Derivations from the even parts into the odd parts for Lie superalgebras $W$ and $S$

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Abstract: Let $W$ and $S$ denote the even parts of the general Witt superalgebra $W$ and the special superalgebra $S$ over a field of characteristic $p > 3$, respectively. In this note, using the method of reduction on $\mathbb{Z}$-gradations, we determine the derivation space $\text{Der}(W, W_1)$ from $W$ into $W_1$ and the derivation space $\text{Der}(S, W_1)$ from $S$ into $W_1$. In particular, the derivation space $\text{Der}(S, S_1)$ is determined.

Keywords: General Witt superalgebra; special superalgebra; derivation space

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1 Introduction

The underlying field $\mathbb{F}$ is assumed of characteristic $p > 3$ throughout. We shall study the derivations from the even parts of the generalized Witt superalgebra $W$ and the special superalgebra $S$ into the odd part of $W$, where $W_\tau$ is viewed as modules for $W_\sigma$ and $S_\sigma$ by means of the adjoint representation. The motivation came from the following observation.

Let $L = L_\sigma \oplus L_\tau$ be a Lie superalgebra. Then $L_\sigma$ is a Lie algebra and $L_\tau$ is an $L_\sigma$ module. Two questions arise naturally: Does the derivation algebra of the even part of $L$ coincide with the even part of the superderivation algebra of $L$? Does the derivation space from $L_\sigma$ into $L_\tau$ coincide with the odd part of the superderivation algebra of $L$? For the generalized Witt superalgebra and the special superalgebra the first question was answered affirmatively in [2]. In this note, the second question will also be answered for these two Lie superalgebras of Cartan type. Speaking accurately, we shall determine the derivation spaces from the even parts of the generalized Witt superalgebra $W$ and the special superalgebra $S$ into the odd part of $W$. As a direct consequence, the derivation space from the even part into the odd part for the special superalgebra is determined.

The author would like to thank the anonymous referee for the paper [1] for posing such an interesting question.

In this note we adopt the notation and concepts used in [2], but here, for convenience and completeness, we repeat certain necessary symbols and notions.
Let $\mathbb{Z}_2 = \{\mathbb{0}, \mathbb{1}\}$ be the field of two elements. For a vector superspace $V = V^1_\mathbb{1} \oplus V^0_\mathbb{0}$, we denote by $p(a) = \theta$ the parity of a homogeneous element $a \in V_\theta, \theta \in \mathbb{Z}_2$. We assume throughout that the notation $p(x)$ implies that $x$ is a $\mathbb{Z}_2$-homogeneous element.

Let $\mathfrak{g}$ be a Lie algebra and $V$ a $\mathfrak{g}$-module. A linear mapping $D : \mathfrak{g} \to V$ is called a derivation from $\mathfrak{g}$ into $V$ if $D(xy) = x \cdot D(y) - y \cdot D(x)$ for all $x, y \in \mathfrak{g}$. A derivation $D : \mathfrak{g} \to V$ is called inner if there is $v \in V$ such that $D(x) = x \cdot v$ for all $x \in \mathfrak{g}$. Following [5], p. 13, denote by $\text{Der}(\mathfrak{g}, V)$ the derivation space from $\mathfrak{g}$ into $V$. Then $\text{Der}(\mathfrak{g}, V)$ is a $\mathfrak{g}$-submodule of $\text{Hom}_F(\mathfrak{g}, V)$. Assume in addition that $\mathfrak{g}$ and $V$ are finite-dimensional and that $\mathfrak{g} = \bigoplus_{r \in \mathbb{Z}} \mathfrak{g}_r$ is $\mathbb{Z}$-graded and $V = \bigoplus_{r \in \mathbb{Z}} V_r$ is a $\mathbb{Z}$-graded $\mathfrak{g}$-module. Then $\text{Der}(\mathfrak{g}, V) = \bigoplus_{r \in \mathbb{Z}} \text{Der}_r(\mathfrak{g}, V)$ is a $\mathbb{Z}$-graded $\mathfrak{g}$-module by setting

$$\text{Der}_r(\mathfrak{g}, V) := \{D \in \text{Der}(\mathfrak{g}, V) \mid D(\mathfrak{g}_i) \subset V_{r+i} \text{ for all } i \in \mathbb{Z}\}.$$ 

In the case of $V = \mathfrak{g}$, the derivation algebra $\text{Der}(\mathfrak{g})$ coincides with $\text{Der}(\mathfrak{g}, \mathfrak{g})$ and $\text{Der}(\mathfrak{g}) = \bigoplus_{r \in \mathbb{Z}} \text{Der}_r(\mathfrak{g}, \mathfrak{g})$ is a $\mathbb{Z}$-graded Lie algebra. If $\mathfrak{g} = \bigoplus_{-r \leq i \leq s} \mathfrak{g}_i$ is a $\mathbb{Z}$-graded Lie algebra, then $\bigoplus_{-r \leq i \leq s} \text{Der}_i(\mathfrak{g}, \mathfrak{g})$ is called the top of $\mathfrak{g}$ (with respect to the gradation).

In the below we review the notions of modular Lie superalgebras $W$ and $S$ of Cartan-type and their gradation structures. In addition to the standard notation $\mathbb{N}$ for the set of positive integers, and $\mathbb{N}_0$ for the set of nonnegative integers. Henceforth, we will let $m, n$ denote fixed integers in $\mathbb{N} \setminus \{1, 2\}$ without notice. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$, we put $|\alpha| = \sum_{i=1}^{m} \alpha_i$. Let $O(m)$ denote the divided power algebra over $\mathbb{F}$ with an $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in \mathbb{N}_0^m\}$. For $\varepsilon_i = (\delta_{1i}, \ldots, \delta_{mi})$, we abbreviate $x^{(\varepsilon_i)}$ to $x_i, i = 1, \ldots, m$. Let $\Lambda(n)$ be the exterior superalgebra over $\mathbb{F}$ in $n$ variables $x_{m+1}, \ldots, x_{m+n}$. Denote the tensor product by $\Lambda(m, n) = O(m) \otimes_\mathbb{F} \Lambda(n)$. Obviously, $\Lambda(m, n)$ is an associative superalgebra with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $O(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$. Moreover, $\Lambda(m, n)$ is super-commutative. For $g \in O(m), f \in \Lambda(n)$, we write $gf$ for $g \otimes f$. The following formulas hold in $O(m, n)$:

$$x^{(\alpha)} x^{(\beta)} = \left( \frac{\alpha + \beta}{\alpha} \right) x^{(\alpha+\beta)} \quad \text{for } \alpha, \beta \in \mathbb{N}_0^m;$$

$$x_k x_l = -x_l x_k \quad \text{for } k, l = m+1, \ldots, m+n;$$

$$x^{(\alpha)} x_k = x_k x^{(\alpha)} \quad \text{for } \alpha \in \mathbb{N}_0^m, k = m+1, \ldots, m+n,$$

where $\left( \frac{\alpha+\beta}{\alpha} \right) := \prod_{i=1}^{m} \left( \frac{\alpha_i + \beta_i}{\alpha_i} \right)$. Put $Y_0 := \{1, 2, \ldots, m\}, Y_1 := \{m+1, \ldots, m+n\}$ and $Y := Y_0 \cup Y_1$. For convenience, we adopt the notation $r' := r + m$ for $r \in Y_1$. Thus, $Y_1 := \{1', 2', \ldots, n'\}$. Set

$$B_k := \{(i_1, i_2, \ldots, i_k) \mid m+1 \leq i_1 < i_2 < \cdots < i_k \leq m+n\}$$

and $B := B(n) = \bigcup_{k=0}^{n} B_k$, where $B_0 := \emptyset$. For $u = (i_1, i_2, \ldots, i_k) \in B_k$, set $|u| := k, |\emptyset| := 0, x^\emptyset := 1$, and $x^u := x_{i_1} x_{i_2} \ldots x_{i_k}$; we use also $u$ to stand for the set $\{i_1, i_2, \ldots, i_k\}$. Clearly, $\{x^{(\alpha)} x^u \mid \alpha \in \mathbb{N}_0^m, u \in B\}$ constitutes an $\mathbb{F}$-basis of $O(m, n)$. Let $D_1, D_2, \ldots, D_{m+n}$ be the linear transformations of $O(m, n)$ such that

$$D_r(x^{(\alpha)} x^u) = \begin{cases} x^{(\alpha - \varepsilon_r)} x^u, & r \in Y_0, \\ x^{(\alpha)} \cdot \partial x^u / \partial x_r, & r \in Y_1. \end{cases}$$
Then $D_1, D_2, \ldots, D_{m+n}$ are superderivations of the superalgebra $\mathcal{O}(m,n)$. Let
\[
W (m,n) = \left\{ \sum_{r \in Y} f_r D_r \mid f_r \in \mathcal{O}(m,n), r \in Y \right\}.
\]
Then $W (m,n)$ is a Lie superalgebra, which is contained in $\text{Der}(\mathcal{O}(m,n))$. Obviously, $p(D_i) = \tau(i)$, where
\[
\tau(i) := \begin{cases} 0, & i \in Y_0 \\ 1, & i \in Y_1. \end{cases}
\]
One may verify that
\[
[f D, g E] = f D(g) E - (-1)^{p(f) p(g)} g E(f) D + (-1)^{p(f) p(g)} f g[D, E]
\]
for $f, g \in \mathcal{O}(m,n)$, $D, E \in \text{Der} \mathcal{O}(m,n)$. Let
\[
\xi := (t_1, t_2, \ldots, t_m) \in \mathbb{N}^m, \quad \pi := (\pi_1, \pi_2, \ldots, \pi_m)
\]
where $\pi_i := p^{i_1} - 1, i \in Y_0$. Let $\mathcal{A} := \mathcal{A}(m; \xi) = \{ \alpha \in \mathbb{N}^m_0 \mid \alpha_i \leq \pi_i, i \in Y_0 \}$. Then
\[
\mathcal{O}(m,n; \xi) := \text{span}_\mathbb{F} \left\{ x^{(\alpha)} x^u \mid \alpha \in \mathcal{A}, u \in \mathbb{B} \right\}
\]
is a finite-dimensional subalgebra of $\mathcal{O}(m,n)$ with a natural $\mathbb{Z}$-gradation $\mathcal{O}(m,n; \xi) = \bigoplus_{r=0}^{\xi} \mathcal{O}(m,n; \xi)_r$ by putting
\[
\mathcal{O}(m,n; \xi)_r := \text{span}_\mathbb{F} \left\{ x^{(\alpha)} x^u \mid |\alpha| + |u| = r \right\}, \quad \xi := |\pi| + n.
\]
Set
\[
W (m,n; \xi) := \left\{ \sum_{r \in Y} f_r D_r \mid f_r \in \mathcal{O}(m,n; \xi), r \in Y \right\}.
\]
Then $W (m,n; \xi)$ is a finite-dimensional simple Lie superalgebra (see [8]). Obviously, $W(\mathcal{O}(m,n; \xi))$ is a free $\mathcal{O}(m,n; \xi)$-module with $\mathcal{O}(m,n; \xi)$-basis $\{ D_r \mid r \in Y \}$. We note that $W(\mathcal{O}(m,n; \xi))$ possesses a standard $\mathbb{F}$-basis $\{ x^{(\alpha)} x^u D_r \mid \alpha \in \mathcal{A}, u \in \mathbb{B}, r \in Y \}$. Let $r, s \in Y$ and $D_{rs} : \mathcal{O}(m,n; \xi) \rightarrow W(\mathcal{O}(m,n; \xi))$ be the linear mapping such that
\[
D_{rs}(f) = (-1)^{\tau(r) \tau(s)} D_r(f) D_s - (-1)^{\tau(r) + \tau(s) + p(f)} D_s(f) D_r \quad \text{for } f \in \mathcal{O}(m,n; \xi).
\]
Then the following equation holds:
\[
[D_k, D_{rs}(f)] = (-1)^{\tau(k) \tau(r)} D_{rs}(D_k(f)) \quad \text{for } k, r, s \in Y; \ f \in \mathcal{O}(m,n; \xi).
\]
Put
\[
S(m,n; \xi) := \text{span}_\mathbb{F} \{ D_{rs}(f) \mid r, s \in Y; \ f \in \mathcal{O}(m,n; \xi) \}.
\]
Then $S(m,n; \xi)$ is a finite-dimensional simple Lie superalgebra (see [8]). Let $\text{div} : W(\mathcal{O}(m,n; \xi)) \rightarrow \mathcal{O}(m,n; \xi)$ be the divergence such that
\[
\text{div}(\sum_{r \in Y} f_r D_r) = \sum_{r \in Y} (-1)^{\tau(r) p(f_r)} D_r(f_r).
\]
Following [6], put
\[
\overline{S}(m,n; \xi) := \{ D \in W(m,n; \xi) \mid \text{div}(D) = 0 \}.
\]
Then $\overline{S}(m, n; \mathfrak{t})$ is a subalgebra of $W(m, n; \mathfrak{t})$ and $S(m, n; \mathfrak{t})$ is a subalgebra of $\overline{S}(m, n; \mathfrak{t})$. The $\mathbb{Z}$-gradation of $\mathcal{O}(m, n; \mathfrak{t})$ induces naturally a $\mathbb{Z}$-gradation structure of $W(m, n; \mathfrak{t}) = \bigoplus_{i=-1}^{\infty} W(m, n; \mathfrak{t})_i$, where

$$W(m, n; \mathfrak{t})_i := \text{span}_F \{ f D_s \mid s \in Y, \ f \in \mathcal{O}(m, n; \mathfrak{t})_{i+1} \}.$$ 

In addition, $S(m, n; \mathfrak{t})$ and $\overline{S}(m, n; \mathfrak{t})$ are all $\mathbb{Z}$-graded subalgebras of $W(m, n; \mathfrak{t})$. In the following sections, $W(m, n; \mathfrak{t})$, $S(m, n; \mathfrak{t})$, $\overline{S}(m, n; \mathfrak{t})$, and $\mathcal{O}(m, n; \mathfrak{t})$ will be denoted by $W, S, \overline{S}$, and $\mathcal{O}$, respectively. In addition, the the even parts of $W, S$ and $\overline{S}$ will be denoted by $W, S$ and $\overline{S}$, respectively.

2 Generalized Witt superalgebras

View $\mathcal{W}_{\overline{1}}$ as a $W$-module by means of the adjoint representation. In this section, our main purpose is to characterize the derivation space $\text{Der}(W, \mathcal{W}_{\overline{1}})$. Note that the $\mathbb{Z}$-gradation of $W$ induces a $\mathbb{Z}$-gradation of $\mathcal{W} = \bigoplus_{i=0}^{\infty} \mathcal{W}_i$. We know that gradation structures provide a powerful tool for the study of (super)derivation algebras of Lie (super)algebra; in particular, the top of a $\mathbb{Z}$-graded Lie (super)algebra plays a predominant role (c.f. [1, 3, 2]).

Following [3], we call $\mathcal{T} := \text{span}_F \{ \Gamma_i \mid i \in Y \}$ the canonical torus of $W$. In the following, we first reduce every nonnegative $\mathbb{Z}$-degree derivation $\phi$ in $\text{Der}(W, \mathcal{W}_{\overline{1}})$ to be vanishing on $\mathcal{W}_{-1}$: that is, we find an inner derivation $\text{ad} x$ such that $(\phi - \text{ad} x)(\mathcal{W}_{-1}) = 0$, where $x \in \mathcal{W}_{\overline{1}}$. In addition, we reduce the derivations in $\text{Der}(W, \mathcal{W}_{\overline{1}})$ to be vanishing on the canonical torus of $W$. In next step, based on these results, we shall reduce the derivations in $\text{Der}(\mathcal{W}, \mathcal{W}_{\overline{1}})$ to be vanishing on the top $\mathcal{W}_{-1} \oplus W_0$.

Set

$$\mathcal{G} := \text{span}_F \{ x^u D_i \mid i \in Y, u \in \mathbb{B}(n), p(x^u D_i) = \overline{1} \}.$$ 

Note that $\mathcal{G} = C_{\mathcal{W}_{\overline{1}}}(\mathcal{W})$. Then $\mathcal{G}$ is a $\mathbb{Z}$-graded subspace of $\mathcal{W}_{\overline{1}}$.

In the sequel we adopt the following notation. Let $P$ be a proposition. Define $\delta_P := 1$ if $P$ is true and $\delta_P := 0$, otherwise. Put $\Gamma_i := x_i D_i$ for $i \in Y$ and $\Gamma := \sum_{i \in Y} \Gamma_i$, $\Gamma' := \sum_{i \in Y_1} \Gamma_i$, and $\Gamma'' := \sum_{i \in Y_0} \Gamma_i$. We call $\text{ad} \Gamma$ the degree derivation of $W$ (or $W$), and $\text{ad} \Gamma'$ and $\text{ad} \Gamma''$ the semi-degree derivations of $W$ (or $W$). The following simple facts will be frequently used in this note:

$$\text{ad} \Gamma(E) = r E \quad \text{for all } E \in W_r, \ r \in \mathbb{Z};$$

$$\text{ad} \Gamma'(x^{(\alpha)} x^u D_j) = (|u| - \delta_j \in Y_1) x^{(\alpha)} x^u D_j \quad \text{for all } \alpha \in \mathbb{A}(m; \mathfrak{t}), u \in \mathbb{B}(n), j \in Y;$$

$$\text{ad} \Gamma''(x^{(\alpha)} x^u D_j) = (|\alpha| - \delta_j \in Y_0) x^{(\alpha)} x^u D_j \quad \text{for all } \alpha \in \mathbb{A}(m; \mathfrak{t}), u \in \mathbb{B}(n), j \in Y.$$

In particular, each standard $F$-basis element $x^{(\alpha)} x^u D_j$ of $W$ is an eigenvector of the degree derivation and the semi-degree derivation of $W$.

Similar to [3 Lemma 2.1.1], we have the following

**Lemma 2.1.** Suppose that $\mathcal{L}$ is a $\mathbb{Z}$-graded subalgebra of $\mathcal{W}$ and $\mathcal{L}_{-1} = \mathcal{W}_{-1}$. Let $E \in \mathcal{L}$ and $\phi \in \text{Der}(\mathcal{L}, \mathcal{W}_{\overline{1}})$ satisfying $\phi(\mathcal{W}_{-1}) = 0$. Then $\phi(E) \in \mathcal{G}$ if and only if $[E, \mathcal{W}_{-1}] \subseteq \ker \phi$.

Analogous to [5 Proposition 8.2, p. 192], we have the following lemma.
Lemma 2.2. Let $k \leq n$, $f_1, \ldots, f_k \in \Lambda(n)$ be nonzero elements and $\Gamma_{q_i} := x_{q_i} D_{q_i}$, $q_i \in Y_1$, $1 \leq i \leq k$. Suppose that

\begin{enumerate}[(a)]
  \item $\Gamma_{q_i}(f_j) = \Gamma_{q_j}(f_i)$ for $1 \leq i, j \leq k$;
  \item $\Gamma_{q_i}(f_i) = f_i$ for $1 \leq i \leq k$.
\end{enumerate}
Then there is $f \in \Lambda(n)$ such that $\Gamma_{q_i}(f) = f_i$ for $1 \leq i \leq k$.

Analogous to [2, Lemma 2.1.6], we have

Lemma 2.3. Suppose that $L$ is a $\mathbb{Z}$-graded subalgebra of $W$ satisfying $L_{-1} = W_{-1}$. Let $\phi \in \text{Der}(L, W_T)$ with $zd(\phi) = t \geq 0$. Then there is $E \in (W_T)_t$ such that

$$(\phi - \text{ad}E)(L_{-1}) = 0.$$}

In view of Lemma 2.3, every nonnegative $\mathbb{Z}$-degree derivation from $W$ into $W$ may be reduced to be vanishing on $W_{-1}$. Thus, next step is to reduce such derivations to be vanishing on the top $W_{-1} \oplus W_0$. To that end, we first consider the canonical torus of $W$, that is, $T := \text{span}_{\mathbb{F}}\{\Gamma_i \mid i \in Y\}$.

The following lemma will simplify our consideration, which tells us that in order to reduce derivations on the canonical torus it suffices to reduce these derivations on $T' := \text{span}_{\mathbb{F}}\{\Gamma_j \mid j \in Y_1\}$.

Lemma 2.4. Suppose that $\phi \in \text{Der}(T, W T)$ with $t \geq 0$ and $\phi(W_{-1} \cup T') = 0$. Then $\phi(\Gamma_i) = 0$ for all $i \in Y_0$.

Proof. (i) First consider the case $t > 0$. From $\phi(W_{-1}) = 0$ and Lemma 2.1 we have $\phi(\Gamma_i) \in G_t$ for all $i \in Y_0$. Thus one may assume that

$$\phi(\Gamma_i) = \sum_{k \in Y, u \in B_{t+1}} c_{u,k} x^u D_k \quad \text{where } c_{u,k} \in \mathbb{F}. \quad (2.1)$$

For arbitrary $l \in Y$ and $v \in B_{t+1}$, noticing that $t + 1 > 1$, one may find $j \in v \setminus \{l\}$. Clearly,

$$[\Gamma_j, \phi(\Gamma_i)] = \sum_{k \in Y_0, u \in B_{t+1}} c_{u,k} [\Gamma_j, x^u D_k]. \quad (2.2)$$

Note that

$$[\Gamma_j, x^u D_k] = (\delta_{j=0} - \delta_{jk}) x^u D_k; \quad (2.3)$$
in particular,

$$[\Gamma_j, x^u D_l] = x^u D_l. \quad (2.4)$$

On the other hand, since $[\Gamma_i, \Gamma_j] = 0$ and $\phi(\Gamma_j) = 0$, one may easily see that

$$[\Gamma_j, \phi(\Gamma_i)] = 0.$$

From (2.2), (2.4) and the equation above, we obtain that $c_{v,l} = 0$. It follows from (2.1) that $\phi(\Gamma_i) = 0$ for all $i \in Y_0$.

(ii) Let us consider the case $t = 0$. In view of Lemma 2.1 we have $\phi(\Gamma_i) \in G_0$. Obviously, $[\Gamma_j, \Gamma_i] = 0$ for all $j \in Y_1, i \in Y_0$. Therefore,

$$[\Gamma_j, \phi(\Gamma_i)] = 0 \quad \text{for all } j \in Y_1.$$
Note that each standard basis element of $W_\Phi$ is an eigenvectors of $\text{ad}\Gamma_j$ for $j \in Y_1$. It follows from the equation displayed above that
\[
\phi(\Gamma_i) = \sum_{k \in Y_1} c_{i,k} \Gamma_k \quad \text{where } c_{i,k} \in \mathbb{F}.
\] (2.5)

For $j, l \in Y_1$, by Lemma 2.4 one gets $\phi(x_jD_l) \in G_0$. Assume that $[\Gamma_i, \phi(x_jD_l)] = \sum_{k \in Y_1} \lambda_k x_k D_i$. From the equation $[\Gamma_i, x_jD_l] = 0$, we obtain that
\[
[\phi(\Gamma_i), x_jD_l] = -\sum_{k \in Y_1} \lambda_k x_k D_i.
\]

Then it follows from (2.5) that $c_{ij} = c_{tl}$ for $j, l \in Y_1$. Write $c_{ij} := c_i$ for all $j \in Y_1$. Then (2.5) shows that
\[
\phi(\Gamma_i) = c_i \Gamma' \quad \text{for } i \in Y_0.
\]

We want to show that $c_i = 0$ for all $i \in Y_0$. Suppose that we are given $i \in Y_0, j, l \in Y_1$. Clearly, $[x_jx_lD_i, \Gamma_i] = x_jx_lD_i$. Applying $\phi$ to this equation, we have
\[
[\phi(x_jx_lD_i), \Gamma_i] - \phi(x_jx_lD_i) = -[x_jx_lD_i, \phi(\Gamma_i)]
\]
\[
= -[x_jx_lD_i, c_i \Gamma']
\]
\[
= 2c_i x_jx_lD_i.
\]

By Lemma 2.4 it is easily seen that $\phi(x_jx_lD_i) \in G_1$. Thus one may assume that $\phi(x_jx_lD_i) = \sum_{k \in Y, u \in B_2} c_{u,k} x^u D_k$. Note that $[x^u D_k, x^u D_l] = \delta_{ki} x^u D_i$. It follows that
\[
\sum_{k \in Y, u \in B_2} (\delta_{ki} - 1) c_{u,k} x^u D_k = 2c_i x_jx_lD_l.
\]

A comparison of the coefficients of $x_jx_lD_l$ in the equation above yields that $2c_i = 0$ for $i \in Y_0$. Since $\text{char} \mathbb{F} \neq 2$, we have $c_i = 0$ for all $i \in Y_0$. So far, we have proved that $\phi(\Gamma_i) = 0$ for all $i \in Y_0$.

Now, by (i) and (ii), we obtain the desired result. \qed

We first consider the homogeneous derivations of odd positive $\mathbb{Z}$-degree.

**Lemma 2.5.** Suppose that $\phi \in \text{Der}_t(W, W_\Phi)$ where $zd(\phi) = t \geq 1$ is odd. If $\phi(W_{-1}) = 0$, then there is $z \in G_t$ such that $(\phi - \text{ad}z)(T') = 0$.

**Proof.** Using Lemma 2.4 and noting that $t$ is odd, we may assume that
\[
\phi(\Gamma_i) = \sum_{r \in Y_1} f_{ri}D_r \quad \text{where } i \in Y_1, f_{ri} \in \Lambda(n).
\] (2.6)

Applying $\phi$ to the equation that $[\Gamma_i, \Gamma_j] = 0$ for $i, j \in Y_1$, we have
\[
\sum_{r \in Y_1} (\Gamma_i(f_{rj}) - \Gamma_j(f_{ri})) D_r + f_{ji}D_j - f_{ij}D_i = 0.
\]

Consequently,
\[
\Gamma_i(f_{rj}) = \Gamma_j(f_{ri}) \quad \text{whenever } r \neq i, j;
\] (2.7)
Accordingly, let $\psi$ be the solution of (2.8) and noticing the fact that $\Gamma$ is linear, using (2.8) and noticing the fact that $\Gamma$ is linear, we have
\[ c_{urj}\delta_{j\in u} = c_{urj}\delta_{i\in u} \quad \text{whenever } r \neq i, j. \]
This implies that $c_{urj} \neq 0$ and $j \in u \iff c_{urj} \neq 0$ and $i \in u$.

Let $r \neq i$ and assume that $c_{urj} \neq 0$. Then the implication relation above shows that $i \in u$.

Accordingly, let $\psi$ be the solution of (2.8) and noticing the fact that $\Gamma$ is linear, we have
\[ \Gamma_i(f_{ri}) = f_{ri} \quad \text{whenever } r \neq i. \]
For any fixed $r \in Y_1$, Lemma 2.2 ensures that there is $f_r \in \Lambda(n)$ such that
\[ \Gamma_i(f_r) = f_{ri} \quad \text{for all } i \in Y_1 \setminus \{r\}. \]

Assert that
\[ \Gamma_i(f_{ii}) = 0 \quad \text{for all } i \in Y_1. \]
Using (2.8) and noticing the fact that $\Gamma_i^2 = \Gamma_i$, we obtain that
\[ \Gamma_j \Gamma_i (f_{ii}) = \Gamma_i \Gamma_j (f_{ii}) = \Gamma_i^2(f_{ij}) - \Gamma_i(f_{ij}) = 0 \quad \text{for } j \neq i. \]
Note that $zd(f_{ii}) = t + 1 \geq 2$ and $\Gamma_i(x^u) = \delta_{i\in u} x^u$. (2.11) follows.

For $r \in Y_1$, put $f_r := -f_{rr} + \Gamma_r(f_r)$. Obviously, $f_r \in \Lambda(n)$. It follows from (2.11) that
\[ \Gamma_r(f_r) - f_r = -\Gamma_r(f_{rr}) + \Gamma_r^2(f_r) + f_{rr} - \Gamma_r(f_r) = f_{rr}. \]

For $i \in Y_1 \setminus r$, by (2.8) and (2.10) we obtain that
\[ \Gamma_i(f_r) = -\Gamma_i(f_{rr}) + \Gamma_i\Gamma_r(f_r) = -\Gamma_i(f_{rr}) + \Gamma_r\Gamma_i(f_r) = (\Gamma_r(f_{ri}) - f_{ri}) + \Gamma_r(f_{ri}) = f_{ri}. \]
Let $z' := -\sum_{r \in Y_1} f_r D_r$. A combination of (2.12) and the equation above yields that for $i \in Y_1$,
\[ [z', \Gamma_i] = -\sum_{r \in Y_1} [f_r D_r, \Gamma_i] = \sum_{r \in Y_1} \Gamma_i(f_r) D_r - f_r D_i = \sum_{r \in Y_1 \setminus i} \Gamma_i(f_r) D_r + (\Gamma_i(f_i) - f_i) D_i = \sum_{r \in Y_1 \setminus i} f_{ri} D_r + f_{ii} D_i = \phi(\Gamma_i). \]

Let $z$ be the $t$-component of $z'$. Since $zd(\phi) = t$, one gets $[z, \Gamma_i] = \phi(\Gamma_i)$ for all $i \in Y_1$.
Putting $\psi := \phi - adz$, then $\psi \in Der(W, W^T)$ and $\psi(\Gamma_i) = 0$ for all $i \in Y_1$. □

**Lemma 2.6.** Suppose that $\phi \in Der(W, W^T)$ and $zd(\phi) = t \geq 0$ is even. If $\phi(W_{-1}) = 0$ then there is $z \in G_t$ such that $(\phi - adz)(T^T) = 0$. 

Proof. Since \( zd(φ) = t \) is even, by Lemma 2.1 one may assume that

\[
φ(Γ_i) = \sum_{r \in Y_0} f_{ri} D_r \quad \text{where } i \in Y_1, \ f_{ri} \in Λ(n).
\]  

(2.13)

Analogous to the proof of Lemma 2.5 one may easily show that

\[
Γ_i(f_{rj}) = Γ_j(f_{ri}) \quad \text{for all } i, j \in Y_1.
\]  

(2.14)

Suppose that

\[
f_{ri} = \sum_{u \in B_{t+1}} c_{u,r,i} x^u \quad \text{where } c_{u,r,i} \in F.
\]  

(2.15)

Then we obtain from (2.14) and (2.15) that

\[
c_{u,r,i} δ_{j \in u} = c_{u,r,j} δ_{i \in u} \quad \text{for all } i, j \in Y_1, \ r \in Y_0.
\]

Consequently, for \( i, j \in Y_1, \ r \in Y_0 \) and \( u \in B_{t+1} \),

\[
c_{u,r,i} ≠ 0 \text{ and } j \in u \iff c_{u,r,j} ≠ 0 \text{ and } i \in u.
\]  

(2.16)

Let us complete the proof of this lemma. Assume that \( zd(φ) = t ≥ 2 \). If \( c_{u,r,i} ≠ 0 \) for \( i \in Y_1 \), one may pick \( j \in u \setminus i \). By (2.10), we have \( i \in u \). Assume that \( zd(φ) = 0 \). Then (2.16) implies that there is at most one nonzero summand \( c_{(i),r,i} x_i \) in the right-hand side of (2.15). Summarizing, every nonzero summand in the right-hand side of (2.15) possesses the factor \( x_i \). Therefore,

\[
Γ_i(f_{ri}) = f_{ri} \quad \text{for all } i \in Y_1, \ r \in Y_0.
\]  

(2.17)

For any fixed \( r \in Y_0 \), by (2.14) and (2.17), \( \{f_{r,m+1}, f_{r,m+2}, \ldots, f_{r,m+n}\} \) fulfills the conditions of Lemma 2.6. Hence, there is \( f_r \in Λ(n) \) such that

\[
Γ_i(f_r) = f_{ri} \quad \text{for } i \in Y_1.
\]

Let \( z' := - \sum_{r \in Y_0} f_r D_r \). Then (2.13) and the equation above show that \( [z', Γ_i] = φ(Γ_i) \) for \( i \in Y_1 \). Let \( z \) be \( t \)-component of \( z' \). Then \( z \in G_t \) and \( (φ - adz)(Γ_i) = 0 \) for all \( i \in Y_1 \). □

Now we come to the following main result.

Proposition 2.7. Let \( φ \) be a homogeneous derivation from \( W \) into \( W_T \) with nonnegative \( Z \)-degree \( t \). Then \( φ \) can be reduced to be vanishing on \( W_{-1} \) and the canonical torus of \( W \); that is, there is \( E \in (W_T)_t \) such that \( (φ - adE)|_{W_{-1} + T} = 0 \).

Proof. By Lemma 2.6 there is \( E' \in (W_T)_t \) such that \( (φ - adE')(W_{-1}) = 0 \). Then by Lemmas 2.4–2.6 there is \( E'' \in G_t \) such that \( (φ - adE' - adE'')(T) = 0 \). Putting \( E := E' + E'' \), then \( (φ - adE)(W_{-1} + T) = 0 \). □

In the following, using Proposition 2.7 we first reduce every nonnegative \( Z \)-degree derivation from \( W \) into \( W_T \) to be vanishing on the top \( W_{-1} \oplus W_0 \) of \( W \); then we determine the \( Z \)-homogeneous components \( \text{Der}_t(W, W_T) \) for \( t ≥ 0 \).

Proposition 2.8. Let \( φ \in \text{Der}_t(W, W_T) \) with \( t ≥ 0 \). Then there is \( E \in (W_T)_t \) such that \( (φ - adE)|_{W_{-1} \oplus W_0} = 0 \).
Proof. By Proposition 2.7 without loss of generality we may assume that \( \phi(W_{-1} + T) = 0. \)

(i) We first consider \( \phi(x_k D_l) \) where \( i, k \in Y_0 \) with \( i \neq k. \) By Lemma 2.1, \( \phi(x_k D_l) \in \mathcal{G}_t. \) Assume that \( \phi(x_k D_l) = \sum_{r \in Y_0, u \in \mathcal{B}_{t+1}} c_{u,r} x^u D_r \) where \( c_{u,r} \in \mathbb{F}. \) If \( t \) is even, then \( \phi(x_k D_l) = \sum_{r \in Y_0, u \in \mathcal{B}_{t+1}} c_{u,r} x^u D_r. \) Note that \( \Gamma_j, \phi(x_k D_l) \) is odd. It follows that \( \phi(x_k D_l) = 0. \) If \( t \) is odd, then \( \phi(x_k D_l) = \sum_{r \in Y_0, u \in \mathcal{B}_{t+1}} c_{u,r} x^u D_r. \) Then

\[
\phi(x_k D_l) = \phi([\Gamma_k, x_k D_l]) = [\Gamma_k, \phi(x_k D_l)] = 0.
\]

(ii) We next consider \( \phi(x_k D_l) \) where \( k, l \in Y_1 \) with \( k \neq l. \)

(a) Suppose that \( t \) is even. Just as in (i) one may assume that

\[
\phi(x_k D_l) = \sum_{r \in Y_0, u \in \mathcal{B}_{t+1}} c_{u,r} x^u D_r \quad \text{where} \quad c_{u,r} \in \mathbb{F}.
\]

Then, from the equation that \( \Gamma_i, \phi(x_k D_l) = 0 \) for all \( i \in Y_0, \) one gets \( c_{u,r} = 0 \) for all \( r \in Y_0, \) \( u \in \mathcal{B}_{t+1} \). Hence, \( \phi(x_k D_l) = 0. \)

(b) Suppose that \( t \) is odd. We proceed in two cases \( t \geq 3 \) and \( 1 \leq t \leq 2 \) to show that \( \phi(x_k D_l) = 0 \) for \( k, l \in Y_1. \) Suppose that \( t \geq 3. \) By Lemma 2.1 one may assume that

\[
\phi(x_k D_l) = \sum_{r \in Y_1, |u| \geq 4} c_{u,r} x^u D_r \quad \text{where} \quad c_{u,r} \in \mathbb{F}.
\] (2.18)

Given \( v \in \mathbb{B} \) with \( |v| \geq 4 \) and \( s \in Y_1, \) choose \( q \in v \setminus \{k, l, s\}. \) Then \( \Gamma_q, x^v D_s = x^v D_s. \) On the other hand, since \( \Gamma_q, x_k D_l = 0, \) we have \( \Gamma_q, \phi(x_k D_l) = 0. \) Note that each standard basis element of \( W \) is an eigenvector of \( \Gamma_q \) and \( \Gamma_q, x^v D_s = x^v D_s. \) It follows from (2.18) that \( c_{v,s} = 0. \) Therefore, \( \phi(x_k D_l) = 0. \)

Finally we consider the case \( t = 1. \) Clearly, \( \Gamma', x_k D_l = 0. \) Consequently, \( \Gamma', \phi(x_k D_l) = 0. \) On the other hand, by Lemma 2.1 \( \phi(x_k D_l) \in \mathcal{G}. \) Thus

\[
0 = [\Gamma', \phi(x_k D_l)] = \phi(x_k D_l).
\]

The proof is complete. \( \square \)

In order to determine the homogeneous derivation subspace \( \text{Der}_t(W, W_T) \) for \( t \geq 0, \) we need the generator set of \( W \) (see [2, Proposition 2.2.1]).

**Lemma 2.9.** \( W \) is generated by \( \mathcal{P} \cup \mathcal{N} \cup \mathcal{M}, \) where

\[
\mathcal{P} := \{ x_k x_l D_i \mid k, l \in Y_1, \; i \in Y_0 \},
\]

\[
\mathcal{N} := \{ x_k x_l D_i \mid k \in Y_0, \; l, i \in Y_1 \},
\]

\[
\mathcal{M} := \{ x^{(k \varepsilon_i)} D_j \mid 0 \leq k \leq \pi_i, \; i, j \in Y_0 \}.
\]

Now we can determine the homogeneous derivations of nonnegative \( \mathbb{Z} \)-degree:

**Proposition 2.10.** Let \( t \in \mathbb{N}_0. \) Then \( \text{Der}_t(W, W_T) = \text{ad}(W_T)_t. \)

Proof. If suffices to show the inclusion “\( \subset \)”. Let \( \phi \in \text{Der}_t(W, W). \) By Proposition 2.8 one may assume that \( \phi(W_{-1} \oplus W_0) = 0. \) In the following we consider the application of \( \phi \) to \( \mathcal{P}, \mathcal{N} \) and \( \mathcal{M}, \) respectively.
If \( t > 0 \), then \((\mathcal{P}^t)\) implies that \( \phi(x_i x_j D_k) = 0 \); if \( t = 0 \), we also obtain from \((\mathcal{P}^t)\) that \( \phi(x_i x_j D_k) = 0 \), since \( \phi \in \text{Der}(\mathcal{W}_\mathcal{W}_{\mathcal{W}_1}) \).

(ii) Let us show that \( \phi \langle \mathcal{N} \rangle = 0 \). Let \( i, j \in \mathcal{Y}_0 \), \( k \in \mathcal{Y}_1 \). By Lemma 2.1, \( \phi(x_i x_j D_k) \in \mathcal{G}_{t+1} \).

Since \( \langle \mathcal{N} \rangle, x_i x_j D_k \rangle = x_i x_j D_k \), as in (i) one may assume that

\[
\phi(x_i x_j D_k) = \sum_{r \in \mathcal{Y}_0, u \in \mathcal{B}_{t+2}} c_{u,r} x^u D_r \quad \text{where} \quad c_{u,r} \in \mathbb{F}.
\]

Then

\[-\phi(x_i x_j D_k) = \langle \mathcal{N} \rangle, \phi(x_i x_j D_k) \rangle = \phi(x_i x_j D_k).\]

Since \( \text{char} \mathbb{F} \neq 2 \), it follows that

\[\phi(x_i x_j D_k) = 0 \quad \text{for all} \quad i, j, k \in \mathcal{Y}_1;\]

that is, \( \phi \langle \mathcal{N} \rangle = 0 \).

(iii) Just as in the proof of [2], Lemma 3.1.4], one may show that \( \phi(\mathcal{M}) = 0 \).

Now, Lemma 2.10 shows that \( \phi = 0 \). \( \square \)

In view of Proposition 2.10, in order to determine the derivation space \( \text{Der}(\mathcal{W}_\mathcal{W}_{\mathcal{W}_1}) \) it suffices to determine the derivations of negative \( \mathcal{Z} \)-degree. We first consider the derivations of \( \mathcal{Z} \)-degree \(-1\).

Lemma 2.11. Suppose that \( \varphi \in \text{Der}_{-1}(\mathcal{W}_\mathcal{W}_{\mathcal{W}_1}) \) and \( \varphi(\mathcal{W}_0) = 0 \). Then \( \varphi = 0 \).

Proof. We first assert that \( \varphi(\mathcal{N}) = \varphi(\mathcal{P}) = 0 \). Given \( i, j \in \mathcal{Y}_1 \), \( k \in \mathcal{Y}_0 \), by Lemma 2.1 \( \varphi(x_i x_j D_k) \in \mathcal{G}_0 \). Thus one may assume that \( \varphi(x_i x_j D_k) = \sum_{r \in \mathcal{Y}_1, s \in \mathcal{Y}_0} c_{r,s} x^r D_s \quad \text{where} \quad c_{r,s} \in \mathbb{F} \).

Then, since \( \langle \mathcal{N} \rangle, x_i x_j D_k \rangle = 2x_i x_j D_k \), we have

\[\varphi(x_i x_j D_k) = \langle \mathcal{N} \rangle, \varphi(x_i x_j D_k) \rangle = 2\varphi(x_i x_j D_k).\]
It follows that \( \varphi(x_ix_jD_k) = 0 \); that is, \( \varphi(P) = 0 \). Similarly, applying \( \varphi \) to the equation 
\[
[\Gamma', x_kx_lD_j] = x_kx_lD_j = x_kx_lD_j.
\]

It follows that \( \varphi(x_kx_lD_j) = 0 \). Hence, \( \varphi(N) = 0 \).

It remains to show that \( \varphi(M) = 0 \). Given \( k \in Y_0 \), just as in the proof of \([2\text{ Lemma 3.2.6}]\), one may prove by induction on \( r \) that
\[
\varphi(x^{(r+1)}D_k) = 0 \quad \text{for all } r \in \mathbb{N}.
\]

From this one may easily prove that \( \varphi(M) = 0 \). Summarizing, by Lemma \([2\text{ Lemma 3.2.9}]\) \( \varphi = 0 \). \( \square \)

**Proposition 2.12.** \( \text{Der}_{-1}(W, W_T) = \text{ad}(W_T)_{-1} \).

**Proof.** The inclusion “\( \supseteq \)" is clear. Let \( \phi \in \text{Der}_{-1}(W, W) \). For \( i \in Y_0, k \in Y_1 \), applying \( \phi \) to the equation that \( [\Gamma_i, \Gamma_k] = 0 \), we have \( [\phi(\Gamma_i), \phi(\Gamma_k)] = 0 \). As \( \phi(\Gamma_i), \phi(\Gamma_k) \in W_{-1} \cap W_T \), we have \( [\Gamma_i, \phi(\Gamma_k)] = 0 \) and therefore, \( \phi(\Gamma_i), \phi(\Gamma_k) = 0 \) for all \( k \in Y_1 \). This implies that \( \phi(\Gamma_i) = 0 \) for \( i \in Y_0 \). It follows that \( \phi(x_iD_j) = 0 \) for all \( i, j \in Y_0 \).

For \( k \in Y_1 \), just as in the proof of \([2\text{ Proposition 3.2.7}]\), one may prove that there are \( c_k \in F \) such that \( \phi(\Gamma_k) = c_kD_k \) and \( \phi(x_kD_l) = c_kD_l \) for all \( k, l \in Y_1 \). By Lemma \([2\text{ Lemma 2.11}]\)
\[
\phi = \sum_{r \in Y_1} c_rD_r \in \text{ad}(W_T)_{-1}.
\]

Analogous to \([2\text{ Lemma 3.2.8}]\), we also have the following

**Lemma 2.13.** Let \( \phi \in \text{Der}_{-q}(W, W_T) \) with \( q > 1 \). If \( \phi(x^{(q\epsilon_i)}D_i) = 0 \) for all \( i \in Y_0 \), then \( \phi = 0 \).

**Proposition 2.14.** Suppose that \( q > 1 \). Then \( \text{Der}_{-q}(W, W_T) = 0 \).

**Proof.** Let \( \phi \in \text{Der}_{-q}(W, W_1) \). In view of Lemma \([2\text{ Proposition 2.11}]\) it is sufficient to show that
\[
\phi(x^{(q\epsilon_i)}D_i) = 0 \quad \text{for all } i \in Y_0.
\]

Note that \( [\Gamma', x^{(q\epsilon_i)}D_i] = 0 \) and \( \phi(x^{(q\epsilon_i)}D_i) \in (W_T)^{-1} \). It follows that
\[
0 = [\Gamma', \phi(x^{(q\epsilon_i)}D_i)] = -\phi(x^{(q\epsilon_i)}D_i).
\]

The proof is complete. \( \square \)

**Theorem 2.15.** \( \text{Der}(W, W_T) = \text{ad}(W_T) \).

**Proof.** By Propositions \([2\text{ Proposition 2.10}]\) \([2\text{ Proposition 2.12}]\) and \([2\text{ Proposition 2.14}]\) “\( \supseteq \)" holds. The converse inclusion is clear. \( \square \)

By \([7\text{ Theorem 2}]\), \([2\text{ Theorem 3.2.11}]\) and Theorem \([2\text{ Theorem 2.15}]\) the even part and the odd part of the superderivation algebra of the finite-dimensional generalized Witt superalgebra \( W \) coincide with the derivation algebra of the even part of \( W \) and the derivation space of the even part into the odd part of \( W \), respectively; that is, \( (\text{Der} W)_{\mathfrak{M}} = \text{Der}(W_{\mathfrak{M}}), (\text{Der} W)_{\mathfrak{T}} = \text{Der}(W_{\mathfrak{T}}, W_T) \).
3 Special superalgebras

Recall the canonical torus $T_S$ of $S$ (c.f. [2]). Clearly,
\[
\{x_r D_r - x_s D_s | \tau(r) = \tau(s); r, s \in Y\} \cup \{x_r D_r + x_s D_s | \tau(r) \neq \tau(s); r, s \in Y\}
\]
is an $\mathbb{F}$-basis of $T_S$ consisting of toral elements.

The following fact is simple.

Lemma 3.1. $S_0 = \operatorname{span}_\mathbb{F} \{T_S \cup \{x_r D_s | \tau(r) = \tau(s), r \neq s; r, s \in Y\}\}$.

Put
\[
Q := \{D_{ij}(x^{(r \varepsilon_j)}) | i, j \in Y_0, r \in \mathbb{N}_0\};
\]
\[
R := \{D_{il}(x^{(2 \varepsilon_l)} x_k) | i \in Y_0, k, l \in Y_1\} \cup \{D_{ij}(x_i x^v) | i, j \in Y_0, v \in \mathbb{B}_2\}.
\]

We need the generator set of $S$ (see [2 Proposition 2.2.3]).

Lemma 3.2. $S$ is generated by $Q \cup R \cup S_0$.

In the following we consider the top of $S$.

Lemma 3.3. Suppose that $\phi \in \operatorname{Der}(S, W_1)$ with $zd(\phi) \geq 0$ and that $\phi(S_{-1} + S_0) = 0$. Then
(i) $\phi(D_{il}(x^{(2 \varepsilon_l)} x_k)) = 0$ for all $i \in Y_0, k, l \in Y_1$.
(ii) $\phi(D_{ij}(x_i x^v)) = 0$ for all $i, j \in Y_0$ and $v \in \mathbb{B}_2$.
(iii) $\phi(D_{ij}(x^{(a \varepsilon_l)})) = 0$ for all $i, j \in Y_0$ and all $a \in \mathbb{N}$.

Proof. (i) The proof is similar to the one of [2 Lemma 4.1.1]. Our discussion here for $zd(\phi)$ being odd is completely analogous to one in [2 Lemma 4.1.1] for $zd(\phi)$ being even; and, the discussion here for $zd(\phi)$ being even is completely analogous to one in [2 Lemma 4.1.1] for $zd(\phi)$ being odd.

Similar to [2 Lemmas 4.1.2, 4.1.3], one may prove (ii) and (iii) in the same way. \(\square\)

As a direct consequence of Lemmas 3.2 and 3.3, we have the following.

Corollary 3.4. Suppose that $\phi \in \operatorname{Der}(S, W_1)$ with $zd(\phi) \geq 0$ and that $\phi(S_{-1} + S_0) = 0$. Then $\phi = 0$.

In order to describe the derivations of nonnegative degree we first give two technical lemmas which will simplify our discussion.

Lemma 3.5. Suppose that $\phi \in \operatorname{Der}_t(S, W_1)$ and $\phi(S_{-1}) = 0$.
(i) If $t = n - 1$ is even, then $\phi(\Gamma_1' - \Gamma_k) = 0$ for all $k \in Y_1 \setminus 1'$.
(ii) If $t = n - 1$ is odd, then there is $\lambda \in \mathbb{F}$ such that
\[
(\phi - \lambda \operatorname{ad}(x^\omega D_1'))(\Gamma_1' - \Gamma_k) = 0 \text{ for all } k \in Y_1 \setminus 1'.
\]
(iii) If $t > n - 1$, then $\phi = 0$.

Proof. (i) The proof is completely analogous to the one of [2 Lemmas 4.2.1(i)].

(ii) The proof is completely analogous to the one of [2 Lemmas 4.2.1(ii)].

(iii) Using Lemma 2.4 and induction on $r$ one may easily prove that $\phi(S_r) = 0$ for all $r \in \mathbb{N}$. \(\square\)
Analogous to [2 Lemmas 4.2.2], we have

**Lemma 3.6.** Suppose that \( \phi \in \text{Der}(S, W_1) \) and \( \text{zd}(\phi) \geq 0 \) is even.

(i) If \( \text{zd}(\phi) < n - 1 \) and

\[
\phi(\Gamma_1 - \Gamma_2) = \phi(\Gamma_1 - \Gamma_3) = \cdots = \phi(\Gamma_1 - \Gamma_m) = 0; \quad \phi(\Gamma_1 + \Gamma_1') = 0.
\]

then

\[
\phi(\Gamma_1 - \Gamma_2) = \phi(\Gamma_1 - \Gamma_3) = \cdots = \phi(\Gamma_1 - \Gamma_m) = 0;
\]

(ii) If \( \text{zd}(\phi) = n - 1 \), then there are \( \lambda_1, \ldots, \lambda_m \in \mathbb{F} \) such that

\[
\left( \phi - \text{ad}\left( \sum_{i \in Y_0} \lambda_i x^\omega D_i \right) \right)(\Gamma_1 - \Gamma_j) = 0 \quad \text{for all } j \in Y_0 \setminus 1;
\]

\[
\left( \phi - \text{ad}\left( \sum_{i \in Y_0} \lambda_i x^\omega D_i \right) \right)(\Gamma_1 + \Gamma_1') = 0.
\]

Recall the canonical torus of \( S \)

\[
\mathcal{T}_S := \text{span}_\mathbb{F}\{\Gamma_1 - \Gamma_2, \ldots, \Gamma_1 - \Gamma_m, \ldots, \Gamma_1 + \Gamma_1', \Gamma_1' - \Gamma_2', \ldots, \Gamma_1' - \Gamma_n'\}.
\]

As a direct consequence of Lemma 3.5 (iii) and Lemma 3.6, we have the following fact:

**Corollary 3.7.** Suppose that \( \phi \in \text{Der}(S, W_1) \) is homogeneous derivation of nonnegative even \( \mathbb{Z} \)-degree such that \( \phi(S_{-1}) = 0 \) and \( \phi(\Gamma_k - \Gamma_1') = 0 \) for all \( k \in Y_1 \setminus 1' \). Then there is \( E \in \mathcal{G} \) such that \( (\phi - \text{ad}E) \) vanishes on the canonical torus \( \mathcal{T}_S \).

Now we prove the following two key lemmas. First, consider the derivations of even \( \mathbb{Z} \)-degree.

**Lemma 3.8.** Suppose that \( \phi \in \text{Der}_t(S, W_1) \), where \( t \geq 0 \) is even. If \( \phi(S_{-1}) = 0 \), then there is \( D \in \mathcal{G}_t \) such that

\[
(\phi - \text{ad}D)(\Gamma_k - \Gamma_1') = 0 \quad \text{for all } k \in Y_1 \setminus 1'.
\]

**Proof.** By Lemma 3.5 (i) and (iii) it suffices to consider the setting \( t < n - 1 \). By Lemma 2.1 one may assume that

\[
\phi(\Gamma_k - \Gamma_1') = \sum_{r \in Y_0} f_{r,k} D_r \text{ where } k \in Y_1 \setminus 1'; \quad f_{r,k} \in \Lambda(n).
\]

Write

\[
f_{r,k} = \sum_{|u|=t+1} c_{u,r,k} x^u \quad \text{where } c_{u,r,k} \in \mathbb{F}.
\]

Discussing just as in the proof of [2 Lemma 4.2.4], we may obtain that

\[
\sum_{|u|=t+1} (\delta_{k,u} - \delta_{1'u,u}) c_{u,r,k} x^u = \sum_{|u|=t+1} (\delta_{l,u} - \delta_{1'u,u}) c_{u,r,k} x^u.
\]

Since \( \{x^u \mid u \in \mathbb{B} \} \) is an \( \mathbb{F} \)-basis of \( \Lambda(n) \), it follows that

\[
(\delta_{k,u} - \delta_{1'u,u}) c_{u,r,k} = (\delta_{l,u} - \delta_{1'u,u}) c_{u,r,k} \quad \text{for } r \in Y_0, \ k, l \in Y_1 \setminus 1'.
\]

(3.3)
Suppose that \( c_{u,r,k} \) is any nonzero coefficient in (3.2), where \(|u| = t + 1 < n\), \( r \in Y_0 \) and \( k \in Y_1 \setminus 1' \). Note that \(|u| \geq 1\). We proceed in two steps to show that \( \delta_{k \in u} + \delta_{1' \in u} = 1 \).

**Case (i):** \(|u| \geq 2\). If \( 1' \not\in u \), one may find \( l \in u \setminus k \). Then (3.3) shows that \( \delta_{k \in u} = 1 \); that is, \( k \in u \). If \( 1' \in u \), noting that \(|u| \leq n - 1\), one may find \( l \in Y_1 \setminus u \). Then (3.3) shows that \( \delta_{k \in u} = 0 \); that is, \( k \not\in u \). Summarizing, for any nonzero coefficient \( c_{u,r,k} \) in (3.2), we have \( \delta_{k \in u} + \delta_{1' \in u} = 1 \).

**Case (ii):** \(|u| = 1\). Since \(|u| = 1\), the case of \( k \in u \) and \( 1' \not\in u \) does not occur. If \( k \not\in u \) and \( 1' \not\in u \), then there is \( l \in u \), since \(|u| = 1\). Then by (3.3), we get \( c_{u,r,k} = 0 \), this is a contradiction. Hence, we have \( \delta_{k \in u} + \delta_{1' \in u} = 1 \).

Then, just like in the proof of \([2, \text{Lemma 4.2.4}]\), we can rewrite (3.2) as follows:

\[
\delta_{r_k} = \sum_{1' \in u, k \not\in u} c_{u, r, k} x^u + \sum_{1' \not\in u, k \in u} c_{u, r, k} x^u.
\]

Now, following the corresponding part of the proof for \([2, \text{Lemma 4.2.4}]\), one may find \( D \in G_t \) such that \((\phi - \text{ad}D)(\Gamma_k - \Gamma_{1'}) = 0\) for all \( k \in Y_1 \setminus 1' \). The proof is complete. \(\square\)

Let us consider the case of odd \( Z \)-degree.

**Lemma 3.9.** Let \( \phi \in \text{Der}_t(S, W_1) \) where \( t > 0 \) is odd. If \( \phi(S_{-1}) = 0 \), then there is \( D \in G_t \) such that

\[
(\phi - \text{ad}D)(\Gamma_1 + \Gamma_k) = 0 \text{ for all } k \in Y_1.
\]

**Proof.** Deleting the part (ii) in the proof of \([2, \text{Lemma 4.2.5}]\), we obtain our proof. \(\square\)

For our purpose, we need still the following three reduction lemmas.

**Lemma 3.10.** Suppose that \( \phi \in \text{Der}_t(S, W_1) \) and \( \phi(S_{-1}) = 0 \), where \( t > 0 \) is odd. If \( \phi(\Gamma_1 + \Gamma_k) = 0 \) for all \( k \in Y_1 \), then \( \phi(S_0) = 0 \).

**Proof.** Following parts (i) and (ii) in the proof of \([2, \text{Lemma 4.2.6}]\), one may show that \( \phi(\Gamma_1 - \Gamma_i) = 0 \) and \( \phi(x_i D_j) = 0 \) for all \( i, j \in Y_0 \) with \( i \neq j \).

To show that \( \phi(x_k D_l) = 0 \) for \( k, l \in Y_1 \) with \( k \neq l \), just as in the part (iii) of the proof of \([2, \text{Lemma 4.2.6}]\), it suffices to consider separately two cases \( zd(\phi) = 1 \) and \( zd(\phi) \geq 3 \).

Now Lemma 3.11 ensures that \( \phi(S_0) = 0 \). \(\square\)

Analogous to \([2, \text{Lemma 4.2.7}]\), one may prove the following

**Lemma 3.11.** Suppose that \( \phi \in \text{Der}(S, W_1) \) is a homogeneous derivation of nonnegative even \( Z \)-degree and \( \phi(S_{-1} + T_S) = 0 \). Then \( \phi(S_0) = 0 \).

Now we are able to characterize the homogeneous derivation space of nonnegative \( Z \)-degree. Using Lemmas 3.9, 3.10, Corollaries 3.4, 3.7 and Proposition 2.8, one may prove the following result (cf. \([2, \text{Proposition 4.2.9}]\)).

**Proposition 3.12.** \( \text{Der}_t(S, W_1) = \text{ad}(W_1)_t \) for \( t \geq 0 \).

As an application of Proposition 3.12, we have:

**Proposition 3.13.** \( \text{Der}_t(S, S_1) = \text{ad}(S_1)_t \) for \( t \geq 0 \).
Applying (3.6), we may obtain

We shall use the following simple fact (by Lemma 2.1):

We first show that

This implies that \( \text{div}(E) \in \Lambda(n)_T \). Similarly, \([E, S] \subset S_T\) implies that \([\text{div}(E), S] = 0\). In particular, \([\text{div}(E), T_S] = 0\). Since \(\text{div}(E) \in \Lambda(n)_T\), one gets \(\text{div}(E), T = 0\). Keeping in mind \(\text{div}(E) \in \Lambda(n)_T\), one may easily deduce that \(\text{div}(E) = 0\) (c.f [2 Proposition 4.2.10]).

\(\square\)

In the following we first determine the homogeneous derivations of negative \(\mathbb{Z}\)-degree from \(S\) into \(W_1\). This combining with Proposition 3.12 will give the structure of the derivation space \(\text{Der}(S, W_1)\). The following lemma tells us that a \(\mathbb{Z}\)-degree \(-1\) derivation from \(S\) into \(W_1\) is completely determined by its action on \(S_0\).

**Lemma 3.14.** Suppose that \(\phi \in \text{Der}_{-1}(S, W_1)\) and that \(\phi(S_0) = 0\). Then \(\phi = 0\).

**Proof.** We first show that \(\phi(R) = 0\). By the definition of \(D_{il}\),

\[
D_{il}(x^{(2e_i)}x_k) = x_ix_kD_l + \delta_{kl}x^{(2e_i)}D_i \quad \text{for all } i \in Y_0, k, l \in Y_1.
\]

(3.5)

We shall use the following simple fact (by Lemma 2.1):

\[
\phi(S_1) \subseteq G_0.
\]

We may assume that

\[
\phi(D_{il}(x^{(2e_i)}x_k)) = \sum_{k,j \in Y_1, r \in Y_0} c_{k,r}x_kD_r.
\]

(3.6)

Given \(i \in Y_0, k, l \in Y_1\), take \(j \in Y_0 \setminus i\). Then

\[
\left[\Gamma_i - \Gamma_j, D_{il}(x^{(2e_i)}x_k)\right] = D_{il}(x^{(2e_i)}x_k).
\]

Applying \(\phi\) to the equation above and then combining that with (3.6), one may obtain by a comparison of coefficients that

\[
c_{k,r} = 0 \quad \text{for } k \in Y_1, r \in Y_0 \setminus j.
\]

Hence, by (3.6), we may obtain

\[
\phi(D_{il}(x^{(2e_i)}x_k)) = \sum_{k \in Y_1} c_{k,j}x_kD_j.
\]

(3.7)

**Case (i):** \(k \neq l\). Then by (3.5), we have \(D_{il}(x^{(2e_i)}x_k) = x_ix_kD_l\) and

\[
\left[\Gamma_i + \Gamma_k, D_{il}(x^{(2e_i)}x_k)\right] = 2D_{il}(x^{(2e_i)}x_k).
\]

(3.8)

Applying \(\phi\) to (3.8) and using (3.7), one may obtain by comparing coefficients that

\[
c_{s,j} = 0 \quad \text{for } s \in Y_1.
\]
Consequently, \( \phi(D_{ik}(x^{(2\varepsilon_i)}x_k)) = 0 \).

**Case (ii):** \( k = l \). Then by (3.5), we have
\[
D_{ik}(x^{(2\varepsilon_i)}x_k) = x^{(2\varepsilon_i)}D_i + x_i x_k D_k. 
\]
and
\[
[\Gamma_i + \Gamma_k, D_{ik}(x^{(2\varepsilon_i)}x_k)] = D_{ik}(x^{(2\varepsilon_i)}x_k). 
\]
Applying \( \phi \) to the equation above and using (3.7), one may obtain by comparing coefficients that
\[
c_{s,j} = 0 \quad \text{for} \quad s \in Y_1 \setminus k. 
\]
Hence, By (3.7), we may obtain
\[
\phi(D_{ik}(x^{(2\varepsilon_i)}x_k)) = c_{k,j} x_k D_j. \tag{3.9}
\]
For \( i \in Y_0, k, l \in Y_1 \), choose \( q \in Y_1 \setminus k \). Then \( m\Gamma_q + \Gamma'' \in S_0 \) and
\[
[m\Gamma_q + \Gamma'', D_{ik}(x^{(2\varepsilon_i)}x_k)] = D_{ik}(x^{(2\varepsilon_i)}x_k). 
\]
Applying \( \phi \) and using (3.9), one gets
\[
\phi(D_{ik}(x^{(2\varepsilon_i)}x_k)) = [m\Gamma_q + \Gamma'', \phi(D_{ik}(x^{(2\varepsilon_i)}x_k))] = -\phi(D_{ik}(x^{(2\varepsilon_i)}x_k)). 
\]
Consequently, \( \phi(D_{ik}(x^{(2\varepsilon_i)}x_k)) = 0 \) for all \( i \in Y_0, k \in Y_1 \).

We want to show that \( \phi(D_{ij}(x_ix_kx_l)) = 0 \) for \( i, j \in Y_0, k, l \in Y_1 \). By a same argument, we can also obtain that
\[
\phi(D_{ij}(x_ix_kx_l)) = c_{kj} x_k D_j \quad \text{for} \quad k \in Y_1, j \in Y_0 \setminus i. \tag{3.10}
\]
Note that \( n\Gamma_q + \Gamma' \in S_0 \) for \( q \in Y_0 \) and that
\[
[n\Gamma_q + \Gamma', D_{ij}(x_ix_kx_l)] = (2 - n\delta_{q,j})D_{ij}(x_ix_kx_l) \quad \text{for} \quad i, j \in Y_0, k, l \in Y_1. \tag{3.11}
\]
Applying \( \phi \) to (3.11) and using (3.10), one gets
\[
c_{kj} = 0 \quad \text{for} \quad k \in Y_1, j \in Y_0 \setminus i. 
\]
Consequently, \( \phi(D_{ij}(x_ix_kx_l)) = 0 \).

It remains to show that \( \phi(Q) = 0 \). But this can be verified completely analogous to the proof of [2, Lemma 4.3.1]. \( \square \)

Using Lemma 3.14 we can determine the derivations of \( Z \)-degree \(-1\).

**Proposition 3.15.** \( \text{Der}_{-1}(S, W_1) = \text{ad}(W_1)_{-1} \). In particular, \( \text{Der}_{-1}(S, S_1) = \text{ad}(S_1)_{-1} \).

**Proof.** Let \( \phi \in \text{Der}_{-1}(S, W_1) \). Let \( k \in Y_1 \). Assume that
\[
\phi(\Gamma_k + \Gamma_1) = \sum_{r \in Y_1} c_{kr} D_r \quad \text{where} \quad c_{kr} \in \mathbb{F}. \tag{3.12}
\]
Let \( l \in Y_1 \setminus k \). Then \([\Gamma_k + \Gamma_1, \phi(\Gamma_l + \Gamma_1)] = [\Gamma_l + \Gamma_1, \phi(\Gamma_k + \Gamma_1)]\). By (3.12), \( c_{kl} = 0 \) whenever \( k, l \in Y_1 \) with \( k \neq l \). It follows that \( \phi(\Gamma_k + \Gamma_1) = c_{kk}D_k \) where \( c_{kk} \in \mathbb{F} \). Obviously, \([\Gamma_k + \Gamma_1, x_kD_l] = x_kD_l \) for \( k \in Y_1 \) with \( k \neq l \). Then

\[
    c_{kk}D_l + [\Gamma_k + \Gamma_1, \phi(x_kD_l)] = \phi(x_kD_l).
\]

Since \( \phi(x_kD_l) \in (W_1)_{-1} \), it follows that \( \phi(x_kD_l) = c_{kk}D_l \). Put \( \psi := \phi - \sum_{r \in Y_1} c_{rr}\text{ad}D_r \).

Then

\[
    \psi(\Gamma_k + \Gamma_1) = \psi(x_kD_l) = 0 \quad \text{for } k, l \in Y_1, k \neq l. \quad (3.13)
\]

We next show that

\[
    \psi(x_iD_j) = 0 \quad \text{for } i, j \in Y_0, i \neq j. \quad (3.14)
\]

Take \( r \in Y_0 \setminus \{i, j\} \). Then \([\psi(\Gamma_r + \Gamma_q), x_iD_j] + [\Gamma_r + \Gamma_q, \psi(x_iD_j)] = 0 \). Since \( \psi(\Gamma_r + \Gamma_q) \in (W_1)_{-1} \), we have \([\psi(\Gamma_r + \Gamma_q), x_iD_j] = 0 \). Consequently, \([\Gamma_q, \psi(x_iD_j)] = 0 \) for all \( q \in Y_1 \). Hence \( \psi(x_iD_j) = 0 \), since \( \psi(x_iD_j) \in (W_1)_{-1} \).

In the same way we can verify that

\[
    \psi(\Gamma_1 - \Gamma_j) = 0 \quad \text{for all } j \in Y_0 \setminus 1. \quad (3.15)
\]

By (3.13)–(3.15), we have \( \psi(S_0) = 0 \). It follows from Lemma 3.14 that \( \psi = 0 \) and hence \( \phi \in \text{ad}(W_1)_{-1} \). This completes the proof.

To compute the derivation of \( Z \)-degree less than \(-1 \) from \( S \) into \( W_1 \), we establish the following lemma.

**Lemma 3.16.** Suppose that \( \phi \in \text{Der}_{-t}(S, W_1) \) with \( t > 1 \) and that \( \phi(D_{ij}(x^{((t+1)\epsilon_i)})) = 0 \) for all \( i, j \in Y_0 \). Then \( \phi = 0 \).

**Proof.** First claim that \( \phi(Q) = 0 \). To that aim, we proceed by induction on \( q \) to show that

\[
    \phi(D_{ij}(x^{(q\epsilon_i)})) = 0 \quad \text{for all } i, j \in Y_0 \text{ with } i \neq j. \quad (3.16)
\]

Without loss of generality suppose that \( q > t + 1 \) in the following. By inductive hypothesis and Lemma 2.1 \( \phi(D_{ij}(x^{(q\epsilon_i)})) \in G_{q-t-2} \). Thus one may write

\[
    \phi(D_{ij}(x^{(q\epsilon_i)})) = \sum_{r \in Y_0, |u| = q-t-1} c_{u,r}x^uD_r \quad \text{where } c_{u,r} \in \mathbb{F}. \quad (3.17)
\]

If \( q - t \geq 3 \), proceeding just as Case (i) in the proof of [2] Lemma 4.3.3, one may show that \( \phi(D_{ij}(x^{(q\epsilon_i)})) = 0 \). Suppose that \( q - t < 3 \). Note that \( q > t + 1 \). Then rewrite (3.17) as

\[
    \phi(D_{ij}(x^{(q\epsilon_i)})) = \sum_{l \in Y_1, r \in Y_0} c_{l,r}x_lD_r \quad \text{where } c_{l,r} \in \mathbb{F}. \quad (3.18)
\]

If \( s = j \), then \([\Gamma_s + \Gamma_1, D_{ij}(x^{(q\epsilon_i)})] = -D_{ij}(x^{(q\epsilon_i)}) \). Applying \( \phi \) to the equation above and then combining that with (3.18), one may obtain by a comparison of coefficients of \( x_lD_r \) that

\[
    c_{l_0,r_0} = 0 \quad \text{for } l_0 \in Y_1, r_0 \in Y_0. \]

Consequently, \( \phi(D_{ij}(x^{(q\epsilon_i)})) = 0 \).
If \( s \neq j \), then \([n\Gamma_s + \Gamma', D_{ij}(x^{(q\varepsilon_i)})]\) = 0. Applying \( \phi \) and then combining that with \( \text{(3.18)} \), one may obtain by a comparison of coefficients of \( x_0D_{r_0} \) that

\[
c_{0}r_{0} = 0 \quad \text{for} \quad l_{0} \in Y_{1}, \quad r_{0} \in Y_{0}.
\]

Consequently, \( \phi(D_{ij}(x^{(q\varepsilon_i)})) = 0 \). Thus \( \text{(3.16)} \) holds for all \( q \) and therefore, \( \phi(\mathcal{Q}) = 0 \).

We next prove that \( \phi(\mathcal{R}) = 0 \). Since \( \mathcal{R} \subseteq \mathcal{S}_{1} \), \( zd(\phi) \leq -2 \), it suffices to consider the case that \( zd(\phi) = -2 \). Note that \( \phi(S_{1}) \subseteq S_{-1} \). For \( k, l \in Y_{1}, i \in Y_{0} \), take \( q \in Y_{0} \setminus \{i, j\} \). Then \( n\Gamma_{q} + \Gamma' \in S_{0} \) and \([n\Gamma_{q} + \Gamma', D_{il}(x^{(2\varepsilon_{i})}x_{k})]\) = 0. Since \( \phi(D_{il}(x^{(2\varepsilon_{i})}x_{k})) \in (W_{1})_{-1} \), it follows that

\[
\phi(D_{il}(x^{(2\varepsilon_{i})}x_{k})) = -[n\Gamma_{q} + \Gamma', \phi(D_{il}(x^{(2\varepsilon_{i})}x_{k}))] = 0 \quad \text{for all} \quad i \in Y_{0}, \quad k, l \in Y_{1}.
\]

Obviously,

\[
[n\Gamma_{q} + \Gamma', D_{ij}(x_{i}x_{k}x_{l})] = 2D_{ij}(x_{i}x_{k}x_{l}).
\]

Applying \( \phi \), one gets,

\[
2\phi(D_{ij}(x_{i}x_{k}x_{l})) = [n\Gamma_{q} + \Gamma', \phi(D_{ij}(x_{i}x_{k}x_{l}))] = -\phi(D_{ij}(x_{i}x_{k}x_{l})),
\]

since \( \phi(D_{ij}(x_{i}x_{k}x_{l})) \in (W_{1})_{-1} \). The assumption \( p \neq 3 \) ensures that \( \phi(D_{ij}(x_{i}x_{k}x_{l})) = 0 \).

By Lemma \( \text{(3.12)} \) \( \phi = 0 \). \( \square \)

Finally, we are to determine the homogeneous derivations of \( \mathbb{Z} \)-degree less than \(-1 \) from \( \mathcal{S} \) into \( W_{1} \).

**Proposition 3.17.** \( \text{Der}_{-t}(\mathcal{S}, W_{1}) = 0 \) for \( t > 1 \). In particular, \( \text{Der}_{-t}(\mathcal{S}, S_{1}) = 0 \) for \( t > 1 \).

**Proof.** Let \( \phi \in \text{Der}_{-t}(\mathcal{S}, W_{1}) \). Assert that

\[
\phi(D_{ij}(x^{(t+1)\varepsilon_{i}})) = 0 \quad \text{for all} \quad i, j \in Y_{0}.
\]

Recall \( \Gamma' = \sum_{r \in Y_{1}} \Gamma_{r} \). Choose \( q \in Y_{0} \setminus \{i, j\} \), since \( m \geq 3 \). Clearly, \( n\Gamma_{q} + \Gamma' \in S_{0} \). Then

\[
[n\Gamma_{q} + \Gamma', D_{ij}(x^{(t+1)\varepsilon_{i}})] = 0.
\]

Applying \( \phi \), one gets

\[
0 = [n\Gamma_{q} + \Gamma', \phi(D_{ij}(x^{(t+1)\varepsilon_{i}}))] = -\phi(D_{ij}(x^{(t+1)\varepsilon_{i}})),
\]

since \( \phi(D_{ij}(x^{(t+1)\varepsilon_{i}})) \in (W_{1})_{-1} \). Consequently, \( \phi(D_{ij}(x^{(t+1)\varepsilon_{i}})) = 0 \). By Lemma \( \text{(3.16)} \) \( \phi = 0 \). The proof is complete. \( \square \)

Now we can describe the derivation spaces \( \text{Der}(\mathcal{S}, W_{1}) \) and \( \text{Der}(\mathcal{S}, S_{1}) \).

**Theorem 3.18.** \( \text{Der}(\mathcal{S}, W_{1}) = \text{ad}W_{1} \).

**Proof.** This is a direct consequence of Propositions \( \text{(3.12)(3.15)(3.17)} \). \( \square \)

**Theorem 3.19.** \( \text{Der}(\mathcal{S}, S_{1}) = \text{ad}S_{1} \).

**Proof.** This is a direct consequence of Propositions \( \text{(3.13)(3.15)(3.17)} \). \( \square \)
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