POINCARÉ’S POLYHEDRON THEOREM FOR COCOMPACT GROUPS IN DIMENSION 4

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ABSTRACT. We prove a version of Poincaré’s polyhedron theorem whose requirements are as local as possible. New techniques such as the use of discrete groupoids of isometries are introduced. The theorem may have a wide range of applications and can be generalized to the case of higher dimension and other geometric structures. It is planned as a first step in a program of constructing compact $C$-surfaces of general type satisfying $c_1^2 = 3c_2$.

To Misha Verbitsky on no particular occasion

1. Introduction

1.1. One of the few powerful tools for constructing compact manifolds (or orbifolds) carrying a prescribed geometry is closely related to Poincaré’s polyhedron theorem (PPT). A compact manifold $X$ in question is the quotient of a model space $M$ by a discrete cocompact group $G$ of automorphisms. One can easily associate to $G$ a polyhedron $P \subset M$ into which $X = M/G$ can be ‘cut and unfolded.’ For instance, at the presence of metric, there is the compact Dirichlet polyhedron $P_c := \{ x \in M | \forall g \in G d(x, c) \leq d(x, gc) \}$ centred at a generic point $c \in M$; it has finitely many codimension 1 faces (contained in hypersurfaces equidistant from two points) and face-pairing isometries between such faces. In general, the settings of PPT include a polyhedron whose face-pairing automorphisms generate $G$ and, a posteriori, the discreteness of $G$ turns out to be a consequence of the tessellation of $M$ by the $G$-copies of the polyhedron. In this way, PPT transforms constructing compact manifolds into verifying tessellation.

Generally speaking, the tessellation of $M$ resides in the mutual position of certain $G$-copies of codimension 1 faces of $P$. In the case of classic geometries [AGr1], the hypersurfaces equidistant from two points are real algebraic and rather simple. They constitute a reasonable choice for codimension 1 faces of $P$. Nevertheless, inferring the tessellation of $M$ from an analysis of the mutual position of some of $G$-copies codimension 1 faces is frequently impossible to perform. In this paper, we follow the strategy taken from [AGr2] and develop a version of PPT whose requirements providing the tessellation are as local as possible, hence, easily verifiable in practice. This allows to avoid the mentioned difficulty.

1.2. The original version of PPT by H. Poincaré is plane. Our version of PPT in [AGr2] is ‘plane-like.’ Surprisingly, Theorems 4.1 and 4.3 in this paper are still ‘plane-like.’ In few vague words, Theorems 4.1 and 4.3 claim that the following requirements provide the tessellation of $M$. A small neighbourhood of a generic point of every codimension 1 or 2 face of $P$ is required to be tessellated by certain ‘expected’ copies of $P$ (Lemmas 3.10 and 3.12 translate such a requirement into truly infinitesimal conditions called ‘interior into exterior condition’ and ‘total angle condition’). Another condition asks the polyhedron to remain locally connected after removing its codimension 3 faces (there are counter-examples if one
drops such a condition).\textsuperscript{1} The last requirement is that the stabilizer in $G$ of every codimension 3 face induces on the face a finite group of isometries.

In the simple case of classic geometries of constant curvature and polyhedra with totally geodesic faces, there existed in the literature a couple of indications that the high-dimensional versions of PPT could be ‘plane-like.’ One is [Ale, p. 184, Theorem 2]. The other can be found in [Thu, p. 127, 11th line from the bottom] :

‘... the condition can be verified by checking that the dihedral angles add up to $360^\circ$ around each edge, and the corners fit together exactly to form a spherical neighborhood of a point in the model space. (Actually, the second condition follows from the first.)’

Yet, we wonder at the absence of requirements concerning the tessellation of a neighbourhood of a point in the interior of codimension $\geq 3$ faces (whose infinitesimal counterpart would be a ‘solid angle condition’).

1.3. In Section 3, we introduce and study a concept of smooth compact polyhedra. On one hand, this concept allows us to proceed by induction on dimension. On the other hand, it suits all possible practical needs.

A crucial feature of the theorems is the extensive use of groupoids caused by the necessity of dealing with nonconnected spaces and polyhedra in the proofs. The lack of sufficient symmetries makes it impossible to apply induction on dimension and to keep simultaneously the symmetries at the level of a group of isometries of the ambient space. Curiously, adapting [AGr2, Section 2] to the case of groupoids required only a few changes.

It does not seem to be difficult to generalize the results of this paper to higher dimension just by following the presented proofs (though this may not be as easy as simply writing this phrase). Our interest lies in dimension 4: this paper is intended as a first step in the program [Ana] of constructing compact C-surfaces of general type satisfying the equality in Bogomolov-Miyaoka-Yau’s inequality $c_2^2 \leq 3c_2$ between Chern numbers. Such surfaces are compact complex hyperbolic manifolds [Yau] and attract considerable interest [Rei]. They are hard to construct. The few known examples include those in [Mos] and the fake projective planes [Mum], [CaS].

1.4. The scope of possible applications of the theorems lies far beyond the originally planned one. As far as compact manifolds are concerned, the ideas involved in the proofs seem to be adaptable to constructing manifolds carrying other geometric structures. For instance, it seems possible to modify them and incorporate the case of compact affine manifolds.

Theorems 4.1 and 4.3 are not free from the defect indicated at the end of [AGr2, Subsection 1.3]. It is still necessary to verify the simplicity of the polyhedron, i.e., to study the mutual position of its codimension 1 faces. Nevertheless, there is some progress in the direction of getting rid of the global requirement of simplicity. Groupoids allow us to work effectively with nonconnected polyhedra. So, we can partially escape the verification that the faces intersect properly by \textit{a priori} cutting the \textit{a posteriori} problematic polyhedron into small pieces and placing them distantly by means of arbitrary isometries.

2. Discrete groupoids of isometries

In this section, we reformulate the corresponding [AGr, Section 2] in a bit more general form as we need to drop the assumption that $M$ or $P$ is connected.

Let $M$ be a locally path-connected metric space with finitely many connected components, denoted by $M_i$, $i \in I$ (in applications, each $M_i$ will be simply-connected). For convenience, we assume that $d(x_1, x_2) = \infty$ for points $x_1, x_2 \in M$ from different components. Denote by $B(x, \varepsilon)$ the open ball of

\textsuperscript{1}We are grateful to the referee who stimulated us to state the conjecture in [AGr2, Subsection 4.3]; unfortunately, our conjecture lacks this condition.
radius $\varepsilon > 0$ centred at $x$ and let $N(X,\varepsilon) := \bigcup_{x \in X} B(x,\varepsilon)$ for $X \subset M$. In this section, we regard a polyhedron in $M$ as being a closed, locally path-connected subspace $P \subset M$ such that

- $P$ has finitely many connected components;
- $P$ is the closure of its nonempty interior: $\overset{\sim}{P} \neq \emptyset$ and $P = \text{Cl} \overset{\sim}{P}$;
- the nonempty boundary of $P$ is decomposed into the union of nonempty path-connected subsets $s \in S$ called faces: $\partial P := P \setminus \overset{\sim}{P} = \bigcup_{s \in S} s$.

The isometry groupoid $\text{Isom} M$ of $M$ is the category whose objects are $M_i$, $i \in I$, and morphisms are all possible isometries between the $M_i$‘s. (By definition, a groupoid is a category whose morphisms are isomorphisms.) When writing $g \in \text{Isom} M$, we mean that $g$ is a morphism whose domain and codomain are therefore prescribed, i.e., $g : M_i \to M_j$ is an isometry for suitable $M_i, M_j$.

A face-pairing of a polyhedron $P$ is an involution $\tau : S \times S \to S$ and a family of isometries $I_s \in \text{Isom} M$ satisfying $I_s = \tau I_s$ and $I_s = I_s^{-1}$ for every face $s \in S$. Since the faces $s, \tau s$ are connected, they determine the corresponding components $M_i, M_j$, hence, we have $I_{s} : M_i \to M_j$. We denote $P_i := P \cap M_i$. Note that $P_i$ is not necessarily connected.

We need a brief elementary introduction to groupoids. Groupoids in this section will always have their reflexivity), we obtain an equivalence relation also denoted by $\sim$. The rule $(g, x) \mapsto gx$ defines an $I$-graded continuous map $\psi : G \ast P \to M$, i.e., $\psi(G \ast P) \subset M_i$. We introduce a relation in $G \ast P$ by putting $(g, x) \sim (h, y)$ exactly when $x \in s$ for some $s \in S$, $I_s x = y$, and $h^{-1} g = I_s$. Taking the closure of this symmetric relation with respect to transitivity (and reflexivity), we obtain an equivalence relation also denoted by $\sim$. Let $J := G \ast P/ \sim$, let $[g, x]$ denote the class of $(g, x)$ in $J$, and let $\pi : G \ast P \to J$, $(g, x) \mapsto [g, x]$, be the quotient map. We equip $J$ with the quotient topology. We have $J = \bigsqcup_i J_i$, where $J_i := G \times_i P/ \sim$. Indeed, $(g, x) \sim (h, y)$ implies that $gx = hy$; hence, $(g, x), (h, y) \in G \times_i P$ for some $i \in I$. The map $\pi$ is clearly $I$-graded continuous.

The groupoid $G$ acts (by homeomorphisms) on $G \ast P$ by means of the composition of (composable) arrows: $h[g, x] := (hg, x)$. If $(g, x) \sim (h, y)$ and $fg$ is defined, then $(fg, x) \sim (fh, y)$. So, we obtain the induced action $h[g, x] := [hg, x]$ of $G$ on $J$. It follows from the definition of quotient topology and from $\pi^{-1}(gp) = g \pi^{-1}p$ that $G$ acts on $J$ by homeomorphisms. We get a commutative diagram of $I$-graded continuous $G$-maps $\psi, \pi$, and $\varphi[g, x] := gx$.

Let

$$[P_i] := \{[1_i, x] \mid x \in P_i\}, \quad [\overset{\sim}{P}_i] := \{[1_i, x] \mid x \in \overset{\sim}{P}_i\}, \quad [P] := \bigcup_i [P_i].$$

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Clearly, $J = \bigcup_{g,i} g[P_i]$ and $g[P_i] \cap h[P_j] \neq \emptyset$ implies $g = h$. In other words, $[P]$ is a fundamental region for the action of $G$ on $J$.

We assume that $\pi^{-1}[1_i, x]$ is finite for every $i \in I$ and $x \in \partial P_i$, hence, for every $x \in P_i$. Take $x \in P_i$. Let $\pi^{-1}[1_i, x] = \{(g_1, x_1), \ldots, (g_n, x_n)\}$ for some $g_k \in G$ and $x_k \in P_{j_k}$ with $(g_k, x_k) \in G \times P$. The polyhedron $g_kP_{j_k}$ are the formal neighbours of $P_i$ at $x$. For $\delta > 0$, define

$$N_{x,\delta} := \{y \in P_{j_k} \mid d(y, x_k) < \delta\} \subset P_{j_k}, \quad N_{x,\delta} := \bigcup_{k=1}^n (g_k, N_{x_k,\delta}) \subset G \times P, \quad W_{x,\delta} := \pi N_{x,\delta} \subset J_i,$$

where $d(\cdot, \cdot)$ stands for the distance function on $M$. Using this notation, we state the

**2.1. Tessellation condition.** A polyhedron $P$ with a given face-pairing satisfies tessellation condition if

1. for all $i \in I$ and $x \in P_i$, there exists some $\delta(x) > 0$ such that $\pi^{-1}W_{x,\delta} = N_{x,\delta}$ and $\varphi W_{x,\delta} = B(x, \delta)$ for any $0 < \delta \leq \delta(x)$;

2. for every $i \in I$, some open metric neighbourhood $N_i$ of $P_i$ in $M_i$ is tessellated; this means that $N(P_i, \varepsilon) \subset N_i$ for some $\varepsilon > 0$ and that there exists a function $f_i : P_i \to \mathbb{R}$ taking positive values such that $\varphi : W_{P_i, f_i} \to N_i$ is bijective, where $W_{P_i, f_i} := \bigcup_{x \in P_i} W_{x, f_i(x)} \subset J_i$.

The proof of the following proposition follows closely that of [AGr, Proposition 2.2].

**2.2. Proposition.** Tessellation condition 2.1 implies that $\varphi$ is a regular covering.

**Proof.** First, using only (the finiteness of $\pi^{-1}[1_i, x]$ and) tessellation condition 2.1 (1), we will show that $\varphi$ is an open map. Indeed, tessellation condition 2.1 (1) and the fact that $G$ acts on $J$ by homeomorphisms imply that $gW_{x,\delta}$ is open in $J$ for every $(g, x) \in G \ast P$ and $0 < \delta \leq \delta(x)$. Moreover, if $(g, x) \in G \ast P$ and $0 < \delta \leq \delta(x)$, then $\varphi(gW_{x,\delta}) = g\varphi W_{x,\delta} = gB(x, \delta) = B(gx, \delta)$ is open in $M$. Hence, it suffices to prove that $B := \{gW_{x,\delta} \mid (g, x) \in G \ast P, \quad 0 < \delta \leq \delta(x)\}$ is a base of the topology on $J$. Let $[g, x] = [1_i, x] \in J_i$ and let $U \subset J_i$ be an open neighbourhood of $[g, x]$, where $g : M_i \to M_j$. The open set $\pi^{-1}U \subset G \times P$ is a union $\pi^{-1}U = \bigcup_{h} (h, U_h)$, where each $U_h$ is open in some $P_i$, $h : M_i \to M_j$. Let $\pi^{-1}[1_i, x] := \{(g_1, x_1), \ldots, (g_n, x_n)\}$. We have $\pi^{-1}[g, x] = \{(gg_1, x_1), \ldots, (gg_n, x_n)\}$. It follows that $x_k \in U_h$ for $h = gg_k$, $k = 1, \ldots, n$. Take $0 < \delta \leq \delta(x)$ such that $N_{x_k,\delta} \subset U_h$ for $h = gg_k$ and all $k = 1, \ldots, n$. Clearly, $gN_{x,\delta} = \bigcup_k (gg_k, N_{x_k,\delta}) \subset \pi^{-1}U$. So, $[g, x] \in gW_{x,\delta} = g\pi N_{x,\delta} = \pi(gN_{x,\delta}) \subset \pi\pi^{-1}U = U$ which implies that $B$ is a base of the topology on $J$ and that $\varphi$ is open.

Using the fact that $B$ is a basis and that $\pi^{-1}[g, x]$ is finite for every $(g, x) \in G \ast P$, it is easy to see that $J$ is Hausdorff.

If $J_i = \emptyset$ for some $i \in I$, then $\varphi|_{J_i} : J_i \to M_i$ is a regular covering of degree zero. So, in order to prove that $\varphi$ is a regular covering, it suffices to show that $\varphi_i := \varphi|_{J_i} : J_i \to M_i$ is a surjective regular covering in the case of $J_i \neq \emptyset$.

We fix such an $i \in I$. Since $\varphi$ is open, $\varphi|_{J_i}$ is open in $M_i$. Let $x \in \text{Cl}(\varphi|_{J_i})$. Then $B(x, \varepsilon) \cap gP_{j} \neq \emptyset$ for some $g : M_j \to M_i$ and $j \in I$. It follows that $x \in N(gP_j, \varepsilon) \subset gN_j = \varphi(gW_{P_j, f_j}) \subset \varphi|_{J_i}$. Hence, $\varphi|_{J_i}$ is closed in $M_i$. Since $M_i$ is connected, $\varphi_i$ is surjective.

Take $x \in M_i$. Define

$$G_x := \{g \in G \mid U_x \cap gP_{j} \neq \emptyset \text{ for } g : M_j \to M_i, \quad j \in I\},$$

where $U_x \subset B(x, \frac{1}{2}\varepsilon)$ is a path-connected open neighbourhood of $x$. For every $G_x \ni g : M_j \to M_i$, let

$$W_g := (\varphi_i^{-1}U_x) \cap gW_{P_j, f_j} \subset J_i.$$
Being a continuous open bijection, \( \varphi_i : gW_{P_j, f_j} \to gN_j \) is a homeomorphism. Since \( U_x \cap gP_j \neq \emptyset \) implies that \( U_x \subset B(x, \frac{1}{2} \varepsilon) \subset N(gP_j, \varepsilon) \subset gN_j \), we conclude that \( \varphi_i : W_g \to U_x \) is a homeomorphism. Moreover,

\[
\varphi_i^{-1}U_x = \bigcup_{g \in G_x} W_g.
\]

Indeed, if \( \varphi_i[g, y] \in U_x \) with \( y \in P_j \) and \( g : M_j \to M_i \), then \( gy \in U_x \), \( g \in G_x \), and \( [g, y] \in gW_{P_j, f_j} \), i.e., \([g, y] \in W_g\).

It remains to show that the distinct \( W_g \)'s are disjoint. Suppose that \( W_{g_1} \cap W_{g_2} \neq \emptyset \) for some \( g_1, g_2 \in G_x \). The projection \( W_{g_1} \times W_{g_2} \to W_{g_1} \) induces a homeomorphism between

\[
X := \{(x_1, x_2) \in W_{g_1} \times W_{g_2} \mid \varphi_i x_1 = \varphi_i x_2\}
\]

and \( W_{g_1} \). The diagonal

\[
\Delta_{W_{g_1} \cap W_{g_2}} = \Delta_J \cap (W_{g_1} \times W_{g_2}) \subset X
\]

is closed in \( X \) as \( J \) is Hausdorff. Therefore, the image \( W_{g_1} \cap W_{g_2} \) of \( \Delta_{W_{g_1} \cap W_{g_2}} \) is closed in \( W_{g_1} \). Since \( W_{g_1} \) is connected, we obtain \( W_{g_1} = W_{g_2} \).  

3. Compact smooth polyhedra with face-pairing

In this section, we introduce compact smooth polyhedra equipped with face-pairing and observe some local properties of such polyhedra.

3.1. Compact smooth polyhedra. Let \( M \) be a (not necessarily connected) smooth manifold. A compact smooth polyhedron \( P \) of dimension \( d = 0 \) is simply a finite subset \( P \subset M \). We define a compact smooth polyhedron \( P \subset M \) and its faces by induction on the dimension \( d > 0 \) of \( P \):

- \( P \) is a finite disjoint union \( P = \bigsqcup F_i \) of its codimension 0 faces \( F_i \). Every \( F_i \) is a connected compact submanifold \( S_i \subset M \) of dimension \( d \), \( F_i \subset S_i \). Every \( F_i \) coincides with the closure of its nonempty interior \( \emptyset \neq \hat{F}_i \subset S_i \), i.e., \( \hat{F}_i \) is open in \( S_i \) and \( F_i = \text{Cl} \hat{F}_i \). We call \( S_i \) a submanifold of the face \( F_i \).
- The boundary \( \partial F_i := F_i \setminus \hat{F}_i \) of \( F_i \) is decomposed into a (possibly empty) finite union \( \partial F_i = \bigsqcup_{s_{ij}} \text{of connected compact smooth polyhedra of dimension } d-1 \), called codimension 1 faces of \( P \). A codimension \( k \) face \( s_{ij} \) of \( s_{ij} \) is called a codimension \( k + 1 \) face of \( P \). A submanifold of the face \( f \) of \( P \) is the same as a submanifold of the face \( f \) of \( s_{ij} \). For any \( j_1 \neq j_2 \), we require that \( s_{ij_1} \cap s_{ij_2} = \partial s_{ij_1} \cap \partial s_{ij_2} \) and that this intersection is a (possibly empty) union of entire faces.

In a Riemannian manifold \( M \), denote by \( S(x, \delta) \) the sphere of radius \( \delta \) centred at \( x \in M \).

3.2. Lemma. Let \( M \) be a complete Riemannian manifold, let \( P \subset M \) be a compact smooth polyhedron, and let \( x \in P \). Then there exists \( \delta(x) > 0 \) such that \( S(x, \delta) \cap P \) is a compact smooth polyhedron for any \( 0 < \delta \leq \delta(x) \). We can take \( \delta(x) \) such that \( S(x, \delta) \) intersects only the faces of \( P \) that properly contain \( x \), and thus that the codimension \( k \) faces of \( S(x, \delta) \cap P \) are the connected components of \( S(x, \delta) \cap f \), where \( f \) is a codimension \( k \) face of \( P \) containing \( x \), and such that the interior of \( S(x, \delta) \cap P \) comes from the interior of the corresponding face of \( P \).

Proof. Given a point in a submanifold \( x \in S \subset M \), the sphere \( S(x, \delta) \) intersects \( S \) transversally for all sufficiently small \( \delta \) and, moreover, \( S(x, \delta) \cap S \) is a smooth sphere. We apply this fact to submanifolds of all faces of \( P \) and choose \( \delta(x) > 0 \) so small that, for any \( 0 < \delta \leq \delta(x) \), the sphere \( S(x, \delta) \) intersects only the faces of \( P \) that properly contain \( x \). In particular, we can assume that \( P \) is connected. By induction on the dimension of \( P \), we assume that \( f_i := S(x, \delta) \cap s_i \) is a compact smooth polyhedron for any codimension 1 face \( s_i \) of \( P \) and \( 0 < \delta \leq \delta(x) \). Denote by \( c_{ij} \) the faces of codimension 0 of \( f_i \), \( f_i = \bigsqcup_{i} c_{ij} \).
By induction, \( \hat{c}_{ij} := c_{ij} \cap \tilde{s}_i \). Let \( S \) and \( S_i \) denote respectively submanifolds of \( P \) and \( s_i \). Then \( S' := S(x, \delta) \cap S_i \) is a submanifold of \( c_{ij} \) and \( S'' := S(x, \delta) \cap S \) is a sphere containing the compact set \( P := S(x, \delta) \cap P \).

Take \( p \in \hat{c}_{ij} \). Note that, in an arbitrarily small open neighbourhood \( U \) of \( p \), there exists a smooth simple path \( c \subset U \cap F \) beginning at \( p \) such that \( c \setminus \{ p \} \subset S(x, \delta) \cap \tilde{P} \). Indeed, we can assume that \( U \) does not intersect the other proper faces of \( P \) and, by routine arguments of transversality, that \( U \) is a smooth chart of \( M \) at \( p \) such that the submanifolds \( S_i, S, \text{ and } S(x, \delta) \) are linear subspaces inside \( U \), with \( p \) as the origin. The linear subspace \( U \cap S_i \) divides \( U \cap S \) into two open half-spaces. One of them \( H \) contains a point \( b \in \tilde{P} \) because \( U \cap S_i \) is contained in the closure of \( U \cap \tilde{P} \). It suffices to show that \( S(x, \delta) \cap H \subset \tilde{P} \). Let \( q \in S(x, \delta) \cap H \setminus \tilde{P} \). As \( H \) is convex, the straight segment \([b, q]\) lies in \( H \), hence, does not intersect \( U \cap S_i \). The first point of \([b, q]\) that does not belong to \( \tilde{P} \) obviously belongs to the closure of \( \tilde{P} \). Consequently, it belongs to a proper face of \( P \). By the choice of \( U \), such a point lies in \( s_i \) and, hence, in \( U \cap S_i \). A contradiction.

By induction, for any \( p \in F \), there exists an arbitrarily short path inside \( F \) that begins at \( p \) and ends at a point in \( S(x, \delta) \cap \tilde{P} \).

We will show that every path-connected component \( C \) of \( F \) is closed in \( S(x, \delta) \). Let \( q \in \text{Cl } C \setminus C \). Then \( q \in F \setminus \tilde{P} \), hence, \( q \) belongs to some \( c_{ij} \). We choose an open path-connected neighbourhood \( V \) of \( q \) in \( S' \) that intersects only those \( c_{ij} \)'s that contain \( q \). Since \( q \in \text{Cl } C \), there is a point \( p \in V \cap C \). By the above, there exist a path inside \( V \cap F \) that begins at \( p \) and ends at a point in \( S(x, \delta) \cap \tilde{P} \). So, we can assume that \( p \in V \cap C \cap \tilde{P} \). There is a simple smooth path \( c \subset V \) from \( p \) to \( q \). The first point \( b \) of \( c \) that does not belong to \( \tilde{P} \) belongs to the closure of \( \tilde{P} \). Hence, \( b \in V \cap C \cap c_{ij} \) for some \( c_{ij} \). By the choice of \( V \), we conclude that \( q \in c_{ij} \). Since \( c_{ij} \) is path-connected, we obtain \( q \in C \).

We can assume that \( C \not\subset \tilde{P} \) for any path-connected component \( C \) of \( F \). Otherwise, being open and closed in the sphere \( S' \), the component \( C \) has to coincide with \( S' \) and we are done. Since every \( c_{ij} \) is path-connected, any path-connected component \( C \) of \( F \) contains some \( c_{ij} \). Therefore, we have finitely many components in \( F \).

Moreover, \( C = \text{Cl } \tilde{C} \), where \( \tilde{C} := C \cap \tilde{P} \) is open in \( S' \). Indeed, suppose that \( C \cap c_{ij} \neq \emptyset \). Then \( \hat{c}_{ij} \subset c_{ij} \subset C \). By the above, we can find a simple smooth path \( c \subset F \) that begins at an arbitrary point \( p \in \hat{c}_{ij} \) and such that \( c \setminus \{ p \} \subset \tilde{C} \). So, \( C \cap c_{ij} \neq \emptyset \) implies \( \hat{c}_{ij} \subset \text{Cl } \tilde{C} \). Since \( \hat{c}_{ij} \) is dense in \( c_{ij} \), we see that \( C \cap c_{ij} \neq \emptyset \) implies \( c_{ij} \subset \text{Cl } \tilde{C} \).

3.3. Tessellation at a point. Let \( M \) be a \( d \)-dimensional smooth manifold.

Let \( P_1, P_2 \subset M \) be \( d \)-dimensional compact smooth polyhedra sharing a common codimension \( 1 \) face \( s \subset P_1 \cap P_2 \), let \( p \in s \) be a point in the interior of \( s \), and let \( p \in B \subset M \) be a neighbourhood of \( p \). We say that \( P_1, P_2 \) tessellate \( B \) if \( B = B_1 \cup B_2 \) and \( B_1 \cap B_2 = B \cap s \), where \( B_i := B \cap P_i \).

Let \( P_i \subset M, i = 1, \ldots, n \), be \( d \)-dimensional compact smooth polyhedra, let \( e \subset \bigcap_i P_i \) be a common codimension \( 2 \) face of the \( P_i \)'s, let \( p \in e \) be a point in the interior of \( e \), and let \( p \in B \subset M \) be a neighbourhood of \( p \). We say that the polyhedra \( P_i \)'s tessellate \( B \) if, for every \( i \) (the indices are modulo \( n \)), there is a common codimension \( 1 \) face \( s_i \) of \( P_{i-1} \) and \( P_i \) containing \( e \) such that \( B = \bigcup_i B_i \), \( B_{i-1} \cap B_i = B \cap s_i \), and \( B_i \cap B_j = B \cap e \) for all \( i, j \) with \( |i - j| > 1 \), where \( B_i := B \cap P_i \).

3.4. Remark. Let \( M \) be a \( d \)-dimensional complete Riemannian manifold, let \( P_i \subset M \) be \( d \)-dimensional compact smooth polyhedra sharing a common codimension \( 1 \) or \( 2 \) face \( f \subset \bigcap_i P_i \), and let \( p \in f \) be a point in the interior of \( f \). Suppose that the \( P_i \)'s tessellate a neighbourhood of \( p \). Let \( S(x, \delta) \) be a sphere related to every polyhedra \( P_i \) as in Lemma 3.2 and let \( p \in S(x, \delta) \). Then the polyhedra \( S(x, \delta) \cap P_i \) tessellate a neighbourhood of \( p \) in \( S(x, \delta) \).
3.5. Smooth polyhedra with face-pairing. Let $M$ be an oriented Riemannian manifold with $d$-dimensional components and let $P \subset M$ be a $d$-dimensional compact smooth polyhedron. Suppose that we are given a face-pairing, i.e., an involution $\pi : S \to S$ on the set $S$ of all codimension 1 faces of $P$ and isometries $I_s \in \text{Isom } M$ satisfying $I_s s = \pi$ and $I_s^{-1} = I_{\pi}^{-1}$ for all $s \in S$. If every codimension 2 face $e$ of $P$ belongs to exactly two codimension 1 faces $s_1, s_2$ of $P$ (in symbols: $s_1 \circ e \circ s_2$) and each $I_s$ maps any face of $s$ onto a face of $\pi s$, we say that $P$ is equipped with face-pairing.

3.6. Geometric cycles. Let $P \subset M$ be a compact smooth polyhedron equipped with face-pairing. Start with $s_0 \circ e \circ s_1$. Applying $I_{s_1}$ to $s_1$ and $e$, we obtain $s_1 \circ I_{s_1} e \circ s_2$. Applying $I_{s_2}$ to $s_2$ and $I_{s_1} e$, we obtain $s_2 \circ I_{s_2} I_{s_1} e \circ s_3$, and so on. Since the number of faces is finite, we eventually arrive back at $s_0 \circ e \circ s_1$.

A cyclic sequence

$$s_n = s_0 \circ e \circ s_1 \circ s_2 \circ s_3 \circ \ldots \circ s_{n-1} \circ s_n,$$

where each term is obtained from the previous one by the above rule, is called a cycle of codimension 2 faces. The number $n$ is the length of the cycle and the isometry $I := I_{s_n} \cdots I_{s_1}$ will be referred to as the cycle isometry. A cycle can be read backwards, i.e., in opposite orientation, which inverts its isometry. If the cycle isometry is the identity and if the cycle is the shortest one with this property, then the cycle is said to be geometric. Clearly, every cycle is a multiple of a shortest, combinatorial one. Note that, in a geometric cycle, a codimension 2 face $e$ may occur several times in the form $s \circ e \circ s'$, where $s, s'$ are codimension 1 faces containing $e$ (this does not happen in a combinatorial cycle).

3.7. Formal neighbours in codimension $\leq 2$. Let $P$ be a compact smooth polyhedron with face-pairing.

It is easy to describe all formal neighbours (see Section 2 for the definition) of $P$ at a point $x \in s$ in the interior of a codimension 1 face $s$. Since $s$ is a unique codimension 1 face containing $x$ and $\pi s$ is a unique codimension 1 face containing $I_s x$, we have $\pi^{-1}[1, x] = \{(1, x), (I_s, I_s x)\}$. Hence, the only formal neighbours of $P$ at $x$ are $P, I_s P$.

Let $x \in e$ be a point in the interior of a codimension 2 face $e$ and suppose that $e$ belongs to a geometric cycle

$$s_n = s_0 \circ I_0 e \circ s_1 \circ I_1 e \circ s_2 \circ I_2 e \circ s_3 \circ \ldots \circ I_{n-1} e \circ s_n,$$

where $I_j := I_{s_j} \cdots I_{s_1}$ for $j = 0, 1, \ldots, n$ (we consider $j$ modulo $n$). Then $\pi^{-1}[1, x] = \{(I_j^{-1}, I_j x) \mid j = 0, 1, \ldots, n-1\}$ and the $I_j^{-1} P$ are all the formal neighbours of $P$ at $x$. Indeed, suppose that $(I_j^{-1}, I_j x) \sim (h, y)$. We can assume that $I_j x \in s'$, $I_s I_j x = y$, and $h^{-1} I_j^{-1} = I_s$ for some $s' \in S$. In particular, the codimension 1 face $s'$ intersects the interior of the codimension 2 face $I_j e$. By 3.1, we have $I_j e \circ s'$. By 3.5, either $s' = \pi s_j$ or $s' = s_{j+1}$. Therefore, either $(h, y) = (I_{j-1}^{-1}, I_{j-1} x)$ or $(h, y) = (I_{j+1}^{-1}, I_{j+1} x)$. It remains to observe that the $I_j, j = 0, 1, \ldots, n-1$, are all distinct because we could otherwise take a shorter cycle whose isometry would be the identity.

3.9. Interior into exterior condition. Let $P$ be a compact smooth polyhedron equipped with face-pairing. Assuming that $M$ is oriented, we orient every codimension 1 face $s$ in such a way that the unit normal vector $n_s$ to $s$ points toward the interior of $P$. We say that the face-pairing isometries of $P$ send interior into exterior if $I_s n_s = -n_{I_s}$ for every codimension 1 face $s \in S$. Since every codimension 1 face is connected, it suffices to verify this condition at a single point of every codimension 1 face.

3.10. Lemma. Let $P$ be a compact smooth polyhedron with face-pairing. Then the face-pairing isometries of $P$ send interior into exterior iff, for every interior point $x \in s$ in any codimension 1 face $s$, the polyhedra $P, I_s P$ tessellate a neighbourhood of $x$. 

**Proof.** The proof follows [AGr, First step of the proof of Theorem 3.5]. Let \( x \in s \) be an interior point in a codimension 1 face of \( P \). We choose \( \delta > 0 \) with the following properties: \( B(x, \delta) \) does not intersect the faces of \( s \), \( B(I_s, x, \delta) \) does not intersect the faces of \( \overline{s} \), \( B(x, \delta) \cap \partial P = B(x, \delta) \cap s \), and \( B(I_s, x, \delta) \cap \partial P = B(I_s, x, \delta) \cap \overline{s} \). Let \( B := B(x, \delta), B_1 := B \cap P \), and \( B_2 := B \cap I_s P = I_s(B(I_s, x, \delta) \cap P) \). Note that \( B \cap s \subset B_1 \cup B_2 \).

Assume that the face-pairing isometries send interior into exterior. Then \( B_1 \neq B_2 \). Pick a point \( q_0 \in B_1 \setminus B_2 \) such that \( q_0 \notin s \). By the choice of \( \delta \), a smooth oriented curve \( \gamma \subset B \) connecting \( q_0 \) and \( q \in B \setminus s \) can intersect \( \partial P \) and \( \partial I_s P \) only along \( (s \setminus \partial s) \cap B \). We can take such intersections as being transversal. Due to the interior into exterior condition, when intersecting \( s \), the curve \( \gamma \) leaves \( B_1 \) and enters \( B_2 \) or vice-versa. Hence, \( q \) belongs to exactly one of \( B_1 \) and \( B_2 \). This implies that \( B = B_1 \cup B_2 \) and \( B_1 \cap B_2 = B \cap s \).

The converse is immediate \( \blacksquare \)

### 3.11. Total angle condition.
Let \( P \) be a compact smooth polyhedron whose face-pairing isometries send interior into exterior. Pick a point \( x \in e \) in a codimension 2 face \( e \) and denote by \( N_x e := (T_x e)^\perp \) the plane normal to \( e \) at \( x \). By 3.5, we have \( \overline{s}_0 \circ e \circ s_1 \) for suitable codimension 1 faces \( \overline{s}_0, s_1 \). Denote by \( n_0, n_1 \) the unit normal vectors to \( \overline{s}_0, s_1 \) at \( x \) that point towards the interior of \( P \). Let \( t_0 \in T_x \overline{s}_0 \cap N_x e \) and \( t_1 \in T_x s_1 \cap N_x e \) stand for the unit vectors that point respectively towards the interiors of \( \overline{s}_0 \) and \( s_1 \). The basis \( t_0, n_0 \) orients \( N_x e \). The **interior angle** \( \alpha \) at \( s_0 \) and \( s_1 \) is the angle \( \angle t_0, n_0, t_1 \). Such an angle takes values in \([0, 2\pi]\). Note that interchanging the faces \( \overline{s}_0, s_1 \) in the definition does not alter \( \alpha \).

Suppose that \( e \) belongs to a geometric cycle (3.8). Denote by \( \alpha_j \) the interior angle between \( \overline{s}_j \) and \( s_{j+1} \) at \( I_j x \). The sum \( \sum_{j=0}^{n-1} \alpha_j \) is the total interior angle of the cycle at \( x \). It is easy to see that altering the orientation of the cycle keeps the same values of the \( \alpha_j \)’s. Using the facts that the face-pairing isometries send interior into exterior and that the cycle (3.8) is geometric, as in [AGr, Subsection 3.3], we conclude that \( \alpha := \sum_{j=0}^{n-1} \alpha_j \equiv 0 \mod 2\pi \). The **total angle condition** consists in the requirement that \( \alpha = 2\pi \). Since any codimension 2 face is connected, the total angle condition for a single point \( x \) implies the same condition for all interior points of the codimension 2 faces involved in the geometric cycle.

### 3.12. Lemma.
Let \( P \) be a compact smooth polyhedron whose face-pairing isometries send interior into exterior, let \( e \) be a codimension 2 face participating in a geometric cycle (3.8), and let \( x \in e \) be an interior point. Then the total angle of the cycle at \( x \) equals \( 2\pi \) iff the polyhedra \( I_j^{-1} P \) tessellate a neighbourhood of \( x \).

**Proof.** The proof is similar to that of Lemma 3.10 (see also [AGr, First step of the proof of Theorem 3.5]). We choose \( \delta > 0 \) with the following properties: \( B(I_j x, \delta) \) does not intersect the faces of \( I_j e \), \( B(I_j x, \delta) \) does not intersect any face of \( \overline{s}_j \) or \( s_{j+1} \) except \( I_j e \), and \( B(I_j x, \delta) \cap \partial P = (B(I_j x, \delta) \cap \overline{s}_j) \cup (B(I_j x, \delta) \cap s_{j+1}) \) for all \( j = 0, 1, \ldots, n-1 \). Let \( B := B(x, \delta) \) and \( B_j := B \cap I_j^{-1} P = I_j^{-1}(B(I_j x, \delta) \cap P) \). Note that \( B \cap I_j^{-1} s_{j+1} \subset B_j \cap B_{j+1} \) and that \( B \cap e \subset B_i \cap B_j \) for all \( i, j \). Let \( T_j \subset N_x e \) stand for the closed sector containing the interior angle of \( I_j^{-1} P \) at \( x \) and let \( F := B \cap \bigcup I_j^{-1} s_{j+1} \).

Assume that the total angle of the cycle at \( x \) equals \( 2\pi \). Since the face-pairing isometries send interior into exterior, we have \( \bigcup_j T_j = N_x e \) and \( \overline{T}_i \cap T_j = \emptyset \) if \( i \neq j \mod n \). Hence, \( B_i \neq B_j \) for \( i \neq j \mod n \). Pick a point \( q_0 \) that lies in exactly one of the \( B_j \setminus F \). By the choice of \( \delta \), a smooth oriented curve \( \gamma \subset B \) connecting \( q_0 \) and \( q \in B \setminus F \) may intersect \( \bigcup_j \partial I_j^{-1} P \) only along \( F \). We can assume that \( \gamma \) does not intersect \( e \) and is transversal to \( F \). Due to the interior into exterior condition, when intersecting \( I_j^{-1} s_{j+1} \), the curve \( \gamma \) leaves \( B_j \) and enters \( B_{j+1} \) or vice-versa. Hence, \( q \) belongs to exactly one of the \( B_j \)’s. It follows that \( B = \bigcup_j B_j \) and, by the description of formal neighbours in 3.7,
that \( B_j \cap B_{j+1} = B \cap I_j^{-1}s_{j+1} \) and \( B_i \cap B_j = B \cap e \) for all \( i, j \) with \(|i - j| > 1\).

Conversely, assume that the polyhedra \( I_j^{-1}P \) tessellate a neighbourhood of \( x \). We can assume (diminishing \( \delta \), if needed) that they tessellate \( B \). Then \( \bigcup_j T_j = N_x \) because \( B = \bigcup_j B_j \). Finally, we have \( \partial T_i \cap \partial T_j = \emptyset \) if \( i \neq j \mod n \) since \( B_j \cap B_{j+1} = B \cap I_j^{-1}s_{j+1} \) and \( B_i \cap B_j = B \cap e \) for all \( i, j \) with \(|i - j| > 1\). In other words, the total angle of the cycle at \( x \) equals \( 2\pi \).

As a corollary to the proofs of Lemmas 3.10 and 3.12, the tessellation by formal neighbours at points in the interior of codimension 1 or 2 faces implies, for such points, a condition that is stronger than tessellation condition 2.1 (1).

### 3.13. Corollary

Let \( P \) be a compact smooth polyhedron such that every codimension 2 face participates in a geometric cycle. Assume that, for every point \( x \) in the interior of a codimension 1 or 2 face, the formal neighbours of \( P \) at \( x \) tessellate a neighbourhood of \( x \). Then, for every such \( x \), there exists \( \delta(x) > 0 \) such that tessellation condition 2.1 (1) holds and \( \varphi : W_{x,\delta} \to B(x, \delta) \) is a bijection for all \( 0 < \delta \leq \delta(x) \).

**Proof.** Let \( x \) be in the interior of a codimension 1 face \( s \). We take \( \delta(x) > 0 \) such that the open ball \( B(x, \delta(x)) \) is tessellated by the formal neighbours \( P, I_s P \) of \( P \) at \( x \) (see 3.7). Let \( 0 < \delta \leq \delta(x) \). As in the proof of Lemma 3.10, we can assume that \( B(x, \delta) \) does not intersect the faces of \( s \), \( B(I_s x, \delta) \) does not intersect the faces of \( s \), \( B(x, \delta) \cap \partial P = B(I_s x, \delta) \cap s \), and \( B(I_s x, \delta) \cap \partial P = B(I_s x, \delta) \cap s \). Let \( B := B(x, \delta) \), \( B_1 := B \cap P \), and \( B_2 := B \cap I_s P \). It follows from \( B = B_1 \cup B_2 \) and \( B_1 \cap B_2 = B \cap s \) that \( \varphi : W_{x,\delta} \to B(x, \delta) \) is a bijection. The choice of \( \delta(x) \) and the description of formal neighbours given in 3.3 imply that \( \pi^{-1}W_{x,\delta} = N_{x,\delta} \).

Let \( x \) be in the interior of a codimension 2 face \( e \) participating in a geometric cycle (3.8). We choose \( \delta(x) > 0 \) such that the open ball \( B(x, \delta(x)) \) is tessellated by the formal neighbours \( I_j^{-1}P \) of \( P \) at \( x \) (see 3.7). Let \( 0 < \delta \leq \delta(x) \). We assume that \( B(x, \delta) \) satisfies the properties in the beginning of the proof of Lemma 3.12: \( B(I_s x, \delta) \) does not intersect the faces of \( I_s e \), \( B(I_s x, \delta) \cap \partial P = B(I_s x, \delta) \cap s \), and \( B(I_s x, \delta) \cap \partial P = B(I_s x, \delta) \cap s \). Let \( B := B(x, \delta) \) and \( B_j := B \cap I_j^{-1}P \). Since \( B = \bigcup_j B_j \) and \( B_j \cap B_{j+1} = B \cap I_j^{-1}s_{j+1} \), and \( B_i \cap B_j = B \cap e \) for all \( i, j \) with \(|i - j| > 1\), we conclude that \( \varphi : W_{x,\delta} \to B(x, \delta) \) is a bijection. The choice of \( \delta(x) \) and the description of formal neighbours given in 3.3 imply the rest.

The above strengthened tessellation condition 2.1 (1) implies tessellation condition 2.1 (2): for every \( x \in P \) and \( 0 < \delta \leq \delta(x) \), where \( \delta(x) \) is the minimal distance from \( x \) to \( \partial P \).

### 3.14. Lemma

Let \( P \) be a compact smooth polyhedron with face-pairing in a complete Riemannian manifold. Assume that \( P \) satisfies tessellation condition 2.1 (1) and that \( \varphi : W_{x,\delta} \to B(x, \delta) \) is a bijection for every \( x \in P \) and \( 0 < \delta \leq \delta(x) \). Then \( P \) satisfies tessellation condition 2.1.

**Proof.** As in Section 2, we put \([P_i] = \{[1_i, x] \mid x \in P_i\} \). Note that \( \varphi : W_{x,\delta} \to B(x, \delta) \) is a homeomorphism for every \( x \in P_i \) and \( 0 < \delta \leq \delta(x) \) since \( \varphi \) is an open map (see the very beginning of the proof of Proposition 2.2). In view of the commutative diagram in Section 2, the map \( \varphi : [P_i] \to P_i \) is injective. Due to the compactness of \( P_i \), there exists a chain of open neighbourhoods \( N_n \) of \( [P_i] \) whose closures are compact, \( C \subset N_n \subset N_n \) for all \( n \), and \( \bigcap_n N_n = [P_i] \).

Suppose that \( \varphi|N_n \) is not injective for every \( n \in \mathbb{N} \). Hence, there are sequences \( \{p_n\}, \{q_n\} \) such that \( p_n, q_n \in N_n \) and \( p_n \neq q_n \) and \( \varphi p_n \neq \varphi q_n \) for every \( n \in \mathbb{N} \). We can assume that the sequences converge. Clearly, their limits belong to \([P_i] \). As the map \( \varphi : [P_i] \to P_i \) is injective, the limits coincide. This contradicts the fact that \( \varphi \) is injective in a neighbourhood \( W_{p,\delta} \) of the limit \([1_i, p]\). Therefore, \( \varphi : N_n \to \varphi N_n \) is a homeomorphism for some \( n \in \mathbb{N} \) which implies tessellation condition 2.1 (2) by the compactness of \( P_i \).

Analyzing the proofs of Theorems 4.1 and 4.3, the interested reader can construct a counter-example if one drops the condition of connectedness of the polyhedron near codimension 3 given in the following
3.15. Definition. A compact smooth polyhedron $P \subseteq M$ is said to be connected near codimension 3 if, for every $v \in F$, there exists $\delta(v) > 0$ such that $B(v, \delta) \cap P \setminus F$ is connected for all $0 < \delta < \delta(v)$, where $F$ for the union of all codimension 3 faces of $P$.

3.16. Lemma. Let $P \subseteq M$ be a connected near codimension 3 compact smooth polyhedron in a Riemannian manifold $M$ of dimension $\leq 4$ and let $v \in F$ be a point in the union $F$ of all codimension 3 faces of $P$. Then there exists $\delta_0 > 0$ such that $S(v, \delta) \cap P \setminus F$ is connected for all $0 < \delta < \delta_0$. In particular, $S(v, \delta) \cap P$ is connected for such $\delta$.

Proof. For sufficiently small $\delta$, the sphere $S(v, \delta)$ intersects only the faces of $P$ that properly contain $v$ and is transversal to the submanifolds of these faces. It follows that $B(v, \delta_0) \cap P \setminus F$ is a (locally) trivial bundle with the fibres $S(v, \delta) \cap P \setminus F$ in the sense of the identification $B(v, \delta_0) \cap P \setminus \{v\} \simeq (S(v, \delta) \cap P) \times (0, \delta_0)$. The converse is obvious.

3.17. Lemma. Let $P \subseteq M$ be a connected near codimension 3 compact smooth polyhedron in a complete Riemannian 4-manifold $M$ and let $v \in F$ be a point in the union $F$ of all codimension 3 faces of $P$. Then there exists $\delta_0 > 0$ such that the compact smooth polyhedron $S(v, \delta) \cap P$ in the 3-manifold $S(v, \delta)$ is connected near codimension 3 for any $0 < \delta < \delta_0$.

Proof. As in the proof of Lemma 3.16, we have a trivial bundle $B(v, \delta_0) \cap P \to (0, \delta_0)$ with the fibres $S(v, \delta) \cap P$. Suppose that $S(v, \delta_0) \cap P$ is not connected near codimension 3 for some $0 < \delta < \delta_0$. Then there exist a codimension 3 face $f$ of $P$, a point $p \in S(v, \delta_0) \cap f$, an open neighbourhood $U \subset S(v, \delta_0)$ of $p$ in $S(v, \delta)$, and path-connected components $C_i$ of $U \cap P \setminus \{p\}$ such that $p \in \text{Cl} C_i$ for $i = 1, 2$. In terms of a suitable identification $B(v, \delta_0) \cap P \setminus \{v\} \simeq (S(v, \delta) \cap P) \times (0, \delta_0)$ such that $\{p\} \times (0, \delta_0) \subset F$, we obtain the path-connected components $\hat{C}_i := C_i \times (0, \delta_0)$ of $U \times (0, \delta_0) \cap P \setminus \{p\} \times (0, \delta_0)$ such that $F \ni \{p\} \times (0, \delta_0) \subset \text{Cl} \hat{C}_i$ for $i = 1, 2$. Taking an open ball $B(p, \varepsilon) \subset U \times (0, \delta_0)$, $\varepsilon > 0$, we arrive at a contradiction with connectedness of $P$ near codimension 3.

4. Main theorems

4.1. Theorem. Let $M$ be an oriented complete Riemannian manifold with 3-dimensional components and let $P \subseteq M$ be a compact smooth polyhedron connected near codimension 3. Suppose that the face-pairing isometries of $P$ send interior into exterior, that every codimension 2 face participates in a geometric cycle, and that, for every such cycle, the total interior angle equals $2\pi$ at some point of a codimension 2 face of the cycle. Then tessellation condition 2.1 is satisfied.

Proof. Let $v_1, \ldots, v_m$ denote the codimension 3 faces of $P$. By combining Lemmas 3.10 and 3.12 with Corollary 3.13 and Lemma 3.14, we reduce the theorem to the following facts:

2We need the completeness of $M$ in order to be able to apply Lemma 3.2 thus guaranteeing that $S(v, \delta) \cap P$ is a compact smooth polyhedron for small $\delta$. 
• For every $i$, there are finitely many formal neighbours of $P$ at $v_i$.
• There exists $\delta_0 > 0$ such that $\pi^{-1}W_{v_i, \delta} = N_{v_i, \delta}$ and $\varphi : W_{v_i, \delta} \to B(v_i, \delta)$ is a bijection for all $i$ and $0 < \delta \leq \delta_0$.

4.2.1. Polyhedron $P_\delta \subset M_\delta$ and its face-pairing. By Lemma 3.2, there exists a small $\delta_0 > 0$ such that $P_\delta := \bigcup_i S(v_i, \delta) \cap P$ is a compact smooth polyhedron in the smooth 2-manifold $M_\delta := \bigcup_i S(v_i, \delta)$ for any $0 < \delta \leq \delta_0$. Moreover, for any $i$, the 2-sphere $S(v_i, \delta)$ intersects only those faces of $P$ that properly contain $v_i$. The codimension $k$ faces of $P_\delta$ are the connected components $(v_i, f)_j$ of $S(v_i, \delta) \cap f$, where $f \subset P$ is a codimension $k$ face of $P$ containing $v_i$. The interior of $(v_i, f)_j$ is the intersection of $(v_i, f)_j$ with the interior of $f$. The only possible faces of $P_\delta$ are those of codimensions 0, 1, 2. By Lemma 3.16, we assume that $S(v_i, \delta) \cap P$ is connected for all $i$ and $0 < \delta \leq \delta_0$. In other words, $S(v_i, \delta) \cap P$ are the codimension 0 faces of $P_\delta$.

We equip $P_\delta$ with face-pairing as follows. Let $(v_i, s)_j$ be a codimension 1 face of $P_\delta$, where $s \subset P$ is a codimension 1 face of $P$ and $v_i \in s$. Since $I_s$ maps $v_i$ to $I_s v_i$, $S(v_i, \delta)$ onto $S(I_s v_i, \delta)$, and $s$ onto $\pi$, it maps the component $(v_i, s)_j$ to $S(v_i, \delta) \cap s$ onto a certain component $(I_s v_i, s)_j$ of $S(I_s v_i, \delta) \cap s$. We define the involution $\pi : S_\delta \to S_\delta$ on the set $S_\delta$ of codimension 1 faces of $P_\delta$ as $(v_i, s)_j := (I_s v_i, s)_j$.

The isometry $I_{(v_i, s)_j} : M \to M$ is the restriction of $I_s \in \text{Isom} M$. Let $G_\delta \subset \text{Isom} M$ denote the subcategory generated by the face-pairing isometries of $P_\delta$.

4.2.2. Diagram. The polyhedron $P$ and the subcategory $G \subset \text{Isom} M$ generated by face-pairing isometries determine the commutative triangle (see Section 2) on the right side of the diagram. Analogously, we obtain from $P_\delta$ and $G_\delta$ the commutative triangle on the left side of the diagram.

The face-pairing isometries that generate $G_\delta$ are restrictions of the face-pairing isometries that generate $G$. Sending the component $S(v_i, \delta)$ of $M_\delta$ to the component of $M$ that contains $S(v_i, \delta)$ and using the fact that the composition of restrictions is the restriction of the composition, we can define a functor $\alpha : G_\delta \to G$. At the level of generators, $\alpha : I_{(v_i, s)_j} \to I_s$. In order to verify the correctness of the definition of $\alpha$ for morphisms, we need only to show that an isometry $I : M_j \to M_j$ of a component $M_j$ of $M$ containing $S(v_i, \delta)$ is the identity if its restriction $I|_{S(v_i, \delta)}$ is the identity. This can be achieved by a suitable choice of $\delta_0$. For sufficiently small $\delta_0$, the sphere $S(v_i, \delta)$ has a unique centre with respect to the radius $\delta$. Hence, $I v_i = v_i$. If $\delta_0$ is less than the radius of injectivity of $M$ at $v_i$, then $I = 1$. Indeed, let $p \in M_j$ and let $\Gamma$ be a shortest geodesic joining $v_i$ and $p$. If necessary, we can extend the geodesic $\Gamma$ so that it intersects $S(v_i, \delta)$, say, at $x \in S(v_i, \delta)$. Since $I v_i = v_i$, $I x = x$, and $v_i, x \in \Gamma$, we obtain $I|_\Gamma = 1$ and, therefore, $I p = p$.

We define a map $\beta : G_\delta * P_\delta \to G * P$, $\beta : (g, x) \mapsto (\alpha g, x)$, in the commutative exterior square of the diagram (we think of $P_\delta$ as being included into $P$). Note that $\beta : G_\delta * P_\delta \to G * P$ is injective. Indeed, $\beta(g, x) = \beta(h, y)$ implies $x = y$, $\alpha g = \alpha h$, and $g(x) = (\alpha g) x = (\alpha h) y = h y$. So, if $x \in S(v_i, \delta)$ and $g x \in S(v_j, \delta)$, then $g, h : S(v_i, \delta) \to S(v_j, \delta)$, $g = \alpha g|_{S(v_i, \delta)}$, and $h = \alpha h|_{S(v_i, \delta)}$, implying the fact.

The equivalence relation $\sim_\delta$ on $G_\delta * P_\delta$ is induced by the equivalence relation $\sim$ on $G * P$, i.e., $(g, x) \sim_\delta (h, y)$ iff $\beta(g, x) \sim \beta(h, y)$ for any $(g, x), (h, y) \in G_\delta * P_\delta$. Indeed, for $x \in (v_i, s)_j$, the relation $(I_{(v_i, s)_j}) x \sim_\delta (1, (v_i, s)_j) x$ is nothing but the relation $(I_s, x) \sim_\delta (1, I_s x)$. The rest follows from $\beta(g(h, x)) = (\alpha g) \beta(h, x)$. We obtain an injective map $\gamma : J_\delta \to J$ in the commutative top square of the diagram of continuous maps. Since $\alpha$ is surjective, the entire diagram is commutative.

4.2.3. Formal neighbours in $M_\delta$ and in $P$. Let $x$ be a point in a codimension 1 face $(v_i, s)_j$ of $S(v_i, \delta) \cap P_\delta$ and let $\pi^{-1}[1, x] = \{ (g_1, x_1), \ldots, (g_n, x_n) \}$. Using the description of formal neighbours given in 3.7, we are going to show that there exist $h_k \in G_\delta$ such that $h_k(x_k) = (g_k, x_k)$ for all $k = 1, \ldots, n$.

Suppose that $x$ belongs to the interior of $s$. Then $\pi^{-1}[1, x] = \{ (1, x), (I^{-1}_s, I_s x) \}$. The point $I_s x$ belongs to a component of $S(I_s v_i, \delta) \cap \pi$. Taking $h_k := I^{-1}_{(v_i, s)_j}$, we have $h_k(I_s x) = (I^{-1}_s, I_s x)$.

Suppose that $x$ lies in the interior of a codimension 2 face $e \subset s$ of $P$. The face $e$ belongs to a geometric cycle and, using the notation from (3.8) with $s_1 := s$, we have $\pi^{-1}[1, x] = \{ (I^{-1}_k, I_k x) | k =$
0, 1, \ldots, n - 1 \right\}. By induction, we get \( h_{k-1} \in G_\delta \) such that \( \beta(h_{k-1}, I_{k-1}x) = (I_{k-1}^{-1}, I_{k-1}x) \) and the point \( I_{k-1}x \in S(I_{k-1}v_1, \delta) \cap I_{k-1}e \subset S(I_{k-1}v_1, \delta) \cap \delta \) belongs to a component of \( S(I_{k-1}v_1, \delta) \cap \delta \). The isometry \( I_{\delta} \) maps this component onto a certain component of \( S(I_{k}v_1, \delta) \cap \delta \). Therefore, the corresponding restriction \( h^{-1} \in G_\delta \) of \( I_{\delta} \) provides \( \beta(h^{-1}, h_{k}x) = (I_{k}^{-1}, I_{k}x) = (I_{k}^{-1}, I_{k}x) \).

In both cases, \( (h_{k}, x_{k}) \in \pi^{-1}_{\delta}[1, x] \) because \( \gamma \) is injective. Conversely, let \( (h, y) \in \pi^{-1}_{\delta}[1, x] \). Then \( \pi \beta(h, y) = \gamma[1, x] = [1, x] \) and \( (h, y) = (h_{k}, x_{k}) \) for some \( k \) as \( \beta \) is injective. In other words, there is a one-to-one correspondence between formal neighbours of \( P \) at \( x \) and formal neighbours of \( P_\delta \) at \( x \).

At the level of \( M \), this correspondence means that the formal neighbours of \( P \) at \( x \) in \( (v_i, s)_j \) are the intersections with \( S(v_i, \delta) \) of the corresponding formal neighbours of \( P \) at \( x \). Indeed, given \( (g_k, x_k) \in \pi^{-1}_{\delta}[1, x] \) and \( (h_k, x_k) \in G_\delta \) as above, let \( S(v_i, \delta) \subset M_\delta \) denote the components of \( M_\delta \) and \( M \) that contain \( x_k \). Since \( g_k x_k = x \), we obtain \( g_k S(v_i, \delta) = S(v_i, \delta) \). By the choice of \( \delta_0 \), we have \( S(v_i, \delta) \cap P_\delta = S(v_i, \delta) \cap P_k \). Hence, \( \varphi_\delta[h_k, S(v_i, \delta) \cap P_\delta] = \varphi[g_k, S(v_i, \delta) \cap P_k] \) by the commutativity of the bottom square. We obtain \( h_k(S(v_i, \delta) \cap P_\delta) = g_k(S(v_i, \delta) \cap P_k) = S(v_i, \delta) \cap g_k P_k \).

4.2.4. Tessellation of \( M_\delta \). Let \( x \in P_\delta \). The above bijection between \( \pi^{-1}_{\delta}[1, x] \) and \( \pi^{-1}_{\delta}[1, x] \) implies that \( \pi^{-1}_{\delta}[1, x] \) is finite. Combining Lemma 3.10, Lemma 3.12, and Remark 3.4 with the fact obtained in 4.2.3 that the formal neighbours of \( P_\delta \) at any \( x \in P_\delta \) are the intersections of the formal neighbours of \( P \) with \( S(v_i, \delta) \), we conclude that formal neighbours of \( P_\delta \) at any \( x \in P_\delta \) tessellate a neighbourhood of \( x \).

Every codimension 2 face \( x \in S(v_i, \delta) \) of \( P_\delta \) participates in a geometric cycle of \( P_\delta \). Indeed, \( x \) lies in the interior of a codimension 2 face \( e \in P \) and \( e \) belongs to a geometric cycle (3.8) of \( P \). As in 4.2.3, we can `restrict’ this cycle to a cycle of \( P_\delta \). Every codimension 2 face \( x \) of \( P_\delta \) is injective. Conversely, let \( (g_j, F_j) \subset C_\delta \) be a component of \( J_\delta \). If \( (g_j, p) \in \pi^{-1}_{\delta} C_\delta \) and \( p \in F_j \) belongs to a codimension 0 face \( F_j \) of \( P_\delta \), then \( (g_j, F_j) \subset \pi^{-1}_{\delta} C_\delta \) because \( F_j \) is path-connected. Moreover, \( \pi^{-1}_{\delta} C_\delta \) is a finite union of such pieces, \( \pi^{-1}_{\delta} C_\delta = \bigcup_{j=1}^{k} (g_j, F_j) \). Indeed, the measure of (the interior of) a codimension 0 face of \( P_\delta \) in \( M_\delta \) is limited from below by a positive constant, \( \varphi_\delta : C_\delta \to S(v_i, \delta) \) is a homeomorphism for some \( i \), and \( \pi_{\delta}(g_j, F_j) \subset C_\delta \) are all disjoint for \( (g_j, F_j) \subset \pi^{-1}_{\delta} C_\delta \).

Note that \( \pi^{-1}_{\delta} C_\delta \) is a minimal nonempty subset of the type \( \bigcup_{j=1}^{k} (g_j, F_j) \) closed with respect to taking \( \sim_{\delta} \)-equivalent elements. Otherwise, \( C_\delta \) is a disjoint union of two nonempty compact sets.

4.2.6. To any \( (g, v) \in \pi^{-1}[1, v_i] \) and \( 0 < \delta \leq \delta_0 \), we associate

\[
N_{(g, v), \delta} := B(v, \delta) \cap P, \quad F_{(g, v), \delta} := S(v, \delta) \cap P \subset P_\delta.
\]

Since the codimension 0 face of \( P \) containing \( v \) is path-connected, by Lemma 3.16, \( F_{(g, v), \delta} \) is a (nonempty) codimension 0 face of \( P_\delta \) for a suitable choice of \( \delta_0 \). Let

\[
N_{v_i, \delta} := \bigcup_{(g, v) \in \pi^{-1}[1, v_i]} (g, N_{(g, v), \delta}) \subset G \ast P, \quad S_{v_i, \delta} := \bigcup_{(g, v) \in \pi^{-1}[1, v_i]} (g, F_{(g, v), \delta}) \subset \beta(G_\delta \ast P_\delta).
\]

The inclusions easily follow from \( (g, v) \in G \ast P \) and from a decomposition of \( g \) in generators \( I_\delta \)’s due to the choice of \( \delta_0 \). Obviously, \( \psi S_{v_i, \delta} \subset S(v_i, \delta) \). We have \( (g, F_{(g, v), \delta}) = \beta(g', F_{(g', v), \delta}) \) for some \( g' \in G_\delta \).
Since \( N_{v_i,δ} = \pi^{-1}[1, v_i] \cup \bigcup_{0 < δ' < δ} S_{v_i, δ'} \), in order to show that \( \pi^{-1}(\pi N_{v_i,δ}) = N_{v_i,δ} \), it suffices to observe that \( \pi^{-1}(\pi S_{v_i,δ}) = S_{v_i,δ} \), i.e., that \( S_{v_i,δ} \) is closed with respect to taking \(~\)equivalent elements.

Let \((g, I_x) ∈ S_{v_i,δ} \), where \( x ∈ s \). We need to show that \((gI_x, x) ∈ S_{v_i,δ} \). For some \( v_j \), we have \((g, v_j) ∈ \pi^{-1}[1, v_i] \) and \( I_x \in S(v_j, δ) \cap s \). Since \( v_j ∈ π \) by the choice of \( δ_0 \), we conclude that \((gI_x, F(v_j)) \sim (g, v_j) ∈ \pi^{-1}[1, v_i] \) and \( x ∈ S(Iπv_j, δ) \cap P \). So, \( x ∈ F(gI_x, F(v_j)), δ) \) and \((gI_x, x) ∈ S_{v_i,δ} \).

Moreover, \( π_δS_δ \) is path-connected, where \( S_δ := \bigcup_{(g, v) ∈ \pi^{-1}[1, v_i]} (g', F(g,v), δ) \). Indeed, \( F(g,v), δ \) is path-connected for any \((g, v) ∈ \pi^{-1}[1, v_i] \) by Lemma 3.16 and by the choice of \( δ_0 \). Let \( v_j ∈ s \). Consider \( F(g,v_j), δ, F(gL_vJ, v_j), δ \), and the corresponding \( g', g'' ∈ G_δ \). Then \( π_δ(g', F(g,v_j), δ) \) and \( π_δ(g'', F(gL_vJ, v_j), δ) \) have a common point because \( S(v_j, δ) \cap s ≠ \emptyset \) due to the path-connectedness of \( s \) and to the choice of \( δ_0 \). In particular, \( π_δS_δ \) lies in a component \( C_δ \) of \( J_δ \) homeomorphic to \( S(v_i, δ) \).

On the other hand, \( S_δ \) is closed with respect to taking \(~\)equivalent elements because \( S_{v_i,δ} = βS_δ \) is closed with respect to taking \(~\)equivalent elements and \(~\) is induced by \( \sim \). By 4.2.5, \( π_δS_δ \) is \( C_δ \) for a suitable component \( C_δ \) of \( J_δ \) such that \( \varphi_δ : C_δ → S(v_i, δ) \) is a bijection.

Summarizing, \( \varphi : πN_{v_i,δ} → B(v_i, δ) \) is a bijection and \( \pi^{-1}(\pi N_{v_i,δ}) = N_{v_i,δ} \). Since \( F(g,v), δ ≠ \emptyset \) for every \((g, v) ∈ \pi^{-1}[1, v_i] \), the finiteness of \( \pi^{-1}[1, v_i] \) follows from 4.2.5.

4.3. Theorem. Let \( M \) be an oriented complete Riemannian manifold with 4-dimensional components and \( P \) be a compact smooth polyhedron connected near codimension 3. Suppose that the face-pairing isometries of \( P \) send interior into exterior, that every codimension 2 face participates in a geometric cycle, and that, for every such cycle, the total interior angle equals \( 2π \) at some point of a codimension 2 face of the cycle. If the stabilizer of every codimension 3 face in the groupoid generated by face-pairing isometries induces on the face a finite group of isometries, then tessellation condition 2.1 is satisfied.

Proof mostly follows the line of the proof of Theorem 4.1.

4.4.1. Extra codimension 4 faces. First, we introduce additional codimension 4 faces inside codimension 3 faces in order to make the latter topological segments (not circles).

Let \( f_i \) be a codimension 3 face of \( P \) inside a codimension 1 face \( s_i \) of \( P, f_i ⊂ s_i \) for \( i = 1, 2 \). We say that \( f_1, f_2 \) are equivalent if \( s_2 = π_1 \) and \( f_2 = I_s f_1 \) and take the closure of this symmetric relation with respect to transitivity.

Let \( f \) be a face in such an equivalence class. By conditions of Theorem 4.3, the group \( Γ \) of isometries of \( f \) induced by the stabilizer of \( f \) in the groupoid is finite. Pick an interior point \( p ∈ f \) and introduce the new codimension 4 faces \( Γ p ⊂ f \). Then, using face-pairing isometries, we copy these new faces into the other codimension 3 faces equivalent to \( f \).

4.4.2. By combining Lemmas 3.10 and 3.12 with Corollary 3.13 and Lemma 3.14, we reduce the theorem to the following facts:

• For every \( p \) in a codimension 3 face of \( P \), there are finitely many formal neighbours of \( P \) at \( p \).
• For every \( p \) in a codimension 3 face of \( P \), there exists \( δ(p) > 0 \) such that \( π^{-1}W_{p, δ} = N_{p, δ} \) and \( \varphi : W_{p, δ} → B(p, δ) \) is a bijection for all \( 0 < δ ≤ δ(p) \).

4.4.3. Polyhedron \( P_{ε, δ} \subset M_{ε, δ} \) and its face-pairing. Fix some \( ε ∈ [0, \frac{1}{2}] \). In every codimension 3 face \( f \subset P \), we take the points that are on the distance \( ε\ell \), measured along \( f \), from one of the vertices of \( f \), where \( \ell \) stands for the length of \( f \). So, we take a couple of points in every codimension 3 face \( f \) if \( ε \neq \frac{1}{2} \) and the middle point of \( f \), otherwise. Denote by \( v_1, \ldots, v_m \) all such points (\( ε \) is fixed). By Lemma 3.2, there exists a small \( δ_0 > 0 \) such that \( P_{ε, δ} := \bigcup_{i=1}^n S(v_i, δ) \cap P \) is a compact smooth polyhedron in the smooth 3-manifold \( M_{ε, δ} := \bigcup_{i=1}^n S(v_i, δ) \) for any \( 0 < δ ≤ δ_0 \). Moreover, for any \( i \), the 3-sphere \( S(v_i, δ) \) intersects only those faces of \( P \) that properly contain \( v_i \). The codimension \( k \) faces

3This condition is obviously satisfied for codimension 3 faces that are not round circles.
of \(P_\delta\) are the connected components \((v_i, f)_j\) of \(S(v_i, \delta) \cap f\), where \(f \subset P\) is a codimension \(k\) face of \(P\) containing \(v_i\). The interior of \((v_i, f)_j\) is the intersection of \((v_i, f)_j\) with the interior of \(f\). The only possible faces of \(P_{\varepsilon, \delta}\) are those of codimensions 0, 1, 2, 3. By Lemma 3.16, we assume that \(S(v_i, \delta) \cap P\) are connected for all \(i\) and \(0 < \delta \leq \delta_0\). In other words, \(S(v_i, \delta) \cap P\) is the codimension 0 faces of \(P_{\varepsilon, \delta}\).

We equip \(P_{\varepsilon, \delta}\) with face-pairing exactly as in 4.2.1 and denote by \(G_{\varepsilon, \delta} \subset \text{Isom } M_{\varepsilon, \delta}\) the subcategory generated by face-pairing isometries of \(P_{\varepsilon, \delta}\).

4.4.4. Diagram. After adapting notation, this subsection is literally the same as Subsection 4.2.2.

4.4.5. Formal neighbours in \(M_{\varepsilon, \delta}\) and in \(M\) in codimension \(\leq 2\). Let \(x \in S(v_i, \delta) \cap P\) be not a codimension 3 face of \(P_{\varepsilon, \delta}\). Then, literally following 4.2.3, we show that the formal neighbours of \(P_{\varepsilon, \delta}\) at \(x\) are the intersections with \(S(v_i, \delta)\) of the corresponding formal neighbours of \(P\) at \(x\).

4.4.6. Tessellation of \(M_{\varepsilon, \delta}\). By Lemma 3.17, the polyhedron \(P_{\varepsilon, \delta} \subset M_{\varepsilon, \delta}\) is connected near codimension 3. By 4.4.5, Lemma 3.10, Lemma 3.12, and Remark 3.4, the polyhedron \(P_{\varepsilon, \delta}\) satisfies the conditions of Theorem 4.1 because the geometric cycles of \(P\) being ‘restricted’ to \(P_{\varepsilon, \delta}\) remain geometric. Indeed, the isometry of a shorter cycle cannot be the identity for the same reason as in 4.2.4. So, by Theorem 4.1 and Proposition 2.2, \(\varphi_{\varepsilon, \delta} : J_{\varepsilon, \delta} \rightarrow M_{\varepsilon, \delta}\) is a surjective regular covering.

4.4.7. As in 4.2.5, we show that \(\pi_{\varepsilon, \delta}^{-1}C_{\varepsilon, \delta} = \bigcup_{j=1}^k (g_j, F_j)\) is a minimal nonempty subset of this type closed with respect to taking \(\sim_{\varepsilon, \delta}\)-equivalent elements for any component \(C_{\varepsilon, \delta}\) of \(J_{\varepsilon, \delta}\), where the \(F_j\)'s are codimension 0 faces of \(P_{\varepsilon, \delta}\).

4.4.8. Here, after adapting notation, we follow literally 4.2.6 using 4.4.7 in place of 4.2.5 ■

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