A MULTI-MATERIAL TRANSPORT PROBLEM AND ITS CONVEX RELAXATION VIA RECTIFIABLE $G$-CURRENTS

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Abstract. In this paper we study a variant of the branched transportation problem, that we call multi-material transport problem. This is a transportation problem, where distinct commodities are transported simultaneously along a network. The cost of the transportation depends on the network used to move the masses, as it is common in models studied in branched transportation. The main novelty is that in our model the cost per unit length of the network does not depend only on the total flow, but on the actual quantity of each commodity. This allows to take into account different interactions between the transported goods. We propose an Eulerian formulation of the discrete problem, describing the flow of each commodity through every point of the network. We provide minimal assumptions on the cost, under which existence of solutions can be proved. Moreover, we prove that, under mild additional assumptions, the problem can be rephrased as a mass minimization problem in a class of rectifiable currents with coefficients in a group, allowing to introduce a notion of calibration. The latter result is new even in the well studied framework of the “single-material” branched transportation.

Keywords: Branched transportation, rectifiable currents, calibrations, multi-material transport problem.

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Introduction

In this paper we study the multi-material transport problem (MMTP). Informally, given two arrays
\[ \mu^- = (\mu_1^-, \ldots, \mu_m^-), \quad \mu^+ = (\mu_1^+, \ldots, \mu_m^+) \]
of discrete positive measures on $\mathbb{R}^d$, we study transportation networks between $\mu^-$ and $\mu^+$ of the form
\[ T = (T_1, \ldots, T_m), \]
where each $T_i$ is a vector valued measure on $\mathbb{R}^d$ (with values also in $\mathbb{R}^d$) having distributional divergence
\[ \text{div}(T_i) = \mu_i^- - \mu_i^+ \quad \forall i = 1, \ldots, m. \quad (0.1) \]
More precisely, we will consider only transportation networks with a certain structure, namely for every $i = 1, \ldots, m$ we require that there exists a 1-rectifiable set $E_i \subset \mathbb{R}^d$ (endowed with a unit tangent field $\tau_i$) and a multiplicity $\theta_i \in \mathbb{Z}$ such that
\[ T_i := \theta_i \tau_i \otimes \mathcal{H}^1 |_{E_i}, \]
where the latter means that, for every continuous and compactly supported vectorfield $v$ on $\mathbb{R}^d$, we have
\[ \langle T_i, v \rangle = \int_{E_i} (\tau_i(x) \cdot v(x)) \theta_i(x) d\mathcal{H}^1(x). \]
Up to changing sign to the multiplicities $\theta_i$ we can assume that there exists a unit vector field $\tau$ defined on $E := \bigcup_{i=1}^m E_i$ such that
\[ \tau(x) = \tau_i(x) \text{ for } \mathcal{H}^1\text{-a.e. } x \in E_i, \text{ for every } i. \]
Then we associate to $T$:
\begin{itemize}
  \item the multiplicity $\theta = (\theta_1, \ldots, \theta_m)$, which is a function on $E$ with values in $\mathbb{Z}^m$,
  \item the vector valued measure on $\mathbb{R}^d$ (with values in $\mathbb{R}^d \times \mathbb{R}^m$)
  \[ T := \theta \tau \otimes \mathcal{H}^1 |_{E}, \]
  \item and the energy
  \[ E(T) := \int_E C(\theta) d\mathcal{H}^1, \quad (0.2) \]
\end{itemize}
where $C : \mathbb{Z}^m \to [0, +\infty)$ is a cost function.
The minimization of this energy, which takes into account a vector-valued multiplicity, is the core of the MMTP.

We briefly step backward for a heuristic introduction to the problem. In optimal transport problems one can focus on specific (concave) costs, which favor the aggregation of moved particles and generate optimal structures with branching. The branched transport problem is named after this peculiar phenomenon. A great interest has been devoted to branched transportation problems in the last years, providing several results concerning existence of solutions [41, 29, 2, 3, 10, 37, 16], regularity and stability [42, 6, 20, 19, 35, 43, 7, 9, 15, 13, 14] and strategies to compute minimizers or to prove minimality of concrete configurations [36, 5, 12, 34, 4, 31, 30, 28].

Nonetheless, to our knowledge only problems involving the transport of one (homogeneous) material have been studied and modelled as variational problems. These models do not apply in planning a network for the transportation of different goods, whose mutual interactions require a formulation which involves several variables. The easiest examples of natural multi-material transport problem concern mixed-use roads (where vehicles of different size and pedestrians are allowed to circulate) and the transport through vehicles for goods and passengers.

Another notable example is given by the power line communications technology (PLC, see [24, 21]), which uses the electric power distribution network for data transmission. PLC has been introduced in the United States of America more than a century ago and used for the communications on moving trains or, more generally, for maintenance operations of the electric power network. Recently, a special type of PLC, the broadband over power lines (BPL) is being studied and improved for high-speed data transmission, being particularly convenient for isolated areas. Electric power and data signals are impossible to be treated as a homogeneous “material” for several reasons the main one being the fact that the electricity and the internet supply are subject to different costs, depending on the users’ concentration and demands.

Similar problems, usually grouped under the name of multi-commodity flow problems, were studied (see e.g. [26, 22]) as minimization problems on graphs, often considering also constraints on the capacity of the network. Up to now, the aim of the research in this area was mainly devoted to study the complexity of the problem and to improve the efficiency of algorithms for numerical solutions.

The main results of the present paper are the existence of solutions to the minimization of the energy (2.2) under the constraint (0.1), with minimal assumptions on the cost $C$ (Theorem 2.3) and, under mild additional assumptions, the equivalence between the MMTP and a mass minimization problem in a class of rectifiable currents with coefficients in a group (Theorem 2.4). The equivalence between the two problems allows us to introduce the notion of calibrations in this context. This was initiated in previous works ([31, 30]) for “single-material” branched transportation problems and for very special choices of the cost functionals (i.e., the Steiner cost and the Gilbert-Steiner $\alpha$-mass, respectively) and the benefit of introducing calibrations in such contexts is witnessed e.g. by [34, 4, 11]. Under our general assumptions on the cost, the equivalence result is new even in the case of “single-material” transportation problems.

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1. Notation

Consider a norm $|\cdot|_\varphi$ on $\mathbb{R}^n$ and its dual norm $|\cdot|_\varphi^*$. The Euclidean norm is instead denoted by $|\cdot|_1$ and we will always denote an orthonormal basis of $(\mathbb{R}^n, |\cdot|)$ by $\{e_1, \ldots, e_n\}$. The scalar product between vectors $v$ and $w$ of $\mathbb{R}^n$ is denoted $\langle v, w \rangle$. Our aim is to define (1-dimensional) currents in $\mathbb{R}^d$ with coefficients in $(\mathbb{R}_+, |\cdot|_\varphi)$. More generally, currents with coefficients in a normed (abelian) group $(G, |\cdot|_\varphi)$ have been introduced in [25] and already studied by several authors (see [39, 40, 18]). Our interest is restricted to the case $(G, |\cdot|_\varphi) = (\mathbb{R}_+, |\cdot|_\varphi)$, and we follow a “non-standard” approach, defining currents by duality with $\mathbb{R}_+$-valued differential forms in $\mathbb{R}^d$ (instead of completing the space of polyhedral $\mathbb{R}^n$-chains). With this approach we obtain an integral representation of currents, (see (1.1))
which allows to introduce calibrations in a natural way. We refer the reader to [32] for a general duality theory for currents with coefficients in Banach spaces.

We introduce now some notation about currents with coefficients in \((\mathbb{R}^n, |\cdot|_p)\). For the rest of this section, we will drop the norm \(|\cdot|_p\), since we are considering it to be fixed. We limit ourselves to define what is strictly necessary for the purposes of our paper. To begin with, we give the following definitions.

**Definition 1.1** (\(\mathbb{R}^n\)-valued 1-covector). A map \(\alpha : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}\) is an \(\mathbb{R}^n\)-valued 1-covector in \(\mathbb{R}^d\) if

(i) \(\forall \tau \in \mathbb{R}^d, \alpha(\tau, \cdot) \in (\mathbb{R}^n)^*\); 
(ii) \(\forall \theta \in \mathbb{R}^n, \alpha(\cdot, \theta) : \mathbb{R}^d \to \mathbb{R}\) is a “classical” 1-covector.

The evaluation of \(\alpha\) on the pair \((\tau, \theta)\) is also denoted by \(\langle \alpha; \tau, \theta \rangle\). The space of \(\mathbb{R}^n\)-valued 1-covectors in \(\mathbb{R}^d\) is denoted by \(\Lambda^1(\mathbb{R}^d; \mathbb{R}^n)\).

Observe that the space \(\Lambda^1(\mathbb{R}^d; \mathbb{R}^n)\) is a normed vector space when endowed with the comass norm
\[
\|\alpha\|_\phi := \sup\{|\alpha(\tau, \cdot)|_\phi : |\tau| \leq 1, \tau \in \mathbb{R}^d\}.
\]
We can write the action of an \(\mathbb{R}^n\)-valued 1-covector \(\alpha\) as
\[
\alpha(\tau, \theta) = \sum_{j=1}^n \alpha_j(\tau)(e_j; \theta),
\]
where, for \(j = 1, \ldots, n\), \(\alpha_j(\cdot; \theta)\) are the components of \(\alpha\).

Fix now an open set \(U \subset \mathbb{R}^d\). Once and for all we assume \(U\) to be simply connected. It is clear that the simple connectedness of \(U\) is not restrictive for most of the reasonable cases (Euclidean space, balls, etc.), nonetheless we remark that another choice of \(U\) would simply change the homology class in which we set the variational problem.

**Definition 1.2** (\(\mathbb{R}^n\)-valued differential 1-form). An \(\mathbb{R}^n\)-valued differential 1-form in \(U\) is a map \(\omega : U \to \Lambda^1(\mathbb{R}^d; \mathbb{R}^n)\). We say that \(\omega\) is smooth if and only if every component \(\omega_j\) belongs to \(C^\infty(U; \Lambda^1(\mathbb{R}^d; \mathbb{R}))\), where the components of an \(\mathbb{R}^n\)-valued differential 1-form are defined similarly to \(\mathbb{R}^n\)-valued 1-covectors.

We denote by \(C^\infty(\mathbb{R}^d; \mathbb{R}^n)\) the vector space of smooth, \(\mathbb{R}^n\)-valued differential 1-forms, with compact support in \(U\).

Finally, we define the **comass norm** of the \(\mathbb{R}^n\)-valued differential 1-form \(\omega\) as
\[
\|\omega\|_\phi := \sup_{x \in U} \|\omega(x)\|_\phi.
\]
The differential of an \(\mathbb{R}^n\)-valued function (0-form) is once again defined using the components.

**Definition 1.3** (Differential of an \(\mathbb{R}^n\)-valued 0-form). Let \(\eta \in C^\infty_0(U; \mathbb{R}^n)\) be an \(\mathbb{R}^n\)-valued 0-form, and, for \(j = 1, \ldots, n\), denote with \(\eta_j\) its components (i.e., \(\eta_j := \langle \eta; e_j \rangle\)). Then the differential of \(\eta\) is the \(\mathbb{R}^n\)-valued differential 1-form which is defined componentwise by
\[
(d\eta)_j := \langle d\eta; e_j \rangle, \quad j = 1, \ldots, n.
\]

We are now ready to define 1-currents with coefficients in \(\mathbb{R}^n\).

**Definition 1.4** (1-currents with coefficients in \(\mathbb{R}^n\)). Let \(T\) be a linear functional on \(C^\infty_0(U; \Lambda^1(\mathbb{R}^d; \mathbb{R}^n))\). By definition, \(T\) is continuous if \(T(\omega^k) \to 0\) for every sequence of \(\mathbb{R}^n\)-valued differential 1-forms \(\omega^k \in C^\infty(U; \Lambda^1(\mathbb{R}^d, \mathbb{R}^n))\) such that

(i) \(\text{spt}(\omega^k) \subset K\), for some compact set \(K \subset U\); 
(ii) every component of \(\omega^k\) converges uniformly to 0 with all its derivatives when \(k \to \infty\).

The space of linear, continuous functionals on \(C^\infty_0(U; \Lambda^1(\mathbb{R}^d, \mathbb{R}^n))\) is the space of 1-currents in \(U\) with coefficients in \(\mathbb{R}^n\). We write \(T^a \rightharpoonup T\) when the sequence of currents \((T^a)_{a \geq 1}\) with coefficients in \(\mathbb{R}^n\) is weakly*-converging to \(T\), i.e., when
\[
T^a(\omega) \rightharpoonup T(\omega) \quad \forall \omega \in C^\infty_0(U; \Lambda^1(\mathbb{R}^d, \mathbb{R}^n)).
\]
Furthermore, if \(T\) is a 1-current with coefficients in \(\mathbb{R}^n\), we define its mass as
\[
\mathcal{M}(T) := \sup\{T(\omega) : \|\omega\|_\phi \leq 1\}.
\]

The **boundary** of \(T\) is the \(\mathbb{R}^n\)-valued distribution \(\partial T\) which fulfills the relation
\[
\partial T(\eta) := T(d\eta) \quad \forall \eta \in C^\infty_0(U; \mathbb{R}^n).
\]
Finally, when we mention the components of the current $T$, we refer to the “classical” currents $T_j$, $j = 1, \ldots, n$, defined as

$$T_j(\omega) := T(\omega e_j) \quad \forall \omega \in C^\infty(U; \Lambda^j(\mathbb{R}^d; \mathbb{R})), $$

where we denoted by $\omega e_j$ the 1-form with coefficients in $\mathbb{R}$ whose components are all null except for the $j$-th component, which is $\omega$.

**Remark 1.5.** Analogously, the definitions of $\mathbb{R}^n$-valued $k$-covectors, $\mathbb{R}^n$-valued $k$-forms, and $k$-currents with coefficients in $\mathbb{R}^n$ are given by specifying their components, i.e., an array made of $n$ “classical” $k$-covectors, $k$-forms, and $k$-currents, respectively. Similarly, the definitions of the differential of an $\mathbb{R}^n$-valued $k$-form and of the boundary of a $k$-current with coefficients in $\mathbb{R}^n$ are understood.

We are going to consider the following special class of currents. We recall that a 1-rectifiable set $E \subset U$ is an $\mathcal{H}^1$-measurable set which can be covered, up to an $\mathcal{H}^1$-null subset, with the images of countably many curves of class $C^1$. A 1-rectifiable set $E$ has a well defined notion of tangent space at $\mathcal{H}^1$-a.e. point $x \in E$, which is denoted $\text{Tan}(E, x)$.

**Definition 1.6 (Rectifiable 1-currents with coefficients in $\mathbb{Z}^n$).** A rectifiable 1-current in $U$, with coefficients in $\mathbb{Z}^n$ is a 1-current with finite mass admitting the integral representation

$$T(\omega) = \int_{\Sigma} \omega(x); \xi(x), \theta(x) \, dH^1(x),$$

where $\Sigma \subset U$ is a countably 1-rectifiable set, $\xi(x) : \Sigma \to \mathbb{Z}^d \cap \text{Tan}(E, x)$ for $\mathcal{H}^1$-a.e. $x$ is called the orientation, and $\theta \in L^1_{\mathcal{H}}(\Sigma; \mathbb{Z}^n)$ is the multiplicity. We denote such current $T$ as $[\Sigma, \xi, \theta]$.

We have the following characterization of the mass of a rectifiable current (see [38, 26.8] for the analogous statement for “classical” currents).

**Lemma 1.7 (Characterization of the mass).** If $T = [\Sigma, \xi, \theta]$ is a rectifiable current with coefficients in $\mathbb{Z}^n$, then

$$M(T) = \int_{\Sigma} [\theta(x)]_\omega \, dH^1(x).$$

For the rest of the paper, we mainly focus on rectifiable 1-currents with coefficients in $\mathbb{Z}^n$ whose boundary has finite mass. With a small abuse of notation we call them integral $\mathbb{Z}^n$-currents. The following structure theorem for integral $\mathbb{Z}^n$-currents is an immediate consequence of its counterpart for “classical” integral 1-currents ([23, 4.2.25]). See also [17, Theorem 2.5]).

**Theorem 1.8 (Structure of integral $\mathbb{Z}^n$-currents).** Let $T$ be an integral $\mathbb{Z}^n$-current in $U$. Then

$$T = \sum_{i=1}^{h} T^i + \sum_{\ell=1}^{\infty} T^\ell,$$

where:

- $T^i = [\hat{\Gamma}_i, \hat{\tau}_i, \hat{\theta}_i]$, where $\hat{\Gamma}_i$ is the image of an injective, Lipschitz, open curve $\gamma_i : [0, 1] \to U$,
  - $\hat{\tau}_i(\gamma_i(t)) = \frac{\hat{\tau}(\gamma_i(t))}{|\gamma_i(t)|}$ for a.e. $t$, and $\hat{\theta}_i \in \mathbb{Z}^n$ is constant on $\hat{\Gamma}_i$,

- $T^\ell = [\hat{\Gamma}_\ell, \hat{\tau}_\ell, \hat{\theta}_\ell]$, where $\hat{\Gamma}_\ell$ is the image of a Lipschitz, closed curve $\gamma_\ell : [0, 1] \to U$, which is injective on $(0, 1)$, $\hat{\tau}_\ell(\gamma_\ell(t)) = \frac{\hat{\tau}(\gamma_\ell(t))}{|\gamma_\ell(t)|}$ for a.e. $t$, and $\hat{\theta}_\ell \in \mathbb{Z}^n$ is constant on $\hat{\Gamma}_\ell$.

### 1.1. Compactness

The following compactness theorem holds:

**Theorem 1.9 (Compactness).** Consider a sequence $(T^i)_{i \geq 1}$ of integral $\mathbb{Z}^n$-currents in $U$ such that

$$\sup_{i \geq 1} (M(T^i) + M(\partial T^i)) < +\infty.$$

Then there exists an integral $\mathbb{Z}^n$-current $T$ in $U$ and a subsequence $(T^{i_h})_{h \geq 1}$ such that

$$T^{i_h} \rightharpoonup T.$$

Moreover it holds

$$\liminf_{h \to +\infty} M(T^{i_h}) \geq M(T).$$
The proof of this result is a straightforward application of the Closure Theorem for integral currents (see [23, 4.2.16] or [27, Theorem 2, §2.4]) to each component $\langle T_i \rangle_{i \geq 1}$ of the sequence $\langle T^n \rangle_{n \geq 1}$. The lower semicontinuity of the mass is straightforward. By direct methods we get the existence of a mass-minimizing rectifiable current for a given boundary.

**Corollary 1.10.** Let $T^\circ$ be an integral $\mathbb{Z}^n$-current in $U$. Then there exists an integral $\mathbb{Z}^n$-current $T^\circ$ in $U$ such that

$$\mathbf{M}(T^\circ) = \min_{\partial T = \partial T^\circ} \mathbf{M}(T),$$

where the minimum is computed among integral $\mathbb{Z}^n$-currents in $U$.

1.2. **Calibrations.** The main advantage of proving the equivalence between the MMTP and a mass minimization problem is that, in the latter case, we can make use of calibrations to prove minimality.

**Definition 1.11** (Calibration). Consider a rectifiable 1-current $T = [\Sigma, \tau, \theta]$ in $U$, with coefficients in $\mathbb{Z}^n$. A smooth $\mathbb{R}^n$-valued differential 1-form $\omega$ in $U$ is a calibration for $T$ if the following conditions hold:

(i) for a.e. $x \in \Sigma$ we have that $\langle \omega(x); \tau(x) \rangle = |\theta(x)|_\varrho$;

(ii) the form is closed, i.e., $d\omega = 0$;

(iii) for every $x \in U$, every unit vector $\tau \in \mathbb{R}^n$ and every $h \in \mathbb{R}^n$ we have that

$$\langle \omega(x); \tau, h \rangle \leq |h|_\varrho.$$

The existence of a calibration is a sufficient condition for minimality.

**Theorem 1.12** (Minimality of calibrated currents). Let $T = [\Sigma, \tau, \theta]$ be a rectifiable 1-current in $U$, with coefficients in $\mathbb{Z}^n$, and let $\omega$ be a calibration for $T$. Then $T$ minimizes the mass among rectifiable 1-currents in $U$, with coefficients in $\mathbb{Z}^n$, with the same boundary $\partial T$.

**Proof.** A competitor $T' = [\Sigma', \tau', \theta']$ fulfills $\partial T' = \partial T$. Since $U$ is simply connected, there exists a 2-dimensional current $R$ in $U$, with coefficients in $\mathbb{Z}^n$, such that $\partial R = T - T'$. As a consequence, together with the properties of $\omega$ listed in Definition 1.11, we obtain that

$$\mathbf{M}(T) = \int_\Sigma |\theta(x)|_\varrho d\mathcal{H}^1(x)$$

$$(i) \int_\Sigma \langle \omega(x); \tau(x), \theta(x) \rangle d\mathcal{H}^1(x) = \partial R(\omega) + T'(\omega)$$

$$(ii) \int_{\Sigma'} \langle \omega(x); \tau'(x), \theta'(x) \rangle d\mathcal{H}^1(x)$$

$$(iii) \int_{\Sigma'} |\theta'(x)|_\varrho d\mathcal{H}^1(x) = \mathbf{M}(T').$$



2. **Multi-material transport problem**

In this section, we define the multi-material transport problem and we state the main result of the paper. First of all, let us introduce some notation. Our ambient is the Euclidean space $\mathbb{R}^d$. Fix also a natural number $m$. For $n = 1, 2, \ldots$, we consider the following partial order on $\mathbb{R}^n$, where the coordinates are always computed with respect to the standard basis $\{e_1, \ldots, e_n\}$. Given two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we write $x \leq y$ if and only if

$$|x_i| \leq |y_i| \quad \text{and} \quad x_i y_i \geq 0, \quad \forall i = 1, \ldots, n.$$  \hfill (2.1)

We say that a norm $\| \cdot \|$ on $\mathbb{R}^n$ is monotone if $\|x\| \leq |y|$, for every $x \leq y \in \mathbb{R}^n$. We say that $\| \cdot \|$ is absolute if $\|x\| = |x|$, $\forall x \in \mathbb{R}^n$, where $x = (|x_1|, \ldots, |x_n|)$.

**Definition 2.1** (Multi-material cost). A multi-material cost is a function $C : \mathbb{Z}^m \to [0, +\infty)$ with the following properties:

(i) $C$ is even, i.e. $C(x) = C(-x)$, and $C(0) = 0$ if and only if $x = 0$;

(ii) $C$ is increasing, i.e. $C(x) \leq C(y)$ for every $x \leq y$;

(iii) $C$ is subadditive, i.e. $C(x + y) \leq C(x) + C(y)$ for every $x, y \in \mathbb{Z}^m$.

In order to prove the equivalence between the MMTP and a mass minimization problem, we will replace (i) with a stronger property, namely
(iii') there exists a monotone norm \( \| \cdot \| \) on \( \mathbb{R}^m \) with respect to which \( C \) is sublinear, i.e., \( \frac{C(x)}{|x|} \leq \frac{C(x')}{|x'|} \) for every \( x, x' \in \mathbb{Z}^m \setminus \{0\} \) with \( x' \leq x \).

**Remark 2.2** (Extension of multi-material costs). If \( C \) is defined only on a rectangle \( R := [-a_1, a_1] \times \ldots \times [-a_m, a_m] \subset \mathbb{Z}^m \) and it satisfies (i), (ii), (iii) (respectively (i), (ii), (iii')) on \( R \), then one can define a new cost \( \bar{C} : \mathbb{Z}^m \to [0, +\infty) \) defining

\[
\bar{C}(x) := \max\{C(y) : y \leq x\}.
\]

One can see immediately that the cost \( \bar{C} \) satisfies (i), (ii), (iii) (respectively (i), (ii), (iii')).

A multi-material cost induces a functional on integral \( Z^m \)-currents, that we denote \( E \). Given an integral \( Z^m \)-current \( T = \lfloor \Sigma, \tau, \theta \rfloor \), we denote its energy by

\[
E(T) := \int_{\mathbb{R}^d} \bar{C}(\theta) \, d\mathcal{H}^1.
\]

Let us now fix a rectifiable 0-current \( B \) on \( \mathbb{R}^d \) with coefficients in \( \mathbb{Z}^m \), which is the boundary of an integral \( Z^m \)-current. In particular \( B \) is represented by the discrete \( \mathbb{R}^m \)-valued measure

\[
B = \sum_{\ell=1}^M \eta_\ell \delta_{p_\ell},
\]

where \( p_\ell \) are points in \( \mathbb{R}^d \), \( \eta_\ell = (\eta_{\ell,1}, \ldots, \eta_{\ell,m}) \in \mathbb{Z}^m \) and \( \sum_{\ell=1}^M \eta_\ell = (0, \ldots, 0) \in \mathbb{Z}^m \).

We are now able to state the multi-material transport problem. Let \( C \) satisfy properties (i), (ii), (iii) of Definition 2.1.

**MMTP:** Among all integral \( Z^m \)-currents \( T = \lfloor \Sigma, \tau, \theta \rfloor \) in \( \mathbb{R}^d \) such that \( \partial T = B \), find one which minimizes the energy \( E(T) \).

**Theorem 2.3** (Existence of solutions). The MMTP admits a solution.

The existence of solutions for the MMTP is a trivial consequence of Theorem 1.9 and the lower semicontinuity of the functional \( \mathcal{E} \), which is stated in [40, [6] and it can be proved with the same technique used in [16]. In [33], we prove the lower semicontinuity in a more general framework to obtain the existence of solutions to a "continuous" version of the MMTP. We remark here that, under the additional assumption (iii') on the cost functional, the existence is also trivial consequence Theorem 2.4 below.

The main result of the paper is the fact that, with the additional assumption (iii') on the cost functional, the MMTP is equivalent to the superposition of a certain number of mass minimization problems among integral currents, with coefficients in a group (which is larger than \( \mathbb{Z}^m \)). Introducing such problems requires some additional notation.

Let \( B \) be as in 2.3. For \( i = 1, \ldots, m \), let

\[
N_i := \frac{1}{2} \sum_{\ell=1}^M |\eta_{\ell,i}| \quad \text{and let} \quad N := \sum_{i=1}^m N_i.
\]

Let \( \sigma = (\sigma_1, \ldots, \sigma_m) \in S_{N_1} \times \cdots \times S_{N_m}, \) where \( S_q \) is the group of permutations on \( q \) elements. We associate to \( \sigma \) and \( B \) a rectifiable 0-current, with coefficients in \( \mathbb{Z}^N \) in the following way.

Firstly we consider two vectors \( P := (P_1, \ldots, P_N) \) and \( D := (D_1, \ldots, D_N) \) in \( \mathbb{R}^{Nq} \), defined via the procedure explained below.

Denote \( N_0 := 0 \). For every \( j \in \{1, \ldots, N\} \), let \( i(j) \) be the first index \( i \) such that \( N_1 + \cdots + N_i \geq j \), and let \( j := \sum_{k=0}^{i(j)-1} N_k \). Moreover let \( \ell^-(j) \) be the first index \( \ell \) such that

\[
\sum_{\ell, i(j) < 0} |\eta_{\ell,i(j)}| \geq j - \bar{j}.
\]

Similarly let \( \ell^+(j) \) be the first index \( \ell \) such that

\[
\sum_{\ell, i(j) > 0} |\eta_{\ell,i(j)}| \geq j - \bar{j}.
\]
Finally we define
\[ P(j) := p_{\ell^-(j)} \quad \text{and} \quad D(j) := p_{\ell^+(j)}. \]
With a small abuse of notation, we write \( \sigma(j) \) for the number \( j + \sigma_{i(j)}(j - \bar{j}) \). Lastly we define our rectifiable 0-current, with coefficients in \( \mathbb{Z}^N \) as
\[ B_\sigma := - \sum_{j=1}^N e_j \delta p_j + \sum_{j=1}^N \epsilon_j \delta D_{\sigma(j)}. \quad (2.5) \]

Now we can state our alternative formulation of the multi-material transport problem, which is simply a mass-minimization problem:

**MMP**: Let \( | \cdot |_{\phi} \) be a norm on \( \mathbb{R}^N \). Among all \( \sigma \in \mathcal{S}_{N_1} \times \cdots \times \mathcal{S}_{N_m} \) and among all integral \( \mathbb{Z}^N \)-currents \( T = [\Sigma, \tau, \theta] \) in \( \mathbb{R}^d \) such that \( \partial T = B_\sigma \) (defined in (2.5)), find one which minimizes the mass \( M(T) \), where the mass is computed with respect to the norm \( | \cdot |_{\phi} \).

The main result of the paper is the following

**Theorem 2.4** (Equivalence between MMTP and MMP). *Let \( B \) be as in (2.3) and \( N \) as in (2.4). Then, for every \( C \) as in Definition 2.1, satisfying (i),(ii),(iii), there exists a norm \( | \cdot |_{\phi} \) on \( \mathbb{R}^N \) such that the problems MMTP and MMP are equivalent. Namely the minima are the same and moreover there is a canonical way to construct a solution of MMTP from a solution of MMP and vice versa.*

**Remark 2.5** (Irrigation-type problems). A corollary of the proof of Theorem 2.4 is the following. If there exists one index \( \tilde{i} \in \{1, \ldots, M\} \) such that \( |\eta_{\tilde{i}}| = \sum_{j \neq \tilde{i}} |\eta_j| \), for every \( i = 1, \ldots, m \), then in the MMP it is not necessary to minimize among the permutations \( \sigma \) (i.e. the minimum is the same for every permutation). In the “single-material” case, the assumption corresponds to the case called “irrigation problem”, where the initial (or the target) measure is a Dirac delta.

3. Equivalence between MMTP and MMP

The aim of this section is to establish the equivalence between MMTP and MMP of Section 2. This follows from Theorem 1.8, once we find a norm \( | \cdot |_{\phi} \) on \( \mathbb{R}^N \) which is monotone and satisfies
\[ C(\theta_1, \ldots, \theta_m) = \left| \frac{1}{\theta_1} \sum_{j=1}^{\theta_1} e_j + \frac{1}{\theta_2} \sum_{j=N_1+1}^{N_1+\theta_2} e_j + \cdots + \frac{1}{\theta_m} \sum_{j=N_{m-1}+1}^{N_m+\theta_m} e_j \right|_{\phi}, \quad (3.1) \]
where \( N \) and \( N_i \) (\( i = 1, \ldots, m \)) are defined in (2.4). The existence of such norm is proved in Theorem 3.2. Firstly we show how to prove Theorem 2.4 using the existence of \( | \cdot |_{\phi} \).

**Proof of Theorem 2.4.** Fix \( B \) as in (2.3). We divide the proof in two steps.

**Step 1:** from MMTP to MMP. Let \( T := [\Sigma, \tau, \theta] \) be an integral \( \mathbb{Z}^m \)-current which is a competitor for MMTP. The aim of this step is to construct from \( T \) a competitor \( \tilde{T} \) for MMP “associated” to \( B \), such that \( M(\tilde{T}) \leq E(T) \).

Consider the components of \( T \)
\[ T_i := [\Sigma, \tau, \theta_1], \ldots, T_m := [\Sigma, \tau, \theta_m]. \]
By [23, 4.2.25] we can write, for \( i = 1, \ldots, m \)
\[ T_i = \sum_{k=1}^{N_i} \tilde{T}_{k}^i + \sum_{h=1}^{\infty} Y_{h}^i, \quad (3.2) \]
with
\[ M(T_i) = \sum_{k=1}^{N_i} M(\tilde{T}_{k}^i) + \sum_{h=1}^{\infty} M(Y_{h}^i) \quad \text{and} \quad M(\partial T_i) = \sum_{k=1}^{N_i} M(\partial \tilde{T}_{k}^i), \quad (3.3) \]
where:
\( \tilde{T}_{k}^i := [\Pi_{\tilde{\gamma}_i}, \tau_{\tilde{\gamma}_i}, 1] \) are integral 1-currents associated to simple, Lipschitz, open curves \( \tilde{\gamma}_i : [0,1] \to \mathbb{R}^d \), where \( \Gamma_i := \text{Im}(\tilde{\gamma}_i) \) and \( \tau_{\tilde{\gamma}_i} := \frac{(\tilde{\gamma}_i)'(\tilde{\gamma}_i)(\tilde{\gamma}_i)'(\tilde{\gamma}_i)}{||\tilde{\gamma}_i'||}. \)
\[ \tilde{T}^i_h := [Z^i_h, \nu^i_h, 1] \] are integral 1-currents associated to simple, Lipschitz, closed curves (cycles) \( \xi^i : [0, 1] \to \mathbb{R}^d \), where \( Z^i_h := \text{Im}(\xi^i_h) \) and \( \nu^i_h := \frac{(\xi^i_h)'}{(\xi^i_h)'} \).

For every \( i = 1, \ldots, m \), denote by \( \Sigma_i := \bigcup_{k=1}^N \Gamma_k^i \) and by \( T^i := [\Sigma, \tau, \theta^i] \) the 1-current

\[ T'_i := \sum_{k=1}^N T^i_h \]

and let \( T' \) be the integral \( \mathbb{Z}^m \)-current whose components are \( (T'_1, \ldots, T'_m) \).

It follows from (3.2) that for every \( i = 1, \ldots, m \) and for \( \mathcal{H}^1 \)-a.e. \( x \in \Sigma \) it holds

\[ \theta_i(x) = \sum_{h=1}^N \chi_{T^i_h}(x) (r^i_h(x); \tau(x)) + \sum_{h=1}^\infty \sum_{k=1}^\infty \chi_{C^i_h}(x) (\nu^i_h(x); \tau(x)), \]

where we denoted by \( \chi_E \) the characteristic function of the set \( E \) taking values 0 and 1. Combining (3.4) and (3.3) we deduce that for every \( i = 1, \ldots, m \) it holds

\[ \tau^i_h(x) = \text{sign}(\theta_i(x)) r^i(x), \quad \text{for } \mathcal{H}^1 \text{-a.e. } x \in \Gamma^i_h, \text{ for every } k \]

and

\[ \nu^i_h(x) = \text{sign}(\theta_i(x)) \tau(x), \quad \text{for } \mathcal{H}^1 \text{-a.e. } x \in C^i_h, \text{ for every } h. \]

Hence, for \( i = 1, \ldots, m \) it holds \( |\theta'_i(x)| \leq |\theta_i(x)| \) and \( \theta'_i(x) \theta_i(x) \geq 0 \), for \( \mathcal{H}^1 \text{-a.e. } x \in \Sigma \), which yields \( E(T') \leq E(T) \). Moreover by (3.2) it holds \( \partial T' = \partial T \).

Next we associate to \( T' \) an integral \( \mathbb{Z}^N \)-current \( T \), simply defining \( T \) to be the current with components \( (T_1, \ldots, T_N) \), where we set, for \( j = 1, \ldots, N \) (recalling the definition of \( i(j) \) and \( j \) from 2)

\[ T_j := \tilde{T}^i_{h(j)}, \quad \text{for } k = j - \bar{j}. \]

Applying the boundary operator to (3.2), it follows that \( T \) is a competitor for MMP. Moreover by (3.4) and (3.1) it follows that \( M(T) = E(T') \leq E(T) \).

**Step 2: from MMP to MMTP**

Let \( \sigma \in \mathcal{S}_{N_1} \times \cdots \times \mathcal{S}_{N_m} \). Let \( \tilde{T} := [\Sigma, \tilde{\tau}, \tilde{\theta}] \) be an integral \( \mathbb{Z}^N \)-current which is a competitor for MMP and in particular \( \partial \tilde{T} = B_\sigma \) (defined in (2.5)). The aim of this step is to construct from \( \tilde{T} \) a competitor \( T \) for MMTP associated to \( B_\sigma \), such that \( E(T) \leq M(\tilde{T}) \).

Let

\[ T_1 := [\Sigma, \tilde{\tau}, \tilde{\theta}_1], \ldots, T_N := [\Sigma, \tilde{\tau}, \tilde{\theta}_N] \]

be the components of \( \tilde{T} \).

As in the previous step, by [23, 4.2.25] and using the fact that \( M(\partial \tilde{T}_j) = 2 \), we can write for \( j = 1, \ldots, N \)

\[ \tilde{T}_j = \tilde{T}_j + \sum_{h=1}^\infty \tilde{T}^j_h, \]

with

\[ M(\tilde{T}_j) = M(\tilde{T}_j) + \sum_{h=1}^\infty M(\tilde{T}^j_h), \]

where:

- \( \tilde{T}_j := [\tilde{\Gamma}_j, \tilde{\tau}_j, 1] \) are integral 1-currents associated to simple, Lipschitz, open curves \( \gamma_j : [0, 1] \to \mathbb{R}^d \), where \( \tilde{\Gamma}_j := \text{Im}(\gamma_j) \) and \( \tilde{r}_j := \frac{(\gamma_j)'}{(\gamma_j)'} \).
- \( \tilde{T}^j_h := [Z^j_h, \nu^j_h, 1] \) are integral 1-currents associated to simple Lipschitz closed curves \( \xi^j_h : [0, 1] \to \mathbb{R}^d \), with \( \xi(0) = \xi(1) \), where \( Z^j_h := \text{Im}(\xi^j_h) \) and \( \nu^j_h := \frac{(\xi^j_h)'}{(\xi^j_h)'} \).

Let \( T' \) be the integral \( \mathbb{Z}^N \)-current whose components are \( (\tilde{T}_1, \ldots, \tilde{T}_N) \). By (3.5) and (3.6), for \( j = 1, \ldots, N \) it holds \( (\tilde{T}_j, \tilde{T}_j) = \text{sign}(\theta_j) \mathcal{H}^1 \)-a.e. in \( \tilde{T}_j \) and hence, since \( |\cdot|_0 \) is a monotone norm, we have \( M(T') \leq M(T) \). Moreover, by (3.5) it holds \( \partial T' = \partial T \).

Let \( T \) be the integral \( \mathbb{Z}^m \)-current with components (recalling the definition of \( i(j) \) from 2)

\[ T_i := \sum_{j=i(j)=1}^\infty \tilde{T}_j. \]

Since \( \partial T' = B_\sigma \), then \( \partial T = B \). Moreover by (3.1) it holds \( E(T) = M(T') \leq M(T) \). \( \square \)
We conclude this section by proving the existence of tone norm $|\cdot|_q$ satisfying (3.1).

In the proof of next theorem, we will use the following fact, which can be found in [1]. Recall the notions of monotone and absolute norm given at the beginning of Section 2, as well as the partial order introduced there.

**Lemma 3.1.** An absolute norm on $\mathbb{R}^n$ is monotone.

We will use the term orlant in $\mathbb{R}^n$ for the following subset of $\mathbb{R}^n$. Consider a vector $\xi \in \mathbb{R}^d$ whose coordinates are only $\pm 1$. The $\xi$-orlant is:

$$\{ x \in \mathbb{R}^d : \xi x \geq 0, \forall \ell = 1, \ldots, n \}.$$ 

Note that an orlant is always closed.

**Theorem 3.2 (Existence of a norm satisfying (3.1)).** Let $C : \mathbb{Z}^m \to [0, +\infty)$ be a function satisfying (i), (ii), (iii') of Definition 2.1. Let $B$ be as in (2.3) and let $N$ and $N_i$ ($i = 1, \ldots, m$) be the natural numbers defined in (2.4). Then there exists a monotone norm $|\cdot|_q$ on $\mathbb{R}^n$ satisfying (3.1).

**Proof. Step 1:** First of all, let us suppose that $C$ has the additional property that

$$C(x) = C(\bar{x})$$

(3.7)

for every $x \in \mathbb{Z}^m$, where we used the notation introduced at the beginning of Section 2. Let us denote by $A$ the set of elements of $\mathbb{Z}^n$ whose coordinates are only 0's and 1's, and denote $\bar{A} := \{ x \in \mathbb{Z}^n : \bar{x} \in A \}$.

We say that the pair $(A, B)$ in $\mathcal{A} \times \mathcal{A}$ is good if $A - B \neq 0$ and the following implications hold, for every $i = 1, \ldots, m$ (again, we set $N_i := 0$):

- if $a_j = 1$ for some $j$ between $N_1 + \cdots + N_{i-1} + 1$ and $N_1 + \cdots + N_i$ then $b_h = 0$ for all indices $h$ in the same range;
- if $b_j = 1$ for some $j$ between $N_1 + \cdots + N_{i-1} + 1$ and $N_1 + \cdots + N_i$ then $a_h = 0$ for all indices $h$ in the same range.

If $(A, B)$ is a good pair with $A = (a_1, \ldots, a_N)$, $B = (b_1, \ldots, b_N) \in A$, we define

$$c_{A, B} := C \left( \sum_{j=1}^{N_1} (a_j - b_j), \ldots, \sum_{j=N-N_{m-1}+1}^N (a_j - b_j) \right).$$

(3.8)

Observing that, if $A - B \neq 0$ we have $c_{A, B} \neq 0$, we can define

$$q_{A, B} := \frac{A - B}{c_{A, B}},$$

for any good pair $(A, B)$. Let now $D \neq 0$ be any vector in $\mathcal{A}$. We define $c_D := c_{D, 0}$, which is well-defined since $(\bar{D}, 0)$ is a good pair. As above, we define $q_D := \frac{D}{c_D}$. Consider the convex hull

$$C := \text{co}\{q_D : D \in \mathcal{A} \setminus \{0\} \} \subset \mathbb{R}^N.$$ 

The lemma is proven if we show three properties of $C$:

1. $C$ is a convex body (i.e. the closure of its non-empty interior) which is bounded and symmetric with respect to the origin;
2. $C$ is a monotone set, i.e. for every $x, y \in \mathbb{R}^N$, with $y \leq x$, if $x \in C$, then also $y \in C$.
3. it holds

$q_{A, B} \in \partial C, \ \forall A, B \in \mathcal{A},$ such that $(A, B)$ is a good pair.

Indeed, if (1) holds, there exists a norm $|\cdot|_q$ on $\mathbb{R}^N$ whose unit ball is the set $C$. Then, (2) and (3) imply respectively that this norm is monotone and that it satisfies (3.1).

To prove (1), notice that for every $j = 1, \ldots, N$, $q_{2^j e_j}$ are contained in $C$, hence $0 \in \text{int}(C)$. The fact that $C$ is symmetric with respect to the origin follows from the fact that the multi-material cost $C$ is even. Finally, the boundedness is trivial, since $C$ the convex hull of a finite set.

We will now prove (2), i.e. that $C$ is a monotone set. To prove it, we show that the norm with ball $C$ is absolute. This implies the monotonicity by Lemma 3.1. Let $x \in \mathbb{R}^N$, with $|x|_q = 1$. Write

$$x = \sum_{k=1}^K k q_D x,$$
where $D^k \in \mathcal{A} \setminus \{0\}$, $\sum_{k=1}^{K} t_k = 1$, with $t_k$ positive. There exists a diagonal matrix $M \in \text{Mat}^{N \times N}$ with entries $1, -1, 0$ such that $Mx = z$. Therefore:

$$|x|_\phi = |Mx|_\phi = \left| \sum_{k=1}^{K} t_k Mq_{D^k} \right|_\phi = \left| \sum_{k=1}^{K} t_k q_{M^k D^k} \right|_\phi \leq \sum_{k=1}^{K} t_k |q_{M^k D^k}|_\phi \leq \sum_{k=1}^{K} t_k \leq 1,$$

where the third equality follows from the fact that $c_{D^k} = c_{M^k D^k}$ and the second inequality follows from the fact that $|q_D|_\phi \leq 1, \forall D \in \mathcal{A}$, by the definition of $C$. This proves that

$$|x|_\phi \leq |x|_\phi, \forall x.$$

The proof of the reverse inequality is analogous.

The proof of (3) is more involved. We can prove equivalently that for every $A, B \in \mathcal{A}$, such that $(A, B)$ is a good pair, and for every $t > 0$ the following implication holds

$$tq_{A,B} \in C \implies t \leq 1. \quad (3.9)$$

Since $tq_{A,B} \in C$, we can write

$$tq_{A,B} = \sum_{k=1}^{K} \lambda_k q_{D^k}, \quad (3.10)$$

where $D^k \in \mathcal{A}$, $\sum_{k=1}^{K} \lambda_k = 1$, with $\lambda_k$ positive. Formula (3.10) can be rewritten componentwise, denoting $D^k = (d_1^k, \ldots, d_N^k)$,

$$\frac{a_j - b_j}{c_{A,B}} = \sum_{k} \lambda_k \frac{d_j^k}{c_{D^k}} \quad \text{for every } j = 1, \ldots, N.$$

For $k = 1, \ldots, K$, we define vectors $E^k := (e_1^k, \ldots, e_N^k) \in \mathcal{A}$ by

$$\begin{cases} e_j^k := 0, & \text{if } a_j - b_j = 0 \\ e_j^k := d_j^k, & \text{otherwise}. \end{cases} \quad (3.11)$$

Note that

$$\frac{a_j - b_j}{c_{A,B}} = \sum_{k} \lambda_k \frac{e_j^k}{c_{D^k}} \quad \text{for every } j = 1, \ldots, N, \quad (3.12)$$

because $e_j^k$ might differ from $d_j^k$ (and it is equal to zero for every $k$) only in correspondence of those indices $j$ where $\sum_k \lambda_k \frac{d_j^k}{c_{D^k}} = 0$. Moreover

$$c_{E^k} \leq c_{D^k} \quad (3.13)$$

by property (ii) in Definition 2.1 (because $E^k \leq D^k$ by definition of $E^k$). Denote, for $i = 1, \ldots, m$,

$$x_i := \sum_{j=N_{i-1}+1}^{N_i} a_j - b_j,$$

and for $k = 1, \ldots, K$,

$$x_i^k := \sum_{j=N_{i-1}+1}^{N_{i-1}+N_i} e_j^k.$$

Define, for every $k = 1, \ldots, K$ and for every $i = 1, \ldots, m$,

$$y_i^k := \begin{cases} x_i^k, & \text{if } x_i^k x_i \geq 0 \\ -x_i^k, & \text{if } x_i^k x_i < 0. \end{cases}$$

Finally, denote $x := (x_1, \ldots, x_m)$, and $y^k := (y_1^k, \ldots, y_m^k)$, for $k = 1, \ldots, K$. By (3.11), we have that

$$y^k \leq x,$$

for every $k$. Moreover, by (3.12), for every $i = 1, \ldots, m$ it holds

$$\frac{\sum_{k} \lambda_k \frac{x_i^k}{c_{D^k}}}{c_{A,B}} = \sum_{k} \lambda_k \frac{x_i^k}{c_{D^k}}.$$
hence the fact that \( \text{sign}(y_i^k) = \text{sign}(x_i) \) implies
\[
\frac{t[x_i]}{c_{A,B}} = \frac{\text{sign}(x_i) x_i^k}{c_{A,B}} = \sum_k \lambda_k \frac{\text{sign}(x_i) x_i^k}{c_{D^k}} = \sum_k \lambda_k \frac{\text{sign}(x_i) \text{sign}(x_i^k) x_i^k}{c_{D^k}} = \sum_k \lambda_k \frac{\text{sign}(x_i^k) y_i^k}{c_{D^k}}.
\]

Now, if \( y_i^k \) is positive, for every \( k \), since \( \text{sign}(x_i^k) \leq 1 \), we get
\[
\frac{t[x_i]}{c_{A,B}} \leq \sum_k \lambda_k \frac{y_i^k}{c_{D^k}}.
\]

Otherwise, using the fact that \( \text{sign}(x_i^k) \geq -1 \),
\[
\frac{t[x_i]}{c_{A,B}} \geq \sum_k \lambda_k \frac{y_i^k}{c_{D^k}}.
\]
We have just proved that
\[
\frac{t[x]}{c_{A,B}} \leq \sum_k \lambda_k \frac{y_i^k}{c_{D^k}}.
\]

Finally, note also that \( \bar{y}^k = y_i^k \). By (3.7) it holds \( c_{A,B} = C(x) \) and \( c_{E^k} = E(y^k) \), this implies, by property (iii') of Definition 2.1 that
\[
\frac{c_{A,B} \|y_i^k\|}{c_{E^k} \|x\|} = \frac{C(x) \|y_i^k\|}{C(y^k) \|x\|} \leq 1, \quad \forall k = 1, \ldots, K \text{ such that } E_i^k \neq 0,
\]
where \( \| \cdot \| \) is the norm appearing in such definition. Using that \( \| \cdot \| \) is monotone, (3.13) and (3.14), we get
\[
t \leq \sum_k \lambda_k \frac{c_{A,B} \|y_i^k\|}{c_{E^k}}.
\]

**Step 2:** Now consider a general cost \( C \), which does not necessarily satisfy (3.7). For any orthant \( O \subset \mathbb{R}^m \), we define a cost \( C_O : \mathbb{R}^m \to [0, +\infty) \), imposing the following properties:

(a) \( C_O(x) = C(x) \) if \( x \in O \);
(b) \( C_O(x) = C(x) \) for every \( x \).

Trivially properties (i),(ii),(iii') of Definition 2.1 are satisfied by \( C_O \). Let \( \| \cdot \|_{\phi(O)} \) be the norm on \( \mathbb{R}^N \) obtained applying Step 1 to the cost \( C_O \) and let \( B_O \) be the unit ball with respect to such norm. Let us take any point \( x \in \text{int}(O) \) and define
\[
\sigma_O := (\text{sign}(x_1), \ldots, \text{sign}(x_m)) \in \mathbb{R}^m.
\]
Let us also denote
\[
\tau_O := (\text{sign}(x_1)(e_1 + \cdots + e_{N_1}), \ldots, \text{sign}(x_m)(e_{N-N_m+1} + \cdots + e_N)) \in \mathbb{R}^N,
\]
and let \( H_O \) be the unique orthant in \( \mathbb{R}^N \) containing the point \( \tau_O \). Finally, consider \( A_O := H_O \cap B_O \) and
\[
C_O := \{ p \in \mathbb{R}^N : \exists q \in A_O \text{ with } (\tau_O)_j(p_j - q_j) \leq 0 \text{ for every } j = 1, \ldots, N \}.
\]
Observe that
\[
C_O \cap H_O = A_O,
\]
by the monotonicity of \( A_O \), which is implied by the monotonicity of \( B_O \) (the intersection of monotone sets is monotone). Lastly we denote
\[
C := \bigcap_{O \subset \mathbb{R}^m} C_O,
\]

**Step 2:** Now consider a general cost \( C \), which does not necessarily satisfy (3.7). For any orthant \( O \subset \mathbb{R}^m \), we define a cost \( C_O : \mathbb{R}^m \to [0, +\infty) \), imposing the following properties:

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\[
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\]
Let us also denote
\[
\tau_O := (\text{sign}(x_1)(e_1 + \cdots + e_{N_1}), \ldots, \text{sign}(x_m)(e_{N-N_m+1} + \cdots + e_N)) \in \mathbb{R}^N,
\]
and let \( H_O \) be the unique orthant in \( \mathbb{R}^N \) containing the point \( \tau_O \). Finally, consider \( A_O := H_O \cap B_O \) and
\[
C_O := \{ p \in \mathbb{R}^N : \exists q \in A_O \text{ with } (\tau_O)_j(p_j - q_j) \leq 0 \text{ for every } j = 1, \ldots, N \}.
\]
Observe that
\[
C_O \cap H_O = A_O,
\]
by the monotonicity of \( A_O \), which is implied by the monotonicity of \( B_O \) (the intersection of monotone sets is monotone). Lastly we denote
\[
C := \bigcap_{O \subset \mathbb{R}^m} C_O,
\]
where the intersection is taken among the $2^m$ orthants in $\mathbb{R}^m$. We claim that $C$ is a closed, convex, and monotone set, with non-empty interior, which is symmetric with respect to the origin, bounded, and satisfies

$$C \cap H_0 = A_0, \quad \text{for every orthant } \mathcal{O} \subset \mathbb{R}^m. \quad (3.17)$$

By Step 1 and the definition of $C_0$, this would imply that the norm on $\mathbb{R}^N$ whose unit ball is $C$ is monotone and satisfies (3.1), which would conclude the proof of Theorem 3.2.

The fact that $C$ is closed, convex and monotone follows from the fact that each set $C_0$ is so, moreover each $C_0$ contains a neighbourhood of the origin, hence $C$ has non-empty interior. The fact that $C$ is bounded and symmetric with respect to the origin follows from the fact that $C_0 \cap C_0^c$ is so for every $\mathcal{O}$, where we denoted by $-\mathcal{O}$ the orthant which is symmetric to $\mathcal{O}$ with respect to the origin. To conclude, we have to prove (3.17). To prove it, we do the following claim:

$$A_{C_0} \cap H_0 = A_{C_0} \cap H_{C_0}, \quad \text{for every pair of orthants } \mathcal{O}, \mathcal{O}^\prime \subset \mathbb{R}^m. \quad (\text{Claim 1})$$

Let us show firstly how (Claim 1) implies (3.17). By the definition of $C$, it is sufficient to show that

$$C_{C_0} \cap A_0 = A_0, \quad \text{for every pair of orthants } \mathcal{O}, \mathcal{O}^\prime \subset \mathbb{R}^m, \quad (3.18)$$

indeed

$$C \cap H_0 = \bigcup_{\mathcal{O} \subset \mathbb{R}^m} C_{C_0} \cap H_{C_0} \bigcup_{\mathcal{O} \subset \mathbb{R}^m} C_{C_0} \cap H_{C_0} \quad \text{by (3.18)}$$

To prove (3.18) using (Claim 1), we write

$$C_{C_0} \cap A_0 = \bigcup_{H} (C_{C_0} \cap H \cap A_0),$$

where $H$ varies among the $2^N$ orthants of $\mathbb{R}^N$. Then (3.18) would follow from

$$C_{C_0} \supseteq H \cap A_0, \quad \forall \mathcal{O}, \mathcal{O}^\prime, H. \quad (3.19)$$

To prove (3.19), consider $z \in H \cap A_0$. We define a new vector, $y \in \mathbb{R}^N$, in this way:

$$y_j = \begin{cases} z_j, & \text{if } (\tau_{C_0})_j z_j \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

It is immediate to see that $(\tau_{C_0})_j (z_j - y_j) \leq 0, \forall j$, and that $y \in H_{C_0}$. Hence, to prove that $z \in C_{C_0}$ it is sufficient to prove that $y \in A_{C_0}$. We observe that $y \in A_0$, because $y \leq z$ and $z \in A_0$. This yields:

$$y \in A_0 \cap H_{C_0} = A_{C_0} \cap H_{C_0},$$

the equality being true by (Claim 1). Therefore $y \in A_{C_0}$ as desired.

To prove (Claim 1), we will prove the more precise formula:

$$A_{C_0} \cap H_0 = \text{co}\{0\} \cup \{q_H : D \in \bar{A} \cap H_0 \cap H_{C_0}\}, \quad \forall \mathcal{O}, \mathcal{O}^\prime \subset \mathbb{R}^m. \quad (3.20)$$

Denote

$$E := \text{co}\{0\} \cup \{q_H : D \in \bar{A} \cap H_0 \cap H_{C_0}\},$$

and observe that $E \subseteq A_{C_0} \cap H_0$, since $q_D \in A_{C_0} \cap H_0$ for every $D \in \bar{A} \cap H_0 \cap H_{C_0}$, by Step 1.

In order to prove (3.20), by Krein-Milman Theorem, we can assume by contradiction that there exists an extreme point $z$ of $A_{C_0} \cap H_0$ that does not belong to $\{0\} \cup \{q_H : D \in \bar{A} \cap H_0 \cap H_{C_0}\}$.

Since $z \in A_{C_0} \subseteq B_{C_0}$, we can write

$$z = \sum_{k=1}^K \lambda_k \frac{D_k}{c_{D_k}}$$

with $D_k \in \bar{A}$, where $c_{D_k} := c_{D_k,0}$ as defined in (3.8), with $C_{C_0}$ in place of $C$. For $k = 1, \ldots, K$, we define vectors $E_k := (e^k_1, \ldots, e^k_N) \in \bar{A}$ by

$$\begin{cases} e^k_j := 0, & \text{if } z_j = 0 \\ e^k_j := d^k_j, & \text{otherwise}. \end{cases} \quad (3.21)$$

Note that for every $k$ it holds $E_k \in \bar{A} \cap \text{span} \{H_0 \cap H_{C_0}\}$ because $z \in H_{C_0} \cap H_0$, and

$$z = \sum_{k=1}^K \lambda_k \frac{E_k}{c_{D_k}}. \quad (3.22)$$
Moreover
\[ c_{E^k} \leq c_{D^k} \] (3.23)
by the monotonicity of the cost \( C_{\Omega'} \). Hence we can write, denoting \( \lambda_k := \lambda_k \frac{c_{E^k}}{c_{D^k}} \leq \lambda_k \),
\[ z = \sum_{k=1}^{K} \lambda_k q_{E^k}. \]

Let now
\[ \eta = (\eta_1, \ldots, \eta_N) := \tau_{\Omega} + \tau_{\Omega'} \]
be a point in the relative interior of \( H_{\Omega} \cap H_{\Omega'} \). For \( k = 1, \ldots, K \), we define vectors \( F^k := (f^k_1, \ldots, f^k_\lambda) \in A \) by
\[ \begin{cases} f^k_j := e^k_j, & \text{if } \eta_j e^k_j \geq 0 \\ f^k_j := -e^k_j, & \text{otherwise}. \end{cases} \] (3.24)

Note that for every \( k \) it holds \( F^k \in A \cap H_{\Omega} \cap H_{\Omega'} \), because \( E^k \in A \cap \text{span}\{H_{\Omega} \cap H_{\Omega'}\} \). Since \( z \in H_{\Omega} \cap H_{\Omega'} \) and, since by the symmetries of the cost \( C \), it holds \( c_{E^k} = c_{F^k} \) for every \( k \), then
\[ z = \sum_{k=1}^{K} \lambda_k q_{E^k} \leq \sum_{k=1}^{K} \lambda_k' \frac{c_{E^k}}{c_{F^k}} = \sum_{k=1}^{K} \lambda_k' q_{F^k} =: z', \] (3.25)
where the inequality is meant with respect to the order relation defined in (2.1). We observe that \( z' \notin E \). This implies that \( z' \neq z \) (otherwise we would have \( z \in E \)) and that \( z' \in A_{\Omega'} \cap H_{\Omega} \), since \( E \subseteq A_{\Omega'} \cap H_{\Omega} \).

Consider now the halfline \( r := \{t(z - z') + z' : t > 1\} \). Observe that \( r \cap A_{\Omega'} \cap H_{\Omega} \neq \emptyset \). Indeed, every \( y \in r \) satisfies \( y \leq z' \), and therefore by monotonicity belongs to \( A_{\Omega'} \cap H_{\Omega} \). We get a contradiction to the fact that \( z \) is an extremal point of \( A_{\Omega'} \cap H_{\Omega} \): consider any \( z'' \in r \cap A_{\Omega'} \cap H_{\Omega} \). Then
\[ z'' = s(z - z') + z', \]
We can rewrite this expression as:
\[ z = \frac{z'' - z'}{s} + z', s > 1. \]
Since \( \frac{1}{s} < 1 \), we get that \( z \) belongs to the internal part of a segment contained in \( A_{\Omega'} \cap H_{\Omega} \), and this contradicts the hypothesis that \( z \) is an extremal point for the set. \( \square \)

As we observed before, Theorem 2.4 provides a proof of the existence of a solution to MMTP, which does not require a proof of the lower semicontinuity of the energy \( E \).

**Corollary 3.3.** Under assumptions (i),(ii),(iii') on the cost functional \( C \), the problems MMTP and MMP admit a solution.

**Proof.** The fact that problem MMP admits a solution follows from Theorem 1.9. The fact that problem MMTP admits a solution then follows from Theorem 2.4. \( \square \)

The property (iii') of Definition 2.1 appears to be the most restrictive. However, at least in the “single-material” case it is also necessary to obtain the equivalence with the mass-minimization problem.

**Theorem 3.4.** If \( m = 1 \), and \( C \) is a cost that fulfills (i),(ii) of Definition 2.1, then (iii') holds if and only if there exists a monotone norm \( |\cdot|_\Omega \) that satisfies (3.1).

**Proof.** One implication has already been proven in Theorem 3.2. Suppose now that there exists a monotone norm \( |\cdot|_\Omega \) on \( \mathbb{R}^N \) that satisfies (3.1). Fix any \( E \in A \), where \( A \) is defined at the beginning of the proof of Theorem 3.2. We can write \( E \) as
\[ E = \sum_{k \in K} e_k, \]
being \( K \) a subset of \( \{1, \ldots, N\} \). We denote with \#\( K \) the cardinality of \( K \). By (3.1), we have:
\[ |E|_{\#} = C(\#K) = |F|_{\#}, \] (3.26)
for any \( F \in A \) such that \( F = \sum_{k \in K'} e_k \) and \( \#K' = \#K \). For every \( \ell \in K \) define \( K_\ell := K \setminus \{\ell\} \). Define \( E_\ell := \sum_{k \in K_\ell} e_k \). Therefore,
\[ (\#K - 1)E = \sum_{\ell \in K} E_\ell. \]
and, by \((3.26)\), we get

\[
(#K - 1)C(#) = (#K - 1)|E|_\phi = \left| \sum_{\ell \in K} E_\ell \right|_\phi \leq \sum_{\ell \in K} |E_\ell|_\phi = \sum_{\ell \in K} C(#) = #KC(#) = #KC(#K - 1).
\]

Since \(K \subset \{1, \ldots, N\}\) is arbitrary, we obtain, \(\forall x \in \{2, \ldots, N\}\),

\[
\frac{C(x)}{x} \leq \frac{C(x - 1)}{x - 1},
\]

and, by induction,

\[
\frac{C(x)}{x} \leq \frac{C(y)}{y}, \text{ if } 1 \leq y \leq x.
\]

\(\square\)

It is well-known that, if \(C : [0, \infty) \to [0, \infty)\) is concave and \(C(0) = 0\), then the quantity \(\frac{C(x)}{x}\) is non-increasing. Hence we obtain the following corollary.

**Corollary 3.5.** In the case \(m = 1\), Theorem 3.2 holds if (iii') is replaced by the request that \(C\) coincide on \(\mathbb{N}\) with a concave function.

**Remark 3.6.** The previous corollary allows us to include in the list of cost functionals for which Theorem 2.4 applies the cost considered in [8], which describes a model for the urban planning (or a discretized version of it, in our case). More precisely the cost is \(C(z) = \min\{azz; z + b\} \) with \(a > 1, b > 0\), which is clearly concave.

4. Examples

In this section, we consider some concrete costs functional \(C\) and we exhibit the norm \(|\cdot|_\phi\) which turns a MMTP associated to such cost into a mass-minimization problem.

1. **Steiner energy.** For \(m = 1\), let

\[
C(z) := \begin{cases} 
0, & z = 0, \\
1, & z \neq 0.
\end{cases}
\]

The minimization of the energy \(E\) associated to such cost corresponds to the minimization of the size functional. Clearly the corresponding norm \(|\cdot|_\phi\) on \(\mathbb{R}^N\) given by Theorem 2.4 is simply the supremum norm.

2. **Gilbert-Steiner energy.** For \(m = 1\), fix \(0 \leq \alpha \leq 1\) and let

\[
C(z) := \begin{cases} 
0, & z = 0, \\
|z|^\alpha, & z \neq 0.
\end{cases}
\]

The minimization of the corresponding energy \(E\) corresponds to the minimization of the \(\alpha\)-mass (see e.g. [41]). As it is shown in [30], the corresponding norm \(|\cdot|_\phi\) on \(\mathbb{R}^N\) is the \(p\)-norm with \(p = \frac{1}{\alpha}\). Note that for \(\alpha = 0\) we recover the Steiner energy.

3. **Linear combinations.** For \(m = 1\), fix \(K \in \mathbb{N}\) and for \(k = 1, \ldots, K\) let \(0 \leq \alpha_k \leq 1\) and let \(\lambda_k > 0\). Define

\[
C(z) := \begin{cases} 
0, & z = 0, \\
\sum_{k=1}^K \lambda_k |z|^\alpha_k, & z \neq 0.
\end{cases}
\]

It is easy to see that \(C\) satisfies properties (i),(ii), and (iii') of Definition 2.1. The corresponding norm \(|\cdot|_\phi\) on \(\mathbb{R}^N\) is \(|z|_\phi = \sum_{k=1}^K \lambda_k |z|^\alpha_k\), where \(p_k = \frac{1}{\alpha_k}\). Such a cost is considered for example in [12] in order to approximate the Steiner energy and to perform numerical simulations.

4. **Supremum of costs.** For \(m = 1\), fix \(K \in \mathbb{N}\) and for \(k = 1, \ldots, K\) let \(C_k\) be a cost functional satisfying properties (i),(ii), and (iii') of Definition 2.1. Define

\[
C(z) := \max_{k=1,\ldots,K} C_k(z).
\]

The corresponding norm \(|\cdot|_\phi\) on \(\mathbb{R}^N\) is the maximum of the norms associated to each \(C_k\).
(5) **PLC technology.** For \( m = 2 \), let \( 0 < \alpha_1 \ll \alpha_2 \leq 1 \). Define
\[
\mathcal{C}((z_1, z_2)) := \max\{\lambda_1 |z_1|^{\alpha_1}; \lambda_2 |z_2|^{\alpha_2}\},
\]
with \( \lambda_1, \lambda_2 > 0 \). A monotone norm \( \cdot \) on \( \mathbb{R}^N \) which satisfies (3.1) is
\[
| (x_1, \ldots, x_N, y_1, \ldots, y_{N_2}) |_{\phi} = \max\{\lambda_1 |(x_1, \ldots, x_N)|_{\phi_1}; \lambda_2 |(y_1, \ldots, y_{N_2})|_{\phi_2}\},
\]
where \( \phi_i = \alpha_i^{-1} \) for \( i = 1, 2 \). The fact that \( \alpha_1 \ll \alpha_2 \) express the idea that once the infrastructure
transporting the second material (i.e., the electricity) is built one can add “almost any” quantity
of the second material (i.e., internet signal) for free.

(6) **Composite multi-material costs.** For general \( m \geq 2 \), consider any monotone norm \( \| \cdot \| \) in
\( \mathbb{R}^m \) and single-material costs \( C_1, \ldots, C_m \), associated to monotone norms \( \| \cdot \|_{\phi_1}, \ldots, \| \cdot \|_{\phi_m} \) on
\( \mathbb{R}^{N_1}, \ldots, \mathbb{R}^{N_m} \) respectively. Define
\[
\mathcal{C}(z_1, \ldots, z_m) := \| (C_1(z_1), \ldots, C_m(z_m)) \|.
\]
A monotone norm \( \cdot \) on \( \mathbb{R}^N \) which satisfies (3.1) for the multi-material cost \( \mathcal{C} \) is
\[
| (x_1, \ldots, x_N) |_{\phi} = \| (|x_1, \ldots, x_N|_{\phi_1}, \ldots, |x_{N-N_m+1}, \ldots, x_N|_{\phi_m}) \|.
\]
Observe that this multi-material cost does not satisfy (3.7). A monotone norm \( \cdot \) on \( \mathbb{R}^N \) which satisfies (3.1) is clearly
\[
| (x_1, \ldots, x_N) |_{\phi} = |x_+|_p + |x_-|_p
\]
where \( p = \alpha^{-1} \) and \( x_+ \) (respectively \( x_- \)) is obtained by \( x \) setting all the negative (respectively positive)
coordinates of \( x \) equal to zero. In a forthcoming paper such cost is used to give a new
description of the discrete mailing problem (see [2]), encoding the fact that, on every branch of
a transportation network, there is a gain in the cost of the transportation in grouping particles
flowing with the same orientation, but there is no gain for two groups of particles flowing with
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