Non–standard Construction of Hamiltonian Structures

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Abstract

Examples of the construction of Hamiltonian structures for dynamical systems in field theory (including one reputedly non–Hamiltonian problem) without using Lagrangians, are presented. The recently developed method used requires the knowledge of one constant of the motion of the system under consideration and one solution of the symmetry equation.

I. INTRODUCTION

Hamiltonian theories have been widely used in almost all areas of physics. The usual approach consists in constructing the momenta (canonically conjugate to the coordinates) and the Hamiltonian starting from a Lagrangian formulation of the system under consideration. Nevertheless, there has been increasing interest in studying non standard procedures to produce Hamiltonian structures starting from the equations of motion only, without using a Lagrangian \cite{1}–\cite{3}. These approaches usually deal with systems which are naturally described in terms of non–canonical variables and the Hamiltonian formulation is consequently written using non–canonical Poisson matrices, which in many cases are singular. Due to the lack of a general procedure to generate Hamiltonian structures from scratch, in most instances the results have been obtained by extremely inspired guesswork.

The purpose of this article is to present examples of the application of a newly devised method \cite{10,11} which constitutes a \textit{general technique} for the construction of Hamiltonian
structures for dynamical systems. We discuss systems described by non–linear equations in field theory and also some linear equations such as the time dependent Schrödinger equations and the heat equation, which according to the usual belief, is clearly non–Hamiltonian [5].

In Section 2 we give a brief outline of the method and the rest of the Sections are devoted to the construction of the examples.

II. BRIEF OUTLINE OF THE METHOD

The contents of this Section summarize partially the results of [10]. We use classical mechanical notation nevertheless, the results may be easily applied to field theory as well, as we will see at the end of this section.

Consider a dynamical system defined by equations which have been cast in first order form,

\[ \frac{dx^a}{dt} = f^a(x^b) \quad a, b = 1, \ldots, N. \]  

(1)

A Hamiltonian structure for it consist of an antisymmetric matrix, \( J_{ab}(x^c) \) and a Hamiltonian \( H(x^c) \) such that \( J_{ab} \) is the Poisson bracket for the variables \( x^a \) and \( x^b \) (which are non–canonical in general) and \( H \) is the Hamiltonian for system (1). In addition to its antisymmetry, the matrix \( J_{ab} \) is required to satisfy the Jacobi identity and to reproduce, in conjunction with the Hamiltonian \( H \), the dynamical equations (1), i.e.,

\[ J_{ab,d}J^{dc} + J_{bc,d}J^{da} + J_{ca,d}J^{db} \equiv 0, \]  

(2)

and,

\[ J^{ab} \frac{\partial H}{\partial x^b} = f^a. \]  

(3)

It has been proved [10] that one solution to the problem of finding a Hamiltonian structure for a given dynamical system is provided by one constant of the motion which may be used as the Hamiltonian \( H \), and a symmetry vector \( \eta^a \) which allows for the construction of a Poisson matrix \( J_{ab} \). The constant of the motion and the symmetry vector satisfy,
\[ \mathcal{L}_f H = 0 , \]  
\[ (\partial_t + \mathcal{L}_f) \eta^a = 0 , \]  
respectively, where \( \mathcal{L}_f \) is the Lie derivative along \( f \) (for a definition, see [12], for instance). In addition, it is required that the deformation \( K \) of \( H \) along \( \eta^a \),
\[ K \equiv \frac{\partial H}{\partial x^a} \eta^a = \mathcal{L}_\eta H , \]
be non-vanishing. The Poisson matrix \( J^{ab} \) is constructed as the antisymmetrized product of the flow vector \( f^a \) and the “normalized” symmetry vector \( \eta^b / K \),
\[ J^{ab} = \frac{1}{K} (f^a \eta^b - f^b \eta^a) . \]

The Poisson matrix so constructed has rank 2 and it is, therefore, singular in many instances. Adding together two Poisson matrices constructed according to (7) will not increase its rank. It will just redefine the symmetry vector used to construct it. One method to increase the rank of such a Poisson matrix is presented in [11].

Let us now deal with systems with infinitely many degrees of freedom defined by some field \( \phi(x, t) \), where \( x \) denotes the coordinates of a point in space. The dynamical equation (1) now is,
\[ \dot{\phi}(x, t) = F[\phi, x] , \]
where \( F \) is a functional of \( \phi \) for every point \( x \) in space. All the above discussion remains valid replacing \( x^a \) by \( \phi(x) \), tensors \( T^{ab...c}(x^a) \) by functionals \( \Theta \) which depend on some spatial coordinates, \( \Theta[\phi, x, y, \ldots, z] \) and partial derivatives \( \partial / \partial x^a \) by functional derivatives \( \delta / \delta \phi(x) \). Details are given in [3].

### III. HEAT EQUATION

Let us consider the heat equation,
\[ u_t = u_{xx} . \]
It is easy to see that
\[ \eta(x) = \Delta , \tag{10} \]
is a symmetry transformation for it, where \( \epsilon \) is any real number.

If we assume periodic boundary conditions in \( x \in [-a/2, a/2] \) then the quantity
\[ H = \int_{-a/2}^{a/2} u \, dx \tag{11} \]
is conserved.

The deformation of \( H \) along \( \eta \) is given by
\[ \mathcal{L}_\eta H = \int_{-a/2}^{a/2} d x \frac{\delta H}{\delta u(x)} \eta(x) = \Delta \int_{-a/2}^{a/2} d x = \Delta a , \tag{12} \]
where all the integrals are taken from \( -a/2 \) to \( a/2 \).

We can now construct a Poisson matrix associated with the Hamiltonian (11) for equation (9),
\[ J(x, y) = \frac{u_{xx}(x) - u_{yy}(y)}{a} . \tag{13} \]

Let us explicitly show that \( J^{ab} \) together with \( H \) provide a Hamiltonian formulation for the heat equation. In fact,
\[ [u(x), H] = \int_{-a/2}^{a/2} d y J(x, y) \frac{\delta H}{\delta u(y)} = \frac{1}{a} u_{xx}(x) \int_{-a/2}^{a/2} d y - \frac{1}{a} \int_{-a/2}^{a/2} u_{yy}(y) d y = u_{xx}(x) . \]
So we see that,
\[ \dot{u} = [u, H] , \tag{14} \]
as required. It is worth noting that according to folk tradition, this equation cannot be endowed with a Hamiltonian structure. In Salmon’s words [5]: “By anyone’s definition, (0.1) is non–Hamiltonian”. In his paper, (0.1) is the heat equation subject to periodic boundary conditions.
Consider the following equations of motion

$$
\psi_t = i(\psi_{xx} + V(x,t)\psi) \equiv f ,
$$
(15)

and its complex conjugate

$$
\psi^*_t = -i(\psi^*_{xx} + V(x,t)\psi^*) \equiv f^* .
$$
(16)

Note that we discuss the case of a time dependent potential. We use one dimensional notation for simplicity only, our results hold irrespective of the dimensionality of space.

It is a straightforward matter to realize that multiplication of the variables $\psi$ and $\psi^*$ by two different constants, $(1 + \lambda)$ and $(1 + \mu \lambda)$ respectively, constitutes a symmetry transformation for the Schrödinger equations (15) and (16). The infinitesimal version of this transformation is

$$
\eta = \psi ,
$$
(17)

and

$$
\eta^* = \mu \psi^* .
$$
(18)

The usual probability conservation statement means that

$$
H = \int \psi^*(x)\psi(x)dx ,
$$
(19)

is a conserved quantity as it may be easily proved. Its deformation $K$ along the symmetry vector defined by (17) and (18) may be written in terms of variational derivatives as

$$
K \equiv \int \frac{\delta H}{\delta \psi(x)}\eta(x)dx + \int \frac{\delta H}{\delta \psi^*(x)}\eta^*(x)dx ,
$$
(20)

and a straightforward calculation yields

$$
K = (1 + \mu)H \neq 0 ,
$$
(21)
for any $\mu$ such that $1 + \mu \neq 0$. The Poisson structure is defined by

$$[\psi(x), \psi(y)] = \frac{1}{K}(f(x)\eta(y) - \eta(x)f(y)),$$  \hspace{1cm} (22)

$$[\psi(x), \psi^*(y)] = \frac{1}{K}(f(x)\eta^*(y) - \eta(x)f^*(y)),$$  \hspace{1cm} (23)

and

$$[\psi^*(x), \psi^*(y)] = \frac{1}{K}(f^*(x)\eta^*(y) - \eta^*(x)f^*(y)).$$  \hspace{1cm} (24)

We have thus constructed a family of Poisson structures which depend on the parameter $\mu$ for a given Hamiltonian $H$. Note that in spite of the time dependent character of the Schrödinger equations (15) and (16) the Hamiltonian (19) is conserved, while the usual Hamiltonian contains the time dependent potential $V(x, t)$ and it is not conserved. On the other hand, our Poisson structures are time dependent. This is, as far as we know, a novel feature for Poisson matrices, which means that, even in the case of a regular matrix, the Hamiltonian structure is not derivable from a Lagrangian.

**V. KORTEWEG–DE VRIES EQUATION**

The equation of motion is

$$u_t = -uu_x - u_{xxx} \equiv f.$$ \hspace{1cm} (25)

It is not difficult to see that symmetry transformation for it is given by

$$\eta = (-2u - xu_x + 3t(uu_x + u_{xxx})).$$ \hspace{1cm} (26)

In fact, to prove that (26) is a symmetry transformation for the KdV equation, it is enough to check that $\eta$ satisfies the symmetry equation (5) with $f$ defined by (25).

To get a Hamiltonian structure for the KdV equation we need to construct constants of the motion which be non trivially deformed by $\eta$. In [10] the energy
\[ H_1 = \int u^2 dx , \]  
(27)

was used as a constant of motion to complete the Hamiltonian structure. Note that the deformation \( K_1 \) of \( H_1 \) along \( \eta \) is non–vanishing

\[ K_1 \equiv \int \frac{\delta H_1}{\delta u(x)} \eta(x)dx = -3H_1 . \]  
(28)

The Poisson structure \( J_1(x,y) \) is given by

\[ J_1(x,y) = \frac{1}{K_1} (f(x)\eta(y) - f(y)\eta(x)) . \]  
(29)

Consider now \( H_2 \)

\[ H_2 = \int (\frac{-u_x^2}{2} + u^3) dx , \]  
(30)

which is also conserved. Its deformation \( K_2 \) along \( \eta \) is

\[ K_2 \equiv \int \frac{\delta H_2}{\delta u(x)} \eta(x)dx = -5H_2 . \]  
(31)

The Poisson structure \( J_2(x,y) \) is given by

\[ J_2(x,y) = \frac{1}{K_2} (f(x)\eta(y) - f(y)\eta(x)) . \]  
(32)

We have been able to construct two different Hamiltonian structures based on one symmetry vector \( \eta \) and two conserved quantities \( H_1 \) and \( H_2 \) as the Hamiltonians of each of the structures. The Poisson structures \( J_1 \) and \( J_2 \) are built as the antisymmetric product of the evolution vector \( f \) and the symmetry vector \( \eta \) normalized using the deformations \( K_1 \) and \( K_2 \) of \( H_1 \) and \( H_2 \), respectively. Note that a different but closely related scheme exists in which no normalization of the antisymmetric product is needed provided that the Hamiltonians \( -(1/3) \log H_1 \) and \( -(1/5) \log H_2 \) be used instead of \( H_1 \) and \( H_2 \) respectively. In this case the Poisson matrix for both Hamiltonian structures is exactly the same one.

A similar construction may be performed with the rest of the constants of the motion which belong to this family, as they appear, for instance, in [13]. We have then constructed
a set of infinitely many Hamiltonian structures for the KdV based on one symmetry transformation and different constants of the motion. Let us remark that as it was mentioned in Section II, the matrices $J(x,y)$ constructed according to (7) have rank 2 hence, there are many Casimir functions which have vanishing Poisson bracket relations with any other dynamical quantity. A few words regarding the construction of Casimir functions for Poisson structures such as these seem in order. Time independent Casimir functions are, of course, constants of the motion. Consider now a different evolution along another parameter (call it $s$) which is given in terms of the symmetry vector $\eta^a$. In other words, consider
\begin{equation}
\frac{d x^a}{ds} = \eta^a(x^b) \quad a, b = 1, \ldots, N .
\end{equation}
Deformation of constants of the motion along $\eta^a$ may be viewed as the evolution of such constants in the parameter $s$. A Casimir function is such that its evolutions (both in time and in the $s$ parameter) vanish. So, the construction of Casimir functions may be viewed as the search of entities which are simultaneously constant for both the time and $s$ evolutions.

To illustrate this, take for instance Eqs. (28) and (31) and rewrite them as
\begin{equation}
\frac{d H_1}{ds} = -3H_1 .
\end{equation}
and
\begin{equation}
\frac{d H_2}{ds} = -5H_2 .
\end{equation}
solve them and eliminate the parameter $s$ to get that $H_1^{1/3}H_2^{-1/5}$ is a Casimir function for both of the Poisson matrices (29) and (32). Similarly, one can get Casimir functions for other Poisson matrices.

VI. BURGERS EQUATION

Consider the Burgers equation,
\begin{equation}
u_t = u_{xx} + 2uu_x \equiv f ,
\end{equation}
being in \([-a/2,a/2]\). It is straightforward to prove that

\[
H_1 = \int_{-a/2}^{a/2} dx u(x) ,
\]

\[
H_2 = \int_{-a/2}^{a/2} dx \exp \left[ \int_{-a/2}^{x} u(y) dy \right] ,
\]

are conserved quantities for it, if we assume that the field vanishes at the boundary \(\pm a/2\).

A symmetry transformation for Eq. (36) is given by

\[
\eta(x) = u(x) \exp \left[ -\int_{-a/2}^{x} u(y) dy \right] .
\]

To see this, it is enough to show that

\[
\mathcal{L}_f \eta = \int_{-a/2}^{a/2} dy \left( \frac{\delta f(x)}{\delta u(y)} \eta(y) - \frac{\delta \eta(x)}{\delta u(y)} f(y) \right) = 0 .
\]

In fact,

\[
\int_{-a/2}^{a/2} dy \frac{\delta f(x)}{\delta u(y)} \eta(y) = \int_{-a/2}^{a/2} dy \frac{\delta \eta(x)}{\delta u(y)} f(y)
\]

\[
= \exp \left( \int_{-a/2}^{x} u(y) dy \right) \left( u_{xx} + uu_x - u^3 \right) .
\]

The deformation of \(H_1\) along \(\eta\) does not vanish,

\[
K_1 = \mathcal{L}_\eta H_1 = \int\frac{\delta H_1}{\delta u(x)} \eta(x)
\]

\[
= \int_{-a/2}^{a/2} dx u(x) \exp \left( -\int_{-a/2}^{x} dw u(w) \right)
\]

\[
= 1 - \exp(-H_1) .
\]

It may be proved that the deformation of \(H_2\) along \(\eta\) is also non vanishing,

\[
K_2 = \mathcal{L}_\eta H_2 = \int\frac{\delta H_2}{\delta u(x)} \eta(x)
\]

\[
= \int_{-a/2}^{a/2} dz \exp \left( \int_{-a/2}^{z} u(w) \left[ \int_{-a/2}^{a/2} dx \exp \left( \int_{-a/2}^{x} dw u(w) \right) \right] \right)
\]

\[
= H_2 - a .
\]

Therefore, we may construct Hamiltonian theories for the Burgers equation using either \(H_1\) or \(H_2\) as Hamiltonians. The appropriate Poisson matrices are
\[ J_1(x, y) = \frac{f(x)\eta(y) - f(y)\eta(x)}{K_1}, \quad (40) \]

and

\[ J_2(x, y) = \frac{f(x)\eta(y) - f(y)\eta(x)}{K_2}, \quad (41) \]

respectively.

It is easy to check, for example, that,

\[ C = \frac{e^{H_1} - 1}{H_2 - a}, \quad (42) \]

is a Casimir for both of the Poisson brackets defined by (40) and (41), where we have used
the approach described at the end of the preceding Section.

**VII. HARRY–DYM EQUATION**

The Harry-Dym equation is

\[ u_t = \left( u^{-1/2} \right)_{xxx} \equiv f, \quad (43) \]

where \( x \in [-a/2, a/2] \). If we assume periodic boundary conditions, \( H_1 \) and \( H_2 \)

\[ H_1 = \int_{-a/2}^{a/2} u \, dx, \quad (44) \]
\[ H_2 = \int_{-a/2}^{a/2} u^{1/2} \, dx, \quad (45) \]

are conserved quantities. Note that \( a \) may be set equal to \( \infty \).

Let us define the vector field

\[ \xi(x) = Au - B xu_x, \quad (46) \]

where \( A \) and \( B \) are real constants. Let us now compute the Lie derivative of it along \( f \),

\[ \mathcal{L}_f \xi = 3 \left( \frac{1}{2} A + B \right) \left( u^{-1/2} \right)_{xxx} = 3 \left( \frac{1}{2} A + B \right) f. \quad (47) \]

Therefore, \( \eta \)
\[
\eta = -3t \left( \frac{1}{2} A + B \right) f + \xi ,
\]

(48)
is a symmetry transformation and can be used to construct a Hamiltonian theory provided it deforms non trivially some Hamiltonian.

This is exactly the case for \( H_1 \) and \( H_2 \). In fact,

\[
K_1 \equiv \mathcal{L}_\eta H_1 = (A + B) H_1 ,
\]

(49)

\[
K_2 \equiv \mathcal{L}_\eta H_2 = \left( \frac{A}{2} + B \right) H_2 ,
\]

(50)
so we have one family of Poisson matrices associated with \( H_1 \) and another one associated with \( H_2 \). They are

\[
J_1(x,y) = \left( u^{-1/2} \right)_{xxx} (Au(y) - Byu_y) - \left( u^{-1/2} \right)_{yyy} (Au(x) - Bxu_x) \quad \frac{1}{K_1},
\]

(51)

and

\[
J_2(x,y) = \left( u^{-1/2} \right)_{xxx} (Au(y) - Byu_y) - \left( u^{-1/2} \right)_{yyy} (Au(x) - Bxu_x) \quad \frac{1}{K_2},
\]

(52)
respectively. Of course, we must be careful to choose \( A \) and \( B \) in such a way that either \( K_1 \) or \( K_2 \) (or both) be non-vanishing.

**VIII. CONCLUSIONS**

We have constructed Hamiltonian structures, without using Lagrangians, for several linear and non–linear systems of partial differential equations (field theory) based on a recently devised method [10,11], which needs only of the knowledge of a constant of motion and a solution of the symmetry equation. The examples include the heat equation which has, up to now, been considered to be non–Hamiltonian [5]. The structures found are singular in the sense that there exist Casimir functions (which have vanishing Poisson bracket with any dynamical variable). This feature is present in many other Hamiltonian theories, and it is sometimes unavoidable (as it is the case of gauge and constrained systems), and it
also appears in Hamiltonian descriptions of some fluids. We should stress that being able to produce a Hamiltonian structure (albeit singular) for a system of differential equations constitutes progress with respect to the situation of having no Hamiltonian structure at all. More examples will be discussed in forthcoming articles.

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