RESTRICTION ESTIMATES FOR THE FREE TWO STEP NILPOTENT GROUP ON THREE GENERATORS

VALENTINA CASARINO AND PAOLO CIATTI

Abstract. Let $G$ be the free two step nilpotent Lie group on three generators and let $L$ be a subLaplacian on it. We compute the spectral resolution of $L$ and prove that the operators arising from this decomposition enjoy a Tomas-Stein type estimate.

1. Introduction

Starting from the observation of E. Stein that, when the Lebesgue exponent $p$ is sufficiently close to 1, the Fourier transform of an $L^p$ function restricts in a sense, that may be made precise, to a compact hypersurface of nonvanishing curvature, various forms of restriction theorems for the Fourier transform became one of the main theme of analysis.

The most satisfactory result obtained so far in this theory is the Stein-Tomas theorem, which concerns the restriction in the $L^2$-sense of the Fourier transform of a function in $L^p(\mathbb{R}^n)$, with $1 \leq p \leq \frac{2n+4}{n+3}$, to the sphere $S^{n-1}$. This result was proved in the mid seventies of the last century and, as any very important theorem, has now a great deal of extensions and applications in different branches of mathematics. In particular, it was observed by Stein himself and by R. Strichartz in [Str] that the Stein-Tomas estimates can be interpreted as bounds concerning the mapping properties between Lebesgue spaces of the operators arising in the spectral decomposition of the Euclidean Laplacian.

This point of view is emphasised in the works of C. Sogge, who studied the boundedness of the spectral projections of the Laplace-Beltrami operator on the spheres in [So1] and more generally on compact Riemannian manifolds in [So2]. Few years later D. Müller proved for the first time a result of this sort for a subelliptic operator, the subLaplacian on the Heisenberg group. In the series of papers [C1], [C2], [CCi2], [CCi4] the authors of this article studied a sort of combination of the previous results considering estimates for the joint spectral projections of a Laplace-Beltrami operator and a subLaplacian.

Some years ago we started in [CCi2] and [CCi3] an investigation devoted to extend the result of Müller to subLaplacians on more general two step nilpotent Lie groups. In those papers we considered groups enjoying a special nondegeneracy condition, according to which the quotient with respect to a codimension one subspace of the center is isomorphic to a Heisenberg group.

In this paper we are instead concerned with the mapping properties of the operators arising in the spectral decomposition of an invariant subLaplacian on the free two step nilpotent Lie group.

2000 Mathematics Subject Classification. 22E25; 22E30, 43A80, 47B40.

Key words and phrases. Free nilpotent Lie groups. Sub-Laplacians. Restriction theorems.

Research supported by the Italian Ministero dell’Istruzione, dell’Università e della Ricerca through PRIN “Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis” and the GNAMPA project 2016 “Calcolo funzionale per operatori subellittici su varietà”.
nilpotent Lie group on three generators. In this group the quotient with respect to a
codimension one subspace of the center is isomorphic to the direct product of the three
dimensional Heisenberg group and the real line. Therefore, the nondegeneracy property
we exploited in the previous works here is absent. However, the analysis on this group
is made easier by the fact that the abelian component in the quotients with respect to
the planes in the center is always one dimensional and by the fact that the combined
action of the rotation group on the two layers of the Lie algebra gives rise to a family of
automorphisms.

We conclude the introduction with a short description of the next sections.
In Section 2 we describe some features of the group $G$ we are concerned with. Decom-
posing its Lie algebra $\mathfrak{g}$ into the direct sum $\mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is the center, we see that once
a nontrivial linear form $\mu$ on $\mathfrak{z}$ has been fixed, the quotient of $\mathfrak{g}$ with respect to the null
space of $\mu$ is the direct product of a three dimensional Heisenberg algebra times the real
line. Moreover, these two components, the Heisenberg and the abelian one, depend only
on the line in $\mathfrak{z}$ spanned by $\mu$.

Then in Section 3 we derive the spectral decomposition of an $L^2$ function $f$ with respect
to the subLaplacian. This decomposition is expressed first of all in terms of the Fourier
transform $f^\mu$ of $f$ on $\mathfrak{z}$, which for every $\mu$ in the dual of $\mathfrak{z}$ is a function living on $\mathfrak{v}$. This
transform is followed by the section of the Fourier transform of $f^\mu$ in the direction of
the radical of $\mu$, which is finally decomposed into eigenfunctions of the two dimensional
twisted Laplacian.

Finally, in Section 4 we prove that the operators arising from the spectral decomposition
of the subLaplacian are bounded for $1 \leq s \leq 6/5$ and $1 \leq p \leq 2$ from the nonisotropic
Lebesgue space $L^s_z L^p_v$, in which the integrations in $\mathfrak{z}$ and $\mathfrak{v}$ are weighted with exponents $s$
and $p$, to $L^s_z L^p_v$. The basic ingredients in these estimates are provided by the Stein-Tomas
theorem, which in fact dictates the range of the exponents concerning the integration
on $\mathfrak{z}$, and by the estimates proved some years ago by H. Koch and F. Ricci in [KR] for
the twisted Laplacian. The range of exponents for which our estimates hold is sharp as
examples analogue to those provided in [CCi2] show.

2. Generalities

Let $G$ be a free two step nilpotent Lie group on three generators. We assume that $G$ is
connected and simply connected.

The Lie algebra, $\mathfrak{g}$, of $G$ splits as a vector space into the direct sum $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the centre. Both $\mathfrak{v}$ and $\mathfrak{z}$ are three dimensional vector spaces. To proceed it is convenient
to introduce on $\mathfrak{g}$ an inner product $\langle \cdot , \cdot \rangle$ with respect to which $\mathfrak{v}$ and $\mathfrak{z}$ are orthogonal
subspaces. The inner product induces a norm on $\mathfrak{g}$ and a norm on $\mathfrak{g}^*$, the space of linear
forms on $\mathfrak{g}$, which we will both denote by $| \cdot |$.

We shall always identify $G$ with its Lie algebra $\mathfrak{g}$ by means of the exponential mapping
and use coordinates $(x, z)$ with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ labeling the points of $\mathfrak{v}$ and $z =
(z_1, z_2, z_3) \in \mathbb{R}^3$ the points of $\mathfrak{z}$. The vector fields

$$X_1 = \partial_{x_1} + \frac{x_2}{2} \partial_{z_2} - \frac{x_2}{2} \partial_{z_3}, \quad X_2 = \partial_{x_2} + \frac{x_1}{2} \partial_{z_2} - \frac{x_3}{2} \partial_{z_1}, \quad X_3 = \partial_{x_3} + \frac{x_2}{2} \partial_{z_1} - \frac{x_1}{2} \partial_{z_2}$$
are left invariant and satisfy
\[ [X_1, X_2] = \partial_{z_3} = Z_3, \quad [X_2, X_3] = \partial_{z_1} = Z_1, \quad [X_3, X_1] = \partial_{z_2} = Z_2. \quad (2.1) \]

Fix a point \( \omega \) in the unit sphere \( S = \{ \nu \in \mathfrak{z}^* : |\nu| = 1 \} \); here \( \mathfrak{z}^* \) denotes the space of linear forms on \( \mathfrak{z} \). We call \( \mathfrak{r}_\omega \) the radical of the skew-symmetric form
\[ \mathbf{v} \times \mathbf{v} \ni (X, Y) \mapsto \omega([X, Y]), \]
that is the space
\[ \mathfrak{r}_\omega = \{ X \in \mathbf{v} : \omega([X, Y]) = 0 \text{ for all } Y \in \mathbf{v} \}, \]
which as follows from (2.1) is one dimensional.

If \( Z_\omega \in \mathfrak{z} \) satisfies \( \omega(Z_\omega) = 1 \), then \( |Z_\omega| = 1 \) and
\[ [X, Y] = \omega([X, Y])Z_\omega, \quad X, Y \in \mathbf{v}. \]
Thus if we fix two unit vectors \( X_\omega \) and \( Y_\omega \) in \( \mathbf{v} \) satisfying \( \omega([X_\omega, Y_\omega]) = 1 \), then the subspace \( \mathfrak{h}_\omega \) spanned by \( \{X_\omega, Y_\omega, Z_\omega\} \) is a Lie algebra isomorphic to the three dimensional Heisenberg algebra.

The subspace \( \mathfrak{e}_\omega \) of \( \mathfrak{z} \) orthogonal to \( Z_\omega \) coincides with the kernel of \( \omega \). Therefore the quotient of \( \mathfrak{g} \) with respect to \( \mathfrak{e}_\omega \) is isomorphic as a Lie algebra to the direct product of \( \mathfrak{h}_\omega \) and \( \mathfrak{r}_\omega \). To equip this algebra with a system of coordinates, we decompose a vector into the sum
\[ v_\omega V_\omega + x_\omega X_\omega + y_\omega Y_\omega + z_\omega Z_\omega, \]
where \( V_\omega \) is a unit vector in \( \mathfrak{r}_\omega \) and \( (v_\omega, x_\omega, y_\omega, z_\omega) \in \mathbb{R}^4 \). We shall identify, with a slight abuse of notation, \( \omega \) with the linear form in \( \mathbf{v}^* \) canonically associated to \( V_\omega \), writing \( v_\omega = \omega(X) \).

Fixing two orthonormal vectors \( \{W_{\omega,1}, W_{\omega,2}\} \) in \( \mathfrak{e}_\omega \), we obtain an orthogonal basis \( \{Z_\omega, W_{\omega,1}, W_{\omega,2}\} \) of \( \mathfrak{z} \). Then any vector in \( \mathfrak{g} \) can be uniquely written as
\[ v_\omega V_\omega + x_\omega X_\omega + y_\omega Y_\omega + z_\omega Z_\omega + w_{\omega,1}W_{\omega,1} + w_{\omega,2}W_{\omega,2}, \]
where \( (v_\omega, x_\omega, y_\omega, z_\omega, w_{\omega,1}, w_{\omega,2}) \in \mathbb{R}^6 \) are coordinates adapted to \( \omega \).

The transformation mapping \( X = (x_1, x_2, x_3) \) to \( (x_\omega, y_\omega, v_\omega) \) is a rotation \( R_\omega \in SO(3) \) fixing \( V_\omega \) and hence
\[ (x_\omega, y_\omega) = x_\omega X_\omega + y_\omega Y_\omega = R_\omega(X) - \omega(X)V_\omega. \]

Since by the universal property rotations of \( \mathbf{v} \) extend to automorphisms of \( \mathfrak{g} \), \( R_\omega \) preserves \( Z_\omega \) and maps \( (z_1, z_2, z_3) \) to \( (w_{\omega,1}, w_{\omega,2}, z_\omega) \).

3. The subLaplacian

The vector fields \( X_1, X_2, X_3 \) generate by (2.1) the entire Lie algebra \( \mathfrak{g} \). Therefore by Hörmander’s theorem the subLaplacian
\[ L = -X_1^2 - X_2^2 - X_3^2 \]
is a hypoelliptic differential operator. It is easy to see that
\[
L = -\partial^2_{x_1} - \partial^2_{x_2} - \partial^2_{x_3} + \frac{i}{2}(x_2 \partial_{x_3} - x_3 \partial_{x_2}) \partial_{z_1} + \frac{i}{2}(x_3 \partial_{x_1} - x_1 \partial_{x_3}) \partial_{z_2} + \frac{i}{2}(x_1 \partial_{x_2} - x_2 \partial_{x_1}) \partial_{z_3}
\]
\[
- \frac{1}{4}(x_2^2 + x_3^2) \partial^2_{z_1} - \frac{1}{4}(x_1^2 + x_2^2) \partial^2_{z_2} - \frac{1}{4}(x_1^2 + x_2^2) \partial^2_{z_3}
\]
and hence that \(L\) is homogeneous with respect to the dilations defined by \(\delta_\epsilon(X, Z) = (\epsilon x, \epsilon^2 z), \epsilon > 0\), that is
\[
L(f \circ \delta_\epsilon) = \epsilon^2 (L f) \circ \delta_\epsilon.
\]

The subLaplacian is a symmetric operator on the Schwartz class and extends to a self-adjoint operator on \(L^2(G)\) with positive spectrum. Our plan in the section consists in deriving its spectral decomposition. Our construction is largely inspired by the analogous derivations in [ACDS] and [M\text{M}]. To accomplish the task, given a Schwartz function \(f\) on \(G\), we take its partial Fourier transform in the central variables
\[
f_\mu(X) = \int_{\mathfrak{g}^*} e^{-ip(Z)} f(X, Z) dZ, \quad \mu \in \mathfrak{z}^*.
\]

We then introduce spherical coordinates in \(\mathfrak{z}^*\), writing \(\mu = \rho \omega\), where \(\rho = |\mu|\) and \(\omega\) belongs to \(S\). In these coordinates the Fourier inversion formula in \(\mathfrak{z}^*\) reads
\[
X_a f(X, Z) = \int_S \int_0^{\infty} e^{ip(Z)} X^\rho_a f^\rho(X) \rho^2 dp d\omega, \quad a = 1, 2, 3, \tag{3.1}
\]
In particular, we have
\[
X_a f(X, Z) = \int_S \int_0^{\infty} e^{ip(Z)} X^\rho_a f^\rho(X) \rho^2 dp d\omega, \quad a = 1, 2, 3, \tag{3.2}
\]
where the vector fields \(X^\rho_a\) are the differential operators on \(\mathfrak{v}\) defined by
\[
X_a (e^{-ip(Z)} g(X)) = e^{-ip(Z)} X^\rho_a g(X), \quad a = 1, 2, 3,
\]
for any Schwartz function \(g\). They are more explicitly given by
\[
X^\rho_1 = \partial_{x_1} + \frac{i}{2} \rho(\omega_3 x_2 - \omega_2 x_3), \quad X^\rho_2 = \partial_{x_2} + \frac{i}{2} \rho(\omega_1 x_3 - \omega_3 x_1), \quad X^\rho_3 = \partial_{x_3} + \frac{i}{2} \rho(\omega_2 x_1 - \omega_1 x_2),
\]
where \(\omega_a = \omega(Z_a), a = 1, 2, 3\). Definining the differential operator \(L^\rho\) on \(\mathfrak{v}\) by
\[
L (e^{-ip(Z)} g(X)) = e^{-ip(Z)} L^\rho g(X), \tag{3.3}
\]
we find
\[
L^\rho = -\partial^2_{x_1} - \partial^2_{x_2} - \partial^2_{x_3} + \frac{i}{2} \rho \omega_1 (x_3 \partial_{x_1} - x_1 \partial_{x_3}) + \frac{i}{2} \rho \omega_2 (x_2 \partial_{x_3} - x_3 \partial_{x_2}) + \frac{i}{2} \rho \omega_3 (x_1 \partial_{x_2} - x_2 \partial_{x_1})
\]
\[
+ \frac{1}{4} \rho^2 \omega_1^2 (x_2^2 + x_3^2) + \frac{1}{4} \rho^2 \omega_2^2 (x_1^2 + x_3^2) + \frac{1}{4} \rho^2 \omega_3^2 (x_1^2 + x_2^2)
\]
In the coordinates \((v_\omega, z_\omega, w_\omega, w_{\omega,1}, w_{\omega,2})\) the operator \(L^\rho\) becomes
\[
L^\rho_{v_\omega, z_\omega, w_\omega} = -\partial^2_{v_\omega} - \partial^2_{z_\omega} - \partial^2_{w_\omega} + \frac{i}{2} \rho (v_\omega \partial_{z_\omega} - z_\omega \partial_{v_\omega}) + \frac{1}{4} \rho^2 (v^2_\omega + z^2_\omega) \tag{3.4}
\]
\[
= -\partial^2_{v_\omega} + \Delta^\rho_{v_\omega},
\]
where

$$\Delta \lambda = -\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial t^2} + \frac{i}{2} \lambda \left( t \frac{\partial}{\partial s} - s \frac{\partial}{\partial t} \right) + \frac{\lambda^2}{4} (s^2 + t^2)$$

is the two dimensional $\lambda$-twisted Laplacian.

The spectrum of $\Delta \lambda$ consists of eigenvalues, which are given by $\lambda(2k + 1)$ with $k = 0, 1, 2, \cdots$. We write the spectral decomposition of $g$ with respect to $\Delta \lambda$ as

$$g(s, t) = \sum_{k=0}^{\infty} \Lambda_k \lambda \lambda(s, t),$$

(3.5)

where $\Lambda_k$ maps $L^2(\mathbb{R}^2)$ onto the eigenspace associated to $\lambda(2k + 1)$. These are integral operators (for more details about them see, for instance, [TK], p.19 ff) satisfying the estimates

$$\|\Lambda_k f\|_{L^2(\mathbb{R}^2)} \leq C\lambda^{\frac{1}{2}}(2k + 1)^{\gamma(\frac{1}{2})}\|f\|_{L^p(\mathbb{R}^2)}, \quad 1 \leq p \leq 2,$$

(3.6)

proved by H. Koch and F. Ricci in [KR]; here $\gamma$ is the piecewise affine function on $[\frac{1}{2}, 1]$ defined by

$$\gamma \left( \frac{1}{p} \right) := \begin{cases} \frac{1}{2} - 1 \quad &\text{if } 1 \leq p \leq \frac{6}{5}, \\ \frac{1}{2} \left( \frac{4}{2} - \frac{1}{p} \right) \quad &\text{if } \frac{6}{5} \leq p \leq 2. \end{cases}$$

(3.7)

In order to decompose $f^{(\lambda)}$ into eigenfunctions of the twisted Laplacian, we define the function $g_{\rho, \omega} = f^{(\lambda)} \circ R^{-1}_\omega$, where $R^{-1}_\omega$ is the inverse of $R_\omega$. Since $(x_\omega, y_\omega, v_\omega) = R_\omega(x_1, x_2, x_3) = R_\omega X$, recalling that $v_\omega = \omega(X)$ and $(x_\omega, y_\omega) = R_\omega(X) - \omega(X)V_\omega$, we have $f^{(\lambda)}(X) = g_{\rho, \omega}(x_\omega, y_\omega, v_\omega) = g_{\rho, \omega}(R_\omega(X) - \omega(X)V_\omega, \omega(X))$. Then it makes sense to decompose the function $g_{\rho, \omega}(x_\omega, y_\omega, v_\omega)$ in eigenfunctions of $\Delta^{(\lambda)}_{x_\omega, y_\omega}$, obtaining

$$f^{(\lambda)}(X) = g_{\rho, \omega}(x_\omega, y_\omega, v_\omega)$$

$$= \sum_{k=0}^{\infty} \Lambda_k \lambda \lambda g_{\rho, \omega}(x_\omega, y_\omega, v_\omega)$$

$$= \sum_{k=0}^{\infty} \Lambda_k \lambda \lambda (f^{(\lambda)} \circ R^{-1}_\omega)(R_\omega(X) - \omega(X)V_\omega, \omega(X)).$$

Taking the partial Fourier transform of $g_{\rho, \omega}$ in $r_\omega$, it follows that

$$\hat{f}_{\omega} g_{\rho, \omega}(x_\omega, y_\omega; \xi) = \int_{-\infty}^{\infty} e^{-i\xi} g_{\rho, \omega}(R_\omega(X) - \omega(X)V_\omega + sV_\omega) ds$$

$$= \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{-i\xi} \Lambda_k \lambda \lambda g_{\rho, \omega}(R_\omega(X) - \omega(X)V_\omega + sV_\omega) ds,$$

and also

$$\hat{f}_{\omega} g_{\rho, \omega}(x_\omega, y_\omega; \xi) = \sum_{k=0}^{\infty} \Lambda_k \lambda \lambda \hat{f}_{\omega} g_{\rho, \omega}(R_\omega(X) - \omega(X)V_\omega; \xi),$$
Thus, inverting the Fourier transform we obtain
\[ g_{\rho,\omega}(x,\omega, y, \omega) = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{i\omega \xi} \lambda_k \mathfrak{F}_\omega (g_{\rho,\omega}(x,\omega, y, \omega; \xi)) d\xi. \]

Hence, we have
\[ f^{\rho\omega}(X) = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{i\omega(\xi)} \lambda_k \mathfrak{F}_\omega (f^{\rho\omega} \circ R_{-\omega}^{-1})(R_\omega(X) - \omega(X)V_\omega; \xi) d\xi, \]
which plugged in (3.1) yields
\[ f(X, Z) = \sum_{k=0}^{\infty} \int_{S} \int_{0}^{-\infty} e^{i\rho(\omega)}(X) \lambda_k \mathfrak{F}_\omega (f^{\rho\omega} \circ R_{-\omega}^{-1})(R_\omega(X) - \omega(X)V_\omega; \xi) d\xi \rho^2 d\rho d\omega. \]

We may now deduce from (3.9) the spectral decomposition of \( f \) with respect to \( L \) by replacing the arguments of the sum and the integrals with generalised eigenfunctions. To do that we notice that by (3.2) we have
\[ Lf(X, Z) = \int_{S} \int_{0}^{-\infty} e^{i\rho(\omega)}(X) L^{\rho\omega} f^{\rho\omega}(X) \rho^2 d\rho d\omega, \]
which by (3.8) and (3.1) implies
\[
\int_{S} \int_{0}^{-\infty} e^{i\rho(\omega)}L^{\rho\omega} \left( \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{i\omega(\xi)} \lambda_k \mathfrak{F}_\omega (f^{\rho\omega} \circ R_{-\omega}^{-1})(R_\omega(X) - \omega(X)V_\omega; \xi) d\xi \right) \rho^2 d\rho d\omega
\]
\[
= \sum_{k=0}^{\infty} \int_{S} \int_{0}^{-\infty} e^{i\rho(\omega)} \left( (-\frac{\partial^2}{\partial \omega^2} + \Delta^{\rho}_{x,\omega}) \int_{-\infty}^{\infty} e^{i\omega(\xi)} \lambda_k \mathfrak{F}_\omega (f^{\rho\omega} \circ R_{-\omega}^{-1})(x,\omega, y_\omega; \xi) d\xi \right) |_{R_\omega(X)} \rho^2 d\rho d\omega
\]
\[
= \sum_{k=0}^{\infty} \int_{S} \int_{0}^{-\infty} e^{i\rho(\omega)} \left( \int_{-\infty}^{\infty} e^{i\omega(\xi)} (\zeta^2 + \Delta^{\rho}_{x,\omega}) \lambda_k \mathfrak{F}_\omega (f^{\rho\omega} \circ R_{-\omega}^{-1})(x,\omega, y_\omega; \xi) d\xi \right) |_{R_\omega(X)} \rho^2 d\rho d\omega
\]
\[
= \sum_{k=0}^{\infty} \int_{S} \int_{0}^{-\infty} e^{i\rho(\omega)} \left( \int_{-\infty}^{\infty} e^{i\omega(\xi)} (\zeta^2 + \rho(2k+1)) \lambda_k \mathfrak{F}_\omega (f^{\rho\omega} \circ R_{-\omega}^{-1})(x,\omega, y_\omega; \xi) d\xi \right) |_{R_\omega(X)} \rho^2 d\rho d\omega
\]
where $|_{R_{\omega}(X)}$ means that the integral must be computed in $x_\omega X_\omega + y_\omega Y_\omega = R_{\omega}(X) - \omega(X)V_\omega$ and $v_\omega = \omega(X)$.

Replacing in the last integral $\rho(2k + 1)$ with $\rho$ we obtain

$$Lf(X, Z) = \int_S \sum_{k=0}^{\infty} (2k + 1)^{-3} \int_{-\infty}^{\infty} e^{i\rho\omega(Z)/(2k+1)} e^{i\omega(X)\xi} (\mu - \xi^2)^2$$

$$\times \Lambda_k^{(2k+1)} \tilde{\mathfrak{g}}_{\omega}(f^{(\omega/(2k+1))} \circ R_{\omega}^{-1})(R_{\omega}(X) - \omega(X)V_\omega; \xi) \rho^2 d\rho d\xi d\omega,$$

from which, setting $\mu = \xi^2 + \rho$ and using Fubini’s theorem, we deduce

$$Lf(X, Z) = \int_S \sum_{k=0}^{\infty} (2k + 1)^{-3} \int_{-\infty}^{\infty} e^{i(\mu - \xi^2)\omega(Z)/(2k+1)} e^{i\omega(X)\xi} (\mu - \xi^2)^2$$

$$\times \Lambda_k^{(2k+1)} \tilde{\mathfrak{g}}_{\omega}(f^{(\omega/(2k+1))} \circ R_{\omega}^{-1})(R_{\omega}(X) - \omega(X)V_\omega; \xi) \xi d\xi d\omega d\mu.$$  

The expression for $Lf$ just derived shows that if we write

$$f(X, Z) = \int_0^\infty \mathcal{P}_\mu f(X, Z) d\mu,$$  

\hspace{1cm} (3.10)

where

$$\mathcal{P}_\mu f(X, Z) = \int_S \sum_{k=0}^{\infty} (2k + 1)^{-3} \int_{-\infty}^{\infty} e^{i(\mu - \xi^2)\omega(Z)/(2k+1)} e^{i\omega(X)\xi} (\mu - \xi^2)^2$$

$$\times \Lambda_k^{(2k+1)} \tilde{\mathfrak{g}}_{\omega}(f^{(\omega/(2k+1))} \circ R_{\omega}^{-1})(R_{\omega}(X) - \omega(X)V_\omega; \xi) d\xi d\omega d\mu,$$  

\hspace{1cm} (3.11)

then

$$L\mathcal{P}_\mu f = \mu \mathcal{P}_\mu f.$$  

Thus, (3.10) provides the spectral resolution of $f$ with respect to $L$.

4. THE RESTRICTION THEOREM

In this section we show that the operators $\mathcal{P}_\mu$ satisfy some restriction estimates. To state and prove this result we introduce nonisotropic norms on $G$ defined, for $1 \leq p \leq \infty$, by

$$\|f\|_{L_p^s L_v^p} = \left( \int_\mathbb{R} \left( \int_3 \left| f(X, Z) \right|^s dZ \right)^\frac{p}{s} dV \right)^\frac{1}{p},$$

with the obvious modifications when $s$ or $p$ equal $\infty$. We shall prove the following theorem.

Theorem 4.1. Let $f$ be a Schwartz function on $G$ and let $1 \leq p \leq 2$ and $1 \leq s \leq \frac{6}{\beta}$, then

$$\|\mathcal{P}_\mu f\|_{L_p^s L_v^p} \leq C \mu^{\frac{1}{2} - \frac{1}{p} + \frac{3}{2}(\frac{1}{s} - \frac{1}{2})} \|f\|_{L_p^s L_v^p}.  \hspace{1cm} (4.1)$$

These estimates are false for $s > \frac{6}{\beta}$. 

\hspace{1cm}
Proof. The sharpness of the range of \( s \) where the estimates hold may be proved with a suitable and easy modification of the example provided in [CC12]. So we prove only the bound \([11]\).

The dependence on \( \mu \) in the right hand side of \([11]\) is dictated by the homogeneity of \( L \). Therefore, it suffices to discuss the case \( \mu = 1 \).

To reduce the complexity of the notation and make the formulas more readable we consider a tensor function \( f(X, Z) = \alpha(Z) \beta(X) \), with \( \alpha \) and \( \beta \) Schwartz functions on \( \mathbb{R} \) and \( \mathbb{V} \). Then \([3.11]\) reduces to

\[
\mathcal{P}_1 f(X, Z) = \int_{\mathbb{S}} \int_{-1}^{1} \sum_{k=0}^{\infty} \frac{(1 - t^2)^2}{(2k + 1)^3} e^{i(1 - t^2)\mathbb{Z} \mathbf{i} t \mathbf{j} \mathbf{k} \mathbf{l}} e^{i\omega(X)\mathbf{i} \mathbf{k} \mathbf{l} \mathbf{j} \mathbf{m} \mathbf{n}} \left( \frac{1 - \xi^2}{2k + 1} \right) \\
\times \Lambda_k^{1/2} \mathcal{F}_\omega(\beta \circ R_{\omega}^{-1}) \left( R_{\omega}(X) - \omega(X)V_\omega; \xi \right) d\xi d\omega,
\]

where \( \mathcal{F} \) denotes the Fourier transform of \( \omega \).

Consider another tensor function \( g(X, Z) = \gamma(Z) \delta(X) \), with \( \|\gamma\|_{L^s_\mathbb{V}} = 1 \) and \( \|\delta\|_{L^s_\mathbb{V}} = 1 \), and compute

\[
\langle \mathcal{P}_1 f, g \rangle = \int_{\mathbb{V}} \int_{\mathbb{S}} \mathcal{P}_1 f(X, Z) g(X, Z) dZ dX.
\]

Then we have

\[
\langle \mathcal{P}_1 f, g \rangle = \int_{\mathbb{V}} \int_{\mathbb{S}} \left( \int_{-1}^{1} \sum_{k=0}^{\infty} \frac{(1 - t^2)^2}{(2k + 1)^3} e^{i(1 - t^2)\mathbb{Z} \mathbf{i} t \mathbf{j} \mathbf{k} \mathbf{l}} e^{i\omega(X)\mathbf{i} \mathbf{k} \mathbf{l} \mathbf{j} \mathbf{m} \mathbf{n}} \left( \frac{1 - \xi^2}{2k + 1} \right) \\
\times \Lambda_k^{1/2} \mathcal{F}_\omega(\beta \circ R_{\omega}^{-1}) \left( R_{\omega}(X) - \omega(X)V_\omega; \xi \right) d\xi d\omega \right) \gamma(Z) \delta(X) dZ dX
\]

\[
= \int_{\mathbb{S}} \int_{-1}^{1} \sum_{k=0}^{\infty} \frac{(1 - t^2)^2}{(2k + 1)^3} \Lambda_k^{1/2} \mathcal{F}_\omega(\beta \circ R_{\omega}^{-1}) \left( R_{\omega}(X) - \omega(X)V_\omega; \xi \right) d\xi d\omega
\]\n
Changing the variables in the integral over \( \mathbb{V} \) we obtain (since \( |\det R_{\omega}| = 1 \))

\[
\langle \mathcal{P}_1 f, g \rangle = \int_{\mathbb{S}} \int_{-1}^{1} \sum_{k=0}^{\infty} \frac{(1 - t^2)^2}{(2k + 1)^3} \left( \frac{1 - \xi^2}{2k + 1} \right) \Lambda_k^{1/2} \mathcal{F}_\omega(\beta \circ R_{\omega}^{-1}) \left( x_\omega, y_\omega; \xi \right) (\delta \circ R_{\omega}^{-1})(x_\omega, y_\omega; v_\omega) dx_\omega dy_\omega d\xi d\omega
\]

\[
= \int_{\mathbb{S}} \int_{-1}^{1} \sum_{k=0}^{\infty} \frac{(1 - t^2)^2}{(2k + 1)^3} \Lambda_k^{1/2} \mathcal{F}_\omega(\beta \circ R_{\omega}^{-1}) \left( x_\omega, y_\omega; \xi \right) \mathcal{F}_\omega(\delta \circ R_{\omega}^{-1})(x_\omega, y_\omega; \xi) dx_\omega dy_\omega d\xi d\omega.
\]
Setting

\[ \Psi(\omega; \xi, k) = \int_{\mathbb{S}} \Lambda_{k}^{1 - \xi^2} \mathfrak{F}_{\omega}(\beta \circ R_{\omega}^{-1})(x, y, \xi) \mathfrak{F}_{\omega}(\delta \circ R_{\omega}^{-1})(x, y, \xi) dx \cdot dy, \tag{4.2} \]

we write

\[ \langle P_{1} f, g \rangle = \sum_{k=0}^{\infty} \frac{(1 - \xi^2)^2}{(2k + 1)^3} \int_{-1}^{1} \left( \int_{S} \hat{\alpha} \left( \frac{(1 - \xi^2)\omega}{2k + 1} \right) \hat{\gamma} \left( \frac{(1 - \xi^2)\omega}{2k + 1} \right) \Psi(\omega; \xi, k) d\omega \right) d\xi. \]

Then the Cauchy-Schwarz inequality yields

\[ |\langle P_{1} f, g \rangle| \leq \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^3} \int_{-1}^{1} (1 - \xi^2)^2 \Phi(\xi, k) \]

\[ \times \left( \int_{S} |\hat{\alpha} \left( \frac{(1 - \xi^2)\omega}{2k + 1} \right)|^2 d\omega \right)^{\frac{1}{2}} \left( \int_{S} |\hat{\gamma} \left( \frac{(1 - \xi^2)\omega}{2k + 1} \right)|^2 d\omega \right)^{\frac{1}{2}} d\xi, \]

where we set

\[ \Phi(\xi, k) = \sup_{\omega \in S} |\Psi(\omega; \xi, k)| \tag{4.3} \]

to simplify the notation. We remind that according to the Tomas-Stein theorem in \( \mathbb{R}^3 \) the inequality

\[ \left( \int_{S} |\hat{\eta} \left( r\omega \right)|^2 d\omega \right)^{\frac{1}{2}} \leq C_{s} r^{-3/s'} \|\eta\|_{L^{1}_{s'}} \]

holds for \( 1 \leq s \leq \frac{4}{3} \) and \( r > 0 \), where \( \frac{1}{s} + \frac{1}{s'} = 1 \). Thus, we obtain (being \( \|\gamma\|_{L^{1}_{s'}} = 1 \))

\[ |\langle P_{1} f, g \rangle| \leq \alpha \|L_{s'} \| \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^3} \int_{-1}^{1} (1 - \xi^2)^2 \Phi(\xi, k) d\xi. \tag{4.4} \]

To estimate the integral in the last formula, we remind the definition (4.3) of \( \Phi \) (see also (1.2)) and use the Cauchy-Schwarz inequality in the integral on \( \mathbb{S} \) to obtain

\[ \int_{-1}^{1} (1 - \xi^2)^{2 - \frac{6}{s'}} \Phi(\xi, k) d\xi \leq \int_{-1}^{1} (1 - \xi^2)^{2 - \frac{6}{s'}} \]

\[ \times \left( \int_{\mathbb{S}} \Lambda_{k}^{1 - \xi^2} \mathfrak{F}_{\omega}(\beta \circ R_{\omega}^{-1})(x, y, \xi) \mathfrak{F}_{\omega}(\delta \circ R_{\omega}^{-1})(x, y, \xi) dx \cdot dy \right) d\xi \]

\[ \leq \int_{-1}^{1} (1 - \xi^2)^{2 - \frac{6}{s'}} \left( \int_{\mathbb{S}} \Lambda_{k}^{1 - \xi^2} \mathfrak{F}_{\omega}(\beta \circ R_{\omega}^{-1})(x, y, \xi) \right)^{2} dx \cdot dy \]

\[ \times \left( \int_{\mathbb{S}} \left| \mathfrak{F}_{\omega}(\delta \circ R_{\omega}^{-1})(x, y, \xi) \right|^{2} dx \cdot dy \right)^{\frac{1}{2}} d\xi. \]
A further application of the Cauchy-Schwarz inequality then implies
\[
\int_{-1}^{1} (1 - \xi^2)^2 \frac{1}{\tilde{F}} \Phi(\xi, k) d\xi \leq \left( \int_{-1}^{1} (1 - \xi^2) \frac{1}{2} \int_{\mathbb{R}} \left| \frac{1}{2\xi^2 + 1} \tilde{g}_\omega(\beta \circ R_\omega^{-1})(x, y, \xi, \eta) \right|^2 dx dy d\eta d\xi \right)^{\frac{1}{2}} \times \left( \int_{-1}^{1} \int_{\mathbb{R}} \left| \tilde{g}_\omega(\delta \circ R_\omega^{-1})(x, y, \xi) \right|^2 dx dy d\xi \right)^{\frac{1}{2}}.
\]

Since the Plancherel theorem applied to \( \tilde{g}_\omega \) yields
\[
\int_{\mathbb{R}} \int_{-\infty}^{\infty} \left| \tilde{g}_\omega(\delta \circ R_\omega^{-1})(x, y, \xi, \eta) \right|^2 dx dy d\eta = \int_{-\infty}^{\infty} \int_{\mathbb{R}} \left| (\delta \circ R_\omega^{-1})(x, y, \xi, \eta) \right|^2 dx dy d\xi = \| \delta \|_{L^2} = 1,
\]
(being \(| \det R_\omega | = 1\)), (4.5) reduces to
\[
\int_{-1}^{1} (1 - \xi^2)^2 \frac{1}{\tilde{F}} \Phi(\xi, k) d\xi \leq \left( \int_{-1}^{1} (1 - \xi^2) \frac{1}{2} \int_{\mathbb{R}} \left| \frac{1}{2\xi^2 + 1} \tilde{g}_\omega(\beta \circ R_\omega^{-1})(x, y, \xi, \eta) \right|^2 dx dy d\eta d\xi \right)^{\frac{1}{2}}.
\]

Then in force of the Koch-Ricci estimates (3.6) we deduce from (4.4) that
\[
\int_{-1}^{1} (1 - \xi^2)^2 \frac{1}{\tilde{F}} \Phi(\xi, k) d\xi \leq C (2k + 1)^{\gamma(\frac{1}{p} - \frac{1}{2})} \left( \int_{-1}^{1} \int_{\mathbb{R}} \left| \tilde{g}_\omega(\beta \circ R_\omega^{-1})(x, y, \xi) \right|^p dx dy d\xi \right)^{\frac{1}{p}}.
\]

Applying the Hölder inequality in (4.7) to the integration in \( \xi \) with exponents \( p'/2 \) and \( p/(p' - 2) = p/(2 - p) \), we deduce that
\[
\int_{-1}^{1} (1 - \xi^2)^2 \frac{1}{\tilde{F}} \Phi(\xi, k) d\xi \leq C (2k + 1)^{\gamma(\frac{1}{p} - \frac{1}{2})} \left( \int_{-1}^{1} (1 - \xi^2) \frac{1}{2\xi^2 + 1} d\xi \right)^{\frac{1}{p} - \frac{1}{2}} \times \left( \int_{-\infty}^{\infty} \left( \int_{\mathbb{R}} \left| \tilde{g}_\omega(\beta \circ R_\omega^{-1})(x, y, \xi) \right|^p dx dy d\xi \right)^{\frac{1}{p'}} d\xi \right)^{\frac{1}{p}}.
\]

The first integral is finite since \( s' \geq 4 \), which implies that
\[
\left( 1 - \frac{3}{s'} \right) \frac{4p}{2 - p} + 1 \geq \frac{2}{2 - p} > 0.
\]
Therefore, we have
\[ \int_{-1}^{1} (1 - \xi^2)^{2 - \frac{p}{2}} \Phi(\xi, k) d\xi \]
\[ \leq C (2k + 1)^{\gamma(\frac{1}{p}) - (\frac{1}{p} - \frac{1}{2})} \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} |\hat{f}(\beta \circ R_w^{-1})(x, y; \xi)|^p d\xi \right)^{\frac{p'}{p}} d\xi \right)^{\frac{1}{p}}. \]

Now, being \( p'/p \geq 1 \), we may use the Minkowski integral inequality, which gives
\[ \int_{-1}^{1} (1 - \xi^2)^{2 - \frac{p}{2}} \Phi(\xi, k) d\xi \]
\[ \leq C (2k + 1)^{\gamma(\frac{1}{p}) - (\frac{1}{p} - \frac{1}{2})} \left( \int_{-\infty}^{\infty} |\hat{f}(\beta \circ R_w^{-1})(x, y, v)|^p d\xi \right)^{\frac{1}{p}}. \]

Then we apply the Hausdorff-Young inequality in the inner integral to \( \hat{f} \), and replace the coordinates \( x, y, v \), with \( x, y, v \),
\[ \int_{-1}^{1} (1 - \xi^2)^{2 - \frac{p}{2}} \Phi(\xi, k) d\xi \]
\[ \leq C (2k + 1)^{\gamma(\frac{1}{p}) - (\frac{1}{p} - \frac{1}{2})} \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left( \beta \circ R_w^{-1}(x, y, v) \right)^p d\xi \right)^{\frac{1}{p}} dx dy dv \right)^{\frac{1}{p}}, \]
deducing that
\[ \int_{-1}^{1} (1 - \xi^2)^{2 - \frac{p}{2}} \Phi(\xi, k) d\xi \leq C (2k + 1)^{\gamma(\frac{1}{p}) - (\frac{1}{p} - \frac{1}{2})} \| \beta \|_{L^p}^p. \]

Finally, plugging (4.8) in (4.4) we obtain
\[ |\langle P_1 f, g \rangle| \leq C \| \alpha \|_{L^1} \| \beta \|_{L^p}^p \sum_{k=0}^{\infty} (2k + 1)^{\frac{p}{2} - 3 + \gamma(\frac{1}{p}) - (\frac{1}{p} - \frac{1}{2})} \]
\[ \leq C \| \alpha \|_{L^1} \| \beta \|_{L^p}^p = C \| f \|_{L^1} \| L^p. \]

The series in fact converges for \( 1 \leq s \leq 4/3 \) and \( 1 \leq p \leq 2 \) since by (3.7), when \( 1 \leq p \leq \frac{6}{5} \), we have
\[ \frac{6}{s'} - 3 + \gamma \left( \frac{1}{p} \right) - \left( \frac{1}{p} - \frac{1}{2} \right) \leq -2 \]
and, when \( \frac{5}{6} \leq p \leq 2 \),
\[ \frac{6}{s'} - 3 + \gamma \left( \frac{1}{p} \right) - \left( \frac{1}{p} - \frac{1}{2} \right) \leq -3 \]

From (4.9) the estimate asserted in the statement follows by duality, proving the theorem.
References

[ACDS] F. Astengo, M. Cowling, B. Di Blasio, M. Sundari, Hardy’s uncertainty principle on certain Lie groups, J. London Math. Soc. (2), 62 (2000), no. 2, 461–472.

[C1] V. Casarino, Norms of complex harmonic projection operators, Canad. J. Math., 55 (2003), no. 6, 1134–1154.

[C2] V. Casarino, Two-parameter estimates for joint spectral projections on complex spheres, Math. Z., 261 (2009), no. 2, 245–259.

[CCi1] V. Casarino and P. Ciatti, Transferring $L^p$ eigenfunction bounds from $S^{2n+1}$ to $\mathfrak{h}_n$, Studia Math., 194 (2009), no. 1, 23–42.

[CCi2] V. Casarino and P. Ciatti, A Restriction Theorem for Métivier Groups, Advances in Mathematics, 245 (2013), 52–77.

[CCi3] V. Casarino and P. Ciatti, Restriction estimates for the full Laplacian on Métivier Groups, Rend. Lincei Mat. Appl. 24 (2013), 165–179.

[CCi4] V. Casarino and P. Ciatti, $L^p$ Joint Eigenfunction Bounds on Quaternionic Spheres, J. Fourier Anal. Appl., electronically published on 14 October 2016, DOI: http://dx.doi.org/10.1007/s00041-016-9506-6 (to appear in print).

[KR] H. Koch and F. Ricci, Spectral projections for the twisted Laplacian, Studia Math. 180 (2007), no. 2, 10–110.

[MMi] A. Martini and D. Müller, $L^p$ spectral multipliers on the free group $N_{3,2}$, Studia Math. 217 (2013), no. 1, 41–55.

[Mi] D. Müller, A restriction theorem for the Heisenberg group, Ann. of Math. (2), 131 (1990), no. 3, 567–587.

[So1] C. Sogge, Oscillatory integrals and spherical harmonics, Duke Math. J. (2), 53 (1986), no. 1, 43–65.

[So2] C. Sogge, Concerning the $L^p$ norm of spectral clusters for second-order elliptic operators on compact manifolds, J. Funct. Anal., 77 (1988), no. 1, 123–138.

[Str] R. Strichartz, Harmonic analysis as spectral theory of Laplacians, J. Funct. Anal., 87 (1989), no. 1, 51–148.

[Th] S. Thangavelu, Harmonic Analysis on the Heisenberg Group, Progress in Mathematics, Vol. 159, Birkhäuser Boston Inc., Boston, MA, 1998.

Università degli Studi di Padova, Stradella san Nicola 3, I-36100 Vicenza, Italy
E-mail address: valentina.casarino@unipd.it

Università degli Studi di Padova, Via Marzolo 9, I-35100 Padova, Italy
E-mail address: paolo.ciatti@unipd.it