On a result of Iwasawa

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Abstract

We recover a result of Iwasawa on the $p$-adic logarithm of principal units of $\mathbb{Q}_p(\zeta_p^{n+1})$ by studying the value at $s = 1$ of $p$-adic $L$-functions.

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Since the nineteenth century, it is known that values of $L$-functions at $s = 1$ contain deep arithmetic information. This result has much importance because it links analytic formulas with arithmetic invariants. Kubota and Leopoldt have defined an analogous $L$-function on the $p$-adic fields with analytic techniques. Iwasawa has shown how to construct this $p$-adic $L$-function algebraically. Our aim in this paper is to give some algebraic interpretations to the analytic formulas giving the value at $s = 1$ of $p$-adic $L$-functions. It leads us to study some properties of the $p$-adic logarithm which enable us to establish the Galois module structure of the logarithm of principal units. The result obtained (theorem 1.10 below) had been discovered first by Iwasawa ([6]) in 1968 using explicit reciprocity laws. The first corollary we state to this theorem is an important result also due to Iwasawa ([5]) which gives the structure of the plus-part of the principal units modulo cyclotomic units.

In the second section we use the theorem 1.10 to study the minus part of the projective limit of these units. The aim is to obtain a theorem which looks like the one of Iwasawa ([5]). But in the minus-part there are no more cyclotomic units. Yet by considering the $p$-adic logarithm of principal units we are able to obtain an Iwasawa-like result (theorem 2.3) and its global counterpart.

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For all the paper, fix an odd prime number $p$ and denote by $v_p$ the normalized valuation at $p$. We are interested in cyclotomic $p$-extensions. We start with $\theta$ an even character of $\text{Gal}(\mathbb{Q}(\mu_{pd})/\mathbb{Q})$ of conductor $f_\theta = d$ or $pd$ with $d \geq 1$ and $p \nmid d$. For $n \geq 0$, let $q_n = p^{n+1}d$. For all integer $m$, we identify both groups

$$\text{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$$

where $\sigma_a(\zeta_m) = \zeta_m^a$.

Fix a primitive $d$th root of unity $\alpha$. We note $\zeta_{p^n} = \alpha\zeta_{p^{n+1}}$ with $\zeta_{p^{n+1}}$ defined such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ and $\zeta_p \notin \mu_p \setminus \{1\}$. Let $F$ be the Frobenius endomorphism of $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ (i.e. $F(\alpha) = \alpha^p$), and $\Delta = \text{Gal}(\mathbb{Q}(\mu_{pd})/\mathbb{Q})$. Then $F$ corresponds to $\sigma_p \in \Delta$. As $\theta \in \Delta$ we write $\theta(F)$ for $\theta(\sigma_p)$.

Let $K_n = \mathbb{Q}_p(\mu_{q_n})$ and $K_\infty = \bigcup_{n \geq 0} K_n$. We know that $\Gamma_n = \text{Gal}(K_n/K_0) \simeq \mathbb{Z}/p^n\mathbb{Z}$ and $\Gamma = \text{Gal}(K_\infty/K_0) \simeq \mathbb{Z}/p\mathbb{Z}$. Moreover $\gamma_0 = \sigma_{1+q_0}$ is a topological generator of $\Gamma$. We have the decomposition

$$\text{Gal}(K_n/\mathbb{Q}) \simeq \Delta \times \Gamma$$

$$\sigma_a \mapsto (\delta(a), \gamma_0(a))$$

Finally, we fix some notations from Iwasawa theory. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}_p(\theta)$ and $\Lambda = \mathcal{O}[T]$. By a fundamental results of Iwasawa there exists a power series $f(T, \theta) \in \Lambda$ such that for all $n \geq 0$, $\chi \in \Gamma_n$, and $s \in \mathbb{Z}_p$

$$L_p(s, \theta\chi) = f(\chi(\gamma_0)(1 + q_0)^s - 1, \theta).$$

1 Value at $s = 1$ of $p$-adic $L$-functions

1.1 A Lemma on the Iwasawa algebra

Let $\Lambda = \mathbb{Z}_p[T]$ be the Iwasawa algebra, $g \in \Lambda$ a formal power series and $\omega_n$ the element $(1+T)^{p^n} - 1$. It is known that the Iwasawa algebra can be described as follows. Let $\Gamma$ be a multiplicative topological group isomorphic to $\mathbb{Z}_p$, $\Gamma_n$ its quotient $\Gamma/\Gamma^n \simeq \mathbb{Z}/p^n\mathbb{Z}$, and $\mathbb{Z}_p[\Gamma] = \lim \mathbb{Z}_p[\Gamma_n]$. Let $\gamma_0$ be a topological generator of $\Gamma$ and $\overline{\gamma_0}$ its image in $\mathbb{Z}_p[\Gamma_n]$. We thus have an isomorphism

$$\mathbb{Z}_p[\Gamma_n] \simeq \mathbb{Z}_p[T]/((1+T)^{p^n} - 1)$$

$$\overline{\gamma_0} \mapsto 1 + T$$

which, passing to the projective limit, gives an isomorphism

$$\mathbb{Z}_p[\Gamma] \simeq \mathbb{Z}_p[[T]]$$

$$\gamma_0 \mapsto 1 + T$$

(see for example [1]). Via this isomorphism the power series $g$ can be written as a sequence of elements $\epsilon_n \in \mathbb{Z}_p[\Gamma_n]$ compatible with the restriction morphisms. The aim of this paragraph is to express $\epsilon_n$ according to $g$.

We can write $g$ as

$$g(T) = \sum_{b=0}^{p^n-1} a_n(b)(1+T)^b + \omega_n(T)Q_n(T)$$
with \( Q_n(T) \in \Lambda \). The canonical isomorphism described above implies that \( \epsilon_n = \sum_{b=0}^{\rho^n-1} a_n(b) \tau_0^b \). Pick a character \( \chi \in \hat{\Gamma}_n = \text{Hom}(\Gamma_n, \mathbb{Q}_p^\times) \), and let \( e_{\chi} = \frac{1}{p^n} \sum_{\gamma \in \Gamma_n} \chi(\gamma) \gamma \in \mathbb{Q}_p[\Gamma_n] \) be the associate idempotent. Thus

\[
e_{\chi} \epsilon_n = e_{\chi} \left( \sum_{b=0}^{\rho^n-1} a_n(b) \tau_0^b \right) = \left( \sum_{b=0}^{\rho^n-1} a_n(b) \chi(\gamma_0)^b \right) e_{\chi} = g(\chi(\gamma_0) - 1) e_{\chi}.
\]

Summing over all characters of \( \Gamma \) we deduce the following result.

**Lemma 1.1.** Let \( g \) be a power series in \( \mathbb{Z}_p[[T]] \) and \( \epsilon_n \) its image in \( \mathbb{Z}_p[[T]]/\omega_n(T)\mathbb{Z}_p[[T]] \). Via the canonical isomorphism between \( \mathbb{Z}_p[[T]] \) and \( \mathbb{Z}_p[\Gamma] \) the element \( \epsilon_n \) in \( \mathbb{Z}_p[\Gamma_n] \) writes as

\[
\epsilon_n = \sum_{\gamma \in \Gamma_n} \left[ \frac{1}{p^n} \sum_{\chi \in \hat{\Gamma}_n} g(\chi(\gamma_0) - 1) \chi(\gamma) \right] \gamma
\]

where \( \gamma_0 \) is a topological generator of \( \Gamma \).

### 1.2 Value of the \( p \)-adic \( L \)-function at \( s = 1 \) : algebraic interpretation

Our aim is to give an algebraic interpretation of the following formula (which can be found in [7], theorem 5.18):

**Proposition 1.2.** Let \( \chi \) be an even nontrivial character of conductor \( f \) and \( \zeta \) be a primitive \( f \)-th root of unity. We have

\[
L_p(1, \chi) = -(1 - \frac{\chi(p)}{p}) \frac{\tau(\chi)}{f} \sum_{a=1}^{f} \chi(a) \log_p(1 - \zeta^a)
\]

where \( \tau(\chi) \) is the Gauss sum \( \tau(\chi) = \sum_{a=1}^{f} \chi(a) \zeta^a \).

We use the notations of the introduction. Recall the well-known relation for all \( n \geq 0 \) and \( i \leq n \)

\[
N_{K_n/K_i}(\zeta_\eta^{p^{i-n}} - 1) = \zeta_\eta - 1.
\]

Assume \( \theta \neq 1 \). By the results of the previous paragraph we know that there exists \( (\epsilon_n(\theta))_{n \geq 0} \in \lim \mathcal{O}[\Gamma_n] \) corresponding to \( f(\frac{1+\theta}{1+\theta}-1, \theta) \) and that for all \( n \geq 0 \) and \( \chi \in \hat{\Gamma}_n \)

\[
e_{\chi} \epsilon_n = L_p(1, \theta \chi) e_{\chi}.
\]

**Remark.** Assume \( \theta \neq 1 \). We have for all \( \chi \in \hat{\Gamma}_n \) and \( n \geq 0 \) \( L_p(1, \theta \chi) \neq 0 \). Thus for all \( n \geq 0 \) \( \epsilon_n \in \text{Frac}(\mathcal{O}[\Gamma_n]^\times) \).

Let \( \chi \in \hat{\Gamma}_n \). We assume that \( \chi \neq 1 \) if \( f_\theta = d \). Then the conductor of the product \( \theta \chi \) is \( f_{\theta \chi} = q_\theta \) for an integer \( 0 \leq k \leq n \).
Lemma 1.3. We have
\[ \frac{1}{p^n} \theta(F^{n-k}) \tau(\theta \chi) L_p(1, \theta \chi) = - \sum_{\delta \in \Delta} \bar{\theta}(\delta) e_\chi \log_p (1 - \zeta_{\eta_n}^\delta). \]

Proof. Start from the formula in proposition 1.2. Using the fact that \( \tau(\chi) = f \) for a character \( \chi \) of conductor \( f \) we find
\[ \tau(\theta \chi) L_p(1, \theta \chi) = - \sum_{\delta \in \Delta} \bar{\theta}(\delta) \tau(\gamma) \log_p (1 - \zeta_{\eta_n}^\delta). \]

Then using the equality (1)
\[
\tau(\theta \chi) L_p(1, \theta \chi) = - \sum_{\delta \in \Delta, \gamma \in \Gamma_n} \bar{\theta}(\delta) \tau(\gamma) \log_p (1 - \zeta_{\eta_n}^\delta)
= -p^n \sum_{\delta \in \Delta, \gamma \in \Gamma_n} \bar{\theta}(\delta) e_\chi \log_p (1 - \zeta_{\eta_n}^\delta)
= -p^n \sum_{\delta \in \Delta} \bar{\theta}(\delta) e_\chi \log_p (1 - \zeta_{\eta_n}^\delta),
\]

which proves the lemma. \( \square \)

Assume now that \( f_\theta = pd \) and let \( \mathcal{F}_n = \sum_{i=0}^n F^{n-i} p^{i-n} \zeta_{\eta_n}^i \).

Proposition 1.4. For all \( n \geq 0 \) we have
\[ \sum_{\delta \in \Delta} \bar{\theta}(\delta) e_\chi (\mathcal{F}_n) = - \sum_{\delta \in \Delta} \bar{\theta}(\delta) \log_p (1 - \zeta_{\eta_n}^\delta). \]

Proof. Recall that for a character \( \chi \) of second kind (i.e. whose conductor is \( f_\chi = p^{k+1} \)) and an integer \( 0 \leq l \leq n \) the \( \chi \)-part \( e_\chi \zeta_{\eta_{l+1}} \) vanishes if and only if \( l \neq k \) (see for example 22). Let \( \chi \) be a character of \( \Gamma_n \). We have already shown that \( \sum_{\delta \in \Delta} \bar{\theta}(\delta) e_\chi (\mathcal{F}_n) = \sum_{\delta \in \Delta} \bar{\theta}(\delta) L_p(1, \theta \chi) e_\chi (\mathcal{F}_n) \).

Moreover
\[ e_\chi (\mathcal{F}_n) = \frac{1}{p^n} \sum_{\gamma \in \Gamma_n} p^{k-n} \chi(\gamma) \zeta_{\eta_n}^{\delta} \]

where \( k \) is such that \( f_\chi = p^{k+1} \) when \( \chi \neq 1 \) and \( k = 0 \) when \( \chi = 1 \). Thus we have \( 0 \leq k \leq n \) and \( f_\theta \chi = q_k \). Then
\[
\sum_{\delta \in \Delta} e_\chi \bar{\theta}(\delta) e_\chi (\mathcal{F}_n) = \frac{L_p(1, \theta \chi)}{p^n} \sum_{\delta \in \Delta} \sum_{\gamma \in \Gamma_n} p^{k-n} \bar{\theta}(\delta) \chi(\gamma) \zeta_{\eta_n}^{\delta} \]
\[
= \frac{\theta(F^{n-k}) L_p(1, \theta \chi)}{p^n} \sum_{\delta \in \Delta} \sum_{\gamma \in \Gamma_n} p^{k-n} \bar{\theta}(\delta) \chi(\gamma) \zeta_{\eta_n}^{\delta} \]
\[
= \frac{\theta(F^{n-k}) L_p(1, \theta \chi)}{p^n} \sum_{\delta \in \Delta} \sum_{\gamma \in \Gamma_n} \bar{\theta}(\delta) \chi(\gamma) \zeta_{\eta_n}^{\delta} \]
\[
= \frac{1}{p^n} \theta(F^{n-k}) L_p(1, \theta \chi) \tau(\theta \chi) \]
\[
= - \sum_{\delta \in \Delta} \bar{\theta}(\delta) e_\chi \log_p (1 - \zeta_{\eta_n}^\delta). \]
We define the following lemma.

Lemma 1.5. If \( f_0 = d \) then

- \( \sum_{\delta \in \Delta} \overline{\theta}(\delta) \log_p (1 - \zeta_{q_0}^\delta) = (\theta(F) - 1) \sum_{y \in (\mathbb{Z}/d\mathbb{Z})^*} \overline{\pi}(y) \log_p (1 - \alpha^y) \) and

- \( \tau(\overline{\theta}) = -\sum_{\delta \in \Delta} \overline{\theta}(\delta) \zeta_{q_0}^\delta \).

It follows that for all \( n \geq 0 \),

\[
\frac{1}{p^n} \tau(\overline{\theta}) L_p(1, \theta) = -(1 - \frac{\theta(F)}{p}) \sum_{\delta \in \Delta} \overline{\theta}(\delta) e_{\chi_0} \log_p (1 - \zeta_{q_n}^\delta)
\]

where \( \chi_0 \) is the trivial character of \( \Gamma_n \). We can now state the theorem which includes all the results of this section.

Theorem 1.6. Let \( \theta \) be an even non-trivial character of \( \text{Gal}(\mathbb{Q}(\mu_{pd})/\mathbb{Q}) \) and \( (e_n(\theta))_{n \geq 0} \in \lim \mathcal{O}[\Gamma_n] \) the element corresponding to \( f(\frac{1 + pd}{p} - 1, \theta) \).

We define

\[
\mathcal{I}_n = \left\{ \sum_{i=0}^n F^{n-i} p^{i-n} \zeta_{q_i} - p^{-n} F^n (f - 1) \zeta_{q_0} \right\}
\]

We thus have for all \( n \geq 0 \),

\[
\sum_{\delta \in \Delta} \overline{\theta}(\delta) e_n(\theta)(\mathcal{I}_n^\delta) = -E(\theta) \sum_{\delta \in \Delta} \overline{\theta}(\delta) \log_p (1 - \zeta_{q_n}^\delta)
\]

where \( E(\theta) \) is a kind of Euler factor: \( E(\theta) = (1 - \frac{\theta(F)}{p}) \) when \( f_0 = d \) and \( E(\theta) = 1 \) when \( f_0 = pd \).

### 1.3 An application

We now apply theorem 1 to the case where \( f_0 = pd \). We slightly change our notations. For the rest of the paper we denote by \( \Delta \) the group \( (\mathbb{Z}/p\mathbb{Z})^* \), \( \theta_1 \) a character of \( \Delta \) and \( \theta_2 \) a character of conductor \( d \) with \( d \) dividing \( p - 1 \) such that \( \theta = \theta_1 \theta_2 \) is even. We assume first that \( \theta_1 \neq 1, \omega \). Both cases will be treated separately in the sequel. Let \( W = \mathbb{Z}[\theta_2] \) and \( (e_n(\theta))_{n \geq 0} \in W[t] \) the element corresponding to \( f(\frac{1 + pd}{p} - 1, \theta) \). Recall that, as usual, \( \alpha \) is a primitive \( d \)th root of unity. In this situation we have the following result.

Lemma 1.7. Let \( \mathcal{I}_n = \sum_{i=0}^n p^{i-n} \zeta_{pd+1}^{i+1} \). Then

\[
\tau(\overline{\theta_2}) e_n(\theta) c_{\theta_1} \mathcal{I}_n = -\sum_{y \in (\mathbb{Z}/d\mathbb{Z})^*} \overline{\pi}_2(y) c_{\theta_1} \log_p (\alpha^y - \zeta_{pd+1}^{n+1}).
\]
Lemma 1.8. Theorem 1 can be restated as

\[ \sum_{\delta \in (\mathbb{Z}/pd\mathbb{Z})^*} \overline{\tau}(\delta)\epsilon_n(\theta)(\hat{\tau}_n^{\delta}) = - \sum_{\delta \in (\mathbb{Z}/pd\mathbb{Z})^*} \overline{\tau}(\delta) \log_p(1 - \zeta_n^{\delta+1}) \]

where \( \hat{\tau}_n = \sum_{i=0}^{n} p^{i-n}\zeta_i \), and \( q_i = dp^{i+1} \). A straightforward calculation gives

\[
\sum_{\delta \in (\mathbb{Z}/pd\mathbb{Z})^*} \overline{\tau}(\delta)\epsilon_n(\theta)(\hat{\tau}_n^{\delta}) = \sum_{\delta \in (\mathbb{Z}/pd\mathbb{Z})^*} \overline{\tau}_2(y)\overline{\tau}_1(\delta)\epsilon_n(\theta)(\sum_{i=0}^{n} p^{i-n}\alpha_y^{\delta} \zeta_{p^{i+1}}^{\delta+1})
\]

\[
= (p-1) \sum_{y \in (\mathbb{Z}/d\mathbb{Z})^*} \overline{\tau}_2(y)\epsilon_n(\theta)\epsilon_{\theta_1}(\sum_{i=0}^{n} p^{i-n}\zeta_{p^{i+1}}^{\delta+1})
\]

\[
= (p-1) \tau(\overline{\tau}_2)\epsilon_n\epsilon_{\theta_1} \mathcal{F}_n.
\]

\[ \square \]

We need a relation between primitive characters and unprimitive ones which is given by the following lemma.

**Lemma 1.8.** Let \( \theta_2 \) be a character whose conductor is \( d_2 \) with \( d_2 | d \). Then

\[
\sum_{y \in (\mathbb{Z}/d_2\mathbb{Z})^*} \overline{\tau}_2(y)\epsilon_{\theta_1}(\alpha_y^{n} - \zeta_{p^{n+1}}^{\delta+1}) = \left( \sum_{y \in (\mathbb{Z}/d_2\mathbb{Z})^*} \overline{\tau}_2(y)\epsilon_{\theta_1}(\alpha_y^{n} - \zeta_{p^{n+1}}^{\delta+1}) \right) x(\theta)
\]

where \( \alpha_2 = \alpha^{d/d_2} \) and \( x(\theta) \in \mathbb{Z}_p[\theta] \).

**Proof.** It is sufficient to consider the case where \( d = ld_2 \) with \( l \) a prime number. Let \( S \) be the left-hand side sum in lemma 1.8.

- First case: \( l | d_2 \). For \( y \in (\mathbb{Z}/d_2\mathbb{Z})^* \), we want to write the elements \( z \) of \( (\mathbb{Z}/d\mathbb{Z})^* \) as \( z = y + kd_2 \) for \( 0 \leq k \leq l-1 \). However, for all \( y \) there exists a \( k_y, 0 \leq k_y \leq l-1 \), such that \( y + k_y d_2 \equiv 0 \mod l \) (i.e. \( y + k_y d_2 \not\equiv 0 \mod l \)). We can write \( S \) as

\[
S = \sum_{y \in (\mathbb{Z}/d_2\mathbb{Z})^*} \overline{\tau}_2(y)\epsilon_{\theta_1} \log_p \prod_{k=0}^{l-1} (\alpha^{kd_2+y} - \zeta_{p^{n+1}}^{\delta+1}).
\]

Note that \( \alpha^{d_2} \) is a \( l \) th root of unity. Since \( \prod_{\zeta \in \mu_1} (X - \zeta Y) = (X^l - Y^l) \) we get \( S \)

\[
S = \sum_{y \in (\mathbb{Z}/d_2\mathbb{Z})^*} \overline{\tau}_2(y)\epsilon_{\theta_1} \log_p \left( \frac{\alpha_y^{l} - \zeta_{p^{n+1}}^{\delta+1}}{\alpha_2^{l} - \zeta_{p^{n+1}}^{\delta+1}} \right)
\]

\[
= (\theta_1(l) - \overline{\tau}_2(\sigma_{y+kd_2})) \sum_{y \in (\mathbb{Z}/d_2\mathbb{Z})^*} \overline{\tau}_2(y)\epsilon_{\theta_1}(\alpha_y^{n} - \zeta_{p^{n+1}}^{\delta+1}).
\]
of unity. Hence the right-hand side of the above is equal to 

\[ S = \sum_{y \in (\mathbb{Z}/d\mathbb{Z})^\times} \overline{\theta}_2(y)e_{\theta_1} \log_p(\alpha_2^y - \zeta_{p^n+1}^l) \]

\[ = \theta_1(l) \sum_{y \in (\mathbb{Z}/d\mathbb{Z})^\times} \overline{\theta}_2(y)e_{\theta_1} \log_p(\alpha_2^y - \zeta_{p^n+1}). \]

For a given \( d \) dividing \( p-1 \) the two previous lemmas yield the equality

\[ \tau(\overline{\theta}_2) x(\theta_1\theta_2) e_{\theta_1} \mathcal{T}_n = - \sum_{y \in (\mathbb{Z}/d\mathbb{Z})^\times} \overline{\theta}_2(y)e_{\theta_1} \log_p(\alpha^y - \zeta_{p^n+1}). \quad (2) \]

**Lemma 1.9.** Let \( d \) be an integer dividing \( p-1 \) and \( \alpha \) a primitive \( d \)th root of unity. Fix \( \theta_1 \in \Delta \) different from \( 1, \omega \). Then there exists \( u_n \in \mathbb{Z}_p[\Gamma_n] \) such that

\[ u_n e_{\theta_1} \mathcal{T}_n = e_{\theta_1} \log_p(\alpha - \zeta_{p^n+1}). \]

**Proof.** Suppose first that \( \theta_1 \) is even. Summing the equality (2) over all the \( \theta_2 \) such that the product \( \theta_1\theta_2 \) is even we obtain

\[ \left( \sum_{\substack{f_{\theta_2} \mid d \\text{ even} \theta_2}} \tau(\overline{\theta}_2) x(\theta_1\theta_2) e_{\theta_1} \mathcal{T}_n \right) e_{\theta_1} \mathcal{T}_n = - \sum_{y \in (\mathbb{Z}/d\mathbb{Z})^\times} \left( \sum_{\substack{f_{\theta_2} \mid d \\text{ even} \theta_2}} \overline{\theta}_2(y) \right) e_{\theta_1} \log_p(\alpha^y - \zeta_{p^n+1}). \]

We have

\[ \sum_{\substack{f_{\theta_2} \mid d \\text{ even} \theta_2}} \overline{\theta}_2(y) = \begin{cases} 0 \text{ when } y \not\equiv \pm 1 \mod d \\ \frac{\varphi(d)}{2} \text{ when } y \equiv \pm 1 \mod d. \end{cases} \]

Hence the right-hand side of the above is equal to 

\[ -\frac{\varphi(d)}{2} e_{\theta_1} \log_p[(\alpha - \zeta_{p^n+1})(\alpha^{-1} - \zeta_{p^n+1})]. \]

Since \( \theta_1 \) is even we have 

\[ e_{\theta_1} \log_p(\alpha^{-1} - \zeta_{p^n+1}) = e_{\theta_1} \log_p(\alpha - \zeta_{p^n+1}). \]

The right-hand side thus equals 

\[ -\varphi(d) e_{\theta_1} \log_p(\alpha - \zeta_{p^n+1}). \]

Set 

\[ u_n = -\frac{1}{\varphi(d)} \left( \sum_{\substack{f_{\theta_2} \mid d \\text{ even} \theta_2}} \tau(\overline{\theta}_2) x(\theta_1\theta_2) e_{\theta_1} \mathcal{T}_n \right). \]

As \( u_n \) is Galois-invariant we have \( u_n \in \mathbb{Z}_p[\Gamma_n] \). The result follows in this case.

When \( \theta_1 \) is odd, the proof runs the same except that 

\[ \sum_{\substack{f_{\theta_2} \mid d \\text{ odd} \theta_2}} \overline{\theta}_2(y) = \begin{cases} 0 \text{ if } y \not\equiv \pm 1 \mod d \\ \pm \frac{\varphi(d)}{2} \text{ if } y \equiv \pm 1 \mod d \end{cases} \]

and 

\[ e_{\theta_1} \log_p(\alpha^{-1} - \zeta_{p^n+1}) = -e_{\theta_1} \log_p(\alpha - \zeta_{p^n+1}). \]

\[ \square \]
Theorem 1.10. Let \( \theta \in \widehat{\Delta}, \theta \neq 1, \omega \). Let \( U_n \) be the group of principal units of \( \mathbb{Q}_p(\zeta_{p^n}) \) defined by \( U_n = 1 + (\zeta_{p^n} - 1)\mathbb{Z}_p[\zeta_{p^n}] \) and \( \mathcal{F}_n = \sum_{i=0}^{n} \mathbb{Z}_p[\zeta_{p^i}] \). Then for all \( n \geq 0 \),
\[
eg \log_p U_n = \mathbb{Z}_p[\Gamma_n]e_\theta \mathcal{F}_n.
\]

Proof. By lemma 1.9 there exists \( u_n \in \mathbb{Z}_p[\Gamma_n] \) such that \( u_ne_\theta \mathcal{F}_n = \log_p(\alpha - \zeta_{p^{n+1}}) \). By the results of [1] section 13.8 there exists an integer \( d \) dividing \( p - 1 \) and a primitive \( d \)-th root of unity \( \alpha \) such that \( \log_p U_n = \mathbb{Z}_p[\Gamma_n]e_\theta \log_p(\alpha - \zeta_{p^{n+1}}) \). Hence \( \log_p U_n \subseteq \mathbb{Z}_p[\Gamma_n]e_\theta \mathcal{F}_n \). The converse inclusion will be proved in the following section (lemma 1.11 and proposition 1.12). Just note it implies \( u_n \in \mathbb{Z}_p[\Gamma_n]^\times \) for such an \( \alpha \). \( \square \)

We now derive some corollaries from theorem 1.10, the first of which is a well-known result of Iwasawa.

Corollary 1 (Iwasawa). ([2] or [7] theorem 13.56) Let \( \theta \) be an even nontrivial character in \( \widehat{\Delta} \). Let \( (\epsilon_n(\theta))_{n \geq 0} \in \lim_{\to} \mathbb{Z}_p[\Gamma_n] \) be the element corresponding to the power series \( f(\frac{1}{1 + \frac{1}{p^2} - 1}, \theta) \). Let \( \overline{C}_n \) the closure of the cyclotomic units in \( U_n \). For all \( n \geq 0 \) we have an isomorphism
\[
eg e_\theta U_n/\overline{C}_n \simeq \mathbb{Z}_p[\Gamma_n]/\epsilon_n(\theta)\mathbb{Z}_p[\Gamma_n].
\]

Proof. By basic results on cyclotomic units we have
\[
eg e_\theta \log_p \overline{C}_n = \mathbb{Z}_p[\Gamma_n]e_\theta \log_p(1 - \zeta_{p^{n+1}}).
\]
Lemma 1.9 allplied to \( d = 1 \) shows that there exists \( u_n \in \mathbb{Z}_p[\Gamma_n] \) such that \( u_ne_\theta \mathcal{F}_n = e_\theta \log_p(1 - \zeta_{p^{n+1}}) \). The definition of \( u_n \) gives \( u_n = -\epsilon_n(\theta) \). Wededuce that \( \log_p \overline{C}_n = \mathbb{Z}_p[\Gamma_n]e_\theta(\theta)e_\theta \mathcal{F}_n \). Moreover theorem 2 shows that \( \log_p U_n = \mathbb{Z}_p[\Gamma_n]e_\theta \mathcal{F}_n \). As \( \theta \) is even and nontrivial we have isomorphisms
\[
eg e_\theta U_n/\overline{C}_n \simeq e_\theta \log_p U_n/\log_p \overline{C}_n \simeq \mathbb{Z}_p[\Gamma_n]/\epsilon_n(\theta)\mathbb{Z}_p[\Gamma_n].
\]

\( \square \)

Corollary 2. Assume \( d \geq 3 \). There exists an odd character \( \theta_2 \) whose conductor divides \( d \) such that there are at least \( \sqrt{p} - 2 \) odd character \( \theta_1 \in \widehat{\Delta}, \theta_1 \neq \omega \) satisfying
\[
eg f(T, \theta_1\theta_2) \in W[T]^\times.
\]

Proof. By a result of Anglèes ([1], theorem 5.4) there are at least \( \sqrt{p} - 2 \) odd characters \( \theta_1 \in \widehat{\Delta}, \) such that \( \theta \neq \omega \) and for all \( n \geq 0 \),
\[
eg c_0 \log_p U_n = c_0 \log_p(\alpha - \zeta_{p^{n+1}}).
\]
For such a character we know that
\[
eg u_n = \sum_{\substack{\theta_2 \mid d \\theta_2 \text{ odd} \\theta_2 \text{ odd}}} \tau(\overline{\theta}_2) x(\theta_1\theta_2)e_n(\theta_1\theta_2) \in \mathbb{Z}_p[\Gamma_n]^\times.
\]
Therefore there exists at least one \( \theta_2 \) such that \( c_0(\theta_1\theta_2) \in W^\times \). Since \( \theta_1\theta_2 \) is a character of the first kind it follows that \( f(T, \theta_1\theta_2) \in W[T]^\times \). \( \square \)
A similar proof yields the following result.

**Corollary 3.** Let \( \theta_1 \in \hat{\Delta}, \theta_1 \neq 1, \omega \). There exists \( \theta_2 \) whose conductor divides \( p-1 \) such that \( \theta_1 \theta_2 \) is even and the generalized Bernoulli number \( B_{1, \theta_1 \theta_2 \omega^{-1}} \) is prime to \( p \)

## 1.4 Some index computations

Let us recall some notations. As usual \( p \) is an odd prime number and \( \mathcal{O}_n \) is the ring of integers of \( \mathbb{Q}_p(\zeta_p^{n+1}) \). Let \( \mathcal{F}_n \) be \( \sum_{i=0}^n p^{\gamma_i} \zeta_{p^{\gamma_i+1}}, \) and for \( 1 \leq d \leq n \)

\[
eq = \sum_{\chi \in \Gamma_\theta} e_\chi.
\]

We first want to compute the index \( [\mathcal{O}_n : \mathcal{O}_n \cap \mathcal{P} \theta] \) with \( \theta \in \hat{\Delta} \).

Let \( \Lambda_\theta \) be the maximal order of \( \mathcal{O}_n \). Then \( \Lambda_\theta[\hat{\Delta}] \) is the maximal order of \( \mathcal{O}_n[\hat{\Delta} \times \hat{\Gamma}] \) and the Leopoldt theorem (see [2]) states that \( \mathcal{O}_n \) is a free \( \Lambda_\theta[\hat{\Delta}] \)-module generated by \( \mathcal{T}_n \). Then \( \Lambda_\theta \mathcal{T}_n = \hat{\Delta} \mathcal{T}_n \).

We can therefore substitute \( \frac{1}{p^n} e_\theta \mathcal{O}_n \) by \( \Lambda_n e_\theta \sum_{i=0}^n \zeta_{p^{i+1}} \) in our calculations. The index \( \frac{1}{p^n} e_\theta \mathcal{O}_n : \mathcal{O}_n \mathcal{T}_n \) with \( \theta \in \hat{\Delta} \) is the product

\[
[\Lambda_n e_\theta \sum_{i=0}^n \zeta_{p^{i+1}} : \mathcal{O}_n e_\theta \sum_{i=0}^n \zeta_{p^{i+1}}] \cdot [\Lambda_n e_\theta \mathcal{T}_n : \mathcal{O}_n \mathcal{T}_n].
\]

By a standard calculation of discriminants we have \( [\Lambda_n e_\theta \mathcal{T}_n : \mathcal{O}_n \mathcal{T}_n] = [\mathcal{O}_n : \mathcal{O}_n] = p^{n+1} \). In order to find \( [\Lambda_n e_\theta \sum_{i=0}^n \zeta_{p^{i+1}} : \mathcal{O}_n e_\theta \sum_{i=0}^n \zeta_{p^{i+1}}] \) we notice that \( \Lambda_n = \bigoplus_{d=0}^n \mathcal{O}_n \mathcal{T}_n [\gamma_n] e_\chi. \) Thus

\[
[\Lambda_n e_\theta \sum_{i=0}^n \zeta_{p^{i+1}} : \mathcal{O}_n e_\theta \sum_{i=0}^n \zeta_{p^{i+1}}] = \prod_{i=0}^n [\mathcal{O}_n [\gamma_i] e_\theta \sum_{i=0}^n \zeta_{p^{i+1}} : \mathcal{O}_n [\gamma_i] e_\theta \sum_{i=0}^n \zeta_{p^{i+1}}] = \prod_{i=0}^n p^{(p^i - p^{i+1})}.
\]

because the \( \mathbb{Z}_p \)-rank of \( \mathcal{O}_n [\gamma_i] e_\theta \sum_{i=0}^n \zeta_{p^{i+1}} \) is \( p^i - p^{i+1} \). Moreover we have the equality

\[
\sum_{i=0}^n d(p^i - p^{i+1}) = np^n - \frac{n}{p-1}.
\]

We have thus obtained the following result.

**Lemma 1.11.** Let \( \theta \in \hat{\Delta} \). We have \( \frac{1}{p^n} e_\theta \mathcal{O}_n : e_\theta \log_p U_n \) = \( p^n e_\theta \).

The next index we want to calculate is \( \frac{1}{p^n} e_\theta \mathcal{O}_n : e_\theta \log_p U_n \).

**Proposition 1.12.** Let \( \theta \in \hat{\Delta} \). We have

\[
[\frac{1}{p^n} e_\theta \mathcal{O}_n : e_\theta \log_p U_n] = \begin{cases} 
p^n & \text{when } \theta \neq 1, \omega 
np^{n+1} & \text{when } \theta = 1 
np^{n+1} & \text{when } \theta = \omega.
\end{cases}
\]
Proof. The proof is based on a result of John Coates, see [3]. We first show the inclusion \( \log_p U_n \subseteq \frac{1}{p} \mathcal{O}_n \). Define \( \pi_n = \zeta_{p^{n+1}} - 1 \) and let \( u \in U_n \). Then there exists \( a \in \mathbb{Z} \) such that

\[
u_p(u) = 1 \mod \pi_n^2.
\]

Therefore \( \log_p U_n = \log_p (1 + \pi_n^2 \mathcal{O}_n) \). Moreover we have \( u \equiv 1 \mod \pi_n^2 \) implies \( v_p(u^p - 1) \geq \frac{2}{p-1} \) for \( u \in U_n \). Then by lemma 5.5 of [7] we have \( \log_p U_n \subseteq \frac{1}{p} \mathcal{O}_n \).

We now calculate the index. We introduce the group \( V = 1 + p \mathcal{O}_n \). By standard properties of the \( p \)-adic logarithm we know that \( \log_p V = p \mathcal{O}_n \). Then

\[
\left[ \frac{1}{p^n} : e_\theta \log_p V \right] = \left[ \frac{1}{p^n} : e_\theta \mathcal{O}_n : p \mathcal{O}_n \right] = p^{n(n+1)}.
\]

It remains to compute \( [e_\theta \log_p U_n : e_\theta \log_p V] \). Consider the morphism

\[
(1 + \pi_n^2 \mathcal{O}_n) / V \xrightarrow{\log_p} \log_p U_n / \log_p V,
\]

the kernel of which consists of the roots of unity. Then for \( \theta \neq \omega \) we have

\[
e_\theta \log_p U_n / \log_p V \equiv e_\theta (1 + \pi_n^2 \mathcal{O}_n) / V = \frac{p^n(p-1)-1}{\prod_{i=2}^{p^n} e_\theta 1 + \pi_n^i \mathcal{O}_n}.
\]

For all integers \( i \geq 1 \) we have an isomorphism of \( \mathbb{Z}_p \)-modules

\[
\begin{array}{ccc}
\frac{1 + \pi_n^i \mathcal{O}_n}{1 + \pi_n^{i+1} \mathcal{O}_n} & \rightarrow & \mathcal{O}_n / \pi_n \mathcal{O}_n \simeq \mathbb{F}_p \\
1 + x \pi_n^i & \mapsto & x \mod \pi_n.
\end{array}
\]

Hence the \( \theta \)-part of \( \frac{1 + \pi_n^i \mathcal{O}_n}{1 + \pi_n^{i+1} \mathcal{O}_n} \) is \( \mathbb{F}_p \) or \( \{0\} \). Moreover for all \( \delta \in \Delta \) we have \( x^\delta \equiv x \mod \pi_n \). Thus for \( \theta \in \Delta \) we have

\[
(1 + x \pi_n^i)^{e_\theta} \equiv \prod_{\delta \in \Delta} \left( 1 + \frac{1}{p-1} \Theta(\delta) x(\pi_n^i)^\delta \right) \mod (\pi_n^{i+1}).
\]

However \( (\pi_n^i)^\delta \equiv \omega^i(\delta) \pi_n^i \mod (\pi_n^{i+1}) \) which yields

\[
(1 + x \pi_n^i)^{e_\theta} \equiv 1 - \left( \sum_{\delta \in \Delta} \Theta(\delta) \omega^i(\delta) \right) x \pi_n^i \mod (\pi_n^{i+1}).
\]

Thus for \( \theta = \omega^k \) we get

\[
e_\theta \frac{1 + \pi_n^i \mathcal{O}_n}{1 + \pi_n^{i+1} \mathcal{O}_n} = \begin{cases}
0 & \text{when } i \neq k \mod (p-1) \\
\mathbb{F}_p & \text{when } i \equiv k \mod (p-1),
\end{cases}
\]

and \( [e_\theta (1 + \pi_n^i \mathcal{O}_n) : e_\theta (1 + p \mathcal{O}_n)] = p^{C(k)} \) where \( C(k) \) is the number of integers \( i \) such that \( 2 \leq i \leq p^n(p-1) - 1 \) and \( i \equiv k \mod (p-1) \). When \( k \neq 0, 1 \) (i.e. \( \theta \neq \omega, 1 \)) we have \( C(k) = p^n \) and when \( k = 0, 1 \) we have \( C(k) = p^n - 1 \). Hence the result for \( \theta \neq \omega \).

The case \( \theta = \omega \) is similar except that \( \zeta_p, \ldots, \zeta_p^n \) are in the kernel of the morphism \( \log_p : (1 + \pi_n^2 \mathcal{O}_n) / V \rightarrow \log_p U_n / \log_p V \). □
Proof.
A careful reading of the proof of theorem 13.54 in [7] shows that
we have the congruences
\[ \sum_{n=0}^{\infty} P_n = \lim_{l} \mathcal{L}_n T_n. \]
Let \( \mathcal{L}_n = t_n \circ \log_p \).

Corollary 4. Let \( \theta \in \Delta \) with \( \theta \neq 1, \omega. \) Then, \( e_\theta \mathcal{L}_n U_n = Z_p[\Gamma_n] e_\theta T_n. \)

1.5 The case of the Teichmüller character

For technical reasons the results in this section are only valid when \( p \geq 5. \)

Let us begin with the following proposition.

Proposition 1.13. There exists \( \alpha \in \mu_{p-1} \setminus \{\pm 1\} \) such that for all \( n \geq 0, \)
\[ e_\omega \log_p U_n = Z_p[\Gamma_n] e_\omega \log_p (\alpha - \zeta_p^n). \]

Proof. A careful reading of the proof of theorem 13.54 in [7] shows that
\[ \lim_{l} e_\omega \log_p U_n \text{ is a free } \Lambda \text{-module of rank 1 and that when } (\epsilon_n)_{n \geq 0} \in \lim_{l} e_\omega \log_p U_n \text{ is such that } e_\omega \log_p U_0 = Z_p[\epsilon_0] \text{ then for all } n \geq 0, \]
\[ e_\omega \log_p U_n = Z_p[\Gamma_n] \epsilon_n. \]
Therefore it is sufficient to show there exists \( \alpha \in \mu_{p-1}, \alpha \neq \pm 1 \) such that
\[ e_\omega \log_p U_0 = Z_p e_\omega \log_p (\alpha - \zeta_p). \quad (3) \]
Let us prove (3). Recall that \( e_\omega \log_p U_0 = e_\omega \pi^2 \Theta_0 = p \tau(\omega^{-1}) Z_p. \) Let \( \pi \)
denote the element \( \zeta_p - 1. \)

Lemma 1.14. Let \( x \in \pi^2 Z_p[\epsilon_p]. \) Then \( \log_p (1 + x) \equiv \frac{(1 + x)^p - 1}{p} \mod (p \pi^2). \)

Proof. We have the congruences
- for all \( n \geq p, v_p \left( \frac{a}{n} \right) \geq p + 1 \) and
- for \( 1 \leq k \leq p - 1, \frac{p}{p} \equiv \frac{(-1)^{k+1}}{k} \mod p. \)

We then have
\[ \log_p (1 + x) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} x^n \equiv \sum_{n \geq 1} \frac{p}{p} x^n \mod (p \pi^2) \equiv \frac{(1 + x)^p - 1}{p} \mod (p \pi^2), \]
and the lemma is proved.

Lemma 1.15. Let \( \alpha \in \mu_{p-1} \setminus \{1\} \) and \( \theta \in \Delta, \theta \neq 1. \) We have
\[ e_\theta \log_p (\alpha - \zeta_p) \equiv -\frac{\theta(-1)}{\omega(\alpha - 1)} \tau(\theta) \left( \sum_{k=1}^{p-1} (-1)^k \frac{p}{p} \theta(k) \alpha^k \right) \mod (p \pi^2). \]
Proof. Let $\gamma = \frac{\alpha - \zeta_p}{(\alpha - 1)\zeta_p p}$. Notice that $\log_p(\gamma) = \log_p(\alpha - \zeta_p)$ and $\gamma \equiv 1 \mod (p^2)$, which allows us to apply lemma 1.14. We get
\[
\log_p(\alpha - \zeta_p) \equiv \frac{(\alpha - \zeta_p)^p - 1}{p} \mod (p^2).
\]
Taking the $p$-part and expanding the sum yields the required result. \qed

Since $e_\omega \log_p U_0 = p \tau(\omega^{-1})Z_p$ and $v_\omega(\tau(\omega^{-1})) = 1$ we deduce from lemma 1.15 the following result.

**Lemma 1.16.** Let $\alpha \in \mu_{p-1} \setminus \{1\}$. The element $e_\omega \log_p(\alpha - \zeta_p)$ is a generator of $e_\omega \log_p U_0$ if and only if
\[
\sum_{k=1}^{p-1} (-1)^k \frac{\binom{p}{k}}{p} \omega(k) \alpha^k \not\equiv 0 \mod (p^2).
\]

**Lemma 1.17.** There exists $\alpha \in \mu_{p-1} \setminus \{\pm 1\}$ such that $\sum_{k=1}^{p-1} (-1)^k \frac{\binom{p}{k}}{p} \omega(k) \alpha^k \not\equiv 0 \mod (p^2)$.

This is precisely what we need to complete the proof of proposition 1.13.

**Proof.** Let us consider the polynomial $P(X) = \sum_{k=1}^{p-1} (-1)^k \frac{\binom{p}{k}}{p} \omega(k) X^k \in Z_p[X]$. We know that $\binom{p}{k}/p \equiv (-1)^{k+1}/k \mod p$ and thus
\[
P(X) \equiv -X \prod_{\alpha \in \mu_{p-1} \setminus \{1\}} (X - \alpha) \mod p.
\]
We can therefore use Hensel’s lemma which shows that $P(X) = -X \prod_{\alpha \in \mu_{p-1} \setminus \{1\}} (X - a(\alpha))$ where $a(\alpha) \in Z_p^\times$ and $a(\alpha) \equiv \alpha \mod p$. Notice that $P(-1) \equiv 0 \mod (p^2)$.

Let us assume the lemma is false. Then for all $\alpha \in \mu_{p-1} \setminus \{1\}$, we have $a(\alpha) \equiv \alpha \mod p^2$. Thus $P(X) \equiv -X \prod_{\alpha \in \mu_{p-1} \setminus \{1\}} (X - \alpha) \mod p^2$. Comparing with the expression of $P(X)$ we deduce that for all $k \in \{1, \ldots, p-1\}$, we have
\[
(-1)^k \frac{\binom{p}{k}}{p} \omega(k) \equiv -1 \mod p^2.
\]

Let us apply this congruence with $k = 2$ and $k = 4$. The assumption $p \geq 5$ is essential in what follows. We obtain on the one hand $(p - 1)\omega(2)/2 \equiv -1 \mod p^2$, that is to say $\omega(2)/2 \equiv p + 1 \mod p^2$. On the other hand we have $\omega(4)/4 \equiv 1 + (11/6)p \mod p^2$. But $\omega(4)/4 = (\omega(2)/2)^2 \equiv 1 + 2p \mod p^2$. This implies $11/6 \equiv 2 \mod p^2$ which is not possible. \qed

This finishes the proof of proposition 1.13.
In order to apply some of the results of section 1.2, we let \( \theta_2 \) be an odd character of conductor \( d \) with \( d \) dividing \( p-1 \). Recall that
\[
\tau(\theta_2)\epsilon_n(\theta_2 \omega)\epsilon_\omega \mathcal{F}_n = - \sum_{y \in \left( \mathbb{Z}/d \mathbb{Z} \right)^*} \overline{\theta_2}(y)\epsilon_\omega \log_p(\alpha^y - \zeta_{p^n+1})
\]
where \( \alpha \) is a primitive \( d \)th root of unity and \( \mathcal{F}_n = \sum_{i=0}^{n} p^i \zeta_{p^n+1} \).

Since \( \theta_2(p) = 1 \) we have for all characters \( \chi \) and all integers \( n \geq 1 \),
\[
L_p(1-n, \chi) = -(1 - \chi \omega^{-n}(p)p^{n-1})\frac{B_n}{\chi \omega^{-n}}
\]
from which we deduce immediately that \( L_p(0, \theta_2 \omega) = 0 \). Thus if \( f(T, \theta_2 \omega) \) denotes the power series associated to \( L_p(s, \theta_2 \omega) \) we have the factorization \( f(T, \theta_2 \omega) = Th(T, \theta_2 \omega) \). Then there exists \( H(T, \theta_2 \omega) \in W[T] \) such that
\[
f\left( \frac{1+pd}{1+p} \right) - 1, \theta_2 \omega) = (T-pd)H(T, \theta_2 \omega).
\]

As usual \( T \) corresponds to \( \sigma_{1+pd} - 1 \in W[\text{Gal}(K_\infty/K_0)] \). Let \( \overline{\epsilon}_n \) be the element of \( \lim_{\leftarrow i} W[\Gamma_n] \) associated to \( H \) via the isomorphism in lemma 1.1. We have
\[
\epsilon_n(\theta_2 \omega)\epsilon_\omega \mathcal{F}_n = \overline{\epsilon}_n(\theta_2 \omega)(\sigma_{1+pd} - 1 - pd)\epsilon_\omega \mathcal{F}_n = \overline{\epsilon}_n(\theta_2 \omega)(\sigma_{1+pd} - 1 - pd)\epsilon_\omega \mathcal{F}_n,
\]
where \( s = \log_p(1+pd)/\log_p(1+p) \in \mathbb{Z}_p^\times \). So we have
\[
\sigma_{1+pd} - 1 - pd
\]
with \( u \in \mathbb{Z}_p[\Gamma_n]^\times \). The same kind of calculation than in the general case shows that there exists a unit \( u_n \in \mathbb{Z}_p[\Gamma_n] \) such that
\[
u_n(\gamma_0 - 1 - p)\epsilon_\omega \mathcal{F}_n = \epsilon_\omega \log_p(\alpha - \zeta_{p^n+1})
\]
where \( \gamma_0 = \sigma_{1+pd} - 1 \) and \( \alpha \) is a primitive \( d \)th root of unity.

**Lemma 1.18.** We have
\[
[\mathbb{Z}_p[\Gamma_n] \epsilon_\omega \mathcal{F}_n : \mathbb{Z}_p[\Gamma_n](\gamma_0 - 1)\epsilon_\omega \mathcal{F}_n] = p^n p^{n+1}.
\]

**Proof.** By lemma 1.11 we have \( \frac{1}{\overline{\epsilon}_n} \mathcal{O}_n : \mathbb{Z}_p[\Gamma_n]\epsilon_\omega \mathcal{F}_n] = p^n p^{n+1} \). Also
\[
[\mathbb{Z}_p[\Gamma_n] \epsilon_\omega \mathcal{F}_n : \mathbb{Z}_p[\Gamma_n](\gamma_0 - 1)\epsilon_\omega \mathcal{F}_n] = [\Lambda/\omega_n \Lambda : (T-p)\Lambda/\omega_n \Lambda] = [\Lambda : (\omega_n, T-p)]
\]
where \( \Lambda \simeq \mathbb{Z}_p[T] \) is the Iwasawa algebra. The \( \Lambda \)-module \( M = \Lambda/(T-p) \) has no \( \mathbb{Z}_p \)-torsion. Its characteristic polynomial is \( T-p \) and it is a standard result that \( |M/\omega_n M| = \prod_{n \in \mathbb{N}, \zeta(n-1)} (\varphi(p) - 1) \). Then
\[
[\Lambda : (\omega_n, T-p)] \sim p \prod_{\zeta \in \mathbb{N}, \zeta \neq 1} (\varphi(p) - 1) \sim p^{n+1}
\]
where \( \sim \) means 'has same \( p \)-adic valuation as'.

\[\square\]
We have thus established the following result.

**Theorem 1.19.** Let \( p \geq 5 \) be a prime number. We have

\[
e_{\omega} \log_p U_n = \mathbb{Z}_p[\Gamma_n](\gamma_0 - 1 - p)e_\omega \mathcal{F}_n
\]

where \( \mathcal{F}_n = \sum_{i=0}^n p_i \zeta_{p_i+1} \). With the notations of section 1.4, this writes as

\[
e_{\omega} \mathcal{L}_n U_n = \mathbb{Z}_p[\Gamma_n](\gamma_0 - 1 - p)e_\omega T_n
\]

where \( T_n = \sum_{i=0}^n \zeta_{p_i+1} \).

### 1.6 The case of the trivial character

The main difference between the trivial character and the other ones is that the power series associated to \( L_p \) has not integral coefficients. Thus we have to work with the power series \( g(T) \) defined by

\[
g(T) = (1 - \frac{\log(1 + T)}{p}) f(1 + T, 1) \in \Lambda.
\]

See [7], chapter 7 for more details on this particularly proposition 7.9. In our situation we have \( q_0 = p \). Let

\[
h(T) = g(1 + p - 1) = -T f(1 + (\frac{1}{p}) - 1, 1)
\]

and \( \eta_n \in \lim_{\leftarrow} \mathbb{Z}_p[\Gamma_n] \) be the element corresponding to \( h \).

This means that we have, as in section 1.1, for all \( \chi \in \hat{\Gamma}_n \) different from 1,

\[
e_{\chi} \eta_n = (1 - \chi(\gamma_0)) L_p(1, \chi)e_\chi.
\]

Let \( T_\Delta = \sum_{\delta \in \Delta} \delta \) the norm element of \( \mathbb{Z}_p[\text{Gal}(K_0)/\mathbb{Q}] \).

**Proposition 1.20.** Let \( E_n \) be the group of units in \( K_n \), \( \mathcal{F}_n \) its closure in \( U_n \) and \( \mathcal{F}_n = \sum_{i=1}^n p_i^{1-n} \zeta_{p_i+1} \) (note that the sum begins at \( i = 1 \)). We have

\[
T_\Delta \log_p E_n = \mathbb{Z}_p[\Gamma_n] T_\Delta \mathcal{F}_n.
\]

**Proof.** Let the field \( \mathbb{F}_n \in \mathbb{Q}(\zeta_{p_n+1}) \) be such that \( \mathbb{Q}_n : \mathbb{Q} = p^n \). Iwasawa has showed [7] that \( p \) does not divide the class number of \( \mathbb{F}_n \). Thus [7], theorem 8.2 shows that

\[
T_\Delta \log_p \mathcal{F}_n = T_\Delta \log_p \mathcal{C}_n.
\]

where \( \mathcal{C}_n \) is the closure of \( C_n \) in \( U_n \).

Moreover we have \( T_\Delta \log_p \mathcal{C}_n = \mathbb{Z}_p[\Gamma_n] T_\Delta \log_p ((1 - \zeta_{p_n+1})^{\gamma_0 - 1}) \). Now fix a character \( \chi \in \hat{\Gamma}_n \) different from 1 whose conductor is \( f_\chi = p^{k+1} \) with \( 1 \leq k \leq n \). We have

\[
\frac{1}{p^n} \tau(\chi) = e_\chi T_\Delta \mathcal{F}_n
\]

and then

\[
\frac{1}{p^n}(1 - \chi(\gamma_0)) \tau(\chi) L_p(1, \chi) = e_\chi \eta_n T_\Delta \mathcal{F}_n.
\]

A little calculation gives

\[
\tau(\chi) L_p(1, \chi) = -\sum_{\delta \in \Delta} \chi(\delta) \log_p (1 - \zeta_{p^\delta+1}^{\chi})
\]

\[
= -p^n e_\chi T_\Delta \log_p (1 - \zeta_{p^{n+1}}).
\]
This proves that for all \( \chi \in \hat{\Gamma}_n, \chi \neq 1 \), we have the following equality
\[
eq T \Delta \log_p (1 - \zeta_{p^{n+1}}) = -e_\chi T \Delta \widehat{\mathcal{F}}_n.
\]
We check that this equality also holds when \( \chi = 1 \) and summing over all the characters gives the required result.

**Lemma 1.21.** We have
\[
[T \Delta \frac{1}{p^d} \mathcal{O}_n : \mathbb{Z}_p[\Gamma_n] \Delta (p \zeta_p + \widehat{\mathcal{F}}_n)] = p^{n^d n + 1}.
\]

**Proof.** We already know that \([\Lambda_n : \mathbb{Z}_p[\Gamma_n]] = p^{n^d n} \) where \( \Lambda_n \) is the maximal order of \( \mathbb{Q}_p[\Gamma_n] \). It remains to calculate \([T \Delta \frac{1}{p^d} \mathcal{O}_n : \Lambda_n \Delta (p \zeta_p + \widehat{\mathcal{F}}_n)]\). Decompose into characters yields
\[
[T \Delta \frac{1}{p^d} \mathcal{O}_n : \Lambda_n \Delta (p \zeta_p + \widehat{\mathcal{F}}_n)] = \prod_{d=0}^{n} [e_d \frac{1}{p^d} T \Delta \mathcal{O}_n : e_d \Lambda_n \Delta (p \zeta_p + \widehat{\mathcal{F}}_n)].
\]
For \( 1 \leq d \leq n \) we have \( e_d \frac{1}{p^d} \mathcal{O}_n = 0 \) and \( e_d \Lambda_n \Delta (p \zeta_p + \widehat{\mathcal{F}}_n) = 0 \). Then when \( d \neq 0 \) we have \( [e_d \frac{1}{p^d} T \Delta \mathcal{O}_n : e_d \Lambda_n \Delta (p \zeta_p + \widehat{\mathcal{F}}_n)] = p^{n^d n - d - 1} \). For \( e_0 \) we notice that \( e_0 \mathcal{O}_n = \mathbb{Z}_p \) and \( e_0 \widehat{\mathcal{F}}_n = 0 \). Then we have \([e_0 \frac{1}{p^d} T \Delta \mathcal{O}_n : e_0 \Lambda_n \Delta (p \zeta_p + \widehat{\mathcal{F}}_n)] = [\frac{1}{p^d} \mathcal{O}_n : p \zeta_p] = p^{n^d n + 1} \). The lemma is now proved.

Recall that proposition 1.12 gives the index
\[
\frac{1}{p^d} T \Delta \mathcal{O}_n : T \Delta \log_p U_n \] = p^{n^d n + 1}.
\]

Moreover note that \( p \zeta_p \in \log_p U_n \) so that \( T \Delta (p \zeta_p + \widehat{\mathcal{F}}_n) \in T \Delta \log_p U_n \). However the value of the index furnished by proposition 1.12 implies that the \( \mathbb{Z}_p[\Gamma_n] \)-module generated by \( T \Delta (p \zeta_p + \widehat{\mathcal{F}}_n) \) cannot be equal to \( T \Delta \log_p U_n \). Let us define \( \mathcal{M} = \mathbb{Z}_p[\Gamma_n] T \Delta (p \zeta_p + \widehat{\mathcal{F}}_n) \). We check that \( \mathcal{M} \subseteq T \Delta \log_p U_n \).

Now we want to calculate the index \([\mathcal{M} : \mathbb{Z}_p[\Gamma_n] T \Delta (p \zeta_p + \widehat{\mathcal{F}}_n)] \). Note that \( T \Delta p \zeta_p \in \mathbb{Z}_p[\Gamma_n] T \Delta (p \zeta_p + \widehat{\mathcal{F}}_n) \). From
\[
p^n T \Delta p \zeta_p = p^n e_0 (T \Delta (p \zeta_p + \widehat{\mathcal{F}}_n)) \in 0 \mathbb{Z}_p[\Gamma_n] T \Delta (p \zeta_p + \widehat{\mathcal{F}}_n)
\]
we deduce that the index is less or equal to \( p^n \). Let \( p \) the order of \( T \Delta p \zeta_p \). Then \( p \) the order of \( T \Delta p \zeta_p = u_n T \Delta (p \zeta_p + \widehat{\mathcal{F}}_n) \) with \( u_n \in \mathbb{Z}_p[\Gamma_n] \). This implies that \( p^h e_0 \in \mathbb{Z}_p[\Gamma_n] \) and \( h \geq n \). Thus we have \([\mathcal{M} : \mathbb{Z}_p[\Gamma_n] T \Delta (p \zeta_p + \widehat{\mathcal{F}}_n)] = p^n \) which proves the following result.

**Theorem 1.22.** The \( \mathbb{Z}_p[\Gamma_n] \)-module \( T \Delta \log_p U_n \) is generated by \( T \Delta \widehat{\mathcal{F}}_n \) and \( p \).

**Corollary 5.** The \( \mathbb{Z}_p[\Gamma_n] \)-module \( T \Delta U_n \) is generated by \( T \Delta \widehat{\mathcal{F}}_n \) and \( (1+p) \).

In particular, let \( U'_n = \{ u \in U_n \mid N_{K_n / K_p}(u) = 1 \} \). Then \( T \Delta \log_p U'_n = T \Delta \log_p E_n = T \Delta \log_p \mathcal{O}_n \).

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2 A result à la Iwasawa in the odd part

We recall that $K_n = \mathbb{Q}(\zeta_{p^{n+1}})$. Let $\pi_n = \zeta_{p^n+1} - 1$. In this section $\epsilon_n(\theta)$ is the element of $\lim \mathbb{Z}_p[\Gamma_n]$ corresponding to the power series $f(\frac{1}{1+T} - 1, \omega \theta)$ where $\theta$ is an odd character of $\Delta$. We assume that $\theta \neq \omega^{-1}$.

2.1 The main result

**Lemma 2.1.** Let $\chi$ be an odd character in $\hat{\Delta}$, with $\chi \neq \omega - 1$. We have $e_{\chi \frac{1}{\pi_n}} \in \mathbb{Z}_p[\zeta_{p^{n+1}}]$.

**Proof.** It is enough to show that $\pi_n e_{\chi \frac{1}{\pi_n}} \equiv 0 \mod \pi_n$. A straightforward calculation gives

$$(p - 1)\pi_n e_{\chi \frac{1}{\pi_n}} = \sum_{\sigma \in \Delta} \frac{\overline{\chi}(\sigma) \pi_n}{\sigma(\pi_n)}$$

$$\equiv \sum_{\sigma \in \Delta} \omega^{-1}(\sigma) \overline{\chi}(\sigma) \mod \pi_n$$

$$\equiv 0 \mod \pi_n$$

because $\chi \neq \omega^{-1}$, which proves the lemma. \qed

Recall that

$$\epsilon_n(\chi) = \frac{1}{p^{n+1}} \sum_{1 \leq a \leq p^{n+1}} \frac{a \theta \omega^{-1}(a)}{p} \gamma_n(a)$$

where $\gamma_n(a)$ is as defined in the introduction.

**Lemma 2.2.** Let $T_n \in K_n$ be the sum $T_n = \sum_{d=0}^{n} \zeta_{p^{d+1}}$. We have

$$e_{\chi \frac{1}{\pi_n}} = e_{\chi}(T_n).$$

**Proof.** Let $f(X) = \frac{X^{p^{n+1}} - 1}{X - 1}$. It is easily checked that

$$Xf'(X) = \sum_{k=1}^{p^{n+1}-1} kX^k$$

and

$$(X - 1)Xf'(X) + Xf(X) = p^{n+1}X^{p^{n+1}}.$$

Letting $X = \zeta_{p^{n+1}}$ we obtain

$$\frac{1}{\pi_n} = \frac{1}{p^{n+1}} \sum_{k=1}^{p^{n+1}-1} k^{p^{n+1}}$$

$$= \frac{1}{p^{n+1}} \sum_{k=1}^{p^{n+1}-1} a_{p^{n+1}} + \frac{1}{p^n} \sum_{k=1}^{p^n-1} k^{p^n}.$$
We now deal with the case where \( \chi = \omega \). Consider the power series
\[
g(T) = (1 - \frac{1}{1+T})f(T,1).
\]
We have
\[
g(\frac{1}{1+T} - 1) = (1 - (1+p)(1+T))f(\frac{1}{1+T} - 1,1).
\]
Let \((\epsilon_n)_{n\geq 0}\) correspond to \(g(\frac{1}{1+T} - 1)\). The same calculation as in lemma
2.3 shows that for all \( n \geq 0 \)
\[
(1 - (1+p)\gamma_0)\epsilon_{\omega^{-1}}\frac{1}{\pi_n} = \epsilon_n \epsilon_{\omega^{-1}} T_n.
\]
Computing the sum \( \sum_{\chi \text{odd}} \epsilon_n \epsilon_{\chi} (T_n - \mathcal{T}_n) \) and using theorem 1.10 we easily obtain the following result.

**Theorem 2.3.** There exists \( \nu_\infty = (\nu_n)_{n\geq 0} \in \lim \mathcal{U}_n^- = \mathcal{U}_\infty^- \) such that each \( \nu_n \in \mathcal{U}_n^- \) is unique modulo \( \mu_{p,n+1} \) and
\[
\log_p \nu_n = \sum_{i=0}^{n} p^{-i} e_\omega(\frac{1}{\pi_n} - \frac{1}{\pi_n} - 2(1+p)\gamma_0 \epsilon_{\omega^{-1}} \frac{1}{\pi_n}).
\]

For all odd characters \( \chi \in \mathcal{\Delta} \setminus \{\omega\} \) we have
\[
\epsilon_{\chi} \mathcal{U}_\infty^- / \Lambda \epsilon_{\chi} \nu_\infty \simeq \Lambda / \tilde{f}(\frac{1}{1+T} - 1, \omega\chi) \Lambda
\]
where \( \tilde{f}(\frac{1}{1+T} - 1, \omega\chi) = f(\frac{1}{1+T} - 1, \omega\chi) \) when \( \chi \neq \omega^{-1} \) and \( \tilde{f}(\frac{1}{1+T} - 1, \omega\chi) = (1 - (1+p)(1+T))f(\frac{1}{1+T} - 1,1) \) when \( \chi = \omega^{-1} \).

For \( \chi = \omega \) we have
\[
\epsilon_{\chi} \log_p \mathcal{U}_\infty^- / \Lambda \epsilon_{\chi} \log_p \nu_\infty \simeq \Lambda / f(\frac{1}{1+T} - 1, \omega^2) \Lambda.
\]

### 2.2 A result à la Stickelberger

The aim of this section is to obtain a global result from theorem 2.3 which is of local nature. In order to achieve this, set
\[
\epsilon_n = \frac{1}{p^{n+1}} \sum_{\substack{1 \leq a \leq p^{n+1} \\text{ p \nmid a}}} a \sigma_a \in \mathbb{Q}[G_n]
\]
where \( G_n = \mathcal{\Delta} \times \Gamma_n \). This element looks like the Stickelberger one and our computation are inspired by this analogy. Notice that the restriction of \( \epsilon_{n+1} \) to \( K_n \) is not \( \epsilon_n \) but \( \epsilon_n + (p - 1)/2N_n \) where \( N_n \) is the norm element of the group algebra \( \mathbb{Z}[G_n] \). We should thus consider the element \((j - 1)\epsilon_n\) where \( j \) is the complex conjugation which is compatible with the canonical morphisms \( \mathbb{Z}[G_{n+1}] \to \mathbb{Z}[G_n] \). By lemma 2.2 we have
\[
\frac{1}{\pi_n} = \sum_{k=0} \epsilon_k (\zeta_{p^{n+1}}).
\]
We deduce that \((j - 1)\frac{1}{\pi_n} = (j - 1)\epsilon_n T_n \).

Let \( \theta_n = \frac{1}{\pi_n} - \frac{1}{\pi_n} \). Define the ideal \( I \) of \( \mathbb{Z}[G_n] \) by \( I = \mathbb{Z}[G_n](\sigma_n - c^*) \) where \( c^* \) is the inverse of \( c \) modulo \( p^{n+1} \). However contrary to the standard case (the one of the Stickelberger element), \( I \) is not the order associated to \( \frac{1}{\pi_n} \). We also define the ideal \( \mathcal{I} = \mathbb{Z}[G_n] \epsilon_n \cap \mathbb{Z}[G_n] \).
We need two more ideals.

\[
\begin{aligned}
\mathcal{E}_n &= \mathbb{Z}[G_n]T_n \text{ and } \\
\mathcal{C}_n &= I \theta_n
\end{aligned}
\]

Notice that \(\mathcal{C}_n = \mathcal{C}_n^-\). The aim of this section is to show the following result.

**Theorem 2.4.** We have \([\mathcal{E}_n^- : \mathcal{C}_n^-] = 2^{[\mathbb{Q}(\zeta(n)) : \mathbb{Q}]} \cdot h_{p^n+1}^-\), where \(h_{p^n+1}^-\) is the quotient of the class number of \(K_n\) by the class number of the maximal real subfield of \(K_n\).

**Proof.** We need several lemmas.

**Lemma 2.5.** We have \(I = I \epsilon_n\).

**Proof.** We want to show that for all \(\beta \in \mathbb{Z}[G_n], \beta \epsilon_n \in \mathbb{Z}[G_n]\) is equivalent to \(\beta \in I\). Let \(\beta \in I\). We can assume that \(\beta = (\sigma_e - c^*)\). Then

\[
(\sigma_e - c^*) = \sum_a \left( \frac{a}{p^n+1} \right) \sigma_{ac} - c^* \left( \frac{a}{p^n+1} \right) \sigma_a
\]

\[
= \sum_a \left( \frac{ac^*}{p^n+1} \right) - c^* \left( \frac{a}{p^n+1} \right) \sigma_a.
\]

As \(cc^* \equiv 1 \text{ mod } p^n+1\) we obtain \((\sigma_e - c^*)\epsilon_n \in \mathbb{Z}[G_n]\). Conversely, notice first that \(p^n+1 \equiv 1 \text{ mod } p^n+1\). Let \(\beta = \sum_a x_a \sigma_a\) with \(x_a \in \mathbb{Z}\) and assume that \(\beta \epsilon_n \in \mathbb{Z}[G_n]\). Then

\[
\beta \cdot \epsilon_n = \sum_a \sum_x x_a \left( \frac{c}{p^n+1} \right) \sigma_{ac}
\]

\[
= \sum b \sum a x_a \left( \frac{a^* b}{p^n+1} \right) \sigma_b.
\]

When \(b = 1\) our assumption implies that \(p^n+1 | \sum_a x_a a^*\) so that \(\sum_a x_a a^* \in I\). Finally we get

\[
\beta = \sum_a x_a a^* = \sum a (\sigma_a - a^*) + \sum x_a a^* \in I.
\]

Let us return to the proof of theorem 2.4. On the one hand, by Leopoldt’s theorem we know that \(\mathcal{E}_n\) is a free \(\mathbb{Z}[G_n]\)-module of rank one. We thus have \(\mathcal{E}_n^- = \mathbb{Z}[G_n]^e \cdot T_n\). A straightforward calculation shows that \(\mathbb{Z}[G_n]^e = (j-1)\mathbb{Z}[G_n]\). On the other hand, we have \((\theta - n) = (j-1)\epsilon_n T_n\) (see beginning os this section). Then we get

\[
\mathcal{C}_n = (j-1)\mathcal{J} \epsilon_n T_n = (j-1)(I)T_n
\]

In order to complete the proof of the theorem we have to express \(\mathcal{J}^-\) according to \((j-1)\mathcal{J}\) and to compute the index of \(\mathcal{J}^-\) in \(\mathbb{Z}[G_n]^e\). By definition we have

\[
\mathcal{J}^- = \mathcal{J} \cap \mathbb{Z}[G_n]^e = \mathbb{Z}[G_n] \cdot \epsilon_n \cap \mathbb{Z}[G_n]^e.
\]

The following result resembles a theorem by Iwasawa (see \cite{Iwasawa} theorem 6.19).
Proposition 2.6. We have $[\mathbb{Z}[G_n]^- : \mathcal{I}^-] = h_p^{n+1}$.

Proof. The proof is the same as the one of Iwasa’s theorem. Let us recall the main steps: first complete and then work at each prime. We define the ideal $\mathcal{I}_q = \mathbb{Z}_q[G_n]\mathcal{I}$ for a prime $q$. We have the following results.

Lemma 2.7. 1. $\mathbb{Z}_q[G_n]^- = (1 - j)\mathbb{Z}_q[G_n]$  
2. $\mathcal{I}_q = \mathbb{Z}_q[G_n] \cdot \epsilon_n \cap \mathbb{Z}_q[G_n]$  
3. $\mathcal{I}_q^- = \mathcal{I}_q \cap \mathbb{Z}_q[G_n] = \mathbb{Z}_q[G_n] \cdot \epsilon_n \cap \mathbb{Z}_q[G_n]$  
4. $\mathcal{I}_q^- = \mathcal{I}^- \cdot \mathbb{Z}_q$  
5. When $p \neq q$, $\mathcal{I}_q = \mathbb{Z}_q[G_n] \epsilon_n$

The proof runs just as in \cite{7}, lemma 6.20. By lemma 2.7 we have an isomorphism $\mathbb{Z}_q[G_n]^- / \mathcal{I}_q^- \simeq (\mathbb{Z}_q[G_n]^- / \mathcal{I}^-) \otimes \mathbb{Z}_q$. It is enough to show the following result.

Proposition 2.8. The index $[\mathbb{Z}_q[G_n]^- : \mathcal{I}_q^-]$ is the $q$-part of $h_p^{n+1}$ for all primes $q$.

Proof. Assume to begin with that $q \neq 2, p$. Then $(1 \pm j)/2 \in \mathbb{Z}_q[G_n]$ so we can separate the plus-part and the minus-part. We obtain 

$$\mathcal{I}_q^- = \frac{1 - j}{2} \mathcal{I} = \mathbb{Z}_q[G_n] \cdot \epsilon_n$$

We have to calculate the index $[\mathbb{Z}_q[G_n]^- : \mathbb{Z}_q[G_n] \cdot \epsilon_n]$ which equals to the $q$-part of the determinant of the map

$$\varphi : \mathbb{Z}_q[G_n]^- \rightarrow \mathbb{Z}_q[G_n]^-$$

$$x \mapsto x \epsilon_n.$$ Compute this determinant in $\mathbb{Q}_q[G_n]^- = \bigoplus_{\chi \text{ odd}} \epsilon_{\chi} \mathcal{Q}_q[G_n]$. However we have $\epsilon_{\chi} \sigma = \chi(\sigma) \epsilon_{\chi}$ for all $\sigma \in G$ from which we deduce that 

$$\epsilon_{\chi} \epsilon_n = B_{1,\chi} \epsilon_{\chi}.$$ 

Then we have 

$$[\mathbb{Z}_q[G_n]^- : \mathcal{I}_q^-] = q - \text{part of det}(\varphi) = q - \text{part of} \prod_{\chi \text{ odd}} B_{1,\chi}$$

$$= q - \text{part of } h_p^{n+1}.$$ 

Let us now deal with the case $p = 2$. The trick is to modify $\epsilon_n$ and to define $\tilde{\epsilon}_n = \epsilon_n - 1/2N$ where $N$ is the norm element of the group algebra $\mathbb{Z}[G_n]$. We easily check that $\frac{1}{2} \tilde{\epsilon}_n = \tilde{\epsilon}_n$ so that $\tilde{\epsilon}_n \in \mathbb{Q}_2[G_n]^-$. 

Lemma 2.9. We have

1. $\mathcal{I}_2^- \subseteq \mathbb{Z}_2[G_n] \tilde{\epsilon}_n$  
2. $[\mathbb{Z}_2[G_n] \tilde{\epsilon}_n : \mathcal{I}_2^-] = 2$
Proof. The first part is obvious. For the second one, notice that if \( x \in \mathbb{Z}_2[G_n] \) then either \( x \epsilon_n \in \mathbb{Z}_2[G_n] \) or \( x \epsilon_n - \epsilon_n \in \mathbb{Z}_2[G_n] \), and that \( \mathbb{Z}_2[G_n] \epsilon_n \cap \mathbb{Z}_2[G_n] = \mathcal{A}_2 \).

The end of the proof for \( p = 2 \) runs the same as previously except that the map \( \varphi \) is now the multiplication by \( \epsilon_n \). Moreover we have \( \epsilon_x \epsilon_n = \epsilon_x \epsilon_n \) when \( \chi \) is odd. Then we get
\[
[\mathbb{Z}_2[G_n]^{-}] = 2 - \text{part of } \det \varphi = 2^{(1/2)|G_n| - 1}(2 - \text{part of } h_{p^n+1}^{-}).
\]
Note that
\[
\mathbb{Z}_2[G_n]^{-} \epsilon_n = (1 - j)\mathbb{Z}_2[G_n] \epsilon_n = \mathbb{Z}_2[G_n](2 \epsilon_n) = 2\mathbb{Z}_2[G_n] \epsilon_n.
\]
It follows that \( \mathbb{Z}_2[G_n] \epsilon_n : \mathbb{Z}_2[G_n] \epsilon_n = 2^{2 - \text{rank of } \mathbb{Z}_2[G_n]} = 2^{1/2|G_n|} \). Together with lemma 2.6 this yields the required result
\[
[\mathbb{Z}_2[G_n]^{-}] : \mathcal{A}_2^{-} = 2 - \text{part of } h_{p^n+1}^{-}.
\]

It remains to deal with the case where \( q = p \). Let us consider again the element \( \epsilon_n = \epsilon_n - \frac{1}{2} \cdot N \). Notice that as in \( \mathbb{Z}_p[G_n] \) we have the equivalence \( x \epsilon_n \in \mathbb{Z}_p[G_n]^{-} \iff x \epsilon_n \in \mathbb{Z}_p[G_n] \) for \( x \in \mathbb{Z}_p[G_n] \). The equality \( \mathbb{Z}_p[G_n] \epsilon_n : \mathbb{Z}_p[G_n] \epsilon_n = \mathbb{Z}_p[G_n] \epsilon_n \cap \mathbb{Z}_p[G_n]^{-} = p^n \) follows. However we also have
\[
\mathbb{Z}_p[G_n] \epsilon_n \cap \mathbb{Z}_p[G_n]^{-} = \mathcal{A}_p^{-} \quad \text{and} \quad \mathbb{Z}_p[G_n] \epsilon_n = \mathbb{Z}_p[G_n]^{-} \epsilon_n,
\]
thus
\[
[\mathbb{Z}_p[G_n]^{-}] : \mathcal{A}_p^{-} = p^n.
\]

We define the map
\[
\varphi : \mathbb{Z}_p[G_n]^{-} \rightarrow \mathbb{Z}_p[G_n]^{-}, \quad x \mapsto p^n \epsilon_n x.
\]

Then \( [\mathbb{Z}_p[G_n]^{-}] : p^n \mathbb{Z}_p[G_n]^{-} \epsilon_n = p^{(n/2)|G_n|} (1/p^n)(p - \text{part of } h_{p^n+1}^{-}) \). We obtain
\[
[\mathbb{Z}_p[G_n]^{-}] : \mathcal{A}_p^{-} = p - \text{part of } h_{p^n+1}^{-}
\]
which concludes the proof of proposition 2.8.

Proposition 2.6 is now proved.

\begin{lemma}

The index \( [\mathcal{A}^{-} : (1 - j)\mathcal{A}] \) equals \( [\mathcal{A}_2^{-} : (1 - j)\mathcal{A}_2] = 2^{1/|G_n| - 1} \).

\end{lemma}

Proof. Localize at each prime by applying part 4 of lemma 2.7. Then
\[
[\mathcal{A}^{-} : (1 - j)\mathcal{A}] = \prod_p [\mathcal{A}_p^{-} : (1 - j)\mathcal{A}_p] = \prod_p [\mathcal{A}_p^{-} : (1 - j)\mathcal{A}_p] \cdot \prod_{q \neq 2, p} [\mathcal{A}_q^{-} : (1 - j)\mathcal{A}_q].
\]

When \( q \neq 2, p \) lemma 2.7 implies \( [\mathcal{A}_q^{-} : (1 - j)\mathcal{A}_q] = 1 \). Moreover \( \frac{1}{2} \in \mathbb{Z}_p[G_n] \) so \( \mathcal{A}_p^{-} = (1/2)\mathcal{A}_p \) and \( \mathcal{A}_p^{-} : (1 - j)\mathcal{A}_p = 1 \). It remains to calculate the index at \( q = 2 \). Notice that \( (1 - j)\mathcal{A}_2 = (1 - j)\mathbb{Z}_2[G_n] \epsilon_n = 2\mathbb{Z}[G_n] \epsilon \). We have \( [\mathbb{Z}_2[G_n] \epsilon_n : \mathcal{A}_2^{-}] = 2 \). Furthermore we have
\[
[\mathbb{Z}_2[G_n] \epsilon_n : 2\mathbb{Z}_2[G_n] \epsilon_n] = 2^{2 - \text{rank of } \mathbb{Z}_2[G_n] \epsilon_n} = 2^{1/2|G_n|}
\]
which concludes the proof of the lemma and of the theorem.
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