Emergence of Space-Time on the Planck Scale within the Scheme of Dynamical Cellular Networks and Random Graphs

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1 Introduction

The following is a very sketchy report about a research program, the first steps of which have been accomplished in [1] and [2]. Starting from the hypothesis that both the physics and the mathematics of the Planck scale should be formed from intrinsically discrete concepts, we develop a framework which is based on the dynamical evolution of a class of 'cellular network models', being capable (as we hope) of performing an 'unfolding phase transition' from a presumed chaotic initial phase into a new phase which acts as an attractor in total phase space. This new phase is assumed to carry a fine or 'super structure' (kind of an 'order parameter manifold') which is then identified as the discrete substratum underlying our ordinary continuous space-time (or rather: the physical vacuum).

The building blocks of this emerging super structure are a particular type of densely entangled subclusters of nodes/bonds of the underlying network, which are conjectured to correspond to the 'physical points' of the space-time continuum living on a much coarser scale; thus reflecting the presumed internal complexity of 'space-time infinitesimals'. Furthermore, these subclusters establish a certain near- and far-order in the network, viz. a certain 'causal structure', which is then again lifted to the ordinary space-time continuum.

As a byproduct of our investigation we develop an arsenal of mathematical tools (typically of a distinctly discrete flavor) which, we think, will have a scientific value of their own well beyond the field of Planck scale physics, as they are designed to analyze complex, self-organizing and unfolding systems in general. On the other side, they may help to create something like discrete analysis, topology, geometry, dimension theory etc.

For further details concerning the various facets of our approach, most notably the physical motivation and the physical/mathematical sources, we refer the reader, due to lack of space, to the papers [1] and [2], the only exception being the extra mentioning of the beautiful book of Bollobas ( [3]) about the theory of 'random graphs', the results
and concepts of which will play an important role in the future analysis of the systems we are having in mind. On the following pages we will mainly review the content of paper [] which is primarily concerned with the analysis of the 'unfolding phase transition' in our cellular network models, leading in the end to the emergence of a kind of (proto) space-time, while in [] we were dealing in some detail with the development of a version of discrete analysis and dimension theory and the like.

2 The Dynamical Cellular Network Universe $QX$

We now introduce one example of a cellular network law from a whole class of possible laws (for more details see e.g. []). The basic constituents are its set of 'nodes', $n_i$, and 'bonds', $b_{ik}$, connecting the nodes $n_i$ and $n_k$ with $i \neq k$; the labels $i, k$ running through some countable index set. At each node $n_i$ sits an internal state $s_i$ taking its values (in our particular example!) in a certain subset of the integers (modulo a possible 'unit charge', say $q$, see below). The direct interaction among the cells (represented by the nodes) is mediated by certain bond states $J_{ik}$. In our example $J_{ik} \in \{+1, 0, -1\}$, where $J_{ik} = 0$ means that the bond is dead or inactive at the time (step) under discussion.

A 'local law' is now introduced as follows: We assume that all the nodes/bonds at '(clock) time' $t + \tau$, $\tau$ an elementary clock time step, are updated according to a certain local rule which relates for each given node $n_i$ and bond $b_{ik}$ their respective states at time $t + \tau$ with the states of the nodes/bonds of a certain fixed local neighborhood at time $t$.

Note: It is important that, generically, such a law does not lead to a reversible time evolution, i.e. there will typically exist attractors in total phase space (the overall configuration space of the node and bond states).

Remark: A crucial ingredient of our network laws is what we would like to call a 'hysteresis interval'. We will assume that our network starts from a densely entangled 'initial phase' $QX_0$, in which virtually every pair of nodes is on average connected by an 'active' bond, i.e. $J_{ik} = \pm 1$. Our dynamical law will have a built-in mechanism which switches bonds off (more properly: $J_{ik} = 0$) if local fluctuations among the node states become too large. There is then the hope that this mechanism may trigger an 'unfolding phase transition', starting from a local seed of spontaneous large fluctuations towards a new phase (an attractor) carrying a certain 'super structure', which we would like to relate to the hidden discrete substratum of space-time (points).

One example of such a peculiar law is the following one:

Local Law: At each clock time step a certain 'quantum' $q$ is transported between, say, the nodes $n_i, n_k$ so that

$$s_i(t + \tau) - s_i(t) = q \cdot \sum_k J_{ki}(t) \quad (1)$$
(i.e. if $J_{ki} = +1$ a quantum flows from $n_k$ to $n_i$ etc.)

The second part of the law describes the back reaction on the bonds (and is, typically, more subtle). This is the place where the so-called 'hysteresis interval' enters the stage. We assume two 'critical parameters' $0 \leq \lambda_1 \leq \lambda_2$ to exist with:

$$J_{ik}(t+\tau) = 0 \quad \text{if} \quad |s_i(t) - s_k(t)| =: |s_{ik}(t)| > \lambda_2$$

$$J_{ik}(t+\tau) = \pm 1 \quad \text{if} \quad 0 < \pm s_{ik}(t) < \lambda_1$$

with the special proviso that

$$J_{ik}(t+\tau) = J_{ik}(t) \quad \text{if} \quad s_{ik}(t) = 0$$

On the other side

$$J_{ik}(t+\tau) = \pm 1 \quad \text{or} \quad 0 \quad \text{if} \quad \lambda_1 \leq \pm s_{ik}(t) \leq \lambda_2$$

provided that $J_{ik}(t) \neq 0$ or $J_{ik}(t) = 0$.

In other words, bonds are switched off if local spacial charge fluctuations are too large and switched on again if they are too small, their orientation following the sign of local charge differences in between or remain inactive.

**Observation:** The above law fulfills two, in our view crucial, properties: it consists of two parts, the one describing the effect of 'geometry' on 'matter', the other the back effect of 'matter' on 'geometry' (in an up to now rather metaphorical sense). It therefore may be viewed as a kind of 'proto gauge theory'.

**Remarks:**

i) The above dynamical law can or has to be supplemented by appropriate boundary conditions.

ii) An important ingredient is the above 'hysteresis law'. One can speculate that, under appropriate conditions, it may be able to catalyze an 'unfolding phase transition' in the course of which a substantial fraction of bonds becomes dead for a larger lapse of time, thus inducing topological/geometrical changes in the primordial network $QX$.

iii) Interrupting connections between nodes if local charge differences become too large enhances the chance of 'pattern formation' as the return to equilibrium is impeded.

### 3 The Random Graph Aspect of the Network Dynamics

We have to be very brief on this important and beautiful topic (for more details see []). As in the study of chaotic systems, where one frequently goes over to a slightly simpler picture by employing the so-called Poincaré map, it may be advantageous to concentrate on the more "graphical" aspects of the underlying network dynamics,
viz. its topological/geometrical changes, in other words: its wiring diagram.

(It goes without saying that, as in statistical mechanics, it would be a fruitful idea to consider the network as a statistical system and employ statistical concepts like 'entropy', 'correlation functions' etc., a route from which we have to refrain due to lack of space; see however []).

At each time step the underlying geometric structure may be considered as a 'graph', \( G(t) \), with 'node set' \( V \) and 'bond set' \( E \), \( |V|, |E| \) their respective cardinalities. This picture becomes a dynamical one if we omit all the bonds \( b_{ik} \) in \( G(t) \) with \( J_{ik}(t) = 0 \) and draw a bond \( b_{ik} \) if \( J_{ik}(t) \neq 0 \). Our hypothesis is now that instead of following the full, complicated time evolution of the deterministically evolving network state, it is sensible to concentrate rather on the behavior of the corresponding time dependent graph \( G(t) \), i.e. to focus attention on its wiring diagram and various typical 'graph observables' as e.g. '(average) degree of a node', 'degree of connectedness', structure and number of certain 'subgraphs', their mutual entanglement, viz. the 'geometrodynamics' of the graph/network.

A, as we think, particularly beautiful tool (which was however designed for an entirely different purpose in pure mathematics) is the 'random graph' concept (see e.g. []). As is the case with the statistical method in general, this concept should be applied to our network scenario with some care, since systems like ours are possibly behaving only in a 'quasirandom' manner, as their time evolution follows a given deterministic law (which, however, may be sufficiently 'ergodic' or 'mixing').

Our probabilistic model is now the following: With \( n \) fixed vertices given we make the set of possible graphs over the \( n \)-set \( V \) a 'probability space' by attributing each possible bond a probability \( 0 \leq p \leq 1 \). In other words: a given graph \( G_m \) over \( V \) with \( m \) bonds (an 'elementary event') has the probability

\[
pr(G_m) = p^m \cdot q^{N-m} \quad \text{with} \quad N = \binom{n}{2}, \quad q = 1 - p
\]  

\( N \) being the maximal possible number of bonds (simplex over \( V \)). On this probability space one can now calculate all sorts of 'graph properties' by introducing and employing the corresponding 'random functions'.

It was a fundamental observation of Erdős and Renyi (for a proof of this deep result see e.g. []) that many graph properties have a so-called 'threshold function' which is very reminiscent to 'phase transitions' in statistical mechanics. To put it sloppily:

**Observation (threshold functions):** A large class of graph properties have a threshold function \( m(n) \), \( m \) the number of edges, \( n \) the number of vertices, s.t. for \( n \to \infty \) the graph under discussion has the property \( Q \) almost surely for \( m > m(n) \) and almost surely not for \( m < m(n) \).
4 The Unfolding Phase Transition

With the help of the random graph concept, briefly described in section 3, we now want to sketch the scenario which, we hope, will lead to the emergence of a (proto)space-time structure (where, to be precise, nothing is said at the moment about the nature of "classical" time, which is postponed to forthcoming work. We rather concentrate on the emergence of the spacial aspect of the super structure under discussion).

We assume that the initial phase $QX_0$ behaves in some sense metastable (with respect to the local law introduced above) and that a sufficiently pronounced spontaneous fluctuation, leading to an avalanche of switched-off bonds, may drive the network $QX$ towards another phase the nature of which we are going to describe now (for more details see []).

Conjecture:
i) Given that sufficiently many bonds are dead, a new superstructure may emerge in $QX$, the building blocks of which are conjectured to be (almost) maximal subsimplices $\subset QX$ or $G(t)$, called $mss$, i.e. maximal subclusters of nodes with (almost) every pair of nodes connected by an active bond.

ii) We want to relate these $mss$ with what we are used to call 'physical points' in ordinary continuum physics, thus endowing them with a rich internal fine structure (which is largely hidden on the continuum level).

iii) The mutual entanglement of these $mss$, i.e. their fabric, is assumed to underlie the causal structure we experience in continuum space-time.

The above $mss$ can be constructively generated for each given graph $G(t)$. Starting from an arbitrary node $n_i$ one selects a node $n_j$ being connected with $n_i$, then a node $n_k$ being connected with both $n_i$ and $n_j$ and so forth until the process terminates, i.e. until there is no further node being connected with all(!) the preceding ones. Note however that at each intermediate step one can possibly make different choices, leading to different $mss$ or not, which, on the other side, will more or less overlap. This makes up the fabric of the $mss$ and leads to a new type of 'hyper graph'. With the help of the random graph approach one is now able to analyze this complicated fabric. One is e.g. interested in the typical size of the $mss$, their cardinality, their mutual overlap, number of non-overlapping $mss$ in $G(t)$ etc.

References

[1] M.Requardt: hep-th/9605103

[2] M.Requardt: hep-th/9610055

[3] B.Bollobas: "Random graphs", Academic Pr., London 1985

[4] B. Bollobas; "Combinatorics", Cambridge Univ.Pr., Cambridge 1994