HODGE STRUCTURES ORBIFOLD HODGE NUMBERS AND A CORRESPONDENCE IN QUASITORIC ORBIFOLDS

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Abstract. We give hodge structures on quasitoric orbifolds. We define orbifold hodge numbers and show a correspondence of orbifold hodge numbers for crepant resolutions of quasitoric orbifolds. In short we extend hodge structures to a non-complex setting.

1. Introduction

The purpose of this paper is to extend hodge structures in a non-complex non-almost complex setting. First we give a canonical hodge structure to quasitoric orbifolds. We compute hodge numbers. Then as an application we define orbifold hodge numbers and show a correspondence of these numbers for crepant resolutions. Since toric manifolds and possibly orbifolds are used widely in physics as non linear sigma models, with canonical hodge structures possible in quasitorics we feel these spaces can also be put into some consideration and inspection by experts of the above areas. Also this is an extension of Deligne’s mixed hodge structures in a non-complex algebraic setting thus will draw interest of Mathematicians. The correspondence which could be a McKay Correspondence generalises the String theoritic hodge number Correspondence of Batyrev and Dias to an non-algebraic non-analytic setting.

2. Quasitoric orbifolds

In this section we describe the combinatorial construction of quasitoric orbifolds. Notations established in this section will be used later.

Take a copy $N$ of $\mathbb{Z}^n$ and form a torus $T_N := (N \otimes \mathbb{Z} \mathbb{R})/N \cong \mathbb{R}^n/N$.

Take a submodule $M$ of $N$ of rank $m$ and construct the torus $T_M := (M \otimes \mathbb{Z} \mathbb{R})/M$ of dimension $m$. Define the map $\zeta_M : T_M \to T_N$ the obvious map generated by the inclusion map $M \to N$.

Definition 2.1. We define the image of $T_M$ under the map $\zeta_M$ as $T(M)$. If $M$ is a submodule of rank 1 and $\lambda$ is the generator then we call the image $T(\lambda)$.

Definition 2.2. A polytope is $P$ is a subset of $\mathbb{R}^n$ which is diffeomorphic as manifolds with corners to a convex hull $C$ to a finite number of points in $\mathbb{R}^n$. The faces of $P$ are images of faces of $C$.

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Definition 2.3. A simple polytope is a polytope where each vertex is an intersection of one faces which are in general position.

Definition 2.4. Codimension one faces of a polytope $P$ are called facets. In a simple polytope every $k$ dimensional face is an intersection of $n-k$ facets. We call $\mathcal{F} = \{F_1, F_2 \ldots F_M\}$ the set of facets of the simple polytope $P$.

Definition 2.5. We define a map $\Lambda : \mathcal{F} \rightarrow \mathbb{Z}^n$ where $F_i$ is mapped to $\Lambda(F_i)$ and if $F_{i_1} \ldots F_{i_k}$ intersect to form a face of the polytope $P$ then the corresponding $\Lambda(F_{i_1}) \ldots \Lambda(F_{i_k})$ are linearly independent. From now onwards we call $\Lambda(F_i)$ as $\lambda_i$ and call it a characteristic vector and $\Lambda$ the characteristic function.

Remark 2.1. In this article we consider only primitive characteristic vectors and call the corresponding quasitoric orbifolds as primitive quasitoric orbifolds. The codimension of the singular locus of these orbifolds is at least 4.

Definition 2.6. For a face $F$ define $\mathcal{I}(F) = \{i : F \subset F_i, F_i \in \mathcal{F}\}$. The set $\Lambda_F = \{\lambda_i : i \in \mathcal{I}(F)\}$ is called the characteristic set of $F$. We call $N(F)$ be the sub module generated $\Lambda_F$. If $\mathcal{I}(F)$ is empty $N(F) = 0$.

For any point $p$ in the polytope we denote $F(p)$ the face whose relative interior contains $p$. We define an equivalence relation in $P \times T_N$ where $(p, t_1) \sim (q, t_2)$ if $p = q$ and $t_2^{-1}t_1 \in T(N(F(p)))$ where $N(F(p))$ is the sub module of $N$ generated by integral linear combinations of vectors of $\Lambda_{F(p)}$. The quotient space $X = P \times T_N / \sim$ has a structure of an $2n$ dimensional orbifold and are called quasitoric orbifolds.

The pair $(P, \Lambda)$ is a model for the above space. If vectors comprising $\Lambda_F$ are unimodular for all faces $F$ we get a quasitoric manifold. The $T_N$ action on $P \times T_N$ induces a torus action on the quotient space $X$, of the equivalence relation, and quotient of this action is the polytope $P$. Let us denote the quotient map by $\pi : X \rightarrow P$. $\pi^{-1}(w)$ for a vertex $w$ of $P$ is a fixed point of the above action and we will denote it by $w$ without confusion.

2.1. Orbifold structure. For every vertex $w$ in $P$ consider open set $U_w$ of $P$ the complement of all faces not containing the vertex $w$. We define

\begin{equation}
X_w = \pi^{-1}(U_w) = U_w \times T_N / \sim.
\end{equation}

For any face $F$ containing the vertex $w$ there is a natural inclusion of $N(F)$ in $N(w)$ and $T_{N(F)}$ in $T_{N(w)}$. We define another equivalence relation $\sim_w$ on $U_w \times T_{N(w)}$ as follows.

For $p \in U_w$, let $F$ be the the face which contains $p$ in its relative interior, by definition $F$ contains $w$. We define the relation as $(p, t_1) \sim_w (q, t_2)$ if $p = q$ and $t_2^{-1}t_1 \in T_{N(F)}$. We define

\begin{equation}
\tilde{X}_w = U_w \times T_{N(w)} / \sim_w.
\end{equation}

The above space is equivariantly diffeomorphic to an open set in $C^n$ with the standard torus action on $C^n$ and $T_{N(w)}$ action on $\tilde{X}_w$. The diffeomorphism will be clear from
the subsequent discussion. The map \( \zeta_{N(w)} : T_{N(w)} \to T_N \) induces a map from \( \zeta_w : \tilde{X}_w \to X_w \) in the following way
\[
\zeta_w((p, t) \sim_w) = (p, \zeta_{N(w)}(t)) \sim .
\]
The kernel of the map \( \zeta_{N(w)} \) is \( G_w = N/N(w) \) is a subgroup \( T_{N(w)} \) and has a smooth action on \( \tilde{X}_w \) and the quotient of this action is \( X_w \). This action is not free and so \( X_w \) is an orbifold and the uniformising chart of \( X_w \) is \( (\tilde{X}_w, G_w, \zeta_w) \).

We define a homeomorphism \( \phi(w) : \tilde{X}_w \to \mathbb{R}^{2n} \) as follows. Assume without loss of generality \( F_1, F_2 \ldots F_n \) are the facets containing \( w \) and \( p_i(w) = 0 \) is the the facet \( F_i \) and in \( U_w \) \( p_i \)'s have non-negative values with positive in interiors of \( U_w \). Let \( \Lambda_w \) be the corresponding set of characteristic vectors represented as follows
\[
\Lambda_w = [\lambda_1 \ldots \lambda_n].
\]
If \( q(w) \) be the representation of the angular coordinates of \( T_N \) in the basis with respect to \( \lambda_1 \ldots \lambda_n \) of \( N \otimes \mathbb{Z} \mathbb{R} \). Then the standard coordinates \( q \) are related in the following manner to \( q(w) \)
\[
q = \Lambda_w q(w).
\]
The homeomorphism \( \phi(w) : \tilde{X}_w \to \mathbb{R}^{2n} \) is
\[
x_i = x_i(w) := \sqrt{p_i(w) \cos(2\pi q_i(w))}, \quad y_i = y_i(w) := \sqrt{p_i(w) \sin(2\pi q_i(w))} \quad \text{for } i = 1, \ldots, n.
\]
We write
\[
z_i = x_i + \sqrt{-1}y_i, \quad \text{and} \quad z_i(w) = x_i(w) + \sqrt{-1}y_i(w).
\]
Now consider the action of \( G_w = N/N(w) \) on \( \tilde{X}_w \). An element \( g \) of \( G_w \) is represented by a vector \( \sum_{i=1}^n a_i \lambda_i \) in \( N \) where each \( a_i \in \mathbb{Q} \). The action of \( g \) transforms the coordinates \( q_i(w) \) to \( q_i(w) + a_i \). Therefore
\[
g \cdot (z_1, \ldots, z_n) = (e^{2\pi \sqrt{-1}a_1}z_1, \ldots, e^{2\pi \sqrt{-1}a_n}z_n).
\]
We define
\[
G_F := ((N(F) \otimes \mathbb{Q}) \cap N)/N(F).
\]
We denote the space \( X \) with the above orbifold structure by \( X \).

2.2. Invaraint Suborbifolds. The \( T_N \) invaraint subset \( \pi^{-1}(F) \) where \( F \) is a face of \( P \) is a quasioric orbifold. The face \( F \) acts as the polytope of \( X(F) \) and the characteristic vectors are obtained by projecting characteristic vectors of \( X \) to \( N/N(F) \) where \( N(F) = N(F) \otimes \mathbb{Q} \cap N \). With this structure \( X(F) \) is a suborbifold of \( X \). The suborbifolds corresponding to the facets are called characteristic suborbifolds. We denote the interior of a face by \( F^\circ \). The interior of a vertex \( w^\circ \) is \( w \).

2.3. Orientation. Quasitoric orbifolds are oriented. For more detailed discussion see section 2.8 of [3]. A choice of orientation of \( T_N \) and a choice of orientation of the polytope \( P \) induces an orientation of the quasitoric orbifold \( X \).
2.4. Omniorientation. A choice of orientations of the normal bundles of the orbifolds corresponding to the facets (which we named as characteristic suborbifolds) is termed as fixing an omniorientation. This is equivalent to fixing the sign of the characteristic vector associated to the facet (note: we call codimension one faces as facets). A quasitoric orbifold with a fixed omniorientation is called an omnioriented quasitoric orbifold. A quasitoric orbifold is positively omnioriented if it has an omniorientation such that for every vertex \( w \), \( \Lambda_w \) has a positive determinant. For more detailed discussion see section 2.9 of [3].

3. Betti numbers of a Quasitoric Orbifolds

Poddar and Sarkar computed the \( \mathbb{Q} \) homology and cohomology of quasitoric orbifolds in [12]. In particular the computation of homology in Section 4 of [12] gives a strong connection between the combinatorics of the polytope \( P \) and the Betti numbers. We discuss the connection in following proposition.

**Proposition 3.1.** Quasitoric orbifolds with combinatorially equivalent polytopes have same Betti numbers.

**Proof.** A brief discussion of the homology computation in [12] is required to establish the above proposition. The computation depends on defining a continuous height function on the polytope \( P \) with following properties.

1. Distinguishes vertices.
2. Strictly increases or decreases on edges.
3. Each face has a unique maximum and minimum vertex.
4. The maximum vertex is the unique vertex of the face where all the edges of the face meeting the vertex have a maximum on the vertex.
5. The minimum vertex is the unique vertex of the face where all the edges of the face meeting the vertex have a minimum on the vertex.

A vertex distinguishing linear functional of \( \mathbb{R}^n \) does the job. Here we assume \( P \) is embedded in \( \mathbb{R}^n \). Once we have such a function we orient the edges of the polytope in increasing direction of the height function and arrange the vertices in increasing order of height. We define index \( i_w \) of a vertex \( w \) as the number of incoming edges. The smallest face containing these incoming edges is the largest face \( F_w \) which has \( w \) as the maximum vertex. Now start attaching \( 2i_w \) \( q \)-cells following the increasing order of vertices. The \( q \) cell covers the entire inverse image of \( F_w \) in the orbifold. For definition and description of \( q \)-cells and the attaching maps we ask the reader to consult [12].

Now each face has a unique maximum vertex \( w \) and interior of the face will be contained in \( F_w \) by points 3 and 4 above. So each face gets covered and each point in the orbifold is in the interior of exactly one \( q \)-cell. Considering the polytope as a face there will be exactly one \( 0 \) \( q \)-cell and one \( 2n \) \( q \)-cell. Thus we get a \( q \) cellular decomposition of the quasitoric orbifold.

Now it is shown in [12] that the \( 2k \) Betti numbers depends on the number of vertices with index \( k \) while the odd betti numbers are zero. Now if we have two quasitoric orbifolds with two combinatorially equivalent polytopes (which means they...
are diffeomorphic as manifold with corners) the height function of one composed with the diffeomorphism gives a height function of the other with identical vertex indices. Thus their Betti numbers will be same by what is done in [12]. □

Corollary 3.2. The dimension of each degree of \( \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) singular cohomology of a quasitoric orbifolds \( X \) and \( X' \) with combinatorially equivalent polytope are same.

Proof. By Universal coefficient theorem. □

Corollary 3.3. The dimension of each degree of \( \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) singular cohomology of a quasitoric orbifold \( X \) is identical with a projective toric orbifold \( X' \).

Proof. Take a quasitoric orbifold \( X \). A slight perturbation makes the polytope \( P \) associated with the orbifold into a rational polytope (see section 5.1.3 in [4]) without changing its combinatorial class, and with suitable dilations makes it into an integral polytope \( P' \) which is combinatorially equivalent to \( P \). Now form \( P' \) taking normal fan we get a projective toric orbifold \( X' \) (the analytic structure determines the orbifold structure so we do not use the bold notation) with polytope \( P' \). Since polytope \( P \) and polytope \( P' \) are combinatorially equivalent by (3.2) the above holds. □

Corollary 3.4. Each degree of the cohomology of the two spaces are isomorphic.

Proof. Since they have the same dimension and the dimensions are finite so the vector spaces are isomorphic. We define the isomorphisms as \( J_k \) where \( k \) is the degree of the cohomology. □

4. Hodge Structure

Definition 4.1. A pure hodge structure of weight \( n \) consists of an abelian group \( H_K \) and a decomposition of its complexification into complex subspaces \( H^{p,q} \) where \( p + q = n \) with the property conjugate of \( H^{p,q} \) is \( H^{q,p} \).

\[
H_C = H_K \otimes \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}.
\]

and

\[
\overline{H^{p,q}} = H^{q,p}.
\]

Definition 4.2. By a hodge structure on a compact space we mean the singular cohomology group of degree \( k \) has a pure hodge structure of weight \( k \) for all \( k \).

Proposition 4.1. Kahler compact orbifolds have a canonical hodge structure.

Proof. By Baily’s hodge decomposition see [1]. □

Proposition 4.2. Projective toric orbifolds coming from integral simple polytopes are Kahler.

Proof. By theorem 8.1 9.1 and 9.2 in [13]. □
Corollary 4.3. Projective toric orbifolds coming from integral simple polytopes have a canonical hodge structure.

Definition 4.3. Let $H^{p,q}$ be the $(p,q)$ hodge component of the canonical hodge structure on Projective toric orbifolds $X'$ coming from integral simple polytopes. We define

(4.3) \[ H^{p,q}(X') = H^{p,q}. \]

and

(4.4) \[ h^{p,q}(X') = \dim(H^{p,q}(X')). \]

Let $X$ be a quasitoric orbifold and $X'$ be the projective toric orbifold whose integral simple polytope $P'$ is combinatorially equivalent to the polytope $P$ of $X$. We assign

(4.5) \[ H^{p,q}(X) = J_k(H^{p,q}(X')). \]

where $p + q = k$ and $J_k$ is the isomorphisms of the degree $k$ cohomologies defined in (3.4).

Theorem 4.4. The above assignment defines a hodge structure on $X$.

Proof. By above and corollary (3.3). \qed

Theorem 4.5. The assignment does not depend on $X'$.

Proof. To show the above we must understand the $E$-polynomial. Let $Y$ be an algebraic variety over $\mathbb{C}$ which is not necessarily compact or smooth. Denote by $h^{p,q}(H^k_c(Y))$ the dimension of the $(p,q)$ Hodge component of the $k$-th cohomology with compact supports. This is a generalization of the hodge structures discussed on the above class of compact projective toric orbifolds and are called mixed hodge structures. For more detailed discussion we ask the reader to consult [14].

We define

(4.6) \[ e^{p,q}(Y) = \sum_{k \geq 0}(-1)^k h^{p,q}(H^k_c(Y)). \]

The polynomial

(4.7) \[ E(Y; u, v) := \sum_{p,q} e^{p,q}(Y) u^p v^q \]

is called $E$-polynomial of $Y$. When we have a proper hodge structure like the above class of compact projective toric orbifold $X'$,

(4.8) \[ e^{p,q}(X') = (-1)^{p+q} h^{p,q}(X'). \]

Now if we have a stratification of an algebraic variety $Y$ by disjoint locally closed subvarieties $Y_i$ (i.e $Y_i \subset Y$ and $Y = \bigcup_i Y_i$) by proposition 3.4 of [3]

(4.9) \[ E(Y; u, v) = \sum_i E(Y_i; u, v) \]

Now in a projective toric orbifold coming from a integral simple polytope as in our case we have a stratification by complex tori corresponding to the interior of
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Let \( X' \) be the concerned orbifold and \( F_i \) be a \( k \) dimensional face of the corresponding polytope then \( \pi^{-1}(F_i^\circ) \) is a \( k \) dimensional complex tori which we denote \( X'_i \). So by (4.3) we have

\[
E(X'; u, v) = \sum_i E(X'_i; u, v).
\]

Here \( i \) runs over all the faces. Now if we have two projective toric orbifolds \( X' \) and \( X'' \) both having combinatorially equivalent polytopes with that of \( X \) by (4.10) we claim they have the same \( E \)-polynomial. This is because since they have combinatorially equivalent polytopes, number of faces of a given dimension will be same for each polytope. So the sum on the right hand side of (4.10) can be partitioned into \( E \)-polynomial of \( k \) dimensional complex tori with a multiplicity of number of faces of dimension \( k \), where \( k \) runs from 0 to dimension of the polytopes. Since same dimensional complex tori have same \( E \)-polynomial the above claim holds.

Thus the hodge numbers of the two projective toric orbifolds will be same by (4.8). So the theorem holds.

**Theorem 4.6.** The hodge numbers of a quasitoric orbifold are as follows

\[
h^{p,q}(X) = 0 \quad \text{if} \quad p \neq q \quad \text{and} \quad h^{p,p}(X) = \dim(H^{2p}(X, \mathbb{C})).
\]

**Proof.** We show this for projective toric orbifolds coming from integral simple polytope. We know that the \( E \)-polynomial of a \( k \)-dimensional complex torus is \((uv - 1)^k\). Since by (4.10) the \( E \)-polynomial of the projective toric orbifold decomposes into sum of \( E \)-polynomial of complex tori and since \( E \)-polynomial of the complex tori have only terms of the form \((uv)^l\), implies that coefficient of \( u^p v^q \) is zero if \( p \neq q \) in the \( E \)-polynomial of the projective toric orbifold. Since these projective toric orbifolds have a proper hodge structure the claim of the theorem is true.

**4.1. Example.** We compute the hodge structure for \( CP^2 \# CP^2 \) which does not have an almost complex structure. We take a projective toric orbifold \( X' \) with a combinatorially equivalent polytope \( P' \). Since the polytope of \( P \) is a four sided polygon it will have four vertices, four edges and one 2-face.

\[
E(X'; u, v) = (uv - 1)^2 + 4(uv - 1) + 4.
\]

\[
E(X'; u, v) = u^2 v^2 + 2uv + 1.
\]

This tallies with the cohomology of \( CP^2 \# CP^2 \) and so we have the decomposition \( h^{2,2} = 1, h^{1,1} = 2 \) and \( h^{0,0} = 1 \).

**5. An application - Orbifold hodge numbers and a correspondence**

**5.1. Orbifold hodge numbers.** Orbifold hodge numbers for closed complex orbifold was defined in [11]. They depend on the twisted sectors of the orbifold. The twisted sector for toric variety was computed in [10]. The determination of twisted sectors of quasitoric orbifolds are similar in essence. Let \( X \) be an omnioriented quasitoric orbifold (i.e the signs of characteristic vectors are fixed). Consider an element \( g \) belonging to the group \( G_F \) defined in equation (2.9). Then \( g \) may be represented...
by the vector $\sum_{j \in \mathcal{I}(F)} a_j \lambda_j$ where $a_j$ is restricted to $[0, 1) \cap \mathbb{Q}$ and $\lambda_j$ belongs to the characteristic set of $F$. We define the degree shifting number or age as

$$\iota(g) = \sum a_j.$$  

(5.1)

For faces $F$ and $H$ of $P$ we write $F \leq H$ if $F$ is a sub-face of $H$, and $F < H$ if it is a proper sub-face. If $F \leq H$ we have a natural inclusion of $G_H$ into $G_F$ induced by the inclusion of $N(H)$ into $N(F)$. Therefore we may regard $G_H$ as a subgroup of $G_F$. Define the set

$$G^*_F = G_F - \bigcup_{F < H} G_H.$$  

Note that $G^*_F = \{ \sum_{j \in \mathcal{I}(F)} a_j \lambda_j | 0 < a_j < 1 \} \cap N$, and $G^*_P = G_P = \{ 0 \}$.

**Definition 5.1.** We define the orbifold dolbeault cohomology groups of an omnioriented quasitoric orbifold $X$ to be $H^{p,q}_{\text{orb}}(X) = \bigoplus_{F \leq P, g \in G^*_F} H^{p-\iota(g),q-\iota(g)}(X(F))$. Here $H^{p-\iota(g),q-\iota(g)}(X(F))$ refers to the components of the hodge structures defined above, when $X(F)$ is considered as a quasitoric orbifold $X(F)$. The pairs $(X(F), g)$ where $F < P$ and $g \in G^*_F$ are called twisted sectors of $X$. The pair $(X(P), 1)$, i.e. the underlying space $X$, is called the untwisted sector.

**Definition 5.2.** We define orbifold hodge numbers as $h^{p,q}_{\text{orb}}(X) = \text{dim}(H^{p,q}_{\text{orb}}(X))$.

Now we introduce some notation. Consider a codimension $k$ face $F = F_1 \cap \ldots \cap F_k$ of $P$ where $k \geq 1$. Define a $k$-dimensional cone $C_F$ in $N \otimes \mathbb{R}$ as follows,

$$C_F = \{ \sum_{j=1}^{k} a_j \lambda_j : a_j \geq 0 \}.$$  

(5.3)

The group $G_F$ can be identified with the subset $\text{Box}_F$ of $C_F$, where

$$\text{Box}_F := \{ \sum_{j=1}^{k} a_j \lambda_j : 0 \leq a_j < 1 \} \cap N.$$  

(5.4)

Consequently the set $G^*_F$ is identified with the subset

$$\text{Box}^*_F := \{ \sum_{j=1}^{k} a_j \lambda_j : 0 < a_j < 1 \} \cap N.$$  

(5.5)

of the interior of $C_F$. We define $\text{Box}_P = \text{Box}^*_P = \{ 0 \}$. Suppose $w = F_1 \cap \ldots \cap F_n$ is a vertex of $P$. Then $\text{Box}_w = \bigcup_{w \leq F} \text{Box}^*_F$. This implies

$$G_w = \bigcup_{w \leq F} G^*_F.$$  

(5.6)
An almost complex orbifold is \( SL \) if the linearization of each \( g \) is in \( SL(n, \mathbb{C}) \). This is equivalent to \( \iota(g) \) being integral for every twisted sector. Therefore, to suit our purposes, we make the following definition.

**Definition 5.3.** An omnioriented quasitoric orbifold is said to be quasi-\( SL \) if the age of every twisted sector is an integer.

5.2. **Blowdowns.** In order to get a blow up of a face we replace a face by a facet with a new characteristic vector. Suppose \( F \) is a face of \( P \). We choose a hyperplane \( H = \{ \hat{p}_0 = 0 \} \) such that \( \hat{p}_0 \) is negative on \( F \) and \( \hat{P} := \{ \hat{p}_0 > 0 \} \cap P \) is a simple polytope having one more facet than \( P \). Suppose \( F_1, \ldots, F_m \) are the facets of \( P \). Denote the facets \( F_i \cap \hat{P} \) by \( F_i \) without confusion. Denote the extra facet \( H \cap P \) by \( F_0 \).

Without loss of generality let \( F = \bigcap_{j=1}^{k} F_j \). Suppose there exists a primitive vector \( \lambda_0 \in \mathbb{N} \) such that

\[
\lambda_0 = \sum_{j=1}^{k} b_j \lambda_j, \quad b_j > 0 \forall j.
\]

Then the assignment \( F_0 \mapsto \lambda_0 \) extends the characteristic function of \( P \) to a characteristic function \( \hat{\Lambda} \) on \( \hat{P} \). Denote the omnioriented quasitoric orbifold derived from the model \( (\hat{P}, \hat{\Lambda}) \) by \( Y \).

**Definition 5.4.** We define blowdown a map \( Y \mapsto X \) which is inverse of a blow-up. Such maps have been constructed in [9].

**Lemma 5.1.** (Lemma 4.2 [9]) If \( X \) is positively omnioriented, then so is a blowup \( Y \).

**Definition 5.5.** A blowdown or blow up is called crepant if \( \sum b_j = 1 \).

**Lemma 5.2.** (Lemma 8.2 [9]) The crepant blowup of a quasi-\( SL \) quasitoric orbifold is quasi-\( SL \).

5.3. **Correspondence of hodge numbers.** The statement of the theorem we are going to prove is as follows

**Theorem 5.3.** For crepant blowdowns(or blowups) orbifold hodge numbers of quasi-\( SL \) quasitoric orbifolds do not change.

**Corollary 5.4.** For crepant resolution orbifold hodge numbers of quasi-\( SL \) quasitoric orbifolds do not change.

We admit the proof is similar to the proof of Mckay Correspondence of Betti-numbers of Chen-Ruan Cohomolgy in the author’s previous paper [15] and motivated by Strong Mckay Correspondence proof [3], but still we give a detailed argument for the convenience of the reader.
5.4. Singularity and lattice polyhedron. Following the discussion in Section 3.1, a singularity of a face $F$ is defined by a cone $C_F$ formed by positive linear combinations of vectors in its characteristic set $\lambda_1, \ldots, \lambda_d$ where $d$ is the codimension of the face in the polytope. The elements of the local group $G_F$ are of the form $g = \text{diag}(e^{2\pi i \sqrt{-1} \alpha_1}, \ldots, e^{2\pi i \sqrt{-1} \alpha_d})$, where $\sum_{i=1}^{d} \alpha_i \lambda_i \in N$, and $0 \leq \alpha_i < 1$. Recall that the age

$$
(5.8) \quad \lambda(g) = \alpha_1 + \ldots + \alpha_d.
$$

is integral in quasi-$SL$ case by definition 5.3.

The singularity along the interior of $F$ is of the form $\mathbb{C}^d/G_F$. These singularities are same as Gorenstein toric quotient singularities in complex algebraic geometry. This means they are toric (coming from a cone) $SL$ orbifold singularity ($SL$ means linearisation of a group element is $SL$, which in our case implies $\lambda(g)$ is integral). Now let $N_w$ be the lattice formed by $\{\lambda_1, \ldots, \lambda_n\}$, the characteristic vectors at a vertex $w$ contained in the face $F$. Let $m_w$ be the element in the dual lattice of $N_w$ such that its evaluation on each $\lambda_i$ is one. Now from Lemma 9.2 of \[3\] we know that the cone $C_w$ contains an integral basis, say $e_1, \ldots, e_n$. Suppose $e_i = \sum a_{ij} \lambda_j$. By (5.4) $e_i$ corresponds to an element of $G_w$, and since the singularity is quasi-$SL$, $\sum a_{ij}$ is integral. Hence $m_w$ evaluated on each $e_j$ is integral. So $m_w$ an element of the dual of the integral lattice $N$.

Consider the $(n - 1)$-dimensional lattice polyhedron $\Delta_w$ defined as $\{x \in C_w \mid \langle x, m_w \rangle = 1\}$. Note that $\Delta_w = \{\sum_{i=1}^{n} a_i \lambda_i \mid a_i \geq 0, \sum a_i = 1\}$. For any face $F$ containing $w$ we define $\Delta_F = \Delta_w \cap C_F$. If $\{\lambda_1, \ldots, \lambda_d\}$ denote the characteristic set of $F$, then $\Delta_F = \{\sum_{i=1}^{d} a_i \lambda_i \mid a_i \geq 0, \sum a_i = 1\}$. Hence $\Delta_F$ is independent of the choice of $w$.

Remark 5.5. An element $g \in G$ of an $SL$ orbifold singularity can be diagonalized to the form $g = \text{diag}(e^{2\pi i \sqrt{-1} \alpha_1}, \ldots, e^{2\pi i \sqrt{-1} \alpha_d})$, where $0 \leq \alpha_i < 1$ and $\lambda(g) = \alpha_1 + \ldots + \alpha_d$ is integral.

We make some definitions following \[3\].

Definition 5.6. Let $G$ be a finite subgroup of $SL(d, \mathbb{C})$. Denote by $\psi_i(G)$ the number of the conjugacy classes of $G$ having $\lambda(g) = i$. Define

$$
(5.9) \quad W(G; uv) = \psi_0(G) + \psi_1(G)uv + \ldots + \psi_{d-1}(G)(uv)^{d-1}.
$$

Definition 5.7. We define height($g$) = rank($g$-I).

Definition 5.8. Let $G$ be a finite subgroup of $SL(d, \mathbb{C})$. Denote by $\tilde{\psi}_i(G)$ the number of the conjugacy classes of $G$ having the height = $d$ and $\lambda(g) = i$. Define

$$
(5.10) \quad \tilde{W}(G; uv) = \tilde{\psi}_0(G) + \tilde{\psi}_1(G)uv + \ldots + \tilde{\psi}_{d-1}(G)(uv)^{d-1}.
$$

Definition 5.9. For a lattice polyhedron $\Delta_F$ defining a $SL$ singularity $\mathbb{C}^d/G_F$, we define the following:

$$
(5.11) \quad W(\Delta_F; uv) = W(G_F; uv).
$$
(5.12) \[ \psi_i(\Delta_F) = \psi_i(G_F). \]

(5.13) \[ \tilde{W}(\Delta_F;uv) = \tilde{W}(G_F;uv). \]

(5.14) \[ \tilde{\psi}_i(\Delta_F) = \tilde{\psi}_i(G_F). \]

Definition 5.10. A finite collection τ = \{θ\} of simplices with vertices in \( \Delta_F \cap N \) is called a triangulation of \( \Delta_F \) if the following properties are satisfied.

1. If \( \theta' \) is a face of \( \theta \in \tau \) then \( \theta' \in \tau \).
2. The intersection of any two simplices \( \theta', \theta'' \in \tau \) is either empty, or a common face of both of them.
3. \( \Delta_F = \bigcup_{\theta \in \tau} \theta \).

5.5. Blowdown and triangulation of polyhedron. A crepant blowup gives rise to triangulation of the polyhedrons corresponding to some of the faces. Suppose we blow up about a face \( F \). Then it is clear that new characteristic vector is an integral vector lying in the interior of the polyhedron \( \Delta_F \). Note that \( \Delta_F \) is a simplex. The crepant blow up induces a triangulation of \( \Delta_F \) with top dimensional simplices being the join of simplices of next lower dimension (in the pretriangulation) with the new characteristic vector. We denote this triangulation of \( \Delta_F \) by \( \tau_F \). For the faces \( F' \) contained in \( F \), \( \Delta_{F'} \) is triangulated as follows. Let \( K_{F'} = \lambda_{F'} - \lambda_F \) be difference of two characteristic sets. The triangulation \( \tau_{F'} \) consists of simplices with vertex set of the form \( \theta \cup \beta \) where \( \theta \) are the vertices of a simplex of \( \tau_F \) and \( \beta \subset K_{F'} \). To see that this process takes care of all the faces lost and created we make the following comments. First of all the faces lost are \( F \) and its subfaces. This means there will be no simplex with vertex set having \( \lambda_F \) as a subset. This is exactly what happens here. The new faces created are subfaces of the intersection of new facet (created by the blowup) with faces having as vertex one of the vertices of \( F \). These faces intersected \( F \) prior to the blow up in some \( F' \) and so the new faces formed correspond to the simplices with vertex set that are subset of the union \( \theta \cup \beta \) discussed above.

5.6. E-polynomial for quasitoric orbifold.

Definition 5.11. We define the E-polynomial of a quasitoric orbifold \( X \) as follows

(5.15) \[ E_{\text{quas}}(X:u,v) = \sum_{p,q}(-1)^{p+q}h^p q^q(X)u^p v^q. \]

If \( X_i \) is the stratification of the quasitoric orbifold by inverse image of the quotient map on interior of faces \( F_i \). Here \( i \) runs over all the faces.

Theorem 5.6.

(5.16) \[ E_{\text{quas}}(X:u,v) = \sum_i E(X_i:u,v). \]

Proof. Let \( X' \) be a projective toric orbifold whose hodge structure has been pulled backed to \( X \). The by proposition 3.4 [3] we have

(5.17) \[ E_{\text{quas}}(X:u,v) = E(X':u,v) = \sum_i E(X'_i:u,v). \]

Where is \( X'_i \) is stratification by inverse images of interiors of faces of the polytope of \( X' \). Since the two orbifolds have combinatorially equivalent polytopes number
of faces of a given dimension is same. And since the stratas are complex tori of
dimension equal to the face covered by it, we can replace $X'_i$ by the corresponding $X_i$
in the right hand most sum. The identification of $X'_i$ with $X_i$ is by the combinatorial
equivalence map. \hfill \Box

**Definition 5.12.** We define

\begin{equation}
E_{orb}(X : u, v) = \sum_{p,q} (-1)^{p+q} h_{orb}^{p,q}(X) u^p v^q.
\end{equation}

From the above discussions it is easy to prove

\begin{equation}
E_{orb}(X : u, v) = \sum_i E_{quas}(X_i : u, v) \tilde{W}(\Delta_{F_i}, uv).
\end{equation}

The following can also be seen from what has been discussed in the previous sub-
section

\begin{equation}
W(\Delta_{F_i}, uv) = \sum_{X_j \supset X_i} \tilde{W}(\Delta_{F_j}, uv).
\end{equation}

\begin{equation}
E_{orb}(X : u, v) = \sum_i E(X_i, u, v) W(\Delta_{F_i}, uv).
\end{equation}

where $X_j \geq X_i$ means $X_i \subset X_j$ and $X$ is a quasi-SL quasitoric orbifold.

5.7. **Ehrhart power series.** Let $\Delta$ be a lattice polyhedron and $k\Delta := \{ kx \mid x \in \Delta \}$. Let $l(k\Delta)$ be the number of lattice points of $k\Delta$. Then the Ehrhart power series

\begin{equation}
P_\Delta(t) = \sum_{k \geq 0} l(k\Delta) t^k.
\end{equation}

**Proposition 5.7.** Let $\Delta_F$ be a $(d-1)$ dimensional lattice polyhedron defining a $d$-
dimensional toric singularity. It is well-known (see, for instance, \cite{3}, Theorem 5.4)
that $P_{\Delta_F}(t)$ can be written in the form,

\begin{equation}
P_{\Delta_F}(t) = \frac{\psi_0(\Delta_F) + \psi_1(\Delta_F) t + \ldots + \psi_{d-1}(\Delta_F) t^{d-1}}{(1-t)^d}.
\end{equation}

where $\psi_0(\Delta_F), \ldots, \psi_{d-1}(\Delta_F)$ are non-negative integers defined in equation (5.12)

5.8. **Proof of the main theorem.** We state the theorem again for the reader’s convenience.

**Theorem 5.8.** For crepant blowdowns (or blowups) orbifold hodge numbers of quasi-
SL quasitoric orbifold do not change.

**Proof.** Let $\rho : \hat{X} \to X$ be a crepant blowdown of omnioriented quasi-SL quasitoric
orbifolds. We set $\hat{X}_i := \rho^{-1}(X_i)$. Then $\hat{X}_i$ has a natural stratification by products

\begin{equation}
\Delta_{F_i} = \cup_{\theta \in \tau_i} \Delta_{\theta}.
\end{equation}

where $\tau_i$ consists of all simplices which intersect the interior of $\Delta_{F_i}$, and $codim(\theta)$
denotes the codimension of $\theta$ in $\Delta_{F_i}$.

Note that the $E$-polynomial of a $k$-dimensional complex torus is $(uv - 1)^k$.

From (5.23) we have

\begin{equation}
W(\Delta_{F_i}; uv) = P_{\Delta_{F_i}}(uv)(1 - uv)^d.
\end{equation}
where $d$ is the dimension of the face $F_i$. Consider the triangulation (5.24) of $\Delta F_i$. By counting lattice points using (5.23) and applying the inclusion exclusion principle we have

\[(5.26)\quad P_{\Delta F_i}(uv) = \sum_{\theta \in \tau_i} (-1)^{\text{codim}(\theta)} P_{\theta}(uv) = \sum_{\theta \in \tau_i} (-1)^{\text{codim}(\theta)} W(\theta, uv)(1-uv)^{-\text{dim}(\theta)}.\]

Multiplying both sides by $(1-uv)^d$, we obtain

\[(5.27)\quad W(\Delta F_i; uv) = \sum_{\theta \in \tau_i} (uv - 1)^{\text{codim}(\theta)} W(\theta, uv).\]

Since we are dealing with simplices $\theta$ which intersect the interior of $\Delta F_i$ each stratum of $\hat{X}$ is counted once. This is because each stratum corresponds to the interior of a face and for each face we have a simplex and it will lie in the interior of exactly one of the original (pre-triangulation) polyhedrons. Thus the equation (5.21) applied to $\hat{X}$ gives

\[(5.28)\quad E_{\text{orb}}(\hat{X} : u, v) = \sum_{i \in I} E(X_i; u, v) \sum_{\theta \in \tau_i} (uv - 1)^{\text{codim}(\theta)} W(\theta, uv).\]

Now using (5.27)

\[(5.29)\quad E_{\text{orb}}(X : u, v) = \sum_{i \in I} E(X_i; u, v) W(\Delta F_i; uv) = E_{\text{orb}}(\hat{X} : u, v).\]

\[\square\]

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