Compactness of Commutators of Integral Operators with Functions in Campanato Spaces on Orlicz-Morrey Spaces

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Abstract
We consider the commutators $[b, T]$ and $[b, I_\rho]$, where $T$ is a Calderón–Zygmund operator, $I_\rho$ is a generalized fractional integral operator and $b$ is a function in the closure of $C_\infty^{\text{comp}}(\mathbb{R}^n)$ with respect to generalized Campanato spaces. We give a necessary and sufficient condition for the compactness of $[b, T]$ and $[b, I_\rho]$ on Orlicz-Morrey spaces. The Orlicz-Morrey spaces unify Orlicz and Morrey spaces, and the Campanato spaces unify BMO and Lipschitz spaces. Therefore, our results contain many previous results as corollaries.

Keywords Orlicz-Morrey space · Campanato space · Singular integral · Fractional integral · Commutator

Mathematics Subject Classification 42B35 · 46E30 · 42B20 · 42B25

1 Introduction

Let $T$ be a Calderón–Zygmund singular integral operator on $\mathbb{R}^n$ and $b \in \text{BMO}(\mathbb{R}^n)$. In 1976 Coifman, Rochberg and Weiss \cite{Coifman} proved that the commutator $[b, T] = bT - Tb$ is bounded on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$), that is,
\begin{equation}
\| [b, T] f \|_{L^p} = \| bTf - Tf(b) \|_{L^p} \leq C \| b \|_{\text{BMO}} \| f \|_{L^p}.
\end{equation}

Dedicated to the 80th anniversary of Professor Stefan Samko.

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where $C$ is a positive constant independent of $b$ and $f$. For the fractional integral operator $I_\alpha$, Chanillo [5] proved the boundedness of $[b, I_\alpha]$ in 1982. Coifman, Rochberg and Weiss [8] and Chanillo [5] also gave the necessary conditions for the boundedness. These results were extended to Orlicz spaces by Janson [13] (1978) and to Morrey spaces by Di Fazio and Ragusa [10] (1991). Further, these results were extended to Orlicz-Morrey spaces by [43]. The commutator $[b, I]$ in (1.1) is of interest since it is not of Calderón–Zygmund type, that is, it is not of weak type $(1, 1)$ as shown by Pérez [37], where, furthermore, it is shown a weak $L \log L$ type estimate, see also Remark 3.2.

On the other hand, Uchiyama [46] considered the compactness of the commutator $[b, T]$ on $L^p(\mathbb{R}^n)$ in 1978, where $T$ is a Calderón–Zygmund singular integral operator with convolution type of smooth kernel $K \not\equiv 0$. He proved that $[b, T]$ is compact on $L^p(\mathbb{R}^n)$ if and only if $b \in \text{CMO}(\mathbb{R}^n)$, where CMO$(\mathbb{R}^n)$ is the closure of $C^\infty_\text{comp}(\mathbb{R}^n)$ in BMO$(\mathbb{R}^n)$.

The theory of commutators plays an important role in harmonic analysis, complex analysis, PDEs and other fields in mathematics. In this paper we investigate the compactness of the commutators $[b, T]$ and $[b, I_\rho]$ on Orlicz-Morrey spaces, where $T$ is a Calderón–Zygmund singular integral operator, $I_\rho$ is a generalized fractional integral operator and $b$ is a function in the closure of $C^\infty_\text{comp}(\mathbb{R}^n)$ with respect to generalized Campanato spaces. The Orlicz-Morrey spaces unify Orlicz and Morrey spaces, and the Campanato spaces unify BMO and Lipschitz spaces. Therefore, our results contain many previous results as corollaries.

Function spaces have been widely used in various areas of analysis such as harmonic analysis and PDEs. In recent years, there has been increasing interest in the Orlicz-Morrey space. It was first studied by [27] in 2004 to extend the Hardy–Littlewood Sobolev theorem for the fractional integral operators to both Orlicz and Morrey spaces simultaneously. In 2012 another kind of Orlicz-Morrey spaces was introduced by Sawano, Sugano and Tanaka [41]. Moreover, in 2014 the third kind of Orlicz-Morrey spaces was introduced by Deringoz, Guliyev and Samko [9]. These three kinds are independent each other and studied actively by many authors, respectively. In this paper we treat the first kind. The difference and equivalence between these kinds were investigated by Gala, Sawano and Tanaka [11] and Karapetyants and Samko [15].

First we recall the Orlicz-Morrey space. We denote by $B(a, r)$ the open ball centered at $a \in \mathbb{R}^n$ and of radius $r$. For a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and a ball $B$, let

$$f_B = \int_B f = \int_B f(y) dy = \frac{1}{|B|} \int_B f(y) dy,$$

where $|B|$ is the Lebesgue measure of the ball $B$.

**Definition 1.1 (Orlicz-Morrey space)** For a Young function $\Phi : [0, \infty] \to [0, \infty]$, a function $\varphi : (0, \infty) \to (0, \infty)$ and a ball $B = B(a, r)$, let

$$\| f \|_{\Phi, \varphi, B} = \inf \left\{ \lambda > 0 : \frac{1}{\varphi(r)} \int_B \Phi \left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.$$  (1.2)
Let \( L^{(\Phi, \varphi)}(\mathbb{R}^n) \) be the set of all functions \( f \) such that the following functional is finite:
\[
\| f \|_{L^{(\Phi, \varphi)}} = \sup_B \| f \|_{\Phi, \varphi, B},
\]
where the supremum is taken over all balls \( B \) in \( \mathbb{R}^n \). (For the definition of the Young function, see the next section.)

Then \( \| f \|_{L^{(\Phi, \varphi)}} \) is a norm and thereby \( L^{(\Phi, \varphi)}(\mathbb{R}^n) \) is a Banach space. If \( \varphi(r) = 1/r^n \), then \( L^{(\Phi, \varphi)}(\mathbb{R}^n) \) coincides with the Orlicz space \( L^{\Phi}(\mathbb{R}^n) \) equipped with the norm
\[
\| f \|_{L^{\Phi}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.
\]

If \( \Phi(t) = t^p, 1 \leq p < \infty \), then \( L^{(\Phi, \varphi)}(\mathbb{R}^n) \) coincides with the generalized Morrey space \( L^{(p, \varphi)}(\mathbb{R}^n) \) equipped with the norm
\[
\| f \|_{L^{(p, \varphi)}} = \sup_{B=B(a, r)} \left( \frac{1}{\varphi(r)} \int_B |f(x) - f_B|^p dx \right)^{1/p}.
\]

Secondly, we recall the definition of the generalized Campanato space.

**Definition 1.2** For \( p \in [1, \infty) \) and a function \( \psi : (0, \infty) \to (0, \infty) \), let \( \mathcal{L}_{p, \psi}(\mathbb{R}^n) \) be the set of all functions \( f \) such that the following functional is finite:
\[
\| f \|_{\mathcal{L}_{p, \psi}} = \sup_{B=B(a, r)} \frac{1}{\psi(r)} \left( \int_B |f(y) - f_B|^p dy \right)^{1/p},
\]
where the supremum is taken over all balls \( B = B(a, r) \) in \( \mathbb{R}^n \).

Then \( \| f \|_{\mathcal{L}_{p, \psi}(\mathbb{R}^n)} \) is a norm modulo constant functions and thereby \( \mathcal{L}_{p, \psi}(\mathbb{R}^n) \) is a Banach space. If \( p = 1 \) and \( \psi \equiv 1 \), then \( \mathcal{L}_{p, \psi}(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n) \). If \( p = 1 \) and \( \psi(r) = r^\alpha (0 < \alpha \leq 1) \), then \( \mathcal{L}_{p, \psi}(\mathbb{R}^n) \) coincides with \( \text{Lip}_\alpha(\mathbb{R}^n) \). If \( \psi \) is almost increasing (see (2.14)), then \( \mathcal{L}_{p, \psi}(\mathbb{R}^n) = \mathcal{L}_{1, \psi}(\mathbb{R}^n) \).

Thirdly, we recall the generalized fractional integral operator \( I_\rho \). For a function \( \rho : (0, \infty) \to (0, \infty) \), the operator \( I_\rho \) is defined by
\[
I_\rho f(x) = \int_{\mathbb{R}^n} \rho(|x - y|) |f(y)| dy, \quad x \in \mathbb{R}^n,
\]
where we always assume that
\[
\int_0^1 \frac{\rho(t)}{t} dt < \infty.
\]

If \( \rho(r) = r^\alpha, 0 < \alpha < n \), then \( I_\rho \) is the usual fractional integral operator \( I_\alpha \). The condition (1.5) is needed for the integral in (1.4) to converge for bounded functions \( f \).
with compact support. In this paper we also assume that there exist positive constants $C$, $K_1$ and $K_2$ with $K_1 < K_2$ such that, for all $r > 0$,

$$\sup_{r \leq t \leq 2r} \rho(t) \leq C \int_{K_1 r}^{K_2 r} \frac{\rho(t)}{t} \, dt. \tag{1.6}$$

The condition (1.6) was considered in [36]. The operator $I_\rho$ was introduced in [24] whose partial results were announced in [23], see also [25–28, 31].

Let $T$ be a Calderón–Zygmund operator (for the definition, see Sect. 3), and let $b \in L^1,\psi(\mathbb{R}^n)$. It is known by [43] that, under suitable assumptions, the commutators $[b, T]$ and $[b, I_\rho]$ are bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $L^{(\Psi, \varphi)}(\mathbb{R}^n)$, i.e. the following norm inequalities hold:

$$\| [b, T] f \|_{L^{(\Psi, \varphi)}} \leq C \| b \|_{L^1, \psi} \| f \|_{L^{(\Phi, \varphi)}},$$

$$\| [b, I_\rho] f \|_{L^{(\Psi, \varphi)}} \leq C \| b \|_{L^1, \psi} \| f \|_{L^{(\Phi, \varphi)}}.$$

It is also known that, if $[b, T]$ or $[b, I_\rho]$ is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $L^{(\Psi, \varphi)}(\mathbb{R}^n)$, then $b$ is in $L^1,\psi(\mathbb{R}^n)$ and the operator norm dominates $\| b \|_{L^1, \psi}$.

In this paper we prove that, with additional assumptions, each of $[b, T]$ and $[b, I_\rho]$ is compact from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $L^{(\Psi, \varphi)}(\mathbb{R}^n)$, if and only if $b$ is in the closure of $C^\infty_{\text{comp}}(\mathbb{R}^n)$ in $L^1,\psi(\mathbb{R}^n)$.

The organization of this paper is as follows: In the next section we state precise definitions of the functions $\Phi$ and $\varphi$ by which we define the Orlicz-Morrey space $L^{(\Phi, \varphi)}(\mathbb{R}^n)$. In Sect. 3 we state known results on the boundedness of the commutators $[b, T]$ and $[b, I_\rho]$. Then we state the main results on the compactness in Sect. 4. To prove the main results we give basic properties of Young functions $\Phi$, growth functions $\varphi$ and the Orlicz and Orlicz-Morrey norms in Sect. 5, and, we show a criterion for the compactness of integral operators on Orlicz-Morrey spaces in Sect. 6. Finally we prove the main results in Sects. 7 and 8.

At the end of this section, we make some conventions. Throughout this paper, we always use $C$ to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as $C_p$, are dependent on the subscripts. If $f \lesssim g$, we then write $f \lesssim g \lesssim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

### 2 Orlicz and Orlicz-Morrey Spaces

In this section we first recall generalized Young functions. Next, we recall the definitions of Orlicz and Orlicz-Morrey spaces with generalized Young functions. The Orlicz space is introduced by [34, 35]. For the theory of Orlicz spaces, see [17–19, 21, 39] for example. The Orlicz-Morrey spaces were investigated in [27–29], etc.

For an increasing (i.e. nondecreasing) function $\Phi : [0, \infty) \to [0, \infty]$, let

$$a(\Phi) = \sup \{ t \geq 0 : \Phi(t) = 0 \}, \quad b(\Phi) = \inf \{ t \geq 0 : \Phi(t) = \infty \}, \quad (2.1)$$
with convention sup ∅ = 0 and inf ∅ = ∞. Then 0 ≤ a(Φ) ≤ b(Φ) ≤ ∞.

Let $\Phi$ be the set of all increasing functions $\Phi : [0, \infty) \rightarrow [0, \infty]$ such that

$$0 \leq a(\Phi) < \infty, \quad 0 < b(\Phi) \leq \infty, \quad (2.2)$$

$$\lim_{t \rightarrow +0} \Phi(t) = \Phi(0) = 0, \quad (2.3)$$

$\Phi$ is left continuous on $[0, b(\Phi))$, 

if $b(\Phi) = \infty$, then $\lim_{t \rightarrow \infty} \Phi(t) = \Phi(\infty) = \infty$, 

if $b(\Phi) < \infty$, then $\lim_{t \rightarrow b(\Phi) - 0} \Phi(t) = \Phi(b(\Phi)) \leq \infty$. 

In what follows, if an increasing and left continuous function $\Phi : [0, \infty) \rightarrow [0, \infty]$ satisfies (2.3) and $\lim_{t \rightarrow \infty} \Phi(t) = \infty$, then we always regard that $\Phi(\infty) = \infty$ and that $\Phi \in \Phi$.

For $\Phi \in \Phi$, we recall the generalized inverse of $\Phi$ in the sense of O’Neil [33,Definition 1.2].

**Definition 2.1** For $\Phi \in \Phi$ and $u \in [0, \infty]$, let

$$\Phi^{-1}(u) = \begin{cases} 
\inf\{t \geq 0 : \Phi(t) > u\}, & u \in [0, \infty), \\
\infty, & u = \infty. 
\end{cases} \quad (2.7)$$

Let $\Phi \in \Phi$. Then $\Phi^{-1}$ is finite, increasing and right continuous on $[0, \infty)$ and positive on $(0, \infty)$. If $\Phi$ is bijective from $[0, \infty]$ to itself, then $\Phi^{-1}$ is the usual inverse function of $\Phi$. Moreover, if $\Phi \in \Phi$, then

$$\Phi(\Phi^{-1}(u)) \leq u \leq \Phi^{-1}(\Phi(u)) \quad \text{for all } u \in [0, \infty], \quad (2.8)$$

which is a generalization of Property 1.3 in [33], see [42,Proposition 2.2].

For $\Phi, \Psi \in \Phi$, we write $\Phi \approx \Psi$ if there exists a positive constant $C$ such that

$$\Phi(C^{-1}t) \leq \Psi(t) \leq \Phi(Ct) \quad \text{for all } t \in [0, \infty].$$

For functions $P, Q : [0, \infty) \rightarrow [0, \infty]$, we write $P \sim Q$ if there exists a positive constant $C$ such that

$$C^{-1}P(t) \leq Q(t) \leq CP(t) \quad \text{for all } t \in [0, \infty].$$

Then, for $\Phi, \Psi \in \Phi$,

$$\Phi \approx \Psi \iff \Phi^{-1} \sim \Psi^{-1}.$$

see [42,Lemma 2.3].

Next we recall the definition of the Young function and give its generalization.

**Definition 2.2** (*Young function and its generalization*) A function $\Phi \in \Phi$ is called Young function (or sometimes also called Orlicz function) if $\Phi$ is convex on $[0, b(\Phi))$.
Let $\Phi_1$ be the set of all Young functions. Let $\overline{\Phi}_1$ be the set of all $\Phi \in \Phi$ such that $\Phi \approx \Psi$ for some $\Psi \in \Phi_1$.

Let $(\Omega, \mu)$ be a measure space, and let $L^0(\Omega)$ be the set of all measurable functions on $\Omega$. Then the Orlicz space is defined by the following.

**Definition 2.3** (*Orlicz space*) For $\Phi \in \overline{\Phi}_1$, let

$$L^\Phi(\Omega) = \left\{ f \in L^0(\Omega) : \int_\Omega \Phi(\epsilon |f(x)|) \, d\mu(x) < \infty \text{ for some } \epsilon > 0 \right\},$$

$$\| f \|_{L^\Phi(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) \, d\mu(x) \leq 1 \right\}.$$

Then $\| \cdot \|_{L^\Phi(\Omega)}$ is a quasi-norm and thereby $L^\Phi(\Omega)$ is a quasi-Banach space. If $\Phi, \Psi \in \overline{\Phi}_1$, then $L^\Phi(\Omega) = L^\Psi(\Omega)$ with equivalent quasi-norms. In the case $\Omega = \mathbb{R}^n$ we always write $\| \cdot \|_{L^\Phi} \text{ instead of } \| \cdot \|_{L^\Phi(\mathbb{R}^n)}$, omitting $(\mathbb{R}^n)$.

**Definition 2.4** (*Orlicz-Morrey space with generalized Young function*) For $\Phi \in \overline{\Phi}_1$ and $\phi : (0, \infty) \to (0, \infty)$, we define the Orlicz-Morrey space $L^{(\Phi, \phi)}(\mathbb{R}^n)$ together with $\| f \|_{L^{(\Phi, \phi)}}$ by (1.3).

For a ball $B = B(a, r)$, let $\mu_B = \frac{dx}{|B(\phi(r))}$. Then we have the following relation:

$$\| f \|_{L^{(\Phi, \phi)}(B)} = \| f \|_{L^{\Phi}(B, \mu_B)}.$$  \hfill (2.10)

Because of the relation (2.10), $\| \cdot \|_{L^{(\Phi, \phi)}}$ is a quasi-norm, and thereby $L^{(\Phi, \phi)}(\mathbb{R}^n)$ is a quasi-Banach space. If $\Phi \in \overline{\Phi}_1$, then $\| \cdot \|_{L^{(\Phi, \phi)}}$ is a norm and thereby $L^{(\Phi, \phi)}(\mathbb{R}^n)$ is a Banach space. If $\Phi \approx \Psi$ and $\phi \sim \psi$, then $L^{(\Phi, \phi)}(\mathbb{R}^n) = L^{(\Psi, \psi)}(\mathbb{R}^n)$ with equivalent quasi-norms.

**Definition 2.5**

(i) A function $\Phi \in \overline{\Phi}_1$ is said to satisfy the $\Delta_2$-condition, denoted by $\Phi \in \Delta_2$, if there exists a constant $C > 0$ such that

$$\Phi(2t) \leq C \Phi(t) \text{ for all } t > 0.$$  \hfill (2.11)

(ii) A function $\Phi \in \overline{\Phi}_1$ is said to satisfy the $\nabla_2$-condition, denoted by $\Phi \in \nabla_2$, if there exists a constant $k > 1$ such that

$$\Phi(t) \leq \frac{1}{2k} \Phi(kt) \text{ for all } t > 0.$$  \hfill (2.12)

(iii) Let $\Delta_2 = \Phi_1 \cap \overline{\Delta}_2$ and $\nabla_2 = \Phi_1 \cap \overline{\nabla}_2$.

Next, we say that a function $\theta : (0, \infty) \to (0, \infty)$ satisfies the doubling condition if there exists a positive constant $C$ such that, for all $r, s \in (0, \infty)$,

$$\frac{1}{C} \leq \frac{\theta(r)}{\theta(s)} \leq C, \text{ if } \frac{1}{2} \leq \frac{r}{s} \leq 2.$$  \hfill (2.13)
We say that \( \theta \) is almost increasing (resp. almost decreasing) if there exists a positive constant \( C \) such that, for all \( r, s \in (0, \infty) \),

\[
\theta(r) \leq C \theta(s) \quad \text{(resp. } \theta(s) \leq C \theta(r)) \text{, if } r < s.
\] (2.14)

In this paper we consider the following class of \( \phi : (0, \infty) \to (0, \infty) \).

**Definition 2.6** (i) Let \( G^{\text{dec}} \) be the set of all functions \( \varphi : (0, \infty) \to (0, \infty) \) such that \( \varphi \) is almost decreasing and that \( r \mapsto \varphi(r)r^n \) is almost increasing. That is, there exists a positive constant \( C \) such that, for all \( r, s \in (0, \infty) \),

\[
C \varphi(r) \geq \varphi(s), \quad \varphi(r)r^n \leq C \varphi(s)s^n, \quad \text{if } r < s.
\]

(ii) Let \( G^{\text{inc}} \) be the set of all functions \( \varphi : (0, \infty) \to (0, \infty) \) such that \( \varphi \) is almost increasing and that \( r \mapsto \varphi(r)/r \) is almost decreasing. That is, there exists a positive constant \( C \) such that, for all \( r, s \in (0, \infty) \),

\[
\varphi(r) \leq C \varphi(s), \quad C \varphi(r)/r \geq \varphi(s)/s, \quad \text{if } r < s.
\]

If \( \varphi \in G^{\text{dec}} \) or \( \varphi \in G^{\text{inc}} \), then \( \varphi \) satisfies the doubling condition (2.13). Let \( \psi : (0, \infty) \to (0, \infty) \). If \( \psi \sim \varphi \) for some \( \varphi \in G^{\text{dec}} \) (resp. \( G^{\text{inc}} \)), then \( \psi \in G^{\text{dec}} \) (resp. \( G^{\text{inc}} \)).

**3 Known Results: Boundedness**

First we recall the definition of Calderón–Zygmund operators following [47]. Let \( \Omega \) be the set of all increasing functions \( \omega : (0, \infty) \to (0, \infty) \) such that \( \int_0^1 \omega(t) \frac{dt}{t} < \infty \).

**Definition 3.1** (standard kernel) Let \( \omega \in \Omega \). A continuous function \( K(x, y) \) on \( \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) \in \mathbb{R}^{2n}\} \) is said to be a standard kernel of type \( \omega \) if the following conditions are satisfied:

\[
|K(x, y)| \leq \frac{C}{|x - y|^n} \quad \text{for } x \neq y, \tag{3.1}
\]

\[
|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \leq \frac{C}{|x - y|^n} \omega \left( \frac{|y - z|}{|x - y|} \right) \quad \text{for } 2|y - z| < |x - y|. \tag{3.2}
\]

**Definition 3.2** (Calderón–Zygmund operator) Let \( \omega \in \Omega \). A linear operator \( T \) from \( \mathcal{S}(\mathbb{R}^n) \) to \( \mathcal{S}'(\mathbb{R}^n) \) is said to be a Calderón–Zygmund operator of type \( \omega \), if \( T \) is bounded on \( L^2(\mathbb{R}^n) \) and there exists a standard kernel \( K \) of type \( \omega \) such that, for \( f \in C_{\text{comp}}^\infty(\mathbb{R}^n) \),

\[
Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy, \quad x \notin \text{supp } f. \tag{3.3}
\]
Remark 3.1 If \( x \notin \text{supp} \, f \), then \( K(x, y) \) is continuous on \( \text{supp} \, f \) with respect to \( y \). Therefore, if (3.3) holds for \( f \in C^\infty_{\text{comp}}(\mathbb{R}^n) \), then (3.3) holds for \( f \in L^1_{\text{comp}}(\mathbb{R}^n) \).

It was known by [47,Theorem 2.4] that any Calderón–Zygmund operator of type \( \omega \in \Omega \) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \). This result was extended to Orlicz–Morrey spaces \( L^{(\Phi,\varphi)}(\mathbb{R}^n) \) by [29] as the following: Let \( \Phi \in \Delta_2 \cap \nabla_2 \). Assume that \( \varphi \in \mathcal{G}^{\text{inc}} \) and that there exists a positive constant \( C \) such that, for all \( r \in (0, \infty) \),

\[
\int_r^\infty \frac{\varphi(t)}{t} \, dt \leq C \varphi(r). \tag{3.4}
\]

For \( f \in L^{(\Phi,\varphi)}(\mathbb{R}^n) \), we define \( Tf \) on each ball \( B \) by

\[
Tf(x) = T(f \chi_{2B})(x) + \int_{\mathbb{R}^n \setminus 2B} K(x, y) f(y) \, dy, \quad x \in B. \tag{3.5}
\]

Then the first term in the right hand side is well defined, since \( f \chi_{2B} \in L^\Phi(\mathbb{R}^n) \subset L^p_{\text{loc}}(\mathbb{R}^n) \) for some \( p \in (1, \infty) \), and the integral of the second term converges absolutely. Moreover, \( Tf(x) \) is independent of the choice of the ball containing \( x \). By this definition we can show that \( T \) is a bounded operator on \( L^{(\Phi,\varphi)}(\mathbb{R}^n) \).

For functions \( f \) in Orlicz–Morrey spaces, we define \([b, T]f\) on each ball \( B \) by

\[
[b, T]f(x) = [b, T](f \chi_{2B})(x) + \int_{\mathbb{R}^n \setminus 2B} (b(x) - b(y))K(x, y)f(y) \, dy, \quad x \in B, \tag{3.6}
\]

see [43,Remark 7.2] for its well definedness. Then the following theorem is known.

Theorem 3.1 [43] Let \( \Phi, \Psi \in \overline{\Phi}_Y \), \( \varphi \in \mathcal{G}^{\text{inc}} \) and \( \psi \in \mathcal{G}^{\text{dec}} \). Let \( T \) be a Calderón–Zygmund operator of type \( \omega \in \Omega \).

(i) Let \( \Phi, \Psi \in \Delta_2 \cap \nabla_2 \) and \( \int_0^1 \frac{\omega(t)}{t} \log(1/t) \, dt < \infty \). Assume that \( \varphi \) satisfies (3.4) and that there exists a positive constant \( C_0 \) such that, for all \( r \in (0, \infty) \),

\[
\psi(r)\Phi^{-1}(\varphi(r)) \leq C_0 \Psi^{-1}(\varphi(r)). \tag{3.7}
\]

If \( b \in \mathcal{L}_{1,\psi}(\mathbb{R}^n) \), then \([b, T]f\) in (3.6) is well defined for all \( f \in L^{(\Phi,\varphi)}(\mathbb{R}^n) \) and there exists a positive constant \( C \), independent of \( b \) and \( f \), such that

\[
\|[b, T]f\|_{L^{(\psi,\varphi)}} \leq C \|b\|_{\mathcal{L}_{1,\psi}} \|f\|_{L^{(\Phi,\varphi)}}.
\]

(ii) Conversely, assume that there exists a positive constant \( C_0 \) such that, for all \( r \in (0, \infty) \),

\[
C_0 \psi(r)\Phi^{-1}(\varphi(r)) \geq \Psi^{-1}(\varphi(r)). \tag{3.8}
\]

If \( T \) is a convolution type such that

\[
Tf(x) = p.v. \int_{\mathbb{R}^n} K(x - y) f(y) \, dy \tag{3.9}
\]
with homogeneous kernel $K$ satisfying $K(x) = |x|^{-n}K(x/|x|)$, $\int_{\mathbb{R}^{n-1}} K = 0$, $K \in C^\infty(S^{n-1})$ and $K \not\equiv 0$, and if $[b, T]$ is bounded from $L^{(\Phi, \psi)}(\mathbb{R}^n)$ to $L^{(\Psi, \phi)}(\mathbb{R}^n)$, then $b$ is in $L_1, \psi(\mathbb{R}^n)$ and there exists a positive constant $C$, independent of $b$, such that

$$\|b\|_{L_1, \psi} \leq C \|[b, T]\|_{L^{(\Phi, \psi)} \to L^{(\Psi, \phi)}},$$

where $\|[b, T]\|_{L^{(\Phi, \psi)} \to L^{(\Psi, \phi)}}$ is the operator norm of $[b, T]$ from $L^{(\Phi, \psi)}(\mathbb{R}^n)$ to $L^{(\Psi, \phi)}(\mathbb{R}^n)$.

**Remark 3.2** In Theorem 3.1, take $\Phi(t) = \Psi(t) = t^p$, $\varphi(r) = 1/r^n$ and $\psi \equiv 1$. Then $L^{(\Phi, \psi)}(\mathbb{R}^n) = L^{(\Psi, \phi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L_1, \psi(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n)$. This case is the result by Coifman, Rochberg and Weiss [8]. Theorem 3.1 also contains the results by Janson [13,Theorem] and Di Fazio and Ragusa [10,Theorem 1]. On the other hand, Theorem 3.1 does not contain the endpoint results. It is known that the commutator $[b, T]$ with $b \in \text{BMO}(\mathbb{R}^n)$ is not of weak type $(1, 1)$ as shown by Pérez [37], where, furthermore, it is shown a weak $L \log L$ type estimate, which was improved in Pérez and Pradolini [38]. See also Lerner et al. [20] in which they used the sparse operators. Moreover, Accomazzo [1] show that $b \in \text{BMO}(\mathbb{R}^n)$ is a necessary condition for the weak $L \log L$ type estimate. In relation to Theorem 3.1, it would be interest to know the endpoint estimate by using Orlicz-Morrey spaces.

Next, for the boundedness of the commutator $[b, I_\rho]$ on Orlicz-Morrey spaces, the following theorem is known, see [43,Remark 7.3] for its well definedness.

**Theorem 3.2** [43] Let $\Phi, \Psi \in \mathcal{G}_Y$, $\varphi \in \mathcal{G}^{\text{dec}}$, $\psi \in \mathcal{G}^{\text{inc}}$.

(i) Let $\rho : (0, \infty) \to (0, \infty)$. Assume that $\rho$ satisfies (1.5) and (1.6). Let $\Phi, \Psi \in \overline{\Delta}_2 \cap \overline{\nabla}_2$. Assume that $\varphi$ satisfies (3.4). Assume also that there exist positive constants $\epsilon, C_\rho, C_0, C_1$ and a function $\Theta \in \overline{\nabla}_2$ such that, for all $r, s \in (0, \infty),

$$C_\rho r^{n-\epsilon} \leq \frac{\rho(r)}{s^{n-\epsilon}}, \quad \text{if } r < s,$$

$$\left| \frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n} \right| \leq C_\rho |r - s| \frac{1}{r^{n-1}} \int_0^r \frac{\rho(t)}{t} \, dt, \quad \text{if } \frac{1}{2} \leq r - s \leq 2,$$

$$\int_0^r \frac{\rho(t)}{t} \, dt \leq C_0 \Theta^{-1}(\varphi(r)) + \int_r^\infty \frac{\rho(t) \Theta^{-1}(\varphi(t))}{t} \, dt \leq C_1 \Psi^{-1}(\varphi(r)).$$

If $b \in L_1, \psi(\mathbb{R}^n)$, then $[b, I_\rho]f$ is well defined for all $f \in L^{(\Phi, \psi)}(\mathbb{R}^n)$ and there exists a positive constant $C$, independent of $b$ and $f$, such that

$$\|[b, I_\rho]f\|_{L^{(\Psi, \phi)}} \leq C \|b\|_{L_1, \psi} \|f\|_{L^{(\Phi, \psi)}}.$$

(ii) Conversely, assume that $0 < \alpha < n$ and that there exists a positive constant $C_0$ such that, for all $r \in (0, \infty),

$$\Psi^{-1}(\varphi(r)) \leq C_0 r^{n-\alpha} \psi(r) \Phi^{-1}(\varphi(r)).$$
If \( b, I_\alpha \) is bounded from \( L^{(\Phi, \psi)}(\mathbb{R}^n) \) to \( L^{(\Psi, \psi)}(\mathbb{R}^n) \), then \( b \) is in \( L^1_{\psi}(\mathbb{R}^n) \) and there exists a positive constant \( C \), independent of \( b \), such that

\[
\| b \|_{L^1_{\psi}} \leq C \| [b, I_\alpha] \|_{L^{(\Phi, \psi)} \to L^{(\Psi, \psi)}},
\] (3.16)

where \( \| [b, I_\alpha] \|_{L^{(\Phi, \psi)} \to L^{(\Psi, \psi)}} \) is the operator norm of \( [b, I_\alpha] \) from \( L^{(\Phi, \psi)}(\mathbb{R}^n) \) to \( L^{(\Psi, \psi)}(\mathbb{R}^n) \).

4 Main Results: Compactness

Now we state our main results. We denote by \( \overline{C_{\text{comp}}^\infty(\mathbb{R}^n)} \bigcap L^{(1, \psi)}(\mathbb{R}^n) \) the closure of \( C_{\text{comp}}^\infty(\mathbb{R}^n) \) with respect to \( L^{(1, \psi)}(\mathbb{R}^n) \). If \( \psi \equiv 1 \), then \( L^{(1, \psi)}(\mathbb{R}^n) = \text{BMO} \) and \( \overline{C_{\text{comp}}^\infty(\mathbb{R}^n)} = \text{CMO}(\mathbb{R}^n) \).

For the compactness of the commutators \( [b, T] \) and \( [b, I_\psi] \), we consider the following condition on \( \psi \): There exists a positive constant \( C \) such that, for all \( r \in (0, \infty) \),

\[
\int_r^\infty \frac{\psi(t)}{t^2} \, dt \leq C \frac{\psi(r)}{r}.
\] (4.1)

Then our main results are the following:

**Theorem 4.1** Let \( \Phi, \Psi \in \Phi_Y, \varphi \in G^{\text{dec}} \) and \( \psi \in G^{\text{inc}} \).

(i) Let \( T \) be a Calderón–Zygmund operator of type \( \omega \in \Omega \). Assume the same condition on \( \Phi, \Psi, \varphi, \psi \) and \( \omega \) as Theorem 3.1 (i). Assume also that \( \psi \) satisfies (4.1) and that, for all \( f \in C_{\text{comp}}^\infty(\mathbb{R}^n) \),

\[
Tf(x) = \lim_{\epsilon \to +0} \int_{|x-y| \geq \epsilon} K(x, y) f(y) \, dy, \quad \text{a.e.} \ x \in \mathbb{R}^n.
\] (4.2)

If \( b \in C_{\text{comp}}^\infty(\mathbb{R}^d) \bigcap L^1_{\psi}(\mathbb{R}^n) \), then the commutator \( [b, T] \) is compact from \( L^{(\Phi, \psi)}(\mathbb{R}^n) \) to \( L^{(\Psi, \psi)}(\mathbb{R}^n) \).

(ii) Conversely, let \( T \) be a Calderón–Zygmund operator of convolution type with kernel \( K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \). Assume the same condition on \( \Phi, \Psi, \varphi, \psi \) and \( T \) as Theorem 3.1 (ii). Assume also that \( \Psi(t) \approx t^{q^*} \) for some \( q^* \in (1, \infty) \) and that \( \psi \) satisfies (4.1). Let \( b \) be a real valued function in \( L^1_{\text{loc}}(\mathbb{R}^n) \). If \( [b, T] \) is compact from \( L^{(\Phi, \psi)}(\mathbb{R}^n) \) to \( L^{(\Psi, \psi)}(\mathbb{R}^n) \), then \( b \) is in \( C_{\text{comp}}^\infty(\mathbb{R}^n) \bigcap L^1_{\psi}(\mathbb{R}^n) \).

**Remark 4.1** (i) It is known by [47] that, if \( T \) is a Calderón–Zygmund operator of type \( \omega \in \Omega \), then the truncated maximal operator \( T_\ast \) of \( T \) is bounded from \( L^p(\mathbb{R}^n) \) to itself and \( L^1(\mathbb{R}^n) \) to \( \text{w}L^1(\mathbb{R}^n) \) (weak-\( L^1 \) space), where

\[
T_\ast f(x) = \sup_{\epsilon > 0} \left| \int_{|x-y| > \epsilon} K(x, y) f(y) \, dy \right|.
\]
Let \( \Phi \in \Delta_2 \cap \nabla_2 \). Then \( T_\ast \) is also bounded from \( L^\Phi(\mathbb{R}^n) \) to itself by the interpolation. Consequently, if (4.2) holds for all \( f \in C^\infty_{\text{comp}}(\mathbb{R}^n) \), then (4.2) also holds for all \( f \in L^\Phi(\mathbb{R}^n) \), since \( C^\infty_{\text{comp}}(\mathbb{R}^n) \) is dense in \( L^\Phi(\mathbb{R}^n) \). Moreover, by [43,Remark 7.2], we see that the equation

\[
[b, T]f(x) = \lim_{\epsilon \to +0} \int_{|x-y|>\epsilon} (b(x) - b(y)) K(x,y) f(y) \, dy \quad \text{a.e.} \ x \in \mathbb{R}^n \tag{4.3}
\]

holds for all \( f \in L^{(\Phi,\varphi)}(\mathbb{R}^n) \) under the assumption of Theorem 4.1.

(ii) If a Calderón–Zygmund operator satisfies (4.2), then it is called a Calderón–Zygmund singular integral operator. It is known that a Calderón–Zygmund operator is equal to a Calderón–Zygmund singular integral operator plus a bounded function times the identity operator, see Grafakos [12,p. 221].

(iii) The Hilbert transform \( (n = 1, \ K(x,y) = (x-y)/|x-y|^2) \) and the Riesz transforms \( (n \geq 2, \ K(x,y) = (x_j-y_j)/|x-y|^{n+1}, j = 1, \ldots, n) \) are Calderón–Zygmund operators satisfying (4.2), i.e., Calderón–Zygmund singular integral operators.

(iv) To prove Theorem 4.1 (ii) we improve the method in [4,Lemma 5.4]. Consequently, we do not need the complicated assumptions (4.15), (4.16) and (4.17) in [4,Theorem 4.5].

Remark 4.2 From the theorem above we have the following several corollaries.

(i) Take \( \Phi(t) = t^p \). Then we have the result for generalized Morrey spaces \( L^{(p,v)}(\mathbb{R}^n) \). This case is known by [4,Theorem 2.1], which is an extension of the result by Chen, Ding and Wang [7]. See also Tan, Yang and Yang [44, 45].

(ii) Take \( \varphi(r) = 1/r^n \). Then we have the result for Orlicz spaces \( L^\Phi(\mathbb{R}^n) \).

(iii) Take \( \Phi(t) = \Psi(t) = t^p \), \( \varphi(r) = 1/r^n \) and \( \Psi \equiv 1 \). Then \( L^{(\Psi,\varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n) = L^1(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n) \). This case is the result by Uchiyama [46]. Moreover, let \( \psi(r) = r^\alpha \). Then, under the assumption that \( -n/q = -n/p + \alpha \in (0, 1) \), for the Riesz transforms \( R_j \), \( j = 1, \ldots, n \), the commutator \( [b, R_1] \) are compact from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) if and only if \( b \in C^\infty_{\text{comp}}(\mathbb{R}^n) \).

Theorem 4.2 Let \( \Phi, \Psi \in \Phi_Y, \varphi \in \mathcal{G}^{\text{dec}}, \psi \in \mathcal{G}^{\text{inc}}. \)

(i) Let \( \rho : (0, \infty) \to (0, \infty) \). Assume the same condition on \( \Phi, \Psi, \varphi, \psi \) and \( \rho \) as Theorem 3.2 (i). Assume also that \( \psi \) satisfies (4.1). If \( b \in C^\infty_{\text{comp}}(\mathbb{R}^d) \), then the commutator \( [b, I_\rho] \) is compact from \( L^{(\Phi,\varphi)}(\mathbb{R}^n) \) to \( L^{(\Psi,\psi)}(\mathbb{R}^n) \).

(ii) Conversely, let \( 0 < \alpha < n \). Assume the same condition on \( \Phi, \Psi, \varphi, \psi \) and \( \alpha \) as Theorem 3.1 (ii). Assume also that \( \Psi(t) \approx t^\alpha \) for some \( q \in (1, \infty) \) and that \( \psi \) satisfies (4.1). Let \( b \) be a real valued function in \( L^1_{\text{loc}}(\mathbb{R}^n) \). If \( [b, I_\alpha] \) is compact from \( L^{(\Phi,\varphi)}(\mathbb{R}^n) \) to \( L^{(\Psi,\psi)}(\mathbb{R}^n) \), then \( b \) is in \( C^\infty_{\text{comp}}(\mathbb{R}^n) \).
Remark 4.3 (i) By [43, Remark 7.3], we see that the equation

\[ [b, I_\rho] f(x) = \lim_{\epsilon \to +0} \int_{|x-y| > \epsilon} (b(x) - b(y)) \frac{\rho(|x-y|)}{|x-y|^n} f(y) \, dy \quad \text{a.e.} \, x \in \mathbb{R}^n \]

holds for all \( f \in L^{(\Phi, \psi)}(\mathbb{R}^n) \) under the Assumption of Theorem 4.2.

(ii) We do not need the complicated Assumptions (4.20), (4.21) and (4.22) in [4, Theorem 4.6] by the same reason as Remark 4.1 (iv).

Remark 4.4 From the theorem above we have the following several corollaries.

(i) Take \( \Phi(t) = t^p \). Then we have the result for generalized Morrey spaces \( L^{(p, \psi)}(\mathbb{R}^n) \). This case is known by [2, Theorem 2.2], which is an extension of the results by Chen, Deng and Wang [6] and Nogayama and Sawano [32].

(ii) Take \( \varphi(r) = 1/r^n \). Then we have the result for Orlicz spaces \( L^\Phi(\mathbb{R}^n) \).

(iii) Take \( \rho(r) = r^\alpha, \Phi(t) = t^p, \Psi(t) = t^q, \varphi(r) = 1/r^n \) and \( \psi \equiv r^\beta \). Then \( L^{(\Phi, \psi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n), L^{(\psi, \Psi)}(\mathbb{R}^n) = L^q(\mathbb{R}^n) \) and \( L_{1, \psi}(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n) \) for \( \beta = 0 \), \( L_{1, \Psi}(\mathbb{R}^n) = \text{Lip}_\beta(\mathbb{R}^n) \) for \( \beta \in (0, 1) \). That is, under the assumption that \( -n/q = -n/p + \alpha + \beta, \alpha \in (0, n), \beta \in [0, 1) \), for the fractional integral operator \( I_\alpha \), the commutator \( [b, I_\alpha] \) is compact from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) if and only if \( b \in \text{CMO}(\mathbb{R}^n) \) for \( \beta = 0, b \in C^\infty_{\text{comp}}(\mathbb{R}^n) \text{Lip}_\beta(\mathbb{R}^n) \) for \( \beta \in (0, 1) \).

At the end of this section we note that, to prove the theorems, we may assume that \( \Phi, \Psi \in \Phi_Y \) instead of \( \Phi, \Psi \in \Phi \). For example, if \( \Phi \) and \( \Psi \) satisfy (3.7) and \( \Phi \approx \Phi_1, \Psi \approx \Psi_1 \), then \( \Phi_1 \) and \( \Psi_1 \) also satisfy (3.7) by the relation (2.9). Moreover, \( L^{(\Phi, \varphi)}(\mathbb{R}^n) = L^{(\Phi_1, \varphi)}(\mathbb{R}^n) \) and \( L^{(\Psi, \varphi)}(\mathbb{R}^n) = L^{(\Psi_1, \varphi)}(\mathbb{R}^n) \) with equivalent quasi-norms.

5 Properties on Young Functions and Orlicz-Morrey Spaces

In this section we state the basic properties of Young functions \( \Phi \), growth functions \( \varphi \) and Orlicz and Orlicz-Morrey norms.

First we recall the complementary function.

Definition 5.1 For a Young function \( \Phi \), its complementary function is defined by

\[ \tilde{\Phi}(t) = \begin{cases} \sup\{tu - \Phi(u) : u \in [0, \infty]\}, & t \in [0, \infty), \\ \infty, & t = \infty. \end{cases} \]

Then \( \tilde{\Phi} \) is also a Young function.

Remark 5.1 (i) Let \( \Phi \in \Phi_Y \). Then \( \Phi \in \nabla_2 \) if and only if \( \tilde{\Phi} \in \Delta_2 \).

(ii) If \( \Phi \in \Delta_2 \), then \( \Phi \) is continuous and bijective from \([0, \infty)\) to itself.

(iii) Let \( \Phi \in \Phi_Y \). Then \( \Phi \in \Delta_2 \) if and only if \( \Phi \approx \Psi \) for some \( \Psi \in \Delta_2 \), and \( \Phi \in \nabla_2 \) if and only if \( \Phi \approx \Psi \) for some \( \Psi \in \nabla_2 \).

(iv) Let \( \Phi \in \Phi_Y \). Then \( \Phi^{-1} \) satisfies the doubling condition by its concavity.
(v) Let $\Phi \in \Phi_Y$. Then $\Phi \in \Delta_2$ if and only if $t \mapsto \frac{\Phi(t)}{t^q}$ is almost decreasing for some $q \in [1, \infty)$.

(vi) Let $\Phi \in \Phi_Y$. Then $\Phi \in \nabla_2$ if and only if $t \mapsto \frac{\Phi(t)}{t^p}$ is almost increasing for some $p \in (1, \infty)$.

Next, we state known lemmas.

Lemma 5.1 ([43, Lemma 4.1]) Let $\Phi \in \Phi_Y$ and $\varphi \in G_{\text{dec}}$. Then there exists a constant $C \geq 1$ such that, for any ball $B = B(a, r)$,

$$\frac{1}{\Phi^{-1}(\varphi(r))} \leq \|\chi_B\|_{L^\infty(\Phi, \varphi)} \leq \frac{C}{\Phi^{-1}(\varphi(r))}. \quad (5.1)$$

Lemma 5.2 ([43, Lemma 4.3]) Let $\Phi \in \Phi_Y$, $\varphi : (0, \infty) \to (0, \infty)$ and $B = B(a, r) \subset \mathbb{R}^n$. Then

$$\int_B |f(x)| \, dx \leq 2\Phi^{-1}(\varphi(r))\|f\|_{\Phi, \varphi, B}. \quad (5.2)$$

Lemma 5.3 ([22, Lemma 2], [30, Lemma 7.1]) Let $\varphi$ satisfy the doubling condition (2.13) and (3.4), that is,

$$\int_r^\infty \frac{\varphi(t)}{t} \, dt \leq C \varphi(r).$$

Then there exist positive constants $\epsilon$ and $C_\epsilon$ such that, for all $r \in (0, \infty)$,

$$\int_r^\infty \frac{\varphi(t) t^\epsilon}{t} \, dt \leq C_\epsilon \varphi(r) r^\epsilon.$$

Moreover, for all $p \in (0, \infty)$, there exists a positive constant $C_p$ such that, for all $r > 0$,

$$\int_r^\infty \frac{\varphi(t)^{1/p}}{t} \, dt \leq C_p \varphi(r)^{1/p}.$$  

Lemma 5.4 ([43, Lemma 4.4]) Let $\Phi \in \Delta_2$ and $\varphi \in G_{\text{dec}}$. If $\varphi$ satisfies (3.4), then there exists a positive constant $C$ such that, for all $r \in (0, \infty)$,

$$\int_r^\infty \frac{\Phi^{-1}(\varphi(t))}{t} \, dt \leq C \Phi^{-1}(\varphi(r)). \quad (5.3)$$

Remark 5.2 If $\varphi$ is in $G_{\text{dec}}$ and satisfies (3.4), then

$$\lim_{r \to +0} \varphi(r) = \infty, \quad \lim_{r \to \infty} \varphi(r) = 0. \quad (5.4)$$

Actually, $\varphi$ satisfies the doubling condition and the following inequalities hold:

$$\varphi(r) \lesssim \int_r^{2r} \frac{\varphi(t)}{t} \, dt \leq \int_r^\infty \frac{\varphi(t)}{t} \, dt \lesssim \varphi(r).$$
Then we have (5.4).

**Lemma 5.5** ([2, Lemma 3.4, Remark 3.1]) Let $k > 0$ and $\rho : (0, \infty) \to (0, \infty)$. Assume that $\rho$ satisfies (1.5). Let

$$
\rho^*(r) = \int_0^r \frac{\rho(t)}{t} dt.
$$

(5.5)

If $r \mapsto \rho(r)/r^k$ is almost decreasing, then $r \mapsto \rho^*(r)/r^k$ is also almost decreasing. In this case $\rho^*$ satisfies the doubling condition (2.13), since $\rho^*$ is increasing.

**6 Compactness Criterion on Orlicz-Morrey Spaces**

We consider the integral operator

$$
T_0 f(x) = \int_{\mathbb{R}^n} K_0(x, y) f(y) \, dy, \quad x \in \mathbb{R}^n,
$$

(6.1)

for a kernel function $K_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$. In this section we prove the following proposition:

**Proposition 6.1** Let $\Phi, \Psi \in \Delta_2 \cap \nabla_2$ and $\varphi \in \mathcal{G}^{\text{dec}}$. Assume that $\varphi$ satisfies (3.4). If $K_0 \in L^\infty_{\text{comp}}(\mathbb{R}^n \times \mathbb{R}^n)$, then $T_0$ defined by (6.1) is a compact operator from $L^{\Phi, \varphi}(\mathbb{R}^n)$ to $L^{\Psi, \varphi}(\mathbb{R}^n)$.

To prove Proposition 6.1 we prepare several lemmas

**Lemma 6.2** Let $\Phi \in \nabla_2$, $\Theta \in \Delta_2$ and $\tilde{\Phi}$ be the complementary function of $\Phi$. If

$$
\|K_0\|_{L^\Phi(\mathbb{R}^n; L^\tilde{\Phi}(\mathbb{R}^n))} := \left\| K_0(x, y) \right\|_{L^\tilde{\Phi}_y} < \infty,
$$

then $T_0$ is compact from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Theta}(\mathbb{R}^n)$ and

$$
\|T_0\|_{L^{\Phi} \to L^{\Theta}} \leq \|K_0\|_{L^\Theta(\mathbb{R}^n; L^{\tilde{\Phi}}(\mathbb{R}^n))},
$$

(6.2)

where $\| \cdot \|_{L^{\Phi} \to L^{\Theta}}$ is the operator norm from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Theta}(\mathbb{R}^n)$.

The proof method of the above lemma is known, see for example [14] or [40]. We give the proof for readers’ convenience.

**Proof of Lemma 6.2** By Hölder’s inequality we have

$$
|T_0 f(x)| \leq \int_{\mathbb{R}^n} |K_0(x, y)| |f(y)| \, dy \leq 2 \|K_0(x, y)\|_{L^\tiel{\Phi}_y} \|f\|_{L^{\Phi}}.
$$

Then

$$
\|T_0 f\|_{L^{\Theta}} \leq 2 \left\| K_0(x, y) \right\|_{L^\tiel{\Phi}_y} \| f \|_{L^{\Phi}}.
$$
This shows (6.2). Next we show the compactness. Since $\Theta, \tilde{\Phi} \in \Delta_2$, the simple functions are dense in both $L^\Theta(\mathbb{R}^n)$ and $L^{\tilde{\Phi}}(\mathbb{R}^n)$. Then, for any $\epsilon > 0$, there exists a finite number of bounded measurable sets $E_1, E_2, \ldots, E_k, F_1, F_2, \ldots, F_k$ and $z_1, z_2, \ldots, z_k \in \mathbb{C}$ such that

$$\|K_0 - K_{0,\epsilon}\|_{L^\Theta(\mathbb{R}^n; L^{\tilde{\Phi}}(\mathbb{R}^n))} < \epsilon, \quad K_{0,\epsilon}(x, y) = \sum_{j=1}^k z_j \chi_{E_j}(x) \chi_{F_j}(y).$$

This shows that $T_0$ can be approximated by a finite rank operator $T_{0,\epsilon}$ whose kernel is $K_{0,\epsilon}$. Therefore, $T_0$ is compact.

**Lemma 6.3** Let $\Phi \in \Delta_2$ and $\varphi \in \mathcal{G}_{\text{dec}}$. Then, for any ball $B_0 = B(a_0, r_0)$, the map $T_1 : f \mapsto f \chi_{B_0}$ is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $L^\Phi(\mathbb{R}^n)$.

**Proof** Since $\Phi \in \Delta_2$, there exists a constant $C_{B_0}$ such that

$$\| \tilde{\Phi}^{-1} \phi(x) \|_{L^\varphi} \leq C_{B_0} \| f \|_{L^{(\tilde{\Phi}, \varphi)}} \leq C_{B_0} \| f \|_{L^\Phi(\mathbb{R}^n)}. \quad \square$$

**Lemma 6.4** ([3, Proposition 3.3]) Let $1 \leq q < \infty$ and $\varphi \in \mathcal{G}_{\text{dec}}$. Assume that $\varphi$ satisfies (5.4). Then there exists a Young function $\Phi_{q, \varphi}$ such that

$$L^{\Phi_{q, \varphi}}(\mathbb{R}^n) \subset L^{(q, \varphi)}(\mathbb{R}^n) \quad \text{and} \quad \| f \|_{L^{(q, \varphi)}} \leq C \| f \|_{L^{\Phi_{q, \varphi}}},$$

where $C$ is a positive constant independent of $f \in L^{\Phi_{q, \varphi}}(\mathbb{R}^n)$.

**Remark 6.1** In the above lemma, by [3, Proposition 3.1] and the proof of [3, Proposition 3.3] we also have that, for any ball $B = B(x, r)$,

$$\Phi_{q, \varphi}^{-1}(1/|B|) \sim \varphi(r)^{1/q}. \quad (6.3)$$

In this case, if $\varphi$ satisfies (3.4), then $\Phi_{q, \varphi} \in \Delta_2$. Indeed, from (3.4) and Lemma 5.3 it follows that, for some $\epsilon > 0$, the function $r \mapsto r^\epsilon \varphi(r)^{1/q}$ is also satisfies (3.4), that is, $r \mapsto r^\epsilon \varphi(r)^{1/q}$ is almost decreasing. By (6.3) we have the relation

$$r^\epsilon \varphi(r)^{1/q} \sim \frac{\Phi_{q, \varphi}^{-1}(1/|B|)}{(1/r)^{\epsilon/n}},$$

which shows that $u \mapsto \Phi_{q, \varphi}^{-1}(u)/u^\epsilon/n$ is almost increasing and that $t \mapsto \Phi_{q, \varphi}(t)/t^{n/\epsilon}$ is almost decreasing. Then $\Phi_{q, \varphi} \in \Delta_2$, see Remark 5.1 (v).
Lemma 6.5 Let $\Psi \in \Delta_2$ and $\varphi \in G^{\text{dec}}$. Assume that $\varphi$ satisfies (3.4). Then there exists $\Theta \in \Delta_2$ such that, for any ball $B_0 = B(a_0, r_0)$, the map $T_3 : f \mapsto f \chi_{B_0}$ is bounded from $L^\Theta(\mathbb{R}^n)$ to $L^{(\Psi, \varphi)}(\mathbb{R}^n)$.

**Proof** Since $\Psi \in \Delta_2$, $t \mapsto \Psi(t)/t^q$ is almost decreasing for some $q \in [1, \infty)$. By Lemma 6.4 and Remark 6.1 there exists $\Theta \in \Delta_2$ such that $\|f\|_{L^{(q, \varphi)}} \lesssim \|f\|_{L^{\Theta}}$. In the following, letting $\|f\|_{L^\Theta} = 1$, we will show that $\|f \chi_{B_0}\|_{L^{(\Psi, \varphi)}} \lesssim 1$. By the almost decreasingness of $t \mapsto \Psi(t)/t^q$ we have

$$
\Psi(t) \leq C t^q, \quad \text{if} \quad t \geq 1,
$$

for some $C > 0$. Then, for any ball $B = B(a, r)$,

$$
\frac{1}{|B|} \int_B \Psi(|(f \chi_{B_0})(x)|) \, dx \\
\leq \frac{1}{|B|} \left( C \int_B |(f \chi_{B_0})(x)|^q \, dx + \Psi(1)|B \cap B_0| \right) \\
\leq C \|f\|_{L^{(q, \varphi)}}^q + \Psi(1) \frac{|B \cap B_0|}{|B|},
$$

Here, we have $\|f\|_{L^{(q, \varphi)}} \lesssim \|f\|_{L^\Theta}^p = 1$. If $r \leq r_0$, then, using the almost decreasingness of $\varphi$, we have

$$
\frac{|B \cap B_0|}{|B|} \lesssim \frac{1}{\varphi(r)} \lesssim \frac{1}{\varphi(r_0)}.
$$

If $r > r_0$, then, using the almost increasingness of $t \mapsto t^n \varphi(t)$, we have

$$
\frac{|B \cap B_0|}{|B|} \lesssim \frac{|B \cap B_0|}{|B_0| \varphi(r_0)} \leq \frac{1}{\varphi(r_0)}.
$$

Hence

$$
\frac{1}{|B|} \int_B \Psi(|(f \chi_{B_0})(x)|) \leq C_{B_0},
$$

where the constant $C_{B_0}$ is independent of $B$. This shows the conclusion.

**Proof of Proposition 6.1** Let $B_0$ be a ball in $\mathbb{R}^n$ such that $\text{supp} \ K_0 \subset B_0 \times B_0$. For $\Psi$ and $\varphi$, take a Young function $\Theta \in \Delta_2$ as in Lemma 6.5. Then we see that $\|K_0\|_{L^\Theta(\mathbb{R}^n; L^\Phi(\mathbb{R}^n))} < \infty$, since $\chi_{B_0}$ is in both $L^\Phi(\mathbb{R}^n)$ and $L^\Theta(\mathbb{R}^n)$. Then $T_3 : L^\Phi(\mathbb{R}^n) \to L^{(\Psi, \varphi)}(\mathbb{R}^n)$ can be factorized as

$$
T_3 : L^\Phi(\mathbb{R}^n) \to L^{\Theta}(\mathbb{R}^n) \to L^{(\Psi, \varphi)}(\mathbb{R}^n),
$$

where

$$
T_1 : f \mapsto \chi_{B_0} f, \quad T_2 : f \mapsto T_0 f, \quad T_3 : f \mapsto \chi_{B_0} f,
$$

since

$$
T_0 f(x) = \chi_{B_0}(x) \int_{\mathbb{R}^n} K(x, y) \chi_{B_0}(y) f(y) \, dy, \quad x \in \mathbb{R}^n.
$$
The operator $T_2$ is compact by Lemma 6.2. The operators $T_1$ and $T_3$ are bounded by Lemmas 6.3 and 6.5, respectively. Then $T_0 = T_3T_2T_1$ is compact. □

7 Proofs of Main Results: Sufficiency

In this section we prove Theorem 4.1 (i) and Theorem 4.2 (i). To do this we first prepare the generalized fractional maximal operator $M_\rho$ and several lemmas in Sect. 7.1. Then we give proofs of Theorem 4.1 (i) and Theorem 4.2 (i) in Subsections 7.2 and 7.3, respectively.

7.1 Generalized Fractional Maximal Operator and Lemmas

For a function $\rho: (0, \infty) \to (0, \infty)$, let

$$M_\rho f(x) = \sup_{B(a, r) \ni x} \rho(r) \int_{B(a, r)} |f(y)| \, dy,$$

where the supremum is taken over all balls $B(a, r)$ containing $x$. If $\rho(r) = r^\alpha$, $0 < \alpha < n$, then $M_\rho$ is the usual fractional maximal operator $M_\alpha$. If $\rho \equiv 1$, then $M_\rho$ is the Hardy–Littlewood maximal operator $M$. Then the following boundedness of $M_\rho$ is known.

**Theorem 7.1** ([43],[16]) Let $\Phi, \Psi \in \Phi_Y, \varphi \in G^{\text{dec}}$ and $\rho: (0, \infty) \to (0, \infty)$. Assume that $\lim_{r \to \infty} \varphi(r) = 0$ or that $\Psi^{-1}(t)/\Phi^{-1}(t)$ is almost decreasing on $(0, \infty)$. If there exists a positive constant $A$ such that, for all $r \in (0, \infty)$,

$$\left( \sup_{0 < t \leq r} \rho(t) \right) \Phi^{-1}(\varphi(r)) \leq A \Psi^{-1}(\varphi(r)), \quad (7.1)$$

then, $M_\rho$ is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $wL^{(\Psi, \varphi)}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then $M_\rho$ is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $L^{(\Psi, \varphi)}(\mathbb{R}^n)$.

Next, we state known lemmas.

**Lemma 7.2** ([3, Lemma 5.3]) If $\psi$ satisfies (4.1), then there exist constants $\theta \in (0, 1)$ and $C \in [1, \infty)$ such that, for all $b \in C^{\text{comp}}_\infty(\mathbb{R}^n)$ and all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \leq C \|\nabla b\|_{L^\infty} |x - y|^\theta \psi(|x - y|), \quad \text{if} \quad |x - y| < 1.$$

**Lemma 7.3** ([3, Lemma 5.5]) Let $\theta \in (0, 1)$. Assume that $\psi$ satisfies the doubling condition (2.13). Let $M_\psi$ be the generalized fractional maximal operator with $\psi$. Then there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $\epsilon \in (0, 1]$,

$$\int_{B(x, \epsilon)} \frac{\psi(|x - y|)|f(y)|}{|x - y|^{n-\theta}} \, dy \leq C \epsilon^\theta M_\psi f(x).$$
Lemma 7.4 ([3,Lemma 5.7]) Let $\theta \in (0, 1]$. Assume that $\psi$ and $\rho^*$ satisfy the doubling condition (2.13). Let $M_{\psi\rho^*}$ be the generalized fractional maximal operator with $\psi\rho^*$. Then there exists a positive constant $C$ such that, for all $x \in \mathbb{R}^n$ and $\epsilon \in (0, 1)$,

$$
\int_{B(x, \epsilon)} \frac{\psi(|x-y|)\rho^*(|x-y|)|f(y)|}{|x-y|^{n-\theta}} \, dy \leq C\epsilon^\theta M_{\psi\rho^*}f(x).
$$

7.2 Proof of Theorem 4.1 (i)

Now we prove Theorem 4.1 (i). For $0 < \epsilon < R < \infty$, let

$$
T_\epsilon f(x) = \int_{|x-y| \geq \epsilon} K(x, y) f(y) \, dy, \quad T_{\epsilon, R} f(x) = \int_{\epsilon \leq |x-y| < R} K(x, y) f(y) \, dy.
$$

From (4.3) and [43,Remark 7.2] it follows that

$$
[b, T] f(x) = \lim_{\epsilon \to +0} [b, T_\epsilon] f(x), \quad [b, T_\epsilon] f(x) = \lim_{R \to \infty} [b, T_{\epsilon, R}] f(x) \quad \text{a.e.} \ x \in \mathbb{R}^n,
$$

for all $f \in L^{(\Phi, \varphi)}(\mathbb{R}^n)$. Since $T_{\epsilon, R}$ is compact by Proposition 6.1, it is enough to show the following proposition to prove Theorem 4.1 (i).

Proposition 7.5 Under the assumption in Theorem 4.1, we have

(i) $\lim_{\epsilon \to +0} \|[b, T_\epsilon] - [b, T]\|_{L^{(\Phi, \varphi)} \to L^{(\Psi, \varphi)}} = 0$,

(ii) $\lim_{R \to \infty} \|[b, T_{\epsilon, R}] - [b, T_\epsilon]\|_{L^{(\Phi, \varphi)} \to L^{(\Psi, \varphi)}} = 0$,

where $\| \cdot \|_{L^{(\Phi, \varphi)} \to L^{(\Psi, \varphi)}}$ is the operator norm from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $L^{(\Psi, \varphi)}(\mathbb{R}^n)$.

Proof (i) Let $f \in L^{(\Phi, \varphi)}(\mathbb{R}^n)$ and $\epsilon \in (0, 1]$. Then, from (4.3) it follows that

$$
[b, T] f(x) - [b, T_\epsilon] f(x) = \lim_{\eta \to 0} \int_{\eta \leq |x-y| < \epsilon} \frac{(b(x) - b(y))}{|x-y|^n} f(y) \, dy, \quad \text{a.e.} \ x.
$$

By Lemmas 7.2 and 7.3 we have

$$
\int_{B(x, \epsilon)} \frac{|b(x) - b(y)|}{|x-y|^n} |f(y)| \, dy \lesssim \int_{B(x, \epsilon)} \frac{\psi(|x-y|)}{|x-y|^{n-\theta}} |f(y)| \, dy \lesssim \epsilon^\theta M_{\psi} f(x),
$$

for some $\theta \in (0, 1]$. Hence, by Theorem 7.1 with the assumption (3.7) we have

$$
\|[b, T] f - [b, T_\epsilon] f\|_{L^{(\Phi, \varphi)}} \lesssim \epsilon^\theta \|M_{\psi} f\|_{L^{(\Phi, \varphi)}} \lesssim \epsilon^\theta \|f\|_{L^{(\Psi, \varphi)}}.
$$

This shows (i).
(ii) Let supp \( b \subset B_0 = B(0, R_0) \). Then

\[
[b, T_{e}]f(x) - [b, T_{e,R}]f(x) \lesssim \int_{|x-y| > R} \frac{|b(x) - b(y)|}{|x-y|^n} |f(y)| \, dy
\]

\[
\lesssim \int_{|x-y| > R} (\chi_{B_0}(x) + \chi_{B_0}(y)) \left( \int_{|x-y|}^\infty \frac{1}{t^{n+1}} \, dt \right) |f(y)| \, dy
\]

\[
= \int_0^\infty \int_{\mathbb{R}^n} \chi_{(R-|x-y| < t)}(y,t)(\chi_{B_0}(x) + \chi_{B_0}(y))
\]

\[
\frac{1}{t^{n+1}} |f(y)| \, dy \, dt
\]

\[
\lesssim \int_R^\infty \left( \int_{B(x,t)} (\chi_{B_0}(x) + \chi_{B_0}(y)) |f(y)| \, dy \right) \frac{1}{t^{n+1}} \, dt.
\]

Let

\[
E_1(x) = \int_R^\infty \left( \int_{B(x,t)} \chi_{B_0}(x) |f(y)| \, dy \right) \frac{1}{t^{n+1}} \, dt,
\]

\[
E_2(x) = \int_R^\infty \left( \int_{B(x,t)} \chi_{B_0}(y) |f(y)| \, dy \right) \frac{1}{t^{n+1}} \, dt.
\]

Then

\[
[b, T_{e}]f(x) - [b, T_{e,R}]f(x) \lesssim E_1(x) + E_2(x).
\]

By Lemmas 5.2 and 5.4 we have

\[
E_1(x) \lesssim \chi_{B_0}(x) \int_R^\infty \left( \Phi^{-1}(\varphi(t)|t^n \| f \|_{L(\Phi,\varphi)} \right) \frac{1}{t^{n+1}} \, dt
\]

\[
\lesssim \chi_{B_0}(x) \Phi^{-1}(\varphi(R)) \| f \|_{L(\Phi,\varphi)}.
\]

Then

\[
\| E_1 \|_{L(\psi,\varphi)} \lesssim \| \chi_{B_0} \|_{L(\psi,\varphi)} \Phi^{-1}(\varphi(R)) \| f \|_{L(\Phi,\varphi)} \to 0 \quad \text{as} \quad R \to \infty,
\]

since \( \varphi(r) \to 0 \) as \( r \to \infty \) and \( \Phi^{-1}(t) \to 0 \) as \( t \to 0 \), see Remark 5.2 and Remark 5.1 (ii), respectively. Next, by Lemmas 5.1 and 5.2 we obtain

\[
\| E_2 \|_{L(\psi,\varphi)} \lesssim \int_R^\infty \| \chi_{B(y,t)}(y) \|_{L(\psi,\varphi)} \chi_{B_0}(y) |f(y)| \, dy \frac{1}{t^{n+1}} \, dt
\]

\[
\lesssim \int_R^\infty \sup_{y \in B_0} \| \chi_{B(y,t)} \|_{L(\psi,\varphi)} \left( \int_{\mathbb{R}^n} \chi_{B_0}(y) |f(y)| \, dy \right) \frac{1}{t^{n+1}} \, dt
\]

\[
\lesssim \int_R^\infty \frac{1}{\psi^{-1}(\varphi(t))t^{n+1}} \, dt \, \Phi^{-1}(\varphi(R)) \| B_0 \| \| f \|_{L(\Phi,\varphi)}.
\]

By Remark 5.1 (vi) we see that \( u \mapsto u^{1/p} / \Psi^{-1}(u) \) is almost increasing for some \( p \in (1, \infty) \), which implies the inequality \( \Psi^{-1}(ut^n) \lesssim t^{n/p} \Psi(u) \) if \( t \geq 1 \). By this
inequality and the almost increasingness of $r \mapsto \varphi(r)r^n$ we have
\[
\int_R^\infty \frac{1}{\Psi^{-1}(\varphi(t)) t^{n+1}} dt \lesssim \int_R^\infty \frac{1}{\Psi^{-1}(\varphi(t)) t^{n-n/p+1}} dt \\
\lesssim \frac{1}{\Psi^{-1}(\varphi(R) R^n)} \int_R^\infty \frac{1}{t^{n-n/p+1}} dt \\
\sim \frac{1}{\Psi^{-1}(\varphi(R) R^n) R^{n-n/p}} \to 0 \text{ as } R \to \infty.
\]
Therefore, we have (ii).

\[\square\]

### 7.3 Proof of Theorem 4.2 (i)

Next we prove Theorem 4.2 (i). For $0 < \epsilon < R < \infty$, let

\[
I_{\rho,\epsilon} f(x) = \int_{|x-y| \geq \epsilon} \frac{\rho(|x-y|)}{|x-y|^n} f(y) \, dy, \\
I_{\rho,\epsilon,R} f(x) = \int_{|x-y| < R} \frac{\rho(|x-y|)}{|x-y|^n} f(y) \, dy.
\]

From (4.4) and [43, Remark 7.3] it follows that

\[
[b, I_{\rho}] f(x) = \lim_{\epsilon \to 0} [b, I_{\rho,\epsilon}] f(x), \quad [b, I_{\rho,\epsilon}] f(x) = \lim_{R \to \infty} [b, I_{\rho,\epsilon,R}] f(x) \quad \text{a.e. } x \in \mathbb{R}^n,
\]

for all $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$. Since $I_{\rho,\epsilon,R}$ is compact by Proposition 6.1, it is enough to show the following proposition to prove Theorem 4.2 (i).

**Proposition 7.6** Under the assumption in Theorem 4.2, we have

(i) $\lim_{\epsilon \to 0} \| [b, I_{\rho,\epsilon}] - [b, I_{\rho}] \|_{L^{(\Phi,\varphi)} \to L^{(\Psi,\varphi)}} = 0,$

(ii) $\lim_{R \to \infty} \| [b, I_{\rho,\epsilon,R}] - [b, I_{\rho,\epsilon}] \|_{L^{(\Phi,\varphi)} \to L^{(\Psi,\varphi)}} = 0,$

where $\| \cdot \|_{L^{(\Phi,\varphi)} \to L^{(\Psi,\varphi)}}$ is the operator norm from $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ to $L^{(\Psi,\varphi)}(\mathbb{R}^n)$.

**Proof** (i) Let $f \in L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and $\epsilon \in (0, 1]$. Then, from (4.4) it follows that

\[
[b, I_{\rho}] f(x) - [b, I_{\rho,\epsilon}] f(x) = \lim_{\eta \to 0} \int_{\eta \leq |x-y| < \epsilon} (b(x) - b(y)) \frac{\rho(|x-y|)}{|x-y|^n} f(y) \, dy, \quad \text{a.e. } x.
\]

Let $\rho^*$ be as in (5.5). Then $\rho^*$ is increasing and satisfies the doubling condition by Lemma 5.5 with (3.10). Hence, from (1.6) it follows that $\rho(r) \lesssim \rho^*(r)$. By Lem-
mas 7.2 and 7.4 we have

\[
\int_{B(x, \epsilon)} |b(x) - b(y)| \frac{\rho(|x - y|)}{|x - y|^n} |f(y)| \, dy \\
\lesssim \int_{B(x, \epsilon)} \frac{\psi(|x - y|) \rho^{\ast}(|x - y|)}{|x - y|^{n-\theta}} |f(y)| \, dy \\
\lesssim \epsilon^\theta M_{\psi, \rho} f(x),
\]

for some \( \theta \in (0, 1) \). By (3.12) and (3.13) we see that

\[
\psi(r) \rho^{\ast}(r) \Phi^{-1}(\varphi(r)) \lesssim \Psi^{-1}(\varphi(r)).
\]

Hence, by Theorem 7.1 we have

\[
\|[b, I_\rho] f - [b, I_{\rho, \epsilon}] f\|_{L^p(\varphi, \rho)} \lesssim \epsilon^\theta \|M_{\psi, \rho} f\|_{L^p(\varphi, \rho)} \lesssim \epsilon^\theta \|f\|_{L^p(\varphi, \rho)}.
\]

This shows (i).

(ii) Let \( \text{supp} \ b \subset B_0 = B(0, R_0) \). Using the relation

\[
\frac{\rho(r)}{r^n} \leq \frac{C}{r^n} \int_{K_1 r}^{K_2 r} \frac{\rho(t)}{t} \, dt \sim \int_{K_1 r}^{K_2 r} \frac{\rho(t)}{t^{n+1}} \, dt \leq \int_{K_1 r}^{\infty} \frac{\rho(t)}{t^{n+1}} \, dt,
\]

we have

\[
\|[b, I_{\rho, \epsilon}] f(x) - [b, I_{\rho, \epsilon, R}] f(x)\|
\leq \int_{|x-y| > R} |b(x) - b(y)| \frac{\rho(|x - y|)}{|x - y|^n} |f(y)| \, dy \\
\lesssim \int_{|x-y| > R} (\chi_{B_0}(x) + \chi_{B_0}(y)) \left( \int_{K_1 |x-y|}^{\infty} \frac{\rho(t)}{t^{n+1}} \, dt \right) |f(y)| \, dy \\
= \int_0^{\infty} \int_{R^2} (\chi_{|x-y| < t/K_1})(y) \left( \chi_{B_0}(x) + \chi_{B_0}(y) \right) \frac{\rho(t)}{t^{n+1}} |f(y)| \, dy \, dt \\
\leq \int_{K_1 R}^{\infty} \left( \int_{B(x, t/K_1)} (\chi_{B_0}(x) + \chi_{B_0}(y)) |f(y)| \, dy \right) \frac{\rho(t)}{t^{n+1}} \, dt.
\]

Let

\[
E_1(x) = \int_{K_1 R}^{\infty} \left( \int_{B(x, t/K_1)} \chi_{B_0}(x) \, f(y) \, dy \right) \frac{\rho(t)}{t^{n+1}} \, dt,
\]

\[
E_2(x) = \int_{K_1 R}^{\infty} \left( \int_{B(x, t/K_1)} \chi_{B_0}(y) \, f(y) \, dy \right) \frac{\rho(t)}{t^{n+1}} \, dt.
\]

Then

\[
|[b, I_{\rho, \epsilon}] f(x) - [b, I_{\rho, \epsilon, R}] f(x)| \lesssim E_1(x) + E_2(x).
\]
By the inequality

\[
\int_{B(x, t/K_1)} |f(y)| \, dy \leq 2\Phi^{-1}(\varphi(t/K_1)) |B(x, t/K_1)| \|f\|_{L^p(\Phi, \varphi)}
\]

\[
\sim \Phi^{-1}(\varphi(t)) t^n \|f\|_{L^p(\Phi, \varphi)}
\]

and the assumptions (3.12) and (3.13) we have that, for large $R$,

\[
E_1(x) \lesssim \chi_{B_0}(x) \int_{K_1 R}^{\infty} \frac{\rho(t) \Phi^{-1}(\varphi(t))}{t} \, dt \|f\|_{L^p(\Phi, \varphi)}
\]

\[
\lesssim \chi_{B_0}(x) \Theta^{-1}(\varphi(R)) \|f\|_{L^p(\Phi, \varphi)}
\]

\[
\lesssim \chi_{B_0}(x) \frac{\Psi^{-1}(\varphi(R))}{\psi(R)} \|f\|_{L^p(\Phi, \varphi)}.
\]

Then

\[
\|E_1\|_{L^p(\Psi, \varphi)} \lesssim \|\chi_{B_0}\|_{L^p(\Psi, \varphi)} \frac{\Psi^{-1}(\varphi(R))}{\psi(R)} \|f\|_{L^p(\Phi, \varphi)} \to 0 \quad \text{as} \quad R \to \infty.
\]

Next we estimate $\|E_2\|_{L^p(\Psi, \varphi)}$. From (1.6), (3.12) and (3.13) it follows that

\[
\rho(r) \Phi^{-1}(\varphi(r)) \lesssim \int_{K_1 r}^{K_2 r} \frac{\rho(t) \Phi^{-1}(\varphi(t))}{t} \, dt \lesssim \Theta^{-1}(\varphi(r)) \lesssim \frac{\Psi^{-1}(\varphi(r))}{\psi(r)}, \quad r > 0,
\]

which implies

\[
\|\chi_{B(y, t/K_1)}\|_{L^p(\Psi, \varphi)} \lesssim \frac{1}{\Psi^{-1}(\varphi(t))} \lesssim \frac{1}{\rho(t) \Phi^{-1}(\varphi(t)) \psi(t)}.
\]

Hence

\[
\|E_2\|_{L^p(\Psi, \varphi)} \lesssim \int_{K_1 R}^{\infty} \left( \int_{\mathbb{R}^n} \|\chi_{B(\cdot, t/K_1)}(y)\|_{L^p(\Psi, \varphi)} \chi_{B_0}(y) |f(y)| \, dy \right) \frac{\rho(t)}{t^{n+1}} \, dt
\]

\[
\lesssim \sup_{y \in B_0} \|\chi_{B(y, t/K_1)}\|_{L^p(\Psi, \varphi)} \left( \int_{\mathbb{R}^n} \chi_{B_0}(y) |f(y)| \, dy \right) \frac{\rho(t)}{t^{n+1}} \, dt
\]

\[
\lesssim \sup_{y \in B_0} \|\chi_{B(y, t/K_1)}\|_{L^p(\Psi, \varphi)} \frac{\rho(t)}{t^{n+1}} \, dt \left( \int_{B_0} |f(y)| \, dy \right)
\]

\[
\lesssim \int_{K_1 R}^{\infty} \frac{1}{\Phi^{-1}(\varphi(t)) \psi(t) t^{n+1}} \, dt \Phi^{-1}(\varphi(R_0)) |B_0| \|f\|_{L^p(\Phi, \varphi)}.
\]
By the same reason as in the proof of Proposition 7.5 we have the inequality 
\[ \Phi^{-1}(ut^n) \lesssim t^{n/p} \Phi(u) \] if \( t \geq 1 \). Then we have

\[
\int_R^\infty \frac{1}{\Phi^{-1}(\varphi(t))\psi(t)t^{n+1}} \, dt \lesssim \frac{1}{\Phi^{-1}(\varphi(R)R^n)\psi(R)} \int_R^\infty \frac{1}{t^{n-n/p+1}} \, dt \\
\lesssim \frac{1}{\Phi^{-1}(\varphi(R)R^n)\psi(R)R^{n-n/p}} \to 0 \quad \text{as} \quad R \to \infty.
\]

Therefore, we have (ii). The proof is complete. \( \square \)

8 Proofs of Main Results: Necessity

In this section we prove Theorem 4.1 (ii) and Theorem 4.2 (ii). In Sect. 8.1 we first prepare a known theorem on the characterization of the closure of \( C^\infty_{\text{comp}}(\mathbb{R}^n) \) in \( L_{1,\psi}(\mathbb{R}^n) \). We also prove some lemmas. Then we give proofs of Theorem 4.1 (ii) and Theorem 4.2 (ii) in Sects. 8.2 and 8.3, respectively.

8.1 Characterization of the Closure of \( C^\infty_{\text{comp}}(\mathbb{R}^n) \) in \( L_{1,\psi}(\mathbb{R}^n) \)

For a function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and a ball \( B \), let

\[ \text{MO}(f, B) = \int_B |f(x) - f_B| \, dx. \]

To prove the necessity, we will use the following characterization.

**Theorem 8.1** ([4]) Let \( \psi \) be in \( G^{\text{inc}} \). Let \( f \in L_{1,\psi}(\mathbb{R}^n) \). Then \( f \in C^\infty_{\text{comp}}(\mathbb{R}^n) \) if and only if \( f \) satisfies the following three conditions:

\[
\begin{align*}
(\text{i}) & \quad \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \frac{\text{MO}(f, B(x, r))}{\psi(r)} = 0, \\
(\text{ii}) & \quad \lim_{r \to \infty} \sup_{x \in \mathbb{R}^n} \frac{\text{MO}(f, B(x, r))}{\psi(r)} = 0, \\
(\text{iii}) & \quad \lim_{|x| \to \infty} \text{MO}(f, B(x, r)) = 0 \quad \text{for each} \ r > 0.
\end{align*}
\]

In the following lemma we used the idea in [6] and [4].

**Lemma 8.2** Let \( \Psi, \Psi \in \Phi_Y \) and \( \varphi \in G^{\text{dec}} \). Assume that (3.8) holds. Let \( b \) be a real valued function. For any ball \( B = B(x, r) \), let

\[
f^B(z) = \frac{\Psi^{-1}(\varphi(r))}{\psi(r)} \left( \text{sgn}(b(z) - b_B) - c_0 \right) \chi_B(z),
\]

where \( c_0 = \int_B \text{sgn}(b(z) - b_B) \, dz \). \( (8.1) \)
Then

\[
\text{supp } f_B \subset B, \quad \int_{\mathbb{R}^n} f_B(z) \, dz = 0, \quad (8.2)
\]
\[
f_B(z)(b(z) - b_B) \geq 0, \quad (8.3)
\]
\[
\int_{\mathbb{R}^n} f_B(z)(b(z) - b_B) \, dz = \frac{\Psi^{-1}(\varphi(r))}{\psi(r)} |B| \text{MO}(b, B), \quad (8.4)
\]
\[
\| f_B \|_{L(\Phi, \psi)} \leq C, \quad (8.5)
\]

where \( C \) is a constant dependent only on \( n \) and \( \varphi \).

**Proof** The first assertion (8.2) is clear. Since \( \int_{B} (b(z) - b_B) \, dz = 0 \), it is easy to check \( |c_0| < 1 \). Then we have

\[
f_B(z)(b(z) - b_B) = \frac{\Psi^{-1}(\varphi(r))}{\psi(r)}\left(|b(z) - b_B| - c_0(b(z) - b_B)\right) \chi_B(z) \geq 0
\]

and

\[
\int_{\mathbb{R}^n} f_B(z)(b(z) - b_B) \, dz = \frac{\Psi^{-1}(\varphi(r))}{\psi(r)} \int_{B} \left(|b(z) - b_B| - c_0(b(z) - b_B)\right) \, dz
\]
\[
= \frac{\Psi^{-1}(\varphi(r))}{\psi(r)} \int_{B} |b(z) - b_B| \, dz
\]
\[
= \frac{\Psi^{-1}(\varphi(r))}{\psi(r)} |B| \text{MO}(b, B).
\]

Finally, we show that, for any \( B' = B(x', r') \),

\[
\| f_B \|_{\Phi, \psi, B'} \leq C.
\]

By (3.8) we have

\[
|f_B| \leq \frac{\Psi^{-1}(\varphi(r))}{\psi(r)} \leq C_0 \Phi^{-1}(\varphi(r)).
\]

Then by (2.8) we have

\[
\Phi\left(\frac{|f_B(z)|}{C_0}\right) \leq \varphi(r).
\]

If \( B \cap B' \neq \emptyset \) and \( r' \leq r \), then \( \varphi(r) \lesssim \varphi(r') \) by the almost decreasingness of \( \varphi \). Hence

\[
\frac{1}{\varphi(r')} \int_{B'} \Phi\left(\frac{|f_B(z)|}{C_0}\right) \, dz \leq \frac{\varphi(r)}{\varphi(r')} \lesssim 1.
\]
If \( B \cap B' \neq \emptyset \) and \( r' > r \), then \( \varphi(r)r^n \lesssim \varphi(2r')\langle 2r' \rangle^n \sim \varphi(r')(r')^n \) by the almost increasingness of \( t \mapsto \varphi(t)t^n \) and the doubling condition (2.13). Hence

\[
\frac{1}{\varphi(r')} \int_{B'} \Phi \left( \frac{|f^B(z)|}{C_0} \right) dz = \frac{1}{\varphi(r')|B'|} \int_B \Phi \left( \frac{|f^B(z)|}{C_0} \right) dz \leq \frac{\varphi(r)|B|}{\varphi(r')|B'|} \lesssim 1.
\]

This shows the conclusion. \( \square \)

**Lemma 8.3** Let \( \Phi, \Psi \in \Delta_2 \cap \nabla_2 \), \( \varphi \in \mathcal{G}^{\text{dec}} \) and \( \psi \in \mathcal{G}^{\text{inc}} \). Assume that (3.8) holds. Let \( T \) be a convolution type singular integral operator such that

\[
Tf(x) = p.v. \int_{\mathbb{R}^n} K(x - y) f(y) \, dy \tag{8.6}
\]

with homogeneous kernel \( K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) satisfying \( K(x) = |x|^{-n} K(x/|x|) \), \( \int_{S^{n-1}} K = 0 \), \( K \in C^\infty(S^{n-1}) \) and \( K \neq 0 \). Assume also that \( \Psi(t) \approx t^q \) for some \( q \in (1, \infty) \) and that \( \Psi \) satisfies (4.1). Let \( b \) be a real valued function and \( \|b\|_{L_1,\Psi} = 1 \).

For any ball \( B = B(x, r) \), define \( f^B \) by (8.1). Then, for any constants \( \epsilon_0, \mu_0 \in (0, \infty) \), there exist constants \( v_1, v_2 \in [2, \infty) \) (\( v_1 < v_2 \)), \( v_3 \in (0, \infty) \) and \( v_4 \in (0, 1) \) such that, for all balls \( B = B(x, r) \) satisfying \( \text{MO}(b, B) / \psi(r) \geq \epsilon_0 \), the following three inequalities hold:

\[
\left( \frac{1}{|B|} \int_{v_2 B \setminus v_1 B} |b, T| f^B(y)|^q \, dy \right)^{1/q} \geq v_3 \varphi(r)^{1/q}, \tag{8.7}
\]

\[
\left( \frac{1}{|B|} \int_{\mathbb{R}^n \setminus v_2 B} |b, T| f^B(y)|^q \, dy \right)^{1/q} \leq \frac{v_3}{4 \mu_0} \varphi(r)^{1/q}, \tag{8.8}
\]

and, for any measurable set \( E \subset v_2 B \setminus v_1 B \) satisfying \( |E|/|B| \leq v_4 \),

\[
\left( \frac{1}{|B|} \int_{E} |b, T| f^B(y)|^q \, dy \right)^{1/q} \leq \frac{v_3}{4} \varphi(r)^{1/q}. \tag{8.9}
\]

The Riesz transforms fall under the scope of Lemma 8.3

**Proof** The proof method is almost the same as [4, Lemma 5.5]. The difference is the part \( \varphi(r)^{1/q} \) in the right-hand side in each (8.7), (8.8) and (8.9). More precisely, in [4, Lemma 5.5], \( \varphi(B)^{1/p} \psi(B) \) was used instead of \( \varphi(r)^{1/q} \). We divide the proof into five steps. However, we will omit Steps 3, 4, 5, since they are the same as Steps 3, 4, 5 in [4, Proof of Lemma 5.5]. We will show (8.11), (8.12) and (8.13) in Step 2, in which we use \( \varphi(r)^{1/q} \) instead of \( \varphi(B)^{1/p} \psi(B) \) in [4, (5.11), (5.12) and (5.13)].

**Step 1.** Since \( K \in C^\infty(S^{n-1}) \) and \( K \neq 0 \), we may assume that \( |K(y') - K(z')| \leq |y' - z'| \) for all \( y', z' \in S^{n-1} \) and that

\[
\sigma(\{x' \in S^{n-1} : K(x') \geq 2\epsilon_1\}) > 0.
\]
for some constant $\epsilon_1 \in (0, 1)$, where $\sigma$ is the area measure on $S^{n-1}$. Let

$$\Lambda = \{ x' \in S^{n-1} : K(x') \geq 2\epsilon_1 \}.$$ 

Then

$$y' \in \Lambda, \ z' \in S^{n-1} \text{ and } |y' - z'| \leq \epsilon_1 \Rightarrow K(z') \geq \epsilon_1,$$

(8.10)

since $K(y') \geq 2\epsilon_1$ and $|K(y') - K(z')| \leq |y' - z'| \leq \epsilon_1$. Set $\ell = 2/\epsilon_1 > 2$.

**Step 2.** We may assume that $\Psi(t) = t^q$. Let $B = B(x, r)$ satisfy $\text{MO}(b, B)/\psi(r) \geq \epsilon_0$. We show that

$$|T((b-b_B)f^B)(y)| \geq \frac{\varphi(r)^{1/q}|B|}{(2|y-x|)^n} \epsilon_1$$

for $y \notin \ell B$ and $\frac{y-x}{|y-x|} \in \Lambda$, (8.11)

$$|T((b-b_B)f^B)(y)| \leq 2^n C_K \frac{\varphi(r)^{1/q}|B|}{|y-x|^n}$$

for $y \notin \ell B$, (8.12)

$$|(b(y) - b_B)T(f^B)(y)| \leq C_K \frac{r|b(y) - b_B|\varphi(r)^{1/q}|B|}{\psi(r)|y-x|^{n+1}}$$

for $y \notin \ell B$, (8.13)

where the constant $C_K$ is dependent only on the kernel $K$.

Now, for $y \notin \ell B$ and $z \in B$, we have

$$\left| \frac{y-x}{|y-x|} - \frac{y-z}{|y-z|} \right| \leq \left| \frac{y-x}{|y-x|} - \frac{y-z}{|y-x|} \right| + \left| \frac{y-z}{|y-x|} - \frac{y-z}{|y-z|} \right| \leq \frac{2|z-x|}{|y-x|} \leq \frac{2}{\ell} = \epsilon_1.$$ 

In this case, if $\frac{y-x}{|y-x|} \in \Lambda$ also, then $K(\frac{y-z}{|y-z|}) \geq \epsilon_1$ by (8.10), and then

$$K(y - z) \geq \epsilon_1 = \frac{\epsilon_1}{|y - z|^n} \geq \frac{\epsilon_1}{(2|y - x|)^n}.$$

Hence, from (8.3) and (8.4) it follows that, for $y \notin \ell B$ and $\frac{y-x}{|y-x|} \in \Lambda$,

$$|T((b-b_B)f^B)(y)| = \int_B K(y - z)(b(z) - b_B)f^B(z) \, dz \geq \frac{\varphi(r)^{1/q}|B|\text{MO}(b, B)}{\psi(r)(2|y-x|)^n} \epsilon_1,$$

which shows (8.11), since $\text{MO}(b, B) \geq \psi(r)\epsilon_0$. On the other hand, for $y \notin \ell B$ and $z \in B$, we have

$$|K(y - z)| \leq \frac{C_K}{|y - z|^n} \leq \frac{2^n C_K}{|y - x|^n}.$$

Then, from (8.3) and (8.4) it follows that, for $y \notin \ell B$,

$$|T((b-b_B)f^B)(y)| \leq 2^n C_K \frac{\varphi(r)^{1/q}|B|\text{MO}(b, B)}{\psi(r)|y-x|^n}.$$
which shows (8.12), since \( \|b\|_{\mathcal{L}_{1, \psi}} = 1 \). Finally, from (8.2) and (8.5) it follows that, for \( y \notin tB \),

\[
| (b(y) - b_B) T(f^B)(y) | = \left| (b(y) - b_B) \int_B (K(y - z) f^B(z) - K(y - x) f^B(z)) \, dz \right|
\leq |b(y) - b_B| \int_B \frac{C_K |z - x|}{|y - x|^{n+1}} |f^B(z)| \, dz
\leq C_K \frac{r |b(y) - b_B| |\varphi(r)^{1/q}| |B|}{|y - x|^{n+1}},
\]

which is (8.13). \( \square \)

**Lemma 8.4** Let \( \Phi, \Psi \in \Delta_2 \cap \nabla_2, \ \varphi \in \mathcal{G}^{\text{dec}}, \ \psi \in \mathcal{G}^{\text{inc}} \) and \( \alpha \in (0, n) \). Assume that \( \Psi(t) \approx t^q \) for some \( q \in (1, \infty) \) and that \( n - \alpha - n/q > 0 \). Assume also that \( \psi \) satisfies (4.1). Let \( b \) be a real valued function and \( \|b\|_{\mathcal{L}_{1, \psi}} = 1 \). For any ball \( B \), define \( f^B \) by (8.1). Then, for any constants \( \epsilon_0, \mu_0 \in (0, \infty) \), there exist constants \( v_1, v_2 \in [2, \infty) \) \( (v_1 < v_2) \), \( v_3 \in (0, \infty) \) and \( v_4 \in (0, 1) \) such that, for all balls \( B \) satisfying \( \text{MO}(b, B)/\psi(r) \geq \epsilon_0 \), the following three inequalities hold:

\[
\left( \frac{1}{|B|} \int_{v_2 B \setminus v_1 B} |[b, I_\alpha] f^B(y)|^q \, dy \right)^{1/q} \geq v_3 \varphi(r)^{1/q} |B|^\alpha/n, \quad (8.14)
\]

\[
\left( \frac{1}{|B|} \int_{\mathbb{R}^n \setminus v_2 B} |[b, I_\alpha] f^B(y)|^q \, dy \right)^{1/q} \leq \frac{v_3}{4 \mu_0} \varphi(r)^{1/q} |B|^\alpha/n, \quad (8.15)
\]

and, for any measurable set \( E \subset v_2 B \setminus v_1 B \) satisfying \( |E|/|B| \leq v_4 \),

\[
\left( \frac{1}{|B|} \int_E |[b, I_\alpha] f^B(y)|^q \, dy \right)^{1/q} \leq \frac{v_3}{4} \varphi(r)^{1/q} |B|^\alpha/n. \quad (8.16)
\]

**Proof** The proof method is almost the same as [4, Lemma 5.6]. The difference is the part \( \varphi(r)^{1/q} \) in the right-hand side in each (8.14), (8.15) and (8.16). More precisely, in [4, Lemma 5.6], \( \varphi(B)^{1/q} \psi(B) \) was used instead of \( \varphi(r)^{1/q} \). Therefore, as a similar way to the proof of Lemma 8.3, we will show (8.17), (8.18) and (8.19) below and omit the rest.

We may assume that \( \Psi(t) = t^q \). Let \( B = B(x, r) \) satisfy \( \text{MO}(b, B)/\psi(r) \geq \epsilon_0 \). For \( y \notin 2B \) and \( z \in B \), we have

\[
\frac{1}{(2|y - x|)^{n-\alpha}} \leq \frac{1}{|y - z|^{n-\alpha}} \leq \frac{1}{(|y - x|/2)^{n-\alpha}}.
\]
From (8.3), (8.4), \( \|b\|_{L_{1,\psi}} = 1 \) and \( \text{MO}(b, B) \geq \psi(r)\epsilon_0 \) it follows that, for \( y \notin 2B \),

\[
|I_{\alpha}((b - b_B)f^B)(y)| = \int_B \frac{(b(z) - b_B)f^B(z)}{|y - z|^{n-\alpha}} \, dz \geq \frac{\varphi(r)^{1/q}|B|}{(2|y - x|)^{n-\alpha}} \epsilon_0, \tag{8.17}
\]

\[
|I_{\alpha}((b - b_B)f^B)(y)| = \int_B \frac{(b(z) - b_B)f^B(z)}{|y - z|^{n-\alpha}} \, dz \leq \frac{\varphi(r)^{1/q}|B|}{(|y - x|/2)^{n-\alpha}}. \tag{8.18}
\]

From (8.2) and (8.5) it follows that, for \( y \notin 2B \),

\[
|(b(y) - b_B)I_{\alpha}(f^B)(y)| = \left| (b(y) - b_B) \int_B \frac{f^B(z)}{|y - z|^{n-\alpha}} \, dz \right|
\]

\[
= \left| (b(y) - b_B) \int_B \left( \frac{f^B(z)}{|y - z|^{n-\alpha}} - \frac{f^B(z)}{|y - x|^{n-\alpha}} \right) \, dz \right|
\]

\[
\leq \frac{r|b(y) - b_B|}{(n-\alpha)(|y - x|/2)^{n-\alpha+1}} \int_B |f^B(z)| \, dz
\]

\[
\leq \frac{r|b(y) - b_B|\varphi(r)^{1/p}|B|}{(n-\alpha)\psi(r)(|y - x|/2)^{n-\alpha+1}}. \tag{8.19}
\]

Then we can show (8.14), (8.15) and (8.16).

\[\Box\]

### 8.2 Proof of Theorem 4.1 (ii)

Firstly, since \([b, T]\) is compact from \(L^{(\Phi,\varphi)}(\mathbb{R}^n)\) to \(L^{(\Psi,\varphi)}(\mathbb{R}^n)\), we see that \(b \in L_{1,\psi}(\mathbb{R}^n)\) by Theorem 3.1 (ii). We may assume that \(\|b\|_{L_{1,\psi}} = 1\). In the following we show that \(b\) must satisfy the conditions (i), (ii) and (iii) in Theorem 8.1.

**Part 1.** Firstly, we show that, if \(b\) does not satisfy the condition (i), then \([b, T]\) is not compact. Since \(b\) does not satisfy the condition (i), there exist \(\epsilon_0 > 0\) and a sequence of balls \(\{B_j\}_{j=1}^{\infty} = \{B(x_j, r_j)\}_{j=1}^{\infty}\) with \(\lim_{j \to \infty} r_j = 0\) such that, for every \(j\),

\[
\frac{\text{MO}(b, B_j)}{\psi(r_j)} > \epsilon_0. \tag{8.20}
\]

For every \(B_j\), we define \(f_j = f^{B_j}\) by (8.1). Then \(\sup_j \|f_j\|_{L^{(\Phi,\varphi)}} \leq C\) by Lemma 8.2. If we can choose a subsequence \(\{f_{j(k)}\}_{k=1}^{\infty}\) such that \(\{[b, T]f_{j(k)}\}_{k=1}^{\infty}\) has no any convergence subsequence in \(L^{(\Psi,\varphi)}(\mathbb{R}^n)\), then we have the conclusion.

Now, let \(\epsilon_0\) be the constant in (8.20). We can also take \(\mu_0 \in (0, \infty)\) such that

\[
\varphi(r)r^n \leq (\mu_0)^q\varphi(s)s^n, \quad \text{if} \quad r \leq s, \tag{8.21}
\]

since \(r \mapsto \varphi(r)r^n\) is almost increasing. For the constants \(\epsilon_0\) and \(\mu_0\), let \(v_i (i = 1, 2, 3, 4)\) be the constants defined by Lemma 8.3. By \(\lim_{j \to \infty} r_j = 0\) we may choose a
subsequence \( \{B_{j(k)}\} \) such that
\[
\frac{|B_{j(k+1)}|}{|B_{j(k)}|} < \frac{\nu_4}{\nu_2^n}. \tag{8.22}
\]

Then the subsequence \( \{f_{j(k)}\} \) associated with \( \{B_{j(k)}\} \) is just what we request. Namely, there exists a positive constant \( \delta \) such that, for any \( k, \ell \in \mathbb{N} \) with \( k < \ell \),
\[
\|[b, T] f_{j(k)} - [b, T] f_{j(\ell)}\|_{L(q, \varphi)} \sim \|[b, T] f_{j(k)} - [b, T] f_{j(\ell)}\|_{L(q, \varphi)} \geq \delta. \tag{8.23}
\]

In the following we will show (8.23). For fixed \( k, \ell \in \mathbb{N} \) with \( k < \ell \), denote
\[
G = v_2 B_{j(k)} \setminus v_1 B_{j(k)}, \quad E = G \cap v_2 B_{j(\ell)}.
\]

Then by (8.22) we have
\[
\frac{|E|}{|B_{j(k)}|} \leq \frac{|v_2 B_{j(\ell)}|}{|B_{j(k)}|} < \nu_4.
\]

From the relation \( G \setminus E = G \setminus v_2 B_{j(\ell)} \subset v_2 B_{j(k)} \cap (v_2 B_{j(\ell)})^c \) it follows that
\[
\left( \int_G \|[b, T] f_{j(k)}\|^q \, dx - \int_E \|[b, T] f_{j(k)}\|^q \, dx \right)^{\frac{1}{q}} = \left( \int_G \|[b, T] f_{j(k)}\|^q \, dx \right)^{\frac{1}{q}} \leq \left( \int_{v_2 B_{j(k)}} \|[b, T] f_{j(k)}\|^q \, dx \right)^{\frac{1}{q}} + \left( \int_{(v_2 B_{j(\ell)})^c} \|[b, T] f_{j(\ell)}\|^q \, dx \right)^{\frac{1}{q}}. \tag{8.24}
\]

By (8.7), (8.8), (8.9) and (8.21) we have
\[
\int_G \|[b, T] f_{j(k)}\|^q \, dx \geq \nu_3^q \varphi(r_{j(k)})|B_{j(k)}|, \tag{8.25}
\]
\[
\left( \int_{(v_2 B_{j(\ell)})^c} \|[b, T] f_{j(\ell)}\|^q \, dx \right)^{\frac{1}{q}} \leq \frac{\nu_3}{4\mu_0} \varphi(r_{j(\ell)})^{1/q} |B_{j(\ell)}|^{1/q} \tag{8.26}
\]
\[
\quad \leq \frac{\nu_3}{4} \varphi(r_{j(k)})^{1/q} |B_{j(k)}|^{1/q},
\]
\[
\int_E \|[b, T] f_{j(k)}\|^q \, dx \leq \left( \frac{\nu_3}{4} \right)^q \varphi(r_{j(k)}) |B_{j(k)}|. \tag{8.27}
\]

Combining (8.24)–(8.27), we have
\[
\left( \nu_3^q - \left( \frac{\nu_3}{4} \right)^q \varphi(r_{j(k)})^{1/q} |B_{j(k)}|^{1/q} \right) \left( \int_{v_2 B_{j(k)}} |[b, T] f_{j(k)} - [b, T] f_{j(\ell)}\|^q \, dx \right)^{\frac{1}{q}} \leq \frac{\nu_3}{4} \varphi(r_{j(k)})^{1/q} |B_{j(k)}|^{1/q}.
\]
which shows

\[
\delta_0 \varphi(r_{j(k)})^{1/q} |B_{j(k)}|^{1/q} \leq \left( \int_{v_2B_{j(k)}} |[b, T]f_{j(k)} - [b, T]f_{j(\ell)}|^q \, dx \right)^{\frac{1}{q}},
\]

where \( \delta_0 = \left( v_3^q - \left( \frac{v_3}{4} \right)^q \right)^{1/q} - \frac{v_3}{4} > 0 \). Thus, using the almost decreasingness of \( \varphi \), we have

\[
\left( \frac{1}{\varphi(v_2r_{j(k)})} \int_{v_2B_{j(k)}} |[b, T]f_{j(k)} - [b, T]f_{j(\ell)}|^q \, dx \right)^{\frac{1}{q}} \geq \delta,
\]

where \( \delta \) is independent on \( m \) and \( \ell \), which shows (8.23).

**Part 2.** Secondly, we show that, if \( b \) does not satisfy the condition (ii), then \([b, T]\) is not compact. Since \( b \) does not satisfy the condition (ii), there exist \( \epsilon_0 > 0 \) and a sequence of balls \( \{B_j\}_{j=1}^{\infty} = \{B(x_j, r_j)\}_{j=1}^{\infty} \) with \( \lim_{j \to \infty} r_j = \infty \) such that, for every \( j \),

\[
\frac{\text{MO}(b, B_j)}{\varphi(r_j)} > \epsilon_0.
\]

For every \( B_j \), we define \( f_j = f^{B_j} \) by (8.1). Then \( \sup_j \|f_j\|_{L^q(\varphi, \psi)} \leq C \) by Lemma 8.2. Let \( \mu_0 \) be as in (8.21). By \( \lim_{j \to 0} r_j = \infty \) we can choose a subsequence \( \{B_{j(k)}\}_{k=1}^{\infty} \) such that

\[
\frac{|B_{j(k)}|}{|B_{j(k+1)}|} < \frac{\nu_4}{\nu_2^{\alpha}}.
\]

Then, in a similar way to Part 1 we conclude that there exists a positive constant \( \delta \) such that, for all \( k, \ell \in \mathbb{N} \) with \( k < \ell \),

\[
\left( \frac{1}{\varphi(v_2r_{j(\ell)})} \int_{v_2B_{j(\ell)}} |[b, T]f_{j(\ell)} - [b, T]f_{j(k)}|^q \, dx \right)^{\frac{1}{q}} \geq \delta.
\]

That is, \([b, T]\) is not compact.

**Part 3.** Finally, we show that, if \( b \) does not satisfy the condition (iii), then \([b, T]\) is not compact. Since \( b \) does not satisfy the condition (iii), there exist \( \epsilon_0 > 0 \) and a sequence of balls \( \{B_j\}_{j=1}^{\infty} = \{B(x_j, r_j)\}_{j=1}^{\infty} \) with \( \lim_{j \to \infty} |x_j| = \infty \) such that, for every \( j \),

\[
\frac{\text{MO}(b, B_j)}{\varphi(r_j)} > \epsilon_0.
\]

Let \( \mu_0 = 1 \). By \( \lim_{j \to 0} |x_j| = \infty \) we may choose a subsequence \( \{B_{j(k)}\}_{k=1}^{\infty} \) such that \( \nu_2B_{j(k)} \cap \nu_2B_{j(\ell)} = \emptyset \) if \( k < \ell \). Then, in a similar way to Part 1 we conclude that \([b, T]\) is not compact.
8.3 Proof of Theorem 4.2 (ii)

We omit the proof of Theorem 4.2 (ii), since we can prove it in the same way as Theorem 4.1 (ii) by using Lemma 8.4 instead of Lemma 8.3.

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