Unconstrained Capacities of Quantum Key Distribution and Entanglement Distillation for Pure-Loss Bosonic Broadcast Channels

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We consider quantum key distribution (QKD) and entanglement distribution using a single-sender multiple-receiver pure-loss bosonic broadcast channel. We determine the unconstrained capacity region for the distillation of bipartite entanglement and secret key between the sender and each receiver, along with the assistance of unlimited LOCC [11]. Even though communication tasks in various network scenarios have been examined [12–17], there has been limited work on the capacity of entanglement and secret key distillation assisted by unlimited LOCC. Only recently in [18] were outer bounds on the achievable rates established for multipartite secret-key agreement and entanglement generation between any subset of the users of a general single-sender m-receiver quantum broadcast channel (QBC) (for any $m \geq 1$), when assisted by unlimited LOCC between all the users. The main idea was to employ multipartite generalizations of the squashed entanglement [19,20] and the methods of [21,22].

We break the proof of the capacity region into two parts. The upper bound (converse) is established by combining the method in [18] and the point-to-point upper bound based on relative entropy of entanglement [23,24], first discussed in [23] and rigorously proven in [24]. The lower bound (achievability) is proved by employing the quantum state merging protocol [25,26]. Our result clearly shows that the rate region considerably improves upon the time-sharing limit, and at the same time, it proves that this is the fundamental limit that cannot be overcome within the same framework. Moreover, we do not leave this result as a purely theoretical development, but we also consider the possible implementation of a QKD protocol overcoming the limit by simple point-to-point protocols for an optical broadcast channel. Our result is thus an important step toward the opening of a new framework of network channel-based quantum communication technology.

**Introduction.** Quantum key distribution (QKD) [1,2] and entanglement distillation (ED) [3,4] are two cornerstones of quantum communication. QKD enables two or more cooperating parties to distill and share unconditionally secure, random bit sequences, which could then be used for secure classical communication. ED, on the other hand, allows them to distill pure maximal entanglement from a quantum state shared via a noisy communication channel, which could then be used to faithfully transfer quantum states by means of quantum teleportation [5]. In both protocols, the parties are allowed to perform (in principle) an unlimited amount of local operations and classical communication (LOCC).

Not only have there been a number of theoretical developments, but also quantum communication technologies have matured tremendously in recent years. In particular, QKD has been commercially available for a number of years and has expanded into real-world networks [6–8], which consist of point-to-point QKD links and trusted nodes.

Another important direction is to go beyond point-to-point links and make use of network channels. In fact the operation of QKD has been proposed for a broadcast channel (single-sender and multiple-receiver) [9] and recently experimentally demonstrated for a multiple access channel [10] (multiple-sender and single-receiver). In [10], the developed system is based on conventional optical-access network protocols (passive optical network), in which the link between each sender and receiver is essentially point-to-point quantum communication and multiple users share the channel, each having a given amount of time to use it. This **time-sharing protocol** has a strong limit on the rate of key that can be generated among the parties: when one sender and one receiver use the channel most of the time, the key or entanglement rates for the other users decrease. Then a natural question arises. Is this a fundamental trade-off limit or can we do better than the time-sharing limit?

In this paper, we answer this question affirmatively by establishing the unconstrained capacity region of a pure-loss bosonic broadcast channel, when used for the distillation of bipartite entanglement and secret key between the sender and each receiver, along with the assistance of unlimited LOCC [11]. Even though communication tasks in various network scenarios have been examined [12–17], there has been limited work on the capacity of entanglement and secret key distillation assisted by unlimited LOCC. Only recently in [18] were outer bounds on the achievable rates established for multipartite secret-key agreement and entanglement generation between any subset of the users of a general single-sender $m$-receiver quantum broadcast channel (QBC) (for any $m \geq 1$), when assisted by unlimited LOCC between all the users. The main idea was to employ multipartite generalizations of the squashed entanglement [19,20] and the methods of [21,22]. We break the proof of the capacity region into two parts. The upper bound (converse) is established by combining the method in [18] and the point-to-point upper bound based on relative entropy of entanglement [23,24], first discussed in [23] and rigorously proven in [24]. The lower bound (achievability) is proved by employing the quantum state merging protocol [25,26]. Our result clearly shows that the rate region considerably improves upon the time-sharing limit, and at the same time, it proves that this is the fundamental limit that cannot be overcome within the same framework. Moreover, we do not leave this result as a purely theoretical development, but we also consider the possible implementation of a QKD protocol overcoming the limit by simple point-to-point protocols for an optical broadcast channel. Our result is thus an important step toward the opening of a new framework of network channel-based quantum communication technology.

**LOCC-assisted distillation in a linear-optical network.** We consider the following general entanglement and secret-key distillation protocol which uses a quantum broadcast channel [18]. The sender $A$ prepares some quantum systems in an initial quantum state and successively sends some of these systems to the receivers $B_1, B_2, \ldots, B_m$ by interleaving $n$ channel uses of the 1-to-$m$ broadcast channel with rounds of LOCC. The goal of the protocol is to distill bipartite maximally entangled states $\Phi_{AB}$, and private states $\gamma_{AB}$, (equivalently, a secret key) [27,28]. After each channel use, they can perform an arbitrary number of rounds of LOCC (in any
direction with any number of parties). The quantity $E_{AB_i}$ denotes the rate of entanglement that can be established between $A$ and $B_i$ (i.e., the logarithm of the Schmidt rank of $\Phi_{AB_i}$ normalized by the number of channel uses) and $K_{AB_i}$ denotes the rate of secret key that can be established between $A$ and $B_i$ (i.e., the number of secret-key bits in $\gamma_{AB_i}$ normalized by the number of channel uses). The parameter $\varepsilon \in (0, 1)$ is such that the fidelity $\Phi$ between the ideal state at the end of the protocol and the actual state is not smaller than $1 - \varepsilon$. The protocol considered here is similar to the one described in [18], except that here the goal is not to establish bipartite entanglement or key among the receivers or multipartite entanglement or key among more than two parties. A rate tuple $(E_{AB_1}, \ldots, E_{AB_m}, K_{AB_1}, \ldots, K_{AB_m})$ is achievable if for all $\varepsilon \in (0, 1)$ and sufficiently large $n$, there exists an $(n, E_{AB_1}, \ldots, E_{AB_m}, K_{AB_1}, \ldots, K_{AB_m}, \varepsilon)$ protocol of the above form. The capacity region is the closure of the set of all achievable rates.

The quantum channel we consider here is a general 1-to-$m$ bosonic broadcast channel $\mathcal{L}_{A^m \rightarrow B_1 \ldots B_m}$ consisting of passive linear optical elements (beam splitters and phase shifters) [30]. An isometric extension of the channel (see, e.g., [31]), denoted by $U^m$, is then given by an $l$-input $l$-output linear optical unitary transformation (see Fig. 1(a)). For $U^m$, one of the inputs is the sender $A$’s input and the others are prepared as vacuum states. Also, $m$ of the outputs ($m \leq l$) are given to the legitimate receivers $\{B_1, \ldots, B_m\}$ (one per receiver), and the rest of the outputs are for the environment, which we allow the eavesdropper to access during the protocol. The model is thus such that all of the light that does not make it to the legitimate receivers is given to the eavesdropper. Let $\{\eta_{B_1}, \ldots, \eta_{B_m}\}$ be a set of power transmittances from the sender to the respective receivers. Each $\eta_{B_i}$ is non-negative and $\sum_{i=1}^m \eta_{B_i} \leq 1$. Let $B = \{B_1, \ldots, B_m\}$, let $T \subseteq B$, and let $\overline{T}$ denote the complement of the set $T$. Then our main result is as follows:

**Theorem 1**: The LOCC-assisted unconstrained capacity region of the pure-loss bosonic QBC $\mathcal{L}_{A^m \rightarrow B_1 \ldots B_m}$ is given by

$$\sum_{T \subseteq B} E_{AB_T} + K_{AB_T} \leq \log_2\left(\left[1 - \eta_{\overline{T}}\right]/\left[1 - \eta_B\right]\right),$$

(1)

for all non-empty $T \subseteq B$, where $\eta_B = \sum_{i=1}^m \eta_{B_i}$ and $\eta_{\overline{T}} = \sum_{B_i \in \overline{T}} \eta_{B_i}$.

The proof of Theorem 1 consists of three steps:

1. **Decomposition of $U^m$**. First we argue that $U^m$ can be rewritten as an equivalent and simpler QBC (see [30] as well for the reduction outlined here). The isometric extension $U^m$ can be represented by an $l \times l$ unitary matrix describing the input-output relation of a set of annihilation operators $\{\hat{a}_1, \ldots, \hat{a}_l\}$ for $l$ input modes. In [32], it was shown that any such $l \times l$ unitary matrix can be decomposed as a sequence of $2 \times 2$ matrices, each realized by a beam splitter and phase shifters combining any two of the $l$ modes (see Fig. 1(b), which contains at most $C_2 \approx 2$ beam splitters). Now recall that all the inputs except for the sender’s are prepared as vacuum states. Then we can remove all the beam splitters that have both inputs set to vacuum states because their outputs are vacuum states as well. In addition, by grouping together all of the eavesdropper’s modes, the channel is simplified to just a sequence of $m$ beam splitters (Fig. 1(c)). In what follows, we consider this simplified, equivalent channel model.

2. **Achievability part**. To achieve the rate region in (1), we consider a distillation protocol which employs quantum state merging. State merging was introduced in [25, 26] and provides an operational meaning for the conditional quantum entropy. For a state $\rho_{AB}$, its conditional quantum entropy is defined as $H(A|B)_\rho = H(AB)_\rho - H(B)_\rho$, where $H(AB)_\rho$ and $H(B)_\rho$ are the quantum entropies of $\rho_{AB}$ and its marginal $\rho_B$, respectively. For many copies of $\rho_{AB}$ shared between Alice and Bob, $H(A|B)_\rho$ is the optimal rate at which two-qubit maximally entangled states need to be consumed to transfer Alice’s systems to Bob’s side via LOCC. If $H(A|B)_\rho$ is negative, the result is that after transferring Alice’s systems, they can gain (i.e., distill) entanglement at rate $-H(A|B)_\rho$. State merging also yields a quantum analog of the Slepian-Wolf theorem concerning classical distributed compression and has been applied to the QBC in [16, 17].

Here we consider the following alternative state-merging-based protocol. Alice first prepares $n$ copies of a two-mode squeezed vacuum (TMSV) state, defined as

$$\Psi(N_S)_{A^m} = \sum_{m=0}^\infty \sqrt{\lambda_m(N_S)} |m\rangle_A |m\rangle_{A'},$$

(2)

where $|m\rangle$ is an $m$-photon state, $\lambda_m(N_S) = N_S^m / (N_S + 1)^{m+1}$, and $N_S$ is the average photon number of one mode of the state. She sends system $A'$ to $B_1 \cdots B_m$ through the broadcast channel in Fig. 1(a). After $n$ uses of the channel, they share $n$ copies of the state $\phi_{A_1B_1 \cdots B_m} = \mathcal{L}_{A' \rightarrow B_1 \ldots B_m} (\Psi(N_S)_{A^m} \Psi(N_S_{A'})_{A'A'})$. Then by using $\phi_{A_1B_1 \cdots B_m}$, they perform state merging to establish entanglement. More precisely, all the receivers successes:
sively transfer their systems back to Alice by LOCC and at the same time generate entanglement with her in the process (similar to reverse reconciliation in the point-to-point scenario). This can be accomplished by applying the point-to-point state merging protocol successively [25] [20].

Then we obtain the achievable rate region as

$$
\sum_{B_i \in \mathcal{T}} E_{AB_i} \leq -H(T|AT)_{\phi},
$$

(3)

where $\Phi_{AB_1 \cdots B_m} = \mathcal{L}_{A'} \rightarrow B_1 \cdots B_m ((\Psi(N_S)) (\Psi(N_S))_{AA'})$. The right-hand side of the inequalities in (3) can be explicitly calculated. Recall that the marginal of the TMSV $\Psi_{A'}(N_S) = \text{Tr}_{A}[\Psi(N_S)|\Psi(N_S)]_{AA'}$ is a thermal state with mean photon number $N_S$. Its entropy is equal to $H(A') = g(N_S)$, where $g(x) = (x+1) \log_2(x+1) - x \log_2 x$. Also a pure-loss channel with transmittance $\eta$ maps a thermal state to another thermal state with reduced average photon number. Then the right-hand side of (3) is calculated as

$$
-H(T|AT)_{\phi} = H(AT)_{\phi} - H(T|AT)_{\phi} = H(T|E)_{\phi} - H(E|\phi) = g((1-\eta_T)N_S) - g((1-\eta_B)N_S).
$$

By taking $N_S \rightarrow \infty$ in the last line above, the limit is equal to $\log_2[(1-\eta_T)/(1-\eta_B)]$. Since one ebit of entanglement can generate one private bit of key, we can replace $E_{AB_i}$, with $E_{AB_i} + K_{AB_i}$, which completes the achievability part.

(3) Converse part. The converse relies upon several tools and is given in terms of the one-shot variant [33] of the relative entropy of entanglement (REE) [34]. The $\varepsilon$-REE for a quantum state $\rho_{AB}$ is defined by

$$
E_{\varepsilon}^R(A;B)_{\rho} = \inf_{\sigma_{AB} \in \text{SEP}} D_{\varepsilon}^R(\rho_{AB}||\sigma_{AB}),
$$

(4)

where $D_{\varepsilon}^R(\rho||\sigma) = -\log_2 \inf_{0 \leq \delta \leq 1: \text{Tr}[\delta_{AB} \geq 1-\varepsilon] \geq 1-\varepsilon} \text{Tr}[\delta]\sigma$ is the hypothesis testing quantum relative entropy [35][37] and SEP denotes the set of separable states. The original LOCC-assisted communication protocol can equivalently be rewritten by using a teleportation simulation argument [4] Section VI (see also [33]) suitably extended to continuous-variable bosonic channels [39]. Teleportation simulation in the case of a point-to-point channel can be understood as a way of reducing a sequence of adaptive protocols involving two-way LOCC to a sequence of non-adaptive protocols followed by a final LOCC [4] [33]. For all ‘teleportation-simulatable channels’ (more precisely the channels arising from the action of a generalized teleportation protocol on a bipartite state) that allow for such a reduction, an upper bound on the entanglement and secret key agreement capacity can be given by the $\varepsilon$-REE [24], because the $\varepsilon$-REE is an upper bound on the one-shot distillable key of a bipartite state [24]. Furthermore, for pure-loss bosonic channels, one can use a concise formula for the REE identified in [23]. With these techniques, an upper bound on the unconstrained capacity of a point-to-point pure-loss channel is given by the REE of the state resulting from sending an infinite-energy TMSV through the channel, explicitly calculated to be $-\log_2(1-\eta)$ [23][24].

Following [24], suppose that the original protocol generates a state $\omega_{AB_1 \cdots B_m}$ which is $\varepsilon$-close to $\hat{\Phi}_{AB_1 \cdots B_m}$:

$$
1 - \varepsilon \leq F(\omega_{AB_1 \cdots B_m}, \hat{\Phi}_{AB_1 \cdots B_m})
$$

(5)

$$
\hat{\Phi}_{AB_1 \cdots B_m} = \Phi_{AB_1}^{\otimes n} \otimes \cdots \otimes \Phi_{AB_m}^{\otimes n} \otimes \gamma_{A^m B^m}^{\otimes n},
$$

(6)

where $A^i$ and $B^i$ are subsystems of $A$ and $B_i$, respectively. Since the pure-loss bosonic QBC is covariant with respect to displacement operations (which are the teleportation corrections for bosonic channels [40]), it is teleportation-simulable [39]. Then the original broadcasting protocol described above can be replaced by the distillation of $n$ copies of $\omega_{AB_1 \cdots B_m} = N_{AB} \rightarrow B_1 \cdots B_m ((\Psi(N_S)) (\Psi(N_S))_{AA'})$ via a single LOCC. This simulation incurs an additional error that depends on the energy $N_S$ of the state $\Psi(N_S)_{AA'}$ and which vanishes in the limit $N_S \rightarrow \infty$. Let us denote the total error by $\varepsilon(N_S)$ and note that $\lim_{N_S \rightarrow \infty} \varepsilon(N_S) = \varepsilon$. Then by using the arguments of [24], we find that

$$
\sum_{B_i \in \mathcal{T}} (E_{AB_i} + K_{AB_i}) \leq \frac{1}{n} E^R_{\varepsilon}(N_S)(T^n; A^n T^n)_{\phi \otimes n}
$$

(7)

for all $T$, where $E^R_{\varepsilon}(N_S)$ denotes the $\varepsilon$-relative entropy of entanglement recalled above.

To find an upper bound on $\frac{1}{n} E^R_{\varepsilon}(N_S)(T^n; A^n T^n)_{\phi \otimes n}$ for each $T$, we use a calculation from [23][24] for a point-to-point pure-loss bosonic channel with transmittance $\eta$. In [24], it was shown that the $\varepsilon$-relative entropy of entanglement for a pure-loss channel is bounded from above by

$$
-\log_2(1-\eta) + C(\varepsilon)/n,
$$

(8)

where $C(\varepsilon) = \log_2 6 + 2 \log_2(1+\varepsilon)/[1-\varepsilon]$. Also it is critical to observe that the order of the beam splitters in Fig. 1(c) is reconfigurable by properly commuting the beam splitting operators. By using this observation and some properties of the TMSV, we obtain the following upper bound on [7]:

$$
\log_2[(1-\eta_T)/(1-\eta_B)] + C(\varepsilon)/n.
$$

(9)

The converse proof is completed by taking the limit $n \rightarrow \infty$. Note that our converse is a strong converse because there is no need to take the limit $\varepsilon \rightarrow 0$ in order to get the upper bound of $\log_2 \left[ \frac{1-\eta_T}{1-\eta_B} \right]$. See Supp. Mat. 1 for more technical details of the calculation. Considering that the converse bound coincides with the achievable rate region, this completes the proof of Theorem 1.

Discussion. The simplest pure-loss broadcast channel is a 1-to-2 broadcast channel with one sender, Alice, and two receivers, Bob and Charlie. The capacity region implied by Theorem 1 is explicitly given by

$$
E_{AB} + K_{AB} \leq \log_2[(1-\eta_C)/(1-\eta_B-\eta_C)],
$$

(10)

$$
E_{AC} + K_{AC} \leq \log_2[(1-\eta_B)/(1-\eta_B-\eta_C)],
$$

(11)

$$
E_{AB} + K_{AB} + E_{AC} + K_{AC} \leq \log_2(1-\eta_B-\eta_C),
$$

(12)
where $\eta_B$ and $\eta_C$ are the transmittances from Alice to Bob and Charlie, respectively.

It is interesting to compare the capacity with a point-to-point protocol based approach such as time-sharing. To do so, we discuss ED and QKD scenarios separately in what follows.

A naive ED protocol in QBC is the time sharing of the optimal point-to-point protocol (i.e., they split $A$ naive ED protocol in QBC is the time sharing of the optimal point-to-point protocol based approach such as time-sharing. To do so, we discuss ED and QKD scenarios separately in what follows.

In the 1-to-2 QBC setting, the simultaneous operation of the point-to-point GC09 protocol generates a pair of key rates $(K_{AB}, K_{AC}) = (I(X;Y) - I(Y;C)_{\rho}, I(X;Z) - I(Z;B')_{\rho})$, where $X$, $Y$, and $Z$ are the classical systems representing the classical data shared by Alice, Bob, and Charlie after $n$ uses of the quantum channel, $B'$ and $C'$ are the quantum systems for possible eavesdroppers which may contain the environment (usually called Eve) and the receiver who is not involved in the key (for example, $C'$ includes Charlie). $I(X;Z)$ and $I(Z;C')_{\rho}$ denote classical and quantum mutual information, respectively. According to the above observation, they can achieve these two key rates simultaneously.

Now we show how to overcome this by using a trick that is similar to the one used in the optimal distillation protocol discussed previously, i.e., sequential operation of successive state merging. Suppose Alice and Charlie first conduct point-to-point key distillation. This operation achieves the key rate $K_{AC} = I(X;Z) - I(Z;B')_{\rho}$ and also reconstructs Charlie’s classical system $Z$ at Alice’s side. After that, Bob distills the key with Alice, where Alice holds $X$ and $Z$ and Bob holds $Y$. Thus, they can achieve the key rate $K_{AB} = I(X;Y) - I(Y;C')_{\rho}$, which can be larger than that in the simple point-to-point protocol. Similarly, by changing the order of distillation, they can achieve the rate pair $(K'_{AB}, K'_{AC}) = (I(X;Y) - I(Y;C')_{\rho}, I(X;Z) - I(Z;B')_{\rho})$. Thus the achievable rate region in this protocol is given by time sharing of these two rate pairs. We refer to this protocol as the broadcast-CVQKD (BC-CVQKD). Figure 3 shows the key rate region for BC-CVQKD, which outperforms the rate regions for the simultaneous point-to-point protocol and the simple time sharing (see Supp. Mat. 3 for an explicit expression of the key rates in a pure-loss QBC). Note that the GC09 protocol is originally proposed as a noise-immune CVQKD protocol (see also [43]). It is therefore an interesting future work to apply these protocols into a noisy bosonic QBC. Also there still remain a huge gap between the key rate region in Fig. 3 and the capacity region in Fig. 2 suggesting that there may exist yet-to-be discovered clever broadcast QKD protocols.

**Conclusion.** We have established the unconstrained capacity region of a pure-loss bosonic broadcast channel for LOCC-assisted entanglement and secret key distillation. The channel we considered here is general in the sense that it includes any (no-repeater) linear optics network as its isometric extension. It could provide a useful benchmark for the broadcasting of entanglement and secret key through such channels. Furthermore, our result stimulates practical protocols for QKD or entanglement distillation over broadcast channels which overcome the time-sharing bound. As an example, we discussed a CVQKD protocol based on the proposal in [42] and show that the BC-CVQKD approach can outperform a simple application of the point-to-point strategy.

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In this section we give more details regarding the calculation of the upper bound on the ε-relative entropy of entanglement (REE) $E_{\phi}^\varepsilon(T;|\Phi^+;T\tilde{T}\rangle)_{\phi^\varepsilon}$ where $\phi_{AB^n}$ is the state with average photon number $N_{S}$. $T$ is some non-empty subset of $\{B_1,\ldots,B_m\}$ and $\tilde{T}$ is its complement.

To proceed with the calculation, it is critical to see that one can reconfigure the channel model described in Eq. (1) of the main text. Recall that the channel has $m+1$ transmittances $\eta_1,\eta_2,\ldots,\eta_m,\eta_{m+1}$. We can order these transmittances in some sequence and label it as $\eta_1,\eta_2,\ldots,\eta_m,\eta_{m+1}$. Then for any ordering, it is possible to describe the channel by a sequence of $m$ beam splitters where the $j$-th beam splitter’s transmittance is given by

$$\tilde{\eta}_j = \frac{1}{1 - \sum_{k=1}^{j} \eta_k} \eta_j.$$  

Now, for each given $T$ involving $t$ parties, consider the following specific ordering. For $\eta_i$ with $1 \leq i \leq t$, assign $\tilde{\eta}_i \in \tilde{T}$, $\eta_{t+1} = \eta_E$, and for $\eta_i$ with $i > t+1$, assign $\eta_i \in T$. Then the transmittance of the $t+1$ beam splitter is $\tilde{\eta}_{t+1} = \eta_T/(1 - \eta_T)$ where $\eta_T = \sum_{B_i \in T} \eta_i$, and $\eta_T = \sum_{B_i \in \tilde{T}} \eta_i$ (see Fig. 4(a)).

In Fig. 4(a), the parties $A_T$ and $\tilde{T}$ are separated by a pure-loss channel with transmittance $\eta_T/(1 - \eta_T)$. To simplify the picture, we consider the local unitary operations at each party, $A_T$ and $\tilde{T}$ (note that such unitaries do not change $E_{\phi}^\varepsilon(T;|\Phi^+;T\tilde{T}\rangle)_{\phi^\varepsilon}$). On $T$, just undo all the beam splitters. Then as shown in Fig. 4(b), the system is described by $B_T$, which is the output sent from $A$, and tensor products of vacuum that can be ignored in the rest of the calculation.

For the other side, $A_T$, we use the fact that our input state from the sender is a TMSV. Let $U_T$ denote the mode before the $(t+1)$-th beam splitter and let $\phi_{A_T}^\varepsilon$ be the TMSV in which only the first $t$ beam splitters are applied (see Fig. 4(b)). Observe that it is a pure state and its marginal $\phi_{B_T}^\varepsilon$ is a thermal state with average photon number $(1 - \eta_{T})N_{S}$. Combining these two observations, we can conclude that the state has the following Schmidt decomposition:

$$|\phi''\rangle_{A_TB_T} = \sum_{m=0}^{\infty} \sqrt{\lambda_m ((1 - \eta_T)N_S)} |\varphi_m\rangle_{A_T}|m\rangle_{B_T},$$

where $\lambda_m(N)$ is a thermal distribution, $|m\rangle$ is a photon number state, and $\{|\varphi_m\rangle_{A_T}\}_m$ is some orthonormal basis. Since $\{|\varphi_m\rangle_{A_T}\}_m$ is an orthonormal set, there exists a local unitary operation acting on $A_T$ such that

$$U_{A_T} : |\varphi_m\rangle_{A_T} \rightarrow |m\rangle_A|\text{aux}\rangle_T,$$

where $|\text{aux}\rangle$ is some constant auxiliary state. Then we have

$$U_{A_T}|\phi''\rangle_{A_TB_T} = |\text{aux}\rangle_T |\Psi((1-\eta_T)N_S)\rangle_{AB_T}$$

\begin{equation}
\equiv |\phi''\rangle_{A_TB_T},
\end{equation}

where $|\Psi((1-\eta_T)N_S)\rangle$ is a TMSV with average photon number $(1 - \eta_{T})N_{S}$.

Let $\phi_{A_TB_T} = \mathcal{L}_{B_T}^{\eta_{t+1}}$ be the $\varepsilon$-REE between $A_T$ and $T$, we have $E_{\phi}^\varepsilon(B_T;A^n\bar{T}^n)_{\phi^\varepsilon} = E_{\phi}^\varepsilon(T\bar{T};A^n\bar{T}^n)_{\phi^\varepsilon}$. Moreover, $E_{\phi}^\varepsilon(B_T^n;A^n\bar{T}^n)_{\phi^\varepsilon}$ is the $\varepsilon$-REE for the TMSV with $(1 - \eta_T)N_S$ followed by $\mathcal{L}_{B_T}^{\eta_{t+1}}$ where $\tilde{\eta}_{t+1} = \eta_T/(1 - \eta_T)$. Then by using the result in [24], the REE bound for $N_S \rightarrow \infty$ turns out to be

$$\sum_{B_i \in T} (E_{AB_i} + K_{AB_i}) \leq \frac{1}{n} E_{R}^{\varepsilon(N_S)}(T;A^n\bar{T}^n)_{\phi^\varepsilon}$$

\begin{equation}
\leq \log_2 \left[ \frac{1}{1 - \frac{1 - \eta_T}{1 - \eta_T}} \right] + C(\varepsilon)/n.
\end{equation}

\begin{equation}
= \log_2 \left[ \frac{1 - \eta_T}{1 - \eta_T} \right] + C(\varepsilon)/n.
\end{equation}

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= \log_2 \left[ \frac{1 - \eta_T}{1 - \eta_T} \right] + C(\varepsilon)/n.
\end{equation}

\begin{equation}
= \log_2 \left[ \frac{1 - \eta_T}{1 - \eta_T} \right] + C(\varepsilon)/n,
\end{equation}

where $\eta_S = \eta_T + \eta_T$.

Supplementary Material 2: The rate sum calculation for the 1-to-m symmetric pure-loss broadcast channel

Here we calculate the achievable rate sum in the 1-to-m symmetric pure-loss channel with equal transmittance $\eta_m \equiv \tilde{\eta}$ to each receiver. We restrict the distillation scenario such that all receivers achieve the same rate. Though only the entanglement distillation is considered in the main text, the same achievable rate sum holds for secret key generation. Thus we here treat both entanglement and secret key. Let $E_{\text{sum}} = \max \sum_i E_{AB_i}$ and $K_{\text{sum}} = \max \sum_i K_{AB_i}$ be the rate sum of the entanglement and secret key distillations,
respectively. Then we show that the maximum achievable $E_{\text{sum}} + K_{\text{sum}}$ is given by

$$E_{\text{sum}} + K_{\text{sum}} = -\log_2(1 - \eta).$$  \hfill (18)

whereas the achievable rate sum for the time sharing strategy is bounded by

$$E_{\text{sum}}^{\text{ts}} + K_{\text{sum}}^{\text{ts}} \leq -\log_2(1 - \eta/m).$$  \hfill (19)

Let us first consider the time-sharing bound. The point-to-point capacity for the sender and each receiver is given by $-\log_2(1 - \eta/m)$. To achieve the same rate for each receiver, the receiver should time share the channel equally. Then each receiver achieves the rate of $-\frac{1}{m} \log_2(1 - \eta/m)$ and sum up for the $m$ receivers, the rate sum is given by $-\log_2(1 - \eta/m)$.

For the maximum achievable rate sum, consider the capacity region for the symmetric pure-loss channel. According to Theorem 1, the capacity region for a pure-loss symmetric QBC is given by a set of inequalities:

$$\sum_{b_i \in \mathcal{T}} E_{AB_i} + K_{AB_i} \leq \log_2 \frac{1 - (m - |\mathcal{T}|)\eta}{1 - m\eta}. \hfill (20)$$

Applying the restriction that all the receivers achieve the same rate $E_{AB}$ and $K_{AB}$, the above rate region expression is reduced to

$$E_{AB} + K_{AB} \leq \frac{1}{l} \log_2 \frac{1 - (m - l)\eta}{1 - m\eta},$$

for all integers $l$ satisfying $1 \leq l \leq m$. Thus the maximum rate sum is bounded by

$$E_{\text{sum}} + K_{\text{sum}} \leq \frac{m}{l} \log_2 \frac{1 - (m - l)\eta}{1 - m\eta},$$

for all $l$. Then to prove Eq. (18), it is sufficient to show that the right hand side of (22) is monotonically decreasing with respect to $l$. Let $f(x) = \log_2\frac{1 - (m - x)\eta}{1 - m\eta}$ for real $x$ with $0 < x \leq 1$. Then

$$f'(x) = \frac{m}{x \ln 2} \left( \frac{\eta}{1 - (m - x)\eta} - \frac{1}{x} \ln \left[ \frac{1 - (m - x)\eta}{1 - m\eta} \right] \right).$$

Since $\frac{1 - (m-x)\eta}{1 - m\eta} \leq 1$, we have

$$\ln y = 2 \left[ \frac{y - 1}{y + 1} + \frac{1}{3} \left( \frac{y - 1}{y + 1} \right)^3 + \frac{1}{5} \left( \frac{y - 1}{y + 1} \right)^5 + \cdots \right] \geq \frac{2(y - 1)}{y + 1}.$$  \hfill (24)

Applying this into the second term of (23), we have

$$f'(x) \leq \frac{m}{x \ln 2} \left( \frac{\eta}{1 - (m - x)\eta} - \frac{2\eta}{2(m\eta) + 2\eta x} \right) = \frac{2m\eta^2}{(2\eta)(1 - m\eta + x\eta)(2(1 - m\eta) + x\eta)} \leq 0.$$  \hfill (25)

Note that $m\eta = \eta \leq 1$. Therefore, the bound (22) is tightest at $l = m$ implying Eq. (18).

---

**FIG. 5.** (a) CVQKD with squeezed state [42] and (b) its’ extension to the 1-to-2 broadcast-CVQKD.

**Supplementary Material 3: Broadcast channel quantum key distribution**

In this section, we give details of the broadcast channel continuous variable quantum key distribution (BC-CVQKD). Our protocol is based on the CVQKD protocol with squeezed state proposed by García-Patrón and Cerf (GC09) [42]. Here we briefly review the GC09 protocol and then extend it to our BC-CVQKD.

Figure 5(a) shows the entanglement-based description of the GC09 protocol (in practice, it is implemented by a randomly modulated single-mode squeezed state). Alice prepares a two-mode squeezed vacuum (TMSV) state in modes $A$ and $A'$, where mode $A$ is measured by a homodyne detector with a random choice of the quadrature basis and mode $A'$ is sent to Bob through a point-to-point quantum channel, possibly attacked by Eve. Bob measures the received state by a heterodyne detector which simultaneously measures two quadratures. After the transmission of $n$ pulses of the TMSV, Alice announces her basis choice for each measurement to Bob through a public classical channel. Bob uses the measurement outcomes for the corresponding bases to make the classical postprocessing, called reverse reconciliation (RR), in order to distill the secret key (thus Bob’s measurement is effectively modeled by a 50% pure-loss followed by a homodyne detector [42]). Let $X$ and $Y$ be the measurement outcomes for Alice and Bob that are used to distill the key, and let $E$ be the quantum system held by Eve, representing the environment of the quantum channel. To simplify the setting, hereafter we consider a pure-loss bosonic channel as the quantum channel. The key generation rate between Alice and Bob is then given by

$$K_{AB} = I(X; Y) - I(Y; E)_{\rho}.$$  \hfill (26)

We extend this GC09 protocol to the broadcast channel scenario. Figure 5(b) depicts the simplest case where one sender, Alice, sends quantum signal to two receivers, Bob and Charlie, through a 1-to-2 pure-loss bosonic broadcast channel. The goal of the protocol is to share bipartite secret keys $K_{AB}$ (Alice-Bob) and $K_{AC}$ (Alice-Charlie). Here we consider a conservative scenario where the key is secure against any third parties. For example, to distill $K_{AB}$, the environment of the channel (Eve) and Charlie are both regarded as eavesdroppers.

Let us first describe a simple operation of the point-to-point GC09 protocol. Alice prepares the TMSV in modes $A$ and $A'$,
measures mode $A$ by a homodyne receiver and sends mode $A'$ to Bob and Charlie through a pure-loss broadcast channel. Bob and Charlie make heterodyne measurements. Then they separately do the key distillation with Alice. Since the key distillation is performed on classical data (measurement outcomes), the key distillation for Alice-Bob and Alice-Charlie can be performed for the same measurement outcomes. Let $X$, $Y$, and $Z$ be the measurement outcomes at Alice, Bob, and Charlie, to be used for the key distillation. Then they can achieve the key rate region

$$K_{AB} \leq I(X;Y) - I(Y;CE)_\rho,$$

$$K_{AC} \leq I(X;Z) - I(Z;BE)_\rho.$$  \hfill (27)

Now the question is if we can surpass the above rate region by an explicit QKD protocol. In the following, we describe the protocol where Charlie cooperate to Alice and Bob to distill $K_{AB}$ and Bob also helps to distill $K_{AC}$, which we call broadcast-CVQKD (BC-CVQKD) protocol. Again Alice sends one part of her TMSVs to Bob and Charlie and then they share $X$, $Y$, and $Z$. Then one of the receivers, for example Charlie, makes the standard point-to-point RR with Alice. After that they can achieve the secret key rate of $K_{AC} = I(X;Z) - I(Z;BE)_\rho$. Note that during this process, Alice reconstructs Charlie’s information $Z$ and thus she holds $X$ and $Z$ as her classical information. Then Bob makes the RR with Alice where Alice can use $X$ and $Z$ as her classical information. Thus they can establish the key rate $K_{AB} = I(XZ;Y) - I(Y;CE)_\rho$. The key rate difference between this BC-CVQKD and the simple point-to-point protocol described above is $I(XZ;Y) - I(X;Y) = I(Y;Z|X) \geq 0$ where the inequality holds since the conditional mutual information is a non-negative function. Thus our BC-CVQKD protocol can always achieve better or at least the same performance as the simple point-to-point-QKD based protocol. In summary, the achievable rate pair is

$$K_{AB} = I(XZ;Y) - I(Y;CE)_\rho,$$

$$K_{AC} = I(X;Z) - I(Z;BE)_\rho.$$  \hfill (28)

Similarly, when Bob first makes the RR, they can achieve the rate pair

$$K_{AB} = I(X;Y) - I(Y;CE)_\rho,$$

$$K_{AC} = I(XY;Z) - I(Z;BE)_\rho.$$  \hfill (29)

Thus the achievable rate region of our protocol is given by the time sharing of (28) and (29).

Each of the information quantities in (28) and (29) are given as follows. The classical mutual informations are derived by calculating the Shannon entropies, $H(X)$, $H(Y)$, $H(Z)$, $H(XY)$, $H(XZ)$, and $H(XYZ)$. Since TMSV and pure-loss channel are Gaussian state and channel, respectively, we use the covariance matrix approach (here we basically use the definition in (44)). Let $\mu$ be the average photon number of the TMSV and $v = 2\mu + 1$. The quantum state held by Alice, Bob, and Charlie before their measurements is described by the covariance matrix

$$\gamma_{ABC} = \begin{bmatrix} a & d & f & 0 & 0 & 0 \\ d & b & e & 0 & 0 & 0 \\ f & e & c & 0 & 0 & 0 \\ 0 & 0 & 0 & a & -d & -f \\ 0 & 0 & 0 & -d & b & e \\ 0 & 0 & 0 & -f & e & c \end{bmatrix},$$  \hfill (30)

where $a = v$, $b = \frac{1}{2}\gamma_B (v - 1) + 1$, $d = \frac{1}{2}\eta_B (v^2 - 1)$, $e = \frac{1}{2}\sqrt{\eta_B}\gamma_C (v - 1)$, and $f = \sqrt{\frac{1}{2}\eta_B (v^2 - 1)}$. Note that the effective 50% loss at the heterodyne measurements are already included. The covariance matrix for the homodyne measurements at Alice, Bob, and Charlie is given by

$$\gamma_M = \begin{bmatrix} e^{-2r}I_3 & 0 & 0 \\ 0 & e^{2r}I_3 & 0 \\ 0 & 0 & e^{2r}I_3 \end{bmatrix},$$  \hfill (31)

where $I_3$ is a $3 \times 3$ identity matrix and $r \to \infty$. The probability density function after the measurement is given by

$$P_{XYZ}(x,y,z) = \frac{1}{\pi^{3/2}\det(\gamma_{ABC} + \gamma_M)} \exp \left[ -x^T \frac{1}{(\gamma_{ABC} + \gamma_M)^{-1}} x \right],$$  \hfill (32)

where $x = [x,y,z]^T$ and $\det \gamma$ is the determinant of $\gamma$. Then we get the differential entropy of this as

$$H(XYZ) = \frac{1}{2} \log_2(\pi e)^3 \det \gamma_{ABC}$$

\hfill (33)

$$= \left( \frac{1}{2} \log_2(\pi e)^3 \right) \left\{ \left( 1 - \frac{\eta_B}{2} \right) (v - 1) + 1 \right\}.$$  \hfill (33)

Since the covariance matrix of the marginal system is obtained by simply taking a corresponding submatrix of $\gamma_{ABC}$, we get the necessary differential entropies in a similar way as

$$H(X) = \frac{1}{2} \log_2(\pi e)v,$$

$$H(Y) = \frac{1}{2} \log_2(\pi e) \left\{ \frac{\eta_B}{2} (v - 1) + 1 \right\},$$

$$H(Z) = \frac{1}{2} \log_2(\pi e) \left\{ \frac{\eta_C}{2} (v - 1) + 1 \right\},$$

$$H(XY) = \frac{1}{2} \log_2(\pi e)^2 \left\{ \left( 1 - \frac{\eta_B}{2} \right) (v - 1) + 1 \right\},$$

$$H(XZ) = \frac{1}{2} \log_2(\pi e)^2 \left\{ \left( 1 - \frac{\eta_C}{2} \right) (v - 1) + 1 \right\}. $$  \hfill (37)

Below, we calculate the quantum mutual information $I(Y;CE)_\rho = H(CE)_\rho - H(CE|Y)_\rho$. Since $C$ and $E$ are regarded as one system, it is useful to rewrite the channel in Fig. 5(b) as the one in Fig. 5(c). Here after system $CE$ is sometimes denoted as $C'$ for simplicity. The covariance ma-
Matrix for the quantum system held by Charlie-Eve and Bob (after the 50% loss in the heterodyne) is

\[
\gamma_{C'B} = \begin{pmatrix}
    a' & c' & 0 & 0 \\
    c' & b' & 0 & 0 \\
    0 & 0 & a' & -c' \\
    0 & 0 & -c' & b'
\end{pmatrix},
\] (39)

where \( a' = (1 - \eta_B)(v - 1) + 1, b' = \frac{2\eta_B}{v}(v - 1) + 1, c' = -\sqrt{\frac{2\eta_B}{v}(1 - \eta_B)}(v - 1) \). Since the von Neumann entropy of a single-mode Gaussian state is given by \( g((\lambda - 1)/2) \), where \( g(x) = (x + 1) \log_2(x + 1) - x \log_2 x \) and \( \lambda \) is a symplectic eigenvalue of the covariance matrix, we have

\[
H(CE)_\rho = g \left( \frac{(1 - \eta_B)(v - 1)}{2} \right).
\] (40)

Eve’s quantum state conditioned on Bob’s measurement is given as follows. By changing the order of the matrix entry, \( \gamma_{C'B} \) is represented as

\[
\gamma_{C'B} = \begin{pmatrix}
    A & C \\
    C^T & B
\end{pmatrix},
\] (41)

where \( A = a'I_2, B = b'I_2 \) and \( I_2 \) is a 2 \times 2 identity matrix, and

\[
C = \begin{pmatrix}
    c' & 0 \\
    0 & -c'
\end{pmatrix}.
\] (42)

Now we apply a homodyne measurement on Bob’s state. The covariance matrix for a single-mode homodyne measurement is given by

\[
\gamma_M = \begin{bmatrix}
    e^{-2r} & 0 \\
    0 & e^{2r}
\end{bmatrix},
\] (43)

with \( r \to \infty \). Then Eve’s state in \( C' \) conditioned on Bob’s homodyne measurement is described by

\[
\gamma_{C'|y} = A - C^T \frac{1}{B + \gamma_M} C = \begin{pmatrix}
    \alpha \\
    0
\end{pmatrix},
\] (44)

where

\[
\alpha = \frac{(1 - \eta_B)(v - 1)}{2\eta_B(v - 1) + 1} + 1,
\] (45)

and \( \beta = (1 - \eta_B)(v - 1) + 1 \). Note that \( \gamma_{C'|y} \) is independent on the homodyne measurement outcome \( y \), i.e. same for any \( y \). As a consequence, we get the conditional quantum entropy as

\[
H(CE|Y)_\rho = g \left( \frac{\sqrt{\alpha\beta} - 1}{2} \right).
\] (46)

Combining (33)–(38), (40), and (46), one can compute (28) and (29). The other mutual information, \( I(Z;BE)_\rho \) is also obtained in a similar way.