NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR
CONVEX FUNCTIONS WITH APPLICATIONS

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ABSTRACT. In this paper, some new inequalities of the Hermite-Hadamard type for functions whose modulus of the derivatives are convex and applications for special means are given. Finally, some error estimates for the trapezoidal formula are obtained.

1. INTRODUCTION

A function \( f : I \rightarrow \mathbb{R} \) is said to be convex function on \( I \) if the inequality

\[
f(ax + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y),
\]

holds for all \( x, y \in I \) and \( \alpha \in [0, 1] \).

One of the most famous inequality for convex functions is so called Hermite-Hadamard’s inequality as follows: Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \), with \( a < b \). Then:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]

In [3], the following theorem which was obtained by Dragomir and Agarwal contains the Hermite-Hadamard type integral inequality.

**Theorem 1.** Let \( f : I^0 \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^0 \), \( a, b \in I^0 \) with \( a < b \). If \( |f'| \) is convex on \([a, b]\), then the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(u)du \right| \leq \frac{(b - a)(|f'(a)| + |f'(b)|)}{8}.
\]

In [4] Kirmaci, Bakula, Özdemir and Pečarić proved the following theorem.

**Theorem 2.** Let \( f : I \rightarrow \mathbb{R}, I \subset \mathbb{R}, \) be a differentiable function on \( I^0 \) such that \( f' \in L[a, b] \), where \( a, b \in I, a < b \). If \( |f'|^q \) is concave on \([a, b]\) for some \( q > 1 \), then:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(u)du \right| \leq \left( \frac{b - a}{4} \right) \left( \frac{q - 1}{2q - 1} \right) \left( \left| f'\left(\frac{a + 3b}{4}\right)\right| + \left| f'\left(\frac{3a + b}{4}\right)\right| \right).
\]

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For recent results and generalizations concerning Hermite-Hadamard’s inequality see [1]-[4] and the references therein.

2. THE NEW HERMITE-HADAMARD TYPE INEQUALITIES

In order to prove our main theorems, we first prove the following lemma:

Lemma 1. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^* \) (the interior of \( I \)), where \( a, b \in I \) with \( a < b \). If \( f' \in L[a,b] \), then the following inequality holds:

\[
\frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du = \frac{(x-a)^2}{b-a} \int_0^1 (t-1)f'(tx + (1-t)a)dt + \frac{(b-x)^2}{b-a} \int_0^1 (1-t)f'(tx + (1-t)b)dt.
\]

Proof. We note that

\[
I = \frac{(x-a)^2}{b-a} \int_0^1 (t-1)f'(tx + (1-t)a)dt + \frac{(b-x)^2}{b-a} \int_0^1 (1-t)f'(tx + (1-t)b)dt.
\]

Integrating by parts, we get

\[
I = \frac{(x-a)^2}{b-a} \left[ (t-1) \frac{f(tx + (1-t)a)}{x-a} \right]_0^1 - \int_0^1 \frac{f(tx + (1-t)a)}{x-a} dt
+ \frac{(b-x)^2}{b-a} \left[ (1-t) \frac{f(tx + (1-t)b)}{x-b} \right]_0^1 + \int_0^1 \frac{f(tx + (1-t)b)}{x-b} dt
\]

\[
= \frac{(x-a)^2}{b-a} \left[ \frac{f(a)}{x-a} - \frac{1}{(x-a)^2} \int_a^x f(u)du \right]
+ \frac{(b-x)^2}{b-a} \left[ \frac{f(b)}{x-b} + \frac{1}{(x-b)^2} \int_b^x f(u)du \right]
\]

\[
= \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du.
\]

Using the Lemma [1] the following result can be obtained.

Theorem 3. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^* \) such that \( f' \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \(|f'|\) is convex on \([a,b]\), then the following inequality holds:

\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right|
\]

\[
\leq \frac{(x-a)^2}{b-a} \left[ \frac{|f'(x)| + 2|f'(a)|}{6} \right] + \frac{(b-x)^2}{b-a} \left[ \frac{|f'(x)| + 2|f'(b)|}{6} \right]
\]

for each \( x \in [a,b] \).
Proof. Using Lemma 1 and taking the modulus, we have
\[
\frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du
\]
\[
\leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt
\]
\[
+ \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt.
\]
Since \(|f'|\) is convex, then we get
\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right|
\]
\[
\leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) \left[ |f'(x)| + (1-t) |f'(a)| \right] dt
\]
\[
+ \frac{(b-x)^2}{b-a} \int_0^1 (1-t) \left[ |f'(x)| + (1-t) |f'(b)| \right] dt
\]
\[
= \frac{(x-a)^2}{b-a} \left[ \frac{|f'(x)| + 2|f'(a)|}{6} \right] + \frac{(b-x)^2}{b-a} \left[ \frac{|f'(x)| + 2|f'(b)|}{6} \right]
\]
which completes the proof. \(\square\)

Corollary 1. In Theorem 3 if we choose \(x = \frac{a+b}{2}\), we obtain
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{b-a}{12} \left( |f'(a)| + |f'(\frac{a+b}{2})| + |f'(b)| \right).
\]

Remark 1. In Corollary 1 using the convexity of \(|f'|\), we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|)
\]
which is the inequality in [12].

Theorem 4. Let \(f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}\) be a differentiable mapping on \(I^o\) such that \(f' \in L[a,b]\), where \(a, b \in I\) with \(a < b\). If \(|f'|^{\frac{p}{p-1}}\) is convex on \([a,b]\) and for some fixed \(p > 1\), then the following inequality holds:
\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u)du \right|
\]
\[
\leq \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{p}}
\]
\[
\times \left[ \frac{(x-a)^2}{b-a} \left[ |f'(a)|^q + |f'(x)|^q \right]^{\frac{1}{p}} + \frac{(b-x)^2}{b-a} \left[ |f'(x)|^q + |f'(b)|^q \right]^{\frac{1}{p}} \right]
\]
for each \(x \in [a,b]\) and \(q = \frac{p}{p-1}\).
The second inequality is obtained using the following fact:

\[
\sum_{k=1}^{n} (a_k + b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s \quad \text{for} \quad 0 \leq s < 1, \quad a_1, a_2, a_3, ..., a_n \geq 0; \quad b_1, b_2, b_3, ..., b_n \geq 0 \quad \text{with} \quad 0 \leq \frac{p-1}{p} < 1, \quad \text{for} \quad p > 1.
\]

**Proof.** From Lemma [1] and using the well-known Hölder integral inequality, we have

\[
\left\| \frac{(b - x) f(b) + (x - a) f(a)}{b - a} - \frac{1}{b - a} \int_{a}^{b} f(u) du \right\| 
\leq \frac{(x - a)^2}{b - a} \int_{0}^{1} (1 - t) |f'(tx + (1 - t)a)| dt
\]

\[
+ \frac{(b - x)^2}{b - a} \int_{0}^{1} (1 - t) |f'(tx + (1 - t)b)| dt
\]

\[
\leq \frac{(x - a)^2}{b - a} \left( \int_{0}^{1} (1 - t)^p dt \right)^{\frac{1}{p}} \left( \int_{0}^{1} |f'(tx + (1 - t)a)|^q dt \right)^{\frac{1}{q}}
\]

\[
+ \frac{(b - x)^2}{b - a} \left( \int_{0}^{1} (1 - t)^p dt \right)^{\frac{1}{p}} \left( \int_{0}^{1} |f'(tx + (1 - t)b)|^q dt \right)^{\frac{1}{q}}.
\]

Since \( |f'|^{\frac{1}{p}} \) is convex, by the Hermite-Hadamard’s inequality, we have

\[
\int_{0}^{1} |f'(tx + (1 - t)a)|^q dt \leq \frac{|f'(a)|^q + |f'(x)|^q}{2}
\]

and

\[
\int_{0}^{1} |f'(tx + (1 - t)b)|^q dt \leq \frac{|f'(b)|^q + |f'(x)|^q}{2},
\]

so

\[
\left\| \frac{(b - x) f(b) + (x - a) f(a)}{b - a} - \frac{1}{b - a} \int_{a}^{b} f(u) du \right\|
\leq \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}}
\]

\[
\times \left[ (x - a)^2 \left[ |f'(a)|^q + |f'(x)|^q \right]^{\frac{1}{q}} + (b - x)^2 \left[ |f'(x)|^q + |f'(b)|^q \right]^{\frac{1}{q}} \right]
\]

\[
\leq \frac{b - a}{4} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} \left[ (|f'(a)| + |f'(b)|) \right].
\]

which completes the proof. 

**Corollary 2.** In Theorem [4] if we choose \( x = \frac{a + b}{2} \), we obtain

\[
\left\| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_{a}^{b} f(u) du \right\|
\leq \frac{b - a}{4} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}}
\]

\[
\times \left[ \left| f'(a) \right|^q + \left| f' \left( \frac{a + b}{2} \right) \right|^q \right]^{\frac{1}{q}} + \left( \left| f'(b) \right|^q + \left| f' \left( \frac{a + b}{2} \right) \right|^q \right) \right]^{\frac{1}{q}}
\]

\[
\leq \frac{b - a}{2} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} \left( |f'(a)| + |f'(b)| \right).
\]

The second inequality is obtained using the following fact: \( \sum_{k=1}^{n} (a_k + b_k)^s \leq \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s \) for \( 0 \leq s < 1, \quad a_1, a_2, a_3, ..., a_n \geq 0; \quad b_1, b_2, b_3, ..., b_n \geq 0 \) with \( 0 \leq \frac{p-1}{p} < 1, \) for \( p > 1. \)
Theorem 5. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I \) such that \( f' \in L[a,b] \), where \( a,b \in I \) with \( a < b \). If \( |f'|^q \) is concave on \([a,b]\), for some fixed \( q > 1 \), then the following inequality holds:

\[
\left| \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\
\leq \left[ \frac{q-1}{2q-1} \right]^{\frac{1}{q-1}} \left[ (x-a)^2 |f' \left( \frac{a+x}{2} \right) | + (b-x)^2 |f' \left( \frac{b+x}{2} \right) | \right]
\]

for each \( x \in [a,b] \).

Proof. As in Theorem 4 using Lemma 1 and the well-known H"older integral inequality for \( q > 1 \) and \( p = \frac{q}{q-1} \); we have

\[
\left| \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\
\leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| dt \\
+ \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| dt \\
\leq \frac{(x-a)^2}{b-a} \left( \int_0^1 (1-t)^{\frac{n}{n-1}} dt \right)^{\frac{n-1}{n}} \left( \int_0^1 |f'(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
+ \frac{(b-x)^2}{b-a} \left( \int_0^1 (1-t)^{\frac{n}{n-1}} dt \right)^{\frac{n-1}{n}} \left( \int_0^1 |f'(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}.
\]

Since \( |f'|^q \) is concave on \([a,b]\), we can use the Jensen’s integral inequality to obtain:

\[
\int_0^1 |f'(tx + (1-t)a)|^q dt = \int_0^1 t^0 |f'(tx + (1-t)a)|^q dt \\
\leq \left( \int_0^1 t^0 dt \right) \left| f' \left( \frac{1}{\int_0^1 t^0 dt} \int_0^1 (tx + (1-t)a) dt \right) \right|^q \\
= \left| f' \left( \frac{a+x}{2} \right) \right|^q.
\]

Analogously,

\[
\int_0^1 |f'(tx + (1-t)b)|^q dt \leq \left| f' \left( \frac{b+x}{2} \right) \right|^q.
\]

Combining all the obtained inequalities, we get

\[
\left| \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) du \right| \\
\leq \left[ \frac{q-1}{2q-1} \right]^{\frac{1}{q-1}} \left[ (x-a)^2 |f' \left( \frac{a+x}{2} \right) | + (b-x)^2 |f' \left( \frac{b+x}{2} \right) | \right]
\]

which completes the proof. \( \square \)
Remark 2. In Theorem 3 if we choose \( x = \frac{a+b}{2} \), we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) \, du \right|
\leq \left| \frac{q-1}{2q-1} \right|^\frac{2q}{q-1} \left( \frac{b-a}{4} \right) \left( \int_a^b \left| f' \left( \frac{3a+b}{4} \right) \right| + \left| f' \left( \frac{a+3b}{4} \right) \right| \right)
\]
which is the inequality in (1.3).

Theorem 6. Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I \) such that \( f'' \in L[a,b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is convex on \( [a,b] \), for some fixed \( q \geq 1 \), then the following inequality holds:
\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) \, du \right|
\leq \frac{1}{2} \left( \frac{1}{3} \right)^\frac{1}{q} \left[ \frac{(x-a)^2}{b-a} \left( \int_0^1 (1-t) |f'(tx + (1-t)a)| \, dt \right)^{\frac{1}{q}} + \frac{(b-x)^2}{b-a} \left( \int_0^1 (1-t) |f'(tx + (1-t)b)| \, dt \right)^{\frac{1}{q}} \right]
\]
for each \( x \in [a,b] \).

Proof. Suppose that \( q \geq 1 \). From Lemma 1 and using the well-known power-mean inequality, we have
\[
\left| \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) \, du \right|
\leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| \, dt
+ \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| \, dt
\leq \frac{(x-a)^2}{b-a} \left( \int_0^1 (1-t) \, dt \right)^{\frac{1}{q}} \left( \int_0^1 (1-t) |f'(tx + (1-t)a)|^q \, dt \right)^{\frac{1}{q}}
+ \frac{(b-x)^2}{b-a} \left( \int_0^1 (1-t) \, dt \right)^{\frac{1}{q}} \left( \int_0^1 (1-t) |f'(tx + (1-t)b)|^q \, dt \right)^{\frac{1}{q}}.
\]
Since \( |f'|^q \) is convex, therefore we have
\[
\int_0^1 (1-t) |f'(tx + (1-t)a)|^q \, dt
\leq \int_0^1 (1-t) \left[ t |f'(x)|^q + (1-t) |f'(a)|^q \right] \, dt
= \frac{|f'(x)|^q + 2 |f'(a)|^q}{6}
\]
Analogously,
\[
\int_0^1 (1-t) |f'(tx + (1-t)b)|^q \, dt \leq \frac{|f'(x)|^q + 2 |f'(b)|^q}{6}.
\]
Combining all the above inequalities gives the desired result. \( \square \)
Corollary 3. In Theorem 6, choosing $x = \frac{a+b}{2}$ and then using the convexity of $|f'|^q$, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(u) \, du \right|$$

$$\leq \left( \frac{b-a}{8} \right) \left( \frac{1}{3} \right)^{\frac{1}{q}} \left[ \left( 2 |f'(a)|^q + |f'(\frac{a+b}{2})|^q \right)^{\frac{1}{q}} + \left( 2 |f'(b)|^q + |f'(\frac{a+b}{2})|^q \right)^{\frac{1}{q}} \right]$$

$$\leq \left( \frac{3^{1-\frac{1}{q}}}{8} \right) (b-a) \left( |f'(a)| + |f'(b)| \right).$$

Theorem 7. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^c$ such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a,b]$, for some fixed $q \geq 1$, then the following inequality holds:

$$\left| \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) \, du \right|$$

$$\leq \frac{1}{2} \left[ (x-a)^2 |f'(\frac{x+2a}{3})| + (b-x)^2 |f'(\frac{x+2b}{3})| \right].$$

Proof. First, we note that by the concavity of $|f'|^q$ and the power-mean inequality, we have

$$|f'(tx + (1-t) a)|^q \geq t |f'(x)|^q + (1-t) |f'(a)|^q.$$  

Hence,

$$|f'(tx + (1-t) a)| \geq t |f'(x)| + (1-t) |f'(a)|,$$

so $|f'|$ is also concave.

Accordingly, using Lemma 4 and the Jensen integral inequality, we have

$$\left| \frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(u) \, du \right|$$

$$\leq \frac{(x-a)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)a)| \, dt$$

$$+ \frac{(b-x)^2}{b-a} \int_0^1 (1-t) |f'(tx + (1-t)b)| \, dt$$

$$\leq \frac{(x-a)^2}{b-a} \left( \int_0^1 (1-t) \, dt \right) \left| f' \left( \frac{\int_0^1 (1-t) (tx + (1-t)a) \, dt}{\int_0^1 (1-t) \, dt} \right) \right|$$

$$+ \frac{(b-x)^2}{b-a} \left( \int_0^1 (1-t) \, dt \right) \left| f' \left( \frac{\int_0^1 (1-t) (tx + (1-t)b) \, dt}{\int_0^1 (1-t) \, dt} \right) \right|$$

$$\leq \frac{1}{2} \left[ (x-a)^2 |f'(\frac{x+2a}{3})| + (b-x)^2 |f'(\frac{x+2b}{3})| \right].$$

□
Corollary 4. In Theorem 7, if we choose \( x = \frac{a+b}{2} \), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(u) du \right| \leq \frac{b - a}{8} \left[ \left| f' \left( \frac{5a+b}{6} \right) \right| + \left| f' \left( \frac{a+5b}{6} \right) \right| \right].
\]

3. APPLICATIONS TO SPECIAL MEANS

Recall the following means which could be considered extensions of arithmetic, logarithmic and generalized logarithmic from positive to real numbers.

1. The arithmetic mean:
   \[ A = A(a,b) = \frac{a+b}{2}; \ a, b \in \mathbb{R} \]

2. The logarithmic mean:
   \[ L(a,b) = \frac{b - a}{\ln |b| - \ln |a|}; \ |a| \neq |b|, \ ab \neq 0, \ a, b \in \mathbb{R} \]

3. The generalized logarithmic mean:
   \[ L_n(a,b) = \left[ \frac{b^{n+1} - a^{n+1}}{(b - a)(n + 1)} \right]^{\frac{1}{n}}; \ n \in \mathbb{Z} \setminus \{-1,0\}, \ a, b \in \mathbb{R}, \ a \neq b \]

Now using the results of Section 2, we give some applications to special means of real numbers.

Proposition 1. Let \( a, b \in \mathbb{R}, \ a < b, \ 0 \notin [a,b] \) and \( n \in \mathbb{Z}, \ |n| \geq 2 \). Then, for all \( p > 1 \)

(a)

\[ |A(a^n,b^n) - L_n(a,b)| \leq |n| (b - a) \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} A \left( |a|^{n-1}, |b|^{n-1} \right) \]

and

(b)

\[ |A(a^n,b^n) - L_n(a,b)| \leq |n| (b - a) \left( \frac{3^{1-\frac{1}{q}}}{4} \right) A \left( |a|^{n-1}, |b|^{n-1} \right). \]

Proof. The assertion follows from Corollary 2 and 3 for \( f(x) = x^n, \ x \in \mathbb{R}, \ n \in \mathbb{Z}, \ |n| \geq 2 \).

Proposition 2. Let \( a, b \in \mathbb{R}, \ a < b, \ 0 \notin [a,b] \). Then, for all \( q \geq 1 \),

(a)

\[ |A(a^{-1},b^{-1}) - L^{-1}(a,b)| \leq (b - a) \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right)^{\frac{1}{q}} A \left( |a|^{-2}, |b|^{-2} \right) \]

and

(b)

\[ |A(a^{-1},b^{-1}) - L^{-1}(a,b)| \leq (b - a) \left( \frac{3^{1-\frac{1}{q}}}{4} \right) A \left( |a|^{-2}, |b|^{-2} \right). \]
4. THE TRAPEZOIDAL FORMULA

Let \( d \) be a division \( a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b \) of the interval \([a, b]\) and consider the quadrature formula

\[
\int_a^b f(x) \, dx = T(f, d) + E(f, d)
\]

where

\[
T(f, d) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i)
\]

for the trapezoidal version and \( E(f, d) \) denotes the associated approximation error.

**Proposition 3.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I \) such that \( f' \in L[a, b] \), where \( a, b \in I \) with \( a < b \) and \( |f'|^q \) is convex on \([a, b]\), where \( p > 1 \). Then in (4.1), for every division \( d \) of \([a, b]\), the trapezoidal error estimate satisfies

\[
|E(f, d)| \leq \left( \frac{1}{p+1} \right)^{\frac{q}{p}} \left( \frac{1}{2} \right) \frac{n-1}{2} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 (|f'(x_i)| + |f'(x_{i+1})|).
\]

**Proof.** On applying Corollary 2 on the subinterval \([x_i, x_{i+1}]\) \((i = 0, 1, 2, \ldots, n - 1)\) of the division, we have

\[
\left| \frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x) \, dx \right| \leq \frac{(x_{i+1} - x_i)}{2} \left( \frac{1}{p+1} \right)^{\frac{q}{p}} \left( \frac{1}{2} \right) \frac{n-1}{2} (|f'(x_i)| + |f'(x_{i+1})|).
\]

Hence in (4.1) we have

\[
\left| \int_a^b \! f(x) \, dx - T(f, d) \right| = \left| \sum_{i=0}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} f(x) \, dx - \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) \right\} \right| \\
\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x) \, dx - \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) \right| \\
\leq \left( \frac{1}{p+1} \right)^{\frac{q}{p}} \left( \frac{1}{2} \right) \frac{n-1}{2} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 (|f'(x_i)| + |f'(x_{i+1})|)
\]

which completes the proof. \( \square \)

**Proposition 4.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I \) such that \( f' \in L[a, b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is concave on \([a, b]\), for some fixed \( q > 1 \). Then in (4.1), for every division \( d \) of \([a, b]\), the trapezoidal error estimate satisfies

\[
|E(f, d)| \leq \left( \frac{q-1}{2q-1} \right)^{\frac{q}{q-1}} \frac{n-1}{2} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left( |f'\left(\frac{3x_i + x_{i+1}}{4}\right)| + |f'\left(\frac{x_i + 3x_{i+1}}{4}\right)| \right).
\]

**Proof.** The proof is similar to that of Proposition 3 and using Remark 2. \( \square \)
Proposition 5. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is concave on $[a, b]$, for some fixed $q \geq 1$. Then in (4.7), for every division $d$ of $[a, b]$, the trapezoidal error estimate satisfies

$$|E(f, d)| \leq \frac{1}{8} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left( \left| f' \left( \frac{5x_i + x_{i+1}}{6} \right) \right| + \left| f' \left( \frac{x_i + 5x_{i+1}}{6} \right) \right| \right).$$

Proof. The proof is similar to that of Proposition 3 and using Corollary 4. □

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