On higher dimensional Einstein spacetimes with a non-degenerate double Weyl aligned null direction

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Abstract

We prove that higher dimensional Einstein spacetimes which possess a geodesic, non-degenerate double Weyl aligned null direction (WAND) $\ell$ must additionally possess a second double WAND (thus being of type D) if either: (a) the Weyl tensor obeys $\mathcal{C}_{abc}[d\ell] \ell^c = 0$ ($\Leftrightarrow$ $\Phi_{ij} = 0$, i.e. the Weyl type is II(abd)); (b) $\ell$ is twistfree. Some comments about an extension of the Goldberg–Sachs theorem to six dimensions are also made.

Keywords: higher-dimensional gravity, WANDs, Weyl tensor

1. Introduction and summary of main results

The Petrov classification has been extended to higher dimensions using the notions of null alignment and boost weight [1] (see also the review [2]). It provides one with a useful tool to classify and construct exact solutions, as well as to understand geometric properties of spacetimes (see [2] for examples of certain applications and for references). A class of spacetimes of particular interest consists of those of Weyl type II. In any dimension $n \geq 4$, the Weyl type II condition of [1] can be expressed as [3]

$$\ell_{[a} C_{b]d} \ell^d = 0,$$

where $\ell$ is a null vector field that defines a multiple Weyl aligned null direction (mWAND). Equivalently, the Weyl type II can be defined by the existence of a null frame in which all positive boost weight (b.w.) components of the Weyl tensor vanish [1, 2]. In four dimensions this is equivalent to the Petrov type II, for which the highest b.w. components (i.e. b.w. 0) are represented by the complex scalar $\Psi_2$. However, in higher dimension $n$ the Weyl tensor admits
\( \frac{1}{12} (n-3)(n-2) n(n-3) + 8 \) real b.w. 0 components. Thus the number of such components increases rapidly with the dimension of the spacetime and, not surprisingly, new subtypes of type II, not occurring in four dimensions, appear. One of these is the subtype \( \text{II}(\text{abd}) \) (considered also in this paper), which can be defined by the existence of a null vector field \( \ell \) obeying

\[
C_{abc} \rho_{\ell d} \ell^c = 0,
\]

(which is stronger than (1)) or equivalently by the vanishing of the part of b.w. 0 Weyl components described by the matrix \( \Phi_{ij} \) (introduced in (5)).

While for \( n = 4 \) large classes of type II Einstein spacetimes are known [4], there appear to exist various restrictions on solutions of genuine type II in more than four dimensions. First, if an Einstein spacetime admits a non-geodesic mW AND, it is necessarily of type D [5] (more precisely, of type D(d), and there exists an infinity of mWANDs, including a geodesic one). Additionally, if \( \ell \) is twistfree and non-degenerate (as defined in (10) below) and \( \Phi_{ij} \neq 0 \) (i.e. the type is not \( \text{II}(\text{abd}) \)), then again the type is necessarily D (more precisely type D(bd), see proposition 4.1 of [6] and see also [7]; this case includes, e.g. Robinson–Trautman solutions with \( \mu \neq 0 \) [8]). In five dimensions, the same conclusion remains true even if one drops the twistfree assumption [9] (while \( \Phi_{ij} \neq 0 \) follows automatically for type II with \( n = 5 \) [10] and need not be assumed). In six dimensions, a similar result has been obtained in [11] in the Ricci-flat case (again without assuming the twistfree condition, but with an additional assumption on the asymptotic fall-off of the Weyl tensor—which implies \( \Phi_{ij} \neq 0 \) and which is automatically satisfied in five dimensions). Restrictions in the \( n = 5 \) degenerate case have been obtained in [12].

The purpose of the present contribution is to present some new results which hold in arbitrary higher dimensions for the case of a non-degenerate mWAND. First, we will prove the following

**Proposition 1.** An \( n > 5 \) dimensional Einstein spacetime of type \( \text{II}(\text{abd}) \) with a geodesic mWAND \( \ell \) such that \( \det \rho \neq 0 \) must be in fact of type D(\( \text{abd} \)). In a frame such that \( L_{i1} = 0 \), the null vector \( n \) is another mWAND.

Examples of such spacetimes with \( n > 6 \) can be found in appendix F of [6] (in particular, in appendix F.4 in the twisting case). Other non-twisting examples (for \( n \geq 6 \)) are given by the Robinson–Trautman solutions with \( \mu = 0 \) [8].

As a further restriction on type II spacetimes, we will additionally prove

**Proposition 2.** An \( n > 4 \) dimensional Einstein spacetime that admits a non-degenerate, non-twisting mWAND \( \ell \) is necessarily of type D(bd) and, in a frame such that \( L_{i1} = 0 \), the null vector \( n \) is another mWAND. If \( \Phi_{ij} \neq 0 \), \( \ell \) must be shearfree [6] and \( \ell \) and \( n \) are the only WANDs. If \( \Phi_{ij} = 0 \), the type further specializes to D(\( \text{abd} \)) and, for \( n > 6 \), the spacetime may admit a continuous infinity of mWANDs.

This is a refinement of proposition 4.1 of [6]. More details are discussed in section 2.3 (including the conditions for having an infinite number of mWANDs in the \( \Phi_{ij} = 0 \) case). We observe that if the non-degeneracy assumption is dropped, examples of Einstein spacetimes of genuine type II can be easily constructed taking direct products or Brinkmann warps of known lower-dimensional type II solutions [2, 14].

As a by-product of the main calculations, we will also show that

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3 The definitions of the Weyl subtypes, such as D(\( d \)) and II(\( abd \)), can be found in [1–3]. The notation and various symbols to be used in this paper are explained at the end of this section (see also [1, 2, 10, 13] for more details).
Proposition 3. In an \( n > 4 \) dimensional Einstein spacetime of type III or \( N \), the \( mWAND \) is necessarily degenerate (i.e. \( \det \mathbf{\rho} = 0 \)).

This result was already known for the type \( N \) \([13]\). For the type III it was known in the following cases: (i) for \( n = 5, 6 \) \([13, 15]\); (ii) for a non-twisting \( mWAND \) (for any \( n > 4 \)) \([13]\); (iii) for a twisting \( mWAND \) with \( n > 6 \) under certain additional ‘genericity’ assumptions on the Weyl tensor \([13]\). (For (i)–(iii) and for the type \( N \), the optical matrix must in fact have rank \( 2 \) \([13, 15]\).)

The proofs of the above propositions are presented in section 2. In connection with these results, in section 3 we study some aspects of an extension of the Goldberg–Sachs theorem to six dimensions. For the case of non-degenerate \( mWANDS \), we arrive at some restrictions on the optical matrix compatible with Einstein spacetimes of type II(abd) (section 3.1). Some of these results can also be extended to the general type II (section 3.2).

1. Notation

Let us summarize the Newman–Penrose notation used in the present paper. The \( n \) frame vectors \( \mathbf{m}(a) \) consist of two null vectors \( \ell \equiv \mathbf{m}(0), \mathbf{n} = \mathbf{m}(1) \) and \( n - 2 \) orthonormal spacelike vectors \( \mathbf{m}(i) \), with \( a, b, \ldots = 0, \ldots, n - 1 \) while \( i, j, \ldots = 2, \ldots, n - 1 \) \([1, 2]\). The Ricci rotation coefficients are defined by \([13]\)

\[
L_{ab} = \ell_{a,b}, \quad N_{ab} = n_{a,b}, \quad M_{a(b)c} = m_{a,b}^{(i)}(c),
\]

and satisfy the identities

\[
L_{0a} = N_{1a} = N_{0a} + L_{1a} = M_{1a} = N_{1a} = M_{ja} + M_{ia} = 0.
\]

Covariant derivatives along the frame vectors are denoted as

\[
D \equiv \ell^a \nabla_a, \quad \Delta \equiv n^a \nabla_a, \quad \delta_i \equiv m_{(i)}^a \nabla_a.
\]

For the non-zero components of a Weyl tensor of type II, we use the symbols \([10, 13]\)

\[
\Phi_{ij} = C_{01ij}, \quad \Phi_{ij}^s = \Phi_{(ij)}, \quad \Phi_{ij}^A = \Phi_{[ij]}, \quad \Phi = \Phi_{ii},
\]

\[
\Psi_i = C_{101i}, \quad \Psi_{ijk} = \frac{1}{2} C_{1kij}, \quad \Psi_{ij} = \frac{1}{2} C_{11ij},
\]

which satisfy the identities

\[
C_{01ij} = 2C_{(i)(j)}, \quad 2\Phi_{ij}^s = 2\Phi_{ij} = -C_{ijk}, \quad 2C_{011} = -C_{1ij} = 2\Phi, \quad \Psi_i = 2\Psi_{ij}, \quad \Psi_{ijk} = 0 \quad \text{(curly brackets denoting cyclicity)}, \quad \Psi_{ijk} = -\Psi_{ijk}, \quad \Psi_{ij} = \Psi_{ji}, \quad \Psi_{ii} = 0.
\]

Throughout the paper, the Einstein equations \( R_{ab} = \frac{8\pi G}{c^4} T_{ab} \) hold, so that \( R_{abcd} = C_{abcd} \). It will be convenient to define the rescaled Ricci scalar

\[
\tilde{R} = \frac{R}{n(n-1)}.
\]

2. Proof of propositions 1–3

2.1. Preliminaries: Einstein spacetimes with a non-degenerate mWAND

Let us consider a \( n \)-dimensional Einstein spacetime of type II or more special (\( n > 4 \)). It follows \([5]\) that there exists a null vector field \( \ell \) that satisfies (1) while being geodesic and affinely parametrized, i.e.
\[ \ell_{ab} \ell^b = 0. \] (8)
The rank of the \((n-2) \times (n-2)\) optical matrix with components
\[ \rho_{ij} \equiv L_{ij} = \ell_{a,b} \delta_{ij} \] (9)
is thus frame-independent [16], for any choice of a frame adapted to \(\ell\).

In this paper we consider the non-degenerate (or full-rank) case, i.e. from now on we assume
\[ \det \rho \neq 0. \] (10)

We will find certain conditions under which the genuine type II cannot occur.

For convenience, we take our frame to be parallelly propagated along \(\ell\) [16], so that (with (8))
\[ L_{00} = 0, \quad L_{10} = 0, \quad i M_{j0} = 0, \quad N_{j0} = 0. \] (11)
Furthermore, thanks to (10), we can perform a null rotation about \(\ell\) such that (see appendix D.2.6 of [17] and lemma 1 of [18])
\[ L_{11} = 0. \] (12)

Given \(\ell\), this uniquely fixes the null direction defined by \(n\) (this choice will be useful in the following). We also define an affine parameter \(r\) such that
\[ \ell = \partial_r. \] (13)

With the above assumptions, the Sachs equation \(D\rho = -\rho^2\) [13, 16] fixes the \(r\)-dependence of \(\rho\) (thanks to the simple identity \(D(\rho^{-1}) = 0\)) [7, 17, 19]
\[ \rho^{-1} = rI - b, \] (14)
where \(I\) is the identity matrix and \(Db = 0\). We can restrict ourselves to the case \(b \neq 0\), for the case \(b = 0\) reduces to the Robinson–Trautman metrics, already known to be of type D [8].

Note that \(\ell\) is twistfree iff \(b_{ij} = 0\) [7]. For later purposes, it is useful to observe [7] that, for large \(r\), equation (14) gives
\[ \rho_{ij} = \frac{1}{r} \delta_{ij} + \frac{1}{r^2} b_{ij} + O(r^{-3}). \] (15)

2.2. Type II(abd) spacetimes: proof of propositions 1 and 3

Let us now focus on Einstein spacetimes of type II(abd), i.e. such that
\[ \Phi_{ij} = 0, \] (16)
in the frame specified above (this case is of interest only for \(n > 5\), since (16) means that all b.w. 0 Weyl components vanish when \(n = 5\) [10]).

The \(r\)-dependence of the Weyl tensor can be determined by using the Bianchi equations containing \(D\)-derivatives. Using (16) and the fact that \(\ell\) is an mWAND, equation (B.12, [13]) (see also (7, [7])) reduces to
\[ DC_{ijk} = -C_{ijkl} \rho_{ln} + C_{ijmn} \rho_{lk}, \] (17)
which, using (14), implies \(D(C_{ijkl} \rho_{ln}^{-1} \rho_{mn}^{-1}) = 0\). By integration one gets
\[ C_{ijklm} = \rho_{kl}\rho_{m}\rho_{ijklm}. \]  
(18)

The integration functions \( c_{ijkl} (= -c_{ijk} = c_{lijk}) \) do not depend on \( r \) and, thanks to \( C_{ijk} = 0 \) (which follows from (16)) and (18) with (15) (see also (27) and (22) of [10]), satisfy
\[ c_{ijkl} = 0, \]
(19)
\[ c_{ijkl} b_{[ij]} = 0, \quad c_{ijkl} b_{[ij]} = 0. \]
(20)

Next, equations (B.1), (B.6), (B.9) and (B.4) of [13] (see also (16)–(18), (22) of [7]) reduce to
\[ D\Psi_i = -2\Psi_s\rho_z\Psi_i, \]
(21)
\[ 2D\Psi_{ijk} = -2\Psi_{sijkl} \rho_z - \Psi_{sijk} \rho_z + \Psi_{sikl} \hat{\psi}^{ijl}, \]
(22)
\[ D\Psi_{jki} = 2\Psi_{kij} + \Psi_{sikl} \hat{\psi}^{ijl}, \]
(23)
\[ 2D\Psi_{ij} = -2\Psi_{sijkl} \rho_z + \delta_j \Psi_i + \Psi_{sijkl} \hat{\psi}^{ijl}. \]
(24)

Thanks to (14), equations (21)–(23) can be easily integrated, giving
\[ \Psi_i = \rho_{kl}\rho_{m}\psi_i, \]
(25)
\[ \Psi_{ijkl} = \rho_{[ijk]} \psi_i + \rho_{ik} \hat{\psi}_{ijkl}, \]
(26)
\[ \Psi_{jki} = \rho_{[ijkl]} \psi_i + \rho_{ik} \hat{\psi}_{ijkl}. \]
(27)

where \( \psi_i, \hat{\psi}_{ijkl} \) and \( \psi_{ijkl} \) are \( r \)-independent quantities. Comparing (26) and (27) (e.g. using (15)) immediately gives \( \hat{\psi}_{ijkl} = 0 \). Using this, the trace of (26) with (25) additionally yields (for \( n > 4 \)) \( \psi_i = 0 \), and therefore
\[ \Psi_i = 0 = \Psi_{ijkl}. \]
(28)

Thanks to these, equation (24) gives
\[ \Psi_{ij} = \rho_{ijkl} \psi_{kl}, \]
(29)
with \( \psi_{[kl]} = 0 = \psi_i \) and \( D\psi_{ijkl} = 0 \).

In order to proceed, we will also need (B.13, [13]) (see also (B11, [11])), which here becomes
\[ -R_{ijklm} = 2\Psi_{ijklmn} + 4\Psi_{ijklmn} = 2\Psi_{ijklmn} + 2C_{ijklmn} + 2C_{ijklmn} M_{[ijkl]} + 2C_{ijklmn} N_{ijklmn}. \]
(30)

The \( r \)-dependence of the last term can be fixed using the Ricci identity (11j, [16]), which with (11) and (16) reduces to \( DN_{ijkl} = -N_{ijkl} \rho_{ijkl} - \hat{R} \delta_{ijkl} \). This gives
\[ N_{ijkl} = \rho_{ijkl} \left( -\frac{1}{2} \hat{R} \delta_{ijkl} + \hat{R} \rho_{ijkl} + \delta_{ijkl} \right), \]
(31)
where \( Dn_{ijkl} = 0 \). The trace of (30) (with (14), (18), (29), (31) and (19)) leads to \( \psi_{ijkl} = 0 \) (thanks to \( n > 4 \)) and thus
\[ \Psi_{ij} = 0. \]
(32)
We have thus proven that all Weyl components of negative b.w. vanish identically, i.e. \( n \) is also an mWAND. Therefore we have proven proposition 1. In passing, the above calculations also prove proposition 3.

### 2.3. Type II spacetimes with a twistfree mWAND: proof of proposition 2

Using proposition 1, as a refinement of proposition 4.1 of [6] one additionally obtains proposition 2.

Let us further observe here that, in that context, a *shearfree* \( \ell \) defines Robinson–Trautman spacetimes [8], for which the type is D(bd) or more special and \( \ell \) and \( n \) define two mWANDs (this follows from [8], see also [6, 20]; \( n \) is fixed by the \( L_{ij} = 0 \) condition). The fact that for \( \Phi_{ij} \neq 0 \), there are no additional (m)WANDs follows from footnote 15 of [20]—where it is also observed that, instead, a Weyl tensor of type D(abd) can admit an infinity of mWANDs (this happens when the equation \( C_{ijkl}z_l = 0 \) admits a solution \( z_l \neq 0 \), see appendix F.4 of [6] for an example; however, in this case a non-zero \( C_{ijkl} \) requires \( n > 6 \)).

### 3. Partial extension of the Goldberg–Sachs theorem to six dimensions

Various results partially extending the Goldberg–Sachs theorem to higher dimensions have been obtained in recent years [5–7, 10, 13, 15, 17, 21]. Here we point out some additional restrictions which apply to the spacetimes of section 2.2 (in section 3.1) as well as to general type II Einstein spacetimes (in section 3.2), at least when \( n = 6 \).

#### 3.1. Type II(abd) spacetimes

Let us consider type II Einstein spacetimes and further assume (10) and (16). By proposition 1, these spacetimes are in fact of type D(abd).

In addition to (19) and (20), the cyclicity \( C_{[ijkl]} = 0 \) (or (21, [10])) further gives

\[
C_{i[}b_{jkl]} = 0, \quad C_{a[}b_{j|m]} = 0. \tag{33}
\]

In any dimension \( n > 5 \), the algebraic equations (19), (20) and (33) can be used to constrain the permitted forms of \( b_{ij} \) and thus of the optical matrix \( \rho_{ij} \). For the sake of definiteness, let us discuss explicitly here only the case of *six dimensions*. Before starting, let us observe that, in this case, a Weyl tensor of type D(abd) admits precisely two mWANDs \( \ell \) and \( n \), which must therefore [5] be both *geodesics*. In an adapted frame, equations (B.7, [13]) and (B.9, [13]) reduce to \( C_{ijkL}L_i = 0 = C_{ijk}N_0 \), which gives (since \( n = 6 \)) \( L_i = 0 = N_0 \) (in agreement with proposition 1).

Now, when \( n = 6 \), one can take as the only (ten) independent components of \( c_{ijkl} \)

\[
\begin{align*}
W_{23} &= c_{2323}, \quad W_{25} = c_{2525}, \\
Z_{23} &= c_{2434}, \quad Z_{24} = c_{2343}, \quad Z_{25} = c_{2535}, \quad Z_{34} = c_{3242}, \quad Z_{35} = c_{3525}, \quad Z_{45} = c_{4252}, \\
Y_{23} &= c_{2345}, \quad Y_{24} = c_{2453}. 
\end{align*}
\]

\( ^4 \) When \( n = 6 \), the condition \( C_{ijkl}z_l = 0 \) with \( C_{ijkl} = 0 \) implies \( C_{ijkl} = 0 \) (this can be seen explicitly, e.g. using equations (34)-(36) given below, if one takes a frame such that \( m_{23} \) is parallel to \( z_3 \)).

\( ^5 \) This applies to any type D(abd) in six dimensions (i.e. even without assuming (10)), see the comments in section 2.3.

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while all the remaining components can be determined using (19) and the first of (33).\footnote{Namely: \( c_{235} = -Z_{25}, c_{245} = -Z_{24}, c_{345} = -Z_{34}, c_{3454} = -Z_{345}, c_{3453} = -Z_{45} \), \( c_{3353} = -W_{23} - W_{25}, c_{344} = W_{25}, c_{445} = W_{25}, c_{3454} = -Y_{24} - Y_{25} \).

Let us perform an \( r \)-independent spin to align the frame vectors \( \mathbf{m}_{(i)} \) to an eigenframe of \( b_{(ij)} \) and denote by \( b_i \) its eigenvalues. From the first of (20) we obtain

\[
(b_3 - b_4)W_{23} + (b_5 - b_6)W_{25} = 0, \quad (b_2 - b_3)W_{23} + (b_4 - b_5)W_{25} = 0, \quad (37)
\]

\[
(b_5 - b_2)W_{23} + (b_3 - b_2)W_{25} = 0, \quad (b_4 - b_3)W_{23} + (b_2 - b_3)W_{25} = 0, \quad (38)
\]

\[
(b_4 - b_3)Z_{23} = 0, \quad (b_3 - b_4)Z_{24} = 0, \quad (b_3 - b_4)Z_{25} = 0, \quad (39)
\]

\[
(b_2 - b_3)Z_{34} = 0, \quad (b_2 - b_4)Z_{35} = 0, \quad (b_2 - b_3)Z_{45} = 0, \quad (40)
\]

while the second of (33) (using the second of (20) to get rid of terms containing \( b_{(ij)} \)) gives

\[
(b_2 - b_4)Y_{23} + (b_4 - b_3)Y_{24} = 0, \quad (b_3 - b_5)Y_{23} + (b_4 - b_3)Y_{24} = 0, \quad (41)
\]

\[
(b_2 - b_4)Y_{23} + (b_4 - b_3)Y_{24} = 0, \quad (b_3 - b_5)Y_{23} + (b_2 - b_3)Y_{24} = 0, \quad (42)
\]

\[
(b_3 - b_5)Z_{23} = 0, \quad (b_4 - b_2)Z_{24} = 0, \quad (b_5 - b_2)Z_{25} = 0, \quad (43)\]

\[
(b_4 - b_3)Z_{34} = 0, \quad (b_3 - b_5)Z_{35} = 0, \quad (b_3 - b_4)Z_{45} = 0. \quad (44)
\]

It can be immediately seen from (37)–(44) that having (at least one) \( Z_{ij} \) non-vanishing requires that at least two pairs of \( b_i \) coincide, and the same conclusion holds also for \( W_{ij} \) and \( Y_{ij} \). Thus, a non-zero Weyl tensor is compatible only with the two following multiplicities of the eigenvalues of \( b_{(ij)} \): \( \{a,a,a,a\} \) or \( \{a,a,b,b\} \). These two cases give, respectively:

1. \( b_{(ij)} = b_0 \delta_{ij} \) (i.e. all the eigenvalues of \( b_{(ij)} \) coincide), which is equivalent (when (10) holds) to the so called \textit{optical constraint} [6, 17]. In this case, equations (37)–(44) are identically satisfied. One can shift the affine parameter \( r \to r + b_0 \), so that \( b_{(ij)} = 0 \). Using \( r \)-independent spins, one can thus arrive at the canonical form (see e.g. [6, 11, 17, 22])

\[
b = \text{diag} \left( \begin{array}{cc} 0 & b_{23} \\ -b_{23} & 0 \end{array} \right), \quad \text{diag} \left( \begin{array}{cc} 0 & b_{25} \\ -b_{25} & 0 \end{array} \right). \quad (45)\]

For \( b_{23} = 0 = b_{25} \), \( \ell \) is expanding, twistfree and shearfree, which corresponds to the Robinson–Trautman spacetimes with \( \mu = 0 \) [8].

2. \( b_3 = b_2 \neq b_5 \neq b_4 \), in which case the Weyl tensor must obey (thanks to (37)–(44))

\[
W_{23} = Z_{24} = Z_{25} = Z_{34} = Z_{35} = Z_{45} = Y_{23} = 0. \quad (46)
\]

(Other possible cases trivially correspond to reordering the frame vectors—these need not be discussed separately.) Using (46), the second of (20) reads

\[
-b_{24}W_{25} - b_{35}Y_{24} + b_{34}Z_{23} + b_{25}Z_{45} = 0, \quad b_{25}W_{25} - b_{34}Y_{24} - b_{35}Z_{23} + b_{24}Z_{45} = 0, \quad (47)
\]

\[
b_{34}W_{25} - b_{25}Y_{24} + b_{24}Z_{33} - b_{35}Z_{45} = 0, \quad b_{35}W_{25} + b_{24}Y_{24} + b_{25}Z_{33} + b_{34}Z_{45} = 0. \quad (48)
\]
A non-zero Weyl tensor requires the determinant associated to the above system to vanish, i.e. \( b_{35} = \mp b_{24} \) and \( b_{34} = \pm b_{25} \). There are therefore two possible subcases.

(a) \( b_{35} = b_{24} = b_{34} = b_{25} = 0 \), in which case (47) and (48) are identically satisfied and

\[
\mathbf{b} = \text{diag} \left( \begin{bmatrix} b_2 & b_{23} \\ -b_{23} & b_2 \end{bmatrix}, \begin{bmatrix} b_4 & b_{24} \\ -b_{24} & b_4 \end{bmatrix} \right),
\]

(49)

The matrices \( \mathbf{b} \) and \( \mathbf{\rho} \) (recall (14)) are normal. Redefining \( r \) one can set \( b_2 = 0 \) (or \( b_4 = 0 \)) or \( b_2 + b_4 = 0 \). In the case of zero twist \( (b_{23} = 0 = b_{45}) \), it has been shown that this case cannot occur \[23\].

(b) \( b_{35} = \mp b_{24}, \ b_{34} = \pm b_{25}, \ b_{24}^2 + \ b_{25}^2 \neq 0 \), in which case (47) and (48) further restrict the Weyl tensor, namely

\[
Y_{24} = \pm W_{25}, \quad Z_{45} = \mp Z_{23}.
\]

(50)

In this case \( \mathbf{b} \) and \( \mathbf{\rho} \) are not normal matrices. It can be verified that the symmetric matrix \( \mathbf{\rho} + \mathbf{\rho}^T \) admits two pairs of repeated eigenvalues (although it is not diagonal in the frame in use). By an \( r \)-independent spin in the plane (23) or (45), one can set \( b_{24} = 0 \) or \( b_{25} = 0 \), but not both (and, as above, redefining \( r \) one can set \( b_2 = 0 \) or \( b_4 = 0 \)).

3.2. Comments for general type II

Proving an extension of the Goldbers–Sachs theorem in the case of general type II (i.e. without assuming (16)) would deserve a separate investigation. However, let us make here some preliminary comments, still assuming (10). The \( r \)-dependence of \( C_{ijkl} \) is not given anymore by (18), nevertheless for \( r \to \infty \) one still has \( C_{ijkl} = c_{ijkl}r^{-2} + \ldots \) with \( c_{ijkl} = 0 \), while

\[\Phi_{ij}^\delta = \tau_{ij}r^{-4} + \ldots, \quad \Phi_{ij}^0 = \pi_{ij}r^{-3} + \ldots\]

(where \( D\tau_{ij} = 0 = D\pi_{ij}, \tau_{ij} = 0 = \tau_{ij} \) [7, 24]). In the special case \( c_{ijkl} = 0 \), it has been already shown that the optical matrix \( \rho_{ij} \) must obey the optical constraint [7], i.e. (45) holds. In the generic case with both \( c_{ijkl} \neq 0 \) and \( \Phi_{ij} \neq 0 \) (so far unexplored), we can adapt some of the arguments of section 3.1 to obtain at least a weaker result which restricts the multiplicity of the eigenvalues of \( b_{(ij)} \). We only sketch the line of the arguments, since the details would be similar to section 3.1.

In this case, the second relations of both (20) and (33) are replaced by \( c_{ijkl}b_i[j] = (n - 4)\pi_{ik} \) and \( c_{im[j]}b_m[i] = -2\pi_{[ij}b_{i]} ), respectively (while the first of (20) and (33) are unchanged; see equations (21,22,27, [10]), (10, [7]) and [24]), nevertheless one can still arrive at (37)–(44) by simple algebraic manipulations. As argued above, it follows that either: (i) \( b_{(ij)} = b_0\delta_{ij} \), i.e. the optical constraint holds or (ii) \( b_3 = b_2 \neq b_5 = b_4 \), in which case the Weyl tensor is constrained by \( W_{23} = Z_{24} = Z_{25} = Z_{34} = Z_{35} = Y_{23} = 0 \) (up to relabeling the frame vectors).

In both cases, \( b_{(ij)} \) possesses at least two pairs of repeated eigenvalues.

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References

[1] Coley A, Milson R, Pravda V and Pravdová A 2004 Classification of the Weyl tensor in higher dimensions Class. Quantum Grav. 21 L35–41
[2] Ortaggio M, Pravda V and Pravdová A 2013 Algebraic classification of higher dimensional spacetimes based on null alignment Class. Quantum Grav. 30 013001
[3] Ortaggio M 2009 Bel–Debever criteria for the classification of the Weyl tensor in higher dimensions Class. Quantum Grav. 26 195015
[4] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 Exact Solutions of Einstein’s Field Equations 2nd edn (Cambridge: Cambridge University Press)
[5] Darkee M and Reall H S 2009 A higher-dimensional generalization of the geodesic part of the Goldberg–Sachs theorem Class. Quantum Grav. 26 245005
[6] Ortaggio M, Pravda V and Pravdová A 2013 On the Goldberg–Sachs theorem in higher dimensions in the non-twisting case Class. Quantum Grav. 30 075016
[7] Ortaggio M, Pravda V and Pravdová A 2009 Asymptotically flat, algebraically special spacetimes in higher dimensions Phys. Rev. D 80 084041
[8] Podol'sky J and Ortaggio M 2006 Robinson–Trautman spacetimes in higher dimensions Class. Quantum Grav. 23 5785–97
[9] de Freitas G B, Godazgar M and Reall H S 2015 Uniqueness of the Kerr–de Sitter spacetime as an algebraically special solution in five dimensions Commun. Math. Phys. 340 291–323
[10] Pravda V, Pravdová A and Ortaggio M 2007 Type D Einstein spacetimes in higher dimensions Class. Quantum Grav. 24 4407–28
[11] Ortaggio M 2017 On the uniqueness of the Myers-Perry spacetime as a type II(D) solution in six dimensions J. High Energy Phys. JHEP06(2017)042
[12] Wyllie L 2015 Finalizing the classification of type II or more special Einstein spacetimes in five dimensions (arXiv:1511.02824)
[13] Pravda V, Pravdová A, Coley A and Milson R 2004 Bianchi identities in higher dimensions Class. Quantum Grav. 21 2873–97
[14] Pravda V, Pravdová A, Coley A and Milson R 2007 Class. Quantum Grav. 24 1691 (corrigendum)
[15] Ortaggio M, Pravda V and Pravdová A 2011 On higher dimensional Einstein spacetimes with a warped extra dimension Class. Quantum Grav. 28 105006
[16] Kubíček J 2015 Gravitace ve vyšších dimenzích Diploma Thesis Charles University in Prague Kubíček J and Pravdová A (in preparation)
[17] Ortaggio M, Pravda V and Pravdová A 2007 Ricci identities in higher dimensions Class. Quantum Grav. 24 1657–64
[18] Ortaggio M, Pravda V and Pravdová A 2009 Higher dimensional Kerr–Schild spacetimes Class. Quantum Grav. 26 025008
[19] Durkee M, Pravda V, Pravdová A and Reall H S 2010 Generalization of the Geroch–Held–Penrose formalism to higher dimensions Class. Quantum Grav. 27 215010
[20] Newman E T and Penrose R 1962 An approach to gravitational radiation by a method of spin coefficients J. Math. Phys. 3 566–78

Newman E and Penrose R 1963 J. Math. Phys. 4 998 (erratum)
[21] Ortaggio M, Podolský J and Žofka M 2015 Static and radiating p-form black holes in the higher dimensional Robinson–Trautman class J. High Energy Phys. JHEP02(2015)045
[22] Ortaggio M, Pravda V, Pravdová A and Reall H S 2012 On a five-dimensional version of the Goldberg–Sachs theorem Class. Quantum Grav. 29 205002
[22] Ortaggio M, Pravda V and Pravdová A 2010 Type III and N Einstein spacetimes in higher dimensions: general properties Phys. Rev. D 82 064043
[23] Reall H S, Graham A A H and Turner C P 2013 On algebraically special vacuum spacetimes in five dimensions Class. Quantum Grav. 30 055004
[24] Ortaggio M and Pravdová A 2014 Asymptotic behaviour of the Weyl tensor in higher dimensions Phys. Rev. D 90 104011