CREATION OPERATORS FOR THE MACDONALD AND JACK POLYNOMIALS

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Abstract. Formulas of Rodrigues-type for the Macdonald polynomials are presented. They involve creation operators, certain properties of which are proved and other conjectured. The limiting case of the Jack polynomials is discussed.

1. Introduction

A formula that gives the Jack polynomials through the application of a string of creation operators on the Jack polynomials of lowest degree has been introduced in [2, 3]. Since the Jack polynomials are a specialization of the Macdonald polynomials which involve two parameters \(q, t\), it was natural to expect that a Rodrigues formula should also exist for Macdonald polynomials. We derive such a formula in this paper. It should be pointed out however, that the construction presented here is much different than the one to be found in [2, 3]; in fact, the latter is not simply the limit of the former. In this connection, we also give in the form of conjecture, the expression of the creation operators that should be the analogs the operators constructed in [2, 3]. If this conjecture is true, it implies in particular, that the expansion coefficients of the Macdonald polynomials in the monomial basis are polynomials in \(q, t\) with integer coefficients. The action of these operators on arbitrary Macdonald polynomials is quite elegant and proves useful to conjecture that families of \(N\) commuting operators can be constructed out of them. The limiting case of the Jack polynomials is presented at the end and the connection is made with the operators given in [2, 3]. The conjectures are seen to be valid in this case also.

2. Definitions

Symmetric polynomials are labelled by partition \(\lambda\) of their degree \(n\), that is sequences \(\lambda = (\lambda_1, \lambda_2, \ldots)\) of non-negative integers in decreasing order \(\lambda_1 \geq \lambda_2 \geq \ldots\) such that \(|\lambda| = \lambda_1 + \lambda_2 + \cdots = n\). The number of non-zero parts in \(\lambda\) is denoted \(\ell(\lambda)\). Let \(\lambda\) and \(\mu\) be two partitions of \(n\). In the dominance ordering, \(\lambda \geq \mu\) if \(\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i\) for all \(i\). We can associate a diagram to each partition \(\lambda\). The diagram is made out of \(\ell(\lambda)\) rows, labelled by the integer \(i\), with \(\lambda_i\) squares in each one of them. The squares are identified by the coordinates \((i, j) \in \mathbb{Z}^2\) with \(i\), the row index, increasing as one goes downwards and \(j\), the column index, increasing as one goes from left to right. For example the diagram of \((5, 4, 4, 1)\) is

\[
\begin{array}{cccc}
\text{(1,1)} & \text{(2,1)} & \text{(3,1)} & \text{(4,1)} \\
\text{(1,2)} & \text{(2,2)} & \text{(3,2)} & \text{(4,2)} \\
\text{(1,3)} & \text{(2,3)} & \text{(3,3)} & \text{(4,3)} \\
\text{(1,4)} & \text{(2,4)} & \text{(3,4)} & \text{(4,4)} \\
\end{array}
\]
For each square \( s = (i, j) \) in the diagram of a partition \( \lambda \), let \( \ell'(s), \ell(s), a(s) \) and \( a'(s) \) be respectively the number of squares in the diagram of \( \lambda \) to the north, south, east and west of the square \( s \). By \( \lambda \supset \mu \) it is meant that the diagram \( \lambda \) contains the diagram \( \mu \), i.e. that \( \lambda_i \geq \mu_i \) for all \( i \geq 1 \). The set-theoretic difference \( \theta = \lambda - \mu \) is called a skew diagram. For example if \( \lambda = (5, 4, 4, 1) \) and \( \mu = (4, 3, 2) \), \( \lambda - \mu \) is made of the dotted squares in the picture.

A skew diagram \( \theta \) is a vertical \( m \)-strip if \( |	heta| = m \) and \( \theta_i \leq 1 \) for each \( i \geq 1 \). In other words, a vertical strip has at most one square in each column.

Let \( \Lambda_N \) denote the ring of symmetric functions in the variables \( x_1, x_2, \ldots, x_N \). Three standard bases for the space of symmetric functions are:

(i) the power sum symmetric functions \( p_\lambda \) which in terms of the power sums
\[
p_i = \sum_k x_k^i,
\]
are given by
\[
p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots .
\]
(ii) the monomial symmetric functions \( m_\lambda \) which are
\[
m_\lambda = \sum_{\text{distinct permutations}} x_1^{\lambda_1} x_2^{\lambda_2} \cdots
\]
(iii) the elementary symmetric functions \( e_\lambda \) which in terms of the i-th elementary function
\[
e_i = \sum_{j_1 < j_2 < \cdots < j_i} x_{j_1} x_{j_2} \cdots x_{j_i} = m_{(1^i)}
\]
are given by
\[
e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots .
\]

The Macdonald polynomials can now be presented as follows. To the partition \( \lambda \) with \( m_i(\lambda) \) parts equal to \( i \), we associate the number
\[
z_\lambda = 1^{m_1} m_1 ! 2^{m_2} m_2 ! \cdots
\]
Let \( q \) and \( t \) be parameters and \( \mathbb{Q}(q, t) \) the field of all rational functions of \( q \) and \( t \) with rational coefficients and define a scalar product \( \langle \cdot, \cdot \rangle_{q,t} \) on \( \Lambda_N \otimes \mathbb{Q}(q, t) \) by
\[
\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda \mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.
\]
where \( \ell(\lambda) \) is the number of parts of \( \lambda \). The Macdonald polynomials \( J_\lambda(x; q, t) \in \Lambda_N \otimes \mathbb{Q}(q, t) \) are uniquely specified by

1. \( \langle J_\lambda, J_\mu \rangle_{q,t} = 0 \), if \( \lambda \neq \mu \), \hspace{1cm} (8)
2. \( J_\lambda = \sum_{\mu \leq \lambda} v_{\lambda \mu}(q, t)m_\mu \), \hspace{1cm} (9)
3. \( v_{\lambda \lambda}(q, t) = c_\lambda(q, t) \), \hspace{1cm} (10)

where

\[
c_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{\alpha(s)} t^{\ell(s) + 1}). \hspace{1cm} (11)
\]

For each 1 \( \leq i \leq N \) the shift operator \( T_{q,x_i} \) is defined by

\[
T_{q,x_i} f(x_1, \ldots, x_N) = f(x_1, \ldots, qx_i, \ldots, x_N) \hspace{1cm} (12)
\]
for any polynomial \( f \in \Lambda_N \otimes \mathbb{Q}(q, t) \).

For \( r = 1, \ldots, N \), let \( M^r_N \) denote the Macdonald operator

\[
M^r_N = \sum_{I \subseteq \{1, \ldots, N\}} A_I(x; t) \prod_{i \in I} T_{q,x_i}, \hspace{1cm} (13)
\]
summed over all \( r \)-element subsets \( I \) of \( \{1, \ldots, N\} \), where

\[
A_I(x; t) = t^{r(r-1)/2} \prod_{i \in I} \frac{t x_i - x_j}{x_i - x_j}, \hspace{1cm} (14)
\]
and take \( M^0_N \equiv 1 \). These operators commute with each other, \([M^r_N, M^l_N] = 0\) and are diagonal on the Macdonald polynomials basis. From the Macdonald operators, one constructs

\[
M_N(X; q, t) = \sum_{r=0}^{N} M^r_N X^r, \hspace{1cm} (15)
\]
with \( X \) an arbitrary parameter. The operator \( M_N \) will play a crucial role in the following. Its action on \( J_\lambda(x; q, t) \) with \( \ell(\lambda) \leq N \) is given, remarkably, by

\[
M_N(X; q, t) J_\lambda(x; q, t) = a_\lambda(X; q, t) J_\lambda(x; q, t), \hspace{1cm} (16)
\]
where

\[
a_\lambda(X; q, t) = \prod_{i=1}^{N} (1 + X q^{\lambda_i} t^{N-i}). \hspace{1cm} (17)
\]

From (15) we see that the eigenvalue of \( M^r_N \) on \( J_\lambda(x; q, t) \) is the coefficient of \( X^r \) in the polynomial (17).

Let us introduce the monic Macdonald polynomials \( P_\lambda = 1/c_\lambda(q, t) J_\lambda \). The formulas giving the action of the elementary symmetric function \( e_k \) on the \( P_\lambda \)'s are known as the Pieri formulas. Explicitly, they are

\[
e_k P_\lambda = \sum_{\mu} \Psi_{\mu/\lambda} P_\mu \hspace{1cm} (18)
\]
summed over partition $\mu \supset \lambda$ (of length $\leq N$) such that $\mu - \lambda$ is a vertical $k$-strip, with

$$\Psi_{\mu/\lambda} = \prod_{s \in C_{\mu/\lambda} \setminus s \not\in R_{\mu/\lambda}} \frac{b_\mu(s)}{b_\lambda(s)}$$

(19)

and

$$b_\lambda(s) = \begin{cases} \frac{1 - q^{a(s)} t^{\ell(s)+1}}{1 - q^{a(s)} t^{\ell(s)}} & \text{if } s \in \lambda \\ 1 & \text{otherwise} \end{cases}$$

(20)

where $C_{\mu/\lambda}$ (resp. $R_{\mu/\lambda}$) denote the union of the columns (resp. rows) that intersect $\mu - \lambda$. For example, with $\mu = (4, 2, 2)$ and $\lambda = (3, 2, 1)$ we have

\[
\begin{array}{ccc}
\bullet & & \\
\times & & \\
\end{array}
\]

so the only $s$ in $C_{\mu/\lambda}$ but not in $R_{\mu/\lambda}$ is in position (2,2).

When restricted to the $N$-dimensional torus $T = \{ \{x_1, \ldots, x_N\} \in \mathbb{C}^N; |x_i| = 1, 1 \leq i \leq N \}$, for $|q| < 1$ and $|t| < 1$, the polynomials $P_\lambda$ are also orthogonal with respect to the scalar product $(,)$ defined by

$$(f, g) = \frac{1}{N} \int_T f(x)\overline{g(x)}\Delta(x; q, t)dx$$

(22)

with $dx$ the normalized Haar measure on $T$ and

$$\Delta(x; q, t) = \prod_{i \neq j} (x_i x_j^{-1} q \cdot (t x_i x_j^{-1} q) \infty, \infty)$$

(23)

where

$$(a; q) \infty = \prod_{i=0}^{\infty} (1 - a q^i).$$

(24)

In fact,

$$(P_\lambda, P_\mu) = \delta_{\lambda \mu} c_N \prod_{s \in \lambda} b_\lambda(s)^{-1} \prod_{s \in \lambda} \frac{1 - q^{a(s)} t^{N - \ell(s)}}{1 - q^{a(s)} t^{N - \ell(s)} + 1},$$

(25)

where $c_N = (1, 1)$.

3. The Rodrigues formula

**Theorem 1.** The Macdonald polynomials $J_\lambda(x; q)$ associated to the partitions $\lambda = (\lambda_1, \ldots, \lambda_N)$ are given by

$$J_\lambda(x; q) = (B^{+}_N)^{N} (B^{+}_{N-1})^{\lambda_{N-1} - \lambda_N} \cdots (B^{+}_1)^{\lambda_1 - \lambda_2} \cdot 1,$$

(26)

with

$$B^{+}_k = \frac{1}{(q^{-1}; t^{-1})_{N-k}} M_N(-t^{k+1-N} q^{-1}; q, t) c_k, \quad k = 1, \ldots, N,$$

(27)

where for $n$ positive integer, $(a; q)_n = (1 - a)(1 - qa) \cdots (1 - q^{n-1}a)$ and $(a; q)_0 \equiv 1$. 
**Proof.** From the Pieri formula, we have the following lemma

**Lemma 2.** The action of \( e_k \) on \( P_\lambda \) with \( \lambda \) a partition with \( \ell(\lambda) \leq k \) is given by

\[
e_k P_\lambda = P_{\lambda+(1^k)} + \sum_{\mu \neq \lambda+(1^k)} \Psi_{\mu/\lambda} P_{\mu},
\]

where all the \( \mu \)'s in the sum are such that \( \mu_{k+1} = 1 \).

This is seen from the fact that \( \mu \) must be a partition which contains \( \lambda \) and \( \mu - \lambda \) a vertical \( k \)-strip. Hence the only way to construct a \( \mu \) with \( \mu_{k+1} \neq 1 \) is to add a 1 in each of the first \( k \) entries of \( \lambda \).

From Lemma 2 and (16), we thus have

\[
M_N(-t^{k+1-N}q^{-1};q,t)e_k P_\lambda = k \prod_{i=1}^k (1 - t^{k+1-i}q^{\lambda_i})(q^{-1};t^{-1})_{N-k}P_{\lambda+(1^k)},
\]

since the eigenvalues of \( M_N(-t^{k+1-N}q^{-1};q,t) \) on the \( P_\mu \)'s in (28) are

\[
a_\mu(-t^{k+1-N}q^{-1};q,t) = \prod_{i=1}^N (1 - t^{k+1-i}q^{\mu_i-1}),
\]

and vanish if \( \mu_{k+1} = 1 \).

**Lemma 3.** If \( \lambda \) is a partition with \( \ell(\lambda) \leq k \),

\[
\frac{c_{\lambda+(1^k)}}{c_\lambda} = k \prod_{i=1}^k (1 - t^{k+1-i}q^{\lambda_i}).
\]

When going from \( \lambda \) to \( \lambda + (1^k) \), what we actually do is add a column at the west of the diagram.

![Diagram](image)

From the fact that \( c_\lambda \) only involves \( a(s) \) and \( \ell(s) \) which do not depend on the number of square at their west, the contribution in \( c_{\lambda+(1^k)} \) of the squares that have been shifted to the east is exactly \( c_\lambda \). Hence we only have to take the product of the contributions of the first column of \( \lambda + (1^k) \), which is \( \prod_{i=1}^k (1 - t^{k+1-i}q^{\lambda_i}) \).

Using Lemma 3 and passing from \( P_\lambda \) to \( J_\lambda \), we have

\[
1 \frac{(q^{-1};t^{-1})_{N-k}}{(q^{-1};t^{-1})_{N-k}} M_N(-t^{k+1-N}q^{-1};q,t)e_k J_\lambda = J_{\lambda+(1^k)}
\]

which gives when \( \ell(\lambda) \leq k \), \( B_k \cdot J_\lambda = J_{\lambda+(1^k)} \) and hence, proves Theorem 1.

Note that the adjoint of \( B_k^+ \) with respect to the scalar product \( ( \cdot , \cdot ) \) is easily found since \( M_N(X;q,t) \) is self-adjoint and \( e_k^\dagger = e_N^{-1}e_{N-k} \). Indeed, one has

\[
B_k^\dagger = (B_k^\dagger)^\dagger = \frac{1}{(q^{-1};t^{-1})_{N-k}}e_N^{-1}e_{N-k}M_N(-t^{k+1-N}q^{-1};q,t).
\]
Using these operators and the fact that \((P_{\mu}, P_{\lambda}) = 0\) if \(P_{\mu} \neq P_{\lambda}\), allows for a straightforward computation of \((P_{\lambda}, P_{\lambda})\). One first observes that if \(\ell(\lambda) \leq k\),

\[
(P_{\lambda+(1^k)}, P_{\lambda+(1^k)}) = \frac{c_\lambda}{c_{\lambda+(1^k)}}(B^+_k P_{\lambda}, P_{\lambda+(1^k)}) = \frac{c_\lambda}{c_{\lambda+(1^k)}}(P_{\lambda}, B^-_k P_{\lambda+(1^k)}).
\] (34)

From (31) and the eigenvalue of \(M_N\) on the R.H.S. of (29), one sees that the constant cancels out and that

\[
(P_{\lambda+(1^k)}, P_{\lambda+(1^k)}) = (P_{\lambda}, e_{N-1}^k P_{\lambda+(1^k)}) = (P_{\lambda+1}, e_{N-1}^k P_{\lambda+(1^k)}).
\] (35)

Finally, using the orthogonality of the \(P_{\lambda}\)'s and the Pieri formula, one finds the following formula

\[
(P_{\lambda+(1^k)}, P_{\lambda+(1^k)}) = \Psi_{\lambda+1/\lambda+(1^k)}(P_{\lambda}, P_{\lambda}),
\] (36)

which gives the norm of \(P_{\lambda}\) through iteration.

4. Conjectures

The Rodrigues formula given in Theorem 1 does not imply that the \(v_{\lambda\mu}(q, t)\)'s of (9) are polynomials in \(q\) and \(t\) with integer coefficients. However, it proves useful to obtain results which once proved, would have this implication. Such formulas which represent generalizations for the Macdonald polynomials of relations that we have proved for the Jack polynomials, are given below in the form of conjectures. Their limits as \(q = t^\alpha\) and \(t \to 1\) will be discussed in the section on the Jack polynomials.

Conjecture 4. The Macdonald polynomials \(J_{\lambda}(x; q, t)\) are given by

\[
J_{\lambda}(x; q, t) = (\hat{B}^+_N)^{\lambda_N}(\hat{B}^+_N)^{\lambda_{N-1}} \ldots (\hat{B}^+_1)^{\lambda_1} \cdot 1,
\] (37)

with

\[
\hat{B}^+_k = \sum_I \hat{A}_I(x; t) x_I M_I(-t; q, t)
\] (38)

summed over all \(k\)-element subsets \(I\) of \(\{1, \ldots, N\}\), where

\[
x_I = \prod_{i \in I} x_i,
\] (39)

\[
\hat{A}_I(x; t) = t^{-(N-k)k} \prod_{i \in I} \frac{tx_i - x_j}{x_i - x_j}
\] (40)

and

\[
M_I(X; q, t) = M_k(X; q, t)
\] (41)

in the \(k\) variables \(x_i \in I\).
Conjecture 5. The Macdonald polynomials $J_\lambda(x; q, t)$ are given by
\[ J_\lambda(x; q, t) = (\bar{B}_N^+)^{\lambda_N} (\bar{B}_{N-1}^+)^{\lambda_{N-1} - \lambda_N} \ldots (\bar{B}_1^+)^{\lambda_1 - \lambda_2} \cdot 1, \tag{42} \]
with
\[ \bar{B}_k^+ = \sum_I \bar{A}_I(x; t)x_I M_I(-t; q, t) \tag{43} \]
summed over all $k$-element subsets $I$ of $\{1, \ldots, N\}$, where
\[ \bar{A}_I(x; t) = \left\{ \prod_{i \in I} x_i - t x_j \right\} \left\{ \prod_{j \not\in I} T_{q, x_j} \right\}. \tag{44} \]

A manifest corollary of Conjecture 5 would be that the $v_{\lambda\mu}(q, t)$’s are polynomials in $q, t$ with integer coefficients. Let us stress however that the operators $\bar{B}_k^+$ of Conjecture 4 appear to be the natural generalizations of the creation operators introduced in [3] in the case of the Jack polynomials. Indeed, as will be confirmed in section 6, the operators (38) and (85) share important properties.

As an indication that Conjecture 4 must be true, we prove that, for partition $\lambda$ with $\ell(\lambda) \leq k$,
\[ \bar{B}_k^+ J_\lambda = J_{\lambda+(1^k)}, \tag{45} \]
in the cases where the number of variables $N = k$ or $k + 1$.

Lemma 6.
\[ M_I(X; q, t)e_N = e_N M_I(Xq; q, t) \tag{46} \]
for all subsets $I$ of $\{1, \ldots, N\}$.

This result follows from the fact that $M_I^* e_N = q^r e_N M_I^*$, which is easily derived from (13).

Lemma 6 immediately ensures that (45) is true for $N = k$, since it implies that $\bar{B}_k^+ = B_k^+$ in this case. The following lemma will be needed in the proof of the special case $N = k + 1$.

Lemma 7. When the number of variables is $k + 1$,
\[ M_{k+1}(-1; q, t)J_{(\lambda_1, \ldots, \lambda_k, 0)} = 0 \tag{47} \]
Since $a_\lambda(-1; q, t) = \prod_{i=1}^{k+1} (1 - q^{\lambda_i} t^{k+1-i})$, setting $\lambda_{k+1} = 0$ implies that $a_\lambda = 0$. From (16), this lemma is then seen to hold.

Given Lemma 7, showing that
\[ \bar{B}_k^+ = B_k^+ + GM_{k+1}(-1; q, t), \tag{48} \]
when $N = k + 1$, with $G$ a certain expression, will prove (45). The following identity obtained by Garsia and Tesler [4] will be used to prove (48):
\[ \sum_I \prod_{i \in I} \frac{x_i - t x_j}{x_i - x_j} x_I = \sum_I x_I = e_k, \tag{49} \]
with the sum over all $k$-element subsets $I$ of $\{1, \ldots, N\}$. 
In the special case \( N = k + 1 \),
\[
B_k^+ = (1 - q^{-1})^{-1} \sum_{r=0}^{k+1} (-q^{-1})^r M_{k+1}^r e_k.
\]

(50)

Taking the part of this equation involving \( M_{k+1}^r x_1 \ldots x_k \), we have
\[
M_{k+1}^r x_1 \ldots x_k = q^{-1} x_1 \ldots x_k M_{k+1}^r + (q^r - q^{-1}) x_1 \ldots x_k \sum_{I \subset \{1, \ldots, k\}} A_I(x; t) \prod_{i \in I} T_{q,x_i},
\]

(51)

remembering that in \( A_I(x; t) \) the total set of variables is \( \{1, \ldots, k+1\} \). The second term in the R.H.S. of (51) is
\[
x_1 \ldots x_k \sum_{I \subset \{1, \ldots, k\}} A_I(x; t) \prod_{i \in I} T_{q,x_i} =
\]
\[
\sum_{I \subset \{1, \ldots, k\}; |I| = r} x_I t^{r(r-1)/2} \prod_{i \in I} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} t_{q,x_i}.
\]

(52)

With \( I \) fixed, knowing that \( M_{k+1}^r e_k \) we have to sum over all \( I' \subset \{1, \ldots, k+1\} \setminus I \) with \( |I'| = k - r \) of which \( I' \) is a special case, and using the following special case of the Garsia-Tesler formula:
\[
\sum_{I' \subset \{1, \ldots, k+1\} \setminus I; |I'| = k - r} x_{I'} = \sum_{I' \subset \{1, \ldots, k+1\} \setminus I} \prod_{i \in I' \setminus I'} \frac{x_i - t^{-1} x_j}{x_i - x_j} \prod_{i \in I'} t_{q,x_i},
\]

(53)

we see that we can replace \( x_{I'} \) by \( t^{r-k} \prod_{i \in I'} \frac{tx_i - x_{k+1}}{x_i - x_{k+1}} x_{I'} \) in (52). Equation (52) thus becomes
\[
x_1 \ldots x_k \sum_{I \subset \{1, \ldots, k\}; |I| = r} A_I(x; t) \prod_{i \in I} T_{q,x_i} =
\]
\[
t^{r-k} \prod_{i \in \{1, \ldots, k\}} \frac{tx_i - x_{k+1}}{x_i - x_{k+1}} x_1 \ldots x_k \sum_{I \subset \{1, \ldots, k\}; |I| = r} \prod_{j \in I' \setminus I} t_{q,x_i} \prod_{i \in I} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} t_{q,x_i}.
\]

(54)

Hence, from (51) and (54), the part of \( B_k^+ \) in (50) involving \( x_1 \ldots x_k \) is
\[
q^{-1} (1 - q^{-1})^{-1} x_1 \ldots x_k M_{k+1}^r(-1; q, t) + x_1 \ldots x_k \tilde{A}_{\{1, \ldots, k\}}(x; t) M_{\{1, \ldots, k\}}(-t; q, t),
\]

(55)

which by symmetry implies (48).

We see that \( B_k^+ \) and \( B_k^+ \) coincide only on the set of Macdonald polynomials with \( \ell(\lambda) \leq k \). The action of \( B_k^+ \) on an arbitrary Macdonald polynomial will be given below in the form of a conjecture. It is much simpler than that of \( B_k^+ \). The operators \( B_k^+ \) further have a number of remarkable properties that the operators \( B_k^+ \) do not share.
5. Properties of the operators $\tilde{B}_k^+$

We extend the definition of a partition to allow real entries:

$$F_{(\beta_1, \ldots, \beta_N)} = e^{\beta_N} F_{(\beta_1-\beta_N, \ldots, \beta_{N-1}-\beta_N, 0)} = e^{\beta_N} F_{\beta-\beta_N}$$  \hspace{1cm} (56)

$\forall \beta_N \in \mathbb{R}$ and $\beta_i - \beta_{i+1}$ an integer $\geq 0$, $i = 1, \ldots, N - 1$. Take the operator $F(\kappa)$ that acts as follows on $P_\beta$:

$$F(\kappa)P_{(\beta_1, \ldots, \beta_N)} = \prod_{i=1}^{N} (t^{\kappa+1-i}q^{\beta_i-1}; q^{-1})_\infty P_{(\beta_1, \ldots, \beta_N)},$$  \hspace{1cm} (57)

From this definition and upon defining $q = t^\alpha$, it follows that

$$F(\kappa)e_N^k = e_N^k F(\kappa + \alpha \rho).$$  \hspace{1cm} (58)

We now form the operators

$$F_{m,\kappa} = F(\kappa)e_m F(\kappa)^{-1},$$  \hspace{1cm} (59)

to see that these have on $P_\beta$ actions that only involve a finite number of products. These are given by

$$F_{m,\kappa} P_\beta = \sum_\delta \Psi_{\delta/\beta} F_{\delta/\beta}(\kappa) P_\delta$$  \hspace{1cm} (60)

with $\delta - \beta$ $m$-vertical strips,

$$F_{\delta/\beta}(\kappa) = \prod_{s \in \delta - \beta_N \sm \beta - \beta_N} F_{\delta - \beta_N}(s; \kappa + \alpha \beta_N),$$  \hspace{1cm} (61)

and where for partitions $\mu$ made of nonnegative integers,

$$F_{\mu}(s; \kappa) = (1 - t^{\kappa - \ell(s)} q^{\delta(s)})^\mu, \forall s \in \mu.$$  \hspace{1cm} (62)

Note that by the same argument as in (21), $\Psi_{\delta/\beta} = \Psi_{\delta - \beta_N / \beta - \beta_N}$. Hence, the action of $F_{m,\kappa}$ on the Macdonald polynomials is very similar to that of $e_m$ on these functions: the action of $F_{m,\kappa}$ differs from that of $e_m$ by the presence of $m$ additional factors in front of the coefficients $\Psi_{\delta/\beta}$. As an example, taking $N = 4$ and $m = 2$,

$$F_{2,\kappa} P_{(1,1,-1,-1)} = \Psi_{(3,3)/(2,2)}(1 - t^{\kappa}q)(1 - t^{\kappa-1}q) P_{(2,2,-1,-1)} + \Psi_{(3,2,1)/(2,2)}(1 - t^{\kappa}q)(1 - t^{\kappa-2}q^{-1}) P_{(2,1,0,-1)} + \Psi_{(2,2,1,1)/(2,2)}(1 - t^{\kappa-2}q^{-1})(1 - t^{\kappa-3}q^{-1}) P_{(1,1,0,0)}$$  \hspace{1cm} (63)

Remarkably it seems that the creation operators $\tilde{B}_k^+$ can be identified with a subset of the operators $F_{m,\kappa}$. Indeed the following conjecture has been arrived at with the aid of the computer.

**Conjecture 8.** The creation operators $\tilde{B}_k^+$ can be written in the form

$$\tilde{B}_k^+ = F(\kappa)e_k F(\kappa)^{-1} = F_{k,k}$$  \hspace{1cm} (64)

This expression immediately provides, through (60), the action of $\tilde{B}_k^+$ on arbitrary Macdonald polynomials. Conjecture 4 must then be a consequence of it. To convince oneself that formula (64) indeed implies Conjecture 4, one uses the same kind of argument as in the proof of Lemma 2. Evaluating the action of $F_{k,k}$ on $P_\lambda$ with the help of (60), one thus shows that all the terms associated to partitions with more than $k$ parts are annihilated. In the framework of this conjecture, the Hermitian conjugate $\tilde{B}_k^-$ of $\tilde{B}_k^+$ with respect to the scalar product defined in (22)
are readily obtained from the fact that $F(\kappa)$ is Hermitian under this scalar product. We thus have

$$F^\dagger_{m,\kappa} = F(\kappa)^{-1} e^\dagger_m F(\kappa)$$

(65)

with $e^\dagger_m = e_N^{-1} e_{N-m}$, which implies that

$$\tilde{B}_k^- = (\tilde{B}_k^\dagger)^\dagger = F(k)^{-1} e^\dagger_k F(k).$$

(66)

It is striking that the set of operators $F_{m,\kappa,\gamma}$ contains a one-parameter family of $N$-dimensional Abelian algebras. From (58), all $F_{m,\kappa}$ can be generated from $F_{m,0}$ by conjugating with powers of $e_N$. More precisely, one has

$$F_{m,\kappa} = e^{-\kappa/\alpha} N F_{m,0} e^\kappa/\alpha.$$  

(67)

With

$$[F_{m,\kappa}, F_{n,\kappa}] = F(\kappa) [e_m, e_n] F(\kappa)^{-1} = 0,$$  

(68)

this immediately shows that we can construct, by proper conjugation of $\tilde{B}_k^+$ with $e_N$, the following set of commuting operators:

$$\{F_{m,\kappa} = e^{(m-\kappa)/\alpha} B_m e^{(\kappa-m)/\alpha}, m = 1, \ldots, N; \kappa \in \mathbb{R}\}$$

(69)

and, for each value of $\kappa$, thus obtain a $N$-dimensional Abelian algebra.

A straightforward extension of Lemma 6 is

$$M_1(X; q, t) e^\rho_N = e_N^\rho M_1(X q^\rho; q, t),$$

(70)

which can in particular be seen from the fact that $M_1 e_{\rho_N} = (q^\rho) e_N M_1$. This last result finally provides the following realization of the operators $F_{m,\kappa}$:

$$F_{m,\kappa} = \sum_l \tilde{A}_l(x; t) x_l M_1(-t^{\kappa-m+1}; q, t)$$

(71)

assuming that Conjecture 8 is valid.

6. JACK POLYNOMIALS

The monic Jack polynomials $P_\lambda(x; \alpha)$ are obtained from the monic Macdonald polynomials $P_\lambda(x; q, t)$ in the limit

$$q = t^\alpha, \quad t \to 1.$$  

(72)

Let $\mathbb{Q}(\alpha)$ denote the field of rational functions of $\alpha$. Taking the above limit in (7) yields the following scalar product on $\Lambda_N \otimes \mathbb{Q}(\alpha)$:

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu} z_{\lambda}^{\ell(\lambda)},$$

(73)

with $\ell(\lambda)$, the number of parts of $\lambda$. The Jack polynomials $J_\lambda(x; \alpha) \in \Lambda_N \otimes \mathbb{Q}(\alpha)$ are given by

$$J_\lambda(x; \alpha) = \lim_{q \to 1} \frac{J_\lambda(x; q, t)}{(1-t)^{\lambda}},$$

(74)

with $|\lambda| = \lambda_1 + \lambda_2 + \ldots$. They are thus uniquely specified by

(i) $\langle J_\lambda, J_\mu \rangle_\alpha = 0$, if $\lambda \neq \mu$,

(ii) $J_\lambda = \sum_{\mu \leq \lambda} v_{\lambda \mu}(\alpha) m_{\mu}$,

(iii) $v_{\lambda \mu}(\alpha) = c_{\lambda}(\alpha)$,

(75)

(76)

(77)
where
\[ c_\lambda(\alpha) = \lim_{q \to t} \frac{c_\lambda(q, t)}{(1-t) \lambda} = \prod_{s \in \lambda} (\alpha a(s) + \ell(s) + 1). \quad (78) \]

In the limit (78) the Pieri formula becomes
\[ e_k P_\lambda^{(\alpha)} = \sum_{\mu} \Psi_{\mu/\lambda}^{(\alpha)} P_\mu^{(\alpha)}, \quad (79) \]
with the sum over partition \( \mu \supset \lambda \) (of length \( \leq N \)) such that \( \mu - \lambda \) is a vertical \( k \)-strip. The coefficients \( \Psi_{\mu/\lambda}^{(\alpha)} \) are
\[ \Psi_{\mu/\lambda}^{(\alpha)} = \prod_{s \in C_{\mu/\lambda}} b_\mu^{(\alpha)}(s), \quad (80) \]
where
\[ b_\lambda^{(\alpha)}(s) = \begin{cases} \frac{\alpha a(s) + \ell(s) + 1}{\alpha a(s) + 1 + \ell(s)} & \text{if } s \in \lambda \\ 1 & \text{otherwise} \end{cases}. \quad (81) \]

We shall now recall the definition of the creation operators entering in the Rodrigues formula for the Jack polynomials that we derived in [3] and shall establish their connection with the operators \( B^+_k \) in the limit (72). The creation operators in this case are constructed from the Dunkl operators
\[ D_i = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{j=1, j \neq i}^N \frac{x_i - x_j}{x_i - x_j} (1 - K_{ij}), \quad i = 1, 2, \ldots, N, \quad (82) \]
where \( K_{ij} = K_{ji} \) is the operator that permutes the variables \( x_i \) and \( x_j \): 
\[ K_{ij} x_i = x_j K_{ij}, \quad K_{ij} D_i = D_j K_{ij}, \quad K_{ij}^2 = 1. \quad (83) \]
Let \( J = \{j_1, j_2, \ldots, j_\ell\} \) be sets of cardinality \( |J| = \ell \) made of integers \( j_\kappa \in \{1, \ldots, N\}, 1 \leq \kappa \leq \ell \) such that \( j_1 < j_2 < \cdots < j_\ell \) and introduce the operators
\[ D_{j,\omega} = (D_{j_1} + \omega)(D_{j_2} + \omega + 1) \cdots (D_{j_\ell} + \omega + \ell - 1), \quad (84) \]
labelled by such sets and a real number \( \omega \). The creation operators \( \tilde{B}^+_k^{(\alpha)} \) are defined by
\[ \tilde{B}^+_k^{(\alpha)} = \sum_{J \subset \{1, \ldots, N\}} x_J D_{J,1}. \quad (85) \]

From the following theorem proved in [3], we see that the operators \( \tilde{B}^+_k^{(\alpha)} \) construct the Jack polynomials.

**Theorem 9.** The Jack polynomials \( J_\lambda(x; \alpha) \) associated to the partitions \( \lambda = (\lambda_1, \ldots, \lambda_N) \) are given by
\[ J_\lambda(x; \alpha) = (\tilde{B}_N^{(\alpha)})^{\lambda_N} (\tilde{B}_{N-1}^{(\alpha)})^{\lambda_{N-1} - \lambda_N} \cdots (\tilde{B}_1^{(\alpha)})^{\lambda_1 - \lambda_2} \cdot 1. \quad (86) \]
We shall now show that the operators $\tilde{B}_k^{+(\alpha)}$ entering in this formula are related to the $\tilde{B}_k^+$ of Conjecture 4 and hence that the conjectures of section 4 on the $\tilde{B}_k^+$ also apply for the operators $\tilde{B}_k^{+(\alpha)}$.

**Lemma 10.** The $\tilde{B}_k^{+(\alpha)}$ are the following limits of the $\tilde{B}_k^+$

$$\operatorname{Res} \tilde{B}_k^{+(\alpha)} = \lim_{q \to t^\alpha} \frac{\tilde{B}_k^+}{(1-t)^k}.$$  

Here $\operatorname{Res} X$ means that $X$ is restricted to symmetric functions of the variables $x_1, \ldots, x_N$. It has been shown in [3] that $\operatorname{Res} D_{J,\omega}$ is symmetric under the permutations of the variables $x_i, i \in J$ and depends only upon these variables. By an argument such as the one given in section 4.4 of [2], we have that

$$\left(\operatorname{Res} D_{J,\omega}\right) J_\lambda(x(J); \alpha) = \prod_{i=1}^\ell (\alpha \lambda_i + \omega - i) J_\lambda(x(J); \alpha),$$

with $x(J) = \{x_i | i \in J\}$. Moreover,

$$\lim_{q \to t^\alpha} \left( M_{\mu}(\omega^e; q, t) J_\lambda(x(J); q, t) \right) = \left( \lim_{q \to t^\alpha} \prod_{i=1}^\ell \frac{(1 - q^{1-\omega + \ell - i})}{(1-t)} \right) J_\lambda(x(J); \alpha)$$

$$= \prod_{i=1}^\ell (\alpha \lambda_i + \omega - i) J_\lambda(x(J); \alpha)$$

$$= \left(\operatorname{Res} D_{J,\omega}\right) J_\lambda(x(J); \alpha).$$

Since $J_\lambda(x(J); \alpha)$ form a basis for the symmetric polynomials in the variables $x_i, i \in J$, we have that

$$\lim_{q \to t^\alpha} \frac{M_{\mu}(\omega^e; q, t)}{(1-t)^\ell} = \operatorname{Res} D_{J,\omega},$$

knowing that differential operators that coincide on a complete set of symmetric functions are identical (see for instance appendix C of [3]). We thus have

$$\lim_{q \to t^\alpha} \frac{\tilde{B}_k^+}{(1-t)^k} = \sum_{|J|=k} x_J \operatorname{Res} D_{J,1} = \operatorname{Res} \sum_{|J|=k} x_J D_{J,1},$$

which proves Lemma 10.

In light of Theorem 1 and (74), we can write in addition to (86) another Rodrigues formula for the Jack polynomials using $\lim_{q \to t^\alpha} \frac{\tilde{B}_k^+}{(1-t)^k}$ as creation operators. With the help of (90), one gets

$$B_k^{+(\alpha)} = \lim_{q \to t^\alpha} B_k^+/ (1 - t)^k = \lim_{q \to t^\alpha} \prod_{k=1}^N \left( \frac{1-t}{1-t^{k+1-jq^{-1}}} \right) M_{N,-t^{k+1-N},q^{-1}; q, t} e_k$$

$$= \prod_{j=k+1}^N (-\alpha + k + 1 - j)^{-1} D_{(1, \ldots, N), k+1-N-\alpha} e_k.$$
These operators $B_k^{+(\alpha)}$ are also such that

$$J_\lambda(x; \alpha) = (B_N^{+(\alpha)})^\lambda (B_{N-1}^{+(\alpha)})^{\lambda_{N-1} - \lambda_N} \ldots (B_1^{+(\alpha)})^{\lambda_1 - \lambda_2} \cdot 1,$$  \hspace{1cm} (93)

for any partition $\lambda$.

As in the case of the Macdonald polynomials, let us extend the definition of a partition to allow real entries:

$$F^{(\alpha)}(\beta_1, \ldots, \beta_N) = e_N^{\beta_N} F^{(\alpha)}(\beta_1 - \beta_N, \ldots, \beta_{N-1} - \beta_N, 0) = e_N^{\beta_N} F^{(\alpha)}_{\beta - \beta_N}$$

\hspace{1cm} (94)

$\forall \beta_N \in \mathbb{R}$ and $\beta_i - \beta_{i+1}$ a non-negative integer, $i = 1, \ldots, N - 1$. Take the operator $F^{(\alpha)}(\kappa)$ that acts as follows on $P^{(\alpha)}_{\beta}$:

$$F^{(\alpha)}(\kappa) P^{(\alpha)}_{\beta} = \prod_{i=1}^{N} \prod_{j=1}^{\infty} (\alpha(\beta_i - j) + \kappa + 1 - i) P^{(\alpha)}_{\beta_i, \ldots, \beta_N},$$ \hspace{1cm} (95)

Again, it follows that

$$F^{(\alpha)}(\kappa) e_N^\beta = e_N^\beta F^{(\alpha)}(\kappa + \alpha \rho).$$ \hspace{1cm} (96)

We now form the operators

$$F^{(\alpha)}_{m,\kappa} = F^{(\alpha)}(\kappa) e_m F^{(\alpha)}(\kappa)^{-1},$$ \hspace{1cm} (97)

to see that these have on $P^{(\alpha)}_{\beta}$ actions that only involve a finite number of products. These actions read

$$F^{(\alpha)}_{m,\kappa} P^{(\alpha)}_{\beta} = \sum_{\delta} \Psi^{(\alpha)}_{\delta/\beta} F^{(\alpha)}_{\delta/\beta}(\kappa) P^{(\alpha)}_{\delta},$$ \hspace{1cm} (98)

with $\delta - \beta$ $m$-vertical strips,

$$F^{(\alpha)}_{\delta/\beta}(\kappa) = \prod_{s \in \delta - \beta_N} F^{(\alpha)}_{\delta - \beta_N}(s; \kappa + \alpha \beta_N),$$ \hspace{1cm} (99)

and where for partitions $\mu$ made of nonnegative integers,

$$F^{(\alpha)}_{\mu}(s; \kappa) = (\alpha(a'(s)) + \kappa - \ell'(s)), \quad \forall s \in \mu.$$ \hspace{1cm} (100)

Again, $\Psi^{(\alpha)}_{\delta/\beta} = \Psi^{(\alpha)}_{\delta - \beta_N/\beta - \beta_N}$. That

$$F^{(\alpha)}_{m,\kappa} = \lim_{q = t^\alpha} \frac{F^{(\alpha)}_{m,\kappa}}{(1 - t)^m},$$ \hspace{1cm} (101)

is established from the fact that

$$\lim_{q = t^\alpha} F^{(\alpha)}_{\delta/\beta}(\kappa) = F^{(\alpha)}_{\delta/\beta}(\kappa),$$ \hspace{1cm} (102)

with $\delta - \beta$ a vertical $m$-strip. It then follows that Conjecture 8 implies, if true, that the operators $\tilde{B}_k^{+(\alpha)}$ and $\tilde{B}_k^{+\alpha}$ share many features.

**Conjecture 11.** The creation operators $\tilde{B}_k^{+(\alpha)}$ share these properties:

(i) $\tilde{B}_k^{+(\alpha)} = F^{(\alpha)}(k) e_k F^{(\alpha)}(k)^{-1} = F^{(\alpha)}_{k,k}$

(ii) $F^{(\alpha)}_{m,\kappa} = \sum_{|J|=m} x_J D_J e_{m-1}^{(\kappa-\alpha)/\alpha} e_{N}\tilde{B}_m^{+(\alpha)} e_{N}^{(\kappa-\alpha)/\alpha}$

(iii) $[F^{(\alpha)}_{m,\kappa}, F^{(\alpha)}_{n,\kappa}] = 0$. \hspace{1cm} (103)
In view of (98), we thus have a conjecture for the action of $\tilde{B}_k^{+\alpha}$ on arbitrary Jack polynomials.

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