Dynamic analysis of noncanonical warm inflation

Xi-Bin Li, Yang-Yang Wang, He Wang, and Jian-Yang Zhu
Department of Physics, Beijing Normal University, Beijing 100875, China
(Dated: April 17, 2018)

We study and analyze the dynamic properties of both canonical and noncanonical warm inflationary models with dissipative effects. We consider different models of canonical warm inflation with different dissipative coefficients and provide that the behaviour at infinity of quadratic dissipative model distinctly differs from the one of constant dissipative model, which inspires that quadratic dissipative coefficient increases the possibility of the occurrence of inflation. We also show that the different choice of combination of the parameters in noncanonical warm inflation will exhibits dramatically different global phase portraits on Poincaré disk. We try to illustrate that noncanonical field will not expans as the regime of inflation, but it will increase the possibility of the occurrence of inflation as well and duration of inflation. Then, by dynamic analysis, we can exclude several inflationary models like warm inflation model with negative dissipative coefficient, and explain that the model without potential is almost impossible. With relevant results, we give the condition when reheating occurs.

PACS numbers: 98.80.Cq, 47.75.+f

I. INTRODUCTION

Inflation is an extremely successful model that provides a graceful method to overcome the shortcomings of standard cosmological model, like horizon problem and flatness problem [1], which is consistent with the cosmological observations of the large-scale CMB [2, 3] and large-scale structure [4, 5]. In this simplest and elegant model, the inflation can be described by a special period when our universe expands rapidly driven by a nearly constant energy density arising from the potential of a scalar field [6, 7].

With the success of inflationary scenario, series of candidate scenarios have been established, among which warm inflation is a model that reckons the early universe has a moderate temperature instead of being cold [1, 8]. From the point of view of this scenario, particles interacts with other fields and decay to other particles, which leads to an effect of a friction term to describe this decay phenomenon during inflation [9, 10]. As a result, the primary source of density fluctuations come from thermal fluctuations [11, 12] rather than via quantum fluctuation. However, it was realized a few years later that the idea of warm inflation was not easy to realize in concrete models and even simply not possible [13, 14]. Soon after successful models of warm inflation have been established, in which the inflaton indirectly interacts with the light degrees of freedom through a heavy mediator fields instead of being coupled with a light field directly [15, 16]. Warm inflation model has been widely studied by a series of methods, like field theory method [18], stability analysis [19, 20]. Dynamic analysis is also an effective method to analyze the dynamical properties of warm inflationary system [21, 22].

Another simplest way to establish the inflationary scenario involves extending the Lagrangian density from a canonical kinetic term to a noncanonical one [23, 24]. The noncanonical inflationary scenario has some interesting features, such as that the equations of motion remain second order and that the slow-roll conditions become easier to realize compared to canonical inflationary theory [23]. Most noncanonical models can drop the tensor-to-scalar ratio considerably [25] and stability analysis shows that many of such models have stable attractors [26]. The work under the frame of noncanonical inflation has been done numerously [27-30]. Recently, relevant researches have shown that noncanonical warm inflationary models still satisfy the stability condition as long as each model control parameters at a moderate stage [19], based on which noncanonical inflationary models are being extended to warm scenario, such as warm-DBI model [25], warm k-inflation [31], and so on [32, 33].

In this paper, we attempt to illustrate the global dynamical behaviours of both canonical and noncanonical warm inflation on planar phase space. We start from the condition of canonical inflation and obtain its singularities at original point and infinite region in the presence of different dissipative coefficients. Based on the result from canonical one, we further study the global dynamical behaviours of warm inflationary models with different Lagrangian density of noncanonical field. Then we extend our work to some models with more complex topological structures in phase space.

The paper is organized as follows. In Sec. II we derive the basic dynamic equations from relevant physical equations and assumptions and define the inflationary region in phase space that interests to our study. In Sec. III we study the canonical warm inflationary models with a constant dissipative coefficient and a quadratic field dependent dissipative coefficient by a mathematical
method. In Sec. IV, we focus on the global dynamic phase portraits for different noncanonical warm inflation and get series of interesting and inspired results. In sec. V, we study some models with some strange topological structures like limit cycle and Hopf bifurcation and use it to exclude some inflationary scenarios and discuss when reheating could realize. In Sec. VI, conclusions and relevant further discussions are given.

II. THE DIFFERENTIAL SYSTEM

The action of noncanonical warm inflation writes

$$S = \int d^4x \sqrt{-g} [\mathcal{L}_{\text{non-con}}(X, \phi) + \mathcal{L}_R + \mathcal{L}_{\text{int}}], \quad (1)$$

where \( X = \frac{1}{2} g_{\mu\nu} \partial_\mu \phi \partial_\nu \phi \), \( \mathcal{L}_{\text{non-con}}(X, \phi) \) is the Lagrangian density of noncanonical field, \( \mathcal{L}_R \) is the Lagrangian density of radiation field, \( \mathcal{L}_{\text{int}} \) is the Lagrangian density of interaction between inflaton and other fields, and \( g \) is the determinant of metric \( g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t)) \). The null energy condition and the physical propagation of perturbations require that \( \mathcal{L}_X \geq 0 \) and \( \mathcal{L}_{XX} \geq 0 \) [24, 26], and the subscript \( X \) denotes a derivative with respect to \( X \). The equation of motion can be obtained by taking the variation of the action:

$$\left[ \frac{\partial \mathcal{L}(X, \phi)}{\partial X} + 2X \frac{\partial^2 \mathcal{L}(X, \phi)}{\partial X^2} \right] \dot{\phi} + \left[ 3H \frac{\partial \mathcal{L}(X, \phi)}{\partial X} + \dot{\phi} \frac{\partial^2 \mathcal{L}(X, \phi)}{\partial X \partial \phi} \right] \dot{\phi} - \frac{\partial \mathcal{L}(X, \phi)}{\partial \phi} = 0, \quad (2)$$

where \( H \equiv \dot{a}/a \) denotes the Hubble parameter and dot means a derivative with respect to cosmic time \( t \). The Lagrangian density of noncanonical field can write a simple form as [30]

$$\mathcal{L}_{\text{non-con}}(X, \phi) = K(\phi)X + \alpha X^2 - V(\phi), \quad (3)$$

where \( K(\phi) \) is called "kinetic function" and \( V(\phi) \) is the potential function of \( \phi \). The energy-momentum tensor writes \( T_{\mu\nu} = (\partial \mathcal{L}/\partial X) \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L} \). Thus the energy density and pressure are respectively

$$\rho_\phi = K(\phi)X + 3\alpha X^2 + V(\phi), \quad (4)$$
$$p_\phi = K(\phi)X + \alpha X^2 - V(\phi), \quad (5)$$

In warm inflation model, there is a dissipation term to describe the inflaton fields coupling with thermal bath. With this assumption and [33], we obtain [31]

$$3\alpha \dot{\phi}^2 + K \dot{\phi} + 3H(\alpha \dot{\phi}^2 + K)\dot{\phi} + \Gamma \dot{\phi} + \frac{1}{2} K\ddot{\phi}^2 + V_\phi = 0, \quad (6)$$

where \( \Gamma(\phi) \) is the dissipative term.

In order to get complete differential dynamic equation, we also need two Einstein equations,

$$H^2 = \frac{8\pi G}{3} (\rho_\phi + \rho_R) - \frac{k}{a^2}, \quad (7)$$
$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -8\pi G(p_\phi + p_R), \quad (8)$$

where \( p_\phi = \frac{1}{3} \rho_\phi \) is the pressure of radiation field. In this paper, we consider only the condition of homogeneous and flat spacetime, i.e., \( k = 0 \) and \( X = \frac{1}{2} \phi^2 \). We have that \( \rho_\phi \) and \( \rho_R \) evolve in time as [34]

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) + \Gamma(\phi)\dot{\phi}^2 = 0, \quad (9)$$
$$\dot{\rho}_R + 4H\rho_R - \Gamma(\phi)\dot{\phi}^2 = 0. \quad (10)$$

From (7). We can consider

$$\rho_R = \frac{3}{8\pi G} H^2 - \rho_\phi = 3 \frac{1}{8\pi G} H^2 - K(\phi)X - 3\alpha X^2 - V(\phi) \quad (11)$$

as the expression of \( \rho_\phi \) in (10). Thus, we can write \( \dot{H} \) as

$$\dot{H} = -2H^2 - \frac{8\pi G}{3} (K(\phi)X - 2V(\phi)), \quad (12)$$

where we have used \( p_\phi \) and \( \rho_\phi \) in (5) and (11).

Now, from equations (6) and (12), together with \( \dot{\phi} = d\phi/dt \), we get a three-dimensional dynamic system in phase space of \((\phi, \dot{\phi}, H)\). After dimensionless treatment, the differential system becomes
\[
\begin{align*}
\dot{x} &= y,  \\
(3K(x) + \alpha y^2)\dot{y} &= -3(3K(x) + \alpha y^2)yz - \Gamma(x) - \frac{1}{2}K_xy^2 - V_x,  \\
\ddot{z} &= -2z^2 - \frac{8\pi G}{3}\left(\frac{1}{2}K(x)y^2 - 2V(x)\right).
\end{align*}
\]

It is noteworthy that any variable or parameter in the equations above is dimensionless and its physical meaning will be introduced in next sections.

The physical region in three-dimensional phase space is defined by the condition \(\rho_R \geq 0\). In general, the dissipative coefficient is not arbitrary but with the form \(\Gamma(\phi) = \Gamma_n\phi^n\), \(n\) is an even number. In this condition, those trajectories that initially lie inside the region \(\rho_R \geq 0\) remain inside the region, in which the trajectories will neither cross the region \(\rho_R = 0\) nor enter the region \(\rho_R \leq 0\) \([21]\). We will also show in this paper, any variables in \(L_{\text{non-con}}(X, \phi)\) or \(\Gamma(\phi)\) must be a positive number or an odd exponential functional form. These forms of relevant functions provide the singularities at infinity is symmetry about the original point, when all trajectories lie in phase planar will not cross the infinite boundary and remain inside the Poincaré disk. The inflationary region is defined by

\[
\frac{\ddot{a}}{a} = H + H^2 > 2
\]

labeled as \(\mathcal{J}\), which must locate in the region with positive curvature

\[
R = 6(\dot{H} + 2H^2) > 0
\]

labeled as \(\mathcal{R}\). Generally the analytical form of \(\mathcal{J}\) is quite complex and sometimes there is no need to know the exact formula, on the contrary, we can plot it on an approximate region locating in \(\mathcal{R}\) which are tangent with each other at infinity. This approximate method is widely used in Sec. \([IV]\).

We attempt, however, to discuss the dynamic system in two-dimensional phase space \((x, y)\) (consider \(z\) as a constant) instead of three-dimension. The reasons are as follows:

- The trajectory of \(z\) is quite simple which is just a monotone decreasing curve, so the trajectory in three-dimensional phase space \((x, y, z)\) is topological equivalent to the trajectory in two-dimensional phase space \((x, y)\).

- Numerical analysis shows that the behaviour of \(z\) trajectories evolve very slow, so the topological structures of them are almost the same.

- What we interest most is the slow-roll condition, i.e., \(\varepsilon \equiv H/H^2 \ll 1\), which is consistent with the reason just above.

- The dynamic system \((13)\) domains by variables \(x\) and \(y\), and the presence of \(z\) will not generate any complicated structure, like chaos or singular closed trajectory.

Next we will show several examples in two-dimensional phase space in different models.

III. DYNAMIC ANALYSIS OF CANONICAL INFLATION

Before analysing the dynamic properties of noncanonical warm inflation, we firstly consider the canonical one, which help us better understanding the properties of the both.

A. \(\Gamma(\phi) = \text{constant}\)

Let’s start from a simple condition. As an easy example, we first consider the canonical warm inflation with a constant dissipation coefficient. Set \(K(\phi) = 1\), \(\alpha = 0\), \(\Gamma(\phi) = \Gamma_0\) and \(V(\phi) = \frac{1}{2}m^2\phi^2\). Redefining variables \(t \rightarrow t/m, \phi \rightarrow M_p x, \dot{\phi} \rightarrow m M_p y, H \rightarrow m \dot{H}\) and \(\Gamma_0 \rightarrow m \Gamma_0\). The expression of such a dynamic dissipative system is quite simple:

\[
\begin{cases}
\dot{x} = y, \\
\dot{y} = -x - (3\dot{H} + \Gamma_0)y.
\end{cases}
\]

To be convenient, we set \(r \equiv 3\dot{H} + \Gamma_0 \gg 1\).

first, we need to research the topological structure at singular point \((9, 0)\). The stability topological structure at \((0, 0)\) is determined by matrix

\[
A = \begin{pmatrix} 0 & 1 \\ -1 & -r \end{pmatrix},
\]

whose eigenvalues are \(\lambda_1 = k_1 \ll -1\) and \(\lambda_2 = k_2 \leq \)
Then system (17) is given by

\[ 2 \text{This transformation maps the point on infinity to } S^2. \]

Set \( k_1 k_2 = 1 \). According to the definitions in Appendix A, singular point \((0, 0)\) is a stable node with topological structure like first pattern in Fig. 1.

Next we analyse the singularity at infinity. To study the orbits which tend to or comes from infinity, we can apply Poincaré compactification, which will tell us the topology in infinite region. Do the coordinate transformation

\[ \phi : (x, y) \mapsto (u, z) = \left( \frac{y}{x}, \frac{1}{x} \right). \]

This transformation maps the point on infinity to \( S^2 \). Then system (17) is given by

\[
\begin{cases}
\dot{u} = -(u^2 + ru + 1), \\
\dot{z} = -zu.
\end{cases}
\] (20)

Set \( z = 0 \) (that means the singular point at infinity), then we get two singularities \((u_1, 0)\) and \((u_2, 0)\) with \( u_1 = k_1 \) and \( u_2 = k_2 \). This result tells us that there are four singularities, which distribute along the directions \( y = u_1x \) and \( y = u_2x \). By studying the stabilities, it is easy to find singular points \( A' \) and \( A'' \) on \( y = u_2x \) are saddles which are symmetry about the original point of planer \((x, y)\), while singular points \( B' \) and \( B'' \) on \( y = u_1x \) are unstable nodes which are also symmetry about the original point (see Fig. 1).

Dynamic system has no more singularities on \( \mathbb{R}^2 \). If we plot the singularities above on one finite plane, which is also called Poincaré disk, we obtain the global phase portrait (also plotted on the panel in Fig. 1). The portrait shows that the directions \( y = u_1x \) and \( y = u_2x \) are not equivalent, i.e., that \( y = u_2x \) is more stable than \( y = u_1x \). So direction along \( y = u_2x \) is a strong direction while \( y = u_1x \) is called weak direction. Detailed calculation shows \( \left\{(x, y) | y = u_2x \right\} \cap J \neq \emptyset \), which means that most trajectories will cross the inflationary region.

B. \( \Gamma(\phi) = \Gamma_2 \phi^2 \)

Let’s start from a simple condition. As an easy example, we first consider the canonical inflation. Now, set \( K(\phi) = 1, \alpha = 0, \Gamma(\phi) = \Gamma_2 \phi^2 \) and \( V(\phi) = \frac{1}{2} m^2 \phi^2 \).

Redefining variables \( t \to t/m, \phi \to M_p x, \dot{\phi} \to m M_p y, \)

\( H \to m H \) and \( \Gamma_2 \to (m/M_p^2) \bar{\Gamma}_2 \), we obtain the dynamic dissipative system:

\[
\begin{cases}
\dot{x} = y, \\
\dot{y} = -x - 3H y - \Gamma_2 x^2 y.
\end{cases}
\] (21)

According to slow-roll condition, set \( 3H \gg 1 \). Obviously, \((0, 0)\) is a singularity. According to theorem in
Appendix A the stability at $(0,0)$ is determined by matrix
\[
A = \begin{pmatrix} 0 & 1 \\ -1 & -3H \end{pmatrix},
\]
whose eigenvalues are $\lambda_1 = k_1 \ll -1$ and $\lambda_2 = k_2 \gg -1$, with $k_1, k_2 = 1$, which is exactly the same as the analysis in Sec. III A.

However the singularities at infinity are quite different from the those of system (17). Do the transformation as above: $u = y/x$ and $z = 1/x$. Thus,
\[
\begin{align*}
\dot{u}' &= -(\Gamma_2 u + z^2 + 3H u z^2 + u^2 z^2), \\
\dot{z}' &= -z^3 u,
\end{align*}
\]
where prime denotes the derivative with respect to $\tau$ and $dr = dt/z^2$. System (22) is not homogeneous and singular point $(0,0)$ is called a semi hyperbolic singularity, so we need to do more treatment about it. Do transformation $\phi = wu$, $z = r\bar{z}$. We first perform a transformation in the $z$-direction by setting $\bar{u} = 1$, which helps us to study the behaviour along $z$-direction. Writing $(u, z) \rightarrow (r, r\bar{z})$, we get
\[
\begin{align*}
\dot{r}' &= -(\Gamma_2 r + r^2 \bar{z}^2 + 3H r^3 \bar{z}^2 + r^4 \bar{z}^2), \\
\dot{\bar{z}}' &= +\left(\bar{\Gamma}_2 \bar{z} + r^3 z + 3H r^3 \bar{z}^3 + 2r^3 z^3\right),
\end{align*}
\]
where $\bar{\Gamma}_2 = \bar{\Gamma}_2 + \frac{3H}{r^3} + \frac{2r^3}{z^3}$.

So, singular point $(0,0)$ is a saddle. Then setting $\bar{u} = -1$, similarly with analysis above, we find $(0,0)$ is a saddle (two saddles are located at negative direction and positive direction, respectively). Moreover, the analysis in the $u$-direction becomes simple: according to semi hyperbolic singularity theorem [12], the vector flow on $z$-axis satisfies $u' = -\Gamma_2 u$. Then, we need to put the vector fields on planar phase into one point. The topological structure at $(0,0)$ is shown on the second panel in Fig. 3 which is just a saddle. Such method is called blow-up. After the calculations above, the singularities $A'$ and $A''$ at infinity are saddles which locate at $x$-axis (because $u = 0$). Similarly, by performing the transformation $v = x/y$, $w = 1/y$, we get another two singularities at infinity which are unstable nodes locating at $y$-axis.

Comparing Fig 1 with Fig 2 we conclude that the topological structures are almost the same at singular point $(0,0)$, and the structures at infinity are also nearly the same except their locations. That means the trajectories in inflationary region $J'$ evolve almost parallel to $y$-axis with $y \sim 0$. This result induces us that warm inflation model with dissipative coefficient $\Gamma(\phi) = \Gamma_2 \phi^2$ dramatically increases the possibility of the occurrence of inflation.

IV. NONCANONICAL WARM INFLATION

From now on, we will discuss the dynamic properties of noncanonical warm inflation. At beginning of this section, let us start from a condition that quite analogous with canonical warm inflation discussed above.

A. $K(\phi) = 1 + k\phi^2$, $\alpha > 0$

Set $K(\phi) = 1 + k\phi^2$, $\alpha > 0$, $\Gamma(\phi) = \Gamma_2 \phi^2$ and $V(\phi) = \frac{1}{2}m^2 \phi^2$. Redefining variables $t \rightarrow t/m$, $\phi \rightarrow M_p x$, $\phi \rightarrow m M_p y$, $H \rightarrow m H$, $\Gamma_2 \rightarrow (m/M_p^2) \Gamma_2$, $k \rightarrow \bar{k}/M_p^2$ and $\alpha \rightarrow \bar{\alpha}/(m^2 M_p^2)$. According to Theorem 2 the expression of such a dynamic dissipative system writes
\[
\begin{align*}
x' &= y(1 + k\bar{x}^2 + 3\bar{\alpha} y^2), \\
y' &= -x - 3\bar{H} x y - 3\bar{H} k x y^2 - \bar{\Gamma}_2 x^2 y - \bar{\alpha} y^2 - 3 \bar{H} \bar{\alpha} y^3,
\end{align*}
\]
where prime denotes the derivative with respect to $\tau$ and $dr = dt/(1 + k\bar{x}^2 + 3\bar{\alpha} y^2)$. Using Theorem 1 in Appendix A once again, it's easy to get the conclusion that the topological structure of system (24) at original point is the same as the one in system (17) or system (21) which are stable nodes and the topological structures at infinity appear identical to the ones in system (21) which are semi hyperbolic singularities locating at $x$-axis and $y$-axis. So the global phase portrait is almost the same as the portrait in Fig. 2.

B. $K(\phi) = k\phi^2$, $\alpha > 0$

Now, let's consider a more complex model. Set $K(\phi) = k\phi^2$, $\alpha > 0$, $\Gamma(\phi) = \Gamma_2 \phi^2$ and $V(\phi) = \frac{1}{2}m^2 \phi^4$. Redefining variables $t \rightarrow t/M_p$, $\phi \rightarrow M_p x$, $\phi \rightarrow M_p^2 y$, $H \rightarrow M_p H$, $\Gamma_2 \rightarrow \Gamma_2 / M_p$, $k \rightarrow \bar{k}/M_p^2$, $\alpha \rightarrow \bar{\alpha}/M_p^4$ and $V_0 = \lambda$, we obtain
\[
\begin{align*}
x' &= k x^2 y + 3\bar{\alpha} y^3, \\
y' &= -3\bar{H} k x^2 y - \bar{\Gamma}_2 x^2 y - 3\bar{H} \bar{\alpha} y^3 - \bar{k} x y^2 - x^3.
\end{align*}
\]
Next it will be seen that there exists completely different topological structures by choosing different combinations of parameters in system (25).

1. Case 1

Set $\bar{k} = \bar{H} = \bar{\Gamma}_2 = V_0 = 1$ and $\bar{\alpha} = 1/3$. We have
\[
\begin{align*}
x' &= x^2 y + y^3, \\
y' &= -4x^2 y - y^3 - x^2 y^2 - x^3.
\end{align*}
\]
The singularity at original point $(0,0)$ is a nonelementary one, and the expression of blow-up map is given by
\[
\varphi : (x, y) \rightarrow (x, ux),
\]
which is also called Briot-Bouquet transformation that maps $(0,0)$ to $xOU$ planar. Thus,
\[
\begin{align*}
dx/d\eta &= ux(1 + u^2), \\
dy/d\eta &= -(u^4 + u^3 + 3u^2 + 4u + 1) = f(u),
\end{align*}
\]
where \( d\eta = x^2d\tau \). There are two singularities in system (29): \((0, u_1 = -1.33)\) and \((0, u_2 = -0.29)\). The linear matrix \( A \) read

\[
\begin{pmatrix}
  u_1(1 + u_1^2) & 0 \\
  0 & f'(u_1) > 0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
  u_2(1 + u_2^2) & 0 \\
  0 & f'(u_2) < 0
\end{pmatrix}
\]

So \((0, u_1)\) is a saddle while \((0, u_2)\) is a stable node whose topological structures on \( xOu \) are plotted on left panel in Fig. 4. Then we can obtain the vector fields near \((0, 0)\) on coordinate \( xOy \) by map \( \varphi^{-1} \), which is drawn on second panel. Next we turn to analysis the singularities at infinity of system (27).

Using coordinate transformation (28), we have

\[
\begin{aligned}
  \frac{du}{d\eta} &= -(u^4 + u^3 + 3u^2 + 4u + 1), \\
  \frac{dz}{d\eta} &= -uz(1 + u^2),
\end{aligned}
\]

where \( d\eta = d\tau/z^2 \). Similar with the analysis above, we immediately see that singularities \( A' \) and \( A'' \) are saddles that locate at \( y = u_2x \) while \( B' \) and \( B'' \) are unstable nodes that locate at \( y = u_1x \).

Finally, we get the global phase portrait of dynamic system (27) in Fig. 6. From (15), we get the region with positive curvature \( R = \{(x, y) | y^2 \leq x^2\} \), which is the same as the one in Sec. IIIB. In other words, the inflationary regions are almost the same with each other. Direction \( y = u_2x \) is repelling while \( y = u_1x \) is an attracting direction. So trajectories evolve leave from the direction \( y = u_2x \) and converge along the direction.
FIG. 5: Blow-up of singularity and local phase portrait of (33).

FIG. 6: Global phase portrait of dynamic system (27).

\[ y = u_1 x. \] Meanwhile, \( \{ (x, y)| y = u_1 x \} \cap \mathcal{J} \neq \emptyset \), which means orbit evolves longer duration in inflationary region \( \mathcal{J} \) compared canonical warm inflationary scenario.

2. Case 2

Set \( H = 1, \alpha = 1/3, \tilde{k} = 1/4, \tilde{\Gamma}_2 = 5/4 \) and \( V_0 = 3/2 \). The dynamic system writes

\[
\begin{align*}
    x' &= \frac{1}{4} x^2 y + y^3, \\
    y' &= -2x^2 y - y^3 - \frac{1}{4} xy^2 - \frac{3}{2} x^3.
\end{align*}
\] (33)

The blow-up at original point is

\[
\begin{align*}
    dx/d\eta &= ux\left(\frac{1}{4} + u^2\right), \\
    du/d\eta &= -(u + 1)^2(u^2 - u + \frac{5}{8}).
\end{align*}
\] (34)

The blowing up topological structure is a little peculiar: any trajectory crossing \( u = u_0 = -1 \) is tangent to it (see left panel in Fig. 5). Transformation \( \varphi^{-1} \) means the trajectories near \((0, 0)\) cross the line \( y = -x \) and tangent to it (see right panel in Fig. 5). However, singularities \( A' \) and \( A'' \) that locate at infinity exhibit a different stability like neither saddles nor nodes, but called saddle-nodes instead (see Fig. 7).

The line \( y = u_0 x \) lies inside the region \( \mathcal{R} = \{ (x, y)| y^2 \leq 6x^2 \} \) and it must intersect with inflationary region \( \mathcal{J} \).
As a result, there exists trajectories crossing through the inflationary region, but the condition is weak than it in Sec. IV B 1, because any trajectory will cross the line $y = u_0 x$ but it is not a strong attracting direction.

3. Case 3

Set $\bar{H} = 1$, $\bar{\alpha} = 1/3$, $\bar{k} = 1/4$, $\bar{\Gamma}_2 = 5/4$ and $V_0 = 2$. The corresponding system with blow-up reads

$$
\begin{align*}
    x' &= \frac{1}{2}x^2 y + y^3, \\
    y' &= -2x^2 y - y^3 - \frac{1}{2}x y^2 - 2x^3,
\end{align*}
$$

(35)

with blowing up

$$
\begin{align*}
    dx/d\eta &= ux(\frac{1}{2} + u^2), \\
    du/d\eta &= -(u^4 + u^3 + \frac{1}{2}u^2 + 2u + 2).
\end{align*}
$$

(36)

Obviously, there is no singular point in (36), which means vector fluids will enter the original point on coordinate $xOy$ along any no special direction, with topological structure something like a stable focus that plotted on the third panel of Fig. 11. By transformation (19), we also see that no singularity at infinity as well. The global phase portrait is plotted in Fig. 8.

Let’s have a brief conclusion of this section. It has been seen that the choice of the nancanonical Lagrangian action determines the behaviours of dynamic system. In Sec. IV A, the topological structure in phase planar is equivalent to the one in canonical warn inflation with dissipative coefficient $\Gamma = \Gamma_2 \phi^2$. In Sec. IV B on the other hand, the topological structure is determined by the choice of the combination of parameters. Although we only plot the portraits of three conditions, there also several other global phase portraits like three or four damped directions towards the original point. However, there always exists the phase trajectories that cross the inflationary region as long as we choose an appropriate initial condition. Finally, based on the accurate analysis above, we reckon the condition in Sec. IV B 1 is more close to the physical reality of inflation model.

V. OTHER PHASE TRAJECTORY STRUCTURES OF WARM INFLATION MODEL

In the sections precious, we introduce both canonical and noncanonical warm inflationary models whose attractors all locate at original point. In this section, we will introduce two models with complex topological structure together with different attractors.

A. Limit cycle

Regularly, the dissipative coefficient in warm inflation model is a positive constant or a function larger than zero, and theoretical calculation has also excluded such conditions that dissipative coefficients range less than zero \[35, 36\]. Now, we will show this conclusion by dynamical system analysis as well. Redefine the variables $t \rightarrow -t$, $\bar{\Gamma}_2 \rightarrow -\bar{\Gamma}_2$ with $\bar{\Gamma}_2 > 0$, which is similar with the transformation in (21). The system is a nonlinear oscillation equation

$$
\frac{d^2 x}{dt^2} - e(1 - rx^2)\frac{dx}{dt} + x = 0
$$

(37)

with $e \equiv 3\bar{H}$ and $r \equiv 3\bar{\Gamma}_2/3\bar{H}$. (37) is just the famous von der Pol equation. According Theorem 3 in Appendix A we find system (37) satisfies all the conditions, which means system has a stable limit cycle, or the $\omega$ limit set of system is a limit cycle. However, we should notice that the result above is obtained under the transformation $t \rightarrow -t$ with the opposite time evolutionary orientation, which means the limit cycle of the initial dynamic equation of warm inflation

$$
\ddot{\phi} + 3H \dot{\phi} + \Gamma(\phi) \dot{\phi} + V_\phi = 0
$$

(38)

is not stable at all, in other words, original point $(0, 0)$ and infinite region are the attractors of system \[35\] instead of a limit cycle (see Fig. 11). Obviously, there can not exist the inflationary region in such model and it represent barely physical meaning in both theoretical practice and observational practice.

B. Bifurcation

Consider the potential function $V(\phi) = \frac{1}{4}(\phi^2 - \sigma^2)^2$, which is widely used in the models of symmetry breaking
FIG. 9: Global phase portrait of dynamic system (38).

in gauge field theory [37] and reheating theory in cosmology [38, 39]. Set $K(\phi) = 0$, $\alpha = 1/3$, $\lambda = 2$, $\bar{H} = 1$, and $\bar{\Gamma}_2 = 1$, which is similar with the parameters in Sec. IV B.

The dynamic system reads

\[
\begin{align*}
x' &= y^3, \\
y' &= -y^3 - x^2y - 2x(x^2 - \bar{\sigma}^2),
\end{align*}
\]

(39)

where $\bar{\sigma} \equiv \sigma/M_p$. Now, assume $\bar{\sigma}^2 < 0$ (mathematical respect), singular point $(0, 0)$ is a strong focus. If $\bar{\sigma} = 0$, singular point $(0, 0)$ is a week focus that is sensitive to a small perturbation. While, when $\bar{\sigma}^2 > 0$, there are three singular points on finite region, $(\pm \bar{\sigma}, 0)$ and $(0, 0)$. We immediately see that $(0, 0)$ is an unstable singularity while two stable focuses $(\pm \bar{\sigma}, 0)$ (see the analysis in Sec. IV B 3) arise near singularity $(0, 0)$. Such singularity is called Hopf bifurcation point [40, 41] and its global dynamical phase portrait is plotted in Fig. 10.

Now, let's have a further discussion. In reheating model, it suggests that the field oscillates at the bottom of potential function (at $\pm \bar{\sigma}$). However, as the discussions previous, some models (dependent on the choice of the parameters) will not oscillate at all but damp to the bottom directly, which means very few new particles will generate during this period. So rehearing appears only under the condition either with a small enough dissipative coefficient (though $\lambda$ is small) or with a large enough $\lambda$ (though the dissipative coefficient is small).

VI. CONCLUSION AND FURTHER DISCUSSION

In this work we derive a dynamic dissipative system in phase space of warm inflationary model and analyze it in both canonical and noncanonical conditions. We first study dynamic systems describing a canonical inflationary dynamic with different dissipative coefficient. We have also distinguished them by global dynamical analysis in planar phase space: (a) If the dissipative coefficient is a constant, it’s a dynamic system just like the standard inflation established initially. The trajectories are attracted along a special direction and exponentially damp to the original point. The singularities at infinity locate at two special directions whose gradients are just the eigenvalues of linear matrix $A$; (b) If the dissipative coefficient is a field dependent function of quadratic exponential, the global dynamical behaviour is quite different. The topological structure at original point is the same as the one with a constant dissipative coefficient, but the singularities at infinity locate at $x$-axis and $y$-axis, which means the trajectories in inflationary region tend more to $y$-axis, i.e., $\dot{\phi} \sim 0$, which increases the possibility of the occurrence of inflation the duration of inflation.

Noncanonical condition is another emphases we mainly discuss in this work. If set $K(\phi) = 1 + k\phi^2$, $\alpha > 0$ in noncanonical Lagrangian action, the system exhibits the same dynamic properties as the model of canonical warm inflation. While, if set $K(\phi) = k\phi^2$, $\alpha > 0$, the systems show dramatically different global phase portraits due to the different combinations of parameters (normalized $H$ to unit). From the physical side, we can also reach some interesting conclusions which may have important meaning in inflationary dynamics. The noncanonical warm inflationary scenarios still have stable attractor of inflationary phase. For the condition $K(\phi) = k\phi^2$, $\alpha > 0$, the inflationary region is almost the same with the region of the canonical, but it keeps a long period during inflationary phase. As an alternative to the standard inflationary model, warm inflationary scenario leads the universe...
to a moderate temperature so that reheating could be avoided. Our results allow us to reach some conclusions that concentrates more on the debate about reheating.

Besides the use of studying the dynamical behaviours of inflationary system, dynamic analysis can also exclude several inflationary models. Canonical warm inflation with dissipative coefficient $\Gamma(\phi) = -\Gamma_0\phi^2$ is the system with an unstable limit cycle that the all trajectories depart from it. There are two attracting points of this dynamic system, original point and infinite region, which means the system is quite sensitive to the initial condition. If the initial point locates outside the limit cycle, the system will evolve to infinity, which cannot occur in the early universe. The model without self-interaction potential is quite difficult to realize. In such model, all trajectories distribute nearly parallel to $y$-axis in phase space, in other words, there exists a quite short period during which a trajectory crosses through the inflationary region. There is another dynamic system that has the Hopf bifurcation structure. This model has a potential as the form $V(\phi) = \frac{1}{4}(\phi^2 - \sigma^2)^2$ which is widely used in the models of symmetry breaking in gauge field theory and reheating theory in cosmology and dynamic analysis on such model will shed some light to study the reheating stage just after the inflationary phase.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grants No. 11575270, No. 11175019, and No. 1235003).

Appendix A: Definitions and properties in dynamic system

We first introduce several definitions and properties in planer dynamic system $[42–44]$. Consider the dynamic system

$$
\begin{align*}
\dot{x} &= P(x, y), \\
\dot{y} &= Q(x, y),
\end{align*}
$$

(A1)

with boundary condition $P(0, 0) = Q(0, 0) = 0$. Then system (A1) become

$$
\begin{align*}
\dot{x} &= \frac{\partial P(0,0)}{\partial x}x + \frac{\partial P(0,0)}{\partial y}y + f(x, y), \\
\dot{y} &= \frac{\partial Q(0,0)}{\partial x}x + \frac{\partial Q(0,0)}{\partial y}y + g(x, y),
\end{align*}
$$

(A2)

where $f(x, y)$ and $g(x, y)$ are the rest part higher than second order. System (A2) can also write in form of vectors:

$$
\dot{x} = Ax + f(x),
$$

(A3)

where $x = (x, y)'$, $f(x) = (f(x, y), g(x, y))'$ and

$$
A = \begin{pmatrix}
\frac{\partial P}{\partial x}(0,0) & \frac{\partial P}{\partial y}(0,0) \\
\frac{\partial Q}{\partial x}(0,0) & \frac{\partial Q}{\partial y}(0,0)
\end{pmatrix}.
$$

(A4)

The linear part

$$
\begin{align*}
\dot{x} &= \frac{\partial P(0,0)}{\partial x}x + \frac{\partial P(0,0)}{\partial y}y, \\
\dot{y} &= \frac{\partial Q(0,0)}{\partial x}x + \frac{\partial Q(0,0)}{\partial y}y,
\end{align*}
$$

(A5)

i.e.,

$$
\dot{x} = Ax,
$$

(A6)

of system (A3) determines the behaviour at elementary singular points.

**Definition 1** If $P(x_0, y_0) = Q(x_0, y_0) = 0$, the point $(x_0, y_0)$ is called singularity. If $\det A \neq 0$, point $(x_0, y_0)$ is called elementary singularity; if $\det A = 0$, point $(x_0, y_0)$ is called nonelementary singularity. If $\lambda_1 = 0$ and $\lambda_2 \neq 0$, point $(x_0, y_0)$ is called semi hyperbolic singularity.

Let $(0, 0)$ be a singular point of the dynamic system and $\lambda_1$ and $\lambda_2$ be the eigenvalues of the linear part matrix $A$.

**Definition 2** If $\lambda_1 < \lambda_2 < 0$, $(0, 0)$ is a stable (unstable) node. If $\lambda_1 \cdot \lambda_2 < 0$, $(0, 0)$ is a saddle. If $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ with $\alpha < 0$, $(0, 0)$ is a stable focus. If $\lambda_1 = i\beta$ and $\lambda_2 = -i\beta$, $(0, 0)$ is called a center (see Fig. 11).

Consider the dynamic system (A3) and (A6), we have

**Theorem 1** Assume that $f(x)$ is continuous on $\mathbb{R}^2$ and satisfies the Lipschitz condition about $x$, if consistently

$$
\lim_{x \to 0} \frac{\|f(x)\|}{\|x\|} = 0
$$

and any eigenvalue of matrix $A$ is not vanish, the stability of system (A3) at $(0, 0)$ is the same as system (A6).
Let $X_1$ and $X_2$ be two vector fields on open subsets $D_1$ and $D_2$ on $\mathbb{R}^2$, respectively.

**Definition 3** If there exist a homeomorphism $h : D_1 \to D_2$ which maps orbits of $X_1$ to $X_2$ by preventing the orientation, in’s said that $X_1$ is topological equivalent to $X_2$.

In this paper, the dynamic system of warm inflation model exhibits different type which do not follow the form as (A1), instead, it satisfies the dynamic system model exhibits different type which do not follow the form

\[
\begin{align*}
\frac{dx}{dt} &= P(x, y), \\
M(x, y) \frac{dy}{dt} &= Q(x, y).
\end{align*}
\tag{A7}
\]

However, is hard to study its stability properties and asymptotic behaviours. We hope that system is topological equivalent to the dynamic system below:

\[
\begin{align*}
\frac{dx}{d\tau} &= P(x, y)M(x, y), \\
\frac{dy}{d\tau} &= Q(x, y),
\end{align*}
\tag{A8}
\]

where $d\tau = dt/M(x, y)$. The theorem below tell us when they are topological equivalent to each other.

**Theorem 2** If $M(x, y)$ is continuous on $\mathbb{R}^2$, $M(x, y) > 0$ on $\mathbb{R}^2/\{0\}$ and

\[
\lim_{t \to \pm\infty} M(x(t), y(t)) = M_0 \geq 0,
\]

then system (A7) is topological equivalent to system (A8).

Consider the nonlinear oscillation equation

\[
\frac{d^2x}{dt^2} + f(x) \frac{dx}{dt} + g(x) = 0,
\tag{A9}
\]

where $-g(x)$ is the restoring force and $f(x)$ is the damping force with $f, g \in C(\mathbb{R})$. Integrate (A9) on the duration from 0 to $t$, we have

\[
\frac{dx}{dt} + \int_0^x f(u)du + \int_0^t g(x)dx = 0.
\]

Set $y = -\int_0^t g(x)dx$ and $F(x) = \int_0^x f(u)du$, then we get the Liénard equations

\[
\begin{align*}
\frac{dx}{d\tau} &= y - F(x), \\
\frac{dy}{d\tau} &= -g(x).
\end{align*}
\tag{A10}
\]

**Theorem 3** Consider Liénard equations (A10), if

1. when $x \neq 0$, $xg(x) > 0$, and

\[
G(x) = \int_0^x g(u)du, \quad G(\pm\infty) = +\infty;
\]

2. when $0 < |x| < 1$, $xF(x) < 0$;

3. there exists constants $M$ and $k > k'$, such that $F(x) > k$ when $x \geq M$ and $F(x) < k'$ when $x \leq -M$,

system (A10) exists a stable limit cycle.

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