Using a large amplitude pendulum in a learning cycle strategy

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Abstract. Experiments with a damped simple pendulum were carried out and modelled for large amplitude motion with physics students in a mechanics laboratory course of the first year of the university. The equation of motion used for this pendulum was the equation of a damped pendulum where the simplification for small angles was avoided. In this investigation, the damping of this pendulum was assumed to be proportional and opposite to its velocity. This hypothesis was verified experimentally and the constant of damping was measured from the experimental data. This constant was introduced in the equation of motion of the pendulum to be solved by a numerical method. The comparison made between the numerical solution and experimental data was successful, giving students confidence in the strategy followed. With the support of the teacher students modelled empirically the period of the pendulum as a quadratic function of its amplitude and made good predictions of the behavior of the pendulum for other amplitudes by using the numerical solution. On completion of the course, in a revision of the students’ reports, it was possible to propose an empirical equation as an analytical solution for this pendulum, an important result of the learning cycle carried out in the study of the pendulum. In this process of teaching, students developed scientific skills that were evaluated and both students and teachers were able to build new knowledge about a pendulum with large amplitude.

1. Introduction
From Galileo to the present day, the pendulum has been studied extensively, but teaching from high school to university level has typically only been about the damped simple pendulum for small amplitude, where the oscillations of the pendulum are isochronous. This is the case because the study of the pendulum for a large amplitude is complex even for university students.

Experimentation with a pendulum of large amplitude is possible because it is easy to register its motion with a video to measure the period. In this way students can find out that the period changes with the amplitude of the pendulum. However, to model the motion of the pendulum is not easy, because although the equation of motion is known there is no simple way for first year of university physics students to solve this equation. It was possible to carry out a complete study of the pendulum for large amplitudes with a small group of seven students by applying a learning cycle, where they experimented with the damped pendulum to get important parameters of its motion that could be used to solve numerically the damped pendulum equation of motion. This solution models this pendulum very well as the comparison with experimental data shows.

This work presents the theory of the damped pendulum for large amplitudes with a numerical solution for its equation of motion, how the study of this pendulum was carried out with physics students, and the results obtained in this research.
2. Theory

The damped simple pendulum is studied in high school and university for a small amplitude of oscillation. In that case, the oscillations of the pendulum are isochronous. In a very precise experiment, it can be measured that the oscillations of a pendulum are not isochronous for large amplitudes.

For small amplitudes, the pendulum approximates a harmonic oscillator and its equation of motion is:

$$m \frac{d^2s}{dt^2} = -mg \theta - bv$$

where $b = 2m\beta$ and $\beta$ is a damping constant. The analytical solution of equation (1) is:

$$\theta(t) = \theta_0 e^{-\beta t} \cos \left( \frac{2\pi}{T_0} t + \delta \right)$$

with $T_0$ the period of the pendulum, $\delta$ a phase angle and $\theta_0$ the initial amplitude. In this case, the period of the pendulum is given by:

$$T_0 = 2\pi \sqrt{\frac{l}{g} - \beta^2}$$

with $l$ the length of the pendulum.

For large amplitudes, the period of the pendulum changes, so it is not constant as predicted by equation (3). With high-speed video, it is not difficult to measure the motion of the pendulum with large amplitudes, and to observe that its period changes with the amplitude of the pendulum.

The equation of motion for a pendulum with large amplitudes is:

$$m \frac{d^2s}{dt^2} = -mg\sin \theta - bv$$

The difference between this equation and equation (1) is that the approximation for small amplitudes of $\sin \theta \approx \theta$ is not used any more. Equation (4) is non-linear, and does not have an analytical solution. In this paper, we solve equation (4) numerically, and thus we find a relationship between the period and the amplitude of the pendulum. With this result, we propose a function relating the period of the pendulum with time, and an empirical analytical equation for the solution of equation (4). All of this was tested with experimental data.

The research methodology followed in this work was applied, with good results, with students in the first year of their studies, in the laboratory course of mechanics of the Science Faculty of the National Autonomous University of Mexico.

3. Equation of motion

Equation (4) changes if the displacement $s$ of the pendulum is considered as a function of time $t$. This displacement is $s = l\theta$, with $\theta$ the angular displacement and $l$ the length of the pendulum. Also, the velocity of the pendulum is $v = \frac{ds}{dt} = l\frac{d\theta}{dt}$, and, since $\omega = \frac{d\theta}{dt}$, $v = l\omega$. If we substitute these equations into equation (4), we get:

$$ml \frac{d\omega}{dt} = -mg \sin \theta - b l \omega$$

or

$$\frac{d\omega}{dt} = -\frac{g}{l} \sin \theta - \frac{b}{m} \omega$$

and

$$\frac{d\omega}{dt} = -\frac{g}{l} \sin \theta - 2\beta \omega$$

if $\beta = \frac{b}{2m}$. Thus equation (4) is equivalent to the following two differential equations:

$$m \frac{d\omega}{dt} = f$$

and

$$\frac{d\theta}{dt} = \omega$$

with $f = -\frac{mg}{l} \sin \theta - 2m\beta \omega$. 

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4. The experiment
The system of the two equations (8) and (9) has a numerical solution which is taken as the model for the real motion of the damped simple pendulum for large amplitudes. This solution is compared with the experimental data, to ascertain if the assumption that the damping term is proportional and opposite to the velocity is valid.

In a damped simple pendulum with small oscillations, the damping is represented by exponential decay of the amplitude according to the function:

$$\theta_A(t) = \theta_0 e^{-\beta t} \quad (10)$$

The assumption is that the damping for large amplitudes of oscillations is the same as for small amplitude oscillations.

The pendulum was built with a ball hanging from two points equidistant from the center O to give stability to a pendulum that swings in a plane (Figure 1a) so that it can swing with a large amplitude (Figure 1b).

![Figure 1. Pendulum setup: (a) lateral view, (b) frontal view.](image)

The experiment was done, and the motion of the pendulum was recorded in a video with a fast digital camera of 300 fps. The video was captured with Tracker, an open source software and this software also was used to collect experimental data of angle of oscillation $\theta$ and time $t$. Data were analyzed with Excel, a spreadsheet software of Microsoft® (see Figure 2).

![Figure 2. Experimental data and graph $\theta_{exp}$ vs $t$ of the pendulum.](image)

The period of the pendulum as a function of time could be determined from the experimental data by the time difference between two nearby maxima of the graph $\theta_{exp}$ vs $t$. If this function is exponential as in equation (10), then equation (11) must hold:

$$\ln \theta_A = -\beta T + \ln \theta_0 \quad (11)$$
The graph of $\theta_A$ vs $T$ derived from experimental data is a straight line, as can be seen in Figure 3.

![Figure 3. The graph of $\theta_A$ vs $T$ is a straight line.](image)

The least squares fitting of the graph in figure 3 is given in Table 1. The slope of the straight line is the negative of the damping constant $\beta$, the y-intercept is $\theta_0$.

| $\beta$ | $\delta \beta$ | $\ln \theta_0$ | $\delta \ln \theta_0$ | $\theta_0$ | $\delta \theta_0$ |
|---------|---------------|----------------|-----------------------|----------|----------------|
| 0.0117  | 0.0002        | -0.212         | 0.002                 | 46.3°    | 0.1°           |

Then, the amplitude of the pendulum of the experiment is the following function of the period of the pendulum:

$$\theta_A(t) = 46.3° e^{-0.0117t}$$ (12)

![Figure 4. Envelope curves of experimental oscillations of the pendulum.](image)

Because the function in equation (11) represents how the amplitude of the pendulum varies with the period, it is possible to extend this dependence to all values of time by the function in equation (12). When this function is graphed as $\theta_A$ vs $t$, it represents the envelope of this curve. Also, the negative of this function is the negative envelope of this curve, as shown in Figure 4.

5. Numerical method of solution

Equations (8) and (9) are solved numerically using an extended Euler method. This method can be built directly from these equations. Defining:

$$m \frac{\Delta \omega}{\Delta t} = \bar{f}$$ (13)

and

$$\frac{\Delta \theta}{\Delta t} = \bar{\omega}$$ (14)
with $\bar{f}$ and $\bar{\omega}$ mean values, and $\Delta \omega = \omega_{i+1} - \omega_i$, $\Delta \theta = \theta_{i+1} - \theta_i$ and $\Delta t = t_{i+1} - t_i$ with $i$ a free index that runs from 0 to $n$, equations (13) and (14) can be written as:

$$\omega_{i+1} = \omega_i + \frac{f_i}{m} \Delta t$$

and

$$\theta_{i+1} = \theta_i + \bar{\omega} \Delta t$$

for $i = 0, \ldots, n - 1$. The new values of $\omega_{i+1}$ and $\theta_{i+1}$ obtained from equations (13) and (14) depend on the initial values of angular position $\theta_i$ and angular velocity $\omega_i$, and the mean values of the angular velocity $\bar{\omega}$ and force $\bar{f}$. Unfortunately, while the initial values are known, the mean values are not. To solve this problem, the mean values $\bar{\omega}$ and $\bar{f}$ are approximated by initial the values $\theta_i$ and $\omega_i$ in the following equations:

$$\omega_{i+1} = \omega_i + \frac{f_i}{m} \Delta t$$

and

$$\theta_{i+1} = \theta_i + \omega_i \Delta t$$

Equations (17) and (18) can be solved iteratively from $i = 0$, where $\theta_0 = \theta(t_0 = 0)$, $\omega_0 = \omega(t_0 = 0)$ and $f_0 = f(\theta_0, \omega_0)$, where $\theta_0$, $\omega_0$, $f_0$ are known from the beginning. This numerical solution is known as the Euler method. It is not a precise method because the mean values are not approximated well by the initial values. A better approximation to the mean values is their substitution by average values. This gives us a new method that begins with initial conditions $\theta_i$, $\omega_i$ and the force:

$$f_i = f(\theta_i, \omega_i)$$

With these values, we calculate:

$$\omega_{i+1}' = \omega_i + \frac{f_i}{m} \Delta t$$

Now, with $\omega_i$ and $\omega_i'$, we have:

$$\omega_{2i+1} = \frac{\omega_i + \omega_i'}{2}$$

where $\omega_{2i+1}$ is an average value centered on the interval from $\omega_i$ to $\omega_i'$. This value can be introduced in equation (18) instead of $\omega_i$, to obtain

$$\theta_{i+1} = \theta_i + \omega_{2i+1} \Delta t$$

With $\theta_i$ and $\theta_{i+1}$, we have:

$$\theta_{2i+1} = \frac{\theta_i + \theta_{i+1}}{2}$$

where $\theta_{2i+1}$ is an average value centered on the interval from $\theta_i$ to $\theta_{i+1}$. Now, with the average values of $\omega_{2i+1}$ and $\theta_{2i+1}$, an approximation of the mean value of $f$ can be made. That is:

$$f_{2i+1} = f\left(\theta_{2i+1}, \omega_{2i+1}\right)$$

where $f_{2i+1}$ is a better approximation of the mean value $\bar{f}$ than the initial value of $f_i$ in equation (17).

This equation is rewritten as:

$$\omega_{i+1} = \omega_i + \frac{f_{2i+1}}{m} \Delta t$$

The numerical method of solution of equations (8) and (9) is given by the iterative calculation of algebraic equations from (19.1) to equation (19.7), incrementing the index $i$ by one unit until it reaches $n$. This method is named the averages method.
6. Programming the averages method

The averages method is programmed in a spreadsheet in an easy way. First, a table with the parameters of the pendulum is written in the spreadsheet, see Table 2.

| Parameters of the pendulum | Initial conditions |
|---------------------------|--------------------|
| m (kg) = 0.077            | t_0 = 0.00        |
| l (m) = 0.645             | l_0 (grad) = 50.00|
| g (m/s^2) = 9.800         | l_0 (rad) = 1.57  |
| \(\beta (1/4) = 0.0117\)  | l_0 (1/4) = 0.00   |

Table 2. Parameters of the pendulum, initial conditions and iteration time interval.

Second, the numerical method of averages is programmed in Excel, Table 3.

| Programming of the averages method in a spreadsheet. The names used in this programming correspond to cells of experimental parameters and initial conditions |
|---|

Table 3. Programming of the averages method in a spreadsheet. The names used in this programming correspond to cells of experimental parameters and initial conditions.

Third, the result of the numerical solution of the averages method is compared graphically with the experimental data in Figure 5. In this graph, the comparison between numerical solution with experimental data is excellent.

Figure 5. Comparison between the numerical solution with the experimental data. Red points are experimental data and the black line is the result of averages method solution.

7. A quadratic function for the period of the pendulum

The period of the damped simple pendulum for large amplitudes varies in each oscillation of the pendulum, as we can see experimentally. This variation of the period can also be studied from the numerical solutions of equations (8) and (9).

If the period of the pendulum is the difference between times of two consecutive amplitudes of the pendulum, this period can be calculated from the numerical solution. The result is a quadratic relation between the period of the pendulum with the amplitude, as we can see in figure 6.
Figure 6. Period as a quadratic function of the amplitude.

The period of the pendulum as a quadratic function of the amplitude has a dependence that is given by the parameters of Table 4.

Table 4. Parameters of the quadratic function of $T$ vs $A$.

| $a_2$       | 0.00000350 | $\delta a_2$ | 0.0000002 |
|-------------|------------|--------------|-----------|
| $a_1$       | 0.0000043  | $\delta a_1$ | 0.0000082 |
| $a_0$       | 1.6108     | $\delta a_0$ | 0.0003    |

The parameter $a_0$ is equal to the period of the pendulum $T_0$ for small oscillations in equation (3). Thus, this dependence is expressed as:

$$T = a_2 \theta_A^2 + a_1 \theta_A + a_0 \quad (20)$$

or

$$T = 3.5 \times 10^{-5} \theta_A^2 + 4.3 \times 10^{-6} \theta_A + T_0 \quad (21)$$

With equation (20) as a relation between the period of the pendulum with the maximum amplitudes, we can find a relation between the period with the time by using the equation (10), as follow:

$$T = a_2 \theta_0 e^{-2 \beta t} + a_1 \theta_0 e^{-\beta t} + T_0 \quad (22)$$

Equation (22) is compared with experimental data with good results.

8. Conjecture about a general solution

We have considered that equation (2) is the solution of the pendulum with small oscillations, with a constant period $T_0$, but in equation (22) the period of the pendulum with large amplitudes is expressed as a continuous function of time. So, we ask ourselves, what happens if we replace in equation (2) the period of equation (22) instead of the period $T_0$ in equation (3)? That is:

$$\theta(t) = \theta_0 e^{-\beta t} \cos\left(\frac{2\pi}{a_2 \theta_0 e^{-2 \beta t} + a_1 \theta_0 e^{-\beta t} + T_0} t + \delta\right) \quad (23)$$

Equation (23) is a conjecture for the solution of a pendulum with large amplitudes that we can test empirically if we graph $\theta$ vs $t$ for values of this equation and compare them with the numerical solution or the experimental data as shown in Figure 7.
Figure 7. Graph of $\theta$ vs $t$ with the comparison between experimental data, numerical model and analytical model.

Because in Figure 7, the coincidence of the experimental graph and the one of the numerical solution is highly satisfactory, we can say that the conjecture that equation (23) is a solution of equation (4) of the pendulum with large amplitudes is acceptable.

9. Teaching strategy

The teaching strategy used with the students for the study of the pendulum’s motion with large amplitudes was a learning cycle of four phases: exploration, experimentation, modeling and predicting. This learning cycle strategy has two methodological approaches: inductive (exploration and experimentation) and deductive (modeling and predicting).

In the study of the motion of the pendulum, students begin the learning cycle with the exploration phase, where they explore the general characteristics of the phenomenon to get used to it. In the case of the pendulum, they examine different kinds of motions to observe if the period is constant or not for different amplitudes. Whatever their observations, students should propose a hypothesis based on these. Next, students need to prove this hypothesis with an experiment, thus the cycle follows with the experimental phase with an experiment design of them, where they do measurements and graphs to analyze the period of the pendulum for small and large amplitudes to have an empirical test of the hypothesis.

With the results of the experiments, students enter the third phase of the learning cycle, the modeling phase. In this phase, students, with the guidance of the professor, get the equation of motion of the pendulum and solve this equation with numerical methods. They compare the numerical solution of the motion equation with the experimental data and if this comparison is successful, the modeling phase finishes. On the contrary, if it is not successful, students need to review very carefully all the steps done to make the necessary corrections to follow with the next phase.

In the prediction phase, students must change the parameters in the numerical solution of the equation of motion to have new results for testing them experimentally or to check extreme cases.

Following this teaching strategy, the understanding of physical phenomena is high in students, because this strategy is like that carried out in real researches by scientists. In the understanding of the motion of the pendulum for great amplitudes, students understood that the period of the pendulum is a quadratic function of its amplitude as in equation (20) and for small oscillations is $T_0$ as in equation (3).

The conjecture of equation (23) as a general solution of the pendulum wasn't discussed with students because it was a result of a reasoning made outside of the laboratory class when it was checking the final reports of students. This conjecture was made by applying the learning cycle to build a model that could predict the behavior of the pendulum in any amplitude.

So, students and teacher learned a lot about the motion of the pendulum by applying the learning cycle.
10. **Scientific skills developed**

The application of the learning cycle in the laboratory class develops scientific skills in students, so important as the scientists apply to their researchers.

In the case of the pendulum's study, the following scientific skills were developed by students in different grades:

- Recognition of a problem to understand how the pendulum motion is for large amplitudes.
- Formulation of hypotheses.
- Designing and carrying out an experiment to research the behavior of this motion.
- Collection of experimental data, production of graphs and their interpretation to find an empirical model.
- Modeling of the pendulum dynamics with a pendulum motion equation for large amplitudes.
- Application of a numerical method to solve the pendulum motion equation with damping.
- Comparison between numerical solution and experimental data.
- Prediction of new results and its comparison with experimental data.
- Drawing conclusions for this study. Confirmation or refutation of hypotheses.

An evaluation of these scientific skills developed by students is given in Figure 8.

![The Learning Cycle evaluation](image)

**Figure 8.** Scientific skills evaluation of the learning cycle strategy applied to the investigation of the damped pendulum with large amplitudes.

According to the evaluation in figure 8, the inductive approach is the best done by students, but they need the conceptual support of teachers for deductive approach and their efficiency decreases in modeling and predicting steps of the learning cycle, however, of seven students, four of them reached the modeling step of the learning cycle, and two could finish the complete cycle.

The evaluation of the teaching cycle strategy carried out by students in the study of the damped pendulum with large amplitudes was made with an oral presentation of the experimental and modeling works in their investigations of this theme. In this presentation, students were able to explain their research to their peers, exposing and answering questions correctly, showing the skills developed by them during the research. Clinical interviews were made to the better students for deepening in the concepts built by them to confirm their learnings about the pendulum.

11. **Conclusions**

In the Mechanics’ Laboratory class, the learning cycle strategy for studying the motion of the pendulum for large amplitudes was applied to seven physics students of the first year of university. In this process, students suggested different types of hypotheses that they tested experimentally successfully, getting results that they used for numerically solving the equation of motion. They also compared the numerical solution and the experimental data successfully and for this good comparison, they accepted with great confidence this solution as a pendulum solution for large amplitudes. They also predicted the behavior of the pendulum for other amplitudes by using the numerical model and they got good results when they
this following a scientific research process, the students were able to achieve good results in the study of oscillation was a complex investigation carried out by the students because it was necessary to find and model scientific facts. The study on the motion of the damped pendulum with large amplitudes of oscillation was a complex investigation carried out by the students because it was necessary to find and interpret a numerical solution not modeled frequently in this way in the scientific literature. However, following a scientific research process, the students were able to achieve good results in the study of this motion.

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