Optimization of NB QC-LDPC Block Codes and Their Performance Analysis

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Abstract

We propose an approach for optimizing nonbinary (NB) quasi-cyclic (QC) LDPC codes. This approach combines constructing of base parity-check matrices by simulated annealing and labeling the obtained base matrices aimed at maximizing the so-called generalized girth of the NB LDPC code Tanner graph. Tightened random coding bounds based on the average binary spectra for ensembles of “almost regular” NB LDPC codes of finite lengths over extensions of the binary Galois field are derived. The simulated FER performance of the sum-product BP decoding of “almost regular” NB QC-LDPC block codes are presented and compared with the derived finite-length random coding bounds as well as with the same performance of the optimized binary QC-LDPC block code in the 5G standard. In the waterfall region our finite-length bounds on the error probability of ML decoding are about 0.1 – 0.2 dB away from the simulated FER performance of BP decoding.

I. INTRODUCTION

Nonbinary (NB) LDPC block codes over an arbitrary fields were introduced and analyzed in \cite{1}, where the average weight spectra for random ensembles of regular NB LDPC codes were derived. After rediscovering LDPC codes in nineties, the generalized belief propagation (BP) decoding for this class of codes was presented in \cite{2}. In this paper, it was also for the first time demonstrated that binary images of NB LDPC codes over extensions of the binary Galois field can significantly outperform binary LDPC codes of the same rate and length. Moreover, it was shown that increasing of the code alphabets improves code performance at the cost of larger decoding complexity. Starting with \cite{2}, the term \textit{NB LDPC codes} is used for binary images of NB LDPC codes over extensions of the binary field unlike the NB LDPC codes in \cite{1}. In the sequel, we use terms binary images of NB LDPC codes and NB LDPC codes interchangeably.

The most attention was paid to NB LDPC codes with only two nonzero elements in each column of their parity-check matrices. They were studied in \cite{3,4}. Experimental
comparison of binary and NB LDPC codes, including NB LDPC codes determined by parity-check matrices with more than two nonzero elements in their columns, was performed in [5].

In these papers, the Galois field extensions GF(2^m) with m = 2 – 10 were considered. It was confirmed that NB LDPC codes of short and moderate length outperform binary ones with the same parameters. Also, it was noticed that increasing m leads, not always monotonically, to improving performance of iterative decoding. This non-monotonicity was observed also in [6], [7], where decoding thresholds for ensembles of random NB LDPC codes were studied.

Similarly to binary LDPC codes, it could be expected that irregular NB LDPC codes, in general, outperform regular ones. Nevertheless, if m is large enough, for example m = 6 – 8, then, typically, NB LDPC codes with only two nonzero elements in each column provide the best performance. However, if m ≤ 4 then NB LDPC codes with larger number of nonzero elements in each column are preferable. For example, in [2] it was shown that for m = 4 the best results can be obtained if the average column weight is 2.4 and for smaller fields larger column weights provide better performance. In [8], the degree distribution of long NB LDPC codes was optimized for NB LDPC codes over GF(3) and GF(4). In the same paper, BP decoding thresholds for rate 1/2 NB LDPC codes over GF(2^m), m = 2,...,6 were obtained. The authors concluded that if column weight is larger than 3, the thresholds for small field sizes are better than for large ones, while for column weight smaller than 2.5 performance improves with increasing m.

Later on, a lot of efforts were put into optimization of practical NB LDPC codes and simplification of their decoding (see, for example, [9], [10], [11] and the references therein). The most research was done for QC and photograph based NB LDPC codes. However, mainly NB LDPC codes with two nonzero elements in each column with m ≥ 6 were studied.

In this paper, we propose a new approach for optimization of NB QC LDPC block codes over GF(2^m) for m ≤ 6. It is based on applying the simulated annealing technique (see [12] and the references therein) to optimization of the code base matrix. Then optimization of the degree matrix is performed by the algorithm in [11]. Labeling of the obtained degree matrix by elements of the field GF(2^m) in a way maximizing the generalized girth of the corresponding Tanner graph, which is equivalent to satisfying the full-rank condition in [13], completes the code constructing procedure.

The simulated FER performance of the generalized BP decoding of the constructed NB QC LDPC codes is compared to the Shannon lower bound [14] and the Poltyrev upper tangential sphere (TS) bound [15] on the error probability of maximum-likelihood (ML) decoding. The upper bound in [15] requires knowledge of the weight spectrum of the code. One of the approaches to using this bound is based on estimating the average code weight spectra for code ensembles and substituting them into the TS bound. As it is mentioned above, for the ensembles of binary regular LDPC codes and LDPC codes over arbitrary nonbinary fields, the average weight spectra were derived
in [1]. A detailed analysis of the asymptotic weight spectrum of the ensemble of NB protograph-based LDPC codes, as well as of NB protograph-based LDPC codes over extensions of the binary Galois field can be found in [16]. Ensembles of irregular NB LDPC codes over extensions of the binary Galois field were analyzed in [7]. Estimates on the thresholds of the ML decoding over an AWGN channel for ensembles of NB LDPC codes over the extension of the binary field were also presented in [16].

Finite length upper bounds on error probability of ML decoding over the AWGN channel obtained by using precise average weight enumerators for both binary random regular LDPC codes and for random regular NB LDPC codes over GF($2^m$) along with the asymptotic ML decoding thresholds were derived in [17]. In this paper by using technique in [18] for computing precise average spectra for ensembles of LDPC codes, we derive a tighter bound on error probability of the ML decoding for the ensemble of ‘almost regular” NB LDPC codes over GF($2^m$). This bound allows analysis of random codes with degree distributions which mimics the degree distribution of practical NB LDPC codes designed using simulated annealing technique.

The main contributions of the paper are

- Random ensemble of almost regular NB LDPC codes
- Finite length random coding bounds for ensembles of almost regular NB LDPC codes
- Simulated annealing-based approach for searching for base matrices of NB LDPC codes

This paper is organized as follows. In Section II necessary definitions are given. We describe our optimization technique in Section III. In Section IV we describe a new ensemble of “almost regular” NB LDPC codes over extensions of the binary field. In the same section, we derive a formula for the average binary weight spectrum of the proposed ensemble. Simulation results are presented and comparisons with tightened bounds based on the computed average binary weight spectrum are performed in Section V. The paper is concluded by a short discussion. For completeness, known bounds on error probability of ML decoding are presented in Appendix.

II. Preliminaries

A rate $R = b/c$ NB QC-LDPC code over GF($2^m$) is defined by its polynomial parity-check matrix of size $(c - b) \times c$

$$H(D) = \{h_{ij}(D)\},$$

where $h_{ij}(D)$ are polynomials of formal variable $D$ with coefficients from GF($2^m$). In the sequel, $h_{ij}(D)$ are either zeros or monomials and

$$H(D) = \{\alpha_{ij}D^{w_{ij}}\}, w_{ij} \in \{0, 1, \ldots, \nu\}, \alpha_{ij} \in \text{GF}(2^m), i = 1, \ldots, c - b, j = 1, \ldots, c,$$

where $\nu$ denote the maximal degree of a monomial. The corresponding $q$-ary parity-check matrix, $q = 2^m$ of the $(Lc, Lb)$ NB QC-LDPC block code is obtained by
replacing $D^{w_{ij}}$, by the $w_{ij}$-th power of a circulant permutation matrix of order $L$. The parameter $L$ is called lifting factor. The parity-check matrix in binary form is obtained by replacing non-zero elements of the $q$-ary, $q = 2^m$ parity-check matrix by binary $m \times m$ matrices which are companion matrices of the corresponding field elements [19].

Let $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{iw_i})$ be a vector consisting of nonzero elements of $i$th row of $H(D)$ and $w_i$ be the number of nonzero elements of this row. After replacing these nonzero elements by their binary $m \times m$ companion matrices we obtain an $m \times mw_i$ parity-check matrix of a linear code which we call the $i$-th constituent code of the NB LDPC code.

To facilitate the low encoding complexity, we consider parity-check matrices having the form (see, for example, [11])

$$H(D) = (H_{\inf}(D) \ h_0(D) \ H_{\text{bd}}(D)),$$

where $H_{\text{bd}}(D)$ is a bidiagonal matrix of size $(c - b) \times (c - b - 1)$, $h_0(D)$ is a column with two nonzero elements, and $H_{\inf}(D)$ can be any monomial submatrix of the proper size. This submatrix corresponds to the information part of a codeword.

Binary matrix $B = \{b_{ij}\}$ of the same size as $H(D)$ is called base matrix for $H(D)$ if $b_{ij} = 1$ iff $h_{ij}(D) \neq 0$.

In the search for optimized parity-check matrices, we represent $H(D)$ in the form of two matrices: degree matrix $H_w = \{w_{ij}\}$ and matrix of field coefficients $H_c = \{\alpha_{ij}\}$ which we obtain by labeling nonzero elements of $B$ by monomial degrees and nonzero field elements, respectively. In these matrices only elements for which elements of base matrix $b_{ij} = 1$ are meaningful. For that reason, in $H_w$ and $H_c$ we write “−1” in positions corresponding to zero elements of $B$.

Constructing the NB QC-LDPC codes requires finding good matrices $B, H_w, H_c$. We start with optimization of the base matrices $B$. If the matrices $B$ and $H(D)$ have $J$ nonzero elements in each column and $K$ nonzero elements in each row, we say about $(J,K)$-regular LDPC codes, otherwise codes are called irregular. In this paper we focus on NB QC-LDPC codes with columns of weight two and three in their matrix $B$. We call such codes almost regular. In the next section we explain our approach to optimization of base matrices, with a given degree distribution, by using the simulated annealing technique.

When searching for base matrices of NB LDPC codes we use notions from graph theory which we define here.

A graph $G$ is determined by a set of vertices $V = \{v_i\}$ and a set of edges $E = \{e_i\}$, where each edge connects exactly two vertices. The degree of a vertex denotes the number of edges that are connected to it. If all vertices have the same degree $l$, the degree of the graph is $l$, or, in other words, the graph is $l$-regular.

Consider the set of vertices $V$ of a graph partitioned into $t$ disjoint subsets $V_k$, $k = 0, 1, \ldots, t - 1$. Such a graph is said to be $t$-partite, if no edge connects two vertices from the same set $V_k$, $k = 0, 1, \ldots, t - 1$. 
A walk of length $N$ in a graph is an alternating sequence of $N + 1$ vertices $v_i$, $i = 1, 2, \ldots, N + 1$, and $N$ edges $e_i$, $i = 1, 2, \ldots, N$, with $e_i \neq e_{i+1}$. If the first and the final vertex coincide, that is, if $v_1 = v_{N+1}$, then we obtain a cycle. A cycle is called simple if all its vertices and edges are distinct, except for the first and final vertex which coincide. The length of the shortest simple cycle is denoted the girth of the graph.

Every parity-check matrix $H$ of a rate $R = k/n$ LDPC block code can be interpreted as the biadjacency matrix of a bipartite graph, the so-called Tanner graph, having two disjoint subsets $V_0$ and $V_1$ containing $n$ and $n - k$ vertices, respectively. The $n$ vertices in $V_0$ are called symbol nodes, while the $n - k$ vertices in $V_1$ are called constraint nodes. If the underlying LDPC block code is $(J, K)$-regular, the symbol and constraint nodes have degree $J$ and $K$, respectively.

Notice, that a parity-check matrix $H$ whose columns have weight two can be considered as the incidence matrix of a graph. In this graph vertices correspond to rows of $H$ and edges correspond to its columns. The corresponding Tanner graph consists of vertices of two types corresponding to rows and columns of $H$ and its edges correspond to nonzero elements of the columns. Consequently, girth of the Tanner graph is two times larger than the girth of the graph with incidence matrix $H$.

A hypergraph is a generalization of a graph in which the hyperedges are subsets of vertices and may connect (contain) any number of vertices. Every parity-check matrix $H$ of a rate $R = k/n$ LDPC block code can be interpreted as the incidence matrix of a hypergraph. For the first time relation between LDPC codes and hypergraphs was discovered in [22]. A hypergraph is called $s$-uniform if every hyperedge connects $s$ vertices. The degree of a vertex in a hypergraph is the number of hyperedges that are connected to (contain) it. If all vertices have the same degree then it is the degree of the hypergraph. The hypergraph is $l$-regular if every vertex has the same degree $l$.

Let the set $V$ of vertices of an $s$-uniform hypergraph be partitioned into $t$ disjoint subsets $V_j$, $j = 1, 2, \ldots, t$. A hypergraph is said to be $t$-partite if no edge contains two vertices from the same set $V_j$, $j = 1, 2, \ldots, t$.

III. SIMULATED ANNEALING TECHNIQUE FOR CONSTRUCTING BASE MATRICES WITH COLUMN WEIGHT TWO AND THREE

As it is mentioned above, when constructing NB LDPC codes over relatively small fields GF$(2^m)$, $m < 6$ the analysis by density evolution technique as well as simulations show that the average column weight of base matrix should be around the interval [2.2, 2.4]. This explains our choice for the structure of the base matrix. We consider base matrices with columns of weight two and three having the aforementioned bi-diagonal structure.

Base matrices with column weight two can be considered as incidence matrices of graphs. Their rows correspond to vertices and columns correspond to edges of these graphs. For base matrices with column weight larger than two their columns are interpreted as hyperedges (edges connecting more than two vertices) and such base
matrices represent incidence matrices of hypergraphs (graphs with hyperedges) (see, [11], [22]).

The problem of constructing graph with a given girth is well-studied in graph theory while constructing hypergraphs is much more complex and is studied in much less degree. The idea behind our approach is to first construct a graph whose incidence matrix determines a higher rate base LDPC code and then to construct hypergraph whose incidence matrix determines a base LDPC code with required parameters by “gluing” edges of the graph. When searching for the graph we aim at maximizing girth of the corresponding code Tanner graph. When searching for hypergraph we are trying to find a base LDPC code with Tanner graph having the same girth as the girth of the initial graph.

Next, we consider how hypergraph can be obtained from a given graph. Then we explain simulated annealing technique and apply this technique to both searching for optimized graphs and searching for the optimal “gluing” steps.

A. Constructing hypergraphs from graphs

The last $(c - b)$ columns of the base matrix $B = H(D)|_{D=1}$ in the form (1) correspond to a cycle passing through all nodes of the corresponding graph (or hypergraph) exactly once. Such a cycle is called Hamiltonian cycle. Therefore, we are going to search for good graphs and hypergraphs with Hamiltonian cycle.

When searching for base hypergraphs we use as search criteria the girth of the corresponding Tanner graph and approximate cycle extrinsic message degree (ACE). Search for hypergraphs with good parameters is computationally infeasible. As mentioned above, in order to reduce the search space we split the search in two steps.

To find a hypergraph with $n_c$ check nodes and $n_v$ variable nodes we first, construct a graph with $n_c$ vertices and $n_v + 2c_3$ edges with good parameters, where $c_3$ is the number of hyperedges containing three vertices. Next, we obtain $c_3$ hyperedges by converting $2c_3$ pairs of edges with common vertex (check node) into hyperedges.

This method of constructing hypergraph is illustrated in Figs 1, 2. In Fig. 1 the graph with girth equal to 3 is shown. Its incidence matrix represents a base matrix whose Tanner graph has girth 6. One hyperedge can be obtained from edges 7 and 9 shown by bold lines, another hyperedge can be obtained from edges 8 and 10 shown by dashed lines. The corresponding elements of parity-check matrix are marked by circles and squares, respectively.

In Fig. 2 we show hypergraph with two hyperedges and the corresponding base matrix with two columns of weight 3. It is easy to verify that girth of the Tanner graph of the obtained base matrix is equal to 6, therefore the girth of the initial graph is preserved (which cannot be guaranteed in general case).

We apply simulating annealing to construct base matrices of size $r_b \times (c_2 + c_3)$ such that:

- there are $c_2$ columns with Hamming weight two;
- there are $c_3$ columns with Hamming weight three;
Fig. 1: Constructing hypergraph from graph.

Fig. 2: Constructing hypergraph from graph.
• the girth of the corresponding Tanner graph is maximized;
• for matrices with equal girth, the number of shortest cycles is minimized.

Equivalently, the goal is to construct a hypergraph with \( r_b \) vertices and \( c_2 + c_3 \) hyperedges such that:
• there are \( c_2 \) hyperedges connecting two vertices;
• there are \( c_3 \) hyperedges connecting three vertices;
• the girth of the Tanner graph corresponding to that hypergraph is maximized;
• among hypergraphs with equal girth of their Tanner graphs, the number of shortest cycles is minimized;
• there exists a Hamiltonian cycle, composed of only hyperedges connecting two vertices.

Consider any pair of incident edges \( \{u, v\} \) and \( \{v, w\} \) in a graph. We replace these two edges by a hyperedge \( \{u, v, w\} \) as in Fig. 3. We call this operation merging the two edges.

Simulated annealing is an optimization technique which allows to escape local extrema. At each iteration of the algorithm, the current solution and a new solution are compared by using an objective function. Improved solutions are always accepted, while a fraction of non-improved solutions are accepted with a probability depending on the so-called temperature parameter. The goal of partly keeping non-improved solutions is to hopefully escape local extrema in the search for the global extremum. The temperature parameter is typically non-increasing with iterations. Terms “energy (objective) function”, “temperature profile” etc. stem from the fact that this algorithm mimics a process of bringing metal to a very high temperature until “melting” of its structure and then cooling it according to a very particular temperature decreasing scheme in order to reach a solid state of minimum energy. For the detailed overview of the algorithm see [12] and the references therein.

A generic description of simulated annealing is presented in a form of pseudo-code in Fig. 4.

This algorithm finds applications in different areas including search for good LDPC codes. In particular, in [23] and [24], simulated annealing technique was used for
Choose: An energy function $E(\cdot)$, a randomized perturbation function $p(\cdot)$, number of iterations $I_{\text{max}}$, and a “temperature profile” $t = (t_1, t_2, \ldots, t_{I_{\text{max}}})$.

Initialization: choose a random point $v_0$ in a given space $V$.

for $I = 1$ to $I_{\text{max}}$ do
    Compute $v'_I = p(v_I)$
    if $E(v'_I) \leq E(v_I)$ then
        set $v_{I+1} = v'_I$
    else
        set $v_{I+1} = v'_I$ with probability $P_T = \exp \left( \frac{E(v'_I) - E(v_I)}{t_I} \right)$
    end if
end for

Fig. 4: Simulated annealing algorithm

labeling of the base matrices of QC-LDPC codes. In [25], simulated annealing is used for decoding of LDPC codes with dynamic schedule.

We construct the hypergraph in two steps:

Step 1. Build a graph with $r_b$ vertices and $c_2 + 2c_3$ edges, containing a fixed Hamiltonian cycle;

Step 2. Pick $c_3$ pairwise disjoint pairs of incident edges, none of them belonging to the Hamiltonian cycle, and merge them pairwise together.

Both steps are done using simulated annealing. We now describe the parameters $V$, $E(\cdot)$, $p(\cdot)$ and $t$, that is, search space, energy function, perturbation function and temperature profile, in the context of both steps. For a (hyper) graph $G$ and natural number $g$, define $N_{G,g}$ as the number of cycles with length $g$ in the Tanner graph. For a graph $G$ and natural number $g$, define $m_{G,g}$ as follows:

- Consider each pair of incident edges $\{u, v\}$ and $\{v, w\}$.
- For each such pair, calculate the shortest walk between $u$ and $w$ that does not visit $v$, i.e. the shortest cycle through $\{u, v, w\}$ if $\{u, v\}$ and $\{v, w\}$ were merged.
- Define $m_{G,g}$ as the number of such pairs ($\{u, v\}$, $\{v, w\}$) for which the length of shortest cycle in the Tanner graph after merging is $g$.

Then, the simulated annealing parameters for Step 1 are:

- Search space $V$ is a set of all graphs with given number of vertices and edges, containing the fixed Hamiltonian cycle.
- Energy function
  \[ E(G) = \sum_g (N_{G,g} + \gamma m_{G,g}) x^{2(g-2)}, \]
  where $0 < x < 1$, $x$ is small. The constant $\gamma$ is picked experimentally. If the graph $G$ is “illegal”, i.e. contains self-loops or parallel edges, then $E(G) = \infty$ instead.
| \(c_3\) | Step 1 | Step 2 |
|------|--------|--------|
| 20   | 1.603  | 10^{-7}| 10 | 10^{-7} |
| 15   | 1.375  | 10^{-7}| 10 | 10^{-7} |
| 10   | 1.298  | 10^{-7}| 10 | 10^{-7} |
| 0    | 0.862  | 10^{-7}| —  | —         |

**TABLE I:** Example set of parameters of simulated annealing for matrices of size 26 × 52

- **Perturbation function** \(p(G)\): pick a random non-cycle edge \(\{u, v\}\); delete \(\{u, v\}\). With probability \(1/2\) swap \(u\) and \(v\). Pick a random vertex \(w\) such that \(\{u, w\}\) is not in the graph, and add \(\{u, w\}\) to the graph.
- **Temperature profile:** \(t_I = t_0 \cdot t_{\text{step}}^I\), \(I = 1, 2, \ldots, I_{\text{max}}\), where
  \[
  t_{\text{step}} = \left(\frac{t_{I_{\text{max}}}}{t_0}\right)^{\frac{1}{I_{\text{max}}}}.
  \]
  The constants \(t_0\) and \(t_{I_{\text{max}}}\) are picked experimentally.

The parameters for Step 2 are:
- Search space \(V\): space of all possible sets \(S\) of pairwise-disjoint pairs to merge;
- Energy function \(E(S) = \sum_g N_{G,g} x^{2(\gamma^g)}\), where \(0 < x < 1\), \(x\) is small. Here \(G\) is the hypergraph obtained after merging the pairs of edges of \(S\). If \(S\) contains some edges in multiple pairs or is otherwise “illegal”, then \(E(S) = \infty\) instead.
- **Perturbation function** \(p(S)\): pick a random element of \(S\) and delete it. Pick a random vertex \(u\) and two neighbors \(v\) and \(w\). If neither \(\{u, v\}\) nor \(\{u, w\}\) are cycle edges, add the pair \(\{\{u, v\}, \{u, w\}\}\) to \(S\); otherwise repeat this step.
- **Temperature profile:** \(t_I = t_0 \cdot t_{\text{step}}^I\), \(I = 1, 2, \ldots, I_{\text{max}}\) where
  \[
  t_{\text{step}} = \left(\frac{t_{I_{\text{max}}}}{t_0}\right)^{\frac{1}{I_{\text{max}}}}.
  \]
  The constants \(t_0\) and \(t_{I_{\text{max}}}\) are picked experimentally.

All random choices above are uniform. Parameters mentioned in the algorithms were chosen as follows: \(I_{\text{max}} = 10^6\), \(x = 0.1\), and \(\gamma = 20\). Parameters \(t_0\) and \(t_{I_{\text{max}}}\) for two optimization steps are presented in Table I.

To finalize description of the search algorithm, we explain the technique for fast computing the energy function. Since we need a large number of iterations in simulated annealing, it is important to calculate the energy functions quickly. For this reason, in the energy functions above, the values \(N_{G,g}\) are computed approximately by using dynamic programming as explained below. In the following, we consider 4-tuples \((\ell, u, e, v)\), where \(\ell\) is a positive integer, \(u\) is a vertex, \(e\) is a (hyper-) edge and \(v\) is a
vertex contained in $e$. Denote by $dp(\ell, u, e, v)$ the number of walks on the graph $G$ with the following properties:
- the length of the walk is $\ell$;
- the first vertex is $u$;
- the last visited (hyper-)edge is $e$;
- the last vertex is $v$;
- the walk never visits any (hyper-)edge twice in a row.

Clearly, if $\ell > 1$, then
\[
dp(\ell, u, e, v) = \sum_{f \neq e, w \neq v, w \in f} dp(\ell - 1, u, f, w).	ag{2}
\]

Here, the sum is taken over all $w \neq v$ that are contained in $e$, and all $f \neq e$ that are incident to $w$. Also, we have
\[
dp(1, u, e, v) = \begin{cases} 1, & \text{if } u, v \in e \text{ and } u \neq v; \\ 0, & \text{in all other cases}. \end{cases}	ag{3}
\]

Using the equations (3) and (2) we can calculate $dp(\ell, u, e, v)$ for all 4-tuples $(\ell, u, e, v)$. We then take
\[
N_{G,g} = \sum_{u,e} dp(g, u, e, u).
\]

This way, the number of shortest cycles will be found exactly. For higher values of $g$, this algorithm additionally counts some closed walks that are not actually cycles. In addition, the number of cycles with length $g$ is multiplied by $2g$, because each cycle with length $g$ has $g$ possible starting points and 2 possible traversal directions.

These inaccuracies do not negatively affect the solutions found by simulated annealing, as the energy function is still mostly “monotone”: better solutions have lower energy functions and the number of shortest cycles (which is exact) always dominates the energy function.

B. Constructing degree matrix and matrix of coefficients

Optimization of monomial parity-check matrix $H(D) = \{\alpha_{ij}D^{w_{ij}}\}$, $i = 1, \ldots, c - b, j = 1, \ldots, c$ includes optimization of base matrix $B$, degree matrix $H_w = \{w_{ij}\}$ and matrix of coefficients $H_c = \{\alpha_{ij}\}$.

As discussed above, $B$ is selected by simulated annealing. Optimization of the degree matrix is performed by using the same techniques which are used for constructing binary QC LDPC codes, for example, by using the algorithm suggested in [11].

In order to construct a matrix of the coefficients $H_c$ we use the following approach. Let us consider the binary image of $(J,K)$-regular $(Lc,Lb)$ NB QC LDPC code over $GF(2^m)$, $m = 4$. It is easy to see that the constituent $(Km, (K - 1)m)$ codes are high-rate codes which cannot have the minimum distance larger than two. We search for the constituent codes with the minimal number of weight two codewords and keep
a set of such codes. Assuming that we have a collection of such codes, we try to select them in such a way that the overall code performance is optimized.

As a search criterion we use a combinatorial characteristic of NB LDPC codes. We call it a generalized girth. We explain this notion by example.

Consider a cycle of length six in the code Tanner graph. The corresponding fragment of the degree matrix can be reduced to the form

\[
\begin{pmatrix}
D^{w_1} & D^{w_2} & 0 \\
D^{w_3} & 0 & D^{w_4} \\
0 & D^{w_5} & D^{w_6}
\end{pmatrix}.
\]

This matrix determines a cycle if and only if

\[w_1 + w_4 + w_5 - w_2 - w_3 - w_6 = 0 \mod L,
\]

where \(L\) is a lifting degree of the QC code. Assume that this condition is fulfilled. By using a corresponding fragment, we obtain the fragment of the polynomial matrix \(H(D)\) labeled by field elements \(\alpha_i\)

\[
\begin{pmatrix}
\alpha_1 D^{w_1} & \alpha_2 D^{w_2} & 0 \\
\alpha_3 D^{w_3} & 0 & \alpha_4 D^{w_4} \\
0 & \alpha_5 D^{w_5} & \alpha_6 D^{w_6}
\end{pmatrix}.
\]

When assuming that all \(\alpha_i\) are nonzero under the condition (4), the submatrix (5) is degenerate if and only if

\[\alpha_1 \alpha_4 \alpha_5 = \alpha_2 \alpha_3 \alpha_6,
\]

where the operations are performed in GF(\(2^m\)). Notice that if (6) is rewritten via degrees of the primitive element of the field, then this condition coincide with (4) up to the notations. Notice that condition (6) is equivalent to the full-rank condition in [13]. Now we can formally define the generalized girth.

Let \(H_w\) and \(H_c\) be the matrices defining the NB QC LDPC code. Consider two Tanner graphs, corresponding to two binary QC codes, one defined by \(H_w\) and the lifting degree \(L\), the second binary QC code defined by the same base matrix labeled by degrees of entries of \(H_c\) and having lifting degree \(q = 2^m\). A sequence of edges is called a nonbinary cycle (generalized cycle) if it is a cycle in the both Tanner graphs. The length of a shortest nonbinary cycle is called a generalized girth of a NB LDPC code.

When searching for matrix of coefficients \(H_c\) we simultaneously maximize the generalized girth and minimize the multiplicity of short nonbinary cycles.

Now we explain our method of selecting candidate constituent codes. Let \(\{K_i\},
\]

\(i = 1, \ldots, J\) be the set of row weights. Before searching for good NB LDPC codes we construct lists of constituent code candidates. The number of lists is equal to the number of different values \(K_i\). For \(q = 2^m\), each candidate code is a linear code with parity-check matrix of size \(m \times mK_i\). In other words, a parity-check matrix of the constituent code represents concatenation of \(K_i\) companion matrices of field
elements. In list sequences, field elements are sorted according to the degrees of the field primitive element. Random permutations of the field elements in sequences are taken into account while overall matrix optimization. To avoid search over equivalent codes, the first element is always equal to 1.

For small \( m \) and low rate codes (small \( K_i \)) the list of candidate codes can be obtained by exhaustive search. Otherwise, codes are selected at random. In both cases the candidate selection criterion is the minimum distance of the linear code and for codes with the same minimum distance we prefer codes with smaller number of minimum weight codewords.

In our code search experiments for each \( K_i \) the list of 50 candidates was constructed. The search algorithm is shown in Fig. 5. The search procedure starts with searching for a base matrix by using simulated annealing technique. The next step is search for a degree matrix by using greedy search [11]. The iterative procedure for labeling the obtained matrix by field elements consists of random assigning the good constituent code-candidates to rows of the parity-check matrix and testing the generalized girth of the obtained code. The newly generated code is considered as the best one if it has either better generalized girth or the same girth but better multiplicity of shortest cycles. The search stops if during the last \( I_{\text{max}} \) attempts a new record was not performed.

IV. Bounds on Error Probability for Ensembles of NB LDPC Codes with Two and Three Nonzero Elements in Each Column of Their Parity-Check Matrices

In this section we compute the tangential-sphere (TS) upper bound [15] on the error probability of the maximum-likelihood (ML) decoding for a random ensemble of “almost regular” NB LDPC codes. For completeness of the paper we present the upper bound as well as the Shannon lower bound on the error probability of ML decoding in the appendix.

A. Computing spectra of the ensembles of almost regular NB LDPC codes

It is easy to see that in order to compute the TS bound (18) – (19), it is necessary to know weight spectrum of the code. In this section, we compute the average binary weight spectrum for an ensemble of NB LDPC codes over GF(2\(^m\)), \( m > 1 \) is an integer, with two and three nonzero elements in each column of their parity-check matrix. We start with a brief overview of ensembles of LDPC codes studied in literature. Then the proposed ensemble is described and its average binary weight spectrum is derived.

B. Ensembles of LDPC codes

Asymptotic distance spectra of ensembles of both regular and irregular binary LDPC codes were studied in [26]. The main idea behind the considered approach
**Input:** Degree distribution, matrix size and lifting degree, maximum number of search attempts $I_{\text{max}}$ until the next new code is found.

**Step 1** Generate base matrix using simulated annealing approach

**Step 2** Assign degrees to monomials using greedy search [11]

**Step 3** Searching for matrix of coefficients

**Initialization:** $I = 0$; Assign a large number $n$ to the generalized girth $g = n$, and zero to multiplicity of cycles with generalized girth $N_g = 0$.

**while** $I \leq I_{\text{max}}$ **do**

$I \leftarrow I + 1$;

Assigning random nonbinary elements row-by-row

**for** $i = 1$ to $c - b$ **do**

Choose at random a constituent code $a = (a_1, \ldots, a_w)$ from the list of good constituent codes

and permute randomly its positions $a' = \text{randperm}(a)$

Assign components of $a'$ to nonzero elements of the $i$th row of parity-check matrix.

**end for**

Compute generalized girth $g_c$ and its multiplicity $N_c$.

**if** $g_c > g$ **or** ($g_c == g$ **and** $N_c < N_g$) **then**

set $g = g_c$, $N_g = N_c$, $I = 0$;

**end if**

**end while**

**Output:** Matrix $H(D)$

---

**Fig. 5:** Code search algorithm

**to computing distance spectra is to determine an LDPC code by its Tanner graph and replace the analysis of the ensemble of irregular LDPC codes by the analysis of the ensemble of irregular bipartite graphs with given degree distributions on variable and check nodes, $\lambda(x) = \sum_{i=1}^{J} \lambda_i x^{i-1}$ and $\rho(x) = \sum_{i=1}^{K} \rho_i x^{i-1}$, where $\lambda_i$ and $\rho_i$ is a fraction of nodes of degree $i$ among variable and check nodes, respectively, and $J$ and $K$ are maximum column and row weights. To each variable node of degree $i$ it is assigned $i$ variable edges and to each check node it is assigned $i$ check edges, where $1 \leq i \leq |E|$ and $|E|$ denotes the total number of edges.

In [26], an ensemble of random graphs with $n$ variable and $r$ check nodes is generated by assigning to each of $\lambda_i n$ variable nodes $J_i$ edges from the set of edges $E$, $i = 1, 2, \ldots$. Next, all edges are randomly permuted by choosing uniformly at random a permutation $\pi = (\pi_1, \pi_2, \ldots, |E|)$ of the set $\{1, 2, \ldots, |E|\}$. Then each of $\rho_j r$ check nodes is connected with $K_j$ edges from the permuted set of edges, $j = 1, 2, \ldots$.

To map the corresponding graph to the code matrix $H$, the element $H_{ij}$ is set to $1$ if there is an odd number of edges between the $j$th variable node and the $i$th check node. Otherwise, $H_{ij}$ is set to $0$. Notice that the described ensemble was introduced
Another ensemble of irregular binary LDPC codes was considered in [28]. Binary LDPC codes of this ensemble are determined by their random parity-check matrices having the following properties. The rows of the size $r \times n$ parity-check matrix are split into $g$ strips, where the $i$th strip contains $r
u_i$ rows, $\sum_{i=1}^{g} \nu_i = 1$. Its columns are split into $h$ strips, where the $i$th strip contains $n\eta_i$ columns, $\sum_{i=1}^{h} \eta_i = 1$. The row sums of the rows in the $i$th strip, $i = 1, 2, ..., g$, is equal to $r_i$ and the column sums in the $i$th strip, $i = 1, 2, ..., h$, is equal to $s_i$, where $r_1, ..., r_g$ and $s_1, ..., s_h$ are nonnegative integers independent on $n$. Asymptotic average weight spectrum for this ensemble is obtained in this paper. Required column and row degree distributions for an irregular code can be obtained by a proper choice of parameters $g, h$, and sequences $\nu_i, \eta_i, s_i, r_i$.

Any ensemble of binary LDPC codes can be straightforwardly generalized to the ensemble of NB LDPC codes by randomly assigning elements of $\text{GF}(q)$ to all nonzero elements of parity-check matrix. Similarly to [26], an ensemble of NB LDPC codes over $\text{GF}(2^m)$ determined by the ensemble of irregular bipartite graphs with given degree distributions on variable and check nodes, where each edge is labeled by an element of $\text{GF}(2^m)$, was studied in [7]. In particular, the average symbol weight and bit weight spectra of the random ensemble of irregular NB LDPC codes were derived. However, for finite-length analysis both the ensemble obtained from random bipartite graphs in [27] and its generalization to the NB case in [7] have the same shortcoming. They do not determine irregular codes with predetermined column and row weight distributions $\lambda(x)$ and $\rho(x)$. Due to unavoidable parallel edges in the code Tanner graph, the true degree distributions will differ from the expected and this phenomenon complicates the finite-length analysis of the ensemble. Finite-length analysis for the ensemble in [28] is even more difficult. Only asymptotic generating functions for code spectra were found in [28].

Ensembles of regular both binary and NB LDPC codes were first analyzed by Gallager in [1]. Later a few different ensembles of binary LDPC codes were studied in [29]. Average weight spectra for the corresponding ensembles of regular LDPC codes were derived in [1] and [29]. In [6] asymptotic average weight spectra for ensembles of regular NB LDPC codes over $\text{GF}(2^m)$ were obtained. In [18], we presented a low-complexity recurrent procedure for computing exact spectra of both binary and NB random ensembles of regular LDPC codes.

Since in this paper we deal with codes having only two and three nonzero elements in each column of their parity-check matrices, we slightly modify the ensemble of regular NB LDPC codes and the corresponding low-complexity procedure for computing the average spectra in [18]. In the next subsection, we describe the new ensemble of “almost regular” NB LDPC codes over $\text{GF}(2^m)$ and present a generalized procedure to computing its average weight spectra.
C. Average binary weight spectrum of the ensemble of almost regular NB LDPC codes over $GF(2^m)$

The binary weight distribution of a linear code from a random ensemble can be represented via its weight generating function

$$G_n(s) = \sum_{w=0}^{n} A_{n,w} s^w,$$

where $A_{n,w}$ is the random variable representing the number of binary words of weight $w$ and length $n$. We aim at computing $E\{A_{n,w}\}$, where $E\{\cdot\}$ denotes the mathematical expectation over the code ensemble. Next we describe a new ensemble of NB LDPC codes over $GF(2^m)$ and derive its weight generating function. This ensemble can be considered as a modification of the Gallager ensemble of $q = 2^m$-ary LDPC codes.

For the most deeply studied Gallager ensemble of binary $(J,K)$-regular codes, the parity-check matrix of rate $1 - r/n$ consists of $J$ strips $H_b^T = (H_1^T | H_2^T | \ldots | H_J^T)^T$, where each strip $H_i$ of width $M = r/J$ is a random permutation of the first strip which can be chosen in the form

$$H_i = (I_M \ldots I_M),$$

where $I_M$ is the identity matrix of order $M$.

First we generalize the Gallager ensemble of binary $(J,K)$-regular codes by allowing a given number $K_i \leq K$ of identity matrices and $K - K_i$ of all-zero $M \times M$ submatrices in strips. Without loss of generality the $i$th strip can be chosen as random permutation $\pi_i(\tilde{H}_i)$ where $\tilde{H}_i$ has the form

$$\tilde{H}_i = (I_M \ldots I_M \ 0_M \ldots 0_M), \quad i = 1, \ldots, J,$$

where $0_M$ is the all-zero matrix of order $M$.

That is, the strips in the generalized ensemble are permuted versions of Gallager’s strip with the same identity matrices replaced by the all-zero matrices of the same order. By choosing $K_i$ we adjust the column weight and row weight distributions. We pay most attention to parity-check matrices with $J = 3$, that is, column weights are equal to 2 or 3.

An example of the parity-check matrix for rate 1/2 LDPC code from this ensemble is

$$H_b = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}, \quad (7)$$
where $K = 6$, $J = 3$, $M = 2$, $K_1 = 5$, this matrix consists of two weight-2 and ten weight-3 columns.

By following the approach in [1] we can write the generating function of the number of binary sequences $x$ of weight $w$ and length $n$ satisfying the equality $xH_i^T = 0$, $i = 1, 2, \ldots, J$

$$G_i(s) = \sum_{w=0}^{n} G_{i,n,w} s^w = g_i(s)^M, \quad i = 1, \ldots, J$$  \hspace{1cm} (8)

where

$$g_i(s) = 2^{K-K_i} \sum_{j=0}^{K_i} g_{ij} s^j = 2^{K-K_i} \left( (1 + s)^{K_i} + (1 - s)^{K_i} \right) / 2$$

$g_{ij} = \binom{K_i}{j}$ if $j$ is even and is equal to 0 otherwise.

The probability that the binary sequence $x$ of weight $w$ and length $n$ satisfies $xH_i^T = 0$ can be expressed as

$$p_i(w) = \frac{G_{i,n,w}}{\binom{n}{w}}.$$  \hspace{1cm} (9)

$$E\{A_{n,w}\} = \binom{n}{w}^{1-J} \prod_{j=1}^{J} G_{j,n,w}.$$  \hspace{1cm} (10)

Consider the same generalization of the Gallager ensemble of $q$-ary LDPC codes, where $q = 2^m$, $m \geq 1$ is an integer. The weight generating function of $q$-ary sequences $x$ of length $n$ satisfying the nonzero part of one $q$-ary parity-check equation can be easily obtained by modifying the generating function in [1]. It has the form

$$f_j(s) = q^{K-K_j} \left( (1 + (q-1)s)^{K_j} + (q-1)(1-s)^{K_j} \right) / q$$

Assuming binomial probability distribution of zeros and ones in the $m$-dimensional binary image of the $q$-ary symbol, we can write the average binary weight generating function for the $i$th strip as

$$F_j(s) = \sum_{w=0}^{nm} F_{j,nm,w} s^w = f_j(\phi(s))^M,$$  \hspace{1cm} (11)

where $F_{j,nm,w}$ denotes the average number of binary sequences $b$ of weight $w$ and length $nm$ satisfying $bB_j^T = 0$, $B_i$ is a binary image of $H_j$ and

$$\phi(s) = \sum_{i=1}^{m} \frac{1}{q-1} \binom{m}{i} s^i = \frac{(1 + s)^m - 1}{q-1}.$$  \hspace{1cm} (12)
Then analogously to (9) and (10), we obtain
\[ p_j(w) = \frac{F_{j,nm,w}}{\binom{nm}{w}} \quad (13) \]

\[ E\{A_{n,w,m}\} = \binom{nm}{w} \prod_{j=1}^{J} p_j(w) = \binom{nm}{w}^{1-J} \prod_{j=1}^{J} F_{j,nm,w}, \quad (14) \]

Thus, computing finite-length average spectra for NB LDPC codes is reduced to computing coefficients of series expansion for functions \( F_j(s) \) in (11). This can be done either directly by multiplying polynomials, or recursively as in [17], [18]. In both cases numerical problems can be overcome by performing computations in logarithmic domain.

V. SIMULATIONS AND COMPARISONS

In this section, first we present the tightened upper bounds on the block error probability for finite lengths NB LDPC codes from the random ensemble described in Section IV-C. These bounds are obtained by substituting the average spectra (14) to the Poltyrev bound (18). Comparison of these bounds with the Shannon lower bound and the Poltyrev bound for the random binary linear code of the same length is performed. In the sequel, we use notation SNRb for signal-to-noise ratio per bit measured in dB, \( w \) is the average column weight, \( J \) and \( K \) denote the maximal number of nonzero elements in each column and each row of the parity-check matrix, respectively.

We consider rate \( R = 1/2 \) NB LDPC codes with maximum column weight of their parity-check matrices \( J = 3 \). The corresponding ensembles of random almost regular NB LDPC codes are determined by base matrix of size \( 3 \times 6 \). In all examples parity-check matrices have row weights \( K_1 = K - \alpha, K_2 = K_3 = K \). Parameter \( \alpha \) characterizes sparseness. Values \( \alpha = 0, 1, 2, 3 \) correspond to the average column weights \( w = 3, w = 17/6 = 2.83, w = 16/6 = 2.67, w = 15/6 = 2.50 \), respectively.

It is expected that sparser codes are weaker in sense of ML decoding performance. However, sparseness is important for improving performance of BP decoding. The goal of our computations and simulations is to evaluate a sparsity factor which allows to stay close to optimal codes in sense of ML decoding and to improve as much as possible BP decoding performance.

The average spectra for the ensembles of NB LDPC codes of length about 2000 with different average column weight \( w \) of their parity-check matrices are shown in Figs. 6–7.

In particular, in Fig. 6 we show the average spectra for the random NB LDPC codes over GF\( (2^4) \) and in Fig. 7 the average spectra for the random NB LDPC codes over GF\( (2^6) \) are shown.

Average binary spectra of NB LDPC codes with \( m = 6 \) are closer than the average spectra of NB LDPC codes for \( m = 4 \) to random linear binary code spectra. Moreover, if \( m = 6 \) then even the ensemble of (2,4)-regular NB LDPC codes has rather large
Fig. 6: The average spectra of rate $R = 1/2$ NB QC LDPC codes of length 2080 bits over $GF(2^4)$

Fig. 7: The average spectra of rate $R = 1/2$ NB QC LDPC codes of length 2080 bits over $GF(2^6)$
average minimum distance (close to 50) which makes it potentially efficient. Also, it follows from these plots that for $m = 4$ denser parity-check matrices are needed for achieving near-optimal performance than in case $m = 6$.

The corresponding random coding upper bounds on the block error probability of ML decoding over the AWGN channel are presented in Figs. 8–9 for $m = 4$ and 6, respectively. For comparison in the same figures the Poltyrev bound for the random linear binary code of the same length and the lower Shannon bound are shown.

It follows from the presented plots that

- For $m = 6$ the random NB LDPC code with only two nonzero elements in each column of its parity-check matrix loses about 0.2 dB in SNRb compared to the random linear code. However, for $m = 4$ the corresponding loss in performance is more than or equal to 0.6 dB

- For $m = 6$ the bound on the FER performance of ML decoding for the random $(3,6)$ regular NB LDPC code coincides with the same bound for the random linear binary code. If $m = 4$ we observe a small gap in performance of the corresponding codes

- For $m = 6$ reducing the column weight influences ML decoding performance in less degree than in case of $m = 4$.

Next, we compare the simulated frame error rate (FER) performance of sum-product BP decoding of rate $R = 1/2$ NB QC-LDPC codes of length about 2000 bits with different average column weight $w$ in their base parity-check matrices with the same performance of the rate $R = 1/2$ standard binary QC-LDPC code of length 2096 bits.
Fig. 9: The Poltyrev bounds on the block error probability of rate $R = 1/2$ random NB LDPC codes of length 2184 bits over GF($2^6$)

from the 5G standard. We consider almost regular NB QC-LDPC codes determined by the base matrix of size $26 \times 52$ with column weights two and three, the lifting factor $L$ is chosen to be equal to 10, 8, and 7 for the code over $2^4$, $2^5$, and $2^6$, respectively. The average column weight $w$ takes on values from the set $\{2.0, 2.19, 2.29, 2.38\}$. Simulations were performed until twenty block errors.

The corresponding plots are shown in Figs. [10]-[12].

From the presented plots we can conclude the following:

- If the field extension size is $m = 4$ then in the low SNRb region the NB QC-LDPC codes are inferior than the optimized binary QC-LDPC code of the same length. Moreover, the NB QC LDPC code having average column weight $w = 2.19$ in its base parity-check matrix loses compared the optimized binary QC-LDPC code in the entire SNRb region. Increasing $w$ almost monotonically improves the FER performance. The NB code with $w = 2.38$ wins at least 0.6 dB in the high SNRb region compared to the binary code.

- When $m$ increases, we observe that the FER performance in the error floor region improves monotonically when $w$ grows. At the waterfall region there exists an optimal value of $w$ providing the best performance. For $m = 5$, the optimal value is $w = 2.19$, if $m = 6$, then the optimal value is $w = 2.0$.

- The gap between theoretical bound and simulation results appears due to, first, imperfectness of the constructed codes, and, second, due to suboptimality of BP decoding algorithm. Since for the all simulated codes we observe, the error floor, our conjecture is that by considering the class of QC LDPC codes only, we
Fig. 10: FER performance of rate $R = 1/2$ NB QC LDPC codes over GF($2^4$) of binary length 2080 bits

Fig. 11: FER and BER performance of rate $R = 1/2$ NB QC LDPC codes over GF($2^5$) of binary length 2080 bits, where $SNR$ is the signal-to-noise ratio per bit in dB
impose restrictions on the achievable performance.

- Binary codes from the 5G standard demonstrate very good FER performance of BP decoding in the waterfall region, where they compete with NB QC-LDPC codes. However, in the error floor region NB LDPC codes are superior their binary counterparts.

- The presented simulation results in the waterfall region are about 0.1 – 0.2 dB away from the tightened random coding bounds on the FER performance of ML decoding.

VI. CONCLUSION

A new optimization technique for constructing NB QC-LDPC codes was proposed and analyzed. The key feature of the new technique is applying simulated annealing approach to optimization of base parity-check matrices of NB QC-LDPC codes.

The new ensemble of irregular NB LDPC codes was introduced and analyzed. The similarity of this ensemble to the Gallager ensemble of regular LDPC codes allowed to use the important advantage of the Gallager ensemble - the simplicity of its analysis. By substituting the computed average binary spectra for the new ensemble to the Poltyrev upper bound the tightened finite-length upper bounds on error probability of ML decoding for irregular NB LDPC codes over $\text{GF}(2^m)$ were derived.
The presented simulation results and comparisons with the bounds on error probability showed that NB QC-LDPC codes outperform known binary QC-LDPC codes but as their binary counterparts suffer from severe error floor. Thus, further improvement of both optimization and decoding for this class of codes can be considered as a subject of the further research.

APPENDIX

A. Lower bound

In the sequel, we use approximation [30] of the Shannon lower bound [14]. Denote by $n$, $R$, and $\sigma$ the code length, code rate and standard noise deviation for an AWGN channel, respectively. We use the notations and formulas in [14] for the cone half-angle $\theta \in [0, \pi]$, which corresponds to the solid angle of an $n$-dimensional circular cone, and for the solid angle of the whole space

$$
\Omega_n(\theta) = \frac{2\pi^{n-1}}{\Gamma(n-1/2)} \int_0^\theta (\sin \phi)^{n-2} d\phi, \quad \Omega_n(\pi) = \frac{2\pi^{n/2}}{\Gamma(n/2)},
$$

respectively. For a given code of length $n$ and cardinality $2^nR$ the parameter $\theta_0$ is selected as a solution of the equation

$$
\frac{\Omega_n(\theta_0)}{\Omega_n(\pi)} = 2^{-nR}.
$$

The approximation [30, Theorem 4.2] for the Shannon lower bound [14] on the FER $P_{sh}(n, R, \sigma)$ is

$$
P_{sh}(n, R, \sigma) \geq \sigma \sqrt{n-1} \frac{\exp \left\{ \frac{3\sigma^2 - (\sigma + 1)^2}{2\sigma^2} \right\} e^{-nF_L(\theta)}}{6n(1 + \sigma)},
$$

where

$$
G(\theta) = \frac{1}{2\sigma} \left( \cos \theta + \sqrt{\cos^2 \theta + 4\sigma^2} \right),
$$

$$
F_L(\theta) = \frac{1}{2\sigma^2} \left( 1 - \sigma G(\theta) \cos(\theta) - \sigma^2 \ln(G(\theta) \sin(\theta)) \right).
$$

B. Upper bound

The Poltyrev bound [15] is the most tight TS-type bound:

$$
P_e \leq \int_{-\infty}^{\sqrt{n}} f \left( \frac{x}{\sigma} \right) \left\{ \sum_{w \leq w_0} S_w \Theta_w(x) + 
\right.
$$

$$
\left. + \ 1 - \chi_{n-1}^2 \left( \frac{r^2}{\sigma^2} \right) \right\} dx + Q \left( \frac{\sqrt{n}}{\sigma} \right).
$$
Here \( f(x) = (1/\sqrt{2\pi}) \exp(-x^2/2) \) is the Gaussian probability density function, \( Q(x) = \int_x^\infty f(x) \, dx \),

\[ \Theta_w(x) = \int_{\beta_w(x)}^{r_x} f \left( \frac{y}{\sigma} \right) \chi_{n-2}^2 \left( \frac{r_x^2 - y^2}{\sigma^2} \right) \, dy, \]

\[ w_0 = \frac{r_0^2}{r_0^2 + n}, \quad r_x = r_0 \left( \frac{1 - x}{\sqrt{n}} \right), \]

\[ \mu_w(r) = \frac{1}{r} \sqrt{\frac{w}{1 - w/n}}, \quad \beta_w(x) = \left( 1 - \frac{x}{\sqrt{n}} \right) \sqrt{\frac{w}{1 - w/n}}, \]

\( S_w \) is the \( w \)-th spectrum coefficient, and \( \chi_n^2 \) denotes the probability density function of chi-squared distribution with \( n \) degrees of freedom.

Parameter \( r_0 \) is a solution with respect to \( r \) of the equation

\[ \sum_{w: \mu_w(r) < 1} S_w \int_0^{\arccos \mu_w(r)} \sin^{n-3} \phi \, d\phi = \sqrt{\pi} \frac{\Gamma \left( \frac{n-2}{2} \right)}{\Gamma \left( \frac{n-1}{2} \right)}. \] (19)

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