Space functions and complexity of the word problem in semigroups.

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Abstract

We introduce the space function $s(n)$ of a finitely presented semigroup $S = \langle A \mid R \rangle$. To define $s(n)$ we consider pairs of words $w, w'$ over $A$ of length at most $n$ equal in $S$ and use relations from $R$ for the transformations $w = w_0 \rightarrow \cdots \rightarrow w_t = w'$; $s(n)$ bounds from above the tape space (or computer memory) sufficient to implement all such transitions $w \rightarrow \cdots \rightarrow w'$. One of the results obtained is the following criterion: A finitely generated semigroup $S$ has decidable word problem of polynomial space complexity if and only if $S$ is a subsemigroup of a finitely presented semigroup $H$ with polynomial space function.

Key words: generators and relations in semigroups, algorithm, space complexity, word problem

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1 Introduction

Let $A$ be an alphabet, $A^*$ the set of all words in $A$, and $A^+$ the set of non-empty words. We will use $|w|$ for the length of a word $w$, in particular the empty word $1$ has length $0$. We write $S = \langle A \mid R \rangle$ for a semigroup (resp., monoid) presentation when $R \subseteq A^+ \times A^+$ (resp., $R \subseteq A^* \times A^*$).

Let $S$ be a semigroup or monoid and $w, w' \in A^*$. A derivation of length $t \geq 0$ from $w$ to $w'$, where $w, w' \in A^+$ or $w, w' \in A^*$, resp., is a sequence of words

$$w = w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_t = w', \quad (1.1)$$

where “$=$” denotes the letter-for-letter equality, and for $0 \leq i < t$, the word $w_{i+1}$ results from $w_i$ after a defining relation from $R$ is applied, i.e., $w_{i+1} = w_i r v, w_{i+1} = w v r''$ for some words $u, v$, and $(r', r'') \in R$ or $(r'', r') \in R$. Two words $w, w'$ represent the same element of $S$ (or they are equal in $S : w =_S w'$) iff there exists a derivation $w \rightarrow \cdots \rightarrow w'$.

The minimal (non-decreasing) function $d(n) : \mathbb{N} \rightarrow \mathbb{N}$ such that for every two words $w, w'$ equal in $S$ and having length $\leq n$, there exists a derivation (1.1) with $t \leq d(n)$, is called the Dehn function of the presentation $S = \langle A \mid R \rangle$ ([13], [2]). For finitely presented $S$ (i.e., both sets $A$ and $R$ are finite), Dehn functions are usually taken up to equivalence to get rid of the dependence on a finite presentation for $S$ (see [18]). To introduce this equivalence $\sim$, we write $f \preceq g$ if there is a positive integer $c$ such that

$$f(n) \leq cg(cn) + cn \quad \text{for any } n \in \mathbb{N} \quad (1.2)$$

For example, we say that a function $f$ is polynomial if $f \preceq g$ for a polynomial $g$. From now on, we use the following equivalence for nondecreasing functions $f$ and $g$ on $\mathbb{N}$.

$$f \sim g \quad \text{if both } f \preceq g \text{ and } g \preceq f \quad (1.3)$$

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It is not difficult to see that the Dehn function \( d(n) \) of a finitely presented semigroup or monoid, or group \( S \) is recursive (or bounded from above by a recursive function) iff the word problem is algorithmically decidable for \( S \) (see [11], [6]). In this case, the word problem can be solved by a primitive algorithm that, given a pair of words \( w, w' \) of length \( \leq n \), just checks if there exists a derivation (1.1) of length \( \leq d(n) \). Therefore the nondeterministic time complexity of the word problem in \( S \) is bounded from above by \( d(n) \). Moreover if \( Q \) is a finitely generated subsemigroup (submonoid, subgroup) of \( S \), then one can use the rewriting procedure (1.1) for \( Q \), and so the nondeterministic time complexity of the word problem for \( Q \) is also bounded by a function equivalent to \( d(n) \).

A converse statement is also true. Assume that the word problem can be solved in a finitely generated semigroup \( S \) by a nondeterministic Turing machine (NTM) with time complexity \( \leq T(n) \), where \( T(n) \) is a superadditive function (i.e. \( T(m+n) \geq T(m) + T(n) \)). Then \( S \) is a subsemigroup of a finitely presented semigroup \( H \) with Dehn function \( O(T(n)^2) \). This is proved in [2] while a similar statement for groups (but with the function \( n^2T(n^2) \) instead of \( T(n)^2 \)) is obtained in [3]. As the main corollary, one concludes that the word problem in a finitely generated semigroup (group) \( H \) has time complexity of class \( NP \) (i.e., there exists a nondeterministic algorithm of polynomial time complexity, which solves the word problem for \( H \)) iff \( H \) is a subsemigroup (resp., subgroup) of a finitely presented semigroup (resp., group) with polynomial Dehn function.

Hence the notion of Dehn function is the (semi)group-theoretical counterpart of the concept of time complexity for algorithms. It turns out that the filling length functions introduced earlier in [13], [12], [3] (or briefly, space functions) of finitely presented groups are counterparts of the concept of space complexity of algorithms. The main theorem of [22] says that for a finitely generated group \( G \) such that the word problem in \( G \) is decidable by a deterministic Turing machine (DTM) with space complexity \( f(n) \), there is an embedding of \( G \) in a a finitely presented group \( H \) with space function equivalent to \( f(n) \). In particular the following criterion is obtained: A finitely generated group \( H \) has decidable word problem of polynomial space complexity if and only if \( H \) is a subgroup of a finitely presented group \( G \) with a polynomial space function.

Thus, on the one hand, theorems from [2] and [4] provide a logical connection between Dehn functions of semigroups and groups and the time complexity of their word problems; and on the other hand, similar interrelation of space functions of groups and the space complexity is obtained in [22]. So it is natural to fill a gap regarding space functions of semigroups and the space complexity of the algorithmic word problem in semigroups.

In the present paper, we say that the derivation (1.1) has space maximum \( \sum_{i=1}^{s} |w_i| \). For two words \( w \) and \( w' \) equal in \( S = \langle A \mid R \rangle \), we denote by space \( s(w, w') \) the minimum of spaces of the derivations connecting \( w \) and \( w' \), and define the value of the space function \( s(n) \) to be equal to \( \max(space(w, w')) \) over all pairs \( (w, w') \) of equal in \( S \) words with \( |w|, |w'| \leq n \). An accurate definition of the space complexity (function) \( f(n) \) for a Turing machine (TM) will be recalled in Subsection 2.1. Now we just note that the space complexities of machines are taken here up to the same equivalence as the space functions of semigroups. Up to this equivalence, the time and space complexities of the word problem do not depend on the choice of a finite generator set, see [2], Prop. 2.1.

**Theorem 1.1.** Let \( S \) be a finitely generated semigroup (monoid) such that the word problem in \( S \) is decidable by a DTM with space complexity \( f(n) \). Then \( S \) is a subsemigroup (resp., submonoid) of a finitely presented monoid \( P \) with space function equivalent to \( f(n) \).

**Remark 1.2.** It follows from [18], [8] that even if \( S \) is finitely presented, one cannot define \( P = S \) in Theorem 1.1. Baumslag’s [1] 1-relator group \( G = \langle a, b \mid (aba^{-1})b(aba^{-1})^{-1} = b^2 \rangle \) is a particular counter-example because the space function of \( G \) is not bounded from above by any multi-exponential function (see [10] and [23]) while the space (and time) complexity of the word problem in \( G \) is polynomial [19].

**Corollary 1.3.** The word problem in a finitely generated semigroup (monoid) \( S \) is polynomial space decidable if and only if \( S \) is a subsemigroup (resp., submonoid) of a finitely presented monoid \( H \) with polynomial space function.

We apply our approach to the realization problem: Which functions \( f(n) : \mathbb{N} \rightarrow \mathbb{N} \) are, up to equivalence, the space functions of finitely presented semigroups? It is not difficult to find examples of
(semi)groups with linear and exponential space functions, but it is not easy even to specify a (semi)group with space function \( n^2 \).

**Corollary 1.4.** The space complexity \( f(n) \) of arbitrary DTM \( M \) is equivalent to the space function of some finitely presented semigroup (or monoid) \( P \).

This corollary reveals an extensive class of space functions of semigroups, including functions equivalent to \([\exp \sqrt{n}], [n^k] (k \in \mathbb{N}), [n^k \log n], [n^k \log(\log n)^m] \), etc. Note that we do not assume in the formulation of Theorem 1.4 that the function \( f(n) \) is superadditive (i.e., \( f(m+n) \geq f(m)+f(n) \)) or grows sufficiently fast. (Compare with theorems in [26] and [2] on Dehn functions of groups and semigroups.) It follows, in particular, that there exists a finitely presented semigroup whose space function is not equivalent to any superadittive function. Recall that it is unknown if the Dehn function of arbitrary finitely presented group is equivalent to a superadditive function; see [14].

**Corollary 1.5.** There is a finitely presented semigroup (and monoid) \( P \) with polynomial space complete word problem and with polynomial space function.

We also describe the functions \( n^\alpha \) which are (up to equivalence) space functions of semigroups (and monoids). As in [22], our approach is based on a modification of the proof of Savitch’s theorem from [9] and the proof from [26], where the similar problem was considered for Dehn functions if \( \alpha \geq 4 \), and close necessary and sufficient conditions were obtained. (See also a dense series of examples with \( \alpha \geq 2 \) presented in [34].) Now we have \( \alpha \geq 1 \) in Corollary 1.6 below. Also it is worth to note that for space functions, the necessary and sufficient conditions just coincide.

To formulate the criterion, we call a real number \( \alpha \) *computable with space* \( \leq f(m) \) if there exists a DTM which, given a natural number \( m \), computes a binary rational approximation of \( \alpha \) with an error \( O(2^{-m}) \), and the space of this computation \( \leq f(m) \).

**Corollary 1.6.** For a real number \( \alpha \geq 1 \), the function \([n^\alpha]\) is equivalent to the space function of a finitely presented semigroup (or monoid) iff \( \alpha \) is computable with space \( \leq 2^m \).

It follows that functions \([n^\alpha]\) with any algebraic \( \alpha \geq 1 \) are all space functions of finitely presented semigroups (and monoids), as well as \([n^e]\), \([n^{\sqrt{7}}]\), etc.

**Remark 1.7.** Some of the above statements sound similar to propositions from [22], but they cannot be deduced from [22] since there, the set of admissible transformations in derivations is larger than here. (After Bridson and Riley [7], we also allowed cyclic permutations and fragmentations of words in [22].) It is an open question if Theorem 1.1 and its corollaries valid for groups as well, provided the derivations are based only on the applications of defining relations, as this is accepted in the present paper.

Recall that Higman proved in [15] that every recursively presented group is embeddable in a finitely presented one. A semigroup analog of this theorem was proved by Murskii in [20]. The approaches to groups and to semigroups are different, since in the group case one can use conjugations and HNN extensions to synchronize applications of all defining relations corresponding to one machine command (see [25]). For semigroup embeddings, Murskii [20] and Birget [2] use one tape symmetric input-output machines. We follow this line but one should look after the space complexity of the constructed machines. Moreover the *generalized* space complexity (see Subsection 2.1 for the definitions) of the latest modification \( M_5 \) must be equal to the space complexity of the initial machine \( M_0 \). The trick used in Subsection 2.3 for this purpose, works if the machine \( M_0 \) is deterministic. (Therefore Theorem 1.1 is formulated in terms of deterministic space complexity while [4] and [2] consider only nondeterministic time complexity.)

As in [20] and [2], our embedding of \( S \) in a finitely presented semigroup \( P \) in Theorem 1.1 is based on the commands of the constructed machine. Unlike [20] and [2], now we should control the space function of \( P \). Some additional technical difficulties appear because we want to obtain a monoid embedding (i.e., \( 1 \to 1 \)) if the semigroup \( S \) is a monoid. Monoid relation \( v = 1 \) is less convenient since the word \( v \) can be inserted before/after any letter of any word \( w \). (Note that only semigroup embedding are under consideration in [2]. Whether the embedding from [20] is a monoid embedding or not, if \( S \) is a monoid, is also left inexplicit.)

In the remaining part of the proof we introduce derivation trapezia. They visualize derivations and make possible to use geometric images, e.g., bands, lenses, cups and caps. Furthermore, one can remove
unnecessary parts of trapezia (e.g., see Lemmas 3.7, 3.11, 3.16). Such parts may not correspond to subderivation, and they can hardly be defined in a different language.

2 Machines

2.1 Definitions

We will use a model of recognizing TM which is close to the model from

Recall that a (multi-tape) TM with k tapes and k heads is a tuple

\[ M = (A, Y, Q, \Theta, \vec{s}_1, \vec{s}_0) \]

where \( A \) is the input alphabet, \( Y = \bigcup_{i=1}^{k} Y_i \) is the tape alphabet, \( Y_1 \supset A \), \( Q = \bigcup_{i=1}^{k} Q_i \) is the set of states of the heads of the machine, \( \Theta \) is a set of transitions (commands), \( \vec{s}_1 \) is the \( k \)-vector of start states, \( \vec{s}_0 \) is the \( k \)-vector of accept states. \((\cup) \) denotes the disjoint union.) The sets \( Y, Q, \Theta \) are finite.

We assume that the machine normally starts working with states of the heads forming the vector \( \vec{s}_1 \), with the head placed at the right end of each tape, and accepts if it reaches the state vector \( \vec{s}_0 \). In general, the machine can be turned on in any configuration and turned off at any time.

A configuration of tape number \( i \) of a TM is a word \( uqv \) where \( q \in Q_i \) is the current state of the head, \( u \) is the word to the left of the head, and \( v \) is the word to the right of the head, \( u, v \in Y^* \). A tape is empty if \( u, v \) are empty words.

A configuration \( U \) of the machine \( M \) is a word

\[ \alpha_1 U_1 \omega_1 \alpha_2 U_2 \omega_2 \ldots \alpha_k U_k \omega_k \]

where \( U_i \) is the configuration of tape \( i \), and the endmarkers \( \alpha_i, \omega_i \) of the \( i \)-th tape are special separating symbols.

An input configuration \( w(u) \) is a configuration, where all tapes, except for the first one, are empty, the configuration of the first tape (let us call it the input tape) is of the form \( wq \), \( q \in Q_1 \), \( u \) is a word in the alphabet \( A \), and the states form the start vector \( \vec{s}_1 \). The accept configuration is the configuration where the state vector is \( \vec{s}_0 \), the accept vector of the machine, and all tapes are empty. (The requirement that the tapes must be empty is often removed for auxiliary machines which are used in construction of bigger machines.)

To every \( \theta \in \Theta \), there corresponds a command (marked by the same letter \( \theta \)), i.e., a pair of sequences of words \( [V_1, \ldots, V_k] \) and \( [V'_1, \ldots, V'_k] \) such that for each \( j \leq k \), either both \( V_j = uqv \) and \( V'_j = u'q'v' \) are configurations of the tape number \( j \), or \( V_j = \alpha_j q v \) and \( V'_j = \alpha_j q' v' \), or \( V_j = uqv \), \( V'_j = \alpha_j q' \omega_j \), or \( V_j = w \omega_j \), \( V'_j = \alpha_j q' \omega_j \) \( (q, q' \in Q_j) \).

In order to execute this command, the machine checks if \( V_i \) is a subword of the current configuration of the machine, and if this condition holds the machine replaces \( V_i \) by \( V'_i \) for all \( i = 1, \ldots, k \). Therefore we also use the notation: \( \theta : [V_1 \rightarrow V'_1, \ldots, V_k \rightarrow V'_k] \), where \( V_j \rightarrow V'_j \) is called the \( j \)-th part of the command \( \theta \).

Suppose we have a sequence of configurations \( w_0, \ldots, w_t \) and a word \( h = \theta_1 \ldots \theta_t \) in the alphabet \( \Theta \), such that for every \( i = 1, \ldots, t \) the machine passes from \( w_{i-1} \) to \( w_i \) by applying the command \( \theta_i \). Then the sequence \( (w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_t) \) is said to be a computation with history \( h \). In this case we shall write \( w_0 \cdot h = w_t \). The number \( t \) will be called the time or length of the computation.

A configuration \( w \) is called accepted by a machine \( M \) if there exists at least one computation which starts with \( w \) and ends with the accept configuration. We do not only consider deterministic TMs, for example, we allow several transitions with the same left side.

A word \( u \) in the input alphabet \( A \) is said to be accepted by the machine if the corresponding input configuration is accepted. (A configuration with the vector of states \( \vec{s}_1 \) is never accepted if it is not an input configuration.) The set of all accepted words over the alphabet \( A \) is called the language \( L_M \) recognized by the machine \( M \).

If a DTM \( M \) halts on an input word \( w \in A^* \) at a non-accepting state \( \vec{s}' \) with all tapes empty, then one says that \( M \) rejects \( w \). Speaking on deterministic TM, we will assume that every input configuration is
either accepted or rejected, i.e., \( M \) may not operate forever being switched on at an input configuration. In other words, we consider DTM-s \( M \) with recursive languages \( \mathcal{L}_M \).

Let \( |w_i|_a \) (\( i = 0, \ldots, t \)) be the number of tape letters (or tape squares) in the configuration \( w_i \). (As in [20], the tape letters are called \( a \)-letters.) Then the maximum of all \( |w_i|_a \) will be called the space of computation \( C: w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_t \) and will be denoted by \( \text{space}_M(C) \). If \( u \in A^* \) then, by definition, \( \text{space}_M(u) \) is the minimal space of the computation that accepts or rejects the corresponding input configuration \( w = w(u) \).

The number \( S(n) = S_M(n) \) is the maximum of the numbers \( \text{space}(u) \) over all words \( u \in A^* \) with \( |u| \leq n \). The function \( S(n) \) will be called the space complexity of the DTM \( M \).

The definition of the generalized space complexity \( S'(n) = S_M'(n) \) is similar to the definition of space complexity but we consider arbitrary pair \( w_0, w_1 \) of configurations which can be connected by a computation \( w_0 \rightarrow \cdots \rightarrow w_t \) (not just input configurations as in the definition of \( S(n) \)). We define \( \text{space}_M(w_0, w_1) \) to be the minimal space of computations connecting \( w_0 \) and \( w_1 \), and \( S'(n) \) is the minimal function that bounds from above all numbers \( \text{space}_M(w_0, w_1) \) under the condition \( |w_0|_a, |w_1| \leq n \). It is clear that \( S(n) \leq S'(n) \).

### 2.2 Input-output machine

Assume that \( S \) is a semigroup (or monoid) generated by a finite set \( A \), and the word problem in \( S \) is decidable by a DTM \( M_0 \) with space function \( S_0(n) \). We define this more exactly as follows. The set of input words of \( M_0 \) consists of the words \( uv' \), where \( u \) is a word in \( A \) and \( v' \) is a word in a disjoint alphabet \( A' \) which is a copy of \( A \). Let \( u \) and \( v \) be two words over \( A \), and \( |u| + |v| \leq n \). Then (1) for a copy \( v' \) of \( v \) in \( A' \), the word \( uv' \) belongs to the language \( L_0 \) of \( M_0 \) iff \( u =_S v \), (2) every input word of \( M_0 \) of length \( \leq n \) is accepted or rejected with space \( \leq S_0(n) \), and (3) \( S_0(n) \) is the minimal function with Property (2).

However to obtain a Higman embedding of \( S \) into a finitely presented monoid we are not able to simulate the work of the recognizing machine \( M_0 \) by semigroup relations but following Murskii [20] and Birget [2] (and preserving the space complexity), we first transform it into an input-output machine \( M_1 \). We will see that if \( u \) is an input word and \( v \) is an output word for a computation of \( M_1 \), then \( u =_S v \).

The definition of an input-output TM is similar to the definition of a recognizing TM, but the first tape is an “input-output tape” that holds the initial input and the final output. To define the space complexity \( S(n) \) of an input-output machine, one consider input-output computations, where both the input word \( u \) and the output word \( v \) are of length at most \( n \). The definition of the generalized space complexity \( S'(n) \) is similar to the definition of space complexity but we consider arbitrary computations \( w_0 \rightarrow \cdots \rightarrow w_t \) with \( \max\{|w_0|_a, |w_1|_a\} \leq n \), not just input-output computations as in the definition of \( S(n) \).

We will assume that the words in \( A^* \) are ShortLex ordered.

**Lemma 2.1.** There exists an input-output DTM \( M_1 \) such that

(a) for every input word \( u \in A^* \), there is an input-output computation \( C \) of \( M_1 \) with input \( u \), and the output is the least word \( v \) equal to \( u \) in \( S \);

(b) the space complexity \( S_1(n) \) of \( M_1 \) is equivalent to the space complexity \( S_0(n) \) of \( M_0 \);

(c) depending on the state, any configuration \( \omega \) of the computation \( C \) contains either (i) a copy of the word \( u \) on one of the tapes or (ii) the output \( v \) on the input-output tape, and to obtain the output configuration \( \omega(v) \) in Case (ii), it remains to erase all other tapes and accept; also we have \( \text{space}(C) = S_1(|u|) \) in Case (ii);

(d) a configuration with the start vector of states \( \vec{s}_1 \) cannot be reached after an application of a command of \( M_1 \) to any configuration.

(e) if a configuration with vector of states \( \vec{s}_0 \) results after an application of a command of \( M_1 \), then this command is the unique accepting command.

**Proof.** The machine \( M_1 \) has two tapes more than \( M_0 \). At first it writes a copy of \( u \) on an extra-tape \( T \). Then it writes a current word \( v \leq u \) (starting with the least \( v; v = 1 \) if \( S \) is a monoid) on another extra-tape \( T' \), and writes down the word \( uv' \), where \( v' \) is a copy of \( v \) in a disjoint alphabet, on the input tape of \( M_0 \) (which is also the input-output tape of \( M_1 \)). Then \( M_0 \) starts working to check whether \( u =_S v \) or not. If “yes”, then \( M_1 \) rewrites \( v \) onto the output tape, cleans up all other tapes, and accepts.
Assume that $h \in \mathcal{M}_1$ cleans up the tapes of $M_0$, replaces the word $v$ by the next word $v_+ \leq u$ on $T'$, and repeats the cycle with $v_+$.

Since $u = \varepsilon$, sooner or later the machine $M_1$ accepts $u$ with Property (a). The first part of (c) follows from the above algorithm as well. Since the current word $v$ is not longer than $u$ and $|uu'| \leq 2|u|$, we have $S_{M_1}(n) \leq 3S_{M_0}(2n)$, and so $S_1(n) \leq S_0(n)$. Now Property (b) and the second part of (c) follow from the inequality $S_0(n) \leq S_1(n)$ which can be easily provided if one forces the machine $M_1$ to check every pair $(\bar{u}, v)$ with $|\bar{u}|, |v| \leq |u|$ (even the shortest $v_0$ with $v_0 = \varepsilon$ is already found). To obtain Property (d), it suffices to add special states for input configuration: the first command changes these states, and the state letters from $\mathcal{S}_1$ do not occur in other commands. Similarly, one obtains Property (e).

\[2.3\] Machine with equal space complexity and generalized space complexity

In this subsection, we construct an NTM $M_2$ which inherits the basic characteristics of the DTM $M_1$ and has equivalent generalized space complexity and space complexity. For this goal we adapt the approach from [22] to input-output machines.

Assume that $M_1$ has $k$ tapes, and let its first tape be the input-output tape. Then we add a tape numbered $k + 1$, which is empty for input/output configurations, and we organize the work of the 3-stage machine $M_2$ as a sequential work of the following machines $M_{21}$, $M_{22}$, and $M_{23}$.

The machine $M_{21}$ uses only one command $\theta_a$ that does not change states and adds one square with an auxiliary letter $\ast$ to the $(k + 1)$-st tape, i.e., the command $\theta_a$ has the form

\[ [q_1 \omega_1 \rightarrow q_1 \omega_1, \alpha_2 \omega_2 \rightarrow \alpha_2 \omega_2, \ldots, \alpha_κq_k \omega_k \rightarrow \alpha_κq_k \omega_k, q_k+1 \omega_{k+1} \rightarrow \ast \] \]

The machine $M_{21}$ can execute this command arbitrarily many times while the tapes numbered $1, \ldots, k$ keep the copy of an input configuration of $M_1$ unchanged. Then a connecting rule $\theta_{22}: [q_1 \rightarrow q'_1, \ldots, \alpha_κq_k \omega_k \rightarrow \alpha_κq'_k \omega_k, q_k+1 \omega_{k+1} \rightarrow q'_{k+1} \omega_{k+1}]$ changes all states of the heads and switches on the machine $M_{22}$. Here $(q'_1, \ldots, q'_k) = (\mathcal{S}_1)$ is the vector of start states for $M_1$.

The work of $M_{22}$ on the tapes with numbers $1, \ldots, k$ copies the work of $M_1$. But the extension $\theta'$ of every command $\theta$ of $M_1$ to the $(k + 1)$-st tape is defined so that its application does not change the current space. More precisely, if a command $\theta$ inserts $m_1$ tape squares and deletes $m_2$ tape squares on the first $k$ tapes, then $\theta'$ inserts $m_2 - m_1$ (deletes $m_1 - m_2$) squares with letter $\ast$ on the $(k + 1)$-st tape if $m_1 - m_2 \leq 0$ (if $m_1 - m_2 \geq 0$). That is the $(k + 1)$-st component of $\theta'$ has the form $q_k+1 \omega_{k+1} \rightarrow s^{m_2 - m_1}q_k+1 \omega_{k+1}$ (resp., $s^{m_1 - m_2}q_k+1 \omega_{k+1} \rightarrow q_k+1 \omega_{k+1}$). Note that one cannot apply $\theta'$ if $m_1 - m_2$ exceeds the current number of squares on the tape numbered $k + 1$.

The connecting command $\theta_{23}$ is applicable when $M_{22}$ reaches the output configuration on the first $k$ tapes. It changes the states and switches on the machine $M_{23}$ erasing all squares on the $(k + 1)$-st tape (one by one).

Let $w$ be a configuration of the machine $M_2$ such that $w \cdot \theta_a$ is defined, or such that $w$ is obtained after an application of the connecting command $\theta_{12}$. Then we have an input configuration on the tapes with numbers $1, \ldots, k$ (plus several $\ast$-s on the $(k + 1)$-st tape). We will denote by $u(w)$ the input word $w$ written on the first tape. It is an input word for the machine $M_1$ as well, and the expression $space_{M_1}u(w)$ makes sense.

The connecting commands $\theta_{12}$ and $\theta_{23}$ are not invertible in $M_2$ by definition. Therefore every non-empty computation of $M_2$ has history of the form $h_1h_2h_3$ or $h_1h_2, h_1, h_2h_3, h_3, h_1$ is the history for $M_{2l}, (l = 1, 2, 3)$. (To simplify notation we attribute the command $\theta_{12}$ (the command $\theta_{23}$) to $h_2$ (to $h_3$).)

**Lemma 2.2.** (a) For every input word $u$, the machine $M_1$ and $M_2$ give out the same output $v$. (b) The space complexity $S_2(n)$ and the generalized space complexity $S_2'(n)$ of $M_2$ are both equivalent to $S_1(n)$.

**Proof.** Assume that $u$ is converted to the output word $v$ by $M_1$. Then $u$ can be converted to $v$ by $M_2$ as well because the machine $M_{21}$ can insert sufficiently many squares (equal to $space_{M_1}(u) - |u|$) so that the input-output computation of $M_1$ can be simulated by $M_{22}$. Also it is clear from the definition of $M_2$, that every accepting computation for $M_2$ having a history $h_1h_2h_3$ as above, simulates, at stage 2, an accepting computation of $M_1$ with history $h_2$. This proves Statement (a) and equality $S_1(n) = S_2(n)$.  

6
Lemma 2.1 (a). We also have space \( w \) corresponding to the accepting command of \( M \) contrary to the determinism of \( \Theta \) the definition of \( M \) \( M \) is positive. Hence the space of this computation is equal to \( |w|_0 \). Similarly, it is \( |w'|_0 \) if \( h_3 \) is empty. Then let both \( h_1 \) and \( h_3 \) be non-empty. It follows that the machine \( M_2 \) starts (ends) working with a copy of an input (resp., output) configuration of the machine \( M_1 \), i.e., the input-output tape of this configuration contains an input word \( u = u(w) \) (output word \( v = v(w') \)) and the additional \((k+1)\)-st tape has \( m \) squares (resp., \( m' \) squares) for some \( m \geq 0 \). We consider two cases.

Case 1. Suppose \( m \geq \text{space}_{M_1}(u) - |u| \). This inequality says that the additional tape has enough squares to enable \( M_{22} \) to simulate the computation of \( M_1 \) with the input word \( u \). Hence there is an \( M_{22} \)-computation \( w_0 \to \cdots \to w_n \) with history of the form \( h'_2 h'_3 \), and so its space, as well as the space of our original computation, is \( |w|_0 \).

Case 2. Suppose \( m < \text{space}_{M_1}(u) - |u| \). Then there is a computation \( w_0 \to \cdots \to w_n \) such that the commands of its \( M_{21} \)-stage insert squares until the total number of squares of the \((k+1)\)-st tape becomes equal to \( \text{space}_{M_1}(u) - |u| \), and then the machines \( M_{22} \) and \( M_{23} \) work in their standard manner. The space of this (and the original) computation is \( \text{space}_{M_1}(u) \).

The estimates obtained in cases 1 and 2 show that \( S_2^2(n) \leq \max(S_1(n), n) \). Hence

\[ S_1(n) = S_2(n) \leq S_2^2(n) \leq \max(S_1(n), n) \sim S_1(n), \]

and statement (b) is completely proved too.

2.4 Symmetric machine \( M_3 \)

For every command \( \theta \) of a \( TM \), given by a vector \([V_1 \to V'_1, \ldots, V_k \to V'_k]\), the vector \([V'_1 \to V_1, \ldots, V'_k \to V_k]\) also gives a command of some \( TM \). These two commands \( \theta \) and \( \theta^{-1} \) are called mutually inverse.

Since the machine \( M_1 \) is deterministic, the machine \( M_2 \) has no invertible commands at all. The definition of the symmetric machine \( M_3 = M_2^{sym} \) is the following. Suppose \( M_2 = (X, Y, Q, \Theta, \vec{s}_1, \vec{s}_0) \). Then by definition, \( M_2^{sym} = (X, Y, Q, \Theta^{sym}, \vec{s}_1, \vec{s}_0) \), where \( \Theta^{sym} \) is the minimal symmetric set containing \( \Theta \), that is, with every command \([V_1 \to V'_1, \ldots, V_{k+1} \to V'_{k+1}]\) it contains the inverse command \([V'_1 \to V_1, \ldots, V'_{k+1} \to V_{k+1}]\); in other words, \( \Theta^{sym} = \Theta^+ \cup \Theta^- \), where \( \Theta^+ = \Theta \) (the set of positive commands) and \( \Theta^- = \{ \theta^{-1} \mid \theta \in \Theta \} \) (the set of negative commands).

A computation \( w_0 \to \cdots \to w_0 \) of \( M_3 \) (or other machine) is called reduced if its history is a reduced word. If the history \( h = \theta_1 \cdots \theta_t \) contains a subword \( \theta_1 \theta_2 \cdots \theta_{t+1} \), where the commands \( \theta_1, \theta_2 \cdots \theta_{t+1} \) are mutually inverse, then obviously there is a shorter computation \( w_0 \to \cdots \to w_{t-1} = w_{t+1} \to \cdots \to w_t \) whose space does not exceed the space of the original one.

**Lemma 2.3.** Let \( C : w_0 \to \cdots \to w_t \) be a reduced computation of \( M_1 \) with history \( h = \tau \tau' \), where \( \tau, \tau' \in \{ \theta_{12}^{\pm 1}, \theta_{23}^{\pm 1} \} \) and every command from \( \tau' \) is a command of \( M_2 \) or its inverse. Then the words \( u = u(w_0) \) and \( v = u(w_t) \) are equal in \( S \) (recall that \( u(w) \) is the subword of \( w \) written on the input-output tape), and \( \text{space}(C) \geq S_1(\max(|u|, |v|)) \). Furthermore, if \( \tau' = \theta_{23} \), then \( \tau = \theta_{12} \) and the word \( \tau' \) is positive.

**Proof.** Note that \( h \) has no 2-letter subwords \( \theta^{-1} \theta' \), where both \( \theta \) and \( \theta' \) are positive commands of \( M_{22} \) since then different commands \( \theta \) and \( \theta' \) would be applicable to the same configuration \( w_0 : \theta^{-1} = w_0 : \theta^{-1} \), and so the corresponding commands of \( M_1 \) would be also applicable to the same configuration contrary to the determinism of \( M_1 \). Hence \( h' = g_1 g_2^{-1} \), where both \( g_1 \) and \( g_2 \) are (positive) histories for \( M_{22} \).

Since one may replace \( C \) by the inverse computation, it suffices to consider three cases: (a) \( \tau = \theta_{12}, \tau' = \theta_{23}, (b) \tau = \theta_{12}, \tau' = \theta_{12}^{-1}, \) and (c) \( \tau = \theta_{23}^{-1}, \tau' = \theta_{23} \).

**Case (a).** In this case \( g_2 \) is empty since \( \theta_{23} \) can be applied only after the unique command of \( M_{22} \) corresponding to the accepting command of \( M_1 \) (see Lemma 2.1 (e)). Thus \( h' \) is the history of an \( M_{22} \)-computation, and by Lemma 2.2(a), the corresponding \( M_1 \)-computation converts \( u \) into \( v \). So \( u = S v \) by Lemma 2.1(a). We also have \( \text{space}(C) \geq S_1(\max(|u|, |v|)) \) by Lemma 2.1(c), since \( v \leq u \) in this case by the definition of \( M_1 \).
Case (c). The same argument shows now that both $g_1$ and $g_2$ are empty, a contradiction. Case (c) is impossible.

Case (b). Note that both $g_1$ and $g_2$ are non-empty since a (positive) $M_{22}$-command cannot follow by $\theta_{12}^{-1}$ by the Property (d) from Lemma 2.4 since the machine $M_{22}$ copies $M_1$. If $u = v$, then two $M_1$-computations $C_1$ and $C_2$ corresponding to the $M_{22}$-computations $w_0 \rightarrow \cdots \rightarrow w_1 \cdot g_1$ and $w_0 \rightarrow \cdots \rightarrow w_1 \cdot g_2$ have equal the first and the last configurations. Since $M_1$ is deterministic it follows that $C_1 = C_2$. But $g_1$ and $g_2$ are completely determined by their $M_1$-parts $C_1$ and $C_2$. Hence we have $g_1 = g_2$, a contradiction. Thus $u \neq v$.

Now by the inequality $u \neq v$ and Lemma 2.4 (c), the configuration $w_0 \cdot g_1 = w_1 \cdot g_2$ must contain (the same) output word on the same input-output tape for the input words $u$ and $v$ of $M_1$. We also have $u = v$ by Lemmas 2.4 (a) and 2.4. Furthermore, by Lemma 2.4 (c), the spaces of $C_1$ and $C_2$ are at least $S_1(|u|)$ and $S_1(|v|)$, respectively. Therefore $\text{space}(C) \geq S_1(\max(|u|, |v|))$.

The claims of the lemma are proved.

We say that a computation $w_0 \rightarrow \cdots \rightarrow w_t$ is an input-input computation of the machine $M_3$ if both $w_0$ and $w_t$ are input configurations of $M_2$ (and of $M_3$ as well).

Lemma 2.4. (a) For two input configurations $w$ and $w'$, of $M_3$, there exists an input-input computation $w \rightarrow \cdots \rightarrow w'$ iff $u(w) = s\ u(w')$. If $w \neq w'$, then $\text{space}(w, w') = S_1(\max(|u(w)|, |u(w')|))$.

(b) Let $C : w \rightarrow \cdots \rightarrow w'$ be a reduced input-output computation of $M_3$ with $u(w) = u$. Then $\text{space}(C) \geq S_1(|u|)$.

Proof. (a) Assume that $u(w) = s\ u(w')$, and let $v$ be the least word equal to $u(w)$ (and to $u(w')$) in $S$. By the definition of $M_1$, we have a computation $C_1$ of $M_1$ connecting the input configuration of $M_1$ with input words $u = u(w')$ and the output configuration of $M_1$ with output $v$. By Lemma 2.4 (c), $\text{space}(C_1) = S_1(|u|)$. Similarly we have $C_2$ with input word $w' = u(w')$ and the same output word $v$. Let $C_3$ and $C_4$, resp., be the corresponding computations of $M_2$ (see Subsection 2.3). Define $C'$ to be the reduced form of the computation $C_3C_4^{-1}$ of $M_3$. Then $C'$ connects $w$ and $w'$ and $\text{space}(C') \leq S_1(\max(|u|, |w'|))$.

Assume now that $u = u(w) \neq u' = u(w')$ and $C$ is an input-input computation $w \rightarrow \cdots \rightarrow w'$ of $M_3$. The history of $C$ is $h = h_0\tau_1h_1\tau_2\cdots \tau_s h_s$, where $\tau_i \equiv \theta_{12}^{\pm i}$ or $\tau_i \equiv \theta_{23}^{\pm i}$ for $i \leq s$, and the subwords $h_i$-s contain no connecting commands. The connecting commands $\tau_i$-s and the subcomputations with histories $h_i$-s whose commands correspond to the commands of $M_{21}$ or to the commands of $M_{23}$ (or to inverses) do not change the content of the input-output tape. By Lemma 2.3, the subcomputations of the form $\tau_{i-1}h_i\tau_i$, where $h_i$ corresponds to $M_{22}$, do not change the content of the input-output tape modulo the relations of $S$. Since $h_0$ and $h_s$ must consist of the commands of $M_{21}$ (or inverses) for an input-input computation, we obtain $u = s\ u'$, as required.

Furthermore, if $u \neq u'$, then $s > 1$, and the subcomputations with histories $\tau_1h_1\tau_2$ and $\tau_s-1h_s-1\tau_s$ satisfy the assumption of Lemma 2.3 whence $\text{space}(C) \geq S_1(|u|, |u'|)$.

The obtained inequalities for $\text{space}(C)$ and $\text{space}(C')$ complete the proof of Statement (a).

(b) Consider the history $h \equiv h_0\tau_1h_1\tau_2\cdots \tau_sh_s$ of $C$. Since $C$ is a reduced input-output computation, we have that $h_0$ must consist of (positive) commands of $M_{21}$, $\tau_1 = \theta_{12}$, and $\tau_s = \theta_{23}$. So $u = u(w) = u(w \cdot h_0)$, $s \geq 2$, and Statement (b) follows from Lemma 2.3 applied to the subcomputation with history $\tau_1h_1\tau_2$.

The notation $\text{space}(C) \leq f(n)$, where $n = n(C)$ depends on the computation $C$, will mean further that for some constants $c_1, c_2, c_3$ independent of $C$, we have $\text{space}(C) \leq c_1f(c_2n) + c_3n$.

Lemma 2.5. Let $C : w_0 \rightarrow \cdots \rightarrow w_t$ be a computation of $M_3$ with the smallest space for the fixed $w_0$ and $w_t$. Then $\text{space}(C) \leq S_1(\max(|w_0|, |w_t|))$.

Proof. Let us say that a configuration $w$ of $M_3$ has type 1 (resp., 2 or 3) if a command of $M_{21}$ (resp., of $M_{22}$ or $M_{23}$) or its inverse is applicable to $w$.

Case 1. Assume that both $w_0$ and $w_t$ are of type 1. Then using the command inverse to the command of $M_{21}$, we can start with $w_0$ and clean up the tape number $k + 1$ preserving the content $u(w_0)$ of the input-output tape: $C_1 : w_0 \rightarrow \cdots \rightarrow w$. Similarly we have $C_2 : w_t \rightarrow \cdots \rightarrow w$. The spaces
of these computations are \(|w_0|_a\) and \(|w_t|_a\), resp. The input configurations \(\bar{w}\) and \(\bar{w}\) can be connected by a computation \(C_{t-1}C_{t-2}\), and by Lemma 2.3 there exists an input-input computation \(C_3: \bar{w} \rightarrow \cdots \rightarrow \bar{w}\) of space \(S_1(\max(|u(\bar{w})|, |u(\bar{w})|)) = S_1(\max(|u(w_0)|, |u(w_t)|))\). The same upper bound holds for the computation \(C_1C_2C_{t-2}^{-1}: w_0 \rightarrow \cdots \rightarrow w_t\), which proves the lemma in this case.

**Case 2.** Assume that \(w_0\) and \(w_t\) have types 1 or 3. Taking into account the previous case, we may assume that \(w_t\) is of type 3. If all \(w_i\)'s have type 3 in \(C\), then their lengths monotonically increase or decrease since \(M_{23}\) has only one command. Hence \(\text{space}(C) \leq \max(|w_0|_a, |w_t|_a)\). Otherwise, by Lemma 2.3, the history of \(C\) must have a suffix \(\theta_{23}h'\theta_{23}h''\), where \(h''\) consists of the commands of \(M_{23}\) and \(h'\) is a history of an \(M_{22}\)-computation. It follows from the definition of \(M_{22}\) that \(v = u(w_1)\) is an output word.

By the definition of \(M_2\), there exists an \(M_2\)-computation \(C'\) which starts with an input configuration \(w'\) with the input word \(v\), and ends with the output configuration with output also \(v\), and has space \(S_2(|v|)\). Also there is a computation \(C''\) of \(M_{23}\) which deletes several letter in \(w_t\) and ends with the same configuration as \(C\). Hence the computation \(C''C'^{-1}\) converts the configuration \(w_t\) into an input configuration \(w''\) of length \(\leq |w_t|\), and has space \(\max(S_2(|v|), |w_{23}|) \leq S_2(|w|_a) \sim S_1(|w|_a)\) by Lemma 2.2(b). Hence it suffices to obtain a desired upper estimate for \(\text{space}(M_{23}(w_0, w'))\). But now \(w''\) is of type 1. Similarly, if \(w_0\) is of type 3, it can be replaced by a word \(w''\) of type 1. Thus Case 2 reduces to Case 1.

**Case 3.** One of the words \(w_0\), \(w_1\) (or both) is of type 2. If all \(w_i\)'s in \(C\) are of type 2, then the commands from \(C\) do not change the lengths, and it is nothing to prove. Otherwise one can find \(j > i\) such that \(w_0, \ldots, w_i\) have equal lengths, \(w_j, \ldots, w_t\) are of the same length too, and the subcomputation \(w_i \rightarrow \cdots \rightarrow w_j\) satisfies the assumptions of Case 1 or of Case 2. This completes the proof. \(\square\)

**Lemma 2.6.** The space function \(S_1(n)\) and the generalized space function \(S_1'(n)\) of the machine \(M_3\) are both equivalent to the space function \(S_1(n)\) of \(M_1\).

**Proof.** By Lemma 2.4 we have \(S_1'(n) \leq S_1(n)\). On the other hand, by the definition of \(S_1(n)\), there is an input-output computation \(C: w_0 \rightarrow \cdots \rightarrow w_t\) of \(M_1\) with the same input and output words \(u = u(w_0) = u(w_t)\) of length \(n\), and with space \(S_1(|u|) = S_2(n)\). Then one can construct a computation \(C': w_0' \rightarrow \cdots \rightarrow w_t'\) of \(M_2\) (and of \(M_3\)) of the same space \(S_1(|u|)\), where the \(M_{23}\)-portion of \(C'\) corresponds to \(C\). By Lemma 2.2(b), any computation of \(M_3\) connecting \(w_0'\) and \(w_t'\) has space at least \(S_1(|u(u_0')|) = S_1(|u|) = S_1(n)\), whence \(S_1'(n) \geq S_1(n)\), and the statement of Lemma 2.6 is proved. \(\square\)

A configuration \(w\) of a machine \(M\) is called *reachable* if there is a computation \(w_0 \rightarrow \cdots \rightarrow w_t = w\), where \(w_0\) is an input configuration of \(M\).

**Lemma 2.7.** If \(w\) is a reachable configuration of the machine \(M_3\), then there is a computation \(w_0 \rightarrow \cdots \rightarrow w_t = w\) of \(M_1\), for any fixed \(w_0, w_t\), and \(w_0, w_t\) are of length \(|w|\), and \(w_t = w\) satisfies the assumptions of Case 1 or of Case 2. This completes the proof. \(\square\)
where every command (or its inverse) is a command of $M_{23}$. It follows that there is an $M_3$-computation $w' \to \cdots \to w'' \to \cdots \to w_s \to \cdots \to w_t$, and the lemma is proved. □

Below we will treat the NTM $M_3$ as a nondeterministic ‘input-input’ machine, i.e., the ‘purpose’ of $M_3$ is to transform an input configuration $w(u)$ to an input configuration $w(v)$. Consider the following relation $u \sim v$ on the set of words in the input alphabet: there exists a (reduced) input-input $M_3$-computation $w(u) \to \cdots \to w(v)$. This is an equivalence relation. Indeed, it becomes reflexive if one adds computations of length 0. The transitivity is obvious, and its symmetry follows from the symmetry of $M_3$. So we use the term equivalence machine (or just $E$-machine) for a symmetric input-input NTM.

2.5 One-tape machine

It is well known that any NTM is equivalent to a one-tape NTM with the same space complexity (see Corollary 1.16 in [9]). But here we take some precautions to preserve the generalized space complexity.

Let $M$ be a $k$-tape E-machine with an input alphabet $A$. We will construct an equivalent (i.e., defining the same equivalence relation on the words over $A$) one-tape E-machine $M'$. $M'$ has the same input alphabet $A$, and at the preliminary stage it inserts the endmarkers $\alpha_2, \ldots, \alpha_k$ and the components $q_{11}, \ldots, q_{1k}$ of the vector of start states $\beta_1$ of $M$, that is, these letters become tape letters of $M'$, and at the first stage, $M'$ converts an input configuration $\alpha_{01}\omega_1 \omega_2 \cdots \omega_m$ of $M'$ into the configuration $\alpha_{01}q_{11}\omega_1 \cdots \alpha_{k1}q_{1k}\omega_k\omega_m$ (i.e., the head of $M'$ runs to $\alpha$, checks the left endmarker, inserts the letter $\alpha_1$, and then returns to $\omega$ inserting the remaining extra-letters $q_{11}\omega_1, \ldots, q_{1k}\omega_k$).

Every configuration $w$ of $M$ is represented by the configuration $W = \alpha\omega_1\omega_2 \cdots \omega_m$, where the state letter $q$ of $M'$ is the vector $(q^{(1)}, \ldots, q^{(k)})$ of states of $w$ (but all the letters of $w$, including the extra-letters, are tape letters for $M'$). For every transition $w \to w' \cdot \theta$ of $M$ with positive command $\theta$, we construct a computation $C(w, \theta) : W \to \cdots \to W'$ of $M'$ as follows. The first command just changes the state $q$ by the state $q_\theta$, i.e., it memorizes $\theta$, and $M'$ will remember $\theta$ until the computation $C(w, \theta)$ ends. This command involves the endmarker $\omega$. Then the head of $M'$ goes to the left and simulates the application of the command $\theta$ when it meets $q^{(i)}$. For example, if the $i$-th part of $\theta$ is $cq^{(i)}a \to dq^{(i)}b$, then the corresponding computation of $M'$ is of the form

$$
\cdots cq^{(i)}q_\theta(1)a \to \cdots cq^{(i)}q_\theta(2)b \to \cdots cq^{(i)}q_\theta(3)q^{(i)}b \to \cdots dq^{(i)}b \to \cdots,$$

where $q_\theta(1), \ldots, q_\theta(4)$ are auxiliary state letters of $M'$. So the head of $M'$ must reach $\alpha$ (there is a command involving $\alpha$) and then returns to $\omega$. The last command of this computation $q^{(i)}b \to q^{(i)}b$ forgets $\theta$, and the state letter $q^{(i)}$ of $W'$ is just the $i$-th vector of states of the configuration $w'$, so that $W'$ corresponds to $w'$.

**Remark 2.8.** Two different positive (or two different negative) commands of $M'$ cannot be applicable to a configuration containing a state letter indexed by some $\theta$.

**Lemma 2.9.** (a) The $E$-machines $M$ and $M'$ recognize the same equivalence relation on the set of input words.

(b) They have equivalent generalized space functions $S'_{M'}(n)$ and $S''_{M'}(n)$.

(c) If $M = M_3$ and $W$ is a reachable configuration of $M'$, then there is a computation $W_0 \to \cdots \to W$, where $W_0$ is an input configuration of $M'$ and $|W_0|_a \leq |W|_a + c$ for a constant $c$ independent of $W$.

(d) If a reduced computation $W_0 \to \cdots \to W_1$ of $M'$ has no commands involving $\alpha$ or has no commands involving $\omega$, then $t$ is bounded from above by $t_c|W_0| + c_2$ for some constants $c_1, c_2$.

(e) If a computation of the form $Uq_1\omega = W_0 \to \cdots \to W_1$, where $q_1$ is the start state of $M'$, has no commands involving $\alpha$ then $|W_0| = |W_1|$ and the computation commands involve tape letters only from the input alphabet. A non-empty computation of the form $Uq_1\omega \to \cdots \to U'q_1\omega$ has a command involving $\alpha$.

(f) Let an $M'$-computation starts with an input configuration $w_0 = \alpha\omega_1\omega_2 \cdots \omega_m$ and ends with $w_1 = \alpha\omega'\omega_2 \cdots \omega_m$. Then $w_1$ is also an input configuration.

**Proof.** Observe that the computation $C(w, \theta) : W \to \cdots$ exists iff one may apply $\theta$ to $w$ and $W$ is the configuration of $M'$ corresponding to $w$. Moreover, if two configurations $W$ and $W'$ of $M'$ represent some configurations $w$ and $w'$ of $M$, then they can be connected by a computation $C(w, \theta)$ iff $w' = w \cdot \theta$. 
Hence to the configuration that at the preliminary stage, the machine reaches and checks the endmarker \( w \) for \( \alpha \mu q \omega \) of \( \alpha q b \), we denote by \( W_{0}, W_{1}, \ldots, W_{s} (0 < i_{0} < \cdots < i_{s} < t) \) the intermediate configurations representing the configurations of \( M \) (i.e., \( M' \) do not remember the commands of \( M \) in these states). Since there are no other configurations with this property between \( W_{i-1} \) and \( W_{i} \) all the commands of the subcomputation \( W_{i-1} \rightarrow \cdots \rightarrow W_{i} \) must correspond to the same command \( \theta \) of \( M \). By Remark 2.8 the history of this subcomputation has no subwords of the form \( \tau^{-1} \tau' \) (of the form \( \tau \tau^{-1} \)), where both \( \tau \) and \( \tau' \) are positive commands of \( M' \). Therefore the subcomputation must be of the form \( C(w_{m-1}, \theta) \) or \( C(w_{m}, \theta)^{-1} \) for a positive command \( \theta \) of \( M \), and \( W_{i-1}, W_{i} \) correspond to \( w_{m-1} \) and to \( w_{m} = w_{m-1} \theta, \) resp., or to \( w_{m-1} = w_{m} \theta \) and to \( w_{m} \), resp.

Since the preliminary stage \( W_{0} \rightarrow \cdots \rightarrow W_{i} \) (resp., \( W_{i} \rightarrow \cdots \rightarrow W_{s} \)) is also deterministic, the pair of input words for \( C' \) coincides with the pair of input words for the computation \( C : w_{1} \rightarrow \cdots \rightarrow w_{s} \) of \( M \), and so \( M \) and \( M' \) recognize the same binary relation.

(b) Let now \( C' : W_{0} \rightarrow \cdots \rightarrow W_{t} \) be an arbitrary reduced computation of \( M' \) with \( \max(|W_{0}|_{\alpha}, |W_{t}|_{\alpha}) \leq n \). We define \( W_{i}, \ldots, W_{s} \) as in Part (a) of the proof. If \( s = 0 \), then for every \( j \), \(|W_{j}|_{\alpha} \leq n + c \) for a constant \( c \) independent of the computation since the computations of the form \( C(w, \theta) \) and the preliminary computations (and their subcomputations), up to a constant, do not change the space. So we will assume that \( s \geq 1 \). Then as in Part (a), the computation \( C' : W_{i} \rightarrow \cdots \rightarrow W_{s} \) corresponds to a computation \( w_{i} \rightarrow \cdots \rightarrow w_{s} \) of \( M \), where \(|w_{j}| \) and \(|W_{i}|_{\alpha} \) are almost (up to an additive constant) equal. Therefore \( w_{i} \) and \( w_{s} \) can be connected by a computation \( C \) of \( M \) of space at most \( S_{M}(n + c) \). There is a computation \( C'' \) of \( M' \) corresponding to \( C \) and having almost the same space. If we replace the subcomputation \( C'' \) by \( C'' \) we get a computation of \( M' \) which connects \( W_{0} \) and \( W_{t} \) and has space \( \leq S_{M}(n + c) + c \). Hence \( S_{M}(n) \leq S_{M}(n) \).

Similarly, if we start with a computation \( C : w_{1} \rightarrow \cdots \rightarrow w_{s} \) of \( M \) with \(|w_{1}|_{\alpha}, |w_{s}|_{\alpha} \leq n \), then we can replace it by a computation of \( M \) of space at most \( S_{M}(n + c) \), whence \( S_{M}(n) \leq S_{M}(n) \), as required.

(c) Assume now that \( W_{i} \) is an input configuration in the computation \( C' \) from (b). Now we consider the computation \( C : w_{0} \rightarrow w_{1} \rightarrow \cdots \rightarrow w_{s} \) of \( M \), where \( w_{j} \) (\( j \leq s \)) corresponds to the configuration \( W_{i} \) of \( M' \). By Lemma 2.7 one can find a computation \( w_{0} \rightarrow \cdots \rightarrow w'_{s} = w_{s} \) of \( M \), such that \( w'_{0} \) is an input configuration and \(|w'_{0}| \leq |w'_{s}| \). Then one can construct a computation \( W_{0}' \rightarrow \cdots \rightarrow W_{i}' \rightarrow \cdots \rightarrow W_{s}' \), where each \( W_{i}' \) represents \( w_{i} \), and therefore \(|W_{0}'| \leq |W_{i}'| \). This gives a computation

\[ W'' \rightarrow \cdots \rightarrow W_{0}' \rightarrow \cdots \rightarrow W_{s}' = W_{s} \rightarrow \cdots \rightarrow W_{i}, \]

where \( W'' \) is an input configuration of \( M' \), and \(|W''| \leq |W_{i}| + c \), as required, because the preliminary subcomputation \( W'' \rightarrow \cdots \rightarrow W_{0}' \) does not decrease the space and the subcomputation \( W_{s} \rightarrow \cdots \rightarrow W_{i} \) is either empty or a part of a computation \( C(w_{s}, \theta) \), and therefore it can remove a bounded number of tape letters.

(d) Follows from the fact that the computation of the form \( C(w, \theta) \) involves both \( \alpha \) and \( \omega \).

(e) The (reduced) work of \( M' \) is deterministic in the beginning: the head goes to the left until it reaches and checks the endmarker \( \alpha \). This implies Property (e).

(f) We must show that \( u' \) is a word in the input alphabet \( A \). For this goal we can (1) assume that the computation is reduced, (2) consider the inverse computation \( w_{i} \rightarrow \cdots \rightarrow w_{0} \), and (3) take into account that at the preliminary stage, the machine \( M' \) verifies (when the head goes to \( \alpha \)) if all the letters of the tape word belong to \( A \).

We will use the doubling of the tape alphabet. This is a well-known trick helpful for simulating of machine commands by (semi)group relations (see 24). Let \( Y \) be a tape alphabet of a one-tape machine \( M \). We denote by \( Y_{l} \) and \( Y_{r} \) two disjoint copies of \( Y \) ('left' and 'right') and replace every configuration \( \alpha u q v \omega \) of \( M \) by \( \alpha u q v, \omega \), where \( u_{l} \) (resp., \( u_{r} \)) is a copy of \( u \) in \( Y_{l} \) (in \( Y_{r} \)). Respectively, one modifies every command, e.g., a command \( a q b \rightarrow a' q' d \) is replaced by \( a q b_{r} \rightarrow c q' d_{r} \). The input alphabet is replaced by its copy \( A_{l} \subset Y_{l} \). Clearly, one obtain one-to-one correspondence between the computations of \( M \) and the computations of the modified TM. The constructed machine inherits the basic properties of \( M \). In particular, it has the same generalized space function.
Lemma 2.10. Assume that a multi-tape DTM $M_0$ solves the word problem in a finitely generated semigroup or monoid $S$ with space function $S_0(n)$. Then there is a one-tape E-machine $M_5$ such that

(a) the equivalence relation recognized by $M_5$ is the set of all pairs of words $(u, v)$ in the generators of $S$ satisfying the equality $u =_S v$;

(b) the generalized space function $S_5(n)$ of $M_5$ is equivalent to $S_0(n)$;

(c) the left and right parts of the tape alphabet of $M_5$ are disjoint;

(d) if $w$ is a reachable configuration of $M_5$, then there is a computation $w_0 \to \cdots \to w$, where $w$ is an input configuration of $M_5$ and $|w_0|_a \leq |w|_a + c$ for an integer $c \geq 1$ independent of $w$;

(e) if a reduced computation $w_0 \to \cdots \to w_t$ of $M_5$ has no commands involving $\alpha$ or has no commands involving $\omega$, then $t$ is bounded from above by $c_1|w_0| + c_2$ for some constants $c_1, c_2$;

(f) if a computation $W_0 = Uq_1\omega \rightarrow \cdots \rightarrow W_t$ of $M_5$ has no commands involving $\alpha$, then $|W_0| = |W_t|$ and the commands of this computation do not involve letters from $Y_1 \setminus A_1$. A non-empty reduced computation $Uq_1\omega \rightarrow \cdots \rightarrow U'q_1\omega$ of $M_5$ has a command involving $\alpha$.

(g) let an $M_5$-computation starts with an input configuration $w_0 = \alpha u q_1 \omega$ and ends with $w_t = \alpha u' q_1 \omega$. Then $w_t$ is also an input configuration of $M_5$.

Proof. Recall that starting with the DTM $M_0$ we have constructed the input-output DTM-s $M_1$, $M_2$, and an E-machine $M_3$. Let us use the construction of this subsection assuming that $M_3 = M$ and and $M_4 = M'$. Doubling the tape alphabet we get a machine $M_5$ providing Property (c). Then the statement (a) follows from Lemmas 2.3(a), 2.9(a), and from the definition of $M_5$. The statement (b) follows from Lemmas 2.1(b), 2.3(b), 2.9(b), and from the definition of $M_5$. Lemma 2.10(c,d,e,f) implies Properties (d), (e), (f) and (g).

3 Defining relations and derivation trapezia

3.1 Embedding homomorphism

Now we define an embedding of $S$ in a finitely presented monoid $H$. Let $A$ be a finite generator set of $S$, and let the machine $M_5$ be given by Lemma 2.10. We have $M_5 = \langle A_1, Y_1 \cup Y_r, Q, \Theta, q_1 \rangle$, where $A_1 \subset Y_1$ is the input alphabet which is the copy of $A$, $Y_1 \cup Y_r$ is the tape alphabet (with left and right parts), $Q$ is the set of states of $M_5$, $\Theta$ is a set of commands, and $q_1 \in Q$ is the start state.

The set of generators of the monoid $H$ is $A_H = A \sqcup Y_1 \sqcup Y_r \sqcup Q \sqcup \{\alpha, \omega, p\}$ where $\alpha$ and $\omega$ are the endmarker symbols of $M_5$, and $p$ is one more generator. The set of defining relations of $H$ is

$$R_H = \{V' = V \text{ for every command } V \to V' \text{ of } M_5\} \cup \{pa = ap \text{ for every } a \in A \text{ and for its copy } a_1 \in A_1\} \cup \{\alpha p = 1, p = q_1\omega\}$$

Lemma 3.1. The identity map on the generator set $A$ of $S$ extends to a homomorphism $\phi : S \to H$. If $S$ has 1 in the signature, then $\phi$ is a monoid homomorphism (i.e. $\phi(1) = 1$).

Proof. Assume that $u =_S v$. We must prove that $u =_H v$.

By Lemma 2.10(a), there is an input-input computation $C$ of $M_5$ starting with $\alpha u q_1 \omega$ and ending with $\alpha v q_1 \omega$, where $u_1$ and $v_1$ are the copies of $u$ and $v$ in the input alphabet $A_1$ of $M_5$. Since the relations $V = V'$ are included in $R_H$ for all the commands $V \to V'$ of $M_5$, all configurations of $C$ are equal in $H$, in particular, $\alpha u q_1 \omega = H \alpha v q_1 \omega$. Using the relation $q_1 \omega = p$, we obtain $\alpha u p = H \alpha v p$. Now applying relations of the form $ap = pa$, we have $\alpha pu = H \alpha pv$. Finally, $u =_H v$ since $ap = 1$ by the definition of $H$.

We will prove in Lemma 3.11 that $\phi$ is an injective homomorphism.

Remark 3.2. It follows from the proof of Lemma 3.1 and from Lemma 2.10(b) that for two equal in $S$ words $u$ and $v$ of length at most $n$, we have $\text{space}_H(u, v) \leq S_5(n) + 3$. 

12
3.2 Derivation trapezia

Assume that \( S = \langle A \mid R \rangle \) is a semigroup or monoid presentation. Then every derivation over this presentation has a visual geometric interpretation in terms of finite connected planar graphs. For group presentations, these graphs are called van Kampen diagrams (see [17]), and semigroup diagrams were introduced by Kashintsev (see [16] and [24]). Below we use a modified approach. Our diagrams uniquely restore derivations, which is preferable when one compares derivations with the computations of a TM.

We call such diagrams derivation trapezia since they look similar to trapezia constructed from bands and associated with group computations (see [20], [26], [21], [2], etc.)

Every cell \( \pi \) is a trapezium in Euclidean plane with horizontal top and bottom. The top and the bottom of a trivial cell are labeled by the same letter from \( A \). In the relation cell \( \pi \) corresponding to a nontrivial relation \( u \rightarrow v \) from \( R \), the bottom is labeled by the word \( u \) and the top is labeled by \( v \). This means that the bottom (the top) is divided into \(|u|\) (resp., \(|v|\)) subsegments of nonzero length, each of the subsegments has a label from \( A \), and one read the word \( u \) (the word \( v \)) on the bottom (on the top) from left to right. The sides of the trapezium \( \pi \) have no labels. Note that \( \pi \) can be a triangle if \(|u| = 0 \) or \(|v| = 0 \); but we will not include the trivial relations of the form \( 1 = 1 \) in \( R \). Also we assume that \( R \) is symmetric, i.e., a relation \( u \rightarrow v \) belongs to \( R \) iff \( v \rightarrow u \) is in \( R \).

For every transition \( w\hat{w}w'' \rightarrow w'v\hat{w}'' \), where \( u = v \) is a defining relation from \( R \), we construct a derivation band as follows. We draw a horizontal parallel paths in the plane, the top and the bottom path directed from left to right. The bottom path (the top path) has \(|\hat{w}|\) (resp., \(|\hat{w}'|\)) edges of nonzero length, each of them is labeled by a letter from \( A \) so that the label of the bottom (the top) path is \( w\hat{w}w'' \) (resp. \( w'v\hat{w}'' \)). Then we connect the initial (the terminal) vertex of the subsegment labeled by \( u \) in the bottom with, respectively, the initial (the terminal) vertex of the subsegment labeled by \( v \) in the top. This gives us the relation cell corresponding to the relation \( u = v \). Finally, we connect the corresponding vertices of the top and the bottom to obtain \(|\hat{w}'| + |\hat{w}''|\) trivial cells of the constructed derivation band. The left-most and the right-most connecting segments are, respectively, the left and the right sides of the derivation band.

It is obvious that every band with at most one transition cell corresponds to an elementary transition \( w \rightarrow w' \). (We allow trivial transitions \( w \rightarrow w \). The corresponding bands have only trivial cells. Note that the band corresponding to the transition \( 1 \rightarrow 1 \) has no cells, but it has unlabeled side edges.)

Let \( w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_t \) be a derivation over \( S \). Then the derivation trapezium \( \Delta \) of height \( t \) corresponding to this derivation is composed of \( t \) derivation bands, where the bottom of the derivation band \( T_{i+1} \) corresponding to the transition \( w_i \rightarrow w_{i+1} \) coincides with the top of the derivation band \( T_i \) corresponding to \( w_{i-1} \rightarrow w_i \) (i = 1, ..., \( t - 1 \)). Thus the label of the bottom (of the top) of \( \Delta \) is \( w_0 \) (resp., \( w_t \)). The left (the right) sides of the derivation bands \( T_i \)-s form the left side (the right side) of \( \Delta \).

We see that every derivation produces a derivation trapezium, and vice versa, every trapezium composed of derivation bands as above, is a derivation trapezium for some derivation (which may admit trivial transitions). Every horizontal edge of a derivation trapezium is labeled, and every vertical one (i.e., connecting the top and the bottom of a derivation band) is unlabeled.

A path is vertical if every its edge is vertical and different edges cross different derivation bands.

We call a derivation trapezium \( \Delta \) indivisible if the only vertical paths connecting the top and the bottom of \( \Delta \) are the left and the right sides of \( \Delta \).
Lemma 3.4. Assume that an \( B \alpha p \) the same \( \alpha p \) bands, derivation band going from left to right and crossing both \( \Delta \) than \( \pi \), \( \pi \) with the same letter labeling the top of \( \pi \) and \( \pi \) bottom (resp., top) of \( \Delta \) or on the boundary of a relation that if a cell \( \pi \), one may switch the order of the corresponding transitions in the derivation \( \pi \) follows:

\[
w_0(1)w_0(2) \rightarrow \cdots \rightarrow w_t(1)w_0(2) \rightarrow \cdots \rightarrow w_t(1)w_t(2),
\]

where the length of this derivation is \( t \) (not \( 2t \)) since some trivial transitions are now omitted in first and in the second parts of (3.6). Thus, the corresponding derivation trapezium \( \Delta' \) has the same height as \( \Delta \) and the same bottom and top labels. The derivation subtrapezia \( \Delta'_1 \) and \( \Delta'_2 \) of \( \Delta' \) are time separated:

the derivation bands corresponding to the nontrivial transitions in \( \Delta'_2 \) follow after the transitions bands corresponding to the nontrivial transitions in \( \Delta'_1 \) (or vice versa). Note that the space of Derivation (3.6) can be greater that the space of the original derivation.

3.3 Vertical bands in trapezia over the monoid \( H \)

derivation bands are horizontal. Now we consider derivation trapezia over the presentation \( H = \langle A_H \mid R_H \rangle \) and define vertical bands, namely \( q \)-bands, \( \alpha \)-bands, \( \omega \)-bands and \( a \)-bands.

By definition, a \( q \)-letter is a letter from \( Q \cup \{ p \} \). A \( q \)-edge is an edge labeled by a \( q \)-letter, a \( q \)-cell is a cell, having a \( q \)-edge in its top or bottom. (So every \( p \)-cell, i.e., having a boundary edge labeled by \( p \), is also a \( q \)-cell.) A \( q \)-band of length \( n \) in a derivation trapezia \( \Delta \) is a sequence of \( q \)-cells \( \pi_1, \ldots, \pi_n \) such that if a cell \( \pi_i \) belongs to a derivation band \( T_j \), then \( \pi_{i+1} \) belongs to \( T_{j+1} \) and these two cells share a \( q \)-edge (\( i = 1, \ldots, n-1 \)).

A \( q \)-band \( C \) is called maximal if it is not contained in a longer \( q \)-band. It follows from the list of defining relations of \( H \) that the first cell \( \pi_1 \) (the last cell \( \pi_n \)) of \( C \) either shares a \( q \)-edge with the bottom of \( \Delta \) (resp. with the top of \( \Delta \)) or it is an \( \alpha \)-cell, i.e., a cell corresponding to the relation \( 1 = \alpha p \) (to the relation \( \alpha p = 1 \), resp.).

Similarly one defines \( \alpha \)- and \( \omega \)-edges, \( \alpha \)-bands and \( \omega \)-bands. The properties of the first and the last cells of maximal \( \alpha \)-bands are similar to the properties of the maximal \( q \)-bands mentioned above. The first cell \( \pi_1 \) (the last cell \( \pi_n \)) of a maximal \( \omega \)-band either shares an \( \omega \)-edge with the bottom of \( \Delta \) (resp. with the top of \( \Delta \)) or it is a \( q_1 \omega \)-cell, i.e., a cell corresponding to the relation \( p = q_1 \omega \) (to the relation \( q_1 \omega = p \), resp.).

An \( a \)-edge is an edge labeled by a letter of the alphabet \( A \cup Y_1 \cup Y_1 \), which, by definition, consists of \( a \)-letters. By definition, an \( a \)-band consists of trivial \( a \)-cells \( \pi_i \)-s with one \( a \)-letter written on the bottom and with the same letter labeling the top of \( \pi_i \)-s. A maximal \( a \)-band must start (end) either on the bottom (resp., top) of \( \Delta \) or on the boundary of a \( q \)-cell having an \( a \)-letter in its top (resp., bottom) label.

The above definitions imply that a cell cannot belong to two different maximal \( q \)-bands (resp., \( \alpha \)-bands, \( \omega \)-bands, \( a \)-bands). If an \( a \)-band \( B \) and a \( q \)-band \( C \) start with the same \( \alpha \)-cell \( \pi \), then any derivation band going from left to right and crossing both \( B \) and \( C \), must first cross \( B \) and then it crosses \( C \) (or it crosses a cell shared by \( B \) and \( C \)). So we may say that the band \( C \) is disposed from the right of \( B \). This simple observation leads to

Lemma 3.4. Assume that an \( a \)-band \( B \) and a \( q \)-band \( C \) of a derivation trapezium \( \Delta \) start or end with the same \( \alpha \)-cell \( \pi \). Then either they end (resp., start) with the same \( \alpha \)-cell \( \pi' \) or they both reach the top (resp., bottom) of \( \Delta \).

Proof. Proving by contradiction, we assume that these two bands start with \( \pi \), and band \( B \) is not longer than \( C \). (The other cases are similar.) Then \( B \) is disposed from the left of \( C \), and the last cell of \( B \) is an \( \alpha \)-cell \( \pi_1 \). Then some \( q \)-band \( C_1 \) must also terminate at \( \pi_1 \), and \( C_1 \) has to be placed from the right of \( B \) and from the left of \( C \). Since it cannot start with \( \pi \), \( C_1 \) has to start with an \( \alpha \)-cell \( \pi_2 \) belonging to a
derivation band situated above the derivation band containing the cell $\pi$. Similarly, an $\alpha$-band $B_1$ must start with $\pi_2$, it is disposed from the left of $C_1$ and from the right of $B$, and its last cell is an $\alpha p$-cell $\pi_3 \neq \pi_1$. Reasoning this way, we can get arbitrarily many cells in $\Delta$, a contradiction. 

Lemma 3.4 implies that there can exist maximal $\alpha$- and $q$-bands of three types in a trapezium $\Delta$:

1. The bands connecting an $\alpha$-edge (or a $q$-edge) of the bottom of $\Delta$ with an $\alpha$-edge (or a $q$-edge) of the top. We call such bands through bands.

2. Pairs formed by an $\alpha$-band and a $q$-band, sharing their the first and the last cells. We call such a pair an $\alpha q$-lens.

3. Pairs formed by an $\alpha$-band and a $q$-band, sharing the first (the last) cell and terminating (resp., starting) on the $\alpha$- and $q$-edges of the top (resp, bottom) of $\Delta$. We say that such a pair form an $\alpha q$-cap (resp., $\alpha q$-cup).

Lemma 3.5. Let $\pi$ and $\pi'$ be, resp., the first cell and the last cell of a maximal $\omega$-band $D$. Then

(a) $\pi$ and $\pi'$ cannot belong to different maximal $q$-bands.

(b) If $\pi$ belongs to the maximal $q$-band $C$ of a $\alpha q$-lens, then $\pi'$ also belongs to $C$.

(c) If an $\alpha q$-cap (or cup) $\Gamma$ surrounds no smaller caps (resp. cups), then $\Gamma$ surrounds no $\omega$-bands starting on the bottom (resp, on the top) of $\Delta$.

Proof. (a) Arguing by contradiction we assume that $D$ is a shortest counter-example. It starts on some maximal $q$-band $C$ and ends on a maximal $q$-band $C'$. Since $D$ is situated from the right of both $C$ and $C'$, and these two $q$-bands do not cross, either $C$ does not reach the top of the trapezium $\Delta$ or $C'$ does not start on the bottom of $\Delta$. Choosing the former case, we deduce that $\pi$ ends with a $\alpha p$-cell $\pi_0$, and the subband $C_1$ of $C$ with the first cell $\pi$ and the lats one $\pi_0$ is shorter than $D$.

Since $\pi$ has a top edge labeled by $q_1$ and $\pi_0$ has a bottom edge labeled by $p$, there must be a cell in $C_1$ which corresponds to the relation $q_1 \omega = p$. Moreover, the number of such cells in $C_1 \setminus \pi$ must be greater
than the number of cells corresponding to the transition $p \to q_1 \omega$. Therefore there is a cell in $C_1$, say $\pi_1$, such that a maximal $\omega$-band $D'$ ends with $\pi_1$ but it does not start on $C_1$. This $\omega$-band $D'$ is situated from the right of $C$ and from the left of $D$. Since the bands $C$ and $D$ have the common cell $\pi$, the band $D'$ is shorter than $C_1$, and consequently, it is shorter than $D$.

We come to a contradiction with the choice of $D$, and Claim (a) is proved.

(b) The assumption that $D$ starts on $C$ and ends on the top of $\Delta$ provides us, as in the proof of (a), with an $\omega$-band $D'$ connecting two different maximal $q$-bands. Thus Property (b) is proved by contradiction.

(c) Follows from (b) since an $\omega$-band cannot start and end on the bottom (resp., on the top) of $\Delta$. \hfill \Box

### 3.4 Minimal trapezia

If a $q$-band $C$ of a derivation trapezium $\Delta$ over $H$ has $k$ $q_1 \omega$-cells, then we say that $C$ has type $k$. Suppose $\Delta$ has $\tau_1$ through $q$-bands of type $i$, $\sigma_i$ maximal $q$-bands of type $i$ in the $\alpha q$-caps and $\omega q$-caps, and $\rho_i$ maximal $q$-bands of type $i$ in the $\alpha q$-lenses, $i = 0, \ldots, k$, and $\Delta$ has no $q$-bands of types $> k$. Then we say that $\Delta$ is a trapezium of type $\tau(\Delta) = (\tau_0, \sigma_0, \rho_0, \ldots, \tau_k, \sigma_k, \rho_k, 0, 0, 0 \ldots)$. Assume that $\tau(\Delta) = (\tau_0', \sigma_0', \rho_0', \ldots, \tau_k', \sigma_k', \rho_k', 0, 0, 0 \ldots)$. Then by definition $\tau(\Delta) > \tau(\Delta')$ if there is $l$ such that $\tau_l > \tau'_l$ or $\tau_l = \tau'_l$ and $\sigma_l > \sigma'_l$, or $\tau_l = \tau'_l$ and $\sigma_l = \sigma'_l$, but $\rho_l > \rho'_l$, and $\tau_m = \tau'_m$, $\sigma_m = \sigma'_m$, $\rho_m = \rho'_m$ for every $m \geq l$.

Clearly, the defined order on derivation trapezia over $H$ satisfies the descending chain condition, and so there is a trapezium having the smallest type among all trapezia with the same bottom an top labels. Such a derivation trapezium is called a minimal trapezium.

**Remark 3.6.** It is easy to see that the time separation trick from Remark 3.3 preserves the numbers of $\alpha q$-cells and $q_1 \omega$-cells in every maximal $q$-band, and so it does not change the types of maximal $q$-bands. Therefore it preserves the minimality of a trapezium. The same is true if one rebuilds two derivation band of a derivation trapezium replacing a subderivation $w \to w \to w'$ by $w \to w' \to w'$ or vice versa.

**Lemma 3.7.** Assume that $p$ is a simple closed path in a minimal trapezium $\Delta$, and every edge of $p$ is unlabeled. Then the closed region $O$ of $\Delta$ bounded by $p$ contains no $q$-edges.

**Proof.** Let us shrink to a point every labeled (horizontal) edge which is inside $O$. If after this surgery some unlabeled (vertical) edges connect the same vertices, we identify such edges.

![Diagram](image)

It is clear that we replace every derivation band of $\Delta$ by a derivation band of the obtained trapezium $\Delta'$ (but the cells belonging to $O$ are removed), $\Delta'$ has the same top and bottom labels as $\Delta$, and $\tau(\Delta') < \tau(\Delta)$ if $O$ has at least one $q$-edge (and therefore contains a maximal $q$-band). Since $\Delta$ is a minimal trapezium, the lemma is proved. \hfill \Box

If a maximal $\omega$-band $D$ starts on the right side of the maximal $q$-band $C$ of a $\alpha q$-lens $E$ then $D$ also terminates on $C$ by Lemma 3.3 (b). Let us attach all such maximal $\omega$-bands to $C$ and call the obtained figure $\Gamma$ a thick lens.
Lemma 3.8. Every edge of the outer boundary component $x$ of a thick lens $\Gamma$ is either unlabeled or labeled by a letter from the alphabet $A$.

Proof. Every edge of an $\alpha$-cell (of an $\omega$-cell) of $\Gamma$ lying on $x$ is unlabeled since $\alpha$ (resp., $\omega$) can occur only as the left-most (resp., the right-most) letter in the relator words of $H$. It follows from the definitions of bands and $\Gamma$ that the edges of $\alpha p$-cells and of $q_1 \omega$-cells belonging to $x$ are unlabeled too.

Since the cells of $C$ corresponding to the relation $p = q_1 \omega$ and to $q_1 \omega = p$ alternate in its maximal $q$-band $C$ of $\Gamma$, every maximal $\omega$-band of $\Gamma$ must start with a cell of $C$ corresponding to $p = q_1 \omega$ and end on the next $q_1 \omega$-cell of $C$ corresponding to $q_1 \omega = p$. Hence the only cells of $C$ having edges in $x$ are $p$-cells, and labeled edges of $x$ are their $a$-edges on the right side of $C$. So they are labeled by letters from $A$ (see Relations (3.5)).

We say that a closed region $O$ in a derivation trapezium is generated by a thick lens $\Gamma$ (or by the $\alpha q$-lens $E$ defining $\Gamma$) if (1) $O$ contains $\Gamma$; (2) if $O$ contains an edge $e$ of a cell $\pi$ and $e$ is labeled by a letter from $A$, then $O$ contains $\pi$; (3) if $O$ contains an edge from the outer boundary of some thick lens $\Gamma'$, then $O$ contains $\Gamma'$ (4) $O$ is minimal with respect to (1)–(3).

Lemma 3.9. Let a region $O$ of a derivation trapezium is generated by a $\alpha q$-lens $E$ or by a thick lens $\Gamma$. Then every edge in the outer boundary component of $O$ is either unlabeled or has a label from $A$.

Proof. By the definition of $O$, it constructed from several thick lenses and several maximal $a$-bands which start/terminate on the thick lenses and correspond to $a$-letters from $A$. So Lemma 3.8 completes the proof.

Lemma 3.10. Let $\Delta$ be a minimal trapezium over $H$. Then

(a) An $\alpha q$-lens $\Gamma$ of $\Delta$ encloses no other $\alpha q$-lenses and no $\omega$-cells.

(b) Let $\Delta$ have a through $q$-band $C$, and assume that the top and bottom edges of $\Delta$ from the left of $C$ are labeled by letters from $\{\alpha\} \cup Y_l$. Then there are no $\alpha q$-lenses and no $\omega$-cells from the left of $C$.

(c) Assume that an $\alpha q$-cap (or cup) $\Gamma$ of $\Delta$ encloses a $\alpha q$-lens $E$. Then the closed region $O$ generated by $E$ does not share any labeled edge with $\Gamma$.

Proof. (a) Assume that an $\alpha q$-lens $E$ is enclosed in $\Gamma$, and there are no bigger $\alpha q$-lenses enclosed in $\Gamma$ and surrounding $E$. Note that the region $O$ generated by the $\alpha q$-lens $E$ is also enclosed in $\Gamma$. By Lemma 3.9 every labeled edge $e$ of the outer boundary component $p$ of $O$ must be connected by a maximal $a$-band $A$ (of length $\geq 0$) with an $a$-edge $f$ on the left side of the maximal $q$-band $C$ of $\Gamma$. However $f$ is labeled by a letter from $Y_l$ while $e$ is labeled by a letter from $A$. This contradicts to the condition $Y_l \cap A = \emptyset$, and so $p$ has no labeled edges.

Then Lemma 3.7 gives another contradiction since $O$ contains $q$-edges of the $\alpha q$-lens $E$. Hence our assumption false, and $\Gamma$ surrounds no $\alpha q$-lenses.

The second assertion of (a) is also true since an $\omega$-band enclosed in $\Gamma$ cannot start/end on $C$.

(b) The same proof as for (a), but now $\Gamma$ is the part of $\Delta$ from the left of $C$.

(c) Follows from Lemma 3.9
Lemma 3.11. (a) Assume that an $a$-band $A$ starts and ends on a $q$-band $C$ of a minimal trapezium, and $C$ has no edges labeled by $p$. Then $A$ and $C$ surrounds no $aq$-lenses and no $\omega$-cells.

(b) Let an $\omega$-band $D$ start and end on a $q$-band $C$ of a minimal trapezium $\Delta$. Then $C$ and $D$ surround no $\omega$, $\alpha$, or $q$-cells.

Proof. (a) It follows from the assumption of the lemma that the $a$-band $A$ corresponds to a letter from $Y_1$ (from $Y_2$) if it is disposed from the left (resp., from the right) of $C$. Now the assumption that $A$ and $C$ surround an $\alpha q$-lens $E$ gives a contradiction as in Lemma 3.10. The second claim is obvious since $C$ has no $p$-edges, and so no $\omega$-band can start on the band $C$.

(b) Let $D$ start with a $q_1\omega$-cell $\pi$ of $C$. Then $\pi$ corresponds to the relation $p = q_1\omega$. Therefore the next $q_1\omega$-cell $\pi'$ of $C$ must correspond to the relation $q_1\omega = p$, and some maximal $\omega$-band $D'$ terminates at $\pi'$. Since by Lemma 3.5 $D'$ must also start on $C$, and different maximal $\omega$-bands cannot cross each other, we conclude that $D' = D$. In other words, if the closed region $\Gamma$ bounded by $C$ and $D$ (where the cells from $C$ and $D$ do not belong to $\Gamma$) encloses a maximal $\omega$-band, then such a band must connect two cells of an $aq$-lens enclosed in $\Gamma$. If $\Gamma$ contains $\alpha$- or $q$-cells, then $\Gamma$ contains $aq$-lenses as well. But such an assumption leads to a contradiction exactly as in the ‘right’ version of Part (a).

\[\square\]

3.5 Types of $q$-bands in minimal trapezia

A derivation trapezium $\Delta$ will be called a machine trapezium if the top or the bottom label $w$ of $\Delta$ is a configuration of the machine $M_5$ and every nontrivial cell corresponds to one of the machine relations (3.4) (i.e., there are no cells corresponding to the auxiliary relations (3.5)).

Let $w$ and $w'$ be the bottom and the top labels of a derivation band of a machine trapezium $\Delta$. Then it follows by the induction on the height of $\Delta$ that either $w' = w$ or this band correspond to a transition $w \rightarrow w'$ of $M_5$. Therefore both the bottom and the top labels of $\Delta$ are configurations of $M_5$, and they can be connected by a computation of $M_5$.

The definition of peeled machine trapezium is similar but now $w$ can be a configuration of $M_5$ without one of the endmarkers $\alpha$ or $\omega$, or without both. Hence the bottom and top labels of a peeled machine trapezium plus the additional letter $\alpha$ in the beginning or the letter $\omega$ at the end of them are connected by a computation of $M_5$ without commands involving $\alpha$ or $\omega$, or both, resp.

Lemma 3.12. Assume that $\Delta$ be a minimal trapezium over $H$.

(a) Let $\Gamma$ be an $aq$-lens in $\Delta$ formed by an $a$-band $B$ and $q$-band $C$. Then the type of $C$ is 2.

(b) Assume that $\Delta$ has a through $q$-band $C$ and a through $\alpha$-band $B$ from the left of $C$. Then there is no horizontal path $x$ starting with an $\alpha$-edge of $B$, ending with a $p$-edge of $C$, and having label $\alpha U p$, where $U$ is a word in the alphabet $A_1$.

(c) Assume that $\Delta$ has an $\alpha q$-cap or an $\alpha q$-cup $\Gamma$ with maximal $\alpha$-band $B$ and maximal $q$-band $C$. Also assume that there are neither $aq$-lenses nor $aq$-caps/cups enclosed in $\Gamma$. Then the type of $C$ is at most 1.

Proof. (a) Denote by $\pi_1$ and $\pi_m$ the first and the last cells of $C$. They correspond to the relations $1 = \alpha p$ and $\alpha p = 1$, resp. Therefore $C$ has equal number of cells corresponding to the relation $p = q_1\omega$ and to $q_1\omega = p$, in particular, the type $t$ of $C$ is even.

Case 1. Assume that $t = 0$. Then each of the cells $\pi_2, \ldots, \pi_{m-1}$ is either trivial or corresponds to a relation $pa = ap$ (or to $ap = pa$). Note that by Lemma 3.10 there are neither $aq$-lenses nor $\omega$-cells enclosed in $\Gamma$. Hence every cell between $B$ and $C$ is a trivial $a$-cell and it belongs to an $a$-band starting and ending on $C$ and corresponding to a letter from $A_t$. A right-most $a$-band $A$ enclosed in $\Gamma$ and a part of $C$ form a derivation (sub)trapezium with top and bottom labels equal to $pa$ ($a \in A_t$). Therefore one can replace this subtrapezium by a trapezium having only trivial cells.
This surgery reduces the number of nontrivial cells in the $q$-band $C$. Finally, we will have an $\alpha q$-lens with unlabeled (outer) boundary. Hence the $\alpha q$-lens can be removed from $\Delta$ by Lemma 3.7. Since $\Delta$ is a minimal trapezium, the case $t = 0$ is not possible.

**Case 2.** Assume now that $t \geq 4$, and let $\pi_i$ and $\pi_j$ be the first cells of $C$ corresponding to the relations $p = q_1 \omega$ and $q_1 \omega = p$, respectively. Let $\Delta_0$ be the trapezium formed by the derivation bands $T_{k}, \ldots, T_{k+j-i}$ of $\Delta$ containing $\pi_i, \ldots, \pi_j$, resp. There are two vertical paths dividing $\Delta_0$: The left side of $B$ and the right side of the $\omega$-band starting with $\pi_i$ and ending with $\pi_j$ (see Lemma 3.5). These paths divide $\Delta_0$ into 3 subtrapezia $\Delta_1, \Delta_2, \Delta_3$ (from left to right).

Applying the time separation trick (see Remarks 3.3 and 3.6) with possible decrease of the length of $C$, we may assume that the derivation corresponding to $\Delta_2$ has no trivial transitions, in particular, the middle part of this derivation of length $j - i - 1$ has this property too. It corresponds to the trapezium $\Delta'_2$ obtained from $\Delta_2$ by the deleting of the first and the last derivation bands.

Observe that by Lemma 3.10 $\Delta'_2$ has no cells corresponding to the auxiliary relations (3.5), and the bottom label of it is of the form $\alpha u q_1 \omega$ for some word $u$ in the tape alphabet of $M_5$. In fact, $u$ is a word in $A$ since by the definition of $\pi_i$, an $a$-band ending on the bottom of $\Delta_2'$ starts on a cell of $C$ having an edge labeled by $p$. Therefore $\alpha u q_1 \omega$ is a configuration of $M_5$, and moreover, it is an input configuration, and $\Delta'_2$ is a machine trapezium.

The derivation trapezium $\Delta'_2$ corresponds to a computation $C$ of $M_5$. By Lemma 2.10 (g), the computation $C$ must be an input-input computation, since it ends with a configuration $\alpha u' q_1 \omega$.

Thus using (3.5) we can construct the following derivation $D$ starting with the bottom label of $T_{k+j-i}$ (this word contains the subword $\alpha u' q_1 \omega$):

\begin{align*}
(\ldots \alpha u' q_1 \omega \ldots) &\rightarrow (\ldots \alpha u' p \ldots) \rightarrow \cdots \rightarrow (\ldots \alpha p u'' \ldots) \rightarrow (\ldots u'' \ldots),
\end{align*}

where $u''$ is the copy of $u'$ in the alphabet $A$. This property makes possible the following surgery with $\Delta$. We cut $\Delta$ along the bottom path of $T_{k+j-i}$, and insert mutually mirror trapezia corresponding to the derivation $D$ and to its inverse. Since $D$ removes the $\alpha$- and $q$-letters in the distinguished subword, this surgery replaces the $\alpha q$-lens $\Gamma$ by an $\alpha q$-lens with maximal $q$-band of type 2 and an $\alpha q$-lens with maximal $q$-band of type $t - 2$. Since all maximal $q$-bands of $\Delta$, except for $C$ are untouched by this surgery (more
precisely, we added several trivial cells to some of them), the obtained trapezium has smaller type than \( \Delta \), a contradiction.

(b) Proving by contradiction, we may assume that the top or bottom of some derivation band contains the subpath \( x \). So its label is of the form \( \ldots \alpha U' p \ldots \). Again due to Relations (3.5), \( \alpha U' p = U' \), where \( U' \) is the copy of \( U \) in \( A \). Hence one can use the same trick as in (a) and replace the through bands \( B \) and \( C \) by a cap and a cup, and the sum of types of their \( q \)-bands is equal to the type of \( C \), contrary to the minimality of \( \Delta \).

(c) We may assume that \( \Gamma \) is a cup. It follows from the assumptions of the lemma that there are no cells corresponding to Relations (3.5) surrounded by \( \Gamma \) and the top of \( \Delta \). If the type of \( \mathcal{C} \) is at least two, then, as in Case (a), one can consider two \( q \)-cells \( \pi_i \) and \( \pi_j \) in \( \mathcal{C} \) and then using similar surgery, replace the cup \( \Gamma \) by a cup of smaller type and an \( \alpha q \)-lens. So our assumption leads to a contradiction with the minimality of \( \Delta \).

The lemma is proved. \( \square \)

**Remark 3.13.** (a) By Lemma 3.12 the maximal \( q \)-band \( \mathcal{C} \) of an \( \alpha q \)-lens \( E \) in a minimal trapezium \( \Delta \) has exactly two \( q \)-cells, say, \( \pi_i \) and \( \pi_j \). By Lemma 3.3 (a), these two cells are connected (from the right of \( \mathcal{C} \)) by a maximal \( \omega \)-band \( D \). We obtain a thick lens \( \Gamma \) by adding \( D \) to \( \Gamma \). The cells of \( \mathcal{C} \) under \( \pi_i \) and above \( \pi_j \) (including \( \pi_i \) and \( \pi_j \)) correspond to the auxiliary relations (3.5). So the edges of the outer boundary of the thick lens are either unlabeled or labeled by letters from \( A \).

(b) One can argue as in Case 2 of Lemma 3.12 (though \( t = 2 \) now) and obtain the subtrapezium \( \Delta_0 \) and its parts \( \Delta'_0 \subset \Delta_2 \). As in the proof of Lemma 3.12, we may assume that \( \Delta'_0 \) is a machine trapezium, and every derivation band of it corresponds to a (non-trivial) command of an input-computation of \( M_0 \), and so the top and the bottom of \( \Delta_0 \) have labels of the form \( w'u_ww'' \) and \( w'v_ww'' \), where \( u_1 \) and \( v_1 \) are words in the alphabet \( A_1 \), and the copies \( u \) and \( v \) of these words in the alphabet \( A \) are equal in \( S \) by Lemma 2.10 (a). We will call \( \Delta'_2 \) the machine part of the thick lens \( \Gamma \); \( \Delta_2 = \bar{M}(\Gamma) \) is the augmented machine part of \( \Gamma \). (It worths to note that it contains nontrivial \( q \)-cells in the first and in the last derivation bands.)

**Lemma 3.14.** The homomorphism \( \phi : S \to H \) defined in Lemma 2.10 is injective.

**Proof.** Let \( w \) and \( w' \) be two words in generators of \( S \), i.e., in the alphabet \( A \). Assuming that \( w =_{H} w' \), we must prove that \( w =_{S} w' \). So we have a derivation \( w = w_0 \to \cdots \to w_t = w' \) over \( H \) and denote by \( \Delta \) the corresponding minimal derivation trapezium. Since the boundary labels \( w \) and \( w' \) of \( \Delta \) have neither \( \alpha \)- nor \( q \)-letters, all maximal \( \alpha \)- and \( q \)-bands (if any) are paired in some \( \alpha q \)-lenses \( E_1, \ldots, E_k \), and neither of the corresponding thick lenses \( \Gamma_1, \ldots, \Gamma_k \) is enclosed in another one by Lemmas 3.10 and 3.11 (b).

We may assume that the first derivation band of \( \Delta \) containing a \( q \)-cell from \( \cup_{l=1}^{k} E_l \) (it exists by Lemma 3.12 if \( k > 0 \)) does contain a cell from \( E_1 \), and so it does not contain other \( q \)-cells. Let \( M(\Gamma_1) \) be the augmented machine part of the lens \( \Gamma_1 \) (see Remark 3.13). As in Remark 3.13, we may use the time separation trick, and have each of the lowest \( q \)-cells of \( \Gamma_2 \ldots \Gamma_k \) disposed in the derivation bands of \( \Delta \) with higher numbers than the derivation bands containing any of the cells of \( \Gamma_1 \). Therefore the time separation trick can now be applied to \( \Gamma_2 \). This reconstruction does not touch \( M(\Gamma_1) \) and creates \( M(\Gamma_2) \) with cells disposed above the derivation bands of \( \Delta \) crossing \( M(\Gamma_1) \). Finally, we replace \( \Delta \) by a minimal trapezium with the same top and bottom labels, where the augmented machine part \( M(\Gamma_i) \) lies above \( \bar{M}(\Gamma_{i-1}) \) for \( i = 2, \ldots, k \).

We will keep the same notation \( \Delta \) for the obtained trapezium. In every word \( w_i \) of the derivation \( w_0 \to \cdots \to w_t \) corresponding to \( \Delta \), we delete all letters which do not belong to \( A \cup A_1 \), replace every letter from \( A_1 \) by its copy from \( A \) and denote the obtained word from \( A^* \) with \( \psi(w_i) = w_i \).

By Remark 3.13, \( W_r =_{S} W_s \) if \( w_r \) and \( w_s \) include, resp., the top and the bottom labels of some \( M(\Gamma_i) \). If \( E \) is a trapezium formed by the derivation bands of \( \Delta \) situated between \( M(\Gamma_{i-1}) \) and \( M(\Gamma_i) \) (or between the bottom (the top) of \( \Delta \) and \( M(\Gamma_1) \) (and \( M(\Gamma_k) \))), and \( W_r \) and \( W_s \) are \( \psi \)-images of the top and the bottom labels of \( E \), then \( W_r = W_s \), because the derivation \( w_s \to \cdots \to w_r \) uses only the auxiliary relation (3.5).

Consequently, \( W_0 =_{S} W_t \), and so \( w_0 = W_0 =_{S} W_t = w_t \), as required. \( \square \)

**Lemma 3.15.** Let \( \Delta \) be a minimal derivation trapezium over \( H \) with the bottom label \( w_0 = (\alpha)U p V \) and the top label \( w_t = (\alpha)U' p V' \), where \( U, U' \) are words in \( A_1 \), \( V, V' \) are words in \( A \), and \( \alpha \) can be absent in
both labels. We assume that $\Delta$ has a through $q$-band $C$. Then using notation of Lemma 3.14 we have $\psi(w_0) =_S \psi(w_l)$.

Proof. Let $\Gamma_1, \ldots, \Gamma_k$ be all the thick lenses of $\Delta$ ($k \geq 0$). By Lemma 3.10 (b) none of them is placed from the left of $C$. If the type of $C$ is equal to $2l \geq 0$, then it has $2l q_1\omega$-cells, and using these cells one can define $l$ (peeled) augmented machine trapezium $\bar{M}_1, \ldots, \bar{M}_l$, where each $\bar{M}_j$ is bounded from the left by a portion of a through $\alpha$-band $B$ (or by the left side of $\Delta$ if there is no $\alpha$-bands in $\Delta$) and bounded from the right by an $\omega$-band connecting some $q_1\omega$-cells of $C$. Note that by Lemma 3.11 (b) and the above observation, these (peeled) trapezia have no lenses.

Now one can apply the time separation trick to the system $\bar{M}_1, \ldots, \bar{M}_l, \bar{M}(\Gamma_1), \ldots, \bar{M}(\Gamma_k)$ as this was done for the augmented machine parts of $\Gamma_1, \ldots, \Gamma_k$ in the proof of Lemma 3.14. So as there, we will have $\psi(w_0) =_S \psi(w_l)$, and the lemma is proved. \hfill \Box

3.6 A-triangles in derivation trapezia

Assume that $x$ is a nontrivial subpath of the bottom (or of the top) of a trapezium $\Delta$, and two vertical paths $y$ and $z$ start at $x_-$ (the original vertex of the path $x$) and $x_+$ (the terminal vertex of $x$), resp. If $y_+ = z_+$ and there are no other common vertices of $y$ and $z$, then we say that $x, y, z$ bound a triangle subtrapezium $\Delta_0$ of $\Delta$. (A triangle trapezium corresponds to a derivation ending or starting with the empty word 1.) If the base $x$ is labeled by a word in $A$, we say that $\Delta_0$ is an $A$-triangle.

Lemma 3.16. (a) Assume that $\Gamma$ is an $aq$-cap or an $aq$-cup in a minimal trapezium $\Delta$, and there are no other $aq$-caps (resp., cups) enclosed in $\Gamma$. Assume that there is an $aq$-lens $E$ enclosed in $\Gamma$. Then there is an $A$-triangle $\Delta_0$ in $\Delta$, containing $E$ and enclosed in $\Gamma$.

(b) Let $\Gamma$ be a triangle in a minimal trapezium $\Delta$. Assume that there are no $aq$-caps or $aq$-cups but there is an $aq$-lens $E$ enclosed in $\Gamma$. Then there is an $A$-triangle $\Delta_0$ containing $E$ and enclosed in $\Gamma$.

(c) Assume that a $q$-band $C$ and an $\omega$-band $D$ start (or end) with the same $q_1\omega$-cell and end (resp. start) on the top (resp., on the bottom) of $\Delta$. If there are no $aq$-caps or $aq$-cups but there is an $aq$-lens surrounded by these two bands and by the top (by the bottom) of $\Delta$, then $\Delta$ has an $A$-triangle containing $E$.

(d) Let $C$ be a through $q$-band without $p$-edges in a minimal indivisible trapezium $\Delta$ and $D$ a through $\omega$-band from the right of $C$. Suppose there are neither $aq$-caps, nor $aq$-cups, nor through bands between $C$ and $D$, but there is an $aq$-lens $E$ between them. Then there is an $A$-triangle $\Delta_0$ containing $E$ between $C$ and $D$.

Proof. (a) We will assume that $\Gamma$ is an $aq$-cap. Let $O$ be the closed region generated by $E$ (see Subsection 3.3). Note that by Lemma 3.10 (c), no labeled edge of $\Gamma$ belongs to $O$, and by Lemma 3.7 and the definition of $O$, every labeled edge of the outer boundary of $O$ belongs to the bottom of $\Delta$.

If $O$ has no edges on the bottom of $\Delta$, then the (outer) boundary of $O$ has no labeled edges. This would contradict Lemma 3.7. So the region $O$ is enclosed in a simple loop $xp$, where $x$ is the minimal subpath of the bottom of $\Delta$ containing all bottom edges belonging to $O$, and $p$ is an unlabeled path on the boundary of $O$.

We select a factorization $p = zy^{-1}$, where the last edges of both $y$ and $z$ go upward, and consider two cases.

Case 1. Assume that both $z$ and $y$ are vertical paths. Then $x, y, z$ bound a triangle trapezium $\Delta_0$. By Lemma 3.3 (b), $x$ has no $\omega$-edges since the maximal $\omega$-band starting on such edge could not end anywhere. So every band starting on $x$ must reach $p$ on (or an edge of a closed region generated by another $aq$-lens). Therefore the label of this edge belongs to $A$ by Lemma 3.4. Hence $\Delta_0$ is an $A$-triangle.

Case 2. One of the paths $y, z$, say $y$ is not vertical. Then there is a subpath $eg_1 \ldots g_lf$ ($l \geq 1$) in $y$, where the edges $e$ and $f$ go upward, but all the edges $g_1, \ldots, g_l$ go downward.
Since both $e^{-1}$ and $g_1$ are directed downward and they start from the same vertex, there must be a cell $\pi$ in $\Gamma$ corresponding to the relation $\alpha p = 1$ and having a common edge with the path $eg_1$. Similarly, there is a cell $\pi'$, corresponding to the relation $1 = \alpha p$ and having an edge from the subpath $qf$. Observe that if $\pi$ belongs to $O$, then $\pi'$ lies outside this region, and vice versa, since $p$ is a part of the boundary of $O$. Let us assume that $\pi$ does not belong to $O$. However the $\alpha p$-cell $\pi$ must belong to one of the $\alpha q$-lenses enclosed in $\Gamma$. This contradicts the definition of the region $O$ since $\pi$ has an edge belonging to the boundary of $O$. Thus Case 2 is impossible, and Statement (a) is proved.

(b,c) The same proof as for (a) since two sides of the triangle are simply unlabeled now, and in Case (c), only $a$-band corresponding to the letters from $Y_r$ can start on $C$.

(d) The proof is similar to that in (a) but there might happen that a segment of the boundary of the closed region $O$ connects the top and the bottom of $\Delta$. As in Case 2 above, this segment must be vertical. But this would imply that $\Delta$ is a divisible trapezium, a contradiction.

Let a minimal trapezium $\Delta$ have no $A$-triangles and have an $\alpha q$-cup (or a cap) $\Gamma$ satisfying the assumptions of Lemma 3.12 (c). We also assume that $\Delta$ has no cups (or caps) of smaller height than $\Gamma$ (i.e., with maximal $q$-band shorter than $C$). We define the base label $b(\Gamma)$ of $\Gamma$ as follows. If the type of the $q$-band $C$ of $\Gamma$ is 0 (and so all cells of $C$ are $p$-cells), then $b(\Gamma)$ is just the word $\alpha Wp$ we read on the top (or on the bottom) of $\Delta$ between the $\alpha$-edge and the $q$-edge of $\Gamma$. If the type of $C$ is 1, then $C$ has one $q_1\omega$-cell, and so one maximal $\omega$-band $D$ starts on $C$ from the right and, by Lemma 3.5 (a), ends on the top (or on the bottom) of $\Delta$. Then $b(\Gamma)$ is the word $\alpha UqV\omega$ we read between the ends of $B$ and $D$ on the top/bottom of $\Delta$.

Lemma 3.17. Under the above restrictions, (1) if $C$ has type 0, then $W$ is a word in the alphabet $A_1$; (2) if the type of $C$ is 1, then the base label $b(\Gamma)$ is a reachable configuration of the machine $M_5$.

Proof. If the type of $C$ is 0, then every maximal band starting on the segment labeled by $W$ ends on a $p$-cell of $C$. This implies the first statement of the lemma. To proof the second one, we consider the derivation bands of $\Delta$ crossing $D$. They form a derivation trapezium $\Delta'$. Let $\Delta''$ be a subtrapezium of $\Delta'$ bounded from the left by the left side of $B$ (which is vertical) and bounded from the right by the right side of $D$ (which is vertical too).

The bottom of $\Delta''$ has label of the form $\alpha W q_1\omega$, where $W$ is a word in $A_1$ since the underlying part of the cup $\Gamma$ has the $q$-band of type 0. The part of $\Delta''$ between $B$ and $C$ has no auxiliary cells (corresponding to Relations (5.5b)) because $\Gamma$ surrounds no $q$-cells. There are no $\alpha q$-cups/caps between $C$ and $D$ by the minimality of the height of $\Gamma$. Also there are no lenses there by Lemma 3.10 (d). Hence the part of $\Delta''$ between $C$ and $D$ has no auxiliary cells too. Therefore $\Delta''$ is a machine trapezium with bottom label $\alpha W q_1\omega$, and so its top label $b(\Gamma)$ is reachable by $M_5$, as required.

4 Indivisible trapezia and completion of proofs

4.1 Upper bounds for spaces of derivations

We will assume in Lemmas 4.1 - 4.7 that $\Delta$ is an indivisible minimal trapezium without caps, cups, and $A$-triangles. Let $\Delta$ correspond to a derivation $D : w_0 \rightarrow \cdots \rightarrow w_t$ over $H$, and $x,y$ are the top and the bottom of $\Delta$, resp.
Lemma 4.1. $\Delta$ has at most one $\alpha$-band (at most one $\omega$-band) connecting $x$ and $y$. If such a band exists, its left side (resp., right side) coincides with the left (resp., right) side of $\Delta$.

Proof. The letter $\alpha$ (resp., $\omega$) can occur in a defining relation $u = v$ of $H$ only as the left-most (the right-most) letter of $u$ or $v$. It follows that the left side of an $\alpha$-band (the right side of an $\omega$-band) connecting $x$ and $y$ is a vertical line. Since $\Delta$ is indivisible, this line must be equal to the left (to the right) side of $\Delta$, and the statement of the lemma follows.  

Lemma 4.2. If $\Delta$ has no through $q$-bands, then $\text{space}_H(w_0, w_t) \leq S_3(\max(|w_0|_a, |w_t|_a) + 3)$.

Proof. We may assume that $\text{space}(D) > 1$. Then there are no $\omega$-bands connecting $x$ and $y$ since either the left or the right side of such a band would make $\Delta$ divisible. Similarly $\Delta$ has no through $\alpha$-bands since any $\alpha q$-cell of $\Delta$ must belong to a lens.

Therefore the top and bottom labels are words in $A \cup Y_l \cup Y_r$. By Remark 3.13 (a), every $\alpha$-edge of the outer boundary of a thick lens is labeled by a letter from $A$. Hence every maximal $\alpha$-band starting on $y$ with an edge labeled by a letter from $Y_l \cup Y_r$ consists of trivial cells only and ends on $x$. This makes $\Delta$ divisible, a contradiction.

Thus the top and bottom labels are words in the alphabet $A$. By Lemma 3.14 we have $w_0 = S w_t$, and Remark 3.2 completes the proof.  

Lemma 4.3. $\Delta$ has at most one through $q$-band.

Proof. Assume that there are two through $q$-bands $C_1$ and $C_2$, where $C_2$ is from the right of $C_1$, and there are no through $q$-bands between them. Note that only maximal $\alpha$-bands corresponding to the letters from $Y_l$ can start on the left side of $C_2$. These $\alpha$-band can end either on $C_2$ or on the top/bottom of $\Delta$. Hence there is a vertical path $z$ connecting $x$ and $y$ whose edges belong either to the left sides of some of these $\alpha$-bands or to the left side of $C_2$. It follows that the derivation trapezium $\Delta$ is divisible by this vertical line, a contradiction. The lemma is proved.  

Lemma 4.4. Assume that $\Delta$ has one through $q$-band $C$. Then

1) each cell from the left of $C$ is a trivial $\alpha$-cell corresponding to a letter from $Y_l$ or it is an $\alpha$-cell;
2) if $\Delta$ has no through $\omega$-bands, and $C$ has no cells having an edge labeled by $p$, then every $\alpha$-cell of $\Delta$ belongs to $C$ and all $\alpha$-edges from the right of $C$ are labeled by letters from $Y_r$.

Proof. We will prove Statement (2) of the lemma. By the assumptions, no $\omega$-bands can start/end on $C$. Let $A_1, \ldots$ be all the maximal $\alpha$-bands starting on $C$ from the right. Since no cell of $C$ has a $p$-edge, all these bands correspond to letters from $Y_r$. Hence each of $A_i$-s must end either on $C$ or on $x$, or on $y$ (but cannot end on the outer boundary of a thick lens by Remark 3.13 (a)). By Lemmas 3.11 (a) and 3.16 (b), there are neither lenses nor $\omega$-cells between any two of these $\alpha$-bands or between some $A_i$ and $C$.

There is a vertical line composed of the side edges of these $\alpha$-bands and of $C$. Since $\Delta$ is indivisible, this line coincides with the right side of $\Delta$, and so every cell in and from the right of $C$ corresponds to a machine relation or is trivial; the trivial $\alpha$-cells are labeled by letters from $Y_r$.

Similarly, each cell from the left of $C$ is a trivial $\alpha$-cell corresponding to a letter from $Y_l$ or it is an $\alpha$-cell.  

23
Lemma 4.5. Under the assumptions of Lemma 4.4 (2), \( \text{space}_H(w_0, w_t) \leq c_3 \max(|w_0|, |w_t|) + c_4 \), where the constants \( c_3, c_4 \) do not depend on the derivation \( D \).

Proof. It follows from Lemma 4.4 that all the cells of \( \Delta \) correspond to the machine relations (3.3), and the derivation \( D \) is a peeled machine derivation, where the letter \( \omega \) is not involved in the commands of the corresponding computation \( C \). Also there is a reduced computation \( C' : w_0 \rightarrow \cdots \rightarrow w_t' = w_t \). Then \( t' \) is bounded by a linear function of \( \min(|w_0|, |w_t'|) \) by Lemma 2.10 (e). Since the set of defining relations of \( H \) is finite, this implies that the space of \( C' \), is bounded by a linear function of \( \max(|w_0|, |w_t'|) \).

Lemma 4.6. Assume that \( \Delta \) has one through \( q \)-band \( C \), has a through \( \omega \)-band \( D \), and \( C \) has no cells having an edge labeled by \( p \). Then

\[
\text{space}_H(w_0, w_t) \leq c_3 \max(|w_0|, |w_t|) + c_4, \quad c_3 \max(|w_0|, |w_t|) + c_4, \quad S'_5(\max(|w_0|, |w_t|))
\]

Proof. By Lemma 4.1 a through \( \alpha \)-band \( B \) (if any exists) consists of the left-most cells of \( \Delta \), and by Lemma 3.10 (d), there are no lenses between \( C \) and \( D \). Therefore, as in the proof of Lemma 4.5, we obtain that \( \Delta \) is a machine or a peeled machine trapezium (depending on the presence of a through \( \alpha \)-band in it). If it is peeled machine, then \( \text{space}_H(w_0, w_t) \leq c_3 \max(|w_0|, |w_t|) + c_4 \), as in Lemma 4.5. If \( \Delta \) is a machine trapezium, then the statement of the lemma follows from the definition of the function \( S'_5(n) \).

Lemma 4.7. Assume that \( \Delta \) has one through \( q \)-band \( C \), and \( C \) has an edge labeled by \( p \). Then \( \text{space}_H(w_0, w_t) \leq S'_5(\max(|w_0|, |w_t|) + 5) \).

Proof. By Lemma 4.1 (1), each cell from the left of \( C \) is a trivial \( a \)-cell corresponding to a letter from \( Y_l \) or it is an \( \alpha \)-cell.

Assume that an \( \omega \)-band \( D \) starts and ends on \( C \) (from the right). Let \( \Gamma \) be the subtrapezium of \( \Delta \) bounded from the right by the right side \( z \) of \( D \) and bounded from the left by a part of the left side of \( \Delta \). By Lemma 3.11 (b), \( \Gamma \) is a (peeled) augmented machine trapezium corresponding to a computation of \( M_5 \). Applying the type separation trick, we may assume that all cells from the right of \( z \) are trivial, and the computation of \( M_5 \) is reduced. Moreover, it is non-empty since otherwise the type of \( C \) could be decreased after removing of two derivation bands containing the \( q \)-cells of \( \Gamma \).

If \( \Gamma \) has no cell having both an \( \alpha \)-edge and a \( q \)-edge, then we have a contradiction with Lemma 2.10 (f). Suppose there is such a cell \( \pi \), and let us choose it to be the closest one to the bottom of \( \Gamma \). Then by Lemma 2.10 (f), the part of \( C \) between the bottom of \( \Gamma \) and \( \pi \) has no edges labeled by letters from \( Y_l \setminus A_t \). Hence only \( \alpha \)-bands corresponding to letters from \( A_t \) can start on the bottom of \( \Gamma \) and end on \( C \) from the left. Therefore the \( a \)-letters from the bottom label of \( \Gamma \) belong to \( A_l \) contrary to Lemma 3.12 (b). So we may assume that no \( \omega \)-band starts and ends on \( C \).

There is at most one \( \omega \)-band \( D \) starting with a cell \( \pi \) of \( C \) and ending on the top of \( \Delta \) and at most one \( \omega \)-band \( D' \) starting on the bottom of \( \Delta \) and ending with a cell \( \pi' \) of \( C \). (We take into account that the \( p \)-cells corresponding to the relation \( p = q_1 \omega \) and to \( q_1 \omega = p \) alternate in \( C \).) Since \( C \) has a \( p \)-cell, \( \Delta \) must consists of the following pieces enumerated from the bottom to the top (some of them may be absent): the
Lemma 4.7. By Lemmas 4.2 and 4.3, we may assume that \( \Delta \) has exactly one through space.

Lemma 4.8. If \( C \) has no \( \omega \)-bands, then \( \omega \)-bands starts/ends on \( C \). Now we consider two cases.

Case 1: \( \Delta \) has no through \( \omega \)-bands.

One can apply the time separation trick to \( \Delta_3 \) and its 'left half' \( \Gamma_3 \). Therefore one can assume that the lenses from the right of \( \Delta \) (if they exist) lie in \( \Delta_2 \). Furthermore, if \( \Delta_2 \) corresponds to a subderivation \( w_i \to \cdots \to w_j \) of \( D \), then \( |w_0| = |w_1| = \cdots = |w_i| \) and \( |w_j| = \cdots = |w_t| \) by Lemma 2.10 (f). Hence it suffices to estimate \( \text{space}_H(w_i, w_j) \).

We have \( w_i = U \pi V \), where \( U \) is a word in \( A \) and \( V \) is a word in \( A \). Indeed, every maximal \( \alpha \)-band \( A \) of \( \Delta_3 \) disposed from the right of \( \Delta \) and corresponding to a non-\( A \)-letter cannot end either on the \( p \)-cell of \( \Delta \) or on an outer boundary of a thick lens of \( \Delta_2 \). So both sides of it divide \( \Delta_2 \), and the sides of the maximal extension of \( A \) in the whole \( \Delta \) divide \( \Delta \), a contradiction. Similar form has \( |w_j| = U' \pi V' \). Hence \( \psi(w_i) = \psi(w_j) \) by Lemma 3.15.

To complete the proof, we first use the relations \( \alpha p = p \alpha \) to replace \( w_i \) by \( w_i' = p U_A V \), where \( U_A \) is the copy of \( U \) in the alphabet \( A \). The derivation \( w_i \to \cdots \to w_i' \) has space \( |w_i| = |w_i'| \). Similarly we obtain \( w_j' \). By Remark 3.2, \( \text{space}_H(U_A V, U_A' V' \pi) \leq S'_0(\text{max}(|U_A V|, |U_A' V'|)) + 3 \). Hence

\[
\text{space}_H(w_i, w_j) = \text{space}_H(w_i', w_j') \leq S'_0(\max(|w_0|, |w_t|)) + 4
\]

Case 2: \( \Delta \) has a through \( \alpha \)-band \( B \) (as in Lemma 3.1). Since there are no \( q_1 \)-\( \omega \)-cells in \( C \), every cell of this band is a cell corresponding to a relation \( \alpha p = p \alpha \) or a trivial \( p \)-cell. In particular, \( C \) does not share cells with \( \omega \)-bands, and therefore \( \Delta \) has no through \( \omega \)-bands since both sides of such a band would be vertical, but \( \Delta \) is indivisible.

Since every \( \alpha \)-band starting/ending on \( C \) from the right or on the outer boundary of a thick lens, correspond to a letter from \( A \), the top and bottom labels of \( \Delta \) from the right of \( C \) are the words in the alphabet \( A \) (again, because \( \Delta \) indivisible). So \( w_0 = \alpha U \pi V, w_i = \alpha U' \pi V' \), where \( V, V' \) are words in \( A \) (and \( U, U' \) are words in \( A_i \)).

Hence, by Lemma 3.15, \( \psi(w_0) = \psi(w_i) \). The relations \( \alpha p = p \alpha \) preserve the value of \( \psi \) and does not change the length. So we may assume that \( w_0 = \alpha p \pi V, w_i = \alpha p \pi V' \), where \( V, V' \) are words in \( A \). By Remark 3.2, the words \( V \) and \( V' \) can be connected by an \( H \)-derivation of space at most \( S'_0(\max(|V|, |V'|)) + 3 \). Therefore \( \text{space}_H(w_0, w_t) \leq S'_0(\max(|w_0|_a, |w_t|_a)) + 5 \), as required.

Summarizing, we obtain

**Lemma 4.8.** Assume that \( \Delta \) is an indivisible minimal trapezium without caps, cups, and \( \alpha \)-triangles, and \( \Delta \) corresponds to a derivation \( D : w_0 \to \cdots \to w_t \) over \( H \). Then

\[
\text{space}_H(w_0, w_t) \leq \max(c_3|w_0| + c_4, c_3|w_t| + c_4, S'_0(\max(|w_0|_a, |w_t|_a)) + 5)
\]

**Proof.** By Lemmas 4.2 and 4.3, we may assume that \( \Delta \) has exactly one through \( q \)-band \( C \). If \( C \) has no \( p \)-cells, then the statement of the lemma follows from Lemmas 4.5 and 4.6. Otherwise it follows from Lemma 4.7.

Now we want to eliminate the restrictions of Lemma 4.8 imposed on \( \Delta \).
Lemma 4.9. The space function of $H$ is bounded from above by a function equivalent to the function $S_5'(n)$.

Proof. We define the function $f(n) = S_5'(n+c) + c_3n + c_4 + 5$ for $n \geq 1$, where $c$ is the constant from Lemma 2.10 (d), $c_3, c_4$ are from Lemma 1.8 (and 1.8), and define $f(0) = 0 (= S_5'(0))$. Obviously, $f(n) \sim S_5'(n)$. We can use the inequality $f(n-k) + k \leq f(n)$ for $0 \leq k \leq n$. Indeed the function $S'(n)$ is non-descending, and one can select $c_3 \geq 1$.

Now we modify the length of a word $w$ : by definition $|w|$ is the number of letters, where every $\alpha$- or $q$-letter is counted with weight $c$, and other letters are counted with weight 1.

Note that $|w| = |w| + c|w|$ for every word in the generators of $H$. Therefore to prove the lemma, it suffices to prove the inequality $space_H(w, w') \leq f(||w|| + ||w'||)$ for any pair of equal in $H$ words $w$ and $w'$. This will be proved by induction on $|w| + ||w'||$ with trivial base $\Sigma = 0$. So we will assume that $\Sigma > 0$ and consider a derivation $D : w = w_0 \rightarrow \cdots \rightarrow w_t = w'$. Let us denote by $\Delta$ the corresponding minimal trapezium. Of course, one may assume that the unique vertical line connecting the endpoints of the left side of $\Delta$ is the left side itself since otherwise one can replace $\Delta$ by a subtrapezium. Similar assumption is taken for the right side of $\Delta$.

First assume that the trapezium $\Delta$ is divisible and use the notation of Remark 3.2. Then (see formula (4.0)) there is a derivation

$$D' : w = w_0(1)w_0(2) \rightarrow \cdots \rightarrow w_t(1)w_0(2) \rightarrow \cdots \rightarrow w_t(1)w_t(2) = w',$$

where $max(||w(0)(1)|| + ||w(1)(1)||, ||w(0)(2)|| + ||w(2)(2)||) < ||w|| + ||w'||$, and so by the inductive hypothesis, the first half (the second half) of the derivation $D'$ can be chosen with space at most $f(||w(0)(1)|| + ||w_t(1)(1)||) + |w(0)(2)|$ (resp., at most $f(||w(0)(2)|| + ||w_t(1)(2)||) + |w_t(1)|$). Hence

$$space(D') \leq \max(f(||w|| + ||w'||) - ||w(0)(2)||) + ||w(0)(2)||, f(||w|| + ||w'|| - ||w_t(1)(1)||) + ||w_t(1)||)$$

Thus we may further assume that the derivation trapezium $\Delta$ is indivisible.

Now assume that there is an $A$-triangle in $\Delta$. It means that the bottom (or the top) label of $\Delta$ is of the form $w = \tilde{w}w\tilde{w}$, where $u$ is a non-empty word in the alphabet $A$ and $u = l_1$ by Lemma 3.11. By Remark 3.2, $space_H(u, 1) \leq S_5'(|u|) + 3$. Therefore there is a derivation $w \rightarrow \cdots \rightarrow \tilde{w}w\tilde{w}$ over $H$ of space at most

$$S_5'(|u|) + 3 + |\tilde{w}| + |\tilde{w}| \leq f(||u|| + |\tilde{w}| + |\tilde{w}| = f(||w - |\tilde{w}| - |\tilde{w}| + |\tilde{w}| + |\tilde{w}| \leq f(||w||) \leq f(||w||))$$

By the inductive hypothesis, there is a derivation $\tilde{w}w\tilde{w} \rightarrow \cdots \rightarrow w'$ of space $\leq f(||\tilde{w}w\tilde{w}|| + ||w'||) \leq f(||w|| + ||w'||)$, hence $space_H(w, w') \leq f(||w|| + ||w'||)$. Thus we may further assume that $\Delta$ has no $A$-triangles.

Assume that $\Delta$ has a cup (or cap). Let $\Gamma$ be a cup of minimal height. By Lemma 3.10 (a), there are no lenses enclosed in $\Gamma$. Therefore the type of the maximal $q$-band $C_\Gamma$ of $\Gamma$ is 0 or 1 by Lemma 3.12 (c).

If the type of $C$ is 0, then by Lemma 3.17, the word $w'$ is of the form $\tilde{w}\alpha \beta \tilde{w}$, where $W$ is a word in $A_l$. Hence there is a derivation $\tilde{w}\tilde{w} \rightarrow \cdots \rightarrow \tilde{w}\alpha \beta \tilde{w} \rightarrow \tilde{w}W\tilde{w}$, where $W'$ is the copy of $W$ in the alphabet $A$ and this derivation has space $|w'| \leq ||w'||$. Since $||\tilde{w}W'\tilde{w}|| = ||w'|| - 2c$, we obtain by the inductive hypothesis, that

$$space_H(w, \tilde{w}W'\tilde{w}) \leq f(||w|| + ||w'|| - 2c) \leq f(||w|| + ||w'||)$$

Hence $space_H(w, w') \leq f(||w|| + ||w'||)$, as desired.

If the type of $C$ is 1, then, by Lemma 3.17, the word $B = b(\Gamma)$ is reachable by the machine $M_5$. By Lemma 2.10 (d), there is a computation $B \rightarrow \cdots \rightarrow B'$ of $M_5$, where $B'$ is an input configuration of $M_5$ and $||B'|| \leq |B| + c$. Denote by $C'$ the corresponding machine derivation over $H$. Furthermore applying the auxiliary relations in the standard way, we can extend $C'$, remove the letters $\alpha, \beta$ from $B'$ and obtain a word $B''$ with $|B''| < ||B'|| - 2c < |B| - c$. Hence we have a derivation $w = \tilde{w}B\tilde{w} \rightarrow \cdots \rightarrow \tilde{w}B''\tilde{w}$ of space at most $S_5'(|B| + c) + ||w'|| - |B|$. By the inductive hypothesis, there is a derivation $w \rightarrow \cdots \rightarrow \tilde{w}B''\tilde{w}$ of space at most $f(||w|| + ||w'|| - c)$. Therefore $space_H(w, w') \leq f(||w|| + ||w'||)$, as desired.

Thus we may assume that $\Delta$ satisfies the assumptions of Lemma 4.8, and therefore $space_H(w, w') \leq f(||w|| + ||w'||)$ by Lemma 4.8 and the definition of $f(n)$. The Lemma is proved.
4.2 Lower bounds and completion of proofs

We define a monoid $H'$ as follows. The set of generators of $H'$ is $A_{H'} = Y_1 \cup Y_r \cup Q \cup \{ \alpha, \omega \}$, i.e., $A_{H'} = A_H \setminus \{ A \cup \{ p \} \}$. The set of defining relations of $H'$ consists of only machine relations of $H$: $R_{H'} = \{ V' = V \text{ for every command } V \to V' \text{ of } M_5 \}$.

**Lemma 4.10.** The space function of $H'$ is equivalent to $S_5(n)$. Proof. The space function of $H'$ is bounded from above by a function equivalent to $S_5(n)$. This statement is a very easy version of Lemma 4.9 since the analogs of Lemmas 4.1-4.6 and 4.8 become trivial when we have no letters from $A$, no $p$, no defining relations of $H'$ with 1 in the left/right sides, and consequently, no triangles, lenses, caps and cups.

By the definition of the set of relations $R_{H'}$, every computation of $M_5$ can be considered as a derivation over $H'$, and vice versa, every derivation $w \to \ldots$ where $w$ is a configuration of $M_5$, is a computation. Let a derivation $C$ over $H'$ connects $w = \alpha U q V \omega$ and $\alpha' U' q' V' \omega'$, where $q, q' \in Q$, the words $U, U'$ are words in the alphabet $Y_1$ and $V, V'$ are words in $Y_r$. Then $C$ is a derivation of $M_5$ since all defining relations of $H'$ are machine relations. Hence if for two configurations $w$ and $w'$ of $M_5$, we have $\text{space}_{M_5}(w, w') = s$ for some $s$, then $w$ and $w'$ cannot be connected by a derivation over $H'$ with space $\leq s$. Hence the space function of $H'$ is at least $S_5(n)$, and the statement of the lemma follows. □

**Proof of Theorem 1.1.** Let a monoid $H''$ be a copy of $H'$ given by a finite presentation with a set of generators $A_{H''}$ disjoint with $A_{H'}$. We define the monoid $P$ announced in Theorem 1.1 as the free product $H \ast H''$ and consider its space function $s(n)$ with respect to the presentation $(A_{H} \cup A_{H''} \mid R_H \cup R_{H''})$.

On the one hand, any derivation over $P$ projects on a derivation over $H''$. One just deletes the letters from $A_{H'}$ in any word from $C$.) Therefore the space function $s(n)$ of $P$ is greater than or equal to the space function of $H''$. Hence $s(n) \geq S_5(n)$ by Lemma 4.10. On the other hand two words $w$ and $w'$ over $A_{H} \cup A_{H''}$ are equal in $P$ iff their corresponding $A_{H'}$- and $A_{H''}$-syllables are equal in $H$ and in $H'$, respectively. Since the derivations between equal words can be define syllable-by-syllable, we see that, up to equivalence, $s(n)$ does not exceed the maximum of the space functions of $H$ and of $H'$. Therefore $s(n) \leq S_5(n)$ by Lemmas 4.9 and 4.10.

Our estimates show that $s(n) \sim S_5(n)$. Recall that $S_5(n) \sim S_0(n)$ by Lemma 2.10 (b). Hence $s(n) \sim S_0(n)$, and by Lemma 3.1 the theorem is proved. □

**Proof of Corollary 1.3.** If the function $S_0(n)$ is bounded by a polynomial, then so is $S_5(n)$ by Lemma 2.10 (b). By Lemma 3.1 $S$ is a subsemigroup (submonoid) of the monoid $H$, and the space function of $H$ is polynomial by Lemma 4.9.

Conversely, assume that a finitely generated semigroup (monoid) $H$ is embedded in a finitely presented semigroup (monoid) $H$ with polynomial space function $s(n)$. Then the word problem in $S$ is solvable by an NTM with space $\leq s(n)$: This machine takes any word $w$ in the generators of $S$, rewrites it in the generators of $H$, and if $w = H 1$, it produces a derivation $w \to \cdots \to 1$. (Recall that an NTM may guess and verify. The head of this NTM can move along a word and can replace a subword $u$ by $v$ if $u = v$ or $v = u$ is one of defining relations of $H$. See details in [2].) By the remarkable theorem of Savitch (see [9], Corollary 3.1), if an NTM has polynomial space complexity, then there exists a DTM solving the same algorithmic problem with a polynomial space as well. Therefore the corollary is proved. □

**Proof of Corollary 1.4.** The word problem in a 1-element group $S$ is linear space decidable. But one can force to solve this problem with space complexity $f(n)$ of given deterministic machine $M$. For this goal, the machine $M_0$ from Subsection 2.2 should do the following extra work. Given an input word $w'$ of length $n$, then in the beginning, $M_0$ let machine $M$ to use extra tapes and to accept or to reject in consecutive order all words $w$ of length $\leq n$ in the tape alphabet of $M$. Clearly the space function of such a machine $M_0$ will be equivalent to $f(n)$. Then we apply Theorem 1.1 to complete the proof. □

**Proof of Corollary 1.5.** There exists a finitely presented semigroup $S$ (even a group, see [27] or [28]) with polynomial space (PSPACE) complete word problem (see [9] for the definition). By Theorem 1.1 $S$ is a submonoid of a finitely presented monoid $P$ with polynomial space function, and so the word problem in $P$ is at least PSPACE hard. On the other hand, there is a polynomial $f(n)$ such that two words $w$ and $w'$ are equal in $P$ if there exists a derivation $w \to \cdots \to w'$ of space $\leq f(\max(|w|, |w'|))$. It
follows (as in the proof of Corollary 1.3) that there is an NTM of space complexity \( \preceq f(n) \) which solves the word problem in \( P \), and so there is a DTM solving the same problem in polynomial time. Thus the corollary is proved. □

**Proof of Corollary 1.6** As we mentioned in Introduction, our notion of space function for semigroups differs from that used in [22] for groups. Nevertheless the proof of Corollary 1.7 [22] can be literally repeated here to deduce the proof of Corollary 1.6 from Corollary 1.4. □

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