UNIRULED VARIETIES WITH SPLIT TANGENT BUNDLE

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1. Introduction

The goal of this paper is to give a partial answer to the following conjecture, initially asked by A. Beauville.

1.1. Conjecture. Let $X$ be a compact Kähler manifold such that $T_X = V_1 \oplus V_2$, where $V_1$ and $V_2$ are vector bundles. Let $\mu : \tilde{X} \to X$ be the universal covering of $X$. Then $\tilde{X} \simeq Y_1 \times Y_2$, where $\dim Y_j = \text{rk} V_j$. If moreover $V_j$ is integrable, $\mu^* V_j \simeq p_{Y_j}^* T_{Y_j}$ (up to an appropriate automorphism of $\tilde{X}$).

The integrability of the vector bundles $V_j$ is not true in general (see example 2.14), but the decomposition of the universal covering $\tilde{X}$ still holds in these examples. The conjecture has been studied before by Beauville [Bea00], Druel [Dru00], Campana-Peternell [CP02] and recently by Brunella-Pereira-Touzet [BPT04]. Their paper contains most of the preceding results, its main result being the

1.2. Theorem. [BPT04, Thm.1] Let $X$ be a compact Kähler manifold. Suppose that its tangent bundle splits as $T_X = V_1 \oplus V_2$, where $V_2 \subset T_X$ is a subbundle of rank $\dim X - 1$. There are two cases:

1.) if $V_2$ is not integrable, $V_1$ is tangent to the fibres of a $\mathbb{P}^1$-bundle.
2.) if $V_2$ is integrable, $\mu : \tilde{X} \to X$, the universal covering of $X$, splits as $Y_1 \times Y_2$, where $Y_2$ is a curve. Moreover $\mu^* V_1 \simeq p_{Y_1}^* T_{Y_1}$ and $\mu^* V_2 \simeq p_{Y_2}^* T_{Y_2}$.

The theorem establishes a surprising link between the existence of rational curves along the foliation $V_1$ and the integrability of the complement $V_2$. We obtain a similar statement in the projective case.

1.3. Theorem. Let $X$ be a projective manifold with split tangent bundle $T_X = V_1 \oplus V_2$. If $X$ is not uniruled, $V_1$ and $V_2$ are integrable.
In this paper we will concentrate on projective uniruled varieties. By the theorem above this is the class of varieties where the integrability of the direct factors $V_j$ might fail, but we will obtain integrability results in special cases. The global strategy is to construct a fibre space structure on $X$ which is related to the decomposition $T_X = V_1 \oplus V_2$. The classical Ehresmann theorem 3.4 allows us to deduce properties of the universal covering of $X$ from this fibre space structure.

The structure of the paper is as follows: in the next section we recall some results about foliations and rational curves. We also give the short proof of theorem 1.3 Section 3 treats rationally connected manifolds, which are known to be simply connected, where we obtain a rather satisfactory answer to the conjecture.

1.4. Theorem. Let $X$ be a rationally connected manifold such that $T_X = V_1 \oplus V_2$. If $V_1$ or $V_2$ is integrable, $X$ is isomorphic to a product $Y_1 \times Y_2$ such that $V_j = p_j^* T_{Y_j}$ for $j = 1, 2$.

We then study the structure of elementary Mori contractions of projective uniruled fourfolds with split tangent bundle. Combining the results of section 4 we obtain

1.5. Theorem. Let $X$ be a projective uniruled fourfold such that $T_X = V_1 \oplus V_2$ where $\text{rk} V_1 = \text{rk} V_2 = 2$. If $V_1$ and $V_2$ are integrable, conjecture 1.1 holds.

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2. Notation and basic results

We work over the complex field $\mathbb{C}$. For standard definitions in complex algebraic geometry we refer to [Har77] or [KK83], for positivity notions of vector bundles we follow the definitions from [Laz04]. Manifolds and varieties are always supposed to be connected.

A fibration is a surjective projective morphism $\phi : X \to Y$ with connected fibres from a quasi-projective manifold to a normal quasi-projective variety $Y$ such that $\dim X > \dim Y$. The $\phi$-smooth locus is the largest Zariski open subset $Y^* \subset Y$ such that for every $y \in Y^*$, the fibre $\phi^{-1}(y)$ is a smooth variety of dimension $\dim X - \dim Y$. The $\phi$-singular locus is its complement. A fibre is always a fibre in the scheme-theoretic sense, a set-theoretic fibre is the reduction of the fibre. We say that a fibration is almost smooth if it is equidimensional and every set-theoretic fibre is a smooth variety.

Let $\phi : X \to Y$ be a morphism from a projective manifold to a normal projective variety $Y$. A line bundle $L$ on $X$ is $\phi$-trivial if for every $\phi$-fibre $F$, we have $L|_F \simeq \mathcal{O}_F$.

2.1. Rational curves and split tangent bundle. We will use the standard terminology from Mori theory. A Mori contraction of a projective manifold $X$ is a morphism with connected fibres $\phi : X \to Y$ to a normal variety $Y$ such that the anticanonical bundle $-K_X$ is $\phi$-ample. We say that the contraction is elementary
if there exists a rational curve $C \subset X$ such that a curve $C' \subset X$ is contracted by $\phi$ if and only if we have an equality in $N_1(X)$

$$[C'] = \lambda [C] \quad \lambda \in \mathbb{Q}^+.$$ 

By $N_1(X)$ we denote the $\mathbb{Q}$-vector space of 1-cycles on $X$ modulo numerical equivalence (cf. [Deb01, 1.3]). The contraction is said to be of fibre type if $d \leq 2$. Proposition.

otherwise it is birational.

2.6. Proposition. [CP02, Prop.1.8.] Let $X$ be a projective manifold such that $T_X = V_1 \oplus V_2$. Let $\phi : X \to Y$ be an elementary extremal contraction and let $F$ be a fibre of $\phi$.

1.) If $L_j = \det V_j$ is not $\phi$-trivial, we have $\dim F \leq \text{rk} V_j$.

2.) $\dim F \leq \max(\text{rk} V_1, \text{rk} V_2)$.

3.) Suppose $F$ contains a rational curve $f : \mathbb{P}^1 \to X$ such that

$$f^*T_X \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \bigoplus_{i=2}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i) \quad \text{with} \quad a_i \leq 1 \quad \forall \ i > 1.$$ 

Then, up to renumbering, $L_1$ is $\phi$-ample, $L_2$ is $\phi$-trivial, and $\mathcal{O}_{\mathbb{P}^1}(2) \subset f^*V_1$.

Remark. If $\phi : X \to Y$ is an elementary contraction of fibre type, a general fibre $F$ always contains a rational curve of splitting type (2.7). Indeed $F$ is a Fano manifold of positive dimension, so by [Deb01, Ex.4.8.3] there exists a rational curve $f : \mathbb{P}^1 \to F \subset X$ such that $f^*T_F$ is nef and has splitting type (2.7). Conclude with lemma 2.8

2.8. Lemma. Let $\phi : X \to Y$ be a morphism from a projective manifold $X$ to a normal variety $Y$. Let $F$ be a smooth irreducible component of a reduced $\phi$-fibre $Z$. Let $f' : \mathbb{P}^1 \to F$ be a rational curve such that $f'^*T_F$ is nef. There exists a deformation $f : \mathbb{P}^1 \to F$ of $f'$ in $F$, such that $f^*T_F$ is nef and

$$f^*T_X \cong f'^*T_F \oplus N_{F/X}.$$ 

Proof. Since $Z$ is reduced, the canonical morphism

$$\mathcal{I}_Z/\mathcal{I}_Z^2 \otimes \mathcal{O}_F \to \mathcal{I}_F/\mathcal{I}_F^2$$

is generically surjective. Since $\mathcal{I}_Z/\mathcal{I}_Z^2$ is globally generated, $\mathcal{I}_F/\mathcal{I}_F^2 = N_{F/X}^*$ is generically generated. Since $F$ is smooth, the deforms of $f'$ cover $F$ (Deb01, Prop.4.8). So for a general deformation $f : \mathbb{P}^1 \to F$, the restriction of the bundle $N_{F/X}$ to $f(\mathbb{P}^1)$ is generically generated and $f^*T_F$ is nef. It follows that $f^*(N_{F/X}^* \otimes T_F)$ is nef, so $H^1(\mathbb{P}^1, f^*(N_{F/X}^* \otimes T_F)) = 0$. Hence the exact sequence

$$0 \to f^*T_F \to f^*T_X \to f^*N_{F/X} \to 0$$

splits. \qed

2.B. Foliations. In this section we introduce the terminology that we will need and state some results about holomorphic foliations. Since the analytic category provides the adapted framework for this theory we state the results for compact Kähler manifolds.

Let $X$ be a compact Kähler manifold. A subbundle $V \subset T_X$ is integrable if it is closed under the Lie bracket. We recall that the Lie bracket

$$[[., .] : V \times V \to T_X$$
is a bilinear antisymmetric mapping that is not $O_X$-linear but induces an $O_X$-linear map $\wedge^2 V \to T_X/V$ that is zero if and only if $V$ is integrable. In particular

$$H^0(X, \mathcal{H}om(\wedge^2 V, T_X/V)) = 0$$

implies that $V$ is integrable. In general we will show this vanishing property using a covering family $S$ (cf. [Cam01] Defn.1.8.] for the terminology of covering families) of subvarieties $(Z_s)_{s \in S}$ of $X$ such that a general member of the family satisfies

$$H^0(Z_s, \mathcal{H}om(\wedge^2 V, T_X/V)|_{Z_s}) = 0.$$  

By the Frobenius theorem an integrable subbundle $V$ of $T_X$ induces a foliation on $X$, i.e. for every $x \in X$ there exists an analytic neighbourhood $U$ and a submersion $q : U \to W$ such that $T_{U|W} = V|_U$. This shows that $V$ can be realised as the tangent bundle of locally closed subsets of $X$. The foliation induces an equivalence relation on $X$, two points being equivalent if and only if they can be connected by chains of smooth (open) curves $C_i$ such that $T_{C_i} \subset V|_{C_i}$. An equivalence class is called a leaf of the foliation. The equivalence relation induces a quotient map $X \to X/V$ onto the so-called leaf space $X/V$ such that the fibres are the leaves of the foliation. This map can always be defined topologically, but in general $X/V$ is not Hausdorff. A subset $X^* \subset X$ is saturated if every leaf of the foliation is either contained in $X^*$ or disjoint from it. We will say that the general leaf of a foliation is compact if there exists a non-empty saturated Zariski open subset $X^* \subset X$ such that every leaf contained in $X^*$ is compact.

2.10. Proposition. Let $X$ be a compact Kähler manifold, and let $V \subset T_X$ be an integrable subbundle. Assume that the general $V$-leaf is compact. Then every $V$-leaf of the foliation is compact. There exists an almost smooth holomorphic map $X \to Y$ whose set-theoretic fibres are $V$-leaves.

Proof. The compactness of the leaves follows from the global stability theorem on Kähler manifolds (cf. [Per01] Thm.1. for a short proof). Holmann [Hol80] Cor.3.2 has shown that in this case the leaf space $X/V$ admits the structure of an analytic space such that the projection is almost smooth. □

2.11. Corollary. Let $X$ be a compact Kähler manifold, and let $V \subset T_X$ be an integrable subbundle such that one leaf is compact and rationally connected. Then there exists a submersion $X \to Y$ onto a smooth variety $Y$ such that $T_{X/Y} = V$.

Remark. Kebekus, Solá and Toma obtained independently similar results ([KSC10] Thm.28) for singular foliations on projective manifolds.

Proof. A rationally connected leaf is simply connected, so by Reeb’s local stability theorem (cf. [CLNS5] IV, §3, Thm.3), the general $V$-leaf is compact. Proposition 2.10 yields an almost smooth map $g : X \to Y$ onto the leaf space $Y := X/V$.

Let $y \in Y$ be an arbitrary point. By [Mol83] Prop.3.7.] there exists a neighbourhood $U$ of $y$ which is isomorphic to $D/G$ where $D$ is the $Y$-dimensional unit disc and $G$ is a finite group. More precisely $G$ is the image of a representation of the fundamental group $\pi_1(g^{-1}(y)_{red})$. Denote now by $q : D \to U$ the quotient map and by $X_U$ the normalisation of $g^{-1}(U) \times_U D$. Let $g' : X_U \to D$ be the induced map and $q' : X_U \to g^{-1}(U)$ the induced étale covering. Then $g'$ is a submersion and the $g'$-fibres are the $g^*V$-leaves.
A $V$-leaf is rationally connected if and only if the corresponding $q^*V$-leaf is rationally connected. Furthermore one $q'$-fibre is rationally connected if and only if all $q'$-fibres are rationally connected (by deformation invariance of rational connectedness, cf. [Ko96] Thm.3.11). Now by hypothesis there exists one rationally connected $V$-leaf, so by connectedness of $Y$ all $V$-leaves are rationally connected. Since rationally connected varieties are simply connected, the group $G$ is trivial. It follows that $U \simeq \mathbb{D}/G = \mathbb{D}$ is smooth and the quotient map $q : \mathbb{D} \to U$ is an isomorphism. Hence $q' : X_U \to g^{-1}(U)$ is also an isomorphism, so $g'$ and $g|_{g^{-1}(U)}$ identify under this isomorphism. In particular $g|_{g^{-1}(U)}$ is a submersion. Since $y$ is arbitrary, this shows the claim. \hfill \Box

2.12. Corollary. Let $X$ be a compact Kähler manifold and $V \subset TX$ a subbundle. Suppose that there exists a fibration $\phi' : X \to Y'$ such that $T_F = V|_F$ for a general fibre $F$. Then $V$ is integrable with compact leaves. The almost smooth map $\phi : X \to Y$ from proposition [2.10] is a factorisation of $\phi'$, i.e. there exists a birational morphism $g : Y \to Y'$ such that $g \circ \phi = \phi'$. In particular if $\phi'$ is equidimensional, it is almost smooth. \hfill \Box

2.C. An integrability result. We show theorem 1.3 and give a counterexample in the uniruled case. This shows that the restriction to the uniruled case is appropriate to the nature of the problem.

2.13. Theorem. [CP04] Thm.1.5] Let $X$ be a projective manifold with split tangent bundle $TX = V_1 \oplus V_2$. Set $L_j := \text{det} V_j^*$ for $j = 1, 2$. If $X$ is not uniruled, $L_j$ is pseudoeffective.

**Proof of theorem 1.3.** By the theorem of Campana-Peternell above, the line bundle $L_1 := \text{det} V_1^*$ is pseudoeffective. Since $V_1^*$ is a direct factor of $\Omega_X$, the vector bundle $\text{det} V_1 \otimes \Lambda^k V_1 \Omega_X$ has a trivial direct factor. If $\theta \in H^0(X, L_1^{-1} \otimes \Lambda^k V_1 \Omega_X)$ is the associated nowhere-vanishing $\text{det} V_1$-valued form, and $\zeta$ a germ of any vector field, a local computation shows that $i_\zeta \theta = 0$ if and only if $\zeta$ is in $V_2$. An integrability criterion by Demailly [Dem02] Thm.] shows that $V_2$ is integrable. The statement for $V_1$ follows completely analogously. \hfill \Box

**Remark.** The integrability theorem 1.3 is optimal, in the sense that there is the following counterexample to the integrability in the uniruled case.

2.14. Example. (A. Beauville) Let $A$ be an abelian surface and $u_1, u_2$ be linearly independent vector fields on $A$. Let $z_1, z_2$ be nonzero vector fields on $\mathbb{P}^1$ such that $[z_1, z_2] \neq 0$. Then $v_1 := p_A^*(u_1) + p_{\mathbb{P}^1}(z_1)$ and $v_2 := p_A^*(u_2) + p_{\mathbb{P}^1}(z_2)$ are everywhere nonzero, linearly independent vector fields on $X := A \times \mathbb{P}^1$. The subbundle $V := \mathcal{O}_X v_1 \oplus \mathcal{O}_X v_2 \subset TX$ is not integrable and $T_X = V \oplus p_{\mathbb{P}^1}^* T_{\mathbb{P}^1}$.

3. Proof of theorem 1.4

The strategy of the proof of theorem 1.4 is in a first step to construct a fibre space structure on $X$ and to show in a second step that this fibre space structure comes from a product structure on $X$. The main technical ingredient is a deep theorem by Bogomolov and McQuillan on the algebraicity of leaves.

3.15. Theorem. [BM01] Thm.0.1.a], [KSCT05] Thm.1.1] Let $X$ be a projective manifold, and let $V \subset TX$ be an integrable subbundle. Let $f : C \to X$ be a curve on $X$ such that $f^*V$ is ample. Then all $V$-leaves meeting $f(C)$ are rationally connected closed submanifolds of $X$. 


3.16. Lemma. Let $X$ be a projective rationally connected manifold such that $T_X = V_1 \oplus V_2$ and $V_1 \subset T_X$ is an integrable subbundle. Then there exists a submersion $X \rightarrow Y$ such that $V_1 = T_{X/Y}$.

Proof. Since $X$ is rationally connected, there exists a (very free) rational curve $f: \mathbb{P}^1 \rightarrow C$ on $X$ such that $f^*T_X$ is ample. Its quotient $f^*V_1$ is then also ample. By theorem 3.15 this implies that the $V_1$-leaves passing through a point of $C$ are rationally connected (closed) subvarieties of $X$. Now apply corollary 2.11.

3.17. Theorem. [CLNS5] V.§2.Prop.1 and Thm.3] Let $\phi: X \rightarrow Y$ be a submersion of manifolds with an integrable connection, i.e., an integrable subbundle $V \subset T_X$ such that $T_X = V \oplus T_{X/Y}$. Suppose furthermore one of the following:

1.) $\phi$ is proper.

2.) the restriction of $\phi$ to every $V$-leaf is a (not necessarily finite) étale map.

Then $\phi: X \rightarrow Y$ is an analytic fibre bundle with typical fibre $F$. More precisely, if $\tilde{Y} \rightarrow Y$ is the universal cover, there is a representation $\rho: \pi_1(Y) \rightarrow \text{Aut}(F)$ such that $X$ is isomorphic to $(\tilde{Y} \times F)/\pi_1(Y)$. Denote by $\tilde{F} \rightarrow F$ the universal cover of $F$; then the map $\mu: \tilde{Y} \times \tilde{F} \rightarrow \tilde{Y} \times F \rightarrow (\tilde{Y} \times F)/\pi_1(Y) \simeq X$ is the universal cover of $X$ and $\mu^*V \simeq p_1^*T_{\tilde{Y}}$ and $\mu^*T_{X/Y} \simeq p_2^*T_F$.

3.18. Lemma. Let $X$ be a projective manifold that admits a submersion $\phi: X \rightarrow Y$ on a projective manifold $Y$. Suppose furthermore that $\phi$ admits a connection, i.e., a vector bundle $V \subset T_X$ such that $T_X = V \oplus T_{X/Y}$. If $Y$ is rationally connected, $V$ is integrable and $X \simeq Y \times F$, where $F$ is a general fibre of $\phi$.

Proof. If $\dim Y = \text{rk} V = 1$ the result is trivial, so we suppose that $\dim Y > 1$. We will construct a covering family of curves on $X$ such that the general member is a smooth rational curve $f: \mathbb{P}^1 \rightarrow C$ that satisfies

$$f^*T_{X/Y} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus (\dim X - \dim Y)}$$

and

$$f^*\bigwedge^2 V \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i) \quad \text{with} \quad a_i \geq 1 \quad \forall \ i \geq 1.$$

Granting the construction for the time being, this implies

$$f^*\text{Hom}(\bigwedge^2 V, T_X/V) \simeq \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(-a_i)^{\oplus (\dim X - \dim Y)},$$

so clearly

$$H^0(C', \text{Hom}(\bigwedge^2 V, T_X/V)|_{C'}) = H^0(\mathbb{P}^1, f^*\text{Hom}(\bigwedge^2 V, T_X/V)) = 0,$$

which shows the integrability of $V$ (see subsection 2.3). Since $\phi$ admits an integrable connection, it follows from the Ehresmann theorem 3.17 that $\phi$ arises as a representation of the fundamental group of $Y$. Since $Y$ is simply connected, this implies $X \simeq Y \times F$, where $F$ is a $\phi$-fibre.

We come to the construction of the family. If $Y$ has dimension at least 3, there exists a covering family of curves on $Y$ such that the general member is a very free smooth rational curve $f: \mathbb{P}^1 \rightarrow C$ (cf. [Ko96 Thm.3.14]). If $Y$ has dimension 2 its relative minimal model is $\mathbb{P}^2$ or a Hirzebruch surface, so there exists a covering
family of free smooth rational curves. In both cases $f^* \wedge^2 T_Y \simeq \bigoplus_i O_{\mathbb{P}^1}(a_i)$ with all $a_i$ positive.

Let $f : \mathbb{P}^1 \to C$ be a general member of the covering family. Since $C$ is smooth and $\phi$ is a submersion, the fibre product $Z := X \times_Y \mathbb{P}^1$ is smooth and admits a submersion $\tilde{\phi} : Z \to \mathbb{P}^1$. Denote by $\mu : Z \to X$ the natural projection; then $\mu^* T_{X/Y} = T_{Z/\mathbb{P}^1}$ and $T_Z \subset \mu^* T_X$ is a subbundle. We consider the sequence of sheaf homomorphisms on $Z$

$$T_{Z/\mathbb{P}^1} \to T_Z \to \mu^* T_X \to \mu^* T_{X/Y}.$$ 

The first two maps are the canonical embeddings, while the last one is the projection along $\mu^* V$. Since $T_{Z/\mathbb{P}^1} = \mu^* T_{X/Y}$ the map $T_Z \to \mu^* T_{X/Y}$ has maximal rank at every point. It follows that $L := \mu^* V \cap T_Z = \ker(T_Z \to \mu^* T_{X/Y})$ is a rank 1 subbundle of $T_Z$ such that

$$T_Z := L \oplus T_{Z/\mathbb{P}^1}.$$ 

Since $L$ has rank 1 it is integrable. This shows that $\tilde{\phi} : Z \to \mathbb{P}^1$ admits an integrable connection, so by the Ehresmann theorem $Z \simeq \mathbb{P}^1 \times F$ where $F$ is a fibre. It follows that for any $a \in F$, we obtain a rational curve $\mu' : \mathbb{P}^1 \times \{a\} \to C''$ on $X$ such that $\mu'^* T_{X/Y}$ is trivial and $\mu'^* \wedge^2 V = \bigoplus_i O_{\mathbb{P}^1}(a_i)$. Since the rational curves we use cover a dense open subset on $Y$, the constructed curves cover a dense open subset on $X$. □

**Proof of theorem 3.14.** Lemma 3.16 yields the existence of a submersion $X \to Y$ with connection one of the direct factors $V_j$. Since $X$ is rationally connected, the manifold $Y$ is rationally connected, so lemma 3.18 applies. □

The following lemma will be useful in the next section.

**3.19. Lemma.** Let $X$ be a projective manifold that admits a submersion $\phi : X \to Y$ with a connection, i.e., a vector bundle $V \subset T_X$ such that $T_X = V \oplus T_{X/Y}$. Then $\phi$ is an analytic fibre bundle.

**Proof.** In general $V$ is not integrable, but if $C \subset Y$ is a smooth curve, the restriction $\phi|_{\phi^{-1}(C)} : \phi^{-1}(C) \to C$ is a smooth map over a curve and $V \cap T_{\phi^{-1}(C)}$ is a rank 1 bundle that provides a connection (cf. the proof of lemma 3.18 above for details). Since the connection has rank 1 it is integrable, so $\phi|_{\phi^{-1}(C)}$ is an analytic bundle. In particular its fibres are isomorphic. Since we can connect any two points in $Y$ by a chain of smooth curves, this shows that all fibres are isomorphic complex manifolds. By the Grauert-Fischer theorem [FG65, p.89] this shows that $\phi$ is a fibre bundle. □

4. Classification of elementary contractions

This section provides the main technical tools for the proof of theorem 3.14. Starting with an elementary observation (lemma 3.20) we classify the structure of elementary Mori contractions of fibre type (proposition 3.21) in dimension 4. One should note that the arguments employed also hold in higher dimensions. The classification of elementary contractions of birational type (proposition 3.23) depends heavily on previous classification results and therefore does not generalise to higher dimensions.

**4.20. Lemma.** Let $X$ be a projective manifold with $T_X = V_1 \oplus V_2$. Let $\phi : X \to Y$ be an elementary extremal contraction of fibre type. Let $F$ be a general fibre, then, after possible renumbering, $T_F \subset V_1|_F$. 

If furthermore \( \dim X - \dim Y = \text{rk} V_1 \) or \( \dim X - \dim Y + 1 = \text{rk} V_1 \), the bundle \( V_1 \) is integrable.

**Proof.** The general fibre \( F \) is a Fano manifold, so there exists a covering family such that the general member \( f : \mathbb{P}^1 \to F \) is a very free rational curve in \( F \), that is

\[
f^*T_F = \oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i) \quad \text{with} \quad a_i \geq 1 \quad \forall i \geq 1.
\]

We use lemma 2.8 to obtain

\[
f^*T_X = f^*T_F \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim Y} = \oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim Y}.
\]

By the remark after proposition 2.6, we may assume after possible renumbering that \( \det V_1 \) is \( \phi \)-ample and \( \det V_2 \) is \( \phi \)-trivial. This implies \( f^*V_2 = \mathcal{O}_{\mathbb{P}^1}^{\oplus \text{rk} V_2} \). We denote by \( \delta : T_F \to V_2|_F \) the projection on \( V_2|_F \) along \( V_1|_F \). Since \( f^*T_F \) is ample and \( f^*V_2 \) is trivial, the restriction of \( \delta \) to \( f(\mathbb{P}^1) \) is zero. Since the very free curves cover a dense set in \( F \) we see that \( \delta \) is zero. This is equivalent to \( T_F \subset V_1|_F \).

Suppose now that \( \dim X - \dim Y = \text{rk} V_1 \) or \( \dim X - \dim Y + 1 = \text{rk} V_1 \). Then we have \( f^*V_1 = \oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i) \) or \( f^*V_1 = \oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i) \oplus \mathcal{O}_{\mathbb{P}^1} \). In both cases \( \wedge^2 V_1 \) is ample, and \( f^*V_2 = f^*T_X/V_1 \) is trivial, so we obtain

\[
H^0(\mathbb{P}^1, \mathcal{H}om(f^* \wedge^2 V_1, f^*T_X/V_1)) = 0.
\]

We saw in subsection 2.B that this implies the integrability of \( V_1 \). \( \square \)

**4.21. Proposition.** Let \( X \) be a smooth projective fourfold with \( T_X = V_1 \oplus V_2 \) and \( \text{rk} V_1 = 2 \). Let \( \phi : X \to Y \) be an elementary extremal contraction of fibre type. One of the following holds:

1.) \( \dim Y = 2 \). Then \( \phi : X \to Y \) is an analytic fibre bundle such that up to renumbering \( T_{X/Y} = V_1 \).

2.) \( \dim Y = 3 \). Then \( \phi : X \to Y \) is a \( \mathbb{P}^1 \)-bundle or conic bundle such that, up to renumbering, \( T_F \subset V_1|_F \) for a general fibre \( F \).

**Proof.** By proposition 2.6 we may suppose up to renumbering that \( \det V_1 \) is \( \phi \)-ample and \( \det V_2 \) is \( \phi \)-trivial. In particular all fibres have dimension at most \( \text{rk} V_1 = 2 \), so \( \dim Y \geq 2 \). By lemma 4.20 we have \( T_F \subset V_1|_F \) for a general fibre \( F \) and the bundle \( V_1 \) is integrable.

Suppose that \( \dim Y = 2 \). A general fibre \( F \) satisfies \( T_F = V_1|_F \). Since the general fibre is a Fano manifold and a \( V_1 \)-leaf, corollary 2.12 shows that \( \phi \) is smooth. The vector bundle \( V_2 \) provides a connection, so by lemma 3.19 we know that \( \phi \) is a fibre bundle.

Suppose that \( \dim Y = 3 \). For a point \( x \in X \), denote by \( V_1^x \) the unique \( V_1 \)-leaf containing \( x \). By Ando’s result ([And85, Thm.3.1]) it is sufficient to show that \( \phi \) is equidimensional. We argue by contradiction and suppose that there exists a fibre that has an irreducible component of dimension 2.

1st case. There exists a 2-dimensional irreducible component \( F \) of a \( \phi \)-fibre that is a \( V_1 \)-leaf.

This implies that the variety \( F_{\text{red}} \) is smooth and the corresponding leaf is compact. Since \( T_{F_{\text{red}}} = V_1|_{F_{\text{red}}} \) and \( \det V_1 \) is \( \phi \)-ample, we see that \( F \) is a Fano manifold. Corollary 2.11 shows that there exists a submersion \( f : X \to Z \) such that all fibres are \( V_1 \)-leaves. Since \( \phi \) contracts \( F \), we can apply the rigidity lemma [Deb08, Lemma 1.15] to obtain a dominant factorisation \( g : Z \to Y \). But this is impossible, since \( 2 = \dim Z < \dim Y = 3 \).
2nd case. For every 2-dimensional irreducible component $F$ of a $\phi$-fibre and every $x \in F$, the set-theoretic intersection $F \cap V^e_1$ is strictly contained in $F$.
Let $Y^* \subset Y$ be the $\phi$-smooth locus which we consider as embedded in the Chow scheme $\mathcal{C}(X)$. Denote by $\bar{Y}$ the closure of $Y^*$ in $\mathcal{C}(X)$, endowed with the reduced structure. Let $\Gamma$ be the reduction of the graph over $\bar{Y}$, and $p_Y : \Gamma \to \bar{Y}$ and $p_X : \Gamma \to X$ the projections. Since every $p_Y$-fibre is contracted by $\phi$ (this depends only on the homology class), the rigidity lemma [Deb01, Lemma 1.15] implies the existence of a factorisation $g : \bar{Y} \to Y$, so that we obtain a commutative diagram

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{p_X} & X \\
\downarrow{p_Y} & & \downarrow{\phi} \\
\bar{Y} & \xrightarrow{g} & Y
\end{array}
$$

We show that $p_X$ is an isomorphism. Since $p_X$ is birational and $X$ normal the fibre $p_X^{-1}(x)$ is connected for every $x \in X$ (Zariski Main Theorem). Suppose that it has positive dimension; then there exists a curve $\Delta \subset \bar{Y}$ such that for every $y \in \Delta$, we have $x \in p_X(p_Y^{-1}(y))$. Consider now the foliation induced by $p_X^*V_1$ on $\Gamma \subset \bar{Y} \times X$. Since a general $p_Y$-fibre is contained in a $p_X^*V_1$-leaf and this is a closed condition, every fibre $p_X^{-1}(y)$ is contained in a $p_X^*V_1$-leaf. Since for $y \in \Delta$ we have $x \in p_X(p_Y^{-1}(y))$ the sets $p_X(p_Y^{-1}(y))$ are contained in the same $V_1$-leaf $V^e_1$. Furthermore they are contained in the $\phi$-fibre $\phi^{-1}(\phi(x))$. This shows that the surface $p_X(p_Y^{-1}(\Delta))$ is contained in the intersection of $V^e_1$ and $\phi^{-1}(\phi(x))$. So there exists a 2-dimensional irreducible component $F$ of $\phi^{-1}(\phi(x))$ that is equal to $V^e_1$, a contradiction. This shows that for every $x \in X$ a point, the fibre $p_X^{-1}(x)$ is a point, so $p_X$ is bijective. Since $X$ is smooth and $\Gamma$ reduced this shows that $p_X$ is an isomorphism.

To simplify the notation we identify $\Gamma$ and $X$ via the isomorphism $p_X$. Let now $F$ be a 2-dimensional component of a fibre. The surface $F$ is not contracted by the equidimensional map $p_Y$. Let $C \subset F$ be a curve that is not contracted by $p_Y$. Since $C \subset F$, it is contracted by $\phi$. Since $\phi$ is elementary the homology class $[C]$ is a multiple of $[D]$, where $D$ is a general $\phi$-fibre. Since the class $(p_Y)_*[D]$ is zero, the homology class $(p_Y)_*[C]$ is zero. So $p_Y$ contracts $C$, a contradiction. □

**Notation.** Let $\phi : X \to Y$ be a fibration between quasi-projective manifolds. The canonical map $\phi^*\Omega_Y \to \Omega_X$ induces a generically surjective sheaf homomorphism $T\phi : TX \to \phi^*TY$. In particular for $S \subset TX$ a quasicoherent subsheaf, we obtain a quasicoherent subsheaf $\phi_*(T\phi(S)) \subset TY$. For a point $y \in Y$ and a point $x \in \phi^{-1}(y)$ we denote by $(T\phi)_x : T_{X,x} \to T_{Y,y}$ the tangent map between the vector spaces $T_{X,x}$ and $T_{Y,y}$.

**4.22. Lemma.** Let $X$ be a quasi-projective manifold with split tangent bundle $TX = V_1 \oplus V_2$. Let $\phi : X \to Y$ be a fibration onto a quasi-projective manifold $Y$ such that for a general fibre $F$, we have

$$T_F = (V_1 \cap T_F) \oplus (V_2 \cap T_F).$$

Suppose furthermore that there exists a Zariski open set $Y^* \subset Y$ such that $Y \setminus Y^*$ has codimension at least 2, and such that for $y \in Y^*$, the following condition is satisfied:

$$\exists x \in \phi^{-1}(y) \text{ such that } \text{rk}((T\phi)_x : T_{X,x} \to T_{Y,y}) = \dim Y.$$
For $j = 1, 2$, the reflexive sheaf $W_j := (\phi_*(T\phi(V_j)))^{**} \subset T_Y$ is a subbundle of $T_Y$ such that $T_Y = W_1 \oplus W_2$.

**Proof.** Step 1. Suppose that $\phi$ is smooth. Then the map $T\phi : T_X \to \phi^*T_Y$ is surjective. Its restriction to the subbundle $V_j$, denoted by $q_j : V_j \to \phi^*T_Y$ is a morphism of sheaves. Since $T_{X/Y} = \ker(T\phi)$, we have $\ker q_j = T_{X/Y} \cap V_j$, so the rank of $q_j$ at a general point is $\rk V_j - \rk(T_F \cap V_j|F)$. By hypothesis this implies $\rk q_1 + \rk q_2 = \dim Y$. Since $T\phi = q_1 \oplus q_2$ and $T\phi$ has rank equal to $\dim Y$ at every point, this implies that $q_j$ is a morphism of vector bundles and $\im(q_1) \oplus \im(q_2) = \phi^*T_Y$. We verify that this induces a splitting of $T_Y$: for every fibre $F$, we have

$$\im(q_1)|_F \oplus \im(q_2)|_F = \phi^*T_Y|_F \simeq \mathcal{O}_F^{\oplus \dim Y},$$

so it is elementary to see that $\im(q_j)|_F$ is trivial. This implies $\im(q_j) = \phi^*E_j$ where $E_j$ is a vector bundle on $Y$, so the splitting pushes down to $Y$.

Step 2. We show the general case. Let $W_j := (\phi_*(T\phi(V_j)))^{**} \subset T_Y$; then $W_j$ is a reflexive sheaf, so the locus $Y' \subset Y$ where $W_1$ and $W_2$ are locally free satisfies $\codim_Y (Y \setminus Y') \geq 2$. Apply the first step to the $\phi$-smooth locus to see that $\rk W_1 + \rk W_2 = \dim Y$.

For $y \in Y' \cap Y^*$, let $x \in \phi^{-1}(y)$ be such that the rank of $(T\phi)_x : T_{X,x} \to T_{Y,y}$ is maximal. Denote by $V_{j,x} \subset T_{X,x}$ and $W_{j,y} \subset T_{Y,y}$ the subspaces induced by the subbundles $V_j$ and $W_j$. Since $(T\phi)_x(V_{j,x}) \subset W_{j,y}$, we have

$$T_{Y,y} = \im(T\phi)_x \subset (T\phi)_x(V_{1,x}) + (T\phi)_x(V_{2,x}) \subset W_{1,y} + W_{2,y} \subset T_{Y,y}.$$ 

Since $\rk W_1 + \rk W_2 = \dim Y$ this implies that $W_{1,y} \oplus W_{2,y} = T_{Y,y}$, hence $T_{Y \cap Y^*} = W_1|_{Y \cap Y^*} \oplus W_2|_{Y \cap Y^*}$. Since $Y \setminus (Y^* \cap Y')$ has codimension 2 and $Y$ is smooth, we have

$$T_Y = W_1 \oplus W_2.$$ 

In particular the sheaves $W_j$ are locally free. \[\square\]

4.23. Proposition. Let $X$ be a projective fourfold such that $T_X = V_1 \oplus V_2$ with $\rk V_1 = \rk V_2 = 2$. Suppose furthermore that $X$ has the structure of a $\mathbb{P}^1$-bundle or conic bundle $\phi : X \to Y$ such that for a general fibre $F$, we have $T_F \subset V_1|_F$. The map $\phi$ induces a splitting $T_Y = (\phi_*(T\phi(V_1)))^{**} \oplus (\phi_*(T\phi(V_2)))^{**}$. If $F$ is a singular fibre, $F$ is isomorphic to two smooth rational curves intersecting transversally in one point.

Furthermore the $\phi$-singular locus $\Delta \subset Y$ is smooth and satisfies

$$T_{\Delta} = (\phi_*(T\phi(V_2)))^{**}|_{\Delta}.\]
Suppose now that φ admits a double line $F = φ^{-1}(y)$ as a fibre. The foliation $L_1$ induces around $y$ a germ of a smooth curve $C$ on $Y$ and the foliation $V_1$ induces around $F_{\text{red}}$ a germ of a smooth surface $S$ on $X$ such that φ induces a morphism $φ|_S : S → C$. The restriction of $N_{F/S}$ to $F_{\text{red}}$ is a non-trivial torsion line bundle. This contradicts the fact that $F_{\text{red}} \cong \mathbb{P}^1$ is simply connected.

The smoothness of the φ-singular locus follows from [Sar82 Prop.1.8]. Let $C$ be an irreducible component of a singular fibre, then $T_X|_C ≃ O_{\mathbb{P}^1}(2) ⊕ O_{\mathbb{P}^1}^{\otimes 2} ⊕ O_{\mathbb{P}^1}(-1)$.

By proposition 2.6 we know that det $V_1$ is φ-ample and det $V_2$ is φ-trivial, so $T_C ⊂ V_1|_C$ and $V_2|_C ≃ O_{\mathbb{P}^1}^{\otimes 2}$. Let $D ⊂ X$ be an irreducible effective divisor such that $\phi(D) ⊂ Δ$. The canonical map $γ : V_2|_D → N_D/X$ is zero since its restriction to every component $C ⊂ D$ of a fibre is given by $γ|_C : O_{\mathbb{P}^1}^{\otimes 2} → O_{\mathbb{P}^1}(-1)$, which is the zero map. Hence we obtain a splitting of the tangent bundle of the nonsingular locus $D_{\text{nons}} ⊂ D$ as

$$T_{D_{\text{nons}}} = (V_1|_{D_{\text{nons}}} ∩ T_{D_{\text{nons}}}) ⊕ V_2|_{D_{\text{nons}}}.$$  

The inclusion $T_C ⊂ V_1|_C$ implies $T_{D_{\text{nons}}}/φ(D_{\text{nons}}) = V_1|_{D_{\text{nons}}} ∩ T_{D_{\text{nons}}}$, so $φ(D) = φ(D_{\text{nons}})$ implies $T_{Δ}|φ(D) = T_{φ(D)} = W_2|φ(D_{\text{nons}}). \ □$

**Remark.** Since elementary contractions $φ : S → C$ from a projective surface to a curve are always $\mathbb{P}^1$-bundles, one might expect that in the situation above there are no singular fibres at all. The following counterexample, for which we thank M. Brunella, shows that this is not true, thereby correcting and completing [CP02 Thm.2.8].

Let $S' := \mathbb{P}^1 × \mathbb{P}^1$, we identify $\mathbb{P}^1 = \mathbb{C} ∪ \{∞\}$ and choose coordinates $(z, w)$ on $S'$. The following map is an involution on $S'' := S' \setminus \{(0, 0), (∞, ∞)\}$.

$$φ' : S'' → S'' \ (z, w) → (z, \overline{w})$$

If $C_z := \{z\} × \mathbb{P}^1$ is a fibre of the projection $pr_1$ on the first factor, then $φ'|_{C_z}$ is the involution on $\mathbb{P}^1$ with fixed points $\sqrt{z}$ and $-\sqrt{z}$. Blow up $S''$ in $(0, 0)$ and $(∞, ∞)$ to resolve the indeterminacies of $φ'$. If we denote by $μ : S → S'$ the blow-up, $φ'$ lifts to an involution $φ$ of $S$. The total transform of $(0) × \mathbb{P}^1$ has two irreducible components $E_1$ and $E_2$ such that $φ(E_1) = E_2$ and $φ(E_2) = E_1$ (an analogous statement holds for the total transform of $(∞) × \mathbb{P}^1$).

Let $T := \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ be an elliptic curve and $ψ : T → T \ t → t + \frac{1}{2}$. Then $T' \cong T/\{id_T, ψ\}$ is an elliptic curve. Define $X := (S × T)/\{id_S × id_T, φ × ψ\}$, then $X$ is a smooth projective variety with split tangent bundle. The map $(pr_1 × μ) × id_T$ induces a morphism

$$f : X → \mathbb{P}^1 × T'.$$

Let us show that $f$ is an elementary contraction. Let $F'_0 = E'_1 + E'_2$ be a singular fibre and let $F_0 = E_1 + E_2$ be its lifting to $S × T$. Then $F_0 ⊂ S × \{t\}$ for some $t ∈ T$ and clearly

$$E_1 = (φ × ψ)(\overline{E})$$

where $\overline{E}$ is the copy of $E_2$ in $S × \{t + \frac{1}{2}\}$. Since $\overline{E}$ is a deformation of $E_2$ in $S × T$, this shows that $E'_2$ is a deformation of $E'_1$ in $X$, hence $E'_1 = E'_2$ in $N_1(X)$. The special fibre $F_0$ is homologous to a general fibre $F$, so

$$F = F_0 = E'_1 + E'_2 = 2E'_1.$$
in $N_1(X)$. This shows that $f$ is the elementary contraction of the ray generated by $E'$. 

**Notation.** Let $\phi : X \to Y$ be an elementary contraction of birational type. If the exceptional locus $E$ of $\phi$ is irreducible, let $k := \dim E$ and $l := \dim \phi(E)$. The birational contraction is then said to be of type $(k, l)$.

**4.24. Proposition.** Let $X$ be a projective fourfold such that $T_X = V_1 \oplus V_2$ with $\text{rk} V_1 = \text{rk} V_2 = 2$, and let $\phi : X \to Y$ be an elementary contraction of birational type. Then $Y$ is smooth and $\phi$ is the blow-up of a smooth 2-dimensional subvariety of the manifold $Y$. Set $W_j := (\phi_*(T\phi(V_j)))^*; \text{ then the tangent bundle of } Y \text{ splits as }$ 

$$T_Y = W_1 \oplus W_2.$$ 

If the universal covering $\mu : \tilde{Y} \to Y$ splits as $\tilde{Y} \simeq Y_1 \times Y_2$ such that $\mu^*W_j = p_{Y_j}^*T_{Y_j}$, the analogous statement holds for $X$.

**Proof.** Let $E$ be the exceptional locus of the contraction and $F$ an irreducible component of a non-trivial fibre. Then by proposition 2.6, we have $\dim F \leq 2$. In particular, the contraction can’t be of type $(3, 0)$. Constructions of type $(3, 1)$ have been classified by Takagi [Tak99, p.316]: the map $\phi_1 : E \to \phi_1(E)$ is a $\mathbb{P}^2$-bundle or quadric bundle, so the general fibre is reduced and isomorphic to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a quadric cone. If $\phi$ is of type $(3, 2)$, a fibre of dimension 2 is isolated, so $F$ is reduced and either $\mathbb{P}^2$, $\mathbb{P}^2 \cup \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a quadric cone [AW97, Thm.4.6]. Elementary fourfold contractions such that the exceptional locus is not an irreducible divisor have been classified by Kawamata [Kaw89, Thm.1.1]: the positive dimensional fibres are reduced and isomorphic to $\mathbb{P}^2$.

We will exclude case by case the existence of 2-dimensional fibres. The strategy is to choose appropriately rational curves $f : \mathbb{P}^1 \to F$ such that $f^*T_F$ is nef so that we can use equation (2.9) to compute $f^*T_X$.

Case 1. $F \simeq \mathbb{P}^2$ or $F \simeq \mathbb{P}^2 \cup \mathbb{P}^2$.

If $F \simeq \mathbb{P}^2$, let $f : \mathbb{P}^1 \to F$ be a line $C = f(\mathbb{P}^1)$, then $f^*N^*_{F/X}$ is nef, so by equation (2.9)

$$f^*T_X = O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-a) \oplus O_{\mathbb{P}^1}(-b),$$

where $a \geq 0$, $b \geq 0$ and $a + b > 0$ (otherwise the deformations of $C$ would cover $X$).

Since the curve is of splitting type (2,7), we have up to renumbering $O_{\mathbb{P}^1}(2) \subset f^*V_1$. Since $C$ is a line in projective space it deforms keeping a point fixed. This shows that $O_{\mathbb{P}^1}(2) \subset f^*V_1$ implies $T_{Y_2} = V_1|_{Y_2}$. In particular $f^*V_1 = O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1}(1)$, so $f^*V_2 = O_{\mathbb{P}^1}(-a) \oplus O_{\mathbb{P}^1}(-b)$. This implies that $f^*\text{det}V_2 = O_{\mathbb{P}^1}(-a - b)$ is not $\phi$-trivial, a contradiction to proposition 2.6. The same argument works in the case $F \simeq \mathbb{P}^2 \cup \mathbb{P}^2$.

Case 2. $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Choose a ruling line $f : \mathbb{P}^1 \to F$ with image $C = f(\mathbb{P}^1)$. Then

$$f^*T_X = O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-a) \oplus O_{\mathbb{P}^1}(-b)$$

where $a \geq 0$ and $b \geq 0$ and $a + b > 0$. Up to renumbering, this implies $O_{\mathbb{P}^1}(2) \subset f^*V_1$. By proposition 2.6, we see that $\text{det}V_1$ is $\phi$-ample and $\text{det}V_2$ is $\phi$-trivial. Now choose a line $f' : \mathbb{P}^1 \to F$ from the second ruling that is transversal to the first one. Then

$$f'^*T_X = O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-a) \oplus O_{\mathbb{P}^1}(-b)$$
with coefficients as above. Then again $O_{p_1}(2) \subset f'' V_1$, since otherwise $f''(c_1(V_1)) = 0$ (for details cf. the proof of [CP02, Lemma 1.3]), which would contradict the $\phi$-ampleness of $\det V_1$. Since the lines are transversal, we obtain

$$O_{p_1}(2) \oplus O_{p_1} = f^* T_F = f^* V_1.$$  

As in the first case we obtain $f^* V_2 = O_{p_1}(-a) \oplus O_{p_1}(-b)$. Clearly $f^* \det V_2 = O_{p_1}(-a - b)$ is not trivial, a contradiction.

Case 3. $F$ is a quadric cone.

Let $f : \mathbb{P}^1 \to F$ be a line passing through the vertex $x$ of the cone $F$. Campana and Peternell have shown in [CP02, Thm.3.6] that $C = f(\mathbb{P}^1)$ is of splitting type (2,7), so up to renumbering, we have $O_{p_2}(2) \subset f^* V_1$. In particular $T_{C,x} \subset V_{1,x}$ for every such line. Since $x$ is a singularity of $F$ and the vector spaces $T_{C,x}$ generate the Zariski tangent space of $F$ in $x$, they generate a subspace of dimension at least 3 in $V_{1,x}$. This contradicts $\rk V_1 = 2$.

We summarize: the morphism $\phi$ is a birational contraction of divisorial type such that all non-trivial fibres are of pure dimension 1. So the structure of $\phi$ follows from a result by Ando [And85, Thm.2.1].

Let $F$ be a positive dimensional fibre; then $T_X|_F \simeq O_{p_1}(2) \oplus O_{p_2}^2 \oplus O_{p_1}(-1)$. Up to renumbering $\det V_1$ is $\phi$-ample and $\det V_2$ is $\phi$-trivial, so $T_F \subset V_1|_F$ and $V_2|_F \simeq O_{p_2}^2$. The splitting of the tangent bundle $T_V$ is a consequence of lemma 4.22. Let $E$ be the exceptional divisor; then the canonical map $\gamma : V_2|_E \to N_{E/X}$ is zero since its restriction to the positive dimensional fibres is given by $\gamma|_F : O_{p_2}^2 \to O_{p_1}(-1)$, which is the zero map. It follows easily that $T_E = (V_1|_E \cap T_E) \oplus V_2|_E$ and $T_F \subset V_1|_F$ implies $T_{E/\phi(E)} = V_1|_E \cap T_E$, so we have $T_{\phi(E)} = W_2|_{\phi(E)}$.

Let now $\mu : \hat{Y} \to Y$ be the universal covering map and suppose that $\hat{Y} \simeq Y_1 \times Y_2$ and $\mu^* W_j = p_{Y_j}^* T_{Y_j}$. Note that this implies the integrability of $W_1$ and $W_2$. Since $\pi_1(X) = \pi_1(Y)$, the pull-back $\hat{X} = X \times_Y \hat{Y}$ is the universal covering of $X$. We have seen that $\phi$ is the blow-up of $Y$ along the $W_2$-leaf $\phi(E)$. It follows that $\hat{\phi}$ is the blow-up of $\hat{Y}$ along a leaf of the foliation $\mu^* W_2 = p_{Y_2}^* T_{Y_2}$. So there exists a $y \in Y$ such that $\hat{X} \simeq Bl_{y \times Y_2} \hat{Y} \simeq Bl_{y} Y_1 \times Y_2 =: X_1 \times X_2$, where $Bl_{A}B$ denotes the blow-up of a manifold $B$ along a submanifold $A$. Since $\hat{\mu}^* V_j = \hat{\phi}^* \circ \mu^* W_j = \hat{\phi}^* p_{Y_j}^* T_{Y_j}$, we have $\hat{\mu}^* V_j = p_{X_1}^* T_{X_1}$ and $\hat{\mu}^* V_2 = p_{X_2}^* T_{X_2}$. □

5. Proof of theorem 1.15

**Proof.** By proposition 4.21 the splitting of $T_X$ is stable under birational contractions. By the general minimal model program it is clear that after finitely many birational contractions $X = X^0 \to X^1 \to \ldots \to X^n$ we obtain either a variety $X^n$ with nef canonical bundle or an elementary contraction of fibre type. Since $X$ is uniruled, the variety $X^n$ is uniruled so we are in the second case.

By proposition 4.21 it is sufficient to show conjecture 1.1 for $X^n$. To simplify the notation we suppose that $X = X^n$. Suppose now that $V_1$ and $V_2$ are integrable. By proposition 4.21 there are the following two cases.

1st case. The variety $X$ is an analytic fibre bundle $\phi : X \to Y$ such that up to renumbering $T_{X/Y} = V_1$. Since $V_2$ is integrable, we conclude with the Ehresmann theorem.

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2nd case. The variety $X$ is a $\mathbb{P}^1$-bundle or conic bundle $\phi : X \to Y$ such that up to renumbering $T_F \subset V_1|_F$ for a general fibre. By proposition 1.23 we have $T_Y = (\phi_*T\phi(V_1))^{**} \oplus (\phi_*T\phi(V_2))^{**}$ where $L := (\phi_*T\phi(V_1))^{**}$ is a line bundle. By theorem 1.2 applied to $Y$, there are two subcases.

Case a. The variety $Y$ is a $\mathbb{P}^1$-bundle $\psi : Y \to Z$ such that $T_{Y/Z} = L$. Then $\psi \circ \phi : X \to Z$ is a proper submersion with integrable connection $V_2$ and we conclude with the Ehresmann theorem 3.17.

Case b. The bundle $V := (\phi_*T\phi(V_2))^{**}$ is integrable, so the universal covering $\mu : \tilde{Y} \to Y$ satisfies $\tilde{Y} \simeq S \times C$ such that $\mu^*V = p_S^*T_S$ and $\mu^*L = p_C^*T_C$. Furthermore we have a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\tilde{\mu}} & X \\
\downarrow{\tilde{\phi}} & & \downarrow{\phi} \\
Y & \xrightarrow{\mu} & \tilde{Y} \\
\downarrow{p_S} & & \downarrow{p_{\tilde{Y}}} \\
S & & \\
\end{array}
\]

where $\tilde{\mu} : X' := X \times_{\tilde{Y}} Y \to X$ is étale. By proposition 1.23 the morphism $\tilde{\phi}$ does not have any multiple fibres, so the almost smooth morphism $q := p_S \circ \tilde{\phi} : X' \to S$ is a submersion such that $\tilde{\mu}^*V_1 = T_{X'/S}$. The morphism $q$ is not necessarily proper, so to apply theorem 3.17 we have to verify that the restriction of $q$ to every $\tilde{\mu}^*V_2$-leaf is a covering map. Let $\frak{M}$ be a $\tilde{\mu}^*V_2$-leaf, then

\[(\tilde{\phi}_*(T\tilde{\phi}(\tilde{\mu}^*V_2)))^{**} = \mu^*V = p_S^*T_S\]

implies that $\tilde{\phi}(\frak{M}) = S \times c$ for some $c \in C$. Set $Z := \tilde{\phi}^{-1}(S \times c)$, then $\tilde{\phi}|_Z : Z \to S \times c$ is proper. By proposition 1.23 we know that $S \times c$ is either disjoint from the $\tilde{\phi}$-singular locus or contained in it. In the first case apply [CLNS85] V.,§2,Prop.1] to see that $\tilde{\phi}|_\frak{M} : \frak{M} \to S \times c$ is a covering map. In the second case let $\nu : Z' \to Z$ be the normalisation of $Z$, then $\tilde{\phi} \circ \nu : Z' \to S \times c$ is a fibre bundle with typical fibre two connected components isomorphic to $\mathbb{P}^1$ and $\nu^*\tilde{\mu}^*V_2 \simeq \nu^*\tilde{\phi}^*T_{S \times c}$. Apply again [CLNS85] V.,§2,Prop.1] to see that the restriction of $\tilde{\phi} \circ \nu$ to every $\nu^*\tilde{\mu}^*V_2$-leaf is a covering map. This implies that $\tilde{\phi}|_\frak{M} : \frak{M} \to S \times c$ is a covering map.

Since $p_S|_{S \times c} : S \times c \to S$ is an isomorphism we obtain in both cases that the restriction of $q$ to $\frak{M}$ is a covering map. □

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