Canonical path integral quantization of Einstein’s gravitational field

October 29, 2018

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Abstract

The connection between the canonical and the path integral formulations of Einstein’s gravitational field is discussed using the Hamilton Jacobi method. Unlike conventional methods, its shown that our path integral method leads to obtain the measure of integration with no δ-functions, no need to fix any gauge and so no ambiguous determinants will appear.

1 Introduction

A Lagrangian $L(q, \dot{q}, t)$, is called regular if the rank of the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j},$$

is $n$. On the other hand the Lagrangian is called singular if the rank of the Hessian is less than $n$ e.g. $(n - r)$, $r > 0$. Systems with this property are equivalently called ”singular Lagrangians”, ”constrained systems” or ”singular systems”.

Studies on singular systems started around 1950’s. Dirac [1,2], initiated the well known method to investigate the Hamiltonian formulation of constrained systems. His formulation became the fundamental tool for the study of classical systems of particles and fields. Bergman [3] and his collaborators work the relationship between invariance principle and constraints in field theories. Their efforts are to quantize Einstein’s theory of gravitation since this theory is a singular due to its general covariance.

The study of Einstein’s theory of gravitation using Dirac’s and Faddeev’s methods has been widely investigated by many authors [4-13]. In this paper we would like to obtain the path integral quantization of Einstein’s theory of gravitation using the canonical path integral method [14-18].
2 A brief review on the canonical path integral method

The canonical formulation [19-22] gives the set of Hamilton-Jacobi partial differential equations (HJPDE) as

\[ H'_\alpha(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_a}) = 0, \]
\[ \alpha, \beta = 0, n - r + 1, ..., n, a = 1, ..., n - r, \]

(2)

where

\[ H'_\alpha = H_\alpha(t_\beta, q_a, p_a) + p_\alpha, \]

(3)

and \( H_0 \) is defined as

\[ H_0 = p_a W_a + p_\mu \dot{q}_\mu \big|_{\mu = n - r} - H_\mu - L(t, q_i, \dot{q}_\nu, \dot{q}_a = W_a), \]
\[ \mu, \nu = n - r + 1, ..., n. \]

(4)

The equations of motion are obtained as total differential equations in many variables as follows:

\[ dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha; \]
\[ dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha; \]
\[ dp_\beta = -\frac{\partial H'_\alpha}{\partial t_\beta} dt_\alpha; \]
\[ dZ = (-H'_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}) dt_\alpha; \]
\[ \alpha, \beta = 0, n - r + 1, ..., n, a = 1, ..., n - r \]

(5-8)

where \( Z = S(t_\alpha; q_a) \). The set of equations (5-8) is integrable [19] if

\[ dH'_0 = 0, \]
\[ dH'_\mu = 0, \mu = n - p + 1, ..., n, \]

(9-10)

or in equivalent form

\[ [H'_\alpha, H'_\beta] = 0 \forall \alpha, \beta. \]

(11)

Equations of motion reveal the fact that the Hamiltonians \( H'_\alpha \) are considered as the infinitesimal generators of canonical transformations given by parameters \( t_\alpha \) and the set of canonical phase-space coordinates \( q_a \) and \( p_a \) is obtained as functions of \( t_\alpha \), besides the canonical action integral is obtained in terms of the canonical coordinates. In this case, the path integral representation may be written as [14-18]

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2
\[ D(q'_a, t'_a; q_a, t_a) = \int_{q_a}^{q'_a} \int_{p_a} Dq^a \, Dp^a \times \exp \{ i \int_{t_a}^{t'_a} \left[ -H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right] dt_\alpha \}, \]
\[ a = 1, \ldots, n - r, \alpha = 0, n - r + 1, \ldots, n. \quad (12) \]

Now we will study the path integral quantization of Einstein’s gravitational theory considering the method given in section 2.

3 An example

Let us consider the Lagrangian density of Einstein’s gravitational field as [4,5]
\[ \mathcal{L} = N^\perp g^{\perp \frac{1}{2}} \left( R + K_{ij} K^{ij} - k^2 \right), \quad (13) \]
where \( g_{\mu\nu} \) are the metric and \( R \) is the Riemann curvature scalar. The four functions \( N_\mu \) are treated as position variables, which are hidden in the extrinsic curvature
\[ K_{ij} = \frac{1}{2N} \left( N_i |_j + N_j |_i - g_{ij,0} \right). \quad (14) \]
The canonical momenta conjugated to \( N^\mu \) are
\[ p_\mu = \frac{\partial \mathcal{L}}{\partial (\partial_0 N^\mu)} = 0, \quad (15) \]
those conjugated to \( g_{ij} \) are
\[ \pi^{ij} = \frac{\partial \mathcal{L}}{\partial (\partial_0 g_{ij})} = -g^{\perp \frac{1}{2}} \left( K^{ij} - kg^{ij} \right). \quad (16) \]
Taking the trace on both sides of relation (16), one gets
\[ \pi = \pi^i_i = 2g^{\perp \frac{1}{2}} k. \quad (17) \]
Hence, equation (16) can be solved for the \( k^{ij} \) as
\[ k^{ij} = -g^{\perp \frac{1}{2}} \pi^{ij} - \frac{1}{2} \pi g^{ij}, \quad (18) \]
in this case the “velocities” \( g_{ij,0} \) can be expressed in terms of the momenta \( \pi^{ij} \) as
\[ g_{ij,0} = -2g^{\perp \frac{1}{2}} N^\perp \left( \pi^{ij} - \frac{1}{2} \pi g^{ij} \right) - N_i |_j - N_j |_i. \quad (19) \]
The canonical Hamiltonian density takes the form
\[ \mathcal{H}_0 = p_\mu \partial_0 N^\mu + \pi^{ij} \partial_0 g_{ij} - \mathcal{L}. \quad (20) \]
Making use of the primary constraint (15) and the expressible velocities $g_{ij,0}$, we have

$$H_0 = 2\pi^{ij} N_{ij} - g^{ij} N_\perp \left( \frac{1}{2} \pi^2 - \pi^{ij} \pi_{ij} + Rg \right).$$

(21)

After partial integration and neglecting the surface term, the total canonical Hamiltonian can be expressed as

$$H_0 = \int d^3x (N_\perp H_\perp + N^i H_i),$$

(22)

where

$$H_\perp = g^{ij} (\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2) - Rg^2,$$

(23)

$$H_i = -2\pi^i_i.$$

(24)

Starting from the Hamiltonian (22) and making use of (15), the canonical method [19-22] lead us to obtain the set of Hamilton Jacobi partial differential equations as

$$H'_0 = p_0 + H_0 = 0;\quad p_0 = \frac{\partial S}{\partial \tau},$$

(25)

$$H' = p_\mu = 0;\quad p_\mu = \frac{\partial S}{\partial N^\mu}. $$

(26)

Thus one calculates the total differential equations as

$$dg_{ij} = \frac{\partial H'_0}{\partial \pi^{ij}} d\tau + \frac{\partial H'}{\partial \pi^{ij}} dN^\mu = \frac{\partial H'_0}{\partial \pi^{ij}} d\tau,$$

(27)

$$d\pi^{ij} = -\frac{\partial H'_0}{\partial g_{ij}} d\tau - \frac{\partial H'}{\partial g_{ij}} dN^\mu = -\frac{\partial H'_0}{\partial g_{ij}} d\tau,$$

(28)

$$dp_\mu = -\frac{\partial H'_0}{\partial N^\mu} d\tau - \frac{\partial H'}{\partial N^\mu} dN^\mu = -H_\mu d\tau,$$

(29)

$$dp_0 = -\frac{\partial H'_0}{\partial \tau} d\tau - \frac{\partial H'}{\partial \tau} d\tau = 0.$$  

(30)

To check whether this set is integrable or not, one should consider the total variations of (25) and (26). In fact, the total variation of $H'_0$ leads to the conditions

$$H_1 = H_\perp = g^{ij} (\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2) - Rg^2,$$

(31)

$$H_2 = H_i = -2\pi^i_i.$$

(32)

Since $H_1$ and $H_2$ are not identically zero, we consider them as new constraints, and one should consider the total variations of $H_1$ and $H_2$ too. Calculations show that they are no further constraints arise.
The set of equations (27-30) is integrable. Hence, the canonical phase space coordinates $g^{ij}$ and $\pi^{ij}$ are obtained in terms of independent parameters $\tau$ and $N^\mu$. In this case the path integral representation for this system is calculate as

$$D(g^{ij}, \tau', N^\perp, N^i; g^{ij}, \tau, N^\perp, N^i) = \int \prod Dg^{ij} D\pi^{ij} \times$$

$$\exp i \left\{ \int_{\tau}^{\tau'} d^3x [N^\perp (g^{ij} \pi^{ij} - \frac{1}{2} \pi^2) - Rg^{ij} + 2N^{i} \pi^{ij}_i + \pi^{ij} g_{ij,0}] d\tau \right\}. \quad (33)$$

The path integral representation (33) is an integration over the canonical phase-space coordinates $g^{ij}$ and $\pi^{ij}$.

4 Conclusion

We have obtained the canonical path integral formulation of Einstein’s theory of gravitation. This treatment leads us to the equation of motion as total differential equations in many variables, which require the investigation of integrability conditions.

The Einstein’s gravitation system is integrable, $H_0'$ and $H'$ can be interpreted as infinitesimal generators of canonical transformations given by parameters $\tau$ and $(N^\mu = (N^\perp, N^i))$ respectively. Although $N^\mu$ are introduced as coordinates in the Lagrangian, the presence of constraints and the integrability conditions force us to treat them as parameters like $\tau$. In this case the path integral is obtained as an integration over the canonical phase-space coordinates $g^{ij}$ and $\pi^{ij}$. Other treatments [6-11] need gauge fixing conditions to obtain the path integral over the canonical variables.

An important point to specified here, is that for the other conventional methods, there is no well defined procedure to obtain the path integral amplitude for Einstein’s theory of gravitation. A formal expression for the amplitude may be written as

$$F = \int dM(g_{\mu\nu}) \exp iS(g_{\mu\nu}). \quad (34)$$

For the measure $dM$, starting with different assumptions, different authors got different results. For example, Faddeev and Popov [8], following Faddeev’s method [12,13] find

$$dM_{FD} = \prod_x (-g)^{\frac{7}{2}} \prod_{\mu \leq \nu} dg^{\mu\nu}, \quad (35)$$

while Fradkin and Vilkovisky [9-11] claim that

$$dM_{FV} = \prod_x (-g)^{\frac{7}{2}} g^{00} \prod_{\mu \leq \nu} dg^{\mu\nu}. \quad (36)$$
Besides, in reference [6] it is shown that, the local measure emerging from canonical quantization of Einstein’s theory of gravitation, may in principle be omitted if the regularization is properly used.

However the problems which arise naturally from identifying the measure do not occur if our canonical path integral method is used. Besides it is obvious that one dose not need to fix any gauge if the canonical path integral method is used. All is needed the set of the Hamilton Jacobi partial differential equations and the set of the equations of motion. Then one should tests whether these equations are integrable or not. If the integrability conditions are not satisfied identically, then the total variation of them should be introduced as new constraints of the theory. Repeating this procedure as many times as needed one may obtain a set of conditions. The number of independent parameters of the theory is determined directly, without imposing any gauge fixing conditions by this set.

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