ℓ-ADIC REPRESENTATIONS ASSOCIATED TO MODULAR FORMS OVER IMAGINARY QUADRATIC FIELDS

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ABSTRACT. Let π be a regular algebraic cuspidal automorphic representation of GL₂ over an imaginary quadratic number field K, and let ℓ be a prime number. Assuming the central character of π is invariant under the non-trivial automorphism of K, it is shown that there is a continuous irreducible ℓ-adic representation ρ of Gal(ℚ/K) such that $L(s, ρ_v) = L(s, π_v)$ whenever v is a prime of K outside an explicit finite set.

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1. Introduction

Let K be an imaginary quadratic field with non-trivial automorphism c, and let π be a cuspidal automorphic representation of GL₂(𝔦_K) with unitary central character ω. If $π_∞$ has Langlands parameter $W_ℂ = ℂ^× \rightarrow GL_2(ℂ)$ given by $z \mapsto \text{diag}(z^{1-k}, z^{-1-k})$ for some integer $k \geq 2$ (that is, in the sense of Clozel [3], π is any regular algebraic cuspidal automorphic representation up to twist), then by the Langlands philosophy π should give rise (for any prime number ℓ) to a continuous irreducible ℓ-adic representation ρ of the Galois group Gal(ℚ/K) such that the associated L-functions agree. In other words, at each prime v of K the Frobenius polynomial of ρ at v agrees with the Hecke polynomial of π at v. Under the assumption that ω = ω^c it is possible to relate π to holomorphic Siegel modular forms via theta lifts and deduce (using ℓ-adic cohomology on Siegel threefolds) some weak version of this predicted correspondence. In fact Taylor [13] managed to obtain the above equality of Frobenius and Hecke polynomials for all v outside a zero density set of places, but he had to make some additional technical assumptions. It is our aim here to describe how the results of Friedberg and Hoffstein [6] on the non-vanishing of certain central L-values and those of Laumon [8, 9] and Weissauer

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[15] on associating Galois representations to Siegel modular forms enable one to remove these technical assumptions and conclude the statement for all \( v \) outside an explicit finite set. Precisely, we shall prove the following

**Theorem 1.1.** Assume that \( \omega = \omega^c \). Let \( S \) denote the set of places in \( K \) which divide \( \ell \) or where \( K/Q \) or \( \pi \) or \( \pi^c \) is ramified. There exists a continuous irreducible representation \( \rho: \text{Gal}(\overline{K}/K) \to \text{GL}_2(\overline{Q}) \) such that if \( v \) is a prime of \( K \) outside \( S \) then \( \rho \) is unramified at \( v \) and the characteristic polynomial of \( \rho(\text{Frob}_v) \) agrees with the Hecke polynomial of \( \pi \) at \( v \), that is \( L(s, \rho_v) = L(s, \pi_v) \).

**Remark 1.2.** This theorem strengthens Theorem A of [13]. The assumption \( \omega = \omega^c \) is inherent to Taylor’s method; one would hope to remove this condition by other methods.

**Remark 1.3.** By [11, Lemma 6] the Galois representation takes values in \( \text{GL}_2(E) \) for a finite extension \( E/Q_\ell \) (see [13, p. 635] for an explicit description).

**Remark 1.4.** There are two cases in which Theorem 1.1 is known to hold (cf. [13, Lemmata 1 and 2]):

1. \( \pi \otimes \delta \cong \pi \) for some nontrivial quadratic character \( \delta \) of \( K \). In this case if \( L/K \) denotes the quadratic extension corresponding to \( \delta \) then there is an algebraic idèle class character \( \psi \) of \( L \) unequal to its conjugate under the non-trivial element of \( \text{Gal}(L/K) \) such that \( \pi \) is the automorphic induction of \( \psi \) to \( K \); the conclusion of Theorem 1.1 follows by work of Serre [10].
2. \( \pi \otimes \nu \cong (\pi \otimes \nu)^c \) for some finite order character \( \nu \) of \( K \). In this case a twist of \( \pi \) is a base change from \( Q \); the conclusion of Theorem 1.1 follows from work of Deligne [4].

The proof of Theorem 1.1 can be briefly outlined as follows. The initial strategy is that of Taylor [13]. We assume we are not in a case covered by Remark 1.4. Using the deep results of [7] and [6] we construct a nonzero theta lift on \( \text{GSp}_4(\mathcal{A}_Q) \) of the twist \( \pi \otimes \mu \) for a dense set of quadratic idèle class characters \( \mu \) of \( K \) (see Definition 2.1 below). The irreducible constituent \( \Pi^\mu \) of such a lift is generated by a vector-valued holomorphic semi-regular cusp form on the Siegel three-space. Using Hasse invariant forms and the theory of pseudo-representations developed by Wiles [16] and Taylor [11, 12], Taylor had shown that one can associate a 4-dimensional representation to \( \Pi^\mu \) if one could associate 4-dimensional Galois representations to regular holomorphic Siegel cusp forms. This is now possible by work of Laumon [8, 9] and Weissauer [15]. We obtain therefore, for each \( \mu \) in some dense set, a 4-dimensional representation of \( \text{Gal}(\overline{Q}/Q) \) with the same partial \( L \)-function as the one associated to \( \Pi^\mu \), and we prove that it is induced from some 2-dimensional representation \( \rho^\mu \) of \( \text{Gal}(\overline{K}/K) \). By exploring global compatibility relations among the various \( \rho^\mu \) we show that they can be replaced by quadratic twists\(^2\) \( \rho \otimes \mu \) of a single 2-dimensional representation \( \rho \) of \( \text{Gal}(\overline{K}/K) \), and we verify that this \( \rho \) has the required property of Theorem 1.1.

**Remark 1.5.** Since the construction of the 4-dimensional Galois representation associated to \( \Pi^\mu \) involves an \( \ell \)-adic limit process one loses information about the geometricity of the Galois representation. For \( \ell \) split in \( K/Q \), Urban [14, Corollaire

\(^1\)Here and later we shift-normalize all \( L \)-functions such that \( s = 1/2 \) is their center.

\(^2\)Here and later we identify finite order idèle class characters with continuous Galois characters.
2. Theta lifts

Theorem 1 and Proposition 5 of [7] show how to construct non-zero theta lifts on $\text{GSp}_4$ of many quadratic twists of $\pi$, conditional on “Conjecture/Theorem 1” [7, p. 403]. The analytic non-vanishing result of [6] implies “Conjecture/Theorem 1”. For completeness, we decided to summarize how [6] implies a strengthening of Theorem 1 of [7].

By assumption the central character of $\pi$ factors through the norm map as $\omega = \tilde{\omega} \circ N_{K/Q}$, where $\tilde{\omega}$ is a character of $Q$. The ratio of the two characters $\omega$ satisfying this equation is the quadratic character corresponding to $K/Q$, hence one of them is odd and the other one is even. We shall consider the character $\omega$ with $\tilde{\omega}_\infty(-1) = (-1)^k$. By Proposition 1 of [7] the pair $(\pi, \tilde{\omega})$ defines a cuspidal automorphic representation of $\text{GO}(\mathbb{A}_Q)$, where $\text{GO}$ is the group of orthogonal similitudes of a certain quadratic space $W_K$ of sign $(3, 1)$ over $Q$. [7] also introduces a signature $\delta = (\delta_v)$, a map from the places of $Q$ to $\{\pm 1\}$ which is 1 at all but finitely many places such that $\delta_v = 1$ whenever $\pi_v \not\cong \pi_v^c$ (here we view $\pi$ as a representation of $R_{K/Q} \text{GL}_2$, the group obtained from $\text{GL}_2$ by restriction of scalars).

By Proposition 2 of [7] the triple $\hat{\pi} := (\pi, \tilde{\omega}, \delta)$ modulo the action of $\{1, c\}$ can be identified with a cuspidal automorphic representation of $\text{GO}(\mathbb{A}_Q)$ which in turn has a theta lift $\Theta(\hat{\pi})$ to $\text{GSp}_4(\mathbb{A}_Q)$. By Proposition 3 of [7] the lift $\Theta(\hat{\pi})$ is contained in the space of cusp forms, and if $\Pi$ denotes an irreducible constituent of $\Theta(\hat{\pi})$ then $\Pi_\infty$ is a holomorphic limit of discrete series representation of weight $(k, 2)$ whenever $\delta_\infty = -1$, while $\Pi_v$ is an unramified irreducible principal series representation with $L(s, \Pi_v) = L(s, \pi_v)$ whenever $\delta_v = 1$ and $v$ is a rational prime which does not lie under a prime in $S$. In addition, $\Pi$ is nonzero assuming there is a character $\varphi$ of $K$ restricting to $\tilde{\omega}$ on $Q$ and satisfying the following two properties:

\[ \pi_v \cong \pi_v^c \quad \Rightarrow \quad \delta_v = \tilde{\omega}_v(-1)\varepsilon(\pi_v \otimes \varphi_v^{-1}, 1/2); \]

\[ L(\pi \otimes \varphi^{-1}, 1/2) \neq 0. \]

Using these as preliminaries we can deduce from the non-vanishing result [6] that $\pi \otimes \mu$ gives rise to suitable $\Pi^\mu$ on $\text{GSp}_4(\mathbb{A}_Q)$ for a dense set of quadratic characters $\mu$ of $K$.

**Definition 2.1.** A set $\mathcal{M}$ of quadratic characters of $K$ is dense if it has the following property. If $\tilde{\mu}$ is a quadratic character of $K$ and $M$ is a finite set of rational primes then there is a character $\mu \in \mathcal{M}$ such that $\mu_v = \tilde{\mu}_v$ for all $v \in M$.

**Definition 2.2.** For a cuspidal automorphic representation $\hat{\pi}$ of $K$ let $S_K(\hat{\pi})$ denote the set of places in $K$ which divide $\ell$ or where $K/Q$ or $\hat{\pi}$ or $\hat{\pi}^c$ is ramified, and let $S_Q(\hat{\pi})$ denote the set of rational places which lie under some place in $S_K(\hat{\pi})$.

**Theorem 2.3.** There exists a dense set $\mathcal{M}$ of quadratic characters of $K$ with the following property. For each $\mu \in \mathcal{M}$ there is a signature $\delta$ such that the representation $(\pi \otimes \mu, \tilde{\omega}, \delta)$ of $\text{GO}(\mathbb{A}_Q)$ gives rise to a cuspidal automorphic representation $\Pi^\mu$ of $\text{GSp}_4(\mathbb{A}_Q)$ satisfying:

- $\Pi^\mu_\infty$ is a holomorphic limit of discrete series representation of weight $(k, 2)$;
- if $v$ is a rational prime outside $S_Q(\pi \otimes \mu)$ then $\Pi^\mu_v$ is an unramified irreducible principal series representation with $L(s, \Pi^\mu_v) = L(s, (\pi \otimes \mu)_v)$. 
To prove this let $\tilde{\mu}$ be a quadratic character of $K$ and let $M$ be a finite set of rational primes. In the light of the above discussion (i.e. by Propositions 1–3 of [7]) it suffices to show that for $\tilde{\pi} := \pi \otimes \tilde{\mu}$ there exist a quadratic character $\eta$ of $K$ with $\eta_v = 1$ for all $v \in M$, a signature $\delta$ with $\delta_{\infty} = -1$ and $\delta_v = 1$ for any rational prime $v \notin S_Q(\tilde{\pi} \otimes \eta)$, and a character $\varphi$ of $K$ with $\varphi|_Q = \tilde{\omega}$, satisfying the additional properties

$$(2.1) \quad \delta_v = \begin{cases} \tilde{\omega}_v(-1)\varepsilon(\tilde{\pi}_v \otimes \eta_v \varphi_v^{-1}, 1/2) & \text{if } \tilde{\pi}_v \otimes \eta_v \cong (\tilde{\pi}_v \otimes \eta_v)^c, \\ 1 & \text{if } \tilde{\pi}_v \otimes \eta_v \not\cong (\tilde{\pi}_v \otimes \eta_v)^c; \end{cases}$$

$$(2.2) \quad L(\tilde{\pi} \otimes \eta \varphi^{-1}, 1/2) \neq 0. \quad \text{The proofs of Lemma 13 and Proposition 5 in [7] provide us with } \eta \text{ and } \varphi \text{ satisfying}$$

$$(2.3) \quad \varepsilon(\pi_{\infty} \otimes \varphi_{\infty}^{-1}, 1/2) = -\tilde{\omega}_{\infty}(-1)$$

and

$$\varepsilon(\tilde{\pi} \otimes \eta \varphi_{\infty}^{-1}, 1/2) = 1. \quad \text{Here we used our initial assumptions for } \pi \text{ and } \tilde{\omega}. \quad \text{Theorem A and the first part of Theorem B in [6] show that } \eta \text{ can be replaced by another quadratic character satisfying (2.2). Now we define } \delta \text{ according to (2.1).} \quad \text{By [7, Lemma 14], } \delta_v = \pm 1 \text{ for all rational primes, and } \delta_{\infty} = 1 \text{ for all rational primes } v \notin S_Q(\tilde{\pi} \otimes \eta). \quad \text{In addition, } \delta_{\infty} = -1 \text{ holds by (2.3) combined with } \tilde{\mu}_{\infty} \eta_{\infty} = 1 \text{ and } \pi_{\infty} \cong \pi_{\infty}. \quad$$

3. 4-DIMENSIONAL GALOIS REPRESENTATIONS OF $\mathbb{Q}$

In the previous section we constructed, for each quadratic character $\mu$ of $K$ in some dense set $\mathcal{M}$, a cuspidal automorphic representation $\Pi^\mu$ of $\text{GSp}_4(\mathbb{A}_Q)$ such that $\Pi^\mu_{\infty}$ is a holomorphic limit of discrete series representation of weight $(k, 2)$. It has the property

$$L^{S(\mu)}(s, \Pi^\mu) = L^{S(\mu)}(s, I^Q_K(\pi \otimes \mu)), \quad \text{where } S(\mu) \text{ abbreviates } S_Q(\pi \otimes \mu), \quad \text{and } I^Q_K \text{ stands for automorphic induction. Note that } S(\mu) \text{ includes all the rational primes where } \Pi^\mu \text{ is ramified.}$$

In this section we shall construct, for each $\mu \in \mathcal{M}$, a continuous semisimple representation

$$\tau^\mu : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_4(\overline{\mathbb{Q}}_\ell)$$

such that

$$L^{S(\mu)}(s, \tau^\mu) = L^{S(\mu)}(s, \Pi^\mu).$$

In other words, we shall show that $\pi \otimes \mu$ is associated to a Galois representation over $\mathbb{Q}$. We shall rely on the following deep result of Weissauer [15] (see also the closely related work of Laumon [8, 9]):

**Theorem 3.1.** Let $\Pi$ be an irreducible cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A}_Q)$ such that $\Pi_{\infty}$ belongs to the holomorphic discrete series of weight $(k_1, k_2)$ with $k_1 \geq k_2 \geq 3$. Let $S$ denote the union of $\{\ell\}$ and the set of rational primes where $\Pi$ is ramified. There exists a continuous semisimple representation

$$\tau : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_4(\overline{\mathbb{Q}}_\ell)$$

such that if $v$ is a rational prime outside $S$ then $\tau$ is unramified at $v$ and the characteristic polynomial of $\tau(Frob_v)$ agrees with the Hecke polynomial of $\Pi$ at $v$. In other words,

$$L^S(s, \tau) = L^S(s, \Pi),$$

where $L^S$ denotes the product of local $L$-factors outside $S$.

This theorem was not available for Taylor in [13]; instead, he utilized the weaker yet powerful results of [12] to conclude $L^S(s, \tau) = L^S(s, \Pi)$ for some exceptional set $S$ of zero Dirichlet density. While the above theorem is not directly applicable to the representations $\Pi^\mu$ we can combine it with Taylor’s method of pseudo-representations [11] to achieve our goal. Our situation is analogous to associating 2-dimensional Galois representations to elliptic cusp forms of weight 1 which was accomplished by Deligne–Serre in the classical paper [5] by a technique involving lifting the weight and then applying a “horizontal” family of congruences.

Let $f$ be a Hecke eigenform belonging to $\Pi^\mu$: it is a vector-valued holomorphic semi-regular Siegel cusp form of weight $(k, 2)$ and some level $N$ (i.e. $\Pi^\mu$ is ramified exactly at the primes dividing $N$). Let $\mathcal{H}_0^N(\mathbb{Z})$ be the $\mathbb{Z}$-algebra generated by the Hecke operators corresponding to primes not dividing $N$ and denote by $\overline{T}_{(k_1, k_2)}(N)$ the image of $\mathcal{H}_0^N(\mathbb{Z})$ in the space of holomorphic Siegel cusp forms of weight $(k_1, k_2)$ and level $N$ (see [11, p. 315] for precise definitions). It is known that $\overline{T}_{(k_1, k_2)}(N) \otimes \mathbb{Q}$ is a semisimple $\mathbb{Q}$-algebra. In particular, we have a homomorphism $\lambda_f : \overline{T}_{(k_2)}(N) \to \mathcal{O}_f$ such that $T(f) = \lambda_f(T)f$ for all $T \in \mathcal{H}_0^N(\mathbb{Z})$, where $\mathcal{O}_f$ is the ring of integers of some (minimally chosen) number field $E_f$.

Using the cup product of $f$ with the $\ell^n$-th power of the “Hasse Invariant” form exhibited by Blasius and Ramakrishnan [2, Proposition 3.6], Taylor [11, Proposition 3] constructs a “vertical” family of morphisms $\lambda_{n, f}$ such that

$$\lambda_{n, f}(T) = \lambda_f(T) \mod \ell^{n+1}, \quad T \in \mathcal{H}_0^N(\mathbb{Z}).$$

Here $m = m(\ell)$ is a positive integer and $n$ is an arbitrary positive integer. Together with Theorem 3.1 this allows us to apply the theory of pseudo-representations, as in Example 1 in [11, §1.3], to piece together the required Galois representation $\tau^\mu$ corresponding to $\Pi^\mu$.

4. 2-DIMENSIONAL GALOIS REPRESENTATIONS OF $K$

We have exhibited, using previous notation, continuous semisimple representations

$$\tau^\mu : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_4(\overline{\mathbb{Q}}_\ell), \quad \mu \in \mathcal{M},$$

satisfying

$$L^S(\mu)(s, \tau^\mu) = L^S(\mu)(s, \text{I}^\mu_\mathbb{K}(\pi \otimes \mu)).$$

3Ramakrishnan has pointed out to us that there is a mistake in [2], but it does not affect the part we are using.

4We assume here that $N \geq 3$, otherwise we replace $N$ by $N\ell^2$, say.
Note that $S(\mu)$ includes all the rational primes where $\Pi^\mu$ is ramified. If $\chi$ denotes the quadratic character of $\mathbb{Q}$ corresponding to $K$, then we have, by the reciprocity formula,

$$L^{S(\mu)}(s, I^Q_K(\pi \otimes \mu) \otimes \chi) = L^{S(\mu)}(s, I^Q_K(\pi \otimes \mu)),$$

since $\chi$ has trivial restriction to $K$. This implies, by (4.1) and the Chebotarev density theorem, that

$$(5.2) \quad \tau^\mu \otimes \chi \cong \tau^\mu.$$

This in turn implies the following

**Lemma 4.1.** For each $\mu \in \mathcal{M}$ there is a continuous semisimple representation

$$\rho^\mu : \text{Gal}(\overline{K}/K) \to \text{GL}_2(\mathbb{Q}_\ell)$$

such that

$$\tau^\mu \cong \text{Ind}_K^Q(\rho^\mu).$$

**Proof.** Suppose first that $\tau^\mu$ is irreducible over $\mathbb{Q}$. Then $\text{Hom}(\chi, \tau^\mu \otimes (\tau^\mu)\vee) \neq 0$ by (4.2). Since $\tau^\mu \otimes (\tau^\mu)\vee = \text{End}(\tau^\mu)$, we see by Schur’s Lemma that $\tau^\mu|_K$ is reducible as $\chi|_K$ is trivial. Let $\rho^\mu$ be an irreducible component of $\tau^\mu|_K$ of minimal dimension (i.e. at most 2). By Frobenius reciprocity $\text{Hom}(\text{Ind}_K^Q(\rho^\mu), \tau^\mu) \neq 0$, hence in fact $\tau^\mu \cong \text{Ind}_K^Q(\rho^\mu)$ since $\tau^\mu$ is irreducible of dimension 4 and $\text{Ind}_K^Q(\rho^\mu)$ is of dimension at most 4.

Suppose now that $\tau^\mu$ is reducible over $\mathbb{Q}$. If $\lambda$ (resp. $\beta$) is a 1-dimensional (resp. 2-dimensional) representation occurring in $\tau^\mu$, then (4.2) shows that $\lambda \chi$ (resp. $\beta \otimes \chi$) also occurs in $\tau^\mu$. Hence there are four cases to consider:

1. $\tau^\mu \cong \beta \oplus (\beta \otimes \chi)$. Then $\tau^\mu \cong \text{Ind}_K^Q(\beta|_K)$.
2. $\tau^\mu \cong \beta \oplus \gamma$, where both $\beta$ and $\gamma$ are $\chi$-invariant. Then $\beta \cong \text{Ind}_K^Q(\kappa)$ and $\gamma \cong \text{Ind}_K^Q(\nu)$ for some 1-dimensional $\kappa$ and $\nu$, so that $\tau^\mu \cong \text{Ind}_K^Q(\kappa \oplus \nu)$.
3. $\tau^\mu \cong \beta \oplus \lambda \otimes \chi$. Then $\beta \cong \beta \otimes \chi$, so $\beta \cong \text{Ind}_K^Q(\kappa)$ for some 1-dimensional $\kappa$ and $\tau^\mu \cong \text{Ind}_K^Q(\kappa \otimes \lambda|_K)$.
4. $\tau^\mu \cong \lambda \otimes \lambda \chi \oplus \nu \otimes \nu \chi$. Then $\tau^\mu \cong \text{Ind}_K^Q(\lambda|_K \oplus \nu|_K)$.

$\square$

5. Compatibility of twists

So far we have constructed, for each quadratic character $\mu$ of $K$ in some dense set $\mathcal{M}$, a continuous semisimple representation

$$\rho^\mu : \text{Gal}(\overline{K}/K) \to \text{GL}_2(\mathbb{Q}_\ell)$$

such that

$$L^{S(\mu)}(s, \text{Ind}_K^Q(\rho^\mu)) = L^{S(\mu)}(s, I^Q_K(\pi \otimes \mu)),$$

where $S(\mu)$ abbreviates $S_\mathbb{Q}(\pi \otimes \mu)$ and both sides involve Euler factors of degree 4 over $\mathbb{Q}$. As we want to compare Euler factors over $K$ it is useful to rewrite the previous equation (using restriction and base change) as

$$(5.1) \quad L^{S(\mu)}(s, \rho^\mu)L^{S(\mu)}(s, (\rho^\mu)^\vee) = L^{S(\mu)}(s, \pi \otimes \mu)L^{S(\mu)}(s, (\pi \otimes \mu)^\vee),$$

where now $S(\mu)$ abbreviates $S_K(\pi \otimes \mu)$ and all $L$-functions involve Euler factors of degree 2 over $K$. Note that $S(\mu)$ includes all the rational primes where $\rho^\mu$ or $(\rho^\mu)^\vee$ is ramified.
Our aim is to show that the Galois representations $\rho^\mu$ are globally compatible in the sense that they can be replaced by twists $\rho \otimes \mu$ of some fixed $\rho$. This will be achieved in three lemmata. Recall our assumption that we are not in a case covered by Remark 1.4.

**Lemma 5.1.** $\rho^\mu|_L$ is irreducible for all $\mu \in \mathcal{M}$ and all quadratic extensions $L/K$.

**Proof.** Assume that $\rho^\mu|_L$ is reducible for some $\mu \in \mathcal{M}$ and some quadratic extension $L/K$. Let $\Psi$ be an irreducible summand of $\rho^\mu|_L$ and for a prime $v$ of $K$ outside $S(\mu)$ let $\{\alpha'_v, \beta'_v\}$ denote the Langlands parameters of $\pi \otimes \mu$. Implicit in the construction of $\rho^\mu$ is the fact that it has image in $GL_2(E)$ for a finite extension of $\mathbb{Q}_\ell$. In particular, $\Psi : \text{Gal}(\overline{T}/L) \to E^\times$ is a continuous character. Applying restriction and base change in (5.2) we see that if $w$ is a place of $L$ lying above a place $v$ of $K$ outside $S(\mu) \cup \text{disc}(L/K)$ then $\Psi(\text{Frob}_w) \in \{(\alpha'_v)^f, (\beta'_v)^f, (\alpha'_v)^f, (\beta'_v)^f\}$, where $f = (L_w : K_v)$. Hence in fact $\Psi(\text{Frob}_w)$ is either one of the Langlands parameters of the base changes $(\pi \otimes \mu)_L$ or $(\pi \otimes \mu)_L^\mu$ at $w$. Applying the results of [13, §3] we conclude that $(\pi \otimes \mu)_L$ is not cuspidal which by [13, Lemma 2] means that we are in Case 1 of Remark 1.4. This contradiction proves the lemma. □

**Lemma 5.2.** Let $\mu \in \mathcal{M}$ and let $\delta$ be a quadratic character of $K$.

1. $\rho^\mu \otimes \delta \not\cong \rho^\mu$ for $\delta$ nontrivial.
2. $\rho^\mu \otimes \delta \not\cong (\rho^\mu)^c$ in all cases.

**Proof.** Assume first that $\delta$ is nontrivial, and denote by $L/K$ the corresponding quadratic extension.

Assume that $\rho^\mu \otimes \delta \cong \rho^\mu$. Then $\text{Hom}(\delta, \rho^\mu \otimes (\rho^\mu)^\vee) \neq 0$. Since $\rho^\mu \otimes (\rho^\mu)^\vee = \text{End}(\rho^\mu)$, we see by Schur’s Lemma that $\rho^\mu|_L$ is reducible as $\delta|_L$ is trivial. This is a contradiction to Lemma 5.1 and establishes the first part of the lemma.

Assume that $\rho^\mu \otimes \delta \cong (\rho^\mu)^c$. Then $(\rho^\mu)^c \otimes \delta^c \cong \rho$, hence in fact $\rho^\mu \otimes (\delta^c)^c \cong \rho$. By the first part of the lemma this forces $\delta^c = 1$, hence $\delta = \delta^c$ since $\delta$ is quadratic. This implies, using (5.2), that

\[
L^T(s, (\rho^\mu \otimes \delta)L^T(s, (\pi \otimes \mu \delta)L^T(s, (\pi \otimes \mu \delta)^c),
\]

where $T$ is a finite set of primes in $K$ containing $S(\mu)$. Using again the assumption $\rho^\mu \otimes \delta \cong (\rho^\mu)^c$ we obtain

\[
L^T(s, (\rho^\mu)^c)L^T(s, \rho^\mu) = L^T(s, (\pi \otimes \mu \delta)L^T(s, (\pi \otimes \mu \delta)^c),
\]

hence by (5.2), multiplicity one, and base change, we have in fact

\[
I^Q_K(\pi \otimes \mu \delta) \cong I^Q_K(\pi \otimes \mu).
\]

Base changing this to $K$ and comparing the cuspidal representations (in the isobaric sums), we are forced to have

\[
(5.3) \quad \pi \otimes \mu \delta \cong (\pi \otimes \mu)^c,
\]

since $\pi \otimes \mu \delta \cong \pi \otimes \mu$ falls under Case 1 of Remark 1.4. Here $\delta = \delta^c$, so there is a quadratic character $\epsilon$ of $\mathbb{Q}$ such that $\delta = \epsilon|_K$. Regarding $\delta$ and $\epsilon$ as idèle class characters, $\delta$ is the pull-back of $\epsilon$ by the norm map $N_{K/\mathbb{Q}}$ and so its restriction to the idèle classes of $\mathbb{Q}$ is the trivial character. Regarding $\delta$ as Galois character this means that its transfer to $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is trivial. Now a standard theorem on
Galois cohomology of idèles (cf. proof of [13, Lemma 1]) implies that we may write \( \delta = (\nu/\nu^c) \) for some character \( \nu \) of \( K \). Plugging this in (5.3) we obtain
\[
\pi \otimes \mu \nu \cong (\pi \otimes \mu \nu)^c,
\]
hence we are in Case 2 of Remark 1.4. This contradiction establishes the second part of the lemma for nontrivial \( \delta \).

It remains to prove the lemma for trivial \( \delta \), i.e. that \( \rho^\mu \not\cong (\rho^\mu)^c \). However, this is immediate from (5.2) and the multiplicity one theorem since we are assuming that \( \pi \otimes \mu \not\cong (\pi \otimes \mu)^c \).

\[\square\]

**Lemma 5.3.** There is a continuous semisimple representation
\[\rho : \text{Gal}(\mathcal{K}/K) \to GL_2(\mathcal{O}_K)\]
which is unramified outside \( S \) and for all \( \mu \in \mathcal{M} \) satisfies
\[\rho^\mu \oplus (\rho^\mu)^c \cong (\rho \otimes \mu) \oplus (\rho \otimes \mu)^c.\]

**Proof.** For any quadratic character \( \lambda \) write \( \delta_\lambda := \lambda \lambda^c \). Let \( \tilde{\rho}^\mu := \rho^\mu \otimes \mu^{-1} \). Our goal is to show that either \( \tilde{\rho}^\mu \) or \( (\tilde{\rho}^\mu)^c \delta_\mu \) is independent of \( \mu \in \mathcal{M} \). For any prime \( v \notin S \) denote by \( \{\alpha_v, \beta_v\} \) the set of inverse roots of the Hecke polynomial of \( \pi \) at \( v \); then for any prime \( v \notin S(\mu) \) the set of eigenvalues of \( (\rho^\mu \oplus (\rho^\mu)^c)(\text{Frob}_v) \cdot (\mu^{-1})(v) = (\tilde{\rho}^\mu \oplus (\tilde{\rho}^\mu)^c)(\text{Frob}_v) \) is
\[\{\alpha_v, \beta_v, \alpha_v \delta_\mu(v), \beta_v \delta_\mu(v)\}\]
by (5.2).

Given \( \mu, \mu' \in \mathcal{M} \) the corresponding set of eigenvalues coincide over the splitting field \( F := K(\delta_\mu, \delta_{\mu'}) \) of degree at most 2. By the Chebotarev density theorem and continuity, so do their characteristic polynomials. Viewing the representations as \( \mathcal{O}_K[\text{Gal}(\mathcal{K}/K)] \)-representations, a general theorem on semisimple modules of algebras over fields of characteristic 0 [1, Ch. 8, Sec. 12.2, Prop. 3] then tells us that their semisimplifications are isomorphic. But since the \( \rho^\mu \) are semisimple, so is the restriction to the normal subgroup \( \text{Gal}(\mathcal{K}/F) \) and we get
\[\rho^\mu \oplus (\rho^\mu)^c \cong (\rho \otimes \mu) \oplus (\rho \otimes \mu)^c.\]

By Lemma 5.1, \( \tilde{\rho}^\mu |_F \) is irreducible so either we have \( \tilde{\rho}^\mu |_F \cong \tilde{\rho}^{\mu'} |_F \) or we have \( (\tilde{\rho}^\mu)^c \delta_{\mu'} |_F \cong \tilde{\rho}^{\mu'} |_F \). Let us fix some element \( \mu_0 \in \mathcal{M} \), and for each \( \mu \in \mathcal{M} \) such that the second case holds for \( \mu' = \mu_0 \) replace \( \tilde{\rho}^{\mu} \) by \( (\tilde{\rho}^{\mu})^c \delta_{\mu} \) (this corresponds to replacing the original \( \rho^\mu \) by \( (\rho^\mu)^c \) which is legitimate). By this change we have achieved that \( \tilde{\rho}^\mu |_F \cong \tilde{\rho}^{\mu_0} |_F \) for all \( \mu \in \mathcal{M} \), that is, \( \tilde{\rho}^\mu \cong \tilde{\rho}^{\mu_0} \otimes \psi_{\mu, \mu_0} \) for some character \( \psi_{\mu, \mu_0} \) of \( \text{Gal}(F/K) \). We shall regard \( \psi_{\mu, \mu_0} \) as a quadratic character of \( \text{Gal}(\mathcal{K}/K) \) trivial on \( \text{Gal}(\mathcal{K}/F) \).

Note that \( \psi_{\mu_0, \mu_0} = 1 \), so that for general \( \mu, \mu' \in \mathcal{M} \) the definition
\[\psi_{\mu, \mu'} := \psi_{\mu, \mu_0} \psi_{\mu', \mu_0}\]
is unambiguous. This character satisfies
\[\tilde{\rho}^\mu \cong \tilde{\rho}^{\mu'} \otimes \psi_{\mu, \mu'},\]
whence Lemma 5.2 tells us that in (5.4) we must have \( \tilde{\rho}^\mu |_F \cong \tilde{\rho}^{\mu'} |_F \) for \( F = K(\delta_\mu, \delta_{\mu'}) \) and \( \psi_{\mu, \mu'} \) must be trivial on \( \text{Gal}(\mathcal{K}/F) \). It follows that \( \psi_{\mu, \mu'} = 1 \) or \( \psi_{\mu, \mu'} = \delta_\mu \delta_{\mu'} \), since these are the only characters of \( \text{Gal}(\mathcal{K}/K) \) trivial on \( \text{Gal}(\mathcal{K}/F) \). We claim that either \( \psi_{\mu, \mu'} = 1 \) for all \( \mu, \mu' \in \mathcal{M} \) or \( \psi_{\mu, \mu'} = \delta_\mu \delta_{\mu'} \) for all \( \mu, \mu' \in \mathcal{M} \).
Assume the first alternative fails then there are \( \mu_1, \mu_2 \in \mathcal{M} \) with \( \psi_{\mu_1, \mu_2} = \delta_\mu \delta_{\mu_2} \neq 1 \). For arbitrary \( \mu \in \mathcal{M} \) we have \( \psi_{\mu_1, \mu} \psi_{\mu_2, \mu} = \psi_{\mu_1, \mu} \psi_{\mu_2, \mu_0} = \psi_{\mu_1, \mu_2} \) by (5.5), hence \( \psi_{\mu_1, \mu_2} \neq \psi_{\mu_2, \mu_2} \). Therefore \( \psi_{\mu_1, \mu_1} = \delta_\mu \delta_{\mu_1} \) or \( \psi_{\mu, \mu_2} = \delta_\mu \delta_{\mu_2} \). In fact these equations are equivalent since the two sides have equal products by (5.5), namely

\[
\psi_{\mu_1, \mu_1} \psi_{\mu_2, \mu_2} = \psi_{\mu_1, \mu_0} \psi_{\mu_2, \mu_0} = \psi_{\mu_1, \mu_2} = \delta_\mu \delta_{\mu_2}.
\]

We infer that \( \psi_{\mu_1, \mu_1} = \delta_\mu \delta_{\mu_1} \) is valid for all \( \mu \in \mathcal{M} \). This implies, for all \( \mu, \mu' \in \mathcal{M} \),

\[
\psi_{\mu, \mu'} = \psi_{\mu_0, \mu_0} \psi_{\mu', \mu_0} = \psi_{\mu_0, \mu'} \psi_{\mu', \mu_0} = \delta_\mu \delta_{\mu'}.
\]

If \( \psi_{\mu, \mu'} = 1 \) for all \( \mu, \mu' \) then (6.4) implies that \( \tilde{\rho}^\mu \) is independent of \( \mu \). If \( \psi_{\mu, \mu'} = \delta_\mu \delta_{\mu'} \) for all \( \mu, \mu' \) then (5.6) implies that \( (\tilde{\rho}^\mu)\delta_\mu \) is independent of \( \mu \). In both cases we denote the common value of these representations by \( \rho \) and verify that it satisfies the required properties. \( \square \)

### 6. End of proof

We have shown that there is a dense set \( \mathcal{M} \) of quadratic characters of \( K \) (see Definition 2.1) and a continuous irreducible semisimple representation

\[
\rho : \text{Gal}(\overline{K}/K) \to \text{GL}_2(\overline{\mathbb{Q}}_\ell)
\]

unramified outside \( S \) such that for any \( \mu \in \mathcal{M} \) we have

\[
(6.1) \quad L^{S(\mu)}(s, \rho \otimes \mu) L^{S(\mu)}(s, (\rho \otimes \mu)^c) = L^{S(\mu)}(s, \pi \otimes \mu) L^{S(\mu)}(s, (\pi \otimes \mu)^c),
\]

where \( S(\mu) \) abbreviates \( S_K(\pi \otimes \mu) \) (see Definition 2.2). Note that \( S(\mu) \) is contained in the union of \( \mathcal{S} \) and the set of primes in \( K \) where \( \mu \) or \( \mu^c \) is ramified.

For any prime \( v \) of \( K \) outside \( S \) denote by \( \{\alpha_v, \beta_v\} \) the set of inverse roots of the Hecke polynomial of \( \pi \) at \( v \) and by \( \{\gamma_v, \delta_v\} \) the inverse roots of the Frobenius polynomial of \( \rho \) at \( v \). We shall regard these as multisets (i.e. sets with multiplicities). By (6.1) for all \( \mu \in \mathcal{M} \) unramified at \( v \) and \( v^c \) we have (as multisets)

\[
\{\gamma_v, \mu(v), \delta_v, \mu(v^c), \gamma_v, \mu(v^c), \delta_v, \mu(v^c)\} = \{\alpha_v, \mu(v), \beta_v, \mu(v), \alpha_v, \mu(v^c), \beta_v, \mu(v^c)\}.
\]

We need to show that (as multisets)

\[
\{\gamma_v, \delta_v\} = \{\alpha_v, \beta_v\} \quad \text{and} \quad \{\gamma_v, \delta_v\} = \{\alpha_{v^c}, \beta_{v^c}\}.
\]

If \( v \) is inert then the statement is trivial by the existence of some \( \mu \in \mathcal{M} \) that is unramified at \( v = v^c \).

If \( v \) is split we can find \( \mu_1, \mu_2 \in \mathcal{M} \) unramified at \( v \) and \( v^c \) such that \( \mu_1(v) = \mu_1(v^c) \) but \( \mu_2(v) \neq \mu_2(v^c) \). It follows that (as multisets)

\[
\{\gamma_v, \delta_v, \gamma_{v^c}, \delta_{v^c}\} = \{\alpha_v, \beta_v, \alpha_{v^c}, \beta_{v^c}\}
\]

and

\[
\{\gamma_v, \delta_v, -\gamma_{v^c}, -\delta_{v^c}\} = \{\alpha_v, \beta_v, -\alpha_{v^c}, -\beta_{v^c}\}.
\]

By forming the sums of both multisets and then adding and subtracting the two resulting equations we conclude that

\[
\gamma_v + \delta_v = \alpha_v + \beta_v \quad \text{and} \quad \gamma_{v^c} + \delta_{v^c} = \alpha_{v^c} + \beta_{v^c}.
\]

By forming the reciprocal sums of both multisets and then adding and subtracting the two resulting equations we conclude that

\[
\gamma_v^{-1} + \delta_v^{-1} = \alpha_v^{-1} + \beta_v^{-1} \quad \text{and} \quad \gamma_{v^c}^{-1} + \delta_{v^c}^{-1} = \alpha_{v^c}^{-1} + \beta_{v^c}^{-1}.
\]
Let us focus on the left hand sides of (6.5) and (6.6). If the left hand side of (6.6) designates a nonzero common value then we divide by it the left hand side of (6.5) and obtain \( \gamma_v \delta_v = \alpha_v \beta_v \). Together with the left hand side of (6.5) this yields \( \{\gamma_v, \delta_v\} = \{\alpha_v, \beta_v\} \), hence in fact the entire (6.2) upon using (6.3) again. In the same way (6.2) follows if the right hand side of (6.6) designates a nonzero common value.

We are left with the subtle case when both sides of (6.6) designate zero as common value. Then (6.3) and (6.4) simplify to the same multiset equation

\[
\{\gamma_v, -\gamma_v, \gamma_v, -\gamma_v\} = \{\alpha_v, -\alpha_v, \alpha_v, -\alpha_v\}
\]

and (6.2) simplifies to

\[
\gamma_v^2 = \alpha_v^2 \quad \text{and} \quad \gamma_v^2 = \alpha_v^c.
\]

We need to deduce (6.8) from (6.7). Taking squares in (6.7) and halving multiplicities we see that it really is equivalent to the multiset equation

\[
\{\gamma_v^2, \gamma_v^c\} = \{\alpha_v^2, \alpha_v^c\}.
\]

Now we observe that in the present situation

\[
\alpha_v^2 = -\omega(v) = -\omega(v^c) = \alpha_v^c,
\]

whence in fact (6.9) yields

\[
\gamma_v^2 = \gamma_v^c = \alpha_v^2 = \alpha_v^c,
\]

so that (6.8) holds as needed.

The proof of Theorem 1.1 is complete.

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