PROBABILITY MEASURES ASSOCIATED TO GEODESICS IN THE SPACE OF KÄHLER METRICS.

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ABSTRACT. We associate certain probability measures on \( \mathbb{R} \) to geodesics in the space \( \mathcal{H}_L \) of positively curved metrics on a line bundle \( L \), and to geodesics in the finite dimensional symmetric space of hermitian norms on \( H^0(X, kL) \). We prove that the measures associated to the finite dimensional spaces converge weakly to the measures related to geodesics in \( \mathcal{H}_L \) as \( k \) goes to infinity. The convergence of second order moments implies a recent result of Chen and Sun on geodesic distances in the respective spaces, while the convergence of first order moments gives convergence of Donaldson’s \( Z \)-functional to the Aubin-Yau energy. We also include a result on approximation of infinite dimensional geodesics by Bergman kernels which generalizes work of Phong and Sturm.

1. INTRODUCTION

Let \( X \) be a compact Kähler manifold and \( L \) an ample line bundle over \( X \). If \( \phi \) is a hermitian metric on \( L \) with positive curvature, then

\[
\omega^\phi := i \partial \bar{\partial} \phi
\]

is a Kähler metric on \( X \) with Kähler form in the Chern class of \( L, c(L) \), and we let \( \mathcal{H}_L \) denote the space of all such Kähler potentials. By the work of Mabuchi, Semmes and Donaldson (see [9], [14], [7]), \( \mathcal{H}_L \) can be given the structure of an infinite dimensional, negatively curved Riemannian manifold, or even symmetric space. With this space one can associate certain finite dimensional symmetric spaces in the following way. Take a positive integer \( k \) and let \( V_k \) be the vector space of global holomorphic sections of \( kL \),

\[
V_k = H^0(X, kL).
\]

(Later we shall consider more generally vector spaces \( H^0(X, kL + F) \) where \( F \) is a fixed line bundle, but for simplicity we omit \( F \) in this introduction.) The finite dimensional symmetric spaces in question are then the spaces \( \mathcal{H}_k \) of hermitian norms on \( V_k \).

There are for any \( k \) natural maps

\[
FS = FS_k : \mathcal{H}_k \mapsto \mathcal{H}_L,
\]

and

\[
\text{Hilb} = \text{Hilb}_k : \mathcal{H}_L \mapsto \mathcal{H}_k,
\]

and a basic idea in the study of Kähler metrics on \( X \) with Kähler form in \( c(L) \) is that under these maps the finite dimensional spaces \( \mathcal{H}_k \) should approximate \( \mathcal{H}_L \) as \( k \) goes to infinity. This will
be explained a bit more closely in the next section of this paper, see also [7], [10] and [5] for excellent backgrounds to these ideas.

The most basic result in this direction is the result of Bouche, [2] and Tian, [17] that for \( \phi \) in \( \mathcal{H}_L \)
\[
\phi_k := FS_k \circ Hilb_k(\phi)
\]
tends to \( \phi \) together with its derivatives. It is natural to ask whether geodesics between points in \( \mathcal{H}_L \) also can be approximated in some sense by geodesics coming from the finite dimensional picture. This question was first addressed by Phong and Sturm in [10], where it is proved that any geodesic in \( \mathcal{H}_L \) is a limit of \( FS_k \) of geodesics in \( \mathcal{H}_k \), in an almost uniform way (see below). Later, this result has been refined in particular cases (like toric varieties) to give convergence of derivatives as well by Song-Zelditch, Rubinstein-Zelditch and Rubinstein, see [16], [13], [12]. (These works also treat more general equations than the geodesic equation.)

In a recent very interesting paper, [5], Chen and Sun have shown that moreover if \( \phi^0 \) and \( \phi^1 \) are two Kähler potentials in \( \mathcal{H}_L \), then the geodesic distance, suitably normalized, between \( Hilb_k(\phi^0) \) and \( Hilb_k(\phi^1) \) in \( \mathcal{H}_k \) tends to the geodesic distance between \( \phi^0 \) and \( \phi^1 \) in \( \mathcal{H}_L \). Hence \( \mathcal{H}_k \) approximates \( \mathcal{H}_L \) as metric spaces in this sense.

In this paper we associate to geodesics, in \( \mathcal{H}_k \) and \( \mathcal{H}_L \) respectively, certain probability measures on \( \mathbb{R} \) from which many quantities related to the geodesic (like length, energy) can be recovered. The main result of the paper is that the measures associated to geodesics in \( \mathcal{H}_k \) converge to their counterparts in \( \mathcal{H}_L \) in the weak *-topology as \( k \) goes to infinity. It follows that their moments converge, which applied to second moments implies the result of Chen and Sun on convergence of geodesic distance.

Let \( H^0_k \) and \( H^1_k \) be two points in \( \mathcal{H}_k \), and let \( H^t_k \) be the geodesic in \( \mathcal{H}_k \) connecting them. The tangent vector to this geodesic
\[
A_{t,k} := (H^1_k)^{-1} \dot{H}^t_k
\]
is then an endomorphism of \( V_k \). The geodesic condition means that it is actually independent of \( t \) so we will omit the \( t \) in the subscript. Since \( A_k \) is hermitian for the scalar products in the curve all its eigenvalues are real. Let \( \nu_k = \nu_{A_k} \) be the normalized spectral measure of \( k^{-1} A_k \). By this we mean that
\[
\nu_k = d_k^{-1} \sum \delta_{\lambda_j},
\]
where \( \lambda_j \) are the eigenvalues of \( k^{-1} A_k \) and \( d_k \) is the dimension of \( V_k \), so that \( \nu_k \) are probability measures on \( \mathbb{R} \).

The second order moment of \( \nu_k \) is precisely the norm squared of the vector \( A_k \) in the tangent space of \( \mathcal{H}_k \), divided by \( d_k \). Since this is independent of \( t \) and \( t \) goes from 0 to 1, the second order moment equals the square of the normalized geodesic distance between \( H^0_k \) and \( H^1_k \). We shall also see in section 2 that the first order moment of \( \nu_k \) equals the Donaldson functional
\[
Z(H^0_k, H^1_k)/d_k
\]
from [8].

We next describe the corresponding objects for the infinite dimensional space \( \mathcal{H}_L \). Let \( \phi^0 \) and \( \phi^1 \) be two points in \( \mathcal{H}_L \) and let \( \phi^t \) be the Monge-Ampere geodesic joining them. By this we
mean that $\phi^t$ is a curve of positively curved metrics on $L$ for $t$ between 0 and 1. We extend the definition of $\phi^t$ to complex $t$ in

$$\Omega := \{0 < \Re t < 1\}$$

by letting it be independent of the imaginary part of $t$. The geodesic equation is then

$$(i\partial \bar{\partial} \phi^t)^{n+1} = 0$$

on $\Omega \times X$.

It was proved by Chen in [4] that such a geodesic always exists and is of class $C^{1,1}$ in the sense that all $(1,1)$-derivatives are uniformly bounded. It is unknown if the geodesic is actually smooth. A 'geodesic in $\mathcal{H}_L$' is therefore not necessarily a curve in $\mathcal{H}_L$ (which consists of smooth metrics), but we will adhere to the common terminology nevertheless. For each $t$ fixed we can now define a probability measure on $\mathbb{R}$ in the following way. Let first $dV_t$ be the normalized volume measure on $X$ induced by $\omega_{\phi^t}$,

$$dV_t := \omega_{\phi^t}/\text{Vol}.$$ 

Here $\omega_n := \omega^n/n!$ for $(1,1)$-forms $\omega$ and $\text{Vol}$ is the volume of $X$

$$\text{Vol} = \int_X c(L)_n.$$ 

Since $\dot{\phi}^t$ is a continuous real valued function, we can consider the direct image (or 'pushforward') of $dV_t$

$$\mu_t = (-\dot{\phi}^t)_*(dV_t)$$

so that $\mu_t$ is a probability measure on $\mathbb{R}$. Concretely, this means that if $f$ is a continuous function on $\mathbb{R}$, then

$$\int_{\mathbb{R}} f(x)d\mu_t(x) = \int_X f(-\dot{\phi}^t)dV_t.$$ 

We shall show in the next section that if $\phi^t$ is a Monge-Ampere geodesic, then $\mu = \mu_t$ is independent of $t$. This is then the measure that corresponds to the spectral measures $\nu_k$ in the infinite dimensional setting, and our main results says that $\nu_k$ converge to $\mu$ in the weak* topology as $k$ goes to infinity.

**Theorem 1.1.** Let $\phi^0$ and $\phi^1$ be two points in $\mathcal{H}_L$, and let

$$H^t_k = \text{Hilb}_k(\phi^t)$$

for $t = 0,1$ be the corresponding norms in $\mathcal{H}_k$. Let for $t$ between 0 and 1 $H^t_k$ be the geodesic in $\mathcal{H}_k$ connecting these two norms and let $\nu_k$ be their normalized spectral measures as defined above. Then

$$\nu_k \longrightarrow \mu,$$

in the weak* topology, where $\mu = \mu_t$ is defined in 1.1.
Just like the spectral measures of the endomorphisms $A_k$ contain part of the properties of the corresponding geodesics in $\mathcal{H}_k$, part of the properties of the Monge-Ampere geodesic can be read off from the measure $\mu$. It is for instance immediately clear that the second order moment of $\mu$ is equal to
\[
\int_X \dot{\phi}^2 dV_t / Vol
\]
which is the length square of the tangent vector to the Monge-Ampere geodesic (which is independent of $t$ as it should be). Since the parameter interval is from 0 to 1 the length of the tangent vector is the length of the geodesic from $\phi^0$ to $\phi^1$. By a theorem of Chen, [4], the length of the geodesic is equal to the geodesic distance, so the convergence of second order moments implies the theorem of Chen and Sun, [5] that normalized geodesic distance in $\mathcal{H}_k$ converges to geodesic distance in $\mathcal{H}_L$. Similarly, we shall see in the next section that the first order moment of $\mu$ is the Aubin-Yau energy of the pair $\phi^0$ and $\phi^1$, and convergence of first order moments therefore says that the Aubin-Yau energy is the limit of Donaldson’s Z-functional (this is a much simpler result).

The proof of our main result is given in section 3; it is based on the curvature estimates from [1]. The basic idea is as follows: The Monge-Ampere geodesic $\phi^t$ induces a certain curve of norms in $\mathcal{H}_k$, $H_{\phi^t,k}$. These are $L^2$-norms on the space of global sections, similar to the curves $Hilb_k(\phi^t)$ but defined slightly differently to fit with the results of [1]. At the end points, $t = 0, 1$,
\[
H_{\phi^t,k} = H^t_k := Hilb_k(\phi),
\]
and we define $H^t_k$ for $t$ between 0 and 1 to be the geodesic in $\mathcal{H}_k$ between these endpoint values.

The main result of [1] immediately implies that
\[
H_{\phi^t,k} \geq H^t_k
\]
for $t$ between 0 and 1, and by definition equality holds at the endpoints. Let
\[
T^t_{k} := H^{-1}_{\phi^t,k} \dot{H}_{\phi^t,k}
\]
Differentiating with respect to $t$ at $t = 0, 1$ we then get that
\[
\langle A_k u, u \rangle_{H^0_k} \leq \langle T^0_{0,k} u, u \rangle_{H^0_k}
\]
and
\[
\langle A_k u, u \rangle_{H^1_k} \geq \langle T^1_{1,k} u, u \rangle_{H^1_k}
\]
This means that we get estimates for the tangent vector to the finite dimensional geodesic in terms of certain operators on $V_k$ defined by the Monge-Ampere geodesic. These operators are Toeplitz operators on $V_k$ with symbol $\dot{\phi}^t$, $t = 0, 1$ and their spectral measures are essentially known to converge to $\mu_t = \mu$. Since $A_k$ is pinched between these two operators it is not hard to see that the spectral measures of $A_k$ have the same limit, which proves the theorem.

In a final section we will give a result on the uniform convergence of $FS_k$ of finite dimensional geodesics to Monge-Ampere geodesics, generalizing the work of Phong-Sturm mentioned earlier. This result is only a small variation of Theorem 6.1 from [1], but it has as a consequence the following theorem which is more natural than Theorem 6.1 in [1] so it seems good to state it explicitly.
**Theorem 1.2.** Let $\phi^0$ and $\phi^1$ be two Kähler potentials in $H_L$ and let $\phi^t$ be the Monge-Ampere geodesic joining them. Let

$$H^t_k = Hilb_k(\phi^t)$$

for $t = 0, 1$ and let $H^t_k$ for $t$ between 0 and 1 be the geodesic in $H_k$ between these two points. Let finally

$$B_{t,k} := FS_k(H^t_k)$$

for $0 \leq t \leq 1$. Then

$$\sup |k^{-1} \log B_{t,k} - \phi^t| \leq C \frac{\log k}{k}.$$  

This theorem strengthens the main result of Phong and Sturm, [10], who proved that

$$\lim_{l \to \infty} \sup_{k \geq l} k^{-1} \log B_{t,k} = \phi^t$$

almost everywhere.

The final parts of this work (the most important parts!) were carried out during the conference on extremal Kähler metrics at BIRS June-July 2009. I am grateful to the organizers for a very stimulating conference. I would also like to thank Jian Song for suggesting that my curvature estimates might be relevant in connection with the Chen-Sun theorem and for encouraging me to write down the details of the proof of Theorem 1.2. Finally I am grateful to Xiuxiong Chen and Song Sun for explaining me their result.

2. Background and Definitions

In the first subsection we will give basic facts about the space $H_L$ and its finite dimensional ‘quantizations’. Since this material is well known (see e.g. [7], [10] or [5]) we will be brief and emphasize a few particularities that are relevant for this paper.

2.1. $H_L$, $H_k$ and its variants. Let $L$ be an ample line bundle over the compact manifold $X$. $H_L$ is the space of all smooth metrics $\phi$ on $L$ with

$$\omega^\phi := i\partial \bar{\partial} \phi > 0.$$  

$H_L$ is an open subset of an affine space and its tangent space at each point equals the space of smooth real valued functions on $X$. The Riemannian norm on this tangent space at the point $\phi$ is the $L^2$-norm

$$\|\psi\|^2 = \int_X |\psi|^2 \omega_\phi^n / Vol$$

(remember we use the notation $\omega_n = \omega^n / n!$ for forms of degree two). A geodesic in $H_L$ is a curve $\phi^t$ for $a < t < b$ that satisfies the geodesic equation

$$\frac{d^2}{dt^2} \phi^t = |\bar{\partial} \frac{d}{dt} \phi^t|^2 \omega^\phi.$$  

(2.1)

It is useful to extend the definition of $\phi^t$ to complex values of $t$ in the strip

$$\Omega = \{t; a < Re t < b\}$$
by taking it to be independent of the imaginary part of $t$. Then 2.1 can be written equivalently on complex form

$$c(\phi^t) := \phi^t_{tt} - |\bar{\partial} \phi^t|^2_{\omega_{\phi^t}} = 0,$$

where $\dot{\phi}^t = \partial \phi^t / \partial t$. On the other hand the expression $c(\phi^t)$ is related to the Monge-Ampere operator through the formula

$$c(\phi^t) \, idt \wedge d\bar{t} \wedge \omega_{\phi^t}^n = (i\partial \bar{\partial} \phi^t)^{n+1},$$

where on the right hand side we take the $\partial \bar{\partial}$-operator on $\Omega \times X$. Geodesics in $\mathcal{H}_L$ are therefore given by solutions to the homogeneous Monge-Ampere equation that are independent of $\text{Im} \, t$. Notice that a geodesic will automatically satisfy

$$i\partial \bar{\partial} \phi^t \geq 0,$$

and we shall refer to any curve with this property as a 'subgeodesic’ even though this term has no meaning in Riemannian geometry in general.

A fundamental theorem of Chen, [4] says that if $\phi^0$ and $\phi^1$ are two points in $\mathcal{H}_L$ they can be connected by a geodesic of class $C^{1,1}$, i.e such that

$$(i\partial \bar{\partial} \phi^t)^{n+1} = 0$$

has bounded coefficients.

One associates with $\mathcal{H}_L$ the vector spaces

$$V_k := H^0(X, kL)$$

of global holomorphic sections of $kL$ for $k$ positive integer. A metric $\phi$ in $\mathcal{H}_L$ is mapped to a hermitian norm $Hilb_k(\phi)$ on $V_k$ by

$$\|u\|_{Hilb_k(\phi)}^2 := \int_X |u|^2 e^{-k\phi} \omega_n^\phi.$$

It will also be useful for us to consider the vector spaces

$$H^0(X, K_X + kL).$$

A metric $\phi$ on $L$ also induces an hermitian norm, $H_{k\phi}$ on these spaces through

$$\|u\|_{H_{k\phi}}^2 := \int_X |u|^2 e^{-k\phi}.$$

An important point is that $|u|^2 e^{-k\phi}$ is a measure on $X$ if $u$ lies in $H^0(X, K_X + kL)$, so the integral of this expression is naturally defined, without the introduction of any extra measure like $\omega_n^\phi$.

In order to treat both these types of spaces simultaneously we let $F$ be an arbitrary line bundle over $X$ and consider spaces

$$H^0(X, K_X + kL + F).$$

Norms on these spaces are then defined by

$$\|u\|_{H_{k\phi+\psi}}^2 := \int_X |u|^2 e^{-k\phi-\psi}.$$
where $\psi$ is some metric on $F$. The two cases we discussed earlier the correspond to $F = -K_X$ and

$$\psi = -\log \omega^\phi_n,$$

and $F = 0$ respectively. In the first case

$$H_{k\phi+\psi} = Hilb_k(\phi)$$

as defined above.

Let now $V$ be any space of sections to some line bundle, $G$, over $X$; it may be any of the choices discussed above, and denote by $\mathcal{H}_V$ the space of hermitian norms on $V$. For such a hermitian norm, $H$, let $s_j$ be an orthonormal basis for the space of sections $H^0(X,G)$, and consider the Bergman kernel

$$B_H = \sum |s_j|^2.$$

The absolute values on the right hand side here are to be interpreted with respect to some trivialization of $G$. When the trivialization changes, $\log B_H$ transforms like a metric on $G$ since

$$|u|^2/B_H$$

is a well defined function if $u$ is a section of $G$. By definition $FS(H)$ is that metric

$$FS(H) = \log B_H.$$

By the well known extremal characterization of Bergman kernels we have

$$B_H(x) = \sup_{u \in H^0(X,G)} \frac{|u(x)|^2}{\|u\|^2_H}.$$

From this we can conclude that the Bergman kernel is a decreasing function of the metric; if we change the metric to a larger one, the Bergman kernel becomes smaller.

Choosing a basis for $V$ we can represent an element in $\mathcal{H}_V$ by a matrix that we slightly abu-
sively also call $H$. A curve in $\mathcal{H}_V$ then gets represented by a curve of matrices $H^t$. Differentiating norms we get

$$\frac{d}{dt} \|u\|^2_{H^t} = \langle A_t u, u \rangle_{H^t},$$

with

$$A_t = (H^t)^{-1} \frac{d}{dt} H^t.$$

$A_t$ is an endomorphism of $V$; the tangent vector to the curve $H^t$. Its norm is

$$\|A_t\|^2 = tr A^* A.$$

Here the $*$ stands for the adjoint with respect to $H$, but since $A$ is selfadjoint for this scalar product, the norm of $A$ is the sum of the squares of its eigenvalues.

Finally, the geodesic equation is

$$\frac{d}{dt} A_t = 0.$$

It is easy to see that any two norms in $\mathcal{H}_V$ can be joined by a geodesic. Explicitly, we can find a basis $s_j$ of $V$ which is orthonormal w r t $H^0$ and diagonalizes $H^1$ with eigenvalues $e^{\lambda_j}$. 
The geodesic is then represented (in this basis) by the diagonal matrix $H^t$ with eigenvalues $e^{t\lambda_j}$. Hence, $A = A_t$ is diagonalized by the same basis and has eigenvalues $\lambda_j$.

Just like in the case of $\mathcal{H}_L$ it is convenient to consider curves $H^t$ defined also for complex values of $t$ in the strip $\Omega$, by letting it be independent of the imaginary part of $t$. We can then write the geodesic equation equivalently as

$$\frac{\partial}{\partial t} H^{-1} \frac{\partial}{\partial t} H.$$ 

This suggests that the geodesic equation can be thought of as the zero-curvature equation for a certain vector bundle. Let $E$ be the trivial bundle over $\Omega$ with fiber $V$. A curve in $\mathcal{H}_V$ is then the same thing as a vector bundle metric on $E$, independent of the imaginary part of $t$, and we see that geodesics correspond to flat metrics on $E$. In analogy with the case of curves in $\mathcal{H}_L$, we will call curves in $\mathcal{H}_V$ that correspond to vector bundle metrics of semipositive curvature 'subgeodesics' in $\mathcal{H}_V$.

A main role in the sequel is played by Theorem 2.1 in [1]. This theorem implies that if $\phi^t$ is a subgeodesic in $\mathcal{H}_L$ (it does not need to be independent of $\text{Im} \ t$), i.e satisfies

$$i\bar{\partial}\partial \phi^t \geq 0,$$

then the induced curve $H_{\phi^t}$ in $\mathcal{H}_V$ for $V = H^0(X, K_X + L)$ has semipositive curvature, so it is a subgeodesic in $\mathcal{H}_V$. Since metrics with semipositive curvature lie above flat metrics having the same boundary values, this gives us a way of comparing $L^2$-norms on $V$ induced by (sub)geodesics in $\mathcal{H}_L$ to finite dimensional geodesics in $\mathcal{H}_V$ (cf Proposition 3.1).

2.2. Measures defined by geodesics. Let us start with the case of a finite dimensional geodesic, $H^t$, in $\mathcal{H}_V$. As we have seen in the previous subsection it can be represented by a diagonal matrix with diagonal elements $e^{t\lambda_j}$ in a suitable basis, and its tangent vector $A$ is then diagonal with diagonal elements $\lambda_j$. The measure we associate to the geodesic is then the (normalized) spectral measure of $A$

$$\nu_A = \frac{1}{d} \sum \delta_{\lambda_j},$$

with $d$ the dimension of $V$. This is defined in terms of eigenvalues of the endomorphism $A$ so it does not depend on the basis we have chosen.

Recall that for any pair of norms in $\mathcal{H}_V$, Donaldson [8] has defined a quantity

$$Z(H^1, H^0) = \log \frac{\det H^1}{\det H^0}$$

(the determinant is the determinant of a matrix representing the norm in some basis, but since we consider quotients of determinants, $Z$ does not depend on which basis). Then

$$\frac{d}{dt} Z(H^t, H^0) = \text{tr} A.$$
Hence we see that, since $A$ is constant and we have chosen our parameter interval to be $[0, 1]$, that

$$\int_{\mathbb{R}} x d\nu_A = tr A / d = Z(H^1, H^0) / d$$

so first moments of the spectral measure gives the Donaldson $Z$-functional. Second order moments are

$$\int_{\mathbb{R}} x^2 d\nu_A = tr A^2 / d = \|A\|^2 / d$$

which in the same way equals the square of the geodesic distance from $H^0$ to $H^1$, again divided by $d$.

We next turn to the corresponding construction for $H_L$. Let $\phi^t$ be a curve in $H_L$ and to fix ideas we think of $t$ as real now. We first assume that $\phi^t$ is smooth and denote by

$$\dot{\phi}^t = \frac{d\phi^t}{dt}$$

the tangent vector (a smooth function on $X$). For ease of notation we also set

$$\omega^t = \omega^{\phi^t}.$$

**Lemma 2.1.** Let $f$ be a compactly supported function on $\mathbb{R}$ of class $C^1$. Then

$$\frac{d}{dt} \int_X f(\dot{\phi}^t) \omega^t_n = \int_X f'(\dot{\phi}^t) c(\phi^t) \omega^t_n.$$

**Proof.** This is just a simple computation.

$$\frac{d}{dt} \int_X f(\dot{\phi}^t) \omega^t_n = \int_X f'(\dot{\phi}^t) \frac{d^2 \phi^t}{dt^2} \omega^t_n + \int_X f(\dot{\phi}^t) i\partial \bar{\partial} \dot{\phi}^t \wedge \omega^t_{n-1}.$$  

By Stokes’ theorem applied to the last term this equals

$$\int_X f'(\dot{\phi}^t) \frac{d^2 \phi^t}{dt^2} \omega^t_n - \int_X f'(\dot{\phi}^t) i\partial \bar{\partial} \dot{\phi}^t \wedge \omega^t_{n-1} = \int_X f'(\dot{\phi}^t) c(\phi^t) \omega^t_n.$$  

Since for smooth geodesics $c(\phi^t) = 0$ it follows that the integrals

$$\int_X f(\dot{\phi}^t) \omega^t_n$$

do not depend on $t$. By approximation we can draw the same conclusion for (say) geodesics of class $C^1$.

**Proposition 2.2.** Let $\phi^t$ be a curve of metrics on $L$ with semipositive curvature which is of class $C^1$ and satisfies

$$(i\partial \bar{\partial} \phi^t)^{n+1} = 0$$

in the sense of currents. Then the integrals

$$\int_X f(\dot{\phi}^t) \omega^t_n$$

do not depend on $t$. 

Proof. Let $K$ be a compact in $\Omega$. We can then approximate $\phi^t$ over $K \times X$ by smooth metrics $\phi^t_\epsilon$ such that
\[ i\partial \bar{\partial} \phi^t_\epsilon \geq 0 \]
and
\[ \int_{K \times X} (i\partial \bar{\partial} \phi^t_\epsilon)^{n+1} \]
tends to 0. In fact, the approximation can be carried out locally by convolution and then patched together with a partition of unity - the patching causes no problem if the initial metric is of class $C^1$. The proposition then follows from the lemma.

For a $C^1$-geodesic we now consider the normalized volume measures on $X$
\[ dV_t = \omega^t_n / Vol \]
where
\[ Vol = \int_X c(L)_n \]
is the volume of $X$, and their direct image measures under the map $-\dot{\phi}^t$
\[ d\mu_t = (-\dot{\phi}^t)_*(dV_t) \].
These are probability measures on $\mathbb{R}$, supported on a compact interval $[-M, M], M = \sup |\dot{\phi}^t|$
and concretely defined by
\[ \int_{\mathbb{R}} f(x)d\mu_t(x) = \int_X f(-\dot{\phi}^t)\omega^t_n / Vol. \]
By the proposition, they do in fact not depend on $t$, so $d\mu = d\mu_t$ is a fixed probability measure on $\mathbb{R}$ associated to the given geodesic.

Recall that the Aubin-Yau energy of a pair of metrics in $H_L$ is defined in the following way:
\[ \frac{d}{dt} \mathcal{E}(\phi^t, \phi^0) = -\int_X \dot{\phi}^t \omega^t_n, \]
and $\mathcal{E}(\phi^0, \phi^0) = 0$. From this we see that the first order moment of $d\mu$
\[ \int x d\mu(x) = -\int_X \dot{\phi}^t \omega^t_n / Vol, \]
is precisely the derivative of the Aubin-Yau energy, which is constant for a geodesic, and hence equal to the Aubin-Yau energy itself if the parameter interval is $(0, 1)$. This corresponds to the relation between the measures $d\nu_k$ and the Donaldson $Z$-functional, and Theorem 1.1 in this case is just the familiar convergence of the $Z$-functionals to the Aubin-Yau energy. Similarly, the second order moments
\[ \int x^2 d\mu(x) = \int_X (\dot{\phi}^t)^2 \omega^t_n / Vol, \]
is the length of the tangent vector to $\phi^t$ squared, so second order moments give geodesic distances. Notice finally that the proposition implies that all $L^p$-norms of $\dot{\phi}^t$ are constant along the
curve, hence also the $L^\infty$-norm. More precisely, since $\sup(-\dot{\phi}^t)$ is the supremum of the support of $\mu$ it follows that $\inf \dot{\phi}^t$ (and $\sup \dot{\phi}^t$) are constant (where we mean essential sup and inf).

**Remark** Notice also that if we define the measures in the same way when $\phi^t$ is a subgeodesic, then the integrals

$$\int_{\mathbb{R}} f(x) d\mu_t(x)$$

increase with $t$ if $f$ is an increasing function. Intuitively, the measures $\mu_t$ move to the right as $t$ increases.

3. **The Convergence of Spectral Measures**

We first state a consequence of the main result from [1]. In the statement of the proposition we shall use the notation

$$\|u\|^2_{H_\phi} = \int_X |u|^2 e^{-\phi}$$

for the hermitian norm on $H^0(X, L + K_X)$ defined by a metric $\phi$ on $L$.

**Proposition 3.1.** Let $L$ be an ample line bundle over $X$ and let $\phi^t$ for $t = 0, 1$ be two elements of $\mathcal{H}_L$. Let for $t = 0, 1$ $H^t$ be the norms $H_{\phi^t}$ on $H^0(X, L + K_X)$ defined by $\phi^0$ and $\phi^1$. Let for $t$ between 0 and 1 $H^t$ be the geodesic in the space of metrics on $H^0(X, L + K_X)$ joining $H^0$ and $H^1$. Let finally $\phi^t$ be any smooth subgeodesic in $\mathcal{H}_L$ connecting $\phi^0$ and $\phi^1$, i.e. any metric with nonnegative curvature on $L$ over $X \times \Omega$, smooth up to the boundary. Then

$$H^t \leq H_{\phi^t}.$$  

**Proof.** If we regard $H^t$ and $H_{\phi^t}$ as vector bundle metrics on the trivial vector bundle over $\Omega$ with fiber $H^0(X, L + K_X)$, then Theorem 2.1 of [1] implies that the second of these metrics has nonnegative curvature. On the other hand the first metric has zero curvature since $H^t$ is a geodesic. Since the two metrics agree over the boundary a comparison lemma from [11] or [14] gives inequality 3.1.

We have been a little bit vague about what ’smoothness’ means in the proposition. The proof of Theorem 2.1 in [1] requires at least $C^2$-regularity, but we claim that $C^1$ regularity is sufficient in the proposition, which can be seen from regularization of the metric (this can be done locally with the aid of a partition of unity in the case that the metric is $C^1$ from the start). This means that we can (and will) apply the proposition to Monge-Ampere geodesics of class $C^{1,1}$.

The next step is to differentiate the inequality 3.1 for $t = 0, 1$ (recall that equality holds at the endpoints). If $u$ lies in $H^0(X, L + K_X)$ we get

$$\frac{d}{dt} \|u\|_{H^t}^2 = \langle A_t u, u \rangle_{H^t},$$

where

$$A_t = (H^t)^{-1} \dot{H}^t.$$
Since $H^t$ is a geodesic, $A_t = A$ is independent of $t$. The derivative of the right hand side of 3.1 is
\[ \frac{d}{dt}\|u\|^2_{H_{\phi^t}} = \langle T_t u, u \rangle_{H_{\phi^t}}, \]
where $T_t$ is the Toepliz operator on $H^0(X, L + K_X)$ defined by
\[ \langle T_t u, u \rangle_{H_{\phi^t}} = -\int_X \dot{\phi} |u|^2 e^{-\phi^t}. \]
The proposition then implies that
(3.2) \hspace{1cm} T_0 \leq A
as operators on the space $H^0(X, L + K_X)$ equipped with the Hilbert norm $H^0$ and
(3.3) \hspace{1cm} A \leq T_1
as operators on the space $H^0(X, L + K_X)$ equipped with the Hilbert norm $H^1$.

We are now going to apply these estimates to multiples $kL$ of the bundle $L$, but in order to accommodate also $L^2$-metrics of the form
\[ \int_X |u|^2 e^{-k\phi_\omega}, \]
we need to generalize the set up first. Let therefore $F$ be an arbitrary line bundle over $X$ and consider line bundles of the form
$K_X + F + kL$.
The main examples will be $F = 0$ and $F = -K_X$, and the reader may find it convenient to focus on the case $F = 0$ first, in which case the argument below is easier, at least notationally. Put now
$V_k = H^0(X, kL + F + K_X)$.

Fix two metrics $\phi^0$ and $\phi^1$ in $H_L$. Let $\chi$ be some fixed metric on $L$ considered as a bundle over $X \times \Omega$, which has positive curvature bounded from below by a positive constant (times say $\omega\phi^0 + idt \wedge dt$), and which equals $\phi^0$ for $\text{Re} t = 0$ and equals $\phi^1$ for $\text{Re} t = 1$. Such a metric $\chi$ can be found on the form
\[ t\phi^1 + (1 - t)\phi^0 + \kappa(\text{Re} t) \]
where $\kappa$ is a sufficiently convex function on the interval $(0, 1)$ which equals 0 at the endpoints.

Let also $\psi$ be an arbitrary metric on $F$, not necessarily with positive curvature, but smooth up to the boundary. Choose a fixed positive constant $a$, sufficiently large so that
\[ ai\bar{\partial}\partial \chi + i\partial \bar{\partial} \psi \geq 0. \]

We next consider the vector spaces
$H^0(X, K_X + F + kL)$
with the induced $L^2$-metrics
\[ \|u\|_{k,t}^2 := \int_X |u|^2 e^{-\phi - a\chi - \psi}. \]
Notice that the metric on the line bundle $F + kL$ that we use here, $(k - a)\phi + a\chi + \psi$ has been chosen so that it has nonnegative curvature, meaning that we can apply the results from 3.1, 3.2 and 3.3. We denote the Toepliz operators arising from differentiation of the norms at $t = 0$ and $t = 1$ by $T_{0,k}$ and $T_{1,k}$ now in order to keep track on how they depend on $k$. By immediate calculation

\begin{equation}
\langle T_{k,t}u, u \rangle_{k,t} = -\int_X [(k - a)\dot{\phi} + a\dot{\chi} + \dot{\psi}]|u|^2 e^{-(k-a)\phi-a\chi-\psi}
\end{equation}

for $t = 0, 1$.

Let now $H^t_k$ be the finite dimensional geodesic in the space of hermitian norms on $H^0(X, K_X + F + kL)$ that connects $\| \cdot \|_{k,t}$ for $t = 0$ and $t = 1$. Let

\[ A_k = (H^t_k)^{-1} \frac{d}{dt} H^t_k \]

be the tangent vector of the finite dimensional geodesic. By 3.2 and 3.3 we have the inequalities

\begin{equation}
T_{0,k} \leq A_k
\end{equation}

with respect to the hermitian scalar product $H^0_k$ and

\begin{equation}
T_{1,k} \geq A_k
\end{equation}

with respect to the hermitian scalar product $H^1_k$. Let $\lambda_j(k)$ be the eigenvalues of $A_k$ arranged in increasing order, and let $\tau_j^0(k)$ be the eigenvalues of the two Toepliz operators, also arranged in increasing order. We then get immediately from 3.5 and 3.6 that

\begin{equation}
\tau_j^0(k) \leq \lambda_j(k) \leq \tau_j^1(k).
\end{equation}

The final step in the argument is the following theorem on the asymptotics of Toepliz operators; it is a variant of a theorem of Boutet de Monvel, [3]. Since the theorem is essentially known, we defer its proof to an appendix.

**Theorem 3.2.** Let $L$ and $F$ be line bundles over $X$ with smooth metrics $\phi$ and $\psi$ respectively. Assume that $\phi$ has strictly positive curvature. Let $\xi$ and $\xi_k$ be continuous real valued functions on $X$ with $\xi_k$ tending uniformly to 0. Define Toepliz operators with symbols $\xi + \xi_k$ on the spaces $H^0(X, K_X + kL + F)$ by

\[ \langle T_k u, u \rangle_{k\phi+\psi} = \int (\xi + \xi_k)|u|^2 e^{-k\phi-\psi}. \]

Let $\mu_k$ be the normalized spectral measure of $T_k$. Then the sequence $\mu_k$ converges weakly to the measure

\[ \mu = \xi_* (\omega^\phi/\text{Vol}), \]

the direct image of the normalized volume element on $X$ defined by $\omega^\phi$ under the map $\xi$. 

We apply this theorem to the Toeplitz operator $k^{-1}T_{k,t}$ for $t = 0, 1$. Its symbol is $-\dot{\phi}^t$ plus a term that goes uniformly to zero. In our operators $k^{-1}T_{k,t}$ the metric on $F$ can be taken to be $\psi + a(\chi - \phi)$ if we take the metric on $L$ to be $\phi$. Theorem 3.2 therefore shows that the spectral measures $d\mu_{k,t}$ of $k^{-1}T_{k,t}$ converge to

$$d\mu_t = (-\dot{\phi}^t)_*(dV_t).$$

By the previous section these two measures are the same (for $t = 0$ and $t = 1$), namely the measure $d\mu$ that we associated to the geodesic in $H_L$. The inequality 3.7 for the eigenvalues shows that

$$\int_{\mathbb{R}} fd\mu_{k,0} \leq \int_{\mathbb{R}} fd\nu_k \leq \int_{\mathbb{R}} fd\mu_{k,1}$$

if $f$ is continuous and increasing (recall that $\nu_k$ is the spectral measure of $A_k$). It follows that

$$\lim \int_{\mathbb{R}} fd\nu_k = \int_{\mathbb{R}} fd\mu$$

for $f$ continuous and increasing. Since any $C^1$-function can be written as a difference of two increasing functions, the previous limit must hold for any $C^1$-function too. But this implies weak convergence of the measures since all the measures involved are probability measures supported on a fixed compact interval. This finishes the proof of our main result:

**Theorem 3.3.** Let $\phi^0$ and $\phi^1$ be two points in $H_L$ and let $\psi$ be an arbitrary smooth metric on the line bundle $F$. Let

$$V_k = H^0(X, K_X + F + kL)$$

and let $\mathcal{H}_k$ be the space of hermitian norms on $V_k$. Let $H^t_k$ be the elements in $\mathcal{H}_k$ defined by

$$\|u\|^2 = \int_X |u|^2 e^{-k\phi^t - \psi}$$

for $t = 0, 1$. Let for $t$ between 0 and 1 $H^t_k$ be the geodesic in $\mathcal{H}_k$ connecting these two norms and let $\nu_k$ be their normalized spectral measures as defined above. Then

$$\nu_k \longrightarrow \mu,$$

in the weak* topology, where $\mu = \mu_t$ is defined in 1.1.

The basic observation in the proof is that the inequality between finite dimensional geodesics and $L^2$-norms coming from Monge-Ampere geodesics in Proposition 3.1 also gives inequality for the first derivatives, since we have equality at the endpoint. The next proposition (cf the sup norm estimate for $\dot{\phi}^t$ from [10]) is another instance of this.

**Proposition 3.4.** With the same notation as in the previous theorem, and

$$A_k = (H^t_k)^{-1}H^t_k,$$

let $\Lambda_{(k)}$ and $\lambda_{(k)}$ be the largest and smallest eigenvalues of $k^{-1}A_k$. Then, for all $k$,

$$\inf -\dot{\phi}^t \leq \lambda_{(k)} \leq \Lambda_{(k)} \leq \sup -\dot{\phi}^t.$$
Proof. This follows immediately from 3.7, since the corresponding inequality for the eigenvalues of the Toeplitz operators is immediate. □

4. APPROXIMATION OF GEODESICS.

Again we consider the spaces

\[ V_k = H^0(X, K_X + F + kL) \]

equipped with metrics

\[ \|u\|^2_{k\phi+\psi} := \int_X |u|^2 e^{-k\phi-\psi} \]

Let

\[ B_{k\phi+\psi} = \sum |s_j|^2, \]

where \( s_j \) is an orthonormal basis for \( V_k \). Since pointwise

\[ |u|^2 / B_{k\phi+\psi} \]

is a function if \( u \) is a section of \( K_X + F + kL \),

\[ \log B_{k\phi+\psi} \]

can be interpreted as a metric on \( K_X + F + kL \). In the proof below we will have use for the following lemma (we formulate it for \( F = 0 \) and \( k = 1 \)), which is a variant on a well known theme. The basic underlying idea, to estimate Bergman kernels using the Ohsawa-Takegoshi theorem is due to Demailly, see e.g. [6].

**Lemma 4.1.** Let \( \omega^0 \) be a fixed Kähler form on \( X \). Let \( \phi \) be a metric (not necessarily smooth) on the line bundle \( L \) satisfying

\[ i\partial\bar{\partial}\phi \geq c_0 \omega^0. \]

Let \( H_\phi \) be the norm

\[ \int_X |u|^2 e^{-\phi} \]

for \( u \) in \( H^0(X, L + K_X) \), and let \( B_\phi \) be its Bergman kernel. Then

\[ B_\phi \geq \delta_0 e^{\phi} \omega^0_n \]

with \( \delta_0 \) a universal constant, if \( c_0 \) is sufficiently large depending on \( X \) and \( \omega^0 \) (only).

**Proof.** By the extremal characterization of Bergman kernels it suffices to find a section \( u \) of \( K_X + L \) with

\[ |u(x)|^2 e^{-\phi(x)} \geq \delta_0 \omega^0_n \int_X |u|^2 e^{-\phi} \]

Choose a coordinate neighbourhood \( U \) centered at \( x \) which is biholomorphic to the unit ball of \( \mathbb{C}^n \). By the Ohsawa-Takegoshi extension theorem we can find a section satisfying the required estimate over \( U \). Let \( \eta \) be a cut-off function, equal to 1 in the ball of radius 1/2 and with compact support in the unit ball. We then solve, using Hörmander’s \( L^2 \)-estimates

\[ \partial v = \partial \eta \wedge u =: g \]
with
\[ \int_X |v|^2 e^{-\phi - 2\eta \log |z|} \leq (C/c_0) \int_X |g|^2 e^{-\phi - 2\eta \log |z|} \]
(\( z \) is the local coordinate). This can be done since
\[ i\partial \bar{\partial} \phi - 2\eta \log |z| \geq c_0 \omega^0 / 2 \]
if \( c_0 \) is large enough. Then \( v(x) = 0 \) since the integral in the left hand side is finite. Then
\[ u - v \]
is a global holomorphic section of \( K_X + L \) satisfying the required estimate. \( \square \)

Let \( \phi^0 \) and \( \phi^1 \) be two points in \( \mathcal{H}_L \), and let \( \psi \) be any smooth metric on \( F \). We abbreviate by \( H_k^t \) the norms \( \| \cdot \|_{k\phi^t + \psi} \) for \( t \) equal to 0 or 1, and let for \( t \) between 0 and 1 \( H_k^t \) be the geodesic in \( \mathcal{H}_k \), the space of hermitian norms on \( V_k \), joining these two endpoints.

**Theorem 4.2.** Let \( \phi^t \) be two points in \( \mathcal{H}_L \) for \( t \) equal to 0 and 1, and let for \( t \) between 0 and 1 \( \phi^t \) be the geodesic in \( \mathcal{H}_t \) joining them. Let \( B_{t,k} \) be the Bergman kernels for the norms \( H_k^t \). Let \( \tau \) be an arbitrary smooth metric on \( K_X + F \) over \( \Omega \times X \). Then
\[ \sup_X |k^{-1} \log B_{t,k} - k^{-1} \tau - \phi^t| \leq C k^{-1} \log k \]
for \( 0 \leq t \leq 1 \)

If \( F = 0 \) this is exactly Theorem 6.1 in [1]; if \( F = -K_X \) (so we can take \( \tau = 0 \)) it is Theorem 1.2 from the introduction.

**Proof.** As just explained \( \log B_{t,k} \) is a metric on \( K_X + F + kL \) and moreover
\[ i\partial \bar{\partial} \log B_{t,k} \geq 0. \]

The last fact follows since \( H_k^t \) are geodesics. Perhaps the easiest way to see it is to use the explicit description
\[ B_{t,k} = \sum |e^{-t\lambda_j}| |s_j|^2 \]
which is immediate from the explicit formula for geodesics in section 2. Thus
\[ k^{-1} (\log B_{t,k} - \tau) \]
is a metric on \( L \). We shall now use the metric \( \chi \) on \( L \) that we introduced in the previous section; it has strictly positive curvature over \( \Omega \times X \) and coincides with \( \phi^0 \) and \( \phi^1 \) respectively when \( (\text{Re}) t \) is 0 or 1. Take \( a \) to be positive and consider
\[ (k - a) k^{-1} (\log B_{t,k} - \tau) + a \chi; \]
it is a smooth metric on \( kL \) and it has positive curvature if \( a \) is sufficiently large. By standard Bergman kernel asymptotics it differs from \( \phi^0 \) and \( \phi^1 \) at most by \( C \log k \) when \( (\text{Re}) t \) equals 0 or 1. Hence
\[ (k - a) k^{-1} (\log B_{t,k} - \tau) + a \chi \leq k \phi^t + C \log k \]
since the geodesic $\phi^t$ is the supremum of all positively curved metrics lying below $\phi^0$ and $\phi^1$ on the boundary (cf \[4\]). Dividing by $(k-a)$ we see that
\[
k^{-1} \log B_{t,k} - k^{-1} \tau - \phi^t \leq Ck^{-1} \log k
\]
since $\chi$, $\tau$ and $\phi^t$ are all uniformly bounded. The crux of the proof is the opposite estimate.

To estimate $B_{t,k}$ from below we first compare it to the Bergman kernel
\[
B_{\phi^t,k},
\]
which is defined using the hermitian norms
\[
\|u\|^2_* = \int_X |u|^2 e^{-(k-a)\phi^t - a\chi - \psi}.
\]
Again, the metric $(k-a)\phi^t + a\chi + \psi$ that we use here has positive curvature if $a$ is sufficiently large. These norms coincide with $H^I_t$ on the boundary and by Proposition 3.1 they are bigger than $H^0_k$ in the interior. This implies (by the extremal characterization of Bergman kernels) that the respective Bergman kernels satisfy the opposite inequality, so we get
\[
\log B_{t,k} \geq \log B_{\phi^t,k}.
\]
To complete the proof it therefore suffices to show that
\[
B_{\phi^t,k} \geq Ce^{k\phi^t + \tau},
\]
or equivalently
\[
B_{\phi^t,k} \geq Ce^{(k-a)\phi^t + a\chi + \tau}
\]
But this follows from Lemma 4.1 since we can take $a$ arbitrarily large so that
\[
i\partial \overline{\partial} (k-a)\phi^t + a\chi + \psi
\]
meets the curvature assumptions of that lemma. \hfill \Box

5. APPENDIX: BACKGROUND ON TOEPLIZ OPERATORS.

We consider Toeplitz operators $T_{k,\xi}$ on the spaces
\[
V_k = H^0(X, K_X + F + kL)
\]
with symbol $\xi$ in $C(X)$. $T_{k,\xi}$ is defined by
\[
\langle T_{k,\xi} u, u \rangle_{k\phi + \psi} = \int_X \xi |u|^2 e^{-k\phi - \psi},
\]
where the inner product is
\[
\langle v, u \rangle_{k\phi + \psi} = \int_X v \overline{u} e^{-k\phi - \psi}.
\]
In other words
\[
T_{k,\xi} u = P_k(\xi u)
\]
where $P_k$ is the Bergman projection.
Recall that if $T$ is any hermitian endomorphism on an $N$-dimensional inner product space, and if we order its eigenvalues
\[ \lambda_1 \leq \lambda_2 \leq \ldots \lambda_N, \]
then
\[ \lambda_j = \inf_{V_j \subset V, \dim V_j = j} \| T|_{V_j} \|. \]
From this it follows that if we perturb the operator $T$ to $T + S$ where $\| S \| \leq \epsilon$, then the eigenvalues shift at most by $\epsilon$. This means that if we consider the spectral measure of
\[ T_{k,\xi + \xi_k} \]
where $\xi_k$ goes uniformly to 0, the limit of the spectral measures is the same as the limit of the spectral measures of
\[ T_{k,\xi}. \]
In other words, in the proof of Theorem 3.2 we may assume that $\xi_k = 0$. By the same token, we may assume that $\xi$ is smooth, since continuous functions can be approximated by smooth functions. The most important part of the proof of Theorem 3.2 is the next lemma.

**Lemma 5.1.** Let $d_k = \dim(V_k)$. Then
\[
\lim \frac{1}{d_k} \text{tr} T_{k,\xi} = \int_X \xi \omega^\phi_n/\text{Vol}.
\]

**Proof.** Let $B_{k\phi+\psi}$ be the Bergman kernel. Then
\[
\frac{1}{d_k} \text{tr} T_{k,\xi} = \frac{1}{d_k} \int_X \xi B_{k\phi+\psi} e^{-k\phi-\psi},
\]
But, by the formula for (first order) Bergman asymptotics
\[
B_{k\phi+\psi} e^{-k\phi-\psi}/d_k
\]
tends to $\omega^\phi_n/\text{Vol}$, so the lemma follows. \[\square\]

**Lemma 5.2.** Let $\xi$ and $\eta$ be smooth functions on $X$. Then
\[
\| T_{k,\xi} T_{k,\eta} - T_{k,\xi\eta} \|^2 \leq C k^{-1}.
\]

**Proof.** Note that if $u$ is in $V_k$ then
\[ T_{k,\xi} u - \xi u =: v_k \]
is the $L^2$-minimal solution to the $\bar{\partial}$-equation
\[ \bar{\partial} v_k = \bar{\partial} \xi \wedge u \]
(this is where we want $\xi$ smooth). By Hörmander $L^2$-estimates
\[
\| T_{k,\xi} u - \xi u \|_{k\phi+\psi}^2 \leq \| \bar{\partial} \xi \wedge u \|_{k\phi+\psi}^2 \leq C k^{-1} \| u \|_{k\phi+\psi}^2
\]
(the last inequality is because the pointwise norm $\| \bar{\partial} \xi \|_2^2 \leq C/k$ when we measure with respect to the Kähler metric $\theta = i \partial \bar{\partial} (k\phi + \psi)$). Therefore, if $u$ is of norm at most 1,
\[
\| T_{k,\xi} T_{k,\eta} u - \xi T_{k,\eta} u \|^2 \leq C k^{-1},
\]
\[
\| \xi T_{k,\eta} u - \xi u \|^2 \leq C k^{-1}
\]
\[ \| T_{k,\xi} u - \xi \eta u \|_2^2 \leq C k^{-1} \]
and the lemma follows. \( \square \)

Let \( \mu_k \) be the normalized spectral measures of \( T_{k,\xi} \). In order to study their weak limits, it is enough to look at their moments
\[ \int_{\mathbb{R}} x^p d\mu_k(x) = \frac{1}{d_k} tr T_{k,\xi}^p. \]
By Lemma 7.2 and induction
\[ \| T_{k,\xi}^p - T_{k,\xi} \|_2^2 \leq C k^{-1}. \]
Hence
\[ \frac{1}{d_k} tr T_{k,\xi}^p = \frac{1}{d_k} tr T_{k,\xi} + O(k^{-1}) \]
and
\[ \lim \frac{1}{d_k} tr T_{k,\xi}^p = \int_X \xi^p \omega_n^e / Vol \]
by Lemma 7.1. Thus,
\[ \lim \int_{\mathbb{R}} x^p d\mu_k(x) = \frac{1}{d_k} tr T_{k,\xi}^p = \int_X \xi^p \omega_n^e / Vol \]
for any power \( x^p \). Taking linear combinations we get the same thing for any polynomial, and therefore for any continuous function. This completes the proof of Theorem 3.2.

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