Driven-dissipative preparation of entangled states in cascaded quantum-optical networks

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Abstract. We study the dissipative dynamics and the formation of entangled states in driven cascaded quantum networks, where multiple systems are coupled to a common unidirectional bath. Specifically, we identify the conditions under which emission and coherent reabsorption of radiation drives the whole network into a pure stationary state with non-trivial quantum correlations between the individual nodes. We illustrate this effect in more detail for the example of cascaded two-level systems, where we present an explicit preparation scheme that allows one to tune the whole network through ‘bright’ and ‘dark’ states associated with different multi-partite entanglement patterns. In a complementary setting consisting of cascaded nonlinear cavities, we find that two cavity modes can be driven into a non-Gaussian entangled dark state. Potential realizations of such cascaded networks with optical and microwave photons are discussed.
1. Introduction

The coupling of a quantum system to an environment is often associated with decoherence. This is exemplified by the field of quantum computation, where the presence of additional reservoirs degrades the performance of quantum algorithms based on unitary operations executed on large many-body systems [1]. However, in many quantum optical settings the environment is actually useful for realizing certain applications. Prominent examples are laser cooling or optical pumping, where quantum systems are prepared in very pure states with the help of a reservoir [2], or continuous measurement schemes which enable the conditioned preparation of quantum states [3]. While quantum control schemes employing engineered unitary evolution are by now standard, the last decade has witnessed an increasing interest in alternative methods based on the concept of ‘quantum reservoir engineering’, i.e. on controlling a quantum system by tailoring its coupling to an environment [4]. Efforts along these lines have lead, e.g. to a broad range of proposals for the dissipative preparation of entangled few-body quantum states [5–11]. However, in recent years attention has in particular been devoted to the study of dissipative many-body systems. In this context, it has been realized that quantum reservoir engineering allows one to dissipatively prepare interesting many-body states [12–16], perform universal quantum computation [14], realize a dissipative quantum repeater [17], or a dissipatively protected quantum memory [18]. Meanwhile, the first experiments demonstrating the dissipative preparation of Greenberger–Horne–Zeilinger (GHZ) states in systems of trapped ions [19] and Einstein–Podolsky–Rosen (EPR) entangled states of two atomic ensembles [20, 21] have been reported. In these experiments the underlying principle has been to carefully design and implement a many-particle master equation (ME), where a fully dissipative dynamics drives the system into a unique steady state representing the entangled state of interest.

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Figure 1. A driven cascaded quantum network which is realized by a set of two-level systems coupled to a unidirectional bath. The two-level systems are driven by classical fields $\sim \Omega$ and the continuously emitted radiation propagates along the waveguide and excites successive nodes. Under specific conditions, all photons emitted in a subsystem A are coherently reabsorbed in subsystem B. In this case the system relaxes into a dark state where no radiation escapes from the network, but a constant stream of photons running from A to B establishes entanglement between the two subsystems.

In this paper, we study entanglement formation in driven few- and many-particle cascaded quantum networks as introduced by Gardiner [22] and Carmichael [23], where the unconventional coupling of multiple systems to a common unidirectional bath offers remarkable new opportunities for dissipative preparation of highly correlated states. As illustrated in figure 1, a cascaded quantum network consists of $N$ systems coupled to a one-dimensional (1D) reservoir that has the unique feature that excitations can only propagate along a single direction, thereby driving successive systems in the network in a unidirectional way. While such a scenario is reminiscent of edge modes in quantum Hall systems [24–28], various artificial and more controlled realizations based on (integrated) non-reciprocal devices for optical [29, 30] and microwave [31, 32] photons have been developed. Our goal is to identify situations where cascaded quantum networks are driven by classical fields in such a way that they exhibit pure and entangled steady states. As shown in figure 1, such dark states emerge if the continuous stream of photons emitted by the first part (‘A’) of the network is coherently reabsorbed by the second part (‘B’), such that no photons escape from the system and the output remains dark. The system hence acts as its own coherent quantum absorber while the constant stream of photons maintains entanglement between its two parts.

We will first discuss the construction of coherent quantum absorbers on general grounds. Given the first part A of the network, we show how to choose the second part B such that the whole system evolves into a dark state. In doing so, we make use of the unidirectionality of the reservoir, which allows us to solve the first part independently of the second. These formal developments are then illustrated by two settings in which the coherent quantum absorber scenario can be realized. The first is a many-body cascaded network, where each of the $N$ nodes consists of a driven two-level system (TLS, ‘spin’). We show that this system exhibits a whole class of multi-partite entangled dark states, whose entanglement structure can be adjusted by tuning local parameters. As a second, complementary system we consider a network consisting of two nonlinear cavities described by bosonic mode operators. By choosing appropriate laser drives and Kerr-type nonlinearities one can ensure that this system evolves into a non-Gaussian dark state in which the two modes are entangled. In both examples, the coherent quantum absorber scenario thus leads to a dissipative state preparation scheme for non-trivial entangled states. In a more general context, these networks realize a novel type of non-equilibrium
(many-body) quantum system, which by changing the system parameters can be tuned between ‘dark’ or ‘passive’ phases (with no scattered photons emerging) and ‘bright’ or ‘active’ phases (with light scattered), while the nodes are driven into pure entangled or mixed states, respectively.

The remainder of this paper is structured as follows. In section 2 we present the general model for an $N$-node cascaded network and show how to construct the coherent quantum absorber subsystem $B$ for some given subsystem $A$ (cf figure 1). Complementing these rather general developments, we discuss the two mentioned realizations of the coherent quantum absorber scenario in subsequent sections. Section 3 presents the driven cascaded spin-system for which we derive and discuss a multi-partite entangled class of dark states and also comment on the influence of various imperfections. Subsequently, section 4 discusses the setup based on cascaded cavities with Kerr-type nonlinearity. In the latter case, we also obtain a purification of the steady state density matrix of the well-known dispersive optical bi-stability problem [33] as an interesting by-product. Implementations of the proposed cascaded networks are discussed in section 5 and concluding remarks can be found in section 6.

2. Coherent quantum absorbers

2.1. Cascaded quantum networks

We consider the general setting of a cascaded quantum network as shown in figure 1. Here, $N \geq 2$ subsystems located at positions $x_i$ are coupled to a 1D continuum of right-propagating bosonic modes $b_\omega$, which represent, for example, photons in an optical or microwave waveguide. The whole network can be modelled by a Hamiltonian ($\hbar = 1$)

$$H = \sum_i H_i + H_{\text{bath}} + \sum_i \int d\omega g_\omega (c_i^\dagger b_\omega e^{i\omega x_i/v} + \text{H.c.}),$$

(1)

where $H_{\text{bath}} = \int d\omega \omega b_\omega^\dagger b_\omega$ is the free Hamiltonian of the bath modes and the integrals run over a broad bandwidth $\Delta \omega$ around the characteristic system frequency $\omega_0$. In this frequency range the bath modes are assumed to exhibit a linear dispersion relation with speed of light $v$. In (1) the $H_i$ describe the dynamics of the individual systems and include classical driving fields (e.g. lasers or microwave fields). The system–bath coupling is determined by the ‘jump operators’ $c_i$ and coupling constants $g_\omega$, which we assume to be approximately constant over the frequency range $\Delta \omega$. Implementations of an effective model of the form given in equation (1) can be achieved with atoms or solid state TLSs coupled to 1D optical or microwave waveguides and will be discussed in more detail in section 5 below.

The system–bath interaction in equation (1) breaks time-reversal symmetry and while photons can be emitted to the right, drive successive subsystems and eventually leave the network, the reverse processes cannot occur. To study the effects of this unconventional coupling, we assume that $\omega_0$ as well as the bandwidth $\Delta \omega$ are large compared to the other relevant frequency scales and eliminate the bath modes in a Born–Markov approximation. This yields a generalized cascaded ME for the reduced system density operator $\rho$ [22, 23, 34],

$$\dot{\rho} = \sum_i \mathcal{L}_i \rho - \gamma \sum_{j>i} ([c_j^\dagger, c_i \rho] + [\rho c_i^\dagger, c_j]).$$

(2)

Here the first part describes the uncoupled evolution of each subsystem $\mathcal{L}_i \rho = -i[H_i, \rho] + \gamma \mathcal{D}(c_i) \rho$, where the Lindblad terms $\mathcal{D}[x] \rho = x \rho x^\dagger - (x^\dagger x, \rho)/2$ model dissipation due to
emission of photons into the waveguide with a rate $\gamma = 2\pi g_{\text{eff}}^2$. The unidirectionality of the bath is reflected by the last term in (2), which accounts for the possibility to reabsorb photons emitted at system $i$ by all successive nodes located at $x_j > x_i$. The explicit Lindblad form of the ME (2) reads

$$\dot{\rho} = -i[H_{\text{cas}}, \rho] + \gamma D[c]\rho,$$

where $H_{\text{cas}} = \sum_i H_i - i\frac{\gamma}{2} \sum_{j > i} (c_j^+ c_i - c_i^+ c_j)$ now includes the non-local coherent part of the environment-mediated coupling, while the only decay channel with collective jump operator $c = \sum_i c_i$ is associated with a photon leaving the system to the right\(^4\). Note that equations (2) and (3) are understood in a rotating frame in order to account for the explicit time-dependence of the driving fields.

2.2. Coherent quantum absorbers

In the following we are interested in steady-state situations where every photon emitted within the system is perfectly reabsorbed by successive nodes in the network, such that there is no spontaneous emission via the waveguide output and the system relaxes to a pure steady state $\rho_0 = |\psi_0\rangle \langle \psi_0|$. To identify the general conditions for the existence of such states, we partition the network into two subsystems, A and B, as indicated in figure 1, with local Hamiltonians $H_A$ and $H_B$, and jump operators $c_A$ and $c_B$, respectively. Specifically, $H_A = \sum_i H_i - i\frac{\gamma}{2} \sum_{j > i} (c_j^+ c_i - c_i^+ c_j)$ and $c_A = \sum_i c_i$, where the primed sums run over the nodes of part A, and corresponding expressions hold for B. Then, in equation (3), $H_{\text{cas}} = H_A + H_B - i\frac{\gamma}{2} (c_A c_B^\dagger - c_B^\dagger c_A)$ and $c = c_A + c_B$, and the conditions for the existence of a pure stationary state are (see [13] and further comments in appendix A):

\begin{align*}
(\text{I}) & \quad (c_A + c_B)|\psi_0\rangle = 0, \quad (\text{II}) \quad [H_{\text{cas}}, \rho_0] = 0.
\end{align*}

The first condition implies that the waveguide output is dark, i.e. $\langle c^+ c \rangle = 0$, and the second one ensures stationarity. Within a quantum trajectory picture, condition (I) means that there are no stochastic quantum jumps, which would lead to a mixed state. In situations where $[H_{A,B}, \rho_0] = 0$ and $c_{A,B} |\psi_0\rangle = 0$ for each subsystem separately, the network can simply be divided into two smaller parts which are then treated independently. In the following, we thus focus on situations where such a division is not possible and where the steady state possesses non-trivial correlations between A and B, as characterized for example by a non-vanishing $C = \langle c_A^\dagger c_B + c_A c_B^\dagger \rangle - 2\text{Re}\{\langle c_A^\dagger \rangle \langle c_B \rangle\}$. In view of (I) this third requirement can be expressed as

$$\text{(III)} \quad C = -2\langle c_A^\dagger c_A \rangle - 2\langle c_A \rangle^2 \neq 0,$$

and directly connects the correlations between A and B with the amount of radiation emitted from the first subsystem. Note that for a pure steady state, $C \neq 0$ also implies that A and B are entangled. Finally, we remark that under stationary conditions, a non-vanishing $C$ implies a constant flow of energy from A to B, while the total scattered light vanishes. In the examples discussed below this ‘coherent’ absorption of energy in B can be understood as a destructive interference of the signals scattered by the two subsystems.

The conditions (I)–(III) will not be satisfied in general. However, given a system A described by a Hamiltonian $H_A$ and jump operator $c_A$ we can construct a perfect coherent

\(^4\) For $N = 2$ this ME stands in contrast to [20, 21], where the dynamics is purely dissipative with no coherent evolution.
absorber system B as follows. First, we point out that due to the unidirectional coupling the dynamics of A is unaffected by B, which can also be shown explicitly by tracing the ME (3) over system B. In particular, the steady state $\rho_A^0$ of A is obtained by solving $L_A \rho_A^0 = 0$, where $L_A \rho_A = -i[H_A, \rho_A] + \gamma D[c_A] \rho_A$, and assuming a unique solution we write its spectral decomposition as $\rho_A^0 = \sum_k p_k |k\rangle \langle k|$. A pure state of the whole system is then given by $|\psi_0\rangle = \sum_k \sqrt{p_k} |k\rangle_A \otimes |\tilde{k}\rangle_B$, where we assumed A and B to have the same Hilbert space dimension and defined $|\tilde{k}\rangle = V |k\rangle$ in terms of an arbitrary unitary $V$ acting on $|k\rangle$. Now, we demonstrate in appendix A that conditions (I) and (II) can be satisfied by the choice

$$c_B = -\sum_{n,m} \sqrt{p_n} p_m \langle m|c_A|n\rangle \langle \tilde{n}|\tilde{m}\rangle_B,$$  

$$H_B = -\frac{1}{2} \sum_{n,m} \left( \sqrt{p_n} A_{mn} + \sqrt{p_m} A_{mn}^* \right) |\tilde{n}\rangle \langle \tilde{m}|_B,$$  

where $A_{mn} = \langle m|H_{A,eff}|n\rangle$ and $H_{A,eff} = H_A - i\frac{\gamma}{2} c_A^i c_A$ is the effective non-Hermitian Hamiltonian associated with $L_A$, and we assumed a positive and non-degenerate spectrum $\{p_k\}$. While equations (6) and (7) define a general absorber system B, we find that for many systems of interest the stationary state $\rho_A^0$ satisfies $\sqrt{p_n} |k\rangle \langle c_A|n\rangle = \sqrt{p_k} \langle n|c_A|k\rangle$ and $\sqrt{p_k} |k\rangle H_{A,eff}|n\rangle = \sqrt{p_n} \langle n|H_{A,eff}|k\rangle$. In this case, the above relations simplify to $c_B = -V c_A V^\dagger$ and $H_B = -V H_A V^\dagger$ such that up to a unitary basis transformation the absorber system is just the negative counterpart of A. In particular, this situation applies to the two examples of cascaded spin systems and cascaded nonlinear cavities, which we describe in more detail in the following sections.

3. Cascaded spin networks

Let us now be more specific and consider a set of $N$ driven spins coupled to a unidirectional bosonic bath as shown in figure 1. In equation (3), the collective jump operator is now $c = \sum_i \sigma_i^-$ and the cascaded Hamiltonian in the frame rotating at the frequency $\omega_d$ of the external driving field reads

$$H_{casc} = \sum_i \left( \frac{\delta_i}{2} \sigma_i^z + \Omega_i \sigma_i^+ \right) - i \frac{\gamma}{2} \sum_{j<i} \left( \sigma_j^+ \sigma_i^- - \sigma_i^- \sigma_j^+ \right).$$  

Here the $\sigma_i^\mu$ are the usual Pauli operators on site $i$, the $\Omega_i$ are local Rabi frequencies, and the $\delta_i = \omega_i - \omega_d$ are the detunings of the individual spin transition frequencies $\omega_i$ from the common classical driving frequency $\omega_d$. In the following, the basis states of the spins are denoted by $|e\rangle$, $|g\rangle$, such that $\sigma_- = |g\rangle \langle e|$. The classical fields which are used to drive the spins can either be applied via additional local channels (assuming that the associated decay rate is much smaller than $\gamma$) or via a coherent field which is sent through the common waveguide. Note that by omitting the cascaded interaction in equation (8) we recover the familiar Dicke model for multiple two-level atoms decaying via a common field mode, where even for $\Omega = 0$ a series

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of dark states can be identified from \( c|\psi_0\rangle = 0 \). In contrast, in our cascaded setting these states are not stationary and non-trivial dark states can only emerge from an interplay between driving and cascading coupled terms.

### 3.1. Construction of dark states

For \( N = 2 \) the dark state condition (I) restricts \( |\psi_0\rangle \) to the subspace spanned by \( |gg\rangle \) and the singlet \( |S\rangle = (|eg\rangle - |ge\rangle)/\sqrt{2} \). Condition (II) can then be satisfied for \( \Omega_1 = \Omega_2 = \Omega \) and any \( \delta_1 = -\delta_2 \equiv \delta \), for which we obtain the unique and pure steady state \( |\psi_0\rangle = |S_0\rangle \), where

\[
|S_0\rangle = \frac{1}{\sqrt{1 + |\alpha|^2}} (|gg\rangle + \alpha |S\rangle), \quad \alpha = \frac{2\sqrt{2}\Omega}{i\gamma - 2\delta}. \tag{9}
\]

The two spins thus realize a source and a matched absorber in the sense introduced in the previous section, and for the matrix representations of the various operators we can identify \( c_A = \sigma_z, c_B = -Vc_A V^\dagger = \sigma_z \) and \( H_B = -V H_A V^\dagger \), using \( V = \sigma_z \). For strong driving, \( |\alpha| \gg 1 \), the state \( |S_0\rangle \) approaches the singlet \( |S\rangle \), where the mutual correlation \( C \to 1 \) is maximized.

While for larger \( N \) a direct search for possible dark states is hindered by the exponential growth of the subspace defined by \( c|\psi_0\rangle = 0 \), we can use the state (9) as a starting point and solve the cascaded system iteratively ‘from left to right’. Suppose that for \( \Omega_i = \Omega \) and \( \delta_1 = -\delta_2 \) the first two spins have evolved into the dark state \( |S_{0i}\rangle \) such that no more photons are emitted into the waveguide. Then, the following two spins effectively see an empty waveguide and evolve into the dark state \( |S_{0i+1}\rangle \), provided that \( \delta_3 = -\delta_4 \). By iterating this argument we see that for any detuning profile with \( \delta_{2i-1} = -\delta_{2i} (i = 1, 2, \ldots) \) the steady state of the ME (3) is given by

\[
|S^0\rangle = |S_{0i}\rangle_{12} \otimes |S_{0i+1}\rangle_{34} \otimes \ldots, \tag{10}
\]

which is shown explicitly in appendix B. In particular, for the homogenous case \( \delta_i \simeq 0 \), a strongly driven cascaded spin system relaxes into a chain of pairwise singlets.

The dimer structure of the state \( |S^0\rangle \) reflects the fact that radiation emitted from one node is immediately reabsorbed by the following one. We now consider situations where this reabsorption occurs by several of the following spins, leading to multi-partite entangled states. To this end, note that by starting from the state in equation (10) we can construct another dark state \( |S'\rangle = U |S^0\rangle \) by any global unitary operation \( U \) with \( [U, c] = 0 \), while implementing the Hamiltonian \( H' = U H_{\text{casc}} U^\dagger \) would ensure stationarity. However, for arbitrary unitary transformations \( H' \) will in general contain additional non-local terms, and we must hence restrict ourselves to unitaries \( U \) under which \( H_{\text{casc}} \) is form invariant. As an example, we write \( H_{\text{casc}} = H_{\text{casc}}(\Delta) \), where \( \Delta = (\delta_1, \delta_2, \ldots) \) is the detuning profile, and introduce the nearest-neighbour operations

\[
U_i(\theta_i) = \exp \left[ \frac{i\theta_i}{4} (\vec{\sigma}_i + \vec{\sigma}_{i+1})^2 \right] \propto \exp \left[ \frac{i\theta_i}{2} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} \right], \tag{11}
\]

where \( \vec{\sigma}_i = (\sigma_i^x, \sigma_i^y, \sigma_i^z) \). Then, by choosing \( \tan(\delta_i) = (\delta_{i+1} - \delta_i)/\gamma \) we obtain

\[
H' = U_i(\theta_i) H_{\text{casc}}(\Delta) U_i^\dagger(\theta_i) = H_{\text{casc}}(\Delta'), \tag{12}
\]

with a new detuning profile \( \Delta' = P_{i,i+1}\Delta \), where \( P_{i,i+1} \) denotes the permutation of \( \delta_i \) and \( \delta_{i+1} \) (this is demonstrated in appendix B). Thus, by starting from a set \( \Delta^0 \) of alternating detunings as defined before equation (10), we can simply swap the detunings of nodes \( i \) and \( i + 1 \) to...
implement a new cascaded spin network with a unique stationary state $|S'\rangle = U_i(\theta_i)|S^0\rangle$. By repeating this argument, we obtain a different pure steady state for each permutation $\Delta'$ of $\Delta^0$. This class of states is given by

$$|S'\rangle = U(\Delta^0 \rightarrow \Delta')|S^0\rangle,$$

where $U(\Delta^0 \rightarrow \Delta')$ is a product of nearest-neighbour operations $U_i(\theta_i)$, specified by the sequence of nearest-neighbour transpositions required for transforming $\Delta^0$ into $\Delta'$. A graphical representation of $U(\Delta^0 \rightarrow \Delta')$ in terms of a circuit model is shown in figure 2(a) for a specific example.

3.2. Discussion

For large detuning differences $|\delta_i - \delta_{i+1}| \gg \gamma$ the unitary transformations given in equation (11) are SWAP operations [1] between neighbouring sites. In this limit the states $|S'\rangle$ remain approximately two-partite entangled, but with singlets shared between arbitrary nodes in the network. In contrast, for $|\delta_i - \delta_{i+1}| \approx \gamma$ the $U_i$ correspond to highly entangling $\sqrt{\text{SWAP}}$ operations. Then, the entanglement structure can be much richer and in general the states $|S'\rangle$ contain multi-partite entanglement between several or even all nodes. While in this case a full characterization is difficult, we point out that the $U_i$ conserve total angular momentum such that the $|S'\rangle$ approach multi-spin singlets in the strong driving limit. The amount of entanglement between subsystems now depends very much on the choice of the detuning profile, as can be seen from the two examples displayed in figure 2(b), showing oscillating and linearly growing block entropy, respectively. More generally, the cascaded network can be driven into different types of states by simply adjusting local detunings. This is illustrated in figure 3, where an adiabatic variation of the detunings in a six-node network is used to prepare pure steady states.
with 2-, 4- and 6-partite entanglement, separated by ‘bright’ (mixed state) phases where the conditions in (4) are violated.

For assessing the relaxation time that characterizes the approach of the system towards the presented steady states it is sufficient to examine the simple detuning profile leading to $|S^0\rangle$, since the spectral properties of the Liouvillian are invariant under the transformations (11). Numerical calculations for small systems suggest that the preparation time for these states scales efficiently with the number of nodes $N$. In particular, the uniqueness of $|S^0\rangle$ implies that effects related to non-unique steady states found in related systems [35] are absent in our case.

3.3. Imperfections

Under realistic conditions various imperfections like onsite decays or losses in the waveguide can violate the exact dark state condition and the system then evolves to a mixed (‘bright’) steady state. This is exemplified by the dashed lines in figure 3 for the case of onsite decays, modelled by adding an additional term $\mathcal{L}' \rho = \kappa_0 \sum_i \mathcal{D}[\sigma_i^+] \rho$ to the ME (3). One clearly observes how the scattered intensity increases, while the purity drops as compared to the ideal case with $\kappa_0 = 0$. To study the entanglement properties in such non-ideal situations, we show in figures 4(a)–(d) the resulting steady state concurrence for various types of imperfections in a two-node network. In addition to onsite decays (a), we also consider intrinsic spin dephasing (b), waveguide losses (c), and small deviations from the ideal detuning profile (d). We see that different sources of imperfections lead to a qualitatively similar behaviour and that in all cases the entanglement is quite robust and optimized for intermediate driving strengths $\Omega$.

For larger networks, the scattering of photons from the first nodes due to imperfections also affects successive spins, as shown in figure 4(c) for a six-spin dimer chain $|S^0\rangle$ in the presence of onsite losses. One clearly observes that the bi-partite entanglement in the dimers decreases with increasing number of previous nodes. To study the robustness of genuine multi-partite entanglement in the presence of imperfections, we employ the entanglement measure proposed
Figure 4. Influence of imperfections on the entanglement properties of the steady state. (a–d) show the steady-state concurrence in a two-node network for various imperfections. (a) On-site decays (see text). (b) Dephasing of the spins modelled by additional Lindblad terms $\mathcal{L} \rho = \sum_{i} \frac{1}{2T_{2}} D[\sigma_{i}^{z}] \rho$. (c) Waveguide losses, where $\eta$ is the fraction of photons that gets lost between the nodes (modelled by adding a factor $\sqrt{1-\eta}$ to the last term in (2)). (d) Deviation from the asymmetric detuning condition modelled by a symmetric detuning offset $\delta_{1} = \delta_{2} = \epsilon$. In (a–c) we have chosen $\delta_{1} = \delta_{2} = 0$. (e) Influence of on-site decays on concurrences $C_{i,i+1}$ of reduced two-spin density matrices $\rho_{i,i+1}$ for $N = 6$ spins with $\delta_{i} = 0$. (f) Influence of on-site decays on the steady state four-partite entanglement in a four-spin network, quantified by the measure of [36], which is bounded by 0.5 in this case. The detuning profile is given by $\Delta' = (0, -\gamma/2, \gamma/2, 0)$.

in [36]. It can be evaluated in a straightforward way and figure 4(f) displays the results for a four-partite entangled steady state in the presence of onsite decays. The behaviour of this measure qualitatively agrees with the results for the concurrence in the two-spin case (cf figure 4(a)). That is, we observe a tradeoff between the maximal achievable entanglement and the robustness of the state. Finally, figure 3 also shows that for a fixed $N$, different bi- and multi-partite entangled states are affected equally.

4. Cascaded nonlinear cavities

The general concept of a coherent quantum absorber introduced in section 2 suggests that the formation of dark entangled states can also exist for systems other than spins. As a second
non-trivial example where this can be shown explicitly, we now discuss a setting where the two cascaded systems A and B are represented by two Kerr nonlinear cavities as depicted in figure 5(a). The resulting distribution scheme for continuous variable entanglement is an alternative to other cascaded settings considered in this context [37]. We denote the two bosonic cavity modes as $a$ and $b$, and the dynamics of the system is governed by the ME (3) with collective jump operator $c = a + b$ and cascaded Hamiltonian

$$H_{\text{casc}} = H_A + H_B - i\frac{\gamma}{2} (b^\dagger a - a^\dagger b). \quad (14)$$

Here, the Hamiltonian of the first cavity (system A, frequency $\omega_{c,A}$) is

$$H_A = \Delta a^\dagger a + Ka^\dagger a^\dagger a + i\Omega (a^\dagger - a), \quad (15)$$

where we have already moved to a frame rotating at the frequency $\omega_d$ of the external driving field, such that $\Delta = \omega_{c,A} - \omega_d$ is the corresponding detuning of the cavity frequency and $\Omega$ the associated driving strength. Further, $K$ denotes the strength of the Kerr nonlinearity. Motivated by our analysis of the cascaded spin system above, we assume that the Hamiltonian for the second cavity is given by

$$H_B = -\Delta b^\dagger b - Kb^\dagger b^\dagger b + i\Omega (b^\dagger - b). \quad (16)$$

Here we have chosen all constants to be identical to those used in $H_A$, such that the first two terms have opposite sign as compared to system A. As demonstrated below, this educated guess for $H_B$ ensures that the cascaded ME exhibits a dark state.

4.1. Steady-state solution

In order to show that the ME (3) indeed exhibits a pure steady state for the local Hamiltonians chosen above we once more exploit the conditions given in (4). From condition (I), i.e.
\((a + b)|\psi_0\rangle = 0\), it is clear that the steady state should contain zero quanta in the symmetric mode\(^6\). Therefore, it is convenient to change to symmetric and anti-symmetric modes \(c_\pm = (a \pm b)/\sqrt{2}\) and to write the dark state ansatz as \(|\psi_0\rangle = |0\rangle_+ \otimes |\chi_0\rangle_-\), with \(|\chi_0\rangle = \sum_n \alpha_n |n\rangle\), where \(|n\rangle\) are Fock states in the occupation number basis. In order to exploit condition (II) we rewrite the cascaded Hamiltonian in terms of the symmetric and anti-symmetric modes,

\[
H_{\text{casc}} = i\sqrt{2}\Omega c_+^\dagger + (\Delta - i\gamma/2 + K (\hat{N} - 1)) c_-^\dagger c_- + \text{H.c.}
\]

(17)

Here \(\hat{N} = c_+^\dagger c_+ + c_-^\dagger c_-\) is the total number of quanta, which is conserved by all terms except for those \(\propto \Omega\). Condition (II) is equivalent to \(H_{\text{casc}} |\psi_0\rangle = \lambda |\psi_0\rangle\), and by projecting this equation onto \(|\psi_0\rangle\) we see that it can only be fulfilled for \(\lambda = 0\) and hence

\[
\alpha_n = \sqrt{\frac{2}{n+1}} \frac{\epsilon}{\sqrt{n}} \alpha_{n-1}, \quad \epsilon = \frac{\Omega}{iK}, \quad x = \frac{i\Delta + \gamma/2}{iK}.
\]

(18)

This recursion is readily solved, such that the unique solution of conditions (I) and (II) is given by

\[
|\psi_0\rangle = |0\rangle_+ \otimes |\chi_0\rangle_- , \quad \text{with} \quad |\chi_0\rangle = \frac{1}{\mathcal{N}} \sum_{n=0}^{\infty} \frac{(\sqrt{2}\epsilon)^n}{\sqrt{n!}} \Gamma(x+n) |n\rangle.
\]

(19)

Here \(\mathcal{N} = \left[ \sum_{0}^{2} F_2(x, x^*; 2|\epsilon|^2) \right]^{1/2}\) and \(\sum_{0}^{2}\) denotes the generalized hyper-geometric function \([38]\).

Due to the simple tensor-product structure of \(|\psi_0\rangle\) its only non-vanishing normally ordered moments are those of the \(c_-\)-mode, i.e.

\[
\langle (c_-^\dagger)^n (c_-)^m \rangle = 2^{n+m} \frac{\Gamma(x+n) \Gamma(x+m)}{\Gamma(x+n+m)} \sum_{0}^{2} F_2(x^* + n, x + m; 2|\epsilon|^2),
\]

and the moments of the original modes thus read

\[
\langle (a^\dagger)^n (b^\dagger)^k (b)^l (a)^m \rangle = \frac{(-1)^{k+l}}{2^{(n+k+l+m)/2}} \langle (c_-^\dagger)^n (c_-)^m \rangle. \quad (20)
\]

Numerical studies suggest that there are no additional mixed steady states.

4.2. Entanglement properties of the steady state

Despite the simple structure of the steady state \(|\psi_0\rangle\) in the \(c_\pm\) representation, it is generally entangled when written in terms of the original modes \(a\) and \(b\) according to

\[
|\psi_0\rangle = \sum_{n,m=0}^{\infty} B_{nm} |n\rangle_A |m\rangle_B, \quad B_{nm} = \alpha_{n+m} \frac{(-1)^m}{2^{(n+m)/2}} \sqrt{\frac{(n+m)!}{n!m!}},
\]

(21)

where the \(\alpha_n\) can be read off from equation (19). We characterize the entanglement with respect to this bipartition by the mixedness of the reduced state \(\rho_A^0 = \text{tr}_B(|\psi_0\rangle\langle\psi_0|) = BB^\dagger\) of the first cavity, as measured by its linear entropy

\[
S_{\text{lin}} = 1 - \text{tr}(\rho_A^0)^2).
\]

(22)

\(^6\) As shown in [13], a pure stationary state could also exist if \(|\psi_0\rangle\) is an eigenstate of the jump operator, i.e. \((a + b)|\psi_0\rangle = \lambda'|\psi_0\rangle\) with \(\lambda' \in \mathbb{C}\). However, in the present example this would imply that the symmetric mode is in a coherent state of amplitude \(\lambda'\). We do not expect this due to the nonlinearity of the problem and it can indeed be shown that such an ansatz does not lead to a stationary solution unless \(\lambda' = 0\).
Figure 6. Characterization of the steady state $|\psi_0\rangle$ for $K/\gamma = 0.01$ (first row) and $K/\gamma = 0.5$ (second row). (a, b) Photon number in the first cavity normalized to resonant linear response. (c, d) Linear entropy $S_{\text{lin}}$ of the first cavity as defined in (22). In all panels the dashed lines indicate the critical driving strength $\Omega_c$, above which the semi-classical response becomes bi-stable (see, e.g., [39]).

The results for two different strengths of the nonlinearity are displayed in figure 6. For a better orientation, we show in (a) and (b) the photon number of the first cavity, normalized to the resonant response in the linear case, i.e. $\Delta = K = 0$. Here, one clearly observes the well-known behaviour of a single Kerr-nonlinear cavity, which is characterized by a deformation of the response curve for increasing $\Omega_c$ above a certain critical driving strength $\Omega_c$, the classical response becomes bi-stable, while the quantum mechanical response curve exhibits a sharp step $[33]$. From (c) and (d) we see that the regions of pronounced nonlinear response are those where the entropy is large, signalling a high degree of entanglement between the two cavities. In general, the features of the response are more washed out for larger nonlinearities, where also the regions of high entropy are more extended. Note that the state $|\chi_0\rangle$, and hence also $|\psi_0\rangle$, is generally non-Gaussian, as can also be shown by evaluating appropriate measures $[40]$. Finally, we remark that for $K \rightarrow 0$ the steady state approaches a product of coherent states, i.e. $|\psi_0\rangle \rightarrow |\beta\rangle_A \otimes |\beta\rangle_B$, with $\beta = \epsilon/\chi$, where $|\beta\rangle$ denotes a coherent state of amplitude $\beta$. We are thus able to recover the classical limit in which no entanglement persists between the cavities.
4.3. The inverse coherent absorber problem

Finally, let us take a different view on the analysis presented so far and emphasize once more that cascaded systems may be solved in a successive fashion ‘from left to right’. This means that tracing the full cascaded ME (3) over the second subsystem B yields a closed equation for the first subsystem A, which in this case reads

\[ \dot{\rho}_A = L_A \rho_A \equiv -i[\Delta a^\dagger a + K a^\dagger a a + i\Omega (a^\dagger - a), \rho_A] + \gamma D[a] \rho_A. \]  

(23)

In section 2 we assumed that the stationary mixed state solution of this equation including its spectral decomposition is known, which allowed us to explicitly construct a corresponding perfect absorber system B, such that the whole system is driven into a pure steady state. However, in many cases solving for the steady state \( \rho_A^0 \) of equation (23) is a non-trivial problem by itself, which in the present case was first accomplished using phase space methods [33]. Note that by using an educated guess for the Hamiltonian \( H_B \) of the absorber system and then solving for the pure steady state \( |\psi_0\rangle \) of the whole network, we have also indirectly solved the mixed-state problem of a single cavity by computing \( \rho_A^0 = \text{tr}_B (|\psi_0\rangle \langle \psi_0|) \). Its explicit matrix-elements in the Fock-state basis agree with the expressions found in the literature [41], as do its moments \( \langle (a^\dagger)^m (a)^n \rangle \) obtained as a special case of (20) [33]. The procedure presented above thus represents an elegant alternative way of obtaining the mixed steady state of the dispersive optical bi-stability problem.

Avoiding working with mixed states by using pure states on larger Hilbert spaces has found wide spread use in the quantum information community, one of the reasons being that pure states can be manipulated more easily [1]. From this point of view, the steady state \( |\psi_0\rangle \) possesses additional relevance as a purification of the density matrix \( \rho_A^0 \). This is an interesting result in itself, since obtaining such a purification generally requires knowledge of the spectral decomposition of \( \rho_A^0 \)—a strong requirement even if \( \rho_A^0 \) is known explicitly. In the above example we were able to circumvent this difficulty by directly constructing the purification \( |\psi_0\rangle \) as a steady state of the cascaded ME. Although so far this procedure is limited to systems where the corresponding absorber Hamiltonian \( H_B \) can be obtained by an educated guess, it is intriguing to think about cascaded quantum systems also as an analytic tool for calculating stationary states of non-trivial open quantum systems.

5. Implementations

The two key ingredients for realizing cascaded networks as discussed in the previous sections are (i) the implementation of low-loss non-reciprocal devices for directional routing of photons and (ii) achieving a coupling of single quantum systems to a 1D waveguide that exceeds local decoherence channels. In the following, we discuss several implementations that fulfill these requirements, where a focus lies on optical and microwave setups. Requirement (i) is crucial for realizing the unidirectional coupling between the nodes and can in principle be fulfilled by standard circulators based on the Faraday effect (see, e.g., [42] for an optical implementation), but also by exploiting unidirectional edge modes in media with broken time-reversal symmetry [25–27]. However, in recent years there has been increasing interest in designing non-reciprocal on-chip devices which are integrable with, e.g. microwave circuitry [31, 32] or nano-fabricated photonic components [29, 30]. Such elements would allow one to build up cascaded networks in a very controlled way in both the optical and the
Figure 7. Possible realizations of a single node of a cascaded spin network as shown in figure 1. (a) Generic setup where the unidirectional coupling is achieved by a cavity connected to a circulator, (b) non-reciprocal superconducting circuit based on the proposal of [31], (c) optomechanical transducer based on toroidal cavities [43, 44].

microwave domain, and thus constitute a promising way of implementing the scenarios discussed in this work.

Concerning the second requirement, we first discuss the case of the cascaded spin network presented in section 3. Realizing a coupling of TLSs to a 1D waveguide could, e.g. be achieved along the lines of the experiments reported in [45, 46], where superconducting qubits or atoms are coupled to transmission lines or hollow core optical fibres, respectively. However, one can also do without a direct coupling of the TLS to the waveguide by using the generic and flexible approach depicted in figure 7(a). Here, the TLS is coupled to a cavity, whose output port is connected to a circulator or another non-reciprocal device. In the bad cavity limit \( \kappa \gg g \), where \( g \) is the TLS–cavity coupling, a series of these nodes results in the desired model (2), with an effective TLS–waveguide decay rate \( \gamma = 2g^2/\kappa \). A particular realization of this scheme in the context of circuit cavity quantum electrodynamics [31, 47] is shown in figure 7(b). Finally, note that other systems like optomechanical transducers (figure 7(c)) have been proposed to realize a similar unidirectional coupling [43, 44]. In the case of cascaded cavities discussed in section 4, the most important requirement is the realization of strong TLS-type nonlinearities, which are of equal magnitude and opposite sign in the two cavities. Such a tunable nonlinearity can be realized by a dispersive coupling of the cavity field to a suitable four-level system, which has been analyzed both for optical [48, 49] as well as superconducting microwave cavities [50].

In summary, we find that the current experimental capabilities for realizing strong TLS–photon interactions in various systems, combined with the development of novel non-reciprocal devices in the optical and microwave domain, enable the implementation and design of various cascaded quantum networks. Here, the dissipative state preparation schemes described in this work could serve as an interesting application exploring the unconventional physical properties of such devices.

6. Conclusions

We have shown that photon emission and coherent reabsorption processes in cascaded quantum systems can lead to the formation of pure and highly entangled steady states. In the case of spin networks, this mechanism provides a tunable dissipative preparation scheme for a whole class of
multi-partite entangled states, and instances of this scheme might serve as a basis for dissipative quantum communication protocols [17]. For the case of two cascaded nonlinear cavities we have identified a dissipative preparation scheme for non-Gaussian entangled states, and we have illustrated how cascaded systems can serve as a new analytic tool to evaluate the stationary states of driven-dissipative systems. More generally, our findings show that such driven cascaded networks realize a novel type of non-equilibrium quantum many-body system, which can be implemented with currently developed integrated optical systems or superconducting devices.

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Appendix A. General construction of coherent quantum absorbers

We present the arguments leading to the results quoted in section 2.2. Given a system A in terms of its Hamiltonian $H_A$ and jump operator $c_A$, we seek a suitable system B described by $H_B$ and $c_B$ which perfectly reabsorbs the output field of A in steady state. In the following, we construct system B by requiring that the total cascaded system evolves into a pure steady state. The dynamics of the network is governed by the ME (3) with Hamiltonian

$$H_{casc} = H_A + H_B - i\frac{\gamma}{2}(c_A c_B^\dagger - c_A^\dagger c_B),$$

(A.1)

and collective jump operator $c = c_A + c_B$, and assuming that $c$ has no eigenvectors a pure state $|\psi_0\rangle$ is a stationary state of (3) if and only if [13]

$$c |\psi_0\rangle = 0, \quad [H_{casc}, |\psi_0\rangle\langle\psi_0|] = 0.$$  

(A.2)

As discussed in section 2.2, the cascaded nature of the interaction allows us to solve for the reduced steady state $\rho_0^A$ of system A without knowing anything about B. Assuming uniqueness, we introduce its spectral decomposition

$$\rho_0^A = \sum_k p_k |k\rangle\langle k|,$$

(A.3)

and assume that the eigenvalues $p_k$ are positive and non-degenerate. A potential pure steady state $|\tilde{\psi}_0\rangle$ of the whole system can then be written as a purification of $\rho_0^A$, i.e. $|\tilde{\psi}_0\rangle = \sum_k \sqrt{p_k} |k\rangle_A \otimes |\tilde{k}\rangle_B$ [1]. Here, $|\tilde{k}\rangle = V |k\rangle$, and $V$ denotes an arbitrary unitary and we assume subsystem B to have the same Hilbert space dimension as A.

In the following, we write out the tensor-products more explicitly, such that the dark-state condition $c |\psi_0\rangle = 0$ reads

$$(c_A \otimes 1 + 1 \otimes c_B) |\psi_0\rangle = 0,$$

(A.4)

with

$$c_A = \sum_{n,m} \langle n | c_A | m \rangle |n\rangle \langle m|, \quad c_B = \sum_{n,m} \langle \tilde{n} | c_B | \tilde{m} \rangle |\tilde{n}\rangle \langle \tilde{m}|.$$

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To proceed, we plug the ansatz for $|\psi_0\rangle$ into equation (A.4), yielding
\[
\sum_{k,n} \sqrt{p_k} (\langle n|c_A |k\rangle |n\rangle \otimes |\tilde{k}\rangle + \langle \tilde{n}|c_B |k\rangle |\tilde{n}\rangle \otimes |n\rangle) = 0,
\]
and after relabelling the indices we obtain
\[
\sum_{k,n} (\sqrt{p_k} |n\rangle |c_A |k\rangle + \sqrt{p_n} |\tilde{k}\rangle |c_B |\tilde{n}\rangle) |k\rangle \otimes |\tilde{n}\rangle = 0.
\] (A.5)
Since the $|k\rangle$ form an orthonormal basis, this condition is equivalent to
\[
\sqrt{p_n} |\tilde{k}\rangle |c_B |\tilde{n}\rangle = -\sqrt{p_k} |n\rangle |c_A |k\rangle,
\] (A.6)
which fixes the operator $c_B$ to be
\[
c_B = -\sum_{n,m} \frac{p_n}{p_m} |m\rangle |c_A |n\rangle \langle \tilde{n}| \langle \tilde{m}|.
\] (A.7)

To construct the Hamiltonian $H_B$ for system B we exploit the second condition $[H_{\text{casc}}, |\psi_0\rangle \langle \psi_0|] = 0$ needed for a pure steady state, which is equivalent to $H_{\text{casc}} |\psi_0\rangle = \lambda |\psi_0\rangle$ for $\lambda \in \mathbb{R}$. Note that $|\psi_0\rangle = 0$ implies
\[
i \frac{\gamma}{2} (c_A \otimes c_B^\dagger - c_A^\dagger \otimes c_B) |\psi_0\rangle = i \frac{\gamma}{2} (c_A^\dagger c_A \otimes 1 - 1 \otimes c_B^\dagger c_B) |\psi_0\rangle,
\]
such that this condition reads
\[
(H_{A,\text{eff}} \otimes 1 + 1 \otimes H_{B,\text{eff}}^\dagger) |\psi_0\rangle = \lambda |\psi_0\rangle,
\] (A.8)
where we have introduced the effective non-Hermitian Hamiltonians $H_{j,\text{eff}} = H_j - i \frac{\gamma}{2} c_j^\dagger c_j$. Since a finite $\lambda$ would only lead to a global shift of $H_B$ below, we can assume $\lambda = 0$ without loss of generality. We write the Hamiltonian as
\[
H_B = \sum_{n,m} \langle \tilde{n}|H_B |\tilde{n}\rangle |n\rangle \langle \tilde{n}| m\rangle,
\] (A.9)
and to determine the matrix elements in this expansion we start from equation (A.8) and proceed as for the dark state condition (A.4) with the replacements $c_A \rightarrow H_{A,\text{eff}}$ and $c_B \rightarrow H_{B,\text{eff}}^\dagger$. As an intermediate result this yields
\[
\langle \tilde{n}|H_B |\tilde{m}\rangle = -\sqrt{p_n} \frac{p_m}{p_m} |m\rangle \langle H_{A,\text{eff}} |n\rangle - i \frac{\gamma}{2} \langle \tilde{n}|c_B^\dagger c_B |\tilde{m}\rangle.
\] (A.10)
To express the right-hand side of this equation fully in terms of operators on A, we employ the identity
\[
\langle \tilde{n}|c_B^\dagger c_B |\tilde{m}\rangle = \sqrt{\frac{1}{p_m p_n}} \langle m|c_A \rho_A^0 c_A^\dagger |n\rangle,
\] (A.11)
which can be derived with the help of equation (A.6), and then make use of the stationarity of system A,
\[
\gamma c_A \rho_A = i (H_{A,\text{eff}} \rho_A^0 - \rho_A^0 H_{A,\text{eff}}^\dagger).
\] (A.12)
Equation (A.10) then becomes
\[
\langle \tilde{n}|H_B |\tilde{m}\rangle = -\frac{1}{2} \left( \sqrt{\frac{p_n}{p_m}} \langle m|H_{A,\text{eff}} |n\rangle + \sqrt{\frac{p_m}{p_n}} \langle m|H_{A,\text{eff}}^\dagger |n\rangle \right),
\] (A.13)
which determines the Hamiltonian of system B. The expressions (A.7) and (A.13) are the results quoted in section 2.2 of the main text.
Appendix B. Dark states of cascaded spin networks

We provide details regarding the cascaded spin network presented in section 3. The system is described by the ME (3) with collective jump operator \( c = \sum_i \sigma_i^+ \) and \( H_{\text{casc}} \) defined in equation (8).

B.1. Uniqueness of the steady state \(|S^0\rangle\)

We show by explicit construction that the state \(|S^0\rangle\) given in equation (10) is the unique steady state of the network, provided that \( \delta_{2i-1} = -\delta_{2i} \) and \( \Omega_i = \Omega \) for \( i = 1, 2, \ldots \). To do so, we exploit the fact that the cascaded interaction allows for a successive construction of the steady state and start with the case \( N = 2 \). In this case, the steady state \( \rho_2^0 \) is obtained by solving \( \mathcal{L}^{(2)} \rho_2^0 = 0 \), where \( \mathcal{L}^{(2)} = L_{12} \) is given by the block-wise Liouvillian

\[
L_{i,j+1} \rho = -i \left[ \frac{\delta_i}{2} (\sigma_z^i - \sigma_z^{i+1}) + \Omega (\sigma_x^i + \sigma_x^{i+1}), \rho \right] + \frac{\gamma}{2} D[c_{i,j+1}] \rho
\]

with \( c_{i,j} = \sigma_j^+ + \cdots + \sigma_j^0 \). We have already seen in the main text that a solution is given by \( \rho_2^0 = |S_{1i}\rangle \langle S_{1i}| \), and its uniqueness can be shown by calculating the characteristic polynomial of \( L_{12} \) and realizing that there is only one zero eigenvalue for \( \gamma > 0 \).

We continue with \( N = 4 \) and write the ansatz for the steady state as \( \rho_4^0 = |S_{i}\rangle \langle S_{i}| \otimes \mu \), where \( \mu \) is a two-node density matrix. The four-node Liouvillian can be rewritten as

\[
\mathcal{L}^{(4)} \rho = L_{12} \rho + L_{34} \rho - \gamma ([c_{34}^\dagger, c_{12} \rho] + [\rho c_{12}^\dagger, c_{34}]),
\]

and we note that \( L_{12} |S_{i}\rangle \langle S_{i}| = 0 \) as well as \( c_{12} |S_{i}\rangle = 0 \), such that the equation \( \mathcal{L}^{(4)} \rho_4^0 = 0 \) simplifies to \( L_{34} \mu = 0 \). However, this is just the two-node problem we have already solved and the unique solution for the four-node network is thus given by \( \rho_4^0 = |S_{i}\rangle \langle S_{i}| \otimes |S_{i}\rangle \langle S_{i}| \). By iterating this argument in blocks of two spins, we obtain the steady state \(|S^0\rangle\) given in (10).

B.2. Unitary form invariance of the master equation

We briefly demonstrate that the cascaded Hamiltonian of the spin network is form-invariant under the unitary transformations (11) as stated in (12). In order to calculate \( U_i H_{\text{casc}} U_i^\dagger \), we write \( j = i + 1 \) for brevity and rearrange \( H_{\text{casc}} \) as follows:

\[
H_{\text{casc}} = \sum_{k \neq i,j} \left( \frac{\delta_i}{2} \sigma_z^k + \Omega \sigma_z^k \right) - \frac{\gamma}{2} \sum_{k \neq i,j} (\sigma_z^k \sigma_-^l - \sigma_-^k \sigma_z^l) - \frac{\gamma}{2} (c_{i,j}^\dagger c_{i-1,j} + c_{j+1,i}^\dagger c_{i,j}) - \text{H.c.}
\]

\[
+ \frac{\delta_i + \delta_j}{2} (\sigma_z^i + \sigma_z^j) + \Omega (\sigma_x^i + \sigma_x^j) + \left[ \frac{\delta_i - \delta_j}{2} (\sigma_z^i - \sigma_z^j) - \frac{\gamma}{2} (\sigma_z^i \sigma_-^l - \sigma_-^i \sigma_z^l) \right],
\]

where we have again used the piecewise jump operator \( c_{k,l} = \sigma_k^+ + \cdots + \sigma_l^0 \). Note that \( U_i(\theta) \) commutes with all terms except for those in brackets in the second line. We abbreviate the operators appearing there by \( A = (\sigma_z^i - \sigma_z^j)/2 \) and \( B = \sigma_x^i \sigma_-^l - \sigma_-^i \sigma_x^l \) and observe that they transform into one another:

\[
U_i(\theta) A U_i^\dagger(\theta) = A \cos 2\theta + iB \sin 2\theta,
\]

\[
U_i(\theta) B U_i^\dagger(\theta) = B \cos 2\theta + iA \sin 2\theta.
\]
Introducing the difference of detunings \( \delta = (\delta_i - \delta_j)/2 \), a non-trivial form-invariance of the Hamiltonian \( H_{\text{casc}} \) under \( U_i(\theta) \) is realized if

\[
U_i(\theta) \left[ \delta A - i\gamma B \right] U_i^\dagger(\theta) = -\delta A - i\gamma B. 
\]

That is, we require the detunings to swap and the cascaded part to remain invariant. It is easy to check that this requirement results in two equations, which are both solved for the choice \( \tan \theta = -2\delta/\gamma = (\delta_{i+1} - \delta_i)/\gamma \), which was to be demonstrated.

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