Classification of binary self-dual $[76, 38, 14]$ codes with an automorphism of order 9

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Abstract

Using the method for constructing binary self-dual codes with an automorphism of order square of a prime number we have classified all binary self-dual codes with length 76 having minimum distance $d = 14$ and automorphism of order 9. Up to equivalence, there are six self-dual $[76, 38, 14]$ codes with an automorphism of type $9-(8, 0, 4)$. All codes obtained have new values of the parameter in their weight enumerator thus more than doubling the number of known values.

Keywords: automorphism; classification; extremal codes; self-dual codes;

1 Introduction

In this paper, we are interested in the classification of the extremal binary self-dual $[76, 38, 14]$ codes with an automorphism of order 9. It was motivated by the following reasons.

Firstly, there are only three known extremal binary self-dual $[76, 38, 14]$ codes, constructed by Dontcheva and Yorgov via an automorphism of order 19 [6]. These three codes are not only shadow optimal but also shortest known self-dual code with minimal distance 14. One of these three codes was the first ever found in the literature and it was discovered by Baartmans and Yorgov [1].

Secondly, Bouyuklieva, et al [4] presented a method for constructing binary self-dual codes with an automorphism of order $p^2$ and classified all optimal binary self-dual codes $[76, 38, 14]$ codes of lengths $44 \leq n \leq 54$ having an automorphism of order 9. The case for the length of an optimal binary self-dual code with

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automorphism of such order was considered by Yankov in [10] where it was proved that a doubly-even self-dual [72, 36, 16] codes with an automorphism of order 9 does not exists.

A linear $[n, k]$ code $C$ is a $k$-dimensional subspace of the vector space $\mathbb{F}_q$, where $\mathbb{F}_q$ is the finite field of $q$ elements. The elements of $C$ are called codewords, and the (Hamming) weight of a codeword $v \in C$ is the number of the non-zero coordinates of $v$. We use $\text{wt}(v)$ to denote the weight of a codeword. The minimum weight $d$ of $C$ is the smallest weight among all its non-zero codewords, and $C$ is called an $[n, k, d]$ code. A matrix whose rows form a basis of $C$ is called a generator matrix of this code and we denote this by $\text{gen}(C)$. Every code satisfies the Singleton bound $d \leq n - k + 1$. A code is maximum distance separable or MDS if $d = n - k + 1$, and near MDS or NMDS if $d = n - k$.

For every $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ from $\mathbb{F}_2^n$, $u.v = \sum_{i=1}^{n} u_i v_i$ defines the inner product in $\mathbb{F}_2^n$. The dual code of $C$ is $C^\perp = \{ v \in \mathbb{F}_2^n \mid u.v = 0, \forall u \in C \}$. If $C \subset C^\perp$, $C$ is called self-orthogonal, and if $C = C^\perp$, we say that $C$ is self-dual. We call a binary code self-complementary if it contains the all-ones vector. Every binary self-dual code is self-complementary.

A self-dual code is doubly even if all codewords have weight divisible by four, and singly even if there is at least one nonzero codeword $v$ of weight $\text{wt}(v) \equiv 2(\text{mod } 4)$. Self-dual doubly even codes exist only if $n$ is a multiple of eight.

The Hermitian inner product on $\mathbb{F}_4^n$ is given by $u.v = \sum_{i=1}^{n} u_i v_i^2$ and we denote by $C^\perp_H$ the dual of $C$ under Hermitian inner product. $C$ is Hermitian self-dual if $C = C^\perp_H$.

The weight enumerator $W(y)$ of a code $C$ is defined as $W(y) = \sum_{i=0}^{n} A_i y^i$, where $A_i$ is the number of codewords of weight $i$ in $C$. Following [8] we say that two linear codes $C$ and $C'$ are permutation equivalent if there is a permutation of coordinates which sends $C$ to $C'$. The set of coordinate permutations that maps a code $C$ to itself forms a group denoted by $\text{PAut}(C)$. Two codes $C$ and $C'$ of the same length over $\mathbb{F}_q$ are equivalent provided there is a monomial matrix $M$ and an automorphism $\gamma$ of the field such that $C = C'M^\gamma$. The field $\mathbb{F}_4$ has an automorphism $\gamma$ given by $\gamma(x) = x^2$.

The set of monomial matrices that maps $C$ to itself forms the group $\text{MAut}(C)$ called the monomial automorphism group of $C$. The set of maps of the form $M^\gamma$, where $M$ is a monomial matrix and $\gamma$ is a field automorphism, that sends $C$ to itself, forms the group $\text{GAut}(C)$, called the automorphism group of $C$. In the binary case all three groups are identical. In general, $\text{PAut}(C) \subseteq \text{MAut}(C) \subseteq \text{GAut}(C)$.

An automorphism $\sigma \in S_n$, $|\sigma| = p^2$ is of type $p^2-(c,t,f)$ if when decomposed to independent cycles it has $c$ cycles of length $p^2$, $t$ cycles of length $p$, and $f$ fixed
points. Obviously, \( n = cp^2 + tp + f \).

This paper is organized in the following way. First in Section 2 we introduce to the reader the main results about the method we use. Section 3 shows the application of the method and the construction of six new binary self-dual [76, 38, 14] codes.

## 2 Construction Method

In [4] a method for constructing binary self-dual codes having an automorphism of order \( p^2 \), where \( p \) is an odd prime, was presented. We consider the case \( p = 3 \).

Let \( C \) be a self-dual [76, 38, 14] code having an automorphism \( \sigma \) of type \( 9-(c, t, f) \). In [3] (Lemma 6) it is proved that \( \sigma \) is of type \( 9-(8, 0, 4) \), i.e. \( c = 8 \), \( t = 0 \) and \( f = 4 \). Thus we have

\[
\sigma = (1, 2, \ldots, 9)(10, 11, \ldots, 18) \ldots (64, 65, \ldots, 72)(73) \ldots (76).
\]

(1)

Denote by \( \Omega_i \), \( i = 1, \ldots, 12 \) the cycles in \( \sigma \). Define

\[
F_{\sigma}(C) = \{ v \in C \mid v\sigma = v \},
\]

\[
E_{\sigma}(C) = \{ v \in C \mid \text{wt}(v|\Omega_i) \equiv 0 \pmod{2} \},
\]

where \( v|\Omega_i \) denotes the restriction of \( v \) to \( \Omega_i \). Clearly \( v \in F_{\sigma}(C) \) iff \( v \in C \) is constant on each cycle. Denote \( \pi : F_{\sigma}(C) \rightarrow F_2^{12} \) the projection map where if \( v \in F_{\sigma}(C) \), \( (\pi(v))_i = v_j \) for some \( j \in \Omega_i \), \( i = 1, \ldots, 12 \). Then the following lemma holds.

**Lemma 1.** [4] \( C = F_{\sigma}(C) \oplus E_{\sigma}(C) \). \( C_{\pi} = \pi(F_{\sigma}(C)) \) is a binary self-dual code of length 12.

Thus each choice of the codes \( F_{\sigma}(C) \) and \( E_{\sigma}(C) \) determines a self-dual code \( C \). So for a given length all self-dual codes with an automorphism \( \sigma \) can be obtained.

Denote with \( E_{\sigma}(C)^* \) the subcode \( E_{\sigma}(C) \) with the last 4 zero coordinates deleted. \( E_{\sigma}(C)^* \) is a self-orthogonal binary code of length 8.3^2 = 72 and dimension \( \frac{3}{2}(3^2 - 1) = 32 \). For \( v \in E_{\sigma}(C)^* \) we let \( v|\Omega_i = (v_0, v_1, \ldots, v_8) \) correspond to the polynomial \( v_0 + v_1x + \cdots + v_8x^8 \) from \( \mathcal{T} \), where \( \mathcal{T} \) is the ring of even-weight polynomials in \( F_2[x]/(x^9 - 1) \). Thus we obtain the map \( \varphi : E_{\sigma}(C)^* \rightarrow \mathcal{T}^8 \). Denote \( C_{\varphi} = \varphi(E_{\sigma}(C)^*) \).

Let \( e_1 = x^8 + x^7 + x^5 + x^4 + x^2 + x \) and \( e_2 = x^6 + x^3 \). In our work [4] we proved that \( \mathcal{T} = I_1 \oplus I_2 \), where \( I_1 = \{ 0, e_1, \omega = xe_1, \overline{\omega} = x^2e_1 \} \) is a field with identity \( e_1 \) and \( I_2 \) is a field with \( 2^6 \) elements with identity \( e_2 \). The element \( \alpha = (x + 1)e_2 \) is a primitive element in \( I_2 \) so \( I_2 = \{ 0, \alpha^k, 0 \leq k \leq 62 \} \).

The following theorem is from [3].
Theorem 2. \[ C_\varphi = M_1 \oplus M_2, \text{ where } M_j = \{ u \in E_\sigma(C) \mid u_i \in I_j, i = 1, \ldots, 8 \}, \quad j = 1, 2. \] Moreover \( M_1 \) and \( M_2 \) are Hermitian self-dual codes over the fields \( I_1 \) and \( I_2 \), respectively. If \( C \) is a binary self-dual code having an automorphism \( \sigma \) of type \( (1) \) then \( E_\sigma(C)^* = E_1 \oplus E_2 \) where \( M_i = \varphi(E_i), i = 1, 2. \)

This proves that \( C \) has a generator matrix of the form
\[
G = \begin{bmatrix}
\varphi^{-1}(M_2) & 0 & 0 & 0 & 0 \\
\varphi^{-1}(M_1) & 0 & 0 & 0 & 0 \\
F_\sigma
\end{bmatrix}.
\] (2)

Let \( B_s \) and \( E_s \) denote the number of words of weight \( s \) in \( E_\sigma(C)^* \) and \( E_\sigma(C)^* \), respectively. Every word of weight \( s \) in \( E_\sigma(C)^* \) is in an orbit of length 3, therefore, \( E_s \equiv 0 \pmod{3} \) and \( A_s \equiv B_s \pmod{3} \) for \( 1 \leq s \leq n. \)

Since the minimum distance of \( C \) is 16 the code \( M_2 \) is a \([8, 4]\) Hermitian self-dual code over \( \mathbb{F}_{64} \), having minimal distance \( d \geq 4 \). Using Singleton bound \( d \leq n - k + 1 \) we have \( d = 5 \) or \( d = 4 \). The case \( d = 5 \) is studied in [3] and there are exactly 96 MDS Hermitian \([8, 4, 5]\) codes such that the minimum distance of \( \varphi^{-1}(M_2) \) is 16. The case for the near MDS codes is completed in [10] and the number of the codes is 26 and we state the following.

Theorem 3 ([3], [10]). Up to equivalence, there are exactly 122 Hermitian \([8, 4]\) self-dual codes such that the minimum distance of \( \varphi^{-1}(M_2) \) is 16.

We denote these codes by \( M_{2,i} \) for \( 1 \leq i \leq 122. \) Their generator parameters can be obtained from [10].

We fix the upper part of \( G \) in (2) to be generated by one of the 122 already constructed Hermitian MDS or NMDS \([8, 4]\) codes. Now we continue with construction of the middle part, i.e. the code \( M_1. \) Theorem 2 states that \( M_1 \) is a quaternary Hermitian self-dual \([8, 4]\) code. There exists a unique such code \( e_8 \) with a generator matrix \( Q_1 = \begin{pmatrix}
10001111 \\
01001011 \\
00101101 \\
00011110
\end{pmatrix}. \) We have to put together the two codes from \( M_2 \) and \( M_1 \) in (2), but we have to examine carefully all transformations on \( Q_1 \) that can lead to a different joined code. The full automorphism group of \( e_8 \) is of order \( 2.3^8(8!) \) and we have to consider the following transformations that preserve the decomposition of the code \( C:\)

(i) a permutation \( \tau \in S_8 \) acting on the set of columns.

(ii) a multiplication of each column by a nonzero element \( e_1, \omega \) or \( \overline{\omega} \) in \( I_1. \)
(iii) a Galois automorphism $\gamma$ which interchanges $\omega$ and $\overline{\omega}$.

The action of (i) and (ii) can be represented by a monomial matrix $M = PD$ for a diagonal matrix $D$ and permutational matrix $P$. Since every column of $Q_1$ consists only of 0 and 1 the action of $PD\gamma$ on $Q_1$ can be obtained via $P\overline{D}$. Thus we apply only transformations (i) and (ii).

Denote by $M_1^\tau$ the code determined by the matrix $Q_1$ with columns permuted by $\tau$. To narrow down the computations we can use $\text{PAut}(M_1) = \langle (47)(56), (45)(67), (12)(3586), (24)(68), (34)(78) \rangle$, $|\text{PAut}(M_1)| = 1344$ and the right transversal $T$ of $S_8$ with respect to $\text{PAut}(M_1)$

$T = \{ (), (78), (67), (687), (68), (56), (56)(78), (567), (5678), (5687),
(568), (576), (5786), (57), (578), (57)(68), (5768), (5876), (586), (587), (58),
(5867), (58)(67), (45678), (4568), (4578), (45768), (458), (458)(67) \}$.

For every one of the 122 codes $M_{2,i}$ and $\tau \in T$ we considered $3^8$ possibilities for $\text{gen}(M_1^\tau)$ and checked the minimum distance in the corresponding binary code $E_\sigma(C)^*$. We state the following result.

**Theorem 4.** There are exactly 36659 inequivalent self-orthogonal $[72, 32, 16]$ codes having an automorphism with 8 cycles of order 9.

Denote the codes obtained by $C_{72,i}$, $i = 1, \ldots, 36659$. In Table 1 and Table 2 we summarize the values of the order of the automorphism groups $|\text{Aut}|$ and the number $A_{16}$ of codewords of weight 16 for these codes.

| Table 1: The cardinality of the automorphism groups of the $[72, 32, 16]$ codes |
|-----------------|---|---|---|---|---|---|
| $|\text{Aut}|$ | 9 | 18 | 27 | 36 | 54 | 72 |
| $\#$ of codes  | 35876 | 730 | 24 | 25 | 2 | 2 |
| $A_{16}$ | # | $A_{16}$ | # | $A_{16}$ | # | $A_{16}$ | # | $A_{16}$ | # |
|---|---|---|---|---|---|---|---|---|---|
| 14751 | 1 | 14967 | 454 | 15183 | 806 | 15399 | 207 | 15615 | 27 |
| 14760 | 2 | 14976 | 479 | 15192 | 787 | 15408 | 212 | 15624 | 28 |
| 14769 | 1 | 14985 | 569 | 15201 | 740 | 15417 | 180 | 15633 | 26 |
| 14778 | 2 | 14994 | 598 | 15210 | 798 | 15426 | 193 | 15642 | 20 |
| 14787 | 6 | 15003 | 635 | 15219 | 654 | 15435 | 148 | 15651 | 20 |
| 14796 | 8 | 15012 | 722 | 15228 | 674 | 15444 | 161 | 15660 | 8 |
| 14805 | 12 | 15021 | 740 | 15237 | 654 | 15453 | 145 | 15669 | 9 |
| 14814 | 17 | 15030 | 760 | 15246 | 687 | 15462 | 118 | 15678 | 8 |
| 14823 | 25 | 15039 | 764 | 15255 | 615 | 15471 | 127 | 15687 | 15 |
| 14832 | 32 | 15048 | 787 | 15264 | 521 | 15480 | 120 | 15696 | 6 |
| 14841 | 45 | 15057 | 807 | 15273 | 544 | 15489 | 116 | 15705 | 13 |
| 14850 | 62 | 15066 | 826 | 15282 | 503 | 15498 | 102 | 15714 | 9 |
| 14859 | 70 | 15075 | 815 | 15291 | 504 | 15507 | 75 | 15723 | 8 |
| 14868 | 93 | 15084 | 889 | 15300 | 424 | 15516 | 75 | 15732 | 8 |
| 14877 | 127 | 15093 | 910 | 15309 | 446 | 15525 | 68 | 15741 | 7 |
| 14886 | 155 | 15102 | 860 | 15318 | 428 | 15534 | 56 | 15750 | 5 |
| 14895 | 179 | 15111 | 962 | 15327 | 416 | 15543 | 60 | 15759 | 8 |
| 14904 | 213 | 15120 | 832 | 15336 | 385 | 15552 | 48 | 15768 | 9 |
| 14913 | 290 | 15129 | 827 | 15345 | 357 | 15561 | 48 | 15777 | 1 |
| 14922 | 264 | 15138 | 863 | 15354 | 340 | 15570 | 39 | 15786 | 2 |
| 14931 | 317 | 15147 | 862 | 15363 | 345 | 15579 | 48 | 15795 | 5 |
| 14940 | 326 | 15156 | 855 | 15372 | 288 | 15588 | 38 | 15804 | 3 |
| 14949 | 401 | 15165 | 847 | 15381 | 267 | 15597 | 39 | 15813 | 2 |
| 14958 | 419 | 15174 | 784 | 15390 | 233 | 15606 | 30 | 15822 | 3 |
3 Construction of new $[76, 38, 14]$ codes with an automorphism of type $9$-$(8, 0, 4)$

The highest attainable minimum weight for length 76 is 14 and there are three possible weight enumerators and shadows \[7\]:

\[
\begin{align*}
W_{76,1} &= 1 + (4750 - 16\alpha)y^{14} + (79895 + 64\alpha)y^{16} + (915800 + 64\alpha)y^{18} + \cdots \\
S_{76,1} &= \alpha y^{10} + (9500 - 14\alpha)y^{14} + (1831600 + 91\alpha)y^{18} + \cdots \\
(0 \leq \alpha \leq 296)
\end{align*}
\]

\[
\begin{align*}
W_{76,2} &= 1 + 2590y^{14} + 106967y^{16} + 674584y^{18} + \cdots \\
S_{76,2} &= y^2 + 8954y^{14} + 1836865y^{18} + 105664452y^{22} + \cdots
\end{align*}
\]

\[
\begin{align*}
W_{76,3} &= 1 + (4750 + 16\alpha)y^{14} + (80919 - 64\alpha)y^{16} + (905560 - 64\alpha)y^{18} + \cdots \\
S_{76,3} &= y^6 + (-16 - \alpha)y^{10} + (9620 + 14\alpha)y^{14} + (1831040 - 91\alpha)y^{18} + \cdots \\
(-296 \leq \alpha \leq -16)
\end{align*}
\]

There are only three known codes with $\alpha = 0$ for $W_{76,1}$ \[6\], possessing an automorphism of type $19$-$(4, 0)$.

Now $C_\pi$ is a binary self-dual $[12, 6]$ code. Up to equivalence there are three such codes $6i_2, 2i_2 + h_8$ and $d_{12}$ \[7\]. In the case of $6i_2$ we can not fix any point since then there will be a codeword of weight 10 in $C$. When $C_\pi \cong 2i_2 + h_8$ we have to take the four fixed points from the $h_8$ summand. Since the automorphism group of $h_8$ is 3-transitive we can take any three points from it and we have to choose one more cyclic point from the last five. We checked all five different splits and found a vector in $F_8(C)$ with weight $d < 14$. Lastly, when $C_\pi \cong d_{12}$, for every 4-weight codeword we have to choose at least two coordinates from its support.

The code $d_{12}$ possesses a cluster \{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\} so we have to choose the four fixed points from different duads. Up to a permutation of the cyclic points or a permutation of the fixed points we have a unique generating matrix

\[
G_2 =
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

for the code $C_\pi$.

By Q-extensions \[2\] we obtained $G'' = \langle (1, 2), (2, 4, 3)(5, 7)(6, 8), (5, 6)(7, 8) \rangle$ the subgroup of the symmetric group $S_8$ that preserves the code generated by $G_2$. 

7
The group $G''$ has cardinality 420. To construct a generator matrix of a $[76, 38]$ self-dual code in form (2) we fix a generator matrix of $E_9(C)^*$ and we use the matrix $G_2$ with columns permuted by $\mu$ for all permutations $\mu \in G''$.

Our exhaustive search gives the following result.

**Theorem 5.** Up to equivalence there exist exactly 6 binary self-dual $[76, 38, 14]$ codes with an automorphism of type 9-$(8, 0, 4)$. All codes have weight enumerators $W_{76,1}$ for $\alpha = 4$ or 13 and automorphism groups of order 9.

The generator parameters and the weight enumerator for the six binary self-dual $[76, 38, 14]$ codes, denoted by $C_{76,i}, 1 \leq i \leq 6$, are displayed in Table 3. The notation $\tau, D$ in Table 3 means that we are using the permutation $\tau \in T$ on $M_{1}^{*}$ and then a multiplication of each column by the corresponding element in $D$. Alternatively the generator matrices of the codes $C_{76,i}$ for $i = 1, 2, \ldots, 6$ can be obtained online at "http://shu.bg/tadmin/upload/storage/2599.txt".

| code   | $C_{72,i}$ | $\tau, D$ | $\text{supp}(C_{\tau})$ | $\alpha$ |
|--------|------------|-----------|--------------------------|---------|
| $C_{70,1}$ | 11 | (4, 5, 7, 8), (1, 1, $\bar{\varepsilon}$, $\bar{\varepsilon}$, $\omega$, 1, 1, $\omega$) | $\{1, 3, 9, 10\}, \{3, 5, 10, 11\}, \{5, 8, 11, 12\}, \{2, 7, 8, 12\}, \{2, 4, 6, 7\}, \{1, 3, 5, 6, 7, 8\}$ | 4 |
| $C_{70,2}$ | 11 | (4, 5, 7, 8), (1, 1, $\bar{\varepsilon}$, $\bar{\varepsilon}$, $\omega$, 1, 1, $\omega$) | $\{1, 3, 9, 10\}, \{3, 5, 10, 11\}, \{5, 8, 11, 12\}, \{2, 7, 8, 12\}, \{2, 4, 6, 7\}, \{1, 3, 4, 5, 7, 8\}$ | 4 |
| $C_{70,3}$ | 36 | (4, 5, 8), (1, $\bar{\varepsilon}$, $\bar{\varepsilon}$, $\bar{\varepsilon}$, $\bar{\varepsilon}$, $\omega$, 1, $\omega$) | $\{2, 4, 9, 10\}, \{4, 6, 10, 11\}, \{6, 8, 11, 12\}, \{1, 7, 8, 12\}, \{1, 3, 5, 7\}, \{2, 4, 5, 6, 7, 8\}$ | 4 |
| $C_{70,4}$ | 36 | (4, 5, 8), (1, $\bar{\varepsilon}$, $\bar{\varepsilon}$, $\bar{\varepsilon}$, $\bar{\varepsilon}$, $\omega$, 1, $\omega$) | $\{2, 4, 9, 10\}, \{4, 6, 10, 11\}, \{6, 8, 11, 12\}, \{1, 7, 8, 12\}, \{1, 3, 5, 7\}, \{2, 3, 4, 6, 7, 8\}$ | 4 |
| $C_{70,5}$ | 106 | (4, 5, 6, 7, 8), (1, $\omega$, 1, $\omega$, $\omega$, $\omega$, $\omega$, 1) | $\{3, 4, 9, 10\}, \{4, 5, 10, 11\}, \{5, 7, 11, 12\}, \{1, 6, 7, 12\}, \{1, 2, 6, 8\}, \{3, 4, 5, 6, 7, 8\}$ | 13 |
| $C_{70,6}$ | 106 | (4, 5, 6, 7, 8), (1, $\omega$, 1, $\omega$, $\omega$, $\omega$, $\omega$, 1) | $\{3, 4, 9, 10\}, \{4, 5, 10, 11\}, \{5, 7, 11, 12\}, \{1, 6, 7, 12\}, \{1, 2, 6, 8\}, \{2, 3, 4, 5, 6, 7\}$ | 13 |

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