On generalized Melvin solution for the Lie algebra $E_6$

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Abstract A multidimensional generalization of Melvin’s solution for an arbitrary simple Lie algebra $G$ is considered. The gravitational model in $D$ dimensions, $D \geq 4$, contains $n$ 2-forms and $l \geq n$ scalar fields, where $n$ is the rank of $G$. The solution is governed by a set of $n$ functions $H_s(z)$ obeying $n$ ordinary differential equations with certain boundary conditions imposed. It was conjectured earlier that these functions should be polynomials (the so-called fluxbrane polynomials). The polynomials $H_s(z)$, $s = 1, \ldots, 6$, for the Lie algebra $E_6$ are obtained and a corresponding solution for $l = n = 6$ is presented. The polynomials depend upon integration constants $Q_s$, $s = 1, \ldots, 6$. They obey symmetry and duality identities. The latter ones are used in deriving asymptotic relations for solutions at large distances. The power-law asymptotic relations for $E_6$-polynomials at large $z$ are governed by the integer-valued matrix $v = A^{-1}(I + P)$, where $A^{-1}$ is the inverse Cartan matrix, $I$ is the identity matrix and $P$ is a permutation matrix, corresponding to a generator of the $Z_2$-group of symmetry of the Dynkin diagram. The 2-form fluxes $\Phi^s$, $s = 1, \ldots, 6$, are calculated.

1 Introduction

In this paper we deal with a multidimensional generalization of the Melvin solution [1] which was considered earlier in Ref. [2]. This solution is governed by a simple finite-dimensional Lie algebra. It is a special case of the so-called generalized fluxbrane solutions from [3]. For generalizations of the Melvin solution, fluxbrane solutions and their applications, see Refs. [4–33] and the references therein.

We remind the reader that Melvin’s original solution in 4d space-time describes the gravitational field of a magnetic flux tube. The multidimensional analog of such a flux tube, supported by a certain configuration of fields of forms, is referred to as a fluxbrane (a “thickened brane” of magnetic flux). The appearance of fluxbrane solutions was motivated by superstring/M-theory models. A physical interest in such solutions is that they supply an appropriate background geometry for studying various processes involving branes, instantons, Kaluza–Klein monopoles, pair production of magnetically charged black holes and other configurations which can be studied via a special kind of Kaluza–Klein reduction (“modding technique”) of a certain multidimensional model in the presence of $U(1)$ isometry subgroup.

The Melvin solution is geodesically complete [34]. Its group of isometry is $U(1) \times P(1, 1)$, where $P(1, 1)$ is 3-dimensional isometry group of 2-dimensional Minkowski space. $P(1, 1)$ is semi-direct product of $O(1, 1)$ and $\mathbb{R}^2$.

In Ref. [2] the electro-vacuum Melvin solution was generalized for the $D$-dimensional model which contains metric $g$, $n$ 2-form fields $F^s = dA^s$ and $l$ scalar fields $\phi^a$. The model also includes $n$ dilatonic coupling vectors belonging to $\mathbb{R}^l$. The $D$-dimensional warped product solution from Ref. [2] comprises two factor spaces: 1-dimensional subspace $M_1$ and a $(D - 2)$-dimensional Ricci-flat subspace $M_2$. Here $M_1$ is either $\mathbb{R}$ or $S^1$. For $M_1 = S^1$ we have a cylindrically symmetric solution with the isometry group $U(1) \times \text{Isom}(M_2)$, where Isom$(M_2)$ is the isometry group of $M_2$.

The generalized fluxbrane solutions from Ref. [2] are governed by functions $H_s(z) > 0$ defined on the interval $(0, +\infty)$ which obey the non-linear differential equations:

$$\frac{d}{dz} \left( \frac{z}{H_s} \frac{d}{dz} H_s \right) = P_s \prod_{s' = 1}^{n} H_{s'}^{-A_{ss'}}$$

with the following boundary conditions:

$$H_s(+0) = 1,$$

$$s = 1, \ldots, n,$$

where $P_s > 0$ for all $s$. Parameters $P_s$ are proportional to $Q_s^2$, where $Q_s$ are integration constants and
\( z = \rho^2 \), where \( \rho \) is a radial parameter. The boundary condition \((1.2)\) guarantees the absence of a conic singularity (in the metric) for \( \rho = +0 \). The integration constants \( Q_s \) are coinciding up to a sign with values of magnetic fields on the axis of the symmetry.

In this paper we assume that \((A_{\alpha\beta})\) is a Cartan matrix for some simple finite-dimensional Lie algebra \( \mathcal{G} \) of rank \( n \) \((A_{\alpha\beta} = 2 \text{ for all } s)\).

According to a conjecture suggested in [3], the solutions to Eqs. \((1.1), (1.2)\) governed by the Cartan matrix \((A_{\alpha\beta})\) are polynomials:

\[
H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k, \quad (1.3)
\]

where \( P_s^{(k)} \) are constants \((P_s^{(1)} = P_s)\). Here \( P_s^{(n_s)} \neq 0 \) and

\[
n_s = 2 \sum_{s'=1}^{n} A^{s's'} \quad (1.4)
\]

where we denote \((A^{s's'}) = (A_{\alpha\beta})^{-1}\). Integers \( n_s \) are components of a twice dual Weyl vector in the basis of simple co-roots [35].

The set of fluxbrane polynomials \( H_s \) defines a special solution to the open Toda chain equations \([36,37]\) corresponding to a simple finite-dimensional Lie algebra \( \mathcal{G} \); see Ref. [38]. In Refs. [2,39] a program (in Maple) for calculation of these polynomials for classical series of Lie algebras (of \( A-, B-, C- \) and \( D-\)series) was suggested.

It should be noted that the open Toda chain corresponding to the Lie algebra \( \mathcal{G} \) has a hidden symmetry group \( G_T = \exp(\mathcal{G}) \). The solution from Ref. [2] corresponding to this group is a special case of solutions from [3]. It may be obtained by using an 1-dimensional sigma-model \([40-42]\) with \((2+l+n)\)-dimensional target space. The isometry group of this target space \( G_{\text{sm}} \) (related to the sigma model) was studied in detail in [43]. For another more general setup with non-diagonal metrics (which is valid for flat \( M_2 \)) see also [9].

The group \( G_{\text{sm}} \) is another hidden symmetry group related to our model. Here the Toda Lagrangian \( L_T \) may be obtained from the sigma-model one after integrating the Maxwell-type equations corresponding to potentials \( \Phi^s(u) = A^s_{\alpha\beta}(u) \), where \( u \) is a radial variable and \( \Phi \) is a coordinate on \( M_1 \) \((0 < \phi < 2\pi \text{ for } M_1 = S^1)\), and obtaining integration constants \( Q_s \). The Toda Lagrangian \( L_T = L_T(x, \dot{x}, Q) \) is responsible for the equations of motion for \( 2 \) scale factors and \( l \) scalar fields described by \( x = (x^s) \) for fixed \( Q = (Q_s) \).

We note also that there are several multidimensional aspects of generalized Melvin solution from Ref. [2]: (1) the space-time dimension \( D \) (for Melvin’s solution \( D = 4 \)), (2) the rank of the Toda group \( G_T \) which is equal to \( n \) (in Melvin’s case \( n = 1 \)) and (3) the dimension of the target space of the corresponding sigma-model which is equal to \( N = n + l + 2 \) (in Melvin’s case \( N = 3 \)).

Here we verify the conjecture from Ref. [3] for the Lie algebra \( E_6 \). In Sect. 2 the generalized Melvin solution for an arbitrary simple finite-dimensional Lie algebra \( \mathcal{G} \) is considered. The exact solution for the Lie algebra \( E_6 \) is presented in Sect. 3, while the fluxbrane polynomials are listed in the appendix. Here duality relations for the polynomials \( H_s(z) \) and asymptotic formulas for \( z \rightarrow +\infty \) are presented, as well as the asymptotics for the solutions at large distances and a calculation of flux integrals. We find that any flux \( \Phi^s \) depends upon the integration constant \( Q_s \), and does not depend upon the other constants \( Q_{s'}, s' \neq s \). The flux \( \Phi^s \) is proportional to \( n_s Q_s^{-1} \), where \( n_s \) are integer numbers \((1.4)\):

\[
n_s = 16, 30, 42, 30, 16, 22 \quad \text{for } s = 1, 2, 3, 4, 5, 6 \text{ respectively.}
\]

2 The main solution

We consider a model governed by the action

\[
S = \int d^D x \sqrt{|g|} \left\{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \frac{1}{2} \sum_{s=1}^{n} \exp(2\lambda_s(\varphi))(F^s)^2 \right\}, \quad (2.1)
\]

where \( g = g_{MN}(x)dx^M \otimes dx^N \) is a metric, \( \varphi = (\varphi^\alpha) \in \mathbb{R}^l \) is a vector of scalar fields, \((h_{\alpha\beta})\) is a constant symmetric non-degenerate \( l \times l \) matrix \((l \in \mathbb{N})\), \( F^s = dA^s = F^{s}_{MN} dx^M \wedge dx^N \) is a 2-form, \( \lambda_s(\varphi) = \lambda_s \varphi^s \), \( s = 1, \ldots, n; \alpha = 1, \ldots, l \).

In \((2.1)\), we denote \(|g| = \left| \det(g_{MN}) \right|\), \((F^s)^2 = F^{s}_{MN} F^{s}_{MN} g^{MN} g_{MN} g_{MN} \) for \( s = 1, \ldots, n \).

Here we consider a family of exact solutions to the field equations corresponding to the action \((2.1)\) and depending on one variable \( \rho \). The solutions are defined on the manifold

\[
M = (0, +\infty) \times M_1 \times M_2, \quad (2.2)
\]

where \( M_1 \) is a one-dimensional manifold (say \( S^1 \) or \( \mathbb{R} \)) and \( M_2 \) is a \((D-2)\)-dimensional Ricci-flat manifold. The solution reads \([2]\)

\[
g = \left( \prod_{s=1}^{n} H_{s}^{2h_s/(D-2)} \right) \left\{ w d\rho \otimes d\rho + \left( \prod_{s=1}^{n} H_{s}^{-2h_s} \right) \rho^2 d\phi \otimes d\phi + g^2 \right\}, \quad (2.3)
\]

\[
\exp(\varphi^\alpha) = \sum_{s=1}^{n} H_{s}^{h_s} \varphi^\alpha, \quad (2.4)
\]
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\[ F_s = -Q_s \left( \prod_{s'=1}^n H_{s'}^{A_{ss'}} \right) \rho d\rho \wedge d\phi. \] (2.5)

\[ \alpha = 1, \ldots, l \text{ and } s = 1, \ldots, n, \text{ where } w = \pm 1, \ g^1 = d\phi \otimes d\phi \text{ is a metric on } M_1 \text{ and } g^2 \text{ is a Ricci-flat metric on } M_2. \]

The functions \( H(z) > 0, \ z = \rho^2, \) obey Eqs. (1.1) with the boundary conditions (1.2) and

\[ P_s = \frac{1}{4} K_s Q_s^2. \] (2.6)

The parameters \( h_s \) satisfy the relations

\[ h_s = K_s^{-1}, \quad K_s = B_{ss} > 0, \] (2.7)

where

\[ B_{ss'} = 1 + \frac{1}{2 - D} + \lambda_{sa} \lambda_{sb} h^{a\beta}, \] (2.8)

\( s, s' = 1, \ldots, n, \) with \((h^{a\beta}) = (h_{ab})^{-1}. \) Here \( \lambda^{a\beta} = h^{a\beta} \lambda_{sb} \) and

\[ (A_{ss'}) = (2B_{s's'}) \] (2.9)

is the Cartan matrix for a simple Lie algebra \( G \) of rank \( n. \)

It may be shown that if the matrix \((h_{a\beta})\) has an Euclidean signature and \( l \geq n, \) there exists a set of co-vectors \( \lambda_1, \ldots, \lambda_n \) obeying (2.9). Thus the solution is valid at least when \( l \geq n \) and the matrix \((h_{a\beta})\) is positive-definite.

The solution under consideration is as a special case of the fluxbrane (for \( w = +1, M_1 = S^1 \)) and \( S \)-brane \((w = -1)\) solutions from [3] and [31], respectively.

If \( w = +1 \) and the (Ricci-flat) metric \( g^2 \) has a pseudo-Euclidean signature, we get a multidimensional generalization of Melvin’s solution [1].

Melvin’s solution (without scalar field) corresponds to \( D = 4, \ n = 1, \ M_1 = S^1 \) \((0 < \phi < 2\pi), \ M_2 = \mathbb{R}^2, \)
\( g^2 = -dt \otimes dt + d\xi \otimes d\xi \) and \( G = A_1. \)

For \( w = -1 \) and \( g^2 \) of Euclidean signature we obtain a cosmological solution with a horizon (as \( \rho = 0 \)) if \( M_1 = \mathbb{R} \)
\((-\infty < \phi < +\infty). \)

### 3 The solution for the Lie algebra \( E_6 \)

Here we deal with the solution for \( n = l = 6, \ w = +1 \) and \( M_1 = S^1, \) which corresponds to the Lie algebra \( E_6. \) We put here \( h_{a\beta} = \delta_{a\beta} \) and denote \((\lambda_{sa}) = (\lambda^a_s) = \lambda_s, s = 1, \ldots, 6. \)

The matrix \( A = (A_{ss'}) \) coincides with the Cartan matrix for the exceptional Lie algebra \( E_6. \)

\[ \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \] (3.1)

This matrix is graphically depicted at Fig. 1 by the Dynkin diagram.

### 3.1 Fluxbrane polynomials for Lie algebra \( E_6 \)

The inverse Cartan matrix for \( E_6, \)

\[ A = (A_{ss'}) = \begin{pmatrix} \frac{4}{3} & \frac{5}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{5}{3} & \frac{10}{3} & 4 & \frac{8}{3} & 4 & \frac{4}{3} \\ 2 & 4 & 6 & 4 & 2 & 3 \\ 4 & 8 & 4 & 10 & \frac{5}{3} & \frac{5}{2} \\ \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{5}{3} & \frac{4}{3} & 1 \\ 1 & 2 & 3 & 2 & 1 & 2 \end{pmatrix} \] (3.2)

implies due to (1.4)

\[ (n_1, n_2, n_3, n_4, n_5, n_6) = (16, 30, 42, 30, 16, 22). \] (3.3)

For the Lie algebra \( E_6 \) we find the set of six fluxbrane polynomials, which are listed in the appendix. Here as in [38] we parametrize the polynomials by using other parameters (here denoted \( B_i \)) instead of \( P_s: \)

\[ P_s = n_s B_s, \] (3.4)

\( s = 1, \ldots, 6. \) This is necessary to avoid huge denominators in monomials of \( H_s. \)

The polynomials have the following structure:

\[ H_1 = 1 + 16B_1z + 120B_1B_2z^2 + \cdots + 120B_1^2B_2B_3B_4B_5B_6z^{14} + 120B_1^2B_2^2B_3B_4B_5B_6z^{15} + B_1^2B_2^3B_3B_4B_5B_6z^{16}. \]

\[ H_2 = 1 + 30B_2z + (120B_1B_2 + 315B_1B_2)z^{2} \cdots + (120B_1^2B_2B_3B_4B_5B_6z^{28} + 30B_1^3B_2B_3B_4B_5B_6z^{29} + B_1^3B_2^2B_3B_4B_5B_6z^{30}. \]
\[ H_3 = 1 + 42 B_3 z + (315 B_3 B_3 + 315 B_4 B_3 + 231 B_6 B_3) z^2 \ldots + (315 B^4 B_3 B_3 B_3 B_6 B_3 B_6 B_3) z^{11} + (315 B^4 B_3 B_3 B_4 B_4 B_6 B_6) z^{11} + 231 B^4 B_3 B_3 B_4 B_4 B_6 B_6 z^{40} + 42 B^4 B_3 B_3 B_4 B_4 B_6 z^{41} + B^4 B_3 B_3 B_4 B_4 B_6 z^{42}. \]

\[ H_4 = 1 + 30 B_4 z + (315 B_3 B_4 + 120 B_5 B_4) z^2 \ldots + (120 B^2 B_4 B_3 B_4 B_5 B_6 z^3 + 315 B^2 B_4 B_3 B_4 B_5 B_6 z^2) z^{28} + 30 B^2 B_4 B_3 B_4 B_5 B_6 z^{29} + B^2 B_4 B_3 B_4 B_5 B_6 z^{30}. \]

\[ H_5 = 1 + 16 B_5 z + 120 B_5 B_4 z^2 + \ldots + 120 B_1 B^2 B_3 B_4 B_5 B_6 z^{14} + 16 B_1 B^2 B_3 B_4 B_5 B_6 z^{15} + B_1 B^2 B_3 B_4 B_5 B_6 z^{16}. \]

\[ H_6 = 1 + 22 B_6 z + 231 B_5 B_6 z^2 + \ldots + 231 B^2 B_4 B_3 B_4 B_5 B_6 z^{30} + 22 B^2 B_4 B_3 B_4 B_5 B_6 z^{31} + B^2 B_4 B_3 B_4 B_5 B_6 z^{32}. \] (3.5)

The powers of polynomials are in agreement with Eq. (3.3). In what follows we denote

\[ H_s = H_s(z) = H_s(z, (B_i)), \quad (3.6) \]

\[ s = 1, \ldots, 6; \] where \((B_i) = (B_1, B_2, B_3, B_4, B_5, B_6).\)

Due to (3.5) the polynomials have the following asymptotical behavior:

\[ H_s = H_s(z, (B_i)) \sim \left( \prod_{i=1}^{6} (B_i)^{\nu_{ij}} \right) z^{n_s} \equiv H_s^a(z, (B_i)), \quad (3.7) \]

\[ s = 1, \ldots, 6, \text{ as } z \to \infty. \]

\[ \nu = (\nu_{ij}) = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 2 \\ 3 & 6 & 8 & 6 & 3 & 4 \\ 4 & 8 & 12 & 8 & 4 & 6 \\ 3 & 6 & 8 & 6 & 3 & 4 \\ 2 & 3 & 4 & 3 & 2 & 2 \\ 2 & 4 & 6 & 4 & 2 & 4 \end{pmatrix}. \] (3.8)

The matrix (3.8) is related to the inverse Cartan matrix as follows:

\[ \nu = A^{-1}(I + P), \quad (3.9) \]

where \(I\) is a 6 \times 6 identity matrix and

\[ P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \] (3.10)

is permutation matrix. This matrix corresponds to the permutation \(\sigma \in S_6\) (\(S_6\) is the symmetric group)

\[ \sigma : (1, 2, 3, 4, 5, 6) \leftrightarrow (5, 4, 3, 2, 1, 6), \] (3.11)

by the relation \(P = (P^i_j) = (\delta^i_j)^\sigma\). Here \(\sigma\) is the generator of the group \(G = \{\sigma, i d\}\) which is the symmetry group of the Dynkin diagram. \(G\) is isomorphic to the group \(Z_2\). \(\sigma\) is a composition of two transpositions: (1 5) and (2 4).

We note that the matrix \(\nu\) is symmetric and

\[ \sum_{s=1}^{6} \nu_{ij} = \eta_{ij}, \quad (3.12) \]

\(l = 1, \ldots, 6.\)

Let us denote \(\hat{B}_i = B_{\sigma(i)}\), \(i = 1, \ldots, 6.\) We call the ordered set \((\hat{B}_i)\) a dual one to the ordered set \((B_i)\). By using the relations for polynomials from the appendix we are led to the following two identities which are verified with the aid of Mathematica.

**Proposition 1** For all \(B_i\) and \(z\)

\[ H_{\sigma(i)}(z, (B_i)) = H_j(z, (\hat{B}_i)), \quad (3.13) \]

\(s = 1, \ldots, 6.\)

**Proposition 2** For all \(B_i \neq 0\) and \(z \neq 0\)

\[ H_s(z, (B_i)) = H_s^a(z, (B_i)) H_s(z^{-1}, (\hat{B}_i^{-1})), \quad (3.14) \]

\(s = 1, \ldots, 6.\)

We call (3.13) symmetry relations, and (3.14) duality ones.

### 3.2 Exact solution for \(E_6\), fluxes and asymptotics

The solution (2.3)–(2.5) in our case reads

\[ g = \left( \prod_{s=1}^{6} H_s^{2s/(D-2)} \right) \left[ d\rho \otimes d\rho \right] + \left( \prod_{s=1}^{6} H_s^{-2s} \right) \rho^2 d\phi \otimes d\phi + g^2 \right], \] (3.15)

\[ \exp(\varphi^a) = \prod_{s=1}^{6} H_s^{(h^a_{ij})}, \] (3.16)

\[ F^a = B^a \rho d\rho \wedge d\phi, \] (3.17)

\(a, s = 1, \ldots, 6,\) where \(g^1 = d\phi \otimes d\phi\) is a metric on \(M_4 = S^4\) (0 < \(\phi < 2\pi\)), \(g^2\) is a Ricci-flat metric on \(M_2\) of signature \((- + , +, +).\) Here

\[ B^a = -Q_s \left( \prod_{l=1}^{s} H_{l}^{-A_{ij}} \right), \] (3.18)

and due to (2.7)–(2.9)

\[ K = K_s = \frac{D - 3}{D - 2} + \mu^2, \] (3.19)
As another (sigma model) hidden group ρ as rank 2 and 3 in Refs. [44] and [45], respectively. We note that phantom scalar fields were considered for Lie algebras of (Γ1, λ) obeying (3.20). Indeed, the matrix (Γ1, λ) is positive-definite for K > K0, where K0 is some positive number. Hence there exists a matrix Λ, such that ΛT Λ = Γ. We put (Λas) = (λs) and get the set of vectors obeying (3.20).

Remark Let us put h_{aβ} = −δ_{aβ}. It may be shown (along a line as was done for h_{aβ} = δ_{aβ}) that, for K < K0, where K0 is some negative number, there exist vectors λs of equal length which obey relations

$$-λ_s λ_{s'} = \frac{1}{2} K A_{s's'} - \frac{D - 3}{D - 2} \equiv Γ_{s's'},$$  (3.20)

s, s' = 1, . . . , 6. For large enough K there exist vectors λs of equal length which obey relations (3.20). Indeed, the matrix (Γ1, λ) is positive-definite for K > K0, where K0 is some positive number. Hence there exists a matrix Λ, such that ΛT Λ = Γ. We put (Λas) = (λs) and get the set of vectors obeying (3.20).

following from (2.8) and (2.9). Thus, for both choices of signatures h_{aβ} = ±δ_{aβ} we get the same algebra (in our case E6) and the same hidden group GT. So, the properties of the matrix (h_{aβ}) are not a priori known from the properties of the group GT. In the case of phantom scalar fields, when h_{aβ} = −δ_{aβ}, we get solutions which are defined for ρ < ρ0, where ρ0 > 0. The cosmological analogs of such solutions with phantom scalar fields were considered for Lie algebras of rank 2 and 3 in Refs. [44] and [45], respectively. We note that another (sigma model) hidden group Gsm (see Introduction) depends upon the choice of the matrix (h_{aβ}) [43].

Now let us consider oriented 2-dimensional manifold Ms = (0, +∞) × S1. The flux integrals

$$Φ^s = \int_{M_s} F^s = 2π \int_0^{+∞} dρ ρ B^s,$$  (3.22)

are convergent since due to

$$H^s ∼ C_s ρ^{2n_s}, \quad C_s = \prod_{i=1}^6 B_i^{v_{sl}},$$  (3.23)

for ρ → +∞, and the equality ∑_i A_i n_i = 2 (following from (1.4)), we get

$$B^s ∼ -Q_s C^s ρ^{-3},$$  (3.24)

as ρ → +∞, where

$$C^s = \prod_{i=1}^6 C_i^{-A_i} = \prod_{k=1}^6 \prod_{l=1}^6 B_i^{-A_i l_{kl}},$$  (3.25)

s = 1, . . . , 6. Due to (3.9) we get Aν = I + P

$$C^s = \prod_{i=1}^6 B_i^{-(I + P)_{si}} = \prod_{i=1}^6 \prod_{l=1}^6 B_i^{-(δ_{s'i} - δ_{s'i})} = B_s^{-1} B_{σ(s)}^{-1},$$  (3.26)

s = 1, . . . , 6.

By using Eq. (1.1) we obtain

$$\int_0^{+∞} dρ ρ B^s = -Q_s P_s^{-1} \frac{1}{2} \int_0^{+∞} dz \frac{d}{dz} \left( \frac{z}{H^s} \frac{d}{dz} H^s \right) = -\frac{1}{2} Q_s P_s^{-1} \lim_{z → +∞} \left( \frac{z}{H^s} \frac{d}{dz} H^s \right) = -\frac{1}{2} n_s Q_s P_s^{-1},$$  (3.27)

which implies (see (2.6))

$$Φ^s = -4π n_s Q_s^{-1} h, \quad h = K^{-1},$$  (3.28)

s = 1, . . . , 6.

It is remarkable that any flux Φ^s depends only upon n_s and the integration constant Q_s, which for D = 4 and g^2 = −dt ⊗ dr + dx ⊗ dx is coinciding up to a sign with the value of the x-component of the magnetic field on the axis of symmetry.

Analogous relations were found recently in Ref. [46] for solutions corresponding to Lie algebras of rank 2; see also Ref. [47].

The asymptotic relations for the solution under consideration for ρ → +∞ read

$$g_{as} = \left( \prod_{i=1}^6 B_i^{n_i} \right)^{2h/(D−2)} ρ^{2A} \left[ dρ ⊗ dρ \right. + \left. \prod_{s=1}^6 B_i^{n_i} \right)^{-2h} ρ^{-2A(D−2)} dφ ⊗ dφ + g^2 \right],$$  (3.29)

$$ψ_{as}^a = h \sum_{s=1}^6 λ_s q_s \left( \sum_{i=1}^6 v_{si} ln B_i + 2n_s ln ρ \right),$$  (3.30)

$$F_{as}^s = -Q_s B_s^{-1} B_{σ(s)}^{-1} ρ^{-3} dρ ∧ dφ,$$  (3.31)

a, s = 1, . . . , 6, where

$$A = (2h/(D-2)) \sum_{s=1}^6 n_s = (312h/(D-2)).$$  (3.32)

In derivation of asymptotic relations Eqs. (3.12), (3.23), (3.24) and (3.26) were used.
4 Conclusions

Here we have obtained a multidimensional generalization of Melvin’s solution for the Lie algebra $E_6$. The solution is governed by a set of six fluxbrane polynomials $H_i(z)$, $s = 1, \ldots, 6$, which are presented in the appendix. These polynomials define special solutions to open Toda chain equations corresponding to the Lie algebra $E_6$.

The polynomials $H_i(z)$ depend also upon parameters $Q_s$, which are coinciding for $D = 4$ (up to a sign) with the values of colored magnetic fields on the axis of symmetry. The symmetry and duality identities for polynomials were verified. The duality identities may be used in deriving $(1/\rho)$-expansion for solutions at large distances \( \rho \), e.g., for asymptotic relations, which are presented in the paper. The power-law asymptotic relations for $E_6$-polynomials at large \( z \) are governed by integer-valued matrix $v$. This matrix is related to the inverse Cartan matrix $A^{-1}$ by the formula $v = A^{-1}(I + P)$, where $I$ is identity matrix and $P$ is permutation matrix. The matrix $P$ corresponds to a permutation $\sigma \in S_6$, which is the generator of the $Z_2$-group of symmetry of the Dynkin diagram.

We have also calculated 2d flux integrals $\Phi^s$, $s = 1, \ldots, 6$. Any flux $\Phi^s$ depends only upon one parameter $Q_s$, while the integrand $F^s$ depends upon all parameters $Q_1, \ldots, Q_6$. An open question is how to apply the approach of this paper to other finite-dimensional simple Lie algebras.

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Appendix

In this appendix we present polynomials corresponding to the Lie algebra $E_6$. The polynomials were calculated by using a certain program in Mathematica. We denote the variable $z$ in bold and capital inside the polynomials for better readability:

\[
H_1 = B_1 B_2 B_3 B_4 B_5 B_6 Z^{16} + 16 B_1^2 B_2^2 B_3^2 B_4 B_5 B_6 Z^{15} + 120 B_1 B_2 B_3 B_4 B_5 B_6 Z^{14} + 560 B_1^2 B_2^3 B_3 B_4 B_5 B_6 Z^{13} + (1050 B_1 B_2 B_3 B_4 B_5 B_6 Z^{12} + (672 B_1 B_2 B_3 B_5 B_6 Z^{11} + (3696 B_1 B_2 B_3 B_5 B_6 Z^{10} + (8800 B_1 B_2 B_3 B_5 B_6 Z^{9} + (660 B_1 B_2 B_3 B_5 B_6 Z^{8} + (240 B_1 B_2 B_3 B_5 B_6 Z^{7} + 8800 B_1 B_2 B_3 B_5 B_6 Z^{6} + 3696 B_1 B_2 B_3 B_5 B_6 Z^{5} + 3696 B_1 B_2 B_3 B_5 B_6 Z^{4} + 1050 B_1 B_2 B_3 B_5 B_6 Z^{3} + 770 B_1 B_2 B_3 B_5 B_6 Z^{2} + 120 B_1 B_2 B_3 B_5 B_6 Z + 1.
\]

\[
H_2 = B_1 B_2 B_3 Z^{10} + 30 B_1 B_2 B_3 B_5 B_6 Z^{9} + (120 B_1 B_2 B_3 B_5 B_6 Z^{8} + + 8085 B_1 B_2 B_3 B_5 B_6 Z^{7} + 315 B_1 B_2 B_3 B_5 B_6 Z^{6} + + 9450 B_1 B_2 B_3 B_5 B_6 Z^{5} + 770 B_1 B_2 B_3 B_5 B_6 Z^{4} + + 4950 B_1 B_2 B_3 B_5 B_6 Z^{3} + 23100 B_1 B_2 B_3 B_5 B_6 Z^{2} + 45360 B_1 B_2 B_3 B_5 B_6 Z + 92400 B_1 B_2 B_3 B_5 B_6.
\]

\[
H_3 = 2695 B_1 B_2 B_3 B_5 B_6 Z^{10} + 8085 B_1 B_2 B_3 B_5 B_6 Z^{9} + 107800 B_1 B_2 B_3 B_5 B_6 Z^{8} + 2695 B_1 B_2 B_3 B_5 B_6 Z^{7} + 443520 B_1 B_2 B_3 B_5 B_6 Z^{6} + 16500 B_1 B_2 B_3 B_5 B_6 Z^{5} + 316800 B_1 B_2 B_3 B_5 B_6 Z^{4} + 44100 B_1 B_2 B_3 B_5 B_6 Z^{3} + 202125 B_1 B_2 B_3 B_5 B_6 Z^{2} + 3256110 B_1 B_2 B_3 B_5 B_6 Z + 242550 B_1 B_2 B_3 B_5 B_6.
\]

\[
H_4 = 32+35 B_1 B_2 B_3 B_5 B_6 Z^{10} + 177750 B_1 B_2 B_3 B_5 B_6 Z^{9} + 1358280 B_1 B_2 B_3 B_5 B_6 Z^{8} + 2674100 B_1 B_2 B_3 B_5 B_6 Z^{7} + 2182950 B_1 B_2 B_3 B_5 B_6 Z^{6} + 1689600 B_1 B_2 B_3 B_5 B_6 Z^{5} + 178200 B_1 B_2 B_3 B_5 B_6 Z^{4} + 231000 B_1 B_2 B_3 B_5 B_6 Z^{3} + 1131900 B_1 B_2 B_3 B_5 B_6 Z^{2} + 3958800 B_1 B_2 B_3 B_5 B_6 Z + 8300600 B_1 B_2 B_3 B_5 B_6.
\]
It should be noted that five polynomials: $H_1$, $H_2$, $H_4$, $H_5$, $H_6$ were found earlier (in a non-ordered form) in Ref. [48]. The biggest key polynomial $H_2$ was not presented in Ref. [48]. We note that the “length” of the polynomial $H_2$ is more than 5/8 of the total “length” of all polynomials.

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