Fuzzy Soc-Semi-Prime Sub-Modules

Saad S. Merie  
SaadSaleem@uokirku.edu.iq  
Department of Mathematics, College of Education of Pure Science, Ibn Al-Haitham, University of Baghdad, Baghdad – Iraq.

Hatam Yahya Khalf  
dr.hatamyahya@yahoo.com  
Department of Mathematics, College of Education of Pure Science, Ibn Al-Haitham, University of Baghdad, Baghdad – Iraq.

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Abstract
In this paper, we study a new concept of fuzzy sub-module, called fuzzy socle semi-prime sub-module that is a generalization the concept of semi-prime fuzzy sub-module and fuzzy of approximately semi-prime sub-module in the ordinary sense. This leads us to introduce level property which studies the relation between the ordinary and fuzzy sense of approximately semi-prime sub-module. Also, some of its characteristics and notions such as the intersection, image and external direct sum of fuzzy socle semi-prime sub-modules are introduced. Furthermore, the relation between the fuzzy socle semi-prime sub-module and other types of fuzzy sub-module presented.

Keyword: ℱ-module, ℱ-sub-module, ℱ-prime sub-module, Socle of ℱ-module.

1. Introduction
The concept of fuzzy sets was introduced by Zadeh in 1965[1]. Many authors indeed presented fuzzy subrings and fuzzy ideals. The concept of fuzzy module was introduced by Negoita and Relescu in 1975 [2]. Since then several authors have studied fuzzy modules. The concept of semi-prime fuzzy sub-module was introduced by Rabi 2004[3]. The concept of approximately semi-prime sub-module was introduced by Ali 2019[4]. The socle of $M$ is a summation of simple sub-modules of an $R$-module $M$ and denoted by $Soc(M)$. But, the fuzzy socle of $ℱ$-module $X$ an $R$-module $M$ is a summation of simple $ℱ$-sub-modules of $X$ and denoted by $F − Soc(X)$.
Preliminaries

There are various definitions and characteristics in this section of ℱ-sets, ℱ-modules, and prime ℱ-sub-modules.

Definition 1.1 [1]
Let D be a non-empty set and I is closed interval [0, 1] of real numbers. An ℱ-set B in D (an ℱ-subset of D) is a function from D into I.

Definition 1.2 [1]
AN ℱ-set B of a set D is said to be ℱ-constant if \( B(x) = t \), \( \forall x \in D \) \( t \in [0, 1] \)

Definition 1.3 [1]
Let \( \chi_t : D \rightarrow [0, 1] \) be an ℱ-set in D, where \( x \in D \), \( t \in [0, 1] \) defined by:
\[
\chi_t(y) = \begin{cases} 
  t & \text{if } x = y \\
  0 & \text{if } x \neq y
\end{cases}
\]
for all \( y \in D \). \( \chi_t \) is said to be an ℱ-singleton or ℱ-point in D.

Definition 1.4 [5]
Let \( B \) be an ℱ-set in D, for all \( t \in [0, 1] \), the set \( B_t = \{ x \in D ; B(x) \geq t \} \) is said to be a level subset of \( B \).

Remark 1.5 [6]
Let \( A \) and \( B \) be two ℱ-sets in S, then:
1- \( A = B \) if and only if \( A(x) = B(x) \) for all \( x \in S \).
2- \( A \subseteq B \) if and only if \( A(x) \leq B(x) \) for all \( x \in S \).
3- \( A = B \) if and only if \( A_t = B_t \) for all \( t \in [0,1] \).

If \( A < B \) and there exists \( x \in S \) such that \( A(x) < B(x) \), then \( A \) is a proper ℱ-subset of \( B \) and written as \( A < B \).

By part (2), we can deduce that \( \chi_t \subseteq A \) if and only if \( A(x) \geq t \).

Definition 1.6 [6]
If \( M \) is an ℛ-module. An ℱ-set X of \( M \) is called ℱ-module of an ℛ-module \( M \) if:
1- \( X(x - y) \geq \min\{X(x), X(y)\} \) for all \( x, y \in M \).
2- \( X(rx) \geq X(x) \) for all \( x \in M \) and \( r \in ℛ \).
3- \( X(0) = 1 \).

Proposition 1.7 [7]
Let \( C \) be an ℱ-set of an ℛ-module \( M \). Then the level subset \( C_t \) of \( M \), \( \forall t \in [0, 1] \) is a submodule of \( M \) if and only if \( C \) is an ℱ-sub-module of an ℱ-module of an ℛ-module \( M \).

Definition 1.8 [8]
Let \( X \) and \( A \) be two ℱ-modules of ℛ-module \( M \). \( A \) is said to be an ℱ-sub-module of \( X \) if \( A \subseteq X \).

Proposition 1.9 [5]
Let $A$ be an $\mathcal{F}$-set of an $\mathcal{R}$-module $M$. Then the level subset $A_t$, $t \in [0, 1]$ is a sub-module of $M$ if $A$ is an $\mathcal{F}$-sub-module of $X$ where $X$ is an $\mathcal{F}$-module of an $\mathcal{R}$-module $M$.

Now, we go over various $\mathcal{F}$-sub-module attributes that will be useful in the next section.

**Lemma 1.10** [6]
If $r_t$ be an $\mathcal{F}$-singleton of $\mathcal{R}$ and $A$ be an $\mathcal{F}$-module of an $\mathcal{R}$-module $M$. Then for any $w \in M$

$$(r_t A)(w) = \begin{cases} 
\sup \{\inf(t, A(x)) : \text{if } w = rx \} & \text{for some } x \in M \\
0 & \text{otherwise}
\end{cases}
$$

Where $r_t : \mathcal{R} \to [0, 1]$, defined by

$$r_t(z) = \begin{cases} 
t & \text{if } r = z \\
0 & \text{if } r \neq z
\end{cases}
$$

For all $z \in \mathcal{R}$

**Definition 1.11** [6]
Let $A$ and $B$ be two $\mathcal{F}$-sub-modules of an $\mathcal{F}$-module $X$ of $\mathcal{R}$-module $M$. The residual quotient of $A$ and $B$ denoted by $(A : B)$ is the $\mathcal{F}$-subset of $\mathcal{R}$ defined by:

$$(A : B)(r) = \sup \{t \in [0, 1] : r_t B \subseteq A \}, \text{ for all } r \in \mathcal{R}. \text{ That is } (A : B) = \{r_t : r_t B \subseteq A; r_t \text{ is an } \mathcal{F} - \text{ singleton of } \mathcal{R} \}. \text{ If } B = \langle x_k \rangle, \text{ then } (A : \langle x_k \rangle) = \{r_t : r_t x_k \subseteq A; r_t \text{ is an } \mathcal{F} - \text{ singleton of } \mathcal{R} \}.
$$

**Lemma 1.12** [9]
Let $A$ be an $\mathcal{F}$-sub-module of $\mathcal{F}$-module $X$, $(A_t : X_t) \geq (A : X)_t$, for all $t \in [0, 1]$. Also, we can prove that by Lemma 2.3.3.,[6].

It follows that if $X = A \oplus B$, where $A, B \leq X$ then $X_t = (A \oplus B)_t = A_t \oplus B_t$.

**Definition 1.13** [10]
Let $f$ be a mapping from a set $M$ into a set $N$ and let $A$ be $\mathcal{F}$-set in $M$. The image of $A$ is denoted by $f(A)$, where $f(A)$ is defined by:

$$f(A)(y) = \begin{cases} 
\sup \{A(z) : z \in f^{-1}(y) \neq \emptyset \} & \text{for all } y \in N \\
0 & \text{otherwise}
\end{cases}
$$

Note that, if $f$ is a bijective mapping, then $f(A)(y) = A(f^{-1}(y))$

**Proposition 1.14** [11]
Let $f$ be a mapping from a set $M$ into a set $N$. Assume that $X$ and $Y$ are $\mathcal{F}$-modules of $M$ and $N$ respectively, let $A$ be an $\mathcal{F}$-sub-module of $X$, then $f(A)$ is an $\mathcal{F}$-sub-module of $Y$.

**Definition 1.15** [12]
An $\mathcal{F}$-subset $K$ of a ring $\mathcal{R}$ is called $\mathcal{F}$-ideal of $\mathcal{R}$, if $\forall x, y \in \mathcal{R}$:

1- $K(x - y) \geq \min \{K(x), K(y)\}.$
2- $K(xy) \geq \max \{K(x), K(y)\}$.

**Definition 1.16** [13]
Let $X$ be an $\mathcal{F}$-module of an $\mathcal{R}$-module $M$, let $A$ be an $\mathcal{F}$-sub-module of $X$ and $K$ be an $\mathcal{F}$ -ideal of $\mathcal{R}$, the product $KA$ of $K$ and $A$ is defined by:

$$KA(x) = \begin{cases} 
\sup \{\inf (K(r_1), \ldots, K(r_n), A(x_1), \ldots, A(x_n)) \} & \text{for some } r_i \in \mathcal{R}, x_i \in M, n \in N \\
0 & \text{otherwise}
\end{cases}
$$
Note that A is an F-sub-module of X, and \((KA)_t = K_t A_t,\forall t \in [0, 1]\).

**Definition 1.17 [9]**

Let X be an F-module of an R-module M, An F-sub-module U of X is called completely prime if whenever \(r_b m_t \subseteq U\), with \(r_b \neq 0_1\) is an F-singleton of R and \(m_t\) is an F-singleton of X implies that \(m_t \subseteq U\) for each \(t, b \in [0, 1]\).

**Definition 1.18 [6]**

Let A and B be two F-sub-modules of an R-module M. The addition \(A + B\) is defined by:

\[(A + B)(x) = \sup\{\min\{A(y), B(z)\} \mid x = y + z, \text{for all } x, y, z \in M\}\]

Furthermore, \(A + B\) is an F-sub-module of an R-module M.

**Corollary 1.19 [8]**

If X is an F-module of an R-module M and \(x_t \subseteq X\), then for all F-singleton \(r_k\) of R,

\[r_k x_t = (rx)_t, \text{ where } \lambda = \min\{t, k\}.\]

**Proposition 1.20 [6]**

Let A and B be two F-sub-modules of an F-module X of an R-module M. Then the residual quotient of A and B (A : B) is an F-ideal of R.

**Proposition 1.21 [14]**

Let \(f: M \rightarrow N\) be an R-homomorphism, then \(f(Soc(M)) \subseteq Soc(N)\).

**Definition 1.22 [15]**

Let X be an F-module of an R-module M, X is called F-simple if and only if X has no proper F-sub-modules (in fact X is F-simple if and only if X has only itself and \(0_1\)).

**Definition 1.23 [16]**

A F-module X is called semi-simple if X is a summation of simple F-sub-modules of X. Moreover, X is called semi-simple if \(X = F - Soc(X)\).

**Definition 1.24 [9]**

Let X be an F-module of an R-module M, X is said to be faithful if \(F - annX = 0_1\).

Where \(F - annX = \{r_t : r_t x_t = 0_1 ; \text{for all } x_t \subseteq X \text{ and } r_t \text{ be an F-singleton of } R\}\).

**Definition 1.25 [17]**

Let X be an F-module of an R-module M, X is said to be cancellative if whenever \(r_t x_t = r_t y_d\) for all \(x_t, y_d \subseteq X\) and \(r_t\) be an F-singleton of R then \(x_t = y_d\).

**Definition 1.26 [3]**

A proper F-sub-module U of an F-module X of an R-module M is called semi-prime F-sub-module of X if whenever \(r^n_b m_t \subseteq U\), where \(r_b\) is an F-singleton of R, \(m_t\) is an F-singleton of X and \(n \in Z^+\) implies that \(r_b m_t \subseteq U\) for each \(t, b \in [0, 1]\).

**Definition 1.27 [4]**

A proper sub-module E of an R-module M is called approximately semi prime (for a short app-semi-prime) sub-module of M if whenever \(am \in E\), for \(a \in R, m \in Min\) implies that \(am \in E + Soc(M)\).
Definition 1.28 [9]
An $\mathcal{F}$-sub-module $N$ of an $\mathcal{F}$-module $X$ of an $\mathcal{R}$-module $M$ is called weakly pure $\mathcal{F}$-sub-module of $X$ if for any $\mathcal{F}$-singleton $r_b$ of $\mathcal{R}$ implies that $r_bN = r_bX \cap N$ with $b \in [0,1]$.

Lemma 1.29 [18]
Let $X$ be an $\mathcal{F}$-module of an $\mathcal{R}$-module $M$ and let $A$, $B$ and $C$ are $\mathcal{F}$-sub-modules of $X$ such that $C \subseteq B$. Then $C + (B \cap A) = (C + A) \cap B$.

Proposition 1.30 [14]
If $M$ be a faithful multiplication $\mathcal{R}$-module, then $\mathcal{Soc}(\mathcal{R})M = \mathcal{Soc}(M)$

Definition 1.31 [15]
Let $X$ be an $\mathcal{F}$-module of an $\mathcal{R}$-module $M$. $X$ is called multiplication $\mathcal{F}$-module if and only if for each $\mathcal{F}$-sub-module $A$ of $X$, there exists an $\mathcal{F}$-ideal $K$ of $\mathcal{R}$ such that $A = KX$.

Proposition 1.32 [15]
An $\mathcal{F}$-module $X$ of an $\mathcal{R}$-module $M$ is a multiplication if and only if every non-empty $\mathcal{F}$-sub-module $A$ of $X$, such that $A = (A_{\mathcal{R}} X)X$.

Definition 1.33 [19]
A sub-module $V$ of $\mathcal{R}$-module $M$ is called essential if $H \cap V \neq 0$. For non-trivial sub-module $H$ of $M$.

Definition 1.34 [9]
Let $X$ be an $\mathcal{F}$-module of an $\mathcal{R}$-module $M$. An $\mathcal{F}$-sub-module $A$ of $X$ is called essential if $A \cap B \neq 0_1$, for nontrivial $\mathcal{F}$-sub-module $B$ of $X$.

Finally, (shortly fuzzy set, fuzzy sub-module, fuzzy ideal, fuzzy module and fuzzy singleton are $\mathcal{F}$-set, $\mathcal{F}$-sub-module, $\mathcal{F}$-ideal , $\mathcal{F}$-module and $\mathcal{F}$-singleton).

$\mathcal{F}$-Soc-semi-prime sub-modules
In this section, we offer the concept of an $\mathcal{F}$-Soc-semi-prime sub-module as a generalization of ordinary concept(approximately semi-prime sub-module). Some characterizations of $\mathcal{F}$-Soc-prime sub-module are introduced.

Definition 2.1
Let $r_b$ be an $\mathcal{F}$-singleton of $\mathcal{R}$ and $m_t$ is an $\mathcal{F}$-singleton of $X$, then a proper $\mathcal{F}$-sub-module $U$ of an $\mathcal{F}$-module $X$ of an $\mathcal{R}$-module $M$ is called an $\mathcal{F}$-Socle semi-prime (for short $\mathcal{F}$-Soc-semi-prime) sub-module(ideal) of $X$ if whenever $r_b^n m_t \subseteq U$ with $n \in \mathbb{Z}^+$ implies that $r_b m_t \subseteq U + \mathcal{F} - \mathcal{Soc}(X)$ for each $t, b \in [0,1]$.

Furthermore, if $r_b$ and $s_h$ are $\mathcal{F}$-singletons of $\mathcal{R}$, then a proper $\mathcal{F}$-ideal $L$ of $\mathcal{R}$ is called an $\mathcal{F}$-Socle semi-prime (for short $\mathcal{F}$-Soc-semi-prime) ideal of $\mathcal{R}$ if whenever $r_b^n s_h \subseteq L$ with $n \in \mathbb{Z}^+$ implies that $r_b s_h \subseteq L + \mathcal{F} - \mathcal{Soc}(\mathcal{R})$ for each $h, b \in [0,1]$.

We will adopt the definition of an $\mathcal{F}$-socle of $X$ in this research as follows:
such that: $\mathcal{F} - Soc(X) : M \to [0,1]$

$$\mathcal{F} - Soc(X)(m) = \begin{cases} 1 & \text{if } m \in Soc(M) \\ h & \text{if } m \notin Soc(M) \end{cases} \quad \text{with } 0 < h < 1$$

Lemma 2.2

for any $\mathcal{F}$-module $X$ for each $t \in (0,1]$ with $(\mathcal{F} - Soc(X))_t \neq (\mathcal{F} - Soc(X))_t = Soc(X_t)$ $X_t$

Proof:

such that: $\mathcal{F} - Soc(X) : M \to [0,1]$

$$\mathcal{F} - Soc(X)(m) = \begin{cases} 1 & \text{if } m \in Soc(M) \\ h & \text{if } m \notin Soc(M) \end{cases} \quad \text{with } 0 < h < 1$$

Now, $(\mathcal{F} - Soc(X))_t = \{m \in M : (\mathcal{F} - Soc(X))(m) \geq t\}$

So, if $t = 1$ then $(\mathcal{F} - Soc(X))_t = Soc(M) = Soc(X_t)$

If $0 < t \leq h$ then $(\mathcal{F} - Soc(X))_t = M = X_t$ that is a contradiction

If $h < t < 1$ then $(\mathcal{F} - Soc(X))_t = Soc(M) = Soc(X_t)$

Lemma 2.3

Let $X$ be an $\mathcal{F}$-module of an $\mathcal{R}$-module $M$ with $X(m) = 1$ for each $m \in M$, if $U$ is an $\mathcal{F}$-sub-module of $X$ is defined by $U: M \to [0,1]$ such that:

$$U(m) = \begin{cases} 1 & \text{if } m \in E \\ k & \text{if } m \notin E \end{cases} \quad \text{with } 0 < k < 1$$

Where $E$ is a sub-module of $M$. Then $U$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X$ if and only if $E$ is an app-semi-prime sub-module of $M$.

Proof:

First of all, we must define $U + \mathcal{F} - Soc(X)$.

$$(U + \mathcal{F} - Soc(X))(m) = \sup\{\min(U(y), \mathcal{F} - Soc(X)(z)) : y + z = m\}$$

So, we have

$$(U + \mathcal{F} - Soc(X))(m) = \begin{cases} 1 & \text{if } m \in E + Soc(M) \\ s & \text{if } m \notin E + Soc(M) \end{cases} \quad \text{with } s = \max(k, h)$$

Where $\mathcal{F} - Soc(X) : M \to [0,1]$ such that:

$$\mathcal{F} - Soc(X)(m) = \begin{cases} 1 & \text{if } m \in Soc(M) \\ h & \text{if } m \notin Soc(M) \end{cases} \quad \text{with } 0 < h < 1$$

Now,

Suppose $E$ is an app-semi-prime sub-module of $M$, to prove that $U$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X$. Let $r_b \subseteq \mathcal{R}$ and $m_t \subseteq X$ for each $t, b \in [0,1]$ such that $(r_b)^n m_t \subseteq U$, thus $(r^n)_b m_t \subseteq U$ that is either $r^n m \in E$ or $r^n m \notin E$. 

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1) If \( r^nm \in E \), then \( rm \in E + Soc(M) \). Hence \((U + F - Soc(X))(rm) = 1\) this implies \( r_b m_t = (rm)_t \subseteq (rm)_{1} \subseteq U + F - Soc(X) \).

2) If \( r^nm \notin E \) then \( U(r^nm) = k \) with \( m \notin E \) thus \( U(m) = k \). Since \((r_b)^nm_t \subseteq U\) then \((r^nm)_{\lambda} \subseteq U\) where \( \lambda = \min\{b, t\} \), that is \( U(r^nm) \geq \lambda \) thus \( k \geq \lambda \). Now, if \( \lambda = t \) this implies \( m_t \subseteq m_k \subseteq U \subseteq U + F - Soc(X) \). That is mean \( r_b m_t \subseteq r_b m_k \subseteq U \subseteq U + F - Soc(X) \) If \( \lambda = b \), \( U(h) \geq k \) for any \( h \in M \), and:

\[
(r^n b X_M)(h) = \begin{cases} b & \text{if } h = r^n a \\ 0 & \text{otherwise} \end{cases} \quad \text{for some } a \in M
\]

Then we get \((r^n b X_M)(h) \leq U(h)\), hence \(r^n b X_M \subseteq U \subseteq U + F - Soc(X)\)

So, each case implies that \( r_b m_t \subseteq U + F - Soc(X) \)

Therefore \( U \) is an \( F \)-Soc-semi-prime sub-module of \( X \).

Conversely

Suppose \( U \) is an \( F \)-Soc-semi-prime of \( X \). Let \( a^n x \in U_t \), with \( a \in R \), \( n \in Z^+ \) and \( x \in X_t \) it follows that \((a^n)x_t \subseteq U\), that is \((a^n)_t x_t = (a_t)_t x_t \subseteq U\). But \( U \) is an \( F \)-Soc-semi-prime of \( X \), then we get \( a_t x_t = (ax)_t \subseteq U + F - Soc(X) \). Thus we get \((U + F - Soc(X))(ax) \geq t\), hence, by (Lemma 1.12) and (Lemma 2.2), we have \( ax \in (U + F - Soc(X))_t = U_t + (F - Soc(X))_t = U_t + Soc(X_t) \). That is mean \( U_t \) is an app-semi-prime sub-module of \( X_t \).

Hence \( U_1 = E \) is an app-semi-prime sub-module of \( M \).

The following example shows that the definition of an \( F \)-socle of \( X \) that we adopt in this research is necessary to prove one side of above lemma.

**Example 2.4**

Let \( M = Z_{12} \) as a \( Z \)-module and \( X: M \to [0,1], U: M \to [0,1] \) defined by:

\[
X(m) = \begin{cases} 1 & \text{if } m \in Z_{12} \\ 0 & \text{otherwise} \end{cases}
\]

\[
U(m) = \begin{cases} 1/2 & \text{if } m \in \langle 0 \rangle \\ 1/4 & \text{otherwise} \end{cases}
\]

And an \( F \)-socle of \( X \) is defined by \( F - Soc(X): M \to [0,1] \) such that:

\[
F - Soc(X)(m) = \begin{cases} 1 & \text{if } x = 0 \\ 2/3 & \text{if } m \in \langle 2 \rangle - \langle 0 \rangle \\ 0 & \text{otherwise} \end{cases}
\]

Where \( Soc(M) = \langle 2 \rangle \). That’s clear \( X \) is an \( F \)-module and \( U \) be an \( F \)-sub-module of \( X \).

We have \( U_t \) is an app-semi-prime sub-module of \( M \) for every \( t > 0 \).

Now,
\[ (U + \mathcal{F} - \text{Soc}(X))(m) = \begin{cases} 
1 & \text{if } x = \overline{0} \\
2/3 & \text{if } m \in \{\overline{2}\} - \{\overline{0}\} \\
1/3 & \text{otherwise} 
\end{cases} \]

But, \( U \) is not an \( \mathcal{F} \)-Soc-semi-prime sub-module of \( X \), since for an \( \mathcal{F} \)-singleton \( 3_2 = X \) and an \( \mathcal{F} \)-singleton \( 2_3 \) of \( \mathcal{R} \) such that \( (2^2)_{3_2} = \overline{0}_2 \), where \( \overline{0}_2 \subseteq U \) since \( U(\overline{0}) = 1 > \frac{3}{4} \). But \( 2_3^{3_2} = \overline{0}_2 \subseteq U + \mathcal{F} - \text{Soc}(X) \) since \( (U + \mathcal{F} - \text{Soc}(X))(\overline{0}) = \frac{2}{3} \geq \frac{3}{4} \).

Hence, \( U \) is not an \( \mathcal{F} \)-Soc-semi-prime of sub-module of \( X \).

**Proposition 2.5**

Let \( U \) and \( V \) are \( \mathcal{F} \)-sub-modules of an \( \mathcal{F} \)-module \( X \) of an \( \mathcal{R} \)-module \( M \) with \( V \) is an \( \mathcal{F} \)- semi-prime sub-module of \( X \). Then \( [U: \mathcal{R} V] \) is an \( \mathcal{F} \)-Soc-semi-prime ideal of \( \mathcal{R} \).

**Proof:**

Suppose that \( r_b^n m_t \subseteq [U: \mathcal{R} V] \) for \( r_b \subseteq \mathcal{R}, m_t \subseteq X \), thus \( r_b^n m_t V \subseteq U \). So we have \( r_b^n (m_t V) \subseteq U \), but \( V \) is an \( \mathcal{F} \)-semi-prime sub-module of \( X \). That is \( r_b (m_t V) \subseteq U \), hence \( r_b m_t V \subseteq U \) that is mean \( r_b m_t \subseteq [U: \mathcal{R} V] \subseteq [U: \mathcal{R} V] + \mathcal{F} - \text{Soc}(\mathcal{R}) \).

**Proposition 2.6**

Let \( U \) and \( V \) are \( \mathcal{F} \)-Soc-semi-prime sub-modules of an \( \mathcal{F} \)-module \( X \) of an \( \mathcal{R} \)-module \( M \) with \( \mathcal{F} - \text{Soc}(X) \subseteq U \), then \( U \cap V \) is an \( \mathcal{F} \)-Soc-semi-prime sub-module of \( X \).

**Proof:**

Let \( r_b^n m_t \subseteq U \cap V \), for \( r_b \subseteq \mathcal{R}, m_t \subseteq X \), that is \( r_b^n m_t \subseteq U \) and \( r_b^n m_t \subseteq V \). But \( U \) and \( V \) are \( \mathcal{F} \)-Soc-semi-prime sub-modules of \( X \), this implies \( r_b m_t \subseteq U + \mathcal{F} - \text{Soc}(X) \) and \( r_b m_t \subseteq V + \mathcal{F} - \text{Soc}(X) \). That is mean \( r_b m_t \subseteq (U + \mathcal{F} - \text{Soc}(X)) \cap (V + \mathcal{F} - \text{Soc}(X)) \), by using modular law we get \( r_b m_t \subseteq (U \cap V) + \mathcal{F} - \text{Soc}(X) \). Hence \( U \cap V \) is an \( \mathcal{F} \)-Soc-semi-prime sub-module of \( X \).

**Remark 2.7**

Every \( \mathcal{F} \)-semi-prime sub-module is an \( \mathcal{F} \)-Soc-semi-prime sub-module, but the converse is not true.

**Proof:**

Suppose \( U \) be an \( \mathcal{F} \)-semi-prime sub-module of an \( \mathcal{F} \)-module \( X \) of an \( \mathcal{R} \)-module \( M \) and \( r_b^n m_t \subseteq U \), for \( r_b \subseteq \mathcal{R}, m_t \subseteq X \). Since \( U \) is an \( \mathcal{F} \)-semi-prime sub-module, then we get \( r_b m_t \subseteq U \subseteq U + \mathcal{F} - \text{Soc}(X) \), thus \( r_b m_t \subseteq U + \mathcal{F} - \text{Soc}(X) \). Therefore \( U \) is an \( \mathcal{F} \)-Soc-semi-prime sub-module.

The following example show that the converse is not true.

**Example 2.8**

Consider \( M = Z_{12} \) as a \( Z \)-module and \( X: M \to [0,1] \), \( U: M \to [0,1] \) defined by:

\[ X(m) = 1 \quad \text{if} \quad m \in Z_{12} \]
Example 2.10

Consider $M = Z$ as a $Z$-module and $X: M \to [0,1]$, $U: M \to [0,1]$ defined by:

$$X(m) = 1 \quad \text{if} \quad m \in Z$$

$$U(m) = \begin{cases} 
1 & \text{if} \quad m \in 2Z \\
1/4 & \text{if} \quad m \notin 2Z 
\end{cases}$$

And an $\mathcal{F}$-socle of $X$ is defined by $\mathcal{F} - Soc(X): M \to [0,1]$ such that:

$$\mathcal{F} - Soc(X)(m) = \begin{cases} 
1 & \text{if} \quad m \in \{0\} \\
1/3 & \text{if} \quad m \notin \{0\} 
\end{cases}$$

$$(U + \mathcal{F} - Soc(X))(m) = \begin{cases} 
1 & \text{if} \quad m \in 2Z \\
1/3 & \text{if} \quad m \notin 2Z 
\end{cases}$$

Where $Soc(M) = \{0\}$. That’s clear $X$ is an $\mathcal{F}$-module and $U$ be an $\mathcal{F}$-sub-module of $X$. 

Remark 2.9

Every completely $\mathcal{F}$-sub-module of an $\mathcal{F}$-module $X$ of an $\mathcal{R}$-module $M$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X$, but the converse is not true.

Proof:

We take $U$ as a completely $\mathcal{F}$-sub-module of $X$ with $r^n m \subseteq U$, for $r \subseteq R$, $m \subseteq X$. Now, if $r = 0$, then $r m = 0 \subseteq 0 \subseteq U$, we get $U$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X$.

If $r \neq 0$, then $(r^n) m \subseteq U$, we get $(r^n) m \subseteq U$, where $d = \min\{b, t\}$. Now, since $U$ is a completely $\mathcal{F}$-sub-module of $X$, then we have $(r m)_d \subseteq U \subseteq U + \mathcal{F} - Soc(X)$, thus $r m \subseteq U + \mathcal{F} - Soc(X)$. Therefore $U$ is an $\mathcal{F}$-Soc-semi-prime sub-module.

The following example shows that the converse is not true.

Example 2.10

Consider $M = Z$ as a $Z$-module and $X: M \to [0,1]$, $U: M \to [0,1]$ defined by:

$$X(m) = 1 \quad \text{if} \quad m \in Z$$

$$U(m) = \begin{cases} 
1 & \text{if} \quad m \in 2Z \\
1/4 & \text{if} \quad m \notin 2Z 
\end{cases}$$

And an $\mathcal{F}$-socle of $X$ is defined by $\mathcal{F} - Soc(X): M \to [0,1]$ such that:

$$\mathcal{F} - Soc(X)(m) = \begin{cases} 
1 & \text{if} \quad m \in \{0\} \\
1/3 & \text{if} \quad m \notin \{0\} 
\end{cases}$$

$$(U + \mathcal{F} - Soc(X))(m) = \begin{cases} 
1 & \text{if} \quad m \in 2Z \\
1/3 & \text{if} \quad m \notin 2Z 
\end{cases}$$

Where $Soc(M) = \{0\}$. That’s clear $X$ is an $\mathcal{F}$-module and $U$ be an $\mathcal{F}$-sub-module of $X$. 

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is an app-semi-prime sub-module of M, so by (Lemma 2.3) we get U is an $\mathcal{F}$-Soc-semi-2Z prime sub-module of X.

But U is not completely $\mathcal{F}$-sub-module of X, since for an $\mathcal{F}$-singleton $\frac{5}{3} \subseteq X$ and an $\mathcal{F}$-singleton $\frac{2}{3}$ of $\mathcal{R}$ such that $\frac{2}{3}, \frac{5}{3} = 10 \frac{1}{3}$ where $10 \frac{1}{3} \subseteq U$ since $U(10) = 1 > \frac{1}{3}$, but $\frac{5}{3} \not\subseteq U$, since $U(5) = \frac{1}{4} \not\geq \frac{1}{3}$.

Hence, U is not completely $\mathcal{F}$-sub-module of X.

**Proposition 2.11**

Let U be an $\mathcal{F}$-Soc-semi-prime sub-module of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M, Then U is an $\mathcal{F}$-Soc-semi-prime sub-module of X if and only if $\forall \mathcal{F}$-sub-module S of X and an $\mathcal{F}$-ideal J of $\mathcal{R}$ with $(J)^n S \subseteq U$ for $n \in \mathbb{Z}^+$ implies that $JS \subseteq U + \mathcal{F} - Soc(X)$.

Proof:

$(\Rightarrow)$ Assume that $(J)^n S \subseteq U$, for S is an $\mathcal{F}$-sub-module of X and J is an $\mathcal{F}$-ideal of $\mathcal{R}$, let $x_t \subseteq JS$ with $t \in [0,1]$ then $x_t = (c_1)_{h_1}(y_1)_{t_1} + (c_2)_{h_2}(y_2)_{t_2} + \cdots + (c_n)_{h_n}(y_n)_{t_n}$, for every $(c_i)_{h_i} \subseteq J$ and $(y_i)_{t_i} \subseteq U$ where $h_i$, $t_i \in [0,1]$ for every i=1,2,...,n. Now, we get $((c_i)_{h_i})^n(y_i)_{t_i} \subseteq (J)^n S \subseteq U$ hence $((c_i)_{h_i})^n(y_i)_{t_i} \subseteq U$. But U is an $\mathcal{F}$-Soc-semi-prime sub-module of X implies that $(c_i)_{h_i}(y_i)_{t_i} \subseteq U + \mathcal{F} - Soc(X)$ for each i=1,2,...,n. So we have $x_t \subseteq U + \mathcal{F} - Soc(X)$, it follows that $JS \subseteq U + \mathcal{F} - Soc(X)$.

$(\Leftarrow)$ Let $(r_b)^n x_t \subseteq U$ for $r_b \subseteq \mathcal{R}$ and $n \in \mathbb{Z}^+$ then $\langle r_b \rangle^n(x_t) \subseteq U$, that is $\langle r_b \rangle^n(x_t) \subseteq U$ then by hypothesis we get $\langle r_b \rangle(x_t) \subseteq U$, hence $r_b x_t \subseteq U$. That is mean U is an $\mathcal{F}$-Soc-semi-prime sub-module of X.

**Corollary 2.12**

Let U be an $\mathcal{F}$-sub-module of an $\mathcal{F}$-module X of an $\mathcal{R}$-module M, Then U is an $\mathcal{F}$-Soc-semi-prime sub-module of X if and only if $\forall \mathcal{F}$-sub-module S of X and every $\mathcal{F}$-singleton $r_b$ of $\mathcal{R}$ with $(r_b)^n S \subseteq U$ implies that $r_b S \subseteq U + \mathcal{F} - Soc(X)$.

Proof:

It is clear from (proposition 2.11).

**Corollary 2.13**

Let L be an $\mathcal{F}$-ideal of $\mathcal{R}$, Then L is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$ if and only if $\forall \mathcal{F}$-sub-ideal J of $\mathcal{R}$ and every $\mathcal{F}$-singleton $r_b$ of $\mathcal{R}$ with $(r_b)^n J \subseteq L$ implies that $r_b J \subseteq L + \mathcal{F} - Soc(\mathcal{R})$.

Proof:

Clearly from (proposition 2.11).

**Proposition 2.14**:
Proof:

Let \( a_t \subseteq U \) this implies \( r_b^n a_t \subseteq r_b^n U \), for \( r_b \) is an \( \mathcal{F} \)-singleton of \( R \). But, \( r_b^n U \) is an \( \mathcal{F} \)-Soc-semi-prime sub-module of \( X \) with \( a_t \subseteq X \), where \( t, b \in [0,1] \). Therefore \( r_b a_t \subseteq r_b^n U + F - Soc(X) \), that is \( r_b^2 a_t \subseteq r_b^{n+1} U + r_b F - Soc(X) \), thus \( r_b^2 a_t \subseteq r_b^2 r_b^{n-1} U + r_b F - Soc(X) \). But, \( X \) is a cancellative \( \mathcal{F} \)-module, we have \( a_t \subseteq r_b^{n-1} U + F - Soc(X) \), that is mean \( U \subseteq r_b^{n-1} U + F - Soc(X) \).

Remark 2.15

Every \( \mathcal{F} \)-semi-prime sub-module is an \( \mathcal{F} \)-Soc-semi-prime sub-module.

Proof:

It is Clear by definition of \( \mathcal{F} \)-semi-prime sub-module.

Remark 2.16

If \( U \) is an \( \mathcal{F} \)-Soc-semi-prime sub-module of \( \mathcal{F} \)-module \( X \), with \( \mathcal{F} - Soc(X) \subseteq U \). Then \( U \) is an \( \mathcal{F} \)-semi-prime sub-module.

Proof:

Assume that \( U \) is an \( \mathcal{F} \)-Soc-semi-prime sub-module of an \( \mathcal{F} \)-module \( X \) of an \( \mathcal{R} \)-module \( M \). Let \( (r^n)_b m_t = (r_b)^n m_t \subseteq U \), for \( r_b \subseteq \mathcal{R}, m_t \subseteq X \), where \( t, b \in [0,1] \). Since \( U \) is an \( \mathcal{F} \)-Soc-semi-prime sub-module, then \( r_b m_t \subseteq U + F - Soc(X) \subseteq U \) but \( \mathcal{F} - Soc(X) \subseteq U \). Hence \( U \) is an \( \mathcal{F} \)-semi-prime sub-module.

Corollary 2.17

If \( U \) is an \( \mathcal{F} \)-Soc-semi-prime sub-module of \( \mathcal{F} \)-module \( X \), with \( U \) be an \( \mathcal{F} \)-essential sub-module of \( X \). Then \( U \) is an \( \mathcal{F} \)-semi-prime sub-module.

Proof:

Since \( U \) be an \( \mathcal{F} \)-essential sub-module of \( X \), then by definition of \( \mathcal{F} \)-socle we have \( \mathcal{F} - Soc(X) \subseteq U \) and by (Remark 2.16) that is complete the proof.

Corollary 2.18

If \( U \) is an \( \mathcal{F} \)-sub-module of \( \mathcal{F} \)-module \( X \), with \( \mathcal{F} - Soc(X) \subseteq U \). Then \( U \) is an \( \mathcal{F} \)-semi-prime sub-module of \( X \) if and only if \( U \) is an \( \mathcal{F} \)-Soc-prime sub-module of \( X \).

Proof:

Consequently from (Remark 2.7) and (Remark 2.16).

Remark 2.19

Let \( U \) and \( V \) are \( \mathcal{F} \)-sub-modules of \( \mathcal{F} \)-module \( X \). If \( U + V \) is an \( \mathcal{F} \)-semi-prime sub-module of \( X \) with \( V \subseteq \mathcal{F} - Soc(X) \), then \( U \) is an \( \mathcal{F} \)-Soc-semi-prime sub-module of \( X \).

Proof:
Let \((r^n)_b x_k = (r_b)^n x_k \subseteq U\), for \(r_b \subseteq R\), \(x_k \subseteq X\), where \(k, b \in [0,1]\). This implies \((r^n)_b x_k \subseteq U + V\). But \(U + V\) is an \(\mathcal{F}\)-semi-prime sub-module of \(X\), hence \(r_b x_k \subseteq U + V \subseteq U + \mathcal{F} - \text{Soc}(X)\) since \(V \subseteq \mathcal{F} - \text{Soc}(X)\). That is \(U\) is an \(\mathcal{F}\)-Soc-semi-prime sub-module of \(X\).

**Theorem 2.20**

Any \(\mathcal{F}\)-sub-module of semi-simple \(\mathcal{F}\)-module \(X\) is an \(\mathcal{F}\)-Soc-semi-prime sub-module of \(X\).

**Proof:**

If \(U\) is an \(\mathcal{F}\)-sub-module of an \(\mathcal{F}\)-module \(X\) of an \(\mathcal{R}\)-module \(M\). Let \((r^n)_b x_k = (r_b)^n x_k \subseteq U\), for \(r_b \subseteq R\), \(x_k \subseteq X\), where \(k, b \in [0,1]\). But, \(X\) is a semi-simple \(\mathcal{F}\)-module, thus \(X = \mathcal{F} - \text{Soc}(X)\). We have \(x_k \subseteq X = \mathcal{F} - \text{Soc}(X) \subseteq U + \mathcal{F} - \text{Soc}(X)\), this implies \(r_b x_k \subseteq r_b X = r_b \mathcal{F} - \text{Soc}(X) \subseteq r_b \big( U + \mathcal{F} - \text{Soc}(X) \big) \subseteq U + \mathcal{F} - \text{Soc}(X)\) that is \(U\) is an \(\mathcal{F}\)-Soc-semi-prime sub-module of \(X\).

**Proposition 2.21:**

If \(U\) is a weakly pure \(\mathcal{F}\)-sub-module of \(\mathcal{F}\)-module \(X\) with \((r^n)_b U\) is an \(\mathcal{F}\)-Soc-semi-prime sub-module of \(X\) for every non-empty \(\mathcal{F}\)-singleton \(r_b\) of \(R\), then \(U\) is an \(\mathcal{F}\)-Soc-semi-prime sub-module of \(X\).

**Proof:**

Suppose that \((r^n)_b x_t \subseteq U\), with \(r_b\) is an \(\mathcal{F}\)-singleton of \(R\) and \(x_t \subseteq X\), where \(t, b \in [0,1]\). Also \((r^n)_b x_t \subseteq (r^n)_b X\) this implies \((r^n)_b x_t \subseteq U \cap (r^n)_b X = (r^n)_b U\) since \(U\) is a weakly pure \(\mathcal{F}\)-sub-module of \(X\), but \((r^n)_b U\) is an \(\mathcal{F}\)-Soc-semi-prime sub-module of \(X\), hence \(r_b x_t \subseteq (r^n)_b U + \mathcal{F} - \text{Soc}(X) \subseteq U + \mathcal{F} - \text{Soc}(X)\). Thus \(U\) is an \(\mathcal{F}\)-Soc-semi-prime sub-module of \(X\).

**Lemma 2.22:**

for every fuzzy sub-\((A \oplus B) + \mathcal{F} - \text{Soc}(X \oplus Y) = (A + \mathcal{F} - \text{Soc}(X)) \oplus (B + \mathcal{F} - \text{Soc}(Y))\)
modules \(A\) and \(B\) of fuzzy modules \(X\) and \(Y\) respectively.

**Proof:**

From (Lemma 2.2) we get \((F - \text{Soc}(X \oplus Y))_t = \text{Soc}((X \oplus Y)_t)\)

For each \(t \in (0,1]\). But, \(\text{Soc}((X \oplus Y)_t) = \text{Soc}(X_t \oplus Y_t)\) and we have \(\text{Soc}(X_t \oplus Y_t) = \text{Soc}(X_t) \oplus \text{Soc}(Y_t)\)

That is \((F - \text{Soc}(X \oplus Y))_t = \text{Soc}(X_t) \oplus \text{Soc}(Y_t) = (F - \text{Soc}(X))_t \oplus (F - \text{Soc}(Y))_t\)

Thus \((F - \text{Soc}(X \oplus Y))_t = [(F - \text{Soc}(X)) \oplus (F - \text{Soc}(Y))]_t\)

Hence from (Remark 1.5) then \(F - \text{Soc}(X \oplus Y) = F - \text{Soc}(X) \oplus F - \text{Soc}(Y)\)

**Proposition 2.23:**

If \(U\) and \(V\) are \(\mathcal{F}\)-sub-modules of \(\mathcal{F}\)-modules \(X\) and \(Y\) respectively, then

1) If \(U \oplus Y\) is an \(\mathcal{F}\)-Soc-semi-prime sub-module of \(X \oplus Y\) thus \(U\) is an \(\mathcal{F}\)-Soc-semi-prime sub-module of \(X\).
2) if $X \oplus V$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X \oplus Y$ thus $V$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X$.

Proof:

1) Suppose that $U \oplus Y$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X \oplus Y$ and $r_b$ is an $\mathcal{F}$-singleton of $R$ and $x_t \subseteq X$ such that $(r^n)b(x_t) \subseteq U$. Then $(r^n)b(x_t, y_p) = (r^n)b(x_t, y_p) \subseteq U \oplus Y$, for any $\mathcal{F}$-singleton $y_p \subseteq Y$, but $U \oplus Y$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X \oplus Y$. Thus $(r_bx_t, r_by_p) \subseteq (U \oplus Y) + F - \text{Soc}(X \oplus Y)$, by (Lemma 2.22) we get $(r_bx_t, r_by_p) \subseteq (U + F - \text{Soc}(X)) \oplus (Y + F - \text{Soc}(Y))$. That is $r_bx_t \subseteq U + F - \text{Soc}(X)$, therefore $U$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X$.

2) Similarly as the idea in (1).

**Lemma 2.24:**

If $X$ is an $\mathcal{F}$-module of an $\mathcal{R}$-module $M$, and $M$ be a faithful multiplication $\mathcal{R}$-module, then:

$$\mathcal{F} - \text{Soc}(X) = X \mathcal{F} - \text{Soc}(\mathcal{R})$$

**Proposition 2.25:**

Let $X$ be a finitely generated multiplication and faithful $\mathcal{F}$-module of an $\mathcal{R}$-module $M$, if $J$ is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$ then $JX$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X$.

Proof:

Assume that $r_b$ is an $\mathcal{F}$-singleton of $\mathcal{R}$ and $x_t \subseteq X$ such that $(r^n)b(x_k) = (r^n)x_k \subseteq JX$, where $k, b \in [0, 1]$, that is $(r^n)b(x_t) \subseteq JX$. But $X$ is a multiplication $\mathcal{F}$-module, thus there exists an $\mathcal{F}$-ideal $L$ of $\mathcal{R}$ with $\langle x_t \rangle = LX$. Then we get $(r^n)bLX \subseteq JX$, so $(r^n)bL \subseteq J + \mathcal{F} - \text{ann}(X) = J$ since $X$ is a faithful $\mathcal{F}$-module. But $J$ is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$, then by (Corollary 2.13) implies that $r_bL \subseteq J + \mathcal{F} - \text{Soc}(\mathcal{R})$. Now, by multiplying both sides with $X$ and using (Lemma 2.24) we have $r_bLX \subseteq JX + \mathcal{F} - \text{Soc}(\mathcal{R})X = JX + \mathcal{F} - \text{Soc}(X)$. Therefore, $JX$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X$.

**Proposition 2.26:**

Suppose that $U$ is an $\mathcal{F}$-Soc-semi-prime sub-module of an $\mathcal{F}$-module $X$ and $V$ is an $\mathcal{F}$-semi-prime sub-module of $X$ with $\mathcal{F} - \text{Soc}(X) \subseteq V$. Then the intersection of $U$ and $V$ is an $\mathcal{F}$-Soc-semi-prime of $X$.

Proof:

If $r_b$ is an $\mathcal{F}$-singleton of $\mathcal{R}$ and $x_t \subseteq X$ where $b, t \in [0, 1]$, such that $(r^n)b(x_k) = (r^n)x_k \subseteq U \cap V$. This implies $(r^n)b(x_t) \subseteq U$ and $(r^n)b(x_t) \subseteq V$, but $U$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X$. So, we have $r_bx_k \subseteq U + \mathcal{F} - \text{Soc}(X)$. Now, since $V$ is an $\mathcal{F}$-semi-prime sub-module of $X$ then $r_bx_k \subseteq V$. We get $r_bx_k \subseteq [U + \mathcal{F} - \text{Soc}(X)] \cap V$, but $\mathcal{F} - \text{Soc}(X) \subseteq V$ then by using (Lemma 1.29) we have $r_bx_k \subseteq (U \cap V) + \mathcal{F} - \text{Soc}(X)$. That is mean $U \cap V$ is an $\mathcal{F}$-Soc-semi-prime of $X$.

**Proposition 2.27**
Let X be a faithful multiplication $\mathcal{F}$-module of an $\mathcal{R}$-module $M$, then a proper $\mathcal{F}$-sub-module $U$ is an $\mathcal{F}$-Soc-semi-prime sub-module of if and only if $[U:R]X$ is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$.

Proof:

Let $(r^n)_b m_t = (r_b)^n m_t \subseteq [U:R]X$ with $m_t$ and $r_b$ are $\mathcal{F}$-singletons of $\mathcal{R}$ where $b, t \in [0, 1]$ implies that $(r^n)_b(m_t X) \subseteq U$. But, $U$ is an $\mathcal{F}$-Soc-semi-prime sub-module, so by (Corollary 2.13) then $\gamma_b(m_t X) \subseteq U + \mathcal{F} - Soc(X)$ . Since $X$ is a multiplication $\mathcal{F}$-module, then by (Preposition 1.32) $U = [U:R]X$ , and since $X$ is a faithful multiplication, so by (Lemma 2.24) $F - Soc(X) = F - Soc(\mathcal{R})X$. Therefore $r_b m_t X \subseteq [U:R]X + \mathcal{F} - Soc(\mathcal{R})X$, this implies $r_b m_t \subseteq [U:R]X + \mathcal{F} - Soc(\mathcal{R})$. Thus $[U:R]X$ is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$.

Conversely

Let $(r^n)_bD = (r_b)^n D \subseteq U$ with $r_b$ be an $\mathcal{F}$-singleton of $\mathcal{R}$ and $D$ is an $\mathcal{F}$-sub-module of $X$. Since $X$ is a multiplication $\mathcal{F}$-module, then $D = JX$ for some an $\mathcal{F}$-ideal of $\mathcal{R}$, we get $(r^n)_bJX \subseteq U$ that is mean $(r^n)_b \subseteq [U:R]X$, but $[U:R]X$ is an $\mathcal{F}$-Soc-semi-prime ideal of $\mathcal{R}$, so by (Corollary 2.12) we have $r_bJ \subseteq [U:R]X + F - Soc(\mathcal{R})$, this implies $r_bJX \subseteq [U:R]X + \mathcal{F} - Soc(\mathcal{R})X$ , then by (Lemma 2.25) we get $r_bJX \subseteq U + \mathcal{F} - Soc(X)$.

Lemma 2.28

Let $f: M \rightarrow \overline{M}$ be isomorphism mapping from an $\mathcal{R}$-module $M$ into an $\mathcal{R}$-module $\overline{M}$ .If $X$ and $\overline{X}$ are $\mathcal{F}$-modules of an $\mathcal{R}$-modules $M$ and $\overline{M}$ respectively. Then $f(F - Soc(X)) \subseteq F - Soc(\overline{X})$.

Proposition 2.29

Let $f: X \rightarrow \overline{X}$ be an $\mathcal{F}$-isomorphism from $\mathcal{F}$-module $X$ into $\mathcal{F}$-module $\overline{X}$, with $U$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X$, such that $ker(f) \subseteq U$. Then $f(U)$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $\overline{X}$.

Proof:

is a proper $\mathcal{F}$-sub-module of $\overline{X}$. If not, then $f(U) = \overline{X}$. Let $x_t \subseteq X$, so $f(x_t) \subseteq \overline{X} = f(U)$ $f(U)$, that is there exists $y_s \subseteq U$ where $s, t \in [0, 1]$ such that $f(x_t) = f(y_s)$ implies that $f(x_t) - f(y_s) = 0_1$ then $f(x_t - y_s) = 0_1$, thus $x_t - y_s \subseteq ker(f) \subseteq U$, it follows that $x_t \subseteq U$.Thus $U = X$ that is a contradiction. Now, Let $(r^n)_b z_c \subseteq f(U)$ with $r_b \subseteq \mathcal{R}$ and $z_c \subseteq \overline{X}$ with $b, c \in [0, 1]$, but $f$ is onto $f(x_t) = z_c$ for some $x_t \subseteq X$, therefore $(r^n)_b z_c = (r^n)_b f(x_t) = f((r^n)_b x_t) \subseteq f(U)$, this implies that there exists $k_h \subseteq U$ with $h \in [0, 1]$ such that $f(k_h) = f((r^n)_b x_t)$, that is $f(k_h) - (r^n)_b x_t = 0_1$, so $k_h - (r^n)_b x_t \subseteq ker(f) \subseteq U$. It follows that $(r^n)_b x_t \subseteq U$. But, $U$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $X$, thus $r_b x_t \subseteq U + \mathcal{F} - Soc(X)$. Then by (Lemma 2.28) we have $r_b z_c = r_b f(x_t) \subseteq f(U) + f(F - Soc(X)) \subseteq f(U) + F - Soc(\overline{X})$ . Hence $f(U)$ is an $\mathcal{F}$-Soc-semi-prime sub-module of $\overline{X}$.

2. Conclusion

Through this research, we were able to know some of the fuzzy algebraic properties of fuzzy socle semi-prime sub-modules and the relationship with other concepts .
The idea of fuzzy socle semi-prime sub-modules is dualized in this study by introducing several characteristics and properties of semi-prime fuzzy sub-modules. This approach has opened up new possibilities for studying the fuzzy dimension. Thus, socle semi-prime module and completely socle semi-prime sub-modules can be defined utilizing the concept of fuzzy socle semi-prime sub-modules.

References

1. Zadeh L. A. 1965. Fuzzy Sets, *Information and control*, 8: 338-353, 1965.
2. Naegoita, C. V.; Ralescu, D. A. Application of Fuzzy Sets in System Analysis, Birkhauser, Basel, Switzerland, 1975.
3. Hadi, I. M. A. Semiprime Fuzzy Sub-modules of Fuzzy Modules, *Ibn-Al-Haitham J. for Pure and Appl. Sci.*, 2004, 17(3), 112-123.
4. Ali S.A. Approximately Prime Sub-modules and Some of Their Generalizations M.Sc. Thesis, University of Tikrit. 2019
5. Martinez, L., *Fuzzy Modules Over Fuzzy Rings in Connection with Fuzzy Ideals of Rings*, J.Fuzzy Math.1996, 4, 843-857.
6. Zahedi, M. M. On L-Fuzzy Residual Quotient Modules and P. Primary Sub-modules, *Fuzzy Sets and Systems*, 1992, 51: 333-344.
7. Mukherjee, T. K.; Sen, M. K.; Roy, D. On Fuzzy Sub-modules and Their Radicals, J. Fuzzy Math., 1996, 4, 549-558.
8. Mashinchi, M.; Zahedi, M.M., 2,"On L-Fuzzy Primary Sub-modules,Fuzzy Sets Systems, 1999,49,231-236.
9. Rabi H. J. Prime Fuzzy Sub-module and Prime Fuzzy Modules, M. Sc. Thesis, University of Baghdad. 2001
10. Zahedi, M. M. A characterization of L-Fuzzy Prime Ideals, *Fuzzy Sets and Systems*, 1991, 44: 147-160.
11. AL-Abege A. M. H, Near-ring, Near Module and Their Spectrum, M.Sc. Thesis, University of Kufa, College of Mathematics and Computers Sciences. 2010
12. Mashinchi, M., Zahedi, M.M., On L-Fuzzy Primary Sub-modules, *Fuzzy Sets Systems*, 1996, 4, 843-857.
13. Gada, A.A., Fuzzy Spectrum of Modules Over Commutative Rings, M.Sc. Thesis, University of Baghdad. 2000
14. Kasch, F., Modules and Rings, Academic press. 1982
15. Hatam Y. K., Fuzzy Quasi-Prime Modules and Fuzzy Quasi-Prime. Sub-modules, M.Sc. Thesis, University of Baghdad. 2001
16. Kalita, M. C A study of fuzzy algebraic structures: some special types, Ph.D Thesis, Gauhati University, Gauhati, India, 2007.
17. Wafaa, H. H.T-ABSO Fuzzy Sub-modules and T-ABSO Fuzzy Modules and Some Their Generalizations, Ph.D. Thesis, University of Baghdad. 2018.
18. Hadi, G. Rashed, Fully Cancellation Fuzzy Modules and Some Generalizations, M.Sc. Thesis, University of Baghdad. 2017
19. Goodreal, K. R. Ring Theory –Non Singular Rings and modules, Marci-Dekker, New York and Basel. 1976
