Planar of special idealization rings

MANAL AL-LABADI
Eman Mohammad Almuhur
Department of Mathematics
University Of Petra
Amman, JORDAN

Department of Mathematics,
Faculty of Basic Sciences and Humanities,
Applied Science Private University,
Amman, JORDAN

Abstract: Let \( R(+)N \) be the idealization of the ring \( R \) by the \( R \)-module \( N \). In this paper, we investigate when \( \Gamma(R(+)N) \) is a Planar graph where \( R \) is an integral domain and we investigate when \( \Gamma(Z_n(+)Z_m) \) is a Planar graph.

Key-Words: The idealization rings \( R \), Planar graph, Zero-divisor graph.

Received: July 27, 2020. Revised: November 23, 2020. Accepted: December 10, 2020. Published: December 23, 2020.

1 Introduction

I. Beck in [6] introduce the concept of zero-divisor graph also, D. D. Anderson and M. Naseer in [3] studied the context of coloring which is an interest concept of graph theory. Anderson and Livingston in [4, Theorem 2.3] give the definition of the zero-divisor graph. For more information in zero-divisor graph see, [5].

Let \( R \) be a commutative ring, the zero-divisor graph is the graph \( \Gamma(R) \) which vertices are the non-zero zero divisors of \( R \), with \( a \) and \( b \) adjacent if \( ab = 0 \). For each ring \( R \), the set of all zero-divisors of the ring \( R \) is \( Z(R) \).

The idealization ring \( R(+)N \) is defined as \( N \) be an \( R \)-module and let \( R(+)N = \{ (a_1, h_1) : a_1 \in R, h_1 \in N \} \) we have two operations \( (a_1, h_1) + (a_2, h_2) = (a_1 + a_2, h_1 + h_2) \) and \( (a_1, h_1)(a_2, h_2) = (a_1a_2, a_1h_2 + a_2h_1) \).

Another concept of interest in the graph theory. The Planar graph is a graph isomorphic to a Plane graph. A Plane graph is graph that can be drawn on the plane without cross edging. If the graph has induced subgraph isomorphic to \( K_5 \) that is not a Planar graph, by Kuratoskies Theorem.

2 When \( \Gamma(R(+)N) \) is a Planar graph?

In this section, we investigate when \( \Gamma(R(+)N) \) is Planar graph where \( R \) is an integral domain and \( N \) be an \( R \)-module.

We begin with the following lemma when \( R \) is an integral domain for the idealization ring \( R(+)N \).

Lemma 1:

[2] Suppose that \( R \) is an integral domain and \( N \) is an \( R \)--module. Then we have the following cases:

- **Case 1.** If \( R \) is an integral domain with \( N \cong Z_2 \) is an \( R \)--module and annihilator of \( Z_2 \) is equal to zero, then the integral domain \( R \cong Z_2 \).

- **Case 2.** If \( R \) be an integral domain with \( N \cong Z_3 \) is an \( R \)--module and annihilator of \( Z_3 \) is equal to zero, then the integral domain \( R \cong Z_3 \).

Theorem 1:

Suppose that \( R \) is an integral domain and \( N \cong Z_2 \) is an \( R \)--module. Then the graph \( \Gamma(R(+)Z_2) \) is a Planar.

Proof:

To proof we have the following two cases to thoughtfulness:

- **Case 1:** If the annihilator of \( Z_2 \) is equal to zero, then \( \Gamma(Z_2(+)Z_2) \) is equal to \( \{(0, 1)\} \) which is a Planar graph.

- **Case 2:** If the annihilator of \( Z_2 \) is not equal to zero, then the graph \( \Gamma(R(+)Z_2) = \{ (0, 1), (k_i, 0), (k_j, 1) : k_i, k_j \in ann(Z_2) \} \). So, the graph \( \Gamma(R(+)Z_2) \) is a star which is a Planar graph.

Theorem 2:

Suppose that \( R \) is an integral domain and \( N \cong Z_3 \) is an \( R \)--module. Then the graph \( \Gamma(R(+)Z_3) \) is a
Planar.

**Proof:**
To proof we must note the following two cases to thoughtfulness:

- **Case 1.** If annihilator of \( \mathbb{Z}_3 \) is equal zero, then \( \Gamma(\mathbb{R}(+)\mathbb{Z}_3) \) is equal to \( \{(0,1),(0,2)\} \) that is a Planar graph.

- **Case 2.** If annihilator of \( \mathbb{Z}_3 \) is not equal zero, then graph \( \Gamma(\mathbb{R}(+)\mathbb{Z}_3) \) is equal to
  \[ \{(0,1),(0,2),(r_1,0),(r_1,1),(r_1,2) : r_1 \in \text{ann}(\mathbb{Z}_3)\} \]. So, that is a Planar graph.

![Figure 1: A graph which is a Planar graph.](image1.png)

The next theorem will discuss when the order of \( \mathbb{N} \) is greater than or equal 5.

**Theorem 4:**
Suppose that \( \mathbb{R} \) is an integral domain and \( |\mathbb{N}| \geq 5 \) is an \( \mathbb{R} \)−module. Then we have the following cases:

- **Case 1.** If the order of \( \mathbb{N} \) is equal 5 and annihilator of \( \mathbb{N} \) is equal to zero, then the graph \( \Gamma(\mathbb{R}(+)\mathbb{N}) \) is a Planar graph.

- **Case 2.** If the order of \( \mathbb{N} \) is equal 5 and annihilator of \( \mathbb{N} \) is not equal to zero, then the graph \( \Gamma(\mathbb{R}(+)\mathbb{N}) \) is not a Planar graph.

- **Case 3.** If the order of \( \mathbb{N} \) is greater than 5 , then the graph \( \Gamma(\mathbb{R}(+)\mathbb{N}) \) is not a Planar graph.

**Proof:**
To proof must note two cases to thoughtfulness:

- **Case 1.** If the order of \( \mathbb{N} \) is equal 4 and annihilator of \( \mathbb{N} \) is equal to zero, then the graph \( \Gamma(\mathbb{R}(+)\mathbb{N}) \) is equal to \( \{(0,l_1),(0,l_2),(0,l_3) : l_i \in \mathbb{N}\} \). That is a Planar graph.

- **Case 2.** If the order of \( \mathbb{N} \) is equal 4 and annihilator of \( \mathbb{N} \) is not equal to zero, then the graph \( \Gamma(\mathbb{R}(+)\mathbb{N}) = \{(r_i,l_1),(0,l_2),(0,l_3),(0,l_4) : l_i \in \mathbb{N}, r_i \in \text{ann}(\mathbb{N})\} \), by previous lemma then the graph is not a Planar graph.

![Figure 2: A graph which is not a Planar graph.](image2.png)


In this section, we consider the planar for the zero-divisor graph of the idealization ring $\mathbb{Z}$.

• Case 3. If the order of $N$ is greater than 5, then graph $\Gamma(\mathbb{R}(+)N)$ is equal to $\{(0, l_1), (0, l_2), (0, l_3), (0, l_4), (0, l_5), \ldots, (0, l_7) : l_i \in N\}$. That has an induced subgraph isomorphic to $K_5$. So, the graph is not a Planar.

3 When $\Gamma(\mathbb{Z}_n(+)\mathbb{Z}_m)$ is Planar graph?

In this section, we consider the Planar for the zero-divisor graph of the idealization ring $\mathbb{Z}_n(+)\mathbb{Z}_m$, $\Gamma(\mathbb{Z}_n(+)\mathbb{Z}_m)$ where $\mathbb{Z}_m$ be $\mathbb{Z}_n$-module.

Al-Labdi [1], she classified the zero-divisor graph of the idealization ring $\mathbb{Z}_n(+)\mathbb{Z}_m$.

We begin with the following lemma, when $n$ is a prime number such that $n = p^\alpha$ and $m = p$.

Lemma 3:

Let $n = p^\alpha$ and $m = p$ where $p$ is a prime number. Then the graph $\Gamma(\mathbb{Z}_n(+)\mathbb{Z}_m)$ have the following cases:

Case 1: If $n$ is equal 4 and $m$ is equal 2, then the graph $\Gamma(\mathbb{Z}_4(+)\mathbb{Z}_2)$ is a Planar.

Case 2: If $n$ is equal $p^\alpha$ and $m$ is equal $p$ where $p$ is a prime number, $\alpha \geq 3$, then the graph $\Gamma(\mathbb{Z}_{p^\alpha}(+)\mathbb{Z}_p)$ is not a Planar.

Proof:

We consider two cases to proof:

Case 1: If $n$ is equal 4 and $m$ is equal 2, then graph $\Gamma(\mathbb{Z}_4(+)\mathbb{Z}_2)$ is equal to $\{(0, 1), (2, 0), (2, 1)\}$. So, that the graph is a Planar.

Case 2: If $n$ is equal $p^\alpha$ and $m$ is equal $p$ where $p$ is a prime number greater than 2, $\alpha \geq 3$, then the graph $\Gamma(\mathbb{Z}_{p^\alpha}(+)\mathbb{Z}_p)$ is equal to $\{(0, 1), (0, 2), \ldots, (0, p - 1), (kp, 0), \ldots, (kp, p - 1) : k \in \mathbb{N}\}$. So, it has an induced subgraph $K_5$ that is not a Planar graph.

Figure 3: A graph which is not a Planar graph.

Figure 4: A graph which is not a Planar graph.

Figure 5: A graph which is not a Planar graph.

Theorem 5:

Let $m$ is a product of powers of prime numbers $m = p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_r^{k_r}$ and $n$ is product power of primes $n = p_1^{s_1} \times p_2^{s_2} \times \ldots \times p_r^{s_r}$ where $p_i$ is a prime number and $l \leq r$. Then the graph $\Gamma(\mathbb{Z}_n(+)\mathbb{Z}_m)$ is not a Planar graph.

Proof:

We consider two cases to proof:

If $m$ is product power of primes $m = p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_r^{k_r}$ and $n$ is product power of primes $n = p_1^{s_1} \times p_2^{s_2} \times \ldots \times p_r^{s_r}$ where $p_i$ is a prime number and $l \leq r$. Then the graph $\Gamma(\mathbb{Z}_{p_1^{s_1} \times p_2^{s_2} \times \ldots \times p_r^{s_r}}(+)\mathbb{Z}_{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_r^{k_r}})$ is equal to $\{(0, h_1), (b_i, h_i) : b_i \in n, h_i \in m\}$ such that $gcd(b_i, n) \neq 1$ or $gcd(b_i, m) \neq 1$. So, it has an induced subgraph $K_5$ that is not a Planar graph.
4 Outcome and questions

In this article, we classify the planarity for the graph of idealization \( \Gamma(R(+)N) \), we conclude in the following theorem.

**Theorem 6:**
Let \( R(+)N \) be an idealization ring. Then the graph \( \Gamma(R(+)N) \) is a Planar graph if the ring \( R \) is an integral domain and the order of \( N \) is less than or equal 4 with \( \text{ann}(N) = 0 \), or the order of \( N \) is equal to 5 with \( \text{ann}(N) = 0 \) and the graph \( \Gamma(Z_n(+)Z_m) \) is a Planar when \( n = 4, m = 2 \).

One can ask the following questions:

1. **When the graph \( \Gamma(R(+)N) \) are Eulerian graph?**
2. **When the complement graph of idealization ring \( \Gamma(R(+)N) \) are Planar graph?**
3. **What is the matching number of the graph \( \Gamma(R(+)N) \)?**

Possible engineering applications of this study can be found in problems of [8] and [9].

References:

[1] M. Allabadi M, Futher results on the diameter of zero-divisor graphs of some special idealizations, *International Journal of Algebra*, Vol. 12 (2010), pp. 609-614.

[2] M. Allabadi, On the Diameter of Zero-Divisor Graphs of Idealizations with Respect to Integral Domain, *Jordan Journal of Mathematics and Statistics*, Vol. 3 (2010), pp. 127-131.

[3] D.D. Anderson, M. Naseer, Beck’s coloring of a commutative ring, *J. Algebra* Vol.159 (1993), pp. 500-514.

[4] D.F. Anderson, P.S. Livingston, The zero-divisor graph of a commutative, *J. Algebra*, Vol.217 (1999), pp. 434-447.

[5] M. Axtell, J. Stickel, The zero-divisor graph of a commutative rings, *Journal of Pure and Applied Algebra*, Vol.204 (2006), pp. 235-243.

[6] I. Beck, Coloring of a commutative ring, *J. Algebra*, Vol. 116 (1988), pp. 208-226.

[7] B. Jackson, Longest cycles in 3-connected cubic, *J. Combin. Theory Ser B*, Vol. 41 (1986), pp. 17-26.

[8] N. Boonsim, Racing Bib Number Localization on Complex Backgrounds, *WSEAS Transactions on Systems and Control*, Vol.13 (2018), pp. 226-231.

[9] T. Ashkan Tashk, H. Jurgen, Esmaeil Nadimi, Automatic Segmentation of Colorectal Polyps based on a Novel and Innovative Convolutional Neural Network Approach, *WSEAS Transactions on Systems and Control*, Vol.14 (2019), pp. 384-391.