Deformation Ricci Flow method in the existence problem of the Kahler metrics

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1 Introduction

In 1986, Hamilton first used the Ricci Flow method in his paper and after a short time, Cao used the same method to research the existence of the Kahler metric on a compact Kahler manifold problem and gave another proof of the existence of the Kahler-Einstein metric problem on a Kahler manifold when the first Chern class is negative. While, in the early 1950s, Yau first used the partial differential equation method to analyze a Monge-Ampere equation then gave a proof of the Calabi’s conjecture. This note illustrates the efficient method based on the Cao.H.D’s paper[1] and Yau.S.T’s paper[4], and tries to explain the method in detail, especially in some estimate which Cao has not proved in his paper. Jian Song and Weinkove’s note[9] used other estimates to obtain the result, this paper will explain their method as well. This note was a seminar lecture note in 2022 summer when the author was giving lectures on the geometry analysis seminar reasearching the classical Ricci flow method. Due to the importance of the Yau’s zero order estimate, the author will illustrate the zero order estimate in detail in another survey paper, hence the zero order estimate here will be explained briefly.

2 The Ricci Flow equation

Let $M$ be a compact Kahler manifold with the Kahler dimension $n$ and the Kahler metric $ds = g_{iar{j}} dz^i \wedge d\bar{z}^j$. We will use the Einstein summation through the whole article.

Let $R_{i\bar{j}} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{i\bar{j}})$ be the Ricci curvature of the corresponding metric, and the Ricci Form, i.e. the $(1,1)$ tensor $\sqrt{-1} 2\pi R_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is closed so we can define the cohomology class of it which is the first Chern class $C_1(M)$ of $M$.

We consider the Ricci Flow equation $\frac{\partial g_{i\bar{j}}}{\partial t} = -2R_{i\bar{j}} + \frac{2}{3} r g_{i\bar{j}}$ and its complex version:

$\frac{\partial \hat{g}_{i\bar{j}}}{\partial t} = -\hat{R}_{i\bar{j}} + \hat{T}_{i\bar{j}}$, $\hat{g}_{i\bar{j}} = g_{i\bar{j}}$ at $t = 0$ (2.1)

where $\hat{R}_{i\bar{j}}$ is the Ricci tensor of the $\hat{g}_{i\bar{j}}$, $\hat{T}_{i\bar{j}}$ is a representation of the first Chern class $C_1(M)$, actually we can choose any representation satisfying our requirement, such as a Kahler-Einstein metric.
Then our idea is first prove the solution exists all the time, then prove when t goes to infinity, \( \tilde{g}_{ij} \) converges to a definite \( \tilde{g}_{ij}(\infty) \) and hence \( \frac{\partial \tilde{g}_{ij}}{\partial t} \) converges to 0, then we get \( -\tilde{R}_{ij} + T_{ij} = 0 \), then we get \( T_{ij} \) will be the Ricci tensor of \( \tilde{g}_{ij}(\infty) \) so the \( \tilde{g}_{ij}(\infty) \) is the metric we want.

However, the equation (2.1) is too abstract to solve, therefore, we replace it. Due to \( \sqrt{\tilde{g}_{ij}} T_{ij}dz^i \wedge d\bar{z}^j \) is in the class of first chern class of M, so is \( \sqrt{\tilde{g}_{ij}} R_{ij}dz^i \wedge d\bar{z}^j \), where by the Hodge theory and the \( \partial \bar{\partial} \) Lemma, the cohomology group \( H^1_{\partial\bar{\partial}}(M, R) \) = \( \{ \text{closed real (1,1) forms} \} \), so they deviate by a term which in the kernel of the boundary operator \( d \), i.e.

\[
\frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} : T_{ij} - R_{ij} = \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}.
\]

So we take \( t = 0 \) in the equation, similarly, we can assume

\[
\tilde{g}_{ij} - g_{ij} = \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}, \quad \text{for } u \in C^\infty(M \times [0, T]) \), \( T \) is nonnegative and less or equal to infinity, in order to satisfy the initial condition, \( u(0) = 0 \), we caculate the new scaler equation replacing \( \tilde{g}_{ij} \) by \( g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \):

The left hand side is: \( g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} - 0 + \frac{\partial^2 g_{ij}}{\partial z^i \partial \bar{z}^j} = \frac{\partial^2 (\frac{\partial u}{\partial t})}{\partial z^i \partial \bar{z}^j} \)

The right hand side is: \( -\tilde{R}_{ij} + R_{ij} + \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} \)

while we can calculate the Ricci tensor explicitly in terms of \( \tilde{g}_{ij} \) and \( g_{ij} \):

\[
R_{ij} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij})
\]

so the right hand side is equal to:

\[
-\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij}) - \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij}) + \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}
\]

\[
= \frac{\partial^2}{\partial z^i \partial \bar{z}^j}((\log \det(g_{ij}) + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}) - \log \det(g_{ij}))) + \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}
\]

So we get:

\[
\frac{\partial^2}{\partial z^i \partial \bar{z}^j}(\frac{\partial u}{\partial t}) = \frac{\partial^2}{\partial z^i \partial \bar{z}^j}((\log \det(g_{ij}) + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}) - \log \det(g_{ij}))) + \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}
\]

Finally, \( \frac{\partial u}{\partial t} - \log \det(g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}) + \log \det(g_{ij} - f) = 0 \), the function \( (\frac{\partial u}{\partial t} - \log \det(g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}) + \log \det(g_{ij} - f) \) satisfies this uniform elliptic equation \( \Delta F = 0 \), then by the strong maximum principle, the maximum and the minimum attain on the boundary, but M is compact, so only non-boundary points exist, then this function can only be a constant relative to \( \partial \bar{\partial} \), this means the term in the partials is equal to a smooth function relative to \( t \):

\[
\frac{\partial u}{\partial t} = \log \det(g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}) - \log \det(g_{ij} + \phi(t))
\]

And the \( \phi(t) \) should satisfy the compatibility condition:

\[
\int_M e^{\frac{n}{n-1} f} d\tilde{V} = e^{\frac{n}{n-1} t} \text{Vol}(M)
\]

Actually, this compatibility condition wants to say that the volume of the compact manifold stays invariant during the deformation of the metric \( \tilde{g}(t) \). From the equation above,

\[
\frac{\partial u}{\partial t} - f = \log \det(g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}) - \log \det(g_{ij} + \phi(t))
\]

we take the exponential of the both sides and integrate them on M:

\[
\int_M e^{\frac{n}{n-1} f} dV = \int_M e^{\log \det(g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}) - \log \det(g_{ij}) + \phi(t)} dV
\]

\[
\int_M e^{\frac{n}{n-1} f} dV = e^{\frac{n}{n-1} t} \int_M \log \det(g_{ij}) dV
\]

While \( d\tilde{V} = \det(g_{ij}) \Lambda_{i=1}^n (\sqrt{\frac{\partial}{\partial x^i}} dz^i \wedge d\bar{z}^j) \) = \( \frac{\omega^n}{n!} \)

\[
d\tilde{V} = \det(g_{ij}) \Lambda_{i=1}^n (\sqrt{\frac{\partial}{\partial z^i}} dz^i \wedge d\bar{z}^j) = \frac{\omega^n}{n!}
\]
so the above equation changes to:
\[ \int_M e^{\varphi - F} dV = e^{\varphi(t)} \int_M dV \]
to make sure the volume of M stay unchanged, there is \[ \int_M e^{\varphi - F} dV = e^{\varphi(t)} \int_M dV = e^{\varphi(t)} \text{Vol}(M). \]

To prove the long time existence of the solution of this parabolic equation, generally, we need to give up to third order estimate. In the proof of the existence and the uniform convergence, we will use these estimates.

3 Existence for the solution in all time

Actually, the goal equation is given initial value:
\[ \frac{\partial u}{\partial t}(x,t) = 0 \quad \text{when} \quad t = 0. \]
And by the initial assumption, the solution exists in the interval \([0,T)\) and the Kahler metric \(\tilde{g}_{ij} = g_{ij} + \frac{\partial^2 u}{\partial x^i \partial x^j}\) is positive definite so it’s a Kahler metric for any \(t \in (0,T].\)

To prove the estimation, we need some notations. We differentiate the equation \((3.1):\)
\[ \frac{\partial}{\partial t} (\log \det(g_{ij} + \frac{\partial^2 u}{\partial x^i \partial x^j})) = \frac{\partial}{\partial t} (f) = 0, \text{and} \frac{\partial}{\partial t} (\log \det(g_{ij})) = 0 \]
then
\[ \frac{\partial}{\partial t} (\log \det(g_{ij} + \frac{\partial^2 u}{\partial x^i \partial x^j})) = \frac{\partial}{\partial t}(\det(g_{ij})) \frac{\partial}{\partial \det(g_{ij})} \frac{\partial}{\partial \det(g_{ij})} \frac{\partial}{\partial \det(g_{ij})} (\frac{\partial^2 u}{\partial x^i \partial x^j}) \]
where \(\frac{\partial}{\partial t} (\log \det(g_{ij}))\) is a Einstein summation, \(\tilde{g}_{ij}\) is the inverse of \(g_{ij}\). It can not be ignored that the \(\tilde{g}_{ij} \frac{\partial}{\partial \det(g_{ij})} (\frac{\partial^2 u}{\partial x^i \partial x^j})\) is a parabolic equation, so by the maximum principle, at the time \(t = 0\), we can consider this manifold with the initial metric is the border of the domain \(M \times [0,T)\). So we have
\[ \max_M |\frac{\partial u}{\partial t}| \leq \max_{[0,T)} |\frac{\partial u}{\partial t}| = \max_M |f|. \]

Let \(\Delta = \det(g_{ij}) \frac{\partial^2}{\partial x^i \partial x^j}\) be the normalized Laplace of \(g_{ij}\), and \(\Delta = g_{ij} \frac{\partial^2}{\partial x^i \partial x^j}\) be the normalized Laplace of the \(g_{ij}\), \(\varphi = \Delta - \frac{\partial}{\partial t}.\)

Zero order estimate

Let \(v = u - \frac{1}{\text{Vol}(M)} \int_M u dV\) be the normalized of \(u\) such that \(\int_M v = 0\) then \(v\) satisfies the equation:
\[ \det(g_{ij} + \frac{\partial^2 v}{\partial x^i \partial x^j}) (\det(g_{ij}))^{-1} = e^F \]
where \(F = \frac{\partial f}{\partial t} - f\) in the paper of Yau, so we can directly use the calculation in Yau’s work, we get the following Lemma:

Lemma 1
\[ \sup_{M \times [0,T]} v \leq C_2, \sup_{M \times [0,T]} \int_M |v| dV \leq C_3 \]

Remark We are here to give a sketch proof of this zero order estimate following Yau’s method. We consider the Green’s function \(G(p,q)\) of the normalized Laplace operator of \(g_{ij}\) \(\Delta\) on \(M\), and let \(K\) be a constant which depends only on \(M\) such that \(G(p,q) + K \geq 0\). Then we find that
\[ \Delta u = \frac{\partial}{\partial t} \text{so} \quad \tilde{g}_{ij} = g_{ij} + \frac{\partial^2 u}{\partial x^i \partial x^j} \]
and \(\int_M v = 0\), then by Yau’s work it shows
\[ v(p) = -\int_M (G(p,q) + K)dq \]
and \(\sup_M v \leq n \sup_{p \in M} \int_M (G(p,q) + K)dq\), while
the right hand side depends only on M, hence we can consider it as a constant $C_2$, and we proceed by this estimate we have
\[
\int_M |v| \leq \int_M |sup_{p \in M} G(p, q) + K| dq.
\]

Similarly, the right hand side depends only on M, hence we can also consider it as a constant $C_3$.

**First order estimate** We want to give an estimate to $|\nabla v|$ by Schauder estimate, here $L = \Delta$ is an elliptic operator, and the equation $Lv = \Delta v$ is a uniformly elliptic equation (see G&T), then by Schauder estimate we have
\[
||v||_{C^{2, \alpha}} = C(\sum_{|\gamma| \leq k} ||D^\gamma v||_{L^\infty} + \sum_{|\gamma| = k} ||D^\gamma v||_{\alpha})
\]
\[
\leq C(||Lv||_{C^{2, \alpha}} + ||v||_{L^\infty})
\]
\[
= C(||\Delta v||_{L^\infty} + ||v||_{L^\infty})
\]
\[
= C(\sup_{M \times [0, T]} ||\Delta v|| + \sup_{M \times [0, T]} ||v||).
\]

While
\[
\sup_{M \times [0, T]} |\nabla v| = ||\nabla v||_{L^\infty} \leq \sum_{|\gamma| \leq k} ||D^\gamma v||_{L^\infty} + \sum_{|\gamma| = k} ||D^\gamma v||_{\alpha},
\]

so
\[
\sup_{M \times [0, T]} |\nabla v| \leq C(\sup_{M \times [0, T]} ||\Delta v|| + \sup_{M \times [0, T]} ||v||)
\]

where $||v||_{C^{2, \alpha}} = \sum_{|\gamma| \leq k} ||D^\gamma v||_{L^\infty} + \sum_{|\gamma| = k} ||D^\gamma v||_{\alpha}$ is the Holder norms and $||v||_{\alpha}$ is the $\alpha$-Holder constant on $\Omega$, i.e. $||f||_{\alpha} = \sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}$.

Therefore, we must give an estimate to $\sup_{M \times [0, T]} ||\Delta v||$ and $\sup_{M \times [0, T]} ||v||$.

**Lemma 2**

$\exists C_1, C_2 > 0$, such that $0 < n + \Delta u \leq C_1 e^{C_0 (u - in f_{M \times [0, T] u})}$, for $\forall t \in [0, T)$

**Proof.**

Because $g_{ij} - g_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, so we get $g^{ij} g_{ij} = g^{ij}(g_{ij} + \frac{\partial^2 u}{\partial x_i \partial x_j}) = n + \Delta u$, while $g_{ij}$ and $g^{ij}$ are both positive definite, so $g^{ij} g_{ij}$ is positive definite, $n + \Delta u$ is its trace so it's positive.

As for the other inequality, first we learn from Yau's work that:

If let $\tilde{g}_{ij} = g_{ij} + \frac{\partial^2 u}{\partial x_i \partial x_j}$, such that the equation holds
\[
det(g_{ij} + \frac{\partial^2 u}{\partial x_i \partial x_j}) (det(\tilde{g}_{ij}))^{-1} = e^F
\]
then we get:
\[
\Delta (e^{-C_F}(n + \Delta \varphi)) \geq e^{-C_F}(\Delta F - n^2 \inf_{\varphi \neq \tilde{R}}(R_{\varphi \varphi})) - C e^{-C_F} n(n + \Delta \varphi)
\]
\[
+ (C + \inf_{\varphi \neq \tilde{R}}(R_{\varphi \varphi})) e^{-C_F} e^{-\frac{C_F}{n}} (n + \Delta \varphi)^{\frac{n}{n-1}}
\]
where $R_{\varphi \varphi}$ is the bisectional curvature of the $g_{ij}$, $C_0$ a positive constant such that
\[
C_0 + \inf_{\varphi = 1}(R_{\varphi \varphi}) > 0.
\]

Then we let
\[
\varphi = u (u satisfies the equation above), F = \frac{\partial u}{\partial t} - f,
\]
so we get:
\[
\Delta (e^{-C_F}(n + \Delta u)) \geq e^{-C_F}(\Delta F - n^2 \inf_{\varphi \neq \tilde{R}}(R_{\varphi \varphi})) - C e^{-C_F} n(n + \Delta u)
\]
\[
+ (C + \inf_{\varphi \neq \tilde{R}}(R_{\varphi \varphi})) e^{-C_F} e^{-\frac{C_F}{n}} (n + \Delta u)^{\frac{n}{n-1}}
\]
and
\[
\frac{\partial}{\partial t} (e^{-C_F}(n + \Delta u)) = e^{-C_F}(-C_0 \frac{\partial u}{\partial t})(n + \Delta u) + e^{-C_F} \Delta \frac{\partial u}{\partial x}
\]

Therefore, we get
\[ (e^{-Cu} (n + \tilde{\Delta}u)) \]
\[ \geq e^{-C_0u(\Delta \frac{\partial}{\partial t} - f)} - n^2\inf \mathcal{F}_{i\neq l}(R_{i\tilde{l}l}) - C_0 e^{-C_0u(n + \Delta u)} \]
\[ + (C_0 + \inf \mathcal{F}_{i\neq l}(R_{i\tilde{l}l})) e^{-C_0u} e^{-\frac{(\partial u)}{(n+\Delta u)^{\frac{1}{n-1}}}} (n + \Delta u) \]
\[ - e^{-C_0u} \Delta \frac{\partial u}{\partial t} \]
\[ = e^{-C_0u}(\Delta f + n^2\inf \mathcal{F}_{i\neq l}(R_{i\tilde{l}l})) - C_0 e^{-C_0u(n - \frac{\partial u}{\partial t})(n + \Delta u)} \]
\[ + (C_0 + \inf \mathcal{F}_{i\neq l}(R_{i\tilde{l}l})) e^{-C_0u} e^{-\frac{(\partial u)}{(n+\Delta u)^{\frac{1}{n-1}}}} (n + \Delta u) \]

Then we assume for any \( t \in (0, T) \), the function \( (e^{-Cu} (n + \tilde{\Delta}u)) \) achieves its maximum at \((p_0, t_0) \in M \times [0, t] \) and \( t_0 > 0 \), so at this point,
\[ (e^{-Cu} (n + \Delta u)) = 0 \]
\[ 0 \geq -\Delta f + n^2\inf \mathcal{F}_{i\neq l}(R_{i\tilde{l}l}) \]
and from \( \max_M |\frac{\partial u}{\partial t}| \leq \max_M |f| \) we get
\[ (n + \Delta u) \rightleftharpoons C \]
while the \( (\Delta f + n^2\inf \mathcal{F}_{i\neq l}(R_{i\tilde{l}l})), C_0(n + \max_M |f|) \) are all independent of \( t \), so there exists a constant \( C' \) such that
\[ (n + \Delta u) \rightleftharpoons C' + C(n + \Delta u) \]
then there exists a constant \( C_1 \) such that
\[ \frac{(n + \Delta u) \rightleftharpoons C_1, \text{ which means } (n + \Delta u) \leq C_1 \text{ for a constant } C_1 \text{ independent of } t. \]

Therefore, on \( M \times [0, T] \) we have \( (e^{-Cu} (n + \tilde{\Delta}u)) \)
\[ \leq C_1 e^{-C_0u(p, t_0)} \text{ for any } n + \Delta u \]
\[ \leq C_1 e^{-C_0u(n - u(p, t_0))} \text{ and } C_0 \text{ and } C_1 \text{ are both independent of } t. \]

**Lemma 3**

There exists a constant \( C_4 \) so that \( \sup_{M \times [0, T]} |v| < C_4. \)

**Proof.**

Let \( \omega = \sqrt{\frac{n}{e}} g_{ij} \; dz^i \wedge d\bar{z}^j \), \( \tilde{\omega} = \sqrt{\frac{n}{e}} \tilde{g}_{ij} \; dz^i \wedge d\bar{z}^j \), these are acutally the Kahler forms of \( g_{ij} \) and \( \tilde{g}_{ij} \) respectively. And the volume forms
\[ dV = \det(g_{ij}) \wedge_{i=1}^{n} (\sqrt{\frac{n}{e}} \; dz^i \wedge d\bar{z}^j) = \omega^n, \]
\[ d\tilde{V} = \det(\tilde{g}_{ij}) \wedge_{i=1}^{n} (\sqrt{\frac{n}{e}} \; dz^i \wedge d\bar{z}^j) = \tilde{\omega}^n \]
and the equation (3.1) we get
\[ \log \det(g_{ij}) - \log \det(g_{ij}) = \frac{\partial u}{\partial t} - f \]
then we get
\[ dV = \det(g_{ij}) e^{\frac{\partial u}{\partial t} - f} \wedge_{i=1}^{n} (\sqrt{\frac{n}{e}} \; dz^i \wedge d\bar{z}^j) = e^{\frac{\partial u}{\partial t} - f} \; dV \]
true, for \( p > 1 \),
\[ -\frac{1}{n!} \int_M \frac{(-v)^{p-1}}{p-1} (\omega^n - \tilde{\omega}^n) = \int_M \frac{(-v)^{p-1}}{p-1} (dV - d\tilde{V}) \]
because of the equation (3.1) we know \( v \) is bounded then we can renormalized it so that \( v < 1 \). Then on the other hand, \(-\int_M \frac{(-v)^{p-1}}{p-1} (\omega^n - \tilde{\omega}^n) \)
= \int_M \frac{(-v)^{p-1}}{p-1} \left( \tilde{\omega} - \omega \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
timeq \int_M \frac{(-v)^{p-1}}{p-1} \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} \psi \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
= \int_M \frac{(-v)^{p-1}}{p-1} \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} v \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
= \int_M \frac{(-v)^{p-1}}{p-1} \left( \frac{\sqrt{-1}}{2} \partial (\partial v + \bar{\partial} v) \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
= \int_M (-v)^{p-2} \frac{dv}{(\frac{\sqrt{-1}}{2} (\partial v + \bar{\partial} v)) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
\text{there we integrate by part because } v \text{ vanishes on the boundary of } M. \\
\text{then the integral above}
\geq \int_M (-v)^{p-2} \left( \frac{\sqrt{-1}}{2} (\partial v \wedge \bar{\partial} v) \right) \wedge \omega^{n-1} \text{ because each term in}
\frac{\sqrt{-1}}{2} (\partial v \wedge \bar{\partial} v) \wedge \omega \wedge \omega^{n-j-1} \text{ is nonnegative.}
\text{Then let } |\nabla v|^2 = g^{j\bar{k}} \frac{\partial v}{\partial x^j} \frac{\partial v}{\partial x^{\bar{k}}} \text{ we have}
\int_M (-v)^{p-2} |\nabla v|^2 \ dV \leq n! \int_M \frac{(-v)^{p-1}}{p-1} (e^{\frac{\omega}{2} - f} - 1) \ dV.
\text{While because } (-v)^{p-2} |\nabla v|^2 = 4p^{2-2} \frac{d}{d\omega} |\nabla (-v)\frac{\omega}{2}|^2 \text{ we replace the corresponding term in the inequality above:}
\int_M 4p^{2-2} |\nabla (-v)^{\frac{p-2}{2}}|^2 \ dV \leq n! \int_M \frac{(-v)^{p-1}}{p-1} (e^{\frac{\omega}{2} - f} - 1) \ dV, \text{ then}
\int_M |\nabla (-v)^{\frac{p}{2}}|^2 \ dV \leq \frac{n!}{2} p^{2-2} \int_M \frac{(-v)^{p-1}}{p-1} (e^{\frac{\omega}{2} - f} - 1) \ dV
\text{while due to the compability condition, we get the term } (e^{\frac{\omega}{2} - f} - 1) \text{ is}
\text{positive and bounded, therefore, the term above :}
\leq C \frac{p^2}{p-1} \int_M (-v)^{p-1} \ dV
\text{and by the norm}
\| (-v)^{\frac{p}{2}} \|^2_{H^1} = \int_M |\nabla (-v)^{\frac{p}{2}}|^2 \ dV + \int_M (-v)^{p} \ dV
\leq \left( C \frac{p^2}{p-1} + 1 \right) \int_M (-v)^p \ dV
\leq C_p \int_M (-v)^p \ dV, \text{ for } p > 1, \text{ there the C has changed but we still use it for}
a constant. When } p = 1 \text{ then we just replace } \frac{(-v)^{p-1}}{p-1} \text{ by the term log}(v) \text{ and the process works well.}
\int_M \frac{(-v)^{p-1}}{p-1} \left( \tilde{\omega} - \omega \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
= \int_M \frac{(-v)^{p-1}}{p-1} \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} \psi \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
= \int_M \frac{(-v)^{p-1}}{p-1} \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} v \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
= \int_M \frac{(-v)^{p-1}}{p-1} \left( \frac{\sqrt{-1}}{2} \partial (\partial v + \bar{\partial} v) \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
= \int_M (-v)^{-1} \frac{dv}{p-1} \left( \frac{\sqrt{-1}}{2} \partial (\partial v + \bar{\partial} v) \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
\text{While } \int_M \frac{(-v)^{p-1}}{p-1} \left( \tilde{\omega} - \omega \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1}
= \int_M \frac{(-v)^{p-1}}{p-1} \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} \psi \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
= \int_M \frac{(-v)^{p-1}}{p-1} \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} v \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
= \int_M \frac{(-v)^{p-1}}{p-1} \left( \frac{\sqrt{-1}}{2} \partial (\partial v + \bar{\partial} v) \right) \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \omega^{n-j-1} \\
\text{So } \int_M (-v)^{p-1} |\nabla v|^2 \ dV \leq n! \int_M \frac{(-v)^{p-1}}{p-1} \left( e^{\frac{\omega}{2} - f} - 1 \right) \ dV.
\frac{(-v)^{p-1}}{p-1} |\nabla v|^2 \ dV = 4 |\nabla (-v)^{\frac{p}{2}}|^2 \ dV
\text{Then } \| (-v)^{\frac{p}{2}} \|^2_{H^1} = \int_M (-v)^{p-1} \ dV.
So the equality works well when \( p \geq 1 \).

Because \( \int_M v dV = 0 \), we learn from the Sobolov inequality (G&T[5] p155)

\[
\| (v) \| L^\frac{p}{p-1} \leq C \| D((v) \| H^1 \\
= C \| (v) \| H^1 \quad \text{while as our assume } v \text{-1 and we have an estimate}
\]

for \( \| \nabla v \| \) so we get \( \| (v) \| \leq \| (v) \| \) and then

\[
\| (v) \| L^2 \leq \| (v) \| L^2 \leq C \| (v) \| L^2 \\
\]

While \( \int_M (v) dV \) gives us the \( L^p \) norm, so we get

\[
\| v \| L^\frac{p}{p-1} \leq C_0 \| v \| L^p \quad \text{for any } p \geq 1
\]

Then we consider \( \gamma = \frac{\gamma}{\gamma-1} \) and \( p = \gamma \) for \( j = 0, 1, 2, \ldots, \ldots \), then

\[
\| v \| L^{\gamma+j} \leq C \| v \| L^{\gamma+j} \quad \text{we continue decrease } j \text{ so:
\]

\[
\| v \| L^{\gamma+j} \leq C \| v \| L^{\gamma+j} \\
\| v \| L^{\gamma+j} \leq C \| v \| L^{\gamma+j} \\
\]

this process continue until \( 0 \), i.e.

\[
\| v \| L^{\gamma+j} < C \sum_{k=0}^{\gamma+j} \sum_{k=0}^{\gamma+j} C_3 \text{ for } L_0 \text{ norm is a constant.}
\]

Finally, we let \( j \) increases to infinity then \( \| v \| L^\infty \leq C_4 \) for \( C_4 \) a constant from the right hand side of the inequality induced above so it’s independent of

\[
\sup_{M \times [0, T]} | v | \leq C_4, \text{ this finishes the proof.}
\]

After the long and complex calculation, we can continue our estimate.

Because \( \frac{1}{v} \int_M u dV \) is not relative to \( \partial \bar{\partial} \), so we get:

\[
0 < n + \Delta v = n + \Delta u \leq C_1 \| v \| \inf e^{n \int_M u \cdot v} = C_1 \| v \| \inf e^{n \int_M u \cdot v} \leq C_5
\]

so the \( \Delta v \) is bounded, the equality holds because the difference is not relevant to the constant \( \| v \| L^\frac{p}{p-1} \) \( \int_M u dV \), then the constant \( C_5 \) appears because \( \sup | v | \) is bounded.

The by the Schauder estimate we presented before

\[
\sup_{M \times [0, T]} | \nabla v | \leq C_6 (\sup_{M \times [0, T]} | \Delta u | \quad + \quad \sup_{M \times [0, T]} | v |)
\]

\[
\leq C_6 (\text{constant} + \text{constant}) \quad \text{by Lemma 2 and Lemma 3 then}
\]

\[
\sup_{M \times [0, T]} | \nabla v | \leq C_7.
\]

This finishes the first order estimate.

**Second order estimate**

From the estimate of \( n + \Delta u \), in Yau’s work the choice of the metric (in page 348) on one hand make the matrix \( (\delta + u_{ij}) \) is positive definite and Hermitian so we know that \( 1 + u_{ij} \) is bounded above for any \( i \), by the metric Yau chose,

\[
\prod_{i=1}^{m} (1 + u_{ij}) = e^{\frac{n}{2} \sum j} \quad \text{gives a lower estimate which is positive, this is because}
\]

of the product of these terms has a upper bound, so each term can not go to the negative infinity, so they all have a lower bound, while the matrix \( (\delta + u_{ij}) \) is positive definite and Hermitian so each term is positive, therefore, there exists two positive constant \( A, B \) such that

\[
A \leq 1 + u_{ij} \leq B \quad \text{for any } i
\]

**Third order estimate**
First let $S = \sum \tilde{g}^{ik} \tilde{g}^{js} \tilde{g}^{kt} v_{ijk} v_{\tilde{r}st}$. Actually in this disturbing definition, Cao said in his paper that he followed E.Calabi and Yau, Yau said he followed E.Calabi, but I have not read E.Calabi’s paper, I have only read Yau and Cao’s paper, so I do not know who I am following, that’s really interesting. To show respect to E.Calabi, I decide to write I’m following E.Calabi here, too. I’ll read his work in a few minutes.

We note $A \simeq B$ if $|A - B| \leq C_1 \sqrt{S} + C_2$ for $C_1$ and $C_2$ are constants can be estimated, further more, we note $A \simeq B$ if $|A - B| \leq C_3 S + C_4 \sqrt{S} + C_5$ for $C_3$ and $C_4, C_5$ are constants can be estimated. And again by the metric Yau defined before, through totally 60 rows calculations in Yau’s appendix,

\[ \Delta S \simeq \sum (1 + v_{ij})^{-1} (1 + v_{jk})^{-1} (1 + v_{ik})^{-1} \times \left\{ | v_{ijk\alpha} - \sum v_{ik\alpha} | \right\}. \]

And $\Delta(\Delta v) \geq \sum (1 + v_{jk})^{-1} (1 + v_{ij})^{-1} | v_{kij} |^2 - C_6$, where $C_6$ is a constant which can be estimated, then without loss generality, we can choose a big $C_7$ such that

\[ \Delta(S + C_7 \Delta v) \geq C_8 S - C_9, \]

for $C_7, C_8, C_9$, are positive constants can be estimated.

Then we assume $p(t)$ is the maximum point of the funcion $S + C_7 \Delta v$, we get

\[ 0 \leq C_8 S - C_9; \]

\[ C_8(S + C_7 \Delta v) \leq C_9 + C_8 S. \]

Due to the $\Delta v$ is bounded from estimate before, this gives an estimate to $\sup_{M \times [0,T]}(S + C_7 \Delta v)$ therefore, we have an estimate of $\sup_{M} S$, as Yau said, this gives an estimate to $v_{ijk}$ and similar term in terms of $g_{ij}$, $\sup |F|, \sup |\nabla F|, \sup_M \sup_i |F_i|$, and $\sup_M \sup_{i,j,k} |F_{ijk}|$.

**Long time existence**

Finally we prove the proposition.

**Proposition of the long existence**

Assume $u$ be the solution of the equation

\[ \frac{\partial u}{\partial t} = \log \det(g_{ij} + \frac{\partial^2 u}{\partial x_i \partial x_j}) - \log \det(g_{ij}) + f \]

where $t \in [0, T]$ which is the maximum time interval. Let $v$ be the normalization of $u$:

\[ v = u - \frac{1}{V_{\partial M}} \int_M u dV. \]

Then the $C_\infty$ norm of $v$ are uniformly bounded for any $t \in (0,T)$, which means $T = \infty$.

Then there exists a sequence $t_n$ increasing to infinity such that $v(x, t_n)$ converges in the topology generated by $C_\infty$ norm to a smooth function $v_\infty(x)$ on $M$ when $n$ increases to infinity.

**Proof.**

Differentiate the equation (3.1):

\[
\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x^i} \right) = g^{ij} \left( \frac{\partial}{\partial x^j} g_{ij} + \partial \delta \left( \frac{\partial u}{\partial x^j} \right) \right) - g^{ij} \frac{\partial}{\partial x^j} g_{ij} + \frac{\partial u}{\partial x^i};
\]

so $\left( \frac{\partial}{\partial t} \right) = g^{ij} \frac{\partial}{\partial x^j} g_{ij} - g^{ij} \frac{\partial}{\partial x^j} g_{ij} + \frac{\partial u}{\partial x^i};$

we have the estimate of $\frac{\partial}{\partial t}$ and $\Delta u$ so the coefficients of the $\phi$ are bounded, and by the $C^{0,\alpha}$ norm is defined as: $\| u \|_{C^{0,\alpha}} = \| u \|_{L^\infty} + \| u \|_\alpha$, according to our 0 order estimate, they are bounded in the $C^{0,\alpha}$ norm; and these term are actually the Holder coefficient so is has bounded $C^{0,\alpha}$ norm. While the right hand side have similar estimate for $C^{0,\alpha}$ norm for $0 < \alpha < 1$ because the
RHS only depends on the Kahler metric and \( f \) which are smooth. By interior regularity theory (see G& T section 6.4) which gives the \( C^{k+2, \alpha} \) estimate when then coefficients and the nonhomogeneous term are of \( C^{k, \alpha} \) estimate, this \( \phi \) is elliptic and the RHS terms are smooth so they have \( C^{0, \alpha} \) estimate, then as the solution of this equation, \( \frac{\partial u}{\partial z} \) has a uniform \( C^{2, \alpha} \) estimate, similarly, \( \frac{\partial u}{\partial \bar{z}} \) also has this estimate, there the \( C^{0, \alpha} \) norm estimate gives the \( C^{2, \alpha} \) norm estimate, then because \( \tilde{g}_{ij} = g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \), then the \( C^{2, \alpha} \) estimate of \( \frac{\partial u}{\partial z} \) actually gives the \( C^{1, \alpha} \) estimate of the RHS and the coefficients of \( \phi \) which are determined by the metrics, therefore, the right hand side and the coefficients of \( \phi \) have uniform \( C^{1, \alpha} \) estimate. Use the Interior regularity theory again, then so \( \frac{\partial u}{\partial t} \) and \( \frac{\partial u}{\partial z} \) have another two orders more estimate...... Then we repeat use the theory, by iteration, then \( v(x, t) \) has uniformly bounded \( C^\infty \) norm for any \( t \in (0, T) \), finally we choose a sequence of \( t \) goes into the infinity such that it has a subsequence \( t_n \) making \( v(x, t_n) \) converge to a smooth function \( v_\infty \), so that solution exists. And then because \( \frac{\partial u}{\partial t} \) is uniformly bounded referring to \( t \), and as \( t \) goes into the infinity \( u \) can not blow up in finite time, this means that our estimates are independent of \( t \) then when solution metric get the upper bound of \( t \), says \( T \), then our estimates still works the estimates above are independent of \( t \) so if we choose \( t_0 \in [0, T] \) then the solution also exists in \([t_0, t_0+\epsilon]\) for \( \epsilon \) independent of \( t \), so we can use this \( \tilde{g}_{ij}(T) \) as the initial condition of the same equation choosing a new initial point and continue to deformation, so the process can continue to infinity because our estimates always work, so \( u \) exists for all time.

Up to here, the long time existence has finally been proved.

4 Argument for uniform convergence

At this time, since we have proved the existence of the long time solution, then to follow our idea, we need to show the uniform convergence of the solution and then let the time goes into the infinity to get the final conclusion.

\[
\frac{\partial u}{\partial t} = \log \det(g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}) - \log \det(g_{ij}) + f
\]

and initial status: \( u(x, t) = 0 \) when \( t = 0 \) on \( M \times [0, \infty) \), similarly, we renormalized \( u \):

\[
v = u - \frac{1}{\text{Vol}(M)} \int_M u dV
\]

And then we’ll show the uniform convergence of \( v(x, t) \) and \( \frac{\partial u}{\partial t} \) when \( t \) goes into the infinity.

We learn from before that \( \frac{\partial u}{\partial t} \) is given by \( \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \) so \( (\Delta - \frac{\partial}{\partial t}) \frac{\partial u}{\partial t} = 0 \)

where \( \frac{\partial}{\partial t} (x, t) = f(x) \) when \( t = 0 \). So in order to analyze \( u \), we should first take care of this equation

\[
(\Delta - \frac{\partial}{\partial t}) \frac{\partial u}{\partial t} = 0
\]
And following Yau’s work, Cao gave a modification of an important theory of this equation.

**Theorem 4.1**

We assume $M$ be a compact manifold whose dimension is $n$, $g_{ij}(t)$ be a family of Riemannian metrics on $M$, such that the following holds:

1. $\exists$ constants $C_1, C_2$ positive and independent of $t$ such that $C_1 g_{ij}(0) \leq g_{ij}(t) \leq C_2 g_{ij}(0)$
2. $\exists$ constants $C_3$ positive and independent of $t$ such that $|\frac{\partial g_{ij}}{\partial t}|(t) \leq C_3 g_{ij}(0)$
3. $\exists$ constants $K$ positive and independent of $t$ such that $R_{ij}(t) \geq -K g_{ij}(0)$

Then we assume $\phi$ is positive and satisfies the equation:

$$\left(\Delta_t - \frac{\partial}{\partial t}\right) \frac{\partial \phi}{\partial t} = 0$$

on $M \times [0, \infty)$ where $\Delta_t$ is the Laplace operator, then $\forall \alpha > 1$,

$$\sup_{t \in [0, \infty)} \phi(x, t_1) \leq \inf_{t \in [0, \infty)} \phi(x, t_2) \left(\frac{1}{C_1} - \frac{d}{\partial t}ight) \left(C_2^2 + \frac{C_3}{C_1} + C_3^2 n + 1\right)^2 (t_2 - t_1)$$

for $d$ is the diameter of $M$ measured by $g_{ij}(0)$, i.e. $d = \sup_{x, y \in M} g_{ij}(0)(x, y)$; and

$$A = \sup ||\nabla^2 \log \phi||; \text{ and } 0 < t_1 < t_2 < \infty.$$

Cao did not show the proof because the proof of the theorem is totally a tough work in Yau’s paper. Now, we can use this conclusion, let $F = \frac{\partial \phi}{\partial t}$, then by the maximum principle we still consider $t=0$ as the boundry of $M \times [0, \infty)$ for this parabolic equation and $t_2 > t_1 > 0$, we get:

$$\sup_{t \in [0, \infty)} F(x, t_2) < \sup_{t \in [0, \infty)} F(x, t_1) < \sup_{t \in [0, \infty)} F(x)$$

$$\inf_{t \in [0, \infty)} F(x, t_2) > \inf_{t \in [0, \infty)} F(x, t_1) > \inf_{t \in [0, \infty)} F(x)$$

here $t_2 > t_1 > 0$ because we can always choose at $t_2$ the $\partial u$ converges more than $t_1$.

**Remark** The conditions above also hold for $\tilde{R}_{ij}$, i.e.

1. $\exists$ constants $C_1, C_2$ positive and independent of $t$ such that $C_1 \hat{g}_{ij}(0) \leq \hat{g}_{ij}(t) \leq C_2 \hat{g}_{ij}(0)$
2. $\exists$ constants $C_3$ positive and independent of $t$ such that $|\frac{\partial \hat{g}_{ij}}{\partial t}|(t) \leq C_3 \hat{g}_{ij}(0)$
3. $\exists$ constants $K$ positive and independent of $t$ such that $\hat{R}_{ij}(t) \geq -K \hat{g}_{ij}(0)$

Actually, $\hat{g}_{ij}(0) = (g_{ij}(0) + \frac{\partial^2 u}{\partial x^2 \partial y^2}(0))$, from the estimate before we know $\frac{\partial^2 u}{\partial x^2 \partial y^2}(0)$ is bounded, then we can always choose a $C_1 \leq \frac{(g_{ij}(0) + \frac{\partial^2 u}{\partial x^2 \partial y^2}(t))}{(g_{ij}(0) + \frac{\partial^2 u}{\partial x^2 \partial y^2}(0))}$ because $g_{ij}(0) \leq C_g \hat{g}_{ij}(c)$ for some $C$, similary, choose $C_2 \geq \frac{(g_{ij}(0) + \frac{\partial^2 u}{\partial x^2 \partial y^2}(t))}{(g_{ij}(0) + \frac{\partial^2 u}{\partial x^2 \partial y^2}(0))}$, so the (1) holds; then

$$|\frac{\partial \hat{g}_{ij}}{\partial t}|(t) \leq |\frac{\partial g_{ij}}{\partial t}|(t) + |\frac{\partial^2 u}{\partial t \partial x^2 \partial y^2}(0)|,$$ while the estimate before tells that $|\frac{\partial^2 u}{\partial t \partial x^2 \partial y^2}(0)|$ is bounded so we can choose a $C_3$ such that $\frac{\partial^2 u}{\partial t \partial x^2 \partial y^2}(0) \geq C_3 |\frac{\partial^2 u}{\partial t \partial x^2 \partial y^2}(t)|$, so (2) holds; (3) Vy the long time existence theorem.
\( \tilde{g}_{ij}(0) = g_{ij} \) so when it holds for \( g_{ij} \) then in the convergence process, \( \tilde{g}_{ij}(t) \) converges to \( \tilde{g}_{ij}(\infty) \), so whenever \( t \in [0, \infty] \), there exists such \( K \).

Then we define
\[
\varphi_n(x,t) = \sup_{x \in M} F(x,n-1) - F(x,n-1+t)
\]
\[
\phi_n(x,t) = F(x,n-1+t) - \inf_{x \in M} F(x,n-1)
\]
\[
\omega(t) = \sup_{x \in M} F(x,t) - \inf_{x \in M} F(x,t)
\]

While because the sup and the inf are constants, and the \( d(n-1)+t = dt \) so the \( \varphi_n(x,t) \) and \( \phi_n(x,t) \) both satisfy the equation and the initial condition, and by the inequality above, they are both positive. Then we take \( t_1 = \frac{1}{2} \) and \( t_2 = 1 \), then we use the Theorem 4.1 in \( \varphi_n(x,t) \) and \( \phi_n(x,t) \) separately:

\[
\sup_{x \in M} \phi_n(x,\frac{1}{2}) \leq \inf_{x \in M} \phi_n(x,1) \gamma
\]

where \( \gamma = 2^{\frac{1}{2}} e^{\frac{1}{2} C_2^2 d^2 + \frac{1}{2} C_3^2 d^2 + C_3 (n+1)} \) is independent of \( t \), then

\[
\sup_{x \in M} F(x,n-1) - \inf_{x \in M} F(x,n-1) \leq \gamma (\sup_{x \in M} F(x,n-1) - \inf_{x \in M} F(x,n))
\]

Similarly,

\[
\sup_{x \in M} F(x,n-\frac{1}{2}) - \inf_{x \in M} F(x,n-1) \leq \gamma (\inf_{x \in M} F(x,n-1) - \inf_{x \in M} F(x,n))
\]

Do not forget we have \( \omega(t) = \sup_{x \in M} F(x,t) - \inf_{x \in M} F(x,t) \) so we add the two inequality together:

\[
\omega(n-1) + \omega(n-\frac{1}{2}) \leq \gamma (\omega(n-1) - \omega(n))
\]

because \( \omega(n) \geq 0 \), so \( \omega(n-1) \leq \gamma (\omega(n-1) - \omega(n)) \)

then \( (\gamma - 1)\omega(n-1) \geq \delta \omega(n-1) \), so \( \omega(n) \leq \delta \omega(n-1) \), for \( \delta = \frac{2}{\gamma} - 1 < 1 \). We repeat this process and we get \( \omega(n) \leq \delta \omega(n-1) \), \( \omega(n-1) \leq \delta^2 \omega(n-2) \), \( \omega(n-2) \leq \delta^3 \omega(n-3) \), ...... finally, \( \omega(n) \leq \delta^n \omega(0) \) for \( \omega(0) = \sup_{x \in M} f - \inf_{x \in M} f \).

While because \( \sup_{x \in M} F(x,t) \) gets smaller when \( t \) gets larger, and \( \inf_{x \in M} F(x,t) \) gets larger when \( t \) gets larger, so \( \omega(t) \) decreases when \( t \) gets larger. Therefore, let \( a = -\log(\delta) \), we can choose a constant \( C_4 \) independent of \( t \) such that \( \omega(t) \leq C_4 e^{-at} \).

Let \( \varphi(x,t) = \frac{\partial u}{\partial t} - V_{ol(M)} \int_M \frac{\partial u}{\partial t} \, dV \). Then in order to show when \( t \) goes into the infinity, we should analyze the behaviour of \( \varphi(x,t) \), if we can prove \( \varphi(x,t) \) goes into 0 then that means \( \frac{\partial u}{\partial t} \) truly converges to some function. We use

\[
E = \frac{1}{2} \int_M \varphi^2 \, dV
\]

and we want to estimate \( E \) in terms of \( t \) to figure out how \( \varphi \) changes. While

\[
\frac{\partial \tilde{g}}{\partial t}(dV) = \frac{1}{2} \int_M \varphi^2 \, dV
\]

we calculate

\[
\frac{\partial \varphi}{\partial t}(x,t) = \frac{\partial u}{\partial t}(x,t) - \frac{1}{V_{ol(M)}} \int_M \frac{\partial u}{\partial t} \, dV - \frac{1}{V_{ol(M)}} \int_M \frac{\partial u}{\partial t} \frac{\partial \tilde{g}}{\partial t} \, dV,
\]

while

\[
\frac{1}{V_{ol(M)}} \int_M \frac{\partial u}{\partial t} \, dV
\]
\[ \begin{align*}
\int_{\partial \Omega} \Delta (\varphi_t) \ dV &= \int_{\Omega} \Delta (\varphi_t) \ dV \\
&= \int_{\Omega} \Delta (\varphi_t) \ dV - \frac{1}{M} \int_{\Omega} M \partial_t \varphi \ dV \\
&= \Delta \left( \frac{\partial u}{\partial t} \right) - \frac{1}{M} \int_{\Omega} M \partial_t \varphi \ dV.
\end{align*} \]

Then we can calculate
\[ \frac{dE}{dt} = \int_{\Omega} \varphi \frac{\partial u}{\partial t} \ dV + \frac{1}{2} \int_{\Omega} \varphi^2 \Delta \left( \frac{\partial u}{\partial t} \right) \ dV \]
and integrating by parts and M is compact so no boundary exists, due to \( \Delta = \nabla |^2 \).

so \( \varphi_t \) = \( \varphi \partial u / \partial t \) - \( \varphi \Delta (\varphi_t) \) \( \partial \Omega \) \( \varphi \), there we use the argument before.

so the term \( \frac{dE}{dt} = \int_{\Omega} \varphi \frac{\partial u}{\partial t} \ dV + \frac{1}{2} \int_{\Omega} \varphi^2 \Delta \left( \frac{\partial u}{\partial t} \right) \ dV \), there we use integrating by parts and M is compact so no boundary exists, due to \( \Delta = \nabla |^2 \), so the term

\[ \begin{align*}
\int_{\Omega} \nabla (\varphi_t) \nabla (\varphi_t) \ dV &+ \frac{1}{2} \int_{\Omega} M \varphi \nabla \varphi \nabla \varphi \ dV \\
&= \int_{\Omega} \nabla (\varphi_t) \nabla (\varphi_t) \ dV + \frac{1}{2} \int_{\Omega} \varphi^2 \nabla \varphi \nabla \varphi \ dV
\end{align*} \]

for \( | \nabla \varphi |^2 = \tilde{g}^{ij}(\varphi) (\varphi)_i (\varphi)_j \) is the square of the gradient.

Because \( \sup_{x \in \Omega} \varphi(x,t) \) is the difference between \( \varphi \) and its average, while \( \omega \) is the largest difference, ans we can choose t large enough such that \( \omega \) less than any value, so \( \exists t \) such that

\[ \sup_{x \in \Omega} \varphi(x,t) < \omega(t) < \frac{1}{2}, \]
then

\[ \frac{dE}{dt} = \int_{\Omega} \nabla (\varphi_t) \nabla \varphi \dV + \frac{1}{2} \int_{\Omega} \varphi \nabla \varphi \nabla \varphi \ dV \]

And by the definition of \( \varphi \), \( \int_{\Omega} \varphi \ dV = 0 \), by the Poincare inequality.
\[ \| \varphi \|_p \leq h \| D \varphi \|_p \text{ then } \int_M \varphi^2 d\tilde{V} \leq h^2 \int_M \| D \varphi \|^2 d\tilde{V} = h^2 \int_M \| \tilde{\nabla} \varphi \|^2 d\tilde{V} \]
for \( h \) can be considered as the diameter of the domain, we get
\[ \int_M \| \tilde{\nabla} \varphi \|^2 d\tilde{V} \geq \lambda_1(t) \int_M \varphi^2 dV, \]
where \( \lambda_1(t) \) is the first eigenvalue of \( \tilde{\Delta} \) at time \( t \). Then there exists a constant \( C_5 \) such that for any \( t, \)
\[ \lambda_1(t) > C_5, \text{ so } \frac{4C_5 t}{\lambda_1(t)} = t \int_M \| \tilde{\nabla} \varphi \|^2 d\tilde{V} \leq C_6 E. \]
That’s really similar to an ordinary differential equation, and the exponential function is monotonic, so we can solve the equation and get the inequality:
\[ E \leq C_6 e^{-C_5 t} \]
while \( d\tilde{V} \) is uniformly equivalent to \( dV \), so there exists constant \( C'_6 \) such that
\[ \int_M \varphi^2 dV \leq C'_6 e^{-C_5 t}, \]
then we can finally prove the uniform convergence theorem.

**Theorem of uniform convergence**

Using the notation in the Proposition of long time existence, as \( t \) goes into the infinity, \( v(x,t) \) converges to the function \( v_{\infty} \) in \( C_{\infty} \) topology, therefore, as \( t \) goes into the infinity, \( \frac{\partial u}{\partial t} \) converges to a constant in \( C_{\infty} \) topology.

**Proof.**

First we prove \( v(x,s) \) is a Cauchy sequence in \( L_1 \) norm, as \( t \) goes into the infinity. For any \( 0 < s < s' \) :
\[ \int_M | v(x,s) - v(x,s') | dV \]
\[ \leq \int_M \int_s^{s'} \frac{\partial u}{\partial t}(x,t) | dt dV \]
\[ \leq \int_M \int_s^{s'} | \frac{\partial u}{\partial t}(x,t) | dt dV \]
\[ = \int_s^{s'} \int_M | \frac{\partial u}{\partial t}(x,t) | dV dt \]
\[ = \int_s^{s'} \int_M | \frac{\partial u}{\partial t} - \frac{1}{\text{Vol}(M)} \int_M \frac{\partial u}{\partial t} dV | dV dt \]
\[ = \int_s^{s'} \int_M | \frac{\partial u}{\partial t} - \frac{1}{\text{Vol}(M)} \int_M \frac{\partial u}{\partial t} dV | dV dt \]
\[ + \frac{1}{\text{Vol}(M)} \int_M \frac{\partial u}{\partial t} dV | \frac{1}{\text{Vol}(M)} \int_M \frac{\partial u}{\partial t} dV | dV dt \]
\[ \leq \int_s^{s'} \int_M | \varphi | dV dt + \int_s^{s'} \int_M \frac{1}{\text{Vol}(M)} \| \tilde{\nabla} \varphi \|_{L^2} \| \tilde{\nabla} \varphi \|_{L^2} dV dt \]
\[ \leq \int_s^{s'} \int_M 1 | \varphi | dV dt \]
\[ + \int_s^{s'} \int_M \frac{1}{\text{Vol}(M)} | \sup_{x \in M} \frac{\partial u}{\partial t} | \int_M \tilde{\nabla} \varphi - \inf_{x \in M} \frac{\partial u}{\partial t} | \int_M dV | dV dt \]
\[ \leq \text{Vol}(M) \int_s^{\infty} (\int_M \varphi^2 dV)^{\frac{1}{2}} dt + \text{Vol}(M) \int_s^{\infty} \omega(t) dt \]
by Cauchy-Schwarz inequality, then by the estimate before we get above term
\[ \leq \text{Vol}(M) \int_s^{\infty} (C'_6 e^{-C_5 t})^{\frac{1}{2}} dt + \text{Vol}(M) \int_s^{\infty} (C'_6 e^{-C_5 t}) dt \]
\[ = C_7 \int_s^{\infty} e^{-C_6 t} dt + C_8 \int_s^{\infty} e^{-C_5 t} dt \]
while the above integral converges in \( s \), so if \( s \) goes into the infinity, the two integral can be very small, so this means \( v(x,s) \) is a Cauchy sequence in \( L_1 \) norm.
So by \( L_1 \) is complete, there exists a function \( v'_{\infty}(x) \) such that \( v(x,t) \) converges uniformly to \( v'_{\infty}(x) \), while in the long time existence theorem we know there exists a sequence \( t_k \) such that \( v(x,t_k) \) converges to the \( v_{\infty}(x) \) there, so here \( v_{\infty}(x) = v'_{\infty}(x) \), so \( v(x,t) \) converges to \( v_{\infty} \) in \( L_1 \) norm. However, what we need is the convergence in \( C_{\infty} \) topology which can be expressed in the equation we study.

We prove it by contradiction. If \( \exists r > 0, \epsilon > 0 \) such that
∀ N, ∃ n>N such that ∥v(x, t_n) − v_∞(x)∥_{C^r} > ε, while N can be chosen arbitrary, so we can find a sequence t_n such that ∥v(x, t_n) − v_∞(x)∥_{C^r} > ε, but the sequence v(x, t_n) is bounded so there exists a subsequence v(x, t_{n_k}) converges to v_∞ ≠ v_∞(x) in C^∞ topology, but we know v(x, t_{n_k}) converges in L_1 norm to v_∞(x), so it’s a contradiction. So finally v(x, t) converges to v_∞(x) in C^∞ topology.

Then we consider the equation again:
\( \frac{∂u}{∂t} = \det(g_{ij} + \frac{∂^2 u}{∂z^i ∂z^j}) - \log det(g_{ij}) + f \) ———-(3.1)
because \( ∂v/∂t = ∂^2 u/∂z^i ∂z^j \), so when t goes into the infinity,
\( \det(g_{ij} + \frac{∂^2 u}{∂z^i ∂z^j}) - \log det(g_{ij}) + f \) converges to
\( \log det(\tilde{g}_{ij} + \frac{∂^2 u}{∂z^i ∂z^j}) - \log det(\tilde{g}_{ij}) + f \), which means \( ∂u/∂t \) converges to \( ∂v/∂t \) in C^∞ topology when t goes into the infinity, while
\( ω(t) = \sup_{x \in M} \frac{∂u}{∂t} - \inf_{x \in M} \frac{∂u}{∂t} ≤ C_4 e^{-at} \), when t goes into the infinity, the right hand side goes into 0, therefore, we can only gets \( ∂u/∂t \) converges to a constant.

5 The final Theorem

According to our idea: First prove the solution exists all the time, then prove when t goes infinitly, \( g_{ij} \) converges to a definite \( \tilde{g}_{ij}(∞) \) and hence \( \frac{∂g_{ij}}{∂t} \) converges to 0, then we get

\( -R_{ij} + T_{ij} = 0 \), then we get \( T_{ij} \) will be the Ricci tensor of \( \tilde{g}_{ij}(∞) \) so the \( \tilde{g}_{ij}(∞) \) is the metric we want. Then we can finish the whole process.

Main Theorem

M be a compact Kahler manifold with the Kahler metric \( g_{ij} dz^i ∧ d\bar{z}^j \), \( C_1(M) \) is the first Chern class of M, consider a presentation of it \( \frac{1}{2π} T_{ij} dz^i ∧ d\bar{z}^j \), while from the initial metric \( g_{ij} \), we consider an equation with changing \( g_{ij} \):

\( \frac{∂g_{ij}}{∂t} = -R_{ij} + T_{ij} \) where \( \tilde{g}_{ij} = g_{ij} \) at \( t = 0 \)

then the equation exists a long time solution and \( \tilde{g}_{ij} \) converges uniformly to another Kahler metric \( \tilde{g}_{ij} \) which is in the same Kahler class of \( g_{ij} \) such that

\( 0 = -R_{ij} + T_{ij} \)

then \( R_{ij} = T_{ij} \), which means \( T_{ij} \) is the Ricci tensor of \( \tilde{g}_{ij} \).

Proof.

While the de Rahm cohomolgy class of the Kahler form \( \frac{1}{2π} R_{ij} dz^i ∧ d\bar{z}^j \) is the first Chern class \( C_1(M) \) of M, where the \( R_{ij} \) is the Ricci curvature of the Kahler metric. Then because \( \frac{1}{2π} T_{ij} dz^i ∧ d\bar{z}^j \) also represents \( C_1(M) \), so

\( T_{ij} - R_{ij} = \frac{∂^2 f}{∂z^i ∂z^j} \), where f is a real-value smooth function on M. By the long time existence Theorem we know the equation

\( \frac{∂u}{∂t} = \det(g_{ij} + \frac{∂^2 u}{∂z^i ∂z^j}) - \log det(g_{ij}) + f \) ———-(3.1)

\( u(x, t) = 0 \) when \( t = 0 \) exists a smooth solution \( u(x, t) \) in all time such that

\( \tilde{g}_{ij}(t) = g_{ij} = \frac{∂^2 u}{∂z^i ∂z^j} \)
Then from the uniformly convergence theorem we get that as \( t \) goes into the infinity, \( u(x,t) \) converges uniformly so \( \tilde{g}^{ij} \) converges in \( C^\infty \) topology to \( \tilde{g}^{ij}(\infty) \), and by

\[
\tilde{g}^{ij}(t) = g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}, \quad \text{so} \quad \frac{\partial \tilde{g}^{ij}}{\partial t} \text{ converges uniformly to 0 for} \quad \frac{\partial u}{\partial t} \text{ converges uniformly.}
\]

Then we differentiate the equation:

\[
\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \left( \frac{\partial u}{\partial t} \right) = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \left( \log \det(g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}) - \log \det(g_{ij}) \right) + \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}
\]

while we can calculate the Ricci tensor explicitly in terms of

\[
g_{ij} \text{ and } \tilde{g}_{ij} : R_{ij} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij}), \quad \text{and} \quad \frac{\partial \tilde{g}^{ij}}{\partial t} = \frac{\partial \tilde{T}^{ij}}{\partial \bar{u}}
\]

we derive the scalar equation:

\[
\tilde{\rho} \text{ represented as:} \quad \rho(X,Y) = \frac{1}{2} \text{ Re}(\text{JX,Y}), \quad \text{and the first Chern class is negative} \quad \text{if the tensor R}c \text{ is negative definite. Here we choose this evolution equation such that when} \quad t \text{ goes into the infinity, the Ricci tensor is negative definite for the Kahler metric is positive definite by the initial condition, that fits our assumption, then we derive the scalar equation:}
\]

\[
g_{ij}(t) = g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}
\]

and \( \frac{\partial g_{ij}(t)}{\partial t} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \left( \frac{\partial u}{\partial t} \right) \)

\[
\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \left( \frac{\partial u}{\partial t} \right) = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}) - \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij}) - \frac{\partial^2}{\partial z^i \partial \bar{z}^j} u
\]

while we can calculate the Ricci tensor explicitly in terms of

\[
r_{ij} \text{ and } \tilde{r}_{ij} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij}), \quad \text{and so the equation changes to:}
\]

\[
\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \left( \frac{\partial u}{\partial t} \right) = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}) - \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij}) - \frac{\partial^2}{\partial z^i \partial \bar{z}^j} u
\]

and similary:

6 The problem of existence of the Kahler-Einstein metric

Here we consider M a compact Kahler manifold with negative first Chern class \( C_1(M) \), if we want to find a Kahler-Einstein metric on M i.e. \( R = \text{kg} \). We consider the evolution function:

\[
\frac{\partial g_{ij}}{\partial t} = -R_{ij} - \tilde{g}^{ij}, \quad g_{ij} = g_{ij} \quad \text{at} \quad t = 0, \quad \text{and} \quad g_{ij} \text{ is positive definite and represents the negative of the first Chern class, here we know that the first Chern class is the de Rahm cohomology class of the Ricci form which is a real (1,1) form represented as:} \quad \rho(X,Y) = \frac{1}{2} \text{ Re}(\text{JX,Y}), \quad \text{and the first Chern class is negative if the tensor R}c \text{ is negative definite. Here we choose this evolution equation such that when} \quad t \text{ goes into the infinity, the Ricci tensor is negative definite for the Kahler metric is positive definite by the initial condition, that fits our assumption, then we derive the scalar equation:}
\]

\[
g_{ij}(t) = g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}
\]

and \( \frac{\partial g_{ij}(t)}{\partial t} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \left( \frac{\partial u}{\partial t} \right) \)

\[
\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \left( \frac{\partial u}{\partial t} \right) = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij} + \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}) - \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{ij}) - \frac{\partial^2}{\partial z^i \partial \bar{z}^j} u
\]
\[ \frac{\partial u}{\partial t} = \log \det (g_{ij} + \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}) - \log \det (g_{ij}) - u + f \]

Differentiating the equation we get
\[ \frac{\partial}{\partial t} (\frac{\partial u}{\partial t}) = \tilde{\Delta} (\frac{\partial u}{\partial t}) - \frac{\partial u}{\partial t} \]

which means that
\[ \frac{\partial}{\partial t} (e^t \frac{\partial u}{\partial t}) = \tilde{\Delta} (e^t \frac{\partial u}{\partial t}) \]

that’s actually a heat equation and we use the maximum principle again, which means
\[ |\frac{\partial u}{\partial t}| \leq |\frac{\partial u}{\partial t}|_{t=0} \]

for when \( t=0 \) it’s actually a boundary of \( M \times [0, \infty) \), then
\[ |\frac{\partial u}{\partial t}|_{t=0} = |\log \det (g_{ij} + \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}) - \log \det (g_{ij}) - u(0) + f| = |f| \]

is bounded in \( C^0(M) \), then there exists a constant \( C > 0 \) such that \( e^t \frac{\partial u}{\partial t} \) is bounded for \( t \in [0, \infty) \), i.e. \( \| u(t) \|_{C^0(M)} \) is uniformly bounded for \( t \in [0, \infty) \). Then from above argument, we obtain the estimates below:

**Lemma 6.1**

1. \( \exists \) uniform constant \( C \) such that \( \forall t \in [0, \infty) \),
\[ \| \frac{\partial u}{\partial t} \| \leq e^{-t} C \]

2. \( \exists \) a continuous real-valued function \( u_\infty \) on \( M \) such that \( \forall t \in [0, \infty) \),
\[ \| u(t) - u_\infty \|_{C^0(M)} \leq e^{-t} C \]

3. \( u(t) \) is uniformly bounded for \( t \in [0, \infty) \)

Here \( \log \det (g_{ij} + \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}) = -\frac{\partial u}{\partial t} - \log \det (g_{ij}) - u + f \) and from estimate above we claim the RHS is uniformly bounded then the \( \tilde{g}^{ij} \) is uniformly bounded with a upper bound and a lower bound such that: there exists a uniformly constant \( C \) such that on \( M \times [0, \infty) \) and
\[ \frac{1}{C} g_{ij} \leq \tilde{g}_{ij} \leq C g_{ij} \]

then we can actually prove the final theorem of the existence of the Kahler-Ricci flow, but the proof of this estimate refers to lots of canonical estimate of the general Kahler Ricci flow, hence in the following sections, we will give more estimates to complete the method.

**7 More details from Kahler-Ricci Flow**

In this section we will figure out some details and problems appeared in the former sections, mainly use some other estimates to figure the problem. First we prove the maximal existence time for the Kahler Ricci flow equation, then similar as before, we will give some important estimates for the normalized Monge-Ampere equation and the corresponding Kahler metric, finally we divide
the problem into three condition: the first Chern class positive, negative, or
equal to zero, and justify the long time convergence.

We consider the Kahler Ricci flow:

$$\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega), \quad \omega = \omega_\infty \text{ when } t = 0$$

where $\omega$ is a family of Kahler form on $M$, we will note $\omega(t)$ as the solution
of the equation. Then we take the cohomology class of the both sides:

$$\frac{\partial}{\partial t} [\omega] = -C_1(M), \quad [\omega] = [\omega_0] \text{ when } t = 0$$

That is an ordinary differential equation because we assume $C_1(M)$ is known,
and the great mathematician S.S.Chern has proved the $C_1(M)$ is independent
of the metric chosen. We solve it and get: $[\omega](t) = [\omega_0] - tC_1(M)$, then if the
solution exists, we should get

$[\omega_0] - tC_1(M) > 0$ because the metric is positive definite, then we set

$T = \sup\{t > 0 | [\omega_0] - tC_1(M) > 0\}$, we will prove in $[0,T)$, the solution exists.

We note that there

$$\omega = \sqrt{-1}g_{ij}dz^i \wedge d\bar{z}^j, \quad \omega^n = n!(\sqrt{-1})^n\det gdz^n \wedge d\bar{z}^n \wedge \ldots \wedge dz^n \wedge d\bar{z}^n,$$

so

$$-\sqrt{-1} \partial \bar{\partial} \log \det g = -\sqrt{-1} \partial \bar{\partial} \log \det \omega^n.$$  

Let $\eta \in [\omega_0] - T'\text{C}_1(M)$ be in a Kahler class, then let $\hat{\omega}_t = \frac{1}{t'}((T' - t)\omega_0 + t\eta)$, it is still in $[\omega_0] - T'\text{C}_1(M)$ because it is in the path from one to the other. We assume $\frac{\partial}{\partial t}(\hat{\omega}_t) = \sqrt{-1}/\pi \partial \bar{\partial} \log \Omega$ in the $-\text{C}_1(M)$, existence of this $\Omega$ is from [9]. Then we find that the solution $\omega(t)$ is equivalent to $\omega(t) = \hat{\omega}_t + \sqrt{-1}/\pi \partial \bar{\partial} \tilde{\varphi}(t)$ exists using the $\partial \bar{\partial}$ Lemma and $\int_M \varphi(t)\omega^n_\infty = 0$, such that $\tilde{\varphi}$ is smooth on $M \times [0, T')$ by the
regularity theorem so the equation is

$$\sqrt{-1}/\pi \partial \bar{\partial} \log \omega^n = -\frac{1}{\pi} \frac{\partial}{\partial t}(\hat{\omega}_t + \sqrt{-1}/\pi \partial \bar{\partial} \tilde{\varphi}(t)) = \sqrt{-1}/\pi \partial \bar{\partial} \log \Omega + \sqrt{-1}/\pi \partial \bar{\partial} \frac{\partial}{\partial t}(\tilde{\varphi}(t));$$

which means

$$c(t) + \log \frac{\omega^n}{\omega^n_\infty} = \frac{\partial}{\partial t},$$

where $c(t)$ is a function just for $t$. Then let $\varphi(t) = \tilde{\varphi}(t) - \int_0^t c(s)ds - \varphi(0)$, we get this $\varphi(t)$ solves the equation

$$\frac{\partial}{\partial t} = \log \frac{\omega^n}{\omega^n_\infty} \text{ for } \varphi(0) = 0,$$

conversely, if this $\varphi(t)$ solves the equation in the $[0, T')$

$$\frac{\partial}{\partial t} = \log \frac{\omega^n}{\omega^n_\infty} \text{ for } \varphi(0) = 0,$$  

then let $\omega(t) = \hat{\omega}_t + \sqrt{-1}/\pi \partial \bar{\partial} \varphi(t)$, then

$$\frac{\partial}{\partial t} + \frac{\partial}{\partial \omega} = \frac{\partial}{\partial t} + \sqrt{-1}/\pi \partial \bar{\partial} \log \Omega \Omega + \sqrt{-1}/\pi \partial \bar{\partial} \frac{\partial}{\partial t}(\log \frac{\omega^n}{\omega^n_\infty})$$

$$= \frac{\partial}{\partial t} \partial \bar{\partial} \log \det g = -\text{Ric}(\omega), \quad \omega(0) = \omega_0,$$  

so the existence of $\varphi$ guarantee the existence of the solution of the original equation. Remark if the solution exits, section has proved the uniqueness of the solution. Just the same as before, we should give estimates to this $\varphi$ and first give the short time existence, but we will assume a $T_{\max}$ such that there is a solution only in $[0, T_{\max})$, for $T_{\max} < T$, then by finding a solution in $[T_{\max}, T)$ conclude a contradiction then prove the maximum of $T$. 

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\[ \frac{\partial \phi}{\partial t} = \log \left( \omega_t + \sqrt{t} \frac{\partial \theta (t)}{\Omega} \right)^n \] for \( \phi (0) = 0 \).

First we need a lemma.

**Lemma 7.1**

Let \( T > 0, f(x, t) \) a smooth function on \( M \times [0, T] \), if \( f \) attains its maximum (minimum) at point \( (x_0, t_0) \), then either \( t_0 = 0 \) or at \( (x_0, t_0) \) we have: \( \frac{\partial f}{\partial t} \geq 0 \) (\( \leq 0 \)), \( df = 0 \), \( \sqrt{-1} \partial \bar{\partial} f \leq 0 \).

**Sketch of Proof.** We know that if a smooth function attains its maximum at a point \( (x_0, t_0) \), then it has zero first derivative and nonpositive Hessian at this point from high dimensional Taylor expansion, if \( t_0 > 0 \), then \( f \) is nondecreasing at \( t_0 \), so if \( \frac{\partial f}{\partial t} \geq 0 \), while if \( t_0 = 0 \), because it is the maximum, so we can only attain \( f \) is nonpositive from \( t_0 \).

Back to the estimate. We should estimate the zero order of the solution \( \varphi (T) \). The argument is due to Song and Weinkove[9].

**Proposition 7.2**

\( \exists C \) constant such that \( \forall t \in [0, T_{\max}) \), \( || \varphi (t) ||_{C^0 (M)} \leq C \).

**Proof.**

Consider \( \theta (t) = \varphi (t) - At \), where \( A \) is to be determined, because \( \varphi (0) = 0 \), so if we can choose an \( A \) to prove if \( \theta \) attains its maximum at \( t = 0 \), then we can give a uniform estimate of \( \varphi (t) \). The \( \theta \) satisfy the equation

\[ \frac{\partial \theta}{\partial t} = \log \left( \omega_t + \sqrt{t} \frac{\partial \theta (t)}{\Omega} \right)^n \] in \( [0, t'] \) for \( 0 < t' < T_{\max} \). We assume \( \theta \) attains its maximum at \( (x_0, t_0) \) in \( M \times [0, t'] \) compact, then we use the Lemma 7.1, if \( t_0 > 0 \) then \( \frac{\partial \theta}{\partial t} \geq 0 \) and \( \sqrt{-1} \partial \bar{\partial} \theta \leq 0 \), so

\[ 0 \leq \frac{\partial \theta}{\partial t} \]

\[ = \log \left( \omega_t + \sqrt{t} \frac{\partial \theta (t)}{\Omega} \right)^n \] - \( A \)

\[ \leq \log (\omega_t)^n - A \], if we choose \( A \) large enough, then this inequality cannot hold, then we choose an appropriate \( A \) such that \( t_0 = 0 \) is the only choice. Then \( \theta (t) \leq \theta (0) = 0 \), which means \( \varphi (t) \leq A t' \leq AT_{\max} \), hence the upper bound. For the lower bound, we choose \( \theta (t) = \varphi (t) + At \), then we consider the minimum of \( \theta (t) \), use the Lemma 7.1 again, similarly if \( t_0 > 0 \) then \( \frac{\partial \theta}{\partial t} \leq 0 \) and \( \sqrt{-1} \partial \bar{\partial} \theta \geq 0 \), then

\[ 0 \geq \frac{\partial \theta}{\partial t} \]

\[ = \log \left( \omega_t + \sqrt{t} \frac{\partial \theta (t)}{\Omega} \right)^n \] + \( A \)

\[ \geq \log (\omega_t)^n + A \],

then we can choose \( A \) large enough such that the inequality does not hold, hence again we get \( \theta (t) \geq \theta (0) = 0 \), then \( \varphi (t) \geq -AT_{\max} \), that is the lower bound. We note that for a real \((1, 1)\) form \( \alpha = \sqrt{\frac{\omega}{2\pi}} \bar{\omega} a_{ij} dz^i \wedge d\bar{z}^j \), the trace with respect to \( \omega \) is defined as

\[ \text{tr}_\omega \alpha = g^{ij} a_{ij} = \sum_{i,j} a_{ij} = \omega^{-1} \omega^{\text{tr}} \alpha \] and \( \Delta f = \text{tr}_\omega (\sqrt{\frac{\omega}{2\pi}} \bar{\omega} f) \), actually this notation we use in the later sections have the same property as the normal trace and laplace.

Next we want to give an estimate of \( \frac{\partial \varphi}{\partial t} \), the argument refers to Tian-Zhang.

**Proposition 7.3**
∃ C positive constant such that on $M \times [0,T_{\text{max}})$,
\[
\frac{1}{C} \Omega \leq \omega^n(t) \leq C \Omega, \text{ equivalently, } \| \frac{\partial}{\partial t} \omega^n \|_{C^0} \text{ is uniformly bounded.}
\]

Proof.
\[
\frac{\partial}{\partial t} \log \omega^n(t) = \frac{\partial}{\partial t} \log \omega^n(0)
\]
\[
= \frac{\partial}{\partial t} \log \det g_{i\bar{j}}
\]
but we take trace of the both sides of the equation $\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega)$; we get
\[
g^{i\bar{j}} \frac{\partial}{\partial t} g_{i\bar{j}} = -R, \text{ where } R \text{ is the scalar curvature, then } \frac{\partial}{\partial t} \log \omega^n(t) \leq R.
\]

Then
\[
\frac{\partial}{\partial t} \log \omega^n(t) = \frac{\partial}{\partial t} \log \omega^n(0) = -R.
\]

While
\[
R = \text{tr}(\text{Ric}), \text{ hence } n |\text{Ric}(\omega)|^2 \geq R^2, \text{ so } \left( \frac{\partial}{\partial t} - \Delta \right) R \geq \frac{2}{n} R^2,
\]
hence
\[
\left( \frac{\partial}{\partial t} - \Delta \right) R \geq 0, \text{ by the maximum principle } | R | \leq C.
\]
Integrating
\[
\frac{\partial}{\partial t} \log \omega^n(t) = -R \text{, then we get } \omega^n(t) \leq e^{Ct} \omega^n(0).
\]
So we finish the upper bound. As for the lower bound, take the derivative of the equation
\[
\frac{\partial}{\partial t} \log \omega^n(t) = \frac{\partial}{\partial t} \log \omega^n(0) = -R.
\]

Then we get
\[
\frac{\partial}{\partial t} \omega^n(t) = \frac{\partial}{\partial t} \log \omega^n(t) = -R.
\]
Integrating
\[
\frac{\partial}{\partial t} \log \omega^n(t) = -R \text{, then we get } \omega^n(t) \leq e^{Ct} \omega^n(0).
\]
So we finish the upper bound. As for the lower bound, take the derivative of the equation
\[
\frac{\partial}{\partial t} \log \omega^n(t) = \frac{\partial}{\partial t} \log \omega^n(0) = -R.
\]

Then we get
\[
\frac{\partial}{\partial t} \omega^n(t) = \frac{\partial}{\partial t} \log \omega^n(t) = -R.
\]
then \[ \| \frac{\partial \omega}{\partial t} \| \geq \frac{1}{\text{tr} \frac{1}{\max} || T \inf \| \omega \| + || \varphi || + n \max, \] by the zero order estimate of \( \varphi \) we conclude the lower estimate of \( \frac{\partial \omega}{\partial t} \), note these norms are uniformly \( C_0(M) \) norm.

**Proposition 7.4**

There exists a uniform positive constant \( C \) such that on \( M \times [0, T_{\max}] \), \( \frac{1}{n} \omega_0 \leq \omega \leq C \omega_0 \).

**Proof.** First we estimate the lower bound. Due to the existence of the normal coordinates, we can always find a system of coordinates for the Hermitian metric \( g \) such that: \( (g_0)_{ij} = \delta_{ij} \), and \( g_{ij} = \lambda_i \delta_{ij} \), for positive eigenvalue \( \lambda_i \). Then
\[
\text{tr} \omega_0 = \sum_{i=1}^{n} \frac{1}{\lambda_i (\sum_{i=1}^{n} \lambda_i)^{n-1}} \leq \frac{1}{(n-1)!} (\text{tr} \omega_0)^{n-1} \omega_0 \leq C, \text{ because from Proposition 7.3 } \omega^n \text{ is uniformly bounded.}
\]

As for the upper bound, we prove there exists a uniform constant \( C \) such that \( \text{tr} \omega_0 \omega \leq C \) on \( M \times [0, T_{\max}] \). We use the traditional method: find a useful quantity using coefficients to be determined, then we prove it satisfies an elliptic equation and use the maximum principle, that method has been used in many of our proof, but how to find a useful quantity is essential, this can only be attained by experience and attempt. We consider \( Q = \log \text{tr} \omega_0 \omega - \lambda \varphi \), where \( \lambda \) is a positive uniform constant to be determined. We fix \( t_0 = 0 \).

First we estimate \( \frac{\partial}{\partial t} \Delta \text{tr} \omega \). Let \( \omega \) be a fixed Kahler form corresponding to the Kahler-Ricci metric \( \hat{g} \) on \( M \), and the same as before, \( \omega \) the solution of the original Kahler-Ricci flow equation, we note \( R_{ijkl} = g^{lp} g^{jq} \hat{R}_{ijkl} \) is the curvature, \( \nabla \) is the connection corresponding to \( g \), \( \hat{R}_{ijkl} = g^{lp} g^{jq} \hat{R}_{ijkl} \) is the curvature, \( \hat{\nabla} \) is the connection corresponding to \( \hat{\omega} \). Then the same as before, we use the normal coordinates for \( \hat{g} \), then
\[
\Delta \text{tr} \omega = g^{ik} \partial_i (\hat{g}^{ij} g_{ij})
\]
\[
= g^{ik} (\partial_i (\hat{g}^{ij} g_{ij}) + g^{ik} \hat{g}^{ji} (\partial_i g_{ij}))
\]
\[
= g^{ik} \hat{R}_{ikj} g_{ij} + \hat{g}^{ji} g^{ik} \hat{g}^{pq} \partial_i g_{pq},
\]
while \( \frac{\partial}{\partial t} \text{tr} \omega = - \hat{g}^{ji} \hat{R}_{ij} \),
\[
\text{we get } \frac{\partial}{\partial t} \Delta \text{tr} \omega = g^{ik} \hat{R}_{ikj} g_{ij} - \hat{g}^{ji} g^{ik} \hat{\nabla} i g_{iq},
\]
\[
\text{Then we use the formular above to calculate } \frac{\partial}{\partial t} \omega \text{, then choose normal coordinates such that } g \text{ is diagonal, by Cauchy-Schwarz inequality we get}
\]
\[
| \partial \text{tr} \omega |^2 \leq (\text{tr} \omega) \sum_{i,j,k} g^{ji} g^{ij} \partial_i g_{ij} \partial_k g_{jk},
\]
and then we define \( C = \inf_{x \in M} \hat{R}_{ijkl}(x) | \partial_{x_1}, \ldots, \partial_{x_n} \), then calculate
\[
g^{ik} \hat{R}_{ikj} g_{ij}
\]
\[
= \sum_{k,i} g^{ik} \hat{R}_{kji} g_{ii}
\]
\[
\geq -C \sum_{k,i} g^{kk} g_{ii}
\]
\[
= -C \sum_{k,i} g_{ii}
\]

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This proof comes from [9], one can obtains more details from [9].

Then at point $(x_0,t_0)$ we use the estimate above.

$0 \leq (\frac{\partial}{\partial t} - \Delta)Q$

$\leq C_0 \text{tr}_{\omega_0} - A \text{tr}_{\omega_1} + A \Delta \varphi$

$= \text{tr}_{\omega}(C_0 \omega_0 - A \text{tr}_{\omega_1}) - A \text{log} \omega + A n + C$, for $C_0$ is a constant only depends on the lower bound of the bisectional curvature of $g_0$. Then we need choose $A$ large enough such that

$A \text{tr}_{\omega} - (C_0 + 1) \omega_0$ is Kähler on $M$, i.e. first it should be positive definite. Then

$\text{tr}_{\omega}(A \text{tr}_{\omega} - (C_0 + 1) \omega_0) \geq 0$, so

$\text{tr}_{\omega}(A \text{tr}_{\omega} - C_0 \omega_0) \leq \text{tr}_{\omega} \omega_0$, when at point $(x_0,t_0)$,

$0 \leq \text{tr}_{\omega}(C_0 \omega_0 - A \text{tr}_{\omega_1}) - A \text{log} \omega + A n + C$, so

$\text{tr}_{\omega} \omega_0 + A \text{log} \omega \leq C$, for a constant $C$. Similarly, find a system of coordinates for the Hermitian metric $g$ such that: $(g_0)_{ij} = \delta_{ij}$, and $g_{ij} = \lambda_i \delta_{ij}$, for positive eigenvalue $\lambda_i$, then it shows

$\sum_{i} \lambda_i + \sum_{i} n \text{Alog} \lambda_i \leq C$. Due to $\lambda_i$ is positive, so there is a uniform upper bound $C$ for $\frac{\partial}{\partial t} + \text{Alog} \lambda_i \leq C$, while $\frac{\partial}{\partial t}$ is positive, we attains a uniform upper bound for $\text{Alog} \lambda_i$, hence a uniform upper bound for $\lambda_i$, i.e. $\lambda_i \leq C$, then we attains an upper bound for $\text{tr}_{\omega} \omega$ at $(x_0,t_0)$, then by the uniform bounded $\varphi$ on $M \times [0,t')$ when $t' \leq T_{\text{max}}$ and $\text{tr}_{\omega} \omega$ at $(x_0,t_0)$, $Q$ is uniformly bounded on $M \times [0,t')$ when $t' \leq T_{\text{max}}$, then use the uniformly bounded $\varphi$ on $[0, T_{\text{max}}]$, we know that $\text{tr}_{\omega} \omega$ has a uniformly upper bound. Therefore, we have prove the uniformly bound for $\frac{\partial}{\partial t} = \text{log} \omega$.

**Lemma 7.5**

If the solution $\omega(t)$ of the Kähler-Ricci flow equation before on $M \times (0,T)$ satisfies there $\exists$ a constant $C_0$ such that: $\frac{\text{tr}_{\omega}}{C_0} \omega_0 \leq \omega \leq C_0 \omega_0$.

Then for any positive integer $m$, there exists corresponding uniform constant $C_m$ such that $\| \omega(t) \|_{C^m(g_0)} \leq C_m$.

**Proof.**

If the solution $\omega(t)$ satisfies there $\exists$ a constant $C_0$ such that: $\frac{\text{tr}_{\omega}}{C_0} \omega_0 \leq \omega \leq C_0 \omega_0$, we prove first there exists constants $C$, $C'$ depending only on $C_0$ and $\omega_0$ such that:

$| \nabla g_{ijkl} |^2 \leq C$, and $(\frac{\partial}{\partial t} - \Delta) | \nabla g_{ijkl} |^2 \leq \frac{1}{2} | R_{ijkl} |^2 + C'$;

then there exists constants $C$, $C'$ depending only on $C_0$ and $\omega_0$ such that:

$| R_{ijkl} |^2 \leq C$ and $(\frac{\partial}{\partial t} - \Delta) | R_{ijkl} |^2 \leq - | \nabla R_{ijkl} |^2 + C'$, where $\nabla$ is the conjugate of $\nabla$, then using the condition above we learn that $\exists$ uniform constants $C_m$ for positive integer $m$ such that $| \nabla R_{ijkl} |^2 \leq C_m$, therefore using the conclusions of the above claim, for $U$ an open subsets of $M$, for any compact...
subset $K$ in $U$, positive integer $m$, there exists constants $C_m'$ depending only on $\omega_0$, $K$, $U$, and $C_m$ such that $\|\omega(t)\|_{C^m(K,g_0)} \leq C_m'$, finally, from the above claim, we conclude the lemma.

Due to the much too long standard proof of this lemma, we give a sketch of proof, the remaining detail one can refer to [9], Theorem 2.13,2.14,2.15.

Now we prove the existence of the Kahler-Ricci flow solution.

**Theorem of the maximal existence time**

The Kahler-Ricci flow equation
\[
\frac{\partial}{\partial t}\omega = -\text{Ric}(\omega), \quad \omega = \omega_0 \text{ when } t = 0
\]
has a unique solution in the maximal time $t \in [0, T)$, then this solution exists for all time.

**Proof.**

By Proposition 7.4 and Lemma 7.5, we conclude the uniform $C^\infty$ estimate for $\omega(t)$ on $[0, T_{\max})$, then as $t$ goes into the $T_{\max}$, by Arzela-Ascoli Theorem and take countable diagonal subsequences, we obtain a solution $g(T_{\max})$ while $g$ converges to it on $[0, T_{\max}]$, then we use the same argument as solving the problem before, our estimates are independent of $t$, then our estimates still works the estimates above are independent of $t$, so if we choose $t_0 \in [0, T)$ then the solution also exists in $[t_0, t_0 + \varepsilon]$ for $\varepsilon$ independent of $t$, we can use this $\tilde{g}_{ij}(T)$ as the initial condition of the same equation choosing a new initial point and continue to deformation, which is contradict to the definition of $T_{\max}$, hence the process can continue to infinity because our estimates always work, so the solution exists for all time.

Then we complete the proof of the condition $C_1(M) < 0$ which remains.

We assume $C_1(M) < 0$, and $[\omega_0]=-C_1(M)$. We consider the normalized Kahler-Ricci flow:
\[
\frac{\partial}{\partial t}\omega = -\text{Ric}(\omega), \quad \omega = \omega_0 \text{ when } t = 0
\]
We use this normalized form to avoid the Kahler class $[\omega(t)]$ given by $(1+t)[\omega_0]$ diverges when $t$ goes into the infinity. While let $s = e^t - 1$, then by some simple calculation we know $\omega(t)$ solves the normalized form equation is equivalent to $\tilde{\omega}(s) = e^s\omega(t)$ solves the original Kahler-Ricci flow equation
\[
\frac{\partial}{\partial t}\omega = -\text{Ric}(\omega).
\]
Therefore, the estimates we have proved before can be used in the proof of the original problem. Using the same method as the former sections, we now prove the existence of the Kahler-Einstein metric problem.

**Theorem of existence of the Kahler – Einstein metric**

The solution to the equation
\[
\frac{\partial}{\partial t}\tilde{g}_{ij} = -\tilde{R}_{ij} - \tilde{g}_{ij}, \quad \tilde{g}_{ij} = g_{ij} \text{ at } t = 0
\]
converges in $C^\infty$ to the unique Kahler-Einstein metric $\tilde{g}_{ij}(\infty)$ which belongs to the negative first Chern class of $M$.

**Proof.**

First it’s a Kahler Ricci flow equation which is a rescaling of $\frac{\partial}{\partial t}\tilde{g}_{ij} = -\tilde{R}_{ij} :$ if $u'(s)$ is a solution of the equation above, then $u(t) = \frac{u'(s)}{t+1}$ for $t = \log(s+1)$, where $s \in [0, \infty)$, then it has a short time solution, and by the Theorem of maximal existence time, it has an all time solution, this is due to the parabolic equation
theory, and since we now have an zero order estimate of $\tilde{g}_{ij}$ by Lemma 7.5 and Lemma 6.1, and an estimate for $\frac{\partial}{\partial t}$ with its exponential decay by Lemma 6.1, then as the same argument as the Proposition before, we use Schauder estimate and Interior regularity theory to obtain a $C^\infty$ estimate of $\tilde{g}_{ij}$, then the $C^0$ estimate of $u(t)$ gives the uniform $C^\infty$ estimate of it. While $u(t)$ converges uniformly to $u(\infty)$ continuously when $t$ goes into the infinity by Lemma 6.1, we prove that $u(t)$ converges to $u(\infty)$ in the $C^\infty$ sense by contradiction. If there exists an integer $k$ and $\epsilon$ positive, and a sequence $t_i$ goes into the infinity such that

$|| u(t_i) - u(\infty) ||_{C^k(M)} \geq \epsilon$ for any positive integer $i$, 

then because $u(t_i)$ has uniform $C^{k+1}$ bound then by Arzela-Ascoli Theorem, there exists a subsequence $u(t_{i_k})$ converges to another limit, says $u'(\infty)$ in the $C^k$ sense, but

$|| u(t_i) - u(\infty) ||_{C^k(M)} \geq \epsilon$ implies $|| u'(\infty) - u(\infty) ||_{C^k(M)} \geq \epsilon$, 

so $u'(\infty) \neq u(\infty)$ which is contradic to the uniqueness of the uniformly convergence of $u(t)$. So we only get $u(t)$ converges to $u(\infty)$ in the $C^\infty$ sense. Then this presents that $\frac{\partial}{\partial t}$ converges to 0 as $t$ goes into the infinity, because $u$ converges to a constant $u_\infty$ therefore, from $\tilde{g}_{ij}(\infty) = g_{ij} + \frac{\partial^2 u(\infty)}{\partial z^i \bar{z}^j}$, we get $\tilde{g}_{ij}$ converges to a constant $t$ when $t$ goes into the infinity, which means that $\frac{\partial \tilde{g}_{ij}(t)}{\partial t} = 0$, then

$0 = -\tilde{R}_{ij}(\infty) - \tilde{g}_{ij}(\infty)$, hence $\tilde{R}_{ij}(\infty) = -\tilde{g}_{ij}(\infty)$,

so that’s the Kahler-Einstein metric $g$ we want.

And the uniqueness follows. If $g'$ is another Kahler-Einstein metric both belonging to the same negative first Chern class, then we can write:

$g' = g + \sqrt{-1} \frac{\partial \bar{\partial} \varphi}{\log \det g}$, then by calculation $\text{Ric}(g') = \text{Ric}(g) - \sqrt{-1} \frac{\partial \bar{\partial} \varphi}{\log \det g}$, so

$\log \det (g + \sqrt{-1} \frac{\partial \bar{\partial} \varphi}{\log \det g}) = \varphi + C$

for $C$ a constant, then by the maximum principle of the function $\varphi + C$, the maximum and the minimum of $\varphi$ attains at the boundary, since by calculation as before argument this $\varphi + C$ satisfies the equation

$\frac{\partial u}{\partial t} = \log \det(g' + \sqrt{-1} \frac{\partial \bar{\partial} \varphi}{\log \det g}) - \log \det(g) - u + f$ for $\varphi + C = 0$ when $t=0$, so we get the maximum and the minimum of $\varphi + C$ are 0, then the LHS is zero, then $g'=g$. The proof of uniqueness actually comes from Calabi. Finally, the proof is completed, and we just justify the deformation method works well in Kahler-Einstein metric existence problem by some important estimates from the Kahler Ricci flow equation.

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