Quantum prescriptions are more ontologically distinct than they are operationally distinguishable

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In what way do quantum prescriptions for physical entities fundamentally deviate from that of classical theories? Such a question necessitates formal notions of classicality and the ontological framework provides a vital ground for such notions. Based on an intuitive generalization of the principle of ‘the identity of indiscernibles’, we introduce such a notion of classicality, called bounded ontological distinctness. Formulated as a principle, bounded ontological distinctness equates the distinguishability of a set of operational physical entities to the distinctness of their ontological counterparts. Employing three instances of two-dimensional quantum preparations, we demonstrate the violation of bounded ontological distinctness or excess ontological distinctness of quantum preparations. Moreover, our methodology enables the inference of tight lower bounds on the extent of excess ontological distinctness of quantum preparations. Similarly, we demonstrate excess ontological distinctness of quantum transformations, using three two-dimensional unitary transformations. However, to demonstrate excess ontological distinctness of quantum measurements, an additional assumption such as outcome determinism or bounded ontological distinctness of preparations is required. Unlike other well-known ontological principles, the operational pre-requisite for the experimental demonstration of excess ontological distinctness i.e. the distinguishability of physical entities is inherently experimentally robust. Furthermore, we show that quantum violations of other well-known ontological principles implicate quantum excess ontological distinctness. Finally, to showcase operational vitality, we introduce two distinct classes of communication tasks powered by excess ontological distinctness.

I. INTRODUCTION

Operational physical theories such as quantum theory serve to instruct experiments and make corresponding predictions. However, owing to their rather abstract mathematical formalism, these operational theories are often incapable (on their own) of facilitating deep insights into the nature and structure of reality. The study of ontology along with the ontological framework enable insights into the structure of reality the operational theories posit. Apart from this, the ontological framework also provides a vital ground for formal notions of classicality. These ontological notions of classicality attempt to make precise the sense in which quantum theory departs from classical physics \cite{1, 2} and are typically philosophically well-substantiated ontological principles which relate certain exclusively operational phenomena to exclusively ontological phenomena. For instance, the ontological principle of Bell’s local causality \cite{3, 4} attributes a local ontological basis to operational non-signaling correlations. Similarly, Kochen-Specker non-contextuality \cite{5-7} provides operational non-disturbance of measurements a non-contextual basis and Spekken’s non-contextuality \cite{8} assigns identical ontological counterparts to operationally indistinguishable preparations, measurements, and transformations.

Subsequently, quantum violation of certain operational consequences of these principles enables significant insights into the ontology of quantum theory revealing (via negation) necessary features of possible ontological models of quantum theory. In order to be of fundamental significance, such a notion of classicality should be applicable in a wide variety of scenarios so that it covers the multitude of experimental tests posited by quantum theory. Furthermore, to reveal the inherent non-classicality of a large class of quantum systems, such a fundamental notion should impose minimal demands on quantum preparations and measurements required to violate it. For instance, Bell’s local causality only applies to spatially separated systems, and in order to violate it, we require at-least two spatially separated two-dimensional entangled quantum systems \cite{4}. Kochen-Specker’s notion of non-contextuality applies to single systems, but in order to violate we require a three dimensional quantum system \cite{9} at the least. Spekken’s notion of non-contextuality reduces the demand on quantum systems required for its violation as it requires at-least four instances of two-dimensional quantum preparations to retrieve a violation \cite{10, 11}. Additionally, a fundamental ontological notion of classicality should have as its implications the other well-known notions of classicality, so that quantum violation of the latter implicate the violation of the former \cite{12}. As the experimental tests of the consequences of these ontological principles allow us to extend the corresponding implications beyond the
particular operational theory to Nature itself [1], such an ontological notion of classicality should be robust to experimental imperfections. Finally, such a notion should be operationally vital so that its quantum violation yields an advantage in computation, communication or information processing tasks.

The extent to which physical entities may be distinguished or discerned apart, termed as their distinguishability, plays a crucial role in physical theories, more generally in natural philosophy, as well as in our very world views [13]. For instance, a principle of analytic ontology, “the Identity of Indiscernibles”, first explicitly formulated by Wilhelm Gottfried Leibniz is inherent in classical intuition. The principle states that operational indistinguishability implies ontological identity i.e. if two objects cannot possibly be distinguished, then they are actually the same (identical) [2, 14]. A natural generalization of the Leibniz principle is the notion that how well a set of objects may be in principle distinguished (i.e. employing all possible measurements) reflects how distinct they actually are. The ontological principle, “bounded ontological distinctness” relies on this notion and states that maximal operational distinguishability reflects ontological distinctness of physical entities. Note that, in this work, we refer to the extent to which physical entities may be distinguished whilst employing an operational physical theory as their “distinguishability” as opposed to their “distinctness”, which refers to how ontologically (actually) distinct these entities are. We mathematically formulate this principle, and demonstrate how quantum preparations, measurements, and transformations violate this principle, demonstrating their “excess ontological distinctness”. Furthermore, we show that bounded ontological distinctness meets all of the aforementioned criteria for a fundamental notion of classicality.

We begin by detailing the general formulation of operational theories and ontological models. In the section that follows, we formalize the definitions of distinguishability of sets of operational physical entities and distinctness of sets of corresponding ontological entities. While distinguishability is the maximal probability of distinguishing the constituent physical entities out of an uniform ensemble, distinctness is the maximal probability of distinguishing the corresponding ontological entities from an uniform ensemble conditioned on fine-grained control and access to the ontic state. Next, we formulate the principle of bounded ontological distinctness for sets of physical entities, which equates their distinguishability to distinctness of corresponding ontological entities. This principle leads to certain operational consequences. Based on the distinguishability of three preparations we obtain an inequality, specifically, an upper-bound on the average pair-wise distinguishability of these preparations, valid in all operational theories that admit ontological models that adhere to bounded ontological distinctness of preparations. Subsequently, we present a set of three two dimensional quantum states which violate this inequality thereby demonstrating excess ontological distinctness of quantum preparations. This forms the most compact instance of quantum ontological incompatibility i.e. incompatibility of predictions of quantum theory with that of a substantial class of ontological models. Additionally, for four preparations, the distinguishability of a pair of disjoint two-preparation mixtures yields an upper-bound on the average distinguishability of the remaining disjoint pairs of two-preparation mixtures, valid in all operational theories that admit ontological models adhering to bounded ontological distinctness of preparations. We then employ four two-dimensional quantum preparations to demonstrate quantum violation of this upper-bound. Moreover, we demonstrate that the extent of quantum violations of these bounds allow inference of lower bounds on the extent of excess ontological distinctness of the quantum preparations under consideration. We substantiate these implications and demonstrate the tightness of these lower bounds employing certain well-known ontological models of quantum theory. Moving on, we demonstrate that unlike the case of preparations, it is not possible to violate bounded ontological distinctness of measurements in quantum theory. However, we show that if a pair of quantum measurements are neither completely indistinguishable nor perfectly distinguishable then outcome-deterministic ontological models violate bounded distinctness of measurements. Next, employing the distinguishability of a set of three transformations we obtain an operational inequality, valid in any theory which admits a ontological models that adhere to bounded ontological distinctness of transformation. Subsequently, employing a set of three two-dimensional quantum unitary transformations we show that quantum theory violates this inequality, thereby demonstrating excess ontological distinctness of quantum transformations. In the following section, we present two distinct classes of communication tasks which benefit from excess ontological distinguishability of preparations. In order to substantiate this, we invoke a generalization of bounded ontological distinctness of preparations so as to apply to arbitrary ensembles of sender’s preparations. We demonstrate that classical communication adheres to generalized bounded ontological distinctness of preparations substantiating its candidature as a notion of classicality. In the penultimate section, we bring forth the fact that excess ontological distinctness of quantum theory is implicate in the violations of other notions ontological classicality such as maximal ψ-epistemicity, Spekken’s non-contextuality, Kochen-Specker’s non-contextuality and Bell’s local causality. Finally, we conclude by laying out the key conceptual insights, implications of our results and tentative avenues for future investigation.
II. PRELIMINARIES

In this section, we lay down the specifics of the experimental scenario employed in this work and revisit the general framework for operational theories and underlying ontological models.

A. Prepare, transform and measure experiments

In prepare, transform and measure experiments, preparation of a physical system is followed by a transformation and finally, a measurement on the same. In each run of the experiment, a preparation, a transformation and a measurement are chosen and the outcomes of the measurement are recorded. The whole process is repeated several times to obtain frequency statistics. Crucially, each run of the experiment is assumed to be statistically independent. However, for the majority of this work, we shall be concerned with prepare and measure experiments, wherein the preparation of a physical system is immediately followed by a measurement on the same.

B. Operational theories

An operational interpretation of a physical theory serves a two-fold purpose: (i) prescription: specification of preparation procedures $P$, transformation procedures $T$ and measurement procedures $M$ and, (ii) prediction of probabilities $p(k|P,T,M)$ of obtaining an outcome $k$ given a measurement $M$ was performed on a preparation $P$ which underwent a transformation $T$.

For instance, quantum formalism prescribes a density matrix $\rho$ to model a preparation, a completely positive map to model a transformation and a set of positive-operator valued measures (POVM) $\{M_k\}$ to model a measurement. However, in this work we need only invoke unitary quantum transformations $U$. The probability of obtaining an outcome $k$ is given by the so-called “Born rule”: $p(k|P,U,M) = \text{Tr}\{U\rho U^\dagger M_k\}$. The operational predictions in prepare and measure experiments are of the form $p(k|P,M)$. Quantum predictions in such experiments are given by: $p(k|P,M) = \text{Tr}\{\rho M_k\}$.

C. Ontological models

An ontological model seeks to explain the predictions of an operational theory, whilst assuming the existence of observer independent attributes associated with physical systems. These attributes describe the “real state of affairs” of a physical system [8, 15]. The complete specification of each such attribute is referred to as the ontic state $\lambda$ of a physical system. The space of values that $\lambda$ may take is referred to as the ontic state space $\Lambda$. Upon a preparation the physical system occupies a specific ontic state $\lambda \in \Lambda$. Crucially, in general, ontological models may not allow for fine-grained operational control over the ontic state the system actually occupies i.e., an operational preparation procedure $P$, might only specify the probabilities $\mu(\lambda|P)$ of the system being in different ontic states. This in turn requires the ontic state space $\Lambda$ to be a measurable space, with a $\sigma$-algebra $\Sigma$ along with $\mu : \Sigma \rightarrow [0,1]$ being a $\sigma$-additive function which satisfies $\mu(\Lambda) = 1$. Summarizing, in an ontological model, every preparation procedure $P$ is associated with an epistemic state (a normalized probability density over the ontic state space) $\mu(\lambda|P)$ such that $\int_{\Lambda} \mu(\lambda|P)d\lambda = 1$. A measurement in an ontological model simply measures the ontic state of a physical system. Consequently, ontological measurements are modelled by conditional probability distributions $\{\xi(k|\lambda)\}$ which specify the probability of obtaining the outcome $k$ given the ontic state $\lambda$. These probability distributions are measurable functions on $\Lambda$ and satisfy: positivity: $\forall \ k, \lambda : \xi(k|\lambda) \geq 0$, completeness: $\sum_{\lambda} \xi(k|\lambda) = 1$ and are referred to as response schemes. However, the outcomes of an operational measurement procedure might not unveil the ontic state uniquely or even allow for an inference of a set of ontic states within which the actual ontic state lies. In other words, the outcome of a measurement procedure might depend on the ontic state on only stochastically due to fundamental in-determinism of nature or dependence of measurement outcome on certain degrees of freedom other than the ontic state of the system being measured. Consequently, every operational measurement procedure $M$ is associated with a response scheme $\{\xi(k|\lambda,M)\}$ which specifies the probability of obtaining the outcome $k$ given a measurement $M$ was performed on a physical system occupying the ontic state $\lambda$. Averaging the conditional probabilities $\xi(k|\lambda,M)$ over our ignorance of the ontic state $\lambda$ yields the predictions of the ontological model for prepare and measure experiments, i.e. $p(k|P,M) = \int_{\Lambda} \mu(\lambda|P)\xi(k|\lambda,M)d\lambda$. Crucially, the response schemes $\{\xi(k|\lambda,M)\}$ associated with operational measurements $M$ form a subset of the set of general response schemes $\{\xi(k|\lambda)\}$ that are constrained only by positivity and completeness.

A transformation in an ontological model simply alters the ontic state of the system. However, an operational transformation procedure may do so only stochastically. To accommodate the same, ontological transformations are modelled by transition schemes $\{\gamma(\lambda'|\lambda)\}$, which represents the probability of transition from the ontic-state $\lambda$ to the ontic-state $\lambda'$. Clearly, the transition schemes are required to be measurable functions $\gamma : \Lambda \rightarrow \Lambda$. Summarizing, in a ontological model every operational transformation $T$ is associated with a transition scheme $\{\gamma(\lambda'|\lambda,T)\}$ such that $\forall \ \lambda : \int_{\lambda'} \gamma(\lambda'|\lambda,T)d\lambda' = 1$. Finally, the predictions of the ontological model in prepare, transform and measure experiments are given by $p(k|P,T,M) = \int_{\Lambda} d\lambda \mu(\lambda|P)\gamma(\lambda'|\lambda,T)\xi(k|\lambda,M)d\lambda'$. 
III. BOUNDED ONTOLOGICAL AND QUANTUM THEORY’S EXCESS ONTOLOGICAL DISTINCTNESS

The ontological principle, “bounded ontological distinctness” is based on the natural generalization of the Leibniz’s principle of “identity of indiscernibles” which states that maximal operational distinguishability reflects ontological distinctness. In this section, we invoke operational measures of distinguishability of sets of preparations, measurements and transformation and ontological measures of distinctness of sets of epistemic states, response schemes and transition schemes. Subsequently, we mathematically substantiate the principle of bounded ontological distinctness of the respective physical entities, derive certain consequences and report on quantum violations of these consequences.

A. Preparations

\[
\{P_x\}_{x=1}^n \rightarrow \text{[Diagram]} \quad k = x
\]

FIG. 1. This figure denotes the set-up for distinguishing a given set of preparations \(\{P_x\}_{x=1}^n\) (dark black device) out of an uniform ensemble. In each run of the experiment, a constituent preparation \(P_x\) is prepared, followed by a \(n\)-outcome measurement producing an outcome \(k\). The (single-shot) distinguishing probability is simply the probability of producing an output \(k = x\) as a result of a \(n\)-outcome measurement. This probability is maximized over all such measurements (light cyan device) yielding the distinguishability \(s_Q\) of the constituent preparations. As a consequence of this maximization, \(s_Q\) is an intrinsic property of this set of preparations and forms the operational condition accompanying bounded ontological distinctness of preparations.

In this work, we consider sets of preparations. We characterize these sets based on how well the constituent preparations can be operationally distinguished (Fig. 1). In an operational theory, a set of preparations \(\mathcal{P} \equiv \{P_x\}_{x=1}^n\) is termed \(p\)-distinguishable if the constituent preparations can be perfectly distinguished from a uniform ensemble with at-most \(p\) probability, i.e.,

\[
s_Q = \max_M \left\{ \frac{1}{n} \sum_x p(k = x|P_x, M) \right\} = p, \tag{1}\]

where \(s_Q\) denotes the maximum operational probability of distinguishing these preparations from a uniform ensemble and the maximization is over the set of all possible \(n\)-outcome measurements in a given operational theory, \(M\) is an instance of such a measurement, and \(k\) is an outcome of the measurement \(M\). Observe that the maximum probability of distinguishing the preparations \(s_Q\), is maximized over all possible measurements available in an operational theory. This maximization relieves \(s_Q\) of its dependence on measurements, deeming it to be a suitable characterizing feature for sets of preparations. In quantum theory, the maximum probability of distinguishing for a set of quantum states \(P_Q\) out of an uniform ensemble has the expression,

\[
s_Q = \max_{\{M_k\}} \left\{ \frac{1}{n} \sum_x \text{Tr}\{\rho_x M_k = x\} \right\}, \tag{2}\]

where the maximization is over the set of all possible \(n\)-outcome POVM \(\{M_k\}\). For a specific set of quantum states \(P_Q\) this maximization can be cast as an efficient semi-definite program which essentially yields the maximum distinguishing probability \(s_Q\), along with the optimal POVM \(\{M_k\}\) [16, 17]. Each set of operational preparations \(\mathcal{P}\) is associated with a set of epistemic states \(\mathcal{P}_\Lambda\). We characterize these sets based on how distinct the constituent epistemic states actually are. In an ontological model, a set of epistemic states \(\mathcal{P}_\Lambda \equiv \{\mu(\lambda|P_x)\}_{x=1}^n\) is termed \(p\)-distinct if the constituent epistemic states can be ontologically distinguished (upon having access to the ontic state \(\lambda\) from an uniform ensemble with at-most \(p\) probability, i.e.,

\[
s_\Lambda = \max_{\{\xi(\kappa|\lambda)\}} \left\{ \frac{1}{n} \sum_x \int_\Lambda \mu(\lambda|P_x) \xi(k = x|\lambda) d\lambda \right\} = p, \tag{3}\]

where \(s_\Lambda\) denotes the maximum probability of distinguishing these epistemic states from an uniform ensemble given the ontic state \(\lambda\), \(\Lambda\) represents the ontic state space, \(\mu(\lambda|P_x)\) is the epistemic state underlying the preparation \(P_x\) and the maximization is over all valid response schemes \(\{\xi(\kappa|\lambda)\}\) which satisfy positivity and completeness. Observe that the set of all possible \(n\)-outcome response schemes forms a convex polytope, the extremal points of this polytope are deterministic response schemes, i.e., each extremal response scheme is of the form: for each \(\lambda\), \(\xi(\kappa|\lambda) = 0\) except for specific \(k = \kappa,\lambda\) for which \(\xi(\kappa,\lambda) = 1\). In lieu of this observation, we can readily solve the above maximization by finding out the optimal extremal response scheme. Clearly, for each \(\lambda\), the optimal response would be to output the index \(x\) of the preparation which assigns the largest probability to \(\lambda\). This in-turn leads us to the following succinct expression for \(s_\Lambda\),

\[
s_\Lambda = \frac{1}{n} \int_\Lambda \max_x \left\{ \mu(\lambda|P_x) \right\} d\lambda. \tag{4}\]

This expression further substantiates the fact that the maximization over response schemes relieves \(s_\Lambda\) from its dependence on response schemes, deeming it a suitable
characterizing feature of the set of epistemic states under consideration and a measure of the actual distinctness of its constituents.

If an ontological model explains the predictions of an operational theory then for a set of preparations \( \mathcal{P} \) one may readily re-express the maximum probability of operationally distinguishing these preparations from an uniform ensemble as,

\[
s_\mathcal{O} = \max_{\{k \in \{1, \ldots, M\}\}} \left\{ \frac{1}{n} \sum_x \mu(\lambda|P_x) \xi(k = x|\lambda, M) \right\},
\]

where the maximization is over only the response schemes \( \{\xi(k|\lambda, M)\} \) associated with operational measurements \( M \). Observe that as the operational measurements may not reveal the ontic state \( \lambda \) and, having access to \( \lambda \) may only enhance the ability to distinguish the preparations under consideration. Therefore, in general we have \( s_\mathcal{O} \leq s_\Lambda \).

**Bounded ontological distinctness (BOD\( \mathcal{P} \)):** The epistemic states \( \mu(\lambda|P_x) \in \mathcal{P}_\lambda \) underlying a set of \( p \)-distinguishable preparations \( \mathcal{P} \equiv \{P_x\}_{x=1}^n \) are \( p \)-distinct (ontologically distinguishable from an uniform ensemble with at-most \( p \) probability), i.e. for a set of \( p \)-distinguishable preparations,

\[
s_\Lambda = \frac{1}{n} \int_\Lambda \max_x \left\{ \mu(\lambda|P_x) \right\} d\lambda = s_\mathcal{O} = p.
\]

When formulated in this way, bounded ontological distinctness serves to connect an exclusively operational phenomenon (distinguishability of operational preparations) to an exclusively ontological property (distinguishability of epistemic states), consequently leading to interesting implications. We explore one such implication via the following proposition, wherein we employ bounded ontological distinctness of a set of three preparations to arrive at an upper bound on operational average pairwise distinguishability of these preparations.

1. **Bounded ontological distinctness of three preparations.**

**Proposition 1.** If an operational theory admits ontological models adhering to the principle of bounded ontological distinctness, then a set of three \( p \)-distinguishable preparations \( \mathcal{P} \equiv \{P_1, P_2, P_3\} \), are pairwise distinguishable with the average probability of success being at-most \( \frac{1+p}{2} \), i.e.,

\[
\frac{1}{3} \left( s^{1,2}_\Lambda + s^{2,3}_\Lambda + s^{3,1}_\Lambda \right) \leq 1 - \frac{1}{6} \int_\Lambda \left( \min \left\{ \mu(\lambda|P_1), \mu(\lambda|P_2) \right\} + \min \left\{ \mu(\lambda|P_2), \mu(\lambda|P_3) \right\} + \min \left\{ \mu(\lambda|P_3), \mu(\lambda|P_1) \right\} \right) d\lambda
\]

\[
\geq \min \left\{ \mu(\lambda|P_1) + \mu(\lambda|P_2), \mu(\lambda|P_2) + \mu(\lambda|P_3), \mu(\lambda|P_3) + \mu(\lambda|P_1) \right\}
\]

\[
\leq \frac{1 + s_\Lambda}{2},
\]

where \( s^{i,j}_\Lambda \) denotes the maximum probability of distinguishing the preparations \( \{P_i, P_j\} \) out of an uniform ensemble, i.e., \( s^{i,j}_\mathcal{O} = \max_M \left\{ \frac{1}{2} \sum_x \mu(k = x|P_x, M) \right\} \).

**Proof.** We proceed in two steps, (i) we obtain an upper bound on the maximal average pairwise distinctness for a set three epistemic states in terms of maximal ontological distinctness and, (ii) we employ the principle of bounded ontological distinctness to port this ontological relation to distinguishability of operational preparations. Recall that, for a set of three epistemic states \( \mathcal{P}_\Lambda = \{\mu(\lambda|P_1), \mu(\lambda|P_2), \mu(\lambda|P_3)\} \), the maximal probability of distinguishing them from an uniform ensemble, upon having access to the ontic state \( \lambda \) has the expression,

\[
s_\Lambda = \frac{1}{3} \int_\Lambda \max_x \left\{ \mu(\lambda|P_x) \right\} d\lambda
\]

\[
= 1 - \frac{1}{3} \int_\Lambda \min \left\{ \mu(\lambda|P_1) + \mu(\lambda|P_2), \mu(\lambda|P_2) + \mu(\lambda|P_3), \mu(\lambda|P_3) + \mu(\lambda|P_1) \right\} d\lambda,
\]

where for the second equality we employed the fact that \( \forall (a \geq 0, b \geq 0, c \geq 0) : \max\{a, b, c\} = a + b + c - \min\{a + b, b + c, c + a\} \) along with the property \( \forall x \in \{1, 2, 3\} \) : \( \int_\Lambda (\lambda|P_x) d\lambda = 1 \). Similarly, for the pair of epistemic states \( \{\mu(\lambda|P_1), \mu(\lambda|P_2)\} \), the maximal probability of distinguishing them from an uniform ensemble, upon having access to the ontic state \( \lambda \) can be expressed as follows,

\[
s^{1,2}_\Lambda = \frac{1}{2} \int_\Lambda \max \left\{ \mu(\lambda|P_1), \mu(\lambda|P_2) \right\} d\lambda
\]

\[
= 1 - \frac{1}{2} \int_\Lambda \min \left\{ \mu(\lambda|P_1), \mu(\lambda|P_2) \right\} d\lambda.
\]
where the inequality follows from the fact that \( \forall (a \geq 0, b \geq 0, c \geq 0) : \min\{a, b\} + \min\{b, c\} + \min\{c, a\} = \min\{a + b, b + c, c + a\} + \min\{a, b, c\} \) and (7). Consequently, the inequality (9) is saturated when \( \forall \lambda : \min\{\mu(\lambda|P_1), \mu(\lambda|P_2), \mu(\lambda|P_3)\} = 0. \) As in general \( s^{1,2}_{\Lambda} \leq s^{3,1}_{\Lambda} \), we have,

\[
\frac{1}{3} \left( s^{1,2}_{O} + s^{2,3}_{O} + s^{3,1}_{O} \right) \leq \frac{1}{3} \left( s^{1,2}_{\Lambda} + s^{2,3}_{\Lambda} + s^{3,1}_{\Lambda} \right) \leq \frac{1 + s_{\Lambda}}{2}
\]

(10)

Now, if an operational theory admits ontological models that adhere to bounded ontological distinguishability, every set of \( p \)-distinct distinguishable preparations is associated with a set of \( p \)-distinct epistemic states i.e. \( s_{\Lambda} = s_{O} = p. \) Inserting this into (10) yields the desired thesis. \( \blacksquare \)

2. Excess ontological distinctness of three quantum preparations

We are now prepared to demonstrate the quantum violation of the principle of bounded ontological distinctness. Consider, three qubit preparations

\[
\rho_i = \frac{I + \vec{n}_i \cdot \vec{\sigma}}{2}, \quad i = 1, 2, 3,
\]

(11)

where \( \vec{\sigma} = [\sigma_x, \sigma_y, \sigma_z] \) is a vector of the Pauli matrices, \( \vec{n}_1 = [1, 0, 0]^T, \vec{n}_2 = [\cos \frac{2\pi}{3}, \sin \frac{2\pi}{3}, 0]^T \) and \( \vec{n}_3 = [\cos \frac{4\pi}{3}, \sin \frac{4\pi}{3}, 0]^T \) are Bloch vectors. Observe that for any three two dimensional quantum preparation the maximum probability of distinguishing them from an uniform ensemble is upper bounded as \( s_{O} = \max_{\{M_k\}} \left\{ \frac{1}{2} \sum_{\{1,2,3\}} \text{Tr}\{\rho_i M_{k=1,2,3}\} \right\} \leq \frac{1}{2} \sum_{\{1,2,3\}} \text{Tr}\{M_k\} = \frac{3}{2}. \) A straightforward semi-definite program yields the optimal measurement \( \{M_k = \frac{2x_1}{3}\} \) which saturates this upper bound. This implies that these states form a set of \( \frac{2}{3} \)-distinguishable quantum preparations. It follows from Proposition 1 that if an operational theory admits ontological models that adhere to bounded ontological distinctness, then these preparations are pairwise distinguishable with at-most average probability \( \frac{5}{6} \), i.e.,

\[
\frac{1}{3}(s^{1,2}_{O} + s^{2,3}_{O} + s^{3,1}_{O}) \leq \frac{5}{6}.
\]

However, the maximum probability of distinguishing the pair of quantum states \( \rho_0, \rho_1 \) out of an uniform ensemble has the following succinct expression,

\[
s_{Q,1}^{1,2} = \frac{1}{2} \left( 1 + T(\rho_1, \rho_2) \right) = \frac{1}{2} \left( 1 + \sqrt{\frac{3}{2}} \right),
\]

(12)

where \( T(\rho_1, \rho_2) \) is the trace distance between the density matrices \( \{\rho_1, \rho_2\} \), and \( s_{Q}^{1,2} \) is the maximum probability for distinguishing quantum states \( \{\rho_1, \rho_2\} \) out of an uniform ensemble. As the states under consideration are symmetrically distributed, the maximum probability of distinguishing \( \{\rho_2, \rho_3\} \) and \( \{\rho_3, \rho_1\} \) remains unchanged i.e. \( s_{Q}^{2,3} = s_{Q}^{3,1} = s_{Q}^{1,2} = \frac{1}{2}(1 + \sqrt{\frac{3}{2}}) \). This in turn leads us to,

\[
\frac{1}{3}(s_{Q}^{1,2} + s_{Q}^{2,3} + s_{Q}^{3,1}) = \frac{1}{2} \left( 1 + \sqrt{\frac{3}{2}} \right),
\]

(13)

which is a clear violation of Proposition 1 and consequently of bounded ontological distinctness. This implies that any ontological model that seeks to explain the predictions of quantum theory must feature excess ontological distinctness. Specifically, in order to explain the maximal average probability for pairwise distinguishing the aforementioned states the set of epistemic states underlying this set of \( \frac{2}{3} \)-distinguishable qubits must be at least \( \frac{\sqrt{3}}{3} \)-distinct. Furthermore, we demonstrate extensive violation of bounded ontological distinctness by randomly sampling triplets of pure qubits and pure qutrits. Remarkably, 72% of randomly picked triplets of pure qubits and 58% of randomly picked triplets of pure qutrits exhibit excess ontological distinctness (Fig. 5). Furthermore, over 46% of randomly picked triplets of mixed qubit states and over 29% of randomly picked triplets of mixed qutrit states exhibit excess ontological distinctness.

3. Examples of quantum ontological models

Next, we discuss three ontological models [18] which exhibit excess ontological distinctness: (i) the Beltrametti-Bugajski ontological model posits that the different possible ontic-states are simply the different possible quantum states [19]. As this model takes the quantum state alone to be a complete description of reality, it is \( \psi \)-complete. Each pure quantum preparation is associated with an unique ontic state in \( \psi \)-complete models and hence ontologically perfectly distinct i.e. all sets of pure non-identical quantum preparations have \( s_{\Lambda} = 1. \) (ii) Similarly, the Bell-Mermin ontological model for a two dimensional Hilbert space is a \( \psi \)-ontic model wherein every ontic state is associated with only one pure quantum preparation [20, 21]. Consequently, in \( \psi \)-ontic models all sets of pure quantum preparation are ontologically perfectly distinct with \( s_{\Lambda} = 1. \) (iii) The more intriguing \( \psi \)-epistemic ontological models posit ontic states that are consistent with more than one pure quantum state. A well known \( \psi \)-epistemic ontological model for a two dimensional Hilbert space was proposed by Kochen and Specker [5]. In this model the ontic-state space \( \Lambda \) is taken to be the unit sphere and each pure qubit preparation \( P \equiv \rho = \frac{1 + \vec{n} \cdot \vec{\sigma}}{2} \) is associated with the epistemic state \( \mu(\lambda|P) = \frac{1}{\pi}(H(\vec{n} \cdot \vec{\lambda}) \vec{n} \cdot \vec{\lambda}) \) where \( H(x) = 1 \) if \( x > 1 \), and \( H(x) = 0 \) if \( x \leq 0 \). Now, we present two intriguing observations pertaining to Kochen-Specker’s ontological model.
Observation 1. Remarkably, the epistemic states prescribed by the Kochen-Specker’s ψ-epistemic ontological model associated with the aforementioned quantum preparations \( \{ \rho_1, \rho_2, \rho_3 \} \) (11) saturate the lower bound on the extent of excess ontological distinct inferred from the violation of (6) i.e. this set is precisely \( \frac{\sqrt{3}}{2} \)-distinct.

Proof. The maximum probability of distinguishing the epistemic-states associated with \( \rho_1, \rho_2 \) and \( \rho_3 \) upon having access to the ontic state \( \lambda \) is,

\[
s_A = \int_{\Lambda} \max_{x \in \{1,2,3\}} \frac{1}{\pi} (H(n^*_x \cdot \hat{x})n^*_x \cdot \hat{x}) d\lambda
\]

\[
= \frac{1}{3\pi} \int_{\Lambda} \max_{x \in \{1,2,3\}} \left\{ H(n^*_x \cdot \hat{x})n^*_x \cdot \hat{x} \right\} d\lambda
\]

\[
= \frac{1}{3\pi} \int_{\Lambda} \max \left\{ x, \cos \frac{2\pi}{3} \lambda_x + \sin \frac{2\pi}{3} \lambda_y, \cos \frac{4\pi}{3} \lambda_x + \sin \frac{4\pi}{3} \lambda_y \right\} d\lambda
\]

\[
= \frac{\sqrt{3}}{2}, \tag{14}
\]

where for the second equality we use the fact that (i) the ontic-state space is the surface of a unit sphere and the ontic state is a real three dimensional unit vector \( \lambda = [\lambda_x, \lambda_y, \lambda_z]^T \) and (ii) because of the particular orientation of the Bloch vectors of the quantum states under consideration, the function \( H(n^*_x \cdot \hat{x}) \) in the integrand is redundant as \( \max \{ n^*_1 \cdot \hat{x}, n^*_2 \cdot \hat{x}, n^*_3 \cdot \hat{x} \} \) is never negative. \( \blacksquare \)

Observation 2. Bounded ontological distinctness of preparations cannot be violated employing a pair of pure two dimensional quantum preparations.

Proof. We have demonstrated violation of bounded ontological distinctness whilst employing triplets of qubits, however, when we restrict ourselves to just two pure qubits we find that the Kochen-Specker’s ontological model satisfies bounded ontological distinctness i.e. in this model, for all pairs of qubits the maximum operational distinguishability \( s_Q \) is exactly the ontological distinctness \( s_A \). To see this, consider two pure qubits \( \rho_1 = \frac{1+i n^*_1}{2}, \rho_2 = \frac{1+i n^*_2}{2} \). Without loss of generality, we can assume the Bloch vector \( n^*_1 \) associated with the first qubit to be \( [1,0,0]^T \) and the Bloch vector \( n^*_2 \) to be \( [\cos \theta_0, \sin \theta_0, 0]^T \) where \( \theta_0 \in [0, \frac{\pi}{2}] \). The maximum operational distinguishing probability for these two qubits turns out to be \( s_Q = \frac{1}{2}(1 + \sin \frac{\theta_0}{2}) \). Now in the Kochen-Specker model the ontological distinctness of corresponding epistemic states turns out to be,

\[
s_A = \frac{1}{2\pi} \int_{\Lambda} \max \left\{ H(n^*_1 \cdot \hat{x})n^*_1 \cdot \hat{x}, H(n^*_2 \cdot \hat{x})n^*_2 \cdot \hat{x} \right\} d\lambda
\]

\[
= \frac{1}{2\pi} \int_{\Lambda} \max \left\{ 0, \lambda_x, \lambda_y, \cos \theta_0 \lambda_x + \sin \theta_0 \lambda_y \right\} d\lambda
\]

\[
= \frac{1}{2\pi} \int_{\theta_\lambda=-\pi}^{\theta_\lambda=\pi} \int_{\phi_\lambda=0}^{\pi} \max \left\{ 0, \cos \theta_\lambda \sin \phi_\lambda, \cos \theta_0 \cos \theta_\lambda \sin \phi_\lambda + \sin \theta_0 \sin \theta_\lambda \right\} \sin \phi_\lambda d\phi_\lambda d\theta_\lambda
\]

\[
= \frac{1}{2\pi} \int_{\theta_\lambda=-\pi}^{\theta_\lambda=\pi} \int_{\phi_\lambda=0}^{\pi} \cos \theta_\lambda \sin^2 \phi_\lambda d\phi_\lambda d\theta_\lambda
\]

\[
= \frac{1}{2} \left( 1 + \sin \frac{\theta_0}{2} \right) = s_Q, \tag{15}
\]

where for the second equality we used the fact that (i) the ontic-state space is the surface of a unit sphere and the ontic state is a real three dimensional unit vector \( \lambda = [\lambda_x, \lambda_y, \lambda_z]^T \) and (ii) the effect of the function \( H(.) \) in the original integral is to assign zero whenever the other two terms are negative. For the third equality we expressed the ontic state in standard spherical coordinates \( [\lambda_x, \lambda_y, \lambda_z]^T \equiv [\cos \theta_\lambda \sin \phi_\lambda, \sin \theta_0 \sin \phi_\lambda, \cos \phi_\lambda]^T \) where \( \theta \in [-\pi, \pi] \) is the azimuthal and \( \phi \in [0, \pi] \) is the polar spherical coordinate. The fourth equality follows from the observations that (i) the second function in the maximization is symmetric with respect to \( [1,0,0]^T \) on the azimuthal plane, (ii) the third function in the maximization is simply the second function rotated by an angle \( \theta_0 \) on the azimuthal plane and, (iii) the functions are mirror symmetric on either side of the plane bisecting \( \theta_0 \) and cross over on the same plane i.e. \( \forall \theta_\lambda \in [-\frac{\pi}{2}, \frac{\pi}{2}], \theta_\lambda' = \theta_0 - \theta_\lambda, \max \{ 0, \cos \theta_\lambda, \cos \theta_0 \cos \theta_\lambda + \sin \theta_0 \sin \theta_\lambda \} = \max \{ 0, \cos \theta_\lambda', \cos \theta_0 \cos \theta_\lambda' + \sin \theta_0 \sin \theta_\lambda' \} = \cos \theta_\lambda. \) \( \blacksquare \)

4. Bounded ontological distinctness of two mixed preparations

With the aid of the following additional property of ontological models we demonstrate excess ontological dis-
tinctness by bounding the ontological distinctness of two mixed preparations.

**Convexity of epistemic states:** The epistemic state underlying an operational mixture of two preparations is the mixture of the respective epistemic states i.e. if \( P_{1+2}^c = cP_1 + (1-c)P_2 \) then \( \mu(\lambda|P_{1+2}^c) = c\mu(\lambda|P_1) + (1-c)\mu(\lambda|P_2) \). For the ease of notation, in what follows, we shall denote the uniform mixture as \( P_{1+2} = P_{1+2}^f = \frac{1}{2}(P_1 + P_2) \) [22].

In the following proposition, we consider four preparations and employ bounded ontological distinctness of a disjoint pair of two-preparation mixtures along with convexity of epistemic states to obtain an upper bound on average distinguishability of other disjoint pairs of two-preparation mixtures.

**Proposition 2.** Consider a set of four operational preparations \( \{P_1, P_2, P_3, P_4\} \) where the set of mixtures \( \{P_{1+2}, P_{3+4}\} \) is \( p \)-distinguishable. Now, if an operational theory admits ontological models adhering to the principle of bounded ontological distinctness along with convexity of epistemic states, then the pairs of mixtures \( \{P_{1+3}, P_{2+4}\} \) and \( \{P_{1+4}, P_{2+3}\} \) are distinguishable with the average probability of success being at-most \( \frac{1+p}{2} \) i.e. if \( s_{1+2}^{1+3,2+4} = s_{1+4,2+3}^{1+4,2+3} = p \) then,

\[
\frac{1}{2} \left( s_{1}^{1+3,2+4} + s_{1}^{1+4,2+3} \right) \leq \frac{1+p}{2},
\]

\[
\frac{1}{2} \left( s_{1}^{1+3,2+4} + s_{1}^{1+4,2+3} \right) = 1 - \frac{1}{8} \int_{\Lambda} \left( \min \left\{ \mu(\lambda|P_3) + \mu(\lambda|P_2) + \mu(\lambda|P_4) \right\} 
\right.
\]

\[
\left. + \min \left\{ \mu(\lambda|P_1) + \mu(\lambda|P_3) + \mu(\lambda|P_4) \right\} \right) d\lambda.
\]

The observation \( \forall \{a \geq 0, b \geq 0, c \geq 0, d \geq 0\} \), \( \min\{a+b, c+d\} + \min\{a+b, c+d\} \geq \min\{a+b, c+d\} \) along with the fact that in general \( s_{1+2}^{1+3,2+4} \leq s_{1+2}^{1+3,2+4} \) yields,

\[
\frac{1}{2} \left( s_{1}^{1+3,2+4} + s_{1}^{1+4,2+3} \right) \leq 1 - \frac{1}{8} \int_{\Lambda} \min \left\{ \mu(\lambda|P_1) + \mu(\lambda|P_2) + \mu(\lambda|P_4) \right\} d\lambda
\]

\[
\leq \frac{1 + s_{1}^{1+2,3+4}}{2}
\]

Now, if an operational theory admits ontological models that adhere to bounded ontological distinguishability, every set \( p \)-distinguishable preparations is associated with a set of \( p \)-distinct epistemic states i.e. \( s_{1+2}^{1+3,2+4} = s_{1+4,2+3}^{1+4,2+3} = p \). Inserting this into (19) yields the desired thesis.

5. **Excess ontological distinctness of a pair of mixed quantum preparations**

To witness quantum violation of (16), consider four pure qubits of the form \( \rho_x = \frac{i\pi_1 + \pi_2}{2} \) where \( \pi_1 = \cos \theta \sin \theta, 0 \rangle \langle \theta, \sin \theta, 0 \rangle \), \( \pi_2 = \begin{bmatrix} -1, 0, 0 \end{bmatrix}^T \), \( \pi_3 = \begin{bmatrix} 0, 1, 0 \end{bmatrix}^T \) and \( \pi_4 = \begin{bmatrix} 0, -1, 0 \end{bmatrix}^T \) and \( \theta \in [0, \frac{\pi}{4}] \). The pair of mixtures \( \{\rho_{1+2}, \rho_{3+4}\} \) is \( \frac{1}{2}(1 + \sqrt{1 - \cos \theta}) \)-distinguishable. It follows from Proposition 2, if an operational theory admits ontological models that adhere to bounded ontological distinctness and convexity of epistemic states then maximum average probability of distinguishing the pairs of mixtures \( \{\rho_{1+3}, \rho_{2+4}\} \) and \( \{\rho_{1+4}, \rho_{2+3}\} \) is upper-bounded by

\[
\frac{1}{2} \left( s_{1}^{1+3,2+4} + s_{1}^{1+3,2+4} \right) \leq \frac{1}{16} \left( \sqrt{2} \sqrt{1 - \cos(\theta)} + 12 \right).
\]
However the pairs of mixtures \( \{ \rho_{1+3}, \rho_{2+4} \} \) and \( \{ \rho_{1+4}, \rho_{2,3} \} \) are \( \frac{1}{2} (1 + \frac{\sqrt{3 + \cos \theta - 2 \sin \theta}}{2\sqrt{2}}) \)-distinguishable and \( \frac{1}{2} (1 + \frac{\sqrt{3 + \cos \theta + 2 \sin \theta}}{2\sqrt{2}}) \)-distinguishable respectively, which leads to violation of (16) for \( 0 < \theta \leq \frac{\pi}{2\sqrt{2}} \) (Fig. 7), since

\[
\frac{1}{2} (s^{1+3,2+4}_Q + s^{1+4,2+3}_Q) = \frac{1}{2} + \frac{\sqrt{3 + \cos \theta + 2 \sin \theta}}{8\sqrt{2}} + \frac{\sqrt{3 + \cos \theta - 2 \sin \theta}}{8\sqrt{2}}.
\]

In particular when \( \theta_0 = 0 \), to explain maximum average probability of distinguishing the pairs of mixtures \( \{ \rho_{1+3}, \rho_{2+4} \} \) and \( \{ \rho_{1+4}, \rho_{2,3} \} \), \( \frac{1}{2} (s^{1+3,2+4}_Q + s^{1+4,2+3}_Q) = \frac{2 + \sqrt{2}}{2\sqrt{2}} \), the epistemic states underlying completely indistinguishable mixtures \( \{ \rho_{1+2}, \rho_{3+4} \} \) must be at least \( \frac{1}{\sqrt{2}} \)-distinct. We demonstrate extensive violation of Proposition 2 employing randomly sampled quadruplets qubits, qutrits, ququarts an qu5its. Over 13%, 9%, 3% and 1% of randomly picked quadruplets of pure qubits, qutrits, ququarts and qu5its respectively exhibit violation of (16) (Fig. 6).

6. Examples of quantum ontological models

In Beltrametti-Bugaiski ontological model (\( \psi \)-complete) and the Bell-Mermin ontological model (\( \psi \)-ontic) and for the aforementioned states \( \rho_{1,2}, \rho_{3,4} \)

\[
\frac{1}{\Lambda} = 1. \quad \text{Yet again Kochen-Specker's \( \psi \)-epistemic model stands out.}
\]

Observation 3. Remarkably, the epistemic states prescribed by the Kochen-Specker's \( \psi \)-epistemic ontological model associated with the aforementioned state of mixed quantum preparations \( \{ \rho_{1+2}, \rho_{3+4} \} \) for \( \theta = 0 \) saturate the lower bound on the extent of excess ontological distinct inferred from the violation of (16) i.e. this pair is precisely \( \frac{1}{\sqrt{2}} \)-distinct.

In the Kochen-Spekker model (\( \psi \)-epistemic), the epistemic states \( \{ \mu(\lambda|P_{1+2}), \mu(\lambda|P_{3+4}) \} \) are ontologically distinguishable with maximum probability,

\[
\begin{align*}
\frac{1}{4\pi} & \int_\Lambda \max \left\{ H(n_1 \cdot \vec{x})n_1 \cdot \vec{x} + H(n_2 \cdot \vec{x})n_2 \cdot \vec{x}, H(n_3 \cdot \vec{x})n_3 \cdot \vec{x} + H(n_4 \cdot \vec{x})n_4 \cdot \vec{x} \right\} d\lambda \\
& = \frac{2}{\pi} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi} \cos \theta \sin^2 \phi d\theta d\phi \\
& = \frac{1}{\sqrt{2}}.
\end{align*}
\]

where for the second equality expressed the ontic state in standard spherical coordinates \( \Lambda(x, y, z) = [\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi]^T \) and employed the observations that (i) \( \forall \theta \in [0, \frac{\pi}{2}] \): \( H(n_2 \cdot \vec{x}) = H(n_4 \cdot \vec{x}) = 0 \) and (ii) the Bloch vectors \( n_1, n_2, n_3, n_4 \) are symmetrically distributed on the azimuthal plane which in turn translates to the same contribution from all four quarters of the azimuthal plane. Now, the second equality follows from the observation that,

\[
\frac{1}{4\pi} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi} \max \{ \cos \theta \sin^2 \phi, \sin \theta \sin^2 \phi \} d\theta d\phi = \frac{1}{2\pi} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi} \cos \theta \sin^2 \phi d\theta d\phi.
\]

B. Measurements

We can readily extended the principle of bounded ontological distinctness so as to apply to measurements. Here, we invoke sets of \( d \)-outcome measurements and characterize these sets on the basis of how well the constituent measurements can be operationally distinguished from an uniform ensemble (Fig. 2). In an operational theory, a set of \( d \)-outcome measurements \( \mathcal{M} \equiv \{ M^n \}_{n=1} \) is called

\( p \)-distinguishable if the constituent measurements can be perfectly distinguished from an uniform ensemble at-most \( p \) probability, i.e.,

\[
m_O = \max_P \left\{ \frac{1}{n} \max_{q(z|k)} \left\{ \sum_{k:y} q(z = y|k) p(k|M^n, P) \right\} \right\} = p,
\]

where \( m_O \) denotes the maximum operational probability of distinguishing these measurements from an uniform ensemble and the first maximization is over the set of all possible operational preparations and the second maximization is over all possible conditional probability distributions \( \{ q(z|k) \} \), \( M^n \in \mathcal{M} \) is a constituent measurement and \( k \) is an outcome of the measurement \( M^n \) [23] (Fig. 2). Clearly, the initial maximization relieves \( m_O \) of its dependence on preparations and the second maximization relieves \( m_O \) of its dependence on classical post-processing schemes, deeming it to be a suitable characterizing feature of sets of measurements. We call the probability distributions \( \{ q(z|k) \} \) classical post-processing schemes which satisfy: positivity: \( \forall k, z : q(k|z) \geq 0 \) and, completeness: \( \forall z : \sum_k q(k|z) = 1 \). Observe that the set of
all possible \( n \)-outcome classical post-processing schemes forms a convex polytope, the extremal points of this polytope are deterministic probability distributions, i.e., each extremal classical post-processing scheme is of the form: for each \( k \), \( q(z|k) = 0 \) except for specific \( z = z_k \) for which \( q(z_k|k) = 1 \). In lieu of this observation, we can readily solve the second maximization by finding out the optimal classical post-processing scheme. Clearly, for each preparation \( P \) and measurement outcome \( k \), the optimal classical post-processing would be to output the index \( y \) of the measurement which assigns the largest probability to \( k \). This in-turn leads us to the following succinct expression for \( m_\mathcal{O} \),

\[
m_\mathcal{O} = \max_{P} \left\{ \frac{1}{n} \sum_k \max_y \left\{ p_k(M^n, P) \right\} \right\}. \tag{24}
\]

Recall that in an ontological model, each operational measurement is associated with a set of response schemes. We characterize the sets of response schemes based on how distinct the constituent response schemes are. In an ontological model, a set of \( d \)-outcome response schemes \( M_d \) is called \( p \)-distinct if the constituent response schemes can be ontologically distinguished (upon having fine-grained control over preparation of the ontic state \( \lambda \)) from an uniform ensemble with at most \( p \) probability, i.e.,

\[
m_A = \max_{\lambda} \left\{ \frac{1}{n} \sum_k \max_y \left\{ \xi(k|\lambda, M^n) \right\} \right\} = p, \tag{25}
\]

where \( m_A \) denotes the maximum probability of distinguishing these response schemes from an uniform ensemble given the ontic state \( \lambda \) and the maximization is over all ontic states \( \lambda \). Now as the operational preparation might not allow fine-grained control on the ontic states, we have in general \( m_\mathcal{O} \leq m_A \).

Equipped with these measures we define the bounded ontological distinctness for measurements \( (\text{BOD}_M) \) as: The response schemes \( \{\xi(k|\lambda, M^n)\} \in M_A \) underlying a set of \( p \)-distinguishable measurements \( M \equiv \{M^n\}_{y=1}^n \) are \( p \)-distinct (ontologically distinguishable from an uniform ensemble with at-most \( p \) probability), i.e. for a set of \( p \)-distinguishable measurements, 

\[
m_A = m_\mathcal{O} = p. \tag{26}
\]

Unlike preparations, the Beltrametti-Bugajski’s \( \psi \)-complete ontological model adheres to the principle of bounded ontological distinctness of measurements. This follows from the observations (i) in this model the response scheme underlying a POVM element \( M_k \) is simply \( \xi(k|\lambda, M) = \text{Tr}(\psi\langle\psi|M_k\rangle) \), and (ii) we have fine-grained control over the ontic state \( \lambda \) which is simply the quantum state \( \lambda \equiv |\psi\rangle \).

However, we find that the ontological models wherein quantum measurements are associated with deterministic response schemes violate the bounded ontological distinguished. In-order to substantiate this, we make the following formal assumption:

**Outcome deterministic response schemes**: In an ontological model each operational measurement \( M \) is associated with a deterministic response scheme \( \{\xi(z|\lambda, M)\} \) such that \( \forall z, \lambda, M : \xi(z|\lambda, M) \in \{0,1\} \).

Consider a pair of two-outcome of measurements \( \{M^1, M^2\} \), then in an outcome deterministic ontological model,

\[
m_A = \frac{1}{2} \max_{\lambda} \left\{ \sum_{k \in \{1,2\}} \max \left\{ \xi(k|\lambda, M^1), \xi(k|\lambda, M^2) \right\} \right\} \tag{27}
\]

belongs to \( \{\frac{1}{2}, 1\} \). If the pair of measurements \( \{M^1, M^2\} \) are neither completely indistinguishable nor perfectly distinguishable, i.e., \( m_\mathcal{O} \notin (\frac{1}{2}, 1) \), then the associated deterministic response schemes must be perfectly distinguishable \( m_A = 1 \). For instance, the Kochen-Specker model prescribes outcome deterministic response schemes to two-outcome two-dimensional projective quantum measurements \( \xi(1|\lambda, \{\phi_1|\phi_1\langle\phi_1|\phi_1\rangle\}) = H(n_\phi \cdot \hat{X}) \) and \( \xi(2|\lambda, \{\phi_1|\phi_1\langle\phi_1|\phi_2\rangle\}) = H(n_\phi \cdot \hat{X}) \). These response schemes return outcome 1 whenever the ontic-state lies in the Bloch hemisphere with central axis \( n_\phi \) and outcome 2 otherwise. Now, consider a pair of two dimensional projective measurements \( \{M^1_1 \equiv |\phi_1|\langle\phi_1|\phi_2\rangle, M^1_2 \equiv |\phi_2\rangle\langle\phi_2|\phi_2\rangle\} \) and \( \{M^2_1 \equiv |\phi_2\rangle\langle\phi_2|\phi_2\rangle, M^2_2 \equiv |\phi_2\rangle\langle\phi_2|\phi_2\rangle\} \).

If \( \phi_1 \neq \phi_2 \) then one can always find suitable ontic state \( \lambda \) such that \( \xi(1|\lambda, \{\phi_1|\phi_1\langle\phi_1|\phi_1\rangle\}) = 1 \) and \( \xi(1|\lambda, \{\phi_2\rangle\langle\phi_2|\phi_2\rangle\}) = 0 \) and similarly for the second outcome, deeming the two response schemes under consideration to be perfectly distinguishable with \( m_A = 1 \).
C. Transformations

\[ \{T_x^n\}_{x=1} = \equiv \{\{\gamma(\lambda'|\lambda,T_x)\}\}_{x=1} \]

Yet again, we invoke sets of transformations. We characterize these sets based on how well the constituent transformations can be operationally distinguished (Fig. 3). In an operational theory, a set of transformations \( \mathcal{T} \equiv \{T_x^n\}_{x=1} \) is called \( p \)-distinguishable if the constituent transformations can be perfectly distinguished from an uniform ensemble with at-most \( p \) probability, i.e.,

\[
t_o = \max_p \max_M \left\{ \frac{1}{n} \sum_x p(k = x|P,T_x,M) \right\} = p, \quad (28)
\]

where \( t_o \) denotes the maximum operational probability of distinguishing these transformations from an uniform ensemble, the initial maximization is over all possible preparations and the second maximization is over the set of all possible \( n \)-outcome measurements in a given operational theory \([24]\) (Fig. 3). Observe that the maximum probability of distinguishing the transformations \( t_o \), is maximized over all possible preparations and measurements available in an operational theory. This maximization relieves \( t_o \) of its dependence on measurements, deeming it to be a suitable characterizing feature of sets of transformations.

Each set of operational transformations \( \mathcal{T} \) is associated with a set of transition schemes \( \{T_x\} \). We characterize these sets based on how distinct the constituent transition schemes actually are.

In an ontological model, a set of transition schemes \( \mathcal{T}_A \equiv \{\{\gamma(\lambda'|\lambda,T_x)\}\}_{x=1} \) is termed \( p \)-distinct if the constituent transition schemes can be ontologically distinguished (upon having fine-grained control over preparation of the ontic state \( \lambda \) and access to the post-transition ontic state \( \lambda' \)) from an uniform ensemble with at-most \( p \) probability, i.e.,

\[
t_A = \max_\lambda \left\{ \max_{\xi(k|x')} \left\{ \frac{1}{n} \sum_x \int_{\Lambda} \gamma(\lambda'|\lambda,T_x)\xi(k = x|\lambda')d\lambda' \right\} \right\} = p, \quad (29)
\]

where \( t_A \) denotes the maximum probability of distinguishing these transition schemes from an uniform ensemble given access to initial ontic state \( \lambda \), \( \Lambda \) represents the ontic state space, \( \{\gamma(\lambda'|\lambda,T_x)\} \) is the transition scheme associated the transformation \( T_x \) and the maximization is over all ontic states \( \lambda \) as well as all valid response schemes \( \{\xi(k|\lambda)\} \) which satisfy positivity and completeness. Yet again, following from the observation that the response schemes are so constrained form a convex polytope, we can readily solve the second maximization which yields the following succinct expression of ontological distinctness,

\[
t_A = \max_\lambda \left\{ \frac{1}{n} \int_{\Lambda} \max_x \gamma(\lambda'|\lambda,T_x)d\lambda' \right\} = t_o = p. \quad (30)
\]

Bounded ontological distinctness for transformations (BODT): The transition schemes \( \{\gamma(\lambda'|\lambda,T_x)\} \in \mathcal{T}_A \) underlying a set of \( p \)-distinguishable transformations \( \mathcal{T} \equiv \{T_x^n\}_{x=1} \) are \( p \)-distinct (ontologically distinguishable from an uniform ensemble with at-most \( p \) probability), i.e. for a set of \( p \)-distinguishable transformations,

\[
t_A = \max_\lambda \left\{ \frac{1}{n} \int \max_x \gamma(\lambda'|\lambda,T_x)d\lambda' \right\} = t_o = p. \quad (31)
\]

Yet again, in this form, bounded ontological distinctness leads to some interesting consequences. We present one such consequence in the form of the following proposition.

1. Bounded ontological distinctness of three transformations

**Proposition 3.** If an operational theory admits ontological models adhering to the principle of bounded ontological distinctness, then a set of three \( p \)-distinguishable transformations \( \mathcal{T} \equiv \{T_1,T_2,T_3\} \), are pairwise distinguishable with respect to the same initial preparation with the average probability of success being at-most \( \frac{1+p}{2} \), i.e.,

\[
\frac{1}{3} \max_p \left\{ t^{1,2}_o(P) + t^{2,3}_o(P) + t^{3,1}_o(P) \right\} \leq \frac{1+p}{2} \quad (32)
\]

where \( t^{i,j}_o(P) \) denotes the maximum probability of distinguishing the transformations \( T_i,T_j \) out of an uniform ensemble given the initial preparation \( P \), i.e., \( t^{i,j}_o(P) = \max_p \{\frac{1}{n} \sum_{x(i,j)} p(k = x|P,T_{i,j},M)\} \).

**Proof.** We follow roughly the same steps as in the proof of Proposition 1. The maximum probability of distinguishing a set of three transition schemes \( \mathcal{T}_A \equiv \{\{\gamma(\lambda'|\lambda,T_x)\}\}_{x=1} \)
\{\{\gamma(\lambda'|T_1, \lambda)\}, \{\gamma(\lambda'|T_2, \lambda)\}, \{\gamma(\lambda'|T_3, \lambda)\}\} upon having access to the initial ontic state \(\lambda\) as well as post-transition ontic state \(\lambda'\) has the expression,

\[
t_\Lambda = \frac{1}{3} \max_{\lambda} \left\{ \int_{\Lambda} \max_x \left\{ \gamma(\lambda'|\lambda, T_x) \right\} d\lambda' \right\} \\
= 1 - \frac{1}{3} \min_{\lambda} \left\{ \int_{\Lambda} \min \left\{ \gamma(\lambda'|T_1, \lambda) + \gamma(\lambda'|T_2, \lambda), \gamma(\lambda'|T_2, \lambda) + \gamma(\lambda'|T_3, \lambda), \gamma(\lambda'|T_3, \lambda) + \gamma(\lambda'|T_1, \lambda) \right\} d\lambda' \right\}
\]

Similarly, the expression for maximal average pairwise ontological distinctness with respect to same initial ontic state is,

\[
t_\Lambda \max \left\{ t_\Lambda^{1,2}(\lambda) + t_\Lambda^{2,3}(\lambda) + t_\Lambda^{3,1}(\lambda) \right\}
\]

\[
= 1 - \frac{1}{6} \min_{\lambda} \left\{ \int_{\Lambda} \left\{ \min \left\{ \gamma(\lambda'|\lambda, T_1), \gamma(\lambda'|\lambda, T_2) \right\} + \min \left\{ \gamma(\lambda'|\lambda, T_2), \gamma(\lambda'|\lambda, T_3) \right\} + \min \left\{ \gamma(\lambda'|\lambda, T_3), \gamma(\lambda'|\lambda, T_1) \right\} \right\} d\lambda' \right\}
\]

\[
\leq 1 - \frac{1}{6} \min_{\lambda} \left\{ \int_{\Lambda} \left\{ \min \left\{ \gamma(\lambda'|\lambda, T_1) + \gamma(\lambda'|\lambda, T_2), \gamma(\lambda'|\lambda, T_2) + \gamma(\lambda'|\lambda, T_3), \gamma(\lambda'|\lambda, T_3) + \gamma(\lambda'|\lambda, T_1) \right\} \right\} d\lambda' \right\}
\]

\[
= 1 + t_\Lambda \frac{2}{2}.
\]

As operational preparations might not allow fine-grained control over the initial ontic-state and operational measurements might not reveal the post-transition ontic state uniquely, we have in general that \(t_\Omega^O(P) \leq t_\Lambda^O(\lambda)\),

\[
\frac{1}{3} \left( \max_{P} \left\{ t_\Omega^{1,2}(P) + t_\Omega^{2,3}(P) + t_\Omega^{3,1}(P) \right\} \right) \\
\leq \frac{1}{3} \max_{\lambda} \left\{ t_\Lambda^{1,2}(\lambda) + t_\Lambda^{2,3}(\lambda) + t_\Lambda^{3,1}(\lambda) \right\}
\]

\[
= \frac{1}{2} + t_\Lambda.
\]

Finally, bounded ontological distinctness of transformations \(t_\Lambda = t_\Omega = p\) yields the desired thesis.

2. Excess ontological distinctness of three quantum transformations

In order to demonstrate quantum violation of Proposition 3, we consider three qubit unitary transformations \(U_1 = I, U_2 = R(\frac{\pi}{2})\) and \(U_3 = R(\frac{\pi}{2})\) where \(R(\theta)\) symbolizes a rotation of angle \(\theta\) in the x-z plane. It is easy to see that for any initial state preparation \(P_\psi\) the post transformations states and consequently the transformations themselves are \(\frac{2}{3}\)-distinguishable i.e. \(t_Q = \frac{2}{3}\). Furthermore, for any initial pure state preparation \(P_\psi\), the post transformation states are pair-wise distinguishable with average probability of success being \(\frac{1}{2}(\max_{P} \left\{ t_\Omega^{1,2}(P) + t_\Omega^{2,3}(P) + t_\Omega^{3,1}(P) \right\}) = \frac{1}{2}(1 + \frac{\sqrt{2}}{2})\), which is a violation of Proposition 3, thereby demonstrating excess ontological distinctness of quantum transformations.

IV. EXCESS ONTOLOGICAL DISTINCTNESS POWERS QUANTUM ADVANTAGE IN COMMUNICATION TASKS

In this section, we lay down a general framework for communication tasks fuelled by excess ontological distinctness. We consider bipartite one-way communication tasks wherein Alice receives an input \(x\) based on a prior probability distribution \(p(x)\), similarly Bob receives an input \(y\) according to a prior probability distribution \(p(y)\) and produces an output \(z\). They repeat the task for several rounds so as to gather frequency statistics in the form of conditional probabilities \(p(z|x, y)\). Their goal is to maximize an associated success metric (figure of merit) \(\text{succ}_{O}\) which has the following generic expression.

\[
\text{succ}_{O} = \sum_{x, y, z} c(x, y, z)p(z|x, y)
\]

where the coefficients \(c(x, y, z)\) are positive reals [25]. For a simple example, consider a task wherein Alice receives a uniformly distributed trit \(x \in \{1, 2, 3\}\) as her input while Bob receives uniformly distributed bit \(y \in \{1, 2\}\) as her input and produces an output bit \(z \in \{1, 2\}\). Their aim is to maximize the associated figure of merit with coefficients:

\[
c(x, y, z) = \begin{cases} 
1 & \text{if } x \in \{1, 2\}, y = 1, z = 1, \\
1 & \text{if } x = 3, y = 1, z = 2, \\
1 & \text{if } x = 1, y = 2, z = 1, \\
1 & \text{if } x = 2, y = 2, z = 2, \\
0 & \text{otherwise.} 
\end{cases}
\]
Alice and Bob can employ various strategies to maximize figure of merit. Here, we need only consider the simplest prepare and measure protocols, wherein Alice’s encodes her input onto an operational preparation $P_x$ and Bob performs an operational measurement $M^y$ to produce an outcome $z$ such that the operational predictions and resulting statistics are of the form $p(z|x,y) = p(z|P_x,M^y)$. Below we describe two different classes of communication characterized by distinct constraints.

**A. Distinguishability constraint**

\[ s_O = \max_P \left\{ \sum_x q_x p(k = x|P_x,M) \right\} \leq p, \]

Next, we present a generalization of bounded ontological distinctness so as to apply to not necessarily uniform ensembles of preparations. The *generalized principle of bounded ontological distinctness for preparations* states that the epistemic states $\mu(\lambda|P_x) \in \mathcal{P}_\Lambda$ underlying a set of $(p,\{q_x\})$-distinguishable preparations $x$ are $(p,\{q_x\})$-distinct i.e. for a set of $(p,\{q_x\})$-distinguishable preparations $\mathcal{P}$ the maximal probability of distinguishing the associated epistemic states upon having access to the ontic-state $\lambda$ is

\[ s_\Lambda = \int_\Lambda \max_x (q_x\mu(\lambda|P_x))d\lambda = p. \]

Consequently, the operational constraint (38) translates to the ontological constraint,

\[ s_\Lambda = \int_\Lambda \max_x (q_x\mu(\lambda|P_x))d\lambda \leq p. \]

This constraint yields a bound on the success metric (37) of the simple communication task described above whilst employing resources pertaining to theories that adhere to bounded ontological distinctness:

**Proposition 4.** If an operational theory admits ontological models adhering to the principle of bounded ontological distinctness, then for a set of three at-most $p$-distinguishable preparations $\mathcal{P} \equiv \{P_1,P_2,P_3\}$, the value of the success metric (37) is upper bounded as follows,

\[ \text{succ}_O = p(z = 1|x = 1,y = 1) + p(z = 1|x = 1,y = 2) \]
\[ + p(z = 1|x = 2,y = 1) + p(z = 2|x = 2,y = 2) \]
\[ + p(z = 2|x = 2,y = 1) \leq 2 + 3p. \]

**Proof.** In an ontological model the maximum value of the operational success metric has the expression,

\[ \text{succ}_O = \max_P \left\{ \int_\Lambda \left[ \mu(\lambda|P_1) + \mu(\lambda|P_2) \right] \xi(k = 1|\lambda,M) + \mu(\lambda|P_3)\xi(k = 2|\lambda,M)d\lambda \right\} \]
\[ + \max_P \left\{ \int_\Lambda \mu(\lambda|P_1)\xi(k = 1|\lambda,M) + \mu(\lambda|P_2)\xi(k = 2|\lambda,M)d\lambda \right\}. \]

This equation facilitates the alternative description of simple communication task under consideration as a distinguishability task, wherein Bob upon receiving $y = 1$ tries to discern the $P_1,P_2$ from $P_3$ and upon receiving $y = 2$ attempts to distinguish $P_1$ from $P_2$. Consequently, upon having access to the ontic state $\lambda$ and in lieu of the observations regarding response schemes employed before (4) the maximum value of the ontological success metric has the
expression,

\[
succ_A = \int_A \left( \max \left\{ \mu(\lambda|P_1) + \mu(\lambda|P_2) , \mu(\lambda|P_3) \right\} + \max \left\{ \mu(\lambda|P_1) , \mu(\lambda|P_2) \right\} \right) d\lambda,
\]

\[
= 5 - \int_A \left( \min \left\{ \mu(\lambda|P_1) + \mu(\lambda|P_2) , \mu(\lambda|P_3) \right\} + \min \left\{ \mu(\lambda|P_1) , \mu(\lambda|P_2) \right\} \right) d\lambda,
\]

(42)

where the second inequality follows from the fact \( \forall (a \geq 0, b \geq 0) : \max\{a, b\} = r + s - \min\{a, b\} \). Observe that as the operational measurements might not reveal the ontic-state \( \lambda \) precisely, we have in general \( succ_O \leq succ_A \).

Moving on, we invoke (7), which yields,

\[
\int_A \min \left\{ \mu(\lambda|P_1) + \mu(\lambda|P_2) , \mu(\lambda|P_3) , \mu(\lambda|P_3) + \mu(\lambda|P_1) \right\} d\lambda \geq 3(1 - s_A)
\]

(43)

where \( s_A \) is the ontological distinctness for the three preparations consideration. Now employing the fact that \( \forall (a \geq 0, b \geq 0, c \geq 0) : \min\{a + b, c\} + \min\{a, b\} \geq \min\{a + b + c, c + a\} \) and plugging (43) into (42) we arrive at,

\[
succ_O \leq 2 + 3s_A.
\]

(44)

The desired thesis follows from bounded ontological distinctness which implies \( s_A = s_O = p \). \( \blacksquare \)

Observe that even the average pair-wise distinguishability of three preparations considered in Proposition 1 can be cast as the success metric of such a communication task wherein Alice and Bob receive trits as inputs \( x, y \in \{1, 2, 3\} \), Bob outputs a bit \( z \in \{1, 2\} \) and the success metric has the following coefficients:

\[
c(x, y, z) = \begin{cases} 
\frac{1}{6} & \text{if } z = x, x \in \{1, 2\}, y = 1 \\
\frac{1}{6} & \text{if } z = x, x \in \{2, 3\}, y = 2, \\
\frac{1}{6} & \text{if } z = x, x \in \{3, 1\}, y = 3, \\
0 & \text{otherwise}.
\end{cases}
\]

(45)

Operationally, Alice and Bob can employ classical or quantum resources and protocols to aid them in these tasks. A classical communication protocol entails Alice encoding her input \( x \) into a classical message \( m \in \{1, \ldots, d\} \) based on an encoding scheme i.e. condition probability distributions of the form \( p_E(m|x) \), Bob upon receiving the message employs a decoding scheme i.e. conditional probability distributions of the form \( p_D(z|m, y) \) to produce the output \( z \). The operational predictions can be summarized as \( p(z|x, y) = \sum_m p_E(m|x) p_D(z|m, y) \). The operational condition (38) translates to the following constraint for classical communication,

\[
s_C = \sum_m \max_x \left\{ q_x p_E(m|x) \right\} \leq p.
\]

(46)

As the dimension of the classical message could be arbitrary large, for an ontological model that adheres to bounded ontological distinctness, sending the ontic state itself as the classical message \( m = \lambda \) would suffice, as even if Bob knew the ontic state the ontological constraint (39) ensures that the operational condition (38) won’t be violated. Specifically, in the limit \( d \to \infty \), the message \( m \) can considered to be the ontic state of the communicated system \( \lambda \equiv m \), the epistemic states are equivalent to the encoding schemes \( \mu(\lambda|P_x) = p_E(m|x) \) and the response schemes are equivalent to decoding schemes \( \xi(k = z|\lambda, M^y) = p_D(z|m, y) \). Finally, replacing the summation \( \sum_m \) with \( \int_A d\lambda \) we find that the operational constrain (46) yields the ontological constrain (39), then it is easy to see that all the operational predictions of ontological models that adhere to bounded ontological distinctness also hold for classical theories that adhere to the operational condition (38). This observation substantiates the candidature of bounded ontological distinctness as a notion of classicality. In particular, this implies that for the classical communication constrained by (46), the maximal value of the success metric is bounded by \( succ_C \leq 2 + 3p \).

Alternatively, Alice and Bob can employ a quantum prepare and measure protocol to aid them in these communication tasks. A prepare and measure quantum protocol entails preparation and transmission of Alice’s quantum state \( \rho_x \) for his input \( x \), followed by a measurement at Bob’s end \( \{M^z\} \) based on his input \( y \). The operational predictions can be summarized as \( p(z|x, y) = \text{Tr}(\rho_x M^z_y) \). The operational condition (38) translates to the following constraint for quantum communication,

\[
s_Q = \max_{\{M^z\}} \sum_x \{q_x \text{Tr}(\rho_x M^z_{x=x})\} \leq p.
\]

(47)

For the simple communication task described above consider the quantum protocol wherein Alice prepares the state \( \rho_x = \frac{1 + \vec{n}_1, \vec{x}}{2} \) based on her input \( x \), where \( \vec{n}_1 = [1, 0, 0]^T, \vec{n}_2 = [1, 0, 0]^T \) and \( \vec{n}_3 = \frac{-\vec{n}_1 - \vec{n}_2}{\sqrt{2}} \). These preparations are \( \frac{3}{4} \)-distinguishable. On the other hand, Bob’s measures \( M^y = s_y^x \vec{\sigma} \) where \( s_1^y = \frac{\vec{n}_1 + \vec{n}_3}{\sqrt{2}} \) and \( s_2^y = \frac{\vec{n}_2 - \vec{n}_3}{\sqrt{2}} \). This protocol achieves \( succ_Q = 3 + \sqrt{2} \) which violates Proposition 4, thereby demonstrating
excess ontological distinctness of quantum preparations and yielding an advantage over classical communication. Similarly, the quantum states and measurements that violate Proposition 1 can be cast as advantageous quantum prepare and measure protocols. In this way, quantum protocols siphon the seemingly in accessible “excess ontological distinctness” of quantum preparations to an advantage in communication tasks.

B. Oblivious communication with bounded leakage

Here we consider a cryptographically significant class of communication tasks called oblivious communication tasks [10, 12, 26]. Specifically, we invoke a generalization of oblivious communication tasks wherein we tolerate certain amount of leakage of information about the oblivious function. We call these tasks oblivious communication tasks with bounded leakage. Additionally, the sender Alice has a certain function of her inputs $f(x) \in \{1, \ldots, d_f\}$ the value of which is to be kept secret. However, we tolerate a bounded amount of leakage. Specifically, the probability of distinguishing the preparations corresponding to different values of $f(x)$ out of an uniform ensemble is constrained in the following way,

$$s^f(x) := s^f(x=1, \ldots, f(x)=d_f) = \frac{1}{d_f} \max_M \left\{ \sum_{f(x)} \sum_x p(x|f(x))P(k=f(x)|P_x, M) \right\} \leq \frac{1}{d_f} + \epsilon \tag{48}$$

where $\epsilon \in (0, \frac{d_f-1}{d_f}]$ is the leakage parameter and,

$$p(x|f(x) = k) = \begin{cases} p(x) \sum_k f(x=k), & \text{if } f(x) = k, \\ 0, & \text{otherwise}. \end{cases} \tag{49}$$

Bounded ontological distinctness along with convexity of epistemic states requires,

$$s^f(x) = \frac{1}{d_f} \int \max_f (x) \left\{ \sum_x p(x|f(x))\mu(\lambda|P_x) \right\}d\lambda \leq \frac{1}{d_f} + \epsilon. \tag{50}$$

The distinguishability inequality in Proposition 2 may posed as an instance of such a communication task. Specifically, consider a oblivious communication task wherein Alice receives a uniformly distributed pair of bits as her input $x = \{a_1, a_2\}$ where $a_1, a_2 \in \{0, 1\}$, $\forall a_1, a_2 : p(a_1, a_2) = \frac{1}{4}$ and the oblivious function is simply the parity of these bits $f(x) = a_1 \oplus a_2 + 1$. Bob on the other hand has to guess Alice’s input bit based on her input bit $y \in \{1, 2\}$, $z = a_y$. The coefficients of the success metric for this task are,

$$c(x, y, z) = \begin{cases} \frac{1}{8}, & \text{if } z = a_y, \\ 0, & \text{otherwise}. \end{cases} \tag{51}$$

This task is based on the well-known parity oblivious multiplexing task [10]. Upon the relabelling, $P_1 = P_{a_1=0,a_2=0}, P_2 = P_{a_1=1,a_2=0}, P_3 = P_{a_1=0,a_2=1}, P_4 = P_{a_1=1,a_2=1}$ along with $p = \frac{1}{2} + \epsilon$, it is easy to see that the inequality (16) is equivalent to a bound on the success metric with coefficients in (51) i.e. for $\epsilon$ leakage bounded ontological distinctness along with convexity of epistemic states implies $\text{succ}_0 \leq \frac{3}{2} + \frac{\epsilon}{2}$.

Moving on, the operational condition (48) translates to the following constraint on classical communication,

$$s^f_C(x) = \frac{1}{d_f} \sum_{m(f(x)} \max_{f(x)} \left\{ \sum_x p(x|f(x))p_\epsilon(m|x) \right\} \leq \frac{1}{d_f} + \epsilon. \tag{52}$$

Yet again, because the reasons stated above this operational constraint on classical communication is equivalent to the ontological constrain (50), which ensures that the predictions of ontological models which adhere to bounded ontological distinctness and convexity of epistemic states also hold for unbounded classical communication constrained by (52). Finally, employing appropriate relabelling one can obtain advantageous quantum prepare and measure protocols from the instances of violation of (16) provided below Proposition 2 [27].

V. THE IMPLICATE QUANTUMNESS

In this section, we show how excess ontological distinctness is implicate in the explicate quantum departure from the well-known ontological notions of classicality. We go about this task by demonstrating that bounded ontological distinctness implies the other ontological notions of classicality as limiting cases such that the violation of the later implies excess ontological distinctness.

First, we consider maximally $\psi$-epistemic ontological model based on a symmetric definition of overlap of epistemic states [28]. Specifically, the existence of such a maximally $\psi$-epistemic ontological model of quan-
tum theory implies that for any two pure quantum preparations \( \{P_{\psi_1}, P_{\psi_2}\} \) corresponding to the pure states \( \{|\psi_1\rangle, |\psi_2\rangle\} \),

\[
\int_{\Lambda} \min\{\mu(\lambda|P_{\psi_1}), \mu(\lambda|P_{\psi_2})\} d\lambda = 1 - \sqrt{1 - |\langle \psi_1 | \psi_2 \rangle|^2}.
\]

(53)

Now recall that the maximal quantum probability of distinguishing \( s_Q \) a pair of quantum preparations \( \{\rho_1, \rho_2\} \) has the expression \( s_Q = \frac{1}{2} (1 + T(\rho_1, \rho_2)) \) where \( T(\rho_1, \rho_2) \) is the trace distance between the argument density matrices. Now bounded ontological distinctness states that operational distinguishability reflects ontological distinctness i.e., if \( \rho_1 \) and \( \rho_2 \) are two operational indistinguishable preparations then\( s_Q = \frac{1}{2} (1 + T(\rho_1, \rho_2)) \). The fact that \( |a| \geq b, b \geq 0 \): \( \max\{a, b\} = a + b - \min\{a, b\} \) yields,

\[
\int_{\Lambda} \min\{\mu(\lambda|P_{\rho_1}), \mu(\lambda|P_{\rho_2})\} d\lambda = 1 - T(\rho_1, \rho_2).
\]

(54)

We refer to the implication of bounded ontological distinctness of two quantum preparations as the existence of maximally \( \rho \)-epistemic ontological models. Furthermore, it is straightforward to see that the above relation reduces to (53) for pure states, thereby leading us to our first implication,

\[
BODP \implies \text{(symmetric) maximally } \psi \text{-epistemic. (55)}
\]

Based directly on the Leibniz’s principle of “identity of indiscernibles”, Spekkens’ notion of preparation non-contextuality states that the epistemic states underlying two operational indistinguishable preparations are identical i.e., if \( \forall \ k, M : p(k|P_1, M) = p(k|P_2, M) \) then preparation non-contextuality implies \( \forall \lambda : \mu(\lambda|P_1) = \mu(\lambda|P_2) \). Clearly, preparation non-contextuality is a limiting case of bounded ontological distinctness of preparations i.e. whenever two given preparations are completely indistinguishable \( s_\Omega = \frac{1}{2} \) then \( BODP \) implies \( \int_{\Lambda} \max\{\mu(\lambda|P_1), \mu(\lambda|P_2)\} d\lambda = 1 \), which only holds whenever \( \forall \lambda : \mu(\lambda|P_1) = \mu(\lambda|P_2) \). Consequently, we have the implication,

\[
BODP \implies \text{preparation non-contextuality. (56)}
\]

Notice that these implications follow from the very definitions of the ontological principles under consideration, without invoking any other formalism dependent assumptions. Now we invoke the following operational condition:

**Self duality (SD):** For any given pure preparation \( P_\psi \) associated with a pure state \( |\psi\rangle \) and every measurement \( M \) which contains \( M_k = |\psi\rangle\langle\psi| \) as a measurement effect corresponding to an outcome \( k \), \( p(k|P_\psi, M_k) = |\langle \psi | \psi \rangle|^2 = 1 \).

This operational condition translates to the following ontological assumption: for a given pure state \( |\psi\rangle \) and corresponding pure preparation \( P_\psi \), we require that for every measurement \( M \) that contains \( |\psi\rangle\langle\psi| \) as a measurement effect corresponding to an outcome \( k \), \( p(k|P_\psi, M_k) = \int_{\Lambda} \mu(\lambda|P_\psi) \xi(k|\lambda, M) = 1 \), which only holds when the corresponding response scheme is of the form \( \forall \lambda \in \{\lambda|\mu(\lambda|P_\psi) > 0\} : \xi(k|\lambda, M) = 1 \).

With the aid of this assumption, we have the implication \([15, 28, 29]\): preparation non-contextuality \( \implies \) Kochen-Specker’s non-contextuality \([30]\). Moreover, Bell’s local causality is limiting case of Kochen-Specker’s non-contextuality. Specifically, spatially separated measurements employed in set-ups for Bell’s local causality imply commutation relations employed in the definition of contexts in Kochen-Specker’s contextuality. Consequently, for non-signaling operational theories we have the implication: Kochen-Specker’s non-contextuality \( \implies \) Bell’s local causality. Consequently, this leads to the following combined implication:

\[
BODP \implies \text{preparation non-contextuality} \implies \text{Kochen-Specker’s non-contextuality} \implies \text{Bell’s local causality. (57)}
\]

Therefore, any quantum violation of Bell’s local causality, Kochen-Specker’s non-contextuality or preparation non-contextuality implies violation of \( BODP \) or excess ontological distinctness of quantum preparations.

Next, we consider Spekkens’ generalized notion of measurement non-contextuality. If \( \forall P : p(k_1|P, M^1) = p(k_2|P, M^2) \) then measurement non-contextuality implies \( \forall \lambda : \xi(k_1|\lambda, M^1) = \xi(k_2|\lambda, M^2) \). Consider two binary outcome measurements \( M^1, M^2 \) which are basically coarse-grained versions of \( M^1, M^2 \), such that we retain the outcomes \( k_1, k_2 \) as their first outcomes, respectively, and relabel all other outcomes as their second outcomes. Now, we invoke the following natural property of ontological models,

**Coarse-graining of response functions**: If a measurement outcome \( k \) is obtained by coarse-graining of two measurement outcomes \( k_1, k_2 \), the response function corresponding to outcome \( k \) is also coarse-graining of two response functions of the outcomes \( k_1, k_2 \). i.e. \( \forall \lambda, \xi(k|\lambda, M) = \xi(k_1|\lambda, M) + \xi(k_2|\lambda, M) \).

Therefore, from the definition of \( M^1, M^2 \) we have,

\[
\forall \lambda, \xi(1|\lambda, M^1) = \xi(k_1|\lambda, M^1) + \xi(k_2|\lambda, M^2) = \xi(k_2|\lambda, M^2).
\]

(58)

The operational condition that \( \forall P : p(k_1|P, M^1) = p(k_2|P, M^2) \) implies that the new measurements
The subject of this work, the principle of bounded ontological distinctness, is accompanied by an operational non-disturbance condition for particular sets of measurements, which is again an operational indistinguishability condition for measurements. Specifically, for such a set of measurements, the condition requires one to not be able to distinguish between the rest of the measurements on the basis of the outcome statistics of any measurement. Moreover, Spekken’s non-contextuality relies directly on an operational equivalence of preparations, measurements and transformations which can clearly be interpreted as an operational indistinguishability condition of respective physical entities. Yet another ontological principle of non-retrocausal time-symmetric ontology [32] requires each temporally separated party to not be able to distinguish between the different (measurement or preparation) settings of the other parties. The subject of this work, the principle of bounded ontological distinctness provides an ontological basis to the maximal probability of distinguishing arbitrary sets of preparations, measurements and transformations, in the distinctness of their ontological counterparts.

In order to demonstrate excess ontological distinctness of quantum preparations only three two-dimensional quantum preparations and two binary outcome measurements are necessary (Proposition 1 and 4). Furthermore, we demonstrate excess ontological distinctness of quantum transformations employing just three two-dimensional quantum unitary transformations (Proposition 3). These form the most compact instances of quantum theory’s incompatibility with a class of ontological models and substantiates the fact that bounded ontological distinctness and its violation captures non-classicality inherent in a broader class of quantum physical systems as compared to all other well-known ontological principles. Moreover, while restricting the operational distinguishability of a pair of mixed quantum preparations we demonstrate violation of bounded ontological distinctness and convexity of epistemic states employing four preparations and two measurements (Proposition 2). Finally, we observed that bounded ontological distinctness for measurements cannot be violated on its own because of the existence of Beltrametti-Bugajski ψ-complete ontological model. However, invoking the additional assumption of outcome determinism, we show that all pairs of quantum projective measurements which are not completely indistinguishable nor perfectly distinguishable violate bounded distinctness for measurements.

This work is based on an overarching perception, namely, the distinguishability, specifically, the maximal operational probability of distinguishing a set of preparations, measurements or transformations forms an intrinsic property of this set. This is so because the maximization involved, relieves the distinguishing probability of a set of a particular type of physical entities of its dependence on other types of physical entities. Similarly, ontological distinctness or the maximal probability of distinguishing sets of epistemic states,
response schemes or transition schemes form an intrinsic property of the set of ontological entities. In lieu of this insight, the consequences of bounded ontological distinctness, specifically the inequalities featured in the Propositions 1, 2, 3 and 4 present a peculiar perspective. Specifically, these propositions relate two distinct operational properties pertaining to distinguishability of respective physical entities. For instance, Proposition 1 bounds the maximal average pair-wise distinguishability, based on maximal distinguishability for a set of three operational preparations. The consequent quantum violation of these propositions implies that the conflict with bounded ontological distinctness lies in the relation between these intrinsic properties of the set of quantum physical entities under consideration.

The particular form of the Propositions 1, 2, 3 and 4 and of the inequalities therein, allow us to infer a lower bound on the extent of excess ontological distinctness. Remarkably, the Kochen-Specker’s maximally $\psi$-epistemic model saturates the lower bounds on the extent of quantum excess ontological distinctness (Observations 1, 3). This in-turn highlights the significance of maximally $\psi$-epistemic models and demonstrates that the inequalities under consideration are tight. Furthermore, more we employ the existence of this model to substantiate the fact bounded ontological distinctness preparations cannot be violated employing just two instances of two-dimensional quantum preparations (Observation 2). We show that bounded ontological distinctness, directly and indirectly, implies the other ontological principles, so that the violation of the latter implies the violation of the former (V). In particular, we show that maximal $\psi$-epistemicity is a restricted case of bounded ontological distinctness of preparations applied to a pair of pure quantum preparations. Similarly, preparation, measurement and transformation non-contextuality emerges from bounded ontological distinctness of the respective physical entities. While these implications follow by the very definitions of these ontological principles, we show that under certain quantum theory dependent ontological assumptions, bounded ontological distinctness for preparations implies Kochen-Specker’s non-contextuality and Bell’s local causality. Consequently, the quantum violations of the other ontological principles is, in essence, a demonstration of the implicate quantum excess ontological distinctness, deeming it to be a fundamental feature of quantum ontology [33].

All of the aforementioned ontological principles including bounded ontological distinctness obtain their primary ontological constraints by requiring ontological models to adhere to corresponding operational conditions on the ontological level i.e. conditioned on the ontic state of the physical system. For instance, the operational non-signaling condition conditioned on the ontic state of the shared physical system forms the primary ontological constraint of Bell’s local causality referred to as parameter independence. However, the primary ontological constraints of the well-known ontological principles are (on their own) not enough to warrant consequences directly in contradiction with the predictions of quantum theory. Therefore, they invoke certain auxiliary constraints to enable a quantum violation. For instance, along with parameter independence Bell’s local causality entails the auxiliary ontological assumption of outcome independence. Similarly, outcome determinism is quintessential to the demonstration of Kochen-Specker’s contextuality, while Spekken’s preparation and transformation non-contextuality require the auxiliary ontological property of convexity of epistemic states and transition schemes respectively to enable their quantum violations. Finally, the quantum demonstration of Spekken’s measurement contextuality requires either outcome determinism or Spekken’s preparation non-contextuality as an additional assumption. The inclusion of these additional assumptions, leads to dilution of the implication of corresponding quantum violations as they could be attributed to violation of the auxiliary assumptions leaving the primary assumptions intact.

Note that, just like measurement non-contextuality, bounded ontological distinctness for measurements requires either outcome determinism or bounded ontological distinctness for preparations to enable a quantum violation. However, unlike these ontological principles, the primary ontological assumption of bounded ontological distinctness for preparations and transformations is enough to warrant a quantum violation on its own. This in-turn leads to an unambiguous ontological implication, that quantum preparations, measurements and transformations are more ontologically distinct than they are operationally distinguishable.

As ontological principles and their violations yield insight into the structure of reality which is not directly accessible via operational theories themselves, the experimental verification of these principles is of key significance. One of the key issues with the experimental tests of the other ontological principles [1, 34, 35], lies in the associated operational conditions that the corresponding physical entities have to adhere. Specifically, the well-known ontological notions of classicality, for their refutation, require corresponding operational conditions to hold. For instance, the refutation of Bell’s local causality requires the experimental data to adhere to non-signaling, the refutation Kochen-Specker’s non-contextuality requires the experimental data to adhere to non-disturbance and for the refutation of Spekken’s preparation non-contextuality an equivalence condition of mixtures of preparations (termed oblivious condition) must hold. As detailed above, these conditions require indistinguishability of the associated physical entities. Consequently, the first step in experimental verification of the other ontological principles, is to decide upon the physical entities that adhere to the associated operational condition. This is followed by experimental composition of the physical entities, which is more often than not inaccurate and subject to errors. Even the slightest of errors in experimental composition may
cause a deviation from the zero measure operational indistinguishability condition. Besides, due to the finite size of experimental data and the scaling of statistical error $\approx \frac{1}{\sqrt{N}}$ with the number of runs $N$, the latter almost always violates the zero measure operational conditions. To address this issue a number of schemes [1, 11, 36–39] have been proposed and implemented but their effects on the theory independent implication of the ontological principles can be scrutinized. However, the operational condition associated with bounded ontological distinctness, namely, distinguishability of physical entities has a considerably larger spectrum. Consequently, the operational condition is robust from the get-go. In this case, experimental composition and tomography precedes deciding on the operational condition. In other words, in experimental tests of bounded ontological distinctness, the operational condition is to be tailored to the erroneous experimental compositions. In order to showcase this, we have provided the results of numerical simulations demonstrating extensive violations of bounded ontological distinctness of preparations. We randomly sampled sets of quantum preparations, followed by tailoring the operational condition to these preparations and finally, calculating the violation of the associated bounded ontological distinctness inequality (Fig. 5, 6).

The success in communication tasks is closely tied to the ability of the receiver to distinguish sender’s preparations. Specifically, the sender encodes her input data into operational preparations and the success depends upon how well the receiver is able to distinguish these preparations so as to figure out the sender’s input [40]. The excess ontological distinctness of the quantum preparations fuels the quantum advantage in communication tasks presented in this work wherein we constrain the distinguishability of the sender’s operational preparations [12]. Specifically, the provided quantum communication protocols siphon the excess ontological distinctness of the associated epistemic states to an advantage over classical unbounded communication protocols and theories that have ontological models that adhere to bounded ontological distinctness. Moreover, as the advantage in oblivious communication tasks witnesses preparation contextuality, the oblivious communication tasks with bounded leakage which are powered by excess ontological distinctness of sender’s preparations demonstrate that the inherent experimental robustness of bounded ontological distinctness may be ported to other ontological principles.

The underlying assumption in our implications is the existence of an ontology adhering to standard ontological framework with epistemic states, response schemes and transition schemes modelling operational preparations, measurements and transformations. Then one of the implications of the violation of the ontological principles could be non-existence on an ontology [15, 41, 42]. More interestingly, in order to preserve bounded ontological distinctness of quantum entities, the prescriptions of the standard ontological framework might be altered, giving way to an exotic quantum ontology. It might be worthwhile to investigate the features of operational theories which enable excess ontological distinctness. Even though it is a fact that the violation of other ontological principles imply excess ontological distinctness, it will be interesting to investigate how certain operational features propagate excess ontological distinctness to the violation of the other well-known ontological principles in the respective scenarios.

On the technical front, it will be interesting to find out if bounded ontological distinctness of a pair of mixed preparation along with convexity of epistemic states can be violated whilst employing just three instances of two-dimensional quantum preparations. Bounded ontological distinctness for preparations and measurements implies preparation and measurement non-contextuality and quantum theory violates consequences of preparation and measurement non-contextuality. Now as quantum theory is known to violate preparation and measurement contextuality, it also violates bounded ontological distinctness for preparations and measurements. As these individual implications are strictly unidirectional, it will be interesting to find an instance where quantum preparations and measurements display excess ontological distinctness but do not violate preparation and measurement contextuality. While we have demonstrated quantum advantage in communication tasks with constraints on the distinguishability of sender’s preparations, we claim that quantum advantage may be retrieved in multiparty communication tasks while restricting the distinguishability of measurements or intermediate transformations. This should be formalized into a semi-device independent framework, equipped with key distribution and randomness certification protocols, based on theory independent departure from the predictions of classical theories. Finally, physical principles such as information causality [43] and macroscopic reality attempt to explain why quantum resources do not violate Bell inequalities to their algebraic maximum and enable insights into peculiarities of the quantum formalism and hierarchies of semi-definite programs aid in obtaining upper-bounds on quantum violations of these Bell inequalities [44]. It would be worthwhile to conjure such physical principles and semi-definite hierarchies that restrict the violation of bounded ontological distinctness inequalities to the quantum maximum, analytically and numerically respectively.

**Conceptual insight:** One of primary functions of the intellect (faculty of thought) is to discern or tell entities apart. The distinguishability or the extent to which the entities under consideration may be distinguished, forms the basis for more complex functions of the intellect such as classification (evident from the from its Latin root “classis” meaning “a distinction”). Similarly, the extent to which physical entities may be distinguished plays a central role in how we employ physical theories in in-
teractions with the physical world. Furthermore, it is straightforward to see that information gain about certain entities is closely tied with the ability to distinguish them. This work specifically deals with the relation between how well “we” can tell a set of entities apart (operational distinguishability) to how these distinct these entities actually are (ontological distinctness). The notion of classicality presented in this work, bounded ontological distinctness, bases itself on the *wysiwng* (read as: “what you see is what you get”) relation of operational distinguishability and ontological distinctness in classical theories. Given the importance of distinguishability and its relation with distinctness, it is remarkable that quantum theory departs from classical theories by positing physical entities that more distinct than they are distinguishable.

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[22] We remark here that the ontological property: convexity of epistemic states, however natural, forms an additional assumption, for it is possible to have ontological models of quantum theory such as a ρ – complete ontological model that do not exhibit this property.
[23] Here, we restrict ourselves to considering simple single-shot measurement distinguishability without the aid of more exotic features of operational theories such as entanglement [46, 47].
[24] Yet again we use the simplest single-shot prepare, transform and measure set-up for transformation distinguishability, without employing more intricate features of operational theories such as entanglement [48].
[25] Observe that the prior probability distributions p(x), p(y) can be readily absorbed into the coefficients so that without loss of generality the inputs may always be considered to be uniformly distributed.
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See related upcoming article(s) on the extent of preparation contextuality in imperfect parity oblivious communication task, by M. S. Liefer et. al. [49].

Matthew S Leifer, “No-go theorems,” Lecture in Summer School - Solstice of Foundations 2017, ETH Zurich (2017).

While we are concerned with self duality in quantum theory, the following implications also hold in general probabilistic theories whenever the operational theory satisfies self-duality [28].

Matthew S Leifer and Matthew F Pusey, “Is a time symmetric interpretation of quantum theory possible without retrocausality?” Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 473, 20160667 (2017).

A conceptual underpinning: The appearance of certain phenomena might appear differently, or might be characterized by varying principal factors, depending on contexts such as scales. The implicate or “enfolded” is the deeper, more fundamental order associated with these phenomena while the explicative or “unfolded” include the abstractions or observations in lieu of specific contexts. “For instance, a circular table looks like an ellipse from various directions. But we know that those are the appearances of a single circular form. So we represent the table as a circle. We say, ‘that what’s it is, a solid circle’ ” [13].

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This is evident in the proof methodology of Proposition 4 and the casting of Proposition 1 and 2 as communication tasks.
This section presents three auxiliary plots. The first two plots contain results of our numerical simulations showcasing extensive violation of (6) and (16). These plots highlight the experimental robustness of distinguishability as an operational pre-requisite, as in these simulations the operational condition and the value of the inequality are decided after randomly sampling arbitrary triplets and quadruplets of two-dimensional quantum preparations respectively. The third plot serves to aid visualization of quantum violation (21) of Proposition 16.

**FIG. 5.** A plot of maximal average pairwise distinguishability $\frac{1}{3}(s_{1+2}^{Q} + s_{2+3}^{Q} + s_{3+1}^{Q})$ vs. maximal distinguishability $s$ for three randomly picked pure qubits (blue circles) and three pure qutrits (orange dots). The yellow line represents the maximal average pairwise distinguishability (6) when the underlying epistemic states adhere to bounded ontological distinctness. Over 72% of randomly picked triplets of pure qubits and over 58% of randomly picked triplets of pure qutrits exhibit excess ontological distinctness.

**FIG. 6.** A plot of maximal average distinguishability of pairs of mixtures $\{\rho_{1+3}, \rho_{2+4}\}$ and $\{\rho_{1+4}, \rho_{2+3}\}, \frac{1}{2}(s_{Q}^{1+3,2+4} + s_{Q}^{1+4,2+3})$ vs. maximal distinguishability of $\{\rho_{1+2}, \rho_{3+4}\}, s_{Q}^{1+2,3+4}$ for four randomly picked pure qubits (blue circles), pure qutrits (orange dots), pure ququarts (yellow crosses) and pure qu5its (purple asterisks). The green line represents the maximal distinguishability of $\{\rho_{1+2}, \rho_{3+4}\}$ (16) when the underlying epistemic states adhere to bounded ontological distinctness and convexity of epistemic states.
FIG. 7. A plot of maximal average distinguishability of the pairs \( \{\rho_{1+3}, \rho_{2+4}\} \) and \( \{\rho_{1+4}, \rho_{2+3}\} \), \( \frac{1}{2}(s_Q^{1+3,2+4} + s_Q^{1+4,2+3}) \) (21) (blue line) along with the bound \( \frac{1+1.3+4}{2} \) (orange dashed line) from Proposition 2 against \( \frac{\theta}{\pi} \). The qubits violate the inequality (16) for the range \( 0 \leq \frac{\theta}{\pi} \leq \frac{1}{2\sqrt{2}} \).