CURVES WITH MORE THAN ONE INNER GALOIS POINT

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Abstract. Let $C$ be an irreducible plane curve in $P^2(k)$ where $k$ an algebraically closed field of characteristic $p \geq 0$. A point $Q \in C$ is an inner Galois point for $C$ if the projection $\pi_Q$ from $Q$ is of Galois. Assume that $C$ has two different inner Galois points $P_1$ and $P_2$, both simple, or more generally unibranch. Let $G_1$ and $G_2$ be the respective Galois groups. Under the assumption that $G_i$ fixes $P_i$, for $i = 1, 2$, we give a complete classification for $G_1, G_2$ and $G = \langle G_1, G_2 \rangle$ and show that each case occurs. Our proofs relies on deeper results from Group theory.

1. Introduction

In this paper, $X$ stands for a (projective, geometrically irreducible, non-singular) algebraic curve defined over an algebraically closed field $k$ of characteristic $p \geq 0$. Also, $C$ stands for a plane model of $X$, that is, for a plane curve $C$ defined over $k$ and birationally equivalent to $X$. Let $\phi$ be a morphism $X \mapsto P^2(k)$ which realizes it, so that $\phi$ is birational onto its image $C$. Further, $\mathcal{K}(X)$ denotes the function field of $X$, and $\text{Aut}(X)$ stands for the automorphism group of $X$ which fixes $k$ element-wise. A point $Q$ in $P^2(k)$ is a Galois point for $C$ if the projection $\pi_Q$ from $Q$ is Galois; more precisely, if the field extension $k(X)/\pi_Q^*(k(P^1(k)))$ is Galois. A Galois point $Q$ is either inner or outer according as $Q \in C$ or $Q \in P^2(k) \setminus C$. An inner Galois point may be a singular point of $C$.

The concept of a Galois point is due to H. Yoshihara and dates back to late 1990s; see [27]. Ever since, several papers have been dedicated to studies on Galois points, especially on the number of Galois points of a given plane model. For non-singular plane models, that number is already known [15, 27]. Nevertheless, for plane models with singularities the picture is much more involved, as it emerges from several recent papers [3, 4, 8, 6, 5, 11, 9, 10, 14, 28] where the authors focused on the problem of determining plane models with at least two Galois points.

In this context, our paper is about plane models $C$ with two different inner Galois points $\varphi(P_1)$ and $\varphi(P_2)$ both simple, or more generally unibranch. Here $C$ is unibranch at its point $Q$ if $\varphi(P) = \varphi(R) = Q$ implies $P = R$. Let $\varphi(P_1), \varphi(P_2) \in C$ be two different unibranch inner points with Galois groups $G_1$ and $G_2$ respectively. Then the following properties hold; see [6], and Lemma [2, 10].

(I) The quotient curves $X/G_1$ and $X/G_2$ are rational.

(II) $G_1$ and $G_2$ have trivial intersection.

(III) In the divisor group of $X$, $P_1 + \sum_{\sigma \in G_1} \sigma(P_2) = P_2 + \sum_{\tau \in G_2} \tau(P_1)$.

From now on we assume that $G_i$ fixes $P_i$, for $i = 1, 2$.

Let $\Omega$ denote the support of the divisor in (III). Then $|\Omega| = |G_i| + 1$, and (III) shows that the action of $G_i$ on $\Omega \setminus \{P_i\}$ is sharply transitive. Therefore, the subgroup $G$ of $\text{Aut}(X)$, generated by $G_1$ and $G_2$, acts 2-transitively on $\Omega$. It should be noticed that some non-trivial element of $G$ may fix $\Omega$ pointwise. In other
words, the kernel $K$ of the permutation representation $\bar{G}$ of $G$ on $\Omega$ may be non-trivial so that $\bar{G} = G/K$ is the 2-transitive permutation group induced by $G$ on $\Omega$. Since all 2-transitive permutation groups are known, as a consequence of the classification of finite simple groups, this gives a chance to determine the possibilities for $\bar{G}$. Unfortunately, the amount of the work needed to deal with all 2-transitive groups appears to be huge and rather complicated. Also, recovering $G$ from $\bar{G}$ may be very demanding although Schur multiplier could give an adequate instrument to deal with. However, important simplifications may be possible whenever some really useful group-theoretic constraints on the structure of $G$ derives from the geometric constraint $G \leq \text{Aut}(\mathcal{X})$. Our main contribution in this direction is to show that $G_1$ is a normal subgroup of the stabilizer of $P_1$ in $G$. A natural idea is to regard $G$ as a 2-transitive group space on $\Omega$ where $G_1$ is a normal subgroup of a 1-point stabilizer of $G$ and $G_1$ is sharply transitive on the remaining points of $\Omega$. Such 2-transitive group spaces were completely determined by Hering; see Result \ref{result:k}. It should be noticed that Hering’s result is independent of the classification of all 2-transitive permutation groups. It is based on the famous Hering-Kantor-Seitz theorem on finite groups with a split BN-pair of rank 1 depending in turn on deep results due to Alperin, Gorenstein, Suzuki and Walter about 2-subgroups in finite simple groups. It turns out that Hering’s result provides a complete list of possibilities for $G$ and its action on $\Omega$. The question of which of these possibilities actually occur for some curve $\mathcal{X}$ is completely answered in our main theorem.

**Theorem 1.1.** Let $\mathcal{C}$ be a plane model of $\mathcal{X}$ associated with the morphism $\varphi : \mathcal{X} \mapsto \text{PG}(2, K)$. Let $P_1, P_2 \in \mathcal{X}$ be two distinct points together with two distinct subgroups $G_1, G_2$ of $\text{Aut}(\mathcal{X})$ such that $\varphi(P_1)$ and $\varphi(P_2)$ are unibranch Galois points of $\mathcal{C}$ with Galois groups $G_1$ and $G_2$, respectively. If $G_i$ fixes $P_i$ for $i = 1, 2$ then $G = (G_1, G_2)$ is isomorphic to one of the following groups:

1. $\text{PSL}(2, q), \text{SL}(2, q), \text{Sz}(q), \text{PSU}(3, q), \text{SU}(3, q), \text{Ree}(q), A_5$, where $q$ is a prime power, and $\text{deg}(\mathcal{C})$ equals $q+1$ in the linear case, $q^2+1$ in the Suzuki case and $q^3+1$ in the unitary and Ree case, and $\text{deg}(\mathcal{C}) = 5$ in the alternating case.
2. $\text{PGL}(2, 8)$, $p = 3$, and $\text{deg}(\mathcal{C}) = 28$.
3. $\text{AGL}(1, m)$ for a prime power $m$ of $p$, $\text{deg}(\mathcal{C}) = m$, and $\mathcal{X}$ is rational.
4. $\text{AGL}(1, 3), p \neq 3$, $\text{deg}(\mathcal{C}) = 3$, and $\mathcal{X}$ is rational.
5. $\text{AGL}(1, 4), p \neq 2$, $\text{deg}(\mathcal{C}) = 4$, and $\mathcal{X}$ is rational.
6. $\text{AGL}(1, m)$, for $m = 3, 4, 5, 7$, $p \neq 2, 3$, $\text{deg}(\mathcal{C}) = m$ and $\mathcal{X}$ is elliptic.
7. $\text{AGL}(1, m)$, for $m = 3, 4, 5, 7$, $p = 3$, $\text{deg}(\mathcal{C}) = m$, and $\mathcal{X}$ is elliptic.
8. $\text{AGL}(1, m)$, for $m = 3, 5, 7$, $p = 2$, $\text{deg}(\mathcal{C}) = m$, and $\mathcal{X}$ is elliptic.
9. $\text{N}(5)$, for $p = 2$, $\text{deg}(\mathcal{C}) = 25$, and $\mathcal{X}$ is elliptic.
10. $\text{SU}(3, 2)$, $p = 2$, and $g(\mathcal{X}) = 10$.

All the above cases occur, see Section \ref{section:k}. A corollary of Theorem \ref{theorem:1} is the following result.

**Theorem 1.2.** Under the hypotheses of Theorem \ref{theorem:1} if $p \nmid |G_1|$, in particular if $p = 0$ or $p > 2g(\mathcal{X}) + 1$, then $\mathcal{X}$ is either rational or elliptic.

Our notation and terminology are standard. In particular, $\text{AGL}(1, m)$ denotes the automorphism group of the affine line over $F_m$. Here, $\text{AGL}(1, 3) \cong S_3$, $\text{AGL}(1, 4) \cong A_4$. Furthermore, $\text{N}(5)$ stands for the sharply 2-transitive group of degree $5^2$ arising from the irregular nearfield of order $5^2$.

2. Background from Function Field theory and some preliminary results

For a subgroup $G$ of $\text{Aut}(\mathcal{X})$, let $\bar{\mathcal{X}}$ denote a non-singular model of $K(\mathcal{X})^G$, that is, a projective non-singular geometrically irreducible algebraic curve with function field $K(\mathcal{X})^G$, where $K(\mathcal{X})^G$ consists of all elements of $K(\mathcal{X})$ fixed by every element in $G$. Usually, $\bar{\mathcal{X}}$ is called the quotient curve of $\mathcal{X}$ by $G$ and denoted by $\mathcal{X}/G$. The field extension $K(\mathcal{X})|K(\mathcal{X})^G$ is Galois of degree $|G|$.

Since our approach is mostly group theoretical, we often use notation and terminology from Finite group theory rather than from Function field theory.
Let $\Phi$ be the cover of $\mathcal{X} \mapsto \bar{\mathcal{X}}$ where $\bar{\mathcal{X}} = \mathcal{X}/G$ is a quotient curve of $\mathcal{X}$ with respect to $G$. A point $P \in \mathcal{X}$ is a ramification point of $G$ if the stabilizer $G_P$ of $P$ in $G$ is nontrivial; the ramification index $e_P$ is \(|G_P|\); a point $\bar{Q} \in \bar{\mathcal{X}}$ is a branch point of $G$ if there is a ramification point $P \in \mathcal{X}$ such that $\Phi(P) = \bar{Q}$; the ramification (branch) locus of $G$ is the set of all ramification (branch) points. The $G$-orbit of $P \in \mathcal{X}$ is the subset of $\mathcal{X}$ \(o = \{ R \mid R = g(P), \, g \in G \}$, and it is long if \(|o| = |G|\), otherwise \(o(P)\) is short. For a point $\bar{Q}$, the $G$-orbit $o$ lying over $\bar{Q}$ consists of all points $P \in \mathcal{X}$ such that $\Phi(P) = \bar{Q}$. If $P \in o$ then \(|o| = |G|/|G_P|\) and hence $\bar{Q}$ is a branch point if and only if $o$ is a short $G$-orbit. It may be that $G$ has no short orbits. This is the case if and only if every non-trivial element in $G$ is fixed–point-free on $\mathcal{X}$, that is, the cover $\Phi$ is unramified. On the other hand, $G$ has a finite number of short orbits. For a non-negative integer $i$, the $i$-th ramification group of $\mathcal{X}$ at $P$ is denoted by $G^{(i)}_P$ (or $G_i(P)$ as in \([19\) Chapter IV]) and defined to be

\[ G^{(i)}_P = \{ g \mid \text{ord}_P(g(t) - t) \geq i + 1, \, g \in G_P \}, \]

where $t$ is a uniformizing element (local parameter) at $P$. Here $G^{(0)}_P = G_P$. The structure of $G_P$ is well known; see for instance \([19\) Chapter IV, Corollary 4] or \([13\) Theorem 11.49].

**Result 2.1.** The stabilizer $G_P$ of a point $P \in \mathcal{X}$ in $G$ has the following properties.

(i) $G^{(i)}_P$ is the unique normal $p$-subgroup of $G_P$;

(ii) For $i \geq 1$, $G^{(i)}_P$ is a normal subgroup of $G_P$ and the quotient group $G^{(i)}_P / G^{(i+1)}_P$ is an elementary abelian $p$-group.

(iii) $G_P = G^{(1)}_P \rtimes U$ where the complement $U$ is a cyclic whose order is prime to $p$.

Let $\bar{g}$ be the genus of the quotient curve $\bar{\mathcal{X}} = \mathcal{X}/G$. The Hurwitz genus formula is the following equation

\[ 2\bar{g} - 2 = |G|(2\bar{g} - 2) + \sum_{P \in \mathcal{X}} d_P. \]

where

\[ d_P = \sum_{i \geq 0} (|G^{(i)}_P| - 1). \]

Here $D(\mathcal{X}|\bar{\mathcal{X}}) = \sum_{P \in \mathcal{X}} d_P$ is the different. For a tame subgroup $G$ of $\text{Aut}(\mathcal{X})$, that is for $p \nmid |G_P|$, \[ \sum_{P \in \mathcal{X}} d_P = \sum_{i=1}^{m} (|G| - \ell_i) \]

where $\ell_1, \ldots, \ell_m$ are the sizes of the short orbits of $G$.

A subgroup of $\text{Aut}(\mathcal{X})$ is a $p'$-group (or a prime to $p$ group) if its order is prime to $p$. A subgroup $G$ of $\text{Aut}(\mathcal{X})$ is tame if the 1-point stabilizer of any point in $G$ is $p'$-group. Otherwise, $G$ is non-tame (or wild). Obviously, every $p'$-subgroup of $\text{Aut}(\mathcal{X})$ is tame, but the converse is not always true. From the classical Hurwitz’s bound, if $|G| > 84(\bar{g}(\mathcal{X}) - 1)$ then $G$ is non-tame; see \([22\) or \([13\) Theorems 11.56]. An orbit $o$ of $G$ is tame if $G_P$ is a $p'$-group for $P \in o$, otherwise $o$ is a non-tame orbit of $G$.

Let $\gamma$ be the $p$-rank of $\mathcal{X}$, and let $\bar{\gamma}$ be the $p$-rank of the quotient curve $\bar{\mathcal{X}} = \mathcal{X}/G$. The Deuring-Shafarevich formula, see \([25\) or \([13\) Theorem 11.62], states for a $p$-subgroup $G$ of $\text{Aut}(\mathcal{X})$ that

\[ \gamma - 1 = |G|(|\bar{\gamma} - 1| + \sum_{i=1}^{k} (|G| - \ell_i) \]

where $\ell_1, \ldots, \ell_k$ are the sizes of the short orbits of $G$.

**Result 2.2.** If $\mathcal{X}$ has zero $p$-rank then $\text{Aut}(\mathcal{X})$ has the following properties:

(i) A Sylow $p$-subgroup of $\text{Aut}(\mathcal{X})$ fixes a point $P \in \mathcal{X}$ but its nontrivial elements have no fixed point other than $P$. 

(ii) The normalizer of a Sylow $p$-subgroup fixes a point of $\mathcal{X}$.
(iii) Any two distinct Sylow $p$-subgroups have trivial intersection.

Claim (i) is [13] Theorem 11.129. Claim (ii) follows from Claim (i). Claim (iii) is [13] Theorem 11.133. For the following results, see [13] Lemmas 11.129, 11.75, 11.60.

**Result 2.3.** Assume that $\text{Aut}(\mathcal{X})$ contains a $p$-subgroup $G$ of order $p'$. If the quotient curve $\mathcal{X}/G$ has $p$-rank zero, and every non-trivial element in $G$ has exactly one fixed point, then $\mathcal{X}$ has $p$-rank zero.

**Result 2.4** (Serre). Let $\alpha \in G_P$ and $\beta \in G_P^{(k)}$, $k \geq 1$. If $\alpha \notin G_P^{(1)}$, then the commutator $[\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}$ belongs to $G_P^{(k+1)}$ if and only if either $\alpha^k \in G_P^{(1)}$ or $\beta \in G_P^{(k+1)}$.

**Result 2.5.** If the order $n$ of $G_P$ is prime to $p$, then $n \leq 4g(\mathcal{X}) + 2$.

Let $\mathcal{E}$ be a non-singular plane cubic curve viewed as a birational model of an elliptic curve $\mathcal{X}$. For an inflection point $O$ of $\mathcal{E}$, the set of points of $\mathcal{E}$ can be equipped by an operation $\bigoplus$ to form an abelian group $G_O$ with zero-element $O$, which is isomorphic to the zero Picard group of $\mathcal{E}$; see for instance [13] Theorem 6.107. The translation $\tau_a$ associated with $a \in \mathcal{E}$ is the permutation on the points of $\mathcal{E}$ with equation $\tau_a : x \mapsto x \oplus a$. Since there exists an automorphism in $\text{Aut}(\mathcal{E})$ which acts on $\mathcal{E}$ as $\tau_a$ does, translations of $\mathcal{E}$ can be viewed as elements of $\text{Aut}(\mathcal{E})$. They form the translation group $J(\mathcal{E})$ of $\mathcal{E}$ which acts faithfully on $\mathcal{E}$ as a sharply transitive permutation group. For every prime $r$, the elements of order $r$ in $J(\mathcal{E})$ are called $r$-torsion points. They together form an elementary abelian $r$-group of rank $h$. Here $h = 2$ for $r \neq p$ while $h$ equals the $p$-rank of $\mathcal{E}$ for $r = p$, that is, $h = 0, 1$ according as $\mathcal{E}$ is supersingular or not.

**Result 2.6.** The translation group $J(\mathcal{E})$ is a normal subgroup of $\text{Aut}(\mathcal{E})$, and $\text{Aut}(\mathcal{E}) = J(\mathcal{E}) \rtimes \text{Aut}(\mathcal{E})_P$ for every $P \in \mathcal{E}$.

**Proof.** For complex cubic curves the claim is known. Here we provide a characteristic free proof based on [20] Theorem 4.8. For an inflection point $O$ of $\mathcal{E}$, the fixed-point-free complete linear series $|3O|$ has dimension 2, and its divisors are cut out on $\mathcal{E}$ by lines. Let $\alpha \in \text{Aut}(\mathcal{E})$ fix $O$. From [13] Lemma 11.20, $\alpha$ preserves $|3O|$. Therefore, any three collinear points of $\mathcal{E}$ are mapped by $\alpha$ to three collinear points of $\mathcal{E}$. In terms of the above group operation $\bigoplus$ where $-a$ stands for the opposite of $a$, this yields $\alpha(-a) \bigoplus \alpha(a) = O$ for every $a \in \mathcal{E}$. Now let $a, b \in \mathcal{E}$. Then the triple $a, b, -(a + b)$ are collinear points of $\mathcal{E}$, and so are $\alpha(a), \alpha(b)$ and $\alpha(-(a + b))$. Thus $O = \alpha(a) + \alpha(b) + \alpha(-(a + b)) = \alpha(a) + \alpha(b) - \alpha(a + b)$ whence $\alpha(a + b) = \alpha(a) + \alpha(b)$. Therefore, $a, \tau_a \alpha^{-1}(a) x = \alpha(\alpha^{-1}(x) + a) = x + \alpha(a) = \tau_{\alpha(a)}$. As $\text{Aut}(\mathcal{E})$ is generated by $J(\mathcal{X})$ together with $\text{Aut}(\mathcal{E})_P$, this shows that $J(\mathcal{E})$ is a normal subgroup of $\text{Aut}(\mathcal{E})$. Since no non-trivial element of $J(\mathcal{E})$ fixes a point of $\mathcal{E}$, $J(\mathcal{E}) \cap \text{Aut}(\mathcal{E})_P = \{1\}$ for any point $P \in \mathcal{E}$. Therefore $\text{Aut}(\mathcal{E}) = J(\mathcal{E}) \rtimes \text{Aut}(\mathcal{E})_P$. $\square$

The following result comes from [20] Theorem 10.1 and [13] Theorem 11.94.

**Result 2.7.** Let $\mathcal{E}$ be an elliptic curve, and $P \in \mathcal{E}$. If the stabilizer $H$ of $P$ in $\text{Aut}(\mathcal{E})$ has order at least 3 then

$$
\begin{align*}
H &\cong C_4, \text{ or } H \cong C_6 \quad &\text{when } p \neq 2, 3; \\
H &\cong C_3 \rtimes C_4, \text{ and } j(\mathcal{E}) = 0 \quad &\text{when } p = 3; \\
H &\cong \text{SL}(2,3) \text{ and } j(\mathcal{E}) = 0 \quad &\text{when } p = 2.
\end{align*}
$$

If $j(\mathcal{E}) = 0$ then $\mathcal{E}$ is birationally equivalent to either the cubic of affine equation $y^2 = x^3 + 1$, $y^2 = x^3 - x$ or $y^2 + y = x^3$, according as $p \neq 2, 3$, $p = 3$ or $p = 2$. Result 2.7 has the following corollary, see [13] Theorem 11.94.

**Result 2.8.** Let $\mathcal{E}$ be an elliptic curve. If $G$ is a subgroup of $\text{Aut}(\mathcal{E})$ and $P \in \mathcal{E}$ then

$$
|G_P| = \begin{cases} 
2, 4, 6 & \text{when } p \neq 2, 3, \\
2, 4, 6, 12 & \text{when } p = 3, \\
2, 4, 6, 8, 24 & \text{when } p = 2.
\end{cases}
$$
For \( p = 2 \), the stabilizer \( G^{(1)}_P \) is cyclic when \( |G^{(1)}_P| \leq 4 \), and it is the quaternion group \( Q_3 \) when \( |G^{(1)}_P| = 8 \).

For a prime \( r \), let \( R \) be the group of \( r \)-torsion points. Since \( R \) is the unique elementary abelian \( r \)-subgroup of \( J(\mathcal{E}) \), and \( J(\mathcal{E}) \) is a normal subgroup of \( \text{Aut}(\mathcal{E}) \), \( R \) is also a normal subgroup of \( \text{Aut}(\mathcal{E}) \).

**Lemma 2.9.** Let \( \mathcal{E} \) be an elliptic curve, and \( \alpha \in \text{Aut}(\mathcal{E}) \) a non-trivial automorphism of prime order \( t \neq p \). If \( \alpha \) has at least two fixed points, then either \( t = 2 \) and \( \alpha \) has exactly 4 fixed points, or \( t = 3 \) and \( \alpha \) has exactly 3 fixed points. Furthermore,

1. if \( t = 3 \), no non-trivial translation of \( J(\mathcal{E}) \) preserving the set of fixed points of \( \alpha \) has order 3;
2. if \( t = 2 \) and, in addition, 4 divides the stabilizer of a fixed point of \( \alpha \) then no non-trivial translation of \( J(\mathcal{E}) \) preserving the set of fixed points of \( \alpha \) has order 2.

**Proof.** From the Hurwitz genus formula applied to the subgroup generated by \( \alpha \)

\[
0 = 2g(\mathcal{E}) - 2 = -2t + \lambda(t - 1)
\]

where \( \lambda \) counts the fixed points of \( \alpha \). From this, the first claim follows. Let \( t = 3 \). Since the 3-torsion group \( R \) of \( \mathcal{E} \) has order 9, \( \alpha \) together with \( R \) generate a subgroup \( M \) of \( \text{Aut}(\mathcal{E}) \) of order 27. For a fixed point \( P \in \mathcal{E} \), let \( \Delta \) be the \( \text{R}-\text{orbit} \) of \( P \). As \( R \) is a normal subgroup of \( M \), \( \Delta \) is left invariant by \( M \). Furthermore, since \(|\Delta| = 9\), the stabilizer \( M_P \) of \( P \) in \( M \) has order 3 and its three fixed points are in \( \Delta \). Therefore, \( M_P = \langle \alpha \rangle \) and the fixed points of \( \alpha \) are in \( \Delta \). Since \( J(\mathcal{E}) \) is sharply transitive on \( \mathcal{E} \), this yields that no non-trivial element of order prime to 3 may takes \( P \) to another fixed point of \( \alpha \) whence (i) follows for \( t = 3 \). Let \( t = 2 \). This time \( R \) is an elementary abelian group of order 4 which together with \( \alpha \) generate a subgroup of \( \text{Aut}(\mathcal{E}) \) of order 8. Also, \( M_P \) has order two and hence again \( M_P = \langle \alpha \rangle \), and \( \alpha \) fixes either two points in \( \Delta \), or all its 4 fixed points are in \( \Delta \). In the latter case, (ii) follows as for \( t = 3 \). To investigate the former case, suppose that \( \alpha = \gamma^2 \) with \( \gamma \in \text{Aut}(\mathcal{E}) \) fixing \( P \). The subgroup \( T \) generated by \( R \) together with \( \gamma \) has order 16 and preserves \( \Delta \). The kernel of the representation of \( T \) on \( \Delta \) is not faithful, as \( |S_4| \) is not divisible by 16. Therefore, \( T \) contains an involution \( \tau \) fixing \( \Delta \) pointwise. Since \( P \in \Delta \) and the stabilizer of \( P \) in \( \text{Aut}(\mathcal{E}) \) is cyclic, \( \tau \) coincides with \( \alpha \) whence (ii) follows.

From previous works on Galois points, we need a very recent result due to Fukasawa; see [1], Theorem 1. We state it for the case of two inner Galois points \( \varphi(P_1), \varphi(P_2) \). We also add some properties in case where the corresponding Galois group \( G_i \) fixes \( P_i \) for \( i = 1, 2 \).

**Lemma 2.10.** Let \( \mathcal{C} \) be a plane model of \( \mathcal{X} \) associated with the morphism \( \varphi : \mathcal{X} \to \text{PG}(2, \mathbb{K}) \). Let \( P_1, P_2 \in \mathcal{X} \) be two distinct points together with two distinct subgroups \( G_1, G_2 \) of \( \text{Aut}(\mathcal{X}) \) such that \( \varphi(P_1) \) and \( \varphi(P_2) \) are Galois points of \( \mathcal{C} \) with Galois groups \( G_1 \) and \( G_2 \), respectively. Then (I) and (II) hold. Furthermore, assume that \( G_i \) fixes \( P_i \) for \( i = 1, 2 \). Then (III) holds if and only if both \( \varphi(P_1) \) and \( \varphi(P_2) \) are unibranch points of \( \mathcal{C} \).

**Proof.** By definition, (I) holds. Since both \( G_1 \) and \( G_2 \) are finite groups, and \( \mathcal{C} \) has a finite number of singular points, there exists a simple point \( \varphi(P) \in \mathcal{C} \) not on the line \( \varphi(P_1)\varphi(P_2) \) which is not fixed by any non-trivial element from either \( G_1 \) or \( G_2 \). To show (II), assume by way of a contradiction that \( g \in G_1 \cap G_2 \) with \( g \neq 1 \). Let \( r \) be the line through \( \varphi(P) \) and \( \varphi(g(P)) \) in \( \text{PG}(2, \mathbb{K}) \). Then the points \( \varphi(P_1), \varphi(P), \varphi(g(P)) \) are three distinct points on the line \( r \), and similarly, \( \varphi(P_2), \varphi(P), \varphi(g(P)) \) are three distinct points on the same line \( r \). This yields that \( P \) lies on the line through \( \varphi(P_1) \) and \( \varphi(P_2) \), a contradiction. Up to a change of the projective frame, \( \varphi(P_1) = Y_\infty \) and \( \varphi(P_2) = X_\infty \). Let \( \mathcal{P}_1 \) the set of all points of \( \mathcal{X} \) which are taken by \( \varphi \) to points of \( \mathcal{C} \) lying on the line \( \ell_\infty \) at infinity. Obviously, \( P_2 \in \mathcal{P}_1 \), and hence the \( G_1 \)-orbit \( \Delta_1 \) of \( P_2 \) is also contained in \( \mathcal{P}_1 \). Furthermore, every point \( P \in \mathcal{X} \) with \( \varphi(P) \neq \varphi(P_1) \) and \( \varphi(P) \in \ell_\infty \) is in \( \Delta_1 \). However, a point \( P \in \mathcal{X} \) with \( \varphi(P) = \varphi(P_1) \) is in \( \Delta_1 \) if and only if \( \varphi(P) \), viewed as a branch of \( \mathcal{C} \) centered at \( \varphi(P_1) \), is tangent to \( \ell_\infty \). Let \( \mathcal{Q}_1 = \mathcal{P}_1 \setminus \Delta_1 \). In the divisor group of \( \mathbb{K}(\mathcal{X}) \), let \( B_1 = \sum_{P \in \mathcal{Q}_1} P \) and \( D_1 = \sum_{P \in \Delta_1} P = \sum_{\sigma \in G_1} \sigma(P_2) \). The analog point sets \( \mathcal{P}_2, \mathcal{Q}_2 \) and divisors \( B_2, D_2 \) are defined interchanging \( P_1 \) with \( P_2 \) and replacing \( G_1 \) by \( G_2 \). Then \( B_1 + D_1 = B_2 + D_2 \), that is,

\[
(\text{III}) \quad B_1 + \sum_{\sigma \in G_1} \sigma(P_2) = B_2 + \sum_{\tau \in G_2} \tau(P_1).
\]
Since $G_t(P_t) = P_t$, we have $P_t \neq \sigma(P_t)$ for every $\sigma \in G_t$. Hence $P_t \in \text{Supp}(B_1)$. Therefore, if $\varphi(P_t)$ is a unibranch point of $C$ then $B_1 = P_t$. Similarly for $B_2$. Thus, (IIIa) becomes (III) if and only if $\varphi(P_1)$ and $\varphi(P_2)$ are unibranch points of $C$.

\begin{lemma}
Let $P_1, P_2$ be two distinct points of $X$ together with two distinct subgroups $G_1, G_2$ of $\text{Aut}(X)$ such that (I), (II), (III) hold. Assume that $G_i$ fixes $P_i$ for $i = 1, 2$. Then $X$ has a birational plane model $C$ such that the associated morphism $\varphi : X \mapsto PG(2, \mathbb{K})$ takes $P_i$ to an inner Galois point $\varphi(P_i)$ of $C$ with Galois group $G_i$. Furthermore, if both $\varphi(P_1)$ and $\varphi(P_2)$ are unibranch points then

- (IV) for $i = 1, 2$, the group $G_i$ is a sharply transitive group on $\text{Supp}(D) \setminus \{P_i\}$;
- (V) the group $G$ generated by $G_1$ and $G_2$ acts on $\text{Supp}(D)$ as a doubly transitive permutation group;
- (VI) $\text{Supp}(D) = \text{deg}(C)$.

Also, $\varphi$ can be chosen in such a way that $C$ has equation $f(X, Y) = 0$ with $f(x, y) = 0$ where

- (VII) $X^{G_1} = \mathbb{K}(x)$ and $X^{G_2} = \mathbb{K}(y)$;
- (VIII) $\varphi(P_1) = Y_\infty$ and $\varphi(P_2) = X_\infty$;
- (IX) the poles of $x$ are the points in $\text{Supp}(D) \setminus \{P_1\}$, each of multiplicity 1, and the poles of $y$ are the points in $\text{Supp}(D) \setminus \{P_2\}$, each of multiplicity 1.

\end{lemma}

\begin{proof}
Take $x, y \in \mathbb{K}(X)$ with $X^{G_1} = \mathbb{K}(x)$ and $X^{G_2} = \mathbb{K}(y)$. Let $f(X, Y) \in \mathbb{K}[X, Y]$ be an irreducible polynomial such that $f(x, y) = 0$. From [12 Proposition 1], the plane curve $C$ with affine equation $f(X, Y) = 0$ is a birational model of $C$. Let $\varphi : X \mapsto PG(2, \mathbb{K})$ be the associated morphism. Up to a change of $x$ by $(x - \varphi(P_2))^{-1}$ when $v_{P_2}(x) \geq 0$, $P_2$ is a pole of $x$. A similar change in $y$ ensures that $P_1$ is a pole of $y$. With this setup, (VII) and (VIII) hold.

Since $\sigma \in \text{Aut}(X)$, each point $\sigma(P_2)$ with $\sigma \in G_1$ is also a pole of $x$. We point out that $\sigma(P_2) = P_2$ with $\sigma \in G_1$ only occurs when $\sigma = 1$. In fact, if both points $\varphi(P_1)$ and $\varphi(P_2)$ are unibranch then (III) Lemma 2.10 holds whence

$$P_1 + \sum_{\sigma \in G_1^*} \sigma(P_2) = \sum_{\tau \in G_2} \tau(P_1)$$

where $G_1^*$ denotes the set of non-trivial elements of $G_1$. Now, if $\sigma(P_2) = P_2$ holds with $\sigma \in G_1^*$ then $P_2$ would be in the support of the divisor on the left hand side, but not on the right hand side as $\sigma(P_2) = P_2$ for every $\tau \in G_2$; a contradiction. Similar $y$ and $\tau(P_1) = P_1$ never holds for $\tau \in G_2$. Therefore (IV) and hence (V) follow from (III). Also, $|\text{Supp}(D)| - 1 = |G_1| = |G_2|$. A further consequence is that the poles of $x$ are exactly the points $\text{Supp}(D) \setminus \{P_1\}$ each with multiplicity 1. The same holds for $y$ when $P_1$ is replaced by $P_2$. From this (IX) follows.

Finally, since $\varphi(P_1)$ is a unibranch point of $C$, (VI) follows from (III).

Assume that $P$ is a pole of $v \in \mathbb{K}(X)$ with multiplicity 1. For a local parameter $t$ of $P$, we have $v = t^{-1} + w$ with $v_P(w) \geq 0$. If $\alpha \in \text{Aut}(\mathbb{K})$ fixes $P$ and has order a power of $p$ then $\alpha(v) = (t + \tilde{w})^{-1} + w_1$ with $v_P(\tilde{w}) \geq 2$ and $v_P(w_1) \geq 0$. Since $(t + \tilde{w})^{-1} = t^{-1} + w_2$ with $v_P(w_2) \geq 1$ this yields $v_P(\alpha(v) - v) \geq 0$, that is, $P$ is not a pole of $\alpha(v) - v$. If $\text{ord}(\alpha) = s$ with $p \nmid s$ and $\alpha$ fixes $P$ then the above argument can be adapted, as $(ut + w)^{-1} = u^{-1}t^{-1} + w_3$ with $v_P(w_3) \geq 0$. It turns out that $P$ is not a pole of $\alpha(v) - u^{-1}v$. This holds true for $\alpha^k$ when $u^{-1}$ is replaced by $u^{-k}$. Therefore, $P$ is not a pole of $\alpha(v) - u^{-1}v$ for any $s$-th root of unity. This gives the following result.

\begin{lemma}
For a pole $P$ of $v \in \mathbb{K}(X)$, let $\alpha \in \text{Aut}(\mathbb{K})$ be a non-trivial automorphism fixing $P$. If $\text{ord}(\alpha)$ is a power of $p$ then $P$ is not a pole of $\alpha(v) - v$. If $\text{ord}(\alpha) = s$ with $p \nmid s$ then $P$ is not a pole of $\alpha(v) - uv$ for all $s$-th roots of unity $u \in \mathbb{K}$.

The following result for curves is well known for complex curves; see [13] Theorem 5.9. It remains valid in any characteristic; see [13] Theorem 11.114.

\end{lemma}
Result 2.13. Let $S$ be a subgroup of $\text{Aut}(\mathcal{X})$ of order $n$ which has a partition with components $S_1, \ldots, S_k$, with $n_i = |S_i|$ for $i = 1, \ldots, k$, and let $g', g'_i$ be the genera of the quotient curves $\mathcal{X}/S$ and $\mathcal{X}/S_i$, for $i = 1, \ldots, k$. Then

$$(k-1)g(\mathcal{X}) + ng' = \sum_{i=1}^{k} n_i g'_i.$$ \hfill (6)

3. Background from Group theory

From Group theory we need properties of Lie type simple groups, namely the projective special group, the projective special unitary group, the Suzuki group $Sz(q)$, and the Ree Group $Ree(q)$. The main reference is [20, Section 3]; see also [13, Appendix A]. Our notation and terminology are standard. In particular, $Z(G)$ stands for the center of a group $G$. The normal closure $S$ of subgroup $H$ of a group $G$ is the subgroup generated by all conjugates of $H$ in $G$. By definition, $S$ is the smallest normal subgroup of $G$ containing $H$.

For $q = r^h$ with $r$ prime, the projective special group $\text{PSL}(2, q)$ has order $(q+1)q(q-1)/\tau$ with $\tau = \text{g.c.d.}(2, q+1)$. $\text{PSL}(2, q)$ is simple for $q \geq 4$, isomorphic to a subgroup of the projective line $\text{PG}(1, q)$ over $\mathbb{F}_q$ and doubly-transitive on the set $\Omega$ of points of $\text{PG}(1, q)$. If $r = 2$ then $\text{PGL}(2, q) = \text{PSL}(2, q)$ whereas, for $r$ odd, $x \to (ax+b)/(cx+d) \in \text{PSL}(2, q)$ if and only if $ad-bc$ is a non-zero square element of $\mathbb{F}_q$.

Result 3.1 (Dickson’s classification of finite subgroups of the projective linear group $\text{PGL}(2, \mathbb{K})$). The finite subgroups of the group $\text{PGL}(2, \mathbb{K})$ are isomorphic to one of the following groups:

(i) prime to $p$ cyclic groups;
(ii) elementary abelian $p$-groups;
(iii) prime to $p$ dihedral groups;
(iv) Alternating group $A_4$;
(v) Symmetric group $S_4$; and $p > 2$
(vi) Alternating group $A_5$;
(vii) Semidirect product of an elementary abelian $p$-group of order $p^n$ by a cyclic group of order $n > 1$ with $n \mid (p^n - 1)$;
(viii) $\text{PSL}(2, p^f)$ for $f \mid m$;
(ix) $\text{PGL}(2, p^f)$ for $f \mid m$.

Here, $A_4 \cong \text{AGL}(1, 4)$, and $A_5 \cong \text{PSL}(2, 5)$.

The special linear group $\text{SL}(2, q)$ has center of order 2, and $\text{SL}(2, q)/Z(\text{SL}(2, q)) \cong \text{PSL}(2, q)$. Moreover, the automorphism group of $\text{PSL}(2, q)$ is the semilinear group $\text{PTL}(2, q)$. Since $Z(\text{PSL}(2, q))$ is trivial, $\text{PSL}(2, q)$ can be viewed as a (normal) subgroup of $\text{PTL}(2, q)$ consisting of all semilinear maps $x \to (ax+b)/(cx+d)$ where $a, b, c, d \in \mathbb{F}_q$ with $ad-bc \neq 0$, and $\sigma \in \text{Aut}(\mathbb{F}_q)$. The quotient group $\text{PTL}(2, q)/\text{PSL}(2, q)$ is either $C_2$, or $C_2 \times C_2$, according as $r = 2$, or $r$ is odd. The “linear subgroup” of $\text{PTL}(2, q)$ is $\text{PGL}(2, q)$ which is isomorphic to $\text{Aut}(\text{PG}(1, q))$, and consists of all linear maps $x \to (ax+b)/(cx+d)$ where $a, b, c, d \in \mathbb{F}_q$ with $ad-bc \neq 0$. Either $\text{PGL}(2, q) = \text{PSL}(2, q)$ or $[\text{PGL}(2, q) : \text{PSL}(2, q) = 2]$ according as $r = 2$ or $r$ is odd.

Lemma 3.2. Let $S_r$ be a Sylow $r$-subgroup of the 1-point stabilizer $M$ of a subgroup $L$ of $\text{PTL}(2, q)$ containing $\text{PSL}(2, q)$. If $S_r$ contains a Sylow $r$-subgroup $T_r$ of $\text{PSL}(2, q)$ then either $S_r = T_r$, or $r \mid h$ and $S_r$ is not a normal subgroup of $M$.

Proof. If $r \nmid h$ then the Sylow $r$-subgroups of $\text{PSL}(2, q)$ are also the Sylow $r$-groups of $\text{PTL}(2, q)$. Therefore, we may assume that $h = r^m v$ with $u \geq 1$, $r \nmid v$. Any Sylow $r$-subgroup $S_r$ of $\text{PTL}(2, q)$ has order $qr^m$. Up to conjugacy, the 1-point stabilizer is the subgroup of $\text{PTL}(2, q)$ fixing the point at infinity $\infty$ of $\text{PG}(1, q)$. Then $T_r$ consists of all transformations $x \to x+b$ with $b \in \mathbb{F}_q$. Furthermore, the transformations $x \to x^\sigma + b$ with $\sigma \in \text{Aut}(GF(q))$, $\sigma^r = 1$, and $b \in GF(q)$, form a group of order $qr^m$ which is a Sylow $r$-subgroup $F$ of $\text{PTL}(2, q)$. By Sylow’s theorem, $S_r$ may be assumed to be a subgroup of $F$. Let $w \in S_r$ be the semilinear
transformation \( w : x \to x^\sigma + a \) with a non-trivial automorphism \( \sigma \) of order \( p^k \) with \( 1 \leq k \leq u \), and \( a \in \mathbb{F}_q \).

Take an element \( \lambda \in \mathbb{F}_q \) of order \( q - 1 \) for \( q \) even and of order \( \frac{1}{3}(q - 1) \) for \( q \) odd. Let \( l(x) = \lambda x \). Then \( l \in \text{PSL}(2, q) \) and \( l \) fixes \( \infty \). Also, \( (l^{-1}wl)(x) = \lambda^{\sigma^{-1}x^\sigma} + \lambda^{-1}a \). By way of contradiction, assume that \( S_r \) is a normal subgroup of \( M \). Then \( l^{-1}wl \in S_r \) which yields \( \lambda^{\sigma} = \lambda \), that is, \( \lambda \) lies in a proper subfield \( \mathbb{F}_{r^s} \) of \( \mathbb{F}_q \). But this contradicts the choice of \( \lambda \).

For \( q = r^h \) with \( r \) prime, the projective special unitary group \( \text{PSU}(3, q) \) has order \( (q^3 + 1)q^3(q^2 - 1) / \mu \) with \( \mu = g.c.d.(3, q + 1) \). \( \text{PSU}(3, q) \) is simple for \( q \geq 3 \), isomorphic to a subgroup of \( \text{Aut}(\mathcal{H}_q) \) and doubly-transitive on the set \( \Omega \) of all \( \mathbb{F}_q \)-rational points of \( \mathcal{H}_q \). Furthermore, its automorphism group is the semilinear group \( \text{PTU}(3, q) \). Since \( Z(\text{PSU}(3, q)) \) is trivial, \( \text{PSU}(3, q) \) can be viewed as a (normal) subgroup of \( \text{PTU}(3, q) \). The “linear subgroup” of \( \text{PTU}(3, q) \) is \( \text{PGU}(3, q) \) which is isomorphic to \( \text{Aut}(\mathcal{H}_q) \). Let \( \infty \) denote the (unique) point at infinity \( \infty \) of \( \mathcal{H}_q \). Then the stabilizer of \( \infty \) in \( \text{PTU}(3, q) \) consists of all transformations \( t \) where \( t(x) = ax^\sigma + c, t(y) = by^\sigma + \alpha y + d \) with \( a, b, c, d \in \mathbb{F}_q, \alpha = a^9, b, c, d \in \mathbb{F}_q, d^3 + \alpha^6 + 1, \) and \( \sigma \in \text{Aut}(\mathbb{F}_q^\times) \).

Here \( t \in \text{PSU}(3, q) \) for \( \sigma = 1 \) and \( a^m = 1 \) where either \( m = \frac{1}{3}(q + 1) \) or \( m = q + 1 \), according as \( 3 \) divides \( q + 1 \) or does not.

The special unitary group \( SU(3, q) \) has center of order \( \mu = g.c.d.(3, q + 1) \), and \( SU(3, q) / Z(SU(3, q)) \cong \text{PSU}(3, q) \).

**Lemma 3.3.** Let \( S_r \) be a Sylow \( r \)-subgroup \( S_r \) of a 1-point stabilizer \( M \) of a subgroup \( L \) of \( \text{PTU}(3, q) \) containing \( \text{PSU}(3, q) \). If \( S_r \) contains a Sylow \( r \)-subgroup \( T_r \) of \( \text{PSU}(3, q) \). Then either \( S_r = T_r \), or \( r \mid h \) and \( S_r \) is not a normal subgroup of \( M \).

**Proof.** We argue as in the proof of Lemma 3.2. By way of contradiction, \( S_r \) may be assumed to contain a transformation \( w \) where \( w(x) = x^\sigma + a, w(y) = y^\sigma + x^\sigma + b \) with \( b = a^9 + a \) and \( \sigma \in \text{Aut}(\mathbb{F}_q^\times) \) of order \( r^k \) with \( 1 \leq k \leq u \). Let \( l \) be a transformation with \( l(x) = \lambda x, l(y) = y \) where \( \lambda \in \mathbb{F}_q^\times \) has order \( q + 1 \) for \( 3 \nmid (q+1) \) and \( 1/3(q+1) \) for \( 3 \mid (q+1) \). Then \( l \in \text{PSU}(3, q) \) and \( l \) fixes \( \infty \). Moreover, \( (l^{-1}wl)(x) = \lambda^{\sigma^{-1}x^\sigma} + \lambda^{-1}a \). As in the proof of Lemma 3.2, this leads to a contradiction.

For \( q = 2^h \) with \( h \geq 3 \) odd, the Suzuki group \( S_z(q) \) has order \( (q^2 + 1)q^3(q - 1) \). It is a simple group, isomorphic to \( \text{Aut}(\mathcal{S}_q) \) where \( \mathcal{S}_q \) stands for the Suzuki curve, see [13] Section 12.2]. \( S_z(q) \) acts faithfully as a doubly transitive permutation group on the set \( \Omega \) of all \( \mathbb{F}_q \)-rational points of \( \mathcal{S}_q \). As \( Z(S_z(q)) \) is trivial, \( S_z(q) \) can be viewed as a normal subgroup of its automorphism group \( \text{Aut}(S_z(q)) \). Furthermore, the quotient group \( \text{Aut}(S_z(q)) / S_z(q) \) is \( C_4 \). Therefore, Lemma 3.2 trivially holds for \( r = 2 \) when \( \text{PSL}(2, q) \) and \( \text{PGU}(2, q) \) are replaced by \( S_z(q) \) and \( \text{Aut}(S_z(q)) \), respectively.

For \( q = 3^h \) with \( h \geq 3 \) odd, the Ree group \( \text{Ree}(q) \) has order \( (q^3 + 1)q^3(q - 1) \). It is simple, isomorphic to \( \text{Aut}(\mathcal{R}_q) \) and doubly-transitive on the set \( \Omega \) of all \( \mathbb{F}_q \)-rational points of the Ree curve \( \mathcal{R}_q \). As \( Z(\text{Ree}(q)) \) is trivial, \( \text{Ree}(q) \) can be viewed as a normal subgroup of its automorphism group \( \text{Aut}(\text{Ree}(q)) \). Furthermore, the quotient group \( \text{Aut}(\text{Ree}(q)) / \text{Ree}(q) \) is \( C_4 \). Therefore, \( \text{Ree}(q) \) has a faithful representation in the six-dimensional projective space \( \text{PG}(6, q) \) as a subgroup of \( \text{PGL}(7, \mathbb{F}_q) \) which preserves the Ree-Tits ovoid \( Q \). The action of \( \text{Ree}(q) \) on \( Q \) is doubly transitive, and it is the same as on \( Q \). We refer to an explicit presentation of \( Q \) in a projective frame \( (X_0, X_1, \ldots, X_6) \) of \( \text{PG}(6, \mathbb{F}_q) \) as given in [18] Appendix A, Example A.13]. Then \( Z_{\infty} = (0, 0, 0, 0, 0, 0) \in Q \). Moreover, a Sylow 3-subgroup \( T_3 \) of \( \text{Ree}(q) \) fixes \( Z_{\infty} \) and consists of all projectivities \( \alpha_{a,b,c} \) associated to the matrices

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a^\sigma & 1 & 0 & 0 & 0 \\
-\alpha & 0 & b - a^\sigma & -1 & 0 & 0 & 0 \\
-a^2 & 1 & 0 & 0 & 0 & 0 & 0 \\
-w_1(a,b,c) & -w_2(a,b,c) & w_3(a,b,c) & w_4(a,b,c) & c & -b + a^\sigma & -a & 1 \\
-w_2(a,b,c) & w_3(a,b,c) & w_4(a,b,c) & c & -b + a^\sigma & -a & 1 & 0 \\
-w_3(a,b,c) & w_4(a,b,c) & c & -b + a^\sigma & -a & 1 & 0 & 0 \\
\end{bmatrix}
\]
for \(a, b, c \in \mathbb{F}_q\). Also, the stabilizer \(\text{Ree}(q)_{Z_{\infty}}\) with \(O = (1, 0, 0, 0, 0, 0) \in Q\) is the cyclic group \(C_{q-1}\) consisting of projectivities \(\beta q\) associated to the diagonal matrices,

\[
\text{diag}(1, d, d^{q+1}, d^{q+2}, d^{q+3}, d^{2q+4})
\]

for \(d \in \mathbb{F}_q\). The stabilizer of \(Z_{\infty}\) in \(\text{Ree}(q)\) is the semidirect product of \(T_3 \rtimes C_{q-1}\). Moreover, the stabilizer of \(Z_{\infty}\) in \(\text{Aut(Ree)}\) consists of all semilinear transformations which are products \(uv\) where \(u \in S_3\) and \(v\) is a \(\sigma\)-Frobenius map of \(PG(6, \mathbb{F}_q)\) where, for every \(\sigma \in \text{Aut}(\mathbb{F}_q)\), the associated \(\sigma\)-Frobenius map is defined by \((X_0, \ldots, X_6) \mapsto (X_0^\sigma, \ldots, X_6^\sigma)\).

**Lemma 3.4.** Let \(S_3\) be a Sylow 3-subgroup of a 1-point stabilizer \(M\) of a subgroup \(L\) of \(\text{Aut(Ree)}\); containing \(\text{Ree}(q)\). If \(S_3\) contains a Sylow 3-subgroup \(T_3\) of \(\text{Ree}(q)\) then \(S_3 = T_3\), or \(3| h \) and \(S_3\) is not a normal subgroup of \(M\).

**Proof.** We argue as in the proofs of Lemmas 3.2 and 3.3. We may assume \(h = 3^u v\) with \(3 \nmid v\). Let \(H_\infty\) be the hyperplane at infinity of equation \(X_0 = 0\) so that the arising affine space \(AG(6, \mathbb{F}_q)\) has coordinates \(x_1 = X_1/X_0, \ldots, x_6 = X_6/X_0\). Look at the 1-point stabilizer of \(Z_{\infty}\). Up to an isomorphism, \(S_3\) consists of products \(\alpha \beta\) where \(\alpha \in T_3\) and \(\beta\) is a Frobenius map \((x_1, \ldots, x_6) \mapsto (x_1^\sigma, \ldots, x_6^\sigma)\) with \(\sigma \in \text{Aut}(\mathbb{F}_q)\). In particular, \(S_3\) contains a transformation \(w\) such that \(w(x) = x^\sigma + a\), and \(\sigma\) of order \(3^k\) with \(1 \leq k \leq u\). For a primitive element \(\lambda \in \mathbb{F}_q^*\), let \(l\) denote a transformation associated with the diagonal matrix \(\text{diag}(1, \lambda, \lambda^{q+1}, \lambda^{q+2}, \lambda^{q+3}, \lambda^{2q+4})\). Computing \(l^{-1}wl(x)\) shows again that \(l^{-1}wl \notin S_3\), a contradiction as in the proof of Lemma 3.2 where \(\infty\) is replaced with \(Z_{\infty}\).

**Remark 3.6.** A transitive group space whose 1-point stabilizer has a subgroup transitive on the remaining points is 2-transitive.

To deal with Case (iii), we need a corollary to Zassenhaus’ classification of finite sharply doubly transitive groups.

**Result 3.7.** (Zassenhaus [10], XII Theorem 9.8) Let \(G\) be a sharply doubly transitive permutation group on a finite set \(\Omega\). Then \(|\Omega|\) is a prime power \(m\), and the elements in \(G\) which have no fixed point in \(\Omega\) together with the identity permutation form an elementary abelian group \(M\) of order \(m\). An example is the group \(\text{AGL}(1, m)\) which acts on the points of the affine line over the finite field \(\mathbb{F}_m\) as a sharply doubly transitive permutation group. For \(m\) prime, there exists no other examples. For \(m = r^2\), with \(r > 2\) prime there exists further examples arising from nearfields of degree \(r^2\).

For \(r = 3\) there exist exactly two sharply doubly transitive permutation groups, namely \(\text{AGL}(1, 9)\) and \(\Lambda \gamma L(1, 9)\), whereas for \(r = 5\) three, namely \(\text{AGL}(1, 25)\), \(\Lambda \gamma L(1, 25)\), and \(N(5)\) arising from the irregular
nearfield of degree 25. The group $A\gamma L(2, r^2)$ arise from the regular nearfield of degree $r^2$ and consists of all permutations on the elements of the finite field $\mathbb{F}_{r^2}$ which are of the form $x \mapsto a \circ x + b$ where $a, b \in \mathbb{F}_{r^2}$ and $a \circ x = ax$ for a square in $\mathbb{F}_{r^2}$ while $ax = a^r$ for non-square $a$ in $\mathbb{F}_{r^2}$. In particular, $A\gamma L(1, 9) \cong PSU(3, 2)$. Furthermore, the 1-point stabilizer of $A\gamma L(1, 9)$ has a subgroup of order 12 while that of $N(5)$ is isomorphic to $SL(2, 3)$ and contains no subgroup of order 12.

4. Doubly Transitive Groups on Curves with Simple Minimal Normal Subgroup

**Theorem 4.1.** Let $G$ be a subgroup of $\text{Aut}(\mathcal{X})$ which has an orbit $\Omega$ such that

(i) $G$ acts on $\Omega$ as a 2-transitive permutation group.

(ii) the action of $G$ on $\Omega$ is faithful.

If $G$ has a simple normal minimal subgroup $W$ then either $G \cong \text{PGL}(2, 8)$ with $|\Omega| = 28$ and $W \cong \text{PSL}(2, 7)$, or $G = W \cong \text{PSL}(2, 7)$ with $|\Omega| = 7$, or one of the following cases occurs: $W \cong \text{PSL}(2, q), \text{Sz}(q), \text{PSU}(3, q), \text{Ree}(q), \text{A}_5$, where $q$ is a power of $p$, and $|\Omega|$ is $q + 1$ in the linear case, $q^2 + 1$ in the Suzuki case, $q^3 + 1$ in the unitary case, and $|\Omega| = 5$ in the alternating case $\text{A}_5$.

**Proof.** Apart from the case $G \cong \text{PGL}(2, 8)$ and $|\Omega| = 28$, the subgroup $W$ is also 2-transitive on $\Omega$. From (iii) of Result 2.1, the 1-point stabilizer of $G$ is solvable. We make a case by case analysis using the list of all classification of doubly transitive permutation groups which the reader is referred to [2] Theorem 5.3.

If $W$ is the Alternating group on $\Omega$ then $|\Omega| \geq 5$ and its 1-point stabilizer is solvable only for $|\Omega| \leq 5$. Hence, $|\Omega| = 5$ and $W = \text{A}_5$.

If $W$ is a projective group then $W \cong \text{PSL}(r + 1, q)$ and $W$ acts on the projective space on $\Omega$ as $\text{PSU}(r + 1, q)$ on the points of $\text{PG}(r, q)$. The 1-point stabilizer of $\text{PSL}(r + 1, q)$ contains the linear group $SL(r, q)$ which is solvable only for $r = 1$.

If $W$ is a symplectic group $S_p(2m, 2)$ then two cases may occur, namely $|\Omega| = 2^m - 1(2^m \pm 1)$, and the 1-point stabilizer is the general orthogonal group $O^\pm(2m)$ which is solvable only for $O^-(2), O^+(2)$, and $O^+(4)$. If $m \leq 2$ then $S_p(2m, 2)$ is not simple, and this rules out the symplectic case.

If $W \cong \text{A}_7$ and $|\Omega| = 15$ then the 1-point stabilizer is isomorphic to the (simple) group $\text{PSL}(2, 7)$, a contradiction.

If $W$ is the Mathieu group $M_{11}$ and $|\Omega| = 11$ then the 1-point stabilizer contains an index 2 subgroup isomorphic to the (simple) group $\text{PSL}(2, 9)$, a contradiction.

If $W$ is the Mathieu group $M_{12}$ and $|\Omega| = 12$ then the 1-point stabilizer is isomorphic to the (simple) group $M_{11}$, a contradiction.

If $W$ is the Mathieu group $M_{22}$ and $|\Omega| = 22$ then the 1-point stabilizer is isomorphic to the (simple) group $\text{PSL}(3, 4)$, a contradiction.

If $W$ is the automorphism group of the Mathieu group $M_{22}$ and $|\Omega| = 22$ then the 1-point stabilizer contains an index 2 subgroup isomorphic to the (simple) group $\text{PSL}(3, 4)$, a contradiction.

If $W$ is the Mathieu group $M_{23}$ and $|\Omega| = 23$ then the 1-point stabilizer is isomorphic to the (simple) group $M_{22}$, a contradiction.

If $W$ is the Mathieu group $M_{24}$ and $|\Omega| = 24$ then the 1-point stabilizer is isomorphic to the (simple) group $M_{23}$, a contradiction.

If $W \cong \text{PSL}(2, q)$ and $|\Omega| = q$ then $q = 5, 7, 9, 11$ and the 1-point stabilizer is solvable for $q = 5, 7$. The former case, $W = \text{PSL}(2, 5) \cong \text{A}_5$, has already been discussed. In the latter case, $G = W \cong \text{PSL}(2, 7)$ and the 1-point stabilizer of $W$ is isomorphic to $\text{S}_4$.

If $W$ is the Mathieu group $M_{11}$ and $|\Omega| = 12$ then the 1-point stabilizer is isomorphic to the (simple) group $\text{PSL}(2, 11)$, a contradiction.

If $W$ is the Higman-Sims group $HS$ and $|\Omega| = 176$ then the 1-point stabilizer contains an index 2-subgroup isomorphic to the (simple) group $\text{PSU}(3, 5)$, a contradiction.
If $W$ is the Conway group $Co_3$ and $|\Omega| = 276$ then the 1-point stabilizer has in index 2 subgroup of to the (simple) McLaughlin group $McL$, a contradiction.

The remaining possibilities for $W$ in the classifications of doubly transitive groups, are those in the theorem, each has a unique 2-transitive permutation representation. Furthermore, up to an isomorphism, $W \leq G \leq \text{Aut}(W)$.

If $W$ is the projective special group $\text{PSL}(2, q)$ with $q \geq 5$ (and $|\Omega| = q + 1$) then 1-point stabilizer has order $\frac{1}{2}q(q - 1)$ and is the semidirect product of subgroup of order $q$ by a cyclic component of order $\frac{1}{2}(q - 1)$. From (iii) of Result $2.1$ $q$ is a power of $p$.

If $W$ is the projective unitary group $\text{PSU}(3, q)$ (and $|\Omega| = q^3 + 1$) then 1-point stabilizer contains a non-cyclic subgroup of order $q^3$. From (iii) of Result $2.1$ $q^3$ is a power of $p$.

If $W$ is the Suzuki group $\text{Sz}(q)$ (and $|\Omega| = q^2 + 1$), then the 1-point stabilizer has order $q^2(q - 1)$ and is the semidirect product of non-cyclic subgroup of order $q^2$ by a cyclic component of order $q - 1$. From (iii) of Result $2.1$ $q$ is a power of $p$. Here $p = 2$.

If $W$ is the Ree group $\text{Ree}(q)$ (and $|\Omega| = q^3 + 1$) then 1-point stabilizer has order $q^3(q - 1)$ and is the semidirect product of non-cyclic subgroup of order $q^3$ by a cyclic component of order $q - 1$. From (iii) of Result $2.1$ $q$ is a power of $p$. Here $p = 3$.

**Proposition 4.2.** Let $G$ be a subgroup of $\text{Aut}(\mathcal{X})$ which has an orbit $\Omega$ such that both (i) and (ii) of Theorem 4.1 are satisfied. Assume that $G$ has a simple minimal normal subgroup $W$. If the 1-point stabilizer $T$ of $G$ has a subgroup $H$ of order $|\Omega| - 1$ that acts (sharply) transitively on the remaining $|\Omega| - 1$ points then $H$ is a normal subgroup of $T$.

**Proof.** Theorem 4.1 applies.

If $G \cong \text{PGL}(2, 8)$ and $|\Omega| = 28$ then the 1-point stabilizer $T$ has order 54 and contains only one subgroup of order 27. Hence, the latter one is $H$, and it is normal in $T$.

If $G \cong \text{PSL}(2, 7)$ and $|\Omega| = 7$ then the 1-points stabilizer of $G$ has a unique subgroup of order 6 but this subgroup is not transitive on the remaining 6 points.

If $W \cong A_5$ (and $|\Omega| - 1 = 4$) then the 1-point stabilizer of $S$ is isomorphic to $A_4$. Since $A_4$ has a unique Sylow 2-subgroup (of order 4) it is a normal subgroup of $A_4$. Moreover, this Sylow 2-subgroup coincides with $H$, and the claim follows. Here $p = 2$ by (iii) of Result 2.1.

If $W \cong \text{PSL}(2, q)$ with $q \geq 5$ (and $|\Omega| = q + 1$ with $q = p^h$) then $\text{PSL}(2, q) \leq G \leq \text{PGL}(2, q)$. Assume that $H$ is not contained in $\text{PSL}(2, q)$, and look at the subgroup $L$ generated by $\text{PSL}(2, q)$ and $H$. The subgroup $\text{PSL}(2, q) \cap H$ is a $p$-subgroup of $\text{PSL}(2, q)$ which fixes $P$. The stabilizer $W_P$ of $P$ in $W$ has a Sylow $p$-subgroup $R$ of $\text{PSL}(2, q)$, and $\text{PSL}(2, q) \cap H$ is contained in $R$. Since $W_P$ is a normal subgroup of $G_P$, $RH$ is a $p$-subgroup of $L$ whose order equals $|R||H|/|R \cap H|$. Thus, $RH$ is a Sylow $p$-subgroup of $L$. From (iii) of Result 2.1 applied to $L_P$, $RH$ is a normal subgroup of $L_P$. From Lemma 3.2 $R = H$.

If $W \cong \text{PSU}(3, q)$ with $q \geq 3$ (and $|\Omega| = q^3 + 1$ with $q = p^h$) then $\text{PSU}(3, q) \leq G \leq \text{PU}(3, q)$. The above argument used for $\text{PSL}(2, q)$ still works with $|H| = q^3$ and Lemma 3.3.

If $W \cong \text{Sz}(q)$ (and $|\Omega| = q^2 + 1$ with $q = 2^h$, $h \geq 3$ odd) then $|H| = 2^{3h}$ but $[\text{Aut}(\text{Sz}(q)) : \text{Sz}(q)] = h$ is odd. Therefore, up to conjugacy, $H \in \text{Sz}(q)$. The 1-point stabilizer of $\text{Sz}(q)$ has a unique (Sylow) 2-subgroup of order $q^2$ which acts transitively on the set of the remaining $|\Omega| - 1$ points. In particular, that Sylow 2-subgroup is normal and coincides with $H$.

If $W \cong \text{Ree}(q)$ (and $|\Omega| = q^3 + 1$ with $q = 3^h$, $h \geq 1$ odd) then $|H| = 3^{3h}$ and $[\text{Aut}(\text{Ree}(q)) : \text{Ree}(q)] = h$.

The above argument used for $\text{PSL}(2, q)$ still works with $|H| = q^3$ and Lemma 3.3.

5. **Doubly transitive groups on curves with solvable minimal normal subgroup**

**Theorem 5.1.** Let $G$ be a subgroup of $\text{Aut}(\mathcal{X})$ which has an orbit $\Omega$ such that both (i) and (ii) in Theorem 4.1 hold. If, in addition,

(iii) $G$ has a solvable minimal normal subgroup $N$,
(iv) the 1-point stabilizer of $G$ has a subgroup $T$ that is sharply transitive on the remaining points of $\Omega$,
(v) the quotient curve $X/T$ is rational,
then $X$ is either rational, or elliptic.

Proof. Let $d = |\Omega|$. Since $N$ is faithful and sharply transitive on $\Omega$, $T \cap N$ is trivial, the subgroup $S = TN$ has order $d(d - 1)$ and hence it is a sharply doubly transitive group on $\Omega$. Therefore, $S$ has a partition whose components are the subgroup $N$ of order $d$ together with the stabilizers $S_i$ in $S$ with $U$ ranging over $\Omega$. Result 2.13 applies to $S$ with $k = 1 + d$, where $S_1 = N$, and, for $i = 2, \ldots, k$, $S_i$ are the conjugates of $T$ in $S$. In particular, the quotient curves $X/S_i$ for $i \geq 2$ are isomorphic. Since one of them, namely $X/T$ is rational, we have $g(X/S_i) = 0$ for $i = 2, \ldots, k$. Also $g(X/S) = 0$, as $T$ is a subgroup of $S$. Now, (iv) reads $mg(X) = mg(X/N)$ whence $g(X) = g(X/N)$. This is only possible when either $g(X) = 0$ or $g(X) = 1$. □

Proposition 5.2. Let $X$ be a rational curve. If $G$ is a subgroup of Aut($X$) such that both (i) and (ii) in Theorem 4.1 hold, and, in addition, $G$ has a solvable minimal normal subgroup then one of the following cases occurs.

(i) $G$ is sharply doubly transitive on $\Omega$, $G \cong AGL(1, m)$ with $|\Omega| = m$ where either $m$ is a power of $p$, or $m = 3$ and $p \neq 3$, or $m = 4$ and $p \neq 2$.
(ii) $|\Omega| = 4$, $G \cong S_4$, $p \neq 2$, and $AGL(1, 4) \cong A_4$ is the unique subgroup of $G$ which is sharply doubly transitive on $\Omega$.

Proof. From the proof of Theorem 4.1, $S = TN$ is a sharply doubly transitive group on $\Omega$. In particular, the order of $S$ is the product of two consecutive integers. From Result 5.1 applied to $S$, we have $S \cong AGL(1, m)$ where either $m$ is a power of $p$, or $m = 3$ and $p \neq 3$, or $m = 4$ and $p \neq 2$. Moreover, if $m$ is a power of $p$ then any solvable subgroup of $PGL(2, \mathbb{K})$ containing $AGL(1, m)$ has an abelian subgroup of order $m' = mp^r$ with $r > 1$. Therefore, $G$ cannot contain $S$ properly. Also, $AGL(1, 3)$ is the only doubly transitive permutation group of degree 3, and hence $G = S$ for $m = 3$ and $p = 3$. Finally, there are two doubly transitive permutation groups of degree 4, one is $AGL(1, 4) \cong A_4$ the other $S_4$, and in the former case $G = S$ but $|G : S| = 2$ in the latter.

Proposition 5.3. Let $E$ be an elliptic curve. If $G$ is a subgroup of Aut($E$) such that both (i) and (ii) in Theorem 4.1 hold, and, in addition, $G$ has a solvable minimal normal subgroup then one of the following occurs.

(i) $G$ is sharply doubly transitive on $\Omega$, $G \cong AGL(1, m)$ with $m = |\Omega|$ where $m = 3, 4, 5, 7$ for $p \neq 2, 3$, and $m = 3, 4, 5, 7$ for $p = 3$, and $m = 3, 5, 7$ for $p = 2$.
(ii) $G$ is sharply doubly transitive on $\Omega$, $G \cong PSU(3, 2)$ where $|\Omega| = 9$ and $p = 2$.
(iii) $G$ is sharply doubly transitive on $\Omega$, $G \cong N(5)$ where $|\Omega| = 25$ and $p = 2$.
(iv) $|\Omega| = 4$, $G \cong S_4$, $p \neq 2$, and $AGL(1, 4) \cong A_4$ is the unique subgroup of $G$ which is sharply doubly transitive on $\Omega$.
(v) $|\Omega| = 9$, $G \cong AGL(1, 9)$ and $A_7L(1, 9)$ is the unique subgroup of $G$ which is sharply doubly transitive on $\Omega$.

Proof. Since a 1-point stabilizer $G_1$ of $G$ has order at least $|\Omega| - 1$, Result 2.8 gives the possibilities for $|\Omega|$, namely $|\Omega| = 3, 5, 7$ for $p \neq 2, 3$, and $|\Omega| = 3, 5, 7, 13$ for $p = 3$, and $3, 4, 5, 9, 25$ for $p = 2$. From Result 6.7 we infer that one of the cases (i), ..., (v) occurs. For this purpose we have to rule out two cases, namely $m = 13$ for $p = 3$, and $m = 4$ for $p = 2$. In the former case, since 13 is a prime, Result 6.7 shows that $G_1$ must be cyclic, but this is not true as $G_1$ is not abelian by Result 2.8. In the latter case, $G \cong AGL(1, 4)$, and hence, from Lemma 2.3 an element $\beta$ of order 3 in $G$ fixes at least one and hence exactly 3 points. Since $j(E) = 0$ its $p$-rank is zero. A Sylow 2-subgroup of $G$ fixes a point by Result 2.2 and hence it contains only one element of order 2 by Result 2.8. On the other hand, $AGL(1, 4)$ contains an elementary abelian subgroup of order 4. This contradiction ends the proof. □
Proposition 5.4. Let $G$ be a subgroup of $\text{Aut}(X)$ which has an orbit $\Omega$ such that both (i) and (ii) in Theorem 7.1 hold. If, in addition,

(iii) $G$ has a solvable minimal normal subgroup $N$,

(iv) the 1-point stabilizer of $G$ has a subgroup $T$ that is sharply transitive on the remaining points of $\Omega$,

(v) the quotient curve $X/T$ is rational,

then $T$ is a normal subgroup of the 1-point stabilizer of $G$.

Proof. In Propositions 5.2 and 5.3 either $T$ coincides with the 1-point stabilizer of $G$, or $T$ is an index 2 subgroup of it. \hfill \Box

6. Auxiliary results for the proof of Theorem 1.1

In this section, $P_1, P_2$ are distinct points of $X$, and $G_1, G_2$ are distinct subgroups of $\text{Aut}(X)$ where $G_1$ fixes $P_1$ and $G_2$ fixes $P_2$. Moreover, $G_1$ and $G_2$ have properties (I),(II),(III). By Lemmas 2.10 and 2.11 properties (IV),(V) and (IX) also hold.

Let $\Omega$ denote the support of the divisor $D$ of $X$ defined in (III). Then (V) states that $G$ acts on $\Omega$ as a doubly transitive permutation group. Actually, the normal closure $S$ of $G_1$ in $G$ still acts doubly transitively on $\Omega$. In fact, there exists $g \in G$ which takes $P_1$ to $P_2$ and the subgroup $H_2 = g^{-1}G_1g$ of $G$ fixes $P_2$ and acts (sharply) transitively on $\Omega \setminus \{P_2\}$. Hence $G_1, H_2$ also have properties (I),(II),(III).

Our aim is to determine all possibilities for $S$. Since $S$ may happen to be not faithful on $\Omega$, we begin by investigating the subgroup $K$ of $S$ consisting of all elements which fix $\Omega$ pointwise.

Lemma 6.1. $K$ is a cyclic group whose order is prime to $p$ and divides $\text{deg}(C)$. Furthermore, $K$ coincides with $Z(G)$.

Proof. From (IX) of Lemma 2.11 the poles of $x$ are the points of $\Omega$ different from $Y_\infty$, each with multiplicity 1. Take a non-trivial element $\alpha \in K$ of order $s$. For any $v \in \mathbb{K}(C)$, $\alpha$ takes a pole of $v$ with multiplicity $m$ to a pole of $\alpha(v)$ with the same multiplicity $m$. Therefore, $\alpha(x)$ has the same poles of $x$.

We show that $p$ does not divide $|K|$. By way of a contradiction, assume $s$ to be a power of $p$. From Lemma 2.12 no point $P \in \Omega$ is a pole of $\alpha(x) - x$. Also, no branch of $C$ centered at an affine point is a pole of $\alpha(x) - x$. Thus $\alpha(x) - x \in \mathbb{K}$. Similarly, $\alpha(y) - y \in \mathbb{K}$. Therefore, $\alpha$ is a translation, that is, $\alpha(x) = x + a, \alpha(y) = y + b$ for $a, b \in \mathbb{K}$, and it has order $p$. Assume that $\alpha \beta \neq \beta \alpha$ for some $\alpha \in K$ and $\beta \in G_1$. Then $\beta^{-1} \alpha \beta(x) = \beta^{-1}(\alpha(x)) = \beta^{-1}(x + a) = \beta^{-1}(x) + \beta^{-1}(a) = x + a$. Therefore $\alpha^{-1} \beta^{-1} \alpha \beta(x) = x$. Since $\mathbb{K}(x)$ is $X^{G_1}$, this yields $\alpha^{-1} \beta^{-1} \alpha \beta \in G_1$. On the other hand $\alpha^{-1} \beta^{-1} \alpha \beta$ fixes $\Omega$ pointwise. Therefore $\alpha^{-1} \beta^{-1} \alpha \beta$ is the identity but this contradicts $\alpha \beta \neq \beta \alpha$. For a translation $\alpha \in K$, let $T$ denote its center. Take a point $P \in \text{Supp}(D)$ different from $T$. Let $\gamma$ be the place (or branch) of $C$ associated with $P$. Then $\gamma$ is centered at $\varphi(P)$, and its tangent $t$ is different from the line at infinity by (IX) of Lemma 2.11. Then $\alpha$ does not leave invariant $t$ and hence $\alpha$ does not fix $P$, a contradiction which shows that $K$ contains no translation. Therefore, $p \nmid |K|$.

For $p \mid s$, the same argument may be used. In fact, if $u$ is a primitive $s$-th root of unity, then Lemma 2.12 shows that no point $P \in \Omega$ is a pole of $\alpha(x) - ux$. Thus $\alpha(x) = ux + b$ with $b \in \mathbb{K}$, and similarly $\alpha(y) = uy + c$ with $c \in \mathbb{K}$. Therefore, $\alpha$ is a homology whose center is in the point $-(b/(u-1), -c/(u-1))$ and $\alpha \beta = \beta \alpha$ follows. This shows that $K \leq Z(G)$. As before, for a point $P \in \text{Supp}(D)$, let $\gamma$ be the place (or branch) of $C$ associated with $P$, centered at $\varphi(P)$, and with tangent $t$ different from the line at infinity. Then the homology $\alpha$ leaves $t$ invariant, and hence $t$ passes through the center of $\alpha$. This shows that the tangents to the branches of $C$ arising from the points in $\text{Supp}(D)$ are concurrent at the center of $\alpha$. Furthermore, since the product of two homologies with different centers is a translation, it turns out that $K$ consists of homologies with the same center $C$. In particular, $K$ is isomorphic to a finite multiplicative subgroup of $\mathbb{K}$. Therefore, $K$ is cyclic and $p \nmid |K|$. Since $G_1$ fixes $\varphi(P) = Y_\infty$ and $Y_\infty$ is a unibranch point of $C$, the tangent to $C$ at $Y_\infty$ contains no point of $C$ other than $Y_\infty$. Therefore $C$ is not a point of $C$. Take a line $\ell$ through $C$ and disjoint from $\Omega$ such that $\ell$ intersects $C$ in non-singular points. From every $K$-orbit $\Delta_\ell$ in $\ell \cap C$, take
a unique point $R_1$. Then the intersection divisor $C \cdot \ell$ is the sum $\sum_j |\Delta_j|R_j$. Therefore, Bézout’s theorem gives $\deg(C) = \deg(C \cdot \ell) = \sum_j |\Delta_j|I(R_j, C \cap \ell)$. Also, $|\Delta_j| = |K|$ as no non-trivial element in $K$ fixes a point in $\ell \cap C$. From this $|K|$ divides $\deg(C)$.

Finally, since any point in $\Omega$ is the only fixed point of a conjugate of $G_1$ in $G$, $Z(G)$ fixes $\Omega$ pointwise. Therefore $Z(G) \leq K$ and hence $K = Z(G)$. □

A crucial ingredient in the proof of Theorem 1.1 is the following result.

**Theorem 6.2.** $G_1$ is a normal subgroup of the stabilizer of $P_1$ in $G$.

**Proof.** By Propositions 5.2 and 5.4 $K$ may be assumed to be non-trivial. Let $G$ be the doubly transitive permutation group induced by $G$ on $\Omega$. Then $G$ acts on $\Omega$ as $G$ does, and no nontrivial element in $G$ fixes $\Omega$ pointwise. Propositions 4.2 and 5.4 apply to the quotient curve $\bar{\mathcal{X}} = \mathcal{X}/K$. Therefore, $G_1 = G_1/K$ is a normal subgroup of the stabilizer of $P_1$ in $G$ where $P_1$ is the point lying under $P_1$ in the cover $\mathcal{X}/\bar{\mathcal{X}}$. Therefore, $G_1$ is a characteristic subgroup of $G_1/K$, and hence $G_1$ is a normal subgroup of $G_1/K$. □

7. **Proof of Theorem 1.1**

Theorem 6.2 allows us to apply Result 3.3 to the group space $(\Omega, G)$ with $Q = G_1$ where $S$ is the normal closure of $G_1$ in $G$.

7.1. **$S$ is of type (i) in Result 3.3.** All cases occur as shown by the examples exhibited in Section 8. Here we observe that $g(\mathcal{X}) \geq 2$ apart from the possibilities where $S \cong \text{PSL}(2, q)$ and $|\Omega| = q + 1$, or $S \cong \text{A}_5$ and $|\Omega| = 5$. This follows by comparison of the list in (i) of Result 3.5 with Result 3.6 (for $g(\mathcal{X}) = 0$) and with Result 2.8 (for $g(\mathcal{X}) = 1$).

7.2. **$S$ is of type (ii) in Result 3.5.** An example is the smallest Ree curve; see Section 8.

7.3. **$S$ is of type (iii) in Result 3.5.** Proposition 5.2 applies to $S$, and the possibilities come from Propositions 5.2 and 5.3. All cases occur; see Section 8.

7.4. **$S$ is of type (iv) in Result 3.5.** Our goal is to show that $S \cong \text{SU}(3, 2)$ (and $g(\mathcal{X}) = 10$). In case (iv) of Result 3.5, $|Z(S)| = d$ with $|\Omega| = d^2$. Furthermore, the quotient curve $\bar{\mathcal{X}} = \mathcal{X}/G_1$ is rational and the quotient group $\bar{Z} = (Z(S) \times G_1)/G_1$ is a subgroup of Aut($\mathcal{X}$) isomorphic to $Z(S)$. Since $Z(S)$ fixes $\Omega$ pointwise whereas $G_1$ has two orbits on $\Omega$, we have that $\bar{Z}$ has at least two fixed points in $\bar{\mathcal{X}}$. Therefore, $p$ is prime to the order of $\bar{Z}$, that is, $p \neq d$. Also, $\bar{Z}$ has no further fixed point. This shows that $\Omega$ coincides with the set of all fixed points of $Z(S)$. Now, look at the quotient curve $\bar{\mathcal{X}} = \mathcal{X}/Z(S)$. From the Hurwitz genus formula, $2g(\mathcal{X}) - 2 = d(2g(\bar{\mathcal{X}}) - 2) + d^2(d - 1)$. Since $\bar{S} = S/Z(S)$ is sharply doubly transitive on $\Omega$, Theorem 5.2 applies to $\bar{\mathcal{X}}$. Thus, $\bar{\mathcal{X}}$ is either rational, or elliptic. In the former case, as $d \neq p$, Result 3.4 yields $\bar{S} \cong \text{A}_4$. This implies $d = 2$, a contradiction.

Therefore, $\mathcal{X}$ is elliptic, and $g(\mathcal{X}) = \frac{1}{2}(d^2(d - 1) + 2)$. Also, the quotient group $\bar{G}_1 = (G_1 \times Z(S))/Z(S)$ is a subgroup of Aut($\mathcal{X}$) fixing the point $\bar{P}_1$ of $\bar{\mathcal{X}}$ lying under $P_1$ in the cover $\mathcal{X}/\bar{\mathcal{X}}$. Since $\bar{G}_1 \cong G_1$ and $|\bar{G}_1| = d^2 - 1$ with $d \geq 3$, Result 2.2 yields $p = d$ and $d = 3, 5$. For $d = 3$, we have $|S| = 216$. More precisely, a MAGMA computation shows that either $S \cong \text{SU}(3, 2) = \text{SmallGroup}(216, 88)$, or $S \cong \text{SmallGroup}(216, 160)$. The latter case cannot actually occur since the 3-Sylow subgroup of $\text{SmallGroup}(216, 160)$ is abelian, and hence is not extra-special.

We are left with the possibility that $p = 2$, $d = 5$, $g(\mathcal{X}) = 51$, and $|S| = 3000$. Since $16 \nmid 3000$, a Sylow 2-subgroup $S_2$ of $G_1$ is also a Sylow 2-subgroup of $S$. Obviously, $S_2$ fixes $P_1$. We show that no non-trivial element in $S_2$ fixes a point other than $P_1$. The quotient group $\bar{S}_2 = (Z(S) \times S_2)/Z(S)$ is isomorphic to $S_2$ and it is a subgroup of Aut($\mathcal{X}$) which fixes $\bar{P}_1$. From Result 2.8 $\bar{S}_2$ (and hence $S_2$) is isomorphic to the
quaternion group $Q_8$ of order 8. The quotient curve $\hat{X} = \hat{X}/S_2$ is rational, and it has zero 2-rank. From Result 2.3, $\hat{X}$ has also zero 2-rank. Therefore, non-trivial element in $S_2$ fixes a point of $X$ other than $P_1$. This yields that $S_2$ fixes $P_1$ but its non-trivial elements fix no point other than $P_1$. To apply the Hurwitz genus formula to $S_2$, compute the ramifications groups of $S_2$ at $P_1$. By definition, $S_2 = S_2(0) = S_2(1)$. From Result 2.4 applied to a generator $\alpha$ of $Z(S)$, we have $S_2(1) = \ldots = S_2(5)$. Since $S_2$ is not an elementary abelian group, (ii) of Result 2.1 yields that $S(0)$ is non-trivial. Therefore, $S(0)$ contains the (unique) subgroup $T$ of $S_2$ of order 2. Since $T$ is in $G_1$ and $G_1$ contains a (cyclic) subgroup $C_{15}$ of order 15, Result 2.4 applies to a generator $\alpha$ of $C_{15}$ whence $S_2^i$ for contains $T$ for $i = 6, \ldots, 15$. Let $\chi' = \chi/S_2$. From the Hurwitz genus formula applied to $S_2$,

\[ 100 = 2(\chi(X) - 1) \geq 16(\chi(X') - 1) + 42 + 10 \]

whence $g(\chi') \leq 4$. Moreover, $(C_{15} \times S_2)/S_2 \cong C_{15}$ is a subgroup of Aut($\chi'$) which fixes the point $P'_1$ lying under $P_1$ in the cover $X'|\chi'$. 

If $\chi'$ is rational, then the subgroup $(Z(S) \times S_2)/S_2 \cong C_5$ of Aut($\chi'$) fixes exactly two points, namely $P'_1$ and $U'$. Therefore, the fixed points of $C_5$ are $P_1$ and some (or all) of the points in the $S_2$-orbit lying over $U'$. This shows that $C_5$ has at most 25 fixed points, a contradiction.

We may assume that $g(\chi') \geq 1$. Result 2.5 yields $15 \leq 4g(\chi') + 2$ whence $g(\chi') = 4$. This shows that equality holds in (7). In particular, $S_2 = S_2(1)$, for $i = 0, 1, \ldots, 5$, and $T = S_2(1)$ for $i = 6, \ldots, 15$, and $S_2(5) = \{1\}$. From (2) applied to $G_1$, we have then $dP_1 = 23 + 5 \cdot 7 + 10 = 68$. Let $C_3$ be the subgroup of $C_{15}$ of order 3. Then the quotient group $C_3' = (S_2 \times C_3)/S_2 \cong C_3$ is a subgroup of Aut($\chi'$). Let $\chi$ be the quotient curve $X'/C_3'$. The Hurwitz genus formula applied to $C_3'$ reads $6 = 2(\chi(X') - 1) = 6(\chi(\chi') - 1) + 2r$ where $r$ counts the fixed points of $C_3$. Here $r \geq 1$ as $C_3'$ fixes $P_1$. From this, $g(\chi(X)) \leq 1$, and $r = 3$ or $r = 6$ according as $\chi$ is elliptic or rational. The former case cannot actually occur by Result 2.8 since $(Z(S) \times (S_2 \times C_3))/S_2 \cong C_5$ is a subgroup of Aut($\chi$) fixing the point lying under the point $P_1$ in the cover $X'/\chi'$. Therefore, $\chi'$ is rational, and $r = 6$. Take a fixed point $U'$ of $C_3'$ other than $P_1'$ and consider the $S_2$-orbit $\Delta$ lying over $U'$. Since $C_3$ leaves $\Delta$ invariant, and $|\Delta| = 8$, $C_3$ has at least two fixed points in $\Delta$. Therefore, $C_3$ has at least 12 fixed points. Moreover, $G_1$ has four (pairwise conjugate) subgroups of order 3. Now, the Hurwitz genus formula applied to $G_1$ reads, $100 = 2(\chi(X) - 1) \geq -48 + 68 + 4 \cdot 24 = 116$ a contradiction.

7.5. $S$ coincides with $G$. By way of a contradiction, assume that some non-trivial element $g \in G_2$ of prime order $r$ does not belong to $S$. Since $S$ is a normal subgroup of $G$, $g$ is in the normalizer of $S$. As $g$ does not fix $P_1$ whereas $P_1$ is the unique fixed point of $G_1$, $g$ does not commute with $G_1$. Therefore $g$ is not in the centralizer of $S$, and hence $g$ is a non-trivial automorphism of $S$. If $\bar{S} = S/Z(S)$ is the doubly transitive permutation group induced by $S$ on $\Omega$ then $\bar{g} = gS/S$ is a non-trivial automorphism of $\bar{S}$. Another useful property of $\bar{g}$ is to have just one fixed point, namely $P_2$.

If $S$ is of type (i) in Result 5.5 then either $d = 5$ and $\bar{S} \cong A_5$, or $d = 4$ a power of $p$ and $\bar{S}$ is isomorphic to one of the groups $PSL(2, q)$, $PSU(3, q)$, $Sz(q)$, or $Ree(q)$. Actually none of these cases are consistent with $r = p$. This is obvious for $\bar{S} \cong A_5$ whereas it has implicitly been proven in the proof of Proposition 4.2. Therefore, $\bar{g} \in \bar{S}$, a contradiction.

If $S$ is of type (ii) in Result 5.5 then $S \cong P\Gamma L(2, 8) = Aut(P\Gamma L(2, 8)) \cong Aut(S)$, and hence $\bar{g} \in \bar{S}$, a contradiction.

If $S$ is of type (iii) in Result 5.5 then $X'$ is either rational, or elliptic and one of the cases in Propositions 5.2 and 5.3 occurs. Let $N$ be the (unique) minimal normal subgroup of $S$. Then $N$ is a characteristic subgroup of $S$, and hence it is a minimal normal subgroup of $G$. Furthermore, $G_1N \leq S$ is a sharply doubly transitive group on $\Omega$. Thus $S = G_1N$. Since $S \leq G$, either $G = S$, or $G > S$ and Lemma 5.3 shows that $G_1N$ is the unique sharply doubly transitive subgroup of $G$ on $\Omega$. Since $G_2N$ is another sharply doubly transitive subgroup of $G$ on $\Omega$, this yields $G_2 \leq S$, that is $G = S$. 
If $S$ is of type (iv) in Result 8.3 then $S \cong SU(3, 2)$ and hence $\tilde{S} \cong PSU(3, 2)$. Also, $\text{Aut}(PSU(3, 2)) \cong PTU(3, 2)$, and every involution in $PTU(3, 2) \setminus PSU(3, 2)$ has more than one fixed points. Again, $\tilde{g}$ cannot be one of them, a contradiction.

8. Examples for Theorem 8.1

For each group $G$ listed in Theorem 8.1 we exhibit an example of a plane curve with two different Galois points $P_1$ and $P_2$ both unibranch, whose automorphism group contains a subgroup isomorphic to $S$. Our basic idea is to find curves with automorphism groups satisfying (I), (II), and (III) in Lemma 2.10. We keep our notation used in Theorem 8.1.

8.1. Case (i). We show that the curves on which $G$ naturally acts provide examples.

8.1.1. Hermitian Curve. Let $q = p^h$. The Hermitian curve (also called the Deligne-Lusztig curve of unitary type) $X$ is the non-singular plane curve $C$ of genus $\frac{1}{2}q(q-1)$ given by the affine equation $x^q + y^{q+1} + 1 = 0$. Furthermore, $\text{PSL}(3, q)$ is isomorphic to a subgroup $G$ of $\text{Aut}(X)$ which acts on the set $\Omega$ of all $\mathbb{F}_{q^2}$-rational points of $X$ as doubly transitive permutation group. Here $|\Omega| = q^2 + 1$, and the stabilizer of $P \in \Omega$ in $G$ contains a normal subgroup $N_P$ which acts on $\Omega \setminus \{P\}$ as a sharply transitive permutation group, and $P$ is a Galois point of $C$ with Galois group $N_P$. For any two distinct points $P_1, P_2 \in \Omega$, define $G_1 = N_{P_1}$ and $G_2 = N_{P_2}$. Then conditions (I), (II), (III) are satisfied, and the subgroup $G = \langle G_1, G_2 \rangle$ is isomorphic to $SU(q) = \text{SU}(3, q)$ and $G$ is in turn the normal closure of $G_1$ in $G$. For $q > 2$, $\text{PSL}(3, q)$ is simple, and $g(X) \geq 2$.

8.1.2. Roquette Curve. Let $q = p^h > 3$ with odd prime $p$. The Roquette curve $X$ is the non-singular model of the irreducible (hyperelliptic) plane curve $C$ of genus $\frac{1}{2}q(q-1)$ given by the affine equation $x^q + x = y^2$. Then either $\text{PSL}(2, q)$ or $\text{SL}(2, q)$ (according as $q \equiv 1 \pmod{4}$ or $q \equiv -1 \pmod{4}$) is isomorphic to a subgroup of $\text{Aut}(X)$ which acts on the set $\Omega$ of all $\mathbb{F}_{q^2}$-rational points of $X$ as doubly transitive permutation group, and $G$ is a Galois point of $C$ with Galois group $N_P$. For any two distinct points $P_1, P_2 \in \Omega$, define $G_1 = N_{P_1}$ and $G_2 = N_{P_2}$. Then conditions (I), (II), (III) are satisfied, and the subgroup $G = \langle G_1, G_2 \rangle$ is isomorphic to $\text{SU}(3, q)$ and $G$ is in turn the normal closure of $G_1$ in $G$. For $q > 2$, $\text{PSL}(3, q)$ is simple, and $g(X) \geq 2$.

8.1.3. Suzuki Curve. Let $p = 2$, $q_0 = 2^s$, with $s \geq 0$ and $q = 2q_0^2 = 2^{2s+1}$. The Suzuki curve (also called the Deligne-Lusztig curve of Suzuki type) $X$ is the non-singular model of the irreducible plane curve $C$ of genus $q_0(q-1)$ given by the affine equation $x^{2q_0}(x^q + x) = y^q + y$. The Suzuki group $Sz(q)$ is isomorphic to a subgroup of $\text{Aut}(X)$ which acts on the set $\Omega$ of all $\mathbb{F}_{q^2}$-rational points of $X$. Here $|\Omega| = q^2 + 1$, and the claim in 8.1.1 on the existence of two internal Galois points remains valid for the Suzuki curve whenever $q^3$ is replaced by $q^2$ and $\text{PSU}(3, q)$ with $Sz(q)$. For $s \geq 1, Sz(q)$ is simple and $g(X) \geq 2$.

8.1.4. Ree Curve. Let $p = 3$, $q = 3q_0^2$, with $q_0 = 3^s$, $s \geq 1$. The Ree curve (also called the Deligne-Lusztig curve of Ree type) $X$ is the non-singular model of the irreducible plane curve $C$ of genus $\frac{1}{2}q_0(q-1)(q+q_0+1)$ given by the affine equation $y^{q^2} - [1 + (x^q - x)^{q-1}]y^q + (x^q - x)^{q-1}y - x^q(x^q - x)^{q+3q_0} = 0$. Let $s \geq 2$. The Ree group $ Ree(q)$ is isomorphic to a subgroup of $\text{Aut}(X)$ which acts on the set $\Omega$ of all $\mathbb{F}_{q^2}$-rational points of $X$ as a doubly transitive permutation group. The claim in 8.1.1 on the existence of two internal Galois points remains valid for the Ree curve whenever $PSU(3, q)$ is replaced by $ Ree(q)$. For $s \geq 2$, Ree$(q)$ is simple and $g(X) \geq 2$.

8.1.5. GK curve. Let $q = p^m$, with $r \geq 1$. The GK curve is the non-singular model of the irreducible plane curve $C$ of genus $\frac{1}{2}(n+1)(n^2 - 2) + 1$ given by the affine equation $y^{q^2+1} - (x^q + x) + (x^n + x) = 0$ where $n = p^r$. Moreover, $SU(3, n)$ is isomorphic to a subgroup of $\text{Aut}(X)$ which acts on the set $\Omega$ of the $n^3 + 1 \mathbb{F}_{q^2}$-rational points of $X$ as a doubly transitive permutation group. The claim in 8.1.1 on the existence of two internal Galois points remains valid for the GK curve whenever $q$ is replaced by $n$ and $PSU(3, q)$ by $SU(3, n)$. For $q > 2$, $SU(3, q)$ is non-solvable, and $g(X) \geq 2$. 

8.1.6. Alternating case $A_5$. Let $p = 2$ and $q = 4$. Look at the Hermitian curve $X$ with genus 6 and affine equation $x^5 + y^5 + 1 = 0$ as a curve in $PG(2, 4)$. Let $\Omega$ be the set consisting of the five points of $X$ at infinity. Also, let $\alpha_x : (x, y) \rightarrow (ux, uy)$ with a primitive fifth root $u$ of unity. Then $\alpha_x \in \text{Aut}(X)$ and it is fixes $\Omega$ pointwise. The centralizer of $\alpha_x$ in $\text{Aut}(X) \cong \text{PGU}(3, 4)$ contains a subgroup $G \cong A_5$ and $G$ acts on $\Omega$ as $A_5$ in its usual permutation representation. For $P \in \Omega$, the stabilizer of $P$ in $G$ is isomorphic to $A_4$. Therefore, the (unique) Sylow 2-subgroup $U_P$ of $G$ is an elementary group of order 4 and it acts on $\Omega \setminus \{P\}$ as a sharply transitive permutation group. We prove that the quotient curve $\hat{X} = X/U_P$ is rational. For this purpose, look at a non-trivial ramification group $U_P(i)$. From (ii) of Result 2.31, $U_P(i)$ is a normal subgroup of $G_P$. Since $G_P \cong A_4$, this gives $U_P = U_P(i)$. Hence, $|U_P(i)| - 1 = 3$. Furthermore, no non-trivial element of $U_P$ fixes a point other than $P$. From the Hurwitz genus formula applied to $U_P$, $10 = 2g(X) - 2 = (2g(X) - 2) + 3k$ where $k \geq 2$ counts the non-trivial ramification groups of $U$ in $P$. From this, $g(X) = 0$ follows. For any two points $P_1, P_2 \in \Omega$, let $G_1 = U_{P_1}$ and $G_2 = U_{P_2}$. Then the pair $\{G_1, G_2\}$ satisfies conditions (I), (II) and (III). Therefore, $P_1$ and $P_2$ are two internal Galois points of $X$ and $G = \langle G_1, G_2 \rangle \cong A_5$.

8.2. Case (ii). Let $p = 3$. The Ree curve $X$ with $s = 1$ provides an example. Indeed, $\text{PTL}(2, 8)$ is isomorphic to a subgroup $G$ of $\text{Aut}(X)$ which acts on the set $\Omega$ of the 27 $\mathbb{F}_q$-rational points of $X$ as a doubly transitive permutation group. Also, the claim in 8.1.1 on the existence of two internal Galois points remains valid for $X$ whenever $SU(3, q)$ is replaced by $\text{PTL}(2, 8)$.

8.3. Cases (iii). The basic tool is Result 3.31.

8.3.1. Case (iii-a). Let $m = p^h$. The rational curve $C$ with homogeneous equation $yz^{n-1} = x^m - xz^{n-1}$ is an example with $G \cong \text{PSL}(2, m)$. To show this, observe that the non-singular points of $C$ defined over $\mathbb{F}_m$ are those lying on the $X$-axis, and they coincide with the points $P_u = (u, 0, 1)$ with $u \in \mathbb{F}_m$. For every non-zero $\lambda \in \mathbb{F}_m$ the transformation $w$ with $w(x) = \lambda x$, $w(y) = \lambda y$ is in $\text{Aut}(X)$ and preserves every line through $P_0$. They form a subgroup $G_1$ of order $m - 1$ fixing $P_0$. Therefore, $P_0$ is a Galois point with Galois group $G_1$. The transformation $\tau$ with $\tau(x) = x - z$, $\tau(y) = y$ is in $\text{Aut}(X)$, and $G_2 = \tau^{-1}G_1\tau$ is a subgroup of order $m - 1$ fixing $P_1$. Therefore, $P_1$ is also a Galois point with Galois group $G_2$. Furthermore, $G_1 \cap G_2 = \{1\}$ and $G = \langle G_1, G_2 \rangle \cong \text{AGL}(1, m)$.

8.3.2. Case (iii-b). Let $p \neq 3$. The rational curve $C$ with equation of degree 3 provides an example with $G \cong \text{AGL}(1, 3)$. To show this, for a subgroup $G \cong \text{AGL}(1, 3)$, take an involution $\alpha \in G$. Let $P \in X$ be one of the fixed points of $\alpha$. Then the orbit of $P$ in $G$ has size 3. In $G$, take two distinct subgroups $G_1$ and $G_2$ of order 2. Let $P_i$ with $i = 1, 2$ be the fixed point of $G_i$. Then conditions (I), (II) and (III) are satisfied. Therefore $P$ is an inner Galois point of $X$ with Galois group $G_1$.

8.3.3. Case (iii-c). Let $p \neq 2$. The quartic curve $C$ with homogeneous equation $2x^2y^2 + y^2z^2 + z^2x^2 = 0$ is rational. For a primitive third root of unity $\varepsilon \in \mathbb{K}$, the cubic transformation $\alpha_1$ with $\alpha_1(x) = y$, $\alpha_1(y) = z$, $\alpha_1(z) = x$ is in $\text{Aut}(C)$ and fixes the point $P_1 = (1 : \varepsilon : \varepsilon^2)$. Also, the involution $\beta$ with $\beta(x) = x$, $\beta(y) = y$, $\beta(z) = z$ is in $\text{Aut}(C)$, and takes $P_1$ to the point $P_2 = (1 : -\varepsilon : \varepsilon^2)$. Therefore, $\alpha_2 = \beta \alpha_1 \beta \in \text{Aut}(C)$ is a cubic transformation such that $\alpha_2(x) = -y$, $\alpha_2(y) = -z$, $\alpha_2(z) = x$ and $\alpha_2(P_2) = P_2$. Let $G_i = \langle \alpha_i \rangle$ for $i = 1, 2$. Then $G = \langle G_1, G_2 \rangle \cong AGL(1, 4)$, and condition (I), (II), and (III) are satisfied. Therefore, $P_1$ and $P_2$ are Galois points with Galois points $G_1$ and $G_2$, respectively.

8.4. Cases (iv). We show a general procedure relying on Lemma 2.44 which provides examples for $p \nmid m$. Let $E$ be an elliptic curve. For a prime $r$ different from $p$, the translations in $\text{Aut}(E)$ associated to the $r$-torsion points together with the identity transformation form an elementary subgroup $R$ of $\text{Aut}(E)$ of order $r^2$. In $\text{Aut}(E)$, the Jacobian subgroup $J(E)$ of $\text{Aut}(X)$ consisting of all translations of $E$ is abelian, and hence $R$ is the unique elementary abelian subgroup of $J(E)$. Since $J(E)$ is a normal subgroup of $\text{Aut}(E)$, this shows that $R$ is also a normal subgroup of $\text{Aut}(X)$. For a point $P_1 \in E$ let $\Omega$ be the $R$-orbit of $P_1$, and $G_1$ the stabilizer of $P_1$ in $\text{Aut}(E)$. For a non-trivial element $\alpha \in R$, the point $P_2 = \alpha(P_1)$ is fixed by $G_2 = \alpha^{-1}G_1\alpha$. 
Therefore, conditions (I) and (II) are satisfied. Moreover, Lemma 2.8 shows that no non-trivial element in $G_1$ fixes a point of $\Omega$ other than $P_1$. Therefore, (III) holds with $\text{Supp}(D) = \Omega$ if and only if $|G_1| = r^2 - 1$. If this is the case then $G = \langle G_1, G_2 \rangle$ is sharply doubly transitive on $\text{Supp}(D)$, and, from Result 2.8, either $G \cong \text{AGL}(1, r^2)$, or $G \cong \text{A}_7 \text{L}(1, r^2)$, or $G$ arises from an irregular nearfield. This together with Result 2.8 provide an example with $m = 4, 9, 25$; more precisely $\text{AGL}(1, 4)$ for $p \neq 2$, and $\text{A}_7 \text{L}(1, 9)$, and $\text{N}(5)$ for $p = 2$. For cases $m = 3, 5$ and $p = 2$, a further step is sufficient since both $\text{A}_7 \text{L}(1, 9)$ and $\text{N}(5)$ contain a unique subgroup isomorphic to $\text{AGL}(1, m)$. In fact, with the above notation, let $L \cong \text{AGL}(1, m)$ be such a subgroup, and take $a$ from $L$. Since $G_1$ is a subgroup of the stabilizer $L_1$ of $P_1$ in $L$, if $|G_1|$ attains $r - 1$, then $G_1$ coincides $L_1$, and hence $G = L$. Therefore, conditions (I), (II) and (III) are satisfied.

To obtain examples for the other cases, namely for $m = 3, 5, 7$ with $p \neq 2, 3$, and for $m = 5, 7$ with $p = 3$, a variant of the above approach is used. For $m = 5$, $R$ is the elementary abelian subgroup of $J(\mathcal{E})$ of order 25, $G_1$ is a cyclic group of order four fixing a point $P \in \mathcal{E}$. From the latter property, $\mathcal{E}$ has $j$-invariant 1728 or 0 according as $p \neq 2, 3$ or $p = 3$ (see [20, Theorem 10.1]), and hence, up to a birational transformation, $\mathcal{E}$ has affine equation $y^2 = x^3 - x$. Furthermore, the group $R \times G_1$ generated by $R$ and $G_1$ has order 100. From the Magma database of small groups, there are six non-abelian group of order 100 which are the semidirect product of an elementary abelian group of order 25 by a cyclic complement $C_4$ of order 4. A case by case analysis shows that each of them has a normal subgroup of order 5 and hence a subgroup of order 20 which is a semidirect product of a subgroup $N$ of order five by a cyclic component $C_4$ of order four. This proves the existence of a subgroup of $\text{Aut}(\mathcal{E})$ isomorphic to $\text{AGL}(1, 5)$ so that conditions (I), (II) and (III) hold where $\text{Supp}(D)$ coincides with the $N$-orbit containing a fixed point $P \in \mathcal{E}$ of $C_4$. The cases $m = 3, 7$ can be treated similarly.

8.5. Case (va). Let $p = 2$. The (smallest) $GK$ curve $C$ has genus 10 and defined over $\mathbb{F}_8$ with homogeneous equation $z^9 + x^8 y + xy^8 + (x^2 y + xy^2)^3 = 0$. $C$ has two Galois points $P_1 = (0 : 1 : 0)$ and $P_2 = (1 : 0 : 0)$ with Galois groups $G_1 \cong G_2$. Here $G = \langle G_1, G_2 \rangle \cong SU(3, 2)$ and $G_1$ is the Sylow 2-subgroup of $P_1$ isomorphic to the quaternion group.

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