Transformation from the nonautonomous to standard NLS equations

Dun Zhao\textsuperscript{1,2}, Xu-Gang He\textsuperscript{1}, and Hong-Gang Luo\textsuperscript{2,3,4}

\textsuperscript{1} School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China
\textsuperscript{2} Center for Interdisciplinary Studies, Lanzhou University, Lanzhou 730000, China
\textsuperscript{3} Key Laboratory for Magnetism and Magnetic Materials of the Ministry of Education, Lanzhou University, Lanzhou 730000, China
\textsuperscript{4} Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100080, China

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Abstract. In this paper we show a systematical method to obtain exact solutions of the nonautonomous nonlinear Schrödinger (NLS) equation. An integrable condition is first obtained by the Painlevé analysis, which is shown to be consistent with that obtained by the Lax pair method. Under this condition, we present a general transformation, which can directly convert all allowed exact solutions of the standard NLS equation into the corresponding exact solutions of the nonautonomous NLS equation. The method is quite powerful since the standard NLS equation has been well studied in the past decades and its exact solutions are vast in the literature. The result provides an effective way to control the soliton dynamics. Finally, the fundamental bright and dark solitons are taken as examples to demonstrate its explicit applications.

PACS. 05.45.Yv Solitons – 42.65.Tg Optical solitons; nonlinear guided waves – 03.75.Lm Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices, and topological excitations

1 Introduction

The standard nonlinear Schrödinger (NLS) equation

\begin{equation}
\frac{\partial}{\partial T} Q(X, T) + \varepsilon \frac{\partial^2}{\partial X^2} Q(X, T) + \delta |Q(X, T)|^2 Q(X, T) = 0,
\end{equation}

where \(\varepsilon\) and \(\delta\) are constants, is a fundamental nonlinear equation to govern system dynamics in many different fields such as Bose-Einstein condensates (BEC)\textsuperscript{2,3} and nonlinear optics\textsuperscript{4,5}. The nature of Eq. (1) has been extensively explored in past decades by many different ways and its exact solutions including soliton\textsuperscript{6} are vast in the literature. For a review, one can refer to Ref. \textsuperscript{7}. When applied to different contexts, Eq. (1) has many different extensions. For example, in BEC an additional harmonic external potential is needed, the resulted equation is well known as Gross-Pitaevskii equation. In addition, the concept of soliton management has been extensively explored in recent years\textsuperscript{8}. The goal is to control effectively the soliton dynamics. In BEC, the nonlinear interaction can be easily tuned by an external magnetic field, namely the Feshbach resonance management\textsuperscript{9,10}. On the other hand, in the context of optical soliton communication, the dispersion management has been explored extensively to improve the optical soliton communication\textsuperscript{11,12}. In both cases, the basic equation to describe the system dynamics should be a generalized nonautonomous NLS equation\textsuperscript{13}, which reads in one-dimensional (1D) case

\begin{equation}
\frac{\partial u(x, t)}{\partial t} + \varepsilon f(x, t) \frac{\partial^2 u(x, t)}{\partial x^2} + \delta g(x, t) |u(x, t)|^2 u(x, t) + V(t) x u(x, t) = 0.
\end{equation}

Here \(\varepsilon\) and \(\delta\) are the same as those in Eq. (1) and \(f(x, t)\) and \(g(x, t)\) are dimensionless control parameters of dispersion and nonlinear interaction, respectively. \(V(t)\) is time-dependent harmonic trap potential in BEC, which is absent in optical transmission line. These coefficients are assumed usually to be real.

Equation (2) and/or its similar versions are very difficult to solve because of the time- and space-dependent dispersion and nonlinear interaction managements and the presence of the external potential in BEC. Some special solutions have been obtained by, for example, the Lax pair method\textsuperscript{13,14,15,16,17}, the similarity transformation\textsuperscript{18,19,20,21}, and so on. However, a general method to find solutions of Eq. (2) has not yet been obtained. Here we obtain a \textit{general} transformation, which can convert all allowed exact solutions of the standard NLS equation (1) to the corresponding solutions of Eq. (2). To our knowledge, the result is reported for the first time in the literature, which provides a straightforward and systematical way to find the exact solution of the generalized NLS
equation \((2)\), as shown by two examples given in the final part of the paper. Below we first show this transformation is consistent with the Painlevé integrability condition of Eq. \((2)\).

### 2 The Painlevé Analysis and Transformation

Motivated by the relationship between complete integrability and the Painlevé property of partial differential equations \([22, 23]\), we perform the WTC test \([22]\) to study possible integrability condition of Eq. \((2)\). Following the Kruskal ansatz \([24]\) and the standard procedure of the Painlevé analysis \([23]\), we obtain a compatibility condition

\[
\frac{g_{t,t}}{g} - \frac{2g_t^2}{g^2} + \frac{f_t^2}{f} + \frac{f_{t,t}}{f} + \frac{g_t f_t}{g f} + 4\varepsilon f V = 0, \tag{3}
\]

where the subscripts denote the time derivatives. Here we should mention that the Painlevé test requires that \(f(x, t)\) and \(g(x, t)\) must be space-independent, i.e., \(f(x, t) = f(t)\) and \(g(x, t) = g(t)\). It is interesting to note that this condition is completely consistent with the integrability condition obtained by Lax pair \([13]\).

The complete integrability of Eq. \((2)\) under the compatibility condition can also be further confirmed through a transformation that reduces Eq. \((2)\) to the standard NLS equation \([1]\). Below we look for such a transformation in a general form of \([25]\)

\[
u(x, t) = Q(X(x, t), T(t))e^{ia(x,t)+c(t)}, \tag{4}\]

where \(X(x, t), T(t), a(x, t)\) and \(c(t)\) are real functions to be determined by the requirement that \(u(x, t)\) and \(Q(X, T)\) are the solutions of Eqs. \((2)\) and \((1)\), respectively. Inserting Eq. \((4)\) into Eq. \((2)\) and comparing with Eq. \((1)\), we obtain a set of differential equations, which have solutions under the condition Eq. \((3)\).

\[
a(x, t) = -\frac{1}{4\varepsilon f(t)} \left( \frac{d}{dt} \ln \frac{f(t)}{g(t)} \right) x^2 + C_1 \frac{g(t)}{f(t)} x - C_2 \varepsilon \int \frac{g(t')^2}{f(t')} dt' + C_2, \tag{5}\n\]
\[
X(x, t) = \frac{g(t)}{f(t)} x - 2 C_1 \varepsilon \int \frac{g(t')^2}{f(t')} dt'. \tag{6}\n\]
\[
T(t) = \int \frac{g(t')^2}{f(t')} dt' + C_3, \tag{7}\n\]
\[
c(t) = \frac{1}{2} \ln \frac{g(t)}{f(t)}. \tag{8}\n\]

where \(C_1, C_2, \text{ and } C_3\) are constants related to the special boundary conditions and the initial state. Here for simplicity, they are set to be zero in the following discussions.

The Painlevé integrability condition Eq. \((3)\) is, in fact, a subtle balance condition to keep the nonautonomous systems integrable. From the management viewpoint of the solitons \([8]\), Eq. \((3)\) also provides an effectively way to manipulate the soliton dynamics. While any two parameters among \(f(t), g(t)\), and \(V(t)\) are set, the remaining one can be tuned according to Eq. \((3)\) in order to control the coherent dynamics of solitons. The applications of Eq. \((3)\) have been extensively explored in Ref. \([13]\). However, the transformation Eqs. \((5) - (8)\) have not been figured out by the Lax pair method. Such transformations are quite systematic in obtaining the exact solutions of the nonautonomous NLS equation. For a given nonautonomous NLS equation, we first check if the coefficients satisfy the compatibility condition Eq. \((3)\). If it is true, then the nonautonomous NLS equation can be reduced to the standard NLS equation \([11]\) All allowed exact solutions, including the canonical solitons, of the standard NLS equation \([1]\) can thus be converted into the corresponding solutions of the nonautonomous NLS equation. In this sense, a canonical soliton can be viewed as a “seed” of the corresponding soliton-like solutions of Eq. \((2)\) under the compatibility condition Eq. \((3)\).

Some remarks are in order. i) If \(f(t) = g(t)\) and \(V(t) = 0\), the nonautonomous NLS equation Eq. \((2)\) have the canonical soliton solutions (up to a phase) regardless of the explicit form of the time-dependent nonlinearity and dispersion. This is because in this case the balance between nonlinearity and dispersion is kept. In this sense the soliton-like solution of Eq. \((2)\) is a quasi-canonical soliton. ii) When \(g(t) \neq f(t)\), the original balance between nonlinearity and dispersion is broken down. In this case the canonical soliton must deform itself to build new balance between nonlinearity and dispersion. In this sense, the soliton-like solution of Eq. \((2)\) is a deformed canonical soliton. The amplitude of the soliton will be scaled by the factor of \(\sqrt{g(t)/f(t)}\), as shown by \(c(t)\). This clearly indicates the influence of the dispersion and nonlinear management to the soliton behavior. iii) It is very interesting to note that the confining harmonic external potential is absent in the transformation equations. However, the presence of the potential affects the balance between nonlinearity and dispersion and builds a deep connection between the optical solitons and the matter-wave ones. iv) If \(V(t) = 0\), the solitons can be quasi-canonical or deformed depending on if \(f(t)\) is equal to \(g(t)\) or not, as mentioned above. On the contrary, if \(V(t) \neq 0\), Eq. \((3)\) indicates that \(f(t) \neq g(t)\). This means that the amplitude of the soliton must change because of Eq. \((8)\). This leads to an important observation that there does not exist the canonical and even quasi-canonical matter-wave solitons under compatibility condition Eq. \((3)\).

It is also helpful to mention some techniques to find the soliton-like solutions of the nonautonomous NLS equation in the literature. The Lax pair analysis is very useful in discussing integrability conditions \([14, 15, 17]\). An widely used method is the similarity transformation \([15, 20]\), which introduces some explicit transformation parameters. These parameters are determined by a set of ordinary differential equations, which in general case can not be solved analytically, as emphasized in Ref. \([20]\). Another similarity transformation reducing the nonautonomous NLS equation to a stationary NLS one has also been intro-
duced [28]. Alternatively, by the Lie point symmetry group analysis [27], Eq. (2) or its similar version can be classified into different classes and each class can be converted into the corresponding representative equation by some allowed transformations. As a result, the exact solutions of the representative equation can be transformed into the corresponding solutions of the equations in the same class. However, it was also pointed out in [27] that in most cases it is difficult to obtain the exact solutions of these representative equations and the integrability of certain representative equations is unclear. Quite different from these techniques, the present work focuses on the integrability of Eq. (2) and builds a deep connection between the nonautonomous NLS equation and its autonomous counterpart, which provides a more systematical way to find solutions of the nonautonomous NLS equation. Moreover, the corresponding transformation formulas are explicit and straightforward. In addition, from the control viewpoint, our method also provides an effective way to control the soliton dynamics, as mentioned above.

3 Applications

Although the transformation obtained can be applied to all allowed exact solutions of the standard NLS equation, the further discussion is purposely restricted to the fundamental bright and dark soliton solutions of the nonautonomous NLS equation without the harmonic external potential, which is enough to show how the soliton dynamics is controlled by corresponding tunable parameters. In this case, Eq. (3) becomes

$$f(t) = g(t) \exp\left(-\alpha \int g(t') dt' \right),$$

where $\alpha$ is a constant and the transformation equations [5]–[8] become

$$a(x, t) = -\alpha \exp(G_\alpha(t)) x^2,$$

$$X(x, t) = \exp(G_\alpha(t)) x,$$

$$T(t) = \int dt' g(t') \exp(G_\alpha(t'))$$

, and

$$c(t) = (1/2)G_\alpha(t),$$

where $G_\alpha(t) = \alpha \int_0^t g(t') dt'$.

When $\varepsilon = 1/2$ and $\delta = 1$ ($\delta \varepsilon > 0$), Eq. (11) has the fundamental canonical bright soliton solution $Q(X, T) = \sech(X) \exp(itT/2)$ and when $\varepsilon = -1/2$ and $\delta = 1$ ($\delta \varepsilon < 0$), the fundamental dark soliton solution of Eq. (11) has the form of $Q(X, T) = \tanh(X) \exp(itT)$. Starting from these two solutions, we show the corresponding soliton solutions of Eq. (2) for four different cases: $g(t) = 1$, $\exp(t)$, $\exp(-t)$, and $\cos(t)$, which represent constant, enhancement, suppression, and periodic modulations of nonlinearity, respectively. We emphasize that these modulations can be readily realized, for example, by the Feshbach resonance technique in the Bose-Einstein condensate context. The corresponding dispersion modulations follow Eq. (9). It is noted that $\alpha = 0$ leads to $G_\alpha(t) = 0$, which is trivial up to a phase, as mentioned above. A nonzero $\alpha$ has nontrivial results and without loss of generality we take $\alpha = 1$ below. In Fig. (b) and Fig. (c) we explicitly present the bright soliton-like and the dark soliton-like solutions for four different nonlinearity modulations, respectively. For comparison, we also plot the canonical solitons, as shown in Fig. (a) and Fig. (d), respectively.

Fig. (b) and (c) show that the canonical bright soliton becomes more and more either sharper or broader depending on enhancement or suppression of the nonlinearity. When the nonlinearity keeps unchanged and the dispersion is suppressed, the canonical bright soliton also becomes more and more sharper, as shown in Fig. (e). This can be understood by the fact that the soliton is due to the balance between dispersion and nonlinearity. Most interesting case is Fig. (d), where the nonlinearity modulation is periodic. As a result, the canonical bright soliton is also modulated periodically. All these results indicate that the bright soliton-like solution and its canonical counterpart has a close relationship. For different nonlinearity modulations, we have checked the integration of $\int |u(x, t)|^2 dx$ and found it keeps unchanged in time, which further shows the nature of the bright soliton-like solutions of Eq. (2). The

![Fig. 1.](image-url)
Fig. 2. The canonical dark soliton of Eq. (1) with $\varepsilon = -1/2$ and $\delta = 1$ and the corresponding dark soliton-like solutions of Eq. (2) for different nonlinearity modulations same as those in Fig. 2. For clarity, the amplitude of the dark soliton-like solutions is normalized by $\eta_0 = \exp \left( \frac{2}{\alpha} G(t) \right)$.

similar result is also true to the dark soliton-like solutions, as shown in Fig. 2. These results shed light on the understanding of the soliton dynamics and provide an exact way to make a dispersion and/or nonlinearity management of solitons. It is expected to have a realistic application to the optical soliton communication technologies and the matter-wave soliton dynamics.

Finally, it should be pointed out that the present analysis can also be applied to all exact solutions of Eq. (1), including the multi-soliton cases. This provides a systematical way to study the dynamics of the nonautonomous NLS equation.

4 Conclusion

We propose a systematical way to find the exact solutions of the nonautonomous NLS equation. The nonautonomous NLS system obtained are completely integrable and the soliton-like solutions result from a balance between dispersion, nonlinearity, and/or an external potential applied, just like the canonical soliton. This result builds a unified picture of the nonautonomous and canonical NLS equations and provides an effective way to control the soliton dynamics.

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