A COMPACT NON-FORMAL CLOSED $G_2$ MANIFOLD WITH $b_1 = 1$

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Abstract. We construct a compact manifold with a closed $G_2$ structure not admitting any torsion-free $G_2$ structure, which is non-formal and has first Betti number $b_1 = 1$. We develop a method of resolution for orbifolds that arise as a quotient $M/\mathbb{Z}_2$ with $M$ a closed $G_2$ manifold under the assumption that the singular locus carries a nowhere-vanishing closed 1-form.

1. Introduction

A $G_2$ structure on a 7-dimensional manifold $M$ is a reduction of its frame bundle to the exceptional Lie group $G_2$. Such a structure determines an orientation, a metric $g$ and a non-degenerate 3-form $\varphi$; these define a cross product $\times$ on $TM$ by means of the expression

$$\varphi(X,Y,Z) = g(X \times Y, Z).$$

The group $G_2$ appears on Berger’s list [2] of possible holonomy groups of simply connected, irreducible and non-symmetric Riemannian manifolds. Non-complete metrics with holonomy $G_2$ were given by Bryant in [5] and complete metrics were obtained by Bryant and Salamon in [6]. The first compact examples were constructed in 1996 by Joyce in [24] and [25]. More compact manifolds with holonomy $G_2$ were constructed later by Kovalev [28], Kovalev and Lee [29], Corti, Haskins, Nordström and Pacini [12] and recently by Joyce and Karigiannis [27].

The torsion of a $G_2$ structure $(M,g,\varphi)$ can be defined as $\nabla \varphi$, the covariant derivative of $\varphi$. Fernández and Gray [19] classified $G_2$ structures into 16 different types according to equations involving the torsion of the structure. In this paper we focus on two of them, namely torsion-free and closed $G_2$ structures. Torsion-free $G_2$ structures are those with holonomy contained in $G_2$; that is $\nabla \varphi = 0$ or equivalently $d\varphi = 0$ and $d\star \varphi = 0$, where $\star$ denotes the Hodge star. Closed $G_2$ structures are those that verify $d\varphi = 0$, and are also named calibrated. Such types of $G_2$ structures have interesting properties; while torsion-free $G_2$ manifolds are Ricci-flat, closed $G_2$ structures have negative scalar curvature and both the scalar-flatness and the Einstein condition are equivalent to the fact that the structure is torsion-free (see [7] and [10]).

This paper contributes to understanding topological properties of compact manifolds with a closed $G_2$ structure that cannot be endowed with a torsion-free $G_2$ structure. First examples of these were provided by Fernández in [10] and [17]; the example in [10] is a nilmanifold and the examples in [17] are solvmanifolds. Nilmanifolds and solvmanifolds arise as compact quotients of a Lie group by lattices; the Lie group is nilpotent in the first

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case and it is solvable in the second. In both examples the $G_2$ structure is induced by a closed left-invariant $G_2$ form on the Lie group. The solvmanifolds in [17] have first Betti number $b_1 = 3$. In [11] the authors classified nilpotent Lie algebras that admit a closed $G_2$ structure; this list provides more examples of compact manifolds with $b_1 \geq 2$ endowed with a closed $G_2$ structure but not admitting torsion-free $G_2$ structures. Recently in [18] the authors constructed another example which has first Betti number $b_1 = 1$. Their starting point is a nilmanifold $M$ with $b_1(M) = 3$ that admits a closed $G_2$ structure $\varphi$ and an involution that preserves $\varphi$. The quotient $X = M/\mathbb{Z}_2$ is an orbifold with $b_1(X) = 1$ whose isotropy locus consists of 16 disjoint tori. Then they resolve the singularities to obtain a smooth manifold.

Being this the geography of such manifolds, this paper provides an example of a compact manifold carrying a closed $G_2$ structure. Its topological properties are different from those that the already mentioned ones have, as we shall discuss later. Our construction consists in resolving an orbifold; for that purpose we first develop a resolution method that is summarized in the following result:

**Theorem 1.** Let $(M, g, \varphi)$ be a closed $G_2$ structure on a compact manifold. Suppose that $j: M \to M$ is an involution such that $j^* \varphi = \varphi$ and consider the orbifold $X = M/j$. Let $L = \text{Fix}(j)$ be the singular locus of $X$ and suppose that there is a nowhere-vanishing closed 1-form $\theta \in \Omega^1(L)$. Then, there exists a compact $G_2$ manifold endowed with a closed $G_2$ structure $(\tilde{X}, \tilde{g}, \tilde{\varphi})$ and a map $\rho: \tilde{X} \to X$ such that:

1. The map $\rho: \tilde{X} - \rho^{-1}(L) \to X - L$ is a diffeomorphism.
2. There exists a small neighbourhood $U$ of $L$ such that $\rho^*(\varphi) = \tilde{\varphi}$ on $\tilde{X} - \rho^{-1}(U)$.

The set $L$ is always an oriented three manifold (see Lemma 10); the assumption on $L$ is equivalent to the fact that each connected component is a mapping torus over an orientable surface by an orientation-preserving diffeomorphism. In our example, the singular locus is formed by 16 disjoint nilmanifolds; whose universal covering is the Heisenberg group.

The resolution method follows the ideas of Joyce and Karigiannis in [27], where they give a method to resolve $\mathbb{Z}_2$ singularities induced by the action of an involution on manifolds having a torsion-free $G_2$ structure in the case that the singular locus $L$ has a nowhere-vanishing harmonic 1-form. Their idea is the following: the local model of the orbifold singularity being $\mathbb{R}^3 \times (\mathbb{C}^2/\{\pm 1\})$, they perform a resolution by cutting a tubular neighbourhood of the singular locus and glueing a bundle over $L$ with fibre the Eguchi-Hanson space. Then they construct a 1-parameter family of closed $G_2$ structures on the resolution; these have small torsion when the value of the parameter is small. Then they apply a theorem of Joyce [26, Th. 11.6.1] which states that if one can find a closed $G_2$ structure $\varphi$ on a compact 7-manifold $M$ whose torsion is sufficiently small in a certain sense, then there exists a torsion-free $G_2$ structure which is close to $\varphi$ and in the same de Rham cohomology class. This method provides a torsion-free $G_2$ structure on the resolution; if its fundamental group is finite then its holonomy is $G_2$.

The main difficulty of their construction relies on the fact that two of the three pieces that they glue, namely an annulus around the singular set of the orbifold and a germ of resolution, do not come naturally equipped with a torsion-free $G_2$ structure. However, there is a canonical way to define a $G_2$ structure and make a perturbation to obtain a closed $G_2$ structure. But its torsion is too large so that they have to make additional
corrections. In this paper we follow the same ideas but the method is simplified because we avoid these technical difficulties.

In this paper we are interested in the interplay between closed $G_2$ manifolds with small first Betti number and the condition of being formal. Formal manifolds are those whose rational cohomology algebra is described by its rational model. This is a notion of rational homotopy theory and has been successfully applied in some geometric situations. The Thurston-Weinstein problem is a remarkable example in the context of symplectic geometry; this consists in constructing symplectic manifolds with no Kähler structure. In [14], Deligne, Griffiths, Morgan and Sullivan proved that compact Kähler manifolds are formal; and thus, non-formal symplectic manifolds are solutions of this problem. Formality is less understood in the case of exceptional holonomy; in particular, the problem of deciding whether or not manifolds with holonomy $G_2$ and $\text{Spin}(7)$ are formal is still open.

There are some partial results for holonomy $G_2$ manifolds; in [15] authors prove that compact non-formal manifolds with holonomy $G_2$ have second Betti number $b_2 \geq 4$. In addition in [9] it is proved that compact manifolds with holonomy $G_2$ are almost formal; this condition means that triple Massey products $\langle \xi_1, \xi_2, \xi_3 \rangle$ are trivial except perhaps for the case that the degree of $\xi_1$, $\xi_2$ and $\xi_3$ is 2. Non-trivial Massey products are obstructions to formality but there are examples of non-formal compact 7-manifolds that have only trivial Massey triple products (see [13]). However, the presence of a geometric structure makes the situation different; for instance in [31] the authors prove that simply-connected 7-dimensional Sasakian manifolds are formal if and only if all the triple Massey products are trivial.

Formal examples of closed $G_2$ manifolds that do not admit any torsion-free $G_2$ structure are the solvmanifold with $b_1 = 3$ provided in [17] and the compact manifold with $b_1 = 1$ provided in [18]. Non-formal examples are the nilmanifolds obtained in [11]; these have $b_1 \geq 2$. In this paper we prove:

**Theorem 2.** There exists a compact non-formal closed $G_2$ manifold with $b_1 = 1$ that cannot be endowed with a torsion-free $G_2$ structure.

The manifold $\tilde{X}$ that we construct is the resolution of a closed $G_2$ orbifold $X$, obtained as the quotient of a nilmanifold $M$ that is provided in [11] by the action of the group $\mathbb{Z}_2$. The orbifold has $b_1(X) = 1$ and a non-trivial Massey product because the group $\mathbb{Z}_2$ preserves a non-trivial Massey product on $M$. The resolution process does not affect the first Betti number; in addition the non-trivial Massey product on $X$ lifts to a non-trivial Massey product on $\tilde{X}$.

This paper is organized as follows. In section 2 we review the necessary preliminaries on orbifolds, $G_2$ structures and formality. Section 3 is devoted to prove Theorem 1 and in section 4 we characterise the cohomology ring of the resolution. With these tools at hand we finally construct in section 5 the non-formal compact closed $G_2$ manifold with $b_1 = 1$.

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2. Preliminaries

2.1. Orbifolds. We first introduce some aspects about orbifolds, which can be found in [8] and [30].

**Definition 3.** An n-dimensional orbifold is a Hausdorff and second countable space X endowed with an atlas \( \{ (U_\alpha, V_\alpha, \psi_\alpha, \Gamma_\alpha) \} \), where \( \{ V_\alpha \} \) is an open cover of \( X \), \( U_\alpha \subset \mathbb{R}^n \), \( \Gamma_\alpha < \text{Diff}(U_\alpha) \) is a finite group acting by diffeomorphisms, and \( \psi_\alpha : U_\alpha \to V_\alpha \subset X \) is a \( \Gamma_\alpha \)-invariant map which induces a homeomorphism \( U_\alpha / \Gamma_\alpha \approx V_\alpha \).

There is a condition of compatibility of charts for intersections. For each point \( x \in V_\alpha \cap V_\beta \) there is some \( V_\gamma \subset V_\alpha \cap V_\beta \) with \( x \in V_\gamma \) so that there are group monomorphisms \( \rho_{\delta\alpha} : \Gamma_\delta \to \Gamma_\alpha \), \( \rho_{\delta\beta} : \Gamma_\delta \to \Gamma_\beta \), and open differentiable embeddings \( \iota_{\delta\alpha} : U_\delta \to U_\alpha \), \( \iota_{\delta\beta} : U_\delta \to U_\beta \), which satisfy \( \iota_{\delta\alpha}(\gamma(x)) = \rho_{\delta\alpha}(\gamma)(\iota_{\delta\alpha}(x)) \) and \( \iota_{\delta\beta}(\gamma(x)) = \rho_{\delta\beta}(\gamma)(\iota_{\delta\beta}(x)) \), for all \( \gamma \in \Gamma_\delta \).

We can refine the atlas of an orbifold \( X \) in order to obtain better properties; given a point \( x \in X \), there is a chart \( (U, V, \psi, \Gamma) \) with \( U \subset \mathbb{R}^n \), \( U/\Gamma \approx V \), so that the preimage \( \psi^{-1}(\{x\}) = \{u\} \), and the group \( \Gamma \) acting on \( U \) leaves the point \( u \) fixed, i.e. \( \gamma(u) = u \) for all \( \gamma \in \Gamma \). We call \( \Gamma \) the isotropy group at \( x \), and we denote it by \( \Gamma_x \). This group is well defined up to conjugation by a diffeomorphism of a small open set of \( \mathbb{R}^n \). The singular locus of \( X \) is the set \( S = \{ x \in X \text{ s.t. } \Gamma_x \neq \{1\} \} \), and of course, \( X - S \) is a smooth manifold.

We now describe the de Rham complex of an n-dimensional orbifold \( X \). First of all, a \( k \)-form \( \eta \) on \( X \) consists of a collection of differential \( k \)-forms \( \{ \eta_\alpha \} \) such that:

1. \( \eta_\alpha \in \Omega^k(U_\alpha) \) is \( \Gamma_\alpha \)-invariant,
2. If \( V_\delta \subset V_\alpha \) and \( \iota_{\delta\alpha} : U_\delta \to U_\alpha \) is the associated embedding, then \( \iota_{\delta\alpha}^*(\eta_\alpha) = \eta_\delta \).

The space of orbifold \( k \)-forms on \( X \) is denoted by \( \Omega^k(X) \). In addition, it is obvious that the wedge product of orbifold forms and the exterior differential \( d \) on \( X \) are well defined. Thus, we have a differential graded algebra \( (\Omega^*(X), d) \) that we call the de Rham complex of \( X \). Its cohomology coincides with the cohomology of the space \( X \) with real coefficients, \( H^*(X) \) (see [8 Proposition 2.13]).

In this paper the orbifold involved is the orbit space of a smooth manifold \( M \) under the action of \( \mathbb{Z}_2 = \{ \text{Id}, j \} \), where \( j \) is an involution. The singular locus of \( X = M/\mathbb{Z}_2 \) is \( \text{Fix}(j) \). In addition, let us denote by \( \Omega^k(M)^{\mathbb{Z}_2} \) the space of \( \mathbb{Z}_2 \)-invariant \( k \)-forms. It is not difficult to check that:

\[
\Omega^k(X) = \Omega^k(M)^{\mathbb{Z}_2},
\]
and both the wedge product and exterior derivative preserve the \( \mathbb{Z}_2 \)-invariance. An averaging argument ensures that \( H^k(X) = H^k(M)^{\mathbb{Z}_2} \).

2.2. \( G_2 \) structures. We now focus on \( G_2 \) structures on manifolds and orbifolds. Basic references are [7], [19], [23], [26] and [36].

Let us identify \( \mathbb{R}^7 \) with the imaginary part of the octonions \( \mathbb{O} \). The multiplicative structure on \( \mathbb{O} \) endows \( \mathbb{R}^7 \) with a cross product \( \times \), which defines a 3-form \( \varphi_0(u, v, w) = \langle u \times v, w \rangle \), where \( \langle \cdot, \cdot \rangle \) denotes the scalar product on \( \mathbb{R}^7 \). In coordinates,

\[
\varphi_0 = v^{127} + v^{347} + v^{567} + v^{135} - v^{236} - v^{146} - v^{245},
\] (1)
where \((v^1, \ldots, v^7)\) is the standard basis of \((\mathbb{R}^7)^*\) and \(v^{ijk}\) stands for \(v^i \wedge v^j \wedge v^k\). The stabilizer of \(\varphi_0\) under the action of \(\text{GL}(7, \mathbb{R})\) on \(\Lambda^3(\mathbb{R}^7)^*\) is the group \(G_2\), a simply connected 14-dimensional Lie group which is contained in \(\text{SO}(7)\).

**Definition 4.** Let \(V\) be a real vector space of dimension 7. A 3-form \(\varphi \in \Lambda^3(V)^*\) is a \(G_2\) form on \(V\) if there is a linear isomorphism \(u : V \to \mathbb{R}^7\) such that \(u^*(\varphi_0) = \varphi\), where \(\varphi_0\) is given by equation (1). A \(G_2\) structure \(\varphi\) determines an orientation because \(G_2 \subset \text{SO}(7)\); the choice of a volume form \(\text{vol}\) on \(V\) compatible with the orientation determines a unique metric \(g_{\text{vol}}\) with associated unit-length volume form \(\text{vol}\) by the formula:

\[
i(x)\varphi \wedge i(y)\varphi \wedge \varphi = 6g_{\text{vol}}(x, y)\text{vol},
\]

which ensures that the metric \(u^*(g_0)\) is determined by the volume form \(u^*(\text{vol}_{\mathbb{R}^7})\). Note that the metric \(u^*(g_0)\) does not depend on the isomorphism \(u\) with \(u^*(\varphi_0) = \varphi\). We say that \(g = u^*(g_0)\) is the associated metric to \(\varphi\). Of course, a \(G_2\) form \(\varphi\) induces a cross product \(\times\) on \(V\) by the formula \(\varphi(u, v, w) = g(u \times v, w)\).

The orbit of \(\varphi_0\) under the action of \(\text{GL}(7, \mathbb{R})\) is an open set of \(\Lambda^3(\mathbb{R}^7)^*\), thus the space of \(G_2\) forms on \(\mathbb{R}^7\) is an open set.

**Definition 5.** Let \(M\) be a 7-dimensional manifold. A \(G_2\) form on \(M\) is a differential 3-form \(\varphi \in \Omega^3(M)\) such that for every \(p \in M\) the 3-form \(\varphi_p\) is a \(G_2\) form.

Let \(X\) be a 7-dimensional orbifold with atlas \(\{(U_\alpha, V_\alpha, \psi_\alpha, \Gamma_\alpha)\}\). A \(G_2\) form on \(X\) is a differential 3-form \(\varphi \in \Omega^3(X)\) such that \(\varphi_\alpha\) is a \(G_2\) form on each \(U_\alpha\).

Let \(\varphi\) be a \(G_2\) form on a manifold \(M\) or an orbifold \(X\). In both cases, \(\varphi\) determines a metric \(g\) and a cross product \(\times\). In this case we say that \((M, \varphi, g)\) or \((X, \varphi, g)\) is a \(G_2\) structure. In addition, \(G_2\) manifolds are of course oriented.

We state a well-known fact about \(G_2\) structures (see for instance [26, Chapter 10, Section 3]).

**Lemma 6.** There exists a universal constant \(m\) such that if \((M, \varphi, g)\) is a \(G_2\) structure and \(\|\phi - \varphi\|_{C^{0, g}} < m\) then \(\phi\) is a \(G_2\) form.

**Proof.** Let \((\mathbb{R}^7, \varphi_0, g_0)\) be the standard \(G_2\) structure. The space of positive forms of \(\mathbb{R}^7\) is open in \(\Lambda^3(\mathbb{R}^7)^*\), so that there exists a constant \(m > 0\) such that if a 3-form \(\phi_0\) verifies that \(\|\phi_0 - \varphi_0\|_{g_0} < m\), then \(\phi_0\) is a \(G_2\) form. We now check that \(m\) is the claimed universal constant. Let \((M, \varphi, g)\) be a \(G_2\) manifold; let \(\phi\) such that \(\|\phi_p - \varphi_p\|_{g_p} < m\) for every \(p \in M\). In order to check that \(\phi_p\) is a \(G_2\) form, let \(A : (T_pM, \varphi_p, g_p) \to (\mathbb{R}^7, \varphi_0, g_0)\) be an isomorphism of \(G_2\) vector spaces, then:

\[
\|A^t\phi_p - \varphi_0\|_{g_0} = \|\phi_p - \varphi_p\|_{g_p} < m
\]

and therefore \(A^t\phi_p\) is a \(G_2\) form. Since \(A\) is an isomorphism, \(\phi_p\) is also a \(G_2\) form. \(\square\)

In [19] Fernández and Gray classified \(G_2\) structures \((M, \varphi, g)\) into 16 types according to \(\nabla\varphi\), where \(\nabla\) denotes the Levi-Civita connection associated to \(g\). The motivation for such classification is the holonomy principle, saying that the holonomy of \(g\) is contained in \(G_2\) if and only if \(\nabla\varphi = 0\). In [19] they also prove that \(\nabla\varphi = 0\) if and only if \(d\varphi = 0\) and \(d(\star\varphi) = 0\), where \(\star\) denotes the Hodge star.
In this paper we are interested in torsion-free and closed $G_2$ structures on manifolds and orbifolds that we now define:

**Definition 7.** Let $(M, g, \varphi)$ or $(X, g, \varphi)$ a $G_2$ structure on a manifold or an orbifold. We say the $G_2$ structure is closed if $d\varphi = 0$. If in addition $d(*\varphi) = 0$ we say that the $G_2$ structure is torsion-free.

**Definition 8.** Let $(X, g, \varphi)$ be a closed $G_2$ structure on a 7-dimensional orbifold. A closed $G_2$ resolution of $(X, \varphi)$ consists of a smooth manifold endowed with a closed $G_2$ structure $(\tilde{X}, \tilde{\varphi})$ and a map $\rho: \tilde{X} \to X$ such that:

1. Let $S \subset X$ be the singular locus and $E = \rho^{-1}(S)$. Then, $\rho|_{\tilde{X}-E}: \tilde{X} - E \to X - S$ is a diffeomorphism.
2. Outside a neighbourhood of $E$, $\rho^*(\varphi) = \phi$.

The subset $E$ is called exceptional locus.

2.2.1. $G_2$ involutions.

**Definition 9.** Let $(M, \varphi)$ be a $G_2$ manifold, we say that $j: M \to M$ is a $G_2$ involution if $j^*(\varphi) = \varphi$, $j^2 = \text{Id}$, and $j \neq \text{Id}$.

In this paper we shall focus on orbifolds that are obtained as a quotient of a closed $G_2$ manifold $(M, \varphi)$ by the action of a $G_2$ involution $j$; that is $X = M/j$. The next result states that the fixed locus $L$ of $j$ is a 3-dimensional submanifold.

**Lemma 10.** The submanifold $L$ is 3-dimensional and oriented by $\varphi|_L$. In addition, $\varphi|_L$ is the oriented unit-length volume form determined by the metric $g|_L$.

**Proof.** The result is deduced from the fact that if $(\mathbb{R}^7, \varphi_0, \langle \cdot, \cdot \rangle)$ is the standard $G_2$ structure on $\mathbb{R}^7$ and if $j \in G_2$ is an involution, $j \neq \text{Id}$, then $j$ is diagonalizable with eigenvalues $\pm 1$ and $\dim(V_1) = 3$, $\dim(V_{-1}) = 4$; denote $V_{\pm 1}$ the eigenspace associated to the eigenvalue $\pm 1$. In addition, $\varphi_0(v_1, v_2, v_3) = \pm 1$ if $(v_1, v_2, v_3)$ is an orthogonal basis of $V_1$.

We now prove this statement; first $j$ is diagonalizable with eigenvalues $\pm 1$ because $j^2 = \text{Id}$, $j \neq \text{Id}$ and $j \in \text{SO}(7)$. Let us take a unit-length vector $v_1 \in V_1$; the vector space $W = \langle v_1 \rangle^\perp$ is fixed by $j$ because $j \in \text{SO}(7)$, and carries in addition an SU(3) structure determined by $\omega = i(v_1)\varphi_0$, $\Re(\Omega) = \varphi_0|_W$ (see [35]). Of course, the SU(3) structure is preserved by $j$. Viewed as a complex map, $j: W \to W$ has three complex eigenvalues $\lambda_1, \lambda_2, \lambda_3$ that verify $\lambda_3^2 = 1$ and $\lambda_1\lambda_2\lambda_3 = 1$ because $j^2 = \text{Id}$ and $j$ preserves the SU(3) structure. Being $j \neq \text{Id}$, we obtain that $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -1$ up to a permutation of the indices; this proves that $\dim(V_1) = 3$ and $\dim(V_{-1}) = 4$. Now observe that $j(u \times v) = j(u) \times j(v)$, where $\times$ is the cross product on $\mathbb{R}^7$ that determines $\varphi$. Thus, let $(v_1, v_2, v_3)$ be an orthogonal basis of $V_1$, then $v_1 \times v_2 \in V_1$; so necessarily, $v_1 \times v_2 = \pm v_3$ and $\varphi_0(v_1, v_2, v_3) = \pm 1$. □

**Remark 11.** If $d\varphi = 0$, Lemma 10 states that $L$ is a calibrated submanifold of $M$ in the sense of [23].

2.2.2. SU(2) structures. Let us identify $\mathbb{R}^4$ with $\mathbb{H}$ and identify SU(2) with Sp(1) as usual. The multiplication by $i$, $j$ and $k$ on the quaternions yields Sp(1)-equivariant
endomorphisms $I$, $J$ and $K$ that determine invariant 2-forms by the contraction of these endomorphism with the scalar product on $\mathbb{R}^4$. In coordinates, these are:

$$
\omega_1^0 = w^{12} + w^{34}, \quad \omega_2^0 = w^{13} - w^{24}, \quad \omega_3^0 = w^{14} + w^{23}.
$$

(2)

where $(w_1, w_2, w_3, w_4)$ denotes the standard basis of $\mathbb{R}^4$.

**Definition 12.** Let $W$ be a real vector space of dimension 4. An SU(2) structure on $W$ is determined by 2-forms $(\omega_1, \omega_2, \omega_3)$ such that there is a linear isomorphism $u : W \to \mathbb{R}^4$ with $u^*(\omega_j^0) = \omega_j$, where the forms $\omega_j^0$ are given by equation (2).

An SU(2) structure on a vector space $W$ determines a $G_2$ structure on $W \oplus \mathbb{R}^3$. To check this we can suppose that $(W, \omega_1, \omega_2, \omega_3) = (\mathbb{R}^4, \omega_1^0, \omega_2^0, \omega_3^0)$. Denote by $(v_5, v_6, v_7)$ the standard basis of $\mathbb{R}^3$ and define $u_5 = v_7$, $u_6 = v_5$ and $u_7 = -v_6$; then

$$
\varphi_0 = u^{567} + \omega_0^0 \wedge u^5 + \omega_0^0 \wedge u^6 + \omega_0^0 \wedge u^7.
$$

(3)

In addition if we fix on $\mathbb{R}^3$ the orientation determined by $u^{567}$, then $W$ is oriented by $\frac{1}{2}(\omega_1^0)^2$.

**Definition 13.** Let $N$ be a 4-dimensional manifold. An SU(2) structure on $N$ consists of 2-forms $(\omega_1, \omega_2, \omega_3) \in \Omega^2(N)$ that determine an SU(2) structure on $T_pN$ for every $p \in N$. In addition, if $d\omega_1 = d\omega_2 = d\omega_3 = 0$ we say that $(\omega_1, \omega_2, \omega_3)$ is a hyperKähler structure.

Let $Y$ be a 4-dimensional orbifold with atlas $\{(U_\alpha, V_\alpha, \psi_\alpha, \Gamma_\alpha)\}$. An SU(2) structure on $Y$ consists of 2-forms $(\omega_1, \omega_2, \omega_3) \in \Omega^2(Y)$ such that $(\omega_1^0, \omega_2^0, \omega_3^0)$ is an SU(2) structure on $U_\alpha$. In addition, if $d\omega_1 = d\omega_2 = d\omega_3 = 0$ we say that $(\omega_1, \omega_2, \omega_3)$ is a hyperKähler structure.

In view of Lemma 10, the local model of $X$ around $L$ is $(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{R}^3$, with $\mathbb{Z}_2 = \langle -\text{Id}, \text{Id} \rangle$. The standard $G_2$ form induces the orbifold hyperKähler SU(2) structure $(\omega_1^0, \omega_2^0, \omega_3^0)$ on $\mathbb{C}^2/\mathbb{Z}_2$. We now detail the hyperKähler resolution of $Y = \mathbb{C}^2/\mathbb{Z}_2$; this will be useful in order to construct the resolution of $X$ in section 5.

The holomorphic resolution of $Y$ is $N = \mathbb{C}^2/\mathbb{Z}_2$; where $\tilde{\mathbb{C}}^2$ is the blow-up of $\mathbb{C}^2$ at 0. That is,

$$
\tilde{\mathbb{C}}^2 = \{(z_1, z_2, \ell) \in \mathbb{C}^2 \times \mathbb{CP}^1 \text{ s.t. } (z_1, z_2) \in \ell\},
$$

and the action of $-\text{Id}$ lifts to $\{(z_1, z_2, \ell) \mapsto (-z_1, -z_2, \ell)\}$. We shall call the exceptional divisor $E = \{0\} \times \mathbb{CP}^1 \subset N$. Note that there is a well-defined projection $\sigma_0 : N \to \mathbb{CP}^1$.

Let us consider $r_0 : Y \to [0, \infty)$ the radial function induced from $\mathbb{C}^2$; one can check taking coordinates that $r_0^2$ is not smooth on $N$, but $r_0^4$ is.

Consider the blow up map, $\chi_0 : N \to Y$. Then, one can check that $\chi_0^*(\omega_2^0)$ and $\chi^*(\omega_3^0)$ are non-degenerate smooth forms on $N$; this holds because $\omega_2^0 + i\omega_3^0 = dz_1 \wedge dz_2$ and the pullback of a holomorphic form under a holomorphic resolution is holomorphic.

A computation in coordinates shows that $\chi_0^*(\omega_1^0)$ has a pole on $\mathbb{CP}^1$. Define $f(x) = g(x) + 2 \log(x)$, where $g(x) = (x^4 + 1)^{1/2} - \log((x^4 + 1)^{1/2} + 1)$. Consider on $Y - E$:

$$
\omega_1^0 = -\frac{1}{4} d\text{Id} f(r_0).
$$
One can check that $(\hat{\omega}_0, \chi^\ast_0(\omega_0^2), \chi^\ast_0(\omega_0^3))$ is a hyperKähler structure on $N - E$; it can be extended as a hyperKähler structure on $N$ because:

$$\frac{1}{4}dId(\log(r_0^2)) = \sigma^\ast_0(\omega_{\mathbb{CP}^1}),$$

where $\omega_{\mathbb{CP}^1}$ stands for the Fubini-Study form of $\mathbb{CP}^1$.

2.3. Formality. In this section we review some definitions and results about formal manifolds and formal orbifolds; basic references are [15], [14] and [34].

We work with commutative differential graded algebras (in the sequel CDGAs); these consist of a pairs $(A, d)$ where $A$ is a commutative graded algebra $A = \oplus_{i \geq 0} A^i$ over $\mathbb{R}$, and $d: A^i \to A^{i+1}$ is a differential, which is a graded derivation that verifies $d^2 = 0$. If $a \in A$ is a homogenous element, we denote its degree by $|a|$, and $\bar{a} = (-1)^{|a|} a$.

The cohomology algebra of a CDGA $(A, d)$ is denoted by $H^\ast(A, d)$; it is also a CDGA with the differential being zero. If $a \in A$ is a closed element we denote its cohomology class by $[a]$. The CDGA $(A, d)$ is said to be connected if $H^0(A, d) = \mathbb{R}$.

In our context, the main examples of CDGAs are the de Rham complex of a manifold or an orbifold. In section 5 we also make use of the Chevalley-Eilenberg CDGA of a Lie group $G$, that consists of the algebra $\Lambda^\ast g^\ast$, the differential of a 1-form is $d\alpha(x, y) = -\alpha[x, y]$, and is extended to $\Lambda^\ast g^\ast$ as a graded derivation.

Definition 14. A CDGA $(A, d)$ is said to be minimal if:

1. $A$ is free as an algebra, that is $A$ is the free algebra $\Lambda V$ over a graded vector space $V = \oplus_i V^i$.
2. There is a collection of generators $\{a_i\}$ indexed by some well ordered set, such that $|a_i| \leq |a_j|$ if $i < j$ and each $da_j$ is expressed in terms of the previous $a_i$ with $i < j$.

Morphisms between CDGAs are required to preserve the degree and to commute with the differential; a morphism of CDGAs $\kappa: (B, d) \to (A, d)$ is said to be a quasi-isomorphism if it induces an isomorphism on cohomology $\kappa: H^\ast(B, d) \to H^\ast(A, d)$.

Definition 15. A CDGA $(B, d)$ is a model of the CDGA $(A, d)$ if there exists a quasi-isomorphism $\kappa: (B, d) \to (A, d)$. If $(B, d)$ is minimal we say that $(B, d)$ is a minimal model of $(A, d)$.

Minimal models of connected DGAs exist and are unique up to isomorphism of CDGAs. So we define the minimal model of a connected manifold or a connected orbifold as the minimal model of its associated de Rham complex.

Definition 16. A minimal algebra $(\Lambda V, d)$ is formal if there exists a quasi-isomorphism,

$$(\Lambda V, d) \to (H^\ast(\Lambda V, d), 0).$$

A manifold or an orbifold is formal if its minimal model is formal.

Triple Massey products detect non-formality of manifolds; we now recall their definition. Let $(A, d)$ be a CDGA and let $\xi_1, \xi_2, \xi_3$ be cohomology classes such that $\xi_1\xi_2 = 0$ and $\xi_2\xi_3 = 0$. Under these assumptions we can define the triple Massey product of these
cohomology classes, that is denoted by \( \langle \xi_1, \xi_2, \xi_3 \rangle \). In order to provide its definition we first introduce the concept of a defining system for \( \langle \xi_1, \xi_2, \xi_3 \rangle \).

**Definition 17.** A defining system for \( \langle \xi_1, \xi_2, \xi_3 \rangle \) is an element \((a_1, a_2, a_3, a_{12}, a_{23})\) such that:

1. \( [a_i] = \xi_i \) for \( 1 \leq i \leq 3 \),
2. \( da_{12} = \bar{a}_1a_2 \), and \( da_{23} = \bar{a}_2a_3 \).

One can check that \( \bar{a}_1a_{23} + \bar{a}_{12}a_3 \) is a closed \((|a_1| + |a_2| + |a_3| - 1)\)-form. The triple Massey product \( \langle \xi_1, \xi_2, \xi_3 \rangle \) is the set formed by the cohomology classes determined by defining systems; that is:

\[
\{ \bar{a}_1a_{23} + \bar{a}_{12}a_3 \text{ s.t. } (a_1, a_2, a_3, a_{12}, a_{23}) \text{ runs over all defining systems} \}
\]

If \( 0 \in \langle \xi_1, \xi_2, \xi_3 \rangle \) we say that the triple Massey product is trivial.

**Theorem 18.** Let \((\Lambda V, d)\) be a formal minimal algebra. Let \(\xi_1, \xi_2, \xi_3\) be cohomology classes such that the triple Massey product \(\langle \xi_1, \xi_2, \xi_3 \rangle\) is defined. Then \(\langle \xi_1, \xi_2, \xi_3 \rangle\) is trivial.

As a consequence, we obtain:

**Corollary 19.** Let \((\Lambda V, d)\) be the minimal model of \((A, d)\). Let \(\xi_1, \xi_2, \xi_3 \in H^*(A, d)\) such that the triple Massey product \(\langle \xi_1, \xi_2, \xi_3 \rangle\) is defined. If \(\langle \xi_1, \xi_2, \xi_3 \rangle\) is not trivial then \((\Lambda V, d)\) is not formal.

**Proof.** Suppose that \((\Lambda V, d)\) is formal and let \(\kappa: (\Lambda V, d) \to (A, d)\) be a quasi-isomorphism. Let us take cohomology classes \(\xi'_1, \xi'_2, \xi'_3 \in H^*(\Lambda V, d)\) with \(\kappa(\xi'_j) = \xi_j\); then the Massey product \(\langle \xi'_1, \xi'_2, \xi'_3 \rangle\) is well-defined and there is a defining system \((a_1, a_2, a_3, a_{12}, a_{23})\) such that

\[
\bar{a}_1a_{23} + \bar{a}_{12}a_3 = da_x.
\]

But of course \(0 = \kappa[\bar{a}_1a_{23} + \bar{a}_{12}a_3] \in \langle \xi_1, \xi_2, \xi_3 \rangle\); yielding a contradiction. \(\square\)

We finally outline some aspects about finite group actions on minimal models. Let \(M\) be a manifold and let \(\kappa: (\Lambda V, d) \to (\Omega(M), d)\) be the minimal model. Let \(\Gamma\) be a finite subgroup of \(\text{Diff}(M)\) acting on the left; the pullback of forms defines a right action of \(\Gamma\) on \((\Omega(M), d)\).

Lifting theorems for CDGAs ensure the existence of a morphism \(\overline{\gamma}: \Lambda V \to \Lambda V\) that lifts up to homotopy the pullback by each \(\gamma \in \Gamma\); that is, \(\kappa \circ \overline{\gamma} \sim \gamma^* \circ \kappa\); in particular, \([\kappa(\overline{\gamma}(a))] = [\gamma^* \kappa(a)]\) if \(da = 0\). This implies that \(\overline{\Id} \sim \Id\) and that \(\overline{\gamma\gamma'} \sim \overline{\gamma} \overline{\gamma'}\); therefore these liftings provide an homotopy action. These liftings can be modified making use of group cohomology techniques (see [33, Theorem 2]) in order to endow \(\Lambda V\) with a right action.

**Theorem 20.** Let \(M\) be a connected manifold and let \(\Gamma\) be a subgroup of \(\text{Diff}(M)\) acting on the left.

There is a right action of \(\Gamma\) on the minimal model \(\kappa: (\Lambda V, d) \to (\Omega(M), d)\) by morphisms of CDGAs such that \([\kappa(a\gamma)] = [\gamma^* \kappa(a)]\) for every closed element \(a \in \Lambda V\) and every \(\gamma \in \Gamma\).

If there is a right action of a finite group \(\Gamma\) on a CDGA \((A, d)\) one can consider the CDGA of \(\Gamma\)-invariant elements \((A^\Gamma, d)\). An average argument leads us to \(H^*(A, d)^\Gamma = \)
$H^*(A^Γ, d)$. In addition, if $Γ$ also acts on $(B, d)$ on the right by morphisms and $ı: (A, d) → (B, d)$ is a morphism such that $[ı(aγ)] = [(ıa)γ]$ for every closed $a ∈ A$ and $γ ∈ Γ$ one can define:

$$1: (A^Γ, d) → (B^Γ, d), \quad ıa = |Γ|^{-1} \sum_{γ ∈ Γ} ı(a)γ,$$

where $|Γ|$ denotes the cardinal number of $Γ$. This verifies that $[ıF(a)] = [F(a)]$ for closed elements $a ∈ A$. In particular if $ı$ is a quasi-isomorphism so is $1$.

**Lemma 21.** Let $Γ$ be a finite group acting on a connected manifold $M$ by diffeomorphisms. If $M$ is formal then $M/Γ$ is also formal.

**Proof.** First of all, the fact that $(Ω(M/Γ), d) = (Ω(M)^Γ, d)$ and our previous argument ensure that $H^*(M/Γ, R) = H^*(M, R)^Γ$. Let $κ: (ΛV, d) → (Ω(M), d)$ be the minimal model of $M$ as constructed in Theorem 20. The CDGA $(ΛV^Γ, d)$ is a model for $(Ω(M/Γ), d)$ because of the quasi-isomorphism $κ: ((ΛV)^Γ, d) → (Ω(M)^Γ, d)$ as defined above. Consider $(ΛW, d)$ the minimal model of $(Ω(M/Γ), d)$ and let $ψ: (ΛW, d) → (ΛV^Γ, d)$ be a quasi isomorphism.

Being $M$ formal one can consider a quasi-isomorphism $ı: (ΛV, d) → (H^*(ΛV, d), 0)$ and define $1: (ΛV^Γ, d) → (H^*(ΛV, d)^Γ, 0) = (H(ΛW, d), 0)$, which is also a quasi-isomorphism. Then we can construct a quasi isomorphism:

$$1 \circ ψ: (AW, d) → (H^*(AW, d), 0).$$

Therefore, $M/Γ$ is formal.

\[ \square \]

3. **Resolution process**

Let $(M, ϕ, g)$ be a closed $G_2$ structure on a compact manifold $M$, let $j: M → M$ be a $G_2$ involution and let $X = M/j$. The singular locus of the closed $G_2$ orbifold $(X, ϕ, g)$ is the set $L = Fix(j)$, a 3-dimensional oriented manifold according to Lemma 10. This section is devoted to construct a resolution $ρ: Y → X$ under the extra assumption that $L$ has a nowhere-vanishing closed 1-form $θ ∈ Ω^1(L)$.

This yields a topological characterisation of $L$ that we now outline. Let us denote by $L_1, \ldots, L_r$ the connected components of $L$; according to Tischler’s Theorem \[37\] each $L_i$ is a fibre bundle over $S^1$ with fibre a connected surface $Σ_i$; that is, $L_i$ is a mapping torus over $Σ_i$ via a diffeomorphism $ψ_i ∈ Diff(Σ_i)$:

$$L_i = Σ_i × [0, 1]/(x, 0) ∼ (ψ_i(x), 1).$$

Let us denote $q_i: Σ_i × [0, 1] → L_i$ the quotient map and $b_i: L_i → S^1$ the bundle map; then we can suppose that $θ|_{L_i} = b_i^*(θ_0)$, where $θ_0$ denotes the angular form on $S^1$. In addition, taking into account that $L_i$ is oriented we obtain that $Σ_i$ is oriented and $ψ_i^* = Id$ on $H^2(Σ_i, Z)$.

The resolution process consists of replacing a neighbourhood of $L$ with a closed $G_2$ manifold. The local model of the singularity is $R^3 × Y$ where $Y = C^2/Z_2$ as we discussed in section \[2\] The closed $G_2$ manifold that we introduce is the blow-up of $ν/j$ at the zero section, where $ν$ denotes the normal bundle of $L$ in $M$. Its local model is $R^3 × N$ where $N = \hat{C}^2/Z_2$. This requires the choice of complex structure on $ν/j$ which is determined by a choice of a unit-length vector $V$ on $L$ by the expression $I(X) = V × X$, where $×$ is the
cross-product associated to \( \varphi \). This vector field exists because \( L \) is parallelizable; but we need to choose \( V = \| \theta \|^{-1} \theta \) in order to guarantee that the \( G_2 \) form is closed.

Before constructing a \( G_2 \) form on the resolution we study the \( O(1) \)-part of \( \exp^*(\varphi) \), where \( \exp : \nu \to M \) denotes the exponential map. We obtain a formula for this part that resembles the standard \( G_2 \) structure on \( \mathbb{R}^3 \times Y \) once we split \( T \nu \) into a horizontal and a vertical bundle with the aid of a connection. The pullback of this form under the blow-up map has a pole at the zero section that can be resolved by mimicking the formulas for the hyperKähler local model. However, we need to consider a closed approximation of the \( O(1) \) part because it is not closed in general. In addition, the resolution process requires the introduction of a parameter \( t \) in order to make a dilation of the vertical part of the bundle that reduces the errors derived from the fact that the curvature of the tautological line bundle of \( \nu \) is not in general a vertical form.

This section is organized as follows: in subsection 3.1 we introduce some notations concerning the normal bundle \( \nu \) of \( L \) and understand its second order Taylor approximation \( \phi_2 \) in subsection 3.2; this is an auxiliary construction. Then in subsection 3.3 we obtain local formulas for the \( O(1) \)-terms and introduce the parameter \( t \); these tools allow us to perform the resolution in 3.4.

### 3.1. Splitting of the normal bundle

We now introduce some notations that will be useful for the resolution process. Let \( \pi : \nu \to L \) be the normal bundle of \( L \). We consider \( R > 0 \) such that the neighbourhood of the 0 section \( Z \), \( \nu_R = \{ v_p \in \nu \text{ s.t. } \| v_p \| < R \} \) is diffeomorphic to a neighbourhood \( U \) of \( L \) on \( M \) via the exponential map. On \( \nu_R \) we consider \( \phi = (\exp)^* \varphi \), which is a closed \( G_2 \) form on \( \nu_R \). In addition, the induced involution on \( \nu \) is \( d\pi(v_p) = -v_p \); but we shall also denote it by \( \jmath \). It will be useful to denote the dilations by \( F_t : \nu \to \nu \), \( F_t(v_p) = tv_p \). We also define the vector field over \( \nu \), \( R(v_p) = \frac{d}{dt}|_{t=0} e^t v_p \).

We shall fix in subsection 3.3 a connection \( \nabla \) on \( \nu \) for our purposes, that induces a splitting \( T \nu = V \oplus H \) where \( V = \ker(d\pi) \cong \pi^* \nu \) and \( d\pi|_\nu : H|_\nu \to T_p L \) is an isomorphism. And since \( TM|_L = \nu \oplus TL \), the connection induces an isomorphism \( \mathcal{T} : T \nu \to \pi^*(TM|_L) \).

Note that any tensor \( T \) on \( TM|_L \) defines a tensor on \( \pi^*(TM|_L) \) because \( \pi^*(TM|_L)|_\nu = T_p M|_L \). Using this we define on \( \nu \):

1. A metric, \( g_1 = \mathcal{T}^*(g|_L) \); that is, \( g_1 \) makes \( (H|_\nu, g_1) \) and \( (T_p L, g) \) isometric, \( H|_\nu \) is perpendicular to \( V|_\nu \) and \( V|_\nu \) isometric to \( \nu_p \).
2. A \( G_2 \) structure \( \phi_1 = \mathcal{T}^*(\varphi|_L) \) with \( g_1 \) as an associated metric.

Of course, \( \mathcal{T} \) is an isometry. These tensors are constant in the fibres in the following sense; under the identification \( \tilde{T}_{v_p} = \mathcal{T}_0^{-1} \circ \mathcal{T}_v : T\nu \to T_0 \nu \) it holds that \( \tilde{T}_{v_p}(g_1) = g_1 \) and \( \tilde{T}_{v_p}^*(\phi_1) = \phi_1 \). Note also that these values coincide with \( \exp^* g|_Z \) and \( \phi \) respectively since \( (d\exp)|_{\nu_p} = \Id \). Thus, these tensors are independent from \( \nabla \) only on \( Z \).

We shall also denote \( W^k_{i,j} = \Lambda^k V^* \otimes \Lambda^j H^* \) where we understand \( V^* = \text{Ann}(H) \) and \( H^* = \text{Ann}(V) \). There are \( g_1 \)-orthogonal splittings \( \Lambda^k T^* \nu = \oplus_{i+j=k} W^k_{i,j} \) and given \( \alpha \in \Lambda^k T^* \nu \) we denote \( [\alpha]_{i,j} \) the projection of \( \alpha \) on \( W^k_{i,j} \).

Observe also that one can restrict each \( \beta \in \Lambda^k V^* \) to the fibre \( \nu_p \), and the restriction \( r_k : \Lambda^k V^* \to \Lambda^k T^* \nu \), \( r_k(\beta)_{v_p} = \beta_{v_p}|_{\nu_p} \) is an isomorphism because \( T_v \nu_p = V_{v_p} \).
We now state some technical observations concerning vertical forms; proofs are computations in terms of local coordinates that we include for completeness.

**Remark 22.** Note that $H^* = \pi^*(T^*L)$ does not depend on the connection but $V^*$ does. More precisely, in local coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, y_4) \in U \times \mathbb{R}^4$ the horizontal distribution at $(x, y)$ is generated by:

$$\partial_{x_i} - \sum_{j=1}^{4} A_i^j(x, y) \partial_{y_j},$$

where $A_i^j(x, y) = \sum_{k=1}^{4} A^j_{i,k}(x) y_k$ for some differentiable functions $A^j_{i,k}$. Then $V^*$ is generated by:

$$\eta_j = dy_j + \sum_{i=1}^{3} A_i^j(x, y) dx_i.$$

Note also that since $A^j_i(x, ty) = t A^j_i(x, y)$ we get that $F^*_t(\eta_i) = t \eta_i$.

**Lemma 23.** The following identities hold:

1. $F^*_t(\phi_1) = [\phi_1]_{0,3} + t^2 [\phi_1]_{2,1}$
2. $F^*_t(g_1) = g_1|_{H \otimes H} + t^2 g_1|_{V \otimes V}$

**Proof.** We shall prove the first equality being the second similar. Note that $\phi_1|_Z$ is a $G_2$ structure whose induced metric makes $V$ perpendicular to $H$ and $H|_Z = T\pi$; thus taking into account formula (3) we can write in local coordinates:

$$\phi_1|_Z = f(p) dx_1 \wedge dx_2 \wedge dx_3 + \sum_{i=1}^{3} \sum_{j<k} f_{ijk}(p) dx_i \wedge dy_j \wedge dy_k.$$

Thus, $\phi_1 = [\phi_1]_{0,3} + [\phi_1]_{2,1}$, where $([\phi_1]_{0,3})_{vp} = f(p) dx_1 \wedge dx_2 \wedge dx_3$ and $([\phi_1]_{2,1})_{vp} = \sum_{i=1}^{3} \sum_{j<k} f_{ijk}(p) dx_i \wedge (\eta_j)_{vp} \wedge (\eta_k)_{vp}$. Therefore, $F^*_t([\phi]_{0,3}) = [\phi]_{0,3}$ and according to the previous remark, $F^*_t([\phi]_{2,1}) = t^2 [\phi]_{2,1}$. \hfill $\square$

**Lemma 24.** Let $\mu \in V^*$ be a form such that $\mu = 0$ on $T\pi|_Z$. Then, $[d\mu]_{1,1} = 0$ and $[d\mu]_{0,2} = 0$ on $T\pi|_Z$.

**Proof.** In local coordinates, $\mu = \sum_{i=1}^{4} f_i(x, y) \eta_i$ with $f_i(x, 0) = 0$ as $\mu = 0$ on $T\pi|_Z$. Then,

$$d\mu = \sum_{i=1}^{4} \sum_{j=1}^{3} \frac{\partial f_i}{\partial x_j}(x, y) dx_j \wedge \eta_i + \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial f_i}{\partial y_j}(x, y) dy_j \wedge \eta_i + \sum_{i=1}^{4} f_i(x, y) d\eta_i.$$
Since $f_i(x, 0) = 0$ and $\eta_i|_{T\nu|_{z}} = dy_i$ the following equalities hold on $T\nu|_{z}$:

$$[d\mu]_{2,0}(x, 0) = \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{\partial f_i}{\partial y_j}(x, 0)dy_j \wedge dy_i,$$

$$[d\mu]_{1,1}(x, 0) = \sum_{i=1}^{4} \sum_{j=1}^{3} \frac{\partial f_i}{\partial x_j}(x, 0)dx_j \wedge \eta_i = 0,$$

$$[d\mu]_{0,2}(x, 0) = 0.$$

\[\square\]

**Lemma 25.** Consider coordinates $(x, y) = (x_1, x_2, x_3, y_1, y_2, y_3, y_4) \in B \times \mathbb{R}^4$ of $\nu$, with $B \subset \mathbb{R}^3$ a closed ball. Let $\eta_j$ be the projection of $dy_j$ to $V^*$ as in Remark 22. Then, $\|\eta_j\|_{(x,0)} = \|\eta_j\|_{(x,y)}$ and $\|(dx_i)_{(x,0)}\|_{g_1} = \|(dx_i)_{(x,y)}\|_{g_1}$.

In addition there exist $C_1 > 0$, $C_2 > 0$ that depend on $B$ such that $\|[d\eta]_{0,2}\|_{g_1} \leq C_1\|y\|_{g}$ and $\|[d\eta]_{1,1}\|_{g_1} \leq C_2$.

**Proof.** The first two equalities are clear taking into account that $T^*(\eta_j) = \eta_j$, $T^*(dx_j) = dx_j$ and that $T$ is a $g_1$-isometry. For the third and fourth equality we first compute $d\eta_i$

$$d\eta_i = \sum_{k=1}^{4} \sum_{j,l=1}^{3} \frac{\partial A_{i,k}^j}{\partial x_l} dx_l \wedge dx_j + \sum_{k,m=1}^{4} \sum_{j=1}^{3} A_{i,k}^j dy_m \wedge dx_j.$$

This implies that:

$$[d\eta]_{0,2} = \sum_{k=1}^{4} \sum_{j,l=1}^{3} \frac{\partial A_{i,k}^j}{\partial x_l} dx_l \wedge dx_j - \sum_{k,m,n=1}^{4} \sum_{j,l=1}^{3} A_{i,k}^j A_{m,n}^j(y_n dx_l \wedge dx_j),$$

$$[d\eta]_{1,1} = \sum_{k,m=1}^{4} \sum_{j=1}^{3} A_{i,k}^j dy_m \wedge dx_j.$$

The choice of the constants $C_1$ and $C_2$ becomes clear when one takes into account that the functions $|A_{i,j}^k|$ are bounded and that the $g_1$-norm of the terms $y_m \wedge dx_j$ and $dx_j \wedge dx_k$ are constant on the fibres as explained before. \[\square\]

### 3.2. Taylor series.

We now introduce the Taylor series of $\phi$ and interpolate it with the second order approximation. This is an auxiliary tool for our resolution process.

Consider the dilation over the fibres $F_t: \nu \to \nu$, and define the Taylor series of $F_t^*\phi$ and $F_t^*g$ near $t = 0$ (note that $F_0^*(\phi)$ and $F_0^*(g)$ are defined on $\nu$). That is,

$$F_t^*\phi \sim \sum_{k=0}^{\infty} t^{2k} \phi^{2k}, \quad F_t^*g \sim \sum_{k=0}^{\infty} t^{2k} g^{2k}.$$

Note that we only wrote even terms because both $\phi$ and $g$ are $J$ invariant and $J = F_{-1}$. In addition, since $F_{ts} = F_t \circ F_s$ we have that $F_{s}^*(\phi^{2k}) = s^{2k} \phi^{2k}$, $F_s^*(g^{2k}) = s^{2k} g^{2k}$. For $i + j = 3$ and $p + q = 2$ we define $\phi^{2k}_{i,j} = [\phi^{2k}]_{i,j}$, $g^{2k}_{p,q} [V^p \otimes H^q]$; here $V^p$ denotes the tensor product of $V$ with itself $p$ times.

We have the following properties:
(1) \( \| \phi^{2k}_{i,j} \|_{g_1} = O(r^{2k-i}) \), where \( r \) is measured with respect to the metric on \( \nu \). To check it let \( \| v_p \|_{g_1} = 1 \); taking into account Lemma 23 and the fact that \( F_r: (\nu, g_1)_{H \otimes H} + t^2 g_1|_{V \otimes V} \rightarrow (\nu, g_1) \) is an isometry we get:

\[
\| (\phi^{2k}_{i,j})_{v_p} \|_{g_1} = \| r^{2k} F_{r^{-1}}(\phi^{2k}_{i,j})_{v_p} \|_{g_1} = r^{2k} \| (\phi^{2k}_{i,j})_{v_p} \|_{g_1|_{H \otimes H} + r^2 g_1|_{V \otimes V}} = r^{2k-i} \| (\phi^{2k}_{i,j})_{v_p} \|_{g_1}.
\]

(2) The previous statement ensures that \( \phi^{2k}_{i,j} = 0 \) if \( i > 2k \).

(3) If \( k \geq 1 \), \( \phi^{2k} \) is exact.

Being \( \phi^{2k} \) homogeneous of order \( 2k \), we have that \( L_\lambda(\phi^{2k}) = 2k \phi^{2k} \); where

\[
\mathcal{R}(v_p) = \frac{d}{dt} \bigg|_{t=0} e^t(v_p) \quad \text{as defined above. In addition, since} \quad \phi \quad \text{is closed we have that} \quad d \phi^{2k} = 0 \quad \text{for every} \quad k. \quad \text{Thus,} \quad 2k \phi^{2k} = d(i(\mathcal{R}) \phi^{2k}).
\]

Taking these properties into account we construct a \( G_2 \) form \( \phi_{3,\varepsilon} \) that interpolates \( \phi \) with the approximation \( \phi_2 = \phi^0 + \phi^2 \). The parameter \( \varepsilon > 0 \) indicates that the interpolation occurs on \( r \leq \varepsilon \) and is done in such a way that \( \phi_{3,\varepsilon}|_{r \leq \frac{\varepsilon}{2}} = \phi_2 \). Of course, this is possible because the difference between \( \phi \) and \( \phi_2 \) is small near the zero section.

**Proposition 26.** Then form \( \phi_2 = \phi^0 + \phi^2 \) is closed and \( \phi = \phi_2 + O(r) \). There exists \( \varepsilon_0 > 0 \) such that for each \( \varepsilon < \varepsilon_0 \) there exists a \( 1 \)-invariant \( G_2 \) form \( \phi_{3,\varepsilon} \) such that \( \phi_{3,\varepsilon} = \phi_2 \) if \( r \leq \frac{\varepsilon}{2} \) and \( \phi_{3,\varepsilon} = \phi \) if \( r \geq \varepsilon \).

**Proof.** The first part is an easy consequence of the previous remark; the zero order terms are \( \phi^0 = \phi^0_{0,3} \) and \( \phi^2_{2,1} \) and thus \( \phi = \phi_2 + O(r) \). In addition, \( \phi_2 \) is closed because each \( \phi^{2k} \) is.

Since \( \phi|_Z = \phi_2|_Z \) Poincaré Lemma for submanifolds ensures that \( \phi = \phi_2 + d \xi \). Note that we can suppose that \( \xi \) is \( 1 \)-invariant because both \( \phi \) and \( \phi_2 \) are. In addition, \( \| \xi \|_{g_1} = O(r^2) \) because \( \| \phi - \phi_2 \|_{g_1} = O(r) \). Let \( \varpi \) be a smooth function such that \( \varpi = 1 \) if \( x \leq \frac{1}{2} \) and \( \varpi = 0 \) if \( x \geq 1 \) and define \( \varpi_{\varepsilon}(x) = \varpi(\frac{\varepsilon}{x}) \). Then, \( \| \varpi_{\varepsilon}' \| \leq \frac{C}{\varepsilon} \) so that the form:

\[
\phi_{3,\varepsilon} = \phi + d(\varpi_{\varepsilon}(r) \xi)
\]

is positive if \( \varepsilon \) is small enough because it is \( O(\varepsilon) \)-near \( \phi \) and it interpolates \( \phi_2 \) with \( \phi \) over the stated domains. Note that \( \phi_{3,\varepsilon} \) is \( 1 \)-invariant because both \( \phi \) and \( \varpi_{\varepsilon}(r) \xi \) are. \( \square \)

### 3.3. Local formulas.

The purpose of this section is making an additional preparation; we first provide a local formula for \( \phi_1 \) that will be useful in order to construct the \( G_2 \) form of the resolution. Later we change \( \phi_2 \) by \( O(r) \) terms so that we control its local formula and introduce the parameter \( t \); these preparations are essential to construct a closed \( G_2 \) form on the resolution.

#### 3.3.1. Formula for \( \phi_1 \).

We first write \( \phi_1 \) and \( g_1 \) in terms of the components of the Taylor series of \( g \) and \( \phi \). This is an easy consequence of the homogeneous behaviour of the tensors involved:

**Lemma 27.** The following equalities hold:

\begin{enumerate}
\item \( \phi_1 = \phi^0 + \phi^2_{2,1} \)
\item \( g_1 = g^0_{0,2} + g^2_{2,0} \)
\end{enumerate}
Proof. We prove the first equality, being the second similar. Using the fact that $\phi^0 = \phi^0_{\nu}$ and $\phi^2_{\nu}$ are homogeneous one can check that these are constant over the fibres. We shall do it for $\phi^2_{\nu}$, write in local coordinates $(x, y)$:

$$\phi^2_{\nu} = \sum_{i=1}^{3} \sum_{j<k} f_{ijk}(x, y) dx_i \wedge (\eta_j)(x, y) \wedge (\eta_k)(x, y).$$

Taking into account that $F^*_t \phi^2_{\nu} = t^2 \phi^2_{\nu}$ and $F^*_t \eta_t = t \eta_t$ we get $f_{ijk}(x, y) = f_{ijk}(x, 0)$. Therefore, $f_{ijk}(x, y) = f_{ijk}(x, 0)$. Since $\phi^1|_{TM|z} = \phi|_{TM|z} = (\phi^0 + \phi^2_{\nu})|_{TM|z}$, we obtain that $[\phi^1]_{0,3}|_{T\nu|z} = \phi^0|_{T\nu|z}$ and $[\phi^1]_{2,1}|_{T\nu|z} = \phi^2_{\nu}|_{T\nu|z}$. But these forms are constant on the fibres of the bundle $T\nu \to \nu$, so that the previous equalities hold on $T\nu$. 

In order to obtain the local formula for $\phi_1$, define $e_1 = \|\theta\|^{-1}\theta$ and complete it to an oriented orthonormal basis, $(e_1, e_2, e_3)$ on a neighbourhood $U \subseteq L$ define $\omega^l_1$, $\omega^l_2$, $\omega^l_3$ by means of the equality:

$$\varphi|_L = e_1 \wedge e_2 \wedge e_3 + e_1 \wedge \omega^l_1 + e_2 \wedge \omega^l_2 + e_3 \wedge \omega^l_3.$$  

As usual, $\omega^l_1, \omega^l_2, \omega^l_3$ induce an SU(2) structure on $\nu$; more precisely, the complex structure is determined by $\omega^l_1 = i(e^l_1)\varphi|_\nu$, that is, $I(X) = e^l_1 \times X$, with $X$ the vector product associated to $\varphi|_L$. The complex volume form is $\omega^l_2 + i\omega^l_3$; note that it is changed by a complex phase under a rotation in $e_2, e_3$.

Using $\mathcal{F}$:

$$\phi_1 = \pi^* e_1 \wedge \pi^* e_2 \wedge \pi^* e_3 + \pi^* e_1 \wedge \omega_1 + \pi^* e_2 \wedge \omega_2 + \pi^* e_3 \wedge \omega_3,$$

where $\omega_j \in \Lambda^2 V^*$ and $\omega_j|_z = \exp^*(\omega^l_1)$. The forms $\omega_1, \omega_2, \omega_3$ are of course $j$-invariant. Since the restriction $r_2$ is an isomorphism, we have that given $p \in L$ the forms $\omega_1|_{\nu_p}$, $\omega_2|_{\nu_p}$, and $\omega_3|_{\nu_p}$ define an SU(2) structure on the 4-manifold $\nu_p$. The associated metric on $T\nu_p$ is $g_1|_{\nu_p}$ and the complex form is the induced by $I$ on $\nu$ under the canonical isomorphism.

Therefore, $\omega_1|_{\nu_p} = -\frac{1}{4}d_{\nu_p}(I|d\nu^2|_{\nu_p})$. In addition, since the complex volume form is given by $dz_1 \wedge dz_2 = \frac{1}{2}d(z_1 dz_2 - z_2 dz_1)$ there is $\mu \in V^*$ such that $d_{\nu_p}(\mu|_{\nu_p}) = (\omega_2 + i\omega_3)|_{\nu_p}$ and $\mu|_{T\nu|z} = 0$. We shall denote $\mu = \mu_1 + i\mu_2$, which is $O(r)$.

Since the restriction to the fibre $r_2$ is a monomorphism,

$$\omega_1 = \frac{1}{4}[d(I\nu^2)|_{2,0}], \quad \omega_2 + i\omega_3 = [d\mu]|_{2,0},$$

here also denoted by $I$ the complex structure on $V^*$ determined by the complex structure $I(X) = e^l_1 \times X$ on $V = \pi^*(\nu)$, this depends on the splitting. Observe that the complex structure $I$ on $\nu$ verifies $j \circ I = I \circ j$ and thus, the complex structure on $V^*$ verifies $j\alpha = I\alpha$; in particular, $I\alpha$ is $j$-invariant if $\alpha$ is. Of course, $\mu$ is also chosen to be $j$-invariant because $\omega_2 + i\omega_3$ is.

3.3.2. Changing $\phi_2$ by $O(r)$ terms. First of all define the 1-parameter family

$$\phi_2^l = \phi^0 + t^2 \phi^2 = F^*_t(\phi_2),$$

which is well-defined on $\nu$ because the forms $\phi^0$ and $\phi^2$ are homogeneous. We now change this 1-parameter family by $O(r)$ terms so that we have an explicit local formula for it.
Consider the exact $j$-invariant form:

$$\beta = -\frac{1}{4} \pi^* \theta \land d((\|\theta\|^{-1} \circ \pi)I[dr^2]_{1,0}) + \sum_{j=2}^{3} d(\pi^* e_j \land \mu_j) \in W_{2,1} \oplus W_{1,2} \oplus W_{0,3},$$

and note that $\phi_1 = \pi^*(e_1 \land e_2 \land e_3) + [\beta]_{2,1}$. In addition, $\beta$ does not depend on the orthonormal oriented basis $(e_2, e_3)$ of $(\theta^*)^\perp$.

We now introduce a 1-parameter family of closed $j$-invariant forms:

$$\hat{\phi}_2^t = \pi^*(e_1 \land e_2 \land e_3) + t^2[\beta].$$

We claim that for each $s > 0$ there exists $t_s$ such that for each $t < t_s$ the form $\hat{\phi}_2^t$ is positive $\nu_{2s}$. To check this we compare $\hat{\phi}_2^t$ with $F_t^* \phi_1$ and use Lemma 6 to conclude. Denote $g_t = F_t^*(g_1)$ and observe that Lemma 27 implies that $F_t^* \phi_1 = \phi^0 + t^2 \phi_2^1$, and $g_t = t^2 g_{2,0} + g_{0,2}$, then:

$$\|F_t^* \phi_1 - \hat{\phi}_2^t\|_{g_t} = t \|[\beta]_{1,2}\|_{g_1} + t^2 \|[\beta]_{0,3}\|_{g_1},$$

so one can bound $\|[\beta]_{1,2}\|_{g_1}$, $\|[\beta]_{0,3}\|_{g_1}$ on $\nu_{2s}$ and chose $t_s > 0$ such that for each $t < t_s$, $t \|[\beta]_{1,2}\|_{g_1} + t^2 \|[\beta]_{0,3}\|_{g_1} < m$ where $m$ is the universal constant given by Lemma 6.

We construct a $G_2$ form $\hat{\phi}_{3,s}$ that interpolates $F_t^* \phi$ with $\phi_2^t$. The parameter $s > 0$ indicates that the interpolation occurs on the disk $r \leq s$ and we require that $\phi_{3,s}|_{r \leq \frac{s}{2}} = \phi_2$.

In subsection 3.3 we employ large values of the parameter.

**Proposition 28.** There is $\xi \in W_{0,2}$ such that $\|\xi\|_{g_1} = O(r^2)$ and $\phi^2 = \beta + d\xi$.

For each $s > 0$ there is $t'_s > 0$ such that for each $t < t'_s$, there is a closed $j$-invariant $G_2$ form $\hat{\phi}_{3,s}$ on $\nu_{2s}$ that coincides with $\hat{\phi}_2^t$ on $r \leq \frac{s}{2}$ and $\phi_2^t$ on $r \geq s$.

**Proof.** Write $\phi^2 = \phi_{2,1}^2 + \phi_{1,2}^2 + \phi_{0,3}^2$; then $\phi_{2,1}^2 = [\beta]_{2,1}$ so that, since $\beta$ and $\phi^2$ are closed: $d(\phi_{2,1}^2 + \phi_{0,3}^2) = d([\beta]_{2,1} + [\beta]_{0,3})$. Poincaré Lemma ensures that $\phi_{1,2}^2 + \phi_{0,3}^2 = [\beta]_{1,2} + [\beta]_{0,3} + d\xi$ with

$$\xi_{v_s} = \int_0^1 i(\mathcal{R}_{tv_s})(\phi_{1,2}^2 - \phi_{0,3}^2 - [\beta]_{1,2} + [\beta]_{0,3}) dt = \int_0^1 i(\mathcal{R}_{tv_s})(\phi_{1,2}^2 - [\beta]_{1,2}) dt,$$

Hence, $\xi \in W_{0,2}$; taking into account that $\phi_{1,2}^2 - [\beta]_{1,2}$ is $j$-invariant and that $\mathcal{R}_0(v_s) = j(\mathcal{R}_{tv_s})$ one can check that $\xi$ is also $j$-invariant.

In addition, $\|\xi\|_{g_1} = O(r^2)$ because $\phi_{1,2}^2 + \phi_{0,3}^2, Z = 0$ (these terms are $O(r)$ and $O(r^2)$ in the $g_1$-norm) and $([\beta]_{1,2} + [\beta]_{0,3}, Z = 0$ according to Lemma 24.

Let $\varpi$ be a smooth function such that $\varpi = 1$ if $x < \frac{r}{2}$ and $\varpi = 0$ if $x \geq 1$. Define $\varpi_s(x) = \varpi\left(\frac{s}{4}\right)$ and, $\hat{\phi}_{3,s} = \phi^0 + t^2 \beta + t^2 d(\varpi_s(r)\xi)$ which is a closed $j$-invariant form that coincides with $\phi_2^t$ on $r \leq \frac{s}{2}$ and $\phi_2^t$ on $r \geq s$.

It is clear that $\hat{\phi}_{3,s}$ is positive on the region $r \geq s$ for $t < t_s$; we now check that this form is positive on $r \leq s$ for some choice of $t$. We are going to compare $\hat{\phi}_{3,s}^t$ with $F_t^* \phi_1$ and use Lemma 6 to conclude the result.
Since \( \varpi_s \xi \in W_{0,2} \) we have that \( d(\varpi_s \xi) \in W_{1,2} \oplus W_{0,3} \). As a consequence \( \|t^2 d(\varpi_s (r) \xi)\|_{g_t} \leq t \|d(\varpi_s (r) \xi)\|_{g_t} = t(O(r^2 s^{-1}) + O(r)) \) so that:

\[
\|\tilde{\phi}_{3,3}^\ell - F_{t}^* \phi_1\|_{g_t} = t(\|\beta\|_{1,2} + t\|\beta\|_{0,3} + O(r^2 s^{-1}) + O(r)) \leq t(\|\beta\|_{1,2} + \|\beta\|_{0,3} + O(r)).
\]

For the last equality we used that \( t < 1 \) and that \( r \leq 2s \). The form \( \tilde{\phi}_{3,3}^\ell \) is positive if we choose \( t < t_s \) such that on \( \nu_{2s} \)

\[
t(m \leq \max_{r \leq s}(\|\beta\|_{1,2} + \|\beta\|_{0,3} + O(r)) < m
\]

where \( m \) is the constant provided by Lemma [6].

3.4. Resolution of \( \nu/\jmath \). For the resolution process, we inspire ourselves in the hyperKähler resolution \( N = \mathbb{C}^2/\mathbb{Z}_2 \) of \( Y = \mathbb{C}^2/\mathbb{Z}_2 \) described in section [2]. Recall that we denoted the blow-up map by \( \chi_0: N \to Y \) and the hyperKähler structure was defined by \( (\tilde{\omega}_0^1, \chi_0^* (\omega_2^0), \chi_0^* (\omega_0^0)) \) where \( \tilde{\omega}_0^1 \) was defined as extension of \( -\frac{i}{4} dId(f_0) \) with \( f_0 \) the radial function on \( \mathbb{C}^2 \) and:

\[
f(x) = g(x) + 2 \log(x), \quad g(x) = (x^4 + 1)^{1/2} - \log((x^4 + 1)^{1/2} + 1).
\]

We now focus in the resolution of \( \nu/\jmath \). For that purpose, recall that we denoted by \( I \) the complex structure on \( \nu \) determined by the 2-form \( \imath(e_1^2)_{\varphi} \), and define \( P \) as the fiberwise blow-up of \( \nu/\jmath \) at \( 0 \); that is, \( P = P_{U(2)}(\nu) \times_{U(2)} N \), where \( P_{U(2)}(\nu) \) denotes the principal \( U(2) \)-bundle associated to \( \nu \). This yield a projection \( \chi: P \to \nu/\jmath \) and \( pr = \tilde{\pi} \circ \chi \); here we denoted by \( \tilde{\pi} \) the map induced by \( \pi: \nu \to L \).

We also define \( Q = \chi^{-1}(0) \), which is a \( \mathbb{CP}^1 \) bundle over \( L \) that can be expressed as \( Q = P_{U(2)}(\nu) \times_{U(2)} \mathbb{CP}^1 \). Note that there is a projection \( \sigma_0: N \to \mathbb{CP}^1 \); and therefore we have a projection \( \sigma: P \to Q \), which is indeed a complex line bundle.

Note that a \( j \)-invariant tensor on \( \nu \) descends to \( \nu/\jmath \) and its pullback via \( \chi \) is smooth over \( P - Q \) but they may not be smooth on \( P \). But it will be smooth on \( P \) if it preserves the complex structure \( I \) on \( P \), because \( P = P_{U(2)}(\nu) \times_{U(2)} N \). Thus, we choose \( \nabla \) such that \( \nabla I = 0 \), so that we can lift \( \nabla \) to \( P \), and define \( TP = V' \oplus H' \) compatible with the splitting \( TV = V \oplus H \). In addition, \( \mu_2, \mu_3, \omega_1, \omega_2, \omega_3 \) induce forms on \( \nu/\jmath \) and \( \chi^* \mu_k, \chi^* \omega_k \) are smooth for \( k = \{2, 3\} \). We shall also consider \( \Lambda^k T^* P = \oplus_{i+j=k} \Lambda^i V' \otimes \Lambda^j H' \) and \( \{\alpha\} = \sum_{i,j} \{\alpha\}_{i,j} \).

In order to define a \( G_2 \) structure on \( P \) we only have to find a resolution of \( \omega_1 \). For that purpose denote by \( r \) the pullback of the radial function on \( \nu \) and define:

\[
\hat{\omega}_1 = -\frac{1}{4} d((\|\theta\|^{-1} \circ pr) I [df(r)]_{1,0}).
\]

Observe that \( g(r) \) is smooth on \( P \) because \( r^4 \) is. In addition, \( -\frac{1}{4} dI [d(\log(r^2))]_{1,0} = \sigma^* (F_Q) \) on \( P - Q \), where \( F_Q \) is the curvature of the line bundle \( \sigma: P \to Q \). Fiberwise it coincides with the Fubini-Study form on \( \mathbb{CP}^1 \). Note also that \( pr^* \theta \wedge [\hat{\omega}_1]_{2,0} = -\frac{1}{4} e_1 \wedge [d(I[df])_{1,0}]_{2,0} \).

We now define a \( G_2 \) form \( \Phi_1^* \) which is near \( \chi^* (F_{t}^* \phi_1) \) on \( r > 1 \), this is:

\[
\Phi_1^* = pr^* (e_1 \wedge e_2 \wedge e_3) + t^2 [\hat{\beta}]_{2,1},
\]

where \( \hat{\beta} = pr^* \theta \wedge [\hat{\omega}_1] + pr^* e_2 \wedge [\chi^* (\omega_2) + pr^* e_3 \wedge [\chi^* (\omega_3)]. \)
Observe that \( \beta \) does not depend on the orthonormal oriented basis \((e_2, e_3)\) of \( \langle \theta^* \rangle \). In addition, the metric induced by \( \Phi_1 \) on \( TP \) has the form \( h_t = h_{2,0} + h_{0,2} \) where \( h_{2,0} \) and \( h_{0,2} \) are metrics on \( V' \) and \( H' \) respectively. In addition, the metric that \( \Phi_1 \) induces is \( h_t = t^2 h_{2,0} + h_{0,2} \). We define a family of closed forms:

\[
\Phi^t_2 = pr^*(e_1 \wedge e_2 \wedge e_3) + t^2 \beta.
\]

Note that \( \Phi^t_2 \) is a \( G_2 \) structure on \( \nu_s \) for some \( t < t'_s \). This is ensured by Lemma 6 because:

\[
\| \Phi^t_2 - \Phi^1_1 \|_{h_t} = t\|\bar{\beta}_{1,2}\|_{h_t} + t^2\|\bar{\beta}_{0,3}\|_{h_t},
\]

and one can bound \( \|\bar{\beta}_{1,2}\|_{h_t} \) and \( \|\bar{\beta}_{0,3}\|_{h_t} \) on \( \chi^{-1}(\nu_s) \).

Observe that the parameter \( t \) is devoted to compensate the error introduced by \( \|\bar{\beta}_{1,2}\|_{h_t} \) and \( \|\bar{\beta}_{0,3}\|_{h_t} \) that mainly come from the terms \( [F^Q]_{1,2} \) and \( [F^Q]_{2,0} \), which are zero if and only if the curvature is vertical. In Lemma 62 we will see that the bundle \( Q \) is trivial, \( Q = L \times \mathbb{CP}^1 \); but it might not happen that \( P \) is the pullback of \( N \) via the projection map \( L \times \mathbb{CP}^1 \to \mathbb{CP}^1 \). In the case that it is, then \( F^Q \in \Lambda^2 V' \).

**Proposition 29.** There exist \( s_0 > 1 \), such that for each \( s > s_0 \) one can find \( t'_s \) such that for each \( t < t'_s \) there is a closed \( G_2 \) structure \( \Phi^t_{3,s} \) such that \( \Phi^t_{3,s} = \Phi^t_2 \) on \( r \geq \frac{\epsilon}{4} \) and \( \Phi^t_{3,s} = \chi^t(\bar{\phi}_{3,s}) \) on \( r \geq \frac{\epsilon}{2} \).

**Proof.** On the annulus \( \frac{\epsilon}{4} < r < \frac{\epsilon}{2} \) we have that:

\[
\Phi^t_2 - \chi^t(\bar{\phi}_{3,s}) = \frac{1}{4} t^2 d(pr^*e_1 \wedge (Id(f(r) - r^2)]_{1,0}).
\]

We now let \( \varpi \) be a smooth function such that \( \varpi = 1 \) if \( x \leq \frac{1}{4} \) and \( \varpi = 0 \) if \( x \geq \frac{1}{2} \) and \( \varpi_s(x) = \varpi(\frac{r}{s}) \). Then, \( \varpi_s' \leq \frac{C}{s} \); define

\[
\xi_s = \varpi_s pr^*e_1 \wedge (Id[\hat{f}(r)]_{1,0}).
\]

The form \( d\xi_s \) lies in \( W_{2,1} \oplus W_{1,2} \oplus W_{0,3} \). In order to analyze the \( h_t \) norm of each component first observe that \( \hat{f}(x) = f(x) - x^2 = \log(x) + \frac{1}{(x+1)^2 + z^2} \) verifies \( |\hat{f}| = O(x^{-1}) \) and \( |\hat{f}'| = O(x^{-2}) \) on \( x > 1 \).

In addition, note that if \( (x, y) = (x_1, x_2, x_3, y_1, y_2, y_3, y_4) \in B \times \mathbb{R}^4 \) is a complex unitary parametrisation; that is, in coordinates \( f(x, y) = (x_1, x_2, x_3, -y_1, y_2, -y_3, y_4) \) and the framing determined by the parametrisation is unitary. Moreover, connection forms verify \( I_{\eta_1} = -\eta_2, I_{\eta_3} = -\eta_4 \); to check this one has to observe that the matrices \( (A_{i,k})_{k,j} \) defined in Remark 22 are complex linear because \( \nabla I = 0 \). Taking this into account, a straightforward computation of the pullback yields the claim.

Taking these observations and Lemma 25 into account we obtain that on \( r > 1 \):

\[
\| [d\xi_s]_{2,1} \|_{h_t} = \| \varpi_s pr^*e_1 \wedge [Id[\hat{f}(r)]_{1,0}]_{2,0} \|_{h_t} + \| d\varpi_s \|_{2,0} \wedge pr^*e_1 \wedge Id[\hat{f}(r)]_{1,0} \|_{h_t} = O(t^{-2}) + O(t^{-1})
\]

\[
\| [d\xi_s]_{1,2} \|_{h_t} = \| \varpi_s pr^*e_1 \wedge [Id[\hat{f}(r)]_{1,0}]_{1,1} \|_{h_t} + \varpi pr^*(de_1) \wedge Id[\hat{f}(r)]_{1,0}
+ \| d\varpi \|_{1,0} \wedge pr^*e_1 \wedge Id[\hat{f}(r)]_{1,0} \|_{h_t} = O(t^{-2}) + O(t^{-1}) + O(r^{-1}s^{-1})
\]

\[
\| [d\xi_s]_{0,3} \|_{h_t} = \| \varpi_s pr^*e_1 \wedge [Id[\hat{f}(r)]_{1,0}]_{0,2} \|_{h_t} = O(1).
\]
We prove that \( \|[d\varpi_\sigma]_{1,0} \wedge pr^*e_1 \wedge Id[\tilde{f}(r)]_{1,0}\|_{h_1} = O(r^{-1} s^{-1}) \) and \( \|[d\varpi_\sigma]_{0,1} \wedge pr^*e_1 \wedge Id[\tilde{f}(r)]_{1,0}\|_{h_1} = O(s^{-1}) \) more explicitly; the rest of terms are similar. We first trivialize \( \nu \) using orthonormal complex coordinates \((x, y)\) and taking into account Lemma 25 we obtain that \( \|[Id\tilde{f}(r)]_{1,0}\|_{h_1} = \|[\sum_{j=1}^4 \tilde{f}(r)\partial y_j]_{1,0}\|_{h_1} = O(r^{-1}) \). On the other hand, \( \varpi_s(x, y) = \varpi_s(y) \) and thus

\[
[d\varpi_s]_{1,0} = \sum_{i=1}^4 \frac{\partial \varpi_s}{\partial y_i} y_i 
\]

Taking into account that \( A^j_i(x, y) = O(r) \) we obtain that \( \|[d\varpi_\sigma]\|_{h_1} = O(s^{-1}), \|[d\varpi_\sigma]_{0,1}\|_{h_1} = O(rs^{-1}) \). A multiplication yields the desired estimates. The rest are obtained taking derivatives of

\[
[Id\tilde{f}]_{1,0} = \frac{\tilde{f}'(r)}{r} (-y_1 \eta_2 + y_2 \eta_1 - y_3 \eta_4 + y_4 \eta_3),
\]

and using Lemma 25. Our estimates yield:

\[
\|t^2 d\xi_s\|_{h_1} = O(r^{-2}) + t(0(r^{-2}) + O(r^{-2}) + O(r^{-1} s^{-1}) + O(s^{-1}) + O(s^{-1})) + t^2 O(1)
\]

Thus, one can take \( s_0 \) such that for each \( 0 < t < 1 \) and \( s > s_0 \) it holds that \( |O(r^{-2}) + t(0(r^{-2}) + O(r^{-2}) + O(r^{-1} s^{-1}))| < m \) on \( \frac{s}{t} < r < \frac{s}{t} \). Thus, let \( s > s_0 \) and take \( t''_s < t_s \) such that \( |t^2 O(1)| < m \) and \( \|\Phi_2 - \Phi'_1\|_{h_1} < \frac{\alpha}{2} \), which is possible as we argued before. Define the closed form

\[
\Phi'_{3,s} = \Phi_2 - \frac{t^2}{4} d\xi_s.
\]

which coincides with \( \Phi'_{2} \) if \( r \leq \frac{s}{4} \) and with \( \chi^*(\Phi'_{3,s}) \) if \( r \geq \frac{s}{4} \). On the neck \( \frac{s}{4} \leq r < \frac{s}{2} \) we have by construction that:

\[
\|\Phi'_{3,s} - \Phi'_{2}\|_{h_1} \leq \|\Phi'_{3,s} - \Phi'_{2}\|_{h_1} + \|\Phi'_2 - \Phi'_1\|_{h_1} < m.
\]

The statement is therefore proved.

The map \( F_t \circ \chi \) allows us to glue an annulus around the zero section on \((\nu/j, \phi_2)\) and an annulus around \( Q \) on \((P, \Phi'_2)\); this yields a resolution.

**Theorem 30.** There exists a closed G\(_2\) resolution \( \tilde{X} \to X \). In addition, let us denote \( D_s(Q) \) the \( s\)-disk of \( P \) centered at \( Q \); then

\[
\tilde{X} = X - \exp(\nu_s/j) \cup_{\exp \circ F_t \circ \chi} D_s(Q)
\]

for some \( \varepsilon > 0 \), \( t > 0 \) and \( s > 0 \).

**Proof.** Let \( \varepsilon_0 < R \) given by Proposition 20 and take \( s > s_0 \), chose \( t < t''_s \) with \( st = \frac{s}{t} \) for some \( \varepsilon < \varepsilon_0 \). The map \( F_t \circ \chi \) identifies \( s \leq r \leq 2s \) on \( P \) with \( \frac{s}{4} \leq r \leq \frac{s}{2} \) on \( \nu/j \).

On \( \chi^{-1}(\nu_2/s/j) \) we consider \( \Phi'_{3,s} \) and on \( \nu_2/s/j \) we consider \( \phi_{3,s} \); on the annulus \( s \leq r \leq 2s \) of \( \chi^{-1}(\nu_2/s/j) \) we have that \( \Phi'_{3,s} = \chi^*(\phi'_{3,s}) = \phi'_2 \) and on \( \frac{s}{4} \leq r \leq \frac{s}{2} \) on \( \nu/j \) we have that \( \phi'_{3,s} = \phi_2 \).

Since \( (F_t \circ \chi)^* \phi_2 = \chi^*(\phi'_2) \), the G\(_2\) structure is well defined on the resolution. \( \square \)
4. Topology of the resolution

This section is devoted to understanding the cohomology algebra of the resolution; we shall make use of real coefficients and denote by $H^*(M)$ the algebra $H^*(M, \mathbb{R})$. We start by describing $H^*(\tilde{X})$ in terms of $H^*(X)$ and $H^*(L)$ and we then compute the induced product on it.

The fibre bundle $\nu$ is topologically trivial; this follows from the fact that every 3-manifold is parallelizable. For a proof see [27] Remark 2.14. However, it might not be trivial as a complex bundle as we shall deduce from the computation of its total Chern class.

Let us suppose for a moment that $L$ is connected; then $L$ is a mapping torus over an orientable surface $\Sigma$ of genus $g$, via a diffeomorphism $\psi: \Sigma \to \Sigma$. We denoted by $q$ the quotient projection; that is, $q: \Sigma \times [0,1] \to L$, and by $b: L \to S^1$ the bundle projection.

In section 3 we chose that $\theta = b^*(\theta_0)$ with $\theta_0$ the angular form on $S^1$.

We compute the total Chern class of $\nu$; this is done by observing that $\nu$ admits a section and thus $\nu = \mathbb{C} \oplus \ker \theta$; where $\mathbb{C}$ denotes the trivial line bundle over $L$. In addition, $\ker(\theta)$ coincides with the tangent space of the fibres because $\theta = b^*(\theta_0)$. Taking this into account the computation of $c(\nu)$ easily follows.

It shall be useful to note that 2-forms on $\Sigma$ determine closed 2-forms on $L$ because $\psi^* = \text{Id}$ on $H^2(\Sigma)$. More precisely, let us consider $\varpi: [0,1] \to \mathbb{R}$ a bump function with $\varpi|_{[0,1/4]} = 0$ and $\varpi|_{[3/4,1]} = 1$. Let $\beta \in \Omega^2(\Sigma)$ and chose $\alpha \in \Omega^1(\Sigma)$ such that $\psi^*(\beta) = \beta + d\alpha$. Then $\hat{\beta} = \beta + d(\varpi(t)\alpha) \in \Omega^2(\Sigma \times [0,1])$ induces a 2-form on $L$ via the push-forward. Of course, one can show that the cohomology class of $\hat{\beta}$ do not depend on $\alpha$. In addition, from the Mayer-Vietoris exact sequence we deduce that $[q_*([\beta])] \neq 0$ if $[\beta] \neq 0$.

We denote by $\omega_{\Sigma} \in \Omega^2(L)$ a closed 2-form induced by a volume form $\text{vol}_{\Sigma}$ of $\Sigma$ that integrates 1 on $\Sigma$. This class represents the Poincaré dual of a circle $C \subset L$ such that $q(\{p_0\} \times [0,1]) \subset C$ and $C - q(\{p_0\} \times \{0\})$ is an embedded line on $q(\Sigma \times \{0\})$ if it is not empty.

**Proposition 31.** The total Chern class of $\nu$ is $c(\nu) = 1 + (2 - 2g)[\omega_{\Sigma}]$.

**Proof.** Let $\times$ be the cross product on $TM|_L$ determined by $\varphi$. Consider on $E = \ker(\theta)$ the complex structure $JW = W \times e_1^\varphi$, where $e_1 = \|\theta\|^{-1}\theta$; which is well-defined because $\times$ defines a cross product on $T_pL$ and if $\theta(X) = 0$, then $X \times e_1^\varphi \perp e_1^\varphi$. Also recall that the complex structure on $\nu$ is: $I(v) = e_1^\varphi \times v$.

We prove that there is an isomorphism of complex line bundles:

$$\mathbb{C} \oplus E \to \nu.$$

Given a nowhere-vanishing section $s: L \to \nu$ which exists because $\dim L = 3 > 4 = \text{rk}(\nu)$, we define the isomorphism $\mathbb{C} \oplus E \to \nu$,

$$(z_1 + iz_2, W) \longmapsto z_1s + z_2e_1^\varphi \times s + W \times s.$$

To see that the complex structure is preserved one uses the equality [35, Lemma 2.9]:

$$u \times (v \times w) + v \times (u \times w) = g(u, w)v + g(v, w)u - 2g(u, v)w.$$ 

where $g$ denotes the restriction to $\nu$ of the metric on $M$. In our case taking $u = e_1^\varphi$, $v = s$ and $w = W$ we obtain that $e_1^\varphi \times (W \times s) = (W \times e_1^\varphi) \times s$. 

From the isomorphism we get that $c(\nu) = c(\Sigma) c(E) = 1 + c_1(E)$. We now compute $c_1(E)$; note that $E$ is the vertical distribution $dq(T\Sigma \times [0,1]) \subset TM$. First consider a compactly-supported 2-form $\varpi \in \Omega^2(T\Sigma)$ representing the Thom class of the bundle $T\Sigma \to S$ that integrates 1 over the fibres. Being the diffeomorphism $d\psi: T\Sigma \to T\Sigma$ volume-preserving we obtain that $(d\psi)^*\varpi$ is also a compactly-supported 2-form that integrates 1 over the fibres. Thus, $(d\psi)^*\varpi = \varpi + d\alpha$ for some compactly-supported $\alpha \in \Omega^1(T\Sigma)$. In addition let $s_0: \Sigma \to T\Sigma$ the zero section; then $[s_0^*(\varpi)] = (2 - 2g)[\omega_{\Sigma}]$.

The push-forward $q_\ast(\varpi + d(\varpi\alpha)) \in \Omega^2(E)$ of course induces the Thom class of $E$. Being $s[p,t] = dq(p,t)(s_0(p,t))$ the zero section of $E$ we obtain:

$$c_1(E) = s^\ast[q_\ast(\varpi + d(\varpi\alpha))] = [q_\ast(s_0^\ast\varpi + d(\varpi s_0^\ast\alpha))] = (2 - 2g)[\omega_{\Sigma}],$$

where we have taken into account the equalities $s_0^\ast(d\varpi) = 0$, $s_0^\ast(\varpi^\ast\varpi) = s_0^\ast\varpi + d(s_0^\ast\alpha)$ and $[s_0^\ast(\varpi)] = (2 - 2g)[\omega_{\Sigma}]$.

\[\square\]

The projectivized bundle of $\nu$ coincides with $Q$ because $\mathbb{P}(\nu) = P_{U(2)}(\nu) \times_{U(2)} \mathbb{CP}^1 = Q$. An obstruction-theoretic argument ensures that it is trivial:

**Lemma 32.** The bundle $Q \to L$ is trivial.

**Proof.** First recall that the spaces $\text{Diff}(S^2)$ and $SO(3)$ have the same homotopy type. Therefore, classifying $S^2$ bundles is equivalent to classifying rank 3 vector bundles. In our case, denoting by $E = \ker(\theta)$ as in the proof of Proposition 31, if $g_{\alpha\beta} \in SO(2)$ are the transition functions of $E$, taking into account the diffeomorphism $\mathbb{CP}^1 \to S^2$ one can compute that the transition functions of $Q$ are

$$h_{\alpha\beta}(x)(v_1, v_2, v_3) = (g_{\alpha\beta}(v_1, v_2), v_3)$$

Therefore, the associated rank 3 vector bundle $V$ has transition functions $g_{\alpha\beta} \times \text{Id} \in SO(3)$. This is trivial if and only if $Q$ is. We now observe that $V$ is trivial if and only if its second Stiefel-Whitney class vanishes. For that purpose consider a CW-decomposition,

$$L = \bigcup_{k=0}^{\infty} L^k.$$ 

Then $V|_{L^1}$ is trivial because $SO(3)$ is connected. The trivialization extends to $L^2$ if the primary obstruction cocycle is exact; this coincides with the second Stiefel-Whitney class (see [22 Proposition 3.21]). If it vanishes, then the last obstruction cocycle lies in $H^2(L, \pi_2(SO(3))) = 0$ and therefore the extends to $L$.

We now compute the second Stiefel-Whitney class of $V$. Regarding the transition functions, $V = E \oplus \mathbb{R}$; thus $w_2(V) = w_2(E)$. Being $E$ a complex vector bundle, we obtain $w_2(E) = c_1(E) \pmod{2} = (2 - 2g)\omega_{\Sigma} \pmod{2} = 0$. \[\square\]

Using Proposition 31 we re-state a well known fact. For that purpose consider the tautological bundle associated to $\nu$:

$$\mathcal{F} = P_{U(2)}(\nu) \times_{U(2)} \mathbb{C}^2.$$

Denote frames in $P_{U(2)}(\nu)$ by $F$. There is a well-defined $\mathbb{Z}_2$ action on $\mathcal{F}$, determined by $[F, (z_1, z_2, \ell)] \mapsto [F, (-z_1, -z_2, \ell)]$. The quotient $\mathcal{F}/\mathbb{Z}_2$ coincides with $P$. We denote by $\varphi: \mathcal{F} \to P$ the projection.
Proposition 33. Let $e(P)$ be the Euler class the line bundle $P \to Q$. Denote by $H^*(L)[x]$ the polynomial algebra with coefficients in $H^*(L)$. The map:

$$F: H^*(L)[x]/(x^2 + (2 - 2g)[\omega_\Sigma]x) \to H^*(Q), \quad F(\beta) = \text{pr}^*\beta, F(x) = e(P),$$

is an isomorphism of algebras.

We denoted the projection by $\text{pr}: P \to L$. Consider $\tau \in \Omega^2(P)$ the Thom 2-form of the line bundle $P \to Q$ and note that we can suppose that $\tau$ is $\mathbb{Z}_2$-invariant because the involution preserves the orientation on the fibres. From Proposition 33 we obtain:

$$[\tau \wedge \tau] = -(2 - 2g)[(\varrho \circ \text{pr})^*\omega_\Sigma \wedge \tau].$$

We also denote by $\tau$ the pushforward $\varrho_*\tau \in \Omega(P)$; on $H^*(P)$ it also verifies that:

$$[\tau \wedge \tau] = -(2 - 2g)[\text{pr}^*\omega_\Sigma \wedge \tau].$$

Of course, we can extend $\tau$ to a 2-form on $\tilde{X}$ and it corresponds to the Poincaré dual of $Q$.

We now compute the cohomology of $\tilde{X}$; for this we do not assume that $L$ is connected and we denote by $L_1, \ldots, L_r$ it connected components. Each $L_i$ is a mapping torus over a surface $\Sigma_i$ of genus $g_i$; we denote by $\omega_i$ the 2-form $\omega_\Sigma_i$ as constructed before. We also denote $Q_i = Q|_{L_i}$, $P_i = P|_{L_i}$ and $\tau_i$ the Thom form of $Q_i \subset P_i$.

Proposition 34. There is a split exact sequence:

$$0 \longrightarrow H^*(X) \overset{\pi^*}{\longrightarrow} H^*(\tilde{X}) \longrightarrow \bigoplus_{i=1}^r H^*(L_i) \otimes \langle x_i \rangle \longrightarrow 0$$

where $x_i$ has degree two.

Proof. The existence of such exact sequence is contained in the proof of [27, Proposition 6.1]; we outline it. Consider the long exact sequence of pairs $(X, L)$ and $(\tilde{X}, Q)$. There is a commutative diagram:

$$
\begin{array}{cccccc}
H^k(X, L) & \longrightarrow & H^k(X) & \overset{\epsilon_L}{\longrightarrow} & \bigoplus H^k(L_i, \mathbb{R}) & \overset{D_1}{\longrightarrow} & H^{k+1}(X, L) \\
\downarrow{\pi^*} & & \downarrow{\pi^*} & & \downarrow{\pi^*} & & \downarrow{\pi^*} \\
H^k(\tilde{X}, Q) & \longrightarrow & H^k(\tilde{X}) & \overset{\epsilon,Q}{\longrightarrow} & \bigoplus H^k(Q_i) & \overset{D_2}{\longrightarrow} & H^{k+1}(\tilde{X}, Q)
\end{array}
$$

Here we denoted the inclusions $e_L: L \to X$ and $e_Q: Q \to \tilde{X}$. The first and fourth columns are isomorphisms; these correspond to the identity map. The third column is injective with cokernel $\bigoplus J H^*(Q_j)/H^*(L_j)$; this is isomorphic to $\bigoplus J H^{k-2}(L_i) \otimes \langle x_i \rangle$, because $Q_j = L_j \times S^2$. Thus we get a commutative diagram with exact columns:
of course, $\bar{e}_Q$ is the action induced by $e^*_Q$ on the quotient. In addition, the fact that first and fourth columns are the identity implies that $\text{Im}(e^*_L) = \text{Im}(e^*_Q)$.

Snake Lemma ensures that there is an exact sequence:

$$0 \to \ker(e^*_L) \to \ker(e^*_Q) \to \ker(\bar{e}_Q) \to \text{Coker}(e^*_L) \to \text{Coker}(e^*_Q) \to \text{Coker}(\bar{e}_Q) \to 0.$$ 

The maps are induced by $\pi^*$, except from $D$: $\ker(\bar{e}_Q) \to \text{Coker}(e^*_L)$ which is a connecting map. But note that $\pi^*$: $\ker(e^*_L) \to \ker(\bar{e}_Q)$ is an isomorphism, because the first row is an isomorphism and the diagram is commutative. In addition, taking into account that the fourth row is an isomorphism and that the diagram is commutative one can also check that $\pi^*$ is an isomorphism between $\text{Im}(D_1)$ and $\text{Im}(D_2)$. But:

$$\text{Im}(D_1) = \oplus_i H^*(Q_i)/\ker(D_1) = \oplus_i H^*(L_i)/\text{Im}(e^*_L) = \text{Coker}(e^*_L),$$

and the isomorphism is induced by the map that $\pi^*$ induces on the quotient. In the same manner, $\text{Coker}(e^*_Q)$ is isomorphic to $\text{Im}(D_2)$ via $\pi^*$. This means that $\ker(\bar{e}_Q) = 0 = \text{Coker}(\bar{e}_Q)$ so,

$$\text{Coker}(\pi^*) = \oplus_i H^{*-2}(L_i) \otimes \langle x_i \rangle.$$ 

Consider $\tau_i$ the Poincaré dual of $Q_i \subset \tilde{X}$ as constructed before. Then,

$$\beta \otimes x_i \longmapsto \text{pr}^*(\beta)\tau_i$$

is a splitting of the previous exact sequence. \qed

This result implies that there is an isomorphism of vector spaces between $H^*(\tilde{X})$ and $H^*(X) \bigoplus \oplus_{i=1}^k H^*(L_i) \otimes \langle x_i \rangle$. The algebra structure of $H^*(\tilde{X})$ induces an algebra structure on $H^*(X) \bigoplus \oplus_{i=1}^k H^*(L_i) \otimes \langle x_i \rangle$ that we compute in Proposition 35. This is necessary in order to decide whether the resolution $\tilde{X}$ is formal or not, because formality condition involves products of cohomology classes.

**Proposition 35.** There is an isomorphism

$$H^*(\tilde{X}) = H^*(X) \bigoplus \oplus_{i=1}^k H^*(L_i) \otimes \langle x_i \rangle.$$ 

Let $\alpha, \beta \in H^*(X)$, $\gamma_i \in H^*(L_i)$, $\gamma_j' \in H^*(L_j)$ and let $e_i: L_i \to X$ be the inclusion. The wedge product on $H^*(\tilde{X})$ determines the following product on the left hand side:
\[(1)\, \alpha \beta = \alpha \wedge \beta,\]
\[(2)\, \alpha(\gamma_i \otimes x_i) = (\epsilon_i^*(\alpha) \wedge \gamma_i) \otimes x_i,\]
\[(3)\, (\gamma_i \otimes x_i)(\gamma_j' \otimes x_i) = 0 \text{ if } i \neq j,\]
\[(4)\, (\gamma_i \otimes x_i)(\gamma_i' \otimes x_i) = -2(\gamma_i \wedge \gamma_i')PD[L_i] - (2 - 2g_i)(\omega_i \otimes x_i).\]

**Proof.** Let \(s: \bigoplus_{i=1}^r H^*(L_i) \otimes \langle x_i \rangle \to H^*(\tilde{X})\) be the splitting map constructed in the proof of Proposition 34. Then, the isomorphism is determined by:

\[T = (\rho^*, s): H^*(X) \bigoplus \bigoplus_{i=1}^r H^*(L_i) \otimes \langle x_i \rangle \to H^*(\tilde{X}).\]

In order to check the product between forms \(\eta, \eta'\) we have to compute \((T)^{-1}(T \eta \wedge T \eta')\). All the statements are evident except for the last one. We only check \(x_i^2 = -2PD[L_i] - (2 - 2g_i)(\omega_i \otimes x_i)\); the announced formula can be deduced from this taking into account that \(H^*(\tilde{X})\) is an algebra. First of all, \(TX_i \wedge TX_i = [\tau_i \wedge \tau_i];\) we now compute \(T^{-1}[\tau_i \wedge \tau_i]\). On the one hand taking into account the equality

\[[\tau_i \wedge \tau_i] = -(2 - 2g_i)[pr^*(\omega_i) \wedge \tau_i],\]

we obtain that the restriction of \(T^{-1}[\tau_i \wedge \tau_i]\) to \(H^*(L_i) \otimes \langle x_i \rangle\) is \(-(2 - 2g_i)(\omega_i \otimes x_i)\). On the other hand, note first that if \(x \in L_i\) then \(\tau_i|_{P_x}\) is the Thom form of \(Q_x \subset P_x\) because \(\tau_i\) is the Thom form of \(Q_i \subset P_i\). Thus:

\[\int_{P_x} \tau_i \wedge \tau_i = [Q_x][Q_x] = -2.\]

Note that the restriction of \(T^{-1}[\tau_i \wedge \tau_i]\) to \(H^*(X)\) has compact support around \(L_i\). Since

\[\int_{\nu_x} \rho^*(\tau_i \wedge \tau_i) = \int_{\nu_x - 0} \rho^*(\tau_i \wedge \tau_i) = \int_{P_x - Q_x} \tau_i \wedge \tau_i = \int_{P_x} \tau_i \wedge \tau_i = -2,\]

this is equal to \(-2PD[L_i]\). \(\square\)

5. Non-formal compact \(G_2\) manifold with \(b_1 = 1\)

Nilpotent Lie algebras that have a closed left-invariant \(G_2\) structure are classified in [11]; from these one can construct nilmanifolds with an invariant closed \(G_2\) structure. Of course, excluding the 7-dimensional torus, these are non-formal and have \(b_2 \geq 2\). From a \(\mathbb{Z}_2\) action on a nilmanifold, in [18] it is constructed a formal orbifold, whose isotropy locus are 16 disjoint 3-tori; then the authors prove that its resolution is also formal. In this section we follow the same process to construct first a non-formal \(G_2\) orbifold with \(b_1 = 1\) from a nilmanifold; its isotropy locus consists of 16 disjoint non-formal nilmanifolds. Later we prove that the resolution is also non-formal and does not admit any torsion-free \(G_2\) structure.

5.1. Orbifold with \(b_1 = 1\). Let us consider the Lie algebra \(\mathfrak{g}\) with structure equations

\[(0, 0, 0, 12, 23, -13, -2(16) + 2(25) + 2(26) - 2(34)),\]

and let us consider \((e_1, e_2, e_3, e_4, e_5, e_6, e_7)\) the generators of \(\mathfrak{g}\) that verify the structure equations, that is, \([e_1, e_2] = -e_4, [e_2, e_3] = -e_5\) and so on. Recall that the simply connected Lie group \(G\) associated to \(\mathfrak{g}\) is the vector space \(\mathfrak{g}\) endowed with the product \(*\) determined by the Baker-Campbell-Hausdorff formula.
Remark 36. The Lie algebra $\mathfrak{g}$ belongs to the 1-parameter family of algebras $147E1$ listed in Gong’s classification [21]; we choose the parameter 2. The associated Lie group admits an invariant closed $G_2$ structure as proved in [11].

Define $u_1 = e_1, u_2 = e_2, u_3 = e_3, u_4 = \frac{1}{2}e_4, u_5 = \frac{1}{2}e_5, u_6 = \frac{1}{2}e_6$ and $u_7 = \frac{1}{6}e_7$.

Proposition 37. If $x = \sum_{k=1}^{7} \lambda_k u_k$ and $y = \sum_{k=1}^{7} \mu_k u_k$ then
\[
x * y = (\lambda_1 + \mu_1)u_1 + (\lambda_2 + \mu_2)u_2 + (\lambda_3 + \mu_3)u_3 + (\lambda_4 + \mu_4 - (\lambda_1\mu_2 - \lambda_2\mu_1))u_4 \\
+ (\lambda_5 + \mu_5 - (\lambda_2\mu_3 - \lambda_3\mu_2))u_5 + (\lambda_6 + \mu_6 + (\lambda_1\mu_3 - \lambda_3\mu_1))u_6 \\
+ (\lambda_7 + \mu_7 + (\lambda_1 - \mu_1 - \lambda_2 + \mu_2)(\lambda_1\mu_3 - \lambda_3\mu_1) - (\lambda_3 - \mu_3)(\lambda_1\mu_2 - \mu_2\lambda_1))u_7 \\
+ (-\lambda_2 + \mu_2)(\lambda_2\mu_3 - \lambda_3\mu_2) + 3(\lambda_1\mu_6 + \lambda_6\mu_1))u_7 \\
+ (-3(\lambda_2\mu_5 - \lambda_5\mu_2) - 3(\lambda_2\mu_6 - \lambda_6\mu_2) + 3(\lambda_3\mu_4 + \lambda_4\mu_3))u_7.
\]

Proof. Since $\mathfrak{g}$ is 3-step, the Baker-Campbell-Hausdorff formula yields:
\[
x * y = x + y + \frac{1}{2} [x, y] + \frac{1}{12} ([x, [x, y]] - [y, [x, y]]).
\]

From this, and taking into account that $u_7 \in Z(\mathfrak{g})$ and that $[u_i, [u_j, u_k]] = 0$ if $i \geq 4$ or $j \geq 4$ or $k \leq 4$, we obtain
\[
x * y = \sum_{k=1}^{7} (\lambda_k + \mu_i)u_i + \frac{1}{2} \sum_{1 \leq i < j \leq 7} (\lambda_i\mu_j - \lambda_j\mu_i)[u_i, u_j] \\
+ \frac{1}{12} \sum_{1 \leq i < j \leq 3} (\lambda_i - \mu_k) \sum_{1 \leq i < j \leq 3} (\lambda_i\mu_j - \lambda_j\mu_i)[u_k, [u_i, u_j]].
\]

The non-zero combinations $[u_i, u_j]$ and $[u_k, [u_i, u_j]]$ are:

$[u_1, u_2] = -2u_4, \quad [u_2, u_5] = -6u_7, \quad [u_3, [u_1, u_2]] = -12u_7$

$[u_1, u_3] = 2u_6, \quad [u_2, u_6] = -6u_7, \quad [u_1, [u_1, u_3]] = 12u_7$

$[u_1, u_6] = 6u_7, \quad [u_3, u_4] = 6u_7, \quad [u_2, [u_1, u_3]] = -12u_7,$

$[u_2, u_3] = -2u_5, \quad [u_2, [u_2, u_3]] = 12u_7.$

The announced formula easily follows from this. \qed

Proposition 37 ensures that
\[
\Gamma = \left\{ \sum_{i=1}^{7} n_i u_i, \text{ s.t. } n_i \in \mathbb{Z} \right\},
\]
is a discrete subgroup of $G$, which is of course co-compact. Indeed, a straightforward computation gives a fundamental domain for the left action of $\Gamma$ on $G$:

Proposition 38. A fundamental domain for left the action of $\Gamma$ on $G$ is
\[
\mathcal{D} = \left\{ \sum_{i=1}^{7} t_i u_i, \text{ s.t. } 0 \leq t_i \leq 1 \right\}.
\]
According to [11] Lemma 5], the group \(G\) admits an invariant closed \(G_2\) structure determined by:

\[\varphi = v^{127} + v^{347} + v^{567} + v^{135} - v^{236} - v^{146} - v^{245}.\]

where:

- \(v^1 = \sqrt{3}(2e^1 + e^5 - e^2 + e^6);\)
- \(v^2 = 3e^2 - e^5 + e^6;\)
- \(v^3 = e^3 + 2e^4;\)
- \(v^4 = \sqrt{3}(e^3 + e^7);\)
- \(v^5 = \sqrt{\frac{3}{2}}(e^6 - e^5);\)
- \(v^6 = \sqrt{6}(e^5 + e^6),\)
- \(v^7 = 2\sqrt{2}(e^4 - e^3).\)

Consider \(M = G/\Gamma\); points of \(M\) will be denoted by \([g]\), for some \(g \in G\). The nilmanifold \(M\) inherits a closed \(G_2\) structure that we also denote by \(\varphi\). We now define an involution \(\gamma\) on \(M\) such that \(\gamma^* \varphi = \varphi\). For that purpose it is sufficient to define an order 2 isomorphism \(\gamma: G \to G\) of \(G\) with \(\gamma^* \varphi = \varphi\), and \(\gamma \Gamma = \Gamma\). The desired map is:

\[\gamma(e_k) = e_k, \quad k \in 3, 4, 7, \quad \gamma(e_k) = -e_k, \quad k \in \{1, 2, 5, 6\}.\]

Looking at the structure constants of \(G\) it becomes clear that \(\gamma\) is an automorphism of \(g\). Baker-Campbell-Hausdorff formula ensures that \(\gamma\) is a homomorphism. In addition, it is clear that \(\gamma(\Gamma) \subset \Gamma\). Finally, one can easily deduce that \(\gamma^*(\varphi) = \varphi\).

We define the orbifold \(X = M/\gamma\), which has a closed \(G_2\) structure determined by \(\varphi\). We now study its singular locus:

**Proposition 39.** The isotropy locus has 16 connected components; these are all diffeomorphic and their universal covering is the Heisenberg group. Let us define \(H_0 = \{\lambda_3 u_3 + \lambda_4 u_4 + \lambda_7 u_7, \text{ s.t. } \lambda_j \in \mathbb{R}\}\) and \(E = \{\varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_5 u_5 + \varepsilon_6 u_6, \text{ s.t. } \varepsilon_j \in \{0, \frac{1}{2}\}\}\). The 16 connected components of the isotropy locus are:

\[H_\varepsilon = [L_\varepsilon H_0], \quad \varepsilon \in E,\]

where \(L_\varepsilon\) denotes the left translation on \(G\) by the element \(\varepsilon \in E\).

**Proof.** It is clear that \(H_0\) is a connected component of \(\text{Fix}(\gamma)\) that contains 0, with is the unit of \(G\). Being \(\gamma\) an homomorphism, we conclude that \(H_0\) is a subgroup of \(G\). It is thus sufficient to prove that the Lie algebra \(h\) of \(H_0\) is the Heisenberg algebra. This is of course true because \(h = \{e_3, e_4, e_7\} \text{ with } [e_3, e_4] = e_7\) and \([e_j, e_7] = 0\) for \(j \in \{3, 4\}\).

Let \(K = \{1, 2, 5, 6\}\), take \(x = \sum_{k \in K} \lambda_k u_k\), we now check that if \(\gamma \ast x = \gamma(x)\) for some \(\gamma\) then \([x] \in E\). Let us denote \(\gamma = \sum_{k=1}^7 n_k u_k\); taking into account Proposition 37 one obtains:

\[\gamma \ast x = (n_1 + \lambda_1)u_1 + (n_2 + \lambda_2)u_2 + n_3 u_3 + (n_4 - n_1 \lambda_2 + n_2 \lambda_1)u_4 + (n_5 + \lambda_5 + n_3 \lambda_2)u_5 + (n_6 + \lambda_6 - n_3 \lambda_1)u_6 + \lambda' u_7,\]

for some \(\lambda' \in \mathbb{R}\). The equation \(x = \gamma \ast x\) yields immediately to \(2\lambda_j = -n_j\) for \(j = \{1, 2\}\) and \(n_3 = 0\). Taking this into account, \(n_4 = n_1 \lambda_2 + n_2 \lambda_1 = n_4, n_5 = \lambda_5 + n_3 \lambda_2 = n_5 + \lambda_5, n_6 + a_6 - n_3 \lambda_1 = n_6 + \lambda_6\) and thus \(n_4 = 0, 2\lambda_5 = -n_5\) and \(2\lambda_6 = -n_6\). Thus, \(x = -\frac{1}{2} \sum_{k \in K} n_k u_k\). Being \(D\) a fundamental domain for the left action of \(\Gamma\) on \(G\) we obtain \([x] = [x']\) for \(x' \in E\).
We now let \([y]\) be an isotropy point; one can write: \(y = x_1 \ast x_2\); with \(x_1 = \sum_{k \in X} \lambda_k u_k\) and \(x_2 = \sum_{k \not\in X} \mu_k u_k \in H_0\). The choice becomes clear from the equality:

\[
x_1 \ast x_2 = \lambda_1 u_1 + \lambda_2 u_2 + \mu_3 u_3 + \mu_4 u_4 + (\lambda_5 - \lambda_2 \mu_3)u_5 + (\lambda_6 + \alpha_1 \mu_3)u_6 + (\mu_7 + (\lambda_1 - \lambda_2)(\lambda_1 \mu_3) + \lambda_2 \mu_3)u_7,
\]

that is of course deduced from Proposition 37.

Using this decomposition we obtain the equality \(\gamma \ast x_1 x_2 = j(y) = j(x_1) x_2\) that implies \(j(x_1) = \gamma x_1\). Take \(x'_1 \in \mathcal{E}\) with \(x_1 = \gamma' x'_1\), then \([y] = [\gamma' x'_1 x_2] = [x'_1 x_2] \in [L_{x'_1} H_0]\). \(\square\)

5.2. Non-formality of the resolution. We start by computing the real cohomology algebra of the orbifold. Nomizu’s theorem [32] ensures that \((\Lambda^* \mathfrak{g}^s, d)\) is the minimal model of \(M\). Taking into account that \(H^*(X) = H^*(M)^{\mathbb{Z}_2}\) we obtain that \(((\Lambda^* \mathfrak{g}^s)^{\mathbb{Z}_2}, d)\) is a model for \(X\). The cohomology of \(X\) is:

\[
\begin{align*}
H^1(X) &= \langle [e^3] \rangle, \\
H^2(X) &= \langle [e^{25}], [e^{15} - e^{26}], [e^{15} - e^{34}] \rangle, \\
H^3(X) &= \langle [e^{235}], [e^{135}], [e^{356}], [e^{124}], [e^{146}], [e^{245}], [e^{127} + 2e^{145}], \\
& \quad [e^{125} + e^{167} - e^{257} - 2e^{456} - e^{347}] \rangle.
\end{align*}
\]

We now prove that \(X\) is not formal.

**Proposition 40.** The triple Massey product \(\langle [e^3], [e^{15} - e^{26}], [e^3] \rangle\) of \(((\Lambda^* \mathfrak{g}^s)^{\mathbb{Z}_2}, d)\) is not trivial. Therefore, \(X\) is not formal.

**Proof.** First of all, one can check that that space of exact 3-forms of \(((\Lambda \mathfrak{g})^{\mathbb{Z}_2}, d)\) is:

\[
B^3((\Lambda^* \mathfrak{g}^s)^{\mathbb{Z}_2}, d) = \langle e^{123}, e^{135} - e^{236}, -e^{136} + e^{236}, e^{236}, e^{127} - 2e^{146} + 2e^{245} + 2e^{246} \rangle,
\]

and the space of closed 2-forms is:

\[
Z^2((\Lambda^* \mathfrak{g}^s)^{\mathbb{Z}_2}, d) = \langle e^{12}, -e^{16} + e^{25} + e^{26} - e^{34}, e^{25}, e^{15} - e^{26}, e^{15} - e^{34} \rangle.
\]

Let us take \(\xi_1 = [e^3] = \xi_3, \xi_2 = [e^{15} - e^{26}]\); the representatives of these cohomology classes are \(\alpha_1 = \alpha_3 = e^3\) and \(\alpha_2 = e^{15} - e^{26} + dx\) for some \(x \in (\mathfrak{g}^s)^{\mathbb{Z}_2}\); our previous computations ensure that the Massey product \(\langle \xi_1, \xi_2, \xi_3 \rangle\) is well defined. Taking into account that \(\bar{\alpha}_1 \wedge \alpha_2 = d(-e^{56} + e^3 x + \beta_1)\) and \(\bar{\alpha}_2 \wedge \alpha_3 = d(e^{56} - e^3 x + \beta_2)\) for every closed forms \(\beta_1\) and \(\beta_2\), we obtain that the defining systems are \((e^3, e^{15} - e^{26} + dx, e^3, -e^{56} + e^3 x + \beta_1, e^{56} - e^3 x + \beta_2)\). Thus, the triple Massey product is

\[
\langle \xi_1, \xi_2, \xi_3 \rangle = \{[2e^{356} + e^3 \beta] \text{ s.t. } d\beta = 0\}.
\]

The zero cohomology class is not an element of this set due to our previous computations. Corollary 13 ensures that \(X\) is not formal. \(\square\)

Let \(\rho: \tilde{X} \to X\) be the closed \(G_2\) resolution constructed in Theorem 30. Lifting this triple Massey product to \(\tilde{X}\) we prove that \(\tilde{X}\) is not formal.

**Proposition 41.** The resolution \(\tilde{X}\) is not formal.
Proof. Let \((\Lambda, V, d)\) be the minimal model of \(\tilde{X}\) with \(V = \bigoplus_{i=1}^{7} V^i\), and let \(\kappa: \Lambda V \to \Omega(\tilde{X})\) be a quasi-isomorphism. From Proposition 35 we deduce that \(H^2(\tilde{X}) = \langle \rho^*(e^3) \rangle\) and that:

\[
H^2(\tilde{X}) = \langle \rho^*(e^{25}), \rho^*(e^{15} - e^{26}), \rho^*(e^{15} - e^{34}), \tau_1, \ldots, \tau_{16} \rangle.
\]

In addition, \(\rho^*(e^3 \wedge (e^{15} - e^{26})) = d\rho^*(e^{56})\) and \(\rho^*[e^{235}]\) are linearly independent on \(H^3(\tilde{X}, \mathbb{R})\). Then, according to Proposition 35 one can choose:

\[
V^1 = \langle a \rangle, \quad V^2 = \langle b_1, b_2, b_3, y_1, \ldots, y_{16}, n \rangle.
\]

with \(da = 0, db_j = dy_j = 0\) and \(dn = ab_2\) and the map \(\kappa\) is:

\[
\kappa(a) = \rho^*(e^3), \quad \kappa(b_2) = \rho^*(e^{15} - e^{26}), \quad \kappa(n) = \rho^*(e^{56}),
\]

\[
\kappa(b_1) = \rho^*(e^{25}), \quad \kappa(b_3) = \rho^*(e^{15} - e^{34}), \quad \kappa(y_j) = \tau_j.
\]

We now define a Massey product. Let us take \(\xi_1 = [a] = [\xi_3], \xi_2 = [b_2]\); the representatives of these cohomology classes are \(\alpha_1 = \alpha_3 = a\) and \(\alpha_2 = b_2\). Then \(\alpha_1 \wedge \alpha_2 = d(-n + \beta_1 + \omega_1)\) and \(\alpha_2 \wedge \alpha_3 = d(n + \beta_2 + \omega_2)\) with \(\beta_1, \beta_2 \in \langle b_1, b_2, b_3 \rangle\) and \(\omega_1, \omega_2 \in \langle y_1, \ldots, y_{16} \rangle\). Therefore, the defining systems of the Massey products \(\langle \xi_1, \xi_2, \xi_3 \rangle\) are \((a, b_2, a, -n + \beta_1 + \omega_1, n + \beta_2 + \omega_2)\) and the Massey product is the set

\[
\{[2an + a\beta + a\omega] \text{ s.t. } \beta \in \langle b_1, b_2, b_3 \rangle, \quad \omega \in \langle y_1, \ldots, y_{16} \rangle \}.
\]

We now observe that \([2an + a\beta + a\omega] = 0\) in \(H^*(\Lambda, V, d)\) if and only if \(\omega = 0\) and \([\kappa(2an + a\beta)] = 0\). This is because \([\kappa(\alpha_3)] = [\rho^*(e^3) \wedge \kappa(\omega)] = 0\) if and only if \(\omega = 0\), and if \(\omega \neq 0\), the elements \([\kappa(\alpha_2)]\) and \([\kappa(2an + a\beta)]\) are linearly independent.

In addition, \(\kappa(2an + a\beta) = \rho^*(2e^{356} + e^3 \wedge \beta')\), with \(\beta' \in \langle e^{25}, e^{15} - e^{26}, e^{15} - e^{34} \rangle\). Taking into account Proposition 35 \([\kappa(2an + a\beta) = 0\) if and only if \([2e^{356} + e^3 \wedge \beta'] = 0\) on \(X\). But \([2e^{356} + e^3 \wedge \beta'] \neq 0\) as shown in Proposition 40.

There is another non-trivial triple Massey product that comes from the isotropy locus. In order to describe it we have to construct the subspace \(V^3\) of our minimal model; it is a direct sum \(V^3 = C \oplus N\); such that \(dC = 0\) and there are not closed elements on \(N\). To construct \(C\) one takes a basis of the space \(H^3(\tilde{X})/H^1(\tilde{X})H^2(\tilde{X})\); for instance:

\[
\langle \rho^*[e^{346}], \rho^*[e^{124}], \rho^*[e^{146}], \rho^*[e^{245}], \rho^*[e^{127} + 2e^{145}],
\]

\[
\rho^*[e^{125} + e^{167} - e^{257} - 2e^{356} - e^{347}]) \oplus \{[e^i \otimes x_i]_{i=1}^{18}\},
\]

Let \(C = \langle c_1, \ldots, c_6, z_1, \ldots, z_{16} \rangle\) with \(dC = 0\) and define \(\kappa(c_1) = \rho^*(e^{346}), \kappa(c_2) = \rho^*(e^{124}), \ldots, \kappa(c_6) = \rho^*(e^{125} + e^{167} - e^{257} - 2e^{356} - e^{347})\) and \(\kappa(z_i) = e^i \otimes x_i\).

With this notation, consider the triple Massey product coming from the singular locus

\[
\langle [a], [z_j], -[a] \rangle.
\]

One can show that \(\langle [a], [z_j], -[a] \rangle\) is not trivial.

Proposition 42. The fundamental group of \(\tilde{X}\) is \(\pi_1(\tilde{X}) = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_6\).

Proof. Let us denote \(\pi: M \to X\) the quotient projection. In order to compute \(\pi_1(X)\) we first observe that \(\pi_1(M)\) is isomorphic to \(\Gamma\) due to the exact sequence \(0 \to \pi_1(G) \to \pi_1(M) \to \Gamma \to 0\). Of course, each generator \(u_i \in \Gamma\) is identified with the homotopy class \(f_i\) determined by the image of the path from 0 to \(u_i\) under the quotient map \(q: \tilde{G} \to \tilde{M}\).
Denote by $[\cdot,\cdot]$ the commutator of two elements on $\pi_1(M)$; then the product structure on $\Gamma$ determines that the non-zero commutators are:

$$
[f_1, f_2] = f_1^{-2}, \quad [f_1, f_2] = f_5^{-2}, \quad [f_2, f_5] = f_7^{-6}, \quad [f_3, f_4] = f_7^6.
$$

Taking into account [4, Corollary 6.3] the map $\pi_*: \pi_1(M) \to \pi_1(X)$ is surjective; we now analyze $\pi_*(f_j)$. First of all, under the projection $\pi$ the image of the loop $f_1$ is the same as the path from 0 to $\frac{1}{2}x_1$ followed by the same path in the reversed direction; this is of course contractible and thus $\pi_*(f_1) = 0$; in the same manner $\pi_*(f_2) = \pi_*(f_3) = \pi_*(f_6) = 0$. Taking into account commutator relations this implies that $\pi_*(f_2^2) = 0$, $\pi_*(f_5^6) = 0$ and that $\pi_*(f_3), \pi_*(f_4), \pi_*(f_7)$ commute. Thus, $\pi_1(X) = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_6$.

We now prove that the resolution process does not alter the fundamental group. For each $\varepsilon \in \mathcal{E}$ consider a small tubular neighbourhood $B^\varepsilon$ of $H_\varepsilon$ and suppose additionally that $B^\varepsilon$ are pairwise disjoint. Take $D^\varepsilon \subset B^\varepsilon$ a smaller tubular neighbourhood of $H_\varepsilon$ Define $U$ a connected open set containing $\cup_\varepsilon B^\varepsilon$ that is homotopy equivalent to $\bigcup_\varepsilon H_\varepsilon$ and $V = X - \cup_\varepsilon D^\varepsilon$.

Seifert-Van Kampen theorem states that $\pi_1(X)$ is the amalgamated product of $\pi_1(V)$ and $\pi_1(U)$ via $\pi_1(U \cap V)$. Define $\tilde{U} = \rho^{-1}(U)$, $\tilde{V} = \rho^{-1}(V)$; note that $\tilde{V}$ and $V$ are diffeomorphic via $\rho$; in addition, $\rho_*: \pi_1(U) \to \pi_1(U)$ is an isomorphism because $\tilde{U}$ is homotopy equivalent to $\bigcup_\varepsilon H_\varepsilon \times S^2$. This observation and a further application of Seifert-Van Kampen theorem ensures that $\pi_1(\tilde{X}) = \pi_1(X)$.

**Proposition 43.** The manifold $\tilde{X}$ does not admit torsion-free $G_2$ structures.

**Proof.** Suppose that $\tilde{X}$ admits a torsion-free $G_2$ structure. Then since $g$ is Ricci flat and $b_1 = 1$, [3] ensures that there is a finite covering $N \times S^1 \to \tilde{X}$ with $N$ a compact simply connected 6-dimensional manifold. Note that the covering is regular because $\pi_1(\tilde{X})$ is abelian; thus $(N \times S^1)/H = \tilde{X}$, where $H$ denotes the Deck group of the covering.

The manifold $N$ is formal because it is simply-connected and 6-dimensional (see [20, Theorem 3.2]); therefore $N \times S^1$ is formal (see [20, Lemma 2.11]). Lemma 21 allows us to conclude that $(N \times S^1)/H = \tilde{X}$ is formal; yielding a contradiction. \qed

**Remark 44.** There exists a finite covering $Y \to \tilde{X}$ such that $\pi_1(Y) = \mathbb{Z}$ because $\pi_1(\tilde{X}) = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_6$. The manifold $Y$ is also non-formal as a consequence of Lemma 21 and of course, it has first Betti number $b_1 = 1$ and admits a closed $G_2$ structure. Arguing as in the proof of Proposition 43 one can conclude that $Y$ does not admit any torsion-free $G_2$ structure.

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