Newtonian-like and anti-Newtonian universes

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Abstract.

In an irrotational dust universe, the locally free gravitational field is covariantly described by the gravito-electric and gravito-magnetic tensors $E_{ab}$ and $H_{ab}$. In Newtonian theory, $H_{ab} = 0$ and $E_{ab}$ is the tidal tensor. Newtonian-like dust universes in general relativity (i.e. with $H_{ab} = 0$, often called ‘silent’) have been shown to be inconsistent in general and unlikely to extend beyond the known spatially homogeneous or Szekeres examples. Furthermore, they are subject to a linearization instability. Here we show that ‘anti-Newtonian’ universes, i.e. with purely gravito-magnetic field, so that $E_{ab} = 0 \neq H_{ab}$, are also subject to severe integrability conditions. Thus these models are inconsistent in general. We show also that there are no anti-Newtonian spacetimes that are linearized perturbations of Robertson-Walker universes. The only $E_{ab} = 0 \neq H_{ab}$ solution known to us is not a dust solution, and we show that it is kinematically Gödel-like but dynamically unphysical.

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1. Introduction

Irrotational dust spacetimes are characterized by vanishing pressure ($p = 0$)‡ and vorticity ($\omega_a = 0$), and positive energy density ($\rho > 0$). They are important arenas for studying both the late universe [1, 2] and gravitational collapse models [3]. Apart from the spatially homogeneous and isotropic Friedmann-Lemaitre-Robertson-Walker (FLRW) case, which is characterized by vanishing shear ($\sigma_{ab} = 0$), these spacetimes have non-zero shear and non-zero locally free gravitational field. This field is represented by the irreducible electric and magnetic parts $E_{ab}$ and $H_{ab}$ of the Weyl tensor (see [4] for a covariant analysis of local freedom in the gravitational field).

‡ It is implicit that the anisotropic stress $\pi_{ab}$ and the energy flux $q_a$ also vanish.
The gravito-electric tensor $E_{ab}$ is the relativistic generalization of the tidal tensor in Newtonian theory, while the gravito-magnetic tensor $H_{ab}$ has no Newtonian analogue [4, 5], and is associated with gravitational radiation. There are no sound waves in dust, and when $H_{ab} = 0$ there can be no gravitational radiation [4, 6]. Dust spacetimes with $H_{ab} = 0$ have therefore been called ‘silent’, since there are no propagating signals [4, 10, 11].§ Within the class of irrotational dust models, the silent ($H_{ab} = 0$) universes may be called ‘Newtonian-like’, since they have a clear Newtonian counterpart. One might therefore expect that there is a broad variety of such Newtonian-like spacetimes, representing the general relativistic generalization of simple Newtonian models. However, there are subtleties involved in the Newtonian limit of general relativity (see for example [6, 12, 13, 14]), and it turns out that the relativistic Newtonian-like spacetimes form a very restricted class.

Independent exact analyses of silent $H_{ab} = 0$ models, using different methods, are given in [15] and [16]. The independent approaches produce the same conclusion. The silent condition $H_{ab} = 0$ leads to an integrability condition whose repeated time differentiation forms in general a non-terminating chain that leads to inconsistencies. It is thus unlikely that consistent silent solutions exist beyond the known special cases of Szekeres spacetimes and spatially homogeneous models [13, 14]. Furthermore, since these integrability conditions are identically satisfied in the linearized theory (with FLRW background), Newtonian-like models have a linearization instability [15], i.e. there are consistent linearized solutions which are not the limit of any consistent solutions in the full, nonlinear theory.

The linearization instability found via an exact covariant analysis in [15] is confirmed by an independent analysis based on a $1/c$ expansion of the field equations and Bianchi identities [13]. The gravito-magnetic tensor vanishes in the Newtonian, but not the post-Newtonian, limit. Forcing $H_{ab} = 0$ at the post-Newtonian level has the consequence that the non-local nature of the Newtonian tidal force cannot be recovered in the Newtonian limit, and reflects the fact that it is incorrect in general to assume $H_{ab} = 0$ in general relativity.

Thus the Newtonian-like silent models ($H_{ab} = 0$) have a very narrow and limited applicability in cosmology and especially in gravitational instability [13, 18, 15]. Realistic collapse scenarios or realistic inhomogeneous models of the late universe require a gravito-magnetic field $H_{ab}$. Spacetimes with $H_{ab} \neq 0$ do not have a Newtonian counterpart [13], and they are consistent in the generic case, i.e. when there is also a gravito-electric field $E_{ab}$ and no external conditions are enforced on $H_{ab}$ or $E_{ab}$ [19].

The most ‘extreme’ non-Newtonian models (in terms of ‘distance’ from Newtonian

§ A weaker covariant condition for the absence of gravitational radiation is that the spatial curls (defined below) of $E_{ab}$ and $H_{ab}$ must vanish [6, 8, 4].
theory) are those with purely gravito-magnetic field, i.e. with $E_{ab} = 0 \neq H_{ab}$. (Note that $E_{ab} = 0$ and $H_{ab} = 0$ together imply that the spacetime is FLRW.) Consequently, we will use the term ‘anti-Newtonian’ for irrotational dust spacetimes with $E_{ab} = 0 \neq H_{ab}$. Purely gravito-magnetic spacetimes (not restricted to irrotational dust) appear to have first been discussed in [20], where it was shown that either the shear or the vorticity must be non-vanishing. In this paper we consider only such models containing irrotational dust. Since $E_{ab} = 0$ also implies no gravitational radiation [7], the anti-Newtonian models are also ‘silent’.

Here we show that irrotational dust spacetimes with $E_{ab} = 0$ are subject to integrability conditions that are even more restrictive than in the $H_{ab} = 0$ case. The integrability conditions once again form in general non-terminating chains that lead to inconsistencies. There may be spatially homogeneous models of this type which satisfy the integrability conditions, but we have been unable to find examples. However, a further result indicates that there are unlikely to be any consistent exact solutions. This further result is that the only linearized irrotational dust solutions with $E_{ab} = 0$ are exactly FLRW (which have $H_{ab} = 0$) – i.e. there are no linearized anti-Newtonian irrotational dust models.

The question arises as to whether there are any known purely gravito-magnetic solutions ($E_{ab} = 0 \neq H_{ab}$) at all. In [21], it is conjectured that there are no such non-flat vacuum solutions. An exact non-vacuum solution is given in [22], apparently the first purely gravito-magnetic solution. However, as pointed out in [22], this solution has unphysical Segre type. We show in an appendix that it is kinematically a magnetic counterpart of the Gödel solution (which is purely gravito-electric [3]), but that dynamically it has an unphysical source, with negative energy density.

In section 2 we discuss briefly the covariant propagation and constraint equations, and the covariant approach that has been developed to analyze consistency [19]. We show in section 3 how the silent condition $H_{ab} = 0$ implies a primary covariant integrability condition, which is the basis for the linearization instability, as well as for the indefinite chain of conditions in the nonlinear case. This discussion forms a prelude to the analysis in section 4 of the $E_{ab} = 0$ case, where there are two primary integrability conditions, each of which produces in general an indefinite chain of further conditions after differentiation. In the linearized theory, the primary conditions themselves lead to the vanishing of anisotropy and inhomogeneity, i.e. to the FLRW case, so that there are no linearized anti-Newtonian models.

We use the notation and conventions of [19, 4]. Units are such that $8\pi G = 1 = c$; $a, b, \cdots$ are spacetime indices; (square) round brackets enclosing indices denote (anti-) symmetrization, while angled brackets denote the spatially projected, symmetric and tracefree part.
2. Covariant dynamical equations

A covariant approach to the propagation and constraint equations and their consistency has been developed in [19]. There it was shown that in generic irrotational dust spacetimes, i.e. without restrictions on $E_{ab}$ and $H_{ab}$, the constraints are consistent and evolve consistently with the propagation equations. (See also [23, 24].) However, as discussed above and shown below, consistency breaks down in general when the Newtonian condition $H_{ab} = 0$ or the anti-Newtonian condition $E_{ab} = 0 \neq H_{ab}$ are imposed.

The dust four-velocity $u^a$ (with $u^a u_a = -1$) provides a unique covariant 1 + 3 splitting, as fully discussed in [5]. Here we follow the streamlined version of the formalism developed in [19, 4]. The Weyl tensor splits covariantly into gravito-electric and gravito-magnetic parts

$$ E_{ab} = C_{acbd} u^c u^d , \quad H_{ab} = \frac{1}{2} \varepsilon_{abcd} C_{cd}^{\ be} u^e , $$

where the spatial permutation tensor is $\varepsilon_{abc} = \eta_{abcd} u^d$, and $\eta_{abcd}$ is the spacetime permutation tensor. The tensor $h_{ab} = g_{ab} + u_a u_b$, where $g_{ab}$ is the metric tensor, projects orthogonal to $u^a$. Then the projected tracefree symmetric part of a rank-2 tensor is $S_{\langle ab \rangle} = h_{ac} h_{bd} S_{\cd} - \frac{1}{3} S_{\cd} h_{cd} h_{ab}$.

The fluid kinematics and dynamics, and the locally free field are described by the covariant scalars $\rho$ (energy density) and $\Theta$ (volume expansion rate), and rank-2 tensors $\sigma_{ab} = \sigma_{\langle ab \rangle}$ (shear), $E_{ab} = E_{\langle ab \rangle}$ and $H_{ab} = H_{\langle ab \rangle}$. The pressure $p$ and the vorticity $\omega_a$ are assumed to vanish in the irrotational dust case we consider here.

The covariant derivative splits into a covariant time derivative $\dot{S}_{\cd} = u^b \nabla_b S_{\cd}$, and a covariant spatial derivative $D_a S_{\cd} = h_a^c h_b^d \cd \nabla_c S_{db\cd}$. Then the latter leads to a covariant spatial divergence and curl [19]:

$$ \text{div} V = D^a V_a , \quad \text{curl} V_a = \varepsilon_{abc} D^b V^c , $$

$$ (\text{div} S)_a = D^b S_{ab} , \quad \text{curl} S_{ab} = \varepsilon_{cd(a} D^c S_{b) d} . $$

Important identities obeyed by these derivatives are collected in Appendix A.

Covariant splitting of the Bianchi identities and the Ricci identity for $u^a$, where the field equations are taken as an algebraic definition of the Ricci tensor, leads to the propagation equations

$$ \dot{\rho} + \Theta \rho = 0 , \quad (1) $$

$$ \dot{\Theta} + \frac{4}{3} \Theta^2 = - \frac{1}{2} \rho - \sigma_{ab} \sigma^{ab} , \quad (2) $$

$$ \dot{\sigma}_{ab} + \frac{2}{3} \Theta \sigma_{ab} + \sigma_{c(a} \sigma_{b) c} = - E_{ab} , \quad (3) $$

$$ \dot{E}_{ab} + \Theta E_{ab} - 3 \sigma_{c(a} E_{b) c} = \text{curl} H_{ab} - \frac{1}{2} \rho \sigma_{ab} , \quad (4) $$

† See [23] for a thorough discussion of gravito-electromagnetism.
\[ \dot{H}_{ab} + \Theta H_{ab} - 3\sigma_{(a}H_{b)}^c = - \text{curl} E_{ab}, \quad (5) \]

and the constraint equations

\[ C^1_a \equiv D^b\sigma_{ab} - \frac{2}{3}D_a\Theta = 0, \quad (6) \]
\[ C^2_{ab} \equiv \text{curl} \sigma_{ab} - H_{ab} = 0, \quad (7) \]
\[ C^3_a \equiv D^bE_{ab} - \frac{1}{3}D_a\rho - \varepsilon_{abc}b^dH^{cd} = 0, \quad (8) \]
\[ C^4_a \equiv D^bH_{ab} + \varepsilon_{abc}\sigma^b_dE^{cd} = 0. \quad (9) \]

The propagation equations (1)–(5) determine the covariant variables uniquely once initial data is specified on a surface \( S(t_0) \), defined by \( t = t_0 \), where \( t \) is comoving proper time. However the constraint equations (6)–(9) place restrictions on the initial data, and must be satisfied on \( S(t) \) for all \( t \). Since we have imposed the conditions \( p = 0 \) (dust) and \( \omega_a = 0 \) (irrotational),† there is no a priori guarantee that the constraints will not lead to inconsistencies. However, consistency has been shown to hold in the generic case \[19\]. Lengthy tensor calculations lead to the following evolution equations of the constraints \( C^A \) along \( u^a \):

\[ \dot{C}^1_a = - \Theta C^1_a + 2\varepsilon_{a}^{\ bc}\sigma^d_bC^2_{cd} - C^3_a, \quad (10) \]
\[ \dot{C}^2_{ab} = - \Theta C^2_{ab} - \varepsilon^{cd}_{(a}\sigma^c_d)C^1_{d}, \quad (11) \]
\[ \dot{C}^3_a = - \frac{4}{3}\Theta C^3_a + \frac{1}{2}\sigma_a^bC^3_b - \frac{1}{2}\rho C^1_a 
+ \frac{3}{2}E_a^bC^1_b - \varepsilon_a^{\ bc}E^d_bC^2_{cd} + \frac{1}{2}\text{curl} C^4_a, \quad (12) \]
\[ \dot{C}^4_a = - \frac{4}{3}\Theta C^4_a + \frac{1}{2}\sigma_a^bC^4_b 
+ \frac{1}{2}H_a^bC^1_b - \varepsilon_a^{\ bc}H^d_bC^2_{cd} - \frac{1}{2}\text{curl} C^3_a. \quad (13) \]

It follows that if \( C^A(t_0) = 0 \), then \( C^A = 0 \) for all time \( t \). Thus, if no further conditions (beyond \( p = 0 \) and \( \omega_a = 0 \)) are imposed, the constraint equations are preserved under evolution in the generic case. We also require that the initial constraints are consistent, i.e. that \( C^A(t_0) = 0 \) is not over-determined. We see this as follows \[19\]. If we freely specify, for example, \( \sigma_{ab}(t_0) \) and \( D_a\rho(t_0) \), then \( C^1 \) determines \( D_a\Theta(t_0) \), \( C^2 \) determines \( H_{ab}(t_0) \), and \( C^3 \) determines \( D^bE_{ab}(t_0) \). It could appear that \( C^4 \) then imposes a consistency condition, but this is not the case, since it can be shown that \[19\]

\[ C^4_a = \frac{1}{2}\text{curl} C^1_a - D^bC^2_{ab}. \quad (14) \]

This means that the constraint equation \( C^4(t_0) = 0 \) is identically satisfied by virtue of \( C^1(t_0) = 0 = C^2(t_0) \). Thus the constraint equations are consistent with each other, and they evolve consistently, in the generic case.

† as well as the implicitly imposed conditions \( \pi_{ab} = 0 \) (no anisotropic stress) and \( q_a = 0 \) (no energy flux)
Note that we can also interpret equation (14) as the statement that no new vector constraint arises from the divergence of the tensor constraint $C^2$. In other words, the tensor constraint is essentially ‘transverse traceless’ in content.

When additional covariant conditions are imposed, then this generally valid consistency may be disturbed and lead to integrability conditions. Consider the additional conditions that the gravito-electromagnetic tensors are divergence-free:

$$D^b H_{ab} = 0 \neq H_{ab} \text{ and } D^b E_{ab} = 0 \neq E_{ab}.$$  

These are covariant necessary conditions for gravitational radiation in linearized theory [7], taken over into the nonlinear regime. Bianchi dust spacetimes were shown to include spatially homogeneous examples of such models in [19, 22], and inhomogeneous $G_2$ solutions have also been found [26]. However, the Newtonian condition $H_{ab} = 0$ and the anti-Newtonian condition $E_{ab} = 0 \neq H_{ab}$ both affect the propagation equations, converting one of them into a new constraint. This feature, which does not arise in the cases $\text{div } H = 0$ or $\text{div } E = 0$, leads to complicated integrability conditions, as described in the following sections.

3. Newtonian-like models ($H_{ab} = 0$)

When $H_{ab} = 0$, the constraints $C^A$ are modified, but are still consistent, since equations (10)–(14) still hold. But there is now an additional constraint

$$C^5_{ab} \equiv \text{curl } E_{ab} = 0,$$

arising from the generic propagation equation (5), that must be satisfied, together with its evolution along $u^a$.

First we consider whether the divergence of the new tensor constraint $C^5$ leads to an additional vector constraint. Using the identities (A1) and (A6), and the constraint equations (8) and (9), we find [17]

$$D^b C^5_{ab} = \frac{1}{2} \text{curl } C^3_a - \frac{1}{3} \Theta C^4_a - \sigma^a_b C^4_b.$$  

Thus no consistency condition arises from the divergence, and $\text{div } C^5$ is determined by $C^A$, where $A = 1, 2, 3$, by virtue of (14).

Now consider the evolution of $C^5$. Using the identities (A5) and (A7), and the propagation equations (4) and (7), we get [17]

$$\dot{C}^5_{ab} = -\frac{1}{3} \Theta C^5_{ab} - \frac{1}{2} \varepsilon^{cd} (a E_{b_c}) C^4_d$$

$$- \frac{1}{2} \rho C^2_{ab} - \frac{3}{2} \varepsilon^{cd} (a \sigma_{b_c}) C^3_d + \frac{3}{2} \mathcal{H}_{ab},$$  

where

$$\mathcal{H}_{ab} = \varepsilon_{cd(a} \left\{ D^c \left[ E_{b_d} \right] \sigma^d_{b_e} \right\} + 2 D^c \left[ (b_{|e|} E^{de}) \right]$$

$$+ \sigma^c_{b|e|} D^c E^d_{|e|} + \frac{1}{3} \sigma^d_{|e|} D^e E_{b_d}.$$  

(17)
It follows that a necessary condition for consistent evolution of the constraints in Newtonian-like (‘silent’) universes is the covariant condition

\[ \mathcal{H}_{ab} = 0. \tag{18} \]

This is the primary integrability condition for Newtonian-like models. Its repeated covariant time derivatives must also be satisfied. In \[15\], these derivatives, up to fourth order, are evaluated in a shear eigenframe, following the tetrad methods developed in \[27\], leading to a set of non-trivial conditions. In general, further derivatives produce further independent conditions, forming an indefinite chain of integrability conditions. The conditions are identically satisfied in Szekeres and spatially homogeneous silent models (including FLRW models, where \( E_{ab} = 0 \)), but not in general. Thus relativistic Newtonian-like models are in general inconsistent. The same result was found independently by different methods in \[16\], and is supported by the further independent results of \[13\].

Furthermore, equation (17) shows that the condition (18) and its time evolution are identically satisfied in the case of covariant linearization (see \[28\]) around an FLRW background characterized by

\[ D_a \rho = D_a \Theta = 0 \quad \text{and} \quad \sigma_{ab} = E_{ab} = H_{ab} = 0. \]

Thus relativistic Newtonian-like models are subject to a linearization instability, in the sense that there exist consistent solutions of the linearized theory of this type which are not the limit of any consistent solution of the exact nonlinear theory. As pointed out in \[15, 16\], these results together cast serious doubt on the validity and usefulness of pursuing exact ‘silent’ solutions as realistic models of the late universe or of gravitational instability. Realistic general relativistic models involve a gravito-magnetic field, which is confirmed by the independent approach of \[13\]. However, as shown in the following section, a purely gravito-magnetic field leads to even more severe restrictions.

4. Anti-Newtonian models \((E_{ab} = 0 \neq H_{ab})\)

One of the nice features of Newtonian-like models, which facilitated extensive investigation of their dynamics, is that the condition \( H_{ab} = 0 \) has the effect of decoupling the curls from the propagation equations. The propagation equations reduce to a coupled system of ordinary differential evolution equations, i.e. to equations (1)–(4) (with curl \( H_{ab} = 0 \) in the last equation). Dynamical analysis of silent models based only on these equations involves the implicit assumption that the constraint equations are automatically satisfied. As shown in \[13, 10\] and outlined in the previous section, this assumption is incorrect.

Interestingly, an analogous situation arises in the anti-Newtonian case \( E_{ab} = 0 \neq H_{ab} \) (although, to our knowledge, these models have not previously been investigated). Once
again, as for silent Newtonian-like models, the curls decouple from the propagation equations, which again reduce to a coupled system of ordinary differential evolution equations. This feature reflects the fact that anti-Newtonian models are also ‘silent’.†

The anti-Newtonian propagation equations are given by (1)–(3) and (5), with $E_{ab} = 0$ in (3) and $\text{curl} E_{ab} = 0$ in (5). The first 3 form a closed system determining the evolution of $\rho$, $\Theta$, and $\sigma_{ab}$, with the last then determining the propagation of $H_{ab}$ without affecting the evolution of the other quantities. Thus the matter propagation is completely decoupled from the Weyl tensor.† Equation (4) with the left hand side vanishing is no longer a propagation equation – it has become a new constraint, as in the Newtonian-like models. One has to investigate the consistency of the new constraint, and this is done below.

However, there is an added complication in the anti-Newtonian case, not present in the Newtonian-like models. The propagation equations are not independent, in the sense that propagation equation (5) for $H_{ab}$ must be consistent with the curl of propagation equation (3) for $\sigma_{ab}$, by virtue of the constraint equation (7). Using identity (A7), we can rewrite (5) as

$$\text{curl} \dot{\sigma}_{ab} + \frac{2}{3} \Theta \text{curl} \sigma_{ab} - \sigma_{e}^{c} \varepsilon_{cd(a} D^{e} \sigma_{b)}^{d} = 0,$$

and we find that the curl of (3) becomes

$$\text{curl} \dot{\sigma}_{ab} + \frac{2}{3} \Theta \text{curl} \sigma_{ab} + \varepsilon_{cd(a} \sigma_{b)}^{d} D_{e} \sigma^{de} + \varepsilon_{cd(a} D^{c} \left[ \sigma^{de} \sigma_{b)e} \right] = 0,$$

where we also used identity (A5) and the constraint equation (6). The difference between these equations leads to the condition

$$\varepsilon_{cd(a} \left\{ D^{c} \left[ \sigma^{de} \sigma_{b)e} \right] + D^{e} \left[ \sigma_{b)}^{d} \sigma^{e} \right] \right\} = 0.$$

This condition turns out to be an identity (satisfied by any tracefree symmetric tensor), which is given in [19]. Thus we can ignore the propagation equation (5) for $H_{ab}$, since it follows from the shear propagation equation (3) and the constraint equation (6), using covariant identities.

The coupled system of ordinary differential evolution equations to be satisfied is then

$$\dot{\rho} = - \Theta \rho,$$

$$\dot{\Theta} = - \frac{1}{3} \Theta^{2} - \frac{1}{2} \rho - \sigma_{ab} \sigma^{ab},$$

$$\dot{\sigma}_{ab} = - \frac{2}{3} \Theta \sigma_{ab} - \sigma_{c(a} \sigma_{b)}^{c}.$$

† Note that within the class of perfect fluid spacetimes, the covariant propagation equations (see [4]) reduce to ordinary differential evolution equations under the more general conditions that $\dot{u}_{a} = 0$ and $\text{curl} E_{ab} = 0 = \text{curl} H_{ab}$.

† There is no Weyl tensor source term in the geodesic deviation equation, cf. [20].
These determine the evolution of the matter variables, which then determine the evolution of the gravito-magnetic field through

$$\dot{H}_{ab} = -\Theta H_{ab} + 3\sigma_{c(a}H_{b)}{}^c.$$  \hfill (22)

The constraint equation (9) shows that anti-Newtonian models have \(\text{div } H = 0\). This is identically satisfied by virtue of equation (14), which continues to hold when \(E_{ab} = 0\). An additional constraint \(C^5\) arises from the gravito-electric propagation equation (4). The system of constraint equations is then

\begin{align*}
C^1_a &\equiv D^b\sigma_{ab} - \frac{2}{3}D_a\Theta = 0, \quad \hfill (23) \\
C^2_{ab} &\equiv \text{curl } \sigma_{ab} - H_{ab} = 0, \quad \hfill (24) \\
C^3_a &\equiv -\frac{1}{3}D_a\rho - \varepsilon_{abc}\sigma^b_dH^{cd} = 0, \quad \hfill (25) \\
C^4_a &\equiv D^bH_{ab} = 0, \quad \hfill (26) \\
C^5_{ab} &\equiv \text{curl } H_{ab} - \frac{1}{2}\rho\sigma_{ab} = 0. \quad \hfill (27)
\end{align*}

We can eliminate \(H_{ab}\) via the constraint equation (24), which expresses the fact that the shear is a covariant gravito-magnetic potential. Constraint equation (25) becomes

$$\varepsilon_{abc}\sigma^b_d\text{curl } \sigma^{cd} = -\frac{1}{3}D_a\rho. \hfill (28)$$

Second-order derivatives then arise in constraint equation (27), which may be rewritten as

$$D^2\sigma_{ab} = \left(\frac{1}{2}\rho - \frac{1}{3}\Theta^2 + \sigma_{cd}\sigma^{cd}\right)\sigma_{ab} - \Theta\sigma_{(a(c}\sigma_{b)c} + \frac{3}{2}D_{(a}D^c\sigma_{b)c},$$  \hfill (29)

after using identity (A8) for the curl of the curl of a tensor, where \(D^2 = D^aD_a\).

The constraint equation (29) is a nonlinear generalization of the covariant Helmholtz equation. It may also be deduced as a special case of the nonlinear wave equation for the shear that is derived in [19].

The propagation equations (19)–(21) provide a unique solution for \(\{\rho, \Theta, \sigma_{ab}\}\), given the values of these quantities on an initial surface \(S(t_0)\). As in the Newtonian-like case, the problem is that the initial data is subject to a system of constraint equations [(23)–(27), or equivalently (23), (28) and (29)], that is in general over-determined. Before showing this in detail, we can see intuitively how it arises, since the solution \(\sigma_{ab}(t_0)\) of equation (23), even though it allows for arbitrary tensors of integration, will in general not satisfy the constraint equation (28).

We consider firstly whether the divergence of the new tensor constraint \(C^5\) leads to an additional vector constraint. By the identity (A6) and the constraint equations (23), (25) and (9), we find

$$D^bC^5_{ab} = -\frac{1}{2}\rho C^1_a + \frac{1}{3}\Theta C^3_a + \frac{1}{2}\text{curl } C^4_a + \frac{1}{9}\mathcal{J}_a,$$  \hfill (30)

where

$$\mathcal{J}_a = \Theta D_a\rho - 3\rho D_a\Theta - \frac{3}{2}\sigma_a{}^bD_b\rho.$$  \hfill (31)
Thus by equation (31) it follows that a necessary condition in anti-Newtonian universes for consistency of the constraints on an initial surface is the covariant condition

\[ \rho D_a \Theta = \frac{1}{3} \Theta D_a \rho - \frac{1}{2} \sigma^b D_b \rho . \]  

(32)

This can be interpreted as an algebraic relation between the spatial gradients of \( \rho \) and \( \Theta \), or we can use the constraint equations (23)–(25) to rewrite it as an algebraic condition on the div and curl of the shear:

\[ 9 \sigma_{ab} \varepsilon^{bcd} \sigma^c_{de} \text{curl} \sigma_{de} - 6 \Theta \varepsilon_{abc} \sigma^b_{d} \text{curl} \sigma^{cd} - 4 \rho D^b \sigma_{ab} = 0 . \]

Equation (32) is a primary integrability condition, whose successive derivatives must also be satisfied. There is at least one special situation where this condition is identically satisfied. If \( D_a \rho = 0 \), as for example in spatially homogeneous models, then by the energy conservation equation (19) and the identity (A2), we find \( D_a \Theta = 0 \). It then follows that \( J_a \) is identically zero. However, note that the new constraint equation (27) itself is not necessarily identically satisfied when \( D_a \rho = 0 \) – only its divergence vanishes identically, as seen from equation (30). Below we will find a further primary integrability condition from (27) which is not identically satisfied when \( D_a \rho = 0 \). We have been unable to find spatially homogeneous solutions that satisfy equation (27) when \( H_{ab} \neq 0 \), and the existence of such solutions remains an open question. (Clearly FLRW solutions, with \( H_{ab} = 0 = \sigma_{ab} \), satisfy equation (27) identically.) Indeed, the only solution with \( E_{ab} = 0 \neq H_{ab} \) known to us is not an irrotational dust solution (see Appendix B).

In general, and especially in the more physically interesting cases, \( D_a \rho \) is nonzero and the condition (32) is not trivial. The evolution of integrability condition (32) along \( u^a \) produces a further integrability condition. Using identity (A2) to commute time and space derivatives, propagation equations (19)–(21) to eliminate time derivative terms, and condition (32) to eliminate \( D_a \Theta \), we get

\[ \rho D_a \left[ (\sigma^2)^b \right] = \left[ -\frac{1}{3} \rho + \frac{5}{42} (\sigma^2)^b \right] D_a \rho - \frac{1}{3} \Theta \sigma_{ab} D_b \rho - \frac{1}{4} (\sigma^2)_{(ab)} D^b \rho , \]

(33)

where \((\sigma^2)_{ab} = \sigma^c_{ac} \sigma_{bc}\) is the contracted tensor product. It is apparent that in general, with \( D_a \rho \neq 0 \), condition (33) is not automatically satisfied if (32) is, i.e. the derived condition (33) is not an automatic consequence of the primary condition (32). A further time derivative gives

\[ \rho D_a \left[ (\sigma^3)^b \right] = \left[ -\frac{1}{8} \Theta (\sigma^2)^b + \frac{7}{10} (\sigma^3)^b \right] D_a \rho - \frac{12}{42} (\sigma^2)^c_{ab} \sigma_{bc} D^b \rho - \frac{1}{8} \Theta (\sigma^2)_{(ab)} D^b \rho - \frac{3}{10} (\sigma^3)_{(ab)} D^b \rho , \]

(34)

where we used the conditions (32) and (33). Clearly the \( N \)-th time derivative leads to an integrability condition of the form

\[ \rho D_a \left[ (\sigma^{N+1})^b \right] = A_{(N+1)} D_a \rho + A_{(N)} \sigma_{ab} D^b \rho + \cdots + A_{(0)} (\sigma^{N+1})_{(ab)} D^b \rho , \]
where $A_{(M)}$ involves in general $\rho, \Theta$ and $(\sigma^I)_b^a, I = 0, 1, \ldots, M$. The algebraic invariants of the shear are $(\sigma^2)_a^a$ and $(\sigma^3)_a^a$ (the maximal three further independent components of the shear correspond to the rotational freedom in the choice of frame). The scalars $(\sigma^N)_a^a$ are not all independent.

Thus there is an indefinite chain of derived integrability conditions on $S(t_0)$, all of which must be satisfied. At each level, the condition does not follow automatically from lower-level conditions. Since each such equation involves only the initial data $\{\rho, \Theta, \sigma_{ab}\}$ on $S(t_0)$, it is clear that in general the chain of conditions is over-determined and will lead to inconsistencies, i.e. relativistic anti-Newtonian universes are in general inconsistent. We conjecture that the new constraint equation (27) and the integrability conditions (32), (33), \ldots that follow from it are only consistent if the shear, and hence the gravito-magnetic field, vanishes. However, we have been unable to prove this conjecture given the complicated nature of the chain of integrability conditions.

This is similar to the situation in silent Newtonian-like models, but in contrast to the case of shear-free rotating dust models. Using tetrad methods as opposed to the covariant approach adopted here, a condition similar to (32) – i.e. an algebraic relation between spatial gradients – was found for shear-free dust in [27], and its successive time derivatives led to the conclusion that $\Theta \omega_a = 0$. Recently, this result was regained via covariant methods in [30] (and then generalized in [14] from dust to vanishing fluid four-acceleration). The crucial simplifying factor in the shear-free case is the central role of the naturally defined vector $\omega_a$, as opposed to the irrotational case, where there is no natural algebraically defined vector, and instead the central role is played by the tensor $\sigma_{ab}$. Furthermore, the propagation equation for $\omega_a$ is linear in $\omega_a$, in contrast to the shear propagation equation (21), which is nonlinear in $\sigma_{ab}$, and which gives rise to the proliferation of terms involving the gradient of the trace of tensor products.

The conclusion arising from the spatial divergence of the anti-Newtonian constraint equation (27) is that in general the models are not consistent. The conjecture is that there are no consistent anti-Newtonian models. Both of these are reinforced by the existence of a further chain of integrability conditions. This arises from the time evolution of (27), which must also be satisfied. Using the identities (A5) and (A7), the propagation equations (19) and (21), and the constraint equations (23) and (24), we find that†

$$\dot{\mathcal{C}}^5_{ab} = -\frac{4}{3} \Theta \mathcal{C}^5_{ab} - \frac{3}{2} \varepsilon^{cd} (aH_b)_c \mathcal{C}^1_d + \mathcal{E}_{ab},$$

where

$$\mathcal{E}_{ab} = \frac{1}{6} \rho \Theta \sigma_{ab} + \frac{1}{2} \rho \sigma_c (a \sigma_b)^c + 3 H_c (a H_b)^c + 3 \text{curl} \left[ \sigma^c (a H_b)_c \right]$$

$$+ \frac{3}{2} \varepsilon^{cd} (a H_b)_c D^e \sigma^d_e - \sigma^c \varepsilon_{cd} (a D^e H_b)_d.$$  \hspace{1cm} (36)

† Instead of constraint equation (24), we can also use the gravito-magnetic propagation equation (22).
It follows that a necessary condition for consistent evolution of the constraints in anti-Newtonian universes is the covariant condition

\[ \mathcal{E}_{ab} = 0. \] (37)

Note that, in contrast to the previous integrability condition (32), this condition involves second derivatives of the shear, given that \( H_{ab} = \text{curl} \, \sigma_{ab} \).

Clearly this condition is identically satisfied in the degenerate case of \( \sigma_{ab} = 0 \), but, in line with the conjecture stated above, it is unlikely to be satisfied for any shearing solutions because of the chain of further conditions that are implied. Specifically, the evolution of this integrability condition must also be satisfied. As before, the propagation equations can be used to eliminate time derivatives and arrive at a chain of derived integrability conditions intrinsic to \( S(t_0) \), which places inconsistent restrictions on the initial data \( \{ \rho, \Theta, \sigma_{ab} \} \) in addition to those remarked on above. Since inconsistency of the anti-Newtonian models in general is already implied by these previous restrictions, we will not give the very complicated condition arising from \( \dot{\mathcal{E}}_{ab} = 0 \) which, together with its time derivatives, strongly reinforces the conclusion arrived at already.

The key role of the condition (37) emerges in the linearized case. Our conjecture that there are in fact no consistent anti-Newtonian models is further reinforced by an examination of the linearized form of the integrability conditions. Linearization about an FLRW universe (see [28]) of (37) shows that, in contrast to the Newtonian-like case, the linearized integrability condition is non-trivial, so that not all linearized anti-Newtonian solutions are consistent. In fact none are consistent, since the linearized integrability conditions are satisfied only when \( H_{ab} = 0 \). Since already \( E_{ab} = 0 \), this is the FLRW case, which is Newtonian-like. This result can be derived as follows. The linearization about an FLRW background of (37) produces

\[ \Theta \sigma_{ab} = 0, \] (38)

implying either \( \Theta = 0 \) or \( \sigma_{ab} = 0 \). The linearized form of the other primary integrability condition (32) is

\[ 3\rho D_a \Theta - \Theta D_a \rho = 0, \]

which is automatically satisfied if \( \Theta = 0 \), and also holds if \( \sigma_{ab} = 0 \), since in that case the spacetime is FLRW. However FLRW spacetimes are not anti-Newtonian, and \( \Theta = 0 \) implies via the linearization of propagation equation (20) that \( \rho = 0 \), and we have ruled out this vacuum case by our definition of dust universes. (Furthermore, the exact nonlinear form of propagation equation (20) shows that \( \Theta = 0 \neq \sigma_{ab} \) leads to the unphysical condition \( \rho < 0 \).)

Thus there are no linearized anti-Newtonian universes. The implication of this result is that it is difficult to see how any consistent exact anti-Newtonian solution can exist. Such a solution would need to have the property that it cannot be linearized about an
FLRW solution – the solution could not admit small gravito-magnetic field or shear. If such solutions existed, the FLRW solution would have to be an isolated point in the space of all irrotational dust solutions with $E_{ab} = 0$.

5. Concluding remarks

In summary, the overall conclusion following from the covariant analysis in sections 3 and 4 is that irrotational dust universes in general relativity with realistic inhomogeneity must have both gravito-electric and gravito-magnetic fields. The Newtonian-like case $H_{ab} = 0$ is too restrictive, supporting the argument that there is not a straightforward relationship between general relativistic and Newtonian universes, while the anti-Newtonian case $E_{ab} = 0 \neq H_{ab}$ is even more restrictive. Newtonian-like universes (i.e. those with $H_{ab} = 0$) are in general inconsistent, and subject to a linearization instability. We showed that anti-Newtonian models are also in general inconsistent by virtue of the indefinite chain of integrability conditions (32), (33), . . . , which arise from the spatial divergence of the new tensor constraint, and which over-restrict the shear. We conjectured that only the degenerate shear-free case, i.e. the FLRW models, satisfy these integrability conditions. This conjecture is strongly reinforced by the second integrability condition (37), which arises from the time evolution of the new constraint, and whose linearized form implies $\Theta \sigma_{ab} = 0$, leading to the non-existence of any anti-Newtonian solutions linearized about FLRW models.

These results extend our previous work on consistency within the class of irrotational dust spacetimes [19, 23, 13, 24], using a streamlined version [19, 4] of the covariant 1+3 formalism [5], and a systematic covariant approach to analysing consistency [19]. This approach parallels the tetrad methods developed in [27] for analysing the consistency of shear-free dust spacetimes, and is well adapted for investigating more general classes of spacetime.

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Appendix A. Covariant Identities

For convenience we collect here the necessary identities from [19]:

\[ \text{curl } D_a f = 0, \]  
\[ (D_a f)' = D_a f - \frac{1}{3} \Theta D_a f - \sigma^b_a D_b f, \]  
\[ (D_a V_b)' = D_a V_b - \frac{1}{3} \Theta D_a V_b - \sigma^c_a D_c V_b + H_a^d \varepsilon_{abc} V_c, \]  
\[ (D^b S_{ab})' = D^b \dot{S}_{ab} - \frac{1}{3} \Theta D^b S_{ab} - \sigma^{bc} D_c S_{ab} + \varepsilon_{abc} H^b_d S^{cd}, \]  
\[ \text{curl } (f S_{ab}) = f \text{curl } S_{ab} + \varepsilon_{cd(a} S_{b)} d^c f, \]  
\[ D^b \text{curl } S_{ab} = \frac{1}{2} \varepsilon_{abcd} D^b \left( D_d S^{cd} \right) + \varepsilon_{abc} S^b d \left( \frac{1}{2} \Theta \sigma^{cd} - E^{cd} \right) 
- \sigma^{cde} \varepsilon^{bcd} \sigma_{cd} S_{e}^d, \]  
\[ (\text{curl } S_{ab})' = \text{curl } \dot{S}_{ab} - \frac{1}{3} \Theta \text{curl } S_{ab} 
- \sigma^{cde} \varepsilon_{cd(a} D^e S_{b)} d + 3 H_{c(a} S_{b)}^c, \]  
\[ \text{curl curl } S_{ab} = -D^2 S_{ab} + \frac{3}{2} D_{(a} D^c S_{b)c} + \left( \rho - \frac{1}{3} \Theta^2 \right) S_{ab} 
+ 3 S_{c(a} \left[ E_{b)}^c - \frac{1}{3} \Theta \sigma_{b)}^c \right] + \sigma_{ab} \sigma^{cd} S_{cd} 
- \sigma^{c} a \sigma_b^d S_{cd} + \sigma^{c} (a S_{b)}^d \sigma_{cd}, \]  

where \( S_{ab} = S_{(ab)} \) and \( D^2 \equiv D^a D_a \) is the covariant Laplacian.
Appendix B. A purely gravito-magnetic solution

In [22], an exact solution of Einstein’s field equations with $E_{ab} = 0 \neq H_{ab}$ is found, apparently the first such solution. The solution is given in the form (reversing the signature to conform with our convention)
\begin{align*}
ds^2 &= -x^{-2}du^2 + (1 - x^2)dy^2 - 2du
dy
+ x^{-2} \exp \left(-4x^2\right) dx^2 + x^{-2}dv^2, \tag{B9}
\end{align*}
where we have labelled the coordinates in the sequence $x^a = (u, y, x, v)$. The four-velocity $u^a$ is proportional to $\ell^a + n^a$ (there is a misprint in [22] that turns the + into a −), where
\begin{align*}
\bar{\ell} &= 2^{-1/2} \left[-x^{-1}du + (1 - x)dy\right], \quad \bar{n} = 2^{-1/2} \left[-x^{-1}du - (1 + x)dy\right],
\end{align*}
are Newman-Penrose null vectors (using the notation $\bar{v} = v_a dx^a$). The Segre type of the nonzero Ricci tensor is given in [22] as $\{1 1 \bar{z} \bar{z}\}$, but no further discussion is given of the properties of (B9).

We find that
\[ \bar{u} = -x^{-1}du - xdy, \]
which implies
\[ \bar{u} \wedge d\bar{u} = 2x^{-1}du \wedge dx \wedge dy \neq 0. \]
It follows that the solution is rotating. Furthermore, it is apparent from (B9) that $\xi^a = \delta^a_0$ is a timelike Killing vector, so that the solution is stationary. (It is not static, since $\dot{\xi} \wedge d\dot{\xi} \neq 0$, or, equivalently, since $\xi^a$ is parallel to $u^a$.) Since $u^a = x\xi^a$, Killing’s equation shows that the solution is non-expanding and non-shearing. Thus it is kinematically characterized by
\[ \Theta = 0, \quad \sigma_{ab} = 0, \quad \omega_a \neq 0. \tag{B10} \]
This solution therefore has similar kinematic characteristics to the Gödel solution, and we can think of it as a gravito-magnetic counterpart of that solution, which is purely gravito-electric ($H_{ab} = 0$) because $\nabla_b \omega_a = 0$ [5].

However, unlike the Gödel solution, the magnetic solution (B9) has no physically significant interpretation by virtue of its pathological Segre type (see [31], p. 72). For this Segre type of the Ricci tensor, the weak energy condition is violated, so that the energy density is negative. Not only does the solution fall outside of the class of irrotational dust models, but it also has an unphysical source.

† Note that the Gödel solution provides a counter-example to the notion that rotating matter always produces a gravito-magnetic field [23].
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