Exceptional Points, Nonnormal Matrices, Hierarchy of Spin Matrices and an Eigenvalue Problem

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Abstract Exceptional points are studied for non-hermitian Hamilton operators given by a hierarchy of spin-operators.

1 Introduction

Kato [1] (see also Rellich [2]) introduced exceptional points for singularities appearing in the perturbation theory of linear operators. Afterwards exceptional points and energy level crossing have been studied for hermitian Hamilton operators [3, 4, 5, 6, 7, 8, 9, 10] and non-hermitian Hamilton operators [11, 12, 13, 14, 15, 16] by many authors. Here we consider the finite dimensional Hilbert space $\mathbb{C}^n$ and the linear operators are $n \times n$ matrices over $\mathbb{C}$.

For hermitian matrices the standard example in literature is

$$H(\epsilon) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\epsilon$ is real. The characteristic polynomial $\det(H(\epsilon) - EI_2) = 0$ is given by $E^2 - E - \epsilon^2 = 0$. When $\epsilon$ is complex, the eigenvalues may be viewed as the 2 values of a single function $E(\epsilon)$ of $\epsilon$, analytic on a Riemann surface with 2 sheets joined at branch point singularities in the complex plane. The exceptional points in the complex $\epsilon$ plane are defined by the solution $\det(H(\epsilon) - EI_2) = 0$ together with $d(\det(H(\epsilon) - EI_2))/d\epsilon = 0$. One finds that the exceptional points are $\epsilon_1 = i/2$ and $\epsilon_2 = -i/2$. 
For non-hermitian systems the standard example is the matrix (Kato [1], Rotter [11], Heiss [12])

\[ \sigma_3 + z\sigma_1 = \begin{pmatrix} 1 & z \\ z & -1 \end{pmatrix} \]

where \( z \in \mathbb{C} \) and \( \sigma_1, \sigma_2, \sigma_3 \) are the Pauli spin matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Let \( z = i \). Then the matrix \( \sigma_3 + i\sigma_1 \) admits the eigenvalue 0 (twice) and the only
normalized eigenvector

\[ \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}. \]

The matrix \( \sigma_3 + i\sigma_1 \) is nonnormal. Let \( z = -i \). Then the nonnormal matrix \( \sigma_3 - i\sigma_1 \) admits the eigenvalue 0 (twice) and the only normalized eigenvector

\[ \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}. \]

We extend this result to arbitrary spin. Since the matrices considered are nonnormal we summarize the properties of nonnormal matrices in section 2. In section 3 we consider the case with spin \( 1/2, 1, 3/2 \) and 2. In section 4 the general case is studied.

### 2 Nonnormal Matrices

An \( n \times n \) matrix \( A \) over \( \mathbb{C} \) is called normal if \( AA^* = A^*A \). Then for a nonnormal matrix we have \( A^*A \neq AA^* \). An example of a nonnormal matrix is the matrix given above \( \sigma_3 + i\sigma_1 \) which only admits the eigenvalue 0 (twice) and only one eigenvector. Note that not all nonnormal matrices are non-diagonalizable, but all non-diagonalizable matrices are nonnormal [17].

If \( A \) is any \( n \times n \) matrix \( A \) over \( \mathbb{C} \), then a classical result due to Schur (Roman [18]) states that there exist a unitary matrix \( U \) and a triangular matrix \( T = (t_{jk}) \) with \( t_{jk} = 0 \) for \( k < j \) such that \( A = UTU^* \). For the matrix \( \sigma_3 + i\sigma_1 \) we find

\[ \sigma_3 + i\sigma_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & 2i \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \]

Let \( A, B \) be hermitian nonzero matrices, i.e. \( A^* = A \) and \( B^* = B \). Consider the matrix \( A + iB \). What are the conditions on \( A \) and \( B \) such that \( A + iB \) is normal?
From
\[(A + iB)^*(A + iB) = (A + iB)(A + iB)^*\]
we find that the commutator of \(A\) and \(B\) must vanish, i.e. \([A, B] = 0\). For the Pauli spin matrices \(\sigma_1\) and \(\sigma_3\) this condition is not satisfied since \([\sigma_3, \sigma_1] = 2i\sigma_2\).

Now the transition from a hermitian matrix to a non-normal matrix can be studied with the matrix
\[\sigma_3 + e^{i\phi}\sigma_1\]
where \(\phi \in [0, \pi/2]\). For \(\phi = 0\) we have the hermitian matrix \(\sigma_3 + \sigma_1\). For \(0 < \phi \leq \pi/2\) we have a nonnormal matrix. The eigenvalues are given by
\[\lambda_\pm = \pm \sqrt{1 + e^{2i\phi}}\]
with the eigenvectors
\[v_\pm = \begin{pmatrix} e^{i\phi} \\ -1 + \lambda_\pm \end{pmatrix} .\]

Note that the commutator of \(\sigma_3 + \sigma_1\) and \(\sigma_3 + e^{i\phi}\sigma_1\) is given by
\[[\sigma_3 + \sigma_1, \sigma_3 + e^{i\phi}\sigma_1] = 2i\sigma_2(e^{i\phi} - 1).\]

Obviously for \(\phi = 0\) the commutator vanishes and for \(\phi = \pi/2\) we have \(2i\sigma_2(i - 1)\). The matrix \(2i\sigma_2(i - 1)\) is normal, but non-hermitian.

Let \(\otimes\) be the Kronecker product and \(\oplus\) the direct sum. Let \(A, B\) be nonnormal matrices. Then \(A \otimes B\) and \(A \oplus B\) are nonnormal. Let \(X, Y\) be non-zero \(n \times n\) matrices. We have
\[(X^*X) \otimes (Y^*Y) = (XX^*) \otimes (YY^*)\]
if and only if \(X^*X = XX^*\) and \(Y^*Y = YY^*\). Note that
\[\exp(\sigma_3 + i\sigma_1) = I_2 + \sigma_3 + i\sigma_1 .\]

This matrix is nonnormal. However, we cannot conclude in general that \(\exp(A)\) of a nonnormal matrix \(A\) is nonnormal. Consider, for example, the matrix
\[A = \begin{pmatrix} i\pi & b \\ 0 & -i\pi \end{pmatrix}\]
with \(b \neq 0\). Then \(\exp(A)\) is a normal matrix. However, if a matrix \(M\) is nonnormal and nilpotent, then \(\exp(M)\) is nonnormal. If \(N\) is a normal matrix, then \(\exp(N)\) is a normal matrix.
3 Spin-$\frac{1}{2}$, 1, 3/2, 2 Cases

For the spin-$\frac{1}{2}$ case we consider the spin matrices for describing a spin-$\frac{1}{2}$ system

$$s_1 = \frac{1}{2} \sigma_1, \quad s_2 = \frac{1}{2} \sigma_2, \quad s_3 = \frac{1}{2} \sigma_3$$

with $s_1^2 + s_2^2 + s_3^2 = \frac{3}{4} I_2$. Consider the matrix $s_3 + is_1$. This is the case given above except for the factor $1/2$. Obviously the matrix $s_3 + is_1$ is nonnormal and the rank is 1. Since $(s_3 + is_1)^2 = 0_2$ the matrix is nilpotent and thus the eigenvalues are 0. The trace of this nonnormal matrix is 0. The eigenvalues of the matrix are 0 (twice) and only normalized eigenvectors of the matrix is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Consider next the spin matrices for describing a spin-1 system

$$s_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with $s_1^2 + s_2^2 + s_3^2 = 2I_3$. For spin-1 the matrix

$$s_3 + is_1 = \begin{pmatrix} 1 \\ i/\sqrt{2} \\ 0 \\ i/\sqrt{2} \\ 0 \\ -1 \end{pmatrix}$$

is nonnormal. The trace of this nonnormal matrix is 0 and the matrix is nilpotent, i.e. we have $(s_3 + is_1)^3 = 0_3$. Thus all three eigenvalues are 0 and the only normalized eigenvector is

$$\frac{1}{2} \begin{pmatrix} -1 \\ -i\sqrt{2} \\ 1 \end{pmatrix}.$$

For spin-3/2 we have the hermitian matrices

$$s_1 = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad s_2 = \frac{1}{2} \begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}.$$
\[
\begin{align*}
    s_3 &= \begin{pmatrix}
        3/2 & 0 & 0 & 0 \\
        0 & 1/2 & 0 & 0 \\
        0 & 0 & -1/2 & 0 \\
        0 & 0 & 0 & -3/2
    \end{pmatrix},
\end{align*}
\]

with \(s_1^2 + s_2^2 + s_3^2 = \frac{15}{4} I_4\). Thus the matrix \(s_3 + is_1\) is given by

\[
\begin{align*}
    s_3 + is_1 &= \begin{pmatrix}
        3/2 & i\sqrt{3}/2 & 0 & 0 \\
        i\sqrt{3}/2 & 1/2 & i & 0 \\
        0 & i & -1/2 & i\sqrt{3}/2 \\
        0 & 0 & i\sqrt{3}/2 & -3/2
    \end{pmatrix}.
\end{align*}
\]

The matrix is nonnormal and nilpotent, i.e. \((s_3 + is_1)^4 = 0\). Thus the trace is equal to 0 and the eigenvalues are 0 (four times). The rank of the matrix is 3. The only normalized eigenvector is

\[
\frac{1}{\sqrt{8}} \begin{pmatrix}
    i \\
    -\sqrt{3} \\
    -i\sqrt{3} \\
    1
\end{pmatrix}.
\]

This eigenvector is entangled, i.e. it cannot be written as a Kronecker product of two vectors in \(\mathbb{C}^2\). The tangle as a measure of entanglement is nonzero.

For spin-2 we have the hermitian 5 \(\times\) 5 matrices

\[
\begin{align*}
    s_1 &= \begin{pmatrix}
        0 & 1 & 0 & 0 & 0 \\
        1 & 0 & \sqrt{6}/2 & 0 & 0 \\
        0 & \sqrt{6}/2 & 0 & \sqrt{6}/2 & 0 \\
        0 & 0 & \sqrt{6}/2 & 0 & 1 \\
        0 & 0 & 0 & 1 & 0
    \end{pmatrix},
    s_2 &= \begin{pmatrix}
        0 & -i & 0 & 0 & 0 \\
        i & 0 & -i\sqrt{6}/2 & 0 & 1 \\
        0 & i\sqrt{6}/2 & 0 & -i\sqrt{6}/2 & 0 \\
        0 & 0 & i\sqrt{6}/2 & 0 & -i \\
        0 & 0 & 0 & i & 0
    \end{pmatrix},
    s_3 &= \begin{pmatrix}
        2 & 0 & 0 & 0 & 0 \\
        0 & 1 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 & 0 \\
        0 & 0 & 0 & -1 & 0 \\
        0 & 0 & 0 & 0 & -2
    \end{pmatrix},
\end{align*}
\]

with \(s_1^2 + s_2^2 + s_3^2 = 6I_5\). Thus the matrix \(s_3 + is_1\) is given by

\[
\begin{align*}
    s_3 + is_1 &= \begin{pmatrix}
        2 & i & 0 & 0 & 0 \\
        i & 1 & i\sqrt{6}/2 & 0 & 0 \\
        0 & i\sqrt{6}/2 & 0 & i\sqrt{6}/2 & 0 \\
        0 & 0 & i\sqrt{6}/2 & -1 & i \\
        0 & 0 & 0 & i & -2
    \end{pmatrix}.
\end{align*}
\]
The matrix is nonnormal and nilpotent, i.e. \((s_3 + is_1)^5 = 0_5\). Thus the trace is equal to 0 and the eigenvalues are 0 (five times). The rank of the matrix is 4. The only normalized eigenvector is

\[
\frac{1}{4} \begin{pmatrix}
1 \\
2i \\
-\sqrt{6} \\
-2i \\
1
\end{pmatrix}.
\]

### 4 General Case

For the general case we look at integer spin, i.e. \(1, 2, 3, \ldots\) and half-integer spin, i.e. \(1/2, 3/2, 5/2, \ldots\) [19]. Let \(s\) (spin quantum number) \(s \in \left\{ \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots \right\}\).

Given a fixed \(s\). The indices \(j, k\) run over \(s, s - 1, s - 2, \ldots, -s + 1, -s\). Consider the \((2s + 1)\) unit vectors (standard basis)

\[
\begin{align*}
e_{s,s} &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, & e_{s,s-1} &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, & \ldots, e_{s,-s} &= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]

Obviously the vectors have \((2s + 1)\) components. The \((2s + 1) \times (2s + 1)\) matrices \(s_+\) and \(s_-\) are defined as

\[
s_+ e_{s,m} := \sqrt{(s-m)(s+m+1)} e_{s,m+1}, \quad m = s - 1, s - 2, \ldots, -s
\]

\[
s_- e_{s,m} := \sqrt{(s+m)(s-m+1)} e_{s,m-1}, \quad m = s, s - 1, \ldots, -s + 1.
\]

The \((2s + 1) \times (2s + 1)\) matrix \(s_3\) is defined as (eigenvalue equation)

\[
s_3 e_{s,m} := me_{s,m}, \quad m = s, s - 1, \ldots, -s.
\]

Thus \(s_3\) is a diagonal matrix with the entries \(s, s - 1, \ldots, -s\) in the diagonal. Let \(s := (s_1, s_2, s_3)\), where \(s_+ = s_1 + is_2\) and \(s_- = s_1 - is_2\). Thus

\[
s_1 = \frac{1}{2}(s_+ + s_-), \quad s_2 = -\frac{i}{2}(s_+ - s_-).
\]
We have

\[(s_+)_j^k = (s_-)_k^j = \sqrt{(s - k)(s + k + 1)} \delta_{j,k+1} = \sqrt{(s + j)(s - j + 1)} \delta_{j,k+1}\]

and

\[(s_-)_j^k = (s_+)_k^j = \sqrt{(s + k)(s - k + 1)} \delta_{j,k-1} = \sqrt{(s - j)(s + j + 1)} \delta_{j,k-1}\]

where \(j, k = s, s - 1, \ldots, -s\). Therefore

\[
s_+ = \begin{pmatrix}
   0 & \sqrt{2s} & 0 & 0 & \ldots & 0 \\
   0 & 0 & \sqrt{2(2s - 1)} & 0 & \ldots & 0 \\
   0 & 0 & 0 & \sqrt{3(2s - 2)} & \ldots & 0 \\
   \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
   0 & 0 & 0 & 0 & \ldots & \sqrt{2s} \\
   0 & 0 & 0 & 0 & \ldots & 0
\end{pmatrix}.
\]

Thus \(s_- = (s_+)^*\). We have \(s_1^2 + s_2^2 + s_3^2 = s(s + 1)I_{2s+1}\) and \([s_1, s_2] = is_3, [s_2, s_3] = is_1, [s_3, s_1] = is_2\). Now the \((2s + 1) \times (2s + 1)\) matrix \(s_3 + is_1\) is nonnormal and nilpotent, i.e.

\[(s_3 + is_1)^{2s+1} = 0_{2s+1}\]

where \(0_{2s+1}\) is the \((2s + 1) \times (2s + 1)\) zero matrix and \(s\) is the spin. Thus all the eigenvalues of \(s_3 + is_1\) are 0. For the eigenvectors \(v\) which is an element of \(\mathbb{C}^{2s+1}\) we set

\[v = (v_s \ v_{s-1} \ \cdots \ v_{-s+1} \ v_{-s})^T\]

Now the eigenvalue equation \((s_3 + is_1)v = 0\) can be easily solved. First we can set without loss of generality the last entry of the eigenvector to \(v_{-s} = 1\). Then using the last row of the matrix \(s_3 + is_1\) we obtain the equation

\[i\frac{1}{2} \sqrt{2s}v_{-s+1} - sv_{-s} = i\frac{1}{2} \sqrt{2s}v_{-s+1} - s = 0\]

with the solution \(v_{-s+1} = -i2s/\sqrt{2s}\). Then the second last row of the matrix \(s_3 + is_1\) provides the equation for \(v_{-s+2}\) and successively we can find the other entries \(v_{-s+3}, \ldots, v_s\) of the eigenvector. All the entries are nonzero. This successive construction also shows that there is only one linearly independent eigenvector.

5 Conclusion

We have studied an eigenvalue problem for a hierarchy of nonnormal matrices constructed from the spin matrices for spin \(1/2, 1, 3/2, 2\) etc. All these matrices have
only one eigenvalue and thus only one eigenvector. The matrices \((2s+1) \times (2s+1)\) matrices \(s_3 + is_2\) have the same properties as \(s_3 + is_1\), i.e. they are nonnormal and nilpotent for all spin \(s\).

Starting from nonnormal matrices we can construct other nonnormal matrices. Let \(c_j^{\dagger}, c_j\ (j = 1, 2)\) be Fermi creation and annihilation operators, respectively. Then we can form the operator

\[
\begin{pmatrix}
  c_1^{\dagger} & c_2^{\dagger} \\
  \frac{1}{i} & -1
\end{pmatrix}
\begin{pmatrix}
  c_1 \\
  c_2
\end{pmatrix}
= c_1^{\dagger}c_1 - c_2^{\dagger}c_2 + i(c_1^{\dagger}c_2 + c_2^{\dagger}c_1).
\]

Using the basis \(|0\rangle, c_1^{\dagger}|0\rangle, c_2^{\dagger}|0\rangle, c_1^{\dagger}c_2^{\dagger}|0\rangle\) with \(\langle0|0\rangle = 1\) we find the matrix representation

\[
\begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 1 & i & 0 \\
  0 & i & -1 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]

for the operator. This matrix has the eigenvalue 0 (fourfold) and the three eigenvectors

\[
\begin{pmatrix}
  1 \\
  0 \\
  0 \\
  0
\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}
  0 \\
  1 \\
  i \\
  0
\end{pmatrix}, \quad \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  1
\end{pmatrix}.
\]

This matrix is nonnormal.

We can also consider the Kronecker product of such matrices. Consider the case of spin-\(\frac{1}{2}\). The \(4 \times 4\) matrix

\[
(\sigma_3 + i\sigma_1) \otimes (\sigma_3 + i\sigma_1)
\]

is nonnormal. However, note that the \(4 \times 4\) matrix

\[
\sigma_3 \otimes \sigma_3 + i\sigma_1 \otimes \sigma_1 = \begin{pmatrix}
  1 & 0 & 0 & i \\
  0 & -1 & i & 0 \\
  0 & i & -1 & 0 \\
  i & 0 & 0 & 1
\end{pmatrix}
\]

is normal, but non-hermitian. The eigenvalues are \((-1)^{1/4}\sqrt{2}, (-1)^{1/4}\sqrt{2}, (-1)^{1/4}i\sqrt{2}, (-1)^{1/4}i\sqrt{2}\).

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