Photon-like flying qubit in the coupled cavity array

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We propose a feasible scheme to realize a spin network via a coupled cavity array with the appropriate arrangement of external multi-driving lasers. It is demonstrated that the linear photon-like dispersion is achievable and this property opens up the possibility of realizing the pre-engineered spin network which is beneficial to quantum information processing.

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\textbf{Introduction.} Performing a perfect quantum state transfer (QST) between two distant qubits is highly desirable for quantum computer architecture. A photon-like “flying qubit” \textsuperscript{12,13} in solid state system is crucial for scalable quantum computation \textsuperscript{14}. The quantum spin lattice is a paradigm in condensed matter physics. It serves as a communication channel to link quantum solid state registers without the need of conversion among different types of qubits since the work of S. Bose \textsuperscript{4}. To perform a high-fidelity state transfer in a quantum spin network the main obstacle is the non-linear dispersion relation which reduces the fidelity of QST due to the dispersive effect for a single magnon, an elementary excitation corresponding to spin wave. It has been found that nonuniform nearest neighbor (NN) coupling can allow the perfect QST \textsuperscript{8,15,16}. Moreover, non-trivial long-range coupling distribution is also possible to achieve the linear photon-like dispersion relation \textsuperscript{9}. In practice, it remains the challenge to realize an optimal coupling strength distributions in quantum device. Recent developments of technology in coupled cavity arrays offer the ability to experimentally observe the quantum many body phenomena and design the controllable quantum devices for quantum information process. The high Q cavity and strong interaction between the cavity mode and atoms have been observed experimentally \textsuperscript{11} \textsuperscript{14}. It has been proposed to realize an effective quasi-spin model through the exchanging of virtual photons \textsuperscript{12,13}. This opens the possibility for the application of the theoretical results in practice.

In this letter, we propose a feasible scheme to realize a quantum channel for a photon-like flying qubit in a coupled cavity array with each cavity containing a single three-level atom \textsuperscript{12}. It is shown that the energy-band broadening for photons can induce a long-range interaction between atoms trapped in different cavities. Moreover, the distribution of the long-range interaction strength is optically controllable, which opens a possibility to pre-engineer a standard XY spin model via tuning the external multi-driving lasers. As an application, using a simple optimal setup, we demonstrate that the linear photon-like dispersion relation for a magnon is achievable.

\textbf{Model setup.} We consider a coupled cavity array with \textit{N} cavities and each cavity has a three-level atom. The level structure of the atom trapped in the \textit{j}th cavity is schemed in Fig. 1(b), where the atom contains two long-lived states \textit{|a\textit{j}\rangle} and \textit{|b\textit{j}\rangle} and an excited state \textit{|e\textit{j}\rangle}. The transition between levels \textit{|a\textit{j}\rangle} and \textit{|e\textit{j}\rangle} is coupled to the cavity mode \textit{ω\textit{c}} with the coupling \textit{g}, while the transition between \textit{|b\textit{j}\rangle} and \textit{|e\textit{j}\rangle} is driven by \textit{n\textit{L}} lasers \{\textit{ω\textit{nl}}\} with the Rabi frequency \{\textit{Ω\textit{n}}\}. The total Hamiltonian is

\begin{equation}
    H = H_a + H_c + H_{ac} + H_{aL}.
\end{equation}

The Hamiltonian \(H_a\) which describes \textit{Λ} level-structure atoms reads

\begin{equation}
    H_a = \sum_j \left( \omega_c |e_j\rangle \langle e_j| + \omega_{ab} |b_j\rangle \langle b_j| \right).
\end{equation}

\(H_c\) describes photons in coupled cavities as

\begin{equation}
    H_c = \sum_j \omega_c a_j^\dagger a_j - T \left( a_j^\dagger a_{j+1} + h.c. \right),
\end{equation}

where \(\omega_c\) is the frequency of photons, \(T\) is the tunnelling rate of photons between two neighboring cavities, and

\textbf{FIG. 1: Schematic diagram of \textit{Λ}-type three-level atoms in coupled cavities. Each atom interacts with a single mode cavity and two external driving lasers. Levels \textit{|a\textit{j}\rangle}, \textit{|b\textit{j}\rangle} and \textit{|e\textit{j}\rangle} have the energy values \textit{0}, \textit{\omega\textit{ab}} and \textit{\omega\textit{c}}, respectively. The transition \textit{|a\textit{j}\rangle} \leftrightarrow \textit{|e\textit{j}\rangle} is driven by photons with the frequency \textit{\omega\textit{c}} and interaction strength \textit{g}. \textit{\omega\textit{1}} and \textit{\omega\textit{2}} are the frequencies of the driving lasers with Rabi frequencies \textit{\Omega\textit{1}} and \textit{\Omega\textit{2}} respectively. When cavities are coupled with each other, the degenerate cavity level \textit{\omega\textit{c}} becomes an energy band with bandwidth \textit{2\textit{T}}. Then the coupled cavity array is equivalent to a multi-mode cavity with frequencies \textit{\omega\textit{L}}.}
\( a_j \) (or \( a_j^\dagger \)) annihilates (creates) a photon in cavity \( j \). \( H_{ac} \) represents the coupling between atoms and the cavity mode as
\[
H_{ac} = \sum_j \left( g |e\rangle_j \langle a| a_j + h.c. \right),
\] (4)
while \( H_{aL} \) describes the coupling between atoms and \( n_L \) driving lasers
\[
H_{aL} = \frac{1}{2} \sum_j \sum_{n=1}^{n_L} \Omega_n e^{-i\omega_n t} |e\rangle_j \langle b| + h.c.\right).
\] (5)

Considering the coupled cavity array as an even \( N \) site ring and taking the Fourier transformation
\[
a_k = \frac{1}{\sqrt{N}} \sum_j e^{ikj} a_j^\dagger,
\] (6)
the Hamiltonian \( H_c \) is diagonalized as
\[
H_c = \sum_k \omega_k a_k^\dagger a_k = \sum_k \left( \omega_c - 2T \cos k \right) a_k^\dagger a_k,
\] (7)
where \( k = 2m\pi/N, \ m = 0, 1, \ldots, N-1 \). Throughout this paper, \( k \) denotes the momentum. Note that such a transformation provides an equivalent view about the coupled cavity array. Due to the tunnelling of photons between neighboring cavities, the degenerate cavity level \( \omega_c \) becomes an energy band with bandwidth \( 2T \). Then the coupled cavity array is equivalent to a multi-mode cavity with frequencies \( \omega_k \). In this sense, it is possible to realize a long-range effective interaction between two atoms placed in distant cavities. We will show that such long-range couplings can be employed to construct the spin network, which possesses the approximate linear photon-like dispersion relation for a magnon.

**Adiabatic elimination of atomic excited and photonic states.** We focus on the case in which the photon excitation in the cavity array is strongly suppressed. In this case, the atomic states are always changed after emitting or absorbing a virtual photon. In the following we will adiabatically eliminate the atomic excited state \( |e\rangle \) and the elimination of photonic states. During these procedures, we only consider the simplest case that each driving laser contributes to the effective Hamiltonian independently. This scheme requires that the parameters satisfy the following conditions
\[
|\delta_k|, |\Delta_n| \gg |g|, \Omega_m, |\Gamma_m^q|, |\omega_m|,
\] (8)
\[
\forall k, q \in [0, 2\pi), \forall m, n, l \in [1, n_L]
\]
and
\[
|\Gamma_k^q|, |\omega_m|, |\Gamma_k^q - \omega_m| \gg \frac{\Omega_m^2}{2\delta_p}, \frac{\Omega_m}{2\delta_p},
\] (9)
\[
\forall k \in [0, 2\pi), \forall m, n, l, p \in [1, n_L],
\]
where
\[
\delta_k = \omega_c - \omega_c + 2T \cos k,
\]
\[
\Delta_n = \omega_c - \omega_m - \omega_m,
\] (10)
\[
\Gamma_m^q = \delta_k - \Delta_n,
\]
\[
\omega_m = \omega_m - \omega_m.
\]

Now turn to the interaction picture with
\[
H_0^e = H_a + H_c
\] (11)
and
\[
H_1^c = \sum_j \left[ |e\rangle_j \langle a| \sum_k g \sqrt{N} e^{i(kj+\delta_k)t} \tilde{a}_k + h.c.\right).
\] (12)

Through adiabatically eliminating the atomic excited state, the effective Hamiltonian of \( H_1^c \) is written as
\[
H_2^e = -iH_1^c (t) \int_{-\infty}^t dt' H_1^c (t')
\] (13)
\[
- [B + \mathfrak{B} (t)] \sum_j \sigma_j^{(j)}
\]
\[
- \sum_{j,k} \left[ g_j (k, t) \sigma_{+}^{(j)} \tilde{a}_k + h.c.\right],
\]
where the pseudo spin operators are
\[
\sigma_j^{(j)} = |b\rangle_j \langle b| - |a\rangle_j \langle a|,
\] (14)
\[
\sigma_{+}^{(j)} = |b\rangle_j \langle a|,
\]
and
\[
B = \sum_n \frac{\Omega_m^2}{8\Delta_n},
\]
\[
\mathfrak{B} (t) = \sum_{m \neq n} \frac{\Omega_m^* \Omega_n}{4(\Delta_n + \Delta_m)} e^{i\omega_m t},
\] (15)
\[
g_j (k, t) = \frac{g}{\sqrt{N}} \sum_n \Omega_m^* e^{i(kj+\Gamma_m^q)t}.
\]

Here an irrelevant constant has been dropped. Note that the Hamiltonian (13) is equivalent to a JC model which describes an ensemble of atoms interacting with a multi-mode cavity.

Next the interaction picture is taken as
\[
H_0^p = H_a + H_c - B \sum_j \sigma_j^{(j)}
\] (16)
and
\[
H_1^p = H_2^e + B \sum_j \sigma_j^{(j)}.
\] (17)
Eliminating the photonic degree of freedom, we have

\[ H^p_2 = -i H^p_1 (t) \int_{-\infty}^{t} dt' H^p_1 (t') \]  

\[ = -\sum_{i,j} \sum_{n=1}^{n_L} \left| \frac{g\Omega_n}{2\Delta_n} \right|^2 S_{ij}^{(n)} \sigma_i^{(i)} \sigma_j^{(j)}, \]

where

\[ S_{ij}^{(n)} = \frac{1}{N} \sum_k e^{i(k(i-j))} \Omega_n \frac{\Delta_n}{D_n - 2T \cos k} \]

and

\[ D_n = \omega_c - \omega_{ab} - \omega_n. \]

Combining \( H^p_0 \) and \( H^p_2 \), the effective Hamiltonian of atoms is obtained as

\[ H_{\text{eff}} = -\sum_{i,j} \frac{J_{ij}}{2} \left( \sigma_i^{(i)} \sigma_j^{(j)} + h.c. \right) \]

\[ + \sum_j \left( \frac{\omega_{1k}}{2} - B \right) \sigma_j^{(j)}, \]

where

\[ J_{ij} = \sum_{n=1}^{n_L} \left| \frac{g\Omega_n}{2\Delta_n} \right|^2 S_{ij}^{(n)}. \]

It is a standard \( XY \) model with the pre-engineered coupling distribution. The long-range interaction is the result of the photonic energy-band broadening. It is also indicated that \( J_{ij} = J_{ji} = J(|i - j|) \).

In the narrow band limit \( |D_n| \gg |2T| \), Eq. \( 19 \) becomes

\[ S_{ij}^{(n)} \simeq \delta_{ij} \frac{1}{D_n} + \delta_{|i-j|, 1} \frac{T}{D_n}, \]

which is reduced to the result of the spin model with NN couplings obtained in Ref. [12]. This setup leads to a cosinusoidal dispersion relation, i.e., a magnon has a quadratic dispersion relation for small momenta but a linear photon-like dispersion relation for momenta around \( k = \pm \pi/2 \) [3, 14].

On the other hand, in the limit \( N \to \infty \), we have

\[ S_{ij}^{(n)} = \sigma_i^{(i-j+1)} \frac{1}{\sqrt{D_n^2 - 4T^2}} \exp \left( \frac{-|i-j|}{\xi_n} \right) \]

with

\[ \xi_n^{-1} = -\ln \left[ \frac{D_n}{2T} - \sqrt{\left( \frac{D_n}{2T} \right)^2 - 1} \right], \]

where \( \xi_n \) is the characteristic length of the long-range effective interaction and \( \sigma_n = \text{sign} (D_n/T) \).

**Designed spin chain.** In quantum information processing, a qubit state is usually transferred by photons via the fiber. In order to implement the scalable quantum computation in solid state systems, a spin chain is of particular interest because it may act as a data bus to link qubits without the need of conversion among different types of qubits. Then realizing a flying qubit in a spin chain is significant. The dynamics of the magnon wave packet has been studied recently [2, 3]. The main obstacle to perform a high-fidelity state transfer in a quantum spin wave. A previous study has shown that a long-range \( XY \)-interaction gives a chance to realize the flying qubit with a linear photon-like dispersion relation [4].

For a standard \( XY \) model,

\[ H_{XY} = -\sum_{i,l} \frac{J(l)}{2} \left( \sigma_i^{(i)} \sigma_{i+l}^{(l)} + h.c. \right), \]

where \( J(l) = J(-l) \). The eigenstates in the subspace with a single spin flipped on a ferromagnetic background are \(|k\rangle = 1/\sqrt{N} \sum_{j=1}^{N} e^{ikj} |j\rangle \) with \(|j\rangle = \sigma_j^{(+)} \prod_{l=1}^{N} |l\rangle_i \).

The dispersion relation for the single magnon is

\[ E_k = -J(0) - \sum_{l>0} 2J(l) \cos kl. \]

Note that, in principle, the system with any dispersion relation can be constructed by an appropriate distribution of \( J(l) \), which can be realized via the external driving lasers. For a linear photon-like dispersion curve \( \varepsilon_k = |k| \), the corresponding Fourier expansion is \( 0 \)

\[ \varepsilon_k = \frac{\pi}{2} - 2 \sum_{l=1}^{\infty} \frac{1}{l^2} \cos kl, \]

which requires

\[ J(0) = -\frac{\pi}{2}, \ J(l \neq 0) = \frac{1}{\pi l^2}. \]

On the other hand, we can use \( n_L/2 \) pairs of driving lasers with

\[ \left| \frac{\Omega_{2m-1}}{\Delta_{2m-1}} \right| = \left| \frac{\Omega_{2m}}{\Delta_{2m}} \right| = G_{2m}; \]

\[ D_{2m-1} = -D_{2m}, \ m \in [1, n_L/2] \]

to simulate the coupling as

\[ J_{ij} = \frac{g}{4} \sum_{m=1}^{n_L/2} G_{2m}^2 \left| S_{ij}^{(2m)} \right| \left[ 1 - (-1)^{i-j} \right]. \]

It is indicated that the photon-like dispersion relation is probably achievable in a system with an optimal arrangement of \( n_L/2 \) pairs of driving lasers. In practice, it
is impossible to set up many external lasers. However, a straightforward calculation shows that a feasible setup with several deriving lasers can work efficiently. Consider the optimal parameters in two simple cases with $n_L = 2$ and 4 where for $n_L = 2, D_2 = 10T/3$ and for $n_L = 4, D_2 = 20T, D_4 = 34T/15, \text{and } [\Omega_2 \Delta_4/\Omega_1 \Delta_2] = 6\sqrt{14}$. Other parameters are set to satisfy the condition (20).

In Table 1, the distribution of the coupling constant $J(l = |i - j|) = J_{ij}$ is displayed in unit $J(1)$ from (31) and from the ideal case (29). The corresponding dispersion curves and group velocities are plotted in Fig. 2 and compared with those of a photon-like cosineoidal dispersion. The obtained results indicate that the ideal distribution of the coupling constant (29) and the photon-like dispersion relation are achievable approximately using our simple and optimal setups with $n_L = 2$ and 4.

To prove our scheme, the time evolution of a single qubit state is considered in a ring with $N = 40$ and with the optimal coupling distribution for $n_L = 4$ as listed in Table 1. The initial state is $|\psi(N_0)\rangle$ with $N_0 = 10$. Driven by the Hamiltonian with the ideal coupling distribution (29), the state separates into two local moving wave packets, and they meet at site $N_0 + N/2$ to form a single qubit state $|\psi(N_0 + N/2)\rangle = |\Psi(30)\rangle$ after a period of time $\tau = N\pi/8$ (9). The profile of the time evolution is plotted in Fig. 3 numerically. It exhibits a “whispering gallery” behavior, which is crucial for the QST and entanglement creation.

| $l$ | 1 | 2 | 3 | 4 | 5 | 7 | 9 | 11 |
|-----|---|---|---|---|---|---|---|----|
| $J(l)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $J_{ij}$ | $\frac{1}{1}$ | $\frac{1}{3}$ | $\frac{1}{5}$ | $\frac{1}{7}$ | $\frac{1}{9}$ | $\frac{1}{11}$ | $\frac{1}{13}$ | $\frac{1}{15}$ |

Table 1. The distribution of the coupling constant in setups with optimal parameters for $n_L = 2, 4$ and in the ideal case (29). It indicates that the ideal coupling distribution is achievable approximately via our scheme.

Summary. In summary, we have shown that the feasible scheme to realize a quantum channel for a photon-like flying qubitis achievable in a coupled cavity array with each cavity containing a single three-level atom. The underlying physics is best understood as that the energy-band broadening for photons can induce a long-range interaction between atoms trapped in different cavities. We also show that a simple optimal setup with several external driving lasers can realize the linear photon-like dispersion relation for a magnon. We have demonstrated that this property opens up the possibility of realizing the pre-engineered spin network which is beneficial to quantum information processing.

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FIG. 2: Dispersions with different coupling distributions for (empty circle) $n_L = 2$, (solid circle) $n_L = 4$, and (solid line) the ideal case, as given in Table 1. The dash line corresponds to the case with NN couplings. The inset shows the derivatives (group velocities) of dispersions.

FIG. 3: Profile of the time evolution of a single-qubit state in a ring with $N = 40$ and with the optimal coupling distribution for $n_L = 4$, as listed in Table 1. The initial state is set as $|\psi(N_0)\rangle$ with $N_0 = 10$. The exhibited “whispering gallery” behavior justifies our scheme.
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