HYPERCYCLIC AND MIXING COMPOSITION OPERATORS ON $H^p$

ZHENG RONG

Abstract. Extending previous results of Bourdon and Shapiro we characterize the hypercyclic and mixing composition operators $C_\varphi$ for the automorphisms of $\mathbb{D}$ on any of the spaces $H^p$ with $1 \leq p < +\infty$.

1. Introduction

Throughout this article, let $\mathbb{N}$ denote the set of nonnegative integers. Let $\mathbb{C}$ denote the complex number field. Let $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ and $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$.

An automorphism of $\mathbb{D}$ is a bijective analytic function $\varphi : \mathbb{D} \to \mathbb{D}$. The set of all automorphisms of $\mathbb{D}$ is denoted by $\text{Aut}(\mathbb{D})$. It is well known that the automorphisms of $\mathbb{D}$ are the linear fractional transformations

$$\varphi(z) = \frac{bz - a}{1 - az}, |a| < 1, |b| = 1.$$ 

Moreover, every $\varphi \in \text{Aut}(\mathbb{D})$ maps $\mathbb{T}$ bijectively onto itself (see [8, pages 131-132]).

For $1 \leq p < +\infty$, let $H^p$ denote the space of all analytic functions on $\mathbb{D}$ for which

$$\sup_{0 \leq r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < +\infty.$$ 

For any $f \in H^p$, let

$$\|f\|_p = \sup_{0 \leq r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$ 

Then $(H^p, \| \cdot \|_p)$ is a Banach space.

Let $\varphi \in \text{Aut}(\mathbb{D})$ and let $C_\varphi(f) = f \circ \varphi(f \in H^p)$ be the corresponding composition operator on $H^p$. It is well known that for any $1 \leq p < +\infty$ and $\varphi \in \text{Aut}(\mathbb{D})$, $C_\varphi$ defines a continuous linear operator on $H^p$ (see [25, pages 220-221]).

A continuous linear operator $T$ on a Banach space $X$ is called hypercyclic if there is an element $x$ in $X$ whose orbit $\{T^n x : n \in \mathbb{N}\}$ under $T$ is dense in $X$; topologically transitive if for any pair $U, V$ of nonempty open subsets of $X$, there exists some nonnegative integer $n$ such that $T^n(U) \cap V \neq \emptyset$; and mixing if for any pair $U, V$
of nonempty open subsets of $X$, there exists some nonnegative integer $N$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

Recently Bourdon and Shapiro \[5, 6\] have done an extensive study of cyclic and hypercyclic linear fractional composition operators on $H^2$. Zorboska \[26\] has determined hypercyclic and cyclic composition operators induced by a linear fractional self map of $\mathbb{D}$, acting on a special class of smooth weighted Hardy spaces $H^2(\beta)$. Gallardo and Montes \[11\] have obtained a complete characterization of the cyclic, supercyclic and hypercyclic composition operators $C_\varphi$ for linear fractional self-maps $\varphi$ of $\mathbb{D}$, acting on any of the spaces $S_\nu$, $\nu \in \mathbb{R}$. In particular, $S_0$ is the Hardy space $H^2$, $S_{-\frac{1}{2}}$ is the Bergman space $A^2$, and $S_{\frac{1}{2}}$ is the Dirichlet space $D$ under an equivalent norm. Since the Hardy space $H^2$ is a particular case of the spaces $H^p$ with $1 \leq p < +\infty$, it is therefore very natural to try to characterize the cyclic, supercyclic and hypercyclic composition operators $C_\varphi$ for linear fractional self-maps $\varphi$ of $\mathbb{D}$ on any of the spaces $H^p$ with $1 \leq p < +\infty$. In this paper we will characterize the hypercyclic and mixing composition operators $C_\varphi$ for the automorphisms of $\mathbb{D}$ on any of the spaces $H^p$ with $1 \leq p < +\infty$, generalizing the corresponding results in \[5, 6\].

**Theorem 1.1.** Let $1 \leq p < +\infty$. Let $\varphi \in \text{Aut}(\mathbb{D})$ and $C_\varphi$ be the corresponding composition operator on $H^p$. Then the following assertions are equivalent:

1. $C_\varphi$ is hypercyclic;
2. $C_\varphi$ is mixing;
3. $\varphi$ has no fixed point in $\mathbb{D}$.

Bourdon and Shapiro \[5, 6\] proved the above result in the case $p = 2$. Hence the above result generalizes the corresponding results in \[5, 6\].

This paper is organized as follows. In Section 2 we characterize the hypercyclic and mixing composition operators $C_\varphi$ for the automorphisms of $\mathbb{D}$ on any of the spaces $H^p$ with $1 \leq p < +\infty$.

2. **HYPERCYCLIC AND MIXING COMPOSITION OPERATORS ON $H^p$**

In this section we characterize the hypercyclic and mixing composition operators $C_\varphi$ for the automorphisms of $\mathbb{D}$ on any of the spaces $H^p$ with $1 \leq p < +\infty$, generalizing the corresponding results in \[5, 6\].

The following propositions are the major techniques we need.

If $f \in H^p(1 \leq p < +\infty)$, then \( \lim_{r \to 1^-} f(re^{i\theta}) \) exists for almost all values of $\theta$ (see \[10\], page 17), thus defining a function which we denote by $f(e^{i\theta})$.

We need the following important properties of the $H^p$-spaces (see \[10\], pages 9, 12, 21).

**Proposition 2.1.** Let $f \in H^p, 1 \leq p < +\infty$. Then

1. \( \|f\|_p = \lim_{r \to 1^-} \left(\frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}; \)
\[ \lim_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta = \int_0^{2\pi} |f(e^{i\theta})|^p d\theta; \]
\[ \lim_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p d\theta = 0. \]

We need the following property of the Taylor coefficients of \( H^p \) functions (see \cite{10}, page 94).

**Proposition 2.2.** Let \( 1 \leq p \leq 2 \) and \( f \in H^p \). Let \( \sum_{n=0}^{\infty} a_n z^n \) be the Taylor series of \( f \) at the origin. Then \( \{a_n\}_{n=0}^{\infty} \in l^q \), where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Our aim now is to characterize when \( C_\varphi \) is hypercyclic on \( H^p \). To this end we need some important dynamical properties of automorphisms \( \varphi \) of \( \mathbb{D} \).

Let \( \varphi(z) = \frac{az + b}{cz + d}, ad - bc \neq 0 \)
be an arbitrary linear fractional transformation, which we consider as a map on the extended complex plane \( \hat{\mathbb{C}} \). Then \( \varphi \) has either one or two fixed points in \( \hat{\mathbb{C}} \), or it is the identity.

Suppose that \( \varphi \) has two distinct fixed points \( z_0 \) and \( z_1 \), and let \( \sigma \) be a linear fractional transformation that maps \( z_0 \) to 0 and \( z_1 \) to \( \infty \). Then \( \psi := \sigma \circ \varphi \circ \sigma^{-1} \) has fixed points 0 and \( \infty \), which easily implies that \( \psi(z) = \lambda z \) for some \( \lambda \neq 0 \). The constant \( \lambda \) is called the **multiplier** of \( \varphi \). Replacing \( \sigma \) by \( 1/\sigma \) one sees that also \( 1/\lambda \) is a multiplier, which, however, causes no problem in the following.

**Definition 2.3.** Let \( \varphi \) be a linear fractional transformation that is not the identity.
(a) If \( \varphi \) has a single fixed point then it is called **parabolic**.
(b) Suppose that \( \varphi \) has two distinct fixed points, and let \( \lambda \) be its multiplier. If \( |\lambda| = 1 \) then \( \varphi \) is called **elliptic**; if \( \lambda > 0 \) then \( \varphi \) is called **hyperbolic**; in all other cases, \( \varphi \) is called **loxodromic**.

We need the following dynamical properties of automorphisms \( \varphi \) of \( \mathbb{D} \) (see \cite{15}, pages 125-126).

**Proposition 2.4.** Let \( \varphi \in \text{Aut}(\mathbb{D}) \), not the identity. Then we have the following:
(i) if \( \varphi \) is parabolic then its fixed point \( z_0 \) lies in \( \mathbb{T} \), and \( \varphi^n(z) \to z_0 \), \( \varphi^{-n}(z) \to z_0 \) for all \( z \in \hat{\mathbb{C}} \);
(ii) if \( \varphi \) is elliptic then it has a fixed point in \( \mathbb{D} \);
(iii) if \( \varphi \) is hyperbolic then it has distinct fixed points \( z_0 \) and \( z_1 \) in \( \mathbb{T} \) such that \( \varphi^n(z) \to z_0 \) for all \( z \in \hat{\mathbb{C}}, z \neq z_1 \), and \( \varphi^{-n}(z) \to z_1 \) for all \( z \in \hat{\mathbb{C}}, z \neq z_0 \);
(iv) \( \varphi \) cannot be loxodromic.

The dynamical properties of \( \varphi \in \text{Aut}(\mathbb{D}) \) imply the dynamical properties of \( C_\varphi \).

Finally we prove Theorem 1.1.

**Proof of Theorem 1.1.**
(2)⇒(1) Assume that $C_{\varphi}$ is mixing. Then $C_{\varphi}$ is topologically transitive. Since the polynomials form a dense set in $H^p$, $H^p$ is separable. Since a continuous linear operator on a separable Banach space is topologically transitive if and only if it is hypercyclic (see [13, page 10]), $C_{\varphi}$ is hypercyclic.

(1)⇒(3) Assume that $C_{\varphi}$ is hypercyclic. We will show that $\varphi$ has no fixed point in $\mathbb{D}$. Suppose $\varphi$ has a fixed point $z_0 \in \mathbb{D}$. Since $C_{\varphi}$ is hypercyclic, there exists a $f \in H^p$ such that $\{(C_{\varphi})^n f : n \geq 0\}$ is dense in $H^p$. We may choose a $g \in H^p$ with $g(z_0) \neq f(z_0)$. Since $\{(C_{\varphi})^n f : n \geq 0\} = H^p$, we may choose a sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers with $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$ such that $\lim_{k \to \infty} (C_{\varphi})^{n_k} f = g$.

Since each point evaluation $k_\lambda : H^p \to \mathbb{C}(\lambda \in \mathbb{D})$ is continuous on $H^p$, where $k_\lambda(f) = f(\lambda)(f \in H^p)$, we have $\lim_{k \to \infty} ((C_{\varphi})^{n_k} f)(z_0) = g(z_0)$. Notice that $((C_{\varphi})^{n_k} f)(z_0) = (f \circ \varphi^{n_k})(z_0) = f(\varphi^{n_k}(z_0)) = f(z_0)$.

Hence $f(z_0) = g(z_0)$, this is a contradiction with $f(z_0) \neq g(z_0)$. Therefore $\varphi$ has no fixed point in $\mathbb{D}$.

(3)⇒(2) Suppose that $\varphi$ has no fixed point in $\mathbb{D}$. It suffices to show that $C_{\varphi}$ satisfies Kitai’s criterion. By Proposition 2.4, $\varphi$ is either parabolic or hyperbolic, and in both cases $\varphi$ has fixed points $z_0$ and $z_1$ in $\mathbb{T}$ (possibly with $z_0 = z_1$) such that $\varphi^n(z) \to z_0$ for all $z \in \mathbb{T}\{z_1\}$ and $\varphi^{-n}(z) \to z_1$ for all $z \in \mathbb{T}\{z_0\}$.

Now, for $X_0$ we will take the subspace of $H^p$ of all functions that are analytic on a neighbourhood of $\overline{\mathbb{D}}$ and that vanish at $z_0$. Since $z_0$ is a fixed point of $\varphi$, $C_{\varphi}$ maps $X_0$ into itself.

Claim 1. For any $1 \leq p < +\infty$ we have $X_0 = H^p$.

We divide it into two cases.

Case i. If $1 < p < +\infty$. First we will show that $X_0^{+} = \{0\}$. Let $1 < p < +\infty$ and $x^* \in X_0^{+}$. We will show that $x^* = 0$. Since $1 < p < +\infty$ and $x^* \in (H^p)^*$, there exists a unique function $g \in H^q$ such that

$$x^*(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{-i\theta}) d\theta (f \in H^p),$$

where $\frac{1}{p} + \frac{1}{q} = 1$ (see [10, pages 112-113]). By Proposition 2.1 we have

$$\lim_{r \to 1^-} \int_0^{2\pi} |g(re^{i\theta}) - g(e^{i\theta})|^q d\theta = 0.$$

Hence for each $f \in H^p$ we have

$$\lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(re^{-i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{-i\theta}) d\theta.$$
Since, for any \( n \geq 0 \), the functions \( g_n : \mathbb{C} \to \mathbb{C} \) defined by \( g_n(z) = z_0 z^n - z^{n+1} \) belong to \( X_0 \) we have that \( x^*(g_n) = 0(n \geq 0) \). Notice that
\[
x^*(g_n) = \frac{1}{2\pi} \int_0^{2\pi} g_n(\epsilon^{i\theta})g(\epsilon^{-i\theta})d\theta
= \lim_{r\to 1^{-}} \frac{1}{2\pi} \int_0^{2\pi} g_n(\epsilon^{i\theta})g(re^{-i\theta})d\theta.
\]
Hence for any \( n \geq 0 \) we have
\[
\lim_{r\to 1^{-}} \frac{1}{2\pi} \int_0^{2\pi} g_n(\epsilon^{i\theta})g(re^{-i\theta})d\theta = 0.
\]
Let \( g(z) = \sum_{n=0}^{\infty} a_n z^n(z \in \mathbb{D}), 0 < r < 1 \) and \( n \geq 0 \). Then
\[
\frac{1}{2\pi} \int_0^{2\pi} g_n(\epsilon^{i\theta})g(re^{-i\theta})d\theta
= \frac{1}{2\pi} \int_0^{2\pi} (z_0 \epsilon^{in\theta} - \epsilon^{i(n+1)\theta})g(re^{-i\theta})d\theta
= \frac{1}{2\pi} \int_0^{2\pi} z_0 \epsilon^{in\theta} g(re^{-i\theta})d\theta - \frac{1}{2\pi} \int_0^{2\pi} \epsilon^{i(n+1)\theta}g(re^{-i\theta})d\theta
= \frac{1}{2\pi} \int_0^{2\pi} z_0 \epsilon^{in\theta} \sum_{k=0}^{\infty} a_k r^k e^{-ik\theta}d\theta - \frac{1}{2\pi} \int_0^{2\pi} \epsilon^{i(n+1)\theta} \sum_{k=0}^{\infty} a_k r^k e^{-ik\theta}d\theta
= \sum_{k=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} z_0 a_k r^k \epsilon^{-ik\theta}d\theta - \sum_{k=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \epsilon^{i(n+1)\theta} a_k r^k e^{-ik\theta}d\theta
= \frac{1}{2\pi} \int_0^{2\pi} z_0 a_n r^n d\theta - \frac{1}{2\pi} \int_0^{2\pi} a_{n+1} r^{n+1} d\theta
= z_0 a_n r^n - a_{n+1} r^{n+1}.
\]
Since \( \lim_{r\to 1^{-}} \frac{1}{2\pi} \int_0^{2\pi} g_n(\epsilon^{i\theta})g(re^{-i\theta})d\theta = 0 \), we have
\[
\lim_{r\to 1^{-}} (z_0 a_n r^n - a_{n+1} r^{n+1}) = z_0 a_n - a_{n+1} = 0.
\]
Hence \( a_n = a_0 z^n_0(n \geq 0) \). Since \( q = \frac{p}{p-1} > 1 \), we may choose \( 1 < q_1 \leq 2 \) with \( q_1 < q \). It is evident that \( H^q \subset H^{q_1} \). Since \( g \in H^q, g \in H^{q_1} \). By Proposition 2.2 we have \( \{a_n\}_{n=0}^{\infty} \subset l^{p_1} \), where \( \frac{1}{p_1} + \frac{1}{q_1} = 1 \). Notice that
\[
\sum_{n=0}^{\infty} |a_n|^{p_1} = \sum_{n=0}^{\infty} |a_0 z^n_0|^{p_1} = \sum_{n=0}^{\infty} |a_0|^{p_1} < +\infty.
\]
Hence \( a_0 = 0 \) and \( a_n = 0 \) for \( n \geq 0 \). Therefore \( g(z) = 0 \) for all \( |z| < 1 \) and \( x^* = 0 \).
Second we will show that $X_0 = H^p$. Since $X_0^\perp = X_0^\perp = X_0$ and $X_0 = \{0\}$, we have $X_0^\perp = \{0\}$. Hence $X_0^{\perp\perp} = (0)^\perp = H^p$. Since $1 < p < +\infty$, $H^p$ is reflexive. Since $X_0$ is norm-closed, $X_0$ is $\sigma(X^*, X)$-closed. Finally we have $X_0^{\perp\perp} = X_0$. Hence $X_0^{\perp\perp} = H^p$. This proves the case $1 < p < +\infty$.

Case ii. If $p = 1$. We will show that $X_0 = H^1$. Let $f \in H^1$ and $\varepsilon > 0$. We will show that there exists a $g \in X_0$ such that $\|f - g\|_1 < \varepsilon$. Since the polynomials form a dense set in $H^1$, there exists a polynomial $h$ such that $\|f - h\|_1 < \frac{\varepsilon}{2}$. By the case $p = 2$, $X_0$ is dense in $H^2$. Hence there exists a $g \in X_0$ such that $\|g - h\|_2 < \frac{\varepsilon}{2}$. Notice that $\|g - h\|_1 \leq \|g - h\|_2$. Then $\|g - h\|_1 < \frac{\varepsilon}{2}$. Hence $\|f - g\|_1 < \|f - h\|_1 + \|h - g\|_1 < \varepsilon$. This proves the case $p = 1$.

Claim 2. $(C_\varphi)^n f \to 0$ for all $f \in X_0$.

Let $f \in X_0$. Since $f \circ \varphi^n$ is continuous on $\overline{D}$, by Proposition 2.1 we have

$$
\|(C_\varphi)^n f\|_p = \frac{1}{2\pi} \int_0^{2\pi} |f(\varphi^n(e^{i\theta}))|^p d\theta.
$$

Since the integrands are uniformly bounded and convergent to $|f(z_0)|^p = 0$, for every $t$ with possibly one exception, the dominated convergence theorem implies that $(C_\varphi)^n f \to 0$. This proves Claim 2.

Next, for $Y_0$ we will take the subspace of $H^p$ of all functions that are analytic on a neighbourhood of $\overline{D}$ and that vanish at $z_1$, and for $S$ we take the map $S = C_{\varphi^{-1}}$. Since $z_1$ is a fixed point of $\varphi^{-1}$, $S$ maps $Y_0$ into itself, and clearly $C_\varphi S = I$. It follows as above that $Y_0$ is dense in $H^p$ and that $S^n f \to 0$ for all $f \in Y_0$. Therefore the conditions of Kitai’s criterion are satisfied, so that $C_\varphi$ is mixing.

\[ \Box \]

Bourdon and Shapiro [5, 6] proved Theorem 1.1 in the case $p = 2$. Hence Theorem 1.1 generalizes the corresponding results in [5, 6].

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Z.R., College of statistics and mathematics, Inner Mongolia University of Finance and Economics, Hohhot 010000, China.

Email address: rongzhenboshi@sina.com