EXISTENCE AND NONEXISTENCE OF EXTREMALS FOR CRITICAL ADAMS INEQUALITIES IN $\mathbb{R}^4$ AND TRUDINGER-MOSER INEQUALITIES IN $\mathbb{R}^2$

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Abstract. Though much work has been done with respect to the existence of extremals of the critical first order Trudinger-Moser inequalities in $W^{1,n}(\mathbb{R}^n)$ and higher order Adams inequalities on finite domain $\Omega \subset \mathbb{R}^n$, whether there exists an extremal function for the critical higher order Adams inequalities on the entire space $\mathbb{R}^n$ still remains open. The current paper represents the first attempt in this direction. The classical blow-up procedure cannot apply to solving the existence of critical Adams type inequality because of the absence of the Pólya-Szego type inequality. In this paper, we develop some new ideas and approaches based on a sharp Fourier rearrangement principle (see [24]), sharp constants of the higher-order Gagliardo-Nirenberg inequalities and optimal poly-harmonic truncations to study the existence and nonexistence of the maximizers for the Adams inequalities in $\mathbb{R}^4$ of the form

$$S(\alpha) = \sup_{\|u\|_{H^2}=1} \int_{\mathbb{R}^4} \left( \exp(32\pi^2|u|^2) - 1 - \alpha|u|^2 \right) dx,$$

where $\alpha \in (-\infty, 32\pi^2)$. We establish the existence of the threshold $\alpha^*$, where $\alpha^* \geq \frac{(32\pi^2)^2 B_2}{2}$ and $B_2 \geq \frac{1}{24\pi^2}$, such that $S(\alpha)$ is attained if $32\pi^2 - \alpha < \alpha^*$, and is not attained if $32\pi^2 - \alpha > \alpha^*$. This phenomena has not been observed before even in the case of first order Trudinger-Moser inequality. Therefore, we also establish the existence and non-existence of an extremal function for the Trudinger-Moser inequality on $\mathbb{R}^2$. Furthermore, the symmetry of the extremal functions can also be deduced through the Fourier rearrangement principle.

Keywords: Trudiner-Moser inequality, Adams inequality, blow up analysis, extremal function, Sharp Fourier rearrangement principle, sharp constants, threshold.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ and $W_0^{m,p}(\Omega)$ denote the usual Sobolev space consisting of functions vanishing on boundary $\partial \Omega$ together with their derivatives of order less than $m - 1$, that
is, the completion of $C_0^\infty(\Omega)$ under the norm
\[\|u\|_{W^{m,p}(\Omega)} = \left( \int_{\Omega} \left( |u|^p + |\Delta^{m/2} u|^p \right) \, dx \right)^{\frac{1}{p}}.\]
If $1 < p < n/m$, the classical Sobolev embedding asserts that $W_0^{m,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ for $p^* = \frac{np}{n-mp}$. However, when $p = n/m$, it is known that $W_0^{m,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ does not hold.

The borderline case of the optimal Sobolev embedding is the well-known Trudinger-Moser inequality ($m = 1$) ([42], [50]) and Adams inequality ($m > 1$) ([2]).

**Trudinger-Moser inequality.** The Trudinger inequality was established independently by Yudovič [52], Pohožaev [46] and Trudinger [50]. In 1971, Trudinger’s inequality was sharpened in [42] by proving
\[
\sup_{u \in W_0^{1,n}(\Omega) \atop \|\nabla u\|_{n(\Omega)} \leq 1} \int_{\Omega} e^{\alpha |u|^n} \, dx < \infty \text{ iff } \alpha \leq \alpha_n = n \omega_{n-1}^{-\frac{1}{n-1}},
\]
for any bounded domain $\Omega \subset \mathbb{R}^n$, where $\omega_{n-1}$ denotes the $n-1$ dimensional surface measure of the unit ball in $\mathbb{R}^n$.

When the volume of $\Omega$ is infinite, there are several extensions of the Trudinger-Moser inequality, see Cao [5] in the case $n = 2$ and for any dimension ($n \geq 2$) by do Ó [11]. Adachi-Tanaka [1] obtained a sharp Trudinger-Moser on $\mathbb{R}^n$. Unlike in the inequality (1.1), the result of [1] has a subcritical form, that is $\alpha < \alpha_n$. In [47] and [30], Li and Ruf showed that the exponent $\alpha_n$ becomes admissible if the Dirichlet norm $\int_{\mathbb{R}^n} |\nabla u|^2 \, dx$ is replaced by Sobolev norm $\int_{\mathbb{R}^n} \left( |u|^2 + |\nabla u|^2 \right) \, dx$, more precisely, they proved that
\[
\sup_{u \in W^{1,n}(\mathbb{R}^n) \atop \int_{\mathbb{R}^n} (|u|^n + |\nabla u|^n) \, dx \leq 1} \int_{\mathbb{R}^2} \Phi_n \left( \alpha |u|^{n/(n-1)} \right) \, dx < +\infty, \text{ iff } \alpha \leq \alpha_n,
\]
where $\Phi_n(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}$.

We should note that all the earlier proofs of both critical and subcritical Trudinger-Moser inequalities rely on the Pólya-Szegö symmetrization argument which is not available in many other non-Euclidean settings. Lam and Lu in [17] developed a symmetrization-free argument using the level sets of the functions under consideration and derive critical Trudinger-Moser inequalities on the Heisenberg groups from local inequalities obtained in [8] to global ones (see also [19], [26]). For such an argument in the subcritical case, see [20]. These also give an alternative proof of both critical and subcritical Trudinger-Moser in the Euclidean space $\mathbb{R}^n$.

**Existence of extremals for Trudinger-Moser inequality.** A classical problem related to Trudinger-Moser inequalities is to investigate the existence of extremal functions. The first proof of the existence of extremals for Trudinger-Moser inequality (1.1) was given by Carlsson and Chang in their celebrated work [6] when the finite domain is a
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ball in $\mathbb{R}^n$. After that, the existence of extremals was proved for any bounded domains in [12] and [31] in $\mathbb{R}^n$. More related results can be found in several works, see e.g. Y. X. Li ([27], [28], [29]) for existence of extremals on compact Riemannian manifold, and Li and Ruf ([47], [30]) for existence of extremals on unbounded domain. Malchiodi and Martinazzi [35] further investigated the blow-up of a sequence of critical points of the Trudinger-Moser functionals on the planar disk.

Adams inequality on bounded domains. In 1988, Adams [2] extended the Trudinger-Moser inequality (1.1) to the higher order space $W^{m, \frac{n}{m}}(\Omega)$ and obtained

$$
\sup_{u \in W^{m, \frac{n}{m}}(\Omega)} \int_{\Omega} \exp(\beta |u(x)|^{\frac{n}{n-m}}) dx \begin{cases} 
\leq c |\Omega| & \text{if } \beta \leq \beta(n, m), \\
+\infty & \text{if } \beta > \beta(n, m),
\end{cases}
$$

where

$$
\Delta^\frac{n}{m} u = \begin{cases} 
\Delta^l u & \text{is } m = 2l, l \in \mathbb{N} \\
\nabla \Delta^l u, \text{is } m = 2l + 1, l \in \mathbb{N}
\end{cases}
$$

and

$$
\beta(n, m) = \begin{cases} 
\frac{n^{n/2m} \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \frac{n}{m}, & \text{when } m \text{ is odd.} \\
\frac{n^{n/2m} \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \frac{n}{m}, & \text{when } m \text{ is even.}
\end{cases}
$$

Later, Tarsi [49] proved that the Adams inequality (1.2) also holds for a larger class of Sobolev functions, i.e. the functions with homogeneous Navier boundary condition:

$$
W^{m, \frac{n}{m}}_N(\Omega) = \{ u \in W^{m, \frac{n}{m}}(\Omega), \text{s.t. } \Delta^j u = 0 \text{ on } \partial\Omega \text{ for } 0 \leq j \leq \lceil (m-1)/2 \rceil \}.
$$

Adams inequality on the entire Euclidean space $\mathbb{R}^n$. In 1995, Ozawa [44] obtained the Adams inequality in Sobolev space $W^{m, \frac{n}{m}}(\mathbb{R}^n)$ on the entire Euclidean space $\mathbb{R}^n$ by using the restriction $\|\Delta^\frac{n}{m} u\|_{\frac{n}{m}} \leq 1$. However, with the argument in [44], one cannot obtain the best possible exponent $\beta$ for this type of inequality. Sharp Adams inequalities on even dimensional space $\mathbb{R}^n$ was proved by Ruf and Sani [48] under the stronger constraint

$$
\{ u \in W^{m, \frac{n}{m}} \|(I - \Delta)^\frac{n}{m} u\|_{\frac{n}{m}} \leq 1 \},
$$

when the order of derivatives $m$ is an even integer. While the order of derivatives $m$ is odd, the inequality was established by Lam and Lu [18]. Moreover, the following Adams inequality on Sobolev spaces $W^{\gamma, \frac{n}{m}}(\mathbb{R}^n)$ of arbitrary positive fractional order $\gamma < n$ was established by Lam and Lu using a rearrangement-free argument [19].

Theorem 1.1. Let $0 < \gamma < n$ be an arbitrary real positive number, $p = \frac{n}{\gamma}$ and $\tau > 0$. There holds

$$
\sup_{u \in W^{\gamma, p}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi \left( \beta_0 (n, \gamma) |u|^p \right) dx < \infty
$$
where
\[ \phi(t) = e^t - \sum_{j=0}^{\frac{n-2}{2}} \frac{t^j}{j!}, \]
\[ j_p = \min \{ j \in \mathbb{N} : j \geq p \} \geq p, \]
and
\[ p' = \frac{p}{p-1}. \]

\[ \beta_0 (n, \gamma) = \frac{n}{\omega_{n-1}} \left[ \frac{\pi^{n/2} \Gamma \left( \frac{\gamma}{2} \right)}{\Gamma \left( \frac{n-\gamma}{2} \right)} \right]^{\frac{1}{p'}}. \]

Furthermore this inequality is sharp, i.e., if \( \beta_0 (n, \gamma) \) is replaced by any \( \beta > \beta_0 (n, \gamma) \), then the supremum is infinite.

The following Adams inequality was established in [19] for \( m = 2 \) and subsequently in [13] for \( m > 2 \):

\[ (1.3) \sup_{u \in W^m, \frac{\omega}{m}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \Phi_{n,m}(\beta |u(x)| \frac{n}{m}) dx \]
\[ \leq C_{m,n} \text{ if } \beta \leq \beta(n, m), \]
\[ = +\infty \text{ if } \beta > \beta(n, m), \]
\[ \|\Delta^\frac{m}{2} u\|_{\frac{n}{m}} + \|u\|_{\frac{n}{m}} \leq 1 \]

where
\[ \Phi_{n,m}(t) = e^t - \sum_{j=0}^{\frac{n-2}{2}} \frac{t^j}{j!}, \]
\[ j_n = \min \{ j \in \mathbb{N} : j \geq \frac{n}{m} \}. \]

We mention that there are sharpened Trudinger-Moser and Adams inequalities with exact growth in \( \mathbb{R}^n \). In 2011, Ibrahim et al [15] discovered a new kind of Trudinger-Moser inequality on \( \mathbb{R}^2 \)–the Trudinger-Moser inequality with the exact growth condition:

\[ (1.4) \sup_{u \in H^1(\mathbb{R}^2)} \int_{\mathbb{R}^2} \frac{\exp(4\pi |u|^2) - 1}{(1 + |u|)^p} dx \leq C_p \int_{\mathbb{R}^2} |u|^2 dx \text{ iff } p \geq 2. \]

Later, (1.4) was extended to the general case \( n \geq 3 \) by Masmoudi and Sani [39] (see [23]) for more general form and to the framework of hyperbolic space by Lu and Tang in [32].

The second order Adams’ inequality with the exact growth condition was obtained by Masmoudi and Sani [38] in dimension 4:

\[ (1.5) \sup_{u \in H^2(\mathbb{R}^4)} \int_{\mathbb{R}^4} \frac{\exp(32\pi^2 |v|^2) - 1}{(1 + |v|)^p} dx \leq C_p \int_{\mathbb{R}^4} |v|^2 dx \text{ iff } p \geq 2, \]

and then established in any dimension \( n \geq 3 \) by Lu et al in [33] (see [40] for higher order case).
Existence of extremals for Adams inequality. The first result of the existence of extremals of Adams’ inequality (1.2) on bounded domain \( \Omega \subset \mathbb{R}^n \) was obtained by Lu and Yang in [34]. We note that the work of Carleson and Chang was based on the rearrangement argument to reduce the problem to the one-dimensional case. However, the symmetrization technique cannot be used for the Adams inequality, since there is no corresponding Pólya-Szegő type inequality in the higher order case. In [34], the authors applied the capacity-type estimates and the Pohozaev identity to obtain the existence of extremals for bounded domains in the case \( n = 4 \) and \( m = 2 \). Recently, DelaTorre and Mancini [9] extended the results of [34] to arbitrary even dimension.

1.1. The main results and Outline of the paper. An interesting and intriguing question is whether the Adams inequality on any unbounded domain has an extremal. As far as we know, nothing is known at the present. In this work, we are devoted to studying this kind of problem for the special case \( n = 4 \) and \( m = 2 \).

Setting

\[
S(\alpha) = \sup_{u \in H} \int_{\mathbb{R}^4} \left( \exp \left( 32\pi^2 |u|^2 \right) - 1 - \alpha |u|^2 \right) dx,
\]

and

\[
(1.6) \quad B_2 = \sup_{u \in H^2(\mathbb{R}^4)} \frac{\|v\|_{H^2}^4}{\|\Delta v\|_2^2},
\]

where \( \alpha \in (-\infty, 32\pi^2) \), and

\[
H := \left\{ u \in H^2(\mathbb{R}^4) \mid \|u\|_{H^2(\mathbb{R}^4)} = \left( \int_{\mathbb{R}^4} |u|^2 + |\Delta u|^2 dx \right)^{\frac{1}{2}} = 1 \right\}.
\]

First, we can prove the following result.

**Theorem 1.2.** If \( 32\pi^2 - \alpha < \frac{(32\pi^2)^2 H_2}{2} \), then \( S(\alpha) \) has a radially symmetric extremal function.

Naturally, one may ask whether extremal functions of critical Adams inequalities must be radially symmetric. Recently, Lenzmann and Sork [24] introduced the Fourier-rearrangement inequalities. Though they did not prove the existence of the Adams inequality on the entire space, they observed that every possible maximizer of \( S(\alpha) \), if exists, must be radially symmetric and real valued (up to translation and constant phase). In fact, assume that \( u \) is a maximizer for \( S(\alpha) \), and define \( u^\sharp \) by \( u^\sharp = F^{-1}\{ (F(u))^* \} \), where \( F \) denotes the Fourier transform on \( \mathbb{R}^4 \) (with its inverse \( F^{-1} \)) and \( u^\sharp \) stands for the Schwarz symmetrization of \( u \). We easily see that \( u^\sharp \) is also a maximizer for \( S(\alpha) \) with \( \|\Delta u^\sharp\|_2 = \|\Delta u\|_2 \). Using the property of the Fourier rearrangement from [24], we conclude that

\[
u(x) = e^{i\alpha} u^\sharp(x - x_0) \quad \text{for any } x \in \mathbb{R}^4
\]

with some constants \( \alpha \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^4 \). That is to say that \( u \) is radially symmetric and real valued up to translation and constant phase. Therefore, combining our Theorem 1.2 with Lenzmann and Sork’s result, we obtain that
Corollary 1.3. If $32\pi^2 - \alpha < \frac{(32\pi^2)^2 B_2}{2}$, the extremals of the Adams inequalities must be radially symmetric and real valued (up to translation and constant phase).

The method developed in this paper on the existence of the extremals of the critical Adams inequality also gives new insight on the existence of extremal functions for the first order critical Trudinger-Moser inequality on the entire space. By adapting the same method as used in the proof of Theorem 1.2, we can also obtain a similar existence result. More precisely, if we define

$$\tilde{S}(\alpha) = \sup_{\|u\|_{H^1} = 1} \int_{\mathbb{R}^2} \left( \exp(4\pi |u|^2) - 1 - \alpha |u|^2 \right) dx,$$

then there exists an extremal function for $\tilde{S}(\alpha)$ when $4\pi - \alpha < \frac{(4\pi)^2 B_1}{2}$, where $B_1 = \sup_{u \in H^1(\mathbb{R}^2)} \frac{\|v\|_4^4}{\|\nabla v\|_2^2 \|v\|_2^2}$.

According to the result in [51], we know $B_1 > \frac{1}{2\pi}$ and $\frac{(4\pi)^2 B_1}{2} > 4\pi$. Therefore, even if we slightly enlarge the coefficient of the first term of the critical Trudinger-Moser functional $\int_{\mathbb{R}^2} \left( \exp(4\pi |u|^2) - 1 \right) dx$, the Trudinger-Moser inequality still has an extremal function. Then we can deduce the following stronger existence result than currently known in the literature.

Theorem 1.4. There exists $\alpha_0 > 0$ such that for any $\beta \in (0, \alpha_0)$, the critical Trudinger-Moser inequality $\sup_{\|u\|_{H^1} = 1} \int_{\mathbb{R}^2} \left( \exp(4\pi |u|^2) - 1 + \beta |u|^2 \right) dx$ has an extremal function. Furthermore, all extremals of the critical Trudinger-Moser inequalities must be non-negative, radially symmetric and real valued up to translation and constant phase.

Since the method to prove Theorem 1.4 is inspired by and similar to that of proving Theorem 1.2 and Corollary 1.3 for the critical Adams inequalities on the entire space $\mathbb{R}^4$, we will be sketchy and only give the outline of proofs in Section 6.

Though the general strategy we use here is the blow up analysis, it is considerably more difficult to deal with than the situation on bounded domains. The failure of the Pólya-Szego inequality for the higher order derivatives will not allow us to take care of the maximizing sequence as in the first order Trudinger-Moser inequality on finite balls. To overcome this difficulty, we first apply the method based on the Fourier rearrangement (see [24]) to obtain the existence of radially symmetric extremals for the subcritical Adams functional on the entire space. Then, we take a sequence $\beta_k \to 32\pi^2$ and find a radially symmetric maximizing sequence $u_k \in H^2(\mathbb{R}^4)$ for $S(\alpha)$. If $u_k$ is bounded in $L^\infty(\mathbb{R}^4)$, i.e. $c_k := \max_{x \in \mathbb{R}^4} |u_k| < \infty$, we can easily show that $u_k$ converges to a function $u$ in $H^2(\mathbb{R}^4)$ by the standard elliptic estimates. If $c_k \to +\infty$, i.e. the blow up arises, we apply the blow up analysis method to analyze the asymptotic behavior of $u_k$ near and far away from the origin, which is the blowing up point, and we are able to derive an upper bound for the Adams functional:

$$S(\alpha) \leq \frac{\pi^2}{6} \exp \left( \frac{5}{3} + 32\pi^2 A \right),$$
where $A$ is the value at 0 of the trace of the regular part of the Green function $G$ for the operator $\Delta^2 + 1$. Finally, we construct a function sequence to show that the upper bound can actually be surpassed, this implies that the concentration phenomenon will not happen.

At first sight, this type of approach may seem to be a straightforward generalization of the existing theories. However, this is not the case. Several substantial difficulties exist and some serious subtlety arises. We are going to describe some of them below.

First of all, unlike the case on a bounded domain, in order to establish the existence of a maximizer of the subcritical Adams functional in the entire space $\mathbb{R}^4$, we need to avoid the lack of compactness. In this case, concentration phenomenon does not occur and vanishing phenomenon is the issue due to the unboundedness of the domain. For this reason, we will impose an extra assumption on $\alpha$ such as $32\pi^2 - \alpha < \frac{(32\pi^2)^2 B_2}{2}$ and adapt the argument used in [16] to rule out the vanishing phenomenon.

Secondly, when we try to analyze the asymptotic behavior of $u_k$, a crucial step is to prove a local estimate for $\Delta u_k$:

\[
(1.7) \quad c_k \int_{B_{Rr_k}} |\Delta u_k| dx \leq C(Rr_k)^2.
\]

When $\Omega$ is a bounded domain, (1.7) can be proved by the following estimate

\[
\int_{\Omega} |\Delta (u_k^2)| dx < c
\]

and a representation formula (see [37]). However, because of the unboundedness of the domain, the argument in [37] cannot be directly applied in our setting. In order to obtain the estimate (1.7), we will try to truncate $u_k$ and adapt the approach in [37] to our situation. But as we know, in the second order Sobolev space $H^2(\mathbb{R}^4)$, one cannot truncate $u_k$ linearly. To overcome this difficulty, we will apply the bi-harmonic type of truncation as used in [9]. We remark that this kind of truncations have many nice properties: on the one hand, they preserve the high order regularity, on the other hand, their behaviors are very similar to the constant in a ball. Nevertheless, we must point out that it is necessary for us to find an optimal bi-harmonic truncation in our case.

When we try to obtain the upper bound of the concentration compactness of the Adams inequality on the entire Euclidean space $\mathbb{R}^4$, firstly, one needs to know the specific value of the upper bound for any blow up function sequences in $H^2_0(B_R)$, but this value cannot be directly derived from the result of Lu and Yang [34] in the case of the finite domain. We will show that the value of that upper bound is $\frac{1}{3} |B_R| \exp \left( -\frac{1}{3} \right)$ by solving the corresponding ODE’s. Secondly, in our situation we cannot truncate $u_k$ directly by the linear truncation as in [30]. Therefore, we have to construct some polynomial truncation functions to preserve some regularity on the boundary of the balls. In the calculation, we find that the polynomial truncation functions will bring some extra energy, which will enlarge the estimate of the upper bound for the normalized concentration sequence of the
Adams inequality, such that the upper bound may not be surpassed by any test function sequence. In order to address this problem, we will construct the optimal polynomial truncation functions which generate the smallest energy, such that the exact upper bound of Adams functional for normalized concentration sequence obtained can be surpassed by the same test function. It should be noted that elliptic estimate of the optimal polynomial truncation is far from the exact upper bound of the concentration compactness sequence, we need the precise expression of polynomial truncation without any error estimate. Therefore, we also need the quantitative estimate with respect to the upper bound of the concentration compactness sequence.

Finally, although we can show that \( u_k \) is radially symmetric by the Fourier rearrangement argument, this function sequence is not necessarily positive, which makes the proof of the existence of a maximizer more complicated.

Our second result is as follows.

**Theorem 1.5.** There exists some constant \( \alpha^{**} > 0 \) such that, when \( 32\pi^2 - \alpha > \alpha^{**} \), \( S \) is not attained.

The proof of this theorem is based on the precise estimates for the upper bounds of the best constants of higher order Gagliardo-Nirenberg inequalities. To calculate the best constants, we will exploit the method of Beckner in [3]. We stress that the upper bound derived here is sharp and can be of independent interest. Indeed, we take \( B_2 \) for example. On the one hand, by calculating the number associate to the trial function \( (1 + |x|)^{-\gamma} \) and letting \( \gamma \to +\infty \), one can found that \( \frac{1}{24\pi^2} \) is a lower bound of \( B_2 \). On the other hand, by the upper bound formula we have \( B_2 < \frac{32}{729\pi^2} \) (see Appendix). Then, we get

\[
\frac{1}{24\pi^2} \left( \sim 4.2217 \times 10^{-3} \right) \leq B_2 < \frac{32}{729\pi^2} \left( \sim 4.4476 \times 10^{-3} \right),
\]

which indicates that our estimates are quite precise.

Define

\[
\alpha^* = \sup \left\{ (32\pi^2 - \alpha) \mid S(\alpha) \text{ is attained} \right\}.
\]

Based on Theorem 1.2 and Theorem 1.5, we have \( \frac{(32\pi^2)^2B_2}{2} \leq \alpha^* \leq \alpha^{**} < +\infty \). Furthermore, we can obtain the following surprising result.

**Theorem 1.6.** When \( 32\pi^2 - \alpha < \alpha^* \), then \( S(\alpha) < \alpha^* \) and \( S(\alpha) \) can be attained, while when \( 32\pi^2 - \alpha > \alpha^* \), \( S(\alpha) = 32\pi^2 - \alpha \), and \( S(\alpha) \) is not attained.

Similarly, we can also obtain the following existence and nonexistence of maximizers for the Trudinger-Moser inequality in \( \mathbb{R}^2 \).

**Theorem 1.7.** When \( 4\pi - \alpha < \beta^* \), then \( \tilde{S}(\alpha) < \beta^* \) and \( \tilde{S}(\alpha) \) can be attained; When \( 4\pi - \alpha > \beta^* \), then \( \tilde{S}(\alpha) = 4\pi - \alpha \) and \( \tilde{S}(\alpha) \) is not attained, where \( \beta^* \) is defined as

\[
\beta^* = \sup \left\{ (4\pi - \alpha) \mid S(\alpha) \text{ is attained} \right\}.
\]

and \( \beta^* \geq \frac{(4\pi)^2B_1}{2} > 4\pi \).
Remark 1.8. The proof of Theorems 1.4 and 1.7 concerning the existence and nonexistence of maximizers for the Trudinger-Moser inequality in \( \mathbb{R}^2 \) is similar in spirit to that of the existence and nonexistence of maximizers for the Adams inequality in \( \mathbb{R}^4 \). Therefore, we have chosen to give the sketch of the proof in Section 6.

It is important to point out that Theorem 1.6 provides a further insight on the existence or nonexistence of extremals for Adams inequality on the whole space. From the proof of Theorem 1.2, we know that the supremum of Adams functional is larger than the upper bound of concentration-compactness sequences. Hence whether \( S(\alpha) \) is attained highly depends on the vanishing phenomena, whose energy level is determined only by the coefficient of the first term of \( S(\alpha) \). Thus, changing the coefficients of finite terms (especially the first term) of \( S(\alpha) \), will not affect on the validity of the Adams inequality and the upper bound of concentration sequences, but will change the existence or nonexistence of extremals. It seems that this phenomenon has not been noticed before, even in the case of Trudinger-Moser inequality.

Once the existence and nonexistence of extremals for Adams inequality in the special case \( m = 2, n = 4 \) are established, a natural, but nontrivial extension is to establish similar results for any arbitrary \( m \geq 2 \). The proof for this extension has some extra difficulties to overcome and we have decided to address this problem in a forthcoming paper.

The following remarks are in order. The problem considered here was initially suggested by the second author to the first and third authors several years ago. We have worked together on the problem since then. This is a revised version of the manuscript posted as arXiv:1812.00413v1 by the first and third authors. A mistake in that version was found recently. In particular, the argument of obtaining the optimal upper bound of the concentration-compactness sequence was incorrect in that version. As we pointed out earlier in the introduction, in the derivation of the optimal upper bound of the concentration-compactness sequence, the polynomial truncation functions will add some extra energy which will enlarge the estimate of the upper bound for the normalized concentration sequence of the Adams inequality. Thus, the upper bound may be too large to be surpassed by the any test function sequence. In the old version, we used the elliptic estimates to get the upper bound of the concentration compactness sequence which were far from being the exact bound. In this new version, we apply the optimal and precise expression of polynomial truncation without any error estimate to address this issue and thus derive the optimal upper bound. In this new version, we have also obtained the existence of extremal functions for the critical Trudinger-Moser supremum \( \sup_{\|u\|_{H^1} = 1} \int_{\mathbb{R}^2} \left( \exp(4\pi |u|^2) - 1 + \beta |u|^2 \right) dx \) for a perturbation term of \( \beta |u|^2 \) which gives more information than those known in the literature.

We finally remark that there is some recent development on the existence and nonexistence of extremal functions for subcritical Trudinger-Moser inequalities established by Lam, Lu and Zhang [21] using the equivalence and identities between the supremums for the critical and subcritical Trudinger-Moser inequalities in \( \mathbb{R}^n \) established by the same
For subcritical Adams inequalities on the entire space, the existence of extremal functions has been proved by Chen, Lu and Zhang [7].

This paper is organized as follows. Section 2 is devoted to proving existence of radially symmetric maximizing sequence for the critical Adams functional; in Section 3, we will analyze the asymptotic behavior of the maximizing sequence, and derive an upper bound for the Adams’ inequality when the blowing up arises; In Section 4, we prove the existence of extremals (Theorem 1.2) by constructing a proper test function sequence. In Section 5, we give the proof for Theorem 1.5 and Theorem 1.6 by estimating the best constant of higher order Gagliardo-Nirenberg inequalities. In Section 6, we establish the existence and nonexistence of extremal functions for the Trudinger-Moser inequality in $\mathbb{R}^2$. For the convenience of the reader, the work of estimating the best constants of Gagliardo-Nirenberg inequalities and some known results concerning elliptic estimates for operator $\Delta^2$ are arranged in the Appendix.

Throughout this paper, the letter $c$ always denotes some positive constant which may vary from line to line.

2. The maximizing sequence for critical Adams functional

2.1. Existence of extremals for the subcritical Adams functionals. In this section, we will establish the existence of extremal functions for subcritical Adams functional. Set

$$I_\beta^\alpha (u) = \int_{B_R} \left( \exp(\beta |u|^2) - 1 - \alpha |u|^2 \right) dx.$$

Lemma 2.1. For any $0 < \beta < 32\pi^2$, there exists a radially symmetric extremal function $u \in H$ such that

$$I_\beta^\alpha (u) = \sup_{u \in H} I_\beta^\alpha (u),$$

provided $\beta - \frac{\beta^2 B_2^2}{2} < \alpha < \beta$.

Remark 2.2. It follows from that Lemma 5.1 in the Appendix that $B_2 < \frac{1}{16\pi^2}$, which leads to that $\beta - \frac{\beta^2 B_2^2}{2} > 0$.

Proof. Define $u^\sharp$ by $u^\sharp = \mathcal{F}^{-1}\{(\mathcal{F}(u))^*\}$, where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^4$ (with its inverse $\mathcal{F}^{-1}$) and $f^*$ stands for the Schwarz symmetrization of $f$. Using the property of the Fourier rearrangement from [24], one can derive that

$$\|\Delta u^\sharp\|_2 \leq \|\Delta u\|_2, \|u^\sharp\|_2 = \|u\|_2, \|u^\sharp\|_q \geq \|u\|_2 (q > 2).$$

Hence

$$\sup_{u \in H} I_\beta^\alpha (u) = \sup_{u \in H_r} I_\beta^\alpha (u),$$

where $H_r$ denotes all radial functions in $H$. Therefore, we may assume that $\{u_k\}_k \in H$ is a radially maximizing sequence for $\sup I_\beta^\alpha (u)$, that is

$$\|u_k\|_{H^2(\mathbb{R}^4)} = 1, \lim_{k \to \infty} I_\beta^\alpha (u_k) = \sup_{u \in H} I_\beta^\alpha (u).$$
EXISTENCE AND NONEXISTENCE OF EXTREMALS FOR CRITICAL ADAMS INEQUALITIES IN $\mathbb{R}^4$

By the Sobolev compact embedding, there exists a subsequence $\{u_k\}_k$ such that
\[ u_k(x) \to u(x), \quad \text{strongly in } L^q(B_R(0)) \quad \text{for any } R > 0, \; q > 1 \]
\[ u_k(x) \to u(x), \quad \text{for a.e. } x \in \mathbb{R}^4. \]

Since $\exp(\beta |u|^2) - 1 - \alpha|u|^2 \in L^p(B_R)$ for some $p > 1$, we have
\[
\lim_{k \to \infty} \int_{B_R} \left( \exp(\beta |u_k|^2) - 1 - \alpha|u_k|^2 \right) dx = \int_{B_R} \left( \exp(\beta |u|^2) - 1 - \alpha|u|^2 \right) dx. \tag{2.1}
\]

On the other hand, it follows from the radial lemma that
\[
\lim_{k \to \infty} \int_{\mathbb{R}^4 \setminus B_R} \left( \exp(\beta |u_k|^2) - 1 - \beta u_k^2 \right) dx
\leq c \lim_{k \to \infty} \int_{\mathbb{R}^4 \setminus B_R} |u_k|^4 dx
\leq c \sup_k \|u_k\|_{H^2(\mathbb{R}^4)}^2 R^{-2}. \tag{2.2}
\]

From (2.1) and (2.2), we derive that
\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp(\beta |u_k|^2) - 1 - \beta u_k^2 \right) dx
= \lim_{R \to \infty} \lim_{k \to \infty} \left( \int_{B_R} + \int_{\mathbb{R}^4 \setminus B_R} \right) \left( \exp(\beta |u_k|^2) - 1 - \beta u_k^2 \right) dx
= \int_{\mathbb{R}^4} \left( \exp(\beta |u|^2) - 1 - \beta |u|^2 \right) dx.
\]

Hence, we have
\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp(\beta |u_k|^2) - 1 - \alpha|u_k|^2 \right) dx
= \lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp(\beta |u_k|^2) - 1 - \beta|u_k|^2 \right) dx + (\beta - \alpha) \lim_{k \to \infty} \int_{\mathbb{R}^4} u_k^2 dx
= \int_{\mathbb{R}^4} \left( \exp(\beta |u|^2) - 1 - \beta |u|^2 \right) dx + (\beta - \alpha) \int_{\mathbb{R}^4} \left( u_k^2 - |u|^2 \right) dx. \tag{2.3}
\]

When $u \neq 0$, we set
\[
\tau^4 = \lim_{k \to \infty} \frac{\int_{\mathbb{R}^4} u_k^2 dx}{\int_{\mathbb{R}^4} u^2 dx},
\]
by Fatou's lemma, we have $\tau \geq 1$. Let $\tilde{u}(x) = \frac{u(x)}{\tau}$, we can easily verify the following fact:
\[
\int_{\mathbb{R}^4} |\Delta \tilde{u}|^2 dx = \int_{\mathbb{R}^4} |\Delta u|^2 dx \leq \lim_{k \to \infty} \int_{\mathbb{R}^4} |\Delta u_k|^2 dx,
\]
\[ \int_{\mathbb{R}^4} \tilde{u}^2 \, dx = \tau^4 \int_{\mathbb{R}^4} u^2 \, dx = \lim_{k \to \infty} \int_{\mathbb{R}^4} u_k^2 \, dx \]

and
\[ \int_{\mathbb{R}^4} |(\Delta \tilde{u})^2 + \tilde{u}^2 | \, dx \leq \lim_{k \to \infty} \int_{\mathbb{R}^4} (|\Delta u_k|^2 + u_k^2) \, dx = 1. \]

Hence, by (2.3) we get
\[
\sup_{u \in H} I_\beta^\alpha (u) \geq \int_{\mathbb{R}^4} \left( \exp(\beta \tilde{u}^2) - 1 - \alpha \tilde{u}^2 \right) \, dx = \tau^4 \int_{\mathbb{R}^4} \left( \exp(\beta u^2) - 1 - \alpha u^2 \right) \, dx
\]
\[= \int_{\mathbb{R}^4} \left( \exp(\beta u^2) - 1 - \alpha u^2 \right) \, dx + (\tau^4 - 1) (\beta - \alpha) \int_{\mathbb{R}^4} u^2 \, dx +
\]
\[(\tau^4 - 1) \int_{\mathbb{R}^4} \left( \exp(\beta u^2) - 1 - \alpha u^2 \right) \, dx \geq \lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp(\beta u_k^2) - 1 - \alpha u_k^2 \right) \, dx + (\tau^4 - 1) \int_{\mathbb{R}^4} \left( \exp(\beta u^2) - 1 - \alpha u^2 \right) \, dx \]
\[= \sup_{u \in H} I_\beta^\alpha (u) + (\tau^4 - 1) \int_{\mathbb{R}^4} \left( \exp(\beta u^2) - 1 - \alpha u^2 \right) \, dx. \]

Since \( \exp(\beta u^2) - 1 - \beta u^2 > 0 \), we have \( \tau = 1 \), and then
\[
\sup_{u \in H} I_\beta^\alpha (u) = \int_{\mathbb{R}^4} \left( \exp(\beta u^2) - 1 - \alpha u^2 \right) \, dx.
\]

Therefore, \( u \) is an extremal function for \( \sup_{u \in H} I_\beta^\alpha (u) \).

Next, it suffices to show that \( u = 0 \) is impossible to happen. Assume by contradiction that \( u = 0 \), we derive from radial lemma that
\[
\sup_{u \in H} I_\beta^\alpha (u) = \lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp(\beta |u_k|^2) - 1 - \alpha u_k^2 \right) \, dx
\]
\[(2.5) = \lim_{R \to \infty} \lim_{k \to \infty} \left( \int_{B_R} + \int_{\mathbb{R}^4 \setminus B_R} \right) \left( \exp(\beta |u_k|^2) - 1 - \alpha u_k^2 \right) \, dx
\]
\[= \lim_{R \to \infty} \lim_{k \to \infty} (\beta - \alpha) \int_{\mathbb{R}^4 \setminus B_R} u_k^2 \, dx \leq \beta - \alpha.
\]

On the other hand, for any \( v \in H^2(\mathbb{R}^4) \) and \( t > 0 \), we introduce a family of functions \( v_t \) by
\[
v_t(x) = t^{\frac{1}{2}} v(t^{\frac{1}{2}} x),
\]
and we easily verify that
\[
\|\Delta v_t\|_2^2 = t \|\Delta v\|_2^2, \|v_t\|_p^p = t^{\frac{p-2}{2}} \|v\|_p^p.
\]
Hence, it follows that

$$
\int_{\mathbb{R}^4} \left( \exp \left( \beta \left( \frac{v_t}{\|v_t\|_{H^2(\mathbb{R}^4)}} \right)^2 \right) - 1 - \alpha \left( \frac{v_t}{\|v_t\|_{H^2(\mathbb{R}^4)}} \right)^2 \right) dx \\
\geq (\beta - \alpha) \frac{\|v_t\|^2}{\|v_t\|^2 + \|v_t\|^2} + \frac{\beta^2}{2} \frac{\|v_t\|}{\|v_t\|^2 + \|v_t\|^2}^2 + \frac{\beta^2}{2} \left( \frac{t\|v\|}{\|v\|^2 + \|v\|^2} \right)
\]

(2.6)

$$
= (\beta - \alpha) \left( \frac{\|v\|^2}{\|v\|^2 + \|v\|^2} \right) + \frac{\beta^2}{2} \left( \frac{t\|v\|}{\|v\|^2 + \|v\|^2} \right)
$$

$$
= (\beta - \alpha) \left( \frac{\|v\|^2}{\|v\|^2 + \|v\|^2} \right)
$$

Note that $g_v(0) = 1$, once we show that $g_v'(t) > 0$ for small $t > 0$, then we have $g_v(t) > g_v(0)$ for small $t > 0$, which leads to $\sup_{u \in H^2(\mathbb{R}^4)} t^2(u) > \beta - \alpha$. Combining (2.5) and (6.1), we obtain a contradiction. This accomplishes the proof of Lemma 2.1.

In the following, we show there exists some $v \in H^2(\mathbb{R}^4)$ such that $g_v'(0) > 0$. Indeed, after a direct calculation, we have

$$
g_v'(0) = -\frac{\|\Delta v\|^2}{\|v\|^2} + \frac{\beta^2}{2} \frac{\|v\|}{\|v\|^2}^2.
$$

(2.7)

$$
= \frac{\|\Delta v\|^2}{\|v\|^2} \left( -1 + \frac{\beta^2}{2} \frac{\|v\|}{\|v\|^2} \right).
$$

It can be shown this supremum

$$
\sup_{v \in H^2(\mathbb{R}^4) \setminus \{0\}} \frac{\|v\|}{\|\Delta v\|^2 + \|v\|^2}
$$

could be attained by some $Q \in H^2(\mathbb{R}^4)$, which must satisfy (after a rescaling $Q \rightarrow \mu Q (\cdot)$, see [4]) the nonlinear equation

$$
\Delta^2 Q + Q - |Q|^2 Q = 0 \text{ in } \mathbb{R}^4.
$$

Set $v = Q$, we get $g_v'(0) = \frac{\|\Delta Q\|^2}{\|Q\|^2} \left( 1 + \frac{\beta^2}{2} \frac{\|Q\|^2}{\|\Delta Q\|^2} \right)$. Thus, if $\frac{\beta^2}{2} > \beta - \alpha$, we have $g_v'(0) > 0$.

\[ \square \]

2.2. The radially symmetric maximizing sequence for critical functional. Let $\{\beta_k\}$ be an increasing sequence which converges to $32\pi^2$. According to Lemma 2.1 we see that there exists a radial function sequence $\{u_k\}_k$ satisfying $\|u_k\|_{H^2(\mathbb{R}^4)} = 1$ such that

$$
\int_{\mathbb{R}^4} \left( \exp (\beta_k |u_k|^2) - 1 - \alpha |u_k|^2 \right) dx = \sup_{u \in H^2} \int_{\mathbb{R}^4} \left( \exp (\beta_k |u|^2) - 1 - \alpha |u|^2 \right) dx.
$$

It is not difficult to see that

$$
\lim_{k \rightarrow \infty} \int_{\mathbb{R}^4} \left( \exp (\beta_k |u_k|^2) - 1 - \alpha |u_k|^2 \right) dx = S(\alpha).
$$
In fact, for any given \( \varphi \in H^2(\mathbb{R}^4) \) with \( \int_{\mathbb{R}^4} (|\varphi|^2 + |\Delta \varphi|^2) \, dx = 1 \), we have
\[
\int_{\mathbb{R}^4} (\exp (\beta_k |\varphi|^2) - 1 - \alpha |\varphi|^2) \, dx \leq \int_{\mathbb{R}^4} (\exp (\beta_k |u_k|^2) - 1 - \alpha |u_k|^2) \, dx.
\]
It follows from Levi’s lemma that
\[
\int_{\mathbb{R}^4} (\exp (32\pi^2 |\varphi|^2) - 1 - \alpha |\varphi|^2) \, dx \leq \lim_{k \to \infty} \int_{\mathbb{R}^4} (\exp (\beta_k |u_k|^2) - 1 - \alpha |u_k|^2) \, dx,
\]
which implies that
\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} (\exp (\beta_k |u_k|^2) - 1 - \alpha |u_k|^2) \, dx = S(\alpha).
\]

An easy computation shows that the Euler–Lagrange equation of \( u_k \) is given by the following bi-harmonic equation in \( \mathbb{R}^4 \):

\[
(2.9) \quad \Delta^2 u_k + u_k = \lambda_k^{-1} u_k \left( \exp (\beta_k u_k^2) - \frac{\alpha}{\beta_k} \right),
\]
where \( \|u_k\|_{H^2(\mathbb{R}^4)} = 1 \) and \( \lambda_k = \int_{\mathbb{R}^4} u_k^2 \left( \exp \{\beta_k u_k^2\} - \frac{\alpha}{\beta_k} \right) \, dx \). Since
\[
\lambda_k^{-1} u_k \left( \exp \{\beta_k u_k^2\} - \frac{\alpha}{\beta_k} \right) \in L_p^p(\mathbb{R}^4)
\]
for any \( 1 \leq p < \infty \), by Lemma 7.6 we know \( u_k \in C^\infty(\mathbb{R}^4) \).

Now, we give the following important observation.

**Lemma 2.3.** \( \inf_k \lambda_k > 0 \).

**Proof.** We assume by contradiction that \( \lambda_k \to 0 \). Since \( \exp t - 1 \leq t \exp t \), we derive that
\[
(2.10) \quad \int_{\mathbb{R}^4} (\beta_k u_k^2 \exp (\beta_k u_k^2)) \, dx \geq \int_{\mathbb{R}^4} (\exp (\beta_k u_k^2) - 1) \, dx.
\]
Hence
\[
\sup_{u \in H^2} \left( \frac{1}{32\pi^2} \int_{\mathbb{R}^4} (\exp (32\pi^2 |u|^2) - 1 - \alpha |u|^2) \, dx \right) = \lim_{k \to \infty} \left( \frac{1}{\beta_k} \int_{\mathbb{R}^4} (\exp (\beta_k u_k^2) - 1 - \alpha |u_k|^2) \, dx \right) \leq \lim_{k \to \infty} \left( \frac{1}{\beta_k} \int_{\mathbb{R}^4} (\beta_k u_k^2 \exp (\beta_k u_k^2) - \alpha |u_k|^2) \, dx \right) = \lim_{k \to \infty} \int_{\mathbb{R}^4} u_k^2 \left( \exp \{\beta_k u_k^2\} - \frac{\alpha}{\beta_k} \right) \, dx = \lim_{k \to \infty} \lambda_k \to 0,
\]
which is a contradiction. \( \square \)
Now, we introduce the following

**Definition 2.4.** We said that \( \{ u_k \}_k \) is a normalized vanishing sequence, if \( \{ u_k \}_k \) satisfies 
\[
\| u_k \|_{H^2(\mathbb{R}^4)} = 1, \quad u_k \rightharpoonup 0 \quad \text{in} \quad H^2(\mathbb{R}^4) \quad \text{and} \\
\lim_{R \to \infty} \lim_{k \to \infty} \int_{B_R} \left( \exp(\beta_k |u_k|^2) - 1 - \alpha |u_k|^2 \right) dx = 0.
\]

Extracting a subsequence and changing the sign of \( u_k \), we can always take a point \( x_k \in \mathbb{R}^4 \) such that 
\[
c_k = \max |u_k| = u_k(x_k).
\]

If \( c_k \) is bounded from above, we have the following

**Lemma 2.5.** If \( \sup_k c_k < +\infty \), then one of the following holds.

(i) \( u \neq 0 \) and \( S(\alpha) \) could be achieved by a radial function \( u \in H^2(\mathbb{R}^4) \),

(ii) \( u = 0 \) and \( \{ u_k \} \) is a normalized vanishing sequence, furthermore, \( S(\alpha) \leq 32\pi^2 - \alpha \).

**Proof.** If \( \sup_k c_k < +\infty \), it follows from the standard elliptic estimates (see Lemma 7.6) that \( u_k \to u \) in \( C^3_{\text{loc}}(\mathbb{R}^4) \). Then for any \( R > 0 \), we have

\[
(2.11) \quad \lim_{k \to \infty} \int_{B_R} \left( \exp(\beta_k |u_k|^2) - 1 - \alpha |u_k|^2 \right) dx = \int_{B_R} \left( \exp(32\pi^2 |u|^2) - 1 - \alpha |u|^2 \right) dx.
\]

On the other hand, according to the radial lemma, we derive that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^4 \setminus B_R} \left( \exp(\beta_k |u_k|^2) - 1 - \beta_k u_k^2 \right) dx \\
\leq c \lim_{k \to \infty} \int_{\mathbb{R}^4 \setminus B_R} |u_k|^4 dx \\
\leq c \sup_k \| u_k \|_{H^2(\mathbb{R}^4)}^2 R^{-2}.
\]

Similar as (2.3), we have

\[
(2.12) \quad \lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp(\beta_k |u_k|^2) - 1 - \alpha |u_k|^2 \right) dx = \int_{\mathbb{R}^4} \left( \exp(32\pi^2 |u|^2) - 1 - \alpha |u|^2 \right) dx \\
+ (32\pi^2 - \alpha) \lim_{k \to \infty} \int_{\mathbb{R}^4} (u_k^2 - u^2) dx.
\]

When \( u \neq 0 \), we set
\[
\tau^4 = \lim_{k \to \infty} \frac{\int_{\mathbb{R}^4} u_k^2 dx}{\int_{\mathbb{R}^4} u^2 dx},
\]
and let \( \tilde{u}(x) = u(\tau x) \), as we did in (2.4), we have
\[ S(\alpha) \geq \int_{\mathbb{R}^4} \left( \exp(32\pi^2 \tilde{u}^2) - 1 - \alpha \tilde{u}^2 \right) dx \]

\[ = \tau^4 \int_{\mathbb{R}^4} \left( \exp(32\pi^2 u^2) - 1 - \alpha u^2 \right) dx \]

\[ \geq S(\alpha) + (\tau^4 - 1) \int_{\mathbb{R}^4} \left( \exp(32\pi^2 u^2) - 1 - 32\pi^2 u^2 \right) dx. \]

Since \( \exp(32\pi^2 u^2) - 1 - 32\pi^2 u^2 > 0 \), we have \( \tau = 1 \), and then

\[ S(\alpha) = \int_{\mathbb{R}^4} \left( \exp(32\pi^2 u^2) - 1 - \alpha u^2 \right) dx. \]

So, \( u \) is an extremal function.

When \( u = 0 \), by (2.11) we know \( \{u_k\} \) is a normalized vanishing sequence. Furthermore, by (2.12), we get

\[ S(\alpha) = \lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp(\beta_k |u_k|^2) - 1 - \alpha |u_k|^2 \right) dx \leq 32\pi^2 - \alpha. \]

\[ \square \]

In the following, we show that the second case of Lemma 2.5 will not happen.

Setting

\[ d_{nv} = \sup_{u_k: (\text{NVS})} \lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp(\beta_k |u_k|^2) - 1 - |u_k|^2 \right) dx, \]

we have the following

**Proposition 2.6.**

\[ d_{nv} = 32\pi^2 - \alpha. \]

**Proof.** Recalling in the proof of Lemma 2.5 we have verified that if \( \{u_k\}_k \) is a radially symmetric normalized vanishing sequence, then

\[ \lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp(\beta_k |u_k|^2) - 1 - |u_k|^2 \right) dx \leq 32\pi^2 - \alpha, \]

that is, \( d_{nv} \leq 32\pi^2 - \alpha \). Next, we show that there exists a radially symmetric normalized vanishing sequence \( \{v_k\} \) such that

\[ \lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp(\beta_k |v_k|^2) - 1 - |v_k|^2 \right) dx = 32\pi^2 - \alpha. \]

Picking a smooth radially symmetric function \( \eta \) satisfying \( \|\Delta \eta\|_2 = \|\eta\|_2 = 1 \) with a compact support. Let \( \omega_k \) be a function defined by \( \omega_k(x) = \rho_k^2 \eta(\rho_k x) \) for \( \rho_k > 0 \), it is easy to check that

\[ \|\Delta \omega_k\|_2 = \rho_k^2 \text{ and } \|\omega_k\|_2 = 1. \]
EXISTENCE AND NONEXISTENCE OF EXTREMALS FOR CRITICAL ADAMS INEQUALITIES IN $\mathbb{R}^4$

Setting $\bar{\omega}_k = \frac{\omega_k}{(1 + \rho_k)^{\frac{4}{3}}}$ and letting $\lim_{k \to \infty} \rho_k = 0$, we can verify that $\|\bar{\omega}_k\|_{H^2(\mathbb{R}^4)} = 1$, $\bar{\omega}_k \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^4)$ and

$$\lim_{k \to \infty} \|\Delta \bar{\omega}_k\|_2 = 0, \quad \lim_{k \to \infty} \|\bar{\omega}_k\|_2 = 1.$$ 

Hence, $\{\bar{\omega}_k\}_k$ is a radially symmetric normalized vanishing sequence. Through the radial lemma and the definition of the normalized vanishing sequence, we have

$$\lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp(\beta_k |\bar{\omega}_k|^2) - 1 - \alpha |\bar{\omega}_k|^2 \right) dx = \left( \frac{32\pi^2}{2} - \alpha \right) g_v(t),$$

where

$$g_v(t) = \left( \frac{\|v\|^2}{\|\Delta v\|^2 + \|v\|^2} + \frac{(32\pi^2)^2}{2 (32\pi^2 - \alpha)} \frac{t \|v\|^4}{\|\Delta v\|^2 + \|v\|^2} \right).$$

Furthermore, one can show that $g'_Q(t) > 0$ ($Q$ is the ground state solution of (2.8)) for small $t > 0$, provided $32\pi^2 - \alpha < \left( \frac{32\pi^2}{2} \right)^2 B_2$, which implies that $S > d_{nv}$. \hfill \Box

Proposition 2.7. It holds that $S > d_{nv}$. 

Proof. For any $v \in H^2(\mathbb{R}^4)$ and $t > 0$, we introduce a family of functions $v_t$ by

$$v_t(x) = t^{\frac{1}{2}} v(t^{\frac{1}{2}} x).$$

then

$$\|\Delta v_t\|^2 = t \|\Delta v\|^2, \quad \|v_t\|^p = t^{\frac{p-2}{2}} \|v\|^p$$

for any $p \geq 2$. Similar as that in Lemma 2.1, we can get

$$\int_{\mathbb{R}^4} \left( \exp \left( \frac{32\pi^2}{2} \left( \frac{v_t}{\|v_t\|_{H^2(\mathbb{R}^4)}} \right)^2 \right) - 1 - \alpha \left( \frac{v_t}{\|v_t\|_{H^2(\mathbb{R}^4)}} \right)^2 \right) dx \geq \left( \frac{32\pi^2}{2} - \alpha \right) g_v(t),$$

where

$$g_v(t) = \left( \frac{\|v\|^2}{\|\Delta v\|^2 + \|v\|^2} + \frac{(32\pi^2)^2}{2 (32\pi^2 - \alpha)} \frac{t \|v\|^4}{\|\Delta v\|^2 + \|v\|^2} \right).$$

3. Blow up analysis

In this section, we are interested the blow-up case, that is,

$$c_k \to +\infty,$$

the method of blow-up analysis will be used to analyze the asymptotic behavior of the radially maximizing sequence $\{u_k\}_k$. By the radial lemma, we have $x_k \to 0 \in \mathbb{R}^4$. We
call 0 the blow-up point. Here and in the sequel, we do not distinguish sequence and subsequence, the reader can understand it from the context.

Since $u_k$ is bounded in $H^2_r(\mathbb{R}^4)$, we have

$$
\begin{align*}
0 & \quad \text{weakly in } H^2_r(\mathbb{R}^4) \\
\beta_k & \quad \to 32\pi^2.
\end{align*}
$$

### 3.1. Asymptotic behavior of $\{u_k\}_k$ near the 0.

Let

$$r^4_k = \frac{\lambda_k}{c_k^2 e^{\beta_k c_k^2}}.$$

We claim that $r^4_k$ converges to zero rapidly. Indeed we have for any $\gamma < 32\pi^2$,

$$
\begin{align*}
& r^4_k e^{c_k^2} \\
& = \frac{\lambda_k}{c_k^2 e^{\beta_k c_k^2}} \int_{\mathbb{R}^4} u_k^2 (\exp(\beta_k u_k^2) - \frac{\alpha}{\beta_k}) dx \\
& \leq \int_{\mathbb{R}^4} u_k^2 \exp(\beta_k u_k^2) \exp((\gamma - \beta_k) u_k^2) dx \\
& = \int_{\mathbb{R}^4} u_k^2 \exp(\gamma u_k^2) dx \\
& = \int_{\mathbb{R}^4} u_k^2 (\exp(\gamma u_k^2) - 1) dx + \int_{\mathbb{R}^4} u_k^2 dx \\
& \leq \left( \int_{\mathbb{R}^4} u_k^2 dx \right)^{\frac{2}{s}} \left( \int_{\mathbb{R}^4} (\exp(\gamma u_k^2) - 1)^{\frac{s-2}{s}} dx \right)^{\frac{s}{s-2}} + \int_{\mathbb{R}^4} u_k^2 dx \\
& \leq c \left( \int_{\mathbb{R}^4} u_k^2 dx \right)^{\frac{2}{s}} \left( \int_{\mathbb{R}^4} (\exp(\gamma s u_k^2 - 2) - 1) dx \right)^{\frac{s}{s-2}} + \int_{\mathbb{R}^4} u_k^2 dx
\end{align*}
$$

provided $s$ large enough, here we have used the Adams inequality in $\mathbb{R}^4$ and (3.2).

To understand the asymptotic behavior of $u_k$ near the blow-up point, we define three sequences of functions on $\mathbb{R}^4$, namely

$$
\begin{align*}
\phi_k(x) &= \frac{u_k(x_k + r_k x)}{c_k}, \\
v_k(x) &= u_k(x_k + r_k x) - u_k(x_k), \\
\psi_k(x) &= c_k (u_k(x_k + r_k x) - c_k),
\end{align*}
$$

where $\phi_k$, $v_k$ and $\psi_k$ are defined on $\Omega_k := \{x \in \mathbb{R}^4 : r_k x \in B_1\}$.

**Lemma 3.1.** $\phi_k(x) \to 1$ in $C^3_{loc}(\mathbb{R}^4)$.
Proof. From equation (2.9), the decay estimate of \( r_k \) and the fact that \( \phi_k \leq 1 \), we know that for any \( R > 0 \), and \( x \in B_R(0) \), \( \phi_k(x) \) satisfy

\[
|\Delta^2 \phi_k(x)| = \frac{r_k^4}{c_k} (\Delta^2 u_k)(x + r_k x)
\]

\[
= r_k^4 \left( \lambda_k^{-1} \phi_k \exp \{ \beta_k u_k^2(x + r_k x) \} - \left( 1 + \frac{\alpha}{\lambda_k \beta_k} \right) \phi_k \right)
\]

\[
\leq r_k^4 \left( \lambda_k^{-1} \phi_k \exp \{ \beta_k c_k^2 \} - \left( 1 + \frac{\alpha}{\beta_k} \lambda_k^{-1} \right) \phi_k \right)
\]

\[
\leq cc_k^{-2} (1 + o(1)) \to 0,
\]

and \( \phi_k(x) \) is bounded in \( L^1_{loc}(\mathbb{R}^4) \). The standard regularity theory gives for any \( R > 0 \) and some \( 0 < \gamma < 1 \), \( \| \phi_k(x) \|_{C^{3,\gamma}(B_R(0))} \) are uniformly bounded with the respect to \( k \).

Through the Arzela-Ascoli theorem, there exists \( \phi \in C^3(\mathbb{R}^4) \) such that \( \phi_k(x) \to \phi \) in \( C^3_{loc}(\mathbb{R}^4) \) with \( \Delta^2 \phi = 0 \) in \( \mathbb{R}^4 \). Since \( \phi_k(0) = 1 \), we have \( \phi = 1 \) in \( \mathbb{R}^4 \) by the Liouville Theorem.

Lemma 3.2. We have

\[
v_k(x) = u_k(x + r_k x) - u_k(x) \to 0 \quad \text{in} \quad C^3_{loc}(\mathbb{R}^4)
\]

as \( k \to \infty \). Hence

\[
|\nabla^i u_k(x)| = o(r_k^{-i}) \quad \text{in} \quad B_{Rr_k}, i = 1, 2, 3,
\]

for any \( R > 0 \).

Proof. It is obvious that \( v_k \) solves the equation

\[
(-\Delta)^2 v_k + r_k^4 u_k(x + r_k x) = \frac{u_k(x + r_k x)}{c_k^2} \exp(\beta_k u_k^2 - \beta_k c_k^2) - \frac{1}{\lambda_k \beta_k} r_k^4 u_k(x + r_k x).
\]

Set \( \Delta v_k = g_k \), and then \( \Delta g_k = f_k \), where

\[
f_k = \frac{u_k(x + r_k x)}{c_k^2} \exp(\beta_k u_k^2 - \beta_k c_k^2) - \left( \frac{1}{\lambda_k \beta_k} + 1 \right) r_k^4 u_k(x + r_k x).
\]

since \( u_k \) is bounded in \( H^2(\mathbb{R}^4) \), directly computations yields that

\[
\int_{\mathbb{R}^4} \left| g_k \right|^2 dx = \int_{\mathbb{R}^4} \left| \Delta v_k \right|^2 dx = \int_{\mathbb{R}^4} \left| \Delta u_k \right|^2 dx < c.
\]

Also, since \( f_k \) is bounded in \( L^p_{loc}(\mathbb{R}^4) \) for any \( p \geq 1 \), by Lemma 7.6 we obtain that for some \( 0 < \gamma < 1 \),

\[
\| g_k \|_{C^{1,\gamma}(B_R)} \leq c,
\]

for any \( R > 0 \). On the other hand, by using Pizzetti’s formula (see Lemma 7.3), we can derive

\[
\int_{B_R(0)} v_k(x) dx = c_0 R^4 v_k(0) + c_1 R^6 \Delta v_k(0) + c_2 R^8 \Delta^2 v_k(\xi),
\]

for some \( \xi \in B_R(0) \).
By (3.4) and observe that $v_k \leq 0$ and $v_k(0) = 0$, then $v_k(x)$ is bounded in $L^1_{loc}(\mathbb{R}^4)$. Hence again by Lemma 7.6 we obtain that there exists some $v \in C^3(\mathbb{R}^4)$ such that

$$v_k(x) \to v \text{ in } C^3_{loc}(\mathbb{R}^4)$$

with $v$ satisfying

$$(-\Delta)^2 v = 0.$$

By the Lemma 7.7 and $v \leq 0$, we know that $v$ is a polynomial degree at most 6. Since

$$\int_{\mathbb{R}^4} |\Delta v|^2 dx \leq \lim_{k \to \infty} \int_{\mathbb{R}^4} |\Delta v_k|^2 dx \leq C,$$

then $v$ must be a constant. This together with $v(0) = 0$ implies $v(x) = 0$. \hfill \square

The following lemma plays an important role in determining the limit behavior of $\psi_k(x)$.

**Lemma 3.3 (Gradient estimate on $B_{R R_k}$).** For any $R > 0$, there holds

$$c_k \int_{B_{R R_k}} |\Delta u_k| dx \leq c(R R_k)^2.$$

Furthermore, we have

$$\int_{B_R} |\Delta \psi_k| dx = c_k R_k^{-2} \int_{B_{R R_k}} |\Delta u_k| dx \leq c R^2.$$

**Proof.** For any $R_0 > 0$, we introduce a sequence of bi-harmonic functions $u^R_k$ solving

$$\begin{cases}
\Delta^2 u^R_k = 0 \text{ in } B_{R_0}(x_k), \\
\partial_i u^R_k = \partial_i^2 u_k \text{ on } \partial B_0(x_k), i = 0, 1.
\end{cases} \tag{3.5}$$

By the elliptic estimates (see Lemma 7.5) and the radial lemma, we derive that

$$\|u^R_k\|_{C^\alpha(B_{R_0})} \leq \frac{c}{R_0^\alpha}$$

for some $\tau > 0$.

Observe that $u_k - u^R_k$ satisfies the following equation

$$\begin{cases}
\Delta^2 (u_k - u^R_k) = \lambda_k^{-1} u_k \left( \exp \{ \beta_k u_k^2 \} - \frac{\alpha}{\beta_k u_k} \right) - u_k \text{ in } B_{R_0}(0) \\
\partial_i (u_k - u^R_k) = 0 \text{ on } \partial B_{R_0}(0), i = 0, 1.
\end{cases} \tag{3.6}$$

Set $f_k = \lambda_k^{-1} u_k \left( \exp \{ \beta_k u_k^2 \} - \frac{\alpha}{\beta_k u_k} \right) - u_k$, and define $L(\log L)^\alpha(B_{R_0})$ as the space

$$L(\log L)^\alpha(B_{R_0}) := \{ f \in L^1(B_{R_0}) : \|f\|_{L(\log L)^\alpha} := \int_{B_{R_0}} |f| (|\log^\alpha(2 + |f|)) dx < \infty \}.$$ 

endowed with the norm $\|f\|_{L(\log L)^\alpha} = \int_{B_{R_0}} |f| (|\log^\alpha(2 + |f|)) dx$.

It is not difficult to check that $f_k$ is bounded in $L(\log L)^\frac{1}{2}(B_{R_0})$. This together with Lemma 7.4 directly leads to

$$\|\nabla^j (u_k - u^R_k)\|_{L^{(4/\beta_j,2)}} \leq C, \quad j = 1, 2, 3. \tag{3.7}$$
EXISTENCE AND NONEXISTENCE OF EXTREMALS FOR CRITICAL ADAMS INEQUALITIES IN $\mathbb{R}^4$

where $\| \cdot \|_{L^{(4/3,2)}}$ is the Lorentz norm. For the definition of Lorentz spaces and their basic properties, we refer the interested reader to [43].

After some computation, we obtain

$$
|\Delta^2((u_k - u_k^{R_0})^2)| \leq |2(u_k - u_k^{R_0})\Delta^2(u_k - u_k^{R_0})| + C \sum_{j=1}^3 |\nabla^j(u_k - u_k^{R_0})| |\nabla^{4-j}(u_k - u_k^{R_0})|.
$$

Thanks to the Lorentz estimates of gradients (3.7) and some Hölder type inequality of O’Neil [43], we know the term $\sum_{j=1}^3 |\nabla^j(u_k - u_k^{R_0})| |\nabla^{4-j}(u_k - u_k^{R_0})|$ is bounded in $L^1(B_{R_0})$.

We now show that $|2(u_k - u_k^{R_0})\Delta^2(u_k - u_k^{R_0})|$ is also bounded in $L^1(B_{R_0})$. In fact, we observe that

$$
\int_{B_{R_0}} |2(u_k - u_k^{R_0})\Delta^2(u_k - u_k^{R_0})| dx \leq 2 \int_{B_{R_0}} |u_k\Delta^2(u_k)| dx + 2 \int_{B_{R_0}} |u_k^{R_0}\Delta^2(u_k)| dx
$$

$$
= I_1 + I_2.
$$

For $I_1$, by equation (2.9), we obtain

$$
\int_{B_{R_0}} |u_k\Delta^2(u_k)| dx \leq \int_{\mathbb{R}^4} \chi_k^{-1} u_k^2 \left( \exp(\beta_k u_k^2) - \frac{\alpha}{\beta_k} \right) dx + \int_{\mathbb{R}^4} |u_k|^2 dx
$$

$$
= \int_{\mathbb{R}^4} |\Delta u_k|^2 dx + 2 \int_{\mathbb{R}^4} |u_k|^2 dx \leq c.
$$

For $I_2$, we have

$$
\int_{B_{R_0}} |u_k^{R_0}\Delta^2(u_k)| dx \leq c \int_{B_{R_0}} |u_k\Delta^2(u_k)| dx + c \int_{B_{R_0} \cap \{|u_k| \leq 1\}} |\Delta^2(u_k)| dx
$$

$$
\leq c(R_0).
$$

Using the above estimates, we conclude that $\int_{B_{R_0}} |\Delta^2((u_k - u_k^{R_0})^2)| dx \leq c$. Carrying out the same procedure as in the proof of Lemma 6 of [37], we have for any $R > 0$,

$$
(3.8) \quad \int_{B_{Rr_k}} \Delta((u_k - u_k^{R_0})^2) dx \leq C(Rr_k)^2.
$$
Combining (3.8) and (3.6), we derive by Lemma 3.2 that

\[ \int_{B_{Rr_k}} |\Delta (u^2_k)| dx \leq c \left\{ \int_{B_{Rr_k}} \Delta \left( (u_k - u_{R_0}^k)^2 \right) dx + \int_{B_{Rr_k}} |\Delta (u_{R_0}^k)^2| dx + \int_{B_{Rr_k}} |u_{R_0}^k \Delta u_k| dx + \int_{B_{Rr_k}} |\nabla u_k \nabla u_{R_0}^k| dx \right\} \]

\[ \leq c \int_{B_{Rr_k}} \Delta \left( (u_k - u_{R_0}^k)^2 \right) dx + o(r_k^2). \]

(3.9)

On the other hand, we also have

\[ c_k |\Delta u_k| \leq cu_k |\Delta u_k| \leq c \left( \Delta (u_k^2) + |\nabla u_k|^2 \right) \leq c \Delta (u_k^2) + o(r_k^{-2}). \]

(3.10)

From (3.9) and (3.10), we conclude that

\[ c_k \int_{B_{Rr_k}} |\Delta u_k| dx \leq c(Rr_k)^2. \]

Hence it follows that for any \( R > 0, \)

\[ \int_{B_R} |\Delta \psi_k| dx = c_k r_k^{-2} \int_{B_{Rr_k}} |\Delta u_k| dx \leq cR^2. \]

This accomplishes the proof. \( \square \)

Now, we are in position to analyze the limit behavior of \( \psi_k(x) \).

**Lemma 3.4.** We have

\[ \psi_k(x) \to \psi \text{ in } C_{\text{loc}}^3(\mathbb{R}^4), \]

where \( \psi \) satisfies the equation

\[ (-\Delta)^2 \psi = \exp(64\pi^2 \psi). \]

Furthermore, we have

\[ \psi(x) = \frac{1}{16\pi^2} \log \frac{1}{1 + \frac{\pi}{\sqrt{6}} |x|^2}, \]

and \( \int_{\mathbb{R}^4} \exp(64\pi^2 \psi(x)) dx = 1. \)

**Proof.** By equation (2.9), we see that \( \psi_k \) satisfies the equation

\[ (-\Delta)^2 \psi_k + c_k r_k^4 u_k(x + r_k x) = \frac{u_k(x + r_k x)}{c_k} \exp(\beta_k u_k^2 - \beta_k c_k^2) - \frac{1}{\lambda_k \beta_k} c_k r_k^4 u_k(x + r_k x). \]

According to Lemma 3.3, we know that

\[ \int_{B_R} |\Delta \psi_k| dx = c_k r_k^{-2} \int_{B_{Rr_k}} |\Delta u_k| dx \leq cR^2. \]
This together with the elliptic estimates (see Lemma 7.6) yields that \( \| \Delta \psi_k \|_{C^{1, \alpha}_{\text{loc}}} \leq c \). As in Lemma 3.2 we know there exists some \( \psi \in C^3(\mathbb{R}^4) \) such that

\[
\psi_k (x) \to \psi \text{ in } C^3_{\text{loc}}(\mathbb{R}^4),
\]

with \( \psi \) satisfying the equation

\[
(-\Delta)^2 \psi = \exp(64\pi^2 \psi).
\]

By Fatou’s lemma, we have

\[
\int_{\mathbb{R}^4} \exp(64\pi^2 \psi) \, dx \leq \frac{1}{\lambda_k} \int_{\mathbb{R}^4} u_k^2 \left( \exp \{ \beta_k u_k^2 \} - \frac{\alpha}{\beta_k} \right) \, dx \leq 1.
\]

We now claim that \( \psi \) must take the form as

\[
(3.11) \quad \psi(x) = \frac{1}{16\pi^2} \log \frac{1}{1 + \frac{\pi}{\sqrt{6}} |x|^2}.
\]

We argue this by contradiction. If \( \psi \) don’t have the form as (3.11), according to [25] (see also [36]), there exists some \( a < 0 \) such that

\[
\lim_{|x| \to +\infty} (-\Delta) \psi(x) = a.
\]

This would imply

\[
\lim_{k \to +\infty} \int_{B_R} |\Delta \psi_k(x)| \, dx = |a| \text{vol}(B_1(0)) R^4 + o(R^4) \text{ as } R \to +\infty,
\]

but this contradicts \( \int_{B_R} |\Delta \psi_k| \, dx < c R^2 \). Hence we have

\[
\psi(x) = \frac{1}{16\pi^2} \log \frac{1}{1 + \frac{\pi}{\sqrt{6}} |x|^2}.
\]

Furthermore, careful computations lead to

\[
\int_{\mathbb{R}^4} \exp \left( 64\pi^2 \psi(x) \right) \, dx = \int_{\mathbb{R}^4} \left( \frac{1}{1 + \frac{\pi}{\sqrt{6}} |x|^2} \right)^4 \, dx \text{ (setting } t = \frac{\pi}{\sqrt{6}} |x|^2\text{)}
\]

\[
= \frac{\omega_3}{4} \int_0^\infty (1 + t)^{-4} \, dt \cdot |x|^4
\]

\[
= \frac{3\omega_3}{\pi^2} \int_0^\infty (1 + t)^{-4} \, dt
\]

\[
= \frac{3\omega_3}{\pi^2} \cdot \frac{1}{6} = 1.
\]

\( \square \)
3.2. Bi-harmonic truncations. In the following, we will need some bi-harmonic truncations \( u_{M}^{k} \) which was studied in [9]. Roughly speaking, the value of truncations \( u_{M}^{k} \) is close to \( c_{k} \) in a small ball centered at \( x_{k} \), and coincides with \( u_{k} \) outside the same ball.

**Lemma 3.5.** [9 Lemma 4.20] For any \( M > 1 \) and \( k \in \mathbb{N} \), there exists a radius \( \rho_{k}^{M} > 0 \) and a constant \( c = c(M) \) such that

1. \( u_{k} \geq \frac{c_{k}}{M} \) in \( B_{\rho_{k}^{M}}(x_{k}) \);
2. \( |u_{k} - \frac{c_{k}}{M}| \leq \frac{c}{c_{k}} \) on \( \partial B_{\rho_{k}^{M}}(x_{k}) \);
3. \( |\nabla u_{k}| \leq \frac{c}{c_{k}(\rho_{k}^{M})} \) on \( \partial B_{\rho_{k}^{M}}(x_{k}) \), for any \( 1 \leq l \leq 3 \);
4. \( \rho_{k}^{M} \to 0 \) and \( \frac{\rho_{k}^{M}}{r_{k}} \to +\infty \), as \( k \to \infty \).

Let \( u_{k}^{\rho_{k}^{M}} \in C^{4}(\overline{B_{\rho_{k}^{M}}(x_{k})}) \) be the unique solution of

\[
\begin{align*}
\Delta^{2} u_{k}^{\rho_{k}^{M}} &= 0 \quad \text{in} \ B_{\rho_{k}^{M}}(x_{k}), \\
\partial^{i} u_{k}^{\rho_{k}^{M}} &= \partial^{i} u_{k} \quad \text{on} \ \partial B_{\rho_{k}^{M}}(x_{k}), \ i = 0, 1.
\end{align*}
\]

We consider the function

\[
\begin{align*}
|u_{k}^{M}| &= \begin{cases} u_{k}^{\rho_{k}^{M}} \quad \text{in} \ B_{\rho_{k}^{M}}(x_{k}), \\
\ u_{k} \quad \text{in} \ \mathbb{R}^{4} \backslash B_{\rho_{k}^{M}}(x_{k}) \end{cases}.
\end{align*}
\]

**Lemma 3.6.** [9 Lemma 4.21] For any \( M > 1 \), we have

\[
u_{k}^{M} = \frac{c_{k}}{M} + O\left(\frac{1}{c_{k}}\right),
\]
uniformly on \( \overline{B_{\rho_{k}^{M}}(x_{k})} \).

**Lemma 3.7.** For any \( M > 1 \), there holds

\[
\limsup_{k \to \infty} \int_{\mathbb{R}^{4}} \left( |\Delta u_{k}^{M}|^{2} + |u_{k}^{M}|^{2} \right) \, dx \leq \frac{1}{M}.
\]

**Proof.** Since \( u_{k} \) converges in \( L^{p}(B_{1}) \) for any \( p > 1 \), by Lemma 3.5 we have

\[
\int_{B_{\rho_{k}^{M}}(x_{k})} |u_{k}^{M}|^{p} \, dx \leq c \int_{B_{\rho_{k}^{M}}(x_{k})} |u_{k}|^{p} \, dx \to 0
\]

and

\[
(3.12) \quad \int_{B_{\rho_{k}^{M}}(x_{k})} |u_{k}^{\rho_{k}^{M}}| \, dx \leq \int_{B_{\rho_{k}^{M}}(x_{k})} |u_{k}^{\rho_{k}^{M}}(u_{k} + O(c_{k}^{-1}))| \, dx \to 0,
\]
as \( k \to \infty \).
Testing (2.9) with \((u_k - u_k^M)\), by Lemma 3.5, for any \(R > 0\), we have

\[
\int_{B_{\rho_k}^M(x_k)} (\Delta u_k \Delta (u_k - u_k^M) + u_k (u_k - u_k^M)) \, dx \\
= \int_{B_{\rho_k}^M(x_k)} \lambda_k^{-1} u_k \left( \exp \{ \beta_k u_k^2 \} - \frac{\alpha}{\beta_k} \right) (u_k - u_k^M) \, dx \\
\geq \int_{B_{R\rho_k}^M(x_k)} \lambda_k^{-1} u_k \exp \{ \beta_k u_k^2 \} (u_k - u_k^M) \, dx \\
= \int_{B_{R\rho_k}^M(x_k)} \lambda_k^{-1} c_k \exp \{ \beta_k u_k^2 \} \left( c_k - \frac{c_k}{M} \right) \, dx + o_k(1) \\
= \int_{B_R} \left( 1 - \frac{1}{M} \right) \exp \{ 2\beta_k \psi_k(x) \} \, dx + o_k(1) \\
\geq \left( 1 - \frac{1}{M} \right) \int_{B_R} \exp \{ 64\pi^2 \psi(x) \} \, dx + o_k(1).
\]

Letting \(R \to \infty\), we get

\[
(3.13) \quad \int_{B_{\rho_k}^M(x_k)} (\Delta u_k \Delta (u_k - u_k^M) + u_k (u_k - u_k^M)) \, dx \geq 1 - \frac{1}{M} + o_k(1).
\]

Observe that

\[
\int_{\mathbb{R}^4} \left( |\Delta u_k^M|^2 + |u_k^M|^2 \right) \, dx \\
= \int_{B_{\rho_k}^M(x_k)} |\Delta u_k^M|^2 \, dx + \int_{\mathbb{R}^4 \setminus B_{\rho_k}^M(x_k)} |\Delta u_k|^2 \, dx + \int_{B_{\rho_k}^M(x_k)} |u_k^M|^2 \, dx + \int_{\mathbb{R}^4 \setminus B_{\rho_k}^M(x_k)} |u_k|^2 \, dx \\
= \int_{B_{\rho_k}^M(x_k)} |\Delta u_k^M|^2 \, dx + \int_{B_{\rho_k}^M(x_k)} |u_k^M|^2 \, dx + 1 - \int_{B_{\rho_k}^M(x_k)} |\Delta u_k|^2 \, dx - \int_{B_{\rho_k}^M(x_k)} |u_k|^2 \, dx \\
= \int_{B_{\rho_k}^M(x_k)} |\Delta u_k^M|^2 \, dx + \int_{B_{\rho_k}^M(x_k)} |u_k^M|^2 \, dx + 1 - \int_{B_{\rho_k}^M(x_k)} \Delta u_k \Delta (u_k - u_k^M) \, dx \\
- \int_{B_{\rho_k}^M(x_k)} u_k (u_k - u_k^M) \, dx - \int_{B_{\rho_k}^M(x_k)} \Delta u_k \Delta u_k^M \, dx - \int_{B_{\rho_k}^M(x_k)} u_k u_k^M \, dx,
\]
by (3.13) and (3.12), we have

\[
\int_{\mathbb{R}^4} \left( |\Delta u_k|^2 + |u_k|^2 \right) dx
\]

\[
\leq \frac{1}{M} + \int_{B_{\rho_k}(x_k)} |\Delta u_k|^2 dx + \int_{B_{\rho_k}(x_k)} |u_k|^2 dx - \int_{B_{\rho_k}(x_k)} \Delta u_k \Delta u_k^M dx - \int_{B_{\rho_k}(x_k)} u_k u_k^M dx
\]

\[
\leq \frac{1}{M} + \int_{B_{\rho_k}(x_k)} \Delta u_k^M \Delta (u_k^M - u_k) dx + \int_{B_{\rho_k}(x_k)} |u_k|^2 dx - \int_{B_{\rho_k}(x_k)} u_k u_k^M dx
\]

\[
= \frac{1}{M} + \int_{B_{\rho_k}(x_k)} |u_k|^2 dx - \int_{B_{\rho_k}(x_k)} u_k u_k^M dx
\]

\[
= \frac{1}{M} + o_k(1).
\]

Hence the lemma is proved. \(\square\)

With the help of bi-harmonic truncations \(u_k^M\), we can show the following result.

**Corollary 3.8.** We have,

\[
\limsup_{k \to \infty} \int_{\mathbb{R}^4 \setminus B_\delta} \left( |u_k|^2 + |\Delta u_k|^2 \right) dx = 0,
\]

for any \(\delta > 0\), and then

(3.14) \( |\Delta u_k|^2 dx \to \delta_0 \) in the sense of measure,

where \(\delta_0\) is the Dirac measure supported at 0.

**Lemma 3.9.** We have

\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) dx
\]

\[
= \lim_{L \to \infty} \lim_{k \to \infty} \int_{B_{\rho_k}(x_k)} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) dx
\]

\[
= \lim_{k \to \infty} \frac{\lambda_k}{c_k^2}
\]

and consequently,

\[
\frac{\lambda_k}{c_k} \to \infty \text{ and } \sup_k \frac{c_k^2}{\lambda_k} < \infty.
\]
Proof. Direct computations yield that
\[
\int_{\mathbb{R}^4} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) dx
\]
\[
= \left( \int_{B_{r_k M}(x_k)} + \int_{\mathbb{R}^4 \setminus B_{r_k M}(x_k)} \right) \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) dx
\]
\[
\leq \int_{B_{r_k M}(x_k)} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) dx + \int_{\mathbb{R}^4} \left( \exp \left( \beta_k (u_k^M)^2 \right) - 1 - \alpha (u_k^M)^2 \right) dx.
\]
Taking some \( L \) such that \( u_k \leq 1 \) on \( \mathbb{R}^4 \setminus B_L \), then we have
\[
\lim_{k \to \infty} \int_{\mathbb{R}^4 \setminus B_L} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) dx \leq \lim_{k \to \infty} c \int_{\mathbb{R}^4} u_k^2 dx = 0.
\]
In view of Lemma 3.7 and the Adams’ inequality with the Navier boundary condition (see \([49]\)), we obtain
\[
\sup_{k \to \infty} \int_{B_L} \left( \exp \left( \beta_k p' \left( u_k^M - u_k(L) \right)^2 \right) - 1 \right) dx < \infty,
\]
for any \( p' < M \). Since
\[
p \left( u_k^M \right)^2 \leq p' \left( u_k^M - u_k(L) \right)^2 + c \left( p, p' \right), \text{if } p < p',
\]
then we get
\[
\sup_{k \to \infty} \int_{B_L} \left( \exp \left( \beta_k p \left( u_k^M \right)^2 \right) - 1 \right) dx < \infty,
\]
for any \( p < M \). The weak compactness of Banach space implies
\[
\lim_{k \to \infty} \int_{B_L} \left( \exp \left( \beta_k \left( u_k^M \right)^2 \right) - 1 \right) dx = 0.
\]
Hence, we get
\[
\lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) dx
\]
\[
= \lim_{k \to \infty} \left( \int_{B_{r_k M}(x_k)} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) dx + o_k(1) \right)
\]
\[
\leq \lim_{k \to \infty} M^2 \frac{\lambda_k}{c_k} \int_{\mathbb{R}^4} \frac{u_k^2}{\lambda_k} \left( \exp \left( \beta_k u_k^2 \right) - \frac{\alpha}{\beta_k} \right) dx
\]
\[
= M^2 \lim_{k \to \infty} \frac{\lambda_k}{c_k}.
\]
On the other hand, we get

$$
\lim_{L \to \infty} \lim_{k \to \infty} \int_{B_{L r_k(x)}} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) dx
= \lim_{L \to \infty} \lim_{k \to \infty} \frac{\lambda_k}{c_k^2} \int_{B_L} \exp \left( \beta_k u_k^2 (r_k x + x_k) - \beta_k c_k^2 \right) dx \\
= \lim_{k \to \infty} \frac{\lambda_k}{c_k^2} \left( \int_{\mathbb{R}^4} \exp \left( 64\pi^2 \psi(x) \right) dx + o_k(1) \right) \\
= \lim_{k \to \infty} \frac{\lambda_k}{c_k^2}.
$$

(3.16)

Combining (3.15) and (3.16), and letting $M \to 1$, we conclude that

$$
\lim_{k \to \infty} \int_{\mathbb{R}^4} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) dx = \lim_{k \to \infty} \frac{\lambda_k}{c_k^2} \\
= \lim_{L \to \infty} \lim_{k \to \infty} \int_{B_{L r_k(x)}} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) dx.
$$

Now, we introduce the following quantities:

$$
b_k = \lim_{R \to \infty} \lim_{k \to \infty} \frac{\lambda_k}{\int_{B_R(x_k)} |u_k| \left( \exp \left( \beta_k u_k^2 \right) - \alpha \beta_k \right) dx}, \quad \tau = \lim_{k \to \infty} \frac{b_k}{c_k} \quad \text{and} \quad \\
\sigma = \lim_{R \to \infty} \lim_{k \to \infty} \frac{\int_{B_R(x_k)} u_k \left( \exp \left( \beta_k u_k^2 \right) - \alpha \beta_k \right) dx}{\int_{B_R(x_k)} |u_k| \left( \exp \left( \beta_k u_k^2 \right) - \alpha \beta_k \right) dx}.
$$

Lemma 3.10. It holds $\sigma = 1$.

Proof. For any $M > 1$ and $R > 0$, we have

$$
\int_{B_R(x_k)} u_k \left( \exp \left( \beta_k u_k^2 \right) - \alpha \beta_k \right) dx = \int_{B_{\rho_k^M(x_k)}} u_k \left( \exp \left( \beta_k u_k^2 \right) - \alpha \beta_k \right) dx \\
+ \int_{B_R(x_k) \setminus B_{\rho_k^M(x_k)}} u_k^M \left( \exp \left( \beta_k \left( u_k^M \right)^2 \right) - \alpha \beta_k \right) dx,
$$

By Lemma 3.7, we know that $\exp \left( \beta_k u_k^2 \right) - \alpha \beta_k$ is bounded in $L^p \left( B_R(x_k) \setminus B_{\rho_k^M(x_k)} \right)$ for some $p > 1$, then we have

$$
\int_{B_R(x_k) \setminus B_{\rho_k^M(x_k)}} u_k^M \left( \exp \left( \beta_k \left( u_k^M \right)^2 \right) - \alpha \beta_k \right) dx \to 0, \quad \text{as} \quad k \to \infty.
$$
This implies that

\[(3.17) \int_{B_{R}(x_k)} u_k \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx = \int_{B_{\rho M}(x_k)} |u_k| \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx + o_k(1).
\]

Similarly, we also have

\[(3.18) \int_{B_{R}(x_k)} |u_k| \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx = \int_{B_{\rho M}(x_k)} |u_k| \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx + o_k(1).
\]

On the other hand, it is not hard to see that

\[c_k \int_{B_{\rho M}(x_k)} \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx \geq \int_{B_{\rho M}(x_k)} |u_k| \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx \]

\[(3.19) \geq \frac{c_k}{M} \int_{B_{\rho M}(x_k)} \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx.
\]

Furthermore, by (4) of Lemma 3.9 and Lemma 3.9, we derive that

\[\lim_{k \to \infty} \int_{B_{\rho M}(x_k)} \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx \geq \lim_{L \to \infty} \lim_{k \to \infty} \int_{B_{L \rho k}(x_k)} \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx \]

\[\geq \lim_{L \to \infty} \lim_{k \to \infty} \int_{B_{L \rho k}(x_k)} \left( \exp \left( \beta_k u_k^2 - 1 - \alpha u_k^2 \right) \right) dx \]

\[(3.20) \geq \int_{\mathbb{R}^4} \left( \exp \left( \beta_k u_k^2 - 1 - \alpha u_k^2 \right) \right) dx > 0.
\]

Therefore, combining (3.17)-(3.20), we get

\[\frac{1}{M} + o_k(1) \leq \frac{\int_{B_{R}(x_k)} u_k \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx}{\int_{B_{R}(x_k)} |u_k| \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx} \leq 1.
\]

Letting \(k \to \infty, R \to \infty\) and \(M \to 1\), we derive that \(\sigma = 1\). \(\square\)

**Lemma 3.11.** It holds \(\tau = 1\).

**Proof.** For any \(R > 0\), similar to (3.19), we have

\[\int_{\mathbb{R}^4} \left| u_k \right|^2 \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx \geq \frac{c_k}{M} \int_{B_{\rho M}(x_k)} \left( \exp \left( \beta_k u_k^2 - \frac{\alpha}{\beta_k} \right) \right) dx + o_k(1)
\]
\[ \int_{B_r(x_k)} |u_k| \left( \exp \left( \beta_k u_k^2 \right) - \frac{\alpha}{\beta_k} \right) \, dx \leq c_k \int_{B_{rM}(x_k)} \left( \exp \left( \beta_k u_k^2 \right) - \frac{\alpha}{\beta_k} \right) \, dx + o_k(1). \]

Thus,
\[
b_k = \lim_{R \to \infty} \frac{\int_{\mathbb{R}^4} |u_k|^2 \left( \exp \left( \beta_k u_k^2 \right) - \frac{\alpha}{\beta_k} \right) \, dx}{\int_{B_R(x_k)} |u_k| \left( \exp \left( \beta_k u_k^2 \right) - \frac{\alpha}{\beta_k} \right) \, dx} \geq \frac{(\alpha \beta_k)^2}{c_k M^2},
\]

letting \( M \to 1 \), we conclude that \( \tau = 1 \).

\[ \square \]

3.3. Asymptotic behavior of \( u_k \) away from the blow-up point 0. In the following, we consider the asymptotic behavior of \( u_k \) away from the blow-up point 0.

We recall that the crucial tool in studying the regularity of higher order equations is the fundamental solution of the operator \( \Delta^2 + 1 \). The fundamental solution \( \Gamma(x,y) \) for \( \Delta^2 + 1 \) in \( \mathbb{R}^4 \) is the solution of
\[ (\Delta^2 + 1) \Gamma(x,y) = \delta_x(y) \quad \text{in} \quad \mathbb{R}^4, \]
and all functions \( u \in H^2(\mathbb{R}^4) \cap C^4(\mathbb{R}^4) \) satisfying \( (\Delta^2 + 1) u = f \) can be represented by
\[ u(x) = \int_{\mathbb{R}^4} \Gamma(x,y) f(y) \, dy. \]

We will need the following useful estimates for \( \Gamma \) :
\[ |\Gamma(x,y)| \leq c \ln \left( 1 + |x-y|^{-1} \right), \quad |\nabla^i \Gamma(x,y)| \leq c |x-y|^{-i}, \quad i \geq 1 \]
for all \( x, y \in \mathbb{R}^4, x \neq y \) with \( |x-y| \to 0 \), and
\[ \nabla^i |\Gamma(x,y)| = o \left( \exp \left( -\frac{1}{\sqrt{2}} |x-y| \right) \right), \quad i = 0, 1, 2. \]
for all \( x, y \in \mathbb{R}^4 \), with \( |x-y| \to +\infty \).

The above properties of \( \Gamma \) can be found in [10].

Lemma 3.12. For any \( 1 < r < 2 \), \( c_k u_k \) is bounded in \( W^{2,r}(\mathbb{R}^4) \).

Remark 3.13. It is quite difficult to prove \( \|c_k u_k\|_{W^{2,r}(\mathbb{R}^4)} \leq c \) directly. But as we have showed the fact \( \lim_{k \to \infty} b_k = 1 \) in Lemma [3.11], we will prove this lemma by showing that \( \|b_k u_k\|_{W^{2,r}(\mathbb{R}^4)} \leq c \). We find it quite easy to obtain the desired result for \( b_k u_k \).
EXISTENCE AND NONEXISTENCE OF EXTREMALS FOR CRITICAL ADAMS INEQUALITIES IN $\mathbb{R}^4$

**Proof.** Let $\eta_k$ be the solution of

$$\Delta^2 \eta_k + \eta_k = \frac{b_k u_k}{\lambda_k} \left( \exp \left\{ \beta_k u_k^2 \right\} - \frac{\alpha}{\beta_k} \right), \quad x \in \mathbb{R}^4.$$ 

By the representation formula, we have

$$\eta_k (x) = \int_{\mathbb{R}^4} \Gamma (x, y) \frac{b_k u_k (y)}{\lambda_k} \left( \exp \left\{ \beta_k u_k^2 (y) \right\} - \frac{\alpha}{\beta_k} \right) dy.$$ 

Then, by Hölder’s inequality, for any $1 < r < 2$, we have

$$\left| \nabla^i \eta_k (x) \right|^r = \left( \frac{b_k}{\lambda_k} \int_{\mathbb{R}^4} \nabla^i \Gamma (x, y) u_k (y) \left( \exp \left\{ \beta_k u_k^2 (y) \right\} - \frac{\alpha}{\beta_k} \right) dy \right)^r \leq \left( \int_{\mathbb{R}^4} \nabla^i \Gamma (x, y) u_k (y) \left( \exp \left\{ \beta_k u_k^2 (y) \right\} - \frac{\alpha}{\beta_k} \right) \frac{dy}{\left( \int_{\mathbb{R}^4} u_k (z) \left( \exp \left\{ \beta_k u_k^2 (z) \right\} \right) dz - \frac{\alpha}{\beta_k} \right) dz} \right)^r \leq \int_{\mathbb{R}^4} \left| \nabla^i \Gamma (x, y) \right|^r \left( \frac{u_k (y) \left( \exp \left\{ \beta_k u_k^2 (y) \right\} - \frac{\alpha}{\beta_k} \right)}{\int_{\mathbb{R}^4} u_k (z) \left( \exp \left\{ \beta_k u_k^2 (z) \right\} \right) dz - \frac{\alpha}{\beta_k} \right) \frac{dy}{\left( \int_{\mathbb{R}^4} u_k (z) \left( \exp \left\{ \beta_k u_k^2 (z) \right\} \right) dz - \frac{\alpha}{\beta_k} \right) dz},$$

where $i = 0, 1, 2$. Applying Fubini’s theorem, we get by (3.22) and (3.23) that

$$\int_{\mathbb{R}^4} \left| \nabla^i \eta_k (x) \right|^r dx = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \left| \nabla^i \Gamma (x, y) \right|^r dx \frac{u_k (y) \left( \exp \left\{ \beta_k u_k^2 (y) \right\} - \frac{\alpha}{\beta_k} \right)}{\int_{\mathbb{R}^4} u_k (z) \left( \exp \left\{ \beta_k u_k^2 (z) \right\} \right) dz - \frac{\alpha}{\beta_k} \right) \frac{dy}{\left( \int_{\mathbb{R}^4} u_k (z) \left( \exp \left\{ \beta_k u_k^2 (z) \right\} \right) dz - \frac{\alpha}{\beta_k} \right) dz} \leq c,$$ 

for $i = 0, 1, 2$, thus, we get

(3.24) \quad $\| \eta_k \|_{W^{2,r} (\mathbb{R}^4)} < c.$

Let $\eta_k = b_k u_k$, then $\eta_k$ satisfies

$$\Delta^2 \eta_k + \eta_k = \frac{b_k u_k}{\lambda_k} \left( \exp \left\{ \beta_k u_k^2 \right\} - \frac{\alpha}{\beta_k} \right) in \mathbb{R}^4.$$ 

By (3.24), we have $\| \eta_k \|_{W^{2,r} (\mathbb{R}^4)} < c$. This accomplishes the proof of Lemma 3.12. \hfill \square

Now, we will show that $c_k u_k$ converges to some Green function.

**Lemma 3.14.** For any $\varphi \in C_0^\infty (\mathbb{R}^4)$, one has

(3.25) \quad $\lim_{k \to \infty} \int_{\mathbb{R}^4} \varphi (x) \frac{c_k u_k}{\lambda_k} \left( \exp \left( \frac{\beta_k u_k^2}{\lambda_k} \right) - \frac{\alpha}{\beta_k} \right) dx = \varphi (0).$
**Proof.** Suppose $\text{supp } \varphi \subset B_\rho$ and we split the integral as follows

$$
\int_{\mathbb{R}^4} \varphi(x) \frac{c_k u_k}{\lambda_k} \left( \exp \left( \beta_k u_k^2 \right) - \frac{\alpha}{\beta_k} \right) \, dx 
= \int_{B_{\rho M}(x_k) \setminus B_{L r_k}} \varphi(x) \frac{c_k u_k}{\lambda_k} \left( \exp \left( \beta_k u_k^2 \right) - \frac{\alpha}{\beta_k} \right) \, dx 
+ \int_{B_{L r_k}} \varphi(x) \frac{c_k u_k}{\lambda_k} \left( \exp \left( \beta_k u_k^2 \right) - \frac{\alpha}{\beta_k} \right) \, dx
+ \int_{B_\rho \setminus B_{\rho M}(x_k)} \varphi(x) \frac{c_k u_k}{\lambda_k} \left( \exp \left( \beta_k u_k^2 \right) - \frac{\alpha}{\beta_k} \right) \, dx 
= I_1^k + I_2^k + I_3^k.
$$

(3.26)

For $I_1^k$, it follows that

$$
I_1^k \leq M \| \varphi \|_{C^0} \int_{B_{\rho M}(x_k) \setminus B_{L r_k}} \frac{u_k^2}{\lambda_k} \left( \exp \left( \beta_k u_k^2 \right) - \frac{\alpha}{\beta_k} \right) \, dx
= M \| \varphi \|_{C^0} \left( 1 - \int_{B_L} \exp \left( 2 \beta_k \psi_k(x) + o_k(1) \right) \, dx \right).
$$

(3.27)

Letting $k \to +\infty$ and $L \to +\infty$, we derive that $\lim_{k \to \infty} I_1^k = 0$.

For $I_2^k$, we have

$$
I_2^k = \int_{B_L} \varphi(r_k x + x_k) \frac{u_k(r_k x + x_k)}{c_k} \exp \left( 2 \beta_k \psi_k(x) + o_k(1) \right) \, dx.
$$

(3.28)

Letting $k \to +\infty$ and $L \to +\infty$, we derive that $\lim_{k \to \infty} I_2^k = \varphi(0)$.

For $I_3^k$, since $\exp(\beta_k |u_k^M|^2)$ is bounded in $L^p(B_\rho)$ for some $p > 1$, choosing $p > 1$ sufficiently close to 1 and by Hölder’s inequality, we derive that

$$
I_3^k \leq \frac{c_k}{\lambda_k} \| \varphi \|_{C^0} \left( \int_{B_\rho} |u_k|^p \, dx \right)^{\frac{1}{p}} \left( \int_{B_\rho} \exp(\beta_k p |u_k^M|^2) \, dx \right)^{\frac{1}{p}}.
$$

(3.29)

Note that $\lim_{k \to \infty} \frac{c_k}{\lambda_k} = 0$, hence $\lim_{k \to \infty} I_3^k = 0$. Combining (3.27), (3.28) and (3.29), we conclude that

$$
\lim_{k \to \infty} \int_{\mathbb{R}^4} \varphi(x) \frac{c_k u_k}{\lambda_k} \left( \exp \left( \beta_k u_k^2 \right) - \frac{\alpha}{\beta_k} \right) \, dx = \varphi(0).
$$

□

**Lemma 3.15.** For any $1 < r < 2$, $c_k u_k \to G \in C^3(\mathbb{R}^4) \setminus \{0\}$ weakly in $W^{2,r}(\mathbb{R}^4)$, where $G$ is a Green function satisfying

$$
\Delta^2 G + G = \delta(x) \text{ in } \mathbb{R}^4.
$$
Also, we have

\[
(3.30) \quad G = -\frac{1}{8\pi^2} \ln |x| + A + \phi(x),
\]

where \(A\) is a constant depending on \(0\), \(\phi(x) \in C^3 (\mathbb{R}^4)\) and \(\phi(0) = 0\). Moreover, we have

\[
\lim_{k \to \infty} \left( \int_{B^4 \setminus B_4(0)} |\Delta (c_k u_k)|^2 \, dx + \int_{\mathbb{R}^4 \setminus B_4(0)} |c_k u_k|^2 \, dx \right) = -\frac{1}{8\pi^2} \ln \delta - \frac{1}{16\pi^2} + A + O(\delta).
\]

**Proof.** By Lemma 3.12, there exists a function \(G \in W^{2,r}(\mathbb{R}^4)\) such that \(c_k u_k \to G\) weakly in \(W^{2,r}(\mathbb{R}^4)\) for any \(1 < r < 2\). For any \(s > 0\), by (3.14), we know \(\exp(\beta_k u_k^2)\) is bounded in \(L^p(B_R \setminus B_s)\), for any \(0 < s < R\). Notice that \(c_k u_k\) satisfies

\[
\Delta^2 (c_k u_k) + c_k u_k = \frac{c_k u_k}{\lambda_k} \left( \exp \left( \frac{\beta_k u_k^2}{\beta_k} \right) - \frac{\alpha}{\beta_k} \right) \text{ in } \mathbb{R}^4.
\]

Then the standard regularity theory gives \(c_k u_k \to G\) in \(C^3_{\text{loc}}(\mathbb{R}^4 \setminus \{0\})\).

For any \(\phi(x) \in C^\infty_0(\mathbb{R}^4)\), in view of Lemma 3.14, we have

\[
\int_{\mathbb{R}^4} \phi(x) \left( \frac{c_k u_k}{\lambda_k} \left( \exp \left( \frac{\beta_k u_k^2}{\beta_k} \right) - \frac{\alpha}{\beta_k} \right) - c_k u_k \right) \, dx = \phi(0) - \int_{\mathbb{R}^4} G(x) \phi(x) \, dx.
\]

Hence, it follows that

\[
\Delta^2 G = \delta(x) - G \text{ in } \mathbb{R}^4.
\]

Fix \(r > 0\), we choose some cutoff function \(\phi \in C^\infty_0(B_{2r}(0))\) such that \(\phi = 1 \text{ in } B_r(0)\), and let

\[
g(x) = G(x) + \frac{1}{8\pi^2} \phi(x) \ln |x|.
\]

Then a direct computation shows that

\[
\Delta^2 g(x) = f \text{ in } \mathbb{R}^4,
\]

where

\[
f(x) = -\frac{1}{8\pi^2} \left( \Delta^2 \phi \cdot \ln |x| + 2\nabla \Delta \phi \cdot \nabla \ln |x| + 2\Delta (\nabla \phi \cdot \nabla \ln |x|) + 2\nabla \phi \cdot \nabla \Delta \ln |x| + \phi \cdot \Delta^2 \ln |x| \right) + \delta(x) - G.
\]

Since \(\frac{1}{8\pi^2} \phi \cdot \Delta^2 \ln |x| = \delta(x)\) in \(\mathbb{R}^4\), direct calculations yield that

\[
f(x) = -\frac{1}{8\pi^2} \left( \Delta^2 \phi \cdot \ln |x| + 2\nabla \Delta \phi \cdot \nabla \ln |x| + 2\Delta (\nabla \phi \cdot \nabla \ln |x|) + 2\nabla \phi \cdot \nabla \Delta \ln |x| \right) - G.
\]

Since \(G \in W^{2,r}(\mathbb{R}^4)\) for any \(1 < r < 2\), we have \(f(x) \in L^p_{\text{loc}}(\mathbb{R}^4)\) for any \(p > 2\). By the standard regularity theory, we get \(g(x) \in C^3_{\text{loc}}(\mathbb{R}^4)\). Let \(A = g(0)\) and

\[
\varphi(x) = g(x) - g(0) + \frac{1}{8\pi^2} (1 - \phi) \ln |x|.
\]
Then we have
\begin{equation}
G = -\frac{1}{8\pi^2} \ln |x| + A + \phi(x),
\end{equation}
where $A$ is constant depending on 0, $\phi(x) \in C^3(\mathbb{R}^4)$ and $\phi(0) = 0$. Then (3.30) follows directly from (3.32).

Setting $U_k = c_k u_k$, then $U_k$ satisfy:

$$
\Delta^2 U_k = \frac{U_k}{\lambda_k} \left( \exp \left( \frac{\beta_k u_k^2}{\alpha} \right) - U_k \right), \text{ in } \mathbb{R}^4.
$$

Testing it with $U_k$, by Proposition 7.2, we get

\begin{equation}
\int_{\mathbb{R}^4 \setminus B_\delta(0)} |\Delta U_k|^2 \, dx + \int_{\mathbb{R}^4 \setminus B_\delta(0)} |U_k|^2 \, dx = \int_{\partial B_\delta(0)} v \left( U_k \Delta^{3/2} U_k - \Delta^{1/2} U_k \Delta U_k \right) \, dx,
\end{equation}

where $v$ is the outer normal vector of $\partial B_\delta(0)$. Then we have

\begin{equation}
\lim_{k \to \infty} \left( \int_{\mathbb{R}^4 \setminus B_\delta(0)} |\Delta U_k|^2 \, dx + \int_{\mathbb{R}^4 \setminus B_\delta(0)} |U_k|^2 \, dx \right) = \int_{\partial B_\delta(0)} v \left( G \Delta^{3/2} G - \Delta^{1/2} G \Delta G \right) \, dx.
\end{equation}

It is known that the fundamental solution of $\Delta^2$ in $\mathbb{R}^4$ is $-\frac{1}{8\pi^2} \ln |x|$, and it satisfies

\begin{equation}
\Delta^{\frac{1}{2}} (\log |x|) = \frac{x}{|x|^2}, \quad \Delta (\log |x|) = \frac{2}{|x|^2}, \quad \Delta^{1+\frac{1}{2}} (\log |x|) = -4 \frac{x}{|x|^3}.
\end{equation}

After some computation, we obtain

\begin{equation}
v \cdot G(\delta) \Delta^{3/2} G(\delta) = \left( -\frac{1}{8\pi^2} \ln |\delta| + A + O(\delta) \right) \left( \frac{1}{2\pi^2} \frac{1}{\delta^3} + O(1) \right)
\end{equation}

and

\begin{equation}
- v \cdot \Delta^{1/2} G(\delta) \Delta G(\delta) = - \left( -\frac{1}{8\pi^2} \frac{1}{\delta^2} + O(1) \right) \left( -\frac{1}{8\pi^2} \frac{2}{\delta^2} + O(1) \right)
\end{equation}

Plugging (3.35) and (3.36) into (3.33), we get

\begin{equation}
\lim_{k \to \infty} \left( \int_{\mathbb{R}^4 \setminus B_\delta(0)} |\Delta U_k|^2 \, dx + \int_{\mathbb{R}^4 \setminus B_\delta(0)} |U_k|^2 \, dx \right) = -\frac{1}{8\pi^2} \ln \delta - \frac{1}{16\pi^2} + A + O(\delta),
\end{equation}

and we are done. \qed
3.4. The upper bound of Adams inequality for normalized concentration sequence. The strategy we will use to obtain the upper bound for the Adams inequality on the entire Euclidean space $\mathbb{R}^4$ is similar to that of Li-Ruf [30]. Firstly, we need to know the specific value of the upper bound for any blow up function sequences in $H_0^2 (B_R)$.

**Lemma 3.16.** Let $B_R$ be the ball with radius $R$ in $\mathbb{R}^4$. Assume that $u_k$ is a bounded sequence in $H_0^2 (B_R)$ with $\int_{B_R} |\Delta u_k|^2 \, dx = 1$. If $u_k \rightharpoonup 0$, then

$$\limsup_{k \to +\infty} \int_{B_R} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) \, dx \leq \frac{1}{3} |B_R| \exp \left( -\frac{1}{3} \right).$$

*Proof.* By the results in [31, (5.23)], we have

$$\limsup_{k \to +\infty} \int_{B_R} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) \, dx \leq \frac{\pi^2}{6} \exp \left( \frac{5}{3} + 32\pi^2 A_0 \right),$$

where $A_0$ is the value at 0 of the trace of the regular part of the Green function $\tilde{G}$ for the operator $\Delta^2$. Actually, when the domain is a ball, by solving the corresponding ODE’s, we have

$$\tilde{G} = -\frac{1}{8\pi^2} \log |x| + \frac{1}{16\pi^2} \frac{|x|^2}{R^2} + \frac{1}{8\pi^2} \log R - \frac{1}{16\pi^2},$$

and the value at 0 of the trace of the regular part of $G$ is $\frac{1}{8\pi^2} \log R - \frac{1}{8\pi^2}$. Therefore we have

$$\limsup_{k \to +\infty} \int_{B_R} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) \, dx \leq \frac{\pi^2}{6} \exp \left( \frac{5}{3} + 32\pi^2 \left( \frac{1}{8\pi^2} \log R - \frac{1}{16\pi^2} \right) \right)$$

$$= \frac{1}{3} |B_R| \exp \left( -\frac{1}{3} \right).$$

*□*

Based on the specific value of the upper bound above, we can obtain the follow upper bound for the Adams inequality on the entire Euclidean space.

**Lemma 3.17.** If $S(\alpha)$ can not be attained, then

$$S(\alpha) = \sup_{u \in H} \int_{\mathbb{R}^4} \left( \exp \left( \beta_k u_k^2 \right) - 1 - \alpha u_k^2 \right) \, dx \leq \frac{\pi^2}{6} \exp \left( \frac{5}{3} + 32\pi^2 A \right),$$

where $A$ is the value at 0 of the trace of the regular part of the Green function $G$ for the operator $\Delta^2 + 1$. 

Proof. Set
\[ \tilde{u}_k(x) = \frac{u_k(x) - u^\delta_k}{\| \Delta (u_k(x) - u^\delta_k) \|_{L^2(B_\delta(x_k))}}, \]
where
\[ u^\delta_k = u_k(\delta) + \frac{u'_k(\delta) r^2}{2\delta} - \frac{u'_k(\delta) \delta}{2}. \]

Now, we compute \( \int_{B_\delta} (\Delta u^\delta_k - \Delta u_k)^2 dx \). For this, we rewrite it as follows
\[
\int_{B_\delta} (\Delta u^\delta_k - \Delta u_k)^2 dx = \int_{B_\delta} (\Delta u_k)^2 dx + \int_{B_\delta} (\Delta u^\delta_k)^2 dx - 2 \int_{B_\delta} \Delta u^\delta_k \Delta u_k dx
\]
(3.38)

By Lemma 3.15, we have
\[
\lim_{k \to \infty} \left( \int_{\mathbb{R}^3 \setminus B_\delta(0)} |c_k \Delta u_k|^2 dx + \int_{\mathbb{R}^3 \setminus B_\delta(0)} |c_k u_k|^2 dx \right) \leq -\frac{1}{8\pi^2} \ln \delta - \frac{1}{16\pi^2} + A + O_k(\delta).
\]

Thus we get
\[
I = \int_{B_\delta(0)} |\Delta u_k|^2 dx = 1 - \int_{\mathbb{R}^3 \setminus B_\delta(0)} |\Delta u_k|^2 dx - \int_{\mathbb{R}^3 \setminus B_\delta(0)} |u_k|^2 dx - \int_{B_\delta(0)} u^2_k dx
\]
(3.39)

By the definition of \( u^\delta_k \), we have
\[
II = \int_{B_\delta} (\Delta u^\delta_k)^2 dx = \int_{B_\delta} \left( \frac{u'_k(\delta)}{\delta^2} \right)^2 dx
\]
(3.40)

In order to estimate \( III \), by Proposition 7.2, we rewrite it as follows:
\[
III = 2 \int_{B_\delta} u^\delta_k \Delta^2 u_k dx - 2 \int_{\partial B_\delta} v \cdot u^\delta_k \Delta^3 u_k d\sigma + 2 \int_{\partial B_\delta} v \cdot \Delta \frac{1}{2} u^\delta_k \Delta u_k d\sigma
\]
\[ = 2 \left( (1) - (2) + (3) \right), \]

By (3.31) and Lemma 3.14, we have
\[
(1) = \int_{B_{\delta}} u_k^\delta \Delta^2 u_k dx = \frac{1}{c_k} \int_{B_{\delta}} u_k^\delta \left( \frac{c_k u_k}{\lambda_k} \left( \exp \left( \beta_k u_k^2 \right) - \frac{\alpha}{\beta_k} \right) - c_k u_k \right) dx
\]
\[
= \frac{1}{c_k} \int_{B_{\delta}} u_k^\delta (\delta (x) - G(x) + o_k (1)) dx
\]
\[
= \frac{1}{c_k} \left( u_k (\delta) - \frac{u_k'(\delta) \delta}{2} + \frac{o_k (\delta)}{c_k} \right)
\]
\[
= \frac{1}{c_k^2} \left( -\frac{1}{8\pi^2} \ln \delta + A + \frac{1}{16\pi^2} + o_k (\delta) \right),
\]

By (3.34), we have

\[
(2) = \int_{\partial B_{\delta}} \delta \cdot u_k^\delta \Delta^\frac{3}{2} u_k d\sigma = \frac{1}{c_k} \left( \int_{\partial B_{\delta}} \delta \cdot u_k^\delta \Delta^\frac{3}{2} G d\sigma + \frac{o_k (\delta)}{c_k} \right)
\]
\[
= -\frac{1}{8\pi^2 c_k} \left( \int_{\partial B_{\delta}} \delta \cdot u_k^\delta \Delta^\frac{3}{2} \ln x d\sigma + \frac{o_k (\delta)}{c_k} \right)
\]
\[
= -\frac{u_k (\delta)}{8\pi^2 c_k} \left( \int_{\partial B_{\delta}} \delta \cdot \left( -\frac{4}{|x|^4} \right) d\sigma + \frac{o_k (\delta)}{c_k} \right)
\]
\[
= \frac{1}{c_k^2} \frac{G (\delta)}{2\pi^2 \delta^3} \cdot 2\pi^2 \delta^3 = \frac{G (\delta) + o_k (\delta)}{c_k^2}
\]

and

\[
(3) = \int_{\partial B_{\delta}} \delta \cdot \Delta^\frac{1}{2} u_k^\delta \Delta u_k d\sigma = \frac{1}{c_k} \int_{\partial B_{\delta}} (u_k' (\delta)) \Delta G d\sigma + \frac{o_k (\delta)}{c_k^2}
\]
\[
= \left( -\frac{1}{8\pi^2} \right)^2 \frac{2}{\delta^2} \frac{1}{\delta c_k^2} 2\pi^2 \delta^3 + \frac{o_k (\delta)}{c_k^2}
\]
\[
= \frac{1}{c_k^2} \left( \frac{1}{16\pi^2} + o_k (\delta) \right).
\]
Thus, we have
\[
\int_{B_r} \Delta u_k^\delta \Delta u_k \, dx \\
= \frac{1}{c_k^2} \left( -\frac{1}{8\pi^2} \ln \delta + A + \frac{1}{16\pi^2} \right) + \frac{1}{c_k^2} \left( -\frac{1}{8\pi^2} \ln \delta + A + \frac{1}{16\pi^2} \right) + \frac{1}{c_k^2} \left( \frac{1}{16\pi^2} + \frac{1}{8\pi^2} \ln \delta - A + O_k(\delta) \right)
\]
\[
= \frac{1}{c_k^2} \left( \frac{1}{8\pi^2} + O_k(\delta) \right)
\]
(3.41)

Combining (3.39) to (3.41), we get
\[
\int_{B_r} (\Delta u_k^\delta - \Delta u_k)^2 \, dx \\
= 1 - \frac{1}{c_k^2} \left( -\frac{1}{8\pi^2} \ln \delta - \frac{1}{16\pi^2} + A + o_k(\delta) \right) + (u_k^\delta(\delta))^2 8\pi^2\delta^2
\]
\[
- \frac{1}{c_k^2} \left( \frac{1}{4\pi^2} + O_k(\delta) \right)
\]
(3.42)

Therefore, we have
\[
\tilde{u}_k^2(x) = \frac{(u_k(x) - u_k^\delta(x))^2}{1 - \frac{1}{8\pi^2} \ln \delta + \frac{1}{16\pi^2} + A + O_k(\delta)}
\]
\[
= u_k^2(x) \left( 1 + \frac{1}{8\pi^2} \ln \delta + \frac{1}{16\pi^2} + A + O_k(\delta) \right) - \left( 2u_k^\delta(x) u_k(x) + (u_k^\delta(x))^2 \right) (1 + o_k(1))
\]
\[
= u_k^2(x) - c \ln \delta^4.
\]

On the other hand, by Lemma 3.9, we have
\[
\lim_{L \to \infty} \lim_{k \to \infty} \int_{B_{\rho} \setminus B_{\rho \cdot k}(x_k)} \exp \left( \beta_k u_k^2 \right) = |B_{\rho}|,
\]
for any \( \rho < \delta \).
Thus we have
\[
\lim_{L \to \infty} \lim_{k \to \infty} \int_{B_\rho \setminus B_L r_k(x_k)} \exp \left( \beta_k \tilde{u}_k^2 \right) dx \leq O \left( \delta^{-4c} \right) \lim_{L \to \infty} \lim_{k \to \infty} \int_{B_\rho \setminus B_L r_k(x_k)} \exp \left( \beta_k u_k^2 \right) dx
\]
\[
= O \left( \delta^{-4c} \right) |B_\rho| \to 0 \text{ as } \rho \to 0.
\]
Also, by (3.14) we get
\[
\lim_{k \to \infty} \int_{B_\rho \setminus B_\rho} \left( \exp \left( \beta_k \tilde{u}_k^2 \right) - 1 - \alpha \tilde{u}_k^2 \right) dx = 0.
\]
Hence, by Lemma 3.16, we derive that
\[
\lim_{L \to \infty} \lim_{k \to \infty} \int_{B_L r_k} \left( \exp \left( \beta_k \tilde{u}_k^2 \right) - 1 - \alpha \tilde{u}_k^2 \right) dx = \lim_{k \to \infty} \int_{B_\delta} \exp(\beta_k \tilde{u}_k^2)dx \leq |B_\delta| \frac{1}{3} \exp \left( \frac{-1}{3} \right).
\]
Now, we fix some \( L > 0 \), then for any \( x \in B_{L r_k(x_k)} \), we have
\[
\beta_k u_k^2 = \beta_k \left( \frac{u_k}{\| \Delta (u_k(x) - u_\delta(x)) \|_{L^2(B_\delta)}} \right)^2 \int_{B_\delta} \left| \Delta (u_k(x) - u_\delta(x)) \right|^2 dx
\]
\[
= \beta_k \left( \tilde{u}_k + \frac{u_\delta(x)}{\| \Delta (u_k(x) - u_\delta(x)) \|_{L^2(B_\delta)}} \right)^2 \int_{B_\delta} \left| \Delta (u_k(x) - u_\delta(x)) \right|^2 dx.
\]
By (3.42), we have
\[
\beta_k u_k^2 = \beta_k \left( \tilde{u}_k + u_\delta + O \left( \frac{1}{c_k^2} \right) \right)^2 \cdot \left( 1 - \frac{1}{c_k^2} \left( -\frac{1}{8 \pi^2} \ln \delta + A + \frac{1}{16 \pi^2} + O_k (\delta) \right) \right)
\]
\[
= \beta_k \tilde{u}_k^2 \left( 1 + \frac{u_\delta}{c_k} + O \left( \frac{1}{c_k^3} \right) \right)^2 \cdot \left( 1 - \frac{1}{c_k^2} \left( -\frac{1}{8 \pi^2} \ln \delta + A + \frac{1}{16 \pi^2} + O_k (\delta) \right) \right),
\]
Observe that
\[
\lim \frac{\tilde{u}_k(x_k + r_k x)}{c_k} = 1, \text{ and } \tilde{u}_k(x_k + r_k x) u_k(\delta) \to G(\delta),
\]
we derive that

$$\beta_k u^2_k = \beta_k \tilde{u}^2_k \left( 1 + \frac{1}{c_k^2} \left( G(\delta) + \frac{1}{16\pi^2} \right) + O\left( \frac{1}{c_k^2} \right) \right)^2 \cdot \left( 1 - \frac{1}{c_k^2} \left( -\frac{1}{8\pi^2} \ln \delta + A + \frac{1}{16\pi^2} + O_k(\delta) \right) \right)$$

$$= \beta_k \tilde{u}^2_k \left( 1 + \frac{2G(\delta)}{c_k^2} + \frac{1}{8\pi^2 c_k^2} - \frac{G(\delta) + \frac{1}{16\pi^2} + O_k(\delta)}{c_k^2} \right)$$

$$= \beta_k \tilde{u}^2_k + \beta_k G(\delta) + \frac{\beta_k}{16\pi^2} + O_k(\delta).$$

Thus we have

$$\lim_{L \to \infty} \lim_{k \to \infty} \int_{B_{Lr_k}(x_k)} \left( \exp \left( \beta_k u^2_k \right) - 1 - \alpha u^2_k \right) dx$$

$$\leq \lim_{L \to \infty} \lim_{k \to \infty} \int_{B_{Lr_k}(x_k)} \exp \left( \beta_k u^2_k \right) dx$$

$$\leq \lim_{L \to \infty} \lim_{k \to \infty} \exp \left( 32\pi^2 G(\delta) + 2 + o_\delta(1) \right) \int_{B_{Lr_k}} \exp \left( \beta_k \tilde{u}^2_k \right) dx$$

$$= \exp \left( 32\pi^2 G(\delta) + 2 + o_\delta(1) \right) |B_\delta| \frac{1}{3} \exp \left( -\frac{1}{3} \right)$$

$$= \exp \left( \ln \delta^{-4} + 32\pi^2 A + \varphi(\delta) + 2 + o_\delta(1) \right) |B_\delta| \frac{1}{3} \exp \left( -\frac{1}{3} \right)$$

$$= \frac{\pi^2}{6} \exp \left( \frac{5}{3} + 32\pi^2 A \right) + o(\delta).$$

Letting $\delta \to 0$, we get

$$S(\alpha) \leq \frac{\pi^2}{6} \exp \left( \frac{5}{3} + 32\pi^2 A \right),$$

and the proof is finished. \(\square\)

### 4. The test function

Let

$$\phi_\varepsilon = \begin{cases} C + \frac{a - \frac{1}{8\pi^2} \ln \left( 1 + \frac{\pi^2}{\varepsilon^2} \right) + A + \varphi(x) + b|x|^2}{\frac{G(x)}{C}} & \text{if } |x| \leq L\varepsilon, \\ G(x)/C & \text{if } |x| \geq L\varepsilon, \end{cases}$$

where $L, C, a, b$ are functions of $\varepsilon$ (which will be defined later) such that

i) $\varepsilon = \exp(-L)$, $\frac{1}{\varepsilon^2} = O \left( \frac{1}{L^2} \right)$ as $\varepsilon \to 0$,

ii) $a = -\frac{1}{8\pi^2} \ln \left( L\varepsilon \right) - C^2 + \frac{1}{16\pi^2} \ln \left( 1 + \frac{\pi^2}{\sqrt{6}} L^2 \right) - bL^2 \varepsilon^2$, 

where $L, C, a, b$ are functions of $\varepsilon$ (which will be defined later) such that

i) $\varepsilon = \exp(-L)$, $\frac{1}{\varepsilon^2} = O \left( \frac{1}{L^2} \right)$ as $\varepsilon \to 0$,

ii) $a = -\frac{1}{8\pi^2} \ln \left( L\varepsilon \right) - C^2 + \frac{1}{16\pi^2} \ln \left( 1 + \frac{\pi^2}{\sqrt{6}} L^2 \right) - bL^2 \varepsilon^2$, 

where $L, C, a, b$ are functions of $\varepsilon$ (which will be defined later) such that

i) $\varepsilon = \exp(-L)$, $\frac{1}{\varepsilon^2} = O \left( \frac{1}{L^2} \right)$ as $\varepsilon \to 0$,

ii) $a = -\frac{1}{8\pi^2} \ln \left( L\varepsilon \right) - C^2 + \frac{1}{16\pi^2} \ln \left( 1 + \frac{\pi^2}{\sqrt{6}} L^2 \right) - bL^2 \varepsilon^2$, 

where $L, C, a, b$ are functions of $\varepsilon$ (which will be defined later) such that

i) $\varepsilon = \exp(-L)$, $\frac{1}{\varepsilon^2} = O \left( \frac{1}{L^2} \right)$ as $\varepsilon \to 0$,

ii) $a = -\frac{1}{8\pi^2} \ln \left( L\varepsilon \right) - C^2 + \frac{1}{16\pi^2} \ln \left( 1 + \frac{\pi^2}{\sqrt{6}} L^2 \right) - bL^2 \varepsilon^2$, 

where $L, C, a, b$ are functions of $\varepsilon$ (which will be defined later) such that

i) $\varepsilon = \exp(-L)$, $\frac{1}{\varepsilon^2} = O \left( \frac{1}{L^2} \right)$ as $\varepsilon \to 0$,

ii) $a = -\frac{1}{8\pi^2} \ln \left( L\varepsilon \right) - C^2 + \frac{1}{16\pi^2} \ln \left( 1 + \frac{\pi^2}{\sqrt{6}} L^2 \right) - bL^2 \varepsilon^2$, 

where $L, C, a, b$ are functions of $\varepsilon$ (which will be defined later) such that

i) $\varepsilon = \exp(-L)$, $\frac{1}{\varepsilon^2} = O \left( \frac{1}{L^2} \right)$ as $\varepsilon \to 0$,
iii) $b = -\frac{1}{16\pi^2 L^2 \epsilon^2 (1 + \frac{\pi}{6} L^2)}$.

By Lemma 3.15 we have

$$\int_{\mathbb{R}^4 \setminus B_{L\epsilon}(0)} (|\Delta \phi_\epsilon|^2 + |\phi_\epsilon|^2) \, dx = \frac{1}{C^2} \int_{\mathbb{R}^4 \setminus B_{L\epsilon}(0)} (|\Delta G|^2 + |G|^2) \, dx$$

$$= \frac{1}{C^2} \left( -\frac{1}{8\pi^2} \ln (L\epsilon) - \frac{1}{16\pi^2} + A + O(L\epsilon) \right),$$

and as 34, one has

$$\int_{B_{L\epsilon}(0)} |\Delta \phi_\epsilon|^2 \, dx = \frac{1}{96\pi^2 C^2} \left( 6 \ln \left( 1 + \frac{\pi}{\sqrt{6}} L^2 \right) + 1 + O \left( \frac{1}{\ln^2 \epsilon} \right) \right).$$

It is easy to check that

(4.1) $\int_{B_{L\epsilon}(0)} |\phi_\epsilon|^2 \, dx = O \left( (L\epsilon)^4 C^4 \right),$

thus it follows that

$$\int_{\mathbb{R}^4} (|\Delta \phi_\epsilon|^2 + |\phi_\epsilon|^2) \, dx$$

$$= \frac{1}{32\pi^2 C^2} \left( 2 \ln \left( 1 + \frac{\pi}{\sqrt{6}} L^2 \right) - \frac{5}{3} - 4 \ln (L\epsilon) + 32\pi^2 A + O \left( \frac{1}{\ln^2 \epsilon} \right) \right)$$

$$= \frac{1}{32\pi^2 C^2} \left( 2 \ln \left( \frac{\pi}{\sqrt{6}} \frac{1 + \sqrt{6} \frac{1}{\pi} L^2}{\epsilon^2} \right) - \frac{5}{3} + 32\pi^2 A + O \left( \frac{1}{\ln^2 \epsilon} \right) \right)$$

$$= \frac{1}{32\pi^2 C^2} \left( 2 \ln \left( \frac{\pi}{\sqrt{6}} \frac{1}{\epsilon^2} \right) + 2 \ln \left( 1 + \frac{\sqrt{6} \frac{1}{\pi} L^2}{\epsilon^2} \right) - \frac{5}{3} + 32\pi^2 A + O \left( \frac{1}{\ln^2 \epsilon} \right) \right)$$

$$= \frac{1}{32\pi^2 C^2} \left( 2 \ln \left( \frac{\pi}{\sqrt{6}} \frac{1}{\epsilon^2} \right) - \frac{5}{3} + 32\pi^2 A + O \left( \frac{1}{\ln^2 \epsilon} \right) \right).$$

Set $\int_{\mathbb{R}^4} (|\Delta \phi_\epsilon|^2 + |\phi_\epsilon|^2) \, dx = 1$ and direct computations yield that

(4.2) $32\pi^2 C^2 = 2 \ln \frac{\pi}{\sqrt{6} \epsilon^2} - \frac{5}{3} + 32\pi^2 A + O \left( \frac{1}{\ln^2 \epsilon} \right),$ then

(4.3) $C^2 \sim \frac{1}{8\pi^2} \ln \frac{1}{\epsilon}.$
For any $x \in B_{L\varepsilon}$, by careful calculation, we also derive that

$$
32\pi^2 \phi^2_x \geq 32\pi^2 \left( C^2 + 2 \left( a - \frac{1}{16\pi^2} \ln \left( 1 + \frac{\pi}{\sqrt{6}} x^2 \right) + A + \varphi(x) + b|x|^2 \right) \right)
$$

$$
= 32\pi^2 \left( -C^2 + 2 \left( -\frac{1}{8\pi^2} \ln (L\varepsilon) + \frac{1}{16\pi^2} \ln \left( 1 + \frac{\pi}{\sqrt{6}} L^2 \right) - bL^2\varepsilon^2 \right) + 2A + 2\varphi(x) + 2b|x|^2 - \frac{1}{8\pi^2} \ln \left( 1 + \frac{\pi x^2}{\sqrt{6} \varepsilon^2} \right) \right)
$$

$$
= 4 \ln \left( 1 + \frac{\pi}{\sqrt{6}} L^2 \right) - 8 \ln (L\varepsilon) - 2 \ln \frac{\pi}{\sqrt{6}\varepsilon^2} + \frac{5}{3} + 32\pi^2 A
$$

$$
= 4 \ln \left( 1 + \frac{\pi}{\sqrt{6}} L^2 \right) - 8 \ln (L\varepsilon) - 2 \ln \frac{\pi}{\sqrt{6}\varepsilon^2} + \frac{5}{3} + 32\pi^2 A
$$

$$
= \ln \left( \left( 1 + \frac{\pi}{\sqrt{6}} L^2 \right)^4 \frac{6\varepsilon^4}{\pi^2} (L\varepsilon)^{-8} \right) + \frac{5}{3} + 32\pi^2 A
$$

$$
= \ln \left( \frac{6\varepsilon^4}{\pi^2} \frac{4\pi^4}{36} + O \left( \frac{1}{\ln^2 \varepsilon} \right) + 32\pi^2 (\varphi(x) + b|x|^2) - 64bL^2\varepsilon^2 \right)
$$

$$
\geq \ln \left( \frac{\pi^2}{6\varepsilon^4} + 32\pi^2 A + \frac{5}{3} - \ln \left( 1 + \frac{\pi r^2}{\sqrt{6}\varepsilon^2} \right)^4 + O \left( \frac{1}{\ln^2 \varepsilon} \right) \right),
$$

where we have used that fact that $\varphi$ is a continuous function and $\varphi(0) = 0$.

Therefore, combining (4.1) and (4.4), we derive that

$$
\int_{B_{L\varepsilon}} \left( \exp \left( 32\pi^2 \phi^2_x \right) - 1 - \alpha \phi^2_x \right) dx
$$

$$
\geq \int_{B_{L\varepsilon}} \exp \left( 32\pi^2 \phi^2_x \right) dx + O \left( (L\varepsilon)^4 C^4 \right)
$$

$$
= \frac{\pi^2}{6\varepsilon^4} \exp \left( 32\pi^2 A + \frac{5}{3} \right) \int_{B_{L\varepsilon}} \left( 1 + \frac{\pi r^2}{\sqrt{6}\varepsilon^2} \right)^{-4} dx + O \left( \frac{1}{\ln^2 \varepsilon} \right).
$$

Since
\( \int_{B_{L\varepsilon}} \left( 1 + \frac{\pi r^2}{\sqrt{6}\varepsilon^2} \right)^{-4} dx \)

\[ = 2\pi^2 \varepsilon^4 \int_0^L \left( 1 + \frac{\pi r^2}{\sqrt{6}} \right)^{-4} r^3 dr \]

\[ = 2\pi^2 \varepsilon^4 \frac{1}{2} \frac{6}{\pi^2} \int_0^{\frac{\pi}{\sqrt{6}} L^2} \frac{u}{(1 + u)^4} du \]

\[ = 2\pi^2 \varepsilon^4 \frac{1}{2} \frac{6}{\pi^2} \left( \frac{1}{6} - \frac{1}{3} \left( 1 + \frac{\pi}{\sqrt{6}} L^2 \right)^{-2} + O \left( L^{-6} \right) \right) \]

\[ = \varepsilon^4 \left( 1 - 2 \left( 1 + \frac{\pi}{\sqrt{6}} L^2 \right)^{-2} + O \left( L^{-6} \right) \right). \]

we have

\[ \int_{B_{L\varepsilon}} \left( \exp \left( 32\pi^2 \phi^2_\varepsilon \right) - 1 - \alpha \phi^2_\varepsilon \right) dx \]

\[ \geq \frac{\pi^2}{6\varepsilon^4} \exp \left( 32\pi^2 A + \frac{5}{3} \right) \varepsilon^4 \left( 1 - 2 \left( 1 + \frac{\pi}{\sqrt{6}} L^2 \right)^{-2} + O \left( L^{-6} \right) \right) + \]

\[ + O \left( C^4 \left( L\varepsilon \right)^4 \right) + O \left( \frac{1}{\ln^2 \varepsilon} \right) \]

\[ = \frac{\pi^2}{6} \exp \left( 32\pi^2 A + \frac{5}{3} \right) + O \left( \frac{1}{\ln^2 \varepsilon} \right). \]

On the other hand, we also have

\[ \int_{R^4 \setminus B_{L\varepsilon}} \left( \exp \left( 32\pi^2 \phi^2_\varepsilon \right) - 1 - \alpha \phi^2_\varepsilon \right) dx \]

\[ \geq \frac{32\pi^2 - \alpha}{C^2} \int_{R^4 \setminus B_{L\varepsilon}} G(x)^2 dx \]

\[ = \frac{32\pi^2 - \alpha}{C^2} \|G(x)\|_{L^2(R^4)}^2 \]

\[ = \frac{32\pi^2 - \alpha}{C^2} \left( \|G(x)\|_{L^2(R^4)}^2 + O \left( L^4 \varepsilon^4 \right) \right). \]

Then we get
\[
\int_{\mathbb{R}^4} \left( \exp \left( 32\pi^2 \phi_x^2 \right) - 1 - \alpha \phi_x^2 \right) dx \\
= \frac{\pi^2}{6} \exp \left( 32\pi^2 A + \frac{5}{3} \right) + \frac{32\pi^2 - \alpha}{C^2} \|G(x)\|^2_{L^2(\mathbb{R}^4)} + O \left( \frac{1}{\ln^2 \varepsilon} \right).
\]

By (4.3), we know \( C^2 \sim \ln \varepsilon \), which implies that
\[
\int_{\mathbb{R}^4} \left( \exp \left( 32\pi^2 \phi_x^2 \right) - 1 \right) dx > \frac{\pi^2}{6} \exp \left( 32\pi^2 A + \frac{5}{3} \right)
\]
for \( \varepsilon \) small enough. This accomplishes the proof.

5. NONEXISTENCE OF EXTREMALS

In this section, we will show that when \( 32\pi^2 - \alpha \) large enough, the supremum \( S(\alpha) \) is not attained. For this aim, we will need the precise estimates for the best constants of Gagliardo-Nirenberg inequalities, the detailed proof is given in the Appendix.

**Lemma 5.1.** Let \( B_k = \sup_{u \in H^2(\mathbb{R}^4)} \frac{\int_{\mathbb{R}^4} u^{2k} dx}{\int_{\mathbb{R}^4} |\Delta u|^2 dx} \), then

\[
B_k \leq \frac{1}{4} \left( 1 + \frac{2k}{2k - 1} \right)^{2k} \left( \frac{2}{2k - 1} \right)^{2k-2} \cdot \sqrt{\left( \frac{k}{k - 1} \right)^{k-1} \left( \frac{k}{32\pi^2} \right)^{k-1/2} \cdot 32\pi^2}.
\]

**Proof of Theorem 1.5.** Let \( u \in H^2(\mathbb{R}^4) \) satisfying \( \|u\|_{H^2(\mathbb{R}^4)} = 1 \), we have

\[
\int_{\mathbb{R}^4} \left( \exp \left( 32\pi^2 |u|^2 \right) - 1 - \alpha |u|^2 \right) dx \\
= \left( 32\pi^2 - \alpha \right) \int_{\mathbb{R}^4} u^2 dx + \frac{(32\pi^2)^2}{2} \int_{\mathbb{R}^4} u^4 dx + \ldots + \frac{(32\pi^2)^k}{k!} \int_{\mathbb{R}^4} u^{2k} dx + \ldots.
\]

By the Gagliardo-Nirenberg inequalities, we have

\[
\frac{(32\pi^2)^k}{k!} \int_{\mathbb{R}^4} u^{2k} dx \leq \frac{(32\pi^2)^k}{k!} B_k \left( \int_{\mathbb{R}^4} |\Delta u|^2 dx \right)^{k-1} \int_{\mathbb{R}^4} u^2 dx,
\]

where \( B_k \) is the best constant. We employ (5.1) to obtain that

\[
B_k \leq \frac{1}{4} \left( 1 + \frac{1}{2k - 1} \right)^{2k-1} \left( \frac{2k}{2k - 1} \right) \left( 1 - \frac{1}{2k - 1} \right)^{2k-1} \left( \frac{2k - 1}{2(2k - 1)} \right).
\]

\[
\sqrt{\left( 1 + \frac{1}{k - 1} \right)^{k-1} \cdot \left( \frac{k}{32\pi^2} \right)^{k-1/2} \cdot 32\pi^2} \sim \frac{8\pi^2 \sqrt{e}}{\sqrt{k}} \left( \frac{k}{32\pi^2} \right)^k,
\]
when $k$ is large enough. Thus, it follows that

$$\frac{(32\pi^2)^k}{k!} B_k \leq c \frac{(32\pi^2)^k}{k!} \frac{8\pi^2 \sqrt{e}}{\sqrt{k}} \left( \frac{k}{32\pi^2} \right)^k$$

$$= c 8\pi^2 \sqrt{e} \frac{k^k}{\sqrt{kk!}} \quad \text{(by the stirling formula)}$$

$$\sim c \frac{8\pi^2 \sqrt{e}}{\sqrt{2\pi}} \frac{e^k}{k}.$$ 

Setting $\int_{\mathbb{R}^4} |\Delta u|^2 \, dx = t$, we have

$$\frac{(32\pi^2)^k}{k!} \int_{\mathbb{R}^4} u^2 \, dx \leq \frac{c 8\pi^2 \sqrt{e}}{\sqrt{2\pi}} \frac{e^k}{k} t^k (1 - t).$$

Since the series

$$\sum_{k=1}^{\infty} \frac{8\pi^2 \sqrt{e}}{\sqrt{2\pi}} \frac{e^k}{k} t^k (1 - t)$$

converges if $t < \frac{1}{e}$, hence, there exists some constant $c > 0$ such that

$$\int_{\mathbb{R}^4} \left( \exp \left( 32\pi^2 |u|^2 \right) - 1 - \alpha |u|^2 \right) \, dx$$

$$\leq F(t) = (32\pi^2 - \alpha) (1 - t) + c \sum_{k=2}^{\infty} \frac{8\pi^2 \sqrt{e}}{\sqrt{2\pi}} \frac{e^k}{k} t^k.$$

Since $F'(t)|_{t=0} = -32\pi^2 + \alpha < 0$ if $\alpha < 32\pi^2$ and $c \sum_{k=2}^{\infty} \frac{8\pi^2 \sqrt{e}}{\sqrt{2\pi}} \frac{e^k}{k} t^k \to 0$ as $t \to 0$, there exists some $t_0$ such that $F(t)$ is decreasing on $[0, t_0]$. Hence, it follows that

$$F(t) \leq F(0) \leq 32\pi^2 - \alpha.$$

Now we consider the case $t \geq t_0$. By the Adams inequality, there exists some $M > 0$ such that

$$\int_{\mathbb{R}^4} \left( \exp \left( 32\pi^2 |u|^2 \right) - 1 - 32\pi^2 |u|^2 \right) \, dx < M.$$ 

Hence

$$F(t) \leq (32\pi^2 - \alpha) (1 - t) + M$$

$$= 32\pi^2 - \alpha + M - (32\pi^2 - \alpha) t_0,$$

and we have $F(t) < 32\pi^2 - \alpha$, if $32\pi^2 - \alpha > \frac{M}{t_0}$. \hfill \Box

Finally, we give the

Proof of Theorem 1.6. We only need to show that if $S(\alpha_1)$ is attained, then for any $\alpha_1 < \alpha_2$, $S(\alpha_2)$ is also attained.
For any $\alpha \in \mathbb{R}$, we denote by $d_{nv}(\alpha)$ and $d_{nc}(\alpha)$ for the upper bounds of Adams' inequality of the normalized vanishing sequences and the normalized concentration sequences, respectively. It is easy to check the following facts:

$$d_{nc}(\alpha_1) = d_{nc}(\alpha_2), \ d_{nv}(\alpha_1) - d_{nv}(\alpha_2) = \alpha_2 - \alpha_1,$$

for any $\alpha_1 < \alpha_2$. Since $S(\alpha_1)$ is attained, we have

$$S(\alpha_1) \geq \max \{d_{nv}(\alpha_1), d_{nc}(\alpha_1)\}.$$

On the other hand, by (4.5), we know that

$$S(\alpha) > d_{nc}(\alpha), \ \text{for any} \ \alpha \in \mathbb{R},$$

thus, we have

$$S(\alpha_1) \geq d_{nv}(\alpha_1).$$

Now, we show that $S(\alpha_2) \geq d_{nv}(\alpha_2)$. Indeed, since $S(\alpha_1)$ is attained by some $\bar{u} \in H^2(\mathbb{R}^4) \setminus \{0\}$ satisfying $\|\bar{u}\|_{H^2(\mathbb{R}^4)} = 1$, that is,

$$S(\alpha_1) = \int_{\mathbb{R}^4} \left( \exp(32\pi^2|\bar{u}|^2) - 1 - \alpha_1 |\bar{u}|^2 \right) dx$$

$$= (32\pi^2 - \alpha_1) \int_{\mathbb{R}^4} \bar{u}^2 dx + G(\bar{u}),$$

where $G(\bar{u}) = \int_{\mathbb{R}^4} \left( \exp(32\pi^2|\bar{u}|^2) - 1 - 32\pi^2|\bar{u}|^2 \right) dx$. By (5.3) and the fact that $\int_{\mathbb{R}^4} \bar{u}^2 dx < 1$, we have

$$S(\alpha_2) \geq (32\pi^2 - \alpha_2) \int_{\mathbb{R}^4} \bar{u}^2 dx + G(\bar{u})$$

$$= (\alpha_1 - \alpha_2) \int_{\mathbb{R}^4} \bar{u}^2 dx + (32\pi^2 - \alpha_1) \int_{\mathbb{R}^4} \bar{u}^2 dx + G(\bar{u})$$

$$= (\alpha_1 - \alpha_2) \int_{\mathbb{R}^4} \bar{u}^2 dx + S(\alpha_1)$$

$$\geq (\alpha_1 - \alpha_2) \int_{\mathbb{R}^4} \bar{u}^2 dx + d_{nv}(\alpha_1)$$

$$> \alpha_1 - \alpha_2 + 32\pi^2 - \alpha_1$$

$$= 32\pi^2 - \alpha_2 = d_{nv}(\alpha_2).$$

Combining (5.2) and (5.4), we have $S(\alpha_2) > \max \{d_{nv}(\alpha_2), d_{nc}(\alpha_2)\}$, and then $S(\alpha_2)$ is attained.

6. Existence and nonexistence of extremal functions for the Trudinger-Moser inequalities in $\mathbb{R}^2$

In this section, we will provide sketchy proofs of Theorems 1.4 and 1.7 all together.
Step 1: Setting

\[ I_{\beta}^\alpha (u) = \int_{B_R} (\exp(\beta |u|^2) - 1 - \alpha |u|^2) \, dx, \]

one can easily check that \( I_{\beta}^\alpha \) could be achieved through combining the subcritical Trudinger-Moser inequality in \( H^1(\mathbb{R}^2) \) and Vitali convergence theorem.

Step 2. We can find a positive radially symmetric maximizing sequence \( \{u_k\} \) for critical functional

\[ \tilde{S}(\alpha) = \sup_{u \in H^1, \|u\|_{H^1} = 1} \int_{\mathbb{R}^2} (\exp(4\pi |u|^2) - 1 - \alpha |u|^2) \, dx. \]

where, \( u_k \) is positive, radial extremals for the Trudinger-Moser inequality \( \int_{B_{R_k}} (\exp(\beta_k |u|^2) - 1 - \alpha |u|^2) \, dx \)

and \( R_k \to \mathbb{R}^2, \beta_k \to 32\pi^2 \).

Step 3. If \( c_k \) is bounded from above, then one of the following holds.

(i) \( u \neq 0 \) and \( \tilde{S}(\alpha) \) could be achieved by a radial function \( u \in H^1(\mathbb{R}^2) \),

(ii) \( u = 0 \) and \( \{u_k\} \) is a normalized vanishing sequence, furthermore, \( \tilde{S}(\alpha) \leq d_{nv} = 4\pi - \alpha \).

where \( d_{nv} \) is the upper bound of Trudinger-Moser inequality for normalized vanishing sequence.

Step 4. Similar to the proof of Lemma 2.1, we can show \( \tilde{S}(\alpha) > d_{nv} \), when \( 4\pi - 8\pi^2 B_2 < \alpha \). Indeed, for any \( v \in H^1(\mathbb{R}^2) \) and \( t > 0 \), we introduce a family of functions \( v_t \) by

\[ v_t(x) = t^{\frac{\alpha}{2}} v(t^{\frac{1}{2}} x), \]

and we easily verify that

\[ \|\nabla v_t\|_2^2 = t \|\nabla v\|_2^2, \|v_t\|_{p^*} = t^{\frac{p-2}{2}} \|v\|_{p^*}. \]

Hence, it follows that

\[ \int_{\mathbb{R}^2} \left( \exp \left( 4\pi \left( \frac{v_t}{\|v_t\|_{H^1(\mathbb{R}^2)}} \right)^2 \right) - 1 - \alpha \left( \frac{v_t}{\|v_t\|_{H^1(\mathbb{R}^2)}} \right)^2 \right) \, dx \]

\[ \geq (4\pi - \alpha) \left( \frac{\|v_t\|_2^2}{\|\nabla v_t\|_2^2 + \|v_t\|_2^2} + \frac{(4\pi)^2}{2} \left( \frac{\|v_t\|_4^4}{\|\nabla v_t\|_2^2 + \|v_t\|_2^2} \right)^2 \right) \]

\[ = (4\pi - \alpha) \left( \frac{\|v\|_2^2}{t \|\nabla v\|_2^2 + \|v\|_2^2} + \frac{(4\pi)^2}{2} \left( \frac{t \|v\|_4^4}{t \|\nabla v\|_2^2 + \|v\|_2^2} \right)^2 \right) \]

\[ = (4\pi - \alpha) g_v(t). \]
Lemma 7.1. For any fixed $a,b,M,N > 0$, let $h(s) := s^aM + s^{-b}N$, then we have
\[
\inf_{s>0} h(s) = h\left(\frac{bN}{aM}\right) = \frac{a + b}{a - b} \left(\frac{b}{a}\right) \frac{1}{\frac{1}{a+b} M \frac{b}{a+b} N \frac{a}{a+b}}.
\]

Proof of Lemma 7.1 Define
\[
C_j = \sup_{u \in H^2(\mathbb{R}^4)} \frac{\left(\int_{\mathbb{R}^4} |u|^{2j} dx\right)^{\frac{1}{j}}}{\int_{\mathbb{R}^4} (|\Delta u|^2 + |u|^2) dx}.
\]
First, we claim that $B_j = \frac{C_j^{j'}}{(j-1)^{j-1}}$. Set
\[
I(u) := \int_{\mathbb{R}^4} (|\Delta u|^2 + |u|^2) \, dx
\]
and
\[
\Lambda := \{u \in H^2(\mathbb{R}^4), \int_{\mathbb{R}^4} |u|^{2j} dx = 1\}.
\]
For any $\tau > 0$, we set
\[
u_{\tau} := \tau^\frac{2}{j} u(\tau x).
\]
Direct calculations lead to
\[ \int_{\mathbb{R}^4} |u_\tau|^{2j} \, dx = \int_{\mathbb{R}^4} |u|^{2j} \, dx \]
and
\[ \int_{\mathbb{R}^4} |\Delta u_\tau|^2 \, dx = \tau_j^4 \int_{\mathbb{R}^4} |\Delta u|^2 \, dx, \quad \int_{\mathbb{R}^4} |u_\tau|^2 \, dx = \tau_j^4 \int_{\mathbb{R}^4} |u|^2 \, dx. \]
It follows from Lemma 7.1 that
\[
\inf_{u \in \Lambda} I(u) = \inf_{u \in \Lambda} \inf_{\tau > 0} I(u_\tau) \\
= \inf_{u \in \Lambda} \inf_{\tau > 0} \tau_j^4 \int_{\mathbb{R}^4} |\Delta u|^2 \, dx + \tau_j^{4-4} \int_{\mathbb{R}^4} |u|^2 \, dx \\
= j(j-1)^{1-j} \left( \int_{\mathbb{R}^4} |\Delta u|^2 \, dx \right)^{1-j} \left( \int_{\mathbb{R}^4} |u|^2 \, dx \right)^{\frac{1}{j}}.
\]
According to the definitions of \( B_j \) and \( C_j \), we derive that
\[
B_j = C_j^{\frac{1}{j}} \sup_{\tau > 0} \tau_j^{4-4} \int_{\mathbb{R}^4} |u|^2 \, dx.
\]
Next, we turn to the estimate of sharp constants \( C_j \). Through Fourier transform and duality, it is easy to check that
\[
\left( \int_{\mathbb{R}^4} |u|^{2j} \, dx \right)^{\frac{1}{2j}} \leq C_j \int_{\mathbb{R}^4} (|\Delta u|^2 + |u|^2) \, dx
\]
is equivalent to
\[
\left( \int_{\mathbb{R}^4} |G \ast u|^{2j} \, dx \right)^{\frac{1}{2j}} \leq C_j^{\frac{1}{j}} \left( \int_{\mathbb{R}^4} |u|^2 \, dx \right)^{\frac{1}{2}}
\]
and
\[
\left( \int_{\mathbb{R}^4} |G \ast u|^2 \, dx \right)^{\frac{1}{2}} \leq C_j^{\frac{1}{j}} \left( \int_{\mathbb{R}^4} |u|^{2j} \, dx \right)^{\frac{2j-1}{2j}},
\]
where \( \hat{G}(\xi) = (16\pi^4 |\xi|^4 + 1)^{-\frac{1}{2}} \). Then it follows that
\[
\int_{\mathbb{R}^4} (G \ast G \ast u) \, dx \leq \left( \int_{\mathbb{R}^4} |G \ast G \ast u|^{2j} \, dx \right)^{\frac{1}{2j}} \left( \int_{\mathbb{R}^4} |u|^{2j} \, dx \right)^{\frac{2j-1}{2j}} \\
\leq C_j^{\frac{1}{j}} \left( \int_{\mathbb{R}^4} |G \ast u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^4} |u|^{2j} \, dx \right)^{\frac{2j-1}{2j}} \\
\leq C_j \left( \int_{\mathbb{R}^4} |u|^{\frac{2j}{2j-1}} \, dx \right)^{\frac{2j-1}{j}}.
\]
(7.1)
On the other hand, for any \( t > 1 \), define \( A_t = \left( \frac{t^j}{t^{j-1}} \right)^{\frac{1}{2}} \), where \( t' \) satisfies \( \frac{1}{p} + \frac{1}{t} = 1 \). Then it follows from sharp convolution Young inequality and Hausdorff Young inequality
that
\[
\int_{\mathbb{R}^4} |(G \ast G \ast u) u| dx \\
\leq \left( \int_{\mathbb{R}^4} |G \ast G \ast u|^{2j} dx \right)^{\frac{1}{2j}} \left( \int_{\mathbb{R}^4} |u|^{\frac{2j}{2j-1}} dx \right)^{\frac{2j-1}{2j}} \\
\leq \left( \frac{A_{2j}^{\frac{2j+1}{2j}} A_{j}}{A_{2j}} \right)^{4} \left( \int_{\mathbb{R}^4} |G \ast G|^j dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^4} |u|^j dx \right)^{\frac{2j-1}{2j}},
\]
which together with (7.1) yields that
\[
C_j \leq \left( \frac{A_{2j}^{\frac{2j+1}{2j}} A_{j}}{A_{2j}} \right)^{4} \left( j^{\frac{1}{2}} j^{-\frac{1}{j}} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^4} |\hat{G} | j \cdot \Delta | dx \right)^{\frac{1}{2}},
\]
and the proof is finished. \(\square\)

7.2. Some useful results.

**Proposition 7.2.** Let \(\Omega \subseteq \mathbb{R}^4\) be a bounded open domain with Lipschitz boundary. Then for any \(u \in H^2(\Omega), \omega \in H^4(\Omega)\), we have
\[
\int_{\Omega} \Delta u \cdot \Delta \omega dx = \int_{\Omega} u \cdot \Delta^2 \omega dx - \int_{\partial \Omega} v \cdot u \Delta \omega dx + \int_{\partial \Omega} v \cdot \Delta^\frac{1}{2} u \Delta \omega dx
\]
where \(v\) denotes the outer normal to \(\partial \Omega\).

**Lemma 7.3** (Pizzetti \[45\]). Let \(u \in C^{2m}(B_R(x_0)), B_R(x_0) \subset \mathbb{R}^n\), for some \(m\) positive integers. Then there are positive constants \(c_i = c_i(n)\) such that
\[
\int_{B_R(x_0)} u(x) dx = \sum_{i=0}^{m-1} c_i R^{n+2i} \Delta^i u(x_0) + c_m R^{n+2m} \Delta^m u(\xi),
\]
for some \(\xi \in B_R(x_0)\).

**Lemma 7.4** (Martinazzi \[37\]). Let \(u\) solve \(\Delta^m u = f \in L(\log L)^\alpha\) in smooth bounded \(\Omega\) with the Dirichlet boundary condition for some \(0 \leq \alpha \leq 1\) and \(n \geq 2m\). Then
\[
\nabla^{2m-l} u \in L^{\left(\frac{\log L}{\log \alpha}\right)}(\Omega), 1 \leq l \leq 2m - 1
\]
and
\[
\|
\nabla^{2m-l} u \|_{L^{\left(\frac{\log L}{\log \alpha}\right)}} \leq C \| f \|_{L(\log L)^\alpha}.
\]

**Lemma 7.5** (\[14\]). Let \(\Omega \subset \mathbb{R}^N\) be a bounded open set with smooth boundary, and take \(k, m \in \mathbb{N}, k \geq 2m\), and \(\gamma \in (0, 1)\). If \(u \in H^m(\Omega)\) is a weak solutions of the problem
\[
\begin{cases}
(-\Delta)^m u = f \text{ in } \Omega \\
\partial_i u = h_i \text{ on } \partial \Omega, 0 \leq i \leq m - 1
\end{cases}
\]
EXISTENCE AND NONEXISTENCE OF EXTREMALS FOR CRITICAL ADAMS INEQUALITIES IN $\mathbb{R}^n$

with $f \in C^{k-2m,\gamma}(\Omega)$ and $h_i \in C^{k-i,\gamma}(\partial\Omega)$, then $u \in C^{k,\gamma}(\Omega)$ and there exists a constant $c = c(\Omega, k, \gamma)$ such that

$$\|u\|_{C^{k,\gamma}(\Omega)} \leq c \left( \|f\|_{C^{k-2m,\gamma}(\Omega)} + \sum_{i=0}^{m-1} \|h_i\|_{C^{k-i,\gamma}(\partial\Omega)} \right).$$

Similarly, If $f \in C^{k-2m,\gamma}(\Omega)$ and $u$ is a weak solution of $(-\Delta)^mu = f$ in $\Omega$, then $u \in C^{k,\gamma}_{loc}(\Omega)$, and for any open set $V \subset \subset \Omega$, then there exists a constant $C = C(k, p, V, \Omega)$ such that

$$\|u\|_{C^{k,\gamma}(V)} \leq C(\|f\|_{C^{k-2m,\gamma}(\Omega)} + \|u\|_{L^1(\Omega)}).$$

Lemma 7.6 ([4]). Let $\Omega \in \mathbb{R}^n$ be a bounded open set with smooth boundary and take $m$, $k \in \mathbb{N}$, $k \geq 2m$, $p > 1$. If $f \in W^{k-2m,p}(\Omega)$ and $u \in H^m(\Omega)$ is a weak solution of $(-\Delta)^mu = f$ in $\Omega$, then $u \in W^{k,p}_{loc}(\Omega)$, and for any open set $V \subset \subset \Omega$, then there exists a constant $C = C(k, p, V, \Omega)$ such that

$$\|u\|_{W^{k,p}(V)} \leq C(\|f\|_{W^{k-2m,p}(\Omega)} + \|u\|_{L^1(\Omega)}).$$

Similarly, If $f \in C^{k-2m,\gamma}(\Omega)$ and $u$ is a weak solution of $(-\Delta)^mu = f$ in $\Omega$, then $u \in C^{k,\gamma}_{loc}(\Omega)$, and for any open set $V \subset \subset \Omega$, then there exists a constant $C = C(k, p, V, \Omega)$ such that

$$\|u\|_{C^{k,\gamma}(V)} \leq C(\|f\|_{C^{k-2m,\gamma}(\Omega)} + \|u\|_{L^1(\Omega)}).$$

Lemma 7.7 ([37]). Suppose that $u$ satisfies the bi-harmonic equation $(-\Delta)^2u = 0$ with $u(x) \lesssim (1 + |x|^l)$ for some $l \geq 0$. Then $u$ is a polynomial of degree at most $\max\{l, 2\}$.

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