Unit Interval Orders and the Dot Action on the Cohomology of Regular Semisimple Hessenberg Varieties

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Abstract
Motivated by a 1993 conjecture of Stanley and Stembridge, Shareshian and Wachs conjectured that the characteristic map takes the dot action of the symmetric group on the cohomology of a regular semisimple Hessenberg variety to $\omega X_G(t)$, where $X_G(t)$ is the chromatic quasisymmetric function of the incomparability graph $G$ of the corresponding natural unit interval order, and $\omega$ is the usual involution on symmetric functions. We prove the Shareshian–Wachs conjecture. Our proof uses the local invariant cycle theorem of Beilinson–Bernstein–Deligne to obtain a surjection from the cohomology of a regular Hessenberg variety of Jordan type $\lambda$ to a space of local invariant cycles; as $\lambda$ ranges over all partitions, these spaces collectively contain all the information about the dot action on a regular semisimple Hessenberg variety. Using a palindromicity argument, we show that in our case the surjections are actually isomorphisms, thus reducing the Shareshian–Wachs conjecture to computing the cohomology of a regular Hessenberg variety. But this cohomology has already been described combinatorially by Tymoczko; we give a bijective proof (using a generalization of a combinatorial reciprocity theorem of Chow) that Tymoczko’s combinatorial description coincides with the combinatorics of the chromatic quasisymmetric function.

1 Introduction

Let $G$ be the incomparability graph of a unit interval order (also known as an indifference graph), i.e., a finite graph whose vertices are closed unit intervals on
the real line, and whose edges join overlapping unit intervals. It is a longstanding conjecture [29] related to various deep conjectures about immanants that if \( G \) is such a graph, then the so-called chromatic symmetric function \( X_G \) studied by Stanley [27] is \( e \)-positive, i.e., a nonnegative combination of elementary symmetric functions. (In fact, Stanley and Stembridge conjectured something seemingly more general, but Guay-Paquet [10] has reduced their conjecture to the one stated here.) Early on, Haiman [11] proved that the expansion of \( X_G \) in terms of Schur functions has nonnegative coefficients, and Gasharov [7] showed that these coefficients enumerate certain combinatorial objects known as P-tableaux. It is well known that if \( \chi \) is a character of the symmetric group \( S_n \),

\[
\text{ch} \chi := \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) \ p_{\text{cycletype}(\sigma)}
\]  

(1)

(where \( p \) here denotes the power-sum symmetric function) is a nonnegative linear combination of Schur functions, with the coefficients giving the multiplicities of the corresponding irreducible characters of \( S_n \). One may therefore suspect that \( X_G \) is the image under \( \text{ch} \) of the character of some naturally occurring representation of \( S_n \), but until recently, there was no candidate, even conjecturally, for such a representation.

Meanwhile, independently and seemingly unrelatedly, De Mari, Procesi, and Shayman [5] inaugurated the study of Hessenberg varieties. Let \( m = (m_1, m_2, \ldots, m_{n-1}) \) be a monotonically increasing sequence of positive integers satisfying \( i \leq m_i \leq n \) for all \( i \), and let \( s : \mathbb{C}^n \to \mathbb{C}^n \) be a linear transformation. The (type A) Hessenberg variety \( \mathcal{H}(m, s) \) is defined by

\[
\mathcal{H}(m, s) := \{ \text{complete flags } F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n : sF_i \subseteq F_{m_i} \text{ for all } i \}.
\]  

(2)

The geometry of a Hessenberg variety depends on the Jordan form of \( s \). If the Jordan blocks have distinct eigenvalues then we say that \( \mathcal{H}(m, s) \) is regular, and if \( s \) is diagonalizable then we say that \( \mathcal{H}(m, s) \) is semisimple. Hessenberg varieties have many interesting properties, but of particular interest to us is that there is a representation, called the dot action, of \( S_n \) on the cohomology of regular semisimple Hessenberg varieties. To the best of our knowledge, this dot action was first defined by Tymoczko, who asked for a complete description of it [32], e.g., one can ask if there is a combinatorial formula for the multiplicities of the irreducible representations and/or for the character values.

A connection between these two apparently unrelated topics has been conjectured by Shareshian and Wachs [24, 23]. Motivated by the \( e \)-positivity conjecture, they have generalized \( X_G \) to something they call the chromatic quasisymmetric function \( X_G(t) \) of a graph, which is a polynomial in \( t \) with power series coefficients that reduces to \( X_G \) when \( t = 1 \). They also noted that if we are given a sequence \( m \) as above, and we let \( G(m) \) be the undirected graph on the vertex set \( \{1, 2, \ldots, n\} \) such that \( i \) and \( j \) are adjacent if \( i < j \leq m_i \), then \( G(m) \) is an indifference graph, and moreover that every indifference graph is
isomorphic to some $G(\mathfrak{m})$. They then made the following conjecture. Let $\omega$ denote the usual involution on symmetric functions [28, Section 7.6].

**Conjecture 3.** If $\chi_{\mathfrak{m},d}$ denotes the dot action on the cohomology group $H^{2d}$ of the regular semisimple Hessenberg variety $\mathcal{H}(\mathfrak{m},s)$, then $\text{ch} \chi_{\mathfrak{m},d}$ equals the coefficient of $t^d$ in $\omega X_{G(\mathfrak{m})}(t)$.

This conjecture is intriguing not only because it would answer Tymoczko’s question, but it would open up the possibility of proving the $e$-positivity conjecture by geometric techniques.

The main result of the present paper is a proof of Conjecture 3 (Theorem 78). The linchpin of our proof is the following result (which is stated more formally later as Theorem 76).

**Theorem 4.** Let $\lambda$ be a partition of $n$. Let $s$ be a regular element with Jordan type $\lambda$, and let $S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}$ be a Young subgroup of the symmetric group $S_n$. Consider the restriction of $\chi_{\mathfrak{m},d}$ to $S_\lambda$. Then the dimension of the subspace fixed by $S_\lambda$ equals the Betti number $\beta_{2d}$ of $H(\mathfrak{m},s)$.

What Theorem 4 does is to reduce the problem of computing the dot action on a regular semisimple Hessenberg variety to computing the cohomology of regular (but not necessarily semisimple) Hessenberg varieties. However, this latter task has already been largely carried out by Tymoczko, who has given a combinatorial description of the Betti numbers $\beta_{2d}$ for all Hessenberg varieties in type A. So with Theorem 4 in hand, all that remains to prove Conjecture 3 is to give a bijection between Tymoczko’s combinatorial description and the combinatorics of $\omega X_{G(\mathfrak{m})}(t)$. More precisely, it is a standard fact (Proposition 8 below) from the representation theory of $S_n$ that the dimension of the subspace fixed by $S_\lambda$ in a representation $\chi$ is the coefficient of $m_\lambda$ in the monomial symmetric function expansion of $\text{ch} \chi$. So the first step of our proof is to compute the coefficients $c_{d,\lambda}(\mathfrak{m})$ of $t^d m_\lambda$ in the monomial symmetric function expansion of $\omega X_{G(\mathfrak{m})}(t)$. We do this with a generalization of a combinatorial reciprocity theorem of Chow (Theorem 26). This yields a description of $c_{d,\lambda}(\mathfrak{m})$ that is almost, but not quite, identical to Tymoczko’s description of $\beta_{2d}$; we show that the descriptions are equivalent by describing an explicit bijection between the two (Theorem 32). As a corollary (Corollary 33), we derive the fact that the Betti numbers of regular Hessenberg varieties form a palindromic sequence (even though the varieties are not smooth), because Shareshian and Wachs have proved that $\omega X_{G(\mathfrak{m})}(t)$ is palindromic.

The idea behind the proof of Theorem 4 is to show that Tymoczko’s dot action coincides with the monodromy action for the family $\mathcal{H}^{rs}(\mathfrak{m}) \to \mathfrak{g}^{rs}$ of Hessenberg varieties over the space of regular semisimple $n \times n$ matrices (Theorem 74). This allows us to apply results from the theory of local systems and perverse sheaves to questions involving the dot action. In particular, the local invariant cycle theorem of Beilinson–Bernstein–Deligne, which is stated in our context as Theorem 10, implies that there is a surjective map from the cohomology of a regular Hessenberg variety to the space of local invariants of the
monodromy action near a regular element \( s \) in the space \( \mathfrak{g} \) of all \( n \times n \)-matrices. In Theorem 48 we prove a general theorem showing that the local invariant cycle map is an isomorphism if and only if the Betti numbers of the special fiber are palindromic in a suitable sense. Then, in Theorem 75 we show that the local invariant cycles near a regular element \( s \) with Jordan type \( \lambda \) coincide with \( S_\lambda \) invariants of the dot action on the regular semisimple Hessenberg variety. The latter fact is proved by a monodromy argument that uses the Kostant section. (This Kostant section argument and some ingredients of the proof of Theorem 74 were inspired by Ngô’s paper on the Hitchin fibration [19].)

Here is a brief description of the contents by section. Section 2 mainly fixes notation and gives preliminary results. Section 3 proves the combinatorial reciprocity theorem, Theorem 26 mentioned above. Section 4 proves Theorem 32 on the Betti numbers of regular Hessenberg varieties, and derives palindromicity as a corollary of a theorem of Shareshian and Wachs. Section 5 reviews the concept of local monodromy and the related notion of a good fundamental system of neighborhoods to a point in topological space. Section 6 proves Theorem 48 on the local invariant cycle map. Section 7 proves Proposition 57 on local monodromy of a Galois cover, which is applied later (in Lemma 62) to compute the local monodromy near a matrix \( s \) of type \( \lambda \). Section 8 introduces the family \( \mathcal{H}(m) \rightarrow \mathfrak{g} \) of Hessenberg varieties. Finally, Section 9 shows that the monodromy action coincides with Tymoczko’s dot action, and uses this fact to prove Theorem 78 which is a restatement of Conjecture 3.

1.1 Previous work and acknowledgments

Prior to our work, Conjecture 3 was already known for some graphs \( G \): a complete graph (trivial), a complete graph minus an edge [30], a complete graph minus a path of length three (Tymoczko, unpublished), and a path (by piecing together known results as explained in [23]). Teff also showed that it would suffice to prove the conjecture for all connected graphs \( G \). In a different direction, Abe, Harada, Horiguchi and Masuda (personal communication) proved that the multiplicity of the trivial representation is indeed as predicted by Conjecture 3 (Hearing about this development and reading the last paragraph of [11], which explains how to compute the multiplicity of the trivial representation in terms of the regular nilpotent Hessenberg variety, partially inspired our own proof.) They also computed the ring structure on regular semisimple Hessenberg varieties of type \((m_1, n, \ldots, n)\), and deduced Conjecture 3 in that case from the computation.

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2 Preliminaries

We fix some notation that will be used throughout the paper.

2.1 General notation

We let \( \mathbb{P} \) denote the positive integers. If \( n \in \mathbb{P} \), we let \([n]\) denote the set \( \{1, 2, \ldots, n\} \).

The vector \( \mathbf{m} = (m_1, \ldots, m_{n-1}) \) will always denote a Hessenberg function, by which we mean a sequence of positive integers satisfying

1. \( m_1 \leq m_2 \leq \cdots \leq m_{n-1} \leq n \), and
2. \( m_i \geq i \) for all \( i \).

We also define

\[
|\mathbf{m}| := \sum_{i=1}^{n-1} (m_i - i). \tag{5}
\]

Given \( \mathbf{m} \), let \( P(\mathbf{m}) \) denote the poset on the vertex set \([n]\) whose order relation \( \prec \) is given by

\[
i \prec j \iff j \in \{m_i + 1, m_i + 2, \ldots, n\}.
\]

Such a poset is called a natural unit interval order. The incomparability graph \( G(\mathbf{m}) \) is the undirected graph on the vertex set \([n]\) in which \( i \) and \( j \) are adjacent if and only if \( i \) and \( j \) are incomparable in \( P(\mathbf{m}) \). In other words, if \( i < j \) then \( i \) and \( j \) are adjacent in \( G(\mathbf{m}) \) if and only if \( j \leq m_i \).

An integer partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) of a positive integer \( n \) is a monotonically decreasing sequence of positive integers that sum to \( n \). Each \( \lambda_i \) is a part of \( \lambda \), and the number of parts of \( \lambda \) is denoted by \( \ell(\lambda) \). The Young diagram of \( \lambda \) comprises \( \ell \) rows of boxes, left-justified, with \( \lambda_i \) boxes in the \( i \)th row from the top.

A composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \) of a positive integer \( n \) is a sequence (not necessarily monotonic) of positive integers that sum to \( n \). Each \( \alpha_i \) is a part of \( \alpha \), and the number of parts of \( \alpha \) is denoted by \( \ell(\alpha) \). It can be useful to visualize a composition of \( n \) by drawing vertical bars in some subset of the \( n - 1 \) spaces between consecutive objects in a horizontal line of \( n \) objects; the parts are then the numbers of objects between successive bars. Motivated by the equivalence between compositions and sets of bars, we define:

- \( |\alpha| \) for the number of bars of \( \alpha \);
- \( \overline{\alpha} \) for the composition that has bars in precisely the positions where \( \alpha \) does not have bars;
- \( \alpha \cup \beta \) for the composition whose bars comprise the union of the bars of \( \alpha \) and the bars of \( \beta \); and
- \( \alpha \leq \beta \) if the bars of \( \alpha \) are a subset of the bars of \( \beta \).
We write $S_n$ for the symmetric group. If $S_n$ acts in the usual way on a set of size $n$, and $\alpha$ is a composition of $n$, then the Young subgroup $S_\alpha$ is the subgroup

$$S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_\ell} \subseteq S_n$$

comprising all the permutations that permute the first $\alpha_1$ elements among themselves, the next $\alpha_2$ elements among themselves, and so on.

An ordered (set) partition $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_\ell)$ of a finite set $S$ is a sequence of pairwise disjoint subsets of $S$ whose union is $S$.

A sequencing $q$ of a finite set $S$ of cardinality $n$ is a bijective map $q : [n] \rightarrow S$. It is helpful to think of $q$ as the sequence $q(1), \ldots, q(n)$ of elements of $S$.

By a digraph we mean a finite directed graph with no loops or multiple edges but that may have bidirected edges, i.e., it may contain both $u \rightarrow v$ and $v \rightarrow u$ simultaneously. If $D$ is a digraph, we write $\overline{D}$ for the complement of $D$, i.e., the digraph with the same vertex set as $D$ but with a directed edge $u \rightarrow v$ if and only if there does not exist a directed edge $u \rightarrow v$ in $D$.

2.2 Symmetric and quasisymmetric functions

We mostly follow the notation of Stanley [28] for symmetric functions. For convenience, we recall some of that notation here. Let $x = \{x_1, x_2, x_3, \ldots\}$ be a countable set of independent indeterminates. If $\kappa : [n] \rightarrow \mathbb{P}$ is a map then we write $x_\kappa$ for the monomial $x_{\kappa(1)}x_{\kappa(2)}\cdots x_{\kappa(n)}$. A formal power series in $x$ is a symmetric function if it is invariant under any permutation of the variables $x$.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is an integer partition, then the monomial symmetric function $m_\lambda$ is the symmetric function of minimal support that contains the monomial $x_{\lambda_1}^1x_{\lambda_2}^2\cdots x_{\lambda_\ell}^\ell$. For example,

$$m_{2,1,1} = x_1^2x_2x_3 + x_2^2x_1x_3 + x_3^2x_1x_2 + x_1^2x_3x_4 + x_3^2x_1x_4 + x_4^2x_1x_3 + \cdots$$

There is an unfortunate conflict between our notation for monomial symmetric functions and our notation $m$ for Hessenberg functions. It should be clear from context which is meant since the subscript of a monomial symmetric function is a partition, whereas the entries of $m$ have integer subscripts.

The characteristic map $\text{ch}$ is a function that sends characters $\chi$ of the symmetric group to symmetric functions via the formula

$$\text{ch}\chi := \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) p_{\text{cycletype}(\sigma)}$$

where cycletype($\sigma$) is the integer partition consisting of the cycle sizes of $\sigma$, listed with multiplicity in monotonically decreasing order, and $p$ denotes the powersum symmetric function. As we explained in the introduction, the following standard fact is an important ingredient in our proof.

**Proposition 8.** Let $\rho$ be a finite-dimensional representation of $S_n$, and let $\chi$ be its character. Let $\text{ch}\chi = \sum_\lambda c_\lambda m_\lambda$ be the monomial symmetric function expansion of $\text{ch}\chi$. Then $c_\lambda$ equals the dimension of the subspace fixed by any Young subgroup $S_\lambda \subseteq S_n$. In particular, knowing $c_\lambda$ for all $\lambda$ uniquely determines $\chi$. 


Proof. Let \(\langle \cdot, \cdot \rangle\) denote the standard inner product on symmetric functions [28, Section 7.9], and let \(\chi^{S_n}_{S_\lambda}\) denote the restriction of \(\chi\) to \(S_\lambda\). Let \(d_\lambda\) be the dimension of the subspace fixed by \(S_\lambda\). Then \(d_\lambda\) equals the multiplicity of the trivial representation \(1\) in \(\chi^{S_n}_{S_\lambda}\), i.e., \(d_\lambda = \langle 1, \chi^{S_n}_{S_\lambda} \rangle\). By Frobenius reciprocity [22, Theorem 1.12.6],

\[\langle 1, \chi^{S_n}_{S_\lambda} \rangle = \langle 1^{S_n}_{S_\lambda}, \chi \rangle,\]  

(9)

where \(1^{S_n}_{S_\lambda}\) is the induction of \(1\) from \(S_\lambda\) up to \(S_n\). But \(1^{S_n}_{S_\lambda}\) is just the homogeneous symmetric function \(h_\lambda\) [28, Corollary 7.18.3]. The monomial symmetric functions and the complete homogeneous symmetric functions are dual bases [28, Equation (7.30)], so \(d_\lambda = \langle h_\lambda, \chi \rangle = c_\lambda\).

Let \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)\) be a composition of \(n\). The \textit{monomial quasisymmetric function} \(M_\alpha\) is the formal power series defined by

\[M_\alpha := \sum_{i_1 < \cdots < i_\ell} x_{i_1}^{\alpha_i} \cdots x_{i_\ell}^{\alpha_\ell},\]  

where the sum is over all strictly increasing sequences \((i_1, \ldots, i_\ell)\) of positive integers. A formal power series is a \textit{quasisymmetric function} if it is a scalar linear combination of monomial quasisymmetric functions. Note that symmetric functions are always quasisymmetric, but not vice versa.

The \textit{fundamental quasisymmetric function} \(F_\alpha\) of Gessel [8] is defined by

\[F_\alpha := \sum_{\beta \geq \alpha} M_\beta.\]  

By inclusion-exclusion,

\[M_\alpha = \sum_{\beta \geq \alpha} (-1)^{\beta - \alpha} F_\beta.\]  

(12)

2.3 Hessenberg varieties

As mentioned in the introduction, if \(m\) is a Hessenberg function and \(s : \mathbb{C}^n \rightarrow \mathbb{C}^n\) is a linear transformation, then we define the \textit{Hessenberg variety} (of type A, which is the only type that we consider in this paper) by

\[\mathcal{H}(m, s) := \{\text{complete flags } F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n : sF_i \subseteq F_m, \text{ for all } i\}.

If the Jordan blocks of \(s\) have distinct eigenvalues then we say that \(\mathcal{H}(m, s)\) is \textit{regular}, if \(s\) is diagonalizable then we say that \(\mathcal{H}(m, s)\) is \textit{semisimple}, and if \(s\) is nilpotent then we say that \(\mathcal{H}(m, s)\) is \textit{nilpotent}. Since \(\mathcal{H}(m, s)\) can equal \(\mathcal{H}(m, s')\) for \(s \neq s'\) (e.g., if \(s' - s\) is a constant), this is a (very minor) abuse of terminology. We adopt the convention of writing \(y\) for \(s\) in the regular semisimple case.
Remark 13. The Hessenberg varieties are defined on affine open subsets of the complete flag variety by fairly obvious equations. So they are closed subschemes of the complete flag variety in a natural way. In general, they are not irreducible. For example, the regular semisimple Hessenberg variety corresponding to the function $\ell = (1, 2, \ldots, n-1)$ is a collection of $n!$ distinct points. They are also not always reduced. For example, when $n = 2$ and $m = \ell$ as above, the regular nilpotent Hessenberg variety is defined by the equation $x^2 = 0$ in $\mathbb{A}^2$. (See [2, Theorem 7.6] for a much more general statement.) Hartshorne defines an abstract variety to be an integral separated scheme of finite type over an algebraically closed field [12, p.105]. So, perhaps, it is unfortunate that Hessenberg varieties are called varieties as they are not, in general, integral. However, it happens that we are only introduced in Betti cohomology in this paper. So the non-reduced structure will not play a role. Moreover, we will reserve the term “Hessenberg scheme” for the families discussed in §8. So we will stick with tradition and continue to call the schemes $\mathcal{H}(m, s)$ Hessenberg varieties.

3 The chromatic quasisymmetric function

Given a graph $G$ whose vertex set is a subset of $\mathbb{P}$, Shareshian and Wachs [23] define the chromatic quasisymmetric function $X_G(x, t)$ of $G$.

**Definition 14.** Let $G$ be a graph whose vertex set $V$ is a finite subset of $\mathbb{P}$. Let $C(G)$ denote the set of all proper colorings of $G$, i.e., the set of all maps $\kappa : V \to \mathbb{P}$ such that adjacent vertices are always mapped to distinct positive integers. Then

$$X_G(x, t) := \sum_{\kappa \in C(G)} t^{\text{asc}\kappa} x^\kappa,$$

(15)

where

$$\text{asc}\kappa := \left| \{ \{u, v\} : \{u, v\} \text{ is an edge of } G \text{ and } u < v \text{ and } \kappa(u) < \kappa(v) \} \right|.$$

For brevity, we sometimes write $X_G(t)$ for $X_G(x, t)$. It will be convenient for us to restate the definition of $X_G(t)$ in terms of monomial quasisymmetric functions.

**Proposition 16.** Let $G$ be a graph whose vertex set $V$ is a finite subset of $\mathbb{P}$. Then

$$X_G(x, t) = \sum_{\sigma \in \text{asc}\sigma} \ell^{\text{asc}\sigma} M_{|\sigma_1|, \ldots, |\sigma_1|},$$

(17)

where the sum is over all ordered partitions $\sigma$ of $V$ such that every $\sigma_i$ is a stable set of $G$ (i.e., there is no edge between any two vertices of $\sigma_i$), and $\text{asc}\sigma$ is the number of edges $\{u, v\}$ of $G$ such that $u < v$ and $v$ appears in a later part of $\sigma$ than $u$ does.

**Proof.** Given a coloring $\kappa \in C(G)$, let $\sigma_i$ be the set of vertices that are assigned the $i$th smallest color. Then it is immediate that
1. \( \sigma = (\sigma_1, \ldots, \sigma_\ell) \) is an ordered partition of the vertex set of \( G \);

2. \( \sigma_i \) is a stable set for all \( i \); and

3. \( \text{asc} \kappa = \text{asc} \sigma \).

It is easy to see that if we sum \( x_\kappa \) over all \( \kappa \in C(G) \) that yield the same ordered partition \( \sigma \), then we obtain the monomial quasisymmetric function \( M_\alpha \) where the \( i \)th part \( \alpha_i \) of the composition \( \alpha \) is the cardinality \( |\sigma_i| \) of \( \sigma_i \). The proposition follows. \( \Box \)

We remark that if we set \( t = 1 \) then the chromatic quasisymmetric function specializes to the chromatic symmetric function \( X_G \) of Stanley \[27\].

### 3.1 Reciprocity

If \( f \) is a symmetric function, then a “reciprocity theorem,” loosely speaking, is a result that gives a combinatorial interpretation of \( \omega f \), where \( \omega \) is a well-known involution on symmetric functions \[28\; \text{Section 7.6}\]. Since Conjecture 3 concerns \( \omega X_G(t) \) rather than \( X_G(t) \) itself, one might expect a reciprocity theorem to be relevant. This is indeed the case. Specifically, the coefficients of the monomial symmetric function expansion of \( \omega X_G(t) \) play an important role in our arguments, so we now introduce some notation for them.

**Definition 18.** Given a Hessenberg function \( m \), we let \( c_{d,\lambda}(m) \) be the coefficients defined by the following expansion of \( \omega X_G(m)(x, t) \) in terms of monomial symmetric functions:

\[
\omega X_G(m)(x, t) = \sum_d t^d \sum_\lambda c_{d,\lambda}(m)m_\lambda.
\] (19)

It is possible to derive a combinatorial interpretation for \( c_{d,\lambda}(m) \) by using the reciprocity theorem of Shareshian and Wachs \[23\; \text{Theorem 3.1}\]. However, as we now explain, we shall take a different route.

Our starting point is the observation that Chow \[4\; \text{Theorem 1}\] has proved a reciprocity theorem for a symmetric function invariant of a digraph called the *path-cycle symmetric function* \( \Xi_D \). There is a certain precise sense in which \( \Xi_D \) is equivalent to Stanley’s \( X_G \) in the case of posets, but the nice thing about reciprocity for \( \Xi_D \) is that it naturally yields a combinatorial interpretation for the coefficients of the monomial symmetric function expansion of \( \omega \Xi_D \), which is not immediately evident from Stanley’s reciprocity theorem \[27\; \text{Theorem 4.2}\] for \( X_G \). This fact suggests the following plan: Generalize \( \Xi_D \) to \( \Xi_D(t) \) (just as Shareshian and Wachs have generalized \( X_G \) to \( X_G(t) \)), prove reciprocity for \( \Xi_D(t) \), and read off the desired combinatorial interpretation of \( c_{d,\lambda}(m) \). This plan works, and we now show how to carry it out.

We define the *path quasisymmetric function* \( \Xi_D(x, t) \) of a digraph \( D \); as its name suggests, it enumerates paths only and not cycles (since for our present purposes we do not care about enumerating cycles), and it has a definition analogous to that of the chromatic quasisymmetric function.
Definition 20. Let $D$ be a digraph whose vertex set $V$ is a subset of $P$. An ordered path cover of $D$ is an ordered pair $(q, \beta)$ such that $q$ is a sequencing of $V$, $\beta = (\beta_1, \ldots, \beta_\ell)$ is a composition of $n := |V|$, and

$$q(\beta_{i-1} + 1) \to q(\beta_{i-1} + 2) \to \cdots \to q(\beta_i)$$

is a directed path in $D$ for all $i \in [\ell]$ (adopting the convention that $\beta_0 = 0$).

Define

$$\Xi_D(x, t) := \sum_{(q, \beta)} t^{\text{asc} q} M_\beta$$

where the sum is over all ordered path covers $(q, \beta)$ of $D$ and $\text{asc} q$ is the number of pairs $\{u, v\}$ of vertices of $D$ such that

1. either $u \to v$ and $v \to u$ are both edges of $D$ or neither one is,
2. $u < v$, and
3. $v$ appears later in the sequencing $q$ than $u$ does.

For brevity, we sometimes write $\Xi_D(t)$ for $\Xi_D(x, t)$. The chromatic quasi-symmetric function and the path quasi-symmetric function coincide for posets. More precisely, we have the following proposition.

Proposition 22. Let $P$ be a poset whose vertex set $V$ is a finite subset of $P$. Let $D(P)$ be the digraph on $V$ that has an edge $u \to v$ if and only if $v \prec u$ in $P$. Let $G(P)$ be the incomparability graph of $P$. Then $\Xi_{D(P)}(x, t) = X_{G(P)}(x, t)$.

Proof (sketch). The proof is mostly a routine verification that the two definitions coincide in this special case. Only a few points require some attention. First, if $S$ is a stable subset in $G(P)$, then $S$ is a totally ordered subset of $P$, and hence there is exactly one directed path in $D(P)$ through the vertices of $S$. Hence ordered partitions $\sigma$ of $V$ such that every $\sigma_i$ is a stable set of $G(P)$ are in bijective correspondence with ordered path covers $(q, \alpha)$ of $D(P)$. Second, because $P$ is a poset, it is not possible for $u \prec v$ and $v \prec u$ simultaneously, so the condition that "either $u \to v$ and $v \to u$ are both edges of $D(P)$ or neither one is" is equivalent to adjacency in $G(P)$. Third, one might worry that $\text{asc} q$ counts some pairs $\{u, v\}$ where $v$ appears later in the sequencing but in the same path while $\text{asc} \sigma$ counts only pairs from different parts, but in fact this cannot happen because vertices in the same path are part of the same totally ordered subset of $P$ and thus have a directed edge between them in exactly one direction.

Although we are ultimately interested in expansions in terms of monomial symmetric functions, it turns out that the proofs are more naturally stated in terms of monomial quasi-symmetric functions. So we need to describe the action of $\omega$ on monomial quasi-symmetric functions.
Definition 23. The linear map \( \omega \) on quasisymmetric functions is defined by the following action on monomial quasisymmetric functions.

\[
\omega M_\beta := (-1)^{|\beta|} \sum_{\alpha \leq \beta} M_\alpha.
\]  \((24)\)

It is known (e.g., see the proof of [27, Theorem 4.2]) that the usual map \( \omega \) is characterized by the equation \( \omega F_\alpha = F_{\overline{\alpha}} \), so the following proposition confirms that our definition of \( \omega \) coincides with the standard one.

Proposition 25. \( \omega F_\alpha = F_{\overline{\alpha}} \).

Proof. Applying \( \omega \) to Equation (11) and invoking Equation (24) yields

\[
\omega F_\alpha = \sum_{\beta \geq \alpha} \omega M_\beta = \sum_{\beta \geq \alpha} (-1)^{|\beta|} \sum_{\gamma \leq \beta} M_\gamma.
\]

So the coefficient of \( M_\gamma \) in \( \omega F_\alpha \) is

\[
\sum_{\beta: (\beta \geq \alpha \text{ and } \beta \geq \gamma)} (-1)^{|\beta|} = \sum_{\beta \geq \alpha \cup \gamma} (-1)^{|\beta|} = \begin{cases} 1, & \text{if } \alpha \cup \gamma = \emptyset; \\ 0, & \text{otherwise}. \end{cases}
\]

But \( \alpha \cup \gamma = \emptyset \) is equivalent to \( \gamma \geq \alpha \), so \( \omega F_\alpha = \sum_{\gamma \geq \alpha} M_\gamma = F_{\overline{\alpha}} \). \( \Box \)

We are ready for the reciprocity theorem for \( \Xi_D(t) \).

Theorem 26. Let \( D \) be a digraph whose vertex set \( V \) is a subset of \( \mathbb{P} \). Then \( \omega \Xi_D(x, t) = \Xi_D(x, t) \).

Proof. Let us apply \( \omega \) to both sides of Equation (21) and invoke the definition of \( \omega \).

\[
\omega \Xi_D(x, t) = \sum_{(q, \alpha)} t^{\text{asc} \ q} \omega M_\beta = \sum_{(q, \alpha)} t^{\text{asc} \ q} (-1)^{|\beta|} \sum_{\alpha \leq \beta} M_\alpha.
\]

Now we interchange the order of summation; i.e., we want to compute the coefficient of \( M_\alpha \) in \( \omega \Xi_D(t) \). This involves a sum over all ordered path covers \((q, \beta)\) such that \( \beta \geq \alpha \). The summands involving a fixed sequencing \( q \) are

\[
\sum_{\beta \geq \alpha} t^{\text{asc} \ q} (-1)^{|\beta|} = t^{\text{asc} \ q} \sum_{\beta \geq \alpha} (-1)^{|\beta|}.
\]  \((27)\)

Now note that if \((q, \alpha)\) is an ordered path cover and \( \beta \) is any composition such that \( \beta \geq \alpha \), then \((q, \beta)\) is also an ordered path cover, because deleting an edge from a directed path simply subdivides it into two smaller directed paths. Therefore the alternating sum in Equation (27) is zero unless the only \( \beta \geq \alpha \) for which \((q, \beta)\) is an ordered path cover is the maximal composition \((\beta_i = 1 \text{ for all } i)\), in which case the alternating sum equals one. But this condition is equivalent to the condition that there is no directed edge in \( D \) between any consecutive vertices in the sequencing \( q \) that are in the same segment of \( \alpha \), i.e., that \((q, \alpha)\) is an ordered path cover of \( \overline{D} \). Finally, note that the definition of \( \text{asc} \ q \) is invariant under taking complements of the digraph. The theorem follows. \( \Box \)
Theorem 26 gives us a nice combinatorial interpretation of $c_{d,\lambda}(m)$.

**Corollary 28.** Let $m$ be a Hessenberg function, and let $D(m)$ denote the digraph on $[n]$ that has an edge $u \to v$ if and only if $v < u$ in $P$. Then for any composition $\alpha$ whose parts are a permutation of the parts of $\lambda$, $c_{d,\lambda}(m)$ equals the number of ordered path covers $(q, \alpha)$ of $D(m)$ with $\text{asc } q = d$.

*Proof.* By Proposition 22, we know that $X_{G(m)}(t) = \Xi_{D(m)}(t)$. Shareshian and Wachs prove [23, Theorem 4.5] that $X_{G(m)}(t)$ is actually a symmetric function (whose coefficients are polynomials in $t$). Therefore $\omega X_{G(m)}(t) = \omega \Xi_{D(m)}(t)$ is also a symmetric function, and the coefficient of $t^d M_{\alpha}$ for any composition $\alpha$ whose parts are a permutation of the parts of $\lambda$. The result then follows from Theorem 26.

**Corollary 29.** For a sequencing $q$ of a digraph whose vertex set is a subset of $P$, let the definition of $\text{des } q$ be the same as the definition of $\text{asc } q$ except with “$u < v$” replaced by “$v < u$.” Then Corollary 28 holds with $\text{des } q$ in place of $\text{asc } q$.

*Proof.* For a proper coloring $\kappa$ of a graph whose vertex set is a subset of $P$, let the definition of $\text{des } \kappa$ be the same as the definition of $\text{asc } \kappa$ except with “$\kappa(u) < \kappa(v)$” replaced by “$\kappa(u) > \kappa(v)$.” Shareshian and Wachs prove [23, Corollary 2.7] that the value of $X_{G}(t)$ is unchanged if “$\text{asc}$” is replaced by “$\text{des}$.” It is readily checked that the proofs of Proposition 22 and Theorem 26 go through if “$\text{asc}$” is replaced by “$\text{des}$” everywhere.

As we remarked before, Corollaries 28 and 29 can be derived from Shareshian–Wachs [23, Theorem 3.1], but we have taken our approach because we believe that Theorem 26 is of independent interest.

### 4 Betti numbers of regular Hessenberg varieties

The main result of this section is that if $H(m, s)$ is a regular Hessenberg variety and $s$ has Jordan type $\lambda$, then its Betti number $\beta_{2d}$ equals $c_{d,\lambda}(m)$.

Tymoczko [31, Theorem 7.1] has already done a lot of the work needed to prove this result, by showing that Hessenberg varieties admit a paving (or cellular decomposition) by affine spaces, and obtaining a combinatorial interpretation of the dimensions of the cells. For regular Hessenberg varieties, Tymoczko’s theorem simplifies as follows. If $\lambda$ is an integer partition of $n$ then by a tableau of shape $\lambda$ we mean any filling of the boxes of the Young diagram of $\lambda$ with one copy each of the numbers $1, 2, \ldots, n$.

**Theorem 30 (Tymoczko).** Let $H(m, s)$ be a regular Hessenberg variety and let the partition $\lambda$ encode the sizes of the Jordan blocks of $s$. Then $H(m, s)$ is paved by affines. The nonempty cells are in bijection with tableaux $T$ of shape $\lambda$ with the property that $k$ appears in the box immediately to the left of $j$ only if $k \leq m_j$. The dimension of a nonempty cell is the sum of:
1. the number of pairs \(i, k\) in \(T\) such that
   (a) \(i\) and \(k\) are in the same row,
   (b) \(i\) appears somewhere to the left of \(k\),
   (c) \(k < i\), and
   (d) if \(j\) is in the box immediately to the right of \(k\) then \(i \leq m_j\);

2. the number of pairs \(i, k\) in \(T\) such that
   (a) \(i\) appears in a lower row than \(k\), and
   (b) \(k < i \leq m_k\).

It remains for us to establish a correspondence between the combinatorics of Theorem 30 and the combinatorics of \(\omega_{XG(m)}(t)\), or equivalently (by the results of the previous section) the combinatorics of ordered path covers.

**Definition 31.** If \(X\) is a topological space and \(i\) is an integer, we write \(\beta_i\) or \(\beta_i(X)\) for the \(i\)-th Betti number \(\dim H^i(X, \mathbb{C})\) of \(X\).

**Theorem 32.** Let \(\mathcal{H}(m, s)\) be a regular Hessenberg variety and let the Jordan type of \(s\) be \(\lambda\). Then the Betti number \(\beta_{2d}\) of \(\mathcal{H}(m, s)\) equals \(c_{d, \lambda}(m)\), and \(\beta_i = 0\) for \(i\) odd.

**Proof.** As Tymoczko [31, Proposition 2.2] mentions, it is well known that if we have a paving by affines, then \(\beta_{2d}\) is just the number of nonempty cells with dimension \(d\). Furthermore, \(\beta_i = 0\) for \(i\) odd [31, Corollary 6.2]. On the other hand, by Corollaries 28 and 29 we know that \(c_{d, \lambda}(m)\) is the number of ordered path covers \((q, \alpha)\) of \(D(m)\) with \(\text{des} q = d\), where we may take the parts of the composition \(\alpha\) to be any permutation of the parts of \(\lambda\). So it suffices to show, firstly, that there is a bijection between nonempty cells and ordered path covers \((q, \alpha)\), and secondly, that under this bijection, the dimension of the nonempty cell is equal to \(\text{des} q\).

First we should specify \(\alpha\). If \(\lambda\) has \(\ell\) parts \(\lambda_1, \ldots, \lambda_\ell\), we set \(\alpha_i := \lambda_{\ell+1-i}\). That is, the parts of \(\alpha\) are the parts of \(\lambda\) in reverse order.

Instead of nonempty cells, we use the tableaux \(T\) of Theorem 30 to describe our bijection. Given an ordered path cover \((q, \alpha)\) of \(\overline{D(m)}\), take the elements of the \(i\)th path

\[q(\alpha_{i-1} + 1) \rightarrow q(\alpha_{i-1} + 2) \rightarrow \cdots \rightarrow q(\alpha_i)\]

and place them from left to right in the \(i\)th row (from the bottom) of \(T\). We need to verify that Tymoczko's condition \(k \leq m_j\) is equivalent to the condition that \(k \rightarrow j\) is a directed edge in \(\overline{D(m)}\). By definition, there is a directed edge \(k \rightarrow j\) in \(\overline{D(m)}\) if and only if there is not a directed edge \(k \rightarrow j\) in \(D(m)\), i.e., if and only if either \(k\) and \(j\) are incomparable in \(P(m)\), or \(k \prec j\) in \(P(m)\). The only way this property can fail is if \(j < k\) in \(P(m)\), i.e., if \(k > m_j\). So indeed the conditions are equivalent.

Let us call a pair \(i, k\) satisfying the conditions in Theorem 30 a "T-inversion." Using the above bijection, we can think of T-inversions as certain pairs \(i, k\) in an
ordered path cover \((q, \lambda)\). The statistic des can also be thought of as counting certain pairs \(i, k\) of \((q, \lambda)\), namely those satisfying

1. either \(i \to k\) and \(k \to i\) are both edges of \(D(m)\) or neither is,

2. \(i > k\), and

3. \(k\) appears later in the sequencing \(q\) than \(i\) does.

Call such a pair an “SW-inversion.” We claim that for any ordered path cover, the number of T-inversions equals the number of SW-inversions. This will prove the theorem.

First let us note that the condition \(k < i\) implies that \(k \leq m\) (since \(m\) is a Hessenberg function) and therefore, by the argument we gave above, \(k \to i\) is an edge of \(D(m)\). That is, if \(k < i\) then it is not possible for neither \(i \to k\) nor \(k \to i\) to be an edge of \(D(m)\), so in fact both must be, and in particular we must have \(i \to k\), or in other words \(i \leq m_k\). Therefore an SW-inversion can be redefined as a pair \(i, k\) such that

1. \(i\) appears earlier in the sequencing \(q\) than \(k\) does, and

2. \(k < i \leq m_k\).

It is now immediate that if \(i\) and \(k\) are in different paths then \(i, k\) is a T-inversion if and only if \(i, k\) is an SW-inversion, because by construction, \(i\) appearing in an earlier path than \(k\) is equivalent to appearing in a lower row than \(k\) in the tableau.

If \(i\) and \(k\) are in the same path then the situation is more complicated because T-inversions and SW-inversions do not necessarily coincide. However, we now give a bijection from the set of SW-inversions to the set of T-inversions, thereby showing that they are equinumerous.

Given an SW-inversion \(i, k\), let \(k_1, k_2, \ldots, k_r\) denote the remaining elements, in order, that succeed \(k\) in the path. For convenience, set \(k_0 := k\) and \(k_{r+1} := \infty\). Now let \(j\) be the smallest number such that \(i \leq m_{k_{j+1}}\). Then we claim that \(i, k_j\) is a T-inversion, and that this is a bijection.

First let us verify that \(i, k_j\) is a T-inversion. Condition 1(d) is satisfied almost by definition because what the construction is doing is scanning to the right until condition 1(d) is satisfied, and it will always succeed, since we just take \(j = r\) in the worst case. So we just need to verify that \(i > k_j\). If \(j = 0\) then we are done, because \((i, k_0) = (i, k)\) is an SW-inversion by assumption, and in particular \(i > k\). Otherwise, by minimality of \(j\), we know that \(i \geq m_{k_j} \geq k_j\).

Thus the construction scans rightwards from \(k\) until the first T-inversion \(i, k_j\) is reached.

To see that this map is injective, observe that by minimality of \(j\), we have \(i > m_{k_j}\) for every \(0 \leq j' \leq j\), so \((i, k_{j'})\) is not an SW-inversion. Thus, as we scan rightwards from \(k\) in search of the first T-inversion \(i, k_j\), we do not encounter any other SW-inversions en route. If more than one SW-inversion were mapped to the same T-inversion, then the leftmost one would have to cross over the other ones en route.
To see that the map is surjective, we can define an inverse map, that scans *leftwards* from a T-inversion until it finds a pair that satisfies \( i \leq m_k \). Such a scan always succeeds because in the worst case it ends up at the successor \( i' \) of \( i \), and \( i \leq m_{i'} \) because they are consecutive elements of a path. Then by minimality, if we arrive at a pair \( i, k \) with \( k' \) being the successor of \( k \), we must have \( k > m_{i'} \geq k \), so what we have arrived at is indeed an SW-inversion.

Let us remark that our proof shows that at least in the case of regular Hessenberg varieties, the two cases of Theorem 30 can be unified, namely that the dimension is just the number of pairs \( i, k \) such \( i \) appears to the left of \( k \) or in a lower row than \( k \), and \( k < i \leq m_k \).

**Corollary 33.** Let \( \mathcal{H}(m, s) \) be a regular Hessenberg variety with \( s \) of type \( \lambda \) as in Theorem 32. Set

\[
q = q_{\mathcal{H}(m, s)} := \sum_{i \in \mathbb{Z}} \beta_i t^{i - |m|}.
\]

Then \( q(t) = q(t^{-1}) \).

**Proof.** First note that \( |m| \) (as defined in Equation (5)) is number \( |E| \) of edges in the incomparability graph \( G = G(m) \) of \( P(m) \). This follows directly from the description of \( G(m) \) given in [22,1] By [23 Corollary 4.6], \( X_G(x, t) \) is palindromic. More precisely, we have \( X_G(x, t) = t^{|m|} X_G(x, t^{-1}) \). Therefore, for each partition \( \lambda \), we have

\[
\sum_d c_{d, \lambda}(m) t^d = t^{|m|} \sum_d c_{d, \lambda} t^{-d}.
\]

So

\[
q(t) = \sum_i \beta_i t^{i - |m|} = \sum_d c_{d, \lambda}(m) t^{2d - |m|} = t^{-|m|} \sum_d c_{d, \lambda}(m) t^{2d} = t^{-|m|} t^{2|m|} \sum_d c_{d, \lambda}(m) t^{-2d} = t^{|m|} \sum_d c_{d, \lambda}(m) t^{-2d} = \sum_d c_{d, \lambda}(m) t^{|m| - 2d} = q(t^{-1}).
\]

\[\Box\]

## 5 Local monodromy and local fundamental groups

### 5.1 Local homotopy type

We begin by defining local homotopy type. This material is probably well-known, but we feel that it will be convenient to review it. Our treatment
follows the book by Looijenga [17], a paper by Kumar [15] and another paper by Prill [21].

Suppose \( X \) is a topological space and \( x \in X \). A fundamental system of neighborhoods \( \mathcal{U} \) of \( x \) is a system of open neighborhoods such that any open neighborhood \( V \) of \( x \) contains a \( U \in \mathcal{U} \).

The following Lemma is [15, Lemma 1.1.].

**Lemma 34.** Suppose \( X \) is a CW complex, \( x \in X \) and \( Y \) is a closed subcomplex of \( X \) containing \( x \). Then there exists a fundamental system \( \{U\}_{U \in \mathcal{U}} \) of open neighborhoods of \( x \) in \( X \) such that the following condition is satisfied:

\[
\text{For any } U, V \in \mathcal{U} \text{ with } V \subset U, \text{ the inclusion } V \setminus Y \hookrightarrow U \setminus Y \text{ is a homotopy equivalence.}\tag{35}
\]

A system of neighborhoods \( \mathcal{U} \) as in Lemma 34 is called a good fundamental system of neighborhoods relative to \( Y \).

**Lemma 36.** Suppose \( C \) is a category and

\[
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D
\]

is a sequence of morphisms. Assume that \( g \circ f \) and \( h \circ g \) are isomorphisms. Then \( f, g \) and \( h \) are all isomorphisms.

**Proof.** Easy exercise. \( \square \)

We have adapted the proof of the following Proposition from Looijenga’s [17] p. 114, and Prill’s Proposition 2 [21].

**Proposition 37.** Suppose \( \{\mathcal{U}_\alpha\}_{\alpha \in I} \) is a non-empty collection of good fundamental systems of neighborhoods as in Lemma 34. Then so is \( \mathcal{U} := \cup_{\alpha \in I} \mathcal{U}_\alpha \). Consequently, the union of all good fundamental systems of neighborhoods is itself a good fundamental system of neighborhoods.

**Proof.** Take \( U \in \mathcal{U}_\alpha \) and \( V \in \mathcal{U}_\beta \) with \( V \subset U \). We can find \( U' \in \mathcal{U}_\alpha \) such that \( U' \subset V \), and \( V' \in \mathcal{U}_\beta \) such that \( V' \subset U' \). Then apply Lemma 36 to the sequence of inclusions

\[
V' \setminus Y \to U' \setminus Y \to V \setminus Y \to U \setminus Y.
\]

This shows that the inclusion \( V \setminus Y \to U \setminus Y \) is a homotopy equivalence. \( \square \)

**Definition 38.** Suppose \( X \) is a CW complex and \( Y \) is a closed subcomplex containing a point \( x \). We say an open neighborhood \( U \) of \( x \) is good relative to \( Y \) if \( U \) is an element of a good fundamental system of neighborhoods. The local homotopy type of \( X \setminus Y \) at \( x \) is the homotopy type of \( U \setminus Y \) where \( U \) is any good neighborhood.
Suppose $U$ is a good neighborhood of $x$ and $W$ is an arbitrary (not necessarily good) open neighborhood of $x$ contained in $U$. Then we can find a good neighborhood $V$ such that $V \subset W$. Since $V$ is good, Proposition 37 shows that the composition

$$V \setminus Y \to W \setminus Y \to U \setminus Y$$

is a homotopy equivalence. In other words, the local homotopy type of $X \setminus Y$ at $x$ is a retract of the homotopy type of $W \setminus Y$.

Now suppose $X$ is an analytic space, $Y$ is a Zariski closed subspace and $x \in Y$. We can find an open neighborhood $W$ of $x$ in $X$ such that $W$ has the topological structure of a CW complex with $W \cap Y$ a subcomplex. (See, for example, [16].) Consequently, there exist good neighborhoods of $x$ in $X$ relative to $Y$.

The following fact is certainly well known (see, e.g., [21, Corollary 1]), but we give a proof because it is short.

Fact 39. Suppose $X$ is a complex manifold and $Y$ is a closed, nowhere dense, analytic subspace of $X$ containing a point $x$. Then $U \setminus Y$ is a non-empty and connected for any good neighborhood of $x$.

Proof. Let $U$ be a good neighborhood of $x$ and let $V$ be any connected neighborhood of $x$ contained in $U$. Then $V \setminus Y$ is connected (for example, by the Criterion for Connectedness on page 133 of [9]). It is also non-empty. But the homotopy type of $U \setminus Y$ is a retract of the homotopy type of $V \setminus Y$. So $U \setminus Y$ is connected and non-empty as well.

In fact, if $X$ is smooth at $x$, we can find a contractible good neighborhood $U$ of $x$. (See [21].) In fact, we can take a sufficiently small ball as in the following theorem, which follows from Theorem 5.1 of Dimca’s [6].

Theorem 40. Suppose $X$ is smooth of dimension $n$ at $x$ and $Y$ is a closed analytic subspace of $X$ containing $x$. For each positive real number $r$, write $B_r$ for the ball of radius $r$ centered at $0$ in $\mathbb{C}^n$. Then there exists a good neighborhood $U$ of $x$ relative to $Y$ and biholomorphism $\varphi : U \to B_1$ such that the following holds: For each $r \in (0, 1)$, $\varphi^{-1}B_r$ is a good neighborhood of $x$.

### 5.2 Local fundamental group

Fact 39 leads to the following definition.

Definition 41. Suppose $X$ is a complex manifold $x$, and suppose $Y$ is a closed, nowhere dense, analytic subspace of $X$ containing a point $x$. Then the local fundamental group of $X \setminus Y$ at $x$ is the isomorphism class of the group $\pi_1(U, p)$ where $U$ is any good neighborhood of $x$ (with respect to $Y$) and $p \in U$.

Since the smoothness of $X$ in the Definition 41 implies that $U$ is connected, the isomorphism class of $\pi_1(U, p)$ is indeed well-defined. But, since we have not given a way to fix a base point, it is only defined up to a non-canonical isomorphism.
On the other hand, suppose $X$ as in Definition 41 is connected. Pick any point $q \in X \setminus Y$. Given a good neighborhood $U$ of $x$ relative to $Y$, we can find a point $p \in U \setminus Y$ and a path $\gamma$ from $p$ to $q$. From this, we get a group homomorphism

$$\varphi_\gamma : \pi_1(U \setminus Y, p) \to \pi_1(X \setminus Y, q).$$

Changing $\gamma$ has the effect of conjugating $\varphi_\gamma$ by an element of $\pi_1(X \setminus Y, q)$. So the conjugacy class of the subgroup $\varphi_\gamma(\pi_1(U \setminus Y, p))$ is independent of $\gamma$.

**Proposition 42.** Suppose $f : X' \to X$ is a morphism of complex analytic spaces admitting a section $\epsilon : X \to X'$. Let $Y$ be a closed, nowhere dense, analytic subspace of $X$ containing $x \in X$, and set $Y' := f^{-1}Y$. Then the local homotopy type of $X \setminus Y$ at $x$ is a retract of the local homotopy type of $X' \setminus Y'$ at $\epsilon(x)$. In particular, if $X$ and $X'$ are complex manifolds, then the local fundamental group of $X \setminus Y$ at $x$ is a retract of the local fundamental group of $X' \setminus Y'$ at $\epsilon(x)$.

**Proof.** Pick a good neighborhood $U$ of $x$. Then find a good neighborhood $V$ of $\epsilon(x)$ contained in $f^{-1}U$. Finally, find a good neighborhood $U'$ of $x$ contained in $\epsilon^{-1}V$. We then have a composition

$$U' \setminus Y \xrightarrow{\varphi} V \setminus Y' \xrightarrow{\epsilon} U \setminus Y$$

which is a homotopy equivalence. The result follows. 

### 5.3 Local systems and local invariant cycles

Suppose $X$ is a CW complex containing a closed subcomplex $Y$ which contains a point $x$, and $\mathcal{L}$ is a local system of complex vector spaces on $X \setminus Y$. For any two good neighborhoods $U_1$ and $U_2$ of $x$ and any integer $k$, the sheaf cohomology groups $H^k(U_i \setminus Y, \mathcal{L}), i = 1, 2$ are canonically isomorphic. To see this, take a good neighborhood $V \subset U_1 \cap U_2$, and note that the restriction maps $H^k(U_i \setminus Y, \mathcal{L}) \to H^k(V \setminus Y, \mathcal{L})$ are isomorphisms. So we write $H^k(x, \mathcal{L})$ for the group $H^k(U \setminus Y, \mathcal{L})$ where $U$ is any good neighborhood of $x$. It is isomorphic to the group colimit $H^k(U, \mathcal{L})$ where the colimit is taken over all open neighborhoods of $x$. The group $H^0(x, \mathcal{L})$ is called the space of local invariants.

If $X$ is a complex manifold and $Y$ is a nowhere dense analytic subspace, then $U \setminus Y$ is connected for any good neighborhood of $x$ relative to $Y$. Pick a basepoint $p \in U \setminus Y$. Then the data of the local system $\mathcal{L}$ defines an action of $\pi_1(U \setminus Y, p)$ on the fiber $\mathcal{L}_p$ at $p$. Moreover, the space of local invariants is given by the invariants of the action:

$$H^0(x, \mathcal{L}) = \mathcal{L}_p^{\pi_1(U \setminus Y, p)}. \quad (43)$$

**Corollary 44.** Suppose $X$ is smooth and $B$ is any connected neighborhood of $x$ contained in a good neighborhood $U$. Then $H^0(x, \mathcal{L}) = H^0(B \setminus Y, \mathcal{L})$.

**Proof.** Pick a point $b \in B \setminus Y$. We have $H^0(B \setminus Y, \mathcal{L}) = \mathcal{L}_b^{\pi_1(B \setminus Y, b)}$. But $\pi_1(U \setminus Y, b)$ is a retract of $\pi_1(B \setminus Y, b)$. So $\mathcal{L}_b^{\pi_1(B \setminus Y, b)} = \mathcal{L}_b^{\pi_1(U \setminus Y, b)} = H^0(x, \mathcal{L})$. \qed
We say a connected neighborhood of a point \( x \) in a complex manifold \( X \) is sufficiently small if it is contained a good neighborhood.

We can also describe the space \( H^k(X, \mathcal{L}) \) sheaf theoretically. Write \( j : X \setminus Y \to X \) for the inclusion. Then the group \( H^k(X, \mathcal{L}) \) is naturally isomorphic to \((R^k j_\ast \mathcal{L})_p\); i.e., to the stalk at \( x \) of the \( k \)-th higher direct image \( R^k j_\ast \mathcal{L} \).

The following is certainly well-known, but we sketch a short proof.

**Lemma 45.** Suppose \( X \) is a connected complex manifold and \( Y \) is a nowhere dense closed analytic subspace. Then, for \( p \in X \setminus Y \), the homomorphism \( \pi_1(X \setminus Y, p) \to \pi_1(X, p) \) is surjective.

**Sketch.** Let \( \tilde{\pi} : \tilde{X} \to X \) denote the universal cover of \( X \). Then \( \tilde{\pi}^{-1}(X \setminus Y) = \tilde{X} \setminus \tilde{\pi}^{-1}(Y) \) is connected because \( \tilde{X} \) is a complex manifold and \( \tilde{\pi}^{-1}(Y) \) is a closed, nowhere dense, complex analytic subspace [9, p. 133]. Therefore the map of fundamental groups is surjective.

Now, suppose \( X \) is a connected, complex manifold, \( Y \) is a closed, nowhere dense, analytic subspace, \( L \) is a local system on \( X \setminus Y \), \( x \in Y \) and \( q \in X \setminus Y \).

The monodromy group of \( L \) is the image \( M = \text{Image} \) of the group homomorphism \( \pi_1(U \setminus Y, p) \to \text{GL}(L_q) \). By Lemma 45, \( M \) is independent of \( Y \). That is, if \( Y' \) contains \( Y \) (but not \( q \)), then the image of the homomorphism \( \pi_1(X \setminus Y', q) \) is also \( M \).

If \( p \) is a point in \( Y \), then the local monodromy group of \( L \) at \( p \) is the image \( H = H(p) \) of the composition

\[
\pi_1(U \setminus Y, p) \xrightarrow{\gamma} \pi_1(X \setminus Y, q) \to M
\]

where \( \gamma \) is a path from \( p \) to \( q \). It depends on the choice of \( \gamma \) and \( p \), but only up to conjugacy by an element of \( M \). Like \( M \) itself, \( H \) is independent of \( Y \) in the sense that enlarging \( Y \) does not change \( H \).

### 6 Symmetric Betti numbers and the local invariant cycle theorem

A crucial tool in our argument is the local invariant cycle theorem of Beilinson, Bernstein, and Deligne, which we state here in the generality relevant to this paper.

**Theorem 46 (\cite{BBD} Corollaire 6.2.9).** Suppose \( f : X \to Y \) is a projective morphism of smooth complex varieties. For \( y \in Y \), set \( X_y := f^{-1}(y) \) and suppose that there is a Zariski dense open subset \( U \) in \( Y \) such that, for all \( y \in U \), \( X_y \) is smooth of dimension \( d \). Then for every sufficiently small connected open neighborhood \( B(y) \) of a closed point \( y \in Y \), there is a surjective map

\[
H^i(X_y, \mathbb{C}) \to H^0(B(y) \cap U, R^if_*\mathbb{C}). \tag{47}
\]

The vector space \( H^0(B(y) \cap U, R^if_*\mathbb{C}) \) is called the space of local invariant cycles. As Beilinson, Bernstein and Deligne point out, it is independent of \( U \).
Moreover, as we have seen above, for a fixed $U$, the spaces $H^0(B(y) \cap U, R^i f_* \mathbb{C})$ are isomorphic for $B(y)$ sufficiently small.

We will devote the remainder of this section to proving the following result.

**Theorem 48.** Suppose that $f : X \to Y$ is a projective morphism between smooth, quasi-projective, complex schemes; and let $y$ be a closed point of $Y$. Assume that $X$ and $Y$ are equidimensional with dimensions $d_X$ and $d_Y$ respectively, and set $d = d_X - d_Y$. Then the local invariant cycle map (47) is an isomorphism for all $i \in \mathbb{Z}$ if and only if

$$\dim H^i(X_y, \mathbb{C}) = \dim H^{2d-i}(X_y, \mathbb{C})$$

for all $i$.

It will be convenient to introduce some notation concerning palindromic polynomials.

**Definition 49.** Suppose $p \in \mathbb{R}[t, t^{-1}]$ is a Laurent polynomial. We say that $p$ is palindromic if $p(t) = p(t^{-1})$.

**Lemma 50.** Suppose $q = \sum_{\ell \geq 0} p_{\ell} t^\ell$ where $q$ and the $p_{\ell}$ are palindromic Laurent polynomials. Assume that, for $\ell > 0$, the coefficients of the $p_{\ell}$ are non-negative. Then $p_{\ell} = 0$ for $\ell > 0$. In particular, $q = p_0$.

**Proof.** Note that a Laurent polynomial $f$ is palindromic only if $f'(1) = 0$. Now, \[ \frac{d}{dt} p_{\ell} t^\ell \bigg|_{t=1} = \ell p_{\ell}(1). \]

It follows that $0 = q'(1) = \sum_{\ell>0} \ell p_{\ell}(1)$. Since the coefficients of the polynomials $p_{\ell}$ are non-negative for $\ell > 0$, it follows that $p_{\ell} = 0$ for $\ell > 0$. \qed

In the proof of Theorem 48, we are going to use the language of Beilinson, Bernstein and Deligne \[3\] regarding perverse sheaves and the derived category $D^b_c(X, \mathbb{C})$ of sheaves with constructible cohomology (with complex coefficients) on a complex scheme $X$. In particular, if $K$ is a complex in $D^b_c(X, \mathbb{C})$ and $i \in \mathbb{Z}$, then $K[i]$ denotes the shift of $K$ by $i$ units to the left.

**Proof of Theorem 48.** By the decomposition theorem of \[3\], we have $Rf_* \mathbb{Q}[d_X] = \oplus E_{i,j}[-i]$ where the $E_{i,j}$ are perverse sheaves on $Y$ supported strictly in codimension $j$. Moreover, by the Hard Lefschetz theorem of \[3\] Théorème 6.2.10], there are isomorphisms $E_{i,j} \cong E_{-i,j}$ for all $i$.

We can write

$$E_{i,j} = \oplus g(ij) \mathcal{F}_{ijh}[d_Y - j]$$

where, for each $h$, $\mathcal{F}_{ijh}$ is a local system on a locally closed subscheme $Z_{ijh}$ of codimension $j$ in $X$ and $g(ij) : Z_{ijh} \to X$ is the inclusion.

For any complex $K$ of sheaves on $Y$, write $H^*(y, K)$ for the cohomology of the stalks of $K$ at $y$. Write $IH^*(y, \mathcal{F}_{ijh})$ for the local intersection cohomology of $\mathcal{F}_{ijh}$ at $y$. So $IH^k(y, \mathcal{F}_{ijh}) = 0$ for $k < 0$ and

$$H^k(y, E_{ij}) = \oplus IH^{k+d_Y-j}(y, \mathcal{F}_{ijh}).$$ (51)
Now, write
\[ q := \sum t^k \dim H^k(X_y, \mathbb{Q}[d]). \tag{52} \]
So, we have \( \dim H^i(X_y, \mathbb{C}) = \dim H^{2d-i}(X_y, \mathbb{C}) \) for all \( i \), if and only if \( q \) is palindromic.

Now, by proper base change, we have
\[
H^k(X_y, \mathbb{Q}[d]) = H^k(y, Rf_* \mathbb{Q}[d]) = H^k(y, \oplus_{ij} E_{ij}[-i - d_Y]) = \oplus_{ij} H^k-i-d_Y(y, E_{ij}) = \oplus_{ijh} H^k-i-j(y, F_{ijh}).
\]

Set
\[
b_{ijk} = \dim IH^k(y, \oplus_h F_{ijh}). \tag{53}\]
We get then that
\[
\dim H^k(X_y, \mathbb{Q}[d]) = \sum b_{i,j,k-i-j}. \tag{54}
\]
So
\[
q = \sum b_{i,j,k-i-j}t^k = \sum b_{i,j,k}t^{k+i+j}.
\]
Now set
\[
p_\ell = \sum_{k+j=\ell} b_{i,j,k}t^i.
\]
Then \( q = \sum_{\ell \geq 0} p_\ell t^\ell \). Moreover, \( b_{i,j,k} = b_{-i,-j,k} \), so each \( p_\ell \) is palindromic. Since each \( b_{i,j,k} \) is a non-negative integer, Lemma \[55\] shows that \( q \) is palindromic if and only if \( p_\ell = 0 \) for \( \ell > 0 \). The local invariant cycles are given by \( \oplus_i IH^0(E_{i0}[-d_Y]) \). So \( p_\ell = 0 \) for all \( \ell > 0 \) if and only if the local invariant cycle map is an isomorphism. This completes the proof of the Theorem \[18\].

7 Galois covers

The purpose of this section is to prove a lemma about the local monodromy groups of Galois covers. We use the concepts of §5, but we have changed some of the notation (partially to avoid running out of capital letters towards the end of the alphabet).

7.1 Covers and monodromy

Suppose \( U \) is a smooth, connected, complex, quasi-projective variety and \( G \) is a finite group acting freely on \( U \). Let \( V = U/G \), and write \( \pi : U \to V \) for the
Moreover, up to conjugation by an element of $G$, we get a surjective group homomorphism

$$\psi_u : \pi_1(V, \pi(u)) \to G.$$ \hspace{1cm} (55)

If $\gamma : [0, 1] \to V$ represents an element of $\pi_1(V, \pi(u))$ and $\hat{\gamma}$ is a lift of $\gamma$ to $U$ with $\hat{\gamma}(0) = u$, then $\psi_u(\gamma)u = \hat{\gamma}(1)$. From this description, we see that, if $\pi(u') = \pi(u)$, then $\psi_u$ and $\psi_{u'}$ differ by conjugation by an element of $\pi_1(V, \pi(u))$. (See [13] for a complete discussion of these matters.)

Now, suppose that $V$ is contained as a Zariski open subset of a smooth, quasi-projective variety $Y$. Set $Z = Y \setminus V$, and suppose $z$ is a closed point of $Z$. Let $W$ be a good neighborhood of $z$ in $Y$ relative to $Z$, and let $w$ be a point in $W \setminus Z$. The choice of a path from $w$ to $\pi(u)$ gives us a sequence of group homomorphisms

$$\pi_1(W \setminus Z, w) \to \pi_1(V, \pi(u)) \to G.$$ \hspace{1cm} (56)

Moreover, up to conjugation by an element of $G$, this map is independent of $u$, the path from $w$ to $\pi(u)$, $W$ and $w$. We call the image $H(z)$ of the composition in (56) the local monodromy subgroup at $z$. (The conjugacy class of $H(z)$ is independent of any choices.) Note that, if we replace $V$ with a any non-empty Zariski open subset $V'$ of $V$ containing $\pi(u)$ and we replace $U$ with $\pi^{-1}(V')$, then $H(z)$ does not change. This follows from Lemma [13].

**Proposition 57.** Let $G$ be a finite group acting on a smooth, quasi-projective variety $X$, and suppose $G$ acts freely on a Zariski dense open subset $U$ of $X$. Suppose $\pi : X \to Y$ is the quotient of $X$ by $G$ and let $V = \pi(U)$. Pick a closed point $x \in X \setminus U$, and suppose that $Y$ is smooth. Then $H(\pi(x))$ is the stabilizer $G_x$ of the point $x$.

**Proof.** Take a good neighborhood $B$ of $y := \pi(x)$ with respect to $Z := Y \setminus V$. Pick $b \in B \cap V$ and set $A = \pi^{-1}B$. Let $A_x$ denote the component of $A$ containing $x$. There exists $a \in A_x$ such that $\pi(a) = b$. Let $H$ denote the image of the composition $\pi_1(B \cap V, b) \to \pi_1(V, b) \to G$, where the last homomorphism is $\psi_a$. Then $H = H(a)$. The group $G$ acts transitively on the connected components of $A \cap U$, and the stabilizer of the component of $A \cap U$ containing $a$ is $H$. Since $A_x \cap U$ is connected, $A_x \cap U$ is this component. So the stabilizer in $G$ of this component is the same as the stabilizer of the $A_x$. But, by possibly shrinking $B$, we can arrange that this is just $G_x$. \hfill \square

**Remark 58.** We use the assumption that $Y$ is smooth because we have only defined the local fundamental group in that case. However, since $Y$ is a quotient of a smooth variety, it is automatically normal. And good neighborhoods of normal quasi-projective varieties are connected. (See Chapter 3 of Mumford’s [18].)

It follows that the assumption that $Y$ is smooth can be dropped.

### 8 Geometry of Hessenberg Schemes

In this section, we study the geometry of the family of Hessenberg varieties over the space of regular matrices. We also study the family of maximal tori defined
by centralizers of regular, semisimple matrices. Ngô’s paper on the Hitchin fibration [19] significantly influenced our thinking about these matters, and we have consequently borrowed Ngô’s notation.

8.1 Regular matrices

Fix a positive integer \( n \) and write \( g \) for the Lie algebra \( \mathfrak{gl}_n \). Recall that a matrix \( s \in g \) is regular if the Jordan blocks of \( s \) have distinct eigenvalues. A matrix \( s \) is regular if and only if its centralizer has dimension \( n \). As above, we say that \( s \) is regular of type \( \lambda \) for a partition \( \lambda \) of \( n \) if the Jordan blocks of \( s \) are of sizes \( \lambda_1, \ldots, \lambda_r \). We write \( g^r \) for the subset of regular matrices and \( g^r_\lambda \) for the subset of regular matrices of type \( \lambda \). We write \( g^{rs} \) for the subset of regular semisimple matrices. This is a dense open subset of \( g \).

8.2 Hessenberg schemes

Fix a Hessenberg function \( m = (m_1, \ldots, m_{n-1}) \) with \( m(i) = m_i \), and set \( m_n = n \). Write \( \mathcal{X} \) for the variety of complete flags in \( \mathbb{C}^n \), and set \( H(m) := \{ (F, s) \in \mathcal{X} \times g : sF_i \subset F_{m_i} \text{ for all } i \} \) \hspace{1cm} (59)

Note that the projection \( pr_1 \) on the first factor makes \( H(m) \) into a vector bundle of rank \( \sum_{i=1}^{n} m_i \) over \( \mathcal{X} \). So \( H(m) \) is a smooth, connected scheme. Let \( \pi : H(m) \to g \) denote the projection on the second factor. Then the fiber of \( \pi \) over a matrix \( s \in g \) is simply the Hessenberg variety \( H(m, s) \). Note that \( \pi \) is smooth over \( g^{rs} \).

8.3 Diagonal matrices and characteristics

Write \( G := \text{GL}_n \) and write \( D \) for the diagonal subgroup of \( G \). Write \( d \) for the Lie algebra of \( D \), and \( d_r \) for the regular elements of \( d \). The symmetric group \( S_n \) acts on \( d = \mathbb{A}^n \) in the obvious way: \( \sigma(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \). The quotient is the characteristic variety \( \text{car} = \text{car}_G = d/S_n \). We can view \( \text{car} \) as the variety of monic polynomials \( t^n + a_{n-1}t^{n-1} + \cdots + a_0 \) of degree \( n \). The Chevalley morphism \( \chi : g \to \text{car} \) is the morphism sending a matrix \( s \in g \) to its characteristic polynomial \( \chi(s) \).

8.4 The smallest Hessenberg scheme

Let \( \ell \) denote the Hessenberg function \( \ell(i) = i \). Then the restriction \( \mathcal{H}^{rs}(\ell) \to g^{rs} \) of \( \pi : \mathcal{H}(\ell) \to g \) to the inverse image of \( g^{rs} \) is an étale cover of degree \( n! \). Since \( \mathcal{H}(\ell) \) is connected and smooth, so is \( \mathcal{H}^{rs}(\ell) \). So, since \( g^{rs} \) is also connected, \( \mathcal{H}^{rs}(\ell) \to g^{rs} \) is an étale cover corresponding to an index \( n! \) subgroup of the fundamental group of \( g^{rs} \).

We call \( \mathcal{H}(\ell) \) the smallest Hessenberg scheme because, for any Hessenberg function \( m \), there is a canonical inclusion \( \mathcal{H}(\ell) \to \mathcal{H}(m) \) (which is a closed immersion).
We have a morphism $\tilde{\chi} : H(\ell) \to d$ sending a pair $(s, F)$ to the diagonal matrix with $\text{Gr}_i^s$'s in the $(i, i)$-entry. This gives rise to a commutative diagram

$$
\begin{align*}
H(\ell) & \xrightarrow{\tilde{\chi}} d \\
\downarrow \pi & \downarrow \chi_{\text{id}} \\
g & \xrightarrow{\chi} \text{car},
\end{align*}
$$

which coincides with Grothendieck’s simultaneous resolution of $\chi$. (For a discussion of Grothendieck’s resolution, see Springer’s [24 §4.1] or Slodowy’s [25 §4.7].) The restriction of (60) to the inverse image of $\text{car}^{rs} := d_r/S_n$ is a pull-back diagram. In other words, $H^{rs}(\ell) = g^{rs} \times_{\text{car}^{rs}} d_r$. This shows that $H^{rs}(\ell)$ is a (connected) Galois cover of $g^{rs}$ with Galois group $S_n$.

We can describe this Galois covering a little bit more explicitly if we introduce the closed subscheme $Z$ of $(\mathbb{P}^{n-1})^n \times g^{rs}$ consisting of ordered tuples $([v_1], \ldots, [v_n]; y)$ where the $v_i$ form a basis of eigenvectors of $y$. Given a point $z = ([v_1], \ldots, [v_n]; y)$ in $Z$, we can define a complete flag $F(z)$ by setting $F_i = \langle v_1, \ldots, v_i \rangle$. This defines a morphism $Z \to H^{rs}(\ell)$ given by $z \mapsto (F(z), y)$. Using the fact that $Z$ and $H^{rs}(\ell)$ are both étale covers of $g^{rs}$ of the same degree, it is easy to see that $Z \to H^{rs}(\ell)$ is an isomorphism. Then $S_n$ acts on $Z$ by permuting the $v_i$: $\sigma([v_1], \ldots, [v_n]; y) = ([v_{\sigma^{-1}(1)}], \ldots, [v_{\sigma^{-1}(n)}], y)$. It is easy to see that the map $\tilde{\chi} : H^{rs}(\ell) \to d_r$ is $S_n$-equivariant.

8.5 The fundamental groups

Suppose $z = (F, y) \in H^{rs}(\ell)$. We get a surjection $\psi_z : \pi_1(g^{rs}, y) \twoheadrightarrow S_n$ corresponding to the Galois covering $H^{rs}(\ell) \to g^{rs}$ with Galois group $S_n$. Similarly, for a regular diagonal matrix $u \in d_r$, we have a surjection $\psi_u : \pi_1(\text{car}^{rs}, \chi(u)) \twoheadrightarrow S_n$. Since

$$
\begin{align*}
H^{rs}(\ell) & \xrightarrow{\tilde{\chi}} d_r \\
g^{rs} & \xrightarrow{\chi} \text{car}^{rs},
\end{align*}
$$

is a pullback diagram of Galois étale covers with Galois group $S_n$, we have $\psi_z = \psi_{\chi(z)} \circ \chi^*$.

**Definition 61.** We say a polynomial $p \in \text{car}$ is of type $\lambda$ if $p = \prod_{i=1}^\ell (x - x_i)^{\lambda_i}$ where $x_1, \ldots, x_\ell$ are distinct.

**Lemma 62.** Suppose $p = \prod_{i=1}^\ell (x - x_i)^{\lambda_i}$ is a polynomial of type $\lambda$. Then the local monodromy subgroup $H(p)$ of $S_n$ at $p$ for the $S_n$-cover $d^r \to \text{car}^{rs}$ is $S_\lambda$.

**Proof.** Let $\tau$ denote the diagonal matrix

$$
\text{diag}(x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_r, \ldots, x_r).
$$

Then the stabilizer in $S_n$ of $\tau$ is precisely $S_\lambda$. The results then follows from Proposition 57. 

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8.6 The Kostant section

The Kostant section is a morphism $\epsilon : \text{car} \to g^t$ which is a section of $\chi$; i.e., $\chi \circ \epsilon = \text{id}$. We give the definition of $\epsilon$ following Ngô’s paper [19, Theorem 2.1]. We remark, however, that, while the general definition makes sense for any reductive Lie algebra, we only discuss it for $gl_n$.

Let $x_-$ (resp. $x_+$) denote the $n \times n$ matrix with 1’s just below (resp. just above) the diagonal and 0’s everywhere else. Then let $g^{x_+}$ denote the centralizer of $x_+$ in $g$. In [14], Kostant showed that the subspace $x_- + g^{x_+}$ is contained in $g^t$. Moreover, he showed that the restriction of $\chi$ to $x_- + g^{x_+}$ induces an isomorphism onto $\text{car}$. The Kostant section is the inverse morphism $\epsilon : \text{car} \to x_- + g^{x_+}$. In the case of $gl_n$:

$$x_- + g^{x_+} = \begin{pmatrix}
    \begin{pmatrix}
0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\
1 & 0 & x_1 & \ldots & x_{n-3} & x_{n-2} \\
0 & 1 & 0 & \ldots & x_{n-4} & x_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & \ldots & 1 & 0
\end{pmatrix}
\end{pmatrix}.$$ 

From this, it is elementary to compute the Kostant section. For example, for $n = 2$, it sends the characteristic polynomial $p = x^2 + a_1 x + a_0$ to the matrix of the form above with $x_0 = -a_1/2$, $x_1 = a_1^2/4 - a_0$.

**Proposition 63.** Suppose $s \in g^t$ is a regular matrix of type $\lambda$. Then the local monodromy $H(s)$ at $s$ for the $S_n$-cover $H^{rs}(l) \to g^{rs}$ is conjugate to the Young subgroup $S_\lambda$.

**Proof.** We can assume that $s = \epsilon(p)$ for some $p \in \text{car}$. Then, by Proposition 12 the local fundamental group at $p$ is a retract of the local fundamental group at $s$. Since the $S_n$-cover $H^{rs}(l) \to g^{rs}$ is a pullback of the $S_n$-cover $d^t \to \text{car}^{rs}$, it follows that the local monodromy subgroup at $s$ is equal to the local monodromy subgroup at $p$. By Lemma 62 this subgroup is $S_\lambda$. \hfill $\square$

8.7 The commuting group scheme

Write $I := \{(g, x) \in G \times g : \text{Ad}\, g(x) = x\}$. The projection $p : I \to g$ is a group scheme in a more or less obvious way. Write $p^r : I \to g^{rs}$ for the restriction of $p$ to the inverse image of $g^{rs}$. Then $I$ is a torus bundle: the fiber over a point $y \in g^{rs}$ is the maximal torus in $G$ centralizing $y$.

Identify the scheme $Z$ from §8.3 with $H^{rs}(l)$ and form a pullback diagram

$$\begin{array}{ccc}
\mathcal{T}_Z & \to & \mathcal{T} \\
\downarrow & & \downarrow p^r \\
Z & \to & g^{rs}.
\end{array}$$

Then $\mathcal{T}_Z$ is equipped with an isomorphism $\mathcal{T}_Z \to G^o_{nZ}$ to the split torus over $Z$. To see this, suppose $z = ([v_1], \ldots, [v_n]; y)$ and $g \in G$ is an element commuting
with $y$. Then $g$ preserves the eigenspaces of $y$. So for each $i = 1, \ldots, n$, there is a unique character $t_i \in X^*(\mathcal{T})$ such that $gv_i = t_i(g)v_i$. The $n$-tuple of characters $t := (t_1, \ldots, t_n) : \mathcal{T} \to \mathbb{G}_m$ is easily seen to give an isomorphism.

Over $\mathfrak{g}^s$, the torus $\mathcal{T}$ is determined up to isomorphism by its group of characters $X^*(\mathcal{T})$ viewed as a local system over $\mathfrak{g}^s$. Moreover, this local system is canonically isomorphic to $\mathbb{R}^1p^*\mathbb{Z}$. For any point $y \in \mathfrak{g}^s$, the fundamental group $\pi_1(\mathfrak{g}^s, y)$ acts on the fiber of $X^*(\mathcal{T})$ lying over $y$ by permuting the characters $t_1, \ldots, t_n$.

9 Monodromy and Tymoczko’s dot action

9.1 Fiberwise cohomology of $B\mathcal{T}$

For each $y \in \mathfrak{g}^s(\mathbb{C})$, we have a torus $\mathcal{T}_y$ and its associated classifying space $B\mathcal{T}_y$. The cohomology of $B\mathcal{T}_y$ is naturally a polynomial ring $\mathbb{C}[t_1, \ldots, t_n] = \mathbb{C}[X^*(\mathcal{T}_y)]$ generated in degree 2 by the characters of $\mathcal{T}_y$. As we vary $y$, these glue together to form a local system $\mathcal{A}$ of polynomial algebras over $\mathfrak{g}^s$. In fact, since $\mathcal{T}$ is etale locally trivial, we can construct a fiber bundle $\alpha : B\mathcal{T} \to \mathfrak{g}^s$ over $\mathfrak{g}^s$ such that the fiber over each $y \in \mathfrak{g}^s$ is $B\mathcal{T}_y$. Then we have $\mathcal{A} = \bigoplus_{k \geq 0} \mathcal{A}^k$ where $\mathcal{A}^k = \mathbb{R}^{2k}\alpha_*\mathbb{C}$. Let $y_0$ denote the regular semisimple matrices with diagonal entries 1, 2, 3, \ldots, $n$ (written in order). Let $F^0$ denote the standard flag

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots$$

where $e_i$ is the standard basis of $\mathbb{C}^n$. Set $z_0 = (F^0, y_0) \in \mathcal{H}(\ell, y_0)$. This gives rise to a surjection

$$\psi : \pi_1(\mathfrak{g}^s, y_0) \to S_n,$$

(64)

where, for simplicity, we write $\psi := \psi_{z_0}$. Let $T$ denote the fiber of $\mathcal{T}$ over $y_0$. So $T$ is simply the diagonal subgroup of $G$. Then, by the discussion in subsection 8.7, $\pi_1(\mathfrak{g}^s, y_0)$ acts on $X^*(T)$ by permuting the characters. Explicitly, if we let $\sigma(t_i) = t_{\sigma^{-1}(i)}$ for $\sigma \in S_n$, then $\gamma(t_i) = (\psi(\gamma))(t_i)$. Consequently, if we let $A = \mathcal{A}_{y_0} = \mathbb{C}[t_1, \ldots, t_n]$, then $\pi_1(\mathfrak{g}^s, y_0)$ acts on the polynomials in $A$ by $\gamma(p) = (\psi(\gamma))(p)$, where $S_n$ acts on $A$ in the standard way:

$$(\sigma p)(t_1, \ldots, t_n) = p(t_{\sigma(1)}, \ldots, t_{\sigma(n)}).$$

(65)

9.2 Fiberwise equivariant cohomology of Hessenberg varieties

Now, for each Hessenberg function $m$, the torus $\mathcal{T}$ acts on the Hessenberg scheme $\mathcal{H}^s(m) \to \mathfrak{g}^s$. So for each $y \in \mathfrak{g}^s$, we can take the equivariant cohomology groups $H^*_\mathcal{T}_y(\mathcal{H}(m), y)$. By localization, we know that $H^*_\mathcal{T}_y(\mathcal{H}(m), y)$ is a free module of rank $n!$ over $\mathcal{A}_y = H^*(B\mathcal{T}_y)$. Moreover, the canonical inclusion $\mathcal{H}(\ell) \to \mathcal{H}(m)$ induces an inclusion $H^*_\mathcal{T}_y(\mathcal{H}(m), y) \to H^*_\mathcal{T}_y(\mathcal{H}(\ell), y)$. 26
Proposition 68. The action of the pullback morphism $\psi: H \to L$ is equivariant inclusion $\pi$ of $\mathcal{A}$-modules. This can be seen explicitly using Tymoczko’s description of the equivariant cohomology of Hessenberg varieties in terms of moment graphs.

**Proof.** This follows from the fact that Hessenberg varieties are GKM spaces. (See [32, §2 and Proposition 5.4]).

For each $m$, localization induces an inclusion $L(m) \to L(\ell)$ of $\mathcal{A}$-modules. Write $L(m)$ for the fiber, $H^*_+(\mathcal{H}(m,y_0))$ of $L(m)$ over $y_0$. Then $L(m)$ is free as an $A$-module, and both $A$ and $L(m)$ are equipped with compatible actions of $\pi_1(g^\mathcal{A},y_0)$. If we write $A_+$ for the ideal of positive degree polynomials, then we have

$$H^*\mathcal{H}(m,y_0) = A/A_+L(m),$$

and the monodromy action of $\pi_1(g^\mathcal{A},y_0)$ on both sides is compatible.

**Proposition 66.** The action of $\pi_1(g^\mathcal{A},y_0)$ on $L(m)$ factors through the homomorphism $\psi: \pi_1(g^\mathcal{A},y_0) \to S_n$.

**Proof.** The pullback $\mathcal{F}$ of $\mathcal{T}$ to the $S_n$-cover $Z \to g^\mathcal{A}$ is a constant group scheme, and the pullback of $\mathcal{H}^*(\ell) = Z$ to $Z$ is disjoint union of copies of $Z$ indexed by elements of $S_n$. It follows that the action of $\pi_1(g^\mathcal{A},y_0)$ on $H^*_+(\mathcal{H}(\ell,y_0))$ is trivial on the image of the map $\pi_1(Z,z_0) \to \pi_1(g^\mathcal{A},y_0)$. In other words, the action of $\pi_1(g^\mathcal{A},y_0)$ on $H^*_+(\mathcal{H}(\ell,y_0))$ factors through $S_n$.

Since $L(m) \to L(\ell)$ is an inclusion of local systems, we have a $\pi_1(g^\mathcal{A},y_0)$-equivariant inclusion $L(m) \to L(\ell)$. The result follows.

**Corollary 69.** The action of $\pi_1(g^\mathcal{A},y_0)$ on $H^*(\mathcal{H}^\mathcal{A}(m,y_0))$ induced from the local system $R^*\pi_1(g^\mathcal{A},y_0)$ factors through $\psi: \pi_1(g^\mathcal{A},y_0) \to S_n$.

**Proof.** This follows directly from Propositions 68 and 69.

**Definition 70.** The action of $S_n$ on $L(m)$ (resp. $H^*(\mathcal{H}^\mathcal{A}(m,y_0))$) coming from Proposition 68 (resp. Corollary 69) is called the monodromy action of $S_n$.

### 9.3 Monodromy action for $\mathcal{H}^\mathcal{A}(\ell)$

To make the monodromy action of $S_n$ on $L(\ell)$ explicit, recall from [37, 4.1] that $z_0$ denotes the element of $Z_0 := \mathcal{H}(\ell,y_0)$ corresponding to $y_0$ with the standard ordering of its eigenspaces. So $z_0 = ([e_1], \ldots, [e_n], y_0)$. Given $\sigma \in S_n$, we have $\sigma z_0 = ([e_{\sigma^{-1}(1)}], \ldots, [e_{\sigma^{-1}(n)}], y_0)$. The cohomology group $H^*Z_0 = H^0Z_0$
is simply the group of functions \( f : Z_0 \to \mathbb{C} \). If, for \( w \in S_n \), we write \( \delta_w \) for the function taking \( wz_0 \) to 1 and all other elements of \( Z_0 \) to 0, then we have \( (\sigma \delta_w)(z) = \delta_w(\sigma^{-1} z) \). From this it easily follows that \( \sigma \delta_w = \delta_{\sigma w} \).

**Lemma 71.** As an \( A \)-module, \( L(\ell) \) is isomorphic to the module \( A^{|S_n|} \) of functions from the set \( S_n \) to \( A \). The monodromy action of \( S_n \) on \( L(\ell) \) is given by

\[
((wp)(v))(t_1, \ldots, t_n) = (p(w^{-1}v))(t_{w(1)}, \ldots, t_{w(n)})
\]

where \( v, w \in S_n \), \( p \in A^{|S_n|} \) and \( t_1, \ldots, t_n \) are variables.

**Proof.** Under the identification \( S_n \to Z_0 \) given by \( w \mapsto wz_0 \), the \( \delta_w \) form a \( \mathbb{C} \)-basis of \( H^0(Z_0) \). Moreover, the map \( H^0(Z_0) \to H^0(Z_0) \) is an \( S_n \)-equivariant isomorphism, and, under this identification, the \( \delta_w \) freely generate \( H^* \) as an \( A \)-module. The result now follows by direct verification using the fact that \( S_n \) acts on \( A \) as in (65).

**Corollary 72.** The monodromy action of \( S_n \) agrees with Tymoczko’s dot action on \( H^*(\mathcal{H}(\ell, y_0)) \).

**Proof.** This follows immediately by comparing the description of the monodromy action in Lemma 71 with Tymoczko’s description of the dot action [32 §3.1].

**Theorem 73.** Let \( m \) be a Hessenberg function. The monodromy action of \( S_n \) on \( H^*(\mathcal{H}(m, y_0)) \) is the same as Tymoczko’s dot action.

**Proof.** Under Tymoczko’s dot action, \( H^*(\mathcal{H}(m, y_0)) \) is an \( S_n \)-equivariant \( A \)-submodule of \( H^*(\mathcal{H}(\ell, y_0)) \). The same is true of the monodromy action of \( S_n \). Therefore, by Corollary 72 the two actions must coincide.

**Corollary 74.** Tymoczko’s dot action of \( S_n \) on the non-equivariant cohomology group \( H^*(\mathcal{H}(m, y_0)) \) coincides with the monodromy action.

**Proof.** Tymoczko defines the dot action on \( H^*(\mathcal{H}(m, y_0)) \) as the dot action on the quotient \( L(m)/A \cdot L(m) \). The monodromy action is also given by this quotient.

Since Tymoczko’s dot action and the monodromy action of \( S_n \) coincide, we will not distinguish between them from now on: it will be the only action of \( S_n \) appearing in the remainder of the paper.

**Theorem 75.** Let \( s \in g' \) be a regular element of type \( \lambda \) and let \( \pi = \pi_m : \mathcal{H}(m) \to g \). Let \( B(s) \) be a sufficiently small ball in \( g \) centered at \( s \). Then, for each \( k \in \mathbb{Z} \), there is a \( \mathbb{C} \)-vector space isomorphism

\[
H^0(B(s) \cap g^*, R^k \pi_\lambda \mathbb{C}) \cong H^k(\mathcal{H}(m, y_0))^S_{\lambda}.
\]
Proof. By (43) applied with $L = R^k \pi_* \mathbb{C}$, we have

$$H^0(B(s) \cap g^s, R^k \pi_* \mathbb{C}) = H^k(\mathcal{H}(m, b))^{\pi_1(B(s) \cap g^s, b)}$$

where $b$ is any point in $B(s) \cap g^s$. The last vector space is isomorphic to the invariants of $H^k(\mathcal{H}(m, y))$ under the local monodromy at $s$. The result then follows from Proposition 63. \qed

**Theorem 76.** Suppose $s \in g^r$ is a regular element of type $\lambda$. Then, for each $k \in \mathbb{Z}$,

$$\dim H^k(\mathcal{H}(m, s)) = \dim H^k(\mathcal{H}(m, y_0))^{S_\lambda}. \quad (77)$$

**Proof.** We are going to apply Theorem 48 to the morphism $\pi : \mathcal{H}(m) \rightarrow g$. Both the source and the target of $\pi$ are smooth, quasi-projective varieties. Moreover, $\pi$ has relative dimension $|m|$. (One way to check this is to use the fact that the projection $pr_1 : \mathcal{H}(m) \rightarrow \mathcal{X}$ has relative dimension $\sum_{i=1}^n m_i$, while $\dim \mathcal{X} = \sum_{i=1}^{n-1} i$. Another way to see it, is to use the fact that the regular semisimple Hessenberg varieties have dimension $|m|$.)

By Corollary 33, we have

$$\dim H^i(\mathcal{H}(m, s), \mathbb{C}) = \dim H^{2|m|-i}(\mathcal{H}(m, s), \mathbb{C})$$

for all $i$.

It follows then from Theorem 48 that the local invariant cycle map

$$H^i(\mathcal{H}(m, s)) \rightarrow H^0(B(s) \cap g^s, R^i \pi_* \mathbb{C})$$

is an isomorphism, where $B(s)$ is any sufficiently small ball centered at $s$ in $g$. The result now follows from Theorem 75. \qed

Finally we can put all the pieces together to prove Conjecture 3.

**Theorem 78.** If $\chi_{m,d}$ denotes the dot action on the cohomology group $H^{2d}$ of the regular semisimple Hessenberg variety $\mathcal{H}(m, s)$, then $\chi_{m,d}$ equals the coefficient of $t^d$ in $\omega_{X_{G(m)}}(t)$.

**Proof.** By Theorem 32 the left-hand side of Equation (77) (in Theorem 76) equals $c_{d,\lambda}(m)$ when $k = 2d$. On the other hand, by Proposition 8 the right-hand side of Equation (77) equals the coefficient of $m_\lambda$ in $\chi_{m,d}$. \qed

## 10 Future work

In this paper, we have considered only Hessenberg varieties of type A, but De Mari, Procesi, and Shayman defined Hessenberg varieties in the context of an arbitrary reductive linear algebraic group [5]. Precup [20] has computed the cohomology of regular Hessenberg varieties in all types. We hope that eventually it will be possible to combine our methods with Precup’s result to
compute Tymoczko’s dot action on the cohomology of Hessenberg varieties in all (or, at least, all classical) types.

The work of Haiman [11] has established a connection between $X_G$ and representation theory, but in seemingly a very different way. It would be interesting to find a direct connection between Haiman’s work and ours.

References

[1] H. Abe, M. Harada, T. Horiguchi, and M. Masuda. The equivariant cohomology rings of regular nilpotent Hessenberg varieties in Lie type A: a research announcement. *ArXiv e-prints*, November 2014, 1411.3065.

[2] Dave Anderson and Julianna Tymoczko. Schubert polynomials and classes of Hessenberg varieties. *J. Algebra*, 323(10):2605–2623, 2010.

[3] A. A. Be˘ılinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[4] Timothy Y. Chow. The path-cycle symmetric function of a digraph. *Adv. Math.*, 118(1):71–98, 1996.

[5] F. De Mari, C. Procesi, and M. A. Shayman. Hessenberg varieties. *Trans. Amer. Math. Soc.*, 332(2):529–534, 1992.

[6] Alexandru Dimca. *Singularities and topology of hypersurfaces*. Universitext. Springer-Verlag, New York, 1992.

[7] Vesselin Gasharov. Incomparability graphs of (3 + 1)-free posets are s-positive. In *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994)*, volume 157, pages 193–197, 1996.

[8] Ira M. Gessel. Multipartite $P$-partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 1984.

[9] Hans Grauert and Reinhold Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984.

[10] M. Guay-Paquet. A modular relation for the chromatic symmetric functions of (3+1)-free posets. *ArXiv e-prints*, June 2013, 1306.2400.

[11] Mark Haiman. Hecke algebra characters and immanant conjectures. *J. Amer. Math. Soc.*, 6(3):569–595, 1993.
[12] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[13] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[14] Bertram Kostant. Lie group representations on polynomial rings. *Amer. J. Math.*, 85:327–404, 1963.

[15] Shrawan Kumar. Finiteness of local fundamental groups for quotients of affine varieties under reductive groups. *Comment. Math. Helv.*, 68(2):209–215, 1993.

[16] S. Lojasiewicz. Triangulation of semi-analytic sets. *Ann. Scuola Norm. Sup. Pisa (3)*, 18:449–474, 1964.

[17] Eduard J. N. Looijenga. *Isolated singular points on complete intersections*, volume 5 of *Surveys of Modern Mathematics*. International Press, Somerville, MA; Higher Education Press, Beijing, second edition, 2013.

[18] David Mumford. *Algebraic geometry. I*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Complex projective varieties, Reprint of the 1976 edition.

[19] Bao Châu Ngô. Fibration de Hitchin et endoscopie. *Invent. Math.*, 164(2):399–453, 2006.

[20] Martha Precup. Affine pavings of Hessenberg varieties for semisimple groups. *Selecta Math. (N.S.)*, 19(4):903–922, 2013.

[21] David Prill. Local classification of quotients of complex manifolds by discontinuous groups. *Duke Math. J.*, 34:375–386, 1967.

[22] Bruce E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.

[23] J. Shareshian and M. L. Wachs. Chromatic quasisymmetric functions. *ArXiv e-prints*, May 2014, 1405.4629.

[24] John Shareshian and Michelle L. Wachs. Chromatic quasisymmetric functions and Hessenberg varieties. In *Configuration spaces*, volume 14 of CRM Series, pages 433–460. Ed. Norm., Pisa, 2012.

[25] Peter Slodowy. *Simple singularities and simple algebraic groups*, volume 815 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[26] T. A. Springer. Quelques applications de la cohomologie d’intersection. In *Bourbaki Seminar, Vol. 1981/1982*, volume 92 of *Astérisque*, pages 249–273. Soc. Math. France, Paris, 1982.
[27] Richard P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111(1):166–194, 1995.

[28] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[29] Richard P. Stanley and John R. Stembridge. On immanants of Jacobi-Trudi matrices and permutations with restricted position. *J. Combin. Theory Ser. A*, 62(2):261–279, 1993.

[30] Nicholas James Teff. *The Hessenberg representation*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–The University of Iowa.

[31] Julianna S. Tymoczko. Linear conditions imposed on flag varieties. *Amer. J. Math.*, 128(6):1587–1604, 2006.

[32] Julianna S. Tymoczko. Permutation actions on equivariant cohomology of flag varieties. In *Toric topology*, volume 460 of *Contemp. Math.*, pages 365–384. Amer. Math. Soc., Providence, RI, 2008.