MULTIPlicity ON A RICHARDSON VARIETY
IN A COMINUSCULE G/P

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Abstract. We show that in a cominuscule partial flag variety $G/P$, the multiplicity of an arbitrary point on a Richardson variety $X_w^v = X_w \cap X^v \subset G/P$ is the product of its multiplicities on the Schubert varieties $X_w$ and $X^v$.

Introduction

Richardson varieties, named after [33], are intersections of a Schubert variety and an opposite Schubert variety inside a partial flag variety $G/P$ ($G$ a connected complex semi-simple group, $P$ a parabolic subgroup). They previously appeared in [9, Ch. XIV, §4] and [36], as well as the corresponding open cells in [6]. They have since played a role in different contexts, such as equivariant K-theory [24], positivity in Grothendieck groups [3], standard monomial theory [4], Poisson geometry [8], positroid varieties [13], and their generalizations [14, 1].

On the other hand, singularities of Schubert varieties have been extensively studied in the last decades. The singular locus of Schubert varieties in Grassmannians has been determined independently in [37] and [27], and more generally in a minuscule $G/P$ in [26]. In the full flag variety of type $A_n$, it has been determined independently in [2], [3], [12], and [29].

Moreover, the multiplicity of a singular point on a Schubert variety is known in several cases: when $G/P$ is minuscule of arbitrary type, or cominuscule of type $C_n$, a recursive formula was given in [26]. A direct determinantal formula was given in [34] for $G/P$ a Grassmannian; it has been subsequently interpreted in terms of non-intersecting lattice paths [17]. The multiplicity problem has also been studied in relationship with Hilbert functions and Gröbner degenerations [7, 16, 18, 23, 31, 32], as well as with $T$-equivariant cohomology [10, 11, 15, 20, 21, 25]. The problem of determining the multiplicity of a point in a Schubert variety in the full flag variety is more complicated; see [39, 28, 40, 41].

For Richardson varieties in a minuscule $G/P$, the multiplicity of a $T$-fixed point ($T \subset P$ a maximal torus in $G$) has been determined by Kreiman and Lakshmibai [22] (for the Gröbner point of view, see also [19] in type $A_n$ and [38] in orthogonal types).

In this paper, we determine the multiplicity of an arbitrary point on a Richardson variety in a cominuscule $G/P$.

Before stating the main result, let us fix some notation. Let $G, P, T$ be as above, with $G$ adjoint. Let $X(T)$ be the character group of $T$, $R \subset X(T)$ the root system,
and $W = N_G(T)/T$ its Weyl group. Let $B \subset G$ be a Borel subgroup such that $T \subset B \subset P$: it determines a system of positive roots $R^+$ and a system of simple roots $S$. Denote by $B^-$ the opposite Borel subgroup (i.e. such that $B \cap B^- = T$).

Let $W_P \subset W$ be the subgroup associated to $P$ (so that $W_G = W$ and $W_\beta$ is the trivial subgroup). In the quotient $W^P = W/W_P$, every coset $wP$ contains a unique minimal element for the Bruhat order $\leq$ on $W$, so we shall identify $W^P$ with the set of minimal representatives. The $B$-orbit (resp. the $B^-$-orbit) of a $T$-fixed point $e_\tau = \tau P$ is called a Schubert cell (resp. an opposite Schubert cell) in $G/P$, and denoted by $C_\tau$ (resp. $C^\tau$). Its closure is the Schubert variety $X_\tau$ (resp. the opposite Schubert variety $X^\tau$).

If $v, w \in W^P$, then the intersection $X_w^v = X_w \cap X^v$ is called a Richardson variety; it is non-empty if and only if $v \leq w$ (note that Schubert varieties are the particular cases $X_w = X_w^v$ and $X^v = X_w^v$, where $e, w_0 \in W$ are the identity and the longest element, respectively).

Now assume $P$ to be maximal, and let $\alpha$ be the associated simple root (so that $W_P$ is generated by the reflections $s_\delta$ with $\delta \in S \setminus \{\alpha\}$). Then $P$ (or $\alpha$) is said to be

- cominuscule if $\alpha$ occurs with a coefficient 1 in the decomposition of the highest root of $R^+$;
- minuscule if $\alpha^\vee$ is cominuscule in the dual root system $R^\vee$.

The main result of this paper is the following:

**Theorem 0.1.** Assume $P$ is cominuscule. Let $m \in X_w^v$ be arbitrary, and denote by $\mu_w$ (resp. $\mu^v$, $\mu_w^v$) the multiplicity of $m$ on $X_w$ (resp. $X^v$, $X_w^v$). Then

$$
(1) \quad \mu^v_w = \mu_w \mu^v.
$$

This result indeed determines the multiplicities on $X_w^v$, since those on $X_w$ and $X^v$ are known: types $A_n$, $D_n$, $E_6$, $E_7$ are covered by [20], Section 3 (since cominuscule is equivalent to minuscule in those types); type $C_n$ is covered by [20], Section 4. The only remaining case, in type $B_n$ (cf. the table below), is elementary, and covered in the Appendix of the present paper for the sake of completeness.

Note that (1) is exactly the result obtained in [22] for a $T$-fixed point in a minuscule $G/P$.

To prove the theorem, we shall use a description of the multiplicity using a central projection: namely, given a projective variety $X \subset P^N$ and a point $m \in X$, we consider the projection $p_m$ of centre $m$, onto a hyperplane not containing $m$. Then the multiplicity of $m$ on $X$ is the difference between the degree of $X$ and the projective degree of $p_m$. Note that the projective degree of $p_m$ is zero when $X$ is a cone. We apply this description for $X$, the projective closure of the affine trace $X_w^v \cap O_\tau$, where $O_\tau$ is an affine open subset of $G/P$ identified with $A^N$. One then needs to know whether the affine traces of $X_w, X^v, X_w^v$ are cones or not. In this setting, we can explain why we assume that $P$ is cominuscule:

- it implies that $X_w \cap O_\tau$ is a cone over any point of the cell $C_\tau$ (although this may not be the case for $X^v \cap O_\tau$);
- we relate the central projection $p_m$ to a map which turns out to be a $C$-action if $P$ is cominuscule. It is this $C$-action which allows to prove all the necessary properties for $p_m$. 


In Section 1, we give a system of local coordinates in which $X_w \cap O_\tau$ is a cone over both $e_\tau$ and $m$, and $X_v \cap O_\tau$ over $e_\tau$. In Section 2, we prove Theorem 0.1 assuming certain formulas for the degrees involved and that $X_v \cap O_\tau$ is not a cone over $m$. These assumptions are summarized in Proposition 2.1 and proved in Sections 4 and 5. The proofs are based on a $C$-action linking the central projections of centres $m$ and $e_\tau$; this action is defined and studied in Section 3.

For the convenience of the reader, we give the minuscule and cominuscule weights in the following table:

| Type | Minuscule | Cominuscule | Both |
|------|-----------|-------------|------|
| $A_n$ | $\bullet$ | $\circ$ | $\bullet$ |
| $B_n$ | $\bullet$ | $\circ$ | $\bullet$ |
| $C_n$ | $\bullet$ | $\circ$ | $\bullet$ |
| $D_n$ | $\bullet$ | $\circ$ | $\bullet$ |
| $E_6$ | $\bullet$ | $\circ$ | $\bullet$ |
| $E_7$ | $\bullet$ | $\circ$ | $\bullet$ |

There are no minuscule nor cominuscule fundamental weights in types $E_8$, $F_4$, $G_2$.

**Assumption.** For the rest of the paper, the parabolic subgroup $P$ is assumed to be cominuscule.

### 1. Local Coordinates

Use the notation as in the Introduction. Moreover, $R_P$ denotes the root system associated with $P$,

$$R^+ \setminus R^+_P = \{ \beta \in R^+ \mid U_\beta \subset R_u(P) \},$$

where $R_u(P)$ is the unipotent radical of $P$, and $U_\beta$ is the root subgroup associated with $\beta$.

Let $m \in X_w^v$. Then $m$ lies in a Schubert cell $C_\tau$ for some $\tau \in W^P$. Let

$$U^-_\tau = \prod_{\beta \in \tau(R^+ \setminus R^+_P)} U_{-\beta}$$

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and $O_\tau = U^- e_\tau$, where $e_\tau = \tau P$. We identify $U_\beta$ with $C$ via an isomorphism $\theta_\beta : C \to U_\beta$ satisfying

$$t \theta_\beta(x) t^{-1} = \theta_\beta \left( \frac{1}{\beta(t)} x \right)$$

for all $t \in T$ and all $x \in C$. Let $N$ be the cardinality of $R^+ \backslash R^+_P$. We identify $O_\tau$ with the affine space $A^N$ via the isomorphism

$$A^N \to O_\tau$$

$$\left( x_\beta \right)_{\beta \in \tau(R^+ \backslash R^+_P)} \to \prod_{\beta \in \tau(R^+ \backslash R^+_P)} \theta_\beta(x_\beta).e_\tau.$$  

(In particular, $N$ is the dimension of $G/P$.)

**Lemma 1.1.** Let $\beta \in R$, and $\tau \in W^P$. Then $U_\beta$ fixes $e_\tau$ if and only if $-\beta \notin \tau(R^+ \backslash R^+_P)$.

**Proof.** Let $\beta \in R$, and $\tau \in W^P$. Then

$$U_\beta.e_\tau = e_\tau \iff \tau^{-1} U_\beta \tau P = P$$

$$\iff U_{-\beta} \subset P$$

$$\iff \tau^{-1} \beta \in R^+ \text{ or } -\tau^{-1} \beta \in R^+_P$$

$$\iff -\beta \notin \tau(R^+) \text{ or } -\beta \in \tau(R^+_P)$$

$$\iff -\beta \notin \tau(R^+ \backslash R^+_P).$$

□

**Lemma 1.2.** The Schubert cell $C_\tau$ is the affine subspace of $O_\tau$ defined by the vanishing of the coordinates $x_{-\beta}$ with $\beta \in R^+$.

**Proof.** Since $B$ is the semi-direct product of $T$ and the unipotent subgroup $U$, we have $C_\tau = U.e_\tau$. Moreover, for any ordering of positive roots $\{\beta_1, \ldots, \beta_p\}$,

$$U = \prod_{i=1}^p U_{\beta_i}.$$ 

We choose an ordering such that the positive roots $\beta$ with $-\beta \notin \tau(R^+ \backslash R^+_P)$ appear at the end. Then, by the preceding lemma, we have

$$C_\tau = \prod_{\beta \in \tau(R^+ \backslash R^+_P)} U_{-\beta}.e_\tau \subset O_\tau.$$ 

□

The following lemma will be useful for the next section.

**Lemma 1.3.** For all $\beta, \gamma \in \tau(R^+ \backslash R^+_P)$ and for all $x, y \in C$, the elements $\theta_\beta(x)$ and $\theta_\gamma(y)$ commute.

**Proof.** We use the following expansion for the commutator (cf. [35], proposition 8.2.3),

$$\theta_\beta(x)\theta_\gamma(y)\theta_\beta(x)^{-1}\theta_\gamma(y)^{-1} = \prod_{i,j \geq 0 \atop i \beta + j \gamma \in R} \theta_{i \beta + j \gamma}(c_{i \beta, j \gamma} x_i^j y^j),$$
where $c_{\delta,\gamma,i,j}$ are some constants in $C$. Since the commutator must lie in $U_{-}^\tau$, it suffices to prove that the roots of the form $i\beta + j\gamma$ do not lie in $\tau(R^+ \setminus R^+_P)$. Now, $P$ is the parabolic subgroup associated with the simple root $\alpha$. Since $\alpha$ is cominuscule, a positive root $\delta$ lies in $R^+ \setminus R^+_P$ if and only if $\alpha$ occurs with coefficient 1 in the expression of $\delta$. Clearly, $\alpha$ occurs with a coefficient $i + j$ in $\tau(\delta_{-\beta} + j\gamma)$. \hfill \Box

Remark 1.4. Identifying $O_\tau$ with $U_{-}^\tau$, it follows from Lemma 1.3 that the isomorphism of algebraic varieties $A_N \to O_\tau$ is also an isomorphism of unipotent groups.

Example 1.5. Let $G = SL_n(C)$. It is a group of type $A_{n-1}$. The torus $T$ is the group of diagonal matrices of determinant 1, and the Borel subgroup $B$ is the group of upper triangular matrices of determinant 1. The roots are denoted $\alpha_{i,j}$, where

$$\alpha_{i,j} : T \to C^* : \begin{pmatrix} t_1 & \cdots & t_i & \cdots & t_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \cdots & t_i & \cdots & t_j \\ \vdots & \cdots & \cdots & \cdots & \vdots \\ 1 & \cdots & \cdots & \cdots & 1 \end{pmatrix} \mapsto \frac{t_i}{t_j}.$$  

The positive roots are the $\alpha_{i,j}$ with $i < j$, and the simple roots are the $\alpha_i = \alpha_{i,i+1}$ ($i = 1, \ldots, n-1$). Let $\omega = \omega_d$ be the fundamental weight associated with the simple root $\alpha_d$. The corresponding parabolic subgroup $P$ is

$$P = \left\{ \begin{pmatrix} \ast & \ast \\ 0_{(n-d) \times d} & \ast \end{pmatrix} \right\}.$$  

The group $G$ acts transitively on the Grassmannian $G_{d,n}$ of $d$-spaces in $C^n$, and $P$ is the isotropy subgroup of the vector space generated by $e_1, \ldots, e_d$, where $(e_1, \ldots, e_n)$ is the canonical basis of $C^n$. The Weyl group $W$ of this root system is $S_n$, and $W_P$ is isomorphic to $S_d \times S_{n-d}$, so

$$W^P = I_{d,n} = \{ i = i_1 \ldots i_d \mid 1 \leq i_1 < i_2 < \cdots < i_d \leq n \}.$$  

The Lie algebra $\mathfrak{g}$ of $G$ is the space of traceless matrices. Let $t$ be the Lie algebra of the torus $T$. We have the weight decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = t \oplus \bigoplus_{i \neq j} \mathbb{C}E_{i,j},$$

where $E_{i,j}$ is the elementary matrix with a 1 on the row $i$ and column $j$, and zero elsewhere. Thus, the root subgroups are given by

$$U_{\alpha_{i,j}} = \{ I_n + xe_{i,j} \mid x \in \mathbb{C} \}$$

and the isomorphism $\theta_{\alpha_{i,j}}$ is just $x \mapsto \exp(xE_{ij})$. Moreover,

$$R^+ \setminus R^+_P = \{ \alpha_{i,j} \mid i \leq d < j \},$$

so in this case, Lemma 1.3 becomes an elementary matrix computation.

Returning to the general case, we denote by $(m_{-\beta} | \beta \in \tau(R^+ \setminus R^+_P))$ the coordinates of $m$, that is,

$$m = \prod_{\beta \in \tau(R^+ \setminus R^+_P)} \theta_{-\beta}(m_{-\beta})e_\tau.$$
Notation 1.6. We set 
\[ Y_w = X_w \cap O_\tau, \quad Y^v = X^v \cap O_\tau, \quad Y^w = X^w \cap O_\tau. \]
These sets are affine varieties, i.e., Zariski-closed in \( O_\tau = A^N \).

We now investigate if these affine varieties are cones over \( m \).

Proposition 1.7. The varieties \( Y_w, Y^v \) and \( Y^w \) are cones over \( e_\tau \).

Proof. Let \( \omega^\vee : C^* \to T \) be the fundamental coweight associated to \( P \). Since \( \omega^\vee \) is minuscule, the pairing \( \langle \omega^\vee, \gamma \rangle \) is equal to 1 if \( \gamma \in R^+ \setminus R^+_P \) and to 0 if \( \gamma \in R^+_P \). Now multiplication in \( A^N \) by a scalar \( \xi \) is then given by conjugation in \( U^- \) by \( \tau(\omega^\vee)(\xi)^{-1} \in T \): indeed, for \( \beta = \tau(\gamma) \) with \( \gamma \in R^+ \setminus R^+_P \), and for \( z \in C \), we have
\[
\tau(\omega^\vee)(\xi)^{-1} \theta_\beta(z) \tau(\omega^\vee)(\xi) = \theta_\beta(\xi^{\tau(\omega^\vee),\beta} z) = \theta_\beta(\xi^{\omega^\vee,\gamma} z). (3)
\]

Let \( x \in Y_w \) (resp. \( x \in Y^v \)), and \( (x_{-\beta}) \) be its coordinates. Then the point that has coordinates \( (\xi x_{-\beta}) \) is \( t.x \), where \( t = \tau(\omega^\vee)(\xi) \in T \). Therefore, this point lies in \( X_w \cap O_\tau \) (resp. in \( X^v \cap O_\tau \)), since \( X_w \) (resp. \( X^v \)) is \( T \)-stable. It follows that \( Y_w, Y^v \), and therefore \( Y^w \) are cones over \( e_\tau \). \( \square \)

Proposition 1.8. The variety \( Y_w \) is a cone over \( m \).

Proof. Consider the translation that maps \( e_\tau \) to \( m \). It is given in coordinates by \( (x_{-\beta}) \mapsto (x_{-\beta} + m_{-\beta}) \). But if \( x \) has coordinates \( (x_{-\beta}) \), then, by Remark 1.2, the point of coordinates \( (x_{-\beta} + m_{-\beta}) \) corresponds to \( b.x \), where \( b = \prod_\beta \theta_\beta(m_{-\beta}) \). Since \( m_{-\beta} = 0 \) for all \( \beta > 0 \), we have \( b \in B \) according to Lemma 1.2. Now \( b \) leaves \( Y_w \) invariant and maps \( e_\tau \) to \( m \). \( \square \)

However, the opposite Schubert variety \( Y^v \) need not be a cone over \( m \).

Example 1.9. We take the same notation as in Example 1.3. In particular, using the identification \( W^F = I_{d,n} \), we denote a Schubert variety in \( G_{d,n} \) by \( X_{i,j} \), and similarly for opposite Schubert and Richardson varieties. In the Grassmannian \( G_{3,7} \), consider the Richardson variety \( X_{3,5}^{125} \). The coordinates on the open set \( O_{256} \) are parametrized by the set \( \{12, 15, 16, 32, 35, 36, 42, 45, 46, 72, 75, 76\} \), where \( ij \) stands for the root \( \alpha_i, j \). More precisely, we have

\[
A^{12} \rightarrow O_{256} \quad \begin{bmatrix} x_{12} & x_{15} & x_{16} \\ 1 & 0 & 0 \\ x_{32} & x_{35} & x_{36} \\ x_{42} & x_{45} & x_{46} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_{72} & x_{75} & x_{76} \end{bmatrix}
\]

Here, a matrix between brackets actually stands for the 3-space in \( C^7 \) generated by its columns. The equations of \( X_{3,5}^{125} \) are
\[
\begin{align*}
& x_{72} = x_{75} = x_{76} = 0, \\
& x_{42} = 0.
\end{align*}
\]
The equations of $X_{125}^\ast$ are
\[
\begin{align*}
&x_{15}x_{36} - x_{35}x_{16} = 0, \\
x_{15}x_{46} - x_{45}x_{16} = 0, \\
x_{35}x_{46} - x_{45}x_{36} = 0.
\end{align*}
\]

Let
\[
m = \begin{bmatrix}
1 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \in X_{356}.
\]

We set
\[
\begin{align*}
y_{16} &= x_{16} - 1, \\
y_{36} &= x_{36} + 1, \\
y_{ij} &= x_{ij}, & \text{if } ij \notin \{16, 36\}.
\end{align*}
\]
The equations in these new coordinates are
\[
\begin{align*}
y_{72} = y_{75} = y_{76} = 0, \\
y_{42} = 0,
\end{align*}
\]
for $X_{356}$ and
\[
\begin{align*}
y_{15}(y_{36} - 1) - y_{35}(y_{16} + 1) &= 0, \\
y_{15}y_{46} - y_{45}(y_{16} + 1) &= 0, \\
y_{35}y_{46} - y_{45}(y_{36} - 1) &= 0,
\end{align*}
\]
for $X_{125}^\ast$. While the equations for $X_{356}$ remain homogeneous, those for $X_{125}^\ast$ do not.

If $Y^\ast$ is indeed a cone over $m$, then we have the following result. The proof is taken from [22], Remark 7.6.6.

**Proposition 1.10.** Assume $Y^\ast$ is a cone over $m$. Let $\mu_w$ (resp. $\mu^v$, $\mu^v_w$) be the multiplicity of $m$ on $X_w$ (resp. $X^v$, $X^v_w$). Then
\[
\mu_w^v = \mu_w \mu^v.
\]

**Proof.** In this case, $Y_w^v = Y_w \cap Y^v$ is a cone (over $m$) as well, so we may consider the projective varieties $P(Y_w)$, $P(Y^v)$ and $P(Y^v_w)$, consisting of lines through $m$. Then $\mu_w$ (resp. $\mu^v$, $\mu^v_w$) is just the degree of $P(Y_w)$ (resp. $P(Y^v)$, $P(Y^v_w)$). We conclude with Bézout’s theorem since $P(Y_w)$ and $P(Y^v)$ intersect transversely (cf. [33], Corollary 1.5).

**Assumption 1.11.** For the rest of the paper, we assume that $Y^\ast$ is not a cone over $m$.

It is not clear however whether $Y_w^v$ is a cone or not. This problem will be solved in Section 4.
2. Central projection and proof of Theorem 0.1

We shall compute the multiplicity of a point \( m \in Y_w \) by relating it to degrees of projections, which requires us to work in a projective setting. More precisely, embed \( \mathbb{A}^N \) into \( \mathbb{P}^N \) via

\[
\iota: \quad \mathbb{A}^N \rightarrow \mathbb{P}^N = \{[\xi : x_{-\beta}]\}
\]

and consider the projective closures

\[
Z_w = \iota(Y_w), \quad Z^v = \iota(Y^v), \quad Z^w = \iota(Y^w).
\]

We also identify \( \mathbb{P}^{N-1} \) with the hyperplane at infinity \( \xi = 0 \) and consider the central projection \( p_m : \mathbb{P}^N \rightarrow \mathbb{P}^{N-1} \), sending any point \( x \neq m \) to the intersection of the line \( (mx) \) with \( \mathbb{P}^{N-1} \). If \( X \subset \mathbb{P}^N \) is any projective variety and \( m \in X \), then we have the following formula (cf. [30], Theorem 5.11),

\[
(5) \quad \deg X - \text{mult}_m X = \begin{cases} \deg(p_m)|_X & \text{if } X \text{ is not a cone over } m, \\ 0 & \text{if } X \text{ is a cone over } m, \end{cases}
\]

where \( \deg X \) is the degree of \( X \), \( \deg(p_m)|_X \) is the degree of the rational map \( p_m \) restricted to \( X \), and \( p_mX \) denotes the Zariski closure of \( p_m(X \setminus \{m\}) \).

Proposition 2.1.

(a) \( \deg Z^v = \deg Z_w \deg Z^v. \)
(b) \( Z^w \) is not a cone over \( m \).
(c) \( \deg(p_m)|Z^v = \deg(p_m)|Z^v. \)
(d) \( \deg(p_m Z^v) = \deg Z_w \deg(p_m Z^v). \)

We defer the proof to Section 4.

Proof of Theorem 0.1. Using (5) and Proposition 2.1, we obtain

\[
\mu^v_w = \deg Z^v_w - \deg(p_m)|Z^v_w \deg(p_m Z^v) = \deg Z_w \deg Z^v - \deg(p_m)|Z^v \deg Z_w \deg(p_m Z^v) = \deg Z_w \deg Z^v - \deg(p_m)|Z^v \deg(p_m Z^v) = \mu_w \mu^v.
\]

\[\square\]

Remark 2.2. In particular, this result enables us to find the singular locus of \( X^w \) in terms of those of \( X_w \) and \( X^v \): the point \( m \) is smooth on \( X_w \) if and only if \( \mu^v_w = 1 \) if and only if \( \mu_w = \mu^v = 1 \), that is, if and only if \( m \) is smooth on both \( X_w \) and \( X^v \). Note that this may also be seen more directly, using the fact that \( X_w \) and \( X^v \) intersect properly and transversely at any point at which \( \mu_w = \mu^v = 1 \) (cf. [33], Corollary 1.5, or [1], Corollary 2.9).

3. C-action on G/P

In this section, we introduce the main tool that will permit us to prove Proposition 2.1 in the next section. Let \( e_\tau, m \in \mathcal{O}_\tau \) be as before. We shall construct an action of (the additive group) \( \mathbb{C} \) on \( G/P \) for which \( e_\tau \) and \( m \) are in the same orbit.
Consider first the map
\[ \varphi^* : \mathbf{C}^* \rightarrow B \]
\[ \xi \mapsto \varphi_\xi = \tau(\omega^\vee)(\xi)^{-1}b\tau(\omega^\vee)(\xi), \]
where \( b \in B \cap U^- \) is the element defined in the proof of Proposition 1.8. The computation (3) shows that this map extends to a group homomorphism \( \varphi : \mathbf{C} \rightarrow B \). The natural \( B \)-action on \( G/P \) thus induces a \( \mathbf{C} \)-action,
\[ \Phi^* : \mathbf{C} \times G/P \rightarrow G/P \]
\[ (\xi, x) \mapsto \varphi_\xi x. \]

Moreover, \( \mathcal{O}_x \) is invariant under this action (again by (3)). Actually, \( \mathbf{C} \) acts on \( \mathcal{O}_x = \mathbf{A}^N \) by translations: indeed, we get the following commutative diagram
\[
\begin{array}{ccc}
\mathbf{C} \times \mathbf{A}^N & \xrightarrow{\Phi} & \mathbf{A}^N \\
(\xi, x_{-\beta}) & \xrightarrow{\Phi} & (x_{-\beta} - \xi m_{-\beta}) \\
\downarrow f & & \downarrow \Phi \downarrow p_m \\
P^N & \xrightarrow{-p_m} & P^{N-1} \\
\end{array}
\]
(6)

Let us now restrict to \( Y_w^u \): since it is a cone over \( e_\tau \), a point \([\xi : x]\) lies in \( Z_w^u \) if and only if \( x \in Y_w^u \). It follows that \( f(\mathbf{C} \times Y_w^u) = Z_w^u \). Thus, the commutative diagram (6) restricts to
\[
\begin{array}{ccc}
\mathbf{C} \times Y_w^u \setminus \{ (\xi, \xi m_{-\beta}) | \xi \in \mathbf{C} \} & \xrightarrow{\Phi} & \Phi(\mathbf{C} \times Y_w^u) \setminus \{ e_\tau \} \\
\downarrow f & & \downarrow \Phi \downarrow p_m \\
Z_w^u \setminus \{ m \} & \xrightarrow{p_m} & P^{N-1}. \\
\end{array}
\]
(7)

Remark 3.1. Since (6) is a fibre product diagram, any fibre \( \Phi^{-1}(\lambda y) \) (for \( \lambda \neq 0 \) and \([y] \in P^{N-1}\)) is mapped isomorphically via \( f \) to the fibre \( p_m^{-1}([y]) \). Since we have the equalities \( f(\mathbf{C} \times Y_w^u) = Z_w^u \), \( f(\mathbf{C} \times Y_w^u) = Z_w^u \) and \( \mathbf{C} \times Y_w = f^{-1}(Z_w^u) \), \( \mathbf{C} \times Y_w^u = f^{-1}(Z_w^u) \), the fibres of \( \Phi|_{\mathbf{C} \times Y_w^u} \), \( \Phi|_{\mathbf{C} \times Y_w^u} \) over a point \( \lambda y \) are isomorphic to the fibres of \( p_m|_{Z_w^u} \), \( p_m|_{Z_w^u} \) over the point \([y]\).

In the next section, this remark will allow us to relate the degree of \( p_m \) in diagram (7) to that of \( \Phi \).

4. PROOF OF PROPOSITION 2.1

Proof of (a). Since \( Y_w^u \), \( Y_v^u \), and \( Y_w^u \) are (affine) cones over \( e_\tau \), it is clear that \( Z_w^u = Z_w \cap Z_v \). In addition, this intersection is proper and generically transverse (34), Corollary 1.5), hence \( \deg Z_w^u = \deg Z_w \deg Z_v^u \) by Bézout’s theorem.

Notation 4.1. We denote by \( F_w^u \) the closure in \( \mathbf{A}^N \) of \( \Phi(\mathbf{C} \times Y_w^u) \), and by \( d_w^u \) the degree of \( p_m : Z_w^u \setminus \{ m \} \rightarrow p_m Z_w^u \) whenever it makes sense (i.e., when \( Z_w^u \) is not a cone). We define \( F_w^u, F_v^u, d_v^u \) in a similar way.

Proposition 4.2. When defined, the degree \( d_w^u \) is equal to the degree of \( \Phi : \mathbf{C} \times Y_w^u \rightarrow F_w^u \).

Proof. This follows from Remark 3.1. \( \square \)
Lemma 4.3. The following properties are equivalent:

- $Z_w^v$ is a cone over $m$,
- $F_w^v = Y_w^v$,
- every fibre of $\Phi : C \times Y_w^v \to F_w^v$ has dimension 1.

In particular, they are true for $v = e$, hence $F_w = Y_w = \Phi(C \times Y_w)$.

Proof. By Remark 3.1, we see that the dimension of a generic fibre of $\Phi$ equals the dimension of a generic fibre of $p_m$. Now $Z_w^v$ is a cone over $m$ if and only if every fibre of $p_m$ has dimension 1, if and only if $\dim F_w^v = \dim Y_w^v$. But $Y_w^v = \Phi(0 \times Y_w^v) \subset F_w^v$ and the varieties $Y_w^v$ and $F_w^v$ are irreducible, so $Z_w^v$ is a cone over $m$ if and only if $F_w^v = Y_w^v$. □

Proof of (b) and (c). By Proposition 4.2, it suffices to compare the degree $d^v$ of $\Phi^v : C \times Y^v \to F^v$ with the degree $d_w^v$ of $\Phi_w^v : C \times Y_w^v \to F_w^v$. First, the fibre of a point $x \in G/P$ for $\Phi$ is

$$\Phi^{-1}(x) = \{(\xi, \Phi(-\xi, x)) | \xi \in C\}.$$ 

In particular, a point lies in $\text{Im}(\Phi^v)$ (resp. in $\text{Im}(\Phi_w^v)$) if and only if its $C$-orbit meets $Y^v$ (resp. $Y_w^v$). There exists an open set $\Omega^v$ of $F^v$ in which the fibre of every point $y$ consists of $d^v$ points. Then $d^v$ is just the number of points in the $C$-orbit of $y$ that belong to $Y^v$. Now set $y = (y_{-\beta})_{\beta \in \tau(R^+ \setminus R^+_{\tau})}$ and let

$$c = \prod_{\beta \in \tau(R^+ \setminus R^+_{\tau})} \theta_{-\beta}(y_{-\beta}), \quad c^- = \prod_{\beta \in \tau(R^+ \setminus R^+_{\tau})} \theta_{-\beta}(-y_{-\beta}),$$

so we have $c.e_{\tau} = c^- . y =: x$. Since $c \in B$, $x \in C_{\tau} \subset Y_w$. Now $c^-$ commutes with $\varphi_{\xi}$ for all $\xi \in C$, hence every point in $c^-(\Omega^v)$ has a $C$-orbit which meets $Y^v$ in exactly $d^v$ points. In particular, $F_w^v \neq Y_w^v$, since otherwise every fibre of $\Phi_w^v$ would have dimension 1 (by Lemma 4.3), which is not the case for the fibre of $x$. This already shows (b), so it makes sense to talk about the degree $d_w^v$ of $\Phi_w^v$. Thus, let $\Omega_w^v$ be an open set of $F_w^v$ such that for every point $z$ in $\Omega_w^v$, the fibre of $z$ consists of $d_w^v$ points. Since $x \in c^-(\Omega^v)$, $c^-(\Omega^v) \cap F_w^v$ and $\Omega_w^v$ are non-empty open sets of the irreducible variety $F_w^v$, so they must meet. Taking $z$ in this intersection, we see that $d_w^v = d^v$, which shows (c). □

Proposition 4.4. The intersection $F_w \cap F^v$ is proper and transverse on an open set of $F_w^v$.

Proof. The transversality of the intersection $F_w \cap F^v$ on a generic point in $F_w^v$ follows from the transversality of the intersection of a direct Schubert variety and an opposite Schubert variety. More precisely, let $(F_w)_{sm}$ be the open set of smooth points of $F_w$. Taking a point smooth on $Y_w^v$ shows that $\Omega_w = (F_w)_{sm} \cap F_w^v$ is a non-empty open set of $F_w^v$. Let $(F^v)_{sm}$ be the open set of smooth points of $F^v$. Again, $\Omega^v = (F^v)_{sm} \cap F^v \neq \emptyset$. Indeed, take a smooth point $x$ of $F^v$ belonging to $\Phi(C \times Y^v)$. We have seen in the previous proof that from $x$ we can construct an isomorphism $c^-$ of $F^v$ mapping $x$ to a point of $F_w^v$, which thus remains smooth on $F^v$. The two non-empty open subsets $\Omega_w$ and $\Omega^v$ of the irreducible variety $F_w^v$ have a non-empty intersection $\Omega_w^v$. Now $\Omega_w^v = \Phi^{-1}(\Omega_w^v) \cap (P^1 \times Y_w^v)_{sm} \neq \emptyset$ since $P^1 \times Y_w^v$ is irreducible. We claim that $\Phi : Q_w^v \to \Omega_w^v$ is dominant. Indeed, we must show that every open subset $U$ of $\Omega_w^v$ meets $\Phi(U)$, since $U$ is open in $F_w^v$. 

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Let $y = \Phi(p) \in \Omega$ be such a point. We view the map $\Phi : C \times A^N \to A^N : (\xi, x) \mapsto \varphi_{-\xi, x}$ as a map $\Phi : C^{N+1} \to C^N$. It is linear and surjective. Thus,

$$C^N \supset T_y(F_w) + T_y(F_v) \supset d\Phi_p(T_p(C \times Y_w)) + d\Phi_p(T_p(C \times Y_v))$$

$$\supset d\Phi_p(C \oplus (T_pY_w + T_pY_v))$$

$$\supset d\Phi_p(C \oplus C^N)$$

$$\supset C^N.$$ 

This transversality result proves that the intersection is proper: indeed, on one hand, $\dim(F_w \cap F_v) \geq \dim(F_w) + \dim(F_v) - N$, but on the other hand,

$$\dim(F_w \cap F_v) \leq \dim(T_y(F_w \cap F_v)) \leq \dim(T_yF_w + T_yF_v)$$

$$\leq \dim(T_yF_w) + \dim(T_yF_v) - \dim(T_yF_w + T_yF_v)$$

$$\leq \dim(F_w) + \dim(F_v) - N.$$

\[ \square \]

**Proposition 4.5.** We have the equality $F_w^v = F_w \cap F_v$. In particular, the intersection $F_w \cap F_v$ is generically transverse.

This result will be proved in the next section.

**Proof of (d).** Since $y = \Phi(\xi, x)$ implies $zy = \Phi(z\xi, zx)$ for all $z \in C$, $\Phi(C \times Y_w^v)$ is a cone over $e_r$, and so is its closure $F_w^v$. But by the commutative diagram 7, 

$$p_{e_r}(F_w^v \setminus \{e_r\}) \subset p_{e_r}(\Phi(C \times Y_w^v) \setminus \{e_r\}) = p_mZ_w^v.$$ 

Comparing dimensions, we see that $p_{e_r}F_w^v = p_mZ_w^v$, i.e. $p_mZ_w^v$ is the projective variety at infinity of the cone $F_w^v$. In particular, $\deg(p_mZ_w^v) = \deg(F_w^v)$, and similarly $\deg(p_mZ_w) = \deg(F_w)$ and $\deg(p_mZ_w^v) = \deg(F_w^v)$. Equality (d) now follows from Proposition 4.4 and Bézout’s theorem, noting that $\deg(p_mZ_w) = \deg(Z_w)$.

\[ \square \]

**5. PROOF OF PROPOSITION 4.5**

Since $\Phi(C \times Y_w^v) \subset \Phi(C \times Y_w) \cap \Phi(C \times Y_v)$, we obtain $F_w^v \subset F_w \cap F_v$. Moreover, the first inclusion is an equality: indeed, if $z = \Phi(\xi, x) \in Y_w$ with $\xi \in C, x \in Y_v$, then $x = \Phi(-\xi, z) \in Y_v$ since $\Phi(C \times Y_w) = Y_v$, so $z = \Phi(\xi, x) \in \Phi(C \times Y_w^v)$. However, the inclusion $F_w \cap F_v \subset F_w^v$ requires some work. Let

$$U = \{(\xi, x, \Phi(\xi, x)) | \xi \in C, x \in G/P\}$$

and $\Gamma$ be its closure in $\mathbb{P}^1 \times G/P \times G/P$ (so $\Gamma$ is the graph of $\Phi$ viewed as a rational map). We have a commutative diagram,

$$\begin{array}{ccc}
\mathbb{P}^1 \times G/P & \xrightarrow{\pi_3} & G/P \\
\downarrow & & \downarrow \\
(\xi, x) & \xrightarrow{\Phi} & \Phi(\xi, x)
\end{array}$$

\[ \Rightarrow \]

$$\begin{array}{ccc}
\mathbb{P}^1 \times G/P & \xrightarrow{\pi_3} & G/P \\
\downarrow & & \downarrow \\
(\xi, x) & \xrightarrow{\Phi} & \Phi(\xi, x)
\end{array}$$

\[ \square \]
The morphism $\pi_1 \times \pi_2 : \Gamma \to \mathbf{P}^1 \times G/P$ is surjective, and restricts to an isomorphism between $U$ and $C \times G/P$. In particular, $\Gamma$ is an irreducible projective variety of dimension $N + 1$.

Likewise, let $U_w = \{ (\xi, x, \Phi(\xi, x)) | \xi \in C, x \in X_w \}$ and $\Gamma_w$ be its closure, and similarly for $U^v, U^w, \Gamma^v, \Gamma^w$. Then $\pi_3(\Gamma_w) = \pi_3(U_w) = \pi_3(U_w)$ in $G/P$, so $\pi_3(\Gamma_w) \cap \mathcal{O}_\tau$ is the closure of $\pi_3(U_w) \cap \mathcal{O}_\tau = \Phi(C \times Y_w)$ in $\mathcal{O}_\tau$. Proceeding similarly with $\Gamma^v$ and $\Gamma^w$, we obtain

$$
\pi_3(\Gamma_w) \cap \mathcal{O}_\tau = F_w, \quad \pi_3(\Gamma^v) \cap \mathcal{O}_\tau = F^v, \quad \pi_3(\Gamma^w) \cap \mathcal{O}_\tau = F^w.
$$

We now need to study the $\pi_3$-fibre of a point in $F_w$. Actually, if $y$ is in $Y_w$, then its fibre lies entirely in $\Gamma_w$. Indeed, $U^-_\tau$ naturally acts on $G/P$ and on $\Gamma$ via $g.(\xi, x, y) = (\xi.g.x, g.y)$ (since $U^-_\tau$ is Abelian), and the morphism $\pi_3$ is $U^-_\tau$-equivariant. It follows that whenever two points in $G/P$ belong to the same $U^-_\tau$-orbit, their fibres are isomorphic. Now since $\pi_3 : \Gamma \to G/P$ is dominant, there is an open set in $G/P$ in which every point has a fibre of pure dimension 1. Since $\mathcal{O}_\tau$ is open in $G/P$, it meets this open set, and since $\mathcal{O}_\tau$ is a $U^-_\tau$-orbit in $G/P$, $y$ itself has a fibre of pure dimension 1.

Now fix an irreducible component $C$ of $\pi_3^{-1}(y)$. Then

$$
(\pi_1 \times \pi_2(C)) \cap (C \times G/P) \subset \Phi^{-1}(y).
$$

If $C \cap U \neq \emptyset$, then the left hand side of this inclusion is non-empty and of dimension 1. Since $\Phi^{-1}(y)$ is isomorphic to the $C$-orbit of $y$, it is itself irreducible of dimension (at most) 1, hence the inclusion becomes an equality. Taking closures, we then obtain $C = \{(\xi, x, y) | (\xi, x) \in \Phi^{-1}(y)\}$; in particular, $C$ is the unique irreducible component of $\pi_3^{-1}(y)$ that intersects $U$. Note also that $C \subset \Gamma_w$.

Now let $C'$ be an irreducible component of $\pi_3^{-1}(y)$ different from $C$, so that $C' \subset \{ \infty \} \times G/P \times \{ y \}$. Let $\Gamma_\infty \subset \Gamma$ be the subvariety $\pi_3^{-1}(\infty)$. We have a $U^-_\tau$-equivariant morphism $\pi : \Gamma_\infty \to G/P : (\infty, x, y) \mapsto y$, so $C'$ is an irreducible subvariety of the fibre $\pi^{-1}(y)$. Since $\Gamma_\infty \subset \Gamma$, its dimension is at most $N$. Because of the equivariance of $\pi$, we see that $\mathcal{O}_\tau$ is in the image of $\pi$, so $\pi$ is surjective.

Decomposing $\Gamma_\infty$ into irreducible components $\Gamma_\infty = C_1 \cup \cdots \cup C_r$, we obtain $G/P = \pi(C_1) \cup \cdots \cup \pi(C_r)$, so that for some $i$, $\pi : C_i \to G/P$ is onto. Renumbering the $C_i$, we may assume that for some $t \geq 1$, $C_1, \ldots, C_t$ are mapped surjectively to $G/P$, and $C_{t+1}, \ldots, C_r$ are not. For $i \leq t$, there is an open set $U_i$ of $G/P$ such that each element on $U_i$ has a finite fibre in $C_i$. For $i > t$, let $U_i$ be the open set $G/P \setminus \pi(C_i)$. Taking the intersection $U = \bigcap_{i=1}^t U_i$, we obtain a non-empty open set of $G/P$ satisfying the following property: for each $z \in U$, the fibre of $z$ in $\Gamma_\infty$ consists of a finite number of points. Again, $U$ meets the open orbit $\mathcal{O}_\tau$, so this property is true for every point in $\mathcal{O}_\tau$, in particular for $y$. So $C'$ is included in the finite fibre $\pi^{-1}(y)$; a contradiction. Therefore, $C'$ cannot exist, i.e. $\pi_3^{-1}(y) = C \subset \Gamma_w$ is irreducible, and not contained in $\{ \infty \} \times G/P \times \{ y \}$.

Assume now that $F_w \neq F_w \cap F^v$. By Proposition 4.4, $F_w$ and $F_w \cap F^v$ have the same dimension, thus $F_w \cap F^v$ is not irreducible. Let $F$ be an irreducible component of the intersection $F_w \cap F^v$ different from $F_w$. Let $y \in F$, and assume that $y \notin F_w$. Then $y \notin \pi_3(U^v)$, so $\pi_3^{-1}(y) \subset \Gamma^v \setminus U^v \subset \{ \infty \} \times G/P \times G/P$. But $y \in F_w$, and we have seen that in this case $\pi_3^{-1}(y)$ is never contained in $\{ \infty \} \times G/P \times G/P$. This gives a contradiction. 

□
APPENDIX. SINGULARITIES OF SCHUBERT VARIETIES IN \( SO(2n + 1)/P_1 \)

In this Appendix, we shall determine the singular locus of Schubert varieties in \( G/P \), where \( G \) is of type \( B_n \) and \( P \) is cominuscule. So let \( V = C^{2n+1} \) together with a non-degenerate symmetric bilinear form \((.,.)\) given in the canonical basis \((e_1,\ldots,e_{2n+1})\) by the anti-diagonal matrix \( E \) with 1’s all along the anti-diagonal. The expression of the quadratic form \( Q \) associated with \((.,.)\) is

\[
Q(x_1,\ldots,x_{2n+1}) = x_{n+1}^2 + 2 \sum_{i=1}^{n} x_{i}x_{2n+2-i}.
\]

Let \( G = SO(V) \), \( B \subset G \) the subgroup of upper triangular matrices, and \( T \subset G \) the subgroup of diagonal matrices. Then \( B \) is a Borel subgroup of \( G \) and \( T \) is a maximal torus of \( G \). The group \( G \) acts naturally on \( V \), and \( e_1 \) is a highest weight vector, of weight \( \omega_1 \) (the unique cominuscule weight of \( G \)), so that \( G/P \) gets identified with the \( G \)-orbit of the line generated by \( e_1 \),

\[
G/P = \{[x_1: \cdots :x_{2n+1}] \mid Q(x_1,\ldots,x_{2n+1}) = 0\}.
\]

In this setting, the Schubert varieties are given by

\[
X_i = \{[x_1: \cdots :x_i :0: \cdots :0] \mid Q(x_1,\ldots,x_i,0,\ldots,0) = 0\},
\]

with \( 1 \leq i \leq 2n+1 \), but \( i \neq n+1 \). Indeed, let \( x = [x_1: \cdots :x_{i-1} :1:0: \cdots :0] \) with \( Q(x) = 0 \), and let us prove that \( x \in C_i \), that is, there exists \( b \in B \) such that \( x = b.c. \). A straightforward calculation shows that we may take the columns \( b_1,\ldots,b_{2n+1} \) of \( b \) as follows:

- **Case 1:** \( i < n+1 \).
  \[
b_j = \begin{cases} 
  e_j, & \text{if } j \neq i \text{ and } j \leq 2n+2-i, \\
  x, & \text{if } j = i, \\
  e_j - x_{2n+2-j}e_{2n+2-i}, & \text{otherwise}.
  \end{cases}
  \]

- **Case 2:** \( i > n+1 \).
  \[
b_j = \begin{cases} 
  e_j, & \text{if } j \leq 2n+2-i, \\
  x, & \text{if } j = i, \\
  e_j - x_{2n+2-j}e_{2n+2-i}, & \text{otherwise}.
  \end{cases}
  \]

The Jacobian criterion easily shows that \( \text{Sing } X_i \) is equal to \( X_{2n+1-i} \) if \( i > n+1 \), and empty if \( i < n+1 \). Moreover, since \( X_i \) is defined by a single quadratic equation, the multiplicity of a singular point must be equal to 2. Hence there are two cases for the multiplicity \( \mu_i(x) \) of a point \( x = [x_1: \cdots :x_i :0: \cdots :0] \) on \( X_i \):

- **Case 1:** \( i < n+1 \). Then \( \mu_i(x) = 1 \).
- **Case 2:** \( i > n+1 \). Then
  \[
  \mu_i(x) = \begin{cases} 
  2, & \text{if } x_1 = \cdots = x_{2n+2-i} = 0, \\
  1, & \text{otherwise}.
  \end{cases}
  \]

Of course, we have the same result for the opposite Schubert varieties

\[
X^J = \{[0: \cdots :0: x_j: \cdots :x_{2n+1}] \mid Q(0,\ldots,0,x_j,\ldots,x_{2n+1}) = 0\}.
\]
There are again two cases for the multiplicity $\mu^j(x)$ of $x = [0 : \cdots : 0 : x_j : \cdots : x_{2n+1}]$ on $X^j$:

- **Case 1:** $j < n + 1$. Then

  $\mu^j(x) = \begin{cases} 2, & \text{if } x_j = \cdots = x_{2n+2-j} = 0, \\ 1, & \text{otherwise.} \end{cases}$

- **Case 2:** $j > n + 1$. Then $\mu^j(x) = 1$.

Note that a Richardson variety $X^j_i$ ($j \leq i$) also is a quadric in a projective space, so the multiplicity of a point $m \in X^j_i$ must be at most 2. But by Theorem 1.1 if $m$ were singular in both $X^j_i$ and in $X^j$, then its multiplicity would be 4. This means that $\text{Sing} X^j_i \cap \text{Sing} X^j = \emptyset$, a fact that can also be verified directly: indeed, if this intersection is non-empty, then $2n + 3 - j \leq 2n + 1 - i$, so $j \leq i \leq j - 2$; a contradiction.

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