Equilibrium states of a variational formulation for the Ginzburg-Landau equation

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Abstract. Periodic boundary value problem for one of the versions of the complex Ginzburg-Landau equation, which is commonly called the variational Ginzburg-Landau equation are studied. Questions of existence and stability in the sense of Lyapunov, and also the local bifurcations problem of spatially nonhomogeneous equilibrium states are investigated. Three types of such solutions for the given problem are indicated. The exact formulas of the solutions for the first two types are suggested. Equilibrium states of the second type are expressed through elliptic functions. The third type of equilibrium states appears as a result of bifurcations of automodel equilibrium states, i.e., solutions of the first type in the case when the stability changes. It is shown that equilibrium states of the second and third types are unstable.

1. Introduction
The cubic complex equation can be written in the form

\[ u_t = u - (1 + ic)u|u|^2 + (d_1 + id_2)u_{xx}, \] (1.1)

where \( u = u_1(t, x) + iu_2(t, x), c, d_1, d_2 \in \mathbb{R}, d_1 > 0 \) is commonly called a quantic time-dependent Ginzburg-Landau equation (QTDGL) [1-3]. More general forms of this equation and also particular variants of this equation are considered. For example, in applications to nonlinear optics and to some model problems of oscillations theory one considers its weakly dissipative form when \( d_1 = 0 \) [3-4].

In [1] the case when equation (1.1) has the form (1.1) is singled out in a separate case

\[ u_t = (1 + ic)[u - u|u|^2 + d_{xx}^2] (d > 0). \] (1.2)

For equation (1.2) in [1] it was suggested a special name - variational QTDGL (see [5]). In the given work it will be considered the following boundary value problem with \( c = 0 \)

\[ u_t = u - u|u|^2 + d_{xx}, \] (1.3)

\[ u(t, x + 2\pi) = u(t, x), \] (1.4)

which was obtained in studying of phenomena in condensed media (\( \Psi^4 \) – Ginzburg-Landau’s model). The questions of existence and stability of nontrivial equilibrium states will be further considered. For such solutions it will be obtained explicit or asymptotic formulas. We will
understand the stability in the sense of the norm of the phase space of solutions for the boundary value problem (1.3), (1.4). We add the following initial condition to the boundary value problem (1.3), (1.4)

\[ u(0, x) = f(x). \]  

(1.5)
The mixed problem (1.3), (1.4), (1.5) is locally solvable if a complex-value function \( f(x) \in H^2_2 \) i.e., belongs to the space of periodic functions with period \( 2\pi \), which have distributional derivatives. They and this function belong to \( L_2(-\pi, \pi) \) (see [6-7]), i.e.,

\[ \int_{-\pi}^{\pi} |f^{(k)}(x)|^2 dx < \infty, \quad k = 0, 1, 2. \]

2. Automodel equilibrium states

The boundary value problem (1.3), (1.4) has the solutions of the form

\[ u_n(x) = \eta_n \exp(inx), \quad n = 0, \pm 1, \ldots, \pm n_0. \]  

(2.1)

where a natural number \( n_0 \) will be defined later. Together with solutions (2.1) there exist the solutions \( u_n(x + h) = \eta_n \exp(inh) \exp(inx), h \in R \). Therefore, considering the solutions (2.1) we suppose that \( \eta_n \in R_+ (\eta_n > 0) \), and a positive constant \( \eta_n \) should be found as the corresponding root of equation \( \eta - \eta^3 - d\eta^3 = 0 \). Except a zero root the last equation has a root \( \eta_n = (1 - dn^2)^{1/2} \) if \( 1 - dn^2 > 0 \). Therefore, the given equilibrium states \( S_n \) exist, if \( |n| \leq n_0 \), where \( n_0 = [1/\sqrt{d}] \) or \( n_0 = [1/\sqrt{d}] - 1 \). The last choice of \( n_0 \) is realized, if \( (1/\sqrt{d}) \in N \).

We remind that as \([a]\) we denote an integer part of the number \( a \).

To analyze the stability of \( S_n (u_n(x) = \eta_n \exp(inx)) \) we set \( u_n(x) = \eta_n \exp(inx)(1 + w(t, x)) \). Then, for \( w(t, x) = w_1(t, x) + iw_2(t, x) \) we obtain the boundary value problem in the form

\[ w_t = -\eta_n^2 (w + \overline{w}) - \eta_n^2 [2w\overline{w} + w^2 + w|w|^2] + dww_{xx} + 2dw_{w_x}, \]  

\[ w(t, x + 2\pi) = w(t, x). \]  

(2.2)

(2.3)

The boundary value problem (2.2), (2.3) for a complex-value function \( w(t, x) = w_1(t, x) + iw_2(t, x) \) may be rewritten in a real form for a vector-function \( w(t, x) = \text{col}(w_1, w_2) \)

\[ w_t = A_n w + F_2(w) + F_3(w), \quad n = 0; \pm 1; \ldots, \pm n_0. \]  

(2.4)

\[ w(t, x + 2\pi) = w(t, x), \]  

\[ A_n w = \left( \begin{array}{c} -2\eta_n^2 w_1 + dw_{1xx} - 2dww_{2x} \\ dw_{2xx} + 2dww_{1x} \end{array} \right), \quad F_2(w) = -\eta_n^2 \left( \frac{3w_1^2 + w_2^2}{2w_1w_2} \right), \]

\[ F_3(w) = -\eta_n^2 \left( \begin{array}{c} w_1(w_1^2 + w_2^2) \\ w_2(w_1^2 + w_2^2) \end{array} \right). \]

A linearized form of the boundary value problem (2.4), (2.5) supposes the analysis of the equation \( w_t = A_n w \) with the boundary conditions (2.5). We should found the eigenfunctions of a linear differential operator \( A_n \) in the following form \( E_m(x, n) = h_m \exp(inx), h_m = \text{col}(h_{1m}, h_{2m}) \) for each \( n \) \((|n| \leq n_0)\).

Standard constructions allow reducing the question about finding the eigenvalues \( A_n \) to the analysis of the characteristic equation

\[ det \left( \begin{array}{cc} -2\eta_n^2 - dm^2 - \lambda & -imdn \\ imdn & -dm^2 - \lambda \end{array} \right) = 0 \]  

or \( \lambda^2 + P_n(m)\lambda + Q_n(m) = 0, \)
where $Q_n(m) = d n^2 (d n^2 + 2 m^2 - 4 d n^2)$, $P_n(m) = 2 (d n^2 + m^2)$. It is clear that $P_n(m) > 0$. The inequality $Q_n(m) > 0$ holds for $m \neq 0$, if $d + 2 - 6 d n^2 > 0$. It is understood that $Q_n(0) = 0$.

**Lemma 1.** Let $n = \pm 1, \ldots, \pm n_0$ and $d < d_n = 2/(6 n^2 - 1)$. Then for the given $d$, the eigenvalues $\lambda_n$, except $\lambda_0 = 0$, are located in the half-lane $Re \lambda \leq -\gamma_0(n) < 0$.

The proof is reduced to the verification of the inequalities $Q_n(m) > 0$. The analysis of characteristic equation shows that eigenvalues $\lambda_n(m) \in \mathbb{R}$ and $\lim_{m \to \infty} \lambda_n(m) = -\infty$.

**Theorem 1.** The boundary value problem (1.3), (1.4) has 2$n_0 + 1$ the one-dimensional invariant manifolds $S_n(\alpha)$ which are formed by solutions $u_n(x + \alpha), \alpha \in R, n = 0, \pm 1, \ldots, n_0$.

Let $n \neq 0$. Then, a one-dimensional invariant manifold $S_n(\alpha)$ is attracting (local attractor), if $d \in (0; d_n)$, unstable (saddle), if $d \in (d_n; \infty)$.

For $n = 0$ the manifold $S_0(\alpha)$ is a local attractor for all $d > 0$. We underline that each of equilibrium states of the family $S_n(\alpha)$, which are existing with $d \in (0; 1/n^2)$ are stable with $d \in (0; d_n)$ and unstable with $d \in (d_n; 1/n^2), d_n < 1/n^2$ for all considering $n(n \neq 0)$.

3. **Equilibrium states of the second kind**

Let $u(t, x) = u_1(t, x) + iu_2(t, x)$ and $w_2(t, x) = 0$. Then for the function $v(t, x) = u_1(t, x)$ we derived the following boundary value problem

$$v_t = v - v^3 + dv_{xx},$$

$$v(t, x + 2 \pi) = v(t, x).$$

The boundary value problem (3.1),(3.2) has a zero equilibrium state, but may have the nonzero equilibrium states. They should be found as nontrivial solutions $v(x)$ of the boundary value problem

$$dv'' + v - v^3 = 0,$$

$$v(t, x + 2 \pi) = v(t, x).$$

The ordinary differential equation (3.3) has the solutions, which can be express through the elliptic functions

$$v(x) = psn(\delta x, k),$$

where $k \in (0; 1)$ is a real parameter (a module of an elliptic sine). We can assume that $p, \delta > 0$. The function (3.5) satisfies the equation (3.3), if $d \delta^2(1 + k^2) = 1$, and $2 d \delta^2 k^2 = p^2$, i.e., $p = \sqrt{2k(\sqrt{1 + k^2})}, \delta = (d(1 + k^2))^{-1/2}$. We should further choose $k \in (0; 1)$ so that the function (3.5) has a period $2 \pi$. It has a period $2 \pi/m$, where $m = 1, 2, \ldots, m$

$$\frac{1}{\sqrt{dm}} = G(k) = \frac{2}{\pi} (1 + k^2)^{1/2} \int_0^{\pi/2} \frac{dy}{\sqrt{1 - k^2 \sin^2 y}}.$$ (3.6)

It is clear that $G(0) = 1, G'(k) > 0$ for all $k \in (0; 1), \lim_{k \to 1} G(k) = \infty$. Furthermore, for each natural number $m$ equation (3.6) has the solution $k_m \in (0; 1)$ if $\sqrt{d}/m < 1$.

**Lemma 2.** Let $m < 1/\sqrt{d}$. For all such natural $m$ the solution (3.5) exists, where

$$p_m = \frac{\sqrt{2k_m}}{\sqrt{1 + k_m^2}}, \delta_m = \frac{1}{\sqrt{d} \sqrt{1 + k_m^2}},$$

$k_m \in (0; 1)$. The corresponding solution to the given $m$ we denote as $v_m(x)$ and investigate the question about its stability. Let $w(t, x) = v_m(x) + w(t, x), w(t, x) = w_1(t, x) + iw_2(t, x)$. A complex-value function $w$ satisfies the boundary value problem

$$w_t = w - v_m^2(2w + \overline{w}) - v_m(2w\overline{w} + w^2) - w^2\overline{w} + dw_{xx},$$

(3.7)
For this problem it should be considered the question of stability of a zero equilibrium state. For that, we consider a linear variant of the boundary value problem (3.7), (3.8). We write the boundary value problem in a real form. For this, we set \( w = \text{colon}(w_1, w_2) \). And as a result, we obtain a linear differential equation in \( R^2 \)

\[
w_t = B_m w,
\]

where \( w(t, x) \) satisfies the periodic boundary conditions, and \( B_m w = \left( \begin{array}{c} w_1 + dw_{1xx} - 3v_m^2 w_1 \\ w_2 + dw_{2xx} - v_m^2 w_2 \end{array} \right) \).

The given linear operator \( B_m \) has the eigenvalue \( \lambda_{0m} = 1 - \frac{1}{1 + k_m^2} > 0 \), corresponding to the eigenvector function \( \text{colon}(0, cn(\delta_m x, k_m)) \), where \( cn(y, k) \) is an elliptic cosine, and the period boundary value problem for the partial differential equation (3.9) has the exponentially growing solution \( w_1 = 0, w_2(t, x) = cn(\delta_m x, k_m) \exp(\lambda_{0m} t) \). The main boundary value problem (1.3), (1.4) with the solution \( v_m(x) \) has a two-parametric family of analogous solutions

\[
u = \exp(ih_2)v_m(x + h_1), \quad h_1, h_2 \in R.
\]

Thus, it was proved the statement.

**Theorem 2.** Let \( dm_0^2 < 1, m_0 \in N \). Then, the boundary value problem (1.3), (1.4) has \( m_0 \) two-dimensional invariant manifolds \( V_2(m) \) filled with the unstable equilibrium states (3.10), \( m = 1, \ldots, m_0 \).

### 4. Bifurcations of the automodel solutions

Consider the local bifurcations problem for the solutions

\[
u_n(x) = \eta_n \exp(inx), \quad \eta_n = (1 - d_n^2)^{1/2}
\]

in the case when the stability changes. Here, \( n = \pm 1, \pm 2, \ldots, \) and the case \( n = 0 \) we do not consider since the corresponding solution \( u_0(x) = 1 \) does not change the stability. We remind that the critical value is \( d = d_n = 2/(6n^2 - 1) \).

In this section it will be shown that for \( d < d_n \) (subcritical bifurcations) the two-dimensional invariant manifolds, which are formed by equilibrium states with more complex structure than \( u_n(x + \alpha), \) bifurcate from the family of equilibrium states \( S_\alpha(\alpha) \). In a boundary value problem (1.3), (1.4) we set

\[
d = d_n(\varepsilon) = d_n(1 + \varepsilon \gamma/2), \quad u = u_n(x)(1 + w(t, x)), \quad u_n(x) = \eta_n(\varepsilon) \exp(inx).
\]

Here, \( \varepsilon \in (0; \varepsilon_0), 0 < \varepsilon_0 << 1, \gamma = \pm 1 \). The value \( \gamma \) will be chosen below. Finally, \( \eta_n^2(\varepsilon) = 1 - d_n^2 - d_n\varepsilon(\gamma/2)n^2 \). As a result, for we obtain the boundary value problem

\[
w_t = -d_n^2(\varepsilon)[w + x + 2u\overline{w} + u^2 + w^2\overline{w}] + dw_{xx} + 2d\text{id}w_x,
\]

\[
w(t, x + 2\pi) = w(t, x).
\]

We note that the substituting \( u_n(x) \) instead of \( u_n(x + h) \), where \( h \in R \), leads to the same boundary value problem (4.2), (4.3) where the right part of equation does not depends on \( h \). We may rewrite the boundary value problem (4.2), (4.3) for the complex-value functions \( w = w_1(t, x) + iw_2(t, x) \) in a real form for \( w = \text{colon}(w_1, w_2) \)

\[
w_t = A_n w + F_2(w) + F_3(w) + \varepsilon B_n w + \varepsilon F_4(w) + \varepsilon F_5(w),
\]
H-invariantly for the solutions of the boundary value problem (4.4), (4.5). In a first component an even function, and as the second one an odd function. This subspace is an eigenvalue with the corresponding eigenfunctions of the theorem of existence and properties of the central invariant manifold. In the space the question about local bifurcations in a neighborhood of a zero equilibrium state. From the results of section 1, it follows that a linear differential operator \( A_n \) has a triple zero eigenvalue with the corresponding eigenfunction of the equation (4.6). We will find the solutions belonging to the functions of \( \varepsilon \). As the phase space of the solutions of the boundary value problem (4.4), (4.5) we assume \( H^2_{2,2} \), i.e., the space of the two-dimensional periodic vector-functions \( f(x) \in H^2_{2,2} \) with a period \( 2\pi \), where \( f_j(x) \in W^2_2[-\pi; \pi] \), i.e., belongs to the Sobolev space.

From the results of section 1, it follows that a linear differential operator \( A_n \) has a triple zero eigenvalue with the corresponding eigenfunction of the equation (4.6). We consider in this space the question about local bifurcations in a neighborhood of a zero equilibrium state. From the theorem of existence and properties of the central invariant manifold \( M_1(\varepsilon) \) (in the given case \( \text{dim} M_1(\varepsilon) = 1 \)) it follows that the question about local bifurcations may be reduced to the analysis of an ordinary differential equation (normal form)

\[
\dot{z} = \varepsilon \Psi_n(z) + o(\varepsilon),
\]

where a form of the function \( \Psi_n(z) \) will be defined below, \( z = z(t) \). Briefly remind the algorithm allowing to form the right part of the equation (4.6). We will find the solutions belonging to \( M_1(\varepsilon) \) in the following form (see [8-9])

\[
w(x, z, \varepsilon) = \varepsilon^{1/2}Q_1(x, z) + \varepsilon Q_2(x, z) + \varepsilon^{3/2}Q_3(x, z) + o(\varepsilon^{3/2}),
\]

where \( z = z(t) \) are solutions of (4.6), \( Q_1(x, z) = z E_1(x) \) vector-functions \( Q_2(x, z), Q_3(x, z) \) as the functions of \( x \) for all \( z \) belong to \( H^* \). These equalities satisfy

\[
\int_{-\pi}^{\pi} (Q_{1k} Q_{11} + Q_{2k} Q_{21}) dx = 0, k = 2, 3, Q_k = Q_k(x, z) = \text{col} (Q_{1k}(x, z), Q_{2k}(x, z)).
\]

The equality (4.7) should be understood as the equation defining the integral (invariant) manifold \( M_1(\varepsilon) \). Substituting the sum (4.7) in the boundary value problem (4.4), (4.5) with (4.6), equating powers of \( \varepsilon \) we obtain the two nonhomogeneous boundary value problems to define \( Q_2(x, z), Q_3(x, z) \)

\[
A_n Q_2 + F_2(Q_1) = 0,
\]

\[
Q_2(x + 2\pi) = Q_2(x, z),
\]

\[
A_n Q_3 + F_3(Q_1) + \Phi_3 - \Psi_n(z) E_1(x) + BQ_1 = 0,
\]

\[
A_n w - \eta_n^2 \left( \begin{array}{c} 2w_1 \\ 0 \end{array} \right) + d_n \left( \begin{array}{c} w_{1xx} \\ w_{2xx} \end{array} \right) + 2\eta_n \left( \begin{array}{c} -w_{2x} \\ w_{1x} \end{array} \right), B_n w = d_n n^2 \gamma \left( \begin{array}{c} w_1 \\ 0 \end{array} \right) +
\]

\[
+ d_n n^2 \frac{\gamma}{2} \left( \begin{array}{c} w_{1xx} \\ w_{2xx} \end{array} \right) + n \eta_n \gamma \left( \begin{array}{c} -w_{2x} \\ w_{1x} \end{array} \right), F_2(w) = -\eta_n^2 G_2(w), F_3(w) = -\eta_n^2 G_3(w),
\]

\[
F_1(w) = d_n n^2 \gamma G_2(w), F_2(w) = \frac{d_n n^2 \gamma G_3(w)}{2},
\]

\[
G_2(w) = \left( \begin{array}{c} 3w_1^2 + w_2^2 \\ 2w_1 w_2 \end{array} \right), G_3(w) = \left( \begin{array}{c} w_1(w_2^2 + w_2^2) \\ 2w_2(w_1^2 + w_2^2) \end{array} \right).
\]

The case \( \text{dim} M_1(\varepsilon) = 1 \)
For $\gamma$ consider the truncated normal form, i.e., the differential equation has the equilibrium states

$$H \text{ value problem (4.4), (4.5) with an additional condition of belonging of solutions to the space equation (4.6) is asymptotically stable.}$$

Both of these equilibrium states are unstable, and a zero equilibrium state of the differential equation (4.11) allows to define $\Psi_n(z)$.

$$\Psi_n(z) = \beta_n z + l_n z^3, \quad l_n = \frac{3}{6n^2 - 1}, \quad \beta_n = \frac{\gamma}{4n^2 + 1}.$$

In particular, $l_n > 0$ for all $n$ ($l_1 = 1.80, l_2 = 1.96, l_3 = 1.98$, $\lim_{n \to \infty} l_n = 2$, $\lim_{n \to \infty} \beta_n = 0$). We now consider the truncated normal form i.e., the differential equation

$$\dot{z} = \varepsilon [\beta_n z + l_n z^3].$$

For $\gamma = -1$ it has the two nonzero equilibrium points (for $\gamma = 1$ they do not exist)

$$z_1 = (\alpha_n)^{-1/2}, \quad z_2 = -(\alpha_n)^{-1/2}, \quad \alpha_n = l_n / |\beta_n|.$$

Both of these equilibrium states are unstable, and a zero equilibrium state of the differential equation (4.6) is asymptotically stable.

Thus, it was proved the statement.

**Lemma 3.** There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0; \varepsilon_0), d = d_n(1 - \varepsilon/2)$ the boundary value problem (4.4), (4.5) with an additional condition of belonging of solutions to the space $H_*$ has the equilibrium states

$$w(x, z, \varepsilon) = \varepsilon^{1/2} Q_1(x, z_j) + \varepsilon Q_2(x, z_j) + o(\varepsilon)$$

where $z_j$ is a coordinate of one of the nonzero equilibrium states of equation (4.6). These equilibrium states are unstable. If $d = d_n(1 + \varepsilon/2), \varepsilon > 0$ then, the nonlinear equilibrium states are absent.
We also note that \( w(x, z_2, \varepsilon) = w(x + \pi, z_1, \varepsilon) \) From the previous statements it follows the theorem for the main boundary value problem (1.3), (1.4) (see [8-9]).

**Theorem 3.** For the indicated \( \varepsilon \) and \( d \) within the formulation of lemma 3 for \( d = d_n(1 - \varepsilon/2) \) the boundary value problem (1.3), (1.4) has the two-parametric family of equilibrium states

\[
\begin{align*}
u &= \eta_n \exp(i n x + i h_1)[1 + \left( \frac{\varepsilon}{\alpha_n} \right)^{1/2} \left( \cos(x + h_2) - 2ni \sin(x + h_2) \right) + \\
&+ \frac{\varepsilon}{\alpha_n} (q_0 + q_1 \cos(2x + 2h_2) + iq_2 \sin(2x + 2h_2)) + o(\varepsilon)], 
\end{align*}
\]

(4.12)

where \( h_1, h_2 \in \mathbb{R} \), and the constants \( q_0 = q_0(n) \), \( q_1 = q_1(n) \), \( q_2 = q_2(n) \), \( \alpha_n \) were defined earlier.

The family of solutions (4.12) forms the two-dimensional integral manifold \( V_2(\varepsilon) \). This manifold is not attracting (local attractor). It is filled with the unstable equilibrium states of the boundary value problem (1.3), (1.4).

### 5. Conclusion

In the present work it was shown that in the periodic boundary value problem for the variational Ginzburg-Landau equation can exist three types of the equilibrium states, which differ from a trivial. These equilibrium states can be divided into two classes: spatially homogeneous and spatially nonhomogeneous, i.e., substantially depending on a variable \( x \). Nontrivial, but spatially homogeneous equilibrium states always exist and they are stable. A different situation exists for spatially nonhomogeneous equilibrium states, and conditions of existence and stability are obtained in the sections 1, 2, 3. We underline that for them the situation when they are unstable is more typical.

### 6. Acknowledgments

The results of the given work were presented on the 6th International conference "The problems of mathematical physics and mathematical modeling" (Moscow, NRNU MEPhi, 25-27 May 2017). Authors thank the participants of the section "Methods of mathematical physics" and professor N.A. Kudryashov for the helpful discussion of the report.

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