Emergent Supersymmetric Many-Body Systems in Doped Z$_2$ Topological Spin Liquid of the Toric-Code Model

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In this paper, we studied the doped Z$_2$ topological spin liquid of the toric-code model. We found that the doped holes become supersymmetric particles. The ground state of the doped Z$_2$ topological spin liquid becomes new matters of quantum states - supersymmetric Bose-Einstein condensation or supersymmetric superfluid. As a result, this system provides a unique example of manipulatable supersymmetric many-body system.

Doped holes in a spin model had become an important issue since the discovery of high Tc superconductivity in cuprates$^1$. As the microscopic model of the high Tc superconductivity - the t – J model has been intensively studied for several decades. Motivated by the experimental facts in the high-Tc cuprates, the spin-charge separation idea$^2$ was a very basic concept by introducing spinless “holon” of charge e and neutral spin-1/2 “spinon” as the essential building blocks of the restricted Hilbert space. However, for a long range antiferromagnetic (AF) order, there is no true spin-charge separation at all. People pointed out that a single doped hole in the AF order can be a charged spin bag$^3$, or Shraiman-Siggia dipole$^4$, or localized object$^5$ from different points of view. The true spin-charge separation occurs only in the quantum disordered spin states (people call them quantum spin liquid states). And the doped spin liquid is always a superconducting order with holon-condensation.

On the other hand, in the last decade, several exactly solvable spin models with Z$_2$ topological spin liquid were found, such as the toric-code model$^2$, the Wen plaquette model$^8$ and the Kitaev model on a honeycomb lattice$^9$. It becomes an interesting issue to study the properties of doped spin liquid by doping holes to these exactly solvable spin models. In Ref.$^{10}$, the fermi liquid nature of doped Z$_2$ topological spin liquid is obtained. People also studied the effect of doped holes in the gapless phase of the Kitaev model in Ref.$^{11}$ and pointed out that the topological superconducting state can be its ground state.

In this paper, after studying the non-perturbative properties of doped holes, we found that a doped hole in the Z$_2$ topological spin liquid of the toric-code model becomes a supersymmetric holon and a universal feature of the doped Z$_2$ topological spin liquid is the emergence of supersymmetry. We found that the ground state of the doped Z$_2$ topological spin liquid is supersymmetric Bose-Einstein condensation (SBEC) state, of which there exist the fermionic Goldstone mode - Goldstino which is the partner of the Bosonic Goldstone mode. And we can manipulate the supersymmetry by tuning the transverse external field to the system. After breaking the supersymmetry by the transverse external field, the ground state becomes a new matter of a quantum state - supersymmetric superfluid. As a result, this system provides a unique example of supersymmetric many-body model.

Our starting point is the so-called t-toric-code model that is described by the following Hamiltonian

$$H = H_t + H_{tc} + H_1,$$  \hspace{1cm} (1)

$$H_t = -t \sum_{\langle ij \rangle, \sigma} P_s c_{i\sigma}^\dagger c_{j\sigma} P_s + \mu \sum_{i, \sigma} c_{i\sigma}^\dagger c_{i\sigma}$$

$$H_{tc} = -A \sum_{i \text{even}} Z_i - B \sum_{i \text{odd}} X_i,$$

$$H_1 = h x \sum_i s_i^x + h y \sum_i s_i^y$$

where $Z_i = s_i^x s_{i+e_x}^x s_{i+e_y}^x s_{i+e_x+e_y}^x s_{i+e_y}^x$, $X_i = s_i^x s_{i+e_y}^x s_{i+e_x+e_y}^x s_{i+e_x}^x s_{i+e_y}^x$ with $A > 0$, $B > 0$. $\sigma$ are the spin-indices representing spin-up ($\sigma = \uparrow$) and spin-down ($\sigma = \downarrow$) for the electrons. $\mu$ is the chemical potential. $\langle ij \rangle$ denote two sites on the nearest-neighbor link. $P_s$ projects the Hilbert space onto the subspace of one electron per-site. $s_i^{x,y,z}$ are the spin operators of the electrons. For simplify, we consider the case of $A = B = g$ in this paper. $H_1$ is the external field terms with $h^x > 0$, $h^y > 0$.

The ground state of $H_{tc}$ denoted by $Z_i \equiv +1$ and $X_i \equiv +1$ at each site is a Z$_2$ topological state. For the toric-code model, the elementary excitations are Z$_2$ vortex ($Z_i = -1$ at even sub-plaquette) and Z$_2$ charge ($X_i = -1$ at odd sub-plaquette). Fermions are the bound states of a pair of Z$_2$ vortex and Z$_2$ charge. When we add the transverse external field, the total spin Hamiltonian $H_{tc} + H_1$ cannot be solved exactly. However, for small value of the external field, the ground state is still Z$_2$ topological order and the topological properties don’t change.

Internal degree of freedom of a hole and emergent supersymmetry: Now we study a single hole doped in the toric-code model. As shown in Fig.1, a hole is a charged...
vacancy to remove the spin degree of freedom at site $i$, as $c^{\dagger}_{i \uparrow} \rightarrow c^{\dagger}_{i \downarrow}$. Then near a hole, four plaquette operators are forced to be zero, i.e. $Z_{i-\hat{x}} = Z_{i-\hat{y}} = X_i = X_{i-\hat{x}-\hat{y}} = 0$. Around this hole, there exist a boundary that separates the topological spin liquid outside and a hollow area around the hole. For a hole in the $Z_2$ topological order, the ground state degeneracy becomes 2 (without being considered the edge states) [13, 17]. To classify the degeneracy of the ground states, as shown in Fig. 2, we define two types of closed string operators, $W_t(C_A) = \prod c_i$ and $W_f(C_B) = \prod c_i$, $W_t(C_A)$ is a closed string operator around a hole. $W_f(C_B)$ is a closed string operator from the boundary of a hole to the boundary of the system. Since $W_t(C_A)$ and $W_f(C_B)$ form the Heisenberg algebra, we can map the Hilbert space of $W_t(C_A)$ and $W_f(C_B)$ onto that of a pseudo-spin ($S = \frac{1}{2}$) and identify $W_t(C_A)$ and $W_f(C_B)$ to spin-1/2 operator $S^x$ and $S^z$ as $W_t(C_A) \rightarrow S^x$ and $W_f(C_B) \rightarrow S^z$. Then, we use $\uparrow$ and $\downarrow$ to denote the pseudo-spin degree of freedom of the quantum states of a hole. These two quantum states can be characterized by the fermion parity (or the flux quanta of the $Z_2$ vortex) of the hole: $\uparrow$ corresponds to the state with even fermion parity, $\downarrow$ corresponds to the state with odd fermion parity. Now the hole’s statistics depends on its fermion parity.

For the zero external field case, $\hbar^x = \hbar^y = 0$, the two quantum states $\uparrow$ and $\downarrow$ are degenerate exactly. When we add the external field, the quantum tunneling effect will lead to an energy splitting. Then we can derive the effective model of a hole by the quantum tunneling theory as $\mathcal{H}_h \simeq \tilde{\hbar}^x S^x + \tilde{\hbar}^z S^z$ where $\tilde{\hbar}^x = \frac{2(h^x)^3}{(-8g)^{1/3}}$ and $\tilde{\hbar}^z = \frac{24(h^z)^3}{(-4g)12}$. Here $L_x$ is the distance between the boundary of the hole and the boundary of the system. See the detailed calculations in Ref. [13, 16]. In general, for an infinite system, we have $\tilde{\hbar}^z = 0$. In addition, for the two-hole case, we have the effective model of two holes as

$$\mathcal{H}_h \simeq J^{xx} S^x_1 S^x_2 + J^{zz} S^z_1 S^z_2 + \sum_{l=1,2} \tilde{\hbar}^i_l S^z_1$$

(2)

where $J^{xx} = \frac{2(h^x)^3}{(-8g)^{1/3}}$, $J^{zz} = \frac{(h^z)^3}{(-4g)^{2}}$. $l$ is the distance between the holes and $L_{zz}$ is the distance for a path surrounding the two holes. Due to $J^{zz} \ll J^{xx}$, the effective model of the two-hole case is reduced into

$$\mathcal{H}_h \simeq J^{xx} S^x_1 S^x_2 + \tilde{\hbar}^1 S^z_1 + \tilde{\hbar}^2 S^z_2.$$  

(3)

Due to the term of $J^{xx} S^x_1 S^x_2$, two holes may exchange their fermion parities.

So one may tune each parameter in above effective Hamiltonian $\mathcal{H}_h$ by controlling the external field along special direction and then manipulate the internal degree of freedom of holes. For example, in the limit $J^{xx} \rightarrow 0$, the quantum state with lowest energy is $|\downarrow\rangle_1 \otimes |\downarrow\rangle_2$ with odd fermion parity for each hole. In the limit, $\hbar^z \rightarrow 0$, the eigenstates are $(|\uparrow\rangle_1 + |\downarrow\rangle_1) \otimes (|\uparrow\rangle_2 + |\downarrow\rangle_2)$ and $(|\uparrow\rangle_1 - |\downarrow\rangle_1) \otimes (|\downarrow\rangle_2 - |\uparrow\rangle_2)$. Now the average fermion parity of each hole is 1/2 for both eigenstates.

Effective supersymmetric model: Because the original statistics of a hole is fermion, a holon (a hole in the $Z_2$ topological state) is a boson for the internal state $|\downarrow\rangle$ with odd fermion parity or a fermion for the internal state $|\uparrow\rangle$ with even fermion parity. Then for a single holon, we can use a two-component supersymmetric operator to describe it as $c_{\uparrow i}$, $c_{\downarrow i}$ plays the role of a two-component spinor with a pseudo-spin where $c_{\uparrow i} = b_i$ and $c_{\downarrow i} = f_i$ are the charged boson and charged fermion operators on site $i$.

Now we consider the t-toric-code model with finite hole-concentration, of which the effective Hamiltonian is $H_{eff} = H_t + \mathcal{H}_h$, where $H_t$ is the hopping term of supersymmetric holons,

$$H_t = -t \sum_{(ij)\tilde{\sigma}} P_{ij} c^\dagger_{\tilde{\sigma} i} c_{\tilde{\sigma} j} P_s + \mu_B \sum_i b_i^\dagger b_i + \mu_F \sum_i f_i^\dagger f_i.$$  

(4)
$\mathcal{P}_s$ projects the Hilbert space onto the subspace of one particle per site. Now the projector $\mathcal{P}_s$ has no effect for the spinless fermionic holon $\hat{f}_i^\dagger$ while guarantees the particle number of bosonic holon $n_i^b = \hat{b}_i^\dagger \hat{b}_i$. $\mu_B$ and $\mu_F$ are the chemical potentials for the bosonic holons and that of fermionic holon, respectively.

To characterize the supersymmetry, we introduce the generators $S_i^x, S_i^y, S_i^z$ that are just the string operators $W_i^f(C_b^+), -iW_i^f(C_b^-)W_i^e(C_a^+), W_i^e(C_a^-)$. And we have $S_i^x = (S_i^+ - S_i^-)/2, S_i^y = (S_i^+ + S_i^-)/2i$. It is clear that $S_i^x$ is a fermionic generator. Physically, $S_i^x$ turns a fermionic holon into a bosonic holon, and $S_i^y$ does the opposite. By the two-component supersymmetric operator, we have $S_i^z = \hat{b}_i^\dagger \hat{f}_i, S_i^+ = \hat{b}_i^\dagger \hat{f}_i^\dagger$ and $S_i^- = (\hat{b}_i^\dagger \hat{f}_i - \hat{f}_i^\dagger \hat{b}_i)/2$. Thus we re-write the term $H_0$ into

$$H_0 \simeq \sum_{ij} J_{ij}^{xx} S_i^x S_j^x + \frac{\hbar^2}{2} \sum_i (\hat{b}_i^\dagger \hat{b}_i - \hat{f}_i^\dagger \hat{f}_i)$$

(5)

where $J_{ij}^{xx} = \frac{2(\mu_B^f)}{(\pi g)^2}$. Because the exchange term $J_{ij}^{xx}$ decays exponentially, we may only consider the shortest case that is $l = 2$ (for $l < 2$, the hollow regions of two holons merge and we cannot define the corresponding quantum tunneling process). As a result, the fermion parity exchange term $\sum_{ij} J_{ij}^{xx} S_i^x S_j^x$ is reduced into

$$J_{ij}^{xx} = \sum_i (S_i^+ S_i^+ + S_i^- S_i^-)$$

$$= \frac{J_{ij}^{xx}}{4} \sum_i [(\hat{b}_i^\dagger \hat{f}_i + \hat{b}_i^\dagger \hat{f}_i^\dagger)(\hat{b}_i^\dagger \hat{f}_i + \hat{b}_i^\dagger \hat{f}_i^\dagger)]$$

Thus we have an exact supersymmetry for the toric code model without the external field, i.e., $[S_i^z, H_{\text{eff}}] = [S_i^z, H_{\text{eff}}^1] = 0$. When there exists the external field ($h_x \neq 0, h_y \neq 0$), the supersymmetry is broken explicitly, i.e., $[S_i^z, H_{\text{eff}}] \neq 0, [S_i^z, H_{\text{eff}}^1] \neq 0$.

**Supersymmetric Bose-Einstein condensation:** We firstly study the many-body system with exact supersymmetry as $[S_i^z, H] = 0$. For the dilute hole limit of holons (the hole-concentration $\delta$ is smaller than 1%), the effective model is

$$H_{\text{eff}} = -t \sum_{(ij)} \hat{b}_i \hat{b}_j - t \sum_{(ij)} \hat{f}_i^\dagger \hat{f}_j + \mu_B \sum_i \hat{b}_i \hat{b}_i + \mu_F \sum_i \hat{f}_i^\dagger \hat{f}_i$$

(7)

where $\mu_F = \mu_B = -4t$. $b_j$ denotes the annihilation operator of the hard-core bosons. Because the small hole-concentration limit, we can release the single-occupied condition of the bosonic holons and consider the holons as non-interacting bosons. For non-interacting bosons we always have $\mu_B = -4t$ (we measure the energy from the bottom of single particle dispersion) at zero temperature, regardless of boson number. Supersymmetry of $H_{\text{eff}}$ requires $\mu_F = \mu_B$, as a result we have 0 or 1 fermionic holon. Now each holon becomes a boson and the ground state is a BEC state with holon-condensation as $\langle 0 | b_{k=0} \rangle = b_{0} e^{i \varphi_0} \neq 0$. Here $| 0 \rangle$ denotes the ground state. Thus the ground state spontaneously breaks both global U(1) symmetry and supersymmetry.

There are two types of collective excitations. One type is the Goldstone mode that describes the phase fluctuations as $\varphi_0 \rightarrow \varphi_0 + \varphi_i$. The effective Hamiltonian of the Goldstone mode is $H = -2t \sum_i (\sum_j \cos(\varphi_j - \varphi_i))$ which is really a two-dimensional XY model. Another type of collective excitation is the Goldstino mode which can be regarded as the fermionic "spin wave" [13]. Due to the holon-condensation, we have $S_i^+ = b_i f_i^\dagger$. From the hopping term of $f_i^\dagger$, we have the effective model of the Goldstino as $H_{\text{Goldstino}} = -t \sum_i (\sum_j \tilde{S}_i^+ \tilde{S}_j^-)$. The dispersion of the Goldstino mode is given by $E_k = |b_0|^{-2} \varepsilon_k$ where $\varepsilon_k = -2t(\cos k_x + \cos k_y)$. Due to the existence of the Goldstino mode, we call the unique ground state to be supersymmetric Bose-Einstein condensation (SBEC).

**Supersymmetric Bose-Einstein condensation with Fermi surface:** For the case of $h_x \neq 0, h_y = 0$, the effective model turns into

$$H_{\text{eff}} = -t \sum_{(ij)} \hat{b}_i \hat{b}_j - t \sum_{(ij)} \hat{f}_i^\dagger \hat{f}_j + \sum_i \mu_B \hat{b}_i$$

$$+ \sum_i \mu_F \hat{f}_i^\dagger \hat{f}_i + \frac{\hbar^2}{2} \sum_i (\hat{b}_i^\dagger \hat{b}_i - \hat{f}_i^\dagger \hat{f}_i)$$

(8)

where $\tilde{h} = \frac{2(\mu_B^f)^2}{(\pi g)^2}$. The supersymmetry is partially broken by $\tilde{h}$ term.

Because there is no couple between fermionic holons and bosonic holons, the ground state can be considered to be a Bose-Fermi mixture. The bosonic holons condense and fermionic holons form Fermi liquid. In the dilute hole limit, we can simplify the Fermi surface of the fermionic holons to be a circle with a radius $k_F$. Now in continuum limit, the effective Hamiltonian can be reduced into

$$H_{\text{eff}} = \sum_k [\frac{\hbar^2}{2m} + (\mu_B)_{\text{eff}}] b_k^\dagger b_k + \sum_k [\frac{\hbar^2}{2m} + (\mu_F)_{\text{eff}}] f_k^\dagger f_k$$

(9)

where the effective chemical potential of bosonic holons $(\mu_B)_{\text{eff}}$ and the effective chemical potential of fermionic holons $(\mu_F)_{\text{eff}}$ are $(\mu_B)_{\text{eff}} = \mu_B + \frac{\tilde{h}}{2}$ and $(\mu_F)_{\text{eff}} = \mu_F - \frac{\tilde{h}}{2}$, respectively. $m$ is the effective mass of holons, $m \simeq \frac{1}{2}$. Then we can estimate the number of fermionic holons and that of bosonic holons by minimizing the total ground state energy. We define the total holon number to be $N_t = N_b + N_f$ where $N_b = \sum_i n_{b_i}^i$ and $N_f = \sum_i n_{f_i}^i$. The total ground state energy is $E_{\text{total}} = E_b + E_f$, where $E_b = \frac{\hbar^2}{2m} N_b = \frac{\hbar^2}{2m} (N_b - N_f)$ is the ground state energy of bosonic holons and $E_f = (\delta f)^2 \frac{\pi N}{m} - \frac{N_f \hbar^2}{2m}$ is the ground state energy of fermionic holons. Here $N$ is the lattice number and $\delta f$ is the concentration of fermionic holons.
From the condition, $\frac{\partial E_{\text{fin}}} {\partial N_f} = 0$, we obtain the concentration of fermionic holons as $\delta_f = \frac{\mu_f}{2g} = \tilde{h}^x_c$. Because the maximum fermion holon’s concentration is $\delta$ which is the hole concentration, we get a critical value of the external field, $(\tilde{h}^x)_c = 4\pi t\delta$ or $(h^x)_c = (4g)^{11/12}(\frac{\pi t\delta}{6})^{1/12}$.

For small external field case, $h^x < (h^x)_c$, there exists bosonic holons with the concentration to be $\delta_b = \delta - \delta_f$. Now the ground state is a mixture of SBEC of bosonic holons and Fermi liquid (FL) of fermionic holons (we call it SBEC+FL state). In general, $\tilde{h}^x$ is very tiny. For example, for the case of $h^x/g = 0.3$, we have $\tilde{h}^x/g \simeq -3 \times 10^{-12}$. So we always have the SBEC state for bosonic holons and the Fermi liquid for fermionic holons with very tiny fermi surface.

For large external field case, $h^x > (h^x)_c$, all holons are fermions. The ground state is a Fermi liquid (FL) state with the number of fermionic holons as $N_f = N_i$. Now the effective chemical potential of fermionic holons is $(\mu_F)_{\text{eff}} \approx 2\pi t\delta$ and the effective chemical potential of bosonic holons is $(\mu_B)_{\text{eff}} \approx \tilde{h}^x/2$. There exists a finite energy gap to excite a bosonic holon. While the fermions have no energy gap.

From these results we plot the phase diagram in Fig.3, of which the red line denotes the quantum phase transition between SBEC+FL and FL. The FL state is only stable in the limit of $t\delta \to 0$ and $h^x/g \to 1$. However, for $h^x > 0.34g$, a topological quantum phase transition occurs and the ground state turns into a spin polarized state without topological order [19-23]. Thus we can only discuss the case of small external field as $h^x < 0.34g$.

**Supersymmetric superfluid**: In this section we study the case of $h^y \neq 0$, $h^x \neq 0$. The supersymmetry is broken completely. Now the effective model is

$$H = -t \sum_{\langle ij \rangle} b_i^\dagger b_j - t \sum_i f_i^\dagger f_j + \mu_B \sum_i b_i^\dagger b_i$$

$$+ \mu_F \sum_i f_i^\dagger f_i + \frac{\tilde{h}^x}{2} \sum_i (b_i^\dagger b_i - f_i^\dagger f_i)$$

$$+ \frac{J^x_{\text{eff}}}{4} \sum_i [(b_i^\dagger f_i + b_i f_i^\dagger)(b_i^\dagger f_{i+3e_x} + b_{i+3e_x} f_i^\dagger)$$

$$+ (b_i^\dagger f_i + b_i f_i^\dagger)(b_i^\dagger f_{i+3e_y} + b_{i+3e_y} f_i^\dagger)]. \quad (10)$$

The ground state is really a BEC of bosonic holons $\langle 0|b_0^\dagger b_0|0 \rangle = b_0 e^{i\pi t\delta}$ with induced pairing of fermionic holons.

Let us estimate the induced SC pairing of fermionic holons. The effective Hamiltonian of fermionic holons is

$$H_f \simeq -t \sum_{\langle i,j \rangle} f_i^\dagger f_j + (\mu_F)_{\text{eff}} \sum_i f_i^\dagger f_i$$

$$+ \frac{\tilde{h}^x}{4} \sum_i (f_i^\dagger f_{i+3e_x} + f_i f_{i+3e_y}) + h.c. \quad (11)$$

Thus, when the bosonic holons condensate, there exists $p_x + p_y$ superconducting order parameter for the non-interacting fermionic holon

$$\langle 0|f_i^\dagger f_{i-k}^\dagger 0 \rangle = \frac{\tilde{h}^x}{4} [(\sin 3k_x + \sin 3k_y)]. \quad (12)$$

Now we have a gapless fermionic holon with $p_x + p_y$ pairing.

Due to the condensation of the bosonic holons and the induced SC pairing of fermionic holons, the effective model of the Goldstino mode turns into

$$H_{\text{Goldstino}} = -t |b_0|^{-2} \sum_{\langle ij \rangle} S_i^+ S_j^- + (\mu_F)_{\text{eff}} |b_0|^{-2} \sum_i S_i^+ S_i^-$$

$$+ J^x_{\text{eff}} \sum_i (S_i^+ S_{i+3e_x} + S_i^+ S_{i+3e_y}) + h.c.. \quad (13)$$

Then the dispersion of the Goldstino mode is derive as $E_k = |b_0|^{-2} \sqrt{\varepsilon^2_k + \Delta_k^2}$ where $\varepsilon_k = -2t(\cos k_x + \cos k_y) + (\mu_F)_{\text{eff}}$ and $\Delta_k^2 = (J^x_{\text{eff}})^2 [(\sin 3k_x + \sin 3k_y)^2]$.  

**Conclusion**: In this paper, we found that for the Z2 topological spin liquid of the toric-code model, a doped hole becomes a charged supersymmetric particle. Thus the doped Z2 topological spin liquid of the toric-code model provides a unique example of supersymmetric many-body system. The ground state is a new matter of quantum state - SBEC, of which there exists fermionic Goldstone mode - Goldstino which is the partner of Bosonic Goldstone mode. And we can tune the supersymmetry by adding transverse external field to the toric-code model. After breaking the supersymmetry by transverse external field, the ground state may be a supersymmetric superfluid - BEC state for bosonic holons and SC state for fermionic holons.
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