ANALYTIC ISOLATION OF NEWFORMS OF GIVEN LEVEL

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Abstract. We describe a method for understanding averages over newforms on $\Gamma_0(q)$ in terms of averages over all forms of some level. The method is simplest when $q$ is divisible by the cubes of its prime divisors.

1. Introduction

Fix a positive integer $k$. For each positive integer $q$, let $A(q)$ denote the space of weight $k$ holomorphic cusp forms on $\Gamma_0(q)$. It is a finite-dimensional inner product space. Let $A^*(q) :\leq A(q)$ denote the Atkin–Lehner newspace; it is the orthogonal complement of the oldspace, which is in turn the span of the forms

$$\varphi|_d(z) := d^{k/2} \varphi(dz)$$

taken over all proper divisors $\ell :\neq q$ of $q$, all divisors $d$ of $q/\ell$, and all $\varphi \in A(\ell)$.

The newspace $A^*(q)$ is of fundamental interest and importance because of its strong interaction with the theory of Hecke operators. In analytic number theory, it is of particular interest to understand averages (of Fourier coefficients, $L$-values, ...) over the newspace. Unfortunately, the basic tools for studying such averages (e.g., trace formulas) apply most directly to the larger spaces $A(q)$.

There arises the problem of relating averages over $A^*(q)$ to those over $A(q)$. Several authors have addressed this problem via the Atkin–Lehner decomposition

$$A(q) = \bigoplus_{\ell | q} \bigoplus_{d | q/\ell} \{ \varphi|_d : \varphi \in A^*(\ell) \} \quad \text{(non-orthogonal direct sum)}$$

followed by a computationally-involved Gram–Schmidt orthogonalization. In this article, we introduce an approach which is more direct for certain values of $q$.

Theorem 1. Suppose $q$ is divisible by the cube of every prime that divides it. Then for $z_1, z_2$ in the upper half-plane,

$$\sum_{\varphi \in B^*(q)} \varphi(z_1) \overline{\varphi(z_2)} = \sum_{d,e | q} \mu(d)\mu(e) \sum_{\varphi \in B(\frac{q}{de})} \varphi|_d(z_1) \overline{\varphi|_d(z_2)},$$

where $B(\ell), B^*(\ell)$ denote arbitrary orthonormal bases for $A(\ell), A^*(\ell)$ defined using Petersson inner products with respect to normalized hyperbolic measures of volume independent of $\ell$, such as $\text{vol}(\Gamma_0(\ell) \backslash \mathbb{H})^{-1} \frac{dx\,dy}{y^2}$, so that $\varphi \mapsto \varphi|_d$ is unitary.

The formulation of Theorem 1 requires only the definition of the newspace as introduced by Atkin–Lehner in 1970, but the simple identity does not appear to have been anticipated prior to this work despite the considerable technical effort expended on the problems that it addresses. The proof is short, simple and...
ANALYTIC ISOLATION OF NEWFORMS OF GIVEN LEVEL

independent of (1). The restriction on \( q \) will be discussed in due course, and is irrelevant for our motivating applications in which it is a large power of a fixed prime.

Theorem 1 applies to problems in which one knows how to average over all forms of given level, but wants to average only over the newforms. For the sake of illustration, we record here three such applications. Some of the consequences to follow are new, some old. Theorem 1 itself is new.

As we explain in detail in the body of the paper (see §2, §3), Theorem 1 may be understood as an identity of hermitian forms (equivalently, self-adjoint operators) on the space of modular forms (compare with [6, Lem 4.1], for instance). The proofs of the applications to follow may then be understood as the result of taking the Hilbert–Schmidt pairing of that identity against a given sesquilinear form.

We begin with an overly simple application which may nevertheless aid orientation. Let \( n \) be a natural number coprime to \( q \). By applying the Hecke operator \( T_n \) to either variable and integrating over the diagonal \( z_1 = z_2 \) of (2), we obtain a relation between traces of Hecke operators acting on the new space and on all forms of given level:

\[
\text{trace}(T_n|A^*(q)) = \sum_{d,e \mid q} \mu(d) \mu(e) \text{trace}(T_n|A(qde)).
\]

(3)

Applying the Eichler–Selberg trace formula to each summand on the RHS of (3) gives a trace formula for newforms, the case \( n = 1 \) of which is a dimension formula for newspaces. Such formulas are not new: they follow (for general \( q \)) from (1) and Möbius inversion (see [17, §5.1] or [9, §2] and [12]). The proof indicated here does not use (1).

Second, write the Fourier expansion of \( \varphi \in A(q) \) as

\[
\varphi(z) = \sum_{n \geq 1} n^{k/2} \rho(n; \varphi) e^{2\pi i n z},
\]

so that \( \rho(n; \varphi|d) = 1_{d|n} \rho(d; \varphi) \). Let \( m, n \) be positive integers. By taking the \( m \)-th (resp. \( n \)-th) Fourier coefficient in the \( z_1 \) (resp. \( z_2 \)) variable of (2), we obtain an identity

\[
\Delta^*(m, n; q) = \sum_{d \mid \gcd(m, n, q)} \mu(d) \mu(e) \Delta(m, n; q),
\]

(4)

expressing the averages of Fourier coefficients

\[
\Delta^*(m, n; q) := \sum_{\varphi \in B(q)} \rho(m; \varphi) \rho(n; \overline{\varphi})
\]

over newforms in terms of the analogous averages \( \Delta(m, n; q) \) over \( B(q) \). Applying the classical Petersson formula to the RHS of (4) gives a Petersson formula for newforms.

The special case of (4) in which the variables \( m, n \) satisfy the coprimality constraint \( (mn, q) = 1 \) and \( q \) is a prime power was established by D. Rouymi [15, Prop. 9, Rmk. 4] in his work on newforms of level \( p^\nu, \nu \to \infty \) after some involved calculations along the lines indicated following (1) (see also [4, §5], [3]). The proof given here by way of Theorem 1 is simpler in that it avoids explicit orthogonalization of the decomposition (1). The uniformity of (4) with respect to the variables \( m, n \) appears to be relevant for applications such as those pursued recently in [14].
Formulas of the shape (4) have already seen diverse applications. The references in footnote 1 contain several such applications, as well as proofs of special cases of (4). Conversely, by Fourier inversion, one can recover Theorem 1 from the general case of (4).

Third, let \( \Psi : \Gamma_0(q) \backslash \mathbb{H} \to \mathbb{C} \) be a measurable function of moderate growth, and let \( n \) be a natural number coprime to \( q \). The Petersson inner products \( \langle \varphi, \Psi \varphi \rangle \) are of basic interest in many questions (quantum unique ergodicity, subconvexity, ...). A formula relating their Hecke–twisted first moments over newforms and over all forms of given level follows from the proof of (3) by weighting the integrand by \( \Psi(z) \) in the final step:

\[
\sum_{\varphi \in B^*} \langle \varphi, \Psi \cdot T_n \varphi \rangle = \sum_{d,e | q} \mu(d)\mu(e) \sum_{\varphi \in B(d,e)} \langle \varphi|_d, \Psi \cdot T_n \varphi|_d \rangle. \tag{5}
\]

The summands on the RHS of (5) may be studied by integrating \( \Psi \) against the holomorphic kernel for \( T_n \) on conjugates of \( \Gamma_0(q) \). By the multiplicity one theorem, the function of \( n \) given by the LHS of (3) determines the family of inner products \( \langle \varphi, \Psi \varphi \rangle \) arising as \( \varphi \) traverses an orthonormal basis of Hecke newforms. The original motivation for the present work is that (3) and its variants constitute the first step in a method to study the quantum variance of newforms of large level (see [13, §7.1]). Formula (5) is new in all cases.

For many problems involving modular forms, the case of squarefree (or even prime) level is often the simplest and hence the natural first case to consider. Our method, perhaps counterintuitively, applies most directly to cubeful levels. It applies also to levels that are not necessarily cubeful, but becomes more complicated to implement and is not clearly superior to existing approaches. The present generality suffices for the depth aspect in which levels are powers of a fixed prime and hence for our motivating applications [13].

The cubeful levels often exhibit representative phenomena. Because of its directness and simplicity, our method may be useful also for problems involving non-cubeful levels as a first step towards understanding the expected truth. This is analogous (in several respects) to studying the asymptotics of smoothly weighted sums \( \sum_n f(n)W(n/x) \) before those of their sharply-truncated counterparts \( \sum_{n \leq x} f(n) \).

To explain the basic ideas behind the proof with minimal notation/prerequisites, we record in [2] a direct proof of a representative special case of Theorem 1. We then formulate in [8] our main result, which may be understood as a local representation-theoretic form of Theorem 1 that applies also to Maass forms, on quotients attached to quaternion algebras, over number fields, and (with minor modifications) in half-integral weight; it consists of constructing an element of the Hecke algebra of \( \mathrm{GL}_2 \) over a non-archimedean local field that projects onto the newvectors of given log-conductor \( \geq 3 \).

To elucidate that result from as many perspectives as possible, we then give three short proofs. Each relies on a novel operator calculus for idempotents in the Hecke algebra (§6) which we verify

1. group-theoretically (§7),
2. by reduction to a probabilistic assertion concerning random non-backtracking walks on the Bruhat-Tits tree (§8), and
3. using the Kirillov model and recurrence relations for Hecke eigenvalues (§9).
In closing, we note that it would be natural and interesting to extend the present work to the setting of newvectors on $\text{GL}_N$.

2. **The proof in a basic but representative case**

We now prove Theorem 1 in the prime-cubed case $q = p^3$, which already captures the key ideas. The general case will be deduced in §5 from our main local results.

The required identity (2) specializes to

$$\sum_{\varphi \in B^*(p^3)} \varphi(z_1)\overline{\varphi(z_2)} = \sum_{\varphi \in B(p^3)} \varphi(z_1)\overline{\varphi(z_2)} - \sum_{\varphi \in B(p^2)} \varphi(z_1)\overline{\varphi(z_2)} - \sum_{\varphi \in B(p)} \varphi(z_1)\overline{\varphi(z_2)}.$$  \[ (6) \]

For $i, j \in \{0, 1, 2, 3\}$ with $i \leq j$, let $E_{ij}: \mathcal{A}(p^3) \rightarrow \mathcal{A}(p^3)$ denote the orthogonal projector onto the subspace $A_{ij} := \{ \varphi|_{\Gamma} : \varphi \in \mathcal{A}(p^{j-i}) \}$. One has $A_{ij} = \{ \varphi \in \mathcal{A}(p^3) : \varphi|_\gamma = \varphi \text{ for all } \gamma \in \Gamma_{ij} \}$ where $\varphi\gamma$ is the slash operator used to define the automorphy of $\varphi$ and $\Gamma_{ij}$ is the group

$$\Gamma_{ij} := \left[ \begin{array}{cc} p^{j-i} & 1 \\ 0 & 1 \end{array} \right] \Gamma_0(p^{j-i}) \left[ \begin{array}{cc} p^i & 1 \\ 0 & 1 \end{array} \right] = \left[ \begin{array}{cc} \mathbb{Z} & p^{-j-i} \mathbb{Z} \\ p^j \mathbb{Z} & \mathbb{Z} \end{array} \right] \cap \text{SL}_2(\mathbb{Q})$$

fitting into the lattice diagram

$$\begin{array}{c}
\Gamma_{00} \quad \Gamma_{11} \\
\Gamma_{01} \quad \Gamma_{12} \quad \Gamma_{22} \quad \Gamma_{33} \\
\Gamma_{02} \quad \Gamma_{13} \\
\Gamma_{03} \\
\end{array}$$

with the smallest group $\Gamma_{03} = \Gamma_0(p^3)$ at the bottom, the largest groups (all conjugates of $\Gamma_{00} = \text{SL}_2(\mathbb{Z})$) along the top, the chain $\Gamma_{0j} = \Gamma_0(p^j)$ along the left edge, and with $\Gamma_{ij} \cap \Gamma_{jk} = \Gamma_{ik}$ for $0 \leq i \leq j \leq k \leq 3$. The projector $E_{ij}$ may be expressed concretely as the averaging operator

$$E_{ij}\varphi = \frac{1}{|\Gamma_{03} \setminus \Gamma_{ij}|} \sum_{\gamma \in \Gamma_{03} \setminus \Gamma_{ij}} \varphi|_\gamma.$$  

Because $\{ \varphi|_{\Gamma} : \varphi \in B(p^{j-i}) \}$ extends to an orthonormal basis of $\mathcal{A}(p^3)$, one has

$$\sum_{\varphi \in B(p^{j-i})} \varphi|_{\Gamma}(z_1)\overline{\varphi|_{\Gamma}(z_2)} = \sum_{\varphi \in B(p^3)} E_{ij}\varphi(z_1)\overline{\varphi(z_2)},$$

so our specialized goal (6) may be rewritten as

$$\sum_{\varphi \in B^*(p^3)} \varphi(z_1)\overline{\varphi(z_2)} = \sum_{\varphi \in B(p^3)} E_{03}\varphi(z_1)\overline{\varphi(z_2)} - \sum_{\varphi \in B(p^2)} E_{02}\varphi(z_1)\overline{\varphi(z_2)} - \sum_{\varphi \in B(p)} E_{13}\varphi(z_1)\overline{\varphi(z_2)} + \sum_{\varphi \in B(p^3)} E_{12}\varphi(z_1)\overline{\varphi(z_2)},$$
Remark 1. The conclusion of Lemma 2 is not altogether formal. For instance, it fails if one replaces “3” by “2.”

Lemma 2. $E^*_0$ defines the orthogonal projector onto the newspace $A^*(p^3)$.

The operators $E_{02}, E_{13}$ and $E_{12}$ are self-adjoint idempotents with image in the oldspace and hence kernel containing the newspace, while $E_0$ is the identity, so $E^*_0$ restricts to the identity on newspace. To conclude that $E^*_0$ orthogonally projects onto the newspace, it remains only to verify that it annihilates the oldspace; as the latter is spanned by the images of $E_{02}$ and $E_{13}$, it suffices to show that $E^*_0 \circ E_{02} = 0$ and $E^*_0 \circ E_{13} = 0$. We verify here the first of these identities, the proof of the second being similar. We claim that

$$E_{03} \circ E_{02} = E_{02}, \quad E_{02} \circ E_{02} = E_{02}, \quad E_{12} \circ E_{02} = E_{12}, \quad (7)$$

and

$$E_{13} \circ E_{02} = E_{12}$$

from which it follows that $E^*_0 \circ E_{02} = E_{02} - E_{02} - E_{12} + E_{12} = 0$, as required. The identities (7) are consequences of the transitivity of orthogonal projection onto nested subspaces. The interesting identity is thus (8), which we may write thanks to the third identity in (7) in the equivalent form $E_{13} \circ E_{02} = E_{12} \circ E_{02}$ and then in terms of averaging operators as the assertion that for all $\varphi \in A_{02}$,

$$\frac{1}{|\Gamma_0 \setminus \Gamma_1|} \sum_{\gamma \in \Gamma_0 \setminus \Gamma_1} \varphi|\gamma = \frac{1}{|\Gamma_0 \setminus \Gamma_1|} \sum_{\gamma \in \Gamma_0 \setminus \Gamma_1} \varphi|\gamma.$$ 

To that end, it suffices to verify that the natural map of coset spaces $\Gamma_0 \setminus \Gamma_1 \rightarrow \Gamma_2 \setminus \Gamma_1$ induced by the inclusions $\Gamma_1 \subseteq \Gamma_2$, $\Gamma_0 \subseteq \Gamma_2$ is bijective. The injectivity follows from the evident identity $\Gamma_1 \cap \Gamma_0 = \Gamma_0$, while the surjectivity, which is the crux of the whole matter, is given as follows:

Lemma 3. The map $\Gamma_0 \times \Gamma_1 \ni (\gamma_1, \gamma_2) \mapsto \gamma_1 \gamma_2 \in \Gamma_0 \setminus \Gamma_1$ is surjective.

Proof. We must show that every $\gamma \in \SL_2(\mathbb{Q})$ satisfying

$$\gamma \in \begin{bmatrix} Z & p^{-1}Z \\ p^2Z & Z \end{bmatrix}$$

arises as the product $\gamma = \gamma_1 \gamma_2$ of two $\gamma_1, \gamma_2 \in \SL_2(\mathbb{Q})$ satisfying

$$\gamma_1 \in \begin{bmatrix} Z & Z \\ p^2Z & Z \end{bmatrix}, \quad \gamma_2 \in \begin{bmatrix} Z & p^{-1}Z \\ p^3Z & Z \end{bmatrix}.$$ 

To simplify calculations, we conjugate by $(p^3)$ and reduce modulo $p^3$. Set $\mathfrak{o} := \mathbb{Z}/p^3$, $p := p\mathbb{Z}/p^3 < \mathfrak{o}$. By the surjectivity of the natural map $\SL_2(\mathbb{Z}) \rightarrow \SL_2(\mathfrak{o})$, we reduce to verifying that every

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathfrak{o} \cap \SL_2(\mathfrak{o})$$
arises as the product $\gamma = \gamma_1\gamma_2$ of some

$$\gamma_1 = \begin{bmatrix} 0 & 1 \\ p & 0 \end{bmatrix} \cap \text{SL}_2(\mathfrak{o}), \quad \gamma_2 = \begin{bmatrix} 0 & a \\ p^2 & 0 \end{bmatrix} \cap \text{SL}_2(\mathfrak{o}).$$

To that end, we note that $\det(\gamma) = 1, c \in \mathfrak{p}$ implies $a \in \mathfrak{o}^\times$ and take

$$\gamma_1 := \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}, \quad \gamma_2 := \begin{pmatrix} a & b \\ 0 & d - bc/a \end{pmatrix}. $$

This completes the proof of Lemma 3 hence of Lemma 2 hence of the $q = p^3$ case of Theorem 1.

\[ \square \]

**Remark 2.** The “standard” approach to proving something like Theorem 1 (see e.g. [15] or [3]) would be to check it one Hecke-irreducible subspace at a time using explicit formulas derived from Atkin–Lehner theory. The present observation is that it is in some cases more efficient to work with the congruence subgroups themselves.

### 3. Statement of the main local result

We now formulate the main result of this article, which as indicated earlier may be understood as a local, flexibly applicable form of Theorem 1 (see [13, §7.1] for an application).

Let $k$ be a non-archimedean local field with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$, uniformizer $\varpi$, and $q := \#\mathfrak{o}/\mathfrak{p}$. Let $G$ be a closed subgroup of $\text{GL}_2(k)$ that contains $\text{SL}_2(k)$. Equip $G$ with some Haar measure $dg$. By a segment, we shall mean a nonempty finite consecutive set of integers, denoted $m..n := \{m, m+1, \ldots, n\}$ for some integers $m, n$ with $m \leq n$. The cardinality of a segment $\ell$ is denoted $\#\ell$, thus $\#m..n := |m - n| + 1$. For each segment $\ell = m..n$, denote by

$$R_\ell := \begin{bmatrix} a & \mathfrak{p}^{n-m} \\ \mathfrak{p}^n & 0 \end{bmatrix}$$

the Eichler order of level $|m - n| = \#\ell - 1$ indexed by $\ell$, regarded as a geodesic segment on the Bruhat–Tits tree (see e.g. [19] or [18] or [7, §1.2]). Let $R_\ell^\ast$ denote its unit group and $K_\ell := R_\ell^\ast \cap G < G$ the intersection of that unit group with $G$. Denote by $1_{K_\ell}$ the characteristic function.

Let $\pi$ be a smooth representation of $G$. For $f \in C_c^\infty(G)$, denote by $\pi(f) \in \text{End}(\pi)$ the operator $\pi(f)v := \int_G f(g)\pi(g)v \, dg$. For each segment $\ell$, denote by $\pi[\ell] := \pi^{K_\ell}$ the subspace of vectors fixed by $K_\ell$. The standard projector onto $\pi[\ell]$ is the averaging operator $e_\ell := \pi(\text{vol}(K_\ell)^{-1}1_{K_\ell}) \in \text{End}(\pi)$: it is an idempotent projector with image $\pi[\ell]$. The standard complement of $\pi[\ell]$ is the kernel of $e_\ell$. One has

$$\pi = \pi[\ell] \oplus \ker(e_\ell).$$

Note that $\ell \supseteq \ell'$ implies $\pi[\ell] \supseteq \pi[\ell']$. Set $\pi[\ell]^\flat := \sum_{\ell' \subsetneq \ell} \pi[\ell']$, with the sum taken over all proper subsegments $\ell'$ of $\ell$. The standard complement of $\pi[\ell]^\flat$ in $\pi[\ell]$ is

$$\pi[\ell]^\sharp := \{v \in \pi[\ell] : e_\ell v = 0 \text{ for all } \ell' \subsetneq \ell\}.$$

One has

$$\pi[\ell] = \pi[\ell]^\sharp \oplus \pi[\ell]^\flat.$$  \hfill (10)

The standard projector onto $\pi[\ell]^\sharp$ is the projector $\pi \to \pi[\ell]^\sharp$ afforded by the decompositions (9) and (10).
If π is unitary, then “standard projector” and “standard complement” have the same meanings as “orthogonal projector” and “orthogonal complement”. Indeed, in that case the invariance under inversion of the Haar measures on the compact groups $K_\ell$ implies that the projectors $e_\ell$ and $e_\ell^*$ are self-adjoint, hence define orthogonal projections (since a projection is orthogonal if and only if it is self-adjoint).

**Theorem 4.** Let $\ell = m..n$ be a segment with $\#\ell - 1 = |m - n| \geq 3$. Then $e_\ell^* := e_{m..n} - e_{m+1..n} - e_{m..n-1} + e_{m+1..n-1}$ is the standard projector onto π[ℓ].

**Remark 3.** The general case of Theorem 4 reduces to the case $G = SL_2(k)$ by convolving against the characteristic function of $\left[ \det(G) \begin{array}{c} 0 \\ 1 \end{array} \right]$. For instance, in the case $G = GL_2$, it does not depend upon the existence of the Kirillov model.

**Remark 4.** We have assumed in Theorem 4 neither that π is irreducible nor generic, hence this result lies somewhat shallower than the fundamental results of local newvector theory [5, 10]. For instance, in the case $G = GL_2$, it does not depend upon the existence of the Kirillov model.

4. **Interpretation for generic representations of GL_2**

We record here what Theorem 4 says when $G = GL_2(k)$ and π is a generic irreducible representation with unramified central character. In that case, local newvector theory [5, 10] says that one may attach to π a nonnegative integer $c(\pi)$, its log-conductor, with the property that $\pi[0..n] \neq 0$ if and only if $n \geq c(\pi)$; one knows then moreover that $\dim \pi[0..n] = \max(0, n + 1 - c(\pi))$ and that $\pi[0..n]^\ell = 0$ unless $n = c(\pi)$, in which case $\dim \pi[0..n]^\ell = 1$. Since the subgroups $K_\ell, K'_\ell$ are conjugate whenever $\#\ell = \#\ell'$, it follows more generally for any segment $\ell$ that $\dim \pi[\ell] = \max(0, \#\ell - c(\pi))$ and

$$\dim \pi[\ell]^\ell = \begin{cases} 1 & \text{if } \#\ell - 1 = c(\pi), \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Theorem 4 thus implies the following:

**Corollary 5.** For $G = GL_2(k)$, π as above, and $\ell$ satisfying the assumptions of Theorem 4, one has $e_\ell^* = 0$ unless $c(\pi) = \#\ell - 1$, in which case $e_\ell^*$ is the standard projector onto the one-dimensional space $\pi[\ell] = \pi[\ell]^\ell$.

**Remark 5.** If π is irreducible and non-generic, then it is one-dimensional, and $e_\ell^* = 0$ under the assumptions of Theorem 4.

**Remark 6.** The formulation and proof of Corollary 5 extend with the usual modifications to representations having ramified central character: Let π be a generic irreducible representation of $G = GL_2(k)$ with central character $\omega : k^\times \to \mathbb{C}^\times$. Denote by $\chi : \mathfrak{o}^\times \to \mathbb{C}^\times$ the restriction of $\omega$. Let $k := c(\chi)$ denote the log-conductor of $\chi$, i.e., the smallest $k \in \mathbb{Z}_{\geq 0}$ for which $\chi$ restricts trivially to $\mathfrak{o}^\times \cap 1 + p^k$. Assume that $k \geq 1$, i.e., that $\chi$ is nontrivial, or equivalently that $\omega$ is ramified. Let $\ell = m..n$ be a segment. If $|m - n| \geq k$, then $\chi$ induces a character $K_\ell \supseteq \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) \to \chi(d)$; by abuse of notation, we denote this character also by $\chi : K_\ell \to \mathbb{C}^\times$. Define $e_{\ell,\chi} := 0$ unless $|m - n| \geq k$, in which case set $e_{\ell,\chi} := \pi(\text{vol}(K_\ell)^{-1} \chi^{-1} 1_{K_\ell})$. Denote by $\pi[\ell, \chi]$ the image of $e_{\ell,\chi}$, thus $\pi[\ell, \chi] = \{ v \in \pi : gv = \chi(g)v \text{ for all } g \in K_\ell \}$. Define $\pi[\ell, \chi]^\ell$ in terms of $\pi[\ell, \chi]$ as in 3 and $e_{\ell,\chi}^*$ in terms of $e_{\ell,\chi}$ as in Theorem 4.

Local newvector theory gives a formula analogous to (11) for $\pi[\ell, \chi]^\ell$. The proof
of Corollary 3 shows in this context that $e^*_\chi \ell = 0$ unless $c(\pi) = \# \ell - 1$, in which case $e^*_\chi \ell$ is the standard projector onto the one-dimensional space $\pi[\ell, \chi] = \pi[\ell, \chi]^\perp$.

(Alternatively, one could reach the same conclusion by running the argument with $K_\ell$ replaced by its subgroup $\{(a_q) \in K_\ell : d \in 1 + p^k\}$.)

5. Deduction of Theorem 1 from Theorem 3

Theorem 1 follows from Theorem 3 via a standard “adelic-to-classical” argument (see e.g. [3]); for the sake of completeness and variety of exposition, we record here a direct “local-to-classical” proof of this implication. Thus, let $q$ be cubefull. For squarefree integers $d, e$ dividing $q$, denote by $E_{d, e}$ the orthogonal projection from $\mathcal{A}(q)$ to the subspace $\{\varphi|d : \varphi \in \mathcal{A}(\frac{d}{q})\}$. As in [2] Theorem 1 amounts to the assertion that the operator $E_{1, q}^\ast := \sum_{d, e|q} \mu(d) \mu(e) E_{d, e}$ defines the orthogonal projection onto $\mathcal{A}^\ast(q)$. For the same reasons as in the proof of Lemma 2, $E_{1, q}^\ast$ acts by the identity on the newspace, so it remains only to verify that it annihilates the oldspace. The oldspace is the sum over all $p | q$ of the subspaces

$$\mathcal{A}(q/p) \text{ and } \{\varphi|p : \varphi \in \mathcal{A}(q/p)\}$$

so we reduce to verifying for each such $p$ that $E_{1, q}^\ast$ annihilates the spaces (12). To that end, denote by $\pi$ the span of the functions $\varphi|\gamma$ taken over all $\varphi \in \mathcal{A}(q)$ and all $\gamma$ in the group $\Gamma := \text{SL}_2(\mathbb{Q}) \cap R_0(q)[1/p]$, where $R_0(q) := \left\lbrack \frac{2}{q}, \frac{2}{2} \right\rbrack \leq M_2(\mathbb{Z})$ is the order for which $\text{SL}_2(\mathbb{Q}) \cap R_0(q) = \Gamma_0(q)$. Regard $\Gamma$ as a subgroup of $G := \text{SL}_2(\mathbb{Q}_p)$. It is dense. Each $\varphi \in \pi$ is invariant under some congruence subgroup of $\Gamma_0(q)$, hence under $\Gamma \cap U$ for some open subgroup $U$ of $G$. Consequently, the left action of $\Gamma$ on $\pi$ given by $\gamma \cdot \varphi := \varphi|\gamma^{-1}$ extends continuously to a smooth unitary representation of $G$, which we continue to denote by $\pi$. Factor $q = q_0 p^n$ where $(q_0, p) = 1$, and denote by $\ell$ the segment $\ell := 0..n$. The subspace $\mathcal{A}(q)$ is recovered from $\pi$ as $\mathcal{A}(q) = \pi[\ell]$, while $\pi[\ell]^\perp$ is the span of (12). By the Chinese remainder theorem, we may factor

$$E_{1, q}^\ast = E_{1, q_0}^\ast \circ e_\ell^* |_{\mathcal{A}(q)}$$

(13)

where $E_{1, q_0}^\ast$ is defined analogously to $E_{1, q}^\ast$ and $e_\ell^*$ is the operator on $\pi$ defined in [3]. By Theorem 3 the subspaces (12) are annihilated by $e_\ell^*$, hence (by (13)) by $E_{1, q_0}^\ast$ as required.

6. Reduction to an operator calculus for idempotents

We reduce here the proof of Theorem 4 to that of the following:

Lemma 6. Let $\ell, \ell'$ be segments. Suppose that $\ell \subseteq \ell'$ or $\ell \supseteq \ell'$ or $\# \ell \cap \ell' \geq 2$. Then $e_\ell \circ e_{\ell'} = e_{\ell \cap \ell'}$.

Remark 7. By analogy to composition formulas arising in microlocal analysis, it may be instructive to think of the characteristic function $1_\ell$ of $\ell$ as a symbol, $e_\ell$ as its quantization, and the conclusion of Lemma 6 as the assertion that the composition of the quantization of two such symbols $1_\ell, 1_{\ell'}$ is the quantization of their product $1_\ell 1_{\ell'} = 1_{\ell \cap \ell'}$ in nice enough cases.

Remark 8. Suppose $G = \text{GL}_2(k)$. The identity $e_\ell \circ e_{\ell'} = e_{\ell \cap \ell'}$ fails in general if $\# \ell \cap \ell' < 2$ and neither segment contains the other, but continues to hold if $\pi$ is irreducible with unramified central character and log-conductor $c(\pi) \geq 2$, the point being that in such cases, one can simultaneously diagonalize the operators $e_\ell$ by a basis of characteristic functions of $\sigma^\times$-cosets in the Kirillov model, corresponding
classically to the Fourier coefficients of newforms of conductor divisible by $p^2$ being supported on integers coprime to $p$; see [9].

Assuming Lemma [6] for the moment, we now deduce Theorem [4] by an argument similar to that in the proof of Lemma [2]. Take $\ell = m..n$ with $|m-n| \geq 3$; we must show that $e_\ell^* = e_\ell^* \circ e_\ell$ and the definition of $\pi(\ell \bmod p)$, we see that $e_\ell^*$ annihilates the standard complement of $\pi(\ell)$ and restricts to the identity on $\pi(\ell)\bmod p$, so our main task is to show that it annihilates $\pi(\ell)$, or equivalently, that

$$e_\ell^* \circ e_\ell^* = 0$$

for all $\ell' \subseteq \ell$. Since $e_{\ell'} = e_{\ell'} \circ e_\ell$ for any $\ell' \supseteq \ell$, it suffices to establish (14) when $\ell'$ is a maximal proper subsegment $\ell' \subseteq \ell$. We verify this when $\ell' = m+1..n$; the case $\ell' = m..n-1$ is similar. Our assumption on $|m-n|$ implies that for each segment $\ell' \in \{m..n, m+1..n, m..n-1, m+1..n-1\}$ arising in the definition of $e_\ell^*$, one has $\#\ell \cap \ell' \geq 2$, so by Lemma [6], we have

$$e_\ell^* \circ e_\ell^* = e_{m..n} - e_{m+1..n} - e_{m+1..n-1} - e_{m+1..n-1} = 0,$$

which simplifies to $e_m = e_{m+1..n} - e_{m+1..n-1} - e_{m+1..n-1} = 0$, as required.

7. GROUP-THEORETIC PROOF

We record here a proof of Lemma [6] similar to that of Lemma [2]. The case in which one of $\ell, \ell'$ contains the other follows from the transitivity of standard projectors onto nested subspaces, so we need only consider the case that $\#\ell \cap \ell' \geq 2$ and neither contains the other. By a symmetry argument, we reduce further to showing that

$$e_m \circ e_{m'} = e_{m'}$$

whenever $m < m' < n < n'$. Note especially that (15) implies $|m'-n| \geq 1$. Since $e_{m..n} = e_{m..n} \circ e_{m..n'}$, we reduce to verifying that $e_{m..n} = e_{m..n'}$ for all $v \in \pi(\ell)\bmod p$. For such $v$, one has

$$e_m = \frac{1}{|K_m/n|} \sum_{\gamma \in K_m/\pi(n)} \pi(\gamma)v,$$

$$e_{m'} = \frac{1}{|K_{m'}/n|} \sum_{\gamma \in K_{m'}/\pi(n)} \pi(\gamma)v.$$

so our task reduces to verifying that the natural map of coset spaces $K_m/n \to K_{m'}/n$ induced by the inclusions $K_m \leq K_{m'} \leq K_{m',n'}$ is bijective. The injectivity follows from $K_m \cap K_{m',n'} = K_m$, the surjectivity from the claim $K_m \cdot K_{m',n'} = K_{m',n'}$ for whose proof we must verify that any

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} 0 & p^{-m} \\ p^n & 0 \end{pmatrix} \cap G$$

arises as $\gamma = \gamma_1 \gamma_2$ for some

$$\gamma_1 \in \begin{pmatrix} 0 & p^{-m} \\ p^n & 0 \end{pmatrix} \cap G, \quad \gamma_2 \in \begin{pmatrix} 0 & p^{-m'} \\ p^n & 0 \end{pmatrix} \cap G.$$

From $\det(\gamma) \in \mathfrak{o}^\times$, $b \in p^{-m'}, c \in p^n$, $n > m'$ we obtain $bc \in p$ and hence $a, d \in \mathfrak{o}^\times$, justifying the choice

$$\gamma_1 := \begin{pmatrix} 1 \\ c/a \end{pmatrix}, \quad \gamma_2 := \begin{pmatrix} a & b \\ 0 & d - bc/a \end{pmatrix}$$
for which $\gamma_1 \in \text{SL}_2(k) < G$ and $\gamma_2 = \gamma_1^{-1}\gamma \in G$. The required congruences are clear. This completes the proof.

8. Probabilistic proof

We record here an alternative proof of the key identity (15). For notational simplicity we suppose that $G = \text{SL}_2(k)$ (cf. Remark 9). Fix an arbitrary linear functional $v^* : \pi \to \mathbb{C}$. It suffices to show that

$$v^*(e_{m..n}v) = v^*(e_{m'.n'}v) \quad \text{for} \quad v \in \pi[m..n']$$

(16)

under the assumptions (15) on the indices. Denote by $X$ the $(q+1)$-regular tree, where $q := \#o/p$, and by $X^{m..n'}$ the set of non-backtracking paths $x = (x_m \to x_{m+1} \to \cdots \to x_{n'})$ on $X$. There is a well-known injection $G/K \to X^{m..n'}$ obtained by identifying $X$ with the set of homothety classes $[L]$ of lattices $L \subseteq \mathbb{L}$ and mapping the coset $gK_{m..n'}$ to the tuple of lattice classes $([gL_m], [gL_{m+1}], \ldots, [gL_n])$ where $L_j := p^{-j} \times o$ has stabilizer $K_{j,j}$. Denote by $X^{m..n'}$ the image of $G/K_{m..n'}$ in $X^{m..n'}$ and by $G \ni g \to [g] \in X^{m..n'}$ the induced surjection. Consider the map $\phi : \pi[m..n'] \to \mathbb{C}^{X^{m..n'}}$ from $\pi$ to the space of complex-valued functions on $X^{m..n'}$ that sends a vector $v \in \pi[m..n']$ to the function $\phi(v) : X^{m..n'} \to \mathbb{C}$ given by

$$\phi(v)([g]) := v^*(\pi(g)v),$$

which is well-defined. Given $x, y \in X^{m..n'}$ and a subsegment $\ell = p..p' \subseteq m..n'$, write $x|\ell = y|\ell$ to denote that the subpaths $(x_p \to \cdots \to x_{p'})$, $(y_p \to \cdots \to y_{p'})$ coincide. The subset $X^{m..n'}$ of $X^{m..n}$ has the property that if $x \in X^{m..n'}$, and $y' \in X^{m..n'}$ satisfy $x|\ell = y|\ell$ for some subsegment $\ell \subseteq m..n'$, then $y \in X^{m..n'}$. For $f : X^{m..n'} \to \mathbb{C}$ and a subsegment $\ell \subseteq m..n'$, denote by $\rho_f : X^{m..n'} \to \mathbb{C}$ the function given by the expectation $\rho_f(y) = \mathbb{E}f(y)$ taken over $y = (y_m \to y_{m+1} \to \cdots \to y_{n'})$ chosen uniformly at random from the finite set of non-backtracking paths for which $x|\ell = y|\ell$. One verifies directly from the definitions that

$$\phi(e_xv) = \rho_x\phi(v),$$

(17)

so to establish (16), our task reduces to showing for all $f : X^{m..n'} \to \mathbb{C}$ that

$$\rho_m..\rho_m'..n'f = \rho_m'..n\rho_f.$$

(18)

It suffices to test this equality on the characteristic function $f := 1_x$ of an arbitrary non-backtracking path $x = (x_m \to \cdots \to x_{n'}) \in X^{m..n'}$. The RHS of (18) is then the uniform distribution on the finite set of non-backtracking paths $y = (y_m \to \cdots \to y_{n'})$ for which $y|m'..n = x|m'..n$, and so (18) follows from:

Lemma 7. Suppose $m \leq m' < n \leq n'$. Then the following probability distributions on $X^{m..n'}$ coincide:

- The uniform distribution $\rho_{m'}..1_x$ on the finite set of non-backtracking paths $y = (y_m \to \cdots \to y_{n'})$ for which $y|m'..n = x|m'..n$.
- The distribution $\rho_{m..n}\rho_{m'..n}1_x$ generated iteratively as follows:
  1. Start with the deterministic subpath $(y_{m'} \to \cdots \to y_{n'}) := (x_{m'} \to \cdots \to x_n)$. (1)
  2. Choose uniformly at random a forward extension $(y_n \to \cdots \to y_{n'})$ satisfying the non-backtracking condition that $(y_n \to y_{n+1})$ not be the inverse of $(x_{n} \to x_{n-1})$. (2)

\[\text{ANALYTIC ISOLATION OF NEWFORMS OF GIVEN LEVEL 10}\]
Independent choice uniformly at random a backward extension \((y_n \to \cdots \to y_{m'})\) for which \((y_{m'} \to y_{m' - 1})\) is not the inverse of \((x_{m'} \to x_{m' + 1})\).

**Proof.** The assumption \(n > m'\) implies that the subpath \((x_{m'} \to \cdots \to x_n)\) contains the (possibly identical) edges \((x_{m'} \to x_{m' + 1})\) and \((x_{n - 1} \to x_n)\), so the non-backtracking condition on the forward path \((y_n \to \cdots \to y_{m'})\) is independent of that on the backward path \((y_m \to \cdots \to y_{m'})\). \(\square\)

9. A THIRD PROOF

We sketch here a proof of Corollary\(^{[5]}\) (thus \(G = \text{GL}_2(k)\)) which is more complicated and less self-contained than the above proofs, but which some readers may find illustrative; it is also similar in spirit to what is typically done classically along the lines discussed in \([14]\). Thus, let notation and assumptions be as in the statement of Corollary\(^{[5]}\). We assume, for clarity of exposition, that \(\pi\) is unitary; a proof for general \(\pi\) may be obtained by working systematically with standard projectors in place of inner products. We may assume that \(\pi\) has unramified central character, as otherwise both sides of the required identity vanish. Let us realize \(\pi\) in its Kirillov model with respect to an unramified additive character (see \([16]\)).

Suppose first that \(c(\pi) \geq 2\), so that \(\pi\) is supercuspidal. Then \(\pi[0..c(\pi)]\) is spanned by the standard newvector \(1_{a}^{\times}\); more generally, for any segment \(\ell\), one has the decomposition

\[
\pi[\ell] = \bigoplus_{n..n+\pi(\ell)} \mathbb{C}1_{\pi^n \circ \pi^c} = \bigoplus_{n..n+\pi(\ell)} \mathbb{C}1_{\pi^n \circ \pi^c}
\]

with respect to which the orthogonal projectors \(e_{\ell'}\) for \(\ell' \subseteq \ell\) are given by projection onto the summands with \(n..n + \pi(\ell) \subseteq \ell'\). (Indeed, by the conjugacy of the \(K_{\ell}\) for \(\ell\) of given length, we may assume that \(\ell = 0..m\) for some \(m \in \mathbb{Z}_{\geq 0}\); in that case, the required conclusion follows from the proof of the “absolutely cuspidal” case of \([5]\) Thm 1), see especially p303-304.) Lemma\(^{[8]}\) and hence Corollary\(^{[5]}\) follows immediately from this description, even without the assumption \#\(\ell \cap \ell' \geq 2\).

It remains to consider the case that \(c(\pi) \in \{0, 1\}\). We explain the proof when \(c(\pi) = 0\), the case \(c(\pi) = 1\) being similar but simpler. To simplify further, we shall prove the conclusion \(e_{\ell} e_{\ell'} = e_{\ell'} e\ell\) of Lemma\(^{[6]}\) only in the special case \(\ell = 0..2, \ell' = 1..3\) as in \([2]\) the general case differs only notionally. Let \(v_0\) be a unit vector in the one-dimensional space \(\pi[0..0]\). Set \(v_n := \pi(a(\pi^n))v_0\), where \(a(y) := \text{diag}(y, 1)\). Then \(v_n\) spans \(\pi[n..n]\). Moreover, \(v_0, v_1, v_2, v_3\) give a linear (non-orthogonal) basis for \(\pi[0..3]\). Our task is to verify that \(e_{0..2} e_{1..3} v_i = e_{1..2} v_i\) for \(i = 0, 1, 2, 3\). For \(i = 1, 2,\) the vector \(v_i\) is preserved under the indicated idempotents, and the required identity follows. It remains to consider the cases \(i = 0, 3\); they are similar to one another, so we consider only the case \(i = 3\). Since \(v_3 = e_{1..3} v_3\), our task is to show that

\[
e_{0..2} v_3 = e_{1..2} v_3.
\]

For an integer \(n\), set \(a_n := \langle v_n, v_0 \rangle\). Then \(a_0 = 1, a_{-n} = a_n\) and \((v_{m+n}, v_m) = a_n\) for all \(m\). The quantity \(a_1\) is the Hecke eigenvalue normalized so that \(|a_1| \leq 2/(q^{1/2} + q^{-1/2})\) holds and is sharp for tempered unitary representations. Since \(\pi\) is generic, it is not one-dimensional, and so \(a_1 \neq \pm 1\); by the determinant test, there exist solutions \(b_1, b_2\) to the system of equations

\[
a_1 = b_1 a_0 + b_0 a_{-1},
\]
\[ a_2 = b_1 a_1 + b_0 a_0. \]  

Set \( w := b_1 v_2 + b_0 v_1 \). We see from (20) that \( v_3 - w \) is orthogonal to \( v_2 \) and from (21) that it is orthogonal to \( v_1 \). The \( a_n \) are known (e.g., by the difference equation for the spherical Whittaker function or the recurrence relation for the Hecke eigenvalues) to satisfy a second order linear difference equation with constants coefficients, so from (20) and (21) we deduce that

\[ a_{n+2} = a_{n+1} b_1 + a_n b_0 \]  

for all \( n \). In particular, (22) holds with \( n = 1 \). It follows that \( v_3 - w \) is orthogonal also to \( v_0 \). Hence \( w \) is the orthogonal projection of \( v_3 \) both to \( \langle v_1, v_2 \rangle = \pi[1..2] \) and to \( \langle v_0, v_1, v_2 \rangle = \pi[0..2], \) giving the required identity (19).

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