New progress in the inverse problem in the calculus of variations

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December 16, 2014

Abstract. We present a new class solutions for the inverse problem in the calculus of variations in arbitrary dimension \( n \). We also provide a number of new theorems concerning the inverse problem using exterior differential systems theory. Our new techniques provide a significant advance in the understanding of the inverse problem in arbitrary dimension. We show that when the eigenvalues of a certain curvature tensor are distinct and with \( n - 1 \) integrable eigen-distributions, the corresponding differential equations are variational only if the non-integrable eigenspace has a certain geometric property. The resulting Lagrangians depend on \( n - 1 \) functions each of 2 variables. We give some non-trivial examples in dimension 3. We finish with a new classification scheme for the inverse problem in arbitrary dimension.

1 Introduction: the inverse problem

The inverse problem in the calculus of variations can be expressed as follows. Given a system of second-order differential equations

\[
\ddot{x}^a = F^a(t, x^b, \dot{x}^b), \quad a, b = 1, \ldots, n,
\]

the question is whether the solutions of this system are also the solutions of the Euler Lagrange equations,

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^a} \right) - \frac{\partial L}{\partial x^a} = 0,
\]

for some regular Lagrangian \( L(t, x^b, \dot{x}^b) \). This problem was first proposed by Helmholtz [11] in 1887. He considered whether the equations in the form presented were Euler-Lagrange. In the case of single equations Helmholtz found necessary conditions for this to be true. It is not well-known that Sonin [23] solved this one-dimensional problem the previous year in a more general form, although Hirsch [12] is credited with the so-called multiplier version of the inverse problem,
which is the focus of this paper. He addressed the uniqueness and existence of a non-degenerate multiplier matrix, \( g_{ab}(t, x^c, \dot{x}^c) \), satisfying

\[
g_{ab}(\dot{x}^b - F^b) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^a} \right) - \frac{\partial L}{\partial x^a}.
\]

Necessary and sufficient conditions for the existence of a regular Lagrangian, according to Doulgas [9] and to Sarlet [19], are that this multiplier satisfy

\[
g_{ab} = g_{ba}, \quad \Gamma(g_{ab}) = g_{ac} \Phi^c_b + g_{bc} \Phi^c_a, \quad g_{ac} \Phi^c_b = g_{bc} \Phi^c_a, \quad \frac{\partial g_{ab}}{\partial \dot{x}^c} = \frac{\partial g_{ac}}{\partial \dot{x}^b} \Phi^c_b,
\]

where

\[
\Gamma^a := -\frac{1}{2} \frac{\partial F^a}{\partial x^b}, \quad \Phi^a_b := -\frac{\partial F^a}{\partial x^b} + \Gamma^a_c \Gamma^c_b - \Gamma^a_b, \quad \Gamma := \frac{\partial}{\partial t} + \dot{x}^a \frac{\partial}{\partial x^a} + F^a \frac{\partial}{\partial \dot{x}^a}.
\]

These conditions, along with non-degeneracy, are commonly referred to as the Helmholtz conditions. If one or more matrices \( g_{ab} \) are found that satisfy these four Helmholtz conditions, then there exists one (or more) Lagrangians related to these matrices by the expression,

\[
\frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b} = g_{ab}.
\]

Integrating this for a given \( g_{ab} \) we see that the related Lagrangian \( L \) is only defined up to the addition of a total time derivative of an arbitrary function of \( t, x^a \).

The multiplier problem was completely solved by Douglas in 1941 for the two dimensional case (see [9]), that is, a pair of second order equations on the plane. He divided the problem into four primary cases (I to IV) according to the properties of the matrix \( \Phi^a_b \). The corresponding solution for higher dimensions, even for dimension 3, remained unsolved until late nineteen nineties when some arbitrary \( n \) subcases were elaborated [5, 20, 3].

The current attacks on this problem, dating back to the 1980s, involve the creation and use of various differential geometric tools. We offer the reader the following references which give some perspective on these developments [3, 6, 7, 10, 14, 16, 17, 21].

The current situation, in the framework of [7], is that the following dimension \( n \) situations are solved in the sense of Douglas.

1. \( \Phi \) is a multiple of the identity. The multiplier is determined by \( n \) arbitrary functions each of \( n + 1 \) variables. This is the extension of Douglas’s case I. See [2, 3, 20].

2. \( \Phi \) is diagonalisable with distinct eigenvalues and “integrable eigenspaces”. The multiplier is determined by \( n \) arbitrary functions each of 2 variables. This is the extension of Douglas’s case IIa1. See [5, 1].

3. A more general case in which, for example, there are no integrable eigenspaces of \( \Phi \). The multiplier is fixed and its existence depends on the closure of a single one form. See [18].

4. There are many non-existence results depending on technical conditions on \( \Phi \). For example, in this paper see theorem [3, 9] and also [18].
In this paper we will tackle the cases where some eigenspaces are integrable and some are not. This generalises Douglas’s case IIa2 in which, for $n = 2$, one eigenspace is integrable and one is not. In doing so we produce a scheme to replace the four cases of Douglas in the treatment of the arbitrary dimension problem. This is detailed in the last section.

Of course, in general, $\Phi$ will not be diagonalisable over the reals. But when it is, the eigenvalues will generally be distinct and none of the eigenspaces will be integrable. Even in this situation there are cases where solutions exist. Indeed, it is remarkable that so many of the cases are well populated with variational equations. We will illustrate our own results with a number of examples in the second last section.

In the context of our current paper Anderson and Thompson [3] made a significant breakthrough by applying the method of exterior differential systems (EDS) to the inverse problem. They illustrated the effectiveness of the method by many concrete examples. In their paper, however only Douglas’s case I where $\Phi$ is a multiple of the identity was generalised to arbitrary $n$. Aldridge [1] and Aldridge et al [2] pursued this EDS approach further using the connection of Massa and Pagani [15]. While the thesis [1] re-investigated Douglas’ case IIa2 in dimension 2 and recovered case IIa1 for arbitrary dimension, the paper [2] focused on the first step of EDS process which is the so-called differential ideal step.

In this paper we will use EDS to generalise to arbitrary dimension Douglas’ case IIa2: the semi-separable case where $\Phi$ is diagonalisable with distinct eigenvalues. We give a set of necessary and sufficient conditions to be able to quickly check the existence of the solution rather than processing the whole EDS procedure over again for each system of SODE. In addition, we identify the importance of an integrable direction in an eigen-distribution for the existence of non-degenerate solutions.

## 2 Geometric formulation

We now briefly describe the basics of our geometrical calculus, for more details we refer to the thesis [1] and the book chapter [14]. We are analysing a system of second-order differential equations

$$\ddot{x}^a = F^a(t, x^b, \dot{x}^b), \quad a, b = 1, \ldots, n, \quad (2)$$

for some smooth $F^a$ on a manifold $M$ with generic local coordinates $(x^a)$. The evolution space $E := \mathbb{R} \times TM$ has adapted coordinates $(t, x^a, \dot{x}^a)$ or $(t, x^a, u^a)$. We construct on $E$ from the system of equations (2) a second-order differential equation field (SODE):

$$\Gamma := \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + F^a \frac{\partial}{\partial u^a}. \quad (3)$$

The SODE produces on $E$ a nonlinear connection with components $\Gamma^a_b := -\frac{1}{2} \frac{\partial F^a}{\partial u^b}$. The evolution space $E$ has a number of natural structures. The contact and vertical structures are combined in the *vertical endomorphism* $S$, a (1,1) tensor field on $E$, in coordinate expression: $S := V_a \otimes \theta^a$, where $V_a := \frac{\partial}{\partial x^a}$ are the vertical basis fields and $\theta^a := dx^a - u^a dt$ are local contact forms. It is shown in [6] the first order deformation $L_\Gamma S$ has eigenvalues 0, 1 and $-1$. The eigenspaces corresponding to eigenvalue 0, 1 and $-1$ are $Sp(\Gamma)$, the *vertical subspace* $V(E) := Sp\{V_a\}$ of the tangent space and the *horizontal subspace* $H(E) := Sp\{H_a\}$ respectively, where

$$H_a = \frac{\partial}{\partial x^a} - \Gamma^b_a \frac{\partial}{\partial u^b}.$$
So an adapted local basis on $E$ is given by \{\Gamma, V_a, H_a\} with dual basis \{dt, \psi^a, \theta^a\} where 
\[ \psi^a := du^a - F^a dt + \Gamma^a_b \theta^b. \]
Following directly from the definitions of $\Gamma, V_a$ and $H_a$ given in [6, 17] we note:

\[ [\Gamma, H_a] = \Gamma^b_a H_b + \Phi^b_a V_b, \quad [\Gamma, V_a] = -H_a + \Gamma^b_a V_b, \quad [V_a, V_b] = 0, \]

and

\[ [H_a, V_b] = -\frac{1}{2} \left( \frac{\partial^2 F^c}{\partial u^a \partial u^b} \right) V_c = V_b(\Gamma^c_a) V_c = V_a(\Gamma^c_b) V_c = [H_b, V_a], \]

where the curvature of the connection is defined by

\[ R^d_{ab} := \frac{1}{2} \left( \frac{\partial^2 F^d}{\partial x^a \partial u^b} - \frac{\partial^2 F^d}{\partial x^b \partial u^a} + \frac{1}{2} \left( \frac{\partial F^c}{\partial u^a} \frac{\partial^2 F^d}{\partial u^c \partial u^b} - \frac{\partial F^c}{\partial u^b} \frac{\partial^2 F^d}{\partial u^c \partial u^a} \right) \right). \]

Following [22] we briefly introduce vector fields and forms along the projection $\pi^0_1 : E \rightarrow \mathbb{R} \times M$. Vector fields along $\pi^0_1$ are sections of the pull back bundle $\pi^0_1(T(\mathbb{R} \times M))$ over $E$. $X(\pi^0_1)$ denotes the $C^\infty(E)$ module of such vector fields. In the current context see [13].

With $X := X^0 \frac{\partial}{\partial t} + X^a \frac{\partial}{\partial x^a}$ and $dt, \theta^a \in X^*(\pi^0_1)$ we define the following lifts from $X(\pi^0_1)$ to $X(E)$:

\[ X^\Gamma := dt(\Gamma)H, \quad X^H := \theta^a(X)H_a, \quad X^V := \theta^a(X)V_a. \]

For an intrinsic treatment of these various lifts see [14].

Any vector $W \in X(E)$ can then be uniquely decomposed as

\[ W = (W^\Gamma)^\Gamma + (W^H)^H + (W^V)^V \]

where $W^\Gamma, W^H, W^V \in X(\pi^0_1)$ with

\[ W^\Gamma := dt(W)\left( \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} \right), \quad W^H := dt(W)\left( \frac{\partial}{\partial t} + dx^a(W) \frac{\partial}{\partial x^a} \right), \quad W^V := \psi^a(W) \frac{\partial}{\partial x^a}. \]

The dynamical covariant derivative and the Jacobi endomorphism $\Phi$ that appear in [11] are defined along the tangent bundle projection through the following formulas

\[ [\Gamma, X^V] = -X^H + (\nabla X)^V, \quad [\Gamma, X^H] = (\nabla X)^H + \Phi(X)^V. \]

In coordinates,

\[ \Phi = \Phi^a_b \frac{\partial}{\partial x^a} \otimes \theta^b, \]

with $\Phi^a_b$ as defined before. $\nabla$ can be extended to act on forms along the projection and is given in coordinates by

\[ \nabla f = \Gamma(f) \text{ on functions}, \quad \nabla \frac{\partial}{\partial x^a} = \Gamma^b_a \frac{\partial}{\partial x^b}, \quad \nabla \theta^a = -\Gamma^a_b \theta^b. \]
Massa and Pagani [15] introduced a linear connection on $E$ by imposing some natural requirements. If we denote the connection by $\hat{\nabla}$, these are that the covariant differentials $\hat{\nabla} dt$, $\hat{\nabla} S$ and $\hat{\nabla} \Gamma$ are all zero and that the vertical sub-bundle is flat. We will produce it here by its coordinate expressions:

$$\begin{align*}
\hat{\nabla}_t \Gamma &= 0, & \hat{\nabla}_t H_a &= \Gamma^b_a H_b, & \hat{\nabla}_t V_a &= \Gamma^b_a V_b, \\
\hat{\nabla}_{H_a} \Gamma &= 0, & \hat{\nabla}_{H_a} H_b &= \frac{\partial \Gamma^c}{\partial u^b} H_c, & \hat{\nabla}_{H_a} V_b &= \frac{\partial \Gamma^c}{\partial u^b} V_c, \\
\hat{\nabla}_{V_a} \Gamma &= 0, & \hat{\nabla}_{V_a} H_b &= 0, & \hat{\nabla}_{V_a} V_b &= 0,
\end{align*}$$

(4)

With any linear connection, $\nabla$ (not to be confused with the dynamical covariant derivative), on a manifold $M$ there is an associated shape map and torsion (see [13]). The shape map $A_Z : \mathcal{X}(M) \to \mathcal{X}(M)$, as given in Jerie and Prince [13], is defined as

$$A_Z(\xi) := \left. \frac{d}{dt} \right|_{t=0} \tau_t^{-1}(\zeta_t \xi), \quad \text{where } \xi \in T_x M.$$ 

where $\tau_t : T_x M \to T_{\zeta_t(x)} M$ is the parallel transport map, defining $\nabla$ along the flow $\{\zeta_t\}$.

More useful representations of this shape map on $M$ are

$$\begin{align*}
A_X Y &= \nabla_X Y - [X, Y], \\
A_X Y &= \nabla_Y X + T(X, Y)
\end{align*}$$

where $X, Y \in \mathcal{X}(M)$ and the torsion is defined as usual by

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

The torsion, $\hat{T}$, of the Massa and Pagani connection captures all the important properties of the SODE $\Gamma$:

$$\begin{align*}
\hat{T}(\Gamma, V_a) &= H_a, & \hat{T}(\Gamma, H_a) &= -\Phi_a^b V_b, & \hat{T}(V_a, V_b) &= 0, \\
\hat{T}(V_a, H_b) &= 0, & \hat{T}(H_a, H_b) &= -R^c_{ab} V_c.
\end{align*}$$

The commutators of the vector fields on $E$ can be written in terms of connection $\hat{\nabla}$ and the shape map as

$$\begin{align*}
[\Gamma, X^V] &= \hat{\nabla}_\Gamma X^V - A_\Gamma(X^V), & (5a) \\
[\Gamma, X^H] &= \hat{\nabla}_\Gamma X^H - A_\Gamma(X^H), & (5b) \\
[X^V, Y^V] &= \hat{\nabla}_{X^V} Y^V - \hat{\nabla}_{Y^V} X^V, & (5c) \\
[X^V, Y^H] &= \hat{\nabla}_{X^V} Y^H - \hat{\nabla}_{Y^H} X^V, & (5d) \\
[X^H, Y^H] &= \hat{\nabla}_{X^H} Y^H - \hat{\nabla}_{Y^H} X^H + R(X^H, Y^H). & (5e)
\end{align*}$$

It can be seen that $\hat{\nabla}$ and $A_\Gamma$ replace $\nabla$ and $\Phi$ respectively.

The curvature tensor $R$ in (5e) is a $(1,2)$ tensor field on $E$ defined as follows

$$R := R^c_{ab} (\theta^a \wedge \theta^b) \otimes V_c.$$ 

We will make no notational distinction between $\Phi$ acting along the tangent bundle projection and acting as an endomorphism on $E$ as $\Phi = \Phi^b_a V_b \otimes \theta^a$.

In the next section we briefly sketch the ideas of the exterior differential systems approach, specifically in the context of the inverse problem.
3 Helmholtz conditions and the EDS process

3.1 The Helmholtz conditions

For a given regular Lagrangian \( L \), there is a unique vector field, called the \emph{Euler-Lagrange field}, \( \Gamma \) on \( E \) such that
\[
\Gamma \cdot d\theta_L = 0 \quad \text{and} \quad dt(\Gamma) = 1.
\]
where \( \theta_L \) is the \emph{Poincaré-Cartan 1-form}
\[
\theta_L := Ldt + dL \circ S = Ldt + \frac{\partial L}{\partial u^a} \theta^a.
\]
This vector field is a SODE, and the equations satisfied by its integral curves are the Euler-Lagrange equations for \( L \). By careful observation of the properties of the Cartan 2-form \( d\theta_L \) the following theorem from [6] gives a transparent geometric version of the Helmholtz conditions.

**Theorem 3.1.** [6] Given a SODE \( \Gamma \), the necessary and sufficient conditions for there to be a Lagrangian, whose Euler-Lagrange field is \( \Gamma \), are that there should exist a 2-form \( \Omega \) satisfying
\[
\begin{align*}
\Omega & \text{ is of maximal rank, i.e., } \wedge^n \Omega \neq 0, \quad (6) \\
\Omega(V_a, V_b) &= 0, \quad \forall V_a, V_b \in V(E), \quad (7) \\
\Gamma \cdot \Omega &= 0, \quad (8) \\
d\Omega &= 0. \quad (9)
\end{align*}
\]

The proof of this theorem can be found in Crampin, Prince and Thompson [6]. We will show how the Helmholtz conditions in (1) arise from this theorem as follows.

The condition (9) gives
\[
d\Omega(\Gamma, V_a, V_b) = 0, \quad d\Omega(\Gamma, H_a, H_b) = 0, \quad d\Omega(\Gamma, V_a, H_b) = 0, \quad d\Omega(V_a, V_b, H_c) = 0, \quad d\Omega(\Gamma, V_a, H_b, H_c) = 0.
\]

In Crampin, Prince and Thompson [6] it is shown by using \( \mathcal{L}_\Gamma \Omega = 0 \) (from (8) and (9)) that
\[
\Omega = g_{ab} \psi^a \wedge \psi^b, \quad |g_{ab}| \neq 0. \quad (11)
\]

Applying the formula for the exterior derivative of a 2-form \( \Omega \):
\[
d\Omega(X, Y, Z) = \sum_{\text{cyclic } X, Y, Z} (X(\Omega(Y, Z)) - \Omega([X, Y], Z)) \quad (12)
\]
to the above conditions we find that the fifth condition is empty by the assumptions (7) and (8) of the theorem and the first four conditions are
\[
\begin{align*}
d\Omega(\Gamma, V_a, V_b) &= 0 \quad \Leftrightarrow \quad g_{ab} = g_{ba}, \\
d\Omega(\Gamma, H_a, H_b) &= 0 \quad \Leftrightarrow \quad g_{ac} \Phi^c_b = g_{bc} \Phi^c_a, \\
d\Omega(\Gamma, V_a, H_b) &= 0 \quad \Leftrightarrow \quad \Gamma(g_{ab}) - g_{ac} \Gamma^c_b - g_{bc} \Gamma^c_a = 0, \\
d\Omega(V_a, V_b, H_c) &= 0 \quad \Leftrightarrow \quad \frac{\partial g_{ab}}{\partial x^c} = \frac{\partial g_{ac}}{\partial x^b}.
\end{align*}
\]
Therefore \( d\Omega = 0 \) is equivalent to the Helmholtz conditions in (11) since two last conditions in (10) are the consequences of the first four as shown in this context by Aldridge [1].

In this paper we focus on the case where the matrix representation, \( \Phi \), of \( \Phi \) is diagonalisable with distinct eigenvalues. For this reason we introduce the eigenvectors of \( \Phi \) on the tangent bundle projection. Let \( X_a = X^b_a \frac{\partial}{\partial x^b} \in X(\pi^0) \) be eigenvectors of \( \Phi \) corresponding to \( n \) distinct eigenvalues \( \lambda_a \), then their vertical lifts \( X^V_a := X^b_a V_b \), and their horizontal lifts \( X^H_a := X^b_a H_b \), along with \( \Gamma \) form a basis for \( X(E) \). The lifted eigenforms \( \phi^V \) and \( \phi^H \) together with \( dt \) form the dual basis \( \{ dt, \phi^V, \phi^H \} \) to the basis \( \{ \Gamma, X^V_a, X^H_a \} \). These are our choices of the basis for \( X(E) \) to work with rather than the conventional ones. This means that we will look for a non-degenerate closed 2-form \( \Omega \in \Sigma := Sp\{ \phi^V \wedge \phi^H \} \), that is \( \Omega = r_{ab} \phi^V \wedge \phi^H \) instead of the one in (11).

Having the basis \( \{ \Gamma, X^V_a, X^H_a \} \), the commutators identities regarding the vector fields belong to this basis derived from (5a)-(5c) are as following:

\[
\begin{align*}
[\Gamma, X^V_a] &= \tau^a \Gamma X^V_a - X^H_a, & (13a) \\
[\Gamma, X^H_a] &= \tau^a \Gamma X^H_a + \lambda_b \delta^a_b X^V_a = \tau^a \Gamma X^H_a + \lambda_b X^V_a, & (13b) \\
[X^V_a, X^V_b] &= (\tau_{bc} - \tau_{cb}) X^V_a, & (13c) \\
[X^V_a, X^H_b] &= \tau^a \Gamma X^H_a - \tau^a \Gamma X^H_b, & (13d) \\
[X^H_a, X^H_b] &= (\tau^a \Gamma - \tau^a \Gamma) X^H_a + \phi^V (R(X^H_a, X^H_b)) X^V_a, & (13e)
\end{align*}
\]

where the \( \tau \)'s are defined through

\[
\begin{align*}
\hat{\nabla}_\Gamma X^V_a &= \tau^a \Gamma X^V_a, & \hat{\nabla}_\Gamma X^H_a &= \tau^a \Gamma X^H_a, \\
\hat{\nabla}_{X^V_a} X^V_b &= \tau_{bc} X^V_a, & \hat{\nabla}_{X^H_a} X^V_b &= \tau_{bc} X^V_a, \\
\hat{\nabla}_{X^H_a} X^V_b &= \tau_{bc} X^H_a, & \hat{\nabla}_{X^H_a} X^H_b &= \tau_{bc} X^H_a.
\end{align*}
\]

We also have

\[
\begin{align*}
A_\Gamma(X^V_a) &= A_\Gamma(X^V_b V_a) = X^a_b H_a = X^H_a, \\
A_\Gamma(X^H_a) &= A_\Gamma(X^H_b H_a) = -X^a_b \phi^b V_c = -\lambda_b X^V_a V_c = -\lambda_b X^V_a.
\end{align*}
\]

We will now calculate the exterior derivatives of the eigenforms \( \phi^V \) and \( \phi^H \) by using the identity \( d\phi(X, Y) = X(\phi(Y)) - Y(\phi(X)) - \phi([X, Y]) \) and the bracket identities (13a)-(13e) as following:

\[
\begin{align*}
d\phi^V(\Gamma, X^V_b) &= -\phi^V( -X^H_b + \tau^a \Gamma X^V_a ) = -\tau^a \Gamma, \\
d\phi^V(\Gamma, X^H_b) &= -\phi^V( \tau^a \Gamma X^H_a + \lambda_b X^V_a ) = -\lambda_b \delta^a_b, \\
d\phi^V(X^V_b, X^V_c) &= -\phi^V( \tau_{bc} X^V_a ) = \tau_{bc} \phi^V, \\
d\phi^V(X^V_b, X^H_c) &= -\phi^V( \tau_{bc} X^H_d - \tau_{cb} X^V_d ) = \tau^a \Gamma, \\
d\phi^V(X^H_b, X^H_c) &= -\phi^V( \tau_{bc} X^H_d - \tau_{cb} X^H_d ) = \tau^a \Gamma, \\
d\phi^V(\Gamma, X^H_b) &= -\phi^V( \tau_{bc} X^H_d - \tau_{cb} X^H_d ) = \tau^a \Gamma, \\
\end{align*}
\]

And similarly for \( d\phi^H \):

\[
\begin{align*}
d\phi^H(\Gamma, X^V_b) &= \delta^a_b, & d\phi^H(\Gamma, X^H_b) &= -\tau^a \Gamma, \\
d\phi^H(X^V_b, X^V_c) &= 0, & d\phi^H(X^V_b, X^H_c) &= -\tau^a \Gamma, \\
d\phi^H(X^H_b, X^H_c) &= -\tau^a \Gamma.
\end{align*}
\]
Putting these components together then we get:

\[
\begin{align*}
\, & d\phi^aV = -\tau^{a\Gamma}_b dt \wedge \phi^bV - \lambda_a dt \wedge \phi^aH + \tau^{aH}_{cb} \phi^bV \wedge \phi^cH + \tau^{aV}_{cb} \phi^bV \wedge \phi^cV \quad (15) \\
& - \frac{1}{2} \phi^aV(R(X^H_b, X^H_c))\phi^{bH} \wedge \phi^{cH} \\
\, & d\phi^{aH} = dt \wedge \phi^{aV} - \tau^{a\Gamma}_b dt \wedge \phi^{bH} + \tau^{aH}_{cb} \phi^{bH} \wedge \phi^{cH} - \tau^{aV}_{bc} \phi^{bV} \wedge \phi^{cH}, \quad (16)
\end{align*}
\]

We will use the following equivalent Frobenius integrability conditions on a co-distribution \( D^\perp_a = Sp\{\phi^aV, \phi^aH\} \) of eigen-forms of \( \Phi \). We use the fact that \( \omega^a := \phi^{aV} \wedge \phi^{aH} \) is a characterising form for \( D^\perp_a \).

\[
\begin{align*}
\, & d\phi^{aV/H} \equiv 0 \pmod{\phi^aV, \phi^aH}, \quad (17) \\
\, & d\omega^a = \xi^a \wedge \omega^a \quad \text{(no sum on } a), \quad \xi^a \in \wedge^1(E) \quad \text{i.e.,} \quad d\omega^a \equiv 0 \pmod{\omega^a}.
\end{align*}
\]

By looking at the expressions for \( d\phi^{aV} \) from (15) and \( d\phi^{aH} \) from (16), together with (17) we get the following proposition.

**Proposition 3.2.** The necessary and sufficient conditions for an eigenform-distribution \( D^\perp_a = Sp\{\phi^aV, \phi^aH\} \) of \( \Phi \) to be (Frobenius) integrable are:

\[
\tau^{a\Gamma}_b = 0, \quad \tau^{aH}_{bc} = 0, \quad \tau^{aV}_{bc} = 0, \quad \phi^{aV}(R(X^H_b, X^H_c)) = 0
\]

for all \( b, c \neq a \).

**Definition 3.3.** Let \( D^\perp_a := Sp\{\phi^aV, \phi^aH\} \) be an eigenform-distribution of \( \Phi \). A 1-form \( \alpha \in D^\perp_a \) is said to be an integrable direction in \( D^\perp_a \) if

\[
\, & d\alpha = \kappa \wedge \alpha, \quad (18)
\]

for some 1-form \( \kappa \).

It can be seen from the expressions of the exterior derivatives of \( \phi^aV \) in (15) and \( \phi^aH \) in (16) that the 1-forms \( \phi^aV \) may be integrable but \( \phi^aH \) can’t be. Thus we can express an integrable direction \( \alpha_a \) in \( D^\perp_a \) as \( \alpha_a = \phi^aV + B_a \phi^aH \).

**Proposition 3.4.** The necessary and sufficient conditions for the existence of an integrable direction \( \alpha_a = \phi^aV + B_a \phi^aH \), that is \( d\alpha_a = \kappa_a \wedge \alpha_a \), in an eigenform-distribution \( D^\perp_a = Sp\{\phi^aV, \phi^aH\} \) of \( \Phi \) are:

\[
\begin{align*}
\, & \tau^{a\Gamma}_b = 0 \quad (19) \\
\, & \tau^{aV}_{cb} - B_a \tau^{aV}_{bc} = 0 \quad (20) \\
\, & \tau^{aH}_{cb} - \tau^{aV}_{bc} = 0 \quad (21) \\
\, & \phi^aV(R(X^H_b, X^H_c)) + B_a(\tau^{aH}_{cb} - \tau^{aV}_{bc}) = 0 \quad (22) \\
\, & \Gamma(B_a) = B^2_a + \lambda_a \quad (23) \\
\, & X_b^V(B_a) = B_a \tau^{aV}_{ab} - \tau^{aH}_{ab} \quad (24) \\
\, & X_b^H(B_a) = \phi^aV(R(X^H_b, X^H_a)) + B_a(\tau^{aV}_{ab} - \tau^{aH}_{ab}) \quad (25)
\end{align*}
\]

and \( \kappa_a \) is given by

\[
\begin{align*}
\, & \kappa_a = (\tau^{a\Gamma}_a + B_a) dt + (\tau^{aV}_{ab} - \tau^{aH}_{ab}) \phi^{bV} + (B_a \tau^{aV}_{ab} - \tau^{aH}_{ab}) \phi^{bH} \\
& + (B_a \tau^{aV}_{ba} - X_b^V(B_a) - \tau^{aH}_{ba}) \phi^{bH} \quad (26)
\end{align*}
\]

for \( b, c \neq a \).
Proof. Let $\alpha_a = \phi^{aV} + B_a \phi^{aH} \in Sp\{\phi^{aV}, \phi^{aH}\}$ be an integrable direction. By definition $d\alpha_a = \kappa_a \wedge \alpha_a$.

Applying the formula for the exterior derivative of a 1-form:

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

to compute the components of $d\alpha_a$ on the basis of vector fields $\{\Gamma, X_a^V, X_a^H - B_a X_a^V, X_b^V, X_b^H; \forall b \neq a\}$. $d\alpha_a(X, Y)$ is zero for basis pairs $(X, Y)$ when neither is $X_a^V$. This gives, for $b, c \neq a$,

$$0 = d\alpha_a(\Gamma, X_b^V) = -\tau_{ab}^\Gamma,$$
$$0 = d\alpha_a(\Gamma, X_c^H) = B_a \tau_{bc}^\Gamma,$$
$$0 = d\alpha_a(X_b^V, X_c^V) = \tau_{bc}^{aV} - \tau_{cb}^{aV},$$
$$0 = d\alpha_a(X_b^H, X_c^V) = B_a \tau_{bc}^{aV} - \tau_{bc}^{aH},$$
$$0 = d\alpha_a(X_b^H, X_c^H) = -(B_a \tau_{bc}^{aH} - \tau_{cb}^{aV}) + \phi^{aV}(R(X_b^H, X_c^H)),$$
$$0 = d\alpha_a(\Gamma, X_a^H - B_a X_a^V) = \Gamma(B_a) - \lambda_a - B_a^2,$$
$$0 = d\alpha_a(X_b^V, X_a^H - B_a X_a^V) = X_b^V(B_a) - B_a \tau_{ab}^{aV} + \tau_{ab}^{aH},$$
$$0 = d\alpha_a(X_b^H, X_a^V - B_a X_a^V) = X_b^H(B_a) - B_a(\tau_{ab}^{aV} - \tau_{ab}^{aH}) - \phi^{aV}(R(X_b^H, X_a^V)).$$

These immediately give the conditions (19)–(23). For the remainder we have:

$$d\alpha_a(\Gamma, X_a^V) = \tau_{a}^{\Gamma} + B_a,$$
$$d\alpha_a(X_b^V, X_a^V) = \tau_{ab}^{aV} - \tau_{ba}^{aV},$$
$$d\alpha_a(X_b^H, X_a^V) = B_a \tau_{ab}^{aV} - \tau_{ba}^{aH},$$
$$d\alpha_a(X_a^H - B_a X_a^V, X_a^V) = -X_a^V(B_a) + B_a \tau_{aa}^{aV} - \tau_{aa}^{aH}.$$

These give the formula for $\kappa_a$ as in (26).

\[ \square \]

### 3.2 Jacobi identities in the inverse problem

The $\tau$’s defined in (14) are not independent and we now examine their relations. To this end we use the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

(We remark in passing that the Bianchi identities for $\nabla$ are redundant in the presence of the Jacobi identities [8] and we do not consider them here.)

Applying this to the triple $\Gamma, X_a^V$ and $X_a^H$ we have,

$$[\Gamma, [X_a^V, X_a^H]] + [X_a^V, [X_a^H, \Gamma]] + [X_a^H, [\Gamma, X_a^V]] = 0.$$

This gives following identities

$$\Gamma(\tau_{bc}^{aV}) = -\tau_{bc}^{aV} \tau_{e}^{\Gamma} + \tau_{bc}^{aH} + X_b^V(\tau_{ec}^{a\Gamma}) - \tau_{ce}^{e\Gamma} \tau_{bc}^{aV} + \tau_{ec}^{eV} \tau_{bc}^{e\Gamma},$$
$$\Gamma(\tau_{cb}^{aH}) = -\tau_{cb}^{aH} \tau_{e}^{\Gamma} + \tau_{ce}^{e\Gamma} \tau_{cb}^{aH} - X_c^H(\tau_{eb}^{a\Gamma}) + \tau_{eb}^{e\Gamma} \tau_{ce}^{aH} - X_b^V(\lambda_c) \delta_{c}^{a},$$

$$+ \lambda_c(\tau_{eb}^{aV} - \tau_{eb}^{aH}) + \lambda_a \tau_{bc}^{aV} + \phi^{aV}(R(X_b^H, X_c^H)).$$
Applying the Jacobi identity to the triple \((\Gamma, X_b^H, X_c^H)\) gives no new identities since the expression that we get from this is
\[
\Gamma(\tau_{bc}^{aV} - \tau_{cb}^{aV}) = \tau_c^{\alpha V} (\tau_{cb}^c - \tau_{bc}^c) + \tau_c^{\alpha H} + \Gamma(\tau_d^{aV}) - \Gamma(\tau_c^{aV}) + \tau_c^{aV} (\tau_{bc}^c - \tau_{cb}^c) - \tau_b^{aV} (\tau_{bc}^c - \tau_{cb}^c)
\]
which can be obtained by using (27a).

Applying the Jacobi identity to other triples we get further identities:
for \((\Gamma, X_b^H, X_c^H)\) we obtain
\[
\Gamma(\phi^{aV}(R(X_b^H, X_c^H))) = \tau_c^{\alpha V}(\phi^{aV}(R(X_b^H, X_c^H))) = \tau_c^{\alpha V}(\phi^{aV}(R(X_b^H, X_c^H)))
\]
(28a)
\[
- \tau_e^{\alpha V}(\phi^{aV}(R(X_b^H, X_c^H))) + (\lambda_c - \lambda_a)\tau_{bc}^{aH} - \lambda_b\tau_{bc}^{aH} - \lambda_c\tau_{cb}^{aH} + X_c^H(\tau_{cb}^c - \tau_{bc}^c) - X_b^H(\tau_{bc}^c - \tau_{cb}^c)
\]
(28b)

Substituting \(\Gamma(\tau_{bc}^{aH})\) and \(\Gamma(\tau_{cb}^{aH})\) from (27b) into (28b) we have
\[
3\phi^{aV}(R(X_b^H, X_c^H)) = X_b^{\alpha V}(\lambda_c)\delta_{bc}^{aH} - X_c^{\alpha V}(\lambda_a)\delta_{bc}^{aH} + \tau_{bc}^{aV}(\lambda_c - \lambda_a) - \tau_{cb}^{aV}(\lambda_b - \lambda_a).
\]
(29)

For the triple \((X_b^H, X_c^H, X_e^H)\) we have
\[
X_a^V(\tau_{ac}^{dV}) - X_b^V(\tau_{ac}^{dV}) = \tau_{ac}^{dV} - \tau_{ac}^{dV} + \tau_{ac}^{dV}(\tau_{ac}^{dV} - \tau_{ac}^{dV}),
\]
(30a)
\[
X_a^V(\tau_{ac}^{dH}) - X_b^V(\tau_{ac}^{dH}) = X_c^H(\tau_{ac}^{dH} - \tau_{ac}^{dH}) - \tau_{ac}^{dH}(\tau_{ac}^{dH} - \tau_{ac}^{dH}) + \tau_{ac}^{dH}(\tau_{ac}^{dH} - \tau_{ac}^{dH})
\]
(30b)

Applying the Jacobi identity to \((X_b^H, X_c^H, X_e^H)\) we get
\[
X_a^H(\tau_{ac}^{dV}) - X_b^H(\tau_{ac}^{dV}) = X_b^V(\tau_{ac}^{dH} - \tau_{ac}^{dH}) + \tau_{ac}^{dV}(\tau_{ac}^{dH} - \tau_{ac}^{dH}) - \tau_{ac}^{dH}(\tau_{ac}^{dH} - \tau_{ac}^{dH}) + \tau_{ac}^{dH}(\tau_{ac}^{dH} - \tau_{ac}^{dH})
\]
(31a)
\[
X_a^H(\tau_{ac}^{dH}) - X_b^H(\tau_{ac}^{dH}) = X_b^V(\phi^{dV}(R(X_b^H, X_c^H))) + \tau_{ac}^{dH}(\phi^{dV}(R(X_b^H, X_c^H)))
\]
(31b)

For the triple \((X_b^V, X_c^V, X_e^V)\) we get only one new identity
\[
X_a^V(\tau_{ac}^{dV} - \tau_{ac}^{dV}) + X_b^V(\tau_{ac}^{dV} - \tau_{ac}^{dV}) + X_c^V(\tau_{ac}^{dV} - \tau_{ac}^{dV})
\]
(32)
Finally, applying the Jacobi identity to $(X^H_a, X^H_b, X^H_c)$ we have two more identities

\[
X^H_a (\tau_{bc}^{dH} - \tau_{cb}^{dH}) + X^H_b (\tau_{ca}^{dH} - \tau_{ac}^{dH}) + X^H_c (\tau_{ab}^{dH} - \tau_{ba}^{dH}) = \tau_{ca}^{aV} (R(X^H_a, X^H_b)) + \tau_{cb}^{aV} (R(X^H_a, X^H_b)) + \tau_{ac}^{aV} (R(X^H_a, X^H_b))
\]

\[
+ \tau_{bc}^{eV} (R(X^H_a, X^H_c)) + \tau_{cb}^{eV} (R(X^H_a, X^H_c)) + \tau_{ca}^{eV} (R(X^H_a, X^H_c))
\]

\[
+ \tau_{ac}^{dH} (\phi^{dV} (R(X^H_b, X^H_c))) + \tau_{bc}^{dH} (\phi^{dV} (R(X^H_b, X^H_c))) + \tau_{ca}^{dH} (\phi^{dV} (R(X^H_b, X^H_c))
\]

\[
- \tau_{ab}^{dH} (\phi^{dV} (R(X^H_c, X^H_b))) - \tau_{ba}^{dH} (\phi^{dV} (R(X^H_c, X^H_b))) - \tau_{ac}^{dH} (\phi^{dV} (R(X^H_c, X^H_a))) - \tau_{bc}^{dH} (\phi^{dV} (R(X^H_c, X^H_a)))
\]

3.3 EDS and the inverse problem

The idea of looking for closed 2-forms leads to the use of EDS method for the inverse problem. The standard EDS reference is the book [4]. The memoir [3] describes the application of EDS to the inverse problem. According to Anderson and Thompson in [3], the EDS process for the inverse problem involves three steps. We start with the submodule of 2-forms $\Sigma$. The first step, namely the differential ideal step, is to look for a final submodule $\Sigma^f$ of $\Sigma$ that generates a differential ideal. The second step is to create an equivalent linear Pfaffian system for the closed 2-forms, and final step is to determine the generality of the solution of the problem by using the Cartan-Kähler theorem.

In the context of this paper where $\Phi$ is diagonalisable with distinct eigenvalues it follows from the theorem [3] that for a given SODE $\Gamma$ the closed 2-form $\omega$ that we are seeking must satisfy the algebraic conditions:

\[
\omega(X^V_a, X^V_b) = 0,
\omega(X^H_a, X^H_b) = 0,
\Gamma \omega = 0,
\omega(X^V_a, X^H_b) = \omega(X^V_b, X^H_a).
\]

So we start the EDS process with the module $\Sigma^0 := Sp\{\omega^{ab}\}$, where $\omega^{ab} := \frac{1}{2}(\phi^{aV} \wedge \phi^{bH} + \phi^{aH} \wedge \phi^{bV})$, $1 \leq a \leq b \leq n$, and look for the (final) differential ideal generated by $\Sigma^f$.

The differential step is a recursive process which produces from $\Sigma = \Sigma^0$ a sequence of submodules $\Sigma^0 \supset \Sigma^1 \supset \Sigma^2 \supset \ldots$. Each stage of the process involves calculating the exterior derivative of forms belonging to some submodule $\Sigma^i$ and then checking whether these three forms belong to the ideal generated by that submodule, which results in a restriction on the admissible 2-forms who form a submodule $\Sigma^{i+1}$ and then the process is repeated from this submodule and so on until a final differential ideal, $(\Sigma^f)$, is found, i.e. $\Sigma^f = \Sigma^{f+1}$, or the trivial set is reached. If it is impossible to create a maximal rank two-form at any stage during this process, the problem has no regular solution.

Before giving a brief description of the second and the third steps of the EDS process, we give some important results related to the initial differential step.
Proposition 3.5. Suppose that $\Phi$ is diagonalisable with distinct eigenvalues and eigenforms $\phi^a$. Let $\Sigma^0$ be $Sp\{\omega^{ab}\}$. Define $\Sigma^1 := \{\omega \in \Sigma^0 : d\omega \in \langle \Sigma^0 \rangle\}$. Then $\omega \in \Sigma^1$ if and only if $\omega := \sum_{a=1}^n r_a \omega^{aa}$ and the curvature satisfies

$$\sum_{\text{cyclic } a,c} r_a \phi^a V(R(X^H_d, X^H_c)) = 0, \quad \text{for all distinct } a, d, c.$$

Proof. Let $\omega \in \Sigma^1 \subset \Sigma^0$, i.e. $\omega = r_{ab} \omega^{ab}$. Calculating $d\omega$ using (15) and (16) gives

$$d\omega = \sum_{a \leq b} dr_{ab} \wedge \frac{1}{2}(\phi^a V \wedge \phi^b H + \phi^b V \wedge \phi^a H)$$

$$- \sum_{a \leq b} r_{ab} \left[ (r^a c + r^a_{dc} \phi^d H + r^a_{dc} \phi^d V) \wedge \frac{1}{2}(\phi^b V \wedge \phi^c H + \phi^c V \wedge \phi^b H) \right]$$

$$- (r^b c + r^b_{dc} \phi^d H + r^b_{dc} \phi^d V) \wedge \frac{1}{2}(\phi^a V \wedge \phi^d H + \phi^d V \wedge \phi^a H)$$

$$\frac{1}{2}(\lambda_b - \lambda_a) dt \wedge \phi^a H \wedge \phi^d H$$

$$- \frac{1}{4} \phi^a V(R(X^H_d, X^H_c)) \phi^d H \wedge \phi^c H \wedge \phi^b H$$

$$= \sum_{a \leq b} r_{ab} \left[ \frac{1}{2}(\lambda_b - \lambda_a) dt \wedge \phi^a H \wedge \phi^b H \right]$$

$$- \frac{1}{4} \phi^a V(R(X^H_d, X^H_c)) \phi^d H \wedge \phi^c H \wedge \phi^b H$$

$$= \sum_{a \leq b} r_{ab} \left[ \frac{1}{2}(\lambda_b - \lambda_a) dt \wedge \phi^a H \wedge \phi^b H \right] \pmod{\langle \Sigma^0 \rangle}.$$

So $d\omega \equiv 0 \pmod{\langle \Sigma^0 \rangle}$ gives

$$r_{ab}(\lambda_b - \lambda_a) dt \wedge \phi^a H \wedge \phi^b H = 0, \quad a < b \quad \text{(no sum)} \quad (34)$$

and

$$\sum_{a \leq b} r_{ab}(\phi^a V(R(X^H_d, X^H_c)) \phi^c H \wedge \phi^b H + \phi^b V(R(X^H_d, X^H_c)) \phi^c H \wedge \phi^a H) = 0 \quad (35)$$

Since $\lambda_b - \lambda_a \neq 0$ by assumption, (34) gives $r_{ab} = 0$ for $a < b$. We will now use $r_a$ instead of $r_{aa}$ and $\omega^a$ instead of $\omega^{aa}$.

The condition (35) becomes

$$\sum_{a \leq b \leq c \leq d} r_{abcd}(\phi^a V(R(X^H_d, X^H_c)) \phi^c H \wedge \phi^b H + \phi^b V(R(X^H_d, X^H_c)) \phi^c H \wedge \phi^a H) = 0$$

for distinct $a, c$ and $d$, and thus,

$$\sum_{\text{cyclic } a,c} r_{a} \phi^a V(R(X^H_d, X^H_c)) = 0, \quad \text{where } a, d \text{ and } c \text{ are distinct.} \quad (36)$$

Therefore $\Sigma^1 = Sp\{\omega^a := \phi^a V \wedge \phi^a H\}$ satisfying the condition (36) as required. \qed
From now on we will begin the EDS process with \( \Sigma^1 = Sp\{\omega^a := \phi^a V \land \phi^a H\} \) without the condition (36) and expect to have it appear at the next differential ideal step because of the nature of the EDS process.

**Proposition 3.6.** Let \( \Sigma^1 := Sp\{\phi^a V \land \phi^a H, a = 1, \ldots, n\} \), where \( \phi^a V \) and \( \phi^a H \) are the vertical and horizontal lifts of eigenforms of a diagonalisable \( \Phi \) with distinct eigenvalues. The necessary and sufficient conditions for \( \omega = \sum_a r_a\phi^a V \land \phi^a H \in \Sigma^1 \) to have its exterior derivative in the ideal \( \langle \Sigma^1 \rangle \) are that, for all distinct \( a, b \) and \( c \) (no summing),

\[
\begin{align*}
& r_a \tau^a_b + r_b \tau^b_a = 0, \\
& r_a(\tau^a_{bc} - \tau^a_{cb}) - r_b \tau^b_{ca} + r_c \tau^c_{ba} = 0, \\
& r_a(\tau^a_{cb} - \tau^a_{cb}) - r_b \tau^b_{ca} + r_c \tau^c_{ba} = 0, \\
& r_a \phi^a V(R(X^H_c, X^H_b)) + r_b \phi^b V(R(X^H_a, X^H_c)) + r_c \phi^c V(R(X^H_b, X^H_a)) = 0.
\end{align*}
\]

**Proof.** Let \( \omega = r_a\phi^a V \land \phi^a H \in \Sigma^1 \). Then

\[
d\omega \in \langle \Sigma^1 \rangle \iff d\omega = \sum_k \beta_k \wedge \phi^k V \land \phi^k H. \quad (37)
\]

By observation (37) is equivalent to

\[
\begin{align*}
& d\omega(\Gamma, X^V_a, X^V_b) = 0, \\
& d\omega(\Gamma, X^H_a, X^H_b) = 0, \\
& d\omega(\Gamma, X^V_a, X^H_b, X^V_c) = 0, \\
& d\omega(\Gamma, X^V_a, X^V_b, X^V_c) = 0, \\
& d\omega(X^V_a, X^V_b, X^V_c) = 0,
\end{align*}
\]

for all distinct \( a, b \) and \( c \).

Applying the formula (12) for \( d\Omega \) to the identities in (38), we can see that only the second part that involves the Lie brackets can contribute. Using the bracket relations (13a)–(13e) in the calculation we find that the first, the second and the sixth condition in (38) are identically satisfied. The third condition gives:

\[
r_a \tau^a_b + r_b \tau^b_a = 0, \quad a \neq b.
\]

The fourth and the fifth condition respectively give

\[
\begin{align*}
& r_a(\tau^a_{bc} - \tau^a_{cb}) - r_b \tau^b_{ca} + r_c \tau^c_{ba} = 0, \\
& r_a(\tau^a_{cb} - \tau^a_{cb}) - r_b \tau^b_{ca} + r_c \tau^c_{ba} = 0,
\end{align*}
\]

for all distinct \( a, b \) and \( c \).

The remaining condition is, that for all distinct \( a, b \) and \( c \),

\[
r_a \phi^a V(R(X^H_c, X^H_b)) + r_b \phi^b V(R(X^H_a, X^H_c)) + r_c \phi^c V(R(X^H_b, X^H_a)) = 0.
\]

This is simply the condition (36) from the proposition 3.5 as expected. \( \square \)

**Corollary 3.7.** For diagonalisable \( \Phi \) with distinct eigenvalues, the necessary and sufficient conditions for \( \Sigma^1 \) to be a differential ideal are that for all distinct \( a, b \) and \( c \),

\[
\begin{align*}
& \tau^a_b = 0, \\
& \tau^a_{bc} = 0
\end{align*}
\]
Proof. $\Sigma^1 := Sp\{\phi^aV \wedge \phi^aH : a = 1..n\}$ generates a differential ideal if and only if the conditions in Proposition 3.6 hold for arbitrary $\tau_a$. This immediately gives the stated conditions, $\tau_{bc}^0 = 0$ and $\phi^aV(R(X_b^H, X_c^H)) = 0$ for all distinct $a, b$ and $c$. But the last two conditions are the consequences of the stated conditions by Jacobi identities (27a) and (27b) for distinct $a, b$ and $c$.

We note here that if we assume $\tau_{ab}^0 = 0$ for all $a \neq b$ so that $\nabla_a X^{V/H}_a = \tau_{ab}^0 X^{V/H}_a$, then all $\tau_{ab}^0$ can be put equal to zero by re-scaling the eigenvectors of $\Phi$. Thus from now on we have that if $\Sigma^1$ is a differential ideal, then $\tau_{ab}^0 = 0$ for all $a, b$.

The following proposition indicates the sufficient condition for degenerate solutions. This can be used to exclude the cases where there are no regular solutions.

**Proposition 3.8.** If a submodule $\Sigma^f \subseteq \Sigma^1 := Sp\{\omega^a := \phi^aV \wedge \phi^aH, a = 1, ..., n\}$ forms a differential ideal; and if there is some $\omega^a$ missing in $\Sigma^f$, then there is no regular solution to the inverse problem.

**Proof.** Let $\omega \in \Sigma^f$. We have $\Gamma \omega = 0$. If $\omega^a$ is missing in $\Sigma^f$, then $X^V_a \omega^a = 0$. It then follows that $\omega$ has kernel of dimension greater than one.

**Theorem 3.9.** Let $\Phi$ be diagonalisable with distinct eigenvalues. Suppose there are $q$ non-integrable eigen-distributions. If the sequence $\langle \Sigma^1 \rangle, ..., \langle \Sigma^q \rangle$ does not contain a differential ideal then there is no non-degenerate solution.

**Proof.** Firstly, if $\langle \Sigma^q \rangle$ is not a differential ideal, then no earlier $\langle \Sigma^p \rangle$ can be a differential ideal. Now each of the $n - q$ integrable $\omega^b := \phi^bV \wedge \phi^bH$ has remained in $\Sigma^q$ since $d\omega^b = \xi^b \wedge \omega^b$ so that $dim(\Sigma^q) \geq n - q$. Now $dim(\Sigma^{q+1}) < dim(\Sigma^p)$ for $p < q + 1$. Hence $dim(\Sigma^q) = n - q + 1$ or there is no non-degenerate solution at this step. But $\langle \Sigma^q \rangle$ is not a differential ideal and hence $dim(\Sigma^{q+1}) = n - q$ and so $\omega^1, ..., \omega^q$ are missing and no solution exists.

We remark that it is not the case that the number of non-integrable eigenspaces matches the terminating differential ideal step. Example 5 in section 5 demonstrates this for $n = 3$.

**Corollary 3.10.** Let $\Phi$ be diagonalisable with distinct eigenvalues. If $\Sigma^f$ is generated by a one-dimensional distribution, and each of $\phi^aV \wedge \phi^aH$ is represented in $\Sigma^f$, then no eigen-distributions of $\Phi$ are integrable.

For the sake of completeness we reproduce two theorems about the limiting cases.

**Theorem 3.11.** (see [2])
The differential ideal step finishes at $\Sigma^0$ if and only if $\Phi$ is a function multiple of the identity.

**Theorem 3.12.** (see [18])
Suppose that the final differential ideal is generated by a one dimensional submodule $\Sigma^f = Sp\{\tilde{\omega}\}$, for non-degenerate $\tilde{\omega}$. That is, there exists $\mu$ such that

$$d\tilde{\omega} = \mu \wedge \tilde{\omega}, \quad \wedge^n \tilde{\omega} \neq 0.$$ 

Then $\langle \tilde{\omega} \rangle$ contains a closed, non-degenerate two-form if and only if $d\mu = 0$.

We characterise non-integrable eigen-distributions in the next theorem.
Theorem 3.13. Let $\Phi$ be diagonalisable with distinct eigenvalues and suppose $\Sigma^1 = Sp\{\phi^V \wedge \phi^H, b = 1, \ldots, n\}$ is a differential ideal. Suppose further that $Sp\{\phi^aV, \phi^aH\}$ is a non-integrable eigen-distribution of $\Phi$ for some $a$. Then
1. there exists at least one non-zero $\tau^a_{bb}$ for some $b \neq a$;
2. let $\tau^a_{bb}, \tau^a_{bb} \neq 0$ for some $a \neq b$ then $\alpha_a = \phi^aV + B_a\phi^aH$ (no sum) is an integrable direction in $Sp\{\phi^aV, \phi^aH\}$ if and only if
   \[ B_a = \frac{\tau^a_{bb}}{\tau^a_{bb}} \quad \text{for all such } b, \] (39)
   and $X^V_b(B_a) = B_a\tau^a_{ab} - \tau^a_{ab}$, $a \neq b$. (40)
3. Let $\tau^a_{bb} = 0$ for all $b \neq a$ then $\phi^aV$ is an integrable direction if and only if $\tau^a_{bb} = 0$ for all $b \neq a$.

Proof. 1. Since the eigen-distribution $Sp\{\phi^aV, \phi^aH\}$ is non-integrable and $\Sigma^1$ is a differential ideal, there is at least one $\tau^a_{bb} \neq 0$ by the proposition 3.2.

2. Since $\alpha_a = \phi^aV + B_a\phi^aH$ is an integrable direction in $Sp\{\phi^aV, \phi^aH\}$, that is (19)-(25) hold. Then from (20) with $c = b$ we get
   \[ B_a = \frac{\tau^a_{bb}}{\tau^a_{bb}} \tau^a_{bb}, \] and (40) is exactly (24).

Now we show that if $B_a = \frac{\tau^a_{bb}}{\tau^a_{bb}}$, and $X^V_b(B_a) = B_a\tau^a_{ab} - \tau^a_{ab}$, then (19)-(25) hold as follows.
The conditions (19), (20), (21) and (22) hold from Corollary 3.7.
To establish (23) we note that
   \[ \tau^a_{bc} = -\Gamma(\tau^a_{bc}) \quad \text{for all } b, c \neq a \] from Jacobi identity (27a) and $\tau^a_{bc} = 0$ for all $a, b,$ and
   \[ \Gamma(\tau^a_{bb}) = \lambda_a \tau^a_{bb} \] from Jacobi identity (27b) together with the conditions in Corollary 3.7.

We have
   \[ \Gamma(B_a) = \Gamma(\frac{\tau^a_{bb}}{\tau^a_{bb}}) \quad \text{for } \tau^a_{bb} \neq 0 \]
   \[ = \frac{\Gamma(\tau^a_{bb}) - B_a\Gamma(\tau^a_{bb})}{\tau^a_{bb}} \]
   \[ = \lambda_a + B^2_a, \]
so the condition (23) holds.
The condition (40) is exactly (24) and implies the condition (25), $X^H_b(B_a) = \phi^aV(R(X^V_b, X^H_a) + B_a(\tau^a_{ab} - \tau^a_{ab})$ as follows.

\[
X^H_b(B_a) = -[\Gamma, X^V_b](B_a) = X^V_b(\Gamma(B_a)) - \Gamma(X^V_b(B_a)) \\
= X^V_b(B^2_a + \lambda_a) - \Gamma(B_a\tau^a_{ab} - \tau^a_{ab}) \\
= 2B_a(\lambda_a) + X^V_b(\lambda_a) - (B^2_a + \lambda_a)\tau^a_{ab} + B_a\tau^a_{ab} + \Gamma(\tau^a_{ab})
\]
By identity \(27\) and Corollary \(3.7\)
\[
\Gamma(\tau_{ab}^H) = \phi^{aV}(R(X_b^H, X_a^H)) + \lambda_a \tau_{ab}^{aV} - X_b^V(\lambda_a)
\]
Substituting this into \(X_b^H(B_a)\) above and then simplifying we get
\[
X_b^H(B_a) = \phi^{aV}(R(X_b^H, X_a^H)) + B_a(\tau_{ab}^{aV} - \tau_{ab}^H)
\]
as required.

3. If \(\phi^{aV}\) is integrable then \(d\phi^{aV} = \mu^a \wedge \phi^{aV}\). By looking at \(15\) along with the assumption of the differential ideal \(\Sigma^1\) we have \(\tau_{ab}^H = 0\) for \(b \neq a\).

Conversely, using identity \(27\) and Corollary \(3.7\) to prove that \(\phi^{aV}(R(X_b^H, X_a^H)) = 0\) and so \(\phi^{aV}(R(X_a^H, X_b^H)) = 0\) because \(R\) is skew, and all other terms apart from the form \(\mu \wedge \phi^{aV}\) in the right hand side of \(15\) go be cause of Corollary \(3.7\) and the assumptions of the theorem. Hence \(\tau_{ab}^H = 0\) together with the stated assumptions and \(15\) imply that \(\phi^{aV}\) is an integrable direction.

\[\square\]

**Corollary 3.14.** Let \(\Phi\) be diagonalisable with distinct eigenvalues and suppose \(\Sigma^1 = Sp\{\phi^{aV} \wedge \phi^{aH}, a = 1, \ldots, n\}\) is a differential ideal. Suppose further that \(Sp\{\phi^{aV}, \phi^{aH}\}\) is a non-integrable eigen-distribution of \(\Phi\).

1. If there is only one \(\tau_{bb}^{aV} \neq 0\) for \(b \neq a\) set \(B_a := \frac{\tau_{bb}^H}{\tau_{bb}^{aV}}\). Then \(\alpha_a = \phi^{aV} + B_a \phi^{aH}\) is an integrable direction in \(Sp\{\phi^{aV}, \phi^{aH}\}\) if and only if
\[
X_b^V(B_a) = B_a \tau_{ab}^{aV} - \tau_{ab}^H,\quad a \neq b.
\]

2. Suppose there exist at least two \(\tau_{bb}^{aV} \neq 0\) for \(b_i \neq a\). Then \(\alpha_a = \phi^{aV} + B_a \phi^{aH}\) is an integrable direction in \(Sp\{\phi^{aV}, \phi^{aH}\}\) if and only if
\[
B_a = \frac{\tau_{bb}^H}{\tau_{bb}^{aV}}\quad\text{for each such }b_i \neq a.
\]

**Proof.**

1. This follows immediately from the theorem \(3.13\)

2. If \(\alpha_a = \phi^{aV} + B_a \phi^{aH}\) is an integrable direction, then
\[
B_a = \frac{\tau_{bb}^H}{\tau_{bb}^{aV}}\quad\text{for all such }b_i \neq a
\]
by theorem \(3.13\) (along with \(40\)). Conversely, we will show that, with the given \(B_a\),
\[
X_b^V(B_a) = B_a \tau_{ab}^{aV} - \tau_{ab}^H,\quad b_i \neq a.
\]
We note that with the conditions in the Corollary \(3.7\) we get
\[
X_b^V(\tau_{bb}^{aV}) = \tau_{bb}^{aV}(2\tau_{bb}^{bV} - \tau_{bb}^{bV}) - \tau_{bb}^{aV} \tau_{bb}^H - \tau_{bb}^{aV} \tau_{bb}^H - \tau_{bb}^{aV}(\tau_{bb}^H - \tau_{bb}^{aV})\quad\text{for distinct }a, b_i, b_j\tag{41}
\]
from the Jacobi identity \(30\), and
\[
X_b^V(\tau_{bb}^{aH}) = \tau_{bb}^{aH}(2\tau_{bb}^{bV} - \tau_{bb}^{bV}) - \tau_{bb}^{aH} \tau_{bb}^H - \tau_{bb}^{aH} \tau_{bb}^H - \tau_{bb}^{aH}(\tau_{bb}^H - \tau_{bb}^{aV})\quad\text{for distinct }a, b_i, b_j\tag{42}
\]
from the Jacobi identity (30b). Now we have

$$X^V_{b_i}(B_a) = X^V_{b_i}(\tau^{aH}_{b_j}) = \frac{X^V_{b_i}(\tau^{aH}_{b_j}) - B_a X^V_{b_i}(\tau^{aV}_{b_j})}{\tau^{aV}_{b_j}}.$$ Substituting $$X^V_{b_i}(\tau^{aV}_{b_j})$$ from (41) and $$X^V_{b_i}(\tau^{aH}_{b_j})$$ from (42) into $$X^V_{b_i}(B_a)$$ above and then simplifying we get

$$X^V_{b_i}(B_a) = B_a \tau^{aV}_{b_i} - \tau^{aH}_{b_i}, \quad b_i \neq a$$ as required.

We now return to the EDS process. Suppose that a differential ideal $$\Sigma^f$$ is found, the next step in the EDS process is to express the problem of finding the closed 2-forms in $$\Sigma^f$$ as a Pfaffian system. We will give a brief outline of this step, see [3], [14] or [4] for details.

Let the differential ideal $$\Sigma^f$$ be generated by the set of 2-forms, not necessary simple, $$\{\tilde{\omega}^k\}, \; k \in \{1, ..., d\}$$, and calculate

$$d\tilde{\omega}^k = \xi^h_k \wedge \tilde{\omega}^h.$$ where the $$\xi^h_k$$'s are now known one-forms.

Since $$\omega \in \Sigma^f = Sp\{\tilde{\omega}^k\}, \; d\omega = \beta_j \wedge \tilde{\omega}^j$$, and because we are looking for those $$\omega$$'s such that $$d\omega = 0$$ the next step is to find all possible $$d$$-tuples of one forms $$(\rho^A_k) = (\rho^1_A, ..., \rho^d_A)$$ such that $$\rho^A_k \wedge \tilde{\omega}^k = 0$$. Once all the $$d$$-tuples $$\rho^A_k \; A \in \{1, ..., e\}$$ have been found, the inverse problem becomes that of finding the functions $$r_k$$ which satisfy the Pfaffian system of equations:

$$dr_k + r_h \xi^h_k + p_A \rho^A_k = 0, \quad (43)$$ for some arbitrary functions $$p_A$$. The freedom in the choice of these $$p_A$$'s will be then exploited in the final part of the EDS procedure.

The general method for finding the solution for this problem in EDS is to define an extended manifold $$N = E \otimes \mathbb{R}^d \otimes \mathbb{R}^e$$ with co-ordinates $$\{x^a, r_k, p_A\}, \; a \in \{1, ..., 2n\}, \; k \in \{1, ..., d\}, \; A \in \{1, ..., e\}$$ and look for $$2n + 1$$ dimensional submanifolds that are sections over $$E$$ and on which the one forms

$$\sigma_k := dr_k + r_h \xi^h_k + p_A \rho^A_k$$ are zero.

To find these manifolds, $$\sigma_k$$ are considered constraint forms for some distribution on $$N$$, and the problem becomes that of looking for integral manifolds arising from this distribution. To find these integral manifolds, we choose a basis of forms on $$N$$, $$\{\alpha_a, \sigma_k, \pi_A\}$$ where $$\{\alpha_a\}$$ are a pulled back basis for $$E$$, $$\pi_A := dp_A$$, and $$\sigma_k$$ as defined above completes the basis.

The condition that we have sections over $$E$$ is that the form

$$\alpha_1 \wedge ... \wedge \alpha_{2n+1}$$

be non-zero on the $$2n + 1$$ dimensional integral manifolds given by the constraint forms.
In the remainder of this section, we will give a brief outline of the process of finding the generality of the solutions to this last problem, see [3] or [4] for details.

According to [3], to determine the existence and generality of the solutions to (43), we calculate the exterior derivatives $d\sigma_k$ modulo the ideal generated by the forms $\sigma_k$.

$$d\sigma_k \equiv \pi^i_k \wedge \alpha_i + t^i_j \alpha_i \wedge \alpha_j \pmod{\sigma} \quad (44)$$

where $\pi^i_k$ are some linear combinations of $\pi_A$. As $d\sigma_k$ expands with no $dp_A \wedge dp_B$ terms, the system is quasi-linear.

As we want the system to be a section over $E$, i.e $\alpha_1 \wedge ... \wedge \alpha_{2n+1} \neq 0$ on the integral manifolds, we need to absorb all the $\alpha_i \wedge \alpha_j$ terms into the $\pi^i_k \wedge \alpha_i$ terms. This is done by changing the basis forms $\pi_A$ to $\bar{\pi}_A := \pi_A - l^j_A \alpha_j$. If any of the $\alpha_i \wedge \alpha_j$ terms can not be absorbed, then asking for $d\sigma_k \equiv 0 \pmod{\sigma}$ is incompatible with the independence condition and therefore there is no solution.

Once the $\alpha_i \wedge \alpha_j$ terms have been removed, the system

$$d\sigma_k \equiv \pi^1_k \wedge \alpha_i \pmod{\sigma} \quad (45)$$

is used to create the tableau $\Pi$ from which the Cartan characters, $s_1, s_2, ..., s_k$, can be calculated allowing us to apply the Cartan test for involution.

$$\Pi = \begin{array}{cccc}
\sigma_1 & \alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\pi^1_1 & \pi^1_2 & \ldots & \pi^1_n \\
\sigma_2 & \pi^2_1 & \pi^2_2 & \ldots & \pi^2_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_d & \pi^d_1 & \pi^d_2 & \ldots & \pi^d_n \\
\end{array}$$

The first character $s_1$ is the number of independent one forms that can be chosen from first column of $\Pi$. $s_2$ is the number of independent forms in the second column that are also independent of all forms in the first column. This is repeated for $s_3$ and onwards until all the independent forms are picked. In computing the Cartan characters, the basis $\{\alpha_i\}$ is chosen such that $s_1$ is as large as possible, and $s_2$ large as possible but less the $s_1$, and so on. In particular, $s_k$ must form a non-increasing sequence of integers.

Once the Cartan characters are found, the Cartan test for involution is performed.

Let $t$ denote the number of ways in which the forms $\pi^i_k$ can be modified by using $\bar{\pi}_A := \pi_A - l^j_A \alpha_j$, without changing (45). That is, $t$ is the dimension of the linear space of $e$-tuples of one forms $(\tau_1, \tau_2, ..., \tau_e)$ of the form: $\tau_A = l^j_A \alpha_j$ such that

$$a^i_j \tau_A \wedge \alpha_i = 0$$

Then, according to Cartan, the differential system (43) is in involution if and only if

$$t = s_1 + 2s_2 + 3s_3 + ... + ks_k.$$
4 Douglas’s case IIa2 and an extension

In this section we give an extension to arbitrary dimension for one of the sub-cases of Douglas’s semi-separated case using the EDS technique. This is the case for dimension $n = 2$, as discussed in [9], where $\Phi$ is diagonalisable with distinct eigenvalues, one eigen-distribution of $\Phi$ is integrable and the other is non-integrable. The extension to dimension $n$ is given by a diagonalisable $\Phi$ with distinct eigenvalues with exactly $n - 1$ eigen-distributions being integrable.

We consider the case where $\Phi$ is diagonalisable with distinct eigenvalues, $n - 1$ eigen-distributions are integrable and one is not. Let suppose that we have differential ideal at the first step, i.e. $\Sigma^1 = Sp\{\phi^k V \land \phi^k H : k = 1, ..., n\}$ generates a differential ideal since otherwise the problem has no non-degenerate solutions by theorem [9]. Without loss of generality, let us assume that the only non-integrable eigen-distribution is $Sp\{\phi^1 V, \phi^1 H\}$ and the other $n - 1$ eigen-distributions $Sp\{\phi^a V, \phi^a H : a = 2, ..., n\}$, are integrable. Then we have:

$$
\begin{align*}
d\omega^a &= d(\phi^a V \land \phi^a H) = \xi^a \land \omega^a \quad \text{(no sum)} \quad a = 2, ..., n, \\
d\omega^1 &= d(\phi^1 V \land \phi^1 H) \\
&= \xi^1 \land \omega^1 + \xi^a \land \omega^a, \quad a \neq 1 \text{ and } \xi^a \neq 0 \text{ for some } a \neq 1
\end{align*}
$$

where, with $A^a_{bc} V/H := \tau^a_{bc} V/H - 2\tau^a_{cb} V/H$ and

$$
\begin{align*}
\xi^a &= A^a_{ac} \phi^c V + A^a_{ac} \phi^c H, \\
\xi^1 &= A^1_{1c} \phi^c V + A^1_{1c} \phi^c H, \\
\xi^1 &= \tau^1_{aa} \phi^1 V + \tau^1_{aa} \phi^1 H.
\end{align*}
$$

We are now looking for 2-forms $\omega \in \Sigma^1$, i.e. $\omega = r_k \omega^k, k = 1, ..., n$ with $d\omega = 0$ and all the $r$’s non-zero for non-degenerate solutions. We have,

$$
d\omega = (dr_a + r_1 \xi^1_a + r_a \xi^a) \land \phi^a V \land \phi^a H + (dr_1 + r_1 \xi^1) \land \phi^1 V \land \phi^1 H, \quad a = 2, ..., n.
$$

Putting $d\omega = 0$, we get a system of Pfaffian equations:

$$
\begin{align*}
(dr_1 + r_1 \xi^1) \land \phi^1 V \land \phi^1 H &= 0, \\
(dr_a + r_1 \xi^1_a + r_a \xi^a) \land \phi^a V \land \phi^a H &= 0, \quad a = 2, ..., n \quad \text{(no sum on } a)\end{align*}
$$

equivalently,

$$
\begin{align*}
&dr_1 + r_1 \xi^1 = -P_1 \phi^1 V - Q_1 \phi^1 H \quad \text{(47)} \\
dr_a + r_1 \xi^1_a + r_a \xi^a = -P_a \phi^a V - Q_a \phi^a H \quad \text{(48)}
\end{align*}
$$

where $P_a$’s, $Q_a$’s, $P_1$ and $Q_1$ are arbitrary functions on $E$.

Following the EDS procedure, we extend $E$ to a new manifold $N$ with coordinates $(t, x^k, u^k, r_k, P_k, Q_k)$ and now the problem is to find the integrable distributions on $N$ with $\sigma_k = 0$ where

$$
\begin{align*}
\sigma_1 := dr_1 + r_1 \xi^1_a + P_1 \phi^1 V + Q_1 \phi^1 H, \quad \text{(49)} \\
\sigma_a := dr_a + r_1 \xi^1_a + r_a \xi^a + P_a \phi^a V + Q_a \phi^a H \quad \text{(50)}
\end{align*}
$$

Continuing the EDS process, set $\pi^V_k := dP_k$ and $\pi^H_k := dQ_k, k = 1, ..., n$. Using this a co-frame on $N$ is $(dt, \phi^k V, \phi^k H, \sigma_k, \pi^V_k, \pi^H_k)$ for $k = 1, ..., n$. So the next step is to calculate $d\sigma_k$ modulo the ideal $(\sigma_k)$.
Taking the exterior derivative of (50) gives

\[ d\sigma_a = dr_1 \land \xi^1_a + r_1 d\xi^1_a + dr_a \land \xi^a_a + r_a d\xi^a_a + \pi^V_a \land \phi^aV + P_a d\phi^aV + \pi^H_a \land \phi^aH + Q_a d\phi^aH \]

\[ = (\sigma_1 - r_1 \xi^1_a - P_1 \phi^1V - Q_1 \phi^1H) \land \xi^1_a + r_1 d\xi^1_a \]

\[ + (\sigma_a - r_1 \xi^1_a - r_a \xi^a_a - P_a \phi^aV - Q_a \phi^aH) \land \xi^a_a + r_a d\xi^a_a \]

\[ + \pi^V_a \land \phi^aV + P_a d\phi^aV + \pi^H_a \land \phi^aH + Q_a d\phi^aH \]

\[ \equiv (-r_1 \xi^1_a - P_1 \phi^1V - Q_1 \phi^1H) \land \xi^1_a + r_1 d\xi^1_a \]

\[ + (-r_1 \xi^1_a - r_a \xi^a_a - P_a \phi^aV - Q_a \phi^aH) \land \xi^a_a + r_a d\xi^a_a \]

\[ + \pi^V_a \land \phi^aV + P_a d\phi^aV + \pi^H_a \land \phi^aH + Q_a d\phi^aH \] (mod \( \sigma_k \))(no sum on \( a \)).

Calculating \( d\xi^1_a \) and \( d\xi^a_a \) from (49) we have:

\[ d\xi^1_a = d\tau^1_{aa} \land \phi^1V + \tau^1_{aa} d\phi^1V + d\tau^1H_a \land \phi^1H + \tau^1H_a d\phi^1H \]

\[ = (\Gamma(\tau^1_{aa})) dt + X^c (\tau^1_{aa}) \phi^cV + X^c_H (\tau^1_{aa}) \phi^cH \land \phi^1V \]

\[ + \tau^1_{aa} (-\lambda_1 dt \land \phi^1H \land \tau^1H_c \phi^cV + \tau^1H_c \phi^cV \land \phi^1V - \frac{1}{2} \phi^1V (R(X^1, X^c_H)) \phi^1H \land \phi^cH) \]

\[ + (\Gamma(\tau^1H_{aa})) dt + X^c (\tau^1H_{aa}) \phi^cV + X^c_H (\tau^1H_{aa}) \phi^cH \land \phi^1H \]

\[ + \tau^1H_{aa} (dt \land \phi^1V \land \tau^1H_c \phi^cH \land \phi^1H - \tau^1V_{aa} \phi^1V \land \phi^cH) \]

We claim that if \( d\omega^a = \xi^a_a \land \omega^a \) (no sum), then

\[ d\xi^a_a = \tilde{\kappa} a \land \phi^aV + \tilde{\kappa} a \land \phi^aH, \]

for some 1-forms \( \tilde{\kappa} a \) and \( \tilde{\kappa} a \).

**Proof.** We already have: \( d(\phi^aV \land \phi^aH) = \xi^a_a \land \phi^aV \land \phi^aH \). Taking the exterior derivative of both sides of this we get

\[ 0 = d\xi^a_a \land \phi^aV \land \phi^aH + \xi^a_a \land d(\phi^aV \land \phi^aH) \]

\[ \iff 0 = d\xi^a_a \land \phi^aV \land \phi^aH \]

\[ \iff d\xi^a_a = \tilde{\kappa} a \land \phi^aV + \tilde{\kappa} a \land \phi^aH \]

\[ \square \]

The next step is to see what terms in \( d\sigma_a \) can be absorbed into \( \pi^V_a \) and \( \pi^H_a \). Given that in each \( d\sigma_a \), any term that can be written as \( \beta \land \phi^aV \) or \( \beta \land \phi^aH \) can be absorbed into terms \( \pi^V_a \land \phi^aV \) and \( \pi^H_a \land \phi^aH \) respectively. After this absorption these terms are denoted as \( \tilde{\pi}^V_a \land \phi^aV \) and \( \tilde{\pi}^H_a \land \phi^aH \) and the remainder that can’t be absorbed represents the ‘torsion’ of the system.

Working through (51), it can be seen that terms \( d\xi^a_a, d\phi^aV, \) and \( d\phi^aH \) give no torsion and terms that contribute to the torsion are

\[ T_a := (r_a (\xi^a_a - \xi^1_a) - P_1 \phi^1V - Q_1 \phi^1H) \land \xi^1_a + d\xi^1_a \] (no sum on \( a \)).

By absorption, it can be seen that for those \( c \in \{2, 3, ..., n\} \) where \( \xi^1_c = 0 \) or equivalently \( \tau^1V_{cc} = 0 \), the torsion \( T_c \) vanishes without any extra conditions. It then follows that

\[ d\sigma_c \equiv \tilde{\pi}^V_c \land \phi^V + \tilde{\pi}^H_c \land \phi^H \] (mod \( \sigma \)).

However, the eigenform co-distribution \( D^1_c = Sp(\phi^1V, \phi^1H) \) is non-integrable by assumption, so there exists at least one \( \xi^1_c \neq 0 \). So we split the problem into two subcases.
i) There is only one fixed $a$ such that $\xi^1_a \neq 0$ or equivalently $\tau^1_{aa} \neq 0$.

ii) There is more than one fixed $a$ such that $\xi^1_a \neq 0$, say there are $a_i \neq 1, i = 1, 2, \ldots$ such that $\tau^1_{aa_i} \neq 0$.

Considering cases where $\xi^1 \neq 0$, computing $T_a$ we get:

$$T_a \equiv (r_1(X^V(\tau^1_{aa})) - X^H(\tau^1_{aa}) + \tau^1_{aa} A^0_{a1} - \tau^1_{aa} A^0_{a1} + \tau^1_{aa} \tau^1_{11} - \tau^1_{aa} \tau^1_{11})$$

$$- P_1 \tau^1_{aa} + Q_1 \tau^1_{aa} \phi^1 V \land \phi^1 H$$

$$+ (\Gamma(\tau^1_{aa}) + \tau^1_{aa} dt \land \phi^1 V)$$

$$+ (\Gamma(\tau^1_{aa}) - \lambda \tau^1_{aa} dt \land \phi^1 H)$$

$$+ (X^V(\tau^1_{aa}) + \tau^1_{aa} A^0_{a1} + \tau^1_{aa} \tau^1_{aa} \phi^1 V \land \phi^1 V)$$

$$+ (X^H(\tau^1_{aa}) + \tau^1_{aa} A^0_{a1} + \tau^1_{aa} (\tau^1_{11} - \tau^1_{11}) + \tau^1_{aa} \tau^1_{11}) \phi^1 H \land \phi^1 V$$

$$+ (\tau^1_{aa} \tau^1_{aa} \tau^1_{11} - \tau^1_{aa} \tau^1_{aa} \phi^1 V \land \phi^1 H, c \neq a \neq 1 \mod \phi^1 V \land \phi^1 H).$$

Using Jacobi identities (27a), (27b), (30a), (31a), (31b) and (31b) respectively, we have that the coefficients of $dt \land \phi^1 V, dt \land \phi^1 H, \phi^1 V \land \phi^1 V, \phi^1 V \land \phi^1 H, \phi^1 H \land \phi^1 V$ and $\phi^1 H \land \phi^1 H$ in $T_a$ vanish. Therefore the torsion is now

$$T_a \equiv (r_1(X^V(\tau^1_{aa})) - X^H(\tau^1_{aa}) + \tau^1_{aa} A^0_{a1} - \tau^1_{aa} A^0_{a1} + \tau^1_{aa} \tau^1_{11} - \tau^1_{aa} \tau^1_{11})$$

$$- P_1 \tau^1_{aa} + Q_1 \tau^1_{aa} \phi^1 V \land \phi^1 H$$

$$+ (\tau^1_{aa} \tau^1_{aa} \tau^1_{11} - \tau^1_{aa} \tau^1_{aa} \phi^1 V \land \phi^1 H, c \neq a \neq 1 \mod \phi^1 V \land \phi^1 H)$$

For the subcase i) there is exactly one $a$ such that $\tau^1_{aa} \neq 0$, then for all $c \neq a$ we have $\tau^1_{cc} = 0$ and so, as discussed above, we get

$$d\sigma \equiv \pi^V_c \land \phi^V + \pi^H_c \land \phi^H \text{ (mod } \sigma)$$

without torsion, and

$$d\sigma \equiv \pi^V_a \land \phi^V + \pi^H_a \land \phi^H \text{ (mod } \sigma)$$

with torsion

$$T_a = (r_1(X^V(\tau^1_{aa})) - X^H(\tau^1_{aa}) + \tau^1_{aa} A^0_{a1} - \tau^1_{aa} A^0_{a1} + \tau^1_{aa} \tau^1_{11} - \tau^1_{aa} \tau^1_{11})$$

$$- P_1 \tau^1_{aa} + Q_1 \tau^1_{aa} \phi^1 V \land \phi^1 H$$

The torsion must be zero for the existence of solutions, and so we get an equation relating $Q_1$ with $P_1$ and $r_1$ as follows

$$Q_1 = P_1 \tau^1_{aa} + r_1 \frac{1}{\tau^1_{aa}} (X^H(\tau^1_{aa}) - X^V(\tau^1_{aa}) - \tau^1_{aa} A^0_{a1} + \tau^1_{aa} A^0_{a1} - \tau^1_{aa} \tau^1_{11} + \tau^1_{aa} \tau^1_{11})$$

(52)

For subcases ii), there exist $a_i$’s such that $\tau^1_{aa_i} \neq 0$, then for each $c \neq a_i$ we have

$$d\sigma \equiv \pi^V_c \land \phi^V + \pi^H_c \land \phi^H \text{ (mod } \sigma)$$
without torsion, and for each $a_i$ that $\tau_{a_i a_i}^{1V} \neq 0$ we have
\[
d\sigma_{a_i} \equiv \widetilde{\pi}_{a_i}^V \wedge \phi^{a_i V} + \widetilde{\pi}_{a_i}^H \wedge \phi^{a_i H} \pmod{\sigma}
\]
with torsion
\[
T_{a_i} \equiv (r_1(X_1^V(\tau_{a_i}^{1H})) - X_1^H(\tau_{a_i}^{1V}) + \tau_{a_i a_i}^{1H} A_{a_i 1}^{a_i V} - \tau_{a_i a_i}^{1V} A_{a_i 1}^{a_i H} + \tau_{a_i a_i}^{1V} \tau_{a_i a_i}^{1H} - \tau_{a_i a_i}^{1H} \tau_{a_i a_i}^{1V})
\]
\begin{equation}
+ P_1 \tau_{a_i a_i}^{1H} + Q_1 \tau_{a_i a_i}^{1V} \phi^{a_i V} \wedge \phi^{1H}
\end{equation}
\begin{equation}
+ (\tau_{a_i a_i}^{1V} \tau_{a_i a_i}^{1H} - \tau_{a_i a_i}^{1H} \tau_{a_i a_i}^{1V}) \phi^{a_i V} \wedge \phi^{a_i H}, \ \tau_{a_i a_i}^{1V} \neq 1 \pmod{\phi^{a_i V/H}}.
\end{equation}
At this stage for the torsion $T_a = 0$ we get extra conditions compared with the subcase i), $Q_1$ satisfies (52) for each $a_i$ together with:
\[
\tau_{a_i a_i}^{1V} \tau_{a_i a_i}^{1H} - \tau_{a_i a_i}^{1H} \tau_{a_i a_i}^{1V} = 0,
\]
or equivalently,
\[
B_1 = \frac{\tau_{a_i a_i}^{1V}}{\tau_{a_i a_i}^{1H}} = \frac{\tau_{a_i a_i}^{1H}}{\tau_{a_i a_i}^{1V}} \text{ for all } a_i \neq a_i \text{ and } a_j \neq 1, \text{ and } \tau_{a_i a_i}^{1V} \neq 0. \tag{53}
\]
Now for each $a$, where $\tau_{aa}^{1V} \neq 0$, we have one equation for $Q_1$ as following
\[
Q_1 = P_1 B_1 + r_1 C_a \tag{54}
\]
with $B_1$ as in (53) and
\[
C_a = \frac{1}{\tau_{aa}^{1V}}(X_1^H(\tau_{aa}^{1V}) - X_1^V(\tau_{aa}^{1V}) - \tau_{aa}^{1H} A_{a 1}^{a V} + \tau_{aa}^{1V} A_{a 1}^{a H} - \tau_{aa}^{1V} \tau_{a a}^{1H} - \tau_{aa}^{1H} \tau_{a a}^{1V}).
\]
We note here that for the subcase ii) there are at least two $C_a$'s needed to satisfy the condition (54) in order for $r_1 \neq 0$, so there are extra conditions for vanishing torsion:
\[
C_a = C_{a_j} = C \text{ for all } a_i, a_j \neq 1 \text{ and } \tau_{a_i a_i}^{1V} \neq 0 \text{ and } \tau_{a_j a_j}^{1V} \neq 0. \tag{55}
\]
At this stage, let us assume that the condition (54) holds for the subcase i) or all conditions (53), (54), and (55) are satisfied for the subcase ii), then for each $a \in \{2, 3, ..., n\}$ we have
\[
d\sigma_a \equiv \tilde{\pi}_a^V \wedge \phi^{a V} + \tilde{\pi}_a^H \wedge \phi^{a H} \pmod{\sigma}.
\]
The consequence of the choice for $Q_1$ in condition (54) affects $d\sigma_1$. Hence by substituting $Q_1$ into $d\sigma_1$ we get
\[
|d\sigma_1 = dr_1 \wedge \xi_1 + r_1 d\xi_1 + dP_1 \wedge \phi^{1V} + P_1 d\phi^{1V} + d(P_1 B_1 + r_1 C) \wedge \phi^{1H} + (P_1 B_1 + r_1 C) d\phi^{1H} \equiv (-r_1 \xi_1 - P_1 \phi^{1V} - (r_1 C + P_1 B_1) \phi^{1H}) \wedge \xi_1 + r_1 d\xi_1 \tag{56}
\]
\begin{equation}
+ dP_1 \wedge \phi^{1V} + P_1 d\phi^{1V} + dP_1 \wedge B_1 \phi^{1H} + P_1 d(B_1 \phi^{1H})
\end{equation}
\begin{equation}
+ C(-r_1 \xi_1 - P_1 \phi^{1V} - (r_1 C + P_1 B_1) \phi^{1H}) \wedge \phi^{1H} + r_1 d(C \phi^{1H}) \pmod{\sigma}.
\end{equation}
Simplifying (56) we get:
\[
d\sigma_1 \equiv (\pi_1^V + P_1(\xi_1 + C \phi^{1H})) \wedge \phi^{1V} + B_1 \phi^{1H} \tag{57}
\begin{equation}
+ r_1 d(\xi_1 + C \phi^{1H}) + P_1 d(\phi^{1V} + B_1 \phi^{1H}) \pmod{\sigma}
\end{equation}
At this point the problem breaks down into two further subcases:
1. If \( d(\phi^1 + B_1\phi^H) = \kappa \wedge (\phi^1 + B_1\phi^H) \), for some 1-form \( \kappa \), then the condition for the existence of non-degenerate solutions is that \( d(\xi_1 + C\phi^H) = \beta \wedge (\phi^1 + B_1\phi^H) \), for some 1-form \( \beta \) and then
\[
d\sigma_1 \equiv \tilde{\pi}^V \wedge (\phi^1 + B_1\phi^H) \pmod{\sigma}.
\]

2. If \( d(\phi^1 + B_1\phi^H) \neq \kappa \wedge (\phi^1 + B_1\phi^H) \), then in order to remove torsion we require \( r_1d(\xi_1 + C\phi^H) + P_1d(\phi^1 + B_1\phi^H) \equiv 0 \pmod{\phi^1 + B_1\phi^H} \). This results in an equation relating \( P_1 \) to \( r_1 \) which would fix \( P_1 \) as a function of \( r_1 \), thus we will have lost flexibility in \( \pi^H = d(P_1) \) to absorb any terms. So in this situation, the problem reduces to finding a solution for \( P_1 \) in term of \( r_1 \) to the equation
\[
(dP_1 + P_1(\xi_1 + C\phi^H)) \wedge (\phi^1 + B_1\phi^H) + r_1d(\xi_1 + C\phi^H) + P_1d(\phi^1 + B_1\phi^H) = 0.
\]
(58)

Thus if there exists a function \( P_1 \) in term of \( r_1 \) satisfying equation (58), then we have
\[
d\sigma_1 \equiv 0 \pmod{\sigma}.
\]

Let assume that we are in case 1, so that there exists an integrable direction \( \alpha_1 = \phi^1 + B_1\phi^H \) and so assume that \( d(\xi_1 + C\phi^H) = \beta \wedge (\phi^1 + B_1\phi^H) \), since otherwise there would be no non-degenerate solution. Then we move onto the calculation of the freedom in the solution to the inverse problem for this case, we have
\[
d\sigma_1 \equiv \tilde{\pi}^V \wedge (\phi^1 + B_1\phi^H),
\]
(59)
\[
d\sigma_a \equiv \tilde{\pi}^V \wedge \phi^a + \tilde{\pi}^H \wedge \phi^aH.
\]
(60)

We change the basis \( \{\phi^V, \phi^H\} \) to the basis \( \{\gamma^V, \gamma^H\} \) using
\[
\phi^{1/V/H} = \gamma^{1/V/H} + \gamma^{2/V/H} + \ldots + \gamma^{n/V/H},
\]
\[
\phi^{c/V/H} = \gamma^{1/V/H} - \gamma^{c/V/H}, \quad c = 2, \ldots, n.
\]

We then get the optimal tableau:

|   | \gamma^1V | \gamma^1H | \gamma^2V | \gamma^2H | \ldots | \gamma^bV | \gamma^bH | \ldots | \gamma^nV | \gamma^nH |
|---|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \sigma_1 | \tilde{\pi}^V | B_1\tilde{\pi}^V | \tilde{\pi}^V | B_1\tilde{\pi}^V | \ldots | \tilde{\pi}^V | B_1\tilde{\pi}^V | \ldots | \tilde{\pi}^V | B_1\tilde{\pi}^V |
| \sigma_2 | \tilde{\pi}^V | \tilde{\pi}^H | -\tilde{\pi}^V | -\tilde{\pi}^H | \ldots | 0 | 0 | \ldots | 0 | 0 |
| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |
| \sigma_n | \tilde{\pi}^V | \tilde{\pi}^H | 0 | 0 | \ldots | 0 | 0 | \ldots | -\tilde{\pi}^H | -\tilde{\pi}^H |

This tableau gives Cartan characters: \( s_1 = n, \ s_2 = n - 1, \ s_i = 0 \) for \( i > 3 \).

The final step is to check for involution. To do this, we let \( t \) be the number of ways that \( \tilde{\pi}_c, \tilde{\pi}_c^H \) and \( \tilde{\pi}_1^V \) can be altered such that (52) and (53) are unchanged. It can be seen that if we write:

\[
\tilde{\pi}_a^V = \tilde{\pi}_a^V + f_a^1\phi^aV + f_a^2\phi^aH,
\]
\[
\tilde{\pi}_a^H = \tilde{\pi}_a^H + f_a^3\phi^aH + f_a^4\phi^aV,
\]
\[
\tilde{\pi}_1^V = \tilde{\pi}_1^V + f_1(\phi^1 + B_1\phi^H),
\]
then \((59)\) and \((60)\) would be unchanged if we replace \(\tilde{\pi}_a^{V/H}\) by \(\tilde{\pi}_a^{V/H}\) and \(\tilde{\pi}_1^V\) by \(\tilde{\pi}_1^V\). Thus for each \(a \neq 1\) we have three degrees of freedom in adding terms to \(\tilde{\pi}_a^V\) and \(\tilde{\pi}_a^H\), giving \(3(n-1)\) degrees of freedom for all \(\tilde{\pi}_a^{V/H}\). We have only 1 degree of freedom in adding terms to \(\tilde{\pi}_1^V\). Therefore in this case, \(t = 3(n-1) + 1 = 3n - 2\), which equal to \(s_1 + 2s_2\) as required for involution. So the solution depends on \(n-1\) functions of two variables in this case.

Now we consider case 2 where there is no integrable direction \(\alpha_1\), and assume that there is solution for equation \((55)\), so we have

\[
d\sigma_1 \equiv 0, \\
d\sigma_a \equiv \tilde{\pi}_a^V \wedge \phi^a^V + \tilde{\pi}_a^H \wedge \phi^1^V. \tag{61}
\]

The tableau corresponding with this system is

\[
\begin{array}{c|cccccccc}
\gamma^1V & \gamma^1H & \gamma^2V & \gamma^2H & \ldots & \gamma^bV & \gamma^bH & \ldots & \gamma^nV & \gamma^nH \\
\hline
\sigma_1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\sigma_2 & \tilde{\pi}_2^V & \tilde{\pi}_2^H & -\tilde{\pi}_2^V & \tilde{\pi}_2^H & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sigma_n & \tilde{\pi}_n^V & \tilde{\pi}_n^H & 0 & 0 & \ldots & 0 & 0 & \ldots & -\tilde{\pi}_n^H & -\tilde{\pi}_n^H \\
\end{array}
\]

This tableau gives Cartan characters: \(s_1 = n - 1, s_2 = n - 1, \) \(s_i = 0 \) for \(i > 3\).
The final step is to check for involution. To do this, we let \(t\) equal the number of ways that \(\tilde{\pi}_c^V\) and \(\tilde{\pi}_c^H\) can be altered such that \((61)\) and \((62)\) is unchanged. It can be seen that if we write

\[
\tilde{\pi}_a^V = \tilde{\pi}_a^V + f_1^1 \phi^a^V + f_2^2 \phi^a^H, \\
\tilde{\pi}_a^H = \tilde{\pi}_a^H + f_3^3 \phi^a^H + f_4^4 \phi^a^V
\]

then \((61)\) and \((62)\) would be unchanged if we replace \(\tilde{\pi}_a^{V/H}\) by \(\tilde{\pi}_a^{V/H}\). Thus for each \(a \neq 1\) we have three degrees of freedom in adding terms to \(\tilde{\pi}_a^V\) and \(\tilde{\pi}_a^H\), giving \(3(n-1)\) degrees of freedom for all \(\tilde{\pi}_a^{V/H}\). Therefore in this case, \(t = 3(n-1)\), which equals \(s_1 + 2s_2\) as required for involution.
So the solution depends on \(n-1\) functions of two variables in this case.
We note here that we follow Anderson and Thompson in reporting only the highest order term in the degree of freedom.

Finally, we examine the possibility of having an integrable direction \(\alpha_1 = \phi^1^V + B_1 \phi^1^H\) in the subcases i) and ii). For the subcase i) there is only one \(a\) such that \(\tau_{aa}^{IV} \neq 0\), putting \(B_1 = \tau_{aa}^{IV} / \tau_{aa}^{IH}\) then according to the corollary \[3.14\] the only one condition needed to be considered for the existence of integrable direction \(\alpha_1 = \phi^1^V + B_1 \phi^1^H\) is

\[
X_a^V(B_1) = B_1 \tau_{aa}^{IV} - \tau_{aa}^{IH}. \tag{63}
\]

For the subcase ii) there more than one \(a\) such that \(\tau_{aa}^{IV} \neq 0\) and as it is shown in the EDS procedure that, in order for torsion to vanish, the ratio condition \((39)\) must hold and so there exists an integrable direction \(\alpha_1\) in the non-integrable eigen-distribution \(Sp\{\phi^1^V, \phi^1^H\}\) by corollary \[3.14\]
5 Examples

Example 1. This is an example of non-existence for \( n = 2 \) because there is not a differential ideal at step 1, see theorem 3.9. Consider the system

\[
\ddot{x} = \dot{y}, \quad \ddot{y} = y
\]

(63)
on an appropriate domain. This example was considered by Prince [17] who showed by direct calculation that no non-degenerate solution exists. \( \Phi \) is given by

\[
\Phi = \begin{pmatrix}
0 & 0 \\
0 & -1 \\
\end{pmatrix}
\]

The eigenvalues and corresponding eigenvectors are:

\[
0 \quad \text{and} \quad X_1 := (a, 0), \\
-1 \quad \text{and} \quad X_2 := (0, b)
\]

for some parameters \( a, b \). We calculate \( \nabla \Gamma X_{1/2} \) to find \( \tau_{ij}^{1/2} \), \( i, j \in \{1, 2\} \). We note here that \( a \) and \( b \) might be chosen so that \( \nabla \Gamma X_{1/2} = 0 \) as discussed in section 3.3. In this case, by choosing \( a = 1 \) we get \( \nabla \Gamma X_1^V = 0 \), i.e. \( \tau_1^V = 0 = \tau_2^V \), but there is no \( b \neq 0 \) such that \( \nabla \Gamma X_2^V = 0 \), in particular we have \( \tau_2^V = -\frac{b}{2} \) and \( \tau_2^H = \frac{\Gamma(b)}{b} \). The functions \( \tau_{ij}^V \) and \( \tau_{ij}^H \) are also easy to compute and apart from other \( \tau_{ij}^{V/H} \) we have

\[
\tau_{12}^V = 0, \quad \tau_{11}^V = 0.
\]

With these results, we conclude that this is the case where \( \Phi \) is diagonalisable with distinct eigenvalues and the eigen-distribution \( Sp\{\phi^V, \phi^H\} \) is integrable and \( Sp\{\phi^V, \phi^V\} \) is not integrable since \( \tau_2^V \neq 0 \) and \( \Sigma^1 \) is not a differential ideal since \( \tau_2^V \neq 0 \) also. Therefore there is no non-degenerate solution of the inverse problem by the theorem 3.9.

Example 2. This is another non-existence example, this time for \( n = 3 \). The example is in subcase i) but fails the condition of the further subcase 1. Consider the system

\[
\ddot{x} = x, \quad \ddot{y} = 0, \quad \ddot{z} = z\dot{y}
\]

(64)
on an appropriate domain. Denoting the derivatives by \( u, v, w \), \( \Phi \) is given by

\[
\Phi = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & \frac{w}{2} & -v
\end{pmatrix}
\]

In this case it is possible to choose the eigenvectors \( X_a \) such that \( \nabla \Gamma X_a = 0 \) and so the eigenvalues and corresponding scaled eigenvectors are

\[
-1 \quad \text{and} \quad X_1 := (1, 0, 0), \\
0 \quad \text{and} \quad X_2 := (0, 2v, w), \\
-v \quad \text{and} \quad X_3 := (0, 0, 1).
\]
The only non-zero $\tau_{bc}^aV$ and $\tau_{bc}^aH$ are
\[
\tau_{22}^{2V} = 2, \quad \tau_{22}^{3V} = -w, \quad \tau_{22}^{3H} = zv, \quad \tau_{32}^{3V} = 1.
\]

These results show we are in the case where $\Phi$ is diagonalisable with distinct eigenvalues and eigen-distributions $Sp\{\phi^{1V}, \phi^{1H}\}$ and $Sp\{\phi^{2V}, \phi^{2H}\}$ are integrable and the third one is not. In particular, we are in the subcase i) where only one $\tau_{22}^{3V} \neq 0$. Now check the condition
\[
X_2^V(B_3) = B_3\tau_{32}^{3V} - \tau_{32}^{3H},
\]
where $X_2^V = 2v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}$ and $B_3 = \frac{\tau_{32}^{3H}}{\tau_{22}^{3V}} = -\frac{zv}{w}$ for an integrable direction $\alpha_3 = \phi^{3V} + B_3\phi^{3H}$.

We find
\[
X_2^V(B_3) = (2v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w})(-\frac{zv}{w}) = -\frac{zv}{w} = B_3\tau_{32}^{3V} - \tau_{32}^{3H}.
\]

Thus there is an integrable direction $\alpha_3$ inside this non-integrable eigen-distribution. Now whether or not there exists non-degenerate solution depends on the remaining condition:
\[
d(\xi_3^3 + C\phi^{3H}) = \kappa \land \alpha_3.
\]

Using the formula for $C$ from (55), we get $C = 0$ and furthermore we have $d\xi_3^3 = d\phi^{2V} = 2\phi^{2V} \land \phi^{2H}$. Therefore there is no non-degenerate solution.

**Example 3.** This example is from Aldridge et al [2] which was modified from one of Douglas’ examples in [9]. They used the differential step to solve the Helmholtz conditions. This example is the candidate for our case i) where only one $\tau_{jj}^{kV} \neq 0$ for $k \neq j$.

Consider the system
\[
\dddot{x} = -x, \quad \dddot{y} = y^{-1}(1 + y^2 + z^2), \quad \dddot{z} = 0 \tag{65}
\]
on an appropriate domain. Again denoting the derivatives by $u, v, w$, we find
\[
\Phi = \frac{1}{y^2} \begin{pmatrix} y^2 & 0 & 0 \\ 0 & 2(1 + w^2) & -2vw \\ 0 & 0 & 0 \end{pmatrix}.
\]

Eigenvalues and corresponding eigenvectors $X_a$ chosen so that $\hat{\nabla}_T X_a^V = 0$ are
\[
\lambda_1 = 0 \quad \text{and} \quad X_1 = (0, vw, 1 + w^2),
\]
\[
\lambda_2 = 1 \quad \text{and} \quad X_2 = (1, 0, 0),
\]
\[
\lambda_3 = 2y^{-2}(1 + w^2) \quad \text{and} \quad X_3 = (0, y, 0).
\]

The only non-zero functions $\tau_{bc}^aV$ and $\tau_{bc}^aH$ are
\[
\tau_{11}^{3V} = \frac{v}{y}, \quad \tau_{11}^{1V} = 2w, \quad \tau_{31}^{3V} = w, \quad \tau_{11}^{3H} = -\frac{1 + w^2}{y^2}.
\]

These results show that our system is in the case that $\Phi$ is diagonalisable with distinct eigenvalues and the eigen-distributions $Sp\{\phi^{1V}, \phi^{1H}\}$ and $Sp\{\phi^{2V}, \phi^{2H}\}$ are integrable and the third one is not. In particular, this is in the subcase i) where there is only one $\tau_{11}^{3V} \neq 0$. Checking the condition for an integrable direction $\alpha_3 = \phi^{3V} + B_3\phi^{3H}$ we get
\[
X_1 \begin{pmatrix} \tau_{11}^{3H} \\ \tau_{11}^{3V} \end{pmatrix} = \frac{\tau_{11}^{3H}}{\tau_{11}^{3V}} \tau_{31}^{3V} - \tau_{31}^{3H},
\]
\[26\]
where \( X^V_1 = vw \frac{\partial}{\partial v} + (1 + w^2) \frac{\partial}{\partial w} \), holds. Now whether or not the system has solution depends on the remaining condition:
\[
d(\xi^3_3 + C\phi^{1H}) = \kappa \land \alpha_3.
\]
Using the formula for \( C \) from (55), we get \( C = 0 \) and furthermore we have \( d\xi^3_3 = 0 \). Therefore the solution of the system depends on 2 functions of 2 variables (plus one function of 1 variable).

**Example 4.** This example is in case ii) and the multiplier depends on two arbitrary functions each of two variables and one function of one variable.

Consider the system
\[
\ddot{x} = z, \quad \ddot{y} = x\dot{x} + z\dot{z}, \quad \ddot{z} = x
\] (66)
on an appropriate domain. We find
\[
\Phi = \begin{pmatrix} 0 & 0 & -1 \\ -\frac{w}{2} & 0 & -\frac{w}{2} \\ -1 & 0 & 0 \end{pmatrix}.
\]

In this case it is possible to choose the eigenvectors \( X_a \) such that \( \hat{\nabla}_r X_a = 0 \) and so the eigenvalues and chosen corresponding eigenvectors are:
\[
0 \quad \text{and} \quad (0,1,0),
1 \quad \text{and} \quad (-2,u-w,2),
-1 \quad \text{and} \quad (2,u+w,2).
\]

The \( \tau_{bc}^V \) and \( \tau_{bc}^H \) are zero except for
\[
\tau_{22}^{1V} = -4, \quad \tau_{33}^{1V} = 4.
\]

These results show that our system is in the case that \( \Phi \) is diagonalisable with distinct eigenvalues and eigen-distribution \( \{\phi^{1V}, \phi^{1H}\} \) is non-integrable and \( Sp\{\phi^{2V}, \phi^{2H}\} \) and \( Sp\{\phi^{3V}, \phi^{3H}\} \) are integrable and \( \Sigma^1 \) is a differential ideal. We are in subcase ii). In addition, the ratio condition (53),
\[
\frac{\tau_{22}^{1H}}{\tau_{22}^{1V}} = 0 = \frac{\tau_{33}^{1H}}{\tau_{33}^{1V}},
\]
and the condition (55), \( C_2 = 0 = C_3 \) are satisfied. We can conclude that there is an integrable direction inside the non-integrable eigen-distribution, which is just \( \phi^{1V} \). Furthermore we have \( \xi^1_1 = 0 \) and so \( d\xi^1_1 + C\phi^{1H} = 0 \). Therefore the conclusion is the solution of the system depends upon two arbitrary functions of 2 variables (plus one function of one variable).

In order to identify the structure of the \( r \)'s, we return to equation (47) and (48). Since in this case we know that \( Sp\{\phi^{1V}, \phi^{1H}\} \) is non-integrable and other two are integrable and \( \phi^{1V} \) is integrable direction, we have
\[
\begin{align*}
\dot{r}_1 + r_1\xi^1_1 &= -P_1\phi^{1V} - Q_1\phi^{1H}, \\
\dot{r}_2 + r_1\xi^1_1 + r_2\xi^2_2 &= -P_2\phi^{2V} - Q_2\phi^{2H}, \\
\dot{r}_3 + r_1\xi^1_3 + r_3\xi^3_3 &= -P_3\phi^{3V} - Q_3\phi^{3H},
\end{align*}
\]
where $P_1, Q_1, P_2, Q_2, P_3$ and $Q_3$ are arbitrary functions. As we know $Q_1 = P_1 B_1 + r_1 C = 0$, and computing $\xi_j^\prime$'s using (66) we get $\xi_1^1 = \xi_2^2 = \xi_3^3 = 0$, $\xi_1^2 = -4\phi^1V$ and $\xi_3^1 = 4\phi^1V$. So we have

$$dr_1 = -P_1\phi^1V,$$

$$dr_2 - 4r_1\phi^1V = -P_2\phi^{2V} - Q_2\phi^{2H},$$

$$dr_3 + 4r_1\phi^1V = -P_3\phi^{3V} - Q_3\phi^{3H}.$$  (67a, 67b, 67c)

Equation (67a) implies $r_1 = G_1(\zeta)$ where $G_1$ is an arbitrary function of $\zeta$ with $d\zeta \in Sp\{\phi^1V\}$. Since $\phi^1V = \frac{1}{2}(-d(uw) + 2dv - xdx - zdz)$, putting $d\zeta = -d(uw) + 2dv - xdx - zdz$ we have $\zeta = 2v - uw - \frac{x^2}{2} - \frac{z^2}{2} = 2\phi^1V$. Now substituting $r_1$ into equation (67b) we get

$$dr_2 = 2G_1(\zeta)d\zeta - P_2\phi^{2V} - Q_2\phi^{2H}. $$  (68)

Since $Sp\{\phi^{2V}, \phi^{2H}\}$ is integrable, we know that there exist functions $u_1^1$ and $u_2^2$ such that $Sp\{du_1^1, du_2^2\} = Sp\{\phi^{2V}, \phi^{2H}\}$. As $\phi^{2V} = \frac{1}{4}(d(w - u) + (z - x)dt)$ and $\phi^{2H} = \frac{1}{4}(d(z - x) + (u - w)dt)$, putting

$$du_2^2 = f_2^2(d(w - u) + (z - x)dt) + g_2^2(d(z - x) + (u - w)dt),$$

and for $f_2 = 2(w - u)$, $g_2^1 = 2(z - x)$, $f_2 = -\frac{z - x}{(w-u)^2+(z-x)^2}$ and $g_2^2 = \frac{w-u}{(w-u)^2+(z-x)^2}$ we get

$$u_1^1 = (w - u)^2 + (z - x)^2$$

$$u_2^2 = \tan^{-1}\left(\frac{z - x}{w - u}\right) - t.$$  (69)

Now equation (68) gives

$$r_2 = 2G(\zeta) + \tilde{r}_2(u_1^1, u_2^2),$$

where $G' = G_1$ and $\tilde{r}_2$ is an arbitrary function of $u_1^1$ and $u_2^2$.

Similarly, $Sp\{\phi^{3V}, \phi^{3H}\}$ is integrable with $\phi^{3V} = \frac{1}{4}(d(u + w) - (x + z)dt)$ and $\phi^{3H} = \frac{1}{4}(d(x + z) - (u + w)dt)$. We find $u_1^3 = (u + w)^2 - (x + z)^2$ and $u_2^3 = \tanh^{-1}\left(\frac{z + x}{w + u}\right) - t$ with $Sp\{du_1^3, du_2^3\} = Sp\{\phi^{3V}, \phi^{3H}\}$. This and equation (67c) give

$$r_3 = -2G(\zeta) + \tilde{r}_3(u_1^3, u_2^3).$$

In summary, the most general Cartan two-form for this example is

$$d\theta_L = \frac{1}{2}G'(\zeta)d\zeta \wedge \phi^{1H} + \frac{1}{32}[(2G(\zeta) + \tilde{r}_2(u_1^1, u_2^2))du_1^2 \wedge du_2^2 + (-2G(\zeta) + \tilde{r}_3(u_1^3, u_2^3))du_1^3 \wedge du_2^3].$$

While this form is beguiling it is not the generic solution for this class.

**Example 5.** This is an example demonstrating that the number of non-integrable eigenspaces is not equal to the number of steps in the differential ideal process. Consider the system

$$\ddot{x} = xw, \quad \ddot{y} = x, \quad \ddot{z} = x$$  (69)

on an appropriate domain. Again denoting the derivatives by $u, v, w$, we find

$$\Phi = \begin{pmatrix} -w & 0 & \frac{v}{2} \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$
Eigenvalues and corresponding eigenvectors $X_a$ chosen so that $\hat{\nabla}_V X^V_a = 0$ are

$$\lambda_1 = \sqrt{-2u + w^2} - w \quad \text{and} \quad X_1 = (-\sqrt{-2u + w^2} + w, 2, 2),$$

$$\lambda_2 = -\sqrt{-2u + w^2} - w \quad \text{and} \quad X_2 = (\sqrt{-2u + w^2} + w, 2, 2),$$

$$\lambda_3 = 0 \quad \text{and} \quad X_3 = (0, 1, 0).$$

The non-zero functions $\tau^a_{Vbc}$ and $\tau^a_{Hbc}$ are

$$\tau^V_{11} = \frac{-2u + w^2}{2(2u - w^2)}, \quad \tau^H_{11} = \frac{x}{2(2u - w^2)},$$

$$\tau^V_{12} = \frac{-3u + w^2 + w}{2(2u - w^2)}, \quad \tau^H_{12} = \frac{-x}{2(2u - w^2)},$$

$$\tau^V_{21} = \frac{-3u + w^2 - w}{2(2u - w^2)}, \quad \tau^H_{21} = \frac{x}{2(2u - w^2)},$$

$$\tau^V_{22} = \frac{-2u + w^2 + w}{2(2u - w^2)}, \quad \tau^H_{22} = \frac{-x}{2(2u - w^2)}.$$

These results indicate that $\Phi$ is diagonalisable with distinct eigenvalues and the eigen-distributions $Sp\{\phi^{1V}, \phi^{1H}\}$ and $Sp\{\phi^{2V}, \phi^{2H}\}$ are non-integrable while the third one is integrable. $\Sigma^1$ is differential ideal.

### Conclusion

We finish this paper with a new proposal for the classification scheme for the inverse problem in dimension $n$:

A. $\Phi = \lambda I_n$. This is equivalent to $\langle \Sigma^0 \rangle$ being a differential ideal (see theorem 3.11 [2]).

B. $\Phi$ is diagonalisable with distinct eigenvalues (real or complex). Further subcases will be divided according to the integrability of the lifted two-dimensional eigenform-distributions of $\Phi$ i.e. $q$ eigenform-distributions are non-integrable and $n - q$ are integrable. According to our theorem 3.9 if up to and including $\langle \Sigma^q \rangle$ there is no differential ideal, then there is no non-degenerate multiplier. Hence, for each $q$, the subcases to be considered are that a differential ideal is generated at step 1, step 2,..., up to step $q$. Subsidiary to this is the consideration of integrable directions within non-integrable eigenspaces. In the event that no eigen-distributions are integrable the existence of a regular solution is given by theorem 3.12 [18]. (The relation of the closed form in this theorem to integrable directions is not known.)

C. $\Phi$ is diagonalisable with repeated eigenvalues. Further subdivision according to integrability will be similar to case B above.

We invite the reader who has persevered thus far to compare this scheme to the geometric translation of Douglas’s scheme for $n = 2$ to be found in [7]. If that scheme was translated into EDS terms the differential ideal conditions would be followed by integrability conditions on the eigenspaces. In the light of theorem 3.9 and our examples we maintain that the integrability of the eigenspaces must be considered first.
Acknowledgements

Thoan Do gratefully acknowledges receipt of a Vietnamese government MOET-VIED scholarship and scholarship support from La Trobe University, and the hospitality of the Australian Mathematical Sciences Institute. Both authors thank Willy Sarlet for useful discussions and his continuing interest.

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