Cobordism of symplectic manifolds and asymptotic expansions

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Abstract. The cobordism ring \( \mathcal{B}_* \) of symplectic manifolds defined by V. L. Ginzburg is isomorphic to the Pontrjagin ring of complex-oriented manifolds with free circle actions. This provides an interpretation of the formal group law of complex cobordism as a composition-law on certain asymptotic expansions. An appendix discusses some related questions about toric manifolds.

Introduction

The theory of geometric quantization is concerned with smooth manifolds \( V \) bearing a complex line bundle \( \mathcal{L} \) with connection \( \nabla_\mathcal{L} \), such that the Chern-Weil form \( \omega_\mathcal{L} \) of the connection is symplectic, i.e. a nondegenerate closed two-form representing an integral class in \( \mathcal{H}^2_{dR}(V) \). A compatible almost-complex structure on a symplectic manifold is an endomorphism \( j \) with square \(-1\) of the tangent bundle of \( V \), such that the bilinear form \( g_\mathcal{L}(-,-) = \omega_\mathcal{L}(j-,-) \) is a Riemannian metric; these structures exist because the unitary group is a retract of the automorphism group of a real symplectic vector space. For the purposes of this note, the quadruple \( V := (V, \mathcal{L}, \nabla_\mathcal{L}, j) \) will be called a prequantized manifold, though this usage is not quite standard.

Analysis on this class of manifolds is very rich [cf. e.g. 17-19] but I will be concerned here with the reflection in topology of some aspects of that subject. I suggest, in particular, that the formal group law on complex cobordism [introduced by thirty years ago [20] by Novikov, which has led to such spectacular advances in our understanding of homotopy theory [22]], has a natural analytic interpretation in terms of asymptotic expansions. The first section of this paper summarizes some well-known facts about heat kernels, which provide a useful class of examples of expansions; the second is an account of the structure of the cobordism ring defined by V.L. Ginzburg [10] for symplectic manifolds, and the third contains some closing remarks. I would like to thank V.L. Ginzburg for very interesting discussions about his work.

It is a pleasure to dedicate this paper to Professor Sergei Petrovich Novikov, in view of his lifelong interest in the frontier between geometry and analysis. This Web version differs from the original, which appears in the Novikov anniversary.

1991 Mathematics Subject Classification. Primary 81S10, Secondary 57R90, 14M25.
The author was supported in part by NSF grant 9504234.
volume of the Proceedings of the Steklov Institute, by the addition of an appendix concerned with related questions about toric manifolds.

§1 Topological invariants of prequantized manifolds

A prequantized manifold, in the terminology adopted here, has a preferred metric as part of its structure, and thus permits the usual constructions of Riemannian geometry. The twisted signature operator on $\Lambda(T^*V) \otimes \mathcal{L}$ defined by a Dirac operator [in the sense of §3.6], i.e. $d + \delta$ acting on forms graded by the Hodge operator] associated to the bundle of Clifford algebras constructed from the Riemannian metric is a particularly accessible example. The Laplace-Beltrami operator $\Delta_L$ on sections of this bundle [§2.4] has a well-behaved heat kernel on $V \times V$, and when $V$ is closed and compact of dimension $2d$, the restriction of that kernel to the diagonal possesses an asymptotic expansion

$$\exp(-t^{-1}\Delta_L) \sim \sum_{k \geq 0} t^{-1} k_i(L) \, d\text{vol}_L$$

as $t \to \infty$; the coefficients $k_i(L)$ are homogeneous local functionals of the metric and $\omega_L$, of great analytic and geometric interest [§4.1]. A theorem of McKean and Singer implies that the index of $\Delta_L$ equals the integral over $M$ of this differential form; after some simplification we recover the Atiyah-Singer formula

$$\text{ind} \, \Delta_L = (\text{ch}(L)L(V))[V],$$

where $L(V)$ is the Hirzebruch $L$-genus, and $[V]$ is the orientation class of $V$.

Now if $V = (V, L, \nabla_L, j)$ is prequantized, and $n$ is a positive integer, then

$$[n](V) := (V, L^\otimes n, \nabla_L^\otimes n, j)$$

is also a prequantized manifold. The behavior of the analytic properties of prequantized manifolds under this ‘Adams operation’ is of considerable interest in the theory of geometric quantization, where $n \to \infty$ is known as the ‘semiclassical limit’, cf. [§34]. The Chern class of $L^\otimes n$ is $n\omega_L$, so $g_{L^\otimes n} = ng_L$; the symbol $\sigma((L^\otimes n))$ thus equals $n^{-1}\sigma(L)$, and homogeneity implies that $k_i(L^\otimes n)$ is polynomial in $n$, while $d\text{vol}_{L^\otimes n} = n^n d\text{vol}_L$. It is a corollary that $\text{ind} \, \Delta_{L^\otimes n}$ is polynomial of degree $d$ in $n$; it is an analogue in some sense of the Hilbert polynomial in algebraic geometry.

This invariant has some interesting properties: it is additive under disjoint unions of manifolds, and it is multiplicative under cartesian product; indeed this is true locally of the heat kernels themselves [§1.8]. Since it is a characteristic number, it is also a cobordism invariant, in a sense which will be made precise below; and its relation to the theory of Feynman path-integrals is rather well-understood [§3]. I have described this example because it is typical of a large class of invariants constructible from heat kernel expansions; but recent striking work of Kórpas and Uribe [17] suggests that the Spin$^c$ Dirac operator has even more natural analytic properties.

The main technical result in this paper is a proof that Ginzburg’s symplectic cobordism is a (cocommutative) Hopf algebra, isomorphic to the complex bordism ring $MU_*BT$ of manifolds with free circle actions; an immediate corollary is that a certain class of additive functionals $\Phi$ on prequantized manifolds (those such that $\Phi([n](V))$ possesses a formal asymptotic expansion in $n$) forms a (cocommutative) Hopf algebra, Cartier dual to Ginzburg’s. This suggests as an interesting question
for the future, the possible existence of some similar kind of algebraic structure on
the heat kernel invariants themselves, rather than on their global integrals.

§2 Cobordism of symplectic and prequantized manifolds

2.1 A (generalized) symplectic manifold is a pair \((V, \omega)\), with \(V\) an oriented manifold
and \(\omega\) a maximally nondegenerate integral closed 2-form on \(V\). This is the usual
definition, if \(V\) is even-dimensional; if not, it entails that the kernel of \(\omega\), viewed
as a homomorphism from the tangent bundle of \(M\) to its cotangent bundle, has an
everywhere one-dimensional kernel. \(\mathcal{B}_*\) denotes the graded ring of even-dimensional
closed compact symplectic manifolds, under the equivalence relation defined by
cobordism of such structures.

Ginzburg [10 § 1.7] defines a homomorphism

\[
\gamma : \mathcal{B}_* \to MU_* BT
\]

from his symplectic cobordism ring to the complex bordism ring of the classifying
space for complex line bundles, which assigns to \((V, \omega)\) a triple \((V, L, j)\), consisting
of a complex line bundle \(L\) over \(M\) with Chern class represented by \(\omega\), together with
an almost-complex structure \(j\) on the tangent bundle of \(V\), such that the bilinear
form

\[
\omega(j-, -)
\]

is positive definite. He then shows that this homomorphism is injective, and that
it becomes an isomorphism after tensoring with the rationals. It will be convenient
here to modify this construction very slightly, by taking \(BT\) to be a classifying
space for bundles with connection [19]; the proof that \(\gamma\) is an isomorphism then
implies that the forgetful map from the cobordism ring of prequantized manifolds
to \(\mathcal{B}_*\) is an isomorphism as well.

The multiplicative monoid \(\mathbb{Z}_\times\) of integers acts by ring automorphisms of \(\mathcal{B}_*\),
inducing from the action \([n] : z \mapsto z^n, z \in T = \{z \in \mathbb{C}||z| = 1\}\)
on line bundles, and the homomorphism \(\gamma\) respects this action. The subring of \(\mathbb{Z}_\times\)
invariants of \(MU_* BT\) is the cobordism ring \(MU_*\) of a point, so there is an induced
monomorphism

\[
\gamma^0 : \mathcal{B}_0^* \to MU_*
\]

It is perhaps surprising that this homomorphism has a relatively large image:

2.2 Proposition The cobordism classes \(\mathbb{C}P_n \in MU_*\) of the complex projective
spaces lie in the image of \(\gamma^0\).

Proof: We will construct polynomials in symplectic manifolds of the form \((\mathbb{C}P_n, k\omega)\),
where \(n \geq m\) and \(k\) is a suitable positive integer, together with a complex two-torus
\(X\) carrying a standard symplectic structure, which map to the class of \(\mathbb{C}P_n \in MU_{2n}\);
here $\omega$ is the standard (Fubini-Study) form. The argument uses characteristic numbers: if $(V, \omega)$ is a compact closed symplectic manifold,
\[ \sum_I (c_I \omega^{-|I|}) [V] t^{|I|} \omega^{|I|} \]
will denote its image under the composition of $\gamma$ with the Chern homomorphism
\[ MU_* BT \to \text{Hom}(H^* MU, H_* BT) = \mathbb{Z}[t_n, \omega_n \mid n \geq 1], \]
where $\omega_n$ is the $n$th divided power of $\omega$. $I = i_1, i_2, \ldots$ is a multiindex of weight $|I| = \sum ki_k$, $c_I$ is the polynomial in Chern classes of $V$ defined by the monomial symmetric function $m_I$, and $t^I = \prod_{k \geq 1} t_k^{i_k}$. For example,
\[ (CP_1, \omega) \mapsto 2t_1 + \omega, \]
\[ (CP_2, \omega) \mapsto 3t_1^2 + 3t_2 + 3t_1\omega + \omega_2, \]
\[ (CP_3, \omega) \mapsto 4t_1^3 + 12t_1t_2 - 20t_3 + (6t_1^2 + 2t_2)\omega + 4t_1\omega_2 + \omega_3, \]
eq etc. The Chern homomorphism is injective on the complex cobordism ring, and it defines an identification of the polynomial $\mathbb{Q}$-algebra generated by the $CP_n$'s with the ring $\mathbb{Q}[t_n \mid n \geq 1]$. In particular, a polynomial in the $t$'s with rational coefficients can be expressed uniquely as a sum of products $CP^I = \prod CP_k^{i_k}$. We now show inductively that we can construct a product of projective spaces, with standard symplectic structures, such that all Chern numbers involving the symplectic class vanish: if $CP_m$ has been shown to lie in $B_n^0$ when $n > m$, we can write
\[ \text{(Chern polynomial of)} \ CP_n + \sum_{n>|I|} b_I CP^I \omega_{n-|I|} \]
for the total symplectic Chern polynomial of $(CP_n, \omega)$, with coefficients $b_I \in \mathbb{Q}$. Let $k$ be the common denominator of the collection
\[ \frac{b_I}{(n-|I|)!} \]
of rational numbers, where $n > |I|$; then
\[ CP_n = (CP_n, k\omega) - \sum_{n>|I|} k^{n-|I|-1} \frac{kb_I}{(n-|I|)!} CP^I X^{n-|I|} \]
expresses $CP_n$ as a sum of elements of $B_\ast$. For example,
\[ CP_1 = (CP_1, \omega) - X, \]
\[ CP_2 = (CP_2, 2\omega) - 3CP_1 X - 2X^2, \]
\[ CP_3 = (CP_3, 6\omega) - (4CP_2 - 6CP_1^2) X - 36CP_1X^2 - 36X^3, \]
etc. This completes the proof of the proposition.

From now on, I will identify the class of the ‘classical’ projective space $CP_n \in MU_{2n}$ with the corresponding class in $B_{2n}^0$; it should be distinguished from the class of a ‘quantum’ projective space $(CP_n, k\omega)$.

**2.3** We now need some standard results about the complex bordism and cobordism of $BT$. If $(V, \mathcal{L})$ is a complex-oriented manifold $V$ together with a complex line bundle $\mathcal{L}$ over it, then Quillen’s Euler class [14] is the cobordism element
\[ q(\mathcal{L}) = [s^{-1}(0_\mathcal{L})] = V_0 \in MU^2(V) \]
defined by the inverse image of the zero-section of $\mathcal{L}$, under a generic smooth section $s$. This defines a hyperplane section homomorphism

$$q : MU_*BT \to MU_{*-2}$$

of $MU_*$-modules, which can also be interpreted as the Kronecker product of $(V, \mathcal{L})$ with the first Chern class in $MU^*BT$; more generally, there are homomorphisms

$$q^i : MU_*BT \to MU_{*-2i}$$

defined as Kronecker products with the $i$th power of the Chern class; in geometric terms this is just the $i$-fold transversal intersection $V_0 \cap \cdots \cap V_0$ of hyperplane sections of $V$. Since $MU_*BT$ and $MU_*BT$ are dual under the Kronecker pairing, there is a basis $b_k$ of the latter module characterized by the formula

$$q^i(b_k) = \delta_{i,k}.$$ 

The $i$th power of the Chern class in $MU_*^{2i}BT$ can be interpreted as the complex projective subspace of $\mathbb{C}P_\infty$ of complex codimension $i$. Let

$$\beta_k : \mathbb{C}P_k \to \mathbb{C}P_\infty$$

denote the element of $MU_{2k}BT$ defined by $(\mathbb{C}P_k, \omega)$; then

$$q^i(\beta_k) = \mathbb{C}P_{k-i},$$

this being zero if $i > k$. It follows that

$$\beta_i = \sum_k \mathbb{C}P_{i-k}b_k,$$

which implies that the classes $b_i$ lie in the subring $\mathcal{B}_*$ of $MU_*BT$: for $\beta_i$ certainly lies in $\mathcal{B}_*$, while $\mathbb{C}P_0 \in \mathcal{B}_0$ by the proposition above. The assertion is thus a consequence of the invertibility of the upper-triangular matrix $\mathbb{C}P_{i-j}$.

The bordism ring of $BT$ (with the Pontrjagin multiplication) is known to be Cartier dual to $MU^*(BT)$: if

$$b(u) = \sum_{n \geq 0} b_n u^n \in MU_*[[u]],$$

then

$$(*) \quad b(u)b(v) = b(u + Fv)$$

in $MU_*[[u, v]]$, where $u + Fv = F(u, v)$ is the formal group sum in $MU_*[[u, v]]$, cf. [23 §3.4]. Now, however, we know that this equation holds in $\mathcal{B}_*$. If we write $\beta(u)$ for the corresponding generating function for the $\beta$’s, then

$$b(u) = \mathbb{C}P(u)^{-1}\beta(u),$$

with $\mathbb{C}P(u)$ the generating function for the classical projective spaces. In this notation, the diagonal for the Hopf algebra $\mathcal{B}_*$ is characterized by

$$\mathbb{C}P(u)\Delta\beta(u) = \beta(u) \otimes \beta(u).$$

2.4 Proposition The completed localization

$$\mathcal{B}_*[[\mathbb{C}P^{-1}_1]] \to (MU_*BT)[[\mathbb{C}P^{-1}_1]]$$

of $\gamma$ is an isomorphism.

Proof: The element

$$b_1 = \beta_1 - \mathbb{C}P_1 = X$$
is a unit in $\mathcal{B}_*(\mathbb{C}P^1)$, so the power series $b(u) - 1$ possesses a formal inverse $\tilde{b}(u)$ with coefficients in $\mathcal{B}_*(\mathbb{C}P^1)$. Formula (*) can thus be restated as

$$\tilde{b}(b(u))b(v) - 1 = u + Fv;$$

it follows that the coefficients of the formal group law $F(u, v)$, which are a priori elements of $MU_*$, all lie in $\mathcal{B}_*(\mathbb{C}P^1)$. But the coefficients of the formal group law [4] generate $MU_*$, so $MU_*$ is a subring of $\mathcal{B}_*(\mathbb{C}P^1)$. This latter ring is then a subring of $(MU_*BT)(\mathbb{C}P^1)$ which contains $MU_*([\mathbb{C}P^1])$ as well as the classes $b_n$ (by the preceding paragraph); but the $b$’s are a basis for $MU_*BT$ over $MU_*$, so $\mathcal{B}_*(\mathbb{C}P^1)$ contains the full ring $(MU_*BT)(\mathbb{C}P^1)$, and is thus isomorphic to it.

2.5 Theorem The homomorphism

$$\gamma : \mathcal{B}_* \to MU_*BT$$

is an isomorphism.

Proof: Since $\mathbb{C}P^1$ is $\mathbb{Z}^*$-invariant, the completion $\mathcal{B}_*(\mathbb{C}P^1)$ inherits a $\mathbb{Z}^*$-action, with invariant subring $\mathcal{B}_0(\mathbb{C}P^1)$ contained in $MU_*([\mathbb{C}P^1])$. The argument above shows that this inclusion is in fact an isomorphism, and to complete the proof of the theorem it is enough to show that

$$\gamma^0 : \mathcal{B}_0 \to MU_*$$

is an isomorphism.

Now consider the map induced by this homomorphism on modules of indecomposables: let $I_B$ be the (graded) ideal of positive-dimensional elements of the domain, and $I_{MU}$ the corresponding ideal in the range; then by Ginzburg’s theorem the induced homomorphism

$$I_B/I_B^2 \to I_{MU}/I_{MU}^2$$

becomes an isomorphism after tensoring with the rationals. Since both of these quotient modules are free of rank one in even dimensions, and zero otherwise, this quotient homomorphism is specified by one nonzero integer (its value on a generator) for each even dimension. On the other hand this map is also an isomorphism after completing with respect to $\mathbb{C}P^1$; thus these integers are all units.

In the time since the first draft of this paper was written, an independent proof for this theorem has been found by A. Baker [2], based on symplectic structures on Milnor’s hypersurface generators.

§3 A Hopf algebra of functionals

The existence of a Hopf algebra structure on the cobordism ring of symplectic manifolds is in some ways rather surprising. This section is concerned with some consequences of this result, expressed in terms not of symplectic manifolds but rather of functionals on them. Perhaps it is best to start with a central example: the formal series

$$h(V) = \sum_{n \geq 1} \frac{\mathbb{C}P^{n-1}}{n} \cap^n V_0$$

[with $\cap^n V_0$ interpreted as the cobordism class of an $n$-fold transversal intersection] defines a homomorphism

$$\mathcal{B}_* \to \mathcal{B}_*^0 \otimes \mathbb{Q}$$
of degree two of graded abelian groups, such that

i) if \( M \in \mathcal{B}_0^* \) is classical, and \( V \in \mathcal{B}_* \) is quantum, then
\[
h(MV) = Mh(V) ,
\]
while

ii) \( h(V, n\omega) = nh(V, \omega) \).

The first of these properties holds for many interesting linear functionals on \( \mathcal{B}_* \), for example the twisted signature considered in the introduction; but the second is rather special. In fact it is a restatement of Quillen’s theorem that the module \( \mathcal{B}_*^* \) of graded \( \mathcal{B}_0^* \)-linear homomorphisms from \( \mathcal{B}_* \) to \( \mathcal{B}_0^* \) is a Hopf algebra isomorphic to \( \mathcal{B}_0^*[q] \), with
\[
q = \text{Exp}(\hbar) ,
\]
where \( \text{Exp} \) is the formal series inverse to Miščenko’s logarithm. It is tempting to regard \( n^{-1}CP_{n-1} \) as a cyclic quotient of \( CP_{n-1} \) defined by the shift \( [z_0 : \cdots : z_{n-1}] \mapsto [z_{n-1} : z_0 : \cdots] \), and to think of the formula for \( h \) as a correction to the hyperplane section construction which keeps track of some kind of bubbling.

In general, any linear functional on cobordism classes of prequantized manifolds which satisfies condition i) factors through \( \mathcal{B}_0^*[q] \); for example, the universal \( K \)-theoretic characteristic number homomorphism \([6]\) can be characterized as the \( \mathcal{B}_0^* \)-linear map which sends \( b_1 = X \) to a unit.

Appendix: some questions about cobordism of toric manifolds

1 Toric varieties are by now familiar objects in algebraic geometry, but this appendix is concerned with variations on that theme, and I will try to be careful about terminology. A toric variety is a kind of orbifold, and hence has mild singularities, but I will use the term toric manifold in the sense of Davis and Januszkiewicz \([7]\); a smooth toric variety thus has an underlying toric manifold, but toric manifolds form a slightly more general class.

2 By symplectic toric manifold I mean a compact \( 2n \)-dimensional symplectic manifold with a Hamiltonian \( \mathbb{T}^n \)-action, which is moreover prequantizable in the sense of Guillemin \([14\, \text{Th. 3.2}] \): such a thing is defined by a Delzant polytope \([14\, \text{Th. 1.8}] \) with vertices at integer lattice points, and the Chern class of its canonical line bundle is calculated in \([14\, \text{appendix 2}] \). If we ignore the torus action, then the underlying symplectic manifold defines an element of Ginzburg’s symplectic cobordism ring \( \mathcal{B}_* \). [There is a small but potentially confusing point here, in that elements of Ginzburg’s ring are equivalence classes of compact oriented manifolds endowed with a nondegenerate closed integral two-form \( \omega \). I will assume that the orientation is defined by \( \omega^n \).]

3 Ginzburg’s cobordism ring is canonically isomorphic to the ring \( MU_*B\mathbb{T} \) defined by complex-oriented manifolds together with a complex line bundle; forgetting this line bundle defines a homomorphism to the complex bordism ring \( MU_*(pt) \). A Delzant polytope thus defines a cobordism class, and Buchstaber and Ray \([5]\) have recently published an elegant proof that this construction is surjective: every complex-oriented manifold is cobordant to a (sum of) toric manifold(s); at least, if we allow polytopes of dimension zero. In fact Conner and Floyd constructed multiplicative generators for the cobordism ring (at least, for each prime \( p \)) as Bott
towers with obvious toric manifold structures [6 §42]; but this happened a few years before the notion of toric variety appeared in print. It seems pretty clear that the Cartesian product of toric manifolds is again a toric manifold; but it doesn’t seem to me so immediate that the Cartesian product of two Delzant polytopes is another such. I don’t know a reference for this presumed fact . . .

4 In light of these observations, it is natural to ask which elements of \( B_\ast \) ‘come from’ symplectic toric actions; but we need to be precise about how this is to be interpreted. If \((V_\Delta, \omega)\) is the class defined by an integral Delzant polytope \( \Delta \), then it seems reasonable to think of the elements of the ray 
\[ \{(V_\Delta, n\omega) \in MU_\ast(BT) \mid n \in \mathbb{Z}_{>0} \} \]
as arising in this way. The disjoint union of two symplectic manifolds defined by torus actions ought also to be in this class. There is thus a kind of **effective cone** \( C \) in \( MU_\ast(BT) \), generated by sums of elements of such rays; this cone will be closed under multiplication, if the remark above is not a mistake. It would be very interesting to know more about \( C \).

I want to thank Ginzburg, Guillemin, and Karshon for very helpful correspondence about this question. In response to some very naive conjectures, they observed that in dimension two, \( B_\ast \) has rank two, with the projective line \( \mathbb{C}P_1 \) and the symplectic torus \( X \) as generators; the latter cannot be toric. In fact it seems plausible that no class in principal ideal \( (X) \) can be toric; if so, then the quotient of the localization \( \mathbb{C}^{-1}B_\ast \) by the image of \( (X) \) will be the quotient field of \( MU_\ast \), and the localization itself would be a local ring. In some ways this ring is a natural context for questions about geometric localization theorems along the lines of [13].

5 A related circle of questions concerns the relation of toric manifolds to the ‘normally split’ cobordism theory \( ML \) defined by manifolds endowed with a stable splitting of the normal bundle as a sum of line bundles [1]. Let \( P \) be a simple convex polytope of dimension \( n \), with \( F \) its set of codimension one faces, and let \( \lambda \) be a characteristic function [i.e. a suitable linear function from the free \( \mathbb{Z} \)-module generated by \( F \), to the free \( \mathbb{Z} \)-module on standard generators \( e_i \)]. The Stanley-Reisner face ring of \( P \) is a (combinatorially-defined) quotient of the polynomial ring on generators \( x_f \) indexed by the faces of \( P \); let 
\[
\lambda^* : H^n_{T^n}(pt) := \mathbb{Z}[e_1, \ldots, e_n] \to \mathbb{Z}[x_f | f \in F]
\]
be the ring homomorphism induced by the dual of \( \lambda \). Then there is a manifold \( M := M(P)_\lambda \) with \( T^n \) action, such that the equivariant cohomology \( H^n_{T^n}(M) \) is isomorphic, as (free [7 §4.12]) \( H^n_{T^n}(pt) \)-algebra, to the face ring of \( P \) [7 §4.8]; moreover, the (equivariant) tangent bundle of \( M \) splits stably, as a sum of complex line bundles indexed by \( F \) [7 §6.6], such that the \( f \)th line bundle has Chern class \( x_f \).

We can thus define an **exhaustive** characteristic class for such a toric manifold as the product 
\[
\prod_{f \in F} (1 - x_f z_f)^{-1} \in H^n_{T^n}(M)[z_f | f \in F],
\]
and we can similarly define an exhaustive characteristic number polynomial of \( M \) as the homogeneous component of degree \( 2n \) of the image of this polynomial, in the quotient of the equivariant cohomology by the ideal \( (e_1, \ldots, e_n) \). Since the
equivariant cohomology is free over $H^*_T(pt)$, this quotient can be identified with the ordinary (nonequivariant) cohomology of $M$, which is isomorphic to $\mathbb{Z}$ in degree $2n$; the polynomial is thus a homogeneous element of $\mathbb{Z}[z_f| f \in F]$, of (homological) degree $2n$. This suggests the question: does a toric manifold define, in some natural way, an $ML$-cobordism class?

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