Research Article

Essential Self-Adjointness of Anticommutative Operators

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The self-adjoint extensions of symmetric operators satisfying anticommutation relations are considered. It is proven that an anticommutative type of the Glimm-Jaffe-Nelson commutator theorem follows. Its application to an abstract Dirac operator is also considered.

1. Introduction and Main Theorem

In this paper, we consider the essential self-adjointness of anticommutative symmetric operators. Let $H$ be a symmetric operator on a Hilbert space $\mathcal{H}$; that is, $H$ satisfies $H \subset H^*$. It is said that $H$ is self-adjoint if $H = H^*$ and $H$ is essentially self-adjoint if its closure $\overline{H}$ is self-adjoint. We are interested in conditions under which a symmetric operator is essentially self-adjoint. The Glimm-Jaffe-Nelson commutator theorem (e.g., [1, Theorem 2.32], [2, Theorem X.36]) is one criterion for the essential self-adjointness of commutative symmetric operators. The commutator theorem shows that if a symmetric operator $H$ and a self-adjoint operator $X$ obey a commutation relation on a dense subspace $D$, which is a core of $X$, then $H$ is essentially self-adjoint on $D$. Historically, Glimm and Jaffe [3] and Nelson [4] investigate the commutator theorem for quantum field models. Fr"ohlich [5] apply it to quantum mechanical models and Fröhlich [6] considers a generalization of the commutator theorem and proves that a multiple commutator formula follows. Here, we overview the commutator theorem.

Let $H$ and $X$ be linear operators on $\mathcal{H}$. Assume the following conditions.

(C.1) $H$ is symmetric and $X$ is self-adjoint.

(C.2) There exists $\delta_X > 0$ such that, for all $\Psi \in \mathcal{D}(X)$,

$$\delta_X (\Psi, \Psi) \leq |(\Psi, X\Psi)|. \quad (1)$$

(C.3) $X$ has a core $\mathcal{D}_0$ satisfying $\mathcal{D}_0 \subset \mathcal{D}(H)$, and there exist constants $a \geq 0$ and $b \geq 0$ such that, for all $\Psi \in \mathcal{D}_0$,

$$\|H\Psi\| \leq a \|X\Psi\| + b \|\Psi\|. \quad (2)$$

Theorem A (Glimm-Jaffe-Nelson commutator theorem). Let $H$ and $X$ be operators satisfying (C.1)–(C.3). Suppose (i) or (ii) as follows.

(i) There exists a constant $c_1 \geq 0$ such that, for all $\Psi \in \mathcal{D}_0$,

$$|(H\Psi, X\Psi) - (X\Psi, H\Psi)| \leq c_1 |(\Psi, X\Psi)|. \quad (3)$$

(ii) There exists a constant $c_2 \geq 0$ such that, for all $\Psi \in \mathcal{D}_0$,

$$c_2 |(\Psi, X\Psi)| \leq |(H\Psi, X\Psi) - (X\Psi, H\Psi)|. \quad (4)$$

Then, $H$ is essentially self-adjoint on $\mathcal{D}_0$.

Remark 1. In the commutator theorem, condition (i) is usually supposed. It is also proven under condition (ii) in a similar way to Theorem 2.

The idea of the proof of the commutator theorem is as follows. Let $X$ and $Y$ be symmetric operators on a Hilbert space. Then, the real part and the imaginary part of the inner product $(X\Psi, Y\Psi)$ for $\Psi \in \mathcal{D}(XY) \cap \mathcal{D}(YX)$ are expressed by

$$\text{Re} (X\Psi, Y\Psi) = \frac{1}{2} \left( \langle \Psi, [X, Y] \Psi \rangle + \langle [X, Y] \Psi, \Psi \rangle \right),$$

$$\text{Im} (X\Psi, Y\Psi) = \frac{1}{2i} \left( \langle \Psi, [X, Y] \Psi \rangle - \langle [X, Y] \Psi, \Psi \rangle \right). \quad (5)$$
respectively, where \( [X, Y] = XY + YX \) and \([X, Y] = XY − YX \).

In the proof of the commutator theorem, the imaginary part is estimated. In Theorem 2, we prove that an anticommutative symmetric operator is essentially self-adjoint on a dense subspace by estimating the real part.

**Theorem 2.** Assume (C.1)–(C.3). In addition, suppose that (I) or (II) holds.

(I) There exists a constant \( d_1 \geq 0 \) such that, for all \( \Psi \in D_0 \),
\[
|\langle H\Psi, \Psi \rangle + \langle X\Psi, H\Psi \rangle| \leq d_1 |\langle \Psi, \Psi \rangle| .
\]

(II) There exists a constant \( d_2 \geq 0 \) such that, for all \( \Psi \in D_0 \),
\[
d_2 |\langle \Psi, \Psi \rangle| \leq |\langle H\Psi, \Psi \rangle + \langle X\Psi, H\Psi \rangle| .
\]

Then, \( H \) is essentially self-adjoint on \( D_0 \).

**Proof of Theorem 2.** We show that, for some \( z \in C \setminus \mathbb{R} \), \( \dim \ker((H|_{z^*})^* + z^*) = 0 \) where \( z^* = z, z^* \). Let \( \Psi \in D((H|_{z^*})^*) \) and let \( \Xi = X^{-1}\Psi \). Then, \((H|_{z^*})^* = H|_{z^*}\), we have
\[
\begin{align*}
\text{Re} \left( \Xi, \left( \left( H|_{z^*} \right)^* + z^* \right) \Psi \right) & = \frac{1}{2} \left( |\langle H|_{z^*} \Xi, \Xi \rangle| + |\langle X\Xi, H|_{z^*} \Xi \rangle| \right) \\
& + \text{Re} \left( \Xi, \left( \left( H|_{z^*} \right)^* + z^* \right) \Xi \right) .
\end{align*}
\]

First, we assume that (I) holds. Let \( z \in C \setminus \mathbb{R} \) satisfy \( |\text{Re} z| > d_2/2 \). Since \( D_0 \) is a core of \( X \), it follows from (C.3) and (I) that \( D(X) \subset D(H|_{z^*}) \) and for all \( \Phi \in D(X) \),
\[
\left| \left( H|_{z^*} \Phi, X\Phi \right) + \left( X\Phi, H|_{z^*} \Phi \right) \right| \leq d_1 |\langle \Phi, X\Phi \rangle| .
\]

By (8) and (9), we have
\[
\left| \text{Re} \left( \Xi, \left( \left( H|_{z^*} \right)^* + z^* \right) \Psi \right) \right| \geq \left( |\text{Re} z| - \frac{d_1}{2} \right) |\Xi, \Xi| \\
\geq \delta_X \left( |\text{Re} z| - \frac{d_1}{2} \right) (\Xi, \Xi) .
\]

(10)

Since \( \Psi \in \ker((H|_{z^*})^* + z^*) \), we have \( \Xi = X^{-1}\Psi = 0 \) from (10). Then, we have \( \Psi = 0 \). Next, we suppose that (II) follows. Let \( z \in C \setminus \mathbb{R} \) satisfy \( |\text{Re} z| < d_2/2 \). Since \( D_0 \) is a core of \( X \), it also follows from (C.3) and (II) that \( D(X) \subset D(H|_{z^*}) \) and for all \( \Phi \in D(X) \),
\[
d_2 |\langle \Phi, X\Phi \rangle| \leq \left| \left( H|_{z^*} \Phi, X\Phi \right) + \left( X\Phi, H|_{z^*} \Phi \right) \right| .
\]

Then from (8) and (II), we have
\[
\left| \text{Re} \left( \Xi, \left( \left( H|_{z^*} \right)^* + z^* \right) \Psi \right) \right| \geq \left( \frac{d_2}{2} - |\text{Re} z| \right) |\Xi, \Xi| \\
\geq \delta_X \left( \frac{d_2}{2} - |\text{Re} z| \right) (\Xi, \Xi) .
\]

(12)

Since \( \Psi \in \ker((H|_{z^*})^* + z^*) \) and \( \Xi = X^{-1}\Psi \), we have \( \Psi = 0 \) from (12). Thus, the proof is obtained.

### 2. Application of Theorem 2

We apply Theorem 2 to abstract Dirac operator theory [1, 7]. Let \( \mathcal{H} \) be a Hilbert space. Let \( H \) and \( \tau \) be self-adjoint operators on \( \mathcal{H} \). Assume that \( \tau \) is bounded, \( \tau^2 = I \) and, \( \tau \mathcal{D}(H) \subset \mathcal{D}(H) \). Then \( H \) is called an abstract Dirac operator on \( \mathcal{H} \) with unitary involution \( \tau \). We construct an abstract Dirac operator by weakly commuting operators. Let \( X \) and \( Y \) be densely defined linear operators on a Hilbert space. The weak commutator of \( X \) and \( Y \) is defined for \( \Phi \in \cap \mathcal{D}(X^*) \cap \mathcal{D}(Y^*) \) and for \( \Psi \in \mathcal{D}(X) \cap \mathcal{D}(Y) \) by
\[
\left[ X, Y \right] (\Phi, \Psi) = \left( X^* \Phi, Y \Psi \right) = \left( Y^* \Phi, X \Psi \right) .
\]

Let \( \{P_j\}_{j=1}^N, N \in \mathbb{N} \), be self-adjoint operators on a Hilbert space \( \mathcal{H} \). Set \( \mathcal{D}_0 = \cap_{j=1}^N D(P_j) \). Assume that \( \{P_j\}_{j=1}^N \) satisfy the following condition.

(S.1) \( \mathcal{D}_0 \) is dense in \( \mathcal{H} \). For all \( \Phi, \Psi \in \mathcal{D}_0, [P_j, P_l]^0(\Phi, \Psi) = 0, j, l = 1, \ldots, N \).

Let \( M \) be a bounded self-adjoint operator \( \mathcal{H} \) satisfying the following condition.

(S.2) For all \( \Phi, \Psi \in \mathcal{D}_0, [M, P_j]^0(\Phi, \Psi) = 0, j = 1, \ldots, N \).

Let \( \mathcal{H} \) be a Hilbert space. Let \( \{\Gamma_j\}_{j=1}^N \) and \( B \) be bounded self-adjoint operators on \( \mathcal{H} \) satisfying the following anticommutation relations:

(S.3) (i) \( \{\Gamma_j, \Gamma_k\} = 2\delta_{jk}, j, k = 1, \ldots, N \), (ii) \( \{\Gamma_j, B\} = 1, j = 1, \ldots, N \), (iii) \( B^2 = I \).

Then, the next assertion holds.

**Theorem 3.** Let \( \mathcal{H}_D = \mathcal{H} \otimes \mathcal{H} \). Assume (S.1)–(S.3). Then,
\[
H_D = \sum_{j=1}^N \Gamma_j \otimes P_j + B \otimes M
\]

is self-adjoint on \( \mathcal{D}(H_D) = \mathcal{H} \otimes \mathcal{D}_0 \).

**Remark 4.** In the case where \( \{P_j\}_{j=1}^N \) strongly commute, Theorem 3 has been proven ([8, Theorem 4.3], [9, Lemma 6.7]) by strongly anticommuting methods [10, 11].

It is seen that \( (I \otimes B)^2 = I \) and \( (I \otimes B)\mathcal{D}(H_D) \subset \mathcal{D}(H_D) \). Then, from Theorem 3, \( H_D \) is an abstract Dirac operator on \( \mathcal{H}_D \) with the unitary involution \( I \otimes B \).

To prove Theorem 3, we show some lemmas.

**Lemma 5.** Let \( \{C_j\}_{j=1}^N, N \in \mathbb{N} \), be closed operators on a Hilbert space \( X \). Suppose that \( \cap_{j=1}^N \mathcal{D}(C_j) \) is dense in \( X \) and for \( j \neq l \), \( \langle C_j \Psi, C_l \Psi \rangle + \langle C_l \Psi, C_j \Psi \rangle = 0, \Psi \in \cap_{j=1}^N \mathcal{D}(C_j) \). Then \( C = \sum_{j=1}^N C_j \) is closed.

**Proof.** We see that \( \langle C \Psi, C \Psi \rangle = \sum_{j=1}^N \|C_j \Psi\|^2 \geq (1/N) \left( \sum_{j=1}^N \|C_j \Psi\|^2 \right)^2 \). Then, \( \sum_{j=1}^N \|C_j \Psi\|^2 \leq \sqrt{N}\|C \Psi\| \). Then from a closedness criterion (e.g., [1, Theorem B1], [12, Proposition 1]), \( C \) is closed. \( \square \)
From an argument of quadratic forms, there exists a self-



adjoint operator \( L \) on \( \mathcal{H} \) such that \( L \geq 1 \), \( \mathcal{D}(\sqrt{L}) = \mathcal{H}_0 \) and for all \( \Phi, \Psi \in \mathcal{D}(\sqrt{L}) \),

\[
(\sqrt{L}\Phi, \sqrt{L}\Psi) = \sum_{j=1}^{N} (P_j\Phi, P_j\Psi) + (\Phi, \Psi).
\] (15)

**Lemma 6.** Assume (S.1). Then, for all \( \Phi, \Psi \in \mathcal{D}(\sqrt{L}) \),

\[
[\sqrt{L}, P_j]^0(\Phi, \Psi) = 0, \quad j = 1, \ldots, N.
\] (16)

**Proof.** Since \( L \) is positive and self-adjoint, it follows that \( \sqrt{L} \Xi = \sum_{j=1}^{N} (1/\sqrt{\lambda}(L + \lambda)^{-1}L \Xi, \Xi) \in \mathcal{D}(L) \). Then, for all \( \Phi, \Psi \in \mathcal{D}(L) \),

\[
[\sqrt{L}, P_j]^0(\Phi, \Psi) = \int_0^{\infty} \frac{1}{\sqrt{\lambda}} [(L + \lambda)^{-1}L, P_j]^0(\Phi, \Psi) d\lambda
\]

\[
= \int_0^{\infty} \sqrt{\lambda} [P_j, L]^0((L + \lambda)^{-1}\Phi, (L + \lambda)^{-1}\Psi) d\lambda
\]

\[
= \sum_{j=1}^{N} \int_0^{\infty} \sqrt{\lambda} \times \left\{ \left[ P_j, P_j \right]^0((L + \lambda)^{-1}\Phi, (L + \lambda)^{-1}\Psi)
\right.
\]

\[
+ \left[ P_j, P_j \right]^0(P_j(L + \lambda)^{-1}\Phi, (L + \lambda)^{-1}\Psi) \right\} d\lambda.
\] (17)

By (S.1) and (17), we have \([\sqrt{L}, P_j]^0(\Phi, \Psi) = 0\) for all \( \Phi, \Psi \in \mathcal{D}(\sqrt{L}) \). Note that \( \mathcal{D}(\sqrt{L}) \) is a core of \( \sqrt{L} \), since \( L \) is self-adjoint. In addition, for all \( \Psi \in \mathcal{D}(\sqrt{L}) \), \( \|P_j\Psi\| \leq \|\sqrt{L}\Psi\| \), \( j = 1, \ldots, N \). Hence, it follows that \([\sqrt{L}, P_j]^0(\Phi, \Psi) = 0\) for all \( \Phi, \Psi \in \mathcal{D}(\sqrt{L}) \). Thus, the proof is obtained. \( \Box \)

**Proof of Theorem 3.** Since \( B \otimes M \) is bounded, it is enough to show that \( H = \sum_{j=1}^{N} \Gamma_j \otimes P_j \) is self-adjoint. Let \( X = B \otimes \sqrt{L} \). We show that \( H \) and \( X \) satisfy (C.1)–(C.3) and (I) in Theorem 2. Since \( H \) is symmetric and \( X \) is self-adjoint, (C.1) is satisfied. Since \( \sigma(B) = \{ \pm 1 \} \) and \( \sqrt{L} \geq 1 \), we see that, for all \( \Psi \in \mathcal{D}(H) \),

\[
|\Psi, X\Psi| = \left( \Psi, (I \otimes \sqrt{L})\Psi \right) \geq (\Psi, \Psi).
\] (18)

Then, (C.2) is satisfied. Since \( \mathcal{D}_0 = \mathcal{D}(\sqrt{L}) \), it follows that \( \mathcal{D}(H) = \mathcal{D}(X) \). Then, by (S.3), we see that, for all \( \Psi \in \mathcal{D}(H) \),

\[
\|H\Psi\|^2 = \sum_{j=1}^{N} \left( (I \otimes P_j)\Psi, (I \otimes P_j)\Psi \right)
\]

\[
\leq \left\| (I \otimes \sqrt{L})\Psi \right\|^2 = \|X\Psi\|^2.
\] (19)

Then, \( \|H\Psi\| \leq \|X\Psi\| \) for all \( \Psi \in \mathcal{D}(H) \), and hence (C.3) is satisfied. By Lemma 6, it is seen that, for all \( \Psi \in \mathcal{D}(H) \),

\[
(H\Psi, X\Psi) + (X\Psi, H\Psi)
\]

\[
= \sum_{j=1}^{N} \left( \left( (\Gamma_j \otimes P_j)\Psi, (B \otimes \sqrt{L})\Psi \right)
\right.
\]

\[
+ \left. \left( (B \otimes \sqrt{L})\Psi, (\Gamma_j \otimes P_j)\Psi \right) \right)
\]

\[
= \sum_{j=1}^{N} \left( \left(I \otimes \sqrt{L})\Psi, (\{|\Gamma_j, B| \otimes P_j\}\Psi \right) \right).
\] (20)

Then by (S.3) and (20), we have \( (H\Psi, X\Psi) + (X\Psi, H\Psi) = 0 \). Then, from (18), it follows that \( (H\Psi, X\Psi) + (X\Psi, H\Psi) \leq \|\Psi, X\Psi\| \) for all \( \Psi \in \mathcal{D}(H) \). Then (I) is satisfied, and hence \( H \) is self-adjoint from Theorem 2. In addition, by (S.1) and (S.3), we see that, for \( j \neq i \),

\[
\left( (\Gamma_j \otimes P_j)\Psi, (\Gamma_i \otimes P_i)\Psi \right) + \left( (\Gamma_i \otimes P_i)\Psi, (\Gamma_j \otimes P_j)\Psi \right)
\]

\[
= \left( (I \otimes B)\Psi, \{|\Gamma_i, \Gamma_j| \otimes P_j\}\Psi \right) = 0.
\] (21)

Then, from Lemma 5, \( H = H \), and hence the proof is obtained. \( \Box \)

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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