COMPLETE ISOMETRIES - AN ILLUSTRATION OF
NONCOMMUTATIVE FUNCTIONAL ANALYSIS

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Abstract. This article, addressed to a general audience of functional analysts, is intended to be an illustration of a few basic principles from ‘noncommutative functional analysis’, more specifically the new field of operator spaces. In our illustration we show how the classical characterization of (possibly non-surjective) isometries between function algebras generalizes to operator algebras. We give some variants of this characterization, and a new proof which has some advantages.

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1. Introduction

The field of operator spaces provides a new bridge from the world of Banach spaces and function spaces, to the world of spaces of operators on a Hilbert space. For researchers in the new field, the philosophical starting point is the combination of the following two obvious facts. Firstly, by the Hahn-Banach theorem any Banach space $X$ is canonically linearly isometric to a closed linear subspace of $C(K)$, where $K$ is the compact space Ball($X^*$). Secondly, $C(K)$ is a commutative $C^*$-algebra. Thus one defines a noncommutative Banach space, or operator space, to be a closed linear subspace $X$ of a possibly noncommutative $C^*$-algebra $A$. This simplistic idea becomes much more substantive with the addition of some additional metric structure. The point is that if $A$ is any $C^*$-algebra, then the $*$-algebra $M_n(A)$ of $n \times n$ matrices with entries in $A$ has a unique norm $\|\cdot\|_n$ making it a $C^*$-algebra (this follows from the well known unicity of $C^*$-norms on a $*$-algebra). If $X \subset A$ then $M_n(X)$ inherits this norm $\|\cdot\|_n$, and more precisely we think of an operator space as the pair $(X, \{\|\cdot\|_n\}_n)$. We usually insist that maps between operator spaces are completely bounded, where the adjective ‘completely’ means that we are applying our maps to matrices too. Thus if $T : X \to Y$, then $T$ is completely contractive if $T_n$ is contractive for all $n \in \mathbb{N}$, where $T_n$ is the map $[x_{ij}] \mapsto [T(x_{ij})]$. Similarly $T$ is completely isometric if $\|T(x_{ij})\| = \|x_{ij}\|$ for all $n \in \mathbb{N}$ and $[x_{ij}] \in M_n(X)$. It is an easy exercise (using one of the common expressions for the operator norm of a matrix in $M_n = M_n(\mathbb{C})$) to prove that a linear map $T : X \to Y$ between subspaces of $C(K)$ spaces is completely contractive if and only if it is contractive. Consequently such a $T$ is isometric if and only if it is completely isometric.

The identification of the term ‘noncommutative Banach space’ with ‘operator space’ may be thought of as a relatively recent entry in the well known ‘dictionary’ translating terms between the ‘commutative’ and ‘noncommutative’ worlds. We spend a paragraph describing some other entries in this dictionary. Although these

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items are for the most part well known to the point of being tedious, it will be helpful to collect them here for the dual purpose of establishing notation, and for ease of reference later in the paper. The most well known item is of course the fact that the noncommutative version of a $C(K)$ space is a unital $C^*$-algebra $B$. The noncommutative version of a unimodular function in $C(K)$ is a unitary $u \in B$ (i.e. $u^*u = uu^* = 1$). The noncommutative version of a function algebra $A \subset C(K)$ containing constant functions is a closed subalgebra $A$ of a $C^*$-algebra $B$, with $1_B \in A$. We call such $A$ a unital operator algebra. For a unital subset $\mathcal{S}$ of a $C^*$-algebra $B$, we will take as a simple noncommutative version of the assertion `$\mathcal{S} \subset C(K)$ separates points of $K$’, the assertion ‘the $C^*$-subalgebra of $B$ generated by $\mathcal{S}$ (namely, the smallest $C^*$-subalgebra of $B$ containing $\mathcal{S}$) equals $B$’. The analogue of a closed subset $E$ of a compact set $K$ is a quotient $B/I$, where $I$ is a closed two-sided ideal in a unital $C^*$-algebra $B$. More generally, unital $*$-homomorphisms $\pi$ between unital $C^*$-algebras are the noncommutative version of continuous functions $\tau$ between compact spaces. Indeed clearly any such $\tau : K_1 \rightarrow K_2$ gives rise to the unital $*$-homomorphism $C(K_2) \rightarrow C(K_1)$ of ‘composition with $\tau$', and conversely it is not much harder to see that any unital $*$-homomorphism $C(K_2) \rightarrow C(K_1)$ comes from a continuous $\tau$ in this way. Moreover such $\pi$ is 1-1 (resp. onto) if and only if the corresponding $\tau$ is onto (resp. 1-1). Thus the noncommutative version of a homeomorphism between compact spaces is a (surjective 1-1) $*$-isomorphism between unital $C^*$-algebras. Coming back to ‘noncommutative functional analysis’, it is convenient for some purposes (but admittedly not for others) to view ‘complete isometries’ as the noncommutative version of isometries. It is very important in what follows that a 1-1 $*$-homomorphism $\pi : A \rightarrow B$ between $C^*$-algebras, is by a simple and well known spectral theory argument, automatically an isometry, and consequently (by the same principle applied to $\pi_n$), a complete isometry. Similarly, a $*$-homomorphism $\pi : A \rightarrow B$ (which is not a priori assumed continuous) is automatically completely contractive, and has a closed range which is a $C^*$-algebra $*$-isomorphic to the $C^*$-algebra quotient of $A$ by the obvious two-sided ideal, namely the kernel of the $*$-homomorphism.

The entries we have just described in this ‘dictionary’ are all easily justified by well known theorems (for example Gelfand’s characterization of commutative $C^*$-algebras). That is, if one applies the noncommutative definition in the commutative world, one recovers exactly the classical object. Similarly one sometimes finds oneself in the very nice ‘ideal situation’ where one can prove a theorem or establish a theory in the noncommutative world (i.e. about operator spaces or operator algebras), which when one applies the theorem/theory to objects which are Banach spaces or function algebras, one recovers exactly the classical theorem/theory. An illustration of this point is the Banach-Stone theorem. The following is a much simpler form of Kadison’s characterization of isometries between $C^*$-algebras [17]:

**Theorem 1.1.** (Folklore) A surjective linear map $T : A \rightarrow B$ between unital $C^*$-algebras is a complete isometry if and only if $T = u\pi(\cdot)$, for a unitary $u \in B$ and a $*$-isomorphism $\pi : A \rightarrow B$.

**Proof.** (Sketch.) The easy direction is essentially just the fact mentioned earlier that 1-1 $*$-homomorphisms are completely isometric. The other direction can be proved by first showing (as with Kadison’s theorem) that $T(1)$ is unitary, so that without loss of generality $T(1) = 1$. The well known Stinespring theorem has as a simple consequence the Kadison-Schwarz inequality $T(a)^*T(a) \leq T(a^*a)$. Applying
this to \( T^{-1} \) too yields \( T(a)^* T(a) = T(a^*a) \), and now the result follows immediately from the ‘polarization identity’ \( a^*b = \frac{1}{4} \sum_{k=0}^{3}(a + i^k b)^*(a + i^k b) \). \( \square \)

Note that if one takes \( A = C(K_1) \) and \( B = C(K_2) \) in Theorem 1.1, and consults the ‘dictionary’ above, then one recovers exactly the classical Banach-Stone theorem. Indeed as we remarked earlier, in this case complete isometries are the same thing as isometries, unitaries are unimodular functions, and a \(*\)-isomorphism is induced by a homeomorphism between the underlying compact spaces.

Indeed consider the following generalization of the Banach-Stone theorem:

**Theorem 1.2.** \( [3, 22, 1, 20] \) Let \( \Omega \) be compact and Hausdorff, and \( A \) a unital function algebra. A linear contraction \( T : A \to C(\Omega) \) is an isometry if and only if there exists a closed subset \( E \) of \( \Omega \), and two continuous functions \( \gamma : E \to \mathbb{T} \) and \( \varphi : E \to \partial A \), with \( \varphi \) surjective, such that for all \( y \in E \)
\[
T(f)(y) = \gamma(y)f(\varphi(y)).
\]

Here \( \partial A \) is the Shilov boundary of \( A \) (see Section 2). We have supposed that \( T \) maps into a ‘selfadjoint function algebra’ \( C(\Omega) \); however since any function algebra is a unital subalgebra of a ‘selfadjoint’ one, the theorem also applies to isometries between unital function algebras. If \( A \) is a \( C(K) \) space too, then \( \partial A = K \) and then the theorem above is called Holsztynski’s theorem. We refer the reader to [4] for a survey of such variants on the classical Banach-Stone theorem.

Often the transition from the ‘classical’ to the ‘noncommutative’ involves the introduction of much more algebra. Next we appeal to our dictionary above to give an equivalent restatement of Theorem 1.2 in more algebraic language.

**Theorem 1.3.** (Restatement of Theorem 1.2) Let \( A, B \) be unital function algebras, with \( B \) selfadjoint. A linear contraction \( T : A \to B \) is an isometry if and only if
\[
(A) \text{ there exists a closed ideal } I \text{ of } B, \text{ a unitary } u \text{ in the quotient } C^*-\text{algebra } B/I, \text{ and a unital 1-1 } *\text{-homomorphism } \pi : A \to B/I, \text{ such that } q_I(T(a)) = u\pi(a) \text{ for all } a \in A.
\]

Here \( q_I \) is the canonical quotient \(*\)-homomorphism \( B \to B/I \).

In light of Theorems 1.1 and 1.2 one would imagine that for any complete isometry \( T : A \to B \) between unital operator algebras, the condition (A) above should hold verbatim. This would give a pretty noncommutative generalization of Theorem 1.3. Indeed if Ran \( T \) is also a unital operator algebra, then this is true (see eg. B.1 in [3]). However, it is quite easily seen that such a result cannot hold generally. For example, let \( M_n = M_n(\mathbb{C}) \); for any \( x \in M_n \) of norm 1, the map \( \lambda \mapsto \lambda x \) is a complete isometry from \( \mathbb{C} \) into \( M_n \). Now \( M_n \) is simple (i.e. has no nontrivial two-sided ideals), and so if the result above was valid then it follows immediately that \( x = u \). This is obviously not satisfactory.

To resolve the dilemma presented in the last paragraph, we have offered in [3] several alternatives. For example, one may replace the quotient \( B/I \) by a quotient of a certain \(*\)-subalgebra of \( B \). The desired relation \( q_I(T(a)) = u\pi(a) \) then requires \( u \) to be a unitary in a certain \( C^* \)-triple system (by which we mean a subspace \( X \) of a \( C^* \)-algebra \( A \) with \( XX^*X \subset X \)). Or, one may replace the quotient \( B/I \) by a quotient \( B/(J + J^*) \), where \( J \) is a one-sided ideal of \( B \). Such a quotient is not an algebra, but is an ‘operator system’ (such spaces have been important in the deep work of Kirchberg (see [8, 9] and references therein). Alternatively, one
may replace such quotients altogether, with certain subspaces of the second dual $B^{**}$ defined in terms of certain orthogonal projections of ‘topological significance’ (i.e. correspond to characteristic functions of closed sets in $K$ if $B = C(K)$) in the second dual $B^{**}$ (which is a von Neumann algebra \[24\]). The key point of all these arguments, and indeed a key approach to Banach-Stone theorems for linear maps between function algebras, $C^*$-algebras or operator algebras, is the basic theory of $C^*$-triple systems and triple morphisms, and the basic properties of the noncommutative Shilov boundary or triple envelope of an operator space. These important and beautiful ideas originate in the work of Arveson, Choi and Effros, Hamana, Harris, Kadison, Kirchberg, Paulsen, Ruan, and others. Indeed our talk at the conference spelled out these ideas and their connection with the Banach-Stone theorem; and the background ideas are developed at length in a book the first author is currently writing with Christian Le Merdy \[7\] (although we do not characterize non-surjective complete isometries there). Moreover, a description of our work from this perspective, together with many related results, may be found in \[12\]. Thus we will content ourselves here with a survey of some related and interesting topics, and with a new and self-contained proof of some characterizations of complete isometries between unital operator algebras which do not appear elsewhere. This proof has several advantages, for example the projections arising naturally with this approach seem to be more useful for some purposes. Also it will allow us to avoid any explicit mention of the theory of triple systems (although this is playing a silent role nonetheless).

We also show how such noncommutative results are generalizations of the older characterizations of into isometries between function algebras or $C(K)$ spaces. We thank A. Matheson for telling us about these results. In the final section we present some evidence towards the claim that (general) isometries between operator algebras are not the correct noncommutative generalization of isometries between function algebras.

For the reader who wants to learn more operator space theory we have listed some general texts in our bibliography.

2. The noncommutative Shilov boundary

At the present time the appropriate ‘extreme point’ theory is not sufficiently developed to be extensively used in noncommutative functional analysis. Although several major and beautiful pieces are now in place, this is perhaps one of the most urgent needs in the subject. However there are good substitutes for ‘extreme point’ arguments. One such is the noncommutative Shilov boundary of an operator space. Recall that if $X$ is a closed subspace of $C(K)$ containing the identity function $1_K$ on $K$ and separating points of $K$, then the classical Shilov boundary may be defined to be the smallest closed subset $E$ of $K$ such that all functions $f \in X$ attain their norm, or equivalently such that the restriction map $f \mapsto f|_E$ on $X$ is an isometry. This boundary is often defined independently of $K$, for example if $A$ is a unital function algebra then we may define the Shilov boundary as we just did, but with $K$ replaced by the maximal ideal space of $A$. In fact we prefer to think of the classical Shilov boundary of $X$ as a pair $(\partial X, i)$ consisting of an abstract compact Hausdorff space $\partial X$, together with an isometry $j : X \to C(\partial X)$ such that $j(1_K) = 1_{\partial X}$ and such that $j(X)$ separates points of $\partial X$, with the following universal property: For any other pair $(\Omega, i)$ consisting of a compact Hausdorff space $\Omega$ and a complete isometry...
and a complete isometry \(i: X \to C(\Omega)\) which is unital (i.e. \(i(1_K) = 1_A\)), and such that \(i(X)\) separates points of \(\Omega\), there exists a (necessarily unique) continuous injection \(\tau: \partial X \to \Omega\) such that \(i(x)(\tau(w)) = j(x)(w)\) for all \(x \in X, w \in \partial X\). Such a pair \((\partial X, i)\) is easily seen to be unique up to an appropriate homeomorphism. The fact that such \(\partial X\) exists is the difficult part, and proofs may be found in books on function algebras (using extreme point arguments).

Consulting our ‘noncommutative dictionary’ in Section 1, and thinking a little about the various correspondences there, it will be seen that the noncommutative version of this universal property above should read as follows. Or at any rate, the following noncommutative statements, when applied to a unital subspace \(X \subset C(K)\), will imply the universal property of the classical Shilov boundary discussed above. Firstly, a unital operator space is a pair \((X, e)\) consisting of an operator space \(X\) with fixed element \(e \in X\), such that there exists a linear complete isometry \(\kappa\) from \(X\) into a unital \(C^*\)-algebra \(C\) with \(\kappa(e) = 1_C\). A ‘noncommutative Shilov boundary’ would correspond to a pair \((B, j)\) consisting of a unital \(C^*\)-algebra \(B\) and a complete isometry \(j: X \to B\) with \(j(e) = 1_B\), and whose range generates \(B\) as a \(C^*\)-algebra, with the following universal property: For any other pair \((A, i)\) consisting of a unital \(C^*\)-algebra and a complete isometry \(i: X \to A\) which is unital (i.e. \(i(e) = 1_A\)), and whose range generates \(A\) as a \(C^*\)-algebra, there exists a (necessarily unique, unital, and surjective) \(*\)-homomorphism \(\pi: A \to B\) such that \(\pi \circ i = j\). Happily, this turns out to be true. The existence for any unital operator space \((X, e)\) of a pair \((B, j)\) with the universal property above is of course a theorem, which we call the Arveson-Hamana theorem (see complete details). As is customary we write \(C_e^*(X)\) for \(B\) or \((B, j)\), this is the ‘\(C^*\)-envelope of \(X\)’.

It is essentially unique, by the universal property. If \(X = A\) is a unital operator algebra (see Section 1 for the definition of this), then \(j\) above is forced to be a homomorphism (to see this, choose an \(i\) which is a homomorphism, and use the universal property). Thus \(A\) may be considered a unital subalgebra of \(C_e^*(A)\). If \(A\) is already a unital \(C^*\)-algebra, then of course we can take \(C_e^*(A) = A\).

To help the reader get a little more comfortable with these concepts, we compute the ‘noncommutative Shilov boundary’ in a few simple examples.

**Example 1.** Let \(T_n\) be the upper triangular \(n \times n\) matrices. This is a unital subspace of \(M_n\), and no proper \(*\)-subalgebra of \(M_n\) contains \(T_n\). Let \((B, j)\) be the \(C^*\)-envelope of \(T_n\). By the universal property of the \(C^*\)-envelope, there is a surjective \(*\)-homomorphism \(\pi: M_n \to B\) such that \(\pi(a) = j(a)\) for \(a \in T_n\). The kernel of \(\pi\) is a two-sided ideal of \(M_n\). However \(M_n\) has no nontrivial two-sided ideals. Hence \(\pi\) is 1-1, and is consequently a \(*\)-isomorphism, and we can thus identify \(M_n\) with \(B\). Thus \(M_n\) is a \(C^*\)-envelope of \(T_n\).

**Example 2.** Consider the linear subspace \(X\) of \(M_3\) with zeroes in the 1-3, 2-3, 2-1, 3-1 and 3-2 entries, and with arbitrary entries elsewhere except for the 3-3 entry, which is the average of the 1-1 and 2-2 entries. It is easy to see that the \(C^*\)-algebra generated by \(X\) inside \(M_3\) is \(M_2 \oplus \mathbb{C}\). However this is not the \(C^*\)-envelope. Indeed it is easy to see that the 3-3 entry here is redundant, since the norm of \(x \in X\) is the norm of the upper left \(2 \times 2\) block of \(x\). This observation can be expanded to show that the canonical projection map \(M_2 \oplus \mathbb{C} \to M_2\) when restricted to \(X\) is a unital complete isometry from \(X\) onto \(T_2\) (see Example 1). This is the same as saying that if one takes the quotient of \(M_2 \oplus \mathbb{C}\) by its ideal \(0_2 \oplus \mathbb{C}\), then one obtains \(M_2\), which by Example 1 is the \(C^*\)-envelope.
Indeed this is typical when calculating the $C^*$-envelope of a unital subspace $X$ of $M_n$. The $C^*$-algebra generated by $X$ is a finite dimensional unital $C^*$-algebra. However such $C^*$-algebras are all $*$-isomorphic to a finite direct sum $B$ of full ‘matrix blocks’ $M_{n_k}$. Some of these blocks are redundant. That is, if $p$ is the central projection in $B$ corresponding to the identity matrix of this block, then $x \mapsto x(1_B - p)$ is completely isometric. If one eliminates such blocks then the remaining direct sum of blocks is the $C^*$-envelope.

**Example 3.** Let $B$ be a unital $C^*$-algebra. Consider the unital subspace $S(B)$ of the $C^*$-algebra $M_2(B)$ consisting of matrices

$$\begin{bmatrix}
\lambda 1 & x \\
y^* & \mu 1
\end{bmatrix}$$

for all $x, y \in B$ and $\lambda, \mu$ complex scalars. We claim that $M_2(B)$ is the $C^*$-envelope $C$ of $S(B)$, and we will prove this using a similar idea to Example 1 above. Namely, first note that $M_2(B)$ has no proper $C^*$-subalgebra containing $S(B)$, Thus by the Arveson-Hamana theorem there exists a $*$-homomorphism $\pi : M_2(B) \to C$ which possesses a property which we will not repeat, except to say that it certainly ensures that $\pi$ applied to a matrix with zero entries except for a nonzero entry in the 1-2 position, is nonzero. It suffices as in Example 1 to show that $\ker \pi = \{0\}$. Suppose that $\pi(x) = 0$ for a $2 \times 2$ matrix $x \in M_2(B)$. Let $E_{ij}$ be the four canonical basis matrices for $M_2$, thought of as inside $M_2(B)$. Then $\pi(E_{ij}x E_{j2}) = \pi(E_{ij})\pi(x)\pi(E_{j2}) = 0$ for $i, j = 1, 2$. Thus by the fact mentioned above about the 1-2 position, we must have $E_{1i}x E_{j2} = 0$. Thus $x = 0$.

In fact a variant of the $C^*$-envelope or ‘noncommutative Shilov boundary’ can be defined for any operator space $X$. This is the *triple envelope* of Hamana (see [14]). This is explained in much greater detail in [10], together with many applications. For example it is intimately connected to the ‘noncommutative $M$-ideals’ recently introduced in [11]. This ‘noncommutative Shilov boundary’ is, as we mentioned in Section 1, a key tool for proving various Banach-Stone type theorems. However in the present article we shall only need the variant described earlier in this section.

### 3. Complete Isometries between Operator Algebras

We begin this section with a collection of very well known and simple facts about closed two-sided ideals $I$ in a $C^*$-algebra $A$, and about the quotient $C^*$-algebra $A/I$.

We have that $I^{\perp \perp}$ is a weak* closed two-sided ideal in the von Neumann algebra $A^{**}$, and there exists a unique orthogonal projection $e$ in the center of $A^{**}$ with $I^{\perp \perp} = A^{**}(1-e)$. The projection $1-e$ is called the *support projection* for $I$, and $1-e$ may be taken to be the weak* limit in $A^{**}$ of any contractive approximate identity for $I$. Thus it follows that $A^{**}/I^{\perp \perp} \cong A^{**}e$ as $C^*$-algebras. Therefore also

$$A/I \subset (A/I)^{**} \cong A^{**}/I^{\perp \perp} \cong A^{**}e$$

as $C^*$-algebras. Explicitly, the composition of all these identifications is a 1-1 $*$-homomorphism taking an $a + I$ in $A/I$, to $\hat{a}e = e\hat{a}e$ in $A^{**}$. Here $e$ is the canonical embedding $A \to A^{**}$ (which we will sometimes suppress mention of). Thus $A/I$ may be regarded as a $C^*$-subalgebra of $A^{**}$, or of the $C^*$-algebra $eA^{**}e$.

We next illustrate the main idea of our theorem with a simple special case. (The following appeared as part of Corollary 3.2 in the original version of [5], with the proof left as an exercise). Suppose that $T : A \to B$ is a complete isometry between
unital $C^*$-algebras, and suppose that $T$ is unital too, that is $T(1) = 1$. Let $C$ be the $C^*$-subalgebra of $B$ generated by $T(A)$. Applying the Arveson-Hamana theorem\footnote{We remark in passing that one does not need the full strength of the Arveson-Hamana theorem here, one may use the much simpler \cite{8} Theorem 4.1.} we obtain a surjective $\ast$-homomorphism $\theta : C \to A$ such that $\theta(T(a)) = a$ for all $a \in A$. If $I$ is the kernel of the mapping $\theta$, then $C/I$ is a unital $C^*$-algebra $\ast$-isomorphic to $A$. Indeed there is the canonical $\ast$-isomorphism $\gamma : A \to C/I$ induced by $\theta$, taking $a$ to $T(a) + I$. The next point is that $C/I$ may be viewed as we mentioned a few paragraphs back, as a $C^*$-subalgebra of $C^{**}$, and therefore also of $B^{**}$. Indeed if $e$ is the central projection in $C^{**}$ mentioned there, then $C/I$ may be viewed as a $C^*$-subalgebra of $eC^{**}e \subset B^{**}$. In view of the last fact, the map $\gamma$ induces an 1-1 $\ast$-homomorphism $\pi : A \to B^{**}$ taking an element $a \in A$ to the element of $B^{**}$ which equals

\begin{equation}
\hat{T}(a)e = e\hat{T}(a) = e\hat{T}(a)e
\end{equation}

(1)

(there are equal because $e$ is central in $C^{**}$). Conversely, if $T : A \to B$ is a complete contraction for which there exists a projection $e \in B^{**}$ such that $e\hat{T}(a)e$ is a 1-1 $\ast$-homomorphism $\pi$, then for all $a \in A$,

$$
\|T(a)\| \geq \|\hat{T}(a)e\| = \|\pi(a)\| = \|a\|
$$

using the fact mentioned earlier that 1-1 $\ast$-homomorphisms are necessarily isometric. Thus $T$ is an isometry, and a similar argument shows that it is a complete isometry. Thus we have characterized unital complete isometries $T : A \to B$.

If $H$ is a Hilbert space on which we have represented the von Neumann algebra $B^{**}$ as a weak$^*$ closed unital $\ast$-subalgebra, then $B$ may be viewed also as a unital $C^*$-subalgebra of $B(H)$, whose weak$^*$ closure in $B(H)$ is (the copy of) $B^{**}$. In this case we shall say that $B$ is represented on $H$ universally. (The explanation for this term is that the well-known ‘universal representation’ $\pi_u$ of a $C^*$-algebra is ‘universal’ in our sense, and conversely if $\pi$ is a representation which is ‘universal’ in our sense then $\pi(B)^\sigma$ is isomorphic to $\pi_u(B)^\sigma \cong B^{**}$. See \cite{26} Section 1.) If, further, $e \in B^{**}$ is a projection for which (1) holds, then with respect to the splitting $H = eH \oplus (1-e)H$ we may write

$$
T(a) = \begin{bmatrix} \pi(\cdot) & 0 \\ 0 & S(\cdot) \end{bmatrix},
$$

for all $a \in A$. We will see that this is essentially true even if $T(1_A) \neq 1_B$:

**Theorem 3.1.** Let $T : A \to B$ be a completely contractive linear map from a unital operator algebra into a unital $C^*$-algebra. Then the following are equivalent:

(i) $T$ is a complete isometry,

(ii) There is a partial isometry $u \in B^{**}$ with initial projection $e \in B^{**}$, and a (completely isometric) 1-1 $\ast$-homomorphism $\pi : C^*_e(A) \to eB^{**}e$ with $\pi(1) = e$, such that for all $a \in A$

$$
\hat{T}(a)e = u\pi(a) \quad \text{and} \quad \pi(a) = u^*\hat{T}(a).
$$

Moreover $e$ may be taken to be a ‘closed projection’ (see \cite{24} 3.11, and the discussion towards the end of our proof).
(iii) If $H$ is a Hilbert space on which $B$ is represented universally, then there exist two closed subspaces $E, F$ of the Hilbert space $H$, a 1-1 $*$-homomorphism $\pi: C_e^*(A) \to B(E)$ with $\pi(1) = I_E$, and a unitary $u: E \to F$, such that

$$T(a)|_{E^\perp} = u\pi(a),$$

and $T(a)|_{E^\perp} \subset F^\perp$, for all $a \in A$. Here $E^\perp$ for example is the orthocomplement of $E$ in $H$.

(iv) If $H$ is as in (iii), then there exists two closed subspaces $E, F$ of $H$, a unital 1-1 $*$-homomorphism $\pi: C_e^*(A) \to B(E)$, a complete contraction $S: C_e^*(A) \to B(E^\perp, F^\perp)$, and unitary operators $U: E \oplus F^\perp \to H$ and $V: H \to E \oplus E^\perp$, such that

$$T(a) = U \begin{bmatrix} \pi(a) & 0 \\ 0 & S(a) \end{bmatrix} V$$

for all $a \in A$.

(v) There is a left ideal $J$ of $B$, a 1-1 $*$-homomorphism $\pi$ from $C_e^*(A)$ into a unital subspace of $B/(J+J^*)$ which is a $C^*$-algebra, and a ‘partial isometry’ $u$ in $B/J$ such that

$$q_J(T(a)) = u\pi(a) \quad \& \quad \pi(a) = u^* q_J(T(a))$$

for all $a \in A$, where $q_J$ is the canonical quotient map $B \to B/J$.

Before we prove the theorem, we make several remarks. First, we have taken $B$ to be a $C^*$-algebra; however since any unital operator algebra is a unital subalgebra of a unital $C^*$-algebra this is not a severe restriction. We also remark that there are several other items that one might add to such a list of equivalent conditions. See [5, 6]. Items (ii)-(iv), and the proof given below of their equivalence with (i), are new. We acknowledge that we have benefitted from a suggestion that we use the Paulsen system to prove the result. This approach is an obvious one to those working in this area (Ruan and Hamana used a variant of it in their work in the ’80’s on complete isometries and triple morphisms [27, 14]). However we had not pushed through this approach in the original version of [5] because this method does not give several of the results there as immediately. Statement (v) above has been simply copied from [5, 6] without proof or explanation. We have listed it here simply because Theorem 1.3 may be particularly easily derived from it as the special case when $A$ and $B$ are commutative (see comments below). Note that (iii) above resembles Theorem 1.2 superficially.

**Proof.** The fact that the other conditions all imply (i) is easy, following the idea in the paragraph above the theorem, namely by using the fact that a 1-1 $*$-homomorphism is completely isometric.

In the remainder of the proof we suppose that $T$ is a complete isometry. We view $A$ as a unital subalgebra of $C_e^*(A)$ as outlined in Section 3. We define a subset $S(B)$ of $M_2(B)$ as in Example 3 in Section 2. Similarly define a subset $S(T(A))$ of $S(B)$ using a similar formula (note that $S(T(A))$ has 1-2 entries taken from $T(A)$ and 2-1 entries taken from $T(A)^*$). Similarly we define the subset $S(A)$ of the $C^*$-algebra $M_2(C_e^*(A))$ (i.e. $S(A)$ has scalar diagonal entries and off diagonal entries from $A$ and $A^*$). We write $1 \oplus 0$ for the matrix in $S(A)$ with 1 as the 1-2 entry and zeroes elsewhere. Similarly for $0 \oplus 1$. We also use these expressions for
the analogous matrices in \( S(B) \). The map \( \Phi : S(A) \to S(T(A)) \subset M_2(B) \) taking

\[
\begin{bmatrix}
\lambda_1 & x \\
y^* & \mu_1
\end{bmatrix}
\mapsto
\begin{bmatrix}
\lambda_1 & T(x) \\
T(y)^* & \mu_1
\end{bmatrix}
\]

is well known to be a unital complete isometry (this is the well known Paulsen lemma, see the proof of 7.1 in [3]). Let \( C \) be the \( C^* \)-subalgebra of \( M_2(B) \) generated by \( S(T(A)) \). The \( C^* \)-envelope of \( S(A) \) is well known to be \( M_2(C_e^*(A)) \) (see Example 3 in Section 2 where we proved this in the case that \( A \) is already a \( C^* \)-algebra, or for example [3] Proposition 4.3 or [29]). Thus by the Arveson-Hamana theorem we obtain a surjective \(*\)-homomorphism \( \theta : C \to M_2(C_e^*(A)) \) such that \( \theta \circ \Phi \) is simply the canonical embedding of \( S(A) \) into \( M_2(C_e^*(A)) \). As in the special case considered above the theorem, we let \( I_0 \) be the kernel of the mapping \( \theta \), then \( C/I_0 \) is a unital \( C^* \)-algebra \(*\)-isomorphic to \( M_2(C_e^*(A)) \). Indeed there is the canonical \(*\)-isomorphism \( \gamma : M_2(C_e^*(A)) \to C/I_0 \) induced by \( \theta \), taking

\[
\begin{bmatrix}
\lambda_1 & x \\
y^* & \mu_1
\end{bmatrix}
\mapsto
\begin{bmatrix}
\lambda_1 & T(x) \\
T(y)^* & \mu_1
\end{bmatrix}
+ I_0.
\]

As in the simple case above the theorem, \( C/I_0 \) may be viewed as a \( C^* \)-subalgebra of \( p_0C^*p_0 \), for a central projection \( p_0 \in C^* \) (namely, the complementary projection to the support projection of \( I_0 \)). Now \( p_0C^*p_0 \subset C^{**} \subset M_2(B)^{**} \), and it is well known that \( M_2(B)^{**} \cong M_2(B^{**}) \) as \( C^* \)-algebras. Thus we may think of \( C^* \) as a \( C^* \)-subalgebra of \( M_2(B^{**}) \). Also, \( C^{**} \) contains \( C \) as a \( C^* \)-subalgebra, and the projections \( 1 \otimes 0 \) and \( 0 \otimes 1 \) in \( C \) correspond to the matching diagonal projections \( 1 \otimes 0 \) and \( 0 \otimes 1 \) in \( M_2(B^{**}) \). These last projections therefore commute with \( p_0 \), since \( p_0 \) is central in \( C^* \), which immediately implies that \( p_0 \) is a diagonal sum \( f \otimes e \) of two orthogonal projections \( f, e \in B^{**} \). Thus we may write the \( C^* \)-algebra \( p_0M_2(B^{**})p_0 \) as the \( C^* \)-subalgebra

\[
\begin{bmatrix}
    fB^{**}f & fB^{**}e \\
    eB^{**}f & eB^{**}e
\end{bmatrix}
\]

of \( M_2(B^{**}) \). We said above that \( C/I_0 \) may be regarded as a \( C^* \)-subalgebra of the subalgebra \( p_0M_2(B^{**})p_0 \) of \( M_2(B^{**}) \). Thus the map \( \gamma \) induces a 1-1 \(*\)-homomorphism \( \Psi : M_2(C_e^*(A)) \to M_2(B^{**}) \). It is easy to check that \( \Psi(1 \otimes 0) = f \otimes 0 \) and \( \Psi(0 \otimes 1) = 0 \otimes e \). Since \( \Psi \) is a \(*\)-homomorphism it follows that \( \Psi \) maps each of the four corners of \( M_2(C_e^*(A)) \) to the corresponding corner of \( p_0M_2(B^{**})p_0 \subset M_2(B^{**}) \). We let \( R : C_e^*(A) \to fB^{**}e \) be the restriction of \( \Psi \) to the ‘1-2-corner’. Since \( \Psi \) is 1-1, it follows that \( R \) is 1-1. If \( \pi \) is the restriction of \( \Psi \) to the ‘2-2-corner’, then \( \pi \) is a \(*\)-homomorphism \( C_e^*(A) \to eB^{**}e \) taking \( 1_A \) to \( e \). Applying the \(*\)-homomorphism \( \Psi \) to the identity

\[
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]

we obtain that \( u = R(1) \) is a partial isometry, with \( u^*u = \pi(1) = e \). Similarly \( uu^* = f \). A similar argument shows that \( R(a) = R(1)\pi(a) \) for all \( a \in C_e^*(A) \). Thus \( u^*R(a) = u^*\pi(a) = \pi(a) \) for all \( a \in C_e^*(A) \).

Next, we observe that \( \Psi \) takes the matrix \( z \) which is zero except for an \( a \) from \( A \) in the 1-2-corner, to the matrix \( w = p_0\Phi(z)p_0 \). Since \( \Phi(z) \in C^{**} \) and \( p_0 \) is in the center of that algebra, we also have \( w = \Phi(z)p_0 = p_0\Phi(z) \). Also \( w \) viewed as
a matrix in $M_2(B^{**})$ has zero entries except in the 1-2-corner, which (by the last sentence) equals

$$fT(a)e = T(a)e = fT(a).$$

Also using these facts and a fact from the end of the last paragraph we have

$$u^*T(\cdot) = R(1)^*\hat{T}(\cdot) = (fT(1)^*\hat{T}(\cdot) = eT(1)^* fT(\cdot) = eT(1)^* \hat{T}(\cdot)e = u^*R(\cdot) = \pi.$$  

Thus

$$\hat{T}(\cdot)e = fT(\cdot) = uu^*T(\cdot) = uu(\cdot).$$

We have now also established most of (ii). One may deduce (iii) from (ii) by viewing $B \subset B^{**} \subset B(H)$, and setting $E = eH$, and $F = (uu^*)H$. We also need to use facts from the proof above such as $u^*u = e$. Clearly (iv) follows from (iii). As we said above, we will not prove (v) here.

Claim: if $e$ is the projection in (ii) above, then $1 - e$ is the support projection for a closed ideal $I$ of a unital $*$-subalgebra $D$ of $B$. Equivalently (as stated at the start of this section), there is a (positive increasing) contractive approximate identity $(b_t)$ for $I$, with $b_t \to 1 - e$ in the weak* topology. This claim shows that $1 - e$ is an ‘open projection’ in $B^{**}$, so that $e$ is a closed projection, as will be obvious to operator algebraists from [24] section 3.11 say. For our other readers we note that for what comes later in our paper, one can replace the assertion about closed projections in the statement of Theorem 3.1 (ii) with the statement in the Claim above.

To prove the Claim, recall from our proof that $p_0 = f \oplus e = 1_C - p_1$, where $p_1$ is the support projection for a closed ideal $I_0$ of $C$. Thus $p_1 = (1 - f) \oplus (1 - e)$. As stated at the start of Section 3, $p_1$ is the weak* limit in $C^{**}$, and hence also in $M_2(B^{**})$, of a contractive approximate identity $(e_t)$ of $I_0$. By the separate weak* continuity of the product in a von Neumann algebra, it follows that the net $b_t = (0 \oplus 1)e_t(0 \oplus 1)$ has weak* limit $(0 \oplus 1)p_1(0 \oplus 1) = 0 \oplus (1 - e)$. Viewing these as expressions in $B$, the above says that $b_t \to 1 - e$ weak* in $B^{**}$. View $(0 \oplus 1)C(0 \oplus 1)$ as a $*$-subalgebra $D$ of $B$, and view $(0 \oplus 1)I_0(0 \oplus 1)$ as a two sided ideal $I$ in $D$. It is easy to see that $(b_t)$ is a contractive approximate identity of $I$. Thus it follows that $1 - e$ is the support projection of the ideal $I$.  

Some applications of results such as Theorem 3.1 may be found in [24].

Next we discuss briefly the relation between our noncommutative characterization of complete isometries (for example Theorem 3.3 above), and Theorem 1.3. Our point is not to provide another proof for Theorem 1.3 - the best existing proof is certainly short and elegant. Rather we simply wish to show that the noncommutative result contains [13]. Indeed Theorem 1.3 quite easily follows from Theorem 3.1 (v). Since however we did not prove Theorem 3.1 (v), we give an alternate proof.

Corollary 3.2. Let $A, B$ be a unital function algebras, with $B$ selfadjoint. Then condition (ii) in Theorem 3.1 implies condition (A) in Theorem 1.3.

Proof. By hypothesis, $T(\cdot)e = uu(\cdot)$, and $u^*u = e = \pi(1)$ so that $u = uu(1) = T(1)e$. Thus $eT(1)^*T(\cdot)e = u^*uu(\cdot) = \pi(1)\pi(\cdot) = \pi$, so that Ran $\pi \subset eBe = Be$ (note $B^{**}$ is commutative in this case). From [24] 3.11.10 for example, the ‘closed projection’ $e$ in $B^{**}$ corresponds to a closed ideal $J$ in $B$ whose support projection is $1 - e$. Alternatively, to avoid quoting facts from [24], we will also deduce this.
from the ‘Claim’ towards the end of the proof of Theorem 3.1. If I is the ideal in that Claim, let J be the closed ideal in \(B\) generated by I. Since \(J = BI\), the contractive approximate identity of \(J\) is a right contractive approximate identity of \(J\). Thus \(J\) has support projection \(1 - e\) too, by the first paragraph of Section 3 above.

By facts in the just quoted paragraph, we have a canonical unital 1-1 map \(\eta : B/J \to B^{**}\) taking the equivalence class \(b + J\) of \(b \in B\) to \(ebe\). Indeed in this commutative case we see by inspection that \(\eta\) is a *-homomorphism from the \(C^*\)-algebra \(B/J\) onto the \(C^*\)-subalgebra \(M = eBe\) of \(B^{**}\). Define \(\theta(a) = \eta^{-1}(\pi(a))\), this is a 1-1 *-homomorphism \(A \to B/J\). Since \(\pi(1) = e\), \(\theta\) is a unital map too. Since \(uu^* = u^*u = e\), \(u\) is unitary in \(M\), and so \(\gamma = \eta^{-1}(u)\) is unitary in \(B/J\). Note also that \(T(a)e = \eta(T(a) + J)\). Applying \(\eta^{-1}\) to the equation \(T(\cdot)e = u\pi(\cdot)\), we obtain \(qJ(T(a)) = \gamma \theta(a)\), that is, condition (A) in Theorem 1.3. □

If one attempts to use the ideas above to find a characterization analogous to condition (A) from Theorem 1.3 but in the noncommutative case, it seems to us that one is inevitably led to a condition such as (v) in Theorem 3.1.

We address a paragraph to experts, on generalizations of the proof of Theorem 3.1. Consider a complete isometry between possibly non-unital \(C^*\)-algebras. Or much more generally, suppose that \(T\) is a complete isometry from an operator space \(X\) into a \(C^*\)-triple system \(W\). One may form the so-called ‘linking \(C^*\)-algebra’ of \(W\), with the identities of the ‘left and right algebras of \(W\)’ adjoined. Call this \(L^t(W)\). As in the proof of Theorem 3.1 we think of \(S(W) \subseteq L^t(W)\). Similarly, if \(Z\) is the ‘triple envelope’ of \(X\) (or if \(X = Z\) is already a \(C^*\)-algebra or \(C^*\)-triple system), then we may consider \(S(X) \subseteq S(Z) \subseteq L^t(Z)\). As in the proof of Theorem 3.1 we obtain firstly a unital complete isometry \(\Phi : S(X) \to S(T(X)) \subseteq L^t(Z)\), and then a unital 1-1 *-homomorphism \(\pi : L^t(Z) \to L^t(W)^{**}\). By looking at the ‘corners’ of \(\pi\) we obtain projections \(e, f\) in certain second dual von Neumann algebras, so that \(\pi(T(\cdot))e\) is (the restriction to \(X\) of a completely isometric) a 1-1 triple morphism into \(W^{**}\). In fact we have precisely such a result in (see Section 2 there), but the key point is that the new proof gives different projections \(e, f\), which are more useful for some purposes.

4. Complete isometries versus isometries

Finally, as promised we discuss why we believe that in this setting of non-surjective maps between \(C^*\)-algebras say, general isometries are not the ‘noncommutative analogue’ of isometries between function algebras. The point is simply this. In the function algebra case we can say thanks to Holsztynski’s theorem that the isometries are essentially the maps composed of two disjoint pieces \(R\) and \(S\), where \(R\) is isometric and ‘nice’, and \(S\) is contractive and irrelevant. However at the present time it looks to us unlikely that there ever will be such a result valid for general non-surjective isometries between general \(C^*\)-algebras. The chief evidence we present for this assertion is the very nice complementary work of Chu and Wong in isometries (as opposed to complete isometries) \(T : A \to B\) between \(C^*\)-algebras. They show that for such \(T\) there is a largest projection \(p \in B^{**}\) such that \(T(\cdot)p\) is some kind of Jordan triple morphism. This appears to be the correct ‘structure theorem’, or version of Kadison’s theorem, for non-surjective isometries. However as they show, the ‘nice piece’ \(R = T(\cdot)p\) is very often trivial (i.e. zero), and is thus certainly not isometric. Thus this approach is unlikely to ever yield a
A good example is $A = M_2$, the smallest noncommutative $C^*$-algebra. Simply because $A$ is a Banach space there exists, as in the discussion in the first paragraph of our paper, a linear isometry of $A$ into a $C(K)$ space. However it is easy to see that there is no nontrivial $*$-homomorphism or Jordan homomorphism from $A$ into a commutative $C^*$-algebra. Such an isometry is uninteresting, because the interesting 'nice part' is zero. Thus we imagine that the 'good noncommutative notions of isometry' are either complete isometries or the closely related class of maps for which the piece $T(\cdot)p$ from $[9]$ is an isometry.

This leads to two questions. Firstly, can one independently characterize the last mentioned class? Secondly, if $T$ is a complete isometry, then is the projection $p$ in the last paragraph equal (or closely related) to our projection $e$ above?

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