On the functoriality of Lott’s secondary analytic index

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Contents

1 Introduction 0

2 Secondary $K$-Groups and analytic push-forward 1

3 Analytic torsion form and iterated fibrations 2

4 Verification of Theorem 1.1 4

1 Introduction

In [2] for a smooth manifold $M$ J. Lott has defined a secondary $K$-group $\tilde{K}^0_R(M)$ of local systems of $R$-modules ($R$ is some ring) such that their complexifications (we must fix a representation $\rho : R \to \text{End}(\mathbb{C}^n)$ in order to define this notion) have explicitly trivialized characteristic classes. If $p : E \to B$ is a smooth fibre bundle with closed fibres, then he defines a push-forward $p^! : \tilde{K}^0_R(E) \to \tilde{K}^0_R(B)$. This construction involves the analytic torsion form of [1]. We will review Lott’s construction in Section 2. In [3] X. Ma has studied the behaviour of the analytic torsion form under iterated fibre bundles. We review his result in Section 3.

The goal of the present note is to verify that Ma’s result indeed implies that the push-forward of Lott is functorial. We now describe this assertion in detail. Let $p : E \to B$ be a fibration with fibre $Z$ which is in fact an iterated fibration. So we assume that there are fibrations $p_1 : E =: E_1 \to E_2$ with fibre $Z_1$ and $p_2 : E_2 \to B$ with fibre $Z_2$ such that $Z \to Z_2$ is a fibration with fibre $Z_1$. We will verify the following:

**Theorem 1.1** We have $p = (p_2)_\circ (p_1)_!$ as maps from $\tilde{K}^0_R(E)$ to $\tilde{K}^0_R(B)$.

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The details of the proof are contained in Section 4. We do now claim any kind of originality here because the argument just amounts to combine the construction of Lott and the result of Ma mentioned above.

2 Secondary $K$-Groups and analytic push-forward

This section reviews Sections 2.1 and 2.2 of [2]. Suppose that $R$ is a right-Noetherian ring which is right-regular, i.e. every finitely generated right-$R$-module has a finite resolution by finitely generated projective right-$R$-modules. We fix a representation $\rho : R \to \text{End}(C^n)$ such that $C^n$ becomes a flat left $R$-module. If $V$ is a right-$R$-module, then $V_C := V \otimes_{R,\rho} C^n$ is called its complexification.

Let $M$ be a connected smooth manifold. If $\mathcal{F}$ is a local system (a locally constant sheaf) of finitely generated right-$R$-modules, then $\mathcal{F}_C$ (the fibrewise complexification) is the sheaf of parallel sections of a flat complex vector bundle which we denote by $(F_C, \nabla^{F_C})$.

Let $(E, \nabla^E)$ be a flat complex vector bundle. If we choose a hermitean metric $h^E$ (we consider $h^E$ is a section of the flat bundle $\text{Hom}_C(E, E^*)$) on $E$, then we can define the characteristic forms

$$\omega(\nabla^E, h^E) := (h^E)^{-1}\nabla^{\text{Hom}(E, E^*)}h^E$$
$$c_k(\nabla^E, h^E) = (2\pi i)^{-\frac{k}{2}}2^{-k}\text{Tr} \omega(\nabla^E, h^E)$$
$$c(\nabla^E, h^E) = \sum_{j=0}^{\infty} \frac{1}{j!} c_{2j+1}(\nabla^E, h^E).$$

The form $c(\nabla^E, h^E)$ is closed and represents the characteristic class $c(\nabla^E) \in H^{\text{odd}}_{dR}(M)$ of the flat vector bundle $(E, \nabla^E)$ which is independent of the choice of $h^E$.

The abelian group $K_R^0(M)$ is generated by triples $f = (\mathcal{F}, h^{F_C}, \eta)$, where

1. $\mathcal{F}$ is a local system of finitely generated right-$R$-modules,
2. $h^{F_C}$ is a hermitean metric of the corresponding flat complex vector bundle $(F_C, \nabla^{F_C})$, and
3. $\eta \in \Omega^{ev}(M)/\text{image}(d)$,

subject to the following relations: If

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence of local systems of finitely generated right-$R$-modules, $h^{(F_i)_C}$ are hermitean metrics, $\eta_i \in \Omega^{ev}(M)/\text{image}(d)$, and we form $f_i := (\mathcal{F}_i, h^{(F_i)_C}, \eta_i)$, then $f_2 \sim f_1 + f_3$ if

$$\eta_2 = \eta_1 + \eta_3 + T(C, h^C),$$
where $\mathcal{T}(\mathcal{C}, h^C)$ is the analytic torsion form associated to the exact complex of flat complex vector bundles $(\mathcal{C}, h^C)$

$$0 \to (F_1)_C \to (F_2)_C \to (F_3)_C \to 0$$
equipped with the metric $h^C$ induced by $h^{(F_i)_C}$. For the details of the definition of the torsion form $\mathcal{T}(\mathcal{C}, h^C)$ we refer to \cite{1}, Sec. 2, or \cite{2}, A.3. The appearence of the torsion form in the equivalence relation is explained by the relation

$$d\mathcal{T}(\mathcal{C}, h^C) = \sum_{i=1}^{3} (-1)^i c((F_i)_C, h^{(F_i)_C}) .$$

Lott shows that $f \mapsto c(\nabla^{F_C}, h^{F_C}) - d\eta$ extends to a map $c' : \hat{K}_R^0(M) \to \Omega^{\text{odd}}(M)$, and he defines

$$\hat{K}_R^0(M) := \ker(c') .$$
The assignment $M \mapsto \hat{K}_R^0(M)$ yields a homotopy invariant contravariant functor from the category of manifolds to abelian groups.

We now consider a smooth fibre bundle $p : E \to B$ with compact fibre $Z$. If $\mathcal{F}$ is a locally constant sheaf of finitely generated right-$R$-modules, then we can form the sheaves $\mathcal{R}^i p_* \mathcal{F}$ on $B$ which are again locally constant sheaves of finitely generated right-$R$-modules.

If we choose a fibrewise Riemannian metric $g^Z$ (i.e. a metric on the vertical bundle $TZ$), then we can compute $(\mathcal{R}^i p_* \mathcal{F})_C = \mathcal{R}^i p_* (\mathcal{F}_C)$ (this equality holds because $C^n$ is a flat $R$-module) using the fibrewise de Rham complex twisted with $F_C$. The metric $g^Z$ and a hermitean metric $h^{F_C}$ induce $L^2$-scalar products. Identifying the stalk $(\mathcal{R}^i p_* (\mathcal{F}_C))_b$ (which is the fibre of the flat complex vector bundle $(\mathcal{R}^i p_* \mathcal{F})_C$) with harmonic forms we obtain metrics $h^{(\mathcal{R}^i p_* \mathcal{F})_C}$.

We further choose a horizontal distribution $T^H E$. It induces a connection $\nabla^{TZ}$ on the vertical bundle. Let $\mathcal{E}(TZ, \nabla^{TZ})$ be the associated Euler form. We refer to \cite{1} for the definition of the analytic torsion form $\mathcal{T}(T^H E, g^Z, h^{F_C}) \in \Omega^{ev}(B)$.

J. Lott defines the push-forward $p_1 : \hat{K}_R^0(M) \to \hat{K}_R^0(B)$ by the assignment:

$$(\mathcal{F}, h^F, \eta) \mapsto \sum_p (\mathcal{R}^i p_* (\mathcal{F}), h^{(\mathcal{R}^i p_* \mathcal{F})_C}, 0) + (0, 0, \int_Z e(TZ, \nabla^{TZ}) \wedge \eta - \mathcal{T}(T^H E, g^Z, h^{F_C}) ) .$$

Lott proves well-definedness and independence of $T^H E$ and $g^Z$.

## 3 Analytic torsion form and iterated fibrations

This section reviews the result of \cite{3}. Let $p : E \to B$ be a fibration with fibre $Z$ which is in fact an iterated fibration. We assume that there are fibrations $p_1 : E := E_1 \to E_2$ with fibre $Z_1$ and
3 ANALYTIC TORSION FORM AND ITERATED FIBRATIONS

$p_2 : E_2 \to B$ with fibre $Z_2$ such that $Z \to Z_2$ is a fibration with fibre $Z_1$. By $TZ$, $TZ_1$, and $TZ_2$, we denote the corresponding vertical bundles. We choose vertical Riemannian metrics $g^Z$, $g^{Z_1}$, $g^{Z_2}$.

Furthermore, we choose horizontal bundles $T^HE$, $T^HE_1$, and $T^HE_2$, for $p$, $p_1$, $p_2$. We obtain connections $\nabla^{TZ}$, $\nabla^{TZ_1}$, and $\nabla^{TZ_2}$. We identify $TZ$ with $TZ_1 \oplus p_1^*TZ_2$ (using $T^HE_1$) and obtain another connection $^0\nabla^{TZ} := \nabla^{TZ_1} \oplus p_1^*\nabla^{TZ_2}$ on $TZ$. By $\tilde{e}(TZ, ^0\nabla^{TZ})$ we denote the corresponding transgression of the Euler form such that

$$d\tilde{e}(TZ, ^0\nabla^{TZ}) = e(TZ, ^0\nabla^{TZ}) - e(TZ, \nabla^{TZ}) .$$

The main result of X. Ma is the formula

$$\mathcal{T}(T^HE, g^Z, h^{Fc}) - \int_{Z_2} e(TZ_2, \nabla^{TZ_2}) \wedge \mathcal{T}(T^HE_1, g^{Z_1}, h^F)$$

$$- \sum_i (-1)^i \mathcal{T}(T^HE_2, g^{Z_2}, h(R^{p,*}F)) c_i$$

$$+ \int_Z \tilde{e}(TZ, ^0\nabla^{TZ}) \wedge c(\nabla^{Fc}, h^{Fc})$$

$$= S \mod \text{image}(d) ,$$

where $S$ is a higher analytic torsion invariant associated to the Leray spectral sequence of the family of fibrations $Z_b \to (Z_1)_b$, $b \in B$.

We now describe $S$ in detail. The spectral sequence $(E_r, d_r)$ is associated to the composition $(p_2)_* \circ (p_1)_*$. Its second term is $E_2^{p,q} = R^p(p_2)_* R^q(p_1)_*(F)$, and it converges to $R^{p+q}p_*(F)$. Since $\mathbb{C}^n$ is a flat $R$-module complexification commutes with taking the spectral sequence. In particular, we obtain flat complex vector bundles $(E_r^{p,q})_\mathbb{C}$. The differentials $d_r$ of the spectral sequence induce corresponding bundle homomorphisms such that we obtain complexes of flat complex vector bundles

$$\ldots \to d^r \to (E_r^{p,q})_\mathbb{C} \to d^{r+1} (E_r^{p+q+1-r})_\mathbb{C} \to \ldots$$

with cohomology $(E_r^{p,q})_\mathbb{C}$. $h^{(R^p(p_+)_*(F))c}$ induces metrics $h^{(E_2^{p,q})_\mathbb{C}}$. Now we obtain inductively metrics on the cohomology groups $h^{(E_r^{p,q})_\mathbb{C}}$.

In order to save notation we denote by $\mathcal{D}_r$ the direct sum of complexes above at the level $r$ and by $h^{\mathcal{D}_r}$ the induced metric.

Let

$$\mathcal{V} : \ldots \to V_{i-1} \to V_i \to V_{i+1} \to \ldots$$

be a finite complex of flat complex vector bundles equipped with hermitean metrics $h^{V_i}$ (we write $h^{\mathcal{V}}$ for the whole collection). We further choose hermitean metrics on the flat cohomology
4 VERIFICATION OF THEOREM ??

The group $K^0_R(E)$ is generated by elements $f := (\mathcal{F}, h^F, \eta)$ with $d\eta = c(F, h^F)$. Let us write out a representative of $(p_1)_!(\{f\})$. We obtain

$$\sum_q (-1)^q (R^q(p_1)_*\mathcal{F}, h^{(R^q(p_1)_*\mathcal{F})}, 0) + (0, 0, \int_{Z_1} e(TZ_1, \nabla TZ_1) \land \eta - \mathcal{T}(H_E, g^{Z_1}, h^F)) .$$

A representative of $(p_2)_! \circ (p_1)_!(\{f\})$ is given by

$$\sum_{p,q} (-1)^{p+q} (R^p(p_2)_*R^q(p_1)_*\mathcal{F}, h^{(E_2)^{p,q}}, 0) + (0, 0, \int_{Z_2} e(TZ_2, \nabla TZ_2) \land \left(\int_{Z_1} e(TZ_1, \nabla TZ_1) \land \eta - \mathcal{T}(H_E, g^{Z_1}, h^F)\right) \right)$$

$$- \sum_q (-1)^q (0, 0, \mathcal{T}(H_E, g^{Z_2}, h^{(R^q(p_1)_*\mathcal{F})})) .$$
We must show that this expression represents the same element as
\[ \sum_i (-1)^i (R^i p_* F, h^{(R^i p_* F)}c, 0) + (0, 0, \int_Z e(TZ, \nabla^{TZ}) \wedge \eta - T(T^H E, g^Z, h^{Fc})). \]

We first compare the terms involving the form \( \eta \). Indeed we have
\[
\int_Z e(TZ_2, \nabla^{TZ_2}) \wedge \left( \int_{Z_1} e(TZ_1, \nabla^{TZ_1}) \wedge \eta \right) - \int_Z e(TZ, \nabla^{TZ}) \wedge \eta
= \int_Z \tilde{e}(TZ, \nabla^{TZ}, 0, \nabla^{TZ}) \wedge d\eta
= \int_Z \tilde{e}(TZ, \nabla^{TZ}, 0, \nabla^{TZ}) \wedge c(\nabla^{Fc}, h^{Fc}) \mod \text{image}(d).
\]

Thus it remains to show that
\[
\sum_{p, q} (-1)^{p+q} (R^p(p_2)_* R^q(p_1)_* F, h^{(R^p p_* F)}c, 0) \\
-(0, 0, \int_{Z_2} e(TZ_2, \nabla^{TZ_2}) \wedge T(T^H E_1, g^Z_1, h^{Fc})) \\
- \sum_q (-1)^q (0, 0, T(T^H E_2, g^Z_2, h^{R^q(p_1)_* F}c)) \\
- \sum_p (-1)^p (R^p p_* F, h^{(R^p p_* F)}c, 0) \\
+(0, 0, T(T^H E, g^Z, h^{Fc}) \\
+ \int_Z \tilde{e}(TZ, \nabla^{TZ}, 0, \nabla^{TZ}) \wedge c(\nabla^{Fc}, h^{Fc})
\]
represents the trivial element in \( \hat{K}_0^R(B) \).

Let
\[ \mathcal{V} : \ldots \to \mathcal{V}_{i-1} \xrightarrow{d_i} \mathcal{V}_i \to \mathcal{V}_{i+1} \xrightarrow{d_{i+1}} \ldots \]
be a finite complex of local systems of finitely generated right-\( R \)-modules over \( B \). We fix hermitian metrics \( h^{(V)c} \) (we write \( h^V \) for the whole collection). Since \( C^n \) is a flat \( R \)-module we can interchange the operation of complexification and of taking fibrewise cohomology. We let \( H^i \) denote the flat complex vector bundle obtained from the complexification of the cohomology sheaves \( H^i(V) \). We further choose hermitian metrics \( h^{H^i} \) (and we write \( h^H \) for this collection). We consider the short exact sequences
\[
\mathcal{C}_i : 0 \to \ker(d_i) \to \mathcal{V}_i \to \text{image}(d_i) \to 0 \\
\mathcal{D}_i : 0 \to \text{image}(d_{i-1}) \to \ker(d_i) \to \mathcal{H}^i(V) \to 0 ,
\]
where the corresponding complexes of flat complex vector bundles \((C_i)_C, (D_i)_C\) have induced hermitean metrics \(h^{(C_i)}_C, h^{(D_i)}_C\). In \(\hat{K}_R^0(B)\) we have
\[
\sum_i (-1)^i (\mathcal{V}_i, h^{(\mathcal{V}_i)}_C, 0) = \sum_i (-1)^i (\mathcal{H}^i, h^{H^i}, -\mathcal{T}(C_i, h^{(C_i)}_C) - \mathcal{T}(D_i, h^{(D_i)}_C))
\]
\[
= \sum_i (-1)^i (\mathcal{H}^i, h^{H^i}, 0) - (0, 0, \mathcal{T}(\mathcal{V}_C, h^{(\mathcal{V})}_C, h^{\mathcal{H}_C})) .
\]

Using this we compute
\[
\sum_{p,q} (-1)^{p+q} (\mathcal{R}^p(p_2)_* \mathcal{R}^q(p_1)_* \mathcal{F}, h^{(\mathcal{E}^p,q)}_C, 0)
\]
\[
= \sum_{p,q} (-1)^{p+q} (\mathcal{E}^p,q, h^{(\mathcal{E}^p,q)}_C, 0) - (0, 0, \mathcal{T}(\mathcal{D}_2, h^{(\mathcal{D}_2)}_C, h^{\mathcal{D}_3}))
\]
\[
= \ldots
\]
\[
= \sum_{p,q} (-1)^{p+q} (\mathcal{E}_\infty^p,q, h^{(\mathcal{E}_\infty^p,q)}_C, 0) - \sum_{r=2}^\infty (0, 0, \mathcal{T}(\mathcal{D}_r, h^{D_r}, h^{D_{r+1}})).
\]

Let now \(\mathcal{V}\) be a local system of finitely generated right-\(R\)-modules which is filtered by local systems of submodules \(0 \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \ldots \subset \mathcal{V}_n = \mathcal{V}\). We fix a hermitean metric \(h^{\mathcal{V}_C}\) which induces metrics \(h^{\mathcal{V}(\mathcal{V})}_C\). Furthermore we fix metrics \(h^{Gr_i(\mathcal{V})_C}\). In \(\hat{K}_R^0(B)\) we have
\[
(\mathcal{V}, h^{\mathcal{V}_C}, 0) = \sum_i (Gr_i(\mathcal{V}), g^{Gr_i(\mathcal{V})}_C, 0) - (0, 0, \mathcal{T}(\mathcal{V}_C, Gr(\mathcal{V})_C, h^{\mathcal{V}_C}, h^{Gr(\mathcal{V})_C})).
\]

Using this observation we further compute
\[
\sum_{p+q=k} (-1)^k (\mathcal{E}_\infty^p,q, h^{(\mathcal{E}_\infty^p,q)}_C, 0)
\]
\[
= \sum_i (Gr_i(\mathcal{R}^k(p_2)_* \mathcal{F}_i), h^{Gr_i(\mathcal{R}^k(p_2)_* \mathcal{F})_C}, 0)
\]
\[
= (\mathcal{R}^k(p_2)_* \mathcal{F}_i, h^{(\mathcal{R}^k(p_2)_* \mathcal{F})_C}, 0) + (0, 0, \mathcal{T}((\mathcal{R}^k(p_2)_* \mathcal{F})_C, \oplus_{p+q=k}(\mathcal{E}_\infty^p,q)_C, h^{(\mathcal{R}^k(p_2)_* \mathcal{F})_C}, h^{Gr(\mathcal{R}^k(p_2)_* \mathcal{F})_C})).
\]

Thus it remains to show that
\[
- \sum_{r=2}^\infty \mathcal{T}(\mathcal{D}_r, h^{D_r}, h^{D_{r+1}})
\]
\[
+ \sum_{k=0}^\infty (-1)^k \mathcal{T}((\mathcal{R}^k(p_2)_* \mathcal{F}_i)_C, \oplus_{p+q=k}(\mathcal{E}_\infty^p,q)_C, h^{(\mathcal{R}^k(p_2)_* \mathcal{F})_C}, h^{Gr(\mathcal{R}^k(p_2)_* \mathcal{F})_C})
\]
\[
- \int_{\mathcal{Z}_2} e(TZ_2, \nabla^{TZ_2}) \wedge \mathcal{T}((T^HE_1, g^{Z_1})_C, h^{F_C})
\]
\[
- \sum_p (-1)^p \mathcal{T}((T^HE_2, g^{Z_2})_C, h^{(R^p(p_2)_* \mathcal{F})_C})
\]
\[
+ \mathcal{T}(T^HE, g^Z, h^{F_C})
\]
\[
+ \int_{\mathcal{Z}} \mathcal{e}(TZ, \nabla^{TZ}, 0 \nabla^{TZ}) \wedge c(\nabla^{F_C}, h^{F_C})
\]
is an exact form. But this is exactly the assertion of X. Ma. This finishes the verification. \hfill \Box

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