A LOWER BOUND FOR EIGENVALUES OF
A CLAMPED PLATE PROBLEM*

QING-MING CHENG AND GUOXIN WEI

Abstract. In this paper, we study eigenvalues of a clamped plate problem. We obtain a lower bound for eigenvalues, which gives an important improvement of results due to Levine and Protter.

1. Introduction

Let $M$ be an $n$-dimensional complete Riemannian manifold. The following is called Dirichlet eigenvalue problem of Laplacian:

$$
\begin{align*}
\Delta u &= -\lambda u, & \text{in } \Omega, \\
u &= 0, & \text{on } \partial \Omega,
\end{align*}
$$

(1.1)

where $\Omega$ is a bounded domain in $M$ with piecewise smooth boundary $\partial \Omega$ and $\Delta$ denotes the Laplacian on $M$. It is well known that the spectrum of this eigenvalue problem (1.1) is real and discrete.

$$
0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty,
$$

where each $\lambda_i$ has finite multiplicity which is repeated according to its multiplicity.

Let $V(\Omega)$ denotes the volume of $\Omega$ and $B_n$ the volume of the unit ball in $\mathbb{R}^n$, then the following Weyl’s asymptotic formula holds

$$
\lambda_k \sim \frac{4\pi^2}{(B_n V(\Omega))^\frac{2}{n}} k^n, \quad k \to \infty.
$$

(1.2)

From this asymptotic formula, one can infer

$$
\frac{1}{k} \sum_{i=1}^{k} \lambda_i \sim \frac{n}{n + 2} \frac{4\pi^2}{(B_n V(\Omega))^\frac{2}{n}} k^n, \quad k \to \infty.
$$

(1.3)

In particular, when $M = \mathbb{R}^n$, Pólya [18] proved

$$
\lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^\frac{2}{n}} k^n, \quad \text{for } k = 1, 2, \cdots,
$$

(1.4)

if $\Omega$ is a tiling domain in $\mathbb{R}^n$. Moreover, he conjectured for a general bounded domain,
Conjecture of Pólya. If $\Omega$ is a bounded domain in $\mathbb{R}^n$, then eigenvalue $\lambda_k$ of the eigenvalue problem (1.1) satisfies

$$\lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{n}{2}}} k^{\frac{n}{2}}, \quad \text{for } k = 1, 2, \ldots. \quad (1.5)$$

On the conjecture of Pólya, Li and Yau [13] (cf. [4], [14]) proved

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_i \geq \frac{n}{n + 2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{n}{2}}} k^{\frac{n}{2}}, \quad \text{for } k = 1, 2, \ldots. \quad (1.6)$$

It is sharp about the highest order term of $k$ in the sense of average according to (1.3). From this formula, one can derive

$$\lambda_k \geq \frac{n}{n + 2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{n}{2}}} k^{\frac{n}{2}}, \quad \text{for } k = 1, 2, \ldots, \quad (1.7)$$

which gives a partial solution for the conjecture of Pólya with a factor $\frac{n}{n+2}$.

Furthermore, Melas [15] obtained the following estimate which is an improvement of (1.6).

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_i \geq \frac{n}{n + 2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{n}{2}}} k^{\frac{n}{2}} + c_n \frac{V(\Omega)}{I(\Omega)}, \quad \text{for } k = 1, 2, \ldots, \quad (1.8)$$

where $c_n$ is a constant depending only on the dimension $n$ and

$$I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x - a|^2 dx$$

is called the moment of inertia of $\Omega$.

For a bounded domain in an $n$-dimensional complete Riemannian manifold, Cheng and Yang [9] have also given a lower bound for eigenvalues, recently.

Our purpose in this paper is to study eigenvalues of the following clamped plate problem. Let $\Omega$ be a bounded domain in an $n$-dimensional complete Riemannian manifold $M^n$. The following is called a clamped plate problem, which describes characteristic vibrations of a clamped plate:

$$\begin{cases}
\Delta^2 u = \Gamma u, & \text{in } \Omega, \\
u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial \Omega,
\end{cases} \quad (1.9)$$

where $\Delta$ is the Laplacian on $M^n$ and $\nu$ denotes the outward unit normal to the boundary $\partial \Omega$. It is well known that this problem has a real and discrete spectrum

$$0 < \Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_k \leq \cdots \to +\infty,$$

where each $\Gamma_i$ has finite multiplicity which is repeated according to its multiplicity.

For eigenvalues of the clamped plate problem, Agmon [1] and Pleijel [17] gave the following asymptotic formula, which is a generalization of Weyl’s asymptotic formula,

$$\Gamma_k \sim \frac{16\pi^4}{(B_n V(\Omega))^{\frac{n}{2}}} k^{\frac{n}{2}}, \quad k \to \infty. \quad (1.10)$$
The average of the eigenvalues satisfies
\[ \frac{1}{k} \sum_{j=1}^{k} \Gamma_j \sim \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^\frac{4}{n}} k^\frac{4}{n}, \quad k \to \infty. \quad (1.11) \]

Furthermore, Levine and Protter \[12\] proved that eigenvalues of the clamped plate problem satisfy
\[ \frac{1}{k} \sum_{j=1}^{k} \Gamma_j \geq \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^\frac{4}{n}} k^\frac{4}{n}. \quad (1.12) \]

The inequality (1.12) is sharp about the highest order term of \( k \) according to (1.11).

In this paper, we give an important improvement of the result due to Levine and Protter \[12\] by adding to its right hand side two terms of the lower order terms of \( k \). In fact, we prove the following:

**Theorem.** Let \( \Omega \) be a bounded domain in an \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). Eigenvalues of the clamped plate problem satisfy
\[
\frac{1}{k} \sum_{j=1}^{k} \Gamma_j \geq \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^\frac{4}{n}} k^\frac{4}{n} \\
+ \left( \frac{n+2}{12n(n+4)} - \frac{1}{1152n^2(n+4)} \right) \frac{V(\Omega)}{I(\Omega)} \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^\frac{4}{n}} k^\frac{4}{n} \\
+ \left( \frac{1}{576n(n+4)} - \frac{1}{27648n^2(n+2)(n+4)} \right) \left( \frac{V(\Omega)}{I(\Omega)} \right)^2,
\]
where \( I(\Omega) \) is the moment of inertia of \( \Omega \).

**Corollary.** Let \( \Omega \) be a bounded domain in an \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). Then eigenvalues \( \Gamma_j \)'s of the clamped plate problem satisfy
\[
\frac{1}{k} \sum_{j=1}^{k} \Gamma_j \geq \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^\frac{4}{n}} k^\frac{4}{n} \\
+ \left( \frac{n+2}{12n(n+4)} - \frac{1}{1152n^2(n+4)} \right) \frac{1}{\sum_{i=1}^{n} \mu_i^{-1}} \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^\frac{4}{n}} k^\frac{4}{n} \\
+ \left( \frac{1}{576n(n+4)} - \frac{1}{27648n^2(n+2)(n+4)} \right) \left( \frac{1}{\sum_{i=1}^{n} \mu_i^{-1}} \right)^2,
\]
where \( \mu_1, \cdots, \mu_n \) are the first \( n \) nonzero eigenvalues of the Neumann eigenvalue problem of Laplacian
\[
\begin{aligned}
\Delta v &= -\mu v, \quad \text{in } \Omega, \\
\frac{\partial v}{\partial \nu} &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]
Remark 1. On universal estimates for eigenvalues of the clamped plate problem, one can see [6], [7], [8], [10] and [20].

2. Proof of results

For a bounded domain $\Omega$, the moment of inertia of $\Omega$ is defined by

$$I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x - a|^2 dx.$$ 

By a translation of the origin and a suitable rotation of axes, we can assume that the center of mass is the origin and

$$I(\Omega) = \int_{\Omega} |x|^2 dx.$$ 

For reader’s convenience, we first review the definition and several properties of the symmetric decreasing rearrangements. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Its symmetric rearrangement $\Omega^*$ is the open ball with the same volume as $\Omega$,

$$\Omega^* = \{ x \in \mathbb{R}^n \mid |x| < \left( \frac{\text{Vol}(\Omega)}{B_n} \right)^{\frac{1}{n}} \}.$$ 

By using a symmetric rearrangement of $\Omega$, we have

$$I(\Omega) = \int_{\Omega} |x|^2 dx \geq \int_{\Omega^*} |x|^2 dx = \frac{n}{n + 2} V(\Omega) \left( \frac{V(\Omega)}{B_n} \right)^{\frac{2}{n}}.$$ \hspace{1cm} (2.1)

Let $h$ be a nonnegative bounded continuous function on $\Omega$, we can consider its distribution function $\mu_h(t)$ defined by

$$\mu_h(t) = \text{Vol}(\{ x \in \Omega \mid h(x) > t \}).$$

The distribution function can be viewed as a function from $[0, \infty)$ to $[0, V(\Omega)]$. The symmetric decreasing rearrangement $h^*$ of $h$ is defined by

$$h^*(x) = \inf\{ t \geq 0 \mid \mu_h(t) < B_n |x|^n \}$$

for $x \in \Omega^*$. By definition, we know that $\text{Vol}(\{ x \in \Omega \mid h(x) > t \}) = \text{Vol}(\{ x \in \Omega^* \mid h^*(x) > t \})$, $\forall t > 0$ and $h^*(x)$ is a radially symmetric function.

Putting $g(|x|) := h^*(x)$, one gets that $g : [0, +\infty) \to [0, \sup h]$ is a non-increasing function of $|x|$. Using the well known properties of the symmetric decreasing rearrangement, we obtain

$$\int_{\mathbb{R}^n} h(x) dx = \int_{\mathbb{R}^n} h^*(x) dx = n B_n \int_0^\infty s^{n-1} g(s) ds$$ \hspace{1cm} (2.2)

and

$$\int_{\mathbb{R}^n} |x|^4 h(x) dx \geq \int_{\mathbb{R}^n} |x|^4 h^*(x) dx = n B_n \int_0^\infty s^{n+3} g(s) ds.$$ \hspace{1cm} (2.3)

Good sources of further information on rearrangements are [3], [19].

One gets from the coarea formula that

$$\mu_h(t) = \int_t^{\sup h} \int_{\{h=s\}} |\nabla h|^{-1} d\sigma_s ds.$$
Since $h^*$ is radial, we have
\[
\mu_h(g(s)) = \text{Vol}\{x \in \Omega | h(x) > g(s)\} = \text{Vol}\{x \in \Omega^* | h^*(x) > g(s)\} = \text{Vol}\{x \in \Omega^* | g(|x|) > g(s)\} = B_n s^n.
\]

It follows that
\[
nB_n s^{n-1} = \mu_h'(g(s))g'(s)
\]
for almost every $s$. Putting $\tau := \sup |\nabla h|$, we obtain from the above equations and the isoperimetric inequality that
\[
-\mu_h'(g(s))g'(s) = \int_{\{h = g(s)\}} |\nabla h|^{-1}d\sigma_{g(s)} \geq \tau^{-1}\text{Vol}_{n-1}\{h = g(s)\}
\]
\[
\geq \tau^{-1}nB_n s^{n-1}.
\]

Therefore, one obtains
\[
-\tau \leq g'(s) \leq 0 \quad (2.4)
\]
for almost every $s$.

Next, we prepare the following lemma in order to prove of our theorem.

**Lemma 2.1.** Let $b \geq 1$, $\eta > 0$ and $\psi : [0, +\infty) \to [0, +\infty)$ be a decreasing smooth function such that
\[
-\eta \leq \psi'(s) \leq 0
\]
and, for a constant $d < 1$,
\[
\frac{\psi(0)^{2b+2}}{6b\eta^2(bA)^{\frac{4}{b}}} < d
\]
with
\[
A := \int_0^\infty s^{b-1}\psi(s)ds > 0.
\]

Then, we have
\[
\int_0^\infty s^{b+3}\psi(s)ds \geq \frac{1}{b+4}(bA)^{\frac{b+4}{b}}\psi(0)^{-\frac{4}{b}}
\]
\[
+ \left( \frac{1}{3b(b+4)\eta^2} - \frac{d}{6(b+2)^2(b+4)\eta^2} \right) (bA)^{\frac{b+2}{b}}\psi(0)^{\frac{2b-2}{b}} \quad (2.5)
\]
\[
+ \left( \frac{1}{36b(b+4)\eta^4} - \frac{d}{36(b+2)^2(b+4)\eta^4} \right) A\psi(0)^4.
\]

**Proof.** Defining
\[
D := \int_0^\infty s^{b+1}\psi(s)ds,
\]
one can prove from the same assertions as in the lemma 1 of [15],
\[
D = \int_0^\infty s^{b+1}\psi(s)ds \geq \frac{1}{b+2}(bA)^{\frac{b+2}{b}}\psi(0)^{-\frac{2}{b}} + \frac{A\psi(0)^2}{6(b+2)\eta^2}. \quad (2.6)
\]
Since the formula (2.6) holds for any constant $b \geq 1$, we have
\[
\int_0^\infty s^{b+3} \psi(s) ds \\
\geq \frac{1}{b+4} \left( (b+2) D \right)^\frac{1}{b+4} \psi(0)^{-\frac{2}{b+2}} + \frac{D \psi(0)^2}{6(b+4) \eta^2} \\
\geq \frac{1}{b+4} \left[ (bA)^\frac{b+2}{b} \psi(0)^{-\frac{2}{b}} + \frac{A \psi(0)^2}{6\eta^2} \right]^{\frac{1}{b+4}} \psi(0)^{-\frac{2}{b+2}} \\
+ \frac{\psi(0)^2}{6(b+4) \eta^2} \left[ \frac{1}{b+2} (bA)^\frac{b+2}{b} \psi(0)^{-\frac{2}{b}} + \frac{A \psi(0)^2}{6(b+2) \eta^2} \right] \\
= \frac{1}{b+4} \left[ (bA)^\frac{b+2}{b} \psi(0)^{-\frac{2}{b}} + \frac{A \psi(0)^2}{6\eta^2} \right]^{\frac{1}{b+4}} \psi(0)^{-\frac{2}{b+2}} \\
\times \left( 1 + \frac{A \psi(0)^\frac{b}{b+2}}{6(bA)^\frac{b}{b+2} \eta^2} \left\{ 2 - \frac{b}{b+2} \frac{A \psi(0)^\frac{b}{b+2}}{6(bA)^\frac{b}{b+2} \eta^2} \right\} \right) \psi(0)^{-\frac{2}{b+2}} \\
\times \left( \text{from the Taylor formula} \right) \\
+ \frac{1}{6(b+2)(b+4) \eta^2} (bA)^\frac{b+2}{b} \psi(0)^\frac{2b-2}{b} + \frac{A \psi(0)^4}{36(b+2)(b+4) \eta^4} \\
\geq \frac{1}{b+4} \left[ (bA)^\frac{b+2}{b} \psi(0)^{-\frac{2}{b}} + \frac{A \psi(0)^2}{6\eta^2} \right]^{\frac{1}{b+4}} \psi(0)^{-\frac{2}{b+2}} \\
\times \left\{ 1 + \frac{1}{b+2} \frac{A \psi(0)^\frac{b}{b+2}}{6(bA)^\frac{b}{b+2} \eta^2} \left( 2 - \frac{b}{b+2} \frac{A \psi(0)^\frac{b}{b+2}}{6(bA)^\frac{b}{b+2} \eta^2} \right) \right\} \psi(0)^{-\frac{2}{b+2}} \\
+ \frac{1}{6(b+2)(b+4) \eta^2} (bA)^\frac{b+2}{b} \psi(0)^\frac{2b-2}{b} + \frac{A \psi(0)^4}{36(b+2)(b+4) \eta^4} \\
= \frac{1}{b+4} (bA)^\frac{b+4}{b} \psi(0)^{-\frac{4}{b}} \\\n+ \left( \frac{1}{3b(b+4) \eta^2} - \frac{d}{6(b+2)^2(b+4) \eta^2} \right) (bA)^\frac{b+2}{b} \psi(0)^\frac{2b-2}{b} \\
+ \left( \frac{1}{36b(b+4) \eta^4} - \frac{d}{36(b+2)^2(b+4) \eta^4} \right) A \psi(0)^4.
\]
This completes the proof of the lemma.
Proof of Theorem. Let \( u_j \) be an orthonormal eigenfunction corresponding to the eigenvalue \( \Gamma_j \), that is, \( u_j \) satisfies

\[
\begin{aligned}
\Delta^2 u_j &= \Gamma_j u_j, & \text{in } \Omega, \\
u_j &= \frac{\partial u_j}{\partial \nu} = 0, & \text{on } \partial \Omega, \\
\int u_i u_j &= \delta_{ij}, & \text{for any } i, j.
\end{aligned}
\] (2.7)

Thus, \( \{u_j\}_{j=1}^\infty \) forms an orthonormal basis of \( L^2(\Omega) \). We define a function \( \varphi_j \) by

\[
\varphi_j(x) = \begin{cases} 
 u_j(x), & x \in \Omega, \\
 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

Denote by \( \hat{\varphi}_j(z) \) the Fourier transform of \( \varphi_j(x) \). For any \( z \in \mathbb{R}^n \), we have by definition that

\[
\hat{\varphi}_j(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi_j(x)e^{ix \cdot z} dx = (2\pi)^{-n/2} \int_{\Omega} u_j(x)e^{ix \cdot z} dx. \quad (2.8)
\]

From the Plancherel formula, we have

\[
\int_{\mathbb{R}^n} \hat{\varphi}_i(z)\hat{\varphi}_j(z) dz = \delta_{ij}
\]

for any \( i, j \). Since \( \{u_j\}_{j=1}^\infty \) is an orthonormal basis in \( L^2(\Omega) \), the Bessel inequality implies that

\[
\sum_{j=1}^k |\hat{\varphi}_j(z)|^2 \leq (2\pi)^{-n} \int_{\Omega} |e^{ix \cdot z}|^2 dx = (2\pi)^{-n}V(\Omega). \quad (2.9)
\]

For each \( q = 1, \ldots, n, \ j = 1, \ldots, k \), we deduce from the divergence theorem and \( u_j|_{\partial \Omega} = \frac{\partial u_j}{\partial \nu}|_{\partial \Omega} = 0 \) that

\[
\begin{aligned}
z_q^2 \hat{\varphi}_j(z) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi_j(x)(-i)^2 \frac{\partial^2 e^{ix \cdot z}}{\partial x_q^2} dx \\
&= -(2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{\partial^2 \varphi_j(x)}{\partial x_q^2} e^{ix \cdot z} dx \\
&= -\frac{\partial^2 \varphi_j}{\partial x_q^2}(z).
\end{aligned}
\] (2.10)
It follows from the Parseval’s identity that
\[
\int_{\mathbb{R}^n} |z|^4 |\hat{\varphi}_j(z)|^2 dz = \int_{\mathbb{R}^n} |z|^2 |\hat{\varphi}_j(z)|^2 dz \\
= \int_{\mathbb{R}^n} \left| \sum_{q=1}^n \frac{\partial^2 \varphi_j}{\partial x_q^2}(z) \right|^2 dz \\
= \int_{\Omega} \left( \sum_{q=1}^n \frac{\partial^2 u_j}{\partial x_q^2} \right)^2 dx \\
= \int_{\Omega} |u_j(x)|^2 dx \\
= \int_{\Omega} \Delta u_j(x) dx \\
= \int_{\Omega} \text{Γ}_j u_j^2(x) dx \\
= \text{Γ}_j. \tag{2.11}
\]

Since
\[
\nabla \hat{\varphi}_j(z) = (2\pi)^{-n/2} \int_{\Omega} ixu_j(x) e^{i<x,z>} dx, \tag{2.12}
\]
we obtain
\[
\sum_{j=1}^k |\nabla \hat{\varphi}_j(z)|^2 \leq (2\pi)^{-n} \int_{\Omega} |ixe^{i<x,z>}|^2 dx = (2\pi)^{-n} I(\Omega). \tag{2.13}
\]

Putting
\[
h(z) := \sum_{j=1}^k |\hat{\varphi}_j(z)|^2,
\]
one derives from (2.9) that \(0 \leq h(z) \leq (2\pi)^{-n} V(\Omega)\), it follows from (2.13) and the Cauchy-Schwarz inequality that
\[
|\nabla h(z)| \leq 2 \left( \sum_{j=1}^k |\hat{\varphi}_j(z)|^2 \right)^{1/2} \left( \sum_{j=1}^k |\nabla \hat{\varphi}_j(z)|^2 \right)^{1/2} \leq 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)} \tag{2.14}
\]
for every \(z \in \mathbb{R}^n\). From the Parseval’s identity, we derive
\[
\int_{\mathbb{R}^n} h(z) dz = \sum_{j=1}^k \int_{\Omega} |u_j(x)|^2 dx = k. \tag{2.15}
\]

Applying the symmetric decreasing rearrangement to \(h\) and noting that \(\tau = \sup |\nabla h| \leq 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)} := \eta\), we obtain, from (2.4),
\[
-\eta \leq -\tau \leq g'(s) \leq 0 \tag{2.16}
\]
for almost every \( s \). According to (2.2) and (2.15), we infer
\[
k = \int_{\mathbb{R}^n} h(z)dz = \int_{\mathbb{R}^n} h^*(z)dz = nB_n \int_0^\infty s^{n-1}g(s)ds.
\] (2.17)

From (2.3) and (2.11), we obtain
\[
\sum_{j=1}^k \Gamma_j = \int_{\mathbb{R}^n} |z|^4 h(z)dz \\
\geq \int_{\mathbb{R}^n} |z|^4 h^*(z)dz \\
= nB_n \int_0^\infty s^{n+3}g(s)ds.
\] (2.18)

In order to apply Lemma 2.1, from (2.17) and the definition of \( A \), we take
\[
\psi(s) = g(s), \quad A = \frac{k}{nB_n}, \quad \eta = 2(2\pi)^{-\frac{n}{2}}\sqrt{V(\Omega)I(\Omega)},
\] (2.19)

from (2.1), we deduce that
\[
\eta \geq 2(2\pi)^{-\frac{n}{2}} \left( \frac{n}{n+2} \right)^{\frac{1}{2}} B_n^{-\frac{n}{2}} V(\Omega)^{-\frac{n+1}{n}}.
\] (2.20)

On the other hand, \( 0 < g(0) \leq \sup h^*(z) = \sup h(z) \leq (2\pi)^{-n}V(\Omega) \), we have from (2.1), (2.19) and (2.20) that
\[
\frac{g(0)^{\frac{2n+2}{n}}}{6n\eta^2(nA)\frac{2}{n}} \leq \frac{(\frac{n}{n+2})^{\frac{1}{2}}}{6n(2\pi)^{-\frac{n}{2}}(\frac{n}{n+2})^{\frac{n}{2}}} B_n^{-\frac{n}{2}} V(\Omega)\frac{n+1}{n} \left( \frac{k}{n} \right)^{\frac{n}{2}}
\]
\[
= \frac{n+2}{24n^2(2\pi)^2k} B_n^{\frac{4}{n}} \leq \frac{n+2}{24n^2(2\pi)^2} B_n^{\frac{4}{n}}.
\]

By a direct calculation, one sees from \( B_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})} \) that
\[
\frac{B_n^{\frac{4}{n}}}{(2\pi)^2} < \frac{1}{2},
\] (2.21)

where \( \Gamma(\frac{n}{2}) \) is the Gamma function. From the above arguments, one has
\[
\frac{g(0)^{\frac{2n+2}{n}}}{6n\eta^2(nA)\frac{2}{n}} \leq \frac{n+2}{48n^2} < 1.
\] (2.22)

Hence we know that the function \( \psi(s) = g(s) \) satisfies the conditions in Lemma 2.1 with \( b = n \) and
\[
\eta = 2(2\pi)^{-\frac{n}{2}}\sqrt{V(\Omega)I(\Omega)}, \quad d = \frac{n+2}{48n^2}.
\]
From Lemma 2.1 and (2.18), we conclude

\[
\sum_{j=1}^{k} \Gamma_j \geq nB_n \int_{0}^{\infty} s^{n+3} g(s) ds \\
\geq \frac{n}{n+4} (B_n)^{-\frac{4}{n+4}} k^{\frac{n+2}{n+4}} g(0)^{-\frac{4}{n+4}} \\
+ \left( \frac{1}{3(n+4)\eta^2} - \frac{1}{288n(n+2)(n+4)\eta^2} \right) k^{\frac{n+2}{n}} (B_n)^{-\frac{2}{n}} g(0)^{\frac{2n-2}{n}} \\
+ \left( \frac{1}{36n(n+4)\eta^4} - \frac{1}{1728n^2(n+2)(n+4)\eta^4} \right) k g(0)^4.
\]

(2.23)

Defining a function \( F \) by

\[
F(t) = \frac{n}{n+4} (B_n)^{-\frac{4}{n+4}} k^{\frac{n+2}{n+4}} t^{-\frac{4}{n+4}} \\
+ \left( \frac{1}{3(n+4)\eta^2} - \frac{1}{288n(n+2)(n+4)\eta^2} \right) k^{\frac{n+2}{n}} (B_n)^{-\frac{2}{n}} t^{\frac{2n-2}{n}} \\
+ \left( \frac{1}{36n(n+4)\eta^4} - \frac{1}{1728n^2(n+2)(n+4)\eta^4} \right) k t^4.
\]

(2.24)

It is not hard to prove from (2.20) that \( \eta \geq (2\pi)^{-n} B_n^{-\frac{1}{n}} V(\Omega)^{-\frac{n+1}{n}} \). Furthermore, it follows from (2.24) that

\[
F'(t) \\
\leq -\frac{4}{n+4} (B_n)^{-\frac{4}{n+4}} k^{\frac{n+2}{n+4}} t^{-\frac{4}{n+4}} \\
+ \left( \frac{2(n-1)}{3n(n+4)} - \frac{(n-1)}{144n^2(n+2)(n+4)} \right) k^{\frac{n+2}{n}} (2\pi)^{2n} V(\Omega)^{-\frac{2(n+1)}{n}} t^{\frac{2n-2}{n}} \\
+ \left( \frac{1}{9n(n+4)} - \frac{1}{432n^2(n+2)(n+4)} \right) k t^3 (2\pi)^{4n}(B_n)^{\frac{4}{n}} V(\Omega)^{-\frac{4(n+1)}{n}} \\
= -\frac{k}{n+4} t^{-\frac{n+2}{n}} \times \left\{ \left( \frac{2(n-1)}{3n} - \frac{(n-1)}{144n^2(n+2)} \right)(2\pi)^{2n} k^{\frac{4}{n}} V(\Omega)^{-\frac{2(n+1)}{n}} t^{\frac{2n-2}{n}} \\
- 4(B_n)^{-\frac{4}{n}} k^{\frac{4}{n}} + \left( \frac{1}{9n} - \frac{1}{432n^2(n+2)} \right)(2\pi)^{4n}(B_n)^{\frac{4}{n}} V(\Omega)^{-\frac{4(n+1)}{n}} t^{\frac{4n-2}{n}} \right\}.
\]

Hence, we have

\[
\frac{n+4}{k} t^{\frac{n+4}{n}} F'(t) \\
\leq \left( \frac{2(n-1)}{3n} - \frac{(n-1)}{144n^2(n+2)} \right)(2\pi)^{2n} k^{\frac{4}{n}} V(\Omega)^{-\frac{2(n+1)}{n}} t^{\frac{2n-2}{n}} \\
- 4(B_n)^{-\frac{4}{n}} k^{\frac{4}{n}} + \left( \frac{1}{9n} - \frac{1}{432n^2(n+2)} \right)(2\pi)^{4n}(B_n)^{\frac{4}{n}} V(\Omega)^{-\frac{4(n+1)}{n}} t^{\frac{4n-2}{n}}.
\]

(2.25)
Since the right hand side of (2.25) is an increasing function of $t$, if it is not larger than 0 at $t = (2\pi)^{-n}V(\Omega)$, that is,
\[
\left(\frac{2(n-1)}{3n} - \frac{(n-1)}{144n^2(n+2)}\right)(2\pi)^{2n}k^{2n} V(\Omega)^{ \frac{2(n+1)}{n}} ((2\pi)^{-n}V(\Omega))^{\frac{2n+2}{n}} \\
+ \left(\frac{1}{9n} - \frac{1}{432n^2(n+2)}\right)(2\pi)^{4n}(B_n)^{\frac{4}{n}} V(\Omega)^{ \frac{4(n+1)}{n}} ((2\pi)^{-n}V(\Omega))^{\frac{4n+4}{n}}
\]
then one has from (2.25) that $F(t) \leq 0$ on $(0, (2\pi)^{-n}V(\Omega)]$. Hence, $F(t)$ is decreasing on $(0, (2\pi)^{-n}V(\Omega)]$. Indeed, by a direct calculation, we have that (2.26) is equivalent to
\[
\left(\frac{(n-1)}{6n} - \frac{(n-1)}{576n^2(n+2)}\right)(2\pi)^{-2}k^{\frac{2n}{n}} \\
+ \left(\frac{1}{36n} - \frac{1}{1728n^2(n+2)}\right)(2\pi)^{-4}(B_n)^{\frac{4}{n}} \leq (B_n)^{-\frac{4}{n}} k^{\frac{4}{n}}.
\]

From (2.21), we can prove that $(2\pi)^{-2}(B_n)^{\frac{4}{n}} < 1$ and
\[
\left(\frac{(n-1)}{6n} - \frac{(n-1)}{576n^2(n+2)}\right)(2\pi)^{-2}k^{\frac{2n}{n}} \\
+ \left(\frac{1}{36n} - \frac{1}{1728n^2(n+2)}\right)(2\pi)^{-4}(B_n)^{\frac{4}{n}} < \frac{1}{6}(2\pi)^{-2}k^{\frac{2n}{n}} + \frac{1}{36n}(2\pi)^{-2} \\
< (2\pi)^{-2}\left\{\frac{1}{6}k^{\frac{4}{n}} + \frac{1}{36n}\right\} < (2\pi)^{-2}k^{\frac{4}{n}} < (B_n)^{-\frac{4}{n}} k^{\frac{4}{n}},
\]
that is, $F(t)$ is a decreasing function on $(0, (2\pi)^{-n}V(\Omega)]$.

On the other hand, since $0 < g(0) \leq (2\pi)^{-n}V(\Omega)$ and the right hand side of the formula (2.23) is $F(g(0))$, which is a decreasing function of $g(0)$ on $(0, (2\pi)^{-n}V(\Omega)]$, then we can replace $g(0)$ by $(2\pi)^{-n}V(\Omega)$ in (2.23) which gives inequality
\[
\frac{1}{k} \sum_{j=1}^{k} \Gamma_j \geq \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}} \\
+ \left(\frac{n+2}{12n(n+4)} - \frac{1}{1152n^2(n+4)}\right)\frac{V(\Omega)}{I(\Omega)} n + 2 (B_n V(\Omega))^{\frac{4}{n}} k^{\frac{4}{n}} \\
+ \left(\frac{1}{576n(n+4)} - \frac{1}{27648n^2(n+4)}\right) \left(\frac{V(\Omega)}{I(\Omega)}\right)^2.
\]
This completes the proof of Theorem.
Proof of Corollary. Let \( v_1, \ldots, v_n \) be \( n \) orthonormal eigenfunctions corresponding to the first \( n \) eigenvalues \( \mu_1, \ldots, \mu_n \) of the Neumann eigenvalue problem of Laplacian, that is,

\[
\begin{cases}
\Delta v_i = -\mu_i v_i, & \text{in } \Omega, \\
\frac{\partial v_i}{\partial \nu} = 0, & \text{on } \partial \Omega, \\
\int_\Omega v_i v_j = \delta_{ij}, & i, j = 1, \ldots, n.
\end{cases}
\]

It then follows from the inequality (2.8) in [2] that

\[
\sum_{i=1}^{n} \frac{1}{\mu_i} \geq \frac{\int_\Omega |x|^2 dx}{V(\Omega)}.
\]  

(2.29)

Combining (1.13) and (2.29), we have the inequality (1.14).

\[ \square \]

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QING-MING CHENG, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ENGINEERING, SAGA UNIVERSITY, SAGA 840-8502, JAPAN, cheng@ms.saga-u.ac.jp

GUOXIN WEI, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ENGINEERING, SAGA UNIVERSITY, SAGA 840-8502, JAPAN, wei@ms.saga-u.ac.jp