Categoricity of Theories in $L_{\kappa^*\omega}$, when $\kappa^*$ is a measurable cardinal. Part II

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Abstract

We continue the work of [KSh 362] and prove that for $\lambda$ successor, a $\lambda$-categorical theory $T$ in $L_{K^*\omega}$ is $\mu$-categorical for every $\mu, \mu \leq \lambda$ which is above the $(2^{LS(T)})^+\text{-beth cardinal}$.

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0 Introduction

We deal here with the categoricity spectrum of theory $T$ in logic: $T \subseteq L_{\kappa^*,\omega}$ with $\kappa^*$ measurable. Makki Shelah [MaSh 285] have dealt with the case $\kappa^*$ a compact cardinal. So $\kappa^*$ measurable is too high compared with the hope to deal with $T \subseteq L_{\omega_1,\omega}$ (or any $L_{\kappa,\omega}$) but seem quite small compared to the compact cardinal in [MaSh 285]. Model theoretically a compact cardinal ensure many cases of amalgamation, whereas measurable cardinal ensure no maximal model. We continue [Sh 300], [MaSh 285], [KlSh 362]: try to imitate [MaSh 285]; a parallel line of research is [Sh 394]. Earlier works are [Sh 48], [Sh 87a], [Sh 87b]; on the upward Loś conjecture, look at [Sh 576] and [Sh 600].

On the situation with the upward direction and generally more see [Sh 576].

This paper continues the tasks begun in [KlSh 362]. We use the results obtained therein to advance our knowledge of the categoricity spectrum of theories in $L_{\kappa^*,\omega}$, when $\kappa^*$ is a measurable cardinal.

The main theorems are proved in section three; section one treats of types and section two described some constructions.

The notation follows [KlSh 362], except in two important details: we reserve $\kappa^*$ for the fixed measurable cardinal and $T$ for the fixed $\lambda$-categorical theory in $L_{\kappa^*,\omega}$ in a given vocabulary $L$. $\kappa$ is any infinite cardinal and $T$ is usually some kind of tree. To recap briefly: $T$ is a $\lambda$-categorical theory in $L_{\kappa^*,\omega}$, $LS(T) \overset{\text{def}}{=} \kappa^* + |T|$, $K = \langle K, \preceq_F \rangle$ is the class of models of $T$, where $F$ is a fragment of $L_{\kappa^*,\omega}$ satisfying $T \subseteq F$, $|F| \leq \kappa^* + |T|$, and for $M, N \in K$, $M \preceq_F N$ means that $M$ is an $F$-elementary submodel of $N$.

The principal relevant results from [KlSh 362] are: $K_{<\lambda}$ has the amalgamation property (5.5 there) and every member of $K_{<\lambda}$ is nice (5.4 there). But this assumption ($T$ categorical in $\lambda$) or its consequences mentioned above will be mentioned in theorems when used.

Let $(M_1, M_0) \preceq_F (M_3, M_2)$ means $M_1 \preceq_F M_3$, $M_0 \preceq_F M_2$.

$(I_1, I_2)$ is a Dedekind cut of the linear order $I$ if

$$I = I_1 \cup I_2, I_1 \cap I_2 = \emptyset, \forall x \in I_1 \forall y \in I_2 (x < y),$$

the two sided cofinality of $I$, $\text{dcf}(I)$ is $(\text{cf} I_1, \text{cf} I_2^*)$ where $I_2^*$ is the order $I_2$ inverted.

Writing proofs we also consider their hopeful rule in the hopeful classification theory. But we have been always careful in stating the assumptions.

Note that [KlSh 362] improve results of [MaSh 285]; but they do not fully recapture the results on the compact case to the measurable case, e.g. there the results work for every $\lambda > \kappa^*$ whereas here we sometimes need “$\lambda$ above the Hanf number of omitting types”, say $\sum(2^{LS(T)})$. 
We thank Oren Kolman for writing and ordering notes from lectures on the subject from spring 90 (you can see his style in the parts with good language).

1 Knowing the right types:

The classical notion of type relates to the satisfaction of sets of formulas in a model. We shall define a post-classical type (following [Sh 300], [Sh:h] which was followed by [MaSh 285] but niceness is involved) and use this to define notions of freeness and non-forking appropriate in the context of a $\lambda$-categorical theory in $L_{\kappa^*,\omega}$. The definitions try to locate a notion which under the circumstances behave as in [Sh:c].

Context 1.1 $T \subseteq L_{\kappa^*,\omega}$ in the vocabulary $L$, $K = \{M : M$ a model of $T\}$, $\preceq_F$ as in the introduction. $K_\mu = \{M \in K : \|M\| = \mu\}$, $K_{<\kappa} = \bigcup_{\mu<\kappa} K_\mu$, and $K = (K, \preceq_F)$ and we stipulate $K_{<\kappa^*} = \emptyset$, e.g. $K_{<\kappa} = \bigcup\{K_\mu : \mu < \kappa$ but $\mu \geq \kappa^*\}$. We let $LS(K) = |F| + \kappa^*$. Remember “$M \in K$ is nice” is defined in [KlSh 362], definitions 3.2, 1.8; nice implies being an amalgamation base in $K_{<\lambda}$ (see 3.7).

Definition 1.2 Suppose that $M \in K_{<\lambda}$ is a nice model of $T$. Define a binary relation, $E_M = E^{<\lambda}_M$, as follows:

$$(\bar{a}_1, N_1)E_M(\bar{a}_2, N_2) \iff \ell = 1,2, N_\ell \in K_{<\lambda}$ is nice and $M \preceq_F N_\ell$, $\bar{a}_\ell \in N_\ell$ (i.e. $\bar{a}_\ell$ a finite sequence of members of $N_\ell$), and there exist a model $N$ and embeddings $h_\ell$ such that $M \preceq_F N$, $h_\ell : N_\ell \rightarrow F N$, $id_M = h_1 \upharpoonright M = h_2 \upharpoonright M$, and $h_1(\bar{a}_1) = h_2(\bar{a}_2)$.

Fact 1.3 1. $E_M$ is an equivalence relation.

2. Let $M \in K_{<\lambda}$, $\bar{a} \in N$, and for $\ell = 1,2$, $N \subseteq N_\ell \preceq_F M$, $\|N_\ell\| < \lambda$ then $(\bar{a}, N_1)E_M(\bar{a}, N_2)$

Proof 1) To prove [1.3], let’s look at transitivity.

Suppose $(\bar{a}_\ell, N_\ell)E_M(\bar{a}_{\ell+1}, N_{\ell+1})$, $\ell = 1,2$. Thus there are models $N_\ell$ and embeddings $h_0^\ell$, $h_1^\ell$ of $N_\ell$, $N_{\ell+1}$ over $M$ into $N^\ell$, with $h_0^\ell(\bar{a}_\ell) = h_1^\ell(\bar{a}_{\ell+1})$, $\ell = 1,2$. W.l.o.g. $N^\ell \in K_{<\lambda}$ (by the Downward Loewenheim Skolem Theorem).

By assumption $N_2$ is nice, hence by [KlSh 362, 3.5] is an amalgamation base for $K_{<\lambda}$, i.e. there is an amalgam $N^* \in K_{<\lambda}$, and embeddings $g_\ell$:
$N^* \xrightarrow{\xi} N^*$, amalgamating $N^1$, $N^2$ over $N^2$ w.r.t $h_1^1$, $h_0^2$. In other words, the following diagram commutes:

\[
\begin{array}{ccc}
N^* & \xrightarrow{g_1} & N^1 \\
\downarrow & & \downarrow h_0^1 \searrow \downarrow h_0^2 \\
N^1 & \xrightarrow{\text{id}} & M \\
\end{array}
\]

Just notice now that $N^*$, $g_1 h_0^1$, $g_2 h_1^2$ witness that $(\bar{a}_1, N_1) E_M (\bar{a}_3, N_3)$, since:

\[g_1 h_0^1(\bar{a}_1) = g_1(h_1^1(\bar{a}_2)) = g_2 h_0^2(\bar{a}_2) = g_2 h_1^2(\bar{a}_3)\]

**Definition 1.4** Suppose that $M \in K_{<\lambda}$ is nice, $a \in N \in K_{<\lambda}$ and $M \preceq_F N$. Then

1. $\text{tp}(a, M, N)$, the type of $a$ over $M$ in $N$, is the $E_M$-equivalence class of $(a, N)$,

$$
(a, N)/E_M = \{(b, N^1) : (a, N) E_M (b, N^1)\}.
$$
We also say “$a \in N$ realizes $p$”. If $\|N\| \geq \lambda$ define $\text{tp}(\bar{a}, M, N)$ by (1.3).

2. If $M' \preceq_F M \in K_{\lambda}$, $p \in S(M)$ (see below) is $(a, N)/E_M$ then $p \restriction M' = (a, N)/E_{M'}$.

3. If $LS(T) < \kappa \leq \mu \leq \lambda$, we call $M \in K_\mu \kappa$-saturated if for every nice $N \preceq_F M$, $\|N\| < \kappa$ and $p \in S(N)$, some $\bar{a} \in M$ realizes $p$ (in $M$) or at least for some $N'$, $N \preceq_F N' \preceq_F M$, some $a' \in N'$ realizes $p$ in $N'$.

4. $S^m(N) = \{ p : p = tp(\bar{a}, N, N_1) \text{ for any } N_1, \bar{a} \text{ satisfying: } N \preceq_F N_1, \|N_1\| \leq \|N\| + LS(K) \text{ and } \bar{a} \in ^m(N_1) \}$

5. $S^{<\omega}(N) = \bigcup_{m<\omega} S^m(N)$.

6. We say $N$ is $\mu$-universal over $M$ when: $M \preceq_F N$, $N \in K_\mu$ and if $M \preceq_F N' \in K_{\leq \mu}$ then there is a $\preceq_F$-embedding of $N'$ into $N$ over $M$.

7. We say $N$ is $(\mu, \kappa)$-saturated over $M$ if there is an increasing continuous sequence $(M_i : i < \kappa)$ such that: $M_0 = M$, $N = \bigcup_{i<\kappa} M_i$, $M_i \in K_\mu$ and $M_{i+1}$ is $\mu$-universal over $M_i$.

8. We say $K$ (or $T$) is stable in $\mu$ if for every $M \in K_\mu$, $M$ is nice and $|S(M)| \leq \mu$.

**Definition 1.5** We shall write $M_1 \bigcup M_2$ to mean: $M_0 \preceq_F M_1 \preceq_F M_3$, $M_0 \preceq_F M_2 \preceq_F M_3$ and there exist suitable operation $(I, D, G)$ and an embedding $h : M_3 \overset{F}{\rightarrow} \text{Op}(M_0, I, D, G)$ such that $h \restriction M_1 = \text{id}_{M_1}$ and $\text{Rang}(h \restriction M_2) \subseteq \text{Op}(M_0, I, D, G)$ (remember that $\text{Op}(M, I, D, G)$ is the limit ultrapower of $M$ w.r.t. $(I, D, G)$; see [KlSh 362, 1.7.4]). We say that $M_1, M_2$ do not fork in $M_3$ over $M_0$ if

$M_3$

$M_1 \bigcup M_2$

$M_0$

If

$M_3$

$M_1 \bigcup M_2$

$M_0$
does not hold, we'll write

\[
M_3 \uplus M_2
\]

and say that \( M_1, M_2 \) fork in \( M_3 \) over \( M_0 \).

**Theorem 1.6** Suppose that \( M_1 \uplus M_2 \) and \( M_2 \uplus M_1 \) (failure of \( \uplus \)-symmetry)

and \( M_0 \preceq_{\text{nice}} M_3 \).

Let \( \mu = \kappa^* + ||T|| + ||M_2|| + ||M_1|| \). Then for every linear order \( (I, <) \) there exists an Ehrenfeucht-Mostowski model \( N = EM(I, \Phi) \) with \( \mu \) (individual) constants \( \{ \tau^0_i : i < \mu \} \) and unary function symbols \( \{ \tau^1_i(x_i) : i < \mu \} \) such that, for \( M = (N \upharpoonright L) \upharpoonright \{ \tau^0_i : i < \mu \} \) (i.e. \( M \) is a submodel of \( N \) with the same vocabulary as \( T \) and universe \( \{ \tau^0_i : i < \mu \} \) i.e. the set of interpretations of these individual constants and for every \( t \in I, \ell = 1, 2 \),

\[
M^t_\ell = (N \upharpoonright L) \upharpoonright \{ \tau^\ell_i(x_i) : i < \mu \},
\]

one has \( M \preceq_F N, M^t_\ell \preceq_F N \) and for \( s \neq t \in I, t < s \) iff \( M^t_1 \uplus M^t_2 \).

**Remark:** Note \( M_0 \preceq_{\text{nice}} M_3 \) is automatic in the interesting case since \( M_0 \in K_{<\lambda} \) and every element of \( K_{<\lambda} \) is nice by [KSh 362, 5.4]. On the operations see [KSh 362].

**Proof** W.l.o.g. \( ||M_3|| = \mu \). Add Skolem functions to \( M_3 \). We know that \( M_0 \preceq_{\text{nice}} M_3 \). So there is \( \text{Op}^1 \) such that \( M_0 \preceq_F M_1 \preceq_F \text{Op}^1(M_0) \) and \( \text{Op}^2 \) such that \( M_1 \preceq_F M_3 \preceq_F \text{Op}^2(M_1) \), \( M_2 \preceq_F \text{Op}^2(M_0) \). Let \( \text{Op} = \text{Op}^2 \circ \text{Op}^1 \). For each \( t \in I \), let \( \text{Op}_t = \text{Op} \). Let \( N \) be the iterated ultrapower of \( M_0 \) w.r.t. \( (\text{Op}_t : t \in I) \). For each \( t \in I \), there is a canonical \( F \)-elementary embedding \( F_t : \text{Op}_t(M_0) \overset{F}{\rightarrow} N \). Let \( M = M_0 \), and \( M^t_\ell = F_t(M_\ell) \) for \( \ell = 1, 2, t \in I \).

For each \( t < s \), we can let \( M^t_\ell = (\text{Op}_v : v < s)(M_0) \), so \( M_0 \preceq_F M^t_\ell \preceq_F M^t_+ \preceq_F \text{Op}^1(M^t_+) \) and we can extend \( F_t \upharpoonright M_1 \) to an embedding of \( \text{Op}^2(M_1) \) into \( \text{Op}^2(\text{Op}_t(M^t_+)) \), so \( (F_t \upharpoonright M_1) \cup (F_s \upharpoonright M_2) \) can be extended to a \( \preceq_F \)-embedding of \( M_3 \) into \( N \). From the definition of the iterated ultrapower it follows that for \( s \neq t \in I, t < s \) implies \( M^t_1 \uplus M^t_2 \) and on the other hand by the assumption it follows that if \( s, t \in I, s < t \) then \( M^t_1 \uplus M^t_2 \).
Corollary 1.7 Assume $T$ categorical in $\lambda$ or just $I(\lambda, T) < 2^\lambda$. Then

$$\bigcup_{\mu^+ < \lambda} K_\mu$$

obeys $\bigcup$-symmetry, i.e.: if $M_1 \bigcup M_2$ holds for $M_0$, $M_1$, $M_2$, $M_3 \in M_0$ then

$$\bigcup_{\mu^+ < \lambda} K_\mu,$$ $M_3$ then $M_2 \bigcup M_1$ holds.

Proof If $\mu^+ < \lambda$, $M_1 \bigcup M_2$ and $M_2 \bigcup M_3$, then theorem 1.6 gives the assumptions of the results at the end of section three in [Sh 300], III (or better [Sh:e], III, §3), yield a contradiction to the $\lambda$-categoricity of $T$ and even $2^\lambda$ pairwise non isomorphic models.

It may be helpful, though somewhat vague, to add the remark that $\bigcup$-asymmetry enables one to define order and to build many complicated models; so 1.7 removes a potential obstacle to a categoricity theorem.

Definition 1.8 Let $A$ be a set. We write $M_1 \bigcup A$ (where $A \subseteq M_3$, $M_0 \preceq_F M_3$) to mean that there exist $M_2$, $M_3'$ such that $A \subseteq |M_2|$, $M_3 \preceq_F M_3'$ and $M_1 \bigcup M_2$. In this situation we say that $A/M_1 = \text{tp}(A, M_1, M_3)$ does not fork over $M_0$ in $M_3$.

We’ll write $M_1 \bigcup a$ to mean $M_1 \bigcup \{a\}$, we then say $\text{tp}(a, M_1, M_3)$ do not fork on $M_0$.

We write $A_1 \bigcup A_2$ if for some $M_3$, $M_3 \preceq_F M_3'$ $\in K_{<\lambda}$, and for some $M_1'$, $A_2 \subseteq M_1' \preceq_F M_3'$, and $M_1' \bigcup A_2$.

Remark 1.9 1. Of particular importance is the case where $A$ is finite. Let us explain the reason. We wish to prove a result of the form:

$$(*) \text{ if } \{M_i : i \leq \delta + 1\} \text{ is a continuous } \preceq_F \text{-chain and } a \in M_\delta, \text{ then there is } i < \delta \text{ such that } M_\delta \bigcup_{M_i} a.$$
This says roughly that the type tp(a, \(M_\delta, M_{\delta+1}\)) is definable over a finite set (or at least in some sense has finite characters). In general the former relation is not obtained. However its properties are correct. Hence it will be possible to define the rank of a over \(M_0\), \(\text{rk}(a, M_0)\), as an ordinal, so that for large enough \(M_3\), if \(M_1 \upharpoonright M_3 \supseteq a\), then \(\text{rk}(a, M_1) < \text{rk}(a, M_0)\).

2. If \(A\) is an infinite set, then we cannot prove (*), in general. For example, suppose that \(\langle M_i : i \leq \omega \rangle\) is (strictly) increasing continuous, \(a_i \in (M_{i+1} \setminus M_i)\) and \(A = \{a_i : i < \omega\}\). Then for every \(i < \omega\), \((\bigcup_{j<\omega} M_j) \upharpoonright M_i A\). Still we can restrict ourselves to \(\delta\) of cofinality \(>|A|\).

3. Notice that quite generally speaking, \(N_1 \bigcup_{\mathcal{N}_0} N_2\) implies that \(N_1 \cap N_2 = \mathcal{N}_0\).

**Definition 1.10** We define

\[
\kappa_\mu(T) = \kappa_\mu(K) = \{\kappa : \text{cf}(\kappa) = \kappa \leq \mu \text{ and there exist a continuous } \prec \text{-chain } \langle M_i : i \leq \kappa + 1 \rangle \subseteq K \leq \mu \text{ and } a \in M_{\kappa+1} \text{ such that for all } i < \kappa, a/M_i \text{ forks over } M_i \text{ in } M_{\kappa+1}\}.
\]

_i.e. for \(\kappa \in \kappa_\mu(T)\) there are \(\langle M_i \in K \leq \mu : i \leq \kappa + 1 \rangle\) and \(a \in M_{\kappa+1}\) such that \(i < \kappa \Rightarrow M_{M_{\kappa+1}} \upharpoonright M_i a\).

**Example 1.11** Fix \(\mu\) and \(\alpha \leq \mu\). Let \((\mu, E_\beta)_{\beta<\alpha}\) be the structure with universe

\[
\mu \omega = \{\eta : \eta \text{ is a function from } \mu \text{ to } \omega\},
\]

\(\eta E_\beta \nu\) iff \(\eta \upharpoonright \beta = \nu \upharpoonright \beta\). Let \(T = \text{Th}(\mu \omega, E_\beta)_{\beta<\alpha}\). Then \(\kappa_\mu(T) = \{\kappa : \text{cf}(\kappa) = \kappa \leq \alpha\}\).

**Why?** If \(\text{cf}(\kappa) = \kappa \leq \alpha\), then there are \(M_i(i \leq \kappa + 1), a \in M_{\kappa+1}\) and \(a_i \in (M_{i+1} \setminus M_i)\) such that \(a_i/E_{i+1} \notin M_i\) (that’s to say, no element of \(M_i\) is \(E_{i+1}\)-equivalent to \(a_i\)) and \(a E_i a_i\).
Definition 1.12 The class $\mathcal{K} = \langle K, \preceq_F \rangle$ is $\chi$-based iff for every pair of continuous $\preceq_F$-chains $\langle N_i \in K_{\leq \chi} : i < \chi^+ \rangle$, $\langle M_i \in K_{\leq \chi} : i < \chi^+ \rangle$, with $M_i \preceq_F N_i$, there is a club $C$ of $\chi^+$ such that

$$(\forall i \in C) \left( M_{i+1} \bigcup_{M_i} N_i \right).$$

Replacing $\chi^+$ by regular $\chi$ we write $(< \chi)$-based. We say synonymously that $T$ is $\chi$-based.

Definition 1.13 The class $\mathcal{K} = \langle K, \preceq_F \rangle$ has continuous non-forking in $(\mu, \kappa)$ iff

(a) whenever $\langle M_i \in K_{\leq \mu} : i \leq \delta \rangle$ is an continuous $\preceq_F$-chain, $|\delta| \leq \mu$, $\text{cf}(\delta) = \kappa$, $M_0 \preceq_F N_0 \preceq_F N^*$, $M_\delta \preceq_F N^*$ and

$$(\forall i < \delta) \left( M_i \bigcup_{M_0} N_0 \right),$$

then $M_\delta \bigcup_{M_0} N_0$;

(b) whenever $\langle M_i \in K_{\leq \mu} : i \leq \delta + 1 \rangle$, $\langle N_i \in K_{\leq \mu} : i \leq \delta + 1 \rangle$ are continuous $\preceq_F$-chains, $M_i \preceq_F N_i$, $|\delta| \leq \mu$, $\text{cf}(\delta) = \kappa$ and

$$(\forall i < \delta) \left( M_{\delta+1} \bigcup_{M_i} N_i \right),$$

then $M_{\delta+1} \bigcup_{M_\delta} N_{\delta}$. 

Again we’ll mean the same thing by saying that $T$ has continuous non-forking in $(\mu, \kappa)$.

Our next goal is to show that if $T$ fails to possess these features for some $\mu < \lambda$ such that $\mu \geq \kappa + \text{LS}(\mathcal{K})$, then $T$ has many models in $\lambda$.

Let us quote in this context a further important result from [Sh 300], II, 2.12:

Theorem 1.14 Assume $T$ be a $\lambda$-categorical theory, or just $K_{<\lambda}$ has amalgamation and every $N \in K_{<\lambda}$ is nice.
1. Let $LS(T) < \mu \leq \lambda$, $M \in K_\mu$. Then TFAE:

(A) $M$ is universal-homogeneous: if $N \preceq_{\mathcal{F}} M$, $\|N\| < \mu$, $N \preceq_{\mathcal{F}} N' \in K_{<\mu}$, then there is an $\mathcal{F}$-elementary embedding $g : N' \xrightarrow{\mathcal{F}} M$ such that $g \upharpoonright N = \text{id}_N$.

(B) if $N \preceq_{\mathcal{F}} M$, $\|N\| < \mu$ and $p \in S(N)$, then $p$ is realized in $M$ i.e. $N$ is saturated.

2. $M$ as in (A) or (B) is unique for fixed $T$, $\mu$.

3. Any two $(\mu, \kappa)$-saturated models are isomorphic (see [Sh 300], II 3.10).

**Proof**
1), 2) See [Sh 300], II 3.10.
3) Easy.

**Claim 1.15** Assume $T$ is $\lambda$-categorical or just $K_{<\lambda}$ has amalgamation.

1. If $LS(T) \leq \mu < \lambda$, $N_0 \preceq_{\mathcal{F}} N_1$ are in $K_\mu$ then TFAE

(A) $N_1$ is $(\mu, \mu)$-saturated over $N_0$

(B) there is a $\preceq_{\mathcal{F}}$-increasing continuous $\langle M_i : i \leq \mu \times \mu \rangle$, such that:

$M_\mu = N_1$, $M_0 = N$ and every $p \in S(M_i)$ is realized in $M_{i+1}$

2. Also TFAE for $\kappa = \text{cf}(\kappa) \leq \mu^+$

(A)$_\kappa$ $N_1$ is $(\mu, \kappa)$-saturated over $N_0$

(B)$_\kappa$ there is a $\preceq_{\mathcal{F}}$-increasing continuous $\langle M_i : i \leq \mu \times \kappa \rangle$ with $M_\mu \times \kappa = N_1$, $M_0 = N$ and every $p \in S(M_i)$ is realized in $M_{i+1}$

3. If $\mathcal{K}$ is stable in $\mu$, $\mu \geq LS(\mathcal{K})$, $\kappa = \text{cf}(\kappa) \leq \mu^+$ then there is a $(\mu, \kappa)$-saturated model.

**Proof**
1) See [Sh 300], II 3.10
2) Same proof.
3) Straight.

**Claim 1.16** (T categorical in $\lambda$)

1. Any $M \in K_\lambda$ is saturated.

2. Every $N \in K_{<\lambda}$ is nice.

3. $K_{<\lambda}$ has $\preceq_{\mathcal{F}}$-amalgamation.
4. $\mathcal{K}$ is stable in $\mu$ for $\mu \in [LS(T), \lambda]$.

Proof 1) By the proof of [KlSh 362. 5.4] (for $\lambda$-regular easier).  
2) [KlSh 362. 5.4].  
3) [KlSh 362. 5.5].  
4) See [KlSh 362].

Intermediate Corollary 1.17  
1. Suppose that $T$ is $\lambda$-categorical. If $\mu < \lambda$, $\mu > LS(T)$ and $T$ is not $\mu$-categorical, then there is an unsaturated model $M \in K_{\mu}$.

2. It now follows that if we show that the existence of an unsaturated model in $K_{\mu}$ implies that of an unsaturated model in $K_{\lambda}$, then $\lambda$-categoricity of $T$ implies $\mu$-categoricity of $T$.

Conclusion 1.18 ($T$ categorical in $\lambda$) If $I$ is a linear order, $I = I_1 + I_2$, $|I| < \lambda$ and $J = I_1 + \omega + I_2$ then every $p \in S(EM(I))$ is realized in $EM(J)$.

Proof $EM(I_1 + \lambda + I_2)$ is in $K_{\lambda}$ hence is saturated hence every $p \in S(EM(I))$ is realized in it, say by $a_p$, for some finite $w \subseteq \lambda$, we have $a_p \in EM(J_1 + w + I_2)$, now we indiscernibility.

Remark 1.19 By changing $\Phi$ we can replace “$\omega$” by “1”.

Conclusion 1.20 ($T$ categorical in $\lambda$)  
1. If $J = \bigcup I_\alpha$, $|J| = \mu \in [LS(T), \lambda]$, $I_\alpha$ increasing continuous, for each $\alpha$ some Dedekind cut of $I_\alpha$ is realized by infinitely many members of $I_{\alpha+1} \setminus I_\alpha$ then $EM(J)$ is $(\mu, \mu)$-saturated over $EM(I_0)$.

2. If $\Phi$ is “corrected” as in [1.13], $I_0 \subseteq J$, $|J \setminus I_0| = |J| = \mu$, $\mu \in [LS(T), \lambda]$, then $EM(J)$ is $(\mu, \mu)$-saturated over $EM(I_0)$ moreover for any $\kappa = \text{cf}(\kappa) \leq \mu$ it is $(\mu, \kappa)$-saturated.

Proof By [1.18+1.13]

Claim 1.21  
1. Suppose $(N_i^\ell : i \leq \alpha)$ is $\leq_{\text{nice}}$-increasing continuous for $\ell = 1, 2$, $N_i^1 \leq_{\text{f}} N_i^2 \in K_{<\lambda}$ and $N_i^2 \bigcup N_{i+1}^1$ for each $i < \alpha$, then $N_0^2 \bigcup N_0^1 \subseteq N_0^2 \bigcup N_0^1$. 





2. The monotonicity properties of $\bigcup\limits_{M_0} M_3$, i.e.: if $M_1 \bigcup\limits_{M_0} M_2$ and for some operation $\text{Op}$ and moduls $M'_1, M'_2, M'_3$ we have $M_3 \preceq_{\mathcal{F}} M'_3 \preceq \text{Op}(M_3)$ and $M_0 \preceq_{\mathcal{F}} M'_1 \preceq_{\mathcal{F}} M_1$ and $M_0 \preceq_{\mathcal{F}} M'_2 \preceq_{\mathcal{F}} M_2$ then $M'_1 \bigcup\limits_{M_0} M'_2$.

3. If $M_1 \bigcup\limits_{M_0} A$ and $M_0 \preceq_{\mathcal{F}} M'_0 \preceq_{\mathcal{F}} M'_1 \preceq_{\mathcal{F}} M'_3 \preceq_{\mathcal{F}} M''_3$ and $M_3 \preceq_{\mathcal{F}} M''_3$ and $A' \subseteq A$ then $M'_1 \bigcup\limits_{M'_0} A'$.

**Proof** Use [KSh 362, 1.11].

**Claim 1.22** If $\mathcal{T}$ is $\lambda$-categorical, if $M_0 \preceq_{\text{nice}} M_1, M_2$ are in $K_{<\lambda}$ then we can find $M_4 \in K_{<\lambda}$, $M_0 \preceq_{\mathcal{F}} M_4$ and $\preceq_{\mathcal{F}}$-embeddings $f_1, f_2$ of $M_1, M_2$ respectively into $M_4$ such that

(a) $f_1(M_1) \bigcup\limits_{M_0} f_2(M_2)$ and

(b) $f_2(M_2) \bigcup\limits_{M_0} f_1(M_1)$.

**Remark 1.23** Note 1.7 deal only with models in $\bigcup\{K_{\mu}: \mu^+ < \lambda\}$ hence (b) is not totally redundant.

**Proof** If we want to get (a) only, use operation $\text{Op}$ such that $\text{Op}(M_0)$ has cardinality $\geq \lambda$, choose $N \preceq_{\mathcal{F}} \text{Op}(M_0)$, $\|N\| = \lambda$, hence $N$ is saturated hence we can find a $\preceq_{\mathcal{F}}$-embedding $f_2 : M_2 \rightarrow N$, let $N_1 = \text{Op}(M_1)$, so $N \preceq_{\mathcal{F}} \text{Op}(M_0) \preceq_{\mathcal{F}} \text{Op}(M_1) = N_1$, and choose $M_4 < N_1, M_4 \in K_{\mu}$ such that $M_1 \bigcup\text{Rang} f_2 \subseteq N$.

By “every $N \in K_{\lambda}$ is saturated” there are an operation $\text{Op}$ and $N \in K_{\lambda}$ such that $M_0 \preceq_{\mathcal{F}} N \preceq_{\mathcal{F}} \text{Op}(M_0)$ hence there are $M^+_1, M^+_2, M^+_3$ in $K_{<\lambda}$ such that:

(*)\(_0\) $(M^+_1, M^+_2) \preceq_{\mathcal{F}} \text{Op}(M_1, M_0), (M^+_2, M^+_3) \preceq_{\mathcal{F}} \text{Op}(M_2, M_0)$ and $M^+_0$ has the form $\text{EM}(I_0)$, $I_0$ a linear order with $|I_0|$ Dedekind cuts with cofinality $(\kappa^*, \kappa^*)$. [Note that by [2.2][2] if $LS(\mathcal{T}) \leq |I_0| \leq \lambda$ then $\text{EM}(I_0)$ is saturated and $N$ is saturated, clearly there is $I_0$ as required.]
Hence we can find $I_1, I_2, I_3$ such that: $I_o \defeq I \subseteq I_1 \subseteq I_3$ $I_0 \subseteq I_2 \subseteq I_3$, $I_1 \cap I_2 = I$, no $t_1 \in I_1 \setminus I_0$, $t_2 \in I_2 \setminus I_0$ realize the same Dedekind cut of $I$, and every $t \in T_3 \setminus I_0$ realize a cut of $I \setminus I_0$ with cofinality $(\kappa^*, \kappa^*)$. Hence $I_0 \subseteq \text{n\textit{ice}} I_\ell$ ($\ell \geq 3$), moreover $I_1 \bigcup I_2$ and $I_2 \bigcup I_1$ hence

$$EM(I_3) \cup EM(I_2) \text{ and } EM(I_2) \cup EM(I_1).$$

Also by \ref{1.20}(2), wlog ($\ell = 1, 2$) $M^+_\ell \preceq_{\mathcal{F}} EM(I_\ell)$.

So

$$EM(I_3) \cup M^+_2 \text{ and } M^+_2 \cup EM(I_3).$$

By $(*)_1 + (*)_2$ and \ref{1.21}(1) (for $\alpha = 2$) we get the conclusions. \hfill \square

\textbf{Claim 1.24} \textit{[$T$ is $\lambda$-categorical]}

1. If $M^\ell_1 \cup M^\ell_2$ for $\ell = 1, 2$, $M^3_\lambda \subseteq K_{< \lambda}$ moreover $\|M^\ell_3\|^+ < \lambda$ and $f_k$

$M^\ell_0$

an isomorphism from $M^\ell_1$ onto $M^\ell_2$ for $k = 0, 1, 2$ such that $f_0 \subseteq f_1$, $f_0 \subseteq f_2$ then there is $M$, $M^3_\lambda \preceq_{\mathcal{F}} M \subseteq K_{< \lambda}$, $\|M\| = \|M^3_\lambda\| + \|M^3_\lambda\|$ and a $\preceq_{\mathcal{F}}$-embedding $f'$ of $M^3_\lambda$ into $M^3_\lambda$ extending $f_1$ and $f_2$.

2. Assume $M^\ell_1 \cup M^\ell_2$ for $\ell = 1, 2$ and $A^\ell_2 \subseteq M^\ell_2 \subseteq M^3_\lambda$, and $M^3_\lambda \subseteq K_{< \lambda}$

$M^\ell_0$

moreover $\|M^\ell_0\|^+ < \lambda$, and $f_k$ is an isomorphism from $M^\ell_1$ onto $M^\ell_2$

$M^\ell_k$

for $k = 0, 1, 2$ such that $f_0 \subseteq f_1$ and $f_0 \subseteq f_2$ and $f_2$ maps $A^\ell_2$ onto $A^\ell_2$

$A^\ell_0$

then there is $M$, $M^3_\lambda \preceq_{\mathcal{F}} M \subseteq K_{< \lambda}$ such that $\|M\| = \|M^3_\lambda\| + \|M^3_\lambda\|$ and a $\preceq_{\mathcal{F}}$-embedding $f'$ of $M^3_\lambda$ into $M^3_\lambda$ extending $f_1$ and $f_2 \upharpoonright A^\ell_2$.

3. If for $\ell = 1, 2$ we have $p_\ell \in S(N)$ does not fork over $M$ (see Definition \ref{1.8}), $M \preceq_{\mathcal{F}} N \subseteq K_\mu$, $\mu^+ < \lambda$ and $p_1 \upharpoonright M = p_2 \upharpoonright M$ then $p_1 = p_2$

\textbf{Remark 1.25} 1. This is uniqueness of non forking amalgamation.

2. The requirement is $\|M^\ell_3\|^+ < \lambda$ rather than $\|M^\ell_3\| < \lambda$ only because of the use of symmetry, i.e. \ref{1.7}.\hfill \square
In this section we’ll attempt to describe some constructions of models of $\mathbf{T}$ relating to the situations in \ref{Def.4.12} and \ref{Def.4.13}, i.e. we want to prove there are “many complicated” models of $\mathbf{T}$ when $\mathbf{T}$ is “on unstable side” of Def.\ref{Def.4.12} or Def.\ref{Def.4.13}. May we suggest that on a first reading the reader be content with the perusal of \ref{2.1} and \ref{2.2}, leaving the heavier work of \ref{2.2.3} until after section three which contains the model-theoretic fruits of the paper. The construction should be meaningful for the classification problem.

What we actually need are \ref{2.2.1}, \ref{2.2.2}, \ref{2.2.3}

\textbf{Construction 2.1} First try

\textbf{Data 2.1.1} Suppose that $\langle M_i : i \leq \kappa + 1 \rangle$ is an continuous $\prec_x$-chain of models of $\mathbf{T}$, $\mu < \lambda$; $T$ is a non empty subset of $(\kappa+1, \text{Ord})$ and

(i) $T$ is closed under initial segments, i.e. if $\eta \in T$ and $\nu < \eta$, then $\nu \in T$,

(ii) if $\eta \in T$ and $\rho(g(\eta)) = \kappa$ then $\eta^\lambda(0) \in T$ and for all $i$, $\eta^\lambda(1 + i) \notin T$.

Let $\lim_\kappa(T) = \{ \eta : \rho(g(\eta)) = \kappa \text{ and } \bigwedge_{i < \kappa} (\eta | i \in T) \}$. Let $\{ \eta_i : i < i^* \}$ be an enumeration of $T$ such that if $\eta_i \prec \eta_j$ ($\eta_i$ is an initial segment of $\eta_j$), then

\textbf{Proof} Wlog $f_0 = \text{id}$, $M_0^1 = M_0^2$ call it $M_0$ and $f_1 = \text{id}_{M_1^1}$, $M_1^1 = M_1^2$ call it $M_1$. For some operation $\text{Op}_\ell$ we have $(M_0^\ell, M_1^\ell) \preceq_x \text{Op}_\ell(M_1^\ell, M_0^\ell)$. Let $\text{Op} = \text{Op}_1 \circ \text{Op}_2$, so $M_2^0 \preceq_x \text{Op}(M_1), M_2^1 \preceq_x \text{Op}(M_0)$. W.l.o.g. $\|\text{Op}(M_0)\| \geq \lambda$ and $\|\text{Op}(M_1)\| \geq \lambda$, so there is $N_0, \bigcup_{\ell = 1}^2 M_\ell^\ell \subseteq N_0 \preceq_x \text{Op}(M_0)$, such that $\|N_0\| = \lambda$, hence $N_0$ is saturated hence there is an automorphism $g_0$ of $N_0$ such that $g_0 \upharpoonright M_1^0 = f_2$ (so $g_0 \upharpoonright M_0 = \text{id}_{M_0}$). So there is $N_2, \bigcup_{\ell = 1}^2 M_\ell^\ell \subseteq N_2 \preceq_x N_0, \|N_2\| < \lambda$, $N_2$ closed under $g_0, g_0^{-1}$. Now there is $N_3, N_0 \cup M_1 \subseteq N_3 \preceq_x \text{Op}(M_1), N_3 \in K_\lambda$, hence $N_3$ is saturated. So $M_1 \cup N_2$ hence $N_2 \cup M_1$ (by symmetry i.e. \ref{1.7}) hence for some $N_3'$, $N_3 \preceq_x N_3' \in K_{<\lambda}$, some automorphism $g_1$ of $N_3$ extend $(g_0 \upharpoonright N_3) \cup \text{id}_{M_1}$. Why? for some $\text{Op}', (N_3, M_1) \preceq_x \text{Op}'(N_1, M_0)$ and $\text{Op}'(N_1), \text{Op}'(g_0 \upharpoonright N_1)$ are as required except having too large cardinality, but this can be rectified.

Clearly we are done.

2) Similarly.

3) Follows.

\hspace{1cm}$\blacksquare$
$i < j$, and if $\eta_i = \nu^\lambda(\alpha)$, $\eta_j = \nu^\lambda(\beta)$, $\alpha < \beta$, then $i < j$. For simplicity $i^*$ is a limit ordinal.

**First Try 2.1.2** From the data of 2.1.1 we shall build a model $N^*$ with Skolem functions, $N^* \upharpoonright L \in K$, and for $\eta \in T$, $M^*_\eta \subseteq N^*$, $f_\eta : M^*_{\eta|y(\eta)} \xrightarrow{\text{onto}} M^*_\eta \upharpoonright L$ such that if $\eta_i \prec \eta_j$, then $f_{\eta_i} \subseteq f_{\eta_j}$, and $M^*_{\eta_i} \subseteq_{\mathcal{F}_k} M^*_{\eta_j}$, where $\mathcal{F}_k \subseteq \mathcal{T}_k$ is a fragment of $(L^k)^{\kappa^*, \omega}$.

Let $M^*_i = \text{Sk}(M_i)$ be a Skolemization of $M_i$ for $\mathcal{F}$, increasing ($\subseteq$) with $i$ i.e. for every formula $(\exists y) \varphi(y, \bar{x}) \in \mathcal{F}$ we choose a function $F_{\varphi(y, \bar{x})}^{M_i}$ from $M_i$ to $M_i$, with $\ell g(\bar{x})$-places such that

$$M_i \models (\exists y) \varphi(y, \bar{a}) \rightarrow \varphi(F_{\varphi(y, \bar{x})}^{M_i}(\bar{a}), \bar{a})$$

such that $j < i \Rightarrow F_{\varphi(y, \bar{x})}^{M_i} | M_j = F_{\varphi(y, \bar{x})}^{M_j}$.

Note: we do not require even $M^*_i < M^*_{i+1}$.

To achieve this, let us define by induction on $i \leq i^*, M^*_i, M^*_j$ and $f_{\eta_j}$. W.l.o.g. $\eta_0 = \emptyset$. Let $N^*_0 = M^*_\eta_0 = \text{Sk}(M_0)$, the Skolemization of $M_0$, $f_{\emptyset} = \text{id}_{M_0}$. If $i$ is a limit ordinal, let $N^*_i = \bigcup_{j<i} N^*_j$. If $i$ is a successor ordinal and $\ell g(\eta_i) = \alpha + 1$, then letting $\eta_j = \eta_i \upharpoonright \alpha$, note that $\eta_j \prec \eta_i$ so $j < i$ and so $M^*_j$ and $f_{\eta_j}$ are defined. We are assuming $M^*_0 \subseteq_{\text{nice}} M^*_\alpha + 1$ hence, there is an operator $\text{Op} = \text{Op}_\alpha$ such that $M^*_\alpha \subseteq_{\text{nice}} \text{Op}(M_0)$. Let $N^*_i = \text{Op}(N^*_i - 1)$, let $\text{Op}(N^*_i - 1, M_\alpha, f_{\eta_j}) = (N^*_i, \text{Op}(M_\alpha), (\text{Op}(f_{\eta_j})), and let $f_{\eta_j} = \text{Op}(f_{\eta_j}) | M^*_{\eta|y(\eta)}$ and $M^*_j = \text{Rang}(f_{\eta_j})$. (We can replace $N^*_i$ by any $N'$ such that $N^*_i \cup M^*_i \subseteq N' \subseteq_{\mathcal{F}} N^*_{i+1}$ so preserving $|N^*_i| \leq \mu + |i|$. Finally, let $N^*_i = \bigcup_{i < \kappa^*} N^*_i$. We are left with the case $i$ successor ordinal, $\ell g(\eta_i)$ a limit ordinal; we let $N^*_i = N^*_{i+1}$, $M^*_i = \bigcup_{\nu \preceq \eta_i} M^*_\nu$ and $f_{\eta_j} = \bigcup_{\nu \preceq \eta_j} f_\nu$.

**Explanation:** In order to use this construction to prove non-structure results, we intend to use property: for every $\eta \in \text{lim}_\alpha T$, it is possible to extend $f_\eta = \bigcup_{\alpha < \kappa} f_{\eta|\alpha}$ to an $\mathcal{F}$-elementary embedding $f^*$ of $M_{\kappa+1}$ into $N^*$ iff $\eta \in T$.

Remark that if for example $\chi$ is a strong limit cardinal of cofinality $\kappa^*$ and $\chi^{<\kappa} \subseteq T \subseteq \chi^{<\kappa} \cap \{\eta^{\nu}(\alpha) : (\exists \alpha < \kappa) \ell g(\eta) = \alpha + 1\}$, then over $\bigcup_{\eta \in \chi^{<\kappa}} M^*_\eta$ for $\chi$ parameters there are $2^\chi$ independent decisions. This is not only a reasonable result, it has been shown, ($\text{Sh:k}$, VIII §1 for $\chi$ as above, $\text{Sh:k}$, III §5 more generally) that this result is sufficient to prove the existence of many models in every cardinality $\lambda > \mu + L\text{S}(T)$.

But to use this construction we have to have some continuity of non forking, which, we have not proved. Hence we shall use another variant of the construction.
Construction 2.2 We modify the construction of 2.1 to suit our purposes.

Modified Data 2.2.1 Let \((M_i ∈ K_{κ+1} : i ≤ κ + 1)\) be an continuous \(≤_{\text{nice}}\)-chain of models of \(T\), \(||M_{κ+1}|| = μ < λ\). Let \(T\) be a subset of \(κ+1≤ (\text{Ord})\), \(≺^{\text{lex}}\) be the lexicographic order on \(T\), it is a linear order of \(T\); suppose that \(T\) is \(＜\)-closed i.e. \(ν ≺ η ∈ T \Rightarrow ν ∈ T\), and if \(η ∈ κ^{\text{Ord}}(T) \cap T\), then \(η^{\langle 0 \rangle}\) is the unique \(≺^{\text{lex}}\)-successor of \(η\) in \(T\). For \(S ⊆ T\) let \(S^{\text{se}} = \{η ∈ S : \ell g(η)\) successor\}. Let \(Ω_{κ+i+1}\) witness \(M_i \leq_{\text{nice}} M_{κ+i+1}\).

We define \(Ω_η = \bigcup_{i < κ} \bigcup_{\ell g(η)} \Omega_i\) for \(η ∈ T^{\text{se}}\). We can iterate the operation \(Ω_η\) w.r.t. \((T^{\text{se}}, ≺^{\text{lex}})\). Also, for each \(S ⊆ T\), we can iterate \(Ω_η\) w.r.t. \((S^{\text{se}}, ≺^{\text{lex}})\). Let us denote the result of this iteration w.r.t. \((S, ≺^{\text{lex}})\) by \(Ω^\text{\text{erm}}\) (see [KlSh 362, 1.11]). Note that for any \(M ∈ K\), if \(S_1 ⊆ S_2 ⊆ T\), then \(M ≺ F Ω^\text{\text{erm}}(M) ≤ F Ω^{\text{S}_2}(M) ≤ F Ω^T(M)\) (by natural embeddings).

More formally:

Claim 2.2.2 There exist operations \(Ω^S\) for \(S ⊆ T\) such that

1. for every \(S ⊆ T\) which is \(≺\)-closed \(M_S = Ω^S(M)\) is defined, and whenever \(S_1 ⊆ S_2 ⊆ T\), then \(M_{S_1} ≤ F M_{S_2}\);
2. for \(η ∈ T\), \(h_η\) is a surjective \(≺^T\)-elementary embedding from \(M^\text{\text{erm}}(η)\) to \(M_η\), \(M_η ≤ F M_{\langle η \rangle}\), and \(\langle η : η ∈ T \rangle\) is a \(≺\)-increasing sequence, i.e. \(h_η ≤ h_ν\) whenever \(η ≺ ν\);
3. for every \(x ∈ M_T\), there exists an \(≺\)-closed \(S ⊆ T\), \(|S| ≤ κ\) such that \(x ∈ M_S\) (in fact \(S\) is the union of finitely many branches);
4. for \(η ∈ T\) with \(\ell g(η) = κ\), and \(α < κ\), letting \(T[η] = \{ν ∈ T : \neg(η ≺ ν)\}, T^≤[η] = \{ν ∈ T[η] : ν ≤ T[η]\}, T^≥[η] = \{ν ∈ T[η] : ν ≤ T[η]\}\) (so \(T[η] = T^≤[η] ∪ T^≥[η]\)) and \(M_T^≤[η] = \bigcup_{\alpha < κ} M_α[η]\) and \(M_T^≥[η] = \bigcup_{\alpha < κ} M_α[η]\) for \(α < κ\);
5. if \(η ∈ \text{lim}_κ(T)\) and \(η ∉ T\), then \(M_T = \bigcup_{\alpha < κ} M_T^{\text{\text{erm}}[η]}\)
6. \(||M_S|| ≤ |S| + ||M_{κ+1}||^{κ^*} + \sup_{\eta ∈ S} ||M^\text{\text{erm}}[η]|\).

Fact 2.2.3 1. By clause (4), if we have the conclusion of 1.7 (and 1.21(1))
then \(M_η \bigcup_{\alpha < κ} M_T^{\text{\text{erm}}[η]}\).
2. Then in fact one can replace clause (4) above by the weaker condition
\[(4)\] for every \( S \subseteq T \), if \( \{ \eta \upharpoonright i : i \leq \alpha \} \subseteq S \subseteq T \), then \( M_\eta \bigcup_{\eta \upharpoonright \alpha} M_S \).

(2) by (4).

**Short Proof of 2.2.2:** As \( \langle M_i : i \leq \kappa +1 \rangle \) is \( \succeq_{\text{nice}} \)-increasing continuous by renaming there is \( \langle M^*_i : i \leq \kappa +1 \rangle \) \( \succeq_{\text{nice}} \)-increasing continuous, \( M^*_0 = M_0 \), \( M^*_i = \text{Op}(M^*_i) \), \( M_i \preceq_F M^*_i \), and \( M^*_i \bigcup_{\eta \in \alpha} M^*_i \) for \( i \leq \kappa \). W.l.o.g.

\[ \| M^*_\kappa \| \leq \| M_i \|^\kappa. \]

Let \( (I_\eta, D_\eta, G_\eta) \) be a copy of \( \text{Op}_{\eta} \) for \( \eta \in T \), and let \( I_\eta \) be pairwise disjoint. Define \( I = \Pi \{ I_\eta : \eta \in T \} \), \( D, G \) as in the proof of [Kri 362, 1.11], so every equivalence relation \( e \in G \) has a finite subset \( \{ \eta_0 <_{\text{lex}} \ldots <_{\text{lex}} \eta_{n(e)-1} \} \subseteq T \) and \( \epsilon e \in G_{\eta e} \) as there. We let \( \text{Op}_{T^e} = (I, D, G) \), \( M_{T^e} = \text{Op}_{T^e}(M_0) \) and for \( S \subseteq T \) we let

\[ M_S = \{ x \in M_T : w[eq(x)] \subseteq S \}. \]

Naturally there are canonical maps \( f^*_\eta \) from \( M^*_{Ig\eta} \) onto \( M_{\{\nu : \nu \subset \eta\}} \) and let \( M_\eta = f^*_{\eta}(M_{Ig\eta}) \).

**Improvement 2.2.4** Improvement in cardinality.

We can replace \( \| M_{\kappa+1} \|^\kappa \) by \( \| M_{\kappa+1} \| + LS(T) \) in part (6) of claim 2.2.2.

After choosing \( \langle M_i : i \leq \kappa +1 \rangle \), let \( M^*_0 = M_0 \), \( M^*_i = \text{Op}(M^*_i) \), \( M^*_i = \bigcup_{i \leq \delta} M^*_i \). Of course \( M^*_S \) (\( S \subseteq T \) is \( \triangleleft \)-closed) are well defined similarly. Let \( N_i \) be the Skolem hull of \( M_i \) in \( M^*_i \). For \( \eta \in T \) let \( N_\eta = f^*_\eta(N_{Ig\eta}) \). Now for any \( \triangleleft \)-closed \( S \subseteq T \) let

\[ N_S = \text{Skolem hull in } M^*_S \text{ of } \cup \{ N_\eta : \eta \in S \}. \]

* * *

There are two different ways to carry on the construction (under Data 2.2.1). We’ll consider each in its turn.

**Construction 2.3** Recall that it is possible to iterate the operation \( \text{Op} \) with respect to the linear order \( (T, <_{\text{lex}}) \) and this iteration can be defined as the direct limit of finite approximations. We shall use different approximations and take the direct limit we obtain the required operation.
Suppose that \( w \subseteq T \) is closed with respect to \( \triangleleft \) (i.e. initial segment) and is \( <_{lex} \)-well-ordered. For each approximation \( w \) of this kind, the iterated ultrapower \( \text{Op}^w(M_0) \) of \( M_0 \) with respect to \( w \) is defined as a limit ultrapower and there are natural elementary embeddings into this limit. The principal difference is that this limit is a little larger than a limit obtained using only finite approximations. For example, if \( \langle \eta_n : n \leq \omega \rangle \) is a \( <_{lex} \)-increasing sequence, then in \( \text{Op}^\omega \left( \ldots \text{Op}^{\eta_0}(M_0) \right) \), the last operation \( \text{Op}^\omega \) adds elements which are dispersed over all \( \text{Op}^{\eta_n}(\ldots \text{Op}^{\eta_0}(M_0)) \). (This is of more interest when the sequence has length \( \kappa \).) Now it is easy to check the symmetry (for \( \eta \in {}^\alpha \lambda, \alpha < \kappa \)) between the \( <_{lex} \)-successors and \( <_{lex} \)-predecessors of \( \eta \).

We define the embeddings \( h_\eta \) for \( \eta \in T \) as follows. For \( \eta = \langle \rangle \), \( h_\eta = \text{id} \upharpoonright M_0 \). If \( \eta = \nu^\lambda \langle i \rangle \), then \( \text{Op}^\eta \) acts on \( M_\nu = h_\nu [M_{\ell^\lambda(\nu)}] \) and we use the commuting diagram:

\[
\begin{array}{ccc}
\text{Op}^\theta(M_{\ell^\lambda(\nu)}) & \longrightarrow & \text{Op}^\theta(M_\nu) \\
\uparrow & & \uparrow \\
M_{\ell^\lambda(\eta)} & \longrightarrow & M_\eta \\
\uparrow & & \uparrow \\
M_{\ell^\lambda(\nu)} & \longrightarrow & M_\nu \\
\downarrow h_\nu & & \downarrow \\
h_\eta & & \end{array}
\]

This completes the construction.

**Construction 2.4** In this approach, we employ the generalized Ehrenfeucht-Mostowski models \( EM(I, \Phi) \) from chapter VII in [Sh:a] or [Sh:c]. For this we need to specify the generators of the model and what the types are.
Let $M_0^+$ be the model obtained from $M_0$ by adding Skolem functions and individual constants for each element of $M_0$. We know that there is an operation $\text{Op}$ such that, for $i \leq \kappa$, $M_i \not\preceq_F M_{i+1} \not\preceq_F \text{Op}(M_i)$. As in [KSh 362, 1.7.4] this means that there are $I, D$ and $G$ such that $\text{Op}(M_i) = \text{Op}(M, I, D, G)$ where $I$ is a non-empty set, $D$ is an ultrafilter on $I$, and $G$ is a suitable set of equivalence relations on $I$, i.e.

(i) if $e \in G$ and $e'$ is an equivalence relation on $I$ coarser than $e$, then $e' \in G$;

(ii) $G$ is closed under finite intersections;

(iii) if $e \in G$, then $D/e = \{ A \subset I/e : \bigcup x \in A x \in D \}$ is a $\kappa^*$-complete ultrafilter on $I/e$.

For each $b \in M_i+1 \setminus M_i$, let $\langle x^b_t : t \in I \rangle/D$ be the image of $b$ in $\text{Op}(M_i)$. We’ll also write $\langle x^b_t : t \in I \rangle/D$ for the canonical image $d(b)$ of $b \in M_i$ in $\text{Op}(M_i)$.

\[
M_{i+1} \ni b \mapsto \langle x^b_t : t \in I \rangle/D \in \text{Op}(M_i)
\]

We define a model $M^+, M_0^+ \preceq_{\text{L}_{\kappa^{*}}, \omega} M^+$, as follows. $M^+$ is generated by the set $\{ x^b_\eta : b \in M_{i+1} \setminus M_i, \eta \in T, \ell g(\eta) = i + 1 \}$. Note that this set does generate a model since $M_0^+$ is closed under Skolem functions. Since functions have finite arity, it is enough to specify, for each finite set of the $x^b_\eta$, what quantifier-free type it realizes. Since there is monotonicity, we shall obtain indiscernibility as in [Sh:a]. The type of a finite set $\langle x^b_\eta^\ell : \ell = 1, \ldots, n \rangle$ depends on the set $\langle b_1, \ldots, b_n \rangle$ and the atomic (i.e. quantifier-free) type of $\langle \eta_1, \ldots, \eta_n \rangle$ in the model $\langle T, \triangleleft, \prec_{\text{lex}}, \lnot \eta \mid i = \nu \mid i \rangle$. Now w.l.o.g. we can allow finite sequences $\vec{b}$ instead of $b$ for $b \in M_{i+1} \setminus M_i$ and thus w.l.o.g.
\(\eta_1, \ldots, \eta_n\) is repetition-free, so w.l.o.g. \(\eta_1 \prec_{\text{lex}} \eta_2 \prec_{\text{lex}} \cdots \prec_{\text{lex}} \eta_n\). Suppose that the lexicographic order \(\prec_{\text{lex}}\) on \(\{\eta_\ell \mid \alpha \leq \ell g(\eta_\ell)\) and \(\ell = 1, \ldots, n\) is a well-order and the sequence \(\{\nu_\zeta : \zeta < \zeta(\ast)\}\) is \(\prec\)-increasing. We define \(N_0 = M_0^\ast, N_{\zeta+1} = \text{Op}(N_\zeta), N_\zeta = \bigcup_{\zeta < \zeta} N_\xi\) (for limit \(\zeta\)). Next, we define \(h_{\nu_\zeta} : M_{\ell g(\nu_\zeta)} \to N_{\zeta+1}, h_{\nu_\zeta} \upharpoonright \beta \subseteq h_{\nu_\zeta}\). If \(\ell g(\nu)\) is a limit ordinal, then \(\alpha < \ell g(\nu) \Rightarrow h_{\nu_\alpha}\) is defined and we let \(h_\nu = \bigcup_{\alpha < \ell g(\nu)} h_{\nu_\alpha}\). If \(\nu_\zeta = \nu_\xi \upharpoonright \gamma\), then \(M_{\zeta+1} = \text{Op}(M_\zeta, I, D, G)\), identifying elements of \(M_\zeta\) with their images in the ultrapower. Now define

\[
h_{\nu_\zeta}(b) = \begin{cases} 
d(H_{\nu_\zeta}(b)) & \text{if } b \in M_i, \\
(h_{\nu_\zeta}(a_i^b)) : t \in I \big/ D & \text{if } b \in M_{i+1} \setminus M_i,
\end{cases}
\]

where \(d(h_{\nu_\zeta}(b))\) is the canonical image of \(H_{\nu_\zeta}(b)\) in the ultrapower. The type of \(\langle x^b_i : \ell = 1, \ldots, n \rangle\) is defined to be the type of \(\langle h_{\eta_\ell}(b) : \ell = 1, \ldots, n \rangle\) in \(N_\zeta\).

**Remark 2.4.1** It is possible to split the construction into two steps. For \(i \leq j < \kappa + 1\), there is an operation \(\text{Op}^{i,j}\), \(M_i \leq M_j \leq \text{Op}^{i,j}(M_i)\), moving \(b\) to \(\langle i^j d_i^b : t \in I\rangle, b \in M_j, i^j d_i^b \in M_i\), with the obvious commutativity and continuity properties. Now the construction is done on a finite tree \(\langle \eta_\ell : \ell = 1, \ldots, n \rangle, \{\eta_\ell \cap \eta_m : \ell, m < \omega\}\). We omit the details of monotonicity.

**Notation 2.4.2** Let \(M_T = M\) be the Skolem closure. If \(S \subseteq T\) is closed with respect to initial segments, let \(M_S = \text{Sk}_M(x^b_\eta : \eta \in S, b \in M_{\ell g(\eta)})\) and \(M^\ast_\eta = M_{\{\eta : \alpha \leq \ell g(\eta)\}}\). Define \(h_\eta : M_{\ell g(\eta)} \to M^\ast_\eta\) by \(h_\eta(b) = x^b_{\eta \upharpoonright \tau(T)}\) and \(N_\eta = h_\eta[M_\eta]\).

**Remark 2.4.3** The construction can be used to get many fairly saturated models. We list the principal properties below.

**Fact 2.4.4** Suppose that \(S_\ell \subseteq T\) is closed with respect to initial segments, \(S_0 = S_1 \cap S_2\) and \([\eta \in S_1 \& \nu \in S_2 \setminus S_1 \Rightarrow \eta \prec_{\text{lex}} \nu]\) then

\[
M_{S_1} \uplus \bigcup_{M_{S_0}^T} M_{S_2}.
\]

**Proof** W.l.o.g. \(S_\ell\) is closed, \(M_{\ell g(S_\ell)} = M_{S_\ell}\). Let \(S_2 \setminus S_0 = \{\nu_\zeta : \zeta < \zeta(\ast)\}\) be a list such that \(\nu_\zeta < \zeta_\xi \Rightarrow \zeta < \xi\); let \(S_2^\xi = S_0 \cup \{\nu_\zeta : \zeta < \zeta(\ast)\}\). Then
1. \( \langle M_{S_2^\xi} : \xi \leq \xi(\ast) \rangle \) is continuous increasing;

2. \( \langle M_{S_2^\xi \cap S_1} : \xi \leq \xi(\ast) \rangle \) is continuous increasing. Hence one has

\[
M_{S_2^\xi+1 \cup S_1} \cup M_{S_2^\xi+1} \cup M_{S_2^\xi} \cup S_1 \cup N_{\nu_{\xi}}
\]

This is immediate from the definitions, because \( M_{S_2^\xi+1 \cup S_1} \) is the Skolem closure of \( M_{S_2^\xi \cup S_1} \cup N_{\nu_{\xi}} \), and so elements of \( N_{\nu_{\xi}} \) can be represented as averages.

3. Categoricity in \( \mu \), when \( LS(T) \leq \mu < \lambda \)

**Hypothesis 3.1** Every \( M \in K_{<\lambda} \) is nice hence has a \( \preceq_F \)-extension of cardinality \( \lambda \) which is saturated and \( K_{<\lambda} \) has amalgamation.

This section contains the principal theorems of the paper: if \( T \) is \( \lambda \)-categorical, \( LS(T) \leq \mu < \lambda \), then \( \kappa_\mu(T) = \emptyset \) when \( \mu \in [LS(T), \lambda) \) and when \( LS(T) \leq \chi = \text{cf}\chi < \lambda \), \( T \) is \( \chi \)-based, (and \( K \) does not have \( (\mu, \kappa) \)-continuous non forking when \( \mu \in [LS(T), \lambda], \kappa \leq \mu \)) also there is a saturated model in \( K_\mu = \langle K_\mu, \preceq_F \rangle \) and \( T \) is \( \lambda \)-categorical. However we first deal with some preliminary results, quoting [Sh 300] extensively.

**Theorem 3.2** Assume the conclusion of 1.7 for \( \mu \) (e.g. \( \mu^+ < \lambda \)). Suppose that the tree \( T \) is as in Claim 2.2.2 and suppose further: \( \langle M_i \in K_{\leq \mu} : i \leq \kappa + 1 \rangle \) is \( \preceq_{\text{nice}} \)-increasing continuous sequence of members of \( K_{\leq \mu} \), and we apply §2 and

\( (*) \) there is no \( \preceq_F \)-increasing continuous sequence \( \langle N_i \in K_{\leq \mu} : i \leq \kappa \rangle \) such that:

\[
M_i \preceq_F N_i \\
M_{\kappa+1} \preceq_F N_\kappa \\
N_{i+1} \cup M_{i+1} \text{ for } i < \kappa
\]

Then TFAE for \( \eta \in \text{Lim}_\kappa(T) \) \( \overset{\text{def}}{=} \{ \eta \in \kappa(\text{Ord}) : \bigwedge_{i<\kappa} (\eta \upharpoonright (i+1) \in T) \} \):

\( (\alpha) \) There is an \( F \)-elementary embedding \( h \) from \( M_{\kappa+1} \) into \( M_T \) such that \( \bigcup_{i<\kappa} h_{\eta \upharpoonright i+1} \subseteq h \).
Claim 3.3 Suppose the conclusion of 1.7 for \( \mu \) and \( \bar{M} = \langle M_i \in K_{\leq \mu} : i \leq \kappa + 1 \rangle \) is given. Then \( \bar{M} \) satisfies (\( \ast \)) of 3.2 if one of the following holds:

(a) there is \( a \in M_{\kappa+1} \) such that \( i < \kappa \Rightarrow M_i \uplus a \), or

(b) \( \kappa = \text{cf}(\kappa) = \mu > \text{LS}(T) \) and \( i < \kappa \Rightarrow ||M_i|| < \kappa \), and there is a continuous 2-elementary chain \( \langle N_i : i \leq \kappa \rangle \),
\[
M_{\kappa+1} = \bigcup_{i \leq \kappa} N_i, \quad \kappa = \chi^{\text{cf}(\kappa)}, \quad \bigwedge_{i < \kappa} (N_i \in K_{< \kappa}),
\]
and \( E = \{ i < \kappa : M_{i+1} \uplus N_i \} \) is a stationary subset of \( \kappa \).

Proof Straight from 3.2, §2.

Remark 3.4 Clause (\( \beta \)) can also be proved using niceness as in the proof of 3.8. This works for any \( \kappa < \lambda \). Also we can imitate 2.2.2 but no need arise.

Corollary 3.5 If \( T \) is a \( \lambda \)-categorical theory\(^1\), then

1. \( T \) is \( \chi \)-based if \( \chi^+ < \lambda \) and \( \chi \geq \text{LS}(T) \); also it is not \( < \mu \)-based if \( \mu = \text{cf}(\mu) \), \( \text{LS}(T) < \mu \), \( \mu^+ < \lambda \);

2. \( \kappa_\mu(T) = \emptyset \) for every \( \mu \), \( \mu^+ < \lambda \) and \( \mu \geq \text{LS}(T) \).

Proof 1), 2) We use 3.3, 3.4 to contradict \( \lambda \)-categoricity.

Case 1: \( \lambda^\mu = \lambda \) By [Sh 300, III, 5.1 = Shc IV, 2.1].

Case 2: \( \lambda \) is regular, \( \lambda > \mu^+ \). We can find a stationary \( W^* \in I[\lambda] \), \( W^* \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \kappa \} \) (by [Sh 420], §1). Hence, possibly replacing \( W^* \) by its intersection with some club of \( \lambda \), there is \( W^+ \), \( W^* \subseteq W^+ \) and \( \langle a_\alpha : \alpha \in W^+ \rangle \) such that: \( \alpha \in a_\beta \) (so \( \beta \in W^+ \)) implies \( \alpha \in W^+ \), \( a_\alpha = a_\beta \cap a_\alpha \) and \( \text{otp}(a_\alpha) \leq \kappa \) and \( \alpha = \sup a_\alpha \iff \text{cf}(\alpha) = \kappa \iff \alpha \in W^* \). Now let \( \eta_\alpha \) enumerate \( a_\alpha \) in increasing order (for \( \alpha \in W^+ \)), and for any \( W \subseteq W^* \) let
\[
T_W = \{ \eta_\alpha : \alpha \in W^+ \text{ but } \alpha \notin W^* \setminus W \} \cup \{ \eta_\alpha^- (0) : \alpha \in W \}.
\]

\[^1\text{or just has } < 2^\lambda \text{ non isomorphic model in } \lambda\]
Now if \( W_1, W_2 \subseteq W \), \( W_1 \setminus W_2 \) is stationary, then \( M_{W_1} \) cannot be \( \preccurlyeq^\mathcal{F} \)-embedded into \( M_{W_2} \) (again by [Sh 300] III, §5 = [Sh:e], IV §2).

**Case 3:** \( \lambda \) singular.

Choose \( \lambda', \lambda > \lambda' = \text{cf}(\lambda') > \mu^+ \) and act as in case 2 (to get \( 2^\lambda \) we need more, see [Sh:e], IV. 3).

**Hypothesis 3.6** The conclusion of 3.5 (in addition to 3.1 of course).

**Conclusion 3.7** Suppose \( \mu \geq \text{LS}(T), \mu^+ < \lambda, M \in K_\mu \).

1. If \( p \in S(M) \) then \( p \) is determined by \( \{ p \restriction N : N \preccurlyeq^\mathcal{F} M \text{ and } \|N\| = \text{LS}(T) \} \)

2. Assume further

\[(*)^M_{\{N_t : t \in I\}} \]

a) \( I \) (a partial order) which is directed (i.e. every finite many elements have a common upper bound)

b) \( N_t \preccurlyeq^\mathcal{F} M \),

c) \( I \models t \leq s \) implies \( N_t \subseteq N_s \) (hence \( N_t \preccurlyeq^\mathcal{F} N_s \) by clause (b))

d) \( \bigcup_{t \in I} M_t = M \).

Then every \( p \in S(M) \) is determined by

\[\{ p \restriction N_t : t \in I \}\]

**Proof**

1) Follows by part (2).

2) Easily (and as [Sh 88] §1):

\( \otimes \) we can choose by induction on \( n < \omega \) for every \( u \in [M]^n \), \( t[u] \in I \) and \( N_u^* \) such that:

\( u \in N_u^*, N_u^* \preccurlyeq^\mathcal{F} N_u, \|N_u^*\| \leq \text{LS}(T) \) and: \( u \subseteq v \in [|M|]^{<\aleph_0} \) implies \( N_u^* \prec N_v^* \) and \( t[u] \leq t[v] \).

Let for \( U \subseteq |M|, N_U^* =: \{ N_u^* : u \subseteq U \text{ finite} \} \) the definitions are compatible. Easily \( U_1 \subseteq U_2 \subseteq |M| \Rightarrow N_{U_1}^* \preccurlyeq^\mathcal{F} N_{U_2}^* \preccurlyeq^\mathcal{F} M \). Now we prove by induction on \( \mu \leq |M| \) that:

\((**):\) if \( U \subseteq \|M\|, |U| = \mu, p \in N_U^* \) then for some \( u \in [U]^{<\aleph_0} \), \( p \) does not fork over \( N_u^* \).
For $\mu$ finite this is trivial, for $\mu$ infinite then $\text{cf}(\mu) \notin \kappa_{\mu+\text{LS}(T)}(T)$ (by 3.52) so (**) holds. Now by 3.24(3), we are done. 

\textbf{Theorem 3.8} Suppose that $\text{cf}(\kappa) = \kappa \leq \mu < \lambda$ and $\text{LS}(T) < \mu$. Then

1. The $(\mu, \kappa)$-saturated model $M$ is saturated (i.e. $N \preceq M$, $\|N\| < \|M\|$, $p \in S(N) \Rightarrow p$ realized in $M$, and hence unique). Hence there is a saturated model in $K_\mu$.

2. The union of a continuous $\preceq_F$-chain of length $\kappa$ of saturated models from $K_\mu$ is saturated.

3. In part (1) we can replace saturated by $(\mu, \mu)$-saturated if $\mu = \text{LS}(T)$.

   We can in part (1) replace saturated by $\chi$-saturated if $\chi > \text{LS}(T)$.

\textbf{Proof} 1), 2) Suppose that $M = M_\kappa$ and $\langle M_i : i \leq \kappa \rangle$ is a continuous $\preceq_F$-chain of members of $K_\mu$ such that for the proof of 1) $M_{i+1}$ is a universal extension of $M_i$ and for the proof of 2) $M_{i+1}$ is saturated. Let $i \leq j \leq \kappa$. Then $M_i \preceq \text{nice} M_j$ (by [KSh 362], 5.4 or more exactly by the hypothesis \textbf{3.1}). So there is an operation $\text{Op}_{i,j}$ such that $M_i \preceq M_j \preceq \text{Op}_{i,j}(M_i)$. It follows that there is an expansion $M_{i,j}^+$ of $M_j$ by at most $\text{LS}(T)$ Skolem functions such that if $N$ is a submodel of $M_{i,j}^+$, then

$$M_i \bigcup_{i \leq j \leq \kappa} M_j \upharpoonright \leq N.$$

[Why? as we use operations coming from equivalence relations with $\leq \kappa^*$ classes and $\text{LS}(T) \geq \kappa^*$ by its definition]. More fully, letting $\text{Op}_{i,j}(N) = N'_{i,j} \downarrow / G$, every element $b \in M_j$ being in $\text{Op}_{i,j}(M_i)$ has a representation as the equivalence class of $\langle t^b_i : t \in I \rangle / D$ under $\text{Op}_{i,j}$, $t^b_i \in M_i$ and $\{\langle t^b_i : t \in I \rangle \} \leq \kappa^*$. The functions of $M_{i,j}^+$ are the Skolem functions of $M_j$ and $M_i$ and functions $F_\zeta (\zeta < \kappa^*)$ such that $\{F_\zeta(b) : \zeta < \kappa^*\} \supseteq \{t^b_i : t \in I\}$.

If $\kappa = \text{cf}(\mu)$, the theorem is immediate. So we’ll suppose that $\kappa < \mu$. Suppose $N \preceq M = M_\kappa$, $\|N\| < \mu$ and $p \in S(N)$. Let $\chi = \|N\| + \kappa + \text{LS}(T)$. W.l.o.g., there is no $N_1$, $N \preceq N_1 < M_\kappa$, $\|N_1\| \leq \chi$ and $p_1$, $p \subseteq p_1 \in S(N_1)$ such that $p$ fork over $N$ (by 3.3). If there is $i < \kappa$ such that $N \subseteq M_i$, then $p$ is realized in $M_{i+1}$. By the choice of the models $M_{i,j}^+$, it is easy to find $N'$ such that $N \preceq N' \preceq M_\kappa$, $\|N'\| = \chi \overset{\text{def}}{=} \|N\| + \kappa + \text{LS}(T)$ and, for every $i \leq \kappa$,

$$M_i \bigcup_{i \leq \kappa} N'. $$
Now let $N_i = N' \cap M_i$ and note that $N_\kappa = N'$. The sequence $\langle N_i : i \leq \kappa \rangle$ is continuous increasing and there is an extension $p'$ of $p$ in $S(N_\kappa) = S(N')$. Hence there exists $i < \kappa$ such that $(i \leq j < \kappa) \Rightarrow (p' \text{ does not fork over } N_j)$. By [Sh:c](4), it is sufficient to find $j \in [i, \kappa)$ and $\langle N^*_\varepsilon : \varepsilon < \chi^+ \rangle$ continuous such that: $N_i \preceq_F N^*_\varepsilon \preceq_F M_j$, $N^*_\varepsilon+1$ is a $\chi$-universal extension of $N^*_\varepsilon$ (recall symmetry and uniqueness of extensions).

3) Similar proof for the second sentence, [Sh:21] for the first sentence.

**Remark:** Using categoricity we can prove 3.8 also by [Sh:21](2) (and uniqueness).

**Conclusion 3.9** Assume $LS(T) \leq \kappa < \mu \in (LS(T), \lambda)$, $M \in K_\mu$ is not $\kappa^+$-saturated; let $\langle N^*_u : u \in \langle |M|^{|< \aleph_0} \rangle \text{ and } N^*_U \text{ (for } U \subseteq |M|) \rangle$ be as in the proof of 3.7(2). Then there is $U \subseteq |M|$, $|U| \leq \kappa$, $p \in S(N^*_U)$ i.e. there are $N^+, N^*_U \preceq_F N^+ \in K_\kappa$, and $a^+ \in N^+$ satisfying $(a^+, N^+)/E_{N^*_U} = p$ such that:

for no $a \in M$ do we have $u \in \langle |U|^{|< \aleph_0} \rangle \Rightarrow \tp(a, N^*_u, M) = \tp(a^+, N^*_u, N^+)$. Equivalently: w.l.o.g. $N^+ \cap M = N^*_U$ and we can define $N^*_u$ for $u \in \langle |N^+|^{|< \aleph_0} \rangle$, such that $\langle N^*_u : u \in \langle |N^+|^{|< \aleph_0} \rangle \rangle$ as in the proof of 3.7(2), and $u \in \langle |U|^{|< \aleph_0} \rangle \Rightarrow N^*_u = N^*_u$ and for no $u_0 \in \langle |M|^{|< \aleph_0} \rangle$, $v_0 \in \langle |N^+|^{|< \aleph_0} \rangle$, $a^+ \in N^*_u$, and $a \in N^*_u$ do we have

$$\bigwedge_{u \in \langle |U|^{|< \aleph_0} \rangle} \tp(a, N^*_u, N^*_u^{|\cup v_0}) = \tp(a^+, N^*_u^{|\cup v_0}).$$

**Corollary 3.10** 1. If $T$ is $\lambda$-categorical and $LS(T) < \mu < \lambda$, $LS(T) \leq \chi$, $\delta(*) = (2^{LS(T)})^+ \text{ and } \exists_\delta(*) \chi \leq \mu$ then every $M \in K_\mu$ is $\chi^+$-saturated. In fact for some $\delta < \delta(*)$ we can replace $\delta(*)$ by $\delta$.

2. If $\mu = \exists_\gamma(2\gamma)^{+\times \delta}$, $\delta$ a limit ordinal then $T$ is $\mu$-categorical.

**Proof** By 3.9 this problem is translated to an omitting type argument + cardinality of a predicts which holds (see [Sh:c], VIII §4, [Sh:c], VII §5). See more on this in [Sh 88].

**Claim 3.11** [T categorical in $\lambda$]

1. If $\langle M_i : i \leq \delta \rangle$ is $\preceq_F$-increasing continuous, $M_i \in K_{\leq \lambda}$, $p \in S(M_\delta)$ then for some $i < \delta$, $p$ does not fork over $M_i$. 


2. If $N \in K_{< \lambda}$ and $p, q \in S(N)$ does not fork over $M$, $M \preceq_F N \in K_{< \lambda}$ then $p = q \iff p \upharpoonright M = q \upharpoonright M$. Moreover if $M \preceq_F N \preceq_F N^+$, $a \in N^+$ then

$$N^+ \bigcup_M a \iff N^+ \bigcup_M a.$$

3. If $M \preceq_F N \in K_{< \lambda}$ and $p \in S(M)$ then there is $q \in S(N)$ extending $p$ not forking over $M$.

4. If $M_0 \preceq_F M_1 \preceq_F M_2$, $p \in S(M_2)$, $p \upharpoonright M_{\ell+1}$ does not fork over $M_{\ell}$ for $\ell = 0, 1$ then $p$ does not fork over $M_0$.

5. If $\mu, \delta < \lambda$, $M_i \in K_{< \mu}$ for $i < \delta$ is $\preceq_F$-increasing continuous, $p_i \in S(M_i)$, $[j < i \implies p_j \subseteq p_i]$ then there is $p \in S(M_\delta)$ such that $i < \delta \implies p_i \subseteq p_\delta$.

**Proof**

1) Otherwise we can find $N$, $M_\delta \preceq_F N \preceq_F \text{Op}(M_\delta)$, $N \in K_\lambda$, $N$ omit $p \bigcup_{i<\delta} \text{Op}(M_i)$; so we get a non-$\lambda$-saturated model of cardinality $\geq \lambda$, contradiction.

2) The first sentence follows from the second. If the second fails then we can contradict stability in $\|N\|$, by a proof just like 1.6, 1.7.

3) we can find an operation Op, $\|\text{Op}(M)\| \geq \lambda$, so in Op($M$) some $\bar{a}$ realizes $p$ so $q = \text{tp}(\bar{a}, N, \text{Op}(N))$ is as required.

4) For some operation Op, some $\bar{a} \in \omega>(\text{Op}(M_0))$ realizes $p \upharpoonright M_0$, so $p_\ell = \text{tp}(\bar{a}, M_\ell, \text{Op}(M_\ell))$ does not fork over $M_0$, and $p_{\ell+1}$ does not fork over $M_\ell$, so by part 2) show $p_1 = p \upharpoonright M_1$ and then $p_2 = p$, but $p_2$ does not fork over $M_0$.

5) **Case 1:** $\text{cf}(\delta) > 8_0$.

For every limit $\alpha < \delta$ for some $i < \delta$ we have $p_\delta$ does not fork over $M_\alpha$. By Fodour lemma, for some $i < \delta$, $j \in [i, \delta) \implies p_j$ does not fork over $M_i$. So the stationarization of $p_i$ in $S(M_\delta)$ (exists by [122]) is as required.

**Case 2:** $\text{cf}(\delta) = 8_0$.

So w.l.o.g. $\delta = \omega$. Here chasing arrows (using amalgamation) suffice.

---

**Lemma 3.12** In $K_{< \lambda}$ we can define $\text{rk}(\text{tp}(a, M, N))$ with the right properties. I.e.

(A) if $M \preceq_F N \in K_{< \lambda}$, $\bar{a} \subseteq N$, $M \in \bigcup_{\mu^+ < \lambda} K_\mu$, $p = \text{tp}(\bar{a}, M, N)$ then

$$\text{rk}(p) \geq \alpha \iff \text{for every } \beta < \alpha \text{ there are } p', M' \text{ such that } M \preceq_F M' \in \bigcup_{\mu^+ < \lambda} K_\mu$$

$$p' \in S(M'), p' \upharpoonright M = p \text{ and } \text{rk}(p') \geq \beta.$$

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(B) for every $M, N, a, p$ as above $\text{rk}(p)$ is an ordinal.

(C) If $M_1 \prec F M_2 \in \bigcup \mathcal{K}_\mu$ and $p_2 \in S(M_2)$, then $\text{rk}(p_2 \upharpoonright M_1) \geq \text{rk}(p_2)$ and equality holds if $p_2$ does not fork over $M_1$ and then $p_2 \upharpoonright M_1$ (and $M_2$) determine $p_2$.

(D) If $(M_i : i \leq \delta)$ is $\preceq_F$-increasing continuous, $M_i \in \bigcup \mathcal{K}_\mu \mu^+ < \lambda$ and $p_\delta \in S(M_\delta)$ then for some $i < \delta$ we have:

\[ j \in [i, \delta] \Rightarrow \text{rk}(p_\delta) = \text{rk}(p_\delta \upharpoonright M_j). \]

**Lemma 3.13** Assume $\mu \geq \text{LS}(T), \mu^+ < \lambda$. If $M \in \mathcal{K}_\mu$ is saturated (for $\mu = \text{LS}(T)$ means $(\mu, \mu)$-saturated), and $p \in S(M)$ then there are $N, a$ such that $N \in \mathcal{K}_\mu$ is saturated, $a \in N$, $\text{tp}(a, M, N) = p$ and $N$ is $\mu$-isolated over $M \cup \{a\}$ (i.e. if $N \preceq_F N^+ \in \mathcal{K}_< \lambda$, and $N^* \preceq_F N^+$, and $\text{tp}(a, N^*, N^+)$ does not fork over $N$ ($\preceq_F N^*$) then $N^* \bigcup_M N$).

**Proof** As in [Sh:h], Ch. V (or Makkai Shelah [MaSh 285], 4.22) because we have 3.5(1) (by 3.6).

**Claim 3.14** For $M, N, a$ as in 3.13 if $N \preceq_F N^+ \in \mathcal{K}_< \lambda$, $A \subseteq N^+$, $|A| \leq \mu$ and $a \bigcup_M A$ then $N \bigcup_M A$.

**Proof** We use the symmetry of $\bigcup$ (hold by 1.7 as $\mu^+ < \lambda$).

**Claim 3.15** If $\mu \in [\text{LS}(T), \lambda)$, $M \in \mathcal{K}_\mu$ is saturated and $p \in S(M)$ then for some saturated $N \in \mathcal{K}_\mu, M \preceq_F N$, $a \in N$ and $(M, N, a)$ satisfies the conclusion of 3.14 for finite $A$.

**Proof** A problem arise only if $\mu^+ = \lambda$. We can find $(M_i : i \leq \mu)$ which is $\preceq_F$-increasing continuous, $\|M_i\| = |i| + \text{LS}(T)$, $M_\mu' = M$, $M_i'$ is saturated, $M_{i+1}$ universal over $M_i'$ and $p$ does not fork over $M_0$.

Now choose by induction on $i \leq \mu$, $(M_i, N_i, a)$ such that:

(a) $M_0 = M_0'$,
(b) $\|M_i\| = \|N_i\| = |i| + \text{LS}(T)$,
(c) for $i$ non limit $(M_i, N_i, a)$ as in 3.13 (with $|i| + \text{LS}(T)$ instead of $\mu$),

(d) $\text{tp}(a, M_0, N_0) = p \upharpoonright M'_0$,

(e) $\langle M_i : i \leq \mu \rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous,

(f) $\langle N_i : i \leq \mu \rangle$ is $\preceq_{\mathcal{F}}$-increasing continuous,

(g) $\text{tp}(a, M_{i+1}, N_{i+1})$ does not fork over $M_i$ (hence is the stationarization of $\text{tp}(a, M_0, N_0) = p \upharpoonright M'_0$),

(h) $M_{i+1}$ is universal over $M_i$.

(i) $N_i \preceq_{\mathcal{F}} N_i$.

(j) $N_{i+1}$ is isolated over $(M_{i+1}, a)$

There is no problem, so as $M_{i+1}$ is saturated and in $K_{\mu}$, $M_0 = M'_0$ has cardinality $< \mu$, w.l.o.g. $M_\mu = M$. For any candidates $N^+$, $A$, as in 3.14 (but $A$ is finite) assume $N \biguplus_{M}^N A$; as $A$ is finite, for some $i < \mu$, the type $\text{tp}(A, M, N^+)$ does not fork over $M_i$, and for some $j < \mu$ the type $\text{tp}(A, N, N^+)$ does not fork over $N_j$, w.l.o.g. $i = j$ is a successor ordinal and $\text{tp}(A \cup \{a\}, M)$ does not fork over $M_i$. So as $N \biguplus_{M}^N A$, necessarily $\text{tp}(A, N_i, N^+)$ forks over $M_i$, hence (by clause (c) above), $a \biguplus_{M_i}^N A$, hence $a \biguplus_{M_i}^N A$ (state the laws of $\biguplus$).

Alternatively repeat the proof of 3.13 using 3.11(2)'s second sentence.

\[3.17\]

**Theorem 3.16** Assume $\lambda$ is a successor cardinal i.e. $\lambda = \lambda_0^+$. Then $T$ is categorical in every $\mu \in \beth\{2\text{LS}(T)^+, \lambda\}$ (really for some $\mu_0 < \beth\{2\text{LS}(T)^+, \mu \in [\mu_0, \lambda) \) suffices).

**Proof** As in [MaSh 285]. By 3.10, for some $\mu_1 < \beth\{2\text{LS}(T)^+, \text{ for every } M \in K_{[\mu_1, \lambda]} \} \text{LS}(T)^+$-saturated. Let $\mu \in [\mu_1, \lambda)$, and assume $M \in K_{\mu}$ is not saturated, so for some $\kappa \in (\text{LS}(T), \mu)$ the model $M$ is $\kappa$-saturated but not $\kappa^+$-saturated. Let $p$, $\langle N^*_u : u \in \beth\{M\}^{<\kappa_0}\rangle$, $U$, $N^+$, $\langle N^*_u : u \in \beth\{N^+\}^{\kappa_0}\rangle$ be as in 3.9. Let $U_0 = U$. W.l.o.g. $N^*_{U_0}$ is saturated, $p$ does not fork over
N_\lambda^*, \ M^* \in [U]^<\aleph_0 \text{ finite, } \text{rk}(p) \text{ minimal under the circumstances. Now let } \ b \in M \setminus N_\lambda^*, \text{ so there is } N_1 \preceq_F M \text{ which is } \mu\text{-isolated over } N_\lambda^* \cup \{b\}. \text{ By defining more } N_\lambda^* \text{ w.l.o.g. } N_1 = N_{\lambda_1}^* . \text{ So } \text{tp}(b, N_{\lambda_0}^*, M), \text{ and } p \text{ are orthogonal (see } [Sh]: \text{ Ch. V}). \text{ Now we deal with orthogonal types and we continue as } [MaSh 285]: \text{ define a } \prec_F \text{-chain } M_i^* (i < \lambda) \text{ of saturated models of cardinality } \lambda_0 \text{ all omitting some fixed } p \in S(M_0^*).

### Discussion 3.17 1) Below \( \Delta_{(2^{LS(T)})^+} \)

A problem is what occurs in \([LS(T), \Delta_{(2^{LS(T)})^+}] \). As T is not necessarily complete, for any \( \psi \) and T we can consider \( T' \) defined as \( \{ \psi \rightarrow \varphi : \varphi \in T \} \), if \( \neg \psi \) has a model in \( \mu \) iff \( \mu < \mu^* \), we get such examples. So we may consider T complete. Hart Shelah [HaSh 323] bound our possible improvement but we may want larger gaps, a worthwhile direction. If \( T \subseteq L_{\kappa^+, \omega} \) is \( L_{\kappa^+, \omega} \)-complete hence \( L_{\kappa^+, \omega} \)-complete, \( LS(T) = \kappa \), we cannot improve.

If \( |T| < \kappa^* \) we may look at what occurs in large enough \( \mu < \kappa^* \).

2) Below \( \lambda \).

If \( \lambda \) is a limit cardinal we get only \( [3.1] \), this is a more serious issue. The problem is that we can get \( \mu \)-saturated not saturated model in \( K_{\mu^+} \), so we get for \( M \in K_{\mu} \text{ saturated, two orthogonal types } p, q \in S(M) \) (not realized in M). We want to build a prime model over \( M \cup \text{(a large indiscernible set for } p) \). Clearly \( P^{-}(n) \)-diagrams are called for.

3) Above \( \lambda \).

In some sense we know every model is saturated: if \( M \in K_{\lambda^+}, N \preceq_F M, \|N\| < \lambda, p \in S(N) \) then \( \text{dim}(p, N, M) = \|M\| \), i.e. if \( N \preceq_F N^+ \preceq_F M \) and : \( \|N^+\| < \|M\| \) when \( \lambda \) is successor, or \( \Delta_{(2^{LS(T)})^+} (\|N^+\|) \) when \( \lambda \) is a limit cardinal.

Another way to say it: the stationarization of \( p \) over \( N^+ \) is realized. But is every \( q \in S(N^+) \) a stationarization of some \( p \in S(N^+) \), \( N' \preceq_F N^+ \), \( \|N'\| \leq LS(T) \)? We can find \( N_0 \preceq_F N^+, \|N_0\| \subseteq (T), \) such that: \( [N_0 \preceq_F N_1 \leq N^+ \& \|N_1\| \leq LS(T) \Rightarrow q \upharpoonright N_1 \text{ does not fork over } N_0] \), we can get it for \( \|N_1\| < \mu, \) but does it hold for \( N_1 = N^+ \)? A central point is

(*) Does K satisfies amalgamation?

Again it seems that \( P^{-}(n) \)-systems are called for. Now Grossberg Shelah have started in the mid eighties to write a paper, which solves the problem but with two drawbacks. It says: if \( T \subseteq L_{\kappa^+, \omega} \) has arbitrarily large models, is categorical in \( \chi^+ \) (for \( n < \omega \)), \( \chi \geq LS(T) \), and \( 2^{\chi^+} < 2^{\chi^+} \) for
$n < \omega$, then $T$ is categorical in every $\lambda' > \chi$. So we need the set theoretic assumption

$$\left( \forall \alpha < (2^{LS(T)})^+ \right) \left( \exists \chi \right) \left[ \exists_\alpha \leq \chi \& \chi^{+n} \leq \lambda \& \bigwedge_n 2^{\chi^{+n}} < 2^{\chi^{+n+1}} \right].$$

4) If $|T| < \kappa^*$ we can do better, as $\text{Op}(EM(I, \Phi)) = EM(\text{Op}(I), \Phi)$, will discuss elsewhere.

5) Elsewhere we shall adopt what is done here to abstract elementary class $K$ categorical in $\lambda \geq \sum_{(2^{LS(K)})^+}$ such that $K_{<\lambda}$ has amalgamation.

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