Edge fluctuations for a class of two-dimensional
determinantal Coulomb gases

David García-Zelada

Abstract

We study the fluctuations of the maxima of some classes of two-dimensional determinantal Coulomb gases. Different behaviors are given when the uniform measure on the circle is the equilibrium measure. This includes exponential fluctuations at quadratic speed and Gumbel fluctuations at linear speed. We also obtain the limiting kernel of certain two-dimensional determinantal Coulomb gases at the origin and at the unit circle. Finally, we explore the relations between Kac’s polynomials and a particular Coulomb gas. We show the independence of their limiting inner and outer process and obtain, in this way, that their limiting behavior is the same far from the unit circle. On the other hand, we characterize the limiting behaviors at the unit circle and remark, in particular, that they are different.

1 Introduction

We will be interested in the fluctuations of the maxima of particles distributed according to two-dimensional determinantal Coulomb gases defined in (1) below. The first result we are aware of is the Gumbel fluctuations at quadratic speed obtained by Rider [12] for the farthest particle of the Ginibre ensemble. Later, Chafai and Péché [5] generalized this result obtaining Gumbel fluctuations at the same speed for a class of strongly confining potentials that includes the Ginibre ensemble. Then, Seo [13] considered the hard edge version of this result by proving exponential fluctuations at linear speed. On a series of articles, [10], [8] and [6], Qi and his collaborators have studied different cases related to matrix models which includes truncated circular unitary matrices and products of matrices from the spherical and from the Ginibre ensemble. Very recently, Butez and the author [3] studied a class of weakly confining potentials generated by probability measures. Despite these efforts, it is not clear yet whether there is a universal behavior or if we can find interesting classes of universality. In this article we show different classes of universality disproving, for instance, the common belief that ‘strongly confining’ may be seen as a universality class. In Section 2 we consider potentials that have the uniform measure on the unit circle as the limit of the empirical measures and we show that different behaviors may arise. We consider weakly confining potentials in Subsection 2.1 except for Theorem 2.4 where the potential may be weakly or strongly confining. The proofs use the results stated and proved in Section 5 about the behavior of the point process near zero. In Subsection 2.2 we consider strongly confining potentials and obtain Gumbel fluctuations at linear speed. The beauty of this model is its integrability as very explicit calculations can be made. In Subsection 2.3 we consider a hard edge potential. The proof involves the calculation of the limiting kernel at the unit circle as in [13]. Once more, we are delighted by the integrability of this model. This even help us obtain in Theorem 6.14 the limiting kernel at the unit circle for some non-radial processes. The proofs of the results stated in Section 2 are found in Section 6. In Section 3 we deal briefly with two classes of potentials...
generated by positive measures. One of them is an example of application for some theorems stated in Section 2 and the other one has a proof that uses the same techniques. In Section 4 we consider the relation between Kac’s polynomials and a particular Coulomb gas. One of the main results of this section is the independence between the inner point process and the outer point process as the number of particles goes to infinity, which implies, in particular, the independence of the minimum and the maximum. The second main result is the behavior of both process at the unit circle. These theorems are proved in Section 7 and 8. Finally, we include two short appendices in Section 9 that have independent interest and are useful for the proof of Theorem 4.1 and Theorem 4.4 and a third appendix that makes a link between the weakly and the strongly confining case.

The general radial determinantal Coulomb gas is given by a positive number \( \chi > 0 \) and a continuous function \( V : [0, \infty) \to \mathbb{R} \) bounded from below. It is the system of particles \( (x_1^{(n)}, \ldots, x_n^{(n)}) \) that follows the law proportional to

\[
\prod_{i<j} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{\infty} V(|x_i|)} d\ell_{\mathbb{C}}(x_1) \cdots d\ell_{\mathbb{C}}(x_n)
\]  

(1)

where \( \ell_{\mathbb{C}} \) denotes the Lebesgue measure on \( \mathbb{C} \). For the integral of (1) to be finite we shall assume that

\[
\lim_{r \to \infty} \{V(r) - \log r\} > \infty.
\]

(2)

If we wish to take \( \chi = 0 \) in (1) we can assume that the potential is strongly confining, i.e.

\[
\lim_{r \to \infty} \{V(r) - \log r\} = \infty.
\]

(3)

If (3) is not satisfied we say that the potential is weakly confining. We shall also consider some degenerate cases such as the hard edge radial determinantal Coulomb gases. They are given by a continuous function \( V : [0, 1] \to \mathbb{R} \) and a real number \( \chi \in \mathbb{R} \). The only difference is that the system of particles \( (x_1^{(n)}, \ldots, x_n^{(n)}) \) lives in \( \bar{D}_1 \), the closed unit disk centered at zero, and follows the law proportional to

\[
\prod_{i<j} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{\infty} V(|x_i|)} d\ell_{\bar{D}_1}(x_1) \cdots d\ell_{\bar{D}_1}(x_n)
\]  

(4)

where \( \ell_{\bar{D}_1} \) denotes the Lebesgue measure restricted to \( \bar{D}_1 \). It may be thought as a particular case of (1) where we let \( V(r) = \infty \) for \( r > 1 \).

The usual motivations for these models are random matrix theory, the fractional quantum Hall effect and the Ginzburg-Landau model. We refer to [14] for further motivations.

2 Circle potentials

This article is mainly focused on what we call circle potentials. These are potentials for which the corresponding empirical measures converge to the uniform measure on the unit circle. More precisely, we will say that a continuous function \( V : [0, \infty) \to [0, \infty) \) (or \( V : [0, 1] \to [0, \infty) \) in the hard edge case) satisfies the circle conditions or that \( V \) is a circle potential if

\[
V(1) = 0 \text{ and } V(r) \geq \max\{0, \log(r)\} \text{ for every } r \geq 0
\]

(5)
In this case, by Frostman’s conditions and well-known large deviation principles (see, for instance, [9], [4] or [7]) we know that the sequence of empirical measures \( \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k^{(n)}} \) will converge towards the uniform measure on the unit circle. In fact, Frostman’s conditions are exactly conditions (5).

In Subsection 2.1 we state four theorems. On the first three, \( V \) is necessarily a weakly confining potential. On the fourth one \( V \) may be weakly or strongly confining. In Subsection 2.2 we recover the Gumbel distribution at linear speed for a family of potentials. Finally, in Subsection 2.3, one theorem about a hard edge potential, where we recover an exponential distribution, is stated.

2.1 Weakly confining circle potentials

In this subsection, the potentials of the first three theorems will satisfy

\[
\lim_{r \to \infty} \{ V(r) - \log(r) \} = 0
\]

as a consequence of the hypotheses. The potential treated on the fourth theorem can be strongly or weakly confining. The proofs follow the same methods Butez and the author recently used in [3]. In the first theorem we find a generalization of the Bergman process (case \( \chi = 1 \)). The second theorem tells us that the limiting process may have a finite number of particles. The third theorem is the infinite particle counterpart of the second theorem. The fourth and final theorem of this subsection is an example of a potential that may be strongly confining and which maximum does not have Gumbel fluctuations.

**Theorem 2.1** (Very weak confinement). Suppose that \( V \) satisfies the circle conditions (5). Suppose there exists \( R \geq 1 \) such that

\[
V(r) = \log(r) \quad \text{for every } r \geq R \quad \text{and} \quad V(r) > \log(r) \quad \text{for every } r \in (1, R).
\]

Then

\[
\lim_{n \to \infty} \{ x_k^{(n)} : k \in \{1, \ldots, n\} \text{ and } |x_k^{(n)}| > R \} = B^R
\]

where \( B^R \) is the determinantal point process in the complement of the closed disk of radius \( R \) associated to the Lebesgue measure and to the kernel

\[
K_{B^R}(z, w) = \frac{1}{\pi |zw|^\chi+1} \sum_{k=0}^{\infty} \frac{(k + \chi)R^{2k+2\chi}}{(zw)^k}.
\]

Furthermore, the maximum of \( |x_1^{(n)}|, \ldots, |x_n^{(n)}| \) converges in law to the maximum of \( B^R \). More explicitly,

\[
\lim_{n \to \infty} \mathbb{P} \left( \max\{ |x_1^{(n)}|, \ldots, |x_n^{(n)}| \} \leq t \right) = \prod_{k=0}^{\infty} \left( 1 - \frac{(R/t)^{2k+2\chi}}{2k+2\chi} \right)
\]

for every \( t \geq R \).

In the last appendix in Proposition 9.4 we will describe the behavior as \( \chi \) goes to infinity of the limiting variable obtained in Theorem 2.1. It may be seen as a connection between the weakly and the strongly confining case.

**Theorem 2.2** (Finite limiting process). Suppose that \( V \) satisfies the circle conditions (5). Take \( \alpha \geq 2\chi \). Suppose that

\[
V(r) > \log(r) \quad \text{for every } r \geq 1 \quad \text{and} \quad \lim_{r \to \infty} r^\alpha (V(r) - \log(r)) = \gamma \in (0, \infty).
\]
Suppose also that there exists $L_+ > 0$ and $L_- \geq 1$ such that

$$
\lim_{r \to 1^+} \frac{V(r)}{r - 1} = L_+ + 1 \quad \text{and} \quad \lim_{r \to 1^-} \frac{V(r)}{1 - r} = L_- - 1.
$$

Then

$$
\lim_{n \to \infty} \{n^{-1/\alpha}x_k^{(n)} : k \in \{1, \ldots, n\}\} = \mathbb{F}_{L_-, L_+}^\alpha
$$

where $\mathbb{F}_{L_-, L_+}^\alpha$ is a determinantal point process on $\mathbb{C} \setminus \{0\}$ associated to the Lebesgue measure and to the kernel

$$
K_{\mathbb{F}_{L_-, L_+}^\alpha}(z, w) = \frac{1}{|zw|^{\chi+1}} \sum_{k=0}^{\infty} \frac{a_k}{(zw)^k} e^{-\gamma |z|^\alpha} e^{-\gamma |w|^\alpha}
$$

where

$$
(a_k)^{-1} = \begin{cases}
\infty & 2k + 2\chi > \alpha \\
\frac{2\pi}{\alpha} \int_0^\infty r^{2k+2\chi-1} e^{-2\gamma r^\alpha} dr & 2k + 2\chi < \alpha \\
\frac{\pi}{\alpha \gamma} + \frac{1}{L_+} + \frac{1}{L_-} & 2k + 2\chi = \alpha
\end{cases}.
$$

This point process has $\lceil \alpha/2 - \chi \rceil$ particles if $\alpha/2 - \chi$ is not an integer. If $N = \lceil \alpha/2 - \chi \rceil$ is an integer, this point process has $N$ particles with probability $\pi(1/L_+ + 1/L_-)a_N$ and $N + 1$ particles with probability $1 - \pi(1/L_+ + 1/L_-)a_N$.

Furthermore, $n^{-1/\alpha}$ times the maximum of $|x_1^{(n)}|, \ldots, |x_n^{(n)}|$ converges in law to the maximum of $\mathbb{F}_{L_-, L_+}^\alpha$. More explicitly, if $\alpha/2 - \chi$ is not an integer,

$$
\lim_{n \to \infty} \mathbb{P} \left( n^{-1/\alpha} \max\{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\} \leq t \right) = \prod_{k=0}^{\lfloor \alpha/2 - \chi \rfloor} \frac{\Gamma \left( \frac{2k + 2\chi}{\alpha}, 2\gamma t^{-\alpha} \right)}{\Gamma \left( \frac{2k + 2\chi}{\alpha} \right)}
$$

and if $\alpha/2 - \chi$ is an integer

$$
\lim_{n \to \infty} \mathbb{P} \left( n^{-1/\alpha} \max\{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\} \leq t \right) = \prod_{k=0}^{\alpha/2 - \chi - 1} \frac{\Gamma \left( \frac{2k + 2\chi}{\alpha}, 2\gamma t^{-\alpha} \right)}{\Gamma \left( \frac{2k + 2\chi}{\alpha} \right)} e^{-2\gamma t^{-\alpha}} + \left( 1 - e^{-2\gamma t^{-\alpha}} \right) \left( \frac{1}{L_+} + \frac{1}{L_-} \right) \left( \frac{1}{\alpha \gamma} + \frac{1}{L_+} + \frac{1}{L_-} \right)^{-1}.
$$

Notice that in the extreme case $\alpha = 2\chi$ the limiting distribution of the maxima is a convex combination of a Fréchet distribution and a Dirac measure at zero. The same techniques can be used to generalize this theorem to a case where $V(r)$ has different kind of singularities at $r = 1$. In those generalizations actual Fréchet distributions can be obtained. An extreme case is considered in the following theorem where there is a very strong singularity which results in an infinite number of particles.

**Theorem 2.3** (Strong singularity at the unit circle). Suppose that $V$ satisfies the circle conditions [3]. Take $\alpha > 0$. Suppose that

$$
V(r) > \log(r) \text{ for every } r \geq 1 \text{ and } \lim_{r \to \infty} r^\alpha (V(r) - \log(r)) = \gamma \in (0, \infty).
$$

Suppose also that

$$
\lim_{r \to 1} \frac{V(r)}{(|r - 1|^{k})} = \infty \text{ for every } k > 0.
$$
Then
\[ \lim_{n \to \infty} \{ n^{-1/\alpha} x_k^{(n)} : k \in \{1, \ldots, n\} \} = \mathbb{I}^\alpha \]
where \( \mathbb{I}^\alpha \) is a determinantal point process in \( \mathbb{C} \setminus \{0\} \) associated to the Lebesgue measure and to the kernel
\[ K_{I^\alpha}(z, w) = \frac{1}{|zw|^{\chi+1}} \sum_{k=0}^{\infty} \frac{a_k}{(zw)^k} e^{-\gamma/|z|^\alpha} e^{-\gamma/|w|^\alpha}, \quad (a_k)^{-1} = 2\pi \int_0^\infty r^{2k+2\chi-1} e^{-2\gamma r^\alpha} dr \]
Furthermore, \( n^{-1/\alpha} \) times the maximum of \( |x_1^{(n)}|, \ldots, |x_n^{(n)}| \) converges in law to the maximum of \( \mathbb{I}^\alpha \). More explicitly,
\[ \lim_{n \to \infty} P\left( n^{-1/\alpha} \max\{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\} \leq t \right) = \prod_{k=0}^{\infty} \frac{\Gamma\left(\frac{2k+2\chi}{\alpha}, 2\gamma t^{-\alpha}\right)}{\Gamma\left(\frac{2k+2\chi}{\alpha}\right)}. \]

The following result involves a potential that does not need to be weakly confining. It tells us that the fact of being strongly confining does not immediately imply a Gumbel fluctuation of the maximum. The limiting process has an infinite number of particles on an annulus that are accumulated in the unit circle.

**Theorem 2.4 (Particles on an annulus).** Suppose that \( V \) satisfies the circle conditions (5). Suppose there exists \( R > 0 \) such that
\[ V(r) = \log(r) \text{ for every } r \in [1, R] \text{ and } V(r) > \log(r) \text{ for every } r \in (R, \infty). \]
Then
\[ \lim_{n \to \infty} \{ x_k^{(n)} : k \in \{1, \ldots, n\} \text{ and } |x_k^{(n)}| > 1 \} = A^R \]
where \( A^R \) is the determinantal point process in the complement of the closed unit disk with kernel
\[ K_{A^R}(z, w) = \begin{cases} \frac{1}{|zw|^{\chi+1}} \sum_{k=0}^{\infty} \frac{k+\chi}{(1-R^{-2k+2\chi})(zw)^k} & \text{if } |z|, |w| \leq R, \\ 0 & \text{otherwise} \end{cases} \]
Furthermore, the maximum of \( |x_1^{(n)}|, \ldots, |x_n^{(n)}| \) converges in law to the maximum of \( A^R \). More explicitly,
\[ \lim_{n \to \infty} P\left( \max\{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\} \leq t \right) = \prod_{k=0}^{\infty} \left( \frac{1-t^{-2k-2\chi}}{1-R^{-2k-2\chi}} \right) \]
for every \( t \in [1, R] \).

Notice that \( A^R \) of Theorem 2.4 is \( B^1 \) of Theorem 2.1 conditioned to live in the annulus \( \{ x \in \mathbb{C} : 1 < |x| < R \} \) and, in particular, the limit of the maxima in Theorem 2.4 is the limit of the maxima in Theorem 2.1 conditioned to live in \( \{ x \in \mathbb{C} : 1 < |x| < R \} \).

### 2.2 Strongly confining circle potentials

Here we consider a potential for which we can make explicit calculations. We obtain the expected Gumbel distribution fluctuation but with a different speed of convergence that the one in [5]. We would like to remark once more that the beauty of this model is the explicitness of the calculations and that it may be considered as a toy model where many conjectures could be tested.
Theorem 2.5 (Strongly confining potential). Take $q > 1$ and $R > 1$. Suppose $V$ is such that

$$V(r) \geq q \log(r) \text{ for every } r > 1 \text{ and } V(r) = \max\{0, q \log(r)\} \text{ for every } r \in [0, R].$$

Define $\varepsilon_n > 0$ as the unique solution to

$$e^{2(q-1)n\varepsilon_n} = 1.$$

Then

$$n \left( \max\{x_1^{(n)}, \ldots, x_n^{(n)}\} - 1 - \varepsilon_n \right) \to G$$

where $G$ has a non-standard Gumbel distribution that satisfies

$$\mathbb{P}(G \leq a) \to e^{-\varepsilon_n} e^{-e^{-\frac{1}{2\pi} \frac{1}{e^{2(q-1)a}}}} = e^{-e^{-\frac{a-\log(2q)}{q-1} - 1}}$$

for every $a \in \mathbb{R}$.

2.3 Hard edge circle potentials

Here we restrict the system of particles to lie on the unit disk. The proof will involve a limit kernel calculation at the edge, such as the one in [13].

Theorem 2.6 (Hard edge potential). Suppose $V$ is such that $V(r) = \infty$ for every $r > 1$ and suppose there exists $R \in (0, 1)$ such that $V(r) = 0$ for every $r \in [R, 1]$. Then

$$n^2 \left( 1 - \max\{|Y_1^{(n)}|, \ldots, |Y_n^{(n)}|\} \right) \to E$$

where $E$ follows a standard exponential distribution, i.e.

$$\mathbb{P}(E \leq t) = 1 - e^{-t}$$

for every $t \geq 0$.

3 Related positive background models

For completeness we shall give positive background model examples for two of our results. Given a positive radial measure $\nu$ in $\mathbb{C}$ we define

$$V^\nu(r) = \int_1^r \frac{\nu(D_s)}{s} ds.$$  \hspace{1cm} (6)

It can be proved that the Laplacian of $z \mapsto V^\nu(|z|)$ is $2\pi \nu$ so that this potential can be thought as some sort of electrostatic potential generated by the charge $-\nu$.

Theorem 3.1 (An example of background model circle potentials). Let $q > 1$. Suppose $\nu = q\delta_{S^1} + \bar{\nu}$ where $\delta_{S^1}$ denotes the uniform measure on the unit circle and $\bar{\nu}$ is a positive measure supported on the complement of a disk $D_R$ with $R > 1$ but which support contains $\partial D_R$. Then, $V^\nu$ satisfies the conditions of Theorem 2.4 if $q = 1$ and it satisfies the conditions of Theorem 2.5 if $q > 1$.

Proof. It is a consequence of the formula (6) for $V^\nu$. \hfill \Box
In fact, Theorem 2.4 admits the following extension.

**Theorem 3.2** (Particles on an annulus for positive background potentials). Suppose \(\nu(D_1) = 1\) and suppose there exists \(R > 1\) such that \(\nu(D_R) = 1\) and \(\partial D_R\) is contained in the support of \(\nu\). Then the same conclusions as in Theorem 2.4 hold for \(V^\nu\).

**Proof.** The proof follows the same steps as the proof of Theorem 2.4. In particular it is a consequence of formula (6) and Theorem 5.3 below. \(\square\)

## 4 The standard circle potential and Kac polynomials

The most interesting case for us is the extreme case of \(V : [0, \infty) \to \mathbb{R}\) defined by

\[
V(r) = \max\{0, \log(r)\}
\]

and \(\chi = 1\). This is an example of a positive background model (6) where \(\nu\) is the uniform probability measure on the unit circle. It is known that the asymptotic of this model has some similarity with the asymptotic of the zeros of standard Gaussian Kac’s polynomials. We can see, for instance, [3]. In Subsection 4.1 we show a further similarity while in Subsection 4.2 we show an compelling difference.

### 4.1 Inner and outer independence

By Proposition 6.1 below, the Coulomb gas model associated to 7 and \(\chi = 1\) is invariant under the inversion \(z \mapsto 1/z\). As such, we know that the inner point process \(\{x^{(n)}_k : k \in \{1, \ldots, n\} \text{ and } |x^{(n)}_k| < 1\}\) converges to the Bergman process on the unit disk and the outer process \(\{x^{(n)}_k : k \in \{1, \ldots, n\} \text{ and } |x^{(n)}_k| > 1\}\) converges to the Bergman process on the complement of the unit disk. A natural question to ask is about the joint limit distribution of the inner and the outer process. This is solved on a greater generality in the next theorem.

**Theorem 4.1** (Inner and outer independence for background Coulomb gases). Suppose that \(\nu\) is a radial probability measure on \(\mathbb{C}\) such that its support is contained in \(\{x \in \mathbb{C} : R \leq |x| \leq \bar{R}\}\) for some \(R, \bar{R} > 0\). Suppose that \(R\) and \(\bar{R}\) are the optimal numbers such that this happens, i.e. suppose the support of \(\nu\) contains \(\partial D_R\) and \(\partial D_{\bar{R}}\). Consider \(V^\nu\) defined by (6). Denote by \(\mathcal{B}\) the Bergman process in the unit disk. More precisely, let \(\mathcal{B}\) be the determinantal point process on the unit disk associated to the Lebesgue measure and to the kernel

\[
K_{\mathcal{B}}(z, w) = \frac{1}{\pi(1 - zw)^2}.
\]

Let \(\bar{\mathcal{B}}\) be an independent copy of \(\mathcal{B}\). Let \(I_n = \{x^{(n)}_k / R : k \in \{1, \ldots, n\} \text{ and } |x^{(n)}_k| < R\}\) and \(O_n = \{\bar{R} / x^{(n)}_k : k \in \{1, \ldots, n\} \text{ and } |x^{(n)}_k| > \bar{R}\}\) be the inner and the outer processes. Then

\[
\lim_{n \to \infty} (I_n, O_n) = (\mathcal{B}, \bar{\mathcal{B}}).
\]

It is a natural question to ask if this also happens in the case of Kac’s polynomials. We answer affirmatively.
Theorem 4.2 (Inner and outer independence for Kac’s polynomials). Let \( \{a_k\}_{k \in \mathbb{N}} \) be an independent sequence of standard complex Gaussian random variables. Consider the Gaussian random polynomials \( p_n \) defined by

\[
p_n(z) = \sum_{k=0}^{n} a_k z^k.
\]

Let \( \mathcal{B} \) and \( \bar{\mathcal{B}} \) be two independent copies of the Bergman process on the unit disk, i.e. the determinantal process in the unit disk associated to Lebesgue measure and to the kernel \( (8) \). Let \( I_n = \{ z \in \mathbb{C} : p_n(z) = 0 \text{ and } |z| < 1 \} \) and \( O_n = \{ 1/z \in \mathbb{C} : p_n(z) = 0 \text{ and } |z| > 1 \} \) be the inner and the outer processes. Then

\[
\lim_{n \to \infty} (I_n, O_n) = (\mathcal{B}, \bar{\mathcal{B}}).
\]

In fact, the same result holds when the coefficients are not Gaussian if we replace \( \mathcal{B} \) and \( \bar{\mathcal{B}} \) by the independent copies of the same limiting process.

4.2 Point process at the unit circle

Having seen that the point processes inside and outside of the unit disk have the same limiting behavior, it may be natural to ask if the behavior in the circle is the same. We answer negatively by describing the limit.

Theorem 4.3 (Coulomb gas at the unit circle). Define \( R : \mathbb{C} \to \mathbb{R} \) by \( R(x) = \min\{\Re(x), 0\} \). If \( V \) is defined by (7) then

\[
\lim_{n \to \infty} \{ n(1 - x_k^{(n)}) : k \in \{1, \ldots, n\} \} = \mathcal{E}
\]

where \( \mathcal{E} \) is a determinantal point process on \( \mathbb{C} \) associated to the Lebesgue measure and to the kernel

\[
K_{\mathcal{E}}(\alpha, \beta) = \frac{1}{\pi(\alpha + \beta)^2} \left( 1 + e^{-(\alpha + \beta)} \right) e^{R(\alpha) + R(\beta)} + \frac{2}{\pi(\alpha + \beta)^3} \left( e^{-(\alpha + \beta)} - 1 \right) e^{R(\alpha) + R(\beta)}.
\]

Theorem 4.4 (Random zeros at the unit circle). Let \( \{a_k\}_{k \in \mathbb{N}} \) be an independent sequence of identically distributed complex centered random variables with variance half the identity. More precisely, \( \mathbb{E}[(a_k)^2] = 0 \) and \( \mathbb{E}[(a_k)^2] = 1 \). Consider the random polynomials \( p_n \) defined by

\[
p_n(z) = \sum_{k=0}^{n} a_k z^k.
\]

Then

\[
\lim_{n \to \infty} \{ n(1 - z) \in \mathbb{C} : p_n(z) = 0 \} = \mathcal{F}
\]

where \( \mathcal{F} \) is a Gaussian analytic function with covariance given by

\[
K_{\mathcal{F}}(z, w) = \frac{1 - e^{-(z + \bar{w})}}{z + \bar{w}}.
\]

That the limiting point processes in Theorem 4.3 and Theorem 4.4 are not the same can be seen by calculating the first intensities \( \rho_{\mathcal{E}} \) and \( \rho_{\mathcal{F}} \). For \( \mathcal{E} \) we have \( \rho_{\mathcal{E}}(x) = K_{\mathcal{E}}(x, x) \) and for \( \mathcal{F} \) we have \( \rho_{\mathcal{F}}(z) = \frac{1}{4\pi} \Delta K_{\mathcal{F}}(z, z) \) by the Edelman-Kostlan formula. See [1] Section 2.4 for a proof of this formula.
5 Results about the minima

The proof of some of our results (namely Theorems 2.1, 2.2, 2.3 and 2.4) use the behavior near zero of an inverted model. The driving idea is that the maximum and the minimum are indistinguishable on the sphere. In fact, Lemma 0.1 is motivated by the regular case where the Laplacian of $V$ is thought as a $(1,1)$-form and $e^{-2V}$ is thought as a metric on the tautological line bundle on the sphere. These objects can be found in the work of Berman [2] who consider analogous processes on complex manifolds. We emphasize that no complex geometry is needed in this article but that the ideas fit nicely in that context.

Having $\chi > 0$ fixed, we will consider a system of particles $(x^{(n)}_1, \ldots, x^{(n)}_n)$ distributed according to the law proportional to

$$
\prod_{i<j} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{\infty} V(|x_i|)} d\Lambda_\chi(x_1) \ldots d\Lambda_\chi(x_n)
$$

where

$$
d\Lambda_\chi(x) = |x|^{2(\chi-1)} d\mathbb{C}(x).
$$

**Theorem 5.1** (Finite limiting process at zero). Suppose $V(r)$ is strictly positive if $r \in (0,1) \cup (1,\infty)$. Take $\alpha > 0$. Suppose that

$$
\lim_{r \to 0} \frac{1}{r^\alpha} V(r) = \lambda \in (0,\infty), \quad \lim_{r \to 1^+} \frac{V(r)}{r-1} = l_+ \in (0,\infty) \quad \text{and} \quad \lim_{r \to 1^-} \frac{V(r)}{1-r} = l_- \in (0,\infty).
$$

Then

$$
\lim_{n \to \infty} \{n^{1/\alpha} x^{(n)}_k : k \in \{1, \ldots, n\} \} = \mathcal{G}^\alpha_{l_+, l_-}
$$

where $\mathcal{G}^\alpha_{l_+, l_-}$ is the determinantal point process in $\mathbb{C}$ associated to the reference measure $\Lambda_\chi$ and to the kernel

$$
K_{\mathcal{G}^\alpha_{l_+, l_-}}(z, w) = \sum_{k=0}^{\infty} a_k z^k w^k e^{-\lambda|z|^{\alpha} e^{-\lambda|w|^{\alpha}}}
$$

where

$$
(a_k)^{-1} = \begin{cases}
\infty & 2k + 2 \chi > \alpha \\
2\pi \int_0^{\infty} r^{2k+2\chi-1} e^{-2\lambda r^\alpha} dr & 2k + 2 \chi < \alpha \\
\pi \left( \frac{1}{\chi} + \frac{1}{\alpha} + \frac{1}{\alpha + \chi - 1} \right) & 2k + 2 \chi = \alpha
\end{cases}
$$

Notice that $\mathcal{G}^\alpha_{l_+, l_-}$ has a finite number of particles. In fact, the number of particles belongs to the interval $[\alpha/2 - \chi, \alpha/2 - \chi + 1]$ and it can be thought as a finite Coulomb gas with power potential. More precisely, $\mathcal{G}^\alpha_{l_+, l_-}$ has $[\alpha/2 - \chi]$ particles if $\alpha/2 - \chi$ is not an integer. If $N = \alpha/2 - \chi$ is an integer, $\mathcal{G}^\alpha_{l_+, l_-}$ has $N$ particles with probability $\pi(1/l_+ + 1/l_-) a_N$ and $N + 1$ particles with probability $1 - \pi(1/l_+ + 1/l_-) a_N$.

**Proof.** Notice that $\{n^{1/\alpha} x^{(n)}_k : k \in \{1, \ldots, n\} \}$ is a determinantal point process associated to the kernel

$$
K_n(z, w) = \sum_{k=0}^{n-1} a^{{(n)}_k} z^k w^k e^{-(n+\chi)V\left(\frac{|z|}{n^{1/\alpha}}\right)} e^{-(n+\chi)V\left(\frac{|w|}{n^{1/\alpha}}\right)}
$$

with respect to $\Lambda_\chi$ where

$$
(a^{{(n)}_k})^{-1} = \int_{\mathbb{C}} |z|^{2k} e^{-2(n+\chi)V\left(\frac{|z|}{n^{1/\alpha}}\right)} d\Lambda_\chi(z) = 2\pi \int_0^{\infty} r^{2k+2\chi-1} e^{-2(n+\chi)V\left(\frac{r}{n^{1/\alpha}}\right)} dr.
$$
By [15, Proposition 3.10], our objective is to prove that $K_n$ converges uniformly on compact sets to $K_{G_{i, i-1}}$. As $\lim_{r \to 0} \frac{1}{r} V(r) = \lambda \in (0, \infty)$ we already have that 

$$(n + \chi) V \left( \frac{|z|}{n^{1/\alpha}} \right) \to \lambda |z|^{\alpha}$$

uniformly on compact sets. Then what is left to prove is that

$$\sum_{k=0}^{n-1} a_k^{(n)} z_k w_k \to \sum_{k=0}^{\infty} a_k z_k w_k$$

uniformly on compact sets. We will proceed by the following steps.

1. Notice that $\lim_{n \to \infty} a_k^{(n)} = a_k$.
2. Find a sequence $\{A_k\}_{k \in \mathbb{N}}$ such that $\sum_{k=0}^{\infty} A_k r^k$ converges for every $r \geq 0$ and such that $a_k^{(n)} \leq A_k$ for every $n$ and $k < n$.
3. Use Lebesgue’s dominated convergence theorem to conclude.

**Step 1.** We want to find the limit, as $n$ goes to infinity, of

$$\left(2\pi a_k^{(n)}\right)^{-1} = \int_0^{\infty} r^{2k+2\chi-1} e^{-2(n+\chi)V\left(\frac{r}{n^{1/\alpha}}\right)} dr = n^{(2k+2\chi)/\alpha} \int_0^{\infty} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr.$$ 

We divide the integral in plenty of intervals

$$[0, \infty) = [0, \varepsilon) \cup [\varepsilon, \varepsilon^*) \cup [\varepsilon^*, 1) \cup [1, M^*) \cup [M^*, M) \cup [M, \infty)$$

where we have chosen

- $\varepsilon > 0$ such that $\frac{1}{2} r^\alpha \leq V(r)$ for $r \leq \varepsilon$,
- $\varepsilon^* \in (\varepsilon, 1)$ such that $\frac{1}{2} (1 - r) \leq V(r) \leq 2(1 - r)$ for $r \in [\varepsilon^*, 1]$,
- $M^* > 1$ such that $\frac{1}{2} (r - 1) \leq V(r) \leq 2(r - 1)$ for $r \in [1, M^*]$ and
- $M > M^*$ such that $\frac{1}{2} \log r \leq V(r)$ for $r \geq M$.

We study the integrals in order.

**Integral over $[0, \varepsilon]$.** As $e^{-2(n+\chi)V\left(\frac{r}{n^{1/\alpha}}\right)} 1_{[0,n^{1/\alpha}\varepsilon]}(r) \leq e^{-\frac{(n+\chi)}{n} \lambda r^{\alpha}} \leq e^{-\lambda r^{\alpha}}$ we can use Lebesgue’s dominated convergence theorem to conclude that

$$n^{(2k+2\chi)/\alpha} \int_0^{\varepsilon} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr = \int_0^{n^{1/\alpha} \varepsilon} r^{2k+2\chi-1} e^{-2(n+\chi)V\left(\frac{r}{n^{1/\alpha}}\right)} dr \to \int_0^{\infty} r^{2k+2\chi-1} e^{-2\lambda r^{\alpha}} dr.$$ 

**Integral over $[\varepsilon, \varepsilon^*)$.** As $V$ is positive lower semicontinuous on $[\varepsilon, \varepsilon^*)$ there exists $C > 0$ such that $C \leq V(r)$ for $r \in [\varepsilon, \varepsilon^*)$. Then

$$n^{(2k+2\chi)/\alpha} \int_{\varepsilon}^{\varepsilon^*} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \leq n^{(2k+2\chi)/\alpha} e^{-2(n+\chi)C} \to 0.$$
Integral over \([\varepsilon^*, 1]\). We write
\[
n^{(2k+2\chi+\alpha)} \int_{\varepsilon^*}^{1} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr = n^{(2k+2\chi+\alpha)} \int_{0}^{1-\varepsilon^*} (1-r)^{2k+2\chi-1} e^{-2(n+\chi)V(1-r)} dr
\]
\[
= n^{(2k+2\chi+\alpha)} \frac{1}{n} \int_{0}^{n(1-\varepsilon^*)} (1-r/n)^{2k+2\chi-1} e^{-2(n+\chi)V(1-\frac{r}{n})} dr.
\]
We use \((1-\frac{r}{n})^{2k+2\chi-1} e^{-2(n+\chi)V(1-\frac{r}{n})} 1_{[0,n(1-\varepsilon^*)]}(r) \leq e^{-\frac{(n+\chi)}{\alpha} l_+ r} \leq e^{-l_- r}\) to apply Lebesgue's dominated convergence theorem and obtain that
\[
\int_{0}^{n(1-\varepsilon^*)} (1-r/n)^{2k+2\chi-1} e^{-2(n+\chi)V(1-\frac{r}{n})} dr \to \int_{0}^{\infty} e^{-2l_- r} dr = \frac{1}{2l_-}.
\]
Then
\[
n^{(2k+2\chi+\alpha)} \int_{\varepsilon^*}^{1} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \to \begin{cases} 
\infty & \text{if } 2k + 2\chi > \alpha \\
\frac{1}{2l_-} & \text{if } 2k + 2\chi = \alpha \\
0 & \text{if } 2k + 2\chi < \alpha
\end{cases}
\]
Integral over \([1, M^*]\). We write
\[
n^{(2k+2\chi+\alpha)} \int_{1}^{M^*} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr = n^{(2k+2\chi+\alpha)} \int_{0}^{M^*-1} (1+r)^{2k+2\chi-1} e^{-2(n+\chi)V(1+r)} dr
\]
\[
= n^{(2k+2\chi+\alpha)} \frac{1}{n} \int_{0}^{n(M^*-1)} (1+\frac{r}{n})^{2k+2\chi-1} e^{-2(n+\chi)V(1+\frac{r}{n})} dr.
\]
We use \((1+\frac{r}{n})^{2k+2\chi-1} e^{-2(n+\chi)V(1+\frac{r}{n})} 1_{[0,n(M^*-1)]}(r) \leq (M^*)^{2k+2\chi-1} e^{-\frac{(n+\chi)}{\alpha} l_+ r} \leq (M^*)^{2k+2\chi-1} e^{-l_- r}\) to apply Lebesgue's dominated convergence theorem and obtain that
\[
\int_{0}^{n(M^*-1)} (1+\frac{r}{n})^{2k+2\chi-1} e^{-2(n+\chi)V(1+\frac{r}{n})} dr \to \int_{0}^{\infty} e^{-2l_+ r} dr = \frac{1}{2l_+}.
\]
Then
\[
n^{(2k+2\chi+\alpha)} \int_{1}^{M^*} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \to \begin{cases} 
\infty & \text{if } 2k + 2\chi > \alpha \\
\frac{1}{2l_+} & \text{if } 2k + 2\chi = \alpha \\
0 & \text{if } 2k + 2\chi < \alpha
\end{cases}
\]
Integral over \([M^*, M]\). As \(V\) is a positive lower semicontinuous on \([M^*, M]\) there exists \(C > 0\) such that \(C \leq V(r)\) for \(r \in [M^*, M]\). Then
\[
n^{(2k+2\chi+\alpha)} \int_{M^*}^{M} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \leq n^{(2k+2\chi+\alpha)} e^{-2(n+\chi)C(M-M^*)} \to 0.
\]
Integral over \([M, \infty)\).
\[
n^{(2k+2\chi+\alpha)} \int_{M}^{\infty} r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} dr \leq n^{(2k+2\chi+\alpha)} \int_{M}^{\infty} r^{2k+2\chi-1} e^{-(n+\chi)\log r} dr \to 0.
\]
In summary, we have obtained that
\[
\lim_{n \to \infty} \left( a_k^{(n)} \right)^{-1} = \begin{cases} 
\infty & 2k + 2 > \alpha \\
2\pi \int_{0}^{\infty} r^{2k+2\chi-1} e^{-2\lambda r} d\lambda & 2k + 2 < \alpha \\
2\pi \int_{0}^{\infty} r^{2k+2\chi-1} e^{-2\lambda r} d\lambda + \pi \left( \frac{1}{r_+} + \frac{1}{r_-} \right) & 2k + 2 = \alpha
\end{cases}
\]
Step 2. Take $\varepsilon > 0$ such that $2\lambda r^\alpha \geq V(r)$ for $r \leq \varepsilon$. Then, as

$$2(n + \chi)V\left(\frac{r}{n^{1/\alpha}}\right) \leq 4 \frac{n + \chi}{n} \lambda r^\alpha \leq 4(1 + \chi)\lambda r^\alpha$$

if $r \leq k^{1/\alpha} \varepsilon \leq n^{1/\alpha} \varepsilon$, we obtain

$$\int_0^\infty r^{2k+2\chi-1} e^{-2(n+\chi)V\left(\frac{r}{n^{1/\alpha}}\right)} dr \geq \int_0^{k^{1/\alpha} \varepsilon} r^{2k+2\chi-1} e^{-4(1+\chi)\lambda r^\alpha} dr.$$

So, if we define $A_k$ by

$$(A_k)^{-1} = 2\pi \int_0^{k^{1/\alpha} \varepsilon} r^{2k+2\chi-1} e^{-4(1+\chi)\lambda r^\alpha} dr = 2\pi k^{(2k+2\chi)/\alpha} \int_0^\varepsilon r^{2k+2\chi-1} e^{-4(1+\chi)\lambda kr^\alpha} dr$$

we get $a_k^{(n)} \leq A_k$. An infinite radius of convergence for the power series $\sum_{k=0}^{\infty} A_k x^k$ is obtained if and only if

$$\lim_{k \to \infty} \frac{1}{k} \log \left[(A_k)^{-1}\right] = \infty.$$ 

This can be seen by noticing that $\lim_{k \to \infty} \frac{1}{k} \log \left[2\pi k^{(2k+2)/\alpha}\right] = \infty$ and that

$$\lim_{k \to \infty} \frac{1}{k} \log \int_0^\varepsilon e^{k(2\log r - 4(1+\chi)\lambda r^\alpha)} r^{2\chi-1} dr = \sup_{r \in [0,\varepsilon]} \left\{2\log r - 4(1+\chi)\lambda r^\alpha\right\} > -\infty$$

where the last equality is obtained by Laplace’s method.

Step 3. If $R > 0$ and $|z|, |w| \leq R$ we have

$$\left|\sum_{k=0}^{n-1} a_k^{(n)} z^k w^k - \sum_{k=0}^{\infty} a_k z^k w^k\right| \leq \sum_{k=0}^{\infty} |a_k^{(n)} - a_k| |z|^k |w|^k \leq \sum_{k=0}^{\infty} |a_k^{(n)} - a_k| R^{2k}$$

where $a_k^{(n)}$ is zero if $k \geq n$. By noticing that $|a_k^{(n)} - a_k| R^{2k} \leq 2A_k R^{2k}$ we apply Lebesgue’s dominated convergence theorem to conclude.

Number of particles. The affirmation about the number of particles is an immediate consequence of [1] Theorem 4.5.3] since $K_{G^\alpha_{\chi+\ell}}$ defines a projection onto a space of dimension $[\alpha/2 - \chi]$ if $\alpha/2 - \chi$ is not an integer and it is almost a projection with only one eigenvalue less than one if $\alpha/2 - \chi$ is an integer.

Theorem 5.2 (A strong singularity). Suppose $V(r)$ is strictly positive if $r \in (0, 1) \cup (1, \infty)$. Take $\alpha \in (0, \infty)$. Suppose that

$$\lim_{r \to 0} \frac{1}{r^\alpha} V(r) = \lambda \in (0, \infty) \quad \text{and} \quad \lim_{r \to 1} \frac{V(r)}{|r - 1|^p} = \infty \text{ for every } p > 0.$$ 

Then

$$\lim_{n \to \infty} \{n^{1/\alpha} x_k^{(n)} : k \in \{1, \ldots, n\}\} = G^\alpha$$

where $G^\alpha$ is the determinantal point process in $\mathbb{C}$ associated to the reference measure $\Lambda_\chi$ and to the kernel

$$K_{G^\alpha}(z, w) = \sum_{k=0}^{\infty} a_k z^k w^k e^{-\lambda |z|^\alpha} e^{-\lambda |w|^\alpha}, \quad (a_k)^{-1} = 2\pi \int_0^{\infty} r^{2k+2\chi-1} e^{-2\lambda r^\alpha} dr$$

12
Proof. The proof follows exactly the same steps as the proof of Theorem 5.1 except for the convergence of \( a_k^{(n)} \). Choose \( p > 0 \) such that \((2k + 2\chi)/\alpha < 1/p\). As in the proof of Theorem 5.1, we decompose the integral defining \( a_k^{(n)} \) in plenty of intervals

\[
[0, \infty) = [0, \varepsilon) \cup [\varepsilon, \varepsilon^*) \cup [\varepsilon^*, M^*) \cup [M^*, M) \cup [M, \infty)
\]

where we chose \( \varepsilon \) and \( M \) as before but \( \varepsilon^* \in (\varepsilon, 1) \) and \( M^* \in (1, M) \) are chosen such that \( V(r) \geq |r - 1|^p \) for every \( r \in [\varepsilon^*, M^*) \). The integral on every interval is dealt in the same way except for the interval \([\varepsilon^*, M^*)\) where a slight change is made.

**Integral over [\( \varepsilon^*, M^* \)]**. We write

\[
n^{(2k+2\chi)/\alpha} \int_{\varepsilon^*}^{M^*} r^{2k+2\chi - 1} e^{-2(n+\chi)V(r)} dr \leq n^{(2k+2\chi)/\alpha} \int_{\varepsilon^*}^{M^*} r^{2k+2\chi - 1} e^{-2(n+\chi)|r-1|^p} dr
\]

\[
\leq n^{(2k+2\chi)/\alpha} (M^*)^{2k+2\chi - 1} \int_{-\infty}^{\infty} e^{-2(n+\chi)|r-1|^p} dr
\]

\[
\leq \frac{n^{(2k+2\chi)/\alpha}}{(n+\chi)^{1/p}} (M^*)^{2k+2\chi - 1} \int_{-\infty}^{\infty} e^{-2|r|^p} dr
\]

\[
\rightarrow 0
\]

Finally, we obtain

\[
\lim_{n \to \infty} (a_k^{(n)})^{-1} = 2\pi \int_0^\infty r^{2k+2\chi - 1} e^{-2\lambda r^{\alpha}} dr
\]

and we conclude the proof following the steps of the proof of Theorem 5.1. \( \Box \)

In the following theorem we must allow \( V \) to have a singularity at zero. In fact, we only need \( V \) to be lower semicontinuous.

**Theorem 5.3** (Particles at zero potential). Suppose \( V : [0, \infty) \to [0, \infty] \) is non-negative and lower semicontinuous. Denote \( A = \{ r \geq 0 : V(r) = 0 \} \) and \( R = \text{ess sup} A \), i.e. \( R \) is such that the Lebesgue measure of \( A \cap (R, \infty) \) is zero but the Lebesgue measure of \( A \cap (R, \infty) \) is different from zero for every \( \bar{R} < R \). Then, if \((x^{(n)}_1, \ldots, x^{(n)}_n)\) follows a law proportional to \((1)\), we have

\[
\lim_{n \to \infty} \{x^{(n)}_k : k \in \{1, \ldots, n\} \text{ and } |x^{(n)}_k| < \bar{R} \} = \mathbb{M}^A
\]

where \( \mathbb{M}^A \) is the (inclusion into the open unit disk of radius \( R \) of the) determinantal point process in \( \{ x \in \mathbb{C} : |x| < R \text{ and } V(|x|) = 0 \} \) associated to the reference measure \( \Lambda_{\chi} \) and to the kernel

\[
K_{\mathbb{M}^A}(z, w) = \sum_{k=0}^{\infty} a_k z^k \bar{w}^k, \quad (a_k)^{-1} = 2\pi \int_A r^{2k+2\chi - 1} dr.
\]

**Proof.** Notice that \( \{x^{(n)}_k : k \in \{1, \ldots, n\} \text{ and } |x^{(n)}_k| < \bar{R} \} \) is a determinantal point process associated to the kernel

\[
K_n(z, w) = \sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k e^{-(n+\chi)V(|z|)} e^{-(n+\chi)V(|w|)}
\]
with respect to $\Lambda_X$ where
\[
\left( a_n^{(n)} \right)^{-1} = 2\pi \int_0^\infty r^{2k+2\chi-1} e^{-2(n+\chi)V(r)} \, dr.
\]
Denote
\[
Z = \{ x \in \mathbb{C} : |x| < R \text{ and } V(|x|) = 0 \}.
\]
We will prove that
\[
\{ x_k^{(n)} : k \in \{1, \ldots, n\} \text{ and } x_k^{(n)} \in Z \} \rightarrow M^d
\]
and that
\[
\# \{ x_k^{(n)} : k \in \{1, \ldots, n\}, |x_k^{(n)}| \leq \bar{R} \text{ and } x_k^{(n)} \not\in Z \} \rightarrow 0
\]
in distribution for every $\bar{R} < R$. Then we conclude by the following lemma.

**Lemma 5.4** (Union with an empty point process). Let $X$ be a Polish space and let $C \subset X$ be a closed subset of $X$. Suppose we have a sequence of random point processes $\{P_n\}_{n \in \mathbb{N}}$ and a random point process $P$ on $C$ such that we have the following convergences in distributions
\[
P_n \cap C \rightarrow P \quad \text{and} \quad \# (P_n \cap K \cap C^c) \rightarrow 0 \quad \text{for every compact set } K \subset X.
\]
Then
\[
P_n \rightarrow P
\]
in distribution where $P$ is seen as a random point process in $X$ by the natural inclusion.

**Proof.** By [11, Theorem 4.11] we have to prove that
\[
\sum_{x \in P_n} f(x) \rightarrow \sum_{x \in P} f(x)
\]
weakly for every continuous function $f : X \rightarrow \mathbb{R}$ with compact support. We already know that
\[
\sum_{x \in P_n \cap C} f(x) \rightarrow \sum_{x \in P} f(x)
\]
so that it is enough, by Slutsky’s theorem, to prove that
\[
\sum_{x \in P_n \cap C^c} f(x) \rightarrow 0.
\]
Let $K = \text{supp} f$. Then, by hypothesis, $\# (P_n \cap K \cap C^c) \rightarrow 0$. We can use that
\[
\left| \sum_{x \in P_n \cap C^c} f(x) \right| \leq \# (P_n \cap K \cap C^c) \|f\|_\infty
\]
to conclude. 

\[\square\]
Our first objective is to prove that \( K_n \) converges uniformly on compact sets of \( Z \times Z \) to \( K_{M^A} \) which would imply, by [15, Proposition 3.10], that
\[
\{ x_k^{(n)} : k \in \{1, \ldots, n\} \text{ and } x_k^{(n)} \in Z \} \rightarrow M^A.
\]
In fact we can prove that
\[
\sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k \rightarrow \sum_{k=0}^{\infty} a_k z^k \bar{w}^k
\]
uniformly on compact sets of \( D_R \times D_R \). We will proceed by the following steps.

1. Notice that \( \lim_{n \to \infty} a_k^{(n)} = a_k \).
2. Find a sequence \( \{ A_k \}_{k \in \mathbb{N}} \) such that \( \sum_{k=0}^{\infty} A_k r^k \) converges for every \( r \in [0, R^2] \) and such that \( a_k^{(n)} \leq A_k \) for every \( n \) and \( k < n \).
3. Use Lebesgue’s dominated convergence theorem to conclude.

**Step 1.** We want to find the limit, as \( n \) goes to infinity, of
\[
\left( 2\pi a_k^{(n)} \right)^{-1} = \int_0^{\infty} r^{2k+2e^{-(n+1)}V(r)} dr.
\]
By Lebesgue’s dominated convergence theorem, using the bound
\[
r^{2k+2e^{-(n+1)}V(r)} \leq r^{2k+2e^{-(k+1)+\chi}V(r)},
\]
we obtain that
\[
\int_0^{\infty} r^{2k+2e^{-(n+1)}V(r)} dr \rightarrow \int_A r^{2k+2e^{-1}} dr.
\]

**Step 2.** By definition of \( A \) we have
\[
\int_0^{\infty} r^{2k+2e^{-(n+1)}V(r)} dr \geq \int_A r^{2k+2e^{-1}} dr.
\]
So, we define \( A_k \) by
\[
(A_k)^{-1} = 2\pi \int_A r^{2k+2e^{-1}} dr
\]
and notice, by Laplace’s method, that
\[
\frac{1}{k} \log \int_A r^{2k+2e^{-1}} = \frac{1}{k} \log \int_A e^{k \log r^2} r^{2e^{-1}} dr \rightarrow \sup \{ \log r^2 \}
\]
where the supremum is taken over the support of the Lebesgue measure on \( A \). By the definition of \( R \) this supremum is \( \log R^2 \) and the radius of convergence of \( \sum_{k=0}^{\infty} A_k r^k \) is \( R^2 \).

**Step 3.** Take \( r \in [0, R] \) and suppose that \( |z|, |w| \leq r \). Then
\[
\left| \sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k - \sum_{k=0}^{\infty} a_k z^k \bar{w}^k \right| \leq \sum_{k=0}^{\infty} |a_k^{(n)} - a_k| |z|^k |\bar{w}|^k \leq \sum_{k=0}^{\infty} |a_k^{(n)} - a_k| r^{2k}
\]
where we have defined \( a_k^{(n)} = 0 \) for \( k \geq n \). As \( |a_k^{(n)} - a_k| r^{2k} \) is bounded by \( 2A_k r^{2k} \) we can use Lebesgue’s dominated convergence theorem to conclude.
Then, to prove that
\[ \# \{ x^{(n)}_k : k \in \{1, \ldots, n\}, |x^{(n)}_k| \leq \bar{R} \text{ and } x^{(n)}_k \notin Z \} \to 0 \]
we notice that
\[
E \left[ \# \{ x^{(n)}_k : k \in \{1, \ldots, n\}, |x^{(n)}_k| \leq \bar{R} \text{ and } x^{(n)}_k \notin Z \} \right] = \int_{Z \cap D_{\bar{R}}} K_n(z, z) d\ell_C(z).
\]
As \( K_n(z, z) \) is bounded by \( \sum_{k=0}^{n-1} a^{(n)}_k |z|^{2k} \), which we know converges uniformly on \( D_{\bar{R}} \), we can use Lebesgue’s dominated convergence theorem to conclude that
\[
E \left[ \# \{ x^{(n)}_k : k \in \{1, \ldots, n\}, |x^{(n)}_k| \leq \bar{R} \text{ and } x^{(n)}_k \notin Z \} \right] \to 0
\]
and then
\[ \{ x^{(n)}_k : k \in \{1, \ldots, n\}, |x^{(n)}_k| \leq \bar{R} \text{ and } x^{(n)}_k \notin Z \} \to 0 \]
in distribution.

6 Proofs of the circle potential theorems

6.1 The weakly confining potentials

The main approach to obtain the results of Subsection 2.1 can be seen in [3]. Here an inversion \( z \mapsto \frac{1}{z} \) is made and we may use the corresponding results of Section 5 along with the following lemma.

Lemma 6.1 (Inversion of Coulomb gases). Let \( V : \mathbb{C} \to (-\infty, \infty] \) be a measurable function. Define \( \tilde{V} : \mathbb{C} \setminus \{0\} \to (-\infty, \infty] \) by
\[
\tilde{V}(x) = V \left( \frac{1}{x} \right) + \log |x|.
\]
Then, the image of the measure
\[
\prod_{i<j} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{\infty} V(x_i)} d\ell_C(x_1) \ldots d\ell_C(x_n)
\]
under the application \( (x_1, \ldots, x_n) \mapsto (1/x_1, \ldots, 1/x_n) \) is the measure
\[
\prod_{i<j} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{\infty} \tilde{V}(x_i)} d\Lambda_\chi(x_1) \ldots d\Lambda_\chi(x_n)
\]
where
\[
d\Lambda_\chi(x) = |x|^{2(\chi-1)} d\ell_C(x)
\]

Proof. To avoid possible mistakes, we divide the change of variables in two steps. Consider the function \( G^V : \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\} \to (-\infty, \infty] \) and the positive measure \( \pi \) defined by
\[
G^V(x, y) = -\log |x - y| + V(x) + V(y) \quad \text{and} \quad d\pi = e^{-2(\chi+1)V} d\ell_C.
\]

16
Then we may write
\[
\prod_{i<j} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{\infty} V(x_i) \, d\ell_C(x_1) \ldots d\ell_C(x_n)}
\]
\[
= \exp \left( -2 \left[ -\sum_{i<j} \log |x_i - x_j| + (n + \chi) \sum_{i=1}^{n} V(x_i) \right] \right) \, d\ell_C(x_1) \ldots d\ell_C(x_n)
\]
\[
= e^{-2 \sum_{i<j} G^V(x_i, x_j) \, d\pi(x_1) \ldots d\pi(x_n)}.
\]
It is enough, then, to notice that the image of \(G^V\) and \(\pi\) under the inversion are \(G^{\tilde{V}}\) and \(\tilde{\pi}\), respectively, defined by
\[
G^{\tilde{V}}(x, y) = -\log |x - y| + \tilde{V}(x) + \tilde{V}(y) \quad \text{and} \quad d\tilde{\pi} = e^{-2(\chi+1)V} d\Lambda_{\chi}.
\]

Then Theorem 2.1 and Theorem 2.4 are consequences of Theorem 5.3 by an inversion. Similarly, Theorem 2.2 is a consequence of Theorem 5.1 and Theorem 2.3 is a consequence of Theorem 5.2. For further details we refer to [3].

6.2 The strongly confining case

Proof of Theorem 2.5. For each natural \(n\) define \(n\) independent non-negative random variables \(X_0^{(n)}, \ldots, X_{n-1}^{(n)}\) such that the law of \(X_k^{(n)}\) is proportional to
\[
r^{2k+1} e^{-2(n+\chi)V(r)} \, dr.
\]
It is known (see for instance [5, Theorem 1.2]) that the law of the point process defined by \(\{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\}\) is the same as the law of the point process defined by \(\{X_0^{(n)}, \ldots, X_{n-1}^{(n)}\}\). Define \(M_n = \max\{X_0^{(n)}, \ldots, X_{n-1}^{(n)}\}\) which has the same law as \(\max\{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\}\).

Let \(q > 1\) and define \(V_q(r) = \max\{0, q \log(r)\}\). We will first study this potential.

Case \(\mathcal{V} = V_q\). Suppose \(\mathcal{V} = V_q\). Let \(m \geq 0\) and let us calculate \(\mathbb{P}(M_n \leq m)\). By the independence we can see that
\[
\mathbb{P}(M_n \leq m) = \prod_{k=0}^{n-1} \mathbb{P}(X_k^{(n)} \leq m)
\]
so that we should calculate \(\mathbb{P}(X_k^{(n)} \leq m)\).
If we suppose \( m \geq 1 \) we have
\[
\int_0^m r^{2k+1} e^{-2q(n+\chi)V(r)} dr = \int_0^1 r^{2k+1} dr + \int_1^m r^{2k+1} e^{-2q(n+\chi)\log(r)} dr
\]
\[
= \int_0^1 r^{2k+1} dr + \int_1^m r^{2k+1-2q(n+\chi)} dr
\]
\[
= \frac{1}{2k+2} + \int_1^m r^{2k+1-2q(n+\chi)} dr
\]
\[
= \frac{1}{2k+2} + \frac{m^{2k+2-2q(n+\chi)}}{2k + 2 - 2q(n + \chi)} - \frac{1}{2k + 2 - 2q(n + \chi)}
\]
\[
= \frac{1}{2k + 2} - \frac{2(q(n + \chi) - (k + 1))}{m^{2k+2-2q(n+\chi)}} + \frac{1}{2(q(n + \chi) - (k + 1))}
\]
\[
= \frac{1}{2} \left[ \frac{q(n + \chi) - (k + 1)m^{2k+2-2q(n+\chi)}}{(k + 1)(q(n + \chi) - (k + 1))} \right].
\]

In particular
\[
\int_0^\infty r^{2k+1} e^{-2q(n+\chi)V(r)} dr = \frac{1}{2} \left[ \frac{q(n + \chi)}{(k + 1)(q(n + \chi) - (k + 1))} \right]
\]
and we get
\[
P(X_k^{(n)} \leq m) = \frac{q(n + \chi) - (k + 1)m^{2k+2-2q(n+\chi)}}{q(n + \chi)} = 1 - \frac{(k + 1)m^{2k+2-2q(n+\chi)}}{q(n + \chi)}
\]
so that we obtain the following cumulative distribution function of \( M_n \).

**Proposition 6.2** (A formula for the cumulative distribution function).

\[
P(M_n \leq m) = \prod_{k=0}^{n-1} \left( 1 - \frac{(k + 1)m^{2k+2-2q(n+\chi)}}{q(n + \chi)} \right).
\]

Suppose \( \{m_n\}_{n \in \mathbb{N}} \) is a sequence of numbers greater than one such that \( m_n \to 1 \). We hope to find the right sequence such that \( \lim_{n \to \infty} P(M_n \leq m_n) \) is not trivial. But, instead of calculating \( \lim_{n \to \infty} P(M_n \leq m_n) \) we will calculate \( \lim_{n \to \infty} \log P(M_n \leq m_n) \). We know that
\[
\log P(M_n \leq m_n) = \sum_{k=0}^{n-1} \log \left( 1 - \frac{(k + 1)m^{2k+2-2q(n+\chi)}}{q(n + \chi)} \right).
\]

If \( \{m_n\}_{n \in \mathbb{N}} \) is such that \( m_n^{-n} \to 0 \) then, by using that \( \log(1 + x) = x + o(x) \), we can prove that
\[
\lim_{n \to \infty} \sum_{k=0}^{n-1} \log \left( 1 - \frac{(k + 1)m^{2k+2-2q(n+\chi)}}{q(n + \chi)} \right) = - \lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{(k + 1)m^{2k+2-2q(n+\chi)}}{q(n + \chi)}.
\]

So, we should study
\[
\sum_{k=0}^{n-1} \frac{(k + 1)m^{2k+2-2q(n+\chi)}}{q(n + \chi)} = \frac{m_n^2}{q(n + \chi)m_n^{2q(n+\chi)}} \sum_{k=0}^{n-1} (k + 1)(m_n^2)^k \sim \frac{1}{q n m_n^{2q(n+\chi)}} \sum_{k=0}^{n-1} (k + 1)(m_n^2)^k.
\]
Define
\[ f(x) = \sum_{k=0}^{n-1} (k + 1)x^k = \frac{(n + 1)x^n}{x - 1} + \frac{x^{n+1} - 1}{(x - 1)^2}. \]

By further simplifications we may obtain the following equivalence.

**Proposition 6.3** (An equivalence for the cumulative distribution function). If \( m_n^2 \to \infty \) then
\[
\log \mathbb{P}(M_n \leq m_n) \sim -\frac{1}{2q m_n^{2(q-1)n}} \frac{1}{(m_n - 1)}.
\]

**Proof.** We have already seen that if \( m_n^2 \to \infty \) then
\[
\log \mathbb{P}(M_n \leq m_n) \sim \frac{1}{q n m_n^{2q_n} f(m_n^2)}.
\]

Write \( f(m_n^2) = \theta_n + \gamma_n \) where
\[
\theta_n = \frac{(n + 1)m_n^{2n}}{m_n^2 - 1} \quad \text{and} \quad \gamma_n = \frac{m_n^{2(n+1)} - 1}{(m_n^2 - 1)^2}.
\]

As \( m_n \to 1 \) and if \( m_n^2 \to \infty \) we have
\[
\theta_n \sim \frac{nm_n^{2n}}{2(m_n - 1)} \quad \text{and} \quad \gamma_n \sim \frac{m_n^{2n}}{4(m_n - 1)^2},
\]

If \( m_n \to 1 \) we have that \( \log(m_n) \sim m_n - 1 \) and then \( m_n^2 \to \infty \) holds if and only if \( n(m_n - 1) \to \infty \) holds. We deduce that
\[
\gamma_n = o(\theta_n)
\]

and conclude the proof of the proposition.

Now let us try to understand the term \( m_n^{2(q-1)n} \). We have
\[
m_n^{2(q-1)n} = e^{2(q-1)n \log(m_n)} = e^{2(q-1)n[(m_n - 1) + O(m_n - 1)^2]}.
\]

So, we obtain the following further simplification.

**Proposition 6.4** (A further equivalence for the cumulative distribution function). If \( m_n^2 \to \infty \) and \( n(m_n - 1)^2 \to 0 \) then
\[
\log \mathbb{P}(M_n \leq m_n) \sim -\frac{1}{2q e^{2(q-1)n(m_n - 1)}} \frac{1}{(m_n - 1)}. \]

Take \( a > 0 \) and define \( m_n = \frac{a}{n} + \varepsilon_n + 1 \).

We notice the following result.

**Proposition 6.5** (Properties of epsilon). The following assertions are true.

- \( \varepsilon_n \to 0 \),
- \( n\varepsilon_n \to \infty \), and
- \( n^{-1+k} \to 0 \) for every \( k > 0 \) or equivalently \( n^p \varepsilon_n \to 0 \) for every \( p \in [0, 1) \).
Proof. Taking the logarithm in the definition of $\varepsilon_n$ we get $2(q-1)n\varepsilon_n = -\log(\varepsilon_n)$. From this we get

$$\varepsilon_n \to 0$$

and

$$n\varepsilon_n \to \infty.$$ 

Then, multiply $2(q-1)n\varepsilon_n = -\log(\varepsilon_n)$ by $\varepsilon_n^k$ for $k > 0$ we get $2(q-1)n\varepsilon_n^{1+k} = -\varepsilon_n^k \log(\varepsilon_n)$ and, taking the limit, we get that

$$n\varepsilon_n^{1+k} \to 0$$

for every $k > 0$ or, equivalently,

$$n^p \varepsilon_n \to 0$$

for every $p < 1$.

This implies the following properties of $m_n$.

**Proposition 6.6** (Properties of $m_n$).

$$m_n - 1 \sim \varepsilon_n$$

In particular, the following assertions are true.

- $m_n \to 1$,
- $n(m_n - 1) \to \infty$, and
- $n(m_n - 1)^{1+k} \to 0$ for every $k > 0$ or equivalently $n^p(m_n - 1) \to 0$ for every $p \in [0,1)$.

Proof. That $m_n - 1 \sim \varepsilon_n$ is a consequence of $n\varepsilon_n \to \infty$. The other assertions follow from the previous proposition.

Finally, we have

$$\log \mathbb{P}(M_n \leq m_n) \sim -\frac{1}{2q e^{2(q-1)n(m_n - 1)} (m_n - 1)}$$

which is the result we were looking for.

**A hard edge case.** Consider $R > 1$ and define

$$V(r) = \max\{0, q \log(r)\} + \infty1_{(R,\infty)}(r) = \begin{cases} \infty & \text{if } r > R \\ V_q(r) & \text{if } r \leq R \end{cases}.$$ 

If $M_n$ denotes the maximum of the moduli, we want to understand the limit of

$$\mathbb{P}(M_n \leq m_n) = \prod_{k=0}^{n-1} \int_0^{m_n} r^{2k+1} e^{-2(n+k)V(r)} dr \int_0^\infty r^{2k+1} e^{-2(n+k)V_q(r)} dr. \quad (9)$$
By the case $V_q$ we already know the limit of
\[
\prod_{k=0}^{n-1} \frac{\int_{0}^{r_{2k+1}} e^{-2(n+\chi)V_q(r)} dr}{\int_{r_{2k+1}}^{\infty} e^{-2(n+\chi)V_q(r)} dr}.
\]
(10)
So, we would like to prove that the limit of the quotient of (10) and (9) is equal to one. As
\[
\int_{0}^{r_{m_n}} e^{-2(n+\chi)V(r)} dr = \int_{0}^{r_{m_n}} e^{-2(n+\chi)V_q(r)} dr
\]
for $n$ large enough, the limit of this quotient becomes the limit of
\[
\prod_{k=0}^{n-1} \frac{\int_{0}^{r_{2k+1}} e^{-2(n+\chi)V(r)} dr}{\int_{r_{2k+1}}^{\infty} e^{-2(n+\chi)V_q(r)} dr} = \prod_{k=0}^{n-1} \frac{\int_{0}^{R_{m_n}} e^{-2(n+\chi)V_q(r)} dr}{\int_{R_{m_n}}^{\infty} e^{-2(n+\chi)V_q(r)} dr}.
\]
But this is the probability that the maximum, for the case $V_q$, is less or equal than $R$ which, as the maximum converges in law to 1, goes to 1.

In other words, as the Coulomb gas defined by $V$ is the Coulomb gas defined by $V_q$ conditioned to live in the disk with center 0 and radius $R$ and as the probability that the particles inside this disk goes to one, the fluctuations are the same.

**End of the proof.** Take $V : [0, \infty) \to [0, \infty]$ such that $V(r) \geq q \log(r)$ for every $r > 1$ and suppose there exists $R > 1$ such that $V(r) = \max\{0, q \log(r)\}$ for every $r \in [0, R]$.

As $\max\{0, q \log(r)\} \leq V \leq \max\{0, q \log(r)\} + \infty 1_{[R, \infty)}(r)$ and as the three potentials are the same for $r \leq R$ we can use a comparison argument to conclude.

\[
\square
\]

**Remark 6.7.** We are able to follow the previous proof to study the potentials $V : [0, \infty) \to [0, \infty]$ defined by
\[
V(r) = \begin{cases} \bar{q} \log(r) & \text{if } r \leq 1 \\ q \log(r) & \text{if } r > 1 \end{cases}
\]
for some $q \in (1, \infty)$ and $\bar{q} \in [0, \infty]$. We would obtain that
\[
\lim_{n \to \infty} \log \mathbb{P}(M_n \leq m_n) = -\frac{1}{2} \frac{1}{\bar{q} + q} \frac{1}{e^{2(q-1)n}}.
\]

**6.3 The hard edge case**

Following the ideas of [13] we use the following theorem, which can be obtained by a straightforward explicit calculation of the kernel.

**Theorem 6.8 (Limiting kernel near the circle).** Take $q \in [1, \infty]$ and define $V_q : [0, \infty) \to [0, \infty]$ by
\[
V_q(r) = \max\{0, q \log(r)\}.
\]
Define
\[
K_n(z, w) = \sum_{k=0}^{n-1} a_k^{(n)} z^k w^k, \quad (a_k^{(n)})^{-1} = 2\pi \int_{0}^{\infty} r e^{-2(n+\chi)V_q(r)} dr.
\]
Then
\[
\lim_{n \to \infty} \frac{1}{n} K_n \left( 1 - \frac{\alpha}{n}, 1 - \frac{\beta}{n} \right) = \frac{1}{\alpha + \beta} e^{-\alpha \bar{\beta}} \left( \frac{1}{q} - 1 \right) + \frac{1}{(\alpha + \beta)^2} \left( 1 + e^{-\alpha \bar{\beta}} \left( \frac{2}{q} - 1 \right) \right) + \frac{2}{q(\alpha + \beta)^2} \left( e^{-\alpha \bar{\beta}} - 1 \right)
\]
uniformly on the compact sets of $\mathbb{C} \times \mathbb{C}$.

21
Proof. Let us calculate $a_k^{(n)}$.

\[
\left( a_k^{(n)} \right)^{-1} = 2\pi \int_0^\infty r^{2k+1} e^{-2(n+\chi)V_q(r)} \,dr
\]

\[
= 2\pi \int_0^1 r^{2k+1} \,dr + 2\pi \int_1^\infty r^{2k+1} e^{-2q(n+\chi)\log(r)} \,dr
\]

\[
= 2\pi \int_0^1 r^{2k+1} \,dr - \frac{\pi}{k+1} + 2\pi \int_1^\infty r^{2k+1-2q(n+\chi)} \,dr
\]

\[
= \frac{\pi}{k+1} - \frac{k+1 - q(n+\chi)}{\pi}
\]

so that

\[
a_k^{(n)} = \frac{1}{\pi} \left[ (k+1) - \frac{(k+1)^2}{q(n+\chi)} \right].
\]

Now, let us define

\[
F_n(x) = \sum_{k=0}^{n-1} \pi a_k^{(n)} x^k.
\]

We would like to prove that the sequence of functions $\{\bar{F}_n : \mathbb{C} \to \mathbb{R}\}_{n \in \mathbb{N}}$ defined by

\[
\bar{F}_n(x) = \frac{1}{n^x} F_n \left( 1 - \frac{x}{n} \right)
\]

converges uniformly on compact sets of $\mathbb{C}$ to

\[
\bar{F}_\infty(x) = \frac{1}{x} e^{-x} \left( \frac{1}{q} - 1 \right) + \frac{1}{x^2} \left( 1 + e^{-x} \left( \frac{2}{q} - 1 \right) \right) + \frac{2}{q x^3} \left( e^{-x} - 1 \right).
\]

In fact, we can find a closed-form expression for $F_n$. To simplify the calculation define, for $|x| < 1,$

\[
f_0(x) = \sum_{k=0}^{\infty} x^k, \quad f_1(x) = \sum_{k=0}^{\infty} (k+1)x^k, \quad \text{and} \quad f_2(x) = \sum_{k=0}^{\infty} (k+1)^2x^k.
\]

Then, we can write

\[
\sum_{k=0}^{n-1} (k+1)x^k = \sum_{k=0}^{\infty} (k+1)x^k - \sum_{k=n}^{\infty} (k+1)x^k = f_1(x) - x^n \sum_{k=0}^{\infty} (n+k+1)x^k
\]

\[
= f_1(x) - x^n (nf_0(x) + f_1(x))
\]

and

\[
\sum_{k=0}^{n-1} (k+1)^2x^k = \sum_{k=0}^{\infty} (k+1)^2x^k - \sum_{k=n}^{\infty} (k+1)^2x^k = f_2(x) - x^n \sum_{k=0}^{\infty} (n+k+1)^2x^k
\]

\[
= f_2(x) - x^n \left( n^2 f_0(x) + 2n f_1(x) + f_2(x) \right).
\]
So, we obtain

\[ F_n(x) = \sum_{k=0}^{n-1} (k+1)x^k - \frac{1}{q(n+\chi)} \sum_{k=0}^{n-1} (k+1)^2 x^k \]
\[ = f_1(x) - x^n (nf_0(x) + f_1(x)) - \frac{1}{q(n+\chi)} f_2(x) + \frac{1}{q(n+\chi)} x^n (n^2 f_0(x) + 2nf_1(x) + f_2(x)) \]
\[ = f_0(x) x^n \left( \frac{n^2}{q(n+\chi)} - n \right) + f_1(x) \left( 1 + x^n \left( \frac{2n}{q(n+\chi)} - 1 \right) \right) + f_2(x) (x^n - 1) \frac{1}{q(n+\chi)}. \]

In fact, we can obtain a closed-form expression for \( f_0, f_1 \) and \( f_2 \). Namely,
\[ f_0(x) = \frac{1}{1-x}, \quad f_1(x) = \frac{1}{(1-x)^2}, \quad \text{and} \quad f_2(x) = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}. \]

As this closed-form expressions are holomorphic functions of \( x \in \mathbb{C} \) except at \( x = 1 \) the formula found for \( F_n \) also works when \( |x| \geq 1 \) (this could have also been done by a straightforward calculation but we consider it is clearer and less messy this way). It is enough then to calculate
\[ \lim_{n \to \infty} \frac{1}{n} f_0 \left( 1 - \frac{x}{n} \right) = \frac{1}{x}, \quad \lim_{n \to \infty} \frac{1}{n^2} f_1 \left( 1 - \frac{x}{n} \right) = \frac{1}{x^2}, \quad \lim_{n \to \infty} \frac{1}{n^3} f_2 \left( 1 - \frac{x}{n} \right) = \frac{2}{x^3} \]
and
\[ \lim_{n \to \infty} \left( 1 - \frac{x}{n} \right)^n = e^{-x} \]
to obtain the convergence of \( \tilde{F}_n(x) \) towards \( F_\infty(x) \) defined in [111] at least when \( x \neq 0 \). What is left to prove is that this convergence is uniform. This can be done directly but it is easier for us to notice that the sequence \( \tilde{F}_n \) is a normal family of holomorphic functions. This can be done, for instance, by noticing that \( |\tilde{F}_n(x)| \leq F_n(-R) \) if \( |x| \leq R \). The pointwise convergence implies then the uniform convergence on compact sets. Finally, we define \( L : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) by
\[ L_n(\alpha, \beta) = n \left( 1 - \left( 1 - \frac{\alpha}{n} \right) \left( 1 - \frac{\beta}{n} \right) \right). \]

which converges uniformly on compact sets of \( \mathbb{C} \times \mathbb{C} \) to \( L_\infty(\alpha, \beta) = \alpha + \beta \) and deduce that \( \tilde{F}_n \circ L_n \) converges uniformly on compact sets of \( \mathbb{C} \times \mathbb{C} \) to \( F_\infty \circ L_\infty \).

**Remark 6.9.** The limit of
\[ \frac{1}{n^2} \sum_{k=0}^{n-1} a_k^{(n)} \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\beta}{n} \right)^k \]
where
\[ a_k^{(n)} = \frac{1}{\pi} \left[ (k+1) - \frac{(k+1)^2}{q(n+\chi)} \right]. \]
can be also obtained by a Riemann sum approximation. Using this we would obtain the limit
\[ \frac{1}{\pi} \int_0^1 (t - t^2/q) e^{-(\alpha+\beta)t} dt \]
which is equal but has a simpler form than the one in Theorem 6.8.
In this subsection, we shall be interested in the infinite case. We will extend Theorem 6.8 to include potentials such as the ones in the hypotheses of Theorem 2.6.

**Theorem 6.10 (Limiting kernel).** Suppose \( V : [0, \infty) \to [0, \infty] \) is a measurable function such that \( V(r) = \infty \) for every \( r > 1 \) and suppose there exists \( R \in (0, 1) \) such that \( V(r) = 0 \) for every \( r \in [R, 1] \). Define

\[
K_n(z, w) = \sum_{k=0}^{n-1} a_k^{(n)} z^k w^k, \quad (a_k^{(n)})^{-1} = 2\pi \int_0^\infty r e^{-2(n+\chi)\alpha V(r)} dr.
\]

Then

\[
\lim_{n \to \infty} \frac{\pi}{n^2} K_n \left( 1 - \frac{\alpha}{n}, 1 - \frac{\beta}{n} \right) = \frac{1}{(\alpha + \beta)^2} \left( 1 - e^{-(\alpha+\beta)} \right) - \frac{1}{\alpha + \beta} e^{-(\alpha+\beta)}
\]

uniformly on the compact sets of \( \mathbb{C} \times \mathbb{C} \).

**Proof.** If we take \( a_k = (k+1)/\pi \) and \( \tilde{a}_k = (k+1)/\pi(1-R^{2k}) \) it is true that

\[
a_k \leq a_k^{(n)} \leq \tilde{a}_k
\]

which can be obtained by noticing that

\[
0 \leq V \leq \infty 1_{[0,R)}
\]

in \([0,1]\) and by taking the respective integrals. By Theorem 6.8 we only need to prove that

\[
D_n(\alpha, \beta) = \frac{1}{n^2} \sum_{k=0}^{n-1} (a_k^{(n)} - a_k) \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\tilde{\beta}}{n} \right)^k \to 0
\]

uniformly on compact sets of \( \mathbb{C} \times \mathbb{C} \). By the Cauchy-Schwarz inequality we get

\[
|D_n(\alpha, \beta)| \leq D_n(\alpha, \alpha)^{1/2} D_n(\beta, \beta)^{1/2}
\]

where we have taken advantage of the fact that \( a_k^{(n)} - a_k \geq 0 \) to simplify notation. So, it would be enough to prove that \( D_n(\alpha, \alpha) \to 0 \) uniformly on compact sets of \( \mathbb{C} \). Furthermore, we obtain, by (12), that

\[
D_n(\alpha, \alpha) \leq \frac{1}{n^2} \sum_{k=0}^{n-1} (\tilde{a}_k - a_k) \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\tilde{\alpha}}{n} \right)^k
\]

so that it is enough to consider the model \( V = \infty 1_{[0,R)} \). Take any \( \varepsilon > 0 \) and consider \( N \) such that

\[
\left| \frac{1}{1 - R^{2k}} - 1 \right| \leq \varepsilon \text{ for every } k \geq N.
\]
Then, if \( n \geq N \),
\[
\frac{1}{n^2} \sum_{k=0}^{n-1} (\tilde{a}_k - a_k) \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\tilde{\alpha}}{n} \right)^k
\]
\[
= \frac{1}{n^2} \sum_{k=0}^{N-1} (\tilde{a}_k - a_k) \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\tilde{\alpha}}{n} \right)^k + \frac{1}{n^2} \sum_{k=N}^{n-1} (\tilde{a}_k - a_k) \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\tilde{\alpha}}{n} \right)^k
\]
\[
\leq \frac{1}{n^2} \sum_{k=0}^{N-1} (\tilde{a}_k - a_k) \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\tilde{\alpha}}{n} \right)^k + \frac{\varepsilon}{n^2} \sum_{k=N}^{n-1} a_k \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\tilde{\alpha}}{n} \right)^k
\]
\[
\leq \frac{1}{n^2} \sum_{k=0}^{N-1} (\tilde{a}_k - a_k) \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\tilde{\alpha}}{n} \right)^k + \frac{\varepsilon}{n^2} \sum_{k=0}^{n-1} a_k \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\tilde{\alpha}}{n} \right)^k.
\]
But we know that, as \( n \to \infty \) (\( N \) fixed),
\[
\frac{1}{n^2} \sum_{k=0}^{n-1} (\tilde{a}_k - a_k) \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\tilde{\alpha}}{n} \right)^k \to 0
\]
uniformly on compact sets of \( \mathbb{C} \) and that
\[
\frac{1}{n^2} \sum_{k=0}^{n-1} a_k \left( 1 - \frac{\alpha}{n} \right)^k \left( 1 - \frac{\tilde{\alpha}}{n} \right)^k
\]
converges uniformly on compact sets to some continuous function in \( \mathbb{C} \). Varying \( \varepsilon \) this implies the required assertion. \( \square \)

If we define \( \rho_n : [0, \infty) \to [0, \infty) \)
\[
\rho_n(r) = \frac{1}{n^2} K_n \left( 1 - \frac{r}{n}, 1 - \frac{r}{n} \right)
\]
we obtain the following consequence.

**Corollary 6.11** (Limiting first intensity). *Suppose \( V \) satisfies the conditions of Theorem 6.10. Then*
\[
\lim_{n \to \infty} \rho_n(r) = -\frac{1}{2\pi r} e^{-2r} + \frac{1}{4\pi r^2} (1 - e^{-2r})
\]
uniformly on compact sets of \( [0, \infty) \).

Now let us prove Theorem 2.6

**Proof of Theorem 2.6** Let \( X_0^{(n)}, \ldots, X_{n-1}^{(n)} \) be \( n \) independent random variables taking values in \( [0, 1] \) such that the law of \( X_k^{(n)} \) is proportional to
\[
r^{2k+1} e^{-2(n+\chi) V(r)} dr.
\]
As explained in the proof of Theorem 2.5, it is known that the law of \( M_n = \max\{X_0^{(n)}, \ldots, X_{n-1}^{(n)} \} \) is the same as the law of \( \max\{|x_1^{(n)}|, \ldots, |x_n^{(n)}| \} \) (see for instance [3, Theorem 1.2]). Let us study the cumulative distribution function of \( M_n \). First, by independence,
\[
\mathbb{P}(M_n \leq m) = \prod_{k=0}^{n-1} \left( 1 - \mathbb{P}(m < X_k^{(n)}) \right).
\]
25
Then, by using that \( V = 0 \) in \([R, 1]\) for the numerator and by comparing to the potential \( \infty \) for the denominator, we can see that, for \( m \geq R \),

\[
\mathbb{P} \left( m < X_k^{(n)} \right) = \int_{m}^{1} r^{2k+1} e^{-2(n+\chi)V(r)} dr \leq \frac{1 - m^{2(k+1)}}{1 - R^{2(k+1)}}.
\]

As

\[
\frac{1 - m^{2(k+1)}}{1 - R^{2(k+1)}} \leq \frac{1 - m^{2n}}{1 - R^{2n}}
\]

we notice that, as soon as \( m^{2n} \to 1 \) and by using that \( \log(1 - x) \sim -x + o(x) \), we have

\[
\log \mathbb{P}(M_n \leq m_n) \sim -n - \sum_{k=0}^{n-1} \mathbb{P} \left( m_n < X_k^{(n)} \right) .
\]

But

\[
\mathbb{P} \left( m_n < X_k^{(n)} \right) = a_k^{(n)} \int_{m_n}^{1} r^{2k+1} e^{-2(n+\chi)V(r)} dr
\]

which implies that

\[
\sum_{k=0}^{n-1} \mathbb{P} \left( m_n < X_k^{(n)} \right) = \int_{m_n}^{1} \sum_{k=0}^{n-1} a_k^{(n)} r^{2k+1} e^{-2(n+\chi)V(r)} 2\pi rdr
\]

\[
= \int_{m_n}^{1} n^2\rho_n(n(1-r)) 2\pi rdr
\]

\[
= \int_{0}^{1-m_n} \rho_n(nr) 2\pi (1-r) d(n^2r)
\]

\[
= \int_{0}^{n^2(1-m_n)} \rho_n \left( \frac{r}{n} \right) 2\pi \left( 1 - \frac{r}{n^2} \right) dr.
\]

So that, if we consider \( m_n \) such that \( n^2(1 - m_n) = a \) we obtain

\[
\sum_{k=0}^{n-1} \mathbb{P} \left( m_n < X_k^{(n)} \right) = \int_{0}^{a} \rho_n \left( \frac{y}{n} \right) 2\pi \left( 1 - \frac{y}{n^2} \right) dy.
\]

As \( \rho_n \to -\frac{1}{2\pi r} e^{-2r} + \frac{1}{4\pi r^2} (1 - e^{-2r}) \) uniformly on \([0, a]\) and since this limit is a continuous function that takes the value \( 1/(2\pi) \) at \( r = 0 \) we obtain that

\[
\sum_{k=0}^{n-1} \mathbb{P} \left( m_n < X_k^{(n)} \right) \to a
\]

which completes the proof of the theorem.

\( \square \)

**Remark 6.12.** By the same method, the limiting kernel at the unit circle for \( V_q \) with \( q < \infty \) allows us to find the fluctuations of the distance to the unit circle. The speed would be quadratic and the fluctuation would be a multiple of the exponential distribution.

We would like to point out that a similar argument as the one in the proof of Theorem 6.10 allows us to treat a general compactly supported measurable function \( V : D_1 \to \mathbb{C} \) defined on the open unit disk \( D_1 \). We explain how in the following theorem.
Theorem 6.13 (Point process at the circle for a non-radial potential). Let $V : D_1 \to [0, \infty]$ be a non-negative measurable function on the open unit disk $D_1$ with compact support. Denote the space of complex polynomials of degree less or equal than $n-1$ by $\mathcal{P}_{n-1}$. Consider $\{p_k^{(n)}\}_{k \in \{0, \ldots, n-1\}}$ any orthonormal basis of $\mathcal{P}_{n-1}$ with respect to the inner product

$$\langle f, g \rangle_n = \int_{D_1} \bar{f} g e^{-2(n+\chi)V} d\ell_C.$$ 

Define

$$K_n(z, w) = \sum_{k=0}^{n-1} p_k(z) \bar{p}_k(w).$$

Then

$$\lim_{n \to \infty} \frac{\pi}{n^2} K_n \left( 1 - \frac{\alpha}{n}, 1 - \frac{\beta}{n} \right) = \frac{1}{(\alpha + \beta)^2} \left( 1 - e^{-(\alpha+\beta)} \right) - \frac{1}{\alpha + \beta} e^{-(\alpha+\beta)}$$

uniformly on the compact sets of $\mathbb{C} \times \mathbb{C}$.

Proof. First, we would like to notice that

$$K_n(z, z) = \sup_{p \in \mathcal{P}_{n-1}} \frac{|p(z)|^2}{\langle p, p \rangle_n}.$$ 

Indeed, using the reproducing property of $K_n$ we have that for every $p \in \mathcal{P}_{n-1}$

$$\int_{D_1} K_n(z, w)p(w)e^{-2(n+\chi)V(w)} d\ell_C(w) = p(z).$$

By the Cauchy-Schwarz inequality we obtain

$$|p(z)|^2 \leq \int_{D_1} |K_n(z, w)|^2 e^{-2(n+\chi)V(w)} d\ell_C(w) \int_{D_1} |f(w)|^2 e^{-2(n+\chi)V(w)} d\ell_C(w).$$

But as $|K_n(z, w)|^2 = K_n(z, w)K_n(w, z)$ we obtain

$$|p(z)|^2 \leq K_n(z, z) \int_{D_1} |f(w)|^2 e^{-2(n+\chi)V(w)} d\ell_C(w).$$

If we choose $p = K_n(\cdot, z)$ we obtain

$$|p(z)|^2 = K_n(z, z) \int_{D_1} |f(w)|^2 e^{-2(n+\chi)V(w)} d\ell_C(w)$$

so that

$$K_n(z, z) = \sup_{p \in \mathcal{P}_{n-1}} \frac{|p(z)|^2}{\langle p, p \rangle_n}.$$ 

But there exists $R \in (0, 1)$ such that $V(z) = 0$ if $|z| \geq R$. Then

$$\int_{D_1 \setminus D_R} |p|^2 d\ell_C \leq \int_{D_1} |p|^2 e^{-2(n+\chi)V} d\ell_C \leq \int_{D_1} |p|^2 d\ell_C$$
for every $p \in \mathcal{P}_{n-1}$. This implies, in particular, that $K_n(z, z)$ takes values between two functions that, after the appropriate recentering and rescaling, converge uniformly on compact sets towards the desired limit. So

$$
\lim_{n \to \infty} \frac{\pi}{n^2} K_n \left( 1 - \frac{\alpha}{n}, 1 - \frac{\alpha}{n} \right) = \left( \frac{1}{\alpha + \bar{\alpha}} \right)^2 \left( 1 - e^{-(\alpha + \bar{\alpha})} \right) - \frac{1}{\alpha + \bar{\alpha}} e^{-(\alpha + \bar{\alpha})}
$$

uniformly on compact sets of $\mathbb{C}$. As $|K_n(z, w)|^2 \leq K_n(z, z)K_n(w, w)$, we obtain, by Montel’s theorem, that $\{K_n(z, \bar{w})\}_{n \in \mathbb{N}}$ is a normal family of holomorphic functions on $\mathbb{C}^2$. By noticing that their limit points are already determined on the set $\{(z, \bar{z}) \in \mathbb{C}^2 : z \in \mathbb{C}\}$ we conclude that they are the same everywhere. This completes the proof.

\[\square\]

### 7 Proof about the inner and outer independence

We begin by proving Theorem 4.1 about the Coulomb gases and immediately after we proceed to the proof of Theorem 4.2 about Kac’s polynomials.

**Proof of Theorem 4.1.** By Lemma 9.2 it is enough to verify a convergence of the kernels. We shall find a simpler kernel for the same process. The usual kernel of $\{x^{(n)}_k : k \in \{1, \ldots, n\}\}$ is

$$
\tilde{K}_n(z, w) = \sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k e^{-(n+1)V^\nu(|z|)} e^{-(n+1)V^\nu(|w|)}
$$

where

$$
(b_k^{(n)})^{-1} = 2\pi \int_0^\infty r^{2k+1} e^{-2(n+1)V^\nu(r)} dr.
$$

Let $R > 0$ and $\tilde{R} > 0$ be such as in the hypotheses. If $|z|, |w| < R$ the potential on the kernel is gone and we may write

$$
\tilde{K}_n(z, w) = \sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k.
$$

If $|z|, |w| > \tilde{R}$ the potential on the kernel is essentially, by (6), a logarithm

$$
\tilde{K}_n(z, w) = \sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k e^{-(n+1)(V^\nu(\tilde{R}) - \log \tilde{R} + \log |z|)} e^{-(n+1)(V^\nu(\tilde{R}) - \log \tilde{R} + \log |w|)}
$$

$$
= \sum_{k=0}^{n-1} b_k^{(n)} e^{-2(n+1)(V^\nu(\tilde{R}) - \log \tilde{R})} \frac{z^k \bar{w}^k}{|z|^{n+1} |w|^{n+1}}.
$$

If $|z| < R$ and $|w| > \tilde{R}$ we have a mixture of both

$$
\tilde{K}_n(z, w) = \sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k e^{-(n+1)(V^\nu(\tilde{R}) - \log \tilde{R} + \log |w|)}
$$

$$
= \sum_{k=0}^{n-1} b_k^{(n)} e^{-(n+1)(V^\nu(\tilde{R}) - \log \tilde{R})} \frac{z^k \bar{w}^k}{|w|^{n+1}}.
$$

28
and similarly for $|z| > \bar{R}$ and $|w| < R$. By inverting the part in $D_R$ we find the kernel of the point process

$$\{x_k^{(n)} : k \in \{1, \ldots, n\} \text{ and } |x_k^{(n)}| < R\} \sqcup \{1/x_k^{(n)} : k \in \{1, \ldots, n\} \text{ and } |x_k^{(n)}| > \bar{R}\}$$

(13)

in the disjoint union $D_R \sqcup D_{\bar{R}^{-1}}$. We obtain that this process is a determinantal point process associated to the sum of Lebesgue measures and to the kernel

$$K_n^I(z, w) = \begin{cases} 
\sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k & z \in D_R, w \in D_R \\
\sum_{k=0}^{n-1} b_k^{(n)} e^{-\nu(R)} z^{k} \bar{w}^{k-n} & z \in D_R, w \in D_{\bar{R}^{-1}} \\
\sum_{k=0}^{n-1} b_k^{(n)} e^{-\nu(R)} \bar{z}^{k} \bar{w}^{k} & z \in D_{\bar{R}^{-1}}, w \in D_R \\
\sum_{k=0}^{n-1} b_k^{(n)} e^{-2\nu(R)} z^{k} \bar{w}^{k-n} & z \in D_{\bar{R}^{-1}}, w \in D_{\bar{R}^{-1}}
\end{cases}$$

The terms $|z|^{n-1}$ and $|w|^{n-1}$ become $z^{n-1}$ and $\bar{w}^{n-1}$ if we consider a conjugation $c(z)K_n(z, w)c(w)^{-1}$ where $c(z) = (z/|z|)^{n-1}$ so the point process (13) is a determinantal point process associated to the sum of Lebesgue measures and to the kernel

$$K_n(z, w) = \begin{cases} 
\sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k & z \in D_R, w \in D_R \\
\sum_{k=0}^{n-1} b_k^{(n)} e^{-\nu(R)} z^{k} \bar{w}^{n-1-k} & z \in D_R, w \in D_{\bar{R}^{-1}} \\
\sum_{k=0}^{n-1} b_k^{(n)} e^{-\nu(R)} \bar{z}^{k} \bar{w}^{k} & z \in D_{\bar{R}^{-1}}, w \in D_R \\
\sum_{k=0}^{n-1} b_k^{(n)} e^{-2\nu(R)} z^{k} \bar{w}^{n-1-k} & z \in D_{\bar{R}^{-1}}, w \in D_{\bar{R}^{-1}}
\end{cases}$$

We already know, by the proof of Theorem 5.3 or by 3, that

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} b_k^{(n)} z^k \bar{w}^k = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(k+1)}{R^{2k}} z^k \bar{w}^k$$

uniformly on compact sets of $D_R \times D_R$ and

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} b_k^{(n)} e^{-2\nu(R)} z^{k} \bar{w}^{n-1-k} = \frac{1}{\pi} \sum_{k=0}^{\infty} R^{2k}(k+1) z^k \bar{w}^k$$

uniformly on compact sets of $D_{\bar{R}^{-1}} \times D_{\bar{R}^{-1}}$. We are going to prove now that

$$\sum_{k=0}^{n-1} b_k^{(n)} e^{-\nu(R)} z^{k} \bar{w}^{n-1-k} \to 0$$

uniformly on compact sets of $D_R \times D_{\bar{R}^{-1}}$. By Cauchy-Schwarz inequality we have

$$|K_n(z, w)| \leq \sqrt{K_n(z, z)K_n(w, w)}$$
where we have taken \( z \in \mathcal{D}_R \) and \( w \in \bar{\mathcal{D}}_{R-1} \). This implies, in particular, that the sequence \( \{ K_n(z, w) \}_{n \in \mathbb{N}} \) is locally uniformly bounded and thus it is a normal family of holomorphic functions in \( \mathcal{D}_R \times \bar{\mathcal{D}}_{R-1} \subset \mathbb{C}^2 \). It is enough to prove that \( K_n(z, w) \) converges pointwise to zero for \((z, w)\) on an open set of \( \mathcal{D}_R \times \bar{\mathcal{D}}_{R-1} \). But we can obtain this pointwise limit, for instance, as soon as \(|z/w| < R\) and \(|w| < e^{V(\bar{R})-\log R}\) which completes the proof. 

\[ \square \]

**Proof of Theorem 4.2.** We have considered

\[ p_n(z) = \sum_{k=0}^{n} a_k z^k. \]

Take

\[ p^*_n(z) = \sum_{k=0}^{n} a_{n-k} \bar{z}^k = z^n p_n(1/z). \]

We defined

\[ I_n = \{ z \in \mathbb{C} : p_n(z) = 0 \text{ and } |z| < 1 \} \]

and

\[ O_n = \{ \bar{z} \in \mathbb{C} : p_n(z) = 0 \text{ and } |z| > 1 \} = \{ z \in \mathbb{C} : p^*_n(z) = 0 \text{ and } |z| < 1 \}. \]

The theorem will be a consequence of the following convergence in law (in the compact-open topology)

\[ \left( \sum_{k=0}^{n} a_k z^k, \sum_{k=0}^{n} a_{n-k} \bar{z}^k \right) \to \left( \sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} \tilde{a}_k \bar{z}^k \right) \]

where \( \tilde{a}_0, \ldots, \tilde{a}_k, \ldots \) is an independent copy of \( a_0, \ldots, a_k, \ldots \). Choose \( N_n = \lfloor n/2 \rfloor \) and \( \tilde{N}_n = \lceil n/2 \rceil \) such that \( N_n + \tilde{N}_n = n \). We have

\[ \sum_{k=0}^{N_n} a_k z^k \to \sum_{k=0}^{\infty} a_k z^k \]

in law as \( n \to \infty \) and

\[ \sum_{k=0}^{\tilde{N}_n-1} a_{n-k} \bar{z}^k \to \sum_{k=0}^{\infty} a_k \bar{z}^k \]

in law as \( n \to \infty \). As \( \sum_{k=0}^{N_n} a_k z^k \) is independent of \( \sum_{k=0}^{\tilde{N}_n-1} a_{n-k} \bar{z}^k \) we get that

\[ \left( \sum_{k=0}^{N_n} a_k z^k, \sum_{k=0}^{\tilde{N}_n-1} a_{n-k} \bar{z}^k \right) \to \left( \sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} \tilde{a}_k \bar{z}^k \right) \]

in law as \( n \to \infty \). We can also notice that

\[ \sum_{k=N_n+1}^{n} a_k z^k = z^{N_n+1} \left( \sum_{k=0}^{n-N_n-1} a_{N_n+1+k} z^k \right) = z^{N_n+1} \left( \sum_{k=0}^{\tilde{N}_n-1} a_{N_n+1+k} z^k \right) \]
and, as $z^{N_n+1}$ goes to zero uniformly on compact sets of $D_1$ and $\sum_{k=0}^{\tilde{N}_n-1} a_{N_n+1+k} z^k$ converges in law to $\sum_{k=0}^{\infty} a_k z^k$ then the product converges in law (in the compact-open topology) to zero so that

$$\sum_{k=N_n+1}^{n} a_k z^k \to 0$$

in law as $n \to \infty$. The same can be said for $\sum_{k=\tilde{N}_n}^{n} a_{n-k} z^k$ and then

$$\left( \sum_{k=N_n+1}^{n} a_k z^k, \sum_{k=\tilde{N}_n}^{n} a_{n-k} z^k \right) \to (0,0)$$

in law as $n \to \infty$. By Slutsky’s theorem we have that

$$\left( \sum_{k=0}^{n} a_k z^k, \sum_{k=0}^{\tilde{N}_n-1} a_{n-k} z^k \right) \to (0,0) + \left( \sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} \tilde{a}_k z^k \right)$$

and we conclude that

$$\left( \sum_{k=0}^{n} a_k z^k, \sum_{k=0}^{\tilde{N}_n-1} a_{n-k} z^k \right) \to \left( \sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} \tilde{a}_k z^k \right)$$

in law as $n \to \infty$.

8 Proof about the behavior near the circle

We begin by proving the limiting behavior of the Coulomb gas at the unit circle and then we proceed to prove the limiting behavior of the zeros of Kac’s polynomials at the unit circle.

Proof of Theorem 4.3. This is a consequence of Theorem 6.8 together with the fact that

$$\lim_{n \to \infty} n \max \left\{ 0, \log \left| 1 - \frac{z}{n} \right| \right\} = -R(z)$$

(14)

uniformly on compact sets of $C$. To prove this we first notice that

$$\lim_{n \to \infty} \frac{n}{2} \log \left| 1 - \frac{z}{n} \right|^2 = -Rz$$

uniformly on compact sets of $C$. This is a consequence of the differentiability of the logarithm and the differentiability of the square of the norm. As $\max\{x, y\} = (|x - y| + x + y)/2$ for every $x, y \in \mathbb{R}$, we obtain that if $f_n$ and $g_n$ converges uniformly on compact sets to $f$ and $g$ respectively then $\max\{f_n, g_n\}$ converges uniformly on compact sets to $\max\{f, g\}$. Then (14) holds and we have completed the proof of the theorem.

Proof of Theorem 4.4. Consider

$$g_n(z) = \frac{1}{\sqrt{n} p_n} \left( 1 - \frac{z}{n} \right).$$
Then, define
\[ K_n(z, w) = \mathbb{E}[q_n(z) \bar{q}_n(w)] = \frac{1}{n} \sum_{k=0}^{n} \left(1 - \frac{z}{n}\right)^k \left(1 - \frac{w}{n}\right)^k \]
that by a straightforward calculation converges uniformly on compact sets to
\[ K(z, w) = 1 - e^{-(z + \bar{w})} \]
With this in hand we may notice that
\[ (q_n(z_1), \ldots, q_n(z_l)) = \left(\frac{1}{\sqrt{n}} \sum_{k=0}^{n} a_k \left(1 - \frac{z_1}{n}\right)^k, \ldots, \frac{1}{\sqrt{n}} \sum_{k=0}^{n} a_k \left(1 - \frac{z_l}{n}\right)^k \right) \]
converges to a Gaussian vector. This is simpler than Lindeberg central limit because all the variables are multiples of each other. Actually it can be obtained by calculating the asymptotic of the characteristic function.

Finally, the tightness of the sequence \( \{q_n\}_{n \in \mathbb{N}} \) can be obtained by Lemma 9.3 because we already know that \( K_n(z, z) \) is uniformly bounded on compact sets of \( \overline{\mathbb{C}} \) and so \( \int_{K} K_n(z, z) d\ell_{\overline{\mathbb{C}}}(z) \) is a bounded sequence for any compact set \( K \subset \overline{\mathbb{C}} \).

9 Appendices

9.1 The correlation functions of the union of point processes

Consider \((A_1, \mu_1)\) and \((A_2, \mu_2)\) two measure spaces. If \( P_1 \) is a point process on \( A_1 \) and \( P_2 \) is a point process on \( A_2 \) independent of \( P_1 \) we consider the union \( P_1 \cup P_2 \) as a point process on the disjoint union \((A_1 \coprod A_2, \mu_1 \oplus \mu_2)\).

**Lemma 9.1** (Correlation function of an independent union). Suppose \( \rho_k^{(1)} \) and \( \rho_k^{(2)} \) are the \( k \)-th correlation function of \( P_1 \) and \( P_2 \) respectively (with respect to the measures \( \mu_1 \) on \( A_1 \) and \( \mu_2 \) on \( A_2 \)). Then the \( n \)-th correlation function of \( P_1 \cup P_2 \) (with respect to the measure \( \mu_1 \oplus \mu_2 \) on \( A_1 \coprod A_2 \)) is
\[ \rho_n = \sum_{k=0}^{n} \rho_k^{(1)} \odot \rho_{n-k}^{(2)} \]
where \( \rho_k^{(1)} \odot \rho_{n-k}^{(2)} \) is defined by
\[ \rho_k^{(1)} \odot \rho_{n-k}^{(2)}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = \rho_k^{(1)}(x_1, \ldots, x_k)\rho_{n-k}^{(2)}(x_{k+1}, \ldots, x_n) \]
if \( x_1, \ldots, x_k \in A_1 \) and \( x_{k+1}, \ldots, x_n \in A_2 \). It is defined by the symmetric property if the argument contains \( k \) points in \( A_1 \) and \( n - k \) points in \( A_2 \) and it is defined as zero in the other cases.

**Proof.** Suppose \( C_1, \ldots, C_n \) are \( n \) measurable sets in \( A_1 \coprod A_2 \). Write \( C_k = C_k^{(1)} \cup C_k^{(2)} \) where \( C_k^{(1)} \subset A_1 \) and \( C_k^{(2)} \subset A_2 \). By the distribution property of multiplication over addition and by
the distribution property of multiplication of sets over union of sets it is enough to suppose that
$C_1, \ldots, C_k \subset A_1$ and $C_{k+1}, \ldots, C_n \subset A_2$ for some $k$. So we want to prove that

\[
\mathbb{E}[\#(C_1 \cap P_1) \ldots \#(C_k \cap P_1) \#(C_{k+1} \cap P_2) \ldots \#(C_n \cap P_2)]
\]

\[
= \int_{C_1 \times \ldots \times C_k \times C_{k+1} \times \ldots \times C_n} \rho^{(1)}_k(x_1, \ldots, x_k) \rho^{(2)}_{n-k}(x_{k+1}, \ldots, x_n) d\mu_1^{\otimes k}(x_1, \ldots, x_k) d\mu_2^{\otimes n-1}(x_{k+1}, \ldots, x_n)
\]

which is a consequence of the independence and Fubini’s theorem.

This translates into a statement about independent union of determinantal point processes.

**Lemma 9.2 (Kernel of an independent union).** Suppose $P_1$ and $P_2$ are independent determinantal point processes with kernels $K_1$ and $K_2$. Then their disjoint union $P_1 \cup P_2$ is a determinantal point process with kernel $K$ defined by

\[
K(x, y) = \begin{cases} 
K_1(x, y) & \text{if } x, y \in A_1 \\
K_2(x, y) & \text{if } x, y \in A_2 \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** It is a consequence of Theorem 9.1 and the calculation of the determinant of a diagonal block matrix.

9.2 Tightness for random analytic functions

We consider an open set $U \subset \mathbb{C}$ and denote by $\mathcal{O}(U)$ the space of holomorphic functions on $U$ endowed with the topology of uniform convergence on compact sets also known as the compact-open topology. By Montel’s theorem we are able to characterize the relatively compact sets of $\mathcal{O}(U)$. In the following lemma we consider a random version of it.

**Lemma 9.3 (Tightness characterization).** Let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a family of random analytic functions in a domain $U$. Then $\{P_\lambda\}_{\lambda \in \Lambda}$ is tight if and only if for any compact set $K \subset U$ and $\varepsilon > 0$ we can find $M > 0$ such that

\[
P\left[\sup_{x \in K} |P_\lambda(x)| > M\right] < \varepsilon
\]

for every $\lambda \in \Lambda$. In particular, if the family $\{I^K_\lambda\}_{\lambda \in \Lambda}$ defined by

\[
I^K_\lambda = \int_K \mathbb{E}[|P_\lambda(z)|^2] \ell_C(z)
\]

is bounded for any compact $K$, then $\{P_\lambda\}_{\lambda \in \Lambda}$ is tight.

**Proof.** The characterization of tightness is a consequence of Montel’s theorem. For the second assertion we notice that for any compact subset $K \subset U$ there exists a compact set $\hat{K}$ that contains $K$ and a constant $C > 0$ such that

\[
\sup_{x \in K} |f(x)|^2 \leq C \int_{\hat{K}} |f(z)|^2 \ell_C(z)
\]

33
for every \( f \in \mathcal{O}(U) \). This is essentially a consequence of the subharmonicity of \(|f|^2\). Then we write
\[
\mathbb{P} \left[ \sup_{x \in K} |P_\lambda(x)|^2 > M^2 \right] \leq M^{-2} \mathbb{E} \left[ \sup_{x \in K} |P_\lambda(x)|^2 \right] \leq M^{-2} C I^K_\lambda.
\]
As \( I^K_\lambda \) is uniformly bounded on \( \lambda \in \Lambda \) we may choose \( M \) large enough such that (15) is satisfied.

9.3 Gumbel distribution and weakly confining fluctuations

Here we establish a connection between the limit of the maxima in the very weakly confining case, Theorem 2.1, and the Gumbel distribution.

**Proposition 9.4.** For each \( \chi > 0 \) let \( X_\chi \) be a random variable with cumulative distribution function
\[
\mathbb{P}(X_\chi \leq t) = \prod_{k=0}^{\infty} \left( 1 - t^{-2k-2\chi} \right)
\]
and let \( \varepsilon_\chi > 0 \) denote the unique solution to
\[
e^{\varepsilon_\chi \varepsilon_\chi} = 1.
\]

Then, as \( \chi \to \infty \), we have
\[
2\chi(X_\chi - 1 - \varepsilon_\chi/2) \to G
\]
where \( G \) has a standard Gumbel distribution, i.e.
\[
\mathbb{P}(G \leq a) = e^{-e^{-a}}
\]
for every \( a \in \mathbb{R} \).

**Proof.** Define \( b_\chi \) by
\[
2b_\chi = \varepsilon_\chi + a/\chi.
\]

We have to prove that
\[
\lim_{\chi \to \infty} \prod_{k=0}^{\infty} \left( 1 - \left(1 + b_\chi\right)^{-2k-2\chi} \right) = e^{-e^{-a}}.
\]

We proceed as in the proof of Theorem 2.5. We notice that, as \( \chi \to \infty \),
\[
\log \prod_{k=0}^{\infty} \left( 1 - \left(1 + b_\chi\right)^{-2k-2\chi} \right) \sim -\sum_{k=0}^{\infty} \left(1 + b_\chi\right)^{-2k-2\chi}
\]
which is due to the fact that \( (1 + b_\chi)^x \to \infty \) and \( \log(1 - x) \sim -x + o(x) \). Then
\[
\sum_{k=0}^{\infty} \left(1 + b_\chi\right)^{-2k-2\chi} = \frac{(1 + b_\chi)^{-2\chi}}{1 - (1 + b_\chi)^{-2}} \sim \frac{e^{-2\chi b_\chi + \chi O(b_\chi)^2}}{2b_\chi} \sim \frac{e^{-\chi \varepsilon_\chi}}{\varepsilon_\chi} e^{-a} = e^{-a}
\]
which concludes the proof. 

\( \square \)
References

[1] John Ben Hough, Manjunath Krishnapur, Yuval Peres and Bálint Virág. Zeros of Gaussian analytic functions and determinantal point processes. Volume 51 of University Lecture Series. American Mathematical Society, Providence, RI, 2009.

[2] Robert J. Berman. Determinantal Point Processes and Fermions on Complex Manifolds: Large Deviations and Bosonization. Communications in Mathematical Physics, vol 327, issue 1, pp. 1-47, 2014.

[3] Raphael Butez and David García-Zelada. Extremal particles of two-dimensional Coulomb gases and random polynomials on a positive background. arXiv preprint arXiv:1811.12225, 2018.

[4] Djalil Chafaï, Nathael Gozlan, and Pierre-André Zitt. First-order global asymptotics for confined particles with singular pair repulsion. The Annals of Applied Probability, vol. 24, no. 6, pp. 2371-2413, 2014.

[5] Djalil Chafaï and Sandrine Péché. A note on the second order universality at the edge of Coulomb gases on the plane. Journal of Statistical Physics, vol. 156, no. 2, pp. 368-383, 2014.

[6] Shuhua Chang, Deli Li and Yongcheng Qi. Limiting Distributions of Spectral Radii for Product of Matrices from the Spherical Ensemble. Journal of Mathematical Analysis and Applications, vol. 461, issue 2, pp. 1165-1176, 2018.

[7] David García-Zelada. A large deviation principle for empirical measures on Polish spaces: Application to singular Gibbs measures on manifolds. arXiv preprint arXiv:1703.02680, 2017.

[8] Wenhao Gui and Yongcheng Qi. Spectral Radii of Truncated Circular Unitary Matrices. Journal of Mathematical Analysis and Applications, vol. 458, issue 1, pp. 536-554, 2018.

[9] Adrien Hardy. A note on large deviations for 2D Coulomb gas with weakly confining potential. Electronic Communications in Probability, vol. 17, paper no. 19, 2012.

[10] Tiefeng Jiang and Yongcheng Qi. Spectral radii of large non-hermitian random matrices. Journal of Theoretical Probability, vol. 30, issue 1, pp. 326-364, 2017.

[11] Olav Kallenberg. Random Measures, Theory and Applications. Volume 77 of Probability Theory and Stochastic Modelling. SpringerVerlag, New York, 2017.

[12] Brian Rider. A limit theorem at the edge of a non-Hermitian random matrix ensemble. Journal of Physics A: Mathematical and General, vol. 36, no. 12, pp. 3401–3409, 2003.

[13] Seong-Mi Seo. Edge scaling limit of the spectral radius for random normal matrix ensembles at hard edge. arXiv preprint arXiv:1508.06591, 2015.

[14] Sylvia Serfaty. Systems of points with Coulomb interactions. arXiv preprint arXiv:1712.04095, 2018.

[15] Tomoyuki Shirai and Yoichiro Takahashi. Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes. Journal of Functional Analysis, vol. 205, issue 2, pp. 414–463, 2003.