DEL PEZZO SURFACES WITH DU VAL SINGULARITIES

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ABSTRACT. In this paper we show that del Pezzo surfaces of degree 1 with Du Val singular points of type

\[ A_4, A_4 + A_3, A_4 + 2A_1, A_4 + A_1, A_3 + 4A_1, \]
\[ A_3 + 3A_1, 2A_3 + 2A_1, A_3 + 2A_1, A_3 + A_1, 2A_3, A_3, \]

admit a Kähler-Einstein metric. Moreover we are going to compute global log canonical thresholds of del Pezzo surfaces of degree 1 with Du Val singularities, and of del Pezzo surfaces of Picard group \( \mathbb{Z} \) with Du Val singularities.

1. Introduction

A result of Demailly-Kollar [4] has recently drawn a lot of attention to global log canonical thresholds of Fano varieties, which are algebraic counterparts of the \( \alpha \)-invariant of Tian for smooth Fano varieties (see [3, Appendix A]). At first, we are going to introduce some definitions from singularity theory while more details could be found in the classical reference [6].

Suppose that \( X \) is a normal variety and \( D = \sum d_i D_i \) is a \( \mathbb{Q} \)-divisor on \( X \) such that \( K_X + D \) is \( \mathbb{Q} \)-Cartier and let \( f : Y \rightarrow X \) be a birational morphism, where \( Y \) is normal. We can write

\[ K_Y \sim_{\mathbb{Q}} f^*(K_X + D) + \sum a(X, D, E)E. \]

Definition 1.1. The discrepancy of the log pair \((X, D)\) is the number

\[ \text{discrep}(X, D) = \inf_E \{a(F, D, E) | E \text{ is exceptional divisor over } X\} . \]

The total discrepancy of the log pair \((X, D)\) is the number

\[ \text{totaldiscrep}(X, D) = \inf_E \{a(F, D, E) | E \text{ is divisor over } X\} . \]

We say that the log pair \( K_X + D \) is

- Kawamata log terminal (or log terminal) iff \( \text{totaldiscrep}(X, D) > -1 \)
- log canonical iff \( \text{discrep}(X, D) \geq -1 \).

Assume, now, that \( X \) is a variety with log terminal singularities, let \( Z \subset X \) be a closed subvariety and let \( D \) be an effective \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor on \( X \). Then the number

\[ \text{lct}_Z(X, D) = \sup \{ \lambda \in \mathbb{Q} | \text{the log pair } (X, \lambda D) \text{ is log canonical along } Z\} \]

is called the log canonical threshold of \( D \) along \( Z \) and is a positive rational number. For \( Z = X \) we use the notation \( \text{lct}(X, D) \), instead of \( \text{lct}_Z(X, D) \).

\[ \text{lct}(X, D) = \sup \{ \lambda \in \mathbb{Q} | \text{the log pair } (X, \lambda D) \text{ is log canonical } \} . \]

Suppose, moreover, that \( X \) is a Fano variety.

Definition 1.2. The global log canonical threshold of \( X \) is the number

\[ \text{lct}(X) = \inf \{|\text{lct}(X, D)|D \text{ effective divisor on } X \text{ such that } D \sim_{\mathbb{Q}} -K_X\} . \]

Another way to see the global log canonical threshold is to take the \( \inf_n \{\text{lct}_n(X)\} \), where

\[ \text{lct}_n(X) = \inf \left\{ |\text{lct}(X, \frac{1}{n}D)|D \text{ effective } \mathbb{Q} \text{-divisor on } X \text{ such that } D \in \mid -nK_X\mid \right\} \]

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In the case when $X$ is a del Pezzo surface with Du Val singularities, such that $K_X^2 = 1$ the number $lct_1(X)$ was computed in [12].

In particular, global log canonical thresholds are related to the existence of Kähler-Einstein metrics on Fano varieties, as we can see in the following result due to [3], [10], [14].

**Theorem 1.3.** Let $X$ be an $n$-dimensional Fano variety with at most quotient singularities. The variety $X$ has a Kähler-Einstein metric if the inequality holds

$$lct(X) > \frac{n}{n+1}.$$ 

For the rest of this paper we are going to assume that $X$ is a del Pezzo surface with at most Du Val singular point\(^1\). The problem of existence of Kähler-Einstein metrics on smooth del Pezzo surfaces was completely settled by Tian in [14].

Moreover the following is due to [3].

**Theorem 1.4.** Let $X$ be a smooth del Pezzo surface. Then

$$lct(X) = \begin{cases} 
1/3 \text{ when } X \cong \mathbb{F}_1 \text{ or } K_X^2 \in \{7, 9\}, \\
1/2 \text{ when } X \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_X^2 \in \{5, 6\}, \\
2/3 \text{ when } K_X^2 = 4 \text{ or } X \text{ is a cubic surface in } \mathbb{P}^3 \text{ with an Eckardt point,} \\
3/4 \text{ when } X \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points,} \\
3/4 \text{ when } K_X^2 = 2 \text{ and } | - K_X | \text{ has a tacnodal curve,} \\
5/6 \text{ when } K_X^2 = 2 \text{ and } | - K_X | \text{ has no tacnodal curves,} \\
5/6 \text{ when } K_X^2 = 1 \text{ and } | - K_X | \text{ has a cuspidal curve,} \\
1 \text{ when } K_X^2 = 1 \text{ and } | - K_X | \text{ has no cuspidal curves.} 
\end{cases}$$

If now $S_3 \subset \mathbb{P}^3$ is a singular cubic surface with Du Val singularities and $S_3$ admits a Kähler-Einstein metric, then according to [5] it can only have points of type $A_1$ or $A_2$.

Moreover on a del Pezzo surface $S_2$ of degree 2 with only $A_1$ or $A_2$ singularities a Kähler-Einstein metric exists due to [3]. In their method they consider $S_2$ as a double cover of $\mathbb{P}^2$ and use a Kähler-Einstein metric on $\mathbb{P}^2$ to construct a Kähler-Einstein metric on $S_2$. A del Pezzo surface $S_1$ of degree 1 can be realised as a double cover of the cone $\mathbb{P}(1,1,2)$, however $\mathbb{P}(1,1,2)$ does not admit a Kähler-Einstein metric. Thus one cannot apply the same idea to prove existence of a Kähler-Einstein metric on $S_1$.

Due to [3] we have the following.

**Theorem 1.5.** Let $X$ be a del Pezzo surface with only Du Val singularities only of type $A_1$ or $A_2$ such that $K_X^2 = 1$. Then

$$lct(X) = \begin{cases} 
1 \text{ when } | - K_X | \text{ does not have cuspidal curves,} \\
2/3 \text{ when } | - K_X | \text{ has a cuspidal curve } C \text{ such that } \text{Sing}(C) = A_2, \\
3/4 \text{ when } | - K_X | \text{ has a cuspidal curve } C \text{ such that } \text{Sing}(C) = A_1 \\
\text{and no cuspidal curve } C \text{ such that } \text{Sing}(C) = A_2, \\
5/6 \text{ in the remaining cases.} 
\end{cases}$$

By Theorem 1.5 and Theorem 1.3 we get that on every del Pezzo surface of degree 1 that has at most ordinary double points a Kähler-Einstein metric exists.

The main purpose of this paper is to prove the following result.

**Theorem 1.6.** Let $X$ be a degree 1 del Pezzo surface having the following type of Du Val singular points:

$$A_4, A_4 + A_4, A_4 + A_3, A_4 + 2A_1, A_4 + A_1, A_3 + 4A_1, A_3 + 3A_1, 2A_3 + 2A_1, A_3 + 2A_1, A_3 + A_1, 2A_3, A_3.$$ 

Then $X$ admits a Kähler-Einstein metric.

\(^1\)All varieties are assumed to be projective, normal and defined over $\mathbb{C}$.  

\[ \]
Furthermore from Table 1 to Table 6 we give a list of all global log canonical thresholds for
• del Pezzo surfaces of degree 1 with Du Val singularities,
• del Pezzo surfaces of Picard group \(\mathbb{Z}\) with Du Val singularities.\(^2\)

We see that Table 6 together with Theorem 1.3 imply the existence of a Kähler-Einstein metric on every del Pezzo surface of degree 1 that has the singularities mentioned in Theorem 1.6.

Remark 1.7. In [14] and [13] the invariant \(\alpha_{m,2}(X)\) was introduced. One can see that \(\alpha_{m,2}(X) \geq \lct(X)\) and \(\alpha_{m,2}(X)\) goes to \(\lct(X)\) as \(m\) goes to \(+\infty\). However it never reaches \(\lct(X)\) if there are only finitely many \(\mathbb{Q}\)-divisors \(D \sim \mathbb{Q} - K_X\), such that \(\lct(X) = \lct(X, D)\). From the proofs of Lemma 3.6, Lemma 3.7 and Theorem 1.5 it follows that this is exactly the case when \(\lct(X) = \frac{2}{3}\) and \(X\) is a del Pezzo of degree 1 with Du Val singularities. It follows from [14] and [13] that a Kähler-Einstein metric exists on a smooth del Pezzo surface \(X\) if \(\alpha_{m,2}(X) > \frac{2}{3}\). It is expected that the same is true in case \(X\) is an orbifold del Pezzo surface.

Therefore we expect to have the following result.

**Conjecture 1.8.** Let \(X\) be a degree 1 del Pezzo surface having only Du Val singularities of type \(A_n\), for \(n \leq 6\), then \(X\) admits a Kähler-Einstein metric.

Apart from their connection to the existence of Kähler-Einstein metrics global log canonical thresholds have a birational application. For the following result we refer the reader to [11] and [2].

**Theorem 1.9.** Let \(V, \bar{V}\) be two varieties and \(Z\) be a smooth curve. Suppose that there is a commutative diagram

\[
\begin{array}{ccc}
V & \xrightarrow{\rho} & \bar{V} \\
\pi \downarrow & & \bar{\pi} \downarrow \\
Z & \xrightarrow{\pi} & \bar{Z}
\end{array}
\]

such that \(\pi\) and \(\bar{\pi}\) are flat morphisms, and \(\rho\) is a birational map that induces an isomorphism

\[
\rho|_{V \setminus F}: V \setminus F \longrightarrow \bar{V} \setminus \bar{F},
\]

where \(F\) and \(\bar{F}\) are scheme fibers of \(\pi\) and \(\bar{\pi}\) over a point \(O \in Z\), respectively. Suppose that

* the varieties \(V\) and \(\bar{V}\) have terminal \(\mathbb{Q}\)-factorial singularities,
* the divisors \(-K_V\) and \(-K_{\bar{V}}\) are \(\pi\)-ample and \(\bar{\pi}\)-ample, respectively,
* the fibers \(F\) and \(\bar{F}\) are irreducible.

Then \(\rho\) is an isomorphism if one of the following conditions hold:

* the varieties \(F\) and \(\bar{F}\) have log terminal singularities, and \(\lct(F) + \lct(\bar{F}) > 1\);
* the variety \(F\) has log terminal singularities, and \(\lct(F) \geq 1\).

2. **Preliminaries**

Let \(X\) be a del Pezzo surface with Du Val singularities. Assume that the global log canonical threshold is

\[
\lct(X) < \omega \leq 1,
\]

where \(\omega\) is a positive rational number such that \(\omega \leq \frac{1}{\lct(X)}\). This means that there is an effective \(\mathbb{Q}\)-divisor \(D\), with \(D \sim \mathbb{Q} - K_X\), such that the log pair \((X, \lambda D)\) is not log canonical for some rational number \(\lambda < \omega \leq 1\).

**Remark 2.1.** Suppose that \(Z\) is an effective \(\mathbb{Q}\)-divisor on \(X\) such that \((X, \lambda Z)\) is log canonical and \(Z \sim \mathbb{Q} - K_X\). Then if \((X, \lambda D)\) not log canonical also the pair

\[
(X, \lambda D - \lambda \alpha Z)
\]

is not log canonical, where \(\lambda \in \mathbb{Q}\) such that \(0 \leq \alpha < 1\).

\(^2\)Global log canonical thresholds of cubic surfaces with Du Val singularities were computed in [2] and \(\lct(\mathbb{P}(1,1,2)) = \frac{4}{3}\) (see [3]).
Denote now by \( \text{LCS}(X, \lambda D) \) the locus of log canonical singularities, that is the set of all points of \( X \) where the pair \((X, \lambda D)\) is not Kawamata log terminal. The following Connectedness Theorem can be found in [7] (Ch.17).

**Theorem 2.2.** If \(- (K_X + \lambda D)\) is ample, then the log canonical locus \( \text{LCS}(X, \lambda D) \) is connected.

From the way log canonicity is defined for the log pair \((X, \lambda D)\), one should understand all resolutions of singularities of the log pair \((X, \lambda D)\). However instead we will use the following condition on multiplicity that follows from [9] (4.5).

**Remark 2.3.** For a smooth point \( P \) of the surface \( X \) the condition that the pair \((X, \lambda D)\) is not log canonical implies that \( \text{mult}_P D > 1 \).

The following Lemma is going to be a key ingredient of the proof of Theorem 1.6.

**Lemma 2.4.** The pair \((X, \lambda D)\) is log canonical everywhere except for a Du Val singular point \( P \), where \((X, \lambda D)\) is not log canonical.

**Proof.** By Theorem 2.2 the log canonical locus \( \text{LCS}(X, \lambda D) \) is connected, since

\[-(K_X + \lambda D) \sim_Q (1 - \lambda)K_X \]

is ample. Suppose now that there is a irreducible curve \( C \) on the surface \( X \), such that \( C \subset \text{LCS}(X, \lambda D) \). Then \( C \in \text{Supp}(D) \) and we can write \( D = mC + \Omega \), where \( m \) is a rational number \( m\lambda \geq 1 \) and \( \Omega \) is an effective \( \mathbb{Q} \)-divisor such that \( C \notin \text{Supp}(\Omega) \).

But then Remark 2.3 implies that

\[K_X^2 = D \cdot (-K_X) = mC \cdot (-K_X) + \Omega \cdot (-K_X) > mC \cdot (-K_X) \geq \frac{1}{\lambda} \deg C > \frac{1}{\omega} > K_X^2,\]

which is a contradiction. Therefore the log canonical locus is zero-dimensional and there is a point \( P \in D \) where the log pair \((X, \lambda D)\) is not log canonical. Moreover we can assume that \( P \) is not a smooth point of \( X \). This follows from [1], where the case of smooth del Pezzo surfaces is treated.

The following theorem is known as adjunction or inversion of adjunction ( [7] ).

**Theorem 2.5.** Let \( X \) be normal and \( S \subset X \) be an irreducible Cartier divisor. Let \( B \) be an effective \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor and assume that \( K_X + S + B \) is \( \mathbb{Q} \)-Cartier and \( S \) is Kawamata log terminal such that \( S \notin \text{Supp} B \). Then

\[(X, S + B) \text{ is log canonical near } S \iff (S, B|_S) \text{ is log canonical.}\]

Throughout this paper we are going to refer to Theorem 2.5 simply as adjunction.

3. Del Pezzo surfaces of degree 1 with exactly one Du Val singularity

All possible combinations of Du Val singularities on a del Pezzo surface of degree 1 are given in [15].

3.1. Del Pezzo surfaces of degree 1 with exactly one \( A_3 \) type singularity. In this section we will prove the following.

**Lemma 3.1.** Let \( X \) be a del Pezzo surface with exactly one Du Val singularity of type \( A_3 \) and \( K_X^2 = 1 \). Then the global log canonical threshold of \( X \) is

\[\text{lct}(X) = 1.\]

**Proof.** Let \( Z \) be the unique curve in the linear system \(|-K_X|\) that contains \( P \), where \( P \) is the Du Val singular point of type \( A_3 \). We take the minimal resolution \( \pi : \tilde{X} \to X \) of \( X \), that contracts the exceptional curves \( E_1, E_2, E_3 \) to the singular point \( P \). The following diagram shows how the exceptional curves intersect each other.

\[A_3 \bullet E_1 ---- \bullet E_2 ---- \bullet E_3\]
Then $$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - E_3.$$ 
Suppose that lct($X$) < lct($X$) = 1, then there exists an effective $\mathbb{Q}$-divisor $D \in X$ such that $D \sim_{\mathbb{Q}} K_X$ and the log pair $(X, \lambda D)$ is not log canonical, where $\lambda < 1$. According to Lemma 2.4 the pair $(X, \lambda D)$ is log canonical everywhere except for a singular point $P \in X$, at which point $(X, \lambda D)$ is not log canonical. Since the curve $Z$ is irreducible we may assume that the support of $D$ does not contain $Z$. Then $$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3,$$
and from the inequalities

\[
0 \leq \tilde{D} \cdot \tilde{Z} = 1 - a_1 - a_3 \\
0 \leq E_1 \cdot \tilde{D} = 2a_1 - a_2 \\
0 \leq E_2 \cdot \tilde{D} = 2a_2 - a_1 - a_3 \\
0 \leq E_3 \cdot \tilde{D} = 2a_3 - a_2
\]
we see that

\[
2a_1 \geq a_2, \quad \frac{3}{2} a_2 \geq a_3 \text{ and } 2a_3 \geq a_2, \quad \frac{3}{2} a_2 \geq a_1 .
\]
Moreover we get the following upper bounds $a_1 \leq \frac{3}{4}$, $a_2 \leq 1$, $a_3 \leq \frac{3}{4}$ . The equivalence

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3$ such that the pair $K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3$ is not log canonical at $Q$.

- If the point $Q \in E_1$ and $Q \notin E_2$ then

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1$$
is not log canonical at the point $Q$ and so is the pair

$$K_{\tilde{X}} + \lambda \tilde{D} + E_1 .$$

By adjunction $(E_1, \lambda \tilde{D}|_{E_1})$ is not log canonical at $Q$ and

$$\frac{4}{3} a_1 \geq 2a_1 - \frac{2}{3} a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D} \cdot E_1) > \frac{1}{\lambda} > 1 ,$$

which is false.

- If $Q \in E_2$ but $Q \notin E_1 \cup E_3$ then

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 E_2$$
is not log canonical at the point $Q$ and so is the pair

$$K_{\tilde{X}} + \lambda \tilde{D} + E_2 , \text{ since } \lambda a_2 \leq 1 .$$

By adjunction $(E_2, \lambda \tilde{D}|_{E_2})$ is not log canonical at $Q$ and

$$2a_2 - \frac{a_2}{2} - \frac{a_2}{2} \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D} \cdot E_2) > \frac{1}{\lambda} > 1 ,$$

which is false.

- If $Q \in E_1 \cap E_2$ then the log pair

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$$
is not log canonical at the point $Q$ and so are the log pairs

$$K_{\tilde{X}} + \lambda \tilde{D} + E_1 + \lambda a_2 E_2 \text{ and } K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + E_2 .$$

By adjunction it follows that

$$2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > 1 - a_2$$

and

$$2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D}|_{E_2}) = \text{mult}_Q(\tilde{D} \cdot E_2) > 1 - a_1 ,$$
which implies that
\[2a_2 - \frac{a_2}{2} \geq 2a_2 - a_3 > 1 \Rightarrow a_2 > \frac{2}{3}.\]

Consider now the blow-up \(\sigma_1: \tilde{X} \to X\) of the surface \(\tilde{X}\) at the point \(Q\) that contracts the \((-1)\)-curve \(F_1\) to the point \(Q\). Then for the strict transforms of the exceptional divisors \(E_1, E_2, E_3\) we have
\[
\tilde{E}_1 \sim_{Q} \sigma_1^*(E_1) - F_1 \\
\tilde{E}_2 \sim_{Q} \sigma_1^*(E_2) - F_1 \\
\tilde{E}_3 \sim_{Q} \sigma_1^*(E_3).
\]

Let now
\[\pi_1: \tilde{X} \xrightarrow{\sigma_1} \tilde{X} \xrightarrow{\pi} X\]
be the composition \(\pi_1 = \pi \circ \sigma_1\). We have
\[K_{\tilde{X}_1} \sim_{Q} \sigma_1^*(K_{\tilde{X}}) + F_1 \sim_{Q} \sigma_1^*(\pi^*(K_X)) + F_1 \sim_{Q} \pi_1^*(K_X) + F_1\]
and
\[\tilde{D}_1 \sim_{Q} \sigma_1^*(\tilde{D}) - m_1F_1 \\
\sim_{Q} \sigma_1^*(\pi^*(\tilde{D}) - a_1E_1 - a_2E_2 - a_3E_3) - m_1F_1 \\
\sim_{Q} \pi_1^*(\tilde{D}) - a_1\tilde{E}_1 - a_2\tilde{E}_2 - a_3\tilde{E}_3 - (a_1 + a_2 + m_1)F_1,
\]
where \(m_1 = \text{mult}_Q \tilde{D}\). Also the strict transform of the anticanonical curve \(Z\) is
\[
\tilde{Z}_1 \sim_{Q} \sigma_1^*(\tilde{Z}) \\
\sim_{Q} \sigma_1^*(\pi^*(Z) - E_1 - E_2 - E_3) \\
\sim_{Q} \pi_1^*(Z) - \tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3 - 2F_1.
\]

From the inequalities
\[
0 \leq \tilde{D}_1 \cdot \tilde{Z}_1 = 1 - a_1 - a_3 \\
0 \leq \tilde{E}_1 \cdot \tilde{D}_1 = 2a_1 - a_2 - m_1 \\
0 \leq \tilde{E}_2 \cdot \tilde{D}_1 = 2a_2 - a_1 - a_3 - m_1 \\
0 \leq \tilde{E}_3 \cdot \tilde{D}_1 = 2a_3 - a_2 \\
0 \leq F_1 \cdot \tilde{D}_1 = m_1
\]
we get that \(m_1 \leq \frac{1}{2}\). The equivalence
\[K_{\tilde{X}_1} + \lambda \tilde{D}_1 + \lambda a_1 \tilde{E}_1 + \lambda a_2 \tilde{E}_2 + \lambda a_3 \tilde{E}_3 + (\lambda(a_1 + a_2 + m_1) - 1)F_1 \sim_{Q} \pi_1^*(K_X + \lambda D)\]
implies that there is a point \(Q_1 \in F_1\) such that the pair
\[K_{\tilde{X}_1} + \lambda \tilde{D}_1 + \lambda a_1 \tilde{E}_1 + \lambda a_2 \tilde{E}_2 + (\lambda(a_1 + a_2 + m_1) - 1)F_1\]
is not log canonical at \(Q_1\).

(i) Suppose \(Q_1 \in \tilde{E}_1 \cap F_1\), then the log pair
\[K_{\tilde{X}_1} + \lambda \tilde{D}_1 + \lambda a_1 \tilde{E}_1 + (\lambda(a_1 + a_2 + m_1) - 1)F_1\]
is not log canonical at \(Q_1\) and so are the pairs
\[K_{\tilde{X}_1} + \lambda \tilde{D}_1 + \tilde{E}_1 + (\lambda(a_1 + a_2 + m_1) - 1)F_1\]
and
\[K_{\tilde{X}_1} + \lambda \tilde{D}_1 + \lambda a_1 \tilde{E}_1 + F_1\].

By adjunction it follows that
\[2a_1 - a_2 - m_1 = \tilde{D}_1 \cdot \tilde{E}_1 \geq \text{mult}_{Q_1}(\tilde{D}_1 \cdot \tilde{E}_1) > 1 - a_1 - a_2 - m_1 + 1,
\]
and
\[m_1 = \tilde{D}_1 \cdot F_1 \geq \text{mult}_{Q_1}(\tilde{D}_1 \cdot F_1) > 1 - a_1,\]
which is false.

(ii) Suppose $Q_1 \in E_1 \setminus (E_1 \cup E_2)$ then the log pair
\[
K_{\tilde{X}_1} + \lambda \tilde{D}_1 + (\lambda(a_1 + a_2 + m_1) - 1)F_1
\]
is not log canonical at $Q_1$ and so is the pair
\[
K_{\tilde{X}_1} + \lambda \tilde{D}_1 + F_1, \text{ since } 0 \leq \lambda(a_1 + a_2 + m_1) - 1 \leq 1
\]
By adjunction it follows that
\[
m_1 = \tilde{D}_1 \cdot F_1 \geq \mult_{Q_1}(\tilde{D}_1 \cdot F_1) > 1,
\]
which is impossible since $m_1 \leq \frac{1}{7}$.

(iii) Suppose $Q_1 \in E_2 \cap F_1$ then the log pair
\[
K_{\tilde{X}_1} + \lambda \tilde{D}_1 + \lambda a_2 \tilde{E}_2 + (\lambda(a_1 + a_2 + m_1) - 1)F_1
\]
is not log terminal at $Q_1$ and so are the pairs
\[
K_{\tilde{X}_1} + \lambda \tilde{D}_1 + \tilde{E}_2 + (\lambda(a_1 + a_2 + m_1) - 1)F_1
\]
and
\[
K_{\tilde{X}_1} + \lambda \tilde{D}_1 + \lambda a_2 \tilde{E}_2 + F_1.
\]
By adjunction it follows that
\[
2a_2 - a_1 - a_3 - m_1 = \tilde{D}_1 \cdot \tilde{E}_2 \geq \mult_{Q_1}(\tilde{D}_1 \cdot \tilde{E}_2) > 1 - a_1 - a_2 - m_1 + 1
\]
and
\[
m_1 = \tilde{D}_1 \cdot F_1 \geq \mult_{Q_1}(\tilde{D}_1 \cdot F_1) > 1 - a_2,
\]
which imply that
\[
3a_2 - \frac{a_2}{2} \geq 3a_2 - a_3 > 2 \Rightarrow a_2 > \frac{4}{5}.
\]

Consider now the blow-up $\sigma_2 : \tilde{X}_2 \to \tilde{X}_1$ of the surface $\tilde{X}_1$ at the point $Q_1$ that contracts the $(-1)$-curve $F_2$ to the point $Q_1$. We then have

\[
\begin{align*}
K_{\tilde{X}_2} & \sim_{Q_2} \pi_2^*(K_X) + F_1 + 2F_2 \\
\tilde{D}_2 & \sim_{Q_2} \pi_2^*(D) - a_1 \tilde{E}_1 - a_2 \tilde{E}_2 - a_3 \tilde{E}_3 - (a_1 + a_2 + m_1)\tilde{F}_1 - (a_1 + 2a_2 + m_1 + m_2)F_2 \\
\tilde{Z}_2 & \sim_{Q_2} \pi_2^*(Z) - \tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3 - 2\tilde{F}_1 - 3F_2,
\end{align*}
\]
and
\[
\begin{align*}
0 & \leq \tilde{D}_2 \cdot \tilde{Z}_2 = 1 - a_1 - a_3 \\
0 & \leq \tilde{E}_1 \cdot \tilde{D}_2 = 2a_1 - a_2 - m_1 \\
0 & \leq \tilde{E}_2 \cdot \tilde{D}_2 = 2a_2 - a_1 - a_3 - m_1 - m_2 \\
0 & \leq \tilde{E}_1 \cdot \tilde{D}_2 = 2a_3 - a_2 \\
0 & \leq \tilde{F}_1 \cdot \tilde{D}_2 = m_1 - m_2 \\
0 & \leq F_2 \cdot \tilde{D}_2 = m_2,
\end{align*}
\]
where $m_2 = \mult_{Q_1} \tilde{D}_2$.

Because of the equivalence
\[
\pi_2^*(K_X + \lambda D) \sim_{Q_2}
\]
\[
K_{\tilde{X}_2} + \lambda \tilde{D}_2 + \lambda a_1 \tilde{E}_1 + \lambda a_2 \tilde{E}_2 + \lambda a_3 \tilde{E}_3 + (\lambda(a_1 + a_2 + m_1) - 1)\tilde{F}_1 + (\lambda(a_1 + 2a_2 + m_1 + m_2) - 2)F_2
\]
there is a point $Q_2 \in F_2$ such that the pair
\[
K_{\tilde{X}_2} + \lambda \tilde{D}_2 + \lambda a_1 \tilde{E}_1 + \lambda a_2 \tilde{E}_2 + \lambda a_3 \tilde{E}_3 + (\lambda(a_1 + a_2 + m_1) - 1)\tilde{F}_1 + (\lambda(a_1 + 2a_2 + m_1 + m_2) - 2)F_2
\]
is not log canonical at $Q_2$. 

7
Suppose $Q_2 \in F_2 \cap \tilde{E}_1$, then the log pair

$$\left(\tilde{X}_2, \lambda \tilde{D}_2 + (\lambda(a_1 + a_2 + m_1) - 1)\tilde{F}_1 + (\lambda(a_1 + 2a_2 + m_1 + m_2) - 2)F_2\right)$$

is not log canonical at $Q_2$ and so are the log pairs

$$K_{\tilde{X}_2} + \lambda \tilde{D}_2 + \tilde{F}_1 + (\lambda(a_1 + 2a_2 + m_1 + m_2) - 2)F_2$$

and

$$K_{\tilde{X}_2} + \lambda \tilde{D}_2 + (\lambda(a_1 + a_2 + m_1) - 1)\tilde{F}_1 + F_2 .$$

By adjunction it follows that

$$m_1 - m_2 = \tilde{D}_2 \cdot \tilde{F}_1 \geq \mult_{Q_2} (\tilde{D}_2 \cdot \tilde{F}_1) > 1 - (a_1 + 2a_2 + m_1 + m_2 - 2)$$

and

$$m_2 = \tilde{D}_2 \cdot F_2 \geq \mult_{Q_2} (\tilde{D}_2 \cdot F_2) \geq 1 - (a_1 + a_2 + m_1 - 1) ,$$

which is false.

Suppose $O \in F_2 \setminus (\tilde{F}_1 \cup \tilde{E}_2)$, then the log pair

$$K_{\tilde{X}_2} + \lambda \tilde{D}_2 + (\lambda(a_1 + 2a_2 + m_1 + m_2) - 2)F_2$$

is not log canonical at $Q_2$ and so is the log pair

$$K_{\tilde{X}_2} + \lambda \tilde{D}_2 + F_2 ,$$

since $\lambda(a_1 + 2a_2 + m_1 + m_2) - 2 \leq 1$.

By adjunction it follows that

$$m_2 = \tilde{D}_2 \cdot F_2 \geq \mult_{Q_2} (\tilde{D}_2 \cdot F_2) > 1 ,$$

which is false.

Suppose $Q_2 \in F_2 \cap \tilde{E}_2$, then the log pair

$$K_{\tilde{X}_2} + \lambda \tilde{D}_2 + (\lambda(a_1 + 2a_2 + m_1 + m_2) - 2)F_2 + \lambda a_2 \tilde{E}_2$$

is not log canonical at $Q_2$ and so are the log pairs

$$K_{\tilde{X}_2} + \lambda \tilde{D}_2 + (\lambda(a_1 + 2a_2 + m_1 + m_2) - 2)F_2 + \tilde{E}_2$$

and

$$K_{\tilde{X}_2} + \lambda \tilde{D}_2 + F_2 + \lambda a_2 \tilde{E}_2 .$$

By adjunction it follows that

$$2a_2 - a_1 - a_3 - m_1 - m_2 = \tilde{D}_2 \cdot \tilde{E}_2 \geq \mult_{Q_2} (\tilde{D}_2 \cdot \tilde{E}_2) > 1 - (a_1 + 2a_2 + m_1 + m_2 - 2)$$

and

$$m_2 = \tilde{D}_2 \cdot F_2 \geq \mult_{Q_2} (\tilde{D}_2 \cdot F_2) > 1 - a_2 ,$$

which implies that

$$4a_2 - \frac{a_2}{2} \geq 4a_2 - a_3 > 3 \Rightarrow a_2 > \frac{6}{7} .$$

Consider now the blow-up $\sigma_k : \tilde{X}_k \to \tilde{X}_{k-1}$ of the surface $\tilde{X}_{k-1}$ at the point $Q_{k-1}$ that contracts the $(-1)$-curve $\mathcal{F}_k$ to the point $Q_{k-1}$. We then have

$$K_{\tilde{X}_k} \sim_{Q_k} \pi_k^*(K_X) + \tilde{F}_1 + 2\tilde{F}_2 + 3\tilde{E}_3 + ... + (k - 1)\tilde{F}_{k-1} + kF_k$$

$$\tilde{D}_k \sim_{Q_k} \pi_k^*(D) - a_1 \tilde{E}_1 - a_2 \tilde{E}_2 - a_3 \tilde{E}_3 - (a_1 + a_2 + m_1)\tilde{F}_1 - (a_1 + 2a_2 + m_1 + m_2)\tilde{F}_2 - ... - (a_1 + (k - 1)a_2 + m_1 + m_2 + ... + m_{k-1})\tilde{F}_{k-1} - (a_1 + ka_2 + m_1 + m_2 + ... + m_k)F_k$$

$$\tilde{Z}_k \sim_{Q_k} \pi_k^*(Z) - \tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3 - 2\tilde{F}_1 - 3\tilde{F}_2 - ... - k\tilde{F}_{k-1} - (k + 1)F_k ,$$
and

\[ 0 \leq \tilde{D}_k \cdot \tilde{E}_k = 1 - a_1 - a_3 \]
\[ 0 \leq \tilde{E}_1 \cdot \tilde{D}_k = 2a_1 - a_2 - m_1 \]
\[ 0 \leq \tilde{E}_2 \cdot \tilde{D}_k = 2a_2 - a_1 - a_3 - m_1 - m_2 - \ldots - m_k \]
\[ 0 \leq \tilde{E}_1 \cdot \tilde{D}_k = 2a_3 - a_2 \]
\[ 0 \leq \tilde{F}_1 \cdot \tilde{D}_k = m_1 - m_2 \]
\[ 0 \leq \tilde{F}_2 \cdot \tilde{D}_k = m_2 - m_3 \]
\[ \ldots \]
\[ 0 \leq \tilde{F}_{k-1} \cdot \tilde{D}_k = m_{k-1} - m_k \]
\[ 0 \leq F_k \cdot \tilde{D}_k = m_k , \]

where \( m_i = \text{mult}_{Q_i} \tilde{D}_i \), for \( i = 1, \ldots, k \).

Because of the equivalence

\[ \pi_k^*(K_X + \lambda D) \sim_{\mathbb{Q}} K_{X_k} + \lambda \tilde{D}_k + \lambda a_1 \tilde{E}_1 + \lambda a_2 \tilde{E}_2 + \lambda a_3 \tilde{E}_3 + (\lambda(a_1 + a_2 + m_1) - 1) \tilde{F}_1 + (\lambda(a_1 + 2a_2 + m_1 + m_2) - 2) \tilde{F}_2 + \ldots + (\lambda(a_1 + (k-1)a_2 + m_1 + m_2 + \ldots + m_{k-1}) - (k-1)) \tilde{F}_{k-1} + (\lambda(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k) - k) F_k \]

there is a point \( Q_k \in F_k \) such that the pair

\[ K_{X_k} + \lambda \tilde{D}_k + \lambda a_2 \tilde{E}_2 + (\lambda(a_1 + (k-1)a_2 + m_1 + m_2 + \ldots + m_{k-1}) - (k-1)) \tilde{F}_{k-1} + (\lambda(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k) - k) F_k \]

is not log canonical at \( Q_k \).

- Suppose \( Q_k \in F_k \cap \tilde{F}_{k-1} \), then the log pair

\[ K_{X_k} + \lambda \tilde{D}_k + (\lambda(a_1 + (k-1)a_2 + m_1 + m_2 + \ldots + m_{k-1}) - (k-1)) \tilde{F}_{k-1} + (\lambda(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k) - k) F_k \]

is not log canonical at \( Q_2 \) and so are the log pairs

\[ K_{X_k} + \lambda \tilde{D}_k + \tilde{F}_{k-1} + (\lambda(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k) - k) F_k \]

and

\[ K_{X_k} + \lambda \tilde{D}_k + (\lambda(a_1 + (k-1)a_2 + m_1 + m_2 + \ldots + m_{k-1}) - (k-1)) \tilde{F}_{k-1} + F_k . \]

By adjunction it follows that

\[ m_{k-1} - m_k = \tilde{D}_k \cdot \tilde{F}_{k-1} \geq \text{mult}_{Q_k} \left( \tilde{D}_k \cdot \tilde{F}_{k-1} \right) > 1 - (a_1 + ka_2 + m_1 + m_2 + \ldots + m_{k-2} + 2m_{k-1}) > k + 1 \]

which is a contradiction. Indeed from the inequality above we have that

\[ a_1 + ka_2 + m_1 + m_2 + \ldots + m_{k-2} + 2m_{k-1} > k + 1 \]

but since \( m_1 \geq m_2 \geq \ldots \geq m_k \) we get that

\[ a_1 + ka_2 + km_1 > k + 1 . \]

However the inequality \( 0 \leq \tilde{E}_1 \cdot \tilde{D}_k = 2a_1 - a_2 - m_1 \) finally gives us

\[ (2k + 1)a_1 > k + 1 \Rightarrow a_3 < \frac{k}{2k + 1} \Rightarrow a_2 \leq 2a_3 < \frac{2k}{2k + 1} . \]
Suppose $Q_k \in F_k \setminus (F_{k-1} \cup E_2)$, then the log pair
\[ K_{\tilde{X}_k} + \lambda \tilde{D}_k + (\lambda(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k) - k) F_k \]
is not log canonical at $Q_k$ and so is the log pair
\[ K_{\tilde{X}_k} + \lambda \tilde{D}_k + F_k, \]
since $(\lambda(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k) - k) \leq 1$.
By adjunction it follows that
\[ m_k = \tilde{D}_k \cdot F_k \geq \text{mult}_{Q_k}(\tilde{D}_k \cdot F_k) > 1, \]
which is false, since $\frac{1}{2} \geq m_1 \geq m_2 \geq \ldots \geq m_k$.

Suppose $Q_k \in F_k \cap E_2$, then the log pair
\[ K_{\tilde{X}_k} + \lambda \tilde{D}_k + (\lambda(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k) - k) F_k + \lambda a_2 E_2 \]
is not log canonical at $Q_k$ and so are the log pairs
\[ K_{\tilde{X}_k} + \lambda \tilde{D}_k + (\lambda(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k) - k) F_k + E_2 \]
and
\[ K_{\tilde{X}_k} + \lambda \tilde{D}_k + F_k + \lambda a_2 E_2. \]
By adjunction it follows that
\[ 2a_2 - a_1 - a_3 - m_1 - m_2 - \ldots - m_k = \tilde{D}_k \cdot E_2 \geq \text{mult}_{Q_k}(\tilde{D}_k \cdot E_2) > 1 - (a_1 + ka_2 + m_1 + m_2 + \ldots + m_k - k) \]
and
\[ m_k = \tilde{D}_k \cdot F_k \geq \text{mult}_{Q_k}(\tilde{D}_k \cdot F_k) > 1 - a_2, \]
which implies that
\[ (k + 2)a_2 - \frac{a_2}{2} \geq (k + 2)a_2 - a_3 > k + 1 \Rightarrow a_2 > \frac{2k + 2}{2k + 3}. \]

Remark 3.2. It remains to be shown that after the $k$-th blow up we have
\[ (\lambda(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k) - k) \leq 1, \]
and for this it is enough to show that
\[ (a_1 + ka_2 + m_1 + m_2 + \ldots + m_k - k) \leq 1. \]
Suppose that we have blown up $k - 1$ times, then $a_2 > \frac{2k}{2k + 1}$. Let us assume on the contrary that
\[ a_1 + ka_2 + m_1 + m_2 + \ldots + m_k - k > 1 \Rightarrow a_1 + ka_2 + m_1 + m_2 + \ldots + m_k > k + 1 \]
\[ a_1 + 2ka_1 \geq a_1 + ka_2 + km_1 > k + 1 \Rightarrow a_1 > \frac{k + 1}{2k + 1} \]
\[ a_3 \leq 1 - a_1 < \frac{k}{2k + 1} \Rightarrow a_2 \leq 2a_3 < \frac{2k}{2k + 1}, \]
which is a contradiction.

3.2. Del Pezzo surfaces of degree 1 with one $\mathbb{A}_4$ type singularity. In this section we will prove the following.

Lemma 3.3. Let $X$ be a del Pezzo surface with one Du Val singularity of type $\mathbb{A}_4$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is
\[ \text{lct}(X) = \frac{4}{5}. \]

\(^1\)I am grateful to Jihun Park for letting me know about a mistake on the upper bound of $\text{lct}(X)$. 

We can assume that $Z$ is not log canonical at the point $Q$ and $\lambda D + a_1 E_1 + a_2 E_2 + a_3 E_3 + a_4 E_4$ is not log canonical at $Q$.

If the point $Q \in E_1$ and $Q \not\in E_2$ then

$$K_X + \lambda D + \lambda a_1 E_1$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda D + E_1,$$

since $\lambda a_1 \leq 1$.

By adjunction $(E_1, \lambda D|_{E_1})$ is not log canonical at $Q$ and

$$1 \geq \frac{5}{4} a_1 \geq 2 a_1 - \frac{3}{4} a_1 \geq 2 a_1 - a_2 = \hat{D} \cdot E_1 \geq \text{mult}_Q(\hat{D} \cdot E_1) > \frac{1}{\lambda} > \frac{6}{5},$$

which is false.
• If $Q \in E_2$ but $Q \not\in E_1 \cup E_3$ then

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 E_2$$

is not log canonical at the point $Q$ and so is the pair

$$K_{\tilde{X}} + \lambda \tilde{D} + E_2$$

By adjunction $(E_2, \lambda \tilde{D}|_{E_2})$ is not log canonical at $Q$ and

$$1 \geq \frac{5}{6} a_2 \geq 2a_2 - \frac{2}{3} a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D} \cdot E_2) > \frac{1}{\lambda} > \frac{6}{5},$$

which is false.

• If $Q \in E_3$ but $Q \not\in E_2 \cup E_4$ then

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_3 E_3$$

is not log canonical at the point $Q$ and so is the pair

$$K_{\tilde{X}} + \lambda \tilde{D} + E_3$$

By adjunction $(E_3, \lambda \tilde{D}|_{E_3})$ is not log canonical at $Q$ and

$$1 \geq \frac{5}{6} a_3 \geq 2a_3 - \frac{2}{3} a_3 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D} \cdot E_3) > \frac{1}{\lambda} > \frac{6}{5},$$

which is false.

• If $Q \in E_4$ but $Q \not\in E_3$ then

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_4 E_4$$

is not log canonical at the point $Q$ and so is the pair

$$K_{\tilde{X}} + \lambda \tilde{D} + E_4$$

By adjunction $(E_4, \lambda \tilde{D}|_{E_4})$ is not log canonical at $Q$ and

$$1 \geq \frac{5}{4} a_4 \geq 2a_4 - \frac{3}{4} a_4 \geq 2a_4 - a_3 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D} \cdot E_4) > \frac{1}{\lambda} > \frac{6}{5},$$

which is false.

• If $Q \in E_1 \cap E_2$ then the log pair

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$$

is not log canonical at the point $Q$ and so are the log pairs

$$K_{\tilde{X}} + \lambda \tilde{D} + E_1 + \lambda a_2 E_2$$

and

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + E_2.$$

By adjunction it follows that

$$2a_2 - a_1 - \frac{2}{3} a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D} \cdot E_2) > \frac{1}{\lambda} - a_1 > \frac{6}{5} - a_1,$$

and

$$2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D} \cdot E_1) > \frac{1}{\lambda} - a_2 > \frac{6}{5} - a_2.$$

These imply that $a_1 > \frac{6}{10}$ and $a_2 > \frac{9}{10}.$

Consider now the blow-up $\sigma_1 : \tilde{X}_1 \to \tilde{X}$ of the surface $\tilde{X}$ at the point $Q$ that contracts the $(-1)$-curve $F_1$ to the point $Q$. Then for the strict transforms of the exceptional divisors $E_1, E_2, E_3, E_4$ we have

$$\tilde{E}_1 \sim_{Q} \sigma_1^*(E_1) - F_1,$$

$$\tilde{E}_2 \sim_{Q} \sigma_1^*(E_2) - F_1,$$

$$\tilde{E}_3 \sim_{Q} \sigma_1^*(E_3),$$

$$\tilde{E}_4 \sim_{Q} \sigma_1^*(E_4).$$

Let now

$$\pi_1 : \tilde{X}_1 \xrightarrow{\sigma_1} \tilde{X} \xrightarrow{\pi} X.$$
be the composition \( \pi_1 = \pi \circ \sigma_1 \). We have
\[
\check{K}_{\check{X}_1} \sim_Q \sigma_1^*(K_{\check{X}}) + F_1 \sim_Q \sigma_1^*(\pi^*(K_X)) + F_1 \sim_Q \pi_1^*(K_X) + F_1
\]
and
\[
\check{D}_1 \sim_Q \sigma_1^*(\check{D}) - m_1 F_1
\]
\[
\sim_Q \sigma_1^*(\pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4) - m_1 F_1
\]
\[
\sim_Q \pi_1^*(D) - a_1 \tilde{E}_1 - a_2 \tilde{E}_2 - a_3 \tilde{E}_3 - a_4 \tilde{E}_4 - (a_1 + a_2 + m_1) F_1 ,
\]
where \( m_1 = \text{mult}_Q \check{D} \). Also the strict transform of the anticanonical curve \( Z \) is
\[
\tilde{Z}_1 \sim_Q \sigma_1^*(\tilde{Z})
\]
\[
\sim_Q \sigma_1^*(\pi^*(Z) - E_1 - E_2 - E_3 - E_4)
\]
\[
\sim_Q \pi_1^*(Z) - \tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3 - \tilde{E}_4 - 2F_1 .
\]
From the inequalities
\[
0 \leq \check{D}_1 \cdot \tilde{Z}_1 = 1 - a_1 - a_4
\]
\[
0 \leq \check{E}_1 \cdot \tilde{D}_1 = 2a_1 - a_2 - m_1
\]
\[
0 \leq \check{E}_2 \cdot \tilde{D}_1 = 2a_2 - a_1 - a_3 - m_1
\]
\[
0 \leq \check{E}_3 \cdot \tilde{D}_1 = 2a_3 - a_2 - a_4
\]
\[
0 \leq \check{E}_4 \cdot \tilde{D}_1 = 2a_4 - a_3
\]
\[
0 \leq F_1 \cdot \tilde{D}_1 = m_1
\]
we get that \( m_1 \leq \frac{1}{3} \). The equivalence
\[
K_{\check{X}_1} + \lambda \check{D}_1 + \lambda a_1 \tilde{E}_1 + \lambda a_2 \tilde{E}_2 + \lambda a_3 \tilde{E}_3 + \lambda a_4 \tilde{E}_4 + (\lambda(a_1 + a_2 + m_1) - 1) F_1 \sim_Q \pi_1^*(K_X + \lambda \check{D})
\]
implies that there is a point \( Q_1 \in F_1 \) such that the pair
\[
K_{\check{X}_1} + \lambda \check{D}_1 + \lambda a_1 \tilde{E}_1 + \lambda a_2 \tilde{E}_2 + (\lambda(a_1 + a_2 + m_1) - 1) F_1
\]
is not log canonical at \( Q_1 \).

- Suppose \( Q_1 \in \check{E}_1 \cap F_1 \), then the log pair
  \[
  K_{\check{X}_1} + \lambda \check{D}_1 + \lambda a_1 \tilde{E}_1 + (\lambda(a_1 + a_2 + m_1) - 1) F_1
  \]
is not log canonical at \( Q_1 \) and so are the pairs
  \[
  K_{\check{X}_1} + \lambda \check{D}_1 + \tilde{E}_1 + (\lambda(a_1 + a_2 + m_1) - 1) F_1
  \]
and
  \[
  K_{\check{X}_1} + \lambda \check{D}_1 + \lambda a_1 \tilde{E}_1 + F_1 .
  \]
By adjunction it follows that
\[
2a_1 - a_2 - m_1 = \check{D}_1 \cdot \tilde{E}_1 \geq \text{mult}_{Q_1}(\check{D}_1 \cdot \tilde{E}_1) \geq \frac{12}{5} - a_1 - a_2 - m_1 ,
\]
and
\[
m_1 = \check{D}_1 \cdot F_1 \geq \text{mult}_{Q_1}(\check{D}_1 \cdot F_1) \geq \frac{1}{\lambda} - a_1 > \frac{6}{5} - a_1 ,
\]
which is false.
- Suppose \( Q_1 \in F_1 \backslash (\check{E}_1 \cup \tilde{E}_2) \) then the log pair
  \[
  K_{\check{X}_1} + \lambda \check{D}_1 + (\lambda(a_1 + a_2 + m_1) - 1) F_1
  \]
is not log canonical at \( Q_1 \) and so is the pair
  \[
  K_{\check{X}_1} + \lambda \check{D}_1 + F_1 ,
  \]
since \( 0 \leq \frac{5}{6}(a_1 + a_2 + m_1) - 1 \leq 1 \).
By adjunction it follows that
\[
m_1 = \check{D}_1 \cdot F_1 \geq \text{mult}_{Q_1}(\check{D}_1 \cdot F_1) \geq \frac{1}{\lambda} > \frac{6}{5} ,
\]
which is impossible since \( m_1 \leq \frac{1}{3} \).
Suppose \( Q_1 \in \tilde{E}_2 \cap F_1 \) then the log pair
\[
K_{\tilde{X}_1} + \lambda \tilde{D}_1 + \lambda a_2 \tilde{E}_2 + (\lambda (a_1 + a_2 + m_1) - 1)F_1
\]
is not log canonical at \( Q_1 \) and so are the pairs
\[
K_{\tilde{X}_1} + \lambda \tilde{D}_1 + \tilde{E}_2 + (\lambda (a_1 + a_2 + m_1) - 1)F_1
\]
and
\[
K_{\tilde{X}_1} + \lambda \tilde{D}_1 + \lambda a_2 \tilde{E}_2 + F_1.
\]
By adjunction it follows that
\[
2a_2 - a_1 - a_3 - m_1 = \tilde{D}_1 \cdot \tilde{E}_2 \geq \text{mult}_{Q_1}(\tilde{D}_1 \cdot \tilde{E}_2) > \frac{12}{5} - a_1 - a_2 - m_1
\]
and
\[
m_1 = \tilde{D}_1 \cdot F_1 \geq \text{mult}_{Q_1}(\tilde{D}_1 \cdot F_1) > \frac{1}{\lambda} - a_2 > \frac{6}{5} - a_2,
\]
which imply that
\[
3a_2 - \frac{a_2}{2} \geq 3a_2 - a_3 > \frac{12}{5} \Rightarrow a_2 > \frac{6}{7} \cdot \frac{6}{5}.
\]
Consider now the blow-up \( \sigma_2 : \tilde{X}_2 \to \tilde{X}_1 \) of the surface \( \tilde{X}_1 \) at the point \( Q_1 \) that contracts the \((-1)\)-curve \( F_2 \) to the point \( Q_1 \). We then have
\[
K_{\tilde{X}_2} \sim_{Q_1} \pi_2^* (K_{\tilde{X}}) + \tilde{F}_1 + 2F_2
\]
\[
\tilde{D}_2 \sim_{Q_1} \pi_2^* (D) - a_1 \tilde{E}_1 - a_2 \tilde{E}_2 - a_3 \tilde{E}_3 - a_4 \tilde{E}_4 - (a_1 + a_2 + m_1)\tilde{F}_1 - (a_1 + 2a_2 + m_1 + m_2)F_2
\]
\[
\tilde{Z}_2 \sim_{Q_1} \pi_2^* (Z) - \tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3 - \tilde{E}_4 - 2\tilde{F}_1 - 3F_2,
\]
and
\[
0 \leq \tilde{D}_2 \cdot \tilde{Z}_2 = 1 - a_1 - a_4
\]
\[
0 \leq \tilde{E}_1 \cdot \tilde{D}_2 = 2a_1 - a_2 - m_1
\]
\[
0 \leq \tilde{E}_2 \cdot \tilde{D}_2 = 2a_2 - a_1 - a_3 - m_1 - m_2
\]
\[
0 \leq \tilde{E}_3 \cdot \tilde{D}_2 = 2a_3 - a_2 - a_4
\]
\[
0 \leq \tilde{E}_4 \cdot \tilde{D}_2 = 2a_4 - a_3
\]
\[
0 \leq \tilde{F}_1 \cdot \tilde{D}_2 = m_1 - m_2
\]
\[
0 \leq F_2 \cdot \tilde{D}_2 = m_2
\]
where \( m_2 = \text{mult}_{Q_1} \tilde{D}_2 \).
Because of the equivalence
\[
\pi_2^* (K_x + \lambda D) \sim_{Q_1}
\]
\[
K_{\tilde{X}_2} + \lambda \tilde{D}_2 + \lambda a_1 \tilde{E}_1 + \lambda a_2 \tilde{E}_2 + \lambda a_3 \tilde{E}_3 + \lambda a_4 \tilde{E}_4 +
\]
\[
(\lambda (a_1 + a_2 + m_1) - 1)\tilde{F}_1 + (\lambda (a_1 + 2a_2 + m_1 + m_2) - 2)F_2
\]
there is a point \( Q_2 \in F_2 \) such that the pair
\[
K_{\tilde{X}_2} + \lambda \tilde{D}_2 + \lambda a_1 \tilde{E}_1 + \lambda a_2 \tilde{E}_2 + \lambda a_3 \tilde{E}_3 + \lambda a_4 \tilde{E}_4 + (\lambda (a_1 + a_2 + m_1) - 1)\tilde{F}_1 + (\lambda (a_1 + 2a_2 + m_1 + m_2) - 2)F_2
\]
is not log canonical at \( Q_2 \).
Suppose \( Q_2 \in F_2 \cap \tilde{F}_1 \), then the log pair
\[
\left( \tilde{X}_2, \lambda \tilde{D}_2 + (\lambda (a_1 + a_2 + m_1) - 1)\tilde{F}_1 + (\lambda (a_1 + 2a_2 + m_1 + m_2) - 2)F_2 \right)
\]
is not log canonical at \( Q_2 \) and so are the log pairs
\[
K_{\tilde{X}_2} + \lambda \tilde{D}_2 + \tilde{F}_1 + (\lambda (a_1 + 2a_2 + m_1 + m_2) - 2)F_2
\]
and
\[
K_{\tilde{X}_2} + \lambda \tilde{D}_2 + (\lambda (a_1 + a_2 + m_1) - 1)\tilde{F}_1 + F_2.
\]
By adjunction it follows that
\[ m_1 - m_2 = \tilde{D}_2 \cdot \tilde{F}_1 \geq \text{mult}_{Q_2} (\tilde{D}_2 \cdot \tilde{F}_1) > \frac{18}{5} - (a_1 + 2a_2 + m_1 + m_2) \]
and
\[ m_2 = \tilde{D}_2 \cdot F_2 \geq \text{mult}_{Q_2} (\tilde{D}_2 \cdot F_2) > \frac{12}{5} - (a_1 + a_2 + m_1), \]
which is false.

• Suppose \( O \in F_2 \setminus (\tilde{F}_1 \cup \tilde{E}_2) \), then the log pair
\[
K_{\tilde{X}_2} + \lambda \tilde{D}_2 + (\lambda(a_1 + 2a_2 + m_1 + m_2) - 2)F_2
\]
is not log canonical at \( Q_2 \) and so is the log pair
\[
K_{\tilde{X}_2} + \lambda \tilde{D}_2 + F_2, \text{ since } \frac{5}{6}(a_1 + 2a_2 + m_1 + m_2) - 2 \leq 1.
\]
By adjunction it follows that
\[
m_2 = \tilde{D}_2 \cdot F_2 \geq \text{mult}_{Q_2} (\tilde{D}_2 \cdot F_2) > \frac{6}{5},
\]
which is false.

• Suppose \( Q_2 \in F_2 \cap \tilde{E}_2 \), then the log pair
\[
K_{\tilde{X}_2} + \lambda \tilde{D}_2 + (\lambda(a_1 + 2a_2 + m_1 + m_2) - 2)F_2 + \lambda a_2 \tilde{E}_2
\]
is not log canonical at \( Q_2 \) and so are the log pairs
\[
K_{\tilde{X}_2} + \lambda \tilde{D}_2 + (\lambda(a_1 + 2a_2 + m_1 + m_2) - 2)F_2 + \tilde{E}_2
\]
and
\[
K_{\tilde{X}_2} + \lambda \tilde{D}_2 + F_2 + \lambda a_2 \tilde{E}_2.
\]
By adjunction it follows that
\[
2a_2 - a_1 - a_3 - m_1 - m_2 = \tilde{D}_2 \cdot \tilde{E}_2 \geq \text{mult}_{Q_2} (\tilde{D}_2 \cdot \tilde{E}_2) > \frac{18}{5} - (a_1 + 2a_2 + m_1 + m_2)
\]
and
\[
m_2 = \tilde{D}_2 \cdot F_2 \geq \text{mult}_{Q_2} (\tilde{D}_2 \cdot F_2) > \frac{6}{5} - a_2,
\]
which implies that
\[
4a_2 - \frac{a_2}{2} \geq 4a_2 - a_3 > \frac{18}{5} \Rightarrow a_2 > \frac{9}{10} \cdot \frac{6}{5}.
\]
Consider now the blow-up \( \sigma_k : \tilde{X}_k \to \tilde{X}_{k-1} \) of the surface \( \tilde{X}_{k-1} \) at the point \( Q_{k-1} \) that contracts the \((-1)\)-curve \( F_k \) to the point \( Q_{k-1} \). We then have

\[
\begin{align*}
K_{\tilde{X}_k} & \sim_{Q} \pi_k^*(K_{\tilde{X}_1}) + \tilde{F}_1 + 2\tilde{F}_2 + 3\tilde{F}_3 + \ldots + (k - 1)\tilde{F}_{k-1} + kF_k \\
\tilde{D}_k & \sim_{Q} \pi_k^*(D) - a_1\tilde{E}_1 - a_2\tilde{E}_2 - a_3\tilde{E}_3 - a_4\tilde{E}_4 - (a_1 + a_2 + m_1)\tilde{F}_1 - (a_1 + 2a_2 + m_1 + m_2)\tilde{F}_2 - \ldots \ - (a_1 + (k - 1)a_2 + m_1 + m_2 + \ldots + m_{k-1})\tilde{F}_{k-1} - (a_1 + ka_2 + m_1 + m_2 + \ldots + m_k)F_k \\
\tilde{Z}_k & \sim_{Q} \pi_k^*(Z) - \tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3 - \tilde{E}_4 - 2\tilde{F}_1 - 3\tilde{F}_2 - \ldots - k\tilde{F}_{k-1} - (k + 1)F_k,
\end{align*}
\]
and
\[
0 \leq \tilde{D}_k \cdot \tilde{Z}_k = 1 - a_1 - a_4 \\
0 \leq \tilde{E}_1 \cdot \tilde{D}_k = 2a_1 - a_2 - m_1 \\
0 \leq \tilde{E}_2 \cdot \tilde{D}_k = 2a_2 - a_1 - a_3 - m_1 - m_2 - ... - m_k \\
0 \leq \tilde{E}_3 \cdot \tilde{D}_k = 2a_3 - a_2 - a_4 \\
0 \leq \tilde{E}_4 \cdot \tilde{D}_k = 2a_4 - a_3 \\
0 \leq \tilde{F}_k \cdot \tilde{D}_k = m_k - m_2 \\
0 \leq \tilde{F}_k \cdot \tilde{D}_k = m_k - m_3 \\
\]
where \( m_i = \text{mult}_{Q_k} \tilde{D}_k \), for \( i = 1, ..., k \).

Because of the equivalence
\[
\pi_k^*(K_X + \lambda D) \sim_Q \\
K_{\tilde{X}_k} + \lambda \tilde{D}_k + \lambda a_1 \tilde{E}_1 + \lambda a_2 \tilde{E}_2 + \lambda a_3 \tilde{E}_3 + \lambda a_4 \tilde{E}_4 + \\
(\lambda a_1 + a_2 + m_1 - 1) \tilde{F}_1 + (\lambda a_1 + 2a_2 + m_1 + m_2 - 2) \tilde{F}_2 + ... + \\
(\lambda a_1 + (k - 1)a_2 + m_1 + m_2 + ... + m_{k-1} - (k - 1)) \tilde{F}_{k-1} + \\
(\lambda a_1 + ka_2 + m_1 + m_2 + ... + m_k - k) \tilde{F}_k
\]
there is a point \( Q_k \in F_k \) such that the pair
\[
K_{\tilde{X}_k} + \lambda \tilde{D}_k + \lambda a_2 \tilde{E}_2 + (\lambda a_1 + (k - 1)a_2 + m_1 + m_2 + ... + m_{k-1} - (k - 1)) \tilde{F}_{k-1} + \\
(\lambda a_1 + ka_2 + m_1 + m_2 + ... + m_k - k) \tilde{F}_k
\]
is not log canonical at \( Q_k \).

- Suppose \( Q_k \in F_k \cap \tilde{F}_{k-1} \), then the log pair
\[
K_{\tilde{X}_k} + \lambda \tilde{D}_k + (\lambda a_1 + (k - 1)a_2 + m_1 + m_2 + ... + m_{k-1} - (k - 1)) \tilde{F}_{k-1} + \\
(\lambda a_1 + ka_2 + m_1 + m_2 + ... + m_k - k) \tilde{F}_k
\]
is not log canonical at \( Q_2 \) and so are the log pairs
\[
K_{\tilde{X}_k} + \lambda \tilde{D}_k + \tilde{F}_{k-1} + (\lambda a_1 + ka_2 + m_1 + m_2 + ... + m_k - k) \tilde{F}_k
\]
and
\[
K_{\tilde{X}_k} + \lambda \tilde{D}_k + (\lambda a_1 + (k - 1)a_2 + m_1 + m_2 + ... + m_{k-1} - (k - 1)) \tilde{F}_{k-1} + \tilde{F}_k.
\]

By adjunction it follows that
\[
m_{k-1} - m_k = \tilde{D}_k \cdot \tilde{F}_{k-1} \geq \text{mult}_{Q_k}(\tilde{D}_k \cdot \tilde{F}_{k-1}) > \frac{1}{\lambda}(k + 1) - (a_1 + ka_2 + m_1 + m_2 + ... + m_k),
\]
which is a contradiction. Indeed from the inequality above we have that
\[
a_1 + ka_2 + m_1 + m_2 + ... + m_{k-2} + 2m_{k-1} > \frac{1}{\lambda}(k + 1)
\]
but since \( m_1 \geq m_2 \geq ... \geq m_k \) we get that
\[
a_1 + ka_2 + km_1 > \frac{1}{\lambda}(k + 1).
\]

However the inequality \( 0 \leq \tilde{E}_1 \cdot \tilde{D}_k = 2a_1 - a_2 - m_1 \) finally gives us
\[
(2k + 1)a_1 > \lambda(k + 1) \Rightarrow a_4 < \frac{4k - 1}{5(2k + 1)} \Rightarrow a_2 \leq 3a_4 < \frac{4k - 1}{4k + 2} \cdot \frac{6}{5}.
\]
which is false since after the \((k - 1)\)-th blow up (and before the \(k\)-th blow up) we have

\[
a_2 > \frac{6}{5} \cdot \frac{3k}{3k + 1} > \frac{4k - 1}{4k + 2} \cdot \frac{6}{5}.
\]

- Suppose \(Q_k \in F_k \setminus (\tilde{F}_{k-1} \cup \tilde{E}_2)\), then the log pair

\[
K_{\tilde{X}_k} + \lambda \tilde{D}_k + (\lambda(a_1 + ka_2 + m_1 + m_2 + ... + m_k) - k) F_k
\]

is not log canonical at \(Q_k\) and so is the log pair

\[
K_{\tilde{X}_k} + \lambda \tilde{D}_k + F_k, \text{ since } (\lambda(a_1 + ka_2 + m_1 + m_2 + ... + m_k) - k) \leq 1.
\]

By adjunction it follows that

\[
m_k = \tilde{D}_k \cdot F_k \geq \text{mult}_{Q_k}(\tilde{D}_k \cdot F_k) > \frac{6}{5},
\]

which is false, since \(\frac{1}{2} \geq m_1 \geq m_2 \geq \ldots \geq m_k\).

- Suppose \(Q_k \in F_k \cap \tilde{E}_2\), then the log pair

\[
K_{\tilde{X}_k} + \lambda \tilde{D}_k + (\lambda(a_1 + ka_2 + m_1 + m_2 + ... + m_k) - k) F_k + \lambda a_2 \tilde{E}_2
\]

is not log canonical at \(Q_k\) and so are the log pairs

\[
K_{\tilde{X}_k} + \lambda \tilde{D}_k + (\lambda(a_1 + ka_2 + m_1 + m_2 + ... + m_k) - k) F_k + \tilde{E}_2
\]

and

\[
K_{\tilde{X}_k} + \lambda \tilde{D}_k + F_k + \lambda a_2 \tilde{E}_2.
\]

By adjunction it follows that

\[
2a_2-a_1-a_3-m_1-m_2-\ldots-m_k = \tilde{D}_k \cdot \tilde{E}_2 \geq \text{mult}_{Q_k}(\tilde{D}_k \cdot \tilde{E}_2) > \frac{6}{5}(k+1)-(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k)
\]

and

\[
m_k = \tilde{D}_k \cdot F_k \geq \text{mult}_{Q_k}(\tilde{D}_k \cdot F_k) > \frac{6}{5} - a_2,
\]

which implies that

\[
(k + 2)a_2 - \frac{2}{3}a_2 \geq (k + 2)a_2 - a_3 > \frac{6}{5}(k + 1) \Rightarrow a_2 > \frac{6}{5} \cdot \frac{3(k + 1)}{3(k + 1) + 1}.
\]

**Remark 3.5.** It remains to be shown that after the \(k\)-th blow up we have

\[
(\lambda(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k) - k) \leq 1,
\]

and for this it is enough to show that

\[
\left(\frac{5}{6}(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k) - k\right) \leq 1.
\]

Suppose that we have blown up \(k - 1\) times, then \(a_2 > \frac{6}{5} \cdot \frac{3k}{3k + 1}\). Let us assume on the contrary that

\[
\frac{5}{6}(a_1 + ka_2 + m_1 + m_2 + \ldots + m_k) - k > 1 \Rightarrow a_1 + ka_2 + m_1 + m_2 + \ldots + m_k > \frac{6}{5}(k + 1)
\]

\[
a_1 + 2ka_1 \geq a_1 + ka_2 + km_1 > \frac{6}{5}(k + 1) \Rightarrow a_1 > \frac{6}{5} \cdot \frac{k + 1}{2k + 1}
\]

\[
a_4 \leq 1 - a_1 < 1 - \frac{6}{5} \cdot \frac{k + 1}{2k + 1} \Rightarrow a_2 \leq 3a_4 < \frac{6}{5} \cdot \frac{4k - 1}{4k + 2},
\]

which is a contradiction.
• If $Q \in E_2 \cap E_3$ then the log pair

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3$$

is not log canonical at the point $Q$ and so are the log pairs

$$K_{\tilde{X}} + \lambda \tilde{D} + E_2 + \lambda a_3 E_3 \text{ and } K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 E_2 + E_3.$$  

By adjunction it follows that

$$2a_2 - \frac{1}{2} a_2 - a_3 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q (\tilde{D} \cdot E_2) > \frac{6}{5} - a_3,$$

and

$$2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q (\tilde{D} \cdot E_3) > \frac{6}{5} - a_2.$$

These imply that $a_2 > \frac{4}{5}$ and $a_3 > \frac{4}{5}$.

Consider now the blow-up $\pi_2 : \tilde{X} \to \tilde{X}$ of the surface $\tilde{X}$ at the point $Q$ that contracts the $(-1)$-curve $E$ to the point $Q$. Then for the strict transforms of the exceptional divisors $E_1, E_2, E_3, E_4$ we have

$$\tilde{E}_1 \sim_Q \pi_2^* (E_1)$$
$$\tilde{E}_2 \sim_Q \pi_2^* (E_2) - E$$
$$\tilde{E}_3 \sim_Q \pi_2^* (E_3) - E$$
$$\tilde{E}_4 \sim_Q \pi_2^* (E_4)$$

Let now

$$\pi : \tilde{X} \xrightarrow{\pi_2} \tilde{X} \xrightarrow{\pi_1} X$$

be the composition $\pi = \pi_1 \circ \pi_2$. We have

$$K_{\tilde{X}} = \pi_2^* (K_{\tilde{X}}) + E \sim_Q \pi_2^* (\pi_1^* (K_X)) + E \sim_Q \pi^* (K_X) + E$$

and

$$\tilde{D} = \pi_2^* (\tilde{D}) - mE$$
$$\sim_Q \pi_2^* (\pi_1^* (D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - b_1 F_1 - b_2 F_2 - b_3 F_3 - b_4 F_4) - mE$$
$$\sim_Q \pi^* (D) - a_1 \tilde{E}_1 - a_2 \tilde{E}_2 - a_3 \tilde{E}_3 - a_4 \tilde{E}_4 - b_1 \tilde{F}_1 - b_2 \tilde{F}_2 - b_3 \tilde{F}_3 - b_4 \tilde{F}_4 - (a_2 + a_3 + m)E,$$

where $m = \text{mult}_Q \tilde{D}$. Also the strict transform of the anticanonical curve $Z$ is

$$\tilde{Z} \sim_Q \pi_2^* (\tilde{Z})$$
$$\sim_Q \pi_2^* (\pi_1^* (Z) - E_1 - E_2 - E_3 - E_4)$$
$$\sim_Q \pi^* (Z) - \tilde{E}_1 - \tilde{E}_2 - \tilde{E}_3 - \tilde{E}_4 - 2E.$$

From the inequalities

$$0 \leq \tilde{D} \cdot \tilde{Z} = 1 - a_1 - a_4$$
$$0 \leq \tilde{E}_1 \cdot \tilde{D} = 2a_1 - a_2$$
$$0 \leq \tilde{E}_2 \cdot \tilde{D} = 2a_2 - a_1 - a_3 - m$$
$$0 \leq \tilde{E}_3 \cdot \tilde{D} = 2a_3 - a_2 - a_4 - m$$
$$0 \leq \tilde{E}_4 \cdot \tilde{D} = 2a_4 - a_3$$
$$0 \leq E \cdot \tilde{D} = m$$

we get that $m = \text{mult}_Q \tilde{D} \leq \frac{1}{2}$.

The equivalence

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 \tilde{E}_2 + \lambda a_3 \tilde{E}_3 + \left(\lambda (a_2 + a_3 + \text{mult}_Q \tilde{D}) - 1\right) E \sim_Q \pi^* (K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3)$$

is valid.
Then the global log canonical threshold of $P$ outside of a point $X$ is

$\lambda D + \lambda a_2 \tilde{E}_2 + \lambda a_3 \tilde{E}_3 + \left( \lambda (a_2 + a_3 + \text{mult}_Q \tilde{D}) - 1 \right) E$

is not log canonical at $R$.

(i) If $Q \in \tilde{E}_2 \cap E$ then the log pair

$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 \tilde{E}_2 + \left( \lambda (a_2 + a_3 + \text{mult}_Q \tilde{D}) - 1 \right) E$

is not log canonical at the point $R$ and so is the log pair

$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 \tilde{E}_2 + E$.

By adjunction and inequality $3.4$ it follows that

$2 - \frac{4}{5} - a_2 \geq 2 - a_2 - a_3 \geq \text{mult}_Q \tilde{D} = \tilde{D} \cdot E \geq \text{mult}_R \left( \tilde{D} \cdot E \right) > \frac{6}{5} - a_2$,

which is false.

(ii) If $Q \in E \setminus (E_2 \cup E_3)$ then the log pair

$K_{\tilde{X}} + \lambda \tilde{D} + \left( \lambda (a_2 + a_3 + \text{mult}_Q \tilde{D}) - 1 \right) E$

is not log canonical at the point $R$ and so is the log pair

$K_{\tilde{X}} + \lambda \tilde{D} + E$.

By adjunction it follows that

$\text{mult}_Q \tilde{D} = \tilde{D} \cdot E \geq \text{mult}_R \left( \tilde{D} \cdot E \right) > \frac{6}{5}$,

which is false.

(iii) If $Q \in \tilde{E}_3 \cap E$ then the log pair

$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_3 \tilde{E}_3 + \left( \lambda (a_2 + a_3 + \text{mult}_Q \tilde{D}) - 1 \right) E$

is not log canonical at the point $R$ and so is the log pair

$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_3 \tilde{E}_3 + E$.

By adjunction and inequality $3.4$ it follows that

$2 - \frac{4}{5} - a_2 \geq 2 - a_2 - a_3 \geq \text{mult}_Q \tilde{D} = \tilde{D} \cdot E \geq \text{mult}_R \left( \tilde{D} \cdot E \right) > \frac{6}{5} - a_3$,

which is false.

3.3. Del Pezzo surfaces of degree 1 with one $A_5$ type singularity. In this section we will prove the following.

**Lemma 3.6.** Let $X$ be a del Pezzo surface with one Du Val singularity of type $A_5$ and $K^2_X = 1$. Then the global log canonical threshold of $X$ is

$\text{lct}(X) = \frac{2}{3}$.

**Proof.** Let $X$ be a del Pezzo surface with at most one Du Val singularity of type $A_5$ and $K^2_X = 1$. Let $\pi_1 : \tilde{X} \to X$ be the minimal resolution of $X$. The following diagram shows how the exceptional curves intersect each other.

\[ A_5. \quad \bullet E_1 \quad \bullet E_2 \quad \bullet E_3 \quad \bullet E_4 \quad \bullet E_5 \]

Suppose that $\text{lct}(X) < \frac{2}{3}$, then there exists an effective $\mathbb{Q}$-divisor $D \in X$ such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical, where $\lambda < \frac{2}{3}$. We derive that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at $P$. Then

$\tilde{D} \sim_{\mathbb{Q}} \pi_1^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5$ and $\tilde{Z} \sim_{\mathbb{Q}} \pi_1^*(Z) - E_1 - E_2 - E_3 - E_4 - E_5$.
Let $Z$ be the curve in $|−K_X|$ that contains $P$. Since the curve $Z$ is irreducible we may assume that the support of $D$ does not contain $Z$.

From the inequalities

\begin{align*}
0 & \leq \tilde{D} \cdot \tilde{Z} = 1 - a_1 - a_5 \\
0 & \leq E_1 \cdot \tilde{D} = 2a_1 - a_2 \\
0 & \leq E_2 \cdot \tilde{D} = 2a_2 - a_1 - a_3 \\
0 & \leq E_3 \cdot \tilde{D} = 2a_3 - a_2 - a_4 \\
0 & \leq E_4 \cdot \tilde{D} = 2a_4 - a_3 - a_5 \\
0 & \leq E_5 \cdot \tilde{D} = 2a_5 - a_4
\end{align*}

we see that

\begin{align*}
a_1 & \leq \frac{5}{6} , \quad a_2 \leq \frac{4}{3} , \quad a_3 \leq \frac{3}{2}, \quad a_4 \leq \frac{4}{3} , \quad a_5 \leq \frac{5}{6}
\end{align*}

and what is more

\begin{align*}
2a_5 & \geq a_4 , \quad \frac{3}{2}a_4 \geq a_3 , \quad \frac{4}{3}a_3 \geq a_2 , \quad \frac{5}{4}a_2 \geq a_1
\end{align*}

Furthermore there exists a curve $L_3 \in X$, that passes through the point $P$, whose strict transform is a $(-1)$-curve that intersects the fundamental cycle as following.

\[\tilde{L}_3 \cdot E_3 = 1\]

and

\[L_3 \cdot E_j = 0 \quad \text{for} \quad j = 1, 2, 4, 5 .\]

Then we easily get that

\[\tilde{L}_3 \sim_\pi (L_3) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - E_4 - \frac{1}{2}E_5 .\]

The image of $L_3$ under involution is either fixed or $L_3$ is mapped to another curve $L'_3$. In either case we can assume that the irreducible line $L_3$ is not contained in $\text{Supp}(D)$ and thus deduce the inequality

\[0 \leq \tilde{L}_3 \cdot \tilde{D} = 1 - a_3 ,\]

The equivalence

\[K_\tilde{X} + \lambda \tilde{D} + a_1 \lambda E_1 + a_2 \lambda E_2 + a_3 \lambda E_3 + a_4 \lambda E_4 + a_5 \lambda E_5 \sim_\pi (K_X + D)\]

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ such that the pair

\[K_\tilde{X} + \lambda \tilde{D} + a_1 \lambda E_1 + a_2 \lambda E_2 + a_3 \lambda E_3 + a_4 \lambda E_4 + a_5 \lambda E_5\]

is not log canonical at $Q$.

- If the point $Q \in E_1$ and $Q \notin E_2$ then

\[K_\tilde{X} + \lambda \tilde{D} + a_1 \lambda E_1\]

is not log canonical at the point $Q$ and so is the pair

\[K_\tilde{X} + \lambda \tilde{D} + E_1 , \quad \text{since} \quad a_1 \lambda \leq 1 .\]

By adjunction $(E_1, \lambda \tilde{D}|_{E_1})$ is not log canonical at $Q$ and

\[1 \geq 2a_1 - \frac{4}{5}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q (\tilde{D} \cdot E_1) > \frac{1}{\lambda} \geq \frac{3}{2} ,\]

which is a contradiction.

- If $Q \in E_1 \cap E_2$ then the log pair

\[K_\tilde{X} + \lambda \tilde{D} + a_1 \lambda E_1 + a_2 \lambda E_2\]

is not log canonical at the point $Q$ and so are the log pairs

\[K_\tilde{X} + \lambda \tilde{D} + E_1 + a_2 \lambda E_2 \quad \text{and} \quad K_\tilde{X} + \lambda \tilde{D} + a_1 \lambda E_1 + E_2 .\]

By adjunction it follows that

\[2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q (\tilde{D}|_{E_2}) = \text{mult}_Q (\tilde{D} \cdot E_2) > \frac{1}{\lambda} - a_1 > \frac{3}{2} - a_1 ,\]
and
\[2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q (\tilde{D}|_{E_1}) = \text{mult}_Q (\tilde{D} \cdot E_1) > \frac{1}{\lambda} - a_2 > \frac{3}{2} - a_2.\]
From the first inequality we get \(a_3 \geq \frac{9}{10}\) and then we see that
\[1 \geq a_1 + a_5 \geq a_1 + \frac{1}{2} \cdot \frac{2}{3}a_3 > \frac{3}{4} + \frac{3}{10} > 1,
which is a contradiction.

- If \(Q \in E_2\) but \(Q \not\in E_1 \cup E_3\) then
  \[K_{\tilde{X}} + \lambda \tilde{D} + a_2 \lambda E_2\]
is not log canonical at the point \(Q\) and so is the pair
  \[K_{\tilde{X}} + \lambda \tilde{D} + E_2, \text{ since } a_2 \lambda \leq 1.\]
By adjunction \((E_2, \lambda \tilde{D}|_{E_2})\) is not log canonical at \(Q\) and
\[1 \geq 2a_2 - \frac{1}{2}a_2 - \frac{3}{4}a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q (\tilde{D} \cdot E_2) > \frac{1}{\lambda} > \frac{3}{2},
which is a contradiction.

- If \(Q \in E_2 \cap E_3\) then the log pair
  \[K_{\tilde{X}} + \lambda \tilde{D} + a_2 \lambda E_2 + a_3 \lambda E_3\]
is not log canonical at the point \(Q\) and so are the log pairs
  \[K_{\tilde{X}} + \lambda \tilde{D} + a_2 \lambda E_2 + E_3, \text{ since } \lambda a_3 \leq 1.\]
By adjunction it follows that
\[2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q (\tilde{D} \cdot E_3) > \frac{1}{\lambda} - a_2 > \frac{3}{2} - a_2.
which, together with the inequality \(a_4 \geq \frac{2}{3}a_3\), implies that \(a_3 > \frac{9}{8}\). However, this is impossible since \(a_3 \leq 1\).

- If \(Q \in E_3\) but \(Q \not\in E_2 \cup E_4\) then
  \[K_{\tilde{X}} + \lambda \tilde{D} + a_3 \lambda E_3\]
is not log canonical at the point \(Q\) and so is the pair
  \[K_{\tilde{X}} + \lambda \tilde{D} + E_3, \text{ since } a_3 \lambda \leq 1.\]
By adjunction \((E_3, \lambda \tilde{D}|_{E_3})\) is not log canonical at \(Q\) and
\[1 \geq \frac{2}{3}a_3 \geq 2a_3 - \frac{2}{3}a_3 - \frac{2}{3}a_3 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q (\tilde{D} \cdot E_3) > \frac{3}{2},
which is false.

We will now show the existence of the curve \(L_3\) which gave us the inequality \(a_3 \leq 1\). If we now contract the curves \(C, E_5, E_4, E_3\) in this order we obtain two curves intersecting each other as following.

\[
\begin{array}{c}
E_1 \\
\bigcap \\
E_2
\end{array}
\]

However the resulting surface is isomorphic to \(\mathbb{P}^2\) blown up at 4 points and we know the configuration of all the -1 curves in this case. Therefore there is always a -1 curve \(L_2\) that intersects the exceptional curve \(E_2\) and not \(E_3\). Indeed the resulting surface is a smooth del Pezzo surface of degree 5 and \(E_2 + E_3\) is its anticanonical divisor. Every -1 curve intersects the anticanonical divisor only at one point by adjunction formula and thus \(L_2\) cannot intersect \(E_1\).
If we now contract the curves $L_2, C, E_5, E_4$ we obtain a smooth del Pezzo surface of degree 6 and we have the following configuration of lines.

Therefore there exist -1 curves $L'_2$ and $L_3$ which intersect the exceptional curves $E_2$ and $E_3$ transversally and do not intersect any other exceptional curve.

Therefore there are two -1 curves $L_2$ and $L'_2$ intersecting the exceptional curve $E_2$ and -1 curve $L_3$ intersecting $E_3$, such that $L_3 \cdot L_3 = L'_2 \cdot L_3 = L_2 \cdot L'_2 = 0$. Involution either fixes the curve $L_3$ or sends it to another line $L'_3$. In any case we deduce the inequality $a_3 \leq 1$.

\[ \square \]

3.4. Del Pezzo surfaces of degree 1 with at most one $\tilde{A}_6$ type singularity. In this section we will prove the following.

**Lemma 3.7.** Let $X$ be a del Pezzo surface with at most one Du Val singularity of type $\tilde{A}_6$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is

\[ \text{lct}(X) = \frac{2}{3}. \]

**Proof.** Let $X$ be a del Pezzo surface with at most one Du Val singularity of type $\tilde{A}_6$ and $K_X^2 = 1$. Suppose that $\text{lct}(X) < \text{lct}_2(X) \leq \frac{2}{3}$, then there exists an effective $\mathbb{Q}$-divisor $D \in X$, such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical, where $\lambda < \frac{2}{3}$.

Let $Z$ be the unique curve in $|-K_X|$ that contains $P$. Since the curve $Z$ is irreducible we may assume that the support of $D$ does not contain $Z$.

We derive that the pair $(X, \lambda D)$ is log canonical outside of the singular point $P \in X$ and not log canonical at $P$. Let $\pi_1 : \tilde{X} \rightarrow X$ be the minimal resolution of $X$. The following diagram shows how the exceptional curves intersect each other.

\[ \tilde{A}_6 : E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_4 \rightarrow E_5 \rightarrow E_6 \]

Then

\[ \tilde{D} \sim_{\mathbb{Q}} \pi_1^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 \quad \text{and} \quad \tilde{Z} \sim_{\mathbb{Q}} \pi_1^*(Z) - E_1 - E_2 - E_3 - E_4 - E_5 - E_6. \]

From the inequalities

\[
\begin{align*}
0 & \leq \tilde{D} : \tilde{Z} = 1 - a_1 - a_6 \\
0 & \leq E_1 : \tilde{D} = 2a_1 - a_2 \\
0 & \leq E_2 : \tilde{D} = 2a_2 - a_1 - a_3 \\
0 & \leq E_3 : \tilde{D} = 2a_3 - a_2 - a_4 \\
0 & \leq E_4 : \tilde{D} = 2a_4 - a_3 - a_5 \\
0 & \leq E_5 : \tilde{D} = 2a_5 - a_4 - a_6 \\
0 & \leq E_6 : \tilde{D} = 2a_6 - a_5
\end{align*}
\]

we see that

\[ 2a_6 \geq a_5, \quad \frac{3}{2}a_5 \geq a_4, \quad \frac{4}{3}a_4 \geq a_3, \quad \frac{5}{4}a_3 \geq a_2, \quad \frac{6}{5}a_2 \geq a_1 \]

and

\[ 2a_1 \geq a_2, \quad \frac{3}{2}a_2 \geq a_3, \quad \frac{4}{3}a_3 \geq a_4, \quad \frac{5}{4}a_4 \geq a_5, \quad \frac{6}{5}a_5 \geq a_6. \]
Furthermore there are four curves $L_2, L_3, L_4, L_5 \in X$ that pass through the point $P$, such that their strict transforms in $\tilde{X}$ are the $(-1)$-curves $\tilde{L}_2, \tilde{L}_3, \tilde{L}_4, \tilde{L}_5$ that intersect the fundamental cycle as following
\[
\tilde{L}_2 \cdot E_2 = \tilde{L}_3 \cdot E_3 = \tilde{L}_4 \cdot E_4 = \tilde{L}_5 \cdot E_5 = 1
\]
and
\[
\tilde{L}_i \cdot E_j = 0 \text{ for all } i = 2, 3, 4, 5 \text{ and } j = 1, ..., 6 \text{ with } i \neq j.
\]
We can easily see that
\[
\begin{align*}
\tilde{L}_2 & \sim_{\mathbb{Q}} \pi^*(L_2) - \frac{5}{7}E_1 - \frac{10}{7}E_2 - \frac{8}{7}E_3 - \frac{6}{7}E_4 - \frac{4}{7}E_5 - \frac{2}{7}E_6, \\
\tilde{L}_3 & \sim_{\mathbb{Q}} \pi^*(L_3) - \frac{4}{7}E_1 - \frac{8}{7}E_2 - \frac{12}{7}E_3 - \frac{9}{7}E_4 - \frac{6}{7}E_5 - \frac{3}{7}E_6, \\
\tilde{L}_4 & \sim_{\mathbb{Q}} \pi^*(L_4) - \frac{3}{7}E_1 - \frac{6}{7}E_2 - \frac{9}{7}E_3 - \frac{12}{7}E_4 - \frac{8}{7}E_5 - \frac{4}{7}E_6, \\
\tilde{L}_5 & \sim_{\mathbb{Q}} \pi^*(L_5) - \frac{2}{7}E_1 - \frac{4}{7}E_2 - \frac{6}{7}E_3 - \frac{8}{7}E_4 - \frac{10}{7}E_5 - \frac{5}{7}E_6.
\end{align*}
\]
We have that $L_2 + L_5 \in | - 2K_X |$ and $L_2 + L_4 \in | - 2K_X |$ and we can assume that at least one member from each pair $L_2 + L_5$ and $L_3 + L_4$ is not contained in the support of $D$. Thus $0 \leq \tilde{L}_3 \cdot D = 1 - a_3$ or $0 \leq \tilde{L}_4 \cdot D = 1 - a_4$. The equivalence
\[
K_{\tilde{X}} + \lambda \tilde{D} + a_1 \lambda E_1 + a_2 \lambda E_2 + a_3 \lambda E_3 + a_4 \lambda E_4 + a_5 \lambda E_5 + a_6 \lambda E_6 = \pi_1^*(K_X + \lambda D)
\]
implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6$ such that the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + a_1 \lambda E_1 + a_2 \lambda E_2 + a_3 \lambda E_3 + a_4 \lambda E_4 + a_5 \lambda E_5 + a_6 \lambda E_6
\]
is not log canonical at $Q$. 

- If the point $Q \in E_1$ and $Q \not\in E_2$ then
\[
K_{\tilde{X}} + \lambda \tilde{D} + a_1 \lambda E_1
\]
is not log canonical at the point $Q$ and so is the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + E_1, \text{ since } a_1 \lambda \leq 1.
\]
By adjunction $(E_1, \lambda \tilde{D}|_{E_1})$ is not log canonical at $Q$ and
\[
1 \geq \frac{7}{6}a_1 \geq 2a_1 - \frac{5}{6}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D} \cdot E_1) > \frac{1}{\lambda} > \frac{3}{2},
\]
which is false.

- If $Q \in E_1 \cap E_2$ then the log pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + a_1 \lambda E_1 + a_2 \lambda E_2
\]
is not log canonical at the point $Q$ and so are the log pairs
\[
K_{\tilde{X}} + \lambda \tilde{D} + E_1 + a_2 \lambda E_2 \text{ and } K_{\tilde{X}} + \lambda \tilde{D} + a_1 \lambda E_1 + E_2.
\]
By adjunction it follows that
\[
2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D} \cdot E_1) > \frac{3}{2} - a_1 \Rightarrow a_1 > 1,
\]
and
\[
2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > \frac{3}{2} - a_2 \Rightarrow a_1 > \frac{3}{4},
\]
which leads to contradiction, as
\[
1 \geq a_1 + a_6 \geq a_1 + \frac{12}{23} \geq \frac{3}{4} + \frac{1}{4} = 1.
\]
• If \( Q \in E_2 \) but \( Q \not\in E_1 \cup E_3 \) then
\[
K_X + \lambda \tilde{D} + a_2 \lambda E_2
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_2 , \text{ since } a_2 \lambda \leq 1 .
\]
By adjunction \((E_2, \lambda \tilde{D}|_{E_2})\) is not log canonical at \( Q \) and
\[
1 \geq \frac{7}{10} a_2 \geq 2a_2 - \frac{4}{5} a_2 - \frac{1}{2} a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mult_Q \left( \tilde{D} \cdot E_2 \right) > \frac{3}{2} ,
\]
which is false.

• If \( Q \in E_3 \) but \( Q \not\in E_2 \cup E_4 \) then
\[
K_X + \lambda \tilde{D} + a_3 \lambda E_3
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_3 , \text{ since } a_3 \lambda \leq 1 .
\]
By adjunction \((E_3, \lambda \tilde{D}|_{E_3})\) is not log canonical at \( Q \) and
\[
1 \geq \frac{7}{12} a_3 \geq 2a_3 - \frac{2}{3} a_3 - \frac{3}{4} a_3 = 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mult_Q \left( \tilde{D} \cdot E_3 \right) > 1 \lambda > \frac{3}{2} ,
\]
which is false.

• If \( Q \in E_3 \cap E_4 \) then the log pair
\[
K_X + \lambda \tilde{D} + a_3 \lambda E_3 + a_4 \lambda E_4
\]
is not log canonical at the point \( Q \) and so are the log pairs
\[
K_X + \lambda \tilde{D} + E_3 + a_4 \lambda E_4 \text{ and } K_X + \lambda \tilde{D} + a_3 \lambda E_3 + E_4 , \text{ since } \lambda a_4 \leq 1 .
\]
By adjunction it follows that
\[
2a_3 - \frac{2}{3} a_3 - a_4 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mult_Q \left( \tilde{D} \cdot E_3 \right) > \frac{1}{\lambda} - a_4 > \frac{3}{2} - a_4 ,
\]
and
\[
2a_4 - a_3 - \frac{2}{3} a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mult_Q \left( \tilde{D} \cdot E_4 \right) > \frac{1}{\lambda} - a_3 > \frac{3}{2} - a_3 .
\]
This means that \( a_3 > 1 \) and \( a_4 > 1 \) which is impossible.

• If \( Q \in E_2 \cap E_3 \) then the log pair
\[
K_X + \lambda \tilde{D} + a_2 \lambda E_2 + a_3 \lambda E_3
\]
is not log canonical at the point \( Q \) and so are the log pairs
\[
K_X + \lambda \tilde{D} + E_2 + a_3 \lambda E_3 \text{ and } K_X + \lambda \tilde{D} + a_2 \lambda E_2 + E_3 , \text{ since } \lambda a_3 \leq 1 .
\]
By adjunction it follows that
\[
2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mult_Q \left( \tilde{D}|_{E_2} \right) > \frac{1}{\lambda} - a_3 > \frac{3}{2} - a_3 \Rightarrow a_2 > 1 ,
\]
and
\[
2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mult_Q \left( \tilde{D}|_{E_3} \right) > \frac{1}{\lambda} - a_2 > \frac{3}{2} - a_2 \Rightarrow a_3 > \frac{6}{5} .
\]
In \( L_2 \not\in \text{Supp}D \) then \( 0 \leq \tilde{D} \cdot L_3 = 1 - a_3 \) which contradicts the above inequalities and the same holds when \( L_2 \not\in \text{Supp}D \). Therefore we assume that \( D = aL_2 + cL_3 + \Omega \).

In the following graph we can see how the exceptional curves intersect each other.
If we now contract the curves \( C, E_6, E_5, E_4, E_3 \) in this order we get a del Pezzo surface of degree 6 and here is how the remaining curves intersect.

However the resulting surface is isomorphic to \( \mathbb{P}^2 \) blown up at 3 points and we know the configuration of all the -1 curves in this case, they form a hexagon. Therefore there is always a -1 curve \( L_2 \) that intersects the exceptional curve \( E_2 \) and not \( E_3 \). Indeed the resulting surface is a smooth del Pezzo surface of degree 6 and \( E_2 + E_3 \) is its anticanonical divisor. Every -1 curve intersects the anticanonical divisor only at one point by adjunction formula and thus \( L_2 \) cannot intersect \( E_1 \).

If we now contract the -1 curve \( L_2 \) in the original setting we have a smooth surface of degree 2 with the following configuration of curves.

If after that we contract the curves \( C, E_6, E_5, E_4 \) we obtain a smooth del Pezzo surface of degree 6 and we have the following configuration of lines.

Therefore there exist -1 curves \( L_2' \) and \( L_3 \) which intersect the exceptional curves \( E_2 \) and \( E_3 \) transversal and do not intersect any other exceptional curve.

Therefore there are two -1 curves \( L_2 \) and \( L_2' \) intersecting the exceptional curve \( E_2 \) and -1 curve \( L_3 \) intersecting \( E_3 \), such that \( \tilde{L}_2 \cdot \tilde{L}_3 = \tilde{L}_2' \cdot \tilde{L}_3 = \tilde{L}_2 \cdot \tilde{L}_2' = 0 \).

Furthermore we also have the involusive images of the curves \( L_2, L_2', L_3 \). In total we get six curves \( L_2, L_2', L_3, L_4, L_5, L_5' \in X \) that pass through the point \( P \), such that their strict transforms in \( \tilde{X} \) are the \((-1)\)-curves \( \tilde{L}_2, \tilde{L}_2', \tilde{L}_3, \tilde{L}_4, \tilde{L}_5, \tilde{L}_5' \) that intersect the
fundamental cycle as following
\[ \tilde{L}_2 \cdot E_2 = \tilde{L}_3 \cdot E_3 = \tilde{L}_4 \cdot E_4 = \tilde{L}_5 \cdot E_5 = 1 \]
and
\[ L_i \cdot E_j = 0 \text{ for all } i = 2, 3, 4, 5 \text{ and } j = 1, \ldots, 6 \text{ with } i \neq j. \]

We can easily see that
\[
\begin{align*}
\tilde{L}_2 & \sim_{\mathbb{Q}} \pi^*(L_2) - \frac{5}{7}E_1 - \frac{10}{7}E_2 - \frac{8}{7}E_3 - \frac{6}{7}E_4 - \frac{4}{7}E_5 - \frac{2}{7}E_6, \\
\tilde{L}_2' & \sim_{\mathbb{Q}} \pi^*(L_2') - \frac{5}{7}E_1 - \frac{10}{7}E_2 - \frac{8}{7}E_3 - \frac{6}{7}E_4 - \frac{4}{7}E_5 - \frac{2}{7}E_6, \\
\tilde{L}_3 & \sim_{\mathbb{Q}} \pi^*(L_3) - \frac{4}{7}E_1 - \frac{8}{7}E_2 - \frac{12}{7}E_3 - \frac{9}{7}E_4 - \frac{6}{7}E_5 - \frac{3}{7}E_6, \\
\tilde{L}_4 & \sim_{\mathbb{Q}} \pi^*(L_4) - \frac{3}{7}E_1 - \frac{6}{7}E_2 - \frac{9}{7}E_3 - \frac{12}{7}E_4 - \frac{8}{7}E_5 - \frac{4}{7}E_6, \\
\tilde{L}_5 & \sim_{\mathbb{Q}} \pi^*(L_5) - \frac{2}{7}E_1 - \frac{4}{7}E_2 - \frac{6}{7}E_3 - \frac{8}{7}E_4 - \frac{10}{7}E_5 - \frac{5}{7}E_6, \\
\tilde{L}_5' & \sim_{\mathbb{Q}} \pi^*(L_5') - \frac{2}{7}E_1 - \frac{4}{7}E_2 - \frac{6}{7}E_3 - \frac{8}{7}E_4 - \frac{10}{7}E_5 - \frac{5}{7}E_6.
\end{align*}
\]

We compute the intersection matrix for the curves $L_2, L_2', L_3$ and we see that these three divisors are linearly independent. We know $\text{Pic}(\tilde{X}) = \mathbb{Z}^9$ and we collapse six exceptional -2 curves, therefore $\text{Pic}(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Therefore this is a basis of the $\text{Pic}(X) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

\[
\begin{align*}
L_2^2 & = \pi^*(L_2) \cdot \pi^*(L_2) = \tilde{L}_2 \cdot \pi^*(L_2) = L_2^2 + \frac{10}{7}\tilde{L}_2 \cdot E_2 = -1 + \frac{10}{7} = \frac{3}{7}, \\
L_2'^2 & = \pi^*(L_2') \cdot \pi^*(L_2') = \tilde{L}_2' \cdot \pi^*(L_2') = L_2'^2 + \frac{10}{7}\tilde{L}_2' \cdot E_2 = -1 + \frac{10}{7} = \frac{3}{7}, \\
L_3^2 & = \pi^*(L_3) \cdot \pi^*(L_3) = \tilde{L}_3 \cdot \pi^*(L_3) = L_3^2 + \frac{12}{7}\tilde{L}_3 \cdot E_3 = -1 + \frac{12}{7} = \frac{5}{7}, \\
L_2' \cdot L_3 & = \pi^*(L_2') \cdot \pi^*(L_3) = \tilde{L}_2' \cdot \pi^*(L_3) = \tilde{L}_2' \cdot \tilde{L}_3 + \frac{8}{7}\tilde{L}_2' \cdot E_2 = \frac{8}{7}, \\
L_2 \cdot L_3 & = \pi^*(L_2) \cdot \pi^*(L_3) = \tilde{L}_2 \cdot \pi^*(L_3) = \tilde{L}_2 \cdot \tilde{L}_3 + \frac{8}{7}\tilde{L}_2 \cdot E_2 = \frac{8}{7}, \\
L_2 \cdot L_2' & = \pi^*(L_2) \cdot \pi^*(L_2') = \tilde{L}_2 \cdot \pi^*(L_2') = \tilde{L}_2 \cdot \tilde{L}_2' + \frac{10}{7}\tilde{L}_2 \cdot E_2 = \frac{10}{7}. 
\end{align*}
\]

Now we would like to calculate $D = aL_2 + bL_2' + cL_3$. We have the following system of equations.

\[
\begin{align*}
\frac{3}{7}a + \frac{10}{7}b + \frac{8}{7}c & = 1, \\
\frac{10}{7}a + \frac{3}{7}b + \frac{8}{7}c & = 1, \\
\frac{8}{7}a + \frac{8}{7}b + \frac{5}{7}c & = 1.
\end{align*}
\]

Therefore our effective divisor $D$ is
\[
D = \frac{1}{3}L_2 + \frac{1}{3}L_2' + \frac{1}{3}L_3.
\]

and
\[
\tilde{D} = \pi^*(D) - \frac{2}{3}E_1 - \frac{4}{3}E_2 - \frac{4}{3}E_3 - E_4 - \frac{2}{3}E_5 - \frac{1}{3}E_6.
\]

We should note here that the divisor $\tilde{D}$ is a simple normal crossings divisor and thus if we blow up more we do not improve the log canonical threshold. There is no need to blow up further and $(X, \lambda D)$ is log canonical which is a contradiction. \[\square\]
3.5. Del Pezzo surfaces of degree 1 with exactly one $A_7$ type singularity. In this section we will prove the following.

**Lemma 3.8.** Let $X$ be a del Pezzo surface with at most one Du Val singularity of type $A_7$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = \frac{1}{2} \text{ or } \frac{8}{15}.$$ 

The del Pezzo surface $X$ can be realised as the double cover

$$X \overset{2:1}{\rightarrow} \mathbb{P}(1,1,2),$$

which is ramified along a sextic curve $R \subset \mathbb{P}(1,1,2)$. Let $\pi_1 : \tilde{X} \rightarrow X$ be the minimal resolution of $X$. The following diagram shows how the exceptional curves intersect each other.

![Diagram of exceptional curves]

If the ramification divisor $R$ is irreducible then this implies the existence of a -1 curve $\tilde{L}_4$ which intersects the fundamental cycle only at the central exceptional curve $E_4$ and this intersection is transversal. In the case the ramification divisor $R$ is reducible no such line exists. Therefore we should consider two cases depending on the existence or not of the -1 curve $\tilde{L}_4$.

3.6. The ramification divisor is irreducible.

**Proof.** Suppose $\text{lct}(X) < \frac{1}{2}$. Then there exists an effective $\mathbb{Q}$-divisor $D \subset X$ and a positive rational number $\lambda < \frac{1}{2}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_\mathbb{Q} -K_X$, where $\lambda < \frac{1}{2}$. Therefore the log pair $(X, \lambda D)$ is also not log canonical.

Let $Z$ be the curve in $| -K_X |$ that contains $P$. Since the curve $Z$ is irreducible we may assume that the support of $D$ does not contain $Z$.

We derive that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at $P$.

Then

$$\begin{align*}
\tilde{D} &\sim_\mathbb{Q} \pi_1^*(D) - a_1E_1 - a_2E_2 - a_3E_3 - a_4E_4 - a_5E_5 - a_6E_6 - a_7E_7 \\
\tilde{Z} &\sim_\mathbb{Q} \pi_1^*(Z) - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7.
\end{align*}$$

From the inequalities

$$0 \leq \tilde{D} \cdot \tilde{Z} = 1 - a_1 - a_7$$
$$0 \leq E_1 \cdot \tilde{D} = 2a_1 - a_2$$
$$0 \leq E_2 \cdot \tilde{D} = 2a_2 - a_1 - a_3$$
$$0 \leq E_3 \cdot \tilde{D} = 2a_3 - a_2 - a_4$$
$$0 \leq E_4 \cdot \tilde{D} = 2a_4 - a_3 - a_5$$
$$0 \leq E_5 \cdot \tilde{D} = 2a_5 - a_4 - a_6$$
$$0 \leq E_6 \cdot \tilde{D} = 2a_6 - a_5 - a_7$$
$$0 \leq E_7 \cdot \tilde{D} = 2a_7 - a_6$$

we get

$$2a_7 \geq a_6 , \quad \frac{3}{2}a_6 \geq a_5 , \quad \frac{4}{3}a_5 \geq a_4 , \quad \frac{5}{4}a_4 \geq a_3 , \quad \frac{6}{5}a_3 \geq a_2 , \quad \frac{7}{6}a_2 \geq a_1$$

and moreover

$$a_1 \leq \frac{7}{8} , \quad a_2 \leq \frac{12}{8} , \quad a_3 \leq \frac{15}{8} , \quad a_4 \leq 2 , \quad a_5 \leq \frac{15}{8} , \quad a_6 \leq \frac{12}{8} , \quad a_7 \leq \frac{7}{8}.$$ 

Furthermore there are lines $L_2, L_4, L_6 \subset X$ that pass through the point $P$ whose strict transforms are $(-1)$-curves that intersect the fundamental cycle as following.

$$L_2 \cdot E_2 = L_4 \cdot E_4 = L_6 \cdot E_6 = 1$$
and
\[ L_i \cdot E_j = 0 \text{ for all } i, j = 2, 4, 6 \text{ with } i \neq j. \]

Then we easily get that
\[
\begin{align*}
\tilde{L}_2 & \sim Q \pi^*(L_2) - \frac{3}{4}E_1 - \frac{3}{2}E_2 - \frac{5}{4}E_3 - E_4 - \frac{3}{4}E_5 - \frac{1}{2}E_6 - \frac{1}{4}E_7 \\
\tilde{L}_4 & \sim Q \pi^*(L_4) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - 2E_4 - \frac{3}{2}E_5 - E_6 - \frac{1}{2}E_7 \\
\tilde{L}_6 & \sim Q \pi^*(L_6) - \frac{1}{4}E_1 - \frac{1}{2}E_2 - \frac{3}{4}E_3 - E_4 - \frac{5}{4}E_5 - \frac{3}{2}E_6 - \frac{3}{4}E_7.
\end{align*}
\]

We observe that \(2L_4\) is a Cartier divisor in the bi-anticanonical linear system \(|-2K_X|\). We will show that \(a_4 \leq 1\) which is a key inequality for what will follow. Indeed, since \(L_4\) is irreducible and \(\tilde{L}_4 \sim Q -K_X\), we can assume that \(L_4 \not\subset \text{Supp}(D)\). Then

\[ 0 \leq \tilde{L}_4 \cdot \tilde{D} = 1 - a_4. \]

The equivalence
\[
K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 \sim Q \pi_1^*(K_X + D)
\]
implies that there is a point \(Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7\), such that the pair
\[
K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7
\]
is not log canonical at \(Q\).

- If the point \(Q \in E_1\) and \(Q \not\subset E_2\) then
  \[
  K_X + \lambda \tilde{D} + a_1 \lambda E_1
  \]
is not log canonical at the point \(Q\) and so is the pair
  \[
  K_X + \lambda \tilde{D} + E_1, \text{ since } a_1 \lambda \leq 1.
  \]

By adjunction \((E_1, \lambda \tilde{D}|_{E_1})\) is not log canonical at \(Q\) and
\[
2 \cdot \frac{7}{6}a_2 - a_2 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D} \cdot E_1) > \frac{1}{\lambda} > 2,
\]
which is false, since \(a_2 \leq \frac{12}{8}\).

- If \(Q \in E_1 \cap E_2\) then the log pair
  \[
  K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2
  \]
is not log canonical at the point \(Q\) and so is the log pair
  \[
  K_X + \lambda \tilde{D} + E_1 + \lambda a_2 E_2.
  \]

By adjunction it follows that
\[
2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > \frac{1}{\lambda} - a_2 > 2 - a_2,
\]
which is false, since \(a_1 \leq \frac{7}{8}\).

- If \(Q \in E_2\) but \(Q \not\subset E_1 \cup E_3\) then
  \[
  K_X + \lambda \tilde{D} + \lambda a_2 E_2
  \]
is not log canonical at the point \(Q\) and so is the pair
  \[
  K_X + \lambda \tilde{D} + E_2, \text{ since } \lambda a_2 \leq 1.
  \]

By adjunction \((E_2, \lambda \tilde{D}|_{E_2})\) is not log canonical at \(Q\) and
\[
2a_2 - \frac{5}{6}a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D} \cdot E_2) > \frac{1}{\lambda} > 2,
\]
which is false, since \(a_2 \leq \frac{12}{8}\).
• If \( Q \in E_2 \cap E_3 \) then the log pair
\[
K_X + \lambda D + \lambda a_2 E_2 + \lambda a_3 E_3
\]
is not log canonical at the point \( Q \) and so is the log pair
\[
K_X + \lambda D + \lambda a_2 E_2 + E_3 , \text{ since } \lambda a_3 < 1 .
\]
By adjunction it follows that
\[
2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) \geq \frac{1}{\lambda} - a_2 > 2 - a_2 ,
\]
which, along with the inequality \( a_4 \geq \frac{1}{\lambda} a_3 \), implies that \( a_4 > 1 \), which is impossible.

• If \( Q \in E_3 \) but \( Q \notin E_2 \cup E_4 \) then
\[
K_X + \lambda D + \lambda a_3 E_3
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda D + E_3 , \text{ since } \lambda a_3 \leq 1 .
\]
By adjunction \((E_3, \lambda \tilde{D}|_{E_3})\) is not log canonical at \( Q \) and
\[
2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) \geq \frac{1}{\lambda} > 2 .
\]
This inequality together with \( a_4 \geq \frac{1}{\lambda} a_3 \) implies that \( a_4 > 1 \), which is impossible.

• If \( Q \in E_3 \cap E_4 \) then the log pair
\[
K_X + \lambda D + \lambda a_3 E_3 + \lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so is the log pair
\[
K_X + \lambda D + \lambda a_3 E_3 + E_4 , \text{ since } \lambda a_4 \leq 1 .
\]
By adjunction it follows that
\[
2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D}|_{E_4}) = \text{mult}_Q(\tilde{D} \cdot E_4) \geq \frac{1}{\lambda} - a_3 > 2 - a_3 ,
\]
which contradicts \( a_4 \leq 1 \).

• If \( Q \in E_4 \) but \( Q \notin E_3 \cup E_5 \) then
\[
K_X + \lambda D + \lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda D + E_4 , \text{ since } \lambda a_4 \leq 1 .
\]
By adjunction \((E_4, \lambda \tilde{D}|_{E_4})\) is not log canonical at \( Q \) and
\[
2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D}|_{E_4}) = \text{mult}_Q(\tilde{D} \cdot E_4) \geq \frac{1}{\lambda} > 2 ,
\]
which is false since \( a_4 \leq 1 \).

\[ \square \]

3.7. The ramification divisor \( R \) is reducible.

Proof. Let \( X \) be a del Pezzo surface with exactly one Du Val singularity of type \( A_7 \) and \( K_X^2 = 1 \). Suppose \( \text{let}(X) < \frac{8}{13} \). Then there exists an effective \( \mathbb{Q} \)-divisor \( D \in X \) and a positive rational number \( \lambda < \frac{8}{13} \), such that the log pair \((X, \lambda D)\) is not log canonical and \( D \sim_{\mathbb{Q}} -K_X \), where \( \lambda < \frac{8}{13} \). Therefore the log pair \((X, \lambda D)\) is also not log canonical.

Let \( Z \) be the curve in \( |-K_X| \) that contains \( P \). Since the curve \( Z \) is irreducible we may assume that the support of \( D \) does not contain \( Z \).

We derive that the pair \((X, \lambda D)\) is log canonical outside of a point \( P \in X \) and not log canonical at \( P \). Then
\[
\tilde{D} \sim_{\mathbb{Q}} \pi_1^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - a_6 E_6 - a_7 E_7 \text{ and}
\]
\[
\tilde{Z} \sim_{\mathbb{Q}} \pi_1^*(Z) - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 .
\]
From the inequalities
\[
\begin{align*}
0 & \leq \tilde{D} \cdot \tilde{Z} = 1 - a_1 - a_7 \\
0 & \leq E_1 \cdot \tilde{D} = 2a_1 - a_2 \\
0 & \leq E_2 \cdot \tilde{D} = 2a_2 - a_1 - a_3 \\
0 & \leq E_3 \cdot \tilde{D} = 2a_3 - a_2 - a_4 \\
0 & \leq E_4 \cdot \tilde{D} = 2a_4 - a_3 - a_5 \\
0 & \leq E_5 \cdot \tilde{D} = 2a_5 - a_4 - a_6 \\
0 & \leq E_6 \cdot \tilde{D} = 2a_6 - a_5 - a_7 \\
0 & \leq E_7 \cdot \tilde{D} = 2a_7 - a_6 \\
\end{align*}
\]
we get
\[
2a_7 \geq a_6, \quad \frac{3}{2}a_6 \geq a_5, \quad \frac{4}{3}a_5 \geq a_4, \quad \frac{5}{4}a_4 \geq a_3, \quad \frac{6}{5}a_3 \geq a_2, \quad \frac{7}{6}a_2 \geq a_1
\]
and moreover
\[
a_1 \leq \frac{7}{8}, \quad a_2 \leq \frac{12}{8}, \quad a_3 \leq \frac{15}{8}, \quad a_4 \leq 2, \quad a_5 \leq \frac{15}{8}, \quad a_6 \leq \frac{12}{8}, \quad a_7 \leq \frac{7}{8}.
\]
Furthermore, there are lines \(L_2, L_5, L_6 \in X\) that pass through the point \(P\) whose strict transforms are \((-1)\)-curves that intersect the fundamental cycle as following.
\[
L_2 \cdot E_2 = L_5 \cdot E_5 = L_6 \cdot E_6 = 1
\]
and
\[
L_i \cdot E_j = 0 \text{ for all } i, j = 2, 5, 6 \text{ with } i \neq j.
\]
Then we easily get that
\[
\begin{align*}
\tilde{L}_2 & \sim_{\mathbb{Q}} \pi^*(L_2) - \frac{3}{4}E_1 - \frac{3}{2}E_2 - \frac{5}{4}E_3 - E_4 - \frac{3}{4}E_5 - \frac{1}{2}E_6 - \frac{1}{4}E_7 \\
\tilde{L}_3 & \sim_{\mathbb{Q}} \pi^*(L_3) - \frac{5}{8}E_1 - \frac{5}{2}E_2 - \frac{15}{8}E_3 - \frac{3}{2}E_4 - \frac{9}{8}E_5 - \frac{3}{4}E_6 - \frac{3}{8}E_7 \\
\tilde{L}_5 & \sim_{\mathbb{Q}} \pi^*(L_5) - \frac{3}{8}E_1 - \frac{1}{2}E_2 - \frac{9}{8}E_3 - \frac{3}{2}E_4 - \frac{15}{8}E_5 - \frac{3}{4}E_6 - \frac{5}{8}E_7 \\
\tilde{L}_6 & \sim_{\mathbb{Q}} \pi^*(L_6) - \frac{1}{4}E_1 - \frac{1}{2}E_2 - \frac{3}{4}E_3 - E_4 - \frac{5}{4}E_5 - \frac{3}{2}E_6 - \frac{3}{4}E_7.
\end{align*}
\]
Since \(L_2 + 2L_3 \in \mid -3K_X \mid\) we can assume that \(L_2 \not\in \text{Supp}D\) or \(L_3 \not\in \text{Supp}D\) and then
\[
0 \leq \tilde{L}_2 \cdot \tilde{D} = 1 - a_2 \Rightarrow a_2 \leq 1 \quad \text{or} \quad 0 \leq \tilde{L}_3 \cdot \tilde{D} = 1 - a_3 \Rightarrow a_3 \leq 1.
\]
In the same way we obtain that
\[
0 \leq \tilde{L}_3 \cdot \tilde{D} = 1 - a_2 \Rightarrow a_3 \leq 1 \quad \text{or} \quad 0 \leq \tilde{L}_5 \cdot \tilde{D} = 1 - a_3 \Rightarrow a_5 \leq 1,
\]
since \(L_3 + L_5 \in \mid -2K_X \mid\) together with Lemma implies that either \(L_3 \not\in D\) or \(L_5 \not\in D\).

The equivalence
\[
K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 \sim_{\mathbb{Q}} \pi^*_1(K_X + \lambda D)
\]
implies that there is a point \(Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7\), such that the pair
\[
K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7
\]
is not log canonical at \(Q\).

- If the point \(Q \in E_1\) and \(Q \not\in E_2\) then
  \[
  K_X + \lambda \tilde{D} + a_1 \lambda E_1
  \]
is not log canonical at the point \(Q\) and so is the pair
  \[
  K_X + \lambda \tilde{D} + E_1, \quad \text{since} \quad a_1 \lambda \leq 1.
  \]
By adjunction \((E_1, \lambda \tilde{D}|_{E_1})\) is not log canonical at \(Q\) and
\[
1 \geq 2a_1 - \frac{6}{7}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D} \cdot E_1) > \frac{1}{\lambda} > \frac{15}{8},
\]
which is false.

- If \( Q \in E_1 \cap E_2 \) then the log pair
  \[ K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 \]
  is not log canonical at the point \( Q \) and so is the log pair
  \[ K_X + \lambda \tilde{D} + E_2 + \lambda a_1 E_1 . \]
  By adjunction it follows that
  \[ 2a_2 - \frac{5}{6}a_2 - a_1 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D} \cdot E_2) > \frac{1}{\lambda} - a_1 > \frac{15}{8} - a_1 , \]
  which is false.

- If \( Q \in E_2 \) but \( Q \not\in E_1 \cup E_3 \) then
  \[ K_X + \lambda \tilde{D} + \lambda a_2 E_2 \]
  is not log canonical at the point \( Q \) and so is the pair
  \[ K_X + \lambda \tilde{D} + E_2 , \text{ since } \lambda a_2 \leq 1 . \]
  By adjunction \((E_2, \lambda \tilde{D}|_{E_2})\) is not log canonical at \( Q \) and
  \[ 1 \geq \frac{2}{3}a_2 \geq 2a_2 - \frac{5}{6}a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D} \cdot E_2) > \frac{1}{\lambda} > \frac{15}{8} , \]
  which is false.

- If \( Q \in E_2 \cap E_3 \) then the log pair
  \[ K_X + \lambda \tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3 \]
  is not log canonical at the point \( Q \) and so are the log pairs
  \[ K_X + \lambda \tilde{D} + \lambda a_3 E_3 + E_2 , \text{ since } \lambda a_2 < 1 \]
  and
  \[ K_X + \lambda \tilde{D} + E_3 + \lambda a_2 E_2 , \text{ since } \lambda a_3 < 1 . \]
  By adjunction it follows that
  \[ 2a_2 - a_3 - a_1 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D} \cdot E_2) > \frac{1}{\lambda} - a_3 > \frac{15}{8} - a_3 \]
  and
  \[ 2a_3 - a_2 - \frac{4}{5}a_3 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D} \cdot E_3) > \frac{1}{\lambda} - a_2 > \frac{15}{8} - a_2 . \]
  This implies that
  \[ \frac{3}{2}a_2 \geq 2a_2 - \frac{a_2}{2} \geq 2a_2 - a_1 > \frac{15}{8} \Rightarrow a_2 \geq \frac{5}{4} \]
  and
  \[ \frac{6}{5}a_3 \geq 2a_3 - \frac{4}{5}a_3 \geq 2a_3 - a_4 > \frac{15}{8} \Rightarrow a_3 \geq \frac{25}{16} \]
  which is false, since either \( a_2 \leq 1 \) or \( a_3 \leq 1 \).

- If \( Q \in E_3 \) but \( Q \not\in E_2 \cup E_4 \) then
  \[ K_X + \lambda \tilde{D} + \lambda a_3 E_3 \]
  is not log canonical at the point \( Q \) and so is the pair
  \[ K_X + \lambda \tilde{D} + E_3 , \text{ since } \lambda a_3 \leq 1 . \]
  By adjunction \((E_3, \lambda \tilde{D}|_{E_3})\) is not log canonical at \( Q \) and
  \[ \frac{8}{15}a_3 \geq 2a_3 - \frac{2}{3}a_3 - \frac{4}{5}a_3 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D} \cdot E_3) > \frac{1}{\lambda} > \frac{15}{8} , \]
  which is impossible.
• If \( Q \in E_3 \cap E_4 \) then the log pair
  \[
  K_X + \lambda \tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4
  \]
is not log canonical at the point \( Q \) and so are the log pairs
  \[
  K_X + \lambda \tilde{D} + \lambda a_3 E_3 + E_4 , \text{ since } \lambda a_4 \leq 1
  \]
and
  \[
  K_X + \lambda \tilde{D} + E_3 + \lambda a_4 E_4 , \text{ since } \lambda a_3 \leq 1 .
  \]
By adjunction it follows that
  \[
  2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D}|_{E_4}) = \text{mult}_Q(\tilde{D} \cdot E_4) > \frac{15}{8} - a_3 ,
  \]
and this implies that
  \[
  2a_4 - \frac{3}{4} a_4 \geq 2a_4 - a_5 > \frac{15}{8} \Rightarrow a_4 > \frac{3}{2}
  \]
We have that either \( a_5 \leq 1 \) or \( a_3 \leq 1 \) and this implies that \( a_4 \leq \frac{4}{3} a_3 \leq \frac{4}{3} \).

• If \( Q \in E_4 \) but \( Q \not\in E_3 \cup E_5 \) then
  \[
  K_X + \lambda \tilde{D} + \lambda a_4 E_4
  \]
is not log canonical at the point \( Q \) and so is the pair
  \[
  K_X + \lambda \tilde{D} + E_4 , \text{ since } \lambda a_4 \leq 1 .
  \]
By adjunction \((E_4, \lambda \tilde{D}|_{E_4})\) is not log canonical at \( Q \) and
  \[
  1 \geq \frac{a_4}{2} \geq 2a_4 - \frac{3}{4} a_4 - \frac{3}{4} a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D} \cdot E_4) > \frac{15}{8} ,
  \]
which is false.

\[\square\]

3.8. **Del Pezzo surfaces of degree 1 with exactly one \(A_8\) type singularity.** In this section we will prove the following.

**Lemma 3.9.** Let \( X \) be a del Pezzo surface with at most one Du Val singularity of type \(A_8\) and \(K_X^2 = 1\). Then the global log canonical threshold of \( X \) is
  \[
  \text{lct}(X) = \frac{1}{2} .
  \]

**Proof.** Let \( X \) be a del Pezzo surface with exactly one Du Val singularity of type \(A_8\) and \(K_X^2 = 1\). Suppose \( \text{lct}(X) < \frac{1}{2} \). Then there exists an effective \(\mathbb{Q}\)-divisor \( D \in X \) and a positive rational number \( \lambda < \frac{1}{2} \), such that the log pair \((X, \lambda D)\) is not log canonical and \( D = -K_X \), where \( \lambda < \frac{1}{2} \). Therefore the log pair \((X, \lambda D)\) is also not log canonical.

Let \( Z \) be the curve in \([-K_X]\) that contains \( P \). Since the curve \( Z \) is irreducible we may assume that the support of \( D \) does not contain \( Z \).

We derive that the pair \((X, \lambda D)\) is log canonical outside of a point \( P \in X \) and not log canonical at \( P \). Let \( \pi_1 : \tilde{X} \to X \) be the minimal resolution of \( X \). The following diagram shows how the exceptional curves intersect each other.

\[
\begin{array}{cccccccc}
A_8 & E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 & E_8 \\
\end{array}
\]

Then
  \[
  \tilde{D} \sim_{\mathbb{Q}} \pi_1^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - a_6 E_6 - a_7 E_7 - a_8 E_8 \text{ and}
  \]
  \[
  \tilde{Z} \sim_{\mathbb{Q}} \pi_1^*(Z) - E_1 - E_2 - E_3 - E_4 - E_5 - E_6 - E_7 - E_8 .
  \]

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From the inequalities

\begin{align*}
0 &\leq \tilde{D} \cdot \tilde{Z} = 1 - a_1 - a_8 \\
0 &\leq E_1 \cdot \tilde{D} = 2a_1 - a_2 \\
0 &\leq E_2 \cdot \tilde{D} = 2a_2 - a_1 - a_3 \\
0 &\leq E_3 \cdot \tilde{D} = 2a_3 - a_2 - a_4 \\
0 &\leq E_4 \cdot \tilde{D} = 2a_4 - a_3 - a_5 \\
0 &\leq E_5 \cdot \tilde{D} = 2a_5 - a_4 - a_6 \\
0 &\leq E_6 \cdot \tilde{D} = 2a_6 - a_5 - a_7 \\
0 &\leq E_7 \cdot \tilde{D} = 2a_7 - a_6 - a_8 \\
0 &\leq E_8 \cdot \tilde{D} = 2a_8 - a_7 
\end{align*}

we get

\[ 2a_8 \geq a_7 , \quad \frac{3}{2} a_7 \geq a_6 , \quad \frac{4}{3} a_6 \geq a_5 , \quad \frac{5}{4} a_5 \geq a_4 , \quad \frac{6}{5} a_4 \geq a_3 , \quad \frac{7}{6} a_3 \geq a_2 , \quad \frac{8}{7} a_2 \geq a_1 \]

and moreover

\[ a_1 \leq \frac{8}{9} , \quad a_2 \leq \frac{14}{9} , \quad a_3 \leq 2 , \quad a_4 \leq \frac{20}{9} , \quad a_5 \leq \frac{20}{9} , \quad a_6 \leq 2 , \quad a_7 \leq \frac{14}{9} , \quad a_8 \leq \frac{8}{9} . \]

Furthermore there are lines \( L_3 , L_6 \in X \) that pass through the point \( P \) whose strict transforms are \((-1)\)-curves that intersect the fundamental cycle as following.

\[ L_3 \cdot E_3 = L_6 \cdot E_6 = 1 \]

and

\[ L_i \cdot E_j = 0 \quad \text{for all} \quad i , j = 3 , 6 \quad \text{with} \quad i \neq j . \]

Then we easily get that

\begin{align*}
\tilde{L}_3 &\sim_Q \pi^*(L_3) - \frac{2}{3} E_1 - \frac{4}{3} E_2 - 2E_3 - \frac{5}{3} E_4 - \frac{4}{3} E_5 - E_6 - \frac{2}{3} E_7 - \frac{1}{3} E_8 \\
\tilde{L}_6 &\sim_Q \pi^*(L_6) - \frac{1}{3} E_1 - \frac{2}{3} E_2 - E_3 - \frac{4}{3} E_4 - \frac{5}{3} E_5 - 2E_6 - \frac{4}{3} E_7 - \frac{2}{3} E_8 .
\end{align*}

We observe that \( L_3 + L_4 \) is a Cartier divisor in the bianticanonical linear system \(| - 2K_X | \). Since \( L_3 \) and \( L_6 \) are irreducible and \( L_3 \sim_Q L_4 \sim_Q -K_X \), we can assume that \( L_3 \notin \text{Supp}(D) \) and \( L_6 \notin \text{Supp}(D) \). Then

\[ 0 \leq \tilde{L}_3 \cdot \tilde{D} = 1 - a_3 \quad \text{and} \quad 0 \leq \tilde{L}_6 \cdot \tilde{D} = 1 - a_6 . \]

The equivalence

\[ K_{\tilde{X}} + \lambda \tilde{D} + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 + \alpha_5 E_5 + \alpha_6 E_6 + \alpha_7 E_7 + \alpha_8 E_8 \sim_Q \pi^*(K_X + D) \]

implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8 \), such that the pair

\[ K_{\tilde{X}} + \lambda \tilde{D} + \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \alpha_4 E_4 + \alpha_5 E_5 + \alpha_6 E_6 + \alpha_7 E_7 + \alpha_8 E_8 \]

is not log canonical at \( Q \).

- If the point \( Q \in E_1 \) and \( Q \notin E_2 \) then

\[ K_{\tilde{X}} + \lambda \tilde{D} + a_1 \lambda E_1 \]

is not log canonical at the point \( Q \) and so is the pair

\[ K_{\tilde{X}} + \lambda \tilde{D} + E_1 , \quad \text{since} \quad a_1 \lambda \leq 1 . \]

By adjunction \((E_1 , \lambda \tilde{D}_{|E_1})\) is not log canonical at \( Q \) and

\[ 2a_1 - \frac{7}{8} a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D} \cdot E_1) > \frac{1}{\lambda} > 2 , \]

which is false since \( a_1 \leq \frac{8}{9} \).

\[ 33 \]
• If $Q \in E_1 \cap E_2$ then the log pair
  \[ K_X + \lambda \bar{D} + \lambda a_1 E_1 + \lambda a_2 E_2 \]
is not log canonical at the point $Q$ and so is the log pair
  \[ K_X + \lambda \bar{D} + E_1 + \lambda a_2 E_2, \text{ since } \lambda a_1 \leq 1. \]
By adjunction it follows that
  \[ 2a_1 - a_2 = \bar{D} \cdot E_1 \geq \text{mult}_Q(\bar{D} \cdot E_1) > \frac{1}{\lambda} - a_2 > 2 - a_2, \]
which is false since $a_1 \leq 1$.

• If $Q \in E_2$ but $Q \not\in E_1 \cup E_3$ then
  \[ K_X + \lambda \bar{D} + \lambda a_2 E_2 \]
is not log canonical at the point $Q$ and so is the pair
  \[ K_X + \lambda \bar{D} + E_2, \text{ since } \lambda a_2 \leq 1. \]
By adjunction $(E_2, \lambda \bar{D}|_{E_2})$ is not log canonical at $Q$ and
  \[ \frac{9}{14} a_2 \geq 2a_2 - \frac{1}{2}a_2 - \frac{6}{7}a_2 \geq 2a_2 - a_1 - a_3 = \bar{D} \cdot E_2 \geq \text{mult}_Q(\bar{D} \cdot E_2) > \frac{1}{\lambda} > 2, \]
which is false, since $a_2 \leq \frac{14}{9}$.

• If $Q \in E_2 \cap E_3$ then the log pair
  \[ K_X + \lambda \bar{D} + \lambda a_2 E_2 + \lambda a_3 E_3 \]
is not log canonical at the point $Q$ and so is the log pair
  \[ K_X + \lambda \bar{D} + \lambda a_2 E_2 + E_3, \text{ since } \lambda a_3 < 1. \]
By adjunction it follows that
  \[ 2a_3 - \frac{5}{6}a_3 - a_2 \geq 2a_3 - a_2 - a_4 = \bar{D} \cdot E_3 \geq \text{mult}_Q(\bar{D}|_{E_3}) > \frac{1}{\lambda} - a_2 > 2 - a_2, \]
which is impossible, since $a_3 \leq 1$.

• If $Q \in E_3$ but $Q \not\in E_2 \cup E_4$ then
  \[ K_X + \lambda \bar{D} + \lambda a_3 E_3 \]
is not log canonical at the point $Q$ and so is the pair
  \[ K_X + \lambda \bar{D} + E_3, \text{ since } \lambda a_3 \leq 1. \]
By adjunction $(E_3, \lambda \bar{D}|_{E_3})$ is not log canonical at $Q$ and
  \[ 2a_3 - \frac{2}{3}a_3 - \frac{5}{6}a_3 \geq 2a_3 - a_2 - a_4 = \bar{D} \cdot E_3 \geq \text{mult}_Q(\bar{D}|_{E_3}) > \frac{1}{\lambda} > 2, \]
and this implies that $a_3 > 4$, which is impossible.

• If $Q \in E_3 \cap E_4$ then the log pair
  \[ K_X + \lambda \bar{D} + \lambda a_3 E_3 + \lambda a_4 E_4 \]
is not log canonical at the point $Q$ and so is the log pair
  \[ K_X + \lambda \bar{D} + E_3 + \lambda a_4 E_4, \text{ since } \lambda a_3 \leq 1. \]
By adjunction it follows that
  \[ 2a_3 - \frac{2}{3}a_3 - a_4 \geq 2a_3 - a_2 - a_4 = \bar{D} \cdot E_3 \geq \text{mult}_Q(\bar{D}|_{E_3}) > \frac{1}{\lambda} - a_4 > 2 - a_4, \]
which contradicts $a_3 \leq 1$. 

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If $Q \in E_4$ but $Q \not\subset E_3 \cup E_5$ then 

$$K_X + \lambda \tilde{D} + \lambda a_4 E_4$$

is not log canonical at the point $Q$ and so is the pair 

$$K_X + \lambda \tilde{D} + E_4, \text{ since } \lambda a_4 \leq \frac{4}{3}\lambda a_3 \leq \frac{8}{9}.$$ 

By adjunction $(E_4, \lambda \tilde{D}|_{E_4})$ is not log canonical at $Q$ and

$$2a_4 - \frac{3}{4}a_4 - \frac{4}{5}a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D}|_{E_4}) > \frac{1}{\lambda} > 2,$$

implies that $a_4 > \frac{40}{9}$ which is false.

- If $Q \in E_4 \cap E_5$ then the log pair 

$$K_X + \lambda \tilde{D} + \lambda a_4 E_4 + \lambda a_5 E_5$$

is not log canonical at the point $Q$ and so is the log pair 

$$K_X + \lambda \tilde{D} + E_4 + \lambda a_5 E_5, \text{ since } \lambda a_4 \leq \frac{4}{3}\lambda a_3 < 1.$$ 

By adjunction it follows that 

$$2a_4 - \frac{3}{4}a_4 - a_5 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D}|_{E_4}) > \frac{1}{\lambda} - a_5 > 2 - a_5,$$

which implies $a_4 > \frac{8}{5}$ and contradicts $a_4 \leq \frac{4}{3}a_3 \leq \frac{4}{5}.$

\[\square\]

3.9. **Del Pezzo surfaces of degree 1 with exactly one $\mathbb{D}_4$ type singularity.** In this section we will prove the following.

**Lemma 3.10.** Let $X$ be a del Pezzo surface with at most one Du Val singularity of type $\mathbb{D}_4$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is 

$$\text{lct}(X) = \frac{1}{2}.$$ 

**Proof.** Suppose $\text{lct}(X) < \frac{1}{2}$. Then there exists an effective $\mathbb{Q}$-divisor $D \in X$ and a rational number $\lambda < \frac{1}{2}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$.

Let $Z$ be the unique curve in $|-K_X|$ that contains $P$. Since the curve $Z$ is irreducible we may assume that the support of $D$ does not contain $Z$.

We derive that the pair $(X, \lambda D)$ is log canonical everywhere outside of a singular point $P \in X$ and is not log canonical at $P$. Let $\pi: \tilde{X} \rightarrow X$ be the minimal resolution of $X$. The following diagram shows how the exceptional curves intersect each other.

$$\begin{array}{c}
\mathbb{D}_4.
\bullet E_1 \quad \bullet E_3 \quad \bullet E_4
\end{array}$$

Then 

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 \quad \text{and} \quad \tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - 2E_3 - E_4.$$ 

From the inequalities

\begin{align*}
0 \leq \tilde{D} \cdot \tilde{Z} &= 1 - a_3 \\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_3 \\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_3 \\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_1 - a_2 - a_4 \\
0 \leq E_4 \cdot \tilde{D} &= 2a_4 - a_3
\end{align*}
we get the following upper bounds $a_1 \leq 1$, $a_2 \leq 1$, $a_3 \leq 1$, $a_4 \leq 1$. The equivalence

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 \sim_{Q^*} \pi^* (K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4$ such that the pair $K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4$ is not log canonical at $Q$.

- If the point $Q \in E_1$ and $Q \not\in E_3$ then
  $$K_X + \lambda \tilde{D} + \lambda a_1 E_1$$
  is not log canonical at the point $Q$ and so is the pair
  $$K_X + \lambda \tilde{D} + E_1 .$$
  By adjunction $(E_1, \lambda \tilde{D}|_{E_1})$ is not log canonical at $Q$ and
  $$2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_{Q} (\tilde{D}|_{E_1}) = \text{mult}_{Q} (\tilde{D} \cdot E_1) > 2 ,$$
  which along with the inequalities $a_3 \leq 2a_1$, $a_3 \leq 2a_2$, $a_1 + a_2 + a_4 \leq 2a_3$, $a_3 \leq 2a_4$ implies that $a_1 > 2$ which is false.

- If $Q \in E_3$ but $Q \not\in E_1 \cup E_2 \cup E_4$ then
  $$K_X + \lambda \tilde{D} + \lambda a_3 E_3$$
  is not log canonical at the point $Q$ and so is the pair
  $$K_X + \lambda \tilde{D} + E_3 ,$$
  since $\lambda a_3 \leq 1$.
  By adjunction $(E_3, \lambda \tilde{D}|_{E_3})$ is not log canonical at $Q$ and
  $$2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_1 \geq \text{mult}_{Q} (\tilde{D}|_{E_3}) = \text{mult}_{Q} (\tilde{D} \cdot E_3) > 2 ,$$
  which along with $a_3 \leq 2a_1$, $a_3 \leq 2a_2$, $a_3 \leq 2a_4$ implies that $a_3 > 4$ which is false.

- If $Q \in E_1 \cap E_3$ then the log pair
  $$K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_3 E_3$$
  is not log canonical at the point $Q$ and so is the log pair
  $$K_X + \lambda \tilde{D} + E_1 + \lambda a_3 E_3 .$$
  By adjunction it follows that
  $$2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_{Q} (\tilde{D}|_{E_1}) = \text{mult}_{Q} (\tilde{D} \cdot E_1) > 2 - a_3 ,$$
  and this implies that $a_1 > 1$ which is not possible.

\[\square\]

3.10. **Del Pezzo surfaces of degree 1 with exactly one $\mathbb{D}_5$ singularity.** In this section we will prove the following.

**Lemma 3.11.** Let $X$ be a del Pezzo surface with exactly one Du Val singularity of type $\mathbb{D}_5$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = \frac{1}{2} .$$

**Proof.** Suppose that $\text{lct}(X) < \frac{1}{2}$, then there exists a $\mathbb{Q}$-divisor $D \in X$ and a rational number $\lambda < \frac{1}{2}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. We derive that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at $P$. Let $\pi : \tilde{X} \rightarrow X$ be the minimal resolution of $X$. The configuration of the exceptional curves is given by the following Dynkin diagram.

\[\mathbb{D}_5, \bullet E_1 \longrightarrow E_3 \longrightarrow \bullet E_4 \longrightarrow \bullet E_5 \longrightarrow \bullet E_2 \]
Then
\[ \tilde{D} \sim_{Q} \pi^*(D) - a_1E_1 - a_2E_2 - 2a_3E_3 - 2a_4E_4 - a_5E_5 \] and \[ \tilde{Z} \sim_{Q} \pi^*(Z) - E_1 - E_2 - 2E_3 - 2E_4 - E_5 \].

From the inequalities
\[
\begin{align*}
0 & \leq \tilde{D} \cdot \tilde{Z} = 1 - 2a_4 \\
0 & \leq E_1 \cdot \tilde{D} = 2a_1 - 2a_3 \\
0 & \leq E_2 \cdot \tilde{D} = 2a_2 - 2a_3 \\
0 & \leq E_3 \cdot \tilde{D} = 4a_3 - a_1 - a_2 - 2a_4 \\
0 & \leq E_4 \cdot \tilde{D} = 4a_4 - 2a_3 - a_5 \\
0 & \leq E_5 \cdot \tilde{D} = 2a_5 - 2a_4
\end{align*}
\]

we see that \( a_1 \leq \frac{4}{5}, a_2 \leq \frac{5}{4}, a_3 \leq \frac{3}{2}, a_4 \leq \frac{1}{2}, a_5 \leq 1 \). The equivalence
\[ K_X + \lambda \tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + 2\lambda a_3E_3 + 2\lambda a_4E_4 + \lambda a_5E_5 \sim \pi^*(K_X + \lambda D) \]
implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \) such that the pair
\[ K_X + \lambda \tilde{D} + \lambda a_1E_1 + \lambda a_2E_2 + 2\lambda a_3E_3 + 2\lambda a_4E_4 + \lambda a_5E_5 \]
is not log canonical at \( Q \).

- If the point \( Q \in E_1 \setminus E_3 \) then
  \[ K_X + \lambda \tilde{D} + \lambda a_1E_1 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_X + \lambda \tilde{D} + E_1 \].

By adjunction \( (E_1, \lambda \tilde{D}|_{E_1}) \) is not log canonical at \( Q \) and
\[
2a_1 - 2a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > 2,
\]
which along with the inequalities \( a_3 \leq a_1, a_3 \leq a_2, a_1 + a_2 + 2a_4 \leq 4a_3, a_4 \leq a_5, 4a_4 - 2a_3 - a_5 \) implies that \( a_1 > \frac{5}{2} \) which is false.

- If \( Q \in E_3 \) but \( Q \notin E_1 \cup E_2 \cup E_4 \) then
  \[ K_X + \lambda \tilde{D} + 2\lambda a_3E_3 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_X + \lambda \tilde{D} + E_3, \text{ since } a_3 \leq 1. \]

By adjunction \( (E_3, \lambda \tilde{D}|_{E_3}) \) is not log canonical at \( Q \) and
\[
4a_3 - a_1 - a_2 - 2a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 2,
\]
which along with the inequalities \( a_3 \leq a_1, a_3 \leq a_2, a_1 + a_2 + 2a_4 \leq 4a_3, a_4 \leq a_5, 4a_4 - 2a_3 - a_5 \) implies that \( a_1 > 3 \) which is false.

- If \( Q \in E_1 \cap E_3 \) then the log pair
  \[ K_X + \lambda \tilde{D} + \lambda a_1E_1 + 2\lambda a_3E_3 \]
is not log canonical at the point \( Q \) and so are the log pairs
  \[ K_X + \lambda \tilde{D} + E_1 + 2\lambda a_3E_3 \text{ and } K_X + \lambda \tilde{D} + \lambda a_1E_1 + E_3. \]

By adjunction it follows that
\[
2a_1 - 2a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > 2 - 2a_3.
\]
and
\[
4a_3 - a_1 - a_2 - 2a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 2 - a_1.
\]
which is not possible.
• $Q \in E_5 \setminus E_4$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_5 E_5$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_5.$$  

By adjunction $(E_5, \lambda \tilde{D}|_{E_5})$ is not log canonical at $Q$ and

$$2a_5 - 2a_4 = \tilde{D} \cdot E_5 \geq \text{mult}_Q \left( \tilde{D}|_{E_5} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_5 \right) > 2,$$

which along with the inequalities

$$a_3 \leq a_1, a_3 \leq a_2, a_1 + a_2 + 2a_4 \leq 4a_3, 2a_3 + a_5 \leq 4a_4, a_4 \leq a_5$$

implies that $a_5 > 2$ which is false.

• $Q \in E_4 \setminus (E_3 \cap E_5)$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_4 E_4$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_4.$$  

By adjunction $(E_4, \lambda \tilde{D}|_{E_4})$ is not log canonical at $Q$ and

$$4a_4 - 2a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q \left( \tilde{D}|_{E_4} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_4 \right) > 2,$$

which along with the inequalities

$$a_3 \leq a_1, a_3 \leq a_2, a_1 + a_2 + 2a_4 \leq 4a_3, 2a_3 + a_5 \leq 4a_4, a_4 \leq a_5$$

implies that $a_4 > 2$ which is false.

• $Q \in E_4 \cap E_5$ then the log pair

$$K_X + \lambda \tilde{D} + 2\lambda a_4 E_4 + \lambda a_5 E_5$$

is not log canonical at the point $Q$ and so is the log pair

$$K_X + \lambda \tilde{D} + E_5 + 2\lambda a_4 E_4.$$  

By adjunction it follows that

$$2a_5 - 2a_4 = \tilde{D} \cdot E_5 \geq \text{mult}_Q \left( \tilde{D}|_{E_5} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_5 \right) > 2 - 2a_4.$$

and we see then that $a_5 > 1$ which is not possible.

\[ \square \]

### 3.11. Del Pezzo surfaces of degree 1 with exactly one $D_6$ singularity.

In this section we will prove the following.

**Lemma 3.12.** Let $X$ be a del Pezzo surface with exactly one Du Val singularity of type $D_6$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = \frac{1}{2}.$$  

**Proof.** Suppose that $\text{lct}(X) < \frac{1}{2}$, then there exists a $\mathbb{Q}$-divisor $D \in X$ and a rational number $\lambda < \frac{1}{2}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_Q -K_X$. We derive that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at $P$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of $X$. The configuration of the exceptional curves is given by the following Dynkin diagram.

\[
\begin{array}{c}
\mathbb{D}_6, \cdot E_1 \quad \cdot E_3 \quad \cdot E_4 \quad \cdot E_5 \quad \cdot E_6 \\
\cdot E_2
\end{array}
\]
Then
\[ \tilde{D} \sim_Q \pi^*(D) - a_1 E_1 - a_2 E_2 - 2a_3 E_3 - 2a_4 E_4 - 2a_5 E_5 - a_6 E_6 \] and \[ \tilde{Z} \sim_Q \pi^*(Z) - E_1 - E_2 - 2E_3 - 2E_4 - 2E_5 - E_6. \]

From the inequalities
\[
\begin{align*}
0 & \leq \tilde{D} \cdot \tilde{Z} = 1 - 2a_5 \\
0 & \leq E_1 \cdot \tilde{D} = 2a_1 - 2a_3 \\
0 & \leq E_2 \cdot \tilde{D} = 2a_2 - 2a_3 \\
0 & \leq E_3 \cdot \tilde{D} = 4a_3 - a_1 - a_2 - 2a_4 \\
0 & \leq E_4 \cdot \tilde{D} = 4a_4 - 2a_3 - 2a_5 \\
0 & \leq E_5 \cdot \tilde{D} = 4a_5 - 2a_4 - a_6 \\
0 & \leq E_6 \cdot \tilde{D} = 2a_6 - 2a_5
\end{align*}
\]
we see that \( a_1 \leq \frac{2}{3}, a_2 \leq \frac{3}{2}, a_3 \leq 1, a_4 \leq \frac{3}{4}, a_5 \leq \frac{1}{2}, a_6 \leq 1 \). The equivalence
\[ K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + 2\lambda a_3 E_3 + 2\lambda a_4 E_4 + 2\lambda a_5 E_5 + \lambda a_6 E_6 \sim_Q \pi^*(K_X + \lambda D) \]
implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \) such that the pair
\[ K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + 2\lambda a_3 E_3 + 2\lambda a_4 E_4 + 2\lambda a_5 E_5 + \lambda a_6 E_6 \]
is not log canonical at \( Q \).

- If the point \( Q \in E_1 \setminus E_3 \) then
  \[ K_X + \lambda \tilde{D} + \lambda a_1 E_1 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_X + \lambda \tilde{D} + E_1. \]
By adjunction \((E_1, \lambda \tilde{D}|_{E_1})\) is not log canonical at \( Q \) and
\[
2a_1 - 2a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > 2,
\]
which is false.

- If \( Q \in E_3 \) but \( Q \not\in E_1 \cup E_2 \cup E_4 \) then
  \[ K_X + \lambda \tilde{D} + 2\lambda a_3 E_3 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_X + \lambda \tilde{D} + E_3, \text{ since } a_3 \leq 1. \]
By adjunction \((E_3, \lambda \tilde{D}|_{E_3})\) is not log canonical at \( Q \) and
\[
4a_3 - a_1 - a_2 - 2a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 2,
\]
which is false.

- If \( Q \in E_1 \cap E_3 \) then the log pair
  \[ K_X + \lambda \tilde{D} + \lambda a_1 E_1 + 2\lambda a_3 E_3 \]
is not log canonical at the point \( Q \) and so are the log pairs
  \[ K_X + \lambda \tilde{D} + E_1 + 2\lambda a_3 E_3 \text{ and } K_X + \lambda \tilde{D} + \lambda a_1 E_1 + E_3. \]
By adjunction it follows that
\[
2a_1 - 2a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > 2 - 2a_3.
\]
and
\[
4a_3 - a_1 - a_2 - 2a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 2 - a_1
\]
which is false.
• If $Q \in E_3 \cap E_4$ then the log pair
\[ K_X + \lambda \tilde{D} + 2\lambda a_3 E_3 + 2\lambda a_4 E_4 \]
is not log canonical at the point $Q$ and so are the log pairs
\[ K_X + \lambda \tilde{D} + E_3 + 2\lambda a_4 E_4 \text{ and } K_X + \lambda \tilde{D} + 2\lambda a_3 E_3 + E_4 . \]
By adjunction
\[ 4a_3 - a_1 - a_2 - 2a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q (\tilde{D} |_{E_3}) = \text{mult}_Q (\tilde{D} \cdot E_3) > 2 - 2a_4 \]
and
\[ 4a_4 - 2a_3 - 2a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q (\tilde{D} |_{E_4}) = \text{mult}_Q (\tilde{D} \cdot E_4) > 2 - 2a_3 . \]
which is false.

• If the point $Q \in E_4 \setminus (E_3 \cup E_5)$ then
\[ K_X + \lambda \tilde{D} + 2\lambda a_4 E_4 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_X + \lambda \tilde{D} + E_4 . \]
By adjunction $(E_4, \lambda \tilde{D} |_{E_4})$ is not log canonical at $Q$ and
\[ 4a_4 - 2a_3 - 2a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q (\tilde{D} |_{E_4}) = \text{mult}_Q (\tilde{D} \cdot E_4) > 2 , \]
which is false.

• If $Q \in E_5 \setminus (E_4 \cup E_6)$ then the log pair
\[ K_X + \lambda \tilde{D} + 2\lambda a_5 E_5 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_X + \lambda \tilde{D} + E_5 . \]
By adjunction $(E_5, \lambda \tilde{D} |_{E_5})$ is not log canonical at $Q$ and
\[ 4a_5 - 2a_4 - 2a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q (\tilde{D} |_{E_5}) = \text{mult}_Q (\tilde{D} \cdot E_5) > 2 . \]
which is false.

• If $Q \in E_4 \cap E_5$ then the log pair
\[ K_X + \lambda \tilde{D} + 2\lambda a_4 E_4 + 2\lambda a_5 E_5 \]
is not log canonical at the point $Q$ and so are the log pairs
\[ K_X + \lambda \tilde{D} + E_4 + 2\lambda a_5 E_5 \text{ and } K_X + \lambda \tilde{D} + 2\lambda a_4 E_4 + E_5 . \]
By adjunction
\[ 4a_4 - 2a_3 - 2a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q (\tilde{D} |_{E_4}) = \text{mult}_Q (\tilde{D} \cdot E_4) > 2 - 2a_5 \]
and
\[ 4a_5 - 2a_4 - a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q (\tilde{D} |_{E_5}) = \text{mult}_Q (\tilde{D} \cdot E_5) > 2 - 2a_6 . \]

• $Q \in E_5 \setminus (E_4 \cup E_6)$ then the log pair
\[ K_X + \lambda \tilde{D} + 2\lambda a_5 E_5 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_X + \lambda \tilde{D} + E_5 . \]
By adjunction $(E_5, \lambda \tilde{D} |_{E_5})$ is not log canonical at $Q$ and
\[ 4a_5 - 2a_4 - a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q (\tilde{D} |_{E_5}) = \text{mult}_Q (\tilde{D} \cdot E_5) > 2 , \]
which is false.
• If $Q \in E_5 \cap E_6$ then the log pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + 2\lambda a_5 E_5 + \lambda a_6 E_6 \]
is not log canonical at the point $Q$ and so is the log pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + 2\lambda a_5 E_5 + E_6 . \]
By adjunction
\[ 2a_6 - 2a_5 = \tilde{D} \cdot E_6 \geq \text{mult}_Q(\tilde{D}|E_6) = \text{mult}_Q(\tilde{D} \cdot E_6) > 2 - 2a_5 , \]
which is false.

• If the point $Q \in E_6 \setminus E_5$ then
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_6 E_6 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + E_6 . \]
By adjunction $(E_6, \lambda \tilde{D}|_{E_6})$ is not log canonical at $Q$ and
\[ 2a_6 - 2a_5 = \tilde{D} \cdot E_6 \geq \text{mult}_Q(\tilde{D}|E_6) = \text{mult}_Q(\tilde{D} \cdot E_6) > 2 , \]
which is false.

\[ \square \]

3.12. **Del Pezzo surfaces of degree 1 with exactly one $D_7$ singularity.** In this section we will prove the following.

**Lemma 3.13.** Let $X$ be a del Pezzo surface with exactly one Du Val singularity of type $D_7$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is
\[ \text{lct}(X) = \frac{2}{5} . \]

**Proof.** Suppose that $\text{lct}(X) < \frac{2}{5}$, then there exist a $\mathbb{Q}$-divisor $D \in X$ and a rational number $\lambda < \frac{2}{5}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. We derive that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at $P$. Let $\pi : \tilde{X} \rightarrow X$ be the minimal resolution of $X$. The configuration of the exceptional curves is given by the following Dynkin diagram.

```
\[ \begin{array}{cccccccc}
D_7: & \bullet & \cdot E_1 & \cdots & \cdot E_3 & \cdots & \cdot E_4 & \cdots & \cdot E_5 & \cdots & \cdot E_6 & \cdots & \cdot E_7 \\
& \bullet & \cdot E_2 \\
\end{array} \]
```

Then
\[ \begin{array}{c}
\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - a_6 E_6 - a_7 E_7 \quad \text{and} \\
\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - E_7 .
\end{array} \]
From the inequalities
\[
\begin{align*}
0 &\leq \tilde{D} \cdot \tilde{Z} = 1 - a_6 \\
0 &\leq E_1 \cdot \tilde{D} = 2a_1 - a_3 \\
0 &\leq E_2 \cdot \tilde{D} = 2a_2 - a_3 \\
0 &\leq E_3 \cdot \tilde{D} = 2a_3 - a_1 - a_2 - a_4 \\
0 &\leq E_4 \cdot \tilde{D} = 2a_4 - a_3 - a_5 \\
0 &\leq E_5 \cdot \tilde{D} = 2a_5 - a_4 - a_6 \\
0 &\leq E_6 \cdot \tilde{D} = 2a_6 - a_5 - a_7 \\
0 &\leq E_7 \cdot \tilde{D} = 2a_7 - a_6
\end{align*}
\]
we see that \( a_1 \leq \frac{7}{4}, a_2 \leq \frac{7}{4}, a_3 \leq \frac{5}{2}, a_4 \leq \frac{8}{4} = 2, a_5 \leq \frac{6}{4}, a_6 \leq \frac{4}{4} = 1, a_7 \leq 1 \). Moreover we get the inequalities
\[
2a_7 \geq a_6, \quad \frac{3}{2}a_6 \geq a_5, \quad \frac{4}{3}a_5 \geq a_4, \quad \frac{5}{4}a_4 \geq a_3
\]
and
\[
2a_1 \geq a_3, \quad 2a_2 \geq a_3, \quad a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7.
\]
The equivalence
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 \sim Q \pi^*(K_{\tilde{X}} + \lambda D)
\]
implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \) such that the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7
\]
is not log canonical at \( Q \).

- If the point \( Q \in E_1 \setminus E_3 \) then
  \[
  K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1
  \]
is not log canonical at the point \( Q \) and so is the pair
  \[
  K_{\tilde{X}} + \lambda \tilde{D} + E_1, \text{ since } \lambda a_1 \leq 1.
  \]
By adjunction \((E_1, \lambda \tilde{D}|_{E_1})\) is not log canonical at \( Q \) and
\[
2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > \frac{1}{\lambda} > \frac{5}{2},
\]
which is false.

- If \( Q \in E_1 \cap E_3 \) then the log pair
  \[
  K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_3 E_3
  \]
is not log canonical at the point \( Q \) and so is the log pair
  \[
  K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + E_3, \text{ since } \lambda a_3 = 1.
  \]
By adjunction \((E_3, \lambda \tilde{D}|_{E_3})\) is not log canonical at \( Q \) and
\[
2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D} \cdot E_3) > \frac{1}{\lambda} > \frac{5}{2} - a_1,
\]
implies that
\[
\frac{7}{4} \geq \frac{7}{10} \geq a_3 \geq \frac{a_3}{2} - \frac{4}{5}a_3 \geq 2a_3 - a_2 - a_4 > \frac{5}{2}
\]
which is a contradiction.
• If \( Q \in E_3 \) but \( Q \not\in E_1 \cup E_2 \cup E_4 \) then
\[
K_X + \lambda \tilde{D} + \lambda a_3 E_3
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_3, \text{ since } \lambda a_3 \leq 1.
\]
By adjunction \((E_3, \lambda \tilde{D}|_{E_3})\) is not log canonical at \( Q \) and
\[
2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mult_Q (\tilde{D} \cdot E_3) > \frac{1}{\lambda} > 2,
\]
implies that
\[
\frac{1}{2} \geq \frac{a_3}{5} \geq \frac{2a_3}{2} - \frac{a_3}{2} - \frac{a_3}{2} \geq \frac{4}{5} a_3 \geq 2a_3 - a_2 - a_4 > \frac{5}{2}
\]
which is a contradiction.

• If \( Q \in E_3 \cap E_4 \) then the log pair
\[
K_X + \lambda \tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so is the log pair
\[
K_X + \lambda \tilde{D} + E_3 + \lambda a_4 E_4, \text{ since } \lambda a_3 \leq 1.
\]
By adjunction
\[
2a_3 - \frac{a_3}{2} - \frac{a_3}{2} - a_4 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mult_Q (\tilde{D} \cdot E_3) > \frac{5}{2} - a_4,
\]
which is false.

• If the point \( Q \in E_4 \backslash (E_3 \cup E_5) \) then
\[
K_X + \lambda \tilde{D} + \lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so is the pair \( K_X + \lambda \tilde{D} + E_4 \). By adjunction
\((E_4, \lambda \tilde{D}|_{E_4})\) is not log canonical at \( Q \) and
\[
2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mult_Q (\tilde{D}|_{E_4}) = \mult_Q (\tilde{D} \cdot E_4) > \frac{5}{2},
\]
which is false.

• If \( Q \in E_4 \cap E_5 \) then the log pair
\[
K_X + \lambda \tilde{D} + \lambda a_4 E_4 + \lambda a_5 E_5
\]
is not log canonical at the point \( Q \) and so are the log pairs
\[
K_X + \lambda \tilde{D} + E_4 + \lambda a_5 E_5 \text{ and } K_X + \lambda \tilde{D} + \lambda a_4 E_4 + E_5.
\]
By adjunction
\[
2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mult_Q (\tilde{D}|_{E_4}) = \mult_Q (\tilde{D} \cdot E_4) > \frac{5}{2} - a_5
\]
and
\[
2a_5 - a_4 - a_6 = \tilde{D} \cdot E_5 \geq \mult_Q (\tilde{D}|_{E_5}) = \mult_Q (\tilde{D} \cdot E_5) > \frac{5}{2} - a_4.
\]
which is false.

• \( Q \in E_5 \backslash (E_4 \cup E_6) \) then the log pair
\[
K_X + \lambda \tilde{D} + \lambda a_5 E_5
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_5.
\]
By adjunction \((E_5, \lambda \tilde{D}|_{E_5})\) is not log canonical at \( Q \) and
\[
2a_5 - a_4 - a_6 = \tilde{D} \cdot E_5 \geq \mult_Q (\tilde{D}|_{E_5}) = \mult_Q (\tilde{D} \cdot E_5) > \frac{5}{2},
\]
which is false.
• If $Q \in E_5 \cap E_6$ then the log pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_5 E_5 + \lambda a_6 E_6 \]
is not log canonical at the point $Q$ and so are the log pairs
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_5 E_5 + E_6 \text{ and } K_{\tilde{X}} + \lambda \tilde{D} + E_5 + \lambda a_6 E_6 . \]
By adjunction
\[ 2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \text{mult}_Q \left( \tilde{D}_{|E_6} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_6 \right) > \frac{5}{2} - a_5 \]
and
\[ 2a_5 - a_4 - a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q \left( \tilde{D}_{|E_5} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_5 \right) > \frac{5}{2} - a_6 . \]
which is false.

• If the point $Q \in E_6 \setminus (E_5 \cup E_7)$ then
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_6 E_6 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_{\tilde{X}} + \tilde{D} + E_6 . \]
By adjunction $(E_6, \lambda \tilde{D}_{|E_6})$ is not log canonical at $Q$ and
\[ 2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \text{mult}_Q \left( \tilde{D}_{|E_6} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_6 \right) > \frac{5}{2} , \]
which is false.

• If $Q \in E_6 \cap E_7$ then the log pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_6 E_6 + \lambda a_7 E_7 \]
is not log canonical at the point $Q$ and so is the log pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_6 E_6 + E_7 . \]
By adjunction
\[ 2a_7 - a_6 = \tilde{D} \cdot E_7 \geq \text{mult}_Q \left( \tilde{D}_{|E_7} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_7 \right) > \frac{5}{2} - a_6 . \]
which is false.

• If the point $Q \in E_7 \setminus E_6$ then
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_7 E_7 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + E_7 . \]
By adjunction $(E_7, \lambda \tilde{D}_{|E_7})$ is not log canonical at $Q$ and
\[ 2a_7 - a_6 = \tilde{D} \cdot E_7 \geq \text{mult}_Q \left( \tilde{D}_{|E_7} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_7 \right) > \frac{5}{2} , \]
which is false.

$\square$
3.13. Del Pezzo surfaces of degree 1 with exactly one $\mathbb{D}_8$ singularity. In this section we will prove the following.

**Lemma 3.14.** Let $X$ be a del Pezzo surface with exactly one Du Val singularity of type $\mathbb{D}_8$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is

$$\text{let}(X) = \frac{1}{3}.$$ 

**Proof.** Suppose that $\text{let}(X) < \frac{1}{3}$, then there exists a $\mathbb{Q}$-divisor $D \in X$ and a rational number $\lambda < \frac{1}{3}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$. We derive that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at $P$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of $X$. The configuration of the exceptional curves is given by the following Dynkin diagram.

$$\mathbb{D}_8, \bullet E_1 \quad E_3 \quad E_4 \quad E_5 \quad E_6 \quad E_7 \quad E_8$$

Then

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - a_6 E_6 - a_7 E_7 - a_8 E_8 \text{ and}$$

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - 2E_7 - a_8 E_8.$$ 

From the inequalities

$$0 \leq \tilde{D} \cdot \tilde{Z} = 1 - a_7$$

$$0 \leq E_1 \cdot \tilde{D} = 2a_1 - a_3$$

$$0 \leq E_2 \cdot \tilde{D} = 2a_2 - a_3$$

$$0 \leq E_3 \cdot \tilde{D} = 2a_3 - a_1 - a_2 - a_4$$

$$0 \leq E_4 \cdot \tilde{D} = 2a_4 - a_3 - a_5$$

$$0 \leq E_5 \cdot \tilde{D} = 2a_5 - a_4 - a_6$$

$$0 \leq E_6 \cdot \tilde{D} = 2a_6 - a_5 - a_7$$

$$0 \leq E_7 \cdot \tilde{D} = 2a_7 - a_6 - a_8$$

$$0 \leq E_8 \cdot \tilde{D} = 2a_8 - a_7$$

we see that $a_1 \leq \frac{8}{7}, a_2 \leq \frac{8}{7}, a_3 \leq 3, a_4 \leq \frac{5}{7}, a_5 \leq 2, a_6 \leq \frac{3}{7}, a_7 \leq 1, a_8 \leq 1$. Moreover we get the inequalities

$$2a_1 \geq a_3, 2a_2 \geq a_3 \geq a_4 \geq a_5 \geq a_6 \geq a_7 \geq a_8$$

and

$$2a_8 \geq a_7, \frac{3}{2} a_7 \geq a_6, \frac{4}{3} a_6 \geq a_5, \frac{5}{4} a_5 \geq a_4, \frac{6}{5} a_4 \geq a_3.$$ 

The equivalence

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 + \lambda a_8 E_8 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8$ such that the pair

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 + \lambda a_8 E_8$$

is not log canonical at $Q$.

- If the point $Q \in E_1 \setminus E_3$ then

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_1,$$ 

since $\lambda a_1 \leq 1$. 

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By adjunction \((E_1, \lambda \tilde{D}|E_1)\) is not log canonical at \(Q\) and
\[
2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|E_1) = \text{mult}_Q(\tilde{D} \cdot E_1) > 3,
\]
which is false.

- If \(Q \in E_1 \cap E_3\) then the log pair
  \[
  K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_3 E_3
  \]
is not log canonical at the point \(Q\) and so is the log pair
\[
K_X + \lambda \tilde{D} + \lambda a_1 E_1 + E_3, \text{ since } \lambda a_3 \leq 1.
\]

By adjunction \((E_3, \lambda \tilde{D}|E_3)\) is not log canonical at \(Q\) and
\[
2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|E_3) = \text{mult}_Q(\tilde{D} \cdot E_3) > 3 - a_1,
\]
implies that
\[
2 \geq \frac{2}{3} a_3 \geq 2a_3 - \frac{a_3}{2} - \frac{5}{6} a_3 \geq 2a_3 - a_2 - a_4 > 3
\]
which is false.

- If \(Q \in E_3\) but \(Q \not\in E_1 \cup E_2 \cup E_4\) then
  \[
  K_X + \lambda \tilde{D} + \lambda a_3 E_3
  \]
is not log canonical at the point \(Q\) and so is the pair
\[
K_X + \lambda \tilde{D} + E_3, \text{ since } \lambda a_3 \geq 1.
\]

By adjunction \((E_3, \lambda \tilde{D}|E_3)\) is not log canonical at \(Q\) and
\[
\frac{1}{2} \geq \frac{a_3}{6} \geq 2a_3 - \frac{a_3}{2} - \frac{5}{6} a_3 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D} \cdot E_3) > 3,
\]
which is false.

- If \(Q \in E_3 \cap E_4\) then the log pair
  \[
  K_X + \lambda \tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4
  \]
is not log canonical at the point \(Q\) and so is the log pair
\[
K_X + \lambda \tilde{D} + E_3 + \lambda a_4 E_4.
\]

By adjunction \((E_3, \lambda \tilde{D}|E_3)\) is not log canonical at \(Q\) and
\[
a_3 - a_4 \geq 2a_3 - \frac{a_3}{2} - \frac{a_3}{2} - a_4 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D} \cdot E_3) > \frac{1}{\lambda} - a_4 > 3 - a_4,
\]
which is false.

- If the point \(Q \in E_4 \backslash (E_3 \cup E_5)\) then
  \[
  K_X + \lambda \tilde{D} + \lambda a_4 E_4
  \]
is not log canonical at the point \(Q\) and so is the pair
\[
K_X + \lambda \tilde{D} + E_4, \text{ since } \lambda a_4 < 1.
\]

By adjunction \((E_4, \lambda \tilde{D}|E_4)\) is not log canonical at \(Q\) and
\[
\frac{1}{4} \geq \frac{1}{5} a_4 \geq 2a_4 - a_4 - \frac{4}{5} a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D} \cdot E_4) > \frac{1}{\lambda} > 3,
\]
which is false.

- If \(Q \in E_4 \cap E_5\) then the log pair
  \[
  K_X + \lambda \tilde{D} + \lambda a_4 E_4 + \lambda a_5 E_5
  \]
is not log canonical at the point \(Q\) and so are the log pairs
\[
K_X + \lambda \tilde{D} + E_4 + \lambda a_5 E_5 \text{ and } K_X + \lambda \tilde{D} + \lambda a_4 E_4 + E_5.
\]
By adjunction
\[ a_4 - a_5 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mult_Q(\tilde{D} \cdot E_4) > \frac{1}{\lambda} - a_5 > 3 - a_5 . \]
which is a contradiction.

• If \( Q \in E_5 \setminus (E_4 \cup E_6) \) then the log pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_5 E_5 \]
is not log canonical at the point \( Q \) and so is the pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + E_5 . \]
By adjunction \((E_5, \lambda \tilde{D}|_{E_5})\) is not log canonical at \( Q \) and
\[ 2a_5 - a_4 - a_6 = \tilde{D} \cdot E_5 \geq \mult_Q(\tilde{D}|_{E_5}) = \mult_Q(\tilde{D} \cdot E_5) > 3 , \]
which is false.

• If \( Q \in E_5 \cap E_6 \) then the log pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_5 E_5 + \lambda a_6 E_6 \]
is not log canonical at the point \( Q \) and so are the log pairs
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_5 E_5 + E_6 \] and \[ K_{\tilde{X}} + \lambda \tilde{D} + E_5 + \lambda a_6 E_6 . \]
By adjunction
\[ 2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \mult_Q(\tilde{D}|_{E_6}) = \mult_Q(\tilde{D} \cdot E_6) > 3 - a_5 \]
and
\[ 2a_5 - a_4 - a_6 = \tilde{D} \cdot E_5 \geq \mult_Q(\tilde{D}|_{E_5}) = \mult_Q(\tilde{D} \cdot E_5) > 3 - a_6 . \]
which is false.

• If the point \( Q \in E_6 \setminus (E_5 \cup E_7) \) then
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_6 E_6 \]
is not log canonical at the point \( Q \) and so is the pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + E_6 . \]
By adjunction \((E_6, \lambda \tilde{D}|_{E_6})\) is not log canonical at \( Q \) and
\[ 2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \mult_Q(\tilde{D}|_{E_6}) = \mult_Q(\tilde{D} \cdot E_6) > 3 , \]
which is false.

• If \( Q \in E_6 \cap E_7 \) then the log pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_6 E_6 + \lambda a_7 E_7 \]
is not log canonical at the point \( Q \) and so is the log pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_6 E_6 + E_7 . \]
By adjunction
\[ 2a_7 - a_6 - a_8 = \tilde{D} \cdot E_7 \geq \mult_Q(\tilde{D}|_{E_7}) = \mult_Q(\tilde{D} \cdot E_7) > 3 - a_6 . \]
which is false.

• If the point \( Q \in E_7 \setminus (E_6 \cup E_8) \) then
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_7 E_7 \]
is not log canonical at the point \( Q \) and so is the pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + E_7 . \]
By adjunction \((E_7, \lambda \tilde{D}|_{E_7})\) is not log canonical at \( Q \) and
\[ 2a_7 - a_6 - a_8 = \tilde{D} \cdot E_7 \geq \mult_Q(\tilde{D}|_{E_7}) = \mult_Q(\tilde{D} \cdot E_7) > 3 , \]
which is false.

- If \( Q \in E_7 \cap E_8 \) then the log pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_7 E_7 + \lambda a_8 E_8 \]
  is not log canonical at the point \( Q \) and so is the log pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_7 E_7 + E_8 . \]
  By adjunction
  \[ 2a_8 - a_7 = \tilde{D} \cdot E_8 \geq \text{mult}_Q \left( \tilde{D} |_{E_8} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_8 \right) > 3 - a_7 . \]
  which is false.

- If the point \( Q \in E_8 \setminus E_7 \) then
  \[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_8 E_8 \]
  is not log canonical at the point \( Q \) and so is the pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + E_8 . \]
  By adjunction \( (E_8, \lambda \tilde{D} |_{E_8}) \) is not log canonical at \( Q \) and
  \[ 2a_8 - a_7 = \tilde{D} \cdot E_8 \geq \text{mult}_Q \left( \tilde{D} |_{E_8} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_8 \right) > 3 , \]
  which is false.

\( \square \)

### 3.14. Del Pezzo surface of degree 1 with exactly an \( E_6 \) type singular point.

In this section we will prove the following.

**Lemma 3.15.** Let \( X \) be a del Pezzo surface with exactly one Du Val singularity of type \( E_6 \) and \( K_X^2 = 1 \). Then the global log canonical threshold of \( X \) is

\[ \text{lct}(X) = \frac{1}{3} . \]

**Proof.** Suppose that \( \text{lct}(X) < \frac{1}{3} \), then there exists a \( \mathbb{Q} \)-divisor \( D \in X \) and a rational number \( \lambda < \frac{1}{3} \), such that the log pair \( (X, \lambda D) \) is not log canonical and \( D \sim_{\mathbb{Q}} -K_X \). We derive that the pair \( (X, \lambda D) \) is log canonical outside of a point \( P \in X \) and not log canonical at \( P \). Let \( \pi : \tilde{X} \rightarrow X \) be the minimal resolution of \( X \). The configuration of the exceptional curves is given by the following Dynkin diagram.

\[ \mathbb{E}_6, \quad E_1 \quad E_2 \quad E_3 \quad E_5 \quad E_6 \]

Then

\[ \tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - 2a_2 E_2 - 3a_3 E_3 - 2a_4 E_4 - 2a_5 E_5 - a_6 E_6 \]

and

\[ \tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - E_1 - 2E_2 - 3E_3 - 2E_4 - 2E_5 - E_6 . \]

The inequalities

\[
\begin{align*}
0 & \leq \tilde{D} \cdot \tilde{Z} = 1 - 2a_4 \\
0 & \leq E_1 \cdot \tilde{D} = 2a_1 - 2a_2 \\
0 & \leq E_2 \cdot \tilde{D} = 4a_2 - a_1 - 3a_3 \\
0 & \leq E_3 \cdot \tilde{D} = 6a_3 - 2a_2 - 2a_4 - 2a_5 \\
0 & \leq E_4 \cdot \tilde{D} = 4a_4 - 3a_3 \\
0 & \leq E_5 \cdot \tilde{D} = 4a_5 - 3a_3 - a_6 \\
0 & \leq E_6 \cdot \tilde{D} = 2a_6 - 2a_5
\end{align*}
\]
imply that \( a_1 = a_6 \leq \frac{4}{3}, a_2 = a_5 \leq \frac{5}{3}, a_3 \leq \frac{2}{3}, a_4 \leq \frac{1}{2} \). The equivalence
\[
K_X + \lambda \tilde{D} + \lambda a_1 E_1 + 2\lambda a_2 E_2 + 3\lambda a_3 E_3 + 2\lambda a_4 E_4 + 2\lambda a_5 E_5 + \lambda a_6 E_6 \sim_Q \pi^*(K_X + \lambda D)
\]
implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \) such that the pair
\[
K_X + \lambda \tilde{D} + \lambda a_1 E_1 + 2\lambda a_2 E_2 + 3\lambda a_3 E_3 + 2\lambda a_4 E_4 + 2\lambda a_5 E_5 + \lambda a_6 E_6
\]
is not log canonical at \( Q \).

- If the point \( Q \in E_1 \) and \( Q \not\in E_2 \) then
  \[
  K_X + \lambda \tilde{D} + \lambda a_1 E_1
  \]
is not log canonical at the point \( Q \) and so is the pair
  \[
  K_X + \lambda \tilde{D} + E_1 \text{ since } \lambda a_1 \leq 1.
  \]
By adjunction \((E_1, \lambda \tilde{D}|_{E_1})\) is not log canonical at \( Q \) and
\[
2a_1 - 2a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > 3,
\]
which is false.

- If the point \( Q \in E_1 \cap E_2 \) then
  \[
  K_X + \lambda \tilde{D} + \lambda a_1 E_1 + 2\lambda a_2 E_2
  \]
is not log canonical at the point \( Q \) and so are the pairs
  \[
  K_X + \lambda \tilde{D} + E_1 + 2\lambda a_2 E_2 \text{ and } K_X + \lambda \tilde{D} + \lambda E_1 + E_2.
  \]
By adjunction
\[
2a_1 - 2a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > 3 - 2a_2 \text{ and}
\]
\[
4a_2 - a_1 - 3a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D}|_{E_2}) = \text{mult}_Q(\tilde{D} \cdot E_2) > 3 - a_1,
\]
which is false.

- If the point \( Q \in E_2 \setminus (E_1 \cup E_3) \) then
  \[
  K_X + \lambda \tilde{D} + 2\lambda a_2 E_2
  \]
is not log canonical at the point \( Q \) and so is the pair
  \[
  K_X + \lambda \tilde{D} + E_2.
  \]
By adjunction
\[
4a_2 - a_1 - 3a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D}|_{E_2}) = \text{mult}_Q(\tilde{D} \cdot E_2) > 3,
\]
which is false.

- If the point \( Q \in E_2 \cap E_3 \) then
  \[
  K_X + \lambda \tilde{D} + 2\lambda a_2 E_2 + 3\lambda a_3 E_3
  \]
is not log canonical at the point \( Q \) and so are the pairs
  \[
  K_X + \lambda \tilde{D} + E_2 + 3\lambda a_3 E_3 \text{ and } K_X + \lambda \tilde{D} + 2\lambda a_2 E_2 + E_3.
  \]
By adjunction
\[
4a_2 - 3a_3 - a_1 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D}|_{E_2}) = \text{mult}_Q(\tilde{D} \cdot E_2) > 3 - 3a_3 \text{ and}
\]
\[
6a_3 - 2a_2 - 2a_4 - 2a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 3 - 2a_2,
\]
which is false.
• If the point $Q \in E_3 \setminus (E_2 \cup E_4 \cup E_5)$ then
  \[ K_{\tilde{X}} + \lambda \tilde{D} + 3\lambda a_3 E_3 \]
is not log canonical at the point $Q$ and so is the pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + E_3 . \]
By adjunction $(E_3, \lambda \tilde{D}|_{E_3})$ is not log canonical at $Q$ and
  \[ 6a_3 - 2a_2 - 2a_4 - 2a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 3 , \]
which is false.
• If the point $Q \in E_3 \cap E_4$ then
  \[ K_{\tilde{X}} + \lambda \tilde{D} + 3\lambda a_3 E_3 + 2\lambda a_4 E_4 \]
is not log canonical at the point $Q$ and so are the pairs
  \[ K_{\tilde{X}} + \lambda \tilde{D} + E_3 + 2\lambda a_4 E_4 \text{ and } K_{\tilde{X}} + \lambda \tilde{D} + 3\lambda a_3 E_3 + E_4 . \]
By adjunction
  \[ 6a_3 - 2a_2 - 2a_4 - 2a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 3 - 2a_4 \]
and
  \[ 4a_4 - 3a_3 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D}|_{E_4}) = \text{mult}_Q(\tilde{D} \cdot E_4) > 3 - 3a_3 , \]
which is false.
• If the point $Q \in E_4 \setminus E_3$ then
  \[ K_{\tilde{X}} + \lambda \tilde{D} + 2\lambda a_4 E_4 \]
is not log canonical at the point $Q$ and so is the pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + E_4 . \]
By adjunction $(E_4, \lambda \tilde{D}|_{E_4})$ is not log canonical at $Q$ and
  \[ 4a_4 - 3a_3 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D}|_{E_4}) = \text{mult}_Q(\tilde{D} \cdot E_4) > 3 , \]
which is false.

3.15. **Del Pezzo surface of degree 1 with exactly one $\mathbb{E}_7$ type singularity.** In this section we will prove the following.

**Lemma 3.16.** Let $X$ be a del Pezzo surface with exactly one Du Val singularity of type $\mathbb{E}_7$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is
  \[ \text{let}(X) = \frac{1}{4} . \]

**Proof.** Suppose that $\text{let}(X) < \frac{1}{4}$, then there exists a $\mathbb{Q}$-divisor $D \in X$ and a rational number $\lambda < \frac{1}{2}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_Q -K_X$. We derive that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at $P$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of $X$. The configuration of the exceptional curves is given by the following Dynkin diagram.

![Dynkin diagram for $\mathbb{E}_7$](image)

Then
  \[ \tilde{D} \sim_Q \pi^*(D) - 2a_1 E_1 - 3a_2 E_2 - 4a_3 E_3 - 2a_4 E_4 - 3a_5 E_5 - 2a_6 E_6 - a_7 E_7 \]
and
\[ \hat{Z} \sim_Q \pi^*(Z) - 2E_1 - 3E_2 - 4E_3 - 2E_4 - 3E_5 - 2E_6 - E_7. \]
The inequalities
\[
\begin{align*}
0 &\leq \hat{D} \cdot \hat{Z} = 1 - 2a_1 \\
0 &\leq E_1 \cdot \hat{D} = 4a_1 - 3a_2 \\
0 &\leq E_2 \cdot \hat{D} = 6a_2 - 2a_1 - 4a_3 \\
0 &\leq E_3 \cdot \hat{D} = 8a_3 - 3a_2 - 3a_5 - 2a_4 \\
0 &\leq E_4 \cdot \hat{D} = 4a_4 - 4a_3 \\
0 &\leq E_5 \cdot \hat{D} = 6a_5 - 4a_3 - 2a_6 \\
0 &\leq E_6 \cdot \hat{D} = 4a_6 - 3a_5 - a_7 \\
0 &\leq E_7 \cdot \hat{D} = 2a_7 - 2a_6
\end{align*}
\]

imply that \( a_1 \leq \frac{1}{2}, \ a_2 \leq \frac{2}{3}, \ a_3 \leq \frac{3}{4}, \ a_4 \leq \frac{7}{8}, \ a_5 \leq \frac{5}{6}, \ a_6 \leq 1, \ a_7 \leq \frac{3}{2}. \) The equivalence
\[ K + \lambda \hat{D} + 2\lambda a_1 E_1 + 3\lambda a_2 E_2 + 4\lambda a_3 E_3 + 2\lambda a_4 E_4 + 3\lambda a_5 E_5 + 2\lambda a_6 E_6 + \lambda a_7 E_7 = \pi^*(K_X + \lambda D) \]
implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \) such that the pair
\[ K + \lambda \hat{D} + 2\lambda a_1 E_1 + 3\lambda a_2 E_2 + 4\lambda a_3 E_3 + 2\lambda a_4 E_4 + 3\lambda a_5 E_5 + 2\lambda a_6 E_6 + \lambda a_7 E_7 \]
is not log canonical at \( Q. \)
- If the point \( Q \in E_1 \) and \( Q \not\in E_2 \) then
  \[ K + \lambda \hat{D} + 2\lambda a_1 E_1 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K + \lambda \hat{D} + E_1 \text{ since } 2\lambda a_1 \leq 1. \]
  By adjunction \( (E_1, \lambda \hat{D}|_{E_1}) \) is not log canonical at \( Q \) and
  \[ 4a_1 - 3a_2 = \hat{D} \cdot E_1 \geq \mult_Q(\hat{D}|_{E_1}) = \mult_Q(\hat{D} \cdot E_1) > 4, \]
  which is false.
- If the point \( Q \in E_1 \cap E_2 \) then
  \[ K + \lambda \hat{D} + 2\lambda a_1 E_1 + 3\lambda a_2 E_2 \]
is not log canonical at the point \( Q \) and so are the pairs
  \[ K + \lambda \hat{D} + E_1 + 3\lambda a_2 E_2 \text{ and } K + \lambda \hat{D} + 2\lambda a_1 E_1 + E_2. \]
  By adjunction
  \[ 4a_1 - 3a_2 = \hat{D} \cdot E_1 \geq \mult_Q(\hat{D}|_{E_1}) = \mult_Q(\hat{D} \cdot E_1) > 4 - 3a_2 \]
  and
  \[ 6a_2 - 2a_1 - 4a_3 = \hat{D} \cdot E_2 \geq \mult_Q(\hat{D}|_{E_2}) = \mult_Q(\hat{D} \cdot E_2) > 4 - 2a_1, \]
  which is false.
- If the point \( Q \in E_2 \setminus (E_1 \cup E_3) \) then
  \[ K + \lambda \hat{D} + 3\lambda a_2 E_2 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K + \lambda \hat{D} + E_2. \]
  By adjunction
  \[ 6a_2 - 2a_1 - 4a_3 = \hat{D} \cdot E_2 \geq \mult_Q(\hat{D}|_{E_2}) = \mult_Q(\hat{D} \cdot E_2) > 4, \]
  which is false.
If the point \( Q \in E_2 \cap E_3 \) then
\[
K_X + \lambda \tilde{D} + 3\lambda a_2 E_2 + 4\lambda a_3 E_3
\]
is not log canonical at the point \( Q \) and so are the pairs
\[
K_X + \lambda \tilde{D} + E_2 + 4\lambda a_3 E_3 \quad \text{and} \quad K_X + \lambda \tilde{D} + 3\lambda a_2 E_2 + E_3 .
\]
By adjunction
\[
6a_2 - 2a_1 - 4a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q (\tilde{D}|_{E_2}) = \text{mult}_Q (\tilde{D} \cdot E_2) > 4 - 4a_3 \quad \text{and}
\]
\[
8a_3 - 3a_2 - 3a_5 - 2a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q (\tilde{D}|_{E_3}) = \text{mult}_Q (\tilde{D} \cdot E_3) > 4 - 3a_2 ,
\]
which is false.

If the point \( Q \in E_3 \setminus (E_2 \cup E_4 \cup E_5) \) then
\[
K_X + \lambda \tilde{D} + 4\lambda a_3 E_3
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_3 .
\]
By adjunction \((E_3, \lambda \tilde{D}|_{E_3})\) is not log canonical at \( Q \) and
\[
8a_3 - 3a_2 - 2a_4 - 3a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q (\tilde{D}|_{E_3}) = \text{mult}_Q (\tilde{D} \cdot E_3) > 4 ,
\]
which is false.

If the point \( Q \in E_3 \cap E_4 \) then
\[
K_X + \lambda \tilde{D} + 4\lambda a_3 E_3 + 2\lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so are the pairs
\[
K_X + \lambda \tilde{D} + E_3 + 2\lambda a_4 E_4 \quad \text{and} \quad K_X + \lambda \tilde{D} + 4\lambda a_3 E_3 + E_4 .
\]
By adjunction
\[
8a_3 - 3a_2 - 2a_4 - 3a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q (\tilde{D}|_{E_3}) = \text{mult}_Q (\tilde{D} \cdot E_3) > 4 - 2a_4 \quad \text{and}
\]
\[
4a_4 - 4a_3 = \tilde{D} \cdot E_4 \geq \text{mult}_Q (\tilde{D}|_{E_4}) = \text{mult}_Q (\tilde{D} \cdot E_4) > 4 - 4a_3 ,
\]
which is false.

If the point \( Q \in E_4 \setminus E_3 \) then
\[
K_X + \lambda \tilde{D} + 2\lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_4 .
\]
By adjunction \((E_4, \lambda \tilde{D}|_{E_4})\) is not log canonical at \( Q \) and
\[
4a_4 - 4a_3 = \tilde{D} \cdot E_4 \geq \text{mult}_Q (\tilde{D}|_{E_4}) = \text{mult}_Q (\tilde{D} \cdot E_4) > 4 ,
\]
which is false.

If the point \( Q \in E_5 \setminus E_3 \) then
\[
K_X + \lambda \tilde{D} + 4\lambda a_3 E_3
\]
is not log canonical at the point \( Q \) and so are the pairs
\[
K_X + \lambda \tilde{D} + E_3 + 3\lambda a_5 E_5 \quad \text{and} \quad K_X + \lambda \tilde{D} + 4\lambda a_3 E_3 + E_5 .
\]
By adjunction
\[
8a_3 - 3a_2 - 2a_4 - 3a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q (\tilde{D}|_{E_3}) = \text{mult}_Q (\tilde{D} \cdot E_3) > 4 - 3a_5 \quad \text{and}
\]
If the point \( Q \in E_5 \setminus (E_3 \cup E_6) \) then

\[
K_{\tilde{X}} + \lambda \tilde{D} + 3\lambda a_5 E_5
\]

is not log canonical at the point \( Q \) and so is the pair

\[
K_{\tilde{X}} + \lambda \tilde{D} + E_5.
\]

By adjunction \((E_5, \lambda \tilde{D}|_{E_5})\) is not log canonical at \( Q \) and

\[
6a_5 - 4a_3 - 2a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q(\tilde{D}|_{E_5}) = \text{mult}_Q(\tilde{D} \cdot E_5) > 4 - 4a_3,
\]

which is false.

- If the point \( Q \in E_5 \cap E_6 \) then

\[
K_{\tilde{X}} + \lambda \tilde{D} + 3\lambda a_5 E_5 + 2\lambda a_6 E_6
\]

is not log canonical at the point \( Q \) and so are the pairs

\[
K_{\tilde{X}} + \lambda \tilde{D} + E_5 + 2\lambda a_6 E_6 \quad \text{and} \quad K_{\tilde{X}} + \lambda \tilde{D} + 3\lambda a_5 E_5 + E_6.
\]

By adjunction

\[
6a_5 - 4a_3 - 2a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q(\tilde{D}|_{E_5}) = \text{mult}_Q(\tilde{D} \cdot E_5) > 4 - 2a_6 \quad \text{and}
\]

\[
4a_6 - 3a_5 - a_7 = \tilde{D} \cdot E_6 \geq \text{mult}_Q(\tilde{D}|_{E_6}) = \text{mult}_Q(\tilde{D} \cdot E_6) > 4 - 3a_5,
\]

which is false.

- If the point \( Q \in E_6 \setminus (E_5 \cup E_7) \) then

\[
K_{\tilde{X}} + \lambda \tilde{D} + 2\lambda a_6 E_6
\]

is not log canonical at the point \( Q \) and so is the pair

\[
K_{\tilde{X}} + \lambda \tilde{D} + E_6.
\]

By adjunction \((E_6, \lambda \tilde{D}|_{E_6})\) is not log canonical at \( Q \) and

\[
4a_6 - 3a_5 - a_7 = \tilde{D} \cdot E_6 \geq \text{mult}_Q(\tilde{D}|_{E_6}) = \text{mult}_Q(\tilde{D} \cdot E_6) > 4,
\]

which is false.

- If the point \( Q \in E_6 \cap E_7 \) then

\[
K_{\tilde{X}} + \lambda \tilde{D} + 2\lambda a_6 E_6 + \lambda a_7 E_7
\]

is not log canonical at the point \( Q \) and so are the pairs

\[
K_{\tilde{X}} + \lambda \tilde{D} + E_6 + \lambda a_7 E_7 \quad \text{and} \quad K_{\tilde{X}} + \lambda \tilde{D} + 2\lambda a_6 E_6 + E_7.
\]

By adjunction

\[
4a_6 - 3a_5 - a_7 = \tilde{D} \cdot E_6 \geq \text{mult}_Q(\tilde{D}|_{E_6}) = \text{mult}_Q(\tilde{D} \cdot E_6) > 4 - a_7 \quad \text{and}
\]

\[
2a_7 - 2a_6 = \tilde{D} \cdot E_7 \geq \text{mult}_Q(\tilde{D}|_{E_7}) = \text{mult}_Q(\tilde{D} \cdot E_7) > 4 - 2a_6,
\]

which is false.
If the point $Q \in E_7 \setminus E_6$, then

$$K_X + \lambda \tilde{D} + \lambda a_7 E_7$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_7.$$ By adjunction $(E_7, \lambda \tilde{D}|_{E_7})$ is not log canonical at $Q$ and

$$2a_7 - 2a_6 = \tilde{D} \cdot E_7 \geq \text{mult}_Q (\tilde{D}|_{E_7}) = \text{mult}_Q (\tilde{D} \cdot E_7) > 4,$$

which is false.

\[\square\]

3.16. **Del Pezzo surface of degree 1 with exactly one $E_8$ type singular point.** In this section we will prove the following.

**Lemma 3.17.** Let $X$ be a del Pezzo surface with exactly one Du Val singularity of type $E_8$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is

$$\text{let}(X) = \frac{1}{6}.$$ 

*Proof.* Suppose that $\text{let}(X) < \frac{1}{6}$, then there exists a $\mathbb{Q}$-divisor $D \in X$, such that the log pair $(X, \lambda D)$ is not log canonical for a rational number $\lambda < \frac{1}{6}$ and $D \sim_{\mathbb{Q}} -K_X$. We derive that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at $P$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of $X$. The configuration of the exceptional curves is given by the following Dynkin diagram.

\[
\begin{array}{cccccccc}
\text{E}_8, & \bullet & E_1 & \bullet & E_2 & \bullet & E_3 & \bullet & E_5 & \bullet & E_6 & \bullet & E_7 & \bullet & E_8 \\
& \bullet & E_4 & \\
\end{array}
\]

Then

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - 2a_1 E_1 - 4a_2 E_2 - 6a_3 E_3 - 3a_4 E_4 - 5a_5 E_5 - 4a_6 E_6 - 3a_7 E_7 - 2a_8 E_8$$

and

$$\tilde{Z} \sim_{\mathbb{Q}} \pi^*(Z) - 2E_1 - 4E_2 - 6E_3 - 3E_4 - 5E_5 - 4E_6 - 3E_7 - 2E_8.$$ 

We have the inequalities

\[
\begin{align*}
0 \leq \tilde{D} \cdot \tilde{Z} &= 1 - 2a_8 \\
0 \leq E_1 \cdot \tilde{D} &= 4a_1 - 4a_2 \\
0 \leq E_2 \cdot \tilde{D} &= 8a_2 - 2a_1 - 6a_3 \\
0 \leq E_3 \cdot \tilde{D} &= 12a_3 - 4a_2 - 5a_5 - 3a_4 \\
0 \leq E_4 \cdot \tilde{D} &= 6a_4 - 6a_3 \\
0 \leq E_5 \cdot \tilde{D} &= 10a_5 - 6a_3 - 4a_6 \\
0 \leq E_6 \cdot \tilde{D} &= 8a_6 - 5a_5 - 3a_7 \\
0 \leq E_7 \cdot \tilde{D} &= 6a_7 - 4a_6 - 2a_8 \\
0 \leq E_8 \cdot \tilde{D} &= 4a_8 - 3a_7 .
\end{align*}
\]

The equivalence

$$K_X + \lambda \tilde{D} + 2\lambda a_1 E_1 + 4\lambda a_2 E_2 + 6\lambda a_3 E_3 + 3\lambda a_4 E_4 + 5\lambda a_5 E_5 + 4\lambda a_6 E_6 + 3\lambda a_7 E_7 + 2\lambda a_8 E_8 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8$ such that the pair

$$K_X + \lambda \tilde{D} + 2\lambda a_1 E_1 + 4\lambda a_2 E_2 + 6\lambda a_3 E_3 + 3\lambda a_4 E_4 + 5\lambda a_5 E_5 + 4\lambda a_6 E_6 + 3\lambda a_7 E_7 + 2\lambda a_8 E_8$$

is not log canonical at $Q$. 

• If the point $Q \in E_1$ and $Q \not\in E_2$ then
\[ K_{\tilde{X}} + \lambda \tilde{D} + 2\lambda a_1 E_1 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_{X} + \lambda \tilde{D} + E_1 \text{ since } 2\lambda a_1 \leq 1. \]
By adjunction $(E_1, \lambda \tilde{D}|_{E_1})$ is not log canonical at $Q$ and
\[ 4a_1 - 4a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q \left( \tilde{D}|_{E_1} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_1 \right) > 6, \]
which is false.
• If the point $Q \in E_1 \cap E_2$ then
\[ K_{\tilde{X}} + \lambda \tilde{D} + 2\lambda a_1 E_1 + 4\lambda a_2 E_2 \]
is not log canonical at the point $Q$ and so are the pairs
\[ K_{X} + \lambda \tilde{D} + E_1 + 4\lambda a_2 E_2 \text{ and } K_{\tilde{X}} + \lambda \tilde{D} + 2\lambda a_1 E_1 + E_2. \]
By adjunction
\[ 4a_1 - 4a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q \left( \tilde{D}|_{E_1} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_1 \right) > 6 - 4a_2 \]
and
\[ 8a_2 - 2a_1 - 6a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q \left( \tilde{D}|_{E_2} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_2 \right) > 6 - 2a_1, \]
which is false.
• If the point $Q \in E_2 \setminus (E_1 \cup E_3)$ then
\[ K_{\tilde{X}} + \lambda \tilde{D} + 4\lambda a_2 E_2 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_{X} + \lambda \tilde{D} + E_2. \]
By adjunction
\[ 8a_2 - 2a_1 - 6a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q \left( \tilde{D}|_{E_2} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_2 \right) > 6, \]
which is false.
• If the point $Q \in E_2 \cap E_3$ then
\[ K_{\tilde{X}} + \lambda \tilde{D} + 4\lambda a_2 E_2 + 6\lambda a_3 E_3 \]
is not log canonical at the point $Q$ and so are the pairs
\[ K_{X} + \lambda \tilde{D} + E_2 + 6\lambda a_3 E_3 \text{ and } K_{\tilde{X}} + \lambda \tilde{D} + 4\lambda a_2 E_2 + E_3. \]
By adjunction
\[ 8a_2 - 2a_1 - 6a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q \left( \tilde{D}|_{E_2} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_2 \right) > 6 - 6a_3 \]
and
\[ 12a_3 - 4a_2 - 5a_5 - 3a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q \left( \tilde{D}|_{E_3} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_3 \right) > 6 - 4a_2, \]
which is false.
• If the point $Q \in E_3 \setminus (E_2 \cup E_4 \cup E_5)$ then
\[ K_{\tilde{X}} + \lambda \tilde{D} + 6\lambda a_3 E_3 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_{X} + \lambda \tilde{D} + E_3. \]
By adjunction $(E_3, \lambda \tilde{D}|_{E_3})$ is not log canonical at $Q$ and
\[ 12a_3 - 4a_2 - 3a_4 - 5a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q \left( \tilde{D}|_{E_3} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_3 \right) > 6, \]
which is false.
• If the point $Q \in E_3 \cap E_4$ then
  $$K_{\tilde{X}} + \lambda \tilde{D} + 6\lambda a_3 E_3 + 3\lambda a_4 E_4$$
is not log canonical at the point $Q$ and so are the pairs
  $$K_{\tilde{X}} + \lambda \tilde{D} + E_3 + 3\lambda a_4 E_4 \quad \text{and} \quad K_{\tilde{X}} + \lambda \tilde{D} + 6\lambda a_3 E_3 + E_4 .$$
By adjunction
  $$12a_3 - 4a_2 - 3a_4 - 5a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q (\tilde{D}|_{E_3}) = \text{mult}_Q (\tilde{D} \cdot E_3) > 6 - 3a_4$$
and
  $$6a_4 - 6a_3 = \tilde{D} \cdot E_4 \geq \text{mult}_Q (\tilde{D}|_{E_4}) = \text{mult}_Q (\tilde{D} \cdot E_4) > 6 - 6a_3 ,$$
which is false.
• If the point $Q \in E_4 \setminus E_3$ then
  $$K_{\tilde{X}} + \lambda \tilde{D} + 3\lambda a_4 E_4$$
is not log canonical at the point $Q$ and so is the pair
  $$K_{\tilde{X}} + \lambda \tilde{D} + E_4 .$$
By adjunction $(E_4, \lambda \tilde{D}|_{E_4})$ is not log canonical at $Q$ and
  $$6a_4 - 6a_3 = \tilde{D} \cdot E_4 \geq \text{mult}_Q (\tilde{D}|_{E_4}) = \text{mult}_Q (\tilde{D} \cdot E_4) > 6 ,$$
which is false.
• If the point $Q \in E_3 \cap E_5$ then
  $$K_{\tilde{X}} + \lambda \tilde{D} + 6\lambda a_3 E_3 + 5\lambda a_5 E_5$$
is not log canonical at the point $Q$ and so are the pairs
  $$K_{\tilde{X}} + \lambda \tilde{D} + E_3 + 5\lambda a_5 E_5 \quad \text{and} \quad K_{\tilde{X}} + \lambda \tilde{D} + 6\lambda a_3 E_3 + E_5 .$$
By adjunction
  $$12a_3 - 4a_2 - 3a_4 - 5a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q (\tilde{D}|_{E_3}) = \text{mult}_Q (\tilde{D} \cdot E_3) > 6 - 5a_5$$
and
  $$10a_5 - 6a_3 - 4a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q (\tilde{D}|_{E_5}) = \text{mult}_Q (\tilde{D} \cdot E_5) > 6 - 6a_3 ,$$
which is false.
• If the point $Q \in E_5 \setminus (E_3 \cup E_6)$ then
  $$K_{\tilde{X}} + \lambda \tilde{D} + 5\lambda a_5 E_5$$
is not log canonical at the point $Q$ and so is the pair
  $$K_{\tilde{X}} + \lambda \tilde{D} + E_5 .$$
By adjunction $(E_5, \lambda \tilde{D}|_{E_5})$ is not log canonical at $Q$ and
  $$10a_5 - 6a_3 - 4a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q (\tilde{D}|_{E_5}) = \text{mult}_Q (\tilde{D} \cdot E_5) > 6 ,$$
which is false.
• If the point $Q \in E_5 \cap E_6$ then
  $$K_{\tilde{X}} + \lambda \tilde{D} + 5\lambda a_5 E_5 + 4\lambda a_6 E_6$$
is not log canonical at the point $Q$ and so are the pairs
  $$K_{\tilde{X}} + \lambda \tilde{D} + E_5 + 4\lambda a_6 E_6 \quad \text{and} \quad K_{\tilde{X}} + \lambda \tilde{D} + 5\lambda a_5 E_5 + E_6 .$$
By adjunction
  $$10a_5 - 6a_3 - 4a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q (\tilde{D}|_{E_5}) = \text{mult}_Q (\tilde{D} \cdot E_5) > 6 - 4a_6$$
and
\[ 8a_6 - 5a_5 - 3a_7 = \tilde{D} \cdot E_6 \geq \text{mult}_Q(\tilde{D}|_{E_6}) = \text{mult}_Q(\tilde{D} \cdot E_6) > 6 - 5a_5 , \]
which is false.

- If the point \( Q \in E_6 \setminus (E_6 \cup E_7) \) then
  \[ K_{\tilde{X}} + \lambda \tilde{D} + 4\lambda a_6 E_6 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + E_6 . \]
By adjunction \((E_6, \lambda \tilde{D}|_{E_6})\) is not log canonical at \( Q \) and
\[ 8a_6 - 5a_5 - 3a_7 = \tilde{D} \cdot E_6 \geq \text{mult}_Q(\tilde{D}|_{E_6}) = \text{mult}_Q(\tilde{D} \cdot E_6) > 6 , \]
which is false.

- If the point \( Q \in E_6 \cap E_7 \) then
  \[ K_{\tilde{X}} + \lambda \tilde{D} + 4\lambda a_6 E_6 + 3\lambda a_7 E_7 \]
is not log canonical at the point \( Q \) and so are the pairs
  \[ K_{\tilde{X}} + \lambda \tilde{D} + E_6 + 3\lambda a_7 E_7 \text{ and } K_{\tilde{X}} + \lambda \tilde{D} + 4\lambda a_6 E_6 + E_7 . \]
By adjunction
\[ 8a_6 - 5a_5 - 3a_7 = \tilde{D} \cdot E_6 \geq \text{mult}_Q(\tilde{D}|_{E_6}) = \text{mult}_Q(\tilde{D} \cdot E_6) > 6 - 3a_7 \]
and
\[ 6a_7 - 4a_6 - 2a_8 = \tilde{D} \cdot E_7 \geq \text{mult}_Q(\tilde{D}|_{E_7}) = \text{mult}_Q(\tilde{D} \cdot E_7) > 6 - 4a_6 , \]
which is false.

- If the point \( Q \in E_7 \setminus (E_6 \cup E_8) \) then
  \[ K_{\tilde{X}} + \lambda \tilde{D} + 3\lambda a_7 E_7 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + E_7 . \]
By adjunction \((E_7, \lambda \tilde{D}|_{E_7})\) is not log canonical at \( Q \) and
\[ 6a_7 - 4a_6 - 2a_8 = \tilde{D} \cdot E_7 \geq \text{mult}_Q(\tilde{D}|_{E_7}) = \text{mult}_Q(\tilde{D} \cdot E_7) > 6 , \]
which is false.

- If the point \( Q \in E_7 \cap E_8 \) then
  \[ K_{\tilde{X}} + \lambda \tilde{D} + 3\lambda a_7 E_7 + 2\lambda a_8 E_8 \]
is not log canonical at the point \( Q \) and so are the pairs
  \[ K_{\tilde{X}} + \lambda \tilde{D} + E_7 + 2\lambda a_8 E_8 \text{ and } K_{\tilde{X}} + \lambda \tilde{D} + 3\lambda a_7 E_7 + E_8 . \]
By adjunction
\[ 6a_7 - 4a_6 - 2a_8 = \tilde{D} \cdot E_7 \geq \text{mult}_Q(\tilde{D}|_{E_7}) = \text{mult}_Q(\tilde{D} \cdot E_7) > 6 - 2a_8 \]
and
\[ 4a_8 - 3a_7 = \tilde{D} \cdot E_8 \geq \text{mult}_Q(\tilde{D}|_{E_8}) = \text{mult}_Q(\tilde{D} \cdot E_8) > 6 - 3a_7 , \]
which is false.
• If the point $Q \in E_8 \setminus E_7$ then

$$K_X + \lambda \tilde{D} + 2\lambda a_8 E_8$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_8.$$

By adjunction $(E_8, \lambda \tilde{D}|_{E_8})$ is not log canonical at $Q$ and

$$4a_8 - 3a_7 = \tilde{D} \cdot E_8 \geq \text{mult}_Q(\tilde{D}|_{E_8}) = \text{mult}_Q(\tilde{D} \cdot E_8) > 6,$$

which is false.

□

4. DEL PEZZO SURFACES OF DEGREE 1 WITH AT LEAST TWO DU VAL SINGULARITIES

Suppose now that $X$ is a del Pezzo surface of degree 1 having at least two Du Val singular points. We have the following result.

**Lemma 4.1.** Suppose that the surface $X$ has at least one singularity of type $D_4, D_5, D_6, E_6$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = \begin{cases} 
\frac{1}{3} & \text{when } E_6 \in \text{Sing}(X) \\
\frac{1}{2} & \text{otherwise.}
\end{cases}$$

**Proof.** We will only treat the case $D_4$, as the rest of the cases are similar. Since the linear system $|-K_X|$ is 1-dimensional there is a unique element $Z \in |-K_X|$ that passes through the singular point $D_4$. This curve $Z$ is irreducible and does not pass through any other singular point of $X$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of $X$. Then

$$\tilde{Z} \sim Q \pi^*(Z) - E_1 - E_2 - 2E_3 - E_4,$$

where $E_1, E_2, E_3, E_4$ are the exceptional curves of $\pi$ that are contracted to the Du Val singular point $D_4$. This means that the global log canonical threshold is

$$\text{lct}(X) \leq \frac{1}{2}.$$

Now we assume that $\text{lct}(X) < \frac{1}{2}$. Then there exists a $Q$-divisor $D \in X$ such that $D \sim Q - K_X$ and the log pair $(X, \lambda D)$ is not log canonical, for some rational number $\lambda < \frac{1}{2}$. According to Lemma 2.4 the pair $(X, \lambda D)$ is not log canonical at a singular point of $X$. If the pair $(X, \lambda D)$ is not log canonical at $D_4$, we proceed as in Lemma 3.10 otherwise we follow the proof of Theorem 1.5. In any case we obtain a contradiction, thus

$$\text{lct}(X) = \frac{1}{2}.$$

□

**Lemma 4.2.** Suppose that the surface $X$ has at least one singularity of type $A_5, A_6$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = \frac{2}{3}.$$

**Proof.** Again we will consider only the case that $X$ has at least one $A_5$ type singular point, as $A_6$ can be treated in a similar fashion. Let $\pi : \tilde{X} \to X$ be the minimal resolution of $X$ and let $E_1, E_2, E_3, E_4, E_5$ be the exceptional curves of $\pi$ that are contracted to the Du Val singular point $A_5$. Then we can always find a -1 curve $\tilde{L}_3$ in $\tilde{X}$ that only intersects $E_3$ transversally among the exceptional curves of the fundamental cycle. Then we have that

$$\tilde{L}_3 \sim Q \pi^*(L_3) - \frac{1}{2}E_1 - E_2 - \frac{3}{2}E_3 - E_4 - \frac{1}{2}E_5,$$

This means that the global log canonical threshold is

$$\text{lct}(X) \leq \frac{2}{3}.$$
Now we assume that $\text{lct}(X) < \frac{2}{3}$. Then there exists a $\mathbb{Q}$-divisor $D \in X$ such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical, for some rational number $\lambda < \frac{2}{3}$. According to Lemma 2.3 the pair $(X, \lambda D)$ is not log canonical at a singular point of $X$. If the pair $(X, \lambda D)$ is not log canonical at $A_5$, we proceed as in Lemma 3.10 otherwise we follow the proof of Theorem 1.5. In any case we obtain a contradiction, thus

$$\text{lct}(X) = \frac{2}{3}.$$  

\[\square\]

**Lemma 4.3.** Suppose that the surface $X$ has at least one singularity of type $A_4$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = \begin{cases} 
2/3 & \text{when } \lvert -K_X \rvert \text{ has a cuspidal curve } C \text{ such that } \text{Sing}(C) = A_2, \\
3/4 & \text{when } \lvert -K_X \rvert \text{ has a cuspidal curve } C \text{ such that } \text{Sing}(C) = A_1 \\
& \text{and no cuspidal curve } C \text{ such that } \text{Sing}(C) = A_2, \\
4/5 & \text{in the remaining cases.}
\end{cases}$$

**Proof.** Let $\pi: \tilde{X} \to X$ be the minimal resolution of $X$ and let $E_1, E_2, E_3, E_4$ be the exceptional curves of $\pi$ that are contracted to the Du Val singular point $A_4$. In all the cases when we have at least an $A_4$ type singularity there exists a unique smooth irreducible element $C$ of the linear system $\lvert -2K_X \rvert$, which passes through the intersection point $E_1 \cap E_2$. For the pull back of the irreducible curve $C$ we have

$$\tilde{C} + E_1 + 2E_2 + 2E_3 + E_4 \in \lvert -K_{\tilde{X}} \rvert.$$ 

If we blow up once more in order to get transversal intersections we see that the global log canonical threshold is

$$\text{lct}(X) \leq \text{lct}(X, C) = \frac{4}{5}.$$ 

Now we assume that $\text{lct}(X) < \frac{4}{5}$. Then there exists a $\mathbb{Q}$-divisor $D \in X$ such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical, for some rational number $\lambda < \frac{4}{5}$. According to Lemma 2.3 the pair $(X, \lambda D)$ is not log canonical at a singular point of $X$. If the pair $(X, \lambda D)$ is not log canonical at $A_4$, we proceed as in Lemma 3.3 otherwise we follow the proof of Theorem 1.5. In any case we obtain a contradiction, and the result follows.  

\[\square\]

**Lemma 4.4.** Suppose that the surface $X$ has at least one singularity of type $A_3$ and no singularity of type $A_4$, $D_4$, $D_5$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = \begin{cases} 
1 & \text{when } \lvert -K_X \rvert \text{ does not have cuspidal curves}, \\
2/3 & \text{when } \lvert -K_X \rvert \text{ has a cuspidal curve } C \text{ such that } \text{Sing}(C) = A_2, \\
3/4 & \text{when } \lvert -K_X \rvert \text{ has a cuspidal curve } C \text{ such that } \text{Sing}(C) = A_1 \\
& \text{and no cuspidal curve } C \text{ such that } \text{Sing}(C) = A_2, \\
5/6 & \text{in the remaining cases.}
\end{cases}$$

**Proof.** Let $\pi: \tilde{X} \to X$ be the minimal resolution of $X$ and let $E_1, E_2, E_3$ be the exceptional curves of $\pi$ that are contracted to the Du Val singular point $A_3$. One can show that in all the cases when we have at least an $A_3$ type singularity we must have $\text{lct}(X) \leq 1$.

Now we assume that $\text{lct}(X) < 1$. Then there exists a $\mathbb{Q}$-divisor $D \in X$ such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical, for some rational number $\lambda < 1$. According to Lemma 2.3 the pair $(X, \lambda D)$ is not log canonical at a singular point of $X$. If the pair $(X, \lambda D)$ is not log canonical at $A_3$, we proceed as in Lemma 3.1 otherwise we follow the proof of Theorem 1.5. In any case we obtain a contradiction, and the result follows.  

\[\square\]

**Lemma 4.5.** Suppose that the surface $X$ has exactly two singularities of type $A_3$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = 1$$

**Proof.** See Lemma 3.1  

\[\square\]
5. Del Pezzo surfaces with Picard group $\mathbb{Z}$

All possible singular points on a del Pezzo surface $X$ that has only Du Val singular points and $\text{Pic}(X) \cong \mathbb{Z}$ are listed in [16].

5.1. Del Pezzo surface of degree 1 with an $E_7$ and an $A_1$ type singularity. In this section we will prove the following.

Lemma 5.1. Let $X$ be a del Pezzo surface with one Du Val singularity of type $E_7$, one of type $A_1$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = \frac{1}{4}.$$ 

Proof. Suppose that $\text{lct}(X) < \frac{1}{4}$, then there exists a $\mathbb{Q}$-divisor $D \in X$ such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_X$ for some rational number $\lambda < \frac{1}{4}$. We derive that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at $P$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of $X$. The configuration of the exceptional curves is given by the following Dynkin diagram.

```
\begin{center}
\begin{tikzpicture}
\node (E7) at (0,0) {$E_7$};
\node (A1) at (-1,0) {$A_1$};
\node (E1) at (-2,0) {$E_1$};
\node (E2) at (-3,0) {$E_2$};
\node (E3) at (-4,0) {$E_3$};
\node (E4) at (-5,0) {$E_4$};
\node (E5) at (-6,0) {$E_5$};
\node (E6) at (-7,0) {$E_6$};
\node (F1) at (-8,0) {$F_1$};
\end{tikzpicture}
\end{center}
```

Then

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - a_6 E_6 - a_7 E_7 - b_1 F_1.$$ 

We should note here that there are two -1 curves $L_1, L_7$ such that

$$L_1 \cdot E_1 = L_7 \cdot E_7 = L_7 \cdot F_1 = 1.$$ 

Therefore we have

$$\tilde{L}_1 \sim_{\mathbb{Q}} \pi^*(L_1) - 2E_1 - 3E_2 - 4E_3 - 2E_4 - 3E_5 - 2E_6 - E_7$$

$$\tilde{L}_7 \sim_{\mathbb{Q}} \pi^*(L_7) - E_1 - 2E_2 - 3E_3 - \frac{3}{2}E_4 - \frac{5}{2}E_5 - 2E_6 - \frac{3}{2}E_7 - \frac{1}{2}F_1.$$ 

and since $L_7 \sim -K_X$ and $L_1 \sim -K_X$ we see that $\text{lct}(X) \leq \frac{1}{4}$.

The inequalities

\begin{align*}
0 &\leq \tilde{D} \cdot \tilde{L}_7 = 1 - a_7 - b_1 \\
0 &\leq \tilde{D} \cdot \tilde{L}_1 = 1 - a_1 \\
0 &\leq E_1 \cdot \tilde{D} = 2a_1 - a_2 \\
0 &\leq E_2 \cdot \tilde{D} = 2a_2 - a_1 - a_3 \\
0 &\leq E_3 \cdot \tilde{D} = 2a_3 - a_2 - a_5 - a_4 \\
0 &\leq E_4 \cdot \tilde{D} = 2a_4 - a_3 \\
0 &\leq E_5 \cdot \tilde{D} = 2a_5 - a_3 - a_6 \\
0 &\leq E_6 \cdot \tilde{D} = 2a_6 - a_5 - a_7 \\
0 &\leq E_7 \cdot \tilde{D} = 2a_7 - a_6 \\
0 &\leq F_1 \cdot \tilde{D} = 2b_1
\end{align*}

imply that $a_1 \leq 1, a_2 \leq 2, a_3 \leq 3, a_4 \leq \frac{7}{2}, a_5 \leq \frac{5}{2}, a_6 \leq 2, a_7 \leq 1, b_1 \leq 1$. Moreover we have

$$2a_1 \geq a_2, \quad \frac{3}{2}a_2 \geq a_3, \quad 2a_4 \geq a_3, \quad \frac{5}{6}a_5 \geq a_5, \quad \frac{4}{5}a_5 \geq a_6$$

and

$$2a_7 \geq a_6, \quad \frac{3}{2}a_6 \geq a_5, \quad \frac{4}{3}a_5 \geq a_3.$$
The equivalence
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 \sim_{\mathbb{Q}} \pi^* (K_{X} + \lambda D) \]
implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \) such that the pair
\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 \]
is not log canonical at \( Q \).

- If the point \( Q \in E_1 \) and \( Q \notin E_2 \) then
  \[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + E_1 \]
since \( \lambda a_1 \leq 1 \).

  By adjunction \((E_1, \lambda \tilde{D}|_{E_1})\) is not log canonical at \( Q \) and
  \[ 2 \geq 2a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q (\tilde{D} \cdot E_1) > 4 \ , \]
  which is false.

- If the point \( Q \in E_1 \cap E_2 \) then
  \[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 \]
is not log canonical at the point \( Q \) and so are the pairs
  \[ K_{\tilde{X}} + \lambda \tilde{D} + E_1 + \lambda a_2 E_2 \text{ and } K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + E_2 \] .

  By adjunction
  \[ 2 - a_2 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q (\tilde{D}|_{E_1}) = \text{mult}_Q (\tilde{D} \cdot E_1) > 4 - a_2 \text{ and} \]
  and
  \[ 2 - a_1 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q (\tilde{D}|_{E_2}) = \text{mult}_Q (\tilde{D} \cdot E_2) > 4 - a_1 \ , \]
  which is false.

- If the point \( Q \in E_2 \setminus (E_1 \cup E_3) \) then
  \[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 E_2 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + E_2 \] .

  By adjunction
  \[ 2a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q (\tilde{D}|_{E_2}) = \text{mult}_Q (\tilde{D} \cdot E_2) > 4 \ , \]
  implies that \( a_2 > 2 \) which is false.

- If the point \( Q \in E_2 \cap E_3 \) then
  \[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3 \]
is not log canonical at the point \( Q \) and so are the pairs
  \[ K_{\tilde{X}} + \lambda \tilde{D} + E_2 + \lambda a_3 E_3 \text{ and } K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 E_2 + E_3 \] .

  By adjunction
  \[ 4 - a_3 \geq 2a_2 - a_3 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q (\tilde{D}|_{E_2}) = \text{mult}_Q (\tilde{D} \cdot E_2) > 4 - a_3 \]
  which is a contradiction.
• If the point \( Q \in E_3 \setminus (E_2 \cup E_4 \cup E_5) \) then
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_3 E_3
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + E_3 .
\]
By adjunction \((E_3, \lambda \tilde{D}|_{E_3})\) is not log canonical at \( Q \) and
\[
\frac{5}{2} \geq \frac{5}{6} a_3 \geq 2a_3 - \frac{2}{3} a_3 = 2a_3 - a_2 - a_4 - a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q \left( \tilde{D} \cdot E_3 \right) > 4 ,
\]
which is false.

• If the point \( Q \in E_3 \cap E_4 \) then
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_3 E_3 + E_4 .
\]
By adjunction
\[
\frac{7}{2} - a_3 \geq 2a_4 - a_3 = \tilde{D} \cdot E_4 \geq \text{mult}_Q \left( \tilde{D}|_{E_4} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_4 \right) > 4 - a_3 ,
\]
which is false.

• If the point \( Q \in E_4 \setminus E_3 \) then
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + E_4 .
\]
By adjunction \((E_4, \lambda \tilde{D}|_{E_4})\) is not log canonical at \( Q \) and
\[
\frac{10}{3} - a_3 \geq \frac{4}{3} a_5 - a_3 \geq 2a_5 - \frac{2}{3} a_5 = 2a_5 - a_3 - a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q \left( \tilde{D} \cdot E_5 \right) > 4 - a_3 ,
\]
which is false.

• If the point \( Q \in E_5 \setminus (E_3 \cup E_6) \) then
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_5 E_5
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + E_5 .
\]
By adjunction \((E_5, \lambda \tilde{D}|_{E_5})\) is not log canonical at \( Q \) and
\[
\frac{1}{3} \geq \frac{2}{15} a_5 \geq 2a_5 - \frac{6}{5} a_5 - \frac{2}{3} a_5 = 2a_5 - a_3 - a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q \left( \tilde{D} \cdot E_5 \right) > 4 ,
\]
which is false.
• If the point $Q \in E_5 \cap E_6$ then
  
  $$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_5 E_5 + \lambda a_6 E_6$$

  is not log canonical at the point $Q$ and so are the pairs
  
  $$K_{\tilde{X}} + \lambda \tilde{D} + E_5 + \lambda a_6 E_6 .$$

  By adjunction
  
  $$2a_5 - \frac{6}{5} a_5 \geq 2a_5 - a_3 - a_6 = \tilde{D} \cdot E_5 \geq \operatorname{mult}_Q \left( \tilde{D} |_{E_5} \right) = \operatorname{mult}_Q \left( \tilde{D} \cdot E_5 \right) > 4 - a_6$$

  implies that $a_5 > 5$ which is false.

• If the point $Q \in E_6 \setminus (E_5 \cup E_7)$ then
  
  $$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_6 E_6$$

  is not log canonical at the point $Q$ and so is the pair
  
  $$K_{\tilde{X}} + \lambda \tilde{D} + E_6 .$$

  By adjunction $(E_6, \lambda \tilde{D} |_{E_6})$ is not log canonical at $Q$ and
  
  $$2a_6 - \frac{5}{4} a_6 - \frac{a_6}{2} \geq 2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \operatorname{mult}_Q \left( \tilde{D} \cdot E_6 \right) > 4$$

  implies that $a_6 > 16$ which is false.

• If the point $Q \in E_6 \cap E_7$ then
  
  $$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_6 E_6 + \lambda a_7 E_7$$

  is not canonical at the point $Q$ and so are the pairs
  
  $$K_{\tilde{X}} + \lambda \tilde{D} + E_6 + \lambda a_7 E_7 .$$

  By adjunction
  
  $$2a_6 - a_7 \geq 2a_6 - a_5 - a_7 = \tilde{D} \cdot E_6 \geq \operatorname{mult}_Q \left( \tilde{D} |_{E_6} \right) = \operatorname{mult}_Q \left( \tilde{D} \cdot E_6 \right) > 4 - a_7$$

  implies that $a_6 > 2$ which is false.

• If the point $Q \in E_7 \setminus E_6$ then
  
  $$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_7 E_7$$

  is not log canonical at the point $Q$ and so is the pair
  
  $$K_{\tilde{X}} + \lambda \tilde{D} + E_7 .$$

  By adjunction $(E_7, \lambda \tilde{D} |_{E_7})$ is not log canonical at $Q$ and
  
  $$2a_7 - a_6 = \tilde{D} \cdot E_7 \geq \operatorname{mult}_Q \left( \tilde{D} |_{E_7} \right) = \operatorname{mult}_Q \left( \tilde{D} \cdot E_7 \right) > 4$$

  implies that $a_7 > 2$ which is false.

• If the point $Q \in F_1$ then
  
  $$K_{\tilde{X}} + \lambda \tilde{D} + \lambda b_1 F_1$$

  is not log canonical at the point $Q$ and so is the pair
  
  $$K_{\tilde{X}} + \lambda \tilde{D} + F_1 \quad \text{since} \quad \lambda b_1 \leq 1 .$$

  By adjunction $(F_1, \lambda \tilde{D} |_{F_1})$ is not log canonical at $Q$ and
  
  $$2 \geq 2b_1 = \tilde{D} \cdot F_1 \geq \operatorname{mult}_Q \left( \tilde{D} \cdot F_1 \right) > 4 ,$$

  which is false.
5.2. Del Pezzo surface of degree 1 with an $E_6$ and an $A_2$ type singularity. In this section we will prove the following.

**Lemma 5.2.** Let $X$ be a del Pezzo surface with one Du Val singularity of type $E_6$, one of type $A_2$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = \frac{1}{3}.$$ 

*Proof.* Suppose that $\text{lct}(X) < \frac{1}{3}$, then there exists a $\mathbb{Q}$-divisor $D \in X$ such that the log pair $(X, \lambda D)$ is not log canonical, where $\lambda < \frac{1}{3}$ and $D \sim_{\mathbb{Q}} -K_X$. We derive that the pair $(X, \lambda D)$ is log canonical everywhere outside of a point $P \in X$ and not log canonical at $P$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of $X$. The configuration of the exceptional curves is given by the following Dynkin diagram.

$$
\begin{array}{cccccccc}
\text{E}_6 + \text{A}_2 . & \bullet_{E_1} & - & \bullet_{E_2} & - & \bullet_{E_3} & - & \bullet_{E_5} & - & \bullet_{E_6} & \bullet_{F_1} & - & \bullet_{F_2} \\
& & & & & & & & & & \bullet_{E_4} \\
\end{array}
$$

Then

$$
\tilde{D} \sim_{\mathbb{Q}} \pi^* (D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - a_6 E_6 - b_1 F_1 - b_2 F_2 .
$$

We should note here that there are two -1 curves $\tilde{L}_4, \tilde{L}_6$ such that

$$
\tilde{L}_4 \cdot E_4 = \tilde{L}_6 \cdot E_6 = \tilde{L}_6 \cdot F_1 = 1 .
$$

Therefore we have

$$
\begin{align*}
\tilde{L}_4 & \sim_{\mathbb{Q}} \pi^* (L_4) - E_1 - 2E_2 - 3E_3 - 2E_4 - 2E_5 - E_6, \\
\tilde{L}_6 & \sim_{\mathbb{Q}} \pi^* (L_6) - \frac{2}{3} E_1 - \frac{4}{3} E_2 - 2E_3 - E_4 - \frac{5}{3} E_5 - \frac{4}{3} E_6 - \frac{2}{3} F_1 - \frac{1}{3} F_2 .
\end{align*}
$$

and since $L_6 \sim_{\mathbb{Q}} -K_X$ and $L_4 \sim_{\mathbb{Q}} -K_X$ we see that $\text{lct}(X) \leq \frac{1}{3}$.

The inequalities

$$
\begin{align*}
0 \leq \tilde{D} \cdot \tilde{L}_6 & = 1 - a_6 - b_1, \\
0 \leq \tilde{D} \cdot \tilde{L}_4 & = 1 - a_4, \\
0 \leq E_1 \cdot \tilde{D} & = 2a_1 - a_2, \\
0 \leq E_2 \cdot \tilde{D} & = 2a_2 - a_1 - a_3, \\
0 \leq E_3 \cdot \tilde{D} & = 2a_3 - a_2 - a_4 - a_5, \\
0 \leq E_4 \cdot \tilde{D} & = 2a_4 - a_3, \\
0 \leq E_5 \cdot \tilde{D} & = 2a_5 - a_3 - a_6, \\
0 \leq E_6 \cdot \tilde{D} & = 2a_6 - a_5, \\
0 \leq F_1 \cdot \tilde{D} & = 2b_1 - b_2, \\
0 \leq F_2 \cdot \tilde{D} & = 2b_2 - b_1
\end{align*}
$$

imply that $a_1 \leq \frac{4}{3}, a_2 \leq \frac{5}{3}, a_3 \leq 2, a_4 \leq 1, a_5 \leq \frac{5}{3}, a_6 \leq 1, b_1 \leq 1, b_2 \leq 2$ . The equivalence

$$
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda b_1 F_1 + \lambda b_2 F_2 \sim_{\mathbb{Q}} \pi^* (K_X + \lambda D)
$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup F_1 \cup F_2$ such that the pair

$$
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda b_1 F_1 + \lambda b_2 F_2
$$

is not log canonical at $Q$.  

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If the point $Q \in E_1$ and $Q \notin E_2$ then

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1$$

is not log canonical at the point $Q$ and so is the pair

$$K_{\tilde{X}} + \lambda \tilde{D} + E_1$$

since $\lambda a_1 \leq 1$.

By adjunction $(E_1, \lambda \tilde{D}|_{E_1})$ is not log canonical at $Q$ and

$$2a_1 - \frac{5}{4}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mult_Q(\tilde{D}|_{E_1}) = \mult_Q(\tilde{D} \cdot E_1) > 3$$

implies that $a_1 > 4$ which is false.

If the point $Q \in E_1 \cap E_2$ then

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$$

is not log canonical at the point $Q$ and so are the pairs

$$K_{\tilde{X}} + \lambda \tilde{D} + E_1 + \lambda a_2 E_2 .$$

By adjunction

$$2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mult_Q(\tilde{D}|_{E_1}) = \mult_Q(\tilde{D} \cdot E_1) > 3 - a_2$$

implies that $a_1 > \frac{3}{2}$ which is false.

If the point $Q \in E_2 \setminus (E_1 \cup E_3)$ then

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 E_2$$

is not log canonical at the point $Q$ and so is the pair

$$K_{\tilde{X}} + \lambda \tilde{D} + E_2 .$$

By adjunction

$$2a_2 - \frac{1}{2}a_2 - \frac{6}{5}a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mult_Q(\tilde{D}|_{E_2}) = \mult_Q(\tilde{D} \cdot E_2) > 3 ,$$

implies that $a_2 > 10$ which is false.

If the point $Q \in E_2 \cap E_3$ then

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3$$

is not log canonical at the point $Q$ and so are the pairs

$$K_{\tilde{X}} + \lambda \tilde{D} + E_2 + \lambda a_3 E_3 .$$

By adjunction

$$2a_2 - \frac{1}{2}a_2 - a_3 \geq 2a_2 - a_3 - a_1 = \tilde{D} \cdot E_2 \geq \mult_Q(\tilde{D}|_{E_2}) = \mult_Q(\tilde{D} \cdot E_2) > 3 - a_3$$

implies $a_2 > 2$ which is false.

If the point $Q \in E_3 \setminus (E_2 \cup E_4 \cup E_5)$ then

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_3 E_3$$

is not log canonical at the point $Q$ and so is the pair

$$K_{\tilde{X}} + \lambda \tilde{D} + E_3 .$$

By adjunction $(E_3, \lambda \tilde{D}|_{E_3})$ is not log canonical at $Q$ and

$$2a_3 - \frac{2}{3}a_3 - \frac{1}{2}a_3 - \frac{2}{3}a_3 \geq 2a_3 - a_2 - a_4 - a_5 = \tilde{D} \cdot E_3 \geq \mult_Q(\tilde{D} \cdot E_3) > 3$$

implies $a_3 > 18$ which is false.
• If the point $Q \in E_3 \cap E_4$ then
  \[ K_X + \lambda \tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4 \]
is not log canonical at the point $Q$ and so is the pair
  \[ K_X + \lambda \tilde{D} + \lambda a_3 E_3 + E_4. \]
By adjunction
  \[ 2a_4 - a_3 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D}|_{E_4}) = \text{mult}_Q(\tilde{D} \cdot E_4) > 3 - a_3, \]
which is false, since $a_4 \leq 1$.
• If the point $Q \in E_4 \backslash E_3$ then
  \[ K_X + \lambda \tilde{D} + \lambda a_4 E_4 \]
is not log canonical at the point $Q$ and so is the pair
  \[ K_X + \lambda \tilde{D} + E_4. \]
By adjunction $(E_4, \lambda \tilde{D}|_{E_4})$ is not log canonical at $Q$ and
  \[ 2a_4 \geq 2a_4 - a_3 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D}|_{E_4}) = \text{mult}_Q(\tilde{D} \cdot E_4) > 3 \]
which is false, since $a_4 \leq 1$.
• If the point $Q \in F_1$ and $Q \notin F_2$ then
  \[ K_X + \lambda \tilde{D} + \lambda b_1 F_1 \]
is not log canonical at the point $Q$ and so is the pair
  \[ K_X + \lambda \tilde{D} + F_1 \text{ since } \lambda b_1 \leq 1. \]
By adjunction $(F_1, \lambda \tilde{D}|_{F_1})$ is not log canonical at $Q$ and
  \[ 2b_1 - b_2 = \tilde{D} \cdot F_1 \geq \text{mult}_Q(\tilde{D}|_{F_1}) = \text{mult}_Q(\tilde{D} \cdot F_1) > 3 \]
implies that $b_1 > \frac{3}{2}$ which is false.
• If the point $Q \in F_1 \cap F_2$ then
  \[ K_X + \lambda \tilde{D} + \lambda b_1 F_1 + \lambda b_2 F_2 \]
is not log canonical at the point $Q$ and so are the pairs
  \[ K_X + \lambda \tilde{D} + F_1 + b_2 F_2. \]
By adjunction
  \[ 2b_1 - b_2 = \tilde{D} \cdot F_1 \geq \text{mult}_Q(\tilde{D}|_{F_1}) = \text{mult}_Q(\tilde{D} \cdot F_1) > 3 - b_2 \]
implies that $b_1 > \frac{3}{2}$ which is false.
• If the point $Q \in F_2$ and $Q \notin F_1$ then
  \[ K_X + \lambda \tilde{D} + \lambda b_2 F_2 \]
is not log canonical at the point $Q$ and so is the pair
  \[ K_X + \lambda \tilde{D} + F_2 \text{ since } \frac{1}{3} b_2 \leq 1. \]
By adjunction $(F_2, \lambda \tilde{D}|_{F_2})$ is not log canonical at $Q$ and
  \[ \frac{3}{2} b_2 \geq 2b_2 - b_1 = \tilde{D} \cdot F_2 \geq \text{mult}_Q(\tilde{D}|_{F_2}) = \text{mult}_Q(\tilde{D} \cdot F_2) > 3 \]
implies that $b_2 > 2$ which is false.
5.3. Del Pezzo surface of degree 1 with an $A_7$ and an $A_1$ type singularity. In this section we will prove the following.

**Lemma 5.3.** Let $X$ be a del Pezzo surface with one Du Val singularity of type $A_7$, one of type $A_1$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = \frac{1}{2}.$$  

**Proof.** Suppose that $\text{lct}(X) < \frac{1}{2}$, then there exists a $\mathbb{Q}$-divisor $D \in X$ such that the log pair $(X, \lambda D)$ is not log canonical, where $\lambda < \frac{1}{2}$ and $D \sim_{\mathbb{Q}} -K_X$. We derive that the pair $(X, \lambda D)$ is log canonical everywhere outside of a point $P \in X$ and not log canonical at $P$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of $X$. The configuration of the exceptional curves is given by the following Dynkin diagram.

\[ A_7 + A_1. \quad \bullet E_1 \longrightarrow \bullet E_2 \longrightarrow \bullet E_3 \longrightarrow \bullet E_4 \longrightarrow \bullet E_5 \longrightarrow \bullet E_6 \longrightarrow \bullet E_7 \longrightarrow \bullet F_1 \]

Then  
\[ \tilde{D} \sim_{\mathbb{Q}} \pi^*(D) = a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - a_6 E_6 - a_7 E_7 - b_1 F_1. \]

We should note here that there are three -1 curves $\tilde{L}_1, \tilde{L}_4, \tilde{L}_6$ such that

\[ \tilde{L}_1 \cdot E_1 = \tilde{L}_1 \cdot E_7 = \tilde{L}_4 \cdot E_4 = \tilde{L}_6 \cdot E_6 = \tilde{L}_6 \cdot F_1 = 1. \]

Therefore we have

\[ \tilde{L}_1 \sim_{\mathbb{Q}} \pi^*(L_1) = E_1 - E_2 - E_3 - E_4 - E_5 - E_6, \]
\[ \tilde{L}_4 \sim_{\mathbb{Q}} \pi^*(L_4) = \frac{1}{2} E_1 - E_2 - \frac{3}{2} E_3 - 2 E_4 - \frac{3}{2} E_5 - E_6 - \frac{1}{2} E_7, \]
\[ \tilde{L}_6 \sim_{\mathbb{Q}} \pi^*(L_6) = \frac{1}{4} E_1 - \frac{1}{2} E_2 - \frac{3}{4} E_3 - E_4 - \frac{5}{4} E_5 - \frac{3}{4} E_6 - \frac{3}{4} E_7 - \frac{1}{2} F_1. \]

and since $L_6 \sim_{\mathbb{Q}} L_1 \sim_{\mathbb{Q}} L_4 \sim_{\mathbb{Q}} -K_X$ we see that $\text{lct}(X) \leq \frac{1}{2}$.

The inequalities

\begin{align*}
0 \leq \tilde{D} \cdot \tilde{L}_1 &= 1 - a_1 - a_7 \\
0 \leq \tilde{D} \cdot \tilde{L}_4 &= 1 - a_4 \\
0 \leq \tilde{D} \cdot \tilde{L}_6 &= 1 - a_6 - b_1 \\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_2 \\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_1 - a_3 \\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_2 - a_4 \\
0 \leq E_4 \cdot \tilde{D} &= 2a_4 - a_3 - a_5 \\
0 \leq E_5 \cdot \tilde{D} &= 2a_5 - a_4 - a_6 \\
0 \leq E_6 \cdot \tilde{D} &= 2a_6 - a_5 - a_7 \\
0 \leq E_7 \cdot \tilde{D} &= 2a_7 - a_6
\end{align*}

imply that $a_1 \leq 1$, $a_2 \leq \frac{3}{2}$, $a_3 \leq \frac{5}{4}$, $a_4 \leq 1$, $a_5 \leq \frac{5}{4}$, $a_6 \leq 1$, $a_7 \leq 1$, $b_1 \leq 1$.

Moreover we get

\[ 2a_7 \geq a_6, \quad \frac{3}{2} a_6 \geq a_5, \quad \frac{4}{3} a_5 \geq a_4, \quad \frac{5}{4} a_4 \geq a_3, \quad \frac{6}{5} a_3 \geq a_2, \quad \frac{7}{6} a_2 \geq a_1. \]

The equivalence

\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 + \lambda b_1 F_1 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D) \]

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup F_1$ such that the pair

\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 + \lambda b_1 F_1 \]

is not log canonical at $Q$. 

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• If the point $Q \in E_1$ and $Q \notin E_2$ then
  
  $$K_X + \lambda \mathcal{D} + a_1 \lambda E_1$$
  
  is not log canonical at the point $Q$ and so is the pair
  
  $$K_X + \lambda \mathcal{D} + E_1 , \text{ since } a_1 \lambda \leq 1 .$$

  By adjunction $(E_1, \lambda \mathcal{D}|_{E_1})$ is not log canonical at $Q$ and

  $$2a_1 - \frac{6}{7} a_1 \geq 2a_1 - a_2 = \mathcal{D} \cdot E_1 \geq \text{mult}_Q(\mathcal{D} \cdot E_1) > \frac{1}{\lambda} > 2$$

  implies that $a_1 > \frac{7}{4}$ which is false.

• If $Q \in E_1 \cap E_2$ then the log pair

  $$K_X + \lambda \mathcal{D} + \lambda a_1 E_1 + \lambda a_2 E_2$$

  is not log canonical at the point $Q$ and so is the log pair

  $$K_X + \lambda \mathcal{D} + E_1 + \lambda a_2 E_2 .$$

  By adjunction it follows that

  $$2a_1 - a_2 = \mathcal{D} \cdot E_1 \geq \text{mult}_Q(\mathcal{D}|_{E_1}) = \text{mult}_Q(\mathcal{D} \cdot E_1) > \frac{1}{\lambda} - a_2 > 2 - a_2 ,$$

  which is false, since $a_1 \leq 1$.

• If $Q \in E_2$ but $Q \notin E_1 \cup E_3$ then

  $$K_X + \lambda \mathcal{D} + \lambda a_2 E_2$$

  is not log canonical at the point $Q$ and so is the pair

  $$K_X + \lambda \mathcal{D} + E_2 , \text{ since } \lambda a_2 \leq 1 .$$

  By adjunction $(E_2, \lambda \mathcal{D}|_{E_2})$ is not log canonical at $Q$ and

  $$2a_2 - \frac{5}{6} a_2 \geq 2a_2 - a_1 - a_3 = \mathcal{D} \cdot E_2 \geq \text{mult}_Q(\mathcal{D} \cdot E_2) > \frac{1}{\lambda} > 2 ,$$

  which is false, since $a_2 \leq \frac{3}{2}$.

• If $Q \in E_2 \cap E_3$ then the log pair

  $$K_X + \lambda \mathcal{D} + \lambda a_2 E_2 + \lambda a_3 E_3$$

  is not log canonical at the point $Q$ and so is the log pair

  $$K_X + \lambda \mathcal{D} + \lambda a_2 E_2 + E_3 , \text{ since } \lambda a_3 < 1 .$$

  By adjunction it follows that

  $$2a_3 - a_2 - a_4 = \mathcal{D} \cdot E_3 \geq \text{mult}_Q(\mathcal{D}|_{E_1}) = \text{mult}_Q(\mathcal{D} \cdot E_3) > \frac{1}{\lambda} - a_2 > 2 - a_2 ,$$

  which, along with the inequality $a_4 \geq \frac{4}{5} a_3$, implies that $a_4 > 1$, which is impossible.

• If $Q \in E_3$ but $Q \notin E_2 \cup E_4$ then

  $$K_X + \lambda \mathcal{D} + \lambda a_3 E_3$$

  is not log canonical at the point $Q$ and so is the pair

  $$K_X + \lambda \mathcal{D} + E_3 , \text{ since } \lambda a_3 \leq 1 .$$

  By adjunction $(E_3, \lambda \mathcal{D}|_{E_3})$ is not log canonical at $Q$ and

  $$2a_3 - a_2 - a_4 = \mathcal{D} \cdot E_3 \geq \text{mult}_Q(\mathcal{D}|_{E_3}) = \text{mult}_Q(\mathcal{D} \cdot E_3) > \frac{1}{\lambda} > 2 .$$

  This inequality together with $a_4 \geq \frac{4}{5} a_3$ implies that $a_4 > 1$, which is impossible.
• If \( Q \in E_3 \cap E_4 \) then the log pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so is the log pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_3 E_3 + E_4 , \text{ since } \lambda a_4 \leq 1 .
\]
By adjunction it follows that
\[
2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q(\tilde{D}|_{E_4}) = \mathrm{mult}_Q(\tilde{D} \cdot E_4) > \frac{1}{\lambda} - a_3 > 2 - a_3 ,
\]
which contradicts \( a_4 \leq 1 \).

• If \( Q \in E_1 \) but \( Q \notin E_3 \cup E_5 \) then
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + E_4 , \text{ since } \lambda a_4 \leq 1 .
\]
By adjunction \((E_4, \lambda \tilde{D}|_{E_4})\) is not log canonical at \( Q \) and
\[
2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \mathrm{mult}_Q(\tilde{D}|_{E_4}) = \mathrm{mult}_Q(\tilde{D} \cdot E_4) > \frac{1}{\lambda} > 2 ,
\]
which is false since \( a_4 \leq 1 \).

• If the point \( Q \in F_1 \) then
\[
K_{\tilde{X}} + \lambda \tilde{D} + b_1 \lambda F_1
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + F_1 , \text{ since } b_1 \lambda \leq 1 .
\]
By adjunction \((F_1, \lambda \tilde{D}|_{F_1})\) is not log canonical at \( Q \) and
\[
2b_1 = \tilde{D} \cdot F_1 \geq \mathrm{mult}_Q(\tilde{D} \cdot F_1) > \frac{1}{\lambda} > 2 ,
\]
which is false, since \( b_1 \leq 1 \).

5.4. Del Pezzo surface of degree 1 with an \( \mathbb{D}_6 \) and two \( \mathbb{A}_1 \) type singularities. In this section we will prove the following.

**Lemma 5.4.** Let \( X \) be a del Pezzo surface with exactly one Du Val singularity of type \( \mathbb{D}_6 \), two of type \( \mathbb{A}_1 \) and \( K_X^2 = 1 \). Then the global log canonical threshold of \( X \) is
\[
\operatorname{let}(X) = \frac{1}{2} .
\]

**Proof.** Suppose that \( \operatorname{let}(X) < \frac{1}{2} \), then there exists a \( \mathbb{Q} \)-divisor \( D \in X \) such that the log pair \((X, \lambda D)\) is not log canonical, where \( \lambda < \frac{1}{2} \) and \( D \sim_Q -K_X \). We derive that the pair \((X, \lambda D)\) is log canonical everywhere outside of a point \( P \in X \) and not log canonical at \( P \). Let \( \pi : \tilde{X} \to X \) be the minimal resolution of \( X \). The configuration of the exceptional curves is given by the following Dynkin diagram.

\[
\mathbb{D}_6 + 2\mathbb{A}_1 . \quad \bullet E_1 \quad \bullet E_3 \quad \bullet E_4 \quad \bullet E_5 \quad \bullet E_6 \quad \bullet F_1 \quad \bullet G_1
\]

\[
\mathbb{D}_6 + 2\mathbb{A}_1 . \quad \bullet E_2
\]

Then
\[
\tilde{D} \sim_Q \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - a_6 E_6 - b_1 F_1 - c_1 G_1 .
\]

We should note here that there are three -1 curves \( L_1, L_2, L_5 \) such that
\[
L_1 \cdot E_1 = L_1 \cdot F_1 = L_2 \cdot E_2 = L_2 \cdot G_1 = L_5 \cdot E_5 = 1 .
\]
Therefore we have

\[ \tilde{L}_1 \sim_Q \pi^*(L_1) - \frac{3}{2}E_1 - E_2 - 2E_3 - \frac{3}{2}E_4 - E_5 - \frac{1}{2}E_6 - \frac{1}{2}F_1 \]
\[ \tilde{L}_2 \sim_Q \pi^*(L_2) - E_1 - \frac{3}{2}E_2 - 2E_3 - \frac{3}{2}E_4 - E_5 - \frac{1}{2}E_6 - \frac{1}{2}G_1 \]
\[ \tilde{L}_5 \sim_Q \pi^*(L_5) - E_1 - E_2 - 2E_3 - 2E_4 - 2E_5 - E_6 . \]

and since \( L_1 \sim_Q L_2 \sim_Q L_5 \sim_Q -K_X \) we see that \( \text{lct}(X) \leq \frac{1}{2} \).

The inequalities

\[
0 \leq \tilde{D} \cdot \tilde{L}_1 = 1 - a_1 - b_1 \\
0 \leq \tilde{D} \cdot \tilde{L}_2 = 1 - a_2 - c_1 \\
0 \leq \tilde{D} \cdot \tilde{L}_5 = 1 - a_5 \\
0 \leq E_1 \cdot \tilde{D} = 2a_1 - a_3 \\
0 \leq E_2 \cdot \tilde{D} = 2a_2 - a_3 \\
0 \leq E_3 \cdot \tilde{D} = 2a_3 - a_1 - a_2 - a_4 \\
0 \leq E_4 \cdot \tilde{D} = 2a_4 - a_3 - a_5 \\
0 \leq E_5 \cdot \tilde{D} = 2a_5 - a_4 - a_6 \\
0 \leq E_6 \cdot \tilde{D} = 2a_6 - a_5 \\
0 \leq F_1 \cdot \tilde{D} = 2b_1 \\
0 \leq G_1 \cdot \tilde{D} = 2c_1
\]

imply that \( a_1 \leq 1, a_2 \leq 1, a_3 \leq 2, a_4 \leq \frac{3}{2}, a_5 \leq 1, a_6 \leq 1, b_1 \leq 1, c_1 \leq 1 \) . Moreover we have

\[ 2a_1 \geq a_3, \ 2a_2 \geq a_3, \ a_3 \geq a_4, \ a_4 \geq a_5, \ a_5 \geq a_6 \]

and

\[ 2a_6 \geq a_5, \ \frac{3}{2}a_5 \geq a_4, \ \frac{4}{3}a_4 \geq a_3 . \]

The equivalence

\( K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda b_1 F_1 + \lambda c_1 G_1 \sim_Q \pi^*(K_X + \lambda D) \)

implies that there is a point \( Q \) such that \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \) or \( Q \in F_1 \) or \( Q \in G_1 \) where the pair

\[ K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda b_1 F_1 + \lambda c_1 G_1 \]

is not log canonical at \( Q \).

- If the point \( Q \in E_1 \setminus E_3 \) then

\[ K_X + \lambda \tilde{D} + \lambda a_1 E_1 \]

is not log canonical at the point \( Q \) and so is the pair

\[ K_X + \lambda \tilde{D} + E_1 . \]

By adjunction \( (E_1, \lambda \tilde{D}|_{E_1}) \) is not log canonical at \( Q \) and

\[ 2a_1 \geq 2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_Q\left( \tilde{D}|_{E_1} \right) = \text{mult}_Q\left( \tilde{D} \cdot E_1 \right) > 2 , \]

implies that \( a_1 > 1 \) which is false.

- If \( Q \in E_3 \) but \( Q \not\in E_1 \cup E_2 \cup E_4 \) then

\[ K_X + \lambda \tilde{D} + \lambda a_3 E_3 \]

is not log canonical at the point \( Q \) and so is the pair

\[ K_X + \lambda \tilde{D} + E_3, \text{ since } \lambda a_3 \leq 1 . \]

By adjunction \( (E_3, \lambda \tilde{D}|_{E_3}) \) is not log canonical at \( Q \) and

\[ 2a_3 - \frac{1}{2}a_3 - \frac{1}{2}a_3 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q\left( \tilde{D} \cdot E_3 \right) > 2 , \]

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implies $a_3 > 2$ which is false.

- If $Q \in E_1 \cap E_3$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \frac{1}{2} a_3 E_3$$

is not log canonical at the point $Q$ and so are the log pairs

$$K_X + \lambda \tilde{D} + E_1 + \lambda a_3 E_3 \text{ and } K_X + \lambda \tilde{D} + \lambda a_1 E_1 + E_3 .$$

By adjunction it follows that

$$2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_Q (\tilde{D} |_{E_1}) = \text{mult}_Q (\tilde{D} \cdot E_1) > 2 - a_3 .$$

and

$$2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q (\tilde{D} |_{E_3}) = \text{mult}_Q (\tilde{D} \cdot E_3) > 2 - a_1 .$$

These inequalities imply that $a_1 > 1$ and $a_3 > 2$ which is a contradiction.

- If $Q \in E_3 \cap E_4$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4$$

is not log canonical at the point $Q$ and so is the log pair

$$K_X + \lambda \tilde{D} + E_3 + \lambda a_4 E_4 .$$

By adjunction

$$2a_3 - \frac{1}{2} a_3 - \frac{1}{2} a_3 - a_4 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q (\tilde{D} \cdot E_3) > 2 - a_4$$

and this implies that $a_3 > 2$ which is false.

- If the point $Q \in E_4 \setminus (E_3 \cup E_5)$ then

$$K_X + \lambda \tilde{D} + \lambda a_4 E_4$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_4 .$$

By adjunction $(E_4, \lambda \tilde{D} |_{E_4})$ is not log canonical at $Q$ and

$$2a_4 - a_4 - \frac{2}{3} a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q (\tilde{D} \cdot E_4) > 2 ,$$

implies that $a_4 > 6$ which is false.

- $Q \in E_5 \setminus (E_4 \cup E_6)$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_5 E_5$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_5 .$$

By adjunction $(E_5, \lambda \tilde{D} |_{E_5})$ is not log canonical at $Q$ and

$$2a_5 - a_5 - \frac{a_5}{2} \geq 2a_5 - a_4 - a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q (\tilde{D} \cdot E_5) > 2 ,$$

implies that $a_5 > 4$ which is false.

- If $Q \in E_4 \cap E_5$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_4 E_4 + \lambda a_5 E_5$$

is not log canonical at the point $Q$ and so are the log pairs

$$K_X + \lambda \tilde{D} + E_4 + \lambda a_5 E_5 \text{ and } K_X + \lambda \tilde{D} + \lambda a_4 E_4 + E_5 .$$

By adjunction

$$2a_4 - a_4 - a_5 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q (\tilde{D} \cdot E_4) > 2 - a_5$$

and

$$2a_5 - a_4 - \frac{a_5}{2} \geq 2a_5 - a_4 - a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q (\tilde{D} \cdot E_5) > 2 - a_4 .$$
imply that $a_4 > 2$ and $a_5 > \frac{4}{3}$ which both lead to a contradiction.

• $Q \in E_5 \setminus (E_4 \cup E_6)$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_5 E_5$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_5 .$$

By adjunction $(E_5, \lambda \tilde{D}|_{E_5})$ is not log canonical at $Q$ and

$$2a_5 - a_5 - \frac{a_5}{2} \geq 2a_5 - a_4 - a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q \left( \tilde{D}|_{E_5} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_5 \right) > 2 ,$$

implies $a_5 > 4$ which is false.

• If $Q \in E_5 \cap E_6$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_5 E_5 + \lambda a_6 E_6$$

is not log canonical at the point $Q$ and so is the log pair

$$K_X + \lambda \tilde{D} + \lambda a_5 E_5 + E_6 .$$

By adjunction

$$2a_6 - a_5 = \tilde{D} \cdot E_6 \geq \text{mult}_Q \left( \tilde{D}|_{E_6} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_6 \right) > 2 - a_5 ,$$

which is false.

• If the point $Q \in E_6 \setminus E_5$ then

$$K_X + \lambda \tilde{D} + \lambda a_6 E_6$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_6 .$$

By adjunction $(E_6, \lambda \tilde{D}|_{E_6})$ is not log canonical at $Q$ and

$$2a_6 \geq 2a_6 - a_5 = \tilde{D} \cdot E_6 \geq \text{mult}_Q \left( \tilde{D}|_{E_6} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_6 \right) > 2 ,$$

which is false.

• If the point $Q \in F_1$ then

$$K_X + \lambda \tilde{D} + \lambda b_1 F_1$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + F_1 .$$

By adjunction $(F_1, \lambda \tilde{D}|_{F_1})$ is not log canonical at $Q$ and

$$2b_1 = \tilde{D} \cdot F_1 \geq \text{mult}_Q \left( \tilde{D}|_{F_1} \right) = \text{mult}_Q \left( \tilde{D} \cdot F_1 \right) > 2 ,$$

implies that $b_1 > 1$ which is false.

\[ \Box \]

5.5. **Del Pezzo surface of degree 1 with an $\mathbb{D}_5$ and an $\mathbb{A}_3$ type singularity.** In this section we will prove the following.

**Lemma 5.5.** Let $X$ be a del Pezzo surface with one Du Val singularity of type $\mathbb{D}_5$, one of type $\mathbb{A}_3$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = \frac{1}{2} .$$
Proof. Suppose that \( \operatorname{lct}(X) < \frac{1}{2} \), then there exists a \( \mathbb{Q} \)-divisor \( D \in X \) such that \( D \sim_{\mathbb{Q}} -K_X \) and the log pair \( (X, \lambda D) \) is not log canonical for some rational number \( \lambda < \frac{1}{2} \). We derive that the pair \( (X, \lambda D) \) is log canonical everywhere outside of a point \( P \in X \) and not log canonical at \( P \). Let \( \pi : \tilde{X} \to X \) be the minimal resolution of \( X \). The configuration of the exceptional curves is given by the following Dynkin diagram.

\[
\begin{array}{cccccccc}
\mathbb{D}_5 + \mathbb{A}_3.
\end{array}
\]

Then
\[
\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - b_1 F_1 - b_2 F_2 - b_3 F_3.
\]
We should note here that there are three -1 curves \( L_1, L_2, L_5 \) such that
\[
L_1 \cdot E_1 = L_1 \cdot F_1 = L_2 \cdot E_2 = L_2 \cdot F_3 = L_4 \cdot E_4 = 1.
\]
Therefore we have
\[
\begin{align*}
\tilde{L}_1 & \sim_{\mathbb{Q}} \pi^*(L_1) - \frac{5}{4} E_1 - \frac{3}{4} E_2 - \frac{3}{2} E_3 - E_4 - \frac{1}{2} E_5 - \frac{3}{4} F_1 - \frac{1}{2} F_2 - \frac{1}{4} F_3, \\
\tilde{L}_2 & \sim_{\mathbb{Q}} \pi^*(L_2) - \frac{3}{4} E_1 - \frac{5}{4} E_2 - \frac{3}{2} E_3 - E_4 - \frac{1}{2} E_5 - \frac{1}{4} F_1 - \frac{1}{2} F_2 - \frac{3}{4} F_3, \\
\tilde{L}_4 & \sim_{\mathbb{Q}} \pi^*(L_4) - E_1 - E_2 - 2E_3 - 2E_4 - E_5.
\end{align*}
\]
and since \( L_1 \sim_{\mathbb{Q}} L_2 \sim_{\mathbb{Q}} L_5 \sim_{\mathbb{Q}} -K_X \) we see that \( \operatorname{lct}(X) \leq \frac{1}{2} \).

The inequalities
\[
\begin{align*}
0 \leq \tilde{D} \cdot \tilde{L}_1 &= 1 - a_1 - b_1, \\
0 \leq \tilde{D} \cdot \tilde{L}_2 &= 1 - a_2 - b_3, \\
0 \leq \tilde{D} \cdot \tilde{L}_5 &= 1 - a_4, \\
0 \leq \tilde{E}_1 \cdot \tilde{D} &= 2a_1 - a_3, \\
0 \leq \tilde{E}_2 \cdot \tilde{D} &= 2a_2 - a_3, \\
0 \leq \tilde{E}_3 \cdot \tilde{D} &= 2a_3 - a_1 - a_2 - a_4, \\
0 \leq \tilde{E}_4 \cdot \tilde{D} &= 2a_4 - a_3 - a_5, \\
0 \leq \tilde{E}_5 \cdot \tilde{D} &= 2a_5 - a_4, \\
0 \leq \tilde{F}_1 \cdot \tilde{D} &= 2b_1 - 1 - b_2, \\
0 \leq \tilde{F}_2 \cdot \tilde{D} &= 2b_2 - b_1 - b_3, \\
0 \leq \tilde{F}_3 \cdot \tilde{D} &= 2b_3 - b_2
\end{align*}
\]
imply that \( a_1 \leq 1, a_2 \leq 1, a_3 \leq \frac{3}{2}, a_4 \leq 1, a_5 \leq 1, b_1 \leq 1, b_2 \leq 2, b_3 \leq 1 \). The equivalence
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda b_1 F_1 + \lambda b_2 F_2 + \lambda b_3 F_3 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)
\]
implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup F_1 \cup F_2 \cup F_3 \) such that the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda b_1 F_1 + \lambda b_2 F_2 + \lambda b_3 F_3
\]
is not log canonical at \( Q \).

- If the point \( Q \in E_1 \setminus E_3 \) then
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + E_1.
\]
By adjunction \( (E_1, \lambda \tilde{D}|_{E_1}) \) is not log canonical at \( Q \) and
\[
2a_1 \geq 2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \operatorname{mult}_Q \left( \tilde{D}|_{E_1} \right) = \operatorname{mult}_Q \left( \tilde{D} \cdot E_1 \right) > 2,
\]
implies that \( a_1 > 1 \) which is false.
• If $Q \in E_1 \cap E_3$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \frac{1}{2} a_3 E_3$$

is not log canonical at the point $Q$ and so is the log pair

$$K_X + \lambda \tilde{D} + E_1 + \lambda a_3 E_3 .$$

By adjunction it follows that

$$2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_Q \left( \tilde{D} |_{E_1} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_1 \right) > 2 - a_3 .$$

which is not possible.

• If $Q \in E_3$ but $Q \not\in E_1 \cup E_2 \cup E_4$ then

$$K_X + \lambda \tilde{D} + \lambda a_3 E_3$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_3 , \text{ since } a_3 \leq \frac{3}{2} .$$

By adjunction $(E_3, \lambda \tilde{D} |_{E_3})$ is not log canonical at $Q$ and

$$2a_3 - \frac{a_3}{2} - \frac{a_3}{2} > 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q \left( \tilde{D} \cdot E_3 \right) > 2 ,$$

implies that $a_3 > 2$ which is false.

• If $Q \in E_3 \cap E_4$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_3 E_3 + \frac{1}{2} a_4 E_4$$

is not log canonical at the point $Q$ and so is the log pair

$$K_X + \lambda \tilde{D} + E_3 + \lambda a_4 E_4 .$$

By adjunction it follows that

$$2a_3 - \frac{a_3}{2} - \frac{a_3}{2} - a_4 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_1 \geq \text{mult}_Q \left( \tilde{D} \cdot E_1 \right) > 2 - a_4 .$$

implies that $a_3 > 2$ which is not possible.

• $Q \in E_4 \setminus (E_3 \cap E_5)$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_4 E_4$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_4 .$$

By adjunction $(E_4, \lambda \tilde{D} |_{E_4})$ is not log canonical at $Q$ and

$$2a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q \left( \tilde{D} |_{E_4} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_4 \right) > 2 ,$$

implies that $a_4 > 1$ which is false.

• $Q \in E_4 \cap E_5$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_4 E_4 + \lambda a_5 E_5$$

is not log canonical at the point $Q$ and so is the log pair

$$K_X + \lambda \tilde{D} + E_5 + \lambda a_4 E_4 .$$

By adjunction it follows that

$$2a_5 - a_4 = \tilde{D} \cdot E_5 \geq \text{mult}_Q \left( \tilde{D} |_{E_5} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_5 \right) > 2 - a_4 .$$

and we see then that $a_5 > 1$ which is not possible.
• $Q \in E_5 \setminus E_4$ then the log pair
\[ K_X + \lambda D + \lambda a_5 E_5 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_X + \lambda D + E_5 . \]
By adjunction $(E_5, \lambda D|_{E_5})$ is not log canonical at $Q$ and
\[ 2a_5 \geq 2a_5 - a_4 = \tilde{D} \cdot E_5 \geq \text{mult}_Q(\tilde{D}|_{E_5}) = \text{mult}_Q(\tilde{D} \cdot E_5) > 2 , \]
implies that $a_5 > 1$ which is false.

• If the point $Q \in F_1 \setminus F_3$ then
\[ K_X + \lambda D + \lambda b_1 F_1 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_X + \lambda D + F_1 . \]
By adjunction $(F_1, \lambda D|_{F_1})$ is not log canonical at $Q$ and
\[ 2b_1 \geq 2b_1 - b_2 = \tilde{D} \cdot F_1 \geq \text{mult}_Q(\tilde{D}|_{F_1}) = \text{mult}_Q(\tilde{D} \cdot F_1) > 2 , \]
implies that $b_1 > 1$ which is false.

• If the point $Q \in F_1 \cap F_2$ then
\[ K_X + \lambda D + \lambda b_1 F_1 + \lambda b_2 F_2 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_X + \lambda D + F_1 + \lambda b_2 F_2 . \]
By adjunction $(F_1, \lambda D|_{F_1})$ is not log canonical at $Q$ and
\[ 2b_1 - b_2 = \tilde{D} \cdot F_1 \geq \text{mult}_Q(\tilde{D}|_{F_1}) = \text{mult}_Q(\tilde{D} \cdot F_1) > 2 - b_2 , \]
implies that $b_1 > 1$ which is false.

• If the point $Q \in F_2 \setminus F_1 \setminus F_3$ then
\[ K_X + \lambda D + \lambda b_2 F_2 \]
is not log canonical at the point $Q$ and so is the pair
\[ K_X + \lambda D + F_2 . \]
By adjunction $(F_2, \lambda D|_{F_2})$ is not log canonical at $Q$ and
\[ 2b_2 - \frac{b_2}{2} - \frac{b_2}{2} \geq 2b_2 - b_1 - b_3 = \tilde{D} \cdot F_2 \geq \text{mult}_Q(\tilde{D}|_{F_2}) = \text{mult}_Q(\tilde{D} \cdot F_2) > 2 , \]
implies that $b_2 > 2$ which is false.

5.6. Del Pezzo surface of degree 1 with two $\mathbb{D}_4$ type singularities. In this section we will prove the following.

Lemma 5.6. Let $X$ be a del Pezzo surface with two Du Val singularity of type $\mathbb{D}_4$ and $K_X^2 = 1$. Then the global log canonical threshold of $X$ is
\[ \text{lct}(X) = \frac{1}{2} . \]
Proof. Suppose that \( \text{lct}(X) < \frac{1}{2} \) then there exists a \( \mathbb{Q} \)-divisor \( D \in X \), such that the log pair \((X, \lambda D)\) is not log canonical, where \( \lambda < \frac{1}{2} \) and \( D \sim_{\mathbb{Q}} -K_X \). We derive that the pair \((X, \lambda D)\) is log canonical outside of a point \( P \in X \) and not log canonical at \( P \). Let \( \pi : \tilde{X} \to X \) be the minimal resolution of \( X \). The configuration of the exceptional curves is given by the following Dynkin diagram.

\[
\begin{array}{ccc}
\mathbb{D}_4 + \mathbb{D}_4. & & \\
\bullet E_1 & \bullet E_3 & \bullet E_4 & \bullet F_1 & \bullet F_3 & \bullet F_4 \\
\bullet E_2 & \bullet F_2 & \\
\end{array}
\]

Then

\[\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - b_1 F_1 - b_2 F_2 - b_3 F_3 - b_4 F_4.\]

We should note here that there are four -1 curves \( \tilde{L}_1, \tilde{L}_2, \tilde{L}_4, \tilde{L}_3 \) such that

\[\tilde{L}_1 \cdot E_1 = \tilde{L}_1 \cdot F_1 = \tilde{L}_2 \cdot E_2 = \tilde{L}_2 \cdot F_2 = \tilde{L}_4 \cdot E_4 = \tilde{L}_4 \cdot F_4 = \tilde{L}_3 \cdot E_3 = 1.\]

Therefore we have

\[
\begin{align*}
\tilde{L}_1 & \sim_{\mathbb{Q}} \pi^*(L_1) - E_1 - \frac{1}{2} E_2 - E_3 - \frac{1}{2} E_4 - F_1 - \frac{1}{2} F_2 - F_3 - \frac{1}{2} F_4 \\
\tilde{L}_2 & \sim_{\mathbb{Q}} \pi^*(L_2) - \frac{1}{2} E_1 - E_2 - E_3 - \frac{1}{2} E_4 - \frac{1}{2} F_1 - F_2 - F_3 - \frac{1}{2} F_4 \\
\tilde{L}_4 & \sim_{\mathbb{Q}} \pi^*(L_4) - \frac{1}{2} E_1 - \frac{1}{2} E_2 - E_3 - E_4 - \frac{1}{2} F_1 - \frac{1}{2} F_2 - F_3 - F_4 \\
\tilde{L}_3 & \sim_{\mathbb{Q}} \pi^*(L_3) - E_1 - E_2 - 2 E_3 - E_4.
\end{align*}
\]

and since \( L_1 \sim_{\mathbb{Q}} L_2 \sim_{\mathbb{Q}} L_3 \sim_{\mathbb{Q}} L_4 \sim_{\mathbb{Q}} -K_X \) we see that \( \text{lct}(X) \leq \frac{1}{2} \).

The inequalities

\[
\begin{align*}
0 \leq \tilde{D} \cdot \tilde{L}_1 &= 1 - a_1 - b_1 \\
0 \leq \tilde{D} \cdot \tilde{L}_2 &= 1 - a_2 - b_2 \\
0 \leq \tilde{D} \cdot \tilde{L}_4 &= 1 - a_4 - b_4 \\
0 \leq \tilde{D} \cdot \tilde{L}_3 &= 1 - a_3 \\
0 \leq E_1 \cdot \tilde{D} &= 2 a_1 - a_3 \\
0 \leq E_2 \cdot \tilde{D} &= 2 a_2 - a_3 \\
0 \leq E_3 \cdot \tilde{D} &= 2 a_3 - a_1 - a_2 - a_4 \\
0 \leq E_4 \cdot \tilde{D} &= 2 a_4 - a_3 \\
0 \leq F_1 \cdot \tilde{D} &= 2 b_1 - b_3 \\
0 \leq F_2 \cdot \tilde{D} &= 2 b_2 - b_3 \\
0 \leq F_3 \cdot \tilde{D} &= 2 b_3 - b_1 - b_2 - b_4 \\
0 \leq F_4 \cdot \tilde{D} &= 2 b_4 - b_3
\end{align*}
\]

imply that \( a_1 \leq 1, a_2 \leq 1, a_3 \leq 1, a_4 \leq 1, b_1 \leq 1, b_2 \leq 1, b_3 \leq \frac{3}{2}, b_4 \leq 1 \). The equivalence

\[K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda b_1 F_1 + \lambda b_2 F_2 + \lambda b_3 F_3 + \lambda b_4 F_4 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D)\]

implies that there is a point \( Q \) such that \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \) or \( Q \in F_1 \cup F_2 \cup F_3 \cup F_4 \) where the pair

\[K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda b_1 F_1 + \lambda b_2 F_2 + \lambda b_3 F_3 + \lambda b_4 F_4\]

is not log canonical at \( Q \).

- If the point \( Q \in E_1 \) and \( Q \not\in E_3 \) then

\[K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1\]

is not log canonical at the point \( Q \) and so is the pair

\[K_{\tilde{X}} + \lambda \tilde{D} + E_1.\]
By adjunction \((E_1, \lambda \tilde{D}|_{E_1})\) is not log canonical at \(Q\) and
\[
2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > 2 ,
\]
implies that \(a_1 > 1\) which is false.
- If \(Q \in E_3\) but \(Q \not\in E_1 \cup E_2 \cup E_4\) then
\[
K_X + \lambda \tilde{D} + \lambda a_3 E_3
\]
is not log canonical at the point \(Q\) and so is the pair
\[
K_X + \lambda \tilde{D} + E_3, \text{ since } \lambda a_3 \leq 1 .
\]
By adjunction \((E_4, \lambda \tilde{D}|_{E_4})\) is not log canonical at \(Q\) and
\[
2a_3 - \frac{a_3}{2} - \frac{a_3}{2} \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D} \cdot E_3) > 2 ,
\]
implies that \(a_3 > 2\) which is false.
- If \(Q \in E_1 \cap E_3\) then the log pair
\[
K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_3 E_3
\]
is not log canonical at the point \(Q\) and so is the log pair
\[
K_X + \lambda \tilde{D} + E_1 + \lambda a_3 E_3 .
\]
By adjunction it follows that
\[
2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > 2 - a_3 .
\]
and we see then that \(a_1 > 1\) which is not possible.

\[\square\]

5.7. Del Pezzo surface of degree 1 with an \(\mathbb{A}_5\), an \(\mathbb{A}_2\) and an \(\mathbb{A}_1\) type singularity. In this section we will prove the following.

**Lemma 5.7.** Let \(X\) be a del Pezzo surface with an \(\mathbb{A}_5\), an \(\mathbb{A}_2\) and an \(\mathbb{A}_1\) type singularity such that \(K_X^2 = 1\). Then the global log canonical threshold of \(X\) is
\[
\text{let}(X) = \frac{2}{3} .
\]

**Proof.** Suppose that \(\text{let}(X) < \frac{2}{3}\), then there exists a \(\mathbb{Q}\)-divisor \(D \in X\) such that the log pair \((X, \frac{2}{3}D)\) is not log canonical, where \(\lambda < \frac{2}{3}\) and \(D \sim_{\mathbb{Q}} -K_X\). We derive that the pair \((X, \lambda D)\) is log canonical outside of a point \(P \in X\) and not log canonical at \(P\). Let \(\pi : \tilde{X} \to X\) be the minimal resolution of \(X\). The configuration of the exceptional curves is given by the following Dynkin diagram.

\[\mathbb{A}_5 + \mathbb{A}_2 + \mathbb{A}_1. \quad \bullet E_1 \quad \bullet E_2 \quad \bullet E_3 \quad \bullet E_4 \quad \bullet E_5 \quad \bullet F_1 \quad \bullet F_2 \quad \bullet G_1\]

Then
\[
\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - b_1 F_1 - b_2 F_2 - c_1 G_1 .
\]
We should note here that there are three -1 curves \(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3\) such that
\[
L_1 \cdot E_1 = L_1 \cdot E_5 = L_2 \cdot E_2 = L_2 \cdot F_1 = L_3 \cdot E_3 = L_3 \cdot G_1 = 1 .
\]
Therefore we have
\[
\tilde{L}_1 \sim_{\mathbb{Q}} \pi^*(L_1) - E_1 - E_2 - E_3 - E_4 - E_5 \]
\[
\tilde{L}_2 \sim_{\mathbb{Q}} \pi^*(L_2) - \frac{2}{3} E_1 - \frac{4}{3} E_2 - \frac{2}{3} E_3 - \frac{2}{3} E_4 - \frac{1}{3} E_5 - \frac{1}{3} F_1 - \frac{1}{3} F_2 \]
\[
\tilde{L}_3 \sim_{\mathbb{Q}} \pi^*(L_3) - \frac{1}{2} E_1 - \frac{3}{2} E_2 - \frac{3}{2} E_3 - \frac{4}{3} E_4 - \frac{1}{2} E_5 - \frac{1}{2} G_1 .
\]
and since $L_1 \sim_Q L_2 \sim_Q L_3 \sim_Q -K_X$ we see that $\operatorname{lc}(X) \leq \frac{2}{3}$. Moreover there are two -1 curves $\tilde{L}_4$ and $\tilde{L}_5$ that intersect the fundamental cycle as following

$$\tilde{L}_4 \cdot F_1 = \tilde{L}_4 \cdot F_2 = 1 \text{ and } \tilde{L}_4 \cdot G_1 = \tilde{L}_4 \cdot E_i = 0 \text{ for all } i = 1, \ldots, 5$$

and

$$\tilde{L}_5 \cdot G_1 = 2 \text{ and } \tilde{L}_5 \cdot E_i = \tilde{L}_5 \cdot F_j = 0 \text{ for all } i = 1, \ldots, 5 \text{ and } j = 1, 2 .$$

The inequalities

$$0 \leq \tilde{D} \cdot \tilde{L}_1 = 1 - a_1 - a_5$$
$$0 \leq \tilde{D} \cdot \tilde{L}_2 = 1 - a_2 - b_1$$
$$0 \leq \tilde{D} \cdot \tilde{L}_3 = 1 - a_3 - c_1$$
$$0 \leq \tilde{D} \cdot \tilde{L}_4 = 1 - b_1 - b_2$$
$$0 \leq \tilde{D} \cdot \tilde{L}_5 = 1 - 2c_1$$
$$0 \leq E_1 \cdot \tilde{D} = 2a_1 - a_2$$
$$0 \leq E_2 \cdot \tilde{D} = 2a_2 - a_1 - a_3$$
$$0 \leq E_3 \cdot \tilde{D} = 2a_3 - a_2 - a_4$$
$$0 \leq E_4 \cdot \tilde{D} = 2a_4 - a_3 - a_5$$
$$0 \leq E_5 \cdot \tilde{D} = 2a_5 - a_4$$
$$0 \leq F_1 \cdot \tilde{D} = 2b_1 - b_2$$
$$0 \leq F_2 \cdot \tilde{D} = 2b_2 - b_1$$
$$0 \leq G_1 \cdot \tilde{D} = 2c_1$$

imply that

$$a_1 \leq \frac{5}{6}, a_2 \leq 1, a_3 \leq 1, a_4 \leq \frac{4}{3}, a_5 \leq \frac{5}{6}, b_1 \leq \frac{2}{3}, b_2 \leq \frac{2}{3}, c_1 \leq \frac{1}{2} ,$$

and what is more

$$2a_5 \geq a_4, \frac{3}{2} a_4 \geq a_3, \frac{4}{3} a_3 \geq a_2, \frac{5}{4} a_2 \geq a_1 .$$

The equivalence

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda b_1 F_1 + \lambda b_2 F_2 + \lambda c_1 G_1 \sim_Q \pi^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup F_1 \cup F_2 \cup G_1$ such that the pair

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda b_1 F_1 + \lambda b_2 F_2 + \lambda c_1 G_1$$

is not log canonical at $Q$.

- If the point $Q \in E_1$ and $Q \notin E_2$ then

$$K_{\tilde{X}} + \lambda \tilde{D} + a_1 \lambda E_1$$

is not log canonical at the point $Q$ and so is the pair

$$K_{\tilde{X}} + \lambda \tilde{D} + E_1$$

since $a_1 \lambda \leq 1$.

By adjunction $(E_1, \lambda \tilde{D}|_E_1)$ is not log canonical at $Q$ and

$$2a_1 - \frac{4}{5} a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \operatorname{mult}_Q(\tilde{D} \cdot E_1) \geq \frac{1}{X} > \frac{3}{2},$$

implies that $a_1 > \frac{5}{4}$ which is a contradiction.

- If $Q \in E_1 \cap E_2$ then the log pair

$$K_{\tilde{X}} + \lambda \tilde{D} + a_1 \lambda E_1 + a_2 \lambda E_2$$

is not log canonical at the point $Q$ and so are the log pairs

$$K_{\tilde{X}} + \lambda \tilde{D} + E_1 + a_2 \lambda E_2$$
and $$K_{\tilde{X}} + \lambda \tilde{D} + a_1 \lambda E_1 + E_2 .$$

By adjunction it follows that

$$2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \operatorname{mult}_Q(\tilde{D}|_{E_2}) = \operatorname{mult}_Q(\tilde{D} \cdot E_2) > \frac{1}{X} - a_1 > \frac{3}{2} - a_1 ,$$
\[2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \mult_Q (\tilde{D}|_{E_1}) = \mult_Q (\tilde{D} \cdot E_1) > \frac{1}{\lambda} - a_2 > \frac{3}{2} - a_2.\]

From the first inequality we get \(a_3 \geq \frac{9}{10}\) and then we see that
\[1 \geq a_1 + a_5 \geq a_1 + \frac{1}{2} \cdot \frac{2}{3}a_3 > \frac{3}{4} + \frac{3}{10} > 1,
\]
which is a contradiction.

- If \(Q \in E_2\) but \(Q \notin E_1 \cup E_3\) then
  \[K_X + \lambda \tilde{D} + a_2 \lambda E_2\]
is not log canonical at the point \(Q\) and so is the pair
\[K_X + \lambda \tilde{D} + E_2, \text{ since } a_2 \lambda \leq 1.\]

By adjunction \((E_2, \lambda \tilde{D}|_{E_2})\) is not log canonical at \(Q\) and
\[2a_2 - \frac{3}{4}a - \frac{a_2}{2} \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \mult_Q (\tilde{D}|_{E_2}) = \mult_Q (\tilde{D} \cdot E_2) > \frac{1}{\lambda} > \frac{3}{2}.\]
This implies that \(a_2 > 2\) which is a contradiction.

- If \(Q \in E_2 \cap E_3\) then the log pair
  \[K_X + \lambda \tilde{D} + a_2 \lambda E_2 + a_3 \lambda E_3\]
is not log canonical at the point \(Q\) and so are the log pairs
\[K_X + \lambda \tilde{D} + a_2 \lambda E_2 + E_3, \text{ since } a_3 \lambda \leq 1.\]

By adjunction it follows that
\[2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mult_Q (\tilde{D} \cdot E_3) > \frac{1}{\lambda} - a_2 > \frac{3}{2} - a_2.\]
which, together with the inequality \(a_4 \geq \frac{2}{3}a_3\), implies that \(a_3 > \frac{9}{8}\). However, this is impossible since \(a_3 \leq 1\).

- If \(Q \in E_3\) but \(Q \notin E_2 \cup E_4\) then
  \[K_X + \lambda \tilde{D} + a_3 \lambda E_3\]
is not log canonical at the point \(Q\) and so is the pair
\[K_X + \lambda \tilde{D} + E_3, \text{ since } a_3 \lambda \leq 1.\]

By adjunction \((E_3, \lambda \tilde{D}|_{E_3})\) is not log canonical at \(Q\) and
\[2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \mult_Q (\tilde{D}|_{E_3}) = \mult_Q (\tilde{D} \cdot E_3) > \frac{1}{\lambda} > \frac{3}{2},\]
which is false as we saw in the previous case.

- If the point \(Q \in F_1\) and \(Q \notin F_2\) then
  \[K_X + \lambda \tilde{D} + b_1 \lambda F_1\]
is not log canonical at the point \(Q\) and so is the pair
\[K_X + \lambda \tilde{D} + F_1, \text{ since } b_1 \lambda \leq 1.\]

By adjunction \((F_1, \lambda \tilde{D}|_{F_1})\) is not log canonical at \(Q\) and
\[2b_1 - \frac{b_1}{2} \geq 2b_1 - b_2 = \tilde{D} \cdot F_1 \geq \mult_Q (\tilde{D} \cdot F_1) > \frac{1}{\lambda} > \frac{3}{2},\]
implies that \(b_1 > 1\) which is a contradiction.
• If \( Q \in F_1 \cap F_2 \) then the log pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + b_1 \lambda F_1 + b_2 \lambda F_2
\]
is not log canonical at the point \( Q \) and so are the log pairs
\[
K_{\tilde{X}} + \lambda \tilde{D} + b_1 \lambda F_1 + F_2
\]
and
\[
K_{\tilde{X}} + \lambda \tilde{D} + F_1 + b_2 \lambda F_2.
\]
By adjunction it follows that
\[
2b_1 - b_2 = \tilde{D} \cdot F_1 \geq \text{mult}_Q (\tilde{D}|_{F_1}) = \text{mult}_Q (\tilde{D} \cdot F_1) > \frac{1}{\lambda} - b_2 > \frac{3}{2} - b_2
\]
and
\[
2b_2 - b_1 = \tilde{D} \cdot F_2 \geq \text{mult}_Q (\tilde{D}|_{F_2}) = \text{mult}_Q (\tilde{D} \cdot F_2) > \frac{1}{\lambda} - b_1 > \frac{3}{2} - b_1.
\]
This implies that
\[
1 \geq b_1 + b_2 > \frac{3}{4} + \frac{3}{4}
\]
which is a contradiction.

• If the point \( Q \in G_1 \) then
\[
K_{\tilde{X}} + \lambda \tilde{D} + c_1 \lambda G_1
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + G_1, \text{ since } c_1 \lambda \leq 1.
\]
By adjunction \((G_1, \lambda \tilde{D}|_{G_1})\) is not log canonical at \( Q \) and
\[
1 \geq 2c_1 = \tilde{D} \cdot G_1 \geq \text{mult}_Q (\tilde{D} \cdot G_1) > \frac{1}{\lambda} > \frac{3}{2},
\]
which is a contradiction.

\[\square\]

5.8. Del Pezzo surface of degree 2 with an \( E_7 \) type singularity. In this section we will prove the following.

**Lemma 5.8.** Let \( X \) be a del Pezzo surface with one Du Val singularity of type \( E_7 \) and \( K_X^2 = 2 \). Then the global log canonical threshold of \( X \) is
\[
\text{lct}(X) = \frac{1}{6}.
\]

**Proof.** Suppose that \( \text{lct}(X) < \frac{1}{6} \), then there exists an effective \( \mathbb{Q} \)-divisor \( D \sim_{\mathbb{Q}} -K_X \) such that the log pair \((X, \lambda D)\) is not log canonical, where \( \lambda < \frac{1}{6} \). We derive that the pair \((X, \lambda D)\) is log canonical everywhere except for a point \( P \in X \) at which it is not log canonical. Let \( \pi : \tilde{X} \to X \) be the minimal resolution of \( X \). The configuration of the exceptional curves is given by the following Dynkin diagram.

![Dynkin diagram](image)

Then
\[
\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - a_6 E_6 - a_7 E_7.
\]
By the way we obtain \( \tilde{X} \) as the blow up of \( \mathbb{P}^2 \) at seven points we can see that there is a \(-1\) curve \( \tilde{L} \) that intersects the exceptional divisor \( E_7 \). In fact we have
\[
\tilde{L} \sim_{\mathbb{Q}} \pi^*(L) - E_1 - 2E_2 - 3E_3 - \frac{3}{2}E_4 - \frac{5}{2}E_5 - 2E_6 - \frac{3}{2}E_7,
\]
and since \( 2L \in | -K_X | \) we get that \( \text{lct}(X) \leq \frac{1}{6} \).
The inequalities

\[
0 \leq \hat{D} \cdot \hat{L} = 1 - a_7 \\
0 \leq E_1 \cdot \hat{D} = 2a_1 - 2a_2 \\
0 \leq E_2 \cdot \hat{D} = 2a_2 - a_1 - a_3 \\
0 \leq E_3 \cdot \hat{D} = 2a_3 - a_2 - a_5 - a_4 \\
0 \leq E_4 \cdot \hat{D} = 2a_4 - a_3 \\
0 \leq E_5 \cdot \hat{D} = 2a_5 - a_3 - a_6 \\
0 \leq E_6 \cdot \hat{D} = 2a_6 - a_5 - a_7 \\
0 \leq E_7 \cdot \hat{D} = 2a_7 - a_6
\]

imply that \(a_1 \leq 2, a_2 \leq 3, a_3 \leq 4, a_4 \leq \frac{7}{3}, a_5 \leq 3, a_6 \leq 2, a_7 \leq 1\). The equivalence

\[
K_{\hat{X}} + \lambda \hat{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 \sim_Q \pi^*(K_X + \lambda D)
\]

implies that there is a point \(Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7\) such that the pair

\[
K_{\hat{X}} + \lambda \hat{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7
\]

is not log canonical at \(Q\).

- If the point \(Q \in E_1\) and \(Q \notin E_2\) then

  \[
  K_{\hat{X}} + \lambda \hat{D} + \lambda a_1 E_1
  \]

  is not log canonical at the point \(Q\) and so is the pair

  \[
  K_{\hat{X}} + \lambda \hat{D} + E_1 \text{ since } \lambda a_1 \leq 1.
  \]

  By adjunction \((E_1, \lambda \hat{D}|_{E_1})\) is not log canonical at \(Q\) and

  \[
  2a_1 - \frac{3}{2} a_1 \geq 2a_1 - a_2 = \hat{D} \cdot E_1 \geq \text{mult}_Q \left( \hat{D} \cdot E_1 \right) > 6,
  \]

  which is false.

- If the point \(Q \in E_1 \cap E_2\) then

  \[
  K_{\hat{X}} + \lambda \hat{D} + \lambda a_1 E_1 + \lambda a_2 E_2
  \]

  is not log canonical at the point \(Q\) and so are the pairs

  \[
  K_{\hat{X}} + \lambda \hat{D} + E_1 + \lambda a_2 E_2 \text{ and } K_{\hat{X}} + \lambda \hat{D} + \lambda a_1 E_1 + E_2.
  \]

  By adjunction

  \[
  2a_1 - a_2 = \hat{D} \cdot E_1 \geq \text{mult}_Q \left( \hat{D}|_{E_1} \right) = \text{mult}_Q \left( \hat{D} \cdot E_1 \right) > 6 - a_2
  \]

  and

  \[
  2a_2 - a_1 - a_3 = \hat{D} \cdot E_2 \geq \text{mult}_Q \left( \hat{D}|_{E_2} \right) = \text{mult}_Q \left( \hat{D} \cdot E_2 \right) > 6 - a_1,
  \]

  which is false.

- If the point \(Q \in E_2 \setminus (E_1 \cup E_3)\) then

  \[
  K_{\hat{X}} + \lambda \hat{D} + \lambda a_2 E_2
  \]

  is not log canonical at the point \(Q\) and so is the pair

  \[
  K_{\hat{X}} + \lambda \hat{D} + E_2.
  \]

  By adjunction

  \[
  2a_2 - a_1 - a_3 = \hat{D} \cdot E_2 \geq \text{mult}_Q \left( \hat{D}|_{E_2} \right) = \text{mult}_Q \left( \hat{D} \cdot E_2 \right) > 6,
  \]

  which is false.
• If the point \( Q \in E_2 \cap E_3 \) then
\[
K_X + \lambda \tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3
\]
is not log canonical at the point \( Q \) and so are the pairs
\[
K_X + \lambda \tilde{D} + E_2 + \lambda a_3 E_3 \quad \text{and} \quad K_X + \lambda \tilde{D} + \lambda a_2 E_2 + E_3 .
\]
By adjunction
\[
2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D}|_{E_2}) = \text{mult}_Q(\tilde{D} \cdot E_2) > 6 - a_3 \quad \text{and}
\]
and
\[
2a_3 - a_2 - a_5 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 6 - a_2 ,
\]
which is false.

• If the point \( Q \in E_3 \setminus (E_2 \cup E_4 \cup E_5) \) then
\[
K_X + \lambda \tilde{D} + \lambda a_3 E_3
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_3 .
\]
By adjunction \((E_3, \lambda \tilde{D}|_{E_3})\) is not log canonical at \( Q \) and
\[
2a_3 - a_2 - a_4 - a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 6 ,
\]
which is false.

• If the point \( Q \in E_3 \cap E_4 \) then
\[
K_X + \lambda \tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so are the pairs
\[
K_X + \lambda \tilde{D} + E_3 + \lambda a_4 E_4 \quad \text{and} \quad K_X + \lambda \tilde{D} + \lambda a_3 E_3 + E_4 .
\]
By adjunction
\[
2a_3 - a_2 - a_4 - a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 6 - a_4 \quad \text{and}
\]
and
\[
2a_4 - a_3 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D}|_{E_4}) = \text{mult}_Q(\tilde{D} \cdot E_4) > 6 - a_3 ,
\]
which is false.

• If the point \( Q \in E_4 \setminus E_3 \) then
\[
K_X + \lambda \tilde{D} + \lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_4 .
\]
By adjunction \((E_4, \lambda \tilde{D}|_{E_4})\) is not log canonical at \( Q \) and
\[
2a_4 - a_3 = \tilde{D} \cdot E_4 \geq \text{mult}_Q(\tilde{D}|_{E_4}) = \text{mult}_Q(\tilde{D} \cdot E_4) > 6 ,
\]
which is false.

• If the point \( Q \in E_3 \cap E_5 \) then
\[
K_X + \lambda \tilde{D} + \lambda a_3 E_3 + \lambda a_5 E_5
\]
is not log canonical at the point \( Q \) and so are the pairs
\[
K_X + \lambda \tilde{D} + E_3 + \lambda a_5 E_5 \quad \text{and} \quad K_X + \lambda \tilde{D} + \lambda a_3 E_3 + E_5 .
\]
By adjunction
\[
2a_3 - a_2 - a_4 - a_5 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 6 - a_5 \quad \text{and}
\]
and
\[ 2a_5 - a_3 - a_6 = \bar{D} \cdot E_5 \geq \text{mult}_Q(\bar{D}|_{E_5}) = \text{mult}_Q(\bar{D} \cdot E_5) > 6 - a_3 , \]
which is false.

- If the point \( Q \in E_5 \setminus (E_3 \cup E_6) \) then
  \[ K_X + \lambda \bar{D} + \lambda a_5 E_5 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_X + \lambda \bar{D} + E_5 . \]
By adjunction \((E_5, \lambda \bar{D}|_{E_5})\) is not log canonical at \( Q \) and
\[ 2a_5 - a_3 - a_6 = \bar{D} \cdot E_5 \geq \text{mult}_Q(\bar{D}|_{E_5}) = \text{mult}_Q(\bar{D} \cdot E_5) > 6 , \]
which is false.

- If the point \( Q \in E_5 \cap E_6 \) then
  \[ K_X + \lambda \bar{D} + \lambda a_5 E_5 + \lambda a_6 E_6 \]
is not log canonical at the point \( Q \) and so are the pairs
  \[ K_X + \lambda \bar{D} + E_5 + \lambda a_6 E_6 \text{ and } K_X + \lambda \bar{D} + \lambda a_5 E_5 + E_6 . \]
By adjunction
\[ 2a_5 - a_3 - a_6 = \bar{D} \cdot E_5 \geq \text{mult}_Q(\bar{D}|_{E_5}) = \text{mult}_Q(\bar{D} \cdot E_5) > 6 - a_6 \text{ and }\]
and
\[ 2a_6 - a_5 - a_7 = \bar{D} \cdot E_6 \geq \text{mult}_Q(\bar{D}|_{E_6}) = \text{mult}_Q(\bar{D} \cdot E_6) > 6 - a_5 , \]
which is false.

- If the point \( Q \in E_6 \setminus (E_5 \cup E_7) \) then
  \[ K_X + \lambda \bar{D} + \lambda a_6 E_6 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_X + \lambda \bar{D} + E_6 . \]
By adjunction \((E_6, \bar{D}|_{E_6})\) is not log canonical at \( Q \) and
\[ 2a_6 - a_5 - a_7 = \bar{D} \cdot E_6 \geq \text{mult}_Q(\bar{D}|_{E_6}) = \text{mult}_Q(\bar{D} \cdot E_6) > 6 , \]
which is false.

- If the point \( Q \in E_6 \cap E_7 \) then
  \[ K_X + \lambda \bar{D} + \lambda a_6 E_6 + \lambda a_7 E_7 \]
is not log canonical at the point \( Q \) and so are the pairs
  \[ K_X + \lambda \bar{D} + E_6 + \lambda a_7 E_7 \text{ and } K_X + \lambda \bar{D} + \lambda a_6 E_6 + E_7 . \]
By adjunction
\[ 2a_6 - a_5 - a_7 = \bar{D} \cdot E_6 \geq \text{mult}_Q(\bar{D}|_{E_6}) = \text{mult}_Q(\bar{D} \cdot E_6) > 6 - a_7 \text{ and }\]
and
\[ 2a_7 - a_6 = \bar{D} \cdot E_7 \geq \text{mult}_Q(\bar{D}|_{E_7}) = \text{mult}_Q(\bar{D} \cdot E_7) > 6 - a_6 , \]
which is false.
• If the point $Q \in E_7 \backslash E_6$ then

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_7 E_7$$

is not log canonical at the point $Q$ and so is the pair

$$K_{\tilde{X}} + \lambda \tilde{D} + E_7.$$ 

By adjunction $(E_7, \lambda \tilde{D}|_{E_7})$ is not log canonical at $Q$ and

$$2a_7 - a_6 = \tilde{D} \cdot E_7 \geq \text{mult}_Q (\tilde{D}|_{E_7}) = \text{mult}_Q (\tilde{D} \cdot E_7) > 6,$$

which is false.

5.9. Del Pezzo surfaces of degree 2 with one $\mathbb{D}_6$ and one $A_1$ singularity. In this section we will prove the following.

**Lemma 5.9.** Let $X$ be a del Pezzo surface with one Du Val singularity of type $\mathbb{D}_6$, one of type $A_1$ and $K^2_X = 2$. Then the global log canonical threshold of $X$ is

$$\text{lct}(X) = \frac{1}{4}.$$ 

**Proof.** Suppose that $\text{lct}(X) < \frac{1}{4}$, then there exists a $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$, such that the log pair $(X, \lambda D)$ is not log canonical for some rational number $\lambda < \frac{1}{4}$. We derive that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at $P$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of $X$. The configuration of the exceptional curves is given by the following Dynkin diagram.

$$\mathbb{D}_6 + A_1. \quad \bullet E_1 \quad \bullet E_3 \quad \bullet E_4 \quad \bullet E_5 \quad \bullet E_6 \quad \bullet F_1 \quad \bullet E_2$$

Then

$$\tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - 2a_3 E_3 - 2a_4 E_4 - 2a_5 E_5 - a_6 E_6 - b_1 F_1.$$ 

From the way we blow up $\mathbb{P}^2$ to obtain $\tilde{X}$ we can see that there exist -1 curves $\tilde{L}_1, \tilde{L}_6$ such that

$$\tilde{L}_1 \cdot E_1 = \tilde{L}_6 \cdot E_6 = \tilde{L}_6 \cdot F_1 = 1$$

and therefore

$$\tilde{L}_1 \sim_{\mathbb{Q}} \pi^*(L_1) - \frac{3}{2} E_1 - E_2 - 2E_3 - \frac{3}{2} E_4 - E_5 - \frac{1}{2} E_6$$

and

$$\tilde{L}_6 \sim_{\mathbb{Q}} \pi^*(L_6) - \frac{1}{2} E_1 - \frac{1}{2} E_2 - E_3 - E_4 - E_5 - E_6 - \frac{1}{2} F_1.$$ 

Since $2L_1 \sim_{\mathbb{Q}} 2L_6 \sim_{\mathbb{Q}} -K_X$ we get that $\text{lct}(X) \leq \frac{1}{4}$. From the inequalities

$$0 \leq \tilde{D} \cdot \tilde{L}_1 = 1 - a_1$$

$$0 \leq \tilde{D} \cdot \tilde{L}_6 = 1 - 2a_5 - b_1$$

$$0 \leq E_1 \cdot \tilde{D} = 2a_1 - 2a_3$$

$$0 \leq E_2 \cdot \tilde{D} = 2a_2 - 2a_3$$

$$0 \leq E_3 \cdot \tilde{D} = 4a_3 - a_1 - a_2 - 2a_4$$

$$0 \leq E_4 \cdot \tilde{D} = 4a_4 - 2a_3 - 2a_5$$

$$0 \leq E_5 \cdot \tilde{D} = 4a_5 - 2a_4 - a_6$$

$$0 \leq E_6 \cdot \tilde{D} = 2a_6 - 2a_5$$

$$0 \leq F_1 \cdot \tilde{D} = 2b_1$$

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we see that

\[ a_3 \leq a_1, \, a_3 \leq a_2, \, a_4 \leq a_3, \, a_5 \leq a_4, \, a_6 \leq 2a_5 \]

and

\[ a_5 \leq a_6, \, a_4 \leq \frac{3}{2} a_5, \, a_3 \leq \frac{4}{3} a_4, \, a_1 \leq \frac{3}{2} a_3, \, a_2 \leq \frac{3}{2} a_3, \, a_1 + a_2 \leq \frac{5}{2} a_3 \]

In particular we get the following upper bounds

\[ a_1 \leq 1, \, a_2 \leq 1, \, a_3 \leq \frac{2}{3}, \, a_4 \leq \frac{3}{4}, \, a_5 \leq \frac{1}{2}, \, a_6 \leq 1, \, b_1 \leq 1. \]

The equivalence

\[ K_X + \lambda \lambda + \lambda a_1 E_1 + \lambda a_2 E_2 + 2\lambda a_3 E_3 + 2\lambda a_4 E_4 + 2\lambda a_5 E_5 + \lambda a_6 E_6 \sim_\lambda K \pi (K_X + \lambda D) \]

implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \) such that the pair

\[ K_X + \lambda \lambda + \lambda a_1 E_1 + \lambda a_2 E_2 + 2\lambda a_3 E_3 + 2\lambda a_4 E_4 + 2\lambda a_5 E_5 + \lambda a_6 E_6 \]

is not log canonical at \( Q \).

- If the point \( Q \in E_1 \setminus E_3 \) then

\[ K_X + \lambda \lambda + \lambda a_1 E_1 \]

is not log canonical at the point \( Q \) and so is the pair

\[ K_X + \lambda \lambda + E_1, \text{ since } \lambda a_1 \leq 1. \]

By adjunction \((E_1, \lambda \lambda |E_1)\) is not log canonical at \( Q \) and

\[ 2a_1 - \frac{4}{3} a_1 \geq 2a_1 - 2a_4 = \lambda \lambda \cdot E_1 \geq \text{mult}_Q \left( \lambda \lambda |E_1 \right) = \text{mult}_Q \left( \lambda \lambda \cdot E_1 \right) > 4, \]

implies that \( a_1 \geq 6 \) which is false.

- If \( Q \in E_3 \) but \( Q \not\in E_1 \cup E_2 \cup E_4 \) then

\[ K_X + \lambda \lambda + 2\lambda a_3 E_3 \]

is not log canonical at the point \( Q \) and so is the pair

\[ K_X + \lambda \lambda + E_3, \text{ since } 2\lambda a_3 \leq 1. \]

By adjunction \((E_3, \lambda \lambda |E_3)\) is not log canonical at \( Q \) and

\[ 4a_3 - a_3 - a_3 - \frac{3}{2} a_3 \geq 4a_3 - a_1 - a_2 - 2a_4 = \lambda \lambda \cdot E_3 \geq \text{mult}_Q \left( \lambda \lambda |E_3 \right) = \text{mult}_Q \left( \lambda \lambda \cdot E_3 \right) > 4, \]

implies that \( a_3 \geq 8 \) which is false.

- If \( Q \in E_1 \cap E_3 \) then the log pair

\[ K_X + \lambda \lambda + \lambda a_1 E_1 + 2\lambda a_3 E_3 \]

is not log canonical at the point \( Q \) and so is the log pair

\[ K_X + \lambda \lambda + \lambda a_1 E_1 + E_3. \]

By adjunction it follows that

\[ \frac{16}{3} a_4 - a_4 - 2a_4 - a_1 \geq 4a_3 - a_1 - a_2 - 2a_4 = \lambda \lambda \cdot E_3 \geq \text{mult}_Q \left( \lambda \lambda |E_3 \right) = \text{mult}_Q \left( \lambda \lambda \cdot E_3 \right) > 4 - a_1 \]

and this implies that \( a_4 > \frac{12}{7} \) which is false.

- If \( Q \in E_3 \cap E_4 \) then the log pair

\[ K_X + \lambda \lambda + 2\lambda a_3 E_3 + 2\lambda a_4 E_4 \]

is not log canonical at the point \( Q \) and so is the log pair

\[ K_X + \lambda \lambda + 2\lambda a_3 E_3 + E_4, \text{ since } 2\lambda a_4 \leq 1. \]

By adjunction

\[ 6a_5 - 2a_5 - 2a_3 \geq 4a_4 - 2a_3 - 2a_5 = \lambda \lambda \cdot E_4 \geq \text{mult}_Q \left( \lambda \lambda |E_4 \right) = \text{mult}_Q \left( \lambda \lambda \cdot E_4 \right) > 4 - 2a_3, \]

and this implies that \( a_5 > 1 \) which is false.
• If the point \( Q \in E_4 \setminus (E_3 \cup E_5) \) then
\[
K_X + \lambda \tilde{D} + 2\lambda a_4 E_4
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_4 .
\]
By adjunction \((E_4, \lambda \tilde{D}|_{E_4})\) is not log canonical at \( Q \) and
\[
4a_4 - 2a_4 - \frac{4}{3} a_4 \geq 4a_4 - 2a_3 - 2a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q \left( \tilde{D}|_{E_4} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_4 \right) > 4 ,
\]
and this implies that \( a_4 > 6 \) which is false.
• \( Q \in E_5 \setminus (E_4 \cup E_6) \) then the log pair
\[
K_X + \lambda \tilde{D} + 2\lambda a_5 E_5
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_5 , \text{ since } 2\lambda a_5 \leq 1 .
\]
By adjunction \((E_5, \lambda \tilde{D}|_{E_5})\) is not log canonical at \( Q \) and
\[
4a_5 - 2a_5 - a_5 \geq 4a_5 - 2a_4 - a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q \left( \tilde{D}|_{E_5} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_5 \right) > 4 ,
\]
and this implies that \( a_5 > 4 \) which is false.
• If \( Q \in E_4 \cap E_5 \) then the log pair
\[
K_X + \lambda \tilde{D} + 2\lambda a_4 E_4 + 2 \frac{1}{4} a_5 E_5
\]
is not log canonical at the point \( Q \) and so is the log pair
\[
K_X + \lambda \tilde{D} + 2\lambda a_4 E_4 + E_5 .
\]
By adjunction
\[
4a_5 - a_5 - 2a_4 \geq 4a_5 - 2a_4 - a_6 = \tilde{D} \cdot E_5 \geq \text{mult}_Q \left( \tilde{D}|_{E_5} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_5 \right) > 4 - 2a_4 .
\]
implies that \( a_5 > \frac{4}{3} \) which is false.
• If \( Q \in E_5 \cap E_6 \) then the log pair
\[
K_X + \lambda \tilde{D} + 2\lambda a_5 E_5 + \lambda a_6 E_6
\]
is not log canonical at the point \( Q \) and so is the log pair
\[
K_X + \lambda \tilde{D} + 2\lambda a_5 E_5 + E_6 , \text{ since } \lambda a_6 \leq 1
\]
By adjunction
\[
2a_6 - 2a_5 = \tilde{D} \cdot E_6 \geq \text{mult}_Q \left( \tilde{D}|_{E_6} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_6 \right) > 4 - 2a_5 .
\]
implies that \( a_6 > 2 \) which is false.
• If the point \( Q \in E_6 \setminus E_5 \) then
\[
K_X + \lambda \tilde{D} + \lambda a_6 E_6
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_6 .
\]
By adjunction \((E_6, \lambda \tilde{D}|_{E_6})\) is not log canonical at \( Q \) and
\[
2a_6 - a_6 \geq 2a_6 - 2a_5 = \tilde{D} \cdot E_6 \geq \text{mult}_Q \left( \tilde{D}|_{E_6} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_6 \right) > 4 ,
\]
implies that \( a_6 > 4 \) which is false.
• If the point $Q \in F_1$ then
  \[ K_X + \lambda \tilde{D} + \lambda b_1 F_1 \]
is not log canonical at the point $Q$ and so is the pair
  \[ K_X + \lambda \tilde{D} + F_1, \text{ since } \lambda b_1 \leq 1. \]
By adjunction $(F_1, \lambda \tilde{D}|_{F_1})$ is not log canonical at $Q$ and
  \[ 2b_1 = \tilde{D} \cdot F_1 \geq \text{mult}_Q(\tilde{D}|_{F_1}) = \text{mult}_Q(\tilde{D} \cdot F_1) > 4, \]
which is false.

\[ \square \]

5.10. Del Pezzo surfaces of degree 2 with one $D_4$ and three $A_1$ type singularities. In this section we will prove the following.

**Lemma 5.10.** Let $X$ be a del Pezzo surface with one Du Val singularity of type $D_4$, three of type $A_1$ and $K_X^2 = 2$. Then the global log canonical threshold of $X$ is
  \[ \text{lct}(X) = \frac{1}{2}. \]

**Proof.** Let $X$ be a del Pezzo surface with one Du Val singularity of type $D_4$, three $A_1$ type singularities and $K_X^2 = 2$. Suppose $\text{lct}(X) < \frac{1}{2}$. Then there exists an effective $\mathbb{Q}$-divisor $D \in X$ such that the log pair $(X, \lambda D)$ is not log canonical for some rational number $\lambda < \frac{1}{2}$ and $D \sim_{\mathbb{Q}} -K_X$.

We derive that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at $P$. Let $\pi : \tilde{X} \to X$ be the minimal resolution of $X$. The following diagram shows how the exceptional curves intersect each other.

```
\[
\begin{array}{cccccccc}
\mathbb{D}_4 & \bullet & E_1 & \bullet & E_3 & \bullet & E_4 & \bullet & F_1 & \bullet & F_2 & \bullet & F_3 \\
 & \bullet & E_2 & & & & & & \\
\end{array}
\]
```

Then
  \[ \tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - b_1 F_1 - b_2 F_2 - b_4 F_4. \]
From the inequalities
  \[
  \begin{align*}
  0 & \leq \tilde{D} \cdot \tilde{L}_1 = 1 - a_1 - b_1 \\
  0 & \leq \tilde{D} \cdot \tilde{L}_2 = 1 - a_2 - b_2 \\
  0 & \leq \tilde{D} \cdot \tilde{L}_4 = 1 - a_4 - b_4 \\
  0 & \leq E_1 \cdot \tilde{D} = 2a_1 - a_3 \\
  0 & \leq E_2 \cdot \tilde{D} = 2a_2 - a_3 \\
  0 & \leq E_3 \cdot \tilde{D} = 2a_3 - a_1 - a_2 - a_4 \\
  0 & \leq E_4 \cdot \tilde{D} = 2a_4 - a_3 \\
  0 & \leq F_1 \cdot \tilde{D} = 2b_1 \\
  0 & \leq F_2 \cdot \tilde{D} = 2b_2 \\
  0 & \leq F_4 \cdot \tilde{D} = 2b_4 
  \end{align*}
\]
we see that
  \[ a_1 \leq 1, \ a_2 \leq 1, \ a_3 \leq 2, \ a_4 \leq 1, \ b_1 \leq 1, \ b_2 \leq 1, \ b_4 \leq 1. \]
We should note here that there are three -1 curves $\tilde{L}_1, \tilde{L}_2, \tilde{L}_4$ such that
  \[ \tilde{L}_1 \cdot E_1 = \tilde{L}_1 \cdot F_1 = \tilde{L}_2 \cdot E_2 = \tilde{L}_2 \cdot F_2 = \tilde{L}_4 \cdot E_4 = \tilde{L}_4 \cdot F_4 = 1. \]
Therefore we have
\[ \tilde{L}_1 \sim_Q \pi^*(L_1) - E_1 - \frac{1}{2}E_2 - E_3 - \frac{1}{2}E_4 - \frac{1}{2}F_1 \]
\[ \tilde{L}_2 \sim_Q \pi^*(L_2) - \frac{1}{2}E_1 - E_2 - E_3 - \frac{1}{2}E_4 - \frac{1}{2}F_2 \]
\[ \tilde{L}_4 \sim_Q \pi^*(L_4) - \frac{1}{2}E_1 - \frac{1}{2}E_2 - E_3 - E_4 - \frac{1}{2}F_4 . \]

The equivalence
\[ K_\tilde{X} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda b_1 F_1 + \lambda b_2 F_2 + \lambda b_4 F_4 \sim \pi^*(K_X + \lambda D) \]
implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup F_1 \cup F_2 \cup F_4 \) such that the pair
\[ K_\tilde{X} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda b_1 F_1 + \lambda b_2 F_2 + \lambda b_4 F_4 \]
is not log canonical at \( Q \).

- If the point \( Q \in E_1 \) and \( Q \not\in E_3 \) then
  \[ K_\tilde{X} + \lambda \tilde{D} + \lambda a_1 E_1 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_\tilde{X} + \lambda \tilde{D} + E_1 . \]
  By adjunction \((E_1, \lambda \tilde{D}|_{E_1})\) is not log canonical at \( Q \) and
  \[ 2a_1 - a_3 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > 2 , \]
  which implies that \( a_1 > 1 \) which is false.

- If \( Q \in E_3 \) but \( Q \not\in E_1 \cup E_2 \cup E_4 \) then
  \[ K_\tilde{X} + \lambda \tilde{D} + \lambda a_3 E_3 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_\tilde{X} + \lambda \tilde{D} + E_3 , \text{ since } \lambda a_3 \leq 1 . \]
  By adjunction \((E_3, \lambda \tilde{D}|_{E_3})\) is not log canonical at \( Q \) and
  \[ a_3 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 2 , \]
  which is false.

- If \( Q \in E_1 \cap E_3 \) then the log pair
  \[ K_\tilde{X} + \lambda \tilde{D} + \lambda a_1 E_1 + \frac{1}{2}a_3 E_3 \]
is not log canonical at the point \( Q \) and so is the log pair
  \[ K_\tilde{X} + \lambda \tilde{D} + \lambda a_1 E_1 + E_3 . \]
  By adjunction it follows that
  \[ a_3 - a_1 \geq 2a_3 - a_1 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D}|_{E_3}) = \text{mult}_Q(\tilde{D} \cdot E_3) > 2 - a_1 . \]
  and we see then that \( a_3 > 1 \) which is not possible.

- If \( Q \in E_1 \) then the log pair
  \[ K_\tilde{X} + \lambda \tilde{D} + \lambda b_1 F_1 \]
is not log canonical at the point \( Q \) and so is the log pair
  \[ K_\tilde{X} + \lambda \tilde{D} + F_1 . \]
  By adjunction it follows that
  \[ 2b_1 = \tilde{D} \cdot F_1 \geq \text{mult}_Q(\tilde{D}|_{F_1}) = \text{mult}_Q(\tilde{D} \cdot F_1) > 2 . \]
  and we see then that \( b_1 > 1 \) which is not possible.

□
5.11. **Del Pezzo surfaces of degree 2 with two \(A_3\) and one \(A_1\) type singularity.** In this section we will prove the following.

**Lemma 5.11.** Let \(X\) be a del Pezzo surface with two Du Val singularities of type \(A_3\), one \(A_1\) type singularity and \(K_X^2 = 2\). Then the global log canonical threshold of \(X\) is

\[
\text{let}(X) = \frac{1}{2}.
\]

**Proof.** Let \(X\) be a del Pezzo surface with two Du Val singularities of type \(A_3\), one \(A_1\) type singularity and \(K_X^2 = 2\). Suppose that \(\text{let}(X) < \frac{1}{2}\), then there exists an effective \(\mathbb{Q}\)-divisor \(D \in X\) such that the log pair \((X, \lambda D)\) is not log canonical for some rational number \(\lambda < \frac{1}{2}\) and \(D \sim_{\mathbb{Q}} -K_X\).

Let \(Z\) be the curve in \(|-K_X|\) that contains \(P\). Since the curve \(Z\) is irreducible we may assume that the support of \(D\) does not contain \(Z\).

We derive that the pair \((X, D)\) is log canonical outside of a point \(P \in X\) and not log canonical at \(P\). Let \(\pi_1 : \tilde{X} \to X\) be the minimal resolution of \(X\). The following diagram shows how the exceptional curves intersect each other.

\[
\begin{array}{cccccc}
A_3 + A_3 + A_1 & \bullet E_1 & \bullet E_2 & \bullet E_3 & \bullet F_1 & \bullet F_2 & \bullet F_3 & \bullet G_1
\end{array}
\]

Then

\[
\tilde{D} \sim_{\mathbb{Q}} \pi_1^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - b_1 F_1 - b_2 F_2 - b_3 F_3 - c_1 G_1.
\]

From the inequalities

\[
\begin{align*}
0 & \leq \tilde{D} \cdot \tilde{L}_1 = 1 - a_1 - b_1 \\
0 & \leq \tilde{D} \cdot \tilde{L}_2 = 1 - a_2 - c_1 \\
0 & \leq \tilde{D} \cdot \tilde{L}_3 = 1 - a_3 - b_3 \\
0 & \leq E_1 \cdot \tilde{D} = 2a_1 - a_2 \\
0 & \leq E_2 \cdot \tilde{D} = 2a_2 - a_1 - a_3 \\
0 & \leq E_3 \cdot \tilde{D} = 2a_3 - a_2 \\
0 & \leq F_1 \cdot \tilde{D} = 2b_1 - b_2 \\
0 & \leq F_2 \cdot \tilde{D} = 2b_2 - b_1 - b_3 \\
0 & \leq F_3 \cdot \tilde{D} = 2b_3 - b_2 \\
0 & \leq G_1 \cdot \tilde{D} = 2c_1
\end{align*}
\]

we see that \(a_1 \leq 1, a_2 \leq 1, a_3 \leq 1, b_1 \leq 1, b_2 \leq 2, b_3 \leq 1, c_1 \leq 1\).

We have three lines \(\tilde{L}_1, \tilde{L}_2, \tilde{L}_3\) intersecting the fundamental cycle as following

\[
\begin{align*}
\tilde{L}_1 \cdot E_1 &= \tilde{L}_1 \cdot F_1 = 1, \\
\tilde{L}_3 \cdot E_3 &= \tilde{L}_3 \cdot F_3 = 1, \\
\tilde{L}_2 \cdot E_2 &= \tilde{L}_2 \cdot G_1 = 1,
\end{align*}
\]

and in particular we have

\[
\begin{align*}
\tilde{L}_1 & \sim_{\mathbb{Q}} \pi_1^*(L_1) - \frac{3}{4} E_1 - \frac{1}{2} E_2 - \frac{1}{4} E_3 - \frac{3}{4} F_1 - \frac{1}{2} F_2 - \frac{1}{4} F_3 \\
\tilde{L}_2 & \sim_{\mathbb{Q}} \pi_1^*(L_2) - \frac{1}{2} E_1 - E_2 - \frac{1}{2} E_3 - \frac{1}{2} G_1 \\
\tilde{L}_3 & \sim_{\mathbb{Q}} \pi_1^*(L_3) - \frac{1}{4} E_1 - \frac{1}{2} E_2 - \frac{3}{4} E_3 - \frac{1}{4} F_1 - \frac{1}{2} F_2 - \frac{3}{4} F_3.
\end{align*}
\]

The equivalence

\[
K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda \tilde{D} + \lambda b_1 F_1 + \lambda b_2 F_2 + \lambda b_3 F_3 + \lambda c_1 G_1 \sim_{\mathbb{Q}} \pi_1^*(K_X + \lambda D)
\]

implies that there is a point \(Q \in E_1 \cup E_2 \cup E_3 \cup F_1 \cup F_2 \cup F_3 \cup G_1\) such that the pair

\[
K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda \tilde{D} + \lambda b_1 F_1 + \lambda b_2 F_2 + \lambda b_3 F_3 + \lambda c_1 G_1
\]

is not log canonical at \(Q\).
If the point \( Q \in E_1 \) and \( Q \not\in E_2 \) then
\[
K_X + \lambda \tilde{D} + \lambda a_1 E_1
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_1.
\]

By adjunction \((E_1, \lambda \tilde{D}|_{E_1})\) is not log canonical at \( Q \) and
\[
\frac{4}{3} a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q \left( \tilde{D}|_{E_1} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_1 \right) > 2,
\]
implies that \( a_1 > \frac{3}{2} \) which is false.

- If \( Q \in E_2 \) but \( Q \not\in E_1 \cup E_3 \) then
\[
K_X + \lambda \tilde{D} + \lambda a_2 E_2
\]
is not log canonical at the point \( Q \) and so is the pair
\[
K_X + \lambda \tilde{D} + E_2, \text{ since } \lambda a_2 \leq 1.
\]

By adjunction \((E_2, \lambda \tilde{D}|_{E_2})\) is not log canonical at \( Q \) and
\[
a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q \left( \tilde{D}|_{E_2} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_2 \right) > 2,
\]
which is false.

- If \( Q \in E_1 \cap E_2 \) then the log pair
\[
K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2
\]
is not log canonical at the point \( Q \) and so is the log pair
\[
K_X + \lambda \tilde{D} + E_1 + \lambda a_2 E_2.
\]

By adjunction it follows that
\[
2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q \left( \tilde{D}|_{E_1} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_1 \right) > 2 - a_2
\]
and this implies that \( a_1 > 1 \) which is false.

- If \( Q \in G_1 \) then the log pair
\[
K_X + \lambda \tilde{D} + \lambda c_1 G_1
\]
is not log canonical at the point \( Q \) and so is the log pair
\[
K_X + \lambda \tilde{D} + G_1.
\]

By adjunction it follows that
\[
2c_1 = \tilde{D} \cdot G_1 \geq \text{mult}_Q \left( \tilde{D}|_{G_1} \right) = \text{mult}_Q \left( \tilde{D} \cdot G_1 \right) > 2
\]
which is false.

\[\square\]

5.12. Del Pezzo surfaces of degree 2 with one \( A_5 \) and one \( A_2 \) type singularity. In this section we will prove the following.

**Lemma 5.12.** Let \( X \) be a del Pezzo surface with one Du Val singularity of type \( A_5 \), one of type \( A_2 \) and \( K_X^2 = 2 \). Then the global log canonical threshold of \( X \) is
\[
\text{lct}(X) = \frac{1}{3}.
\]
Proof. Suppose \( \text{lct}(X) < \frac{1}{3} \). Then there exists an effective \( \mathbb{Q} \)-divisor \( D \in X \) such that the log pair \( (X, \lambda D) \) is not log canonical and \( D \sim_{\mathbb{Q}} -K_X \), where \( \lambda < \frac{1}{3} \). Therefore the log pair \( (X, \lambda D) \) is also not log canonical.

Let \( Z \) be the curve in \( |-K_X| \) that contains \( P \). Since the curve \( Z \) is irreducible we may assume that the support of \( D \) does not contain \( Z \).

We derive that the pair \( (X, \lambda D) \) is log canonical outside of a point \( P \in X \) and not log canonical at \( P \). Let \( \pi_1 : \tilde{X} \to X \) be the minimal resolution of \( X \). The following diagram shows how the exceptional curves intersect each other.

\[
\begin{array}{cccccccc}
A_5 + A_2: & E_1 & \bullet & E_2 & \bullet & E_3 & \bullet & E_4 & \bullet & E_5 & \bullet & F_1 & \bullet & F_2 \\
\end{array}
\]

Then

\[
\tilde{D} \sim_{\mathbb{Q}} \pi_1^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - b_1 F_1 - b_2 F_2.
\]

We have three lines \( \tilde{L}_1, \tilde{L}_3, \tilde{L}_5 \) intersecting the fundamental cycle as following

\[
\tilde{L}_1 \cdot E_1 = \tilde{L}_1 \cdot F_1 = \tilde{L}_3 \cdot E_3 = \tilde{L}_5 \cdot E_5 = \tilde{L}_5 \cdot F_2 = 1
\]

Therefore

\[
\begin{align*}
\tilde{L}_1 & \sim_{\mathbb{Q}} \pi_1^*(L_1) - \frac{1}{6} E_1 - \frac{1}{3} E_2 - \frac{1}{2} E_3 - \frac{2}{3} E_4 - \frac{5}{6} E_5 - \frac{1}{3} F_1 - \frac{2}{3} F_2, \\
\tilde{L}_3 & \sim_{\mathbb{Q}} \pi_1^*(L_3) - \frac{5}{6} E_1 - \frac{2}{3} E_2 - \frac{1}{2} E_3 - \frac{1}{3} E_4 - \frac{1}{6} E_5 - \frac{2}{3} F_1 - \frac{1}{3} F_2, \\
\tilde{L}_5 & \sim_{\mathbb{Q}} \pi_1^*(L_5) - \frac{1}{2} E_1 - E_2 - \frac{3}{2} E_3 - E_4 - \frac{1}{2} E_5.
\end{align*}
\]

Since \( 2L_1 \sim_{\mathbb{Q}} 2L_3 \sim_{\mathbb{Q}} 2L_5 \sim_{\mathbb{Q}} -K_X \) we see that \( \text{lct}(X) \leq \frac{1}{3} \).

From the inequalities

\[
\begin{align*}
0 \leq \tilde{D} \cdot \tilde{L}_1 &= 1 - a_1 - b_1, \\
0 \leq \tilde{D} \cdot \tilde{L}_3 &= 1 - a_3, \\
0 \leq \tilde{D} \cdot \tilde{L}_5 &= 1 - a_5 - b_2, \\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_2, \\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_1 - a_3, \\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_2 - a_4, \\
0 \leq E_4 \cdot \tilde{D} &= 2a_4 - a_3 - a_5, \\
0 \leq E_5 \cdot \tilde{D} &= 2a_5 - a_4, \\
0 \leq F_1 \cdot \tilde{D} &= 2b_1 - b_2, \\
0 \leq F_2 \cdot \tilde{D} &= 2b_2 - b_1,
\end{align*}
\]

we see that

\[
a_1 \leq 1, \ a_2 \leq \frac{4}{3}, \ a_3 \leq 1, \ a_4 \leq \frac{4}{3}, \ a_5 \leq 1, \ b_1 \leq 1, \ b_2 \leq 1
\]

and what is more

\[
2a_5 \geq a_4, \ \frac{3}{2}a_4 \geq a_3, \ \frac{4}{3}a_3 \geq a_2, \ \frac{5}{4}a_2 \geq a_1.
\]

The equivalence

\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda b_1 F_1 + \lambda b_2 F_2 \sim_{\mathbb{Q}} \pi_1^*(K_X + D)
\]

implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup F_1 \cup F_2 \) such that the pair

\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda b_1 F_1 + \lambda b_2 F_2
\]

is not log canonical at \( Q \).
• If the point $Q \in E_1$ and $Q \not\in E_2$ then
\[
K_{\tilde{X}} + \lambda \tilde{D} + a_1 \lambda E_1
\]
is not log canonical at the point $Q$ and so is the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + E_1 , \text{ since } \lambda a_1 \leq 1 .
\]
By adjunction $(E_1, \lambda \tilde{D}|_{E_1})$ is not log canonical at $Q$ and
\[
2a_1 - \frac{4}{5}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_{Q}(\tilde{D} \cdot E_1) > \frac{1}{\lambda} > 3 ,
\]
implies that $a_1 > \frac{5}{2}$ which is a contradiction.

• If $Q \in E_1 \cap E_2$ then the log pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + a_1 \lambda E_1 + a_2 \lambda E_2
\]
is not log canonical at the point $Q$ and so are the log pairs
\[
K_{\tilde{X}} + \lambda \tilde{D} + E_1 + a_2 \lambda E_2 \text{ and } K_{\tilde{X}} + \lambda \tilde{D} + a_1 \lambda E_1 + E_2 .
\]
By adjunction it follows that
\[
2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_{Q}(\tilde{D}|_{E_2}) = \text{mult}_{Q}(\tilde{D} \cdot E_2) > \frac{1}{\lambda} - a_1 > 3 - a_1 ,
\]
and
\[
2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_{Q}(\tilde{D}|_{E_1}) = \text{mult}_{Q}(\tilde{D} \cdot E_1) > \frac{1}{\lambda} - a_2 > 3 - a_2 .
\]
From the second inequality we get that $a_1 \geq \frac{3}{2}$ which is a contradiction.

• If $Q \in E_2$ but $Q \not\in E_1 \cup E_3$ then
\[
K_{\tilde{X}} + \lambda \tilde{D} + a_2 \lambda E_2
\]
is not log canonical at the point $Q$ and so is the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + E_2 , \text{ since } a_2 \lambda \leq 1 .
\]
By adjunction $(E_2, \lambda \tilde{D}|_{E_2})$ is not log canonical at $Q$ and
\[
2a_2 - \frac{a_2}{2} - \frac{3}{4}a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_{Q}(\tilde{D}|_{E_2}) = \text{mult}_{Q}(\tilde{D} \cdot E_2) > \frac{1}{\lambda} > 3 .
\]
Then we get $a_2 > 4$ which is a contradiction.

• If $Q \in E_2 \cap E_3$ then the log pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + a_2 \lambda E_2 + a_3 \lambda E_3
\]
is not log canonical at the point $Q$ and so are the log pairs
\[
K_{\tilde{X}} + \lambda \tilde{D} + a_2 \lambda E_2 + E_3 , \text{ since } \lambda a_3 \leq 1 .
\]
By adjunction it follows that
\[
2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_{Q}(\tilde{D} \cdot E_3) > \frac{1}{\lambda} - a_2 > 3 - a_2 ,
\]
which, together with the inequality $a_4 \geq \frac{2}{3}a_3$, implies that $a_3 > \frac{4}{3}$. However, this is impossible since $a_3 \leq 1$.

• If $Q \in E_3$ but $Q \not\in E_2 \cup E_4$ then
\[
K_{\tilde{X}} + \lambda \tilde{D} + a_3 \lambda E_3
\]
is not log canonical at the point $Q$ and so is the pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + E_3 , \text{ since } a_3 \lambda \leq 1 .
\]
By adjunction $(E_3, \lambda \tilde{D}|_{E_3})$ is not log canonical at $Q$ and
\[
2a_3 - \frac{2}{3}a_3 - \frac{2}{3}a_3 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_{Q}(\tilde{D}|_{E_3}) = \text{mult}_{Q}(\tilde{D} \cdot E_3) > \frac{1}{\lambda} > 3 ,
\]
implies that $a_3 > \frac{9}{2}$ which is false.
• If $Q \in F_1$ then the log pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda b_1 F_1 \]
  is not log canonical at the point $Q$ and so is the log pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + F_1 . \]
  By adjunction it follows that
  \[
  \frac{3}{2} b_1 \geq 2b_1 - b_2 = \tilde{D} \cdot F_1 \geq \text{mult}_Q \left( \tilde{D} |_{F_1} \right) = \text{mult}_Q \left( \tilde{D} \cdot F_1 \right) > 3 .
  \]
  and we see then that $b_1 > 2$ which is not possible.

• If $Q \in F_1 \cap F_2$ then the log pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda b_1 F_1 + \lambda b_2 F_2 \]
  is not log canonical at the point $Q$ and so is the log pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + F_1 + \lambda b_1 F_1 . \]
  By adjunction it follows that
  \[
  2b_1 - b_2 = \tilde{D} \cdot F_1 \geq \text{mult}_Q \left( \tilde{D} |_{F_1} \right) = \text{mult}_Q \left( \tilde{D} \cdot F_1 \right) > 3 - b_2 .
  \]
  and we see then that $b_1 > \frac{3}{2}$ which is false.

• If $Q \in F_2$ then the log pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda b_2 F_2 \]
  is not log canonical at the point $Q$ and so is the log pair
  \[ K_{\tilde{X}} + \lambda \tilde{D} + F_2 . \]
  By adjunction it follows that
  \[
  \frac{3}{2} b_2 \geq 2b_2 - b_1 = \tilde{D} \cdot F_2 \geq \text{mult}_Q \left( \tilde{D} |_{F_2} \right) = \text{mult}_Q \left( \tilde{D} \cdot F_2 \right) > 3 .
  \]
  and we see then that $b_2 > 2$ which is not possible.

\[ \Box \]

5.13. Del Pezzo surfaces of degree 2 with exactly one $A_7$ type singularity. In this section we will prove the following.

Lemma 5.13. Let $X$ be a del Pezzo surface with at most one Du Val singularity of type $A_7$ and $K_{\tilde{X}}^2 = 1$. Then the global log canonical threshold of $X$ is

\[ \text{lct}(X) = \frac{1}{3} . \]

Proof. Suppose that $\text{lct}(X) < \frac{1}{3}$, then there exists an effective $\mathbb{Q}$-divisor $D \in X$ and a positive rational number $\lambda < \frac{1}{3}$, such that the log pair $(X, \lambda D)$ is not log canonical and $D \sim_{\mathbb{Q}} -K_{\tilde{X}}$, where $\lambda < \frac{1}{3}$.

We derive that the pair $(X, \lambda D)$ is log canonical outside of a point $P \in X$ and not log canonical at $P$. Let $\pi_1 : \tilde{X} \to X$ be the minimal resolution of $X$. The following diagram shows how the exceptional curves intersect each other.

\[ \tilde{A}_7 . \quad \bullet E_1 \cdots \bullet E_2 \cdots \bullet E_3 \cdots \bullet E_4 \cdots \bullet E_5 \cdots \bullet E_6 \cdots \bullet E_7 \]

Then

\[ \tilde{D} \sim_{\mathbb{Q}} \pi_1^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5 - a_6 E_6 - a_7 E_7 . \]

Furthermore there are lines $\tilde{L}_2, \tilde{L}_6 \in X$ that pass through the point $P$ whose strict transforms are $(-1)$-curves that intersect the fundamental cycle as following.

\[ \tilde{L}_2 \cdot E_2 = \tilde{L}_6 \cdot E_6 = 1 \]
and 
\[ \tilde{L}_i \cdot E_j = 0 \text{ for all } i, j = 2, 6 \text{ with } i \neq j . \]

Then we easily get that 
\[ \begin{align*}
\tilde{L}_2 &= \pi^*(L_2) - \frac{3}{4}E_1 - \frac{3}{2}E_2 - \frac{5}{4}E_3 - E_4 - \frac{3}{4}E_5 - \frac{1}{2}E_6 - \frac{1}{4}E_7 \\
\tilde{L}_6 &= \pi^*(L_6) - \frac{1}{4}E_1 - \frac{1}{2}E_2 - \frac{3}{4}E_3 - E_4 - \frac{5}{4}E_5 - \frac{3}{2}E_6 - \frac{3}{4}E_7 .
\end{align*} \]

Because \( 2L_2 \sim Q 2L_6 \sim Q -K_X \) we have that \( \text{lct}(X) \leq \frac{1}{3} \).

From the inequalities 
\[ \begin{align*}
0 \leq \tilde{D} \cdot \tilde{L}_2 &= 1 - a_2 \\
0 \leq \tilde{D} \cdot \tilde{L}_6 &= 1 - a_6 \\
0 \leq E_1 \cdot \tilde{D} &= 2a_1 - a_2 \\
0 \leq E_2 \cdot \tilde{D} &= 2a_2 - a_1 - a_3 \\
0 \leq E_3 \cdot \tilde{D} &= 2a_3 - a_2 - a_4 \\
0 \leq E_4 \cdot \tilde{D} &= 2a_1 - a_3 - a_5 \\
0 \leq E_5 \cdot \tilde{D} &= 2a_5 - a_4 - a_6 \\
0 \leq E_6 \cdot \tilde{D} &= 2a_6 - a_5 - a_7 \\
0 \leq E_7 \cdot \tilde{D} &= 2a_7 - a_6
\end{align*} \]

we get 
\[ 2a_7 \geq a_6, \frac{3}{2}a_6 \geq a_5, \frac{4}{3}a_5 \geq a_4, \frac{5}{4}a_4 \geq a_3, \frac{6}{5}a_3 \geq a_2, \frac{7}{6}a_2 \geq a_1 \]

and 
\[ 2a_1 \geq a_2, \frac{3}{2}a_2 \geq a_3, \frac{4}{3}a_3 \geq a_4, \frac{5}{4}a_4 \geq a_5, \frac{6}{5}a_5 \geq a_6, \frac{7}{6}a_6 \geq a_7 . \]

Therefore 
\[ a_1 \leq \frac{7}{6}, a_2 \leq 1, a_3 \leq \frac{3}{2}, a_4 \leq 2, a_5 \leq \frac{3}{2}, a_6 \leq 1, a_7 \leq \frac{7}{6} . \]

The equivalence 
\[ K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 \sim Q \pi_1^*(K_X + D) \]

implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \), such that the pair 
\[ K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 + \lambda a_6 E_6 + \lambda a_7 E_7 \]
is not log canonical at \( Q \).

- If the point \( Q \in E_1 \) and \( Q \not\in E_2 \) then 
  \[ K_X + \lambda \tilde{D} + a_1 \lambda E_1 \]
is not log canonical at the point \( Q \) and so is the pair 
  \[ K_X + \lambda \tilde{D} + E_1 , \text{ since } a_1 \lambda \leq 1 . \]

By adjunction \( (E_1, \lambda \tilde{D}|_{E_1}) \) is not log canonical at \( Q \) and 
\[ \frac{8}{7}a_1 \geq 2a_1 - \frac{6}{7}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q \left( \tilde{D} \cdot E_1 \right) > \frac{1}{\lambda} > 3, \]

which is false, since \( a_1 \leq \frac{7}{6} \).

- If \( Q \in E_1 \cap E_2 \) then the log pair 
  \[ K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 \]
is not log canonical at the point \( Q \) and so is the log pair 
  \[ K_X + \lambda \tilde{D} + E_1 + \lambda a_2 E_2 . \]

By adjunction it follows that 
\[ 2a_2 - 5 \frac{a_2 - a_1}{a_2 - a_1} \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q \left( \tilde{D} \cdot E_2 \right) = \text{mult}_Q \left( \tilde{D} \cdot E_2 \right) > \frac{1}{\lambda} > a_2 > 3 - a_1 , \]

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which is false, since \( a_2 \leq 1 \).
• If \( Q \in E_2 \) but \( Q \not\in E_1 \cup E_3 \) then
  \[
  K_X + \lambda \tilde{D} + \lambda a_2 E_2
  \]
  is not log canonical at the point \( Q \) and so is the pair
  \[
  K_X + \lambda \tilde{D} + E_2 , \quad \text{since } \lambda a_2 \leq 1 .
  \]
  By adjunction \( (E_2, \lambda \tilde{D}|_{E_2}) \) is not log canonical at \( Q \) and
  \[
  2a_2 - \frac{5}{6} a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q\left(\tilde{D} \cdot E_2\right) > \frac{1}{\lambda} > 3 ,
  \]
  which is false, since \( a_2 \leq 1 \).
• If \( Q \in E_2 \cap E_3 \) then the log pair
  \[
  K_X + \lambda \tilde{D} + \lambda a_2 E_2 + \lambda a_3 E_3
  \]
  is not log canonical at the point \( Q \) and so is the log pair
  \[
  K_X + \lambda \tilde{D} + \lambda a_2 E_2 + E_3 , \quad \text{since } \lambda a_3 < 1 .
  \]
  By adjunction it follows that
  \[
  2a_3 - a_2 - \frac{4}{5} a_3 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q\left(\tilde{D} \cdot E_3\right) = \text{mult}_Q\left(\tilde{D} \cdot E_3\right) > \frac{1}{\lambda} - a_2 > 3 - a_2 ,
  \]
  which implies that \( a_3 > \frac{5}{7} \), which is impossible.
• If \( Q \in E_3 \) but \( Q \not\in E_2 \cup E_4 \) then
  \[
  K_X + \lambda \tilde{D} + \lambda a_3 E_3
  \]
  is not log canonical at the point \( Q \) and so is the pair
  \[
  K_X + \lambda \tilde{D} + E_3 , \quad \text{since } \lambda a_3 \leq 1 .
  \]
  By adjunction \( (E_3, \lambda \tilde{D}|_{E_3}) \) is not log canonical at \( Q \) and
  \[
  2a_3 - \frac{4}{5} a_3 \geq 2a_3 - a_2 - a_4 = \tilde{D} \cdot E_3 \geq \text{mult}_Q\left(\tilde{D} \cdot E_3\right) > \frac{1}{\lambda} > 3 .
  \]
  This inequality implies that \( a_3 > \frac{5}{7} \), which is impossible.
• If \( Q \in E_3 \cap E_4 \) then the log pair
  \[
  K_X + \lambda \tilde{D} + \lambda a_3 E_3 + \lambda a_4 E_4
  \]
  is not log canonical at the point \( Q \) and so is the log pair
  \[
  K_X + \lambda \tilde{D} + \lambda a_3 E_3 + E_4 , \quad \text{since } \lambda a_4 \leq 1 .
  \]
  By adjunction it follows that
  \[
  2a_4 - a_3 - \frac{3}{4} a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q\left(\tilde{D} \cdot E_4\right) = \text{mult}_Q\left(\tilde{D} \cdot E_4\right) > \frac{1}{\lambda} - a_3 > 3 - a_3 ,
  \]
  which contradicts \( a_4 \leq 2 \).
• If \( Q \in E_4 \) but \( Q \not\in E_3 \cup E_5 \) then
  \[
  K_X + \lambda \tilde{D} + \lambda a_4 E_4
  \]
  is not log canonical at the point \( Q \) and so is the pair
  \[
  K_X + \lambda \tilde{D} + E_4 , \quad \text{since } \lambda a_4 \leq 1 .
  \]
  By adjunction \( (E_4, \lambda \tilde{D}|_{E_4}) \) is not log canonical at \( Q \) and
  \[
  2a_4 - \frac{3}{4} a_4 \geq 2a_4 - a_3 - a_5 = \tilde{D} \cdot E_4 \geq \text{mult}_Q\left(\tilde{D} \cdot E_4\right) = \text{mult}_Q\left(\tilde{D} \cdot E_4\right) > \frac{1}{\lambda} > 3 ,
  \]
  which is false since \( a_4 \leq 2 \).
5.14. **Del Pezzo surfaces of degree 6 with one $A_2$ and one $A_1$ type singularity.** In this section we will prove the following.

**Lemma 5.14.** Let $X$ be a del Pezzo surface with one Du Val singularity of type $A_2$, one of type $A_1$ and $K_X^2 = 6$. Then the global log canonical threshold of $X$ is

$$\text{let}(X) = \frac{1}{6}.$$ 

**Proof.** Suppose that $\text{let}(X) < \frac{1}{6}$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for a rational number $\lambda < \frac{1}{6}$.

We derive that the pair $(X, \lambda D)$ is log canonical everywhere except for a singular point $P$, at which point $P$ it is not log canonical. Let $\pi_1 : \widetilde{X} \to X$ be the minimal resolution of $X$. The following diagram shows how the exceptional curves intersect each other.

$A_2 + A_1 \quad E_1 \quad \bullet E_2 \quad \bullet E_3$

Then

$$\tilde{D} \sim_{\mathbb{Q}} \pi_1^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3.$$ 

We have a $-1$ curve $\tilde{L}_1$ intersecting the fundamental cycle as following

$$\tilde{L}_1 \cdot E_2 = \tilde{L}_1 \cdot E_3 = 1 \quad \tilde{L}_1 \cdot E_1 = 0$$

and

$$\tilde{L}_1 \sim_{\mathbb{Q}} \pi_1^*(L_1) - \frac{1}{3} E_1 - \frac{2}{3} E_2 - \frac{1}{2} E_3.$$ 

Since $6L_1 \sim_{\mathbb{Q}} -K_X$ we see that $\text{let}(X) \leq \frac{1}{6}$.

From the inequalities

$$0 \leq \tilde{D} \cdot \tilde{L}_1 = 1 - a_2 - a_3$$

$$0 \leq E_1 \cdot \tilde{D} = 2a_1 - a_2$$

$$0 \leq E_2 \cdot \tilde{D} = 2a_2 - a_1$$

$$0 \leq E_3 \cdot \tilde{D} = 2a_3$$

we see that $a_1 \leq 2$, $a_2 \leq 1$, $a_3 \leq 1$.

The equivalence

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 \sim_{\mathbb{Q}} \pi_1^*(K_X + \lambda D)$$

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3$ such that the pair

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3$$

is not log canonical at $Q$.

- If the point $Q \in E_1$ and $Q \notin E_2$ then

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_1,$$

since $\lambda a_1 \leq 1$.

By adjunction $(E_1, \lambda \tilde{D}|_{E_1})$ is not log canonical at $Q$ and

$$2a_1 - \frac{a_1}{2} \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D}|_{E_1}) = \text{mult}_Q(\tilde{D} \cdot E_1) > 6,$$

implies that $a_1 > 4$ which is false.

- If $Q \in E_1 \cap E_2$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$$

is not log canonical at the point $Q$ and so are the log pairs

$$K_X + \lambda \tilde{D} + E_1 + \lambda a_2 E_2$$

and

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1 + E_2$$,

and
since \( \lambda a_2 \leq 1 \). By adjunction it follows that
\[
2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q \left( \tilde{D}|_{E_1} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_1 \right) > 6 - a_2
\]
and
\[
2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q \left( \tilde{D}|_{E_2} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_2 \right) > 6 - a_1 .
\]
This implies that \( a_1 > 3, a_2 > 3 \) which is false.

• If \( Q \in E_3 \) then the log pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_3 E_3
\]
is not log canonical at the point \( Q \) and so is the log pair
\[
K_{\tilde{X}} + \lambda \tilde{D} + E_3 .
\]
By adjunction it follows that
\[
2 \geq 2a_3 = \tilde{D} \cdot E_3 \geq \text{mult}_Q \left( \tilde{D}|_{E_3} \right) = \text{mult}_Q \left( \tilde{D} \cdot E_3 \right) > 6
\]
which is a contradiction.

5.15. **Del Pezzo surfaces of degree 5 with exactly one \( A_4 \) type singularity.** In this section we will prove the following.

**Lemma 5.15.** Let \( X \) be a del Pezzo surface with exactly one Du Val singularity of type \( A_4 \) and \( K_X^2 = 5 \). Then the global log canonical threshold of \( X \) is
\[
\text{lct}(X) = \frac{1}{6}.
\]

**Proof.** Suppose \( \text{lct}(X) < \frac{1}{6} \). Then there exist an effective \( \mathbb{Q} \)-divisor \( D \sim Q - K_X \) and a positive rational number \( \lambda < \frac{1}{6} \), such that the log pair \((X, \lambda D)\) is not log canonical.

We derive that the pair \((X, \lambda D)\) is log canonical everywhere except for a Du Val point \( P \), at which point the pair is not log canonical. Let \( \pi_1 : \tilde{X} \rightarrow X \) be the minimal resolution of \( X \). The following diagram shows how the exceptional curves intersect each other.

\[
\begin{array}{cccc}
A_4 & \cdot E_1 & \cdot E_2 & \cdot E_3 & \cdot E_4 \\
\end{array}
\]

Then
\[
\tilde{D} \sim \mathbb{Q} \quad \pi_1^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 .
\]

Furthermore there is a line \( L_1 \in X \) that passes through the point \( P \), whose strict transform intersects the fundamental cycle as following
\[
\tilde{L}_1 \cdot E_2 = 1 \text{ and } \tilde{L}_1 \cdot E_j = 0 \text{ for all } j = 1, 3, 4 .
\]

Then we easily get that
\[
\tilde{L}_1 \sim \mathbb{Q} \quad \pi^*(L_1) - \frac{3}{5} E_1 - \frac{6}{5} E_2 - \frac{4}{5} E_3 - \frac{2}{5} E_4 .
\]

Because \( 5L_1 \sim \mathbb{Q} - K_X \) we have that \( \text{lct}(X) \leq \frac{1}{6} \).

Since \( L_1 \) is irreducible we can assume that \( L_1 \not\subseteq \text{Supp} D \). Then from the inequalities
\[
0 \leq \tilde{D} \cdot \tilde{L}_1 = 1 - a_2 \\
0 \leq E_1 \cdot \tilde{D} = 2a_1 - a_2 \\
0 \leq E_2 \cdot \tilde{D} = 2a_2 - a_1 - a_3 \\
0 \leq E_3 \cdot \tilde{D} = 2a_3 - a_2 - a_4 \\
0 \leq E_4 \cdot \tilde{D} = 2a_4 - a_3
\]
we get
\[
2a_4 \geq a_3 , \quad \frac{3}{2} a_3 \geq a_2 , \quad \frac{4}{3} a_2 \geq a_1
\]
and
\[ 2a_1 \geq a_2 , \quad \frac{3}{2}a_2 \geq a_3 , \quad \frac{4}{3}a_3 \geq a_4 . \]
Therefore
\[ a_1 \leq \frac{4}{3} , \quad a_2 \leq 1 , \quad a_3 \leq \frac{3}{2} , \quad a_4 \leq 2 . \]
The equivalence
\[ K_X + \lambda D + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 \sim_Q \pi^*_1(K_X + \lambda D) \]
implies that there is a point \( Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \), such that the pair
\[ K_X + \lambda D + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 \]
is not canonical at \( Q \).
- If the point \( Q \in E_1 \) and \( Q \notin E_2 \) then
  \[ K_X + \lambda D + a_1 \lambda E_1 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_X + \lambda D + E_1 , \quad \text{since } a_1 \lambda \leq 1 . \]
By adjunction \((E_1, \lambda D|E_1)\) is not log canonical at \( Q \) and
\[ \frac{5}{3} \geq 2a_1 - \frac{3}{4}a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q \left( \tilde{D} \cdot E_1 \right) > \frac{1}{\lambda} > 6 , \]
which is a contradiction.
- If \( Q \in E_1 \cap E_2 \) then the log pair
  \[ K_X + \lambda D + \lambda a_1 E_1 + \lambda a_2 E_2 \]
is not log canonical at the point \( Q \) and so is the log pair
  \[ K_X + \lambda D + E_1 + \lambda a_2 E_2 . \]
By adjunction it follows that
\[ 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q \left( \tilde{D}|E_1 \right) = \text{mult}_Q \left( \tilde{D} \cdot E_1 \right) > \frac{1}{\lambda} - a_2 > 6 - a_2 , \]
and
\[ 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q \left( \tilde{D} \cdot E_2 \right) > \frac{1}{\lambda} > 6 - a_1 . \]
This implies that \( a_1 > 3 \) which is false.
- If \( Q \in E_2 \) but \( Q \notin E_1 \cup E_3 \) then
  \[ K_X + \lambda D + \lambda a_2 E_2 \]
is not log canonical at the point \( Q \) and so is the pair
  \[ K_X + \lambda D + E_2 , \quad \text{since } \lambda a_2 \leq 1 . \]
By adjunction \((E_2, \lambda D|E_2)\) is not log canonical at \( Q \) and
\[ 2a_2 - \frac{a_2}{2} - \frac{2}{3}a_2 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q \left( \tilde{D} \cdot E_2 \right) > \frac{1}{\lambda} > 6 , \]
which is false, since \( a_2 \leq 1 . \)
- If \( Q \in E_2 \cap E_3 \) then the log pair
  \[ K_X + \lambda D + \lambda a_2 E_2 + \lambda a_3 E_3 \]
is not log canonical at the point \( Q \) and so is the log pair
  \[ K_X + \lambda D + \lambda a_2 E_2 + E_3 , \quad \text{since } \lambda a_3 < 1 . \]
By adjunction it follows that
\[ 2a_2 - \frac{a_2}{2} - a_3 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q \left( \tilde{D} \cdot E_2 \right) > \frac{1}{\lambda} > 6 - a_3 . \]
and
\[2a_3 - a_2 - a_4 = \hat{D} \cdot E_3 \geq \text{mult}_Q(\hat{D}|_{E_3}) = \text{mult}_Q(\hat{D} \cdot E_3) > \frac{1}{\lambda} - a_2 > 6 - a_2.\]

This implies \(a_2 > 4, a_3 > 4\), which is false. 

\[ \square \]

5.16. Del Pezzo surfaces of degree 4 with one \(A_3\) and two \(A_1\) type singularities. In this section we will prove the following.

**Lemma 5.16.** Let \(X\) be a del Pezzo surface with two Du Val singular points of type \(A_3, A_1\) type singular points and \(K_X^2 = 4\). Then the global log canonical threshold of \(X\) is
\[ \text{lct}(X) = \frac{1}{4}. \]

**Proof.** Suppose that \(\text{lct}(X) < \frac{1}{4}\), then there exists an effective \(\mathbb{Q}\)-divisor \(D\) such that \(D \sim_{\mathbb{Q}} -K_X\) and the log pair \((X, \lambda D)\) is not log canonical for some rational number \(\lambda < \frac{1}{4}\).

We derive that the pair \((X, \lambda D)\) is log canonical everywhere except for a singular point \(P \in X\), where \((X, \lambda D)\) is not log canonical. Let \(\pi_1 : \tilde{X} \to X\) be the minimal resolution of \(X\). The following diagram shows how the exceptional curves intersect each other.

\[
\begin{array}{ccccccc}
A_3 & + & 2A_1 & \cdot E_1 & \cdot E_2 & \cdot E_3 & \cdot F_1 & \cdot G_1 \\
\end{array}
\]

Then
\[
\tilde{D} \sim_{\mathbb{Q}} \pi_1^* (D) = a_1 E_1 - a_2 E_2 - a_3 E_3 - b_1 F_1 - c_1 G_1.
\]

We have two lines \(L_1, L_3\) intersecting the fundamental cycle as following
\[
\tilde{L}_1 \sim_{\mathbb{Q}} \pi_1^* (L_1) = -\frac{3}{4} E_1 - \frac{1}{2} E_2 - \frac{1}{4} E_3 - \frac{1}{2} F_1
\]
\[
\tilde{L}_3 \sim_{\mathbb{Q}} \pi_1^* (L_3) = -\frac{1}{4} E_1 - \frac{1}{2} E_2 - \frac{3}{4} E_3 - \frac{1}{2} G_1.
\]

Since \(4L_1 \sim_{\mathbb{Q}} 4L_3 \sim_{\mathbb{Q}} -K_X\) we see that \(\text{lct}(X) \leq \frac{1}{4}\). Moreover we can assume that \(L_1 \not\subset \text{Supp} D\) and \(L_3 \not\subset \text{Supp} D\). From the inequalities
\[
\begin{align*}
0 \leq \tilde{D} \cdot \tilde{L}_1 & = 1 - a_1 - b_1 \\
0 \leq \tilde{D} \cdot \tilde{L}_3 & = 1 - a_3 - c_1 \\
0 \leq E_1 \cdot \tilde{D} & = 2a_1 - a_2 \\
0 \leq E_2 \cdot \tilde{D} & = 2a_2 - a_1 - a_3 \\
0 \leq E_3 \cdot \tilde{D} & = 2a_3 - a_2 \\
0 \leq F_1 \cdot \tilde{D} & = 2b_1 \\
0 \leq G_1 \cdot \tilde{D} & = 2c_1
\end{align*}
\]

we see that
\[a_2 \leq 2a_1, a_3 \leq \frac{3}{2} a_2\] and \(a_2 \leq 2a_3, a_1 \leq \frac{3}{2} a_2\).

Therefore we get the bounds
\[a_1 \leq 1, a_2 \leq 2, a_3 \leq 1, b_1 \leq 1, c_1 \leq 1.\]

The equivalence
\[K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 \sim_{\mathbb{Q}} \pi_1^* (K_X + \lambda D)\]
implies that there is a point \(Q \in E_1 \cup E_2 \cup E_3\) such that the pair
\[K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3\]
is not log canonical at \(Q\).
• If the point $Q \in E_1$ and $Q \notin E_2$ then

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + E_1,$$

since $\lambda a_1 \leq 1$.

By adjunction $(E_1, \lambda \tilde{D}|_{E_1})$ is not log canonical at $Q$ and

$$\frac{4}{3} a_1 \geq 2a_1 - \frac{2}{3} a_1 \geq 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D} \cdot E_1) > 4,$$

implies that $a_1 > 3$ which is false.

• If $Q \in E_1 \cap E_2$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2$$

is not log canonical at the point $Q$ and so are the log pairs

$$K_X + \lambda \tilde{D} + E_1 + \lambda a_2 E_2$$

and

$$K_X + \lambda \tilde{D} + \lambda a_1 E_1 + E_2.$$

By adjunction it follows that

$$2a_1 - a_2 = \tilde{D} \cdot E_1 \geq \text{mult}_Q(\tilde{D} \cdot E_1) > 4 - a_2$$

and

$$\frac{3}{2} a_2 - a_1 \geq 2a_2 - \frac{a_2}{2} - a_1 \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq \text{mult}_Q(\tilde{D} \cdot E_2) > 4 - a_1.$$

The first inequality implies that $a_1 > 2$, which is false.

• If $Q \in E_2$ and $Q \notin E_1 \cup E_3$ then the log pair

$$K_X + \lambda \tilde{D} + \lambda a_2 E_2$$

is not log canonical at the point $Q$ and so is the log pair

$$K_X + \lambda \tilde{D} + E_2.$$

By adjunction it follows that

$$2a_2 - \frac{a_2}{2} - \frac{a_2}{2} \geq 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_3 \geq \text{mult}_Q(\tilde{D} \cdot E_3) > 4$$

and this implies that $a_2 > 4$, which is false.

• If the point $Q \in F_1$ then

$$K_X + \lambda \tilde{D} + \lambda b_1 F_1$$

is not log canonical at the point $Q$ and so is the pair

$$K_X + \lambda \tilde{D} + F_1,$$

since $\lambda b_1 \leq 1$.

By adjunction $(F_1, \lambda \tilde{D}|_{F_1})$ is not log canonical at $Q$ and

$$2b_1 = \tilde{D} \cdot F_1 \geq \text{mult}_Q(\tilde{D}|_{F_1}) = \text{mult}_Q(\tilde{D} \cdot F_1) > 4,$$

implies that $b_1 > 2$ which is false.

$\square$
5.17. **Del Pezzo surfaces of degree 4 with one $\mathbb{D}_5$ type singularity.** In this section we will prove the following.

**Lemma 5.17.** Let $X$ be a del Pezzo surface with one Du Val singularity of type $\mathbb{D}_5$ and $K_X^2 = 4$. Then the global log canonical threshold of $X$ is

\[ \text{lct}(X) = \frac{1}{6}. \]

**Proof.** Suppose that $\text{lct}(X) < \frac{1}{6}$, then there exists a $\mathbb{Q}$-divisor $D \in X$ such that $D \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda D)$ is not log canonical, for some rational number $\lambda < \frac{1}{6}$. We derive that the pair $(X, \lambda D)$ is log canonical everywhere except for a singular point $P \in X$, where $(X, \lambda D)$ is not log canonical. Let $\pi: \tilde{X} \to X$ be the minimal resolution of $X$. The configuration of the exceptional curves is given by the following Dynkin diagram.

\[ \begin{array}{c}
\mathbb{D}_5, \\
\bullet E_1 \quad \bullet E_3 \quad \bullet E_4 \quad \bullet E_5 \quad \bullet E_2
\end{array} \]

Then

\[ \tilde{D} \sim_{\mathbb{Q}} \pi^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3 - a_4 E_4 - a_5 E_5. \]

We have a line $L_1$ intersecting the fundamental cycle as following

\[ \tilde{L}_1 \sim_{\mathbb{Q}} \pi_1^*(L_1) - \frac{5}{4} E_1 - \frac{3}{4} E_2 - \frac{3}{2} E_3 - E_4 - \frac{1}{2} E_5. \]

Since $4L_1 \sim_{\mathbb{Q}} -K_X$ we see that $\text{lct}(X) \leq \frac{1}{6}$ and moreover we can assume that $L_1 \not\in \text{Supp} D$. From the inequalities

\[
\begin{align*}
0 & \leq \tilde{D} \cdot \tilde{L}_1 = 1 - a_1 \\
0 & \leq E_1 \cdot \tilde{D} = 2a_1 - a_3 \\
0 & \leq E_2 \cdot \tilde{D} = 2a_2 - a_3 \\
0 & \leq E_3 \cdot \tilde{D} = 2a_3 - a_1 - a_2 - a_4 \\
0 & \leq E_4 \cdot \tilde{D} = 2a_4 - a_3 - a_5 \\
0 & \leq E_5 \cdot \tilde{D} = 2a_5 - a_4
\end{align*}
\]

we see that

\[ a_3 \leq 2a_1, \ a_3 \leq 2a_2, \ a_4 \leq a_3, \ a_5 \leq a_4 \]

and

\[ a_4 \leq 2a_5, \ a_3 \leq 3a_4, \ a_2 \leq \frac{5}{6} a_3, \ a_1 \leq \frac{5}{6} a_3. \]

In particular we get the following upper bounds

\[ a_1 \leq 1, \ a_2 \leq \frac{5}{3}, \ a_5 \leq a_4 \leq a_3 \leq 2. \]

The equivalence

\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D) \]

implies that there is a point $Q \in E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ such that the pair

\[ K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_1 E_1 + \lambda a_2 E_2 + \lambda a_3 E_3 + \lambda a_4 E_4 + \lambda a_5 E_5 \]

is not log canonical at $Q$. 

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• If the point $Q \in E_1 \setminus E_3$ then

$$K_{X_1} + \lambda \hat{D} + \lambda a_1 E_1$$

is not log canonical at the point $Q$ and so is the pair

$$K_{\hat{X}} + \lambda \hat{D} + E_1.$$ 

By adjunction $(E_1, \lambda \hat{D}|_{E_1})$ is not log canonical at $Q$ and

$$\frac{4}{5} \geq \frac{4}{5} a_1 \geq 2a_1 - \frac{6}{5} a_1 \geq 2a_1 - a_3 = \hat{D} \cdot E_1 \geq \text{mult}_Q(\hat{D} \cdot E_1) > 6,$$

which is contradiction.

• If $Q \in E_1 \cap E_3$ then the log pair

$$K_{\hat{X}} + \lambda \hat{D} + \lambda a_1 E_1 + \lambda a_3 E_3$$

is not log canonical at the point $Q$ and so is the log pair

$$K_{\hat{X}} + \lambda \hat{D} + E_1 + \lambda a_3 E_3.$$ 

By adjunction it follows that

$$2a_1 - a_3 = \hat{D} \cdot E_1 \geq \text{mult}_Q(\hat{D}|_{E_1}) = \text{mult}_Q(\hat{D} \cdot E_1) > 6 - a_3,$$

and this implies that $a_3 > 3$, which is false.

• If $Q \in E_3$ but $Q \not\in E_1 \cup E_2 \cup E_4$ then

$$K_{\hat{X}} + \lambda \hat{D} + \lambda a_3 E_3$$

is not log canonical at the point $Q$ and so is the pair

$$K_{\hat{X}} + \lambda \hat{D} + E_3, \text{ since } \lambda a_3 \leq 1.$$ 

By adjunction $(E_3, \lambda \hat{D}|_{E_3})$ is not log canonical at $Q$ and

$$2a_3 - \frac{a_3}{2} - \frac{a_3}{2} - a_3 \geq 2a_3 - a_1 - a_2 - a_4 = \hat{D} \cdot E_3 \geq \text{mult}_Q(\hat{D} \cdot E_3) > 6,$$

implies that $a_3 > 18$ which is false.

• If $Q \in E_3 \cap E_4$ then the log pair

$$K_{\hat{X}} + \lambda \hat{D} + \lambda a_3 E_3 + \lambda a_4 E_4$$

is not log canonical at the point $Q$ and so are the log pairs

$$K_{\hat{X}} + \lambda \hat{D} + E_3 + \lambda a_4 E_4 \text{ and } K_{\hat{X}} + \lambda \hat{D} + \lambda a_3 E_3 + E_4.$$ 

By adjunction it follows that

$$2a_4 - a_3 - a_5 = \hat{D} \cdot E_4 \geq \text{mult}_Q(\hat{D}|_{E_4}) = \text{mult}_Q(\hat{D} \cdot E_4) > 6 - a_3$$

and

$$2a_3 - a_2 - a_1 - a_4 = \hat{D} \cdot E_3 \geq \text{mult}_Q(\hat{D}|_{E_3}) = \text{mult}_Q(\hat{D} \cdot E_3) > 6 - a_4.$$ 

The last inequality implies that $a_3 > 6$ which is false.

• $Q \in E_4 \setminus (E_3 \cap E_5)$ then the log pair

$$K_{\hat{X}} + \lambda \hat{D} + \lambda a_4 E_4$$

is not log canonical at the point $Q$ and so is the pair

$$K_{\hat{X}} + \lambda \hat{D} + E_4.$$ 

By adjunction $(E_4, \lambda \hat{D}|_{E_4})$ is not log canonical at $Q$ and

$$2a_4 - a_3 - a_5 = \hat{D} \cdot E_4 \geq \text{mult}_Q(\hat{D}|_{E_4}) = \text{mult}_Q(\hat{D} \cdot E_4) > 6,$$

implies that $a_4 > 12$ which is false.
• $Q \in E_5 \setminus E_4$ then the log pair

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_5 E_5$$

is not log canonical at the point $Q$ and so is the pair

$$K_{\tilde{X}} + \lambda \tilde{D} + E_3 .$$

By adjunction $(E_5, \lambda \tilde{D}|_{E_5})$ is not log canonical at $Q$ and

$$2a_5 - a_4 = \tilde{D} \cdot E_5 \geq \text{mult}_Q(\tilde{D}|_{E_5}) = \text{mult}_Q(\tilde{D} \cdot E_5) > 6 ,$$

implies that $a_5 > 6$ which is false.

• $Q \in E_4 \cap E_5$ then the log pair

$$K_{\tilde{X}} + \lambda \tilde{D} + \lambda a_4 E_4 + \lambda a_5 E_5$$

is not log canonical at the point $Q$ and so is the log pair

$$K_{\tilde{X}} + \lambda \tilde{D} + E_5 + \lambda a_4 E_4 .$$

By adjunction it follows that

$$2a_5 - a_4 = \tilde{D} \cdot E_5 \geq \text{mult}_Q(\tilde{D}|_{E_5}) = \text{mult}_Q(\tilde{D} \cdot E_5) > 6 - a_4 .$$

and we see then that $a_5 > 3$ which is not possible.

\[\square\]
6. Tables of Global Log Canonical Thresholds

Table 1. Del Pezzo surfaces of degree 1 and Pic$(X) \cong \mathbb{Z}$

| Singularity Type | lct$(X)$ |
|------------------|----------|
| $E_8$            | $\frac{1}{6}$ |
| $E_7 + A_1$      | $\frac{1}{4}$ |
| $E_6 + A_2, D_8$ | $\frac{1}{3}$ |
| $A_8, A_7 + A_1, D_6 + 2A_1, D_5 + A_3, D_4 + D_4$ | $\frac{1}{2}$ |
| $2A_4$           | $\frac{4}{5}$ |
| $A_5 + A_2 + A_1$ | $\frac{2}{3}$ |
| $4A_2$ and $| - K_X |$ has no cuspidal curves | 1 |
| $4A_2$ and $| - K_X |$ has a cuspidal curve, but no cuspidal curve $C$ such that Sing$(C) = A_2$ | $\frac{5}{6}$ |
| $4A_2$ and $| - K_X |$ has a cuspidal curve such that Sing$(C) = A_2$ | $\frac{2}{3}$ |
| $2A_3 + 2A_1$ and $| - K_X |$ has no cuspidal curves | 1 |
| $2A_3 + 2A_1$ and $| - K_X |$ has a cuspidal curve, but no cuspidal curve $C$ such that Sing$(C) = A_1$ | $\frac{5}{6}$ |
| $2A_3 + 2A_1$ and $| - K_X |$ has a cuspidal curve with Sing$(C) = A_1$ | $\frac{3}{4}$ |
Table 2. Del Pezzo surfaces of degree 2 and $\text{Pic}(X) \cong \mathbb{Z}$

| Singularity Type | $\text{lct}(X)$ |
|------------------|----------------|
| $E_7$            | $\frac{1}{6}$  |
| $D_6 + A_1$      | $\frac{1}{4}$  |
| $A_7$            | $\frac{1}{3}$  |
| $D_4 + 3A_1$     | $\frac{1}{2}$  |
| $A_5 + A_2$      | $\frac{1}{3}$  |
| $2A_3 + A_1$     | $\frac{1}{2}$  |

Table 3. Del Pezzo surfaces of degree 3 and $\text{Pic}(X) \cong \mathbb{Z}$

| Singularity Type | $\text{lct}(X)$ |
|------------------|----------------|
| $E_6$            | $\frac{1}{6}$  |
| $A_5 + A_1$      | $\frac{1}{4}$  |
| $3A_2$           | $\frac{1}{3}$  |
Table 4. Del Pezzo surfaces of degree 4 and Pic\( (X) \cong \mathbb{Z} \)

| Singularity Type | lct\( (X) \) |
|------------------|--------------|
| \( D_5 \)       | \( \frac{1}{6} \) |
| \( A_3 + 2A_1 \) | \( \frac{1}{3} \) |

Table 5. Del Pezzo surfaces of degree 5 and Pic\( (X) \cong \mathbb{Z} \)

| Singularity Type | lct\( (X) \) |
|------------------|--------------|
| \( A_4 \)       | \( \frac{1}{6} \) |

Table 6. Del Pezzo surfaces of degree 6 and Pic\( (X) \cong \mathbb{Z} \)

| Singularity Type | lct\( (X) \) |
|------------------|--------------|
| \( A_2 + A_1 \)  | \( \frac{1}{6} \) |
| Singularity Type                      | lct($X$) |
|--------------------------------------|----------|
| $E_8$                                | $\frac{1}{6}$ |
| $E_7, E_7 + A_1$                     | $\frac{1}{4}$ |
| $E_6, E_6 + A_2, E_6 + A_1$          | $\frac{1}{3}$ |
| $D_8$                                | $\frac{1}{3}$ |
| $D_7$                                | $\frac{2}{3}$ |
| $D_6, D_6 + 2A_1, D_6 + A_1$         | $\frac{1}{2}$ |
| $D_5, D_5 + A_3, D_5 + A_2, D_5 + 2A_1, D_5 + A_1$ | $\frac{1}{2}$ |
| $D_4, D_4 + D_4, D_4 + A_3, D_4 + A_2, D_4 + 4A_1,$ | $\frac{1}{2}$ |
| $D_4 + 3A_1, D_4 + 2A_1, D_4 + A_1$  | $\frac{1}{2}$ |
| $A_8$                                | $\frac{1}{2}$ |
| $A_7, A_7 + A_1$                     | $\frac{1}{2}$ |
| $A'_7$                               | $\frac{8}{15}$ |
| $A_6, A_6 + A_1$                     | $\frac{2}{3}$ |
| $A_5, A_5 + A_1, A_5 + 2A_1, A_5 + A_2, A_5 + A_2 + A_1$ | $\frac{2}{3}$ |
| Singularity Type | $\text{ltc}(X)$ |
|------------------|----------------|
| $\mathbb{A}_4, \mathbb{A}_4 + \mathbb{A}_4, \mathbb{A}_4 + \mathbb{A}_3$ | $\frac{4}{5}$ |
| $\mathbb{A}_4 + \mathbb{A}_2 + \mathbb{A}_1, \mathbb{A}_4 + 2\mathbb{A}_1, \mathbb{A}_4 + \mathbb{A}_2, \mathbb{A}_4 + \mathbb{A}_1$ | $\frac{4}{5}$ |
| If $| - K_X|$ has no cuspidal curve $C$ such that $\mathbb{A}_1 = \text{Sing}(C)$ or $\mathbb{A}_2 = \text{Sing}(C)$ | |
| $\mathbb{A}_4 + \mathbb{A}_2 + \mathbb{A}_1, \mathbb{A}_4 + 2\mathbb{A}_1, \mathbb{A}_4 + \mathbb{A}_2$ | $\frac{2}{3}$ |
| If $| - K_X|$ has a cuspidal curve $C$ such that $\mathbb{A}_2 = \text{Sing}(C)$ | |
| $\mathbb{A}_3, 2\mathbb{A}_3$ | $1$ |
| $\mathbb{A}_3 + 4\mathbb{A}_1, \mathbb{A}_3 + 3\mathbb{A}_1, 2\mathbb{A}_3 + 2\mathbb{A}_1, \mathbb{A}_3 + 2\mathbb{A}_1, \mathbb{A}_3 + \mathbb{A}_1, \mathbb{A}_3 + \mathbb{A}_2 + 2\mathbb{A}_1$ | $1$ |
| If $| - K_X|$ has no cuspidal curves | |
| $\mathbb{A}_3 + 4\mathbb{A}_1, \mathbb{A}_3 + 3\mathbb{A}_1, \mathbb{A}_3 + 2\mathbb{A}_1, \mathbb{A}_3 + \mathbb{A}_1, \mathbb{A}_3 + \mathbb{A}_2 + \mathbb{A}_1$ | $\frac{3}{4}$ |
| If $| - K_X|$ has a cuspidal curve such that $\text{Sing}(C) = \mathbb{A}_1$, but no cuspidal curve $C$ such that $\mathbb{A}_2 = \text{Sing}(C)$ | |
| $\mathbb{A}_3 + 4\mathbb{A}_1, \mathbb{A}_3 + 3\mathbb{A}_1, \mathbb{A}_3 + 2\mathbb{A}_1, \mathbb{A}_3 + \mathbb{A}_1, \mathbb{A}_3 + \mathbb{A}_2 + \mathbb{A}_1$ | $\frac{5}{6}$ |
| If $| - K_X|$ has cuspidal curves $C$, but $\text{Sing}(C) \neq \mathbb{A}_1$ and $\text{Sing}(C) \neq \mathbb{A}_2$ | |
| $\mathbb{A}_3 + \mathbb{A}_2, \mathbb{A}_3 + \mathbb{A}_2 + \mathbb{A}_1$ | $\frac{2}{3}$ |
| If $| - K_X|$ has a cuspidal curve $C$ such that $\text{Sing}(C) = \mathbb{A}_2$ | |
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