On the Limiting Solution of the Bartnik-McKinnon Family

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Received: 10 May 1994

Abstract: We analyze the limiting solution of the Bartnik-McKinnon family and show that its exterior is an extremal Reissner-Nordström black hole and not a new type of non-abelian black hole as claimed in a recent article by Smoller and Wasserman.

The purpose of this short communication is to correct some erroneous statements made in a recent article by J.A. Smoller and A.G. Wasserman [1]. This article concerns the limiting behaviour of an infinite discrete family of regular, static, spherically symmetric solutions of the Einstein-Yang-Mills equations (gauge group SU(2)), whose first few members were discovered by Bartnik and McKinnon [2]. A general existence proof for this family was given by Smoller and Wasserman [3] and by the present authors together with P. Forgács [4].

In their article [1] the authors claim that a suitable subsequence of the infinite family converges to some limiting solution for all values of the radial coordinate $r \neq 1$. The part of this limit defined for $r > 1$ is interpreted as a new type of black hole solution with event horizon at $r = 1$. According to their claim the function $W(r)$ parametrizing the Yang-Mills potential is non-trivial, i.e., $W \neq 0$ and tends to $+1$ or $-1$ for $r \to \infty$. In contrast we claim that the limiting solution for $r > 1$ is given by the extremal Reissner-Nordström (RN) solution with $W \equiv 0$. This can be easily derived from the results of our article [4] and is also strongly supported by numerical calculations. Subsequently we shall give a proof of this claim using the results of [4].

First we recall some definitions and results of [4]. The variables $T$, $A$, $\mu$, $w$, and $\lambda$ used in [1,3] correspond to the quantities $(AN)^{-1}$, $\mu$, $2m$, $W$, and $2b$ in [4] and in this article. We parametrize the line element in the form

$$ds^2 = A^2(r)\mu(r)dt^2 - \frac{dr^2}{\mu(r)} - r^2d\Omega^2,$$

and use the ‘Abelian gauge’
\[ W_{\mu} T_{\nu} dx^\nu = W(r)(T_1 d\theta + T_2 \sin \theta d\varphi) + T_3 \cos \theta d\varphi , \]  

(2)

for the static, spherically symmetric $SU(2)$ Yang-Mills field.

The field equations for $A$, $\mu$, and $W$ (see, e.g., Eqs. (6) in [4]) are singular at $r = 0$ and $r = \infty$ as well as for $\mu(r) = 0$. In order to desingularize them when $\mu \to 0$ we introduce $N = \sqrt{\mu}$, $U = N W$, a new independent variable $\tau$ (with $\dot{\tau} = d/d\tau$), and $\kappa = (\ln r AN)$ as additional dependent variable. The field equations are then equivalent to the autonomous first order system

\[ \begin{align*}
\dot{r} &= r N , \\
W &= r U , \\
\dot{U} &= \frac{W(W^2 - 1)}{r} - (\kappa - N)U , \\
\dot{N} &= (\kappa - N)N - 2U^2 , \\
\dot{\kappa} &= 1 + 2U^2 - \kappa^2 , \\
(AN)' &= (K - N)AN ,
\end{align*} \]  

(3a) - (3f)

subject to the constraint

\[ 2\kappa N = 1 + N^2 + 2U^2 - (W^2 - 1)^2 / r^2 . \]  

(4)

If the initial data satisfy this constraint then it remains true for all $\tau$.

There exists a one-parameter family of local solutions with regular origin where $W(r) = 1 - br^2 + O(r^4)$, $\mu(r) = 1 + O(r^2)$ such that $W(r)$ and $\mu(r)$ are analytic in $r$ and $b$. If we adjust $\tau$ such that $\tau = \ln r + O(r^2)$ we obtain a one-parameter family of local solutions of the system (3) which satisfy the constraint (4) and are analytic in $\tau$ and $b$.

Similarly there exists a two-parameter family of local black hole solutions with $W(r) = W_h + O(r - r_h)$, $\mu(r) = O(r - r_h)$ such that $W(r)$ and $\mu(r)$ are analytic in $r$, $r_h$, and $W_h$. If we adjust $\tau$ such that $\tau = 0$ at the horizon we obtain a two-parameter family of solutions of (3,4) analytic in $\tau$, $r_h$, and $W_h$ except for a simple pole in $\kappa(\tau)$ at the horizon.

Both types of initial data satisfy $\kappa \geq 1$ and this relation remains true for all $\tau$ due to the form of Eq. (3e).

In the following we exclude the case $W \equiv 0$ and can therefore assume $(W, U) \neq (0, 0)$ for all (finite) $\tau$. Integrating Eqs. (3) with regular initial data $\tau(\tilde{\tau}) > 0$, $N(\tilde{\tau}) > 0$, $\kappa(\tilde{\tau}) \geq 1$ satisfying the constraint (4) we obtain solutions analytic for all $\tau > \tilde{\tau}$ as long as $N > -\infty$. There are three possible cases:

i) $N(\tau)$ has a zero at some $\tau = \tau_0$, the generic case. Then

\[ (W^2(\tau_0) - 1)^2 = (1 + 2U^2(\tau_0)) r^2(\tau_0) , \]  

(5)

and $r$ has a maximum at $\tau = \tau_0$. For $\tau > \tau_0$ we find that $N < 0$ and $r$, $W$, $U$, $\kappa$, $r N$, and $r AN$ remain analytic at least as long as $\tau \geq 0$.

ii) $N(\tau) > 0$ for all $\tau$ and $r(\tau)$ tends to infinity for $\tau \to \infty$. These are the asymptotically flat solutions with $(W, U, N, \kappa) \to (\pm 1, 0, 0, 1)$.

iii) $N(\tau) > 0$ for all $\tau$ and $r(\tau)$ remains bounded. This is a new type of ‘oscillating’ solution with $(r, W, U, N, \kappa, A) \to (1, 0, 0, 1, \infty)$ for $\tau \to \infty$ first discussed in detail in [4].
Analyzing the solutions with regular origin and their dependence on $b$ we have shown in [4]:

1. For each positive integer $n$ there exists a globally regular and asymptotically flat solution with $n$ zeros of $W$ for at least one value $b = b_n$ and there is at most a finite number of such values $b_n$.
2. There exists an oscillating solution for at least one value $b = b_{\infty}$ and there is at most a finite number of such values $b_{\infty}$.
3. The values $b_n$ have at least one accumulation point for $n \to \infty$ and each such accumulation point is one of the values $b_{\infty}$.

Completely analogous results hold for black hole solutions with fixed $r_h < 1$ and their dependence on $W_h$.

Let us analyze the oscillating solutions in some detail. Near the singular point $(r, W, U, N, \kappa) = (1, 0, 0, 0, 1)$ we introduce the parametrization (with $\bar{W} = \frac{W}{r}$ and $\bar{\kappa} = \kappa - 1$)

$$
\begin{align*}
\bar{W}(\tau) &= C_1 e^{-\frac{1}{2}r} \sin(\frac{\sqrt{3}}{2} \tau + \theta), \\
U(\tau) &= C_1 e^{-\frac{1}{2}r} \sin(\frac{\sqrt{3}}{2} \tau + \frac{2\pi}{3} + \theta), \\
N(\tau) &= C_2 e^\tau + \frac{2}{7}(\bar{W}^2 - U\bar{W} + 2U^2), \\
\bar{\kappa}(\tau) &= C_4 e^{-2\tau} + \bar{W}^2 + 2U\bar{W} + 2U^2,
\end{align*}
$$

as in [4] and compute $r$ from the constraint (4)

$$
r^{-2} = \rho + \sqrt{\rho^2 - \bar{W}^4}, \quad \text{where} \quad \rho = \frac{1}{2}(1 - N)^2 + \bar{W}^2 + U^2 - \bar{\kappa}N. $$

The functions $\theta, C_1, C_2$, and $C_4$ satisfy differential equations,

$$
\begin{align*}
\dot{\theta} &= f_0, \\
(C_1^2e^{-\tau}) &= C_1^2 e^{-\tau}(-1 + f_1), \\
(C_2 e^{\tau}) &= C_2 e^{\tau} + f_2, \\
(C_4 e^{-2\tau}) &= -2C_4 e^{-2\tau} + f_4,
\end{align*}
$$

with ‘non-linear’ terms $f_t$ that can be expressed as homogeneous polynomials in $C_1^2 e^{-\tau}, C_2 e^{\tau},$ and $C_4 e^{-2\tau}$ of degree one for $f_0$ and $f_1$ and of degree two for $f_2$ and $f_4$ with $(r, \theta)$-dependent coefficients that are bounded as long as $r$ is bounded.

We can apply a general result for perturbed linear systems (see, e.g., [5] p.330) stating the existence of a stable manifold. The system (8) has one unstable mode, $C_2 e^{\tau}$, and hence there exists a three-dimensional stable manifold of initial data, i.e., quadruples $Y = (\bar{W}, U, N, \bar{\kappa})$ such that $Y \to 0$ for $\tau \to \infty$. Eliminating the freedom to add a constant to $\tau$ we are left with a two-parameter family of oscillating solutions. In [4] we have derived the stronger result that $\theta$ and $C_1$ have a limit for $\tau \to \infty$ (with $C_1(\infty) \neq 0$) whereas $C_2 e^{2\tau} \to 0$ and $C_4 e^{-\tau} \to 0$ for each member of this two-parameter family. Consequently these oscillating solutions have infinitely many zeros of $W$ and infinitely many minima of $N$ as $r \to 1$.

Conversely there exists a one-dimensional ‘unstable manifold’ (i.e., stable manifold for decreasing $\tau$) of initial data such that $Y \to 0$ for $\tau \to -\infty$. These initial data $Y = (0, 0, N, 0)$ describe the extremal RN black hole with $r = (1 - N)^{-1}$. 


In the following we analyze the behaviour of solutions for $b$ near (one of the values) $b_\infty$ and in particular the behaviour of globally regular solutions with $n$ zeros of $W$ in the limit $b_n \to b_\infty$ for $n \to \infty$. In view of the analytic dependence of the solutions on $b$ and $r$ the trajectories reach any given neighbourhood of the singular point $Y = 0$ for $b$ sufficiently close to $b_\infty$. Trajectories missing the singular point cannot stay near it, they must start to ‘run away.’ They will, however, remain close to the unstable manifold. In the limit $b_n \to b_\infty$ they converge to the unstable manifold, i.e., extremal RN solution.

We can decompose $Y$ into its parts parallel and perpendicular to the unstable manifold and measure the distance from the singular point $Y = 0$ by

$$|Y| = \max(|Y_\parallel|, |Y_\perp|), \quad \text{with} \quad |Y_\parallel| = |N|, \quad |Y_\perp| = \max(C_i^2 e^{-\tau}, |\kappa|).$$

(9)

Using the distance function $|\cdot|$ we get from the smooth dependence of the solutions on $b$ and $r$ that all solutions with $b \approx b_\infty$ must come close to the singular point $Y = 0$ for some $r = \tau_0$.

**Lemma 1.** Given $b_\infty$ and any $\epsilon > 0$ there exist some $\delta > 0$ and $\tau_0$ such that all solutions with $|b - b_\infty| < \delta$ satisfy $|Y|(< \epsilon$ and $0 < 1 - r(\tau_0) < \epsilon$.

Let us analyze the behaviour of these trajectories in the neighbourhood of $Y = 0$. The general result [5] also states the existence of some $\eta > 0$ such that trajectories missing the singular point cannot stay within $|Y| < \eta$ for all $\tau$. Due to the structure of Eqs. (3), resp. (8) this runaway is caused by the growth of $N$. The trajectories can therefore be characterized by three possibilities: They either run into the singular point $Y = 0$ or miss it on one or the other side; in the latter case either $N$ stays positive and $r$ grows beyond $r = 1$ or $N$ has a zero while $r < 1$ and $r$ runs back to $r = 0$. This is expressed by

**Lemma 2.** There exists some $\eta > 0$ such that for any solution of Eqs. (3a – e, 4) with $|Y| < \epsilon < \eta$ and $0 < 1 - r < \epsilon$ at some $\tau = \tau_0$ there are three possible cases:

a) $r < 1$, $N > 0$ for all $\tau > \tau_0$ and $Y \to 0$ for $\tau \to \infty$.
b) $r = 1$ for some $\tilde{\tau} > \tau_0$, $N = \eta$ for some $\tau_1 > \tilde{\tau}$, $\tilde{N}(\tau_1) > 0$, and $0 < N < \eta$,

$$|Y_\perp| < \epsilon < \tilde{N}(\tau_1) < \tau < \tau_1,$$
c) $N = 0$ for some $\tilde{\tau} > \tau_0$, $N = -\eta$ for some $\tau_1 > \tilde{\tau}$, $\tilde{N}(\tau_1) < 0$, and $\tau < 1$, $|N| < \eta$,

$$|Y_\perp| < \epsilon < \tilde{N}(\tau_1) < \tau < \tau_1.$$

**Proof.** The general result [5] mentioned above states the existence of some $\eta > 0$ such that either $Y \to 0$ (case a) or $|Y| = \eta$ for some $\tau_1$ (case b and c). Choosing $\eta$ small enough, Eq. (8b) shows that the ‘amplitude’ $|C_i|e^{-\tau/2}$ decreases as long as $|Y| < \eta$. Moreover Eq. (3e) implies that $|\kappa| < \epsilon$ remains true as long $U^2 < \epsilon$. Therefore $|Y_\perp| < \epsilon$ as long as $|N| < \eta$.

Next, if $|N| \gg |Y_\perp|$ then $\tilde{N} \approx (1 - N)N$ due to Eq. (3d) and $r \approx (1 - N)^{-1}$ due to Eq. (7), i.e., $r > 1$ and $\tilde{N} > 0$ for $N > \epsilon$, resp. $r < 1$ and $\tilde{N} < 0$ for $N \ll -\epsilon$. Finally, Eq. (5) implies that $N$ can vanish only when $r < 1$.

To conclude the argument we analyze what happens to the solutions in the limit $b \to b_\infty$.

**Proposition 3.** Given $b_\infty$ and $\eta$ as defined above there exists some $\delta > 0$ such that the solutions with regular origin and $|b - b_\infty| < \delta$ satisfy:

1. Case a of Lemma 2 holds if and only if $b = b_\infty$. There exist continuous functions $\tilde{\tau}(b) < \tau_1(b)$ defined for $b \neq b_\infty$ such that the same case either b or c holds for all
$b < b_\infty$ and for all $b > b_\infty$ (with $|b - b_\infty| < \delta$); case $b$ holds in particular for the globally regular solutions with $n$ zeros of $W$ as $b_n \to b_\infty$ for $n \to \infty$.

2. In the limit $b \to b_\infty$ both $\bar{\tau}$ and $\tau_1 - \bar{\tau}$ diverge. The part of the solution defined for $\tau < \bar{\tau}$ converges for any fixed $\tau$ or $r < 1$ to the oscillating solution. The part defined for $\tau > \bar{\tau}$ converges for any fixed $\tau - \tau_1$ or $r \neq 1$ to the exterior, resp. interior of the extremal RN solution with $W = 0$ in case $b$, resp. $c$.

Proof.

1. Since an oscillating solution exists only for finitely many values of $b$, we can choose $\delta > 0$ in Lemma 1 such that the interval $|b - b_\infty| < \delta$ contains only one of them, namely $b_\infty$. The existence of $\bar{\tau}$ and $\tau_1$ for $b \neq b_\infty$ was shown in Lemma 2. The rest follows from the continuity of the solutions in $b$.

2. The convergence of the solutions follows from the convergence of the initial data, i.e., quadruples $Y$ at an arbitrary regular point. The initial data for any fixed $\tau$ converge to those of the oscillating solution. At the same time $\bar{\tau}$ (with $r(\bar{\tau}) = 1$, resp. $N(\bar{\tau}) = 0$) diverges. On the other hand $Y(\tau_1)$ converges to $(0, 0, \pm \eta, 0)$, i.e., to initial data for the exterior or interior of the extremal RN black hole and $\bar{\tau} - \tau_1 \to -\infty$. Convergence for fixed $r$ requires in addition $N \neq 0$; given $r \neq 1$ this is satisfied for $b$ sufficiently close to $b_\infty$.

Using exactly the same arguments one obtains

Corollary. Analogous results hold true for black hole solutions with any fixed $r_h < 1$ and $W_h, W_{hn}, W_{h\infty}$ replacing $b, b_n, b_\infty$.

Having shown the incorrectness of the statements made by Smoller and Wasserman in [1] about the limiting solution one may ask for the source of this error. Looking at their arguments one finds that they use Prop. 3.2 of their earlier work [3] in an essential way. This proposition is, however, wrong as it stands; its validity requires the further assumption of a uniformly bounded rotation number (as made for their Prop. 3.1). This additional assumption is not satisfied for the Bartnik-McKinnon family.

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Communicated by S.-T. Yau

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