| \leq n - 1$ and $y \in X_\Lambda$. When $\pi_\Lambda(y) > 0$, the underlying Markov chain is reversible with respect to $\pi_{-\Lambda|\Lambda}(\cdot \mid y)$. Taking $\Lambda = \emptyset$ in Algorithm 2 yields Algorithm 1.
Algorithm 2 One step of the Gibbs sampler targeting $\pi_{-\Lambda|\Lambda}(\cdot | y)$, block size $l$, where $l \in \{1, \ldots, n - |\Lambda|\}$:

**Input:** Current state $z \in X_{-\Lambda}$.

Let $x \in X$ be such that $x_{\Lambda} = y$ and $x_{-\Lambda} = z$.

Randomly and uniformly choose a set of indices $\Gamma \subset -\Lambda$ under the constraint $|\Gamma| = l$.

Draw $w \in X_{\Gamma}$ from $\pi_{\Gamma|-\Gamma}(\cdot | x_{-\Gamma})$.

Let $x' \in X$ be such that $x'_{\Gamma} = w$ and $x'_{-\Gamma} = x_{-\Gamma}$.

**Return:** New state $z' = x'_{-\Lambda}$.

Algorithm 3 One step of the recursive Gibbs sampler targeting $\pi_{-\Lambda|\Lambda}(\cdot | y)$, block size $l$, where $l \in \{1, \ldots, n - |\Lambda|\}$:

**Input:** Current state $z \in X_{-\Lambda}$.

Let $x \in X$ be such that $x_{\Lambda} = y$ and $x_{-\Lambda} = z$.

if $|\Lambda| = n - l$ then

Draw $z' \in X_{-\Lambda}$ from $\pi_{-\Lambda\Lambda}(\cdot | x_{\Lambda})$.

**Return:** New state $z'$.

else

Randomly and uniformly choose a coordinate $i \in -\Lambda$.

Draw $w \in X_{-(\Lambda \cup \{i\})}$ by running one step of the recursive Gibbs sampler targeting $\pi_{-(\Lambda \cup \{i\})|\Lambda\cup\{i\}}(\cdot | x_{\Lambda \cup \{i\}})$ with block size $l$ and current state $x_{-(\Lambda \cup \{i\})}$.

Let $x' \in X$ be such that $x'_{-(\Lambda \cup \{i\})} = w$ and $x'_{\Lambda \cup \{i\}} = x_{\Lambda \cup \{i\}}$.

**Return:** New state $z' = x'_{-\Lambda}$.

end if
Our analysis begins with the observation that Algorithm 2 has a hierarchical, or recursive structure. Indeed, following arguments given in Section 1.1, we see that Algorithm 2 can be written into a recursive form as in Algorithm 3. In particular, Algorithm 1 is equivalent to Algorithm 3 for $\Lambda = \emptyset$.

Consider the significance of the recursive representation. It connects the Gibbs sampler targeting $\pi_{-\Lambda|\Lambda}$ to ones targeting lower dimensional distributions, given by $\pi_{-\Lambda'|\Lambda'}$ where $\Lambda' \supset \Lambda$. While one would rarely implement the recursive algorithm in practice, based on it we can establish multiple intriguing relations concerning the convergence rate and spectral gap of the standard Algorithm 1. We now list these relations. The detailed derivation is given in Section 5.

3.2 The spectral telescope

For $\Lambda \subset [n]$ such that $|\Lambda| \in \{0, \ldots, n-1\}$, $y \in X_\Lambda$, and $l \in \{1, \ldots, n - |\Lambda|\}$, let $\text{gap}(\Lambda, y, l)$ be the spectral gap associated with Algorithm 2. For $l \in \{1, \ldots, n\}$ and $m \in \{l, \ldots, n\}$, let

$$\text{Gap}(m, l) = \min_{\Lambda \subset [n]} \inf_{y \in X_\Lambda} \text{gap}(\Lambda, y, l).$$

In particular, $\text{Gap}(n, l)$ is simply the spectral gap of Algorithm 1. Our main result is a consequence of the hierarchical structure of Gibbs samplers.

**Theorem 2** (Spectral Telescope). For $l \in \{1, \ldots, n-1\}$ and $m \in \{l+1, \ldots, n\}$,

$$\text{Gap}(m, l) \geq \text{Gap}(m, m-1) \text{Gap}(m-1, l).$$

In particular, for $l \in \{1, \ldots, n-1\}$,

$$\text{Gap}(n, l) \geq \prod_{m=l+1}^{n} \text{Gap}(m, m-1).$$
Theorem 2 describes a quasi-telescoping property of the sequence \( \text{Gap}(n, n - 1), \ldots, \text{Gap}(2, 1) \). We dub it “the spectral telescope.” From here we see that it is possible to bound \( \text{Gap}(n, l) \) from below via lower bounds on \( \text{Gap}(m, m - 1) \) for \( m \in \{l + 1, \ldots, n\} \). All other major results in this section are obtained via this strategy.

3.3 Spectral gaps and correlation coefficients

Let \((Y_1, \ldots, Y_m)\) be a vector of random elements taking values in a product space \( Y_1 \times \cdots \times Y_m \). For \( i = 1, \ldots, m \), let \( \varpi_i \) be the marginal distribution of \( Y_i \), and note that \( L_0^2(\varpi_i) \) represents the collection of real functions \( f \) on \( Y_i \) such that

\[
\mathbb{E}[f(Y_i)] = \int_{Y_i} f(y) \varpi_i(dy) = 0, \quad \mathbb{E}[f(Y_i)^2] = \int_{Y_i} f(y)^2 \varpi_i(dy) < \infty.
\]

Define the summation-based correlation coefficient of \((Y_1, \ldots, Y_m)\) to be

\[
s^*(Y_1, \ldots, Y_m) = \sup_{f_i \in L_0^2(\varpi_i) \forall i \exists i \text{ s.t. } \mathbb{E}[f_i(Y_i)^2] > 0} \frac{\mathbb{E}\left[\left|\sum_{i=1}^{m} f_i(Y_i)\right|^2\right]}{m \sum_{i=1}^{m} \mathbb{E}[f_i(Y_i)^2]}.
\]

\( s^*(Y_1, \ldots, Y_m) \) can range from \( 1/m \) to 1. To establish the lower bound, take \( f_i = 0 \) for \( i \geq 2 \). To establish the upper bound, use Cauchy-Schwarz. If \( Y_1, \ldots, Y_m \) are independent, then this coefficient is \( 1/m \). If there exists a sequence of functions \( f_1, \ldots, f_m \) such that \( f_i \in L_0^2(\varpi_i) \) and \( f_i \neq 0 \) for \( i \in [m] \), and \( f_i(Y_i) = f_j(Y_j) \) for \( i, j \in [m] \), then the correlation coefficient is 1.

For \( \Lambda \subset [n] \) such that \( |\Lambda| \leq n - 1 \) and \( y \in X_\Lambda \), let

\[
s(\Lambda, y) = s^*(Y_1, \ldots, Y_{n-|\Lambda|}),
\]

where \((Y_1, \ldots, Y_{n-|\Lambda|})\) is distributed according to \( \pi_{-\Lambda|\Lambda}(\cdot \mid y) \), i.e., the conditional
distribution of $X_\Lambda$ given $X_\Lambda = y$. For $m \in \{2, \ldots, n\}$, let

$$S(m) = \max_{\Lambda \subseteq [n]} \sup_{y \in X_\Lambda} s(\Lambda, y).$$

Then the following holds.

**Corollary 3.** For $m \in \{2, \ldots, n\}$,

$$\text{Gap}(m, m - 1) \geq 1 - S(m).$$

As a result, by Theorem 2 for $l \in \{1, \ldots, n - 1\}$,

$$\text{Gap}(n, l) \geq \prod_{m=l+1}^{n} [1 - S(m)].$$

This result relates the convergence rate of Algorithm 1 to the dependence structure of $\Pi$.

### 3.4 Spectral gaps and random walks

Let $\Lambda \subseteq [n]$ be such that $|\Lambda| \leq n - 1$, and let $y \in X_\Lambda$. We can define a random walk on the space $\bigcup_{i \in -\Lambda} \{i\} \times X_i$, given by Algorithm 4. Whenever $\pi_\Lambda(y) > 0$, this is a Markov chain reversible with respect to the probability measure $\varphi_{\Lambda, y}$ given by

$$\varphi_{\Lambda, y}(\{i\} \times A) = \frac{1}{n - |\Lambda|} \int_A \pi_{\{i\} \setminus \Lambda}(x \mid y) \, dx,$$

where $dx$ is a short-hand notation for $\mu_i(dx)$. Let $g(\Lambda, y)$ be the spectral gap of this chain. For $m \in \{2, \ldots, n\}$, let

$$G(m) = \min_{\Lambda \subseteq [n]} \inf_{y \in X_\Lambda} g(\Lambda, y).$$

We then have the following result:
Algorithm 4 One step of a random walk associated with \( \Lambda \subset [n] \) and \( y \in X_{\Lambda} \):

**Input:** Current state \( (j, x) \in \bigcup_{i \in -\Lambda} \{i\} \times X_i \).

Let \( x \in X \) be such that \( x_{\Lambda} = y \) and \( x_{(j)} = x \).

Randomly and uniformly choose a coordinate \( j' \in -\Lambda \).

**if** \( j' = j \) **then**

Set \( x' = x \).

**else**

Draw \( x' \in X_{\{j'\}} \) from \( \pi_{\{j'\} | \Lambda \cup \{j\}}(\cdot \mid x_{\Lambda \cup \{j\}}) \).

**end if**

**Return:** New State \( (j', x') \).

**Corollary 4.** For \( m \in \{2, \ldots, n\} \),

\[
\text{Gap}(m, m - 1) \geq G(m).
\]

As a result, by Theorem 2, for \( l \in \{1, \ldots, n - 1\} \),

\[
\text{Gap}(n, l) \geq \prod_{m=l+1}^{n} G(m).
\]

This extends a result in Alev and Lau (2020), which concerns random walks on pure simplicial complexes, a discrete structure frequently studied in computer science.

### 3.5 Spectral independence

Let \( \Lambda \subset [n] \) be such that \( |\Lambda| \leq n - 2 \), and let \( y \in X_{\Lambda} \) be such that \( \pi_{\Lambda}(y) > 0 \). Suppose that, for \( i \in -\Lambda \), there is a measurable “distance-like” function \( d_{\Lambda, X, i} : X_i \times X_i \to [0, \infty) \) such that (i) \( x = y \) if and only if \( d_{\Lambda, X, i}(x, y) = 0 \), and (ii) \( d_{\Lambda, X, i}(x, y) = d_{\Lambda, X, i}(y, x) \) for \( x, y \in X_i \). For \( j \in -\Lambda \) such that \( i \neq j \) and \( x \in X_i \), let \( \Pi_{\Lambda, X, i, x} \) be the probability measure associated with \( \pi_{\{j\} | \Lambda \cup \{i\}}(\cdot \mid x_{\Lambda \cup \{i\}}) \), where \( x_{\Lambda} = y \) and...
\( x_{(i)} = x \). In other words, \( \Pi^j_{\Lambda, y, i, x} \) is the conditional distribution of \( X_j \) given \( X_i = x \) and \( X_\Lambda = y \). Assume that the following conditions hold:

**H1** For \( i \in -\Lambda \),
\[
\int_{X_i} \left[ \int_{X_i} d_{\Lambda, y, i} (x, x') \, \pi_{\{i\}|\Lambda} (dx \mid y) \right]^2 \pi_{\{i\}|\Lambda} (dx' \mid y) < \infty.
\]

**H2** There exists \( k < \infty \) such that, for \( i, j \in -\Lambda \) satisfying \( i \neq j \) and \( \pi_{\{i\}|\Lambda} (\cdot \mid y) \)-almost every \( x, x' \in X_i \),
\[
d_{TV}(\Pi^j_{\Lambda, y, i, x}, \Pi^j_{\Lambda, y, i, x'}) \leq kd_{\Lambda, y, i}(x, x'),
\]
where \( d_{TV} \) denotes the total variation distance, which is the maximal difference between the probabilities of a measurable set assigned by the two probability measures. The constant \( k \) may depend on \( (\Lambda, y) \) but not on \( (i, j, x, x') \).

Note that if, for \( i \in -\Lambda \), \( d_{\Lambda, y, i} \) is the discrete metric, i.e., \( d_{\Lambda, y, i}(x, x') = 1_{x \neq x'} \), then \( \text{(H1)} \) and \( \text{(H2)} \) are satisfied.

For two probability distributions \( \nu_1 \) and \( \nu_2 \) on \( B_i \), where \( i \in [n] \), denote by \( C(\nu_1, \nu_2) \) the collection of couplings of \( \nu_1 \) and \( \nu_2 \). That is, \( \nu \in C(\nu_1, \nu_2) \) if and only if \( \nu \) is a probability measure on \( B_i \times B_i \) such that \( \nu(A \times X_i) = \nu_1(A) \) and \( \nu(X_i \times A) = \nu_2(A) \) for \( A \in B_i \).

A coupling kernel associated with \( (\Lambda, y, i, j) \), where \( i, j \in -\Lambda \) and \( i \neq j \), is a Markov transition kernel \( K_{i,j} : X_i \times X_i \to B_j \times B_j \) such that \( K_{i,j}((x, x'), \cdot) \) is a probability measure in \( C(\Pi^j_{\Lambda, y, i, x}, \Pi^j_{\Lambda, y, i, x'}) \) for \( x, x' \in X_i \). (Of course, \( K_{i,j} \) also depends on \( \Lambda \) and \( y \), but to suppress notation we do not include them in the subscript. The same goes for \( \phi_{i,j} \) given below.) We say that a contraction condition holds for \( (\Lambda, y, i, j) \) with coefficient \( \phi_{i,j} \in [0, \infty) \) if there exists a coupling kernel \( K_{i,j} \) associated
with \((\Lambda, y, i, j)\) such that
\[
\int_{X_j \times X_j} d_{\Lambda,y,j}(x'', x''') K_{i,j} ((x, x'), d(x'', x''')) \leq \phi_{i,j} d_{\Lambda,y,i}(x, x')
\] (1)
for \(\pi_{\{i\}\Lambda}(. \mid y)\)-almost every \(x, x' \in X_i\). Note that (1) implies that the Wasserstein divergence induced by \(d_{\Lambda,y,j}\) between \(\Pi_{\Lambda,y,i,x}^j\) and \(\Pi_{\Lambda,y,i,x'}^j\) is upper bounded by \(\phi_{i,j} d_{\Lambda,y,i}(x, x')\). In particular, if \(d_{\Lambda,y,i}\) and \(d_{\Lambda,y,j}\) are the discrete metric, then (1) is equivalent to contraction in the total variation distance, i.e.,
\[
d_{TV}(\Pi_{\Lambda,y,i,x}^j, \Pi_{\Lambda,y,i,x'}^j) \leq \phi_{i,j} 1_{x \neq x'}.
\]

An influence matrix associated with \((\Lambda, y)\), denoted by \(\Phi(\Lambda, y)\), is a square matrix of dimension \(n - |\Lambda|\) whose \(i,j\)th element (where \(i \neq j\)) is the contraction coefficient \(\phi_{i,j}\) given above, assuming that a contraction condition holds for \((\Lambda, y, i, j)\). The diagonal elements of \(\Phi(\Lambda, y)\) are set to be zero. Let \(r(\Phi(\Lambda, y))\) be the spectral radius of \(\Phi(\Lambda, y)\).

Now, allow \(\Lambda\) and \(y\) to vary. Given \(l \in \{1, \ldots, n - 1\}\) and \((\eta_{l+1}, \ldots, \eta_n) \in \mathbb{R}^{n-l}\) such that \(\eta_m < m - 1\) for each \(m\), we say that the full joint distribution \(\Pi\) is \((\eta_{l+1}, \ldots, \eta_n)\)-spectrally independent if the following holds: For every \(m \in \{l + 1, \ldots, n\}\), \(\Lambda \subset [n]\) such that \(|\Lambda| = n - m\), and \(y \in X_\Lambda\) such that \(\pi_\Lambda(y) > 0\), there exists an influence matrix \(\Phi(\Lambda, y)\) associated with \((\Lambda, y)\) such that \(r(\Phi(\Lambda, y)) \leq \eta_m\).

Recently, spectral independence has received tremendous attention in the theoretical computer science community. It is regarded as a potentially powerful tool for bounding the spectral gaps of Gibbs chains. All existing works on this topic focus on chains on finite state spaces. Moreover, the distance-like function \(d_{\Lambda,y,i}\) is always set to be the discrete metric. Our next corollary extends existing results, particularly Feng et al.’s (2021) Theorem 3.1, with regard to these two aspects.
Corollary 5. Let $m \in \{2, \ldots, n\}$. Suppose that, for $\Lambda \subset [n]$ such that $n - |\Lambda| = m$ and $y \in X_\Lambda$ such that $\pi_\Lambda(y) > 0$, there is an influence matrix $\Phi(\Lambda, y)$ associated with $(\Lambda, y)$ such that

$$r(\Phi(\Lambda, y)) \leq \eta,$$

where $\eta < m - 1$. Then

$$\text{Gap}(m, m - 1) \geq \frac{m - 1}{m} - \frac{\eta}{m}.$$

In particular, it follows from Theorem 2 that, for $l \in \{1, \ldots, n-1\}$, if $\Pi$ is $(\eta_{l+1}, \ldots, \eta_n)$-spectrally independent, then

$$\text{Gap}(n, l) \geq \prod_{k=l+1}^{n} \left( \frac{k - 1}{k} - \frac{\eta_k}{k} \right) = \frac{l}{n} \prod_{k=l+1}^{n} \left( 1 - \frac{\eta_k}{k - 1} \right).$$

As will be seen from Section 5, Corollary 5 is derived from Corollary 4, which is in turn derived from Corollary 3, which is in turn derived from Theorem 2.

3.6 Additional remarks

We observe the lower bounds on $\text{Gap}(n, l)$ in Corollaries 3 and 5 are at most $l/n$. The quantity $l/n$ is in fact an upper bound on the spectral gap of the random-scan Gibbs sampler targeting $\pi$ with block size $l$. To see this, let $K$ be the Markov operator associated with the algorithm. Then, for $f \in L_0^2(\Pi)$ and $x \in X$, 

$$Kf(x) = \frac{1}{\binom{n}{l}} \sum_{\Gamma \subset [n]} E[f(X) \mid X_{-\Gamma} = x_{-\Gamma}].$$
Let \( f \in L^2_0(\Pi) \) be such that \( \|f\|_\Pi = \mathbb{E}[f(X)^2] = 1 \). Suppose that \( f(x) \) depends on \( x \in X \) only through \( x_{\{1\}} \). One can verify that, whenever \( n \geq l + 1 \),

\[
\langle f, Kf \rangle_\Pi = \frac{1}{\binom{n}{l}} \sum_{\Gamma \subset [n], |\Gamma| = l} \mathbb{E} \left\{ \mathbb{E} [f(X) \mid X_{-\Gamma}]^2 \right\} \\
\geq \frac{1}{\binom{n}{l}} \sum_{\Gamma \subset [n], |\Gamma| = l, 1 \notin \Gamma} \mathbb{E} \left\{ \mathbb{E} [f(X) \mid X_{-\Gamma}]^2 \right\} \\
= \frac{1}{\binom{n}{l}} \sum_{\Gamma \subset [n], |\Gamma| = l, 1 \notin \Gamma} \mathbb{E} [f(X)^2] \\
= \frac{n - l}{n}.
\]

Then the spectral gap satisfies

\[
1 - \|K\|_\Pi \leq 1 - \langle f, Kf \rangle_\Pi \leq \frac{l}{n}.
\]

Our framework leaves several interesting open questions and directions for further extension. It is unclear when the lower bound in Theorem 2 will give the exact spectral gap. Moreover, the existing bound relies on uniform lower bounds on the spectral gap of lower-dimensional Gibbs samplers. Generalizing existing results to position-dependent lower bounds may increase the applicability of our method.

Perhaps more importantly, in many models, \( \Pi \) does not have a Radon-Nikodym derivative \( \pi \). It seems that many of our results could still hold if we replace the existence of \( \pi \) with some weaker regularity conditions. However, establishing this rigorously would likely require extremely careful (and possibly tedious) analysis. This is an important topic for future studies.
4 An Example

The relations derived in Section 3 can be used to construct convergence bounds for Gibbs algorithms. The following example illustrates the strengths and limitations of this framework.

Let \( X_1 = \cdots = X_n = (0, 1) \), and let \( \mu_1 = \cdots = \mu_n \) be Lebesgue measures. Let

\[
\pi(x_1, \ldots, x_n) \propto \begin{cases} 
1 & \sum_{i=1}^{n} x_i < 1, \\
0 & \text{otherwise}.
\end{cases}
\]

That is, \( \pi \) corresponds to the uniform distribution on the corner of an \( n \)-cube given by

\[
\left\{ (x_1, \ldots, x_n) \in (0, 1)^n : \sum_{i=1}^{n} x_i < 1 \right\}.
\]

Consider the random-scan Gibbs algorithm targeting \( \pi \) with block size \( l = 1 \). In each iteration of the algorithm, given the current state \( x = (x_1, \ldots, x_n) \in X = (0, 1)^n \), where \( \sum_{i=1}^{n} x_i < 1 \), one randomly and uniformly selects \( i \in [n] \), then updates the value of \( x_{(i)} = x_i \) by drawing from the density

\[
\pi_{\{i\}|\{-i\}}(x \mid x_{\{-i\}}) = \frac{1}{1 - \sum_{j \in \{-i\}} x_j}, \quad x < 1 - \sum_{j \in \{-i\}} x_j.
\]

We will use Corollary 3 to construct a sharp lower bound on the spectral gap of this chain. We then briefly illustrates how Corollary 5 can be used to construct a similar but looser bound.

4.1 A spectral gap bound based on correlation coefficients

We will prove the following result for the chain in question.
Proposition 6. Let $m \in \{2, \ldots, n\}$. Let $\Lambda \subset [n]$ be such that $|\Lambda| = n - m$, and let $x = (x_1, \ldots, x_n) \in X$. Assume that $\sum_{i \in \Lambda} x_i < 1$. Then

$$s(\Lambda, x_\Lambda) \leq \begin{cases} 3/4 & m = 2, \\ 1/m + 2(m-1)/[(m+1)m^2] & m \geq 3. \end{cases}$$

By Corollary 3 when $n \geq 3$, the spectral gap satisfies

$$\text{Gap}(n, 1) \geq \frac{1}{4} \prod_{m=3}^{n} \left[ 1 - \frac{1}{m} - \frac{2(m-1)}{(m+1)m^2} \right].$$

Note that $1/m + 2(m-1)/[(m+1)m^2] \leq 1/(m-2)$. Thus, if $n \geq 4$, then

$$\text{Gap}(n, 1) \geq \frac{5}{36} \prod_{m=4}^{n} \frac{m-3}{m-2} = \frac{5}{36(n-2)}.$$

Recall that the spectral gap is upper bounded by $1/n$. Thus, the bound here gives the correct order as $n \to \infty$.

To prove Proposition 6, fix $m \in \{2, \ldots, n\}$, $\Lambda \subset [n]$ such that $|\Lambda| = n - m$, and $x = (x_1, \ldots, x_n) \in X$. Suppose that $\sum_{i \in \Lambda} x_i < 1$. Without loss of generality, assume that $-\Lambda = \{1, \ldots, m\}$. Let $Y_1, \ldots, Y_m$ be distributed as $\pi_{-\Lambda \mid \Lambda}(\cdot \mid x_\Lambda)$. For $i = 1, \ldots, m$, let $f_i \in L_0^2(\pi_i)$, where $\pi_i$ denotes the distribution given by the density

$$\pi_{(i)|\Lambda}(x \mid x_\Lambda) = \frac{m \left(1 - \sum_{j=m+1}^{n} x_j - x\right)^{m-1}}{\left(1 - \sum_{j=m+1}^{n} x_j\right)^m}, \quad x < 1 - \sum_{j=m+1}^{n} x_j. \tag{2}$$

It suffices to prove that

$$\mathbb{E} \left\{ \left[ \sum_{i=1}^{m} f_i(Y_i) \right]^2 \right\} \leq A_m \sum_{i=1}^{m} \mathbb{E} \left[ f_i(Y_i)^2 \right], \tag{3}$$

where

$$A_m = \begin{cases} 3/2 & m = 2, \\ 1 + 2(m-1)/[(m+1)m] & m \geq 3. \end{cases} \tag{4}$$
We will prove this using orthogonal polynomials. The techniques we employ are similar to those in Diaconis et al. (2008).

Let \( i, j \in -\Lambda \) be such that \( i \neq j \). Then, for \( x \in X_i \) such that \( \pi_{\{i\}|\Lambda}(x | x_\Lambda) > 0 \) and \( x' \in X_j \), the conditional density of \( Y_j \) given \( Y_i = x \) is

\[
\pi_{\{j\}|\Lambda \cup \{i\}}(x' | x, x_\Lambda) = \frac{(m-1)\left(1 - \sum_{a=m+1}^n x_a - x'\right)^{m-2}}{(1 - \sum_{a=m+1}^n x_a - x)^{m-1}}, \quad x' < 1 - \sum_{a=m+1}^n x_a - x,
\]

where \( (x, x_\Lambda) = (x, x_{m+1}, \ldots, x_n) \). For \( f \in L^2_0(\mathcal{W}_j) \), let \( P_{i,j}f \) be a function on \( X_i \) such that

\[
P_{i,j}f(x) = \int_{X_j} f(x')\pi_{\{j\}|\Lambda \cup \{i\}}(x' | x, x_\Lambda) \, dx', \quad x \in X_i.
\]

Using Cauchy-Schwarz, it is easy to show that \( P_{i,j}f \in L^2_0(\mathcal{W}_i) \). In fact, \( P_{i,j} : L^2_0(\mathcal{W}_j) \to L^2_0(\mathcal{W}_i) \) is a bounded linear transformation. Let \( P_{i,i} \) be the identity on \( L^2_0(\mathcal{W}_i) \). Then, for \( i, j \in -\Lambda \),

\[
\langle f_i, P_{i,j}f_j \rangle_{\mathcal{W}_i} = \langle P_{j,i}f_i, f_j \rangle_{\mathcal{W}_j} = \mathbb{E} [f_i(Y_i)f_j(Y_j)]. \tag{6}
\]

It follows that

\[
\sum_{i=1}^m \mathbb{E} [f_i(Y_i)^2] = \sum_{i=1}^m \langle f_i, P_{i,i}f_i \rangle_{\mathcal{W}_i} = \sum_{i=1}^m \|f_i\|_{\mathcal{W}_i}^2, \quad \sum_{i=1}^m \sum_{j=1}^m \langle f_i, P_{i,j}f_j \rangle_{\mathcal{W}_i}, \tag{7}
\]

Now, for a positive integer \( k \) and \( i, j \in -\Lambda \) such that \( i \neq j \), the following holds if \( \pi_{\{i\}|\Lambda}(x | x_\Lambda) > 0 \):

\[
\int_{X_j} x'^k \pi_{\{j\}|\Lambda \cup \{i\}}(x' | x, x_\Lambda) \, dx = \zeta_k x^k + q_{k-1}(x), \tag{8}
\]

where

\[
\zeta_k = \frac{(-1)^k k!(m-1)!}{(m+k-1)!},
\]

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and \( q_{k-1}(x) \) is a polynomial of \( x \) whose degree is \( k - 1 \). Standard arguments show that, for \( i \in \Lambda \), \( L_0^2(\varpi_i) \) has an orthonormal basis \( \{p_{i,k}\}_{k=1}^\infty \), where \( p_{i,k} \) is a polynomial function of degree \( k \). By (3), when \( i \neq j \),

\[
P_{i,j}p_{j,k} = \zeta_k p_{i,k} + r_{i,j,k-1},
\]

where \( r_{i,j,k-1} \) is in the span of \( \{p_{i,1}, \ldots, p_{i,k-1}\} \). Claim: for \( k \geq 1 \) and \( i \neq j \), \( r_{i,j,k} = 0 \). This can be proved through induction. The claim holds for \( k = 1 \), because the only polynomial of order 0 in \( L_0^2(\varpi_i) \) is 0. Assume that it holds for \( k = k' - 1 \geq 1 \). Then, by (6) and the fact that \( \{p_k\} \) is an orthonormal basis, for \( k = 1, \ldots, k' \) and \( i \neq j \),

\[
\langle r_{i,j,k'}, p_{i,k} \rangle_{\varpi_i} = \langle P_{i,j}p_{j,k'+1} - \zeta_{k'+1}p_{i,k'+1}, p_{i,k} \rangle_{\varpi_i} = \langle P_{i,j}p_{j,k'+1}, p_{i,k} \rangle_{\varpi_i} = \langle p_{j,k'+1}, p_{i,k} \rangle_{\varpi_j} = \zeta_k \langle p_{j,k'+1}, p_{j,k} \rangle_{\varpi_j} = 0.
\]

This implies that \( r_{i,j,k'} = 0 \). Thus, for \( k \geq 1 \) and \( i \neq j \),

\[
P_{i,j}p_{j,k} = \zeta_k p_{i,k}.
\]

For \( i \in \Lambda \), we can decompose \( f_i \in L_0^2(\varpi_i) \) into \( f_i = \sum_{k=1}^\infty a_{i,k}p_{i,k} \). Then (7) can be written as

\[
\sum_{i=1}^m \mathbb{E} \left[ f_i(Y_i)^2 \right] = \sum_{i=1}^m \sum_{k=1}^\infty a_{i,k}^2,
\]

\[
\mathbb{E} \left\{ \left[ \sum_{i=1}^m f_i(Y_i) \right]^2 \right\} = \sum_{i=1}^m \sum_{j=1}^m \sum_{k=1}^\infty a_{i,k} a_{j,k} [1_{i=j} + 1_{i \neq j} \zeta_k].
\]

Elementary matrix algebra shows that, given a positive integer \( k \),

\[
\sum_{i=1}^m \sum_{j=1}^m a_{i,k} a_{j,k} [1_{i=j} + 1_{i \neq j} \zeta_k] \leq \max \{1 - \zeta_k, 1 + (m-1)\zeta_k\} \sum_{i=1}^m a_{i,k}^2.
\]
It follows that
\[
\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{\infty} a_{i,k} a_{j,k} [1_{i=j} + 1_{i \neq j} \zeta_k] \leq \left( \sup_k \max \{1 - \zeta_k, 1 + (m - 1)\zeta_k\} \right) \sum_{k=1}^{\infty} \sum_{i=1}^{m} a_{i,k}^2 \\
= [\max\{1 - \zeta_1, 1 + (m - 1)\zeta_2\}] \sum_{k=1}^{\infty} \sum_{i=1}^{m} a_{i,k}^2 \\
= A_m \sum_{k=1}^{\infty} \sum_{i=1}^{m} a_{i,k}^2,
\]
where \(A_m\) is given in (4). This establishes (3), and in turn, Proposition 6.

### 4.2 A spectral gap bound based on influence matrices

One can also use Corollary 5 to bound the spectral gap. However, the bound would be looser than the one obtained in the previous subsection. Hence, we will not present the full calculation for this alternative bound. Instead, we only present parts of the calculation to illustrate how influence matrices are computed.

We will establish the following result for our example.

**Proposition 7.** Assume that \(n \geq 4\). Let \(m \in \{4, \ldots, n\}\). Then, for \(\Lambda \subset [n]\) such that \(|\Lambda| = n - m\) and \(y \in X_{\Lambda}\) such that \(\pi_{\Lambda}(y) > 0\), there is an influence matrix \(\Phi(\Lambda, y)\) associated with \((\Lambda, y)\) such that \(r(\Phi(\Lambda, y)) = (m - 1)/(m - 2)\).

By Corollary 5, Proposition 7 implies that
\[
\text{Gap}(m, m - 1) \geq \frac{m - 1}{m} - \frac{(m - 1)}{m(m - 2)}
\]
for \(m \geq 4\). If one can obtain a non-trivial lower bound \(c > 0\) on \(\text{Gap}(3, 2)\) and \(\text{Gap}(4, 3)\) (which can be achieved through Corollary 5 but requires some tedious calculations), then, by Theorem 2
\[
\text{Gap}(n, 1) \geq c^2 \prod_{m=4}^{n} \left[ \frac{m - 1}{m} - \frac{(m - 1)}{m(m - 2)} \right] = \frac{3c^2}{n(n - 2)}.
\]
Let us now prove Proposition 7. Assume that $n \geq 4$ and let $m \in \{4, \ldots, n\}$. Fix $\Lambda \subset [n]$ such that $|\Lambda| = n - m$. Let $y \in X_\Lambda$, and let $x = (x_1, \ldots, x_n) \in X$ be such that $x_\Lambda = y$. Assume that $\sum_{i \in \Lambda} x_i < 1$, so that $\pi_\Lambda(y) > 0$. Without loss of generality, assume that $-\Lambda = \{1, \ldots, m\}$.

For $i \in -\Lambda$, define a distance-like function

$$d_{\Lambda, y, i}(x, x') = \frac{|x - x'|}{1 - \sum_{j \in \Lambda} x_j - x \vee x'}, \quad x, x' \in X_i,$$

where $x \vee x' = \max\{x, x'\}$. We first need to verify that (H1) and (H2), which are given in Section 3.5, hold. Establishing (H1) through (2) is rather straightforward. To establish (H2), recall that the total variation distance between two distributions equals half of the integration of the absolute difference of their density functions. Then, based on (5), one can find that, for $i, j \in -\Lambda$ such that $i \neq j$ and $x, x' < 1 - \sum_{a \in \Lambda} x_a$,

$$d_{\text{TV}}(\Pi^j_{\Lambda, y, i, x}, \Pi^j_{\Lambda, y, i, x'}) = \frac{|x - x'|^{m-1}}{(1 - \sum_{a \in \Lambda} x_a - x)^{(m-1)/(m-2)} - (1 - \sum_{a \in \Lambda} x_a - x')^{(m-1)/(m-2)}}^{m-2}
\leq \left(\frac{m - 2}{m - 1}\right)^{m-2} d_{\Lambda, y, i}(x, x').$$

This establishes (H2).

It remains to establish a set of appropriate contraction conditions. For $i, j \in -\Lambda$ such that $i \neq j$, define a coupling kernel associated with $(\Lambda, y, i, j)$, denoted by $K_{i,j}$, as follows. Let $X$ follow the distribution given by the density function

$$x \mapsto (m - 1)(1 - x)^{m-2}, \quad x \in (0, 1).$$

For $x, x' \in X_i$, let $K_{i,j}((x, x'), \cdot)$ be the distribution of

$$\left(\left(1 - \sum_{a \in \Lambda} x_a - x\right) X, \left(1 - \sum_{a \in \Lambda} x_a - x'\right) X\right).$$
One can verify that this is a valid coupling kernel. In particular, $K_{i,j}((x, x'), \cdot)$ is a coupling of $\Pi_{\Lambda, y, i, x}$ and $\Pi_{\Lambda, y, i, x'}$. Now,

$$
\int_{X_i \times X_j} \int_{X_i \times X_j} d_{\Lambda, y, j}(x'', x''') K_{i,j}((x, x'), d(x'', x'''))
$$

$$= \mathbb{E} \left[ \frac{|x - x'| X}{1 - \sum_{a \in \Lambda} x_a - \left(1 - \sum_{a \in \Lambda} x_a - x \wedge x'\right) X} \right]
$$

$$\leq \frac{|x - x'|}{1 - \sum_{a \in \Lambda} x_a - x \wedge x'} \mathbb{E} \left( \frac{X}{1 - X} \right)
$$

$$\leq \frac{d_{\Lambda, y, i}(x, x')}{m - 2},$$

where $x \wedge x' = \min\{x, x'\}$. The above calculation shows that there exists an influence matrix $\Phi(\Lambda, y)$ associated with $(\Lambda, y)$ whose non-diagonal elements are $1/(m - 2)$. Then $r(\Phi(\Lambda, y)) = (m - 1)/(m - 2)$. This proves Proposition 7.

If one instead use the discrete metric to construct the influence matrix $\Phi(\Lambda, y)$, then all the off-diagonal entries of $\Phi(\Lambda, y)$ would be 1. The spectral gap obtained through Corollary 5 would then be trivial.

4.3 Discussion

We see from this example that both Corollaries 3 and 5 are capable of giving reasonably sharp bounds on the spectral gap. However, to construct these bounds, we need sufficient information on $\pi_{\{j\}\setminus \Lambda \cup \{i\}}$ for every $\Lambda \subset [n]$ such that $|\Lambda| \in \{0, \ldots, n-2\}$ and $i, j \in -\Lambda$ such that $i \neq j$. For many practical problems, $\pi_{\{j\}\setminus \Lambda \cup \{i\}}$ is intractable, especially when $\Lambda \cup \{i\} \cup \{j\} \neq [n]$. Indeed, even for chains on finite state spaces, spectral independence is often non-trivial to establish. A subject for future research would be to apply spectral telescope to analyze Gibbs chains used in various fields. Our results may be useful to study certain physics models, similar to those studied in Janvresse (2001), Carlen et al. (2003), Johnson and Jones (2015), and Pillai and Smith (2018);
and statistical models such as the de Finitti’s priors for almost exchangeable data (Gerencsér, 2019; Gerencsér and Ottolini, 2020).

5 Derivation of Main Results

5.1 Hierarchical Structure of the Spectral Gap

In this subsection, we derive Theorem 2. It suffices to prove the following lemma, which, as we will see, follows from the recursive representation of Algorithm 2 given in Algorithm 3.

Lemma 8. Let \( l \in \{1, \ldots, n - 1\} \) and \( m \in \{l + 1, \ldots, n\} \). Let \( \Lambda \subset [n] \) be such that \( |\Lambda| = n - m \), and let \( y \in X_\Lambda \) be such that \( \pi_\Lambda(y) > 0 \). Then

\[
\text{gap}(\Lambda, y, l) \geq \text{Gap}(m, m - 1) \text{Gap}(m - 1, l).
\]

To begin our analysis, fix \( l \in \{1, \ldots, n - 1\} \), \( \Lambda \subset [n] \) such that \( |\Lambda| = n - m \) where \( m \in \{l + 1, \ldots, n\} \), and \( y \in X_\Lambda \) where \( \pi_\Lambda(y) > 0 \). Denote by \( \varpi \) the probability measure given by \( \pi_{-\Lambda\mid \Lambda}(\cdot \mid y) \). For \( i \in -\Lambda \) and \( x \in X_i \), let \( \varpi_{i,x} \) be the probability measure given by \( \pi_{-(\Lambda\cup\{i\})\mid \Lambda\cup\{i\}}(\cdot \mid x_{\Lambda\cup\{i\}}) \) where \( x \in X \) satisfies \( x_{\Lambda} = y \) and \( x_{\{i\}} = x \).

Denote the Mtk of Algorithm 2 targeting \( \varpi \) with block size \( l \) by \( K(\cdot, \cdot) \). Then, for \( f \in L^2(\varpi) \) and \( x \in X \),

\[
Kf(x_{-\Lambda}) = \frac{1}{\binom{n}{l}} \sum_{\substack{\Gamma \subset \Lambda \mid |\Gamma| = l}} \mathbb{E} \left[ f(X_{-\Lambda}) \mid X_{-(\Lambda\cup\Gamma)} = x_{-(\Lambda\cup\Gamma)}, X_\Lambda = y \right].
\]

It is straightforward to check that \( K \) defines a positive semi-definite operator on \( L_0^2(\varpi) \). Its spectral gap is \( 1 - \|K\|_{\varpi} = \text{gap}(\Lambda, y, l) \).
Denote the Mtk of Algorithm 2 targeting $\mathcal{W}$ with block size $m - 1$ by $\bar{K}(\cdot, \cdot)$. Then, for $f \in L^2(\mathcal{W})$ and $x \in X$,

$$\bar{K}f(x_{-\Lambda}) = \frac{1}{m} \sum_{i \in -\Lambda} \mathbb{E} \left[ f(x_{-\Lambda}) \mid X_i = x_{\{i\}}, X_\Lambda = y \right].$$

$\bar{K}$ defines a positive semi-definite operator on $L^2_0(\mathcal{W})$, and its spectral gap satisfies

$$1 - \|\bar{K}\|_{\mathcal{W}} \geq \text{Gap}(m, m - 1). \quad (9)$$

Denote the Mtk of Algorithm 2 targeting $\mathcal{W}_i,x$ with block size $l$, where $i \in -\Lambda$ and $x \in X_i$, by $K_{i,x}(\cdot, \cdot)$. Then, for $f \in L^2(\mathcal{W}_i,x)$ and $x \in X$,

$$K_{i,x}f(x_{-(\Lambda \cup \{i\})}) = \frac{1}{(m - 1)} \sum_{\Gamma \subseteq -(\Lambda \cup \{i\}) \mid |\Gamma| = l} \mathbb{E} \left[ f(x_{-(\Lambda \cup \{i\})}) \mid x_{-(\Lambda \cup \{i\})} = x_{-(\Lambda \cup \{i\}) \cup \{i\}}, X_i = x, X_\Lambda = y \right].$$

One can check that, for $\pi_\{i\}|_{-\Lambda}(\cdot \mid y)$-almost every $x \in X_i$, $K_{i,x}$ defines a positive semi-definite operator on $L^2_0(\mathcal{W}_i,x)$, and its spectral gap satisfies

$$1 - \|K_{i,x}\|_{\mathcal{W}_i,x} \geq \text{Gap}(m - 1, l). \quad (10)$$

For $f \in L^2(\mathcal{W})$, $i \in -\Lambda$, and $x \in X_i$, let $f_{i,x} : X_{-(\Lambda \cup \{i\})} \to \mathbb{R}$ be such that $f(x_{-\Lambda}) = f_{i,x}(x_{-(\Lambda \cup \{i\})})$ whenever $x_{\{i\}} = x$. In other words, $f_{i,x}$ is just $f$ with the $X_i$-component of its argument fixed at $x$. For instance, if $f$ is a function on $X_1 \times X_2$, then $f_{1,x}(x_2) = f(x, x_2)$ for $x \in X_1$ and $x_2 \in X_2$, whereas $f_{2,x}(x_1) = f(x_1, x)$ for $x_1 \in X_1$ and $x \in X_2$. Given $f \in L^2(\mathcal{W})$, for $\pi_\{i\}|_{-\Lambda}(\cdot \mid y)$-almost every $x \in X_i$,
\( f_{i,x} \in L^2(\omega_{i,x}) \). For \( f \in L^2(\omega) \), the following holds \( \omega \)-almost everywhere on \( X_{-\Lambda} \):

\[
\frac{1}{m} \sum_{i \in -\Lambda} K_{i,x(i)} f_{i,x(i)}(x_{-(\Lambda \cup \{i\})}) = \frac{1}{m} \sum_{i \in -\Lambda} \frac{1}{m-l} \sum_{\Gamma \subset -(\Lambda \cup \{i\}) \mid |\Gamma| = l} \mathbb{E} \left[ f_{i,x(i)}(x_{-(\Lambda \cup \Gamma)}) \mid x_{-(\Lambda \cup \{i\})} = x_{-(\Lambda \cup \{i\})}, X_i = x_{\{i\}}, X_\Lambda = y \right]
\]

\[
= \frac{1}{m} \sum_{i \in -\Lambda} \frac{1}{m-l} \sum_{\Gamma \subset -(\Lambda \cup \{i\}) \mid |\Gamma| = l} \mathbb{E} \left[ f_{i,x(i)}(x_{-(\Lambda \cup \Gamma)}) \mid x_{-(\Lambda \cup \Gamma \cup \{i\})} = x_{-(\Lambda \cup \Gamma \cup \{i\})}, X_i = x_{\{i\}}, X_\Lambda = y \right]
\]

\[
= \frac{1}{m} \sum_{i \in -\Lambda} \frac{1}{m-l} \sum_{\Gamma \subset -(\Lambda \cup \{i\}) \mid |\Gamma| = l} \mathbb{E} \left[ f(x_{-\Lambda}) \mid x_{-(\Lambda \cup \Gamma \cup \{i\})} = x_{-(\Lambda \cup \Gamma \cup \{i\})}, X_i = x_{\{i\}}, X_\Lambda = y \right]
\]

\[
= \frac{1}{m} \sum_{i \in -\Lambda} \frac{1}{m-l} \sum_{\Gamma \subset -(\Lambda \cup \{i\}) \mid |\Gamma| = l} \mathbb{E} \left[ f(x_{-\Lambda}) \mid x_{-(\Lambda \cup \Gamma \cup \{i\})} = x_{-(\Lambda \cup \Gamma \cup \{i\})}, X_\Lambda = y \right]
\]

\[
= \frac{1}{m} \sum_{i \in -\Lambda} \frac{1}{m-l} \sum_{\Gamma \subset -(\Lambda \cup \{i\}) \mid |\Gamma| = l} \mathbb{E} \left[ f(x_{-\Lambda}) \mid x_{-(\Lambda \cup \Gamma)} = x_{-(\Lambda \cup \Gamma)}, X_\Lambda = y \right]
\]

\[
= K f(x_{-\Lambda}).
\]

This formula gives precisely the equivalence between Algorithms 2 and 3.

To prove Lemma 8 fix \( f \in L^2_0(\omega) \). For \( i \in -\Lambda \), let \( \Delta_i f = f - P_i f \), where

\[
P_i f \in L^2_0(\omega)
\]

satisfies

\[
P_i f(x_{-\Lambda}) = \mathbb{E} \left[ f(x_{-\Lambda}) \mid X_i = x_{\{i\}}, X_\Lambda = y \right].
\]

Then, for \( \pi_{\{i\}|\Lambda}(\cdot \mid y) \)-almost every \( x \in X_i \), \( (P_i f)_{i,x} \in L^2(\omega_{i,x}) \), and \( (\Delta_i f)_{i,x} \in L^2(\omega_{i,x}) \). (In fact, \( f_{i,x} \mapsto (P_i f)_{i,x} \) corresponds to the orthogonal projection on \( L^2(\omega_{i,x}) \) associated with the subspace of constant functions.) By the tower prop-
The integrand in the last line equals

$$\langle (P_i f), i,x + (P_i f), i,x \rangle \omega_{i,x}$$

$$= \langle (P_i f), i,x, (P_i f), i,x \rangle \omega_{i,x} + \langle (P_i f), i,x, K_{i,x} (\Delta_i f), i,x \rangle \omega_{i,x} +$$

$$\langle (P_i f), i,x, K_{i,x} (\Delta_i f), i,x \rangle \omega_{i,x} + \langle (P_i f), i,x, (P_i f), i,x \rangle \omega_{i,x}$$

$$= \langle (P_i f), i,x, (P_i f), i,x \rangle \omega_{i,x} + \langle (P_i f), i,x, K_{i,x} (\Delta_i f), i,x \rangle \omega_{i,x} + 2 \langle (P_i f), i,x, (P_i f), i,x \rangle \omega_{i,x},$$

where the last equality follows from the fact that $K_{i,x}$ is self-adjoint and that $K_{i,x} (P_i f), i,x = (P_i f), i,x$.

Let us examine the three terms in the last line of (13). Firstly, one can verify that, for $\pi_{\{i\}|\Lambda}(\cdot \mid y)$-almost every $x \in X_i$,

$$\langle (P_i f), i,x, (P_i f), i,x \rangle \omega_{i,x} = \mathbb{E} \{ f(X_{-\Lambda}) \mathbb{E} [ f(X_{-\Lambda}) \mid X_i = x, X_{\Lambda} = y] \mid X_i = x, X_{\Lambda} = y \}$$

$$= \langle f_{i,x}, (P_i f), i,x \rangle \omega_{i,x}. \quad (14)$$
It follows that
\[
\frac{1}{m} \sum_{i \in -\Lambda} \int_{X_i} \langle (P_i f)_{i,x}, (P_i f)_{i,x} \rangle \omega_{i,x} \pi_{\{i\}\Lambda}(x \mid y) \, dx
\]
\[= \frac{1}{m} \sum_{i \in -\Lambda} \int_{X_i} \langle f_{i,x}, (P_i f)_{i,x} \rangle \omega_{i,x} \pi_{\{i\}\Lambda}(x \mid y) \, dx
\]
\[= \frac{1}{m} \sum_{i \in -\Lambda} \langle f_i, P_i f \rangle \omega
\]
\[= \langle f, \bar{K} f \rangle \omega. \tag{15}\]

Secondly, by \((10)\), for \(\pi_{\{i\}\Lambda}(\cdot \mid y)\)-almost every \(x \in X_i\), since \((\Delta_i f)_{i,x} \in L^2(\omega_{i,x})\),
\[
\langle (\Delta_i f)_{i,x}, K_{i,x} (\Delta_i f)_{i,x} \rangle \omega_{i,x} \leq \| K_{i,x} \omega_{i,x} \| (\Delta_i f)_{i,x} \| \omega_{i,x} \|
\]
\[\leq [1 - \text{Gap}(m - 1, l)] (\| (\Delta_i f)_{i,x} \|) \omega_{i,x}.\]

By \((14)\),
\[
\| (\Delta_i f)_{i,x} \| \omega_{i,x}^2
\]
\[= \langle f_{i,x}, f_{i,x} \rangle \omega_{i,x} + \langle (P_i f)_{i,x}, (P_i f)_{i,x} \rangle \omega_{i,x} - 2 \langle f_{i,x}, (P_i f)_{i,x} \rangle \omega_{i,x}
\]
\[= \| f_{i,x} \|^2 \omega_{i,x} - \langle f_{i,x}, (P_i f)_{i,x} \rangle \omega_{i,x}.
\]
Therefore,
\[
\frac{1}{m} \sum_{i \in -\Lambda} \int_{X_i} \langle (\Delta_i f)_{i,x}, K_{i,x} (\Delta_i f)_{i,x} \rangle \omega_{i,x} \pi_{\{i\}\Lambda}(x \mid y) \, dx
\]
\[\leq [1 - \text{Gap}(m - 1, l)] \left[ \| f \|^2 \omega - \frac{1}{m} \sum_{i \in -\Lambda} \int_{X_i} \langle f_{i,x}, (P_i f)_{i,x} \rangle \omega_{i,x} \pi_{\{i\}\Lambda}(x \mid y) \, dx \right] \tag{16}\]
\[= [1 - \text{Gap}(m - 1, l)] \left( \| f \|^2 \omega - \langle f, \bar{K} f \rangle \omega \right),\]
where the final equality follows from \((15)\).

Finally, by \((14)\), for \(\pi_{\{i\}\Lambda}(\cdot \mid y)\)-almost every \(x \in X_i\),
\[
\langle (\Delta_i f)_{i,x}, (P_i f)_{i,x} \rangle \omega_{i,x} = \langle f_{i,x}, (P_i f)_{i,x} \rangle \omega_{i,x} - \langle (P_i f)_{i,x}, (P_i f)_{i,x} \rangle \omega_{i,x} = 0,
\]

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so
\[
\frac{1}{m} \sum_{i \in -\Lambda} \int_{X_i} \langle (\Delta_i f)_{i,x}, (P_i f)_{i,x} \rangle_{\pi_{i|\Lambda}(x \mid y)} \, \pi_{i|\Lambda}(x \mid y) \, dx = 0.
\] (17)

Combining (9) and (12) to (17) shows that
\[
\langle f, Kf \rangle_{\pi} \leq [1 - \text{Gap}(m - 1, l)] \|f\|_{\pi}^2 + \text{Gap}(m - 1, l) \langle f, \bar{K}f \rangle_{\pi} \\
\leq [1 - \text{Gap}(m - 1, l)] \|f\|_{\pi}^2 + \text{Gap}(m - 1, l) [1 - \text{Gap}(m, m - 1)] \|f\|_{\pi}^2 \\
= [1 - \text{Gap}(m - 1, l) \text{Gap}(m, m - 1)] \|f\|_{\pi}^2.
\]

Since $K$ is positive semi-definite, and $f \in L_0^2(\pi)$ is arbitrary, Lemma 8 holds.

### 5.2 Spectral gap and correlation coefficients

In this subsection, we derive Corollary 3. It suffices to show the following.

**Lemma 9.** Let $m \in \{2, \ldots, n\}$. Then, for $\Lambda \subset [n]$ such that $|\Lambda| = n - m$ and $y \in X_\Lambda$ such that $\pi_{\Lambda}(y) > 0$,
\[
\text{gap}(\Lambda, y, m - 1) \geq 1 - s(\Lambda, y).
\]

In particular,
\[
\text{Gap}(m, m - 1) \geq 1 - S(m).
\]

To prove the lemma, fix $\Lambda \subset [n]$ such that $|\Lambda| = n - m$ where $m \in \{2, \ldots, n\}$ and $y \in X_\Lambda$ such that $\pi_{\Lambda}(y) > 0$. As in the previous section, denote by $\pi$ the probability measure given by $\pi_{-\Lambda|\Lambda}(\cdot \mid y)$, and let $\bar{K}(\cdot, \cdot)$ be the Mtk of Algorithm 2 targeting $\pi$ with block size $m - 1$. Then
\[
\text{gap}(\Lambda, y, m - 1) = 1 - \|\bar{K}\|_{\pi}.
\]

For $f \in L_0^2(\pi)$,
\[
\bar{K}f = \frac{1}{m} \sum_{i \in -\Lambda} P_i f,
\]

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where $P_i : L^2_0(\varpi) \to L^2_0(\varpi)$ satisfies

$$P_i f(x_\Lambda) = \mathbb{E} \left[ f(X_{\Lambda}^i) \mid X_i = x_i^i, X_\Lambda = y \right].$$

For $i \in -\Lambda$, $P_i^2 = P_i$, and for $f, g \in L^2_0(\varpi)$,

$$\langle P_i f, g \rangle_\varpi = \langle P_i f, P_i g \rangle_\varpi = \langle f, P_i g \rangle_\varpi.$$

In fact, $P_i$ is the orthogonal projection onto the space of $L^2_0(\varpi)$ functions $x_\Lambda \mapsto f(x_\Lambda)$ that depend on $x_\Lambda$ only through $x_i^i$. Denote the range of $P_i$ by $L_i$. Let $L = \sum_{i \in -\Lambda} L_i$. That is, $L \subset L^2_0(\varpi)$ consists of functions that are sums of functions from $L_i$. Obviously, $\bar{K}$ maps a function in $L^2_0(\varpi)$ to a function in $L$. Let $\bar{K}|_L$ be $\bar{K}$ restricted to $L$. We then have the following lemma.

**Lemma 10.**

$$\| \bar{K} \|_\varpi = \| \bar{K}|_L \|_\varpi. \tag{18}$$

**Proof.** It is clear that $\| \bar{K} \|_\varpi \geq \| \bar{K}|_L \|_\varpi$. It remains to prove the reverse inequality. Since $\bar{K}$ is self-adjoint, its norm equals its spectral radius. Then, by Gelfand’s formula,

$$\| \bar{K} \|_\varpi = \lim_{t \to \infty} \| \bar{K}^t \|^{1/t}_\varpi. \tag{19}$$

For any $f \in L^2_0(\varpi)$ and positive integer $t$ such that $t \geq 2$,

$$\| \bar{K}^t f \|_\varpi = \| \bar{K}|_L (\bar{K}^t)^{-1} \bar{K}^t f \|_\varpi \leq \| \bar{K}|_L \|^{t-1}_\varpi \| \bar{K} \|_\varpi \| f \|_\varpi.$$

Then

$$\lim_{t \to \infty} \| \bar{K}^t \|^{1/t}_\varpi \leq \lim_{t \to \infty} \| \bar{K}|_L \|^{(t-1)/t}_\varpi \| \bar{K} \|^{1/t}_\varpi = \| \bar{K}|_L \|_\varpi.$$

It then follows from (19) that

$$\| \bar{K} \|_\varpi \leq \| \bar{K}|_L \|_\varpi.$$
To derive Lemma 9, we combine Lemma 10 with a simple result from Bjørstad and Mandel (1991) concerning norms of sums of orthogonal projections.

**Lemma 11.** (Bjørstad and Mandel, 1991, Theorem 3.2)

\[
\|ar{K}_L\|_\infty \leq \sup_{f_i \in L_i \forall i \text{ s.t. } f_i \neq 0} \sum_{i \in -\Lambda} \|f_i\|_2^2 \leq \sup_{f_i \in L_i \forall i \text{ s.t. } f_i \neq 0} \frac{\left\| \sum_{i \in -\Lambda} f_i \right\|_2^2}{m \sum_{i \in -\Lambda} \|f_i\|_2^2}.
\]

Note that for \(f \in L_0^2(\varpi)\),

\[
\|f\|_\infty^2 = \mathbb{E} \left[ f(X_{-\Lambda})^2 \mid X_{\Lambda} = y \right].
\]

Moreover, if we let \(\varpi_i\) be the probability measure on \((X_i, \mathcal{B}_i)\) given by \(\pi_{\{i\}_\Lambda}(\cdot \mid y)\), then there is a natural isomorphism from \(L_i\) to \(L_0^2(\varpi_i)\). It follows that

\[
\sup_{f_i \in L_i \forall i \text{ s.t. } f_i \neq 0} \frac{\left\| \sum_{i \in -\Lambda} f_i \right\|_2^2}{m \sum_{i \in -\Lambda} \|f_i\|_2^2} = \sup_{f_i \in L_i \forall i \text{ s.t. } \mathbb{E}[f_i^2(\varpi_i) \mid X_{\Lambda} = y] > 0} \frac{\mathbb{E} \left\{ \left[ \sum_{i \in -\Lambda} f_i(X_i) \right]^2 \mid X_{\Lambda} = y \right\}}{m \sum_{i \in -\Lambda} \mathbb{E}[f_i(X_i)^2 \mid X_{\Lambda} = y]} = s(\Lambda, y).
\]

Lemma 9 then follows from Lemmas 10 and 11.

### 5.3 Spectral gap and random walks

In this section, we derive Corollary 4. In light of Corollary 3, it suffices to prove the following result.

**Lemma 12.** Let \(m \in \{2, \ldots, n\}\). Then, for \(\Lambda \subset [n]\) such that \(|\Lambda| = n - m\) and \(y \in X_{\Lambda}\) such that \(\pi_{\Lambda}(y) > 0\),

\[
g(\Lambda, y) = 1 - s(\Lambda, y).
\]

In particular,

\[
G(m) = 1 - S(m).
\]
To prove Lemma 12, fix $\Lambda \subset [n]$ such that $|\Lambda| = n - m$, where $m \in \{2, \ldots, n\}$, and $y \in X_\Lambda$ such that $\pi_\Lambda(y) > 0$.

Consider Algorithm 4 associated with $\Lambda$ and $y$. Recall that the underlying random walk Markov chain has $\tilde{X} = \bigcup_{i \notin -\Lambda} (\{i\} \times X_i)$ as its state space. The chain is reversible with respect to the probability measure $\varphi$ given by

$$\varphi(\{i\} \times A) = \frac{1}{m} \varpi_i(A), \quad i \in -\Lambda, \ A \in \mathcal{B}_i,$$

where $\varpi_i$ is the probability measure on $(X_i, \mathcal{B}_i)$ given by $\pi_{\{i\}|\Lambda}(\cdot | y)$. A measurable function on $\tilde{X}$ has the form $(i, x) \mapsto f(i, x)$, where $i \in -\Lambda$ and $x \in X_i$. For such a function $f$, we can identify $m$ functions $(T_i f)_{i \in -\Lambda}$, such that $T_i f(x) = f(i, x)$ for $i \in -\Lambda$ and $x \in X_i$. Then $f \in L^2_0(\varphi)$ if and only if $T_i f \in L^2(\varpi_i)$ for each $i$, and

$$\sum_{i \in -\Lambda} \mathbb{E}[T_i f(X_i) \mid X_\Lambda = y] = 0. \quad (20)$$

For $f \in L^2_0(\varphi)$,

$$\|f\|^2_\varphi = \frac{1}{m} \sum_{i \in -\Lambda} \mathbb{E}\{[T_i f(X_i)]^2 \mid X_\Lambda = y\}. \quad (21)$$

Let $R(\cdot, \cdot)$ be the Mtk of Algorithm 4 associated with $\Lambda$ and $y$. $R(\cdot, \cdot)$ defines the following operator on $L^2_0(\varphi)$: For $f \in L^2_0(\varphi)$, $i \in -\Lambda$, and $x \in X_i$,

$$Rf(i, x) = \frac{1}{m} \sum_{j \in -\Lambda} \mathbb{E}[T_j f(X_j) \mid X_i = x, X_\Lambda = y]. \quad (22)$$

It follows that, for $f \in L^2_0(\varphi)$,

$$\langle f, Rf \rangle_\varphi = \frac{1}{m^2} \sum_{i, j \in -\Lambda} \mathbb{E}[T_i f(X_i) T_j f(X_j) \mid X_\Lambda = y]$$

$$= \frac{1}{m^2} \mathbb{E} \left\{ \left[ \sum_{i \in -\Lambda} T_i f(X_i) \right]^2 \mid X_\Lambda = y \right\}. \quad (23)$$

From this formula, we can see that $R$ is positive semi-definite.
Let $\mathcal{L}'$ be the space of functions $f$ in $L_0^2(\varphi)$ such that

$$
\mathbb{E}[T_i f(X_i) \mid X_\Lambda = y] = 0
$$

for $i \in -\Lambda$. In other words, $f \in \mathcal{L}'$ if and only if $T_i f \in L_0^2(\varphi_i)$ for $i \in -\Lambda$. By (20) and (22), for $f \in L_0^2(\varphi)$, $R f \in \mathcal{L}'$. Just like in Lemma 10 one can argue that

$$
1 - g(\Lambda, y) = \|R\|_\varphi = \|R|_{\mathcal{L}'}\|_\varphi,
$$

where $R|_{\mathcal{L}'}$ is $R$ restricted to $\mathcal{L}'$. It then follows from (21), (23), and the fact that $R$ is positive semi-definite that

$$
1 - g(\Lambda, y) = \sup_{f \in \mathcal{L}' \setminus \{0\}} \frac{\langle f, R f \rangle_\varphi}{\|f\|_\varphi^2} \leq s(\Lambda, y). \tag{24}
$$

It remains to show the reverse inequality. To this end, let $(f_i)_{i \in -\Lambda}$ be such that $f_i \in L_0^2(\varphi_i)$ for each $i$, and that $f_i \not= 0$ for some $i$. One can find a function $f \in L_0^2(\varphi)$ such that

$$
f(i, x) = T_i f(x) = f_i(x)
$$

for $i \in -\Lambda$ and $x \in X_i$. Then, by (21) and (23),

$$
\mathbb{E} \left\{ \left[ \sum_{i \in -\Lambda} f_i(X_i) \right]^2 \mid X_\Lambda = y \right\} = \mathbb{E} \left\{ \left[ \sum_{i \in -\Lambda} T_i f(X_i) \right]^2 \mid X_\Lambda = y \right\} = \frac{\mathbb{E} \left\{ \left[ \sum_{i \in -\Lambda} T_i f(X_i) \right]^2 \mid X_\Lambda = y \right\}}{\mathbb{E} \left\{ f_i(X_i)^2 \mid X_\Lambda = y \right\}} \leq \|R\|_\varphi.
$$

This shows that

$$
s(\Lambda, y) \leq 1 - g(\Lambda, y).
$$

In summary, Lemma 12 holds.
5.4 Spectral gap and spectral independence

In this section, we prove Corollary 5. In light of Corollary 4, it suffices to prove the following.

**Lemma 13.** Let \( \Lambda \subset [n] \) be such that \( |\Lambda| = n - m \), where \( m \in \{2, \ldots, n\} \), and let \( y \in X_\Lambda \) be such that \( \pi_\Lambda(y) > 0 \). Suppose that there is an influence matrix \( \Phi(\Lambda, y) \) associated with \((\Lambda, y)\) such that

\[
r(\Phi(\Lambda, y)) \leq \eta,
\]

where \( \eta < m - 1 \). Then

\[
g(\Lambda, y) \geq \frac{m - 1}{m} - \frac{\eta}{m}.
\]

The proof is divided into several steps. We first define an altered version of the random walk and relate the \( L^2 \) norm of its Markov operator to \( g(\Lambda, y) \), the spectral gap of the original random walk. We then incorporate a coupling argument, somewhat similar to that used in Feng et al. (2021), to construct a convergence bound for the altered random walk in a Wasserstein divergence. Next, we use one-shot coupling (Roberts and Rosenthal, 2002; Madras and Sezer, 2010) to translate the bound to one in total variation distance. Finally, we use a result in Roberts and Rosenthal (1997) to further translate the convergence bound in total variation distance to a bound on the \( L^2 \) norm of the chain’s Markov operator.

Throughout this subsection, fix \( \Lambda \subset [n] \) such that \( |\Lambda| = n - m \), where \( m \in \{2, \ldots, n\} \), and let \( y \in X_\Lambda \) be such that \( \pi_\Lambda(y) > 0 \). Assume that the assumptions of Lemma 13 hold. In particular, for \( i, j \in -\Lambda \) such that \( i \neq j \), there is a coupling kernel \( K_{i,j} \) and \( \phi_{i,j} < \infty \) such that

\[
\int_{X_i \times X_j} d_{\Lambda, y}((x'', x''')) K_{i,j}((x, x'), d((x'', x'''), (x, x'))) \leq \phi_{i,j} d_{\Lambda, y}(x, x')
\]  

(25)
for $\pi_{(i\mid \Lambda)}(\cdot \mid y)$-almost every $x, x' \in X_i$. For $i \in -\Lambda$, let $\phi_{i,i} = 0$, and let $K_{i,i} : X_i \times X_i \to B_i \times B_i$ be a Markov transition kernel such that

$$K_{i,i}((x, x'), A) = \int_{A'} \pi_{(i\mid \Lambda)}(x'' \mid x') \, dx'' ,$$

where $A' = \{(x'', x'') : x'' \in A\}$. ($A'$ is measurable when $A$ is since the former is the intersection of $A \times A$ and the set of points $(x'', x''')$ such that $d_{\Lambda, y,i}(x'', x''') = 0$.) Then (25) holds even when $i = j$. Moreover, the influence matrix $\Phi(\Lambda, y)$ can be written as $(\phi_{i,j})$.

### 5.4.1 An altered random walk

It is convenient to consider a random walk chain that is a slight alteration of Algorithm 4. Just like Algorithm 4, Algorithm 5 defines a Markov chain that is reversible with respect to a distribution of the form

$$\varphi(\{i\} \times A) = \frac{1}{m} \varpi_i(A), \quad i \in -\Lambda, \ A \in B_i,$$

with $\varpi_i$ being the measure given by $\pi_{(i\mid \Lambda)}(\cdot \mid y)$.

Let $R(\cdot, \cdot)$ be the transition kernel for Algorithm 4 and $\tilde{R}(\cdot, \cdot)$, that for Algorithm 5. Each kernel defines a self-adjoint operator on $L^2_0(\varphi)$. Indeed, $R$ is given by (22), i.e., for $f \in L^2_0(\varphi)$, $i \in \Lambda$, and $x \in X_i$,

$$Rf(i, x) = \frac{1}{m} \sum_{j \in -(\Lambda \cup \{i\})} \mathbb{E}[T_j f(X_j) \mid X_i = x, X_\Lambda = y] + \frac{f(i, x)}{m} ,$$

where $T_i f(x) = f(i, x)$. On the other hand,

$$\tilde{R}f(i, x) = \frac{1}{m} \sum_{j \in -(\Lambda \cup \{i\})} \mathbb{E}[T_j f(X_j) \mid X_i = x, X_\Lambda = y] + \frac{1}{m} \mathbb{E}[T_i f(X_i) \mid X_\Lambda = y] .$$

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Algorithm 5 One step of an altered random walk associated with $\Lambda \subseteq \mathbb{N}$ and $y \in X_{\Lambda}$:

**Input:** Current state $(j, x) \in \bigcup_{i \in -\Lambda} \{i\} \times X_i$.

Let $x \in X$ be such that $x_{\Lambda} = y$ and $x_{\{j\}} = x$.

Randomly and uniformly choose a coordinate $j' \in -\Lambda$.

if $j' = j$ then
   Draw $x' \in X_{\{j\}}$ from $\pi_{\{j\}|\Lambda}(\cdot \mid x_{\Lambda})$.
else
   Draw $x' \in X_{\{j'\}}$ from $\pi_{\{j'\} | \Lambda \cup \{j\}}(\cdot \mid x_{\Lambda \cup \{j\}})$.
end if

Return: New State $(j', x')$.

Let $L'$ be the space of functions $f$ in $L_0^2(\varphi)$ such that $T_i f \in L_0^2(\omega_i)$ for $i \in -\Lambda$. $R|_{L'}$ and $\tilde{R}|_{L'}$, the restrictions of $R$ and $\tilde{R}$ to $L'$, are related by the following formula:

$$R|_{L'} = \tilde{R}|_{L'} + \frac{\text{Id}}{m},$$

where $\text{Id}$ is the identity on $L'$. It follows that

$$g(\Lambda, y) = 1 - \sup_{f \in L', f \neq 0} \frac{\langle f, Rf \rangle_{\varphi}}{\|f\|_{\varphi}^2} = \frac{m - 1}{m} - \sup_{f \in L', f \neq 0} \frac{\langle f, \tilde{R}f \rangle_{\varphi}}{\|f\|_{\varphi}^2} \geq \frac{m - 1}{m} - \|\tilde{R}\|_{\varphi},$$

where the first equality is part of (24) derived in Section 5.3. Hence, to prove Lemma 13, it suffices to show that

$$\|\tilde{R}\|_{\varphi} \leq \frac{\eta}{m}. \quad (26)$$

5.4.2 Convergence in a Wasserstein divergence

According to Lemma 1, to say that (26) holds is to say that the altered random walk chain converges geometrically in the $L^2$ distance at a rate of $\eta/m$. To prove this, we
first show that the chain converges geometrically in some Wasserstein divergence.

Let \( D : \bigcup_{i \in -\Lambda} (\{i\} \times X_i) \times \bigcup_{i \in -\Lambda} (\{i\} \times X_i) \to [0, \infty] \) be such that, for \( i, j \in -\Lambda \) and \( x \in X_i, x' \in X_j \),

\[
D((i, x), (j, x')) = \begin{cases} 
  d_{\Lambda, y_i}(x, x') & i = j, \\
  \infty & i \neq j.
\end{cases}
\]

A measure in \( L^2_\ast(\varphi) \) has the form

\[
\omega(\{i\} \times A) = a_i \omega_i(A), \quad i \in -\Lambda, \ A \in B_i,
\]

where, \( \sum_{i \in -\Lambda} a_i = 1 \), and, for \( i \in -\Lambda, a_i \geq 0 \) and \( \omega_i \in L^2_\ast(\varpi_i) \).

The following lemma implies that the altered random walk chain converges geometrically in the Wasserstein divergence induced by \( D \).

**Lemma 14.** Let \( \omega \in L^2_\ast(\varphi) \) be as in (27). Then there exist a pair of random walk chains associated with Algorithm \( \mathfrak{A} \), denoted by \((I(t), X(t))_{t=0}^\infty \) and \((I'(t), X'(t))_{t=0}^\infty \), that satisfy the following properties:

(P1) \((I(0), X(0)) \sim \omega, \) and independently, \((I'(0), X'(0)) \sim \varphi. \)

(P2) \( I(t) = I'(t) \) for \( t \geq 1. \)

(P3) Given \( I(t) = i_t \in -\Lambda, \) the distribution of \( X(t) \) is absolutely continuous with respect to \( \varpi_{i_t}. \)

(P4) There exists a constant \( C_\omega < \infty \) such that, for each positive integer \( t, \)

\[
\mathbb{E} [D((I(t), X(t)), (I'(t), X'(t)))] \leq C_\omega \|\Psi^{t-1}\|_\infty,
\]

where \( \Psi = \Phi(\Lambda, y) / m, \) and, for any matrix \( A = (a_{i,j}), \ |A|_\infty = \max_i \sum_j |a_{i,j}|. \)
Proof. Construct the two chains according the following Markovian procedure.

1. Let \((I(0), X(0)) \sim \omega\), and independently, \((I'(0), X'(0)) \sim \varphi\).

2. Draw \(I(1) = I'(1)\) randomly and uniformly from \(-\Lambda\). Denote the observed values of \(I(0), I'(0), X(0), X'(0),\) and \(I(1) = I'(1)\) by \(i_0, i'_0, z_0, z'_0,\) and \(i_1\) respectively.

3. For \(i, j \in -\Lambda\) and \(x \in X_i\), let \(\bar{\Pi}^i_{\Lambda,y,i,x} = \Pi^i_{\Lambda,y,i,x}\) if \(i \neq j\), and let \(\bar{\Pi}^i_{\Lambda,y,i,x}\) be the probability measure associated with \(\pi_{(j)a} \cdot | y\) if \(i = j\). Independently, draw \(X(1)\) from \(\bar{\Pi}^i_{\Lambda,y,i_0,z_0}\), and \(X'(1)\) from \(\bar{\Pi}^i_{\Lambda,y,i_0,z'_0}\).

4. For a positive integer \(t\), given \((I(t), X(t)) = (i_t, z_t)\) and \((I'(t), X'(t)) = (i'_t, z'_t)\), draw \((I(t+1), X(t+1), I'(t+1), X'(t+1))\) as follows. Randomly and uniformly draw \(I(t+1) = I'(t+1)\) from \(-\Lambda\), and denote the observed value by \(i_{t+1}\). Then, draw \((X(t+1), X'(t+1))\) using the coupling kernel \(K_{i_t,i_{t+1}}((z_t, z'_t), \cdot)\).

It is easy to see that \((I(t), X(t))\) and \((I'(t), X'(t))\) are both Markov chains whose transition laws follow Algorithm 5 and that they satisfy (P1) and (P2). Let us establish (P3) and (P4). Fix a positive integer \(t\). Let \(i_s \in -\Lambda\) for \(s = 0, \ldots, t\), and let \(i'_0 \in -\Lambda\). Given \(I(s) = I'(s) = i_s\) for \(s = 0, \ldots, t\) and \(I'(0) = i'_0\), the distribution of \((X(t), X'(t))\), denoted by \(\nu_{i_0,\ldots,i_t;i'_0}\), is given by the following recursive formula:

\[
\nu_{i_0;i'_0}(dz_0, dz'_0) = \omega_{i_0}(dz_0) \varpi_{i'_0}(dz'_0),
\]

\[
\nu_{i_0,i_1;i'_0}(dz_1, dz'_1) = \int_{X_i \times X_{i_0}} \bar{\Pi}^i_{\Lambda,y,i_0,z_0}(dz_1) \bar{\Pi}^i_{\Lambda,y,i'_0,z'_0}(dz'_1) \nu_{i_0,i'_0}(dz_0, dz'_0),
\]

\[
\nu_{i_0,\ldots,i_{s+1};i'_0}(dz_{s+1}, dz'_{s+1}) = \int_{X_{i,s+1}} K_{i_s,i_{s+1}}((z_s, z'_s), (dz_{s+1}, dz'_{s+1})) \nu_{i_0,\ldots,i_s;i'_0}(dz_s, dz'_s),
\]

where \(s \geq 1\). One can check that, for \(s \geq 0\), the distribution given by \(A \mapsto \nu_{i_0,\ldots,i_s;i'_0}(A \times X_{i,s})\) is absolutely continuous with respect to \(\varpi_{i,s}\), while that given.
by \( A \mapsto \nu_{i_0, \ldots, i_t; i_0'}(X_s \times A) \) is \( \varpi_{i_s} \) itself. This implies that (P3) holds. Moreover, for \( s \geq 1 \) and \( \nu_{i_0, \ldots, i_t; i_0'} \)-almost every \((z_s, z_s') \in X^2_s\),

\[
\int_{X^2_{s+1}} d_{\Lambda, Y, i_{s+1}} (z_{s+1}, z_{s+1}') K_{i_s, i_{s+1}} ((z_s, z_s'), d(z_{s+1}, z_{s+1}')) \leq \phi_{i_s, i_{s+1}} d_{\Lambda, Y, i_s} (z_s, z_s').
\]

Thus,

\[
\mathbb{E} \left[ D((I(t), X(t)), (I'(t), X'(t))) \mid I(s) = I'(s) = i_s \text{ for } s = 0, \ldots, t; I'(0) = i_0' \right]
\]

\[
= \int_{X^2_{t+1}} d_{\Lambda, Y, i_t} (z_t, z_t') \nu_{i_0, \ldots, i_t; i_0'} (dz_t, dz_t')
\]

\[
= \int_{X^2_{t-1}} \int_{X^2_{t}} d_{\Lambda, Y, i_t} (z_t, z_t') K_{i_{t-1}, i_t} ((z_{t-1}, z_{t-1}'), (dz_t, dz_t')) \nu_{i_0, \ldots, i_{t-1}; i_0'} (dz_{t-1}, dz_{t-1}')
\]

\[
\leq \phi_{i_{t-1}, i_t} \int_{X^2_{t-1}} d_{\Lambda, Y, i_{t-1}} (z_{t-1}, z_{t-1}') \nu_{i_0, \ldots, i_{t-1}; i_0'} (dz_{t-1}, dz_{t-1}')
\]

\[
\leq \left( \prod_{s=1}^{t-1} \phi_{i_s, i_{s+1}} \right) \int_{X^2_{1}} d_{\Lambda, Y, i_1} (z_1, z_1') \nu_{i_0, i_1; i_0'} (dz_1, dz_1').
\]

(If \( t = 1 \), then \( \prod_{s=1}^{t-1} \phi_{i_s, i_{s+1}} \) is interpreted as 1.) Then

\[
\mathbb{E} \left[ D_{\Lambda, Y} ((I(t), X(t)), (I'(t), X'(t))) \right]
\]

\[
\leq \frac{1}{m^{t+1}} \left( \sum_{i_2, \ldots, i_t \in -\Lambda} \prod_{s=1}^{t-1} \phi_{i_s, i_{s+1}} \right) \sum_{i_0, i_1, i_1' \in -\Lambda} a_{i_0} \int_{X^2_{1}} d_{\Lambda, Y, i_1} (z_1, z_1') \nu_{i_0, i_1; i_0'} (dz_1, dz_1')
\]

\[
\leq C_\omega \| \Psi^{t-1} \|_{\infty},
\]

where

\[
C_\omega = \frac{1}{m^2} \sum_{i_0, i_1 \in -\Lambda} a_{i_0} \int_{X^2_{1}} d_{\Lambda, Y, i_1} (z_1, z_1') \nu_{i_0, i_1; i_0'} (dz_1, dz_1')
\]

\[
= \frac{1}{m^2} \sum_{i_0, i_1 \in -\Lambda} a_{i_0} \int_{X^2_{1}} \int_{X^2_{1}} d_{\Lambda, Y, i_1} (z_1, z_1') \Pi_{\Lambda, Y, i_0; i_0'} (dz_1) \omega_{i_1} (dz_1') \omega_{i_0} (dz_0).
\]

It remains to show that \( C_\omega < \infty \). Fix \( i_0, i_1 \in -\Lambda \). Recall that (H1) and (H2) in Section 3.5 are assumed. By (H1),

\[
z_1 \mapsto \int_{X^2_{1}} d_{\Lambda, Y, i_1} (z_1, z_1') \omega_{i_1} (dz_1')
\]

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is in $L^2(\varpi_{i_1})$. By Cauchy-Schwarz,
\[ z_0 \mapsto f(z_0) = \int_{X_{i_1}} \int_{X_{i_1}} d_{A,Y,i_1}(z_1, z_1') \varpi_{i_1} (dz_1') \Pi_{A,Y,i_0,z_0}^{i_1} (dz_1) \]
is in $L^2(\varpi_{i_0})$. Thus,
\[ \int_{X_{i_0}} \int_{X_{i_1}} f(z_0) \frac{d\omega_{i_0}}{d\varpi_{i_0}} (z_0) \varpi_{i_0} (dz_0) \]
\[ < \infty. \]

This concludes the proof. \qed

5.4.3 Convergence in total variation

To continue, we use the one-shot coupling technique to show that the altered random walk chain converges in total variation distance. To be specific, we show the following.

**Lemma 15.** Let $\omega \in L^2_2(\varphi)$. For $t \geq 0$, denote by $\omega \bar{R}^t$ the distribution of the $t$th element of a Markov chain associated with Algorithm 5 assuming that the chain’s starting distribution (i.e., distribution of its zeroth element) is $\omega$. Then, there exists a constant $C_\omega < \infty$ such that, for $t \geq 2$,
\[ d_{TV}(\omega \bar{R}^t, \varphi) \leq C_\omega \|\Psi^{t-2}\|_\infty, \]
where $\Psi = \Phi(\Lambda, y)/m$, and $\| \cdot \|_\infty$ is defined in Lemma 14.

**Proof.** Let $(I(t), X(t))_{t=0}^\infty$ and $(I'(t), X'(t))_{t=0}^\infty$ be a pair of chains associated with Algorithm 5 that satisfy (P1) to (P4) in Lemma 14.

Fix $t \geq 2$. Given $(I(t-1), X(t-1)) = (i_{t-1}, z_{t-1})$ and $(I'(t-1), X'(t-1)) = (i_{t-1}', z_{t-1}'),$ proceed as follows. Draw $J$ randomly and uniformly from $-\Lambda$, and call
the observed value \( j \). If \( j = i_{t-1} \), draw \( Z \) from \( \pi_{(j)\setminus i} : x \mapsto -p_{(i_{t-1}, j, z_{t-1}, z'_{t-1})} \)

\[
= \int_{x_j} \min \left\{ \pi_{(j)\setminus i_{t-1}}(x | x_{\psi_{i_j}}), \pi_{(j)\setminus i_{t-1}}(x | x'_{\psi_{i_j}}) \right\} dx.
\]

Then

\[
1 - p = d_{TV} \left( \Pi^{j}_{\Lambda, y, i_{t-1}, z_{t-1}}, \Pi^{j}_{\Lambda, y, i_{t-1}, z'_{t-1}} \right).
\]

With probability \( p \), draw \( Z = Z' \) from the density

\[
x \mapsto q(x) = \frac{1}{p} \min \left\{ \pi_{(j)\setminus i_{t-1}}(x | x_{\psi_{i_j}}), \pi_{(j)\setminus i_{t-1}}(x | x'_{\psi_{i_j}}) \right\}.
\]

With probability \( 1 - p \), draw \( Z \) from the density

\[
x \mapsto \frac{\pi_{(j)\setminus i_{t-1}}(x | x_{\psi_{i_j}}) - p q(x)}{1 - p},
\]

and independently, draw \( Z' \) from the density

\[
x \mapsto \frac{\pi_{(j)\setminus i_{t-1}}(x | x'_{\psi_{i_j}}) - p q(x)}{1 - p}.
\]

Then \((J, Z) \sim \omega R^i\), while \((J, Z') \sim \varphi\). Moreover, given \((I(t-1), X(t-1)) = (i_{t-1}, z_{t-1}), (I'(t-1), X'(t-1)) = (i'_{t-1}, z'_{t-1})\), and \( J = j \), the probability of the event \( Z = Z' \) is precisely \( p \).

By (H2) and (P3) along with (P2), there is a constant \( k < \infty \) such that, almost surely,

\[
p(I(t-1), J, X(t-1), X'(t-1)) = 1 - d_{TV} \left( \Pi^{j}_{\Lambda, y, I(t-1), X(t-1)}, \Pi^{j}_{\Lambda, y, I(t-1), X'(t-1)} \right)
\geq 1 - kd_{\Lambda, y, I(t-1)}(X(t-1), X'(t-1))
\geq 1 - kD ((I(t-1), X(t-1)), (I'(t-1), X'(t-1))).
\]

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It then follows from (P4) that there exists a constant $C'_\omega$ unrelated to $t$ such that
\[
\mathbb{P}((J, Z) \neq (J, Z')) = 1 - \mathbb{E} [p(I(t - 1), J, X(t - 1), X'(t - 1))]
\leq k \mathbb{E} [D ((I(t - 1), X(t - 1)), (I'(t - 1), X'(t - 1)))]
\leq k C'_\omega \| \Psi^{-2} \|_\infty.
\]

By the well-known coupling inequality,
\[
d_{TV}(\omega \tilde{R}^t, \varphi) \leq \mathbb{P}((J, Z) \neq (J, Z')) \leq k C'_\omega \| \Psi^{-2} \|_\infty.
\]

\[
5.4.4 \text{ Convergence in the } L^2 \text{ distance}
\]

To establish (26) and thus Lemma [13], we use Lemma [15] to derive a convergence bound in the $L^2$ distance.

Recall that it is assumed that the spectral radius of $\Phi(\Lambda, y)$ is no greater than $\eta \in [0, m - 1)$. Then the spectral radius of $\Psi = \Phi(\Lambda, y)/m$ is no greater than $\eta/m \in [0, (m - 1)/m)$. Since $\| \cdot \|_\infty$, as given in Lemma [14] is a matrix norm, by Gelfand’s formula,
\[
\lim_{t \to \infty} \| \Psi^{-2} \|_\infty^{1/t} \leq \frac{\eta}{m},
\]
This implies that, for $\rho > \eta/m$, one can find a constant $c_\rho$ such that
\[
\| \Psi^{-2} \|_\infty \leq c_\rho \rho^t
\]
for $t \geq 2$.

Fix $\rho \in (\eta/m, 1)$. For $\omega \in L^2_*(\varphi)$ and $t \geq 0$, let $\omega \tilde{R}^t$ be as defined in Lemma [15]. Then the said lemma implies that there is a constant $C_\omega < \infty$ such that
\[
d_{TV}(\omega \tilde{R}^t, \varphi) \leq C_\omega \rho^t.
\]
for $t \geq 0$. By Theorem 2.1 in Roberts and Rosenthal (1997), the $L^2$ distance between $\omega \tilde{R}^t$ and $\varphi$ also decreases at a rate of $\rho^t$ or faster; moreover, $\| \tilde{R} \|_\varphi \leq \rho$. Since $\rho \in (\rho/m, 1)$ is arbitrary, (26) holds.

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