THE 2-ADIC VALUATION OF A SEQUENCE ARISING FROM A RATIONAL INTEGRAL

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Abstract. We analyze properties of the 2-adic valuation of an integer sequence that originates from an explicit evaluation of a quartic integral. We also give a combinatorial interpretation of the valuations of this sequence.

1. Introduction

Wallis’s formula
\begin{equation}
\int_0^\infty \frac{dx}{(x^2 + 1)^{m+1}} = \frac{\pi}{2^{2m+1}} \binom{2m}{m}
\end{equation}
is one of the earlier instances of evaluation of definite integrals where the result contains interesting arithmetical and combinatorial properties. In this paper we examine such connection for the integral
\begin{equation}
N_{0,4}(a; m) = \int_0^\infty \frac{dx}{(x^4 + 2ax^2 + 1)^{m+1}}.
\end{equation}
The condition \( a > -1 \) is imposed for convergence. The evaluation
\begin{equation}
N_{0,4}(a, m) = \frac{\pi}{2} \frac{P_m(a)}{[2(a + 1)]^{m+\frac{1}{2}}}
\end{equation}
where
\begin{equation}
P_m(a) = \sum_{l=0}^{m} d_l(m) a^l
\end{equation}
with
\begin{equation}
d_l(m) = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m - 2k}{m - k} \binom{m + k}{m} \binom{k}{l}, \quad 0 \leq l \leq m,
\end{equation}
appeared in [4]. The reader will find in [2] a survey of the different proofs of (1.3) and an introduction to the many issues involved in the evaluation of definite integrals in [8].

The study of combinatorial aspects of the sequence \( d_l(m) \) was initiated in [3] where the authors show that \( d_l(m) \) form a unimodal sequence, that is, there exists and index \( l^* \) such that \( d_0(m) \leq \ldots \leq d_{l^*}(m) \) and \( d_{l^*}(m) \geq \ldots \geq d_m(m) \). The fact that \( d_l(m) \) satisfies the stronger condition of logconcavity \( d_{l-1}(m) d_{l+1}(m) \leq d_l^2(m) \).
has been recently established in [6]. We consider here arithmetical properties of the sequence \(d_{l,m}\). It is more convenient to analyze the auxiliary sequence

\[
A_{l,m} = l! \cdot m! \cdot 2^{m+l} d_{l,m} = \frac{l! \cdot m!}{2^{m-l}} \sum_{k=l}^{m} 2^k \binom{2m - 2k}{m-k} \binom{m+k}{k} \binom{k}{l}
\]

for \(m \in \mathbb{N}\) and \(0 \leq l \leq m\). The integral (1.2) is then given explicitly as

\[
N_{0,4}(a; m) = \frac{\pi}{\sqrt{2} m! (4(2a+1))^{m+1/2}} \sum_{l=0}^{m} A_{l,m} \frac{a^l}{l!}
\]

In [5] it is shown that \(A_{l,m} \in \mathbb{N}\). Observe that the computation of \(A_{l,m}\) using (1.6) is more efficient if \(l\) is close to \(m\). For instance,

\[
A_{m,m} = 2^{m} (2m)! \quad \text{and} \quad A_{m-1,m} = 2^{m-1} (2m-1)! (2m+1).
\]

A second method to compute \(A_{l,m}\), efficient now when \(l\) is small, has been discussed in [5]. There, it is shown that \(A_{l,m}\) is a linear combination (with polynomial coefficients) of

\[
\prod_{k=1}^{m} (4k - 1) \quad \text{and} \quad \prod_{k=1}^{m} (4k + 1).
\]

For example,

\[
A_{0,m} = \prod_{k=1}^{m} (4k - 1) \quad \text{and} \quad A_{1,m} = (2m + 1) \prod_{k=1}^{m} (4k - 1) - \prod_{k=1}^{m} (4k + 1).
\]

The results described in this paper started with some empirical observations on the behavior of the 2-adic valuation of \(A_{l,m}\), i.e. \(\nu_2(A_{l,m})\). Recall that, for \(x \in \mathbb{N}\), the 2-adic valuation \(\nu_2(x)\) is the highest power of 2 that divides \(x\). This is extended to \(x = a/b \in \mathbb{Q}\) via \(\nu_2(x) = \nu_2(a) - \nu_2(b)\). From (1.10) it follows that \(A_{0,m}\) is odd, so \(\nu_2(A_{0,m}) = 0\). Moreover,

\[
\nu_2(A_{1,m}) = \nu_2(m(m + 1)) + 1,
\]

i.e., the main result of [5]. We present as Theorem 2.1, an expression for \(\nu_2(A_{l,m})\) that generalizes (1.11).

The study of the sequence

\[
X(l) := \{\nu_2(A_{l,l+m-1}) : m \geq 1\}
\]

requires the introduction of two operators, \(F\) and \(T\), defined in (4.1) and (4.2), respectively. The iteration of these operators creates an integer vector

\[
\Omega(l) := \{n_1, n_2, n_3, \ldots, n_{\omega(l)}\}, \quad \text{with} \quad n_i \in \mathbb{N},
\]

associated to the index \(l \in \mathbb{N}\). We call \(\Omega(l)\) the reduction sequence of \(l\). See (4.2) for the precise definition of the integers \(n_j\). The structure of \(X(l)\) motivates the following definition.

**Definition 1.1.** Let \(s \in \mathbb{N}\), \(s \geq 2\). We say that a sequence \(\{a_j : j \in \mathbb{N}\}\) is simple of length \(s\) (or \(s\)-simple) if \(s\) is the largest integer such that for each \(t \in \{0, 1, 2, \ldots\}\), we have

\[
a_{st+1} = a_{st+2} = \cdots = a_{s(t+1)}.
\]

The sequence \(\{a_j : j \in \mathbb{N}\}\) is said to have a block structure if it is \(s\)-simple for some \(s \geq 2\).
Section 2 presents two proofs of the expression for \( \nu_2(A_{l,m}) \). Section 3 shows that \( X(l) \) is a simple sequence of length \( 2^{l+\nu_2(l)} \). In Section 4 an algorithm generating the vector \( \Omega(l) \) is described in detail. A combinatorial interpretation of \( \Omega(l) \), as the composition of \( l \), is provided in Section 5. Theorem 5.5 gives \( \Omega(l) \) in terms of the dyadic expansion of \( l \). More precisely, if \( \{k_1, \ldots, k_n : 0 \leq k_1 < k_2 < \cdots < k_n \} \) is the unique collection of distinct nonnegative integers such that \( l = \sum_{i=1}^{n} 2^{k_i} \), then the reduction sequence \( \Omega(l) \) of \( l \) is \( \{k_1 + 1, k_2 - k_1, \ldots, k_n - k_{n-1} - 1 \} \). Finally, the last section contains a conjecture on symmetries of the graph of \( \nu_2(A_{l,m}) \).

2. The 2-adic valuation of \( A_{l,m} \)

In this section we prove that \( \nu_2(A_{l,m}) \) agrees with \( \nu_2((m+1-l)2l) + l \). The first proof actually produces the latter term in a natural way starting from the former. The second proof employs the WZ-machinery [9] to prove the identity (2.1).

**Theorem 2.1.** The 2-adic valuation of \( A_{l,m} \) satisfies

\[
\nu_2(A_{l,m}) = \nu_2((m+1-l)2l) + l,
\]

where \((a)_k = a(a+1) \cdots (a+k-1)\) is the Pochhammer symbol for \( k \geq 1 \). For \( k = 0 \), we define \((a)_0 = 1\).

**Proof. First proof.** We have

\[
\nu_2(A_{l,m}) = l + \nu_2 \left( \sum_{k=l}^{m} T_{m,k} \frac{(m+k)!}{(m-k)! (k-l)!} \right),
\]

where

\[
T_{m,k} = \frac{(2m-2k)!}{2^{m-k} (m-k)!}.
\]

The identity

\[
T_{m,k} = \frac{(2(m-k))!}{2^{m-k} (m-k)!} = (2m-2k-1)(2m-2k-3) \cdots 3 \cdot 1
\]

shows that \( T_{m,k} \) is an odd integer. Then (2.2) can be written as

\[
\nu_2(A_{l,m}) = l + \nu_2 \left( \sum_{k=0}^{m-l} T_{m,l+k} \frac{(m+k+l)!}{(m-k-l)! k!} \right)
\]

\[
= l + \nu_2 \left( \sum_{k=0}^{m-l} T_{m,l+k} \frac{(m-k-l+1)2k+2l}{k!} \right).
\]

The term corresponding to \( k = 0 \) is singled out as we write

\[
\nu_2(A_{l,m}) = l + \nu_2 \left( T_{m,l}(m-l+1)2l + \sum_{k=1}^{m-l} T_{m,l+k} \frac{(m-k-l+1)2k+2l}{k!} \right).
\]

The claim

\[
\nu_2 \left( \frac{(m-k-l+1)2k+2l}{k!} \right) > \nu_2((m-l+1)2l)
\]

for any \( k, 1 \leq k \leq m-l \), will complete the proof.
To prove (2.5) we use the identity
\[
\frac{(m-k-l+1)_{2k+2}}{k!} = (m-l+1)_{2l} \cdot \frac{(m-l-k+1)_{k}}{k!} (m+l+1)_{k}
\]
and the fact that the product of \(k\) consecutive numbers is always divisible by \(k!\).

This follows from the identity
\[
(2.6) \quad \left(\begin{array}{c}
\alpha
\end{array}\right)_{k} = \left(\begin{array}{c}
\alpha + k - 1
\end{array}\right).
\]

Now if \(m + l\) is odd,
\[
(2.7) \quad \nu_2 \left(\frac{(m-l-k+1)_{k}}{k!}\right) \geq 0 \text{ and } \nu_2((m+l+1)_{k}) > 0,
\]
and if \(m + l\) is even
\[
(2.8) \quad \nu_2 \left(\frac{(m+l+1)_{k}}{k!}\right) \geq 0 \text{ and } \nu_2((m-l-k+1)_{k}) > 0.
\]

This proves (2.5) and establishes the theorem.

**Second proof.** Define the numbers
\[
B_{l,m} := A_{l,m} \cdot 2^{(m+1-l)_{2l}}.
\]

We need to prove that \(B_{l,m}\) is odd. The WZ-method [9] provides the recurrence
\[
B_{l-1,m} = (2m+1)B_{l,m} - (m-l)(m+l+1)B_{l+1,m}, \quad 1 \leq l \leq m - 1.
\]

Since the initial values \(B_{m,m} = 1\) and \(B_{m-1,m} = 2m + 1\) are odd, it follows that \(B_{l,m}\) is an odd integer. \(\square\)

### 3. Properties of the function \(\nu_2(A_{l,m})\)

Let \(l \in \mathbb{N} \cup \{0\}\) be fixed. In this section we describe properties of the function \(\nu_2(A_{l,m})\). In particular, we show that each of these sequences has a block structure.

**Theorem 3.1.** Let \(l \in \mathbb{N} \cup \{0\}\) be fixed. Then for \(m \geq l\), we have
\[
(3.1) \quad \nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) = \nu_2(m + l + 1) - \nu_2(m - l + 1).
\]

**Proof.** From (2.1) and \((a)_k = (a + k - 1)!/(a - 1)!\), we have
\[
(3.2) \quad \nu_2(A_{l,m}) = \nu_2 \left(\frac{(m + l)!}{(m - l)!}\right) + l.
\]

This implies
\[
\nu_2(A_{l,m+1}) - \nu_2(A_{l,m}) = \nu_2 \left(\frac{(m + l + 1)!}{(m - l + 1)!}\right) - \nu_2 \left(\frac{(m + l)!}{(m - l)!}\right)
\]
\[
= \nu_2 \left(\frac{(m + l + 1)!}{(m - l + 1)!}\right) \cdot (m - l + 1)!
\]
\[
= \nu_2(m + l + 1) - \nu_2(m - l + 1).
\]

\(\square\)

The next corollary is a special case of Theorem 3.1.
**Corollary 3.2.** The sequence $\nu_2(A_{l,m})$ satisfies
1) $\nu_2(A_{l,t+1}) = \nu_2(A_{l,t})$.
2) For $l$ even,
$$\nu_2(A_{l,t+3}) = \nu_2(A_{l,t+2}) = \nu_2(A_{l,t+1}) = \nu_2(A_{l,t}).$$
3) The sequence $\nu_2(A_{1,m})$ is 2-simple; i.e., $\nu_2(A_{1,m+1}) = \nu_2(A_{1,m})$ for $m$ odd. In fact,
$$A_{1,m} = \{2, 2, 3, 2, 4, 2, 2, 2, 2, 2, \ldots\}.$$

Fix $k, l \in \mathbb{N}$ and let $\mu := 1 + \nu_2(l)$. Define the following sets
$$(3.3) \quad C_{k,l} := \{l + k \cdot 2^\mu + j : 0 \leq j \leq 2^\mu - 1\},$$
which will be instrumental in proving the main result of this section; i.e., $\{\nu_2(A_{l,m})\}$ is $2^{1+\nu_2(l)}$-simple.

We begin by showing that these sets form a partition of $\mathbb{N}$. Moreover, for fixed $k, l \in \mathbb{N}$ the set $C_{k,l}$ has cardinality $2^\mu$ and the 2-adic valuation of $\{A_{l,m} : m \in C_{k,l}\}$ is constant. For example, if $l \in \mathbb{N}$ is odd, then $\mu = 1$ and
$$C_{k,l} = \{l + 2k, l + 2k + 1\}.$$

The next result is immediate.

**Lemma 3.3.** Let $l \in \mathbb{N}$ be fixed. The sets $\{C_{k,l} : k \geq 0\}$ form a disjoint partition of $\mathbb{N}$; namely,
$$(3.5) \quad \{m \in \mathbb{N} : m \geq l\} = \bigcup_{k \geq 0} C_{k,l},$$
and $C_{r,l} \cap C_{t,l} = \emptyset$, whenever $r \neq t$.

**Lemma 3.4.** Fix $l \in \mathbb{N}$ and let $\mu = \nu_2(2l)$.
1) The sequence $\{\nu_2(A_{l,m}) : m \in C_{k,l}\}$ is constant. We denote this value by $\nu_2(C_{k,l})$.
2) For $k \geq 0$, $\nu_2(C_{k+1,l}) \neq \nu_2(C_{k,l})$.

Proof. Suppose $0 \leq j \leq 2^\mu - 2$. Since $\nu_2(2l) = \mu \leq \nu_2(k \cdot 2^\mu)$, then
$$\nu_2(2l + k \cdot 2^\mu) \geq \nu_2(2l) = \mu > \nu_2(j + 1),$$
because $j + 1 < 2^\mu$. Therefore
$$\nu_2(2l + k \cdot 2^\mu + j + 1) = \nu_2(j + 1) = \nu_2(k \cdot 2^\mu + j + 1).$$
Using these facts and (3.1), we obtain
$$\nu_2(A_{l,t+k \cdot 2^\mu + j+1}) - \nu_2(A_{l,t+k \cdot 2^\mu + j}) = \nu_2(2l + k \cdot 2^\mu + j + 1) - \nu_2(k \cdot 2^\mu + j + 1) = \nu_2(j + 1) - \nu_2(j + 1) = 0$$
for consecutive values in $C_{k,l}$. This proves part 1). To prove part 2), it suffices to take elements $l + k \cdot 2^\mu + 2^\nu - 1 \in C_{k,l}$ and $l + (k + 1) \cdot 2^\mu \in C_{k+1,l}$ and compare their 2-adic values. Again by (3.1), we have
$$\nu_2(A_{l,t+(k+1) \cdot 2^\mu}) - \nu_2(A_{l,t+(k+1) \cdot 2^\mu - 1}) = \nu_2(2l + (k + 1) \cdot 2^\mu) - \nu_2((k + 1) \cdot 2^\mu) = \mu + \nu_2(2l \cdot 2^{-\mu} + k + 1) - \mu - \nu_2(k + 1) = \nu_2(2l \cdot 2^{-\mu} + k + 1) - \nu_2(k + 1) \neq 0.$$
The last step follows from $2l \cdot 2^{-\mu}$ being odd and thus $2l \cdot 2^{-\mu} + k + 1$ and $k + 1$ having opposite parities. This completes the proof.
Theorem 3.5. For each \( l \geq 1 \), the set \( \{ \nu_2(A_{l,m}) : m \geq l \} \) is an \( s \)-simple sequence, with \( s = 2^{1+\nu_2(l)} \).

Proof. From Lemma 3.3 and Lemma 3.4, we know that \( \nu_2(\cdot) \) maintains a constant value on each of the disjoint sets \( C_{k,l} \). The length of each of these blocks is \( 2^{1+\nu_2(l)} \). \( \Box \)

4. The Algorithm and its Combinatorial Interpretation

In this section we describe an algorithm that extracts from the sequence \( X(1) := \{ \nu_2(A_{1,m}) : m \geq 1 \} \) its combinatorial information. We begin with the definition of the operators \( F \) and \( T \) mentioned in the Introduction.

Definition 4.1. The maps \( F \) and \( T \). These are defined by

\[
F(\{a_1, a_2, a_3, \cdots \}) := \{a_1, a_1, a_2, a_3, \cdots \},
\]
and

\[
T(\{a_1, a_2, a_3, \cdots \}) := \{a_1, a_3, a_5, a_7, \cdots \}.
\]

We employ the notation

\[
c := \{ \nu_2(m) : m \geq 1 \} = \{0, 1, 0, 2, 0, 1, 0, 3, 0, \cdots \}.
\]

The algorithm:

1) Start with the sequence \( X(l) := \{ \nu_2(A_{l,l+m-1}) : m \geq 1 \} \).

2) Find \( n \in \mathbb{N} \) so that the sequence \( X(l) \) is \( 2^n \)-simple. Define \( Y(l) := T^n(X(l)) \).

At the initial stage, Theorem 3.5 ensures that \( n = 1 + \nu_2(l) \).

3) Introduce the shift \( Z(l) := Y(l) - c \).

4) Define \( W(l) := F(Z(l)) \).

If \( W(l) \) is a constant sequence, then STOP; otherwise go to step 2) with \( W \) instead of \( X \). Define \( X_k(l) \) as the new sequence at the end of the \( (k-1) \)th cycle of this process, with \( X_1(l) = X(l) \).

Section 5 contains the justification for the steps of this algorithm. In particular, we prove that the sequences \( X_k(l) \) have a block structure, so they can be used back in step 1 after each cycle. Theorem 5.3 states that the algorithm finishes in a finite number of steps and that \( W(l) \) is essentially \( X(j) \), for some \( j < l \).

Definition 4.2. Let \( \omega(l) \) be the number of cycles required for the algorithm to yield a constant sequence and denote by \( n_j \) the integers appearing in Step 2 of the algorithm. The integer vector

\[
\Omega(l) := \{n_1, n_2, n_3, \cdots, n_{\omega(l)}\}
\]

is called the reduction sequence of \( l \). The number \( \omega(l) \) will be called the reduction length of \( l \). The constant sequence obtained after \( \omega(l) \) cycles is called the reduced constant.
Table 1. Reduction sequence for \(1 \leq l \leq 15\).

| \(l\) | binary form | \(\Omega(l)\) |
|-------|-------------|--------------|
| 4     | 100         | 3            |
| 5     | 101         | 1, 2         |
| 6     | 110         | 2, 1         |
| 7     | 111         | 1, 1, 1      |
| 8     | 1000        | 4            |
| 9     | 1001        | 1, 3         |
| 10    | 1010        | 2, 2         |
| 11    | 1011        | 1, 1, 2      |
| 12    | 1100        | 3, 1         |
| 13    | 1101        | 1, 2, 1      |
| 14    | 1110        | 2, 1, 1      |
| 15    | 1111        | 1, 1, 1, 1   |

In Corollary 5.8 we enumerate \(\omega(l)\) as the number of ones in the binary expansion of \(l\). Therefore the algorithm yields a constant sequence in a finite number of steps. In fact, the algorithm terminates after \(O(\log_2(l))\) cycles as will follow directly from Corollory 5.8. Table 1 shows the results of the algorithm for \(4 \leq l \leq 15\).

We now provide a combinatorial interpretation of \(\Omega(l)\). This requires the composition of the index \(l\).

**Definition 4.3.** Let \(l \in \mathbb{N}\). The composition of \(l\), denoted by \(\Omega_1(l)\), is defined as follows: write \(l\) in binary form. Read the sequence from right to left. The first part of \(\Omega_1(l)\) is the number of digits up to and including the first 1 read in the corresponding binary sequence; the second one is the number of additional digits up to and including the second 1 read, and so on.

**Example 4.4.** Reading off the values from Table 1, we obtain \(\Omega_1(13) = \{1, 2, 1\}\) and \(\Omega_1(14) = \{2, 1, 1\}\). Therefore \(\Omega_1(13) = \Omega(13)\) and \(\Omega_1(14) = \Omega(14)\). Corollary 5.6 shows that this is always true.

The next result describes the formation of \(\Omega_1(l)\) from \(\Omega_1(\lfloor l/2 \rfloor)\).

**Lemma 4.5.** Given the values of \(\Omega_1(l)\) for \(2^j \leq l \leq 2^{j+1} - 1\), the list for \(2^{j+1} \leq l \leq 2^{j+2} - 1\) is formed according to the following rule:

- \(l\) is even: add 1 to the first part of \(\Omega_1(l/2)\) to obtain \(\Omega_1(l)\);
- \(l\) is odd: prepend a 1 to \(\Omega_1 \left( \frac{l-1}{2} \right)\) to obtain \(\Omega_1(l)\).

**Proof.** Let \(x_1x_2 \cdots x_k\) be the binary representation of \(l\). Then \(x_1x_2 \cdots x_k0\) corresponds to \(2l\). Thus, the first part of \(\Omega_1(2l)\) is increased by 1, due to the extra 0 on the right. The relative position of the remaining 1s stays the same. A similar argument takes care of \(\Omega_1(2l + 1)\). The extra 1 that is placed at the end of the binary representation gives the first 1 in \(\Omega_1(2l + 1)\). \(\square\)

We now relate the 2-adic valuation of \(A_{l,m}\) to that of \(A_{\lfloor l/2 \rfloor, m}\).
Proposition 4.6. Let

\[ \lambda_l := \frac{1 - (-1)^l}{2}, \quad M_0 := \left\lfloor \frac{m + \lambda_l}{2} \right\rfloor. \]

Then

\[ \nu_2(A_{l,m}) = 2l - \lfloor l/2 \rfloor + \lambda_l \nu_2(M_0 - \lfloor l/2 \rfloor) + \nu_2(A_{\lfloor l/2 \rfloor, M_0}). \]

Proof. We present the details for \( \nu_2(A_{2l,2m}) \). Theorem 2.1 gives

\[ \nu_2(A_{2l,2m}) = \nu_2((2m - 2l + 1)2l) + \nu_2(2l) \]

\[ = \nu_2((2m - 2l + 1)(2m - 2l + 2) \cdots (2m + 2l - 1)(2m + 2l)) + 2l \]

\[ = \nu_2(2l(m - l + 1)(m - l + 2) \cdots (m + l)) + 2l \]

\[ = 4l + \nu_2((m - l + 1)2l) \]

\[ = 3l + \nu_2(A_{l,m}). \]

A similar calculation shows that

\[ \nu_2(A_{2l+1,2m}) = 3l + 2 + \nu_2(A_{l,m}) + \nu_2(m - l). \]

The general case then follows from Theorem 3.1. □

Corollary 4.7. The 2-adic valuation of \( A_{l,m} \) satisfies

\[ \nu_2(A_{l,m}) = 2l + \nu_2(l!) + \sum_{k \geq 0} \lambda_{\lfloor l/2^k \rfloor} \nu_2(M_k - \lfloor l/2^{k+1} \rfloor) \]

where

\[ M_k = \left\lfloor \frac{m + \lambda_l + 2\lambda_{\lfloor l/2 \rfloor} + \cdots + 2^k\lambda_{\lfloor l/2^k \rfloor}}{2^{1+k}} \right\rfloor = \left\lfloor \frac{m + \sum_{n=0}^{k} 2^n \lambda_{\lfloor l/2^n \rfloor}}{2^{1+k}} \right\rfloor. \]

Proof. This is a repeated application of Proposition 4.6. The first term results from

\[ \sum_{k \geq 0} \left( 2^\lfloor \frac{l}{2^k} \rfloor - \lfloor \frac{l}{2^{k+1}} \rfloor \right) = 2l + \sum_{k \geq 1} \left\lfloor \frac{l}{2^k} \right\rfloor \]

\[ = 2l + \nu_2(l!). \]

□

5. Verification of the Algorithm and the Reduction sequence

In this section we show that the algorithm presented in Section 4 terminates after a finite numbers of cycles. Moreover, we prove that \( \Omega(l) \), the reduction sequence of \( l \), is identical to the composition sequence of \( l \).

Notation: The constant sequences will be denoted by \( (t) = \{t, t, t, \ldots \} \).

Definition 5.1. A sequence \( (a) = \{a_1, a_2, a_3, \ldots \} \) is a translate of \( (b) = \{b_1, b_2, b_3, \ldots \} \) if \( (a) = (b) + (t) \), for some constant sequence \( (t) \). Addition of sequences is performed term by term.

We first consider the base case \( l = 1 \).
Lemma 5.2. The initial case \( l = 1 \) satisfies
\[
W(1) = F(T(X(1)) - c) = (2),
\]
where \( (c) \) is given in (4.3).

Proof. Since \( \nu_2(A_{1,m}) = \nu_2(m(m + 1)) + 1 \) and \( \nu_2(2m - 1) = 0 \), we have
\[
T(X(1)) = \{\nu_2((2m - 1)(2m)) + 1 : m \geq 1\} = \{\nu_2(m) + 2 : m \geq 1\} = c + (2).
\]
Then the assertion follows from \( F((t)) = (t) \) for a constant \( (t) \).

□

Theorem 5.3. The algorithm terminates after finitely many iterations. Furthermore, in each cycle, \( W(l) \) is a translate of \( X(j) \), for some \( j < l \).

Proof. Start by rewriting the terms in \( X(l) \) as
\[
\nu_2\left(\frac{(m-1+2l)!}{(m-1)!}\right) + l = \nu_2((m-1+2l)(m-2+2l)\cdots(m+1)m) + l, \quad m \geq 1.
\]
Then, the operator \( T \) acts on these to yield (for \( m \geq 1 \))
\[
\nu_2((2m-2+2l)(2m-3+2l)\cdots(2m)(2m-1)) + l
= \nu_2((m-1+l)\cdots(m)) + 2l
\]
(5.2)

Case I: \( l \) is even. From (5.2), we can easily obtain the relation
\[
T(X(l)) = \{\nu_2\left(\frac{(m-1+l)!}{(m-1)!}\right) + l/2 + t : m \geq 1\} = X(l/2) + (t), \quad t = 3l/2.
\]

Case II: \( l \) is odd. Upon subtracting the sequence \( c = \{\nu_2(m) : m \geq 1\} \) from (5.2) we get that
\[
\nu_2\left(\frac{(m+l-1)!}{m!}\right) + 2l = \nu_2\left(\frac{(m+l-1)!}{m!}\right) + \frac{l-1}{2} + \frac{3(l-1)}{2} + 2,
\]
for \( m \geq 1 \). Then, apply the operator \( F \) to the last sequence and find
\[
W(l) = \{\nu_2\left(\frac{(m-2+l)!}{(m-1)!}\right) + \frac{l-1}{2} + t : m \geq 1\} = X\left(\frac{l-1}{2}\right) + (t), \quad t = (3l+1)/2.
\]
Here, we have utilized the property that \( \nu_2(r!) = \nu_2((r-1)!) \), when \( r \geq 1 \) is odd. This justifies that the first term augmented in the sequence, as a result of the action of \( F \), coincides with the next term (these are values at \( m = 1 \) and \( m = 2 \), respectively).

We can now conclude that in either of the two cases (or a combination thereof), the index \( l \) shrinks dyadically. Thus the reduction algorithm must end in a finite step into a translate of \( X(1) \). Since Lemma 5.2 handles \( X(1) \), the proof is completed.

□

Corollary 5.4. For general \( k \in \mathbb{N} \), the sequence \( X_k(l) \) is \( 2^{nk} \)-simple for some \( n_k \in \mathbb{N} \).
Theorem 5.5. Let \( \{k_1, \cdots, k_n : 0 \leq k_1 < k_2 < \cdots < k_n\} \) be the unique collection of distinct nonnegative integers such that

\[
(5.3) \quad l = \sum_{i=1}^{n} 2^{k_i}.
\]

Then the reduction sequence \( \Omega(l) \) of \( l \) is \( \{k_1 + 1, k_2 - k_1, \cdots, k_n - k_{n-1}\} \).

Proof. The argument of the proof is to check that the rules of formation for \( \Omega_1(l) \) also hold for the reduction sequence \( \Omega(l) \). The proof is divided according to the parity of \( l \). The case \( l \) odd starts with \( l = 1 \), where the block length is 2. From Theorem 2.1 we obtain a constant sequence after iterating the algorithm once. Thus the algorithm terminates and the reduction sequence for \( l = 1 \) is \( \Omega(1) = \{1\} \).

Now consider the general even case: \( X(2l) \). Theorem 5.3 shows that applying \( T \) to this sequence yields a translate of \( X(l) \). This does not affect the reduction sequence \( \Omega(l) \), but the doubling of block length increases the first term of \( \Omega(l) \) by 1. Therefore

\[
(5.4) \quad \Omega(2l) = \{k_1 + 2, k_2 - k_1, \cdots, k_n - k_{n-1}\}.
\]

This is precisely what happens to the binary digits of \( l \): if

\[
(5.5) \quad l = \sum_{i=1}^{n} 2^{k_i}, \quad \text{then} \quad 2l = \sum_{i=1}^{n} 2^{k_i+1}.
\]

This concludes the argument for even indices.

For the general odd case, \( X(2l+1) \), we apply \( T \), subtract \( c \) and then apply \( F \). Again, by Theorem 5.3, this gives us a translate of \( X(l) \). We conclude that, if the reduction sequence of \( l \) is

\[
(5.6) \quad \{k_1 + 1, k_2 - k_1, \cdots, k_n - k_{n-1}\},
\]

then that of \( 2l + 1 \) is

\[
\{1, k_1 + 1, k_2 - k_1, \cdots, k_n - k_{n-1}\}.
\]

This is precisely the behavior of \( \Omega_1 \). The proof is complete. \( \Box \)

Corollary 5.6. The reduction sequence \( \Omega(l) \) associated to an integer \( l \) is the sequence of compositions of \( l \), that is,

\[
(5.7) \quad \Omega(l) = \Omega_1(l).
\]

Corollary 5.7. The reduced constant is \( 2l + \nu_2(2l!) = \nu_2(A_{l,l}) \).

Proof. In Corollary 4.7, subtract the last term as per the reduction algorithm. \( \Box \)

Corollary 5.8. The set \( \Omega(l) \) has cardinality

\[
(5.8) \quad s_2(l) = \text{the number of ones in the binary expansion of } l.
\]

Note. The function \( s_2(l) \) defined in (5.8) has recently appeared in a different divisibility problem. Lengyel [7] conjectured, and De Wannemacker [10] proved, that the 2-adic valuation of the Stirling numbers of the second kind \( S(n, k) \) is given by

\[
(5.9) \quad \nu_2(S(2^n, k)) = s_2(k) - 1.
\]

The reader will find in [1] a general study of the 2-adic valuation of Stirling numbers.
6. A Symmetry Conjecture on the Graphs of $\nu_2(A_{l,m})$

The graphs of the function $\nu_2(A_{l,m})$, where we take every other $2^{1+\nu_2(l)}$-element to reduce the repeating blocks to a single value, are shown in the next figures. We conjecture that these graphs have a symmetry property generated by what we call an *initial segment* from which the rest is determined by adding a *central piece* followed by a *folding rule*. We conclude with sample pictures of this phenomenon.

**Example 6.1.** For $l = 1$, the first few values of the reduced table are

\[ \{2, 3, 2, 4, 2, 3, 5, 2, 3, \ldots \}. \]

![Figure 1. The 2-adic valuation of $A_{1,m}$](image)

The ingredients are:

- **initial segment**: $\{2, 3, 2\}$,
- **central piece**: the value at the center of the initial segment, namely 3.
- **rules of formation**: start with the initial segment and add 1 to the central piece and reflect.

This produces the sequence

\[ \{2, 3, 2\} \rightarrow \{2, 3, 2, 4\} \rightarrow \{2, 3, 2, 4, 2, 3, 2\} \rightarrow \{2, 3, 2, 4, 2, 3, 2, 5\} \rightarrow \{2, 3, 2, 4, 2, 3, 2, 5, 2, 3, 2, 4, 2, 3, 2\}. \]

The details are shown in Figure 1.

**Remark.** We have found no way to predict the initial segment nor the central piece. Figure 2 shows the beginning of the case $l = 9$. From here one could be tempted to anticipate that this graph extends as in the case $l = 1$. This is not correct however, as can be seen in Figure 3. In fact, the initial segment is depicted in Figure 3 and its extension is shown in Figure 4.

The initial pattern can be quite elaborate. Figure 5 illustrates the case $l = 53$ and Figure 6 shows it for $l = 59$. A complete description of these initial segments is open to further exploration.
Figure 2. The beginning for \( l = 9 \)

Figure 3. The continuation of \( l = 9 \)

Figure 4. The pattern for \( l = 9 \) persists

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Figure 5. The initial pattern for $l = 53$

Figure 6. The initial pattern for $l = 59$

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