Classical and quantum time dependent solutions in string theory

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Using the ontological interpretation of quantum mechanics in a particular sense, we obtain the classical behaviour of the scale factor and two scalar fields, derived from a string effective action for the FRW time dependent model. Besides, the Wheeler-DeWitt equation is solved exactly. We speculate that the same procedure could also be applied to S-branes.

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I. INTRODUCTION

There are several attempts to understand diverse aspects of time dependent models in the framework of string theory and branes [for a review see Ref. [1]]. In this context, new interesting results arise: Starting with an $AdS_5$ metric one can specify the localization of a brane and by these means a 4D FRW cosmology emerges on the brane [2], D-Brane inflation has also been proposed [3], the interaction between D-branes and anti-D-branes lead to the study of Tachyon condensation and its role in inflation [4]. The intersection of branes at different angles [5] generates an inflation potential. The rolling tachyon has also been considered in connection with inflation [6]. Compactified M-theory on an interval $S^1/Z_2$ and further compactified in a six-dimensional Calabi-Yau leads to two 4-D worlds at the ends of the interval in the 5D bulk. It turns out that besides boundary branes, those at the end of the intervals, there are also 5-D branes that can move through the bulk. A very interesting proposal was made [7] assuming that a bulk brane going from one boundary of the interval to the other end, would collide with the second boundary brane and produce the big bang. It has been claimed that the density fluctuations measured in the CMB could be a consequence of quantum fluctuations of the bulk brane. Also, there is no need for an inflation potential, an attractive potential is proposed describing the attraction of the branes and as a consequence the scale factor depends on the position of the brane in the interval. This model has, however, been very much debated in the literature [8].

On the other hand, it is well known that relativistic theories of gravity such as general relativity or string theory are invariant under reparametrization of time. The quantization of such theories presents a number of problems of principle loosely known as “the problem of time” [9–11]. This problem occurs in all system whose classical version is invariant under reparametrization of the time parameter, which leads to the absence of this parameter at the quantum level. The formal question is how to handle the classical Hamiltonian constraint, $H \approx 0$, in the quantum theory. Also, connected with the problem of time is the “Hilbert space problem” [9,10], it is not at all obvious which inner product of states one has to use in quantum gravity, and whether there is a need for such a structure at all.

In some situations, the notion of time can be recovered in quantum cosmology. Basically, the approaches to address the problem of time can be classified as to whether an appropriate time variable is identifiable already at the classical level, or only after quantization. The
former try, for example, to cast the hamiltonian constraint through an appropriate canonical transformation into the reduced form \( P_T + H \approx 0 \), where \( P_T \) denotes the momentum conjugate to the time variable \( T \) or, in other words \( P_T \) is the momentum associated with a variable, that appears linearly in the hamiltonian constraint, and after quantization this implies a Schrödinger equation.

If this were possible, the problem of time would be solved, since the new form of the constraints would be transformed into a Schrödinger type of equation upon quantization, and a standard inner product could be proposed to tackle the Hilbert space problem. However, it is known that this cannot work for the full configuration space [12]. One uses then a Klein-Gordon type of inner product [13], since this inner product is not positive definite, many problems arise which have lead some authors to invoke a “third quantization” of the theory.

The purpose of this paper is to present a mechanism to obtain classical solutions for time dependent models from the quantum regime, without having to solve directly the field equations of motion. The idea is that when the behaviour of the wave function with respect to the gravitational part leads to an increasing function, this type of solution cannot correspond to a classical trajectory. Therefore, there is a forbidden region in the evolution of our universe. Then, we need a wave function having a decreasing behaviour with respect to the scale factor \( A \). The behaviour of the scale factor may be determined in two different forms from the quantum regime: we can calculate the expectation value of the scale factor, in the spirit of the many worlds interpretation of quantum mechanics [14]

\[
<A>_t = \frac{\int_{0}^{\infty} \mathcal{W}(A) \Psi^*(A, t) A(t) \Psi(A, t) \, dA}{\int_{0}^{\infty} \mathcal{W}(A) \Psi^*(A, t) \Psi(A, t) \, dA},
\]

where \( \mathcal{W}(A) \) is a weight function that normalizes the expectation value. The other way corresponds to use the WKB semiclassical approximation, evaluating the Bohmian trajectories in the ontological formulation of quantum mechanics [15], where the system follows a real trajectory given by the equation

\[
\Pi_q = \frac{\partial \Phi}{\partial q},
\]

where the index \( q \) designates one of the degrees of freedom of the system, and \( \Phi \) is the phase of the wave function

\[
\Psi = W e^{i\Phi},
\]

where \( W \) and \( \Phi \) are real functions.

A classical solution was found using the last approach for a gravity model coupled to barotropic perfect fluid, as matter field, and cosmological constant [16]. We will show that the solution found for the cosmological model under consideration, in a particular effective string theory, has the same kind of behaviour than in ref. [16].

On the other hand, recently a new class of objects were introduced in string theory named spacelike or S-brane [17], objects that exist only for a moment of time. The main motivation for the introduction of the S-branes was the conjectured dS/CFT correspondence [18]. The expectative is that, using the analogy with p branes, S-branes can also be found as
explicit solutions of Einstein equations (coupled to scalar fields), the S-branes solutions are then \textit{time-dependent} backgrounds of the theory. Therefore, the approach we shall follow for a cosmological model could also be analyzed for a S-brane solution (see [19] and references therein).

We will consider the canonical quantization of a low-energy string effective action for a particular cosmological metric. This procedure is interesting by its own right and being the quantization (for a model dependent on time) of the effective theory of strings one would expect to gain some information on the quantization of the objects depending on time in string theory.

We begin by considering the compactification of the NS-NS sector of string effective action which contains the dilaton field, the graviton and a 2-form potential and is common to both type II and heterotic theories. Consider the form of the \((D+d)\)dimensional NS-NS action compactified on \(T^d\) (\(d\) torus). In \((D+d)\) dimensions, the action is given by

\[
S = \int d^{D+d}x \sqrt{g_{D+d}} e^{-\Phi} \left[ R_{D+d} + g^{AB} \nabla_A \Phi \nabla_B \Phi - \frac{1}{12} H_{M1M2M3} H_{N1'N2'M3'} g^{M1'M1'} g^{M2'M2'} g^{M3'M3'} \right].
\]  

(4)

In our case we compactify on 6-torus \(T^6\) with internal metric

\[
h_{ab} = \text{diag} \left( e^{-2\sigma}, e^{-2\sigma}, e^{-2\sigma}, 1, 1, 1 \right).
\]  

(5)

where \(a, b = (4, 5, 6, 7, 8, 9)\). Choosing \(H_{456} = F = \text{constant.}\), and by compactifying the coupling parameter \(\Phi\) becomes

\[
\Phi = 2\phi - \frac{1}{2} \ln(\det h_{ab}),
\]  

(6)

where \(d = 6, D = 4\). In the internal space we have \(h_{44} = h_{55} = h_{66} = e^{-2\sigma}\), thus by means of straightforward calculations and choosing for simplicity \(F = 0\), we arrive to an action in the string frame with two scalar fields (dilatonic and moduli field).

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} e^{-2\phi - 3\sigma} \left[ R + 4g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 12g^{\mu\nu} \partial_\mu \phi \partial_\nu \sigma + 6g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \right].
\]  

(7)

The static model with \(H^{\mu\nu \beta} \neq 0\) has been considered in the literature [20] in relation with black hole solutions. In [21] exact solutions for homogeneous, anisotropic cosmologies in four dimensions were obtained for the low-energy string effective action including a homogeneous dilaton \(\phi\) and antisymmetric tensor field \(B_{\mu\nu}\) (axion field).

Under the dynamical compactification scheme, one expect that the radius depending on the scalar field \(\sigma\) should be a small constant or vanishing, in such a way that the time dependence of \(\sigma\), keeps \(\sigma\) large.

The standard Friedmann-Robertson-Walker (FRW) metric is

\[
ds^2 = -N^2(t)dt^2 + A^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],
\]  

(8)
where $N$ is the lapse function, $A$ is the scale factor of the model, and $\kappa$ is the curvature index of the universe ($\kappa = 0, +1, -1$ flat, close and open, respectively). From this metric, the scalar curvature becomes

$$R = -\frac{6\ddot{A}}{AN^2} + \frac{6N\dot{A}}{N^3A} - \frac{6k}{A^2} - 6\left(\frac{\dot{A}}{NA}\right)^2,$$

where the overdot denotes time derivative. Thus, the action (7) is written as

$$S = \frac{1}{2\kappa^2} \int d^3x \sqrt{g} \int dt \ e^{-2\phi - 3\sigma} \left\{ 6 \left[ -\left(\frac{A^2\ddot{A}}{N}\right)^\prime + \frac{A(\dot{A})^2}{N} - NkA \right] - \frac{A^3}{N} \left[ 4\dot{\phi}^2 + 12\dot{\sigma}\dot{\phi} + 6\dot{\sigma}^2 \right] \right\}.$$

In the Einstein frame, we perform the conformal transformation into the metric components

$$\bar{g}_{\mu\nu} = e^{-(2\phi+3\sigma)}g_{\mu\nu},$$

getting

$$N = e^{\phi + \frac{3}{2}\sigma}\bar{N}; \quad A = e^{\phi + \frac{3}{2}\sigma}\bar{A} \quad \dot{A} = e^{\phi + \frac{3}{2}\sigma}\left[ \bar{A} + (\dot{\phi} + \frac{3}{2}\dot{\sigma})\bar{A} \right],$$

and substituting the equations (12) in the action (10), we get

$$S = \frac{1}{2\kappa^2} \int d^3x \sqrt{\bar{g}} \int dt \ \left\{ \left( \frac{6\ddot{A}\bar{A}}{N} - 6k\bar{N}\bar{A} \right) - \frac{A^3}{N} \left( 10\dot{\phi}^2 + 30\dot{\phi}\dot{\sigma} + \frac{39}{2}\dot{\sigma}^2 \right) \right\}.$$

Given (13), the work is organized as follows. In the next section we will get the Hamiltonian, using the procedure given by Arnowitt-Deser-Misner (ADM), obtaining the Wheeler-DeWitt (WDW) equation for the three coordinate fields, in Sec. III we solve this WDW equation. The classical time behaviour of the coordinate fields is discussed in Sec. IV, the solutions are written in quadratures with respect to the scale factor, due to the fact that the solution was obtained in terms of elliptic integrals of first and second class. We take the particular case of a flat universe, where the solutions are found in a closed way. Sec. V is devoted to remarks.

**II. THE HAMILTONIAN**

The Lagrangian density of our model is

$$\mathcal{L} = \frac{6A\ddot{A}}{N} - 6kNA - \frac{A^3}{N} \left( 10\dot{\phi}^2 + 30\dot{\phi}\dot{\sigma} + \frac{39}{2}\dot{\sigma}^2 \right),$$

(14)
where the bars have been omitted. The canonical momenta to coordinate fields are defined in the usual way

\[ \Pi_A \equiv \frac{\partial L}{\partial \dot{A}} = \frac{12A\dot{A}}{N}, \]

\[ \Pi_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = -\frac{A^3}{N}(20\dot{\phi} + 30\dot{\sigma}), \]

\[ \Pi_\sigma \equiv \frac{\partial L}{\partial \dot{\sigma}} = -\frac{A^3}{N}(30\dot{\phi} + 39\dot{\sigma}), \]

from which we can obtain the temporal derivative of \( \phi \) and \( \sigma \) fields

\[ \dot{\sigma} = \frac{N}{A^3} \left( -\frac{3}{12} \Pi_\phi + \frac{1}{6} \Pi_\sigma \right), \]

\[ \dot{\phi} = \frac{N}{A^3} \left( \frac{13}{40} \Pi_\phi - \frac{1}{4} \Pi_\sigma \right). \]

Using (15, 16, 17), we can rewrite the Lagrangian density (14) in the canonical form

\[ L = \Pi_A \dot{A} + \Pi_\phi \dot{\phi} + \Pi_\sigma \dot{\sigma} - \frac{N}{24A^3} \left( \frac{\Pi_A^2}{24A} + 6kA + \frac{13 \Pi_\phi^2}{80 A^3} + \frac{\Pi_\sigma^2}{12A^3} - \frac{1}{4} \Pi_\phi \Pi_\sigma \right). \]

We can see that the Hamiltonian is given by

\[ H = \Pi_A \dot{A} + 6kA + \frac{13 \Pi_\phi^2}{80 A^3} + \frac{\Pi_\sigma^2}{12A^3} - \frac{1}{4} \Pi_\phi \Pi_\sigma. \]

Performing the variation of \( N \) in the Lagrangian density (18), we get the equation \( H = 0 \). Following the standard procedure to get the quantum version of this Hamiltonian, we promote to operators the canonical momenta \( \Pi_\lambda \) that satisfy the commutation relation \( [\Pi_\lambda, \lambda] = -i \) with the representation \( \Pi_\lambda = -i \frac{\partial}{\partial \lambda} \). We then apply the operator \( \hat{H} \) to the wave function \( \Psi \), i.e. \( \hat{H}\Psi = 0 \).

Notice that in principle the ambiguity order in equation (19) should be taken into account. This is quite a difficult problem to be treated in all its generality. We will, however, consider the proposal in [22] (there are other possibilities depending on different considerations on the operators, see refs. [23,24])

\[ \Pi_A^2 = -A^{-B} \frac{\partial}{\partial A} A^B \frac{\partial}{\partial A} = -\frac{\partial^2}{\partial A^2} - \frac{B}{A} \frac{\partial}{\partial A}, \]

where \( B \) is a real parameter that measure this ambiguity. For the other operators this problem does not arise

\[ \Pi_\phi^2 = -\frac{\partial^2}{\partial \phi^2}; \quad \Pi_\sigma^2 = -\frac{\partial^2}{\partial \sigma^2}. \]

In this way (19) can be written as

\[ \frac{1}{24A} \left( -\frac{\partial^2}{\partial A^2} - \frac{B}{A} \frac{\partial}{\partial A} \right) \Psi + 6kA \Psi - \frac{13}{80 A^3} \frac{\partial^2}{\partial \phi^2} \Psi - \frac{1}{12A^3} \frac{\partial^2}{\partial \sigma^2} \Psi + \frac{1}{4A^3} \frac{\partial^2}{\partial \phi \partial \sigma} \Psi = 0, \]

and multiplying (22) by \( 24A^3 \), we obtain the equation

\[ -A^3 \frac{\partial^2}{\partial A^2} \Psi - BA \frac{\partial}{\partial A} \Psi + 144kA^4 \Psi - \frac{39}{10} \frac{\partial^2}{\partial \phi^2} \Psi - 2 \frac{\partial^2}{\partial \sigma^2} \Psi + 6 \frac{\partial^2}{\partial \phi \partial \sigma} \Psi = 0. \]
III. QUANTUM SOLUTION

Employing the separation of variables method to solve (23), \( \Psi(A, \phi, \sigma) = A(A)C(\phi, \sigma) \) and rearranging, we get

\[
\frac{1}{A} \left( -A^2 \frac{d^2 A}{dA^2} - BA \frac{dA}{dA} + 144kA^4 A \right) + \frac{1}{C} \left( -\frac{39}{10} \frac{\partial^2 C}{\partial \phi^2} - 2 \frac{\partial^2 C}{\partial \sigma^2} + 6 \frac{\partial^2 C}{\partial \phi \partial \sigma} \right) = 0. \tag{24}
\]

This equation is equivalent to the set of partial differential equations

\[
A^2 \frac{d^2 A}{dA^2} + BA \frac{dA}{dA} - \left( 144kA^4 \pm \nu^2 \right) A = 0, \tag{25}
\]
\[
\frac{39}{10} \frac{\partial^2 C}{\partial \phi^2} + 2 \frac{\partial^2 C}{\partial \sigma^2} - 6 \frac{\partial^2 C}{\partial \phi \partial \sigma} = \pm \nu^2 C, \tag{26}
\]

where \( \nu \) is a separation constant.

Equation (25) can be transformed to a Bessel differential equation for the function \( \Phi \), performing the transformations \( z = 6\sqrt{\nu}A \), \( A = A^{1-B} \Phi(z) \), whose solution for \( k \neq 0 \) becomes

\[
A = A^{1-B} Z_\alpha \left( 6\sqrt{-kA^2} \right), \tag{27}
\]

where \( Z_\alpha \) is a generic Bessel function, with order \( \alpha = \frac{1}{4} \sqrt{(1-B)^2 - 4(\pm \nu^2)} \). For \( k = 1 \), we have the modified Bessel functions \( I_\alpha \) and \( K_\alpha \). For \( k = -1 \), if \( \alpha \) is not integer, the solutions become the ordinary Bessel function \( J_{\pm \alpha} \), in other case we have a combination of the Bessel functions \( J_\alpha \) and \( Y_\alpha \) [25].

At this point, we need some conditions on the wave function, in such a way that the classical solutions are not forbidden for the scale factor \( A \). Then, we need a wave function having a decreasing behaviour with respect to the scale factor \( A \), it can be shown that this is the case when the parameter \( B \geq 1 \) and the order of the generic Bessel function \( \alpha > 0 \); besides we choose that the generic Bessel function will be \( (K_\alpha \) or \( J_\alpha \)), the modified Bessel function or ordinary Bessel function, according to case. With these conditions on the parameters, we postulate that the classical behaviour will be obtained, at least in the semiclassical approximation.

For the particular case of \( k = 0 \), the solution for the scale factor has the behaviour

\[
A = C_1 A^{1-B+\mu} + C_2 A^{1-B-\mu}, \tag{28}
\]

where \( \mu \) is given by

\[
\mu = \sqrt{(1-B)^2 - 4(\pm \nu^2)} \neq 0, \tag{29}
\]

with \( B > 2 + \nu \) in order to get a decreasing behaviour, and when \( \mu = 0 \) the quantum solution becomes \( A = A^{\frac{B}{2}} (C_1 + C_2 \log A) \).

To solve the equation (26) for the fields \( \phi, \sigma \), we propose the following ansatz

\[
C = G e^{m_1 \phi} e^{m_2 \sigma}, \tag{30}
\]
where $G$ is a constant, and $m_1, m_2$ are two complex parameters. Introducing (30) into (26), one gets

$$\frac{39}{10}m_1^2 + 2m_2^2 - 6m_1m_2 = \pm \nu^2.$$  

(31)

Solving $m_2$ as a function of $m_1$, we get

$$m_2 = \frac{3}{2}m_1 \pm \frac{1}{2}\sqrt{\frac{6}{5}}m_1^2 \pm 2\nu^2.$$  

(32)

So, (30) can be written as

$$C = e^{m_1(\phi + \frac{3}{2}\sigma)} \left( A_1 e^{\frac{1}{2}\sqrt{\frac{6}{5}}m_1^2 \pm 2\nu^2\sigma} + A_2 e^{-\frac{1}{2}\sqrt{\frac{6}{5}}m_1^2 \pm 2\nu^2\sigma} \right).$$  

(33)

A possible way to construct a "Gaussian state" is

$$C = \int_{-\infty}^{+\infty} \left[ A_1(m_1) \sinh \left( \frac{1}{2} \sqrt{\frac{6}{5}}m_1^2 \pm 2\nu^2\sigma \right) + A_2(m_1) \cosh \left( \frac{1}{2} \sqrt{\frac{6}{5}}m_1^2 \pm 2\nu^2\sigma \right) \right] \times$$  

$$\times e^{m_1(\phi + \frac{3}{2}\sigma)} \, dm_1.$$  

(34)

This will be solution of (26) under the ansatz (30).

In this way, a more general solution to (23) than (33) is obtained

$$\Psi(A, \phi, \sigma) = \left[ A^{1-B} Z_\alpha \left( 6\sqrt{-kA^2} \right) \right] \int_{-\infty}^{+\infty} \left[ A_1(m_1) \sinh \left( \frac{1}{2} \sqrt{\frac{6}{5}}m_1^2 \pm 2\nu^2\sigma \right) 

+ A_2(m_1) \cosh \left( \frac{1}{2} \sqrt{\frac{6}{5}}m_1^2 \pm 2\nu^2\sigma \right) \right] e^{m_1(\phi + \frac{3}{2}\sigma)} \, dm_1.$$  

(35)

**IV. CLASSICAL SOLUTIONS A LA WKB**

In the WKB approximation, we propose a solution as $\Psi = \exp(iS(A, \phi, \sigma))$, where the function $S = S_1(A) + S_2(\phi) + S_3(\sigma)$ is known as the superpotential function, being a real function, and fulfilling the usual following conditions

$$\left( \frac{\partial S}{\partial A} \right)^2 \gg \left| \frac{\partial^2 S}{\partial A^2} \right|, \quad \left( \frac{\partial S}{\partial \phi} \right)^2 \gg \left| \frac{\partial^2 S}{\partial \phi^2} \right|, \quad \left( \frac{\partial S}{\partial \sigma} \right)^2 \gg \left| \frac{\partial^2 S}{\partial \sigma^2} \right|,$$

(36)

in this way, the Einstein-Hamilton-Jacobi equation (EHJ) is obtained

$$A^2 \left( \frac{dS_1}{dA} \right)^2 + 144kA^4 + \frac{39}{10} \left( \frac{dS_2}{d\phi} \right)^2 + 2 \left( \frac{dS_3}{d\sigma} \right)^2 - 6 \frac{dS_2 \, dS_3}{d\phi \, d\sigma} = 0.$$  

(37)

This same equation is recovered directly, when we introduce the transformation on the canonical momentas in equation (19) multiplied by $24A^3$.
\[ \Pi_A = \frac{\partial S}{\partial A} = \frac{dS_1}{dA}, \quad \Pi_\phi = \frac{\partial S}{\partial \phi} = \frac{dS_2}{d\phi}, \quad \Pi_\sigma = \frac{\partial S}{\partial \sigma} = \frac{dS_3}{d\sigma}. \quad (38) \]

Employing the separation of variables method, we have

\[ A^2 \left( \frac{dS_1}{dA} \right)^2 + 144kA^4 = -\frac{39}{10} \left( \frac{dS_2}{d\phi} \right)^2 - 2 \left( \frac{dS_3}{d\sigma} \right)^2 + 6 \frac{dS_2}{d\phi} \frac{dS_3}{d\sigma} = \alpha, \quad (39) \]

where \( \alpha \) is a separation constant

\[ A^2 \left( \frac{dS_1}{dA} \right)^2 + 144kA^4 = \alpha, \quad (40) \]

\[ -\frac{39}{10} \left( \frac{dS_2}{d\phi} \right)^2 - 2 \left( \frac{dS_3}{d\sigma} \right)^2 + 6 \frac{dS_2}{d\phi} \frac{dS_3}{d\sigma} = \alpha. \quad (41) \]

Taking into account that the canonical momentum \( \Pi_A \) was defined as \( \Pi_A = \frac{12}{N} A \frac{dA}{dt} \), the equations (38) and (40) give us the following relation

\[ \frac{dA}{Ndt} = \frac{\sqrt{\alpha - 144kA^4}}{12A^2}. \quad (42) \]

Defining \( d\tau = Ndt \) as a physical time, we can rewrite the solution for the scale factor \( A \) with respect to the coordinate \( \tau \)

\[ \tau = \int d\tau = \int \frac{12A^2 dA}{\sqrt{\alpha - 144kA^4}}. \quad (43) \]

The relation between \( A \) and \( \tau \), (43) can be expressed in terms of elliptic integrals [25], in this way by rewriting the integral, we have

\[ \tau = \frac{1}{\sqrt{k}} \int_0^A \frac{z^2 dz}{\sqrt{\sqrt{\frac{\alpha}{144k}} - z^2 \sqrt{\frac{\alpha}{144k} + z^2}}} \]

\[ = \frac{1}{\sqrt{k}} \left( \sqrt{2} \sqrt[4]{\frac{\alpha}{144k}} E(\gamma, r) - \sqrt[4]{\frac{\alpha}{144k}} F(\gamma, r) - A \sqrt{\frac{\alpha}{144k} + A^2} \right), \quad (44) \]

where \( F \) and \( E \) are the elliptic integrals of first and second class, respectively, and the parameters

\[ \gamma = \arccos \left( \frac{A}{\sqrt[4]{\frac{\alpha}{144k}}} \right) \quad \sqrt[4]{\frac{\alpha}{144k}}, \quad (45) \]

\[ r = \frac{1}{\sqrt{2}}. \quad (46) \]

For the fields \( \phi \) and \( \sigma \), we use the equation (41), where we recognize a quadratic equation in the associated momenta. This equation can diagonalized, defining the parameters \( x = \frac{dS_2}{d\phi} \),
and \( y = \frac{dS}{d\sigma} \), where \( x \) and \( y \) will be taken as constants. The solution show us that \( x \) is proportional to \( y \). Solving the quadratic equation (41) we get

\[
y = \frac{3x \pm \sqrt{6x^2 - 2\alpha}}{2}.
\]  

(47)

Now, we have two integration constants \((x, \alpha)\), which could be determined with appropriate initial boundary conditions.

Using \( \frac{d\phi}{dt} \) and \( \frac{d\sigma}{dt} \) as function of \( x = \Pi \phi \), \( y = \Pi \sigma \) (see equations (16), (17) and (38)) and integrating, we obtain:

\[
\sigma - \sigma_0 = \int d\sigma = (-\frac{3}{12}x + \frac{y}{6}) \int \frac{d\tau}{A^3(\tau)},
\]

(48)

\[
\phi - \phi_0 = \int d\phi = (\frac{13}{40}x - \frac{y}{4}) \int \frac{d\tau}{A^3(\tau)}.
\]

(49)

For \( k = 0 \), a flat universe, equation (43) is very simple, its solution is

\[
\tau = \frac{4A^3}{\sqrt{\alpha}}.
\]

(50)

So, the scale factor \( A \) has the following behaviour

\[
A = \left( \frac{\sqrt{\alpha}}{4} \right)^{\frac{1}{3}} \tau^{\frac{4}{3}}.
\]

(51)

Introducing this result into (48, 49) we have

\[
\sigma - \sigma_0 = (-\frac{3}{12}x + \frac{y}{6}) \left( \frac{4}{\sqrt{\alpha}} \right)^{\frac{1}{3}} \int \tau^{-1} d\tau = (-\frac{3}{12}x + \frac{y}{6}) \left( \frac{4}{\sqrt{\alpha}} \right) \ln \tau,
\]

(52)

\[
\phi - \phi_0 = (\frac{13}{40}x - \frac{y}{4}) \left( \frac{4}{\sqrt{\alpha}} \right) \ln \tau.
\]

(53)

In this manner, we can obtain the behaviour of the fields \( \phi \) and \( \sigma \) as functions of the scale factor \( A \), as follows

\[
\sigma - \sigma_0 = \left( -\frac{3}{12}x + \frac{y}{6} \right) \left( \frac{4}{\sqrt{\alpha}} \right) \left( 3 \ln A + \ln \frac{4}{\sqrt{\alpha}} \right),
\]

(54)

\[
\phi - \phi_0 = \left( \frac{13}{40}x - \frac{y}{4} \right) \left( \frac{4}{\sqrt{\alpha}} \right) \left( 3 \ln A + \ln \frac{4}{\sqrt{\alpha}} \right).
\]

(55)

In a similar way, for the cases, \( k = \pm 1 \), we have

1. \( k = -1 \).

\[
\sigma - \sigma_0 = -\left( -\frac{3}{12}x + \frac{y}{6} \right) \frac{6}{\sqrt{\alpha}} \ln \left( \frac{A^2}{\sqrt{\alpha} + \sqrt{\alpha} + A^4} \right),
\]

(56)

\[
\phi - \phi_0 = -\left( \frac{13}{40}x - \frac{y}{4} \right) \frac{6}{\sqrt{\alpha}} \ln \left( \frac{A^2}{\sqrt{\alpha} + \sqrt{\alpha} + A^4} \right).
\]

(57)
2. \( k = 1 \).

\[
\sigma - \sigma_0 = \left( -\frac{3}{12} x + \frac{y}{6} \right) \frac{6}{\sqrt{x^2}} \ln \left( \frac{A^2}{\frac{\sqrt{A}}{12} + \frac{\sqrt{A}}{144} - A^4} \right), \quad (58)
\]

\[
\phi - \phi_0 = \left( \frac{13}{40} x - \frac{y}{4} \right) \frac{6}{\sqrt{x^2}} \ln \left( \frac{A^2}{\frac{\sqrt{A}}{12} + \frac{\sqrt{A}}{144} - A^4} \right). \quad (59)
\]

V. FINAL REMARKS

It is well known that the Wheeler-DeWitt cosmological equation is not an evolution equation and therefore the associated quantum states do not evolve in time. A possible way to connect some parameters of the ‘quantum’ WDW solutions with classical Einstein ones is by phenomenological restrictions imposed on the superpotential functions or final conditions over the wave function as we did. By these means we get a decreasing function in the gravitational part of the wave function of the Universe. Using this method, we find the classical behaviour for the scale factor, scalar fields \( \phi \) and \( \sigma \) in terms of \( A \). So even not knowing a time WDW quantum equation, our physical assumptions allow us to connect the quantum behaviour with the classical one. Under dynamical compactification condition, the moduli scalar \( \sigma \) will be an increasing function, in such away that the radius for the extra dimension vanish, see Eq. (5). On the other hand, our procedure applied to a time dependent object in the context of a low-energy string effective action (7) provides a quantization interesting in its own right and that is expected to provide information on the quantum objects depending on time in string theory. Their classical behaviour was also presented. It is a matter of future work to follow a similar procedure for S-branes models.

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