Skeleton decomposition of linear operators in
the theory of degenerate differential equations

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Abstract. We suggest method based on the skeleton decomposition of
linear operators in order to reduce ill-posed degenerate differential equation
to the non-classic initial-value problem enjoying unique solution.

Key words: IVP, DAE, degenerate operator, ill-posed problem, regularization, skeleton decomposition.

1 Problem statement

This brief paper concerns Cauchy problem
\[
\begin{aligned}
B \frac{dx}{dt} &= x(t) + f(t) \\
x(0) &= x_0,
\end{aligned}
\]
where \( B : X \rightarrow X \) is non-invertible linear operator, \( X \) is banach space, e.g. \( B \) is Fredholm operator, \( \ker B \neq \{0\} \). The Cauchy problem (1)-(2) is not solvable for arbitrary \( x_0 \) making it ill-posed problem in general settings. Let us outline here that high-dimensional differential-algebraic-equations (DAEs) are the special case of such problem. DAEs are in the core of electromagnetic models of power systems.

If (1) is solvable then it’s necessary
\[
\langle x(0) + f(0), \psi \rangle = 0,
\]
where \( B^{*} \psi = 0 \). This problem has been addressed by many authors. The approach described in [1, 2] appears to be productive in practical applications. Nevertheless, due to condition (3), Cauchy problem (1)-(2) is not solvable in most of the cases and it is ill-posed problem in sense of theory [3]. Here readers may also refer to ch. 5 of textbook [4]. In monograph [1] it’s demonstrated that in sufficiently general settings, Cauchy problem (1)-(2) is solvable in class of generalized functions.

In the next section we suggest to employ another initial condition for eq. (1) and propose the constructive approach which is easy of implement. We constract operator \( M \subset \mathcal{L}(X \rightarrow X_{p}) \), where \( X_{p} \) is linear normed space. Th. 1 and Th. 2 make it possible to implement structure regularization of solution

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of eq. (1) with non-invertible operator $B$ using the special initial condition selection

$$Mx(t)|_{t=0} = c_0,$$

where $c_0 \in X_0$ instead of conventional Cauchy condition. IVP (1), (4) enjoy unique classic solution crossing the hyperplanes specified in proposed condition (4). Solution of such regularized IVP (1), (4) depends continuously on selection of $c_0 \in X_p$.

## 2 Structure Regularization

Let us introduce linear operators

$$A_{2i} : X_{i-1} \to X_i, \quad A_{2i-1} : X_i \to X_{i-1}, \quad B^i \triangleq A_{2i}A_{2i-1} : X_i \to X_i, \quad i = \overline{1,p},$$

where $X_i$ are linear normed spaces, $X_0 \triangleq X$.

Let

1) $B = A_1A_2, \quad A_{2i}A_{2i-1} = A_{2i+1}A_{2i+2}, \quad i = \overline{1,p}.$

**Property 1** $i$-th operator of the sequence $\{B^i\}_{i=0}^p$ is constructed by permutation of skeleton decomposition of operators of $i$-1th operator. Here $B^0 \triangleq B$.

Let us impose the following condition

2) operators $B^0, B^1, ..., B^{n-1}$ are noninvertible, and $B^p : X_p \to X_p$ is continuously invertible or non zero.

**Definition 1** Let the sequence $\{B^i\}_{i=0}^p$ meets conditions 1) and 2). Then this sequence we call attached skeletal chain of finite length of the operator $B$. Moreover, if $B^p$ is invertible then we call this chain as regular chain and we call it degenerate chain is $B^p$ is zero operator.

**Property 2** Any finite non-invertible operator generates skeletal chain of finite length [6, ch. II, §7].

**Remark 1** If $B$ is invertible, then its skeletal chain is reducible to operator $B$ itself.

If conditions 1) and 2) are satisfied then we can introduce functions

$$x_i(t) = \prod_{j=1}^i A_{2j}x_0(t),$$

where $x_0(t)$ is solution of eq. (1), $x_i(t) = A_{2i}x_{i-1}$,

$$i = \overline{1,p}.$$ Let $\{B_i\}_{i=0}^p$ be regular skeleton chain. Then function $x_p(t)$ satisfies the regular Cauchy problem

$$\begin{cases}
B^p \frac{dx}{dt} = x_p + \prod_{j=1}^p A_{2j}f(t), \\
x_p(0) = c_0,
\end{cases}$$

2
Once we have functions $x_p(t)$ determined, the rest of the functions $x_{p-1}, \ldots, x_0$ we construct following recursion

$$x_i = -\prod_{j=1}^{i} A_{2j} f(t) + A_{2i} \frac{dx_{i+1}}{dt}, \quad i = p - 1, \ldots, 1,$$

$$x_0 = -f(t) + A_1 \frac{dx_1}{dt},$$

where $x_0$ is solution of initial problem (1), (4).

**Theorem 1** If operator $B$ has regular skeleton chain of length $p$, $f(t)$ is $p$ times differentiable then eq. (1) with initial condition

$$\prod_{j=1}^{p} A_{2j} x(t)|_{t=0} = c_0,$$

where $c_0 \in X_p$ enjoys unique classic solution $x_0(t, c_0)$. If spectrum of operator $B^p$ lies in the left half-plane, then solution of homogeneous IVP

$$\begin{cases} B \frac{dx}{dt} = x, \\ M x(t)|_{t=0} = c_0 \end{cases}$$

will be asymptotically stable. Here $M := \prod_{j=1}^{p} A_{2j}$, $0 \leq t < \infty$.

**Theorem 2** If operator $B$ has degenerate skeleton chain, i.e. $B^p$ are zero operator, then homogeneous equation $B \frac{dx}{dt} = x$ has only trivial solution and unique solution of non-homogeneous equation (1) can be constructed as follows:

$$x(t) = -f(t) + A_1 \frac{dx_1}{dt},$$

where $x_1(t)$ can be constructed as following recursion

$$x_p(t) = -\prod_{j=1}^{p} A_{2j} f(t),$$

$$x_i(t) = -\prod_{j=1}^{i} A_{2j} f(t) + A_{2i} \frac{dx_{i+1}}{dt}, \quad i = p - 1, \ldots, 1.$$  

**Remark 2** If operator $B$ has degenerate skeleton chain then $B$ is nilpotent operator and unique solution of nonhomogeneous eq. (1) can be also constructed as iteration $u_n(t) = -f(t) + B \frac{dx}{dt} u_{n-1}(t)$, $n = 1, \ldots, p$, $u_0(t) = 0$, $u_p(t)$ satisfies a given eq. (1).
Such regularization scheme based on skeleton decomposition is applied in [5] for the construction of trajectories passing through the hyperplanes $Mx(0) = c_0$.

Possible Generalizations.
As footnote let us outline that similar results are valid for degenerate nonlinear Volterra integral-differential equations

$$B \frac{dx(t)}{dt} = u(t) + \int_0^t K(t, s)g(s, x(s)) ds,$$

where $B$ is noninvertible linear operator. Of particular interest is the following case

$$B = \sum_{i=1}^N a_i(t) \langle \cdot, \gamma_i \rangle,$$

where $\gamma_i \in X^*$, $a_i(t) : \mathbb{R}^n \to X$.

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