MAXIMALLY HOMOGENEOUS PARA-CR MANIFOLDS OF SEMISIMPLE TYPE

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Abstract. An almost para-CR structure on a manifold $M$ is given by a distribution $HM \subset TM$ together with a field $K \in \Gamma(\text{End}(HM))$ of involutive endomorphisms of $HM$. If $K$ satisfies an integrability condition, then $(HM, K)$ is called a para-CR structure. The notion of maximally homogeneous para-CR structure of a semisimple type is given. A classification of such maximally homogeneous para-CR structures is given in terms of appropriate gradations of real semisimple Lie algebras.

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1. Introduction and notation

Let $M$ be a $2n$-dimensional manifold. An almost paracomplex structure on $M$ is a field of endomorphisms $K \in \text{End}(TM)$ of the tangent bundle $TM$ of $M$ such that $K^2 = \text{id}$. It is called an (almost) paracomplex structure in the strong sense if its $\pm 1$-eigenspace distributions

$$T^\pm M = \{ X \pm KX \mid X \in \Gamma(M, TM) \}$$

have the same rank (see e.g. [13], [9]). An almost paracomplex structure $K$ is called a paracomplex structure, if it is integrable, i.e.

$$S(X, Y) = [X, Y] + [KX, KY] - K[X, KY] - K[KX, Y] = 0$$

for any vector fields $X, Y \in \Gamma(TM)$. This is equivalent to say that the distributions $T^\pm M$ are involutive.

Recall that an almost CR-structure of codimension $k$ on a $2n + k$-dimensional manifold $M$ is a distribution $HM \subset TM$ of rank $2n$ together with a field of endomorphisms $J \in \text{End}(HM)$ such that $J^2 = -\text{id}$. An almost CR-structure is called CR-structure, if the $\pm i$-eigenspace subdistributions $H^\pm M$ of the complexified tangent bundle $T^C M$ are involutive.

We define an almost para-CR structure in a similar way.

**Definition 1.1.** An almost para-CR structure of codimension $k$ on a $2n + k$-dimensional manifold $M$ (in the weak sense) is a pair $(HM, K)$, where $HM \subset TM$ is a rank $2n$ distribution and $K \in \text{End}(HM)$ is a field of endomorphisms such that $K^2 = \text{id}$ and $K \neq \pm \text{id}$. An almost para-CR structure is said to be a para-CR structure, if the eigenspace subdistributions $H^\pm M$ of the complexified tangent bundle $T^C M$ are involutive.

A manifold $M$, endowed with an (almost) para-CR structure, is called an (almost) para-CR manifold. We define an almost para-CR structure in a similar way.

If the eigenspace distributions

$$H^\pm M = \{ X \pm KX \mid X \in \Gamma(M, HM) \}$$

of an almost para-CR structure have the same rank, then $(HM, K)$ is called an almost para-CR structure in the strong sense. A straightforward computation shows that the integrability condition is equivalent to the involutiveness of the distributions $H^\pm M$ and $H_- M$. A manifold $M$, endowed with an (almost) para-CR structure, is called an (almost) para-CR manifold.

Note that a direct product of (almost) para-CR manifolds is an (almost) para-CR manifold.

One can associate with a point $x \in M$ of a para-CR manifold $(M, HM, K)$ a fundamental graded Lie algebra $\mathfrak{m}$. A para-CR structure is said to be regular if these Lie algebras $\mathfrak{m}_x$ do not depend on $x$. In this case, a para-CR
structure can be considered as a Tanaka structure (see [3] and section 4). A regular para-CR structure is called a structure of semisimple type if the full prolongation

$$g = m^\infty = m^{-d} + \cdots + m^{-1} + g^0 + g^1 + \cdots$$

of the associated non-positively graded Lie algebra $g^{-d} + \cdots + g^{-1} + g^0$ (which is an analogue of the generalized Levi form of a CR structure) is a semisimple Lie algebra. Such a para-CR structure defines a parabolic geometry and its group of automorphisms $\text{Aut}(M, H M, K)$ is a Lie group of dimension $\leq \dim g$.

Recently in [16] P. Nurowski and G. A. J. Sparling consider the natural para-CR structure which arises on the 3-dimensional space $M$ of solutions of a second order ordinary differential equation $y'' = Q(x, y, y')$. Using the Cartan method of prolongation, they construct the full prolongation $G \to M$ of $M$ with a $\mathfrak{sl}(3, \mathbb{R})$-valued Cartan connection and a quotient line bundle over $M$ with a conformal metric of signature $(2, 2)$. This is a para-analogue of the Feffermann bundle of a CR-structure. They apply these bundles to the initial ODE and get interesting applications.

In [2] we proved that a para-CR structures of semisimple type on a simply connected manifold $M$ with the automorphism group of maximal dimension $\dim g$ can be identified with a (real) flag manifold $M = G/P$ where $G$ is the simply connected Lie group with the Lie algebra $g$ and $P$ the parabolic subgroup generated by the parabolic subalgebra $p = g^0 + g^1 + \cdots + g^d$.

We gave a classification of maximally homogeneous para-CR structures of semisimple type such that the associated graded semisimple Lie algebra $g$ has depth $d = 2$. In the present paper we classify all maximally homogeneous para-CR structures of semisimple type in terms of graded real semisimple Lie algebras.

2. Graded Lie algebras associated with para-CR structures

2.1. Gradations of a Lie algebra. Recall that a gradation (more precisely a $\mathbb{Z}$-gradation) of depth $k$ of a Lie algebra $g$ is a direct sum decomposition

$$g = \sum_{i \in \mathbb{Z}} g^i = g^{-k} + g^{-k+1} + \cdots + g^0 + \cdots + g^j + \cdots$$

such that $[g^i, g^j] \subset g^{i+j}$, for any $i, j \in \mathbb{Z}$, and $g^{-k} \neq \{0\}$. Note that $g^0$ is a subalgebra of $g$ and each $g^i$ is a $g^0$-module.

We say that an element $x \in g^j$ has degree $j$ and we write $d(x) = j$. The endomorphism $\delta$ of $g$ defined by

$$\delta_{|g^j} = j \cdot id$$

is a semisimple derivation of $g$ (with integer eigenvalues), whose eigenspaces determine the gradation. Conversely, any semisimple derivation $\delta$ of $g$ with integer eigenvalues defines a gradation where the grading space $g^j$ is the eigenspace of $\delta$ with eigenvalue $j$. If $g$ is a semisimple Lie algebra, then any
derivation $\delta$ is inner, i.e. there exists $d \in g$ such that $\delta = \text{ad}_d$. The element $d \in g$ is called the grading element.

**Definition 2.1.** A gradation $g = \sum g^i$ of a Lie algebra (or a graded Lie algebra $g$) is called

1. fundamental, if the negative part $m = \sum_{i<0} g^i$ is generated by $g^{-1}$;
2. (almost) effective or transitive, if the non-negative part
$$g^{\geq 0} = p = g^0 + g^1 + \cdots$$
contains no non-trivial ideal of $g$;
3. non-degenerate, if
$$X \in g^{-1}, \quad [X, g^{-1}] = 0 \implies X = 0.$$

2.2. **Fundamental algebra associated with a distribution.** Let $\mathcal{H}$ be a distribution on a manifold $M$. We recall that to any point $x \in M$ it is possible to associate a Lie algebra $m(x)$ in the following way.

First of all, we consider a filtration of the Lie algebra $\mathcal{X}(M)$ of vector fields defined inductively by
$$\Gamma(\mathcal{H})_{-1} = \Gamma(\mathcal{H}), \quad \Gamma(\mathcal{H})_{-i} = \Gamma(\mathcal{H})_{-i+1} + [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})_{-i+1}], \text{ for } i > 1.$$

Then evaluating vector fields at a point $x \in M$, we get a flag
$$T_x M \supset \mathcal{H}_{-d-1}(x) = \mathcal{H}_{-d}(x) \supset \mathcal{H}_{-d+1}(x) \supset \cdots \supset \mathcal{H}_{-2}(x) \supset \mathcal{H}_{-1}(x) = \mathcal{H}_x$$
in $T_x M$, where
$$\mathcal{H}_{-i}(x) = \{X|_x \mid X \in \Gamma(\mathcal{H})_{-i}\}.$$

Let us assume that $\mathcal{H}_{-d}(x) = T_x M$. The commutators of vector fields induce a structure of fundamental negatively graded Lie algebra on the associated graded space
$$m(x) = \text{gr}(T_x M) = m^{-d}(x) + m^{-d+1}(x) + \cdots + m^{-1}(x),$$
where $m^{-j}(x) = \mathcal{H}_{-j}(x)/\mathcal{H}_{-j+1}(x)$. Note that $m^{-1}(x) = \mathcal{H}_x$.

A distribution $\mathcal{H}$ is called a regular distribution of depth $d$ and type $m$ if all graded Lie algebras $m(x)$ are isomorphic to a given graded fundamental Lie algebra
$$m = m^{-d} + m^{-d+1} + \cdots + m^{-1}.$$

In this case $m$ is called the Lie algebra associated with the distribution $\mathcal{H}$. A regular distribution $\mathcal{H}$ is called non-degenerate if the associated Lie algebra is non-degenerate.
2.3. Para-CR algebras and regular para-CR structures. We recall the following

**Definition 2.2.** A pair \((m, K_0)\), where \(m = m^{-d} + \cdots + m^{-1}\) is a negatively graded fundamental Lie algebra and \(K_0\) is an involutive endomorphism of \(m^{-1}\), is called a para-CR algebra of depth \(d\). If, moreover, the \(\pm 1\)-eigenspaces \(m_{\pm 1}^{-1}\) of \(K_0\) on \(m^{-1}\) are commutative subalgebras of \(m\), then \((m, K_0)\) is called an integrable para-CR algebra.

**Definition 2.3.** Let \((m, K_0)\) be a para-CR algebra of depth \(d\). An almost para-CR structure \((H M, K)\) on \(M\) is called regular of type \((m, K_0)\) and depth \(d\) if, for any \(x \in M\), the pair \((m(x), K_x)\) is isomorphic to \((m, K_0)\).

We say that the regular almost para-CR structure is non-degenerate if the graded algebra \(m\) is non-degenerate.

Note that a regular almost para-CR structure of type \((m, K_0)\) is integrable if and only if the Lie algebra \((m, K_0)\) is integrable.

3. Prolongations of graded Lie algebras

3.1. Prolongations of negatively graded Lie algebras. The full prolongation of a negatively graded fundamental Lie algebra \(m = m^{-d} + \cdots + m^{-1}\) is defined as a maximal graded Lie algebra \(g(m) = g^{-d}(m) + \cdots + g^{-1}(m) + g^0(m) + g^1(m) + \cdots\) with the negative part

\[ g^{-d}(m) + \cdots + g^{-1}(m) = m \]

such that the following transitivity condition holds:

\[ [X, g^{-1}(m)] = 0, \quad \text{if } X \in g^k(m), \quad k \geq 0, \quad \text{then } X = 0. \]

In [17], N. Tanaka proved that the full prolongation \(g(m)\) always exists and it is unique up to isomorphisms. Moreover, it can be defined inductively by

\[ g^i(m) = \begin{cases} m^i & \text{if } i < 0, \\ \{ A \in \text{Der}(m, m) : A(m^j) \subset m^j, \forall j < 0 \} & \text{if } i = 0, \\ \{ A \in \text{Der}(m, \sum_{h<i} g^h(m)) : A(m^j) \subset g(m)^{i+j}, \forall j < 0 \} & \text{if } i > 0, \end{cases} \]

where \(\text{Der}(m, V)\) denotes the space of derivations of the Lie algebra \(m\) with values in the \(m\)-module \(V\).

Note that

\[ g^i(m) = \left\{ A \in \text{Hom}_R(m, \sum_{h<i} g^h(m)) \left| A(g^h(m)) \subset g^{h+i}(m) \forall h < 0, \right. \right\} \]

and \([A(Y), Z] + [Y, A(Z)] = A([Y, Z]) \forall Y, Z \in m\).
3.2. Prolongations of non-positively graded Lie algebras. Consider now a non-positively graded Lie algebra \( m + g^0 = m^{-d} + \cdots + m^{-1} + g^0 \). The full prolongation of \( m + g^0 \) is the subalgebra

\[(m + g^0)_{\infty} = m^{-d} + \cdots + m^{-1} + g^0 + g^1 + g^2 + \cdots\]

of \( g(m) \), defined inductively by

\[g^i = \{X \in g(m)^i : [X, m^{-1}] \subset g^{i-1}\}, \text { for any } i \geq 1.\]

**Definition 3.1.** A graded Lie algebra \( m + g^0 \) is called of finite type if its full prolongation \( g = (m + g^0)_{\infty} \) is a finite dimensional Lie algebra and it is called of semisimple type if \( g \) is a finite dimensional semisimple Lie algebra.

We have the following criterion (see [18], [3])

**Lemma 3.2.** Let \( (m = \sum_{i<0} m^i, K_0) \) be an integrable para-CR algebra and \( g^0 \) the subalgebras of \( g^0(m) \) consisting of any \( A \in g^0(m) \) such that \( A|_{m^{-1}} \) commutes with \( K_0 \). Then the graded Lie algebra \( (m + g^0) \) is of finite type if and only if \( m \) is non-degenerate.

The following result will be used in the last section (see e.g. [14], Theorem 3.21)

**Lemma 3.3.** Let \( g = \sum g_i \) be a fundamental effective semisimple graded Lie algebra such that \( m + g^0 \) is of finite type. Then \( g \) coincides with the full prolongation \( (m + g^0)_{\infty} \) of \( m + g^0 \).

4. Standard almost para-CR manifolds

4.1. Maximally homogeneous Tanaka structures. A regular para-CR structure of type \( (m, K_0) \) is of finite type or, respectively, of semisimple type, if the Lie algebra \( (m + g^0)_{\infty} \) is finite-dimensional or, respectively, semisimple. Recall that \( g^0 = \text{Der}(m, K_0) \) is the Lie algebra of Lie group \( \text{Aut}(m, K_0) \).

We recall the following (see [3])

**Definition 4.1.** Let \( m = m^{-d} + \cdots + m^{-1} \) be a negatively graded Lie algebra generated by \( m^{-1} \) and \( G^0 \) a closed Lie subgroup of (grading preserving) automorphisms of \( m \). A Tanaka structure of type \( (m, G^0) \) on a manifold \( M \) is a regular distribution \( \mathcal{H} \subset TM \) of type \( m \) together with a principal \( G^0 \)-bundle \( \pi : Q \to M \) of adapted coframes of \( \mathcal{H} \). A coframe \( \varphi : \mathcal{H}_x \to m^{-1} \) is called adapted if it can be extended to an isomorphism \( \varphi : m_x \to m \) of Lie algebra.

We say that the Tanaka structure of type \( (m, G^0) \) is of finite type (respectively semisimple type \( (m, G^0) \)), if the graded Lie algebra \( m + g^0 \) is of finite type (respectively semisimple type). Let \( P \) be a Lie subgroup of a connected Lie group \( G \) and \( p \) (respectively \( g \)) the Lie algebra of \( P \) (respectively \( G \)).

**Theorem 4.2.** Let \( (\pi : Q \to M, \mathcal{H}) \) be a Tanaka structure on \( M \) of semisimple type \( (m, G^0) \). Then the Tanaka prolongation of \( (\pi, \mathcal{H}) \) is a \( P \)-principal
bundle $\mathcal{G} \to M$, with the parabolic structure group $\mathcal{P}$, equipped with a Cartan connection $\kappa : TG \to \mathfrak{g}$, where $\mathfrak{g}$ is the full prolongation of $\mathfrak{m} + \mathfrak{g}^0$ and $\text{LieP} = \mathfrak{p} = \sum_{i \geq 0} \mathfrak{g}_i$. Moreover, $\text{Aut}(\mathcal{H}, \pi)$ is a Lie group and

$$\dim \text{Aut}(\mathcal{H}, \pi) \leq \dim \mathfrak{g}.$$ 

Let $(\mathcal{H}, \pi : Q \to M)$ be a Tanaka structure of semisimple type $(\mathfrak{m}, G^0)$ and $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty = \mathfrak{m} + \mathfrak{p}$ be the full prolongation of the non-positively graded Lie algebra $\mathfrak{m} + \mathfrak{g}^0$.

**Definition 4.3.** A semisimple Tanaka structure $(\mathcal{H}, \pi : Q \to M)$ is called maximally homogeneous if the dimension of its automorphism group $\text{Aut}(\mathcal{H}, \pi)$ is equal to $\dim \mathfrak{g}$.

### 4.2. Tanaka structures of semisimple type.

We construct maximally homogeneous Tanaka structures of semisimple type $(\mathfrak{m}, G^0)$ as follows. Let $G = \text{Aut}(\mathfrak{g})$ be the Lie group of automorphisms of the graded Lie algebra $\mathfrak{g}$. Recall that $G^0$ is a closed subgroup of the automorphism group of the graded Lie algebra $\mathfrak{g}^- = \mathfrak{m}$. Since the Lie algebra $\mathfrak{g}$ is canonically associated with $\mathfrak{m}$, we can canonically extend the action of $G^0$ on $\mathfrak{m}$ to the action of $G^0$ on $\mathfrak{g}$ by automorphisms. In other words, we have an embedding $G^0 \hookrightarrow \text{Aut}(\mathfrak{g}) = G$ as a closed subgroup. We denote by $G^+$ the connected (closed) subgroup of $G$ with Lie algebra $\mathfrak{g}_+ = \sum_{p > 0} \mathfrak{g}_p$. Then $P = G^0 \cdot G^+ \subset G$ is a (closed) parabolic subgroup of $G$. Let $G/P$ be the corresponding flag manifold. We have a decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{p}$ and we identify $\mathfrak{m}$ with the tangent space $T_m(G/P)$. Then the natural action of $G^0$ on $\mathfrak{m}$ is the isotropy representation of $G^0$. We have a natural Tanaka structure $(\mathcal{H}, \pi : Q \to G/P)$ of type $(\mathfrak{m}, G^0)$, where $\mathcal{H}$ is the $G$-invariant distribution defined by $\mathfrak{m}^{-1}$ and $Q$ is the $G^0$-bundle of coframes on $\mathcal{H}$.

Hence, the flag manifold $G/P$ carries a natural maximally homogeneous Tanaka structure $(\mathcal{H}, \pi : Q \to G/P)$.

The universal covering $F$ of the manifold $G/P$ also has the induced Tanaka structure $(\mathcal{H}_F, \pi_F : Q_F \to F)$ of type $(\mathfrak{m}, G^0)$ and the simply connected (connected) Lie group $\tilde{G}$ with the Lie algebra $\mathfrak{g}$ acts transitively and almost effectively on $F$ as a group of automorphisms of this Tanaka structure. Moreover, the stabilizer in $\tilde{G}$ of an appropriate point $o \in F$ is the (connected) parabolic subgroup $\tilde{P}$ generated by the subalgebra $\mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots + \mathfrak{g}^d$.

The Tanaka structure $(\mathcal{H}, \pi : Q \to \tilde{G}/\tilde{P})$ is obviously maximally homogeneous and it is called the standard (simply connected maximally homogeneous) Tanaka structure of type $(\mathfrak{m}, G^0)$. We can state the following (see e.g. [2], Theor. 4.8)

**Theorem 4.4.** Any maximally homogeneous Tanaka structure of semisimple type $(\mathfrak{m}, G^0)$ is isomorphic to the standard Tanaka structure on the simply connected flag manifold $F = \tilde{G}/\tilde{P}$ where $\tilde{G}$ is the simply connected semisimple Lie group with the Lie algebra $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty$ and $\tilde{P}$ is the parabolic subgroup generated by the subalgebra $\mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots + \mathfrak{g}^d$. 

Let \((HM, K)\) be a regular almost para-CR structure of type \((m, K_0)\). Assume that it has finite type, i.e. \(m = \dim (m + g^0)\infty < \infty\). According to the above definition, \((HM, K)\) is \textit{maximally homogeneous}, if it admits a (transitive) Lie group of automorphisms of dimension \(m\).

By Theorem 4.4, a maximally homogeneous almost para-CR structure of semisimple type is locally equivalent to the standard structure associated with a gradation of a semisimple Lie algebra. In the following subsection we describe this correspondence in more details.

4.3. Models of almost para-CR manifolds. Let \(g = \sum_{d} g^i = g^- + g^0 + g^+\) be an effective fundamental gradation of a semisimple Lie algebra \(g\) with negative part \(m = g^- = \sum_{i<0} g^i\) and positive part \(g^+ = \sum_{i>0} g^i\).

Denote by \(\mathcal{F} = \tilde{G}/\tilde{P}\) the simply connected real flag manifold associated with the graded Lie algebra \(g\) where \(\tilde{G}\) is the simply connected Lie group with Lie algebra \(g\) and \(\tilde{P} = G^0G^+\) is the connected subgroup generated by the Lie subalgebra \(g^0 + g^+\).

We will identify the tangent space \(T_o\mathcal{F}\) at the point \(o = eP\) with the subspace \(g/p \cong m\).

Since the subspace \((g^- + p)/p \subset T_o\mathcal{F}\) is invariant under the isotropy representation of \(P\), it defines an invariant distribution \(\mathcal{H}\) on \(\mathcal{F}\). Since the gradation is fundamental, one can easily check that, for any \(x \in \mathcal{F}\), the negatively graded Lie algebra \(m(x)\) associated with \(\mathcal{H}\) is isomorphic to the Lie algebra \(m\). Moreover, let

\[(5)\quad g^- = g^+_1 + g^-_1\]

be a decomposition of the \(G^0\)-module \(g^-\) into a sum of two submodules and \(K_0\) the associated \(ad_{m}\)-invariant endomorphism such that \(g^+_1, g^-_1\) are \(\pm 1\)-eigenspaces of \(K_0\).

The decomposition \((5)\) defines two invariant complementary subdistributions \(\mathcal{H}_\pm\) of the distribution \(\mathcal{H} \subset TF\) associated with \(g^-\) and \(K_0\) defines \(\tilde{G}\)-invariant para-CR structure \((HF, K)\) on \(F\). It is the standard para-CR structure associated with the graded Lie algebra \(g\) and the decomposition \((5)\). We get the following theorem (see also [2, Theor. 5.1])

**Theorem 4.5.** Let \(F = \tilde{G}/\tilde{P}\) be the simply connected flag manifold associated with a (real) semisimple effective fundamental graded Lie algebra \(g\). A decomposition \(g^- = g^+_1 + g^-_1\) of \(g^-\) into complementary \(G^0\)-submodules \(g^+_1, g^-_1\) determines an invariant almost para-CR structure \((HM, K)\) such that \(\pm 1\)-eigenspaces \(H_\pm M\) of \(K\) are subdistributions of \(HM\) associated with \(g^+_1, g^-_1\).

Conversely, any standard almost para-CR structure \((HM, K)\) on \(F\) can be obtained in such a way.

Moreover, \((HM, K)\) is:

1. an almost para-CR structure if \(g^+_1\) and \(g^-_1\) have the same dimensions,
(2) a para-CR structure if and only if \( g_+^{-1} \) and \( g_-^{-1} \) are commutative subalgebras of \( g \).

(3) non-degenerate if and only if \( g \) has no graded ideals of depth one.

By Theorem 4.5, the classification of maximally homogeneous para-CR structures of semisimple type, up to local isomorphisms (i.e. up to coverings), reduces to the description of all gradation of semisimple Lie algebras \( g \) and to decomposition of the \( g^0 \)-module \( g^{-1} \) into irreducible submodules. We will give such a description for complex and real semisimple Lie algebras in the next two sections.

5. Fundamental gradations of a complex semisimple Lie algebra

We recall here the construction of a gradation of a complex semisimple Lie algebra \( g \). Let \( h \) be a Cartan subalgebra of a semisimple Lie algebra \( g \) and

\[
    g = h \oplus \sum_{\alpha \in R} g_\alpha
\]

be the root decomposition of \( g \) with respect to \( h \). We denote by

\[
    \Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset R
\]

a system of simple roots of the root system \( R \) and associate to each simple root \( \alpha_i \) (or corresponding vertex of the Dynkin diagram) a non-negative integer \( d_i \). Using the label vector \( \vec{d} = (d_1, \ldots, d_\ell) \), we define the degree of a root \( \alpha = \sum_{i=1}^\ell k_i \alpha_i \) by

\[
    d(\alpha) = \sum_{i=1}^\ell k_i d_i.
\]

This defines a gradation of \( g \) by the conditions

\[
    d(h) = 0, \quad d(g_\alpha) = d(\alpha), \quad \forall \alpha \in R,
\]

which is called the gradation associated with the label vector \( \vec{d} \).

We denote by \( d \in h \) the corresponding grading element. Then \( d(\alpha) = \alpha(d) \).

Any gradation of a complex semisimple Lie algebra \( g \) is conjugated to a gradation of such a type (see [11]). In particular, it has the form

\[
    g = g^{-k} + \cdots + g^0 + \cdots + g^k,
\]

where \( g^0 \) is a reductive subalgebra of \( g \) and the grading spaces \( g^{-i} \) and \( g^i \) are dual with respect to the Killing form. It is clear now that any graded semisimple Lie algebra is a direct sum of graded simple Lie algebras. Hence, it is sufficient to describe gradations of simple Lie algebras.

We need the following (see [19])

**Lemma 5.1.** The gradation of a complex semisimple Lie algebra \( g \) associated with a label vector \( \vec{d} = (d_1, \ldots, d_\ell) \) is fundamental if and only if all labels \( d_i \in \{0, 1\} \).
Let $\Pi^1 \subset \Pi$ be a set of simple roots. We denote by $\vec{d}_{\Pi^1}$ the label vector which associates label one to the roots in $\Pi^1$ and label zero to the other simple roots.

Now we describe the depth of a fundamental gradation. Let $\mu$ be the maximal root with respect to the fundamental system $\Pi$. It can be written as a linear combination

$$\mu = m_1 \alpha_1 + \cdots + m_\ell \alpha_\ell$$

of fundamental roots, where the coefficient $m_i$ is a positive integer called the Dynkin mark associated with $\alpha_i$.

**Lemma 5.2.** Let $\Pi^1 = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\} \subset \Pi$ be a set of simple roots. Then the depth $k$ of the fundamental gradation defined by the label vector $\vec{d}_{\Pi^1}$ is given by

$$k = m_{i_1} + m_{i_2} + \cdots + m_{i_s}.$$ 

**Proof.** The depth $k$ of the gradation is equal to the maximal degree $d(\alpha)$, $\alpha$ being a root. If $\alpha = k_1 \alpha_1 + \cdots + k_\ell \alpha_\ell$ is the decomposition of a root $\alpha$ with respect to simple roots, then

$$d(\alpha) = k_{i_1} + \cdots + k_{i_s} \leq d(\mu) = m_{i_1} + \cdots + m_{i_s} = k.$$ 

\[\square\]

**Irreducible submodules of the $g^0$-module $g^1$.** Let $g = \sum g^i$ be a fundamental gradation of a complex semisimple Lie algebra $g$, defined by a label vector $\vec{d}$. Following [11], we describe the decomposition of a $g^0$-module into irreducible submodules. Set

$$R^i = \{\alpha \in R \mid d(\alpha) = i\} = \{\alpha \in R \mid g_\alpha \subset g^i\}$$

and

$$\Pi^i = \Pi \cap R^i = \{\alpha \in \Pi \mid d(\alpha) = i\}.$$ 

For any simple root $\gamma \in \Pi$, we put

$$R(\gamma) = \{\gamma + (R^0 \cup \{0\})\} \cap R = \{\alpha = \gamma + \phi^0 \in R, \ \phi^0 \in R^0 \cup \{0\}\}.$$ 

We associate to any set of roots $Q \subset R$ a subspace

$$g(Q) = \sum_{\alpha \in Q} g_\alpha \subset g.$$ 

**Proposition 5.3.** (11) The decomposition of a $g^0$-module $g^1$ into irreducible submodules is given by

$$g^1 = \sum_{\gamma \in \Pi^1} g(R(\gamma)).$$ 

Moreover, $\gamma$ is a lowest weight of the irreducible submodule $g(R(\gamma))$. In particular, the number of the irreducible components is equal to the number $\#\Pi^1$ of the simple roots of degree 1.

Since the $g^0$-modules $g^i$ and $g^{-i}$ are dual, Proposition 5.3 gives also the decomposition of the $g^0$-module $g^{-1}$ into irreducible submodules.
6. Fundamental gradations of a real semisimple Lie algebra

6.1. Real forms of a complex semisimple Lie algebra. Now we recall the description of a real form of a complex semisimple Lie algebra in terms of Satake diagrams. It is sufficient to do this for complex simple Lie algebras.

Any real form of a complex semisimple Lie algebra \( \mathfrak{g} \) is the fixed points set \( \mathfrak{g}^\sigma \) of an antilinear involution \( \sigma \), that is, an antilinear map \( \sigma : \mathfrak{g} \to \mathfrak{g} \), which is an automorphism of \( \mathfrak{g} \) as a real algebra, such that \( \sigma^2 = \text{id} \). We fix a Cartan decomposition \( \mathfrak{g}^\sigma = \mathfrak{k} + \mathfrak{m} \) of the real form \( \mathfrak{g}^\sigma \), where \( \mathfrak{k} \) is a maximal compact subalgebra of \( \mathfrak{g}^\sigma \) and \( \mathfrak{m} \) is its orthogonal complement with respect to the Killing form \( B \). Let

\[
\mathfrak{h}^\sigma = \mathfrak{h}_k + \mathfrak{h}_m
\]

be a Cartan subalgebra of \( \mathfrak{g}^\sigma \) which is consistent with this decomposition and such that \( \mathfrak{h}_m = \mathfrak{h} \cap \mathfrak{m} \) has maximal dimension. Then the root decomposition of \( \mathfrak{g}^\sigma \), with respect to the subalgebra \( \mathfrak{h}^\sigma \), can be written as

\[
\mathfrak{g}^\sigma = \mathfrak{h}^\sigma + \sum_{\lambda \in \Sigma} \mathfrak{g}_{\lambda}^\sigma,
\]

where \( \Sigma \subset (\mathfrak{h}^\sigma)^* \) is a (non-reduced) root system. The number \( m_\lambda = \dim \mathfrak{g}_\lambda \) is the multiplicity of a root \( \lambda \in \Sigma \).

Denote by \( \mathfrak{h} = (\mathfrak{h}^\sigma)^C \) the complexification of \( \mathfrak{h}^\sigma \) which is a \( \sigma \)-invariant Cartan subalgebra. We denote by \( \sigma^* \) the induced antilinear action of \( \sigma \) on \( \mathfrak{h}^* \) given by

\[
\sigma^* \alpha = \overline{\alpha \circ \sigma}, \quad \alpha \in \mathfrak{h}^*.
\]

Consider the root space decomposition

\[
\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \mathfrak{R}} \mathfrak{g}_\alpha
\]

of the Lie algebra \( \mathfrak{g} \) with respect to the Cartan subalgebra \( \mathfrak{h} \). Note that \( \sigma^* \) preserves the root system \( \mathfrak{R} \), i.e. \( \sigma^* \mathfrak{R} = \mathfrak{R} \). Now we relate the root space decomposition of \( \mathfrak{g}^\sigma \) and \( \mathfrak{g} \). We define the subsystem of compact roots \( \mathfrak{R}_c \) by

\[
\mathfrak{R}_c = \{ \alpha \in \mathfrak{R} \mid \sigma^* \alpha = \overline{\alpha} \} = \{ \alpha \mid \alpha(\mathfrak{h}_m) = 0 \}
\]

and denote by \( \mathfrak{R}' = \mathfrak{R} \setminus \mathfrak{R}_c \) the complementary set of non-compact roots. We can choose a system \( \Pi \) of simple roots of \( \mathfrak{R} \) such that the corresponding system of positive roots \( \mathfrak{R}_+ \) satisfies the condition: \( \mathfrak{R}_+ = \mathfrak{R}' \cap \mathfrak{R}_c \) is \( \sigma \)-invariant. In this case, \( \Pi \) is called a \( \sigma \)-fundamental system of roots. We denote by \( \Pi_c = \Pi \cap \mathfrak{R}_c \) the set of compact simple roots (which are also called black) and by \( \Pi' = \Pi \setminus \Pi_c \) the non-compact simple roots (called white). The action of \( \sigma^* \) on white roots satisfies the following property:
for any $\alpha \in \Pi'$ there exists a unique $\alpha' \in \Pi'$ such that $\sigma^*\alpha - \alpha'$ is a linear combination of black roots, i.e.

$$\sigma^*\alpha = \alpha' + \sum_{\beta \in \Pi^\bullet} k_\beta \beta, \quad k_\beta \in \mathbb{N}.$$ 

In this case, we say that the roots $\alpha, \alpha'$ are $\sigma$-equivalent and we will write $\alpha \sim \alpha'$. The information about fundamental system $(\Pi = \Pi^\bullet \cup \Pi')$ together with the $\sigma$-equivalence can be visualized in terms of the Satake diagram, which is defined as follows.

On the Dynkin diagram of the system of simple roots $\Pi$, we paint the vertices which correspond to black roots into black and we join the vertices which correspond to $\sigma$-equivalent roots $\alpha, \alpha'$ by a curved arrow. By a slight abuse of notation, we will refer to the $\sigma$-fundamental system $\Pi = \Pi^\bullet \cup \Pi'$, together with the $\sigma$-equivalence $\sim$, as the Satake diagram. This diagram is determined by the real form $\mathfrak{g}^\sigma$ of a complex simple Lie algebra $\mathfrak{g}$ and does not depend on the choice of a Cartan subalgebra and a $\sigma$-fundamental system. The list of Satake diagram of real simple Lie algebras is known (see e.g. [11]).

Conversely, Satake diagram $(\Pi = \Pi^\bullet \cup \Pi', \sim)$ allows to reconstruct the action of $\sigma^*$ on $\Pi$, hence on $\mathfrak{h}^*$. This action can be canonically extended to the antilinear involution $\sigma$ of the complex Lie algebra $\mathfrak{g}$. Hence, there is a natural $1 - 1$ correspondence between Satake diagrams subordinated to the Dynkin diagram of a complex semisimple Lie algebra $\mathfrak{g}$, up to isomorphisms, and real forms $\mathfrak{g}^\sigma$ of $\mathfrak{g}$, up to conjugations.

6.2. Gradations of a real semisimple Lie algebra. Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{g}^\sigma$ be a real form of $\mathfrak{g}$ with a Satake diagram $(\Pi = \Pi^\bullet \cup \Pi', \sim)$. Let $\vec{d} = (d_1, \ldots, d_\ell)$ be a label vector of the simple roots system $\Pi$ and $\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}^i$ be the corresponding gradation of $\mathfrak{g}$, with the grading element $d \in \mathfrak{h} \subset \mathfrak{g}$.

The following theorem gives necessary and sufficient conditions in order that this gradation induces a gradation

$$\mathfrak{g}^\sigma = \sum_{i \in \mathbb{Z}} \mathfrak{g}^\sigma \cap \mathfrak{g}^i$$

of the real form $\mathfrak{g}^\sigma$. This means that the grading element $d$ belongs to $\mathfrak{g}^\sigma$. We denote by $\Pi^0 \subset \Pi$ the set of simple roots with label zero.

**Theorem 6.1.** ([11]) A gradation of a complex semisimple Lie algebra $\mathfrak{g}$, associated with a label vector $\vec{d} = (d_1, \ldots, d_\ell)$, induces a gradation of the real form $\mathfrak{g}^\sigma$, which corresponds to a Satake diagram $(\Pi = \Pi^\bullet \cup \Pi', \sim)$ if and only if the following two conditions hold:

i) $\Pi^\bullet \subset \Pi^0$, i.e. any black vertex of the Satake diagram has label zero;
ii) if $\alpha \sim \alpha'$ for $\alpha, \alpha' \in \Pi \setminus \Pi_\bullet$, then $d(\alpha) = d(\alpha')$, i.e. white vertices of the Satake diagram which are joint by a curved arrow have the same label.

A label vector $\vec{d} = (d_1, \ldots, d_\ell)$ of a Satake diagram $(\Pi = \{\alpha_1, \ldots, \alpha_\ell\} = \Pi_\bullet \cup \Pi', \sim)$ and the corresponding gradation of $g$ are called of real type if they satisfy conditions i) and ii) of the theorem above, that is black vertices have label zero and vertices related by a curved arrow have the same label. Hence, we can state Theorem 6.1 as follows.

**Corollary 6.2.** There exists a natural 1 – 1 correspondence between label vectors $\vec{d}$ of real type of a Satake diagram of a real semisimple Lie algebra $g^\sigma$ and gradations of $g^\sigma$. The gradation of $g^\sigma$ is fundamental if and only if the corresponding gradation of $g$ is fundamental, i.e. $\vec{d} = \vec{d}_1$.

**Irreducible submodules of the $g^0$-module $g^1$.** Let $g = \sum g^i$ be a grading of a complex semisimple Lie algebra $g$ with grading element $d$ and $g^\sigma = \sum (g^\sigma)^i = \sum g^i \cap g^\sigma$ be a real form of $g$, consistent with this gradation. We denote by $(\Pi = \Pi_\bullet \cup \Pi', \sim)$ the Satake diagram of $g^\sigma$.

By Proposition 5.3, the decomposition of $g^1$ into irreducible $g^0$-submodules is given by $g^1 = \sum_{\gamma \in \Pi} g(R(\gamma))$, where $\Pi^1$ is the set of simple roots of label one. The following obvious proposition describes the decomposition of $(g^\sigma)^0$-module $(g^\sigma)^1$ into irreducible submodules.

**Proposition 6.3.** For any simple root $\gamma \in \Pi^1$ of label one, there are two possibilities:

i) $\sigma^*\gamma = \gamma + \sum_{\beta \in \Pi_\bullet} k_\beta \beta$. Then $\sigma^*\gamma \in R(\gamma)$ and the $g^0$-module $g(R(\gamma))$ is $\sigma$-invariant;

ii) $\sigma^*\gamma = \gamma' + \sum_{\beta \in \Pi_\bullet} k_\beta \beta$, where $\gamma \neq \gamma' \in \Pi^1$. Then, $\sigma^* R(\gamma) = R(\gamma')$ and the two irreducible $g^\sigma$-modules $g(R(\gamma))$ and $g(R(\gamma'))$ determine one irreducible submodule $g^\sigma \cap (g(R(\gamma)) + g(R(\gamma')))$. 

**Corollary 6.4.** Let $g^\sigma = \sum (g^\sigma)^i$ be the gradation of a real semisimple Lie algebra $g^\sigma$, associated with a label vector $\vec{d}$ of real type. Then irreducible submodules of the $(g^\sigma)^0$-module $(g^\sigma)^{-1}$ correspond to vertices $\gamma$ with label one without curved arrow and to pairs $(\gamma, \gamma')$ of vertices with label one related by a curved arrow. In particular, a decomposition of the $(g^\sigma)^0$-module $(g^\sigma)^{-1}$ is determined by a decomposition of the set $\Pi^1$ of vertices with label 1 into a disjoint union $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ such that equivalent vertices belong to the same component. The corresponding submodules $(g^\sigma)^{-1}_+$ and $(g^\sigma)^{-1}_-$ are given by

$$ \tag{7} (g^\sigma)^{-1}_\pm = g^\sigma \cap \sum_{\gamma \in \Pi^1_\pm} g(R(-\gamma)). $$

We will always assume that a decomposition of $\Pi^1$ satisfies the above property.
7. Classification of Maximally Homogeneous para-CR Manifolds

Let $\mathfrak{g}^{\sigma}$ be a real semisimple Lie algebra associated with a Satake diagram $(\Pi = \Pi_{+} \cup \Pi_{-}, \sim)$ with the fundamental gradation defined by a subset $\Pi^{1} \subset \Pi$ and $F = \tilde{G} / \tilde{P}$ be the associated flag manifold.

By Theorem 4.5, an almost para-CR structure on $F = \tilde{G} / \tilde{P}$ associated with a decomposition $\Pi^{1} = \Pi_{+}^{1} \cup \Pi_{-}^{1}$ is integrable (i.e. a para-CR structure) if and only if the $(\mathfrak{g}^{\sigma})_{0}$-submodules $(\mathfrak{g}^{\sigma})_{+}^{1}$ and $(\mathfrak{g}^{\sigma})_{-}^{1}$ given by (7) are Abelian subalgebras of $\mathfrak{g}^{\sigma}$. In order to give an integrability criterion, we introduce the following definitions.

**Definition 7.1.** Let $R$ be a system of roots and $\Pi$ be a system of simple roots. A subset $\Pi^{1} \subset \Pi$ is said to be admissible if $\Pi^{1}$ contains at least two roots and there are no roots of $R$ of the form

$$2\alpha + \sum k_{i}\phi_{i}, \text{ with } \alpha \in \Pi^{1}, \phi_{i} \in \Pi_{0} = \Pi \setminus \Pi^{1}. \quad (8)$$

**Definition 7.2.** Let $\mathfrak{g}^{\sigma}$ be a real semisimple Lie algebra with a fundamental gradation defined by a subset $\Pi^{1} \subset \Pi$. We say that a decomposition $\Pi^{1} = \Pi_{+}^{1} \cup \Pi_{-}^{1}$ is alternate if the following conditions hold:

i) if $\alpha \in \Pi_{+}^{1}$ and $\alpha' \sim \alpha$, then $\alpha' \in \Pi_{-}^{1}$;

ii) the vertices in $\Pi_{+}^{1}$ and $\Pi_{-}^{1}$ appear in the Satake diagram in alternate order. This means that each connected component of the graph obtained deleting vertices in $\Pi_{+}^{1}$ (respectively in $\Pi_{-}^{1}$) has not more than one vertex in $\Pi_{+}^{1}$ (respectively in $\Pi_{-}^{1}$).

We are ready to state the following

**Proposition 7.3.** Let $\mathfrak{g}^{\sigma}$ be a semisimple real Lie algebra with the fundamental gradation associated with a subset $\Pi^{1} \subset \Pi$ and $F = \tilde{G} / \tilde{P}$ the associated flag manifold. A decomposition $\Pi^{1} = \Pi_{+}^{1} \cup \Pi_{-}^{1}$ defines a para-CR structure on the flag manifold $F$ if and only if the subset $\Pi^{1}$ is admissible and the decomposition of $\Pi^{1}$ is alternate.

For the proof we need the following lemma.

**Lemma 7.4.** The subspace $g_{+}^{1} = \sum_{\gamma \in \Pi_{+}^{1}} \mathfrak{g}(R(\gamma))$ (hence also the subspace $(\mathfrak{g}^{\sigma})_{+}^{1} = \mathfrak{g}^{\sigma} \cap g_{+}^{1}$) which corresponds to a subset $\Pi_{+}^{1} \subset \Pi^{1}$ is an Abelian subalgebra if and only if there is no root $\beta$ of the form

$$\beta = \alpha + \alpha' + \sum k_{i}\phi_{i} \quad (9)$$

where $\alpha, \alpha' \in \Pi_{+}^{1}$ and $\phi_{i} \in \Pi_{0}$. The case $\alpha = \alpha'$ is allowed.

**Proof.** If such a root $\beta$ exists, then $[\mathfrak{g}(R(\alpha)), \mathfrak{g}(R(\alpha'))] \neq 0$ and $g_{+}^{1}$ is not an Abelian subalgebra. The converse is also clear. \(\square\)

**Proof of Proposition 7.3.** Let $\Pi^{1} = \Pi_{+}^{1} \cup \Pi_{-}^{1}$ be a decomposition of $\Pi^{1}$. The condition (9) for $\alpha = \alpha'$ is fulfilled if and only if $\Pi^{1}$ is admissible. Assume now that two different vertices $\alpha, \alpha'$ in $\Pi_{+}^{1}$ are connected in the Satake
is admissible $(\Pi)$. Then there is a root of the form $(\sigma)$ and $g_1^+$ is not a commutative subalgebra. This shows that the decomposition which defines a para-CR structure on $F$ must be alternate.

Conversely, assume that the decomposition is alternate. Then any two vertices $\alpha, \alpha' \in \Pi^1_+$ belong to different connected components of the Satake graph with deleting $\Pi^1_+$. This implies that there is no root of the form $(\sigma)$ for $\alpha \neq \alpha'$. Then Lemma 7.4 shows that $(g^\sigma)_+^1$ is a commutative subalgebra. The same argument is applied also for $(g^\sigma)_-^1$. □

We enumerate simple roots of complex simple Lie algebra $g$ algebras as in [3]. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be the simple roots of $g$, which are identified with vertices of the corresponding Dynkin diagram. We denote the elements of a subset $\Pi_1 \subset \Pi$ (respectively $\Pi_1 \subset \Pi'$) which defines a fundamental gradation of $g$ (respectively $g^\sigma$) by

$$\alpha_{i_1}, \ldots, \alpha_{i_k}, \quad i_1 < i_2 < \cdots < i_k.$$ 

**Proposition 7.5.** Let $\Pi$ be a system of simple roots of a root system $R$ of a complex simple Lie algebra $g$. Then a subset $\Pi_1 \subset \Pi$ of at least two elements is admissible (see Definition 7.2) in the following cases:

- for $g = A_\ell$, in all cases;
- for $g = B_\ell$, under the condition: $i_k = i_{k-1} + 1$;
- for $g = C_\ell$, under the condition: $i_k = \ell$;
- for $g = D_\ell$, under the condition: if $i_k < \ell - 1$, then $i_k = i_{k-1} + 1$;
- for $g = E_6$, in all cases except the following ones:
  $$\{\alpha_1, \alpha_4\}, \{\alpha_1, \alpha_5\}, \{\alpha_3, \alpha_6\}, \{\alpha_4, \alpha_6\}, \{\alpha_1, \alpha_4, \alpha_6\};$$
- for $g = E_7$, in all cases except the following ones:
  $$\{\alpha_1, \alpha_4\}, \{\alpha_1, \alpha_5\}, \{\alpha_3, \alpha_6\}, \{\alpha_4, \alpha_6\}, \{\alpha_1, \alpha_6\}, \{\alpha_2, \alpha_7\}, \{\alpha_3, \alpha_7\}, \{\alpha_4, \alpha_7\}, \{\alpha_5, \alpha_7\}, \{\alpha_1, \alpha_4, \alpha_6\}, \{\alpha_1, \alpha_4, \alpha_7\}, \{\alpha_1, \alpha_5, \alpha_7\}, \{\alpha_3, \alpha_6, \alpha_7\}, \{\alpha_4, \alpha_6, \alpha_7\}, \{\alpha_1, \alpha_4, \alpha_6, \alpha_7\};$$
- for $g = E_8$, in all cases except the following ones:
  $$\{\alpha_1, \alpha_4\}, \{\alpha_1, \alpha_5\}, \{\alpha_3, \alpha_6\}, \{\alpha_4, \alpha_6\}, \{\alpha_1, \alpha_6\}, \{\alpha_2, \alpha_7\}, \{\alpha_3, \alpha_7\}, \{\alpha_4, \alpha_7\}, \{\alpha_5, \alpha_7\}, \{\alpha_1, \alpha_4, \alpha_6, \alpha_7\}, \{\alpha_1, \alpha_7\}, \{\alpha_1, \alpha_8\}, \{\alpha_2, \alpha_8\}, \{\alpha_3, \alpha_8\}, \{\alpha_4, \alpha_8\}, \{\alpha_5, \alpha_8\}, \{\alpha_6, \alpha_8\}, \{\alpha_1, \alpha_4, \alpha_8\}, \{\alpha_1, \alpha_5, \alpha_8\}, \{\alpha_3, \alpha_6, \alpha_8\}, \{\alpha_4, \alpha_6, \alpha_8\}, \{\alpha_4, \alpha_8\}, \{\alpha_2, \alpha_7, \alpha_8\}, \{\alpha_3, \alpha_7, \alpha_8\}, \{\alpha_4, \alpha_7, \alpha_8\}, \{\alpha_5, \alpha_7, \alpha_8\}, \{\alpha_1, \alpha_4, \alpha_6, \alpha_8\}, \{\alpha_1, \alpha_4, \alpha_7, \alpha_8\}, \{\alpha_1, \alpha_5, \alpha_7, \alpha_8\}, \{\alpha_3, \alpha_6, \alpha_7, \alpha_8\}, \{\alpha_4, \alpha_6, \alpha_7, \alpha_8\}, \{\alpha_1, \alpha_4, \alpha_6, \alpha_7, \alpha_8\};$$
• for \( g = F_4 \), in all cases except the following ones:
  \( \{\alpha_1, \alpha_3\}, \{\alpha_1, \alpha_4\}, \{\alpha_2, \alpha_4\}, \{\alpha_3, \alpha_4\}, \{\alpha_1, \alpha_3, \alpha_4\} \);

• for \( g = G_2 \), in the case \( \{\alpha_1, \alpha_2\} \).

In cases different from \( D_\ell, E_6, E_7 \) and \( E_8 \), for any \( \Pi^1 \) given as above it is possible to give an alternate decomposition \( \Pi^1 = \Pi^1_1 \cup \Pi^1_2 \).

For \( D_\ell \), an alternate decomposition of \( \Pi^1 \) can be given in the following cases:

• \( \alpha_{\ell - 2} \in \Pi^1 \),
• \( \Pi^1 \) is contained in at most two of the branches issuing from \( \alpha_{\ell - 2} \).

For \( E_6, E_7 \) and \( E_8 \), an alternate decomposition of \( \Pi^1 \) can be given in the following cases:

• \( \alpha_4 \in \Pi^1 \),
• \( \Pi^1 \) is contained in at most two of the branches issuing from \( \alpha_4 \).

**Proof.** We have to describe all subsets \( \Pi^1 \) of \( \Pi \) which satisfy (8). This condition can be reformulated as follows. For any \( \alpha \in \Pi^1 \), denote by \( \Pi_\alpha \) the connected component of the subdiagram of the Dynkin diagram \( \Pi \) obtained by deleting vertices in \( \Pi^1 \setminus \{\alpha\} \) and containing \( \alpha \). Then the root system associated with \( \Pi_\alpha \) has no roots of the form

\[
\beta = 2\alpha + \sum_{\phi \in \Pi_\alpha \setminus \{\alpha\}} k_\phi \phi.
\]

Using this condition and the decomposition of any root into a linear combination of simple roots, one can prove the proposition.

In the case of \( A_\ell \), any root has coefficient 0, 1 in the decomposition into simple roots. Hence, any decomposition satisfies the property (8).

In the case of \( B_\ell \), any root which has coefficient 2 has the form

\[
\sum_{i \leq h < j} \alpha_h + 2 \sum_{j \leq h \leq \ell} \alpha_h, \quad (1 \leq i < j \leq \ell).
\]

Hence the condition (8) holds if and only if the last two roots in \( \Pi^1 \) are consecutive, i.e. \( i_{k-1} + 1 = i_k \).

In the case of \( C_\ell \), the roots with a coefficient 2 are given by

\[
\sum_{i \leq h < j} \alpha_h + 2 \sum_{j \leq h < \ell} \alpha_h + \alpha_\ell, \quad (1 \leq i < j \leq \ell),
\]

\[
2 \sum_{i \leq h < \ell} \alpha_h + \alpha_\ell, \quad (1 \leq i < \ell).
\]

The second formula implies that there are no roots of the form given in (8) if and only if \( i_k = \ell \).

In the case of \( D_\ell \), the roots with a coefficient 2 are

\[
\sum_{i \leq h < j} \alpha_h + 2 \sum_{j \leq h < \ell - 1} \alpha_h + \alpha_{\ell - 1} + \alpha_\ell, \quad (1 \leq i < j < \ell - 1).
\]
The condition (8) fails if and only if the last two roots $\alpha_{i_{k-1}}, \alpha_{i_k}$ satisfy $i_{k-1} < i_k - 1$ and $i_k < \ell - 1$.

The case of exceptional Lie algebras can be treated in a similar way, by using tables in [4]. □

Let $\Pi^1 \subset \Pi'$ be an admissible subset which defines a fundamental gradation of $\mathfrak{g}^\sigma$. An alternate decomposition of $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ can be given if the conditions of Proposition 7.5 are satisfied and, in addition, the following ones hold:

- for $\mathfrak{su}(p,q)$, it has to be $q = p$ and $\alpha_p \in \Pi^1$;
- for $\mathfrak{so}(\ell - 1,\ell + 1)$, it has to be $\Pi^1 \cap \{\alpha_{\ell-1},\alpha_{\ell}\} = \emptyset$ or $\{\alpha_{\ell-2},\alpha_{\ell-1},\alpha_{\ell}\} \subset \Pi^1$;
- for $E_6\Pi$, it has to be $\alpha_4 \in \Pi^1$ and if $\alpha_2 \notin \Pi^1$, then $\{\alpha_3,\alpha_5\} \subset \Pi^1$;
- while for $\mathfrak{so}^*(2\ell)$ and $E_6\Pi$ III there is no alternate decomposition of $\Pi^1$.

Proposition 7.3 implies the following final theorem.

**Theorem 7.6.** Let $(\Pi = \Pi_+ \cup \Pi_-, \sim)$ be a Satake diagram of a simple real Lie algebra $\mathfrak{g}^\sigma$ and $\Pi^1 \subset \Pi'$ be an admissible subset as described above. Let $\tilde{G}$ be the simply connected Lie group with the Lie algebra $\mathfrak{g}^\sigma$ and $\tilde{P}$ be the parabolic subgroup of $\tilde{G}$ generated by the non-negatively graded subalgebra

$$p = \sum_{i \geq 0} (\mathfrak{g}^\sigma)^i$$

associated with the grading element $\tilde{d}_{\Pi^1}$. Then the alternate decomposition $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ defines a decomposition

$$(\mathfrak{g}^\sigma)^1 = (\mathfrak{g}^\sigma)_+^1 + (\mathfrak{g}^\sigma)_-^1$$

of the $(\mathfrak{g}^\sigma)^0$-module $(\mathfrak{g}^\sigma)^1$ into a sum of two commutative subalgebras. This decomposition determines an invariant para-CR structure on the simply connected flag manifold $F = \tilde{G}/\tilde{P}$. Moreover, any simply connected maximally homogeneous para-CR manifolds of semisimple type is a direct product of such manifolds.

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