Self-force on a non-minimally-coupled static scalar charge outside a Schwarzschild black hole

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Abstract. In this work we compute the self-force experienced by a static non-minimally-coupled scalar charge outside a Schwarzschild black hole. We find that the finite part of this self-force is zero. Our no-self-force result is in disagreement with a previous result of Zel’nikov and Frolov, who have suggested that there is a non-zero finite-part of the self-force on a non-minimally-coupled charge in Schwarzschild spacetime. We are able to show that the omission of an unusual flux in their calculation was responsible for their (incorrect) result.

1. Introduction
Several years ago, Zel’nikov and Frolov [11] published a result suggesting that the self-force on a non minimally coupled scalar charge held fixed in a Schwarzschild spacetime, contained a finite part that depended upon the value of the coupling constant ($\xi$). In this work, we show that there is no finite part of the self-force for a non-minimally-coupled scalar charge held fixed in a Schwarzschild spacetime. With the method we use for calculating the force, we are able to isolate the error and show that it is subtle and very interesting. The resolution requires that we address physical processes that did not enter the self-force calculations for other types of charges.

2. Summary of results
Consider a point particle with a small bare mass $m_o$, a small electric charge $e$ and a small scalar charge $q$. The charge is held stationary on the $z$ axis at coordinate radius $b$ by a strut. The force the strut must exert to hold the charge stationary is

$$F_{\text{strut}} = m_{\text{ren}} \frac{M}{b^2} \left(1 - \frac{2M}{b}\right)^{-\frac{1}{2}} - \frac{e^2 M}{b^3} + C_{sf} \frac{q^2 M}{b^3},$$

where $m_{\text{ren}}$ is the renormalized mass $m_{\text{renorm}} = m_o + \frac{e^2}{2\epsilon} - \frac{q^2}{2\epsilon}$, $m_o$ is the bare mass of the particle, and the terms involving $e$ and $q$ represent the contribution to the mass of the particle from the electric and scalar charges. Although these are made finite, by the introduction of the

1 Point particles will be modeled as shells of proper radius $\epsilon$, and the charges and $\epsilon$ can be adjusted to insure the field is weak everywhere.
cut-off radius $\epsilon$, they are often referred to as the infinite or divergent part of the self force. [See e.g. DeWitt and Brehme [5,1] for discussion.]

The first term in Eq. (1) is just the (red-shifted) weight of the particle, including contributions from the electric and scalar charges.

The second term is the finite contribution to the self-force due to the electric charge. Notice that this is repulsive and therefore partially supports the charge. This term has been derived by Smith and Will [10] and Lohiya [12], and first appeared in the weak field calculation of [1].

The final term represents a possible finite contribution to the self force due to the scalar charge of the particle, i.e. the scalar analogue of the previous electric term. It was shown by Zel’nikov and Frolov [11] and Wiseman [13] that for a minimally coupled scalar field $C_{sf}(\xi = 0) = 0$. The central theme of this work is to demonstrate that $C_{sf}(\xi) = 0$ for any value of $\xi$, in disagreement with the result obtained by Zel’nikov and Frolov [11], which is $C_{sf}(Zel’nikov-Frolov) = 2\xi$. As we go through our calculation, we will show that a flux of radiation into the black hole was omitted in the Zel’nikov and Frolov calculation and it was this omission that led to their incorrect self-force.

3. Basic equations

All quantities will be derived from the following action

$$A = \int \left[ f(\Phi) \frac{R}{16\pi} + \frac{1}{8\pi} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi - \rho \Phi + L_{em} + L_{bare} + L_{strut} \right] \sqrt{-g} d^4x .$$

(2)

Here $R$ is the Ricci curvature scalar, $\Phi$ is the scalar field, and $\rho$ is the scalar charge density, and $\sqrt{-g}$ is the determinant of the covariant components of the metric. The term $L_{em}$ represents the electromagnetic contribution to the action. The term $L_{bare}$ represents the contribution to the action from the bare mass $m_o$ of the particle. This term results in the leading order term in the Eq.(1): the bare weight of the particle. The term $L_{strut}$ represents the matter in the strut that will hold the particle stationary. Also $f(\Phi) = 1 - 2\xi \Phi^2$ where $\xi$ is the “coupling constant”.

Requiring the action to be stationary with respect to variations in $\Phi$ gives the field equation

$$g^{\alpha\beta} \Phi_{\alpha\beta} - \xi R \Phi = 4\pi \rho , \quad \rho = \frac{q}{4\pi} \frac{\delta^3(x-b\hat{z})}{\sqrt{-g}} .$$

(3)

Here, $g^{\alpha\beta}$ are the metric coefficients, $\rho$ is the charge density, and is taken to be a point charge with strength $q$ at a fixed coordinate $b$ on the z-axis. Notice that in Schwarzschild spacetime (where $R = 0$) the field equations do not depend on the coupling to the curvature. The solution to the field equation for a static charge in Schwarzschild spacetime is

$$\Phi = -q \sqrt{\frac{b_h - M}{b_h + M}} \frac{1}{\sqrt{r_h^2 - 2r_h b_h \cos \theta + b_h^2 - M^2 \sin^2 \theta}} .$$

(4)

In this equation, $b_h$ and $r_h$ are the radial harmonic [16] coordinate positions of the field point and source point respectively, and $\theta$ is angular position of the field point measured from the z-axis. As noted above, this solution is valid for minimal- ($\xi = 0$) and non-minimal-coupling ($\xi \neq 0$).

Requiring the action to be stationary under variations of the metric coefficients, results in the following gravitational field equations

$$8\pi T^{\alpha\beta} = 8\pi \left[ T^{\alpha\beta}_{(\xi=0)} + T^{\alpha\beta}_{(interaction)} + T^{\alpha\beta}_{(\xi)} + T^{\alpha\beta}_{(em)} + T^{\alpha\beta}_{(bare)} + T^{\alpha\beta}_{(strut)} \right] = (1 - 2\xi \Phi^2) G^{\alpha\beta} .$$

(5)
where

\[ T^{\alpha\beta}_{(\xi=0)} = \frac{1}{4\pi} \left[ \nabla^\alpha \Phi \nabla^\beta \Phi - \frac{1}{2} g^{\alpha\beta} g^{\sigma\tau} \nabla^\sigma \Phi \nabla^\tau \Phi \right], \] (6)

\[ T^{\alpha\beta}_{(\text{int})} = -g^{\alpha\beta} \rho \Phi, \] (7)

\[ T^{\alpha\beta}_{(\xi)} = \frac{\xi}{2\pi} \left[ g^{\alpha\beta} \nabla^\sigma (\Phi \nabla_\sigma \Phi) - \nabla^\alpha (\Phi \nabla^\beta \Phi) \right], \] (8)

\[ T^{\alpha\beta}_{(\text{em})} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\alpha\beta}} [ L_{\text{em}} \sqrt{-g} ], \] (9)

\[ T^{\alpha\beta}_{(\text{bare})} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\alpha\beta}} [ L_{\text{strut}} \sqrt{-g} ] \] (10)

\[ T^{\alpha\beta}_{(\text{strut})} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\alpha\beta}} [ L_{\text{strut}} \sqrt{-g} ]. \] (11)

4. The divergence of the stress-tensor and a local force calculation

Taking the divergence of the full set of field equations we find that the equation of motion for all the matter is

\[ -\nabla^\beta T^{\alpha\beta}_{(\text{strut})} = +\Phi \nabla^\alpha \rho + \nabla^\beta T^{\alpha\beta}_{(\text{em})} + \nabla^\beta T^{\alpha\beta}_{(\text{bare})} \] (12)

where we have used Eqs.(3), (5-11) to simplify the result. We wish to emphasize that Eq.(12) is exact. In the derivation, we have not discarded any higher order terms, and yet there is no explicit \( \xi \)-dependence. The only way the coupling constant can affect the self-force is through the Ricci scalar in the scalar field equation or through corrections to the metric; these will all produce higher-order effects that we are discarding in this calculation.

Terms such as we see in Eq.(12) have already been addressed by Smith and Will [10] and Wiseman [13]. The method they used is to transform to a freely falling frame and integrate each term on the right (i.e. the force density) in a small ball around the charge. The net force on the right side must be counteracted by the force on the left due to the strut. The result is the force given in Eq.(1).²

5. Global method

We now determine the self-force using a Gedankenexperiment. Consider a freely falling observer in a Schwarzschild spacetime. At the apex of her trajectory at coordinate radius \( b \), she momentarily comes to rest with respect to the charge supported on the strut. The observer lowers (or raises) the charge at constant speed through a small proper distance \( \delta b \) and places the charge back on the next rung of the strut. The assumption is that the force measured by the locally freely falling observer is the same force that strut was exerting until she grabbed the charge. After she returns the charge to the strut, she converts the energy to a photon and fires the photon to infinity. The energy received by a distant observer will be red-shifted and the change in energy must be manifested as a change in the asymptotically measured mass of the system. We now use Carter’s total mass variation law ([18] Eq.(6.342)) to relate the change in asymptotic mass to the changes in the central black hole and the matter content of the spacetime. It is

\[ \delta M_{\text{asymp}} = \frac{\kappa}{8\pi} \delta A_{\text{bh}} - \frac{1}{8\pi} \delta \int G^0_0 \sqrt{-g} \, d^3 x = \delta M + \delta \mathcal{E}_{\text{self}} \] (13)

² In the local calculation in [10] and [13], the point charge was not replaced by a shell, and thus the contribution to the renormalized mass is somewhat different than what is shown in Eq.(1).
where
\[ \mathcal{E}_{\text{self}} = - \int T^0_0 \sqrt{-g} \, d^3x \ . \] (14)

We have neglected terms involving the angular momentum and the metric perturbations. Also we used the fact that the area of the Schwarzschild black hole is \( A_{bh} = 16\pi M^2 \) and the surface gravity is \( \kappa = \frac{1}{4M} \).

The force measured by the freely falling observer is seen to be
\[ F^z = \frac{1}{\sqrt{-g_{00}(b)}} \frac{\delta M_{\text{asymp}}}{\delta b} = \frac{1}{\sqrt{-g_{00}(b)}} \left( \frac{\delta M}{\delta b} + \frac{\delta \mathcal{E}_{\text{self}}}{\delta b} \right) . \] (15)

where
\[ \mathcal{E}_{\text{self}} = \left( m_o - \frac{q^2}{2\epsilon} \right) \sqrt{1 - (2M/b) - \xi M q^2 / b^2} . \] (16)

The contribution of this part to the force in Eq.(4.27) is
\[ \frac{1}{\sqrt{-g_{00}(b)}} \frac{\delta \mathcal{E}_{\text{self}}}{\delta b} = \frac{\delta \mathcal{E}_{\text{self}}}{\delta b} = \left( m_o - \frac{q^2}{2\epsilon} \right) \frac{M}{b^2} \left( 1 - \frac{2M}{b} \right)^{-1/2} + 2\xi q^2 \frac{M}{b^3} . \] (17)

Although we still need to evaluate the \( \delta M \)-term in Eq.(15) in order to determine the true self-force, we pause to note that Eq.(17) is the value of the self-force given by Zel’nikov and Frolov. As it stands now, it does appear that the last term is a finite contribution to the self-force with the same functional form as the electric piece in Eq.(1).

We now turn to the task of evaluating the first term in Eq.(15). When the freely falling observer slowly lowers the charge on the strut, the mass of the black hole changes
\[ \frac{dM}{dt} = \left[ \text{rate energy flows down the hole} \right] = \lim_{r \to 2M} \int T^t_0 \sqrt{-g} \, d\theta d\phi \] (18)
\[ = \frac{\xi M}{2\pi} \lim_{r \to 2M} \oint \Phi \Phi_t \, d\Omega = \frac{\xi M}{4\pi} \left( \frac{\delta}{\delta b} \oint_{\text{horizon}} \Phi^2 d\Omega \right) \frac{\delta b}{\delta t} . \] (19)

Notice that for a slowly moving charge, the only flux down the hole comes from the terms proportional to \( \xi \), i.e. the “non-minimal” part of the stress-energy. We use this result to construct the mass variation term that enters the force calculation
\[ \frac{1}{\sqrt{-g_{00}(b)}} \frac{\delta M}{\delta b} = \frac{\xi M}{4\pi} \frac{\delta}{\delta b} \oint_{\text{horizon}} \Phi^2 d\Omega = -2\xi q^2 \frac{M}{b^3} . \] (20)

Collecting all terms we find the force exerted by the charge
\[ F^z = \left( m_o - \frac{q^2}{2\epsilon} \right) M \frac{M}{b^2} \left( 1 - \frac{2M}{b} \right)^{-1/2} \] (21)

Notice that there is no finite part to the self-force: the last term in Eq.(17), that appeared to give a finite contribution to the self-force, exactly canceled a similar term arising from the flux into the hole in Eqs.(19) and (20). Therefore, the self-force does not depend on the non-minimal-coupling constant \( \xi \). Collecting Eq.(21) with its electromagnetic counterpart from Smith and Will [10] yields Eq.(1), with \( C_{sf} \equiv 0 \).
6. Entropy and reversibility

In the previous section we showed that when the charge is raised (and $\xi > 0$) the mass ($M$) of the hole decreases. This implies that the area of the hole ($A_{bh} = \frac{1}{4} A_{bh}$) decreases. Since one generally thinks of the black hole entropy as $S_{bh} = \frac{1}{4} A_{bh}$ it appears that we have found a process by which to lower the entropy of a black hole. In fact Iyer and Wald ([19] Eq.(1.2)) have shown that for a gravitational theory with an action of the form we have used the appropriate generalized entropy is

$$S_{bh} = \frac{1}{4} \oint_{\text{horizon}} (1 - 2\xi \Phi^2) dA.$$  \hfill (22)

The change in entropy as we displace the charge is

$$\frac{\delta S_{bh}}{\delta \bar{b}} = 8\pi M \frac{\delta M}{\delta \bar{b}} - 2\xi \left[ 2M \frac{\delta M}{\delta \bar{b}} \oint_{\text{horizon}} \Phi^2 d\Omega + M^2 \frac{\delta}{\delta \bar{b}} \oint_{\text{horizon}} \Phi^2 d\Omega \right].$$  \hfill (23)

Now the second term is $O(q^4)$ thus can be discarded. The first term identically cancels the last, thus we are left with the expected $\frac{\delta S_{bh}}{\delta \bar{b}} = 0$ Therefore the generalized entropy of a black hole bathed in the non-minimally-coupled scalar field of a charge at Schwarzschild radius $b$ is

$$S_{bh} = \frac{1}{4} A_{bh} \left[ 1 - 2\xi \left( \frac{q}{b} \right)^2 \right].$$  \hfill (24)

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