BOUNDS ON THE INDIVIDUAL BETTI NUMBERS OF
COMPLEX VARIETIES, STABILITY AND ALGORITHMS

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Abstract. We prove graded bounds on the individual Betti numbers of affine
and projective complex varieties. In particular, we give for each $p, d, r$, explicit
bounds on the $p$-th Betti numbers of affine and projective subvarieties of $C^d$,
$P^k$, as well as products of projective spaces, defined by $r$ polynomials of degrees
at most $d$ as a function of $p, d$ and $r$. Unlike previous bounds these bounds
are independent of $k$, the dimension of the ambient space. We also prove as
consequences of our technique certain homological and representational sta-
bility results for sequences of complex projective varieties which could be of
independent interest. Finally, we highlight differences in computational com-
plexities of the problem of computing Betti numbers of complex as opposed to
real projective varieties.

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1. Introduction

Throughout the paper we denote by $R$ a real closed field, and $C$ the algebraic
closure of $R$. We can even assume that $R = \mathbb{R}$, and $C = \mathbb{C}$ without any loss of
generality, since all the results of the paper follow for arbitrary real closed fields.

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be the set of common zeros of \( P \). We denote by \( Z(P, K^n) \) the set of common zeros of \( P \) in \( K^n \). Similarly, for finite subsets of homogeneous polynomials \( P \subset K[X_0, \ldots, X_n] \), we denote by \( Z(P, P^n_K) \) the set of common zeros of \( P \) in \( P^n_K \). Given a real or complex variety \( X \), we denote by \( H^i(X, F) \) (resp. \( H^i_c(X, F) \)) the \( i \)-th cohomology group (resp. \( i \)-th homology, \( i \)-th cohomology group with compact support) with coefficients in the field \( F \). We refer the reader to [7, Chapter 6] for the definition of homology/cohomology groups of semi-algebraic sets defined over arbitrary real closed fields, noting that they are isomorphic to the singular homology/cohomology groups in the special case of \( R = \mathbb{R} \).

We denote by \( b^i(X, F) \) (resp. \( b^i_c(X, F) \)) the dimension of \( H^i(X, F) \) (resp. \( H^i_c(X, F) \)), and by \( b(X, F) = \sum_{i \geq 0} b^i(X, F) \) (resp. \( b_c(X, F) = \sum_{i \geq 0} b^i_c(X, F) \)).

1.2. History and prior results. The problem of bounding explicitly the Betti numbers of real and complex varieties have been considered for a long time and there have been many applications of these bounds in combinatorics and discrete geometry (see for example, [6] for a survey).

The first results are due to Oleinik and Petrovskii [29], Thom [34] and Milnor [27] who proved the following bounds. Slightly refined bounds of the same asymptotic nature occur in [12] and [9].

**Theorem 1.** [29, 34, 27] Let \( P \subset \mathbb{R}[X_1, \ldots, X_k] \) be a finite set polynomials of degrees at most \( d \), and let \( V = Z(P, \mathbb{R}^k) \). Then,

\[
b(V, \mathbb{R}) \leq \text{Aff}_\mathbb{R}(k, d) := d(2d - 1)^{k-1} = (O(d))^k.
\]

The bound in Theorem 1 also holds in the projective case.

**Theorem 2.** [29, 34, 27] Let \( P \subset \mathbb{R}[X_0, \ldots, X_k] \) be a finite set homogeneous polynomials of degrees at most \( d \), and let \( V = Z(P, \mathbb{P}^k_\mathbb{R}) \). Then,

\[
b(V, \mathbb{P}_\mathbb{R}) \leq \text{Proj}_\mathbb{R}(k, d) := d(2d - 1)^{k-1} = (O(d))^k.
\]

The bounds in Theorems 1 and 2 are (for every fixed \( d \)) singly exponential in the number of variables \( k \). Moreover, this exponential dependence on \( k \) is unavoidable even if the variety \( V \) is a non-singular hypersurface defined by one polynomial, and we consider just a single Betti number of \( V \) (for example \( b_0(V, \mathbb{R}) \) or \( b_{k-1}(V, \mathbb{R}) \)) instead of their sum, as the following examples show.

**Example 1.** Let \( P = \sum_{i=1}^k X_i^2(X_i - 1)^2 - \varepsilon\), with \( 0 < \varepsilon \ll 1 \), and \( P^h \) denote the homogenization of \( P \). Let \( V = Z(P, \mathbb{R}^k) \), and \( V^h = Z(P^h, \mathbb{P}^k_\mathbb{R}) \). Now notice that \( \deg(P) = 4 \), and

\[
b^0(V, \mathbb{R}) = b^{k-1}(V, \mathbb{R}) = b^{k-1}(V^h, \mathbb{R}) = 2^k,
\]

\[
b^0(V^h, \mathbb{R}) = 2^{k+1}.
\]
While Theorems 1 and 2 deal only with real varieties, they can be used to bound the Betti numbers of complex varieties, since every complex affine variety in $\mathbb{C}^k$ defined by $r$ polynomials of degrees bounded by $d$, can be considered after separating the real and imaginary parts of the defining polynomials as a real affine variety in $\mathbb{R}^{2k}$ defined by $2r$ polynomials of degree at most $d$. It then follows directly from Theorem 1 that:

**Theorem 3.** Let $\mathcal{P} \subset \mathbb{C}[Z_1, \ldots, Z_k]$ be a finite set of polynomials of degrees at most $d$, and let $V = Z(\mathcal{P}, \mathbb{C}^k)$. Then,

$$b(V, \mathbb{F}) \leq \text{Aff}_{\mathbb{C}}(k, d) := d(2d - 1)^{2k - 1} = (O(d))^{2k}. \quad (1.3)$$

Using an argument involving the Hopf fibration and the Gysin exact sequence one also derives a similar bound in the projective case.

**Theorem 4.** Let $\mathcal{P} \subset \mathbb{C}[Z_0, \ldots, Z_k]$ be a set of homogeneous polynomials of degrees at most $d \geq 2$, and let $V = Z(\mathcal{P}, \mathbb{P}^k_{\mathbb{C}})$. Then,

$$b(V, \mathbb{F}) \leq \text{Proj}_{\mathbb{C}}(k, d) := kd(2d - 1)^{2k + 1} = (O(d))^{2k+2}. \quad (1.4)$$

**Proof.** Let $S^{2k+1} \subset \mathbb{C}^{k+1} = \mathbb{R}^{2k+2}$ denote the unite sphere defined by $|Z_0|^2 + \cdots + |Z_k|^2 = 1$. Consider the Hopf fibration $\phi : S^{2k+1} \to \mathbb{P}^{k}_{\mathbb{C}}$ defined by $(z_0, \ldots, z_k) \mapsto (z_0 : \cdots : z_k)$. We denote by $\tilde{V} = \phi^{-1}(V)$. We have the following commutative diagram:

$$\begin{array}{ccc}
\tilde{V} & \longrightarrow & S^{2k+1} \\
\downarrow \phi|_{\tilde{V}} & & \downarrow \phi \\
V & \longrightarrow & \mathbb{P}^{k}_{\mathbb{C}}
\end{array}$$

Note that $\tilde{V}$ is a $S^1$-bundle over $V$. It follows from the Gysin exact sequence [33, page 260] of this bundle that for each $n \geq 0$,

$$b^n(V, \mathbb{F}) \leq b^{n-2}(V, \mathbb{F}) + b^n(\tilde{V}, \mathbb{F}) \leq b^{n-4}(V, \mathbb{F}) + b^{n-2}(\tilde{V}, \mathbb{F}) + b^n(\tilde{V}, \mathbb{F}) \leq \sum_{i \geq 0} b^{n-2i}(\tilde{V}, \mathbb{F}).$$

It follows that

$$b(V, \mathbb{F}) \leq \sum_{i = 0}^{2k} \lceil (2k - i)/2 \rceil b^i(\tilde{V}, \mathbb{F}) \leq kb(\tilde{V}, \mathbb{F}).$$

The theorem now follows from Theorem 1. \qed

**Remark 1.** With a little more care (for example, using [9, Theorem 32] instead of Theorem 1 as in the proof above), it is possible to prove a bound of $(O(d))^{2k}$ on $b(V, \mathbb{F})$. This would also improve the bound on $b(V, \mathbb{F})$ in Theorem 4 to $(O(d))^{2k}$. 

1.3. Upper bounds on \(\ell\)-adic Betti numbers. The bounds mentioned above follow essentially from Morse theoretic considerations by counting critical points. By following a completely different approach, and using bounds on exponential sums due to Bombieri [14], and Adolphson and Sperber [1], Katz [26] gave analogous bounds for \(\ell\)-adic Betti numbers of complex varieties for any prime \(\ell\).

We first recall here the definition of \(\ell\)-adic cohomology groups of a complex variety \(V\) for any prime \(\ell\). (Note that the notation for the \(\ell\)-adic cohomology is in conflict with the notation used above for cohomology groups with coefficients in the field \(\mathbb{Q}_\ell\). However, this abuse of notation is standard in literature.)

\[
H^*(V, \mathbb{Z}_\ell) = \operatorname{proj lim}_n H^*(V, \mathbb{Z}/\ell^n \mathbb{Z}),
\]

\[
H^*_c(V, \mathbb{Z}_\ell) = \operatorname{proj lim}_n H^*_c(V, \mathbb{Z}/\ell^n \mathbb{Z}),
\]

\[
H^*(V, \mathbb{Q}_\ell) = H^*(V, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell,
\]

\[
H^*_c(V, \mathbb{Q}_\ell) = H^*_c(V, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell,
\]

and denote

\[
b_i(V, \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_\ell} H^i(V, \mathbb{Q}_\ell),
\]

\[
b_i(V, \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_\ell} H^*_c(V, \mathbb{Q}_\ell),
\]

\[
b(V, \mathbb{Q}_\ell) = \sum_i b_i(V, \mathbb{Q}_\ell),
\]

\[
b_c(V, \mathbb{Q}_\ell) = \sum_i b^*_c(V, \mathbb{Q}_\ell).
\]

The following bounds appear in [26].

**Theorem 5.** [26] Let \(\mathcal{P} = \{P_1, \ldots, P_r\} \subset C[Z_1, \ldots, Z_k]\) be a finite set polynomials of degrees at most \(d\), and let \(U = Z(\mathcal{P}, C^k)\). Then,

\[
b_c(U, \mathbb{Q}_\ell) \leq \operatorname{Aff}_{C, \ell}(k, r, d) := 5 \cdot 2^{r-2} (4dr + 13)^{k+2} = 2^r (O(rd))^{k+2}.
\]

**Theorem 6.** [26] Let \(\mathcal{P} \subset C[Z_0, \ldots, Z_k]\) be a set of homogeneous polynomials of degrees at most \(d \geq 2\), and let \(V = Z(\mathcal{P}, \mathbb{P}_C^k)\). Then,

\[
b(V, \mathbb{Q}_\ell) \leq \operatorname{Proj}_{C, \ell}(k, r, d) := 1 + \sum_{j=1}^k \operatorname{Aff}_{C, \ell}(j, r, d) = 2^r (O(rd))^{k+2}.
\]

**Remark 2.** Notice that unlike bounds in Theorems 3 and 4 (which are in turn derived from Theorem 1), the bounds in Theorems 5, and 6 depend on the number of polynomials \(r\) occurring in the definition of the variety.

Like their counterparts in the real case (namely, Theorems 1 and 2) these bounds are singly exponential in \(k\) for any fixed \(d\). However, complex varieties differ topologically from real ones in one important respect. The Betti numbers of a non-singular, projective variety of dimension \(\ell\) which is a complete intersection are concentrated at dimension \(\ell\). More precisely, suppose that \(V \subset \mathbb{P}_C^k\) be a complete intersection non-singular variety of dimension \(\ell > 0\). Then,
(1.7) \[ b^i(V, \mathbb{F}) = \begin{cases} 0, & \text{if } i > 2\ell \text{ or if } i \neq \ell \text{ and } i \text{ is odd,} \\ 1, & \text{if } i \neq \ell \text{ and } i \text{ is even.} \end{cases} \]

The above behavior regarding the distribution of Betti numbers is thus very different from the real case (cf. Example 1). The concentration of the Betti numbers in the middle dimension as shown in Eqn. (1.7) is clearly not true if \( V \) is singular or not a complete intersection. For example, a cubic hypersurface \( V_3 \subset \mathbb{P}^4_{\mathbb{C}} \) having 10 nodes satisfies \( b_3(V_3, \mathbb{F}) = 10 \) (see [19, §4, Example 4.6]) showing that the odd Betti numbers of such varieties can be non-zero.

1.4. Summary of new results. Nevertheless, we prove (Theorems 7, 9, 8 and 10) that some of the concentration in the non-singular case extends to singular varieties as well. In fact, we show that for every fixed \( p \) and \( d \) and \( r \), the \( p \)-th Betti numbers of complex affine and projective subvarieties of \( C^k \) (resp. \( \mathbb{P}^k_C \)) can be bounded by an explicit function of \( d, p \) and \( r \), independent of \( k \).

The proofs of these results rely on well-known results on Lefschetz hyperplane section theorems for singular varieties [21, 22], Poincaré-Lefschetz duality, the Stein property of smooth affine varieties, as well as standard tools from algebraic topology such as the Mayer-Vietoris sequence and the Oleinik-Petrovskii-Thom-Milnor bounds cited earlier.

Next, we extend our results to sub-varieties of a product of projective spaces (Theorem 12). In fact, we only deal with the case of the product of two projective spaces, but the same method can be used for more general products as well. This extension is not immediate and needs some subtle book-keeping. As an application we prove a bound on the small dimensional Betti numbers of varieties obtained as images under projection of a subvariety of two projective spaces (Theorem 13). Aside from their importance in algebraic geometry, the topological complexity of such images play an important role in computational complexity theory [15, 5].

1.5. Applications. Aside from the quantitative estimates on the individual Betti numbers of complex varieties, we give two other applications of the method.

The first application (§5 below) is to prove a homological stability theorem which answers in the complex case a question related to representational stability of the cohomology groups of certain symmetric varieties which was raised in [8]. Roughly speaking the question (is still unresolved) asks whether the multiplicities of the Specht modules corresponding to some fixed partition appearing in some fixed dimensional cohomology groups of certain natural sequences of symmetric real varieties \( V_n \) are eventually expressible as a polynomial in \( n \). We prove in this paper (Theorem 15) that in the complex projective case, this is trivially true since the cohomology groups, \( H^p(V_n, \mathbb{Q}) \) become eventually isomorphic as \( n \) grows for every fixed \( p \) (see Theorem 16). This last statement is not true over the reals (cf. Remark 12).

The second application (§6 below) is related to computational complexity theory. The problem of computing the Betti numbers of varieties as well as semi-algebraic sets occupies a very important place in the hierarchy of computational complexity classes especially in the Blum-Shub-Smale (henceforth B-S-S) [13] model of computation (see for example [15, 11, 5]). Indeed, the problem of computing the Poincaré
polynomial of the fibers of certain sequences of maps plays a critical role in the
definition of the B-S-S analog of classical counting complexity class \#P (see [11, 5]).
It is not a surprise that the problem of computing Betti numbers of real and com-
plex varieties have been investigated thoroughly from both from the point of view
of hardness, as well as from the point of view of designing efficient algorithms to
solve the problem. Currently, for every fixed degree \( d > 2 \), the best algorithm for
computing all the (possibly non-zero) Betti numbers of a real or complex variety
(affine or projective) has complexity which is doubly exponential in the dimension
of the ambient space.

However, since the Betti numbers of varieties are bounded singly exponentially
it has been conjectured that there should exist singly exponential complexity algo-
rithms for computing them (see for example [3]). Such algorithms exist in certain
special situations – for example, for computing the first \( \ell \) (for any constant \( \ell \))
Betti numbers of general semi-algebraic sets [2], or for computing the Betti num-
bers of smooth complex projective varieties [31]. There also exist algorithms with
polynomially bounded complexity for computing the top few Betti numbers of semi-
algebraic sets defined by quadratic inequalities, or all the Betti numbers of semi-
algebraic sets defined by few quadratic inequalities [4, 10] – reflecting polynomial
bounds on these quantities. In this paper, we prove the existence of algorithms with
polynomially bounded complexity for computing the first \( \ell \) (i.e. for any constant \( \ell \))
Betti numbers of sub-varieties of complex projective spaces defined by a constant
number of equations (Theorem 17), or more generally by first-order formulas of
certain special kind (Theorem 18).

The rest of the paper is organized as follows. We state the new quantitative
bounds on Betti numbers of complex projective and affine varieties in \( \S 2 \). We
prove these bounds in \( \S 3 \). In \( \S 4 \), we deal with the multi-projective case and its
application. We give the applications to homological and representational stability
in \( \S 5 \). Finally, we describe the algorithmic results in \( \S 6 \).

2. Main Results

We bound the smallest as well as the largest Betti numbers of both projective
and affine varieties. In the projective case we have the following bounds.

**Theorem 7** (Bounds on the smallest Betti numbers of a projective variety). Let
\( \mathcal{P} = \{P_1, \ldots, P_r\} \subset \mathbb{C}[X_0, \ldots, X_k] \) be a set of homogeneous polynomials of degrees
at most \( d \geq 2 \), and let \( V = Z(\mathcal{P}, \mathbb{P}^k_\mathbb{C}) \). Then for all \( 0 \leq p < k - r \) and \( \ell \) prime,

\[
\begin{align*}
\beta^p(V, \mathbb{F}) & \leq \text{Proj}_{\mathbb{C}}(p + r, d) \\
& = (O(d))^{2(p+r)+1}, \\
\beta^p(V, \mathbb{Q}_\ell) & \leq \text{Proj}_{\mathbb{C}, \ell}(p + r, d) \\
& = (O(rd))^{p+r+2}.
\end{align*}
\]

With the same notation as in Theorem 7 we also have:
Theorem 8 (Bounds on the largest Betti numbers of a projective variety). For all $0 \leq p < k/2$ and $\ell$ prime,

$$b_{2k-p}(V, F) \leq 2^r + \sum_{j=1}^{p} \binom{r}{j} (1 + \lfloor p/2 \rfloor) \text{Aff}_{C}(p, jd + 1)$$

$$= 2^r \cdot (O(pd))^{2p},$$

$$b_{2k-p}(V, Q_{\ell}) \leq 2^r + \sum_{j=1}^{p} \binom{r}{j} (1 + \lfloor p/2 \rfloor) \text{Aff}_{C,\ell}(p, jd + 1, 1)$$

$$= 2^r \cdot (O(pd))^{p+2}.$$

In the affine case we have the following bounds. Notice that the bounds are on Betti numbers with compact support.

For the small Betti numbers we have in fact a vanishing result.

Theorem 9 (Vanishing of the small dimensional cohomology groups with compact support for affine varieties). Let $\mathcal{P} = \{P_1, \ldots, P_r\} \subset \mathbb{C}[X_1, \ldots, X_k]$ be a set of polynomials of degrees at most $d \geq 2$, and let $U = Z(\mathcal{P}, \mathbb{C}^k)$. Then for all $0 \leq p < k - r$ and $\ell$ prime,

$$b^c_p(U, F) = 0,$$

$$b^c_p(U, Q_{\ell}) = 0.$$

Remark 3. Notice that the vanishing interval of the cohomology with compact support implied by Theorem 9 cannot be improved. Take for example the affine part, $C \subset \mathbb{C}^2$, of a non-singular projective curve in $\mathbb{P}^2_{\mathbb{C}}$ of positive genus. Then, $b^c_1(C, F) > 0$. In this case $k = 2, r = 1$, and $p = 1 = k - r$.

With the same notation as in Theorem 9 we have:

Theorem 10 (Bounds on the largest Betti numbers with compact support of an affine variety). For all $0 \leq p < k/2$ and $\ell$ prime,

$$b^{c,k-p}(U, F) \leq 2^{r+1} + \sum_{j=1}^{p} \binom{r}{j} (1 + \lfloor p/2 \rfloor) \text{Aff}_{C}(p, jd + 1) +$$

$$\sum_{j=1}^{p+1} \binom{r}{j} (1 + \lfloor (p+1)/2 \rfloor) \text{Aff}_{C}(p+1, jd + 1)$$

$$= 2^r \cdot (O(pd))^{2p},$$

$$b^{c,2k-p}(U, Q_{\ell}) \leq 2^{r+1} + \sum_{j=1}^{p} \binom{r}{j} (1 + \lfloor p/2 \rfloor) \text{Aff}_{C,\ell}(p, jd + 1, 1) +$$

$$\sum_{j=1}^{p+1} \binom{r}{j} (1 + \lfloor (p+1)/2 \rfloor) \text{Aff}_{C,\ell}(p+1, jd + 1, 1)$$

$$= 2^r (O(pd))^{p+2}.$$

While Theorems 9 and 10 only provide bounds on the Betti numbers with compact support, in the case the affine variety is smooth we can extend these bounds to the ordinary Betti numbers using Poincaré-Lefschetz duality. We have the following corollary.
Corollary 1. Let $\mathcal{P} = \{P_1, \ldots, P_r\} \subset C[X_1, \ldots, X_k]$ be a set of polynomials of degrees at most $d \geq 2$, and let $U = Z(\mathcal{P}, C^k)$. Suppose that $U$ is non-singular and of co-dimension $q$ in $C^k$. Then, for all $0 \leq p < k/2 - 2q$,
\begin{equation}
\beta_p(U, \mathbb{F}) \leq 2^r (O((2q + p)d))^{2(2q + p)}.
\end{equation}

Question 1. Is it possible to extend the bound (2.3) in Corollary 1 to the case of singular affine varieties as well?

Remark 4. Notice that the bounds in Theorems 3 and 4 do not hold in the real case. Example 1 shows that for a real hypersurface $V \subset \mathbb{R}^k$ defined by a polynomial of degree $d$, the extremal Betti numbers, $b_0^c(V, \mathbb{F}), b_{k-1}^c(V, \mathbb{F})$, can both grow exponentially in $d$.

Remark 5. One method that has been used for obtaining bounds on the $\mathbb{Z}_2$-Betti numbers of real varieties is to first bound the sum of the $\mathbb{Z}_2$-Betti numbers of the complex varieties defined by the polynomials defining the real variety (such varieties are naturally equipped with an involution – namely complex conjugation), and then use Smith inequalities (see for example [12, 9]). Unfortunately, the bounds in Theorems 7, 8, 9, and 10 do not give any interesting new bound on the middle (i.e. the $k$-th Betti number) for complex subvarieties of $C^k$ or $\mathbb{P}^k_C$. As a result applying the bounds in these theorems with $\mathbb{F} = \mathbb{Z}_2$ do not yield any new bounds on the $\mathbb{Z}_2$-Betti numbers in the real case. This is because applying Smith inequality to get a bound on any Betti number of a real algebraic variety in $\mathbb{R}^k$ or $\mathbb{P}^k_R$ invariably involves bounding a sum of certain Betti numbers of the corresponding complex variety that invariably includes the middle (i.e. the $k$-th) Betti number.

The exponents in the bounds in the above Theorems are probably not tight. However, unlike in the non-singular projective case, the fact that these bounds go to infinity with $d$ is necessary as shown in the following example.

Example 2. Let $V \subset \mathbb{P}^k_C$ be the union of $d$ generic hyperplanes in $\mathbb{P}^k_C$. Then a standard argument involving weight purity and the Mayer-Vietoris spectral sequence gives:
\begin{equation}
b_{2k-2}^c(V, \mathbb{F}) = d,
\end{equation}
which clearly grows polynomially with $d$, and is independent of $k$.

3. Proofs of the main theorems

We prove the theorems only in the case of coefficients in $\mathbb{F}$. The proofs in the $\ell$-adic cases are identical except that instead of using the bounds $\text{Aff}_{C}(k, d)$ and $\text{Proj}(k, d)$ (defined in Eqns. (1.1) and (1.2) above), one has to use the corresponding bounds in the $\ell$-adic case – namely, $\text{Aff}_{C, \ell}(k, d, r)$ and $\text{Proj}_{C, \ell}(k, d, r)$ (defined in Eqns. (1.5) and (1.6) above).

We first prove Theorem 9 which gives the vanishing of the lowest cohomology groups with compact support for affine varieties.

Proof of Theorem 9. For $i = 1, \ldots, r$, let
\begin{align*}
U_i &= Z(P_i, C^k), \\
W_i &= C^k \setminus U_i, \\
W &= C^k \setminus U.
\end{align*}
Then,
\[ U = \bigcap_{i=1}^{r} U_i, \]
\[ W = \bigcup_{i=1}^{r} W_i. \]

We have the exact sequence (see [25, page 185, (7.6)] or [22, Theorem 1.2]):
\[ \cdots \rightarrow H^{c}_{c}(W, F) \rightarrow H^{c}_{c}(C^{k}, F) \rightarrow H^{c}_{c}(U, F) \rightarrow H^{c}_{c}(W, F) \rightarrow H^{c}_{c+1}(C^{k}, F) \rightarrow \cdots. \]

Since,
\[ H^{c}_{c}(C^{k}, F) \cong F, \text{ for } i = 2k, \]
\[ = 0, \text{ otherwise}, \]
it follows that for \( 0 \leq i < 2k \),
\[
(3.1) \quad b^{c}_{c}(U, F) \leq b^{c}_{c+1}(W, F).
\]

For \( I \subset \{1, \ldots, r\} \), we denote by \( W_I = \cap_{i \in I} W_i. \)

The \( E^{p,q}_2 \) term of Mayer-Vietoris spectral sequence corresponding to the covering of \( W \) by the open subsets \((W_i)_{1 \leq i \leq r}\) is given by
\[
(3.2) \quad E^{p,q}_2 \cong \bigoplus_{I \subset \{1, \ldots, r\}, \text{card}(I) = p+1} H^{p}(W_I, F),
\]
and moreover,
\[
(3.3) \quad H^{n}(W, F) \cong \bigoplus_{p+q=n} E^{p,q}_\infty.
\]

Notice that it follows from (3.2) that
\[
(3.4) \quad E^{p,q}_2 = 0, \text{ for } p \geq r.
\]

Now let \( I \subset \{1, \ldots, r\} \).

Notice that \( W_I \) is homeomorphic to a smooth affine variety \( \tilde{W}_I \subset C^{k+\text{card}(I)} \) defined by the polynomial equations,
\[
T_i P_i(X_1, \ldots, X_k) - 1 = 0, \text{ for } i \in I.
\]

Using Poincaré-Lefschetz duality [25, page 282, Theorem 6.6] (with \( Z = X = \tilde{W} \)) we have for \( 0 \leq i \leq 2k \),
\[
(3.5) \quad H^{2k-i}_{c}(\tilde{W}, F) \cong H^{i}_{c}(\tilde{W}, F).
\]

Furthermore, since \( \tilde{W}_I \) being a complex affine algebraic variety is a Stein space of complex dimension \( k \), we have that [28]:
\[
(3.6) \quad H^{j}_{c}(\tilde{W}_I, F) = 0, \text{ for } j > k.
\]

It follows from (3.6), (3.2), and (3.3), that
\[
(3.7) \quad H^{j}_{c}(W, F) = 0, \text{ for } j > k + r - 1.
\]
Hence, for all \( i < k - r + 1 \), we have using (3.5)

\[
\text{H}^i_c(\mathbb{W}, \mathbb{F}) \cong \text{H}^i_c(\tilde{\mathbb{W}}, \mathbb{F}) \cong \text{H}_{2k-i}(\tilde{\mathbb{W}}, \mathbb{F}) = 0.
\]

It now follows from (3.8) and (3.1) that

\[
\text{H}^i_c(U, \mathbb{F}) = 0, \quad \text{for } 0 \leq i < k - r.
\]

The proof in the \( \ell \)-adic case is similar and omitted.

\( \square \)

**Proof of Theorem 7.** For \( 0 \leq i \leq k \), we denote by \( H_i \) the linear space defined by \( X_0 = \cdots = X_{k-i-1} = 0 \), and we denote

\[
V_i = V_k \cap H_i, \quad U_i = V_i \setminus V_{i-1}.
\]

In particular, \( V_{k-1} = V_k \cap H_{k-1} \), where \( H_{k-1} \) is the hyperplane defined by \( X_0 = 0 \), and \( U_k = V_k - V_{k-1} \).

We have the following exact sequence (see [25, page 185, (7.6)]) or [22, Theorem 1.2]):

\[
\cdots \to \text{H}^i_c(U_k, \mathbb{F}) \to \text{H}^i_c(V_k, \mathbb{F}) \to \text{H}^i_c(V_{k-1}, \mathbb{F}) \to \text{H}^{i+1}_c(U_k, \mathbb{F}) \to \text{H}^{i+1}_c(V_k, \mathbb{F}) \to \cdots
\]

Consequently, we have the inequalities

\[
\text{b}^i_c(U_k, \mathbb{F}) \leq \text{b}^i_c(V_{k-1}, \mathbb{F}) + \text{b}^i_c(U_k, \mathbb{F}).
\]

We identify the affine space \( \mathbb{C}^k \) with \( \mathbb{P}^k_{\mathbb{C}} \setminus H_{k-1} \), and notice that \( U_k \subset \mathbb{C}^k \).

Using Theorem 9 we have that for all \( 0 \leq i < k - r \),

\[
\text{b}^i_c(U_k, \mathbb{F}) \leq \text{b}^i_c(V_{k-1}, \mathbb{F}) + \text{b}^i_c(U_k, \mathbb{F})
\]

\[
\leq \text{b}^i_c(V_{k-1}, \mathbb{F}).
\]

Now let \( 0 \leq p < k - r \). Using (3.11) repeatedly we obtain

\[
\text{b}^p_p(U_k, \mathbb{F}) \leq \text{b}^p_p(V_{p+r}, \mathbb{F}).
\]

Now notice that since each \( V_i \subset \mathbb{P}^i_{\mathbb{C}} \) is compact, \( \text{H}^*_c(V_i, \mathbb{F}) \cong \text{H}^*(V_i, \mathbb{F}) \), and \( V_{p+r} \subset \mathbb{P}^r_{\mathbb{C}} \) is a projective variety defined by \( r \) homogeneous polynomials of degree \( \leq d \). Using Theorem 4 we get

\[
\text{b}^p_p(V_{p+r}, \mathbb{F}) = \text{b}^p_p(V_{p+r}, \mathbb{F})
\]

\[
\leq \text{ProjC}(p + r, d)
\]

\[
= (p + r)d(2d - 1)^{2(p+r)+1}.
\]

The theorem now follows from (3.12) and (3.13).

\( \square \)

**Proof of Theorem 8.** We use the same notation as in the proof of Theorem 7, and we first consider the case \( r = 1 \).

Applying the affine hyperplane section theorem [21, Theorem 5] repeatedly starting with the smooth affine hypersurface \( \tilde{\mathbb{W}}_k \subset \mathbb{C}^{k+1} \), we obtain that for all \( p, 0 \leq p \leq k \), we have that the restriction homomorphism, \( \text{H}^p(\tilde{\mathbb{W}}_k, \mathbb{F}) \to \text{H}^p(\tilde{\mathbb{W}}_p, \mathbb{F}) \) induced by the inclusion \( \tilde{\mathbb{W}}_p \hookrightarrow \tilde{\mathbb{W}}_k \), is injective (here \( \tilde{\mathbb{W}}_p \) is the intersection of \( \tilde{\mathbb{W}}_k \) with a generic affine subspace in \( \mathbb{C}^{k+1} \) of dimension \( p + 1 \)). Notice that for each \( p, 0 \leq p \leq k \), \( \tilde{\mathbb{W}}_p \) is an affine hypersurface in \( \mathbb{C}^{p+1} \) defined by \( r \) polynomials of degrees bounded by \( d + 1 \).
This implies furthermore that,
\[ b^p(W_k, F) \leq b^p(\tilde{W}_p, F) \leq \text{Aff}_C(p + 1, d + 1). \] (3.14)

(using Theorem 3).

It now follows from inequality (3.14) and Poincaré-Lefschetz duality that for \( 0 \leq p \leq k \),
\[ b^{2k-p}(\tilde{W}_k, F) = b^p(\tilde{W}_p, F). \] (3.15)

Using inequalities (3.1), and (3.10), we obtain,
\[ b^{2k-p}(V_k, F) \leq b^{2k-p}(V_{k-1}, F) + b^{2k-p+1}(\tilde{W}_k, F) \]
\[ \leq b^{2k-p}(V_{k-2}, F) + b^{2k-p+1}(\tilde{W}_{k-1}, F) + b^{2k-p+1}(\tilde{W}_k, F) \]
\[ \vdots \]
\[ \leq b^{2k-p}(V_{k-\lfloor p/2 \rfloor}, F) + \sum_{i=0}^{\lfloor p/2 \rfloor} b^{2k-p+1}(\tilde{W}_{k-i}, F) \] (3.16)
Using (3.16) iteratively we obtain for \( 0 < p < k \),
\[ b^{2k-p}(V_k, F) \leq b^{2k-p}(V_{k-\lfloor p/2 \rfloor}, F) + \sum_{i=0}^{\lfloor p/2 \rfloor} b^{2k-p+1}(\tilde{W}_{k-i}, F) \]
\[ = 0 + \sum_{i=0}^{\lfloor p/2 \rfloor} b^{2k-p+1}(\tilde{W}_{k-i}, F) \]
\[ = \sum_{i=0}^{\lfloor p/2 \rfloor} b^{2(k-i)-(p-2i-1)}(\tilde{W}_{k-i}, F) \]
\[ = \sum_{i=0}^{\lfloor p/2 \rfloor} b^p-2i-1(\tilde{W}_{k-i}, F) \text{(using Poincaré-Lefschetz duality)} \]
\[ = \sum_{i=0}^{\lfloor p/2 \rfloor} b^p-2i-1(\tilde{W}_{p-2i-1}, F) \text{ (using (3.15))} \]
\[ \leq \sum_{i=0}^{\lfloor p/2 \rfloor} \text{Aff}_C(p - 2i, d + 1) \text{ (using (3.14))} \]
\[ \leq (1 + \lfloor p/2 \rfloor) \text{Aff}_C(p, d + 1) \] (3.17)

noticing that \( p - 2i - 1 < k - i \) whenever \( 0 \leq i < \lfloor p/2 \rfloor \) and \( 0 \leq p \leq k/2 \), and hence (3.15) is applicable.

It follows that
\[ b^{2k-p}(V_k, F) = b^{2k-p}(V_k, F) \leq \sum_{i=0}^{\lfloor p/2 \rfloor} \text{Aff}_C(p - 2i, d + 1) \leq (1 + \lfloor p/2 \rfloor) \text{Aff}_C(p, d + 1), \]
for \( 0 \leq p < k/2 \).
The proof in the case of $r > 1$, follows from the case $r = 1$, and the Mayer-Vietoris inequality ([7, Proposition 7.33 b]).

More precisely, let for $j \in [1, r],$

$$V^j_k = Z(P_j, \mathbb{P}^k_C),$$

and for $J \subset [1, r]$, let

$$V^J_k = \bigcup_{j \in J} V^j_k.$$

Note that for each $J \subset [1, r],$

$$V^J_k = Z(\prod_{j \in J} P_j, \mathbb{P}^k_C),$$

and thus $V^J_k$ is a projective hypersurface in $\mathbb{P}^k_C$ defined by one polynomial of degree bounded by $\text{card}(J) \cdot d$.

Using Mayer-Vietoris inequality ([7, Proposition 7.33 b]) we have that,

$$b^{2k-p}(V_k, F) \leq \binom{r}{p} + \sum_{j=1}^{p} \sum_{J \subset [1, r], \text{card}(J) = j} b^{2k-(p-j+1)}(V^J_k, F)$$

$$\leq 2^r + \sum_{j=1}^{p} \binom{r}{j} (1 + \lceil p/2 \rceil) \text{Aff}_C(p, jd + 1) (\text{using (3.17)}).$$

(3.18)

**Proof of Theorem 10.** Let $\mathcal{P}^h \subset \mathbb{C}[X_0, \ldots, X_k]$ denote the homogenization of $\mathcal{P}$, and let $V_k = Z(\mathcal{P}^h, \mathbb{P}^k_C)$. Let $V_{k-1} = V_k \cap H_{k-1}$, where $H_{k-1}$ is the hyperplane defined by $X_0 = 0$, and $U = U_k = V_k - V_{k-1}$.

We have from the exact sequence (3.9) that

$$b^i_c(U_k, F) \leq b^i(V_k, F) + b^{i-1}(V_{k-1}, F).$$

Since $V_k$ (resp. $V_{k-1}$) is a subvariety of $\mathbb{P}^k_C$ (resp. $\mathbb{P}^{k-1}_C$) defined by $r$ homogeneous polynomials of degrees at most $d$, we have using Theorem 7 that for $p < k/2$,

$$b^{2k-p}_c(V_k, F) \leq 2^r + \sum_{j=1}^{p} \binom{r}{j} \left(1 + \left\lceil \frac{p}{2} \right\rceil \right) \text{Aff}_C(p, jd + 1),$$

(3.19)

$$b^{2k-p-1}_c(V_{k-1}, F) \leq 2^r + \sum_{j=1}^{p+1} \binom{r}{j} \left(1 + \left\lceil \frac{p + 1}{2} \right\rceil \right) \text{Aff}_C(p + 1, jd + 1).$$

(3.20)

The theorem follows from (3.18), (3.19), and (3.20). □

**Proof of Corollary 1.** Since the variety $U$ is smooth of (complex) dimension $k - q$, it follows from Poincaré-Lefschetz duality that for $0 \leq i \leq 2(k - q)$, $H_i(U, F) \cong H^{2(k-q)-i}_e(U, F)$. Now (2.3) follows from Theorem 10. □
4. Multi-projective bounds and application

In this section we extend our bounds on the small dimensional Betti numbers of complex projective varieties to sub-varieties of products of projective spaces. As an application we obtain bounds on the small dimensional Betti numbers of the image under the projection maps of a subvariety of the product of two projective spaces. We restrict our attention to the bi-homogeneous case for simplicity, noting that the method should be extendable to more general multi-homogeneous cases as well.

Let \( X = (X_0, \ldots, X_k), Y = (Y_0, \ldots, Y_\ell) \) and let \( P = \{P_1, \ldots, P_r\} \subset C[X; Y] \) be a set of bi-homogeneous polynomials, such that the degrees of each \( P_i \) (i.e. the total degree in \( X \) and \( Y \)) is bounded by \( d \). We consider the variety \( V_{k,\ell} = Z(P, \mathbb{P}_C^k \times \mathbb{P}_C^\ell) \) and have the following bound on \( b(V_{k,\ell}, \mathbb{F}) \).

**Theorem 11.**

\[
4.1 \quad b(V_{k,\ell}, \mathbb{F}) \leq \text{Proj}_C(k, \ell, d) := (O(d))^{2(k+\ell+1)}.
\]

**Proof.** Let \( \tilde{V}_{k,\ell} \) denote the intersection of \( Z(P, \mathbb{R}^{2(k+1)} \times \mathbb{R}^{2(\ell+1)}) \) with \( S^{2k+1} \times S^{2\ell+1} \) defined by \( ||X||^2 = 1, ||Y||^2 = 1 \). There is an obvious surjection \( \pi_1 : S^{2k+1} \times S^{2\ell+1} \to \mathbb{F}_C^k \times \mathbb{F}_C^\ell \), and \( \pi_2 : \mathbb{F}_C^k \times \mathbb{F}_C^\ell \to \mathbb{F}_C^k \times \mathbb{F}_C^\ell \).

The restrictions of \( \pi_1 \) (resp. \( \pi_2 \)) to \( \tilde{V}_{k,\ell} \) (resp. to \( \pi_1(\tilde{V}_{k,\ell}) \)) define \( S^1 \)-bundles.

It now follows from the Gysin exact sequence that,

\[
\begin{align*}
&b(\pi_1(\tilde{V}_{k,\ell}), \mathbb{F}) \leq k \times b(\tilde{V}_{k,\ell}, \mathbb{F}), \\
&b(\pi_2 \circ \pi_1(\tilde{V}_{k,\ell}), \mathbb{F}) \leq \ell \times b(\pi_1(\tilde{V}_{k,\ell}, \mathbb{F})).
\end{align*}
\]

Notice that \( \pi_2 \circ \pi_1(\tilde{V}_{k,\ell}) = V_{k,\ell} \). The theorem now follows by applying Theorem 1 to the real variety \( \tilde{V}_{k,\ell} \). \( \square \)

**Remark 6.** Similar to Theorem 4 one can improved with a little more care (for example, using [9, Theorem 32] instead of Theorem 1 as in the proof above) the bound of \((O(d))^{2(k+\ell)}\) on \( b(V, \mathbb{F}) \). This in turn would also improve the bound on \( b(V, \mathbb{F}) \) in Theorem 4 to \((O(d))^{2(k+\ell)}\).

However, we can get a more refined bound on the individual Betti numbers as follows. Using the same notation as above:

**Theorem 12.**

1. For every \( 0 \leq p < k+\ell-r \), \( b^p(V_{k,\ell}, \mathbb{F}) \) is bounded by the minimum of the following three quantities:

\[
\text{Proj}_C(k, \ell, d, r),
\text{Proj}_C(p + r - N_1, \ell - N_1, d, r) + \sum_{i=1}^{N_1} \text{AffProj}_C(p + r - i, \ell - i - 1, d, r),
\text{Proj}_C(k - N_2, p + r - N_2, d, r) + \sum_{j=1}^{N_2} \text{AffProj}_C(k - i - 1, p + r - i, d, r),
\]

where \( N_1 = \min(p + r, \ell - p - r), N_2 = \min(k - p - r, p + r) \).

In particular,

\[
b^p(V_{k,\ell}, \mathbb{F}) \leq (O(d))^{2(p+r)}.
\]

2. For every \( 0 \leq p < (k + \ell)/2 \),

\[
b^{2(k+\ell)-p}(V_{k,\ell}, \mathbb{F}) \leq 3^p 2^p (O(pd))^2p.
\]
Remark 7. Note that Theorem 12 cannot be proved by simply taking the Segre embedding of $\mathbb{P}_C^k \times \mathbb{P}_C^\ell$ into $\mathbb{P}_C^{(k+1)(\ell+1)-1}$ and then using Theorem 7, since the image of the Segre map requires too many polynomials to be cut out in $\mathbb{P}_C^{(k+1)(\ell+1)-1}$.

We introduce some notation that we are going to use in the proof of Theorem 12:

We identify the hyperplane in $\mathbb{P}_C^k \times \mathbb{P}_C^\ell$ defined by $X_k = 0$ with $\mathbb{P}_C^{k-1} \times \mathbb{P}_C^\ell$, and that defined by $Y_\ell = 0$ with $\mathbb{P}_C^k \times \mathbb{P}_C^{\ell-1}$. Note that with these identifications,

\[
\mathbb{P}_C^{k-1} \times \mathbb{P}_C^\ell \cap \mathbb{P}_C^k \times \mathbb{P}_C^{\ell-1} = \mathbb{P}_C^{k-1} \times \mathbb{P}_C^{\ell-1},
\]

\[
\mathbb{P}_C^k \times \mathbb{P}_C^\ell \setminus (\mathbb{P}_C^{k-1} \times \mathbb{P}_C^\ell \cup \mathbb{P}_C^k \times \mathbb{P}_C^{\ell-1}) = C^{k+\ell}.
\]

We denote,

\[
V_{k-1,\ell} = V_{k,\ell} \cap \mathbb{P}_C^{k-1} \times \mathbb{P}_C^\ell,
\]

\[
V_{k,\ell} = V_{k,\ell-1} \cup V_{k-1,\ell},
\]

\[
V_{k-1,\ell-1} = V_{k,\ell} \cap \mathbb{P}_C^{k-1} \times \mathbb{P}_C^{\ell-1},
\]

\[
U_{k,\ell} = V_{k,\ell} \setminus W_{k,\ell} = V_{k,\ell} \cap C^{k+\ell}.
\]

Observe that $W_{k,\ell} \cap V_{k-1,\ell-1}$ is a disjoint union of two locally closed subvarieties, $W_{k,\ell-1}^1 \subset C^k \times \mathbb{P}_C^{\ell-1}$ and $W_{k-1,\ell}^2 \subset \mathbb{P}_C^{k-1} \times C^\ell$. Let $U_{k,\ell}'$ denote $W_{k,\ell-1}^1 \cup W_{k-1,\ell}^2$.

The following corollary of Theorem 11 will be needed in the proof of Theorem 12 below.

Corollary 2.

\[
b_c(W_{k,\ell-1}^1, F) + b_c(W_{k-1,\ell}^2, F) \leq \text{AffProj}_C(k, \ell, d, r) = \text{Proj}_C(k, \ell - 1, d, r) + \text{Proj}_C(k - 1, \ell, d, r) + 2 \cdot \text{Proj}_C(k - 1, \ell - 1, d, r).
\]

Proof. First observe that $W_{k,\ell}$ is compact, and $V_{k-1,\ell-1}$ is closed in $W_{k,\ell}$. Hence, we have the exact sequence. It follows from the exact sequence

\[
\cdots \to H^j_c(U_{k,\ell}', F) \to H^j_c(W_{k,\ell}, F) \to H^j_c(V_{k-1,\ell-1}, F) \to H^{j+1}_c(U_{k,\ell}', F) \to \cdots
\]

from which it follows that

\[
b_c(U_{k,\ell}', F) = b_c(W_{k,\ell-1}^1, F) + b_c(W_{k-1,\ell}^2, F) \leq b_c(W_{k,\ell}, F) + b_c(V_{k-1,\ell-1}, F).
\]

Moreover, since $W_{k,\ell} = V_{k,\ell-1} \cup V_{k-1,\ell}$, and $V_{k,\ell-1} \cap V_{k-1,\ell} V_{k-1,\ell-1}$, the Mayer-Vietoris exact sequence yields

\[
b_c(W_{k,\ell}, F) \leq b_c(V_{k,\ell-1}, F) + b_c(W_{k,\ell-1}, F) + b_c(V_{k-1,\ell-1}, F).
\]

The corollary now follows from (4.2), (4.3), and (4.1).

Proof of Theorem 12. We first prove item (1). From the exact sequence,

\[
\cdots \to H^j_c(U_{k,\ell}, F) \to H^j_c(W_{k,\ell}, F) \to H^j_c(V_{k,\ell}, F) \to H^{j+1}_c(U_{k,\ell}, F) \to \cdots
\]

and the fact that

\[
H^j_c(U_{k,\ell}, F) = H^{j+1}_c(U_{k,\ell}, F) = 0,
\]

we have

\[
H^j_c(W_{k,\ell}, F) = H^{j+1}_c(W_{k,\ell}, F) = 0.
\]
for $0 \leq j < k + \ell - r$ (using Theorem 9), we obtain that

\[
(4.4) \quad b^r_c(V_{k, \ell}, \mathbb{F}) = b^r_c(W_{k, \ell}, \mathbb{F}).
\]

It follows from the exact sequence

\[
\cdots \rightarrow H^j_c(U^r_{k, \ell}, \mathbb{F}) \rightarrow H^j_c(W_{k, \ell}, \mathbb{F}) \rightarrow H^j_c(V_{k-1, \ell-1}, \mathbb{F}) \rightarrow H^{j+1}_c(U^r_{k, \ell}, \mathbb{F}) \rightarrow \cdots
\]

that,

\[
(4.5) \quad b^r_c(W_{k, \ell}, \mathbb{F}) \leq b^r_c(V_{k-1, \ell-1}, \mathbb{F}) + b^r_c(U^r_{k, \ell}, \mathbb{F}) \leq b^r_c(V_{k-1, \ell-1}, \mathbb{F}) + b^r_c(W^1_{k-1, \ell-1}, \mathbb{F}).
\]

We claim that for $0 \leq j < k - r$,

\[
(4.6) \quad b^r_j(W^1_{k, \ell-1}, \mathbb{F}) = 0,
\]

and similarly for $0 \leq j < \ell - r$.

\[
(4.7) \quad b^r_j(W^2_{k-1, \ell-1}, \mathbb{F}) = 0.
\]

To see (4.6), let $\pi$ be the restriction to $W^1_{k, \ell-1}$ of the projection map $\mathbb{P}^k_{\mathbb{C}} \times C^{\ell-1} \rightarrow \mathbb{P}^k_{\mathbb{C}}$. Note that $\pi(W^1_{k, \ell-1})$ is locally compact subset of $\mathbb{P}^k_{\mathbb{C}}$. There exists a first quadrant spectral sequence $E^{p,q}_2$ abutting to $H^1_c(W^1_{k, \ell}, \mathbb{F})$ whose $E_2$-term is given by,

\[
E^{2,j}_2 \cong H^j_c(W^1_{k, \ell-1}, R^j \pi_1 \mathbb{F}_{W^1_{k, \ell-1}})
\]

(see for example [20, III, §7] taking for $G$ and $F$ the functor of sections with proper support). In particular, this implies that for each $n \geq 0$,

\[
H^n(W^1_{k, \ell-1}, \mathbb{F}) \cong \bigoplus_{i+j=n} E^{i,j}_\infty,
\]

and thus the vanishing of $E^{2,j}_2$ for all $i, j \geq 0$ with $i + j = n$ implies the vanishing of $H^n_c(W^1_{k, \ell-1}, \mathbb{F})$ as well.

Now, the stalks

\[
(4.9) \quad (R^j \pi_1 \mathbb{F}_{W^1_{k, \ell-1}})_x \cong H^j_c(\pi^{-1}(x), \mathbb{F}),
\]

for all $x \in \mathbb{P}^k_{\mathbb{C}}$, and hence

\[
(4.10) \quad (R^j \pi_1 \mathbb{F}_{W^1_{k, \ell-1}})_x = 0,
\]

for all $x \in \mathbb{P}^k_{\mathbb{C}}$, and $0 \leq j \leq k - r$, using Theorem 9. The claim now follows from (4.8), (4.9), and (4.10). The proof of (4.7) is similar and omitted.

For $1 \leq i \leq \ell$ denote by $W^1_{k,\ell-i} \subset C^k \times \mathbb{P}^{\ell-i}_{\mathbb{C}}$, the intersection of the variety $W^1_{k,\ell-1}$ with the variety defined by $Y_{i} = \cdots = Y_{\ell-i+1} = 0$. Then, in case $k \leq p + r$ and $\ell > p + r$, we have

\[
(4.11) \quad b^r_c(W^1_{k, \ell-1}, \mathbb{F}) = b^r_p(W^1_{k,p+r-k}, \mathbb{F}).
\]

To see this, first note that $W^1_{k,\ell-i}$ is locally compact, and $W_{k,\ell-i-1}$ is a closed subspace of $W^1_{k,\ell-i}$. Now, define $U^1_{k,\ell-i} = W^1_{k,\ell-i} \setminus W_{k,\ell-i-1}$, and consider the exact sequence

\[
\cdots \rightarrow H^j_c(U^1_{k,\ell-i}, \mathbb{F}) \rightarrow H^j_c(W^1_{k,\ell-i}, \mathbb{F}) \rightarrow H^j(W^1_{k-1,\ell-i-1}, \mathbb{F}) \rightarrow H^{j+1}_c(U^1_{k,\ell-i}, \mathbb{F}) \rightarrow \cdots
\]
noting that $H^j(U_{k,i-1},\mathbb{F}) = H^j(U_{k,i},\mathbb{F}) = 0$ whenever $j < k + \ell - i - r$ (using Theorem 9). The last inequality holds for $j = p$, and $(\ell - i) \geq p + r - k + 1$, which gives for all $i$ satisfying $\ell - i \geq p + r - k + 1$,

\begin{equation}
  b^p_\ell(W_{k,\ell-i},\mathbb{F}) = b^p_\ell(W_{k,\ell-i-1},\mathbb{F}).
\end{equation}

The equation (4.11) follows from applying (4.14) repeatedly.

Similarly, denoting for $1 \leq i \leq k$, by $W_{k-i,\ell} \subset \mathbb{P}^d_{k-i} \times C^\ell$, the intersection of the variety $W_{k-1,\ell}$ with the variety defined by $X_k = \cdots = X_{k-i+1} = 0$, we have that in case $\ell \leq p + r$ and $k > p + r$,

\begin{equation}
  b^p_\ell(W_{k-1,\ell},\mathbb{F}) = b^p_\ell(W_{p+r,\ell},\mathbb{F}).
\end{equation}

We are now in a position to finish the proof of the theorem. If both $k, \ell \leq p + r$, then it follows from Theorem 11 that

$$b^p(V_{k,\ell},\mathbb{F}) \leq \text{Proj}_C(k, \ell, d).$$

Otherwise, if $k \leq p + r$, $\ell > p + r$, then using (4.5) and (4.7) repeatedly we obtain

\begin{equation}
  b^p(V_{k,\ell},\mathbb{F}) = b^p(V_{k-1,\ell-1},\mathbb{F}) + b^p_\ell(W_{k-1,\ell-1},\mathbb{F})
  = b^p(V_{k-2,\ell-2},\mathbb{F}) + b^p_\ell(W_{k-2,\ell-2},\mathbb{F}) + b^p_\ell(W_{k,\ell-1},\mathbb{F})
  = b^p(V_{k-N,\ell-N},\mathbb{F}) + \sum_{i=0}^{N} b^p_\ell(W_{k-i,\ell-i-1},\mathbb{F}),
\end{equation}

where $N = \min(k, \ell - p + r)$. The Theorem follows in this case from (4.14), (4.11), and Theorem 11.

Similarly, if $k > p + r$, $\ell \leq p + r$, then using (4.5) and (4.7) repeatedly we obtain

\begin{equation}
  b^p(V_{k,\ell},\mathbb{F}) = b^p(V_{k-1,\ell-1},\mathbb{F}) + b^p_\ell(W_{k-1,\ell},\mathbb{F})
  = b^p(V_{k-2,\ell-2},\mathbb{F}) + b^p_\ell(W_{k-2,\ell-1},\mathbb{F}) + b^p_\ell(W_{k,\ell},\mathbb{F})
  = b^p(V_{k-N,\ell-N},\mathbb{F}) + \sum_{i=0}^{N} b^p_\ell(W_{k-i,\ell-i},\mathbb{F}),
\end{equation}

where $N = \min(k - p + r, \ell)$. The Theorem follows in this case from (4.15), (4.13), and Theorem 11.

Finally, if both $k, \ell > p + r$, we apply (4.5), (4.6) and (4.7) repeatedly to reduce to one of the three cases described above. This completes the proof of the theorem.

We now prove item (2). From the exact sequence,

$$\cdots \rightarrow H^j_c(U_{k,\ell},\mathbb{F}) \rightarrow H^j_c(V_{k,\ell},\mathbb{F}) \rightarrow H^j_c(W_{k,\ell},\mathbb{F}) \rightarrow H^j_{c+1}(U_{k,\ell},\mathbb{F}) \rightarrow \cdots,$$

we obtain the inequality

$$b^{2(k+\ell)-p}(V_{k,\ell},\mathbb{F}) \leq b^{2(k+\ell)-p}(U_{k,\ell},\mathbb{F}) + b^{2(k+\ell)-p}(W_{k,\ell},\mathbb{F}),$$

Also, since $W_{k,\ell} = V_{k-1,\ell} \cup V_{k,\ell-1}, V_{k-1,\ell-1} = V_{k-1,\ell} \cap V_{k,\ell-1}$, it follows from the Mayer-Vietoris exact sequence that

\begin{align*}
  b^{2(k+\ell)-p}(W_{k,\ell},\mathbb{F}) &\leq b^{2(k+\ell)-p}(V_{k-1,\ell},\mathbb{F}) + b^{2(k+\ell)-p}(V_{k,\ell-1},\mathbb{F}) + \nonumber \\
  &\quad b^{2(k+\ell)-p-1}(V_{k-1,\ell-1},\mathbb{F}) \
  &\leq b^{2(k+\ell-1)-(p-2)}(V_{k-1,\ell},\mathbb{F}) + b^{2(k+\ell-1)-(p-2)}(V_{k,\ell-1},\mathbb{F}) + \nonumber \\
  &\quad b^{2(k+\ell-2)-(p-3)}(V_{k-1,\ell-1},\mathbb{F}).
\end{align*}
The required inequality now follows from a double induction on \(k + \ell\) and \(p\), and Theorem 10, noting that \(H^{2(m+n)-(p-j)}(V_{m,n}, \mathbb{F}) = 0\) for \(j > p\).

We now give an application of Theorem 12 in bounding the small dimensional Betti numbers of the image under projection maps of subvarieties of products of projective spaces. We keep the same notation as in Theorem 12, let \(\pi : \mathbb{P}_C^k \times \mathbb{P}_C^\ell \to \mathbb{P}_C^k\) denote the projection on the first factor. In the following, we obtain a bound on the small dimensional Betti numbers of the image \(\pi(V_{k,\ell})\) — and as before our bound depends only on \(d, p\) and \(r\) (and independent of \(k\) and \(\ell\)).

We prove the following theorem.

**Theorem 13.** Let \(s = r - \ell > 0\). For all \(p\) satisfying, \(0 \leq p < \frac{k-2s+1}{1+s}\),

\[
 b^p(\pi(V_{k,\ell}), \mathbb{F}) \leq (O(d))^{2(p+(p+2)(s+\ell))}.
\]

Remark 8. Note that if \(r \leq \ell\), then \(\pi(V_{k,\ell}) = \mathbb{P}_C^k\). Thus, studying the topology of \(\pi(V_{k,\ell})\) is interesting only for \(r > \ell\).

Before proving Theorem 13 we recall some topological properties of complex join fibered over a projection proved in [5].

Suppose that \(V, W\) are finite dimensional \(C\)-vector spaces and \(A \subset \mathbb{P}(V) \times \mathbb{P}(W)\) a subset. Let \(\pi : \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V)\) denote the projection on the first component. Then, for \(p \geq 0\), the \(p\)-fold complex join of \(A\) fibered over the projection \(\pi\) is defined by:

**Definition 1** (Complex join fibered over a projection in terms of co-ordinates). Let \(A \subset \mathbb{P}_C^k \times \mathbb{P}_C^\ell\) be a constructible set defined by a first-order bi-homogeneous formula,

\[
 \Phi(X_0, \ldots, X_k; Y_0, \ldots, Y_{\ell})
\]

and let \(\pi : \mathbb{P}_C^k \times \mathbb{P}_C^\ell \to \mathbb{P}_C^k\) be the projection map to the first component. For \(p \geq 0\), the \(p\)-fold complex join of \(A\) fibered over the map \(\pi\), \(J_{C,\pi}^p(A) \subset \mathbb{P}_C^k \times \mathbb{P}_C^{(\ell+1)(p+1)-1}\), is defined by the formula

\[
 J_{C,\pi}^p(A) = \bigcap_{i=0}^p \phi(X_0, \ldots, X_k; Y_0^i, \ldots, Y_{\ell}^i).
\]

Remark 9. The projection map

\[
 \pi : \mathbb{P}_C^k \times \mathbb{P}_C^{(\ell+1)(p+1)-1} \to \mathbb{P}_C^k
\]
clearly restricts to a surjection

\[
 \pi^p : J_{C,\pi}^p(A) \to \pi(A)
\]

sending \((x_0 : \cdots : x_k; y_0^0 : \cdots : y_{\ell}^p) \in J_{C,\pi}^p(A)\) to \((x_0 : \cdots : x_k) \in \pi(A)\).

Now, let \(V_{k,\ell} \subset \mathbb{P}_C^k \times \mathbb{P}_C^\ell\) be a variety as before, and \(\pi : \mathbb{P}_C^k \times \mathbb{P}_C^\ell \to \mathbb{P}_C^k\) be the projection on the first component.

**Notation 1.** For any constructible set \(X\) we denote \(P_X(T, \mathbb{F}) = \sum_i b^i(X, \mathbb{F})T^i\) (the Poincaré polynomial of \(X\)).
The following theorem relates the Poincaré polynomial of $J^{p+1}_{C,\pi}(V_{k,\ell})$ to that of the image $\pi(V_{k,\ell})$.

The main result that we need is the following.

**Theorem 14.** [5] For every $p \geq 0$, we have that
\[
P_{\pi(V_{k,\ell})} = (1 - T^2)P_{J^{p+1}_{C,\pi}(V_{k,\ell})} \mod T^{p+1}.
\]

We are now in a position to prove Theorem 13.

**Proof of Theorem 13.** Note that $J^{p+1}_{C,\pi}(V_{k,\ell})$ is a subvariety of $\mathbb{P}^{(p+2)(\ell+1) - 1} \times \mathbb{P}^k$, defined by $(p+2)r$ bi-homogeneous polynomials of degrees bounded by $d$. Applying Theorem 12 to the bi-projective variety $J^{p+1}_{C,\pi}(V_{k,\ell})$, we obtain that for all $p \geq 0$ satisfying $p < (p+2)(\ell+1) + k - (p+2)r - 1$,
\[
b^p(J^{p+1}_{C,\pi}(V_{k,\ell}), \mathbb{F}) \leq O(d^{2(p+2)r}).
\]

Observe that
\[
0 \leq p < (p+2)(\ell+1) + k - (p+2)r - 1 \iff 0 \leq p < \frac{k - 2s + 1}{1 + s},
\]
with $s = r - \ell$. Now apply Theorem 14 to finish the proof. \qed

**Remark 10.** A possible alternative method for obtaining a bound on $b^p(\pi(V_{k,\ell}), \mathbb{F})$ could be to first bound the degrees and the number of polynomials in a generating set of the elimination ideal of the ideal generated by $\mathcal{P}$ eliminating the variables $Y$. This ideal defines $\pi(V_{k,\ell})$, and we can then use Theorem 7. While there exists effective bounds on the degrees and the number of generators of the elimination ideal, via classical elimination theory [23] or using Gröbner basis methods [17], in general these bounds depend on $k$ (as well as on $d, r, \ell$), while the bound in Theorem 13 is independent of $k$, the dimension of the ambient space of $\pi(V_{k,\ell})$.

5. Application to stability

One of the motivation for us to revisit the results on explicit bounds on the Betti numbers of varieties is to understand better the complex analog of the following question related to representational stability raised in [8].

5.1. **Representational stability.** In [8], the authors study the isotypic decomposition of cohomology modules of symmetric real varieties into irreducible representations. Certain natural sequences of symmetric varieties occur in this study. The following sequence is a key example.

Let $K = \mathbb{R}$ or $\mathbb{C}$, and $F_1, \ldots, F_r \in K[Z_1, \ldots, Z_d]$ be weighted homogeneous polynomials with weight vector $(1, \ldots, d)$.

Further, for each $j > 0$, set
\[
\psi_j^{(n)} = X_0^j + X_1^j + \cdots + X_n^j,
\]
and for each $n > 0$, we define for each $1 \leq i \leq r$, the polynomial
\[
P_i^{(n)} = F_i(\psi_1^{(n)}, \ldots, \psi_d^{(n)}).
\]

The polynomials $P_i^{(n)}, 1 \leq i \leq r$ define for each $n > 0$ a projective variety $V_n = Z((P_1^{(n)}, \ldots, P_r^{(n)}), \mathbb{P}^n_K)$. 

Taking $\mathbb{F} = \mathbb{Q}$, this yields for each fixed $p$ a sequence $(H^p(V_n, \mathbb{Q}))_{n \geq 0}$ of the corresponding cohomology modules. Notice that since the polynomials $\psi_j^{(n)}$ are symmetric, it follows that for all $n > 0$, the cohomology module $H^p(V_n, \mathbb{Q})$ is an $S_{n+1}$-representation, which can be decomposed into irreducible $S_{n+1}$-representations.

To be more precise, for any partition $\lambda \vdash n + 1$, there exists a unique irreducible representation $S_{\lambda}$ (the so-called Specht-module), and the number $m_{p, \lambda}(V_n) = \dim_{\mathbb{Q}} \text{hom}_{S_{n+1}}(S^\lambda, H^p(V_n, \mathbb{Q}))$

is the multiplicity of $S^\lambda$ in the module $H^p(V_n, \mathbb{Q})$. Now starting with a fixed partition $\mu = (\mu_1, \ldots, \mu_\ell) \vdash n_0$ one can define for all $n \geq n_0 + \mu_1$ the unique partition $\{\mu\}_n = (n - n_0, \mu_1, \mu_2, \ldots, \mu_\ell) \vdash n$.

It is interesting to observe that the dimensions of the irreducible representations corresponding to the partitions $\{\mu\}_n$ are connected to the dimension of $S_{\mu}$ via the so-called hook-length formula as

$$\dim_{\mathbb{Q}}(S^\{\mu\}_n) = \frac{\dim_{\mathbb{Q}}(S_{\mu})}{|\mu|!} P_{\mu}(n),$$

where $P_{\mu}(T)$ is a monic polynomial having distinct integer roots, and $\deg(P_{\mu}) = |\mu|$ (see [18, 7.2.2]).

The following question was asked in [8] (albeit in the real affine case).

**Question 2.** Does there exist a polynomial $P_{F,p,\mu}(n)$ such that for all sufficiently large $n$, $m_{p,\{\mu\}_n}(V_n) = P_{F,p,\mu}(n)$?

**Remark 11.** Note that it follows from the results in [8], that in the case $K = \mathbb{R}$, that there exists a polynomial $P_{F,p,\mu}(n)$ of degree $O(d^2)$, with the property that $m_{p,\{\mu\}_n}(V_n, F) \leq P_{F,p,\mu}(n)$ for all $n \geq 0$.

While Question 2 was asked for $K = \mathbb{R}$, it is natural to try to first resolve it in the case $K = \mathbb{C}$. The techniques of the current paper show that in the case $K = \mathbb{C}$, and for projective varieties, the answer to Question 2 is positive and follows from a stronger more general homological stability result explained below. We have the following theorem.

**Theorem 15.** With the notation defined above we have that for every fixed $p$, the sequence $(i_{n,n+1}^p : H^p(V_{n+1}, \mathbb{F}) \to H^p(V_n, \mathbb{F}))_{n \geq 0}$ are eventually isomorphisms.

Before proving Theorem 15 we first prove a homological stability result that might be of independent interest.

5.2. Homological stability. The topic of homological stability of sequences of spaces and its connection with representational stability – particularly sequences of configurations spaces $(\text{Conf}_n(M))_{n \geq 0}$ of some fixed manifold $M$ is an extremely active area of research (see for example [16]). Recently algebro-geometrical methods have been used to prove homological stability of $(\text{Conf}_n(M))_{n \geq 0}$ for $M = \mathbb{C}^d$ for some fixed $d$ (see for example [24]). The sequences of varieties that we consider
below are very natural but are not isomorphic to the sequence of configuration
spaces of any fixed manifold \( M \).

Let \( K \) be a field, and let \( A_n(K) \) denote the polynomial ring \( K[X_1, \ldots, X_n] \), and
for \( m > n \), let \( \phi_{m,n} : A_m(K) \to A_n(K) \) the homomorphism defined by,

\[
\phi_{m,n}(f) = f(X_1, \ldots, X_n, 0, \ldots, 0).
\]

Let \( A(K) = \varprojlim A_n(K) \), and \( \phi_n : A(K) \to A_n(K) \) the corresponding homomorphisms. We say that \( f \in A(K) \) is homogeneous of degree \( d \) if each polynomial \( \phi_{n}(f) \)
is homogeneous and of degree \( d \). We say that an ideal \( I \subset A(K) \) is homogeneous
and finitely generated if there exists \( f_1, \ldots, f_r \in A(K) \) such that \( I = (f_1, \ldots, f_r) \),
and each \( f_i \) is homogeneous.

Notice that in this case for each \( k > 0 \), \( I_n = \phi_n(I) \) is a homogeneous ideal of
\( A_n(K) \). Denote by \( V_n(I) \subset \mathbb{P}^{n-1}_K \) the projective variety defined by \( I_n \). Notice that
for \( n < m \), there is a natural inclusion \( i_{n,m} : V_n(I) \to V_m(I) \).

We have the following stability result.

**Theorem 16.** Let \( K = \mathbb{C} \), and \( I \subset A(K) \) be a f.g. homogeneous ideal of \( A(K) \).
Then for every \( p \geq 0 \), there exists some \( N = N(I,p) \geq 0 \), such that for \( N < n \leq m \), the restriction homomorphisms, \( i^p_{n,m} : H^p(V_n(I), \mathbb{F}) \to H^p(V_m(I), \mathbb{F}) \) are isomorphisms.

**Proof.** Let \( I \) be generated by \( r \) homogeneous elements of \( A(K) \). Following the same
notation as in the proof of Theorem 7 we have for each \( k > 1 \) the exact sequence

\[
\cdots \to H^i_c(U_k, \mathbb{F}) \to H^i_c(V_k, \mathbb{F}) \to H^i_c(V_{k-1}, \mathbb{F}) \to H^{i+1}_c(U_k, \mathbb{F}) \to H^{i+1}_c(V_k, \mathbb{F}) \to \cdots.
\]

We also have from Theorem 9 that

\[
H^i_c(U_k, \mathbb{F}) = 0 \quad \text{for all } i < k - r.
\]

It follows from the exact sequence (5.3) and (5.4) that the restriction homomorphism \( i^p_{k-1,k} : H^j_c(V_k, \mathbb{F}) \to H^j_c(V_{k-1}, \mathbb{F}) \) is an isomorphism for all \( j \) satisfying
\( 0 \leq j < k - r \), and hence \( i^p_{n,m} : H^p(V_m(I), \mathbb{F}) \to H^p(V_n(I), \mathbb{F}) \) are isomorphisms for
all \( m, n \) satisfying
\[
p + r + 1 = N(I,p) < n \leq m.
\]

**Remark 12.** In the case, \( K = \mathbb{R} \), the stability statement in Theorem 16 is demonstratively false. Consider for example the ideal \( I = (P) \subset A(K) \), where

\[
\phi_n(P) = \sum_{j=2}^{n} X_j^2(X_1 - X_j)^2.
\]

Then, for each \( n > 0 \),

\[
b_0(V_n(I), \mathbb{F}) = 2^{n-1}.
\]

**Proof of Theorem 15.** For \( i \geq 1 \), let \( \Psi_i \in A(K) \) denote the element defined by,
\( \phi_n(\Psi_i) = \psi_i^{(n)} \), where

\[
\psi_i^{(n)} = X_1^i + \cdots + X_n^i.
\]
Setting \( I = (F_1(\Psi_1, \ldots, \Psi_d), \ldots, F_r(\Psi_1, \ldots, \Psi_d)) \subset A(K) \). Applying Theorem 16 we obtain that for sufficiently large \( n \), the restriction homomorphism \( i_{n,n+1}^*: H^p(V_{n+1}, \mathbb{Q}) \to H^p(V_n, \mathbb{Q}) \) is an isomorphism.

Thus, as an \( S_{n+1} \) representation

\[
H^p(V_n, \mathbb{Q}) \cong_{S_{n+1}} \text{Res}_{S_{n+1}}^S(H^p(V_{n+1}, \mathbb{Q})).
\]

Now since the dimension of \( H^p(V_n, \mathbb{Q}) \) does not grow with \( n \), the only irreducible representations that can occur in \( H^p(V_n, \mathbb{Q}) \) are the trivial or the sign representations (the dimensions of all other irreducible representations of \( S_{n+1} \) grows with \( n \) by the hook-length formula). Since the trivial (resp. sign) representation of \( S_{n+2} \) restricts to the trivial (resp. sign) representation of \( S_{n+1} \), it follows from (5.5) that the multiplicity of the trivial (resp. sign) representation in \( H^p(V_n, \mathbb{Q}) \) are eventually constant for large enough \( n \), and the multiplicities of all other irreducible representations are zero. This proves the theorem. \( \square \)

6. Algorithms

As mentioned in the Introduction, the algorithmic problem of computing the Betti numbers of real and complex varieties has been studied from the point of view of computational complexity [15].

In the classical Turing machine model the problem of computing Betti numbers (indeed just the number of connected components) of a real variety defined by a polynomial of degree 4 is \textbf{PSPACE}-hard. On the other hand it follows doubly exponential algorithms for semi-algebraic triangulation (see [7] for definition) of real varieties, it follows that there exists a doubly exponential complexity algorithm for computing the Betti numbers of real varieties. More precisely, there exists an algorithm in the B-S-S model over \( R \), that computes the Betti numbers of the real sub-variety of \( R^k \) (resp. \( \mathbb{P}^k_R \)) defined by a polynomial (homogeneous polynomial) in \( R[X_1, \ldots, X_k] \) (resp. \( R[X_0, \ldots, X_k] \)) whose degree is bounded by \( d \) with complexity \( d^{O(n^k)} \) [32].

By separating a complex polynomial into real and imaginary parts, the above results hold true even for complex varieties in the real B-S-S model. The problem of computing the Betti numbers of a complex projective variety is \textbf{PSPACE}-hard [30].

We have the following theorems.

\textbf{Theorem 17.} For every fixed \( \ell \) and \( r \), there exists an algorithm (for a B-S-S machine over \( R \)) with polynomially bounded complexity, that takes as input a finite set \( \mathcal{P} = \{ P_1, \ldots, P_r \} \), where each \( P_i \in C[X_0, \ldots, X_k] \) (each coefficient of \( P_i \) being given as \( a + b\sqrt{-1}, a, b \in R \)), and computes \( b^*(\mathbb{Z}(\mathcal{P}, \mathbb{P}^k_R), \mathbb{Q}) \), \( 0 \leq i \leq \ell \).

\textbf{Proof.} For \( 0 \leq i \leq k \), we denote by \( H_i \) the linear space defined by \( X_0 = \cdots = X_{k-i-1} = 0 \), and we denote

\[
V_i = V_k \cap H_i, U_i = V_i \setminus V_{i-1}.
\]

For \( n \leq m \) let \( i_{n,m} : V_n \leftrightarrow V_m \) denote the inclusion map.

It follows from the exact sequence (5.3) and (5.4) that the restriction homomorphism \( i_{k-r,k-r}^*: H^p_c(V_k, \mathbb{Q}) \to H^p_c(V_{k-1}, \mathbb{Q}) \) is an isomorphism for all \( j \) satisfying \( 0 \leq j < k - r - 1 \), and \( i_{n,m}^* : H^p(V_m, \mathbb{Q}) \to H^p(V_n, \mathbb{Q}) \) are isomorphisms for all \( m, n \).
satisfying
\[ p + r < n \leq m. \]

Thus, it suffices to compute \( b^i(V_{\ell+r},\mathbb{Q}), 0 \leq i \leq \ell \). Since, \( V_{\ell+r} \) is a projective variety in \( \mathbb{P}^{\ell+r}_\mathbb{C} \) defined by \( r \) homogeneous polynomials of degrees bounded by \( d \), the Betti numbers of \( V_{\ell+r} \) can be computed via semi-algebraic triangulation with complexity \( d^{2O(\ell+r)} \). \( \square \)

Before stating the next theorem we first introduce a notation.

**Notation 2** (Realization). For a first order formula \( \Phi(X_0, \ldots, X_k) \) whose atoms are of the form \( P = 0 \), \( P \in \mathbb{C}[X_0, \ldots, X_k] \), \( P \) homogeneous, we will denote
\[ R(\Phi, \mathbb{P}^k_\mathbb{C}) = \{ (x_0 : \cdots : x_k) \in \mathbb{P}^k_\mathbb{C} \mid \Phi(x_0, \ldots, x_k) = \text{True} \}. \]

**Remark 13.** Notice that if \( \Phi \) is negation free then \( R(\Phi, \mathbb{P}^k_\mathbb{C}) \) is a projective variety.

**Theorem 18.** For every fixed \( \ell, d, r, s \), there exists an algorithm in the real B-S-S model that takes as input a first-order formula \( \Phi \) either of the form
\[ (6.1) \quad \bigwedge_{i=1}^r \bigvee_{j=1}^n (P_{i,j} = 0) \]
or of the form
\[ (6.2) \quad \bigvee_{i=1}^s \bigwedge_{j=1}^r (P_{i,j} = 0), \]
where each \( P_{i,j} \in \mathbb{C}[X_0, \ldots, X_k] \) is homogeneous of degree at most \( d \), and computes \( b^i(R(\Phi, \mathbb{P}^k_\mathbb{C}), \mathbb{Q}), 0 \leq i \leq \ell \). The complexity of the algorithm is bounded by a polynomial in \( n \) and \( k \).

**Proof.** If the formula \( \Phi \) is of the form (6.1), then \( R(\Phi, \mathbb{P}^k_\mathbb{C}) = Z(\mathbb{Q}, \mathbb{P}^k_\mathbb{C}) \), where \( \mathbb{Q} = \{Q_1, \ldots, Q_r\} \), and
\[ Q_j = \prod_{i=1}^n P_{i,j}, 1 \leq j \leq r. \]
We have that \( \deg(Q_j) \leq nd, 1 \leq j \leq r \). Now apply the algorithm from Theorem 17 to compute the first \( \ell \) Betti numbers of \( Z(\mathbb{Q}, \mathbb{P}^k_\mathbb{C}) \). The complexity is bounded by \( (nd)^{2O(\ell+r)} + O(n kd) \).

If the formula \( \Phi \) is of the form (6.2), then \( R(\Phi, \mathbb{P}^k_\mathbb{C}) = Z(\mathbb{Q}, \mathbb{P}^k_\mathbb{C}) \), where \( \mathbb{Q} = \{Q_I \mid I \in [1, s]^{1, r}\} \), and
\[ Q_I = \prod_{j=1}^r P_{j,I(j)}, I \in [1, s]^{1, r}. \]
Now apply the algorithm in the previous case, noting that \( \text{card} (\mathbb{Q}) = s^r \), and the degree of each polynomial in \( \mathbb{Q} \) is bounded by \( rd \). \( \square \)

Theorems 17 and 18 should be contrasted with the PSPACE-hardness of the general problem of computing all the Betti number of a given projective varieties (with no restrictions on the number of equations) [30], as well the PSPACE-hardness of the problem of even computing the zero-th Betti number of a real variety.
Also notice that the algorithms described in the proofs of Theorems 17 and 18 are deterministic and do not require any randomness (for example, for choosing generic projections or sections).

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