Overfullness of edge-critical graphs with small minimal core degree

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Abstract
Let $G$ be a simple graph. Let $\Delta(G)$ and $\chi'(G)$ be the maximum degree and the chromatic index of $G$, respectively. We call $G$ overfull if $E(G)/|V(G)|/2 > \Delta(G)$, and critical if $\chi'(H) < \chi'(G)$ for every proper subgraph $H$ of $G$. Clearly, if $G$ is overfull then $\chi'(G) = \Delta(G) + 1$. The core of $G$, denoted by $G_{\Delta}$, is the subgraph of $G$ induced by all its maximum degree vertices. We believe that utilizing the core degree condition could be considered as an approach to attack the overfull conjecture. Along this direction, we in this paper show that for any integer $k \geq 2$, if $G$ is critical with $\Delta(G) \geq \frac{2}{3}n + \frac{3k}{2}$ and $\delta(G_{\Delta}) \leq k$, then $G$ is overfull.

KEYWORDS
extended Vizing fan shifting, overfull conjecture, Vizing fan

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1 | INTRODUCTION

We will mainly follow the notation from [9]. Graphs in this paper are simple, that is, finite, undirected, without loops or multiple edges. Let $G$ be a graph and let $[k] = \{i \mid 1 \leq i \leq k \text{ and } i \in \mathbb{Z}\}$ for a nonnegative integer $k$. A $k$-edge-coloring of $G$ is a mapping $\varphi: E(G) \rightarrow [k]$ that assigns to every edge $e$ of $G$ a color $\varphi(e) \in [k]$ such that no two adjacent edges receive the same color. Denote by $\mathcal{C}^k(G)$ the set of all $k$-edge-colorings of $G$. The chromatic index $\chi'(G)$ is the least integer $k \geq 0$ such that $\mathcal{C}^k(G) \neq \emptyset$. Denote by $\delta(G)$ and
Δ(G) the minimum and maximum degree of G, respectively. In the 1960s, Vizing [12] and, independently, Gupta [6] proved that Δ(G) ≤ χ′(G) ≤ Δ(G) + 1. This leads to a natural classification of graphs. Following Fiorini and Wilson [4], we say a graph G is of class 1 if χ′(G) = Δ(G) and of class 2 if χ′(G) = Δ(G) + 1. Holyer [8] showed that it is NP-complete to determine whether an arbitrary graph is of class 1. The problem of deciding which graphs are of class one, and which are of class two, is known as the Classification Problem [4, 9].

A graph G is critical if χ′(H) < χ′(G) for every proper subgraph H of G. In investigating the Classification Problem, critical graphs are of particular interest. A critical class 2 graph is called Δ-critical if G has at most two vertices then G is class 1. Fournier [5] generalized Vizing’s result by showing that if G is acyclic then G is class 1. Thus a necessary condition for a graph to be class 2 is to have a core that contains cycles. A long-standing conjecture of Hilton and Zhao [7] claims that for a connected class 2 graph G with Δ ≥ 4, if Δ(Ga) ≤ 2, then G is overfull. This conjecture was recently confirmed by the authors [1].

Along this direction, we prove the following result and verify the overfull conjecture for critical graphs with a more general minimum core degree condition, and we hope to use similar ideas to attack the overfull conjecture in the future. For example, if we can improve the coefficient of k in Theorem 1.1 from 3/2 to 1/12, then the overfull conjecture holds for all graphs with maximum degree Δ ≥ 3n/4.

**Theorem 1.1.** Let k ≥ 2 be a positive integer and G be a Δ-critical graph of order n. If Δ ≥ 2n/3 + 3k/2 and δ(Ga) ≤ k, then G is overfull.

2 | PRELIMINARIES

This section is divided into two subsections. In Section 2.1 we introduce some basic notation and terminologies. In Section 2.2 we introduce the traditional Vizing fan and generalize it to a larger structure.

2.1 | Basic notation and terminologies

Let G be a graph with maximum degree Δ, let e ∈ E(G) be a critical edge, and let φ ∈ CΔ(G − e). For a vertex v ∈ V(G), define the two color sets

φ(v) = {φ(f) : f ≠ e is incident to v} and φ(v) = [Δ] \ φ(v).
We call \( \varphi(v) \) the set of colors present at \( v \) and \( \overline{\varphi}(v) \) the set of colors missing at \( v \). If \( |\overline{\varphi}(v)| = 1 \), we will also use \( \overline{\varphi}(v) \) to denote the color missing at \( v \). Let \( N(v) \) be the collection of all the neighbors of \( v \), \( N_{< \Delta}(v) \) be the collection of neighbors of \( v \) with degree less than \( \Delta \), and \( N_{\geq \Delta}(v) \) be the collection of neighbors of \( v \) with degree exactly \( \Delta \).

For a vertex set \( X \subseteq V(G) \), define \( \overline{\varphi}(X) = \bigcup_{v \in X} \overline{\varphi}(v) \) to be the set of missing colors of \( X \). The set \( X \) is called elementary w.r.t. \( \varphi \) or simply called elementary if \( \varphi(u) \cap \varphi(v) = \emptyset \) for every two distinct vertices \( u, v \in X \). In the rest of this paper, we may not always mention the coloring \( \varphi \) if it is clearly understood.

For a color \( \alpha \), the edge set \( E_{\alpha} = \{ e \in E(G) | \varphi(e) = \alpha \} \) is called a color class. Clearly, \( E_{\alpha} \) is a matching of \( G \) (possibly empty). For two distinct colors \( \alpha, \beta \), the subgraph of \( G \) induced by \( E_{\alpha} \cup E_{\beta} \) is a union of disjoint paths and even cycles, which are referred to as \((\alpha, \beta)\)-chains of \( G \) w.r.t. \( \varphi \). For a vertex \( v \), let \( C_v(\alpha, \beta, \varphi) \) denote the unique \((\alpha, \beta)\)-chain containing \( v \). If \( C_v(\alpha, \beta, \varphi) \) is a path, we just write it as \( P_{\alpha}(\alpha, \beta, \varphi) \). The latter is commonly used when we know that \( |\overline{\varphi}(v) \cap \{\alpha, \beta\}| = 1 \). If we interchange the colors \( \alpha \) and \( \beta \) on an \((\alpha, \beta)\)-chain \( C \) of \( G \), we briefly say that the new coloring is obtained from \( \varphi \) by an \((\alpha, \beta)\)-swap on \( C \), and we write it as \( \varphi/C \). This operation is called a Kempe change. If \( \alpha \in \overline{\varphi}(v) \), by doing operation \( \alpha \to \beta \) at \( v \) we mean the Kempe change \( \varphi/P_{\alpha}(\alpha, \beta, \varphi) \). Note that \( P_{\alpha}(\alpha, \beta, \varphi) \) could be empty when \( \alpha, \beta \in \varphi(v) \) and \( \alpha \to \beta \) at \( v \) does nothing in this case. We say two vertices \( x \) and \( y \) are \((\alpha, \beta)\)-linked if they belong to the same \((\alpha, \beta)\)-chain. Moreover, when \( x = y \), for convenience we still say \( x \) and \( y \) are \((\alpha, \beta)\)-linked even if \( \alpha, \beta \in \overline{\varphi}(x) \).

### 2.2 Linear sequence, shifting, and extended Vizing fan

The fan argument was introduced by Vizing [10, 11] in his proof of the classic results on the upper bounds for chromatic indices. Let \( G \) be a class 2 graph with maximum degree \( \Delta \), \( e = rs \) be a critical edge of \( G \), and let \( \varphi \in \mathcal{C}^\Delta(G - e) \). For an integer \( p \geq 0 \), a sequence \( F = (r, e_0, s_0, e_1, s_1, ..., e_p, s_p) \) alternating between distinct vertices and edges is called a Vizing fan at \( r \) with respect to \( e \) and \( \varphi \) if \( s_0 = s, e_0 = e \) and for each \( i \in \{p\} \), the edge \( e_i = rs_i \) satisfies \( \varphi(e_i) \in \varphi(s_h) \) for some \( 0 \leq h \leq i - 1 \). For the purpose of generalization in this paper, we include the vertex \( r \) in \( F \) comparing to the definition of a Vizing fan in the book [9].

Let \( q \) be a nonnegative integer. A linear sequence at \( r \) from \( s_0 \) to \( s_q \) in \( G \), denoted by \( L = (r, e_0, s_0, e_1, s_1, ..., e_q, s_q) \), is a sequence of distinct vertices and edges such that \( \varphi(e_i) \in \overline{\varphi}(s_{i-1}) \) for every \( i \in \{q\} \). Denote by \( V(L) \) and \( E(L) \), respectively, the set of vertices and edges contained in \( L \). A shifting from \( s_i \) to \( s_j \) in the linear sequence \( L = (r, e_0, s_0, e_1, s_1, ..., e_q, s_q) \) is an operation that replaces the current color of \( e_i \) by the color of \( e_{i+1} \) for each \( i \leq t \leq j - 1 \) with \( 0 \leq i < j \leq q \). Note that shifting from \( s_i \) to \( s_j \) does not change the color of \( e_i \) where \( e_j = rs_j \), so the resulting coloring will not be a proper coloring. In our proof we will treat \( e_i \) separately to avoid this problem. The following result regarding a Vizing fan can be found in [9, Theorem 2.1].

**Lemma 2.1.** Let \( G \) be a class 2 graph, \( e = rs_0 \) be a critical edge and \( \varphi \in \mathcal{C}^\Delta(G - e) \). If \( F \) is a Vizing fan w.r.t. \( e \) and \( \varphi \), then \( V(F) \) is elementary.

Note that \( e_0 \) may not be \( e \) in a linear sequence, but a linear sequence with \( e_0 = e \) is also a Vizing fan at \( r \). Moreover, for any \( s_i \in V(F) \) with \( i \in \{p\} \), the Vizing fan \( F = (r, e_0, s_0, e_1, s_1, ..., e_p, s_p) \) contains a linear sequence at \( r \) from \( s_0 \) to \( s_i \).
at \( r \) with \( \varphi(e_0) = \tau \) is called a \( \tau \)-sequence. In our proof we will add some linear sequences not contained in a Vizing fan to enlarge it. We say a Vizing fan \( F \) at \( r \) is maximal w.r.t. \( e \) and \( \varphi \) if there is no Vizing fan at \( r \) w.r.t. \( e \) and \( \varphi \) containing \( F \) as a proper subsequence. We say a Vizing fan \( F \) at \( r \) is maximum w.r.t. \( e \) and \( \varphi \) if there is no Vizing fan at \( r \) w.r.t. \( e \) and \( \varphi \) containing \( F \) as a proper subsequence. We say a Vizing fan \( F \) at \( r \) is maximal w.r.t. \( e \) and \( \varphi \) if there is no Vizing fan at \( r \) w.r.t. \( e \) and \( \varphi \) containing \( F \) as a proper subsequence. We say a Vizing fan \( F \) at \( r \) is maximum w.r.t. \( e \) if \( VF(\tau) = \varphi(v_t) \) is maximum among all Vizing fans at \( r \) w.r.t. \( e \) over all colorings \( \varphi \in \mathcal{C}^A(G - e) \). Clearly if \( F \) is maximum at \( r \) w.r.t. \( e \), it is also maximal w.r.t. \( e \) and the coloring \( \varphi \) where \( F \) is obtained. Let \( F \) be a maximal Vizing fan at \( r \) w.r.t. \( e \) and \( \varphi \). A \( \tau \)-sequence \( L \) at \( r \) is said to be outside of \( F \) if \( V(L) \cap V(F) = \emptyset \). For an integer \( t \geq 0 \), we say a \( \tau \)-sequence \( L = (r, f_0, v_0, f_1, v_1, \ldots, f_t, v_t) \) at \( r \) outside of \( F \) is extremal if \( v_t \) is the only vertex \( v_j \) with index \( 0 \leq j \leq t \) such that either \( \varphi(v_j) \cap (\cup_{i=0}^{t-1} \varphi(v_i) \cup \varphi(V(F)) \cup \{\tau\}) \neq \emptyset \) or \( \varphi(v_j) = \emptyset \). Since a \( \tau \)-sequence cannot be enlarged forever, it must be a subsequence of some extremal \( \tau \)-sequence. Moreover, exactly one of the followings must happen for an extremal \( \tau \)-sequence \( L \):

(a) \( V(L) \cup V(F) \) is elementary and \( \{\tau\} = \varphi(v_t) \). In this case we say \( L \) is of Type A.
(b) \( \varphi(v_t) \cap \varphi(V(F)) \neq \emptyset \). In this case we say \( L \) is of Type B.
(c) \( \varphi(v_i) \cap \varphi(V(F)) = \emptyset \) for all \( 0 \leq i \leq t \), and \( V(L) \) is not elementary. In this case there exists a color \( \alpha \in (\varphi(v_i) \cap \varphi(v_j)) - \varphi(V(F)) \) for some \( 0 \leq i \leq j \leq t \) and we say \( L \) is of Type C.
(d) \( \varphi(v_i) = \emptyset \) and \( V(L) \cup V(F) \) is elementary. In this case \( d(v_t) = \Delta \) and we say \( L \) is of Type D.

See the following Figure 1 for examples of four types of extremal \( \tau \)-sequences, where a dash line represents a color missing at a vertex.

From now on we will not mention “at \( r \)” when we refer to a Vizing fan or a linear sequence if it creates no confusion. Additionally, when we refer to a linear sequence outside of \( F \), we always mean an extremal one unless specified otherwise.

Let \( G \) be a class 2 graph, \( e = rs_0 \) be a critical edge and \( \varphi \in \mathcal{C}^A(G - e) \). Let \( F = (r, e_0, s_0, e_1, s_1, \ldots, e_p, s_p) \) be a Vizing fan centered at \( r \) under the coloring \( \varphi \). Clearly a

![FIGURE 1 Examples of \( \tau \)-sequences.](image)
linear sequence \( L \) at \( r \) from \( s_0 \) to \( s_q \) with \( q \in [p] \) defines a linear order \( \leq_L \) on vertices in \( L \). By \( s_a <_L s_b \), we mean \( s_a \leq_L s_b \) and \( s_a \neq s_b \). Since \( V(F) \) is elementary by Lemma 2.1, it is easy to see that all the linear sequences at \( r \) starting from \( s_0 \) to \( s_q \) for some \( q \in [p] \) together induce a partial order \( \leq_F \) by \( \alpha \leq_F \beta \) for every color \( \alpha \in \varphi(V(F)) \), and \( \alpha \leq_F \beta \) for two different colors \( \alpha, \beta \in \varphi(V(F)) \) if there exists a linear sequence \( L \) at \( r \) starting from \( s_0 \) to some \( s_q \) with \( q \in [p] \) such that an edge \( e' \in E(L) \) with \( \varphi(e') = \alpha \) comes before a vertex \( v \in V(L) \) with \( \beta \in \varphi(v) \) along \( L \). Moreover, for any color \( \alpha \in \varphi(V(F)) \), there is a unique vertex \( v \in V(F) \) such that \( \alpha \in \varphi(v) \) since \( V(F) \) is elementary. Let \( v_F(\alpha) \) denote such vertex \( v \). We have the following lemma as a direct consequence of Lemma 2.1.

**Lemma 2.2.** Let \( G \) be a class 2 graph, \( e = rs_0 \) be a critical edge and \( \varphi \in \mathcal{C}^2(G - e) \). Let \( F \) be a maximal Vizing fan at \( r \) w.r.t. \( e \) and \( \varphi \). Then for any two colors \( \alpha, \beta \in \varphi(V(F)) \), we have the following statements:

(a) If \( v_F(\alpha) = r \), then \( v_F(\alpha) \) and \( v_F(\beta) \) are \((\alpha, \beta)\)-linked.

(b) If \( \alpha \) and \( \beta \) are incomparable along \( \leq_F \), then \( v_F(\alpha) \) and \( v_F(\beta) \) are \((\alpha, \beta)\)-linked.

(c) If \( \alpha \leq_F \beta \) and \( v_F(\alpha) \) and \( v_F(\beta) \) are not \((\alpha, \beta)\)-linked, then \( P_{v_F(\beta)}(\alpha, \beta, \varphi) \) must contain the vertex \( r \).

(d) If \( v \in V(F) \) and \( v \neq r \), then \( F \) contains at least \( |\varphi(v)| \) many \( \Delta \)-degree neighbors of \( r \).

**Proof.** To prove (a) we assume \( v_F(\alpha) = r \). Note that if \( v_F(\beta) = r \), we are done by definition. So we may assume that \( v_F(\beta) \neq r \). If \( v_F(\alpha) \) and \( v_F(\beta) \) are not \((\alpha, \beta)\)-linked, then by \( \beta \to \alpha \) at \( v_F(\beta) \), we have a nonelementary Vizing fan \( F' \) from \( r \) to \( v_F(\beta) \) contradicting Lemma 2.1. Thus (a) holds.

For (b) we assume that \( \alpha \) and \( \beta \) are incomparable along \( \leq_F \). Note that there are two linear sequences \( L_1 \) and \( L_2 \) at \( r \) from \( s_0 \) to \( v_F(\alpha) \) and \( v_F(\beta) \), respectively. Since \( \alpha \) and \( \beta \) are incomparable along \( \leq_F \), and \( v_F(\alpha) \) and \( v_F(\beta) \) are the last vertices for \( L_1 \) and \( L_2 \), respectively, \( L_1 \) and \( L_2 \) do not contain any edge colored by \( \alpha \) or \( \beta \). Now by \( \beta \to \alpha \) at \( v_F(\beta) \), we have a new coloring and we denote the new coloring by \( \varphi_1 \). Since no edge in \( L_1 \) and \( L_2 \) is colored by either \( \alpha \) or \( \beta \) under \( \varphi \), \( L_1 \) and \( L_2 \) are still linear sequences under \( \varphi_1 \). Let \( F' \) be a maximal Vizing fan w.r.t. \( e \) and \( \varphi_1 \). Then \( L_1 \) and \( L_2 \) are all contained in \( F' \), giving a nonelementary Vizing fan contradicting Lemma 2.1.

If (c) fails, since \( \alpha \leq_F \beta \), we can just do \( \beta \to \alpha \) at \( v_F(\beta) \) to get a nonelementary Vizing fan contradicting Lemma 2.1.

To see (d), we assume \( \alpha \in \varphi(v) \) with \( v \in V(F) \) and \( v \neq r \). Since \( V(F) \) is elementary by Lemma 2.1 and \( F \) is maximal, every color \( \alpha \) in \( \varphi(v) \) induces at least one maximal \( \alpha \)-sequence \( L_\alpha \) ending with a unique \( \Delta \)-degree vertex in \( F \), giving at least \( |\varphi(v)| \) many \( \Delta \)-degree neighbors of \( r \) in \( F \).

The following Vizing’s Adjacency Lemma (VAL) is a direct consequence of Lemma 2.2(d).

**Lemma 2.3 (VAL).** Let \( G \) be a class 2 graph with maximum degree \( \Delta \). If \( e = xy \) is a critical edge of \( G \), then \( x \) is adjacent to at least \( \Delta - d(y) + 1 \) \( \Delta \)-vertices from \( V(G) \) \( \setminus \{y\} \).

Let \( G \) be a class 2 graph, \( e = rs_0 \) be a critical edge and \( F = (r, e_0, s_0, e_1, s_1, \ldots, e_p, s_p) \) be a maximum Vizing fan at \( r \) w.r.t. \( e \), and let \( \varphi \in \mathcal{C}^2(G - e) \) be the coloring where \( F \) is obtained. We call a color \( \beta \) a stopping color at \( r \) if \( r \) has a \( \Delta \)-degree neighbor \( x \) with \( \varphi(rx) = \beta \). Let \( K \) be
the set of all stopping colors at \( r \). Since \( G \) is class 2, \( e \) is critical, and \( F \) is maximum and elementary, \( F \) must contain some \( \Delta \)-degree neighbors of \( r \). So there exists a vertex \( s_h \in V(F) \) and stopping color \( \beta \) such that \( \beta \in \varphi(s_h) \). We let \( K_F = K - \varphi(V(F)) \) and call colors in \( K_F \) \textit{stopping colors outside} of \( F \). By a slightly abuse of notation, in this paper, a \textit{union} of two sequences \( A \) and \( B \), denoted by \( A \cup B \), is the sequence obtained by joining the sequence \( B \) to \( A \) after the last element of \( A \). We now fix a vertex \( s_h \in V(F) \) with a stopping color \( \beta \in \varphi(s_h) \). Let \( F' \) be the union of all the \( \varphi(rv) \)-sequences outside of \( F \), where \( v \) is any vertex in the set \( N_\Delta(s_h) \cap N(r) \) with \( \varphi(vs_h) \notin K_F \). Then we call the sequence \( F \cup F' \) an \textit{extended Vizing fan} w.r.t. \( F \) and \( s_h \). See Figure 2 for an extended Vizing fan with \( F' \) being a single Type A \( \tau \)-sequence with \( \varphi(vs_h) = \pi \), where a dash line represents a color missing at a vertex. For simplification of notation, we did not indicate \( s_h \) in the notation \( F' \), even though \( F' \) relies on a fixed vertex \( s_h \). The following Lemma 2.4 is a key lemma in our proof and it is a natural generalization of Lemmas 2.1 and 2.2 on \( F \cup F' \). It is worth pointing out that Lemma 2.4 can be easily generalized further along this direction if we allow \( F' \) to be the union of all the \( \varphi(rv) \)-sequences outside of \( F \) such that \( v \in N_\Delta(s_h) \cap N(r) \) with \( \varphi(vs_h) \notin K_F \) for every vertex \( s_h \) having any stopping color \( \beta \in \varphi(s_h) \).

**Lemma 2.4.** Let \( G \) be a class 2 graph, \( e = rs_0 \) be a critical edge and \( F = (r, e_0, s_0, e_1, s_1, \ldots, e_p, s_p) \) be a maximum Vizing fan at \( r \) w.r.t. \( e \), and let \( \varphi \in C^\Delta(G - e) \) be the coloring where \( F \) is obtained. Let \( F \cup F' \) be an extended Vizing fan w.r.t. \( F \) and \( s_h \in V(F) \), where \( \beta \) is a stopping color with \( \beta \in \varphi(s_h) \). Then the following holds.

(a) \( V(F \cup F') \) is elementary.

(b) For two colors \( \gamma \in \varphi(F \cup F') - K_F \), the vertices \( r \) and \( v_{F\cup F'}(\gamma) \) are \((1, \gamma)\)-linked, where \( v_{F\cup F'}(\gamma) \) is the unique vertex in \( V(F \cup F') \) with \( \gamma \in \varphi(v_{F\cup F'}(\gamma)) \).

![Figure 2](image.png)
(c) For each color \( \gamma \in \Phi(V(F')) \), the vertices \( s_\gamma \) and \( v_{F'}(\gamma) \) are \((\beta, \gamma)\)-linked, where \( v_{F'}(\gamma) \) is the unique vertex in \( V(F') \) with \( \gamma \in \Phi(v_{F'}(\gamma)) \).

(d) Let \( \gamma \) be a color in \( \Phi(V(F')) \cap K_F \) and let \( L = (r, v_{l_0}, v_{l_0}, ..., v_{l_l}) \) be a \( \Phi(v_{l_0}) \)-sequence in \( F' \) such that \( v_{F'}(\gamma) \in V(L) \). If \( \Phi(s_{\gamma}v_{l_0}) \neq 1 \), then \( r \) is \((1, \gamma)\)-linked to \( v_{F'}(\gamma) \). If \( \Phi(s_{\gamma}v_{l_0}) = 1 \), then \( v_{F'}(\gamma) \) is \((\zeta, \gamma)\)-linked to \( v_{F'}(\zeta) \) for any color \( \zeta \in \Phi(V(F)) \cap K \).

The proof of Lemma 2.4 will be given in Section 4. We call a vertex \( r \) light if \( d(r) = \Delta \) and \( d_{Gk}(r) = \delta(Gk) \). The next lemma is the main tool used in the proof of Theorem 1.1.

**Lemma 2.5.** Let \( G \) be a critical class 2 graph with \( \delta(Gk) = k \), \( |V(G)| = n \), and \( \Delta \geq \frac{2}{3}n + \frac{3k}{2} \), let \( r \) be a light vertex, and let \( e = rs \) be a critical edge with \( d(s) \leq \Delta - 1 \). Let \( \varphi \in C^2(G - e) \) be a coloring under which there is a maximum Vizing fan centered at \( r \). Then all vertices of degree at least \( \Delta - k + 1 \) form an elementary set under \( \varphi \).

3 | PROOF OF THEOREM 1.1

**Proof.** Let \( G \) be a \( \Delta \)-critical graph of order \( n \) and \( k \geq 2 \) be a positive integer. Furthermore, we assume \( \Delta \geq \frac{2}{3}n + \frac{3k}{2} \) with \( \delta(Gk) \leq k \). Since \( \Delta \geq \frac{2}{3}n + \frac{3k}{2} \geq \frac{2}{3}n + \frac{3\delta(Gk)}{2} \), we will just take \( \delta(Gk) = k \) in this proof. Let \( r \) be a light vertex of \( G \) and \( s \) be a neighbor of \( r \) with \( d(s) \leq \Delta - 1 \). Then the edge \( rs \) is a critical edge. Let \( \varphi \) be a \( \Delta \)-edge-coloring of \( G - rs \) and \( F \) be a maximum Vizing fan centered at \( r \). We first claim that if \( V(G) \) is elementary under \( \varphi \), then \( G \) is overfull. Indeed, if \( G \) is elementary, then each color can only be missing at most once for vertices in \( V(G) \). Since \( r \) has at least one missing color, \( n \) must be odd as any color missing at \( r \) induces a perfect matching of \( G - r \). Therefore, each color must be missing exactly once in \( G \) as \( n \) is odd. Thus \( G \) has exactly \( \left( \frac{n-1}{2} \right) \Delta + 1 \) many edges since we have \( \Delta \) many color classes and the edge \( rs \) is uncolored. So \( G \) is overfull as we claimed.

Now we shall show that \( V(G) \) is elementary to confirm that \( G \) is overfull in the remainder of this section. By Lemma 2.5, all vertices with degree at least \( \Delta - k + 1 \) form an elementary set, so we are done if there’s no vertex of degree less than \( \Delta - k + 1 \). Thus we assume otherwise that there is a vertex \( x \) with \( d(x) \leq \Delta - k \). Since \( |N_x(r)| = k \), all the vertices in \( N(r) \) have degree at least \( \Delta - k + 1 \) by applying Lemma 2.3 (VAL) to the edge \( xr \). Since \( d(x) \leq \Delta - k \), we have \( x \not\in N(r) \).

We claim that \( d(x) \geq \frac{n}{3} + 2k \). Since every edge in \( G \) is critical, \( x \) is adjacent to at least one maximum degree vertex in \( G \) by Lemma 2.3 (VAL). Let \( u \) be a maximum degree vertex with \( ux \in E(G) \). Then \( u \neq r \) as \( x \not\in N(r) \). Since \( d(u) = \Delta \), we have \( N(u) \cap N(r) \geq d(u) + d(r) - |N(u) \cup N(r)| \geq \Delta + \Delta - n \geq \frac{4n}{3} + 3k - n \geq \frac{n}{3} + 3k \). Since \( |N_x(r)| = k \), we have \( |N_{<\Delta}(u)| \geq \frac{n}{3} + 2k \), and therefore \( N_{\Delta}(u) \leq \Delta - \frac{n}{3} - 2k \). Since \( ux \) is a critical edge, we have \( N_{\Delta}(u) \geq \Delta - d(x) + 1 \). So \( d(x) \geq \frac{n}{3} + 2k + 1 \geq \frac{n}{3} + 2k \) as claimed.

Since \( N_{\Delta}(r) = k \), we have \( d(v) \geq \Delta - k + 1 \) for each vertex \( v \in N_{<\Delta}(r) \) by Lemma 2.3(VAL). Recall that by Lemma 2.5, all vertices with degree at least \( \Delta - k + 1 \)
form an elementary set. As \( s \in N_{<\Delta}(r) \), we have \(|\mathcal{F}(N_{<\Delta}(r))| \geq |N_{<\Delta}(r)| + 1 \geq \Delta - k + 1 \).

Since \( d(x) \geq \frac{n}{3} + 2k \), we have \( |N(r) \cap N(x)| \geq |N_{<\Delta}(r)| + 1 \geq \Delta - k + 1 \). Because \(|N_{<\Delta}(r)| = k\), it follows that \(|N_{<\Delta}(r) \cap N_{<\Delta}(x)| \geq \frac{k^2}{2}\). Since \(|\mathcal{F}(N_{<\Delta}(r))| \geq \Delta - k + 1\) and all edges connecting \( x \) to vertices in \( N_{<\Delta}(r) \cap N_{<\Delta}(x) \) are colored differently, there is a vertex \( v \in N_{<\Delta}(r) \cap N_{<\Delta}(x) \) such that \( \varphi(vx) = \beta \in \mathcal{F}(w) \) where \( w \in N_{<\Delta}(r) \). Since \( d(x) \leq \Delta - k, |\mathcal{F}(x)| \geq k \). Since \( |\mathcal{F}(N_{<\Delta}(r))| \geq \Delta - k + 1 \), \( |\mathcal{F}(x) \cap \mathcal{F}(N_{<\Delta}(r))| \geq 1 \). Thus, there exists \( \alpha \in \mathcal{F}(x) \cap \mathcal{F}(u') \) where \( u' \in N_{<\Delta}(r) \). So \( d(u') \geq \Delta - k + 1 \). Let \( 1 \in \mathcal{F}(r) \). We claim that \( u' \) is \((1, \alpha)\)-linked to \( r \). Otherwise, we have \( u' \not\in V(F) \) by Lemma 2.2(a). Thus \( F \) stays as a maximum Vizing fan after \( 1 \rightarrow \alpha \) at \( u' \). However, we have a contradiction to Lemma 2.5 as now \( 1 \in \mathcal{F}(r) \cap \mathcal{F}(u') \) and \( d(u') \geq \Delta - k + 1 \). Thus we have as claimed. Then \( x \) is not \((1, \alpha)\)-linked to \( r \), as \( x \not\in V(F) \) (\( x \not\in N(r) \)). Thus after doing \( \alpha \rightarrow 1 \) at \( x \), \( F \) stays as a maximum Vizing fan. Now we let \( \gamma \in \mathcal{F}(v) \). Similarly as earlier, we see that \( v \) is \((1, \gamma)\)-linked to \( r \), as otherwise we can do \( \gamma \rightarrow 1 \) at \( v \) and reach a contradiction with Lemma 2.5. Hence \( x \) is not \((1, \gamma)\)-linked to \( r \), as \( x \not\in V(F) \) (\( x \not\in N(r) \)). Thus we do \( 1 \rightarrow \gamma \) at \( x \). Now \( \gamma \in \mathcal{F}(x) \cap \mathcal{F}(v) \) and we recolor the edge \( vx \) by \( \gamma \). Note that \( F \) stays as a maximum Vizing fan after these two operations. As a result, we have \( \beta \in \mathcal{F}(v) \cap \mathcal{F}(w) \), a contradiction to Lemma 2.5. Therefore, \( G \) has no vertex of degree less than \( \Delta - k + 1 \) and \( V(G) \) is elementary by Lemma 2.5, as desired. \( \square \)

4 Proof of Lemma 2.4

Proof. Let \( G \) be a class 2 graph, \( e = rs_0 \) be a critical edge and \( F = (r, e_0, s_0, e_1, s_1, \ldots, e_p, s_p) \) be a maximum Vizing fan at \( r \) w.r.t. \( e \), and let \( \varphi \in C^2(G - e) \) be the coloring where \( F \) is obtained. Let \( F \cup F' \) be an extended Vizing fan as defined earlier using the vertex \( s_h \) with \( 0 \leq h \leq p \). Let \( 1 \in \mathcal{F}(r) \) and \( \beta \in \mathcal{F}(s_h) \cap K \), where \( K \) is the set of stopping colors.

We first prove (a). Assume otherwise that there exist \( \alpha \in \mathcal{F}(x_1) \cap \mathcal{F}(x_2) \) with \( x_1, x_2 \in V(F \cup F') \). We first assume that \( \alpha \not\in K_F \). Since only one of \( x_1, x_2 \) is \((1, \alpha)\)-linked to \( r \), so we assume that \( x_1 \) is not \((1, \alpha)\)-linked to \( r \). By Lemma 2.1, \( x_1 \not\in V(F) \). By the definition of \( F' \), \( x_1 \) must be a vertex along a \( \tau \)-sequence \( L \) such that \( \varphi(v) = \tau \) for a vertex \( v \in N_{<\Delta}(r) \) and \( \varphi(vs_h) \not\in K_F \), where \( K_F \) is the set of stopping colors outside of \( F \). In this case, we do \( \alpha \rightarrow 1 \) at \( x_1 \). Note that \( \varphi(vs_h) \) may be changed when we did \( \alpha \rightarrow 1 \) at \( x_1 \), but we still have \( \varphi(vs_h) \not\in K_F \) because \( 1, \alpha \not\in K_F \). Moreover, if \( L \) contains an edge colored by \( \alpha \) and a vertex \( x_3 <_L x_1 \) with \( \alpha \in \mathcal{F}(x_3) \) such that \( x_1 \) and \( x_3 \) are \((1, \alpha)\)-linked, \( x_1 \) may no longer belong to the corresponding \( \tau \)-sequence \( L \) after the color switching at \( x_1 \). Nonetheless, we can still shift \( L \) from \( v \) to \( x_3 \) and recolor \( rx_3 \) by 1 if the earlier mentioned \( x_3 \) exists, and we shift \( L \) from \( v \) to \( x_1 \) and recolor \( rx_1 \) by 1 if otherwise. Now \( \tau \in \mathcal{F}(v) \cap \mathcal{F}(r) \) and \( F \) is still a Vizing fan. Recall that \( \beta \in \mathcal{F}(s_h) \cap K \). Let \( \gamma = \varphi(vs_h) \). So \( \gamma \not\in K_F \). By Lemma 2.2(a), \( s_h \) and \( r \) are \((\tau, \beta)\)-linked. Then we do \( \tau \rightarrow \beta \) at \( v \). As a result, \( \beta \in \mathcal{F}(s_h) \cap \mathcal{F}(v) \) and the edge \( vs_h \) is still colored by \( \gamma \). If \( \gamma \not\in \mathcal{F}(F) \), then we do \( \beta \rightarrow \gamma \) at \( s_h \). Recall that \( \beta \in K_F \) gives us a \( \Delta \)-degree neighbor of \( r \), say \( w \). Now since \( \gamma \not\in K_F \), under the new coloring a \( \gamma \)-sequence of at least two vertices can be added to \( F \) with removing \( w \) from \( F \), resulting in a larger Vizing fan, which is a contradiction to \( F \) being maximum w.r.t. \( e \). Thus we may assume \( \gamma \in \mathcal{F}(s_i) \) for a vertex \( s_i \in V(F) \). Now we recolor the edge \( vs_h \) by \( \beta \), and \( \gamma \) becomes a missing color at \( s_h \). Since \( \beta \in K \), we then
have a nonelementary Vizing fan containing both \( s_h \) and \( s_i \), a contradiction to Lemma 2.1.

Now we assume that \( \alpha \in K_F \). Recall that \( \beta \in \varphi(s_h) \cap K \). Similarly as earlier, we assume that \( x \) is not \((\alpha, \beta)\)-linked to \( s_h \), and \( x \) is added to \( F' \) through a \( \tau \)-sequence at \( r \) starting from \( rv \). We do \( \alpha \to \beta \) at \( x \). Since \( x \) is not \((\alpha, \beta)\)-linked to \( s_h \) and \( \alpha, \beta \in K \), this process does not change the colors on \( vs_h \) and \( rv \), and \( x \) still belongs to a \( \tau \)-sequence at \( r \) starting from \( rv \). Since \( s_h \) is a \((1, \beta)\)-linked to \( r \), we can do \( \beta \to 1 \) at \( x \) and this process still keeps the colors of \( vs_h \) and \( rv \), and \( x \) still belongs to a \( \tau \)-sequence at \( r \) starting from \( rv \). We then have \( 1 \in \varphi(x) \cap \varphi(r) \) and returned to the previous case of \( \alpha \notin K_F \) with 1 in place of \( \alpha \).

To see (b), we just do \( \gamma \to 1 \) at \( v_{F \cup F'}(\gamma) \) if (b) fails. Since \( v_{F \cup F'}(\gamma) \) and \( r \) are not \((1, \gamma)\)-linked, this Kempe change does not involve any edge of \( F \cup F' \). Although this Kempe change may change the color on some edge \( vs_h \) where a \( \varphi(rv) \)-sequence is contained in \( F' \) by the definition of \( F' \), since \( 1, \gamma \notin K_F \), we still have that the color on the edge \( nv \) is not contained in \( K_F \), and as a result, \( v_{F \cup F'}(\gamma) \) now belongs to a nonelementary extended Vizing fan (it may be different from \( F \cup F' \), but it still contains the vertex \( v_{F \cup F'}(\gamma) \)), giving a contradiction to (a).

The proof of (c) is similar to the proof of (b), as we can do \( \gamma \to \beta \) at \( v_{F'}(\gamma) \) if (c) fails. Since \( \beta \in K, \beta \in \varphi(s_h) \) and \( v_{F'}(\gamma) \) is not \((\beta, \gamma)\)-linked to \( s_h \), \( v_{F'}(\gamma) \) now belongs to a nonelementary extended Vizing fan (again it may be different from \( F \cup F' \), a contradiction to (a).

If the first part of (d) fails, then by \( \gamma \to 1 \) at \( v_{F'}(\gamma) \), we have a nonelementary extended Vizing fan containing the vertex \( v_{F'}(\gamma) \), a contradiction to (a). If the second part of (d) fails, then similarly we do \( \gamma \to \zeta \) at \( v_{F'}(\gamma) \) and get a nonelementary extended Vizing fan containing the vertex \( v_{F'}(\gamma) \). Note that in both parts after the operation on \( v_{F'}(\gamma) \), the extended Vizing fan may be different from \( F \cup F' \), but it will still contain the vertex \( v_{F'}(\gamma) \).

5 PROOF OF LEMMA 2.5

Proof: Let \( G, k, s = s_0, r \), and the maximum Vizing fan \( F = (r, e_0, s_0, e_1, s_1, ..., e_p, s_p) \) be as defined in Lemma 2.5 under the coloring \( \varphi \in C^A(G - e_0) \). Recall that \( K \) is the set of stopping colors at \( r \) and \( K_F = K - \varphi(V(F)) \) is the set of stopping colors outside of \( F \). Then \( |K| = k \) as \( r \) has core degree \( k \). We denote \( K_F \) by \( k' \). Let \( 1 \in \varphi(r) \). To prove Lemma 2.5, we assume otherwise that there are two vertices \( x, x' \) with degree at least \( \Delta - k + 1 \) such that \( \alpha \in \varphi(x) \cap \varphi(x') \). Since \( r \) is \((1, \alpha)\)-linked to exactly one vertex, we may assume that \( x \) is not \((1, \alpha)\)-linked to \( r \). By Lemma 2.1, \( x \notin V(F) \). Thus we do \( \alpha \to 1 \) at \( x \). As a result, \( 1 \in \varphi(x) \). Assume \( \beta \in K \cap \varphi(s_h) \) for some \( h \) with \( 0 \leq h \leq p \). Let \( F \cup F' \) be an extended Vizing fan defined with \( s_h \) where \( F' \) is a collection of all the \( \tau \)-sequences outside of \( F \) such that \( \tau = \varphi(rv) \) and \( \varphi(vs_h) \notin K_F \). Since \( N_{\alpha}(r) = k \) and \( rs \) is critical, all neighbors of \( r \) have degree at least \( \Delta - k + 1 \) by Theorem 2.3(VAL). Thus all vertices in \( V(F \cup F') \) have degree at least \( \Delta - k + 1 \). We have the following claim.

Claim 1. \( |N_{\Delta}(x) \cap V(F \cup F')| \geq k + 1 \).

By Lemma 2.2(d), \( F \) contains at least \( |\varphi(s_h)| \) many \( \Delta \)-neighbors of \( r \). Thus we have \( k' \geq k - |\varphi(s_h)| \). Since \( |\varphi(s_h)| = \Delta - d_G(s_h) \) when \( h > 0 \) and \( |\varphi(s_0)| = \Delta - d_G(s_0) + 1 \), we have
when $h = 0$, we have $d_G(s_h) \geq \Delta - |\varphi(s_h)|$, and therefore $d_G(s_h) \geq \Delta - (k - k') = \Delta - k + k'$. So $|\mathcal{N}(r) \cap \mathcal{N}(s_h)| \geq d_G(r) + d_G(s_h) - |V(G)| = \Delta + \Delta - k + k' - n \geq n/3 + 2k - k$. Since there are at most $k'$ neighbors of $s_h$ that are joined to $s_h$ by colors in $K_F$, we have $|V(F \cup F')| \geq n/3 + 2k$. Because $r$ has exactly $k$ many $\Delta$-neighbors, $F \cup F'$ has at least $n/3 + 2k - k = n/3 + k$ many vertices with degree less than $\Delta$. As $d_G(x) \geq \Delta - k + 1 \geq 2n/3 + k/2 + 1$, we have $|N_{<\Delta}(x) \cap V(F \cup F')| \geq 2n/3 + k/2 + 1 + n/3 + 2k - n \geq k + 1$, as desired.

By considering the colors on edges joining $x$ and vertices in $N_{<\Delta}(x) \cap V(F \cup F')$, we have the following three cases.

Case 1. There is a vertex $u \in N_{<\Delta}(x) \cap V(F \cup F')$ such that $\tau = \varphi(xu) \in \varphi(V(F \cup F'))$. We first assume that $u \in V(F)$. Now if $\tau \in \varphi(V(F))$ and there is a color $\gamma \in \varphi(u)$ such that $\gamma$ and $\tau$ are incomparable along $\leq_F$, we do $1 \rightarrow \gamma$ at $x$. Since $u$ is $(1, \gamma)$-linked to $r$ by Lemma 2.2(a) before the operation, we see that $\gamma$ is missing at both ends of the edge $ux$ colored by $\tau$ after the operation. So $u$ is $(\gamma, \tau)$-linked to $x$ in the resulting coloring. However, this is a contradiction, as $u$ should be $(\gamma, \tau)$-linked to the vertex in $F$ with missing color $\tau$ by Lemma 2.2(b) and the fact that $\gamma$ and $\tau$ are incomparable along $\leq_F$. If $\tau \in \varphi(V(F))$ and there is a color $\gamma \in \varphi(u)$ such that $\tau \prec_F \gamma$ along $\leq_F$, we similarly do $1 \rightarrow \gamma$ at $x$ and get a contradiction with Lemma 2.2(c). In the case $\tau \in \varphi(V(F))$ and there is a color $\gamma \in \varphi(u)$ such that $\gamma \prec_F \tau$ along $\leq_F$, we simply do a shifting from $rs$ to $r\varphi(\tau)$ and uncolor the edge $rv_F(\tau)$. As a result, there exists a color in $\varphi(u)$ that is incomparable with $\tau$ and we reach an earlier case.

We now consider the case that $\tau \in \varphi(V(F'))$. If $\tau \not\in K_F$, then we do $1 \rightarrow \tau$ at $x$. Since $r$ is $(1, \tau)$-linked to $v_F(\tau)$ by Lemma 2.4(b), and the set $\{s \in N(s_h) : \varphi(ss_h) \not\in K_F\}$ stays the same, it is easy to see that $F \cup F'$ is still an extended Vizing fan. Similarly by Lemma 2.4(c), $s_h$ is $(\beta, \tau)$-linked to $v_F(\tau)$. We then do $\tau \rightarrow \beta$ at $x$. Note that $F$ is still a Vizing fan under this new coloring. So by $\beta \rightarrow 1$ at $x$, we reach the earlier case of $\varphi(xu) \in \varphi(V(F'))$, because $s_h$ is $(1, \beta)$-linked to $r$. For readers’ convenience, in the remainder of this paper we will only give operations performed without repeating each time in details Lemmas 2.2 and 2.4, the set $\{s \in N(s_h) : \varphi(ss_h) \not\in K_F\}$, and the resulting extended Vizing fan. Now if $\tau \in K_F$, we do $1 \rightarrow \beta \rightarrow \tau$ at $x$. Since $|\varphi(s_0)| \geq 2$, by Lemma 2.2(d), there exists a color $\zeta \in \varphi(V(F)) \cap K$ with $\tau \not\sim \zeta$. If $v_F(\tau)$ is obtained through a linear sequence with the first vertex $v$ joined to $s_h$ by the color $1$, we do $\tau \rightarrow \zeta \rightarrow 1$ at $x$ following Lemmas 2.4(d) and 2.2(a), where we reach the previous case of $\varphi(xu) \in \varphi(V(F'))$. Note that here $\varphi(u\tau)$ might be changed to $\zeta$ and the set $\{s \in N(s_h) : \varphi(ss_h) \not\in K_F\}$ might change, but $u$ stays in an extended Vizing fan in the new coloring. If $v_F(\tau)$ is obtained through a linear sequence with the first vertex $v$ joined to $s_h$ by a color other than $1$, we do $\tau \rightarrow 1$ at $x$ following Lemma 2.4(d), where we reach the previous case of $\varphi(xu) \in \varphi(V(F'))$. Here the set $\{s \in N(s_h) : \varphi(ss_h) \not\in K_F\}$ might change, but $u$ stays in an extended Vizing fan in the new coloring.

We then assume that $u \in V(F')$. Let $\gamma$ be a color in $\varphi(u)$. If $\gamma \in K_F$, we can just do $1 \rightarrow \beta \rightarrow \gamma \rightarrow \tau$ at $x$ and get a nonelementary extended Vizing fan, a contradiction to Lemma 2.4(a). Therefore, we may assume $\gamma \not\in K_F$. If $v_F(\tau)$ and $u$ do not belong to the same linear sequence $L$ added to $F'$ with $u \prec_L v_F(\tau)$, then we have a nonelementary extended Vizing fan by $1 \rightarrow \gamma \rightarrow \tau$ at $x$. Thus we may assume $v_F(\tau)$ and $u$ are both belong to a linear sequence $L$ added to $F'$ with $u \prec_L v_F(\tau)$. Let the first vertex of $L$ be $v$. Since $|\varphi(s_0)| \geq 2$, by Lemma 2.2(d), there exists a color $\zeta \in \varphi(V(F)) \cap K$ with $\zeta \not\sim \beta$. We
first do \(1 \rightarrow \beta \rightarrow \tau \) at \(x\). Now similarly as before, if \(\varphi (v_{s_h}) = 1\), we then do \(\tau \rightarrow \zeta \rightarrow 1 \) at \(x\) following Lemma 2.4(d) to reach the previous case as \(\varphi (xu) = \beta \in \bar{\varphi}(V(F))\). If \(\varphi (v_{s_h}) \neq 1\), we then do \(\tau \rightarrow 1 \) at \(x\) following Lemma 2.4(d) to reach the previous case as \(\varphi (xu) = \beta \in \bar{\varphi}(V(F))\).

Case 2. There is a vertex \(u \in N_{< \Delta} (x) \cap V(F \cup F')\) such that \(\tau = \varphi (xu) \notin \bar{\varphi}(V(F \cup F'))\) and not all the \(\tau\)-sequence outside of \(F\) is of Type D.

By the assumption of this case, there is a \(\tau\)-sequence \(L\) outside of \(F\) which is of Type A, B, or C. Let \(y \in \bar{\varphi}(u)\). We first assume that \(u \in V(F)\). Now if \(L\) is of Type A or C, then by doing \(1 \rightarrow \gamma \rightarrow \tau \) at \(x\), we have a nonelementary Vizing fan contradicting Lemma 2.5. If \(L\) is of Type B with \(\bar{\varphi}(V(L)) \cap \{1, \gamma\} \neq \emptyset\), then by doing \(1 \rightarrow \gamma \rightarrow \tau \) at \(x\), we still have \(\bar{\varphi}(V(L)) \cap \{1, \gamma\} \neq \emptyset\), reaching a contradiction by resulting either a larger Vizing fan or a nonelementary Vizing fan. Now the remaining case is that \(L\) is of Type B, and there exists a color \(\eta \in \varphi(V(F)) \cap \varphi(V(L))\) with \(\eta \notin \{1, \gamma\}\). If \(u \leq F v_F(\eta)\) along \(\leq F\) does not happen, then by doing \(1 \rightarrow \gamma \rightarrow \tau \) at \(x\), we have a nonelementary Vizing fan contradicting Lemma 2.5. If \(u \leq F v_F(\eta)\) along \(\leq F\), then we simply do a shifting from \(r\) to \(n v_F(\eta)\) and uncolor the edge \(n v_F(\eta)\), reaching the earlier case of \(u \leq F v_F(\eta)\) not happening.

We then assume \(u \in V(F')\). We first consider the case that \(L\) is not of Type B with \(\{1\} = \bar{\varphi}(V(L)) \cap \bar{\varphi}(V(F'))\), or \(L\) is of Type B with \(\{1\} = \bar{\varphi}(V(L)) \cap \bar{\varphi}(V(F'))\) and \(x \in V(L)\). If \(\gamma \notin K_F\), we just do \(1 \rightarrow \gamma \rightarrow \tau \) at \(x\) to get a nonelementary extended Vizing fan, a contradiction to Lemma 2.4(a). Therefore, we have \(\gamma \in K_F\). If \(L\) is not of Type B with \(\{\beta\} = \bar{\varphi}(V(L)) \cap \bar{\varphi}(V(F'))\), we have a nonelementary extended Vizing fan by \(1 \rightarrow \beta \rightarrow \gamma \rightarrow \tau \) at \(x\), a contradiction to Lemma 2.2(a). Thus we may assume \(L\) is of Type B with \(\{\beta\} = \varphi(V(L)) \cap \varphi(V(F))\). Suppose that \(u\) is added to \(F'\) by a linear sequence \(L'\) with first vertex \(v\). Again since \( \bar{\varphi}(s_0) \geq 2\), by Lemma 2.2(d), there exists a color \(\zeta \in \varphi(V(F)) \cap K\) with \(\zeta \neq \beta\). Now if \(\varphi(v_{s_h}) \neq 1\), we do \(1 \rightarrow \gamma \rightarrow \tau\) at \(x\) to get a nonelementary extended Vizing fan, and if \(\varphi(v_{s_h}) = 1\), we do \(1 \rightarrow \zeta \rightarrow \gamma \rightarrow \tau\) at \(x\) to get a nonelementary extended Vizing fan, both give contradictions to Lemma 2.2(a).

Thus we can assume \(L\) is of Type B with \(\{1\} = \bar{\varphi}(V(L)) \cap \bar{\varphi}(V(F))\) and there is a vertex \(z \in V(L)\) with \(1 \in \bar{\varphi}(z)\) and \(z \neq x\). Clearly \(\tau \notin K_F\) as \(L\) is of Type B. By the definition of extremal linear sequences outside of \(F\), \(z\) is the last vertex of \(L\). Let the first vertex of \(L\) be \(w\). Note that here \(z\) and \(w\) could be the same vertex. Clearly \(F'\) and \(L\) do not share common vertices, as otherwise \(z \in V(F')\) and we have a nonelementary extended Vizing fan. Recall that \(y \in \bar{\varphi}(u)\). Now we do \(1 \rightarrow \beta\) at both \(x\) and \(z\) following Lemma 2.2(a). As a result, \(\beta \in \bar{\varphi}(z) \cap \bar{\varphi}(x)\). Note that \(\varphi(wr) = \tau\) and \(d(w) < \Delta\), so \(\tau \notin K_F\). Therefore, \(s_h\) and \(r\) must be \((\beta, \tau)\)-linked, as otherwise we have a larger Vizing fan by interchanging \(\beta\) and \(\tau\) along \(C_r(\beta, \tau)\). Now if \(s_h\) and \(x\) are \((\beta, \tau)\)-linked, we can do \(\beta \rightarrow \tau \) at \(z\) and then do \(\beta \rightarrow 1 \) at \(x\), reaching the earlier case that \(L\) is not of Type B. We then assume \(s_h\) and \(z\) are \((\beta, \tau)\)-linked. In this case, we first do \(\beta \rightarrow \tau\) at \(x\) and then do \(\beta \rightarrow 1 \) at \(z\). Now \(\varphi(xu) = \beta\), \(\tau \in \bar{\varphi}(x)\), and \(1 \in \bar{\varphi}(z)\). We then do a shift along \(L\) from \(w\) to \(z\), and color the edge \(r \xi_2\) by 1. For reader’s convenience, we switch labels for color 1 and \(\tau\) to meet the notations we used earlier. After the switching of 1 and \(\tau\), we now still have \(1 \in \bar{\varphi}(r) \cap \bar{\varphi}(x), \gamma \in \bar{\varphi}(u), \varphi(ux) = \beta,\) and \(F \cup F'\) is still an extended Vizing fan as \(\tau \notin K_F\), which returns to Case 1. Finally we may assume that \(s_h \in (\beta, \tau)\)-linked to neither \(z\) nor \(x\). We then do \(\beta \rightarrow \tau\) at both \(z\) and \(x\). As a result, \(\tau \in \bar{\varphi}(x) \cap \bar{\varphi}(z)\) and \(\varphi(ux) = \beta\). Recall that \(\tau \notin K_F\). If \(r\) is \((1, \tau)\)-linked to \(z\), then by doing \(\tau \rightarrow 1 \) at \(x\), we reach Case 1. If \(r\)
is \((1, \tau)\)-linked to \(x\), then by doing \(\tau \rightarrow 1\) at \(z\), we reach the earlier case where we shift along \(L\) from \(w\) to \(z\), and color the edge \(rz\) by 1. If \(r\) is \((1, \tau)\)-linked to neither \(z\) nor \(x\), then by doing \(\tau \rightarrow 1\) at both \(z\) and \(x\), we reach Case 1. This finishes Case 2.

**Case 3.** All the \(\tau\)-sequence outside of \(F\) is of Type D, where \(\tau = \varphi(xu) \notin \varphi(V(F) \cup V(F'))\) and \(u \in N_{\leq \Delta}(x) \cap V(F \cup F')\).

Since \(|N_{\leq \Delta}(x) \cap V(F \cup F')| \geq k + 1\) by Claim 1 and \(|K_{\varphi}| < k\), there must exist two vertices \(u\) and \(u^*\) in \(N_{\leq \Delta}(x) \cap V(F \cup F')\) such that \(\tau = \varphi(xu) \notin \varphi(V(F) \cup V(F'))\) and \(\tau^* = \varphi(xu^*) \notin \varphi(V(F) \cup V(F'))\) where the \(\tau\)-sequence \(L_1\) and \(\tau^*-\)sequence \(L_2\) are both of Type D ending with the same stopping color of \(F\) in \(K_F\).

We claim that one of \(L_1\) and \(L_2\) is a subsequence of the other one. Otherwise, since \(L_1\) and \(L_2\) are of Type D both ending with the same stopping color, there exists \(\theta \in \varphi(v_1) \cap \varphi(v_2)\) such that \(v_1 \in V(L_1), v_2 \in V(L_2)\), and \(v_1 \neq v_2\). Because \(L_1\) and \(L_2\) are of Type D, \(\theta \notin \varphi(V(F))\). Since \(\beta \in \varphi(s_h)\), at most one of \(v_1\) and \(v_2\) is \((\beta, \theta)\)-linked to \(s_h\). Thus we may assume that \(v_1\) is not \((\beta, \theta)\)-linked to \(s_h\). Note that if \(\theta \notin K_F\), \(r\) and \(s_h\) must be \((\beta, \theta)\)-linked, as otherwise switching \(\beta\) and \(\theta\) along \(C_r(\beta, \theta, \varphi)\), we would have a larger Vizing fan. Now by \(\theta \rightarrow \beta\) at \(s_1\), we have reached Case 2 as the subsequence of \(L_1\) ending at \(v_1\) is of Type B and this operation will not change the set \(\{s \in N(s_h) : \varphi(s_h) \notin K_F\}\). Thus we have as claimed.

Now by the above claim, we may assume \(L_2\) is a subsequence of \(L_1\), \(\tau^* \in \varphi(z)\) with \(z \in V(L_1)\), and \(\tau \notin K_F\). We are going to consider the following three cases depending on which one of \(u\) and \(u^*\) is in \(V(F')\).

We first assume that both \(u\) and \(u^*\) are in \(V(F)\). Similarly as earlier, we may assume that there exist \(\gamma \in \varphi(u)\) and \(\gamma^* \in \varphi(u^*)\) such that \(\gamma \neq \gamma^*\) are incomparable along \(\leq_F\). Otherwise, \(\gamma \leq_F \gamma^*\), then by shifting from \(s_0\) to \(v_F(\gamma^*)\) and uncolor the edge \(rv_F(\gamma^*)\), we have as desired. Now we first do \(1 \rightarrow \gamma \rightarrow \tau\) at \(x\) following Lemma 2.2(a) and consider a maximal Vizing fan \(F^*\) under this new coloring \(\varphi\). Since now \(\tau \in \varphi(u)\), and \(\gamma\) and \(\gamma^*\) were incomparable along \(\leq_F\) earlier, we have \(u, u^*, z \in V(F')\), \(\tau \in \varphi(u) \cap \varphi(x)\), \(\tau^* \in \varphi(z)\), \(\varphi(xu^*) = \tau^*\), and \(\tau^*\) and \(\gamma^*\) are incomparable along \(\leq_F\). Following Lemma 2.2(a), we can do \(\tau \rightarrow 1 \rightarrow \gamma^*\) at \(x\). As a result, \(F^*\) is still a Vizing fan and now \(u^*\) and \(x\) are \((\gamma^*, \tau^*)\)-linked, a contradiction to Lemma 2.2(b).

We then assume that \(u \in V(F')\) and \(u^* \in V(F)\), or both \(u, u^* \in V(F')\) but \(u\) and \(u^*\) do not satisfy \(u \leq_{L^*} u^*\) for a linear sequence \(L^*\) in \(F'\). Note that \(\{\gamma, \gamma^*\} \cap \{\tau, \tau^*\} = \emptyset\) as \(\tau = \varphi(xu) \notin \varphi(V(F) \cup V(F'))\) and \(\tau^* = \varphi(xu^*) \notin \varphi(V(F) \cup V(F'))\). In the case that \(\gamma \notin K_F\), we do \(1 \rightarrow \gamma \rightarrow \tau\) at \(x\) following Lemma 2.4(b), and in the case that \(\gamma \in K_F\), we do \(1 \rightarrow \beta \rightarrow \gamma \rightarrow \tau\) at \(x\) following Lemmas 2.2(b) and 2.4(c). Note that the set \(\{s \in N(s_h) : \varphi(s_h) \notin K_F\}\) does not change after the above operations and \(\tau \in \varphi(u)\) now. Thus if \(\gamma\) was in \(\varphi(V(L_1))\) and \(\gamma^*\) was changed to \(1\) or \(\beta\) by the above operations in \(\varphi(V(L_1))\), we then have a nonelementary extended Vizing fan contradicting Lemma 2.4(a). Otherwise, since \(L_2\) is a subsequence of \(L_1\), \(\tau^* \in \varphi(z)\) with \(z \in V(L_1)\) and \(u \leq_{L^*} u^* \) is not satisfied for any \(L^*\) in \(F'\), the new extended Vizing fan will contain \(z, u, u^*\) while \(\tau^* \in \varphi(z)\), \(\tau \in \varphi(x)\) and \(\varphi(u^*x) = \tau^*\). Now since \(\tau \notin K_F\), we just do \(\tau \rightarrow 1\) at \(x\) following Lemma 2.4(b) to reach Case 1 as \(\varphi(u^*x) = \tau^* \in \varphi(z)\).

Finally we assume that \(u^* \in V(F')\) and \(u \notin V(F)\), or both \(u, u^* \in V(F')\) but \(u\) and \(u^*\) do not satisfy \(u^* \leq_{L^*} u\) for a linear sequence \(L^*\) in \(F'\). Note that \(\{\gamma, \gamma^*\} \cap \{\tau, \tau^*\} = \emptyset\) as \(\tau = \varphi(xu) \notin \varphi(V(F) \cup V(F'))\) and \(\tau^* = \varphi(xu^*) \notin \varphi(V(F) \cup V(F'))\). Similar as before,
in the case that \( \gamma^* \notin K_F \), we do \( 1 \rightarrow \gamma^* \rightarrow \tau^* \) at \( x \) following Lemma 2.4(b), and in the case that \( \gamma^* \in K_F \), we do \( 1 \rightarrow \beta \rightarrow \gamma^* \rightarrow \tau^* \) at \( x \) following Lemmas 2.2(b) and 2.4(c). Note that the set \( \{ s \in N(s_h) : \varphi(s_h) \notin K_F \} \) does not change after the above operations, and \( \tau^* \in \varphi(u^*) \) and \( \tau^* \in \varphi(x) \). Moreover, we now have \( \tau^* \in \varphi(u^*), \tau^* \in \varphi(z), \varphi(u^*x) = \gamma^*, \) and vertices \( u, u^* \) and the subsequence of \( L_1 \) after \( z \) is contained in an extended Vizing fan after the above operations. Thus if \( \gamma^* \) was a missing color in the subsequence of \( L_1 \) after \( z \) and \( \gamma^* \) was changed to 1 or \( \beta \) by the above operations as a missing color in the subsequence of \( L_1 \), we then have a nonelementary extended Vizing fan contradicting Lemma 2.4(a). If \( \gamma^* \) was a missing color in the subsequence of \( L_1 \) until \( z \) and \( \gamma^* \) was changed to 1 or \( \beta \) by the above operations as a missing color in the subsequence of \( L_1 \), we then do \( \tau^* \rightarrow 1 \) at \( x \) when \( \tau^* \notin K_F \) and \( \tau^* \rightarrow \beta \rightarrow 1 \) at \( x \) when \( \tau^* \in K_F \). Now \( 1 \in \varphi(x), \tau^* \in \varphi(u), \varphi(xu) = \tau, u \) and \( u^* \) are in an extended Vizing fan while there is \( \tau \)-sequence of either Type A or B, reaching Case 1. If the above two possibilities did not happen, then we still have \( \tau^* \in \varphi(u^*) \cap \varphi(z) \cap \varphi(x), \tau = \varphi(u^*x), u \) and \( u^* \) are still in an extended Vizing fan, and the \( \tau \)-sequence \( L_1 \) stays the same containing the vertex \( z \). Now if \( \tau^* \notin K_F \), we do \( \tau^* \rightarrow 1 \) at both \( x \) and \( z \) following Lemma 2.4(b) to reach Case 1, as now there is a \( \tau \)-sequence of Type B. In the case that \( \tau^* \in K_F \), we do \( \tau^* \rightarrow \beta \rightarrow 1 \) at both \( x \) and \( z \) following Lemmas 2.4(c) and 2.2(b) to reach Case 1, as now there is a \( \tau \)-sequence of Type B. □

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