EXPOSITIONAL MIXING, KAM AND SMOOTH LOCAL RIGIDITY

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Abstract. Consider actions of $\mathbb{Z}^r$ by ergodic automorphisms on a compact nilmanifolds for $r \geq 2$. We show that small $C^k$ perturbations of such higher rank partially hyperbolic actions are smoothly conjugate to the original action, using a KAM scheme. The driving force for convergence of this iteration is the exponential mixing of the original action.

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1. Introduction

Smooth rigidity of actions by commuting diffeomorphisms and flows has attracted significant attention over the last three decades. There are many such actions coming from homogeneous dynamics, and in particular automorphisms of tori and nilmanifolds, or possibly an action by commuting translations on a homogeneous space $G/\Gamma$ for $G$ a Lie group, $\Gamma$ a lattice in $G$. These are called algebraic or also homogeneous models. The goal is to completely classify such actions up to smooth conjugacy or at least prove local smooth rigidity for the algebraic models.

The case when $G$ is noncompact semisimple is possibly best understood, thanks to the abundance of tools from hyperbolic dynamics, representation theory, an abundance of relations between various one-parameter subgroups and the many rigidity properties of such groups $G$. We refer to the works by Damjanovic, Hurder, Kalinin, Katok, Lewis, Spatzier, Vinhage and Wang (cf. e.g. [Hur94, KS97, DK05, DK11, VW19]). They were at first largely motivated by applications to the Zimmer program of studying actions of higher rank semisimple Lie groups and their lattices on compact manifolds, cf. e.g. [Hur94, KL96, KL91, KLZ96, KS97, GS99] amongst others.

* Supported in part by NSF grants DMS-1607260 and DMS-2003712.
** Supported in part by NSFC grant 11801384 and the Fundamental Research Funds for the Central Universities YJ201769.
In this paper, we will concentrate on higher rank actions of $\mathbb{Z}^k$ by automorphisms on nilmanifolds, the last major open case for local rigidity. Let us first though recall some prior work. Both local and global rigidity has been achieved for actions on compact nilmanifolds $G/\Gamma$ where $G$ is a simply connected nilpotent Lie group and $\Gamma$ a discrete cocompact subgroup of $G$ with the following proviso. If the action is generated by a single Anosov diffeomorphism $f$, one can easily perturb $f$ to another Anosov diffeomorphism by a local perturbation at a periodic point which changes the derivative at that point. One can then argue that the perturbed Anosov diffeomorphism cannot be smoothly (not even $C^1$) conjugate to an algebraic model. A similar remark applies to products of Anosov diffeomorphisms on product manifolds. One coins the notion of rank 1 factor of an arbitrary $\mathbb{Z}^k$ action, i.e. a smooth factor of the action on which some $\mathbb{Z}^{r-1} \subset \mathbb{Z}^r$ acts trivially. We further recall that any Anosov diffeomorphism on a compact nilmanifold is $C^0$-conjugate to an algebraic model by a classical result of Franks and Manning [Fra69, Man73]. This conjugacy will automatically extend to the commuting diffeomorphisms as well. We call the resulting algebraic model the linearization of the original action.

The ultimate global rigidity result for $\mathbb{Z}^r$ Anosov actions on nilmanifolds was obtained by Federico Rodriguez Hertz and Zhiren Wang in [RHW14]. More precisely, they assume that the action has at least one Anosov diffeomorphism and that the linearization has no rank 1 factors. This extended earlier work by Fisher, Kalinin and the first author who had to assume existence of many Anosov diffeomorphisms [FKS13].

Note that the global rigidity results for higher rank Anosov actions strongly depend on the existence of the Franks-Manning Hölder conjugacy which requires the presence of an Anosov diffeomorphism. In consequence, global rigidity results for higher rank actions of commuting partially hyperbolic diffeomorphisms on nilmanifolds are currently out of reach. Instead, we investigate local rigidity results for the algebraic models. The following theorem settles the nilmanifold case. It generalizes work by D. Damjanovic and A. Katok for higher rank actions of commuting ergodic toral automorphisms [DK10a] assuming the action does not have a rank 1 factor via an action by nil-automorphisms on a compact nilmanifold $G^*/\Gamma^*$. To be precise, we require that the action $\rho_0$ by nil-automorphisms of $\mathbb{Z}^r$ on $G/\Gamma$ does not have a factor action by nil-automorphisms on a nilmanifold $G^*/\Gamma^*$ which is virtually cyclic and for which the factor map is affine, i.e. given by the composition of a left translation with a homomorphism. This is clearly the weakest assumption we can make, is algebraic in nature and therefore checkable. In this situation, we will say that $\rho_0$ does not have an algebraic rank 1 factor.

**Theorem 1.1.** Let $M = G/\Gamma$ be a compact nilmanifold with an action $\rho_0$ of $\mathbb{Z}^r$ for $r \geq 2$ by automorphisms of $G$. Assume that $\rho_0$ does not have an algebraic rank 1 factor. Then for some positive integer $k$ (depending only on the dimension of $G$), any $C^k$-small $C^\infty$ perturbation $\rho$ of $\rho_0$ is $C^\infty$-conjugate to $\rho_0$.

Rank 1 factors are closely tied to ergodicity properties. Indeed, Rodriguez Hertz and Wang show in [RHW14] that non-existence of an algebraic rank 1 factor condition for actions by commuting nil-automorphisms on a compact nilmanifold is equivalent to the existence of a $\mathbb{Z}^2$ subgroup of $\mathbb{Z}^k$ for which all non-trivial elements act ergodically. This extends results by Starkov in the toral case [Sta99] - cf. Proposition 2.1.

To handle nilmanifolds we introduce a new method which combines exponential mixing of the unperturbed action with a KAM approach. The latter usually is done for dynamical systems on tori and heavily uses Fourier analysis, especially the fast decay of Fourier coefficients of smooth functions on tori. This is not available to us. In fact the harmonic analysis for nilmanifolds is complicated and not easily used. Instead, we use the exponential mixing results for commuting groups of...
ergodic automorphisms of nilmanifolds by Gorodnik and the first author [GS15] which are based on the polynomial equidistribution results of Green and Tao [GT12a].

We use a KAM iteration as in [DK10a] - with some important differences. First, the iteration involves solving approximate linearized conjugacy equations by suitably smooth functions. With exponential mixing, a priori we can only get solutions which are dual to suitable spaces of Hölder functions. The exponential mixing however gives us partial derivatives of arbitrarily high order which are still dual to the same spaces. This is motivated by arguments from [FKS13, RHW14]. There however, the cohomology equation is solved over the non-linear action and the regularity of the distributions is controlled along certain foliations. In the partially hyperbolic case, we do not know exponential mixing for the perturbed action as there is no Hölder conjugacy between perturbed and original action a priori. When trying to do this for the KAM scheme, the transversal regularity unfortunately becomes uncontrollable. This forced us to consider derivatives along the whole manifold.

Acknowledgements: We are deeply grateful to Jeffrey Rauch who helped us understand regularity properties of distributions, and even proved general results on our behalf. Regrettably, in the end we did not use these though. We further are grateful to Danijela Damjanovic for a number of helpful conversations. We thank Yunfeng Shi for his reference to a paper by Salamon on an interpolation result, Jianjun Liu for help with KAM arguments and Zhiyuan Zhang for bringing this problem to our attention.

1.1. Notation. In this paper, we employ the Vinogradov and Bachmann–Landau notation: for functions $f$ and positive-valued functions $g$, we write $f \ll g$ or $f = O(g)$ if there exists a constant $C$ such that $|f| \leq Cg$ pointwise, and write $f \asymp g$ if $f \ll g$ and $g \ll f$.

We denote by $d$ the dimension of $G$ and by $\mathfrak{g}$ the Lie algebra of $G$; $\text{exp} : \mathfrak{g} \to G$ and $\text{log} : G \to \mathfrak{g}$ will denote the exponential map and logarithm map respectively.

2. Preliminaries

In this section we recall some facts about algebraic actions as well as basic facts on interpolation, Sobolev spaces and smoothing operators.

2.1. Algebraic Preliminaries and Exponential Mixing. We will recall some general facts about actions by nil-automorphisms.

First, Rodriguez-Hertz and Wang [RHW14, Lemma 2.9] generalized work by Starkov [Sta99] to nilmanifolds:

**Proposition 2.1.** Suppose $\rho_0$ is a $\mathbb{Z}^r$ action on a nilmanifold $G/\Gamma$ by nil-automorphisms without algebraic rank 1 factor. Then there is a $\mathbb{Z}^2$ subgroup of $\mathbb{Z}^r$ for which all non-trivial elements act ergodically.

In the proof of the main result, Theorem 1.1, we will combine this characterization of existence of algebraic rank 1 factors, with the following exponential mixing result from [GS15]. It will allows us to define converging series in a suitable space of distributions which will be the basis for starting the KAM procedure.

**Proposition 2.2.** Suppose $\rho_0$ is a $\mathbb{Z}^r$ action on a nilmanifold $G/\Gamma$ by nil-automorphisms such the all nontrivial elements of $\mathbb{Z}^r$ act ergodically wrt Haar measure. Then there exists a constant $\eta = \eta(\rho_0) > 0$ such that for any $f, g \in C^0(G/\Gamma)$ with zero integrals and any $a \in \mathbb{Z}^r$,

$$\langle f \circ a, g \rangle \ll e^{-\eta \|a\|} \|f\|_{C^0} \|g\|_{C^0}.$$
Finally, we recall Walter’s characterization of centralizers of ergodic nil-automorphisms and more generally affine maps [Wal70, Corollary 1]. We recall that an affine map of a nilmanifold $N/\Lambda$ is the composition of an automorphism with a left translation.

**Proposition 2.3.** A homeomorphism centralizing an ergodic affine map of a compact nilmanifold $G/\Gamma$ is affine.

We will need this to compare solutions in different Sobolev spaces so that they all have the same regularity in the end (cf. Remark 3.2).

2.2. **Interpolation of Pointwise Derivatives:** Interpolation theory gives multiplicative control of intermediate derivatives $D_j^j f$ of an $k$ times differentiable function $f : \mathbb{R} \to \mathbb{R}$ in terms of $f$ and $D^\alpha f$ where $j < k$, generalizing Landau’s famous inequality $|f'|^2 \leq 4|f| \cdot |f''|$. While this is usually generalized using Sobolev norms, we will use a more direct estimate from [Sal04, Lemma 5], inspired by work of Moser.

We let $\partial_j$ denote the $j$'th partial derivative, and for a multi-index $\alpha = (i_1, \ldots, i_k)$, let $|\alpha| := k$, and denote by $D^\alpha$ the concatenation of all the $\partial_i$ derivatives. Denote by $C^s(\mathbb{R}^n)$ the space of $s$-times continuously differentiable functions on $\mathbb{R}^n$ with bounded derivatives $D^\alpha f$ for $\alpha \leq s$. Denoting the sup norm by $|f|_{C^s}$, we endow $C^s(\mathbb{R}^n)$ with the $| \cdot |_{C^s}$ norm:

$$|f|_{C^s} := \sum_{\alpha \leq s} |D^\alpha f|_{C^0}.$$ 

**Lemma 2.4.** For every $n \in \mathbb{N}$ and every $l > 0$, there is a constant $c = c(l, n) > 0$ such that for all $f, g \in C^l(\mathbb{R}^n)$ and all $k \leq m \leq l$:

$$|f|_{C^m}^{l-k} \leq c |f|_{C^k}^{l-m} |f|_{C^m}^{m-k}.$$ 

**Proof.** We mimic the proof of [Sal04, Lemma 5]. The main difference is that our functions are defined on $\mathbb{R}^n$, not $T^n$. To deduce the claim for a function $f : \mathbb{R}^n \to \mathbb{R}$, consider $p \in \mathbb{R}^n$ and pick a bump function $g$ supported in a unit box $B$ around $p$. Then $g \cdot f$ is supported on $B$, and naturally defines a function on $T^n$. Then

$$|g \cdot f|_{C^m}^{l-k} \leq c |g \cdot f|_{C^k}^{l-m} |g \cdot f|_{C^m}^{m-k}.$$ 

Since $|g \cdot f|_{C^j} \leq G |f|_{C^j}$ for a constant $G$ depending on $g$ and $f$ for all $0 \leq j \leq l$, we get

$$|g \cdot f|_{C^m}^{l-k} \leq cG^2 |f|_{C^k}^{l-m} |f|_{C^m}^{m-k}.$$ 

Since $g \equiv 1$ in a neighborhood of $p$, the claim follows easily where $c$ is replaced by $cG^2$. \hfill $\Box$

**Note:** In principle one can keep track of the dependence of the constant $c = c(l, n)$ on $l$. One simply has to estimate derivatives of bump functions. For example if $f(x) = \exp\left(\frac{a}{1-x}\right)$, then $|f^{(n)}|_\infty \leq (2n\sqrt{15})^{2n} \cdot n!$. Similarly if $n > 1$, we can take products of the one-dimensional bump functions.

This could be used directly in the KAM convergence arguments that the solutions are smooth. We will argue this more indirectly, show that KAM scheme convergence on any fixed level of differentiability, and then use essential uniqueness of the solution to conclude the solution is actually $C^\infty$.  

2.3. Preliminaries on Sobolev Spaces: We refer the reader to [Tay11] and [Zim90] for detailed expositions on Sobolev theory.

We denote the space of essentially bounded measurable functions on $\mathbb{R}^n$ by $L^\infty(\mathbb{R}^n)$ and endow it with the usual $L^\infty$ norm. For a multi-index $\beta = (\beta_1, \ldots, \beta_n)$ of non-negative integers and $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, set $x^\beta := x_1^{\beta_1} \cdots x_n^{\beta_n}$. As usual, we call a smooth function $f \in C^\infty(\mathbb{R}^n)$ rapidly decreasing if $x^\beta D^\alpha f \in L^\infty(\mathbb{R}^n)$ for multi-indices $\alpha, \beta$ with all $\alpha_i, \beta_i \geq 0$. We denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ by $S(\mathbb{R}^n)$. We topologize $S(\mathbb{R}^n)$ by the family of semi-norms $p_k(f) := \sum_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} (x^k | D^\alpha f |)$ to obtain a Fréchet space. We call continuous linear functionals on $S(\mathbb{R}^n)$ a tempered distributions, denote the space of all such by $\mathcal{S}(\mathbb{R}^n)$ and endow it with the weak* topology.

Let $L^2(\mathbb{R}^d)$ denote the space of square integrable functions defined on $\mathbb{R}^d$. Then $L^2(\mathbb{R}^d)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{L^2}$:

$$\langle f, g \rangle_{L^2} := \int_{\mathbb{R}^d} f(x) g(x) dx, \text{ for } f, g \in L^2(\mathbb{R}^d).$$

For a positive integer $m$, we define

$$H^m(\mathbb{R}^d) := \{ f \in L^2(\mathbb{R}^d) : D^\alpha f \in L^2(\mathbb{R}^d) \text{ for any } |\alpha| \leq m \}.$$  

Then $H^m(\mathbb{R}^d)$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{H^m}$:

$$\langle f, g \rangle_{H^m} := \sum_{|\alpha| \leq m} \langle D^\alpha f, D^\alpha g \rangle_{L^2} \text{ and norm } \| f \|_{H^m} := (\langle f, f \rangle_{H^m})^{1/2}.$$  

The usual Fourier transform defines a continuous linear operator $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}^n)$ and by duality on $\mathcal{S}'(\mathbb{R}^n)$. Fourier transform converts differentiation into multiplication. This allows us to extend the definition of $H^m(\mathbb{R}^d)$ from positive integers to all real values $s \in \mathbb{R}$:

$$H^s(\mathbb{R}^d) := \{ f \in \mathcal{S}'(\mathbb{R}^d) : (1 + \| \xi \|^2)^{s/2} \hat{f} \in L^2(\mathbb{R}^d) \},$$

where $\hat{f}$ denotes the Fourier transform of $f$ and $\| \xi \| = (\xi_1^2 + \cdots + \xi_n^2)^{1/2}$. Then $H^s(\mathbb{R}^d)$ is a Hilbert space where the inner product $\langle \cdot, \cdot \rangle_{H^s}$ is given as follows:

$$\langle f, g \rangle_{H^s} := \langle (1 + \| \xi \|^2)^{s/2} \hat{f}, (1 + \| \xi \|^2)^{s/2} \hat{g} \rangle_{L^2}.$$  

Moreover, we have that

$$H^s(\mathbb{R}^d)^* = H^{-s}(\mathbb{R}^d).$$

From (2.5) it is easy to see the following:

**Lemma 2.5.** For any $s \in \mathbb{R}$ and positive integer $k$, if $f \in H^s(\mathbb{R}^d)$ satisfies that for any differential operator $D^\alpha$ of order $|\alpha| \leq k$, $D^\alpha f \in H^s(\mathbb{R}^d)$ and $\| D^\alpha f \|_{H^s} \leq g(k)$ for some constant $g(k) > 0$, then $f \in H^{s+k}(\mathbb{R}^d)$ and $\| f \|_{H^{s+k}} \ll g(k)$ where the implicit constant only depends on $d$.

**Proof.** The assumption that $f \in H^s(\mathbb{R}^d)$ and $D^\alpha f \in H^s(\mathbb{R}^d)$ for any differential operator $D^\alpha$ of order $|\alpha| \leq k$ implies by definition that $f \in H^{s+k}(\mathbb{R}^d)$. Control of the $H^{s+k}$-norms is achieved by their definition and expanding $(1 + \| \xi \|^2)^{s/2+k/2}$ on the Fourier transform side. The latter becomes a sum of products $\xi_{i_1} \cdots \xi_{i_l}$ with $l \leq s + k$ and coefficients coming from the binomial theorem. Since all norms $\| f \|_{H^{s+l}}$, $l \leq s$ are bounded by $g(k)$, the claim is immediate.

We will further need the Sobolev embedding theorem:
**Theorem 2.6** (see [Zim90, Theorem 5.2.4]). For any \( k \in \mathbb{N} \), if \( f \in H^{k+1+d/2}(\mathbb{R}^d) \), then \( f \in C^k(\mathbb{R}^d) \) and for

\[
\|f\|_{C^k} \ll \|f\|_{H^{k+1+d/2}}
\]

where the implicit constant does not depend on \( f \).

2.4. **Convolution and Smoothing operators:** Below we will use a KAM scheme in our arguments. This will employ smoothing procedures using convolution. Let us briefly review some basic facts.

Let \( \phi \in C^\infty(\mathbb{R}^d) \) be a smooth function supported on \( B_1(0) \) with \( \int \phi \, dx = 1 \). For \( J > 1 \), let us denote \( \phi_J(x) := J^d \phi(Jx) \). For \( f \in L^2(\mathbb{R}^d) \), let us denote

\[
S_J f := f * \phi_J
\]

to be the convolution of \( f \) and \( \phi_J \). Then by standard facts on convolution (cf. e.g. [GS14, Lemma 2.3] and its proof), we have the following:

if \( f \in C^s(\mathbb{R}^d) \), then for any \( J > 0 \), any \( s' > 0 \) and \( 0 < s'' < s \),

\[
\|S_J f\|_{C^{s+s'}} \ll J^{d+s'} \|f\|_{C^s},
\]

and

\[
\|S_J f - f\|_{C^{s-s'}} \ll J^{d-s''} \|f\|_{C^s},
\]

where in both inequalities, the implicit constants only depend on \( \phi \) and \( s' \).

3. Outline of Argument

Let \( M = G/\Gamma \) be a compact nilmanifold with an action \( \rho_0 \) of \( \mathbb{Z}^r \) for \( r \geq 2 \) by automorphisms of \( G \), as in Theorem 1.1. Since \( \rho_0 \) does not admit an algebraic rank 1 factor, by Proposition 2.1, there exists a subgroup \( \Sigma \subset \mathbb{Z}^r \) isomorphic to \( \mathbb{Z}^2 \), such that every element in \( \Sigma \) acts ergodically on \( G/\Gamma \). Then by Proposition 2.2, there exists a constant \( \eta > 0 \) such that for any \( a \in \Sigma \), and any \( f, g \in C^\theta(G/\Gamma) \) with zero integrals,

\[
\langle f \circ a, g \rangle \ll e^{-\eta |a|} \|f\|_{C^\theta} \|g\|_{C^\theta}.
\]

We will use this exponential mixing on \( \Sigma \) to define distributions (generalized functions) which will converge in a suitable Sobolev space (dual to Hölder functions).

3.1. **Basic Overall KAM Strategy.** Let us first review the KAM approach to prove local rigidity. Rather than directly constructing a conjugacy, one successively finds conjugacies which bring the action closer to the given algebraic action with an error term that is quadratically small in the previous one. One shows that this process converges to the unperturbed action.

To be more precise, fix a right invariant Riemannian metric \( g_0 \) on \( G \) and endow \( G/\Gamma \) with the quotient metric, also denoted by \( g_0 \). Fix a finite generating set \( \Omega \) of \( \mathbb{Z}^r \). We consider a \( C^k \)-close \( C^\infty \) perturbation \( \rho \) of \( \rho_0 \). By this we mean any \( C^\infty \) action \( \rho \) of \( \mathbb{Z}^r \) such that for all \( a \in \Omega \), \( \rho(a) \) and \( \rho_0(a) \) are \( C^k \)-close for a \( k \) to be determined later.

The goal is to show that \( \rho \) is \( C^\infty \)-conjugate to \( \rho_0 \). Since \( G/\Gamma \) is compact, the injectivity radius \( i_0 \) of \( G/\Gamma \) is positive. We will first assume that for all \( a \in \Omega \), \( \rho(a) \) are \( 1/2 \) close to \( \rho_0(a) \) in the \( C^0 \)-metric (w.r.t. the given fixed metric \( g_0 \)).

For \( a \in \Omega \) and \( x \in G/\Gamma \), let us denote

\[
\rho(a)x = \exp(R(a,x))\rho_0(a)x
\]
where exp : g → G is the Lie theoretic exponential map and R(a, ·) : G/Γ → g is the unique error term with d(exp(R(a, x)), 1) < i₀/2. Note that R(a, ·) is a C∞-function on G/Γ. Let us consider h of the form

\[ h(x) = \exp(\omega(x))x, \]

where ω : G/Γ → g. Assume that h is a diffeomorphism. Then we obtain a new action by conjugating \( ρ_1 = h^{-1} \circ ρ \circ h \) with a new error term

\[ R^{(1)}(a, x) = \log(\rho_1(a)x(\rho_0(a)x)^{-1}) \]

for \( a \in \mathbb{Z}^r \) and \( x \in G/\Gamma \). We want to find a smooth ω such that the new error term \( R^{(1)} \) is much smaller than \( R \), for \( a \in Ω \).

By repeating this process, we will get a sequence of \( C^∞ \) conjugacies, \( \{h_1, h_2, \ldots, h_n, \ldots\} \). We then show that the limit

\[ h_∞ := \lim_{n \to \infty} h_1 \circ h_2 \circ \ldots \circ h_n \]

is a \( C^∞ \) diffeomorphism and that

\[ ρ_0 = h_∞^{-1} \circ ρ \circ h_∞. \]

It will clearly suffice to do this for \( a \in Ω \).

### 3.2. Exponential Mixing

The main difference to the more usual KAM approaches lies in combining the general overall strategy above with exponential mixing. This extends the use of Fourier series for tori from [DK10a]. However, it does not use the representation theory of nilpotent groups. Rather we use the exponential mixing results for commuting automorphisms of nilmanifolds [GS15, GS14]. There the Diophantine conditions are hidden as they follow from Diophantine properties of unstable eigenspaces of such automorphisms. In the end, they use the equidistribution properties of polynomial sequences from [GT12b] which depend on Diophantine conditions. Furthermore, in this paper we work with the Lie algebra rather than the nilpotent group. The main advantage is that the Lie algebra is a direct sum of the common generalized eigenspaces of the automorphisms.

Let us first set up some notation. For \( F : G/\Gamma \to g \cong \mathbb{R}^d \) and \( a \in \mathbb{Z}^r \), we set

\[ \Delta_a F = ρ_0(a)F - F \circ ρ_0(a). \]

For any two maps \( F_1, F_2 : G/\Gamma \to g \cong \mathbb{R}^d \) and \( a, b \in \mathbb{Z}^r \), we set

\[ L_{a,b}(F_1, F_2) := \Delta_b F_1 - \Delta_a F_2. \]

The induced action of \( ρ_0 \) on \( g \) admits the following decomposition into generalized eigenspaces:

\[ g = \bigoplus_{λ \in Φ} V_λ \]

where \( Φ \) is the collection of characters \( λ : \mathbb{Z}^r \to \mathbb{R} \) such that

\[ V_λ = \left\{ v \in g : \lim_{\|a\| \to \infty} \frac{\log \|ρ_0(a)v\| - λ(a)}{\|a\|} = 0 \right\} \]

is nontrivial. For a function \( F : G/\Gamma \to g \), let \( F_λ \) denote its projection onto \( V_λ \).

Recall that Proposition 2.1 gives a subgroup \( Σ \cong \mathbb{Z}^2 \) of \( \mathbb{Z}^r \) consisting of ergodic elements. For \( λ \in Φ \), we choose \( a_λ \in Σ \) such that

\[ 1 \leq \|a_λ\|_{V_λ} \leq e^{η/2}. \]

where

\[ exp : g \to G \]
We will work with the same elements \( a_\lambda \) at each step of the KAM iteration. This is fine since the condition on the \( a_\lambda \) is in terms of the unperturbed action \( \rho_0 \). Let \( \Omega_0 := \Omega \cup \{ a_\lambda | \lambda \in \Phi \} \). For any norm \( \| \cdot \| \) on functions on \( G/\Gamma \), set
\[
(3.6) \quad \| R \| := \max_{a \in \Omega_0} \| R(a, \cdot) \|
\]
where \( R(a, \cdot) \) is the error term defined as in equation 3.2. Denoting, as above, the projection to \( V_\lambda \) by \( R_\lambda(a_\lambda, \cdot) \), choose a large \( J > 1 \) and set
\[
(3.7) \quad S_R := S_J R_\lambda(a_\lambda, \cdot).
\]
Note that \( S_R \) still takes values in \( V_\lambda \). Then for any \( k' \in \mathbb{N} \), we get from the smoothing bound 2.8
\[
(3.8) \quad \| S_R \|_{C^{k+k'}} \ll J^{d+k'} \| R \|_{C^k}
\]
where the implied constants depend only on the smoothing and \( k' \).

**Remark:** We note that the implied constants may explode uncontrollably as \( k' \) increases. Therefore, we work with a fixed smoothness \( k + k' \), prove convergence of the KAM sequence in the \( C^{k+k'} \) topology. This will give us a conjugation \( \phi_{k+k'} \) in \( C^{k+k'} \). Since conjugations are unique up to affine maps by Proposition 2.3, the \( \phi_{k+k'} \) conjugacy will agree with \( \phi_{k+k'+1} \) up to an affine map. Hence \( \phi_{k+k'} \) is \( C^{k+k'+1} \), and thus \( C^\infty \) by induction.

Let us choose \( \epsilon = \min(\| R \|_{C^{0, \epsilon}}, \epsilon^{n/4} - 1) \) where \( n \) is chosen as in Equation 3.1. Then define the following:
\[
\omega_+^\lambda = - \sum_{i=0}^{\infty} \rho_0(-(i+1)a_\lambda) S_R \circ \rho_0(ia_\lambda) \quad \text{for} \quad \lambda(a_\lambda) \neq 0,
\]
and
\[
\omega_+^\lambda = - \sum_{i=0}^{\infty} (1 + \epsilon)^{-(i+1)} \rho_0(-(i+1)a_\lambda) S_R \circ \rho_0(ia_\lambda) \quad \text{for} \quad \lambda(a_\lambda) = 0;
\]
and
\[
\omega_-^\lambda = \sum_{i=-\infty}^{-1} \rho_0(-(i+1)a_\lambda) S_R \circ \rho_0(ia_\lambda) \quad \text{for} \quad \lambda(a_\lambda) \neq 0,
\]
and
\[
\omega_-^\lambda = \sum_{i=-\infty}^{-1} (1 + \epsilon)^{-(i+1)} \rho_0(-(i+1)a_\lambda) S_R \circ \rho_0(ia_\lambda) \quad \text{for} \quad \lambda(a_\lambda) = 0.
\]

At this point, these definitions are formal. We can however view them as distributions using the exponential mixing from Lemma 3.1. Moreover, each \( \omega_+^\lambda \) defines a Hölder function. Indeed, for \( \lambda(a_\lambda) > 0 \), the power series defining \( \omega_+^\lambda \) converges in the space of Hölder functions (for some exponent \( \theta \)). For \( \lambda(a_\lambda) = 0 \) this remains true thanks to the extra \( (1 + \epsilon)^{-(i+1)} \) factor on each term.

**Remark:** If we chose \( \epsilon = 0 \) in the latter case, we would still get distributions \( \omega_+^\lambda \) and \( \omega_-^\lambda \). However, in the proof of Lemma 3.1 we need to use exponential mixing which we only have for Hölder functions, not for distributions.

**Lemma 3.1.** We have
\[
\sum_{i=-\infty}^{\infty} (1 + \beta(\epsilon))^{-(i+1)} \rho_0(-(i+1)a_\lambda) S_R \circ \rho_0(ia_\lambda) = 0,
\]
where \( \beta(\epsilon) = \epsilon \) if \( \lambda(a_\lambda) > 0 \) and \( = 0 \) otherwise. Therefore we get \( \omega_+^\lambda = \omega_-^\lambda \), as distributions. Hence each \( \omega_-^\lambda \) is also a Hölder function. Henceforth we let \( \omega_\lambda = \omega_+^\lambda = \omega_-^\lambda \) and set \( \omega = \sum_{\lambda \in \Phi} \omega_\lambda = \sum_{\lambda \in \Phi} \omega_+^\lambda \).
Proof. By choice of \(a_\lambda\), each \(\omega^+_\lambda\) is a Hölder function. First note by telescoping sums that 
\[
\sum_{i=-N}^{N} (1 + \beta(\epsilon))^{-i-1} \rho_0(-(i+1)a_\lambda)SR_\lambda \circ \rho_0(ia_\lambda)
\]
\[
= (1 + \beta(\epsilon))^N \rho_0(Na_\lambda)\omega^+_\lambda \circ \rho_0(-Na_\lambda) - (1 + \beta(\epsilon))^{-N-1} \rho_0(-(N+1)a_\lambda)\omega^+_\lambda \circ \rho_0((N+1)a_\lambda) \to 0,
\]
by the exponential mixing of Proposition 2.2, choice of \(a_\lambda\) and since \(\omega^+_\lambda\) is Hölder .

\[\square\]

Lemma 3.2. Let \(S = \{X_1, \ldots, X_d\}\) be a basis of \(g\) such that every \(X_i\) belongs to some generalized eigenspace. Then every differential operator \(D^\alpha\) of order \(|\alpha|\) on \(M\) can be written as 
\[D^\alpha = \sum a_j D^{\alpha_i}\]
where every \(a_j\) is uniformly bounded and every \(D^{\alpha_i}\) is of the form \(D_{X_{1_i}} \cdots D_{X_{|\alpha|}}\) for suitable \(X_{1_i} \in S\).

Proof. This easily follows from an induction, expressing general vectorfields in terms of the \(X_j\). If 
\[X = \sum b_j X_j\]
and \(Y = \sum c_j X_j\) for suitable \(C^\infty\) functions \(b_j\) and \(c_j\), then \(D_X D_Y = \sum a_i(X_i(b_j)) D_Y + a_i b_j D_X D_X\).

The following lemma is the key to our main result:

Lemma 3.3. For any differential operator \(D^\alpha\), we have 
\[D^\alpha \omega_\lambda \in H^{-(2+\frac{d}{2})}(G/\Gamma).\]
Moreover, for any given \(k' > 0\) and any \(D^\alpha\) with order \(|\alpha| \leq k' - 1\),
\[\|D^\alpha \omega_\lambda\|_{H^{-(2+\frac{d}{2})}} \ll \|SR_\lambda\|_{C^{k'}}.\]

Proof. As the second statement implies the first, it suffices to prove the second statement. Moreover, note that it suffices to prove the statement for \(\omega_\lambda\) for any \(\lambda\).

By Lemma 3.2, it suffices to show the statement for \(D^\alpha\) of the form \(D_{X_{1_i}} \cdots D_{X_{|\alpha|}}\) with \(X_{1_i} \in V_{\lambda_i}\) and \(|\alpha| \leq k' - 1\). We can assume that 
\[|\prod_{j=1}^{l} X_j(a_{\lambda_j})| \leq 1,\]
without loss of generality. Indeed, if the opposite inequality holds, by Lemma 3.1 we can use the expression \(\omega^{-}_\lambda\) for \(\omega_\lambda\) and iterate backwards. Then for any \(f \in C^\infty(G/\Gamma)\) and any \(i \in \mathbb{Z}\), we have
\[\langle D^\alpha \omega_\lambda, f \rangle = -\sum_{i=0}^{\infty} (1 + \beta(\epsilon))^{-i-1} \langle D^\alpha \rho_0(-(i+1)a_\lambda)SR_\lambda \circ \rho_0(ia_\lambda), f \rangle.\]
As \(\rho_0(-(i+1)a_\lambda)\) acts on \(V_\lambda\) linearly, we have that
\[D^\alpha \rho_0(-(i+1)a_\lambda)SR_\lambda \circ \rho_0(ia_\lambda) = \rho_0(-(i+1)a_\lambda)\rho_0(ia_\lambda).\]
Note that \(D\rho_0(ia_\lambda)\) acts linearly on \(V_{X_{|\alpha|}}\) with generalized eigenvalue \(\chi^i_{|\alpha|}(\lambda)\). Then it follows from the Jordan normal form that 
\[\rho_0(ia_\lambda)X_{|\alpha|} = \chi^i_{|\alpha|}(\lambda) p_{|\alpha|}(i) Y^i_{|\alpha|}\]
where \(p_{|\alpha|}(i)\) is a polynomial of degree bounded by \(d\) and \(Y^i_{|\alpha|} \in V_{\chi^i_{|\alpha|}}\) is a vector of norm 1. Therefore
\[DX_{|\alpha|} \rho_0(ia_\lambda) = \chi^i_{|\alpha|}(\lambda) p_{|\alpha|}(i) (D_{Y^i_{|\alpha|}} \rho_0(ia_\lambda)).\]
By repeating this argument for $D_{X_{[\alpha]+1}}$, we get
\[ D^\alpha(SR_\lambda \circ \rho_0(i\alpha)) = \left( \prod_{j=1}^{[\alpha]} \chi_j(a_\lambda) \right)^i p_i(i)(D^\alpha_i SR_\lambda) \circ \rho_0(i\alpha), \]
where $p_i(i)$ is a polynomial with degree bounded by $d|\alpha|$ and $D^\alpha_i$ is a differential operator of order $|\alpha|$. Let us denote $\chi = \prod_{j=1}^{[\alpha]} \chi_j(a_\lambda)$. Noting that $|\chi| \leq 1$ and $||\rho_0(-(i+1)a_\lambda)||_{V_\lambda} \leq e^{i\eta/2}$ by Equation 3.1, we have for a Hohlder function $f$ on $M$ with $\int_M f = 0$
\[ ||(D^\alpha \omega_\lambda, f)|| \leq \sum_{i=0}^{\infty} ||\rho_0(-(i+1)a_\lambda)||_{V_\lambda} ||(D^\alpha (SR_\lambda \circ \rho_0(i\alpha)), f)|| \leq \sum_{i=0}^{\infty} e^{i\eta/2}|\chi|^i ||p_i(i)|| ||(D^\alpha_i SR_\lambda) \circ \rho_0(i\alpha), f)|| \leq \sum_{i=0}^{\infty} e^{i\eta/2}||p_i(i)||||D^\alpha_i SR_\lambda||_{C^0} ||f||_{C^0} \leq C||R_\lambda||_{C^{\bar{k}'}} ||f||_{C^0}.
\]
Combining this with the fact that $||f||_{C^0} \ll ||f||_{H^{(2+\frac{d}{2})}}$ by Theorem 2.6, we have that $D^\alpha \omega_\lambda \in H^{-(2+\frac{d}{2})}(G/\Gamma)$ and
\[ ||D^\alpha \omega_\lambda||_{H^{-(2+\frac{d}{2})}} \ll ||SR_\lambda||_{C^{\bar{k}'}}. \]

Lemma 3.3 easily implies the following

**Proposition 3.4.** The distribution $\omega$ is $C^\infty$. Letting $\sigma := (2+\frac{d}{2})+2d > 0$ we get for any $\bar{k}, \tilde{k} \geq 0$,
\[ ||\omega||_{C^{\bar{k}+\tilde{k}}} \ll J^{\sigma+\bar{k}} ||R||_{C^{\bar{k}}}. \]

We remark again that the implied constants can get worse. While they give us that $\omega$ is $C^\infty$ they will not be good enough to apply KAM for arbitrarily high derivatives forcing us later to work with fixed but arbitrary $\bar{k}, \tilde{k}$.

**Proof.** It suffices to prove the statement for $\omega_\lambda$.

By Lemma 3.3, we have that for any differential operator $D^\alpha$ of order $\leq \bar{k}'$, $D^\alpha \omega_\lambda \in H^{-(2+\frac{d}{2})}(G/\Gamma)$, and $||D^\alpha \omega_\lambda||_{H^{-(2+\frac{d}{2})}} \ll ||SR_\lambda||_{C^{\bar{k}'}}$. By Lemma 2.5, this implies that
\[ \omega_\lambda \in H^{\bar{k}'}(G/\Gamma), \text{ and } ||\omega_\lambda||_{H^{\bar{k}'}(G/\Gamma)} \ll ||SR_\lambda||_{C^{\bar{k}'}}. \]
Let us take $k' = \bar{k} + \tilde{k} + (2 + \frac{d}{2}) + d$, we have
\[ \omega_\lambda \in H^{k+k+d}, \text{ and } ||\omega_\lambda||_{H^{k+k+d}} \ll ||SR_\lambda||_{C^{k+k+d}}. \]
Since $||\omega_\lambda||_{C^{\bar{k}+\tilde{k}}} \leq ||\omega_\lambda||_{H^{k+k+d}}$, we have that
\[ \omega_\lambda \in C^{\bar{k}+\tilde{k}}(G/\Gamma), \text{ and } ||\omega_\lambda||_{C^{\bar{k}+\tilde{k}}} \ll ||SR_\lambda||_{C^{\bar{k}+\tilde{k}+(2+\frac{d}{2})+d}}. \]
This implies that $\omega_\lambda \in C^{\infty}(G/\Gamma)$.

By (3.8), $||SR_\lambda||_{C^{\bar{k}+\tilde{k}+(2+\frac{d}{2})+d}} \ll J^{\sigma+\bar{k}} ||R||_{C^{\bar{k}}}$ where $\sigma = (2+\frac{d}{2})+2d$. Thus, we have $||\omega_\lambda||_{C^{\bar{k}+\tilde{k}}} \ll J^{\sigma+\bar{k}} ||R||_{C^{\bar{k}}}$. This completes the proof.  \[ \square \]
We will now follow the basic KAM Strategy 3.1 and conjugate \( \rho \) by the candidate suggested by our previous discussion. Setting \( h(x) = \exp(\omega(x))x \). If \( ||\omega||_{C^1} \) is small enough then \( h \) is a diffeomorphism. That this can be arranged for a small enough perturbation follows from Proposition 3.4. We obtain a new action \( \rho_1 = h^{-1} \circ \rho \circ h \). We will estimate the new error term

\[
R^{(1)}(a, x) = \log(\rho_1(a)x(\rho_0(a)x^{-1})^{-1})
\]

for \( a \in \Omega_0 \) and \( x \in G/\Gamma \). As for the definition of the initial error term \( R(a, x) \), we assume that \( R^{(1)}(a, \cdot) : G/\Gamma \rightarrow g \) is the unique term with \( d(\exp(R^{(1)}(a, x)), 1) < i_0/2 \).

We first consider the \( C^0 \) norm of \( R^{(1)}(a, \cdot) \). First note that

\[
\rho_1(a)x = \exp(R^{(1)}(a, x))\rho_0(a)x,
\rho(a)x = \exp(R(a, x))\rho_0(a)x,
\]

\[
h(x) = \exp(\omega(x))x
\]

and \( h(\rho_1(a)x) = \rho(a)(h(x)) \). We get

\[
\exp(\omega(\rho_1(a)x))(\exp(R^{(1)}(a, x)))\rho_0(a)x = \exp(R(a, h(x))) \exp(\rho_0(a)\omega(x))\rho_0(a)x.
\]

Therefore,

\[
\exp(\omega(\rho_1(a)x))(\exp(R^{(1)}(a, x))) = \exp(R(a, h(x))) \exp(\rho_0(a)\omega(x)).
\]

Note that \( G \) is nilpotent. By taking logarithm on both sides, we have

\[
\omega(\rho_1(a)x) + R^{(1)}(a, x) + I = R(a, h(x)) + \rho_0(a)\omega(x) + II.
\]

Here, by the well-known Baker-Campbell-Hausdorff formula

\[
I = [\omega(\rho_1(a)x), R^{(1)}(a, x)] + \cdots
\]

has finitely many terms consisting of Lie brackets of \( \omega(\rho_1(a)x) \) and \( R^{(1)}(a, x) \), and

\[
II = [R(a, h(x)), \rho_0(a)\omega(x)] + \cdots
\]

similarly has finitely many terms consisting of Lie brackets of \( R(a, h(x)) \) and \( \rho_0(a)\omega(x) \). Note that \( ||\omega||_{C^0} \ll J^\sigma ||R||_{C^0} \) by Proposition 3.4 and the definition of \( R^{(1)}(a, \cdot) \). Since \( ||x, y|| \leq ||x||||y|| \) for any \( x, y \in g \), we have

\[
||I||_{C^0} \ll ||\omega||_{C^0} ||R^{(1)}(a, \cdot)||_{C^0} \ll J^\sigma ||R||_{C^0} ||R^{(1)}(a, \cdot)||_{C^0}
\]

and

\[
||II||_{C^0} \ll ||\omega||_{C^0} ||R||_{C^0} \ll J^\sigma ||R||_{C^0}^2.
\]

Given that \( ||R||_{C^0} \) is small enough, we get that

\[
||R^{(1)}(a, \cdot)||_{C^0} \ll ||R(a, h(x)) + \rho_0(a)\omega(x) - \omega(\rho_1(a)x)||_{C^0} + J^\sigma ||R||_{C^0}^2.
\]

Note that

\[
R(a, h(x)) + \rho_0(a)\omega(x) - \omega(\rho_1(a)x) = (\omega(\rho_0(a)x) - \omega(\rho_1(a)x)) + (R(a, h(x)) - R(a, x))
\]

\[+ (R(a, x) - \omega(\rho_0(a)x) + \rho_0(a)\omega(x)).
\]

Therefore,

\[
||R(a, h(x)) + \rho_0(a)\omega(x) - \omega(\rho_1(a)x)||_{C^0} \leq ||\omega(\rho_0(a)x) - \omega(\rho_1(a)x)||_{C^0} + ||R(a, h(x)) - R(a, x)||_{C^0}
\]

\[+ ||R(a, x) - \omega(\rho_0(a)x) + \rho_0(a)\omega(x)||_{C^0}.
\]

To sum up, we have
Lemma 3.5. We have
\[ \|R(1)(a, \cdot)\|_{C^0} \ll \|\omega(\rho_0(a)x) - \omega(\rho_1(a)x)\|_{C^0} + \|R(a, h(x)) - R(a, x)\|_{C^0} \]
\[ + \|R(a, x) - \omega(\rho_0(a)x) + \rho_0(a)\omega(x)\|_{C^0} + J^\sigma \|R\|_{C^0}^2. \]

Lemma 3.6.
\[ \|\omega(\rho_0(a)x) - \omega(\rho_1(a)x)\|_{C^0} \ll \|\omega\|_{C^1} \|R(1)(a, \cdot)\|_{C^0}. \]

Proof. The estimate easily follows from mean value theorem. In fact, by the mean value theorem,
\[ \|\omega(\rho_0(a)x) - \omega(\rho_1(a)x)\|_{C^0} \ll \|\omega\|_{C^1} d(\rho_0(a)x, \rho_1(a)x) \ll \|\omega\|_{C^1} \|R(1)(a, \cdot)\|_{C^0}. \]

By the mean value theorem, we also have the following estimate: Since we are close to the identity, the derivative of the exponential map can be controlled by a constant as close to 1 as we wish.

Lemma 3.7.
\[ \|R(a, h(x)) - R(a, x)\|_{C^0} \ll \|R\|_{C^1} \|\omega\|_{C^0}. \]

To estimate \(\|R(a, x) - (\omega(\rho_0(a)x) - \rho_0(a)\omega(x))\|_{C^0}\), we first consider its projections onto \(V_\lambda\):

Lemma 3.8.
\[ \|R_\lambda(a, x) - (\omega_\lambda(\rho_0(a)x) - \rho_0(a)\omega_\lambda(x))\|_{C^0} \ll J^{-k+\sigma} \|R\|_{C^k} + \|R\|_{C^{(2+\frac{3}{2})d+1}}^2. \]

Remark We need the following estimates to prove Lemma 3.8. We strongly use commutativity of the acting group here as it allows us to compare the error terms for the \(a_\lambda\) and \(a\). They are all quadratically small in the error terms of the previous step. This will force the sequence of actions in the KAM scheme to converge for all \(a \in \Omega_0\), and thus yield a \(C^k\) conjugacy between \(\rho\) and \(\rho_0\).

Lemma 3.9.
\[ \|L_{a_\lambda,a} \omega_\lambda(a_\lambda, \cdot), R_\lambda(a, \cdot)\|_{C^{(2+\frac{3}{2})d+1}} \ll \|R\|_{C^{(2+\frac{3}{2})d+1}}^2. \]

Proof. Note that \(\rho(a)\rho(a_\lambda)x = \rho(a_\lambda)\rho(a)x\). The left hand side equals
\[
\exp(R(a, \rho(a_\lambda)x))\rho_0(a_\lambda)x = \exp(R(a, \rho(a_\lambda)x))\rho_0(a_\lambda) \exp(R(a_\lambda, x))\rho_0(a_\lambda)x
\]
\[ = \exp(R(a, \rho(a_\lambda)x)) \exp(\rho_0(R(a_\lambda, x)))\rho_0(a_\lambda)\rho_0(a_\lambda)x. \]

Similarly, the right hand side equals
\[
\exp(R(a_\lambda, \rho(a)x)) \exp(\rho_0(a_\lambda)R(a, x))\rho_0(a_\lambda)\rho_0(a)x.
\]
Since \(\rho_0(a_\lambda)\rho_0(a)x = \rho_0(a)\rho_0(a)x\), we have
\[
\exp(R(a, \rho(a_\lambda)x)) \exp(\rho_0(R(a_\lambda, x)))\exp(\rho_0(a_\lambda)R(a, x)).
\]
By taking logarithm on both sides and applying the Baker-Campbell-Hausdorff formula, we get
\[ R(a, \rho(a_\lambda)x) + \rho_0(a)R(a_\lambda, x) + I = R(a_\lambda, \rho(a)x) + \rho_0(a_\lambda)R(a, x) + \Pi, \]
where
\[ I = [R(a, \rho(a_\lambda)x), \rho_0(a_\lambda)R(a_\lambda, x)] + \cdots \]
has finitely many terms of Lie brackets of \(R(a, \rho(a_\lambda)x)\) and \(\rho_0(a)R(a_\lambda, x)\), and
\[ \Pi = [R(a_\lambda, \rho(a)x), \rho_0(a_\lambda)R(a, x)] + \cdots \]
Let us denote 

\[ \tilde{L}_{a,\lambda}(R_{\lambda}(a, x), R_{\lambda}(a, x)) := R_{\lambda}(a, \rho(a)x) + \rho_0(a)R_\lambda(a, x) - R_\lambda(a, \rho(a)x) - \rho_0(a)R_\lambda(a, x). \]

Then

\[ \| \tilde{L}_{a,\lambda}(R_{\lambda}(a, x), R_{\lambda}(a, x)) \|_{C^{(2 + \frac{d}{2}) + d}} \ll \| R \|_{C^{(2 + \frac{d}{2}) + d}}^2. \]

Note that

\[ L_{a,\lambda}(R_{\lambda}(a, x), R_{\lambda}(a, x)) = \Delta_{a} R_{\lambda}(a, x) - \Delta_{a,\lambda} R_{\lambda}(a, x) \]

\[ = \rho_0(a)R_\lambda(a, x) - R_\lambda(a, \rho_0(a)x) - \rho_0(a)R_\lambda(a, x) + R_\lambda(a, \rho_0(a)x) \]

\[ = \tilde{L}_{a,\lambda}(R_{\lambda}(a, x), R_{\lambda}(a, x)) + (R_{\lambda}(a, \rho(a)x) - R_\lambda(a, \rho_0(a)x)) \]

\[ + (R_\lambda(a, \rho_0(a)x) - R_\lambda(a, \rho(a)x)). \]

By the mean value theorem, we have

\[ \| R_{\lambda}(a, \rho(a)x) - R_\lambda(a, \rho_0(a)x) \|_{C^{(2 + \frac{d}{2}) + d}} \ll \| R \|_{C^{(2 + \frac{d}{2}) + d}} \| R \|_{C^{(2 + \frac{d}{2}) + d + 1}} \leq \| R \|_{C^{(2 + \frac{d}{2}) + d + 1}}^2 \]

and

\[ \| R_{\lambda}(a, \rho_0(a)x) - R_\lambda(a, \rho(a)x) \|_{C^{(2 + \frac{d}{2}) + d}} \ll \| R \|_{C^{(2 + \frac{d}{2}) + d}} \| R \|_{C^{(2 + \frac{d}{2}) + d + 1}} \leq \| R \|_{C^{(2 + \frac{d}{2}) + d + 1}}^2. \]

Combining these estimates with

\[ \| \tilde{L}_{a,\lambda}(R_{\lambda}(a, x), R_{\lambda}(a, x)) \|_{C^{(2 + \frac{d}{2}) + d}} \ll \| R \|_{C^{(2 + \frac{d}{2}) + d}}^2, \]

and applying the triangle inequality, we complete the proof.

The above lemma implies the following:

**Lemma 3.10.** For any \( \lambda \) and any \( a \in \Delta_0 \) and for any sufficiently large \( k \) (coming from the KAM scheme on p. 6) we have

\[ \| L_{a,\lambda}(SR_\lambda, R_{\lambda}(a, \cdot)) \|_{C^{(2 + \frac{d}{2}) + d}} \ll J^{-k + \sigma} \| R \|_{C^k} + \| R \|_{C^{(2 + \frac{d}{2}) + d + 1}}^2. \]

**Proof.** First we note that

\[ L_{a,\lambda}(SR_\lambda, R_{\lambda}(a, \cdot)) = \Delta_{a} SR_\lambda - \Delta_{a,\lambda} R_{\lambda}(a, \cdot) \]

\[ = \Delta_{a} R_{\lambda}(a, \cdot) - \Delta_{a,\lambda} R_{\lambda}(a, \cdot) - (\Delta_{a}(R_{\lambda}(a, \cdot) - SR_\lambda)) \]

\[ = L_{a,\lambda}(a, \rho(a)x) - \Delta_{a}(R_{\lambda}(a, \cdot) - SR_\lambda). \]

We first estimate \( \| \Delta_{a}(R_{\lambda}(a, \cdot) - SR_\lambda) \|_{C^{(2 + \frac{d}{2}) + d}} \) by applying equation 2.9 with \( s = k \) and \( s'' = k - \sigma + d \).

Note that \( s'' < s \) since \( \sigma = (2 + \frac{d}{2}) + 2d \):

\[ \| \Delta_{a}(R_{\lambda}(a, \cdot) - SR_\lambda) \|_{C^{(2 + \frac{d}{2}) + d}} \ll \| R_{\lambda}(a, \cdot) - SR_\lambda \|_{C^{(2 + \frac{d}{2}) + d}} \ll J^{-k + \sigma} \| R \|_{C^k}. \]

Then let us consider \( \| L_{a,\lambda}(R_{\lambda}(a, \cdot), R_{\lambda}(a, \cdot)) \|_{C^{(2 + \frac{d}{2}) + d}} \). By Lemma 3.9, we have

\[ \| L_{a,\lambda}(SR_\lambda, R_{\lambda}(a, \cdot)) \|_{C^{(2 + \frac{d}{2}) + d}} \ll \| R \|_{C^{(2 + \frac{d}{2}) + d + 1}}^2. \]
This implies that
\[ \| L_{\alpha,a}(SR_{\lambda}, R_{\alpha}(a, \cdot)) \|_{C^{(2 + \frac{d}{2}) + d}} \ll J^{-k + \sigma} \| R \|_{C^{k}} + \| R \|_{C^{(2 + \frac{d}{2}) + d+1}}^{2}. \]

\[ \square \]

**Proof of Lemma 3.8.** First note that \( \omega_{\lambda}(\rho_{0}(a)x) - \rho_{0}(a)\omega_{\lambda}(x) = -\Delta_{a}\omega_{\lambda}, \) and
\[ R_{\lambda}(a, x) = \sum_{i=0}^{\infty} (1 + \beta(\epsilon))^{-i-1} \rho_{0}(-(i+1)a_{\lambda}) \Delta_{a}R_{\lambda}(a, \cdot) \circ \rho_{0}(ia_{\lambda}), \]
where
\[ \tilde{\Delta}_{a}R_{\lambda}(a, \cdot) = (1 + \beta(\epsilon))\rho_{0}(a_{\lambda})R_{\lambda}(a, \cdot) - R_{\lambda}(a, \cdot) \circ \rho_{0}(a_{\lambda}) = \Delta_{a}\lambda R_{\lambda}(a, \cdot) + \beta(\epsilon)\rho_{0}(a_{\lambda})R_{\lambda}(a, \cdot). \]
Then the quantity in the statement of Lemma 3.7 is equal to
\[ R_{\lambda}(a, x) + \Delta_{a}\omega_{\lambda} = R_{\lambda}(a, x) - \sum_{i=0}^{\infty} (1 + \beta(\epsilon))^{-i-1} \rho_{0}(-(i+1)a_{\lambda}) \Delta_{a}SR_{\lambda} \circ \rho_{0}(ia_{\lambda}) \]
\[ = \sum_{i=0}^{\infty} (1 + \beta(\epsilon))^{-i-1} \rho_{0}(-(i+1)a_{\lambda}) (\Delta_{a}\lambda R_{\lambda}(a, \cdot) - \Delta_{a}SR_{\lambda}) \circ \rho_{0}(ia_{\lambda}) \]
\[ = \sum_{i=0}^{\infty} (1 + \beta(\epsilon))^{-i-1} \rho_{0}(-(i+1)a_{\lambda}) (L_{a,a\lambda}(R_{\lambda}(a, \cdot), SR_{\lambda}) + \beta(\epsilon)\rho_{0}(a_{\lambda})R_{\lambda}(a, \cdot)) \circ \rho_{0}(ia_{\lambda}). \]

Using exponential mixing property of \( \rho_{0}(a_{\lambda}) \) (Proposition 2.2) applied to the series in the last equation, we can use the same argument as the proof of Lemma 3.12 to show that for any differential operator \( D^{\alpha} \) of order \( \leq (2 + \frac{d}{2}) + d \), we have
\[ \| D^{\alpha}(R_{\lambda}(a, \cdot)) + \Delta_{a}\omega_{\lambda} \|_{H^{-2}(2+\frac{d}{2})} \ll \| L_{a,a}(R_{\lambda}(a, \cdot), SR_{\lambda}) + \beta(\epsilon)\rho_{0}(a_{\lambda})R_{\lambda}(a, \cdot) \|_{C^{(2 + \frac{d}{2}) + d}} \]

Note that \( |\beta(\epsilon)| \leq \| R \|_{C^{0}} \leq \| R \|_{C^{(2 + \frac{d}{2}) + d}}. \) Then we have that
\[ \| R_{\lambda}(a, \cdot) + \Delta_{a}\omega_{\lambda} \|_{H^{d}} \ll \| L_{a,a\lambda}(R_{\lambda}(a, \cdot), SR_{\lambda}) \|_{C^{(2 + \frac{d}{2}) + d}} + \| R \|_{C^{(2 + \frac{d}{2}) + d}}^{2}. \]

Therefore,
\[ \| R_{\lambda}(a, \cdot) + \Delta_{a}\omega_{\lambda} \|_{C^{0}} \ll \| L_{a,a\lambda}(R_{\lambda}(a, \cdot), SR_{\lambda}) \|_{C^{(2 + \frac{d}{2}) + d}} + \| R \|_{C^{(2 + \frac{d}{2}) + d}}^{2}. \]

By Lemma 3.10, we get (subsuming a factor 2 in \( \| R \|_{C^{(2 + \frac{d}{2}) + d+1}}^{2} \) in the implicit constant)
\[ \| R_{\lambda}(a, x) - (\omega_{\lambda}(\rho_{0}(a)x) - D\rho_{0}(a)\omega_{\lambda}(x)) \|_{C^{0}} \ll J^{-k + \sigma} \| R \|_{C^{k}} + \| R \|_{C^{(2 + \frac{d}{2}) + d+1}}^{2}. \]

\[ \square \]

**Proposition 3.11.** There exists a constant \( c > 0 \) such that if \( \| \omega \|_{C^{1}} \leq c \), then
\[ \| R^{(1)}(a, \cdot) \|_{C^{0}} \ll J^{-k + \sigma} \| R \|_{C^{k}} + J^{\sigma} \| R \|_{C^{(2 + \frac{d}{2}) + d+1}}^{2}. \]

**Proof.** By Lemma 3.5, we have
\[ \| R^{(1)}(a, \cdot) \|_{C^{0}} \ll \| \omega(\rho_{0}(a)x) - \omega(\rho_{1}(a)x) \|_{C^{0}} + \| R(a, h(x)) - R(a, x) \|_{C^{0}} \]
\[ + \| R(a, x) - \omega(\rho_{0}(a)x) + \rho_{0}(a)\omega(x) \|_{C^{0}} + J^{\sigma} \| R \|_{C^{0}}^{2}. \]
By estimates we get from Lemmas 3.6, 3.7, and 3.8, we have

\[ \|R^{(1)}(a, \cdot)\|_{C^0} \ll J^\sigma \|R\|_{C^0}^2 + \|\omega\|_{C^1}\|R^{(1)}(a, \cdot)\|_{C^0} + \|R\|_{C^1}\|\omega\|_{C^0} + J^{-k+\sigma}\|R\|_{C^k} + \|R\|_{C^{(2+\frac{d}{2})+d+1}}^2. \]

Let \( C > 0 \) denote the implicit constant in the estimate above. Then if \( \|\omega\|_{C^1} \leq C/2 \), we have

\[ 1/2\|R^{(1)}(a, \cdot)\|_{C^0} \ll J^{-k+\sigma}\|R\|_{C^k} + J^\sigma\|R\|_{C^0}^2 + \|R\|_{C^{(2+\frac{d}{2})+d+1}}^2 + \|R\|_{C^1}\|\omega\|_{C^0}. \]

Note that \( \|\omega\|_{C^0} \ll J^\sigma\|R\|_{C^0} \). We have

\[ \|R^{(1)}(a, \cdot)\|_{C^0} \ll J^{-k+\sigma}\|R\|_{C^k} + \|R\|_{C^{(2+\frac{d}{2})+d+1}}^2 + J^\sigma\|R\|_{C^1}\|R\|_{C^0} \]

\[ \ll J^{-k+\sigma}\|R\|_{C^k} + J^\sigma\|R\|_{C^{(2+\frac{d}{2})+d+1}}^2. \]

This completes the proof. \( \square \)

We then consider the \( C^{k+\bar{k}} \) norm of \( h \) and \( R^{(1)}(a, \cdot) \) for any given \( \bar{k}, \bar{k} \geq 0 \). It is here that we use crucially that the underlying group is nilpotent.

**Lemma 3.12.** Let \( \bar{k}, \bar{k} \geq 0 \) be nonnegative integers. We have

\[ \|h\|_{C^{\bar{k}+\bar{k}}} \ll J^{\bar{k}+\bar{k}}\|R\|_{C^{\bar{k}}} \]

and for \( a \in \Omega_0 \)

\[ \|R^{(1)}(a, \cdot)\|_{C^{\bar{k}+\bar{k}}} \leq J^{\bar{k}+\bar{k}}\|R\|_{C^{\bar{k}}}. \]

**Proof.** Note that we use coordinate system on \( g \). Since \( G \) is nilpotent, the product

\[ \exp(\omega(x))x = \exp(P(\omega(x), \log x)) \]

where \( P(\cdot) \) is a polynomial in coordinates of \( \omega(x) \) and \( \log x \). Hence we get the \( C^{k+\bar{k}} \) norm of \( \log h(x) = P(\omega(x), \log x) \) is \( \ll \|\omega\|_{C^{k+\bar{k}}} \). By Proposition 3.4, we can estimate the last term further by \( \ll J^{\bar{k}+\bar{k}}\|R\|_{C^{\bar{k}}} \).

Then since \( p_1 = h^{-1} \circ \rho \circ h \) and \( R^{(1)}(a, x) = \log((p_1(a)x)^{-1}p_0(a)x) \), we can use the same argument as in the error estimate for the inductive step [DK10b, Section 5.2] to conclude that for \( a \in \Omega_0 \)

\[ \|R^{(1)}(a, \cdot)\|_{C^{\bar{k}+\bar{k}}} \ll J^{\bar{k}+\bar{k}}\|R\|_{C^{\bar{k}}}. \]

\( \square \)

4. KAM Iteration

In this section, we will use a fairly standard KAM scheme and the estimates from previous section to finish the proof.

Starting from \( \rho, \) we can use the previous argument to construct \( \omega_1(x), h_1(x) = \exp(\omega_1(x))x \) and \( p_1 = h_1^{-1} \circ \rho \circ h_1 \). For \( n \geq 1 \), suppose \( \rho_n \) has been constructed. Then by repeating the same construction with \( \rho \) replaced by \( \rho_n \), we get \( \omega_{n+1}(x), h_{n+1}(x) = \exp(\omega_{n}(x))x, \) and \( \rho_{n+1} = h^{-1}_{n+1} \circ \rho_n \circ h_{n+1} \).

For any fixed \( k' \geq 0 \), let us denote \( \epsilon_n = \epsilon^{(3/2)^n}, k_n = \min\{k + n, k + k'\} \). Suppose \( \rho_n(a)x = \exp(R^{(n)}(a, x))p_0(a)x \) and we have the following estimates:

\[ ||R^{(n)}||_{C^0} \ll \epsilon_n; \]  
\[ ||R^{(n)}||_{C^{k_n}} \ll \epsilon_n^{-1}. \]
Then by interpolation (Lemma 2.4), we have for $k > (2 + \frac{d}{2}) + d + 1$
\[ \|R^{(n)}\|_{C^0} \leq \|R^{(n)}\|_{C^0}^{(k - (2 + \frac{d}{2}) - d - 1)/k_n} \|R^{(n)}\|_{C^0}^{((2 + \frac{d}{2}) + d + 1)/k_n} \ll \epsilon_n^{1 - 2((2 + \frac{d}{2}) + d + 1)/k_n}. \]

If $k$ is even larger such that $k \geq 20((2 + \frac{d}{2}) + d + 1)$, we get
\[(4.3) \quad \|R^{(n)}\|_{C^0} \ll \epsilon_n^{3/10}. \]

Let us define $J_n = \epsilon_n^{\frac{5}{2(k_n - \sigma)}}$. Let us define $ω_n(x)$ using the construction from the previous section with $R(a, x)$ being $R^{(n)}(a, x)$ and $J$ being $J_n$, and define $h_{n+1}(x) = \exp(ω_n(x))x$, $ρ_{n+1} = h_{n+1}^{-1} \circ ρ_n \circ h_{n+1}$ and $R^{(n+1)}(a, x) = \log((ρ_{n+1}(a)x)^{-1}ρ_0(a)x)$. Then by Proposition 3.11 applied with $R$ being $R^{(n)}$, $k$ being $k_n$, we have that
\[ \|R^{(n+1)}\|_{C^0} \ll J_n^\sigma \|R^{(n)}\|^2_{C^0} + J_n^{-k_n + \sigma} \|R^{(n)}\|_{C^0}. \]

It is easy to see that
\[ J_n^{-k_n + \sigma} \|R^{(n)}\|_{C^0} \ll \epsilon_n^{-1 + 5/2} = \epsilon_n^{\frac{3}{2}}. \]

Given $k$ large enough such that $k \geq 101\sigma$, we have
\[ J_n^\sigma \|R^{(n)}\|^2_{C^0} \ll \epsilon_n^{\frac{18}{10} - 1/10} \leq \epsilon_n^{\frac{3}{2}}. \]

Therefore, we have
\[(4.4) \quad \|R^{(n+1)}\|_{C^0} \ll \epsilon_n^{\frac{3}{2}} = \epsilon_n^{1}. \]

By Lemma 3.12 with $R$ being $R^{(n)}$, $J$ being $J_n$, and $(\tilde{k}, \tilde{k})$ being $(k_{n+1} - k_n, k_n)$ (note that $k_{n+1} - k_n = 0$ or 1), we have that
\[ \|R^{(n+1)}\|_{C^0} \ll J_n^{\tilde{k} + 1} \|R^{(n)}\|_{C^0}. \]

Given that $k$ is large enough such that $k \geq 101(\sigma + 1)$, we have
\[ \|R^{(n+1)}\|_{C^0} \ll \epsilon_n^{-1 - 1/2} = \epsilon_n^{1}. \]

This shows that (4.1) and (4.2) hold for every $n \in \mathbb{N}$.

For any $0 < \ell \leq \frac{k + k'}{200}$ and $n \geq k'$,
\[ \|R^{(n)}\|_{C^{\ell}} \ll \|R^{(n)}\|_{C^0}^{(k_n - \ell)/k_n} \|R^{(n)}\|_{C^0}^{\ell/k_n} \ll \epsilon_n^{1 - 2\ell/k_n} \ll \epsilon_n^{1/4}. \]

By Proposition 3.4, we have
\[ \|\omega_n\|_{C^{\ell}} \ll J_n^{\tilde{k} + 1} \|R^{(n)}\|_{C^{\ell}} \leq \epsilon_n^{1/8}. \]

Since $h_{n+1}(x) = \exp(\omega_n(x))x$,
\[ \log(h_{n+1}(x)) = \log x + \omega_n(x) + [\Box], \]

where $[\Box]$ consists of finitely many terms of Lie brackets of $\log x$ and $\omega_n(x)$. This implies that
\[ \|h_{n+1} - \text{Id}\|_{C^{\ell}} \ll \|\omega_n\|_{C^{\ell}} \ll \epsilon_n^{1/8}. \]

This implies that the limit $h_{k', \infty} = \lim_{n \to \infty} h_{k' + 1} \circ \cdots \circ h_{n+1}$ converges in $C^{\ell}(G/T)$. Since $h_n \in C^\infty(G/T)$ for all $n \in \mathbb{N}$, we have that $h_\infty = h_1 \circ \cdots \circ h_{k'} \circ h_{k', \infty} \in C^{\ell}(G/T)$. Since $\|R^{(n)}\|_{C^0} \to 0$, we have that
\[ \rho_0 = h_\infty^{-1} \circ \rho \circ h_\infty. \]

By repeating the above KAM scheme with larger $k'$, we can get a conjugacy $h_\infty$ with arbitrarily high regularity. Since such a conjugacy is unique up to an affine automorphism by Proposition 2.3, we have that $h_\infty \in C^\infty(G/T)$ and thus finish the proof.
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