Global SSS space-time models: \( M_a \) and \( Q \)

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Abstract

To make sense of a global space-time model and to give a meaning to the coordinates that we use, a choice of a constant curvature space-metric of reference it is as much necessary as it is a choice of units of mass, length and time. The choice we make leads to contradict the belief that the exterior domain of a Static Spherically Symmetric (SSS) space-time model of finite radius \( R \) depends only on the active mass \( M_a \) of the source. In fact it depends on two parameters \( M_a \) and a new one \( Q \). We prove that both can be calculated as volume integrals extended over the whole space.

We integrate Einstein’s equations numerically in two simple cases: assuming either that the source of perfect fluid has constant proper density or that the pressure depends linearly on the proper density. We confirm a preceding paper showing that very compact objects can have active masses \( M_a \) much greater than their proper masses \( M_p \), and we conjecture that the mass point Fock’s model can be understood as the limit of a sequence of compact models when both \( Q \) and its radius shrink to zero and the pressure equals the density.

1 Global SSS space-time models

Using Weyl’s like decomposition, and obvious notations, we consider the line-element

\[
ds^2 = -A^2 dt^2 + A^{-2} ds^2, \quad A = A(r)
\]

with:

\[
ds^2 = B^2 dr^2 + B C r^2 d\Omega^2, \quad B = B(r), \quad C = C(r)
\]

Reference space-model. We shall refer to this 3-dimensional metric as the space-model, and to restrict it as well as the coordinates used we shall require the condition:

\[
C' = \frac{2}{r} (B - C)
\]

Introducing the Euclidean reference metric:

\[
ds^2 = dr^2 + r^2 d\Omega^2,
\]

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\footnote{c = 1, \( G = 1 \) are used throughout
we can write the equation above as follows:

\[(\tilde{\Gamma}_{jk}^i - \tilde{\Gamma}_{jk}^i)g^{jk} = 0, \quad i, j, k = 1, 2, 3 \]  \hspace{1cm} (5)

where \( \tilde{\Gamma} \) and \( \tilde{\Gamma} \) are the connection symbols of [2] and [4] in which case this equation becomes a restriction on the model but remains true whatever space coordinates we use. In particular Cartesian coordinates of [4] become harmonic coordinates of [2] and this implies, as it is well known, that they are also harmonic coordinates of [1].

**Einstein’s tensor.** We write below the non identically zero components of the Einstein’s tensor:

\[
S_0 = \frac{2AA''}{B^2} - \frac{A^2B''}{B^3} - \frac{3A'^2}{B^2} + \frac{4AA'}{rBC} - \frac{A^2}{r^3BC} + \frac{A^2}{r^2BC} + \frac{5A^2B''}{4B^4} \quad (6)
\]

\[
S_1 = -\frac{A^2B'}{rBC} + \frac{A^2}{rBC} - \frac{A^2}{r^2BC} - \frac{A^2B'^2}{4B^4} \quad (7)
\]

\[
S_2 = S_3 = -\frac{A^2B''}{2B^3} + \frac{3A^2B'^2}{4B^4} - \frac{A'^2}{B^2} - \frac{A^2}{r^2BC} + \frac{A^2B'}{rB^2C} + \frac{A^2}{r^2C^2} \quad (8)
\]

where a prime means a derivative with respect to \( r \). They have been somewhat simplified eliminating the derivatives of \( C \) using (3) and its derivative:

\[
C'' = \frac{2}{r^2}(3C - 3B + rB') \quad (9)
\]

**The global model.** the global perfect fluid models that we consider will be solutions of Einstein’s equations:

\[
a : S_0^0 = 8\pi \rho, \quad b : S_1^1 = -8\pi P, \quad c : S_2^2 = S_3^3 = -8\pi P \quad (10)
\]

A more convenient system of differential equations equivalent to the above one is the following:

\[
\frac{2AA''}{B^2} - \frac{2A'^2}{B^2} + \frac{4AA'}{BCr} = 8\pi(\rho + 3P) \quad (11)
\]

\[
\frac{1}{2} \frac{A^2B''}{B^3} + \frac{A^2}{B^2} + \frac{3A^2B'^2}{4B^4} - \frac{A^2}{BCr^2} + \frac{A^2B'}{B^2Cr} = -8\pi P \quad (12)
\]

\[
P' = -\frac{A'}{A}(\rho + P) \quad (13)
\]

\[
C' = \frac{2}{r}(B - C) \quad (14)
\]

Eq. (11) is a linear combination of Eqs. a and c. Eq. (12) is Eq. c left unchanged. Eq. (13) is known to be equivalent to Eq. b as a consequence of the conservation equations satisfied by the Einstein’s tensor provided that \( S_1^1 = -8\pi P \) for \( r = 0 \). Eq. (14) is the same as Eq. (6)

As we shall see it is important to realize that Eq. (11) can be written as:

\[
\Delta U = 4\pi(\rho + 3P)A^{-2}, \quad U = \ln(A) \quad (15)
\]

where \( \Delta \) is the Laplace operator of the space-model [2].
\[\Delta U = \frac{1}{\sqrt{g}} \partial_i (\sqrt{\bar{g}} g^{ij} \partial_j U), \quad \bar{g} = \det [\bar{g}_{ij}]\]  
(16)

or explicitly:

\[\Delta U = \frac{1}{B^2 C r^2} \frac{d}{dr} \left( C r^2 \frac{dU}{dr} \right)\]  
(17)

## 2 Boundary conditions

To define a particular class of SSS models demands to be specific about the boundary conditions. This requires i) To state regularity conditions at the origin \(r = 0\). ii) To implement Lichnerowicz’s continuity conditions at the boundary of the source, \(r = R\), that we assume to be compact. And iii) to guarantee that from \(r = R\) onwards \(\rho\) and \(P\) are zero and the solution has the correct asymptotic behavior.

### Regularity at the origin.

To have continuity at the origin requires to have:

\[A'_0 = 0, \quad B'_0 = 0, \quad C'_0 = 0.\]  
(18)

and taking into account (3) there follows that we have to have also:

\[B_0 = C_0\]  
(19)

This means in particular that in a neighborhood of the origin we can write:

\[A = A_0 + \frac{1}{2} A_2 r^2 + O(r^3), \quad B = B_0 + \frac{1}{2} B_2 r^2 + O(r^3), \quad C = B_0 + \frac{1}{2} C_2 r^2 + O(r^3)\]  
(20)

with \(C_2 = 1/2B_2\) to satisfy Eq. (14). It follows then from (7) and (8) that at the origin we always have:

\[(S^1_1)_0 = (S^2_2)_0 = (S^3_3)_0 = -\frac{5}{4} A_2^2 B_2\]  
(21)

that guarantees that the solutions of (11)-(14) will satisfy Eq. b of (10).

### Matching conditions at \(r = R\).

The models will be matched in the sense of Lichnerowicz, requiring the continuity of \(A, B\) and \(C\) and its first derivatives at \(r = R\), \(R\) being the radius of a compact source.

### Asymptotic conditions.

They will be the general asymptotic conditions of Schwarzschild’s model compatible with (3). They were derived in [5] and shall be used here [3].

\[A = 1 - \frac{M_0}{r} + \frac{1}{2} \frac{M_2^2}{r^2} - \frac{1}{2} \frac{M_3^2}{r^3} + O(1/r^4)\]  
(22)

\[B = 1 - \frac{2}{3} \frac{Q}{r^3} + O(1/r^4)\]  
(23)

\[C = 1 - \frac{M_2^2}{r^2} + \frac{4}{3} \frac{Q}{r^3} + O(1/r^4)\]  
(24)

\(^2Q\) in this paper is \(QM_2^2\) in [5].
where:

\[ M_a = \lim_{r \to \infty} A' r^2 \]  (25)

is by definition the active mass of the source, and:

\[ Q = \lim_{r \to \infty} \frac{1}{2} B' r^4 \]  (26)

is a new parameter that has already been considered in other contexts [2–5].

3 Integral expressions of \( M_a \) and \( Q \)

We define the proper mass of the source by:

\[ M_p = 4\pi \int_0^\infty \rho r^2 \, dr \]  (27)

Multiplying both members of (15) by \( B^2 Cr^2 \) and integrating from 0 to \( \infty \) we get:

\[ \left( Cr^2 \frac{dU}{dr} \right)_\infty - \left( Cr^2 \frac{dU}{dr} \right)_0 = 4\pi \int_0^\infty (\rho + 3P) A^{-2} BCr^2 \, dr \]  (28)

that using (22) and (24) becomes the well-known Tolman’s formula [8]:

\[ M_a = 4\pi \int_0^R (\rho + 3P) A^{-2} BCr^2 \, dr \]  (29)

On the other hand, integrating by parts the first two terms of \( S_1' r^2 \) over all space, and taking into account the conditions stated above we obtain:

\[ M_a = M_p + 4\pi \int_0^\infty \sigma p r^2 \, dr \]  (30)

where:

\[ \sigma_p = \frac{5A^2}{B^2} - \frac{6AA'B'}{B^3} - \frac{4AA'}{rB^2} + \frac{7A^2B'^2}{4B^4} - \frac{2A^2B'}{rB^3} + \frac{4AA'}{BCr^2} - \frac{A^2}{C^2r^2} + \frac{3A^2B'}{B^2Cr} \]  (31)

Similarly integrating by parts the first term of \( (S_2^0 - S_1^0) r^4 \) and using the boundary conditions above we obtain:

\[ Q = \int_0^\infty \sigma_p r^2 \, dr \]  (32)

where:

\[ \sigma_p = \frac{2r^2 A'^2}{B^2} - \frac{2A^2}{BC} - \frac{2r^2 A^2 B'^2}{B^3} + \frac{2A^2}{C^2} + \frac{r^2 AA'B'}{B^3} + \frac{2rA^2B'}{B^3} \]  (33)
4 Numerical models: Examples

We consider two cases based on the following relationships between the density $\rho$ and the pressure $P$.

- Case I: we assume that the density $\rho$ is constant and positive from $r = 0$ to $r = R$, where $P = 0$;
- Case II: we assume that $P = k\rho - \eta$ from $r = 0$ to $r = R$, where $P$ is a small fraction of an assumed $P_0$ value at the origin. $k = 1/3$ or $k = 1$ and $\eta$ is a positive constant included to guarantee that the pressure reaches the value zero at some radius $R$;

The initial initial conditions will be:

$$A'_0 = B'_0 = C'_0 = 0 \quad B_0 = C_0 = 1, \quad \rho_0 = (8\pi)^{-1}$$

compatible with the regularity conditions discussed above.

The following coordinate transformations:

$$t = \mu \bar{t}, \quad r = \nu \bar{r}$$

modifies $A$, $B$ and $C$ as follows:

$$\bar{A} = \mu A, \quad \bar{B} = \nu \mu B, \quad \bar{C} = \nu \mu C,$$

but leaves invariant (3) as well as (6)-(8) and therefore $\rho$ and $P$. A covariant property that will be used below.

The integration proceeds in two steps and two runs. First of all we integrate Eqs. (11)-(14) until $P$ reaches a chosen estimated small value at some $r = R$. Beyond that we set $\rho = 0$, $P = 0$ and the integration proceeds until the quantity $\epsilon = rA\rho$ reaches a chosen estimated small value. Let $A_{\epsilon}$ and $B_{\epsilon}$ be the corresponding values of $A$ and $B$.

The second round consists in integrating in two steps the same differential Eqs. (11)-(14) with the same initial values of $\rho_0$ and $P_0$ but with the following different initial conditions:

$$A'_0 = \frac{A_0}{A_\epsilon}, \quad B'_0 = \frac{A_0}{B_\epsilon}$$

which is one of the coordinate transformations considered in (38), with:

$$\mu = A_\epsilon^{-1}, \quad \lambda = A_{\epsilon}B_\epsilon^{-1}$$

Examples.- The parameter $\eta$ in Case II is 0.01 for $k = 1/3$ and 0.10 for $k = 1$, and Eqs. (11)-(14) have been integrated until $P$ became of the order of $1.0 \epsilon - 8$.

The following tables where $\lambda$ is the compactness parameter:

$$\lambda = \frac{M_\nu}{R}$$

summarize some of the informative results:

Case I:
Case II:

|   |   |   |   |   |
|---|---|---|---|---|
| 1/3 |  .92 | .13 | .36 | .39 | .02 |
| 1   | .91 | .13 | .57 | .62 | .43 |

Appendix

This appendix is an update of our reference [7] that is meant to prove that Fock’s model [6] of the Schwarzschild’s exterior solution can be considered as a legitimate limit model of regular SSS models when the proper volume that contains the proper mass shrinks to zero.

For this model we have:

\[ A = \sqrt{\frac{r - M_a}{r + M_a}}, \quad B = 1, \quad C = 1 - \frac{M_a^2}{r^2} \]  

(40)

and evaluating (31) and (33) we obtain:

\[ \sigma_\rho = \frac{4M_a^2}{r(r + m)}, \quad \sigma_p = 0 \]  

(41)

and therefore using (30) and:

\[ 4\pi \int_{M_a}^{\infty} \sigma \rho r^2 dr = \frac{3}{4} M_a \]  

(42)

we get:

\[ M_p = \frac{1}{4} M_a, \quad Q = 0. \]  

(43)

The first of these formulas suggest that the source of Fock’s model is:

\[ \rho = \frac{1}{4} M_a \delta(r - M_a) \]  

(44)

where the Dirac density will be defined by the two properties:

\[ r > M_a \Rightarrow \delta(r - M_a) = 0, \quad \text{and} \quad 4\pi \int_{M_a}^{\infty} \delta(r - M_a)r^2 dr = 1 \]  

(45)

Tolman’s formula [29] for general regular models and Eq. (44) above suggest that if the proper density for Fock’s model contributes to only one third of the active mass then the pressure must be:

\[ P = \frac{1}{4} M_a \delta(r - M_a) \]  

(46)
so that:

\[ P = \rho \quad \text{and} \quad \rho + 3P = M_a \delta(r - M_a) \quad (47) \]

Now, let us take a look to the geometry (2) corresponding to the coefficients \( B \) and \( C \) above. We have:

\[ \sqrt{\det \bar{g}_r} = B^2 C \sin \theta = (r^2 - M_a^2) \sin \theta \quad (48) \]

that becomes zero when \( r = M_a \). This means that the system of coordinates whose meaning is that derived from the metric of reference (1) is not to be trusted when discussing properties of the model in a neighborhood of \( r = M_a \).

What we do below is more than we need to discuss this point but we prefer here to deal with a global condition instead of a local one as we did in [7] and introduce a system of coordinates:

\[ y_1 = u(r) \sin \theta \cos \phi, \quad y_2 = u(r) \sin \theta \sin \phi, \quad y_3 = u(r) \cos \theta \quad (49) \]

\( u \) being a function of \( r \) such that:

\[ \sqrt{\det \bar{g}_u} = 1 \Leftrightarrow \sqrt{\det \bar{g}_u} = u^2 \quad (50) \]

Using the polar form of the condition, a short calculation proves that the function \( u \) will be given by:

\[ u = (r^3 - 3M_a^2r + 2M_a^3)^{1/3} \quad (51) \]

so that:

\[ u(M_a) = 0 \quad (52) \]

The other way around, to get the inverse function \( r(u) \) we need to solve the third degree algebraic equation:

\[ r^3 - 3M_a^2r + 2M_a^3 - u^3 = 0 \quad (53) \]

The discriminant of this equation is positive if \( u > 2^{2/3}M_a \) and negative otherwise down to \( u = 0 \). In the first case there exist only one real solution, namely:

\[ r = \left( -M_a^3 + \frac{1}{2} u^3 + \frac{1}{2} \sqrt{-4M_a^3u^3 + u^6} \right)^{1/3} + \left( -M_a^3 + \frac{1}{2} u^3 - \frac{1}{2} \sqrt{-4M_a^3u^3 + u^6} \right)^{1/3} \quad (54) \]

that has an asymptote \( r = u \), while in the second case there are three real functions. But there is only one for which \( r \) is a positive increasing function of \( u \), namely:

\[ r = 2M_a \sin \left( \frac{1}{6} \pi + \frac{1}{3} \arccos \left( 1 - \frac{1}{2} \frac{u^3}{M_a^3} \right) \right) \quad (55) \]

whose behavior near \( u = 0 \) is:

\[ r = M_a + \frac{1}{3} \sqrt[3]{u^{3/2}} + \frac{1}{18} \frac{u^3}{M_a^3} + O(u^{3/2}) \quad (56) \]
The two solutions here considered joint smoothly at \( u = 2^{2/3}M_a \).

Using the radial coordinate \( u \) instead of \( r \) the space-model becomes:

\[
ds^2 = B_u^2 du^2 + B_u^{-1} u^2 d\Omega^2
\]

where:

\[
B_u = \frac{dr}{du} = \frac{\cos \left( \frac{\pi}{6} + \frac{1}{2} \arccos \left( 1 - \frac{u^3}{2M_a^3} \right) \right) u^2}{M_a^2 \sqrt{1 - \left( 1 - \frac{u^3}{2M_a^3} \right)^2}}
\]

and \( B_u^2 \) can be approximated in a neighborhood of \( u = 0 \) by:

\[
B_u^2 = \frac{3}{4} \frac{u}{M_a} - \frac{1}{6} \sqrt{3} \frac{u^{5/2}}{M_a^{5/2}} + O(u^4)
\]

while \( U_u \) and \( dU_u/du \) can be approximated by:

\[
U_u = \frac{1}{2} \ln \left( \frac{\sqrt{3}}{6} \frac{M_a^2}{u^2} \right) + \ln(u) - \frac{\sqrt{3}}{9} \frac{u^{3/2}}{m^{3/2}} + O(u^3)
\]

and:

\[
dU_u = \frac{3}{4} \frac{u}{M_a} - \frac{1}{6} \sqrt{3} \frac{u}{M_a^{1/2}} + O(u^2)
\]

Let us now consider again Eq. (17) that becomes now, for Fock’s model, the Laplace equation:

\[
\Delta U_u = \frac{1}{u^2} \frac{d}{du} \left( u^2 B_u^{-2} \frac{dU_u}{du} \right) = 0
\]

that holds for \( u > 0 \) but it is actually undefined for \( u = 0 \).

Let us define \( U_u \) as the linear functional:

\[
< U_u, \varphi > = 4\pi \lim_{\epsilon \to 0} \int_\epsilon^\infty U_u \varphi u^2 du
\]

where \( \varphi \) is any function such that itself and all its derivatives become zero beyond some finite value of \( u \). As a distribution we shall have then:

\[
< \Delta U_u, \varphi > = 4\pi \lim_{\epsilon \to 0} \int_\epsilon^\infty U_u \Delta \varphi u^2 du
\]

or:

\[
< \Delta U_u, \varphi > = 4\pi \lim_{\epsilon \to 0} \int_\epsilon^\infty U_u \frac{d}{du} \left( u^2 B_u^{-2} \frac{d\varphi}{du} \right) du
\]

Using now this particular case of Green’s formula:

\[
U_u \frac{d}{du} \left( u^2 B_u^{-2} \frac{d\varphi}{du} \right) - \varphi \frac{d}{du} \left( u^2 B_u^{-2} \frac{dU_u}{du} \right) = \frac{d}{du} \left( u^2 B_u^{-2} U_u \frac{d\varphi}{du} \right) - \frac{d}{du} \left( u^2 B_u^{-2} \varphi \frac{dU_u}{du} \right)
\]
and taking into account (65) we have:
\[
< \Delta U_u, \varphi > = 4\pi \lim_{u \to 0} \left( u^2 B_u^{-2} \left( U_u \frac{d\varphi}{du} - \varphi \frac{dU_u}{du} \right) \right)
\] (67)

Now since:
\[
\lim_{u \to 0} (u^2 B_u^{-2} U_u) = 0 \quad \text{and} \quad \lim_{u \to 0} \left( u^2 B_u^{-2} \frac{dU_u}{du} \right) = M_a
\] (68)

we conclude that:
\[
< \Delta U_u, \varphi > = 4\pi M_a \varphi(0), \ or \ \Delta U_u = 4\pi \frac{1}{2} M_a \delta(u)
\] (69)

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