String Network from M-theory

Morten Krogh\textsuperscript{1} and Sangmin Lee\textsuperscript{2}

Joseph Henry Laboratories
Princeton University
Princeton, NJ 08544, USA

Abstract

We study the three string junctions and string networks in Type IIB string theory by explicitly constructing the holomorphic embeddings of the M-theory membrane that describe such configurations. The main feature of them such as supersymmetry, charge conservation and balance of tensions are derived in a more unified manner. We calculate the energy of the string junction and show that there is no binding energy associated with the junction.

December, 1997

\textsuperscript{1} krogh@princeton.edu
\textsuperscript{2} sangmin@princeton.edu.
1. Introduction

Recently there has been a lot of interest in three string junctions in Type IIB string theory (IIB) \cite{1-7}. A three string junction is a configuration where three strings of different type \((p, q)\) meet as shown in the figure. The configuration is planar.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{3-point-junction.png}
\caption{The 3-point junction}
\end{figure}

The BPS nature of this configuration was conjectured in \cite{2} and proven in \cite{4,5}. It was shown that if the strings are of type \((p_i, q_i), i = 1, 2, 3\) then charge conservation requires

\[ \sum_{i=1}^{3} p_i = \sum_{i=1}^{3} q_i = 0. \]  
\hspace{1cm} (1.1)

Furthermore the angles between the strings is determined purely by their type. The angles are such that the total force on the vertex is zero. The tensions of the 3 strings counted with direction add up to zero. In \cite{4} these results were derived by considerations of the Super-Yang-Mills (SYM) theory on a D-string. In \cite{5} more general configurations, networks or lattice of strings, were considered. All these configurations were argued to be BPS configurations preserving one fourth of the supersymmetry (SUSY).

In this paper we will derive all these results by lifting the picture to M-theory. IIB can be viewed as M-theory on a torus in the limit where the torus shrinks. The strings in IIB are the membrane of M-theory with one direction wrapped on the torus. The \((p, q)\) type of the string is determined by which homology cycle of the torus the membrane wraps. We find a smooth configuration of the membrane which corresponds to the three string junction in type IIB. The singularity at the vertex in the IIB description is removed by going to M-theory. In the M-theory description there is no special point.

Essentially the same technique has been used extensively to analyze five dimensional field theories by lifting webs of \((p, q)\) 5-branes of IIB to M-theory \cite{8-11}. The holomorphic curves in those works appear as the low energy solution of the gauge theories.
The organization of the paper is as follows. In section 2, we derive the criterion for a membrane to be a BPS state. The result is that to preserve some supersymmetry the membrane has to be embedded as a holomorphic curve. This result is well known, but since it is usually not explained we include a proof of it. The result is true not only for membranes in flat Minkowski space but also in more general settings like Calabi-Yau compactifications. In section 3, we explain the relation between \((p, q)\) strings and membranes of M-theory. We especially see that the orientation of the \((p, q)\) string in spacetime is correlated with its type in agreement with [4]. In section 4, we derive the equation for the membrane configuration corresponding to the three string junction, thereby giving an alternative proof for the BPS nature of the three string junction. The method of this section is generalized to construct string networks in section 5. In section 6, we calculate the energy of the membrane. It is seen to be exactly equal to the sum of the energies of each string, in accordance with the previous analysis of fundamental strings ending on D-p-branes \((p \geq 3)\) using Born-Infeld action [12-17]. We briefly discuss possible applications of our construction in section 7.

2. Supersymmetric embedding of membranes

In this section we will derive the condition for a membrane configuration in M-theory to preserve some supersymmetry. We are interested in static configurations, in other words the time axis on the worldvolume is directed along the time axis of 11 dimensional spacetime and the 2 spatial directions on the worldvolume are embedded in the 10 spatial dimensions of spacetime. We want to figure out which embeddings give rise to BPS states. To do that we closely follow the discussion of [18], where a similar question was considered with the difference that the membrane was embedded as an Euclidean instanton.

The action for a membrane in M-theory is [19]

\[
S = T_2 \int d^3 \sigma \sqrt{-h} \left( \frac{1}{2} h^{\alpha \beta} \partial_\alpha X^M \partial_\beta X^N G_{MN} - \frac{1}{2} - i \bar{\Theta} \partial^\alpha X^M \Gamma_M \nabla_\alpha \Theta + ... \right) \tag{2.1}
\]

Here \(X^M(\sigma)\), \(M = 0, ..., 10\) describes the membrane configuration. \(\Theta\) is an 11 dimensional Majorana spinor. \(h_{\alpha \beta}, \alpha, \beta = 0, 1, 2\) is an auxiliary worldvolume metric. The dots denote terms of higher power in the fermi fields. The 3-form of M-theory has been set to zero. \(G_{MN}\) is the metric of spacetime. \(\Gamma_M\) are gamma matrices satisfying \(\{\Gamma_M, \Gamma_N\} = 2G_{MN}\). In the applications in this paper we are solely interested in the case \(G_{MN} = \eta_{MN}\), but for the time being we can be more general and take spacetime to be of the form \(\mathbb{R}^{1,0} \times K^{10}\), where \(\mathbb{R}^{1,0}\) is time and \(K^{10}\) is a Kähler manifold. This would cover both flat spacetime and Calabi-Yau compactifications.
The equation of motion for $h_{\alpha\beta}$ sets it equal to the induced metric

$$h_{\alpha\beta} = \partial_\alpha X^M \partial_\beta X^N G_{MN} \quad (2.2)$$

The action has two fermionic symmetries. One is the global SUSY transformation

$$\delta_\varepsilon \Theta = \varepsilon$$
$$\delta_\varepsilon X^M = i \varepsilon \Gamma^M \Theta \quad (2.3)$$

where $\varepsilon$ is a covariantly constant anticommuting 11 dimensional spinor. The other symmetry is the local $\kappa$ symmetry

$$\delta_\kappa \Theta = 2 P_+ \kappa(\sigma)$$
$$\delta_\kappa X^M = 2i \Theta \Gamma^M P_+ \kappa(\sigma) \quad (2.4)$$

where $\kappa$ is an 11 dimensional spinor and $P_\pm$ are projection operators

$$P_\pm = \frac{1}{2} (1 \pm \frac{1}{3!} \varepsilon^{\alpha\beta\gamma} \partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P \Gamma_{MNP}) \quad (2.5)$$

obeying

$$P_\pm^2 = P_\pm$$
$$P_+ P_- = P_- P_+ = 0 \quad (2.6)$$
$$P_+ + P_- = 1$$

Here $\Gamma_{MNP} = \frac{1}{3!} (\Gamma_M \Gamma_N \Gamma_P \pm 5\text{permutations.})$

For a bosonic membrane configuration ($\Theta = 0$) the condition for unbroken SUSY is that $\delta_\varepsilon \Theta = 0$. From (2.3) this seems to be impossible. However configurations that differ by a $\kappa$ transformation are to be identified so a supersymmetry generated by $\varepsilon$ is unbroken if there exists a function $\kappa(\sigma)$ such that

$$\delta_\varepsilon \Theta + \delta_\kappa \Theta = \varepsilon + 2 P_+ \kappa(\sigma) = 0 \quad (2.7)$$

Since $P_+$ is a projection operator this equation for $\kappa(\sigma)$ has a solution if and only if $\varepsilon = P_+ \varepsilon$ or equivalently

$$P_- \varepsilon = 0 \quad (2.8)$$

Since we are only interested in static configurations we take $X^0(\sigma^0, \sigma^1, \sigma^2) = \sigma^0$ and $X^M, M = 1, .., 10$ to be a function of only $\sigma^1$ and $\sigma^2$. For this configuration the condition on $\varepsilon$ becomes

$$(1 - \frac{1}{2} \varepsilon^{\alpha \beta} \partial_\alpha X^M \partial_\beta X^N \Gamma_{MN} \Gamma_0) \varepsilon = 0 \quad (2.9)$$
with $\alpha, \beta = 1, 2$, $M, N = 1, \ldots, 10$ and $\Gamma_{MN} = \frac{1}{2}(\Gamma_M \Gamma_N - \Gamma_N \Gamma_M)$. Now the 10 dimensional space is a Kähler manifold with metric $g$. In many cases there are several choices of complex structure which makes the manifold Kähler. We will return to this point later. For now let us assume no more than the space being Kähler. We have gamma matrices $\Gamma_i, \Gamma_\overline{i}, i = 1, .. 5$ which obey

\begin{align}
\{\Gamma_i, \Gamma_j\} &= 2g_{ij} \\
\{\Gamma_i, \Gamma_j\} &= \{\Gamma_\overline{i}, \Gamma_\overline{j}\} = 0 \\
\Gamma_\overline{\overline{i}} &= (\Gamma_i)^\dagger
\end{align}

(2.10)

The 32 complex dimensional representation of this Clifford algebra can be built from a highest weight vector $\varepsilon$ satisfying

$$\Gamma_i \varepsilon = 0 \quad i = 1, .., 5$$

(2.11)

by applying the lowering operators $\Gamma_\overline{i}$. This $\varepsilon$ is not Majorana. In Calabi-Yau compactifications we know that this $\varepsilon$, together with others, is unbroken by the compactification. Thus it makes sense to ask which membrane configurations preserve this $\varepsilon$. We have to solve the problem of which configurations, $X^M(\sigma^1, \sigma^2)$, solve (2.9) for this $\varepsilon$. First we are free to change coordinate system on the worldvolume. It is well known, from string theory for instance, that we can choose coordinates, at least locally, such that the metric $h_{\alpha\beta}$ is on the form

$$h_{\alpha\beta} = g(\sigma^1, \sigma^2)\delta_{\alpha\beta}$$

(2.12)

Here we are just displaying the spatial part of the metric. Define a complex structure on the worldvolume by $u = \sigma_1 + i\sigma_2$. The statement is now that the supersymmetry generated by $\varepsilon$ is preserved if and only if the configuration is a holomorphic map, i.e. $X^i(u)$ is holomorphic. To prove this we should prove that (2.9) is true if and only if $X^i(u)$ is holomorphic.

First we note that (2.9) is an equation on each point of the membrane. In a given point we can always, for simplicity, choose coordinates in spacetime such that $g_{ij} = \frac{1}{2}\delta_{ij}$. Writing $z_k = x_k + iy_k$ the condition (2.11) becomes $\Gamma_{x_k} \Gamma_{y_k} \varepsilon = i\varepsilon$. This implies

$$\Gamma_{x_1} \Gamma_{y_1} \ldots \Gamma_{x_5} \Gamma_{y_5} \varepsilon = i\varepsilon$$

(2.13)

Working in conventions with $\Gamma_0 \ldots \Gamma_{10} = -1$ this implies $\Gamma_0 \varepsilon = i\varepsilon$. (2.9) now becomes

$$\frac{1}{2}\varepsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \Gamma_{MN} \varepsilon = \varepsilon$$

(2.14)
Using (2.11) this splits into several equations. From the coefficient of $\Gamma_i \Gamma_j$ we get

$$\partial_1 X^i \partial_2 X^j = \partial_1 X^j \partial_2 X^i$$  \hspace{1cm} (2.15)

From the coefficient of the unit matrix we get

$$\frac{1}{2} i \frac{1}{h_{11}} (\partial_1 X^i \partial_2 X^\bar{i} - \partial_2 X^i \partial_1 X^\bar{i}) = 1$$ \hspace{1cm} (2.16)

We also get equations from (2.2) and (2.12)

$$h_{11} = h_{22} = \partial_1 X^i \partial_1 X^\bar{i} = \partial_2 X^i \partial_2 X^\bar{i}$$ \hspace{1cm} (2.17)

$$0 = h_{12} = \frac{1}{2} (\partial_1 X^i \partial_2 X^\bar{i} + \partial_1 X^\bar{i} \partial_2 X^i)$$ \hspace{1cm} (2.18)

Combining (2.16), (2.17) and (2.18), we get

$$i \partial_1 X^i \partial_2 X^\bar{i} = \partial_1 X^i \partial_1 X^\bar{i}$$ \hspace{1cm} (2.19)

or

$$i \partial_2 X^\bar{j} \partial_1 X^i \partial_2 X^\bar{i} = \partial_1 X^i \partial_1 X^\bar{i} \partial_2 X^\bar{j}$$ \hspace{1cm} (2.20)

valid for all $j$. Using (2.13) we get

$$i \partial_2 X^\bar{j} \partial_1 X^i \partial_2 X^\bar{i} = \partial_1 X^i \partial_1 X^\bar{i} \partial_2 X^\bar{j}$$ \hspace{1cm} (2.21)

or

$$\partial_1 X^\bar{j} - i \partial_2 X^\bar{j} = 0$$ \hspace{1cm} (2.22)

Here we used that $\partial_1 X^i \partial_2 X^\bar{i} \neq 0$ which follows from (2.16). (2.22) exactly tells us that $X^\bar{j}$ is antiholomorphic or equivalently $X^j$ is holomorphic in $u = \sigma^1 + i \sigma^2$.

The $\varepsilon$ which satisfied the equation is not Majorana. However $P_- \varepsilon = 0$ is solved by the real and imaginary part separately. Alternatively the complex conjugate of $\varepsilon$ also satisfies $P_- \varepsilon^* = 0$. $\varepsilon^*$ is the highest weight vector with respect to the complex conjugate complex structure. This complex structure gives the manifold the opposite orientation.

The result of the discussion above is as follows. Consider M-theory on $\mathbb{R}^{1,0} \times K^{10}$, where the first factor is time and $K^{10}$ is an oriented Riemannian manifold which admits a complex structure compatible with the orientation and which makes it Kähler. Let $\varepsilon$ be a covariantly constant spinor satisfying

$$\Gamma_i \varepsilon = 0 \quad i = 1, \ldots, 5$$ \hspace{1cm} (2.23)
Furthermore consider a membrane with its time direction along time and its spatial part embedded in $M^{10}$. Then this configuration preserves the SUSY given by the real and imaginary part of $\varepsilon$ if and only if the spatial part of the membrane is a holomorphic curve in $M^{10}$.

This prescription allows us to determine the unbroken SUSY of a membrane configuration. Below we will consider various special cases. Let us first consider $\mathbb{R}^{1,10}$ with standard metric, $\eta_{ab}$. Define $z_1 = x_1 + ix_2$, ..., $z_5 = x_9 + ix_{10}$. The Dirac spinors constitute a 32 dimensional complex vectorspace with a basis given by

$$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_5) \quad \varepsilon_i = \pm 1$$

(2.24)

where each $\varepsilon_i$ is 1 or -1 depending on whether $\Gamma_i \varepsilon = 0$ or $\Gamma_i \varepsilon = 0$ respectively. Obviously $(1, 1, 1, 1, 1)$ is highest weight vector for the complex structure $(z_1, z_2, z_3, z_4, z_5)$ and $(1, -1, -1, 1, -1)$, for example, is the highest weight vector for the complex structure $(z_1, \bar{z}_2, \bar{z}_3, z_4, \bar{z}_5)$ and so on. A membrane configuration is holomorphic in both $z$ and $\bar{z}$ if and only if $z$ is constant along the membrane. We can now consider several cases.

1. The planar membrane, where $z_5$ is holomorphic and $z_1, \ldots, z_4$ are constant. We see that the preserved supersymmetries are given by

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, 1) \quad \text{with} \quad \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 = 1$$

(2.25)

where the last condition came from the requirement that the complex structure is compatible with the orientation. Remembering the complex conjugate of $\varepsilon$ we see that 16 supersymmetries are unbroken and they satisfy

$$\Gamma_1 \ldots \Gamma_8 \varepsilon = \varepsilon$$

(2.26)

This is, of course, the expected result for the planar membrane.

2. Consider now a membrane holomorphically embedded in $z_4, z_5$ and constant in $z_1, z_2, z_3$. Furthermore assume the embedding is non-degenerate, i.e., the membrane is not embedded in a 2-plane inside $z_4, z_5$. The preserved SUSYs are

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, 1, 1) \quad \text{with} \quad \varepsilon_1 \varepsilon_2 \varepsilon_3 = 1$$

(2.27)

Remembering the complex conjugate

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, -1, -1) \quad \text{with} \quad \varepsilon_1 \varepsilon_2 \varepsilon_3 = -1$$

(2.28)
we see that there are 8 unbroken supersymmetries and they satisfy
\[
\begin{align*}
\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6 \Gamma_7 \Gamma_8 \epsilon &= \epsilon \\
\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 \Gamma_6 \Gamma_9 \Gamma_{10} \epsilon &= \epsilon
\end{align*}
\] (2.29)
This is the same as two perpendicular planar membranes. This last case is exactly the case we are interested in, namely the membrane embedded in a four real dimensional plane.

3. It is obvious to extend to the case where the membrane is embedded in 6,8 and 10 real dimensions. We preserve respectively 4,2 and 2 SUSYs and the unbroken SUSY is the same as for intersecting membranes.

Finally let us briefly discuss the case of a Calabi-Yau three fold with the membrane wrapped around a holomorphic 2-cycle. Let \( z_3, z_4, z_5 \) be holomorphic coordinates for the Calabi-Yau. The unbroken SUSY by the compactification is the real part of \((\epsilon_1, \epsilon_2, 1, 1, 1)\) and \((\epsilon_1, \epsilon_2, -1, -1, -1)\), i.e. there are 8 unbroken supersymmetries. With the membrane the unbroken supersymmetries are
\[
\begin{align*}
(\epsilon_1, \epsilon_2, 1, 1, 1) & \quad \text{with} \quad \epsilon_1 \epsilon_2 = 1 \\
(\epsilon_1, \epsilon_2, -1, -1, -1) & \quad \text{with} \quad \epsilon_1 \epsilon_2 = -1
\end{align*}
\] (2.30)
,i.e., there are 4 unbroken supersymmetries. In the next sections we will use the results of this section to construct BPS configurations of the membrane.

3. \((p, q)\) Strings from Membranes

Type IIB string theory (IIB) with complexified string coupling \( \tau = \tau_1 + i \tau_2 \) is obtained by compactifying M-theory on a torus with complex structure \( \tau \). Consider M-theory on \( \mathbb{R}^{1,8} \times T^2 \) parametrized by \((X^0, X^1, \cdots, X^9, X^{10})\) with the identifications
\[
(X^9, X^{10}) \sim (X^9 + 2\pi R, X^{10}) \sim (X^9 + 2\pi R \tau_1, X^{10} + 2\pi R \tau_2).
\] (3.1)
For finite \( R \) this describes IIB on a circle. In the limit \( R \to 0 \), we recover IIB in ten dimensions.

The \((p, q)\) strings in IIB are easily described in this setting. They are simply the membranes with one circle wrapped on the torus along the \((p, q)\) homology cycle. Specifically, a \((p, q)\) string oriented along the \(X^1\)-axis, say, is described by a membrane embedded as follows.
\[
\begin{align*}
X^1 &= s, \quad X^9 = 2\pi R t (p \tau_1 + q), \quad X^{10} = 2\pi R t (p \tau_2) \\
s &\in \mathbb{R}, \quad t \in [0, 1].
\end{align*}
\] (3.2)
The overall sign of \((p, q)\) depends on a choice of the orientation of both the string and the membrane.

We are interested in the 3-string junction, which is located in a plane. In M-theory this junction is described by a single membrane. This membrane has a nontrivial behavior in the plane of the junction and in the torus. From the previous section, we know that a BPS configuration is given by choosing a complex structure in these four dimensions and embedding the membrane along a holomorphic curve. The position of the membrane is fixed in the other six spatial dimensions. We also saw that this BPS configuration preserves \(1/4\) of the SUSY.

Let the junction lie in the \((X^1, X^2)\) plane and choose the complex structure.

\[
\begin{align*}
  z^1 &= X^1 + iX^9, \\
  z^2 &= X^2 + iX^{10}.
\end{align*}
\]  

(3.3)

The identifications defining the torus are now

\[
\begin{align*}
  (z^1, z^2) &\sim (z^1 + i2\pi R, z^2) \sim (z^1 + i2\pi R\tau_1, z^2 + i2\pi R\tau_2).
\end{align*}
\]  

(3.4)

Define

\[
\begin{align*}
  u &= \exp \left( \frac{z^1}{R} - \frac{\tau_1}{R} \frac{z^2}{\tau_2} \right), \\
  v &= \exp \left( \frac{z^2}{\tau_2 R} \right).
\end{align*}
\]  

(3.5)

We see that \((u, v) \in (\mathbb{C} - \{0\})^2\) are single valued and constitute a global coordinate system on our two complex dimensional manifold \(\mathbb{R}^2 \times T^2\).

What is the equation for a \((p, q)\) string? To be a \((p, q)\) string, the membrane has to be oriented along the \((p, q)\) homology cycle on the \(T^2\). In other words, the membrane embedding obeys

\[
p\tau_2 X^9 = (p\tau_1 + q)X^{10} + \text{const.}, \quad \text{or } \text{Im} \left( p\tau_2 z^1 - (p\tau_1 + q)z^2 \right) = \text{const.}
\]  

(3.6)

Since the embedding has to be holomorphic, the equation must be

\[
p\tau_2 z^1 - (p\tau_1 + q)z^2 = \text{const.}
\]  

(3.7)

The real part of this equation shows that the \((p, q)\) string has a fixed orientation in the \((X^1, X^2)\) plane given by

\[
p\tau_2 X^1 = (p\tau_1 + q)X^2 + \text{const.}
\]  

(3.8)

In other words, the \((p, q)\) string is directed along the unit vector

\[
\frac{1}{\sqrt{(p\tau_1 + q)^2 + (p\tau_2)^2}} (p\tau_1 + q, p\tau_2).
\]  

(3.9)
in the \((X^1, X^2)\) plane. Specifically, the \((0, 1)\) string (D-string) is oriented along the \(X^1\) axis. We recover the observation in \([4,5]\) that the type of the string is correlated with its orientation. We can write the equation (3.7) for a single \((p, q)\) string in terms of \(u\) and \(v\) as

\[ u^p v^{-q} = \lambda, \]  

where the nonzero complex constant \(\lambda\) determines the position of the string on \(\mathbb{R}^2 \times T^2\).

From this discussion we also see that all BPS saturated string network are planar. This is because the internal torus is 2 dimensional. Fixing the type of a string is the same as fixing the behaviour of the membrane in the internal torus. Since this is the imaginary part of an equation the direction of the string in space is then fixed and must lie in the 2 plane which together with the torus makes a 2 complex dimensional space.

Before we find the equation describing a 3-string junction, let us digress to discuss the metric on \(\mathbb{R}^2 \times T^2\) since we need it later in order to calculate the area of the membrane configuration. The metric is

\[ ds^2 = dx_1^2 + dx_2^2 + dx_9^2 + dx_{10}^2 \]  

(3.11)

In our complex structure \((z^1, z^2)\) the manifold is Kähler with the Kähler form equal to

\[ \omega = \frac{i}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2) \]  

(3.12)

In terms of \((u, v)\), \(\omega\) is

\[ \omega = \frac{i}{2} R^2 \left\{ \frac{du \wedge d\pi}{|u|^2} + |\tau|^2 \frac{dv \wedge d\bar{\pi}}{|v|^2} + \tau_1 \left( \frac{du \wedge d\bar{\pi}}{uv} + \frac{dv \wedge d\pi}{v\bar{\pi}} \right) \right\} \]  

(3.13)

4. 3-String Junction

In this section, we will derive the equation describing the membrane corresponding to a 3-string junction in IIB. We will start with the simplest case which is a junction with a \((1, 0)\), \((0, 1)\) and a \((-1, -1)\) string. This is the same case as was considered in \([4]\). Later we will present the general case. The membrane configuration is given by a holomorphic curve which is the zero locus of a holomorphic function,

\[ f(u, v) = 0. \]  

(4.1)

To find the function \(f(u, v)\), we use the fact that it should look like one of the three strings away from the junction. We expect the vertex to be smoothed out. We want the
(0, 1) string to be recovered for large $X^1$ and $X^2 \approx \text{const}$. This means that for $u$ fixed at a very large value we want exactly one solution in $v$. Similarly, the (1, 0) string is recovered for large $v$, so for fixed large $v$, we want exactly one solution in $u$. The most general form of the equation with these two properties is

$$uv + au + bv + c = 0, \quad \text{or} \quad (u - \lambda_1)(v - \lambda_2) = \lambda_3$$

(4.2)

with $\lambda_1, \lambda_2, \lambda_3$ complex constants. If $\lambda_3 = 0$, the curve is reducible and the equation describes two intersecting planar membranes. This is not what we want, so $\lambda_3 \neq 0$.

Let us analyze the curve described by (4.2) for $u \to \infty$. In this limit, $v \to \lambda_2$ which, according to (3.10), describes a (0, 1) string extended along the $X^1$-direction. We also see the geometrical significance of the parameter $\lambda_2$. $\lambda_2$ gives the location of the (0, 1) string in $\mathbb{R}^2 \times T^2$. For $v \to \infty$, we similarly get $u = \lambda_1$, which is the (1, 0) string at a position given by $\lambda_1$.

What about the $(-1, -1)$ string? We expect to see this for $u, v \to 0$. This is only possible if $\lambda_3 = \lambda_1\lambda_2$. Now the equation becomes

$$uv - \lambda_1 v - \lambda_2 u = 0.$$  

(4.3)

For $u, v \to 0$, the first term is negligible and the equation becomes

$$uv^{-1} = \frac{\lambda_1}{\lambda_2}$$

(4.4)

Combining with (3.10), we conclude that this describes a (1, 1) or $(-1, -1)$ string. Since it is oriented towards small $u$ and $v$ we see from (3.5) and (3.9) that it is a $(-1, -1)$ string.

The solution (4.3) thus has the property that there are exactly 3 ways to go to infinity where the solution becomes respectively a (0, 1), (1, 0) and $(-1, -1)$ string. This is enough to conclude that (4.3) is the M-theory description of the 3-string junction.

There is one immediate advantage in this description of the 3-string junction. Eq. (4.3) is easily seen to describe a smooth curve. In other words, the junction has no singularities associated with it. This is in contrast to the SYM theory on the D-string [4] which is not well-suited to capture the nature of the vertex.

Strictly speaking the above analysis is only valid when the torus is large compared the 11 dimensional Planck scale since we have used a low energy action to describe the membrane. The 10 dimensional type IIB theory is only recovered in the limit where the torus shrinks to zero size. However usual BPS arguments show that the state will remain BPS for all values of the area of the torus, thereby proving that the three string junction in IIB is a BPS state.
Having explained the junction with a (1, 0), (0, 1) and (−1, −1) string in detail we will just state the equation for the general junction. Consider the junction made of a \((p_1, q_1), (p_2, q_2)\) and \((-p_1 - p_2, -q_1 - q_2)\) string. Assume furthermore that \(p_1 q_2 - p_2 q_1 > 0\). If this is not the case we can always relable the indices. The equation for this junction is

\[
\lambda_1 u^{-p_1 q_1} + \lambda_2 u^{p_2 q_2} = 1 \tag{4.5}
\]

where \(\lambda_1\) and \(\lambda_2\) are two non-zero complex numbers specifying the position of the junction. To see that this equation really describes the junction one checks, as before, that there are 3 ways of going to infinity and that the equation here describes a string. By setting \((p_1, q_1) = (1, 0)\) and \((p_2, q_2) = (0, 1)\) we recover the special case (4.3). This proves that any three string junction obeying charge conservation (1.1) exists. These configurations are not all S-duality transforms of each other. Many of them are genuinely different.

5. String Network

Having set up the formalism, it is easy to generate other curves and see what they correspond to in the IIB picture. Clearly, there are very many possibilities. We will not attempt to classify all possible configurations. Instead, we will give two examples of string networks to illustrate the idea. For simplicity, we will only use \((0, \pm 1), (\pm 1, 0), (\pm 1, \pm 1)\) strings to build the networks. The figures will be drawn for \(\tau = i\).

![String Networks](image)

**Fig. 2:** String Networks. The dotted lines show how the shape of the lattice (b) changes with the parameter \(e\).
The simplest network is drawn in Fig. 2(a). The angles are fixed by the types of strings in the network, but we are free to change three lengths \( A, B, C \). We can write down the equation for the network for arbitrary number of unit cells. Infinite lattice is obtained by taking an appropriate limit. For, \((2j - 1) \times (2k - 1)\) hexagonal cells, the equation is

\[
\sum_{l=-k}^{k} \left\{ v^l P_l(b, c) u^{-j} \prod_{m=1}^{2j} (u - b^j a^{2m-2j-1}) \right\} = 0, \tag{5.1}
\]

where \( P_l(c, d) \) is defined by

\[
\sum_{l=-k}^{k} v^{l+k} P_l(b, c) = \prod_{m=1}^{2k} (v - b^j c^{2m-2k-1}). \tag{5.2}
\]

The parameters in the equation are related to the lengths by

\[
A = 2R \ln a, \quad C = 2R \ln c, \quad B = \sqrt{2R} \ln(ac/b). \tag{5.3}
\]

To see that (5.1) indeed describes the network, let us look at the case \( j = k = 1 \) in some detail. The equation becomes

\[
(u - ba)(u - ba^{-1})u^2 - b(c + c^{-1})(u - a)(u - a^{-1})v + b^2(u - b^{-1} a)(u - b^{-1} a^{-1}) = 0. \tag{5.4}
\]

First, note that the asymptotics of the equation correctly produce the eight external lines. The polynomials in \( u \) multiplying each \( v^l \), \((0 \leq l \leq 2k)\) have the factors that specify the positions of the internal lines.

In the second example depicted in Fig. 2(b), there are five independent lengths we can change. \( A, B, C \) and \( D \) are shown in the figure. The fifth parameter, \( e \), represents the freedom to deform the shape of the lattice without changing the asymptotics. The equation for this network is as follows.

\[
\sum_{l=0}^{2k-1} (v^l + v^{-l})Q_l(c, d)S \left( u; a, \frac{b + e}{2} + (-1)^{2k-1-l} \frac{b - e}{2} \right), \tag{5.5}
\]

where \( S(u; a, t) \) and \( Q_l(c, d) \) are defined by

\[
S(u; a, t) \equiv u^{-(2j+1)} \prod_{m=j+1}^{j-1} (u - a^m t)(u - a^m t^{-1}), \tag{5.6}
\]

\[
\sum_{l=0}^{2k-1} (v^l + v^{-l})Q_l(c, d) = v^{-2k+1} \prod_{m=-k+1}^{k-1} (v - c^m d)(v - c^m d^{-1}).
\]

The parameters \( a, b, c, d \) are related to the physical parameters by

\[
A = R \ln a, \quad B = R \ln b, \quad C = R \ln c, \quad D = R \ln d. \tag{5.7}
\]

Apart from the correct asymptotics, note that (5.5) factorizes to give intersecting fundamental and D-strings when \( e = b \).
6. Energy of the 3-String Junction

The Born-Infeld (BI) description of the fundamental string ending on D-p-branes shows that such configurations have no binding energy \[12,13\] for \( p \geq 3 \). For \( p = 2 \), the D-p-branes do not become flat asymptotically and the binding energy is not well-defined. For \( p = 1 \), the BI theory becomes singular at the vertex and is not appropriate to calculate the binding energy. In this section, we calculate the energy of the 3-string junction using the M-theory description derived in the previous section. The binding energy is shown to be zero as expected.

In M-theory, the energy of a 2-brane in its ground state is simply the area of the 2-brane multiplied by the 2-brane tension \( T_{M2} = \frac{1}{(2\pi)^2 l_{11}^2} \). The area is obtained by integrating the Kähler form of the 2-brane, which is the pull-back of the Kähler form of the \( \mathbb{R}^2 \times T^2 \) given by (3.13).

To be definite, let us work with the simplest junction given by (4.3). Set also \( \lambda_1 = \lambda_2 = 1 \). This amounts to locating the junction at the origin in the \((X^1, X^2)\) plane as well as fixing the position in the internal torus. If we choose \( u \) as the coordinate on the 2-brane, the Kähler form becomes

\[
\omega = \frac{i}{2} R^2 \operatorname{Re} \left\{ \frac{1}{u(u-1)} + |\tau|^2 \frac{1}{|u-1|^2 u} + |\tau + 1|^2 \frac{1}{|u|^2 (1-u)} \right\} du \wedge d\bar{u}. \quad (6.1)
\]

Note that each of the three terms gives a divergent integral at \( u = \infty, 1, 0 \), respectively. The divergence comes from the infinite length of the three strings. We will introduce cutoffs, \( \Lambda_{(p,q)} \gg 1 \), for each \((p,q)\) string. Specifically, the integration will be limited to the regions \(|u| \leq \Lambda_{(0,1)}, |u-1| \geq \Lambda_{1,0}^{-1}, |u| \geq \Lambda_{-1,-1}^{-1}\) for the three terms, respectively. The integral is easy to evaluate and the result is

\[
A = 2\pi R^2 \sum_{p,q} |p\tau + q|^2 \ln \Lambda_{(p,q)} \quad (6.2)
\]

In order to understand this result in IIB, recall that the tension of a \((p,q)\) string is the length of the \((p,q)\) homology cycle of the torus times the tension of the membrane (M2),

\[
T_{(p,q)} = 2\pi R |p\tau + q| T_{M2}. \quad (6.3)
\]

We also need to know the relation between the cutoff in the \( u \)-plane and the length of each string. From (3.5) and (3.10), it is clear that \( \ln \Lambda_{(0,1)} \) is the length of the D-string divided by \( R \). In the same way, one can show that the length of the \((p,q)\) string is given by

\[
L_{(p,q)} = R |p\tau + q| \ln \Lambda_{(p,q)} + O(\Lambda^{-1}). \quad (6.4)
\]

\(^3\) Our convention for \( l_{11} \) is that \( 16\pi G_{11} = (2\pi)^8 l_{11}^9 \), where \( G_{11} \) is the 11-dimensional Newton’s constant.
The $O(\Lambda^{-1})$ correction becomes negligible in the uncompactified IIB limit ($R \to 0$). Combining (6.2), (6.3) and (6.4), we obtain

$$E = T_{M2}A = \sum_{p,q} L_{(p,q)} T_{(p,q)}.$$  

We see that the energy is precisely the sum of the energy of the three strings and there is no binding energy.

7. Discussions

We have constructed the M-theory realization of the 3-string junction and string networks. All the properties of the static configurations are easily derived in this formulation. This approach is also suitable for analyzing the dynamics of these systems\[6\]. The propagation of wave through the junction could be studied by solving the equation of motion (e.o.m.) that follows from the membrane action in the static background. For example, when the fluctuation is transverse to both $(X^1, X^2)$ and $(X^9, X^{10})$ planes, the linearized approximation to the e.o.m. is simply a Helmholtz’s equation on the (curved) membrane.

Strictly speaking, the membrane description cannot be trusted when the compactification torus is smaller than the 11 dimensional Planck scale. However, it may worth comparing the result from the membrane action with other approaches such as the Born-Infeld theory \[6\], or boundary conformal field theory.

Acknowledgements

We are grateful to Y. K. E. Cheung and L. Thorlacius for discussions. The work of MK was supported by the Danish Research Academy. The work of SL was supported in part by a DOE grant DE-FG02-91ER40671.
References

[1] O. Aharony, J. Sonnenschein and S. Yankielowicz, Interactions of Strings and D-branes from M-theory, \texttt{hep-th/9603003}, Nucl. Phys. \textbf{B474} (1996) 309.

[2] J.H. Schwarz, Lectures on Superstring and M theory Dualities, \texttt{hep-th/9607201}, Nucl. Phys. Proc. Suppl. \textbf{55B} (1997) 1.

[3] M. Gaberdiel and B. Zwiebach, Exceptional groups from open strings, \texttt{hep-th/9709013}.

[4] K. Dasgupta and S. Mukhi, BPS Nature of 3-String Junctions, \texttt{hep-th/9711094}.

[5] A. Sen, String Network, \texttt{hep-th/9711130}.

[6] S.-J. Rey and J.-T. Yee, BPS Dynamics of Triple (p,q) String Junction, \texttt{hep-th/9711202}.

[7] J. P. Gauntlett, J. Gomis, P. K. Townsend, BPS Bounds for Worldvolume Branes, \texttt{hep-th/9712203}.

[8] O. Aharony, A. Hanany, Branes, Superpotentials and Superconformal Fixed Points, \texttt{hep-th/9704170}, Nucl. Phys. \textbf{B504} (1997) 239.

[9] B. Kol, 5d Field Theories and M Theory, \texttt{hep-th/9705031}.

[10] A. Brandhuber, N. Itzhaki, J. Sonnenschein, S. Theisen, On the M-Theory approach to (Compactified) 5D Field Theories, \texttt{hep-th/9709010}.

[11] O. Aharony, A. Hanany, B. Kol, Webs of (p,q) 5-branes, Five Dimensional Field Theories and Grid Diagrams, \texttt{hep-th/9710116}.

[12] C. Callan and J. Maldacena, Brane Dynamics from the Born-Infeld Action, \texttt{hep-th/9708147}.

[13] G. Gibbons, Born-Infeld Particles and Dirichlet p-Branes, \texttt{hep-th/9709027}.

[14] S. Lee, A. Peet and L. Thorlacius, Brane-Waves and Strings, \texttt{hep-th/9710097}.

[15] L. Thorlacius, Born-Infeld String as a Boundary Conformal Field Theory, \texttt{hep-th/9710181}.

[16] A. Hashimoto, The Shape of Branes Pulled by Strings, \texttt{hep-th/9711097}.

[17] R. Emparan, Born-Infeld Strings Tunneling to D-branes, \texttt{hep-th/9711106}.

[18] K. Becker, M. Becker and A. Strominger, Fivebranes, Membranes and Non-Perturbative String Theory, \texttt{hep-th/9507158}, Nucl. Phys. \textbf{B456} (1995) 130.

[19] P.K. Townsend, Three Lectures on Supermembranes, Proceedings of the Trieste Spring School, 11.-19. April 1988.