Oscillations in the cumulative individual income distribution have been found in the data of various countries studied by different authors at different time periods, but the dynamical origins of this behavior are currently unknown. These datasets can be fitted by different functions at different income ranges, but recently the Tsallis distribution has been found capable of fitting the whole distribution by means of only two parameters, procedure which showed even more clearly such oscillatory features in the entire income range. This behavior can be described by assuming log-periodic functions, however a different approach to naturally disclose such oscillatory characteristics is to allow the Tsallis $q$-parameter to become complex. In this paper we have used these ideas in order to describe the behavior of the complementary cumulative distribution function of the personal income of Brazil recently studied empirically by Soares et al. (2016). Typical elements of periodic motion, such as amplitude and angular frequency coupled to this income analysis, were obtained.

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The study of the individual income distribution of populations has a long history. Vilfredo Pareto (1848-1927), the pioneer of this type of analysis, studied the distribution of personal income at the end of 19th century for some regions and countries in specific years and sets of years. Pareto looked at the problem systematically and concluded that, individually speaking, the richest people in a society have the complementary cumulative distribution function (CCDF) of income obeying a power law function [1]. Consequently, the probability density function (PDF) \( p(x) \) of the personal income \( x \) of the richest persons may be given by,

\[
p(x) = \beta x^{-(1+\alpha)},
\]

where \( \beta \) is a normalization constant. Through the years, this power law behavior has become known as the Pareto power-law and, consequently, the exponent \( \alpha \) is now known as the Pareto index. This law has been interpreted later as being a classic example of a fractal distributions, where the Pareto index plays the role of the single fractal dimension of the distribution [2]. Higher values of the Pareto index imply more uneven distribution of the personal income. The interesting detail is that this result has not been disputed by different investigations carried out since then, which considered several different samples obtained at different times for different populations in distinct countries or groups of countries [3–13, and references therein].

Despite its empirical success, the Pareto power-law does not work for the overwhelmingly majority, and less rich, part of the population. Namely, it only describes well the income data of those belonging to the narrow “slice” of the richest population. To consider the income data of the group composed by the less rich people, the method that has been used since shortly after Pareto’s time is to fit the data by various other function, like the exponential, the log-normal, the gamma function, the Gompertz curve, as well as other ones [3–5, 7, 12, 13]. There are also successful approaches for analyzing the whole data range using less than simple functions with many parameters, but usually such approaches require four or more parameters in order to fit the entire distribution. In addition, these methodologies mean the assumption that societies are divided in two classes only: on one side the very rich, formed by about 1% of the whole population, and on the other side the remaining 99%. One basic problem of this methodology is the absence of the middle class, which certainly exists in between these two groups. In addition, the question remains of whether or not this class division is a real feature of societies or basically a mathematical artifact convenient for data fitting. Considering these objections, a simple function able to fit the entire income distribution which, at the same time, allows for various features to emerge at different income range is certainly preferable.

A recent approach for representing the whole income data is to fit the data using the Tsallis functions instead of the combination of functions depicted above. The individual income distribution was classified in terms of the well known Tsallis parameter, \( q \), and another normalization constant. Borges [14] used two \( q \)-parameter, where one controls the slope of the intermediate income range and the other describes the tail of the distribution. He was then able to analyze the income distribution concerning some counties of the USA from 1970 to 2000, Brazil from 1970 to 1996, Germany from 1992 to 1998 and the United Kingdom from 1993 to 1998. The conclusion was that an increase in \( q \) with time points to growing inequality. Greater values of \( q \) indicate greater probability to find counties much richer than others. Ferrero [15, 16] used the Tsallis function to fit the entire income data of several counties, but only at single years, not being able though to indicate an evolution in the Tsallis parameters.
Recently three of us [17] have used the Tsallis formalism to empirically study the income distribution of Brazil during a relatively large yearly time window, from 1978 to 2014. The results showed that the two fitted parameters of the Tsallis function follow a cycling behavior. Moreover, a linear fit of the distribution in each year showed that the data oscillate periodically around the fitted straight line with an amplitude that grows with the income values. A closer look at the fitted data made by other authors using different methods applied to different samples collected at different time periods showed a similar oscillatory pattern, which means that there must be indeed a second order dynamical effect not previously identified in the income data [18, p. 164]. This kind of oscillatory behavior has not been noted before in the income distribution data, although it has been observed in financial markets and other systems [18].

In this work we have analyzed this periodic oscillation through the alternative approach of allowing the Tsallis parameter to become complex, as suggested by Ref. [19]. Under this methodology the $q$-parameter discloses such periodic behavior in the income distribution curves, allowing us to defined periodic elements that are ordinary in physics, like the amplitude and angular frequency. This paper is organized as follows. Sect. 2 presents the Tsallis functions and some of their properties required in our analysis. Sect. 3 presents and discusses the complexification process and its influence in the analysis of the individual income distribution. Sect. 4 presents our conclusions.

II. TSALLIS FUNCTIONS

It is well known that the Tsallis thermostatistics [20, 21] is based on both the $q$-logarithm and $q$-exponential functions, given by,

$$\ln_q x \equiv \frac{x^{(1-q)} - 1}{1 - q},$$

$$e_q^x \equiv \left[ 1 + (1 - q)x \right]^{1/(1-q)}.$$

These functions are defined such that for $q = 1$ both expressions become the standard logarithm and exponential functions, namely, $e_1^x = e^x$ and $\ln_1 x = \ln x$. Hence, the Tsallis $q$-functions are in fact the usual exponential and logarithmic expressions twisted in such a way as to be used in Tsallis’ theory of nonextensive statistical mechanics [21].

At this point it should be noted that there are other ways to deform these two common functions viewing other application, such as the personal income distribution. This is the case of the $\kappa$-generalized exponential, introduced by Ref. [22], which can be used to fit the entire income data range in similar manner as the Tsallis $q$-functions. This is especially significant because both of them have the power-law and exponential as their limiting cases. The interested reader can find more applications of the $\kappa$-generalized function in Refs. [23–25].

From the definitions above it is clear that,

$$e_q^{(\ln_q x)} = \ln_q \left( e_q^x \right) = x.$$

Moreover, $\ln_q 1 = 0$ for any $q$. Hence, if we have a value $x_0$ such that $x/x_0 = 1$, as a result $\ln_q \left( x/x_0 \right) = 0$. Two other properties of the $q$-exponential useful for our purposes here are as follows [26],

$$\left[ e_q^{f(x)} \right]^a = e_q^{a f(x) \frac{1}{1-(1-q)/a}},$$
\[
\frac{d}{dx} \left[ e_q^{f(x)} \right] = \frac{[e_q^{f(x)}]^q}{f'(x)}. \tag{6}
\]

These results will be useful when we explore the fact that the Tsallis parameter can be represented in the complex plane, as discussed by Wilk and Włodarczyk in Ref. [19]. It is worth noting that from Eqs. (3), (5) and (6), it is not obvious that a complex \( q \) can help us disclose some income distribution details that are hidden in these functions, such as a periodic behavior, as we shall show below.

### III. \( q \)-PARAMETER COMPLEXIFICATION

#### A. Complex heat capacity

The fact that the nonextensivity parameter \( q \) can be represented by a complex formulation is not new. In Ref. [27] the \( q \)-parameter can be seen as a measure of the thermal bath heat capacity \( C \), where

\[
C = \frac{1}{q - 1}. \tag{7}
\]

Moreover, such complex \( C \) is well known in the literature [28] and can be written as follows,

\[
C = C_\infty + \frac{C_0 - C_\infty}{1 + (\omega \tau)^2} (1 - i \omega \tau), \tag{8}
\]

where \( C_\infty \) is the heat capacity for infinitely fast degrees of freedom (DOF), \( \omega \) is the frequency, \( C_0 \) is the heat capacity at equilibrium of DOF where the frequency is set to zero and \( \tau \), the time constant, is the kinetic relaxation time constant of a certain DOF. The form of Eq. (8) suggests us that we can write it as \( C = C' + i C'' \), where

\[
C' = C_\infty + \frac{C_0 - C_\infty}{1 + (\omega \tau)^2} \tag{9}
\]

and

\[
C'' = \frac{(C_0 - C_\infty)\omega \tau}{1 + (\omega \tau)^2}. \tag{10}
\]

Hence, let us write a complex form of \( q \) as \( q = q_r + i q_i \). Substituting in Eq. (7) and associating with Eq. Eq. (8), we have that

\[
\frac{q_r - 1}{q_i} = \frac{C_0 + C_\infty (\omega \tau)^2}{(C_0 - C_\infty)\omega \tau}, \tag{11}
\]

and for \( C_0 \ll C_\infty \) we have \((1 - q_r)/q_i \approx \omega \tau\), which shows that the relationship between the real and imaginary parts of \( q \) is proportional to the frequency. More details of this discussion can be found in Ref. [27].

#### B. Complex income distribution means periodic behavior

Let us start with the suggestion of Wilk and Włodarczyk [19] for the complexification of the \( q \)-parameter. A three-parameters Tsallis distribution (TD) may be written as follows,

\[
f(x) = C e^{-x/T} = C \left( 1 - \frac{x}{mT} \right)^{-m}, \tag{12}
\]
where
\[ m = \frac{1}{(q - 1)}. \]  

is a real power index, \( T \) is a scale parameter identified in thermodynamical applications, in general the standard temperature, and \( C \) is a normalization constant. The proposal is to consider \( m \), or \( q \), complex. Hence, the TD keeps its main quasi-power like form, however, that brings about some log-periodic oscillations. As examples, one can mention that such behavior has been encountered in many subjects, such as earthquakes \([29]\), chaos \([30]\), tracers on random systems \([31]\), random quenched and fractals \([32, 33]\), specific heat \([34]\), clusters \([35]\), growth models \([36]\), stock markets \([37]\) and, finally, non-extensive statistical mechanics log-periodic oscillations \([38]\). When \( m \to \infty \), namely, \( q \to 1 \), we have that this power-like distribution is analogous to the standard exponential distribution \( f(x) = Ce^{-x/T} \).

The complexification proposal means turning \( m \) (or \( q \)) complex in Eq. \( 12 \), yielding,
\[ m = m' + im''. \]  

which means that we can also have a complex nonextensive \( q \)-parameter written as below,
\[ q = 1 + \frac{1}{m} = q' + iq''. \]  

It is simple to see that
\[ q' = 1 + \frac{m'}{|m|^2} \quad \text{and} \quad q'' = -\frac{m''}{|m|^2}. \]  

where
\[ |m|^2 = m'^2 + m''^2. \]  

Hence, the goal here is to analyze the results obtained in Ref. \([18]\), where the personal income distribution of Brazil shows a periodic behavior as a function of the income variable for each yearly sample, in the light of the complexification of the \( q \)-parameter. Therefore, we wish to describe mathematically such an oscillatory behavior.

C. Nonextensive analysis of the income distribution of Brazil

Let us now turn our attention to the main issue of this article. As discussed by Soares \textit{et al.} \([18]\), if the entire income distribution range can be fitted by just one function using only two parameters, a well-defined two-classes-base income structure implicitly assumed when the income range is described by two distinct functions may be open question. Hence, such income-class division could possibly be only a result of fitting choices and not of an intrinsic feature of societies.

The TD is known to become a pure power-law for large values of its independent variable \( x \), and exponential when \( x \) tends to zero. However, it has not the same behavior as assuming from the start a two-classes approach to the income distribution problem because the TD will only have power-law and exponential like behaviors as limiting cases. Thus, a possible different behavior at the intermediate level might not be described by neither of these functions. Hence, the TD does not necessarily imply in two very distinct classes based on well-defined income domain ranges, but possibly having an intermediate income range of unknown size which might behave as neither of them. With these points in mind, let us now proceed with the description of the income distribution in terms of the TD and its subsequent complexification.
Let \( F(x) \) be the cumulative distribution function (CDF) of the personal income, representing the proportionality, or probability, that a person receives an income less than or equal to \( x \). Let us now denote its complementary version, the CCDF, by \( F(x) \), which then describes the probability that a person receives an income greater or equal to \( x \). It is then clear that, 

\[
F(x) + F(x) = 100. \tag{18}
\]

Here the maximum probability is normalized to 100% instead of the standard unity value. The boundary conditions involved in both functions are \( F(x = \infty) = 0 \) and \( F(x = 0) = 100 \). In addition, the following properties apply to these income functions,

\[
\frac{dF(x)}{dx} = -\frac{dF(x)}{dx} = f(x), \tag{19}
\]

\[
\int_{0}^{\infty} f(x) \, dx = 100, \tag{20}
\]

where \( f(x) \) is the PDF \([5, 12]\).

The empirical suggestions that the income distribution can be modeled by the TD comes from the fact that when \( F(x) \) is obtained from income data and plotted in a log-log scale, its functional curve decreases as the income \( x \) increases. In addition, the general shape of the empirical CCDF function, particularly its “belly”, is analogous to the behavior of \( e^{-x^q} \) for \( q > 1 \) when plotted in a log-log scale [see Fig. 3.4 at p. 40 of Ref. 21]. Besides, as mentioned above, the Tsallis functions behave like a power-law for high income values, agreeing then with the Pareto power-law. Taking together these observations into account, Ref. [18] advanced the following description for the individual income distribution,

\[
F(x) = A e^{-Bx}, \tag{21}
\]

where \( A \) and \( B \) are positive parameters. Since we have a boundary condition in the form of \( F(0) = 100 \), from Eq. (21) it is straightforward to conclude that \( A = 100 \). Hence, Eq. (21) becomes as below,

\[
F(x) = 100 e^{-Bx}. \tag{22}
\]

which will be the starting point for the complexification procedure. Note that a cursory examination of Eq. (22) does not present any obvious evidence of a periodic behavior, or that the complexification of \( q \) will disclose any oscillatory feature, since the \( q \)-parameter is just an index. We shall show below that allowing \( q \) to become complex will expose such features.

D. Periodic behavior of income distribution function

Let us start with the definition (3) in order to rewrite Eq. (22) as below,

\[
F(x) = 100 \left[ 1 + (1 - q)(-Bx)^{1/(1-q)} \right]. \tag{23}
\]

This equation can also be expressed as a function of \( m \) using Eq. (13),

\[
F(x) = 100 \left[ 1 + \frac{B}{m} x \right]^{-m}. \tag{24}
\]
Following the complexification suggested in Eqs. (14) to (17), Eq. (24) may be written as below,

\[ F(x) = 100 \left[ a(x) + i b(x) \right]^{(m'+im'')} , \quad (25) \]

where

\[ a(x) = 1 + \frac{Bm'}{|m|^2} x , \quad (26) \]

\[ b(x) = -\frac{Bm''}{|m|^2} x . \quad (27) \]

Expanding the terms in Eq. (25) results in the following expression,

\[ F(x) = 100 \mathcal{A}(x) e^{-i\omega(x)} = 100 \mathcal{A}(x) \left[ \cos \omega(x) - i \sin \omega(x) \right] , \quad (28) \]

where

\[ \begin{align*}
\mathcal{A}(x) &= \exp \left[ m'' \varphi(x) - m' \theta(x) \right], \\
\omega(x) &= m' \varphi(x) + m'' \theta(x),
\end{align*} \quad (29, 30) \]

and

\[ \begin{align*}
\varphi(x) &= \tan^{-1} \left[ \frac{-Bm''x}{(|m|^2 + Bm'x)} \right], \\
\theta(x) &= \ln \sqrt{1 + \frac{Bx}{|m|^2} (Bx + 2m')} .
\end{align*} \quad (31, 32) \]

Now, if we consider only the real part of Eq. (28), the original description of the income data in terms of the TD, as given by Eq. (22), turns out to be written as follows,

\[ F(x) = 100 e^{-Bx} = 100 \mathcal{A}(x) \cos \omega(x). \quad (33) \]

This result shows clearly that the periodic behavior empirically observed by Ref. [18] in the income data seems to be described by the expression above. Or that the observed oscillatory behavior can at least be expected, since it is built in the TD. Note that the original \( q \)-parameter is present in both \( \mathcal{A}(x) \) and \( \omega(x) \) through \( m' \) and \( m'' \), which are the respective real and imaginary parts of \( q \).

The fact that the individual income distribution function can be written in the standard exponential form given by Eq. (28) shows us that we can define a periodic function where \( \mathcal{A}(x) \) can be seen as the amplitude of this oscillatory motion and \( \omega(x) \) its angular frequency. Considering the \( q \)-parameter only by a real term hides this periodic behavior. The complexification adds new components such as the kind derived above so that it is able of revealing a periodic behavior that does appear in the empirical curves obtained from the income distribution data.

The issue of always considering the \( q \)-parameter as an entire complex number, namely, with both real and imaginary parts, is an open question although we believe that it depends on the problem we are dealing with. We can imagine an analogy of this feature with the dual characteristic of the electron in quantum physics, since it has a wave-particle duality behavioral feature that depends on the experience that we are analyzing. In this way we could talk about a \( q \)-duality, which would deserves alternative interpretations and could certainly be a target of further research.
IV. CONCLUSIONS

Since the Pareto’s work the study of the income distribution of the whole population has been a target for the economic experts, and for some time now, to the econophysics literature. Despite the relative great number of parameters used in several functions to fit the income data, from three to five in some cases, some mathematical approaches have thrived in the description of the entire data range. In this work we have used Tsallis’ nonextensive point of view with a complexification mode where its $q$-parameter is represented by a complex number. The objective here was to use this complexification in order to justify analytically the results obtained in Ref. [18], which show a periodic behavior in the income data.

As shown above, in doing this we, however, increased the number of parameters required to fit the data, from the original two to three. Although increasing the number of unknown parameters in a problem is a bad procedure, in our case the complexification has disclosed an extra behavior of the income distribution in the form of a periodic motion in the income distribution, motion which was already present in several studies of income distribution, but was only explicitly acknowledged by Soares et al. [18]. By interpreting this oscillatory motion of the income distribution as a typical periodic motion allowed us to define commonly used oscillatory parameters such as amplitude and angular frequency.

So, from the results obtained here we can say that “the highs and lows” of the yearly income distribution samples are an expected behavior, confirmed using the complex form of the Tsallis $q$-parameter. However, it not all clear that this periodic oscillation could lead us to the understanding of the real nature of the $q$-parameter, namely, if it is complex or not depends on the features of the problem we are dealing with.

Finally, this work shows us a glimpse of the task ahead as far as the empirical studies of income distribution are concerned. The data give us $F(x), x, B, q$ and $m$, so the empirical task is to determine both $m'$ and $m''$ from the data in order to end up with only three parameters required to characterize the whole yearly distribution, oscillatory feature included, namely $B, m'$ and $m''$.

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