INTEGRALS OF MOTION AND QUANTUM GROUPS

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CONTENTS

1. Introduction

2. Classical Toda field theories associated to finite-dimensional simple Lie algebras
   2.1. The case of $\mathfrak{sl}_2$ – classical Liouville theory
   2.2. General case
   2.3. BGG resolution
   2.4. Extended complex and its cohomology

3. Classical affine Toda field theories
   3.1. The case of $\hat{\mathfrak{sl}}_2$ – classical sine-Gordon theory
   3.2. General case

4. Quantum Toda field theories
   4.1. Vertex operator algebras
   4.2. The VOA of the Heisenberg algebra
   4.3. Quantum integrals of motion
   4.4. Liouville theory
   4.5. Quantum groups and quantum BGG resolutions
   4.6. Toda field theories associated to finite-dimensional simple Lie algebras
   4.7. Affine Toda field theories
   4.8. Concluding remarks

References

1. Introduction.

1.1. In these lectures we propose a new approach to the study of local integrals of motion in the classical and quantum Toda field theories. Such a theory is associated to a Lie algebra $\mathfrak{g}$, which is either a finite-dimensional simple Lie algebra or an affine Kac-Moody algebra.

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The classical Toda field theory, associated to $\mathfrak{g}$, revolves around the system of equations
\begin{equation}
\partial_r \partial_t \phi_i(t, \tau) = \frac{1}{2} \sum_{j \in S} (\alpha_i, \alpha_j) \exp[\phi_j(t, \tau)], \quad i \in S
\end{equation}
(1.1.1)
where each $\phi_i(t, \tau)$ is a family of functions in $t$, depending on the time variable $\tau$, $S$ is the set of simple roots of $\mathfrak{g}$, and $(\alpha_i, \alpha_j)$ is the scalar product of the $i$th and $j$th simple roots. The simplest examples of the Toda equations are the Liouville equation
\begin{equation}
\partial_r \partial_t \phi(t, \tau) = e^{\phi(t, \tau)},
\end{equation}
(1.1.2)
corresponding to $\mathfrak{g} = \mathfrak{sl}_2$, and the sine-Gordon equation
\begin{equation}
\partial_r \partial_t \phi(t, \tau) = e^{\phi(t, \tau)} - e^{-\phi(t, \tau)},
\end{equation}
(1.1.3)
corresponding to $\mathfrak{g} = \hat{\mathfrak{sl}}_2$.

Many aspects of the classical Toda field theories have been studied by both physicists and mathematicians (cf., e.g., [1, 113, 96, 97, 102, 103, 32, 33, 86, 116, 90, 38, 7, 3, 4, 5, 104] and references therein): realization as a zero-curvature equation, complete integrability, soliton solutions, dressing transformations, connection with generalized KdV hierarchies, lattice analogues, etc. There are also many interesting works devoted to the quantum Toda field theory, cf., e.g., [118, 112, 74, 13, 77, 35, 106, 23, 115, 39, 100] and references therein.

1.2. In this paper we will study a Hamiltonian formalism for the Toda field theories. By that we mean constructing a Hamiltonian space $M$ and a hamiltonian $H$, such that the system of equations (1.1.1) can be rewritten in the Hamiltonian form:
\begin{equation}
\partial_r U = \{U, H\}.
\end{equation}
(1.2.1)
Here $\{\cdot, \cdot\}$ stands for a Poisson bracket on the space $F(M)$ of functions on $M$. We will be primarily interested in the integrals of motion for the equation (1.2.1). An integral of motion is an element $X$ of the space $F(M)$, which satisfies the equation
\[ \{X, H\} = 0. \]
It is conserved with respect to the evolution of the Hamiltonian system, defined by the equation (1.2.1).

Note that this definition does not require $H$ to be an element of $F(M)$, it merely requires the Poisson bracket with $H$ to be a well-defined linear operator, acting from $F(M)$ to some other vector space. Given $H$, we can define the space of integrals of motion of the system (1.2.1) as the kernel of this linear operator. If this operator preserves the Poisson bracket, then the space of integrals of motion is itself a Poisson algebra.

1.3. For the Toda equation (1.1.1) we choose as the Hamiltonian space, the space $L \mathfrak{h}$ of polynomial functions on the circle with values in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and as the space of functions, the space $\mathcal{F}_0$ of local functionals on $L \mathfrak{h}$. Such a functional can be presented in the form of the residue
\[ F[u(t)] = \int P(u, \partial_t u, \ldots) dt, \]
where $P$ is a polynomial in the coordinates $u^i(t)$ of $u(t) \in L \mathfrak{h}$ with respect to the basis of the simple roots, and their derivatives (cf. § 3 for the precise definition). The Poisson structure on $L \mathfrak{h}$ has an interpretation as a Kirillov-Kostant structure, because $L \mathfrak{h}$ can be
viewed as a hyperplane in the dual space to the Heisenberg Lie algebra \( \hat{h} \) – the central extension of \( \mathcal{L} \). This defines a Poisson bracket on \( \mathcal{F}_0 \).

The space \( \mathcal{F}_0 \) was one of the first examples of Poisson algebras of functions on infinite-dimensional hamiltonian spaces. It has been studied since the discovery of integrability of the KdV equation and its generalizations, and exhaustive literature is devoted to it, cf., e.g. \([68, 57, 117, 88, 59, 71, 2, 33, 110, 37, 27]\). We essentially follow the approach of Gelfand and Dickey and treat the space \( \mathcal{F}_0 \) in a purely algebraic way. In addition, we also consider the spaces \( \mathcal{F}_{\alpha_1} \), consisting of functionals of the form

\[
\int P(u, \partial_t u, \ldots) e^{\phi_i(t)} dt,
\]

where \( \phi_i(t) \) is such that \( \partial_t \phi_i(t) = u^i(t) \). It is possible to extend the Poisson bracket \( \mathcal{F}_0 \times \mathcal{F}_0 \to \mathcal{F}_0 \) to a bilinear map \( \mathcal{F}_0 \times \mathcal{F}_{\alpha_i} \to \mathcal{F}_{\alpha_i} \), cf. \([50, 113, 33]\), and this allows to write the Toda equation (1.1.1) in the Hamiltonian form

\[
\partial_t u(t) = \{u(t), H\}.
\]

Here the hamiltonian \( H \) is given by

\[
H = \frac{1}{2} \sum_i \int e^{\phi_i(t)} dt.
\]

It is an element of \( \oplus_i \mathcal{F}_{\alpha_i} \), and the Poisson bracket with \( H \) is a well-defined linear operator, acting from \( \mathcal{F}_0 \) to \( \oplus_i \mathcal{F}_{\alpha_i} \).

So we can define the space of local integrals of motion of the Toda equation (1.1.1) as the kernel of the operator \( \{\cdot, H\} \), or, in other words, as the intersection of the kernels of the operators \( Q_i = \{\cdot, \int e^{\phi_i(t)} dt\} : \mathcal{F}_0 \to \mathcal{F}_{\alpha_i} \). These operators preserve the Poisson structure, and hence the space of integrals of motion is a Poisson subalgebra of \( \mathcal{F}_0 \).

1.4. The crucial observation, which will enable us to compute this space, is that, roughly speaking, the operators \( \bar{Q}_i \) satisfy the Serre relations of the Lie algebra \( \mathfrak{g} \), or, in other words, they generate the nilpotent subalgebra \( \mathfrak{n}_+ \) of \( \mathfrak{g} \). Using this fact, we will be able to interpret the space of local integrals of motion as a cohomology space of a certain complex \( F^*(\mathfrak{g}) \). To construct this complex, we will use the so-called Bernstein–Gelfand–Gelfand (BGG) resolution, which is the resolution of the trivial representation of \( \mathfrak{n}_+ \) by Verma modules. The cohomologies of this complex coincide with the cohomologies of \( \mathfrak{n}_+ \) with coefficients in some \( \mathfrak{n}_+ \)-module.

More precisely, we can lift the operators \( \bar{Q}_i \) to certain linear operators \( Q_i \), acting on the space \( \pi_0 \) of differential polynomials in \( u^i(t) \). These operators give us an action of the Lie algebra \( \mathfrak{n}_+ \) on \( \pi_0 \).

In the case, when \( \mathfrak{g} \) is a finite-dimensional simple Lie algebra, the 0th cohomology of \( \mathfrak{n}_+ \) with coefficients in \( \pi_0 \) can be identified with the space of differential polynomials in \( W^{(1)}, \ldots, W^{(l)} \in \pi_0 \) of degrees \( d_1 + 1, \ldots, d_l + 1 \), respectively. Here the \( d_i \)'s are the exponents of \( \mathfrak{g} \), and the grading is defined on \( \pi_0 \) in such a way that the degree of \( \partial_t^i u^i(t) \) is equal to \( n + 1 \). We prove that the integrals of motion of the Toda field theory, corresponding to \( \mathfrak{g} \), coincide with all residues of differential polynomials in the \( W^{(i)} \)'s. They form a Poisson subalgebra of \( \mathcal{F}_0 \), which is called the Adler–Gelfand–Dickey algebra, or the classical \( \mathcal{W} \)-algebra, associated with the Lie algebra \( \mathfrak{g} \).

The space of integrals of motion of the affine Toda field theory, associated to an affine algebra \( \mathfrak{g} \), can be identified with the first cohomology of the nilpotent subalgebra \( \mathfrak{n}_+ \) of \( \mathfrak{g} \) with coefficients in \( \pi_0 \). This space is naturally embedded into the space of integrals.
of motion of the Toda theory, associated to the finite-dimensional Lie algebra $\mathfrak{g}$, whose Dynkin diagram is obtained by deleting the 0th nod of the Dynkin diagram of $\mathfrak{g}$ (or any other nod).

We can compute the latter cohomology and obtain the well-known result that the integrals of motion of the affine Toda theory have degrees equal to the exponents of $\mathfrak{g}$ modulo the Coxeter number. These integrals of motion commute with each other. This is especially easy to see in the case when all the exponents of the corresponding affine algebra are odd, and the Coxeter number is even (this excludes $A_n^{(1)}$, $n > 1$, $D_n^{(1)}$, $E_6^{(1)}$ and $E_7^{(1)}$).

In such a case the degrees of all integrals of motion are odd, so that the Poisson bracket of any two of them should be an integral of motion of an even degree, and hence should vanish. The set of local integrals of motion of the affine Toda field theory, associated to $\mathfrak{g}$, coincides with the set of hamiltonians of the corresponding generalized KdV system. These integrals of motion generate a maximal abelian subalgebra in the Poisson algebra of integrals of motion of the corresponding finite-dimensional Toda field theory.

In our next paper [51] we explain further the geometric meaning of higher KdV hamiltonians. Namely, we will identify the vector space with the coordinates $\partial^n u^i, i = 1, \ldots, l, n \geq 0$, with a homogeneous space of the nilpotent subgroup $N_+$ of the corresponding affine group. This homogeneous space is the quotient of the group $N_+$ by its principal commutative subgroup – the Lie group of $\mathfrak{a}$. The vector fields on this space, which, by Gelfand-Dickey formalism correspond to the KdV hamiltonians, coincide with the vector fields of the infinitesimal action of the opposite principal abelian subalgebra $\mathfrak{a}_- \subset \mathfrak{n}_-$ on this homogeneous space. In particular, this identification enables us to prove the mutual commutativity of hamiltonians in general case.

1.5. Thus we obtain an interpretation of the integrals of motion of the classical Toda field theories as cohomologies of certain complexes. This formulation not only allows us to describe the spaces of classical integrals of motion, but also to prove the existence of their quantum deformations.

The quantum integrals of motion are defined as elements of the quantum Heisenberg algebra, which is a quantization of the Poisson algebra $\mathcal{F}_0$. More precisely, we define a Lie algebra $\mathcal{F}_0^\beta$ of all Fourier components of vertex operators from the vertex operator algebra of the Heisenberg algebra, cf. § 4.2. For completeness, we include in § 4.1 a survey of vertex operator algebras, which closely follows § 3 of [58].

The Lie bracket in $\mathcal{F}_0^\beta$ is polynomial in the deformation parameter $\beta^2$ with zero constant term, and the linear term coincides with the Poisson bracket in $\mathcal{F}_0$, so that $\mathcal{F}_0^\beta$ degenerates into $\mathcal{F}_0$ when $\beta \to 0$. We can also interpret the operators $\bar{Q}_i$ as classical limits of integrals of bosonic vertex operators, $Q_i^\beta$. Therefore it is natural to define the space of quantum integrals of motion of the affine Toda field theory as the intersection of kernels of the operators $\bar{Q}_i^\beta$.

Thus, if $x = x^{(0)} + \beta^2 x^{(1)} + \cdots \in \mathcal{F}_0^\beta$ is a quantum integral of motion, then $x^{(0)}$ is a classical integral of motion. It remains to be seen however, whether for each classical integral of motion there exists its quantum deformation, and whether such deformed integrals of motion commute with each other.

Such quantum deformations do not necessarily exist in general. Indeed, if we have a family of linear operators acting between two vector spaces, then the dimension of the kernel may increase for a special value of parameter. Thus, the space of classical integrals of motion, which is defined as the kernel of the operator $\sum_i \bar{Q}_i^\beta$ for the special value $\beta = 0$,
may well be larger than the space of quantum integrals of motion, which is defined as the kernel of the operator \( \sum_i \bar{Q}_i^\beta \) for generic values of \( \beta \).

1.6. In order to prove the existence of quantum integrals of motion we will use higher cohomologies. The usefulness of higher cohomologies can be illustrated by the following toy example.

Suppose, we have two finite-dimensional vector spaces, \( A \) and \( B \), and a family of linear operators \( \phi_\beta \) depending on a parameter \( \beta \). Assume that for \( \beta = 0 \) the 1st cohomology of the complex \( A \to B \) (\( = \) the cokernel of the operator \( \phi_0 \)) is equal to 0. One can show then that the 0th cohomology of this complex (\( = \) the kernel of \( \phi_0 \)) can be deformed.

Indeed, vanishing of the 1st cohomology of the complex \( A \to B \) for \( \beta = 0 \) entails vanishing of the 1st cohomology for generic \( \beta \), because the dimension of cohomology stays the same for generic values of parameter, and it may only increase for special values. But the Euler characteristics of our complex, i.e. the difference between the dimension of the kernel and the dimension of the cokernel, is also equal to \( \dim A - \dim B \) and hence does not depend on \( \beta \). Since the 1st cohomology vanishes for generic \( \beta \) and \( \beta = 0 \), we see that the dimension of the 0th cohomology for generic \( \beta \) is the same as for \( \beta = 0 \).

We can apply this idea in our situation. Although our spaces are infinite-dimensional, they are \( \mathbb{Z} \)-graded with finite-dimensional homogeneous components, and our operator \( H \) preserves the grading. So, our infinite-dimensional linear problem splits into a set of finite-dimensional linear problems. In the simplest case of \( g = sl_2 \), these finite-dimensional problems can be solved in the same way as in the example above. By proving vanishing of the cokernel of our operator \( \bar{Q}_1 \), we can prove that all classical local integrals of motion in the Liouville theory can be quantized. The quantum algebra of integrals of motion in this case is the quantum Virasoro algebra, which is a well-known fact.

In general, the cokernel of our operator is not equal to 0 for \( \beta = 0 \), so that this simple trick does not work. However, a deformation of our extended complex \( F^*(g) \) will do the job. Roughly speaking, it turns out that the operators \( \bar{Q}_i^\beta \) generate the quantized universal enveloping algebra \( U_q(n_+) \) of the nilpotent subalgebra \( n_+ \) with \( q = \exp(\pi i \beta^2) \). A quantum analogue of the BGG resolution will allow us to construct such a deformed complex, \( F^*_\beta(g) \), for generic \( \beta \) in the same way as for \( \beta = 0 \).

1.7. In the case when \( g \) is finite-dimensional, we prove that all higher cohomologies of our complex vanish when \( \beta = 0 \), so they also vanish for generic \( \beta \). Using Euler characteristics we then prove that all classical integrals of motion of the corresponding Toda field theory can be quantized. These quantum integrals of motion are Fourier components of vertex operators from a certain vertex operator algebra – the so-called \( W \)-algebra. A comprehensive review of the theory of \( W \)-algebras and references can be found in [20].

The \( W \)-algebra, corresponding to \( g = sl_2 \), is the vertex operator algebra of the Virasoro algebra. The \( W \)-algebra, corresponding to \( g = sl_3 \), was constructed by Zamolodchikov [119]. For \( A \) and \( D \) series of simple Lie algebras the \( W \)-algebras were constructed by Fateev and Lukyanov [10, 11, 12]. We give a general proof of the existence of \( W \)-algebras, associated to arbitrary finite-dimensional simple Lie algebras. A similar construction also appeared in [99] for \( g = sl_n \).

The \( W \)-algebra can also be defined by means of quantum Drinfeld-Sokolov reduction [13, 18, 59] as the 0th cohomology of a certain BRST complex. The definition via quantum Drinfeld-Sokolov reduction is related to the definition in this paper, because the first term of a spectral sequence associated to the BRST complex coincides with our quantum
complex $F_\beta^*(\mathfrak{g})$ for generic values of $\beta$ \cite{48, 57}. Therefore the cohomologies of the BRST complex and the complex $F_\beta^*(\mathfrak{g})$ coincide for generic $\beta$. Recently these cohomologies were computed using the opposite spectral sequence of the BRST complex \cite{23}. This gives an alternative proof of existence of $\mathcal{W}$--algebras.

If $\mathfrak{g}$ is affine, our extended complex has non-trivial higher cohomologies when $\beta = 0$. However, in the case when all exponents of $\mathfrak{g}$ are odd and the Coxeter number is even, we can still derive that all cohomology classes can be deformed, from the Euler character argument. We then immediately see that the corresponding quantum integrals of motion commute with each other. In the remaining cases we can prove these results by a more refined argument. When $\mathfrak{g} = \hat{\mathfrak{sl}}_2$, we obtain a proof of the existence of quantum KdV hamiltonians. In this setting, it was conjectured in \cite{72} and some partial results were obtained in \cite{120, 35, 85, 29, 30, 108}.

1.8. The $\mathcal{W}$--algebra is the chiral algebra of a certain two-dimensional conformal field theory; this explains its importance for quantum field theory. Our construction gives a realization of this chiral algebra in terms of the simplest chiral algebra, the chiral algebra of the free fields. Such a realization, which is usually referred to as a free field realization, is very important for computation of the correlation functions in the corresponding models of quantum field theory. The free field realization is an attempt to immerse a complicated structure into a simpler one and then give the precise description of the image of the complicated structure by finding the constraints, to which it satisfies inside the simple one. Our construction resembles the construction of the Harish-Chandra homomorphism, which identifies the center of the universal enveloping algebra of a simple Lie algebra $\mathfrak{g}$ (complicated object) with the polynomials on the Cartan subalgebra $\mathfrak{h}$ -- an abelian Lie algebra (simple object), which are invariant with respect to the action of the Weyl group. In other words, the center can be described as the subalgebra in the algebra of polynomials on $\mathfrak{h}$, which are invariant with respect to the simple reflections $s_i$. Likewise, we have been able to embed the $\mathcal{W}$--algebra of $\mathfrak{g}$ into the vertex operator algebra of the Heisenberg algebra $\hat{\mathfrak{h}}$, which is something like the algebra of polynomials on $L\mathfrak{h}$. The image of this embedding coincides with the kernel of the operators $\bar{Q}_i^\beta$, which play the role of simple reflections from the Weyl group.

The quantum integrals of motion of the affine Toda field theory, associated to an affine algebra $\mathfrak{g}$, constitute an infinite-dimensional abelian subalgebra in the $\mathcal{W}$--algebra, associated to the finite-dimensional Lie algebra $\hat{\mathfrak{g}}$. This abelian subalgebra consists of the local integrals of motion of a deformation of this conformal field theory. The knowledge of the existence of infinitely many integrals of motion and of their degrees (or spins) is very important for understanding this non-conformal field theory, and in many cases it allows to construct the S-matrix of this theory explicitly \cite{120}.

Some of the results of this paper have previously appeared in our papers \cite{45, 48, 57, 50}. An earlier version of this paper appeared in September of 1993 as a preprint YITP/K-1036 of Yukawa Institute of Kyoto University, and also as hep-th/9310022 on hep-th computer net.

2. Classical Toda field theories associated to finite-dimensional simple Lie algebras.

2.1. The case of $\mathfrak{sl}_2$ -- classical Liouville theory.
2.1.1. **Hamiltonian space.** Denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{sl}_2$ — the one-dimensional abelian Lie algebra, and by $L\mathfrak{h}$ the abelian Lie algebra of polynomial functions on the circle with values in $\mathfrak{h}$. This will be our hamiltonian space. It is isomorphic to the space of Laurent polynomials $\mathbb{C}[t, t^{-1}]$. We would like to introduce a suitable space of functions on $L\mathfrak{h}$, which we will denote by $\mathcal{F}_0$, together with a Poisson bracket.

First let us introduce the space $\pi_0$ of differential polynomials, i.e. the space of polynomials in variables $u, \partial u, \partial^2 u, \ldots$. It is equipped with an action of derivative $\partial$, which sends $\partial^n u$ to $\partial^{n+1} u$ and satisfies the Leibnitz rule.

We define $\mathcal{F}_0$ as the space of local functionals on $L\mathfrak{h}$. A local functional $F$ is a functional, whose value at a point $u(t) \in L\mathfrak{h}$ can be represented as the formal residue

$$F[u(t)] = \int P(u(t), \partial_t u(t), \ldots) dt,$$

where $P \in \pi_0$ is a differential polynomial, and $\partial_t = \partial/\partial t$. In words: we insert $u(t), \partial_t u(t), \ldots$ into $P$; this gives us a Laurent polynomial, and we take its residue, i.e. the $(-1)$st Fourier component.

We can represent local functionals as series of the form

$$\sum_{i_1 + \cdots + i_m = -m+1} c_{i_1 \ldots i_m} \cdot u_{i_1} \ldots u_{i_m}, \tag{2.1.1}$$

where the coefficients $c_{i_1 \ldots i_m}$ are polynomials in $i_1, \ldots, i_m$. Here $u_i$'s are the Fourier components of $u(t) : u(t) = \sum_{i \in \mathbb{Z}} u_i t^{-i-1}$. For example, $\int u(t) dt = u_0$, $\int u(t)^2 dt = \sum_{i+j=-1} u_i u_j$.

Note that since we deal with Laurent polynomials, only finitely many summands of the series (2.1.1) can be non-zero for a given $u(t)$.

We have a map $\int : \pi_0 \to \mathcal{F}_0$, which sends $P \in \pi_0$ to $\int P dt \in \mathcal{F}_0$. The following Lemma, for the proof of which cf. [88, 69], shows that the kernel of the residue map consists of total derivatives and constants.

**2.1.2. Lemma.** The sequence

$$0 \longrightarrow \pi_0/\mathbb{C} \xrightarrow{\partial} \pi_0/\mathbb{C} \xrightarrow{\int} \mathcal{F}_0 \longrightarrow 0$$

is exact.

2.1.3. **Poisson bracket.** We can now define the Poisson bracket of two local functionals $F$ and $G$, corresponding to two differential polynomials, $P$ and $R$, as follows:

$$\{F, G\}[u(t)] = -\int \frac{\delta P}{\delta u} \frac{\delta R}{\delta u} dt, \tag{2.1.2}$$

where

$$\frac{\delta P}{\delta u} = \frac{\partial P}{\partial u} - \partial_t \frac{\partial P}{\partial (\partial u)} + \partial^2 \frac{\partial P}{\partial (\partial^2 u)} - \ldots$$

denotes the variational derivative. Note that the variational derivative of a differential polynomial, which is a total derivative, is equal to 0, and so formula (2.1.2) defines a well-defined bracket map $\mathcal{F}_0 \times \mathcal{F}_0 \to \mathcal{F}_0$. 
This bracket satisfies all axioms of the Lie bracket and so it defines a structure of Lie algebra on $F_0$. One can prove this in terms of differential polynomials \[88, 69\], or in terms of Fourier components \[67\]. We will recall the latter proof following \[95\], §§7.21-7.23, since we will use it later in §4.

Note that we can extend our Poisson algebra of local functionals $F_0$ by adjoining all Fourier components of differential polynomials, not only the $(−1)$st ones. Let $\hat{F}_0$ be the space of functionals on $Lh$, which can be represented as residues of differential polynomials with explicit dependence on $t$:

$$F[u(t)] = \int P(\partial^n u(t); t)dt.$$ 

We can define a Poisson bracket on $\hat{F}_0$ by the same formula (2.1.2). We will prove now that this bracket makes $\hat{F}_0$ into a Lie algebra. This will imply that $F_0$ is a Lie algebra as well, because the bracket of two elements of $F_0$ is again an element of $F_0$.

Any element of $\hat{F}_0$ can be presented as a finite linear combination of the infinite series of the form

$$(2.1.3) \sum_{i_1 + \ldots + i_m = N} c_{i_1 \ldots i_m} \cdot u_{i_1} \ldots u_{i_m},$$

where the $u_i$'s are the Fourier components of $u(t) = \sum_{i \in \mathbb{Z}} u_i t^{−i−1}$, $c_{i_1 \ldots i_m}$ is a polynomial in $i_1, \ldots, i_m$, and $N$ is an arbitrary integer.

We can consider these elements as lying in a certain completion $\bar{A}$ of the polynomial algebra $A = \mathbb{C}[u_n]_{n \in \mathbb{Z}}$. In order to define this completion, introduce a $\mathbb{Z}$-grading on $A$ by putting $\deg u_n = −n$. We have

$$A = \bigoplus_{N \in \mathbb{Z}, m \geq 0} A_{N,m},$$

where $A_{N,m}$ is the linear span of monomials of degree $N$ and power $m$. Denote by $I^M, M > 0$ the ideal of $A$ generated by $u_n, n \geq M$ and by $I^M_{N,m}$ the intersection $I^M \cap A_{N,m}$. Let $\hat{A}_{N,m}$ be the completion of $A_{N,m}$ with respect to the topology, generated by the open sets $I^M_{N,m}, M > 0$. Clearly,

$$\bar{A} = \bigoplus_{N \in \mathbb{Z}, m \geq 0} \hat{A}_{N,m}$$

is a commutative algebra. It consists of finite linear combinations of infinite series of the form (2.1.3), where $c_{i_1 \ldots i_m}$ is an arbitrary function of $i_1, \ldots, i_m$. In particular, we have an embedding $\hat{F}_0 \to \bar{A}$.

Now, $u_i, i \in \mathbb{Z},$ are elements of $\hat{F}_0$ and $\bar{A}$. We find from formula (2.1.2):

$$(2.1.4) \{u_n, u_m\} = n\delta_{n,−m}.$$ 

By the Leibnitz rule

$$(2.1.5) \{xy, z\} = \{x, z\} + \{y, z\},$$

we can extend this bracket to a bracket, defined for any pair of monomials in $u_n$. We can then formally apply this formula by linearity to any pair of elements of $\bar{A}$, using their presentation in the form (2.1.3). This will give us a well-defined bracket $[\cdot, \cdot, \cdot]$ on $\bar{A}$. One can check directly that the restriction (2.1.4) of this formula to the subspace $\bigoplus_{n \in \mathbb{Z}} \mathbb{C}u_n$ is antisymmetric and satisfies the Jacobi identity. Therefore, by construction, $[\cdot, \cdot]$ is antisymmetric and satisfies the Jacobi identity on the whole $\bar{A}$. Thus $[\cdot, \cdot]$ defines a Lie algebra structure on $\bar{A}$.
It is shown in [95], §7.23 that the restriction of the Lie bracket \([\cdot, \cdot]\) to \(\hat{\mathcal{F}}_0\) coincides with the bracket (2.1.2) (for instance, this is clear for the subspace \(\oplus_{n \in \mathbb{Z}} \mathbb{C}u_n\) of \(\hat{\mathcal{F}}_0\)). Therefore \(\hat{\mathcal{F}}_0\) is a Lie algebra, and \(\mathcal{F}_0\) is its Lie subalgebra.

2.1.5. Remark. A Poisson algebra is usually defined as an object, which carries two structures: associative commutative product and a Lie bracket, which are compatible in the sense that the Leibnitz rule (2.1.5) holds. It is clear that the product of two local functionals is not a local functional, so that \(\mathcal{F}_0\) and \(\hat{\mathcal{F}}_0\) are not Poisson algebras in the usual sense, but merely Lie algebras (however, \(\hat{\mathcal{A}}\) is a Poisson algebra in the usual sense). We could, of course, take the algebra of all polynomials in local functionals and extend our Poisson bracket to it by the Leibnitz rule. This would give us a Poisson algebra in the usual sense. But this would not make any difference for us, because we will never use the product structure, only the Lie bracket. For this reason we will work with \(\mathcal{F}_0\) and \(\hat{\mathcal{F}}_0\), but we will still refer to them as Poisson algebras and will call the bracket (2.1.2) the Poisson bracket, because of another meaning of the term “Poisson structure” – as the classical limit of some quantum structure. In §4 we will define this quantum structure.

2.1.6. Kirillov-Kostant structure. The Poisson structure on \(\mathcal{F}_0\), defined above, has a nice interpretation as a Kirillov-Kostant structure.

Let us introduce an anti-symmetric scalar product \(\langle \cdot, \cdot \rangle\) on \(L\mathfrak{h}\):

\[
\langle u(t), v(t) \rangle = \int u(t) dv(t).
\]

Note that this scalar product does not depend on the choice of coordinate \(t\) on the circle and that its kernel consists of constants. Using this scalar product, we can define a Heisenberg Lie algebra \(\hat{\mathfrak{h}}\) as the central extension of \(L\mathfrak{h}\) by the one-dimensional center with generator \(I\). The commutation relations in \(\hat{\mathfrak{h}}\) are

\[
[u(t), v(t)] = \langle u(t), v(t) \rangle I, \quad [u(t), I] = 0.
\]

In the natural basis \(b_j = t^{-j}, j \in \mathbb{Z}\), of \(L\mathfrak{h}\) they can be rewritten as

(2.1.6) \[
[b_n, b_m] = n\delta_{n,-m}I, \quad [b_n, I] = 0.
\]

The (restricted) dual space \(\hat{\mathfrak{h}}^*\) of \(\hat{\mathfrak{h}}\) consists of pairs \((y(t)dt, \mu)\), which define linear functionals on \(\hat{\mathfrak{h}}\) by formula

\[
(y(t)dt, \mu)[u(t) + \nu I] = \mu \nu + \int u(t)y(t)dt.
\]

One has the Kirillov-Kostant Poisson structure on \(\hat{\mathfrak{h}}^*\). Since \(I\) generates the center of \(\hat{\mathfrak{h}}\), we can restrict this structure to a hyperplane \(\hat{\mathfrak{h}}^*_\mu\), which consists of the linear functionals, taking value \(\mu\) on \(I\).

If we choose a coordinate \(t\) on the circle, then we can identify \(\hat{\mathfrak{h}}^*_\mu\) with the space \(L\mathfrak{h}\). This gives us a Poisson bracket on various spaces of functionals on \(L\mathfrak{h}\), e.g., on the space \(A\) of polynomial functionals or its completion \(\hat{A}\). Formulas (2.1.4) and (2.1.6) show that this bracket coincides with the bracket \([\cdot, \cdot]\), defined in §2.1.4. The restrictions of this Poisson bracket to \(\hat{\mathcal{F}}_0\) and \(\mathcal{F}_0\) coincide with the ones, defined by formula (2.1.2).
2.1.7. The action of $F_0$ on $\pi_0$. The Poisson bracket $F_0 \times F_0 \to F_0$, defined by formula (2.1.2), can be thought of as the adjoint action of the Lie algebra $F_0$ on itself. We can lift this action to an action of $F_0$ on the space $\pi_0$ of differential polynomials. Namely, we can rewrite formula (2.1.2) as follows:

$$\{F,G\} = -\int \frac{\delta P}{\delta u} \frac{\delta R}{\partial t} dt = -\int \sum_{n \geq 0} (-\partial_t)^n \frac{\partial P}{\partial (\partial^n u)} \partial_t \frac{\delta R}{\delta u} dt,$$

using the fact that the integral of a total derivative is 0. This suggests to define an action of the functional $G = \int R dt$ on the space $\pi_0 \simeq \mathbb{C}[\partial^n u]_{n \geq 0}$ by the vector field

$$(2.1.7) \quad -\sum_{n \geq 0} \left( \frac{\partial^{n+1} \delta R}{\delta u} \right) \frac{\partial}{\partial (\partial^n u)}.$$

The map $\pi_0 \to \pi_0$, given by this formula, manifestly commutes with the action of the derivative $\partial$, and the projection of this map to a map from $F_0 = \pi_0/(\partial \pi_0 \oplus \mathbb{C})$ to itself coincides with the adjoint action.

We will use the same notation $\{\cdot, G\}$ for the action of $G \in F_0$ on $\pi_0$. Note that the action of $\partial$ on $\pi_0$ coincides with the action of $\frac{1}{2} \int u^2(t) dt$ and that it commutes with the action of any other $G \in F_0$.

2.1.8. Now we define another space, $F_1$, of functionals on $Lh$ and extend our Poisson bracket. We follow [10].

Let $\pi_1$ be the tensor product of the space of differential polynomials $\pi_0$ with a one dimensional space $\mathbb{C} v_1$. Let us define an action of $\partial$ on $\pi_1$ as $(\partial + u) \otimes 1$, where $u$ stands for the operator of multiplication by $u$ on $\pi_0$. Define the space $F_1$ as the cokernel of the homomorphism $\partial : \pi_1 \to \pi_1$. We have the exact sequence:

$$0 \to \pi_1 \to \pi_1 \to F_1 \to 0.$$

To motivate this definition, introduce formally $\phi(t) = \int u(s) ds$, so that $\partial_t \phi(t) = u(t)$. Consider the space of functionals on $Lh$, which have the form

$$(2.1.8) \quad \int P(u(t), \partial u(t), \ldots) e^{\phi(t)} dt.$$

There is a map, which sends $P \otimes v_1 \in \pi_1$ to the functional (2.1.8), so that the action of derivative $\partial_t$ on $P e^{\phi(t)}$ coincides with the action of $\partial$ on $P \in \pi_1$. The kernel of this map consists of the elements of $\pi_1$, which are total derivatives; therefore $F_1$ can be interpreted as the space of functionals of the form (2.1.8).

Note that if $P$ is a differential polynomial in $u$, then

$$-\partial \frac{\delta P}{\delta u} = \frac{\delta P}{\delta \phi}.$$

This formula allows us to extend the Poisson bracket (2.1.2), which was defined for two differential polynomials in $u$, to the case, when $P$ is a differential polynomial in $u$, and $R$ depends explicitly on $\phi$. In particular, we obtain a well-defined map $F_0 \times F_1 \to F_1$:

$$(2.1.9) \quad \{\int P dt, \int R e^{\phi} dt\} = \int \frac{\delta P}{\delta u} \left[ \frac{\delta R}{\delta \phi} e^{\phi} \right] dt = \int \frac{\delta P}{\delta u} \left[ R e^{\phi} - \partial_t \left( \frac{\delta R}{\delta u} e^{\phi} \right) \right] dt.$$
This bracket satisfies the Jacobi identity for any triple $F, G \in \mathcal{F}_0, H \in \mathcal{F}_1$. In other words, $\mathcal{F}_1$ is a module over the Lie algebra $\mathcal{F}_0$.

This statement can be proved in the same way as in § 2.1.4. Namely, we can extend the space $\mathcal{F}_1$ to the space $\hat{\mathcal{F}}_1$ by adjoining elements of the form $\int Re^\phi dt$, where $R$ is a polynomial in $\partial^mu, m \geq 0$, and $t$. Consider the elements $w_m = \int t^m e^\phi dt \in \hat{\mathcal{F}}_1$. By formula (2.1.9), the action of $u_m = \int t^m u(t) dt \in \mathcal{F}_0$ (cf. § 2.1.4) on $w_n$ is given by

$$\{u_m, w_n\} = w_{n+m}.$$  

This formula defines a map $U \times W \to W$, where $U = \oplus_{m \in \mathbb{Z}} Cu_m$ and $W = \oplus_{n \in \mathbb{Z}} w_n$. Recall that $U$ is a Lie algebra, with the commutation relations given by formula (2.1.4). One can check directly that this map defines a structure of a $U$-module on $W$.

In the same way as in § 2.1.4, we can define a completion $\hat{B}$ of the space $B = \mathbb{C}[u_m]_{m \in \mathbb{Z}} \otimes W$, which contains $\hat{\mathcal{F}}_1$ and $\mathcal{F}_1$. Using the Leibnitz rule, we can extend the map $U \times W \to W$ given by (2.1.10) to a map $\hat{A} \times \hat{B} \to \hat{B}$. By construction, this map defines a structure of an $\hat{A}$-module on $\hat{B}$. The restriction $\hat{\mathcal{F}}_0 \times \hat{\mathcal{F}}_1 \to \hat{\mathcal{F}}_1$ of this map coincides with the map $\{\cdot, \cdot\}$ given by (2.1.9). Therefore it defines on $\hat{\mathcal{F}}_1$ a structure of a module over the Lie algebra $\hat{\mathcal{F}}_0$. Hence it makes $\mathcal{F}_1$ into a module over $\mathcal{F}_0$.

2.1.9. The Liouville hamiltonian. We now introduce the hamiltonian $H$ of the Liouville model by the formula $H = \int e^\phi(t) dt \in \mathcal{F}_1$. We can rewrite the Liouville equation (1.1.2) in the hamiltonian form as

$$\partial_t U(t) = \{U(t), H\}.$$ 

Here $U(t)$ stands for the delta-like functional on $Lh$, whose value on a function from $Lh$ is equal to the value of this function at the point $t$. We can rewrite it as $U(t) = \int \delta(t-s) u(s) ds$. The formula (2.1.9) can be extended to such functionals as well. Applying this formula, we obtain

$$\left\{ \int \delta(t-s) u(s) ds, \int e^\phi(s) ds \right\} = \int \delta(t-s) e^\phi(s) ds = e^\phi(t).$$

2.1.10. Definition. The kernel of the linear operator

$$(2.1.11) \quad \bar{Q} = \{\cdot, \int e^\phi dt\} : \mathcal{F}_0 \to \mathcal{F}_1$$

will be called the space of local integrals of motion of the classical Liouville theory and will be denoted by $I_0(sl_2)$.

2.1.11. Our goal is to compute the space $I_0(sl_2)$. Note that by the Jacobi identity, it is closed with respect to the Poisson bracket.

It is more convenient to work with the spaces $\pi_0$ and $\pi_1$, than with $\mathcal{F}_0$ and $\mathcal{F}_1$. We want to define a linear operator $\bar{Q} : \pi_0 \to \pi_1$, which commutes with the action of $\partial$ on these spaces, and descends down to the operator $\bar{Q}$.

To define such an operator, we will use the same approach as in § 2.1.7. According to formula (2.1.9), we have:

$$\{\int P dt, \int e^\phi dt\} = \int \sum_{n \geq 0} \left(-\partial_t\right)^n \frac{\partial P}{\partial (\partial^n u)} \cdot e^\phi dt = \int \sum_{n \geq 0} \frac{\partial P}{\partial (\partial^n u)} \cdot \partial_t^n e^\phi dt,$$
where we used the fact that the integral of a total derivative is 0. We find: \( \partial_t e^\phi = ue^\phi, \partial_t^2 e^\phi = (u^2 + \partial_t u)e^\phi, \) etc. In general, \( \partial_t^n e^\phi = B_n e^\phi, \) where the \( B_n \)'s are certain differential polynomials in \( u(t), \) which are connected by the recurrence relation

\[
B_{n+1} = uB_n + \partial_t B_n.
\]

(2.1.12)

Then we obtain

\[
\{ \int P dt, \int e^\phi dt \} = \sum_{n \geq 0} \int \frac{\partial P}{\partial (\partial^n u)} \cdot B_n e^\phi dt.
\]

(2.1.13)

Therefore we can define a map \( \tilde{Q} : \pi_0 \to \pi_1 \) as follows:

\[
\tilde{Q} \cdot P = \sum_{n \geq 0} B_n \frac{\partial P}{\partial (\partial^n u)} \otimes v_1.
\]

It commutes with the action of \( \partial \) and descends down to the operator \( \bar{Q} : \mathcal{F}_0 \to \mathcal{F}_1. \)

In fact, in the same way we can define for any \( G \in \mathcal{F}_1 \) a map \( \{ \cdot, G \} : \pi_0 \to \pi_1, \) which commutes with \( \partial \) and descends down to the map \( \mathcal{F}_0 \to \mathcal{F}_1, \) given by formula (2.1.9):

\[
\{ P, \int e^{\phi(t)} dt \} = \sum_{n \geq 0} \frac{\partial P}{\partial (\partial^n u)} \cdot \partial^n (R \otimes v_1) - \partial^{n+1} \left( \frac{\delta}{\delta u} R \otimes v_1 \right).
\]

We can also extend the action of \( G \in \mathcal{F}_0 \) on \( \pi_0, \) given by (2.1.7), to an action on \( \pi_1 \) by adding to the vector field (2.1.7) the term \( \frac{\delta R}{\delta u} \frac{\partial}{\partial \phi}, \) where \( \frac{\partial}{\partial \phi} \) acts on \( \pi_n \) by multiplication by \( n. \)

It is convenient to pass to the new variables \( x_n = \partial^{-n-1} u/(-n-1)!, \) \( n < 0. \) Then \( \pi_0 = \mathbb{C}[x_n]_{n<0}, \pi_1 = \mathbb{C}[x_n]_{n<0} \otimes \mathbb{C}v_1. \) In these variables the action of the derivative \( \partial \) on these spaces is given by

\[
\partial = -\sum_{n<0} nx_{n-1} \frac{\partial}{\partial x_n} + x_{-1} \frac{\partial}{\partial \phi}.
\]

Let \( T : \pi_0 \to \pi_1 \) be the translation operator, which sends \( P \in \pi_0 \) to \( P \otimes v_1 \in \pi_1. \) In the new variables the operator \( \tilde{Q} : \pi_0 \to \pi_1 \) is given by the formula

\[
\tilde{Q} = T \sum_{n<0} S_{n+1} \frac{\partial}{\partial x_n},
\]

where the polynomials \( S_n \) are the Schur polynomials, defined via the generating function:

\[
\sum_{n \geq 0} S_n z^n = \exp(\sum_{m<0} \frac{-x_m}{m} z^m).
\]

(2.1.15)

One can check that \( \partial S_n = -x_1 S_n - (n-1) S_{n-1}, \) and therefore, by formula (2.1.12), the differential polynomial \( B_{-n} \) coincides with \( S_n \) in the new variables \( x_m. \) We summarize these results in the following Lemma.

2.1.12. Lemma. The operator \( \tilde{Q}, \) given by formula (2.1.14), commutes with the action of the derivative \( \partial \) and the corresponding operator \( \mathcal{F}_0 \to \mathcal{F}_1 \) coincides with the operator \( \bar{Q}, \) given by formula (2.1.11).
2.1.13. Let us put $\deg 1 = 0$, $\deg v_1 = 1$, $\deg x_n = -n$. Since the $x_n$’s generate the spaces $\pi_0$ and $\pi_1$ from 1 and $v_1$, respectively, this defines a $\mathbb{Z}$–grading on these spaces such that the homogeneous components are finite-dimensional. The operator $\partial$ is homogeneous of degree 1, and we can define grading on the spaces $\mathcal{F}_0$ and $\mathcal{F}_1$ by subtracting 1 from the grading on the spaces $\pi_0$ and $\pi_1$, respectively. The operators $\tilde{Q}$ and $\bar{Q}$ are homogeneous of degree 0 and therefore their kernels and cokernels are $\mathbb{Z}$–graded with finite-dimensional homogeneous components.

2.1.14. To find the space of local integrals of motion of the classical Liouville equation we have to find the 0th cohomology of the complex

\[ \mathcal{F}_0 \xrightarrow{\bar{Q}} \mathcal{F}_1. \]

Consider the following double complex.

\[ \begin{array}{ccc}
\mathbb{C} & \uparrow & \\
\pi_0 & \xrightarrow{\tilde{Q}} & \pi_1 \\
\theta & \uparrow & -\theta \\
\pi_0 & \xrightarrow{\bar{Q}} & \pi_1 \\
\uparrow & & \\
\mathbb{C} & & \\
\end{array} \]

\[ (2.1.16) \]

We can calculate its cohomology by means of two spectral sequences (as general references on spectral sequences, cf., e.g., [93, 15]). In one of them the 0th differential is vertical. Therefore, in this spectral sequence the 1st term coincides with our complex with the degrees shifted by 1. Hence the 1st cohomology of the double complex coincides with the space of integrals of motion.

In the other spectral sequence the first differential is horizontal. We have two identical complexes

\[ \pi_0 \xrightarrow{\tilde{Q}} \pi_1. \]

Let us calculate the cohomology of this complex.

2.1.15. Proposition. The operator $\tilde{Q} : \pi_0 \to \pi_1$ is surjective, so that its cokernel is equal to 0. The kernel $\mathcal{W}_0(\mathfrak{sl}_2)$ of the operator $\tilde{Q}$ contains an element $W_{-2}$ of degree 2, such that $\mathcal{W}_0(\mathfrak{g})$ coincides with the polynomial algebra $\mathbb{C}[W_n]_{n \leq -2}$, where $W_n = \partial^{-n-2}W_{-2}/(-n-2)!$.

Proof. The operator $Q = T^{-1}\tilde{Q}$ is a linear combination of the vector fields $S_{n+1}\partial/\partial x_n$, $n < 0$, where $S_0 = 1$ and $S_{n+1}$ is a polynomial in $x_{-1}, \ldots, x_{n+1}, n < -1$. The operator

\[ \sum_{-m \leq j \leq -1} S_{j+1} \frac{\partial}{\partial x_j} \]

is therefore a well-defined linear operator $Q(m)$ from $\mathbb{C}[x_j]_{j=-1,\ldots,-m}$ to itself.
The operator $Q(m)$ is surjective. To see that, consider the dual operator $Q(m)^*$, acting on the space dual to $\mathbb{C}[x_j]_{j=-1,...,n}$. Since our operator is homogeneous, it is sufficient to consider the restricted dual space, which we can identify with itself, choosing the monomials $x_{-j_1}^{k_1}...x_{-j_n}^{k_n}/(k_1!...k_n!)^{1/2}$ as the orthonormal basis. The operator $Q(m)^*$ then has the form

$$Q(m)^* = \sum_{-m \leq j \leq -1} x_j S_j^*,$$

where $S_j^*$ is obtained from the polynomial $S_j$ by replacing $x_i$ with $\partial/\partial x_i$. We see that the operator $Q(m)^*$ is the sum of multiplication by $x_{-1}$, which increases the power of any polynomial by 1, and other operators, which do not change or decrease the power. Since the operator of multiplication by $x_{-1}$ has no kernel, the operator $Q(m)^*$ is injective. Therefore, the operator $Q(m)$ is surjective. Hence, the operators $Q$ and $\bar{Q}$ are also surjective.

For any $n < -1$, there exist polynomials $W_n = (n + 1)x_n + W'_n$, where $W'_n$ is a linear combination of terms of power greater than 1 and of degree $n$, such that $\bar{Q} \cdot W_n = 0$. Indeed, $Q(n) \cdot x_n = S_{n+1}$ is an element of $\mathbb{C}[x_j]_{j=-1,...,n+1}$. The operator $Q(n+1)$ is surjective on this space. Therefore, there exists such $W'_n \in \mathbb{C}[x_j]_{j=-1,...,n+1}$ that $Q(n+1) \cdot W'_n = -(n+1)x_n$. But then $Q(n) \cdot ((n+1)x_n + W'_n) = 0$, and hence $\bar{Q} \cdot W_n = TQ(n) \cdot W_n = 0$.

In the coordinates $x_{-1}, W_n, n < -1$, the operator $\bar{Q}$ is equal to

$$\bar{Q} = T \left( \frac{\partial}{\partial x_{-1}} + \sum_{n < -1} (\bar{Q} \cdot W_n) \frac{\partial}{\partial W_n} \right) = T \frac{\partial}{\partial x_{-1}}.$$

The $W_n$’s are algebraically independent by construction, therefore the kernel of the operator $\bar{Q}$ coincides with $\mathbb{C}[W_n]_{n < -1}$.

Finally, we can choose as $W_n$ the polynomial $\partial^{-n-2}W_{-2}/(-n-2)!$. Indeed, the latter lies in the kernel of $\bar{Q}$, because $[\partial, \bar{Q}] = 0$. Its linear term is equal to $(n + 1)x_n$, because the linear term of $W_{-2}$ is $-x_{-2}$, and the derivative preserves the power of a polynomial.

The Proposition is proved.

2.1.16. Remark. For any $\mathbb{Z}$–graded vector space

$$V = \oplus_{m \in \mathbb{Z}} V(m)$$

with finite-dimensional homogeneous components, introduce its character as

$$\text{ch} V = \sum_{m \in \mathbb{Z}} \dim V(m) q^m.$$

The Euler character of the complex (2.1.17) is equal to

$$\text{ch} \pi_0 - \text{ch} \pi_1 = \text{ch} \text{Ker} \bar{Q} - \text{ch} \text{Coker} \bar{Q} = \prod_{n \geq 1} (1 - q^n)^{-1} - q \prod_{n \geq 1} (1 - q^n)^{-1} = \prod_{n \geq 2} (1 - q^n)^{-1} = 1 + q^2 + \ldots.$$

This formula shows that there exists an element of degree 2 in the kernel of $\bar{Q}$. Denote it by $W_{-2}$. The operator of multiplication by $W_{-2}$, acting on $\pi_0$ and $\pi_1$, commutes with the action of $\bar{Q}$. Further, since $[\partial, \bar{Q}] = 0$, the operators of multiplication by $W_n = \partial^{-n-2}W_{-2}/(-n-2)!$, $n \leq -2$, also commute with $\bar{Q}$. Therefore any polynomial in the $W_n$’s, constructed this way, lies in the kernel of the operator $\bar{Q}$. The algebraic independence of these $W_n$’s and the surjectivity of the operator $\bar{Q}$ allowed us to identify the polynomial algebra in the $W_n$’s with the kernel of $\bar{Q}$. 


2.1.17. Proposition. The space $I_0(\mathfrak{sl}_2)$ of local integrals of motion of the classical Liouville theory coincides with the quotient of $\mathcal{W}_0(\mathfrak{sl}_2)$ by the total derivatives and constants.

Proof. We have to prove that the kernel of the operator $\bar{Q}$ coincides with the quotient of $\mathcal{W}_0(\mathfrak{sl}_2)$ by the total derivatives and constants. As we explained before, this kernel is the same as the 1st cohomology of the double complex (2.1.16). By Proposition 2.1.15, the first term of the spectral sequence, associated to this double complex, looks as follows:

$$\mathbb{C} \rightarrow \mathcal{W}_0(\mathfrak{sl}_2) \rightarrow \mathcal{W}_0(\mathfrak{sl}_2) \rightarrow \mathbb{C}.$$ 

The 1st cohomology of this complex is equal to the quotient of $\mathcal{W}_0(\mathfrak{sl}_2)$ by the total derivatives and constants.

2.1.18. We can write down explicit formulas for the $W_n$'s as follows:

$$W_{-2} = \frac{1}{2} x_1^2 - x_{-2}, \quad W_n = \frac{1}{(-n-2)!} \partial^{-n-2}W_{-2}, \quad n < -2.$$ 

Thus, the space $I_0(\mathfrak{sl}_2)$ consists of local functionals, which are defined by differential polynomials, depending on $W = \frac{1}{2} u^2 - \partial u$.

These local functionals constitute a Poisson subalgebra in $\mathcal{F}_0$. It is known that this Poisson subalgebra is isomorphic to the classical Virasoro algebra.

The Virasoro algebra is the central extension of the Lie algebra of vector fields on the circle. Its dual space is equipped with the Kirillov-Kostant Poisson structure. This structure can be restricted to the hyperplane, which consists of the linear functionals, whose value on the central element is 1. If we choose a coordinate on the circle, then this hyperplane can be identified with the space of Laurent polynomials $W(t)$. The local functionals on this space form a Poisson algebra, which is isomorphic to $I_0(\mathfrak{sl}_2)$.

There is a map from a hyperplane in the dual space to the Heisenberg algebra to a hyperplane in the dual space to the Virasoro algebra, which sends $u(t)$ to $W(t) = \frac{1}{2} u^2(t) - \partial_t u(t)$ and preserves the Poisson structure. This map is called the Miura transformation.

In the case of the Liouville theory, which is the simplest Toda field theory, we were able to find explicit formulas for the local integrals of motion and identify their Poisson algebra with the classical Virasoro algebra. However, in general explicit formulas are much more complicated, and we will have to rely on homological algebra to obtain information about the integrals of motion.

2.2. General case.

2.2.1. Hamiltonian space. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. It is equipped with the scalar product $(\cdot, \cdot)$, which is the restriction of the invariant scalar product on $\mathfrak{g}$, normalized as in [81]. In what follows we will identify $\mathfrak{h}$ with its dual by means of this scalar product.

We choose as the Hamiltonian space the space $L\mathfrak{h}$ of Laurent polynomials on the circle with values in $\mathfrak{h}$, that is the space $\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]$.

Each element $u(t)$ of $L\mathfrak{h}$ can be represented by its coordinates $(u^1(t), \ldots , u^l(t))$ with respect to the basis of simple roots $\alpha_1, \ldots , \alpha_l$, where $l$ is the rank of $\mathfrak{g}$. Let $\pi_0$ be the space of differential polynomials of $u(t)$, i.e. the space of polynomials in $\partial^n u^i, i = 1, \ldots , l, n \geq 0$. 

We define the space $F_0$ of local functionals as the space of functionals on $L\mathfrak{h}$, which can be represented as formal residues

$$F[u(t)] = \int P(\partial^n u^i(t))dt,$$

where $P \in \pi_0$. Again, we have the exact sequence

$$0 \longrightarrow \pi_0/\mathbb{C} \xrightarrow{\partial} \pi_0/\mathbb{C} \xrightarrow{\int} F_0 \longrightarrow 0.$$

Introduce the Poisson structure on $F_0$ by the formula

$$(2.2.1) \{F, G\}[u(t)] = -\int \left( \frac{\delta P}{\delta u} \partial \delta R \right) dt.$$

Here $\frac{\delta P}{\delta u}$ and $\partial \frac{\delta R}{\delta u}$ are vectors in the dual space to $\mathfrak{h}$, and we take their scalar product.

In coordinates, we can rewrite it as

$$(2.2.2) \{F, G\}[u(t)] = -\int \sum_{i,j=1}^l (\alpha_i, \alpha_j) \frac{\delta P}{\delta u^i} \partial_t \frac{\delta R}{\delta u^j} dt.$$  

This Poisson structure has an interpretation as a Kirillov-Kostant structure on a hyperplane in the dual space to the Heisenberg algebra $\widehat{\mathfrak{h}}$, which is the central extension of $L\mathfrak{h}$ (cf. § 2.1.6).

Note that our conventions in the case of $\mathfrak{g} = \mathfrak{sl}_2$ (cf. § 2.1) differ from our conventions in general by a factor of 2, since for $\mathfrak{sl}_2$ we have $(\alpha, \alpha) = 2$.

2.2.2. Let us define other spaces of functionals on $L\mathfrak{h}$. For each element $\gamma$ of the weight lattice $P \subset \mathfrak{h}^* \simeq \mathfrak{h}$ we define the space $\pi_\gamma = \pi_0 \otimes \mathbb{C}v_\gamma$, equipped with the action of the derivative by the formula $(\partial + \gamma) \otimes 1$, where $\gamma$ denotes the operator of multiplication by $\gamma$ (as an element of $\pi_0$) on $\pi_0$.

Let $F_\gamma$ be the quotient of $\pi_\gamma$ by the image of the operator $\partial$, i.e. by the total derivatives. We have the exact sequence

$$(2.2.3) 0 \longrightarrow \pi_\gamma \xrightarrow{\partial} \pi_\gamma \longrightarrow F_\gamma \longrightarrow 0.$$  

As in § 2.1.8, we can interpret $F_\gamma$ as the space of functionals on $L\mathfrak{h}$ of the form

$$\int P(\partial^n u^i(t))e^{\gamma(t)}dt,$$

where $\gamma(t)$ is such that $\partial_t \gamma(t) = \gamma(t)$.

In the same way as in § 2.1.8, we can extend our Poisson bracket (2.2.1) to a map

$$F_0 \times \bigoplus_{\gamma \in P} F_\gamma \rightarrow \bigoplus_{\gamma \in P} F_\gamma$$

by the formula

$$(2.2.4) \{ \int P dt, \int Re^{\tilde{\gamma}} dt \} = \int \left[ \left( \frac{\delta P}{\delta u}, \gamma \right) Re^{\tilde{\gamma}} - \left( \frac{\delta P}{\delta u}, \partial_t \frac{\delta R}{\delta u} e^{\tilde{\gamma}} \right) \right] dt.$$  

One can check that this bracket satisfies the Jacobi identity for any triple $F, G \in F_0, H \in F_\gamma$.  

2.2.3. *The Toda Hamiltonian.* We can now introduce the Hamiltonian of the Toda field theory, associated to \( \mathfrak{g} \), by the formula

\[
H = \frac{1}{2} \sum_{i=1}^{l} \int e^{\phi_i(t)} dt \in \bigoplus_{i=1}^{l} \mathcal{F}_{\alpha_i},
\]

where \( \phi_i(t) = \bar{\alpha}_i(t) \). Note that \( \int e^{\phi_i(t)} dt \) stands for the image of \( v_{\alpha_i} \in \pi_{\alpha_i} \) under the projection \( \pi_{\alpha_i} \to \mathcal{F}_{\alpha_i} \). The equation

\[
U(t) = \{ U(t), H \},
\]

where \( U(t) \) denotes the delta-like functional as in §2.1.9, coincides with the Toda equation (1.1.1).

The operator \( \bar{Q}_i = \{ \cdot, \int e^{\phi_i(t)} dt \} \) is a well-defined linear operator, acting from \( \mathcal{F}_0 \) to \( \mathcal{F}_{\alpha_i} \). We can therefore give the following definition.

2.2.4. Definition. The kernel of the linear operator

\[
\frac{1}{2} \sum_{i=1}^{l} \bar{Q}_i : \mathcal{F}_0 \to \bigoplus_{i=1}^{l} \mathcal{F}_{\alpha_i}
\]

will be called the space of local integrals of motion of the classical Toda field theory associated to \( \mathfrak{g} \) and will be denoted by \( I_0(\mathfrak{g}) \).

2.2.5. Clearly, \( I_0(\mathfrak{g}) \) is equal to the intersection of the kernels of the operators \( \bar{Q}_i : \mathcal{F}_0 \to \mathcal{F}_{\alpha_i} \). By Jacobi identity, \( I_0(\mathfrak{g}) \) is a Poisson subalgebra of \( \mathcal{F}_0 \).

Let us write down an explicit formula for the operator \( \bar{Q}_i \).

It is convenient to pass to the new variables \( x_n^i = \partial^{-n-1} u_i^{(-n-1)!}, i = 1, \ldots, l, n < 0 \).

Then \( \pi_{\gamma} = \mathbb{C}[x_n^i] \otimes \mathbb{C} v_\gamma \) (here \( v_0 = 1 \)).

In these new variables the action of the derivative \( \partial \) can be written as

\[
\sum_{i=1}^{l} \left( -\sum_{n<0} n x_{n-1}^i \frac{\partial}{\partial x_n^i} + x_{n-1}^i \frac{\partial}{\partial \phi_i} \right),
\]

where the action of \( \partial/\partial \phi_i \) on \( \pi_\gamma \) with \( \gamma = \sum_{i=1}^{l} l_i \alpha_i \) is given by multiplication by \( l_i \).

Let \( T_i : \pi_\gamma \to \pi_{\gamma+\alpha_i} \) be the translation operator, which maps \( P \otimes v_\gamma \in \pi_\gamma \) to \( P \otimes v_{\gamma+\alpha_i} \in \pi_{\gamma+\alpha_i} \).

Introduce the operators \( \bar{Q}_i : \pi_\gamma \to \pi_{\gamma+\alpha_i} \) by the formula

\[
\bar{Q}_i = T_i \sum_{n<0} S_{n+1}^i \partial_n^{(i)},
\]

where

\[
\partial_n^{(i)} = \sum_{j=1}^{l} (\alpha_i, \alpha_j) \frac{\partial}{\partial x_n^j},
\]

and the Schur polynomials \( S_n^i \) are given by the generating function

\[
\sum_{n<0} S_n^iz^n = \exp(\sum_{m<0} \frac{x_m^i z^m}{m}).
\]

2.2.6. Lemma. The operator \( \bar{Q}_i : \pi_0 \to \pi_{\alpha_i} \) commutes with the action of derivative \( \partial \) and the corresponding operator \( \mathcal{F}_0 \to \mathcal{F}_{\alpha_i} \) coincides with the operator \( \bar{Q}_i \).

Proof. The same as in Lemma 2.1.12.
2.2.7. Our task is to compute the kernel of the operator \( \sum_{i=1}^{l} \tilde{Q}_{i} \), or, in other words, the 0th cohomology of the complex

\[
\mathcal{F}_0 \longrightarrow \oplus_{i=1}^{l} \mathcal{F}_{\alpha_i}.
\]  

(2.2.5)

The cohomology of this complex is very difficult to compute. Indeed, it is clear that the 1st cohomology is very large, because the first group of the complex is “\( l \) times larger” than the 0th group. So, we can not use the argument we used in the proof of Proposition 2.1.15.

What is even worse is that as was explained in the Introduction, for such a complex it is virtually impossible to prove that the cohomology classes can be quantized.

To fix this situation we will extend this complex further to the right. Clearly, by doing so we will not change the 0th cohomology, but we will be able to kill all higher cohomologies. We will then use the resulting complex to compute the 0th cohomology, and to prove that it can be quantized.

First of all, it is convenient to realize our complex as the double complex

\[
\begin{array}{ccc}
\pi_0 & \longrightarrow & \oplus_{i=1}^{l} \pi_{\alpha_i} \\
\downarrow & & \downarrow \\
\tilde{Q}_i & \longrightarrow & \oplus_{i=1}^{l} \pi_{\alpha_i} \\
\downarrow & & \downarrow \\
\pi_0 & \longrightarrow & \oplus_{i=1}^{l} \pi_{\alpha_i} \\
\downarrow & & \downarrow \\
C & & C
\end{array}
\]  

(2.2.6)

in the same way as in § 2.1.14.

We can compute the cohomology of the double complex (2.2.6) by means of the spectral sequence, whose first term consists of two identical complexes

\[
\pi_0 \longrightarrow \oplus_{i=1}^{l} \pi_{\alpha_i}.
\]  

(2.2.7)

We will extend both complexes (2.2.7) in such a way that the higher differentials will commute with the derivative \( \partial \). We will then be able to form a double complex, which will give us an extension of the complex (2.2.5) that we are looking for.

The key observation, which will enable us to do that, is as follows. Introduce the operators \( Q_i : \pi_0 \rightarrow \pi_0 \) as \( T^{-1} \tilde{Q}_i \). We will use the notation \( \text{ad} A \cdot B = [A, B] \).

2.2.8. Proposition. The operators \( Q_i \) satisfy the Serre relations of the Lie algebra \( \mathfrak{g} \)

\[
(\text{ad} Q_i)^{-a_{ij}+1} \cdot Q_j = 0,
\]

where \( \|a_{ij}\| \) is the Cartan matrix of \( \mathfrak{g} \).

Proof. The Proposition follows from the following formula:

\[
(\text{ad} Q_i)^m \cdot Q_j = C_m \cdot (-a_{ij} - m + 1) \sum_{n_1, \ldots, n_{m+1} < 0} S_{n_1}^{i} \cdots S_{n_m}^{i} S_{n_{m+1}}^{j} \frac{1}{n_1 \ldots n_m} \cdot \left( \sum_{l=1}^{m} \frac{n_l}{n_1 + \cdots + n_l + n_{m+1}} \partial^{(j)}_{n_1 + \cdots + n_{m+1}} - \partial^{(j)}_{n_1 + \cdots + n_{m+1}} \right).
\]
relations

\[ C \] (here we put \( S_m^j = 0 \), if \( m > 0 \)) and the simple identity

\[ \frac{1}{a(a + b)} + \frac{1}{b(a + b)} = \frac{1}{ab} \]

2.2.9. Remark. In the proof of Proposition 2.2.8 we never used the fact that \( |a_{ij}| \) is the Cartan matrix of a simple Lie algebra. In fact, we could associate the main objects, defined in this section, such as \( \pi \), the Cartan matrix of a simple Lie algebra. In fact, we could associate the main objects, defined in this section, such as \( \pi \), the Cartan matrix of a simple Lie algebra. In fact, we could associate the main objects, defined in this section, such as \( \pi \), the Cartan matrix of a simple Lie algebra. In fact, we could associate the main objects, defined in this section, such as Proposition 2.2.8, remain valid.

2.2.10. Proposition 2.2.8 shows that the operators \( Q_i \) generate an action of the nilpotent subalgebra \( \mathfrak{n}_+ \) of \( \mathfrak{g} \) on \( \pi_0 \). In order to extend the complex (2.2.7), we will use the Bernstein-Gelfand-Gelfand (BGG) resolution of the trivial representation of \( \mathfrak{g} \) by Verma modules. We will recall the relevant facts about this resolution in the next subsection.

2.3. BGG resolution.

2.3.1. Verma modules. Recall that the Lie algebra \( \mathfrak{g} \) has the Cartan decomposition \( \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \). For \( \lambda \in \mathfrak{h}^* \) denote by \( \mathbf{C}_\lambda \) the corresponding one-dimensional representation of \( \mathfrak{h} \). We can extend it trivially to a representation of \( \mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_- \). The induced module over \( \mathfrak{g} \),

\[ M_\lambda = U(\mathfrak{g}) \otimes U(\mathfrak{b}_-) \quad \mathbf{C}_\lambda, \]

is called the Verma module with lowest weight \( \lambda \). It is freely generated from the lowest weight vector \( 1_\lambda = 1 \otimes 1 \) by the action of the nilpotent subalgebra \( \mathfrak{n}_+ \) of \( \mathfrak{g} \).

A vector \( w \) in \( M_\lambda \) is called a singular vector of weight \( \mu \), if it satisfies the properties:

\[ \mathfrak{n}_- \cdot w = 0, \quad y \cdot w = \mu(y)w, \quad y \in \mathfrak{h}. \]

In particular, \( 1_\lambda \) is a singular vector of weight \( \lambda \). A singular vector of weight \( \mu \) generates a submodule of \( M_\lambda \), which is isomorphic to the Verma module \( M_\mu \).

Consider the Verma module \( M_0 \). It is known that the singular vectors of \( M_0 \) are labeled by the elements of the Weyl group of \( \mathfrak{g} \) \[ \Pi \]. Such a vector \( w_s \), corresponding to an element \( s \) of the Weyl group, has the weight \( \rho - s(\rho) \), where \( \rho \in \mathfrak{h}^* \) is the half-sum of the positive roots of \( \mathfrak{g} \). Let us fix these vectors once and for all.

2.3.2. The definition of the resolution. The BGG resolution is a complex, i.e. a \( \mathbb{Z} \)-graded vector space

\[ B_s(\mathfrak{g}) = \oplus_{j \geq 0} B_j(\mathfrak{g}), \]

together with differentials \( d_j : B_j(\mathfrak{g}) \rightarrow B_{j-1}(\mathfrak{g}) \), which are nilpotent: the composition of two consecutive differentials \( d_{j-1}d_j \) is equal to 0.

The vector space \( B_j(\mathfrak{g}) \) is the direct sum of the Verma modules \( M_{\rho - s(\rho)} \), where \( s \) runs over the set of elements of the Weyl group of length \( j \) \[ \Pi \].

Once we fixed the vectors \( w_s \), we have canonical embeddings \( M_{\rho - s(\rho)} \rightarrow M_0 \). Therefore the module \( M_{\rho - s(\rho)} \) can be thought of as a submodule of the module \( M_0 \), generated by the vector \( w_s \). It is known that the vector \( w_{s'} \) belongs to the module \( M_{\rho - s(\rho)} \) if and only if \( s \preceq s' \) with respect to the Bruhat order on the Weyl group \[ \Pi \]. It is clear that these vectors are singular vectors of the module \( M_{\rho - s(\rho)} \) and that there are no other singular
vectors in this module. In that case we have an embedding $i_{s',s} : M_{\rho - s' (\rho)} \to M_{\rho - s (\rho)}$, which sends the lowest weight vector of $M_{\rho - s' (\rho)}$ to the singular vector of $M_{\rho - s (\rho)}$ of weight $\rho - s' (\rho)$.

2.3.3. Lemma. \[1\]

(a) Let $s$ and $s''$ be two elements of the Weyl group, such that $s \prec s''$ and $l(s'') = l(s) + 2$. Then there are either two or no elements $s'$, such that $s \prec s' \prec s''$.

(b) Let us call a square a set of four elements of the Weyl group, satisfying the conditions of the part (a).

It is possible to attach a sign $\epsilon_{s',s''}$, or $-\epsilon_{s',s''}$, to each pair of elements of the Weyl group $s, s'$, such that $s \prec s', l(s') = l(s) + 1$, so that the product of signs over any square is $-1$.

2.3.4. The differential. We are now ready to define the differential of the BGG resolution $d_j : B_j (\mathfrak{g}) \to B_{j-1} (\mathfrak{g})$ as

$$d_j = \sum_{l(s) = j-1, l(s') = j, s \prec s'} \epsilon_{s',s} \cdot i_{s',s}.$$

In other words, we take the sum of all possible embeddings of the Verma modules, which are direct summands of $B_j (\mathfrak{g})$, with the special choice of signs from Lemma 2.3.3. By definition, these differentials commute with the action of $\mathfrak{g}$.

2.3.5. Theorem. \[1\]

(a) The differentials $d_j, j > 0$, are nilpotent: $d_{j-1}d_j = 0$, and so $B_\bullet (\mathfrak{g})$ is a complex.

(b) The 0th homology of the complex $B_\bullet (\mathfrak{g})$ is the trivial one-dimensional representation of $\mathfrak{g}$, and all higher homologies vanish.

Proof. In the notation of § 2.3.3, we have $w_s = P_{s',s} \cdot w_s$, for some element $P_{s',s}$ of $U(\mathfrak{n}_+)$. If we have a square $s, s', s'', s'''$ of elements of the Weyl group, such that $s \prec s', s'' \prec s'''$, then we can write: $w_{s''} = P_{s'',s'} P_{s',s} w_s$ and $w_{s'''} = P_{s''',s''} P_{s'',s'} w_s$. Therefore we obtain the following identity

$$P_{s''',s''} P_{s'',s'} P_{s',s} w_s = P_{s''',s''} P_{s',s} w_s$$

in $U(\mathfrak{n}_+)$. By definition, $i_{s',s} : (u \cdot 1_{\rho - s' (\rho)}) = (u P_{s',s}) \cdot 1_{\rho - s (\rho)}$. So, we obtain from formula (2.3.2): $i_{s',s} \circ i_{s'',s'} = i_{s'',s'} \circ i_{s',s}$. Thus, because of our sign convention (cf. Lemma 2.3.3 (b)), such terms in the composition of two consecutive differentials $d_{j-1}d_j$ will cancel out. This proves part (a) of the Theorem.

The proof of part (b) is rather complicated; it uses the so-called weak BGG resolution, which is obtained from the de Rham complex on the big cell of the flag manifold of $\mathfrak{g}$ (cf. \[1\]).

2.3.6. Remarks. (1) One can define analogous resolutions for arbitrary finite-dimensional representations of $\mathfrak{g}$.

(2) There are generalizations of the BGG resolutions to arbitrary symmetrizable Kac-Moody algebras \[107\]. We will use such resolutions for affine algebras in § 3.

2.4. Extended complex and its cohomology. One of the main applications of the BGG resolution is to computation of the cohomologies of the nilpotent Lie algebra $\mathfrak{n}_+$ of $\mathfrak{g}$. In this subsection we will use this resolution to extend our complex (2.2.7), and to compute the cohomology of the resulting complex.
2.4.1. The $j$th component $F^j(\mathfrak{g})$ of our extended complex
\[ F^*(\mathfrak{g}) = \bigoplus_{j \geq 0} F^j(\mathfrak{g}) \]
will be the direct sum of the spaces $\pi_{\rho-s(\rho)}$, where $s$ runs over the set of elements of the Weyl group of length $j$.

Now let us define the differentials. The algebra $U(\mathfrak{n}_+)$ is generated by $e_i$, $i = 1, \ldots, l$, which satisfy the Serre relations. So any element of $U(\mathfrak{n}_+)$ can be expressed in terms of $e_i$. Let $P_{s',s}(Q): \pi_{\rho-s(\rho)} \rightarrow \pi_{\rho-s'(\rho)}$ be the map, obtained by inserting into $P_{s',s} \in U(\mathfrak{n}_+)$ the operators $Q_i$ instead of $e_i$. We can then introduce the differential $\delta^j: F^{j-1}(\mathfrak{g}) \rightarrow F^j(\mathfrak{g})$ of our complex by the formula
\[ \delta^j = \sum_{l(s) = j - 1, l(s') = j, s < s'} \epsilon_{s',s} \cdot P_{s',s}(Q). \]

2.4.2. Lemma. The differentials $\delta^j, j > 0$, are nilpotent: $\delta^{j+1} \delta^j = 0$, and so $F^*(\mathfrak{g})$ is a complex.

Proof. Just as in the proof of part (a) of Theorem 2.3.4, we have to check that $P_{s',s'} \delta^j = P_{s',s'} \delta^{j-1} = 0$, since, according to Proposition 2.2.8, the operators $Q_i$, and therefore the operators $Q_i$, satisfy the defining relations of the algebra $U(\mathfrak{n}_+)$. However, this follows at once from (2.3.2), since $P_{s',s'} \delta^{j-1} = 0$.

2.4.3. We introduce a $\mathbb{Z}$-grading on the complex $F^*(\mathfrak{g})$ by putting $\deg x_i^j = -n$, and $\deg v_{\rho-s(\rho)} = (\rho', \rho - s(\rho))$, where $\rho' \in h^*$ is defined by the property $(\rho', \alpha_i) = 1, i = 1, \ldots, l$. Clearly, all homogeneous subspaces of $\pi_{\gamma}$ have finite dimensions. With respect to this grading, the operator $\partial$ is homogeneous of degree 1, and we can define a grading on the spaces $\mathcal{F}_{\gamma}$ by subtracting 1 from the grading on the space $\pi_{\gamma}$. The differentials $\delta^j$ are homogeneous of degree 0 with respect to this grading. Therefore our complex $F^*(\mathfrak{g})$ decomposes into a direct sum of finite-dimensional subcomplexes, corresponding to its different graded components.

2.4.4. Example of $\mathfrak{sl}_3$. In this case the Weyl group consists of six elements. It is generated by two reflections: $s_1$ and $s_2$ with the relation $s_1s_2s_1 = s_2s_1s_2$. The complex $F^*(\mathfrak{sl}_3)$ is shown on Fig. 1.

The vertices represent the spaces $\pi_{\rho-s(\rho)}$, and arrows represent the maps of the differential. There are four squares. The anti-commutativity of maps, associated to one of them, reads:
\[ \tilde{Q}_1 \tilde{Q}_2 = -A \tilde{Q}_1. \]

To find a solution $A$ to this equation, let us consider the Serre relation, which the operators $\tilde{Q}_1$ and $\tilde{Q}_2$ satisfy (cf. Proposition 2.2.8):
\[ (\text{ad}(\tilde{Q}_1))^2 \cdot \tilde{Q}_2 = \tilde{Q}_1^2 \tilde{Q}_2 - 2\tilde{Q}_1 \tilde{Q}_2 \tilde{Q}_1 + \tilde{Q}_2 \tilde{Q}_1^2 = 0. \]

Therefore, $A = -2\tilde{Q}_1 \tilde{Q}_2 + \tilde{Q}_2 \tilde{Q}_1$ is such a solution. Similarly, if we put $B = -2\tilde{Q}_2 \tilde{Q}_1 + \tilde{Q}_1 \tilde{Q}_2$; then all squares will be anti-commutative.
2.4.5. Proposition. The cohomologies of the complex $F^*(\mathfrak{g})$ are isomorphic to the cohomologies of the Lie algebra $\mathfrak{n}_+$ with coefficients in the module $\pi_0$, $H^*(\mathfrak{n}_+, \pi_0)$.

Proof. The complex $F^*(\mathfrak{g})$ is isomorphic to $\text{Hom}_{\mathfrak{n}_+}(B_*(\mathfrak{g}), \pi_0)$. Indeed, for any $\lambda$ the module $M_\lambda$ is isomorphic to a free $\mathfrak{n}_+$-module $M$ with one generator. Therefore the space of $\mathfrak{n}_+$-homomorphisms $\text{Hom}_{\mathfrak{n}_+}(M, \pi_0)$ is canonically isomorphic to $\pi_0$. Indeed, any non-zero homomorphism $x \in \text{Hom}_{\mathfrak{n}_+}(M, \pi_0)$ defines a non-zero element in $\pi_0$: the image of the lowest weight vector of $M$. The embedding $i_{s',s}$ of $M$ into itself then induces the homomorphism from $\pi_0$ to $\pi_0$, which sends $y \in \pi_0$ to $P_{s',s} \cdot y$. This is precisely the homomorphism $P_{s',s}(Q)$. Hence the differentials $d_j$ of the BGG resolution $B_*(\mathfrak{g})$ map to the differentials $\delta^j$ of the complex $F^*(\mathfrak{g})$.

According to part (b) of Theorem 2.3.3, the BGG resolution $B_*(\mathfrak{g})$ is the resolution of the trivial $\mathfrak{n}_+$-module by free $\mathfrak{n}_+$-modules. The cohomologies of the $\text{Hom}_{\mathfrak{n}_+}$ of such a resolution to an $\mathfrak{n}_+$-module, are, by definition, the cohomologies of $\mathfrak{n}_+$ with coefficients in this module, cf., e.g., [10]. Therefore the cohomologies of the complex $F^*(\mathfrak{g})$ coincide with $H^*(\mathfrak{n}_+, \pi_0)$.

2.4.6. Proposition. All higher cohomologies of the complex $F^*(\mathfrak{g})$ vanish.

Proof. Each of the root generators $e_\alpha$ of $\mathfrak{n}_+$ acts on $\pi_0$ by a certain vector field. This vector field has a shift term, which is a linear combination of $\partial/\partial x^i_\alpha$ (cf. the proof of Proposition 3.1.10). It follows from the proof of Proposition 3.2.5 that the shift terms of the root generators of $\mathfrak{n}_+$ are linearly independent. Therefore the dual operators to $e_\alpha$ are equal to the sum of some linear combination of $x^i_\alpha$ and some differential operators which do not change or decrease the power of a polynomial (cf. the proof of Proposition 2.1.15). Hence the dual module to the module $\pi_0$ is a free $\mathfrak{n}_+$-module. But then the module $\pi_0$ is injective, and so all higher cohomologies of $\mathfrak{n}_+$ with coefficients in $\pi_0$ must vanish. Proposition 2.4.5 then implies that all higher cohomologies of the complex $F^*(\mathfrak{g})$ vanish.

2.4.7. Proposition. There exist elements $W_{-d_1-1}^{(1)}, \ldots, W_{-d_i-1}^{(i)}$ of $\pi_0$ of degrees $d_1+1, \ldots, d_i+1$, where the $d_i$’s are the exponents of $\mathfrak{g}$, such that the $0$th cohomology $\mathcal{W}_0(\mathfrak{g})$ of the
complex $F^*(\mathfrak{g})$ is isomorphic to the polynomial algebra

$$\mathbb{C}[W_{n_i}^{(i)}]_{1 \leq i \leq l, n_i < -d_i},$$

where $W_{n_i}^{(i)} = \partial^{-n_i - d_i - 1} W_{-d_i - 1}^{(i)} / (-n_i - d_i - 1)!$.

**Proof.** The algebra $\pi_0$ is the inductive limit of the free commutative algebras with the generators $x_n, -M \leq n \leq -1$. Hence the spectrum of $\pi_0$ is the inverse limit of the affine spaces $R_M, M > 0$, with the coordinates $x_n, -M \leq n \leq -1$. From the explicit formula for the action of the generators of the Lie algebra $\mathfrak{n}$ on $\pi_0$ we see that the algebras $\mathbb{C}[R_M]$ are preserved under the action of $\mathfrak{n}_+$ (cf. the proof of Proposition 2.1.15). The infinitesimal action of $\mathfrak{n}_+$ on $R_M$ by vector fields can be integrated to an action of the Lie group $N_+$ by means of the exponential map $\mathfrak{n}_+ \to N_+$, which is an isomorphism. The action of $N_+$ commutes with the projections $R_{M+K} \to R_M$.

At each point of the spectrum of $\pi_0$ the vector fields of the infinitesimal action of the root generators $e_\alpha$ of $\mathfrak{n}_+$ are linearly independent (cf. the proof of Proposition 2.4.6). Hence the action of $N_+$ on $R_M$ is free for $M$ large enough. The orbits of this action are isomorphic to the affine space $C^{\dim N_+}$. Therefore the quotient space $R_M/N_+$ is also an affine space and the algebra of functions on this space is a free commutative algebra.

Since the projections $R_{M+K} \to R_M$ are compatible with the action of $N_+$, we can take the inverse limit of the quotient spaces $R_M/N_+$. The algebra of functions on this inverse limit is the inductive limit of the free polynomial algebras of functions on $R_M/N_+$ and therefore it is a free polynomial algebra with infinitely many generators. It consists of all $N_+$-invariant elements of $\pi_0$, which are the same as the $\mathfrak{n}_+$-invariant elements. Hence this algebra coincides with the 0th cohomology of our complex.

The algebra $\pi_0$ is $\mathbb{Z}$-graded and the action of $\mathfrak{n}_+$ preserves this grading, if we introduce the principal grading on $\mathfrak{n}_+$ by putting $\deg e_i = 1$. Therefore the 0th cohomology of our complex is also $\mathbb{Z}$-graded, and it is easy to compute the degrees of the generators of this algebra by computing its character.

By Proposition 2.4.6, all higher cohomologies of the complex $F^*(\mathfrak{g})$ vanish. Therefore, the character of the 0th cohomology is equal to the Euler character of the complex. The latter is equal to

$$\sum_{j \geq 0} (-1)^j \sum_{l(s) = j} \text{ch}_{\pi, s - \rho(s)} = \prod_{n > 0} (1 - q^n)^{-l} \sum_{s} (-1)^{l(s)} q^{l(s) - s}.$$  

From the specialized Weyl character formula we deduce

$$\sum_{s} (-1)^{l(s)} q^{s - \rho(s)} = \prod_{1 \leq i \leq l, n_i \leq d_i} (1 - q^{n_i}).$$

This gives for the character of the 0th cohomology, $\mathcal{W}_0(\mathfrak{g}),$

$$\text{ch}\mathcal{W}_0(\mathfrak{g}) = \prod_{1 \leq i \leq l, n_i > d_i} (1 - q^{n_i})^{-1}.$$  

This formula shows that the 0th cohomology is the free commutative algebra with generators $W_{n_i}^{(i)}$ of degree $-n_i$, where $1 \leq i \leq l, n_i < -d_i$. In the same way as in the proof of Proposition 2.1.15 we can see that as the generators $W_{n_i}^{(i)}$ we can take $\partial^{-n_i - d_i - 1} W_{-d_i - 1}^{(i)} / (-n_i - d_i - 1)!$. The Proposition follows.
2.4.8. Lemma. Let $P$ be a homogeneous element of the algebra $U(n_+)$ of weight $\gamma$, such that $P \cdot 1_\lambda$ is a singular vector of the Verma module $M_\lambda$ of weight $\lambda + \gamma$. Then the operator $P(Q): \pi_\lambda \to \pi_{\lambda+\gamma}$ commutes with the action of the derivative $\partial$.

Proof. The action of the derivative $\partial$ on $\pi_\lambda$ differs from its action on $\pi_0$ by the operator of multiplication by $\lambda - 1$. We have: $[\lambda - 1, \tilde{Q}_i] = (\alpha_i, \lambda)T_i$. Therefore, by Lemma 2.2.7, the commutator of the operator $\tilde{Q}_i: \pi_\lambda \to \pi_{\lambda+\alpha_i}$ with $\partial$ is equal to $(\alpha_i, \lambda)T_i$. Hence, the commutator of the monomial $\tilde{Q}_{i_m} \ldots \tilde{Q}_{i_1}: \pi_\lambda \to \pi_{\lambda+\gamma}$, where $\gamma = \sum_{j=1}^m \alpha_{i_j}$, with $\partial$ is equal to

$$\sum_{j=1}^m (\alpha_{i_j}, \lambda + \alpha_{i_1} + \ldots + \alpha_{i_{j-1}})\tilde{Q}_{i_m} \ldots T_{i_j} \ldots \tilde{Q}_{i_1}.$$ 

This precisely coincides with the action of

$$\sum_{i=1}^l \frac{(\alpha_{i_1}, \alpha_i)}{2} f_i,$$

where the $f_i$, $i = 1, \ldots, l$, are the generators of the Lie algebra $n_-$, on the vector $e_{i_m} \ldots e_{i_1}1_\lambda$ of the Verma module $M_\lambda$. If $P1_\lambda$ is a singular vector in $M_\lambda$, then

$$\sum_{i=1}^l \frac{(\alpha_{i_1}, \alpha_i)}{2} f_i \cdot P1_\lambda = 0,$$

and so $[\partial, P(Q)] = 0$.

2.4.9. Corollary. The higher differentials $\delta^j, j > 1$, of the complex $F^*(g)$ commute with the action of the derivative $\partial$.

Proof. Each $\delta^j$ is a linear combination of maps $P_{s',s}(Q)$. Since by definition $P_{s',s}$ defines a singular vector, such a map commutes with $\partial$ by Lemma 2.4.8.

2.4.10. Theorem. The space $I_0(g)$ of local integrals of motion of the classical Toda field theory, associated with $g$, coincides with the quotient of $W_0(g)$ by the total derivatives and constants.

Proof. Using Corollary 2.4.9, we can construct the double complex $\mathbb{C} \to F^*(g) \to F^{*+}(g) \to \mathbb{C}$, which is shown on Fig. 2.

By Corollary 2.4.9, the total differential of this complex is nilpotent. If we compute the cohomology of this complex by means of the spectral sequence, in which the 0th differential is vertical, then in the first term we obtain the complex $\tilde{F}^*(g) = \oplus_{j>0} \tilde{F}^j(g)$, where

$$\tilde{F}^j(g) = \oplus_{l(s) = j} F_{\rho - s(\rho)}.$$ 

By definition, the 0th cohomology of the complex $F^*(g)$ is the space $I_0(g)$. Therefore it coincides with the 1st cohomology of our double complex.

But we can compute this cohomology by means of the other spectral sequence, in which the 0th differential is the horizontal one. Then the Theorem follows from Proposition 2.4.7 in the same way as in the proof of Proposition 2.1.17.
2.4.11. The Adler-Gelfand-Dickey algebra. According to Theorem 2.4.10, the Poisson algebra $I_0(g)$ of local integrals of motion of the Toda field theory associated to $g$ is the algebra of local functionals on a certain hamiltonian space, $H(g)$.

This Poisson algebra coincides with the Adler-Gelfand-Dickey (AGD) algebra, or the classical $W$-algebra [71, 2].

The Drinfeld-Sokolov reduction [32, 33] produces the hamiltonian space of the AGD algebra as the result of a hamiltonian reduction of a hyperplane in the dual space to the affinization $\hat{g}$ of $g$. Following the standard technique of hamiltonian reduction [83], one can obtain this algebra as the 0th cohomology of the corresponding (classical) BRST complex.

It was explained in [48, 56, 57] that the complex $F^*(g)$ appears as the first term of a spectral sequence, associated to the BRST complex of the Drinfeld-Sokolov reduction. Therefore $I_0(g)$ is precisely the AGD algebra. We also see that higher cohomologies of the BRST complex vanish.

Usually, one constructs a map, which is called the Miura transformation, from the hamiltonian space $Lh$ to $H(g)$, which preserves the Poisson structures. The image of the inverse map of the spaces of functionals embeds the AGD algebra into $F_0$. As we have explained, the image of this map coincides with the algebra of local integrals of motion of the corresponding Toda field theory, and can be characterized in very simple terms as the intersection of the kernels of certain linear operators, acting from $F_0$ to the spaces $F_{\alpha_i}$.

For the classical simple Lie algebras explicit formulas for the Miura transformation map are known [33]. They give explicit formulas for the generators $W_n^{(i)}$ of the Poisson algebra $I_0(g)$.

For example, the AGD hamiltonian space $H(sl_n)$ is isomorphic to the space of differential operators on the circle of the form

$$\partial^n_t + \sum_{i=1}^{n-1} W^{(i)}(t)\partial^{n-i-1}_t.$$ 

The Miura transformation from the space $Lh$, which consists of functions on the circle
with values in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_n$, $u(t) = (u^1(t), \ldots, u^{n-1}(t))$, to $H(\mathfrak{sl}_n)$ can be constructed as follows.

Introduce new variables $v^1(t), \ldots, v^n(t)$, such that $\sum_{i=1}^n v^i(t) = 0$, and $u^i(t) = v^i(t) - v^{i+1}(t)$. Then put

$$\partial^i_v + \sum_{i=1}^{n-1} W^{(i)}(t) \partial^{n-i-1} = (\partial_t + v^1(t)) \ldots (\partial_t + v^n(t)).$$

These formulas allow to express $W^{(i)}(t)$ as a differential polynomial in $u^j(t)$ (cf. §2.1.15 for the case of $\mathfrak{sl}_2$, when the AGD algebra is isomorphic to the classical Virasoro algebra). One can find other generators $W^{(i)}(t)$ of $I_0(\mathfrak{sl}_n)$, which transform as tensor fields on the circle under changes of variables [28].

2.4.12. Integrals of motion in the extended space of local functionals. As in §2.1.4, we can extend our Poisson algebra of local functionals $\mathcal{F}_0$ by adjoining all Fourier components of differential polynomials, not only the $(-1)$st ones. Let $\hat{\mathcal{F}}_0$ be the space of functionals on $L\mathfrak{h}$, which can be represented as residues of differential polynomials with explicit dependence on $t$:

$$F[u(t)] = \int P(\partial^n u^i(t); t) dt.$$

We can define the Poisson structure on $\hat{\mathcal{F}}_0$ by the same formula (2.2.1). Thus, $\hat{\mathcal{F}}_0$ is a Poisson algebra, and $\mathcal{F}_0 \subset \hat{\mathcal{F}}_0$ is its Poisson subalgebra.

Analogously, one can define the spaces $\hat{\mathcal{F}}_\gamma$, $\gamma \in P$ by allowing differential polynomials to depend on $t$.

One has the analogue of the exact sequence (2.2.2):

$$0 \longrightarrow \pi_\gamma \otimes \mathbb{C}[t, t^{-1}] \overset{\partial}{\longrightarrow} \pi_\gamma \otimes \mathbb{C}[t, t^{-1}] \longrightarrow \hat{\mathcal{F}}_\gamma \longrightarrow 0,$$

where the action of $\partial$ on $\pi_\gamma \otimes \mathbb{C}[t, t^{-1}]$ is given by $\partial \otimes 1 + 1 \otimes \partial$. If $\gamma = 0$, we have to replace the first $\pi_0 \otimes \mathbb{C}[t, t^{-1}]$ by $\pi_0 \otimes \mathbb{C}[t, t^{-1}]/\mathbb{C} \otimes \mathbb{C}.$

The Poisson bracket with $\int e^{\phi_i(t)} dt$ defines a linear operator $\hat{\mathcal{F}}_0 \rightarrow \hat{\mathcal{F}}_{\alpha_i}$, which we also denote by $Q_i$. We can then define the space of integrals of motion of the Toda field theory as the intersection of kernels of the operators $Q_i : \hat{\mathcal{F}}_0 \rightarrow \hat{\mathcal{F}}_{\alpha_i}, i = 1, \ldots, l$.

We can use the methods of this section to compute this space. Indeed, consider the tensor product of the complex $F^*(\mathfrak{g})$ with $\mathbb{C}[t, t^{-1}]$, with the differentials acting on the first factor as $\partial^j$ and identically on the second factor. Such differentials commute with the action of the derivative, and therefore we can use the double complex

$$\mathbb{C} \otimes \mathbb{C} \longrightarrow F^*(\mathfrak{g}) \otimes \mathbb{C}[t, t^{-1}] \longrightarrow F^*(\mathfrak{g}) \otimes \mathbb{C}[t, t^{-1}] \longrightarrow \mathbb{C} \otimes \mathbb{C}$$

to compute the space of integrals of motion.

The cohomologies of the complex $F^*(\mathfrak{g}) \otimes \mathbb{C}[t, t^{-1}]$ are equal to the cohomologies of the complex $F^*(\mathfrak{g})$ tensored with $\mathbb{C}[t, t^{-1}]$. We deduce from Proposition 2.4.7 that the space of integrals of motion is isomorphic to the quotient of $\mathcal{W}_0(\mathfrak{g}) \otimes \mathbb{C}[t, t^{-1}]$ by the total derivatives and constants. This Poisson algebra is the algebra of local functionals on the AGD hamiltonian space $H(\mathfrak{g})$, extended in the same way – by adjoining all Fourier components of differential polynomials. It contains $I_0(\mathfrak{g})$ as a Poisson subalgebra. Sometimes it is this algebra, which is called the classical $\mathcal{W}$–algebra.

3. Classical affine Toda field theories.

3.1. The case of $\mathfrak{sl}_2$ – classical sine-Gordon theory.
3.1.1. **Hamiltonian structure.** The Hamiltonian space of the classical sine-Gordon theory is the same as the Hamiltonian space of the classical Liouville theory, namely, the space $L\mathfrak{h}$ of polynomial functions on the circle with values in the one-dimensional Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_2$, cf. §2.1.1. As the space of functions on this space, we again take the space $\mathcal{F}_0$ of local functionals. The Poisson structure on $\mathcal{F}_0$ is given by formula (2.1.2).

We also define spaces $\pi_n$, $n \in \mathbb{Z}$, as the tensor products $\pi_0 \otimes \mathbb{C}v_n$, where $\pi_0$ is the space of differential polynomials (cf. §2.1.1), and extend the action of derivative $\partial$ from $\pi_0$ to $\pi_n$ by the formula $(\partial + n \cdot u) \otimes 1$. We then put: $\mathcal{F}_n = \pi_n / \partial \pi_n$. This space has an interpretation as the space of functionals of the form

$$\int P(u(t), \partial u(t), \ldots) e^{n\phi(t)} dt.$$ 

As in §2.1.8, we can extend the Poisson structure on $\mathcal{F}_0$ to well-defined maps $\mathcal{F}_0 \times \mathcal{F}_n \to \mathcal{F}_n$ given by the formula

$$(3.1.1) \{ \int P dt, \int Re^{n\phi} dt \} = \int \frac{\delta P}{\delta u} \frac{\delta [Re^{n\phi}]}{\delta \phi} dt = \int \frac{\delta P}{\delta u} \left[ nRe^{n\phi} - \partial_t \left( \frac{\delta R}{\delta u} e^{n\phi} \right) \right] dt.$$

These brackets satisfy the Jacobi identity for any triple $F, G \in \mathcal{F}_0, H \in \mathcal{F}_n$.

3.1.2. **The sine-Gordon Hamiltonian and local integrals of motion.** The Hamiltonian $H$ of the sine-Gordon model is given by

$$H = \int e^{\phi(t)} dt + \int e^{-\phi(t)} dt \in \mathcal{F}_1 \oplus \mathcal{F}_{-1}.$$ 

In other words, it is equal to the sum of the projections of the vectors $v_{\pm 1} \in \pi_{\pm 1}$ to $\mathcal{F}_{\pm 1}$. One can check that the corresponding Hamiltonian equation

$$\partial_t U(t) = \{ U(t), H \}$$ 

coincides with the sine-Gordon equation (1.1.3).

We then define the space of local integrals of motion $I_0(\mathfrak{sl}_2)$ of the sine-Gordon theory as the kernel of the linear operator

$$\hat{Q}_1 + \hat{Q}_0 : \mathcal{F}_0 \to \mathcal{F}_1 \oplus \mathcal{F}_{-1},$$

where $\hat{Q}_1 = \{ \cdot, \int e^{\phi(t)} dt \} : \mathcal{F}_0 \to \mathcal{F}_1$, and $\hat{Q}_0 = \{ \cdot, \int e^{-\phi(t)} dt \} : \mathcal{F}_0 \to \mathcal{F}_{-1}$. In other words, $I_0(\mathfrak{sl}_2)$ is the intersection of the kernels of the operators $\hat{Q}_1$ and $\hat{Q}_0$. By the Jacobi identity, $I_0(\mathfrak{sl}_2)$ is closed with respect to the Poisson bracket. Note that the operator $\hat{Q}_1$ coincides with the operator $\hat{Q}$, defined by (2.1.11), and therefore $I_0(\mathfrak{sl}_2)$ is a Poisson subalgebra in the Poisson algebra $I_0(\mathfrak{sl}_2)$ of local integrals of motion of the Liouville theory, that is in the classical Virasoro algebra.

3.1.3. Now introduce the operators $\hat{Q}_1 : \pi_m \to \pi_{m+1}$ and $\hat{Q}_0 : \pi_m \to \pi_{m-1}$ by the formulas:

$$\hat{Q}_1 = T \sum_{n<0} S^+_{n+1} \frac{\partial}{\partial x_n}, \quad \hat{Q}_0 = -T^{-1} \sum_{n<0} S^-_{n+1} \frac{\partial}{\partial x_n},$$

where $T_{\pm} : \pi_m \to \pi_{m\pm 1}$ are the translation operators, and $S^\pm_{n}$ are the Schur polynomials:

$$\sum_{n \leq 0} S^n z^n = \exp \left( \sum_{m<0} \frac{x_m}{m} z^m \right).$$
We will also need the operators $Q_1 = T^{-1} \tilde{Q}_1$ and $Q_0 = T \tilde{Q}_0$, which are linear endomorphisms of $\pi_0$ of degree 1. Note that $S_n^+ \cong S_n$, given by formula (2.1.15), and so $\tilde{Q}_1$ coincides with $\tilde{Q}$, given by formula (2.1.14).

In the same way as in Lemma 2.1.12, one shows that the operators $\tilde{Q}_1$ and $\tilde{Q}_0$, acting from $\pi_0$ to $\pi_{\pm 1}$ commute with the action of derivative $\partial$, and that the corresponding operators $\mathcal{F}_0 \to \mathcal{F}_{\pm 1}$ coincide with $\tilde{Q}_1$ and $\tilde{Q}_0$.

The following Proposition will enable us to compute the space of local integrals of motion $I_0(\hat{\mathfrak{sl}}_2)$.

3.1.4. Proposition. The operators $Q_1$ and $Q_0$ satisfy the Serre relations of the Lie algebra $\hat{\mathfrak{sl}}_2$:

$$(\text{ad} \ Q_1)^3 \cdot Q_0 = 0, \quad (\text{ad} \ Q_0)^3 \cdot Q_1 = 0.$$ 

Proof follows from Proposition 2.2.8 and Remark 2.2.9.

3.1.5. Thus, operators $Q_0$ and $Q_1$ generate an action of the nilpotent subalgebra $\mathfrak{n}_+$ of $\hat{\mathfrak{sl}}_2$ on the space $\pi_0$. We will use this fact and the BGG resolution for $\hat{\mathfrak{sl}}_2$ to extend the complex

$$\mathcal{F}_0 \to \mathcal{F}_1 \oplus \mathcal{F}_{-1},$$

whose 0th cohomology is, by definition, the space $I_0(\hat{\mathfrak{sl}}_2)$, to a larger complex with nicer cohomologies.

Again, as in the previous section, we will be using the double complex, which consists of the spaces $\pi_n$. So, our task is to extend the complex

$$\pi_0 \to \pi_1 \oplus \pi_{-1}$$

in such a way that all higher differentials commute with the action of the derivative.

3.1.6. BGG resolution of $\hat{\mathfrak{sl}}_2$. As in the case of finite-dimensional Lie algebras, there exists a BGG resolution $B_\bullet(\hat{\mathfrak{sl}}_2)$ of $\hat{\mathfrak{sl}}_2$ [107]. It is a complex, consisting of Verma modules over $\hat{\mathfrak{sl}}_2$, whose higher homologies vanish, and the 0th homology is one-dimensional.

The $j$th group $B_j(\hat{\mathfrak{sl}}_2)$ of this complex is equal to $M_0$, if $j = 0$, and $M_{2j} \oplus M_{-2j}$, if $j > 0$. Here $M_\lambda$ stands for the Verma module over $\hat{\mathfrak{sl}}_2$ of level 0 and lowest weight $\lambda$ (that is the weight of the Cartan subalgebra of $\mathfrak{sl}_2$, embedded into $\hat{\mathfrak{sl}}_2$ as the constant subalgebra). Such a module is defined in exactly the same way as a Verma module over a finite-dimensional Lie algebra (cf. § 2.3.1).

The module $M_0$ contains the singular vectors, labeled by the elements of the Weyl group of $\mathfrak{sl}_2$. Let us denote them by $w_0$, and $w_j, w_j', j > 0$. The weight of $w_j$ (respectively, $w_j'$) is equal to $2j$ (respectively, $-2j$), and it generates the submodule of $M_0$, which is isomorphic to $M_{2j}$ (respectively, $M_{-2j}$). We have $w_1 = Y_1' w_0, w_1' = Y_1 w_0$, and $w_j = X_j w_{j-1}, w_j' = Y_j w_{j-1}, w_j = Y_j' w_{j-1},$ for $j > 1$, where $X_j, Y_j, X_j'$ and $Y_j'$ are certain elements from $U(\mathfrak{n}_+)$. The differential $d_j : B^q_j \to B^q_{j-1}, j > 0$, is given by the alternating sum of the embeddings of $M_{2j}$ and $M_{-2j}$ into $M_{2(j-1)}$ and $M_{-2(j-1)}$, which map the lowest weight vectors to the corresponding singular vectors. The nilpotency of the differential, $d_{j-1}d_j = 0$, is ensured by the commutativity of the embeddings, corresponding to the “squares” in the Weyl group and a special sign convention, analogous to the one from Lemma 2.3.3.
3.1.7. Extended complex. We are ready now to define the extended complex $F^*(\overset{\sim}{\mathfrak{sl}}_2)$ (cf. Fig. 3). The $j$th group $F^*(\overset{\sim}{\mathfrak{sl}}_2)$ of this complex is equal to $\pi_0$, if $j = 0$, and $\pi_j \oplus \pi_{-j}$, if $j > 0$. The differential $\delta^j : F^{j-1}(\mathfrak{g}) \to F^j(\mathfrak{g})$ is given by the formula $\delta^1 = Y_0(Q) + Y'_0(Q), \delta^j = X_j(Q) - (-1)^jY_j(Q) - (-1)^jY'_j(Q) + X'_j(Q), j > 1$, where we insert into these elements of $U(n_+, \pi_0)$ the operators $\tilde{Q}_1$ and $\tilde{Q}_0$ instead of the generators $e_1$ and $e_0$.

The nilpotency of the differential, $\delta^j\delta^{j-1} = 0$, follows from the relations in $U(n_+, \pi_0)$: $X'_jY_{j-1} = Y_jX_{j-1}, X_jY'_{j-1} = Y'_jX_{j-1}, X_jX_{j-1} = Y'_jY_{j-1}$, and $X'_jX'_{j-1} = Y_jY'_{j-1}$, and Proposition 3.1.4.

Note that $Y'_j(Q) = \tilde{Q}_1^{2j-1}, Y_j(Q) = \tilde{Q}_0^{2j-1}$. Other operators are more complicated, but explicit formulas for them can be obtained in principle from the commutativity relations above, using the Serre relations, in the same way as in §2.4.4.

Let us introduce a $\mathbb{Z}$-grading on our complex, by putting $\deg v_n = n^2$, and $\deg x_m = -m$. One can check that with respect to this grading the differentials $\delta^j$ are homogeneous of degree 0. Therefore our complex is a direct sum of finite-dimensional subcomplexes, corresponding to various graded components.

3.1.8. Proposition. The operators $\tilde{Q}_1$ and $\tilde{Q}_0$ define an action of the nilpotent Lie subalgebra $n_+$ of $\overset{\sim}{\mathfrak{sl}}_2$ on $\pi_0$. The cohomologies of the complex $F^*(\overset{\sim}{\mathfrak{sl}}_2)$ are isomorphic to the cohomologies of $n_+$ with coefficients in $\pi_0$, $H^*(n_+, \pi_0)$.

Proof. The same as in Proposition 2.4.5.

3.1.9. Principal commutative subalgebra. Recall that in the realization of $\overset{\sim}{\mathfrak{sl}}_2$ as the central extension of the loop algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$, the nilpotent subalgebra $n_+$ of $\overset{\sim}{\mathfrak{sl}}_2$ is identified with the Lie algebra $(\overset{\sim}{n}_+ \otimes 1) \oplus (\mathfrak{sl}_2 \otimes t\mathbb{C}[t])$, where $\overset{\sim}{n}_+$ is the nilpotent subalgebra of $\mathfrak{sl}_2$. Let $e, h$ and $f$ be the standard generators of $\mathfrak{sl}_2$. If $y$ is one of them, we will denote by $y(m)$ the element $y \otimes t^m$ of $\overset{\sim}{\mathfrak{sl}}_2$. The basis of $n_+$ consists of $e(m), m \geq 0, h(m), m > 0$, and $f(m), m > 0$.

Let $\mathfrak{a}$ be the commutative subalgebra of $n_+$, which is linearly generated by $e(m) + f(m+1), m \geq 0$. We call $\mathfrak{a}$ the principal commutative subalgebra.

3.1.10. Proposition. The cohomologies of the complex $F^*(\overset{\sim}{\mathfrak{sl}}_2)$ are isomorphic to the exterior algebra $\Lambda^*(\mathfrak{a}^*)$ of the dual space to the principal commutative subalgebra of $n_+$.
Proof. If $X$ is an operator on $\pi_0$ of the form $\sum_i X_i \partial/\partial x_i$, where $X_i$ are polynomials in $x_n$, then we can define its shift term as the sum of terms $X_i \partial/\partial x_i$, for which $X_i$ is a constant. According to the definition of the operators $Q_1$ and $Q_0$, their shift terms are equal to $\partial/\partial x_1$ and $-\partial/\partial x_1$, respectively. Thus the shift term of the operator $p = e(0) + f(1) = Q_0 + Q_1$ is equal to 0. This operator has the form $\sum_{n>0} A_n \frac{\partial}{\partial x_{n-1}}$, where $A_n = S_n^+ - S_n^-$ is a certain polynomial in $x_m$; $A_n$ is equal to $-\frac{2}{n} x_n + \text{higher power terms}$.

It is known that the Lie algebra $\mathfrak{n}_+$ splits into the direct sum $\text{Ker}(\text{ad} p) \oplus \text{Im}(\text{ad} p)$, and that $\text{Ker}(\text{ad} p) = \mathfrak{a}$. In the principal grading of $\mathfrak{n}_+$, in which $\text{deg} e(0) = 1$, $\text{deg} f(1) = 1$,

$\text{Im}(\text{ad} p)$ is linearly generated by vectors of all positive degrees, and $\text{Ker}(\text{ad} p)$ is linearly generated by vectors of all positive odd degrees.

The element $(\text{ad} p)^m \cdot e_1$, $m \geq 0$, can be chosen as a generator $y_{m+1}$ of $\text{Im}(\text{ad} p)$ of degree $m + 1$. By induction one can check that the shift term of the operator $(\text{ad} p)^m \cdot Q_1$ is equal to a non-zero multiple of $\partial/\partial x_{m-1}$. Indeed, suppose that we have shown this for $m = 0, 1, \ldots, n - 1$. Since $p$ does not have a shift term, the shift term of $y_{n+1} = [p, y_n]$ is equal to the commutator of the shift term $\partial/\partial x_{-n}$ (times a constant) of $y_n$ and a linear term of the form $x_{-n} \partial/\partial x_j$ from $p$. There is only one such summand in $p$, namely,

$$\frac{2}{n} x_{-n} \frac{\partial}{\partial x_{-n-1}}.$$ 

Its commutator with $\partial/\partial x_{-n}$ is equal to $\partial/\partial x_{-n-1}$ up to a non-zero constant. Therefore the shift term of $y_{n+1}$ equals $\partial/\partial x_{-n-1}$ up to a non-zero constant.

In the same way we can show that the shift term of any element of $\text{Ker}(\text{ad} p)$ has to be 0, because otherwise the commutator of this element with $p$ would be non-trivial.

Let us consider the module $\pi^*_0$ over $\mathfrak{n}_+$, dual to $\pi_0$. One can identify $\pi_0$ with $\pi^*_0$ as vector spaces. We can then obtain the formulas for the action of $\mathfrak{n}_+$ on $\pi^*_0$ from the formulas for its action on $\pi_0$ by interchanging $x_m$’s and $\partial/\partial x_m$’s (cf. the proof of Proposition 2.1.15). Since the Lie algebra $\mathfrak{a}$ acts on $\pi_0$ by vector fields, which have no shift terms, this Lie algebra acts by 0 on the vector $1^* \in \pi^*_0$, dual to $1 \in \pi_0$. Let $L$ be the $\mathfrak{n}_+$-module, induced from the trivial one-dimensional representation of the Lie subalgebra $\mathfrak{a}$ of $\mathfrak{n}_+$:

$$L = U(\mathfrak{n}_+) \otimes_{U(\mathfrak{a})} \mathbb{C}.$$ 

Since the vector $1^* \in \pi^*_0$ is $\mathfrak{a}$-invariant, there is a unique $\mathfrak{n}_+$-homomorphism: $L \rightarrow \pi_0$, which sends the generating vector of $L$ to $1^*$.

Under this homomorphism, a monomial $y_{i_1} \cdots y_{i_m} \otimes 1 \in L$ maps to a vector of $\pi^*_0$, which is equal to a non-zero multiple of $x_{-i_1} \cdots x_{-i_m} + \text{lower power terms}$. Therefore this map has no kernel. On the other hand, the character of the module $L$ in the principal gradation is equal to

$$\prod_{n>0} (1 - q^n)^{-1}.$$ 

This coincides with the character of the module $\pi^*_0$. Therefore, $\pi^*_0$ is isomorphic to $L$ as an $\mathfrak{n}_+$-module.

Going back, we see that the module $\pi_0$ is isomorphic to the module $L^*$, which is the $\mathfrak{n}_+$-module coinduced from the trivial representation of $\mathfrak{a}$.

By “Shapiro’s lemma” (cf. [67], §1.5.4, [76], §II.7), $H^*(\mathfrak{n}_+, \pi_0) \simeq H^*(\mathfrak{a}, \mathbb{C})$. But $\mathfrak{a}$ is an abelian Lie algebra, hence $H^*(\mathfrak{a}, \mathbb{C}) = \wedge^*(\mathfrak{a}^*)$. The Proposition now follows from Proposition 3.1.8.
3.1.11. Theorem. The space $I_0(\hat{\mathfrak{sl}}_2)$ of local integrals of motion of the sine-Gordon model is linearly generated by mutually commuting local functionals $\mathcal{H}_{2i+1}, i \geq 0$, of all positive odd degrees.

Proof. According to Proposition 3.1.10, the 1st cohomology of the complex $F^*(\hat{\mathfrak{sl}}_2)$ is isomorphic to $\mathfrak{sl}^*$. As a $\mathbb{Z}$-graded space it is a direct sum of one-dimensional subspaces, generated by some elements $h_j$ of all positive odd degrees. In the same way as in Proposition 2.4.3 one checks that the higher differentials of the complex $F^*(\hat{\mathfrak{sl}}_2)$ commute with the action of the derivative $\partial$. Therefore we can form the double complex

$$\mathbb{C} \longrightarrow F^*(\hat{\mathfrak{sl}}_2) \longrightarrow F^*(\hat{\mathfrak{sl}}_2) \longrightarrow \mathbb{C}$$

(cf. Fig. 2). The 1st cohomology of this double complex is isomorphic to the space $I_0(\hat{\mathfrak{sl}}_2)$.

We can calculate the cohomologies of this double complex $F^*(\hat{\mathfrak{sl}}_2)$ by means of the spectral sequence, in which the 0th differential is horizontal. The first term of this spectral sequence consists of two copies of the cohomologies of the complex $F^*(\hat{\mathfrak{sl}}_2)$, and the first differential coincides with the action of the derivative on them.

The 0th cohomology of the complex $F^*(\hat{\mathfrak{sl}}_2)$ is generated by the vector $1 \in \pi_0$. Clearly, the corresponding cohomology classes in the 1st term of the spectral sequence are canceled by the two spaces $\mathbb{C}$ in the double complex (cf. Fig. 2).

On the other hand, the derivative $\partial$ acts by 0 on $h_{2j+1} \in H^1(\hat{\mathfrak{sl}}_2)$, because $\partial$ has degree 1 and so it should send cohomology classes of odd degrees to cohomology classes of even degrees, which we do not have.

Since we only have two rows in our double complex, the spectral sequence collapses in the first term. Therefore, the 1st cohomology of the double complex is equal to the 1st cohomology $H^1$ of the complex $F^*(\hat{\mathfrak{sl}}_2)$, which is equal to $\bigoplus_{j \geq 0} \mathfrak{ch}_{2j+1}$, by Proposition 3.1.10.

Each $h_{2j+1}$ gives rise to a local integral of motion $\mathcal{H}_{2j+1} \in \mathcal{F}_0$ as follows. It is clear that $\partial h_{2j+1}$ is also a cocycle in $F^1(\hat{\mathfrak{sl}}_2)$, of even degree. Therefore it must be a coboundary: $\partial h_{2j+1} = \delta^1 H_{2j+1}$, for some element $H_{2j+1}$ from $F^0(\hat{\mathfrak{sl}}_2) = \pi_0$. This element is not a total derivative, because otherwise $h_{2j+1}$ would also be a coboundary. Hence $H_{2j+1} = \int H_{2j+1}(t) dt$ is non-zero and it lies in the kernel of $Q_1 + \tilde{Q}_0$, because $(Q_1 + \tilde{Q}_0) H_{2j+1} = \delta^1 H_{2j+1}$ is a total derivative.

Recall that $I_0(\hat{\mathfrak{sl}}_2)$ is closed with respect to the Poisson bracket. So, it is a Poisson subalgebra in $\mathcal{F}_0$. One can easily check that the Lie bracket in $\mathcal{F}_0$ is compatible with the $\mathbb{Z}$-grading, introduced in § 2.4.3. Therefore the degree of $\{H_{2i+1}, H_{2j+1}\}$ should be even, and we obtain $\{H_{2i+1}, H_{2j+1}\} = 0$.

3.1.12. Examples of integrals of motion. In this subsection we will analyze our complex $F^*(\hat{\mathfrak{sl}}_2)$ in low degrees and give explicit formulas for local integrals of motion of degrees 1 and 3.

Let us consider the homogeneous components of our complex $F^*(\hat{\mathfrak{sl}}_2)$ of degrees up to 4. Since $\deg v_n > 4$ for $|n| > 2$ (cf. § 3.1.7), only $F^0(\hat{\mathfrak{sl}}_2) = \pi_0, F^1(\hat{\mathfrak{sl}}_2) = \pi_1 \oplus \pi_{-1}$ and $F^2(\hat{\mathfrak{sl}}_2) = \pi_2 \oplus \pi_{-2}$ contain subspaces of degrees less than or equal to 4, cf. Fig. 4.

On Fig. 4 dots represent basis vectors in $F^0(\hat{\mathfrak{sl}}_2), F^1(\hat{\mathfrak{sl}}_2)$, and $F^2(\hat{\mathfrak{sl}}_2)$, and the dots, corresponding to vectors of the same degree, are situated at the same horizontal level.

We can see that the vector $1 \in \pi_0$ is the only element of degree 0, therefore it is necessarily in the kernel of the differential $\delta^1 : F^0(\hat{\mathfrak{sl}}_2) \to F^1(\hat{\mathfrak{sl}}_2)$. It is, as we know, the
only cohomology class in the 0th cohomology of our complex.

The second group of the complex $F^2(\hat{\mathfrak{sl}}_2)$ has degrees greater than or equal to 4. Therefore, all vectors of $\pi_1 \oplus \pi_{-1}$ of degrees 1, 2 and 3 necessarily lie in the kernel of the differential $\delta^1 : F^1(\hat{\mathfrak{sl}}_2) \to F^2(\hat{\mathfrak{sl}}_2)$. But some of them lie in the image of the differential $\delta^1$.

The component of degree 1 of the space $\pi_0$ is one-dimensional, and of $\pi_1 \oplus \pi_{-1}$ is two-dimensional, therefore, the 1st cohomology in degree 1 is one-dimensional. As a representative of this cohomology class we can take, for instance, vector $2v_{-1}$. In degree 2 both spaces have two-dimensional components, so there are no cohomologies of degree 2. In degree 3, the component of the space $\pi_0$ is three-dimensional, and the component of the space $\pi_1 \oplus \pi_{-1}$ is four-dimensional, so again we have a cohomology class. As a representative we can choose vector $x_2v_{-1}$.

In degree 4 the vectors $v_2$ and $v_{-2}$ from $F^2(\hat{\mathfrak{sl}}_2)$ are both in the kernel of the differential $\delta^3 : F^2(\hat{\mathfrak{sl}}_2) \to F^3(\hat{\mathfrak{sl}}_2)$. But some linear combination of them is in the image of the differential $\delta^2$. Indeed, we know that the differential $\delta^1$ has no kernel in degree 4. Therefore its image in the subspace of $F^2(\hat{\mathfrak{sl}}_2)$ of degree 4 is 5-dimensional. But the 1st cohomology of degree 4 is trivial. Hence the kernel of the differential $\delta^2$ is also 5-dimensional. But the space $F^1(\hat{\mathfrak{sl}}_2)$ has dimension 6 in degree 4. Therefore the image of $\delta^3$ is one-dimensional. Thus the second cohomology is one-dimensional in degree 4. It can be represented by the vector $v_{-2}$.

In view of Proposition 3.1.10, the cohomology classes that we constructed in $F^1(\hat{\mathfrak{sl}}_2)$, correspond to the generators of degrees 1 and 3 of the space $\alpha^*$, and the class that we constructed in $F^2$ corresponds to their exterior product. The encircled dots represent these classes on the picture.

Now let us assign to the 1st cohomology classes of degrees 1 and 3 local integrals of motion $H_1$ and $H_3$ (cf. Theorem 3.1.11).

The cohomology class $h_1$ of degree 1 was represented by the vector $2v_1$. According to the general procedure, described in Theorem 3.1.11, we have to take $\partial v_1 = -2x_{-1}v_1$ and find a vector $H_1$ from $\pi_0$, such that $\delta^1 H_1 = -2x_{-1}v_1$. One checks that $\frac{1}{2}x_{-1}^2 - x_{-2} \in \pi_0$ is such a vector. Therefore, we can take as the integral of motion

$$
H_1 = \int \left( \frac{1}{2} u(t)^2 - \partial_t u(t) \right) dt.
$$

According to § 2.1.7, we can define an action of $H_1$ on $\oplus_{n \in \mathbb{Z}} \pi_n$, and this action will coincide with the action of $\partial$.

The cohomology class $h_3$ of degree 3 was represented by vector $x_{-1}^2v_{-1}$. We have: $\partial h_3 = (-x_{-1}^3 + 2x_{-1}x_{-2})v_{-1}$, and one can check that $\delta^1 \frac{1}{2}(\frac{1}{2}x_{-1}^2 - x_{-2})^2 = \partial h_3$. Therefore,

$$
H_3 = \frac{1}{2} \int \left( \frac{1}{2} u(t)^2 - \partial_t u(t) \right)^2 dt
$$

is an integral of motion.
3.1.13. Connection with the KdV and mKdV systems. According to Theorem 3.1.11, the integrals of motion $H_{2j+1} \in F_0, j \geq 0$, mutually commute with respect to the Poisson bracket. In particular, we have

$$\{H_1, H_{2j+1}\} = 0, \quad \{H_3, H_{2j+1}\} = 0,$$

where $H_1$ and $H_3$ are given by formulas (3.1.2) and (3.1.3), respectively. Since $H_1 = \partial$, the first equation is satisfied automatically for any element of $F_0$ (cf. §2.1.7). However, the second equation is non-trivial, and it shows that the higher integrals of motion $H_{2j+1}, j > 1$, coincide with the higher hamiltonians of the mKdV hierarchy. In fact, this equation can be taken as a definition of these integrals of motion.

This is because $H_3$ is the hamiltonian of the modified Korteweg–de Vries (mKdV) equation. In other words, the mKdV equation

$$\partial_t U(t) = \partial^3_t U(t) - \frac{3}{2} U(t)^2 \partial_t U(t)$$

(3.1.4)

can be written in the hamiltonian form as

$$\partial_t U(t) = \{U(t), H_3\}.$$

The elements of $F_0$, which commute with $H_3$ are called the higher mKdV hamiltonians. It is known that they exist precisely for all odd degrees, therefore they coincide with our $H_{2j+1}$. Thus, we see that the local integrals of motion of the sine-Gordon model coincide with the hamiltonians of the mKdV hierarchy, and that the property of commutativity with $H_3$ defines these local integrals of motion uniquely. The mKdV hamiltonians define hamiltonian flows on $Lh$, which commute with the flow, defined by $H_3$. Altogether they define the mKdV hierarchy of partial differential equations.

The mKdV hierarchy is closely connected with the KdV hierarchy. Namely, as was explained in §2.1.18, the kernel of the operator $Q_1$ is isomorphic to the classical Virasoro algebra, which is the Poisson algebra of local functionals, depending on $W(t) = \frac{1}{2} u^2(t) - \partial_t u(t)$. Since $I_0(\mathfrak{sl}_2)$ was defined as the intersection of the kernels of the operators $Q_1$ and
$Q_0$, it is a subspace in the classical Virasoro algebra. For example, we can rewrite $H_1$ and $H_3$ via $W(t)$ by the formulas

$$H_1 = \int W(t) dt, \quad H_3 = \frac{1}{2} \int W^2(t) dt.$$  

The functional $H_3$ is the KdV hamiltonian, therefore the commutativity condition

$$\{H_3, H_{2j+1}\} = 0$$

implies that the $H_{2j+1}$'s, rewritten in terms of $W(t)$, coincide with the higher KdV hamiltonians.

These hamiltonians define the KdV hierarchy of mutually commuting flows on a hyperplane in the dual space to the Virasoro algebra, cf., e.g., [111]. The Miura transformation (cf. §2.1.18) maps the hierarchy of these flows to the mKdV hierarchy. In particular, the Miura transformation maps the mKdV equation (3.1.4) to the KdV equation

$$\partial_t W(t) = \partial_x^3 W(t) + 3W(t) \partial_t W(t).$$

3.2. General case. Let $g$ be an arbitrary affine algebra, twisted or untwisted. We will denote by $\bar{g}$ the finite-dimensional simple Lie algebra, whose Dynkin diagram is obtained by deleting the 0th node of the Dynkin diagram of $g$.

3.2.1. Hamiltonian structure. As the hamiltonian space, the spaces of functionals and the Poisson brackets of the affine Toda field theory, associated to $g$, we will take the objects, corresponding to the Toda field theory, associated to $\bar{g}$. They were defined in §2.2.1 and §2.2.2. Throughout this section we will use notation, introduced in §2.2.

3.2.2. The hamiltonian and integrals of motion. The imaginary root $\delta$ of the Lie algebra $g$ has the decomposition

$$\delta = \sum_{i=0}^l a_i \alpha_i,$$

where $\alpha_i, i = 0, \ldots, l$, are the simple roots of $g$, and the $a_i$'s are the labels of the Dynkin diagram of $g$ [81]. In particular, $a_0 = 1$ for all affine algebras, except $A^{(2)}_{2n}$, in which case $a_0 = 2$. Somewhat abusing notation, we will introduce a vector $\alpha_0$ in the Cartan subalgebra $\mathfrak{h}$ of $g$ by the formula

$$\alpha_0 = -\frac{1}{a_0} \sum_{i=1}^l a_i \alpha_i.$$

We can now define the hamiltonian of the affine Toda field theory, associated to $g$, by the formula

$$H = \frac{1}{2} \sum_{i=0}^l \int e^{\phi_i(t)} dt \in \bigoplus_{i=0}^l \mathcal{F}_{\alpha_i},$$

where $\phi_i(t) = \bar{\alpha}_i(t)$ (cf. §2.2.3). The corresponding hamiltonian equation coincides with the Toda equation (1.1.1), associated to $g$.

We define the space $I_0(g)$ of local integrals of motion as the kernel of the operator

$$\{\cdot, H\} : \mathcal{F}_0 \to \bigoplus_{i=0}^l \mathcal{F}_{\alpha_i},$$

or, in other words, as the intersection of the kernels of the operators

$$Q_i = \{\cdot, \int e^{\phi_i(t)} dt\} : \mathcal{F}_0 \to \mathcal{F}_{\alpha_i}, i = 0, \ldots, l.$$
As before, we will use the operators \( \tilde{Q}_i : \pi_0 \to \pi_\alpha, i = 0, \ldots, l \), given by formula (2.2.4), where we put

\[
x_n^0 = -\frac{1}{a_0} \sum_{i=1}^{l} a_i x_n^i.
\]

These operators commute with the action of the derivative and descend down to the operators \( \bar{Q}_i, i = 0, \ldots, l \). We will also need the operators \( Q_i = T_{i-1} \tilde{Q}_i \), acting on \( \pi_0 \).

According to Proposition 2.2.8 and Remark 2.2.9, the operators \( \tilde{Q}_i, i = 0, \ldots, l \), as well as \( Q_i, i = 0, \ldots, l \), satisfy the Serre relations of the affine algebra \( g \). Thus, the operators \( Q_i \) generate an action of the nilpotent subalgebra \( n_+ \) of \( g \) on the space \( \pi_0 \).

In the rest of this subsection we will go through the main steps of §2 and §3.1 to describe the space \( I_0(g) \). Most of the proofs are the same as in those sections.

### 3.2.3. The complex \( F^*(g) \)

For an affine algebra \( g \) there also exists a BGG resolution, which is defined in the same way and has the same properties as in the case of finite-dimensional simple Lie algebras [107].

Using this resolution, we can define the complex

\[
F^*(g) = \bigoplus_{j \geq 0} F^j(g)
\]

in the same way as in §2.4, by putting

\[
F^j(g) = \bigoplus_{l(s) = j} \pi_{\rho - s(\rho)},
\]

where \( s \) runs over the affine Weyl group. The differential \( \delta^j : F^{j-1}(g) \to F^j(g) \) is given by formula (2.4.1). In the same way as in the case of finite-dimensional simple Lie algebras (cf. Lemma 2.4.2), it follows that this differential is nilpotent.

Introduce a \( \mathbb{Z} \)-grading on this complex similar to the one introduced in §2.4.3. Namely, we put \( \deg v_{\rho - s(\rho)} = (\rho^\vee, \rho - s(\rho)) \), where \( \rho^\vee \) is an element in the dual space to the Cartan subalgebra of \( g \), such that \( (\rho^\vee, \alpha_i) = 1, \alpha_i = 0, \ldots, l \), and \( \deg x_n^i = -n \). With respect to this grading, the differentials of our complex have degree 0, and our complex decomposes into a direct sum of finite-dimensional subcomplexes.

One has the analogue of Proposition 2.4.5: the cohomologies of the complex \( F^*(g) \) are isomorphic to the cohomologies of \( n_+ \) with coefficients in \( \pi_0 \).

### 3.2.4. Principal commutative subalgebra

Consider the element \( p = \sum_{i=0}^{l} a_i e_i \) in the Lie algebra \( n_+ \). It is known that \( n_+ = \text{Ker}(\text{ad} p) \oplus \text{Im}(\text{ad} p) \), where \( \text{Ker}(\text{ad} p) \) is an abelian Lie subalgebra \( a \) of \( n_+ \), which we will call the principal commutative subalgebra.

In the principal gradation of \( n_+ \), which is obtained by assigning degree 1 to the generators \( e_i \), the Lie algebra \( a \) has a basis of homogeneous elements of degrees equal to the exponents \( d_i, i = 1, \ldots, l \), of \( g \) modulo the Coxeter number \( h \).

The space \( \text{Ker}(\text{ad} p) \) splits into the direct sum \( \text{Ker}(\text{ad} p) = \bigoplus_{j>0} n_j^+ \) of homogeneous components of degree \( j \) with respect to the principal grading. Each of these components has dimension \( l \) and the operator \( \text{ad} p : n_j^+ \to n_{j+1}^+ \) is an isomorphism.

For the proof of these facts, cf. [80], Proposition 3.8 (b). We will use them in the proof of the following statement, which is a generalization of Proposition 3.1.10.
3.2.5. Proposition. The cohomologies of the complex $F^*(\mathfrak{g})$ are isomorphic to the exterior algebra $\wedge^*(\mathfrak{a}^*)$ of the dual space to the principal commutative subalgebra $\mathfrak{a}$.

Proof. From the formulas, defining the operators $Q_i$, one can see that the operator $p = \sum_{i=0}^{l} a_i Q_i$ does not have a shift term (cf. the proof of Proposition 3.1.10). Therefore, other elements of the Lie subalgebra $\mathfrak{a}$ coincides with $\mathfrak{V} \partial/\partial x$ and $\mathfrak{P}$ check that the operator $\text{ad} p$ isomorphically maps $V_j$ to $V_{j+1}$. The space of shift terms of the operators from $\mathfrak{n}_+ \subset \mathfrak{n}_+$ coincides with $V_1$. By induction, in as in the proof of Proposition 3.1.10, one can show that the space of shift terms of operators from $\mathfrak{n}_+ \subset \mathfrak{n}_+$ coincides with $V_j$.

In the same way as in the proof of Proposition 3.1.10, we deduce from these facts that the $\mathfrak{n}_+$-module $\pi_0$ is isomorphic to the module, conduced from the trivial representation of $\mathfrak{a}$. Therefore the cohomology $H^*(\mathfrak{n}_+, \pi_0)$, are equal, by “Shapiro’s lemma”, to $H^*(\mathfrak{a}, \mathbb{C}) \simeq \wedge^*(\mathfrak{a}^*)$, because $\mathfrak{a}$ is an abelian Lie algebra. But $H^*(\mathfrak{n}_+, \pi_0)$ coincides with the cohomology of the complex $F^*(\mathfrak{g})$. The Proposition is proved.

3.2.6. Theorem. The space $I_0(\mathfrak{g})$ of local integrals of motion of the Toda field theory, associated to an affine algebra $\mathfrak{g}$, is linearly generated by mutually commuting local functionals of degrees equal to the exponents of $\mathfrak{g}$ modulo the Coxeter number.

Proof. If the exponents of $\mathfrak{g}$ are odd and the Coxeter number is even, then the proof simply repeats the proof of Theorem 3.1.11. We can again identify the space $I_0(\mathfrak{g})$ with the 1st cohomology of the double complex $\mathbb{C} \rightarrow F^*(\mathfrak{g}) \rightarrow F^*(\mathfrak{g}) \rightarrow \mathbb{C}$. According to Proposition 3.2.3, the 1st cohomology of the complex $F^*(\mathfrak{g})$ is linearly spanned by elements of certain odd degrees. We can then compute this cohomology using the spectral sequence in the same way as in the proof of Theorem 3.1.11.

The proof in other cases ($A_n^{(1)}, n > 1, D_2^{(1)}, E_6^{(1)}$ and $E_7^{(1)}$) will be published in [51].

3.2.7. Integrals of motion in the extended space of local functionals. We can define the space of integrals of motion of an affine Toda field theory as the intersection of the kernels of operators $Q_i : \tilde{\mathcal{F}}_0 \rightarrow \mathcal{F}_{\alpha_i}$, cf. § 2.4.12. We can use the tensor product of the complex $F^*(\mathfrak{g})$ with $\mathbb{C}[t, t^{-1}]$ to compute this space. By repeating the proof of Theorem 3.2.6, we conclude that the space of integrals of motion in the larger space $\tilde{\mathcal{F}}_0$ of local functionals coincides with the space $I_0(\mathfrak{g})$ of integrals of motion in the small space $\mathcal{F}_0$.

3.2.8. Remark. One usually defines a conservation law of the Toda field theory (in the light cone coordinates $x_+$ and $x_-$) as a pair $(P^-, P^+) \in \pi_0 \oplus (\oplus_i \pi_{\alpha_i})$, such that
\[
\frac{\partial P^-}{\partial x_+} = \frac{\partial P^+}{\partial x_-}.
\]
In our language, $\partial/\partial x_+ = \sum_i Q_i = \delta^1$ is the Poisson bracket with the Toda hamiltonian, and $\partial/\partial x_- = \partial$. Thus, the pair $(P^-, P^+)$, where $P^+$ is a 1-cocycle of the complex $F^*(\mathfrak{g})$ and $P^-$ is the density of our integral of motion, i.e. the 0-cocycle of the quotient complex $F^*(\mathfrak{g})/\partial F^*(\mathfrak{g})$ such that $\delta^1 P^- = \partial P^+$, is a conservation law. The important observation, which enabled us to find all such conservation laws was that in order to be a component of a conservation law, $P^+$ should satisfy a certain equation, namely, $\delta^2 \cdot P^+ = 0$.

The space $I_0(\mathfrak{g})$ constitutes a maximal abelian Poisson subalgebra in the classical $\mathcal{W}$-algebra $I_0(\mathfrak{g})$. One can show that the first generators of $I_0(\mathfrak{g})$ of degrees $d_i, i = 1$,
We say that the composition \( \sum \) with coordinates \( \partial \eta \) theory therefore define certain vector fields \( \langle \phi \rangle \) called the space of states. The first of them, \( \pi \) of conformal dimension \( \Delta \), then the power series \( \partial \pi \) that \( N \) field of conformal dimension \( \Delta + 1 \).

The theory of vertex operator algebras was started by Borcherds [14] and then further developed by I. Frenkel, Lepowsky and Meurman in [62] (cf. also [63, 75, 91]). In our paper [51] we identify the vector field \( \eta \) with the infinitesimal right action of a generator of degree \(-j\) of the opposite abelian subalgebra \( a_- \) of \( g \). Indeed, since \( g \) infinitesimally acts on \( N_+ \) from the right as on the big cell of the flag manifold, and \( a_- \) commutes with \( a \), the Lie algebra \( a_- \) acts on the coset space \( N_+/A \) by mutually commuting vector fields. In particular, the action of the vector field \( \partial \) corresponds to the action of the element

\[
\sum_{i=0}^{1} \frac{1}{2} f_i \in a_-,
\]

in accordance with Lemma 2.4.8.

In the next section we will show how to quantize the classical integrals of motion.

4. QUANTUM TODA FIELD THEORIES.

4.1. Vertex operator algebras. The theory of vertex operator algebras was started by Borcherds [14] and then further developed by I. Frenkel, Lepowsky and Meurman in [62] (cf. also [63, 75, 91]).

In this section we will give the definition of vertex operator algebras and study some of their properties following closely § 3 of [58]. After that we will introduce the vertex operator algebra (VOA) of the Heisenberg algebra (or the VOA of free fields), which we will need in the study of quantum integrals of motion of Toda field theories.

4.1.1. Fields. Let \( V = \bigoplus_{n=0}^{\infty} V_n \) be a \( \mathbb{Z}_+ \)-graded vector space, where \( \dim V_n < \infty \) for all \( n \), called the space of states. A field on \( V \) of conformal dimension \( \Delta \in \mathbb{Z} \) is a power series \( \phi(z) = \sum_{j \in \mathbb{Z}} \phi_j z^{-j-\Delta} \), where \( \phi_j \in \text{End} V \) and \( \phi_j V_n \subset V_{n-j} \). Note that if \( \phi(z) \) is a field of conformal dimension \( \Delta \), then the power series \( \partial \phi(z) = \sum_{j \in \mathbb{Z}} (-j-\Delta) \phi_j z^{-j-\Delta-1} \) is a field of conformal dimension \( \Delta + 1 \).

If \( z \in \mathbb{C}^\times \) is a non-zero complex number, then \( \phi(z) \) can be considered as a linear operator \( V \to V \), where \( V = \prod_{n=0}^{\infty} V_n \).

We have a natural pairing \( \langle , \rangle : V_n^* \times V_n \to \mathbb{C} \). A linear operator \( P : V \to V \) can be represented by a set of finite-dimensional linear operators \( P^j_i : V_i \to V_j, i,j \in \mathbb{Z} \), such that \( \langle A, P \cdot B \rangle = \langle A, P^j_i \cdot B \rangle \) for \( B \in V_i, A \in V_j^* \). Let \( P, Q \) be two linear operators \( V \to V \). We say that the composition \( PQ \) exists, if for any \( i,k \in \mathbb{Z}, B \in V_i, A \in V_k^* \) the series \( \sum_{j \in \mathbb{Z}} \langle A, P^j_k Q^i_j \cdot B \rangle \) converges absolutely.
4.1.2. Definition. Two fields $\phi(z)$ and $\psi(z)$ are called local with respect to each other, if

- for any $z, w \in \mathbb{C}^\times$, such that $|z| > |w|$, the composition $\phi(z)\psi(w)$ exists and can be analytically continued to a rational operator-valued function on $(\mathbb{C}^\times)^2 \setminus \text{diagonal}, R(\phi(z)\psi(w));$
- for any $z, w \in \mathbb{C}^\times$, such that $|w| > |z|$, the composition $\psi(w)\phi(z)$ exists and can be analytically continued to a rational operator-valued function on $(\mathbb{C}^\times)^2 \setminus \text{diagonal}, R(\psi(w)\phi(z));$
- $R(\phi(z)\psi(w)) = R(\psi(w)\phi(z)).$

In other words, fields $\phi(z)$ and $\psi(z)$ are local with respect to each other, if for any $x \in V_n$ and $y \in V_m^*$ both matrix coefficients $\langle y|\phi(z)\psi(w)|x \rangle$ for $|z| > |w|$ and $\langle y|\psi(w)\phi(z)|x \rangle$ for $|z| < |w|$ converge to the same rational function in $z$ and $w$ which has no poles outside the lines $z = 0, w = 0$ and $z = w$.

4.1.3. Definition. A VOA structure on $V$ is a linear map $Y(\cdot, z) : V \longrightarrow \text{End } V[[z, z^{-1}]]$ which associates to each $A \in V_n$ a field of conformal dimension $n$ (also called a vertex operator) $Y(A, z) = \sum_{j \in \mathbb{Z}} A_j z^{-j-n}$, such that the following axioms hold:

(A1) (vacuum axiom) There exists an element $|0\rangle \in V_0$ (vacuum vector) such that $Y(|0\rangle, z) = \text{Id}$ and $\lim_{z \to 0} Y(A, z)|0\rangle = A$.

(A2) (translation invariance) There exists an operator $T \in \text{End } V$ such that $\partial_z Y(A, z) = [T, Y(A, z)]$ and $T|0\rangle = 0$.

(A3) (locality) All fields $Y(A, z)$ are local with respect to each other. A VOA $V$ is called conformal of central charge $c \in \mathbb{C}$ if there exists an element $\omega \in V_2$ (called the Virasoro element), such that the corresponding vertex operator $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfies the following properties:

(C) $L_{-1} = T, L_0|_{V_n} = n \cdot \text{Id}$, and $L_2 \omega = \frac{1}{2} c|0\rangle$.

4.1.4. Proposition. A VOA $V$ automatically satisfies the associativity property: for any $A, B \in V$,

\[ R(Y(A, z)Y(B, w)) = R(Y(Y(A, z - w)B, w)), \]

where the left-hand (resp. right-hand) side is the analytic continuation from the domain $|z| > |w|$ (resp. $|w| > |z - w|$).

Proof which was communicated to us by V. Kac follows from the following two Lemmas.

4.1.5. Lemma. For any $z \in \mathbb{C}^\times$ and $w \in \mathbb{C}$, such that $|w| < |z|$, the composition $e^{wT} Y(A, z) e^{-Tw}$, where

\[ e^{wT} = \sum_{n=0}^{\infty} \frac{(wT)^n}{n!}, \]

exists and can be analytically continued to a rational operator-valued function, which is equal to $Y(A, z + w)$, i.e.

\[ R(e^{wT} Y(A, z) e^{-Tw}) = Y(A, z + w). \]

Proof. By axiom (A2), $[T, Y(A, z)] = \partial_z Y(A, z)$. Hence as a formal powers series,

\[ e^{wT} Y(A, z) e^{-Tw} = \sum_{n=0}^{\infty} \frac{w^n}{n!} \partial_z^n Y(A, z). \]
The right hand side is the Taylor expansion of $Y(A, z + w)$. Since $Y(A, z)$ is holomorphic everywhere except the origin, $e^{wT}Y(A, z)e^{-Tw}$ converges to $Y(A, z + w)$ if $|w| < |z|$.

4.1.6. Lemma. For any $z \in \mathbb{C}^\times$, $Y(A, z)B = e^{zT}Y(B, -z)A$.

Proof. By Lemma 4.1.5,

$$R(e^{(z+w)^T}Y(B, -z)A) = R(Y(B, w)e^{(z+w)^T}A).$$

We can derive from axioms (A1) and (A2) that $e^{(z+w)^T}A = Y(A, z + w)|0\rangle$. This and the previous formula give:

$$R(e^{(z+w)^T}Y(B, -z)A) = R(Y(B, w)Y(A, z + w)|0\rangle).$$

By locality,

$$R(e^{(z+w)^T}Y(B, -z)A) = R(Y(A, z + w)Y(B, w)|0\rangle).$$

Both sides of the last formula have well-defined limits when $w \to 0$, hence these limits coincide and we obtain using axiom (A1): $e^{Tz}Y(B, -z)A = Y(A, z)B$.

4.1.7. Proof of Proposition 4.1.4. For any vector $C \in V$ we have:

$$Y(A, z)Y(B, w)C = Y(A, z)e^{wT}Y(C, -w)B$$

for $|z| > |w|$, where we applied Lemma 4.1.6 to $Y(B, w)C$. By applying $1 = e^{Tw}e^{-Tw}$ to both sides of this formula (we can do this, because $e^{\pm Tw}$ is a series infinite in only one direction) and using Lemma 4.1.6 in the right hand side of the previous formula we obtain

$$Y(A, z)Y(B, w)C = e^{wT}Y(A, z - w)Y(C, -w)B$$

for $|z| > |w|$. On the other hand, consider

$$Y(Y(A, z - w)B, w)C = \sum_{n \in \mathbb{Z}}(z - w)^{-\Delta^A}Y(A_n \cdot B, w)C$$

as a formal power series in $(z - w)$. By Lemma 4.1.6,

$$Y(A_n \cdot B, w)C = e^{wT}Y(C, -w)A_n B.$$}

Hence

$$Y(Y(A, z - w)B, w)C = e^{wT}Y(C, -w)Y(A, z - w)B,$$

as formal power series in $(z - w)$. But the right hand side of (4.1.3) converges when $|w| > |z - w|$ and can be analytically continued to a rational function in $z$ and $w$, by axiom (A3). Therefore the left hand side of (4.1.3) has the same properties.

By applying axiom (A3) to the analytic continuations of the right hand sides of (4.1.2) and (4.1.3), we obtain the equality of the analytic continuations of the left hand sides:

$$R(Y(A, z)Y(B, w))C = R(Y(A, z - w)B, w)C$$

for any $C \in V$, and Proposition 4.1.4 follows.
4.1.8. **Operator product expansion.** We may rewrite formula (4.1.1) as

\[ Y(A, z)Y(B, w) = \sum_{n \in \mathbb{Z}} (z - w)^{-n-\deg A} Y(A_n \cdot B, w), \]

using the formula \( Y(A, z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-\deg A} \). Here and further on to simplify notation we omit \( R(\cdot) \) in formulas for analytic continuation of functions. Such an identity is called an operator product expansion (OPE). There exists such \( M \in \mathbb{Z} \) that \( A_n \cdot B = 0 \) for any \( n > M \). Therefore, the right hand side of this formula has only finitely many terms with negative powers of \( (z - w) \). Combining axioms (3), (4), and the Cauchy theorem we obtain the following identity [62]:

\[
\int_{C^w} \int_{C^z} Y(A, z)Y(B, w)f(z, w) \, dz \, dw - \int_{C^w} \int_{C^z} Y(B, w)Y(A, z)f(z, w) \, dz \, dw = \\
\int_{C^w} \int_{C^z} \sum_{n \in \mathbb{Z}} (z - w)^{-n-\deg A} Y(A_n \cdot B, w)f(z, w) \, dz \, dw,
\]

where \( C^x \) denotes a circle of radius \( x \) around the origin, \( R > \rho > r \), \( C^\delta(w) \) denotes a small circle of radius \( \delta \) around \( w \), and \( f(z, w) \) is an arbitrary rational function on \((C^x)^2 \setminus \text{diagonal})\.

This formula can be used in order to compute commutation relations between Fourier components of vertex operators. Indeed, if we choose \( f(z, w) = z^{m+\deg A - 1}w^{k+\deg B - 1} \), then we obtain:

\[
[A_m, B_k] = \sum_{n > -\deg A} \frac{dw}{w} \int_{C^w} \int_{C^z} \sum_{n \in \mathbb{Z}} \frac{z^{m+\deg A - 1}}{(z - w)^{n+\deg A}} Y(A_n \cdot B, w) = \\
\sum_{-\deg A < n \leq m} \left( \frac{m+\deg A - 1}{n+\deg A - 1} \right) (A_n \cdot B)_{m+k}.
\]

In particular, we see that only the terms in the OPE, which are singular at the diagonal \( z = w \), contribute to the commutator. This formula also shows that the commutator of Fourier components of two fields is a linear combination of Fourier components of other fields (namely, the ones, corresponding to the vectors \( A_n \cdot B \)). Therefore we obtain the following result.

4.1.9. **Theorem.** The space of all Fourier components of vertex operators defined by a VOA is a Lie algebra.

In particular we derive from formula (4.1.1) and axiom (C) that

\[
Y(\omega, z)Y(\omega, w) = \frac{c/2}{(z-w)^4} + \frac{2Y(\omega, w)}{(z-w)^2} + \frac{\partial_w Y(\omega, w)}{z-w} + O(1),
\]

which implies using (4.1.4) the following commutation relations between the Fourier coefficients \( L_n, n \in \mathbb{Z} \), of \( Y(\omega, z) \):

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n,-m}c.
\]

These are the defining relations of the Virasoro algebra with central charge \( c \).
4.1.10. Remark. The axioms of VOA may look rather complicated, but in fact they are quite natural generalizations of the axioms of a \( \mathbb{Z} \)-graded associative commutative algebra with a unit. Indeed, such an algebra is defined as a \( \mathbb{Z} \)-graded vector space \( V \) along with a linear operator \( Y : V \to \text{End}(V) \) of degree 0 and an element \( 1 \in V \), such that \( Y(1) = \text{Id} \). The linear operator \( Y \) defines a product structure by the formula \( A \cdot B = Y(A)B \). The axioms of commutativity and associativity of this product then read as follows: 
\[
Y(A)Y(B) = Y(B)Y(A), \text{ and } Y(Y(A)B) = Y(A)Y(B).
\]

On a VOA \( V \) the operator \( Y(\cdot, z) \) defines a family of “products” – linear operators \( V \to V \), depending (in the formal sense) on a complex parameter \( z \). The axiom (A3) and formula (4.1.1) can be viewed as analogues of the axioms of commutativity and associativity (with a proper regularization of the compositions of operators), and the first half of the axiom (A1) is the analogue of the axiom of unit. A rather surprising aspect of the theory of VOA is that the “associativity” property (4.1.1) follows from the “commutativity” (locality) (A3) together with (A1) and (A2).

Another novelty is vector \( \omega \). The meaning of this vector is the following. The Fourier components of \( Y(\omega, z) \) define on \( V \) an action of the Virasoro algebra, which is the central extension of the Lie algebra of vertex fields on a punctured disc. The existence of such an element inside \( V \) means that all infinitesimal changes of the coordinate \( z \) can be regarded as “interior automorphisms” of \( V \).

4.1.11. Ultralocal fields. Let us call two fields \( \phi(z) \) and \( \psi(z) \) \textit{ultralocal} with respect to each other if there exists an integer \( N \), such that for any \( v \in V_n \) and \( v^* \in V^*_m \), both series \( \langle v^*|\phi(z)\psi(w)|v\rangle(z-w)^N \) and \( \langle v^*|\psi(w)\phi(z)|v\rangle(z-w)^N \) are equal to the same finite polynomial in \( z^{\pm 1} \) and \( w^{\pm 1} \). Clearly, ultralocality implies locality. Moreover, in a vertex operator algebra any two vertex operators are automatically ultralocal with respect to each other according to formula (4.1.1) and the fact that the \( \mathbb{Z} \)-gradation on \( V \) is bounded from below.

Given two fields \( \phi(z) \) and \( \psi(z) \) of conformal dimensions \( \Delta_\phi \) and \( \Delta_\psi \) one defines their \textit{normally ordered product} as the field
\[
\phi(z)\psi(z) := \sum_{n \in \mathbb{Z}} \left( \sum_{m < -\Delta_\phi} \phi_m \psi_{n-m} + \sum_{m \geq -\Delta_\phi} \psi_{n-m} \phi_m \right) z^{-n-\Delta_\phi-\Delta_\psi}
\]
of conformal dimension \( \Delta_\phi+\Delta_\psi \). The Leibniz rule holds for the normally ordered product:
\[
\partial_z : \phi(z)\psi(z) := : \partial_z \phi(z)\psi(z) : + : \phi(z)\partial_z \psi(z) : .
\]

The following proposition proved in [58], Proposition 3.1, allows one to check easily the axioms of a VOA.

4.1.12. Proposition. Let \( V \) be a \( \mathbb{Z}_+ \)-graded vector space. Suppose that to some vectors \( a^{(0)} = |0\rangle \in V_0, a^{(1)} \in V_{\Delta_1}, \ldots, \) one associates fields \( Y(|0\rangle, z) = \text{Id}, Y(a^{(1)}, z) = \sum_j a^{(1)}_j z^{-j-\Delta_1}, \ldots, \) of conformal dimensions \( 0, \Delta_1, \ldots, \) such that the following properties hold:

\begin{enumerate}
\item all fields \( Y(a^{(i)}, z) \) are ultralocal with respect to each other;
\item \( \operatorname{lim}_{z \to 0} Y(a^{(i)}, z)|0\rangle = a^{(i)}; \)
\item the space \( V \) has a linear basis of vectors
\end{enumerate}
\[
a^{(k_1)}_{-j_1-\Delta_{k_1}} \cdots a^{(k_s)}_{-j_s-\Delta_{k_s}}|0\rangle, \quad j_1, \ldots, j_s \in \mathbb{Z}_+;
\]
there exists an endomorphism \( T \) of \( V \) such that
\[
[T, a_{-j-\Delta_k}^{(k)}] = (j + 1)a_{-j-\Delta_k-1}^{(k)}, \quad T([0]) = 0.
\]
Then letting
\[
Y(a_{-j_1-\Delta_{k_1}}^{(k_1)} \cdots a_{-j_r-\Delta_{k_r}}^{(k_r)} | 0), z)
= (j_1! \cdots j_r!)^{-1} : \partial z^1 Y(a^{(k_1)}, z) \cdots \partial z^r Y(a^{(k_r)}, z) :
\]
(where the normal ordering of more than two fields is nested from right to left), gives a well-defined VOA structure on \( V \).

Proof. We define the map \( Y(\cdot, z) \) by formula (4.1.9). It is clear that axiom (A1) holds. Given two fields \( \phi(z) \) and \( \psi(z) \), if \([T, \phi(z)] = \partial_z \phi(z)\) and \([T, \psi(z)] = \partial_z \psi(z)\), then from (4.1.5) and (4.1.6) it follows that
\[
[T : \phi(z) \psi(z) :] = \partial_z : \phi(z) \psi(z) :.
\]
Hence the axiom (A2) follows inductively from (4.1.9) and (4.1.10).

Using an argument of Dong (cf. [11], Proposition 3.2.7), one can show that if three fields \( \chi(z), \phi(z) \) and \( \psi(z) \) are ultralocal with respect to each other, then : \( \phi(z) \psi(z) : \) and \( \chi(z) \) are ultralocal.

Indeed, by assumption, there exists such \( r \in \mathbb{Z}_+ \) that
\[
(w - z)^s \phi(z) \psi(w) = (w - z)^s \psi(w) \phi(z),
\]
(4.1.12)
\[
(u - z)^s \phi(z) \chi(u) = (u - z)^s \chi(u) \phi(z),
\]
(4.1.13)
\[
(u - w)^s \psi(w) \chi(u) = (u - w)^s \chi(u) \psi(w),
\]
for any \( s \geq r \).

Consider the formal power series in \( z, w \) and \( u \):
\[
(w - u)^3 [(z - w)^{-1} \phi(z) \psi(w) - (z - w)^{-1} \psi(w) \phi(z)] \chi(u),
\]
where \( (z - w)^{-1} \) is considered as a power series in \( w/z \) in the first summand and as a power series in \( z/w \) in the second summand.

This series is equal to
\[
(w - u)^r \sum_{s=0}^{2r} \binom{2r}{s} (w - z)^s (z - u)^{2r-s} \left[ (z - w)^{-1} \phi(z) \psi(w) - (z - w)^{-1} \psi(w) \phi(z) \right] \chi(u).
\]
The terms with \( r < s \leq 2r \) in the last formula vanish by (4.1.11). Hence we can rewrite it as
\[
(w - u)^r \sum_{s=0}^{r} \binom{2r}{s} (w - z)^s (z - u)^{2r-s} \left[ (z - w)^{-1} \phi(z) \psi(w) - (z - w)^{-1} \psi(w) \phi(z) \right] \chi(u),
\]
and further as
\[
(w - u)^r \sum_{s=0}^{r} \binom{2r}{s} (w - z)^s (z - u)^{2r-s} \chi(u) \left[ (z - w)^{-1} \phi(z) \psi(w) - (z - w)^{-1} \psi(w) \phi(z) \right],
\]
using (4.1.12) and (4.1.13). By the same trick as above, we see that (4.1.14) is equal to
\[
(w - u)^3 \chi(u) \left[ (z - w)^{-1} \phi(z) \psi(w) - (z - w)^{-1} \psi(w) \phi(z) \right].
\]
Now it follows from the definition of normal ordering (4.1.5) that the series \( \phi(w)\psi(w) : \) is the coefficient of \( z^{-1} \) in the series

\[
(z - w)^{-1}\phi(z)\psi(w) - (z - w)^{-1}\psi(w)\phi(z),
\]

where again \( (z - w)^{-1} \) is considered as a power series in \( w/z \) in the first summand and as a power series in \( z/w \) in the second summand.

Hence if we take the coefficients in front of \( z^{-1} \) in formulas (4.1.14) and (4.1.15), we obtain the following equality of formal power series in \( w^{\pm 1} \) and \( u^{\pm 1} \):

\[
(w - u)^{3r} : \phi(w)\psi(w) : \chi(u) = (w - u)^{3r} \chi(u) : \phi(w)\psi(w) : .
\]

But since the gradation on \( V \) is bounded from below, the matrix coefficients of the left hand side are finite in \( u^{-1} \) and \( w \), whereas the matrix coefficients of the right hand side are finite in \( w^{-1} \) and in \( u \). Therefore both are polynomials in \( w^{\pm 1} \) and \( u^{\pm 1} \), and so \( : \phi(z)\psi(z) : \) and \( \chi(z) \) are ultralocal.

It is also clear that if \( \phi(z) \) and \( \psi(z) \) are ultralocal, then \( \partial_z \phi(z) \) and \( \psi(z) \) are ultralocal. This implies axiom (A3) and completes the proof.

4.2. The VOA of the Heisenberg algebra.

4.2.1. Let \( \mathfrak{h} \) be the Cartan subalgebra of a simple finite-dimensional Lie algebra \( \mathfrak{g} \). In § 2.1.4 we defined the Heisenberg algebra \( \hat{\mathfrak{h}} \). It has generators \( b^i_n, i = 1, \ldots, l, n \in \mathbb{Z} \), and relations

\[
[b^i_n, b^j_m] = n (\alpha_i, \alpha_j) \delta_{n, -m}.
\]

In the case \( \mathfrak{g} = \mathfrak{sl}_2 \) we will have generators \( b_n, n \in \mathbb{Z} \), and relations

\[
[b_n, b_m] = n \delta_{n, -m}.
\]

For \( \lambda \in \mathfrak{h}^* \), let \( \pi_\lambda \) be the Fock representation of \( \hat{\mathfrak{h}} \), which is freely generated by the operators \( b^i_n, i = 1, \ldots, l, n \in \mathbb{Z} \), from a vector \( v_\lambda \), such that

\[
b^i_n v_\lambda = 0, n > 0, \quad b^i_0 v_\lambda = (\lambda, \alpha_i) v_\lambda.
\]

The fact that we used the same notation \( \pi_\lambda \) for these representations as for the spaces of differential polynomials in § 3 will be justified later on.

We want to introduce a structure of VOA on \( \pi_0 \).

First we introduce a \( \mathbb{Z} \)-grading on \( \pi_0 \) by putting \( \deg v_0 = 0 \), \( \deg b(n) = -n \), so that \( \deg b(n_1) \ldots b(n_m) v_0 = -\sum_{i=1}^m n_i \).

Next we introduce a linear map \( Y(\cdot, z) \) from \( \pi_0 \) to \( \text{End}_{\pi_0}[[z, z^{-1}]] \).

Defining the operator \( Y(\cdot, z) \) amounts to assigning to each homogeneous vector \( A \in \pi_0 \) a formal power series

\[
Y(A, z) = \sum_{n \in \mathbb{Z}} A_n z^{-n - \deg A}.
\]

In this formula, the Fourier component \( A_n \) stands for some linear operator of degree \(-n\), acting on \( \pi_0 \).

Let us define these operators by explicit formulas. The degree 0 subspace of the module \( \pi_0 \) is spanned by one vector: \( v_0 \). The corresponding operator \( Y(v_0, z) \) is equal to the identity operator (times 1, and all other Fourier components are equal to 0).

In degree one we have \( l \) linearly independent vectors: \( b^i_1 v_0, i = 1, \ldots, l \). We assign to them the fields

\[
Y(b^i_{-1} v_0, z) = b^i(z) = \sum_{n \in \mathbb{Z}} b^i_n z^{-n - 1}.
\]
In degree 2 the module $\pi_0$ has two types of vectors: $b_{-2}^i v_0$, to which we assign

$$Y(b_{-2}^i v_0, z) = \partial_z b^i(z),$$

where $\partial_z = \partial/\partial z$; and $b_{-1}^i b_{-1}^j v_0$, to which we assign

$$Y(b_{-1}^i b_{-1}^j v_0, z) =: b^i(z) b^j(z) := \sum_{n \in \mathbb{Z}} \left( \sum_{m \in \mathbb{Z}} :b^i(m) b^j(n - m): \right) z^{-n-2}.$$

Here the columns denote the normal ordering. If we have a monomial in $b_n^i$'s, its normal ordering is the ordering, in which all “creation” operators $b_n^i$, $n < 0$, are to the left of all “annihilation” operators $b_n^i$, $n \geq 0$.

The general formula for a monomial basis element $b_{n_1}^{i_1} \ldots b_{n_m}^{i_m} v_0$ of $\pi_0$ is the following:

$$Y(b_{n_1}^{i_1} \ldots b_{n_m}^{i_m} v_0, z) = \frac{1}{(-n_1 - 1)!} \ldots \frac{1}{(-n_m - 1)!} :\partial_z^{-n_1-1} b^{i_1}(z) \ldots \partial_z^{-n_m-1} b^{i_m}(z) :.$$

By linearity, we can extend the map $Y(\cdot, z)$ to any vector of $\pi_0$.

Finally, we put $|0\rangle = v_0$ and

$$T = \frac{1}{2} \sum_{i=1}^l \sum_{n \in \mathbb{Z}} :b_n^i b_n^{i*} :,$$

where $b_n^{i*}$ are the dual generators of the Heisenberg algebra, i.e. the following commutation relations hold:

$$[b_n^i, b_m^{i*}] = n \delta_{i,j} \delta_{n,-m}.$$

We also have:

$$[\partial, b_n^i] = -nb_n^{i-1}.$$

4.2.2. **Theorem.** The map $Y(\cdot, z)$ defined by formula (4.2.1) together with the vector $|0\rangle$ and the operator $T$ define the structure of VOA on $\pi_0$.

**Proof.** Consider the space $\pi_0$ and put $a^{(0)} = v_0$, $a^{(1)} = b(-1) v_0$. Define also $T$ by formula (4.5.2). Then all conditions of Proposition 4.1.12 will be satisfied. In particular, the fact that $Y(a^{(1)}, z) = b(z)$ is ultralocal with itself follows from the computation in §4.2.3. Therefore, by Proposition 4.1.12, the map $Y(\cdot, z)$ defined in §4.2 satisfies the axioms of vertex operator algebra.

The VOA $\pi_0$ can be given a structure of conformal VOA with an arbitrary central charge. Define for any $\gamma \in \mathfrak{h}$ the elements $\omega_\gamma$ by the formula

$$\omega_\gamma = \left( \frac{1}{2} \sum_{i=1}^l b_{i-1}^i b_{i*} + \gamma_{-2} \right) v_0.$$

These elements satisfy the axiom (C) with $c = 1 - 12\|\gamma\|^2$. 
4.2.3. Commutation relations. Now we are going to compute the commutation relations in the Heisenberg algebra $\hat{\mathfrak{h}}$ using formula (4.1.4). First let us compute $b^i(z)b(w)$ for $|z| > |w|$. We have to rewrite this composition as a linear combination of well-defined, i.e. normally ordered, linear operators on $\pi_0$. We have

\[
b^i(z)b^j(w) = \sum_{n,m \in \mathbb{Z}} b^i(n)b^j(m)z^{-n-1}w^{-m-1} = \sum_{n<0,m \in \mathbb{Z}} b^i(n)b^j(m)z^{-n-1}w^{-m-1} + \sum_{n \geq 0,m \in \mathbb{Z}} b^i(m)b^j(n)z^{-n-1}w^{-m-1} =: b^i(z)b^j(w) + \sum_{n \geq 0} n(\alpha_i, \alpha_j) \frac{1}{z^2} \left( \frac{w}{z} \right)^{n-1} =: b^i(z)b^j(w) + \frac{\alpha_i, \alpha_j}{(z-w)^2},
\]

We can further rewrite this as

\[
(4.2.3) \quad b^i(z)b^j(w) = \frac{(\alpha_i, \alpha_j)}{(z-w)^2} + \sum_{n \geq 0} \frac{1}{n!} (z-w)^n : \partial_w^n b^i(w)b^j(w) :,
\]

by Taylor’s formula.

On the other hand, we have: $b^i(n) \cdot b^i(-1)v_0 = 0$, for $n > 0$ or $n = 0$, $b^i(1) \cdot b^i(-1)v_0 = (\alpha_i, \alpha_j)v_0$, and $b^i(n) \cdot b^i(-1)v_0 = b^i(n)b^i(-1)v_0$, $n < 0$. Hence we obtain

\[
(4.2.4) \quad Y(Y(b^i(-1)v_0, z - w) \cdot b^i(-1)v_0, w) = \sum_{n \in \mathbb{Z}} (z-w)^{-n-1}Y(b^i(n) \cdot b^i(-1)v_0, w) = \frac{(\alpha_i, \alpha_j)}{(z-w)^2} + \sum_{n \geq 0} \frac{1}{n!} (z-w)^n : \partial_w^n b^i(w)b^i(w) :,
\]

which coincides with formula (4.2.3), in accordance with formula (4.1.1). Note that the second computation is simpler.

Now we obtain using formula (4.1.4):

\[
[b^i(n), b^j(m)] = \int_{C^i_0} dw w^m \int_{C^j_0(w)} dz z^n \frac{(\alpha_i, \alpha_j)}{(z-w)^2} = n(\alpha_i, \alpha_j) \int dw w^{n+m-1} = n(\alpha_i, \alpha_j) \delta_{n,-m},
\]

in accordance with the commutation relations of $\hat{\mathfrak{h}}$.

4.2.4. Bosonic vertex operators. Other Fock representations $\pi_\lambda$ carry the structure of a module over the VOA $\pi_0$ (for general definition, cf. [88]). The Fourier components of the vertex operators $Y(A, z)$, given by formula (4.2.1), define linear operators on any of the modules $\pi_\lambda$, and so we obtain maps $\pi_0 \to \text{End}(\pi_\lambda)[[z, z^{-1}]]$, which satisfy axioms similar to the axioms of VOA.

There is also another structure: a map $\pi_\lambda \to \text{Hom}(\pi_0, \pi_\lambda)[[z, z^{-1}]]$, in other words, to each vector of $\pi_\lambda$ we can assign a field, whose Fourier components are linear operators, acting from $\pi_0$ to $\pi_\lambda$.

Let us define this map by explicit formulas. To the highest weight vector $v_\lambda \in \pi_\lambda$ we associate the following field, which is called the bosonic vertex operator:

\[
(4.2.5) \quad \tilde{V}_\lambda(z) = \sum_{n \in \mathbb{Z}} V_\lambda(n)z^{-n} = T_\lambda \exp \left( -\sum_{n<0} \frac{\lambda_n z^{-n}}{n} \right) \exp \left( -\sum_{n>0} \frac{\lambda_n z^{-n}}{n} \right),
\]

where $T_\lambda: \pi_0 \to \pi_\lambda$ is the shift operator, which maps $v_0$ to $v_\lambda$ and commutes with the operators $b^i_n$, $n < 0$, and $\lambda_n = \lambda \otimes t^n$ is the element of the Heisenberg algebra $\mathfrak{h} = h \otimes \mathbb{C}[t, t^{-1}]$, corresponding to the element $\lambda \in h$ ($\mathfrak{h}$ is identified with $\mathfrak{h}^*$ by means of the scalar product).
The field $\tilde{V}_\lambda(z)$ can be viewed as the normally ordered exponential of
\[
\tilde{\chi}(z) = \int^z \lambda(w)dw = -\sum_{n \neq 0} \frac{\lambda_n z^{-n}}{n} + \lambda_0 \log z.
\]

We can then define the fields for other elements of $\pi_\lambda$ as follows:
\[
Y(Pv_\lambda, z) := Y(Pv_0, z)\tilde{V}_\lambda(z) ;,
\]
where $P$ is a polynomial in $b_n^i, n < 0$.

These fields satisfy the axioms (A1) and (A2). The axiom (A3) and formula (4.1.1) are also satisfied for any pair $A \in \pi_0, B \in \pi_\lambda$, or $B \in \pi_0, A \in \pi_\lambda$.

Using formula (4.1.1), it is easy to compute the commutation relations between $b_n^i$ and $V_\lambda(m)$. Indeed, from the fact that $b_n^i v_\lambda = (\lambda, \alpha_i) v_\lambda$ we derive the following OPE:
\[
b_n^i(z) V_\lambda(w) = \frac{(\lambda, \alpha_i) V_\lambda(w)}{z - w} + \text{regular terms},
\]
which gives (using the technique of § 4.2.3) the commutation relations
\[
[ b_n^i, V_\lambda(m) ] = (\lambda, \alpha_i) V_\lambda(n + m).
\]

In the next subsection we will show that the operators $\int V_{\beta \alpha_i}(z)dz$, where $\beta$ is a deformation parameter, can be viewed as quantizations of the classical operators $\tilde{Q}_i$ from § 4.2 corresponding to the value $\beta = 0$. We will use these operators to define the space of quantum integrals of motion of the Toda field theories and related structures.

4.3. Quantum integrals of motion.

4.3.1. Quantization of the Poisson algebra. Let us introduce a deformation parameter $\beta$ and modify the commutation relations in our Heisenberg algebra by putting
\[
[ b_n^i, b_m^j ] = n(\alpha_i, \alpha_j) \delta_{n,-m} \beta^2.
\]
If $g = sl_2$, we put
\[
[ b_n, b_m ] = n \delta_{n,-m} \beta^2.
\]

We also modify the definition of the Fock module $\pi_\lambda$ by putting $b_0^i v_\lambda = \beta(\alpha_i, \lambda) v_\lambda$. Of course, if $\beta \neq 0$, this algebra and these modules are equivalent to the original Heisenberg algebra and Fock modules if we multiply the old generators by $\beta$.

According to Theorem 4.1.3, the vector space of all Fourier components of fields from a VOA is a Lie algebra. It contains a Lie subalgebra, which consists of residues of fields, i.e. their (-1)st Fourier components. Indeed, according to formula (4.1.4),
\[
[ \int Y(A, z)dz, \int Y(B, z)dz ] = \int Y(\int Y(A, w)dw \cdot B, z)dz.
\]

Let us denote the Lie algebra of all Fourier components of fields from the VOA $\pi_0$, by $\tilde{\mathcal{F}}_0^\beta$, and its Lie subalgebra of the residues of the fields – by $\mathcal{F}_0^\beta$. They depend on $\beta$, because $\beta$ enters the defining commutation relations (4.3.1). Since the commutation relations in these Lie algebras are polynomial in $\beta^2$, we can consider $\tilde{\mathcal{F}}_0^\beta$ and $\mathcal{F}_0^\beta$ as Lie algebras over the ring $\mathbb{C}[\beta^2]$, which are free as $\mathbb{C}[\beta^2]$–modules.

The constant term in the commutation relations in $\tilde{\mathcal{F}}_0^\beta$ is always equal to 0. We can therefore define the structure of Lie algebra on the space $\tilde{\mathcal{F}}_0^\beta/(\beta^2 \cdot \tilde{\mathcal{F}}_0^\beta)$, by taking the $\beta^2$–linear term in the commutator. That is for any pair $A, B \in \tilde{\mathcal{F}}_0^\beta/(\beta^2 \cdot \tilde{\mathcal{F}}_0^\beta)$ consider their
arbitrary liftings \( \tilde{A}, \tilde{B} \in \hat{F}_0 \) and define \( \{A, B\} \) as the \( \beta^2 \)-linear term in the commutator of \( \tilde{A} \) and \( \tilde{B} \):
\[
[\tilde{A}, \tilde{B}] = \beta^2 \{A, B\} + \beta^4 (\ldots).
\]

Clearly this bracket does not depend on the liftings of \( A \) and \( B \) and satisfies the axioms of Lie bracket.

As vector spaces \( \hat{F}_0^\beta / (\beta^2 \cdot \hat{F}_0^\beta) \) and \( \mathcal{F}_0^\beta / (\beta^2 \cdot \mathcal{F}_0^\beta) \) are isomorphic to \( \hat{F}_0 \) and \( \mathcal{F}_0 \), respectively:
\[
f : P(\partial_i^n b^i(z)) : z^n \, dz \text{ is identified with } \int P(\partial_i^n u^i(t)) t^n \, dt.
\]

4.3.2. \textbf{Lemma.} The Poisson structure on \( \hat{F}_0^\beta / (\beta^2 \cdot \hat{F}_0^\beta) \), induced by the Lie algebra structure on \( \hat{F}_0^\beta \), coincides with the Poisson structure on \( \hat{F}_0 \), given by formula (2.2.1).

\textbf{Proof.} Recall that the Poisson structure on \( \hat{F}_0 \) is uniquely determined by the Poisson bracket of the functionals \( u^i_n = \int u^i(t) t^n \, dt \), which is equal to
\[
\{u^i_n, u^j_m\} = n(\alpha_i, \alpha_j) \delta_{n,-m}.
\]

Indeed, any local functional can be represented by an infinite sum
\[
\sum_{n_1 + \cdots + n_m = M} c_{n_1 \ldots n_m} \cdot u^{i_1}_{n_1} \cdots u^{i_m}_{n_m},
\]
and we can compute the Poisson bracket of two such sums term by term, using the Leibniz rule (cf. § 2.1.4).

Likewise, the commutation relations in the Lie algebra \( \hat{F}_0^\beta \) are uniquely determined by formula (4.3.1), and we can again compute the Lie bracket of two Fourier components term by term. But then we immediately see that the \( \beta^2 \)-linear term in the Lie bracket in \( \hat{F}_0^\beta \) coincides with the Poisson bracket in \( \hat{F}_0 \).

4.3.3. Thus, the Lie algebras \( \hat{F}_0^\beta \) and \( \mathcal{F}_0^\beta \) can be viewed as quantizations of the Poisson algebras \( \hat{F}_0 \) and \( \mathcal{F}_0 \), respectively.

Let \( \mathcal{F}_\lambda^\beta \) be the space of linear operators \( \pi_0 \rightarrow \pi_\lambda \) of the form \( \int Y(A, z) dz, A \in \pi_\lambda \).

Formula (4.3.2) shows that the Lie algebra \( \mathcal{F}_\lambda^\beta \) acts on \( \mathcal{F}_\lambda^\beta \) by commutation. In the same way as in Lemma 4.3.2 we can check that this action is a quantization of the action of \( \mathcal{F}_0 \) on \( \pi_\lambda \), defined in § 2.2.2. In other words, we can check that the \( \beta^2 \)-linear term in the commutator between operators from \( \mathcal{F}_0^\beta \) and \( \mathcal{F}_\lambda^\beta \) coincides with the Poisson bracket between the corresponding elements of \( \mathcal{F}_0 \) and \( \mathcal{F}_\lambda \).

Moreover, the actions of \( \mathcal{F}_\lambda^\beta \) from \( \pi_0 \) to \( \pi_\lambda \) and of \( \mathcal{F}_0^\beta \) from \( \pi_\lambda \) to \( \pi_\lambda \) are quantizations of the actions of \( \mathcal{F}_\lambda \) from \( \pi_0 \) to \( \pi_\lambda \) and of \( \mathcal{F}_0 \) from \( \pi_0 \) to \( \pi_0 \), defined in § 2.1.11. More precisely, let us consider the Fock representation \( \pi_\lambda \) as a free module \( \pi_\lambda^\beta \) over \( \mathbb{C}[[\beta^2]] \). The operator \( \int Y(A, z) dz \) can be considered over the ring \( \mathbb{C}[[\beta^2]] \). It has the form \( \int Y(A, z) dz = \beta^2 \cdot Y(A)^{(0)} + \beta^4 \cdot (\ldots) \). Hence it induces the operator \( Y(A)^{(0)} \) on the quotients \( \pi_\lambda^\beta / \beta^2 \cdot \pi_\lambda \). Such a quotient can be identified with the space \( \pi_\lambda \) of differential polynomials by identifying \( b^i_n, n < 0 \), with \( x^i_n \). Note that the actions of derivative on both spaces coincide. We have
\[
Y(A)^{(0)} dz = \{ \int \hat{A} dt, \}
\]
where \( \hat{A} = A \mod \beta^2 \). A particular example of this formula, when \( A = v_{\alpha_i} \), will be considered in Lemma 4.3.4.
Consider the map $\pi_\lambda \to \mathcal{F}_\lambda^\beta$, which sends $A \in \pi_\lambda$ to $\int Y(A,z)dz$. From the classical result of §2.2.2 we derive that the kernel of this map consists of total derivatives, and therefore we have the exact sequences

\begin{equation}
(4.3.3) \quad 0 \to \pi_\lambda \overset{\partial}{\to} \pi_\lambda \to \mathcal{F}_\lambda^\beta \to 0
\end{equation}

(if $\lambda = 0$, then $\pi_0$ should be replaced by $\pi_0/C\pi_0$).

Analogously, one obtains the exact sequence

\begin{equation}
(4.3.4) \quad 0 \to \pi_\lambda \otimes \mathbb{C}[z, z^{-1}] \overset{\partial}{\to} \pi_\lambda \otimes \mathbb{C}[z, z^{-1}] \to \bar{\mathcal{F}}_\lambda^\beta \to 0
\end{equation}

(if $\lambda = 0$, then the first $\pi_0 \otimes \mathbb{C}[z, z^{-1}]$ should be replaced by $\pi_0 \otimes \mathbb{C}[z, z^{-1}]/(C\pi_0 \otimes \mathbb{C})$).

Now we can quantize the Toda hamiltonian. Let us define the map $\widehat{Q}_i^\beta : \pi_0 \to \pi_\lambda^\beta$ as $\int \bar{V}_i(z)dz$. For any $A \in \pi_\lambda$ the operator $\int Y(A,z)dz$ commutes with the derivative $\partial$. Hence it defines a map $\mathcal{F}_0 \to \mathcal{F}_\lambda$. Denote the map $\mathcal{F}_0^\beta \to \mathcal{F}_\lambda^\beta$, corresponding to $\widehat{Q}_i^\beta$, by $\bar{Q}_i^\beta$.

4.3.4. Lemma. The $\beta^2$–linear terms of the operators $\widehat{Q}_i^\beta$ and $\bar{Q}_i^\beta$ coincide with the operators $-\bar{Q}_i$ and $-Q_i$, respectively.

Proof. According to formula (4.3.1), the operator $b_i^n, n > 0$, acts on $\pi_0$ as

$$
\beta^2 \sum_{j=1}^l (\alpha_i, \alpha_j) \frac{\partial}{\partial b_{i-j}^n}.
$$

Thus, we obtain from formula (4.2.5):

$$
\bar{Q}_i^\beta = \int \exp \left( - \sum_{n<0} \frac{b_i^n z^{-n}}{n} \right) \exp \left( - \sum_{n>0} \frac{b_i^n z^{-n}}{n} \right) dz =
$$

$$
\int \exp \left( - \sum_{n<0} \frac{b_i^n z^{-n}}{n} \right) dz + \int \exp \left( - \sum_{n<0} \frac{b_i^n z^{-n}}{n} \right) \left( - \sum_{n>0} \frac{b_i^n z^{-n}}{n} \right) dz + \ldots =
$$

$$
-\beta^2 \sum_{n<0} \bar{V}_i(n+1) \sum_{j=1}^l (\alpha_i, \alpha_j) \frac{\partial}{\partial b_i^n} + \beta^4 (\ldots),
$$

where $\bar{V}_i(n+1)$ are the Schur polynomials in $b_i^n, n < 0$. If we replace $b_i^n, n < 0$ by $x_i^n$, then the linear term in this formula will coincide with formula (2.2.4), which defines the operator $\bar{Q}_i$, with the sign minus. Therefore, the $\beta^2$–linear term in $\bar{Q}_i^\beta$ coincides with the operator $-\bar{Q}_i$.

4.3.5. Definition. The intersection of kernels of the linear operators $Q_i^\beta : \mathcal{F}_0^\beta \to \mathcal{F}_{\alpha_1}^\beta, i \in S$, will be called the space of local integrals of motion of the quantum Toda field theory, associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$ (in this case $S = \{1, \ldots, l\}$), or an affine Lie algebra $\mathfrak{g}$ (in this case $S = \{0, \ldots, l\}$), and will be denoted by $I_\beta(\mathfrak{g})$.

4.3.6. Definition. The intersection of kernels of the operators $\bar{Q}_i^\beta : \pi_0 \to \pi_{\alpha_i}, i = 1, \ldots, l$, will be called the $\mathcal{W}$-algebra of a finite-dimensional simple Lie algebra $\mathfrak{g}$ and will be denoted by $\mathcal{W}_\beta(\mathfrak{g})$. 
4.3.7. We will say that $U$ is a vertex operator subalgebra of a vertex operator algebra $V$, if there is an embedding of vector spaces $i: U \rightarrow V$, which preserves the $\mathbb{Z}$-gradings, such that $i(|0\rangle_U) = |0\rangle_V$, and that for any $A, B \in U$

\[(4.3.5) \quad i \cdot [Y(A, z) \cdot B] = Y(i \cdot A, z) \cdot (i \cdot B).\]

We will say that $U$ is a conformal vertex operator subalgebra of $V$, if in addition $i(\omega_U) = \omega_V$.

4.3.8. Lemma. Let $A(1), \ldots, A(N)$ be homogeneous vectors of $\pi_{\lambda_1}, \ldots, \pi_{\lambda_N}$, respectively. Then

(a) The intersection of kernels of the operators $\int Y(A(j), z)dz : \pi_0 \rightarrow \pi_{\lambda_j}$, $j = 1, \ldots, N$, is a vertex operator subalgebra of $\pi_\lambda$;

(b) The intersection of kernels of the operators $\int Y(A(j), z)dz : \mathcal{F}_0 \rightarrow \mathcal{F}_{\lambda_j}$, $j = 1, \ldots, N$, is a Lie subalgebra of $\mathcal{F}_0$.

**Proof.** (a) Denote by $X$ the intersection of kernels of the operators $\int Y(A(j), z)dz$ for $j = 1, \ldots, N$, and by $i$ the embedding of $X$ into $\pi_0$. Since the elements $A^{(j)}$ are homogeneous, these operators are homogeneous, and hence $X$ is $\mathbb{Z}$-graded. Clearly, $v_0 \in \pi_0$ lies in $X$, so we can put $|0\rangle_X = v_0$. Thus, we have $i(|0\rangle_X) = |0\rangle_{\pi_0}$. We have to show that for any $B, C \in X$ the vector $B_k \cdot C$, where $B_k$ is a Fourier component of the field $Y(B, z)$, lies in $X$ for any $k \in \mathbb{Z}$. In other words, we have to show that

$$\int Y(A^{(j)}, z)dz \cdot (B_k \cdot C) = 0, \quad j = 1, \ldots, N.$$

But this follows from vanishing of the commutators

$$[\int Y(A^{(j)}, z)dz, B_k] = 0, \quad j = 1, \ldots, N,$$

and the fact that $\int Y(A^{(j)}, z)dz \cdot C = 0$. The commutator vanishes according to formula (4.1.4): $\int Y(A^{(j)}, z)dz = A^{(j)}_{-\deg A^{(j)}+1}$, so there is only one term in the commutator, which vanishes, because $A^{(j)}_{-\deg A^{(j)}+1} \cdot B = \int Y(A^{(j)}, z)dz \cdot B = 0$.

(b) follows at once from formula (4.1.4).

Thus we see that $I_\beta(g)$ is a Lie algebra and $W_\beta(g)$ is a VOA.

4.3.9. A quantum integral of motion can be represented as $x = x^{(0)} + \beta^2 x^{(1)} + \cdots \in \mathcal{F}_0^\beta$. By Lemma 4.3.4, the constant term $x^{(0)}$ should be a classical integral of motion. We will call $x$ a deformation or a quantization of $x^{(0)}$. Clearly, if exists, it is uniquely defined up to adding $\beta^2$ times other quantum integrals of motions. In other words, the dimension of the space of quantum integrals of motion of a given degree is less than or equal to the dimension of the space of classical integrals of motion of the same degree. In the rest of this section we will show that the dimension of the space of integrals of motion of a given degree of a Toda field theory for generic values of $\beta$ is the same as for $\beta = 0$. Since the quantum integrals of motion depend algebraically on $\beta$, this will imply that all classical integrals of motion of the Toda field theories can be deformed. The same is true for the VOA $W_\beta(g)$.

4.4. Liouville theory.
4.4.1. In the quantum Liouville theory the space of integrals of motion is defined as the kernel of the operator $\tilde{Q}^\beta : \pi_0 \to \pi_1$. We will proceed in the same way as in §2.1, by computing the kernel of the operator $\tilde{Q}^\beta : \pi_0 \to \pi_1$ and then using the spectral sequence. Note that since the operator $\tilde{Q}^\beta$ is a residue of a field, its kernel is a VOA, by Lemma 4.3.8.

Thus, we consider the complex

$$\pi_0 \xrightarrow{\beta^{-2}\tilde{Q}^\beta} \pi_1,$$

where we normalized $Q^\beta = \int \tilde{V}(z)dz$ by $\beta^{-2}$. According to Lemma 4.3.4, this complex makes sense even when $\beta = 0$, when it coincides with the classical complex (2.1.17). We have therefore a family of complexes, depending on a complex parameter $\beta$. Moreover, as a vector space our complex does not change, and the differential depends on $\beta$ polynomially in each homogeneous component. The following simple observation will enable us to prove that all classical integrals of motion can be quantized.

4.4.2. **Lemma.** Suppose, one is given a family of finite-dimensional complexes, depending on a complex parameter $\beta$, which are the same as vector spaces, and the differentials depend analytically on $\beta$. Then for any $n$ the dimension of the $n$th cohomology group of the complex is the same for generic values of $\beta$, and it may only increase for special values of $\beta$. In particular, if a certain cohomology group of the complex vanishes, when $\beta = 0$, then it also vanishes for generic values of $\beta$.

**Proof.** The $n$th cohomology is the quotient of the kernel of the $(n + 1)$st differential by the image of the $n$th differential. The dimension of the kernel (respectively, the image) of a finite-dimensional linear operator, depending analytically on a parameter $\beta$, is the same for generic values of $\beta$, and it may only increase (respectively, decrease) for special values of $\beta$.

4.4.3. **Corollary.** Under the conditions of Lemma 4.4.2, suppose also that all higher cohomologies of the complex vanish, when $\beta = 0$. Then the dimension of the 0th cohomology for generic $\beta$ is the same as for $\beta = 0$.

**Proof.** Vanishing of the higher cohomology groups for generic values of $\beta$ follows from vanishing for $\beta = 0$ and Lemma 4.4.2. Since the complex does not depend on $\beta$ as a vector space, its Euler characteristics, which is equal to the alternating sum of the dimensions of the groups of the complex, also does not depend on $\beta$. But the Euler characteristics is also equal to the alternating sum of the dimensions of the cohomology groups. Since the dimensions of higher cohomology groups are 0 for generic $\beta$ and for $\beta = 0$, the Euler characteristics is equal to the dimension of the 0th cohomology group for generic $\beta$ and for $\beta = 0$. Hence the dimension of the 0th cohomology group for generic $\beta$ is the same as for $\beta = 0$.

4.4.4. **Proposition.** The operator $\tilde{Q}^\beta : \pi_0 \to \pi_1$ has no cokernel for generic $\beta$. Its kernel, $W_\beta(\mathfrak{sl}_2)$, is a conformal vertex operator algebra. It contains a Virasoro element $W_\beta^\beta v_0$, and $W_\beta(\mathfrak{sl}_2)$ is freely generated from $v_0$ under the action of the Fourier components $W_n^\beta$, $n < -1$, of the field

$$Y(W_\beta^\beta v_0, z) = \sum_{n \in \mathbb{Z}} W_n^\beta z^{-n-2}.$$  

This VOA is isomorphic to the VOA of the Virasoro algebra with central charge $c = c(\beta) = 13 - 3\beta^2 - 12\beta^{-2}$.  


Proof. By Proposition 2.1.15, the operator $-\tilde{Q}$, which, according to Lemma 4.3.4, is the limit of the operator $\beta^{-2}\tilde{Q}$, when $\beta \to 0$, has no cokernel. In other words, the 1st cohomology of the complex (4.4.1) vanishes, when $\beta = 0$. This complex decomposes into a direct sum of finite-dimensional subcomplexes with respect to the $\mathbb{Z}$-grading. By Corollary 4.4.3, the character of the 0th cohomology for generic $\beta$ ( = the kernel of the operator $\tilde{Q}$) is the same as for $\beta = 0$:

$$\prod_{n \geq 2} (1 - q^n)^{-1}.$$

This formula means that there is a vector $W^\beta_{-2}v_0$ of degree 2 in the kernel of $\tilde{Q}$. One can deduce from the axioms of VOA and the fact that the kernel of $\tilde{Q}$ is a VOA (cf. Lemma 4.3.8) that this vector must be a Virasoro element in $\pi_0$. But we can also find an explicit formula for this vector:

(4.4.2) $W^\beta_{-2}v_0 = \left(\frac{1}{2\beta^2}b_{-1}^2 + \left(\frac{1}{2} - \frac{1}{\beta^2}\right)b_{-2}\right)v_0$.

The Fourier components $W^\beta_n$ of the corresponding field generate an action of the Virasoro algebra with central charge $c(\beta)$ on $\pi_0$ and $\pi_1$. This action commutes with the action of the differential $\tilde{Q}$. The operators $W^\beta_n, n < -1$, act freely on $\pi_0$ for any value of $\beta$, because they act freely for $\beta = 0$ (cf. Proposition 2.1.15). Therefore, by applying these operators to $v_0$ we obtain a subspace of the kernel, which has the same character as the kernel; hence it coincides with the kernel.

Clearly, this is the vertex operator algebra of the Virasoro algebra, which is defined, e.g., in [44].

4.4.5. Corollary. The space $I_\beta(\mathfrak{s}l_2)$ of local integrals of motion of the quantum Liouville model is isomorphic to the Lie algebra of residues of fields of the Virasoro vertex operator algebra.

Proof. By definition, $I_\beta(\mathfrak{s}l_2)$ coincides with the 1th cohomology of the complex $\mathcal{C} \to \mathcal{F}_0^\beta \to \mathcal{F}_1^\beta \to \mathbb{C}$. We can use the spectral sequence (2.2.6) to compute this cohomology, in the same way as in the proof of Proposition 2.1.17. The Corollary then follows from Proposition 4.4.4 and the fact that for any VOA $V$ the quotient of $V$ by the total derivatives and constants is isomorphic to the space of residues of fields.

4.4.6. General case. We now want to show that all classical integrals of motion of the classical Toda field theory, associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$, can be deformed. In order to establish that, we will construct a quantum deformation $F^\beta_\mathfrak{g}(\mathfrak{g})$ of the complex $F^\ast(\mathfrak{g})$ and then use vanishing of higher cohomologies in the classical limit $\beta = 0$.

Roughly speaking, this can be achieved as follows. We will first construct a resolution $B^q_\mathfrak{g}(\mathfrak{g})$ over the quantized universal enveloping algebra (quantum group) of $\mathfrak{g}$, $U_q(\mathfrak{g})$, which is a deformation of the standard BGG resolution $B_\mathfrak{g}(\mathfrak{g})$. The resolution $B^q_\mathfrak{g}(\mathfrak{g})$ consists of Verma modules $M^q_{\rho - s(\rho)}$ over $U_q(\mathfrak{g})$, and the differentials are given by linear combinations of embeddings of Verma modules, defined by singular vectors $P^q_{s'}\cdot 1_{\rho - s'(\rho)} \in M^q_{\rho - s(\rho)}$, where $P^q_{s',s} \in U_q(\mathfrak{n}_+)$. In the limit $q \to 1$ this resolution coincides with the BGG resolution of $\mathfrak{g}$.

We will define the quantum deformations of the operators $P^q_{s'}(\mathfrak{g}) : \pi_{\rho - s(\rho)} \to \pi_{\rho - s'(\rho)}$ as linear combinations of certain multiple integrals of products of bosonic vertex operators
then in the limit $q \rightarrow 0$ it turns out that the operators $\tilde{Q}_i^\beta$ satisfy the $q-$deformed Serre relations, which are the relations in the quantized universal enveloping algebra of the nilpotent subalgebra of $\mathfrak{g}$, $U_q(\mathfrak{n}_+)$, with $q = \exp(\pi i \beta^2)$. This will allow us to define differentials on the quantum complex.

Using the quantum complex, we will show that all classical integrals of motion can be quantized.

### 4.5. Quantum groups and quantum BGG resolutions.

#### 4.5.1. Quantum group $U_q(\mathfrak{g})$.

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra associated to a symmetric Cartan matrix $\|a_{ij}\|, i, j \in S$. The quantum group $U_q(\mathfrak{g})$ \cite{BFST, FF}, where $q \in \mathbb{C}^\times, q \neq \pm 1$, is the associative algebra over $\mathbb{C}$ with generators $e_i, f_i, K_i, K_i^{-1}, i \in S$, and the relations:

\[
K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,
\]

\[
K_i e_j K_i^{-1} = q^{(\alpha_i, \alpha_j)} e_j, \quad K_i f_j K_i^{-1} = q^{-(\alpha_i, \alpha_j)} f_j,
\]

\[
[e_i, f_j] = \delta_{ij} q^{(\alpha_i, \alpha_i)/2} - q^{-(\alpha_i, \alpha_i)/2},
\]

and the so-called $q$-Serre relations, which can be defined as follows.

Introduce a grading on the free algebra with generators $e_i, i = 1, \ldots, l$, with respect to the weight lattice $P$, by putting $\deg e_i = \alpha_i$. If $x$ is a homogeneous element of this algebra of weight $\gamma$, put

\[
ad_q e_i \cdot x = e_i x - q^{(\alpha_i, \gamma)} x e_i.
\]

Likewise, we can introduce operators $\ad_q f_i$ on the free algebra with generators $f_i, i = 1, \ldots, l$. Then the $q$-Serre relations read:

\[(\ad_q e_i)^{-a_{ij} + 1} \cdot e_j = 0, \quad (\ad_q f_i)^{-a_{ij} + 1} \cdot f_j = 0.\]  \hspace{1cm} (4.5.1)

If we put

\[
h_i = \frac{K_i - K_i^{-1}}{q^{(\alpha_i, \alpha_i)/2} - q^{-(\alpha_i, \alpha_i)/2}},
\]

then in the limit $q \rightarrow 1$ these relations coincide with the standard relations of $\mathfrak{g}$ in terms of $e_i, f_i, h_i, i \in S$.

Denote by $U_q(\mathfrak{n}_+)$ the subalgebra of $U_q(\mathfrak{g})$, generated by $e_i, i = 1, \ldots, l$, with the relations (4.5.1). This algebra is a quantum deformation of the universal enveloping algebra of the nilpotent subalgebra $\mathfrak{n}_+$ of $\mathfrak{g}$. Let $U_q(\mathfrak{b}_-)$ be the subalgebra of $U_q(\mathfrak{g})$, generated by $f_i, K_i, K_i^{-1}, i = 1, \ldots, l$.

#### 4.5.2. Verma modules.

Verma modules over $U_q(\mathfrak{g})$ are defined in the same way as Verma modules over $\mathfrak{g}$ (cf. \S 2.3.3).

Let $\mathbb{C}_\lambda$ be the one-dimensional representation of $U_q(\mathfrak{b}_-)$, which is spanned by vector $1_\lambda$, such that

\[
f_i \cdot 1_\lambda = 0, \quad K_i \cdot 1_\lambda = q^{(\lambda, \alpha_i)} 1_\lambda, \quad i = 1, \ldots, l.
\]

The Verma module $M_\lambda^q$ over $U_q(\mathfrak{g})$ of lowest weight $\lambda$ is the module induced from the $U_q(\mathfrak{b}_-)$-module $\mathbb{C}_\lambda$:

\[
M_\lambda^q = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b}_-)} \mathbb{C}_\lambda.
\]
4.5.3. **Singular vectors.** We want to construct a resolution $B_j^q(g)$ over $U_q(g)$, which would coincide with the standard BGG resolution in the limit $q \to 1$. Therefore as a vector space, the $j$th group $B_j^q(g)$ of this resolution should be the direct sum of Verma modules

$$B_j^q(g) = \oplus_{l(s)=j} M^q_{p-s}(\rho).$$

Now we have to construct the differentials.

We want to show first that the structure of singular vectors in the modules $M^q_{p-s}(\rho)$ for generic $q$ is the same as the structure of the singular vectors for $q = 1$, which is described in §2.3.2. We will then proceed in the same way as in the case $q = 1$.

The existence of these singular vectors can be derived from the determinant formula for the Shapovalov form on $M^q_\lambda$. This formula has been established in [26] (cf. formula (1.9.3)) for finite-dimensional simple Lie algebras. In [79] (cf. Lemma 6.4) the irreducible factors of the formula were found for arbitrary symmetrizable Kac-Moody algebras. This is sufficient for our purposes.

The above cited results show that the Shapovalov form on $M^q_\lambda$ at weight $\eta$ vanishes, if one of the following equations is satisfied:

$$\lambda - \rho, \gamma + \frac{m}{2}(\gamma, \gamma) = 0,$$

where $\gamma$ runs over the set of positive roots of $g$, and $m$ runs over the set of positive integers, such that $m\gamma < \eta$. Note that in this formula we have signs different from those in [26, 79], because we work with modules of lowest weight.

Consider now the module $M^q_{p-s}(\rho)$. It is known that for any $s'$, which satisfies $s \prec s'$ and $l(s') = l(s) + 1$, there exist $\gamma$ and $m$, such that $\rho - s'(\rho) = \rho - s(\rho) + m\gamma$ and $(s(\rho), \gamma) - \frac{m}{2}(\gamma, \gamma) = 0$. Therefore the equation (4.5.2) is satisfied and hence the determinant is equal to 0 at weight $\rho - s'(\rho)$.

4.5.4. **Remark.** In fact, in order to prove this statement, we do not need the exact formula for the determinant. We can proceed along the lines of [82], using the Casimir operators, constructed in [34] or [79] and the limit $q \to 1$ of the determinant formula from [82].

4.5.5. On the other hand, the determinant is not equal to 0 at any level $\eta < \rho - s'(\rho)$ for $q = 1$ [82], and hence for generic $q$. Therefore in $M^q_{p-s}(\rho)$ there should exist a singular vector of weight $\rho - s'(\rho)$.

This singular vector defines a map

$$i^q_{s', s} : M^q_{\rho-s'(\rho)} \to M^q_{\rho-s(\rho)}$$

by sending the lowest weight vector $1^q_{\rho-s'(\rho)}$ of $M^q_{\rho-s'(\rho)}$ to the singular vector $P^q_{s', s} \cdot 1^q_{\rho-s(\rho)}$ of $M^q_{\rho-s(\rho)}$ of weight $\rho - s'(\rho)$. This map is an embedding of Verma modules for generic $q$, because it is an embedding for $q = 1$.

We can map singular vectors, constructed this way, inductively to $M^q_0$. Thus, we obtain in $M^q_0$ singular vectors of weights $\rho - s(\rho)$ for arbitrary elements $s$ of the Weyl group.

We know that for $q = 1$ there is only one singular vector of weight $\rho - s(\rho)$ in $M_0$. But the dimension of the space of singular vectors of a Verma module $M^q_\lambda$ of a certain weight for generic $q$ is greater than or equal to that for $q = 1$. Therefore there is only one singular vector of such weight in $M^q_\lambda$ for generic $q$.

Uniqueness implies the relation

$$P^q_{s', s_1} P^q_{s_1, s} = P^q_{s', s_2} P^q_{s_2, s}$$

for
in \( U_q(\mathfrak{n}_+). \) Indeed, both \( P_{s,t,s}^q P_{s,t,s}^q 1_q \) and \( P_{s,t,s}^q P_{s,t,s}^q 1_q \) are singular vectors in \( M_{\rho-s(\rho)} \) of weight \( \rho - s''(\rho). \) Therefore they coincide for generic \( q. \)

4.5.6. The differential. Now we can construct a differential on the complex \( B^j_q(\mathfrak{g}). \) For any pair \( s, s' \) of elements of the Weyl group of \( \mathfrak{g}, \) such that \( s \prec s', \) we have the embeddings of \( \mathfrak{g} \)-modules \( i_{s',s}^q : M_{\rho-s'(\rho)}^q \to M_{\rho-s(\rho)}^q. \) They satisfy: \( i_{s',s}^q \circ i_{s',s}^q = i_{s',s}^q \circ i_{s',s}^q, \) according to (4.5.3). We define the differential \( d^j_q : B^j_q(\mathfrak{g}) \to B^j_{q-1}(\mathfrak{g}) \) by the formula

\[
d^j_q = \sum_{l(s)=j-1, l(s')=j, s<s'} \epsilon_{s',s} i_{s',s}^q, \tag{4.5.4}
\]

(cf. formula (2.3.1)). By construction, this differential is nilpotent (cf. Theorem 2.3.3).

Note that since higher cohomologies of \( B^j_q(\mathfrak{g}) \) vanish for \( q = 1, \) they also vanish for generic \( q, \) by Lemma 4.4.2. Therefore the 0th cohomology is one-dimensional for generic \( q, \) by Corollary 4.4.3. Thus, for generic \( q, B_*(\mathfrak{g}) \) is a free resolution of the trivial representation of \( U_q(\mathfrak{n}_+). \)

4.5.7. Remark. In the same way we can \( q \)-deform the BGG resolution of an arbitrary integrable representation \( V_\lambda, \lambda \in P, \) of a Kac-Moody algebra. Such a resolution has also been constructed in [94] by other methods.

Vanishing of higher cohomologies of this resolution for \( q = 1 \) implies that they vanish for generic \( q. \) But then the 0th cohomology for generic \( q \) is a module over \( U_q(\mathfrak{g}), \) which is irreducible, since \( V_\lambda \) is irreducible, and whose character is the same as the character of \( V_\lambda \) for \( q = 1, \) by Lemma 4.4.3. This gives an alternative proof of the fact that any integrable representation of \( \mathfrak{g} \) can be \( q \)-deformed, previously proved by Lusztig [92] by other methods.

4.6. Toda field theories associated to finite-dimensional simple Lie algebras.

4.6.1. Now we can use the resolution \( B_*(\mathfrak{g}) \) to define the quantum complex \( F_\beta^j(\mathfrak{g}) = \bigoplus_{j \geq 0} F_\beta^j(\mathfrak{g}). \) The \( j \)th group \( F_\beta^j(\mathfrak{g}) \) of the complex is the same as the \( j \)th group of the classical complex \( F^*(\mathfrak{g}): \)

\[
F_\beta^j(\mathfrak{g}) = \bigoplus_{l(s)=j} \pi_{\rho-s(\rho)}. \tag{4.6.1}
\]

To define the differentials of the complex, we have to quantize the operators \( P_{s',s}(Q) : \pi_{\rho-s'(\rho)} \to \pi_{\rho-s(\rho)} \). In order to do that we should first learn how to compose the operators \( Q_s^\beta. \)

4.6.2. Let \( p = (p_1, \ldots, p_m) \) be a permutation of the set \( (1, 2, \ldots, m). \) We define a contour of integration \( C_p \) in the space \((\mathbb{C}^\times)^m\) with the coordinates \( z_1, \ldots, z_m \) as the product of one-dimensional contours along each of the coordinates, going counterclockwise around the origin starting and ending at the point \( z_i = 1, \) and such that the contour of \( z_{p_i} \) is contained inside the contour of \( z_{p_j} \) for \( i < j. \)

Denote by \( i = (i_1, \ldots, i_m) \) a sequence of numbers from 1 to \( l, \) such that \( i_1 \leq i_2 \leq \ldots \leq i_m. \) We can apply a permutation \( p \) to this sequence to obtain another sequence \( j = (j_1, \ldots, j_m) = (i_{p_1}, \ldots, i_{p_m}). \) Let us define an operator \( V_j^\beta \) as the integral

\[
\int_{C_p} dz_1 \ldots dz_m \prod_{1 \leq k < l \leq m} (z_l - z_k) e^{2(\alpha_{ik}, \alpha_{ik})} \prod_{1 \leq k \leq m} z_k^{\beta(\lambda, \alpha_{ik})} : \nabla_{\alpha_{i_1}}(z_1) \ldots \nabla_{\alpha_{i_m}}(z_m) :=\]

\[
\sum_{n_1, \ldots, n_m \in \mathbb{Z}} \Gamma_j^{n_1, \ldots, n_m} V_{\alpha_{i_1}}(n_1) \ldots V_{\alpha_{i_m}}(n_m), \tag{4.6.1}
\]
where the coefficient $\Gamma_{j}^{n_1 \ldots n_m}$ is given by

$$(4.6.2) \quad \Gamma_{j}^{n_1 \ldots n_m} = \int_{C_p} dz_1 \ldots dz_m \prod_{1 \leq k < l \leq m} (z_k - z_l)^{\beta^2(\alpha_{ik}, \alpha_{il})} \prod_{1 \leq k \leq m} z_k^{\beta^2(\lambda, \alpha_{ik})-n_k}.$$  

In the integrals above, we choose the branch of the power function, which takes real values for real values of the $z_i$'s, such that $z_{j_1} > z_{j_2} > \ldots > z_{j_m}$. Thus, $C_p$ should be viewed as an element of the group of relative $m$-chains in $(\mathbb{C}^\times)^m$ modulo the diagonals, with values in the one-dimensional local system $\xi_i$, which is defined by the multivalued function

$$\prod_{1 \leq k < l \leq m} (z_k - z_l)^{\beta^2(\alpha_{ik}, \alpha_{il})} \prod_{1 \leq k \leq m} z_k^{\beta^2(\lambda, \alpha_{ik})}.$$  

The integral of the type (4.6.2) over any such relative chain is well-defined for generic values of $\beta$. Indeed, the integral

$$\int_{C} dz_1 \ldots dz_m \prod_{1 \leq k < l \leq m} (z_k - z_l)^{\mu_{kl}} \prod_{1 \leq k \leq m} z_k^{\nu_k}$$

over such a chain $C$ converges in the region $\text{Re} \mu_{kl} \geq 0$, and can be uniquely analytically continued to other values of $\mu_{kl}$, which do not lie on certain hyperplanes, cf. [14], especially, Theorem (10.7.7), for details. These hyperplanes are defined by setting some linear combinations of $\mu_{kl}$'s, with integral coefficients, to negative integers. For generic $\beta$ the exponents in our integral do not lie on those hyperplanes, therefore the integral is well-defined.

Note that the integral in (4.6.2) depends on $j$ and not on $p$. For any $N \in \mathbb{Z}$ the operator

$$\sum_{n_1 \ldots + n_m = N} \Gamma_{j}^{n_1 \ldots n_m} : V_{\alpha_{i_1}}(n_1) \ldots V_{\alpha_{i_m}}(n_m) :$$

is a well-defined homogeneous operator acting from $\pi_\lambda$ to $\pi_{\lambda + \gamma}$, where $\gamma = \sum_{j=1}^{m} \alpha_{ij}$. Therefore, the operator $V_j^{\beta}$ is a linear operator from $\pi_\lambda$ to the completion $\bar{\pi}_{\lambda + \gamma}$ of $\pi_{\lambda + \gamma}$.

4.6.3. One can interpret the operator $V_j^{\beta}$ as a suitably defined composition operator $\bar{Q}_{j_1}^{\beta} \ldots \bar{Q}_{j_m}^{\beta}$. Indeed, let us define the bosonic vertex operator $V_{\alpha_i}(z)$, acting from $\pi_\lambda$ to $\pi_{\lambda + \alpha_i}$ as $z^{\beta^2(\lambda, \alpha_i)}V_{\alpha_i}(z)$. One has (cf., e.g., [12], formulas (8.4.25) and (A.2.9))

$$(4.6.3) \quad V_{\alpha_{j_1}}(z_1) \ldots V_{\alpha_{j_m}}(z_m) = \prod_{1 \leq k < l \leq m} (z_k - z_l)^{\beta^2(\alpha_{jk}, \alpha_{jl})} : V_{\alpha_{j_1}}(z_1) \ldots V_{\alpha_{j_m}}(z_m) :$$

for $|z_1| > \ldots > |z_m|$. Note that unlike the Fourier components of $V_{\alpha_{j_1}}(z_1) \ldots V_{\alpha_{j_m}}(z_m)$, the Fourier components of $V_{\alpha_{j_1}}(z_1) \ldots V_{\alpha_{j_m}}(z_m)$ are well-defined linear operators, analytically depending on $z_1, \ldots, z_m$ in the region $\mathbb{C}^m \setminus \text{diag}$, which do not change under the permutations of coordinates.

We have for $j_1 = (j_1^1, \ldots, j_1^m)$ and $j_2 = (j_2^1, \ldots, j_2^n)$:

$$(4.6.4) \quad V_j^{\beta_1}V_j^{\beta_2} = V_{(j_1,j_2)}^{\beta}$$

where $V_j^{\beta_1} : \pi_\lambda \rightarrow \bar{\pi}_{\lambda + \gamma_1}$ and $V_j^{\beta_2} : \pi_{\lambda + \gamma_1} \rightarrow \bar{\pi}_{\lambda + \gamma_1 + \gamma_2}$. Indeed, we can choose the contour $C_p$ in such a way that $|z_{p_i}| \geq 1$ for $i = 1, \ldots, m$ and $|z_{p_i}| \leq 1$ for $i = m+1, \ldots, m+n$. Then the composition in the left hand side of (4.6.4) is well-defined, and coincides with the right hand side.

According to Lemma [3.4], the $\beta^2$-linear term of the operator $\bar{Q}_j^{\beta}$ coincides with the operator $-\bar{Q}_j$, therefore the leading $(\beta^2m)$th term of $V_j^{\beta}$ coincides with $\bar{Q}_j^{m} \ldots \bar{Q}_j^{m}$. 
The operators $\tilde{Q}_i$ satisfy the Serre relations. We want to show that the operators $V_{i}^\beta$ satisfy the $q$-Serre relations, where $q = \exp(\pi i \beta^2)$.

4.6.4. Consider a free algebra $A$ with generators $g_i, i = 1, \ldots, l$. We can assign to each monomial $g_{i_1} \cdots g_{i_m}$ the contour $C_j$ and hence the operator $V_{j}^\beta$. This gives us a map $\Delta$ from $A$ to the space of linear combinations of such contours. Given such a linear combination $C$, we define $V_{C}^\beta$ as the linear combination of the corresponding operators $V_{j}^\beta$.

Consider the two-sided ideal $S_q$ in $A$, which is generated by the $q$-Serre relations $(\text{ad} g_i)q^{-a_{ij}+1} \cdot g_j, i \neq j$, where $q = \exp(\pi i \beta^2)$.

4.6.5. **Lemma.** If $C$ belongs to $\Delta(S_q)$, then $V_{C}^\beta = 0$.

**Proof** is given in [19]. It is based on rewriting the integrals over the contours $C_j$ as integrals over the contours, where all variables are on the unit circle with some ordering of their arguments.

We have to prove that $(\text{ad} \tilde{Q}_i)^m \cdot \tilde{Q}_j = 0$. Let $V_{\beta(i)}$ be the operator, defined by formula (4.6.1), where the contour $C_p$ is replaced by contour $C_p = \{ (z_1, \ldots, z_m) | z_i = 1, 0 < \arg z_{p_1} < \ldots < \arg z_{p_m} < 2\pi \}$.

By induction one can prove [19] that

$$(\text{ad} \tilde{Q}_i)^m \cdot \tilde{Q}_j = I_m V_{(i, \ldots, i, j)},$$

where

$$I_m = \prod_{j=0}^{m-1} \frac{(1 - q^{j(\alpha_i, \alpha_i)+2(\alpha_i, \alpha_i)})(1 - q^{(j+1)(\alpha_i, \alpha_i)})}{1 - q^{(\alpha_i, \alpha_i)}}.$$

and Lemma follows.

Thus, we obtain a well-defined map, which assigns to each element $P$ of the algebra $U_q(n_+) \simeq A/S_q$ the operator $V_{P}^\beta$.

4.6.6. **Lemma.** Let $P \in U_q(n_+)$ be such that $P \cdot 1_\lambda$ is a singular vector of $M^\lambda$ of weight $\lambda + \gamma$. Then the operator $V_{P}^\beta$ is a homogeneous linear operator $\pi_\lambda \to \pi_{\lambda+\gamma}$.

**Proof.** It is known that under the conditions of the Lemma, the contour $\Delta(P)$ is a cycle in the group $H^m((\mathbb{C}^\times)^m, \text{diag}; \xi_1)$, cf. [17, 18, 19, 109, 114]. Therefore, only the degree 0 integrands (with respect to the grading $\deg z_i = \deg dz_i = 1$) of the integral over $\Delta(P)$ give non-zero results. In other words, all coefficients

$$\int_{\Delta(P)} dz_1 \cdots dz_n \prod_{1 \leq k < \ell \leq m} (z_k - z_\ell)^{\beta^2(\alpha_{i_k}, \alpha_{i_\ell})} \prod_{1 \leq k \leq m} z_k^{\beta^2(\lambda, \alpha_{i_k}) - n_k}$$

vanish, unless $\sum_{i=1}^{m} n_i = -m$. It means that in the operator $V_{P}^\beta$ all homogeneous components, except one, vanish, and Lemma follows.
4.6.7. Remark. The statements of Lemma 4.6.5 and Lemma 4.6.6 follow from more general results on the remarkable and not yet fully understood correspondence between quantum groups and local systems of the type $\xi_i$ on configuration spaces, established in the works of Schechtman and Varchenko [109, 114, 110] (cf. also [54]).

Let us also remark that in [114] Varchenko constructed explicitly certain absolute chains in $(\mathbb{C}^*) \setminus \text{diag}$, which have all the nice properties of the relative chains $C_p$, including the factorization property (4.6.4) (cf. §14 of [114]). One can use these chains instead of $C_p$ in the definition of our operators.

4.6.8. Quantum differentials. We are ready now to define the differentials $\delta^j_\beta : F^j_{\beta} (g) \to F^j_{\β} (g)$ of the quantum complex $F^*_{\beta} (g)$. Recall that for any pair $s, s'$ of elements of the Weyl group there exists a singular vector $P^q_{s', s} \cdot 1^q_{\rho - s(\rho)} \in M_{\rho - s(\rho)}$ of weight $\rho - s'(\rho)$, where $P^q_{s', s} \in U_q(n_+)$. We put:

$$
\delta^j_\beta = \sum_{l(s) = j - 1, l(s') = j, s < s'} \epsilon_{s', s} \cdot P^q_{s', s},
$$

where $q = \exp(\pi i \beta^2)$. By Lemma 4.6.6, the differentials $\delta^j_\beta$ are well-defined homogeneous linear operators. From the nilpotency of the differential of the quantum BGG resolution and Lemma 4.6.5 we deduce that these differentials are nilpotent.

Thus, we obtain a family of complexes $F^*_{\beta} (g)$, depending on the parameter $\beta$. As was explained above, we can rescale the differentials by some powers of $\beta$, so that when $\beta = 0$ the complex $F^*_{\beta} (g)$ becomes the classical complex $F^* (g)$.

4.6.9. Theorem. For generic $\beta$ higher cohomologies of the complex $F^*_{\beta} (g)$ vanish. The 0th cohomology, $W_0 (g)$, is a conformal vertex operator algebra. There exist elements $W^{(1), \beta}_{-d_i - 1} v_0, \ldots, W^{(1), \beta}_{-d_i - 1} v_0$ of $W_0 (g)$ of degrees $d_i + 1, \ldots, d_i + 1$, where the $d_i$ are the exponents of $g$, such that $W_0 (g)$ is freely generated (in the sense explained below) from $v_0$ under the action of the Fourier components $W^{(i), \beta}_{n_i} v_0$, $1 \leq i \leq l, n_i < 0$, of the corresponding fields

$$
W^i (z) = Y(W^{(i), \beta}_{-d_i - 1} v_0, z) = \sum_{n \in \mathbb{Z}} W^{(i), \beta}_{n} z^{-n-d_i-1}.
$$

Moreover, $W^{(1), \beta}_{-2} v_0$ is the Virasoro element, and the other elements $W^{(i), \beta}_{-d_i - 1} v_0$ can be chosen so as to be annihilated by the corresponding Virasoro generators $W^{(1), \beta}_{n} v_0, n > 0$.

Proof. Lemma 4.4.2 together with Proposition 2.4.7 lead us to conclude that for generic $\beta$ all higher cohomologies of the complex $F^*_{\beta} (g)$ vanish.

Therefore, by Corollary 4.4.3, the 0th cohomology, $W_0 (g)$, of the complex $F^*_{\beta} (g)$ for generic $\beta$ has the same character as the 0th cohomology of the complex $F^* (g)$ (cf. the proof of Proposition 2.4.7):

$$
\prod_{1 \leq i \leq l, n_i > d_i} (1 - q^{n_i})^{-1}.
$$

This formula shows that there is a vector of degree 2, $W^{(1), \beta}_{-2} v_0$, in $W_0 (g)$, which is a Virasoro element. It is given by the following formula:

$$
W^{(1), \beta}_{-2} v_0 = \left( \frac{1}{2\beta^2} \sum_{i=1}^l b^i_{-1} b^i_{-1} + \rho - \frac{1}{\beta^2} \rho_2 \right) v_0.
$$
The Fourier components \( L_n = W_n^{(1)\beta}, n \in \mathbb{Z} \), of the field corresponding to \( W_{-2}^{(1)\beta} v_0 \), generate an action of the Virasoro algebra on \( \pi_{\lambda} \). They commute with the action of the differential \( \delta_3^\lambda \) and hence they preserve the space \( \mathcal{W}_\beta(g) \), which is the kernel of \( \delta_3^\lambda \).

Recall that a singular vector is a vector, which is annihilated by the positive Virasoro generators \( L_n, n > 0 \). Verma module is a module over the Virasoro algebra, which is freely generated from a singular vector by \( L_n, n < 0 \). The degree of the singular vector, from which it is generated is called highest weight. The vacuum Verma module is the module, freely generated by \( L_n, n < -1 \), from a vector, which is annihilated by \( L_n, n \geq -1 \). From the structural theory of Verma modules over the Virasoro algebra [52] we know that Verma modules of integral non-zero highest weight and the vacuum Verma module are irreducible for generic central charge. Moreover, there can be no non-trivial extensions between such modules for generic central charge.

Therefore for generic \( \beta \) each singular vector of \( \pi_0 \), except for \( v_0 \), generates an irreducible Verma module of positive integral highest weight under the free action of the generators \( L_n, n < 0 \). The vector \( v_0 \) generates the irreducible vacuum Verma module under the free action of the generators \( L_n, n < -1 \). As a module over the Virasoro algebra, \( \pi_0 \) is a direct sum of the vacuum Verma module generated from \( v_0 \) and some Verma modules of positive integral highest weights. Since \( \mathcal{W}_\beta(g) \) is a submodule of \( \pi_0 \), it is also a direct sum of such modules. In particular, we see that the character of the space of singular vectors in \( \mathcal{W}_\beta(g) \) is equal to

\[
(1 - q) \prod_{2 \leq i \leq \lambda > \delta} (1 - q^{m_i})^{-1} + q.
\]

From this fact and the character formula (4.6.6) we can derive that there exist singular vectors \( W_{-d_i-1}^{(1)\beta} v_0 \) of degrees \( d_i + 1 \) in the 0th cohomology, which in the limit \( \beta = 0 \) can be chosen as polynomial generators \( W_{-d_i-1}^{(1)\beta} \) of \( \mathcal{W}(g) \) from Proposition 4.4.

This can be proved by induction. Suppose, we have proved this fact for \( i < j \). Thus, we have constructed singular vectors \( W_{-d_i-1}^{(1)\beta} v_0, i = 1, \ldots, j - 1 \), satisfying the conditions above. Let \( \mathcal{W}'(g) \) be the subspace of \( \mathcal{W}(g) \), which consists of all polynomials in \( W_{n_i}^{(1)\beta}, n_i < -d_i, i = 1, \ldots, j - 1 \). Consider the component \( \mathcal{W}(g)_{d_j+1} \) of \( \mathcal{W}(g) \) of degree \( d_j + 1 \). In this component the subspace \( \mathcal{W}(g)'_{d_j+1} = \mathcal{W}'(g) \cap \mathcal{W}(g)_{d_j+1} \) has codimension 1. Now consider \( \beta^2 \) as a formal variable and the space \( \mathcal{W}_\beta(g) \) as a free module over the ring \( \mathbb{C}[[\beta^2]] \) as in §4.3. Denote by \( S_{d_j+1}^\beta \) the space of singular vectors of \( \mathcal{W}_\beta(g) \) of degree \( d_j + 1 \). We have a natural projection \( \mathcal{W}_\beta(g) \rightarrow \mathcal{W}_\beta(g)/\beta^2 \cdot \mathcal{W}_\beta(g) \simeq \mathcal{W}(g) \), the classical limit. We will show that the image of \( S_{d_j+1}^\beta \) in \( \mathcal{W}(g)_{d_j+1} \) is not contained in \( \mathcal{W}(g)'_{d_j+1} \).

Indeed, note that the action of the operators \( L_n, n \geq -1 \), on \( \pi_0 \) is well-defined in the limit \( \beta = 0 \), and denote the corresponding operators by \( L_n^{(0)} \). These are derivations of \( \pi_0 \), which generate a Lie subalgebra of the Virasoro algebra. The action of the operator \( L_{-1} \) does not depend on \( \beta \) and coincides with the action of the derivative \( \partial \). From our inductive assumption we already know the commutation relations between \( L_n \) and \( W_n^{(1)\beta}, i = 1, \ldots, j - 1 \). They are given by formula (4.6.9) below. These relations give us in the limit \( \beta = 0 \):

\[
L_n^{(0)} \cdot W_n^{(i)} = (nd_i - n_i)W_n^{(i)} + W_{n+n_i}^{(i)},
\]

where we put \( W_m^{(i)} = 0 \), if \( m \geq -d_i \). From these formulas it is clear that the polynomial algebra \( A^{(j)} \), generated by \( W_{n_i}^{(1)\beta}, i = 2, \ldots, j - 1 \), is preserved by the action of \( L_n^{(0)}, n \geq -1 \). But the operator \( L_{-1}^{(0)} = \partial \) acts freely on \( \pi_0 / \mathbb{C} \) and hence on \( A^{(j)} / \mathbb{C} \). If \( X \) is a singular
vector in $W(g)'$, i.e. if it is annihilated by the operators $L^{(0)}_n$, $n > 0$, it should lie in $A^{(j)}$. But if $X$ is a singular vector, then $L^{(0)}_{-1}X$ cannot be a singular vector. Therefore the character of the space of singular vectors, contained in $W(g)'$, is less than or equal to the character of the quotient of $A^{(j)}$ by the total derivatives, which is equal to

\[ (1 - q) \prod_{2 \leq i \leq j-1, n_i > d_i} (1 - q^{n_i})^{-1} + q \]  

(in fact, it can be shown that it is equal to (4.6.8)).

The image of $S^{\beta} \in W_{\beta}(g)_{d_j+1}$ in $W(g)_{d_j+1}$ lies in the space of singular vectors of $W(g)_{d_j+1}$. Formulas (4.6.7) and (4.6.8) show that the dimension of the space $S^{\beta}_{d_j+1}$ is greater than the dimension of the space of singular vectors of $W(g)'_{d_j+1}$. Hence there should exist a singular vector in $W_{\beta}(g)_{d_j+1}$, whose image in $W(g)_{d_j+1}$ is linearly independent from the subspace $W(g)'_{d_j+1}$. Denote such a vector by $W^{(j),\beta}_{-d_j-1}v_0$. By Proposition 2.4.7, its image in $W(g)$ is algebraically independent from the previously constructed $W^{(i)}_{-d_i-1}, i < j$, and hence it can be chosen as a generator $W^{(j)}_{-d_j-1}$ of $W(g)$. This completes our inductive argument.

The Fourier components $W^{(i),\beta}_{n_i}, n_i < -d_i$, of the fields corresponding to the singular vectors $W^{(i),\beta}_{-d_i-1}v_0$ act on $\pi_\lambda$ and commute with the differential $\delta^{1\beta}_\lambda$. Therefore they act on $W_{\beta}(g)$. In the limit $\beta = 0$, their action coincides with the action by multiplication by the polynomials $W^{(i)}_{n_i}$.

The polynomials $W^{(i)}_{n_i}, n_i < -d_i$, were shown in Proposition 2.4.7 to be algebraically independent. Therefore monomials

$$W^{(i_1)}_{n_{i_1}} \ldots W^{(i_m)}_{n_{i_m}} \in W(g),$$

where $i_1 \leq \ldots \leq i_m$ and $n_{i_j} < n_{i_{j+1}}$ for $i_j = i_{j+1}$ are linearly independent. But these monomials are images (classical limits) of monomial elements

$$W^{(i_1,\beta)}_{n_{i_1}} \ldots W^{(i_m,\beta)}_{n_{i_m}} v_0 \in W_{\beta}(g)$$

ordered so that $i_1 \leq \ldots \leq i_m$ and $n_{i_j} < n_{i_{j+1}}$ for $i_j = i_{j+1}$. Therefore the latters are linearly independent in $W_{\beta}(g)$.

Hence such monomials linearly span a subspace in the $W_{\beta}(g)$, whose character is given by (4.6.6). But we know that this is the character of $W_{\beta}(g)$. Hence these monomials form a basis of $W_{\beta}(g)$. By analogy with the case of universal enveloping algebras, where one can choose the Poincare-Birkhoff-Witt basis, one can say that the operators $W^{(i,\beta)}_{n_i}, 1 \leq i \leq l, n_i < -d_i$, freely generate $W_{\beta}(g)$ from $v_0$.

4.6.10. Remark. A vector in a VOA, which is a singular vector with respect to the action of the Virasoro algebra, gives rise to a field, which is called a primary field of conformal dimension equal to the degree of this vector.

Suppose, $A$ is such a vector, of degree $\Delta$. Then we have the following OPE:

$$Y(T, z)Y(A, w) = \frac{\Delta Y(A, z)}{(z-w)^2} + \frac{\partial_z Y(A, z)}{z-w} + \text{regular terms}.$$ 

This gives us the following formula for the commutation relations between the generators $L_n, n \in \mathbb{Z}$, of the Virasoro algebra, and the Fourier components, $A_m$, of the field $Y(A, w) = \sum_{m \in \mathbb{Z}} A_m z^{-m-\Delta}$:

$$[L_n, A_m] = (n(\Delta - 1) - m) A_{n+m}.$$
These commutation relations show that the Fourier components of a primary field of conformal dimension $\Delta$ behave with respect to the generators $L_n = t^{-n+1} \partial_t$ as the $(1 - \Delta)$-differentials on the circle $t^{-m+\Delta-1} dt^1-\Delta$.

Theorem 4.6.9 shows that the VOA $W_{\beta}(\mathfrak{g})$ is “generated” by $l$ fields: $W^\beta_1(z), i = 1, \ldots, l$. The first of them, $W^\beta_1(z)$, is the Virasoro field, and the others are primary fields of conformal dimensions equal to the exponents of $\mathfrak{g}$ plus 1 with respect to this Virasoro field for generic $\beta$. This means that we have the following commutation relations for generic $\beta$:

$$[L_n, W^{(i),\beta}_m] = (nd_i - m)W^{(i),\beta}_{n+m}.$$  

The fields $W^\beta_1(z), i = 1, \ldots, l$, generate the VOA $W_{\beta}(\mathfrak{g})$ in the sense that the OPE of any two of these fields can be expressed through normally ordered expressions of the same fields and their derivatives. This is equivalent to the property that $W_{\beta}(\mathfrak{g})$ is freely generated from $v_0$ under the action of the non-negative Fourier components of the fields $W^\beta_1(z)$.

The vertex operator algebras $W_{\beta}(\mathfrak{g})$ were constructed by Fateev and Zamolodchikov in the case of $\mathfrak{g} = \mathfrak{sl}_3$ [13], and by Fateev and Lukyanov in the cases $\mathfrak{g} = \mathfrak{sl}_n$ [10] and $\mathfrak{so}_2n$ [11]. They found explicit expressions of $W^\beta_1(z)$ through the free fields and then verified that the OPE closes. It was conjectured that such vertex operator algebras exist for arbitrary finite-dimensional simple Lie algebras. In Theorem 4.6.9 we prove this conjecture (this was first announced in [57]). A similar construction for $\mathfrak{g} = \mathfrak{sl}_n$ was proposed in [59].

One can also define $W$-algebra $W_{\beta}(\mathfrak{g})$ through the quantum Drinfeld-Sokolov reduction of the affine algebra $\hat{\mathfrak{g}}$ of level $k$ [13, 18, 77]. In this setting, $W_{\beta}(\mathfrak{g})$ with $\beta = -(k+h^\vee)^{-1/2}$ is the 0th cohomology of the corresponding BRST complex. We have shown in [18, §4, and §3] that the complex $F^\beta\hat{\mathfrak{g}}$ appears as the first term of a spectral sequence of this BRST complex for generic $k$.

One can also use the opposite spectral sequence of the BRST complex to prove the existence of $W$-algebras, cf. [25].

4.6.11. **Theorem.** The Lie algebra $I_{\beta}(\mathfrak{g})$ of local integrals of motion of the quantum Toda field theory, associated to $\mathfrak{g}$, is isomorphic to the Lie algebra of residues of fields from the $W$-algebra $W_{\beta}(\mathfrak{g})$.

**Proof.** As explained in Lemma 4.6.6, the differentials of the complex $F^\ast_{\beta}\hat{\mathfrak{g}}$ are integrals over cycles. Therefore, the differentials of the complex $F^\ast_{\beta}\hat{\mathfrak{g}}$ commute with the derivative, and we can form the double complex

$$\mathbb{C} \hookrightarrow F^\ast_{\beta}\hat{\mathfrak{g}} \rightarrow F^\ast_{\beta}\hat{\mathfrak{g}} \rightarrow \mathbb{C}$$

with $\pm \partial$ as the vertical differentials. By definition, the space $I_{\beta}(\mathfrak{g})$ coincides with the 1st cohomology of this double complex. The Theorem now follows from Theorem 4.6.9 and the analogue of the exact sequence (4.3.3) for the VOA $W_{\beta}(\mathfrak{g})$. Thus, the Lie algebra $I_{\beta}(\mathfrak{g})$ is a quantum deformation of the Poisson algebra $I_0(\mathfrak{g})$ in the same sense as before: the Lie brackets in $I_{\beta}(\mathfrak{g})$ in the $\beta^2$-expansion have no constant term, and the linear term coincides with the Poisson bracket in $I_0(\mathfrak{g})$.

In the same way we can show that the space of integrals of motion in the larger space $\hat{\mathfrak{F}}_0^\beta$ coincides with the Lie algebra of all Fourier components of fields from $W_{\beta}(\mathfrak{g})$.

4.7. **Affine Toda field theories.** In this subsection we will extend the methods of the previous subsection to the quantum affine Toda field theories.
4.7.1. Our task is again to construct the deformed complex $F^*_\beta(g)$, which becomes $F^*(g)$ in the limit $\beta \to 0$, for an affine Lie algebra $g$.

According to § 4.5.6 the quantum BGG resolution $B^q_\beta(g)$ exists for the quantized affine algebra $U_q(g)$. Lemma 4.6.5 and Lemma 4.6.6 also hold in the affine case. Thus, we can define the complex $F^*_\beta(g)$ in the same way as in the case of finite-dimensional simple Lie algebras.

As a vector space, the $j$th group of the complex $F^*_\beta(g)$ is the direct sum of the modules $\pi_{\rho-s(\rho)}$, where $s$ runs over the set of elements of the Weyl group of $g$ of length $j$. The differential $\delta^j_\beta : F^j_{\beta-1}(g) \to F^j_\beta(g)$ is given by formula (4.6.5). Note that this complex is $\mathbb{Z}$-graded with finite-dimensional homogeneous components and the differentials are homogeneous of degree 0.

After proper rescaling of the differentials, we obtain a family of complexes, defined for generic $\beta \in \mathbb{C}$, such that for $\beta = 0$ we obtain our classical complex $F^*(g)$. Let us restrict ourselves with the affine algebras, whose exponents are odd and the Coxeter number is even. It turns out that using the result of Proposition 3.2.5, in which the cohomologies ourselves with the affine algebras, whose exponents are odd and the Coxeter number is even, it can be shown that using the result of Proposition 3.2.5, in which the cohomologies of the complex $F^*_\beta(g)$ were computed, and the fact that the Euler characteristics of the cohomologies does not depend on $\beta$, we can prove that the cohomologies of the complex $F^*_\beta(g)$ for generic $\beta$ are the same as for $\beta = 0$.

4.7.2. Proposition. For generic $\beta$ the cohomologies of the complex $F^*_\beta(g)$ are isomorphic to the exterior algebra $\wedge^*(\mathfrak{a}^*)$ of the dual space to the principal commutative subalgebra $\mathfrak{a}$ of $\mathfrak{n}_+$.

Proof. Let $F^j_{\beta}(g)_m$ and $H^j_{\beta}(g)_m$ be the $m$th homogeneous components of the $j$th group of the complex $F^*_\beta(g)$ and of its $j$th cohomology group, respectively. The Euler characteristics of the $m$th homogeneous component of the complex $F^*_\beta(g)$,

$$\sum_{j \geq 0} (-1)^j \dim F^j_{\beta}(g)_m = \sum_{j \geq 0} (-1)^j \dim H^j_{\beta}(g)_m,$$

does not depend on $\beta$.

From Proposition 3.2.5 we know that the Euler character of the complex $F^*(g)$ is equal to that of $\wedge^*(\mathfrak{a}^*)$:

$$\prod_{n \equiv d_i \mod h} (1 - q^n),$$

where $d_1, \ldots, d_l$ are the exponents of $g$ and $h$ is the Coxeter number. In fact, one can compute the Euler character of our complex by a different method. By definition, it is equal to

$$\sum_{j \geq 0} (-1)^j \text{ch} F^j_{\beta}(g) = \prod_{n > 0} (1 - q^n)^{-l} \sum_{s} (-1)^{l(s)} q^{(\rho - s(\rho), \rho^\vee)},$$

Using the Weyl-Kac character formula for the trivial representation of $g$ in the principal gradation $[80]$, we can reduce it to the product formula above.

We restrict ourselves with the case when all the exponents of $g$ are odd and the Coxeter number is even (the general proof is technically more complicated and it will be published separately). Then the subcomplex $F^*_\beta(g)_m$ has cohomologies of only even degrees, if $m$ is even, and of only odd degrees, if $m$ is odd. It means, according to Lemma 4.4.2, that the same property holds for the subcomplex $F^*_\beta(g)_m$ for generic values of $\beta$.

But then we have:

$$\sum_{j \geq 0} \dim H^j_{\beta}(g)_m = \sum_{j \geq 0} \dim H^j_{\beta}(g)_m.$$
for even \( m \) and

\[
\sum_{j \geq 0} \dim H^{2j-1}_\beta(g)_m = \sum_{j \geq 0} \dim H^{2j-1}_\beta(g)_m
\]

for odd \( m \). If the value \( \beta = 0 \) of the parameter were not generic, then for generic \( \beta \) we would have \( \dim H^{i}_\beta(g)_m \geq \dim H^{i}_\beta(g)_m \) for all \( l \), and there would exist such \( i \) that \( \dim H^{i}_\beta(g)_m > \dim H^{i}_\beta(g)_m \). But this would contradict the equalities above. Therefore for any \( m \) and \( j \)

\[
\dim H^{j}_\beta(g)_m = \dim H^{j}_\beta(g)_m,
\]

and Proposition follows.

4.7.3. Theorem. All local integrals of motion of the classical affine Toda field theory can be deformed, and so the space \( I_\beta(g) \) of quantum integrals of motion is linearly generated by mutually commuting elements of degrees equal to the exponents of \( g \) modulo the Coxeter number.

Proof. The same as in Theorem 3.2.3, in particular, since the Lie bracket preserves the grading, from the fact that they all have odd degrees follows that they commute with each other.

4.8. Concluding remarks.

4.8.1. The duality \( \beta \to -\frac{\beta}{2}/\beta \) and the limit \( \beta \to \infty \). There is a remarkable duality in \( \mathcal{W} \)-algebras [15, 52, 53]. Let \( g \) be a simple Lie algebra and \( g^L \) be the Langlands dual Lie algebra, whose Cartan matrix is the transpose of the Cartan matrix of \( g \). Let \( r^\vee \) be the maximal number of edges, connecting two vertices of the Dynkin diagram of \( g \).

For generic values of \( \beta \) the vertex operator algebra \( \mathcal{W}_\beta(g) \) is isomorphic to the vertex operator algebra \( \mathcal{W}_{\beta^L}(g^L) \), where \( \beta^L = -\frac{\beta}{2}/\beta \).

Accordingly, under these conditions \( I_\beta(g) \simeq I_{\beta^L}(g^L) \).

Clearly, \( g \simeq g^L \), unless \( g \) is of types \( B_n \) or \( C_n \), in which case they are dual to each other. The duality means then that there is only one family of \( \mathcal{W} \)-algebras associated to the Lie algebras \( B_n \) and \( C_n \).

The proof of this duality [15, 28] is based on the explicit computation in the rank one case, which follows from the proof of Proposition 4.4.4. Indeed, it is clear that the Virasoro element in \( \mathcal{W}_\beta(g) \) given by formula (4.4.2) is invariant under the transformation \( \beta \to -2/\beta \). Therefore for generic \( \beta \) we have the isomorphism \( \mathcal{W}_\beta(sL_2) \simeq \mathcal{W}_{-2/\beta}(sL_2) \).

General case can be reduced to the case of \( sL_2 \). Let \( \pi_0^{(i)} \) be the subspace in \( \pi_0 \), which is generated from \( v_0 \) by the operators \( b^*_n, n < 0, j \neq i \). These operators commute with \( b^*_m, m \in \mathbb{Z} \), and hence with \( \tilde{Q}_i^\beta = \int V^{\beta}_{3\beta}(z)dz \). Therefore the kernel of the operator \( \tilde{Q}_i^\beta \) on \( \pi_0 \) coincides with the tensor product of \( \pi_0^{(i)} \) and the kernel of the operator \( \tilde{Q}_i^\beta \) on the subspace of \( \pi_0 \), generated from \( v_0 \) by the operators \( b^*_n, n < 0 \). But the latter is isomorphic to \( \mathcal{W}_{|\alpha_i|^2}(sL_2) \), and hence does not change, if we replace \( \beta \) by \( -2/\beta \) instead.

Thus we see that the kernel of the operator \( \tilde{Q}_i^\beta = \int V^{\beta}_{3\beta}(z)dz \) coincides with the kernel of the operator \( \int V^{-\alpha_i^\vee/\beta}(z)dz \), where \( \alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i) \). It is known [31] that the scalar product \( (\alpha_i^\vee, \alpha_j^\vee) \) in \( h^* \subset g \) equals \( r^\vee \) times the scalar product \( (\alpha_i^L, \alpha_j^L) \) of the simple roots \( \alpha_i^L \in h^L \subset g^L \) of the Langlands dual Lie algebra \( g^L \). We can therefore identify the Heisenberg algebras \( \hat{h} \) and \( \hat{h}^L \) by identifying \( \alpha_i^\vee \) with \( (r^\vee)^\frac{1}{2}\alpha_i^L \). But then the operator \( \int V^{\alpha_i^\vee/\beta}(z)dz \) becomes the operator \( \tilde{Q}_i^{\alpha_i^L} \), where \( \beta^L = -(r^\vee)^\frac{1}{2}/\beta \). Therefore for generic
values of $\beta$ the kernel of the operator $\tilde{Q}^\beta_i$ coincides with the kernel of the operator $\tilde{Q}^{\beta_L}_i$ of the Langlands dual Lie algebra. Thus, $\mathcal{W}_\beta(g) \simeq \mathcal{W}_{\beta_L}(g^L)$.

This duality has a remarkable limit when $\beta \to \infty$. Of course, in this case the operator $\tilde{Q}^\beta_i$ is not well-defined and has to be regularized.

It suffices describe this regularization in the rank one case. To this end, consider the Heisenberg algebra with generators $\tilde{Q}$ elements of the operator commutation relations described in Proposition 2.1.15. 

prove that $\mathcal{W}$ the Langlands dual Lie algebra. Thus, $\mathcal{W}$ which follow from formula (4.2.6). So, when we apply $\tilde{Q}$ isomorphic to the $\beta$ where for finite (4.8.1) the kernel of the operator $\tilde{Q}$ we obtain a linear combination of terms of the form (4.8.1), but with $\beta$ values of $\beta$ the kernel of the operator $\tilde{Q}$ is not well-defined and has to be regularized.

This identity can be obtained by applying to $v_0$ the negative Fourier components of the formula 

$$b(z)V_\beta(z) := \frac{1}{\beta} \frac{\partial}{\partial z} V_\beta(z).$$

But then see that the matrix elements of the operator $\tilde{Q}^\beta$ in this new basis are polynomials in $\beta^{-2}$, and therefore they define a certain linear operator, when $\beta^{-2} = 0$. We will denote this operator by $\tilde{Q}^\infty$.

We can show that the kernel of the operator $\tilde{Q}^\infty$ coincides with the $\beta \to \infty$ limit of the kernel of the operator $\tilde{Q}$ for generic $\beta$ [54]. We can then check that this kernel is isomorphic to the $\beta \to 0$ limit of the kernel of the operator $\tilde{Q}^\beta$ for generic $\beta$, which is described in Proposition 2.1.15.

By extending this result to higher rank case in the same way as for generic $\beta$, we can prove that $\mathcal{W}_\infty(g) \simeq \mathcal{W}_0(g^L)$, where $\mathcal{W}_\infty(g)$ denotes the intersection of kernels of the operators $\tilde{Q}^\infty$ associated to $g$ [48, 53, 49].

The quotient $I_\infty(g)$ of $\mathcal{W}_\infty(g)$ by total derivatives and constants has a Poisson bracket, which is equal to the $\beta^{-2}$-linear term in the commutator in $I_\beta(g)$. This isomorphism implies that $I_\infty(g) \simeq I_0(g^L)$.

In [48, 49] we showed that $I_\infty(g)$ is isomorphic to the center $Z(\widehat{g})$ of a certain completion of the universal enveloping algebra of the affine algebra $\widehat{g}$ at the critical level. Thus, we see that the center $Z(\widehat{g})$ is isomorphic to the classical $W$-algebra of the Langlands dual Lie algebra $g^L$. 
In the same way we can prove the isomorphism $I_{\infty}(\mathfrak{g}) \simeq I_0(\mathfrak{g}^L)$, where $\mathfrak{g}$ is an affine Kac-Moody algebra and $\mathfrak{g}^L$ is the affine algebra, whose Cartan matrix is obtained by transposing the Cartan matrix of $\mathfrak{g}$.

4.8.2. Explicit formulas. It is an interesting problem to find explicit formulas for the quantum integrals of motion of the affine Toda field theories.

Explicit formulas for the classical ones are known in many cases. For instance, there are many effective methods to compute explicitly the KdV hamiltonians, which are the local integrals of motion of the sine-Gordon model. However, it seems none of those methods can be used to produce the quantum integrals, cf. e.g. [35]. So far, only partial results have been obtained in this direction.

First of all, a few quantum integrals of motion of the sine-Gordon theory of low degrees are known for any value of the central charge of the Virasoro algebra, cf. e.g. [108]. When the central charge is equal to $1 - 3(2n - 1)^2/(2n + 1)$ (the $(2, 2n + 1)$ minimal model of the Virasoro algebra), it is known that the quantum integral of motion of degree $2n - 1$ can be obtained as the residue of the field, corresponding to the singular vector of degree $2n$ in the vacuum Verma module of the Virasoro algebra [63, 36, 29]. An explicit formula is known for this singular vector, and this allows one to write down the corresponding integral of motion for this value of the deformation parameter. A similar phenomenon has been observed in other theories [29].

Finally, the quantum integrals of motion are known for the central charge $c = -2$, which corresponds to $\beta = 2$. The reason for that is that in this case the operators $\tilde{Q}_1$ and $\tilde{Q}_0$ have a simple realization in terms of the Clifford algebra with the generators $\psi_i, \psi_i^*, i \in \mathbb{Z}$ and the anti-commutation relations

$$[\psi_i, \psi_j^*]_+ = \delta_{i,-j}.$$  

Indeed, let $\Lambda^* = \oplus_{n \in \mathbb{Z}} \Lambda^n$ be the Fock representation of this algebra with the vacuum vector $v$, satisfying

$$\psi_i v = 0, i \geq 0, \quad \psi_i^* v = 0, i > 0.$$  

This representation is $\mathbb{Z}$-graded in accordance with the convention $\deg \psi_i = 1, \deg \psi_i^* = -1$. One can introduce an action of this Clifford algebra on the space $\oplus_{n \in \mathbb{Z}} \pi_n$ using the vertex operators by the formulas

$$\psi(z) = \sum_{m \in \mathbb{Z}} \psi_m z^{-m-1} = V_1(z), \quad \psi^*(z) = \sum_{m \in \mathbb{Z}} \psi_m^* z^{-m} = V_{-1}(z).$$  

Since $\beta$ is an integer, all Fourier components of these vertex operators are well-defined on any of the modules $\pi_n$. This boson-fermion correspondence allows us to identify our complex $F^*_2(\hat{\mathfrak{sl}}_2) \simeq \oplus_{n \in \mathbb{Z}} \pi_{2n}$ with the even part of $\Lambda^*$. The operators $\tilde{Q}_1$ and $\tilde{Q}_0$ then become

\begin{equation}
\tag{4.8.3}
\tilde{Q}_1 = \int V_2(z)dz = \int \psi(z) \partial_z \psi(z)dz, \quad \tilde{Q}_0 = \int V_{-2}(z)dz = \int \psi^*(z) \partial_z \psi^*(z)dz.
\end{equation}

It is not difficult to prove directly that they satisfy the Serre relations with $q = 1$. Recall that for generic values of $\beta$ only the compositions of the operators $\tilde{Q}_1^\beta$ and $\tilde{Q}_0^\beta$, corresponding to the singular vectors in the Verma modules over the quantum group, are well-defined as linear operators acting between the spaces $\pi_n$. The operators (4.8.3) are always well-defined and they generate an action of the nilpotent Lie subalgebra $\mathfrak{n}_+^\perp$ of $\mathfrak{sl}_2$ on $\Lambda^*$. In fact, this action can be extended to an action of the whole Lie algebra $\mathfrak{sl}_2$ [89].
It is possible to write down explicit formulas for the integrals of motion for $\beta = 2$ in terms of the fermions $\psi(z), \psi^*(z)$ [30]:

$$H_{2n+1} = \int \psi(z) \partial_z^{2n-1} \psi^*(z) dz.$$ 

These formulas can be converted into nice formulas in terms of the generators of the Virasoro algebra, which first appeared in [108]. Other features of the case $\beta = 2$ have been studied in [89].

Similar, but more complicated is the case when $\beta = N$, a positive integer. Then the operators $\tilde{Q}_1$ and $\tilde{Q}_0$ can be written in terms of $\psi(z)$ and $\psi^*(z)$ as follows:

$$\tilde{Q}_1 = \int \psi(z) \partial_z \psi(z) \cdots \partial_z^{N-1} \psi(z) dz,$$

$$\tilde{Q}_0 = \int \psi^*(z) \partial_z \psi^*(z) \cdots \partial_z^{N-1} \psi^*(z) dz.$$ 

In these cases explicit formulas for the integrals of motion are still lacking. Finding such formulas for infinitely many integers $N$ would lead to an independent proof of the existence of quantum integrals of motion.

4.8.3. **Special values of $\beta$ for finite-dimensional $g$.** So far, we have only been interested in the generic values of the deformation parameter $\beta$. In this subsection we will discuss briefly what happens for special values of $\beta$, that is the values, for which the kernel of the operator $\sum_i \tilde{Q}_i^\beta$ on $F_0^\beta$ becomes larger.

Let us first look at the case of finite-dimensional $g$. In the simplest case of $g = sl_2$ our complex is $\pi_0 \rightarrow \pi_1$, and we have proved that for generic values of $\beta$ the first cohomology of this complex is trivial. In fact, this statement can be proved directly, using the description of the structure of the modules $\pi_0$ and $\pi_1$ over the Virasoro algebra from [32]. According to this description, the module $\pi_1$ is irreducible for generic $\beta$, while the module $\pi_0$ contains an irreducible submodule, such that the quotient by this submodule is isomorphic to $\pi_1$. If $\beta^2$ is a positive rational number, however, the modules $\pi_0$ and $\pi_1$ may become highly reducible, and that leads to the appearance of the first cohomology and the enlargement of the 0th cohomology.

The most interesting situation occurs when $\beta^2 = 2p/q$, where $p, q > 1$ are two relatively prime integers. The corresponding central charge $c = 1 - 6(p-q)^2/pq$ is the central charge of the $(p,q)$ minimal model [8]. In that case the composition structure of the modules $\pi_0$ and $\pi_1$ becomes very complicated [72], and our complex has very large cohomology groups. It turns out, however, that one can extend this complex to an infinite two-sided complex, whose cohomologies are concentrated in one dimension and are isomorphic to the irreducible representation of the Virasoro algebra of highest weight 0. This representation is the quotient of the vacuum Verma module of the Virasoro algebra by its submodule, generated by a unique singular vector, which it contains. Note that this irreducible representation is at the same time the VOA of the corresponding minimal model.

Such a complex was constructed by Felder [53]. It has one Fock space $\pi_\lambda$ with an appropriate $\lambda$ in each group, and the only cohomology occurs in the 0th group of the complex.

The situation with the $W$-algebras, associated to general finite-dimensional Lie algebras, is apparently very similar. The vacuum Verma module $W_{\beta}(g)$, freely generated from the vacuum vector by the operators $W_{n_i}^{(1),\beta}, n_i < -d_i$ (cf. Theorem 4.6.9), which is irreducible for generic values of $\beta$, may contain singular vectors, if $\beta^2$ is a positive rational
number. The quotient $L_\beta(\frak{g})$ of $\mathcal{W}_\beta(\frak{g})$ by the submodule, generated by these singular vectors, is irreducible. One should be able to construct two-sided complexes, which consist of the Fock spaces $\pi_\lambda$, labeled by elements of the affine Weyl group of $\frak{g}$, in which the 0th cohomology would be isomorphic to $L_\beta(\frak{g})$ and all other cohomologies would be trivial.

Such complexes have been conjectured in [59]. Conjecture 3.5.2. These complexes should appear [60] as the result of the quantum Drinfeld-Sokolov reduction of similar complexes (two-sided BGG resolutions) over the corresponding affine Kac-Moody algebra $\hat{\frak{g}}$. This has been proved for $\frak{g} = \frak{sl}_2$ in [16, 12].

These complexes are closely connected with certain complexes [18] of modules over the quotient of the quantum group $U_q(\frak{g})$ with $q$ a root of unity, $q = \exp(2\pi ip/q)$, by a big central subalgebra, cf. [26].

There is also another interesting value of $\beta$, namely, $\beta = 1$ for a simply-laced Lie algebra $\frak{g}$ [10, 12, 18, 58]. In this case $q = -1$, and slightly redefined operators $\hat{Q}_i^\beta$ generate the nilpotent Lie algebra $\frak{n}_+$. This nilpotent subalgebra lies in the constant subalgebra of the whole affine algebra $\hat{\frak{g}}$ acting on the direct sum of the Fock modules $\pi_\lambda$ where the summation is over the root lattice $Q$ of $\frak{g}$, by vertex operators [51]. The $\mathcal{W}$–algebra $\mathcal{W}_1(\frak{g})$ can then be interpreted as the space of invariants of the constant subalgebra of $\hat{\frak{g}}$ in $\pi = \oplus_{\lambda \in Q} \pi_\lambda$. This implies that $I_1(\frak{g})$ (for which the central charge is the rank of $\frak{g}$) is the commutant of $\frak{g}$ in $\hat{\mathcal{F}}_0^1$. A version of $I_1(\frak{g})$ was defined for the first time by I. Frenkel in [60].

It was proved in [58], Theorem 4.2 (cf. also [16, 20]), that $\mathcal{W}_1(\frak{g})$ has the same character as $\mathcal{W}_\beta(\frak{g})$ for generic $\beta$, i.e. that the intersection of kernels of the operators $\hat{Q}_i^\beta$ does not increase at the point $\beta = 1$. Let us show that the higher cohomologies of the complex $F^*_1(\frak{g})$ vanish. This has been conjectured (and proved for $\frak{g} = \frak{sl}_2$) in [60].

Consider the complex $\hat{F}^*(\frak{g})$, in which the $j$th group consists of $\# \{ w | l(w) = j \}$ copies of $\pi$, and the differentials are given by the same formulas as the differentials of the complex $F^*_1(\frak{g})$. The cohomology of the complex $\hat{F}^*(\frak{g})$ it is the cohomology of the Lie algebra $\frak{n}_+$ with coefficients in $\pi$. The complex $F^*_1(\frak{g})$ is a subcomplex of $\hat{F}^*(\frak{g})$. It is easy to show that its cohomology is the subspace of the cohomology of $\hat{F}^*(\frak{g})$ of weight 0 with respect to the Cartan subalgebra of $\frak{g}$ acting on $\hat{F}^*(\frak{g})$ and commuting with the differentials. But $\pi$ is a direct sum of finite-dimensional $\frak{g}$–modules. By Borel-Weil-Bott-Kostant theorem, weight 0 cohomology classes in $\hat{F}^*(\frak{g})$ can occur only in dimension 0 and those are the invariants of $\frak{g}$ in $\pi$. Therefore the higher cohomologies of $F^*_1(\frak{g})$ vanish. This implies that the character of $\mathcal{W}_1(\frak{g})$ coincides with the character of $\mathcal{W}_\beta(\frak{g})$ for generic $\beta$, and thus we obtain an alternative proof of Theorem 4.2 of [58].

It was shown in [58] that $I_1(\frak{sl}_N)$ is the quotient of the local completion of the universal enveloping algebra of the Lie algebra $\mathcal{W}_\infty$ with central charge $N - 1$.

4.8.4. Special values of $\beta$ for affine $\frak{g}$. Now let us turn to the space $I_\beta(\frak{g})$ of the integrals of motion of the affine Toda field theory associated to an affine algebra $\frak{g}$. The space of these integrals was defined as the intersection of kernels of the operators $\hat{Q}_i^\beta$, $i = 0, \ldots, l$, on $\mathcal{F}_0^\beta$. For generic values of $\beta$ the intersection of kernels of the operators $\hat{Q}_i^\beta$ with $i = 1, \ldots, l$ coincides with the $\mathcal{W}$–algebra $I_\beta(\frak{g})$, and so the space $I_\beta(\frak{g})$ can be defined as the kernel of the operator $\hat{Q}_0^\beta$ on $I_\beta(\frak{g})$.

Recall that the $\mathcal{W}$–algebra $I_\beta(\frak{g})$ is the quotient of the vacuum Verma module $\mathcal{W}_\beta(\frak{g})$ by the total derivatives and constants. If $\frak{g}$ is untwisted, the operator $\hat{Q}_0^\beta$ can be interpreted as the residue $\int \Phi_{1,1,A\beta}(z)dz$ of a certain primary field $\Phi_{1,1,A\beta}(z)$, acting from $\mathcal{W}_\beta(\frak{g})$ to
another module $M_\beta(\mathfrak{g})$ over the $\mathcal{W}$-algebra \([120, 12, 35, 77]\), so that the operator $\tilde{Q}_0^\beta$ is the corresponding operator on the quotients by the total derivatives and constants. Therefore, for generic $\beta$ the space $I_\beta$ consists of the elements $P^- \in \mathcal{W}_\beta(\mathfrak{g})$, for which $\tilde{Q}_0^\beta \cdot P^-$ is a total derivative in $M_\beta(\mathfrak{g})$:

\[
\int \Phi_{1,1,Adg}(z)dz \cdot P^- = \partial P^+.
\]

This equation shows that the pair $(P^-, P^+)$ can be interpreted as a conservation law (compare with Remark 3.2.8) in the deformation of the corresponding conformal field theory obtained by adding $\lambda \int \Phi_{1,1,Adg}(z)dz$, where $\lambda$ is a parameter of deformation, to the action \([120]\).

When $\beta^2$ is a positive rational number, the module $\mathcal{W}_\beta(\mathfrak{g})$ may become reducible. Because of that, the cohomologies of the complex $F^*(\mathfrak{g})$ increase. In such a case it is appropriate to redefine integrals of motion as elements $P^-$ of the irreducible module $L_\beta(\mathfrak{g})$, which satisfy the equation \((4.8.4)\) \([120]\). This may result in dropping out of some of the “generic” integrals of motion. At the same time some new ones may appear.

For instance, for $\mathfrak{g} = \widehat{\mathfrak{sl}_2}, \beta^2 = 4/(2n + 1)$ (the $(2,2n + 1)$ model) the density of the integral of motion of the quantum sine-Gordon theory of degree $2n - 1$ coincides with the field, corresponding to the singular vector of degree $2n$. Since we take the quotient by the submodule, generated by this vector, this integral of motion drops out (cf. the previous section). It has been argued that the integrals of motion of degrees, which are divisible by $2n - 1$, also drop out in this case \([65, 84, 36]\).

Another example of dropping out of integrals of motion is (in our terminology) the Toda field theory associated to the twisted algebra $A_2^{(2)}$ for the value $\beta^2 = 3/2$. The integrals of motion of this Toda theory for generic values of $\beta$ have all positive integral degrees, which are not divisible by 2 and 3. They are elements of the Virasoro algebra $I_\beta(\mathfrak{sl}_2)$, because for $\mathfrak{g} = A_2^{(2)}, \mathfrak{g} = \mathfrak{sl}_2$. They have the property \((4.8.4)\) with the field $\Phi_{(1,3)}(z) = \Phi_{1,1,Adg}(z)$ replaced by $\Phi_{(1,2)}(z)$ \([120]\). The value $\beta^2 = 3/2$ corresponds to the Ising model $(3, 4)$ with central charge $c = 1/2$. It was found in \([120, 44]\) that the integral of motion of degree 5 drops out for this special value of parameter. It was conjectured that the degrees of the integrals of motion which should occur are relatively prime with 30, so that they are the exponents of $E_8^{(1)}$ modulo the Coxeter number.

On the other hand, in the same theory with $\beta^2 = 8/5$ (the Ising tri-critical point, the $(4,5)$ minimal model) the appearance of an integral of motion of degree 9 was observed \([14, 24]\) and it was conjectured that there should also be integrals of motion of degrees $9n$, where $n$ is an arbitrary positive odd integer.

In these examples, the dropping out or appearance of new integrals of motion is caused by the existence of a larger vertex operator algebra of symmetries of the model. For instance, it is known that the $(2, 2n + 1)$ minimal model of the Virasoro algebra coincides with $(2n - 1, 2n + 1)$ minimal model of the $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{sl}_{2n-1})$ \([24]\), and that the Ising model has a hidden symmetry of $\mathcal{W}(E_8)$ \([27]\). Therefore one should expect that the degrees of integrals of motion in such a model should satisfy “exclusion rules” of the larger symmetry algebra as well. It is interesting whether there are other reasons for dropping out or appearance of new integrals of motion.

It seems plausible that for the special values of $\beta$ one can construct a two-sided complex, consisting of the modules $\pi_\lambda$, whose first cohomology would give the space of integrals of motion, corresponding to the irreducible representation $L_\beta(\mathfrak{g})$. We have constructed a
candidate for such a complex for the \((2, 2n + 1)\) model. The computation of the Euler character of this complex suggests that its first cohomology is indeed generated by elements of all odd degrees, which are not divisible by \(2n - 1\). We will discuss this complex elsewhere.

4.8.5. \textit{Spectrum of the integrals of motion.} Our integrals of motion, both classical and quantum, act on the spaces \(\pi_\lambda\). They are not diagonalizable, since they are all of negative degrees (in particular, the first of them is the operator of derivative \(\partial\)). However, one can define a transformation on the space \(\hat{\mathcal{F}}^\beta_0\) of Fourier components of fields \([98]\), which maps the set of integrals of motion to a set of mutually commuting elements of degree 0 (for example, the first integral of motion, \(\mathcal{H}_1 = \partial = L_{-1}\) maps to \(L_0 - c/24\), cf. \([101]\).

It would be very interesting to find the spectrum of these operators on the modules \(\pi_\lambda\).

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