Four NP-complete problems about generalizations of perfect graphs

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Abstract

We show that the following problems are NP-complete.

• Can the vertex set of a graph be partitioned into two sets such that each set induces a perfect graph?

• Is the difference between the chromatic number and clique number at most 1 for every induced subgraph of a graph?

• Can the vertex set of every induced subgraph of a graph be partitioned into two sets such that the first set induces a perfect graph, and the clique number of the graph induced by the second set is smaller than that of the original induced subgraph?

• Does a graph contain a stable set whose deletion results in a perfect graph?

The proofs of the NP-completeness of the four problems follow the same pattern: Showing that all the four problems are NP-complete when restricted to triangle-free graphs by using results of Maffray and Preissmann [3] on 3-colorability and 4-colorability of triangle-free graphs.

1 Introduction

All graphs considered in this article are finite and simple. Let $G$ be a graph. The complement $G^c$ of $G$ is the graph with vertex set $V(G)$ and such that two vertices are adjacent in $G^c$ if and only if they are non-adjacent in $G$. For two graphs $H$ and $G$, $H$ is an induced subgraph of $G$ if $V(H) \subseteq V(G)$, and a pair of vertices $u, v \in V(H)$ is adjacent if and only if it is adjacent in $G$. We say that $G$ contains $H$ if $G$ has an induced subgraph isomorphic to $H$. If $G$ does not contain $H$, we say that $G$ is $H$-free. For a set $X \subseteq V(G)$ we denote by $G\!\![X]$ the induced subgraph of $G$ with vertex set $X$. A hole in a graph is an induced subgraph that is isomorphic to the cycle $C_k$ with $k \geq 4$, and $k$ is the length of the hole. A hole is odd if $k$ is odd, and even otherwise. The chromatic number of a graph $G$ is denoted by $\chi(G)$ and the clique number by $\omega(G)$. $G$ is called perfect if for every induced subgraph $H$ of $G$, $\chi(H) = \omega(H)$. $G$ is said to be perfectly divisible if for all induced subgraphs $H$ of $G$, $V(H)$ can be partitioned into two sets $A, B$ such that $H[A]$ is perfect and $\omega(B) < \omega(H)$. $G$ is said to be nice if for every induced subgraph $H$ of $G$, $\chi(H) - \omega(H) \in \{0, 1\}$. $G$ is said to be
2-perfect if \( V(G) \) can be partitioned into two sets \( A, B \) such that both \( G[A] \) and \( G[B] \) are perfect. \( G \) is said to be stable-perfect if \( G \) contains a stable set \( S \) such that \( G \setminus S \) is perfect. Note that perfect graphs are stable-perfect, and stable-perfect graphs are 2-perfect, perfectly divisible, and nice. In this note, we show that the recognition problems for the four classes (2-perfect, nice, perfectly divisible, stable-perfect) are NP-complete, a stark contrast to the existence of a polynomial-time recognition algorithm for perfect graphs [1].

## 2 Four NP-complete problems

We need the following results from [3].

**Theorem 2.1** (Maffray-Preissmann). It is NP-complete to determine whether a triangle-free graph is 3-colorable.

**Theorem 2.2** (Maffray-Preissmann). It is NP-complete to determine whether a triangle-free graph is 4-colorable.

The following is a basic fact about perfect graphs.

**Lemma 2.1.** A triangle-free graph is perfect if and only if it is bipartite.

**Proof.** Since bipartite graphs are perfect, one direction is trivial. To prove the other direction, let \( G \) be a triangle-free perfect graph. Since \( G \) contains neither a triangle nor an odd hole, it contains no odd cycle as a subgraph. Hence \( G \) is bipartite. \( \square \)

We first prove the NP-completeness of recognizing 2-perfect graphs. First we need a lemma.

**Lemma 2.2.** A triangle-free graph is 2-perfect if and only if it is 4-colorable.

**Proof.** This follows easily from Lemma 2.1. \( \square \)

**Theorem 2.3.** It is NP-complete to determine whether a graph is 2-perfect.

**Proof.** We show that the restricted problem of determining whether a triangle-free graph is 2-perfect is NP-complete. Let \( G \) be a triangle-free graph. By Lemma 2.2 \( G \) is 2-perfect if and only if \( G \) is 4-colorable. By Theorem 2.2 it is NP-complete to determine whether a triangle-free graph is 4-colorable, We thus conclude that it is NP-complete to determine whether a triangle-free graph is 2-perfect. \( \square \)

We now move on to the classes of perfectly divisible graphs, stable-perfect, and nice graphs. Problem 32 in [4] asks whether nice graphs can be recognized in polynomial time. The recognition problem for nice graphs turns out to be NP-complete. The following lemma tells that for triangle-free graphs, the three classes mentioned above are equivalent to the class of 3-colorable graphs.

**Lemma 2.3.** For a triangle-free graph \( G \), the following are equivalent:

(i) \( G \) is 3-colorable.

(ii) \( G \) is perfectly divisible.
(iii) $G$ is stable-perfect.

(iv) $G$ is nice.

Proof. We prove the following chain of implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$: Suppose $G$ is 3-colorable. Let $H$ be an induced subgraph of $G$. Note that $H$ is also 3-colorable. We may assume that $H$ has clique number 2. Let $(S_1, S_2, S_3)$ be a partition of $V(H)$ into three stable sets. Now $(S_1 \cup S_2, S_3)$ is a partition of $V(G)$ as in the definition of being perfectly divisible. We conclude that $G$ is perfectly divisible.

$(ii) \Rightarrow (iii)$: Suppose $G$ is perfectly divisible. Hence there is a partition of $V(G)$ into sets $A, B$ such that $G[A]$ is perfect and $\omega(B) < \omega(G)$. Since $G$ has no triangles, this implies that $B$ is a stable set. Thus $G$ is stable-perfect.

$(iii) \Rightarrow (iv)$: Suppose $G$ is stable-perfect. Let $H$ be an induced subgraph of $G$. We may assume that $H$ has clique number 2. Thus $H$ contains a stable set $S$ such that $H \setminus S$ is perfect. Since $H$ is also triangle-free, by Lemma 2.1, $H \setminus S$ is bipartite. Hence the chromatic number of $H$ is at most 3. We conclude that $G$ is nice.

$(iv) \Rightarrow (i)$: Suppose $G$ is nice. Since $G$ is triangle-free, its clique number is at most 2. Since $G$ is nice, we conclude that its chromatic number is at most 3. Thus $G$ is 3-colorable.

This concludes the proof of all the implications, and proves the theorem.

**Theorem 2.4.** The following problems are NP-complete:

1. Given a graph, is it perfectly divisible?
2. Given a graph, is it stable-perfect?
3. Given a graph, is it nice?

Proof. By Lemma 2.3 and Theorem 2.1, the problems are already NP-complete when restricted to triangle-free graphs.

3 Open problems

$G$ is said to be 2-divisible if for all induced subgraphs $H$ of $G$, $V(H)$ can be partitioned into two sets $A, B$ such that $\omega(A) < \omega(H)$ and $\omega(B) < \omega(H)$.

**Conjecture 3.1.** It is NP-complete to determine whether a graph is 2-divisible.

There is a nice conjecture about 2-divisible graphs:

**Conjecture 3.2** (Hoang-McDiarmid [2]). A graph is 2-divisible if and only if it is odd-hole-free.

The complexity of the recognition of odd-hole-free graphs is also unknown.

**Conjecture 3.3.** It is NP-complete to determine whether a graph contains an odd hole.
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References

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