ON THE NONEXISTENCE OF FAT PARTIALLY HYPERBOLIC HORSESHOES

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Abstract. We show that there are no partially hyperbolic horseshoes with positive Lebesgue measure for diffeomorphisms whose class of differentiability is higher than 1. This generalizes a classical result by Bowen for uniformly hyperbolic horseshoes.

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1. INTRODUCTION

Let $M$ be a Riemannian manifold. We use Leb to denote a normalized volume form defined on the Borel sets of $M$ that we call Lebesgue measure. Given a submanifold $\gamma \subset M$ we use Leb$\gamma$ to denote the measure on $\gamma$ induced by the restriction of the Riemannian structure to $\gamma$.

Let $f : M \to M$ be a $C^1$ diffeomorphism, and let $\Lambda \subset M$ be a compact invariant set, i.e. $f(\Lambda) \subset \Lambda$. We say that $\Lambda$ is a hyperbolic set if there is a $Df$-invariant splitting $T_{\Lambda}M = E^s \oplus E^u$ of the tangent bundle restricted to $\Lambda$ and a constant $\lambda < 1$ such that (for some choice of a Riemannian metric on $M$) for every $x \in \Lambda$

$$\|Df \mid E^s_x\| < \lambda \quad \text{and} \quad \|Df^{-1} \mid E^u_x\| < \lambda.$$ 

We say that an embedded disk $\gamma \subset M$ is an unstable manifold, or an unstable disk, if $\text{dist}(f^{-n}(x), f^{-n}(y)) \to 0$ exponentially fast as $n \to \infty$, for every $x, y \in \gamma$. Similarly, $\gamma$ is called a stable manifold, or a stable disk, if $\text{dist}(f^n(x), f^n(y)) \to 0$ exponentially fast as $n \to \infty$, for every $x, y \in \gamma$. It is well-known that every point in a hyperbolic set possesses a local stable
manifold $W^s_{loc}(x)$ and a local unstable manifold $W^u_{loc}(x)$ which are disks tangent to $E^s_x$ and $E^u_x$ at $x$ respectively.

A hyperbolic set $\Lambda$ is said to be a horseshoe if local stable and local unstable manifolds through points in $\Lambda$ intersect $\Lambda$ in a Cantor set. Horseshoes were introduced by Smale and appear naturally when one unfolds a homoclinic tangency associated to some hyperbolic periodic point of saddle type.

It follows from [4, Theorem 4.11] that a $C^{1+\alpha}$ diffeomorphism cannot have a fat hyperbolic horseshoe, i.e. a hyperbolic horseshoe $\Lambda$ with $\text{Leb}(\Lambda) > 0$; actually the result in [4] is proved for basic sets. Nevertheless, here we obtain a generalization of that result to a much more general situation. Let us remark that fat hyperbolic horseshoes exist for $C^1$ diffeomorphisms, as shown in [3].

We say that a compact invariant set $\Lambda$ has a dominated splitting if there exists a continuous $\mathcal{D}f$-invariant splitting $\mathcal{T}_\Lambda \mathcal{M} = E^{cs}_x \oplus E^{cu}_x$ of the tangent bundle restricted to $\Lambda$, and a constant $0 < \lambda < 1$ such that (for some choice of a Riemannian metric on $\mathcal{M}$) for every $x \in \Lambda$

$$\|Df \mid E^{cs}_x\| \cdot \|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda.$$ 

We call $E^{cs}$ the centre-stable bundle and $E^{cu}$ the centre-unstable bundle. We say that $f$ is non-uniformly expanding along the centre-unstable direction for $x \in \Lambda$ if

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1} \mid E^{cu}_{f^j(x)}\| < 0. \quad (\text{NUE})$$

Condition NUE means that the derivative has expanding behavior in average over the orbit of $x$. This implies that $x$ has dim($E^{cu}$) positive Lyapunov exponents in the $E^{cu}_x$ direction. As shown in [2, Theorem C], if condition NUE holds for every point in a compact invariant set $\Lambda$, then $E^{cu}$ is necessarily uniformly expanding in $\Lambda$, i.e. there is $0 < \lambda < 1$ such that

$$\|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda, \quad \text{for every } x \in \Lambda.$$ 

A class of diffeomorphisms with a dominated splitting $T\mathcal{M} = E^{cs} \oplus E^{cu}$ for which NUE holds Lebesgue almost everywhere in $\mathcal{M}$ and $E^{cu}$ is not uniformly expanding can be found in [1, Appendix A].

**Theorem A.** Let $f : \mathcal{M} \to \mathcal{M}$ be a $C^{1+\alpha}$ diffeomorphism and let $\Lambda \subset \mathcal{M}$ have a dominated splitting. If there is $H \subset \Lambda$ with $\text{Leb}(H) > 0$ such that NUE holds for every $x \in H$, then $\Lambda$ contains some local unstable disk.

We say that a compact invariant set $\Lambda$ is partially hyperbolic if it has a dominated splitting $T\mathcal{M} = E^{cs} \oplus E^{cu}$ for which $E^{cs}$ is uniformly contracting or $E^{cu}$ is uniformly expanding, meaning that there is $0 < \lambda < 1$ such that $0 < \lambda < 1$ such that $\|Df \mid E^{cs}_x\| \leq \lambda$ for every $x \in \Lambda$, or $\|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda$ for every $x \in \Lambda$.

The next result is a direct consequence of Theorem A whenever $E^{cu}$ is uniformly expanding. If, on the other hand, $E^{cs}$ is uniformly contracting, then we just have to apply Theorem A to $f^{-1}$. 


Corollary B. Let \( f : M \to M \) be a \( C^{1+\alpha} \) diffeomorphism and let \( \Lambda \subset M \) be a partially hyperbolic set with \( \text{Leb}(\Lambda) > 0 \).

1. If \( E^{cs} \) is uniformly contracting, then \( \Lambda \) contains a local stable disk.
2. If \( E^{cu} \) is uniformly expanding, then \( \Lambda \) contains a local unstable disk.

In particular, \( C^{1+\alpha} \) diffeomorphisms cannot have partially hyperbolic horseshoes with positive Lebesgue measure. The same holds for partially hyperbolic sets intersecting a local stable or a local unstable disk in a positive Lebesgue measure subset, as Corollary D below shows.

Theorem C. Let \( f : M \to M \) be a \( C^{1+\alpha} \) diffeomorphism and let \( \Lambda \subset M \) have a dominated splitting. Assume that there is a local unstable disk \( \gamma \) with \( \text{Leb}_{\gamma}(\gamma \cap \Lambda) > 0 \) such that NUE holds for every \( x \in \gamma \cap \Lambda \). Then \( \Lambda \) contains some local unstable disk.

The next result is a direct consequence of Theorem C in the case that \( E^{cu} \) is uniformly expanding, and a consequence of the theorem applied to \( f^{-1} \) in the case that \( E^{cs} \) is uniformly contracting.

Corollary D. Let \( f : M \to M \) be a \( C^{1+\alpha} \) diffeomorphism and let \( \Lambda \subset M \) be a partially hyperbolic set.

1. If \( E^{cs} \) is uniformly contracting and there is a local stable disk \( \gamma \) such that \( \text{Leb}_{\gamma}(\gamma \cap \Lambda) > 0 \), then \( \Lambda \) contains a local stable disk.
2. If \( E^{cu} \) is uniformly expanding and there is a local unstable disk \( \gamma \) such that \( \text{Leb}_{\gamma}(\gamma \cap \Lambda) > 0 \), then \( \Lambda \) contains a local unstable disk.

Theorems A and C are in fact corollaries of a slightly more general result that we present at the beginning of Section 4.

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2. Hölder control of tangent direction

This section is a survey of results in \([1\text{, Section 2]}\) concerning the Hölder control of the tangent direction of submanifolds. As observed in \([1\text{, Remark 2.3]}\) those results are valid for diffeomorphisms of class \( C^{1+\alpha} \). In this section we only use the existence of a dominated splitting \( T_{\Lambda}M = E^{cs} \oplus E^{cu} \). We fix continuous extensions of the two bundles \( E^{cs} \) and \( E^{cu} \) to some neighborhood \( U \) of \( \Lambda \), that we denote by \( \tilde{E}^{cs} \) and \( \tilde{E}^{cu} \). We do not require these extensions to be invariant under \( Df \). Given \( 0 < a < 1 \), we define the centre-unstable cone field \( C^{cu}_{a} = (C^{cu}_{a}(x))_{x \in U} \) of width \( a \) by

\[
C^{cu}_{a}(x) = \{ v_{1} + v_{2} \in \tilde{E}^{cs}_{x} \oplus \tilde{E}^{cu}_{x} \text{ such that } \|v_{1}\| \leq a\|v_{2}\| \}.
\]

We define the centre-stable cone field \( C^{cs}_{a} = (C^{cs}_{a}(x))_{x \in U} \) of width \( a \) in a similar way, just reversing the roles of the subbundles in \([1]\). We fix \( a > 0 \) and \( U \) small enough so that, up to slightly increasing \( \lambda < 1 \), the domination condition remains valid for any pair of vectors in the two cone fields:

\[
\|Df(x)v^{cs}\| \cdot \|Df^{-1}(f(x))v^{cu}\| \leq \lambda\|v^{cs}\|\|v^{cu}\|
\]
for every $v^{cs} \in C^a_{cs}(x)$, $v^{cu} \in C^a_{cu}(f(x))$, and any point $x \in U \cap f^{-1}(U)$. Note that the centre-unstable cone field is positively invariant:

$$Df(x)C^a_{cu}(x) \subset C^a_{cu}(f(x)), \quad \text{whenever } x, f(x) \in U.$$  

Indeed, the domination property together with the invariance of $E^{cu} = (E^{cu} \mid \Lambda)$ imply that

$$Df(x)C^a_{cu}(x) \subset C^a_{cu}(f(x)) \subset C^a_{cu}(f(x)),$$

for every $x \in K$. This extends to any $x \in U \cap f^{-1}(U)$ just by continuity.

We say that an embedded $C^1$ submanifold $N \subset U$ is tangent to the centre-unstable cone field if the tangent subspace to $N$ at each point $x \in N$ is contained in the corresponding cone $C^a_{cu}(x)$. Then $f(N)$ is also tangent to the centre-unstable cone field, if it is contained in $U$, by the domination property.

Our aim now is to express the notion of Hölder variation of the tangent bundle in local coordinates. We choose $\delta_0 > 0$ small enough so that the inverse of the exponential map $\exp_x$ is defined on the $\delta_0$ neighbourhood of every point $x$ in $U$. From now on we identify this neighbourhood of $x$ with the corresponding neighbourhood $U_x$ of the origin in $T_x N$, through the local chart defined by $\exp_x^{-1}$. Reducing $\delta_0$, if necessary, we may suppose that $\tilde{E}^{cs}_x$ is contained in the centre-stable cone $C^a_{cs}(y)$ of every $y \in U_x$. In particular, the intersection of $C^a_{cs}(y)$ with $\tilde{E}^{cs}_x$ reduces to the zero vector. Then, the tangent space to $N$ at $y$ is parallel to the graph of a unique linear map $A_x(y) : T_x N \to \tilde{E}^{cs}_x$. Given constants $C > 0$ and $0 < \zeta \leq 1$, we say that the tangent bundle to $N$ is $(C, \zeta)$-Hölder if for every $y \in N \cap U_x$ and $x \in V_0$

$$\|A_x(y)\| \leq Cd_x(y)^{\zeta},$$

where $d_x(y)$ denotes the distance from $x$ to $y$ along $N \cap U_x$, defined as the length of the shortest curve connecting $x$ to $y$ inside $N \cap U_x$.

Recall that we have chosen the neighbourhood $U$ and the cone width $a$ sufficiently small so that the domination property remains valid for vectors in the cones $C^a_{cs}(z)$, $C^a_{cu}(z)$, and for any point $z$ in $U$. Then, there exist $\lambda_1 \in (\lambda, 1)$ and $\zeta \in (0, 1]$ such that

$$\|Df(z)v^{cs}\| \cdot \|Df^{-1}(f(z))v^{cu}\|^{1+\zeta} \leq \lambda_1 < 1$$

for every norm 1 vectors $v^{cs} \in C^a_{cs}(z)$ and $v^{cu} \in C^a_{cu}(z)$, at any $z \in U$. Then, up to reducing $\delta_0 > 0$ and slightly increasing $\lambda_1 < 1$, condition (3) remains true if we replace $z$ by any $y \in U_x$, $x \in U$ (taking $\|\cdot\|$ to mean the Riemannian metric in the corresponding local chart).

We fix $\zeta$ and $\lambda_1$ as above. Given a $C^1$ submanifold $N \subset U$, we define

$$\kappa(N) = \inf\{C > 0 : \text{the tangent bundle of } N \text{ is } (C, \zeta)-\text{Hölder}\}.$$  

The next result appears in [1, Corollary 2.4].

**Proposition 2.1.** There exists $C_1 > 0$ such that, given any $C^1$ submanifold $N \subset U$ tangent to the centre-unstable cone field,
(1) there exists \( n_0 \geq 1 \) such that \( \kappa(f^n(N)) \leq C_1 \) for every \( n \geq n_0 \) such that \( f^k(N) \subset U \) for all \( 0 \leq k \leq n \);

(2) if \( \kappa(N) \leq C_1 \), then the same is true for every iterate \( f^n(N) \) such that \( f^k(N) \subset U \) for all \( 0 \leq k \leq n \);

(3) in particular, if \( N \) and \( n \) are as in (2), then the functions

\[
J_k : f^k(N) \ni x \mapsto \log |\det (Df | T_x f^k(N))|, \quad 0 \leq k \leq n,
\]

are \((L, \zeta)\)-Hölder continuous with \( L > 0 \) depending only on \( C_1 \) and \( f \).

3. HYPERBOLIC TIMES AND BOUNDED DISTORTION

The following notion will allow us to derive uniform behaviour (expansion, distortion) from the non-uniform expansion.

**Definition 3.1.** Given \( \sigma < 1 \), we say that \( n \) is a \( \sigma \)-hyperbolic time for \( x \in \Lambda \) if

\[
\prod_{j=n-k+1}^{n} \| Df^{-1} | E^c_{f^j(x)} \| \leq \sigma^k, \quad \text{for all } 1 \leq k \leq n.
\]

In particular, if \( n \) is a \( \sigma \)-hyperbolic time for \( x \), then \( Df^{-k} | E^c_{f^n(x)} \) is a contraction for every \( 1 \leq k \leq n \):

\[
\| Df^{-k} | E^c_{f^n(x)} \| \leq \prod_{j=n-k+1}^{n} \| Df^{-1} | E^c_{f^j(x)} \| \leq \sigma^k.
\]

Moreover, if \( \alpha > 0 \) is taken sufficiently small in the definition of our cone fields, and we choose \( \delta_1 > 0 \) also small so that the \( \delta_1 \)-neighborhood of \( \Lambda \) should be contained in \( U \), then by continuity

\[
\| Df^{-1}(f(y))v \| \leq \frac{1}{\sqrt{\sigma}} \| Df^{-1}|E^c_{f(x)}\| \|v\|,
\]

whenever \( x \in \Lambda \), \( \text{dist}(x, y) \leq \delta_1 \), and \( v \in E^c_{f^n(y)} \).

Given any disk \( \Delta \subset M \), we use \( \text{dist}_\Delta(x, y) \) to denote the distance between \( x, y \in \Delta \), measured along \( \Delta \). The distance from a point \( x \in \Delta \) to the boundary of \( \Delta \) is \( \text{dist}_\Delta(x, \partial\Delta) = \inf_{y \in \partial\Delta} \text{dist}_\Delta(x, y) \).

**Lemma 3.2.** Take any \( C^1 \) disk \( \Delta \subset U \) of radius \( \delta \), with \( 0 < \delta < \delta_1 \), tangent to the centre-unstable cone field. There is \( n_0 \geq 1 \) such that if \( x \in \Delta \) with \( \text{dist}_\Delta(x, \partial\Delta) \geq \delta/2 \) and \( n \geq n_0 \) is a \( \sigma \)-hyperbolic time for \( x \), then there is a neighborhood \( V_n \) of \( x \) in \( \Delta \) such that:

1. \( f^n \) maps \( V_n \) diffeomorphically onto a disk of radius \( \delta_1 \) around \( f^n(x) \) tangent to the centre-unstable cone field;
2. for every \( 1 \leq k \leq n \) and \( y, z \in V_n \),

\[
\text{dist}_{f^n(V_n)}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2} \text{dist}_{f^n(V_n)}(f^n(y), f^n(z)).
\]
Proof. First we show that $f^n(\Delta)$ contains some disk of radius $\delta_1$ around $f^n(x)$, as long as
\[ n > 2 \frac{\log(\delta/(2\delta_1))}{\log(\sigma)}. \] (6)
Assume that there is $y \in \partial \Delta$ with $\text{dist}_{f^n(\Delta)}(f^n(x), f^n(y)) < \delta_1$. Let $\eta_0$ be a curve of minimal length in $f^n(\Delta)$ connecting $f^n(x)$ to $f^n(y)$. For $0 \leq k \leq n$ we write $\eta_k = f^{n-k}(\eta_0)$. We prove by induction that $\text{length}(\eta_k) < \sigma^{k/2} \delta_1$, for $0 \leq k \leq n$. Let $1 \leq k \leq n$ and assume that
\[ \text{length}(\eta_j) < \sigma^{j/2} \delta_1, \quad \text{for } 0 \leq j \leq k - 1. \]
Denote by $\dot{\eta}_0(w)$ the tangent vector to the curve $\eta_0$ at the point $w$. Then, by the choice of $\delta_1$ in (5) and the definition of $\sigma$-hyperbolic time,
\[ \|Df^{-k}(w)\dot{\eta}_0(w)\| \leq \sigma^{-k/2} \|\eta_0(w)\| \prod_{j=n-k+1}^n \|Df^{-1}(E_{f^j(x)})\| \leq \sigma^{k/2} \|\dot{\eta}_0(w)\|. \]
Hence,
\[ \text{length}(\eta_k) \leq \sigma^{k/2} \text{length}(\eta_0) < \sigma^{k/2} \delta_1. \]
This completes our induction. In particular, we have $\text{length}(\eta_n) < \sigma^{n/2} \delta_1$. Note that $\eta_n$ is a curve in $\Delta$ connecting $x$ to $y \in \partial \Delta$, and so $\text{length}(\eta_n) \geq \delta/2$. Thus we must have
\[ n < 2 \frac{\log(\delta/(2\delta_1))}{\log(\sigma)}. \]
Hence $f^n(\Delta)$ contains some disk of radius $\delta_1$ around $f^n(x)$ for $n$ as in (6).

Let now $\Delta_1$ be the disk of radius $\delta_1$ around $f^n(x)$ in $f^n(\Delta)$ and let $V_n = f^{-n}(\Delta_1)$, for $n$ as in (5). Take any $y, z \in V_n$ and let $\eta_n$ be a curve of minimal length in $\Delta_1$ connecting $f^n(y)$ to $f^n(z)$. Defining $\eta_k = f^{n-k}(\eta_0)$, for $1 \leq k \leq n$, and arguing as before we inductively prove that for $1 \leq k \leq n$
\[ \text{length}(\eta_k) \leq \sigma^{k/2} \text{length}(\eta_0) = \sigma^{k/2} \text{dist}_{f^n(V_n)}(f^n(y), f^n(z)), \]
which implies that for $1 \leq k \leq n$
\[ \text{dist}_{f^n(V_n)}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2} \text{dist}_{f^n(V_n)}(f^n(y), f^n(z)). \]
This completes the proof of the lemma. \hfill \Box

We shall sometimes refer to the sets $V_n$ as \textit{hyperbolic pre-balls} and to their images $f^n(V_n)$ as \textit{hyperbolic balls}. Notice that the latter are indeed balls of radius $\delta_1$.

\textbf{Corollary 3.3 (Bounded Distortion).} There exists $C_2 > 1$ such that given $\Delta$ as in Lemma 3.2 with $\kappa(\Delta) \leq C_1$, and given any hyperbolic pre-ball $V_n \subset \Delta$ with $n \geq n_0$, then for all $y, z \in V_n$
\[ \frac{1}{C_2} \leq \frac{|\det Df^n|_{T_y \Delta}|}{|\det Df^n|_{T_z \Delta}} \leq C_2. \]
Proof. For $0 \leq i < n$ and $y \in \Delta$, we denote $J_i(y) = |\det Df| \cdot |T_{f^i(y)} f^i(\Delta)|$. Then,

$$\log \frac{|\det Df^n| \cdot |T_{y} \Delta|}{|\det Df^n| \cdot |T_{z} \Delta|} = \sum_{i=0}^{n-1} \left( \log J_i(y) - \log J_i(z) \right).$$

By Proposition 2.1, $\log J_i$ is $(L, \zeta)$-H"older continuous, for some uniform constant $L > 0$. Moreover, by Lemma 3.2, the sum of all $\text{dist}_\Delta(f^j(y), f^j(z))^{\sigma}$ over $0 \leq j \leq n$ is bounded by $\delta_1/(1 - \sigma^{\zeta/2})$. Now it suffices to take $C_2 = \exp(L\delta_1/(1 - \sigma^{\zeta/2}))$. $\square$

4. A local unstable disk inside $\Lambda$

Now we are able to prove Theorems A and C. These will be obtained as corollaries of the next result, as we shall see below.

**Theorem 4.1.** Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism and let $\Lambda \subset M$ have a dominated splitting. Assume that there is a disk $\Delta$ tangent to the centre-unstable cone field with $\text{Leb}_\Delta(\Delta \cap \Lambda) > 0$ such that NUE holds for every $x \in \Delta \cap \Lambda$. Then $\Lambda$ contains some local unstable disk.

Assume that there is $H \subset \Lambda$ with $\text{Leb}(H) > 0$ such that NUE holds for every $x \in H$. Choosing a Leb density point of $H$, we laminate a neighborhood of that point into disks tangent to the centre-unstable cone field contained in $U$. Since the relative Lebesgue measure of the intersections of these disks with $H$ cannot be all equal to zero, we obtain some disk $\Delta$ as in Theorem 4.1 under the assumption of Theorem A. For Theorem C observe that local stable manifolds are tangent to the centre-unstable spaces and these vary continuously with the points in $\Lambda$, thus being locally tangent to the centre-unstable cone field.

Let us now prove Theorem 4.1. Let $\Delta$ be a disk tangent to the centre-unstable cone field intersecting $\Lambda$ in a positive Leb$_\Delta$ subset such that NUE holds for every $x \in \Delta \cap \Lambda$. Let $H = \Delta \cap \Lambda$. Taking a subset of $H$, if necessary, still with positive Leb$_\Delta$ measure, we may assume that there is $c > 0$ such that for every $x \in H$

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df^{-1} \cdot E_{f^j(x)}^{cu}\| \leq -c. \tag{7}$$

Since condition (7) remains valid under iteration, by Proposition 2.1 we may assume that $\kappa(\Delta) < C_1$. It is no restriction to assume that $H$ intersects the sub-disk of $\Delta$ of radius $\delta/2$, for some $0 < \delta < \delta_1$, in a positive Leb$_\Delta$ subset, and we do so.

The following lemma is due to Pliss [6], and a proof of it in the present form can be found in [1, Lemma 3.1].
**Corollary 4.3.** There is \( \sigma > 0 \) such that every \( x \in H \) has infinitely many \( \sigma \)-hyperbolic times.

**Proof.** Given \( x \in H \), by (7) we have infinitely many values \( N \) for which

\[
\sum_{j=1}^{N} \log \| Df^{-1}|E_{j}(x) \| \leq -\frac{\sigma}{2} N.
\]

Then it suffices to take \( c_1 = c/2 \), \( c_2 = c \), \( A = \sup \| Df^{-1}|E_{j}(x) \| \), and \( a_j = -\log \| Df^{-1}|E_{j}(x) \| \) in the previous lemma. \( \square \)

Note that under assumption (7) we are unable to prove the existence of a positive frequency of hyperbolic times at infinity. This would be possible if we had \( \limsup \) instead of \( \liminf \) in (7), as shown in [1, Corollary 3.2]. The existence of infinitely many hyperbolic times is enough for what comes next.

**Lemma 4.4.** There are an infinite sequence of integers \( 1 \leq k_1 < k_2 < \cdots \) and, for each \( n \in \mathbb{N} \), a disk \( \Delta_n \) of radius \( \delta_1/4 \) tangent to the centre-unstable cone field such that the relative Lebesgue measure of the set \( f^{k_n}(H) \) in \( \Delta_n \) converges to 1 as \( n \to \infty \).

**Proof.** Let \( \epsilon > 0 \) be some small number. Let \( K \) be a compact subset of \( H \) and \( A \) be an open neighborhood of \( H \) in \( \Delta \) such that

\[ \text{Leb}_{\Delta}(A \setminus K) < \epsilon \text{Leb}_{\Delta}(K). \]

It follows from Corollary 4.3 and Lemma 3.2 that we can choose for each \( x \in K \) a \( \sigma \)-hyperbolic time \( n(x) \) and a hyperbolic pre-ball \( V_x \) such that \( V_x \subset A \). Here \( V_x \) is the neighborhood of \( x \) associated to the hyperbolic time \( n(x) \) constructed in Lemma 3.2 which is mapped diffeomorphically by \( f^{n(x)} \) onto a ball \( B_{\delta_1}(f^{n(x)}(x)) \) of radius \( \delta_1 \) around \( f^{n(x)}(x) \) tangent to the centre-unstable cone field. Let \( W_x \subset V_x \) be the pre-image of the ball \( B_{\delta_1/4}(f^{n(x)}(x)) \) of radius \( \delta_1/4 \) under this diffeomorphism.

By compactness we have \( K \subset W_{x_1} \cup \cdots \cup W_{x_m} \) for some \( x_1, \ldots, x_m \in K \). Writing

\[ \{n_1, \ldots, n_s\} = \{n(x_1), \ldots, n(x_m)\}, \quad \text{with } n_1 < n_2 < \cdots < n_s, \quad (8) \]

let \( \mathcal{U}_1 \subset \mathbb{N} \) be a maximal set of \( \{1, \ldots, m\} \) such that if \( u \in \mathcal{U}_1 \) then \( n(x_u) = n_1 \) and \( W_{x_u} \cap W_{x_a} = \emptyset \) for all \( a \in \mathcal{U}_1 \) with \( a \neq u \). Inductively we define \( \mathcal{U}_j \)
for $1 < j \leq s$ as follows. Suppose that $U_{j-1}$ has already been defined. Let $U_j \subset \mathbb{N}$ be a maximal set of $\{1, \ldots, m\}$ such that if $u \in U_j$ then $n(x_u) = n_j$ and $W_{x_u} \cap W_{x_a} = \emptyset$ for all $a \in U_j$ with $a \neq u$, and also $W_{x_u} \cap W_{x_a} = \emptyset$ for all $a \in U_1 \cup \ldots \cup U_{j-1}$.

Let $U = U_1 \cup \ldots \cup U_s$. By maximality, each $W_{x_i}$, $1 \leq i \leq m$, intersects some $W_{x_u}$ with $u \in U$ and $n(x_i) \geq n(x_u)$. Thus, given any $1 \leq i \leq m$ and taking $u \in U$ such that $W_{x_i} \cap W_{x_u} \neq \emptyset$ and $n(x_i) \geq n(x_u)$, we get

$$f^{n(x_u)}(W_{x_i}) \cap B_{\delta_1/4}(f^{n(x_u)}(x_u)) \neq \emptyset.$$ 

Lemma 3.2 assures that

$$\text{diam}(f^{n(x_u)}(W_{x_i})) \leq \frac{\delta_1}{2} \sigma^{(n(x_i) - n(x_u))/2} \leq \frac{\delta_1}{2},$$

and so

$$f^{n(x_u)}(W_{x_i}) \subset B_{\delta_1}(f^{n(x_u)}(x_u)).$$

This implies that $W_{x_i} \subset V_{x_u}$. Hence $\{V_{x_u} \}_{u \in U}$ is a covering of $K$.

It follows from Corollary 3.3 that there is a uniform constant $\gamma > 0$ such that

$$\frac{\text{Leb}_\Delta(W_{x_u})}{\text{Leb}_\Delta(V_{x_u})} \geq \gamma, \quad \text{for every } u \in U.$$

Hence

$$\text{Leb}_\Delta \left( \bigcup_{u \in U} W_{x_u} \right) = \sum_{u \in U} \text{Leb}_\Delta(W_{x_u})$$

$$\geq \sum_{u \in U} \gamma \text{Leb}_\Delta(V_{x_u})$$

$$\geq \gamma \text{Leb}_\Delta \left( \bigcup_{u \in U} V_{x_u} \right)$$

$$\geq \gamma \text{Leb}_\Delta(K).$$

Setting

$$\rho = \min \left\{ \frac{\text{Leb}_\Delta(W_{x_u} \setminus K)}{\text{Leb}_\Delta(W_{x_u})} : u \in U \right\},$$

we have

$$\varepsilon \text{Leb}_\Delta(K) \geq \text{Leb}_\Delta(A \setminus K)$$

$$\geq \text{Leb}_\Delta \left( \bigcup_{u \in U} W_{x_u} \setminus K \right)$$

$$\geq \sum_{u \in U} \text{Leb}_\Delta(W_{x_u} \setminus K)$$

$$\geq \rho \text{Leb}_\Delta \left( \bigcup_{u \in U} W_{x_u} \right)$$

$$\geq \rho \gamma \text{Leb}_\Delta(K).$$

This implies that $\rho < \varepsilon / \gamma$. Since $\varepsilon > 0$ can be taken arbitrarily small, we may always choose $W_{x_u}$ such that the relative Lebesgue measure of $K$ in $W_{x_u}$ is arbitrarily close to 1. Then, by bounded distortion, the relative Lebesgue measure of $f^{n(x_u)}(H) \supset f^{n(x_u)}(K)$ in $f^{n(x_u)}(W_{x_u})$, which is a disk of radius $\delta_1/4$ around $f^{n(x_u)}(x_u)$ tangent to centre-unstable cone field, is
also arbitrarily close to 1. Observe that since points in $H$ have infinitely many hyperbolic times, we may take the integer $n(x_u)$ arbitrarily large, as long as $n_1$ in \( \mathcal{N} \) is also taken large enough.

\[ \square \]

**Proposition 4.5.** There is a local unstable disk $\Delta_\infty$ of radius $\delta_1/4$ inside $\Lambda$.

**Proof.** Let $(\Delta_n)_n$ be the sequence of disks given by Lemma 4.4 and consider $(x_n)_n$ the sequence of points at which these disks are centered. Up to taking subsequences, we may assume that the centers of the disks converge to some point $x$. Using Ascoli-Arzelà, the disks converge to some disk $\Delta_\infty$ centered at $x$. By construction, every point in $\Delta_\infty$ is accumulated by some orbit of a point in $H \subset \Lambda$, and so $\Delta_\infty \subset \Lambda$.

Note that each $\Delta_n$ is contained in the $k_n$-iterate of $\Delta$, which is a disk tangent to the centre-unstable cone field. The domination property implies that the angle between $\Delta_n$ and $E_{cu}$ goes to zero as $n \to \infty$, uniformly on $\Lambda$. In particular, $\Delta_\infty$ is tangent to $E_{cu}$ at every point in $\Delta_\infty \subset \Lambda$. By Lemma 3.2, given any $k \geq 1$, then $f^{-k}$ is a $\sigma^{k/2}$-contraction on $\Delta_n$ for every large $n$. Passing to the limit, we get that $f^{-k}$ is a $\sigma^{k/2}$-contraction on $\Delta_\infty$ for every $k \geq 1$.

In particular, we have shown that the subspace $E^c_{xu}$ is uniformly expanding for $Df$. The fact that the $Df$-invariant splitting $T_\Lambda M = E^{cs} \oplus E^{cu}$ is a dominated splitting implies that any expansion $Df$ may exhibit along the complementary direction $E^c_{xu}$ is weaker than the expansion in the $E^c_{xu}$ direction. Then, by [5], there exists a unique unstable manifold $W^u_{loc}(x)$ tangent to $E^{cu}$ and which is contracted by the negative iterates of $f$. Since $\Delta_\infty$ is contracted by every $f^{-k}$, and all its negative iterates are tangent to centre-unstable cone field, then $\Delta_\infty$ is contained in $W^u_{loc}(x)$. \[ \square \]

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