LIPSCHITZ NORMALLY EMBEDDED SET AND TANGENT CONES AT INFINITY

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Abstract. We prove that any analytic set in $\mathbb{C}^n$ with a unique tangent cone at infinity is an algebraic set. We prove that the degree of a complex algebraic set in $\mathbb{C}^n$, which is Lipschitz normally embedded at infinity, is equal to the degree of its tangent cone at infinity.

1. Introduction

This note is motivated by the arguments presented in [6].

Let $X$ be a path connected subset of $\mathbb{C}^n$. Given $x, y \in X$, we set the inner distance on $X$ between $x$ and $y$, denoted by $d_X(x, y)$, as the infimum of the length($\gamma$) where $\gamma$ varies on the set of continuous paths on $X$ connecting $x$ to $y$. We recall the following definition from [2].

Definition 1.1. A path connected subset $X$ of $\mathbb{C}^n$ is called Lipschitz normally embedded if there exists a positive real number $C$ such that

$$d_X(x, y) \leq C\|x - y\|,$$

for all $x, y \in X$.

Lipschitz normal embedding and necessary conditions for a set to be Lipschitz normally embedded have been studied by many authors; see for instance [1, 4, 5, 9, 10, 11] and the references cited therein.

In a recent work, Fernandes and Sampaio [6] introduced the following definition in the global case.

Definition 1.2. A path connected subset $X$ of $\mathbb{C}^n$ is called Lipschitz normally embedded at infinity if there exists a compact subset $K \subset X$ such that $X \setminus K$ is Lipschitz normally embedded.

When a pure dimensional analytic set $X \subset \mathbb{C}^n$ has a unique tangent cone at infinity, we denote it by $C_\infty(X)$ and call it the tangent cone of $X$ at infinity; see Definitions 2.6 and 2.7 for details. The main theorem of [6] is the following one:
Theorem 1.3 ( )[6, Theorem 1.1]. Let \( X \subset \mathbb{C}^n \) be a closed and pure \( d \)-dimensional analytic subset. Suppose \( X \) has a unique tangent cone at infinity and this cone is a \( d \)-dimensional complex linear subspace of \( \mathbb{C}^n \). If \( X \) is Lipschitz normally embedded at infinity, then \( X \) is an affine linear subspace of \( \mathbb{C}^n \).

Let \( V \subset \mathbb{C}^n \) be an algebraic set. Bearing in mind that \( \text{deg} V = 1 \) if and only if \( V \) is a linear subspace of \( \mathbb{C}^n \); see for instance Proposition 3.3 of [6]. We prove in this note that Theorem 1.3 holds true with the following more general statement.

Theorem 1.4. Let \( X \subset \mathbb{C}^n \) be a closed and pure \( s \)-dimensional analytic subset. Suppose \( X \) has a unique tangent cone at infinity and this cone is a pure \( k \)-dimensional complex algebraic set of \( \mathbb{C}^n \). Then \( X \) is an algebraic set and \( s = k \). Moreover, if \( X \) is Lipschitz normally embedded at infinity, then \( \text{deg} X = \text{deg} C_\infty(X) \).

We remark that from equality (2.1) of [7], it follows that \( \text{deg} X \geq \text{deg} C_\infty(X) \) is always true for any pure dimensional complex algebraic sets \( X \) and \( C_\infty(X) \).

This note is organized as follows. In section 2, we recall the definition of algebraic region after [12] and the definition of tangent cones at infinity. In Theorem 2.9, we prove that if an unbounded set \( V \subset \mathbb{C}^n \) has a unique tangent cone at infinity, we say \( C_\infty(V) \), and this cone lies in an algebraic region, then there exists an algebraic region \( \Omega \) such that \( V \) and \( C_\infty(V) \) lie in \( \Omega \).

As a consequence of Theorem 2.9 it follows that \( \dim X = \dim C_\infty(X) \) for any complex algebraic set (Proposition 2.10). This last result can be seen as a version at infinity of a result of Whitney on tangent cones at a point of an analytic variety.

In Section 3 we recall the definition of degree of a complex algebraic set and we present the proof of Theorem 1.4. The proof is motivated by the arguments of [6].

Finally, we observe that a local version of Theorem 1.4 follows from Corollary 3.13 and Remark 3.8 of [5]. Precisely, let \( X \subset \mathbb{C}^n \) be a pure analytic set and \( 0 \in X \). We say that \( X \) is Lipschitz normally embedded at 0 if there exists a neighborhood \( U \subset \mathbb{C}^m \) of 0 such that \( X \cap U \) is Lipschitz normally embedded. The tangent cone of \( X \) at 0 is denoted by \( C(X,0) \); see for instance section 8.1 of [3]. We denote by \( \mu_0(X) \) and \( \mu_0(C(X,0)) \) the multiplicity at 0 of \( X \) and \( C(X,0) \) respectively; see for instance section 11 of [3]. Thus, a local version of Theorem 1.4 is the next result, whose proof follows from Corollary 3.13 and Remark 3.8 of [7].

Theorem 1.5. Let \( X \subset \mathbb{C}^m \) be a pure dimensional complex analytic set with \( 0 \in X \). If \( X \) is a Lipschitz normally embedded set at 0, then \( \mu_0(X) = \mu_0(C(X,0)) \).

We point out that another proof of Theorem 1.5 may be given by similar argument of the proof of Theorem 1.4.
Definition 2.1 (See p. 672 of [12]). Let \( n \geq 2 \). A set \( \Omega \subset \mathbb{C}^n \) is called an algebraic region of type \((k, n)\) if there exist vector subspaces \( V_1, V_2 \subset \mathbb{C}^n \) and positive real numbers \( A, B \) such that the following conditions hold: \( \dim V_1 = k, \dim V_2 = n - k; \mathbb{C}^n = V_1 + V_2 \) and \( \Omega \) consists of those \( z \in \mathbb{C}^n \) for which:

\[
\|z''\| \leq A(1 + \|z'\|)^B,
\]

where \( z = z' + z'' \), with \( z' \in V_1 \) and \( z'' \in V_2 \).

A geometric characterization of analytic sets of \( \mathbb{C}^n \) which are algebraic is given by the next theorem.

Theorem 2.2 (See Theorem 2 of [12]). A closed analytic subset \( V \subset \mathbb{C}^n \) of pure dimension \( k \) is algebraic if and only if \( V \) lies in some algebraic region \( \Omega \) of type \((k, n)\).

Let \( \Omega \subset \mathbb{C}^n \) be an algebraic region of type \((k, n)\), with \( V_1, V_2 \subset \mathbb{C}^n \) and constants \( A, B \) as in Definition 2.1. We may choose linear coordinates \((x, y)\) in \( \mathbb{C}^n \) such that \( x \in V_1 \) and \( y \in V_2 \). In this coordinate system, we denote by \( \pi_\Omega \) the canonical projection from \( \mathbb{C}^n \) to \( V_1 \), i.e., \( \pi_\Omega(x, y) = x \). In the proof of Proposition 2.4 we need a fact concerning the projection \( \pi_\Omega \), as follows.

Lemma 2.3. If \( V \) is a closed set in \( \mathbb{C}^n \) such that \( V \subset \Omega \), then the restriction \( \pi_\Omega : V \to V_1 \) is a proper mapping.

Proof. Let \( \{z_j\}_{j \in \mathbb{N}} \) be any sequence in \( V \) with \( \lim_{j \to \infty} \|z_j\| = \infty \). We have to show that \( \lim_{j \to \infty} \|\pi_\Omega(z_j)\| = \infty \). We may write \( z_j = (x_j, y_j) \), with \( x_j \in V_1 \) and \( y_j \in V_2 \). By hypothesis, \((x_j, y_j)\) satisfies the inequality (1) and, therefore, \( \lim_{j \to \infty} \|x_j\| = \infty \). This finishes the proof, since \( \pi_\Omega(z_j) = x_j \). \( \square \)

Let \( V \subset \mathbb{C}^n \) be an algebraic set. It follows from [12, p. 677] that if \( \dim V \leq k \), then \( V \) lies in an algebraic region of type \((k, n)\). The converse is also true, by the next result.

Proposition 2.4. Let \( V \) be an algebraic set in \( \mathbb{C}^n \). If \( V \) lies in an algebraic region of type \((k, n)\), then \( \dim V \leq k \).

Proof. Let \( \Omega \subset \mathbb{C}^n \) be an algebraic region of type \((k, n)\), with \( V_1, V_2 \subset \mathbb{C}^n \) and constants \( A, B \) as in Definition 2.1 and that \( V \subset \Omega \). From Lemma 2.3 the projection \( \pi_\Omega : V \to V_1 \) is a proper mapping. Then \( \dim V = \dim \pi_\Omega(V) \); see for instance Propositions 1 and 2 of [31, p. 31]. Since \( \dim V_1 = k \), it follows that \( \dim V \leq k \). \( \square \)

2.2. Tangent cone at infinity. Let us start this section recalling two equivalent definitions of tangent cone at infinity of algebraic sets after Theorem 1 of [13].

Definition 2.5. Let \( V \) be an algebraic set. The geometric tangent cone \( C_{g, \infty}(V) \) of \( V \) at infinity is defined by the set of tangent vectors of \( V \) at infinity, in the sense that \( v \in C_{g, \infty}(V) \) if and only if there exist sequences \( \{x_j\}_{j \in \mathbb{N}} \subset V, \{t_j\}_{j \in \mathbb{N}} \subset \mathbb{C} \), such that \( \lim_{j \to \infty} \|x_j\| = \infty \) and \( \lim_{j \to \infty} t_j x_j = v \).
The algebraic tangent cone $C_{a,\infty}(V)$ of $V$ at infinity is defined by the set

$$C_{a,\infty}(V) = \{ v \in \mathbb{C}^n \mid f^*(v) = 0 \text{ for all } f \in I(V) \},$$

where $I(V)$ denotes the ideal defining $V$, and for each polynomial $f \in I(V)$, $f^*$ denotes its homogeneous component of highest degree.

It follows by Theorem 1 of [13] that $C_{g,\infty}(V) = C_{a,\infty}(V)$ for any algebraic set $V \subset \mathbb{C}^n$. In particular, Theorem 1 of [13] can be seen as a version at infinity of a well-known theorem of H. Whitney [15] on tangent cones at a point of an analytic variety.

For any unbounded subset $V$ of $\mathbb{C}^n$, it is defined in [6] the tangent vectors at infinity with respect to a sequence of real positive numbers, as follows.

**Definition 2.6.** Let $V \subset \mathbb{C}^n$ be an unbounded subset. Given a sequence of real positive numbers $\{t_j\}_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} t_j = \infty$, we say that $v \in \mathbb{C}^n$ is tangent to $V$ at infinity with respect to $\{t_j\}_{j \in \mathbb{N}}$ if there exists a sequence of points $\{x_j\}_{j \in \mathbb{N}} \subset V$ such that $\lim_{j \to \infty} \frac{1}{t_j} x_j = v$.

**Definition 2.7.** Let $V \subset \mathbb{C}^n$ be an unbounded subset and $T = \{t_j\}_{j \in \mathbb{N}}$ be a sequence of real positive numbers such that $\lim_{j \to \infty} t_j = \infty$. Denote by $E_T(V)$ the set of $v \in \mathbb{C}^n$ which are tangent to $V$ at infinity with respect to $T$. We call $E_T(V)$ a tangent cone of $V$ at infinity. When $V$ has a unique tangent cone at infinity, we denote it by $C_{\infty}(V)$ and we call $C_{\infty}(V)$ the tangent cone of $V$ at infinity.

In general, the problem of determining the uniqueness of tangent cones at infinity for unbounded sets is a difficult problem. For instance, a conjecture by Meeks [14, Conjecture 3.15] states that any embedded minimal surface in $\mathbb{R}^3$ of quadratic area growth has a unique tangent cone at infinity. See for instance [8] for a partial answer of this conjecture.

**Remark 2.8.** Let $X \subset \mathbb{C}^n$ be an unbounded algebraic set. From Corollary 2.16 of [6], we know that $X$ has a unique tangent cone at infinity and, with Theorem 1 of [13], $C_{\infty}(X) = C_{a,\infty}(X) = C_{g,\infty}(X)$. See also Proposition 2.21 of [6]. In particular, if $v \in C_{\infty}(X)$ then $\lambda v \in C_{\infty}(X)$, for any $\lambda \in \mathbb{R}$.

**Theorem 2.9.** Let $V \subset \mathbb{C}^n$ be an unbounded subset. Suppose that $V$ has a unique tangent cone at infinity and this cone lies in an algebraic region $\Omega$ of type $(k, n)$. Then there exists an algebraic region $\Omega$ of type $(k, n)$ such that $V$ and $C_{\infty}(V)$ lie in $\Omega$.

**Proof.** It follows from Definition 2.1 that there exist vector spaces $V_1$ and $V_2$ in $\mathbb{C}^n$, with $\dim V_1 = k$, $\dim V_2 = n - k$, $\mathbb{C}^n = V_1 + V_2$, and positive real numbers $\tilde{A}, B$ such that for any $z \in \tilde{\Omega}$ (hence, in particular, for $z \in C_{\infty}(V)$) we have:

$$\|z''\| < \tilde{A}(1 + \|z'\|)^B,$$

(2)

where $z = z' + z''$, with $z' \in V_1$ and $z'' \in V_2$. Since $(1 + t)^s \leq (1 + t)$, for any $t \geq 0$ and $0 \leq s \leq 1$, we may assume in (2) that $B \geq 1$.

We claim that there exists a positive real number $R$ such that, for any $w \in V$, we have:

$$\|w''\| < R(1 + \|w'\|)^B,$$

(3)
where \( w = w' + w'' \), with \( w' \in V_1 \) and \( w'' \in V_2 \).

If the claim is not true, this means that there exists a sequence \( \{w_j\}_{j \in \mathbb{N}} \subset V \) such that
\[
\|w_j'\| > k(1 + \|w_j''\|)^B
\]
and, up to a subsequence, we may suppose that \( \lim_{j \to \infty} \frac{w''_j}{\|w''_j\|} = y_0 \in V_2 \), with \( \|y_0\| = 1 \). It follows that \( \lim_{j \to \infty} \|w_j''\| = \infty \) and therefore \( \lim_{j \to \infty} \|w_j\| = \infty \) since \( \mathbb{C}^n \) is a direct sum of \( V_1 \) and \( V_2 \). We have
\[
\frac{1}{k} > \frac{(1 + \|w_j'\|)^B}{\|w_j''\|} \geq \frac{(1 + \|w_j''\|)^B}{\|w''_j\|} \geq \frac{\|w_j''\|}{\|w''_j\|}.
\]

Let \( t_j := (\tilde{A})^{-1}\|w_j''\| \). We have \( \lim_{j \to \infty} t_j = \infty \) and
\[
\lim_{j \to \infty} \frac{1}{t_j} w_j = \lim_{j \to \infty} \frac{1}{t_j} (w_j' + w_j'') = \tilde{A} y_0.
\]

It follows that \( \tilde{A} y_0 \in C_\infty(V) \) by Definition 2.7. Now, since \( V_2 \) is a linear space and \( y_0 \in V_2 \), we have \( \tilde{A} y_0 \in (V_2 \cap C_\infty(V)) \). This is a contradiction, since (2) does not hold for \( \tilde{A} y_0 \).

Therefore, if follows that there exists a positive real number \( R \) such that for any \( w \in V \) we have:
\[
\|w''\| < R(1 + \|w'\|)^B
\]
where \( z = w' + w'' \), with \( w' \in V_1 \) and \( w'' \in V_2 \). Let \( A := \max\{R, \tilde{A}\} \), then \( V \) and \( C_\infty(V) \) lie in an algebraic region \( \Omega \) of type \((k, n)\), with vector spaces \( V_1 \) and \( V_2 \) in \( \mathbb{C}^n \), with \( \dim V_1 = k \), \( \dim V_2 = n - k \) and positive real numbers \( A \) and \( B \). This finishes the proof. \( \square \)

Bearing in mind that \( C_\infty(V) = C_{d, \infty}(V) = C_{g, \infty}(V) \) for any unbounded algebraic set \( V \subset \mathbb{C}^n \) (see Remark 2.8) and that, by definition of dimension, \( \dim_{\mathbb{R}}(V) = 2 \dim_{\mathbb{C}}(V) \), we finish this section with the following result.

**Proposition 2.10.** Let \( V \subset \mathbb{C}^n \) be an unbounded algebraic set. Then \( \dim V = \dim C_\infty(V) \).

**Proof.** It follows from Corollary 2.18 of [6] that \( \dim C_\infty(V) \leq \dim V \).

On the other hand, let \( k = \dim C_\infty(V) \), which is an algebraic set. By [12, page 677], \( C_\infty(V) \) lie in an algebraic region \( \tilde{\Omega} \) of type \((k, n)\) and by Theorem 2.9, \( V \) and \( C_\infty(V) \) lie in a common algebraic region \( \Omega \) of type \((k, n)\). Then, it follows from Proposition 2.4 that \( \dim V \leq k = \dim C_\infty(V) \). \( \square \)

The previous result can be seen as a version at infinity of a result of Whitney on tangent cones at point of an analytic variety; see for instance Lemma 8.11 of [15].

### 3. Proof of main theorem

We recall the definition of degree of an algebraic set. We follow section 11.3 of [3]. The complex projective space is denoted by \( \mathbb{P}^n \) and \( \mathbb{C}^n \) is identified with the open set \( \{x_0 : x_1 : \ldots : x_n \in \mathbb{P}^n \mid x_0 \neq 0\} \) of \( \mathbb{P}^n \) through the mapping \( \tau : \mathbb{C}^n \to \mathbb{P}^n, \tau(x_1, \ldots, x_n) = [1 : x_1 : \ldots : x_n] \).
Definition 3.1. Let $X \subset \mathbb{C}^n$ be an algebraic set. The degree of $X$, denoted by $\deg X$, is the degree of its closure in $\mathbb{P}^n$.

In the proof of Theorem 1.4 we consider the following. Let $X \subset \mathbb{C}^n$ be a pure $k$-dimensional algebraic set. We have that $X$ has a unique tangent cone at infinity and it is an algebraic set; see Remark 2.8. From Proposition 2.10 and Theorem 2.9 there exists an algebraic region $\Omega$ of type $(k, n)$ such that $X$ and $C_\infty(X)$ lie in $\Omega$. Then, there exist vector spaces $V_1, V_2 \subset \mathbb{C}^n$ and constants $A, B$ as in Definition 2.1. We may choose linear coordinates $(x, y)$ in $\mathbb{C}^n$ such that $x \in V_1$ and $y \in V_2$. In this coordinate system, we denote by $\pi_\Omega$ the canonical projection from $\mathbb{C}^n$ to $V_1$, i.e., $\pi_\Omega(x, y) = x$. Moreover, we may assume that $V_1$ and $V_2$ are orthogonal to each other and that $\pi_\Omega$ is an orthogonal projection; see Theorem 3 of page 78 of [3]. By similar argument as in proof of Lemma 3.2, it follows that the closures of $X$ and $V_2$ (respectively, the closures of $C_\infty(X)$ and $V_2$) in $\mathbb{P}^n$ do not have points at infinity in common. Then, it follows by Corollary 1 of [3, p. 126] the following fact concerning the projection $\pi_\Omega$:

Lemma 3.2. The restriction $\pi_\Omega : X \to V_1$ (respectively, $\pi_\Omega : C_\infty(X) \to V_1$) is a ramified covering with number of sheets equal to $\deg X$ (respectively, $\deg C_\infty(X)$).

\[
\square
\]

3.1. Proof of Theorem 1.4. First, we prove that $X$ is an algebraic set and that $\dim X = \dim C_\infty(X)$. By Theorem 2.2, $C_\infty(X)$ lies in an algebraic region of type $(k, n)$. It follows from Theorem 2.9 that $X$ and $C_\infty(X)$ lie in an algebraic region $\Omega$ of type $(k, n)$. Then, by Theorem 2.2 we have that $X$ is an algebraic set. That $\dim X = \dim C_\infty(X)$, it follows by Proposition 2.10.

Now, we assume that $X$ is Lipschitz normally embedded at infinity. Thus, there exist a compact subset $K \subset X$ and a positive real number $C$ such that

\[ d_X(x_1, x_2) \leq C\|x_1 - x_2\|, \text{ for all } x_1, x_2 \in X \setminus K. \]

We denote $\deg X = d_1$ and $\deg C_\infty(X) = d_2$.

From Theorem 2.9 and Proposition 2.10 there exists an algebraic region $\Omega$ of type $(k, n)$ containing $X$ and $C_\infty(X)$. We may consider $\Omega$ as in the paragraph before Lemma 3.2. Thus, there exist vector spaces $V_1$ and $V_2$ in $\mathbb{C}^n$, with $\dim V_1 = k$, $\dim V_2 = n - k$, $\mathbb{C}^n = V_1 + V_2$, and positive real numbers $A, B$ such that for any $z \in (X \cup C_\infty(X))$ the following holds:

\[ \|z''\| < A(1 + \|z'\|)^B, \]

where $z = z' + z''$, with $z' \in V_1$ and $z'' \in V_2$.

Let $\pi_\Omega : \mathbb{C}^n \to V_1$ be the projection. As in the paragraph before Lemma 3.2, we may assume that $V_1, V_2$ are orthogonal to each other and that $\pi_\Omega$ is an orthogonal projection.

It follows by Lemma 3.2 that the restriction $\pi_\Omega : X \to V_1$ (respectively, $\pi_\Omega : C_\infty(X) \to V_1$) is a ramified covering with number of sheets equal to $d_1$ (respectively, $d_2$).
Let $\Sigma_1 \subset V_1$ and $\Sigma_2 \subset V_1$ be the ramification locus of the restriction of $\pi_\Omega$ to $X$ and to $C_\infty(X)$ respectively.

We have that $\Sigma_i$ is a complex algebraic set such that $\dim \Sigma_i < k$, for $i = 1, 2$. Then, there exists a vector $v \in V_1 \setminus (C_\infty(\Sigma_1) \cup C_\infty(\Sigma_2))$. In particular, for big enough $t \in \mathbb{R}_{> 0}$, we have $tv \notin (\Sigma_1 \cup \Sigma_2)$.

Let $\lambda_1(t), \ldots, \lambda_d_1(t) \in X$ be the different liftings of $tv$ by $\pi_\Omega$ restricted to $X$, i.e., $\pi_\Omega(\lambda_i(t)) = tv$, $i = 1, \ldots, d_1$.

Since $\mathbb{C}^n = V_1 + V_2$, we may write $\lambda_i(t) = tv + y_i(t)$, with $y_i(t) \in V_2$, for $i = 1, \ldots, d_1$.

We have that $\frac{||y_i(t)||}{t}$ is bounded when $t \to \infty$, for $i = 1, \ldots, d_1$. Otherwise, there exists an index $i \in \{1, \ldots, d_1\}$ and there exists a sequence of positive numbers $\{t_l\}_{l \in \mathbb{N}}$ such that $\frac{||y_i(t_l)||}{t_l} > l$, for any $l$. We may assume that $\lim_{l \to \infty} \frac{y_i(t_l)}{||y_i(t_l)||} = w_i$, with $w_i \in V_2$. It follows that

$$\lim_{l \to \infty} \frac{A}{||y_i(t_l)||} \lambda_i(t_l) = A \lim_{l \to \infty} \frac{t_l v + y_i(t_l)}{||y_i(t_l)||} = 0 + A w_i.$$

It follows by definition of $C_\infty(X)$ and (3) that $Aw_i \in C_\infty(X)$ and, therefore, $Aw_i \in (C_\infty(X) \cap V_2)$. This contradicts $C_\infty(X) \subset \Omega$.

Therefore, for $i \in \{1, \ldots, d_1\}$, we have that $\frac{||y_i(t)||}{t}$ is bounded when $t \to \infty$. Thus, we may assume that there exists a sequence of real positive numbers $\{t_j\}_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} t_j = \infty$ and, for $i = 1, \ldots, d_1$, $\lim_{j \to \infty} \frac{y_i(t_j)}{t_j} = v_i$, with $v_i \in V_2$.

Then,

$$\lim_{j \to \infty} \frac{1}{t_j} \lambda_i(t_j) = \lim_{j \to \infty} \left( \frac{t_j v + y_i(t_j)}{t_j} \right) = v + v_i, \text{ for } i = 1, \ldots, d_1.$$

By definition of $C_\infty(X)$, we have $v + v_i \in C_\infty(X)$, for $i = 1, \ldots, d_1$. It follows that the curves $\beta_i(t) = t(v + v_i)$ lie in $C_\infty(X)$, for $i = 1, \ldots, d_1$; see Remark 2.8. Therefore, $\beta_1, \ldots, \beta_{d_1}$ are liftings of $tv$ by $\pi_\Omega$ restricted to $C_\infty(X)$, i.e., $\pi_\Omega(\beta_i(t)) = tv$, $i = 1, \ldots, d_1$.

If we suppose that $d_1 > d_2$, then there exists two distinct indexes $s_1, s_2 \in \{1, \ldots, d_1\}$ such that $\beta_{s_1} = \beta_{s_2}$, which implies $v_{s_1} = v_{s_2}$ by definitions of $\beta_{s_1}$ and $\beta_{s_2}$. Then, by (7), it follows that

$$\lim_{j \to \infty} \frac{1}{t_j} (\lambda_{s_1}(t_j) - \lambda_{s_2}(t_j)) = 0.$$

Since $v \in V_1 \setminus (C_\infty(\Sigma_1) \cup C_\infty(\Sigma_2))$, it follows from Lemma 2.4 of [13] that there exists $\delta > 0$ such that, for $j$ big enough, the following hold:

$$\text{dist}(t_j v, \Sigma_1) > \delta t_j.$$

Now, we follow Lemma 2.2 of [4] and proof of Theorem 1.1 of [6]. For any path $\omega$ in $X$ joining $\lambda_{s_1}(t_j)$ to $\lambda_{s_2}(t_j)$, the path $\pi_\Omega(\omega)$ is a loop in $V_1$ with base point $t_j v$ which is not contractible in $V_1 \setminus \Sigma_1$. Then, with (9), it follows that the length of $\pi_\Omega(\omega)$ is at least $2\delta t_j$. Now, since $\pi_\Omega$ is an orthogonal projection, it follows that
\begin{equation}
\label{eq:10}
d_X(\lambda_{s_1}(t_j), \lambda_{s_2}(t_j)) \geq 2\delta t_j.
\end{equation}

Then, by (\ref{eq:4}) and (\ref{eq:10}), we have:

\begin{equation}
\label{eq:11}
\frac{1}{t_j} C \|\lambda_{s_1}(t_j) - \lambda_{s_2}(t_j)\| \geq \frac{1}{t_j} d_X(\lambda_{s_1}(t_j), \lambda_{s_2}(t_j)) \geq 2\delta, \text{ for } j \text{ big enough.}
\end{equation}

From (\ref{ineq:3}) and (\ref{ineq:11}), as \( j \to \infty \), it follows the following contradiction:

\[ 0 \geq 2\delta. \]

Therefore, we have \( d_1 \leq d_2 \). Now, as mentioned in the Introduction, \( d_1 \geq d_2 \) is always true for any pure dimensional complex algebraic sets \( X \) and \( C_\infty(X) \); see equality (2.1) of \cite{Fernandez2018}. This ends the proof. \( \square \)

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