Quantum causal correlations and non-Markovianity of qubit evolution

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A non-Markovianity measure for qubit channels is introduced based on causality measure - a monotone of causal (temporal) correlations - arising out of the pseudo-density matrix (PDM) formalism which treats quantum correlations in space and time on an equal footing. Using the well known damped Jaynes-Cummings model of a two-level system interacting with a bosonic reservoir at zero temperature as an example, it is shown that breakdown of monotonicity of the causality measure is associated with the revival of temporal (causal) correlations hence with the negativity of the decay rate. Also, a note on the comparison of causality measure with other geometric measures such as trace distance is given.

\textbf{I. INTRODUCTION}

Quantum non-Markovian dynamics has received significant attention over the last few years. Defining, characterizing and quantifying memory effects in open quantum system dynamics has been a rich topic of interest \cite{1, 2}. Any quantum communication scheme requires the use of quantum channels \cite{3}. Generally, quantum channels involve noise and losses, which can deteriorate the information contained in quantum states and quantum correlations. Contrary to conventional understanding, noise can sometimes be beneficial \cite{4–6}. On top of that, non-Markovian nature of quantum channels can give an advantage in information processing tasks \cite{7}, e.g., non-local memory effects can help teleport a mixed state with perfect fidelity and that it can give rise to higher quantum capacity for even for longer channels \cite{8}. On the other hand, it has been shown that indefinite causal order and superposition of causal orders can give rise to similar results \cite{9, 10}. Lately, a number of approaches have been formulated to address quantum correlations in space and time in a unified picture \cite{11, 12}. Among them, pseudo-density matrix (PDM) formalism \cite{13}, which will be introduced shortly, has been shown to be equivalent to a number approaches, e.g., process matrix formalism which deals with quantum theory without a definite global causal order although obeying quantum mechanics locally, for studying temporal correlations in non-relativistic quantum mechanics \cite{11}.

A number of non-Markovianity measures, based on distance \cite{14, 15}, correlations \cite{16–18}, channel capacity \cite{8} have been proposed to name a few. Recently, non-Markovian dynamics has been studied using temporal steering and quantified using temporal steerable weight \cite{19}. This work is focused on quantifying non-Markovianity using a recently introduced causality measure arising out of PDM formalism, hence unravel the relation between memory effects and causality. Interestingly, a quantum channel with memory, arising due to correlation between successive use of channel does not violate causality: outputs of channel at \( t \) do not depend on the input of the channel at \( t' > t \) \cite{20}. However, in the PDM formalism, application of channels preserves causality, but the causal order may be indefinite \cite{13}. See, e.g., \cite{11}, where a tripartite PDM is shown to be equivalent to quantum switch - the only known higher order linear operation that gives rise to indefinite causal order. Interestingly, violation of temporal Bell inequalities, i.e., temporal Tsirelson bound, as an indication of completely positive (CP-) indivisibility was noted in \cite{21}. In this work, we present a measure of non-Markovianity based on causality, which interestingly, for a maximally mixed initial states, upper bounds the quantum channel capacity based non-Markovianity measure \cite{8}. We make use of causality monotone, which is a measure of causal or temporal correlations, introduced in \cite{13, 22} and show that breakdown of its monotonicity is associated with non-Markovianity of a channel. Studying open system dynamics by considering spacial and temporal correlations on a equal footing may open up an arena to study memory effects with indefinite causal order and help understand a possible intimate connection between non-Markovianity and causal order \cite{23}. As an added note, we shall be using the terms temporal correlations and causal correlations (or relationships) interchangeably, since both mean the same in PDM formalism.

\textbf{II. THE PSEUDO DENSITY MATRIX}

The framework of pseudo-density matrix was introduced in \cite{13}, in which spatial and temporal correlations can be studied in a unified way. A hierarchy of temporal correlations namely temporal non-separability, temporal steering, and temporal non-local correlations has been presented in \cite{24}. The former corresponds to the framework of PDM which will have at least one negative eigenvalue, implying that there exists temporal correlations between two time-like separated (non-simultaneous) events. This gives rise to a causality monotone which shares similar properties of an entanglement monotone except for classical communication \cite{25}.

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Among them, most notable for our purpose here is that it is non-increasing under local quantum operations [26], which are generally given by completely positive trace preserving (CPTP) maps or trace non-increasing maps. Breakdown of such monotonicity is important to note that causal correlations due to PDM are stronger form of quantum direct cause and it was shown [24] that temporal steering can be a weaker form of quantum direct cause. In this light, non-Markovianity measure proposed in this work can be thought to be stronger than that proposed using temporal steerable weight [19].

A PDM, for any channel $\mathcal{E}$, is given by [22]

$$
\mathcal{P} = (I \otimes \mathcal{E}) \left( \rho \otimes \frac{I}{2} \otimes Q_{\text{swap}} \right)
$$

(1)

where $Q_{\text{swap}} := \frac{1}{2} \sum_{i=0}^{3} \sigma_i \otimes \sigma_i$ and $\{a, b\} = ab + ba$ and $\sigma_i$ are Pauli-X,Y and Z operators with $\sigma_0 = I$. See Appendix (A) for a pedagogical overview of PDM formalism. Throughout the paper we consider one time use of channel acting on the initial qubit state $\rho$ with measurements before and after the use of the channel. This essentially creates two-point temporal correlations. E.g., measurements before and after the use of the channel acting on the initial qubit state $\rho$.

In Eq. (1), the function $f_{\text{cm}}$ is a small time increment. Also, as it is shown [13] that $f_{\text{cm}}$ satisfies the convexity condition $f_{\text{cm}}(\sum_j p_j \mathcal{P}_j) \leq \sum_j p_j f_{\text{cm}}(\mathcal{P}_j)$, it can be argued that it is also strongly monotonic under (incoherent) CPTP maps [30]. One may define a measure of non-Markovianity based on $f_{\text{cm}}$ also, but $f_{\text{cm}}$ is not additive in which case the relation with the quantum capacity may not be possible for the non-Markovianity measure based on $f_{\text{cm}}$. Whereas $\mathcal{F}$ is additive and therefore, convexity of $\mathcal{F}$ can be sacrificed in favor of additivity which will in fact be beneficial to relate the resulting causality measure (4) with quantum capacity through Choi-Jamiolkowski isomorphism [22].

The condition Eq. (6) implies that the CPTP channel $\mathcal{E}$ is CP-divisible in the sense that the condition

$$
\mathcal{E}(t + \tau, t) \mathcal{E}(t, 0) = \mathcal{E}(t + \tau, 0)
$$

(7)

is satisfied. In other words, if the channel $\mathcal{E}$ is non-Markovian then the intermediate map $\mathcal{E}(t + \tau, t)$ is not completely positive (NCP) in the sense that the Choi matrix $(I \otimes \mathcal{E}(t + \tau, t)) |00 \rangle \langle 11|$ will have at least one negative eigenvalue, however the total map $\mathcal{E}(t + \tau, 0)$ remains CP. It is known that non-Markovian dynamics may give rise to information back-flow incuring revivals of quantum resources back to the system from the environment. Since $\mathcal{F}$ is a monotone, violation of condition (6) hence of (7) implies that monotonicity will be broken under a CP-indivisible channel hence signal non-Markovianity through revivals of temporal quantum correlations.

Based on the above considerations, one can define a measure using causality monotone $\mathcal{F}$:

$$
\mathcal{M} := \max_{\rho} \int_{\sigma(\rho, \mathcal{E}, t) > 0} dt \sigma(\rho, \mathcal{E}, t)
$$

(8)
where $\sigma(\rho, \mathcal{E}, t) = \frac{d\mathcal{F}}{dt}$ for channel $\mathcal{E}(t)$ acting on an initial state $\rho$, where the integration is done over positive slope of $\mathcal{F}$. Eq. (8) is, in fact, computationally equivalent to

$$M := \int_{t_0}^{t_{\text{max}}} \left| \frac{d\mathcal{F}}{dt} \right| dt + (\mathcal{F}_{\text{max}} - \mathcal{F}_{t_0}) \quad (9)$$

for a given initial state $\rho$. In PDM formalism, $t_{\text{max}}$ may be thought of as waiting time [See Figure (1) and (2) of Ref. [13]]. Note the similarity of $\mathcal{F}$ with logarithmic negativity of a bipartite entangled state, which is a well known entanglement monotone used to define a measure of non-Markovianity [16]. Interestingly, Eq. (4), for a maximally mixed initial state, gives exactly the same results as does Eq. (5) in quantifying non-Markovianity of quantum dynamics, and it has been shown [22] to upper bound the quantum capacity of the channel $\mathcal{E}$.

The method may be extended to $n$ parallel use of non-Markovian channel $\mathcal{E}$ so that one may relate the causality measure to quantum capacity thereby allowing for a more general capacity based non-Markovianity measure. The measure does not require optimization in such case (i.e., with a maximally mixed initial state), else requires optimization over all initial input states. We may omit the discussion in this work.

IV. EXAMPLE 1: THE DAMPED JAYNES-CUMMINGS MODEL

Let us consider damped Jaynes-Cummings (JC) model [31] of a two level system interacting with a dissipative bosonic reservoir at zero temperature, with a Lorentzian spectral density function $J(\omega) = \gamma_0 b^2 / 2\pi[(\omega_0 - \Delta - \omega)^2 + b^2]$, where $\Delta = \omega - \omega_0$ is the detuning parameter which governs the shift in frequency $\omega$ of the qubit from the central frequency $\omega_0$ of the bath. Here, $\gamma_0$ quantifies the strength of the system-environment coupling and $b$ the spectral bandwidth.

It can modeled as an amplitude damping (AD) channel $\mathcal{E}_{\text{AD}}[\rho] = \sum_j A_j \rho A_j^\dagger$ with the Kraus operators

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{;} \quad A_2 = \begin{pmatrix} 0 & \sqrt{r(t)} \\ 0 & 0 \end{pmatrix} \quad (10)$$

Here, for damped JC model, $r(t)$ takes the form $r(t) = 1 - |G(t)|^2$, with

$$G(t) = e^{-\frac{b_+ - b_- t}{d}} \left( \frac{b - i\Delta}{d} \right) \sinh \left( \frac{d t}{2} \right) + \cosh \left( \frac{d t}{2} \right), \quad (11)$$

where $d = \sqrt{b^2 - 2i\Delta + \frac{2\omega_0^2 + \Delta^2}{4}}$. The map (10) satisfies the completeness condition $\sum_j A_j A_j^\dagger = I$.

Consider a general state $\rho = \begin{pmatrix} \sin^2(\theta) & \frac{1}{2} \sin(2\theta) \\ \frac{1}{2} \sin(2\theta) & \cos^2(\theta) \end{pmatrix}$. From Eq. (1) for an initial pseudo-pure state $\rho = |0\rangle \langle 0|$, which is obtained when $\theta = \frac{\pi}{2}$, one obtains PDM for the above channel as below:

$$\mathcal{P}_{\text{AD}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{1-r(t)}}{2} & 0 \\ 0 & \frac{\sqrt{1-r(t)}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (12)$$

which has the eigenvalues $e_1 = -\frac{\sqrt{1-r(t)}}{2}$, $e_2 = \frac{\sqrt{1-r(t)}}{2}$, $e_3 = 1$, $e_4 = 0$. Note that $e_1$ is always negative indicating that $\mathcal{P}_{\text{AD}}$ is not positive semi-definite. The measure (4) for reads

$$F_{\text{AD}} = \log_2 \left( 1 + \sqrt{1-r(t)} \right) \quad (13)$$

which is plotted in Fig. (1) for the considered noise in its Markov and non-Markov regimes. Note that, for the particular case of initial state $\rho = |0\rangle \langle 0|$, when $G(t) = \sqrt{1-r(t)} = 1$, then $F = 1$. That is when there is no noise acting on the system, the temporal correlations are maximum. And when $r(t) = 1$, i.e., when the system is maximally damped, both $e_1 = e_2 = 0$, hence the causality measure $F = 0$, which implies that causal correlations die out at that point. One may observe the revivals of causal correlations which happens when $G(t)$ becomes negative; see Fig. (1). When all the eigenvalues are positive, then the PDM is called acausal in the sense that there are no causal relationships, and if at least one eigenvalue is negative then $F > 0$ hence there exist causal correlations and the PDM then is called causal.

Note that when the system interacts with the environment in weak-coupling regime, the causal correlations die slower than that compared with in the strong-coupling regime. The reason is that the stronger the coupling, the rapid the process. Interestingly, strong-coupling regimes make PDM to become acausal at times, but because of non-Markovian nature, the correlations can revive making PDM causal again. The normalized measure $C$, normalized as $C = \frac{M}{1+M}$ from Eq. (9) for AD, is plotted in Fig. (2).

Comparison with the decay rate measure. A measure of non-Markovianity based on canonical decay rate was given in [32]. The canonical master equation of the considered process is given by

$$\frac{d\rho_s(t)}{dt} = \gamma(t)[\sigma_- \rho_s(t) \sigma_+ - \frac{1}{2}(\sigma_+ \sigma_- - \rho_s(t))], \quad (14)$$

where $\gamma(t) = -2R[G(t)]$ is the time-dependent decoherence rate, and $G$ is the decoherence function given Eq. (11). As was noted earlier, the measure is defined over positive slope of $\frac{d\mathcal{F}}{dt}$, which from Eq. (13), is found to be

$$\frac{d\mathcal{F}}{dt} = \gamma(t) \frac{G(t)}{2(1 + \sqrt{G^2(t)})} \quad (15)$$
FIG. 1. (Color online) Breakdown of monotonicity of causality measure (4) for AD for a range of waiting times. For the values $\gamma_0 = 3$ and $b = 0.6$, the curves with $\theta = 0$ (dotted, yellow), $\theta = \frac{\pi}{4}$ (dot-dashed, blue), and $\theta = \frac{\pi}{2}$ (bold, orange) represent non-Markovian (NM) processes with revivals of temporal (causal) correlations. And for $\gamma_0 = 0.6$ and $b = 3$ the dashed purple curve, for an initial state with $\theta = \frac{\pi}{4}$, represents time-dependent Markovian (TDM) process, where the correlations fall monotonically. All the plots are for a simple case of $\Delta = 0$.

where $\gamma(t) = 2R\left(\sqrt{1 - \frac{2\gamma_0}{\Delta}} \cosh\left(\frac{\gamma_0}{2}t\right)\right)$ is the decay rate with $\Delta = 0$. Note that the denominator of (15) is always positive. The fraction $\frac{d\gamma}{dt} > 0$ when $G(t) < 0$ i.e., when the revivals occur, and that is when $\gamma(t) < 0$. Therefore, negativity of decay rate gives rise to CP-indivisibility of $\mathcal{E}$ which is when the monotonicity condition (6) of causality measure $\mathcal{F}$ will be broken.

Comparison with trace distance. One may consider comparing the measure (3 and 8) with trace distance between a pair of initial states $\{\rho_1, \rho_2\}$ under a channel $\mathcal{E}$, given by $\text{TD} = \frac{1}{2}\text{tr}|\mathcal{E}[\rho_1] - \mathcal{E}[\rho_2]|$. The distinguishability of a pair of initial states, represented on a Bloch sphere, undergoing a CPTP map decreases over time. It can be shown [14] that $|+\rangle$ and $|−\rangle$ are an optimal pair for an AD process given in Eq. (10). However, for a qubit undergoing AD, if the temporal correlations are set up, meaning that the future qubit under AD gets correlated with itself in the past such that both qubit states (in time) can not be written in a tensor product form, then it is seen in this work that the initial state $|0\rangle$ maximizes the measure (9) for AD. Interestingly, TD under AD for initial pair of states $|+\rangle$ and $|−\rangle$ reads $\sqrt{1 - r(t)}$, which is exactly equal to the causality measure (3) for an input state $|0\rangle$ under AD i.e., $f_{cm} = \sqrt{1 - r(t)}$. Hence we have

$$\mathcal{F}_{\rho = |0\rangle\langle 0|} = \log_2(1 + \text{TD}_{\rho_+, \rho_-}),$$

where $\rho_{\pm} = |\pm\rangle\langle \pm|$ are the initial pair of orthogonal states on the Bloch sphere and $\rho$ is the initial input the PDM.

It is important to note that in this framework, PDM, representing the temporally non-separable state, undergoes AD. One may say that revival of temporal correlations can be interpreted as equivalent to information back-flow via recurrence of trace-distance, given that the measures under comparison are optimized over initial input states. Therefore, one must keep in mind the optimization over input states while comparing the causality based non-Markovianity measure with that of others.

As a second example, let us consider non-Markovian generalized amplitude damping channel (GADC) introduced in [15], details of which are given in appendix (B). This example is given in order to exemplify a unique feature of the causality measure that it detects non-Markovianity of the channel while trace distance doesn’t when memory effects are coming solely from non-unitary part.

V. CONCLUSIONS

A measure of non-Markovianity is introduced based on causality monotone given in Eq. (4) arising out of pseudo-density-matrix (PDM) formalism [13]. A more generalized monotone [22], which is just logarithm of the trace norm of PDM, is used to define measure. The function $\mathcal{F}$ in Eq. (5) is non-increasing for local operations which are CPTP maps in general. Breakdown of monotonicity of $\mathcal{F}$ is an indication that the corresponding map is CP-indivisible. A suitable quantifier of non-Markovianity is given in Eq. (9).

Its comparison with the decay rate measure for the case of damped JC model is also done showing that negativity of decay rate entails breakdown of monotonicity of causality measure hence of CP-divisibility of a channel. Since causality measure, for certain input states of PDM, is shown to upper bound the quantum capacity of a channel, it can be said that it upper bounds the capacity based measures introduced [8].

It is worth noting that treating quantum mechanics beyond a definite causal structure is under intense investigation [11]. PDM formalism, as proposed, is only
for qubits rather than for states of arbitrary dimensions across time [33]. Thus, the proposed measure is only for qubit channels. Therefore, limitations of PDF framework would apply to the measure presented in this work too. Nevertheless, the framework, as it stands, is also for multiple qubits across space and time, hence use of, say n independent qubit channels is possible, which in fact enables one to quantify quantum capacity of a channel using causality measure [22]. Interesting future direction would be to investigate its applicability and compare it with the recently introduced non-Markovianity measure based on temporal steering [19].

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Appendix A: Pedagogical overview of PDM formalism

A density matrix can be given a representation based on expectation values of Pauli operators \( \sigma_0 = I, \sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z \):

\[
\rho = \frac{1}{2} \sum_{j=0}^{3} \alpha_j \sigma_j,
\]

where \( \alpha_j = \langle \sigma_j \rangle = \text{Tr}(\rho \sigma_j) \). Similarly, an entangled state of two spatially separated systems A and B can also be given a Pauli representation as:

\[
\rho^{AB} = \frac{1}{4} \sum_{i,j=0}^{3} \alpha_{ij} (\sigma_i \otimes \sigma_j),
\]

where \( \alpha_{ij} = \langle \sigma_i \otimes \sigma_j \rangle = \text{Tr}(\rho^{AB} \sigma_i \otimes \sigma_j) \) are the joint expectation values of two Pauli operators pertaining to symbols \( i \) and \( j \). Here, the system density operators \( \rho^A \) and \( \rho^B \) act on different Hilbert spaces \( \mathcal{H}^A \) and \( \mathcal{H}^B \) respectively, each of dimension 2. The 4-dimensional state \( \rho^{AB} \) is separable if \( |\alpha_{11}| + |\alpha_{22}| + |\alpha_{33}| \leq 1 \), otherwise entangled. This approach can be extended to define correlations in time by associating different qubit Hilbert spaces \( \mathcal{H}_{t_1} \) and \( \mathcal{H}_{t_2} \) to the density matrix \( \rho_A \) at time \( t_1 \) and \( \rho_B \) at time \( t_2 \) respectively, each of dimension 2. Time evolution is generally obtained by using a qubit completely positive trace preserving map \( \mathcal{E}_{B\rightarrow A} \) such that \( \rho_B = \mathcal{E}_{B\rightarrow A}[\rho_A] \). The state \( \rho_B \) may be thought of as an input to the quantum channel \( \mathcal{E} \). This scenario corresponds to a simple case of two Pauli measurements and the density matrix reads

\[
\mathcal{P}_{AB} = \frac{1}{4} \sum_{i,j=0}^{3} \langle \sigma_i \otimes \sigma_j \rangle (\sigma_i \otimes \sigma_j)
\]

This can be extended to k sequence of Pauli measurements representing k-qubit temporal state as below:

\[
P = \frac{1}{2^k} \sum_{i_1=0}^{3} \cdots \sum_{i_k=0}^{3} \langle \sigma_{i_1} \rangle_{j=1}^{k} \otimes \sigma_{i_j}
\]

where \( \langle (\sigma_{i_j})_{j=1}^{k} \rangle \) is the joint expectation value of k Pauli observables where each measurement event is denoted by \( j \) for each \( i \). In other words it is nothing but the correlation function of a k sequence of Pauli measurements \( \sigma_{i_j} \) on the system [28]. Above state (A4) shares all the features of a proper density matrix except that it may not be positive semi-definite. Hence it is called pseudo-density matrix (PDM). In this work, we consider the simplest case of two measurements where for \( k = 2 \), the above equation reduces to (A3). For convenience, Eq. (1) is used as an alternative representation of a two-point PDM (A3). Interestingly, through Choi-Jamiołkowski isomorphism, Eq. (1) in fact can be represented as

\[
\mathcal{P}_{AB} = \{ \rho_A \otimes \frac{I}{2}, \chi_{AB} \},
\]

where \( \chi_{AB} := \sum_{i,j} \langle I_A \otimes \mathcal{E}_{B\rightarrow A} \rangle \langle i | A \rangle \langle j | B \rangle \), which is known as Jordan product representation. It is important to note that PDM represents both spatial and temporal correlations. Whenever PDM is not positive semi-definite, i.e., when it has at least one negative eigenvalue, then there exist causal (temporal) correlations and PDM represents the time-like correlated system state. In other words, the correlations are purely temporal as they are obtained for a system between time \( t_1 \) and the same system at a later time \( t_2 \) evolved through the channel \( \mathcal{E}_{B\rightarrow A} \). When PDM is positive semi-definite, it is equivalent to the state of a space-like correlated systems. Hence, PDM formalism accounts for spatial and temporal correlations in unified way. However, it has its own limitations such as it is, as proposed, only for qubit systems across space and time.

Appendix B: Example 2: non-Markovian GADCC

In [15], it was pointed out that trace distance may fail to detect non-Markovianity for non-unital channels. Here, we show that causality measure detects memory originating from both unital and non-unital parts. We consider a generalized amplitude damping (GAD) chan-
FIG. 3. (Color online) Breakdown of monotonicity of causality measure (4) for $\omega = 3$, (bold, red curve) and the behavior of trace distance (TD) under non-Markovian GAD [15].

The channel given by the Kraus operators representation [15, 34]

\[ E_1 = \sqrt{1 - p(t)} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1 - \lambda(t)} \end{bmatrix}; \]
\[ E_2 = \sqrt{1 - p(t)} \begin{bmatrix} 0 & \sqrt{\lambda(t)} \\ \sqrt{1 - \lambda(t)} & 0 \end{bmatrix}; \]
\[ E_3 = \sqrt{p(t)} \begin{bmatrix} \sqrt{1 - \lambda(t)} & 0 \\ 0 & 1 \end{bmatrix}; \]
\[ E_4 = \sqrt{p(t)} \begin{bmatrix} 0 & \sqrt{\lambda(t)} \\ \lambda(t) & 0 \end{bmatrix}. \]  

(B1)

with $p(t) = \sin^2(\omega t)$ and $\lambda(t) = 1 - e^{-t}$.

From Eq. (1), for an initial state $\rho = |0\rangle \langle 0|$, one obtains PDM for GAD as below

\[ \mathcal{P}^{\text{GAD}} = \begin{pmatrix} 1 - p\lambda & 0 & 0 & 0 \\ 0 & p\lambda & \sqrt{1 - x} \sqrt{1 - x} & 0 \\ 0 & \sqrt{1 - x} \sqrt{1 - x} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]  

(B2)

which has the eigenvalues $h_1 = 0$, $h_2 = (1 - \lambda p)$, and $h_{3,4} = \frac{1}{2} \left( \lambda p \pm \sqrt{1 - \lambda + \lambda^2 p^2} \right)$.

The measure (4) for GAD reads

\[ \mathcal{F}^{\text{GAD}} = \log_2 \left( \sqrt{1 + 2x + y} + \sqrt{1 - 2x + y + 2z} \right) - 1. \]  

(B3)

where $x := \sqrt{\lambda^2 p^2 (\lambda (\lambda p^2 - 1) + 1)}$ and $y := \lambda (2 \lambda p^2 - 1)$ and $z := \sqrt{(\lambda p - 1)^2}$.

Note the revivals of temporal correlations for $\omega = 3$ in Figure (3), which is evident from the observation that $p(t)$ appears in the measure. Whereas, the trace distance (TD) between a pair of initial states $|+\rangle$ and $|-\rangle$ for the channel (B1) evaluates to $\sqrt{1 - \lambda(t)}$ (plotted in Fig. (3)) which is independent of $p(t)$. It can be shown that TD is independent of $p(t)$ for all initial pairs of orthogonal states under the action of GAD.

One may obtain a measure of non-Markovianity using Eq. (9), which we do not do here, since the purpose of this example was to show that the causality measure can detect non-Markovianity of non-unital channels.

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