ON THE HOFER GEOMETRY INJECTIVITY RADIUS CONJECTURE

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ABSTRACT. We verify here a version of the injectivity radius conjecture in Hofer geometry in the case of Ham($S^2$, $\omega$) and Ham($\Sigma$, $\omega$), for $\Sigma$ a closed genus at least 1 surface, and partly verify a version of the conjecture stated in [9], for Ham($S^2$, $\omega$). In particular we show that the inclusion map of any sufficiently small Hofer metric epsilon-ball into Ham($S^2$, $\omega$) is $k$-connected for any $k$, (depending on the size of the ball). Our theorems have some surprising dynamical consequences. Here is one: Suppose $\phi \in$ Ham($S^2$, $\omega$) is non-degenerate with exactly 2 fixed points, and Hofer $\delta'$ close to $id$, then there are no non-zero index contractible Ustilovsky geodesics $\gamma$ from $id$ to $\phi$, for the positive Hofer length functional, with length less than area($S^2$, $\omega$) $- \delta$, for any $0 < \delta' < \delta < \text{area}(S^2, \omega)$, where contractible means: for some path $p$ from $\phi$ to $id$, with length less than $\delta$, the concatenation $p \cdot \gamma$ is contractible. We also discuss a parallel application of Floer theory to the study of the topology of non-degenerate area preserving diffeomorphisms of $S^2$, and this leads to another partial verification of a version of the injectivity radius conjecture.

1. Introduction

1.1. Injectivity radius conjecture. One of the most fundamental objects associated to a Finsler manifold is its injectivity radius function. For the Finsler group Ham($M, \omega$) with its Hofer metric, despite this metric having no well defined exponential map, we may still ask for a kind of injectivity, respectively weak injectivity radius, by asking for the size of the largest metric epsilon-ball which is contractible, respectively has null-homotopic inclusion map into the total space Ham($M, \omega$). The group Ham($M, \omega$) and all the associated spaces are always considered with respect to $C^\infty$ topology.

In this formulation we studied the question of injectivity radius for Ham($M, \omega$) in [9], and formulated there the conjecture that the injectivity/weak injectivity radius of Ham($M, \omega$) is positive. In this note by Hofer length we shall exclusively mean the length functional obtained from the $L^\infty$ norm on lieHam($M, \omega$), as opposed to the oscillation norm, see Section 2.1. However to avoid confusion we shall use the name $L^\infty$ Hofer length functional.

The problem of injectivity radius out to be a very difficult and interesting problem even for $M = S^2$. Here we partly verify the weak injectivity radius conjecture for Ham($S^2$, $\omega$), and verify another version of the injectivity radius conjecture which sits roughly between the two versions mentioned above. The main ingredient other than classical technology of Gromov-Witten theory, and the theory of coupling forms of [4], is a kind of “volume” flow for connection type symplectic forms on surface Hamiltonian fibrations over $S^2$, which exists in the presence of certain foliations by holomorphic curves, using which we also obtain a curve shortening
algorithm. It is interesting to speculate whether this is at all related to symplectic Ricci type flows, but to emphasize our "flow" is of very elementary nature.

Using these developments we obtain some dynamical applications, which are formulated in terms of certain non-existence results of non-zero index Ustilovsky geodesics, and this uses the author's (virtual) Morse theory for the Hofer length functional.

Set $h = \text{area}(S^2, \omega)$, and let $B_{\delta,N}$ denote the open metric ball $B(id, h/N - \delta)$, for a $0 < \delta < h/N$, with respect to the $L^\infty$ Hofer length functional, considered with respect to the induced $C^\infty$ topology, (note that it is then a Frechet manifold since the metric ball is open in the $C^\infty$ topology). We say that a map $f : X \to Y$ is $k$-connected if it is vanishing on homotopy groups up to and including degree $k$.

**Theorem 1.1.** If $N \geq 27$ then the inclusion map of $B_{\delta,N}$ into $\text{Ham}(S^2, \omega)$ is 3-connected, in particular vanishes on $\pi_3$. Moreover for any $k > 0$ there is an $N_k$ so that the inclusion map of $B_{\delta,N_k}$ into $\text{Ham}(S^2, \omega)$ is $k$-connected.

That the inclusion map $B_{\delta,2} \to \text{Ham}(S^2, \omega)$ vanishes on $\pi_3$ is shown in [9], note that this also requires Gromov-Witten theory. And of course $\pi_2(\text{Ham}(S^2, \omega)) = \pi_2(SO(3)) = 0$ so that the first part of the theorem is just about $\pi_3$. Note that by Hurewitz theorem this implies in particular that the inclusion map of $B_{\delta,N}$ into $\text{Ham}(S^2, \omega)$ vanishes on $H_3$ with $\mathbb{Z}$ coefficients. In [9] we showed that the inclusion map cannot hit the $\mathbb{Z}$ generator of $H_3(\text{Ham}(S^2, \omega), \mathbb{Z})$, which can be proved by a much more elementary argument, but fails for the even multiples of the generator of $H_3(\text{Ham}(S^2, \omega), \mathbb{Z})$.

Let $\Omega^c \text{Ham}(M, \omega)$, respectively $\Omega^{c+} \text{Ham}(M, \omega)$ denote the space of contractible loops based at $id$, with Hofer $L^\infty$ length, respectively positive Hofer length less than $c$.

**Conjecture 1.2.** Let $\text{injrad}_{\Omega}(M, \omega)$, respectively $\text{injrad}_{\Omega^+}(M, \omega)$ be the supremum over $c$ so that

$$\Omega^c \text{Ham}(M, \omega),$$

respectively

$$\Omega^{c+} \text{Ham}(M, \omega)$$

is intrinsically contractible, then $\text{injrad}_{\Omega}(M, \omega) > 0$ and $\text{injrad}_{\Omega^+}(M, \omega) > 0$.

Let us call these the $\Omega$, respectively $\Omega^+$ injectivity radius conjectures.

**Theorem 1.3.** The $\Omega^+$ and $\Omega$ injectivity radius conjectures hold for $M = S^2$, and $M = \Sigma$ a genus at least 1 closed Riemann surface. Specifically

\begin{align}
\text{injrad}_{\Omega}(S^2) &= \text{injrad}_{\Omega^+}(S^2) = h, \\
\text{injrad}_{\Omega}(\Sigma) &= \text{injrad}_{\Omega^+}(\Sigma) = \text{injrad}(\Sigma) = \infty.
\end{align}

We can readily see that $\text{injrad}_{\Omega}(S^2, \omega) \leq h$ as we have a natural representative $f_{\min}$ of $\pi_2(\text{OmHam}(S^2, \omega))$ all of whose loops have $L^\infty$ Hofer length at most $h$. Let us make this more explicit. Take the natural Lie group homomorphism $\tilde{f} : S^3 \to \text{Ham}(S^2, \omega)$, representing the generator of $\pi_3(\text{Ham}(S^2, \omega))$. Deloop this to a map $f_{\min} : S^2 \to \text{OmHam}(S^2, \omega)$, by taking the natural $S^2$ family of based at $id$ loops on $S^3$, forming the unstable manifold for a great index 2 geodesic $\gamma$ in $S^3$ (for the Riemannian bi-invariant metric), in other words a simple great geodesic. Then $f(\gamma)$ is the longest loop in the image of $S^2$, and its positive Hofer, and $L^\infty$ Hofer
length is $2\hbar/2 = \hbar$. Consequently (1.1) follows once we show that $\Omega^{\hbar}\text{Ham}(S^2, \omega)$ is contractible.

We also get:

**Corollary 1.4.** The map $f_{\min}$ cannot be homotoped into $\Omega^{\hbar}+\text{Ham}(S^2, \omega)$.

We also proved this using a more general theory of quantum characteristic classes in [18]. Theorem 1.1 is deduced from the Theorem 1.3. For a general $(M, \omega)$ the strong injectivity radius conjecture clearly implies the $\Omega$ injectivity radius conjecture, which in turn by the proof of Theorem 1.1 implies the partial weak injectivity radius conjecture clearly implies the $\Omega$ injectivity radius conjecture stating that the inclusion of a $L^\infty$ Hofer metric $r_k$-ball into $\text{Ham}(M, \omega)$ is $k$-connected, if the radius $r_k$ of the ball is small enough. The weak injectivity radius conjecture does not in general imply the other versions.

1.1.1. Dynamical consequences. In [8] Lalonde and McDuff show that there elements $\phi \in \text{Ham}(S^2, \omega)$ which cannot be connected to $id$ via a locally (in the path space) minimizing geodesic. Their notion of a geodesic is somewhat more general than the classical notion coming from calculus of variations. But in particular this implies that for some $\phi$ there is no Morse index 0 geodesic $\gamma$ in the path space from $id$ to $\phi$: $P(\phi)$, for $\gamma$ a differentiable critical point of the Hofer length functional: an Ustilovsky geodesic. The Morse index always makes sense for an Ustilovsky geodesic and is finite as shown in [20], where the index is also related to the Conley-Zehnder index. For what follows we also need that our Ustilovsky geodesics are robust, which is a mild Floer theoretical assumption, which we review in Section 2.3.

**Theorem 1.5.** Suppose $\phi \in \text{Ham}(S^2, \omega)$ is Hofer $\delta'$ close to $id$, then there are no non-zero index robust contractible Ustilovsky geodesics from $id$ to $\phi$, for the positive Hofer length functional, with length less than $\hbar - \delta$, for any $0 < \delta' < \delta < \hbar$, where contractible means: for some path $p$ from $\phi$ to $id$, with length less than $\delta$, the concatenation $p \cdot \gamma$ is contractible. The same statement holds for $\phi \in \text{Ham}(\Sigma, \omega)$ with no restrictions on $\phi$, (and so without contractible assumption).

This readily follows by the author’s [19, Theorem 1.6] and Theorem 1.3. Let us be a bit more explicit in the case of $\text{Ham}(S^2, \omega)$. If such an index $k > 0$ geodesic $\gamma$ exists then by [19, Theorem 1.6] the group

$$\pi_{k-1}(P^{h-\delta, +}(\phi'), P^{h-\delta-E, +}(\phi')) \neq 0,$$

for some $\phi'$ arbitrarily $C^\infty$ close to $\phi$, $E > 0$, and $P^{c, +}(\phi')$ denoting the space of paths with positive Hofer length less than $c$. Consequently using the contractability hypothesis we get:

$$\pi_{k-1}(\Omega^{h-\delta+\delta', +}\text{Ham}(S^2, \omega), \Omega^{h-\delta-E+\delta', +}\text{Ham}(S^2, \omega)) \neq 0,$$

but this contradicts Theorem 1.3. A small caveat is that we state [19, Theorem 1.6] for monotone symplectic manifolds, however this is just an unfortunate omission, the argument works perfectly well for aspherical symplectic manifolds. In particular the case of $\text{Ham}(\Sigma, \omega)$ likewise follows.

**Remark 1.6.** We may heuristically interpret the statement for $\text{Ham}(\Sigma, \omega)$ as the statement that the “curvature” of $\text{Ham}(\Sigma, \omega)$ with respect to the positive Hofer length functional is non-positive, see Milnor’s [15] for background on connections of curvature and topology of loop spaces. Similarly we may interpret the statement for $\text{Ham}(S^2, \omega)$ as a certain upper bound on the positivity of “curvature”. Thus these
results on geodesics are qualitatively very different from Lalonde-McDuff’s result, which just says that $\text{Ham}(S^2, \omega)$, is far from being a geodesic metric space. However what is “curvature”? We can try to think of this in terms of coarse geometry, the dream is something like:

**Conjecture 1.7.** The space $\text{Ham}(\Sigma, \omega)$ is a rough $\text{CAT}(0)$ space and $\text{Ham}(S^2, \omega)$ is a rough $\text{CAT}(k)$ space for some $k > 0$, see [2], with respect to the positive and/or $L^\infty$ Hofer metric.

Let us also point out that as a very special case that if $\phi$ has only even CZ index, non-degenerate fixed points, than an Ustilovsky geodesic $\gamma$ is automatically robust.

Here is then a more elementary corollary stated in the abstract.

**Theorem 1.8.** Suppose $\phi \in \text{Ham}(S^2, \omega)$ is $\delta'$ close to $\text{id}$, and has exactly 2 fixed points, which are non-degenerate then there are no non-zero index contractible Ustilovsky geodesics $\gamma$ from $\text{id}$ to $\phi$, for the positive Hofer length functional, with length less than $h - \delta$, for any $0 < \delta' < \delta < h$.

### 1.1.2. Path spaces.

**Notation 1.9.** Let $P_a(\phi)$ denote the space of paths in $\text{Ham}(S^2, \omega)$ from $\text{id}$ to $\phi$ minimizing the $L^\infty$ Hofer length up to $a$, that is:

$$\text{Length}^\infty(\text{id}, \phi) < d^\infty(\text{id}, \phi) + a.$$  
Define $P_{a,+}(\phi)$ similarly but with respect to $L^+$. And let $P(\phi)$ denote the space of all paths.

**Corollary 1.10.** If $\phi \in B_{\delta,2}$, then the inclusion map $P_{\delta/3}(\phi) \to P(\phi)$ is null-homotopic.

This follows by the Theorem 1.3 and by the elementary Lemma 4.1. In [9] we have used the theory of quantum characteristic classes to show that if $\phi \in B_{\delta,2}$ the inclusion map $P_{\delta/3,+}(\phi) \to P(\phi)$ vanishes on rational homotopy groups, in fact we proved a similar statement for $M = \mathbb{CP}^2$. So the above corollary is a certain improvement for $M = S^2$.

### 1.2. Topology of non-degenerate Hamiltonian diffeomorphisms of $S^2$ and more on the injectivity radius conjecture.

Floer homology has had phenomenal applications in symplectic geometry, giving in particular a proof of the (weak) Arnold conjecture itself. We now give one elementary application of Floer theory which a priori has nothing to do with symplectic geometry, and give yet another partial verification of the weak injectivity radius conjecture for $S^2$.

Let

$$\text{Ham}^{ndg}(S^2) \subset \text{Ham}(S^2, \omega) = \text{DiffVol}(S^2)$$

denote the subspace of area preserving diffeomorphisms $\phi$ of $S^2$ with isolated, non-degenerate fixed points, where non-degenerate means that the graph of $\phi$ in $S^2 \times S^2$ intersects the diagonal transversely. We may further split $\text{Ham}^{ndg}(S^2)$ into components $\text{Ham}(S^2, 2k)$ with $2k \geq 2$, denoting the number of fixed points. Note that these components are not connected, and their topological structure seems rather intricate to completely determine, in particular the homotopy groups are not known at the moment, (to the author). We note however that there are natural fibrations

$$\text{Ham}(S^2, 2k) \to C_{2k}(S^2),$$
with the latter denoting the configuration space of unordered points on $S^2$, where points are not allowed to collide. While much seems to be known about the topology of the spaces $C_{2k}(S^2)$, see for example McDuff [11], the fiber may be tricky to understand for higher $k$, although for $k = 1$ it is contractible, (we don’t have a proof of this claim, but it seems rather evident).

**Theorem 1.11.** For $i : \text{Ham}^{ndg}(S^2) \to \text{Ham}(S^2, \omega)$ the inclusion map, and for $j : B \to \text{Ham}^{ndg}(S^2)$ a continuous map with $B$ having the homotopy type of a CW complex, s.t. the induced map $i \circ j : B \leftrightarrow \text{Ham}(S^2, \omega)$ is vanishing on $\pi_1$, $i \circ j$ is null-homotopic.

In particular $i$ vanishes on higher homotopy groups. Our proof involves elliptic techniques in symplectic geometry, and Floer theory in particular, but in a rather transparent way. Note that the above corollary fails if we partially compactify the space $\text{Ham}^{ndg}(S^2)$ by adding $\text{id}$. Since the natural Lie group homomorphism $S^3 \to \text{Ham}(S^2, \omega)$ would lie in the corresponding space, and represents the generator.

Let $0 < \delta < h/2$ be given. Let $B(id, \delta)^{ndg}$ denote the metric ball in $\text{Ham}^{ndg}(S^2)$ with respect to the Hofer length functional $L$. As a corollary we also get the following very special case of the weak injectivity radius conjecture.

**Corollary 1.12.** The natural inclusion map $B(id, \delta)^{ndg} \to \text{Ham}(S^2, \omega)$ is null-homotopic.

This follows by 1.11 as the inclusion $B(id, \delta) \to \text{Ham}(S^2, \omega)$ is vanishing on $\pi_1$, see [9].

**Remark 1.13.** A technically identical argument but dealing with Hamiltonian fibrations over strips/discs, rather than the cylinder/$S^2$, gives an analogue of the above theorems for the space $\text{Lag}(S^2)$ of Lagrangians submanifolds in $S^2$ Hamiltonian isotopic to equator, i.e. simple closed smooth unparametrized loops partitioning $S^2$ into a pair of regions with equal area. The basic ingredient for this is that a Hamiltonian loop of Lagrangian submanifolds, that is a loop obtained by a Hamiltonian flow, gives rise to a sub-fibration of $M \times D^2$ over the boundary $\partial D^2$ with fiber $L_\theta \subset M_\theta$ over $\theta$ the element of the loop. We may then analogously study moduli spaces of holomorphic disks with boundary on this Lagrangian sub-fibration, for suitable almost complex structures, see for instance [1] for a similar story, or [6] for the monotone case.

For example we have:

**Claim 1.14.** For $i : \text{Lag}^{ndg}(S^2) \to \text{Lag}(S^2)$ the inclusion map, and for $j : B \to \text{Lag}^{ndg}(S^2)$ a map of a CW complex, s.t. the induced inclusion map $i \circ j : B \leftrightarrow \text{Lag}(S^2)$ is vanishing on $\pi_1$, $i \circ j$ is null-homotopic. Where the superscript $ndg$ means we take the subspace of those Lagrangians intersecting the standard equator at isolated points.

It should be much easier to prove this by elementary techniques than 1.11, but this is still not known.

## 2. Some preliminaries and notation

### 2.1. The group of Hamiltonian symplectomorphisms and the Hofer metric.

Given a smooth function $H : M^{2n} \times (S^1 = \mathbb{R}/\mathbb{Z}) \to \mathbb{R}$, there is an associated
time dependent Hamiltonian vector field $X_t$, $0 \leq t \leq 1$, defined by
\begin{equation}
\omega(X_t, \cdot) = -dH_t(\cdot).
\end{equation}
The vector field $X_t$ generates a path $\gamma : [0, 1] \to \text{Diff}(M)$, starting at $id$. Given such a path $\gamma$, its end point $\gamma(1)$ is called a Hamiltonian symplectomorphism. The space of Hamiltonian symplectomorphisms forms a group, denoted by $\text{Ham}(M, \omega)$. In particular the path $\gamma$ above lies in $\text{Ham}(M, \omega)$. It is well-known that any smooth path $\gamma$ in $\text{Ham}(M, \omega)$ with $\gamma(0) = id$ arises in this way (is generated by $H : M \times [0, 1] \to \mathbb{R}$ as above). Given a general smooth path $\gamma$, the classical Hofer length, $L(\gamma)$ is defined by
\begin{equation}
L(\gamma) := \int_0^1 \max_M H^\gamma_t - \min_M H^\gamma_t dt,
\end{equation}
where $H^\gamma$ is a generating function for the path $t \mapsto \gamma(0)^{-1} \gamma(t)$, $0 \leq t \leq 1$. The Hofer distance $\rho(\phi, \psi)$ is defined by taking the infimum of the Hofer length of paths from $\phi$ to $\psi$. It is a deep and interesting theorem that the resulting metric is non-degenerate, (cf. [5, 7]). This gives $\text{Ham}(M, \omega)$ the structure of a Finsler manifold.

As already mentioned in this paper we shall make use of different but closely related functionals. The $L^\infty$ Hofer length functional is defined by:
\begin{equation}
\text{Length}^{\infty}(\gamma) = \int_0^1 \max_M |H^\gamma_t| dt,
\end{equation}
where $H^\gamma$ is the generating function for $\gamma$ normalized to have zero mean at each moment. We denote the associated (genuine metric) distance function by $d^\infty$.

The positive Hofer length functional is defined by:
\begin{equation}
L^+(\gamma) = \int_0^1 \max_M H^\gamma_t dt,
\end{equation}
where $H^\gamma$ is likewise the generating function normalized to have zero mean at each moment.

2.2. Floer chain complex. Classically, the generators of the Hamiltonian Floer chain complex associated to $H : M \times S^1 \to \mathbb{R}$, are pairs $(o, \overline{o})$, with $o$ a time 1 periodic orbit of the Hamiltonian flow generated by $H$, and $\overline{o}$ a homotopy class of a disk bounding the orbit. The function $H$ determines a Hamiltonian connection $A_H$ on the bundle $M \times S^1 \to S^1$. The horizontal spaces for $A_H$ are the $\overline{\Omega}_H$ orthogonal spaces to the vertical tangent spaces of $M \times S^1$, where
\begin{equation}
\overline{\Omega}_H = \omega - d(Hd\theta).
\end{equation}

**Notation 2.1.** If $H$ generates $p$ then we may also denote $A_H$ by $A_p$.

The horizontal (a.k.a flat) sections for $A_H$ correspond to periodic orbits of $H$, in the obvious way. The homotopy classes $\overline{\sigma}$, induce homotopy classes of bounding disks in $M \times D^2$ of the corresponding flat sections. The connection $A_H$ induces an obvious $\mathbb{R}$-translation invariant connection on $M \times \mathbb{R} \times S^1$, trivial in the $\mathbb{R}$ direction, that we also call $A_H$. For an $\mathbb{R}$ translation invariant family $\{j_\theta\}$ of almost complex structures on the vertical tangent bundle of $M \times \mathbb{R} \times S^1$, $A_H$ induces an almost complex structure $J_H$ on $M \times \mathbb{R} \times S^1$, by declaring $A_H$ horizontal spaces to be
J_H invariant, requiring that J_H coincides with \{j_0\} on the vertical tangent bundle, and asking the projection to (\mathbb{R} \times S^1, j_H) be J_H holomorphic.

Assuming the pair (H,\{j_0\}) is regular, (which is a generic property in both variables in the so called semi-positive case), the differential in the classical Hamiltonian Floer chain complex is obtained via count (modulo natural action of \mathbb{R}) of J_H-holomorphic sections u of M \times \mathbb{R} \times S^1 \to \mathbb{R} \times S^1, whose projections to M \times S^1 are asymptotic in backward, forward time to generators (o_-, \bar{\sigma}_-), respectively (o_+, \bar{\sigma}_+) of CF(H), such that \bar{\sigma}_- + u - \bar{\sigma}_+ = 0 \in \pi_2(M \times S^2), with the obvious interpretation of this equation, and such that

\[ CZ(o_+, \bar{\sigma}_+) - CZ(o_-, \bar{\sigma}_-) = 1. \]

We shall omit further details. The corresponding chain complex will be denoted by CF(p,\{j_0\}), if p is the path from id to \phi \in \text{Ham}(M,\omega) generated by H or just CF(p) if \{j_0\} is implicit.

Grading does not play an important role in this note except in order to specify in which degree the fundamental class lies. We shall grade the Floer chain complex using Conley-Zehnder index so that for a C^2 small, Morse, time independent Hamiltonian H : M \to \mathbb{R}, the CZ index of a flat section of \mathcal{A}_H corresponding to a local maximizer is 2n.

### 2.3. Robust Ustilovsky geodesics.

For the reader’s convenience we review here the definition of robust Ustilovsky geodesic, which appeared in [19] and which we need to state our dynamical application.

It is shown by Ustilovsky [22] that \gamma is a smooth critical point of

\[ L^+: \mathcal{P}_\phi \to \mathbb{R}, \]

if there is a unique point x_{max} \in M maximizing the generating function \gamma_0^t at each moment t, and such that H^t_\gamma is Morse at x_{max}, at each moment t.

**Definition 2.2.** We call such a \gamma a **Ustilovsky geodesic**.

**Definition 2.3.** Given a chain complex (A_d) with some distinguished basis, and the inner product induced by this this basis, we say that a chain \gamma is **semi-homologically essential** if \gamma is orthonormal to d(\gamma) for any \gamma.

**Definition 2.4.** We will say that an Ustilovsky geodesic \gamma \in \mathcal{P}_\phi is **robust**, if the fixed points of \phi are non-degenerate and the constant, period one orbit o_{max} at x_{max} for the flow \gamma is semi-homologically essential in CF(\gamma,\{j_0\}), for some family \{j_0\} so that the pair (\gamma,\{j_0\}) is regular.

**Theorem 2.5.** [20] Let \gamma \in \mathcal{P}_\phi be an Ustilovsky geodesic, then the Morse index of \gamma with respect to L_+ is

\[ |CZ(o_{max}) - CZ([M])|, \]

where CZ is the Conley-Zehnder index and CZ([M]) denotes the Conley-Zehnder degree of the fundamental class, under the PSS isomorphism.

### 3. Proof of Theorem 1.11

Fix b_0 \in B. We consider the path Serre fibration \mathcal{P}^B over B, with fiber \mathcal{P}^B(b) over b the space of paths in B from b_0 to b. We shall always assume that our paths are constant near end points. While each fiber \mathcal{P}^B(b) may not be contractible, its induced map into \mathcal{P}(j(b)): the space of all paths in Ham(S^2,\omega) from j(b_0) to j(b),
will be shown to be null-homotopic. This uses the non-degeneracy assumption to construct a natural foliation by holomorphic curves of a Hamiltonian fibration \( Cp \) over \( \mathbb{CP}^1 \) associated to each path \( p \in \mathcal{P}^B(b) \) and some reference path \( p_0 \in \mathcal{P}^B(b) \), via a clutching construction after pushing the paths to paths in \( \text{Ham}^{ndg}(S^2) \). This in turn determines a natural smooth trivialization of \( C_p \). The construction of this foliation also uses an elementary application of Cerf theory, together with classical positivity of intersections ideas of Gromov-McDuff, particularly following [3]. Given this it is a matter of simple topology to deduce the main result.

**Proposition 3.1.** The natural map
\[
i_* j_* : \mathcal{P}^B(b) \to \mathcal{P}(j(b))
\]
is vanishing on all homotopy groups.

**Proof.** We may suppose that \( B \) is connected so that \( j \) maps into \( \text{Ham}(S^2, 2k) \) for some \( 2k \). Fix \( p_0 \in \mathcal{P}^B(b) \), so that the loop \( f(p) = j_* p_0 \cdot j_* p^{-1} \) is contractible in \( \mathcal{P}(j(b)) \) for some and hence every \( p \). Then for each \( p \in \mathcal{P}^B(b) \) we get a Hamiltonian \( S^2 \) fibration over \( S^1 \times \mathbb{R} \) that we call \( X_p \) by using \( f(p) \) as a clumping map:

\[
X_p = S^2 \times (-\infty, 0] \times S^1 \sqcup S^2 \times [0, \infty) \times S^1 / \sim,
\]

for \( \sim \) the equivalence relation under which

\[
(x, (0, \theta)) \in S^2 \times \partial(-\infty, 0] \times S^1
\]
is equivalent to

\[
(f^{-1}(p)(x), (0, \theta)) \in S^2 \times \partial[0, \infty) \times S^1,
\]

using the polar coordinates \((r, \theta), 0 \leq \theta \leq 1\). Fix any path \( p_{b_0} \in \mathcal{P}(j(b_0)) \). This determines a translation invariant Hamiltonian connection over the ends \((-\infty, -1]\) and \([1, \infty)\) of \( X_p \), which over \( \{r\} \times S^1, r \leq -1, r \geq 1\) is the Hamiltonian connection with holonomy path generated by \( H^{p_{b_0}} \) the generating function for \( p_{b_0} \), and is trivial in the \( r \) direction. The paths \( p_0, p \) partially determine a connection \( A_{p_0, p} \) extending the above connection over all \( \mathbb{R} \times S^1 \), so that for \( r \in [-1, 0] \) the flat sections of \( A_{p_0, p} \) over \( \{r\} \times S^1 \) correspond to fixed points of \( j_* p (1 + r) \), and for \( r \in [0, 1] \) the flat sections of \( A_{p_0, p} \) over \( \{r\} \times S^1 \) correspond to fixed points of \( j_* p (r - 1) \). In the \( r \) direction \( A_{p_0, p} \) is specified arbitrarily.

By assumption for each \( p \) the pair \((X_p|\{r\} \times S^1, A_{p_0, p})\) is Floer non-degenerate for all \( r \), meaning the holonomy map of the connection over \( S^1 \) has non-degenerate fixed points. Consequently we have \( 2k \) smooth sections \( \{\sigma_i\} \) of \( X_p \) whose restrictions over \( \{r\} \times S^1 \) for each \( r \) are flat sections of \( A_{p_0, p} \), corresponding to the fixed points of the associated holonomy map. We shall call these *spectral sections*. The asymptotic constraint at \( \pm \infty \) for each \( \sigma_i \) will be denoted by \( o_{i, \pm} \). Let us label the generators of \( CF(p_{b_0}) \) by \( \omega_{i,-} = (o_{i,-}, \varpi_{i,-}) \). Then we may further specify that the asymptotic constraints of each \( \sigma_i \) are \( o_{i, \pm}, \) i.e. taking homotopy classes of bonding disks into account, with \( \omega_{i,+} \) also in \( CF(p_{b_0}) \), this makes sense because each loop \( f(p) \) is contractible by assumptions, so that starting with a homotopy class of a bounding disk for \( o_{i,-} \) in \( M \times D^2 \), and trivializing \( X_p \) relative to the ends, \( \sigma_i \) induces a homotopy class of a bounding disk for \( o_{i,+} \) and it is immediate that resulting CZ index of \( \omega_{i,+} \) coincides with the CZ index for \( \omega_{i,-} \). The CZ index 2 generators will be denoted by \( \{\omega_{j, \pm}\} \) with \( \{\omega_{j, \pm}\} \) a subset of \( \{o_{i, \pm}\} \).

Let \( F(p) \) denote the space of Hamiltonian connections on \( X_p \) coinciding with \( A_{p_0, p} \) over all the slices \( \{r\} \times S^1 \) and for which all the spectral sections are flat.
This is an affine (and non empty) space and hence contractible. A Hamiltonian connection gives rise to an almost complex structure as described in the preliminaries.

The total space of the family $F(p)$ fibers over $P^B(p)$ and this is clearly a Serre fibration. The fiber is contractible as previously observed so there are sections. Let $g : S^m \to P^B(b)$ be a continuous map. Pullback $\{F(p)\}$ by $g$ and take any section $\{A_s\}$, $s \in S^m$ and let $\{J_s\}$ denote the corresponding family of almost complex structures on $\{X_s\} = \{X_{g(s)}\}$. Cap off both ends of $X_s$ via gluing, for a sufficiently small gluing parameter (to be described shortly) with $(M \times \mathbb{C}, A^{cap}_p)$ for $A^{cap}_p$ the Hamiltonian connection with holonomy path $p_0$ over radius $r$ circles in $\mathbb{C}$ for $r$ sufficiently large, and trivial in the radial direction for $r$ large. We call the resulting glued structure $C_s$, which is a Hamiltonian $S^2$ fibration over $\mathbb{CP}^1$, and the resulting Hamiltonian connection on $C_s$ will be called $A^{cap}_s$.

We may suppose that the path $p_0$ was chosen so that the induced Hamiltonian connection $A^{p_0}_p$, is regular that is so that $CF(p_0)$ is defined. We likewise choose $\tilde{A}^{p_0}_p$ extending $A^{p_0}_p$ in the natural sense, so that the associated moduli spaces

$$\mathcal{M}(\tilde{A}^{p_0}_p, \tilde{o}_{j,\pm})$$

of $J(\tilde{A}^{p_0}_p)$-holomorphic sections of $M \times \mathbb{C}$ asymptotic to $\tilde{o}_{j,\pm}$ are regular. As a remark we note that the Fredholm index will be 0 for elements of $\mathcal{M}(\tilde{A}^{p_0}_p, \tilde{o}_{j,-})$ and Fredholm index 2 for $\mathcal{M}(\tilde{A}^{p_0}_p, \tilde{o}_{j,+})$ as we must reverse the orientation on $\mathbb{C}$ for the $+ \end{eqnarray}$ to do the gluing, which affects how the Fredholm index is determined by Atiyah-Singer index theorem via the Conley-Zehnder index. This is as expected of course since we must get 2 dimensional moduli space after gluing. We shall not elaborate as this is a rather elementary technical point and more of a remark to the reader.

Consequently we obtain a family $\{C_s, J^{cap}_s\}$ of Hamiltonian fibrations over $\mathbb{CP}^1$ each of which is Hamiltonian equivalent to the trivial bundle $S^2 \times \mathbb{CP}^1$, by the assumption that $i \circ j$ vanishes on $\pi_1$.

**Lemma 3.2.** If the gluing parameter was taken to be sufficiently small the moduli space $\overline{\mathcal{M}}(s, J^{cap}_s)$ of $J^{cap}_s$ stable holomorphic sections of $C_s$, in the class of a constant section $[\text{const}]$ contains no nodal curves. Where $J^{cap}_s$ denotes the almost complex structure induced by $A^{cap}_s$.

**Proof.** Let

$$\sum_j c_j \tilde{o}_{j,-},$$

represent a fundamental class in $CF_2(p_0)$. Then for some $j'$ with $c_{j'} \neq 0$ there must be an element $\sigma'_{j'}$ of $\mathcal{M}(\tilde{A}^{p_0}_p, \tilde{o}_{j',-})$ since the PSS map from quantum homology to Floer homology is obtained by count of elements of these moduli spaces and so must hit this fundamental chain. The chain

$$\sum_j c_j \tilde{o}_{j,+},$$

must also represent the fundamental class in $CF_2(p_0)$. This fact is an extremely special case of Cerf-Floer continuation map theory, which in this case is without bifurcations, and follows by classical induced cobordisms of moduli spaces technique. If however the reader is curious about the general framework for Cerf-Floer theory
with bifurcations, we refer to [10] or Oh [16] for a more recent treatment. So there likewise must be an element $\sigma'$ in the moduli space $\mathcal{M}(A_{p_0}, \tilde{o})$.

Now, the spectral section $\sigma_j'$ is regular by automatic transversality, [14, Appendix C], consequently if the gluing parameter was small we may analytically glue $\sigma_j'$, $\sigma_j'$ and $\sigma_j'$ to get a smooth $J$ cap holomorphic $[const]$ class section $\sigma$ cap of $C$. Finally, if there are nodal curves in our moduli space $\mathcal{M}(p, J$ cap $p)$ then taking the principal component of the corresponding nodal section, we would have a smooth holomorphic section $\sigma'$ in class $[const] + A$ for some spherical fiber class $A$ with $c_1(A) < 0$. Consequently $\sigma'$ would have negative intersection number with $\sigma$ cap, which is impossible by positivity of intersections as $\sigma$ cap is embedded, see [14, Section 2.6].

$\square$

**Lemma 3.3.** $\overline{\mathcal{M}}(s, J$ cap $s)$ is also regular, (in particular has the expected dimension).

**Proof.** By automatic transversality the top stratum is regular, as since all elements are in class $[const]$, the associated real linear CR operator is automatically surjective, see for example [14, Appendix C]. But we just showed that there are no nodal curves and hence no other strata. $\square$

By regularity and compactness the map

$$ev_s : \overline{\mathcal{M}}(s, J$ cap $s) \to \pi_s^{-1}(z_0) \cong S^2,$$

taking a section to its value over some $z_0 \in \mathbb{C}$ is a degree 1 map. Together with positivity of intersections we get that the sections of $\overline{\mathcal{M}}(s, J$ cap $s)$ induce a canonical foliation of $C_s$, with all leaves diffeomorphic to $\mathbb{C}P^1$, and as they are sections we get a canonical smooth trivialization

$$C_s \rightarrow S^2 \times \mathbb{C}P^1,$$

by identifying all fibers with a fixed fiber over $0 \in \mathbb{C}P^1$ using the leaves of the foliation.

We get a bundle $P_g \rightarrow S^m$ with fiber over $s$: $C_s$ and with structure group reducible to the group of Hamiltonian bundle automorphisms of $S^2 \times \mathbb{C}P^1$ fixing the fiber over $0$, with $0$ identified with the origin of $\mathbb{C}$ before gluing in the cap $S^2 \times \mathbb{C}$ at the $-\infty$ end of $X_s$. In other words this is the group:

$$\Omega^2\text{Ham}(S^2, \omega),$$

with $\Omega^2$ denoting the based spherical mapping space. We drop the subscript id from now on. We have an analogous group

$$\Omega^2\text{Diff}(S^2)$$

consisting of smooth orientation preserving bundle automorphisms fixing the fiber over $0$. By the argument above the bundle $P_g$ is trivializable as a structure group $\Omega^2\text{Diff}(S^2)$ bundle. The inclusion

$$\Omega^2\text{Ham}(S^2, \omega) \rightarrow \Omega^2\text{Diff}(S^2),$$

is a homotopy equivalence by a classical theorem of Smale, [21], we can also use Moser argument. Consequently $P_g$ is trivializable as a structure group $\Omega^2\text{Ham}(S^2, \omega)$ bundle.

By construction $P_g$ is equivalent to the pullback by the sequence

$$S^m \rightarrow \Omega_j(b_0)\text{Ham}^{nd}(S^2) \rightarrow \Omega_j(b_0)\text{Ham}(S^2, \omega) \rightarrow \Omega_{id}\text{Ham}(S^2, \omega) \rightarrow \Omega^2B\text{Ham}(S^2, \omega)$$
of the universal structure group $\Omega^2\text{Ham}(S^2, \omega)$ bundle over $\Omega^2\text{BHam}(S^2, \omega)$. The last pair of maps are homotopy equivalences, so if $P$ is trivial as a structure group $\Omega^2\text{Ham}(S^2, \omega)$ bundle the map $\tilde{i} \circ (f \circ j \circ g)$ is null-homotopic. It immediately follows that $i_*j_*$ is vanishing on all homotopy groups.

**Proof of Theorem 1.11.** Consider the map of Serre fibrations from $P \rightarrow B$ to $\mathcal{P} \rightarrow \text{Ham}(S^2, \omega)$, induced by $i \circ j$. The total spaces are contractible and the induced map of fibers vanishes on all homotopy groups by our proposition above, consequently we see that $B \rightarrow \text{Ham}(S^2, \omega)$ vanishes on all homotopy groups, by considering the associated map of long exact sequences of homotopy groups. Since $\text{Ham}(S^2, \omega)$ has the homotopy type of a CW complex the conclusion follows.

**4. Proof of Theorem 1.1, 1.3**

**Proof of 1.1.** We assume theorem 1.3. Let $g : S^3 \rightarrow B_{\delta, N}$ be a smooth map. Set

$$P_g \equiv \{ p \in \mathcal{P}(g(s)) | s \in S^k \}.$$

We have a canonical projection

$$pr : P_g \rightarrow S^k,$$

and we construct a section. Given such a section we may use the corresponding family of paths to contract $g$ inside $\text{Ham}(S^2, \omega)$. Define the section $sec : S^3 \rightarrow P_g$ over $0 \in S^3$ to be some $p_0 \in \mathcal{P}(\delta/6)(g(0))$. We need to extend this over all of $S^3$. We shall consider $S^3$ as the suspension of $S^2$, i.e. the quotient

$$S^2 \times [0, 1]/S^2 \times 0 \sqcup S^2 \times 1,$$

and first construct a section of $pr^*P_g$ for $pr : S^2 \times [0, 1] \rightarrow S^3$ the projection. Let $h : S^2 \times [0, 1] \rightarrow [0, 1]$ be the natural projection $(x, t) \mapsto t$. The preimage spheres of $0, 1$ will be denoted by $S_0$, respectively $S_\infty$. Partition $[0, 1]$ into $n$ sub-intervals $\{e_i\}$:

$$e_i = \left[ \frac{i-1}{n}, \frac{i}{n} \right]$$

so that for each path

$$\gamma : [0, 1] \rightarrow S^2 \times [0, 1],$$

with the form

$$\gamma(t) = \{x\} \times \{t\}$$

and $\gamma_i$ the segment of $\gamma$ with $h(\gamma_i) \subset e_i$, we have

$$\text{Length}^\infty(g \circ pr \circ \gamma_i) \leq \delta/6.$$

We shall call these paths: $\gamma$ arcs. Likewise triangulate $S^2$ into $m$ triangles $\{a_j\}$,

$$a_j : \Delta^2 \rightarrow S^2,$$

so that for any linear

$$b : [0, 1] \rightarrow \Delta^2,$$ we have

$$\text{Length}^\infty(g \circ pr \circ (a_j \times \{t\}) \circ b) \leq \delta/12,$$
where for each $t \in [0, 1]$, $a_j \times \{t\}$ is the map
\[
\Delta^2 \to S^2 \times [0, 1] \\
a_j \times \{t\}(x) = (a_j(x), t).
\]
Let
\[
R_{i,j} \subset S^2 \times [0, 1],
\]
denote the images of the maps
\[
r_{i,j} : \Delta^2 \times e_i \to S^2 \times [0, 1] \\
(x, t) \mapsto (a_j(x), t).
\]
Using the $\gamma$ arcs we have a canonical extension of $pr^* sec$ to a section $sec'_{1,j}$ over each $R_{1,j}$, with
\[
\text{image}(sec'_{1,j}) \subset pr^* \mathcal{P}_{g,\delta/3},
\]
with the subscript $\delta/3$ as usual meaning that these are $\delta/3$-minimizing paths. We want to modify $sec'_{1,j}$ to $sec_{1,j}$ so that over the top bounding triangle $R_{1,j}^{\text{top}}$ of $R_{1,j}$ (image of $\Delta^2 \times \{1/n\}$) the image of $sec_{1,j}$ is in $pr^* \mathcal{P}_{g,\delta/6}$. Pick a
\[
p_{1} \in pr^* \mathcal{P}_{\delta/12}(g \circ pr \circ r_{1,j}(\text{center}, \frac{1}{n})),
\]
where $\text{center}$ is the center of mass of $\Delta^2$. This naturally determines a section, by fixing a geodesic (i.e. linear) deformation retraction of $\Delta^2$ to $\text{center}$: $sec_{1,j}^{\text{top}}$ over $R_{1,j}^{\text{top}}$ with
\[
\text{image } sec_{1,j}^{\text{top}} \subset pr^* \mathcal{P}_{g,\delta/6},
\]
using Theorem 1.3 we may homotopy
\[
sec'_{1,j} \big|_{R_{1,j}^{\text{top}}}
\]
to $sec_{1,j}$ through sections of $pr^* \mathcal{P}_{g,3h/N} \big|_{R_{1,j}^{\text{top}}}$. To do this we need the following lemma which can be readily deduced by [Lemma 1.4D, [17]] which says that the generating Hamiltonian for the product $f \cdot g$ of a pair of paths in $\text{Ham}(M, \omega)$ starting at $id$ is
\[
F(x, t) + G(f_t^{-1}(x), t),
\]
and by the fact that $f^{-1}$ is generated by
\[
-F(f_t(x), t).
\]
\textbf{Lemma 4.1.} For $p, p' \in \mathcal{P}(x)$,
\[
\text{Length}^\infty(p \cdot p'^{-1}) \leq \text{Length}^\infty(p) + \text{Length}^\infty(p'),
\]
where $\cdot$ is the point-wise product of paths with respect to the topological group structure.

This lemma is also implicitly used further on. The factor of 3 in $3h/N$ appears because we need to use the lemma twice.

Use homotopy extension to obtain a section $sec_{1,j}$ over $R_{1,j}$, of $pr^* \mathcal{P}_{g,3h/N}$. Then similarly extend $sec_{1,j}$ to a section $sec_{2,j}$ over $R_{1,j} \cup R_{2,j}$, of $pr^* \mathcal{P}_{g,3h/N}$. Iterate the process above to obtain a section $sec_j$ over the region
\[
R_j = \cup_i R_{i,j}.
\]
Note that by construction,
\[ \text{image } \sec_j |_{\text{image } a_j \times \{1\} \subset S_\infty} \subset \text{pr}^* \mathcal{P}_{g, \delta/6}. \]

As constructed the sections \( \{ \sec_j \} \) will not match over common boundary components, of the regions
\[ R_j = \text{image } r_j, \]
\[ r_j : \Delta^2 \times [0, 1] \to S^2 \times [0, 1] \]
\[ r_j(x, t) = a_j(x) \times \{t\}. \]

We may first deform them to match up over common vertices in \( S_\infty \), (they already match up over \( S_0 \)) using Theorem 1.3 and homotopy extension property, since by construction, over the vertices the corresponding paths are \( \delta/6 \) minimizing. Note that the deformed sections will still be in \( \text{pr}^* \mathcal{P}_{g, \delta/6} \).

Next we again use Theorem 1.3, to deform the sections so that they match up over the common edges, (leaving them fixed over the common vertices) This involves contracting loops in the based loop space of \( \text{Ham} (S^2, \omega) \) using Theorem 1.3. The sections are then in \( \text{pr}^* \mathcal{P}_{g, 9\delta/6} \).

Finally we deform the sections to match up over the common 2-faces of the form \( \text{image } (r_j |_{\text{edge } \Delta^2 \times [0, 1]} \) leaving them unchanged where they have already been deformed to match up. This involves contracting spheres in the based loop space of \( \text{Ham} (S^2, \omega) \) again using Theorem 1.3. The final result is a section in \( \text{pr}^* \mathcal{P}_{g, 27\delta/6} \) which is in \( \text{pr}^* \mathcal{P}_{g, 9\delta/6} \) over \( S_\infty \). Deform this using Theorem 1.3 one final time so that this section is constant over \( S_\infty \). Then this gives a single valued continuous section of \( \mathcal{P}_{g, 9\delta/6} \) over \( S_3 \), as required.

The second part of the theorem follows by the proof of the first part. The reason we need to take smaller and smaller metric balls to insure higher connectivity is because of increasing topological complexity of a triangulation of \( S^k \). That is we have to match sections in the construction over vertices, edges, faces, etc. At each step in the process we require more space. □

**Proof of 1.3.** We first verify the case of \( M = S^2 \). The following argument works the same way for any \( 0 < c \leq \hbar \), we shall just do it with \( c = \hbar \), moreover we shall just do the case of \( \Omega \) injectivity, i.e. the case of \( L^\infty \) Hofer length, as the \( \Omega^+ \) injectivity follows by the same argument. To recall, our conventions for the Hamiltonian flow and compatible almost complex structures are:
\[
\omega(X_H, \cdot) = -dH(\cdot)
\]
\[
\omega(v, Jv) > 0, \text{ for } v \neq 0.
\]

For every \( l \in L_{\hbar, \delta} \) we get a Hamiltonian \( S^2 \) fibration \( X_l \) over \( \mathbb{C}P^1 \), by using \( l \) as a clutching map,
\[
P_l = S^2 \times D_\omega \sqcup S^2 \times D_\infty \sim,
\]
as in (3.1). We have the coupling form \( \tilde{\Omega}_l \) on \( S^2 \times D_\omega \) defined by
\[
\tilde{\Omega}_l = \omega - d(\eta(r) \cdot H^l d\theta),
\]
where \( 0 \leq r \leq 1, 0 \leq \theta \leq 1 \), (we are using modified polar coordinates) \( H^l \) is the normalized generating function for \( l \), and \( \eta : [0, 1] \to [0, 1] \) is a smooth function satisfying:
\[
0 \leq \eta'(r),
\]
and

\[
\eta(r) = \begin{cases} 
  r^2 & \text{if } 0 \leq r \leq 1 - 2\kappa \\
  1 & \text{if } 1 - \kappa \leq r \leq 1,
\end{cases}
\]

for a small \(\kappa > 0\). Under the gluing relation \(\sim\), \(\tilde{\Omega}_l\) corresponds to the form \(\omega\) near the boundary of \(S^2 \times D_+^2\), so we may extend trivially over \(D_+^2\) to get a coupling form \(\tilde{\Omega}_l\) on \(X_l\). The coupling form \(\tilde{\Omega}_l\) determines a Hamiltonian connection \(A_l\) by declaring horizontal spaces to be \(\tilde{\Omega}_l\) orthogonal spaces to the vertical tangent spaces in the bundle \(TX_p\), and an almost complex structure \(J_l, \pm\) by first fixing a family \(\{j_l(z)\}, z \in \mathbb{CP}^1\) smooth in \(z\): \(j_l(z)\) is an almost complex structure on the fiber \(\pi_p^{-1}(z)\) compatible with \(\omega\), and then defining \(J_l, \pm\) to coincide with \(j_l\) on the vertical tangent bundle, to preserve the horizontal distribution of \(A_l\) and to have a holomorphic projection map to \((\mathbb{CP}^1, j_\pm)\), where \(j_\pm\) are the almost complex structures which preserve, respectively reverse the standard orientation on \(\mathbb{CP}^1\). As each \(X_l\) is trivializable we may consider the moduli spaces \(\overline{M}(J_l, \pm)\) consisting of stable \(J_l, \pm\) holomorphic sections \(\sigma\) of \(X_l\), in the class of the constant section \([\text{const}]\), in particular satisfying:

\[
\langle [\tilde{\Omega}_l], [\sigma]\rangle = 0.
\]

We first note that \(\overline{M}(J_l, \pm)\) have no nodal curves. For otherwise there is a stable \(J_l, \pm\) holomorphic section \(\sigma\) of \(X_l\), with total homology class \([\text{const}]\), and consequently having a principal component \(\sigma_{princ}\) which is a smooth \(J_l, \pm\) holomorphic section of \(X_l\) with:

\[
\langle [\tilde{\Omega}_l], [\sigma_{princ}]\rangle \leq -\hbar
\]

as \(\hbar\) is the minimal energy of a non-constant holomorphic sphere in \(S^2\), i.e. \(\text{area}(S^2, \omega)\). However in this case the classical energy inequality for holomorphic curves gives:

\[
\hbar \leq \text{area}^\infty(\tilde{\Omega}_l),
\]

where \(\text{area}^\infty\) is the functional on the space of coupling forms

\[
\text{area}^\infty(\tilde{\Omega}) = \inf_\alpha \{ \int_{\mathbb{CP}^1} \alpha [\tilde{\Omega} + \pi^*(\alpha) \text{ and } \tilde{\Omega} - \pi^*(\alpha) \text{ is symplectic}] \},
\]

for \(\alpha\) a 2-form on \(\mathbb{CP}^1\), with positive integral. On the other hand by direct calculation we have

\[
\text{area}^\infty(\tilde{\Omega}_l) = \text{Length}^\infty(l) < \hbar.
\]

This gives a contradiction.

By automatic transversality see [14, Appendix C] \(\overline{M}(J_l, \pm)\) is regular i.e. the associated real linear Cauchy-Riemann operator is transverse for all \(\sigma \in \overline{M}(J_l, \pm)\), and by using positivity of intersections as in the proof of Theorem 1.11 we infer that \(\overline{M}(J_l, \pm)\) determine folliations of \(X_l\) by \([\text{const}]\) class holomorphic sections. In particular

\[
ev: \overline{M}(J_l, \pm) \to S^2,
\]

obtained by evaluating a section at \(0 \in \mathbb{CP}^1\), are diffeomorphisms, and determine canonical smooth trivializations of \(X_l\). Let \(\Theta_{l, \pm}\) denote the corresponding horizontal distributions.
Consequently for an appropriately smooth family:
\[ f : S^k \to \Omega^h \text{Ham}(S^2, \omega), \]
we get natural (up to some choices of almost complex structures) smooth trivializations of the bundle
\[ P_f \to S^k, \]
with fiber over \( s \in S^k \): \( X_s = X_{f(s)} \). This is a trivialization of a bundle with structure group \( \Omega^2 \text{Diff}(S^2) \).

Let
\[ tr^0_{\pm} : P_f \to (S^2 \times \mathbb{C} P^1) \times S^k, \]
denote these trivializations. Set \( \{ F^0_s = (tr^0_{s,+})^* \omega \} \), where \( tr_{s,+} \) denote the restriction of \( tr_+ \) to the fiber \( X_s \).

This is a family of closed 2-forms on the family \( \{ X_s \} \), such that by construction each \( F^0_s \) vanishes on the horizontal distribution \( \Theta_{f(s),+} \), and such that
\[
\tau^0_{+} = F^0_s + \rho n^*_s \tau, \\
\tau^0_{-} = F^0_s - \rho n^*_s \tau,
\]
are symplectic forms on \( X_s \) for a small \( \rho > 0 \), where \( n^*_s : X_s \to \mathbb{C} P^1 \) is the projection, and \( \tau \) a fixed area one area form on \( \mathbb{C} P^1 \). Moreover the restriction of \( F^0_s \) to the fibers \( S^{2,s}_z \) of \( X_s \to \mathbb{C} P^1 \) over \( z \), is cohomologous to the restriction of \( \Omega_{f(s)} \) to \( S^{2,s}_z \). Likewise
\[
\alpha_s,= \Omega_{f(s)} + \text{area}^\infty (\Omega_{f(s)}) \pi^* \tau + \rho n^*_s \tau, \\
\alpha_s,= \Omega_{f(s)} - \text{area}^\infty (\Omega_{f(s)}) \pi^* \tau - \rho n^*_s \tau,
\]
are symplectic, and \( \alpha_s,= \) is positive on the horizontal distribution \( \Theta_{f(s),+} \).

Let \( \alpha^0_{s,t,\pm}, \ t \in [0,1] \) denote the convex linear combination
\[ \alpha^0_{s,t,\pm} = (t)\tau^0_{\pm} + (1-t)\alpha_{s,\pm}. \]
By construction \( \alpha^0_{s,t,\pm} \) is symplectic for every \( t \), however \( \alpha^0_{s,t,-} \) is only symplectic in some interval \( [0,t_0] \subset [0,1] \), this is because the distributions \( \Theta_{\pm f(s)} \) do not in general coincide. (They do coincide if all the elements of \( \mathcal{M}(J_{f(s)}) \) are actually flat, that is if \( \mathcal{A}_{f(s)} \) has no curvature.)

By Lemma 4.2, and the energy/compactness argument above we get a trivialization maps \( tr^1_{\pm} \) as before induced by \( \mathcal{M}(J(\alpha^0_{s,t_0}, \pm)) \), for \( J(\alpha^0_{s,t_0}, \pm) \) the almost complex structure induced by the connection corresponding to \( \alpha^0_{s,t_0} \), with respect to \( j_{\pm} \). Let \( F^1_s = (tr^1_{s,+})^* \omega \), and let
\[ \tau^1_{\pm} = F^1_s \pm \rho n^*_s \tau. \]
We define:
\[ \alpha^1_{s,t,\pm}, \ t \in [0,1], \]
to be the family coinciding with \( \alpha^0_{s,t,\pm} \) for \( 0 \leq t \leq t_1 \), and which for \( t > t_1 \) is defined as the convex linear combination:
\[ (t-t_1)/(1-t_1) \cdot \tau^1_{\pm} + (1-t)/(1-t_1) \cdot \alpha^0_{s,t_0,\pm}. \]
The family \( \{ \alpha^1_{s,t,\pm} \} \) is symplectic in an interval \( [0,t_1] \subset [0,1] \) with \( t_1 > t_0 \).

We may iterate this procedure to get families
\[ \{ \alpha^i_{s,t,\pm} \}. \]
which are defined and symplectic in intervals $[0,t_i]$ with $\{t_i\}$ an increasing sequence. We claim that we may iterate a finite number of times until $t_n = 1$. Suppose not so that we must accumulate to some $t' \in (0,1)$. Let $\mathcal{A}(s,t_i,+)\) denote the connection induced by $\alpha_{s,t_i,}^+$, and likewise denote the associated covariant derivative operator. Let

$$\text{angle}(f(s),t_i),$$

denote

$$\sup\{\left| F_s (\mathcal{A}(s,t_i,+)z u(v), \mathcal{A}(s,t_i,+)z u(w)) \right| \text{ s.t. } u \in \mathcal{M}(J(\alpha_{s,t_i},-)), z \in \mathbb{CP}^1 \}$$

where $\mathcal{A}(s,t,+)(u)$ denotes the operator:

$$T_{\mathbb{CP}^1} \rightarrow T_{\text{vert}} X_{f(s)},$$

obtained by the covariant derivative of the section $u$, and where $v, w \in T_z \mathbb{CP}^1$ is an orthonormal pair.

Clearly $\text{angle}(f(s),t_i)$ cannot remain bounded as $i \rightarrow \infty$, since otherwise we may use this bound to get that $\alpha_{s,t_i,}^{i+1}$ is symplectic for $t \in [t_i,t'_i]$, for any $t'_i$ s.t.

$$(t'_i - t_i) \cdot C < \rho,$$

but this is a contradiction, since we may then extend our family beyond $t'$.

Thus $\text{angle}(f(s),t_i)$ must go to infinity, but then the $L^\infty$ norm

$$\sup_{z \in \mathbb{CP}^1} ||du_i(z)||$$

is blowing up for a sequence $\{u_i\}$ of sections in either

$$\mathcal{M}(J(\alpha_{s,t_i,}^-, -)) \text{ or } \mathcal{M}(J(\alpha_{s,t_i,}^+, +))$$

as $i \rightarrow \infty$. But the almost complex structures $\{J(\alpha_{s,t_i}^\pm)\}$ must clearly form a $C^\infty$ Cauchy sequence, so that vertical bubbling is disallowed by Gromov compactness and the previous energy bound argument, as we have

**Lemma 4.2.**

$$\text{area}^\infty(\alpha_{s,t_i}^\pm) \leq L^\infty(f(s)) + \rho < h,$$

for some $\rho$ small enough, every $s$, every $i$ and every $t \in [0,t_i]$.

**Proof.** Define

$$\text{area}(\alpha) \equiv \text{Vol}(X_s, \alpha) / \text{Vol}(S^2, \omega).$$

Then by construction

$$\text{area}(\alpha_{s,t_i}^\pm) \leq L^\infty(f(s)) + \rho < h,$$

for some $\rho$ small enough, every $s$ and every $i$, since $s \in S^k$ varies in a compact family, and since $f$ maps into $\Omega^b \text{Ham}(S^2, \omega)$.

**Lemma 4.3.**

$$\text{area}^\infty(\tilde{\alpha}_{s,t}^i) \leq \max(\text{area}(\alpha_{s,t_i}^+, \text{area}(\alpha_{s,t_i}^-)), \text{area}(\alpha_{s,t_i}^+, \text{area}(\alpha_{s,t_i}^-))),$$

where $\tilde{\alpha}_{s,t}^i$ denotes the coupling form determined by $\alpha_{s,t_i}^\pm$.\]
Proof. By construction the forms $\alpha_{s,t,\pm}^i$ determine the same connections. Hence by the uniqueness property for coupling forms [13, Theorem 6.21], we may write

$$\alpha_{s,t,\pm}^i = \tilde{\alpha}_{s,t}^i \pm \pi_s^* \tau'_\pm$$

for some area forms $\tau'_\pm$, from which the claim immediately follows. □

So we get a contradiction.

Let us then denote by $\alpha_{s,t,\pm}$, $t \in [0,1]$ the family of symplectic forms obtained by the volume flow algorithm above. Then clearly $\text{area}(\alpha_{s,t,\pm})$ is non-increasing with $t$ and

$$\text{area}(\alpha_{s,1,\pm}) = \text{Length}^\infty(f(s)) + \rho.$$  

In particular $\text{area}^\infty(\tilde{\alpha}_{s,t})$ is bounded from above by $\text{Length}^\infty(f(s)) + \rho$.

Each $\tilde{\alpha}_{s,t}$ determines a loop $f_{s,t}$ in $\text{Ham}(S^2, \omega)$, defined by first identifying the fiber over $0 \in \mathbb{CP}^1$ with the fiber over $\infty \in \mathbb{CP}^1$ by $\tilde{\alpha}_{s,t}$ parallel transport map over the $\theta = 0$ ray from $0$ to $\infty$ ($0 = 0 \in D^2$, $\infty = 0 \in D^2$ in our coordinates from before), and then for each other $\theta$ ray from $0$ to $\infty$ getting an element $f_{s,t}(\theta) \in \text{Ham}(S^2, \omega)$ by $\tilde{\alpha}_{s,t}$ parallel transport over this ray. Clearly when $t = 1$ $f_{s,1} = f(s)$, and $f_{s,0}$ is the constant loop at $id$. On the other hand by the proof of [12, Lemma 2.2]:

$$\text{Length}^\infty(f_{s,t}) \leq \text{area}^\infty(\tilde{\alpha}_{s,t}).$$

Clearly the family $\{f_{s,t}\}$ is invariant under choice of $\rho$. Consequently after passing to the limit as $\rho \to 0$, we get that $\text{Length}^\infty(f_{s,t})$ is bounded from above by $\text{Length}^\infty(f(s))$.

By classical arguments $\Omega^h\text{Ham}(S^2, \omega)$ has the homotopy type of a CW complex, since $f$ was arbitrary this implies our theorem in the case of $S^2$.

For a $\Sigma$ as in the statement of the theorem, we may proceed with exactly the same argument upon noting that now there is no bubbling at all, so that we don’t need to restrict to short loops. □

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