A SIMPLE GENERALISED PLANCHEREL FORMULA FOR COMPACTLY
INDUCED CHARACTERS

CHAITANYA AMBI

Abstract. The aim of this article is to present a simple generalized Plancherel formula for
a locally compact unimodular topological group $G$ of type I. This formula applies to the
functions representing $c$-$\text{Ind}_{G}^{U}\psi$ for a unitary character $\psi$ of a closed unimodular subgroup $U$
of $G$. This specializes to the Whittaker-Plancherel formula for a split reductive $p$-adic group
of Sakellaridis-Venkatesh and differs from that of a quasi-split $p$-adic group due to Delorme.
Furthermore, it also applies to certain metaplectic groups and other interesting situations
where the local theory of distinguished representations has been studied.

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1. Introduction and statement of the main theorem

Let $G$ be a unimodular, locally compact topological group of type I with the unitary dual $\hat{G}$.
Fix a Haar measure on $G$. Let $L^2(G)$ denote the space of complex-valued functions on $G$ which
are square integrable with respect to the left Haar measure. The left regular representation of $G$
on $L^2(G)$ can be disintegrated into irreducible representations. This permits one to express a
function within a suitable class in $L^2(G)$ in terms of functions on $\hat{G}$ via the Fourier transform.
The Plancherel formula reflects the structure of $\hat{G}$.

For a closed subgroup $U$ of $G$ having a continuous unitary character $\psi$, a function in the
space representing the compact induction $c$-$\text{Ind}_{G}^{U}\psi$ also affords a generalised Plancherel for-
mula which provides further information about $\hat{G}$. Such formulae have been worked out in
various settings, especially when $G$ is a $p$-adic group or a Lie group with certain properties.
and $U$ is the unipotent radical of a Borel subgroup of $G$ (notably, the Whittaker-Plancherel formulae by Baruch and Mao [1], Delorme [3] and also Sakellaridis and Venkatesh [10]). In this article, we derive a generalised Plancherel formula for a broad class of groups which generalises such Whittaker-Plancherel type formulae in the pointwise case. We prove the existence of a locally integrable kernel for the distribution character and compute it in terms of the kernel of the ordinary Plancherel formula for $G$ whenever the latter exists. We also establish the absolute continuity of the corresponding Plancherel measure with respect to that on $\hat{G}$. Thus, our result presented below facilitates an explicit derivation of a generalised Plancherel formula in several situations including those where the formulae known so far do not apply (such as certain metaplectic groups).

Assume that $U$ is unimodular. For $f \in C_c(G)$ and $g \in L^\infty(U)$, set

\[
(g *_U f)(x) = \int_U g(u)f(u^{-1}x)du.
\]

Let $C_c(G)$ denote the space of compactly supported, complex-valued continuous functions on $G$. For $f \in C_c(G)$, define the operator

\[
\hat{f}(\pi) = \int_G f(x)\pi(x)dx,
\]

which is known to be compact and Hilbert-Schmidt (hence of the trace class). Since we have fixed a Haar measure on $G$, there exists a unique Plancherel measure $\mu_\pi$ on $\hat{G}$ such that the following pointwise Inversion Formula holds (see [4]):

\[
h(1) = \int_{\hat{G}} \Theta_\pi(h)d\mu_\pi, \quad \forall h \in C_c(G),
\]

where the distribution $\Theta_\pi$ is defined as

\[
\Theta_\pi(h) = Tr[\hat{h}(\pi)].
\]

For a continuous unitary character $\psi$ of $U$, define the space

\[
c-Ind_{U}^G \psi := \{W : G \to \mathbb{C} : W \text{ is compactly supported modulo } U \text{ and satisfies } W(ug) = \psi(u)W(g)\},
\]

($u \in U$). This space serves as a model for compact induction of $\psi$. Define

\[
(W_\psi f)(g) = (\psi *_U f)(g).
\]

It can be shown (see [4]) that the map $f \to W_\psi f$ is surjective. In many situations, there exists (except for a set of measure zero in $\hat{G}$) a locally integrable function $\theta_\pi : G \to \mathbb{C}$ ($\pi \in \hat{G}$) which is constant on conjugacy classes of $G$ and satisfies

\[
\Theta_\pi(f) = \int_G f(g)\theta_\pi(g)dg, (f \in C_c(G)).
\]

This occurs in the case of real reductive groups as well as reductive $p$-adic groups (see [6],[7] and [8]). The existence of $\theta_\pi$, which we shall term as the kernel of $\Theta_\pi$, permits us to derive a simple generalised Plancherel formula as follows:
Theorem 1.8 (Generalised Plancherel formula for a type-I group). Assume that the distribution character Θ_{π} of G has a locally integrable function θ_{π} as its kernel. Then, there exists a distribution Φ_{π}^{ψ} (defined except on a set of measure zero for π ∈ Ĝ) such that the function W_{ψ}f ∈ c-Ind_{G}^{U}ψ satisfies

\[(1.9) \quad (W_{ψ}f)(1) = \int_{Ĝ} \Phi_{π}^{ψ}(f)dμ_{π}.\]

The distribution Φ_{π}^{ψ} is given explicitly by

\[(1.10) \quad Φ_{π}^{ψ}(f) = \int_{G} f(g)(\overline{ψ}∗Uθ_{π})(g)dg,\]

where kernel \(\overline{ψ}∗Uθ_{π}\) of Φ_{π}^{ψ} exists as a locally integrable function on G.

(See also the Remark in Section (3.1)).

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2. Proof of the generalised Plancherel theorem for a type-I unimodular group

We shall maintain the notation as in the Introduction.

Since U is closed in G, it is also locally compact and second countable. Hence, U is σ-compact and we can select a sequence \(K_{n}\) of symmetric compact subsets of U such that

\[(2.1) \quad K_{i} ⊂ K_{i+1} \quad \forall i ∈ \mathbb{N} \text{ and } \bigcup K_{i} = U.\]

Define a sequence of bounded, compactly supported functions \(ψ_{n} : n ∈ \mathbb{N}\) by

\[(2.2) \quad ψ_{n}(u) = ψ(u)1_{K_{n}}(u) \quad (u ∈ U)\]

For each function \(f ∈ C_{c}(G)\) and \(x, y ∈ G\), we have the estimate

\[(2.3) \quad |ψ_{n}∗Uf(x) − ψ_{n}∗Uf(y)| ≤ ||ψ_{n}||_{L_{∞}(U)} \int_{U} |f(u^{-1}x) − f(u^{-1}y)|du.\]

Noting the uniform continuity of f on its support and that no \(ψ_{n}\) exceeds 1 in magnitude, we conclude that each \(ψ_{n}∗Uf\) is continuous on G. Further, both \(ψ_{n}\) and f are compactly supported and hence \(ψ_{n}∗Uf ∈ C_{c}(G)\) for each n. Hence, the formula (2.2) applies to \(ψ_{n}∗Uf\) and we have

\[(2.4) \quad (ψ_{n}∗Uf)(1) = \int_{Ĝ} Θ_{π}(ψ_{n}∗Uf)dμ_{π}.\]

\[(2.5) \quad = \int_{Ĝ} \int_{G} θ_{π}(g)(ψ_{n}∗Uf)(g)dgdμ_{π}.\]

\[(2.6) \quad = \int_{Ĝ} \int_{U} θ_{π}(g)f(u^{-1}g)ψ_{n}(u)du dg dμ_{π}.\]
Now, we use Fubini’s Theorem to interchange the inner two integrals. This stands justified as the whole integrand is integrable with respect to the product measure \( dgdu \) (as \( f \) is compactly supported) and both \( dg \) and \( du \) are \( \sigma \)-finite (owing to the local compactness and second countability of \( G \) and \( U \), respectively).

\[
(2.7) = \int_G \int_U \psi_n(u)(\int_G \theta_\pi(g)f(u^{-1}g)dg)dud\mu_\pi.
\]

Substituting \( g = ux \) in the innermost Haar integral, we get

\[
(2.8) = \int_G \int_U (\int_G \theta_\pi(ux)f(x)dx)\psi_n(u)dud\mu_\pi.
\]

Since we have not changed the integrand, Fubini’s Theorem applies as before.

\[
(2.9) = \int_G \int_G f(x)(\int_U \theta_\pi(ux)\psi_n(u)du)dxd\mu_\pi.
\]

Noting the unimodularity of \( U \) and that \( \bar{\psi}_n(u) = \psi_n(u^{-1}) \), (bar denotes complex conjugation throughout this article), we obtain

\[
(2.10) = \int_G \int_G f(x)(\int_U \bar{\psi}_n(u)\theta_\pi(u^{-1}x)du)dxd\mu_\pi,
\]

We recognise the above expression as

\[
(2.11) = \int_G \int_G f(x)(\bar{\psi}_n * U \theta_\pi)(x)dxd\mu_\pi.
\]

Note that \( \bar{\psi}_n * U \theta_\pi \) exists as a locally integrable function on \( G \) because \( \psi_n \) is compactly supported and \( \theta_\pi \) is locally integrable (see Prop. 5.4.25 on p.283 in [5]).

Now, we restrict ourselves to an arbitrary compact subset \( K \) of positive measure in \( G \). By Prop. 4.4.24 on p.282 in [5], \( \bar{\psi} * U \theta_\pi \) exists as an integrable function on \( K \).

It remains to establish the convergence of \( \bar{\psi}_n * U \theta_\pi \) to \( \bar{\psi} * U \theta_\pi \) in \( L^1(K) \). This follows from the Dominated Convergence Theorem once we observe that \( \bar{\psi}_n \) converges to \( \psi \) pointwise and that the integrable function \( 1 + |\theta_\pi| \) dominates each \( \bar{\psi}_n * U \theta_\pi \) almost everywhere on \( K \). This proves the existence of \( \bar{\psi} * U \theta_\pi \) as a locally integrable function on \( G \), which we have denoted by \( \phi_\psi^\theta \) in the statement of Theorem (1.8). If we denote the distribution with \( \phi_\psi^\theta \) as its kernel by \( \Phi_\psi^\theta \), we have

\[
(2.12) (W_\psi f)(1) = \int_G \Phi_\psi^\theta(f)d\mu_\pi \quad (f \in C_c(G)),
\]

which is the pointwise generalised Plancherel formula for \( W_\psi f \). If we define the convolution of \( \psi \) with a distribution in a similar manner, \( \Phi_\psi^\theta \) equals

\[
(2.13) \Phi_\psi^\theta = \psi * U \Theta_\pi.
\]

This completes the proof of Theorem (1.8) once we note that the Plancherel measure \( d\mu_\pi \) remains unaltered throughout the computation.
3. Comparison with other known Whittaker-Plancherel formulae

It is well-known that a real or $p$-adic reductive group $G$ is of type I (see [2]). The unipotent radical $U$ of its standard Borel subgroup is unimodular as well as closed. Hence, $U$ is also locally compact as well as second countable. Theorem (1.8) is applicable and provides a broader framework for the following Whittaker-Plancherel formulae:

3.1. Sakellaridis-Venkatesh. Sakellaridis and Venkatesh [10] have obtained a Whittaker-Plancherel formula (more precisely, the Parseval’s Identity of isometry) for Whittaker functions on a split reductive algebraic group. They also prove the absolute continuity of the Whittaker-Plancherel measure with respect to the Plancherel measure for the group. Theorem (1.8) not only improves the formula of Sakellaridis and Venkatesh to a Whittaker-Plancherel pointwise inversion formula, but also allows us to find the Radon-Nikodym derivative of the involved measure, provided the measure is suitably normalised (namely, equal to 1).

Remark: Since Plancherel measures are defined only up to absolute continuity, we could have adopted the alternative viewpoint that the Plancherel measure changes to $\nu^\psi_\pi$ while the distribution $\Theta_\pi$ remains unchanged in the Whittaker-Plancherel case, i.e.,

\begin{equation}
\Theta_\pi d\nu^\psi_\pi = \Phi^\psi_\pi d\mu_\pi.
\end{equation}

We conjecture that the Radon-Nikodym derivative would then be equal to

\begin{equation}
d\nu^\psi_\pi / d\mu_\pi = \text{mult}(\pi, c-\text{Ind}^G_U \psi),
\end{equation}

where $\text{mult}(\pi, c-\text{Ind}^G_U \psi)$ denotes the multiplicity of $\pi$ in $c-\text{Ind}^G_U \psi$.

The multiplicity is known to be finite in several cases, such as that of degenerate Whittaker models (see [9]).

3.2. Delorme. Delorme [3] has obtained an inversion formula for a Whittaker function using the matrix coefficients of certain representations induced parabolically and a version of the Schur Orthogonality relation for such coefficients. Assuming a character $\psi$ to be nondegenerate, Delorme’s formula expresses a $\psi$-Whittaker function in terms of certain transforms of the function itself. Theorem (1.8) expresses the value of a $\psi$-Whittaker function at unity in terms of the compactly supported function which generates it and the kernel $\theta_\pi$ of a representation $\pi \in \hat{G}$. This is substantially different from the formula obtained by Delorme.

3.3. Baruch-Mao. Baruch and Mao [1] have obtained a Whittaker-Plancherel formula for $\text{SL}(2, \mathbb{R})$ by means of its characters expressed explicitly in terms of Bessel functions. Since $\text{SL}(2, \mathbb{R})$ is a connected semisimple Lie group, Theorem (1.8) evidently applies to it.

3.4. The Whittaker-Plancherel formula for metaplectic groups. We now discuss the applicability of Theorem (1.8) to metaplectic groups. Let $\text{Sp}_{2n}(\mathbb{F})$ be the symplectic group over a local field $\mathbb{F}$ of characteristic zero ($n \in \mathbb{N}$). Consider a group $\text{Mp}_{2n}(\mathbb{F})$ which is a finite cover of $\text{Sp}_{2n}(\mathbb{F})$ (and thus is second countable and locally compact). Further, $\text{Mp}_{2n}(\mathbb{F})$ is semisimple and hence of type I. We conclude this article by mentioning that Theorem (1.8) is applicable to $\text{Mp}_{2n}(\mathbb{F})$ as well.
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Indian Institute of Science Education and Research, Dr. Homi Bhabha Road, Pashan, Pune 411008, India.

E-mail address: chaitanya.ambi@students.iiserpune.ac.in