Conformal and Weyl-Einstein gravity: Classical geometrodynamics

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We present a new formulation for the canonical approach to conformal (Weyl-squared) gravity and its extension by the Einstein-Hilbert term and a nonminimally coupled scalar field. For this purpose we use a unimodular decomposition of the three-metric and introduce unimodular-conformal canonical variables. The important feature of this choice is that only the scale part of the three-metric and the rescaled trace part of the extrinsic curvature change under a conformal transformation. This significantly simplifies the constraint analysis and manifestly reveals the conformal properties of a theory that contains the conformally invariant Weyl-tensor term. The conformal symmetry breaking which occurs in the presence of the Einstein-Hilbert term and a nonconformally coupled scalar field can then be interpreted directly in terms of this scale and this trace. We also discuss in detail the generator for the conformal transformations. This new Hamiltonian formulation is especially suitable for quantization, which will be the subject of a separate paper.

I. INTRODUCTION

Gravitational theories beyond general relativity (GR) are addressed for various reasons. One is the conceptual need to accommodate gravity into the quantum framework [1]. Another is the attempt to describe cosmological features, notably Dark Matter and Dark Energy, by generalized classical theories [2]. In this paper, we deal with conformal (Weyl-squared) gravity (also called W theory below) both as a pure gravitational theory and as part of an action containing also an Einstein-Hilbert (EH) part and an action describing a (in general nonminimally coupled) scalar field. Our main reason for doing so is quantum gravity. Researchers have often entertained the idea that at a fundamental level, for example at high energy, Nature can be described by a scale free theory, with scales emerging only at lower energy (see e.g. [3]). In order to study the consequences of a scale free theory we investigate here conformal (Weyl-squared) gravity described by an action containing the square of the Weyl tensor. Historically, this has emerged as an offspring from Weyl’s original gauge theory published in 1918; see for example [4] for a brief historical review. Conformal gravity was later used (and still is today) as an alternative classical theory to GR with potential astrophysical implications [5] and as an emerging contribution to the full power of the conformal invariance associated with the Einstein-Hilbert action at the one-loop level of quantum field theory in curved spacetime [6, 7]. These are not the aspects we are interested in here. We take conformal gravity as a model for a conformally invariant theory whose quantum version might be relevant at the most fundamental level.

An approach especially suited for conceptual questions and for cosmological applications is the canonical (Hamiltonian) approach [1]. This is the subject of this paper. We take it as a preparation for the quantum theory discussed in a forthcoming contribution, but find the classical discussion interesting in its own right because, as we shall see, interesting conceptual and mathematical structures appear there. These will mainly concern the structure of the constraints and the generator for conformal transformations.

The Hamiltonian formalism of conformal gravity has already been the subject of various investigations. Important earlier contributions include [8, 9, 10, 11]. In [11], the Hamiltonian formalism of $f(R)$ theories as well as of Weyl gravity was studied. More recently, the authors of [12] extended the analysis to the Hamiltonian formulation of the Weyl action plus conformally coupled scalar fields as well as their extension by including the Einstein-Hilbert term and an $R^2$ term; they analyzed, in particular, the constraint algebra in great detail. The authors of [13] investigated a particular model within Weyl-squared gravity, with an important step being the derivation of the generator of gauge conformal transformations using the Castellani algorithm [14].

In our paper here, we shall develop a new version of the Hamiltonian formalism, which is especially suited for quantization. We shall use an irreducible decomposition of the three-metric into its scale (determinant) part and its conformally invariant (unimodular) part. In this way, new canonical variables will be identified. A similar procedure will be applied to the lapse function and the shift vector, leading to densities. This will then induce a decomposition of the extrinsic curvature into its rescaled traceless and trace parts [15]. The resulting variables are all densities and are all conformally invariant except the scale (determinant) and the rescaled trace of the extrinsic curvature. One may refer to this approach as the Hamiltonian formalism in unimodular-conformal variables. It will turn out that the use of these new variables reveals the full power of the conformal invariance associated with

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1 In some of the earlier work, e.g. [8] and [10], the extrinsic curvature was decomposed in this way by hand, but the unimodular decomposition was not used.
the Weyl-tensor part of the full action and simplifies the discussion of conformal symmetry breaking induced by other terms.

Our paper is organized as follows. In Sec. II, we present a brief review of the Weyl-tensor action and some of its implications. Section III contains a summary of the $3+1$ decomposition for Weyl-squared gravity. Our own contributions start with Sec. IV. There, we introduce the unimodular-conformal variables in configuration space, thus dividing the variables into a conformally invariant and a noninvariant sector. Section V treats pure Weyl gravity. It is divided into four parts. Part A is devoted to the Hamiltonian formalism and the constraint analysis. Part B deals with the constraint algebra and part C with the generator of conformal transformations. Part D presents the Hamilton-Jacobi functional. Section VI is devoted to Weyl gravity plus the EH term. It is divided into two parts. In part A, we present the Hamiltonian formulation and in part B the Hamilton-Jacobi functional and nongauge transformations. The addition of the EH term explicitly leads to conformal symmetry breaking, for which only two variables containing a scale are responsible: the determinant of the three-metric and the trace of the extrinsic curvature. In that section we shall also argue that, in spite of some of the constraints being second class, it is still possible to define the generator of conformal but nongauge transformations. In Sec. VII, a nonminimally coupled scalar field is added, with the Hamiltonian formalism presented in part A and the Hamilton-Jacobi functional together with the generator of conformal transformations presented in part B. Section VIII contains our conclusions. We also have some Appendices. The first Appendix contains a short discussion of the physical dimensions; the remaining three Appendices present technical details for the discussion in the body of our paper.

II. PRELIMINARIES

In this section, we shall present a short summary of the covariant formulation for conformal gravity. This theory is defined by the action

$$S_{\text{w}} := -\frac{\alpha_w}{4} \int d^4x \sqrt{-g} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}, \tag{1}$$

where $\alpha_w$ is a coupling constant with the dimension of an action, and

$$C_{\mu\nu\lambda\rho} = R_{\mu\nu\lambda\rho} - \left( \delta_{\mu\lambda} R_{\nu\rho} - g_{\nu\lambda} R_{\mu\rho} \right) - \frac{1}{3} \delta_{\mu\rho} g_{\nu\lambda} R \tag{2}$$

is the Weyl tensor, which is invariant under conformal transformations of the metric,

$$g_{\mu\nu}(x) \rightarrow \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x). \tag{3}$$

We refer to this model simply as “Weyl-squared theory” or “W theory”. The field equations following from (1) read

$$\left( \nabla_\mu \nabla_\nu + \frac{1}{2} R_{\mu\nu} \right) C^\mu_{\lambda\nu\rho} = 0. \tag{4}$$

In order to study the breaking of conformal symmetry, we shall also investigate below the extension of (1) by the EH term and the action for a nonminimally coupled scalar field $\phi$,

$$S_{\text{wE}} := \int d^4x \sqrt{-g} \left[ -\frac{\alpha_w}{4} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} + \frac{1}{2\kappa} R - \frac{1}{2} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi R \phi^2 \right) \right], \tag{5}$$

where $\kappa := 8\pi G$, and $\xi$ is the dimensionless nonminimal coupling constant. We will refer to the extended models as “Weyl-Einstein” (WE) and WE$\chi$ theories, respectively. Note that for $\xi = 1/6$, the scalar field is conformally invariant (see e.g. [6], Sec. II B). We use units with $c = 1$ throughout.

For a general field $\phi_{A}(x)$, the conformal transformation is implemented by

$$\phi_A(x) \rightarrow \tilde{\phi}_A(x) = \Omega^{n_A}(x) \phi_A(x), \tag{6}$$

where $A$ denotes a collection of spacetime and/or internal indices, $\Omega(x)$ is a positive function of the spacetime coordinates, and $n_A$ is a rational number that is characteristic for each field and called “conformal weight”. More appropriately, this transformation is called local Weyl rescaling or local dilatational transformation because it is a transformation of the fields themselves and not of coordinates. We are thus not talking about the 15-parameter conformal group; see, for example, [17] for details. The transformation (3) expresses the fact that the conformal metric tensor is of conformal weight 2.

Weyl gravity is an example of a theory with higher derivatives. For such theories, various subtleties occur. When linearizing the pure W theory, one finds that it contains a massless spin-2 state (not yet the graviton), a massless spin-1 state, and also a massless spin-2 ghost (negative-energy) state, adding up to six degrees of freedom in total as shown by Riegert [18]. The existence of ghost states is, in fact, not surprising. In the canonical formalism, there is in the Hamiltonian a term linear in the momentum, which signals that the energy is unbounded from below. This is usually called “Ostrogradski instability”; see for example [19]. It can also be

\[\chi\] because below we shall introduce a rescaled field $\chi$.\footnote{\[\chi\] also referred to as Theorem of Ostrogradski, which states that any nondegenerate Lagrangian containing second or higher (but even-order) time derivatives gives rise to the existence of both positive and negative energy states. It has recently been shown that this conclusion can be extended to odd-order derivatives as well, including the case of a degenerate Lagrangian [20].}
deduced from the corresponding propagator in a perturbative approach \[21\]. It follows, in particular, that the W theory supplemented by the EH-term contains two degrees of freedom in one massless spin-2 propagator and five degrees of freedom in one massive spin-2 negative energy propagator, amounting to seven in total. Negative energy states can be traded for positive ones, with the price of unitarity violation \[22\]. Which representation of Ostrogradski instability one takes depends on the context of the theory and the approach in question. The issue of Ostrogradski instabilities and ghosts has been addressed so far on a number of occasions, for example partial masslessness \[23\], critical gravity \[24\], and by in-
troducing a PT-symmetric Hamiltonian (where P stands for parity reversal and T for time reversal) \[25\]. It is still an open problem and we do not solve it here either, but we shall reveal a new perspective by which it could be eventually solved.

### III. 3 + 1 DECOMPOSITION OF WEYL-SQUARED GRAVITY

We shall employ here a 3 + 1 decomposition of spacetime, for which the foliation is performed in terms of three-dimensional spacelike hypersurfaces \(\Sigma\), parametrized by a time function \(t\); see, for example, \[1\]. Or more covariant four-gradient of the time function is used to define a timelike unit covariant four-vector \(n_\mu = -N \nabla_\mu t\), where \(N > 0\) is the lapse function, and we have the normalization \(g_{\mu\nu}n^\mu n^\nu = -1\). In Arnowitt-Deser-Misner (ADM) variables, the covector has components \(n_\mu = (-N, 0, 0, 0)\), while its contravariant version is given by \(n^\mu = (1/N, -N^i/N)\), where \(N^i\) is the shift vector. The decomposition of the four-metric is then given by

\[
g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu, \tag{7}\]

where \(h_{\mu\nu}\) is the metric induced to the hypersurface \(\Sigma\). The timelike vector \(n^\mu\) is orthogonal to the hypersurface \(\Sigma\), that is, \(h_{\mu\nu} n^\nu = 0\). This is the core of the 3 + 1 decomposition: \(g^{\alpha\beta} h_{\alpha\mu} = h^\mu_{\nu}\) projects the components of four-tensors to the spatial hypersurface \(\Sigma\), while \(n^\mu\) projects them to the direction orthogonal to it. Using these projections, a four-tensor \(T_{\mu\nu}\), for example, can be decomposed in the following way:

\[
T_{\mu\nu} = (h^\alpha_{\mu} - n^\alpha n_\mu) (h_\nu^\beta - n^\beta n_\nu) T_{\alpha\beta} = \frac{1}{2} (T_{\mu\nu} - |T_{\mu\bot}| - |T_{\nu\bot}| + T_{\bot\bot}), \tag{8}\]

where \(\bot\) denotes the Greek indices are projected to the hypersurface using \(h^\mu_{\nu}\), while \(\bot\) denotes the position of an index that has been projected along the orthogonal vector \(n^\mu\).

The decomposition (7) implies that the four-metric and its determinant decompose as

\[
g_{\mu\nu} = \begin{pmatrix} -N^2 + N_i N^i & N_i \\ N_i & h_{ij} \end{pmatrix}, \quad \sqrt{-g} = N \sqrt{h}, \tag{9}\]

where \(h_{ij}\) is now the three-metric as directly formulated with spatial indices, which is used to raise and lower spatial indices; we denote \(h := \det h_{ij}\). The “local three-volume” \(\sqrt{h}\) is often referred to as an intrinsic time because it makes the kinetic term of the Hamiltonian indefinite; see, for example \[27\] and \[1\] [see there in particular Eq. (5.21)]. The inverse of the four-metric has the form

\[
g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^i/N^2 \\ N^i/N^2 & h^{ij} - N^i N^j/N^2 \end{pmatrix}. \tag{10}\]

With these definitions, the time component of objects projected to the hypersurface vanish; in \[8\], for example, all components with “\(\bot\)” are now spatial, and the “\(\bot\)” can be dropped with the understanding that Greek indices can there be turned into Latin ones \(i, j\), etc.: \(|T_{\mu\nu}| \rightarrow (\nabla T_{ij}), |T_{L\nu}| \rightarrow T_{Lj}\), etc., where objects denoted with a left superscript “(3)” are intrinsic to the hypersurface.

The Riemann tensor, Ricci tensor, and Ricci scalar can be decomposed in a manner similar to \[8\]. The resulting expressions are well known and can be found, for example, in \[1, 20, 28\]; for the decomposition of the Weyl tensor we refer to \[12\] and \[13\] for a derivation. Here, we only state the final expressions for the Ricci scalar and the squared Weyl tensor,

\[
R = (\nabla R + K_{ij} K^{ij} + K^2 + 2\mathcal{L}_n K - \frac{2}{N} D^i D_i N) \tag{11}\]

\[
\mathcal{L}_n K_{ij} = (\nabla R + K_{ij} K^{ij} + K^2 + 2\nabla (n^\mu K) - \frac{2}{N} D^i D_i N, \tag{12}\]

\[
C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} = 8 C_{i\bot j\bot} C^{i\bot j\bot} - 4 C_{ijk} C^{ijk}, \tag{13}\]

where

\[
K_{ij} = \frac{1}{2} \mathcal{L}_n h_{ij} = \frac{1}{2N} \left( h_{ij} - 2D_i N_j \right), \tag{14}\]

\[
K = h^{ij} K_{ij} = \mathcal{L}_n \sqrt{h} = \frac{1}{N} \left( \sqrt{h} - D_i N_j \right) \tag{15}\]

are the extrinsic curvature (second fundamental form), \(K = h^{ij} K_{ij}\) its trace, and \(D_i\) the spatial covariant derivatives with respect to \(h_{ij}\). The quantities

\[
C^{ij}_\bot := -2 C_{i\bot j\bot} = \frac{1}{2} \mathcal{L}_n K_{ab} - (\mathcal{L}_n K_{ab} - K_{ab} K - \frac{1}{N} D_{ab} N) \tag{16}
\]

\[
C^{ij}_{ijk} := \frac{1}{2} \mathcal{L}_n K_{ab} - (\mathcal{L}_n K_{ab} - K_{ab} K - \frac{1}{N} D_{ab} N), \tag{17}\]

are related to the “electric” and “magnetic” parts of the

\[4\] A superscript “\(\bot\)” always denotes a traceless object.
Weyl tensor \( \mathit{L} \), where \( \mathbb{1}^{ab}_{(ij)} \) and \( S^{def}_{ijk} \) are defined as
\[
\mathbb{1}^{ab}_{(ij)} := \delta^{a}_{(i} \delta^{b}_{j)} - \frac{1}{3} h_{ij} h^{ab},
\]
\[
S^{def}_{ijk} := \delta^{[e}_{i} \delta^{f]}_{k} (\varepsilon^{d}_{j} \varepsilon^{g]}_{k} - h_{jk} h^{e][g}) ,
\]
and \( D^{r}_{ij} \equiv \mathbb{1}^{ab}_{(ij)} D_{ab} \equiv \mathbb{1}^{ab}_{(ij)} D_{a}D_{b} \). Note that the term \( C^{CT}_{ij}C^{SJ} \) in \( \mathit{L} \) contains only traceless quantities and does not contain velocities of the trace \( K \), but contains the trace \( K \) itself. This is an important observation to be referred to in our Hamiltonian formulation below.

It is evident from \( \mathit{L} \), \( \mathit{E} \), and \( \mathit{H} \) that the Lagrangian of the W theory is of second order in the time derivatives of the three-metric and that the order cannot be reduced using partial integration. In order to formulate the theory canonically, one needs to introduce a new variable in order to “hide” the first derivative. For this, we add to the original Lagrangian density \( \mathit{L}^{W} \) a term that implements the relation \( \mathit{L}^{W} \),
\[
\mathit{L}^{W} \rightarrow \mathit{L}^{W} = \mathit{L}^{W} - \lambda^{ij} (2K_{ij} - \mathcal{L}_{n}h_{ij}) ,
\]
with Lagrange multipliers \( \lambda^{ij} \). This will be the starting point for the Hamiltonian formulation of the Weyl-squared theory.

The addition of the EH term changes nothing regarding the promotion of \( K_{ij} \) to a canonical variable, even if the usual boundary term is subtracted from it, as will be discussed in more detail in Sec. VI

We are now ready for the Hamiltonian formulation of the two theories.

### IV. UNIMODULAR-CONFORMAL 3 + 1 CONFIGURATION VARIABLES

The Hamiltonian formulation of Weyl gravity is by no means new. The previous works \( 8, 13, 20 \) have addressed such a formulation in several ways, some of which are more similar to each other than others. But there are still some gaps in understanding the constraint structure and the conformal symmetry. One of the most important ones is the following. If the trace of the extrinsic curvature \( K \) is indeed an arbitrary object in the theory (as originally observed by Kaku \( 8 \) and Boulware \( 9 \)), one can conclude that the local volume element (intrinsic time) \( \sqrt{h} \), in which the scale degree of freedom of the three-metric is contained, should be arbitrary, too, since in the conformally invariant Weyl-tensor theory all scales are irrelevant. Consequently, it should be possible to formulate the Hamiltonian constraint in a conformally invariant way. We believe that the formal reason for not implementing this fact so far lies in not making use of an irreducible unimodular-conformal decomposition for all 3+1 variables. It is the purpose of this section to introduce such variables and to use them to perform the Hamiltonian analysis in the following sections, which will show that the Hamiltonian constraint can indeed be made conformally invariant. This is also suitable for the discussion of the quantum theory for which we can observe a connection between conformal symmetry and the absence of intrinsic time \( 4, 30 \).

In the following, we shall define the unimodular-conformal 3 + 1 canonical variables with which we will separate the full set of canonical variables into a conformally invariant and a conformally noninvariant part. The irreducible nature of the unimodular-conformal decomposition is crucial for this. It was already mentioned in \( 11 \) in connection with the Hamiltonian formulation of \( f(\text{Riem}) \) theories that the canonical description of the W theory would be more transparent if one could isolate the determinant of the three-metric as a canonical variable. The present paper puts this into practice and relates the unimodular decomposition of the three-metric to the conformal decomposition of the extrinsic curvature. Namely, we shall decompose the three-metric into its scale part, \( \sqrt{h}^{1/3} \), and its unimodular (conformal) part, \( h_{ij} \). The usefulness of such a unimodular decomposition can be seen in other situations; see, for example, \( 31 \). We shall, however, go beyond the decomposition of only the three-metric and decompose also the lapse function \( N \) into its scale part and a scale free lapse density \( \bar{N} \). We observe that the contravariant shift vector \( N^{i} \) is already scale free, that is, conformally invariant, while its covariant version can be decomposed into a scale part and a scale free density part \( \bar{N}_{i} \). In explicit form, the decomposition of lapse, shift, and three-metric reads
\[
N^{i} =: \bar{N}^{i} , \quad N_{i} =: (\sqrt{h})^{\frac{2}{3}} \bar{N}_{i} , \quad N =: (\sqrt{h})^{\frac{1}{3}} \bar{N} , \quad h_{ij} =: (\sqrt{h})^{\frac{2}{3}} \bar{h}_{ij} ,
\]
All barred objects, being scale free, are invariant under conformal transformations (see Appendix 13). This decomposition then suggests for the hypersurface-orthogonal unit four-vector the definitions
\[
n^{\mu} := (\sqrt{h})^{-\frac{1}{2}} \bar{n}^{\mu} = (\sqrt{h})^{-\frac{1}{2}} \left( \frac{1}{N} - \frac{N^{i}}{N} \right) ,
\]
\[
n_{\mu} := (\sqrt{h})^{\frac{1}{3}} \bar{n}_{\mu} = (\sqrt{h})^{\frac{1}{3}} \left( -\bar{N}, 0 \right) ,
\]
which further implies
\[
\mathcal{L}_{n} T = (\sqrt{h})^{-\frac{1}{3}} \mathcal{L}_{n} T
\]
for the Lie derivative along \( \bar{n}^{\mu} \) with respect to the usual Lie derivative along \( n^{\mu} \) of any tensor (density) \( T \).

Using \( 21 \), the extrinsic curvature \( 14 \) can be decom-
posed as

\[ K_{ij} = \frac{\sqrt{h}}{2N} \left( \dot{h}_{ij} - 2 [D_i \bar{N}_j]^T \right) 
\]

\[ + \frac{1}{3} h_{ij} \frac{1}{(\sqrt{h})^{\frac{3}{2}N}} \left( \frac{\sqrt{h}}{\sqrt{h}} - D_i \bar{N}^i \right), \tag{24} \]

where the superscript “c” denotes that the expression in the brackets is traceless. Notice from the structure of Eq. (24) that we can identify explicitly the traceless and trace parts of \( K_{ij} \) (corresponding to “shear” and “expansion”), each of which can be decomposed suitably such that the resulting objects have a simplified conformal transformation law.\(^6\)

\[ K_{ij} = K^T_{ij} + \frac{1}{3} h_{ij} K, \quad \text{where} \quad K^T_{ij} = \frac{\sqrt{h}}{2N} \left( \dot{h}_{ij} - 2 [D_i \bar{N}_j]^T \right) =: (\sqrt{h})^{\frac{1}{2}} K^T_{ij}, \tag{25} \]

\[ K = \frac{1}{(\sqrt{h})^{\frac{3}{2}N}} \left( \frac{\sqrt{h}}{\sqrt{h}} - D_i \bar{N}^i \right) =: 3(\sqrt{h})^{-\frac{3}{2}} K. \tag{26} \]

We thus have arrived at the interesting conclusion that the irreducible unimodular decomposition of the three-metric induces an irreducible decomposition of the extrinsic curvature into it traceless and trace parts; to our knowledge, this has not been remarked before in the literature. Notice that the trace density \( \bar{K} \), unlike the traceless extrinsic curvature density \( K^T_{ij} \), still contains the scale and transforms inhomogeneously under conformal transformations, see Eq. (13) in Appendix C, it represents the evolution of the scale (or local three-volume). One can put together (26), (27), and write

\[ K_{ij} = (\sqrt{h})^{\frac{1}{2}} \bar{K}_{ij} = (\sqrt{h})^{\frac{1}{2}} \left( K^T_{ij} + \dot{h}_{ij} K \right), \tag{28} \]

where the combination in the parentheses could be referred to as extrinsic curvature density.

The conformal nature of most of the variables resulting from the unimodular-conformal decomposition with (21), (20), and (27) motivates us to choose the following \textit{canonical} variables for studying (1) and its extension (5), see also Appendix B.

\[ N^i = N^i, \tag{29} \]

\[ \bar{N}_i = a^{-2} N_i, \quad \bar{N} = a^{-1} N, \tag{30} \]

\[ \dot{h}_{ij} = a^{-2} h_{ij}, \quad a := (\sqrt{h})^{\frac{1}{2}}, \tag{31} \]

\[ \dot{K}^T_{ij} = a^{-1} K^T_{ij}, \quad \bar{K} = \frac{aK}{3}. \tag{32} \]

Except \( a \) and \( \bar{K} \), all the new variables are deprived of scale density and are thus conformally invariant; \( a \) and \( \bar{K} \) transform under conformal transformations as

\[ a \to \bar{a} = \Omega a, \tag{33} \]

\[ \bar{K} \to \bar{K} = \bar{K} + \mathcal{L}_\bar{a} \log \Omega, \tag{34} \]

where according to (23)

\[ \mathcal{L}_\bar{a} \log \Omega = \bar{n}^\mu \partial_\mu \log \Omega = (\sqrt{h})^{-1/3} n^\mu \partial_\mu \log \Omega, \]

and \( \bar{K}^T_{ij} \) and \( \bar{K} \) are given by

\[ \bar{K}^T_{ij} = \frac{1}{2N} \left( \dot{h}_{ij} - 2 [D_i \bar{N}_j]^T \right) \tag{35} \]

\[ = \frac{1}{2N} \left( \bar{h}_{ij} - 2 [\bar{D}_i \bar{N}_j]^T \right) \tag{36} \]

\[ \bar{K} = \frac{1}{N} \left( \frac{\dot{a}}{a} - \frac{1}{3} \bar{D}_i \bar{N}^i \right) \tag{37} \]

\[ = \frac{\bar{n}^\mu \partial_\mu a}{a} - \frac{\partial_\mu \bar{N}^i}{3N}. \tag{38} \]

Equations (36) and (38) were derived using the results from Appendix C and they manifestly reveal that the extrinsic curvature transforms inhomogeneously due to the first term in (38), while (36) is conformally invariant. Note that the “bar derivative” \( \bar{D}_i \) is defined with respect to the conformal part of the metric \( \bar{h}_{ij} \) and is not of covariant nature (see Appendix C). It should be noted that all new variables are now tensor densities, except the contravariant shift vector \( \bar{N}^i = N^i \), whose bar we omit from now on.

Note that similar decompositions are used in a number of instances related, in particular, to the Cauchy problem \( 32, 33 \) and to conformal-traceless decompositions of Einstein equations \( 34 \). We refer to \( \bar{N}, \bar{K}^T_{ij}, \) and \( \bar{K} \) as the “lapse density” \( \bar{N} \), “traceless extrinsic curvature density”, and “trace density”, respectively.

One now expects that any conformally invariant theory — such as the W theory — will be deprived of scale density \( a \) and trace density \( \bar{K} \). Moreover, any conformal symmetry breaking of a theory would be connected to only these two variables. In other words, their absence reflects conformal invariance. Accordingly, we expect the configuration space structure to be much simpler and the constraints easier to understand with the choice of unimodular-conformal variables as \textit{canonical} variables. This is investigated in the following sections.

V. PURE WEYL-SQUARED GRAVITY

A. Hamiltonian formulation and constraint analysis

The starting point is the Weyl-tensor action \( 11 \), for which the \( 3 + 1 \) decomposition \( 13 \) of the Weyl tensor

\[ \text{\textit{The tracelessness of } } \dot{h}_{ab} \text{ \textit{can be seen by using } } \delta h = hh^{ab} \delta h_{ab} \text{ \textit{to show that } } h^{ab} \delta h_{ab} = 0. \]

\[ \text{\textit{Also called } } \text{“densitized lapse” or “Taub function” } 32. \]
as well as the added constraint in (20) are used. Recall that the latter is needed to take care of the additional degrees of freedom introduced by promoting $K_{ij}$ to an independent variable. This leads to the action

$$S^w = \int dt d^3x \sqrt{\gamma} \left\{ -\frac{\alpha_w}{2} C_{ij}^T C_{ij}^{\text{tr}} + \alpha_w C_{ij} C^{ij} - \lambda^{ij} (2K_{ij} - \mathcal{L}_n h_{ij}) \right\}. \tag{39}$$

Unlike previous works, we now implement the unimodular-conformal variables introduced in the last section. Term by term, this leads us to the following expressions.

Using in (16) Eq. (14) from Appendix 3, the electric part of the Weyl tensor becomes

$$C_{ij}^T = \mathcal{L}_n K_{ij}^T - \frac{1}{3} K_{ij}^T K - \frac{2}{3} h_{ij} K_{ab} K_{^b^a^T} - \frac{1}{3} D_{ij}^T N, \tag{40}$$

where $D_{ja} = 0$ was used. Using (28), the rightmost identity in (20), and (14), the first two terms on the right-hand side of (10) reduce to

$$\mathcal{L}_n K_{ij}^T - \frac{1}{3} K_{ij}^T K = \mathcal{L}_n K_{ij}^c. \tag{41}$$

Thus not only the velocity $\dot{K}$, but also the trace of the extrinsic curvature itself disappears explicitly from the Lagrangian of the Weyl-squared theory. Furthermore, the scale density $\alpha$ disappears as well, because $\frac{(\omega)R_{ij}^c + \frac{2}{3} D_{ij}^T N}$ can be shown not to depend on $\alpha$ (see Appendix D). In fact, $K$ and $\alpha$ were never there in the first place, owing to the conformally invariant nature of the Weyl tensor, but this fact is obscured if the original variables are used. This is directly related to the fact that the Weyl tensor is traceless. One can thus make the following statement: In the Weyl-squared theory, traces and scales do not propagate and should thus not appear explicitly in the constraints resulting from the Hamiltonian formulation – they are arbitrary.

We will write an overbar to $C_{ij}^c$, $\bar{C}_{ij}^c \equiv C_{ij}^c$ in order to mark that it is expressed in terms of the new variables,

$$\bar{C}_{ij}^c = \mathcal{L}_n \bar{K}_{ij}^c \equiv \mathcal{L}_n K_{ij}^c - \frac{2}{3} \bar{h}_{ij} \bar{K}_{ab}^{\text{tr}} \bar{h}_{^a^b^m} \bar{K}_{^n^m}^{\text{tr}} - \frac{1}{N} [D_{ij} \bar{N}]^c. \tag{42}$$

Note that the trace of the first term on the right-hand side of (12) is canceled by the second term, leaving only the traceless part of $\mathcal{L}_n \bar{K}_{ij}^c$, while the combination $\bar{R}_{ij}^c - \frac{1}{N} [D_{ij} \bar{N}]^c$ is conformally invariant; see (D6)–(D9).

Using (28) and (17), the $C_{ij}^c$-term in (39) is seen to scale with $a^{-4}$,

$$C_{ij} C^{ij} = C_{ij} h^a h^b h^c C_{abc} = a^{-4} \bar{C}_{ij} \bar{h}^a \bar{h}^b \bar{h}^c \bar{C}_{abc} \equiv a^{-4} \bar{C}_{ij}^2, \quad \text{for short.} \tag{43}$$

In the last term of the Lagrangian in (39), we simply split all objects, using (24) and (29)–(32), as well as (36) and (38), and finally obtain the following Lagrangian, which is the starting point for our canonical formalism,

$$L^w_c = N \left\{ -\frac{\alpha_w}{2} \bar{h}^a \bar{h}^b \bar{C}_{ij}^c \bar{C}_{^a^b^c} + \alpha_w \bar{C}_{ij}^2 \right. \right.$$

$$\left. \left. - a^5 \lambda^{ij} \left[ 2\bar{K}^T_{ij} - \frac{1}{N} \left( \bar{h}_{ij} - 2 [D_{ij} \bar{N}]^c \right) \right] \right. \right.$$

$$\left. \left. - 2a^3 \lambda \left( \bar{K} \right. - \frac{1}{N} \left( \dot{\bar{a}} - \frac{2}{3} D_{a} N^a \right) \right) \right. \right\}. \tag{44}$$

We note that the scale density $a = (\sqrt{\gamma})^{1/3}$ and the trace $\bar{K}$ have vanished from the Weyl-tensor part of this Lagrangian, as is expected for a conformally invariant theory. It then seems unnecessary to introduce $\bar{K}$ as an independent variable, but since we want to start from the full configuration space, not the subspace, we will take the full $\bar{K}_{ij}$ as independent degrees of freedom. This provides a deep insight into the structure of the theory.

The canonical momenta conjugate to our unimodular-conformal variables are then defined as follows:

$$p_S = \frac{\partial \mathcal{L}^w_c}{\partial \dot{\bar{N}}} \approx 0, \quad p_i = \frac{\partial \mathcal{L}^w_c}{\partial \dot{\bar{N}}^i} \approx 0, \tag{45}$$

$$\bar{p}^{ij} = \frac{\partial \mathcal{L}^w_c}{\partial \dot{\bar{h}}_{ij}} = a^5 \lambda^{ij}, \tag{46}$$

$$p_a = \frac{\partial \mathcal{L}^w_c}{\partial \dot{a}} = 2a^2 \lambda, \tag{47}$$

$$\bar{P}^{ij} = \frac{\partial \mathcal{L}^w_c}{\partial \dot{\bar{K}}_{ij}} = -\alpha_w \bar{h}^a \bar{h}^b \bar{C}_{ab}, \tag{48}$$

$$P = \frac{\partial \mathcal{L}^w_c}{\partial \dot{\bar{K}}} \approx 0, \tag{49}$$

where the “≈” is Dirac’s “weak equality” 30. It can be shown 12 that it is unnecessary to include variables $\lambda_{ij}^c, \lambda$ and their conjugate momenta as canonical variables, and then (10) and (17) are strong equalities. Note that the momenta (10)–(48) are tensor densities of scale weight 9, $w_a = 5$, $w_a = 2$, and $w_a = 4$, respectively. Note

9 The “scale weight” $w_a$, which is related to the weight $w$ of a tensor density by $w_a = 3w$, is introduced in Appendix C.
that the momenta (16) and (18) are traceless. One can conclude from the above that there are three primary constraints of which

\[ \bar{P} \approx 0 \quad (50) \]

is a new one compared to the standard ADM formulation of GR where we only have the two corresponding to (15) [1]. This new constraint is signaling the arbitrariness of \( \bar{K} \) (and implicitly \( K \)) if it turns out (as it will) to be a first class constraint. Within the formalism of (12), where the original variables are used, one cannot conclude that the trace \( \bar{K} \) is arbitrary because \( P \) and \( K \) are there not canonical variables, but a combination of them, namely \( P = h_{ij}p^{ij} \) and \( K = h^{ij}K_{ij} \), so they cannot be identified one to one with an arbitrary degree of freedom. Here, however, \( \bar{P} \approx 0 \) is similar in nature to (45): if \( p^i \) and \( p^i \) imply that the lapse \( \bar{N} \) and shift \( \bar{N}^i \) do not appear in the constraints, then \( \bar{P} \approx 0 \) implies that \( \bar{K} \) should not appear, too. Hence, \( \bar{K} \) is an arbitrary degree of freedom. Recall the definition (27) of \( \bar{K} \), where we have explicitly spelled out the dependence on the spatial coordinates only for the first term under the integral.

The transformation (20)–(32) is, in fact, a canonical one. The Poisson brackets among the canonical variables are given by

\[ \{ q^A_{ij}(x), \Pi_B^{ij}(y) \} = \delta_{ij} \delta_B^A \delta(x, y) \quad (52) \]

for the conformally invariant pairs \( q^A_{ij} = (\bar{h}_{ij}, \bar{K}^T_{ij}), \Pi^B_{ij} = (\bar{p}^{ab}, \bar{P}^{ab}) \), and

\[ \{ q^A(x), \Pi_B(y) \} = \delta^A_B \delta(x, y) \quad (53) \]

for the scale and trace pairs \( q^A = (a, \bar{K}), \Pi_B = (p_a, \bar{P}) \). Recall that \( T_{ij}^{br} \) is defined in (15); its presence guarantees that both sides of (22) are traceless, that is, compatible with each other. Expressions similar to (22) and (33) hold for lapse, shift, and their canonical momenta, while all other Poisson brackets vanish.

In terms of the canonical variables, the constrained Lagrangian (11) reads

\[ \mathcal{L}^W_c = \bar{N} \left[ -\frac{\bar{h}_{ia}\bar{h}_{jb}\bar{P}^{ij}}{2\alpha_w} - 2\bar{K}_{ij}^T\bar{P}^{ij} - a\bar{K}p_a + \alpha_w \bar{C}^2_{ij} \right] + \bar{h}_{ij}\bar{p}^{ij} + \dot{a} p_a - 2\dot{D}_i\bar{N}_j\bar{p}^{ij} + \frac{1}{3} \alpha p_a D_i N^i , \quad (54) \]

where the symmetrization and the “-” were dropped in the next-to-last term because \( \bar{P}^{ij} \) is symmetric and traceless. The fact that \( \alpha_w \) does enter the various terms in a different way has important consequences for the quantum theory (4, 30).

For the Hamiltonian, we need in addition to express \( \bar{K}^T_{ij}\bar{P}^{ij} \) in terms of the new canonical pairs. Using (22) and (48) [1], we get

\[ \dot{\bar{K}}^T_{ij} \bar{P}^{ij} = \bar{N} \left[ -\frac{\bar{h}_{ia}\bar{h}_{jb}\bar{P}^{ij}}{\alpha_w} + (\bar{R})_{ij} + \bar{D}_j \bar{N}^{ij} \right] + \mathcal{L}_{\bar{N}} \bar{K}^T_{ij} \right] + \bar{D}_i \left( D_j \bar{N} \bar{P}^{ij} - \bar{N} \bar{D}_j \bar{P}^{ij} \right) , \quad (55) \]

where \( \mathcal{L}_{\bar{N}} \) is the Lie derivative with respect to the shift vector \( N^i \), and we have used the Leibniz rule twice to avoid the double covariant derivative for the lapse. From the traceless nature of \( \bar{P}^{ij} \) it is clear that only the traceless part of the parentheses contributes, which is denoted by attaching the superscript “-”.

The total Hamiltonian is found by a Legendre transform of the constrained Lagrangian supplemented by all

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10 In terms of the original variables, this would correspond to \( p = h_{ij}p^{ij} \approx 0 \), which obviously did not appear as one of our primary constraints. In earlier works, e.g. (12) [1], one can find such a term only as a part of their conformal (or dilatational) secondary constraint.

11 Note that the second term on the right-hand side of (12) vanishes when contracted with the traceless \( \bar{P}^{ij} \).
primary constraints,
\[
H^w = \int d^3 x \left\{ \dot{h}_{ij} \dot{p}^{ij} + \dot{a} p_a + \kappa_{ij} \dot{p}^{ij} - L_c^w \\
+ \lambda_X p_N + \lambda_p \dot{p}^i + \lambda_P \dot{P} \right\}
\]
\[
= \int d^3 x \left\{ -\frac{h_{ik} \dot{h}_{ij} \dot{p}^{ij} \dot{p}^{kl}}{2 \alpha_w} + \left( \kappa^{ij}_a \dot{\partial}_i \dot{D}_j \right) \dot{p}^{ij} \\
+ 2 \kappa^{ij}_i \dot{p}^{ij} + a \dot{K} p_a - \alpha_w c_{ij}^2 \\
+ 2 \dot{D}_k \left( \kappa^{ij}_a \dot{p}^{kj} \right) + \dot{p}^{jk} \dot{D}_i \kappa_{ij}^{T} \\
+ \lambda_X p_N + \lambda_p \dot{p}^i + \lambda_P \dot{P} \right\} + H_{surf},
\]
(56)
where \(H_{surf}\) are the surface terms arising from partial integration,
\[
H_{surf} = 2 \int d^3 x \left( \dot{\partial}_i \left( \dot{N} \dot{p}^{i} \right) + \dot{\partial}_j \left( \dot{K}_{ia}^{T} \dot{p}^{ij} N^a \right) \right) \\
+ \int d^3 x \dot{\partial}_i \left( \dot{\partial}_j \dot{N} \dot{p}^{ij} - \dot{N} \dot{D}_j \dot{p}^{ij} \right).
\]
(57)
Except for the term \(D_i \left( a p_a \right)\), all covariant derivatives in [50] reduce to barred derivatives \(\dot{D}_i\), and \((\kappa^{ij}_a \dot{\partial}_i \dot{D}_j) \dot{p}^{ij}\) reduces to \((\kappa^{ij}_a \dot{\partial}_i \dot{D}_j) \dot{P}^{ij}\) when using \((22) - (25)\) from Appendix [4].

This concludes our preparation for the constraint analysis.

Let us now address the constraints. At first glance, one could interpret the expressions in front of \(\dot{N}\) and \(\dot{N}^i\) in [55] as the Hamiltonian and momentum constraints, respectively, arising from the conservation of the primary constraints [4] in time. But let us be more careful. We note that the scale density \(a\) appears only in two terms in [50]. One of these terms (the term \(a \dot{K} p_a\)) is the only term containing the trace \(\dot{K}\). At this point, it seems to be part of the Hamiltonian constraint. But since \(\dot{P} \approx 0\) implies that \(\dot{K}\) is arbitrary and should vanish from the constraints explicitly, as stated above, \(\dot{K}\) should be (part of) a Lagrange multiplier. [12] Here, this fact follows from the conservation of the constraint [59] in time,
\[
\dot{P} = \{ P, H^w \} = -\frac{\delta H^w}{\delta K} = -\dot{N} a p_a \approx 0,
\]
(58)
which introduces a secondary constraint \(Q_w^w\) given by
\[
Q_w^w := a p_a \approx 0.
\]
(59)
One could, of course, have concluded from this that we have \(p_a \approx 0\) instead of \(a p_a \approx 0\), since we know nothing about \(a\) as a configuration variable, while \(p_a\) is somewhat artificial. This would indicate that \(a\) is arbitrary, as suspected, but since the expression is \textit{weakly} vanishing, we keep it as a whole. Another reason to support taking \(a p_a\) as the constraint is that it will turn out to be part of the generator of conformal transformations; see below.

We call the constraint [59] the “scaling constraint”, because it implies the conformal transformation of the scale only, as we shall see, and differs from the corresponding constraint derived in [8, 11, 13] and called there the “conformal (or dilatational) constraint”. The conformal constraint in these earlier papers contains an additional term of the form \(K_{ij} P^{ij}\), which in our case is absent because the use of conformally invariant variables leaves in [59] only those variables that are affected by a conformal transformation.

Note that the consistency condition [55] is another way of stating that the total Hamiltonian should not depend on \(K\). This is very important because this statement is here formally realized by the use of the unimodular-conformal canonical variables; in the original variables used in most of the earlier work, this conclusion cannot be drawn.

The demand for the temporal conservation of \(Q_w^w\) gives no further constraints,
\[
\dot{Q}^w = \{ a, P, H^w \} = 0.
\]
(60)
In addition, the Poisson bracket between \(Q_w^w\) and \(\dot{P}\) trivially vanishes because \(a, p_a\) and \(\dot{P}\) are independent variables. The result [60] may also be understood from the fact that
\[
\dot{a} = \{ a, H^w \} = a \left( \dot{N} K + \frac{1}{3} D_i N^i \right)
= -a \frac{\dot{p}_a}{p_a} = a p_a = f(x),
\]
(61)
meaning that \(a p_a\) is a constant of motion. Note that this calculation is done completely at the level of Poisson brackets, without imposing any constraints. This result is valid only in the pure Weyl theory (see the next subsection).

By examining the two terms in [53] that contain \(a p_a\), [59] suggests that \(\dot{N} K + D_i N^i / 3\) is the Lagrange multiplier (using one of the surface terms and partial integration on the second of the two terms) for \(Q_w^w\), so we can isolate \((\dot{N} K + D_i N^i / 3) Q_w^w\) from the rest of the terms. Recalling [27], one can easily see that this Lagrange multiplier is effectively \(\dot{a} / a\). One may then conclude that we do not need to add the secondary constraint \(Q_w^w\) to the total Hamiltonian by hand with an additional Lagrange multiplier, as was done in [13]; we merely need to isolate

\[\text{At this point, Kaku’s prescription [8] would be to isolate terms with } \dot{K} \text{ and interpret it as a } \text{l Lagrange multiplier times a constraint } \text{where the constraint stems from } \dot{P}. \text{ Having achieved this, Kaku employed his argument that } K \text{ is arbitrary and arrived at his “dilatational constraint”. A similar treatment can be found in [13].}\]
it from the rest of the Hamiltonian, with the Lagrange multiplier essentially being $\bar{K}$. Such a prescription was used in [8]. Moreover, adding it by hand would break the equivalence with the Lagrange formulation.

Demanding that the primary constraints be preserved in time, and keeping in mind that $\delta p_a \approx 0$, we find that the Hamiltonian and the momentum constraints are given by (adding $Q^W$ for completeness)

$$\mathcal{H}^W_1 = -\bar{h}_{ij} \bar{h}_{kl} \bar{D}^{ij} + \bar{D}_{ij} \bar{D}^{kl} + \partial_i \bar{D}_j \bar{P}^{ij} + 2 \bar{K}^{ij} \bar{P}^{ij} - \alpha_w \bar{C}_{ij}^2 \approx 0, \quad (62)$$

$$\mathcal{H}^W_i = -2 \bar{D}_j (\bar{h}_{ij} \bar{P}^{jk}) - 2 \bar{D}_k (\bar{K}^{ij} \bar{P}^{jk}) + \bar{P}^{jk} \bar{D}_i \bar{K}^{jk} \approx 0, \quad (63)$$

$$Q^W = \delta p_a \approx 0. \quad (64)$$

Here, we have used (62) and (63) and the density nature of the variables to reduce the momentum constraint from (63) to (64). It can now easily be seen that no scale density $a$ or trace density $\bar{K}$ appear in the Hamiltonian and momentum constraints. Every object in these two constraints is conformally invariant, which confirms our earlier claim that no constraint should depend on the trace $\bar{K}$. The conformal constraint has to contain the scale density because it carries information about the conformal transformation, as we discuss below.

Thus one can conclude that the “intrinsic time” $a$ is absent from the $W$ theory, along with its time evolution encoded in $\bar{K}$. In the case of GR, the Hamiltonian constraint is much simpler, and one cannot get rid of the scale density $a$. The momentum constraints in the $W$ theory have a structure similar to the one in GR, and additional terms are due to the phase space being extended by the canonical pair $(\bar{K}^{ij}, \bar{P}^{ij})$. The most important difference is that while the extrinsic curvature leads in GR to the momentum conjugate to the three-metric, it is in the $W$ theory an independent canonical variable.

B. Constraint algebra and number of degrees of freedom

Using unimodular-conformal variables, the algebra of constraints is expected to be simple. We do not give a proof for the first three Poisson brackets below, but take the results from [11], where it is shown that the hypersurface algebra for a general f(Riem) theory is the same as for GR. This can be understood as a consequence of the reparametrization invariance for such theories ([9], Sec. 1.5). Writing the smeared versions of the constraints (62–64) and (60) (summarized as $\mathcal{C}^W$) as

$$\mathcal{C}^W_\eta = \int \bar{d}^3x \, \eta(x) \cdot \mathcal{C}^W(x), \quad (66)$$

where $\eta(x)$ is an arbitrary (vector or scalar) function, the Poisson brackets among them are

$$\{ \mathcal{H}^W_\eta | \varepsilon_1, \mathcal{H}^W_\eta | \varepsilon_2 \} = \mathcal{H}^W_\eta | \varepsilon_1 \partial^j \varepsilon_2 - \varepsilon_2 \partial^j \varepsilon_1, \quad (67)$$

$$\{ \mathcal{H}^W_\eta | \varepsilon_1, \mathcal{H}^W_\eta | \varepsilon_2 \} = \mathcal{H}^W_\eta | \varepsilon_1 \partial^j \varepsilon_2, \quad (68)$$

$$\{ \mathcal{H}^W_\eta | \varepsilon_1, \mathcal{H}^W_\eta | \varepsilon_2 \} = \mathcal{H}^W_\eta | \varepsilon_1 \partial^j \varepsilon_2, \quad (69)$$

$$\{ \mathcal{H}^W_\eta | \varepsilon_1, \mathcal{P}^W | \varepsilon_2 \} = 0, \quad (70)$$

$$\{ \mathcal{H}^W_\eta | \varepsilon_1, \mathcal{P}^W | \varepsilon_2 \} = 0, \quad (71)$$

$$\{ \mathcal{H}^W_\eta | \varepsilon_1, \mathcal{Q}^W | \varepsilon_2 \} = 0, \quad (72)$$

$$\{ \mathcal{H}^W_\eta | \varepsilon_1, \mathcal{Q}^W | \varepsilon_2 \} = 0, \quad (73)$$

$$\{ \mathcal{P}^W | \varepsilon_1, \mathcal{Q}^W | \varepsilon_2 \} = 0. \quad (74)$$

and all constraints are first class. We note, however, that in [12] and [13] the foliation algebra contains an additional term of the form $P[(\varepsilon_1 D^i \varepsilon_2 - \varepsilon_2 D^i \varepsilon_1)(D_j K^j i - D_i K)]$. The consequence of its presence is unclear, but its origin lies in the fact that the $P$-constraint was there not taken into account when defining the Hamiltonian, momentum, and conformal constraints. Using unimodular-conformal variables, in which neither $P$ nor $\bar{K}$ enter any of the secondary constraints, it is not surprising that our results for the foliation algebra give those of [11]. Since the transformation to the unimodular-conformal variables is canonical, the hypersurface foliation algebra should not change. We stress that the same hypersurface foliation algebra appears both in GR and higher order theories [11]. It would therefore be worth investigating the possibility of a “seventh route to higher derivative theories”, in analogy to the “seventh route to geometrodynamics” [10] since it seems that the same “seventh route” could lead under different assumptions to a theory different from GR, namely to the whole class of higher derivative theories of gravity.

It is instructive to count the number of degrees of freedom, which we can do in three different ways, cf. [1], Sec. 4.2.3 for the general counting procedure. The first way is based on the original phase space and proceeds as follows. There are 32 phase space variables — 12 in the three-metric sector, 12 in the extrinsic curvature sector, and eight in the lapse-shift sector. There are 10 first class constraints — five primary and five secondary — for which one can invoke 10 gauge fixing conditions. This leaves $(32 - 10 - 10)/2 = 6$ degrees of freedom, which agree with earlier results and with the content of the linearized theory presented in [13]. The second way of counting is based on our unimodular-conformal configuration variables. We have seen that scale density $a$ and trace density $\bar{K}$ are absent from the theory. We can thus
go to the subspace spanned by $\tilde{N}, N^i, \tilde{h}_{ij}, \tilde{K}^i_{ij}$ and their canonical conjugates. These are 28 degrees of freedom in phase space. Since $\tilde{P}$ and $Q^\omega$ were already taken into account to expel $a$ and $K$ from the formalism, we have to take into account only eight constraints, leading again to $(28 - 8 - 8)/2 = 6$ degrees of freedom.

There is yet another method of counting the degrees of freedom. Let us consider the configuration space instead of the phase space to count the physical, propagating, degrees of freedom. We have ten configuration variables, five in $\tilde{h}_{ij}$ and five in $\tilde{K}^i_{ij}$. Since we do not have any restrictions on these variables with respect to conformal transformations, there are only four constraints left to reduce the number of variables to $10 - 4 = 6$ degrees of freedom.

A simple thought about an alternative derivation of the Hamiltonian analysis is of use in understanding the previous statement. Suppose we first used unimodular-conformal variables at the unconstrained Lagrangian level. It would then follow from the evidence presented so far that scale density $a$ and trace density $K$ were completely absent at the Lagrangian level, too. The scale constraint (65) would not be present at all, since there would be no variables which transform under conformal transformations. This is clear once one takes into account (17), which with (65) implies that the Lagrange multiplier $\lambda$ is unnecessary. The counting then takes place in such a way that it does not include arbitrary variables. Namely, there would be four vanishing momenta, implying that the lapse density and shift are arbitrary. We are then left with ten variables ($\tilde{h}_{ij}$ and $\tilde{K}^i_{ij}$) which are not constrained by any scale constraint, since they are conformally invariant. Reparametrization invariance (Hamiltonian and momentum constraints) constrains the ten configuration variables to $10 - 4 = 6$ degrees of freedom, which agrees with the above counting.

Note that the linearized theory possesses ghost degrees of freedom, which cannot be deduced from the constraint analysis by a simple counting of degrees of freedom. One can see, however, that the Hamiltonian is linear in momenta, hence seeming unstable (unbounded from below), and one may expect to deal either with negative energies or with nonunitarity (in quantum theory). When referring to such an instability, one must, however, keep in mind that we have a Hamiltonian constraint. This means that any negative part arising from linearity in momenta is compensated by a corresponding positive term, so the negative term is tamed and most likely harmless. In fact, already in GR, part of the kinetic term in the Hamiltonian constraint (the one connected with intrinsic time) is negative definite, without causing instabilities; in fact, this term is needed for good reasons.\[13\]

C. Conformal symmetry and the generator of conformal transformation.

It was claimed in \[8\] \[11\] \[12\] that the constraint arising from $\dot{P} \approx 0$, that is, the conformal constraint [in our case, this is the scale constraint \[65\]], is the generator of conformal transformations on the original variables. Kluson et al. \[12\], for example, arrived at a conformal constraint of the following form:\[14\]

$$ Q := 2h_{ij} P^{ij} + K_{ij} P^{ij} \approx 0, \quad (75) $$

where $P^{ij}$ was not decomposed and is thus not traceless. They claimed, in particular, that $Q$ generates a conformal transformation of $K_{ij}$,

$$ \{K_{ij}, Q[\omega]\} = \omega K_{ij}. \quad (76) $$

This is, however, in contradiction to the actual conformal transformation of $K_{ij}$ given by \[13\] \[12\], because the above Poisson bracket cannot produce the inhomogeneous term containing the derivatives of $\omega$. It would be a correct transformation if one had split $K_{ij}$ into traceless and trace parts and noticed that the term $KP \approx 0$ in \[76\] already vanishes weakly, so that only

$$ Q = 2h_{ij} P^{ij} + K_{ij}^T P^{ij} \approx 0 \quad (77) $$

should be demanded as a constraint.\[15\] In this case, the action of $Q$ could be considered as generating a conformal transformation, but of $K_{ij}^T$ only,

$$ \{K_{ij}^T, Q[\omega]\} = \omega K_{ij}^T. \quad (78) $$

A problem would still remain for the trace $K$, because the Poisson bracket of $K$ with $Q$ would be trivially 0, in contradiction with \[13\] \[10\]. Thus, the action of $Q$ alone cannot fully recover the transformation law for all variables—a piece of information is missing, and it has to involve derivatives of the conformal transformation parameter, $\partial_i \omega$, $\partial^i \omega$.

This apparent contradictory nature of the conformal transformation was first noticed by Irakleidou et al. \[12\], who used Castellani’s algorithm \[14\] (more precisely, the Anderson-Bergmann-Castellani (ABC) algorithm \[14\] \[42\]) to derive the generator of gauge transformations and proved, among other things, that its correct form contains both the primary constraint $P$ (corresponding to $P$ in our case) and the secondary constraint arising from the consistency condition $\dot{P} \approx 0$. The generator also contains the momentum constraints in order to include conformal transformations of the lapse. Another way to identify this contradiction is to consider the

---

\[13\] We use here the original variables.

\[14\] Note that the first term is just (twice) the trace of the momentum, which is related to $p_a$. 
second piece of information, which is the lack of physical interpretation of the isolated constraint \( P \approx 0 \), as noticed by [12], namely, the counterintuitive conformal-like nature of transformations on \( p^{ij} \) and \( K_{ij} \),

\[
\{ K_{ij}, P^i \} = \epsilon h_{ij}, \quad \{ p^{ij}, P^i \} = -\epsilon p^{ij}. \tag{79}
\]

Checking whether the trace of the first equation in (79) yields the correct conformal transformation for \( K \), one finds \( \{ K, P^i \} = 3\epsilon \), which is obviously a contradiction. The resolution of the problem can be found in [88, 43], where the ABC algorithm for diffeomorphism and internal gauge transformations is carefully analyzed. Here, we use a simplified version of their results, as emphasized by Pitts [43] (see also the references therein), which states that primary and secondary constraints have to “work together in a tuned sum” in order to give the correct generator of gauge transformations. Roughly speaking, they are put together in such a way that the transformation parameter of the primary constraint \( \epsilon \) and the one corresponding to the secondary constraint \( \omega \) have to be related in a specific way, namely

\[
\epsilon = -\bar{\omega}. \tag{80}
\]

It is easy to see that in the present case this is not enough, since the spatial derivatives to complete the Lie derivative of \( \omega \) are missing. A rigorous treatment following [88, 43] could solve the problem, and we refer the reader to these papers for further insight. Here, we extend Pitts’ reasoning to the spatial variations and make the following identification:

\[
\epsilon = -\mathcal{L}_n \omega. \tag{81}
\]

Putting the constraints \( P \) and \( Q \) together, we propose the following expression as the generator for conformal transformations:

\[
G_\omega[\omega, \dot{\omega}] := \int d^3x (Q \cdot \omega + P \cdot \mathcal{L}_n \omega). \tag{82}
\]

To check if this gives the correct result, we calculate the Poisson bracket of \( K_{ij} \) with \( G \) to find

\[
\{ K_{ij}, G_\omega[\omega, \dot{\omega}] \} = \omega K_{ij} + h_{ij} \mathcal{L}_n \omega, \tag{83}
\]

which exactly agrees with (82).

However, as stated in [13], the generator of conformal transformations should include the transformation of the lapse, in accordance with (77), for which the simple “sum tuning” does not work. The authors of [13] used the ABC recipe to derive the following generator:

\[
G_\omega[\omega, \dot{\omega}] = \int d^3x \left( \frac{1}{N} \dot{\omega} \bar{P} + \omega \left( Q + N p_n + \mathcal{L}_n \frac{P}{N} \right) \right), \tag{84}
\]

which generates correctly the conformal transformations (83, 810).

Returning to our case, only \( a, p_n \), and \( \bar{K} \) change under a conformal transformation, due to the choice of unimodular-conformal variables (\( \bar{P} \) does not change since it vanishes). Therefore, we expect that the appropriate generator of the conformal transformation reproduces (819). Leaving a rigorous derivation via the ABC algorithm for another time, we take the primary constraint \( \bar{P} \) and the secondary constraint \( Q^w \) and form a tuned sum. Namely, if we let \( \bar{P}[\epsilon] + Q^w[\omega] \) act on \( \bar{K} \) and \( a \), we get

\[
\{ \bar{K}, \bar{P}[\epsilon] + Q^w[\omega] \} = \epsilon, \quad \{ a, \bar{P}[\epsilon] + Q^w[\omega] \} = a \omega. \tag{85}
\]

By comparing with (819), it can be seen that one should make the identification (81) (with \( n \) replaced by \( \bar{n} \)) to define the generator of the conformal transformation as follows:

\[
G_\omega^w[\omega, \dot{\omega}] = \int d^3x (\omega P^a + \bar{P} \mathcal{L}_{\bar{n}} \omega), \tag{86}
\]

Comparing with the result (84) from [13], one can see that the lapse momentum term is missing [apart from the last term differing by a boundary term from (85)]. This is because \( \bar{N} \) is conformally invariant. One can now check that the conformal variations of all variables vanish trivially, except for the scale density \( \bar{a} \) and the trace density \( \bar{K} \), for which one gets

\[
\delta \omega \bar{a} = \{ a, G_\omega^w[\omega, \dot{\omega}] \} = \{ a, \int d^3x \omega a p_n \} = \omega a, \tag{87}
\]

\[
\delta \omega \bar{K} = \{ \bar{K}, G_\omega^w[\omega, \dot{\omega}] \} = \{ \bar{K}, \int d^3x \bar{P} \mathcal{L}_{\bar{n}} \omega \} = \mathcal{L}_{\bar{n}} \omega, \tag{87}
\]

in agreement with (819). It is now clear why one should use \( Q^w = a p_n \) in (84) instead of \( Q^w = p_n \) only; \( \delta \omega \bar{a} \) has to be proportional to \( a \), and our generator \( G_\omega^w[\omega, \dot{\omega}] \) incorporates this demand, giving the correct transformation laws for \( a \) and \( \bar{K} \). Using (819), one can easily check that our results are in agreement with [12]. In this sense, (77) along with (814) can be considered as “building blocks” of conformal transformations in the 3 + 1 formalism. The result is that all other secondary constraints are conformally invariant, which improves the previous result [14, 12] that the Hamiltonian constraint is conformally covariant. Moreover, the absence of \( a \) and \( \bar{K} \) from the theory means that these variables do not need to be gauge fixed. We thus have exploited the full power of the conformal symmetry present in the Weyl theory.

To summarize, we have constructed here the generator of gauge transformations by putting primary and secondary constraints, \( \bar{P} \) and \( Q^w \), into a tuned sum, using (21), to derive the correct form of the conformal transformation generator in unimodular-conformal variables. It should be emphasized that this identification (with \( n \) replaced by \( \bar{n} \)) is not general and only works in the present case. Our treatment is possible because we work with a conformally invariant lapse density \( \bar{N} \),...
and because the gauge transformations are of a particular simple form: for a derivation of all gauge generators in GR and the Weyl-tensor theory in original variables, see [13] and [13], respectively. A rigorous derivation of gauge generators can be found in [28, 13]. We believe that the method presented there gives different results for the explicit form of the generators in the original and the unimodular-conformal variables, due to their different behavior under conformal transformations.

D. Weyl-Hamilton-Jacobi functional

In 1962, Peres has shown [13] that the Hamiltonian constraint of GR can be written as a Hamilton-Jacobi equation, by introducing a functional $S^w[h_{ij}]$ as its solution, such that the ADM momentum is defined as $p_{ADM}^{ij} = \delta S^w / \delta h_{ij}$. The resulting equation became known as the “Einstein-Hamilton-Jacobi equation” (EHJ) and reads

$$\frac{2\kappa}{\sqrt{h}} G_{ijkl} \frac{\delta S^w}{\delta h_{ij}} \frac{\delta S^w}{\delta h_{kl}} - \frac{\sqrt{h}}{2\kappa} R = 0, \quad (88)$$

where

$$G_{ijkl} = \frac{1}{2} \left( h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl} \right) \quad (89)$$

is the inverse of the DeWitt metric [27, 41],

$$G^{ijkl} = \frac{1}{2} \left( h^{ik} h^{jl} + h^{il} h^{jk} - 2 h^{ij} h^{kl} \right). \quad (90)$$

Gerlach [42] then showed that the EHJ is equivalent to all ten Einstein field equations. In analogy to [42], we introduce a functional

$$S^w = S^w[h_{ij}, a, K^c_{ij}, \bar{K}], \quad (91)$$

which is defined on the full configuration space in the unimodular-conformal basis, such that the conjugate momenta $\bar{p}^{ij}, p_a, P^i$, and $\bar{P}$ are given by

$$\bar{p}^{ij} = \frac{\delta S^w}{\delta h_{ij}}, \quad p^{ij} = \frac{\delta S^w}{\delta K^c_{ij}}, \quad \quad (92)$$

$$p_a = \frac{\delta S^w}{\delta a}, \quad \bar{P} = \frac{\delta S^w}{\delta \bar{K}}. \quad (93)$$

The Hamiltonian constraint turns into what we call the Weyl-Hamilton-Jacobi (WHJ) equation [3],

$$\mathcal{H}^w_{\perp} = -\frac{1}{2\alpha_w} \bar{h}_{ik} \bar{h}_{jl} \frac{\delta S^w}{\delta \bar{K}^c_{ij}} \frac{\delta S^w}{\delta K^c_{kl}} + \left( \frac{\delta \bar{K}^c_{ij}}{\delta K^c_{ij}} + \partial_i \bar{D}_j \right) \frac{\delta S^w}{\delta K^c_{ij}} + 2 \bar{K}^c_{ij} \frac{\delta S^w}{\delta h_{ij}} - \alpha_w \bar{C}_{ijk}^c \approx 0. \quad (94)$$

The momenta (92) and (93) should be replaced in all other constraints, too. One may, in analogy to [42], try to prove that the WHJ equation is equivalent to the

Bach equations (equations of motion for the Weyl action); see, for example, [47] for some solutions to this equation. Here, however, we only focus on the functional $S^w[h_{ij}, a, K^c_{ij}, \bar{K}]$ itself and discuss its invariance under gauge transformations.

The Hamiltonian and momentum constraints reflect the fact that $S^w$ is invariant under time reparametrizations and three-diffeomorphisms, respectively. The generator of conformal transformations suggested by [30] should imply conformal invariance of the functional $S^w$, as we show now.

Conformal variations of the functional $S^w$ only act along the $\delta\alpha$ and $\delta \bar{K}$ directions in the unimodular-conformal basis of the configuration space [using (31)],

$$\delta \omega S^w = \int d^3 x \left( \delta \omega \frac{\delta S^w}{\delta a} + \delta \omega \bar{K} \frac{\delta S^w}{\delta \bar{K}} \right) = \int d^3 x \left( \omega a \frac{\delta S^w}{\delta a} + \mathcal{L}_a \omega \frac{\delta S^w}{\delta \bar{K}} \right). \quad (95)$$

On the one hand, one expects to have $\delta \omega S^w = 0$. On the other hand, we have seen from the discussion so far that the correct generator of conformal transformation is given by [30], and we have

$$G^w_{\omega, \bar{\omega}} = 0 \quad (96)$$

on the constraint surface. Now one can easily see that [30] coincides with [15] when the momenta are expressed in terms of $S^w$. Thus, one concludes that

$$G^w_{\omega, \bar{\omega}} = 0 \land G^w_{\omega, \bar{\omega}} = \delta \omega S^w \Rightarrow \delta \omega S^w = 0, \quad (97)$$

that is, that the evaluation of $G_{\omega, \bar{\omega}}$ on the constraint surface expresses the invariance of $S^w$ under conformal transformation. It follows that $S^w$ is independent of scale density $a$ and trace density $\bar{K}$,

$$S^w = S^w[h_{ij}, K^c_{ij}], \quad (98)$$

as expected. One might ask which relation holds instead of (97) and what its meaning is if one attempts to construct such expressions in a theory with second class constraints, as a consequence of conformal symmetry breaking. This is discussed in the following section.

VI. ADDING THE EINSTEIN-HILBERT TERM: WEYL-EINSTEIN GRAVITY

A. Hamiltonian formulation

Supplementing the Weyl-tensor action [11] by the EH term using the expression [11] for the Ricci scalar, the
velocity $\mathcal{L}_n K$ is explicitly introduced into the action.\textsuperscript{15}

\[ \tilde{S}^{\text{WE}} = \int dt \, d^3 x \, N \sqrt{h} \left\{ -\frac{\alpha_w}{2} C_{ij}^2 C^{ij} + \alpha_0 C_{ij}^2 \right. \right. $$
\[ + \frac{1}{2\kappa} \left( (3)R + K_{ij} K^{ij} + K^2 \right) \right. \right. $$
\[ - \lambda^{ij} (2K_{ij} - \mathcal{L}_n h_{ij}) \right\}. \tag{99} \]

As a consequence, the constraint $P \approx 0$ no longer holds, which is an explicit signal for the conformal symmetry breaking. Instead of using (11) in (99), one could alternatively perform a partial integration of the term $\mathcal{L}_n K$ to arrive at the usual expression for the EH action and the well-known boundary term [this is equivalent to using (12)]

\[ \frac{1}{\kappa} \int d^3 x \sqrt{h} K, \tag{100} \]

where $\sigma = -1$ or $+1$ for the timelike or spacelike boundary, respectively.

In GR, it is customary to supplement the EH action with the negative of the above term in order to get rid of the second time derivative of the metric (which translates to the first time derivative of $K$) and provide second order equations of motion upon variation with respect to the metric.\textsuperscript{19} Here, we are however dealing with a higher order equations of motion which cannot be reduced and whose equations of motion are fourth order. Note that the boundary term [100] vanishes upon variation, since $K_{ij}$ is also required besides $h_{ij}$ to be fixed on the boundary. Therefore, a partial integration of the $K$-velocity term is not needed in squared curvature actions supplemented by the EH term, and it is harmless; see [12] and references therein. We choose to use the latter (i.e., the partially integrated) version of the EH action, because the EH term then leaves the constraint $\dot{P} \approx 0$ intact. This choice does not make the breaking of conformal symmetry obvious at first glance, but it simplifies the resulting Poisson brackets. In which way the choice of (not) using the boundary term affects the quantum theory is left to be investigated elsewhere.

The simplified action takes the following form:

\[ S^{\text{WE}} = \int dt \, d^3 x \, N \sqrt{h} \left\{ -\frac{\alpha_w}{2} C_{ij}^2 C^{ij} + \alpha_0 C_{ij}^2 \right. \right. $$
\[ + \frac{1}{2\kappa} \left( (3)R + K_{ij} K^{ij} - K^2 \right) \right. \right. $$
\[ - \lambda^{ij} (2K_{ij} - \mathcal{L}_n h_{ij}) \right\}. \tag{101} \]

Using unimodular-conformal variables, the conjugate momenta for the above action are the same as in [15 – 19]. The total Hamiltonian is given by

\[ H^{\text{WE}} = \int d^3 x \left\{ \dot{\bar{N}} \left[ -\frac{\bar{h}_{ij} \bar{h}_{ij} \bar{P}^{ij} \bar{P}^{kl}}{2\alpha_w} + \left( (3)\bar{R}^{ij} + \partial_i \bar{D}_j \right) \bar{P}^{ij} \right. \right. $$
\[ + 2\bar{K}_{ij} \bar{P}^{ij} + a\bar{K}_{pa} - \alpha_0 C_{ij}^{2} \right. \right. $$
\[ - \frac{1}{2\kappa} a^4 \left( (5)R + a^{-2} \bar{K}_{ij}^{2} - 6 a^{-2} \bar{K}^2 \right) \right. \right. $$
\[ + N^i \left[ -2D_k (\bar{h}_{ij} \bar{p}^{jk}) - \frac{1}{3} D_l (a \bar{p}_a) \right. \right. $$
\[ - 2D_k (\bar{K}_{ij} \bar{P}^{jk}) + D_i \bar{K}_{jk} \bar{P}^{jk} \right. \right. $$
\[ + \lambda_{n} \bar{p}_N + \lambda_{i} \bar{p}^i + \lambda_{p} \bar{P} \right\} + H_{\text{surf}}, \tag{102} \]

where $\bar{K}_{ij}^{2} = \bar{K}_{ij} \bar{h}^{ab} \bar{h}_{ab}$, but $(3)R$ is not yet decomposed. Let us again first discuss the conservation of the $P \approx 0$ constraint, which leads to the requirement

\[ \dot{P} = - \frac{\delta H}{\delta K} = - \bar{N} \left( a \bar{p}_a + \frac{6a^2}{\kappa} \bar{K} \right) \approx 0, \]

\[ \Rightarrow Q^{\text{WE}} := a \bar{p}_a + \frac{6a^2}{\kappa} \bar{K} \approx 0, \tag{103} \]

where we have included $a$ in the definition of $Q^{\text{WE}}$ for reasons similar to the ones in the pure Weyl case. The “spoiled” conformal constraint turns out to form a pair of second class constraints with $\dot{P} \approx 0$, since the constraints do not commute,

\[ \{ \dot{P}(x), Q^{\text{WE}}(y) \} = - \frac{6a^2}{\kappa} \delta(x - y). \tag{104} \]

It is precisely the trace density $\bar{K}$ (which is proportional to $\bar{u}$) that is responsible for this, due to the expression

\[ Q^{\text{E}} := \frac{6a^2}{\kappa} \bar{K}, \tag{105} \]

which depends on $\kappa$. Observe that for $\kappa^{-1} \to 0$ (which would correspond to a high energy regime beyond the Planck scale where the Weyl-tensor term dominates) the constraints become first class and the conformal symmetry is restored. It is now clear that this behavior obviously needs to be treated within a quantum theory of gravity, and that is exactly what we are going to do in a follow-up paper [31]. In addition, it is clear that $Q^{\text{WE}}$ is not automatically conserved in time, since the consistency condition produces terms which are not weakly 0.

\textsuperscript{15} We ignore the total divergence coming from $D_i D_j N / N$.

\textsuperscript{16} This was already noticed by Einstein [18].
Demanding the time derivative to be 0,
\[ \dot{Q}_\text{WE} = \{ Q_\text{WE}, H_\text{WE} \} \]
\[ = \frac{\bar{N}}{2\kappa} \left( 2a^2 \bar{R} - 4a \partial_i (\bar{h}^{ij} \partial_j a) + 12a^2 \bar{K}^2 \right. \]
\[ + 2a^2 K_{ij}^T \right) + \frac{4a^2}{\kappa} \bar{K} D_i N^i + \frac{6a^2}{\kappa} \lambda_p \approx 0, \quad (106) \]
determines the Lagrange multiplier \( \lambda_p \),
\[ \lambda_p = -\frac{\bar{N}}{6a^2} \left( a^2 \bar{R} - 2a \partial_i (\bar{h}^{ij} \partial_j a) + 6a^2 \bar{K}^2 \right. \]
\[ + a^2 K_{ij}^T \right) - \frac{2}{3} \bar{K} D_i N^i, \quad (107) \]
effectively determining \( \dot{\bar{K}} \), which is undetermined in the pure W theory. Note that our result for this Lagrange multiplier differs from the one obtained in [12], because we use unimodular-conformal canonical variables, which expel the variable \( \bar{P} \) from the constraints.

We do not insert this value for the Lagrange multiplier into (102), but calculate instead Dirac brackets. Let us first derive from (102) the Hamiltonian and the momentum constraints. Conservation of the constraint \( p_\perp \approx 0 \) in time leads to
\[ \dot{p}_\perp = \{ p_\perp, H_\text{WE} \} \]
\[ = -\bar{h}_{ik} \bar{h}_{jl} \dot{\bar{P}}^{ij} \frac{\partial \bar{P}^{kl}}{2a_w} \left( \bar{R}_{ij}^{T} + \partial_i \tilde{D}_j \right) \dot{\bar{P}}^{ij} \]
\[ + 2 \bar{K}_{ij} \dot{\bar{P}}^{ij} + a \bar{K} p_a - \alpha W \bar{C}^2_{ijk} \]
\[ - \frac{1}{2\kappa} a^4 \left( \bar{R} + a^2 \bar{K}^2 + 6a^2 \bar{K}^2 \right) \approx 0. \]
\[ (108) \]
We already know from (103) that \( a p_a + 6a^2 \bar{K}/\kappa \approx 0 \); if we use this relation in (108), we arrive at the following expression for the Hamiltonian constraint:
\[ \mathcal{H}_\perp \text{WE} = -\bar{h}_{ik} \bar{h}_{jl} \dot{\bar{P}}^{ij} \frac{\partial \bar{P}^{kl}}{2a_w} \left( \bar{R}_{ij}^{T} + \partial_i \tilde{D}_j \right) \dot{\bar{P}}^{ij} \]
\[ + 2 \bar{K}_{ij} \dot{\bar{P}}^{ij} - \alpha W \bar{C}^2_{ijk} \]
\[ - \frac{1}{2\kappa} a^4 \left( \bar{R} + a^2 \bar{K}^2 + 6a^2 \bar{K}^2 \right) \]
\[ = \mathcal{H}_\perp \text{WE} - \frac{1}{2\kappa} a^4 \left( \bar{R} + a^2 \bar{K}^2 + 6a^2 \bar{K}^2 \right) \approx 0, \]
\[ (109) \]
where \( \mathcal{H}_\perp \text{WE} \) is the Hamiltonian constraint of the pure W theory, which is given in (62). It is clear that due to the term
\[ \mathcal{H}_\perp \text{WE} := -\frac{1}{2\kappa} a^4 \left( \bar{R} + a^2 \bar{K}^2 + 6a^2 \bar{K}^2 \right), \]
\[ (110) \]
the Hamiltonian constraint is now explicitly not invariant under conformal transformations. The momentum constraint in the WE theory contains an additional term \( D_i (a p_a) \) which is scale dependent, as is the case with the scale constraint. The constraints read
\[ \mathcal{H}_i \text{WE} = -2\partial_k \left( \bar{h}_{ij} \bar{P}^{jk} \right) + \partial_i \bar{h}_{jk} \bar{P}^{jk} - \frac{1}{3} \partial_i (a p_a), \]
\[ -2 \partial_k \left( \bar{K}_{ij}^T \bar{P}^{jk} \right) + \partial_i \bar{K}_{jk}^T \bar{P}^{jk} \approx 0 \]
\[ (111) \]
\[ Q_\text{WE} = a p_a + \frac{6a^2}{\kappa} \bar{K} \approx 0. \]
\[ (112) \]
The theory is now manifestly not conformally invariant. The use of our unimodular-conformal variables makes this clear by revealing the scale density- and trace density-dependent terms which are responsible for conformal symmetry breaking in the WE theory.

We can now formulate the total Hamiltonian as a linear combination of all constraints:
\[ H_\text{WE} = \int d^3 x \left\{ \bar{N} \mathcal{H}_\perp \text{WE} + N^i \mathcal{H}_i \text{WE} + (\bar{N} \bar{K}) Q_\text{WE} \right\} \]
\[ + \lambda \bar{K} p_a + \lambda \bar{P} + \lambda p_a \}
\[ (113) \]
where \( \bar{N} \cdot \bar{K} \) can be interpreted as the Lagrange multiplier for the \( Q_\text{WE} \) constraint. If we had, instead, eliminated \( a p_a \) from (111) using (112) (in the spirit of what we did in the pure Weyl case), the momentum constraint would read \( \mathcal{H}_\perp \text{WE} + 2a^2 D_i \bar{K}/\kappa \approx 0 \), and the Lagrangian multiplier to \( Q_\text{WE} \) would have the same form as in the pure Weyl case. It is unclear to us which procedure is the more appropriate one.

What can we say about the constraint algebra in the WE theory? Comparing with the algebra (67)–(74) in the pure Weyl theory, we expect that the first three Poisson brackets remain unchanged because the hypersurface foliation algebra should be preserved (the Weyl and EH terms in the action fulfill this symmetry separately). Concerning the other constraints, we observe that because of (105) we now have
\[ \{ P[\epsilon], Q_\text{WE}[\omega] \} = -\frac{6}{\kappa} \int d^3 x \epsilon \omega a^2, \]
\[ (114) \]
and the commutation of the constraints is thus spoiled. This is, of course, a consequence of the conformal symmetry breaking.

\[ \text{Compare this with the result from [12] where } Q_\text{WE} \text{ was added by hand with a Lagrange multiplier } \lambda_Q, \text{ which of course led the authors to the conclusion that } \lambda_Q = 0. \text{ The reason why they got such a result is that there is no need to add the secondary second class constraint by hand—it is already present in the total Hamiltonian, and one simply needs to identify it from the appropriate consistency condition.} \]
Since we are dealing here with a system that has both first and second class constraints, leading to one determined Lagrange multiplier, the Poisson brackets should be replaced by Dirac brackets. For a general function $F(x)$ and $G(x)$, the Dirac bracket reads

$$\{F(x), G(y)\}_D = \{F(x), G(y)\} - \int d^3z d^3z' \{F(x), \phi_A(z)\} \mathcal{M}^{AB} \{\phi_B(z'), G(y)\},$$

where the sum is understood as running over the second class constraints here labelled by $A, B = (1, 2)$ with $\phi_1(z) = \bar{P}(z)$ and $\phi_2(z) = (aq_a + \frac{1}{2}a^2\bar{K})(z)$, and $\mathcal{M}^{AB}$ is the inverse matrix to

$$\mathcal{M}_{AB} = \begin{pmatrix} \phi_1(z), \phi_1'(z') & \phi_1(z), \phi_2(z') \\ \phi_2(z), \phi_1'(z') & \phi_2(z), \phi_2(z') \end{pmatrix} = \begin{pmatrix} \phi_1(z), \phi_1'(z') & \phi_1(z), \phi_2(z') \\ \phi_2(z), \phi_1'(z') & \phi_2(z), \phi_2(z') \end{pmatrix} = \begin{pmatrix} 6a^2 \kappa & 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so

$$\mathcal{M}^{AB} = \begin{pmatrix} \kappa & 6a^2 \\ 6a^2 & 0 \end{pmatrix}.$$

The form of $\phi_1$ and $\phi_2$ is such that only brackets between $\bar{K}$ and $a, p_a, \bar{P}$ are modified. This concerns the following three brackets:

$$\{\bar{K}, \bar{P}\}_D = 0,$$

$$\{\bar{K}, a\}_D = \frac{\kappa}{6a^2},$$

$$\{\bar{K}, p_a\}_D \approx -\frac{\bar{K}}{a},$$

where in the last bracket we have used the constraint $a^2 = 0$. Because of the use of unimodular-conformal variables, the resulting Dirac brackets are of a much simpler form than the ones derived in [12]. This clearly shows which part of the configuration space is affected by conformal symmetry breaking. The above brackets show that $\bar{K}$ can no longer be treated as a “true” canonical variable—it is determined by other variables in the theory. Counting the degrees of freedom proceeds as follows: we have 12 degrees of freedom to start, including $a$ and $\bar{K}$, and four first class constraints, which gives eight degrees of freedom. We also have one pair of second class constraints, which eliminates one more degree of freedom, so we end up with a total of seven physical degrees of freedom, which agrees with previous results [12].

B. Hamilton-Jacobi functional and the generator of nongauge conformal transformations

In the same way as in the pure Weyl case, a functional $S^{W}[\bar{h}_{ij}, a, \bar{K}_{ij}^{T}, \bar{K}]$ can be introduced to express the momenta as functional derivatives. The Hamilton-Jacobi equation that follows from (109) is then given by

$$-\frac{1}{2\kappa} \bar{h}_{ij} \bar{h}_{ij} \frac{\delta S^{WE}}{\delta \bar{h}_{ij}} + \frac{1}{\kappa} \bar{K}_{ij}^{T} \frac{\delta S^{WE}}{\delta \bar{K}_{ij}^{T}} + (\bar{K}_{ij}^{T} + \partial_i \bar{D}_j) \frac{\delta S^{WE}}{\delta \bar{K}_{ij}^{T}} + 2 \bar{K}_{ij}^{T} \frac{\delta S^{WE}}{\delta \bar{K}_{ij}} - \sigma \bar{C}_{ij} + 2 = 0.$$

One may refer to this equation as the “Weyl-Einstein-Hamilton-Jacobi equation” (WEHJ). We see that the WEHJ is qualitatively different from both the EHJ and the WHJ, owing to the different structure of the Hamiltonian constraint. The signature of symmetry breaking is the explicit presence of scale density $a$ and trace density $\bar{K}$, making $S^{WE}$ noninvariant under conformal transformation. Thus, it would seem that $S^{WE}$ now remains to live on the full configuration space. We recognize, however, from the Dirac brackets that $\bar{K}$ is not to be treated as a configuration space variable at all—it is fixed by the EH term, and it is not a true degree of freedom. This is clarified below.

At first glance, the fact that

$$\bar{P} \approx 0 \Rightarrow \frac{\delta S^{WE}}{\delta \bar{K}} = 0$$

seems to be contradictory with the explicit appearance of $\bar{K}$ in (119). How can this be? Equation (120) should be interpreted merely as a statement that $S^{WE}$ should functionally not depend on $\bar{K}$ as a configuration variable. Based only on (120), nothing more can be said—additional information is required. The secondary constraint $Q^{WE}$ when evaluated with the HJ functional provides the second piece of information,

$$Q^{WE} \approx 0 \Rightarrow \frac{\delta S^{WE}}{\delta a} = -\frac{6a^2}{\kappa} \bar{K}.$$
“downgrades” $\tilde{K}$ from a configuration space variable to a function with a fixed form.

What does all this imply for the generator of gauge conformal transformations? Since the involved constraints are of second class, such a generator cannot be defined in the WE theory, at least not using the ABC algorithm which determines only gauge generators using first class constraints. However, a closer look at the action \cite{10} of the generator in W theory suggests that the form of the generator in the W theory may be used as a general definition independent of the context of that theory because the conformal variations $\delta_\omega a = \omega a$ and $\delta_\omega K = \mathcal{L}_n \omega$ are determined only by the definitions of the respective variables (the only variables which change under conformal transformation) and the specific form of the conformal transformation for the metric. One may thus use

$$G_\omega[\omega, \dot{\omega}] = \int \! d^3x \left( \omega \cdot a \, p_a + \bar{P} \cdot \mathcal{L}_n \bar{\omega} \right)$$

as an ansatz for the generator of conformal transformation in the Hamiltonian formulation, meaning that such a generator may be defined independently of a particular theory and studied in any theory expressed in the $3 + 1$ formalism in unimodular-conformal canonical variables (hence the superscript “W” is omitted). With this in mind, we can write \cite{10} again, with the difference that in the WE theory it does not vanish,

$$G_\omega[\omega, \dot{\omega}] = \delta_\omega S^{\text{WE}} \neq 0.$$  

The above statement is not a postulate, but follows directly when \cite{122} is evaluated on the constraint hypersurface of the WE theory with \cite{120} and \cite{121}, that is,

$$\delta_\omega S^{\text{WE}} = \int \! d^3x \left( \omega a \frac{\delta S^{\text{WE}}}{\delta a} + \mathcal{L}_n \bar{\omega} \frac{\delta S^{\text{WE}}}{\delta K} \right) = \frac{6}{\kappa} \int \! d^3x \, \omega a^2 \tilde{K} \neq 0,$$

from which \cite{123} reads

$$G_\omega[\omega, \dot{\omega}] = -\frac{6}{\kappa} \int \! d^3x \, \omega a^2 \tilde{K}.$$  

This is an important result. It means that evaluating the generator of conformal transformation \cite{122} on the constraint hypersurface in the configuration space of the WE theory fails to give a vanishing variation of the Hamilton-Jacobi functional, which implies that the conformal symmetry in the WE theory is broken. Since $S^{\text{WE}}$ functionally does not depend on $\tilde{K}$ (see \cite{120}), and variation of $S^{\text{WE}}$ with respect to scale density $a$ is nonarbitrary and relates the trace density $\tilde{K}$ to the momentum $p_a$ [see \cite{121}], the HJ functional of the WE theory depends on scale density, that is,

$$S^{\text{WE}} = S^{\text{WE}}[\bar{h}_{ij}, a, \tilde{K}_{ij}].$$

Since scale density is the variable affected by conformal transformation, the HJ functional of the WE theory is not conformally invariant, in contrast to the HJ functional of the pure W theory [see \cite{118}].

To summarize, a theory has a symmetry under conformal (rescaling) transformation only if the generator of conformal transformations \cite{122} vanishes on the constraint hypersurface in the configuration space of the theory. We think that a similar conclusion may be generalized to hold for other symmetries.

We conclude this subsection with a remark on the generator of conformal transformations for fields. As mentioned earlier, in the presently discussed case of the WE theory the ABC algorithm cannot be applied to find the generator of gauge conformal transformation, since constraints are second class and a conformal transformation is not a gauge one. However, suppose we naively use the prescription of a tuned sum in this case, we might think of putting the primary-secondary pair of second class constraints into the following expression:

$$\tilde{G}_\omega[\omega, \dot{\omega}] = \int \! d^3x \left( \omega \cdot Q^{\text{WE}} + \bar{P} \cdot \mathcal{L}_n \bar{\omega} \right) = \int \! d^3x \left( \omega \cdot \left( a \, p_a + \frac{6a^2}{\kappa} \tilde{K} \right) + \bar{P} \cdot \mathcal{L}_n \bar{\omega} \right).$$

This object does, of course, not have the meaning of a gauge generator of conformal transformations, but it is interesting to note that it still produces correct conformal transformations for the scale density $a$ and the trace density $\tilde{K}$ (note that this may not be true in a setting more general than the WE theory). Moreover, we note that the requirement for this object to vanish on the constraint hypersurface in the configuration space,

$$\tilde{G}_\omega[\omega, \dot{\omega}] = 0,$$

implies the result found earlier in \cite{123}, which is obvious if one rewrites \cite{127} and \cite{128} as

$$\tilde{G}_\omega[\omega, \dot{\omega}] = G_\omega[\omega, \dot{\omega}] + \frac{6}{\kappa} \int \! d^3x \, \omega a^2 \tilde{K} = 0.$$  

It would then be of interest to investigate whether it is possible to formulate such an object $\tilde{G}$ in any constrained system containing second class constraints, and whether they are always connected with the breaking of some symmetry, that is, whether one can always impose the requirement \cite{128}, but this is beyond the scope of the present work.

We are now ready to include matter into the theory and discuss symmetries and generators of transformations of the full WE theory with matter.
VII. ADDING MATTER: WEYL-EINSTEIN GRAVITY WITH NONMINIMALLY COUPLED SCALAR FIELD

A. Hamiltonian formulation

For a more realistic picture, it is necessary to extend the WE theory by including a nongravitational (matter) sector. We consider the Lagrangian of a nonminimally coupled scalar field (including the case of minimal coupling as a special case) given by

\[ \mathcal{L} = -\frac{1}{2} \sqrt{-g} \left[ g^{\mu \nu} \partial_\mu \varphi \partial_\nu \varphi + \xi R \varphi^2 + V(\varphi) \right] \]

\[ = \frac{1}{2} N \sqrt{h} \left[ (n^\mu \partial_\mu \varphi)^2 + \xi \left( \frac{2}{3} K^2 - 2 \nabla_\mu (n^\mu K) \right) \varphi^2 \right. \]

\[ - h^{ij} \partial_i \varphi \partial_j \varphi - \xi \left( -\frac{1}{2} D^i D_i N \varphi^2 \right) \varphi^2 \]

\[ - \xi K^2 \chi^2 - V(\varphi) \],

where the nonminimal coupling constant \( \xi \) is dimensionless, and where we have used (12) simply because it is easier to work with than with (11). It is known that for \( \xi = 1/6 \) the above action is invariant under conformal transformation if the scalar field is transformed as

\[ \varphi \to \tilde{\varphi} = \Omega^{-1} \varphi \],

along with the metric, and if the potential term is either 0 or \( \beta \varphi^4 \), with \( \beta \) being a dimensionless coupling. If the scalar field is massive, the conformal symmetry is broken, and if \( \xi = 0 \) we have the usual minimally coupled scalar field, for which the conformal symmetry is also lost. The essential features of our investigation can be seen already for the case \( V(\varphi) = 0 \) to which we restrict ourselves from now on. It is clear that, depending on the value of \( \xi \), we will either have only first or first and second class constraints, but in both cases we will be able to define a generator of conformal transformation.

The 3+1 decomposition of the above action can be found, for example, in [12, 51] and the references therein. It is convenient to rescale the scalar field to \( \varphi \to a \varphi \), such that the resulting variable does not transform under a conformal transformation. We can thus introduce a new scalar density by

\[ \chi := a \varphi \],

which is of scale weight \( w_a = 1 \). It is also possible to define this scalar density as \( \chi := a^{6\xi} \varphi \) for which it was shown in [51] for vanishing shift that the interaction term \( \varphi K \) is eliminated from the \( \varphi \)-Lagrangian, cf. Eq. (2.3) there. However, we choose to use (132) because in that case the scaling of the scalar field \( \varphi \) with \( a \) reflects its physical dimension (inverse length) and \( \chi \) is dimensionless (see Appendix A), while a more general discussion is possible if the scaling is independent of \( \xi \). Rescaling the scalar field in this way is also important in cosmological perturbation theory; see, for example, [51].

This choice of scaling is particularly suggestive because it compensates in (132) the necessary conformal rescaling of \( \varphi \) in (131), resulting in a conformally invariant scalar density variable \( \chi \).

Employing unimodular-conformal variables, along with (132), the Lagrangian can be decomposed in the following way:

\[ \mathcal{L}^\chi = \frac{1}{2} N \left[ (\bar{h}^{\mu} \partial_\mu \chi - (1 - 6\xi) \bar{K} \chi - \partial_i N^i) \right]^2 \]

\[ + 6\xi (1 - 6\xi) \bar{K}^2 \chi \]

\[ - \xi (\bar{K}^2 \chi^2 + 4\chi \partial_i (\bar{h}^{ij} \partial_j \chi) + (4\xi - 1) \bar{h}^{ij} \partial_i \chi \partial_j \chi \]

\[ - \xi \bar{K}^2 \chi^2 + (1 - 6\xi) S(a) \right] + \xi \partial B, \]

where \( \bar{K}^{ij} = \bar{K}^{ij} \bar{h}^{ia} \bar{h}^{jb} \bar{K}^{ab} \), as before, and

\[ \partial B := \partial_i \left( \bar{h}^{ij} \left( \partial_j \bar{N} \chi^2 \right) \right) \right] + 3 \partial_i \left( N \bar{h}^{ij} \partial_j \log a \chi^2 \right) \]

\[ - 3 \partial_i \left( N \bar{n}^{\mu} \bar{K}^2 \chi^2 \right) \]

is a collection of total divergences resulting from several applications of the Leibniz rule to extract the lapse density from the derivatives. The third line in the above decomposed Lagrangian (133), along with the last term \( (1 - 6\xi) S(a) \), results from the combination \( h^{ij} \partial_i \varphi \partial_j \varphi - \xi (\bar{K}^2 \chi^2 + 4\chi \partial_i (\bar{h}^{ij} \partial_j \chi) + (4\xi - 1) \bar{h}^{ij} \partial_i \chi \partial_j \chi \)

\[ - \xi \bar{K}^2 \chi^2 + (1 - 6\xi) S(a) \] (134)

and is a collection of all scale-dependent terms arising from the decomposition of \( \bar{K}^2 \bar{R} \) [see (C19)] and the rest of the terms in the third line of (130). The first and the second lines in (133) result from the first line of (130) and from \( \xi \bar{K}^2 \chi^2 \).

It is obvious that all terms in the Lagrangian (133) are separately conformally invariant, except for three terms proportional to \( (1 - 6\xi) \). These are the only terms which depend either on \( a \) or \( \bar{K} \). It is clear that the latter three terms have something in common: they vanish for \( \xi = 1/6 \), that is, for conformal coupling. Hence, if these terms were absent, the Lagrangian of the nonminimally coupled scalar field for \( \xi = 1/6 \) would be conformally invariant.

19 Defined in Appendix [C]
term by term,
\[ \mathcal{L}^X = \frac{1}{2N} \left[ \left( \bar{n}^\mu \partial_\mu \chi - \frac{\partial_i N^i}{3N} \chi \right)^2 - \frac{1}{6} \left( \xi \bar{R} + \bar{K}^{ij} \right) \chi^2 \right] \]
\[ - \frac{1}{3} \left( 2\chi \partial_i \left( \bar{h}^{ij} \partial_j \chi \right) - \bar{h}^{ij} \partial_i \partial_j \chi \right) + \frac{1}{6} \partial B, \quad (136) \]
up to a total divergence \( \partial B \). It can then be concluded that the conformal symmetry requires the absence of scale density \( a \) and trace density \( \bar{K} \) from the action, up to a surface integral. In other words, absence of \( a \) and \( \bar{K} \) is the fingerprint of conformal invariance. It may now be understood in the context of the \( 3+1 \) decomposition why the nonminimal coupling term \( \xi \bar{R} \phi^2 \) is necessary for the conformal invariance of the scalar field action: it serves to eliminate \( a \) and \( \bar{K} \) from those terms in \( g^\mu\nu \partial_\mu \phi \partial_\nu \phi \) for \( \xi = 1/6 \) that contain scales and traces. Note that this cancellation was made possible by the particular choice of scaling (132) for the scalar field (see, for example, Appendix D in [20]) which reflects its physical dimension of inverse length.

After this preparation, we can address the Hamiltonian formulation. The canonical momentum conjugate to \( \chi \) is given by
\[ p_\chi = \frac{\partial \mathcal{L}^X}{\partial \dot{\chi}} = \bar{n}^\mu \partial_\mu \chi - (1 - 6\xi) \bar{K} \chi - \frac{\partial_i N^i}{3N} \chi, \quad (137) \]
and the total matter Hamiltonian follows to read
\[ H^X = \int d^3 x \left( \dot{\chi} p_\chi - \mathcal{L}^X \right) \]
\[ = \int d^3 x \left\{ \tilde{N} H_\chi^X + N^i H_\chi^X \right\} + H_{\text{surf}}^X, \quad (138) \]
where
\[ H_\chi^X := \frac{1}{2} \left[ p_\chi^2 + \xi \left( \bar{R} + \bar{K}^{ij} \right) \chi^2 \right] \]
\[ - 4\xi \chi \partial_i \left( \bar{h}^{ij} \partial_j \chi \right) + (1 - 4\xi) \bar{h}^{ij} \partial_i \partial_j \chi \]
\[ + (1 - 6\xi) \left( 2\bar{K} \chi p_\chi - 6\xi \bar{K}^2 \chi^2 - S(a) \right) \right\}, \quad (139) \]
\[ H_\chi^X := -\frac{1}{3} \left[ \chi \partial_i p_\chi - 2\partial_i \chi p_\chi \right], \quad (140) \]
\[ H_{\text{surf}}^X := \xi \int d^3 x \left[ 2\partial_i \left( N^i \chi p_\chi \right) - \partial B \right] \quad (141) \]
are the matter contributions to the Hamiltonian constraint, the momentum constraints, and the surface integral, respectively. They contribute to the Hamiltonian and momentum constraints of the Weyl-Einstein-\( \chi \) theory (WE\( \chi \) for short). Note that for \( \xi = 1/6 \) the expressions (139) and (140) are both manifestly conformally invariant, so this feature also holds for the full Hamiltonian (138) (up to the surface term). For the sake of completeness, we give here the matter contribution to the Hamiltonian constraint in the conformally invariant case:
\[ H_\chi^X \left( \xi = \frac{1}{6} \right) = \frac{1}{2} p_\chi^2 + \frac{1}{12} \left( \xi \bar{R} + \bar{K}^{ij} \right) \chi^2 \]
\[ - \frac{1}{6} \left( 4\chi \partial_i \left( \bar{h}^{ij} \partial_j \chi \right) - 2\bar{h}^{ij} \partial_i \partial_j \chi \right). \quad (142) \]
This expression does not contain any scales or traces. On the other hand, note that in the case of minimal coupling (\( \xi = 0 \)), both \( a \) and \( \bar{K} \) are present, reflecting the conformally noninvariant nature of such coupling.

The reason why we can simply add the various Hamiltonians is because we treat \( h_{ij} \), \( a \), \( \bar{K}_{ij} \), \( \bar{K} \) as independent variables (below to be introduced for the full WEX theory) and because (138) does not contain the velocities \( \bar{\lambda} \) (application of the Leibniz rule led to a boundary term). For this reason, we do not need to consider the term \( \tilde{K} \bar{P} \) in the Hamiltonian (138). (This was also the case for the W and WEH theories above.) If, instead, we had chosen to keep the term \( \bar{\lambda} \), there would have been a nonzero contribution to \( \bar{P} \) due to the presence of \( \bar{\lambda} \bar{K} \chi^2 \). This would lead to a different formulation that deserves further study.

We can now put the Lagrangians of the WE and \( \chi \) theories together and proceed to the Hamiltonian formulation using the definitions (136), (140) and (157) for the momenta and replacing \( \mathcal{L}^W \rightarrow \mathcal{L}_{\text{WEW}}^X = \mathcal{L}_{\text{WE}}^X + \mathcal{L}^X \). The Legendre transform to the total Hamiltonian then gives
\[ H_{\text{WEW}} = \int d^3 x \left\{ \bar{h}_{ij} \bar{P}^{ij} + \bar{a} p_a + \bar{K}^{ij} \bar{p}^{ij} \right\} + \bar{\chi} p_\chi - \mathcal{L}_{\text{WEW}}^X \]
\[ + \lambda_S p_S + \lambda_I p^I + \lambda_P \bar{P}, \quad (143) \]
from which one can determine the following secondary constraints
\[ \mathcal{H}_{\text{W}} + \mathcal{H}_{\text{E}} + \mathcal{H}_{\text{W}}^X \approx 0, \quad (144) \]
\[ \mathcal{H}_{\text{I}} + \mathcal{H}_{\text{W}}^X + \mathcal{H}_{\text{I}}^X \approx 0, \quad (145) \]
\[ Q_{\text{W}} + Q_{\text{E}} + Q_{\text{X}} \approx 0. \quad (146) \]
It is obvious from this formulation of the constraints that \( \mathcal{H}_{\text{W}} \) comes from (62), \( \mathcal{H}_{\text{I}} \) comes from (109), \( \mathcal{H}_{\text{W}}^X \) comes from (130), and that the momentum constraints consist of (111) and (140); the expressions \( Q_{\text{W}} \) and \( Q_{\text{E}} \) are defined in (59) and (105), respectively, while \( Q_{\text{X}} \) originates

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20 Recall again that we talk about the local rescaling of fields, not conformal coordinate transformations.

21 Here, we ignore the gravitational variables in the Legendre transformation because we want to focus on the expressions arising solely from matter.
from the matter part of the theory and reads

\[ Q^x := (1 - 6\xi)\chi p_\chi - 6\xi (1 - 6\xi) \tilde{K} \chi^2. \]  

(147)

The explicit form of the constraints is determined as follows. It will prove convenient to introduce

\[ \frac{1}{\tilde{\kappa}} := \frac{1}{\kappa} - \xi (1 - 6\xi) \frac{\chi^2}{a^2}, \]  

(148)

which could be interpreted as coupling between \( \chi \) and the scale and trace density \( a, \tilde{K}, \) see below. The consistency condition for \( \tilde{P} \approx 0 \) is such that it requires the expression \[ Q^{\text{WE}_x} = ap_a + (1 - 6\xi)\chi p_\chi + 6a^2 \tilde{K}/\kappa \] to vanish. It is now not possible to use this constraint in order to completely eliminate \( \tilde{K} \) from the remaining constraints, in contrast to the pure Weyl case. Namely, the consistency condition for \( p_\chi \) leads to a Hamiltonian constraint which depends on \( a \) and \( \tilde{K} \); the constraints read

\[ H^{\text{WE}_x}_\perp = H^\perp_W + \frac{1}{2} i^2 \chi + 3 \xi^2 (\tilde{R} + \tilde{K}^{ij}) \chi^2 \]
\[ - \frac{1}{2} (4\xi \partial_i (\tilde{h}^{ij}\partial_j \chi) - (1 - 4\xi) \tilde{h}^{ij}\partial_i \partial_j \chi) \]
\[ - \frac{a^2}{2\kappa} (\tilde{R} + \tilde{K}^{ij} + 6\tilde{K}^2) \]
\[ + \frac{a^2}{\kappa} (2\partial_i (\tilde{h}^{ij}\partial_j a) + \tilde{h}^{ij}\partial_i a \partial_j a) \]
\[ + (1 - 6\xi) (4\xi \partial_i (\tilde{h}^{ij}\partial_j a) \chi^2 - \tilde{h}^{ij}\partial_i \partial_j a) \]
\[ + (1 + 2\xi) \frac{\tilde{h}^{ij}\partial_i \partial_j a \partial_j a \chi^2}{2} \approx 0. \]  

(149)

\[ H^{\text{WE}_x}_i = -2\partial_k (\tilde{h}_{ijk}\tilde{p}^{jk}) + \partial_i \tilde{h}_{jk} \tilde{p}^{jk} - \frac{1}{3} D_i (a p_a) \]
\[ - 2\partial_k (\tilde{K}^{ij} \tilde{p}^{jk}) + \partial_i \tilde{K}^{jk} \tilde{p}^{jk} \]
\[ - \frac{1}{3} (\chi \partial_i p_\chi - 2\tilde{\kappa} \partial_i a \chi) \approx 0 \]  

(150)

\[ Q^{\text{WE}_x} = ap_a + (1 - 6\xi)\chi p_\chi + \frac{6a^2}{\kappa} \tilde{K} \approx 0. \]  

(151)

Note the appearance of \( 1/\tilde{\kappa} = a(\alpha, \chi) \) and \( \tilde{K} = a(\alpha, \chi) \) in the Hamiltonian constraint in terms of the couplings \( \xi \) and \( 1/\tilde{\kappa} \). We choose to work with \( \tilde{\kappa} \) instead of \( \kappa \) because one can then easily recognize the features of the Hamiltonian constraint in terms of the conformal and nonconformal versions of the theory. For example, \( 1/\tilde{\kappa} = 0 \)

22 After identifying \( Q^{\text{WE}_x} \approx 0 \), similarly to the step from (108) to (109) in the WE case.

eliminates the trace density \( \tilde{K} \) from all the constraints and also eliminates the scale part of the Ricci scalar, while \( \xi = 1/6 \) eliminates the coupling of \( \chi \) to the scale density \( a \) and the trace density \( \tilde{K} \). Let us therefore now have a closer look at these couplings.

We can distinguish between vanishing and nonvanishing \( 1/\tilde{\kappa} \). This reflects the commutation properties between the constraints,

\[ \{ \tilde{P}, Q^{\text{WE}_x} \} = - \frac{6a^2}{\kappa}. \]  

(152)

This commutator can vanish \( (1/\tilde{\kappa} = 0) \) in the following three cases:

1. \( \frac{1}{\kappa} = 0 \) and \( \xi = \frac{1}{6} \),
2. \( \frac{1}{\kappa} = 0 \) and \( \xi = 0 \),
3. \( \chi^2 = \chi^2_{\perp} := a^2 (\kappa \xi (1 - 6\xi))^{-1} \).

It is important to realize that the commutator vanishes because \( 1/\tilde{\kappa} = 0 \) eliminates \( \tilde{K}^2 \) from the total Hamiltonian, making all constraints independent of \( \tilde{K} \). We also note that \( \kappa \to \infty \) is the strong-coupling limit \( G \to \infty \).

The constraints are first class only in the first case; this corresponds to the conformal version \( \xi = 1/6 \) and \( 1/\kappa = 0 \). The former turns \( \tilde{\kappa} / \kappa \) and eliminates the coupling of \( \chi \) with scale density \( a \), while the latter excludes the EH contribution. Then one is left with \( H^{\text{WE}_x}_\perp = H^\perp_W + H^\perp_i (\xi = 1/6) \), see (142), that is, a scalar density field conformally coupled to the Weyl gravity. In this case the constraints are first class.

In the second case, we are dealing with a minimally coupled scalar field in Weyl-squared gravity, namely \( \xi = 0 \) eliminates the nonminimal coupling. One would expect that the conformal symmetry is broken due to conformal noninvariance of the minimally coupled scalar field. However, constraint analysis needs to be investigated further. Namely, the secondary constraint (151) simplifies to \( Q^{\text{WE}_x} = ap_a + \chi p_\chi \), and one needs to check again its consistency condition. Several further constraints appear, since term \( \chi p_\chi \) and the absence of \( \tilde{K} \) from this constraint prevent the Dirac algorithm from terminating until several further steps. We do not investigate this further here, but we emphasize that this is an interesting and simple case for studying conformally noninvariant scalar fields.

The third case is a very interesting one. It says that if the scalar density field adopts a certain critical value \( \chi_c \), then the contribution of the trace density \( \tilde{K} \) from the EH part is canceled, along with some of the spatial derivatives of the scale density, but the scale density is still present and the theory is not conformally invariant. Thus, one expects again a second class system with further constraints, a matter into which we do not go here. This critical value was already met in [50] [see there Eq. (2.12)] and turned out to be the limiting case between an indefinite and a positive definite kinetic term in the Hamiltonian constraint.
If $1/k \neq 0$ and $\xi \neq 0, 1/6$, we have a nonconformal scalar density nonminimally coupled to WE gravity—the most general case. The consistency requirement for $Q_{\text{WEX}}$ gives a nontrivial result from which, in this most general case, the Lagrange multiplier $\lambda_P$ is determined from part of the second and part of the third term in

$$\dot{Q}_{\text{WEX}} = \{Q^w, H_{\text{WEX}}\} + \{Q^e, H_{\text{WEX}}\} + \{Q^x, H_{\text{WEX}}\},$$

which give a nonvanishing term proportional to $\langle \delta Q_{\text{WEX}} / \delta \bar{\chi} (\delta H_{\text{WEX}} / \delta \bar{P}) \rangle \sim \lambda_P$, similarly to the WE case, see (106). The Dirac algorithm has then arrived at its end, and all the constraints are determined. We do not present the explicit expression here and only give a list of the Dirac brackets,

$$\{\bar{K}, \bar{P}\}_D = 0,$$

$$\{\bar{K}, a\}_D = \frac{\bar{k}}{6a},$$

$$\{\bar{K}, p_\chi\}_D \approx \frac{\bar{k}}{a} + (1 - 6\xi) \frac{\bar{k}}{6a^2} \chi p_\chi,$$

$$\{\bar{K}, \chi\}_D = (1 - 6\xi) \frac{\bar{k}}{6a^2} \chi,$$

$$\{\bar{K}, p_\chi\}_D = (1 - 6\xi) \frac{\bar{k}}{6a^2} (p_\chi + 12\bar{k} \chi).$$

(154)

The first three Dirac brackets are analogous to the ones derived for the WE theory. The fourth and fifth ones are there because of the matter degree of freedom and support the conclusion that $\bar{K}$ is no longer a variable in configuration space, but is fixed. The number of degrees of freedom is now 8, seven gravitational and one matter.

The results from this subsection support the claim that a conformally invariant theory has to be independent of scale density $a$ and trace density $\bar{K}$.

### B. Hamilton-Jacobi functional and the generator of (non-)gauge conformal transformations

We saw that only for particular values of couplings $1/k = 0$ and $\xi = 1/6$, all constraints are first class, and only in this case can one formulate the generator of gauge conformal transformation by using the ABC algorithm. However, one may nevertheless consider the most general case with $\bar{P}$ and $Q_{\text{WEX}}$ being the only second class constraints, by logically extending the discussion from Sec. VIIB and thereby investigating the action of the generator of conformal transformations (122) in the configuration space.

We thus focus our discussion here on the HJ functional for the general case of nonvanishing couplings and derive its change under conformal transformations. The configuration space is now extended by the $\chi$ variable, so that we also have here the matter momentum $p_\chi = \delta S_{\text{WEX}} / \delta \chi$, with $S_{\text{WEX}} = S_{\text{WEX}}[h_{ij}, a, K_{ij}, \bar{K}, \chi]$ being the HJ function of the WE$_X$ theory. This functional is determined by the Hamilton-Jacobi equation resulting from (139) by substituting all the momenta.

As in the WE theory, one realizes that $\bar{P} \approx 0$ and that the relations

$$Q_{\text{WEX}} \approx 0 \Rightarrow a \delta S_{\text{WEX}} / \delta a = -(1 - 6\xi) \chi p_\chi - \frac{6a^2}{k} \bar{K},$$

imply that $S_{\text{WEX}}$ functionally does not depend on $\bar{K}$; hence we actually have

$$S_{\text{WEX}} = S_{\text{WEX}}[\bar{h}_{ij}, a, K_{ij}, \chi],$$

(156)

with $\bar{K}$ fixing the variation of the HJ functional with respect to the scale density. The interpretation of independence of $S_{\text{WEX}}$ on $\bar{K}$ is analogous to the case of the vacuum WE theory, and we may again say that conformal symmetry breaking gives rise to the time evolution of scale density $a$ by determining it through $\bar{K}$. In the present case, both the EH term and the nonminimally coupled scalar field are responsible for this. Therefore, $\bar{K}$ is again not a configuration variable, but a function which fixes the variation of the HJ functional with respect to scale density $a$.

Since we first do not impose any restrictions on the couplings, we are dealing with broken conformal symmetry due to the presence of both nonminimally coupled matter field $\chi$ and the EH term. Then the generator of conformal transformations as defined generically in (122) determines a nonvanishing conformal variation of the HJ functional $\delta_\omega S_{\text{WEX}}$ when evaluated on the constraint hypersurface,

$$G_\omega[\omega, \dot{\omega}] = \delta_\omega S_{\text{WEX}} \neq 0,$$

similarly to the action of the generator in the vacuum WE theory, see (133). It follows from

$$\delta_\omega S_{\text{WEX}} = \int d^3 x \left( \omega a \frac{\delta S_{\text{WEX}}}{\delta a} + \bar{L}_\omega \frac{\delta S_{\text{WEX}}}{\delta \bar{K}} \right)$$

$$= - \int d^3 x \omega \left( 1 - 6\xi \right) \chi p_\chi + \frac{6a^2}{k} \bar{K} \neq 0$$

(158)

that (161) does not vanish in general,

$$G_\omega[\omega, \dot{\omega}] = - \int d^3 x \omega \left( 1 - 6\xi \right) \chi p_\chi + \frac{6a^2}{k} \bar{K},$$

(159)

thus signaling the breaking of conformal symmetry.

This formulation of the generator of conformal transformations as defined here assumes that the definition of such a generator is independent of the theory in question — and thus of the number of constraints arising from the Dirac algorithm for the primary constraint $\bar{P}$, but we saw in Sec. VIIB that both the class and the number of constraints depend on the value of the couplings. How the values of the couplings influence the above discussion is not investigated here. Instead, let us restrict ourselves
to the case where $1/\kappa = 0$ and $\xi = 1/6$, which is the case of scalar density field conformally coupled to the Weyl gravity. The choice of couplings implies $1/\kappa = 0$, which makes $\tilde{P}$ and $Q^{\text{WE}} = Q^N$ first class constraints and the generator of conformal transformations vanishes on the constraint hypersurface, that is, (\ref{159}) becomes

$$1/\kappa = 0 \land \xi = 1/6 \Rightarrow G_\omega[\omega, \dot{\omega}] = \delta_\omega S^{\text{WE}} = 0 , \quad (160)$$

which is the statement of conformal invariance of the HJ functional $S^{\text{WE}}$. The HJ functional now loses dependence on both scale density $a$ and trace density $K$, that is, $S^{\text{WE}} \rightarrow S^W = S^W \{ h_{ij}, K_{ij}, \chi \}$, which solves the WHJ with conformally coupled scalar density field. In this case, $G_\omega[\omega, \dot{\omega}]$ is the generator of gauge conformal transformations, but since no $a$ or $K$ appear in the theory, its action on the Hamiltonian is trivial. Thus, we have confirmed that in the case of nonminimally coupled scalar field the conformal symmetry is present if the scale density $a$ and trace density $K$ are absent from the theory, which does happen for a particular choice of couplings.

Note in passing that, in analogy to (128) and (129), one could define an object $\tilde{G}_\omega[\omega, \dot{\omega}]$ such that it satisfies

$$\tilde{G}_\omega[\omega, \dot{\omega}] = G_\omega[\omega, \dot{\omega}]
+ \int d^3x \omega \left( (1 - 6\xi)\chi p_\chi + \frac{6a^2}{\kappa} K \right) = 0 , \quad (161)$$

which would follow from

$$\tilde{G}_\omega[\omega, \dot{\omega}] \equiv 0 . \quad (162)$$

If such an object could be formulated via a well-defined procedure, Eqs. (128) and (129) could be identified as a universal statement providing the answer to the question of (non-)vanishing variation of the HJ functional and thus the question of symmetries of an underlying theory. We do not pursue here this discussion further.

Collecting the results from the current and the previous two sections, one can derive the following conclusion:

The conformal symmetry of a theory implies that the corresponding Hamilton-Jacobi functional is independent of scale density $a$ and trace density $K$. Conversely, if the Hamilton-Jacobi functional does not depend on scale density $a$ and trace density $K$, the theory is conformally invariant, up to a total divergence.

It remains to be seen how general this statement is.

VIII. CONCLUSIONS

In our paper, we have developed a Hamiltonian formalism for Weyl-squared gravity, for Weyl-squared gravity plus Einstein-Hilbert term (WE theory), and for Weyl-squared gravity plus Einstein-Hilbert term plus generally coupled scalar field (WEH theory). The new feature compared to earlier work is the systematic use of unimodular-conformal variables, that is, the consistent decomposition into the scale part and conformal part for all variables. This makes especially transparent the features of conformal invariance for Weyl-squared gravity and of conformal symmetry breaking in the other cases.

We have derived and discussed all constraints and their algebra. In the Weyl theory, the Hamiltonian and momentum constraints have been formulated exclusively in terms of scale free variables. There is thus no scale factor $a$ appearing and, consequently, no part that corresponds to an intrinsic time, as is the case for general relativity \cite{1}. This is especially important for the quantum theory. An interesting question for future research concerning the constraint algebra is whether one can derive this algebra by a “seventh route” to the geometrodynamics of higher-derivative theories, in analogy to the seventh route for Einstein’s theory \cite{40}.

We have formulated and discussed in detail the generators of symmetry transformations. In the pure Weyl case, all constraints are of first class. Forming a tuned sum in the spirit of \cite{44}, they together generate conformal and diffeomorphism transformations; the individual constraints do not generate appropriate transformations. Using again a tuned sum in the WE and WEH theories, we have argued in favor of the possibility of introducing a generator of nongauge conformal transformations in these theories, where two constraints are of second class and where conformal invariance does not exist. This generator is able to produce appropriate conformal transformations of the configuration space variables $a$ and $K$, while producing a nonvanishing change of the Hamilton-Jacobi functional in the WE and WEH theories.

The next step in our investigation of those theories is quantization \cite{33}, see also \cite{1} for some preliminary results. The classical theories are now available in a form in which the scheme of canonical quantization \cite{1} can be employed in a straightforward way. We are led to the analogues of Wheeler–DeWitt equation and quantum momentum constraints, but also have to take into account the new constraints. In addition, the semiclassical approximation to those quantum constraints needs to be investigated. The final goal is, of course, be the application of such a theory of quantum gravity to the early Universe and to find out whether it is of empirical relevance or not.

Appendix A: A note on physical units

We list here for convenience the physical dimensions of some relevant quantities. Since we choose $c = 1$ throughout, all variables can be expressed in units of mass ($M$) and length ($L$). In order to provide the scale part of the three-metric, $a \equiv (\sqrt{\bar{h}})^{1/3}$, with unit of length, we choose the dimension for the three-metric components to be $L^2$ and of the inverse metric components to be $L^{-2}$. This means that the spatial coordinates are dimensionless. We keep, however, the dimension $L$ for time, which means that the lapse function $N$ is dimensionless.
All “barred” (conformally invariant) configuration space variables (those constructed from the three-metric and the second fundamental form) including $\chi$ are then dimensionless. For other relevant quantities we have the dimensions

$$[a] = L, \quad [N] = 1, \quad [\bar{N}] = L^{-1}, \quad (A1)$$

$$[N_i] = L, \quad [N^i] = [\bar{N}^i] = L^{-1}, \quad (A2)$$

$$[K_{ij}] = [K^T_{ij}] = L, \quad [K^{ij}] = L^{-3}, \quad [K] = L^{-1}, \quad (A3)$$

$$\alpha_w = M \cdot L, \quad [C_{ijk}] = L, \quad [\bar{C}_{ijk}] = L^{-5}, \quad (A4)$$

$$[\bar{p}^{ij}] = [\bar{P}^{ij}] = [\bar{P}] = M \cdot L, \quad [p_a] = M, \quad [\varphi] = L^{-1}. \quad (A5)$$

Consequently, the total Hamiltonian then has the physical dimension of a mass, as it should.

**Appendix B: Conformal transformations and unimodular-conformal variables**

Throughout the paper we refer to the following transformation as conformal transformation,

$$g_{\mu\nu}(x) \rightarrow \tilde{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x), \quad (B1)$$

$$\sqrt{-g} \rightarrow \sqrt{-\tilde{g}} = \Omega^4\sqrt{-g}, \quad (B2)$$

emphasizing that this is not a transformation of coordinates, but of the four-metric itself. It is also referred to as local dilatations, or local rescaling, or Weyl rescaling. It is easy to understand from (9) that the above conformal transformation in four dimensions induces conformal transformations for the components of the $3+1$ decomposed metric, namely,

$$h_{ij} \rightarrow \tilde{h}_{ij} = \Omega^2 h_{ij}, \quad (B3)$$

$$\sqrt{\tilde{h}} = \sqrt{h} = \Omega^3\sqrt{h} \quad \text{(B4)}$$

$$n_{\mu} \rightarrow \tilde{n}_{\mu} = \Omega n_{\mu} = \Omega(-N, 0), \quad \text{(B5)}$$

$$n^\mu \rightarrow \tilde{n}^\mu = \Omega^{-1} n^\mu = \Omega^{-1} \left( \frac{1}{N}, -N^i \right), \quad (B6)$$

where, by comparing (B5) with (B6), one can deduce that lapse and shift functions transform as

$$N \rightarrow \tilde{N} = \Omega N, \quad (B7)$$

$$N^i \rightarrow \tilde{N}^i = N^i \quad \text{and} \quad \tilde{N}_i = \tilde{h}_{ij}\tilde{N}^j = \Omega^2 N_i. \quad (B8)$$

From the definition of extrinsic curvature (14), it can be seen that it transforms nontrivially,

$$K_{ij} \rightarrow \tilde{K}_{ij} = \Omega K_{ij} + \frac{1}{\Omega} h_{ij} L_n \Omega, \quad (B9)$$

which is due to the inhomogeneous transformation of its trace,

$$\tilde{K} = \frac{1}{\Omega} (K + 3 L_n \Omega), \quad (B10)$$

while the traceless component transforms covariantly,

$$\tilde{K}^T_{ij} \rightarrow \tilde{K}^T_{ij} = \Omega K^T_{ij}. \quad (B11)$$

The infinitesimal conformal variation $\delta_\omega$ of $K_{ij}$ and its trace $K$ are then, respectively, given by

$$\delta_\omega K_{ij} = \omega K_{ij} + h_{ij} L_n \omega, \quad (B12)$$

$$\delta_\omega K = -\omega K + 3 L_n \omega. \quad (B13)$$

It follows from (B2) that the numerical power of $\Omega$, that is, the conformal weight, corresponds to the same power of $(\sqrt{h})^{1/3}$ in each of the objects $\Omega, \bar{\Omega}, \tilde{\Omega}$. Therefore, if one rescales these objects by the appropriate power of $(\sqrt{h})^{1/3}$, one ends up with zero conformal weight objects,

$$\tilde{N}_i = N_i, \quad (B14)$$

$$\tilde{N} = (\sqrt{h})^{-1/3} N, \quad (B15)$$

$$\tilde{h}_{ij} = (\sqrt{h})^{-1/3} h_{ij}, \quad (B16)$$

$$\tilde{K}^T_{ij} = (\sqrt{h})^{-1/3} K^T_{ij}, \quad (B17)$$

which are then all conformally invariant, except the scale $(\sqrt{h})^{1/3}$ itself (which is of conformal weight 1), and the rescaled trace of the extrinsic curvature $K$ (of zero conformal weight),

$$\tilde{K} = \frac{1}{3} (\sqrt{h})^{1/3} K, \quad (B18)$$

because the scale is essential in their definition; see (5). Thus, the only nonvanishing infinitesimal conformal variations $\delta_\omega$ of the new variables are $\delta_\omega \sqrt{h}$ and $\delta_\omega \tilde{K}$,

$$\delta_\omega (\sqrt{h})^{1/3} = \omega (\sqrt{h})^{1/3}, \quad \delta_\omega \tilde{K} = \tilde{n}^i \partial_\mu \omega. \quad (B19)$$

An important conclusion from here is that if a Lagrangian does not depend on velocities $\sqrt{h}$ and $\tilde{K}$, the theory is conformally invariant in the metric sector. Therefore, the tensor densities (B14)–(B18) are natural choices for canonical variables in theories in which conformal invariance or its breaking is of interest. Introducing $a := (\sqrt{h})^{1/3}$, we end up with (29)–(32).

If matter is present, it is appropriate to introduce new, rescaled matter fields by

$$\chi_\lambda = a^\alpha \phi_\lambda, \quad (B20)$$

such that $\chi$ does not transform under conformal transformations if the conformal weight of the matter field $\phi$ is $-n$. This has already proven to be a useful substitution on many occasions in the literature, for example in the use of Mukhanov-Sasaki variables for $n = 1$; see, for example, (51). The meaning of such a substitution is the separation of the gravitational degrees of freedom from the pure matter degree of freedom. It follows that the conformal transformation of any field originates in the rescaling of the gravitational scale degree of freedom only.
Appendix C: Unimodular decomposition of the connection and the $d$-dimensional Ricci tensor

We also need to know how to decompose the covariant derivative, because the Christoffel symbols also decompose. One expects that the covariant derivatives within the definitions of the extrinsic curvature variables contain scale parts which are solely responsible for the conformal transformation. Indeed, Christoffel symbols can be decomposed into a part $\Gamma^k_{ij}$ which is determined solely by the unimodular part of the metric, and a part $\sigma^k_{ij}$ which is solely responsible for the conformal transformation of the connection:

$$\Gamma^k_{ij} = \Gamma^k_{ij} + \sigma^k_{ij}, \quad \text{where}$$

$$\Gamma^k_{ij} = \frac{1}{2} h^{kl} \left( \partial_i h_{lj} + \partial_j h_{il} - \partial_l h_{ij} \right), \quad \text{(C2)}$$

$$\sigma^k_{ij} = \left( 2\delta^k_{(i} \delta^l_{j)} - h^k_{ij} h^l_{ij} \right) \partial_l \log a.$$

(C3)

The individual parts have the properties

$$\Gamma^i_{ij} = 0,$$  (C4)

$$\sigma^i_{ij} = 3 \partial_j \log a,$$  (C5)

because $h^{ij} \partial_i h_{ij} = 0$. Thus, the conformal transformation of the connection is solely due to its scale part $\sigma^k_{ij}$, which transforms as

$$\sigma^k_{ij} \rightarrow \delta^k_{ij} = \sigma^k_{ij} + \left( 2\delta^k_{(i} \delta^l_{j)} - h^k_{ij} h^l_{ij} \right) \partial_l \log \Omega.$$  (C6)

We call the quantity $\Gamma^k_{ij}$ we call the conformal Riemannian connection, or simply conformal Christoffel symbol, because it is invariant under conformal transformations of the metric. The introduction of the conformal connection dates back to Thomas [32, 53], who investigated conformal invariants and derived the Weyl and Schouten tensors using the transformation properties of the conformal connection.

Let us now address the behavior of the covariant derivative of a covariant vector density $V_i$ of weight under the unimodular decomposition; we have

$$D_i V_j = \bar{D}_i V_j - \sigma^k_{ij} V_k - w \partial_i \log \sqrt{h} V_j \quad \text{or}$$

$$\bar{D}_i V_j = \bar{D}_i V_j - \sigma^k_{ij} V_k - w_a \partial_i \log a V_j.$$  (C7)

(C8)

The term $\bar{D}_i V_j$ above will be explained shortly. In (C8), we have introduced $w_a := 3w$, which we call “the scale weight”, that is, the weight with respect to the scale density $a$. For example, the unimodular part of the three-metric $h_{ij}$ is a density of weight $w = -2/3$, or a density of scale weight $w_a = -2$. In general, if an object has a conformal weight $m$, then the scale weight of its conformally rescaled part is just its negative [compare (32) with (33, 311)]. Due to (C3), the second term in (C8) is proportional to $\partial_i \log a$, and together with the third term (which is absent for absolute tensors) yields the only scale part of the connection transforming under conformal transformations.

The overbar on the covariant derivative in the first terms in (C7) and (C8) indicates that it is completely determined by the unimodular part of the three-metric only, that is, it is given in terms of (C2) by

$$\bar{D}_i V_j := \partial_i V_j - \bar{\Gamma}^k_{ij} V_k.$$  (C9)

This could be called “conformal covariant derivative”, but in fact, the resulting quantity does not seem to be a tensor in general; hence we say it is the “scaleless part” of the covariant derivative, or the “conformal part” of the covariant derivative. We will call it derivative, for short. In a similar way, we can define these bar derivatives on a tensor density of a general rank and weight—since the scale part does not enter its definition, there is no difference between cases for tensors of the same rank but different weight. For example, for rank-2 tensors we have

$$\bar{D}_k T^{ij} = \partial_k T^{ij} + \bar{\Gamma}^k_{ij} T^{il} + \bar{\Gamma}^l_{kj} T^{il}, \quad (C10)$$

$$\bar{D}_k T^i_j = \partial_k T^i_j + \bar{\Gamma}^k_{ji} T^l_j - \bar{\Gamma}^l_{kj} T^i_j. \quad (C11)$$

Note that the bar derivative of an arbitrary tensor density is not, in general, scale independent and thus not conformally invariant. However, an interesting case of (C7) is its symmetrized traceless part, for a vector density of the scale weight $w_a = -2$, which turns out to be independent of the scale density,

$$[\bar{D}_i V_j]^T \equiv \frac{\bar{1}^{ab}}{\bar{a}^{ij}} \bar{D}_i (V_j) = \left[ \bar{D}_i (V_j) \right]^T,$$  (C12)

and is thus conformally invariant. Such is the case with the second term in (C8), leading to (C9), which makes $K^a_{ij}$ conformally invariant. Note in passing that it can be easily shown that the above identity is valid in any dimension $d$ if the vector density is of weight $w = -2/d$.

Using (C3), the covariant gradient of the contravariant vector is simply given by

$$D_i V^i = \partial_i V^i + (3 - w_a) \partial_i \log a V^i.$$  (C13)

Note that the factor of 3 comes from the conversion $\partial_i \log \sqrt{h} = 3(\partial_i \log a)$ in the special case of $w_a = 3$, corresponding to a vector density of weight $w = 1$, one obtains the well-known formula $D_i V^i = \partial_i V^i$. Because of (C4), we have in this case $D_i V^i = \bar{D}_i V^i$. Similarly, for tensor densities of higher rank, the following expressions follow from (C10) and (C11):

$$\bar{D}_i T^{ij} = \partial_i T^{ij} + \bar{\Gamma}^k_{ij} T^{ik},$$

$$\bar{D}_i T^i_j = \partial_i T^i_j - \bar{\Gamma}^k_{ij} T^i_k.$$  (C14)

(C15)

The difference between the barred covariant derivative and the usual covariant derivative consists of all the scaleful terms arising from the connection. Application to tensor densities of a general rank is straightforward. One should, however, be aware of an exception, which is $\bar{D}_a$. 

This expression does not vanish, unlike its “full metric” analog; this is because $\bar{D}$ is insensitive to scale density and does not induce the weight-dependent term of the covariant derivative, therefore leaving $\bar{D}_{ij}a = \partial_{\bar{a}}a$. Additionally, we have $\bar{D}_{ij}g_{kl} = 0$, which follows easily from $D_{ijkl}g_{kl} = 0$. In summary, the barred covariant derivative does not recognize the difference between absolute tensors and tensor densities of general weight.

We are now equipped with all we need to decompose the Ricci tensor. We do this for general dimension $d$, pretending that indices and labels are not restricted to the three-dimensional case discussed in this paper [from now on, Latin indices assume values $i, j = 0, 1, 2, 3...d-1$, and scale density is defined as $\sigma = (\sqrt{\Omega})^{1/d}$]. Using (C11) and (C14), we have for the Ricci tensor

$$R_{ij} = \partial_{ij}h^{k}_{ij} + 2\Gamma^{l}_{ij}h^{k}_{kl} = \bar{R}_{ij} + 2\partial_{ij}h^{k}_{ij} + 2\Gamma^{l}_{ijh}a_{kl} + \Gamma^{l}_{ijh}a_{kl}$$

$$= \bar{R}_{ij} + 2\partial_{ij}h^{k}_{ij} + 2\Gamma^{l}_{ijh}a_{kl} + \Gamma^{l}_{ijh}a_{kl}, \tag{C16}$$

where

$$\bar{R}_{ij} = \partial_{ij}h^{k}_{ij} - \Gamma^{l}_{ijh}a_{kl} \tag{C17}$$

is the part of the Ricci tensor that does not depend on the scale density at all; hence, it is conformally invariant.\(^{23}\)

Explicitly expressed in terms of scale, the Ricci tensor is

$$R_{ij} = \bar{R}_{ij} - (d - 2) \left( \delta_{ij} \log \Omega - \frac{1}{d - 2} \bar{h}_{ij} \bar{h}^{bc} \right) \bar{D}_{ij} \partial_{a} \log \Omega$$

$$+ (d - 2) \left( \delta_{ij} \bar{h}^{bc} - \bar{h}_{ij} \bar{h}^{bc} \right) \partial_{a} \log \Omega \partial_{a} \log \Omega. \tag{C18}$$

The Ricci scalar is obtained by contracting the above expression with $h^{ij}$; we, however, contract it with $\bar{h}^{ij}$, which corresponds to a rescaled Ricci scalar $a^{2}R$, namely,

$$a^{2}h^{ij}R_{ij} = \bar{h}^{ij}R_{ij}$$

$$= \bar{R} - 2(d - 1)\bar{h}^{ij} \left[ \bar{D}_{ij} \partial_{a} \log \Omega + \frac{(d - 2)}{2} \partial_{ij} \partial_{a} \log \Omega \right] \tag{C19}$$

$$= \bar{R} - \frac{2(d - 1)}{a^{2}} \left[ a \partial_{a} \left( \bar{h}^{ij} \partial_{a} \log \Omega + \frac{(d - 4)}{2} \bar{h}^{ij} \partial_{a} \partial_{a} \log \Omega \right) \right], \tag{C20}$$

where $\bar{R} \equiv \bar{h}^{ij}R_{ij}$ is conformally invariant. The expression in the last line above is obtained by employing $\partial_{a} \partial_{a} \log \Omega = \frac{1}{a} \partial_{a} \partial_{a} \log \Omega + \partial_{a} \partial_{a} \log \Omega$. Therefore, only the second and third terms transform under conformal transformations, producing the following expression:

$$\Delta_{ij} \left( a^{2}R \right) := \bar{a}^{2}\bar{R} - a^{2}R$$

$$= -2(d - 1) \left( D_{ij} \left( \bar{h}^{ij} \partial_{a} \log \Omega \right) \right.$$  

$$+ \frac{1}{2} \left( (d - 2) \bar{h}^{ij} \partial_{a} \log \Omega \partial_{a} \log \Omega \right) \right) \tag{C21}$$

$$= -2(d - 1) \frac{2}{\Omega^{2}} \left[ \Omega \bar{D}_{ij} \left( \bar{h}^{ij} \partial_{a} \log \Omega \right) + \frac{(d - 4)}{2} \bar{h}^{ij} \partial_{a} \partial_{a} \log \Omega \right], \tag{C22}$$

which turns out to be exactly the expected well-known difference after a conformal transformation of the rescaled Ricci scalar. A rule of thumb can be used to quickly determine the conformal transformation of an object in question: simply make a substitution $\bar{D}_{ij} \rightarrow D_{ij}$ in the first term in (C19) or $\bar{D}_{ij} \rightarrow D_{ij}$ in the first term in (C20), along with $a \rightarrow \Omega$.

It is useful to take a look at the traceless Ricci tensor, since it is of importance in the W theory. It turns out to be

$$R^{i}_{ij} = R_{ij} - \frac{1}{d} \bar{h}_{ij} R = \bar{R}_{ij} - \frac{1}{d} \bar{h}_{ij} \bar{R}$$

$$= \bar{R}_{ij} - (d - 2) \left( \bar{D}_{ij} \log \Omega - \partial_{ij} \log \Omega \partial_{ij} \log \Omega \right), \tag{C23}$$

where, again, “$\tau$” denotes that the trace has been projected away from an object by contraction with traceless identity $\Omega I$.

Note that one could define a quantity $R^{i}_{jklt}$ which would be the part of the Riemann tensor determined only by the conformal connection $\Gamma^{i}_{hk} \tag{22}$, in analogy to $\bar{R}_{ij}$. It is expected that the Weyl tensor—due to its conformal invariance—could be determined from $\bar{h}_{ij}$ only as well, but what is its relationship with $R^{i}_{jklt}$? One may then seek a relationship from the decomposition of the Riemann tensor. Namely, the irreducible decomposition of the Riemann tensor under the Lorentz group is

$$R^{i}_{jklt} = C^{i}_{jklt} = \left( \delta_{[k}R_{lj]} - g_{lj}R^{i}_{[k]} \right) + \frac{1}{3} \delta_{[k}g_{lj]}R, \tag{C24}$$

but it can also be written in terms of the Schouten tensor $S_{ij}$,

$$R^{i}_{jklt} = C^{i}_{jklt} + \frac{2}{d - 2} \left( \delta_{[k}S_{lj]} - g_{lj}S^{i}_{[k]} \right), \tag{C25}$$

where the Schouten tensor is defined by

$$S_{ij} = R_{ij} - \frac{1}{2(d - 1)} \bar{h}_{ij} R. \tag{C26}$$

Expressing now the Weyl tensor from (C25),

$$C^{i}_{jklt} = R^{i}_{jklt} - \frac{2}{d - 2} \left( \delta_{[k}S_{lj]} - g_{lj}S^{i}_{[k]} \right) \tag{C27}$$
and recalling its independence on scale, a separate scale dependence from the two terms on the right-hand side should cancel. Therefore, it should be possible to define the Weyl tensor equivalently in terms of the barred version of the quantities on the right-hand side only. Indeed, by defining $R^i_{jkl}$ as explained at the beginning of this paragraph, with decomposing the Schouten tensor into its conformally invariant part $S_{ij}$, as the barred version of the Lie derivative of $K_{ij}$ and scale part $W_{ij}$,

$$S_{ij} = \bar{S}_{ij} + W_{ij},$$

$$S_{ij} := \bar{R}_{ij} - \frac{1}{2(d-1)}\bar{h}_{ij}\bar{R}$$

one arrives at the same definition of the Weyl tensor obtained (using transformation properties of the conformal connection and eliminating the scale dependent terms from its antisymmetrized derivatives) by Thomas $^{[53]}$,

$$C^{i}_{jkl} = \bar{R}^{i}_{jkl} - \frac{2}{d-2} \left( \delta^{[i}_{[k} \bar{S}^{j]}_{l]} - g_{kij} \bar{S}_{lj}^{i} \right).$$

Note that both terms on the right-hand side are separately conformally invariant, but are not tensorial quantities. Therefore, the Weyl tensor may be interpreted as the traceless part of the unimodular part $\bar{R}^{i}_{jkl}$ of the Riemann tensor. One concludes that the conformal transformation of the Riemann tensor arises solely from the expression

$$W^{i}_{jkl} := \frac{2}{d-2} \left( \delta^{[i}_{[k} W^{j]}_{lj} - g_{kij} W^{i}_{lj} \right),$$

where $W_{ij}$ is given by (C31).

**Appendix D: Various identities**

Starting from the split of the velocity $\mathcal{L}^{n}K_{ij}$ into its traceless and trace part,

$$\mathcal{L}^{n}K_{ij} = (\mathcal{L}^{n}K_{ij})^{T} + \frac{1}{3} h_{ij} h^{ab} \mathcal{L}^{n}K_{ab},$$

using the traceless-trace decomposition of the extrinsic curvature $^{[25]}$ as well as the following identity,

$$\mathcal{L}^{n}K = h^{ab} \mathcal{L}^{n}K_{ab} - 2K_{ab} K^{ab},$$

where $K^{ab} = -\mathcal{L}^{n}h_{ab}/2$, one can show that the traceless part of $\mathcal{L}^{n}K_{ij}$ can be expressed in terms of the Lie derivative of $K^{ij}$,

$$\left(\mathcal{L}^{n}K_{ij}\right)^{T} = \mathcal{L}^{n}K^{T}_{ij} + \frac{2}{3} K^{T}_{ij} K - \frac{2}{3} h_{ij} K^{T}_{ab} K^{abT}. $$

This expression is then used to rewrite (10) in terms of the Lie derivative of $K^{ij}_{ij}$, namely,

$$(\mathcal{L}^{n}K_{ab})^{T} - K^{T}_{ab} K = \mathcal{L}^{n}K^{T}_{ij} - \frac{1}{3} h_{ij} K^{T}_{ij} K^{abT} = \frac{2}{3} h_{ij} K^{T}_{ab} K^{abT},$$

which is then used with unimodular-conformal variables in Sec. $^{[53]}$ to arrive at (10). Also note that by taking the trace of the above equation, we obtain

$$h^{ij} \mathcal{L}^{n}K_{ij} = 2K^{T}_{ij} K^{ijT},$$

which actually simply follows also from $\mathcal{L}^{n}h^{ij} K^{T}_{ij} = 0$.

The appearance of $R_{ij}$ and covariant derivatives in an expression introduces a dependence on the scale density $a$, but there is a particular type of operator that can be constructed with these two objects such that this dependence vanishes when they are contracted with some particular tensor densities. Such operators are met in many places in physics, and it appears in the present paper, too. The motivation for seeking such an operator in general can be found in the expression for $C^{ij}_{ij}$ given by (10). Since the square of $C^{ij}_{ij}$ is the kinetic term (of the electric part) of the Weyl action, it is expected that it is itself conformally invariant, that is, scaleless and independent of $K$. Now, all terms except $(3)R^{i}_{ij}$ and $(D_{i} \partial_{j} N)^{T}/N$ are manifestly scaleless (and $K$-less). To show that $(3)R^{i}_{ij} + (D_{i} \partial_{j} N)^{T}/N$ is conformally invariant, we use the unimodular-conformal decomposition to prove that the scale density $a$ indeed vanishes from this expression, and the proof will be given for any dimension [again, $i, j = 0, 1, 2, 3,...d - 1$], and the scale density is defined as $a = (\sqrt{h})^{1/d}$. We also introduce a conformally invariant scalar density $\phi = a^{-1}\bar{\phi}$, where $\bar{\phi}$ is a general scalar of conformal weight $-1$, in order to generalize the case with the lapse function. We do not go here into the question of whether or not the result is valid for a tensor of arbitrary rank and density, but leave this for another place.

Let us take a look at the two objects $(3)R^{i}_{ij}$ and $(D_{i} \partial_{j} \bar{\phi})/\bar{\phi}$ separately. We already have $(3)R^{i}_{ij}$ in (C18). Then,

$$\frac{D_{i} \partial_{j} \bar{\phi}}{\bar{\phi}} = \frac{D_{i} D_{j} \bar{\phi}}{\bar{\phi}} = \frac{1}{\bar{\phi}} D_{i} \partial_{j} \bar{\phi} = \frac{1}{\bar{\phi}}(D_{i} \partial_{j} \bar{\phi} + \bar{h}_{ij} h^{bc} \bar{\partial}_{b} \log a \partial_{c} \bar{\phi} - (2\delta^{b}_{(i} \delta^{c}_{j)} - \bar{h}_{ij} h^{bc}) \bar{\partial}_{b} \log a \partial_{c} \log a.$$
which means that the traceless part of (D6) does not contain it. Therefore, this “coincidence” can be used to form a traceless operator from traceless parts \( R_{ij}^T \) and \((D_i \partial_j \phi)^T / \phi\),

\[
R_{ij}^T + (d - 2) \frac{1}{\phi} D_{ij}^T \phi = \bar{R}_{ij}^T - (d - 2) \left( \bar{D}_i \partial_j \phi + \bar{D}_j \partial_i \phi + a \partial_k \log a \partial_k \phi \right) + (d - 2) \frac{1}{\phi} \left[ \bar{D}_i \partial_j \phi \right]^{T},
\]

which is indeed manifestly conformally invariant. Let us now write this operator in a more recognizable form, by dividing it with \( (d - 2) (D_i \partial_j \phi)^T / \phi \) and switching the order of terms,

\[
\left( (D_i D_j)^T + \frac{1}{d - 2} R_{ij}^T \right) \phi = \left( \bar{D}_i \partial_j + \frac{1}{d - 2} \bar{R}_{ij} \right)^T \phi.
\]

The same operator appears in (40), with \( d = 3 \). Namely, the traceless momentum density \( P^{ij} \) of scale weight \( \omega_a = 4 \), contracted with \( \left( (D_i D_j)^T + \frac{1}{d - 2} R_{ij}^T \right) \) gives

\[
\left( (D_i D_j)^T + \frac{1}{d - 2} R_{ij}^T \right) P^{ij} \overset{d=3}{=} (\partial_i \bar{D}_j + \bar{R}_{ij})^T \bar{P}^{ij},
\]

which matches the expression in (62). These derivations add to the power of the method of using the unimodular-conformal decomposition.

We finally make the interesting observation that the very same operator considered above is precisely the one that appears in the Bach equations, which are conformally invariant. Setting \( d = 4 \), contraction of \( (D_i D_j + \frac{1}{2} R_{ij}) \) with the Weyl tensor ensures that the operator is traceless, thus eliminating all the scale-dependent terms from it. That is why the Bach equations (41) can be simplified to

\[
\left( \nabla_k \nabla_l + \frac{1}{2} R_{kl} \right) C_{i j}^{k l} = \left( \nabla_k \nabla_l + \frac{1}{2} \bar{R}_{kl} \right) C_{i j}^{k l} = 0,
\]

which is manifestly scaleless and thus conformally invariant.

However, a question of generality poses itself. The above examples refer to a scalar density of scale weight \( \omega_a = -1 \), (D8), a traceless rank 2 tensor density of scalar weight 4, (D10), and the Weyl tensor, which is a rank 4 traceless tensor, but is also conformally invariant, (D11). One could, in principle, derive a general expression for the operator \( (D_i D_j + \frac{1}{2} R_{ij}) \) contracted with a tensor of arbitrary rank, symmetry and density, in arbitrary dimensions, such that the resulting expression is conformally invariant, but that question deserves a separate study.

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