Limit theorems for maximum flows on a lattice

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Abstract

We independently assign a non-negative value, as a capacity for the quantity of flows per unit time, with a distribution $F$ to each edge on the $\mathbb{Z}^d$ lattice. We consider the maximum flows through the edges of two disjoint sets, that is from a source to a sink, in a large cube. In this paper, we show that the ratio of the maximum flow and the size of source is asymptotic to a constant. This constant is denoted by the flow constant.

1 Introduction of the model and results.

We consider the $\mathbb{Z}^d$ lattice, $d \geq 2$, with integer vertices and edges between $u = (u_1, \cdots, u_d)$ and $v = (v_1, \cdots, v_d)$ when

$$\sum_{i=1}^{d} |u_i - v_i| = 1.$$ 

Two vertices $u$ and $v$ with an edge connecting them are said to be $\mathbb{Z}^d$-adjacent or $\mathbb{Z}^d$-connected. The edge is identified as a $\mathbb{Z}^d$-edge $e = (u, v)$, or simply, an edge, with the open line segment in $\mathbb{R}^d$ from $u$ to $v$. Two vertices $u$ and $v$ are said to be $\mathbb{L}^d$-adjacent or $\mathbb{L}^d$-connected if

$$\max_{1 \leq i \leq d} |u_i - v_i| = 1.$$ 

Clearly, if $u$ and $v$ are $\mathbb{Z}^d$-connected, then they are also $\mathbb{L}^d$-connected.

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For two vertices $u$ and $v$, let $\text{dist}(u, v)$ be the Euclidian distance between the two vertices. For any two vertex sets $A, B \subset \mathbb{Z}^d$, the distance between $A$ and $B$ is also defined by

$$\text{dist}(A, B) = \min\{\text{dist}(u, v) : u \in A \text{ and } v \in B\}.$$ 

Now we assign independently to each $\mathbb{Z}^d$-edge $e$ a non-negative value $\tau(e)$ with a distribution $F$. More formally, we consider the following probability space. As sample space, we take $\Omega = \prod_{e \in \mathbb{Z}^d}[0, \infty)$, points of which are represented as configurations. If we want to emphasize a particular configuration $\omega$, for a random set $A$ or a random variable $N$, we may write $A(\omega)$ or $N(\omega)$ for each, respectively. Let $P$ be the corresponding product measure on $\Omega$. The expectation with respect to $P$ is denoted by $E(\cdot)$. For simplicity, we assume that $\tau(e)$ has a short tail

$$E\exp(\eta \tau(e)) = \int_0^{\infty} e^{\eta x} dF(x) < \infty$$

for some $\eta > 0$. For each finite graph $B$, we may think of $\tau(e)$ as the non-negative capacity for the quantity of fluid that may flow along $e \in B$ in unit time, where an edge in a set means that the two vertices of the edge belong to the set. Let $S$ and $T$ be two disjoint sets in $B$, called the source and the sink. A flow (see Kesten (1988); Grimmett (1999)), from a vertex set $S$ to another vertex set $T$ in $B$, is an assignment of a non-negative number $f(e)$ and an orientation to each edge $e = (v, w)$ of $B$ such that

$$I(v) = \sum_{w \in B : v \rightarrow w} f((v, w)) - \sum_{w \in B : w \rightarrow v} f((v, w))$$

satisfies $I(v) = 0$ for all vertices $v \notin S \cup T$, where the first summation (with respect to the second summation) is calculated over all neighbors $w$ of $v$, which $e(v, w)$ is oriented away from (respectively toward) $v$. Thus fluid is conserved at all vertices except, possibly, at sources and sinks. In other words, the current flowing into a vertex $v \notin S \cup T$ must equal to the current flowing out. This basic assumption is called Kirchhoff’s law in physics. A flow is admissible if

$$f(e) \leq \tau(e) \text{ for all edges } e,$$

and the value of such a flow is defined to be $\sum_{v \in S} I(v)$, the aggregate amount of fluid entering $B$ at source vertices. The maximum flow is the largest value of all admissible flows. One of the fundamental questions of this physics topic concerns understanding how the maximum flow depends on the source and sink. It is believed (see Kesten (1988); Grimmett (1999)) that the maximum flow approximately equals the “size” of $\min\{|S|, |T|\}$ with a certain ratio for a convex set $B$, where $|A|$ is the number of vertices in $A$. We write the ratio as the flow constant, which only depends on $F$. The main purpose of this paper is to demonstrate the existence of the flow constant.
To study the maximum flow, we need to understand cutsets. To define a cutset from \( S \) to \( T \) on \( B \), we may first define a path on \( \mathbb{Z}^d \) as follows. For any two vertices \( u \) and \( v \) of \( \mathbb{Z}^d \), a \( \mathbb{Z}^d \)-path, or simply a path, \( \gamma \) from \( u \) to \( v \) is an alternating sequence \((v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)\) of vertices \( v_i \) and edges \( e_i = (v_{i-1}, v_i) \) in \( \mathbb{Z}^d \), with \( v_0 = u \) and \( v_n = v \). \( u \) and \( v \) are called \( \mathbb{Z}^d \)-connected. A \( \mathbb{Z}^d \)-connected vertex set is called a cluster. An edge set \( X \) of \( B \) is called an \( S \)-\( T \) cutset if all paths on \( B \) from \( S \) to \( T \) use at least one edge of \( X \). For convenience, we also add all the vertices of edges in \( X \) to have an edge and a vertex set. We still denote the set by \( X \).

A cutset \( X \) is said to be self-avoiding if \( X \) is a cutset, and \( X \setminus \{e\} \) will no longer be a cutset for every \( e \in X \). Note that there might be many self-avoiding cutsets. For any edge set \( E \), we denote the passage time of \( E \) by \( \tau(E) = \sum_{e \in E} \tau(e) \).

One of the fundamental problems in percolation is to study the cutset. Cutsets are also related to the boundary of clusters. For each edge \( e \), it is said to be open or closed if \( \tau(e) > 0 \) or \( \tau(e) = 0 \). Clearly, for each \( e \),
\[
P[e \text{ is closed}] = F(0), \quad \text{and} \quad P[e \text{ is open}] = 1 - F(0) = p.
\]

Note that if \( e \) is closed, its passage time is zero, so we also sometimes denote it as a zero-edge. Let \( C(x) \) be an open cluster containing \( x \) and let
\[
\theta(p) = P[|C(0)| = \infty], \quad \text{and} \quad p_c = p_c(d) = \sup\{p : \theta(p) = 0\}.
\]

If \( F(0) < 1 - p_c \), there exists an infinite open cluster from the origin with a positive probability. If \( |C(0)| \) is finite, there exists a closed cutset that cuts the origin from \( \infty \). An edge \( e \) is called the boundary edge of \( C(0) \) if \( e \not\in C(0) \), but \( e \) is \( \mathbb{Z}^d \)-adjacent to \( C(0) \). \( \Delta C(0) \) is defined as all the boundary edges of \( C(0) \). If \( |C(0)| \) is finite, then \( \Delta C(0) \) is a finite closed cutset. Here we define the more general boundary edges of open clusters, starting at a large set. Let
\[
B(k, m) = \prod_{i=1}^{d-1} [0, k_i] \times [0, m] \quad \text{for} \quad k = (k_1, \cdots, k_{d-1}).
\]
We may also assume without loss of generality that
\[
0 \leq k_1 \leq k_2 \leq \cdots \leq k_{d-1}. \quad (1.2)
\]
When \( k_1 = k_2 = \cdots = k_{d-1} = m = 0 \), \( B(k, m) \) is the origin. We also denote the volume of \([0, k_1] \times \cdots [0, k_{d-1}]\) by
\[
\|k\|_v = k_1 \times k_2 \times \cdots \times k_{d-1}.
\]
As we defined above, a set is said to be a cutset that cuts \( B(k, m) \) from \( \infty \) if any path from \( B(k, m) \) to \( \infty \) uses at least one edge of the set. We select, from these cutsets, a cutset \( X(k, m) \) with the minimum passage time among all the cutsets. We also denote by \( \chi(k, m) \) the passage time of \( X(k, m) \):

\[
\tau(X(k, m)) = \chi(k, m).
\]

There might be many such cutsets. If so, we select the one with the minimum number of edges among all such cutsets by using a unique method. We still denote it by \( X(k, m) \). With this selection, \( X(k, m) \) must be self-avoiding. In this paper, the unique method of selecting cutsets always involves using the same selection rule for each configuration.

Furthermore, a set \( X(k, m) \) is said to be a zero-cutset (or closed cutset) that cuts \( B(k, m) \) from \( \infty \) if

\[
\tau(X(k, m)) = 0.
\]

In this case, any path from \( B(k, m) \) to \( \infty \) must use at least one closed edge of the set. In other words, there is no open path from \( B(k, m) \) to \( \infty \). Let \( N(k, m) \) be the number of edges in \( X(k, m) \). We have the following fundamental geometric theorem to show that \( N(k, m) \) cannot be much larger than \( \|k\|_v \) when \( F(0) < 1 - p_c \).

**Theorem 1.** If \( F(0) < 1 - p_c \), then there exist constants \( \beta = \beta(F,d) \) and \( C_i = C_i(F,\beta,d) \) for \( i = 1, 2 \) such that for all \( n \geq \beta\|k\|_v \) and \( m \leq \min_{1 \leq i \leq d-1} k_i \),

\[
P[N(k, m) \geq n] \leq C_1 \exp(-C_2 n).
\]

In this paper, we always denote by \( C \) or \( C_i \) a large or a small positive constant that will be used for some upper or lower bound in inequalities. \( C \) and \( C_i \) do not depend on \( k_1, \ldots, k_{d-1}, m, l, n, w_1, \ldots, w_{d-1} \). In addition, both values of \( C \) and \( C_i \) may change from appearance to appearance. For a finite open cluster \( C(0) \), its exterior boundary edges are the edges in \( \Delta C(0) \) such that there is a \( \mathbb{Z}^d \)-path from the vertices of the edges to \( \infty \) without using any edges of \( C(0) \). We denote by \( \Delta_e C(0) \) the exterior boundary of \( C(0) \). Kesten and Zhang (1990) showed that there exists a constant \( \sigma \) such that

\[
\lim_{n \to \infty} -n^{-1} \log P[|\Delta_e C(0)| = n] = \sigma(F(0)).
\]

By (6.18) in Grimmett (1999), we know that

\[
\sigma(1 - p_c) = 0. \tag{1.3}
\]

It also follows from (6.13) in Grimmett (1999) that if \( F(0) > 1 - p_c \), then

\[
\sigma(F(0)) > 0. \tag{1.4}
\]
It is natural to ask whether (1.4) holds when \( F(0) < 1 - p_c \). Note that \( \Delta_e C(0) \) is a closed cutset for \( C(0) \), so on the event that there is no infinite open cluster,

\[
\text{the number of edges in } \Delta_e C(0) \geq N(0,0).
\]

However, the size of \( N(0,0) \) may be much less than the size of \( \Delta_e C(0) \) when \( C(0) \) is finite. Thus, we still do not know whether \( \sigma(F(0) > 0 \text{ when } F(0) < 1 - p_c \).

Now we focus on a specific cutset. In fact, one of most interesting questions (see Kesten (1988); Grimmett (1999)) is to understand the behavior of the cutsets on \( B(k, m) \) that cut its bottom face from its top face. We denote by

\[
F_0 = F_0(k,m) = \{(x_1, \ldots, x_d) \in B(k,m) : x_d = 0\}
\]

and

\[
F_m = F_m(k,m) = \{(x_1, \cdots, x_d) \in B(k,m) : x_d = m\}
\]

the bottom and the top faces of the box, respectively. We select \( W(k, m) \) as a cutset, cutting the bottom face of \( B(k,m) \) from its top face, with minimal passage time. Similarly, if there is more than one such cutsets, we use a unique method to select one with the minimum number of edges among all such cutsets. We still denote it by \( W(k,m) \). Let \( \bar{N}(k,m) \) be the number of vertices in cutset \( W(k,m) \). We now show the fundamental geometric theorem for this cutset.

**Theorem 2.** If \( F(0) < 1 - p_c \), and if \( m = m(k_1, \cdots, k_{d-1}) \to \infty \) as \( k_1, k_2, \cdots, k_{d-1} \to \infty \) in such a way that

\[
\log m \leq \|k\|_v,
\]

then there exist constants \( \beta = \beta(F,d) \geq 1 \) and \( C_i = C_i(F, \beta, d) \) for \( i = 1,2 \) such that for all \( n \geq \beta\|k\|_v \),

\[
P[\bar{N}(k,m) \geq n] \leq C_1 \exp(-C_2n).
\]

**Remark 1.** In the proof of Theorem 2, we can use a weak condition that \( \log m \leq C\|k\|_v \) to replace (1.5).

**Remark 2.** In Theorems 1 and 2, we consider a cutset that cuts \( B(k, m) \) from \( \infty \) or from \( F_0 \) to \( F_m \). The same proof can be shown for a general set rather than \( B(k,m) \).

With Theorem 2, the number of vertices for each cutset is proportional to the size of \( F_0 \). We call the results in Theorem 2 the **linearity**. When \( F(0) > 1 - p_c \), it is known (see
chapter 6 in Grimmett (1999)) that Theorems 1 and 2 hold. On the other hand, it is known by Aizenman et al. (1983) that if $F(0) < 1 - p_c$ and $m$ satisfies (1.5), then

$$\mathbb{P}[\exists \, W(k, m) \text{ with } \tau(W(k, m)) = 0] \leq \exp(-C\|k\|_v).$$

(1.7) is called the area law. Clearly, $\bar{N}(k, m)$ is always larger than $\|k\|_v$. Therefore, Theorem 2 implies the area law. In fact, we may view the cutset (see Aizenman et al. (1983)) as a surface between $F_m$ and $F_0$. (1.6) tells us that it costs probability $\exp(-Ct)$ whenever the surface increases $t$ units. We call (1.6) the surface law. The surface law has proved to hold (Kesten (1986) and (1988)) when $d = 2$ and $F(0) < 1 - p_c(2)$, and when $d = 3$ and $F(0) < 1/27$. As the main conjecture, Kesten believed that the surface law should hold for all $d$ and all $F(0) < 1 - p_c(d)$. In Theorem 2, we answer Kesten’s conjecture affirmatively.

When $F(0) = 1 - p_c$, the closed cutsets are very chaotic. For example, we believe that $\bar{N}(k, m)$ should be much larger than $\|k\|_v$.

Now we focus on the maximum flow problem to discuss the existence of the flow constant. Without loss of generality (see Kesten (1988); Grimmett (1999)), we discuss the maximum flow on $B(k, m)$ from $F_0$ to $F_m$. The max-flow min-cut theorem characterizes the maximum flow through the network in terms of the sizes of cutsets. The size of the $(F_0, F_m)$-cutset $W(k, m)$ is defined to be the sum of the capacities of edges in $W(k, m)$. As we mentioned, one of fundamental questions (see Kesten (1988); Grimmett (1999)) is how to understand the limit behavior of the flow from $F_0$ in $B(k, m)$. Let $\phi_{\text{max}}(k, m)$ denote the maximum flow through the edges of $Z^d$ in $B(k, m)$ from $F_0$ to $F_m$. Let

$$\tau_{\text{min}}(k, m) = \tau(W(k, m)).$$

By the max-flow min-cut theorem, we have

$$\tau_{\text{min}}(k, m) = \phi_{\text{max}}(k, m).$$

In particular, if $\tau(e)$ only takes 0 or 1, the maximal flow $\phi_{\text{max}}(k, m)$ is the number of disjoint open paths from $F_0$ to $F_m$ in $B(k, m)$.

With these definitions, let us introduce the developments in this field. When $F(0) = 1 - p_c$, the so-called critical case, it has been proved (see Zhang (2000)) that

$$\lim_{k_1, \ldots, k_{d-1}, m \to \infty} \frac{\tau_{\text{min}}(k, m)}{\|k\|_v} = 0 \text{ a.s. and in } L_1.$$  

(1.8)

When $F(0) > 1 - p_c$, the so-called supercritical case, we also have

$$\lim_{k_1, \ldots, k_{d-1}, m \to \infty} \frac{\tau_{\text{min}}(k, m)}{\|k\|_v} = 0 \text{ a.s. and in } L_1.$$  

(1.9)
In fact, as we mentioned before (see chapter 6 in Grimmett (1999)), with a large probability,

$$\tau_{\text{min}}(k, m) = 0 \text{ when } m \geq k_\delta^{\delta} \text{ for } \delta > 0.$$ 

In other words, the flow constant will vanish in the supercritical and critical cases.

The most interesting case is understanding the limit behavior when $$F(0) < 1 - p_c$$, the subcritical case. With the moment assumption in (1.1), we have

$$\limsup_{k \to \infty} \frac{E\tau_{\text{min}}(k, m)}{\|k\|_v} < \infty. \quad (1.10)$$

In fact, by a standard large deviation estimate, we can show that

$$P(\tau_{\text{min}}(k, m) \geq C\|k\|_v) \leq C_1 \exp(-C_2\|k\|_v). \quad (1.11)$$

On the other hand, it can be shown (see Chayes and Chayes (1986)) that when $$F(0) < 1 - p_c$$,

$$0 < \liminf_{k \to \infty} \frac{E\tau_{\text{min}}(k, m)}{\|k\|_v}. \quad (1.12)$$

With (1.10) and (1.12), it is natural to ask what the limit behavior is. If the limit exists, then the flow constant exists. When $$d = 2$$, Grimmett and Kesten (1984) showed that

$$\lim_{k_1, m \to \infty} (k_1)^{-1}\tau_{\text{min}}(k_1, m) = \nu(F) \text{ a.s. and in } L_1 \quad (1.13)$$

when $$k_1 \to \infty, m \to \infty$$ such that

$$\log m/k \to 0. \quad (1.14)$$

In fact, when $$d = 2$$, the min-cutset is just a dual path from the left to the right in $$B(k_1, m)$$. The techniques to handle paths have been well developed since Hammersley and Welsh created the first passage percolation model in 1965.

When $$d = 3$$, Kesten (1988) used a surface consisting of a plaquette (see Aizenman et al. (1983); Kesten (1988)) to work on the limit behavior of $$\tau_{\text{min}}(k, m)$$. He showed, in an extensive proof, that if the surface law holds, then

$$\lim_{k_1, k_2, m \to \infty} (k_1 \times k_2)^{-1}\tau_{\text{min}}((k_1, k_2), m) = \nu(F) \text{ a.s. and in } L_1 \quad (1.15)$$

when $$k_1, k_2 \to \infty, m(k_1, k_2) \to \infty$$ as $$k_1 \leq k_2$$ in such a way for some $$\delta > 0$$ such that

$$\log m \leq k_1^{1-\delta}. \quad (1.16)$$

Furthermore, he showed that the surface law holds when $$d = 3$$ and $$F(0) < 1/27$$. Therefore, his result implies that the flow constant exists when $$d = 3$$ and $$F(0) < 1/27$$. Kesten conjectured that the surface law should hold for all $$F(0) < 1 - p_c$$. In Theorem 2, we show the
surface law. Thus, the flow constant exists for all $F(0) \leq 1 - p_c$ when $d = 3$. In addition, Kesten also conjectured that the flow constant should exist for all $d \geq 2$. In this paper, we answer the conjectures affirmatively to show the existence of the flow constant for all $F$.

**Theorem 3.** If (1.1) holds, and if $m = m(k_1, \ldots, k_{d-1}) \to \infty$ as $k_1, k_2, \ldots, k_{d-1}, m \to \infty$ in such a way for some $0 < \delta \leq 1$ that

$$
\log m \leq \max_{1 \leq i \leq d-1} \{k_i^{1-\delta}\},
$$

then there exists a flow constant $\nu(F)$ such that

$$
\lim_{k_1, k_2, \ldots, k_{d-1}, m \to \infty} \frac{\tau_{\text{min}}(k, m)}{\|k\|_v} = \lim_{k_1, k_2, \ldots, k_{d-1}, m \to \infty} \frac{\phi_{\text{max}}(k, m)}{\|k\|_v} = \nu(F) \text{ a.s. and in } L_1.
$$

**Remark 3.** In the proof of Theorem 3, we have to use the short tail assumption in (1.1). Recall that Grimmett and Kesten’s proof for Theorem 3, when $d = 2$, only requires that $E(\tau(e))^2 < \infty$. Kesten believes (open problem 2.24 in Kesten (1988)) that the second moment condition should imply Theorem 3. However, we are unable to show the conjecture.

**Remark 4.** As we discussed before, if $\nu(F)$ exists in Theorem 3, $\nu(F)$ is positive if and only if $F(0) < 1 - p_c$. Kesten (1988) also asked the large deviation behaviors for this limit when $F(0) < 1 - p_c$. We will attempt to answer this question in a separate paper. In addition, we can also estimate a convergence rate for the limits in Theorem 3. With this convergence rate, we can show the continuity of the flow constant $\nu(F)$ in $F$. We will also expose these results in the separate paper. Note that the continuity of $\nu(F)$, when $F(0) = 1 - p_c$, is proved by Zhang (2000).

**Remark 5.** Theorem 3 can be generalized to any periodic lattice (see the definition in Kesten (1982)) with the $d$ coordinate axes.

## 2 A construction for a linear cutset.

In this section, we will construct a special zero-, or closed, cutset about the linear size of $\|k\|_v$. Since we only consider the cutset surrounding $B(k, m)$, for convenience, we will assume that all edges inside $B(k, m)$ are open edges in this section. Now our probability measure is on the edges in $\mathbb{Z}^d \setminus B(k, m)$.

For a finite $\mathbb{Z}^d$-connected set $A$, $\partial A$ is a vertex set, called the *boundary* of $A$, that is $L^d$-adjacent to $A$ but is not in $A$. We also denote by $\partial_i A$ the vertex set, called the *interior*
Figure 1: A is a cluster containing edges (dashed lines) and vertices (solid circles). The solid circles are also interior boundary $\partial_i(A)$. $\partial A$ is a vertex set (circles and otimes). $\partial_e A$ is a subset of $\partial A$ (only circles). The four corners in $\partial_e A$ are $L^d$-adjacent to $A$. $\Delta A$ is an edge set (all solid lines), and $\Delta_e A$ is a subset of $\Delta A$ (the edges are not adjacent to otimes).

boundary of $A$, that is in $A$ and is $L^d$-adjacent to $\partial A$. Furthermore, we name $\partial_e A$ as its exterior boundary if its vertex $v \in \partial A$ and there is a $Z^d$-connected path from $v$ to $\infty$ that does not use vertices of $A$ (see Fig. 1). Note that

$$\partial_e A \subset \partial A.$$  

Recall from section 1 that $\Delta A$ and $\Delta_e A$ are defined as the $Z^d$-edges from $\partial A$ and from $\partial_e A$, respectively, to $\partial_i A$. They are called boundary edges and exterior boundary edges. By the definition,

$$\Delta_e A \subset \Delta A.$$ 

In addition, we denote the interior vertex set of $A$ by

$$\text{int}(A) = A \setminus \partial_i A.$$ 

With these definitions, the following lemma is well known (see Lemma 2.23 Kesten (1986)).

**Lemma 1** (Kesten). If $A$ is a $Z^d$-connected finite set, then $\partial_e A$ is a $Z^d$-connected graph.
Let
\[ C(k, m) = \{ v \in \mathbb{Z}^d : v \text{ is } \mathbb{Z}^d\text{-connected by an open path to } B(k, m) \}. \]

Note that \( B(k, m) \) is a \( \mathbb{Z}^d \)-connected open set as we defined, so
\[ C(k, m) \text{ is a } \mathbb{Z}^d\text{-connected open cluster, and } B(k, m) \subset C(k, m). \quad (2.0) \]

As we have defined, if there is a cutset that cuts \( B(k, m) \) from \( \infty \), then any path from \( B(k, m) \) to \( \infty \) must use at least an edge of the cutset. Furthermore, if there is a zero-cutset, then a path from \( B(k, m) \) to \( \infty \) must not only use at least an edge of the cutset, but also a zero-edge of the cutset. If there is a zero-cutset, then there is no open path from \( B(k, m) \) to \( \infty \), so
\[ |C(k, m)| < \infty. \quad (2.1) \]

On the other hand, if (2.1) holds, then there exists such a zero-cutset. Let \( G(k, m) \) be the event that (2.1) occurs. In this section, we will always discuss particular fixed configurations in \( G(k, m) \).

For each configuration in \( G(k, m) \), it follows from the definitions that the boundary edges of \( \Delta C(k, m) \) are all closed and they cut \( B(k, m) \) from \( \infty \). However, \( \Delta C(k, m) \) may contain too many extra edges (see Fig. 1), so we would like to focus on \( \Delta_e C(k, m) \). Since \( C(k, m) \) is uniquely determined for each configuration in \( G(k, m) \), \( \partial_e C(k, m) \) (see Fig. 1) is also uniquely determined. With these definitions, we have the following lemma.

**Lemma 2.** For all configurations in \( G(k, m) \), \( \Delta_e C(k, m) \) is a finite zero-cutset cutting \( B(k, m) \) from \( \infty \).

**Proof.** As we mentioned above, \( \Delta_e C(k, m) \) is a zero-edge set. Since each vertex of \( \partial_e C(k, m) \) is \( \mathbb{L}^d \)-connected to \( C(k, m) \), \( \Delta_e C(k, m) \) is finite. It remains to show that \( \Delta_e C(k, m) \) is a cutset. Since \( C(k, m) \) is finite, for any \( \mathbb{Z}^d \)-path \( \gamma \) from \( B(k, m) \) to \( \infty \), \( \gamma \) must be outside of \( C(k, m) \). Let \( u \) be the last vertex in \( C(k, m) \) such that after \( u \), the remaining piece of \( \gamma \) never uses another edge of \( C(k, m) \). Suppose that after \( u \), \( \gamma \) uses the edge \( e \). Thus, \( e \) will be a zero-edge, otherwise, \( e \in C(k, m) \). By the definition, \( e \in \Delta C(k, m) \). On the other hand, the remaining piece of \( \gamma \) from \( e \) will not return to \( C(k, m) \) again as we defined. Thus, \( e \in \Delta_e C(k, m) \). Since any path must use an edge of \( \Delta_e C(k, m) \), \( \Delta_e C(k, m) \) will be a cutset cutting \( B(k, m) \) from \( \infty \), so Lemma 2 follows. \( \square \)

Furthermore, by Lemma 1 (see Fig. 1), we know that
\[ \partial_e C(k, m) \text{ is } \mathbb{Z}^d\text{-connected.} \quad (2.2) \]
By Lemma 2, we know that $\Delta_\epsilon C(k, m)$ is a zero-cutset. However, we cannot use this cutset to show Theorem 1, since this cutset might be too tangled. We need to eliminate the tangled parts of $\Delta_\epsilon C(k, m)$ to construct another zero-cutset. To construct such a zero-cutset, we use the idea of renormalization in Kesten and Zhang (1990). We define, for integer $t \geq 1$ and $u = (u_1, \ldots, u_d) \in \mathbb{Z}^d$, the cube

$$B_t(u) = \prod_{i=1}^d [tu_i, tu_i + t].$$

Here we need to take $t$ large, but much smaller than $m$ and $k_1, \ldots, k_d$. Also, without loss of generality, we assume that $k_i/t$ for $i = 1, 2, \ldots, d - 1$ and $m/t$ are integers. Usually, we consider the $\mathbb{Z}^d$-vertices in $B_t(u)$. In addition, we can also consider the edges in $B_t(u)$ if their two vertices belong to $B_t(u)$.

Two cubes, $B_t(u)$ and $B_t(v)$, for $u \neq v$, are said to be $\mathbb{Z}^d$-adjacent or $L^d$-adjacent if $u$ and $v$ are $\mathbb{Z}^d$- or $L^d$-adjacent. If $(v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n)$ is a $\mathbb{Z}^d$-path, then $B_t(v_0), B_t(v_1), \ldots, B_t(v_n)$ is a cubic $\mathbb{Z}^d$-path. With these cubic paths, we can define a cubic $\mathbb{Z}^d$-cluster. Similarly, if we replace the $\mathbb{Z}^d$-path by a $L^d$-path, we can define a cubic $L^d$-cluster. Let $C_t(k, m)$ be a cube cluster defined as

$$C_t(k, m) = \{ B_t(u) : B_t(u) \cap C(k, m) \neq \emptyset \}.$$ 

Note that $C(k, m)$ is $\mathbb{Z}^d$-connected and that our cubes contain their inside boundaries, so $C_t(k, m)$ is $\mathbb{Z}^d$-connected in the sense of the connection of cubes. A boundary cube of $C_t(k, m)$ is also defined as (see Fig. 2)

$$\partial C_t(k, m) = \{ B_t(u) : B_t(u) \text{ contains a vertex of } \partial C(k, m) \cup \partial \bar{C}(k, m) \}.$$ 

Note that $C_t(k, m)$ and $\partial C_t(k, m)$ may have a common cube. For convenience, we account $B_t(u)$ in $\partial C_t(k, m)$ if $B_t(u)$ is a common cube in both $C_t(k, m)$ and $\partial C_t(k, m)$. Note also that $\partial C_t(k, m)$ contains all boundary edges of $C(k, m)$.

As we proved, on $G(k, m)$, $\Delta_\epsilon C(k, m)$ is a zero-cutset. Therefore,

there is a zero-cutset in $\partial C_t(k, m)$. \hfill (2.3)

In addition, the exterior cube-boundary of $C_t(k, m)$ is defined as (see Fig. 2)

$$\partial_e C_t(k, m) = \{ B_t(u) \in \partial C_t(k, m) : B_t(u) \text{ is connected by a cubic } \mathbb{Z}^d \text{-path to } \infty \text{ outside } \partial C_t(k, m) \}.$$ 

Note that each cube in $\partial_e C_t(k, m)$ contains at least one vertex of $\partial_e C(k, m)$. Thus, by (2.2),

$\partial_e C_t(k, m)$ is $\mathbb{Z}^d$-connected. \hfill (2.4)
If we consider $\partial_e C_t(k, m)$ as an edge set, it follows from the same proof of Lemma 2 to show that

$$\partial_e C_t(k, m)$$

is a finite cutset that cuts $B(k, m)$ from $\infty$. (2.5)

By (2.5), we know that $\partial_e C_t(k, m)$ is a cutset. On the other hand, as the main task, Kesten and Zhang (1990) showed that $\partial_e C_t(k, m)$ is not very tangled. However, unlike $\partial C_t(k, m)$, $\partial_e C_t(k, m)$ may not contain a zero-cutset (see Fig. 2). The main task in this section is to combine $\partial_e C_t(k, m)$ with additional edges to construct a zero-cutset. This construction is much easier to understand through Fig. 2, than through rigorous written descriptions. We suggest that readers refer to Fig. 2 while reading the following definitions.

For configurations in $G(k, m)$, $\partial_e C_t(k, m)$ is a cutset. Each path from $\infty$ to $B(k, m)$ must meet a vertex of $\partial_e C_t(k, m)$ and then go to $\partial B(k, m)$ from the vertex. We name these vertices the *surface* of $\partial_e C_t(k, m)$ and denote them by

$$U(\partial_e C_t(k, m)) = \{v \in \partial_e C_t(k, m) : \exists \text{ a } \mathbb{Z}^d\text{-path from } v \text{ to } \infty \text{ without using vertices, except } v, \text{ in } \partial_e C_t(k, m)\}.$$ 

By (2.5) and the definition of the surface,

$$U(\partial_e C_t(k, m)) \cap C(k, m) = \emptyset.$$ (2.6)

Since $\partial_e C_t(k, m)$ cuts $B(k, m)$ from $\infty$, it divides $\mathbb{R}^d$ into two parts: the inside and outside parts, where the inside part, containing $B(k, m)$, is enclosed by $\partial_e C_t(k, m)$. We denote the inside part by

$$L_t(k, m) = \{B_t(u) : \text{any cubic } \mathbb{Z}^d\text{-path from } B_t(u) \text{ to } \infty \text{ must use a cube of } \partial_e C_t(k, m)\}.$$ 

Furthermore, we denote by

$$\bar{L}_t(k, m) = L_t(k, m) \cup \partial_e C_t(k, m).$$

Next, we will focus on the cubes in $L_t(k, m)$ that contain neither a vertex of $\partial C(k, m)$ nor a vertex of $C(k, m)$. More precisely, let (see Fig. 2)

$$Q'_t(k, m) = \{B_t(u) \in L_t(k, m) : B_t(u) \cap [\partial C(k, m) \cup C(k, m)] = \emptyset\}.$$ 

Note that $Q'_t(k, m)$ may be empty. If it is empty, then we will stop our process. We collect all the cubic $L^d$-clusters in $Q'_t(k, m)$ to denote them by

$$Q'_t(k, m, 1), Q'_t(k, m, 2), \cdots, Q'_t(k, m, \tau').$$ (2.7)
Figure 2: This graph shows how to construct linear cubes that contain a zero-, or closed, cutset. The large dotted line, $\partial_e C(k,m)$, is a zero-boundary for open cluster $B(k,m)$. The cubes $\{B_t(u)\}$ that contain $\partial_e C(k,m)$ are $\partial C_t(k,m)$. Part of the cubes from $\partial C_t(k,m)$ enclose a cubic circuit $\partial_e C_t(k,m)$. The circuit, $\partial e C_t(k,m)$, divides $\mathbb{Z}^d$ into two parts: the inside part including $B(k,m)$ and the outside part. There are three ponds $Q_t(k,m,i)$ for $i = 1, 2, 3$ in the inside part. Two are live ponds, named $Q_t(k,m,1)$ and $Q_t(k,m,2)$, but the third pond is dead. On the cubic circuit, there are three exits such that open paths (boldfaced lines) in $S(k,m)$ penetrate $\partial_e C_t(k,m)$ from the outside. The first exit to a pond is blocked by closed edges, so the pond is called a dead pond. In exit 2, an open cluster (boldfaced lines) in $S(k,m)$ goes into a good component containing live ponds $Q_t(k,m,1)$ and $Q_t(k,m,2)$, where $Q(k,m,1)$ is connected to $Q(k,m,2)$ by an open cluster in $S(k,m)$. In fact, the open cluster can further extend to the inside of the live ponds (lightfaced lines), but we ignore them. Exit 3 does not connect to a pond.
We will name the cubic $L^d$-clusters **ponds** (see Fig. 2). Note that $L_t(k, m)$ only contains three kinds of cubes: $\partial C_t(k, m)$, $Q'_t(k, m)$, and $C_t(k, m)$.

Let us analyze the geometric structure of the ponds. For a pond $Q'_t(k, m, i)$, let $\partial_t Q'_t(k, m, i)$ be its exterior boundary cubes (see Fig. 2). More precisely, these are the cubes that do not belong to $Q'_t(k, m, i)$, but are $L^d$-adjacent to $Q'_t(k, m, i)$. Furthermore, there exists a cubic $Z^d$-connected path from each boundary cube to $\infty$ without using the cubes of $Q'_t(k, m, i)$. By Lemma 1 in Kesten and Zhang (1990), we know that

$$\partial_t Q'_t(k, m, i) \text{ is } Z^d\text{-connected.} \quad (2.8)$$

We want to remark that $\partial_t Q'_t(k, m, i)$ and $\partial_t Q'_t(k, m, j)$ may have common cubes even though

$$Q'_t(k, m, i) \cap Q'_t(k, m, j) = \emptyset.$$  

We also want to remark that a cube is in $\partial_t Q'_t(k, m, i)$, but it may also be in $\partial_t C_t(k, m)$. Note that $Q'_t(k, m, i) \cap [C(k, m) \cup \partial C(k, m)] = \emptyset$, so $\partial_t Q'_t(k, m, i) \cap [C(k, m) \cup \partial C(k, m)] \neq \emptyset$.

Thus, $\partial_t Q'_t(k, m, i)$ either contains a vertex of $\partial C(k, m)$ or all its vertices must be in $C(k, m)$. However, the latter case is impossible; otherwise, the adjacent cube of $B_t(u)$ in the pond would then contain $C(k, m)$. Therefore,

$$\bigcup_i \partial_t Q'_t(k, m, i) \subset \partial C_t(k, m). \quad (2.9)$$

Recall our definition of the inside boundary from the beginning of this section. With this definition, we define all the open clusters in $\tilde{L}(k, m)$ (see Fig. 2) starting from the inside boundary of the ponds and $U(\partial_t C_t(k, m))$ without using edges of ponds. More precisely, let $D'_1, \ldots, D'_\nu$ be all open clusters in $\tilde{L}(k, m) \setminus \bigcup_i \text{int}(Q'_t(k, m, i))$. Now we only collect the open clusters from $D'_1, \ldots, D'_\nu$ such that they contain at least a vertex of the inside boundary of $Q'_t(k, m, i)$ or a vertex of $U(\partial_t C_t(k, m))$, but we ignore the other open clusters (see Fig. 2). We denote the collected open clusters by $D'_1, \ldots, D'_t$. Let

$$S'(k, m) = \bigcup_{i=1}^t D_i;$$

$$S'_t(k, m) = \{B_t(u) : B_t(u) \text{ contains a vertex of } S'(k, m), B_t(u) \notin \partial_t C_t(k, m) \cup \partial_t Q'_t(k, m)\}.$$ 

Note that it is possible for $S'_t(k, m)$ to be empty.

For each pond $Q'_t(k, m, i)$, let $Q'(k, m, i)$ be all the vertices of $Q'_t(k, m, i)$ together with the open clusters in $S'(k, m)$ starting from the inside boundary of the pond. $Q'(k, m, i)$ and $Q'(k, m, j)$ are connected if they have a common vertex. This connection decomposes $\{Q'(k, m, i)\}$ into several connected components. A component is said to be **good** if it contains
a vertex of $U(\partial_eC_t(k,m))$; otherwise, it is bad. The ponds in a good component are called live ponds; otherwise, they are called dead ponds (see Fig. 2). We collect all the live ponds in good components to denote them by

$$Q_t(k, m, 1), Q_t(k, m, 2), \ldots, Q_t(k, m, \tau).$$

In addition, we also eliminate all the open clusters from $S'(k, m)$ in the bad components (see Fig. 2). We denote the remaining open clusters in $S'(k, m)$ by $S(k, m)$. We also denote by $S_t(k, m)$ the cubes in $S'_t(k, m)$ that contain a vertex of $S(k, m)$. Finally, we denote by

$$\Gamma_t(k, m) = \partial_eC_t(k, m) \cup S_t(k, m) \cup \bigcup_i \partial_eQ_t(k, m, i).$$

With these definitions, we have the following lemmas.

**Lemma 3.** For all configurations in $G(k, m)$,

$$S(k, m) \cap C(k, m) = \emptyset, \quad \text{(2.10)}$$

and for each $i$,

$$Q_t(k, m, i) \cap C(k, m) = \emptyset. \quad \text{(2.11)}$$

**Proof.** If (2.10) does not hold, then there exists a common vertex $v$ in $S(k, m) \cap C(k, m)$. Note that $v$ is connected by an open path to a pond or to $U(\partial_eC_t(k, m))$, so the pond or $U(\partial_eC_t(k, m))$ will contain a vertex of $C(k, m)$. This will contradict the assumption of ponds or (2.6). By the same argument, (2.11) holds. □

By Lemma 3, on $G(k, m)$,

$$S(k, m) \cap B(k, m) = \emptyset, \text{ and } Q_t(k, m) \cap B(k, m) = \emptyset. \quad \text{(2.12)}$$

**Lemma 4.** For all configurations in $G(k, m)$, $\Gamma_t(k, m)$ is a $Z^d$-connected cube set. In addition,

$$\Gamma_t(k, m) \subset \partial C_t(k, m).$$

**Proof.** By (2.4), $\partial_eC_t(k, m)$ is $Z^d$-connected. By our definition for components and (2.8), the cubes in $\partial_eQ_t(k, m, i)$ and the cubes containing open clusters in $S(k, m)$ starting at $\partial_eQ_t(k, m, i)$ are $Z^d$-connected. Note that we only focus on good components, so all these cubes are also $Z^d$-connected to $\partial_eC_t(k, m)$. Therefore, this shows that $\Gamma_t(k, m)$ is $Z^d$-connected.
Now we show the second argument in Lemma 4. For a cube $B_t(u) \in \partial C_t(k, m)$, we know $B_t(u) \in \partial C_t(k, m)$. For a cube $B_t(u) \in \partial Q_t(k, m, i)$, by (2.9), $B_t(u) \in \partial C_t(k, m)$. For a cube $B_t(u) \in S_t(k, m)$, note that

$$B_t(u) \notin \partial C_t(k, m) \cup \bigcup_i \partial Q_t(k, m, i).$$

In this case, as we mentioned earlier, $L_t(k, m)$ only contains three kinds of cubes: $\partial C_t(k, m)$, $Q_t(k, m)$, and $C_t(k, m)$. Thus, $B_t(u)$ either contains a vertex of $\partial C(k, m)$ or all its vertices are in $C(k, m)$. Note that the latter case will contradict Lemma 3, so $B_t(u) \in \partial C_t(k, m)$. Therefore,

$$\Gamma_t(k, m) \subset \partial C_t(k, m). \quad \Box$$

With these definitions and lemmas, we would like to show the following fundamental geometric lemma.

**Lemma 5.** For all configurations in $G(k, m)$, $\Gamma_t(k, m)$ contains a zero-cutset.

**Proof.** To show Lemma 5, we need to show that there is a cutset in $\Gamma_t(k, m)$ and that all its edges are closed. If we collect all closed $Z^d$-edges in $\Gamma_t(k, m)$, for any $Z^d$-path $\gamma$ from $\infty$ to $B(k, m)$, we only need to show that $\gamma$ must use one of these closed edges. We will now go along $\gamma$ from $\infty$ to $B(k, m)$. By Lemma 2 and the definition of the surface, $\gamma$ must reach the surface $U(\partial e C_t(k, m))$. Let $v_1$ be the last vertex of $\gamma$ at the surface such that the remaining piece of $\gamma$ from $v_1$ to $B(k, m)$, denoted by $\gamma_1$, will not have common vertices with the surface.

We suppose that in the following case, case (a), $v_1 \notin S(k, m)$. By the definition of $S(k, m)$, the edge next to $v_1$ in $\gamma_1$ must be closed. Thus, $\gamma_1$ will use a closed edge in $\partial e C_t(k, m)$, so Lemma 5 follows for case (a).

Now we suppose the following case, case (b), where $v_1 \in S(k, m)$. We only have the following two subcases:

1. $\gamma_1$ follows from the open paths in $S(k, m)$ to reach the inside boundary of a live pond $Q_t(k, m, i)$.
2. $\gamma_1$ does not.

By Lemma 3, we know that $S(k, m) \cap B(k, m) = \emptyset$. Thus, $\gamma_1$ will use a boundary edge of $S(k, m)$. Note that $S(k, m)$ is an open set (not necessarily an open cluster), so its boundary edges, except the edges in ponds or outside of $\bar{L}(k, m)$, are closed. Therefore, in case (b), (2), the boundary edge is closed. In other words, $\gamma_1$ must use a closed boundary edge of $S(k, m)$,
Lemma 5 follows for case (b), (2).

Now we focus on case (b), (1). By (2.12), $\gamma_1$ must leave from $Q_t(k, m, i)$ and never comes back before reaching $B(k, m)$. Let $v_2$ be the last vertex such that $\gamma_1$ will be out of $Q_t(k, m, i)$ at $v_2$ and will never return after $v_2$. We denote by $\gamma_2$ the subpath of $\gamma_1$ from $v_2$ to $B(k, m)$. Now we assume the following two sub-cases:

(1) $\gamma_2$ follows from open paths in $S(k, m)$ from $v_2$ to meet another pond $Q_t(k, m, j)$ for $i \neq j$.

(2) $\gamma_2$ does not.

For case (b), (1), (4), we can use the same argument as case (a) or as case (b), (2) to show that $\gamma_2$ either uses a closed edge in $\partial C_t(k, m)$ or in $S_t(k, m) \cup \partial C_t(k, m)$. Therefore, Lemma 5 follows for case (b), (1), (4).

For case (b), (1), (3), we can use the same argument as case (b), (1), (4) to show that either $\gamma_2$ will reach another pond $Q_t(k, m, l)$, or it will use a closed edge in $\partial C_t(k, m, j)$ or in $S_t(k, m) \cup \partial C_t(k, m)$. Note that the number of live ponds is finite, so by simple induction, we can show that $\gamma_1$ must use a closed edge in $\Gamma_t(k, m)$. With this observation, if we collect all the closed edges in $\Gamma_t(k, m)$, these closed edges consist of a closed cutset that cuts $B(k, m)$ from $\infty$. Therefore, Lemma 5 follows. □

Since $\Gamma_t(k, m)$ contains a closed cutset for $B(k, m)$, we select a closed self-avoiding cutset inside $\Gamma_t(k, m)$ using a unique method and denote it by $\Gamma(k, m)$ with $Z^d$-edges. Now we will show another geometric property for $\Gamma_t(k, m)$. For a cube $B_t(u)$, we denote by $\bar{B}_t(u)$ the cube $B_t(u)$ and its $L^d$-adjacent neighbor cubes. We call $B_t(u)$ a $t$-cube and call $\bar{B}_t(u)$ a 3$f$-cube. Through a simple computation, $B(k, m)$ contains 9 or 27 $t$-cubes when $d = 2$ or 3. In general,

\[ \text{the number of } t\text{-cubes in } \bar{B}_t(u) \leq 2^{2d}. \] (2.13)

Also, a $Z^d$-connected neighbor of $B_t(u)$ and $B_t(v)$ have common vertices. We simply name these vertices the surface of $B_t(u)$. A cube has $2d$ surfaces. In particular, two surfaces of $B_t(u)$ with a distance $t$ are called opposite surfaces.

We divide cubes $\Gamma_t(k, m)$ into two groups:

\[ (a) \ \partial C_t(k, m) \bigcup_{i=1}^{\tau} \partial Q_t(k, m, i) \text{ and } (b) \ S_t(k, m). \] (2.14)

A cube $B_t(u)$ is said to have a blocked property if (1) there are two surfaces of $t$-cubes in $\bar{B}_t(u)$ that are not connected by open paths in $\bar{B}_t(u)$, or (2) there is an open path inside $\bar{B}_t(u)$ from one surface of $B_t(u)$ to the boundary of $\bar{B}_t(u)$, without connecting in $\bar{B}_t(u)$ to one of the surfaces of $t$-cubes in $\bar{B}_t(u)$. For an independent purpose, we require that the
above open paths will only use the edges in int($\bar{B}(\mathbf{u})$). Intuitively, open paths are blocked to reach certain surfaces. Note that $B_t(\mathbf{u})$ is a blocked cube that only depends on configurations of edges in int($\bar{B}_t(\mathbf{u})$). If a cube $B_t(\mathbf{u})$ is in the cubic set of (2.14) (a), then it is either a boundary cube of live ponds or a cube in $\partial_e C_t(k, m)$. If one of its $L^d$-neighbor cubes belongs to $B(k, m)$, then there are two surfaces of the $t$-cubes in $\bar{B}_t(\mathbf{u})$: one is of the cube in $B(k, m)$ and another one is of the cube in the live ponds or in $U(\partial_e C_t(k, m))$. By Lemma 3, the two surfaces cannot be connected by an open path in $\bar{B}_t(\mathbf{u})$. The cube has a blocked property.

If $B_t(\mathbf{u})$ is not next to $B(k, m)$, then, by Lemma 4 and the definition of cluster boundary, there is an open path in $C(k, m)$ from $B_t(\mathbf{u})$ to $B(k, m)$. However, the open path cannot be connected to the surfaces of live ponds or to $U(\partial_e C_t(k, m))$. It also has a blocked property.

In either case, $B_t(\mathbf{u})$ has a blocked property.

For a fixed cube $B_t(\mathbf{u})$, we say it has a disjoint property if there exist two disjoint open paths in $\bar{B}_t(\mathbf{u})$ from cube $B_t(\mathbf{u})$ to $\partial \bar{B}_t(\mathbf{u})$. Similarly, for an independent purpose, we require that the above open paths will use the edges in int($\bar{B}_t(\mathbf{u})$). With this definition, $B_t(\mathbf{u})$ has a disjoint property depending only on the configurations of edges in int($\bar{B}_t(\mathbf{u})$). If $B_t(\mathbf{u})$ is in the cubic set in (2.14) (b), but not (2.14) (a), we may assume that

$$B_t(\mathbf{u}) \in S_t(k, m) \text{ but } B_t(\mathbf{u}) \notin \partial_e C_t(k, m) \bigcup \tau \partial_e Q_t(k, m, i).$$

We will show that it has either a blocked or a disjoint property. To see this, we suppose that one of its $L^d$-neighbor cubes belongs to $B(k, m)$. Since $S(k, m)$ must connect to the inside boundary of live ponds or to $U(\partial_e C_t(k, m))$ by an open path, then there is an open path from $B_t(\mathbf{u})$ to $\partial \bar{B}_t(\mathbf{u})$. By (2.12), this open path cannot further connect to the cube in $B(k, m)$. Therefore, $B_t(\mathbf{u})$ has a blocked property. If $B_t(\mathbf{u})$ is not next to $B(k, m)$, by Lemma 4, there is another open path from $B_t(\mathbf{u})$ to $B(k, m)$ in $C(k, m)$, so that there is an open path from $B_t(\mathbf{u})$ to $\partial B_t(\mathbf{u})$. Moreover, by Lemma 3, these two open paths, one in $S(k, m)$ and another in $C(k, m)$ cannot be connected. This shows that $B_t(\mathbf{u})$ has a disjoint property. We summarize this geometric property as the following lemma.

**Lemma 6.** For all configurations in $G(k, m)$, the cubes in $\Gamma_t(k, m)$ have either a blocked or a disjoint property.

### 3 Probability estimates for the linear zero-cutset.

In section 3, we will first estimate the probabilities of events on Lemma 6.
Lemma 7. If $F(0) < 1 - p_c$, then there exist $C_i = C_i(F(0), d)$ for $i = 1, 2$ such that for each cube $B_t(u)$,

$$
P[B_t(u) \text{ has a disjoint property}] \leq C_1 \exp(-C_2t).$$

Proof. The proof of Lemma 7 follows from Lemma 7.89 in Grimmett (1999). □

Lemma 8. If $F(0) < 1 - p_c$, then there exist $C_i = C_i(F(0), d)$ for $i = 1, 2$ such that for each cube $B_t(u)$,

$$
P[B_t(u) \text{ has a blocked property}] \leq C_1 \exp(-C_2t).$$

Proof. By Lemma 7.104 in Grimmett (1999),

$$
P\left[ \text{any two surfaces in the cubes of } B_t(u) \text{ are connected by open paths in } B_t(u) \right] \geq 1 - C_1 \exp(C_2t).$$

Here we want to remark that Grimmett’s lemma only deals with the connection between two opposite surfaces of a cube, but the same proof can be carried out to show the above inequality. Now we suppose that there is an open path from $B_t(u)$ to $B_t(u)$ for some $u$, but the path cannot be further connected to one of the surfaces in the cubes of $B_t(u)$. We denote this event by $B_t(u)$. By the above inequality and Lemma 7,

$$
P[\mathcal{B}_t(u)] 
\leq P[\mathcal{B}_t(u), \text{ any two surfaces in the cubes of } B_t(u) \text{ are connected by open paths in } B_t(u)] 
+ C_1 \exp(-C_2t) 
\leq P[B_t(u) \text{ has a disjoint property}] + C_1 \exp(-C_2t) 
\leq C_3 \exp(-C_4t).$$

Lemma 8 follows from the two inequalities above. □

For a configuration $\omega$, recall that $X(k, m)$ is the selected cutset with passage time $\chi(k, m)$. We also set the following edge set (see Fig. 3) as the surface edges of $B(k, m)$:

$$\alpha(k, m) = \{ e : e \text{ is a } \mathbb{Z}^d\text{-edge in } \Delta B(k, m) \}.$$ 

Clearly, $\alpha(k, m)$ is a cutset that cuts $B(k, m)$ from $\infty$. Therefore,

$$\chi(k, m) \leq \tau(\alpha(k, m)).$$ (3.1)
Note that if $m \leq \min_{1 \leq i \leq d-1} k_i$, there are at most $2d\|k\|_v \mathbb{Z}^d$-edges in $\alpha(k, m)$, so

$$E\chi(k, m) \leq E\tau(\alpha(k, m)) \leq 2d\|k\|_v E\tau(e).$$

(3.2)

Also, with our moment assumption in (1.1), by a standard large deviation result, there exist $C_i = C_i(F, d)$ for $i = 1, 2$ such that for all $u \geq 4dE\tau(e)\|k\|_v$,

$$P[\tau(\alpha(k, m)) \geq u] \leq C_1 \exp(-C_2u).$$

(3.3)

With these observations, we have the following lemma.

**Lemma 9.** If the conditions in Theorem 1 hold, and $u \geq 4d\|k\|_v E\tau(e)$, then

$$P[\chi(k, m) \geq u] \leq P[\tau(\alpha(k, m)) \geq u] \leq C_1 \exp(-C_2u).$$

4  **Connectedness of cutsets.**

In section 4, we need to show that each self-avoiding cutset is connected. Beforehand, we will show a lemma.

**Lemma 10.** If $Z(k, m)$ is a self-avoiding cutset with $\mathbb{Z}^d$-edges that cuts $B(k, m)$ from $\infty$, then for each edge $e \in Z(k, m)$ with two vertices $v_1(e)$ and $v_2(e)$, there exist disjoint paths $\gamma_1$ and $\gamma_2$ from $v_1(e)$ to $B(k, m)$ and from $v_2(e)$ to $\infty$ without using $Z(k, m)$.

**Proof.** For each $e \in Z(k, m)$, note that the cutset is self-avoiding, so $Z(k, m) \setminus e$ is not a cutset. There exists a path $\gamma$ without using $Z(k, m) \setminus e$ from $B(k, m)$ to $\infty$. If $\gamma$ does not pass through $e$, then $\gamma$, without using $Z(k, m)$, connects $B(k, m)$ to $\infty$. This contradicts the assumption that $Z(k, m)$ is a cutset. Therefore, $e \subset \gamma$ and $e$ is the only edge of $Z(k, m)$ contained in $\gamma$. Let $v_1(e)$ and $v_2(e)$ be the two vertices of $e$. By this observation, there exist paths $\gamma_1$ and $\gamma_2$ from $v_1(e)$ to $B(k, m)$, and from $v_2(e)$ to $\infty$, respectively, such that

$$\gamma_1 \cap Z(k, m) = v_1(e) \text{ and } \gamma_2 \cap Z(k, m) = v_2(e).$$

Therefore, Lemma 10 is proved. □

For $Z(k, m)$ defined in Lemma 10, let $\hat{Z}(k, m)$ be all the vertices that are connected by $\mathbb{Z}^d$-paths to $B(k, m)$ without using $Z(k, m)$. Note that $B(k, m)$ is $\mathbb{Z}^d$-connected, and so is $\hat{Z}(k, m)$. Recall that $\Delta_\epsilon \hat{Z}(k, m)$ is denoted by the $\mathbb{Z}^d$-edges between $\partial_\epsilon \hat{Z}(k, m)$ and
∂\(\hat{Z}(k, m)\). For each edge \(e \in Z(k, m)\), as we proved in Lemma 10, there exist \(\gamma_1\) and \(\gamma_2\) from \(v_1(e)\) to \(B(k, m)\), and from \(v_2(e)\) to \(\infty\), respectively, without using \(Z(k, m)\). Note also that
\[
\gamma_2 \cap \hat{Z}(k, m) = \emptyset;
\]
otherwise, \(Z(k, m)\) would not be a cutset. Therefore, \(v_1(e) \in \hat{Z}(k, m)\), but \(v_2(e) \not\in \hat{Z}(k, m)\) and is connected by a path \(\gamma_2\) without using an edge of \(Z(k, m)\) from \(v_2(e)\) to \(\infty\). This implies that \(e \in \Delta_e \hat{Z}(k, m)\), so
\[
\text{all edges of } Z(k, m) \subset \Delta_e \hat{Z}(k, m). \tag{4.1}
\]
For \(e \in \Delta_e \hat{Z}(k, m)\), by the definition of \(\hat{Z}(k, m)\), \(v_1(e)\) is connected to \(B(k, m)\) by \(\gamma_1\) without using an edge of \(Z(k, m)\), and \(v_2(e) \not\in \hat{Z}(k, m)\). This tells us that \(e \in Z(k, m)\), since, otherwise, \(\gamma_1 \cup \{e\}\) would be a path that does not use \(Z(k, m)\) from \(v_2(e)\) to \(B(k, m)\). So \(v_2(e) \in \hat{Z}(k, m)\). Therefore,
\[
\Delta_e \hat{Z}(k, m) \subset \text{all edges of } Z(k, m). \tag{4.2}
\]
By (4.1) and (4.2), we have
\[
\Delta_e \hat{Z}(k, m) = \text{all edges of } Z(k, m). \tag{4.3}
\]
By Lemma 1, \(\partial_e \hat{Z}(k, m)\) is \(Z^d\)-connected. By (4.3), each vertex of \(\partial_e \hat{Z}(k, m)\) is either \(Z^d\)-adjacent to \(\hat{Z}(k, m)\) by a \(Z^d\)-edge in \(Z(k, m)\) or \(L^d\)-adjacent to \(\hat{Z}(k, m)\). Suppose that \(v \in \partial_e \hat{Z}(k, m)\) is only \(L^d\)-adjacent to \(\hat{Z}(k, m)\), but is not \(Z^d\)-adjacent. It is easy to verify (see Fig. 1) that one of its \(L^d\)-neighbors is \(Z^d\)-adjacent to \(\hat{Z}(k, m)\). In other words, one of its \(L^d\)-neighbors is adjacent to \(\hat{Z}(k, m)\) by an edge in \(Z(k, m)\). Let us account for the number of \(L^d\)-neighbors for a vertex. Without loss of generality, we account for the origin. We assume that \((x_1, \cdots, x_d)\) is an \(L^d\)-neighbor of the origin. Thus, \(x_i\) can take either \(\pm 1\) and zero. Hence, there are at most \(3^d\) \(L^d\)-neighbors for the origin. With this observation, for each \(Z^d\)-edge \(e\) in \(Z(k, m)\), there are at most \(3^{d+1}\) vertices in \(\partial_e \hat{Z}(k, m)\) that are \(L^d\)-adjacent to \(e\). With Lemma 1 and with these observations above, we have the following lemma to show the connectedness of cutsets.

**Lemma 11.** If \(Z(k, m)\) is a self-avoiding cutset that cuts \(B(k, m)\) from \(\infty\), then \(\partial_e \hat{Z}(k, m)\) is \(Z^d\)-connected and
\[
\text{the number of } Z^d\text{-edges in } Z(k, m) \geq |\partial_e \hat{Z}(k, m)|/3^{d+1}. \tag{4.4}
\]
Now we focus on the connectedness of the cutsets that cut \(F_0\) from \(F_m\). Let \(V(k, m)\) be a self-avoiding cutset that cuts \(F_0\) from \(F_m\). Similarly, let \(V(k, m)\) be the all vertices
in $B(k, m)$ that are connected by $\mathbb{Z}^d$-paths in $B(k, m)$ to $F_0$ without using $V(k, m)$. In addition, let $\Delta \hat{V}(k, m)$ be the all boundary edges of $\hat{V}(k, m)$ in $B(k, m)$. For each edge $e \in \Delta \hat{V}(k, m)$, if there exists a path in $B(k, m)$ from one of its vertices to $F_m$ without using edges in $V(k, m)$, then $e$ is an exterior boundary edge of $\hat{V}(k, m)$. We denote by $\Delta_e \hat{V}(k, m)$ all the exterior boundary edges of $\hat{V}(k, m)$. By the same proof as (4.3), we can show that $\Delta_e \hat{V}(k, m) = \text{all edges of } V(k, m)$. (4.5)

Kesten (Lemma 3.17 in Kesten (1988)) showed that for $d = 3$, $\partial_e \hat{V}(k, m)$ is $\mathbb{Z}^d$-connected; but his proof can be directly adapted to apply for all $d \geq 3$. On the other hand, it can also use the same proof of Lemma 1 to show the $\mathbb{Z}^d$-connectedness of $\partial_e \hat{V}(k, m)$. By the same discussion of (4.4), we can work on the number of $\mathbb{Z}^d$-edges of $V(k, m)$. We summarize the above results as the following lemma.

**Lemma 12.** If $V(k, m)$ is a self-avoiding cutset that cuts $F_0$ from $F_m$ in $B(k, m)$, then $\partial_e \hat{V}(k, m)$ is $\mathbb{Z}^d$-connected and

$$\text{the number of } \mathbb{Z}^d\text{-edges in } V(k, m) \geq |\partial_e \hat{V}(k, m)|/3^{d+1}.$$ (4.6)

### 5 Proof of Theorem 1.

In this section, we assume that $F(0) < 1 - p_c$. For each $0 < \epsilon < 1$, $e$ is said to be an $\epsilon^+$-edge or $\epsilon^-$-edge if $\tau(e) > \epsilon$ or $0 < \tau(e) \leq \epsilon$. Let $N^+(k, m)$ and $N^-(k, m)$ be the numbers of $\epsilon^+$-edges and $\epsilon^-$-edges in $X(k, m)$, respectively. Note that

$$\epsilon N^+(k, m) \leq \chi(k, m),$$

so if we take $\beta_1 = 4dE\tau(e)$, by Lemma 9 for $n \geq \epsilon^{-2}\beta_1\|k\|_v$ there exist $C_i = C_i(F, d, \epsilon)$ for $i = 1, 2$,

$$P\left[N^+(k, m) \geq \epsilon n\right] \leq P\left[\chi(k, m) \geq \epsilon^2 n\right] \leq C_1 \exp(-C_2 n).$$ (5.1)

Now we take care of the $\epsilon^-$-edges in the cutset. By our definition,

$$P[e \text{ is an } \epsilon^- \text{-edge}] \leq F(\epsilon) - F(0) = \delta_1 = \delta_1(\epsilon),$$ (5.2)

where $\delta_1 \to 0$ as $\epsilon \to 0$. We need to fix a vertex of $X(k, m)$. Since $X(k, m)$ is a cutset, it must intersect the line $L$:

$$L = \{(x_1, x_2, \cdots, x_d) : x_i = 0 \text{ for } i \geq 2\}.$$
We let \( z = (x_1, 0, \ldots, 0) \) be the intersection vertex of \( X(k, m) \) and \( L \). If there are many intersections, we select one with the largest \( x_1 \)-coordinate and still denote it by \( z \). Note that if \( \text{dist}(0, z) = l \) for some \( l \), then

the number of edges \( X(k, m) \) is larger than \( l \). \hspace{1cm} (5.3)

To show (5.3), simply note that each layer between the hyperplanes \( x_1 = i \) and \( x_1 = (i + 1) \) for \( i \leq l \) contains at least one edge of \( X(k, m) \).

Now we estimate the following probability for small \( \delta_1 \) defined in (5.2) and for a constant \( D = D(d) \) selected later:

\[
P \left[ N(k, m) \geq n, N^-(k, m) \geq -\left( D \log^{-1}(\delta_1) \right) N(k, m) \right].
\]

By (5.3), we fix \( z \) to have

\[
P \left[ N(k, m) \geq n, N^-(\epsilon, k, m) \geq -\left( D \log^{-1}(\delta_1) \right) N(k, m) \right]
\]

\[
= \sum_{j=n}^{\infty} \sum_{i=0}^{j} P \left[ N(k, m) = j, x_1 = i, N^-(k, m) \geq -Dj \log^{-1}(\delta_1) \right].
\]

Recall \( \hat{X}(k, m) \) and \( \partial_e \hat{X}(k, m) \) defined above (see (4.2)). As we defined before, \( X(k, m) \) is unique for each configuration, and so is \( \partial_e \hat{X}(k, m) \). Thus, for two different fixed sets \( \Gamma_1 \) and \( \Gamma_2 \), we have

\[
\{ \partial_e \hat{X}(k, m) = \Gamma_1 \} \text{ and } \{ \partial_e \hat{X}(k, m) = \Gamma_2 \} \text{ are disjoint.}
\]

If \( \partial_e \hat{X}(k, m) = \Gamma_1 \), we say that \( \partial_e \hat{X}(k, m) \) has a choice \( \Gamma_1 \).

If \( x_1 = i \), then by Lemma 11, \( \partial_e \hat{X}(k, m) \) is \( \mathbb{Z}^d \)-connected and

\[
|\partial_e \hat{X}(k, m)| \leq 3^{d+1}j.
\]

Thus, by using (4.24) in Grimmett (1999), there are at most \( 7^{3^{d+1}j} \) choices of these cutsets for \( \partial_e \hat{X}(k, m) \) when \( X(k, m) \) has \( j \) edges. After \( \partial_e \hat{X}(k, m) \) is a fixed vertex set, we select the vertices in \( \partial_e \hat{X}(k, m) \) with \( \mathbb{Z}^d \)-edges of \( X(k, m) \). We next select the \( \mathbb{Z}^d \)-edges of \( X(k, m) \) adjacent to these vertices. Note that each vertex has at most \( 2d \) adjacent edges, so there are at most

\[
\sum_{k=1}^{3^{d+1}j} \binom{3^{d+1}j}{k} (2d)^j \leq 2^{3^{d+1}j} (2d)^j
\]

for the selections. Thus, if \( |\Gamma|_e \) is denoted by the number of edges in \( \Gamma \),

\[
P \left[ N(k, m) \geq n, N^-(k, m) \geq -\left( D \log^{-1}(\delta_1) \right) N(k, m) \right]
\]

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\[ = \sum_{j=n}^{\infty} \sum_{i=0}^{j} P \left[ N(k, m) = j, x_1 = i, N^{-1}(k, m) \geq -D j \log^{-1}(\delta_1) \right] \]
\[ \leq \sum_{j=n}^{\infty} j^{d^d + 1} 2^{d^d + 1} (2d)^j \max_{\Gamma} P \left[ |\Gamma|_e = j, \Gamma \text{ contains more than } -D j \log^{-1}(\delta_1) \epsilon^- \text{-edges} \right], \quad (5.4) \]

where \( \Gamma \) is a fixed cutset that cuts \( B(k, m) \) from \( \infty \) with a number of edges \( |\Gamma|_e = j \), and the maximum is taking over all the possible \( \Gamma \). Let us estimate the probability in (5.4) for a fixed set \( \Gamma \) with more than \( -D j \log^{-1}(\delta_1) \epsilon^- \)-edges for a large number \( D \).

\[ P \left[ |\Gamma|_e = j, \Gamma \text{ contains more than } -(D j) \log^{-1}(\delta_1) \epsilon^- \text{-edges} \right] \leq \sum_{i \geq -D j \log^{-1}(\delta_1)}^{j} \binom{j}{i} \delta_1^{i}. \quad (5.5) \]

By using Corollary 2.6.2 in Engle (1997), there are at most

\[ \left( \frac{j}{i} \right) \leq \exp \left( jH \left( \frac{i}{j} \right) \right), \quad (5.6) \]

where

\[ H(x) = -x \log x - (1 - x) \log(1 - x). \]

By (5.6), we have

\[ \sum_{i \geq -D j \log^{-1}(\delta_1)}^{j} \binom{j}{i} \delta_1^{i} \leq \sum_{i \geq -D j \log^{-1}(\delta_1)}^{j} \exp(jH(i/j)) \delta_1^{i}. \quad (5.7) \]

If \( 0 < x < 1 \), we have

\[ H(x) \leq 2x \log(1/x). \quad (5.8) \]

By (5.7) and (5.8),

\[ \sum_{i \geq -D j \log^{-1}(\delta_1)}^{j} \binom{j}{i} \delta_1^{i} \leq \sum_{i \geq -D j \log^{-1}(\delta_1)}^{j} \exp \left[ 2i \log(j/i) + i \log(\delta_1) \right]. \quad (5.9) \]

Note that \( -(D j \log^{-1}(\delta_1)) \leq i \leq j \) and \( \log(j/i) \) is decreasing when \( i \) is increasing until \( j \), so for small \( \delta_1 \) and \( D \geq 1 \), we have

\[ \log \frac{j}{i} \leq \log \frac{j}{-D j \log^{-1}(\delta_1)}. \quad (5.10) \]

Note also that if \( \delta_1 \to 0 \), then

\[ \frac{\log(- \log(\delta_1))}{\log \delta_1} \to 0 \text{ from the left}, \]

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so we take a small \( \delta_1 \),

\[
\log(-\log(\delta_1)) \leq -(\log \delta_1)/4.
\]

Hence, by (5.9)

\[
\sum_{i \geq -Dj \log^{-1}(\delta_1)}^j \left( \begin{array}{c} j \vspace{1ex} \\
\end{array} \begin{array}{c} i \vspace{1ex} \\
\end{array} \right) \delta_1^i \leq \sum_{i \geq -Dj \log^{-1}(\delta_1)}^j \exp\left( \frac{i}{2} \log(\delta_1) \right).
\]

(5.11)

We use (5.11) in (5.5) to produce

\[
\begin{align*}
\text{P} & \left[ |\Gamma|_e = j, \Gamma \text{ contains more than } -Dj \log^{-1}(\delta_1) \epsilon^- \text{-edges} \right] \\
& = \sum_{i \geq -Dj \log^{-1}(\delta_1)}^j \left( \begin{array}{c} j \vspace{1ex} \\
\end{array} \begin{array}{c} i \vspace{1ex} \\
\end{array} \right) \delta_1^i \\
& \leq \sum_{i \geq -Dj \log^{-1}(\delta_1)}^\infty \exp(i \log(\delta_1)/2) \\
& \leq 2 \exp(-Dj/2).
\end{align*}
\]

With this observation and (5.4),

\[
\begin{align*}
\text{P} & \left[ N(k, m) \geq n, N^-(k, m) \geq -(D \log^{-1}(\delta_1))N(k, m) \right] \\
& = \sum_{j=0}^\infty \sum_{i=0}^j \text{P} \left[ N(k, m) = j, x_1 = i, N^-(k, m) \geq -Dj \log^{-1}(\delta_1) \right] \\
& \leq \sum_{j=0}^\infty j^{3d^2+1}j2^{3d^2+1}j(2d)^j \max_{\Gamma} \text{P} \left[ |\Gamma|_e = j, \Gamma \text{ contains more than } -Dj \log^{-1}(\delta_1) \epsilon^- \text{-edges} \right] \\
& \leq \sum_{j=0}^\infty j^{3d^2+1}j2^{3d^2+1}j(2d)^j \exp(-Dj/2).
\end{align*}
\]

By taking \( D = D(d) \) and a small \( \delta_1 \), there are \( C_i = C_i(F, d, \delta_1) \) for \( i = 1, 2 \) such that

\[
\begin{align*}
\text{P} & \left[ N(k, m) \geq n, N^-(k, m) \geq -(D \log^{-1}(\delta_1))N(k, m) \right] \leq C_1 \exp(-C_2 n).
\end{align*}
\]

(5.12)

For a configuration \( \omega \), we denote edges \( \epsilon^\pm \) in the cutset by \( e_1, \cdots, e_J \). Therefore,

\[
\chi(k, m) = \sum_{i=1}^J \tau(e_i).
\]

(5.13)

By (5.1) and (5.12), for all small \( \epsilon \) and corresponding \( \delta_1 \) and \( \beta_1 \), and for all \( n \geq \epsilon^{-2}\beta_1\|k\|_v \), there are \( C_i = C_i(F, d, \epsilon) \) for \( i = 1, 2 \) such that

\[
\begin{align*}
\text{P} & \left[ N(k, m) \geq n \right] \\
& = \sum_{j \geq n} \text{P} \left[ N(k, m) = j, N^+(k, m) \leq \epsilon j, N^-(k, m) \leq -(Dj) \log^{-1}(\delta_1) \right] + C_1 \exp(-C_2 n).
\end{align*}
\]

(5.14)
On \( \{N^+(k, m) \leq \epsilon j, N^-(k, m) \leq -(Dj) \log^{-1}(\delta_1)\} \), we have

\[
J \leq \left( \epsilon - D \log^{-1}(\delta_1) \right) j.
\]

(5.15)

For a large number \( t \), by (5.14), we take \( \epsilon \) small and then \( n \) large such that

\[
P[N(k, m) \geq n] \leq \sum_{j \geq n} P[N(k, m) = j, J \leq j/(4t^d)] + C_1 \exp(-C_2n)
\]

(5.16)

for \( C_i = C_i(F, d, \epsilon, t) \) with \( i = 1, 2 \). Note that the edges in \( X(k, m) \) other than \( e_i \) for \( i = 1, \ldots, J \) are zero-edges, so if we change these \( \epsilon^\pm \)-edges from positive to zero, we will have a closed cutset corresponding to another configuration \( \omega' \). More precisely, for each configuration \( \omega \), if we make the changes for these \( \epsilon^\pm \)-edges, then \( \omega \) will change to another configuration \( \omega' \). Since \( X(k, m)(\omega) \) is uniquely selected, \( \omega' \) is determined uniquely for each \( \omega \). If there exists a closed cutset for \( \omega \), there exists the zero-cutset \( \Gamma(k, m)(\omega') \), constructed in section 2 inside \( \Gamma_t(k, m)(\omega') \). By Lemma 4, \( \Gamma_t(k, m)(\omega') \) is \( \mathbb{Z}^d \)-connected. Therefore, for each configuration \( \omega \), \( e_1, \ldots, e_J \) exist. So we have \( \omega' \) and \( \Gamma(k, m)(\omega') \). If we change these edges in \( \{e_1, \ldots, e_J\} \) from zero back to the original values, \( \Gamma(k, m) \), as a vertex set, exists corresponding to \( \omega \), but it will no longer be a closed cutset. We denote it by \( \Gamma(k, m)(\omega) \), as this vertex set for the configuration \( \omega \). Note that \( \Gamma(k, m)(\omega) \) is uniquely determined for each \( \omega \). We claim that for each configuration,

\[
\tau(\Gamma(k, m)(\omega)) = \chi(k, m)(\omega).
\]

(5.17)

To see this, note that \( \Gamma(k, m)(\omega) \) is a cutset, so

\[
\tau(\Gamma(k, m)(\omega)) \geq \chi(k, m)(\omega).
\]

(5.18)

On the other hand, the other edges in \( \Gamma(k, m)(\omega) \), except for \( e_1, \ldots, e_J \), are all zero-edges, and \( X(k, m) \) uses all the edges \( e_1, \ldots, e_J \), so

\[
\tau(\Gamma(k, m)(\omega)) \leq \tau(X(k, m)) = \chi(k, m)(\omega).
\]

(5.19)

Therefore, (5.17) follows. By the definition of \( N(k, m) \) and (5.17), note that \( \Gamma(k, m) \) as a vertex set is the same for either \( \omega \) or \( \omega' \), so we have

\[
|\Gamma(k, m)(\omega')|_e = |\Gamma(k, m)(\omega)|_e \geq N(k, m)(\omega).
\]

(5.20)

If these edges \( e_1, \ldots, e_J \) are zero-edges, as we mentioned above, \( \Gamma(k, m) \) is a zero-cutset contained inside \( \Gamma_t(k, m) \). Since \( \Gamma_t(k, m) \) is also a cutset, \( \Gamma_t(k, m) \) and \( \mathbb{L} \) must intersect. We denote by \( B_t(z)(\omega') \) the cube in \( \Gamma_t(k, m)(\omega') \) that intersects \( \mathbb{L} \). If there are many such
cubes, we simply select $z$ with the largest $x_1$ value. By the same argument of (5.3), if $l$ is denoted the number of cubes with the lower corners at $L$ from the origin to $z$, then

the number of cubes in $\Gamma_t(k, m)(\omega')$ is larger than $l$. \hfill (5.21)

For $\omega'$ and each $t$-cube in $\Gamma_t(k, m)(\omega')$, by Lemma 6, it has either the blocked or disjoint property. By our definition, if $B_t(u)$ and $B_t(v)$ for fixed $u$ and $v$ satisfy that

$$\text{int}(B_t(u)) \cap \text{int}(B_t(v)) = \emptyset,$$

then

$$\{B_t(u) \text{ has a blocked or disjoint property}\} \text{ and } \{B_t(v) \text{ has a blocked or disjoint property}\}$$

are independent. Therefore, we need to decompose $\Gamma_t(k, m)(\omega')$ into $3t$-cubes such that their center cubes belong to $\Gamma_t(k, m)(\omega')$. By a standard estimation (see Grimmett and Kesten, page 345 (1984) or Zhang, pages 21 (2007), or Steele and Zhang (2003), Lemma 6 by using Turan’s theorem), if the number of cubes in $\Gamma_t(k, m)(\omega')$ is $s$, then

$$\exists \text{ at least } s/2^{2d} \text{ disjoint } 3t\text{-cubes with their center cubes belong to } \Gamma_t(k, m)(\omega'). \tag{5.22}$$

Let $M_{3t}(k, m)(\omega')$ be all the disjoint $3t$-cubes and $M_{3t}(k, m)(\omega')$ be the number of the $3t$-cubes in $M_{3t}(k, m)(\omega')$. Note that each $t$-cube has $2t^d$ edges, so by (5.20) and (5.22), if $N(k, m)(\omega) = j$, then

$$M_{3t}(k, m)(\omega') \geq \frac{\# \text{ cubes in } \Gamma_t(k, m)(\omega')}{2^{2d}} \geq \frac{|\Gamma(k, m)(\omega')|_e}{2^{2d+1}2^d} \geq \frac{N(k, m)(\omega)}{2(4t)^d} \geq \frac{j}{2(4t)^d}. \tag{5.23}$$

Furthermore, if

$$J(\omega) \leq j/(4(4t)^d) \text{ and } N(k, m)(\omega) = j \text{ and } \# \text{ cubes in } \Gamma_t(k, m)(\omega') = s,$$

then by (5.22) and (5.23), for each $\omega'$, there are at least

$$M_{3t}(k, m) = \frac{j}{(4(4t)^d)} = \frac{M_{3t}(k, m)}{2} + \frac{M_{3t}(k, m)}{2} - \frac{j}{(4(4t)^d)} \geq \frac{M_{3t}(k, m)}{2} \geq \frac{s}{2^{2d+1}} \tag{5.24}$$

disjoint $3t$-cubes in $M_{3t}(k, m)(\omega')$ such that their center $t$-cubes have either the blocked or disjoint property, and these interior $3t$-cubes do not contain $e_1, \cdots, e_J$. Note that these disjoint $s/2^{2d+1}$ cubes always have the blocked or disjoint property whether $\tau(e_i)$ is positive or zero for $i = 1, 2, \cdots, J$, since they do not contain these edges in their interiors. We call them permanent blocked or disjoint cubes. Now we change these edges in $\{e_1, \cdots, e_J\}$ from zero back to the original values. We still have $s/2^{2d+1}$ permanent block or disjoint $3t$-cubes.
In summary, for each $\omega$, if $N(k, m)(\omega) = j$, and $J(\omega) \leq j/(4(4t)^d)$, note that by (5.20),
\[
\# \text{ cubes in } \Gamma_t(k, m)(\omega') = s \geq |\Gamma(k, m)(\omega)|/((2t)^d) \geq N(k, m)(\omega)/(2t)^d = j/(2t)^d, \quad (5.25)
\]
so by (5.21)–(5.24), there are $\mathbb{Z}^d$-connected $s \geq j/(2t)^d$ $t$-cubes containing $B_t(z)$ for $z \in \mathbb{L}$ with $\|z\| \leq s$ such that
(a) there are at least $s/2^{2d+1}$ disjoint $3t$-cubes containing the above $t$-cubes as their center cubes and
(b) each center $t$-cube in these $3t$-cubes in (a) has the blocked or disjoint property, where the blocked or disjoint property in (b) corresponds to the configuration $\omega$. We denote the event of (a) and (b) by $\mathcal{E}(s, j, z)$.

Now we try to estimate $\mathcal{E}(s, j, z)$ by fixing these $3t$-cubes in the following steps. We suppose that the connected $t$-cubes in event $\mathcal{E}(s, j, z)$ is $\Gamma_t$ with $s$ cubes in $\Gamma_t$. First, we fix $B_t(z)$. By (5.21), there are at most $s$ choices for this cube. With this cube, note that $\Gamma_t$ is $\mathbb{Z}^d$-connected, by using a standard computation technique (see (4.24) in Grimmett (1999)), there are at most $s7^{2ds}$ choices for this set $\Gamma_t$. If $\Gamma_t$ is fixed, we select these disjoint $3t$-cubes in $\Gamma_t$. There are at most
\[
\sum_{i=1}^{s} \binom{s}{i} = 2^s
\]
choices for these $3t$-cubes. With these $3t$-cubes, we select these disjoint $s/2^{2d+1}$ $3t$-cubes such that their center cubes are permanent blocked or disjoint $t$-cubes. There are at most
\[
\sum_{i=1}^{s} \binom{s}{i} = 2^s
\]
choices. Therefore, by Lemmas 7 and 8, there are $C_i = C_i(F, d, \epsilon, t)$ for $i = 1, 2$ and $C_i = C_i(F(0), d)$ for $i = 3, 4$ such that
\[
\Pr[N(k, m) \geq n] \leq \sum_{j \geq n} \Pr[N(k, m) = j, J \leq j/(2(4t)^d)] + C_1 \exp(-C_2n).
\]
\[
\leq \sum_{j \geq n} \sum_{s \geq j/(2t)^d} \Pr[\mathcal{E}(s, j, z)]
\]
\[
\leq \sum_{j \geq n} \sum_{s \geq j/(2t)^d} s7^{2ds}2^s[3s/C_3 \exp(-C_4t)]^{s/2^{2d+1}} + C_1 \exp(-C_2n).
\]
If we take $t$ large, there exist $C_i = C_i(F, d, \beta, \epsilon, t)$ for $i = 1, 2$ such that
\[
\Pr[N(k, m) \geq n] \leq C_1 \exp(-C_2n). \quad (5.26)
\]
Therefore, Theorem 1 follows.
6 Proof of Theorem 2.

Since section 6 focuses on the edges inside \( B(k, m) \), we use \( P_{k,m}(\cdot) \) to denote the probability measure. In addition, we assume that \( F(0) < 1 - p_c \) in this section. Let \( \alpha(k, m) \) be vertical edges between \( \prod_{i=1}^d [0, k_i] \times \{0\} \) and \( \prod_{i=1}^d [0, k_i] \times \{1\} \) inside \( B(k, m) \). Note that \( \alpha(k) \) is a cutset that cuts \( \alpha \) edges in \( \bar{\alpha} \). By a similar large deviation result for edges between \( \prod F \) measure. In addition, we assume that such that so if we take \( z \) for \( \geq \) we select the intersection \( z \) where \( \delta \). Similar to (5.4), we need to fix a vertex in \( W(k, m) \), since \( \geq \). Let \( N^+(k, m) \) and \( N^-(k, m) \) be the numbers of \( \epsilon^+ \)-edges and \( \epsilon^- \)-edges in \( W(k, m) \), respectively. Note that

\[
\epsilon N^+(k, m) \leq \tau(W(k, m)),
\]

so if we take \( \beta_1 = 2E \tau(e) \), by (6.1) for \( n \geq \epsilon^{-2} \beta_1 \|k\|_v \)

\[
P_{k,m}[N^+(k, m) \geq \epsilon n] \leq C_1 \exp(-C_2 n).
\]

Now we take care of the \( \epsilon^- \)-edges in the cutset. As in (5.2), we assume that

\[
P_{k,m}[e \text{ is an } \epsilon^- \text{ edge}] \leq F(\epsilon) - F(0) = \delta_1 = \delta_1(\epsilon),
\]

where \( \delta_1 \to 0 \) as \( \epsilon \to 0 \). With a small \( \delta_1 \), we estimate the following probability:

\[
P_{k,m}[\bar{N}(k, m) \geq n, N^-(k, m) \geq -(D \log^{-1}(\delta_1)) \bar{N}(k, m)].
\]

Similar to (5.4), we need to fix a vertex in \( W(k, m) \cap L \). Since \( L \) must intersect \( W(k, m) \), we select the intersection \( z \) with the largest \( x_i \)-coordinate. There are at most \( m \) choices for \( z \), since \( W(k, m) \) stays inside \( B(k, m) \). By our assumption in Theorem 2, note that \( n \geq \beta \|k\|_v \geq \|k\|_v \),

\[
m \leq \exp(\|k\|_v) \leq \exp(n).
\]

When \( z \) at \( W(k, m) \) is fixed, by Lemma 12 and the same estimate in (5.4), we have

\[
P_{k,m}[\bar{N}(k, m) \geq n, N^-(k, m) \geq -(D \log^{-1}(\delta_1)) \bar{N}(k, m)]
\]

\[
= \sum_{j=n}^{\infty} P_{k,m}[\bar{N}(k, m) = j, N^-(k, m) \geq -Dj \log^{-1}(\delta_1)]
\]

\[
\leq \sum_{j=n}^{\infty} \exp(n)7^{d^d+1} j^{2d+1} 2^{d+1} j (2d)^j
\]

\[
\times \max_{\Gamma} P_{k,m}[|\Gamma|_e = j, \Gamma \text{ contains more than } -Dj \log^{-1}(\delta_1) \epsilon^- \text{-edges}]
\]
where $\Gamma$ is a fixed cutset that cuts $F_0$ from $F_m$ such that the number of its edges $|\Gamma|_e = j$, and the maximum takes over all possible fixed vertex sets $\Gamma$. For a fixed set $\Gamma$, by the same estimate from (5.5)–(5.11), we have for a small $\delta_1 > 0$,

$$\mathbf{P}_{k,m} \left[ \Gamma, |\Gamma|_e = j, \Gamma \text{ contains more than } -Dj \log^{-1}(\delta_1) \epsilon^- - \text{edges} \right]$$

$$= \sum_{i \geq -Dj \log^{-1}(\delta_1)} \binom{j}{i} \delta_1^i$$

$$\leq \sum_{i \geq -Dj \log^{-1}(\delta_1)} \exp(Di \log(\delta_1)/2)$$

$$\leq 2 \exp(-Dj/2).$$

With this observation, by taking $D = D(d)$, there exists $\beta \geq 1$, and $\epsilon$, and $C_i = C_i(F, d, \epsilon)$ for $i = 1, 2$ such that for all $n \geq \beta \|k\|_v$,

$$\mathbf{P}_{k,m} \left[ \bar{N}(k, m) \geq n, N^-(k, m) \geq -D \log^{-1}(\delta_1) \bar{N}(k, m) \right]$$

$$\leq \exp(n) \sum_{j=n}^{\infty} 7^{d+1} j 2^{d+1} j(2d)^j \exp(-Dj/2)$$

$$\leq C_1 \exp(-C_2 n). \tag{6.5}$$

Therefore, for a small $\epsilon$ and corresponding $\delta_1$, by (6.2) and (6.5), there exists $\beta = \beta(\epsilon) \geq \epsilon^{-2}/\beta_1$ such that for $n \geq \beta \|k\|_v$,

$$\mathbf{P}_{k,m} \left[ \bar{N}(k, m) \geq n \right]$$

$$\leq \sum_{j \geq n} \mathbf{P}_{k,m} \left[ \bar{N}(k, m) = j, N^+(k, m) \leq \epsilon j, N^-(k, m) \leq -Dj \log^{-1}(\delta_1) j \right] + C_1 \exp(-C_2 n).$$

Similarly, we denote by $J$ the number of all $\epsilon^\pm$-edges in $W(k, m)$, and $\{e_1, \ldots, e_J\}$ are these $\epsilon^\pm$ edges. On

$$\left\{ N^+(k, m) \leq \epsilon j, N^-(k, m) \leq -Dj \log^{-1}(\delta_1) \right\},$$

we have

$$J \leq \left( \epsilon - D \log^{-1}(\delta_1) \right) j. \tag{6.6}$$

Therefore, for any large $t$, we take $\beta$ large such that for all $n \geq \beta \|k\|_v$,

$$\mathbf{P}_{k,m} \left[ \bar{N}(k, m) \geq n \right] \leq \sum_{j \geq n} \mathbf{P}_{k,m} \left[ \bar{N}(k, m) = j, J \leq j/(2(4t)^d) \right] + C_1 \exp(-C_2 n). \tag{6.7}$$

For a configuration $\omega$, since $e_1, \ldots, e_J$ are the only non-zero edges in $W(k, m)$,

$$\tau_{\min}(k, m) = \sum_{i=1}^{J} \tau(e_i). \tag{6.8}$$
Figure 3: The surface edges of $B(k, m)$, denoted by $\alpha(k, m)$, are the dotted lines outside $B(k, m)$. We can use the surface together with $W(k, m)(\omega')$ to construct a cutset that cuts $F_0$ from $\infty$. Thus, any open path from $F_m$ to $F_0$ must use an edge of $W(k, m)$. $\Gamma'(k, m)$ only uses the edges inside $B(k, m)$.

To use the proof of Theorem 1, we need to construct a cutset that cuts $B(k, 0) = F_0$ from $\infty$. We need to use the surface edges $\alpha(k, m)$ defined in section 3 (see Fig. 3) and $W(k, m)$. In particular, the surface edges of $\alpha(k, m)$ adjacent to $F_m$ are called the top surface edges. Moreover, let all the surface edges be closed. Note that the surface edges are outside of $B(k, m)$, so it will not affect our measure $P_{k,m}(\cdot)$. With the closed surface edges, any path from $B(k, 0)$ to $\infty$ must use at least one surface edge. Thus, the closed surface consists of a zero-cutset, so $\mathcal{G}(k, 0)$ occurs. Therefore, $\Gamma_t(k, 0)$ defined in section 2 exists and it contains a zero-cutset $\Gamma(k, 0)$. Note that $\partial_C(k, 0)$ cannot be outside of the surface boundary, so we may choose our $\Gamma(k, 0)$ such that

$$\Gamma(k, 0) \text{ uses only edges of the surface and the edges in } B(k, m).$$  \hfill (6.9)

Let

$$\Gamma'(k, 0) = \Gamma(k, 0) \cap B(k, m) \text{ and } \Gamma'_t(k, 0) = \{B_t(u) : B_t(u) \cap \Gamma'(k, 0) \neq \emptyset\}.$$

For each configuration $\omega$, if we change all $e_1, e_2, \ldots, e_J$ from $\epsilon^\pm$ to zero, we have another configuration $\omega'$. With these changes, $W(k, m)(\omega')$ is a zero-cutset that cuts $F_0$ from $F_m$. Furthermore, we will show that

$$\Gamma'(k, 0)(\omega') \text{ is a closed set inside } \Gamma'_t(k, 0)(\omega') \text{ that also cuts } F_0 \text{ from } F_m.$$  \hfill (6.10)
Before showing (6.10), we first show that for \( \omega \) from \( \mathcal{F} \), \( \Gamma(0)(\omega) \) is a zero-cutset that cuts \( F \) from \( F_0 \). Intuitively, the surface edges in \( \alpha(k, m) \) and the edge of \( \mathcal{W}(k, m)(\omega') \) consist of a zero-cutset, so \( \Gamma'(k, 0)(\omega') \) only uses the edges inside \( \mathcal{B}(k, m)(\omega') \) (see Fig. 3). If \( \Gamma(k, 0)(\omega') \) is not a cutset that cuts \( F_0 \) from \( F_m \), then there exists a path (not necessarily open) from \( F_0 \) to \( F_m \) without using an edge of \( \Gamma(k, 0)(\omega') \). The path must reach a vertex of an edge, denoted by \( e \), in the top surface. By (6.9), \( e \in \Gamma(k, 0)(\omega') \); otherwise, we can construct a path from \( \mathcal{B}(k, 0) \) to \( \infty \) without using an edge of \( \Gamma(k, 0)(\omega') \). In other words, it reaches an edge in \( \Gamma(k, 0)(\omega') \) and the edge is also adjacent to \( F_m \) from outside of \( \mathcal{B}(k, m) \). Let \( B_t(u) \in \Gamma_t(k, m)(\omega') \) be the t-cube that contains the edge. Since \( k_i/t \) and \( m/t \) are integers, \( B_t(u) \) and \( \mathcal{B}(k, m) \) do not have other vertices in common, except for vertices at \( F_m \). By Lemma 4, there exists an open path from \( B_t(u) \) to \( \mathcal{B}(k, 0) = F_0 \). Note that the surface is closed, so the open path must go from \( F_m \) to \( F_0 \) inside \( \mathcal{B}(k, m) \) (see Fig. 3). However, this situation contradicts the fact that \( \mathcal{W}(k, m)(\omega') \) is a zero-cutset. This contradiction shows that \( \Gamma(k, 0)(\omega') \) is indeed a cutset that cuts \( F_0 \) from \( F_m \). Furthermore, note that \( \omega' \) has more zeros than \( \omega \)'s and all edges in \( \Gamma(k, 0)(\omega) \) are all zero-edges, so \( \Gamma(k, 0)(\omega') \) is a zero-cutset that cuts \( F_0 \) from \( F_m \). Note that the edges of \( \Gamma(k, 0)(\omega') \) outside of \( \mathcal{B}(k, m) \) will not affect whether or not \( \Gamma(k, 0)(\omega') \) cuts \( F_0 \) from \( F_m \) inside \( \mathcal{B}(k, m) \), so (6.10) follows. In addition, \( \Gamma'(k, m)(\omega') \) can be easily shown as a self-avoiding cutset, since \( \Gamma(k, m)(\omega') \) is self-avoiding.

If we change \( \omega' \) back to \( \omega \), \( e_i \) changes from zero back to original values. \( \Gamma'(k, m) \), as a vertex set, exists. But \( \Gamma'(k, m) \) will no longer be a zero-cutset. We denote by \( \Gamma'(k, m)(\omega) \) as the set corresponding to configuration \( \omega \). Note that the other edges except for \( e_i \) are all zero-edges in both \( \Gamma'(k, m)(\omega) \) and \( \mathcal{W}(k, m)(\omega) \), so

\[
\tau(\Gamma'(k, 0)(\omega)) = \tau(\mathcal{W}(k, m)(\omega)) = \tau_{\text{min}}(k, m)(\omega). \tag{6.11}
\]

Therefore, for each \( \omega \),

\[
|\Gamma'(k, 0)(\omega')|_e = |\Gamma'(k, 0)(\omega)|_e \geq \bar{N}(k, m)(\omega). \tag{6.12}
\]

For each \( \omega \), we focus on \( \omega' \). As we mentioned above, \( \Gamma'(k, 0)(\omega') \) is a self-avoiding zero-cutset contained inside \( \Gamma'(k, 0)(\omega') \). Note that \( L \), defined as the line below (5.2), must intersect \( \Gamma'(k, 0)(\omega') \) inside \( \mathcal{B}(k, m)(\omega') \), otherwise \( \Gamma'(k, 0)(\omega') \) will not be a cutset. We denote by \( z \) the intersection vertex. If it is not unique, we select the one with the largest \( x_1 \)-coordinate. Thus, there are at most \( m \) choices in \( \Gamma'(k, 0)(\omega') \) for the cube that contains \( z \), since \( \Gamma'(k, 0)(\omega') \) must stay inside \( \mathcal{B}(k, m) \). As we discussed in the proof of Theorem 1, if the number of cubes of \( \Gamma'(k, 0)(\omega') \) is \( s \), then

\[
\exists \text{ at least } s/2^{2d} \text{ disjoint } 3t \text{-cubes with their center cubes belong to } \Gamma'(k, 0)(\omega'). \tag{6.13}
\]
For each center $t$-cube in these $3t$-cubes, by Lemma 6, it has either the blocked or disjoint property. Let $M_{3t}(k, m)(\omega')$ be all $3t$-cubes and $M_{3t}(k, m)(\omega')$ be the number of these $3t$-cubes in $M_{3t}(k, m)(\omega')$. Note that each $t$-cube has $2t^d$ edges, so by (6.12), if $\tilde{N}(k, m)(\omega) = j$, by (6.12)

$$M_{3t}(k, m)(\omega') \geq \frac{\# \text{ cubes in } \Gamma_t(k, 0)(\omega')}{2^{2d}} \geq \frac{\Gamma(k, 0)(\omega')_{e}}{2^{2d+1}t^d} \geq \frac{\tilde{N}(k, m)(\omega)}{2(4t)^d} = \frac{j}{2(4t)^d}. \quad (6.14)$$

Furthermore, if

$$J(\omega) \leq j/(2(4t)^d) \text{ and } \tilde{N}(k, m)(\omega) = j \text{ and the number of cubes in } \Gamma_t'(k, 0)(\omega') \text{ is } s,$$

by (6.14), for each $\omega'$, there are at least

$$M_{3t}(k, m) - j/4(4t)^d = M_{3t}(k, m)/2 + M_{3t}(k, m)/2 - j/(4(4t)^d) \geq M_{3t}(k, m)/2 \geq s/2^{2d+1}$$

center cubes in $\Gamma'_t(k, 0)(\omega')$ with either the blocked or disjoint property and they do not contain $e_1, \ldots, e_J$ in their interiors. Recall that they are called the permanent blocked or disjoint cubes. Now we change these edges in $\{e_1, \ldots, e_J\}$ from zero back to the original values. We still have $s/2^{2d+1}$ permanent blocked or disjoint $t$-cubes. Also, by (6.12),

$$\# \text{ cubes in } \Gamma_t(k, 0)(\omega') = s \geq \frac{\Gamma(k, 0)(\omega')_{e}}{2(4t)^d} \geq \frac{\tilde{N}(k, m)(\omega)}{2(4t)^d} = \frac{j}{2(4t)^d}. \quad (6.15)$$

Finally, by Lemma 12,

$$\Gamma'_t(k, 0)(\omega') \text{ is } \mathbb{Z}^d\text{-connected}. \quad (6.16)$$

In summary, for each $\omega$, if $N(k, m)(\omega) = j$ and $J(\omega) \leq j/(2(4t)^d)$, then there are $s \geq j/(2t)^d$ and $\|z\| \leq m \leq \exp(\|k\|_v)$ such that $E(s, j, z)$ occurs, where $E(s, j, z)$ is the event defined in section 5 after (5.25). Therefore, by the same estimate as (5.26), there are $C_i = C_i(F, d, \epsilon, t)$ for $i = 1, 2$ and $C_i = C_i(F(0), d)$ for $i = 3, 4$ such that

$$P_{k, m}\left[ \tilde{N}(k, m) \geq n \right] \leq \sum_{j \geq n} \sum_{s \geq \frac{1}{(2t)^d}} \exp(s/2^{2d}) 4^s [C_3 \exp(-C_4 t)]^{s/2^{2d+1}} + C_1 \exp(-C_2 n)$$

$$\leq \sum_{j \geq n} \sum_{s \geq \frac{t}{(2t)^d}} \exp(s/\beta) 4^s [C_3 \exp(-C_4 t)]^{s/2^{2d+1}} + C_1 \exp(-C_2 n).$$

If we take $t$ large and $\beta$ large, there exist $C_i = C_i(F, d, t, \epsilon, \beta)$ for $i = 1, 2$ such that for all $n \geq \beta \|k\|_v$,

$$P_{k, m}[\tilde{N}(k, m) \geq n] \leq C_1 \exp(-C_2 n). \quad (6.17)$$

Therefore, Theorem 2 follows.
7 Patching cutsets.

Given a cutset $W(k, m)$ as we defined in section 1, we shall now discuss a few basic properties of this cutset. Let $k' = (k'_1, \cdots, k'_{d-1})$ and $k = (k_1, \cdots, k_{d-1})$ be two vectors. We say

$$k' \leq k \text{ if } 0 \leq k'_i \leq k_i \text{ for all } i = 1, \cdots, d - 1.$$

We also denote by $F'_0$ and $F'_m$ the bottom and the top faces of the box $B(k', m)$. With these definitions we have the following lemma.

**Lemma 13.** If $k' \leq k$, then

(a) $W(k, m) \cap B(k', m)$ is a cutset that cuts $F'_0$ from $F'_m$ in $B(k', m)$,

(b) $\tau(W(k', m)) \leq \tau(W(k, m))$,

(c) $\tau(W(k, m)) \leq \tau(W(k', m)) + \sum_{e \in B(k, m) \setminus B(k', m)} \tau(e)$.

**Proof.** To prove (a), we only need to show that any path in $B(k', m)$ from $F'_0$ to $F'_m$ must use at least one edge of $W(k, m) \cap B(k', m)$. Note that such a path is also a path from $F_0$ to $F_m$ in $B(k, m)$ and note also that $W(k, m)$ is a cutset, so any such path must use at least one edge of $W(k, m)$. On the other hand, any such path must stay in $B(k', m)$, so it must use at least one edge of $W(k, m) \cap B(k', m)$. Therefore, (a) follows. With (a), (b) follows from the definitions of $W(k', m)$ and $W(k, m)$ directly.

Now we show (c). By the same argument as (a), we can show $W(k', m) \cup [B(k, m) \setminus B(k', m)]$ is a cutset for $B(k, m)$, so (c) follows. □

Now we want to patch two smaller cutsets into a larger cutset. To do it, we need to study the traces of the cutset in the boundary of the box $B(k, m)$. We denote the hyperplane by

$$L_n = \{(x_1, \cdots, x_d) : x_1 = n\}.$$

For a cutset $W(k, m)$, we define its trace in the hyperplane $L_{k_1}$ by

$$I(k, m) = W(k, m) \cap L_{k_1}.$$

Let edges in $I(k, m)$ be $I_e(k, m)$. If we remove all the edges of $I_e(k, m)$ from $L_{k_1}$, but leave the vertices of these edges, the new graph, after removing these edges, consists of several clusters on $L_{k_1}$. Note that there might be a few clusters with only one isolated vertex. We now analyze these clusters on the hyperplane.
Figure 4: This graph shows the exits of upper tunnels and lower tunnels on hyperplane $L_{k_1}$. The middle plane is the cutset $W(k, m)$. There are four exits of tunnels. $S$ below and $T$ above the cutset are two trivial exits. The circled $S'$ and $T'$, above and below the cutset, respectively, are the other two exits. One can use the exits of the circled tunnels $S'$ or $T'$ from $F_m$ or $F_0$ to $T'$ or $S'$ to cross the middle surface without using its edges.

We denote by $T_{k_1,1}(k, m), \ldots, T_{k_1,t}(k, m) \subset L_{k_1} \setminus I_e(k, m)$ (see Fig. 4) the first kind of clusters such that each of their vertices is connected to $F_m$ by a path lying in $B(k, m)$ without using any edge of $W(k, m)$ (see Fig. 4). Note that $t \geq 1$ since

$$L_{k_1} \cap F_m \neq \emptyset.$$

Here we may view $T$ as both a vertex and an edge set.

We also denote by $S_{k_1,1}(k, m), \ldots, S_{k_1,s}(k, m) \subset L_{k_1} \setminus I_e(k, m)$ (see Fig. 4) as the second kind of clusters such that each of their vertices is connected to $F_0$ by a path lying in $B(k, m)$ without using any edge of $W(k, m)$. Similarly, we have $s \geq 1$. We write these $T$ and $S$ for the exits of upper tunnels and exits of lower tunnels, respectively. If we do not work on a specific box, we may just write $T_{k_1,j}$ and $S_{k_1,i}$ rather than $T_{k_1,j}(k, m)$ and $S_{k_1,i}(k, m)$ as the exits of the upper and the lower tunnels. With these definitions, we have the following lemma.

**Lemma 14.** For all configurations,

$$S_{k_1,i}(k, m) \cap T_{k_1,j}(k, m) = \emptyset$$

for all $i = 1, \ldots, s$ and $j = 1, \ldots, t$.

**Proof.** If there exists a common vertex belonging to $S_{k_1,i} \cap T_{k_1,j}$ for some $i$ and $j$, then there exist paths from $F_0$ to $F_m$ in $B(k, m)$ without using an edge of the cutset $W(k, m)$.
Figure 5: This cross graph shows that if all the upper and lower tunnels of \( B(k, m) \) and of \( B'(k, m) \), respectively, in \( L_{k_1} \) are the same, and if there is a path from \( F_0 \cup F'_0 \) to \( F_m \cup F'_m \) without using \( W(k, m) \cup W'(k, m) \), then the path, from \( v_{11} \) to \( v'_{l} \), must go around the tunnels. Hence, the result is that the exits of the lower and upper tunnels for \( W(k, m) \) have a common vertex, which is a contradiction.

This contradicts the definition of \( W(k, m) \). □

Also by our definitions of \( S \) and \( T \), we have the following lemma.

**Lemma 15.** (a) The boundary edges of \( T_{k_1,j} \) and \( S_{k_1,i} \), on \( L_{k_1} \cap B(k, m) \), belong to \( W(k, m) \).

(b) For any vertex \( v \in L_{k_1} \cap B(k, m) \), if there exists a path from \( v \) to \( S_{k_1,i} \) (to \( T_{k_1,j} \)) in \( B(k, m) \) without using any edge of \( W(k, m) \), then \( v \in S_{k_1,i} \) (\( v \in T_{k_1,j} \)).

(c) \( W(k, m), T_{k_1,j} \) and \( S_{k_1,i} \) only depend on the configurations of edges in \( B(k, m) \).

With these observations, we are ready to patch two cutsets on two adjacent boxes. Before we state our result, we would like introduce more definitions (see Fig. 5). Let \( W(k, m) \) and \( W'(k, m) \) be two cutsets from \( F_0 \) to \( F_m \) and from \( F'_0 \) to \( F'_m \) of \( B(k, m) \) and \( B'(k, m) \), respectively, where

\[
B'(k, m) = [k_1 + 1, 2k_1 + 1] \times [0, k_2] \times \cdots \times [0, k_{d-1}] \times [0, m]
\]

and \( F'_0 \) and \( F'_m \) are the bottom and top faces of \( B'(k, m) \). Here we select \( W'(k, m) \) to be a self-avoiding cutset with the minimal passage time in \( B'(k, m) \). If there is more than one
such cutset, we simply select one with the unique method, but it may not necessarily follow
the same rule as the selection for \( W(k, m) \).

Similarly, we define the hyperplane of \( B'(k, m) \) next to \( L_{k_1} \) by
\[
L_{k_1+1} = \{(x_1, \cdots, x_d) : x_1 = k_1 + 1\}.
\]
Let \( E'(k, m) \) be the edge set with vertices in
\[
W'(k, m) \cap L_{k_1+1}.
\]
We denote
\[
B(k, m) \cup B'(k, m) = [0, 2k_1 + 1] \times [0, k_2] \times \cdots \times [0, k_{d-1}] \times [0, m]
\]
and use \( F_0 \cup F'_0 \) and \( F_m \cup F'_m \) to denote its bottom and top faces.

Similarly, let \( T'_{k_1+1,j} \) and \( S'_{k_1+1,i} \) be the exits of the upper or lower tunnels of \( W'(k, m) \)
on the hyperplanes \( L_{k_1+1} \). We say \( T \) is shifted \( l \) units invariance as \( T' \) if
\[
T' = \{ (u_1 + l, u_2, \cdots, u_d) : (u_1, \cdots, u_d) \in T \}.
\]
We write \( T \triangleq T' \) for the above \( T \) and \( T' \).

**Lemma 16.** Let \( \{T_{k_1,1}, T_{k_1,2}, \cdots, T_{k_1,t}\} \) and \( \{S_{k_1,1}, S_{k_1,2}, \cdots, S_{k_1,s}\} \) be the exits of the upper and the lower tunnels in \( L_{k_1} \) for \( W(k, m) \). Let \( \{T'_{k_1,1}, T'_{k_1,2}, \cdots, T'_{k_1,t}\} \) and \( \{S'_{k_1,1}, S'_{k_1,2}, \cdots, S'_{k_1,s}\} \) be the exits of the upper and the lower tunnels in \( L_{k_1+1} \) for \( W'(k, m) \). If \( T_{k_1,j} \triangleq T'_{k_1+1,j} \) and \( S_{k_1,i} \triangleq S'_{k_1,i} \) for all \( i \) and \( j \), then \( W(k, m) \cup W(k'm) \) is a cutset that cuts from \( F_0 \cup F'_0 \) to \( F_m \cup F'_m \) in the box \( B(k, m) \cup B'(k, m) \).

**Proof.** Under the hypotheses of Lemma 16, we suppose that there is a path \( \gamma \) in \( B(k, m) \cup B'(k, m) \) from \( F_0 \cup F'_0 \) to \( F_m \cup F'_m \) without using any edge of \( W(k, m) \cup W(k'm) \). Since \( W(k, m) \) and \( W(k'm) \) are cutsets of \( B(k, m) \) and \( B'(k, m) \), \( \gamma \) cannot lie in \( B(k, m) \) or in \( B'(k, m) \), respectively. The path \( \gamma \) should be a snake-shaped between two boxes \( B(k, m) \) and \( B'(k, m) \) (see Fig. 5). We then go along \( \gamma \) from \( F_0 \cap F'_0 \) to \( F_m \cup F'_m \). Without loss of generality, we assume that \( \gamma \) starts at \( F_0 \) and ends at \( F'_m \). With this definition, \( \gamma \) must go out of the hyperplane \( L_{k_1} \). Let \( v_1 \) be the first vertex that \( \gamma \) exits from \( L_{k_1} \). After that, \( \gamma \) must go through \( L_{k_1+1} \) at \( v'_1 \). Let \( e_{v_1,v'_1} \) be the edge with vertices \( v_1 \) and \( v'_1 \) (see Fig. 5). Note that \( e_{v_1,v'_1} \) is neither in \( B(k, m) \) nor in \( B(k', m) \), but just between these two boxes. We then continue following \( \gamma \) from \( v'_1 \). If it can reach \( F'_m \) inside \( B(k', m) \) directly, then we stop our trip. If it cannot, let \( v''_2 \) be the vertex in \( L_{k_1+1} \) that \( \gamma \) first goes out of \( B'(k, m) \). Similarly, we
will have the vertex \( v_2 \in \gamma \cap L_{k_1} \) such that \( \gamma \) first goes back at \( v_2 \) from \( B'(k, m) \). Let \( e_{v_2,v_2} \) be the edge with the vertices \( v_2 \) and \( v_2' \) between these two boxes. We continue this process until \( \gamma \) reaches \( F'_m \). Let \( v_0 \in F_0 \) and \( v'_0 \in F'_m \) be the starting vertex and the ending vertex of \( \gamma \). Our \( \gamma \) contains the following vertices and edges between \( B(k, m) \) and \( B'(k, m) \):

\[
v_0, v_1, e_{v_1,v_1'}, v_1', v_2, e_{v_2,v_2'}, v_2', v_3, e_{v_3,v_3'}, v_3', \ldots, v'_l.
\]

Note that \( \gamma \) never uses an edge of \( W(k, m) \), so \( v_1 \in \bigcup_j S_{k_1,i} \). By the assumption of Lemma 16, \( v'_1 \in \bigcup_j S'_{k_1+1,i} \). By Lemma 15 (b), \( v'_2 \in \bigcup_j S'_{k_1+1,i} \) so \( v_2 \in \bigcup_j S_{k_1,i} \). If we iterate this way, we finally have \( v'_l \in \bigcup_j S'_{k_1+1,i} \). However, note that \( \gamma \) never uses an edge of \( W(k, m) \), so \( v_l \in \bigcup_j T'_{k_1+1,j} \). Therefore, this result shows that \( \bigcup_j S'_{k_1,i} \) and \( \bigcup_j T'_{k_1,j} \) have a common vertex, but it contradicts Lemma 14, so Lemma 16 follows. □

8 Estimates for the boundary size of a cutset.

A cutset \( W_r(k, m) \) in \( B(k, m) \) cutting \( F_0 \) from \( F_m \) is said to be regular if

\[
|W_r(k, m)| \leq \beta \|k\|_v, \tag{8.0}
\]

where \( \beta = 2d\beta \) for the \( \beta \) defined in Theorem 2. We select a regular cutset, still denoted by \( W_r(k, m) \), with the minimum passage time. We may also denote \( \tau(W_r(k, m)) = \tau_r(k, m) \). If \( W_r(k, m) \) is not unique, we select it with the minimum number of edges using the unique method of selection. In particular, if \( W(k, m) \), defined in section 1, satisfies (8.0), we only select

\[
W_r(k, m) = W(k, m).
\]

Clearly,

\[
\tau_{\min}(k, m) \leq \tau_r(k, m).
\]

If \( |W(k, m)| \geq \beta \|k\|_v \), then

\[
|W(k, m)|_v \geq \beta \|k\|_v.
\]

By this observation and Theorem 2, there exist \( C_i = C_i(\beta, F) \) such that

\[
P[W(k, m) \neq W_r(k, m)] \leq P[N(k, m) \geq \beta \|k\|_v] \leq C_1 \exp(-C_2 \|k\|_v). \tag{8.1}
\]

Now we only focus on regular cutsets. Under (8.0), note that there are \( k_1 \) disjoint hyperplanes in \( B(k, m) \) perpendicular to the first coordinate, so the number of vertices of \( W_r(k, m) \) on a few of these hyperplanes should be much less than \( \beta \|k\|_v \). Recall that \( L_i \) is defined in section
7 as the hyperplane of \( \{ x_1 = i \} \). Now we try to find two such hyperplanes. We account for the size of \( \{ W_r(k, m) \cap L_{k_i} \} \cup \{ W_r(k, m) \cap L_0 \} \) to see whether

\[
|\{ W_r(k, m) \cap L_{k_i} \} \cup \{ W_r(k, m) \cap L_0 \}| \leq \bar{\beta}k_1^{\delta/2}k_2 \cdots k_{d-1}, \tag{8.2}
\]

where \( \delta \) is defined in (1.11). If the cutset satisfies (8.2), we select \( L_0 \) and \( L_{k_1} \). If it does not, we account for the size of \( W_r(k, m) \cap L_1 \) and \( W_r(k, m) \cap L_{k_1-1} \) to see whether

\[
|\{ W_r(k, m) \cap L_{k_1-1} \} \cup \{ W_r(k, m) \cap L_1 \}| \leq \bar{\beta}k_1^{\delta/2}k_2 \cdots k_{d-1}. \tag{8.3}
\]

If the cutset satisfies (8.3), we select \( L_1 \) and \( L_{k_1-1} \). If it does not, we continue this process until we find the first hyperplanes \( L_r \) and \( L_{k_1-\tau} \) such that

\[
|\{ W_r(k, m) \cap L_{k_1-\tau} \} \cup \{ W_r(k, m) \cap L_r \}| \leq \bar{\beta}k_1^{\delta/2}k_2 \cdots k_{d-1}. \tag{8.4}
\]

Note that the total number of vertices in a regular cutset is less than \( \bar{\beta}k_1k_2 \cdots k_{d-1} \), so we need to do this process at most \( k_1^{1-\delta/2} \) times to find the hyperplanes. In other words,

\[
\tau \leq k_1^{1-\delta/2}. \tag{8.5}
\]

By (8.5), there exists \( 0 < l < k_1^{1-\delta/2} \) such that

\[
P[\mathcal{B}(k, m, l)] \geq \frac{1}{2k_1^{1-\delta/2}}, \tag{8.6}
\]

where \( \mathcal{B}(k, m, l) \) is the event that \( l \) is the first hyperplane with the property (8.4).

For the fixed \( l \leq k_1^{1-\delta/2} \) defined in (8.6), we collect all cutsets \( \{ W_r(k, m, l) \} \) in

\[
\mathcal{B}(k, m, l) = [l, k_1 - l] \times [0, k_2] \times \cdots [0, k_{d-1}] \times [0, m]
\]

cutting the bottom from the top of \( \mathcal{B}(k, m, l) \) such that

\[
|W_r(k, m, l)| \leq \bar{\beta}\|k\|_v, |\{ W_r(k, m, l) \cap L_l \} \cup \{ W_r(k, m, l) \cap L_{k_1-1} \}| \leq \bar{\beta}k_1^{\delta/2}k_2 \cdots k_{d-1}. \tag{8.7}
\]

We select one from these cutsets, still denoted by \( W_r(k, m, l) \), with the minimum passage time:

\[
\tau_r(k, l, m) = \tau(W_r(k, l, m))
\]

If \( W_r(k, m, l) \) is not unique, we select \( W_r(k, m, l) \) with the minimum number of edges in a unique method of selection. By our definition,

\[
\tau_r(k, m, l) \text{ only depends on the configurations of the edges in } \mathcal{B}(k, m, l). \tag{8.8}
\]
Lemma 17. On $\mathcal{B}(k, m, l)$,

$$\tau_r(k, m, l) \leq \tau_r(k, m).$$

Proof. By Lemma 13 (a),

$$\mathcal{W}_r(k, m) \cap \mathcal{B}(k, m, l)$$

is a cutset that cuts the bottom from the top of $\mathcal{B}(k, m, l)$. On the other hand, on $\mathcal{B}(k, m, l)$, the cutset in (8.9) satisfies (8.7). Therefore, Lemma 17 follows. □

We use $\{T_{(i,j)}\}, \{T_{(k_1-l,i)}\}, \{S_{(i,j)}\}$, and $\{S_{(k_1-l,i)}\}$ to denote all the exits of the upper and the lower tunnels on the hyperplanes $L_l$ and $L_{k_1-l}$ for the cutset $\mathcal{W}_r(k, m, l)$, respectively. For given positive integers $t_1, t_2, s_1, s_2$, we now define the events

$$\{\mathcal{I}_{t_1, t_2}\} = \{\exists t_1 \text{ exits of the upper tunnels } T_{(i,1)}, \cdots, T_{(i,t_1)} \text{ on } L_l \text{ and } \exists t_2 \text{ exits of the upper tunnels } T_{(k_1-l,1)}, \cdots, T_{(k_1-l,t_2)} \text{ on } L_{k_1-l}\}$$

$$\{\mathcal{J}_{s_1, s_2}\} = \{\exists s_1 \text{ exits of the lower tunnels } S_{(i,1)}, \cdots, S_{(i,s_1)} \text{ on } L_l \text{ and } \exists s_2 \text{ exits of the lower tunnels } S_{(k_1-l,1)}, \cdots, S_{(k_1-l,s_2)} \text{ on } L_{k_1-l}\}.$$

On $\{\mathcal{I}_{t_1, t_2}\} \cap \{\mathcal{J}_{s_1, s_2}\}$, note that $\mathcal{W}_r(k, m, l)$ is uniquely selected, so the exits of the lower and upper tunnels are also uniquely determined. Thus, we decompose the exits of the tunnels to fixed sets:

$$1 = P[\exists \text{ cutset } \mathcal{W}_r(k, m, l)]$$

$$= \sum_{t_1, t_2} \sum_{s_1, s_2} \sum_{\Gamma_{(i,1)}, \cdots, \Gamma_{(i,t_1)} \cdots, \Gamma_{(k_1-l,1)}, \cdots, \Gamma_{(k_1-l,t_2)} \cdots, \beta_{(i,1)}, \cdots, \beta_{(i,s_1)} \cdots, \beta_{(k_1-l,1)}, \cdots, \beta_{(k_1-l,s_2)}} P[\exists \mathcal{W}_r(k, m, l), \mathcal{I}_{t_1, t_2}, \mathcal{J}_{s_1, s_2}, \bigcap_{j=1}^{t_1} \{T_{(i,j)} = \Gamma_{(i,j)}\}, \bigcap_{j=1}^{t_2} \{T_{(k_1-l,j)} = \Gamma_{(k_1-l,j)}\}, \bigcap_{i=1}^{s_1} \{S_{(i,i)} = \beta_{(i,i)}\}, \bigcap_{j=1}^{s_2} \{S_{(k_1-l,j)} = \beta_{(k_1-l,j)}\}],$$

where the first two sums above take over all possible $t_1, t_2, s_1,$ and $s_2$, and the last two sums take all possible groups of fixed clusters such that each group of clusters

$$\Gamma_{(i,1)}, \Gamma_{(i,2)}, \cdots, \Gamma_{(i,t_1)}, \beta_{(i,1)}, \beta_{(i,2)}, \cdots, \beta_{(i,s_1)} \subset L_l \cap \mathcal{B}(k, m, l)$$

and

$$\Gamma_{(k_1-l,1)}, \Gamma_{(k_1-l,2)}, \cdots, \Gamma_{(k_1-l,t_2)}, \beta_{(k_1-l,1)}, \beta_{k_1-l,2}, \cdots, \beta_{(k_1-l,s_2)} \subset L_{k_1-l} \cap \mathcal{B}(k, m, l).$$

(8.11)
For simplicity, we denote each group of clusters by,

\[ \Gamma_{(I, I)} = \{ \Gamma_{l,1}, \cdots, \Gamma_{l,t_1} \}, \quad \Gamma_{(I, II)} = \{ \Gamma_{k_1-l,1}, \cdots, \Gamma_{l,t_2} \}, \]
\[ \beta_{(I, I)} = \{ \beta_{l,1}, \cdots, \beta_{l,s_1} \}, \quad \beta_{(I, II)} = \{ \beta_{k_1-l,1}, \cdots, \beta_{k_1-l,s_2} \}. \]

We also denote the event in the probability of the right side of (8.10) for the group of clusters in (8.11) by

\[ \mathcal{D}_1(t_1, t_2, s_1, s_2, \Gamma_{(I, I)}, \Gamma_{(k_1-l,II)}, \beta_{(I, I)}, \beta_{(k_1-l,II)}) \].

Note that for some groups of clusters, we have

\[ P \left[ \mathcal{D}_1(t_1, t_2, s_1, s_2, \Gamma_{(I, I)}, \Gamma_{(k_1-l,II)}, \beta_{(I, I)}, \beta_{(k_1-l,II)}) \right] = 0. \]

In these cases, the groups of clusters are trivial and we will not account for these terms in the four sums in (8.10). Note also that the four sums only take finitely many terms, so there is a term with the largest probability among the others. We denote this largest term with the indexes \( t_1, t_2, s_1, s_2 \) and denote the group of clusters by

\[ \Gamma_{(I, I)} = \{ \Gamma_{l,1}, \cdots, \Gamma_{l,t_1} \}, \quad \Gamma_{(I, II)} = \{ \Gamma_{k_1-l,1}, \cdots, \Gamma_{l,t_2} \}, \]
\[ \beta_{(I, I)} = \{ \beta_{l,1}, \cdots, \beta_{l,s_1} \}, \quad \beta_{(I, II)} = \{ \beta_{k_1-l,1}, \cdots, \beta_{k_1-l,s_2} \}. \]

We also define

\[
\max_{\substack{t_1, t_2, s_1, s_2, \Gamma_{l,1}, \cdots, \Gamma_{l,t_1}, \beta_{l,1}, \cdots, \beta_{l,s_1} \\
\Gamma_{k_1-l,1}, \cdots, \Gamma_{k_1-l,t_2}, \beta_{k_1-l,1}, \cdots, \beta_{k_1-l,s_2}}}

P \left[ \mathcal{J}_{t_1, t_2}, \mathcal{J}_{s_1, s_2}, \bigcap_{j=1}^{t_1} \{ T_{i,j} = \Gamma_{l,j} \}, \bigcap_{i=1}^{s_1} \{ S_{l,i} = \beta_{l,i} \} \right]
\left[ \bigcap_{j=1}^{t_2} \{ T_{k_1-l,j} = \Gamma_{k_1-l,j} \}, \bigcap_{i=1}^{s_2} \{ S_{k_1-l,i} = \beta_{k_1-l,i} \} \right]
:= P \left[ \mathcal{D}_1(t_1, t_2, s_1, s_2, \Gamma_{(I, I)}, \Gamma_{(k_1-l,II)}, \beta_{(I, I)}, \beta_{(k_1-l,II)}) \right].
\]

It is possible that there is another group of clusters with the same largest probability. If this occurs, we select one group in a unique method. We will account for the number of non-trivial groups of the clusters in the four sums in (8.10). In other words, we need to account for all possible groups of clusters on \( L_{k_1-l} \cup B(k, m, l) \) and \( L_{k_1-l} \cup B(k, m, l) \) such that they are the exits of upper or lower tunnels for \( W_r(k, m, l) \). Let \( N_r(k, m, l) \) be the number of all the possible non-trivial groups of clusters above. We will then give an upper bound estimate. For fixed \( l \leq k^{1-\delta/2} \), let

\[ I(k, m, l) = \{ W_r(k, m, l) \cap L_t \} \cup \{ W_r(k, m, l) \cap L_{k_1-l} \}. \]
By the definition,

\[ |I(k, m, l)| \leq \bar{\beta}_1^{\delta/2}k_2 \cdots k_{d-1}. \]

If we use \( I_e(k, m, l) \) to denote the edge set in \( I(k, m, l) \), then

the number of edges in \( I_e(k, m, l) \leq 2d\bar{\beta}_1^{\delta/2}k_2 \cdots k_{d-1}. \) \hspace{1cm} (8.12)

Note that the total number of vertices of \( B(k, m) \) on the two faces

\[ |B(k, m) \cap L_l \cup B(k, m) \cap L_{k_1-l}| \leq 2k_1 \cdots k_{d-1}m. \]

Therefore, the total number of choices of all possible \( I(k, m, l) \) is less than

\[ \sum_{r=1}^{\bar{\beta}_1^{\delta/2}k_2 \cdots k_{d-1}} \left( \begin{array}{c} 2k_1k_2 \cdots k_{d-1}m \\ r \end{array} \right) \leq \bar{\beta}_1^{\delta/2}k_2 \cdots k_{d-1} \left( \begin{array}{c} 2k_1k_2 \cdots k_{d-1}m \\ \bar{\beta}_1^{\delta/2}k_2 \cdots k_{d-1} \end{array} \right) . \] \hspace{1cm} (8.13)

Note that for a cluster on \( L_l \) and on \( L_{k_1-l} \), if its boundary edges are fixed, then the precise location of the cluster is uniquely fixed. By Lemma 15 (a), the boundary edges of the exits of the upper and the lower tunnels belong to \( I_e(k, m, l) \). Note that if we remove \( I_e(k, m, l) \) from both \( L_l \) and \( L_{k_1-l} \), we can view the remaining edges as many clusters. These clusters are the exits of the upper and the lower tunnels. With these clusters, we need to identify the upper or the lower exists from them. Given a fixed \( I_e(k, m, l) \), suppose that there are \( q \) clusters, as the exits of the upper and the lower tunnels on both \( L_l \) and \( L_{k_1-l} \), after removing the edges of \( I_e(k, m, l) \) from \( L_l \) and \( L_{k_1-l} \). Note that if we remove one edge, it can separate one cluster into at most two clusters. Therefore, by (8.12) after removing \( I_e(k, m, l) \), the total number of the clusters of these exits of the upper and the lower tunnels is

\[ q \leq 2|I_e(k, m, l)| \leq 4d\bar{\beta}_1^{\delta/2}k_2 \cdots k_{d-1}. \] \hspace{1cm} (8.14)

Among these \( q \) fixed clusters, we select some of them as the exits of the upper and lower tunnels on \( L_l \) and on \( L_{k_1-l} \). By (8.12) and (8.14), the number of selections is at most

\[ \sum_{t_1=1}^{q} \sum_{t_2=1}^{q} \left( \begin{array}{c} q \\ t_1 \end{array} \right) \left( \begin{array}{c} q \\ t_2 \end{array} \right) \leq 2^{2q} \leq 2^{8d\bar{\beta}_1^{\delta/2}k_2 \cdots k_{d-1}}. \] \hspace{1cm} (8.15)

With these observations, we first select \( I(k, m, l) \), defined above, on \( L_l \) and \( L_{k_1-l} \). With the first selection, \( I_e(k, m, l) \) is determined. After removing \( I_e(k, m, l) \), the remaining clusters, the exits of the upper and the lower tunnels, are determined. We then select the exits for the upper and for the lower tunnels from these clusters. With these selections and
(8.12)–(8.15), the total number, \(N_r(k, m, l)\), of all the possible exits of the upper and the lower tunnels is at most

\[
N_r(k, m, l) \leq \beta k_1^{\delta/2} k_2 \cdots k_{d-1} \left(\frac{2k_1 k_2 \cdots k_{d-1} m}{\beta k_1^{\delta/2} k_2 \cdots k_{d-1}}\right)^{2^{8d+5k_1^{\delta/2} k_2 \cdots k_{d-1}}}.
\]

(8.16)

By using Corollary 2.6.2 in Engle (1997),

\[
N_r(k, m, l) \leq C_1 \exp \left[ C_2 k_1^{\delta/2} \cdots k_{d-1} \log(k_1 \cdots k_{d-1} m) \right]
\]

(8.17)

for \(C_i = C_i(F, d, \beta, \delta)\), \(i = 1, 2\). Furthermore, if we assume that \(k_1 \leq k_2 \leq \cdots k_{d-1}\) with \(k_{d-1} \leq 2 \exp(10 k_1^{1-5\delta/6})\) and \(\log m \leq k_{d-1}^{1-\delta}\) (the assumptions for \(m\) in (1.17)), there exist \(C_i = C_i(F, d, \delta)\) such that

\[
N_r(k, m, l) \leq C_1 \exp \left[ C_2 k_1^{\delta/2} k_2 \cdots k_{d-1} \log(k_{d-1} m) \right] \leq C_3 \exp \left[ C_4 k_1^{1-\delta/3} k_2 \cdots k_{d-1} \right].
\]

(8.18)

Therefore, the number of all terms in these four sums in (8.10) is at most \(N_r(k, m, l)\). With these observations, by (8.18),

\[
1 = P[\exists \text{ the cutset } W(k, m, l)]
\]

\[
= \sum_{t_1, t_2} \sum_{s_1, s_2} \sum_{\Gamma(t, l, t') \Gamma(t, l, t'_1)} \sum_{\beta(t, l, t')} \sum_{\beta(t, l, t'_1)} P[D_1(t_1, t_2, s_1, s_2, \Gamma(t, l), \Gamma(k-l, l), \beta(t, l), \beta(k-l, l))]
\]

\[
\leq P[D_1(\bar{t}_1, \bar{t}_2, \bar{s}_1, \bar{s}_2, \bar{\Gamma}(l, l), \bar{\Gamma}(k-l, l), \bar{\beta}(l, l), \bar{\beta}(k-l, l))] N_r(k, m, l)
\]

\[
\leq P[D_1(\bar{t}_1, \bar{t}_2, \bar{s}_1, \bar{s}_2, \bar{\Gamma}(l, l), \bar{\Gamma}(k-l, l), \bar{\beta}(l, l), \bar{\beta}(k-l, l))] C_1 \exp \left( C_2 k_1^{1-\delta/3} k_2 \cdots k_{d-1} \right).
\]

If we simply denote by

\[
D_1(\bar{t}_1, \bar{t}_2, \bar{s}_1, \bar{s}_2, \bar{\Gamma}(l, l), \bar{\Gamma}(k-l, l), \bar{\beta}(l, l), \bar{\beta}(k-l, l)) = D_1,
\]

we summarize the above result as the following lemma.

**Lemma 18.** If \(k_1 \leq k_2 \leq \cdots \leq k_{d-1}\) with \(k_{d-1} \leq 2 \exp(10 k_1^{1-5\delta/6})\), and \(\log m \leq k_{d-1}^{1-\delta}\), then there are constants \(C_i = C_i(F, d, \beta, \delta)\) for \(i = 1, 2\) such that

\[
C_1 \exp \left( -C_2 k_1^{1-\delta/3} k_2 \cdots k_{d-1} \right) \leq P[D_1].
\]

If we work on \(k_j\)’s direction rather than \(k_1\)’s, similar to \(D_1\), let \(D_j\) be the event corresponding to the \(j\)-th coordinate. By the same estimate, we have the following the result in Lemma 18 for \(D_j\) holds.
Lemma 19. If \( k_1 \leq k_2 \leq \cdots \leq k_{d-1} \) with \( k_{d-1} \leq 2 \exp(10k_1^{1/3}/6) \), and \( \log m \leq k_{d-1}^{1/3} \), then there are constants \( C_i = C_i(F, d, \beta, \delta) \) for \( i = 1, 2 \) such that
\[
C_1 \exp \left( -C_2 k_1 \cdots k_{d-1}^{1/3} \right) \leq P[D] .
\]
In particular, if \( k_1 \leq k_2 \leq \cdots \leq k_{d-1} \) and \( \log m \leq k_1^{1/3} \), then there are constants \( C_i = C_i(F, d, \beta, \delta) \) for \( i = 1, 2 \) such that
\[
C_1 \exp \left( -C_2 k_1 \cdots k_{d-1}^{1/3} \right) \leq P[D_{d-1}] .
\]

9 Concentration of \( \tau_r(k, m) \) from its mean.

In general, there are two major methods to estimate the concentration inequalities. Kesten (1993) has investigated the concentration for the first passage percolation by using a martingale argument. Later, Talagrand (1995) obtained a better result by using the isoperimetric inequality. Both ways can be carried out to investigate the concentration for the passage time of a minimal cutset from its mean. We use the Talagrand method in this paper. Denote by \( S \) the sets of all regular cutsets \( \{Z_r(k, m)\} \), defined in section 8, with the minimum passage time. Let
\[
\alpha = \sup_{Z_r(k, m) \in S} |Z_r(k, m)| .
\]
It follows from this definition
\[
\alpha \leq \beta \|k\|_v .
\]
(9.0)

Denote by \( M \) a median of \( \tau_r(k, m) \). By Theorem (8.3.1) (see Talagrand (1995)) there exist constants \( C \) and \( C_1 \) such that
\[
P[|\tau_r(k, m) - M| \geq u] \leq C \exp \left( -C_1 \min \left\{ \frac{u^2}{\alpha}, u \right\} \right) .
\]
(9.1)

By (9.0) and (9.1), for all \( u > 0 \),
\[
P[|\tau_r(k, m) - M| \geq u] \leq C \exp \left( -C_1 \min \left\{ \frac{u^2}{\beta \|k\|_v}, u \right\} \right) .
\]
(9.2)

If we select \( u \) satisfying
\[
(\|k\|_v)^{2/3} \leq u ,
\]
(9.3)
then
\[
P[|\tau_r(k, m) - M| \geq u] \leq C_1 \exp \left( -C_2 \|k\|_v^{1/3} \right) .
\]
(9.4)
By (9.4),
\[ |E[r(k, m)] - M| \leq E[|r(k, m) - M|] = \sum_{i=1}^{P} P(|r(k, m) - M| \geq i) \leq C(||k||_{v})^{2/3}. \tag{9.5} \]

Thereofore, for all large $k_1, \ldots, k_{d-1}$, and for $u$ with
\[ \max\{2C(||k||_{v})^{2/3}, \beta(||k||_{v})^{2/3}\} \leq u \text{ for the } C \text{ in (9.5)}, \tag{9.6} \]
then by (9.2) and (9.5),
\[
P[|r(k, m) - E[r(k, m)]| \geq u] \leq P[|r(k, m) - M| + |M - E[r(k, m)]| \geq u] \leq C \exp \left(-C_1 \frac{u^2}{||k||_{v}}\right). \tag{9.7} \]

If we focus on $r(k, m, l)$, the passage time of cutsets $W_r(k, m, l)$, then by the same estimates in (9.1)–(9.7), for all large $k_1, \ldots, k_{d-1}$, and for the $u$ satisfying (9.6), we have
\[
P[|r(k, m, l) - E[r(k, m, l)]| \geq u] \leq C \exp \left(-C_1 \frac{u^2}{||k||_{v}}\right). \tag{9.8} \]

Now we will try to use the concentration property to estimate the means of $r(k, m)$ and $r(k, m, l)$ on some event $\mathcal{E}$ that may depend on $k$ and $m$.

**Lemma 20.** Under (1.1), there exist $C'_i = C'_i(F, d, \beta, \delta)$ for $i = 1, 2$ such that for each $k_j, j = 1, 2, \ldots, d-1$ and $0 < \delta \leq 1$,
\[
|E[r(k, m)] - E[r(k, m) | \mathcal{E}]| \\
\leq C_1 k_1 \cdots k_j^{(1-\delta/8)} \cdots k_{d-1} + C_1 \{P(\mathcal{E})\}^{-1} \exp \left(-C_2 k_1 \cdots k_j^{(1-\delta/4)} \cdots k_{d-1}\right)
\]
and
\[
|E[r(k, m, l)] - E[r(k, m, l) | \mathcal{E}]| \\
\leq C_1 k_1 \cdots k_j^{(1-\delta/8)} \cdots k_{d-1} + C_1 \{P(\mathcal{E})\}^{-1} \exp \left(-C_2 k_1 \cdots k_j^{(1-\delta/4)} \cdots k_{d-1}\right).
\]

**Proof.** Without loss of generality, we show Lemma 20 for $j = 1$. We begin with an estimate of
\[
E[|E[r(k, m)] - r(k, m) | \mathcal{E}] = 0.
\]
Denote the event $\mathcal{L}(k, m)$ by
\[
\mathcal{L}(k, m) = \{|E[r(k, m)] - r(k, m)| > \beta k_1^{(1-\delta/8)} k_2 \cdots k_{d-1}\}.
\]
We divide
\[ E \left[ |E_{\tau_r}(k, m) - \tau_r(k, m)| \mid \mathcal{E} \right] = E \left[ |E_{\tau_r}(k, m) - \tau_r(k, m)| I(\mathcal{L}(k, m)) \mid \mathcal{E} \right] + E \left[ |E_{\tau_r}(k, m) - \tau_r(k, m)|(1 - I(\mathcal{L}(k, m))) \mid \mathcal{E} \right] = I + II, \]
where \( I(\mathcal{A}) \) is the indicator for the event \( \mathcal{A} \). By the definition of \( \mathcal{L}(k, m) \),
\[ II \leq \beta k_1^{(1-\delta)/8} k_2 \cdots k_{d-1}. \] (9.9)

We estimate \( I \). By (9.7), there exist \( C \) and \( C_1 \) such that
\[ I \leq \sum_{i \geq \beta k_1^{(1-\delta)/8} k_2 \cdots k_{d-1}} P \left[ |E_{\tau_r}(k, m) - \tau_r(k, m)| \geq i \mid \mathcal{E} \right] \]
\[ = \sum_{i \geq \beta k_1^{(1-\delta)/8} k_2 \cdots k_{d-1}} \frac{P \left[ |E_{\tau_r}(k, m) - \tau_r(k, m)| \geq i \right]}{P[\mathcal{E}]} \leq C\{P[\mathcal{E}]\}^{-1} \exp \left( -C_1 k_1^{(1-\delta/4)} k_2 \cdots k_{d-1} \right). \] (9.10)

Combining (9.9) and (9.10), we have
\[ E \left[ |E_{\tau_r}(k, m) - \tau_r(k, m)| \mid \mathcal{E} \right] \leq Ck_1^{(1-\delta)/8} k_2 \cdots k_{d-1} + C\{P[\mathcal{E}]\}^{-1} \exp \left( -C_1 k_1^{(1-\delta/4)} k_2 \cdots k_{d-1} \right). \] (9.11)

Using the same estimate of (9.11) together with (9.8),
\[ E \left[ |E_{\tau_r}(k, m, l) - \tau_r(k, m, l)| \mid \mathcal{E} \right] \leq Ck_1^{(1-\delta)/8} k_2 \cdots k_{d-1} + C\{P[\mathcal{E}]\}^{-1} \exp \left( -C_1 k_1^{(1-\delta/4)} k_2 \cdots k_{d-1} \right). \] (9.12)

With (9.11) and (9.12), let us show Lemma 20. We then have
\[ E_{\tau_r}(k, m) = E \left[ \tau_r(k, m) \mid \mathcal{E} \right] + E \left[ E_{\tau_r}(k, m) - \tau_r(k, m) \mid \mathcal{E} \right]. \] (9.13)

By (9.13), we have
\[ |E \left[ \tau_r(k, m) \right] - E \left[ \tau_r(k, m) \mid \mathcal{E} \right]| \leq E \left[ |E_{\tau_r}(k, m) - \tau_r(k, m)| \mid \mathcal{E} \right]. \] (9.14)

Therefore, by (9.11) and (9.14), there exists \( C_i = C_i(F, d, \beta, \delta) \) for \( i = 1, 2 \) such that
\[ |E \left[ \tau_r(k, m) \right] - E \left[ \tau_r(k, m) \mid \mathcal{E} \right]| \leq C_1 k_1^{(1-\delta/8)} k_2 \cdots k_{d-1} + C_1 \{P[\mathcal{E}]\}^{-1} \exp \left( -C_2 k_1^{1-\delta/4} k_2 \cdots k_{d-1} \right). \] (9.15)
The same estimate in (9.15) also shows that
\[
|E[\tau_r(k, m, l)] - E[\tau_r(k, m, l) \mid \mathcal{E}]| \\
\leq C_1 k_1^{(1-\delta/8)} k_2 \cdots k_{d-1} + C_1 \{P[\mathcal{E}]\}^{-1} \exp \left( -C_2 k_1^{(1-\delta/4)} k_2 \cdots k_{d-1} \right).
\] (9.16)

Lemma 20, for \( j = 1 \), follows from (9.15) and (9.16). \( \square \)

\section{Proof of Theorem 3.}

As we pointed out in section 1, we only need to show Theorem 3 when \( F(0) < 1 - p_c \). Thus, we assume that \( F(0) < 1 - p_c \) in this section. Note that
\[
0 \leq E_{\tau_{\min}}(k, m) \leq E\tilde{\alpha}(k, m) \leq C\|k\|_v E(\tau(e)),
\] (10.0)
so we assume that there exist \( 0 \leq \nu_1 \leq \nu_2 < \infty \) such that
\[
\nu_1 = \liminf_{k_1, \ldots, k_{d-1}, m \to \infty} \left( \frac{E_{\tau_{\min}}(k, m)}{\|k\|_v} \right) \leq \nu_2 = \limsup_{k_1, \ldots, k_{d-1}, m \to \infty} \left( \frac{E_{\tau_{\min}}(k, m)}{\|k\|_v} \right). \] (10.1)

We first show that \( \nu_1 = \nu_2 \). The key proof of this argument is to show a multiple subadditive property for \( E_{\tau_{\min}}(k, m) \).

Now we assume that
\[
k_1 \leq k_2 \leq \cdots \leq k_{d-1} \text{ with } k_{d-1} \leq 4k_1.
\] (10.2)

Besides \( B(k, m, l) \) defined in section 8, we also denote by (see Fig. 5)
\[
B'(k, m, l) = [k_1 - l + 1, 2k_1 - 3l + 1] \times [0, k_2] \times \cdots \times [0, k_{d-1}] \times [0, m],
\]
\[
B''(k, m, l) = [2k_1 - 3l + 2, 3k_1 - 5l + 2] \times [0, k_2] \times \cdots \times [0, k_{d-1}] \times [0, m].
\]

We denote by \( \omega(B(k, m, l)), \omega(B'(k, m, l)), \) and \( \omega(B''(k, m, l)) \) the configurations on \( B(k, m, l), B'(k, m, l), \) and \( B''(k, m, l), \) respectively. For each \( \omega(B'(k, m, l)) \) and \( \omega(B''(k, m, l)) \), we can select the unique cutsets \( W'_r(k, m, l) \) and \( W''_r(k, m, l) \) in \( B'(k, m, l) \) and \( B''(k, m, l), \) respectively, using the same rule for selecting \( W_r(k, m, l) \) on \( B(k, m, l). \)

Recall that \( D_1 \), defined in section 8, is the event with the largest probability for the fixed exits of the upper and the lower tunnels for \( W_r(k, m, l) \). Similarly, let \( D'_1 \) and \( D''_1 \) be the events with the largest probabilities for the fixed exits of the upper and the lower tunnels for \( W'_r(k, m, l) \) and \( W''_r(k, m, l), \) respectively, the same as for \( W_r(k, m, l) \) in the sense of translation.
For each set \( A \subseteq B'(k, m, l) \), we define a mirror reflection about \( L_{k_1-l+0.5} \) as follows. For \( u = (u_1, \ldots, u_d) \in A \), let
\[
\sigma_1(u) = (2k_1 - 2l + 1 - u_1, u_2, \ldots, u_d).
\]
After the mirror reflection, \( \sigma_1(A) \subseteq B(k, m, l) \). We move \( \sigma_1(A) \) along the first coordinate \( k_1 - 2l + 1 \) units back to \( B'(k, m, l) \). More precisely, for each \( u \in \sigma_1(A) \), let
\[
\sigma_2(u) = (u_1 + k_1 - 2l + 1, u_2, \ldots, u_d).
\]
We denote by
\[
\pi(A) = \sigma_2(\sigma_1(A)).
\]
With these changes, we have another vertex set, denoted by \( \pi(A) \) in \( B'(k, m, l) \). For each edge \( e = (u, v) \) with a configuration \( \omega(e) \), let \( \pi(\omega(e)) \) be the same value \( \omega(e) \) on the edge \((\pi(u), \pi(v))\). Thus, for \( \omega(B'(k, m, l)) \), \( \pi(\omega(B'(k, m, l))) \) will be the configuration by changing each configuration \( \omega(e) \) at \( e \) to \( \pi(\omega(e)) \). With configurations \( \{\pi(\omega(B'(k, m, l)))\} \), we consider \( \pi(W'_r(k, m, l)) \) (see Fig. 6). By our definition, \( \pi(W'_r(k, m, l)) \) is still a self-avoiding regular cutset that cuts the bottom face from the top face of \( B'(k, m, l) \). Also, it has the minimum passage time among all the other regular cutsets. Since the selection of \( W'_r(k, m, l) \) is unique, the selection of \( \pi(W'_r(k, m, l)) \) is also unique. In addition, let (see Fig. 6)
\[
\pi(D'_1) = \{\pi(\omega(B'(k, m, l))) : \omega(B'(k, m, l)) \in D'_1\}.
\]
By symmetry, we have
\[
E[\tau(\pi(W'_r(k, m, l)))] = E[\tau_r(k, m, l)] = E[\tau(W''_r(k, m, l))]
\]
\[
P[\pi(D'_1)] = P[D_1] = P[D''_1].
\]
By the definition of the mirror reflection \( \sigma_1 \) and the horizontal move \( \sigma_2 \), on \( D_1 \cap \pi(D'_1) \), the upper and the lower tunnels for \( W'_r(k, m, l) \) and \( \pi(W'_r(k, m, l)) \) on \( L_{k_1-l} \) and on \( L_{k_1-l+1} \) are matched, so by Lemma 16, these two cutsets consist of a larger cutset (see Fig. 6):
\[
W_r(k, m, l) \cup \pi(W'_r(k, m, l)) \text{ is a cutset that cuts the bottom from the top of}
\]
\[
[l, 2k_1 - 3l + 1] \times [0, k_2] \times \cdots \times [0, k_{d-1}] \times [0, m].
\]
Note also that the new cutset consists of \( W_r(k, m, l) \) and \( \pi(W'_r(k, m, l)) \). Therefore, it is still regular. With this observation,
\[
E[\tau_r((2k_1 - 2l + 1, k_2, \ldots, k_{d-1}), m, l) | D_1 \cap \pi(D'_1)]
\]
\[
\leq E[\tau(W_r(k, m, l)) + \tau(\pi(W'_r(k, m, l))) | D_1 \cap \pi(D'_1)].
\]
Note that $W_r$ and $\mathcal{D}_1$, and $\pi(W'_r)$ and $\pi(D'_1)$, only depend on the configurations of edges in different boxes, so
\[(W_r, \mathcal{D}_1), (\pi(W'_r), \pi(D'_1)) \text{ are independent.} \quad (10.6)\]

By (10.5)–(10.6) and symmetry,
\[\mathbb{E}[\tau_r((2k_1-2l+1, k_2, \ldots, k_{d-1}), m, l) \mid \mathcal{D}_1 \cap \pi(\mathcal{D}'_1)] \leq 2\mathbb{E}[\tau_r(k, m, l) \mid \mathcal{D}_1]. \quad (10.7)\]

We need to use Lemma 20 to change conditional expectations in (10.7) to unconditional expectations. By (10.2)–(10.3) and Lemma 18, we have
\[C_1 \exp[-C_2k_1^{1-\delta/3}k_2\cdots k_{d-1}] \leq \mathbb{P}[\mathcal{D}_1] = \mathbb{P}[\pi(\mathcal{D}'_1)]. \quad (10.8)\]

By (10.8) and Lemma 20, there exist $C_i = C_i(F, d, \beta, \delta)$ for $i = 1, 2, 3$ such that for all $k_1$,
\[\mathbb{E}[\tau_r(k, m, l) \mid \mathcal{D}_1] \leq \mathbb{E}[\tau_r(k, m, l) + C_1k_1^{1-\delta/8}k_2\cdots k_{d-1} + C_1 \exp(-C_2k_1^{1-\delta/4}k_2\cdots k_{d-1})].\]

Now we find $C = C(F, d, \beta, \delta)$ such that
\[\mathbb{E}[\tau_r(k, m, l) \mid \mathcal{D}_1] \leq \mathbb{E}[\tau_r(k, m, l) + Ck_1^{1-\delta/8}k_2\cdots k_{d-1}]. \quad (10.9)\]
As we mentioned in Lemma 17,

\[
E[\tau_r(k, m) \mid B(k, m, l)] \leq E[\tau_r(k, m) \mid B(k, m, l)].
\]

(10.10)

By using Lemma 20 twice, (10.2), and (8.6), there exist constants \(C_i = C_i(F, d, \beta, \delta)\) for \(i = 1, 2, 3\) such that for all large \(k_1, \ldots, k_{d-1}\),

\[
E[\tau_r(k, m)] \\
\leq E[\tau_r(k, m) \mid B(k, m, l)] \\
+ C_1 k_1^{1-\delta/8} k_2 \cdots k_{d-1} + C_1 k_1^{1-\delta/2} \exp\left(-C_2 k_1^{1-\delta/4} k_2 \cdots k_{d-1}\right) \\
\leq E[\tau_r(k, m) \mid B(k, m, l)] \\
+ C_1 k_1^{1-\delta/8} k_2 \cdots k_{d-1} + C_1 k_1^{1-\delta/2} \exp\left(-C_2 k_1^{1-\delta/4} k_2 \cdots k_{d-1}\right) \\
\leq E\tau_r(k, m) + 2C_1 k_1^{1-\delta/8} k_2 \cdots k_{d-1} + 2C_1 k_1^{1-\delta/2} \exp\left(-C_2 k_1^{1-\delta/4} k_2 \cdots k_{d-1}\right) \\
\leq E\tau_r(k, m) + C_3 k_1^{1-\delta/8} k_2 \cdots k_{d-1}.
\]

(10.11)

Let \(\mathcal{H}(k, m)\) be the event that

\[\tau_r(k, m) \leq 2E\tau(e)\|k\|_v.\]

Note that

\[\tau_r(k, m) \leq \bar{\alpha}(k, m)\] and \(E\bar{\alpha}(k, m) = E\tau(e)\|k\|_v,\]

where \(\bar{\alpha}(k, m)\) is defined in (6.0). Thus, by (1.1) and a standard large deviation estimate,

\[E\tau_r(k, m) \leq E[\tau_r(k, m)I(\mathcal{H}(k, m))] + C_1 \exp(-C_2\|k\|_v).\]

(10.12)

By Theorem 2,

\[
E[\tau_r(k, m)] \\
\leq E[\min(k, m)] \\
+ E[\tau_r(k, m)I(\mathcal{H}(k, m)) \mid \bar{N}(k, m) \geq \beta\|k\|_v] \mathbf{P}[\bar{N}(k, m) \geq \beta\|k\|_v] + C_1 \exp(-C_2\|k\|_v) \\
\leq E[\min(k, m)] + C_3 E[\tau(e)]\|k\|_v \exp(-C_4\|k\|_v) + C_1 \exp(-C_2\|k\|_v),
\]

where \(\bar{N}(k, m)\) and \(\beta\) are defined in Theorem 2. Therefore,

\[E[\tau_r(k, m)] \leq E[\min(k, m)] + C_1 \exp(-C_2\|k\|_v).\]

(10.13)

Together with (10.7)–(10.13), there is \(C_1 = C_1(F, d, \beta, \delta)\) such that

\[E[\tau_r(k, m, l) \mid D_1] \leq E[\tau_{\min}(k, m) + C_1 k_1^{1-\delta/8} k_2 \cdots k_{d-1}.\]

(10.14)
Now we work on the lower bound of (10.5). By the independent discussion and (10.8),

\[
P[\mathcal{D}_1 \cap \pi(\mathcal{D}_1')] = (P[\mathcal{D}_1])^2 \geq C_1^2 \exp \left(\frac{-2C_1(\delta/8)k_2 \cdots k_{d-1}}{k_1}\right). \quad (10.15)
\]

By Lemma 18 and (10.15), and by translation invariance, we may use the same \(C_1\) in (10.14) to have

\[
E[\tau_r((2k_1 - 2l + 1, k_2, \cdots, k_{d-1}), m, l) \mid \mathcal{D}_1 \cap \pi(\mathcal{D}_1')] \geq E[\tau_r((2k_1 - 2l + 1, k_2, \cdots, k_{d-1}), m, l)] - C_1(2k_1 - 4l + 1)(\delta/8)k_2 \cdots k_{d-1}
\]

\[
\geq E[\tau_{\min}((2k_1 - 4l + 1, k_2, \cdots, k_{d-1}), m)] - 2C_1k_1^{(\delta/8)}k_2 \cdots k_{d-1}. \quad (10.16)
\]

Therefore, by (10.5), (10.13), and (10.16),

\[
E[\tau_{\min}((2k_1 - 4l + 1, k_2, \cdots, k_{d-1}), m)] \leq 2E\tau_{\min}(k, m) + 2Ck_1^{(\delta/8)}k_2 \cdots k_{d-1}. \quad (10.17)
\]

We then use the same proof for \(W''(k, m, l)\) on \(B''(k, m)\). On the event \(\mathcal{D}_1 \cap \pi(\mathcal{D}_1') \cap \mathcal{D}_{1}'\), we know that (see Fig. 6)

\[
W_r(k, m, l) \cup \pi(W_r'(k, m, l)) \cup W''(k, m, l) \text{ is a cutset that cuts the bottom from the top of } [l, 3k_1 - 5l + 2] \times [0, k_2] \times \cdots \times [0, k_{d-1}] \times [0, m].
\]

By the same discussion from (10.5)–(10.17), there is \(C = C(F, d, \beta, \delta)\) such that

\[
E[\tau_{\min}((3k_1 - 6l + 2, k_2, \cdots, k_{d-1}), m)] \leq 3E\tau_{\min}(k, m) + 3Ck_1^{(\delta/8)}k_2 \cdots k_{d-1}. \quad (10.18)
\]

With the same method, (10.2), and Lemmas 18 and 20 by replacing \(2\) with \(w_1\) in (10.5)–(10.17), we patch \(w_1\) cutsets on adjacent boxes together along the first coordinate to show

\[
E[\tau_{\min}((w_1(k_1 - 2l) + w_1, k_2, \cdots, k_{d-1}), m)] \leq w_1E\tau_{\min}(k, m) + Cw_1k_1^{(\delta/8)}k_2 \cdots k_{d-1},
\]

where \(C = C(F, d, \beta, \delta)\) is a constant. Note that by (8.5), \(l \leq k_1^{(\delta/2)}\), so by Lemma 13 (b), for all large \(k_1\) satisfying (10.2) and \(w_1k_1 \leq 2\exp(k_{d-1}^{1-5\delta/6})\),

\[
E\left[\tau_{\min}\left((w_1(k_1 - \lfloor k_1^{(\delta/3)}\rfloor), k_2, \cdots, k_{d-1}), m\right)\right] \\
\leq w_1E\tau_{\min}(k, m) + Cw_1k_1^{(\delta/8)}k_2 \cdots k_{d-1}. \quad (10.19)
\]

We want to remark that \(k_{d-1}\) cannot be arbitrarily larger than \(k_i\) for \(i \leq d - 2\) in (10.17), since we need Lemma 18. So (10.2) is good enough for (10.17). However, if we work on the \(d-1\)-th direction, by using Lemma 19, we do not need a restriction for \(k_{d-1}\). More precisely, for all \(k_1 \leq \cdots \leq k_{d-1}\),

\[
E\left[\tau_{\min}\left((k_1, k_2, \cdots, k_{d-2}, 2(k_{d-1} - \lfloor k_{1}^{(\delta/3)}\rfloor)), m\right)\right] \\
\leq 2E\tau_{\min}(k, m) + 2Ck_1k_2 \cdots k_{d-1}^{(\delta/8)}. \quad (10.20)
\]
We next work on $w_1$ cutsets along the first coordinate and $w_2$ cutsets along the second coordinate. Along the second coordinate, we have $w_2$ strips with a width $k_2$ for each strip. We first use (10.19) to patch $w_1$ cutsets in each strip. After the first patching, we use the same method of (10.19) to patch $w_2$ patched cutsets in each strip to a cutset. Note that the size of each cutset, after the first patching, is $w_1(k_1 - \lfloor k_1^{(1-\delta/3)} \rfloor)$ along the first coordinate. Thus, Lemma 19 may not be applied for large $w_1 k_1$. So we need to make an extra assumption:

$$w_1 k_1 \leq 2 \exp \left( k_{d-1}^{1-5\delta/6} \right) \leq 2 \exp \left( 4 k_2^{1-5\delta/6} \right).$$

With this assumption, Lemma 19, and the same method of (10.19), we have

$$E \left[ \tau_{\min} \left( \left( w_1(k_1 - \lfloor k_1^{(1-\delta/3)} \rfloor), w_2(k_2 - \lfloor k_2^{(1-\delta/3)} \rfloor), k_3, \ldots, k_{d-1}, m \right) \right]$$

$$\leq w_1 w_2 E \tau_{\min}(k, m) + C \left[ w_1 w_2 k_1\left( 1-\delta/8 \right) k_2 \cdots k_{d-1} + w_1 w_2 k_1 \left( 1-\delta/8 \right) \cdots k_{d-1} \right]. \quad (10.21)$$

If we continue to iterate this way for the third, ..., the $d-1$-th coordinates, we can show that for integers $w_1, w_2, \ldots, w_{d-1}$ with $w_j k_j \leq 2 \exp(k_{d-1}^{1-5\delta/6})$ for $j = 1, \ldots, d-1$, there exists $C = C(F, d, \beta, \delta)$ such that for all large $k_1 \leq k_2 \leq \cdots \leq k_{d-1}$ with $k_{d-1} \leq 4k_1$, and $m$ that satisfies (1.17),

$$E \left[ \tau_{\min} \left( \left( w_1(k_1 - \lfloor k_1^{(1-\delta/3)} \rfloor), w_2(k_2 - \lfloor k_2^{(1-\delta/3)} \rfloor), \ldots, w_{d-1}(k_{d-1} - \lfloor k_{d-1}^{(1-\delta/3)} \rfloor), m \right) \right]$$

$$\leq w_1 w_2 \cdots w_{d-1} E \tau_{\min}(k, m)$$

$$+ C w_1 w_2 \cdots w_{d-1} \left[ k_1^{(1-\delta/8)} k_2 \cdots k_{d-1} + k_1 k_2^{(1-\delta/8)} \cdots k_{d-1} + \cdots + k_1 k_2 \cdots k_{d-2} k_{d-1}^{(1-\delta/8)} \right]. \quad (10.22)$$

By (10.1), we pick large numbers $k_1', \ldots, k_{d-1}'$, and $m$, given their precise values later, such that for $\epsilon > 0$,

$$\left( \frac{E \tau_{\min}(k', m)}{\|k'\|_v} \right) \leq \nu_1 + \epsilon. \quad (10.23)$$

Now we need to justify the values of these $k_j'$'s such that they satisfy (10.2). If $k_{d-1}' \geq 4k_1'$, we may choose $0 \leq s$ and $0 \leq t \leq k_1'$ such that

$$2^s(k_1' + t) \leq k_{d-1}' \leq 2^s(k_1' + t + 1). \quad (10.24)$$

We divide $[0, 2^s(k_1' + t)]$, in the $d-1$-th coordinate, to $2^s$ equal subsegments:

$$D_1, \ldots, D_{2^s}.$$

We consider

$$\mathbf{T}(j) = [0, k_1'] \times [0, k_2'] \times \cdots \times [0, k_{d-2}'] \times D_j \times [0, m]$$

for $j = 1, 2, \ldots, 2^s$. 

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By using Lemma 13 (a), we know that $W(k, m) \cap T(j)$ is a cutset that cuts the bottom from the top of $T(j)$. By translation invariance and Lemma 13 (b), we have

$$2^s \mathbb{E}_{\tau_{\min}}((k_1', \cdots, k_{d-2}', (k_1' + t)), m) \leq \mathbb{E}_{\tau_{\min}}(k', m). \quad (10.25)$$

If we divide (10.25) by $\|k'\|_v$ and use (10.23), for all $k_1' \geq \kappa_1$, then

$$\frac{\mathbb{E}_{\tau_{\min}}((k_1', \cdots, k_{d-2}', (k_1' + t)), m)}{k_1' \cdots k_{d-2}'(k_1' + t)} \leq \frac{\mathbb{E}_{\tau_{\min}}(k', m)}{\|k'\|_v} (1 + \epsilon) \leq (\nu_1 + \epsilon)(1 + \epsilon). \quad (10.26)$$

We use the same argument of (10.26) for the second, ..., the $d-2$-th coordinates. Thus, by symmetry, there are $k_1 \leq k_2 \leq \cdots \leq k_{d-1}$ with $k_{d-1} \leq 4k_1$ such that for all $k_1 \geq \kappa_1,$

$$\left(\frac{\mathbb{E}_{\tau_{\min}}(k, m)}{\|k\|_v}\right) \leq (\nu_1 + \epsilon)(1 + \epsilon)^d. \quad (10.27)$$

By the assumption in Theorem 3, we can take

$$m \leq \exp(k_1^{1-\delta}). \quad (10.28)$$

Now we assume that

$$\lim_{n, m'} \frac{\mathbb{E}_{\tau_{\min}}(n, m')}{\|n\|_v} = \nu_2$$

for a subsequence in $(n, m')$. We select $n = (n_1, \cdots, n_{d-1})$ and $m'$ such that, for

$$m \leq m' \quad \text{and} \quad 2 \exp\left(k_1^{1-\delta/6}\right) \leq n_j \quad \text{for} \quad j = 1, 2, \cdots, d-1,$$

$$\nu_2 - \epsilon \leq \frac{\mathbb{E}_{\tau_{\min}}(n, m')}{\|n\|_v}.$$

Also, by symmetry, we take $n_1 \leq n_2 \leq \cdots \leq n_{d-1}$. Note that

$$\mathbb{E}_{\tau_{\min}}(n, m') \leq \mathbb{E}_{\tau_{\min}}(n, m) \quad \text{for the} \quad m \quad \text{in (10.28)}. \quad (10.29)$$

We assume that

$$2^{s_j}[\exp(k_1^{1-\delta/6}) + t_j - 1] \leq n_j \leq 2^{s_j}[\exp(k_1^{1-\delta/6}) + t_j] \quad (10.30)$$

for $1 \leq s_j$ and $0 \leq t_j \leq \exp(k_1^{1-\delta/6})$. Let

$$l_j = \exp(k_1^{1-\delta/6}) + t_j \quad \text{and} \quad L = (l_1, \cdots, l_{d-1}).$$

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Here we assume that $l_j$ is an integer; otherwise, we just use $\lfloor l_j \rfloor$ to replace $l_j$. By Lemma 13 (b), for all $k_1 \geq \kappa_1$,

$$
\frac{\mathbb{E}_{\tau_{\min}}(n, m)}{\|n\|_v} \leq \frac{\mathbb{E}_{\tau_{\min}}((2^{s_1}l_1, \ldots, 2^{s_{d-1}}l_{d-1}), m)}{2^{s_1+\cdots+s_{d-1}}\|L\|_v}(1 + \epsilon).
$$

(10.31)

Let

$$
q - \lfloor q^{1-\delta/3} \rfloor = 2^{s_{d-1}-1}l_{d-1}.
$$

We may take $k_1 \geq \kappa_2$ such that

$$
q \leq 2 \times 2^{s_{d-1}-1}l_{d-1} = 2^{s_{d-1}}l_{d-1}.
$$

Thus

$$
q = 2^{s_{d-1}-1}l_{d-1} + \lfloor q^{1-\delta/3} \rfloor \leq 2^{s_{d-1}-1}l_{d-1} + (2^{s_{d-1}}l_{d-1})^{1-\delta/3}.
$$

Under these observations, by (10.20) and Lemma 13 (b),

$$
\begin{align*}
\mathbb{E}_{\tau_{\min}}((2^{s_1}l_1, \ldots, 2^{s_{d-1}}l_{d-1}), m) \\
= \mathbb{E}_{\tau_{\min}}((2^{s_1}l_1, \ldots, 2(q - \lfloor q^{1-\delta/3} \rfloor)), m) \\
\leq 2\mathbb{E}_{\tau_{\min}}((2^{s_1}l_1, \ldots, q), m) \\
+ 2C \left[2^{s_1+\cdots+s_{d-2}}l_1 \ldots l_{d-2}(2^{s_{d-1}}l_{d-1})^{1-\delta/8}\right] \\
\leq 2\mathbb{E}_{\tau_{\min}}((2^{s_1}l_1, \ldots, 2^{s_{d-2}}l_{d-2}, 2^{s_{d-1}}l_{d-1} + [2^{s_{d-1}}l_{d-1}]^{1-\delta/3}), m) \\
+ 2C \left[2^{s_1+\cdots+s_{d-2}}l_1 \ldots l_{d-2}(2^{s_{d-1}}l_{d-1})^{1-\delta/8}\right]. 
\end{align*}
$$

(10.32)

Note that $m \leq \exp(k_{d-1}^{1-\delta})$ and $l_{d-1} \geq \exp(k_{d-1}^{-5\delta/6})$, so we may take $k_1 \geq \kappa_3$ such that

$$
m \leq \exp(k_{d-1}^{1-\delta}) \leq \exp \left( \frac{\delta}{100}k_{d-1}^{1-5\delta/6} \right) \leq \left[ k_{d-1}^{1-\delta/3} \right]^{\delta/100}.
$$

(10.33)

By Lemma 13 (c) and (10.33),

$$
\begin{align*}
2\mathbb{E}_{\tau_{\min}}((2^{s_1}l_1, \ldots, 2^{s_{d-2}}l_{d-2}, 2^{s_{d-1}}l_{d-1} + (2^{s_{d-1}}l_{d-1})^{1-\delta/3}), m) \\
\leq 2\mathbb{E}_{\tau_{\min}}((2^{s_1}l_1, \ldots, 2^{s_{d-1}}l_{d-1}), m) + 2d(\mathbb{E}_\tau(e))2^{s_1+\cdots+s_{d-1}l_1} \ldots l_{d-2}(2^{s_{d-1}}l_{d-1})^{1-\delta/3} \cdot m \\
\leq 2\mathbb{E}_{\tau_{\min}}((2^{s_1}l_1, \ldots, 2^{s_{d-1}}l_{d-1}), m) + 2d(\mathbb{E}_\tau(e))2^{s_1+\cdots+s_{d-1}l_1} \ldots l_{d-2}(2^{s_{d-1}}l_{d-1})^{1-\delta/4}. 
\end{align*}
$$

(10.34)

Together with (10.32) and (10.34), there exists $C = C(F, d, \beta, \delta)$ such that

$$
\begin{align*}
\mathbb{E}_{\tau_{\min}}((2^{s_1}l_1, \ldots, 2^{s_{d-1}}l_{d-1}), m) \\
\leq 2\mathbb{E}_{\tau_{\min}}((2^{s_1}l_1, \ldots, 2^{s_{d-1}}l_{d-1}), m) + C2^{-\delta s_{d-1}/8}2^{s_1+\cdots+s_{d-1}}(l_1 \ldots l_{d-2})l_{d-1}^{1-\delta/8}. 
\end{align*}
$$

(10.35)
If
$$2^j l_j = \max\{2^s l_1, \ldots, 2^{s-d-1} l_{d-1}\}$$
for \(j = 1, 2, \ldots, d - 2\),
then we continue the process of (10.35) in \(j\)-th coordinate; otherwise, we still work on the \(d-1\)-th coordinate. With this iteration, we can show that for all \(k_1 \geq \max\{\kappa_2, \kappa_3\}\),
\[
E_{\tau_{\text{min}}}(\{2^s l_1, \ldots, 2^{s-d-1} l_{d-1}\}, m) \\
\leq 2^{s-d-1} E_{\tau_{\text{min}}}(L, m) + C \sum_{j=1}^{d-1} \left[ 2^{s+\delta/2} (l_1 \cdots l_{j-1}) \left( l_j^{1-\delta/8} l_{j+1} \cdots l_{d-1} \right) \left( \sum_{i=1}^{s_j} 2^{-i/8} \right) \right].
\]

With these observations and (10.31), for all \(k_1 \geq \kappa_4\), we have
\[
\frac{E_{\tau_{\text{min}}}(n, m)}{\|n\|_v} \leq \frac{E_{\tau_{\text{min}}}(L, m)}{\|L\|_v} (1 + \epsilon).
\]

Now we need to investigate the relationship between \(E_{\tau_{\text{min}}}(L, m)\) and \(E_{\tau_{\text{min}}}(k, m)\). We select \(w_i\) and \(r_i\) for \(i = 1, 2, \ldots, d - 1\) such that
\[
l_i = w_i (k_i - \lfloor k_i^{(1-\delta/3)} \rfloor) + r_i \text{ for } r_i \leq k_i - \lfloor k_i^{(1-\delta/3)} \rfloor.
\]
As we defined,
\[
W\left( w_1(k_1 - \lfloor k_1^{(1-\delta/3)} \rfloor), w_2(k_2 - \lfloor k_2^{(1-\delta/3)} \rfloor), \ldots, w_{d-1}(k_{d-1} - \lfloor k_{d-1}^{(1-\delta/3)} \rfloor), m \right)
\]
is a cutset that cuts the bottom from the top of
\[
[0, w_1(k_1 - \lfloor k_1^{(1-\delta/3)} \rfloor)] \times \cdots \times [0, w_{d-1}(k_{d-1} - \lfloor k_{d-1}^{(1-\delta/3)} \rfloor)] \times [0, m].
\]

By Lemma 13 (c),
\[
E_{\tau_{\text{min}}}(L, m) \\
\leq E\left[ \tau_{\text{min}}\left( (w_1(k_1 - \lfloor k_1^{(1-\delta/3)} \rfloor), w_2(k_2 - \lfloor k_2^{(1-\delta/3)} \rfloor), \ldots, w_{d-1}(k_{d-1} - \lfloor k_{d-1}^{(1-\delta/3)} \rfloor), m) \right) \right] \\
+ 2d(E(\tau(e))) \left[ r_1 l_2 \cdots l_{d-1} m + l_1 r_2 l_3 \cdots l_{d-1} m + \cdots + l_1 \cdots l_{d-2} r_{d-1} m \right].
\]

Note that \(l_i \leq 2 \exp(k_i^{1-5\delta/6})\) for \(i = 1, \ldots, d - 1\), so by (10.22) and (10.37),
\[
E_{\tau_{\text{min}}}(L, m) \\
\leq E\left[ \tau_{\text{min}}\left( (w_1(k_1 - \lfloor k_1^{(1-\delta/3)} \rfloor), w_2(k_2 - \lfloor k_2^{(1-\delta/3)} \rfloor), \ldots, w_{d-1}(k_{d-1} - \lfloor k_{d-1}^{(1-\delta/3)} \rfloor), m) \right) \right] \\
+ 2d(E(\tau(e))) \left[ r_1 l_2 \cdots l_{d-1} m + l_1 r_2 l_3 \cdots l_{d-1} m + \cdots + l_1 \cdots l_{d-2} r_{d-1} m \right]
\]
\[
\leq w_1 w_2 \cdots w_{d-1} E_{\tau_{\text{min}}}(k, m) \\
+ C w_1 w_2 \cdots w_{d-1} \left[ k_1^{(1-\delta/8)} k_2 \cdots k_{d-1} + k_1 k_2^{(1-\delta/8)} \cdots k_{d-1} + \cdots + k_1 k_2 \cdots k_{d-2} k_{d-1}^{(1-\delta/8)} \right] \\
+ 2d(E(\tau(e))) \left[ r_1 l_2 \cdots l_{d-1} m + l_1 r_2 l_3 \cdots l_{d-1} m + \cdots + l_1 \cdots l_{d-2} r_{d-1} m \right].
\]
Therefore, we divide $\|L\|_v$ on both sides of (10.38). Now we work on the left side of (10.38). Note that $l_i \geq \exp(k_{d-1}^{1-5\delta/6})$, so for all $k_1 \geq \kappa_5$,

$$\frac{w_ik_i}{l_i} \leq (1 + \epsilon).$$

(10.39)

Thus, the first term in the left side of (10.38), divided by $\|L\|_v$, is

$$\frac{w_1w_2 \cdots w_{d-1}E_{\tau_{\min}}(\mathbf{k}, m)}{\|L\|_v} \leq \frac{E_{\tau_{\min}}(\mathbf{k}, m)}{\|\mathbf{k}\|_v}(1 + \epsilon)^{d-1}. \quad (10.40)$$

By (10.39), the second sum in the left side of (10.38), divided by $\|L\|_v$, is

$$Cw_1w_2 \cdots w_{d-1}\left[k_1^{(1-\delta/8)}k_2 \cdots k_{d-1} + k_1^2k_2^{(1-\delta/8)} \cdots k_{d-1} + \cdots + k_1k_2 \cdots k_{d-2}k_{d-1}^{(1-\delta/8)}\right]$$

$$\leq (1 + \epsilon)^{d-1}\sum_{i=1}^{d-1} \frac{C}{k_1^{(1-\delta/8)}} \leq (d - 1)\epsilon(1 + \epsilon)^{d-1} \quad (10.41)$$

for all $k_1 \geq \kappa_6$. Finally, note that $r_i \leq k_i$ and

$$m \leq \exp(k_{d-1}^{1-\delta}) \text{ and } \exp(k_{d-1}^{1-5\delta/6}) \leq l_j$$

for all $j = 1, \cdots, d - 1$. Thus, for all $k_1 \geq \kappa_7$,

$$mr_j \leq r_j \exp(k_{d-1}^{1-\delta}) \leq k_j \exp(k_{d-1}^{1-\delta}) \leq \exp(2k_{d-1}^{1-\delta}) \leq \exp(k_{d-1}^{1-5\delta/6}/2) \leq l_j^{1/2}.$$

With this observation,

$$l_1 \cdots l_{j-1}r_jl_{j+1} \cdots l_{d-1}m \leq l_1 \cdots l_{j-1}l_j^{1/2}l_{j+1} \cdots l_{d-1}. \quad (10.42)$$

Therefore, the third sum in the left side of (10.38), divided by $\|L\|_v$, is

$$\frac{2dE(\tau(\epsilon))[r_1l_2 \cdots l_{d-1}m + l_1r_2l_3 \cdots l_{d-1}m + \cdots + l_1 \cdots l_{d-2}r_{d-1}m]}{\|L\|_v} \leq \frac{2d^2}{k_1^{1/2}} \leq C\epsilon \quad (10.43)$$

for all $k_1 \geq \kappa_8$.

We now select $k_1 \geq \max\{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6, \kappa_7, \kappa_8\}$ such that (10.27) holds. Finally, if we put (10.29)–(10.43) together, we show that

$$\nu_2 - \epsilon \leq \left(\frac{E_{\tau_{\min}}(\mathbf{n}, m)}{\|\mathbf{n}\|_v}\right) \leq \left(\frac{E_{\tau_{\min}}(\mathbf{k}, m)}{\|\mathbf{k}\|_v}\right) (1 + \epsilon)^{d-1} + C\epsilon \leq \nu_1(1 + \epsilon)^{d-1} + C_1\epsilon. \quad (10.44)$$

This shows that $\nu_1 = \nu_2 = \nu$. 

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Next we need to show the pointwise and $L_1$ convergence. By simply using a Borel-Cantelli lemma together with the mean convergence, the concentration property in (9.7), and (8.1), we have

$$\lim_{k_1,\ldots,k_{d-1},m\to\infty} \frac{\tau_{\min}(k,m)}{\|k\|_v} = \nu \text{ a.s. and in } L_1.$$ (10.45)

Therefore, Theorem 3 follows.

References

Aizenman, M., Chayes, J. T., Chayes, L., Frohlich, J., and Russo, L. (1983). On a sharp transition from area law to perimeter law in a system of random surfaces. *Comm. Math. Phys.* **92** 19–69.

Chayes, L., and Chayes, J. (1986). Bulk transport properties and exponent inequalities for random resistor and flow networks. *Comm. Math. Phys.* **105** 133–152.

Engle, E. (1997). *Sperner Theory*. Cambridge University Press, Cambridge, UK.

Grimmett, G. (1999). *Percolation*. Springer, Berlin.

Grimmett, G., and Kesten, H. (1984). First-passage percolation, network flows, and electrical resistances. *Z. Wahrsch. verw. Gebiete.* **66** 335–366.

Hammersley, J. M., and Welsh, D. J. A. (1965). First-passage percolation, subadditive processes, stochastic networks and generalized renewal theory. In *Bernoulli, Bayse, Laplace Anniversary Volume* (J. Neyman and L. LeCam, eds.) 61–110. Springer, Berlin.

Kesten, H. (1982). *Percolation theory for mathematicians*. Birkhauser, Berlin.

Kesten, H. (1986). Aspects of first-passage percolation. *Lecture Notes in Math.* **1180** 126–264. Springer, Berlin.

Kesten, H. (1988). Surfaces with minimal random weights and maximal flows: A higher-dimensional version of first-passage percolation. *Illinois J. Math.* **31** 99–166.

Kesten, H. (1993). On the speed of convergence in first passage percolation. *Ann. Appl. Probab.* **3** 296–338.

Kesten, H., and Zhang, Y. (1990). The probability of a large finite cluster in supercritical Bernoulli percolation. *Ann. Probab.* **18** 537–555.

Steele, M., and Zhang, Y. (2003). Nondifferentiability of the time constants of first-passage percolation. *Ann. Probab.* **31** 1028–1051.

Talagrand, M. (1995). Concentration of measure and isoperimetric inequalities in product spaces. *Inst. Hautes Publ. Math. Etudes Sci.* **81** 73–205.

Zhang, Y. (2000). Critical behavior for maximal flows on the cubic lattice. *J. Stat. Phys.* **98** 799–811.

Zhang, Y. (2007). Shape fluctuations are different in different directions. To appear *Ann.*
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