GLOBAL SMALL SOLUTIONS OF 2-D INCOMPRESSIBLE MHD SYSTEM

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Abstract. In this paper, we consider the global wellposedness of 2-D incompressible magneto-hydrodynamical system with smooth initial data which is close to some non-trivial steady state. It is a coupled system between the Navier-Stokes equations and a free transport equation with an universal nonlinear coupling structure. The main difficulty of the proof lies in exploring the dissipative mechanism of the system. To achieve this and to avoid the difficulty of propagating anisotropic regularity for the free transport equation, we first reformulate our system (1.1) in the Lagrangian coordinates (2.19). Then we employ anisotropic Littlewood-Paley analysis to establish the key *a priori* $L^1(R^+; Lip(R^2))$ estimate for the Lagrangian velocity field $Y_t$. With this estimate, we can prove the global wellposedness of (2.19) with smooth and small initial data by using the energy method. We emphasize that the algebraic structure of (2.19) is crucial for the proofs to work. The global wellposedness of the original system (1.1) then follows by a suitable change of variables.

Keywords: Inviscid MHD system, Anisotropic Littlewood-Paley Theory, Dissipative estimates, Lagrangian coordinates

AMS Subject Classification (2000): 35Q30, 76D03

1. Introduction

In this paper, we investigate the global wellposedness of the following 2-D incompressible magneto-hydrodynamical system:

\[
\begin{align*}
\partial_t \phi + u \cdot \nabla \phi &= 0, \quad (t, x) \in R^+ \times R^2, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p &= -\text{div}[\nabla \phi \otimes \nabla \phi], \\
\text{div} u &= 0, \\
\phi|_{t=0} &= \phi_0, \quad u|_{t=0} = u_0,
\end{align*}
\]

with initial data $(\phi_0, u_0)$ smooth and close enough to the equilibrium state $(x_2, 0)$. Here $\phi$ denotes the magnetic potential and $u = (u^1, u^2)^T$, $p$ the velocity and scalar pressure of the fluid respectively.

Recall that the general MHD system in $R^d$ reads

\[
\begin{align*}
\partial_t b + u \cdot \nabla b &= b \cdot \nabla u, \quad (t, x) \in R^+ \times R^d, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p &= -\frac{1}{2} \nabla |b|^2 + b \cdot \nabla b, \\
\text{div} u &= \text{div} b = 0, \\
b|_{t=0} &= b_0, \quad u|_{t=0} = u_0,
\end{align*}
\]

where $b = (b^1, \ldots, b^d)^T$ denotes the magnetic field, and $u = (u^1, \ldots, u^d)^T$, $p$ the velocity and scalar pressure of the fluid respectively. This MHD system (1.2) with zero diffusivity in the equation for the magnetic field can be applied to model plasmas when the plasmas are strongly collisional, or the resistivity due to these collisions are extremely small. It often

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applies to the case when one is interested in the k-length scales that are much longer than the ion skin depth and the Larmor radius perpendicular to the field, long enough along the field to ignore the Landau damping, and time scales much longer than the ion gyration time \([14, 20, 4]\). In the particular case when \(d = 2\) in (1.2), \(\text{div} \, b = 0\) implies the existence of a scalar function \(\phi\) so that \(b = (\partial y \phi, -\partial x \phi)^T\), and the corresponding system becomes (1.1).

It is a long standing open problem that whether or not classical solutions of (1.2) can develop finite time singularities even in the 2-D case. Except with full magnetic diffusion in (1.2), the corresponding 2-D system possesses a unique global smooth solution (see \([16, 31]\) and \([1]\) for initial data in the critical spaces). With mixed partial dissipation and additional (artificial) magnetic diffusion in the 2-D MHD system, Cao and Wu \([5]\) (see also \([6]\)) proved its global wellposedness for any data in \(H^2(\mathbb{R}^2)\). In \([25]\), we proved the global wellposedness of a three dimensional version of (1.1) with smooth initial data which is close to a non-trivial steady state. The aim of this paper is to establish the global existence and uniqueness of solutions to the MHD equation (1.1) in the 2-D case with the same class of the initial data.

We note that the system (1.1) has appeared in many problems, see the recent survey article \([22]\). For the inviscid, incompressible MHD equations (1.2), it is an important problem that if it possesses a dissipation mechanism even though the magnetic diffusivity is close to zero. The heating of high temperature plasmas by MHD waves is one of the most interesting and challenging problems of plasma physics especially when the energy is injected into the system at the length scales much larger than the dissipative ones. Indeed it has been conjectured that in the MHD systems, energy is dissipated at a rate that is independent of the ohmic resistivity \([10]\). In other words, the diffusivity for the magnetic field equation can be zero yet the whole system may still be dissipative. We shall justify this conjecture for initial data sufficiently close to a non-trivial equilibrium state in the two-dimensional case. Here the dissipation property is closely related to a partial dissipative property (in spatial directions) of magnetic waves due to non-trivial background magnetic fields. Our global existence results are solely based on the latter property. For this reason we also conjecture that such global well posedness results would not be possible without non-trivial background magnetic fields.

Notice that, after substituting \(\phi = x_2 + \psi\) into (1.1), one obtains the following system for \((\psi, u)\) :

\[
\begin{align*}
\partial_t \psi + u \cdot \nabla \psi + u^2 &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\partial_t u^1 + u \cdot \nabla u^1 - \Delta u^1 + \partial_1 \psi &= -\partial_1 p - \text{div} [\partial_1 \psi \nabla \psi] \overset{\text{def}}{=} f^1, \\
\partial_t u^2 + u \cdot \nabla u^2 - \Delta u^2 + (\Delta + \partial_2^2) \psi &= -\partial_2 p - \text{div} [\partial_2 \psi \nabla \psi] \overset{\text{def}}{=} f^2, \\
\text{div} \, u &= 0, \\
(\psi, u)|_{t=0} &= (\psi_0, u_0).
\end{align*}
\]

Starting from (1.3), standard energy estimate gives rise to

\[
\frac{1}{2} \frac{d}{dt} (\|\nabla \psi(t)\|^2_{L^2} + \|u(t)\|^2_{L^2}) + \|\nabla u(t)\|^2_{L^2} = 0
\]

for smooth enough solutions \((\psi, u)\) of (1.3). The main difficulty to prove the global existence of small smooth solutions to (1.3) is thus to find a dissipative mechanism for \(\psi\). Motivated by the heuristic analysis in Subsection 2.1, we shall first write the system (1.1) in the Lagrangian formulation (2.19). There is, however, a subtle technical difficult for MHD equations in this formulation unlike many other fluid dynamic problems. We need to introduce a notion of admissible initial data (see Definition 1.1 below). Next, we employ anisotropic Littlewood-Paley theory to capture the delicate dissipative property of \(Y_1\) in Section 3. It turns out that \(\partial_{y_1} Y\) decays faster than \(\partial_{y_2} Y\). This, in some sense, also justifies the necessity of using
Remark 1.1. Let $b = (b_1, b_2)^T$ be a smooth enough vector field. We define its trajectory $X(t, x)$ by
\begin{align}
\frac{dX(t, x)}{dt} = b(X(t, x)), \quad X(t, x)|_{t=0} = x.
\end{align}
We call that $f$ and $b$ are admissible on a domain $D$ of $\mathbb{R}^2$ if there holds
\begin{align}
\int_{\mathbb{R}} f(X(t, x)) \, dt = 0 \quad \text{for all} \quad x \in D.
\end{align}

Remark 1.1. The condition that $f$ and $b$ are admissible on $\mathbb{R}^2$ (or some subsets of $\mathbb{R}^2$) is to guarantee that
\begin{align}
b \cdot \nabla \psi = f \quad \text{on} \quad \mathbb{R}^2,
\end{align}
has a solution $\psi$ so that $\lim_{|x| \to \infty} \psi(x) = 0$. Let us take $b = (1, 0)^T$ for example. In this case, (1.6) becomes $\partial_{x_1} \psi = f$, which together with the condition $\lim_{|x_1| \to \infty} \psi(x) = 0$ ensures that
\begin{align}
\psi(x_1, x_2) = -\int_{x_1}^{\infty} f(s, x_2) \, ds = \int_{-\infty}^{x_1} f(s, x_2) \, ds.
\end{align}
We thus obtain that $\int_{\mathbb{R}} f(s, x_2) \, ds = 0$, that is, $f$ and $(1, 0)^T$ are admissible on $\{0\} \times \mathbb{R}$.

In what follows, for $X_1, X_2$ being two Banach spaces, we always denote the norms $\| \cdot \|_{X_1 \cap X_2}$ defined as $\| \cdot \|_{X_1} + \| \cdot \|_{X_2}$ and $\| \cdot \|_{LP(\mathbb{R}^+; X_1 \cap X_2)}$ defined as $\| \cdot \|_{LP(\mathbb{R}^+; X_1)} + \| \cdot \|_{LP(\mathbb{R}^+; X_2)}$ for $p \in [1, \infty]$.

We now present our main result in this paper:

Theorem 1.1. Let $s_1 > 1$, $s_2 \in (-1, -\frac{1}{2})$ and $s \geq s_1 + 2$. Given $(\psi_0, u_0)$ satisfying $\nabla \psi_0 \in H^s \cap H^{s+2}(\mathbb{R}^2)$, $u_0 \in H^s \cap H^{s+2}(\mathbb{R}^2)$ and
\begin{align}
\| \nabla \psi_0 \|_{H^{s+1+\gamma} \cap H^{s+2}} + \| \partial_{x_2} \psi_0 \|_{H^{s+1+\gamma} \cap H^{s+1+\gamma}} \leq c_0
\end{align}
for some $c_0$ sufficiently small. We assume moreover that $\partial_{x_2} \psi_0$ and $(1 + \partial_{x_2} \psi_0, -\partial_{x_1} \psi_0)^T$ are admissible on $\{0\} \times \mathbb{R}$ and $\text{Supp} \ \partial_{x_2} \psi_0(\cdot, x_2) \subset [-K, K]$ for some positive number $K$. Then (1.3) has a unique global solution $(\psi, u, p)$ (up to a constant for $p$) so that
\begin{align}
\nabla \psi \in C([0, \infty); H^s \cap H^{s+2}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+; H^{s+1} \cap H^{s+1+\gamma}(\mathbb{R}^2)),
\end{align}
\begin{align}
u \in C([0, \infty); H^s \cap H^{s+2}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+; H^{s+1} \cap H^{s+1+\gamma}(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; Lip(\mathbb{R}^2)),
\end{align}
\begin{align}
\nabla u \in L^2_{\text{loc}}(\mathbb{R}^+; H^s(\mathbb{R}^2)), \quad \nabla p \in C([0, \infty); H^{s-1}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+; H^{s} \cap H^{s+2}(\mathbb{R}^2))
\end{align}
Furthermore, there holds
\begin{align}
\| u \|_{L^\infty(\mathbb{R}^+; H^{s+1} \cap H^{s+2})} + \| \nabla \psi \|_{L^\infty(\mathbb{R}^+; H^{s+1} \cap H^{s+2})} + \| u \|_{L^2(\mathbb{R}^+; H^{s+1} \cap H^{s+1+\gamma})} + \| \nabla \psi \|_{L^2(\mathbb{R}^+; H^{s+1} \cap H^{s+1+\gamma})} + \| \nabla \psi \|_{L^\infty(\mathbb{R}^+; H^{s+2})} + \| \nabla \psi \|_{L^\infty(\mathbb{R}^+; H^{s+1} \cap H^{s+2})}
\leq C(\| \nabla \psi \|_{H^{s+1} \cap H^{s+2}} + \| \partial_{x_2} \psi_0 \|_{H^{s+1+\gamma}} + \| u_0 \|_{H^{s+1+\gamma}}).
\end{align}

Remark 1.2. We can replace the condition that $\text{Supp} \ \partial_{x_2} \psi_0(\cdot, x_2) \subset [-K, K]$ for some positive number $K$, in Theorem 1.1 by assuming appropriate decay of $\partial_{x_2} \psi_0(x)$ with respect to $x_1$ variable. To make the presentation more transparent, we would not emphasize this technical point here.
Remark 1.3. As ψ is a scalar function in (1.3), we can not apply the ideas and analysis developed in [23, 24, 9, 21] for the viscoelastic fluid system to solve (1.3). Though it may sound to be a technical reason, there is, in fact, a fundamental difference between the viscoelastic fluid system and (1.3) which one can see from (2.19) and (2.20). We would like also to point out that it is more tricky to find a mechanism for dissipation in (1.3) than the case of the classical isentropic compressible Navier-Stokes system (CNS), see for example [15].

Remark 1.4. For the three-dimensional version of the system (1.1), we [25] introduced the function spaces $B^{s_{1},s_{2}}$ (see Definition 3.2 below). Due the difficulty in understanding the propagating of anisotropic regularity for the transport equations, where we first established the a priori $L^{1}(\mathbb{R}^{+};B_{2}^{2,-\delta,\delta})$ $(\delta \in \left(\frac{1}{2},1\right))$ estimate for $u^{3}$, the third component of the velocity field, before proving the key $L^{1}(\mathbb{R}^{+};\text{Lip}((\mathbb{R}^{3})))$ estimate of $u^{3}$. In doing so, it was essential that $f^{\alpha} \overset{\text{def}}{=} -\sum_{i,j=1}^{3} \partial_{\alpha}(-\Delta)^{-1}[\partial_{i}u^{j}(\partial_{j}u^{i} + \partial_{i}\partial_{j}(\partial_{i}\psi\partial_{j}\psi))] - \sum_{j=1}^{3} \partial_{j}(\partial_{j}\psi\partial_{j}\psi)$ belongs to $L^{1}(\mathbb{R}^{+};B_{2,1}^{0,\frac{1}{2}}(\mathbb{R}^{3}))$. This latter requirement is essentially equivalent to that $f^{\alpha} \in L^{1}(\mathbb{R}^{+};B_{2,1}^{\frac{1}{2},-\delta}(\mathbb{R}^{3}))$ for $\delta \in \left(\frac{1}{2},1\right)$. In the 2-D case, this would require $f^{2}$ given by (1.3) belonging to $L^{1}(\mathbb{R}^{+};B_{2,1}^{\frac{1}{2},-\delta}(\mathbb{R}^{3}))$ for $\delta \in \left(\frac{1}{2},1\right)$. The latter, however, is impossible due to the product laws in Besov spaces for the vertical variable. This gives another good reason why we will use the Lagrangian formulation of (1.1) in this paper.

Let us complete this section by the notations we shall use in this paper.

Notations. For any $s \in \mathbb{R}$, we denote by $H^{s}(\mathbb{R}^{2})$ the classical $L^{2}$ based Sobolev spaces with the norm $\|u\|_{H^{s}} \overset{\text{def}}{=} \left(\int_{\mathbb{R}^{2}}(1 + |\xi|^{2})^{s}|\hat{u}(\xi)|^{2}d\xi\right)^{\frac{1}{2}}$, while $H^{s}_{\text{loc}}(\mathbb{R}^{2})$ the classical homogenous Sobolev spaces with the norm $\|u\|_{H^{s}_{\text{loc}}(\mathbb{R}^{2})} \overset{\text{def}}{=} \left(\int_{\mathbb{R}^{2}}|\xi|^{|s|2}\hat{u}(\xi)|^{2}d\xi\right)^{\frac{1}{2}}$. Let $A, B$ be two operators, we denote $\|A;B\| = AB - BA$, the commutator between $A$ and $B$. For $a \lesssim b$, we mean that there is a uniform constant $C$, which may be different on different lines, such that $a \leq Cb$, and $a \sim b$ means that both $a \lesssim b$ and $b \lesssim a$. We shall denote by $(a \mid b)$ the $L^{2}(\mathbb{R}^{2})$ inner product of $a$ and $b$. $(d_{j,k})_{j,k \in \mathbb{Z}}$ (resp. $(c_{j})_{j \in \mathbb{Z}}$) will be a generic element of $\ell^{1}(\mathbb{Z}^{2})$ (resp. $\ell^{2}(\mathbb{Z})$) so that $\sum_{j,k \in \mathbb{Z}}d_{j,k} = 1$ (resp. $\sum_{j \in \mathbb{Z}}c_{j}^{2} = 1$). Finally, we denote by $L_{T}^{p}(L_{h}^{q}(L_{r}^{q}(\mathbb{R}_{x})))$ the space $L^{p}([0,T];L^{q}(\mathbb{R}_{x}^{2});L^{r}(\mathbb{R}_{x}))$.

2. Lagrangain Formulation of the System (1.1)

As in [11], [12], [13], [17], [27], [30], [32] and various earlier references therein, we shall use the Lagrangian formulation. At first, we solve the couple system between (1.1) and the following additional transport equation:

\begin{equation}
\partial_{t}\tilde{\phi} + \mathbf{u} \cdot \nabla x\tilde{\phi} = 0, \quad \tilde{\phi}|_{t=0} = \tilde{\phi}_{0},
\end{equation}

where $\tilde{\phi}_{0} = -x_{1} + \tilde{\psi}_{0}$, and $\tilde{\psi}_{0}$ is determined by

\begin{equation}
\det U_{0} = 1 \quad \text{for} \quad U_{0} = \begin{pmatrix}
1 + \partial_{x_{2}}\psi_{0} & \partial_{x_{2}}\tilde{\psi}_{0} \\
-\partial_{x_{1}}\psi_{0} & 1 - \partial_{x_{1}}\tilde{\psi}_{0}
\end{pmatrix}.
\end{equation}

The existence of $\tilde{\psi}_{0}$ will be a consequence of Lemma 6.1.

Setting $\tilde{\phi} = -x_{1} + \tilde{\psi}$ in (2.1) yields

\begin{equation}
\partial_{t}\tilde{\psi} + \mathbf{u} \cdot \nabla \tilde{\psi} - u^{1} = 0, \quad \text{and} \quad \tilde{\psi}|_{t=0} = \tilde{\psi}_{0}.
\end{equation}

The main result concerning the global small solutions to the coupled system (1.1) and (2.1) can be stated as follows:
Theorem 2.1. Let \( s_1 > 1, s_2 \in (-1, -\frac{1}{2}) \). Given \((\phi_0, \tilde{\phi}_0, u_0) \) defined as \((x_2 + \psi_0, -x_1 + \tilde{\psi}_0, u_0) \) satisfying \((\nabla \psi_0, u_0) \in H^{s_1+1}(\mathbb{R}^2) \cap H^{s_2}(\mathbb{R}^2)\), \((\nabla \tilde{\psi}_0, u_0) \in H^{s_1+1}(\mathbb{R}^2) \cap H^{s_2+1}(\mathbb{R}^2)\), the coupled system (1.1) and (2.1) has a unique global solution \((\phi, \psi, u, p) = (x_2 + \psi, -x_1 + \tilde{\psi}, u, p)\) (up to a constant for \(\psi, \psi, p\)) so that

\[
\nabla \psi \in C([0, \infty); \dot{H}^{s_1+1}(\mathbb{R}^2) \cap \dot{H}^{s_2}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1+1}(\mathbb{R}^2) \cap \dot{H}^{s_2+1}(\mathbb{R}^2)),
\]

\[
\nabla \tilde{\psi} \in C([0, \infty); \dot{H}^{s_1+1}(\mathbb{R}^2) \cap \dot{H}^{s_2+1}(\mathbb{R}^2)),
\]

(2.4)

\[
u \in C([0, \infty); \dot{H}^{s_1+1}(\mathbb{R}^2) \cap \dot{H}^{s_2}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1+2}(\mathbb{R}^2) \cap \dot{H}^{s_2+1}(\mathbb{R}^2))
\]

\[
\nabla p \in L^2(\mathbb{R}^+; \dot{H}^{s_1}(\mathbb{R}^2) \cap \dot{H}^{s_2}(\mathbb{R}^2),
\]

provided that

(2.5) \[
\|\nabla \psi\|_{\dot{H}^{s_1+1} \cap \dot{H}^{s_2}} + \|\nabla \tilde{\psi}\|_{\dot{H}^{s_1+1} \cap \dot{H}^{s_2+1}} + \|u_0\|_{\dot{H}^{s_1+1} \cap \dot{H}^{s_2}} \leq c_0
\]

for some \(c_0\) sufficiently small. Furthermore, there holds

(2.6) \[
\|
u\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2})} + \|
abla \psi\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1})} + \|
abla \tilde{\psi}\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1})} + \|u\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{H}^{s_2+1})} + \|
abla u\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2})} + \|
abla p\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2})} \leq C(\|
abla \psi\|_{\dot{H}^{s_1+1} \cap \dot{H}^{s_2}} + \|
abla \tilde{\psi}\|_{\dot{H}^{s_1+1} \cap \dot{H}^{s_2+1}} + \|u_0\|_{\dot{H}^{s_1+1} \cap \dot{H}^{s_2}}).
\]

In order to avoid the difficulty of propagating anisotropic regularity for the free transport equation in the coupled system (1.1) and (2.1), we shall first write them in the Lagrangian coordinates. Indeed let \(U \) defined as \((\partial_x \phi, \partial_x \tilde{\phi})\), we deduce from (1.1) and (2.1) that \(U\) solves

(2.7) \[
\partial_t U + u \cdot \nabla_x U = \nabla_x U, \quad U|_{t=0} = U_0,
\]

with \(U_0\) being given by (2.2).

To write the nonlinear term, \(\operatorname{div}[\nabla \phi \otimes \nabla \phi]\), in the momentum equation of (1.1), into a clear form in the Lagrangian formulation, we need the following lemma concerning the structure of \(U_0\).

Lemma 2.1. Let \(\phi_0(x) = x_2 + \psi_0(x), \tilde{\phi}_0(x) = -x_1 + \tilde{\psi}_0, \) with \(\psi_0 \in \dot{H}^{\tau_1+1}(\mathbb{R}^2) \cap \dot{H}^{\tau_2+1}(\mathbb{R}^2), \tilde{\psi}_0 \in \dot{H}^{\tau_1+1}(\mathbb{R}^2) \cap \dot{H}^{\tau_2+2}(\mathbb{R}^2)\) for \(\tau_1 \in (2, \infty), \tau_2 \in (-1, 0), \) and \(\|
abla \psi_0\|_{L^\infty} + \|
abla \tilde{\psi}_0\|_{L^\infty} \leq \varepsilon_0\) for some \(\varepsilon_0\) sufficiently small. Let \(U_0\) satisfy (2.2). Then there exists \(Y_0 = (Y_0^1, Y_0^2)^T \in \dot{H}^{\tau_1+1}(\mathbb{R}^2) \cap \dot{H}^{\tau_2+2}(\mathbb{R}^2)\) with \(\partial_t Y_0 \in \dot{H}^{\tau_2}(\mathbb{R}^2)\) so that \(X_0(y) \) defined as \(y + Y_0(y)\) satisfies

(2.8) \[
U_0 \circ X_0(y) = \nabla_y X_0(y) = I + \nabla_y Y_0(y).
\]

Moreover, there holds

(2.9) \[
\|Y_0\|_{\dot{H}^{\tau_1+1} \cap \dot{H}^{\tau_2+2}} + \|\partial_t Y_0\|_{\dot{H}^{\tau_2}} \leq C(\|
abla \psi_0\|_{\dot{H}^{\tau_1+1} \cap \dot{H}^{\tau_2+2}} + \|
abla \tilde{\psi}_0\|_{\dot{H}^{\tau_1+1} \cap \dot{H}^{\tau_2+2}} + \|
abla \psi_0\|_{\dot{H}^{\tau_1+1} \cap \dot{H}^{\tau_2+2}} + \|
abla \tilde{\psi}_0\|_{\dot{H}^{\tau_1+1} \cap \dot{H}^{\tau_2+2}}),
\]

where \(C(\lambda_1, \lambda_2)\) is a constant depending on \(\lambda_1\) and \(\lambda_2\) non-decreasingly.
Proof. Let $Y = (Y^1, Y^2)^T$, we denote
\[
F(y, Y) \overset{\text{def}}{=} \psi_0(y + Y) + Y^2, \quad G(y, Y) \overset{\text{def}}{=} \tilde{\psi}_0(y + Y) - Y^1.
\]
Notice from the assumption $\det U_0 = 1$ that
\[
\det \frac{\partial (F, G)}{\partial (Y^1, Y^2)} = \det \left( \begin{array}{cc}
\partial_{x_1} \psi_0 & 1 + \partial_{x_2} \psi_0 \\
\partial_{x_2} \psi_0 & \partial_{x_2} \psi_0
\end{array} \right) \bigg|_{x = y + Y} = \det U_0(y + Y) = 1,
\]
from which and the classical implicit function theorem, we deduce that around every point $y$, the functions $F(y, Y) = 0$ and $G(y, Y) = 0$ determines a unique function $Y_0(y) = (Y_0^1(y), Y_0^2(y))^T$ so that
\[
F(y, Y_0(y)) = 0 = G(y, Y_0(y)),
\]
or equivalently
\[
Y_0^1(y) = \tilde{\psi}_0(y + Y_0(y)) \quad \text{and} \quad Y_0^2(y) = -\psi_0(y + Y_0(y)).
\]
Moreover, there holds
\[
\begin{cases}
\partial_{y_1} Y_0^1(y) = \partial_{x_1} \tilde{\psi}_0 \circ (y + Y(y))(1 + \partial_{y_1} Y_0^1(y)) + \partial_{x_2} \tilde{\psi}_0 \circ (y + Y(y)) \partial_{y_1} Y_0^2(y), \\
\partial_{y_2} Y_0^1(y) = \partial_{x_1} \tilde{\psi}_0 \circ (y + Y(y)) \partial_{y_2} Y_0^1(y) + \partial_{x_2} \tilde{\psi}_0 \circ (y + Y(y))(1 + \partial_{y_2} Y_0^2(y)), \\
\partial_{y_1} Y_0^2(y) = -\partial_{x_1} \psi_0 \circ (y + Y(y))(1 + \partial_{y_1} Y_0^1(y)) - \partial_{x_2} \psi_0 \circ (y + Y(y)) \partial_{y_1} Y_0^2(y), \\
\partial_{y_2} Y_0^2(y) = -\partial_{x_1} \tilde{\psi}_0 \circ (y + Y(y)) \partial_{y_2} Y_0^1(y) - \partial_{x_2} \psi_0 \circ (y + Y(y))(1 + \partial_{y_2} Y_0^2(y)).
\end{cases}
\]
Thanks to (2.12) and $\det U_0 = 1$, we infer
\[
\begin{cases}
\partial_{y_1} Y_0^1(y) = \partial_{x_2} \psi_0 \circ (y + Y_0(y)), \\
\partial_{y_2} Y_0^1(y) = \partial_{x_2} \tilde{\psi}_0 \circ (y + Y_0(y)), \\
\partial_{y_1} Y_0^2(y) = -\partial_{x_1} \psi_0 \circ (y + Y_0(y)), \\
\partial_{y_2} Y_0^2(y) = -\partial_{x_1} \tilde{\psi}_0 \circ (y + Y_0(y)),
\end{cases}
\]
which implies (2.8). Moreover, it follows from (2.13) that
\[
\nabla_x (X_0^{-1}(x) - x) = (I + \nabla_y Y_0)^{-1} \circ X_0^{-1}(x) - I = \left( \begin{array}{cc}
-\partial_{x_1} \tilde{\psi}_0 & -\partial_{x_2} \tilde{\psi}_0 \\
\partial_{x_1} \psi_0 & \partial_{x_2} \psi_0
\end{array} \right).
\]
Then using Lemma A.1 with $\Phi = X_0^{-1}(x)$, we deduce from (2.13) that for $\tau_1 \in (2, \infty)$ and $\tau_2 \in (-1, 0)$,
\[
\begin{align*}
\|\nabla_y Y_0\|_{H^\tau_1} & \leq C(\|\nabla_x \psi_0\|_{L^\infty}, \|\nabla_x \tilde{\psi}_0\|_{L^\infty})(1 + \|\Delta_x \psi_0\|_{H^{\tau_1 - 2}} + \|\Delta_x \tilde{\psi}_0\|_{H^{\tau_1 - 2}}) \times (\|\Delta_x \psi_0\|_{H^{\tau_1 - 1}} + \|\Delta_x \tilde{\psi}_0\|_{H^{\tau_1 - 1}}), \\
\|\nabla_y Y_0\|_{H^{\tau_2 + 1}} & \leq C(\|\nabla_x \psi_0\|_{L^\infty}, \|\nabla_x \tilde{\psi}_0\|_{L^\infty})(\|\nabla_x \psi_0\|_{H^{\tau_2 + 1}} + \|\nabla_x \tilde{\psi}_0\|_{H^{\tau_2 + 1}}), \\
\|\partial_{y_1} Y_0\|_{H^{\tau_2}} & \leq C(\|\nabla_x \psi_0\|_{L^\infty}, \|\nabla_x \tilde{\psi}_0\|_{L^\infty})(\|\nabla_x \psi_0\|_{H^{\tau_2}} + (\|\nabla_x \psi_0\|_{H^{\tau_2 + 1}} + \|\nabla_x \tilde{\psi}_0\|_{H^{\tau_2 + 1}})\|\nabla_x \psi_0\|_{L^2}),
\end{align*}
\]
which along with Sobolev imbedding theorem ensures (2.9). This concludes the proof of Lemma 2.1. \qed
With Lemma 2.1, for $(\psi_0, \dot{v}_0)$ given by Theorem 2.1 and $U_0$ by (2.2), there exists $Y_0 = (Y_0^1, Y_0^2)^T$ so that $\nabla Y_0 \in H^{s+1}(\mathbb{R}^2) \cap \dot{H}^{s+1}(\mathbb{R}^2)$, $\partial_t Y_0 \in \dot{H}^{s}(\mathbb{R}^2)$, and there hold (2.8) and (2.9). With $X_0(y) = y + Y_0(y)$ thus obtained, we define the flow map $X(t, y)$ by

$$
\begin{cases}
\frac{dX(t, y)}{dt} = u(t, X(t, y)), \\
X(t, y)|_{t=0} = X_0(y),
\end{cases}
$$

and $Y(t, y)$ through

$$X(t, y) = X_0(y) + \int_0^t u(s, X(s, y)) \, ds \overset{\text{def}}{=} y + Y(t, y).$$

Then thanks to Lemma 1.4 of [26] and (2.8), we deduce from (2.7) that

$$U(t, X(t, y)) = \nabla_y X(t, y) = I + \nabla_y Y(t, y) \quad \text{and} \quad \det (I + \nabla_y Y(t, y)) = 1. \quad (2.15)$$

Denoting $U(t, X(t, y)) \overset{\text{def}}{=} (a_{ij})_{i,j=1,2}$ and $U^{-1}(t, X(t, y)) = (I + \nabla_y Y(t, y))^{-1} \overset{\text{def}}{=} (b_{ij})_{i,j=1,2}$. It is easy to check that, as det $U = 1$, $(b_{ij})_{i,j=1,2} = A_Y$ with

$$A_Y \overset{\text{def}}{=} \begin{pmatrix}
1 + \partial_{y_2} Y^2 & -\partial_{y_2} Y^1 \\
-\partial_{y_1} Y^2 & 1 + \partial_{y_2} Y^1
\end{pmatrix}, \quad (2.16)$$

is the adjoint matrix of $(I + \nabla_y Y)$, and $\sum_{i=1}^2 \partial_{y_i} Y^i = 0$. Furthermore, let $b \overset{\text{def}}{=} (\partial_x \phi, -\partial_y \phi)^T$, it follows from (2.15) that

$$b \circ X(t, y) = (1 + \partial_{y_1} Y^1, \partial_{y_1} Y^2)^T \quad \text{and} \quad A_Y(b \circ X) = (1,0)^T, \quad (2.17)$$

and consequently one has

$$[-\text{div}_x (\nabla_x \phi \otimes \nabla_x \phi) + \nabla_x(|\nabla_x \phi|^2)] \circ X(t, y) = [-\text{div}_x (b \otimes b)] \circ X(t, y) = \nabla_y \cdot [A_Y(b \circ X) \otimes (b \circ X)] = \partial_{y_1} (b \circ X) = \partial_{y_1}^2 Y(t, y),$$

that is

$$[-\text{div}_x(\nabla_x \phi \otimes \nabla_x \phi) + \nabla_x(|\nabla \phi|^2)](t, X(t, y)) = \partial_{y_1}^2 Y(t, y). \quad (2.18)$$

With (2.14) and (2.18), we can reformulate (1.1) and (2.1) as

$$
\begin{cases}
Y_{tt} - \nabla_Y \cdot \nabla_Y Y_t - \partial_{y_1}^2 Y + \nabla Y q = 0, \\
\nabla_Y \cdot Y_t = 0, \\
Y|_{t=0} = Y_0, \quad Y_t|_{t=0} = u_0 \circ X_0(y) \overset{\text{def}}{=} Y_1,
\end{cases}
$$

where $q(t, y) = (p + |\nabla \phi|^2) \circ X(t, y)$ and $\nabla_Y = A_Y^T \nabla_y$ with $A_Y$ being given by (2.16). Here and in what follows, we always assume that $||\nabla Y||_{L_\infty} \leq \frac{1}{2}$. Under this assumption, we rewrite (2.19) as

$$
\begin{cases}
Y_{tt} - \Delta_y Y_t - \partial_{y_1}^2 Y = f(Y, q), \\
\nabla_y \cdot Y = \rho(Y), \\
Y|_{t=0} = Y_0, \quad Y_t|_{t=0} = Y_1,
\end{cases}
$$

where $f(Y, q) = q - \partial_{y_1}^2 Y - (p + |\nabla \phi|^2) \circ X(t, y)$.
where
\[ f(Y, q) = (\nabla Y \cdot \nabla Y - \Delta_y) Y_t - \nabla Y q, \]
\[ \rho(Y) = \nabla Y \cdot Y_0 - \int_0^t (\nabla Y - \nabla Y) \cdot Y_s ds = \partial_{y_1} Y^2 \partial_{y_2} Y^1 - \partial_{y_1} Y^1 \partial_{y_2} Y^2. \]

Here we have used the fact that \( \det (I + \nabla Y Y_0) = \det U_0 = 1 \) and (2.16) to derive the second equality of (2.21). Indeed, thanks to (2.16) and \( \det (I + \nabla Y Y_0) = 1 \), one has
\[ (\nabla Y - \nabla Y) \cdot Y_t = \frac{d}{dt} (\partial_{y_1} Y^1 \partial_{y_2} Y^2 - \partial_{y_1} Y^2 \partial_{y_2} Y^1) \]
\[ = \frac{d}{dt} (\det (I + \nabla Y Y) - 1 - \nabla Y \cdot Y). \]

Furthermore, (2.22) ensures that the equation \( \nabla Y \cdot Y = \rho(Y) \) would imply \( \det (I + \nabla Y Y) = 1 \) and \( \nabla Y \cdot Y_t = 0 \).

For notational convenience, we shall neglect the subscripts \( x \) or \( y \) in \( \partial, \nabla \) and \( \Delta \) in what follows. We make the convention that whenever \( \nabla \) acts on \( (\psi, u, p) \), which is a solution to (1.3), we understand \( (\nabla \psi, \nabla u, \nabla p) \) as \( (\nabla_x \psi, \nabla_x u, \nabla_x p) \). While \( \nabla \) acts on \( (Y, q) \), the solution to (2.20), we understand \( (\nabla Y, \nabla q) \) as \( (\nabla_y Y, \nabla_y q) \). Similar conventions for \( \partial \) and \( \Delta \).

For (2.20)-(2.21), we have the following global wellposedness result:

**Theorem 2.2.** Let \( s_1 > 1, s_2 \in (-1, -\frac{1}{2}) \). Let \( (Y_0, Y_1) \) satisfy \( \partial_{\hat{t}} Y_0, \Delta Y_0 \in \dot{H}^{s_1}(\mathbb{R}^2) \cap \dot{H}^{s_2}(\mathbb{R}^2) \), \( Y_1 \in \dot{H}^{s_1+1}(\mathbb{R}^2) \cap \dot{H}^{s_2}(\mathbb{R}^2) \) and
\[ \det (I + \nabla Y_0) = 1, \quad \nabla Y_0 \cdot Y_1 = 0, \quad \text{and} \]
\[ \| \partial_{\hat{t}} Y_0 \|_{\dot{H}^{s_2}} + \| \Delta Y_0 \|_{\dot{H}^{s_1} \cap \dot{H}^{s_2}} + \| Y_1 \|_{\dot{H}^{s_1+1} \cap \dot{H}^{s_2}} \leq \epsilon_0 \]
for some \( \epsilon_0 \) sufficiently small. Then (2.20)-(2.21) has a unique global solution \( (Y, q) \) (up to a constant for \( q \)) so that
\[ Y \in C([0, \infty); \dot{H}^{s_1+2} \cap \dot{H}^{s_2+2}(\mathbb{R}^2)) \quad \text{and} \quad Y^2 \in C([0, \infty); \dot{H}^{s_2+1}(\mathbb{R}^2)), \]
\[ \partial_{\hat{t}} Y \in C([0, \infty); \dot{H}^{s_2}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1}(\mathbb{R}^2)), \]
\[ Y_1 \in C([0, \infty); \dot{H}^{s_1+1} \cap \dot{H}^{s_2}(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{H}^{s_2+1}(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; Lip(\mathbb{R}^2)), \]
\[ \nabla q \in L^2(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2} \cap \dot{H}^{s_2}(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^2)). \]

Moreover, there hold \( \det (I + \nabla Y) = 1, \quad \nabla Y \cdot Y_t = 0, \quad \text{and} \)
\[ \| Y \|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{H}^{s_2+2})} + \| \partial_{\hat{t}} Y \|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_2})} + \| Y^2 \|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_2+1})} \]
\[ + \| Y_1 \|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2})} + \| \partial_{\hat{t}} Y \|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{H}^{s_2+1})} + \| Y_1 \|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+3} \cap \dot{H}^{s_2+2})} \]
\[ + \| \nabla q \|_{L^2(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2})} + \| \nabla^2 q \|_{L^2(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2})} + \| Y_1 \|_{L^1(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{H}^{s_2})} \]
\[ \leq C(\| \partial_{\hat{t}} Y_0 \|_{\dot{H}^{s_2+2}} + \| \Delta Y_0 \|_{\dot{H}^{s_1} \cap \dot{H}^{s_2}} + \| Y_1 \|_{\dot{H}^{s_1+2} \cap \dot{H}^{s_2+2}}). \]

**Remark 2.1.** As for the Eulerian formulation (1.3), it is also technical to explore the delicate mechanism of partial dissipations in (2.20). We overcome this difficulty by applying the anisotropic Littlewood-Paley theory. Since we use the Lagrangian formulation (2.20) instead of Eulerian one (1.3), we can avoid the difficulty concerning the propagation anisotropic regularity for the transport equation, which we encountered in [25]. In general, it is interesting to study how the anisotropic Littlewood-Paley theory can be applied to these evolution equations with degenerations of certain ellipticity (parabolicity) in phase variables.
Remark 2.2. We note once again that the equation $\nabla \cdot Y = \rho(Y) = \partial_1 Y^2 \partial_2 Y^1 - \partial_1 Y^1 \partial_2 Y^2$, which implies the incompressibility condition $\det (I + \nabla Y) = 1$, plays a key role in the proof of Theorem 2.2. In particular, this equation ensures the global in time $L^1$-estimates of $\nabla q$ and $\nabla Y_t$, which are crucial in order to close the energy estimates for (2.20)-(2.21). Furthermore, this equation provides better estimates for the component $Y^2$ than that of $Y^1$.

Scheme of the proofs.

To avoid the difficulty caused by propagating anisotropic regularity for the free transport equation, we shall first prove Theorem 2.2, which concerns the global wellposedness in the Lagrangian formulation (2.20)-(2.21) for the coupled system (1.1) and (2.1).

Indeed let $(Y, q)$ be a smooth enough solution of (2.20), applying standard energy estimate to (2.20) gives rise to

$$\frac{d}{dt} \left\{ \frac{1}{2} \left( \|Y_t\|_{H^s}^2 + \|Y_t\|_{H^{s+1}}^2 + \|\partial_1 Y^2\|_{H^s}^2 + \|\partial_1 Y^1\|_{H^{s+1}}^2 + \frac{1}{4} \|Y^1\|_{H^{s+2}}^2 \right) - \frac{1}{4} \langle Y_t, \nabla Y \rangle_{H^s} \right\}$$

(2.27)

$$= (f | Y_t - \frac{1}{4} \Delta Y - \Delta Y_t)_{H^s},$$

where $f$ is given by (2.21) and $(a | b)_{H^s}$ denotes the standard $H^s$ inner product of $a$ and $b$. (2.27) shows that $\partial_1 Y$ belongs to $L^2(\mathbb{R}^+; H^{s+1}(\mathbb{R}^2))$. After a careful check, to close the energy estimate (2.27), we need also the $L^1(\mathbb{R}^+; Lip(\mathbb{R}^2))$ estimate of $Y_t$. Toward this, we investigate first the spectrum properties to the following linearized system of (2.20)-(2.21):

$$\begin{cases}
Y_t - \Delta Y_t - \partial_1^2 Y = f, \\
Y|_{t=0} = Y_0, \quad Y_t|_{t=0} = Y_1.
\end{cases}$$

(2.28)

Simple calculation shows that the symbolic equation of (2.28) has eigenvalues $\lambda_\pm(\xi)$ given by (3.1) and they satisfy (3.2). This shows that smooth solution of (2.28) decays in a very subtle way. In order to capture this delicate decay property for the solutions of (2.28), we will have to decompose our frequency analysis into two parts: $\{ \xi = (\xi_1, \xi_2) : |\xi|^2 \leq 2|\xi_1| \}$ and $\{ \xi = (\xi_1, \xi_2) : |\xi|^2 > 2|\xi_1| \}$. It suggests us to use anisotropic Littlewood-Paley analysis to obtain the $L^1(\mathbb{R}^+; Lip(\mathbb{R}^2))$ estimate of $Y_t$.

With Theorem 2.2, we can prove Theorem 2.1 through coordinate transformation, namely from Lagrangian coordinates to Eulerian ones. Finally thanks to Lemma 6.1, for $s > 2$ and $\nabla \psi_0 \in H^s(\mathbb{R}^2)$, there exists $\tilde{\psi}_0$ so that there hold (2.2) and (6.16). We thus obtained $(\psi_0, \tilde{\psi}_0)$ and the initial velocity field $u_0$ given by Theorem 1.1, we infer from Theorem 2.1 that the coupled system between (1.1) and (2.1) has a unique solution $(\phi, \tilde{\phi}, u, p)$, which satisfies (2.4) and (2.6). In particular, $(\phi, u, p)$ solves (1.1) and we complete the proof of Theorem 1.1.

The rest of the paper is organized as follows. In the first part of Section 3, we shall present a heuristic analysis to the linearized system of (2.20)-(2.21), which motivates us to use anisotropic Littlewood-Paley theory below, then we shall collect some basic facts on Littlewood-Paley analysis in Subsection 3.2. In Section 4, we apply anisotropic Littlewood-Paley theory to explore the dissipative mechanism for a linearized model of (2.20)-(2.21). In Section 5, we present the proof of Theorem 2.2 and then Theorems 2.1 and 1.1 in Section 6. Finally, we present the proofs of some technical lemmas in the Appendices.
3. Preliminary

3.1. Spectral analysis to the linearized system of (2.20). Before dealing with the full system (2.20)-(2.21), we shall make some heuristic analysis to the linearized system (2.28). Observe that the symbolic equation of (2.28) reads
\[ \lambda^2 + |\xi|^2 \lambda + \xi_1^2 = 0 \quad \text{for} \quad \xi = (\xi_1, \xi_2). \]
It is easy to calculate that this equation has two different eigenvalues
\[ \lambda_{\pm} = -\frac{|\xi|^2 \pm \sqrt{|\xi|^4 - 4\xi_1^2}}{2}. \]
The Fourier modes corresponding to \( \lambda_{\pm} \) decays like \( e^{-t|\xi|^2} \). Whereas the decay property of the Fourier modes corresponding to \( \lambda_{\pm} \) varies with directions of \( \xi \) as
\[ \lambda_-(\xi) = -\frac{2\xi_1^2}{|\xi|^2(1 + \sqrt{1 - \frac{4\xi_1^2}{|\xi|^4}})} \rightarrow -1 \quad \text{as} \quad |\xi| \rightarrow \infty \]
only in the \( \xi_1 \) direction. This simple analysis shows that the dissipative properties of the solutions to (2.28) may be more complicated than that for the linearized system of isentropic compressible Navier-Stokes system in [15]. It also suggests us to employ the tool of anisotropic Littlewood-Paley theory, which has been used in the study of the global wellposedness to 3-D anisotropic incompressible Navier-Stokes equations [7, 8, 18, 28, 29, 33], and in [25] to explore the dissipative properties to the three-dimensional case of (1.3). One may check Section 4 below for the detailed rigorous analysis corresponding to this scenario.

3.2. Littlewood-Paley theory. The proof of Theorem 2.1 requires a dyadic decomposition of the Fourier variables, or the Littlewood-Paley decomposition. For the convenience of the readers, we recall some basic facts on Littlewood-Paley theory from [2]. Let \( \varphi \) and \( \chi \) be smooth functions supported in \( C \equiv \{ \tau \in \mathbb{R}^+, \ \frac{3}{4} \leq \tau \leq \frac{4}{3} \} \) and \( B \equiv \{ \tau \in \mathbb{R}^+, \ \tau \leq \frac{4}{3} \} \) such that
\[ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1 \quad \text{for} \quad \tau > 0 \quad \text{and} \quad \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1. \]
For \( a \in \mathcal{S}'(\mathbb{R}^2) \), we set
\[ \Delta_k a \equiv \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_1|)\hat{a}), \quad S_k a \equiv \mathcal{F}^{-1}(\chi(2^{-k}|\xi_1|)\hat{a}), \quad \Delta_k a \equiv \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_2|)\hat{a}), \quad S_k a \equiv \mathcal{F}^{-1}(\chi(2^{-k}|\xi_2|)\hat{a}), \quad \Delta_j a \equiv \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\hat{a}), \quad S_j a \equiv \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\hat{a}), \]
where \( \mathcal{F}a \) and \( \hat{a} \) denote the Fourier transform of the distribution \( a \). The dyadic operators defined in (3.3) satisfy the property of almost orthogonality:
\[ \Delta_k \Delta_j a \equiv 0 \quad \text{if} \quad |k - j| \geq 2 \quad \text{and} \quad \Delta_k (S_{j-1} a \Delta_j b) \equiv 0 \quad \text{if} \quad |k - j| \geq 5. \]
Similar properties hold for \( \Delta_k^\gamma \) and \( \Delta_j^\gamma \).

In what follows, we shall frequently use the following anisotropic type Bernstein inequalities:

**Lemma 3.1.** Let \( B_h \) (resp. \( B_v \)) a ball of \( \mathbb{R}_h \) (resp. \( \mathbb{R}_v \)), and \( C_h \) (resp. \( C_v \)) a ring of \( \mathbb{R}_h \) (resp. \( \mathbb{R}_v \)); let \( 1 \leq p_2 \leq p_1 \leq \infty \) and \( 1 \leq q_2 \leq q_1 \leq \infty \). Then there holds:
If the support of $\hat{a}$ is included in $2^k B_h$, then
$$\|\partial_{x_1}^\alpha a\|_{L^2 \left( \mathbb{R}^2 \right)} \lesssim 2^{k \left( \alpha + \left( \frac{1}{p} - \frac{1}{r} \right) \right)} \|a\|_{L^2 \left( \mathbb{R}^2 \right)}.$$ 

If the support of $\hat{a}$ is included in $2^l B_v$, then
$$\|\partial_{x_2}^\beta a\|_{L^2 \left( \mathbb{R}^2 \right)} \lesssim 2^{l \left( \beta + \left( \frac{1}{q} - \frac{1}{p} \right) \right)} \|a\|_{L^2 \left( \mathbb{R}^2 \right)}.$$ 

If the support of $\hat{a}$ is included in $2^k C_h$, then
$$\|a\|_{L^2 \left( \mathbb{R}^2 \right)} \lesssim 2^{-kN} \|\partial_{x_1}^\alpha a\|_{L^2 \left( \mathbb{R}^2 \right)}.$$ 

If the support of $\hat{a}$ is included in $2^l C_v$, then
$$\|a\|_{L^2 \left( \mathbb{R}^2 \right)} \lesssim 2^{-lN} \|\partial_{x_2}^\beta a\|_{L^2 \left( \mathbb{R}^2 \right)}.$$ 

**Definition 3.1.** Let $(p, r) \in [1, +\infty]^2$, $s \in \mathbb{R}$ and $u \in S'(\mathbb{R}^2)$, we set
$$\|u\|_{\dot{B}_{p, r}^s} \overset{\text{def}}{=} \left( 2^{qs} \|\Delta_j u\|_{L^p} \right)_{j \in \mathbb{Z}}.$$ 

- For $s < \frac{2}{p}$ (or $s = \frac{2}{p}$ if $r = 1$), we define $\dot{B}_{p, r}^s(\mathbb{R}^2) \overset{\text{def}}{=} \{ u \in S'(\mathbb{R}^2) \mid \|u\|_{\dot{B}_{p, r}^s} < \infty \}.$
- If $k \in \mathbb{N}$ and $\frac{2}{p} + k - 1 \leq s < \frac{2}{p} + k$ (or $s = \frac{2}{p} + k$ if $r = 1$), then $\dot{B}_{p, r}^s(\mathbb{R}^2)$ is defined as the subset of distribution $u \in S'(\mathbb{R}^2)$ such that $\partial^\beta u \in \dot{B}_{p, r}^{-k}(\mathbb{R}^2)$ whenever $|\beta| = k$.

**Remark 3.1.** (1) It is easy to observe that $\dot{B}_{2, 2}^s(\mathbb{R}^2) = \dot{H}^s(\mathbb{R}^2)$.

(2) Let $(p, r) \in [1, +\infty]^2$, $s \in \mathbb{R}$ and $u \in S'(\mathbb{R}^2)$. Then $u \in \dot{B}_{p, r}^s(\mathbb{R}^2)$ if and only if there exists $\{c_{j, r}\}_{j \in \mathbb{Z}}$ such that $\|c_{j, r}\|_{c} = 1$ and
$$\|\Delta_j u\|_{L^p} \leq C c_{j, r} 2^{-j s} \|u\|_{\dot{B}_{p, r}^s} \quad \text{for all} \quad j \in \mathbb{Z}.$$ 

(3) Let $s, s_1, s_2 \in \mathbb{R}$ with $s_1 < s < s_2$ and $u \in \dot{H}^{s_1}(\mathbb{R}^2) \cap \dot{H}^{s_2}(\mathbb{R}^2)$. Then $u \in \dot{B}_{2, 1}^{s_1}(\mathbb{R}^2)$, and there holds
$$\|u\|_{\dot{B}_{2, 1}^{s_1}} \lesssim \|u\|_{\dot{H}^{s_1}} \|u\|_{\dot{H}^{s_2}} \lesssim \|u\|_{\dot{H}^{s_1}} + \|u\|_{\dot{H}^{s_2}}.$$ 

To derive the $L^1(\mathbb{R}^+; Lip(\mathbb{R}^2))$ estimate of $Y_1$ determined by (2.28), we need the following two-dimensional version of the anisotropic Besov type space introduced in [25]:

**Definition 3.2.** Let $s_1, s_2 \in \mathbb{R}$ and $u \in S'(\mathbb{R}^2)$, we define the norm
$$\|u\|_{\dot{B}_{1, 2}^{s_1, s_2}} \overset{\text{def}}{=} \sum_{j, k \in \mathbb{Z}^2} 2^{js_1 + ks_2} \|\Delta_j \Delta_k^u\|_{L^2}.$$ 

Then motivated by the proof of Lemma 2.1 in [25], we have the following improved version:

**Lemma 3.2.** Let $s_1, s_2, \tau_1, \tau_2 \in \mathbb{R}$, which satisfy $s_1 < \tau_1 + \tau_2 < s_2$ and $\tau_2 > 0$. Then for $a \in \dot{H}^{s_1}(\mathbb{R}^2) \cap \dot{H}^{s_2}(\mathbb{R}^2)$, $a \in \dot{B}_{\tau_1, \tau_2}^{s_1, s_2}(\mathbb{R}^2)$ and there holds
$$\|a\|_{\dot{B}_{1, 2}^{s_1 + s_2}} \lesssim \|a\|_{\dot{B}_{\tau_1 + \tau_2}^{s_1 + s_2}} \lesssim \|a\|_{\dot{H}^{s_1}} + \|a\|_{\dot{H}^{s_2}}.$$
Proof. Indeed thanks to Definition 3.2 and the fact: \( j \geq k - N_0 \) for some fixed positive integer \( N_0 \) in dyadic operator \( \Delta_j \Delta^h_k \), we have
\[
\|a\|_{B^{1, \tau_2}} \leq \sum_{k \leq j + N_0} 2^{2j+2k\tau_2} \|\Delta_j \Delta^h_k a\|_{L^2} \\
\leq \sum_{j \in \mathbb{Z}} 2^{j\tau_2} \|\Delta_j a\|_{L^2} \sum_{k \leq j + N_0} 2^{k\tau_2} \\
\leq \sum_{j \in \mathbb{Z}} 2^{j(\tau_1 + \tau_2)} \|\Delta_j a\|_{L^2} \lesssim \|a\|_{B^{2, \tau_1 + \tau_2}}.
\]
which together with (3.5) completes the proof of the lemma.

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we will use Chemin-Lerner type spaces \( \tilde{L}^\lambda_T(B^s_{p,r} (\mathbb{R}^2)) \) (see [2] for instance).

**Definition 3.3.** Let \( (r, \lambda, p) \in [1, +\infty]^3 \) and \( T \in (0, +\infty] \). We define the \( \tilde{L}^\lambda_T(B^s_{p,r} (\mathbb{R}^2)) \) norm by
\[
\|u\|_{\tilde{L}^\lambda_T(B^s_{p,r})} = \left( \sum_{j \in \mathbb{Z}} 2^{jrs} \left( \int_0^T \|\Delta_j u(t)\|_{L^p}^\lambda \, dt \right)^{\frac{1}{\lambda}} \right)^{\frac{1}{s}} < \infty,
\]
with the usual change if \( r = \infty \). For short, we just denote this space by \( \tilde{L}^\lambda_T(B^s_{p,r}) \).

**Remark 3.2.** Corresponding to Definitions 3.2 and 3.3, we define the norm
\[
\|u\|_{\tilde{L}^\lambda_T(\tilde{B}^{s, \tau_2}_{r, r})} = \left( \sum_{j,k \in \mathbb{Z}} 2^{jks} \|\Delta_j \Delta^h_k u\|_{L^2_{\tilde{J}}(L^2)} \right)^{\frac{1}{s}} < \infty.
\]
Then it follows from the proof of Lemma 3.2 that
\[
\|u\|_{\tilde{L}^\lambda_T(\tilde{B}^{s, \tau_2}_{r, r})} \lesssim \|u\|_{\tilde{L}^\lambda_T(\tilde{B}^{s, \tau_1 + \tau_2}_{r, r})} \lesssim \|u\|_{\tilde{L}^\lambda_T(\tilde{B}^{s, \tau_1}_{r, r})} + \|u\|_{L^2(\tilde{B}^{s, \tau_2}_{r, r})},
\]
with \( \tau_1, \tau_2 \) and \( s_1, s_2 \) satisfying the assumptions of Lemma 3.2.

We also need both the isotropic and anisotropic versions of para-differential decomposition of Bony [3]. We first recall the isotropic para-differential decomposition from [3]: let \( a, b \in \mathcal{S}'(\mathbb{R}^2) \),
\[
ab = T(a, b) + R(a, b), \quad \text{or} \quad \ab = T(a, b) + \bar{T}(a, b) + R(a, b), \quad \text{where}
T(a, b) = \sum_{j \in \mathbb{Z}} S_{j-1} a \Delta_j b, \quad \bar{T}(a, b) = T(b, a), \quad R(a, b) = \sum_{j \in \mathbb{Z}} \Delta_j a S_{j+2} b, \quad \text{and}
\]
\[
R(a, b) = \sum_{j \in \mathbb{Z}} \Delta_j a \bar{\Delta} j b, \quad \text{with} \quad \bar{\Delta} j b = \sum_{\ell = j}^{j+1} \Delta \ell b.
\]
We shall also use the following anisotropic version of Bony’s decomposition for the horizontal variables:
\[
ab = T^h(a, b) + R^h(a, b), \quad \text{or} \quad \ab = T^h(a, b) + \bar{T}^h(a, b) + R^h(a, b), \quad \text{where}
T^h(a, b) = \sum_{k \in \mathbb{Z}} S^h_{k-1} a \Delta^h_k b, \quad \bar{T}^h(a, b) = T^h(b, a), \quad R^h(a, b) = \sum_{k \in \mathbb{Z}} \Delta^h k a S^h_{k+2} b, \quad \text{and}
\]
\[
R^h(a, b) = \sum_{k \in \mathbb{Z}} \Delta^h k a \bar{\Delta}^h k b, \quad \text{with} \quad \bar{\Delta}^h k b = \sum_{\ell = k}^{k+1} \Delta^h \ell b.
\]
Considering the special structure of the functions in $B^{s_1,s_2}(\mathbb{R}^2)$, we sometimes use both (3.7) and (3.8) simultaneously.

As an application of the above basic facts on Littlewood-Paley theory, we prove the following product law in space $B^{s_1,s_2}(\mathbb{R}^2)$ given by Definition 3.2.

**Lemma 3.3.** Let $s_1, s_2, \tau_1, \tau_2 \in \mathbb{R}$, which satisfy $s_1, s_2 \leq \frac{1}{2}$, $\tau_1, \tau_2 \leq \frac{1}{2}$ and $s_1 + s_2 > 0$, $\tau_1 + \tau_2 > 0$. Then for $a \in B^{s_1, \tau_1}(\mathbb{R}^2)$ and $b \in B^{s_2, \tau_2}(\mathbb{R}^2)$, $ab \in B^{s_1+s_2-\frac{3}{2}, \tau_1+\tau_2-\frac{3}{2}}(\mathbb{R}^2)$ and there holds

\[
(3.9) \quad \|ab\|_{B^{s_1+s_2-\frac{3}{2}, \tau_1+\tau_2-\frac{3}{2}}} \lesssim \|a\|_{B^{s_1, \tau_1}}\|b\|_{B^{s_2, \tau_2}}.
\]

**Proof.** We first get by using Bony’s decompositions (3.7) and (3.8) that

\[
(3.10) \quad ab = (TT^h + TT^h + TR^h + \overline{TT}^h + \overline{TR}^h + RT^h + RT^h + RR^h)(a, b).
\]

We shall present the detailed estimates to typical terms above. Indeed applying Lemma 3.1 gives

\[
\|\Delta_j \Delta_k^h (TR^h(a, b))\|_{L^2} \lesssim 2^j \sum_{|j'| \leq j \leq k} \|S_{j'-1} \Delta_k^h a\|_{L^\infty} \|\Delta_j \Delta_k^h b\|_{L^2}
\]

\[
\lesssim 2^j \sum_{|j'| \leq j \leq k} d_{j', k} 2^{j'(\frac{1}{2}-s_1-s_2)} 2^{-k'(\tau_1+\tau_2)} \|a\|_{B^{s_1, \tau_1}} \|b\|_{B^{s_2, \tau_2}}
\]

\[
\lesssim d_{j, k} 2^{-j(s_1+s_2-\frac{1}{2})} 2^{-k(\tau_1+\tau_2-\frac{1}{2})} \|a\|_{B^{s_1, \tau_1}} \|b\|_{B^{s_2, \tau_2}},
\]

due to the fact: $s_1 + s_2 > 0$ and $\tau_1 + \tau_2 > 0$. The estimate to the remaining terms in (3.10) is identical, and we omit the details here.

Whence thanks to (3.10), we arrive at

\[
\|\Delta_j \Delta_k^h (ab)\|_{L^2} \lesssim d_{j, k} 2^{-j(s_1+s_2-\frac{1}{2})} 2^{-k(\tau_1+\tau_2-\frac{1}{2})} \|a\|_{B^{s_1, \tau_1}} \|b\|_{B^{s_2, \tau_2}},
\]

which implies (3.9). This concludes the proof of Lemma 3.3. \qed

We finish this section by some product laws in $\dot{H}^s(\mathbb{R}^2)$ and $\dot{B}^{s}_{2,1}(\mathbb{R}^2)$ (see [2] for instance):

**Lemma 3.4.** For any $s \in (-1, 0)$, there hold

(i) $\|ab\|_{\dot{H}^s} \lesssim \|a\|_{L^\infty} \|b\|_{\dot{H}^s} + \|a\|_{\dot{H}^s} \|b\|_{L^\infty}$, for $s > 0$;

(ii) $\|ab\|_{\dot{H}^s} \lesssim \|a\|_{\dot{B}^s_{2,1}} \|b\|_{\dot{H}^s} + \|a\|_{\dot{H}^s} \|b\|_{\dot{B}^s_{2,1}}$, for $s > 1$;

(iii) $\|ab\|_{\dot{H}^s} \lesssim \|a\|_{L^\infty} \|b\|_{\dot{H}^s} + \|a\|_{\dot{H}^s} \|b\|_{L^\infty}$,

where the notation $A_s = B_s + \langle C_s \rangle_{s > 0}$ means $A_s = B_s$ if $s \leq s_0$ and $A_s = B_s + C_s$ if $s > s_0$.
Lemma 3.5. For any $s > -1$, there hold

\begin{enumerate}[(i)]
\item $\|ab\|_{B^s_{2,1}} \lesssim \|a\|_{B^s_{2,1}} \|b\|_{B^s_{2,1}} + \langle \|a\|_{B^s_{2,1}} \|b\|_{B^s_{2,1}} \rangle_{s>1}$;
\item $\|ab\|_{B^s_{2,1}} \lesssim \|a\|_{B^s_{2,1}} \|b\|_{B^s_{2,1}} + \|a\|_{B^s_{2,1}} \|b\|_{L^2}$.
\end{enumerate}

4. $L^1(T; Lip)$ estimate of $Y_t$

As it is well-known, the existence of solutions to a nonlinear partial differential equation follows essentially from the uniform estimates for its appropriate approximate solutions. In Subsection 5.3, we shall present the uniform global estimates to the approximate solutions. Toward this, a key ingredient used in Section 5.3 will be the $L^1(\mathbb{R}^+; Lip(\mathbb{R}^2))$ estimate for solutions of (2.28). In this section, we shall present the uniform global estimates to the approximate solutions.

4.1. The $\|\nabla Y_t\|_{L^1_T(L^\infty)}$ estimate for solutions of (2.28). We first investigate the $\|\nabla Y_t\|_{L^1_T(L^\infty)}$ estimate on solutions of the linear equation (2.28).

Proposition 4.1. Let $Y$ be a sufficiently smooth solution of (2.28) on $[0, T]$. Then there holds

\begin{align}
(4.1) \quad \|\nabla Y_t\|_{L^1_T(L^\infty)} &\lesssim \|Y_t\|_{L^1_T(B^{1 \frac{1}{2}}_2)} \lesssim \|Y_t\|_{B^{\frac{1}{2}}} + \|\partial_t Y_0\|_{B^{\frac{1}{2}}} + \|\Delta Y_0\|_{B^{\frac{1}{2}}} + \|f\|_{L^1_T(B^{0,0})}.
\end{align}

Proof. Applying operator $\Delta_j \Delta_k^h$ to (2.28), we first get that

\begin{align}
(4.2) \quad \Delta_j \Delta_k^h Y_{tt} - \Delta \Delta_j \Delta_k^h Y_t - \partial_t^2 \Delta_j \Delta_k^h Y = \Delta_j \Delta_k^h f.
\end{align}

Taking the $L^2$ inner product of (4.2) with $\Delta_j \Delta_k^h Y_t$ gives

\begin{align}
(4.3) \quad \frac{1}{2} \frac{d}{dt} \left( \|\Delta_j \Delta_k^h Y_t\|^2_{L^2} + \|\partial_t \Delta_j \Delta_k^h Y_t\|^2_{L^2} \right) + \|\nabla \Delta_j \Delta_k^h Y_t\|^2_{L^2} = (\Delta_j \Delta_k^h f, \Delta_j \Delta_k^h Y_t).
\end{align}

While taking the $L^2$ inner product of (4.2) with $\Delta \Delta_j \Delta_k^h Y$ leads to

\begin{align}
(4.4) \quad \|\nabla \Delta_j \Delta_k^h Y_t\|^2_{L^2} - \frac{1}{4} \frac{d}{dt} \|\Delta_j \Delta_k^h Y_t\|^2_{L^2} = (\Delta_j \Delta_k^h f, \Delta \Delta_j \Delta_k^h Y).
\end{align}

Notice that

\begin{align}
(4.5) \quad \frac{d}{dt} g_{j,k}(t) + 3 \frac{d}{dt} \|\nabla \Delta_j \Delta_k^h Y_t\|^2_{L^2} + \|\partial_t \nabla \Delta_j \Delta_k^h Y_t\|^2_{L^2} = (\Delta_j \Delta_k^h f, \Delta \Delta_j \Delta_k^h Y).
\end{align}

Summing up (4.3) with $\frac{1}{4} \times (4.4)$ gives

\begin{align}
(4.6) \quad \frac{d}{dt} g_{j,k}(t) + 3 \frac{d}{dt} \|\nabla \Delta_j \Delta_k^h Y_t\|^2_{L^2} + \|\partial_t \nabla \Delta_j \Delta_k^h Y_t\|^2_{L^2} = (\Delta_j \Delta_k^h f, \Delta \Delta_j \Delta_k^h Y - \frac{1}{4} \Delta \Delta_j \Delta_k^h Y),
\end{align}

where

\begin{align}
(4.7) \quad g_{j,k}(t) \overset{\text{def}}{=} \frac{1}{2} \left( \|\Delta_j \Delta_k^h Y_t(t)\|^2_{L^2} + \|\partial_t \Delta_j \Delta_k^h Y(t)\|^2_{L^2} + \|\Delta_j \Delta_k^h Y(t)\|^2_{L^2} \right)
\end{align}

and

\begin{align}
(4.8) \quad g_{j,k}(t) \overset{\text{def}}{=} -\frac{1}{4} \left( \|\Delta_j \Delta_k^h Y_t(t)\|_{L^2} + \|\Delta_j \Delta_k^h Y(t)\|_{L^2} \right).
\end{align}
It is easy to check that
\[
\frac{1}{4} |(\Delta_j \Delta_k^h Y_i | \Delta \Delta_j \Delta_k^h Y)| \leq \frac{1}{16} \|\Delta \Delta_j \Delta_k^h Y\|_{L^2}^2 + \frac{1}{4} \|\Delta_j \Delta_k^h Y_i\|_{L^2}^2,
\]
which implies
\[
\frac{1}{4} \|\Delta_j \Delta_k^h Y_i(t)\|_{L^2}^2 + \frac{1}{2} \|\partial_t \Delta_j \Delta_k^h Y(t)\|_{L^2}^2 + \frac{1}{16} \|\Delta \Delta_j \Delta_k^h Y(t)\|_{L^2}^2
\]
\[
\leq g_{j,k}(t)^2 \leq \frac{3}{4} \|\Delta_j \Delta_k^h Y_i(t)\|_{L^2}^2 + \frac{1}{2} \|\partial_t \Delta_j \Delta_k^h Y(t)\|_{L^2}^2 + \frac{3}{16} \|\Delta \Delta_j \Delta_k^h Y(t)\|_{L^2}^2,
\]
or equivalently
\[
(4.6) \quad g_{j,k}(t)^2 \sim \|\Delta_j \Delta_k^h Y_i(t)\|_{L^2}^2 + \|\partial_t \Delta_j \Delta_k^h Y(t)\|_{L^2}^2 + \|\Delta \Delta_j \Delta_k^h Y(t)\|_{L^2}^2.
\]

With (4.5) and (4.6), and according to the heuristic analysis in Section 2, we shall divide the proof of (4.1) into two cases: one is when \( j \leq \frac{k+1}{2} \), and the other one is when \( j > \frac{k+1}{2} \).

**Case (1):** when \( j \leq \frac{k+1}{2} \). In this case, we infer from Lemma 3.1 and (4.6) that
\[
g_{j,k}(t)^2 \sim \|\Delta_j \Delta_k^h Y_i(t)\|_{L^2}^2 + \|\partial_t \Delta_j \Delta_k^h Y(t)\|_{L^2}^2,
\]
and
\[
\|\nabla \Delta_j \Delta_k^h Y_i(t)\|_{L^2}^2 + \|\partial_t \nabla \Delta_j \Delta_k^h Y(t)\|_{L^2}^2
\]
\[
\geq c 2^{2j} (\|\Delta_j \Delta_k^h Y_i(t)\|_{L^2}^2 + \|\partial_t \Delta_j \Delta_k^h Y(t)\|_{L^2}^2) \geq c 2^{2j} g_{j,k}(t)^2,
\]
from which, for any \( \varepsilon > 0 \), dividing (4.5) by \( g_{j,k}(t) + \varepsilon \), then taking \( \varepsilon \to 0 \) and integrating the resulting equation over \([0,T]\), we deduce
\[
\|\Delta_j \Delta_k^h Y_i\|_{L^p(L^2)} + \|\partial_t \Delta_j \Delta_k^h Y\|_{L^p(L^2)}
\]
\[
+ c 2^{2j} (\|\Delta_j \Delta_k^h Y_i\|_{L^2(L^2)} + \|\partial_t \Delta_j \Delta_k^h Y\|_{L^2(L^2)})
\]
\[
\leq \|\Delta_j \Delta_k^h Y_i\|_{L^2} + \|\partial_t \Delta_j \Delta_k^h Y_0\|_{L^2} + \|\Delta_j \Delta_k^h Y_0\|_{L^2},
\]

**Case (2):** when \( j > \frac{k+1}{2} \). In this case, we apply Lemma 3.1 to obtain that
\[
g_{j,k}(t)^2 \sim \|\Delta_j \Delta_k^h Y_i(t)\|_{L^2}^2 + \|\Delta \Delta_j \Delta_k^h Y(t)\|_{L^2}^2,
\]
and
\[
\|\nabla \Delta_j \Delta_k^h Y_i(t)\|_{L^2}^2 + \|\partial_t \nabla \Delta_j \Delta_k^h Y(t)\|_{L^2}^2
\]
\[
\geq c 2^{2j} (\|\Delta_j \Delta_k^h Y_i(t)\|_{L^2}^2 + \|\Delta \Delta_j \Delta_k^h Y(t)\|_{L^2}^2) \geq c 2^{2j} g_{j,k}(t)^2,
\]
from which and (4.5), we deduce by a similar argument for (4.7) that
\[
\|\Delta_j \Delta_k^h Y_i\|_{L^p(L^2)} + \|\Delta \Delta_j \Delta_k^h Y\|_{L^p(L^2)}
\]
\[
+ c 2^{2j} (\|\Delta_j \Delta_k^h Y_i\|_{L^2(L^2)} + \|\Delta \Delta_j \Delta_k^h Y\|_{L^2(L^2)})
\]
\[
\leq \|\Delta_j \Delta_k^h Y_i\|_{L^2} + \|\Delta \Delta_j \Delta_k^h Y_0\|_{L^2} + \|\Delta_j \Delta_k^h Y_0\|_{L^2}.
\]

On the other hand, via (4.3), one has
\[
\frac{1}{2} \frac{d}{dt} \|\Delta_j \Delta_k^h Y_i(t)\|_{L^2}^2 + \|\nabla \Delta_j \Delta_k^h Y_i(t)\|_{L^2}^2 = (\partial_t^2 \Delta_j \Delta_k^h Y + \Delta_j \Delta_k^h f | \Delta_j \Delta_k^h Y_i),
\]
from which, Lemma 3.1 and (4.8), we conclude
\[
\|\Delta X Y_l\|_{L^\infty_T(L^2)} + c 2^{j}\|\Delta Y_l\|_{L^\infty_T(L^2)}
\leq \|\Delta Y_l\|_{L^2} + C(2^{k}\|\Delta Y_l\|_{L^\infty_T(L^2)} + \|\Delta Y_l Y_l\|_{L^2})
\leq \|\Delta Y_l\|_{L^2} + \|\Delta Y_l Y_l\|_{L^2} + \|\Delta Y_l Y_l\|_{L^2} \text{ for } j > \frac{k+1}{2}.
\]

By Definition 3.2, we get, by combining (4.7) with (4.9), that
\[
\|Y_l\|_{L^2_T(B^{\frac{3}{2}})} \leq \|Y_l\|_{B^{-\frac{1}{2}}^2} + \|\partial_t Y_l\|_{B^{-\frac{1}{2}}^2} + \|\Delta Y_l\|_{B^{-\frac{1}{2}}^2} + \|f\|_{L^2_T(B^{0,0})},
\]

which together with the fact
\[
\|\nabla Y_l\|_{L^2_T(L^\infty)} \leq \sum_{j, k \in \mathbb{Z}} \|\Delta Y_l\|_{L^2_T(L^\infty)}
\leq \sum_{j, k \in \mathbb{Z}} 2^{j+2k}\|\Delta Y_l\|_{L^2_T(L^2)}
\leq \sum_{j, k \in \mathbb{Z}} 2^{j+2k}\|\Delta Y_l\|_{L^2_T(L^2)} \leq \frac{1}{2} \|\nabla Y_l\|_{L^2_T(B^{\frac{3}{2}})}
\]

implies (4.1). This finishes the proof of Proposition 4.1.

\[ \square \]

### 4.2. \(L^1_T(B^{0,0})\) estimate of \(f(Y, q)\) given by (2.21).

With (4.1), to derive the \(L^1_T(Lip(\mathbb{R}^2))\) estimate of \(Y_l\) for smooth enough solutions of (2.20), we need to estimate the \(L^1_T(B^{0,0})\) norm of \(f(Y, q)\) given by (2.21). This will be done in this subsection.

**Proposition 4.2.** Let \((Y, q)\) be a smooth enough solution of (2.19) (or equivalently (2.20)-(2.21)) on \([0, T]\), with the initial data satisfying (2.23). We assume further that
\[
\|\nabla Y\|_{L^\infty_T(B^{1,1})} \leq c_0
\]
for some \(c_0\) sufficiently small. Then one has
\[
\|\nabla q\|_{L^1_T(B^{0,0})} \leq \sum_{j, k} \|\nabla Y_l\|_{L^2_T(B^{\frac{3}{2}})} + \|\partial_t Y\|_{L^2_T(B^{\frac{3}{2}})} \|\nabla Y\|_{L^2_T(B^{\frac{3}{2}})}\}
\]

where
\[
\nabla Y_l = -\partial_t A_T Y_l \nabla Y_l.
\]

While for \(c_0\) sufficiently small so that
\[
\|\nabla Y\|_{L^\infty_T(L^\infty)} \leq C \|\nabla Y\|_{L^\infty_T(B^{1,1})} \leq C c_0 \leq \frac{1}{2},
\]

\(X(t, y)\) determined by (2.14) has a smooth inverse map \(X^{-1}(t, x)\) with \(X(t, X^{-1}(t, x)) = x\) and \(X^{-1}(t, X(t, y)) = y\). Then it follows from \(\nabla Y \cdot Y_l = 0\) that
\[
\nabla Y \cdot \nabla Y_l = [\nabla x \cdot \Delta x (Y_l \circ X^{-1}(t, x))] \circ X(t, y) = \nabla Y \cdot \nabla Y_l \cdot \nabla Y_l = 0,
\]

from the latter and (4.12), we obtain by taking \(\nabla Y\) to the first equation of (2.19) that
\[
\nabla Y \cdot \nabla Y_l = \partial_t A_T Y_l \nabla Y_l + \nabla Y \cdot \partial_t^2 Y
\]
or equivalently
\[
\Delta q = -(\nabla Y - \nabla) \cdot \nabla q - \nabla \cdot (\nabla Y - \nabla) q + \partial_t A_T Y_l \nabla Y_l + \nabla Y \cdot \partial_t^2 Y
\]
\[
= -\nabla \cdot ((\nabla Y - I) A_T Y_l \nabla q) - \nabla \cdot ((\partial_t A_T Y_l + \nabla Y \cdot \partial_t^2 Y) \nabla q).
\]
The above, Lemma 3.1 and Definition 3.2 lead to
\[
\|\nabla q\|_{L^2_t(L^1_x(B^0,0))} \leq \|(A_Y - I)A_Y^{T} \nabla q\|_{L^2_t(L^1_x(B^0,0))} + \|(A_Y^{T} - I) \nabla q\|_{L^2_t(L^1_x(B^0,0))} + \|\partial_t A_Y Y_t\|_{L^2_t(L^1_x(B^0,0))} + \|\nabla (\Delta)^{-1} (\nabla Y \cdot \partial_t^2 Y)\|_{L^2_t(L^1_x(B^0,0))}. \tag{4.14}
\]
Applying Lemma 3.3 and (2.16), one has
\[
\|(A_Y^{T} - I) \nabla q\|_{L^2_t(L^1_x(B^0,0))} \lesssim \|\nabla Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|\nabla q\|_{L^2_t(L^1_x(B^0,0))}, \tag{4.15}
\]
\[
\|(A_Y - I)A_Y^{T} \nabla q\|_{L^2_t(L^1_x(B^0,0))} \lesssim (1 + \|\nabla Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})}) \|\nabla Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|\nabla q\|_{L^2_t(L^1_x(B^0,0))} \quad \text{and}
\]
\[
\|\partial_t A_Y Y_t\|_{L^2_t(L^1_x(B^0,0))} \lesssim \|\nabla Y_t\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|Y_t\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})}. \tag{4.16}
\]
While by (2.20) and (2.21), we have \(\nabla \cdot Y = \rho(Y) = \partial_1 Y^2 \partial_2 Y^1 - \partial_1 Y^1 \partial_2 Y^2\), from which, we deduce
\[
\nabla \cdot \partial_t^2 Y = \nabla \cdot ((A_Y - I) \partial_t^2 Y) + \partial_t^2 \rho(Y)
\]}
\[= \partial_1 (\partial_1 Y^2 \partial_2 Y^1 - \partial_1 Y^1 \partial_2 Y^2) + \partial_2 (\partial_1 Y^2 \partial_2 Y^1 - \partial_1 Y^1 \partial_2 Y^2).
\]
Applying Lemma 3.1 and Lemma 3.3 leads to
\[
\|\nabla (\Delta)^{-1} (\nabla Y \cdot \partial_t^2 Y)\|_{L^2_t(L^1_x(B^0,0))} \lesssim \|\partial_1 Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|\partial_1 Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})}. \tag{4.17}
\]
Substituting (4.15) and (4.17) into (4.14) and using Lemma 3.2, we conclude that
\[
\|\nabla q\|_{L^2_t(L^1_x(B^0,0))} \lesssim (1 + \|\nabla Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})}) \|\nabla Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|\nabla q\|_{L^2_t(L^1_x(B^0,0))} + \|\nabla Y_t\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|Y_t\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} + \|\partial_t Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|\partial_1 Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})},
\]
which together with (4.10) ensures (4.11). This completes the proof of Proposition 4.2. \(\square\)

We now turn to the estimate of \(f(Y, q)\) given by (2.21). The main result is as follows:

**Proposition 4.3.** Let \(f(Y, q)\) be given by (2.21). Under the assumptions of Proposition 4.2, and, in addition, if
\[
\|\nabla Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|\nabla Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \leq 1,
\]
we have
\[
\|f(Y, q)\|_{L^2_t(L^1_x(B^0,0))} \lesssim \|\nabla Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|Y_t\|_{L^1_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|\nabla Y_t\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} + \|\partial_t Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|\partial_1 Y\|_{L^1_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \]}
\[+ \big(\|\partial_2 Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|\nabla Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} + \|\nabla Y_t\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} + \|\partial_2 Y^2\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})}\big) \|\partial_1 Y\|_{L^1_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})} \|\partial_1 Y\|_{L^2_t(\mathbb{R}^{\frac{1}{2}} + \frac{1}{2})}.
\tag{4.18}
\]

**Proof.** Via (2.21), we split \(f(Y, q)\) as
\[
f(Y, q) = \tilde{f}(Y) + \check{f}(Y, q) \quad \text{with}
\]
\[
\tilde{f}(Y) \overset{\text{def}}{=} (\nabla Y \cdot \nabla Y - \Delta) Y_t, \quad \text{and} \quad \check{f}(Y, q) \overset{\text{def}}{=} -\nabla Y q.
\]
Notice from the definition of \(A_Y = (b_{ij})_{i, j=1, 2}\) given by (2.16) that
\[
\check{f}(Y) = \sum_{\ell, m, j=1}^2 \partial_\ell (b_{\ell j} b_{mj} \partial_m Y_t) - \Delta Y_t \overset{\text{def}}{=} \partial_1 F_1 + \partial_2 F_2,
\]
where \(F_1\) and \(F_2\) are given by (4.9) and (4.10), respectively.
Lemma 4.1. Under the assumptions of Proposition 4.3, one has
\[ \| \partial_2(\partial_2 Y^1 \partial_1 Y_t) \|_{L_t^2(B^{0,0})} \lesssim \| \nabla Y \|_{L_t^\infty(B^{\frac{1}{2},1})} \| Y_t \|_{L_t^2(B^{\frac{3}{2},1})} + \| \partial_1 Y \|_{L_t^2(B^{\frac{3}{2},1})} \| Y_t \|_{L_t^2(B^{\frac{3}{2},1})}. \]

Lemma 4.2. Under the assumptions of Proposition 4.3, one has
\[ \| \partial_1(\partial_2 Y^1 \partial_2 Y_t) \|_{L_t^2(B^{0,0})} \lesssim \| \partial_2 Y^1 \|_{L_t^2(B^{\frac{1}{2},1})} \| Y_t \|_{L_t^2(B^{\frac{3}{2},1})}. \]
Combining (4.25)-(4.28) and using Lemma 3.2 and (4.22), we obtain
\[
\|\partial_{k} F_{2}\|_{L_{T}^{2}(B^{0,0})} \lesssim (1 + \|\nabla Y\|_{L_{T}^{\infty}(B^{1,1})}) \left\{ \|\nabla Y\|_{L_{T}^{\infty}(B^{1,1})} \|Y_{1}\|_{L_{T}^{2}(B^{\frac{3}{2},\frac{1}{2}})} \right. \\
\quad + \left( \|\partial_{1} Y\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{4}})} + \|\partial_{1} \nabla Y\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{4}})} \right) \left( \|\partial_{2} Y_{1}\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{4}})} + \|\partial_{2} Y\|_{L_{T}^{2}(B^{1,1})} \right) \\
\quad + (1 + \|\partial_{2} Y\|_{L_{T}^{2}(B^{1,1})} + \|\partial_{2} Y\|_{L_{T}^{2}(B^{1,1})}) \\
\times \left( \|\partial_{1} Y\|_{L_{T}^{2}(B^{1,1})} + \|\partial_{1} Y\|_{L_{T}^{2}(B^{1,1})} \right) \|Y_{1}\|_{L_{T}^{2}(B^{\frac{3}{2},\frac{1}{2}})}.
\]
(4.29)

On the other hand, via (4.22), we deduce from Lemma 4.2 along with its proof that
\[
\|\partial_{1} F_{1}\|_{L_{T}^{2}(B^{0,0})} \lesssim (1 + \|\nabla Y\|_{L_{T}^{\infty}(B^{1,1})}) \|\nabla Y\|_{L_{T}^{\infty}(B^{1,1})} \|Y_{1}\|_{L_{T}^{2}(B^{\frac{3}{2},\frac{1}{2}})}.
\]
(4.30)

Therefore, under the assumptions (4.10) and (4.18), we arrive at (4.19) by summing up (4.11), (4.21), (4.24), (4.29) and (4.30). This concludes the proof of Proposition 4.3.

4.3. $L_{T}^{1}(Lip(\mathbb{R}^{2}))$ estimate of $Y_{1}$. We now conclude this section with

**Proposition 4.4.** Under the assumption of Proposition 4.3, one has
\[
\|\nabla Y_{1}\|_{L_{T}^{\infty}(L^{\infty})} \lesssim \|Y_{1}\|_{L_{T}^{1}(B^{\frac{1}{2},\frac{1}{2}})} + \|\partial_{1} Y_{0}\|_{B^{\frac{1}{2},\frac{1}{2}}} + \|Y_{0}\|_{B^{\frac{1}{2},\frac{1}{2}}} + \|Z_{1}\|_{L_{T}^{1}(B^{\frac{1}{2},\frac{1}{2}})}^{2} \\
+ \|Y_{1}\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})}^{2} + \|\partial_{1} Y\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})}^{2} + \|\partial_{1} Y\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})}^{2}.
\]
(4.31)

**Proof.** Thanks to Propositions 4.1 and 4.3, and the assumption (4.10), we infer
\[
\|\nabla Y_{1}\|_{L_{T}^{\infty}(L^{\infty})} \lesssim \|Y_{1}\|_{L_{T}^{1}(B^{\frac{1}{2},\frac{1}{2}})} \lesssim \|Y_{1}\|_{B^{-\frac{1}{2},\frac{1}{2}}} + \|\partial_{1} Y_{0}\|_{B^{-\frac{1}{2},\frac{1}{2}}} + \|\Delta Y_{0}\|_{B^{-\frac{1}{2},\frac{1}{2}}} \\
+ \|Y_{1}\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})} \|\nabla Y_{1}\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})} + \left( \|\partial_{1} Y\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})} + \|\partial_{1} Y\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})} \right) \|Y_{1}\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})} \\
+ (\|\partial_{1} Y\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})} + \|\partial_{1} \nabla Y\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})} \left( \|\nabla Y_{1}\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})} + \|\nabla^{2} Y_{1}\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})} \right) \\
+ \|\partial_{1} Y\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})} \|\partial_{1} \nabla Y\|_{L_{T}^{2}(B^{\frac{1}{2},\frac{1}{2}})} \\
\right).
\]
(4.32)

which along with Lemma 3.2 and (3.6) lead to (4.31). This completes the proof of Proposition 4.4.

5. The proof of Theorem 2.2

5.1. A priori estimate of (2.19). With $L_{T}^{1}(Lip(\mathbb{R}^{2}))$ estimate of $Y_{1}$, we can now proceed with the energy method.

**Proposition 5.1.** Let $Y$ be a sufficiently smooth solution of (2.20) on $[0, T]$. Then one has
\[
\|\Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} + \|\nabla \Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} + \|\partial_{1} \Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} + \|\nabla \Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} + \|\Delta \Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} \\
+ \|\partial_{1} \nabla \Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} + \|\Delta \Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} \\
\lesssim \|\Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} + \|\nabla \Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} + \|\partial_{1} \Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} + \|\Delta \Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} + \|\partial_{1} \nabla \Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} \\
+ \|\Delta \Delta_{j} Y\|_{L_{T}^{2}(L^{2})}^{2} + \left| \int_{0}^{T} \langle \Delta_{j} f, \Delta_{j} Y_{i} - \frac{1}{4} \Delta \Delta_{j} Y - \Delta \Delta_{j} Y_{i} \rangle \, dt \right|.
\]
(5.1)
Proof. Applying $\Delta_j$ to (2.20) gives
\[ \Delta_j Y_t - \Delta \Delta j Y_t - \partial_i^2 \Delta j Y = \Delta_j f, \]
from which, and along the same line of the proof of (4.5), we get by taking the $L^2$ inner product of (5.2) with $\Delta_j Y_t - \frac{1}{4}\Delta \Delta j Y - \Delta \Delta_j Y_t$ that
\[
\frac{d}{dt} \left( \frac{1}{2} \left( \| \Delta_j Y_t \|_{L^2}^2 + \| \nabla \Delta_j Y_t \|_{L^2}^2 + \| \partial_i \nabla \Delta_j Y \|_{L^2}^2 + \| \partial_i \nabla \Delta_j Y \|_{L^2}^2 + \frac{1}{4} \| \Delta \Delta_j Y \|_{L^2}^2 \right) \right) \\
+ \frac{1}{4} (\Delta_j Y_t \mid \Delta \Delta j Y)^2 \right) + \frac{3}{4} \| \nabla \Delta_j Y_t \|_{L^2}^2 + \| \Delta \Delta_j Y_t \|_{L^2}^2 + \frac{1}{4} \| \partial_i \nabla \Delta_j Y \|_{L^2}^2 \\
= (\Delta_j f \mid \Delta_j Y_t - \frac{1}{4} \Delta \Delta j Y - \Delta \Delta_j Y_t).
\]
While it deduces from $\text{div}Y = \rho(Y)$ the followings
\[
\| \nabla \Delta_j Y^2 \|_{L^2 (L^2)} \lesssim \| \partial_i \Delta_j Y \|_{L^2 (L^2)} \| \Delta_j \rho \|_{L^2 (L^2)}, \]
\[
\| \Delta \Delta_j Y^2 \|_{L^2 (L^2)} \lesssim \| \partial_i \nabla \Delta_j Y \|_{L^2 (L^2)} + \| \Delta_j \partial_2 \rho \|_{L^2 (L^2)},
\]
And similar to (4.6), here we have
\[
\frac{1}{2} \left( \| \Delta_j Y_t \|_{L^2}^2 + \| \nabla \Delta_j Y_t \|_{L^2}^2 + \| \partial_i \Delta_j Y \|_{L^2}^2 + \| \partial_i \nabla \Delta_j Y \|_{L^2}^2 + \frac{1}{4} \| \Delta \Delta_j Y \|_{L^2}^2 \right) \\
- \frac{1}{4} (\Delta_j Y_t \mid \Delta \Delta j Y)^2 \sim \| \Delta_j Y_t \|_{L^2}^2 + \| \nabla \Delta_j Y_t \|_{L^2}^2 + \| \partial_i \Delta_j Y \|_{L^2}^2 + \| \Delta \Delta j Y \|_{L^2}^2.
\]
Hence by integrating (5.3) over $[0, T]$ and using (5.4), we obtain (5.1). This completes the proof of Proposition 5.1.

The next proposition is concerned with the a priori estimate to the pressure function $q$ in (2.19).

**Proposition 5.2.** Let $(Y, q)$ be a smooth enough solution of (2.19) on $[0, T]$. Then under the assumption (4.10), for any $s > -1$, we have
\[
\| \nabla q \|_{L^2_s (H^s)} \lesssim \| Y \|_{L^2_s (H^{s+2})} \| \nabla q \|_{L^2_s (L^2)} + \| \nabla q \|_{L^2_s (H^{s+1})} \| Y \|_{L^2_s (H^{s+1})} \| \partial_1 \nabla Y \|_{L^2_s (L^2)},
\]
and
\[
\| \nabla q \|_{L^2_s (H^s)} \lesssim \| Y \|_{L^2_s (H^{s+2})} \| \nabla q \|_{L^2_s (L^2)} + \| \nabla q \|_{L^2_s (H^{s+1})} \| Y \|_{L^2_s (H^{s+1})} \| \partial_1 \nabla Y \|_{L^2_s (L^2)}.
\]

**Proof.** We first deduce from (4.13) that
\[
\| \nabla q(t) \|_{H^s} \leq \|((A_Y - I)A_Y^2 \nabla q)(t)) \|_{H^s} + \|((A_Y^2 - I) \nabla q)(t)) \|_{H^s} \\
+ \|((\partial_1 A_Y Y_1(t)) \|_{H^s} + \| \nabla (-\Delta)^{-1} (\nabla Y \cdot \partial_1^2 Y)(t)) \|_{H^s}.
\]
Thanks to (2.16), Lemmas 3.4 and 3.5, we see that for any $s > -1$,
\[
\|((A_Y - I)A_Y^2 \nabla q)(t)) \|_{H^s} \lesssim (1 + \| \nabla Y(t) \|_{B_{2,1}^1}) (\| \nabla Y(t) \|_{B_{2,1}^1} \| \nabla q(t) \|_{H^s} \\
+ \| \nabla Y(t) \|_{H^{s+1}} \| \nabla q(t) \|_{L^2}).
\]
By a similar argument, we obtain that for any $s > -1$,
\[
\|((A_Y^2 - I) \nabla q)(t)) \|_{H^s} \lesssim \| \nabla Y(t) \|_{B_{2,1}^1} \| \nabla q(t) \|_{H^s} + \| \nabla Y(t) \|_{H^{s+1}} \| \nabla q(t) \|_{L^2},
\]
Lemma 5.1. Present the related estimates for (5.14) which along with (5.11) implies for any $t \in (0, T)$.

\[
\| (\partial_t \mathcal{A}_Y Y_t)(t) \|_{H^s} \lesssim \| Y_t(t) \|_{B^{1,1}_{2,1}}. \tag{5.10}
\]

On the other hand, due to (4.16), we deduce from Lemma 3.4 that for any $s > -1$,
\[
\| \nabla (\Delta)^{-1}(\nabla Y \cdot \partial_t^2 Y)(t) \|_{H^s} \lesssim \| (\partial_t Y \partial_t \nabla Y)(t) \|_{H^s} + \| \partial_1 Y(t) \|_{H^{s+1}} \| \partial_1 \nabla Y(t) \|_{L^2}.
\]

This combines with (5.7), (5.8), (5.9), (5.10) and the assumption (4.10) ensures that for any $s > -1$,
\[
\| \nabla q(t) \|_{H^s} \lesssim \| \nabla Y(t) \|_{H^{s+1}} \| \nabla q(t) \|_{L^2} + \| Y_t(t) \|_{B^{1,1}_{2,1}} \| Y_t(t) \|_{H^{s+1}} + \| \partial_1 Y(t) \|_{H^s} + \| \partial_1 \nabla Y(t) \|_{H^{s+1}} \| \partial_1 \nabla Y(t) \|_{L^2}, \quad \text{for} \quad t \in (0, T).
\]

Integrating the above inequality over $(0, T)$ leads to (5.5), whereas taking its $L^2$ norm with respect to time on $(0, T)$ gives rise to (5.6). This proves Proposition 5.2. \hfill \Box

5.2. The Estimate to the source terms in (5.1). With Propositions 5.1 and 5.2, to close the a priori estimate for smooth enough solutions of (2.20)-(2.21), we need to deal with the estimate of $\rho$ and $\int_0^T (\Delta_j f \mid \Delta_j Y_t - \frac{1}{2} \Delta \Delta_j Y - \Delta_j Y_t) \, dt$ for $\rho, f$ given by (2.21). We first present the related estimates for $\rho$, which is a direct consequence of Lemma 3.4 and (2.21).

Lemma 5.1. Let $Y$ be a sufficiently smooth function on $[0, T] \times \mathbb{R}^2$. Then for any $s > -1$, $\rho(Y) = \partial_1 Y^2 \partial_2 Y^1 - \partial_1 Y^1 \partial_2 Y^2$ satisfies
\[
\| \Delta_j \rho \|_{L^\infty_T(L^2)} \lesssim c_j 2^{-js} \left( \| \partial_2 Y \|_{L^\infty_T(B^{1,1}_{2,1})} \| \partial_2 Y \|_{L^\infty_T(H^s)} + \| \partial_2 Y \|^2_{L^\infty_T(H^{s+1})} \right),
\]
\[
\| \nabla \Delta_j \rho \|_{L^2_T(L^2)} \lesssim c_j 2^{-js} \left( \| \partial_2 Y \|^2_{L^\infty_T(B^{1,1}_{2,1})} \| \partial_1 Y \|_{L^2_T(H^s)} + \| \partial_2 Y \|^2_{L^\infty_T(H^{s+1})} \right) \| \partial_1 Y \|_{L^2_T(B^{1,1}_{2,1})}.
\]

Lemma 5.2. Let $(Y, q)$ be a smooth enough solution of (2.20)-(2.21). For $s > -1$, we assume, in addition, that
\[
\| \nabla Y \|_{L^\infty_T(B^{1,1}_{2,1})} + \| Y \|_{L^\infty_T(H^{s+2})} \leq 1,
\]
then one has
\[
\left| \int_0^T (\Delta_j f \mid \Delta_j Y_t) \, dt \right| \lesssim c_j^2 2^{-2js} \left\{ \left( \| \nabla q \|_{L^2_T(H^s)} + \| \nabla q \|_{L^2_T(L^2)} \right) \| Y_t \|_{L^\infty_T(H^s)} + \left( \| \nabla Y \|^2_{L^\infty_T(B^{1,1}_{2,1})} + \| Y_t \|_{L^2_T(H^s)} \right) \| \nabla q \|^2_{L^2_T(H^{s+2})} + \| Y_t \|^2_{L^2_T(H^{s+1})} \right\}.
\]

Proof. For $\tilde{f}$ given by (4.23), it follows from Lemma 3.4 that
\[
\| \tilde{f} \|_{L^1_T(H^s)} \lesssim (1 + \| \nabla Y \|_{L^\infty_T(B^{1,1}_{2,1})}) \| \nabla q \|_{L^2_T(L^2)} + \| \nabla Y \|_{L^\infty_T(H^{s+1})} \| \nabla q \|_{L^2_T(L^2)},
\]
which along with (5.11) implies for any $s > -1$
\[
\left| \int_0^T (\Delta_j f \mid \Delta_j Y_t) \, dt \right| \lesssim c_j^2 2^{-2js} \left( \| \nabla q \|_{L^2_T(H^s)} + \| \nabla q \|_{L^2_T(L^2)} \right) \| Y_t \|_{L^\infty_T(H^s)}.
\]

While via (4.21), we get by using integration by parts that
\[
\int_0^T (\Delta_j f \mid \Delta_j Y_t) \, dt = -\sum_{i=1}^2 \int_0^T (\Delta_j F_i \mid \Delta_j \partial_t Y_i)_{L^2} \, dt,
\]
with \( \mathbf{F}_1, \mathbf{F}_2 \) given by (4.22). For any \( s > -1 \), applying Lemma 3.4, one has

\[
\| \partial_t \mathbf{Y}^1 \mathbf{Y} i \|_{L^2_t (\dot{H}^s)} \lesssim \| \partial_t \mathbf{Y}^1 \|_{L^2_t (\dot{H}^{s+1})} \| i \|_{L^2_t (\dot{H}^s)} + \| \partial_t \mathbf{Y}^1 \|_{L^2_t (\dot{H}^{s+1})} \| Y \|_{L^2_t (L^2)}.
\]

Applying Lemma 3.4 twice and Lemma 3.5, it leads to

\[
\| \partial_t \mathbf{Y}^2 \partial_t \mathbf{Y}^2 \|_{L^2_t (\dot{H}^s)} \lesssim \| \partial_t \mathbf{Y}^2 \partial_t \mathbf{Y}^2 \|_{L^2_t (\dot{H}^{s+1})} \| \partial_t \mathbf{Y} \|_{L^2_t (L^2)} \lesssim \| \nabla \mathbf{Y}^2 \|_{L^2_t (\dot{H}^{s+1})} \| \partial_t \mathbf{Y} \|_{L^2_t (\dot{H}^1)} + \| \nabla \mathbf{Y}^2 \|_{L^2_t (\dot{H}^{s+1})} \| \partial_t \mathbf{Y} \|_{L^2_t (L^2)}.
\]

Similar estimates hold for the other terms in \( \mathbf{F}_1, \mathbf{F}_2 \) given by (4.22). Therefore, under the assumption (5.11), we obtain

\[
\| i \|_{L^2_t (\dot{H}^{s+2})} \lesssim e^{c_2^2 t^{2j}} \left\{ (\| \nabla \mathbf{Y} \|_{L^2_t (\dot{H}^1)}, \| \mathbf{Y} \|_{L^2_t (\dot{H}^1)}) \right\}.
\]

This together with (4.20) and (5.14) implies (5.12). We complete the proof of this Lemma.

**Lemma 5.3.** Let \( (Y, q) \) be a smooth enough solution of (2.20)-(2.21). For \( s > -1 \), we assume also that (4.18) and (5.11), then one has

\[
\| \mathbf{f} \|_{L^2_t (\dot{H}^s)} \lesssim \sum_{i=1}^2 \| \mathbf{F}_i \|_{L^2_t (\dot{H}^{s+1})}
\]

and

\[
\| \mathbf{f} \|_{L^2_t (\dot{H}^s)} \lesssim \sum_{i=1}^2 \| \mathbf{F}_i \|_{L^2_t (\dot{H}^{s+1})}
\]

and

\[
\| \mathbf{f} \|_{L^2_t (\dot{H}^s)} \lesssim (1 + \| \nabla \mathbf{Y} \|_{L^2_t (\dot{H}^1)}) \| \mathbf{Y} \|_{L^2_t (\dot{H}^s)} + \| \nabla \mathbf{Y} \|_{L^2_t (\dot{H}^{s+1})} \| \mathbf{Y} \|_{L^2_t (L^2)}
\]

which together with (4.20) implies (5.15).
On the other hand, we infer from (4.18), (5.11), (5.13), (5.17) and (5.18) that for any \( s > -1 \)

\[
\int_0^T (\Delta_j \tilde{f} \mid \Delta \Delta_j Y) \, dt \lesssim c_2^2 2^{-2js}(\|\nabla q\|_{L_x^1(H^s)} + \|\nabla q\|_{L_x^1(L^2)}) \|\Delta Y\|_{L_x^p(H^s)},
\]

\[
\int_0^T (\Delta_j \tilde{f}^2 \mid \Delta \Delta_j Y^2) \, dt \lesssim c_2^2 2^{-2js}(\|\nabla Y\|_{L_x^p(B^1_2)} \|\nabla Y\|_{L_x^2(H^{s+1})})
+ \|\nabla Y\|_{L_x^p(H^{s+1})} \|\nabla Y\|_{L_x^2(B^1_2)}) \|\Delta Y\|_{L_x^p(H^s)}.
\]

Whereas via (5.17) and using integration by parts, we can obtain for \( s > -1 \) the following

\[
\int_0^T (\Delta_j \partial_1 F_1^1 \mid \Delta \Delta_j Y^1) \, dt = \int_0^T (\Delta_j \nabla F_1^1 \mid \partial_1 \nabla \Delta_j Y^1) \, dt
\]

\[
\lesssim c_2^2 2^{-2js}(\|\nabla Y\|_{L_x^p(B^1_2)} \|\nabla Y\|_{L_x^2(H^{s+1})})
+ \|\nabla Y\|_{L_x^p(H^{s+1})} \|\nabla Y\|_{L_x^2(B^1_2)}) \|\partial_1 \nabla Y^1\|_{L_x^p(H^s)}.
\]

Because of (4.21), (5.19) to (5.21), to complete the proof of (5.16), we only need to deal with the term \( \int_0^T (\Delta_j \partial_2 F_2^1 \mid \Delta \Delta_j Y^1) \, dt \). For this purpose, we split \( F_2^1 \) as \( G^1 + G^2 \) with

\[
G^1 = (2\partial_1 Y^1 + |\partial_1 Y|^2)\partial_2 Y_t^1 - (1 + \partial_2 Y^2)\partial_1 Y^2\partial_1 Y_t^1, \quad G^2 = -(1 + \partial_1 Y^1)\partial_2 Y^1\partial_1 Y_t^1.
\]

It follows from Lemmas 3.4 and 3.5 and (4.18) that

\[
\|\partial_2 G^1\|_{L^1_x(H^s)} \lesssim \|\partial_1 Y\|_{L^p_x(B^1_2)} \|\nabla Y\|_{L^2_x(H^{s+1})}
+ \left( \|\partial_1 Y\|_{L^2_x(H^{s+1})} + \|\nabla Y^2\|_{L^2_x(B^1_2)} \right) \|\nabla Y\|_{L^2_x(B^1_2)}
\]

for \( s > -1 \), this along with the fact that \( \|\partial_2 G^1\|_{L^1_x(H^s)} \lesssim \|\partial_2 G^1\|_{L^1_x(H^s)} \) implies

\[
\int_0^T (\Delta_j \partial_2 G^1 \mid \Delta \Delta_j Y^1) \, dt \lesssim c_2^2 2^{-2js} \left( \|\partial_1 Y\|_{L^2_x(B^1_2)} \|\nabla Y\|_{L^2_x(H^{s+1})} \right)
+ \left( \|\partial_1 Y\|_{L^2_x(H^{s+1})} + \|\nabla Y^2\|_{L^2_x(B^1_2)} \right) \|\nabla Y\|_{L^2_x(B^1_2)} \|\Delta Y\|_{L^p_x(H^s)}
\]

for any \( s > -1 \).

To handle \( G^2 \) in (5.22), we first do a Bony’s decomposition (3.7) so that

\[
\partial_2 Y^1\partial_1 Y_t^1 = T_{\partial_2 Y^1}\partial_1 Y_t^1 + T_{\partial_1 Y_t^1}\partial_2 Y^1 + R(\partial_2 Y^1, \partial_1 Y_t^1).
\]

Using integration by parts, we have

\[
\int_0^T (\Delta_j (T_{\partial_2 Y^1}\partial_1 Y_t^1) \mid \Delta \Delta_j \partial_2 Y^1) \, dt
= -\int_0^T (\Delta_j (T_{\partial_1 Y_t^1}\partial_2 Y^1) \mid \Delta \Delta_j \partial_2 Y^1) \, dt - \int_0^T (\Delta_j (T_{\partial_2 Y^1} Y_t^1) \mid \Delta \Delta_j \partial_2 Y^1) \, dt,
\]

where
from which and Lemma 3.1, we conclude

$$
\left| \int_0^T (\Delta_j(T_{\partial_2Y^1} \partial_1Y_t^1) \mid \Delta \Delta_j \partial_2Y^1) \ dt \right|
\leq \sum_{|j-j'| \leq 4} \left( \|S_{j-j}' \partial_1 \partial_2 Y^1 \|_{L^2_\ast_t(L^\infty)} \| \Delta_j Y^1_t \|_{L^2_{x,t}(L^2)} \| \Delta \Delta_j \partial_2Y^1 \|_{L^\infty_{x,t}(L^2)} \right)
+ \|S_{j-j}' \partial_2 Y^1 \|_{L^\infty_{x,t}(L^\infty)} \| \Delta_j Y^1_t \|_{L^2_{x,t}(L^2)} \| \Delta \Delta_j \partial_2Y^1 \|_{L^2_{x,t}(L^2)}
\lesssim c_j^2 2^{-2js} (\| \partial_1 Y^1_t \|_{L^2_{x,t}(L^\infty)} \| \partial_2 Y^1 \|_{L^2_{x,t}(L^2)} + \| \partial_2 Y^1 \|_{L^2_{x,t}(L^\infty)} \| \partial_1 Y^1 \|_{L^2_{x,t}(L^2)} + \| \partial_1 Y^1_t \|_{L^2_{x,t}(L^\infty)} \| \partial_2 Y^1 \|_{L^2_{x,t}(L^2)} + \| \partial_2 Y^1 \|_{L^2_{x,t}(L^\infty)} \| \partial_1 Y^1 \|_{L^2_{x,t}(L^2)}).
$$

While it is easy to verify that for \( s > -1 \),

$$
\left| \int_0^T (\Delta_j(R(\partial_2Y^1, \partial_1Y_t^1)) \mid \Delta \Delta_j \partial_2Y^1) \ dt \right|
\leq \sum_{|j-j'| \geq j-N_0} \| \Delta_j \partial_2 Y^1 \|_{L^\infty_{x,t}(L^2)} \| \Delta \Delta_j \partial_1 Y_t^1 \|_{L^\infty_{x,t}(L^\infty)} \| \Delta \Delta_j \partial_2 Y^1 \|_{L^\infty_{x,t}(L^2)}
\lesssim c_j^2 2^{-2js} (\| \partial_1 Y^1_t \|_{L^2_{x,t}(L^\infty)} \| \partial_2 Y^1 \|_{L^2_{x,t}(L^2)}).
$$

The same estimate holds for \( \int_0^T (\Delta_j(T_{\partial_1 Y_t^1} \partial_2 Y^1)) \mid \Delta \Delta_j \partial_2Y^1) \ dt \). Consequently we obtain for any \( s > -1 \) that

$$
\left| \int_0^T (\Delta_j(\partial_1 Y^1_t \partial_2 Y^1) \mid \Delta \Delta_j \partial_2Y^1) \ dt \right|
\lesssim c_j^2 2^{-2js} \left( \| \partial_1 Y^1_t \|_{L^2_{x,t}(L^\infty)} \| \partial_2 Y^1 \|_{L^2_{x,t}(L^2)} + \| \partial_1 Y^1 \|_{L^2_{x,t}(L^\infty)} \| \partial_2 Y^1 \|_{L^2_{x,t}(L^2)} \right).
$$

(5.24)

Finally, under the assumption (5.11), we deduce from Lemmas 3.4 and 3.5 the following

$$
\left| \int_0^T (\Delta_j(\partial_1 Y^1_t \partial_2 Y^1 \partial_1 Y_t^1) \mid \Delta \Delta_j \partial_2Y^1) \ dt \right|
\lesssim c_j^2 2^{-2js} \left( \| \partial_1 Y^1_t \|_{L^2_{x,t}(B^1_{2,1})} \| \partial_1 Y_t^1 \|_{L^2_{x,t}(B^1_{2,1})} + \| \partial_1 Y^1_t \|_{L^2_{x,t}(B^1_{2,1})} \right) \| \partial_2 Y^1 \|_{L^2_{x,t}(B^1_{2,1})} \| \partial_1 Y^1 \|_{L^2_{x,t}(B^1_{2,1})} \| \partial_1 Y^1 \|_{L^2_{x,t}(B^1_{2,1})} \| \partial_2 Y^1 \|_{L^2_{x,t}(B^1_{2,1})} \| \partial_2 Y^1 \|_{L^2_{x,t}(B^1_{2,1})} \text{ for } s > -1,
$$

from which and (5.22), (5.24), we arrive at

$$
\left| \int_0^T (\Delta_j \partial_2 G^2 \mid \Delta \Delta_j Y^1) \ dt \right|
\lesssim c_j^2 2^{-2js} \left( \| \partial_1 Y^1_t \|_{L^2_{x,t}(L^\infty)} + \| \partial_1 Y^1_t \|_{L^2_{x,t}(B^1_{2,1})} \right)
+ \| \partial_1 Y^1 \|_{L^2_{x,t}(B^1_{2,1})} + \| \nabla Y \|_{L^\infty_{x,t}(B^1_{2,1})} + \| \nabla Y \|_{L^2_{x,t}(B^1_{2,1})} \left( \| \partial_1 Y^1 \|_{L^2_{x,t}(B^1_{2,1})} + \| \partial_1 Y^1_t \|_{L^2_{x,t}(B^1_{2,1})} \right) + \| Y \|_{L^2_{x,t}(B^1_{2,1})} \| \partial_1 Y^1 \|_{L^2_{x,t}(B^1_{2,1})} + \| \partial_1 Y^1_t \|_{L^2_{x,t}(B^1_{2,1})} \right)
+ \| Y \|_{L^2_{x,t}(B^1_{2,1})} \| \partial_1 Y^1 \|_{L^2_{x,t}(B^1_{2,1})} + \| \partial_1 Y^1_t \|_{L^2_{x,t}(B^1_{2,1})} \right),
$$

(5.25)

for any \( s > -1 \) and with the assumption (5.11).

Using (4.21), and by summing up (5.19), (5.20), (5.21), (5.23) and (5.25), we conclude the proof of (5.16). This finishes the proof of Lemma 5.3. \(\square\)
5.3. The proof of Theorem 2.2. The proof of Theorem 2.2 is based on the following proposition:

**Proposition 5.3.** Let \( s_1 > 1 \) and \( s_2 \in (-1, -\frac{1}{2}) \). Let \((Y, q)\) be a smooth enough solution of (2.20)-(2.21) on \([0, T]\). We denote \( E^{s_1, s_2}_T(Y, q) \) as

\[
E^{s_1, s_2}_T(Y, q) \overset{\text{def}}{=} E^{s_1}_T(Y, q) + E^{s_2}_T(Y, q) \quad \text{and} \quad E^{s_1, s_2}_0 \overset{\text{def}}{=} E^{s_1}_0 + E^{s_2}_0,
\]

with

\[
E^{s}_T(Y, q) \overset{\text{def}}{=} \|Y_t\|^2_{L^\infty_T(B^{s}_2)} + \|Y_t\|^2_{L^\infty_T(\dot{B}^{s}_2)} + \|\partial_t Y\|^2_{L^2_T(\dot{B}^{s+1}_2)} + \|\nabla Y\|^2_{L^2_T(\dot{B}^{s+1}_2)} + \|\partial_t^2 Y\|^2_{L^2_T(\dot{B}^{s+1}_2)} + \|\partial_t Y\|^2_{L^2_T(\dot{B}^{s+1}_2)} + \|\nabla q\|^2_{L^2_T(\dot{B}^{s}_2)} + \|\nabla q\|^2_{L^2_T(\dot{B}^{s}_2)},
\]

and

\[
E^{s}_0 \overset{\text{def}}{=} \|Y_t\|^2_{H^{s}} + \|Y_t\|^2_{H^{s+1}} + \|\partial_t Y_0\|^2_{H^{s}} + \|Y_0\|^2_{H^{s+2}}.
\]

Then under the assumption (4.10) and

\[
\|\nabla Y\|_{L^\infty_T(\dot{B}^{1}_2)} + \|\nabla Y\|_{L^\infty_T(\dot{B}^{2}_2)} + \|Y\|_{L^\infty_T(\dot{B}^{s+1}_2)} + \|Y\|_{L^\infty_T(\dot{B}^{s+2}_2)} \leq 1,
\]

we have

\[
E^{s_1, s_2}_T(Y, q) \leq C_1 E^{s_1, s_2}_0 + C_1 ((E^{s_1, s_2}_0)^{1/2} + E^{s_1, s_2}_T(Y, q)^{1/2} + E^{s_1, s_2}_T(Y, q)) E^{s_1, s_2}_T(Y, q),
\]

for some positive constant \( C_1 \).

**Proof.** Under the assumptions (4.10) and (5.27), for \( s = s_1 \) and \( s = s_2 \), we can deduce from Propositions 5.1 and 5.2 that

\[
E^{s}_T(Y, q) \lesssim E^{s}_0 + \|\rho\|^2_{L^\infty_T(\dot{B}^{s}_2)} + \|\rho\|^2_{L^2_T(\dot{B}^{s+1}_2)} + \|Y\|^2_{L^\infty_T(\dot{B}^{s+2}_2)} (\|\nabla q\|_{L^2_T(\dot{B}^{s}_2)} + \|\nabla q\|_{L^2_T(\dot{B}^{s}_2)})
\]

\[
+ (\|Y_t\|^2_{L^2_T(\dot{B}^{s}_2)} + \|\partial_t Y\|^2_{L^2_T(\dot{B}^{s}_2)} + \|\partial_t \nabla Y\|^2_{L^2_T(\dot{B}^{s}_2)} + \|Y_t\|^2_{L^2_T(\dot{B}^{s}_2)})
\]

\[
+ \|\nabla Y\|^2_{L^2_T(\dot{B}^{s}_2)} E^{s}_T(Y, q) + \sum_{j \in \mathbb{Z}} \left( E^{s}_j f \mid \Delta_j Y_t - \frac{1}{4} \Delta \Delta_j Y - \Delta \Delta_j Y_t \right) dt.
\]

Thanks to Lemma 5.1, there holds

\[
\|\rho\|^2_{L^\infty_T(\dot{B}^{s}_2)} + \|\rho\|^2_{L^2_T(\dot{B}^{s+1}_2)} \lesssim (\|\nabla Y\|^2_{L^\infty_T(\dot{B}^{s}_2)} + \|\partial_t Y\|^2_{L^2_T(\dot{B}^{s}_2)} + \|\partial_t Y\|^2_{L^2_T(\dot{B}^{s}_2)}) \quad \text{for} \quad s = s_1, s_2.
\]

While it follows from Lemmas 5.2 and 5.3 that

\[
\sum_{j \in \mathbb{Z}} \left( E^{s}_j f \mid \Delta_j Y_t - \frac{1}{4} \Delta \Delta_j Y - \Delta \Delta_j Y_t \right) dt
\]

\[
\lesssim (\|Y\|^2_{L^\infty_T(\dot{B}^{s+2}_2)} + \|Y_t\|^2_{L^\infty_T(\dot{B}^{s+2}_2)} + \|\partial_t Y\|^2_{L^\infty_T(\dot{B}^{s+2}_2)} + \|\nabla q\|^2_{L^\infty_T(\dot{B}^{s+2}_2)} + \|\partial_t \nabla Y\|^2_{L^\infty_T(\dot{B}^{s+1}_2)} + \|\partial_t Y\|^2_{L^\infty_T(\dot{B}^{s+1}_2)} + \|\nabla Y\|^2_{L^\infty_T(\dot{B}^{s+1}_2)} + \|\nabla Y\|^2_{L^\infty_T(\dot{B}^{s+1}_2)} + \|\partial_t Y\|^2_{L^\infty_T(\dot{B}^{s+1}_2)} + \|\partial_t Y\|^2_{L^\infty_T(\dot{B}^{s+1}_2)}) \quad \text{for} \quad s = s_1, s_2.
\]
As a consequence and with Proposition 5.2, we obtain for \( s = s_1, s_2 \), that
\[
E^s_T(Y, q) \lesssim E^0_T + \left( \left( E^s_T(Y, q) \right)^{1/2} + E^0_T(Y, q) + E^s_T(Y, q) \right)^{1/2} + E^0_T(Y, q) + E^s_T(Y, q) + \left( E^0_T(Y, q) + E^s_T(Y, q) + \| \partial_t Y \|^2_{L^2(B_{2,1})} \right).
\]
(5.29)

Notice from (3.5), one can easily deduce
\[
\| \partial_t Y \|^2_{L^2(B_{2,1})} \lesssim \| \partial_t Y \|^2_{L^2(H^{s+1})} + \| \partial_t Y \|^2_{L^2(H^{s+1})},
\]
for any \( s_1 > 1, s_2 \in (-1, -\frac{1}{2}) \). Thus by taking \( s = s_1, s_2 \) in (5.29) and summing up the resulting inequality yields
\[
E^{s_1, s_2}_T(Y, q) \lesssim E^0_T + \left( \| \partial_t Y \|^2_{L^2(L^\infty)} + E^{s_1, s_2}_T(Y, q) \right)^{1/2} + E^{s_1, s_2}_T(Y, q) + \left( E^0_T(Y, q) + E^{s_1}_T(Y, q) \right).
\]
(5.30)

On the other hand, it follows from Proposition 4.4 and (3.5) that
\[
\| \partial_t Y \|^2_{L^2(L^\infty)} \lesssim \left( E^0_T + E^{s_1}_T(Y, q) \right)^{1/2} + E^{s_1, s_2}_T(Y, q).
\]
(5.31)

Substituting (5.31) into (5.30), one concludes the proof of (5.28).

\[\square\]

**Remark 5.1.** We should mention that the restriction for \( s_2 \in (-1, -\frac{1}{2}) \) in Theorem 2.2 is due to the following fact:
\[
\| Y_t \|^2_{L^2(B_{2,1})} + \| \partial_t Y \|^2_{L^2(B_{2,1})} \lesssim \| Y_t \|^2_{L^2(H^{s+1})} + \| \partial_t Y \|^2_{L^2(H^{s+1})} + \| \partial_t Y \|^2_{L^2(H^{s+1})},
\]
which has been used in the proof of (5.31).

Now we are in position to complete the proof of Theorem 2.2.

**Proof of Theorem 2.2.** Given initial data \((Y_0, Y_1)\) satisfying the assumptions listed in Theorem 2.2, we deduce by a standard argument that (2.20)-(2.21) has a unique solution \((Y, q)\) on \([0, T]\), which satisfies (2.25) on \([0, T]\). Let \( T^* \) be the largest possible time so that (2.25) holds. Then to complete the proof of Theorem 2.2, we only need to show that \( T^* = \infty \) and there holds (2.26) provided that (2.23) and (2.24) hold. Toward this, we denote
\[
\bar{T} \overset{\text{def}}{=} \max \{ T < T^* : E^{s_1, s_2}_T(Y, q) \leq \eta_0^2 \},
\]
for \( \eta_0 \) so small that \( C_1(2\eta_0 + \eta_0^2) \leq \frac{1}{2} \), and
\[
\| \nabla Y \|^2_{L^2(B_{2,1})} + \| \nabla Y \|^2_{L^2(B_{2,1})} + \| Y \|^2_{L^2(H^{s+1})} + \| Y \|^2_{L^2(H^{s+1})} \lesssim C_2 E^{s_1, s_2}_T(Y, q) + C_2 \eta_0 \leq 1,
\]
(5.33)

for the same \( C_1 \) as that in (5.28).

We shall prove that \( \bar{T} = \infty \) provided that \( \varepsilon_0 \) is sufficiently small in (2.24). Otherwise, by (5.33), we can apply Proposition 5.3 to conclude that
\[
E^{s_1, s_2}_T(Y, q) \leq 2C_1E^{s_1, s_2}_T(Y, q).
\]
(5.34)
In particular, if we take $\varepsilon_0$ so small that $2C_1\varepsilon_0^2 \leq \frac{1}{2}\eta_0^2$, (5.34) contradicts with (5.32) if $T < \infty$. This in turn shows that $T = T^* = \infty$, and (2.26) is valid. This completes the proof of Theorem 2.2.

6. THE PROOF OF THEOREM 2.1 AND THEOREM 1.1

With Theorem 2.2 and Lemma A.1 in the Appendix A in hand, we can now present the proof of Theorem 2.1.

Proof of Theorem 2.1. Given $\psi_0, \tilde{\psi}_0$ satisfying the assumptions of Theorem 2.1, we get by using Lemma 2.1 that there exists a vector-valued function $Y_0(y) = (Y_0^1(y), Y_0^2(y))^T$ so that

\[
\|Y_0\|_{H^{1+2\cap H^{2+2}}}^2 + \|\partial_t Y_0\|_{H^{2+2}} \leq C(\|\nabla \psi_0\|_{H^{1+1\cap H^{2+1}}}, \|\nabla \tilde{\psi}_0\|_{H^{1+1\cap H^{2+1}}}) (\|\nabla \psi_0\|_{H^{1+1\cap H^{2+1}}} + \|\nabla \tilde{\psi}_0\|_{H^{1+1\cap H^{2+1}}}) ,
\]

and $X_0(y) \overset{\text{def}}{=} I + Y_0(y)$ satisfies

\[
\nabla_y Y_0(y) = \left( \begin{array}{cc}
\partial_{x_2} \psi_0 & \partial_{x_2} \tilde{\psi}_0 \\
-\partial_{x_1} \psi_0 & -\partial_{x_1} \tilde{\psi}_0 
\end{array} \right) \circ X_0(y).
\]

Let $Y_1(y) \overset{\text{def}}{=} u_0(X_0(y))$. Applying Lemma A.1 (vi) and (ii) with $\Phi = X_0^{-1}(x)$ gives

\[
\|Y_1\|_{H^{1+1\cap H^{2+2}}} \leq C(\|\nabla_x \Psi_0\|_{L^\infty}) (1 + \|\Delta_x \Psi_0\|_{H^{1+1\cap H^{2+1}}}) \|\nabla_x u_0\|_{H^{1+1\cap H^{2+1}}},
\]

\[
\|Y_2\|_{H^{2+2}} \leq C(\|\nabla_x \Psi_0\|_{L^\infty}) \|\nabla_x \psi_0\|_{H^{2+1\cap H^{2+1}}},
\]

where $\Psi_0 \overset{\text{def}}{=} (\psi_0, \tilde{\psi}_0)^T$. Therefore, thanks to (2.5), we conclude that

\[
\|Y_0\|_{H^{1+2\cap H^{2+2}}} + \|\partial_t Y_0\|_{H^{2+2}} + \|Y_1\|_{H^{1+1\cap H^{2+2}}} + \|Y_2\|_{H^{2+2}} \leq \|\nabla \psi_0\|_{H^{1+1\cap H^{2+1}}} + \|\nabla \tilde{\psi}_0\|_{H^{1+1\cap H^{2+1}}} + \|u_0\|_{H^{1+1\cap H^{2+2}}},
\]

from which, (2.5) and Theorem 2.2, we deduce that the system (2.19) has a unique global solution $(Y, q)$ which satisfies (2.25) and (2.26) provided that $c_0$ in (2.5) is sufficiently small.

Let $X(t, y) \overset{\text{def}}{=} y + Y(t, y)$, it follows from (2.26) that $X(t, y)$ is invertible with respect to $y$ variable and we denote its inverse mapping by $X^{-1}(t, x)$. Let

\[
(a_{ij}(t, y))_{i,j=1,2} \overset{\text{def}}{=} I + \nabla_y Y(t, y) \quad \text{and} \quad (b_{ij}(t, y))_{i,j=1,2} \overset{\text{def}}{=} (a_{ij}(t, y))_{i,j=1,2}^{-1}.
\]

Then as det$(I + \nabla_y Y) = 1$, $(b_{ij})_{i,j=1,2}$ equals to the adjoint matrix of $(a_{ij})_{i,j=1,2}$ and $\sum_{i=1}^2 \partial_i b_{ij} = 0$. With the notations above, we can write

\[
\partial_{x_2} (-\partial_1 Y^2(t, X^{-1}(t, x)) - \partial_{x_1} (\partial_1 Y^1(t, X^{-1}(t, x)))
\]

\[
= -\partial_{x_2} (a_{21}(t, X^{-1}(t, x)) - \partial_{x_1} (a_{11}(t, X^{-1}(t, x)))
\]

\[
= -\sum_{j=1}^2 [\partial_j (b_{2j} a_{21} + b_{1j} a_{11})(t, X^{-1}(t, x))] = \sum_{j=1}^2 [\partial_j \delta_{j1}] (t, X^{-1}(t, x)) = 0.
\]

By a similar argument, we have

\[
\partial_{x_2} (-\partial_2 Y^2(t, X^{-1}(t, x)) - \partial_{x_1} (\partial_2 Y^1(t, X^{-1}(t, x))) = 0,
\]
so that we can define \((\psi(t, x), \tilde{\psi}(t, x))\) through

\[
\begin{align*}
\nabla_x \psi(t, x) &= (-\partial_1 Y^2(t, X^{-1}(t, x)), \partial_1 Y^1(t, X^{-1}(t, x)))^T, \quad \psi(t, x) \to 0 \text{ as } |x| \to \infty, \\
\nabla_x \tilde{\psi}(t, x) &= (-\partial_2 Y^2(t, X^{-1}(t, x)), \partial_2 Y^1(t, X^{-1}(t, x)))^T, \quad \tilde{\psi}(t, x) \to 0 \text{ as } |x| \to \infty,
\end{align*}
\]

and we define \((u(t, x), p(t, x))\) via

\[
(6.3) \quad u(t, x) \equiv Y_1(t, X^{-1}(t, x)), \quad p(t, x) \equiv q(t, X^{-1}(t, x)) - |\nabla_x(x_2 + \psi(t, x))|^2.
\]

Then according to Section 2, \((\phi, \tilde{\phi}, u, p) = (x_2 + \psi, -x_1 + \tilde{\psi}, u, p)\) thus defined satisfies (2.4) and globally solves the coupled system between (1.1) and (2.1). Then to complete the proof of Theorem 2.1, it suffices to prove (2.6).

For this, we first notice from (6.3) that

\[
(\nabla u) \circ X(t, y) = \nabla_y Y_1(t, y)(I + \nabla_y Y(t, y))^{-1},
\]

which leads to

\[
(6.4) \quad \|\nabla u\|_{L^1(\mathbb{R}^+; L^\infty)} \lesssim (1 + \|\nabla_y Y\|_{L^\infty(\mathbb{R}^+; L^\infty)})\|\nabla_y Y_1\|_{L^1(\mathbb{R}^+; L^\infty)}.
\]

Again thanks to (6.3) and (6.2), we get by applying Lemma A.1 (ii) with \(\Phi = X(t, y)\) that

\[
(6.5) \quad \|u\|_{L^\infty(\mathbb{R}^+; H^{s+2})} + \|\nabla \psi\|_{L^\infty(\mathbb{R}^+; H^{s+2})} \leq C(\|
abla Y\|_{L^\infty(\mathbb{R}^+; L^\infty)}(\|Y_1\|_{L^2(\mathbb{R}^+; H^{s+1})} + \|\partial_1 Y\|_{L^\infty(\mathbb{R}^+; H^{s+1})} + \|\partial_1 Y\|_{L^\infty(\mathbb{R}^+; L^2)}).)
\]

Whereas applying Lemma A.1 (iii) yields

\[
(6.6) \quad \|u\|_{L^2(\mathbb{R}^+; H^{s+1})} + \|\nabla \psi\|_{L^\infty(\mathbb{R}^+; H^{s+1})} \leq C(\|
abla Y\|_{L^\infty(\mathbb{R}^+; L^\infty)}(\|Y_1\|_{L^2(\mathbb{R}^+; H^{s+1})} + \|\partial_1 Y\|_{L^\infty(\mathbb{R}^+; H^{s+1})} + \|\partial_1 Y\|_{L^\infty(\mathbb{R}^+; H^{s+1})})).
\]

In the same manner, we get by applying Lemma A.1 (vi) to (6.3) and (6.2) that

\[
(6.7) \quad \|u\|_{L^\infty(\mathbb{R}^+; H^{s+1})} + \|\nabla \psi\|_{L^\infty(\mathbb{R}^+; H^{s+1})} \leq C(\|
abla Y\|_{L^\infty(\mathbb{R}^+; L^\infty)}(1 + \|\Delta Y\|_{L^\infty(\mathbb{R}^+; H^{s+1})} + \|\partial_1 Y\|_{L^\infty(\mathbb{R}^+; H^{s+1})} + \|\partial_1 Y\|_{L^\infty(\mathbb{R}^+; H^{s+1})}).
\]

Consequently, we deduce from (2.26), (3.5), (6.1), and (6.4) to (6.7) that

\[
(6.8) \quad \|u\|_{L^\infty(\mathbb{R}^+; H^{s+1} \cap H^{s+2})} + \|\nabla \psi\|_{L^\infty(\mathbb{R}^+; H^{s+1} \cap H^{s+2})} \lesssim \|\nabla \psi_0\|_{H^{s+1} \cap H^{s+2}} + \|\nabla \psi_0\|_{H^{s+1} \cap H^{s+2}} + \|u_0\|_{H^{s+1} \cap H^{s+2}},
\]

provided that (2.5) holds for \(c_0\) sufficiently small.

Next, it follows from (6.3) that

\[
\nabla_x p = ((I + \nabla Y)^{-1} \nabla q) \circ X^{-1} - 2\nabla_x \partial_{x_2} \psi - \nabla_x(|\nabla_x \psi|^2).
\]
Applying Lemma A.1 (ii) and Lemma 3.4 gives rise to
\[ \| \nabla_x p(t) \|_{\dot{H}^2} \leq C(\| \nabla Y \|_{L^\infty(\mathbb{R}^+; L^\infty)}) \{(1 + |\nabla Y|)^{-\gamma} \} \left( \| (I + \nabla Y)^{-\gamma} \nabla u(t) \|_{\dot{H}^2} + \| \nabla Y(t) \|_{\dot{H}^{2+1}} \right) \times \| (I + \nabla Y)^{-\gamma} \nabla u(t) \|_{L^2} + \| \nabla \psi(t) \|_{\dot{H}^{2+1}} + \| (\nabla \psi)^2(t) \|_{\dot{H}^{2+1}} \right) \leq C(\| \nabla Y \|_{L^\infty(\mathbb{R}^+; L^\infty)}) \{(1 + |\nabla Y(t)|_{\dot{H}^1}) \{(\nabla q(t))_{\dot{H}^2} + \| \nabla Y(t) \|_{\dot{H}^{2+1}} \| \nabla q(t) \|_{L^2} \} \right) \times \| (I + \nabla Y)^{-\gamma} \nabla u(t) \|_{L^2} + \| \nabla \psi(t) \|_{\dot{H}^{2+1}} \| \nabla \psi(t) \|_{\dot{H}^{2+1}} \right) \}
\]
which along with (2.26), (3.5), (6.1) and (6.8) implies that
\[ \| \nabla_x p(t) \|_{L^2(\mathbb{R}^+; \dot{H}^2)} \leq \| \nabla \psi_0(t) \|_{\dot{H}^{2+1} \cap \dot{H}^2} + \| \nabla \tilde{\psi}_0(t) \|_{\dot{H}^{2+1} \cap \dot{H}^2} + \| \tilde{u}_0 \|_{\dot{H}^{2+1} \cap \dot{H}^2} \]
provided that (2.5) holds for \( c_0 \) sufficiently small.

Finally, applying Lemma A.1 (v) and (iv), along with (2.26), (3.5), (6.1), (6.8) and Lemma 3.4 yields
\[ \| \nabla_x p(t) \|_{L^2(\mathbb{R}^+; \dot{H}^{2+1})} \leq \| \nabla \psi_0(t) \|_{\dot{H}^{2+1} \cap \dot{H}^2} + \| \nabla \tilde{\psi}_0(t) \|_{\dot{H}^{2+1} \cap \dot{H}^2} + \| \tilde{u}_0 \|_{\dot{H}^{2+1} \cap \dot{H}^2} \]
This completes the proof of (2.6) and thus Theorem 2.1. \( \square \)

Before we present the proof of Theorem 1.1, we shall first prove the following blow-up criterion for smooth enough solutions of (1.3).

**Proposition 6.1.** Given \( \nabla \psi_0 \in H^s(\mathbb{R}^2) \) and \( u_0 \in H^s(\mathbb{R}^2) \) for \( s > 1 \), (1.3) has a unique solution \((\psi, u)\) on \([0, T]\) for some \( T > 0 \) so that
\[ \nabla \psi \in C([0, T]; H^s(\mathbb{R}^2)), \quad u \in C([0, T]; H^s(\mathbb{R}^2)), \quad \nabla u \in L^2((0, T); H^s(\mathbb{R}^2)), \]
\[ \nabla p \in C([0, T]; H^{s-1}(\mathbb{R}^2)). \]
Moreover, if \( T^* \) is the lifespan to this solution and \( T^* < \infty \), then
\[ \int_0^{T^*} (\| \nabla u(t) \|_{L^\infty} + \| \nabla \psi(t) \|_{L^\infty}^2) \, dt = \infty. \]

**Proof.** Given initial data \((\psi_0, u_0)\), it is standard to prove that (1.3) has a unique solution \((\psi, u)\) on \([0, T]\) for some \( T > 0 \), so that there holds the first line of (6.9). While we get by taking div to the momentum equation of (1.3) that
\[ \nabla p = -2\nabla \partial_2 \psi + \nabla (\Delta)^{-1} \text{div} \left\{ u \cdot \nabla u + \text{div}(\nabla \psi \otimes \nabla \psi) \right\}, \]
which along with the first line of (6.9) implies that \( \nabla p \in C([0, T]; H^{s-1}(\mathbb{R}^2)) \). This proves (6.9).

It remains to verify the blow-up criterion (6.10). Toward this, for any \( t < T^* \), we get by using a standard energy estimate for (1.3) that
\[ \frac{1}{2} \frac{d}{dt} (\| \nabla \psi(t) \|_{L^2}^2 + \| u(t) \|_{L^2}^2) + \| \nabla u(t) \|_{L^2}^2 = 0, \]
which implies that for any \( t < T^* \)
\[ \| \nabla \psi(t) \|_{L^2}^2 + \| u(t) \|_{L^2}^2 \leq \| \nabla \psi_0 \|_{L^2}^2 + \| u_0 \|_{L^2}^2. \]
While acting \( \Delta_j \) to \( u^1 \) equation of (1.3) and then taking the \( L^2 \) inner product of the resulting equation with \( \Delta_j u^1 \), it leads to
\[ \frac{1}{2} \frac{d}{dt} \| \Delta_j u^1(t) \|_{L^2}^2 + \| \nabla \Delta_j u^1 \|_{L^2}^2 \]
\[ = - (\Delta_j \partial_1 (p + \partial_2 \psi) \cdot \Delta_j u^1) - (\Delta_j (u \cdot \nabla u^1) \cdot \Delta_j u^1) - (\text{div} \Delta_j (\partial_1 \psi \cdot \nabla \psi) \cdot \Delta_j u^1). \]
Similarly acting $\Delta_j$ to $u^2$ equation of (1.3) and then taking the $L^2$ inner product of the resulting equation with $\Delta_j u^2$ leads to
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\Delta_j u^2(t)\|_{L^2}^2 + \|\nabla \Delta_j u^2\|_{L^2}^2 + (\Delta_j \Delta \psi | \Delta_j u^2) \\
= - (\Delta_j \partial_2(p + \partial_2\psi) | \Delta_j u^2) - (\Delta_j(u \cdot \nabla u^2) | \Delta_j u^2) - (\text{div} \Delta_j(\partial_2\psi \nabla \psi) | \Delta_j u^2).
\end{equation}

However by the transport equation of (1.3) and using integration by parts, one has
\[ (\Delta_j \Delta \psi | \Delta_j u^2) = - (\Delta_j \Delta \psi | \Delta_j(\partial_2\psi + u \cdot \nabla \psi)) \]
\[ = \frac{1}{2} \frac{d}{dt} \|\nabla \Delta_j \psi(t)\|_{L^2}^2 + (\Delta_j \nabla \psi | \nabla \Delta_j(u \cdot \nabla \psi)). \]

Hence by combining (6.12) with (6.13) and using $\text{div} \ u = 0$, we obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\Delta_j u(t)\|_{L^2}^2 + \|\nabla \Delta_j \psi(t)\|_{L^2}^2 + \|\nabla \Delta_j u\|_{L^2}^2 \\
= - (\Delta_j(u \cdot \nabla u) | \Delta_j u) - (\Delta_j \nabla \psi | \nabla \Delta_j(u \cdot \nabla \psi)) - (\text{div} \Delta_j(\nabla \psi \otimes \nabla \psi) | \Delta_j u).
\end{equation}

Next for $s > 0$, we claim that
\begin{equation}
| (\Delta_j(u \cdot \nabla b) | \Delta_j b) | \lesssim c_j(t)^2 2^{-2js} (\|\nabla u(t)\|_{L^\infty} \|b(t)\|_{\dot{H}^s} + \|\nabla b(t)\|_{L^\infty} \|u(t)\|_{\dot{H}^s} \|b(t)\|_{\dot{H}^s}) \text{ or } \\
| (\Delta_j(u \cdot \nabla b) | \Delta_j b) | \lesssim c_j(t)^2 2^{-2js} (\|\nabla u(t)\|_{L^\infty} \|b(t)\|_{\dot{H}^s}) \plusim \|b(t)\|_{L^\infty} \|u(t)\|_{\dot{H}^s} \|b(t)\|_{\dot{H}^s}).
\end{equation}

Indeed applying Bony’s decomposition (3.7) for $u \cdot \nabla b$ and then using a standard commutator’s argument, we can write
\[ (\Delta_j(u \cdot \nabla b) | \Delta_j b) = \sum_{|j-\ell| \leq 5} (|\Delta_j; S_{\ell-1} u| \cdot \nabla \Delta_j b + (S_{\ell-1} u - S_{j-1} u) \cdot \nabla \Delta_j b | \Delta_j b) \\
+ (S_{j-1} u \cdot \nabla \Delta_j b | \Delta_j b) - (\Delta_j(\mathcal{R}(u, \nabla b)) | \Delta_j b). \]

It follows from the classical commutator’s estimate (see [2] for instance) that
\[ \sum_{|j-\ell| \leq 5} |(|\Delta_j; S_{\ell-1} u| \cdot \nabla \Delta_j b | \Delta_j b)| \lesssim \sum_{|j-\ell| \leq 5} \|\nabla S_{\ell-1} u\|_{L^\infty} \|\Delta_j b\|_{L^2} \|\Delta_j b\|_{L^2} \\
\lesssim c_j(t)^2 2^{-2js} \|\nabla u(t)\|_{L^\infty} \|b(t)\|_{\dot{H}^s}. \]

The same estimate holds for $\sum_{|j-\ell| \leq 5} (|S_{j-1} u - S_{j-1} u| \cdot \nabla \Delta_j b | \Delta_j b)$ and $(S_{j-1} u \cdot \nabla \Delta_j b | \Delta_j b)$.

Whereas applying Lemma 3.1 gives
\[ \|\Delta_j \mathcal{R}(u, \nabla b)\|_{L^2} \lesssim \sum_{|j-\ell| \leq 5} \|\Delta_j u\|_{L^2} \|S_{j+2} \nabla b\|_{L^\infty}, \]

which can be controlled by $c_j(t)^2 2^{-js} \|\nabla b(t)\|_{L^\infty} \|u(t)\|_{\dot{H}^s}$ or $c_j(t)^2 2^{-js} \|b(t)\|_{L^\infty} \|\nabla u(t)\|_{\dot{H}^s}$ as long as $s > 0$. This completes the proof of (6.15).

Now we go back to (6.14). In fact, applying Lemma 3.4 (i) and (6.15) to (6.14) gives
\[ \frac{1}{2} \frac{d}{dt} (\|\Delta_j u(t)\|_{L^2}^2 + \|\nabla \Delta_j \psi(t)\|_{L^2}^2 + \|\nabla \Delta_j u\|_{L^2}^2) \\
\lesssim c_j(t)^2 2^{-2js} \{ \|\nabla u\|_{L^\infty} (\|u\|_{\dot{H}^s}^2 + \|\nabla \psi\|_{\dot{H}^s}^2) + \|\nabla \psi\|_{L^\infty} \|\nabla u\|_{\dot{H}^s} \|\nabla \psi\|_{\dot{H}^s} \} \text{ for } s > 0. \]
The above implies for any $s > 0$

$$
\|u(t)\|_{H^s}^2 + \|\nabla \psi(t)\|_{H^s}^2 + \|\nabla u\|_{L^2_t(H^s)}^2
\leq \|u_0\|_{H^s}^2 + \|\nabla \psi_0\|_{H^s}^2 + C \int_0^t (\|\nabla u(t')\|_{L^\infty} + \|\nabla \psi(t')\|_{L^\infty}^2)(\|u(t')\|_{H^s} + \|\nabla \psi(t')\|_{H^s}^2) \, dt'.
$$

Applying Gronwall’s inequality yields

$$
\|u(t)\|_{H^s}^2 + \|\nabla \psi(t)\|_{H^s}^2 + \|\nabla u\|_{L^2_t(H^s)}^2
\leq (\|u_0\|_{H^s}^2 + \|\nabla \psi_0\|_{H^s}^2) \exp\left\{C \int_0^t (\|\nabla u(t')\|_{L^\infty} + \|\nabla \psi(t')\|_{L^\infty}^2) \, dt' \right\} \quad \text{for } t < T^*,
$$

which together with (6.11) implies (6.10). This completes the proof of Proposition 6.1. \qed

In order to apply Theorem 2.1 to prove Theorem 1.1, we also need the following lemma concerning the existence of $\tilde{\psi}_0$ so that there holds (2.2).

**Lemma 6.1.** Under the assumptions of Theorem 1.1, (2.2) has a solution $\tilde{\psi}_0 \in H^{s_1+2}(\mathbb{R}^2)$ so that there holds

$$
\|\tilde{\psi}_0\|_{H^{s_1+2}} \leq C(K, \|\nabla \psi_0\|_{H^{s_1+2}}) \|\partial_{x_2} \psi_0\|_{H^{s_1+2}}.
$$

The proof of Lemma 6.1 will be postponed in the Appendix C. We now turn to the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Under the assumption of Theorem 1.1, we deduce from Lemma 6.1 that there exists a $\tilde{\psi}_0$ so that there holds (6.16) and (2.2). Notice that for $s_2 \in (-1, -\frac{1}{2})$, then it is easy to observe that

$$
\|\nabla \tilde{\psi}_0\|_{H^{s_1+1} \cap \dot{H}^{s_2+1}} \lesssim \|\tilde{\psi}_0\|_{H^{s_1+2}} \leq C(\|\nabla \psi_0\|_{H^{s_1+2}}) \|\partial_{x_2} \psi_0\|_{H^{s_1+2}}
$$

Therefore, under the assumption of (1.7), we infer from Theorem 2.1 that the coupled system (1.1) and (2.1) has a unique global solution $(\phi, \hat{\phi}, u, p) = (x_2 + \psi, -x_1 + \hat{\psi}, u, p)$ so that there holds (2.4) and (2.6). Then according to the discussions at the beginning of Section 2, $(\hat{\psi}, u, p)$ thus obtained solves (1.3), which is in fact the unique solution of (1.3) with initial data $(\psi_0, u_0)$, and there holds (1.9).

On the other hand, thanks to Proposition 6.1, given initial data $(\psi_0, u_0)$ with $\nabla \psi_0 \in H^s(\mathbb{R}^2)$, $u_0 \in H^s(\mathbb{R}^2)$, (1.3) has a unique solution $(\psi, u, p)$ with $\nabla \psi \in C([0, T]; H^s(\mathbb{R}^2))$, $u \in C([0, T]; H^s(\mathbb{R}^2))$, $\nabla u \in L^2(\mathbb{R}^2)$, $\nabla p \in C([0, T]; H^{s-1}(\mathbb{R}^2))$ for any given $T < T^*$. Moreover, if $T^* < \infty$, there holds (6.10). Due to the uniqueness, this solution must coincide with the one obtained in the last paragraph. By virtue of (1.9), (6.10) can not be true for any finite $T^*$. Therefore $T^* = \infty$ and there holds (1.8). This completes the proof of Theorem 1.1. \qed

**Appendix A. The Sobolev estimates to a function composed with a measure preservation mapping**

**Lemma A.1.** Let $\Phi(y) = y + \Psi(y)$ be a diffeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$ with $\det(\nabla y \Psi) = \det(I + \nabla y \Phi) = 1$, and $\Phi^{-1}(x)$ be its inverse mapping. Then for any smooth function $u, v$, there hold

(i) if $s = 0$,

$$
\|u \circ \Phi\|_{H^0} = \|u\|_{H^0} \quad \text{and} \quad \|v \circ \Phi^{-1}\|_{H^0} = \|v\|_{H^0};
$$

(ii) if $s > 0$,

$$
\|u \circ \Phi\|_{H^s} \leq C(\|u\|_{H^s}) \quad \text{and} \quad \|v \circ \Phi^{-1}\|_{H^s} \leq C(\|v\|_{H^s}).
$$
(ii) if \(-1 < s < 0\),
\[
\|u \circ \Phi\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{s+3}\|u\|_{H^s} + (1 + \|\nabla_y \Psi\|_{L^\infty})\|\nabla_y \Psi\|_{H^{s+1}}\|u\|_{L^2} \quad \text{and}
\|v \circ \Phi^{-1}\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{s+3}\|v\|_{H^s} + (1 + \|\nabla_y \Psi\|_{L^\infty})\|\nabla_y \Psi\|_{H^{s+1}}\|v\|_{L^2};
\]

(iii) if \(0 < s < 1\),
\[
\|u \circ \Phi\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{s+1}\|u\|_{H^s} \quad \text{and}
\|v \circ \Phi^{-1}\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{s+1}\|v\|_{H^s};
\]

(iv) if \(s = 1\),
\[
\|u \circ \Phi\|_{H^1} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})\|u\|_{H^1} \quad \text{and}
\|v \circ \Phi^{-1}\|_{H^1} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})\|v\|_{H^1};
\]

(v) if \(1 < s \leq 2\),
\[
\|u \circ \Phi\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{s+1}\|u\|_{H^s} + \|\nabla_y \Psi\|_{H^s}\|\nabla_x u\|_{L^2} \quad \text{and}
\|v \circ \Phi^{-1}\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{s+1}\|v\|_{H^s} + \|\nabla_y \Psi\|_{H^s}\|\nabla_y v\|_{L^2};
\]

(vi) if \(s > 2\),
\[
\|u \circ \Phi\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{s+1}(1 + \|\Delta_y \Psi\|_{H^{s-2}})\|\nabla_x u\|_{H^{s-1}} \quad \text{and}
\|v \circ \Phi^{-1}\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{s+1}(1 + \|\Delta_y \Psi\|_{H^{s-2}})\|\nabla_y v\|_{H^{s-1}}.
\]

**Proof.** We denote
\[
\mathcal{A} = (a_{ij})_{i,j=1,2} \overset{\text{def}}{=} I + \nabla_y \Psi, \quad \mathcal{B} = (b_{ij})_{i,j=1,2} \overset{\text{def}}{=} (I + \nabla_y \Psi)^{-1}.
\]
Due to \(\det \mathcal{A} = 1\), the matrix \(\mathcal{B}\) equals to the adjoint matrix of \(\mathcal{A}\). This leads to
\[
(A.1) \quad (\partial_x u) \circ \Phi = \sum_{j=1}^{2} b_{ij} \partial_y (u \circ \Phi) \quad \text{and} \quad (\partial_y v) \circ \Phi^{-1} = \sum_{j=1}^{2} a_{ij} \circ \Phi^{-1} \partial_x (v \circ \Phi^{-1}).
\]

In what follows, we shall only present the proof of the related estimates involving \(u \circ \Phi\), and the ones involving \(v \circ \Phi^{-1}\) is identical. Firstly it follows from \(\det (\nabla_y \Phi) = 1\) that
\[
\|u \circ \Phi\|_{H^0} = \|u\|_{H^0}.
\]
When \(s \in (0,1)\), we obtain from
\[
\|f\|_{H^s}^2 \sim \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|f(x) - f(y)|^2}{|x - y|^{2+2s}} \, dx \, dy,
\]
that
\[
(A.2) \quad \|u \circ \Phi\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{s+1}\|u\|_{H^s}.
\]

For the case \(s \in (-1,0)\), we get, by using (A.1), that
\[
u \circ \Phi = -(\nabla_x \cdot \nabla_x (-\Delta_x)^{-1} u) \circ \Phi = -B^T \nabla_y \cdot ((\nabla_x (-\Delta_x)^{-1} u) \circ \Phi),
\]
which combining with (iii) of Lemma 3.4 leads to
\[
\|u \circ \Phi\|_{H^s} \lesssim \|\nabla_y \cdot ((\nabla_x (-\Delta_x)^{-1} u) \circ \Phi)\|_{H^s} + \|((B^T - I) \nabla_y \cdot ((\nabla_x (-\Delta_x)^{-1} u) \circ \Phi))\|_{H^s} \\
\lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})\|\nabla_x (-\Delta_x)^{-1} u \circ \Phi\|_{H^{s+1}} + \|\nabla_y \Psi\|_{H^{s+1}} \|\nabla_y (\nabla_x (-\Delta_x)^{-1} u) \circ \Phi\|_{L^2}.
\]
Applying (A.1) and (A.2), one thus obtains for $s \in (-1,0)$ that
\begin{equation}
\|u \circ \Phi\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{s+\frac{3}{2}}u\|_{H^s} + (1 + \|\nabla_y \Psi\|_{L^\infty})\|\nabla_y \Psi\|_{H^{s+1}}\|u\|_{L^2}.
\end{equation}
Whereas we deduce from (A.1) that
\begin{equation}
\|u \circ \Phi\|_{H^1} \lesssim \|\nabla_x u \circ \Phi(I + \nabla_y \Psi)\|_{L^2} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})\|u\|_{H^1}.
\end{equation}
To handle the case that $1 < s \leq 2$, we first use (A.1) and then Lemma 3.4 (iii) to deduce
\begin{equation}
\|u \circ \Phi\|_{H^s} \lesssim [(\nabla_x u \circ \Phi(I + \nabla_y \Psi))\|_{H^{s-1}} + \|\nabla_y \Psi\|_{H^s}\|\nabla_x u \circ \Phi\|_{L^2},
\end{equation}
which along with (A.2) and (A.4) ensures
\begin{equation}
\|u \circ \Phi\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{s+\frac{1}{2}}\|u\|_{H^s} + \|\nabla_y \Psi\|_{H^s}\|\nabla_x u\|_{L^2}.
\end{equation}
For $k < s - 1 \leq k + 1$ ($k \in \mathbb{N}$), applying (A.5) repeatedly, we obtain
\begin{equation}
\|u \circ \Phi\|_{H^{s-1}} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^j\|\nabla_x^j u \circ \Phi\|_{H^{s-1}},
\end{equation}
On the other hand, thanks to (A.1) and (i) of Lemma 3.4, one has
\begin{equation}
\|u \circ \Phi\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})\|\nabla_x u \circ \Phi\|_{H^s} + \|\nabla_y \Psi\|_{H^s}\|\nabla_x u\|_{L^\infty},
\end{equation}
this combining with (A.7) yields for $k + 1 < s \leq k + 2$ ($k \in \mathbb{N}$) that
\begin{equation}
\|u \circ \Phi\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{k+\frac{1}{2}}\|\nabla_x^{k+1} u \circ \Phi\|_{H^{s-(k+1)}}
\end{equation}
\begin{equation}
+ \sum_{j=2}^{k+1} (1 + \|\nabla_y \Psi\|_{L^\infty})^{j-1}\|\nabla_y \Psi\|_{H^{s-1-j}}\|\nabla_x^j u\|_{L^2} + \|\nabla_y \Psi\|_{H^{s-1}}\|\nabla_x u\|_{L^\infty}
\lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{k+1}\|\nabla_x^{k+1} u \circ \Phi\|_{H^{s-(k+1)}} + \|\Delta_y \Psi\|_{H^{s-2}}\|\nabla_x u\|_{H^{s-1}},
\end{equation}
where we used the embedding inequality
\begin{equation}
\|\nabla_x u\|_{L^\infty} \lesssim \|\nabla_x u\|_{H^{s-1}} \quad \text{for} \quad s > 2.
\end{equation}
By (A.2) and (A.8), we finally obtain
\begin{equation}
\|u \circ \Phi\|_{H^s} \lesssim (1 + \|\nabla_y \Psi\|_{L^\infty})^{s+1}(1 + \|\Delta_y \Psi\|_{H^{s-2}})\|\nabla_x u\|_{H^{s-1}}.
\end{equation}
This completes the proof of Lemma A.1.

\section*{Appendix B. The Proof of Lemmas 4.1 and 4.2}

\textbf{Proof of Lemma 4.1.} We first get by applying Bony’s decomposition (3.7) and (3.8) that
\begin{equation}
\partial_t Y^1 \partial_1 Y_{i} = (TT^{h} + TR^{h} + T\bar{T}^{h} + \bar{T}T^{h} + \bar{T}R^{h} + RT^{h} + R\bar{R}^{h})(\partial_2 Y^1, \partial_1 Y_{i}).
\end{equation}
Since
\begin{equation}
\|\Delta_{y}^{j} S_{k_i^{+2}} \partial_1 Y_{i}(t)\|_{L^2} \lesssim d_{j',k'}(t)2^{-\frac{j'j}{2}}\|Y_{i}(t)\|_{B^{\frac{j'}{2}, \frac{j'}{2}}},
\end{equation}
and since
\begin{equation}
\|S_{j'-1} \Delta_{k'}^{h} \partial_2 Y^1(t)\|_{L^2_{y}(L^\infty)} \lesssim d_{k'}(t)2^{-\frac{j'}{2}}\|\partial_2 Y^1(t)\|_{B^{\frac{j'}{2}, \frac{j'}{2}}},
\end{equation}

\end{document}
by applying Lemma 3.2 and Lemma 3.1, we have
\[ \| \Delta_j \Delta_{k}^{h}(TT^{h}(\partial_{2}Y^{1}, \partial_{1}Y_{1})) (t) \|_{L^{2}} \]
\[ \lesssim 2^{j} \sum_{|j'| \leq 4} \| S_{j'-1} \Delta_{k}^{h}(\partial_{2}Y^{1}) (t) \|_{L^{2}_{t}(L^{\infty}_{x})} \| \Delta_{j'} S_{k'-1}^{h}(\partial_{1}Y_{1}) (t) \|_{L^{2}}. \]
\[ \lesssim 2^{j} \sum_{|j'-1| \leq 4} \| d_{j',k'}(t) 2^{-j'} 2^{-j} \|_{\tilde{B}_{2,1}^{1}} \| Y_{1}(t) \|_{\tilde{B}_{2,1}^{1}}. \]

The same estimate holds for \( \| \Delta_{j} \Delta_{k}^{h}(TT^{h}(\partial_{2}Y^{1}, \partial_{1}Y_{1}))(t) \|_{L^{2}}. \)

Following the same line of the arguments, one has
\[ \| \Delta_{j} \Delta_{k}^{h}(RR^{h}(\partial_{2}Y^{1}, \partial_{1}Y_{1}))(t) \|_{L^{2}} \]
\[ \lesssim 2^{j} \sum_{|j' \leq 1} \| \Delta_{j'} \Delta_{k}^{h}(\partial_{2}Y^{1}) (t) \|_{L^{2}} \| \tilde{\Delta}_{j'} S_{k'}^{h}(\partial_{1}Y_{1}) (t) \|_{L^{2}}. \]
\[ \lesssim 2^{j} \sum_{|j' \leq 1} \| d_{j',k'}(t) 2^{-j'} 2^{-j} \|_{\tilde{B}_{2,1}^{1}} \| Y_{1}(t) \|_{\tilde{B}_{2,1}^{1}}. \]

Similar estimate holds for \( \| \Delta_{j} \Delta_{k}^{h}(TT^{h}(\partial_{2}Y^{1}, \partial_{1}Y_{1}))(t) \|_{L^{2}}, \| \Delta_{j} \Delta_{k}^{h}(TT^{h}(\partial_{2}Y^{1}, \partial_{1}Y_{1}))(t) \|_{L^{2}}, \)
and \( \| \Delta_{j} \Delta_{k}^{h}(TT^{h}(\partial_{2}Y^{1}, \partial_{1}Y_{1}))(t) \|_{L^{2}}. \)

Finally as
\[ \| S_{j'-1} S_{k'-1}^{h}(\partial_{1}Y_{1})(t) \|_{L^{\infty}} \lesssim d_{k'}(t) 2^{j'} \| Y_{1}(t) \|_{\tilde{B}_{2,1}^{1}}, \]
we get by applying Lemma 3.1 that
\[ \| \Delta_{j} \Delta_{k}^{h}(\tilde{T}T^{h}(\partial_{2}Y^{1}, \partial_{1}Y_{1}))(t) \|_{L^{2}} \lesssim \sum_{|j' \leq 4} \| \Delta_{j'} \Delta_{k}^{h}(\partial_{2}Y^{1}) (t) \|_{L^{2}} \| S_{j'-1} S_{k'-1}^{h}(\partial_{1}Y_{1})(t) \|_{L^{\infty}}. \]
\[ \lesssim \sum_{|j' \leq 4} \| d_{j',k'}(t) 2^{-j'} \|_{B_{2,1}^{1}} \| Y_{1}(t) \|_{B_{2,1}^{1}}. \]
\[ \lesssim d_{j,k}(t) 2^{-j} \| \partial_{1}Y_{1}(t) \|_{B_{2,1}^{1}} \| Y_{1}(t) \|_{B_{2,1}^{1}}. \]

Substituting the above estimates into (B.1) and integrating the resulting inequality over \((0, T), \) we complete the proof of Lemma 4.1. \( \square \)

**Proof of Lemma 4.2.** Thanks to Definition 3.2 and Lemmas 3.2, 3.3, we obtain that
\[ \| \partial_{1} \partial_{2}Y^{1}(\partial_{2}Y_{1}) \|_{L^{1}_{t}(L^{6}_{x})} = \| \partial_{1} \partial_{2}Y^{1}(\partial_{2}Y_{1} + \partial_{2}Y^{1}(\partial_{1} \partial_{2}Y_{1}) \|_{L^{1}_{t}(L^{6}_{x})}. \]
\[ \lesssim \| \partial_{1} \partial_{2}Y^{1} \|_{L^{\infty}_{t}(L^{6}_{x})} \| \partial_{2}Y_{1} \|_{L^{1}_{t}(L^{1}_{x})} + \| \partial_{2}Y^{1} \|_{L^{\infty}_{t}(L^{6}_{x})} \| \partial_{1} \partial_{2}Y_{1} \|_{L^{1}_{t}(L^{6}_{x})}. \]
\[ \lesssim \| \partial_{2}Y^{1} \|_{L^{\infty}_{t}(B_{2,1}^{1})} \| Y_{1} \|_{L^{1}_{t}(B_{2,1}^{1})}, \]
which finishes the proof of Lemma 4.2. \( \square \)
Lemma C.1. Let $s > 2$ and $f \in H^s(\mathbb{R}^2)$ with $\text{Supp } f(\cdot, x_2) \subset \lbrack -K, K \rbrack$ for some positive constant $K$. Let $b = (b^1, b^2)^T$ be a divergence free vector field with $\nabla b \in H^{s-1}(\mathbb{R}^2)$ and $b^1 \geq \frac{1}{2}$. We assume moreover that $f$ and $b$ are admissible on $\{0\} \times \mathbb{R}$ in the sense of Definition 1.1. Then (1.6) has a solution $\psi \in H^s(\mathbb{R}^2)$ so that there holds
\begin{equation}
\|\psi\|_{H^s} \leq C(K, \|\nabla b\|_{H^{s-1}})\|f\|_{H^s}.
\end{equation}

Proof. Since $\nabla b \in H^{s-1}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$, (1.5) has a unique global solution on $\mathbb{R}$ so that for all $t \in \mathbb{R}$,
\begin{equation}
\|\nabla X(t, \cdot)\|_{L^\infty} \leq \exp\left(\|\nabla b\|_{L^\infty}|t|\right) \quad \text{and} \quad \det\left(\frac{\partial X(t, x)}{\partial x}\right) = 1.
\end{equation}
While it follows from (1.5) and (1.6) that
\[ \frac{d}{dt} \psi(X(t, x)) = f(X(t, x)), \]
from which, we define
\[ \psi(x) = \begin{cases} -\int_{-\infty}^{\infty} f(X(t, x)) \, dt & \text{if } x_1 \geq 0, \\ \int_{-\infty}^{\infty} f(X(t, x)) \, dt & \text{if } x_1 \leq 0. \end{cases} \]
Thanks to the assumption that $f$ and $b$ are admissible on $\{0\} \times \mathbb{R}$ in the sense of Definition 1.1, the values of $\psi(x)$ at $(0, x_2)$ are compatible. We remark that the equation (1.6), $b^1 \partial_2 \psi = -b^2 \partial_2 \psi + f$, and $b^1 \geq \frac{1}{2}$ implies that the derivatives of $\psi$ in the $x_2$ variable yields the derivatives of $\psi$ with respect to $x_1$ variable. Therefore, we do not require any admissible condition for the derivatives of $f$ and $b$.

On the other hand, as $b^1 \geq \frac{1}{2}$, we deduce from (1.5) that
\begin{align*}
X^1(t, x) &\geq x_1 + \frac{t}{2} \geq K \quad \text{if } t \geq 2K, \ x_1 \geq 0, \\
X^1(t, x) &\leq x_1 + \frac{t}{2} \leq -K \quad \text{if } t \leq -2K, \ x_1 \leq 0,
\end{align*}
which together with the assumption: $\text{Supp } f(\cdot, x_2) \subset \lbrack -K, K \rbrack$ for some positive constant $K$, implies that
\begin{equation}
\psi(x) = \begin{cases} -\int_{-\infty}^{2K} f(X(t, x)) \, dt & \text{if } x_1 \geq 0, \\ \int_{-2K}^{\infty} f(X(t, x)) \, dt & \text{if } x_1 \leq 0. \end{cases}
\end{equation}
It remains to prove (C.1). Indeed for any $s > 2$, we deduce from (1.5) and product laws in Sobolev spaces that
\[ \|\nabla_x X(t, \cdot) - I\|_{H^{s-1}} \lesssim \int_0^t \left( \|\nabla b\|_{L^\infty} \|\nabla_x X(t', \cdot) - I\|_{H^{s-1}} \\
+ \|\nabla(X(t', \cdot)\right)\|_{H^{s-1}} (1 + \|\nabla_x X(t', \cdot) - I\|_{L^\infty}) \right) \, dt', \]
from which, (C.2) and Lemma A.1, we infer, for $|t| \leq 2K$, that
\[ \|\nabla_x X(t, \cdot) - I\|_{H^{s-1}} \leq C(K, \|\nabla b\|_{H^{s-1}}) \left( \|\nabla b\|_{H^{s-1}} + \int_0^t \|\nabla_x X(t', \cdot) - I\|_{H^{s-1}} \, dt' \right). \]
Applying Gronwall’s Lemma gives rise to
\[
(C.4) \max_{|t| \in [0,2K]} \|\nabla_x X(t,\cdot) - I\|_{H^{s-1}} \leq C(K, \|\nabla b\|_{H^{s-1}}) \|\nabla b\|_{H^{s-1}}.
\]

By virtue of Lemma A.1 and (C.2), we thus deduce from (C.3) that
\[
\|\psi\|_{H^s} \lesssim \int_{-2K}^{2K} C(\|\nabla_x X(t',\cdot) - I\|_{L^\infty})(1 + \|\nabla_x X(t',\cdot) - I\|_{H^{s-1}}) \|f\|_{H^{s}} dt',
\]
which along with (C.2) and (C.4) implies (C.1). And Lemma C.1 is proved. \(\square\)

We now turn to the proof of Lemma 6.1.

**Proof of Lemma 6.1.** We first deduce from (2.2) that \(\tilde{\psi}_0\) satisfies
\[
(C.5) \quad (1 + \partial_{x_2} \psi_0)\partial_{x_1} \tilde{\psi}_0 - \partial_{x_1} \psi_0 \partial_{x_2} \tilde{\psi}_0 = \partial_{x_2} \psi_0, \quad \text{on} \quad \mathbb{R}^2.
\]
It is easy to observe that \(\text{div}(1 + \partial_{x_2} \psi_0, -\partial_{x_1} \psi_0)^T = 0\), and \(\|\partial_{x_2} \psi_0\|_{L^\infty} \leq C\rho_0 \leq \frac{1}{2}\) for \(\rho_0\) sufficiently small in (1.7). Applying Lemma C.1 then ensures that (C.5) has a solution \(\tilde{\psi}_0\) which satisfies (6.16). This completes the proof of Lemma 6.1. \(\square\)

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