Neutral functional differential equations and systems of conservation laws

Daniela Danciu and Vladimir Rășvan

Department of Automation and Electronics, University of Craiova, A.I.Cuza, 13, Craiova, RO-200585, Romania
(e-mail: {ddanciu,vrasvan}@automation.ucv.ro)

Abstract: A natural way of introducing neutral functional differential equations is integration along the characteristics of the Riemann invariants of the hyperbolic partial differential equations. Up to now the problem has been discussed by A.D. Myshkis, K.L. Cooke and their followers in the linear or quasilinear case. The conservation laws are nonlinear hyperbolic partial differential equations whose study goes back to the fifties of the previous century, being due to the pioneering papers of O.A. Oleinik and P.D. Lax. These equations describe important phenomena in Physics and Engineering, being also subject to control issues. The present paper is an attempt to extend the method of integrating along the characteristics to the systems of conservation laws. The main purpose of this attempt is to emphasize the occurring nonlinear neutral functional differential equations and to construct the augmented validation (existence, uniqueness, continuous data dependence, Stability Postulate) at least at the level of the classical solutions.

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1. MOTIVATING EXAMPLES

The conservation laws are basic in physics of the continuous media. But this is not the place to study physics development or related mathematical aspects. We shall rather focus on engineering applications with particular reference to control and modeling for control. To be still more specific, there will be considered here the boundary control which is more common in applications for engineering. With respect to this we shall mention traffic control described by conservation laws Kachroo (2007) and the applications to hydraulic and thermal power engineering.

A quite well studied application is modeling and stabilization of flows in open canals where the basics of the modeling are given by the equations of Saint Venant. The reader is sent e.g. to Leufering and Schmidt (2002), de Halleux et al. (2003), Coron et al. (2007), Petre and Rășvan (2009). The literature cited by these papers can give more insight to this problem which has broad applications in irrigation control via control gates.

We shall give here, for the sake of the completeness, the equations of a controlled hydraulic channel Bastin et al. (2005)

\[
\frac{\partial H}{\partial t} + \frac{\partial Q}{\partial x} = 0
\]

\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{H} + g \frac{H^2}{2} \right) = 0
\]

\[Q(0,t) = Q_d(t)\]

\[Q(L,t) = \gamma(H(t,L) - \mu(t))^{3/2}\]

\[T_s \frac{d\mu}{dt} = \sigma, \quad T_s \frac{d\sigma}{dt} = -\sigma - k_s \mu + \eta(t)\]

where \(H\) is the water level and \(Q\) is the water flow, \(\mu(t)\) represents the level control signal downstream, while the upstream flow \(Q(0,t)\) is assumed not regulated. The control aim is to maintain constant level \(H(t,L) \equiv \bar{H}\). The constant \(\gamma > 0\) characterizes the down stream spillway.

On the other hand, the dynamics of hydraulic power plants also involves hydraulics but through galleries and pipes i.e. closed media. The equations are very similar to those of Saint Venant Wylie and Streeter (1978), Popescu (2008). However, the distributed parameters are taken into account for waterhammer computation only.

Another class of applications arises from thermal power engineering. Here it is interesting to cite three pioneering papers in this field Sokolov (1946), Kabakov (1946), Kabakov and Sokolov (1946) where there are used the equations of the isentropic (barotropic) flow under the form

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho w}{\partial x} = 0
\]

\[\rho \frac{\partial w}{\partial t} + \rho w \frac{\partial w}{\partial x} + \frac{\partial p}{\partial x} = 0, \quad p = f(\rho)\]

with \(p\) – the pressure, \(\rho_w\) – the mass flow through the area unit, \(\rho\) – the mass density and \(w\) – the flow velocity; here \(p = f(\rho)\) is the thermodynamic state equation for the barotropic compressible fluid. In the aforementioned papers the approach is that of the brute force linearization in order to reduce the model to a boundary value problem for linear hyperbolic partial differential equations (PDEs).

Equations (2) can however be given the form of a system of conservation laws (we skip the manipulations)
\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial l} (\rho w) &= 0 \\
\frac{\partial}{\partial l} (\rho w) + \frac{\partial}{\partial l} \left( f(\rho) + \frac{1}{\rho} (\rho w)^2 \right) &= 0
\end{align*}
\]

Now the engineering approach to modeling is based on the use of the rated variables: the rating is done with respect to some reference (steady state, maximal) values. Introducing the rated variables (including the rated length of the pipe) the conservation laws (3) take the form

\[
\begin{align*}
T_c \frac{\partial \xi_p}{\partial t} + \frac{\partial \xi_w}{\partial t} &= 0, \quad t > 0, \quad 0 < \lambda < 1 \\
\psi_t^2 T_c \frac{\partial \xi_w}{\partial t} + \frac{\partial}{\partial \lambda} \left( \xi_p + \psi_t^2 \frac{\xi_w}{\xi_p} \right) &= 0
\end{align*}
\]

where \( \xi_p \) and \( \xi_w \) account for the rated variables of the steam pressure (for isothermal flow) and steam flow respectively, \( T_c = L/w_r \) – the propagation time constant for the pipe of length \( L \) under the so-called reduced steam velocity \( w_r \) – a rather constant reference under various steady states. The coefficient \( \psi_t = w_r/c_0 \) denotes the ratio of the aforementioned reduced velocity to the sound velocity at maximal flow; normally \( 0 < \psi_t < 1 \).

The boundary conditions (BCs) for (4) – in rated variables – are as follows

\[
\begin{align*}
\xi_w(0,t) &= \pi_i(t) \Phi(\pi_i(t)/\xi_p(0,t)) \\
\xi_w(1,t) &= \psi_t \xi_p(1,t)
\end{align*}
\]

Here also some explanation are necessary and useful. The variable \( \pi_i(t) \) is a rated pressure of the steam delivered by the steam turbine at the regulated steam extraction. The flow being sub-critical but also critical (especially during the transients), there was used the Saint Venant formula, with the Saint Venant function given by

\[
\Phi(x) = \begin{cases} 
(1/x)\sqrt{2\ln x}, & 1 \leq x \leq \sqrt{e} \\
1/\sqrt{e}, & x \geq \sqrt{e}
\end{cases}
\]

At the steam consumer (\( \lambda = 1 \)) the flow is critical and \( \psi_t \) accounts for a coefficient that describes the consumer. We add to the aforementioned equations (4)–(6) the equations describing the steam turbine dynamics

\[
\begin{align*}
T_s \frac{ds}{dt} &= \alpha \pi_1 + (1 - \alpha) \pi_2 - v_g, \quad 0 < \alpha < 1 \\
T_{\mu_1} \frac{d\mu_1}{dt} &= \mu_1 - \pi_1, \quad 0 \leq \mu_1 \leq 1 \\
T_{\mu_2} \frac{d\mu_2}{dt} &= \pi_1 - \beta \mu_2 \pi_2 - (1 - \beta \delta) \xi_w(0,t) \\
T_{\mu_2} \frac{d\mu_2}{dt} &= \mu_2 \pi_2 - \pi_2, \quad 0 < \mu_{2\text{min}} \leq \mu_2 \leq 1.
\end{align*}
\]

where \( s \) is the rated rotating speed deviation with respect to the synchronous speed, \( v_g \) is the mechanical load, \( \pi_1 \) and \( \pi_2 \) are the steam pressures in the high pressure (HP) and low pressure (LP) turbine cylinders respectively and \( \mu_1, \mu_2 \) are the control signals (rated values).

Looking at (5) – the first equation – and (7) – the third equation – one can observe some kind of internal feedback. As pointed out in Neymark (1978) an internal feedback may be the source of some instability. On the other hand, this feedback shows that (5) and (7) have to be considered together. Consequently, equations (4) – (5) will define a boundary value problem for a system of conservation laws with some boundary conditions that are both non-standard and nonlinear.

Under these circumstances the so-called augmented model validation Răsvan (2014) that integrates existence of positive evolutions and inherent stability of equilibria (Stability Postulate of Četaev (1931, 1936a,b)) to the usual well posedness in the sense of Hadamard appears as both necessary and useful.

An additional remark concerns the small parameters. In the pioneering papers Sokolov (1946), Kabakov (1946), Kabakov and Sokolov (1946) some terms in (2) had been neglected from the beginning and the conservation law character of the equations had been lost (the PDEs had become linear). Under the rated variables in (4), one can observe the parameter \( \psi_t \), which might be small. Under the data of Kabakov and Sokolov (1946) \( \psi_t \approx 0.1 \) hence \( \psi_t^2 \approx 0.01 \) and the term containing it might be neglected. But in this case also \( \psi_t^2 T_c \) has to be considered as small (while we do not know very well what are singular perturbations for such systems of partial differential equations); therefore \( \xi_p(\lambda,t) \approx \xi_p(t) \) – it is independent of \( \lambda \). We integrate the first equation of (4) from 0 to 1 and take into account the boundary conditions to find

\[
T_c \frac{d\xi_p}{dt} + \psi_t \xi_p - \pi_1 \Phi(\pi_1/\xi_p) = 0
\]

to obtain, together with (7), a system of nonlinear ordinary differential equations.

For this system existence, uniqueness and data dependence have known standard issues. The proof of the existence of positive solutions goes as in Răsvan (1981) while the inherent stability property requires association of a Liapunov function. In the bilinear case – with \( \Phi(\cdot) \) being a constant function – the aforementioned problem was considered in Răsvan (1984). Also, in Danciu et al. (2015) a numerical solution is obtained by means of a computational procedure based on a “convergent” Method of Lines combined with the paradigm of cell-based neural networks.

2. THE FUNCTIONAL EQUATIONS OF THE CONTROLLED HYDRAULIC CHANNEL.

Among the approaches that are currently used in order to tackle the boundary value problems such as (1) or (4)–(7) is the integration of the Riemann invariants of the PDEs along the characteristics. The most general cases of this approach were considered in the papers of A.D. Myshkis and his co-workers Myshkis and Shilopak (1957), Abolinia and Myshkis (1960), Myshkis and Filimonov (1981, 2008). A simpler case, that of lossless/distorsionless systems can be found in the papers of Cooke and Krumme (1968), Cooke (1970) (the complete proof of their main result is to be found in Răsvan (2014). The specific feature of the aforementioned papers is that they deal with linear partial differential equations or with quasi-linear at most, while containing hints for iterative approach of the nonlinear case.
The conservation laws are highly nonlinear partial differential equations: they can be approached either by linearization or by other methods that can avoid this linearization. We shall refer for these techniques to Petre and Răsvan (2009), Răsvan (2015) for the hydraulic and for the thermal power case respectively.

Consider first the equations (1) of the controlled hydraulic channel. The partial differential equations can be written as

\[
\frac{\partial}{\partial t} \left( \frac{H}{Q} \right) + \left( \frac{0}{gH - \frac{Q^2}{H^2}} \right) \frac{\partial}{\partial x} \left( \frac{H}{Q} \right) = 0
\]

for classical solutions at least. The eigenvalues of the system matrix are

\[
\lambda_{1,2} = \frac{Q}{H} \pm \sqrt{gH}
\]

and if the fluiviality assumption \( Q < H\sqrt{gH} \) holds then

\[
\frac{Q}{H} - \sqrt{gH} < 0 < \frac{Q}{H} + \sqrt{gH}
\]

and the system is hyperbolic. The matrix can be diagonalized as follows

\[
\left[ \left( \frac{\sqrt{gH} \pm \frac{Q}{H}}{\sqrt{gH}} \right) \frac{\partial H}{\partial t} \pm \frac{\partial Q}{\partial t} + \left( \frac{\sqrt{gH} \pm \frac{Q}{H}}{\sqrt{gH}} \right) \frac{\partial H}{\partial x} \pm \frac{\partial Q}{\partial x} \right] = 0
\]

Solving a differential equation with exact differential by finding an integrating factor, the Riemann invariants are obtained:

\[
\xi^\pm(x,t) = 2\sqrt{gH} \pm \frac{Q}{H}
\]

The integrating factor being \( 1/H \), (12) becomes

\[
\frac{\partial}{\partial t} \xi^\pm(x,t) \pm \left( \sqrt{gH} \pm \frac{Q}{H} \right) \frac{\partial}{\partial x} \xi^\pm(x,t) = 0
\]

or

\[
\frac{\partial}{\partial t} \xi^+(x,t) + \left[ 3\xi^+(x,t) - \xi^-(x,t) \right] \frac{\partial}{\partial x} \xi^+(x,t) = 0 \quad \text{(14)}
\]

\[
\frac{\partial}{\partial t} \xi^-(x,t) - \left[ 3\xi^-(x,t) - \xi^+(x,t) \right] \frac{\partial}{\partial x} \xi^-(x,t) = 0
\]

hence the partial differential equations are expressed in Riemann invariants only. The boundary conditions become

\[
\left( \xi^+(0,t) - \xi^-(0,t) \right) \left( \xi^+(0,t) + \xi^-(0,t) \right)^2 = 32gQ_0(t)
\]

\[
\left( \xi^+(L,t) - \xi^-(L,t) \right) \left( \xi^+(L,t) + \xi^-(L,t) \right)^2 = 4g^{3/4} \left[ \left( \xi^+(L,t) + \xi^-(L,t) \right)^2 - 4\sqrt{g\mu}(t) \right]^{3/2}
\]

We shall now illustrate the integration along the characteristics Răsvan (2014). The problem is here nonlinear, hence all results will have a local character. Therefore let \((\tilde{H}(x,t), \tilde{Q}(x,t))\) be a solution of (1) defined by some given \( u(t) \) and by the initial conditions \( \tilde{H}(x,0) = H_0(x) \), \( \tilde{Q}(x,0) = Q_0(x) \), defined on \([0,L]\) as sufficiently smooth functions. We assume also \( H_0(x) > 0 \), \( Q_0(x) > 0 \) and such that (11) holds for these functions i.e.

\[
\frac{Q_0(x)}{H_0(x)} - \sqrt{gH_0(x)} < 0 < \frac{Q_0(x)}{H_0(x)} + \sqrt{gH_0(x)}
\]

If \((\tilde{H}(x,t), \tilde{Q}(x,t))\) is a continuous classical solution, then (17) holds for these functions provided \( t > 0 \) is small enough. Based on (13) we can stress that

\[
\tilde{\xi}^+(x,t) = \tilde{\xi}^+(x,0) > 0; \quad \tilde{\xi}^-\tilde{\xi}^+(x,t) = \tilde{\xi}^-(x,0) > 0
\]

and \( \tilde{\xi}^+(x,t) > 0 \) provided \( t > 0 \) is small enough. Also, for \( t > 0 \) small enough we shall have

\[
3\tilde{\xi}^\pm(x,t) - \tilde{\xi}^\mp(x,t) > 0
\]

In order to emphasize the role of the initial conditions we make further notations

\[
\tilde{\xi}^\pm(x,t) := \tilde{\xi}^\pm(\tilde{\xi}^\mp(\cdot);x,t)
\]

The characteristics associated to this solution are defined by

\[
\frac{dr}{dx} = \pm \frac{1}{3\tilde{\xi}^\pm(\tilde{\xi}^\mp(\cdot);x,t) - \tilde{\xi}^\mp(\tilde{\xi}^\mp(\cdot);x,t)}
\]

where \( 3\tilde{\xi}^\pm(\tilde{\xi}^\mp(\cdot);x,t) - \tilde{\xi}^\mp(\tilde{\xi}^\mp(\cdot);x,t) > 0 \) as a consequence of (17). For each \( x \) within the considered domain \( 0 < x < L \) and \( t > 0 \) sufficiently small) system (21) is defining an increasing characteristic curve \( t^+(\sigma;x,t) \) and a decreasing one \( t^-\sigma;x,t; \). Consequently, \( t^+(\cdot;x,t) \) can be extended “to the right” up to \( x = L \) while \( t^-\sigma;x,t; \) can be extended “to the left” up to \( x = 0 \). We write down the forward wave along the increasing characteristic and the backward wave along the decreasing characteristic

\[
\Phi^\pm(\sigma;x,t) := \tilde{\xi}^\pm(\sigma;\xi^+(\cdot;x,t))
\]

The functions \( \Phi^\pm(\cdot;x,t) \) are constant along the characteristics hence

\[
\Phi^+(x;x,t) = \Phi^+(L;x,t)
\]

\[
\Rightarrow \tilde{\xi}^+(x,t) = \tilde{\xi}^+(L,t;x,t)
\]

\[
\Phi^-\sigma;x,t) = \Phi^-(0;x,t)
\]

\[
\Rightarrow \tilde{\xi}^-\sigma;x,t; = \tilde{\xi}^-\sigma;0;x,t;)
\]

In the cases when the increasing characteristic curve can be extended also backwards up to \( x = 0 \) and the decreasing one forwards up to \( x = L \) we can obtain from (23)

\[
\tilde{\xi}^+(0,t) = \tilde{\xi}^+(L,t^+(L,0,t))
\]

\[
\tilde{\xi}^-\sigma;0; = \tilde{\xi}^-\sigma;0;x,t;)
\]

We can now define the propagation times of the two waves
Here we made use of (20) to emphasize the dependence of the propagation times on the initial conditions. Now, we re-write (24) as

$$T^+(\xi_0^+ (\cdot), t) = t^+(L; 0, t) - t$$

$$T^-(\xi_0^+ (\cdot), t) = t^-(0; L, t) - t$$  \hspace{1cm} (25)$$

and introduce the functions $\tilde{\xi}^\pm(t)$ by

$$\tilde{\xi}^+(0, t) = \tilde{\xi}^+(L, t + T^+), \quad \tilde{\xi}^-(L, t) = \tilde{\xi}^-(0; t + T^-)$$  \hspace{1cm} (26)$$

If the boundary conditions (16) are taken into account, it is found that $\tilde{\xi}^\pm(t)$ are subject to the nonlinear difference equations

$$\frac{(\tilde{\xi}^+(t + T^+) - \tilde{\xi}^-(t)) (\tilde{\xi}^+(t + T^+) + \tilde{\xi}^-(t))^2}{(\tilde{\xi}^+(t) - \tilde{\xi}^-(t + T^+)) (\tilde{\xi}^+(t) + \tilde{\xi}^-(t + T^-))^2} = 32gQ_d(t)$$

$$= 4g^{3/4} \gamma \left[ (\tilde{\xi}^+(t) + \tilde{\xi}^-(t + T^-))^2 - 4\sqrt{g\mu(t)} \right]^{3/2}$$ \hspace{1cm} (27)$$

with time varying $T^\pm$ also dependent on the initial conditions of the considered solution of (15) – (16). The initial conditions for (28) can be determined by considering those characteristics which cannot be extended up to $x = 0$ (the increasing one) or to $x = L$ (the decreasing one) because they cross the abscissa of the definition rectangle $[0, L] \times [0, \tilde{t})$ before reaching the verticals $x = 0$ or $x = L$.

The problem is now to construct by steps the solution of (28) where the time delays $T^\pm(t)$ are time dependent (the initial conditions $\xi_0(\cdot)$ are fixed for a given solution). Moreover, the difference equations are non-linear.

An additional problem would be to obtain from the construction by steps positiveness of the solutions, knowing that the initial conditions were assumed to be positive.

To end this section we send again to Petre and Răsvan (2009) where the same problem has been discussed under more general assumptions on the hydraulics. The variables in the conservation laws had been chosen the flow and the cross-section area. The associated functional differential equations were somehow more tractable. The explanation is the nonlinear dependence of cross-section area versus water level.

The aforementioned paper as well as the results of this section deal nevertheless with nonlinear models unlike almost all cited literature where the models are linearized around a steady state.

Two final remarks are useful here. First, the control is a boundary control and by control synthesis it is possible to linearize the controlled boundary condition. Consequently, the associated difference equations may be simpler. The second remark is that the aforementioned control synthesis based e.g. on the energy integral identity can be accomplished at the formal level. The mathematical background can be proved a posteriori for the closed loop system.

### 3. THE FUNCTIONAL DIFFERENTIAL EQUATIONS ASSOCIATED TO THE CO-GENERATION MODEL

We shall consider here the co-generation model defined by (3) – (7) including a steam turbine with one regulated steam pipe with distributed parameters connecting the turbine steam extraction with the steam consumer. This model is highly nonlinear: the partial differential equations are nonlinear systems of conservation laws for the isentropic steam, the boundary conditions at the steam extraction chamber is nonlinear since the steam flow is sub-critical subject to the Saint Venant non-linear dependence (6) and the turbine model is bilinear since the controlled flow within the turbine is critical.

Since previous studies dealt with linearized model only, we shall consider the nonlinear conservation laws (4) and look for the Riemann invariants. We write down these equations in a vector-matrix form (assuming the solutions to be smooth enough)

$$\frac{\partial}{\partial t} \begin{pmatrix} \xi_p \\ \xi_w \end{pmatrix} + \frac{1}{T_c} \begin{pmatrix} \frac{1}{\psi_c} & \frac{1}{\xi_p} \\ 2 & \frac{1}{\xi_w} \end{pmatrix} \frac{\partial}{\partial \lambda} \begin{pmatrix} \xi_p \\ \xi_w \end{pmatrix} = 0$$ \hspace{1cm} (29)$$

The eigenvalues of the matrix in (29) are

$$\lambda_{1,2} = \frac{\xi_w}{\xi_p} \pm \frac{1}{\psi_c}$$ \hspace{1cm} (30)$$

and if we want $\lambda_{\min} < 0 < \lambda_{\max}$, some kind of fluviality assumption is also required here. We compute the diagonalization matrix and obtain the equations

$$\begin{pmatrix} \frac{1}{\psi_c} \mp \frac{\xi_w}{\xi_p} \\ \xi_w \end{pmatrix} \frac{\partial \xi_w}{\partial t} \pm \frac{\partial \xi_w}{\partial \lambda}$$

$$+ \frac{1}{T_c} \begin{pmatrix} \frac{1}{\psi_c} & \frac{1}{\xi_p} \\ 2 & \frac{1}{\xi_w} \end{pmatrix} \begin{pmatrix} \frac{1}{\psi_c} & \frac{1}{\xi_p} \\ 2 & \frac{1}{\xi_w} \end{pmatrix} \begin{pmatrix} \frac{\partial \xi_p}{\partial \lambda} \\ \frac{\partial \xi_w}{\partial \lambda} \end{pmatrix} = 0.$$ \hspace{1cm} (31)$$

Computing the integrating factor we find the Riemann invariants

$$\xi^\pm(\xi_p, \xi_w) = \frac{1}{\psi_c} \ln \xi_p \pm \frac{\xi_w}{\xi_p}$$ \hspace{1cm} (32)$$

deducing that

$$\frac{\xi_w}{\xi_p} = \frac{1}{T_c} (\xi^+ - \xi^-).$$ \hspace{1cm} (33)$$

Therefore, the Riemann invariants are subject to the equations

$$\frac{\partial \xi^+}{\partial t} + \frac{1}{2T_c} \left( \xi^+ - \xi^+ \right) + \frac{1}{\psi_c} \frac{\partial \xi^-}{\partial \lambda} = 0$$

$$\frac{\partial \xi^-}{\partial t} + \frac{1}{2T_c} \left( \xi^- - \xi^- \right) - \frac{1}{\psi_c} \frac{\partial \xi^-}{\partial \lambda} = 0$$ \hspace{1cm} (34)$$

and the boundary conditions
These equations have to be coupled to (7) by substituting \( \xi_w(0, t) \) from the boundary conditions. However, in the critical flow case it will result \( \xi_w(0, t) = (1/\sqrt{e}) \pi_s(t) \) hence the turbine differential equations will be decoupled from (41) that might be discussed independently, starting form a construction by steps. Equations (41) can be given the form

\[
\tilde{y} + (t + T^+) e^{(\psi_s/2)\tilde{y} + (t+T^+)} = \tilde{y}^-(t) + \frac{1}{\sqrt{e}} \pi_s(t) e^{-(\psi_s/2)\tilde{y}^-(t)} \tag{42}
\]

We have to take into account however that the propagation times are time dependent also solution dependent.

4. SOME CONCLUSIONS AND HINTS ON FUTURE RESEARCH

We have sketched in this paper some connection between the description by conservation laws of the controlled systems with distributed parameters and some functional equations associated by integration of the Riemann invariants along the characteristics. It is a common knowledge fact that for linear hyperbolic partial differential equations this association of functional equations by integration along the characteristics represents the most natural way of introducing equations with deviated arguments of retarded, neutral, even advanced type. The neutral equations (including difference equations with continuous time) are most frequent, as a consequence of the character of the boundary conditions; moreover, if these boundary conditions are nonlinear, the resulting equations are also nonlinear. The systems of conservation laws are nonlinear hyperbolic partial differential equations and display such phenomena as shock and rarefaction waves. Integration along the characteristics in this case leaves aside generalized solutions = only classical, at most discontinuous solutions are tractable by this method and the characteristics themselves are solution dependent, the deviating argument of the functional equations being thus time varying. Since these functional equations appear also as highly nonlinear – see e.g. the applications tackled in the present paper – their mathematical tractability appears as quite doubtful. Consequently we stress that the basic theory for model validation (well posedness in the sense of Hadamard) would be better served by the standard results of the field, incorporated in some widely cited monographs - see Bressan (2000), Dafermos (2010), Godunov and Romenskii (2003), Lax (1987, 2006), Li (1994), Li and Wang (2009), Rozhdestvensky and Yanenko (1983), Serre (2000).

We have to add here a remark made throughout the paper concerning the approach to be taken in control synthesis: one may proceed in a formal (mathematically speaking) way in order to synthesize the controller and obtain the structure and the model of the closed loop system. In this way some subsystems might result simplified e.g. linearized – for instance the boundary conditions since the control is mainly boundary control. The mathematical theory can thus start afterwards and concern the closed loop system (where obviously stability is a fundamental issue due to the feedback structure).

Finally a remark about numerics: it is useful to obtain a sound mathematical basis for the Method of Lines (MoL) for conservation laws Danciu et al. (2015) in the nonlinear case. At their...
turn, the numerical results can be useful to see if consideration of nonlinear partial differential equations makes indeed a significant difference with respect to the linearized versions.

Summarizing – quite a program to follow.

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