The Evolution of Distorted Rotating Black Holes I: Methods and Tests

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Abstract

We have developed a new numerical code to study the evolution of distorted, rotating black holes. We discuss the numerical methods and gauge conditions we developed to evolve such spacetimes. The code has been put through a series of tests, and we report on (a) results of comparisons with codes designed to evolve non-rotating holes, (b) evolution of Kerr spacetimes for which analytic properties are known, and (c) the evolution of distorted rotating holes. The code accurately reproduces results of the previous NCSA non-rotating code and passes convergence tests. New features of the evolution of rotating black holes not seen in non-rotating holes are identified. With this code we can evolve rotating black holes up to about $t = 100M$, depending on the resolution and angular momentum. We also describe a new family of black hole initial data sets which represent rotating holes with a wide range of distortion parameters, and distorted non-rotating black holes with odd-parity radiation. Finally, we study the limiting slices for a maximally sliced rotating black hole and find good agreement with theoretical predictions.

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I. INTRODUCTION

In the last few years there has been much study of black holes. They are expected to be an important source of gravitational waves. The LIGO and VIRGO interferometers should begin taking data later this decade [1], and sensitive bar detectors are already on line and will improve [2]. At the same time computer power is increasing dramatically, permitting well resolved numerical simulations of axisymmetric black holes and the first 3D simulations of black holes. Collectively this work will lead ultimately to numerical simulations of 3D spiraling, coalescing black holes, which will be essential to interpreting the gravitational waveforms expected to be detected.

Astrophysical black holes in the universe are expected to possess angular momentum, yet due to the difficulty of black hole simulations most numerical calculations of vacuum black hole initial data sets, such as those discovered by Bowen and York [3] nearly 15 years ago, have not been attempted until now. (Recently, however, rotating matter configurations have been successfully collapsed to the point where a horizon forms [4].) Rotation adds a new degree of freedom in the system, complicating matters significantly. Even in the stationary case, the Kerr solution is much more complex than the Schwarzschild solution. As all black hole systems with net angular momentum must eventually settle down to a perturbed Kerr spacetime, it is essential to develop techniques to study distorted rotating black holes numerically.

Such studies are interesting not only because they allow one to examine the nonlinear evolution of distorted, rotating single black holes, but also because they should be useful in understanding the intermediate and late coalescence phase of the general collision of two rotating black holes, as both cases correspond to highly distorted Kerr spacetimes. This parallel between distorted single black holes and the collision of two black holes was striking in the non-rotating case [5–7], and we expect the same to be true for the rotating case.

In this series of papers we show how one can construct and evolve vacuum, distorted, rotating black holes. In this first paper in this series we present details of a numerical code designed to evolve rotating black hole initial data sets, such as Kerr, Bowen and York [3], and a new class of data sets we have developed [8]. With such a code we can now study the dynamics of highly distorted, rotating black holes, paralleling recent work of the NCSA group on non-rotating holes [9,10]. The first studies of the physics of these evolving systems are presented in a companion paper [11], referred to henceforth as Paper II, and a complete study of the initial data sets will be discussed in Ref. [8], which we refer to as Paper III.

The paper is organized as follows: In section II we review the basic formulation of the equations, discuss our choice of variables, and analyze symmetries present in our spacetimes. The initial data sets we have used are discussed in section III, and the numerical code and gauge choices are described in section IV. In section V we describe the behavior of the metric functions in a rotating code and discuss the differences with the non-rotating code.

II. PRELIMINARIES
A. 3+1 Formalism

We use the well known ADM [12] formulation of the Einstein equations as the basis for our numerical code. Pertinent details are summarized here, but we refer the reader to [13] for a complete treatment of this formalism. In the ADM formalism spacetime is foliated into a set of non-intersecting spacelike surfaces. There are two kinematic variables which describe the evolution between these surfaces: the lapse $\alpha$, which describes the rate of advance of time along a timelike unit vector $n^\mu$ normal to a surface, and the spacelike shift vector $\beta^\mu$ that describes the motion of coordinates within a surface. The choice of lapse and shift is essentially arbitrary, and our choices will be described in section IV A below.

The line element is written as

$$ds^2 = -(\alpha^2 - \beta_a \beta^a)dt^2 + 2\beta_a dx^a dt + \gamma_{ab} dx^a dx^b, \quad (1)$$

where $\gamma_{ab}$ is the 3–metric induced on each spacelike slice. Given a choice of lapse $\alpha$ and shift vector $\beta^b$, the Einstein equations in 3+1 formalism split into evolution equations for the 3–metric $\gamma_{ab}$ and constraint equations that must be satisfied on any time slice. The evolution equations, in vacuum, are

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \quad (2)$$
$$\partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \alpha \left(R_{ij} + K_i K_{ij} - 2K_{im} K^m_{\ j} \right) + \beta^m \nabla_m K_{ij} + K_{im} \nabla_j \beta^m + K_{mj} \nabla_i \beta^m \quad (3)$$

where $K_{ij}$ is the extrinsic curvature of the 3–dimensional time slice. The Hamiltonian constraint equation is

$$R + K^2 - K^{ij} K_{ij} = 0 \quad (4)$$

and the three momentum constraint equations are

$$\nabla_i \left(K^{ij} - \gamma^{ij} K \right) = 0. \quad (5)$$

Note that in the above, $K = K_{ij} \gamma^{ij}$, $\beta_i = \beta^j \gamma_{ij}$, and $\nabla_i$ is the covariant three-space derivative. As we discuss in section II, the constraint equations are used to obtain the initial data, and the evolution equations are used to advance the solution in time.

B. Definition of Variables

We build on earlier work of Ref. [9,10] in defining the variables used in our code. Additional metric and extrinsic curvature variables must be introduced to allow for the odd-parity modes present now that the system allows for rotation. We define the variables used in our evolutions as follows:

$$\gamma_{ij} = \begin{pmatrix} \gamma_{\eta \eta} & \gamma_{\eta \theta} & \gamma_{\eta \phi} \\ \gamma_{\theta \eta} & \gamma_{\theta \theta} & \gamma_{\theta \phi} \\ \gamma_{\phi \eta} & \gamma_{\phi \theta} & \gamma_{\phi \phi} \end{pmatrix} = \Psi^4 \begin{pmatrix} A & C & E \sin^2 \theta \\ C & B & F \sin \theta \\ E \sin^2 \theta & F \sin \theta & D \sin^2 \theta \end{pmatrix} \quad (6a)$$

and
\[ K_{ij} = \Psi^4 H_{ij} = \Psi^4 \begin{pmatrix} H_A & H_C & H_E \sin^2 \theta \\ H_C & H_B & H_F \sin \theta \\ H_E \sin^2 \theta & H_F \sin \theta & H_D \sin^2 \theta \end{pmatrix} \]  

(6b)

In these expressions \( \eta \) is a logarithmic radial coordinate, and \((\theta, \phi)\) are the usual angular coordinates. The relation between \( \eta \) and the standard radial coordinates used for Schwarzschild and Kerr black holes is discussed in section III. As in Ref. [9], the conformal factor \( \Psi \) is determined on the initial slice and, since we do not use it as a dynamical variable, it remains fixed in time afterwards. The introduction of \( \Psi \) into the extrinsic curvature variables simplifies the evolution equations somewhat. For the purposes of our numerical evolution we will treat \( A, B, D, F \), all six \( H \)'s, and all three components of the shift as dynamical variables to be evolved. Two of the shift components are used to eliminate metric variable \( C \) (this will be discussed in section IV A 2), and one shift component is used to eliminate \( E \). The various factors of \( \sin \theta \) are included in the definitions to explicitly account for some of the behavior of the metric variables near the axis of symmetry and the equator.

Within this coordinate system, the ADM mass and angular momentum about the \( z \)-axis are defined to be [14]

\[ M_{ADM} = -\frac{1}{2\pi} \oint_S \nabla_a (\Psi e^{-\eta/2}) dS^a \]  

(7a)

\[ P_a = \frac{1}{8\pi} \oint_S (H^b_a - \gamma^b_a H) dS_b. \]  

(7b)

In terms of the variables defined in this paper these expressions yield

\[ M_{ADM} = -\int_0^\pi e^{\eta/2} (\partial_\eta \Psi - \Psi / 2) \sin \theta d\theta, \]  

(8a)

\[ J = P_\phi = \frac{1}{4} \int_0^\pi \Psi^6 H_E \sqrt{BD/A} \sin^3 \theta d\theta \]  

(8b)

Because of this, the variable \( H_E \) is extremely important. It determines whether angular momentum is present in the spacetime. Although the ADM mass is defined strictly only at spatial infinity \( I^0 \), in practice we evaluate it at the edge of the spatial grid. As we use a logarithmic radial coordinate \( \eta \), this is in the asymptotic regime. While the angular momentum is, in principle, also measured at \( I^0 \), the presence of the azimuthal Killing vector makes it possible to evaluate \( J \) at any radius. We compute this quantity during our evolution and use it as a test of the accuracy of our code. This will be discussed in a later section.

C. Symmetries

Symmetries are an important consideration in the evolution code for the setting of boundary conditions. This is important not only for solving numerical elliptic equations, but also for appropriate finite differencing of our variables. Most of the appropriate conditions can be derived merely by considering the behavior of metric functions near the boundaries or
the symmetry operations one can perform on a spinning object, without appealing to the Einstein equations themselves.

The principal symmetries are:

(i) Axisymmetry. We chose to study axisymmetric spacetimes as a first step towards understanding general rotating black holes. Even in the stationary, Kerr case, rotating black holes are already an inherently 2D problem (i.e., they cannot be treated as a spherical system, as Schwarzschild is.) The fact that any rotating black hole should settle down to a Kerr black hole, and thus an axisymmetric solution, makes this a good choice for understanding the late time behavior of any rotating black hole system. One physical restriction imposed by this symmetry is that angular momentum cannot be radiated (see, e.g., p. 297 of Ref. [15]).

Axisymmetry requires that the transformation $\phi \rightarrow \phi + \text{constant}$ leaves the problem unchanged, and therefore all variables in the problem must be independent of $\phi$. However, it is important to realize that $\phi \rightarrow -\phi$ is not a symmetry of this spacetime, since performing this transformation would amount to reversing the direction of spin of the hole. In polar coordinates, there is an additional consequence of axisymmetry: the transformation $\phi \rightarrow \phi + \pi$ produces the same result as $\theta \rightarrow -\theta$, providing a boundary condition on the symmetry axis $\theta = 0$. These considerations require the following variables to be symmetric across the axis: $A$, $B$, $D$, $F$, $H_A$, $H_B$, $H_D$, $H_F$, $H_E$, $\alpha$, $\beta^\eta$, and $\beta^\phi$. The remaining variables $C$ and $H_C$ are antisymmetric across the axis.

(ii) Equatorial Plane Symmetry. We require the spacetime to be identical when reflected through the equator defined by $\theta = \pi/2$ ($z = 0$). This symmetry was not strictly necessary, but adopting it reduces the complexity of the problem slightly. One might think that the transformation $z \rightarrow -z$ (or equivalently $\theta \rightarrow \pi - \theta$) would result in a hole spinning in the opposite direction, but it does not. Inverting through the equator in this fashion is actually equivalent to rotating the hole by $\pi$ radians about the $x$ axis, then by sending $\phi \rightarrow -\phi$. Both of these latter operations clearly result in reversing the sense of the hole’s spin, and so performing both of them leaves the direction of the hole’s spin unchanged. One can easily check that the Kerr solution itself is manifestly unchanged by the operation $z \rightarrow -z$.

These considerations require the following variables to be symmetric across the equator: $A$, $B$, $D$, $H_A$, $H_B$, $H_D$, $H_E$, $\alpha$, $\beta^\eta$, and $\beta^\phi$. The remaining variables $C$ and $H_C$, $H_F$, and $F$ are antisymmetric across the equator.

There is an alternate equatorial boundary condition to consider. In the “cosmic screw” (the collision of two black-holes on axis with equal and opposite angular momenta) the sense of the rotation of the two holes is changed upon inversion through the equatorial plane. The appropriate isometry is thus $(\theta, \phi) \rightarrow (\pi - \theta, -\phi)$. $H_E$ and $\beta^\phi$ are now antisymmetric, and $H_F$ and $F$ are symmetric. These conditions will be considered in a future paper.

(iii) Time/rotation symmetry. This is a symmetry of the initial slice. It simplifies the initial data for the extrinsic curvature by requiring that all but the values $H_E$ and $H_F$ be zero. This symmetry says that the transformation $(\phi, t) \rightarrow (-\phi, -t)$ leaves the problem unchanged.

(iv) Inversion through the throat. Building on previous work [4][10], we construct our spacetimes to be inversion symmetric through the black hole “throat”. This Einstein-Rosen bridge [16] construction has two geometrically identical sheets connected smoothly at the throat, located at $\eta = 0$. This symmetry requires the metric to be invariant under the transformation $\eta \rightarrow -\eta$. For the case of the Kerr spacetime, it can be expressed as
\[ \bar{r} \to \left( m^2 - a^2 \right) / (4\bar{r}) , \] (9)

where \( \bar{r} \) is a generalization of the Schwarzschild isotropic radius, \( a \) is the Kerr rotation parameter, and \( m \) is the mass of the Kerr black hole. The variable \( \bar{r} \) is defined as

\[ \bar{r} = \frac{\sqrt{m^2 - a^2}}{2} e^{\eta} \] (10a)

and is related to the usual Boyer-Lindquist radial coordinate via

\[ r = \bar{r} \left( 1 + \frac{m + a}{2\bar{r}} \right) \left( 1 + \frac{m - a}{2\bar{r}} \right) . \] (10b)

Note that in the Kerr spacetime the horizon, located at \( r = m + \sqrt{m^2 - a^2} \), is at \( \bar{r} = \sqrt{m^2 - a^2}/2 \) in the \( \bar{r} \) coordinates, or at \( \eta = 0 \), just as in previous studies of the Schwarzschild spacetime [18].

This symmetry is perhaps the most important because it is what makes our spacetime a black hole spacetime. When combined with symmetry (iii) above and the differential equation for a trapped surface it tells us that \( \eta = 0 \) is a trapped surface on the initial slice. To impose this symmetry, one requires that metric variables with a single \( \eta \) index will be antisymmetric across the throat, and all others will be symmetric across the throat. The extrinsic curvature variables will have the opposite symmetry on the throat as the corresponding metric variables when the lapse is antisymmetric and the same symmetry when the lapse is symmetric. Thus, the following are symmetric: \( A, B, D, F, H_E, H_C, \beta^\theta \), and \( \beta^\phi \). The following are antisymmetric: \( C, E, H_A, H_B, H_D, H_F, \beta^\eta \) (this is the symmetry of the \( H \)'s when \( \alpha \) is antisymmetric).

It is possible to slice the Kerr spacetime with a symmetric lapse across the throat, but a different symmetry at the throat needs to be employed for certain extrinsic curvature variables. Kerr initial data specifies that \( H_E \) and \( H_F \) are symmetric and antisymmetric across the throat, respectively; Bowen and York [3] initial data sets, discussed below, specify that \( H_E \) is symmetric across the throat. These conditions are incompatible with an \( \eta \to -\eta \) boundary condition if a symmetric lapse is employed (\( \gamma_{ij} \) must have the same symmetry as \( \alpha H_{ij} \) for the evolution equations to be consistent). This problem could be removed if we use \( (\eta, \phi) \to (-\eta, -\phi) \) instead of \( \eta \to -\eta \), resulting in consistent evolution equations on both sides of the throat. We point out that this condition is consistent with the Kerr initial data, and is the generalization of the technique of evolving Schwarzschild with a symmetric lapse [10].

III. INITIAL DATA

We can use the constraints to construct initial data. An especially convenient formulation of the initial data problem was given by Bowen and York [3]. They define a conformal spacetime in which

\[ \gamma_{ij} = \Psi^4 \tilde{\gamma}_{ij} \] (11a)

\[ K_{ij} = \Psi^{-2} \tilde{H}_{ij} . \] (11b)
In terms of the conformal extrinsic curvature, we write the $K_{ij}$ as

$$K_{ij} = \Psi^{-2} \hat{H}_{ij} = \Psi^{-2} \begin{pmatrix} \hat{H}_A & \hat{H}_C & \hat{H}_E \sin^2 \theta \\ \hat{H}_C & \hat{H}_B & \hat{H}_F \sin \theta \\ \hat{H}_E \sin^2 \theta & \hat{H}_F \sin \theta & \hat{H}_D \sin^2 \theta \end{pmatrix}. \quad (12)$$

If the initial data set is maximally sliced ($\text{tr} K = 0$), or if the value of $\text{tr} K$ is held constant, then the conformal factor drops out of the momentum constraint equations. This decouples the constraints and allows one to solve for the momentum constraints first and then to solve the Hamiltonian constraint for the conformal factor. For more detail, see Ref. [13].

The initial data sets we have developed to study distorted rotating black holes are based on the both the “Brill Wave plus Black Hole” solutions of the NCSA group [19] and the rotating black holes of Bowen and York [3]. A complete discussion and analysis of these rotating data sets will be published in Paper III. Here we provide only the basic construction of the initial data sets. Generalizing the conformally flat approach of Bowen and York [3], and following earlier work of Ref. [19], we write the metric with a free function $q(\eta, \theta)$:

$$dl^2 = \Psi^4 \left[ e^{2(q-q_0)} \left(d\eta^2 + d\theta^2\right) + \sin^2 \theta d\phi^2. \right] \quad (14)$$

The function $q_0$ is chosen so that if the function $q(\eta, \theta)$ is chosen to vanish, we are left with the Kerr 3-metric in these coordinates, as shown below. (We note that Kerr is not conformally flat.) For other choices of $q(\eta, \theta)$ we obtain another spacetime. As we will see, appropriate choices of this function (along with appropriate solutions to the momentum constraint for the extrinsic curvature terms) can lead to Schwarzschild, the NCSA distorted non-rotating black hole (as in Ref. [13]), the Bowen and York rotating black hole [3], or a distorted Bowen and York black hole.

As in Ref. [19], the function $q$, representing the Brill wave, can be chosen somewhat arbitrarily, subject to symmetry conditions on the throat, axis, and equator, and falloff conditions at large radii [3,19]. Often the function $q$ will be chosen to have an inversion symmetric gaussian part given by:

$$q = \sin^n \theta q_G, \quad (15a)$$

$$q_G = Q_0 \left( e^{-s_+} + e^{-s_-} \right), \quad (15b)$$

$$s_\pm = (\eta \pm \eta_0)^2 / \sigma^2. \quad (15c)$$

This form of the Brill wave will be characterized by several parameters: $Q_0$ (its amplitude), $\sigma$ (its width), $\eta_0$ (its coordinate location), and $n$, specifying its angular dependence, which must be positive and even. We note that Eqs. (15) simply provide a convenient way to parameterize the initial data sets, and to allow us to easily adjust the “Brill wave” part of the initial data. Many other devices are possible.

In this generalization, we may now interpret the parameter $q_0$ as the Brill wave $q$ required to make the spacetime conformally flat. A distorted Bowen and York spacetime can be made
by setting $q = q_G + q_0$ and a distorted Kerr spacetime can be made by setting $q = \sin^n \theta q_G$, when appropriate solutions are taken for the momentum constraints.

As a consequence of this generalization of the metric, we must now solve both the Hamiltonian and momentum constraints. The Hamiltonian constraint equation is

\[
\frac{\partial^2 \Psi}{\partial \eta^2} + \frac{\partial^2 \Psi}{\partial \theta^2} + \frac{\partial \Psi}{\partial \theta} \cot \theta - \frac{\Psi}{4} = -\frac{\Psi}{4} \left( \frac{\partial^2}{\partial \eta^2} (q - q_0) + \frac{\partial^2}{\partial \theta^2} (q - q_0) \right) + \frac{\Psi^{-7}}{4} \left( \hat{H}_E \sin^2 \theta + \hat{H}_F \right),
\]

(16)

and the $\phi$ component of the momentum constraint (the only non-trivial momentum constraint equation for the initial data) is

\[
\partial_\eta \hat{H}_E \sin^3 \theta + \partial_\theta \left( \hat{H}_F \sin^2 \theta \right) = 0.
\]

(17)

Based on these ideas, we consider several solutions to the initial value problem for evolution in this paper. Complete details of these data sets and extensions of them will be discussed in Paper III. The first solution we consider is the Kerr solution, where

\[
\Psi_0^4 = g_{\phi\phi}^{(K)}/\sin^2 \theta
\]

(18a)

\[
e^{-2\theta_0} = g_{rr}^{(K)} \left( \frac{dr}{d\eta} \right)^2 = g_{\theta\theta}^{(K)}
\]

(18b)

\[
\hat{H}_F = -2\Psi_0^2 a^3 m^* r \cos \theta \sin^3 \theta \rho^{-4}
\]

(18c)

\[
\hat{H}_E = \Psi_0^2 a m \sin^2 \theta \left( \rho^2 \left( r^2 - a^2 \right) + 2r^2 \left( r^2 + a^2 \right) \right) \Delta^{-1/2} \rho^{-4},
\]

(18d)

\[
\rho^2 = r^2 + a^2 \cos^2 \theta,
\]

(18e)

\[
\Delta = r^2 - 2m^* r + a^2.
\]

(18f)

Here $\Psi_0$ is the value of $\Psi$ for the Kerr spacetime, and $g_{ij}^{(K)}$ denotes a Kerr metric element in Boyer-Lindquist coordinates, as given in Ref. [17]. Although this may not look familiar, one may check that with the transformations

\[
r = r_+ \cosh^2 \left( \eta/2 \right) - r_- \sinh^2 \left( \eta/2 \right),
\]

(19a)

and

\[
r_\pm = m \pm \sqrt{m^2 - a^2}
\]

(19b)

the Kerr solution in Boyer-Lindquist coordinates actually results, as detailed in Paper III. Note that this is the same transformation given in Eqs.(10). Note also that if $a = 0$ then $q_0 = 0$ and we recover the Schwarzschild 3-metric. This metric is now in the form used in the previous NCSA black hole studies [5,9,10,19,20].

Another solution we study is related to the original solution of Bowen and York [3]. Following them, we choose

\[
\hat{H}_E = 3J
\]

(20a)

\[
\hat{H}_F = 0
\]

(20b)
where \( J = a \cdot m \) is the total angular momentum of the spacetime. But then we may distort it by choosing
\[
q = q_G \sin^n \theta + q_0
\] (21)
and solving the Hamiltonian constraint.

A third solution is the odd-parity distorted non-rotating black hole, which we create by choosing
\[
\hat{H}_E = q_G' \left( (n' + 1) - (2 + n') \sin^2 \theta \right) \sin^{n' - 3} \theta
\] (22a)
\[
\hat{H}_F = -\partial_\eta q_G' \cos \theta \sin^{n' - 1} \theta,
\] (22b)
which has “odd-parity” radiation but no rotation, as one can verify by checking Eq. (8b). Because \( \hat{H}_E \) vanishes in the large \( \eta \) limit, the ADM integral for the angular momentum must also vanish (See Eq. (8b)). Also note that \( n' \) must be odd and have a value of at least 3.

Thus, in some sense this third solution is a distortion from a Schwarzschild black hole, but one that is fundamentally different from the “Brill wave plus black hole” data sets of Ref. [10]. This data set describes a non-rotating spacetime, parts of which are “twisting and untwisting,” but without any angular momentum. Some of these data sets will be evolved and studied in detail in Paper II.

For all these data sets we need to solve the nonlinear elliptic Hamiltonian constraint equation. We accomplish this by replacing \( \Psi \) with \( \Psi + \delta \Psi \) and expanding in \( \delta \Psi \) to create a linear elliptic equation and iteratively solving for \( \delta \Psi \). The linear elliptic equation is solved by employing finite differencing and the multigrid elliptic solver described in section IV A.

For a more detailed discussion, see Paper III.

### IV. THE EVOLUTION CODE

In this section we discuss our evolution code, including various gauge choices and computational details.

#### A. Gauge Choices

1. **Lapse**

Maximal slicing is used, with a lapse that is antisymmetric across the throat. Maximal slicing is the condition that the trace of the extrinsic curvature is zero throughout the evolution. We note that the Kerr solution in standard form is maximally sliced with antisymmetric lapse. Setting \( \text{tr} K = 0 \) in the evolution for \( \text{tr} K \) gives
\[
0 = -\nabla^a \nabla_a \alpha + \alpha R.
\] (23)
We find that the code is more stable if we use the energy constraint equation to eliminate the Ricci scalar \( R \) from Eq. (23) since it contains second derivatives of the metric functions.
The behavior of second derivatives can be troublesome near the peaks in metric functions that can develop near black holes. The evolution equation for $\text{tr}K$ becomes, therefore,

$$0 = -\nabla^a \nabla_a \alpha + \alpha K^{ij} K_{ij}. \quad (24)$$

This equation is solved numerically on each time slice during the evolution using a multigrid solver to be discussed in section IV B. Maximal slicing is an important slicing condition for its singularity avoiding properties. As the black hole evolves, constant $\eta$ observers are really falling into the hole. Observers at the same value of $\eta$ but different values of $\phi$ along the equator will, for example, find that they are closer together as the evolution proceeds. If we were not careful, our code would follow these observers as they fall in until the distance between them becomes zero (as they reach the singularity) causing our code to crash. For Schwarzschild and distorted non-rotating spacetimes maximal slicing prevents the evolution from reaching the singularity. Instead, when the slice approaches the radius $r = \frac{3}{2}M$ the lapse goes to zero and proper time will not advance at that location [21].

For rotating black holes the singularity structure inside the horizon is significantly different. For these reasons one might worry that the singularity avoidance properties could be different. The interior region of a rotating black hole has a timelike singularity, Cauchy horizons, and an infinite patchwork of “other universes” [22]. However, this problem was investigated by Eardley and Smarr [23], who showed that maximal slicing does in fact avoid the singularity inside the Kerr black hole (in fact, it avoids the Cauchy horizon), although in a different manner from Schwarzschild. They estimated the location of a limit surface, which was found with more precision by Duncan [24] who obtained

$$r_{\text{lim}} \approx 3M \left[ 1 + \left( 1 - 8a^2/9M^2 \right)^{1/2} \right] / 4, \quad (25)$$

where $r_{\text{lim}}$ is given in terms of the standard Boyer-Lindquist coordinates. However, we will measure “the circumferential radius,” not the Boyer-Lindquist $r$ coordinate, so the above equation must be converted using the following formula:

$$r_c = \sqrt{r^2 + a^2 + 2a^2m/r}. \quad (26)$$

This is simply the result of evaluating $r_c = \Psi^2 \sqrt{D}$ at $\theta = \pi/2$ for the analytic Kerr metric.

Thus the limit surface depends on the rotation parameter. Note that if the rotation parameter $a$ vanishes, we recover the well known limiting surface $r_c = 3M/2$ for Schwarzschild. Actual time slices will approach but do not fall inside this radius interior to the horizon. However, we usually use an antisymmetric slicing condition at the throat, causing the lapse to vanish there, preventing $r_c$ from evolving on throat. Further out from the throat, however, the coordinate system will evolve toward this limiting surface.

We have tested these ideas by numerically evolving rotating black holes with maximal slicing using our code, described in detail below. First we evolve a non-rotating hole, but with our new black hole construction described above. In Fig. 1 we plot the limit surface using the circumferential radius for an odd-parity distorted non-rotating hole, and we see that it locks onto the familiar $r_c = \frac{3}{2}M$ surface. The parameters describing the initial data for this calculation are $(Q_0', n', \eta_0', \sigma') = (2.0, 3.0, 1.0, 1.0)$. This run is described as run o1 in Paper II.
Next, Fig. 2 shows the computed value of $r_c$ for a rapidly rotating ($a/m = .70$) black hole. The calculation of $r_c$ from the data on the slice $t = 60M$ is given with a solid line, and the limit surface predicted by Duncan with a dashed line. In terms of the parameters in section III the initial data for the $a/m = .70$ spacetime is described by $(Q_0, \eta_0, \sigma, J, n) = (1.0, 1.0, 1.0, 10.0, 2)$. This same calculation is labeled run $r_4$ in Paper II, where its evolution is analyzed in detail.

As one can see, maximal slicing does indeed serve to limit the advance of proper time inside a rotating black hole, in spite of the fact that the singularity structure is so different from that of a non-rotating black hole. The slices do converge to the correct radius, as predicted by Duncan, near the horizon ($\eta \approx 2.7$). Outside this region the radius is, of course, much larger. This spacetime is not expected to have exactly the same limiting structure as the symmetrically sliced Kerr spacetime, since it has been strongly distorted by gravitational waves and since an antisymmetric lapse has been employed (freezing the evolution at the throat); hence the variations about Duncan’s prediction. We have tested this prediction for limit surfaces for a number of rotating black holes and find similar results in all cases. We note that this observation could be used to extract the rotation parameter, $a/m$, from the maximally sliced spacetime. We also note that these figures are very useful for estimating the location of the apparent horizon, even without an apparent horizon finder, since the slices will wrap up close to the limit surface inside, but move quickly away from it as one moves out, crossing the horizon.

2. Shift

In previous work on non-rotating black holes a shift vector was chosen to make the off-diagonal component of the 3–metric $\gamma_{\eta \theta}$ vanish (in that system no other off-diagonal components were present). This condition was found to be crucial to suppress a numerical instability occurring near the axis of symmetry. An elliptic equation for the two components of the shift ($\beta^\eta$ and $\beta^\theta$) present in that system provided a smooth shift and stable evolution. Following this philosophy we have developed a shift condition that generalizes this approach to the rotating case.

The gauge condition used in our evolutions is $C = 0$ and $E = 0$. Our choice has the property of reducing to the NCSA gauge used in the non-rotating work as the rotation goes to zero, and reducing to the Kerr shift as $Q_0 \rightarrow 0$. Since the Kerr shift allows the stationary rotating black hole metric to be manifestly time independent, one expects that for the dynamical case a similar shift will be helpful. The evolution equations for $C$ and $E$ can be used to construct a differential equation for the shifts. Other off-diagonal terms may be eliminated through appropriate shift choices, and as with the shift used in the non-rotating system, there is additional gauge degree of freedom that has not been exploited. We note that the quasi-isotropic shift has been used successfully in a recent study of rotating matter collapse [4].

Let us now consider how to implement the condition $C = 0$ and $E = 0$. First, the relevant metric evolution equations are:

$$
\partial_t C = -2\alpha H_C + A\partial_\eta \beta^\theta + B\partial_\eta \beta^n + F \sin \theta \partial_\eta \beta^\phi
$$

(27a)

$$
\partial_t E = -2\alpha H_E + D\partial_\eta \beta^\phi + F \frac{\beta^n}{\sin \theta}.
$$

(27b)
These equations can be combined to produce a single equation involving $\beta^n$ and $\beta^\theta$:

$$2\alpha \left( H_C - \frac{F \sin \theta}{D} H_E \right) = A \partial_\eta \beta^\theta + \left( B - \frac{F^2}{D} \right) \partial_\theta \beta^n.$$  \hspace{1cm} (28)

We can solve this equation by introducing an auxiliary function $\Omega$ through the definitions:

$$\beta^n = \partial_\eta \Omega, \hspace{1cm} (29a)$$

$$\beta^\theta = \partial_\eta \Omega, \hspace{1cm} (29b)$$

(following Ref. [19]), producing an elliptic equation for the function $\Omega$:

$$2\alpha \left( H_C - \frac{F \sin \theta}{D} H_E \right) = A \partial^2_\eta \Omega + \left( B - \frac{F^2}{D} \right) \partial^2_\theta \Omega.$$ \hspace{1cm} (30)

This equation is then solved by finite differencing using a numerical elliptic equation solver discussed in the next section. The solution $\Omega$ is then differentiated by centered derivatives to recover the shift components $\beta^n$ and $\beta^\theta$ according to Eqs. (29). In practice, these shifts remain fairly small during the evolution. Their main function is to suppress the axis instability, as noted in Ref. [19] where a similar shift was used.

Once $\Omega$ is known, $\beta^\phi$ can be calculated by integrating Eq. (27b):

$$\beta^\phi = \int_{\eta_{\text{max}}}^{\eta} \frac{d\eta}{D} \left( 2\alpha H_E - F \partial_\eta \beta^\theta / \sin \theta \right).$$ \hspace{1cm} (31)

Only one boundary condition needs to be set (the outer boundary condition is most convenient), and it is generally set equal to the Kerr value. The inner boundary condition, that $\beta^\phi$ must be symmetric across the throat, is guaranteed by Eq. (27b). This shift component is needed to keep the coordinates from becoming “tangled up” as they are dragged around by the rotating hole. Without such a shift the coordinates would rotate, leading to metric shear [25]. This shift component, $\beta^\phi$, is typically larger than $\beta^n$ or $\beta^\theta$.

This method for obtaining the shift has proved effective, although there are some numerical difficulties that should be mentioned. First, since $\beta^\phi$ is computed by integrating an ODE, errors tend to accumulate as the integration progresses inward. This can cause trouble when integrating across the sharp peaks that develop in the metric functions near the horizon of a black hole (See section \textbf{V} below for discussion of these peaks). On the other hand, this occurs in the region where the lapse collapses significantly (near the inner portion of the grid), and is not noticeable before the axis instability sets in (see section \textbf{V} for a discussion of the instability), so it has not been a serious problem. For Kerr initial data, the numerically computed shift had a maximum deviation from the analytic solution of about 0.1% on a $200 \times 55$ grid. We show a plot of the shift in Fig. \textbf{4} for a Kerr black hole with a rotation parameter of $a/m = .676$. For dynamic black holes, the shift takes on a similar form.

In addition, each $\theta = \text{constant}$ line of integration is independent of the others. As a result of this decoupling, there can be fluctuations in the shift across different angular zones near
the axis. This becomes more apparent at late times when the axis instability, common in most axisymmetric codes, sets in. As noted above this problem is not noticeable until after the instability sets in, and so it is not a cause of difficulty. An example of this problem is shown in the next section.

B. Computational Issues and Numerical Issues

The code was developed using MathTensor, a package that runs under Mathematica, to convert the equations to Fortran readable form. Scripts have been written that automatically produce Fortran code given symbolic input in MathTensor, so different variable choices and gauge conditions can be tested fairly easily.

The actual evolution code was written in Fortran 77 to run on the Cray Y-MP and the Cray C-90. Currently, for a $200 \times 55$ grid it obtains about 160 MFlops on a single processor C-90. About 70% of the time to evolve a spacetime with this code is spent solving elliptic equations (two elliptic equations must be solved on each time slice). The multigrid linear system solver used by our code for the elliptic equations was provided by Steven Schaffer of New Mexico Technical Institute [26] (UMGS2). We find that although it does not achieve the highest code performance in the traditional sense (measured by MFlops), it produces a solution in very few iterations, so the time to solution is often less than we achieve with “higher performance” iterative solvers [27]. UMGS2 is a semi-coarsening multigrid code. This differs from full multigrid (described in detail in the above reference) by only performing the coarsening along the angular dimension of the grid. This is quite useful because the spacetimes vary much less in the angular direction than in the radial direction.

The evolution method that we have chosen is leapfrog. Leapfrog is an explicit evolution scheme that requires us to keep the metric variables on two time steps, and the extrinsic curvature data on two other time steps sandwiched between the metric variables. Schematically the evolution looks like this:

\[
\begin{align*}
\tilde{\gamma}_{t+\Delta t/2} &= \frac{3}{2}\gamma_t - \frac{1}{2}\gamma_{t-\Delta t} \\
\gamma_{t+\Delta t} &= \gamma_t + \Delta t\tilde{\gamma}\left(K_{t+\Delta t/2}, \tilde{\gamma}_{t+\Delta t/2}\right) \\
K_{t+\Delta t} &= \frac{3}{2}K_{t+\Delta t/2} - \frac{1}{2}K_{t-\Delta t/2} \\
K_{t+3\Delta t/2} &= K_{t+\Delta t/2} + \Delta t\dot{K}\left(\tilde{K}_{t+\Delta t}, \gamma_{t+\Delta t}\right)
\end{align*}
\]

Spatial derivatives needed in the above equations were calculated using centered, second order finite differencing. This scheme is essentially the same as that discussed in Ref. [10].

As in most asymmetric black hole codes to date, our code has an axis instability (See Ref. [10] for a detailed discussion of the axis instability). This instability grows worse as rotation increases. An effective strategy to slow the growth of this instability is to reduce the number of angular zones, thus keeping them farther away from the axis. As mentioned above, we have also used the gauge freedom in the equations to eliminate off-diagonal elements in the 3-metric that tend to exacerbate this instability. However, at late times, ($t \approx 70-100M$),
when peaks in metric functions become large, strong instabilities can develop, causing the code to crash. These next two plots show the function $B$ for the run labeled $r4$, a distorted Bowen and York (rotating) black hole. In Fig. 4(a) we see the function at $t = 100M$ for a grid resolution of $150 \times 24$. In Fig. 4(b) we see the same function for $t = 70M$ with a grid resolution of $300 \times 48$ (all angular zones are shown in the figure). The instability is clearly visible at earlier times in the higher resolution calculation, while the lower resolution calculation is still stable at later times. Note that it is the region in the interior of the black hole that develops difficulty (the horizon is located near $\eta = 3$, where the dip in the metric function $B$ occurs).

Because the radial resolution is most crucial, we often perform calculations with resolutions of $300 \times 30$ zones, providing both good accuracy and late time stability. It is the angular resolution that is most crucial in determining when the numerical instability becomes serious. Higher angular resolution does give more accurate results, but also leads to instabilities at earlier times.

We have performed a series of convergence tests on our code. Convergence was measured along the line $\theta = \pi/4$ for a number of metric functions, and for the conformal factor. Because we did not have data placed along this value of $\theta$ we interpolated it from our existing data using a third order interpolation scheme. The convergence rate of a given quantity was calculated by comparing results obtained at three resolutions in a similar manner as reported in Ref. [10]. The basic principle is to assume the error in a given quantity is proportional to $(\Delta \eta)^\sigma$, and then the convergence rate $\sigma$ is determined experimentally. For a formally second order accurate numerical scheme, such as ours, one expects $\sigma \approx 2$.

For the purposes of these tests we required that $\Delta \theta = \Delta \eta$. As in previous work [10] we performed most of the tests with a unit lapse. We found, in general, that all quantities converged to second order with slight variations throughout the domain. This applied to both high amplitude Brill wave data sets and rapidly rotating data sets. In Fig. 5 we show the result of a convergence test for a pure Kerr spacetime with $J = 5$ (this turns out to have $a/m = .676$). We evolved it at three different resolutions corresponding to 75, 150, and 300 radial zones and checked the convergence of the radial variable $A$. The results are shown after $4.8M$. In two places the direction of the convergence changes, making the convergence almost impossible to measure there. These points are each labeled “crossing point” in the figure. These results are typical for a variety of convergence tests we have performed.

V. DISCUSSION

A. Comparison with 1D codes

In this section we discuss the evolution of the metric functions, both to compare the present code to the non-rotating NCSA code, and to show what effect rotation has on the system. For these purposes we compare and contrast the evolution of two different black hole spacetimes representing a wide range of the kinds of problems our code can evolve. The first is a pure Schwarzschild black hole, evolved with a symmetric lapse. This has become a classic test problem for black hole codes [9,10,18]. Because the system can be evolved very accurately in 1D (given sufficient resolution), and also because it is a difficult problem due to very large peaks that develop in the solution [28], it is a strong testbed calculation. In Fig. 6
we show a comparison of our new code for rotating spacetimes with a 1D code described in Ref. [10]. Both codes were run with the same radial resolution, $\Delta \eta \approx 0.020$, with the same time steps $\Delta t = \Delta \eta$. In the 2D case the angular resolution was taken to be $\Delta \theta \approx 0.033$. The radial metric function $A$ is shown at several times for both codes. Only a single angular zone is shown: The initial data are spherically symmetric, and our codes maintain this symmetry to a high degree of accuracy throughout the evolution. All other angular zones are indistinguishable. The agreement between the two codes is excellent, through the time $t = 100M$ shown here. It is important to note that the 2D evolution was performed as a full 2D problem without specializing it in any way. For example, the maximal slicing equation was solved as a full 2D elliptic equation in the 2D code, whereas it is solved as a simple ordinary differential equation in the 1D code.

A key feature of this and the other black hole spacetimes studied here is that constant $\eta$ observers fall inward toward the singularity, actually passing through the horizon in finite time. As a result, more and more grid zones represent regions inside the horizon as the evolution progresses. Because of the differential rate at which these observers fall through the horizon the grid stretches, creating a sharp peak in the radial metric function $A$ as the evolution continues, as shown in Fig. 6. This effect assures us that the code will eventually become inaccurate and crash. This is the bane of all black hole evolution evolution codes at present, and is the main motivation for considering apparent horizon boundary conditions that can, in principle, eliminate this problem [28,29].

We have also evaluated the error in the Hamiltonian constraint. This error is in the form of a mass density which is given by

$$\rho = \frac{1}{16\pi} \left( R - K_{ab} K^{ab} + K^2 \right). \quad (33)$$

The error in this quantity is dominated by the axis instability. However, because this instability grows most rapidly near the horizon its effect on the code is partially cancelled by the collapse of the lapse. Because we are not as interested in violations of the Hamiltonian constraint inside the horizon, since this should not affect what happens outside for hyperbolic evolution, we will be more interested in the quantity $\alpha |\rho|$. (Although we do solve elliptic equations, which propagate information instantaneously, these equations are for gauge conditions. An “error” in a gauge condition, which is arbitrary, does not affect physical results in principle.) The maximum error in this quantity as a function of time for various runs is plotted in Fig. 7. Our evolutions are generally unable to proceed past the point where this maximum density passes unity. The Hamiltonian constraint violation illustrated by the maximum error is dominated by the axis instability, and highly localized.

Because of the strong locality of this error and the hyperbolic nature of our evolution, we do not feel, however, that the maximum error provides us with the best understanding of the overall accuracy of the code. For this reason we also consider the average value of $|\rho|$ over the grid (weighted by the lapse function $\alpha$). We provide a plot of this measure of the error for the same runs in Fig. 8. It should be noted that this quantity is generally 4-8 orders of magnitude smaller than the maximum error on the grid for all runs considered. The runs that represent distorted spacetimes are initially less smooth, and for this reason the Hamiltonian violation, is larger than for Schwarzschild. However, it is small and grows at a rate comparable to that of the Schwarzschild evolution.
B. Comparison with the 2D codes

Although the 1D problem is a good test of the longitudinal component of the field it does not test the code’s ability to handle a highly non-spherical, distorted black hole. The next case we consider is a distorted, non-rotating black hole with a Brill wave, labeled run \( r\theta \) and is specified by \((Q_0, \eta_0, \sigma, J, n) = (1.0, 1.0, 1.0, 0.0, 2)\). Initially, this data set is a highly distorted and nonspherical black hole that evolves in an extremely dynamic way, so it is a very strong test case for this new code. In Fig. 9 we show the evolution of this distorted non-rotating black hole spacetime obtained with our code, and compare it to the same evolution obtained with the code described in Ref. [9,10]. The radial metric function \( A = g_{\eta\eta}/\Psi^4 \) is shown at time \( t = 60M \). Only a single angular zone is plotted \((\theta = \pi/2)\), as by this time the developing peak is quite spherical and nearly independent of \( \theta \). In the figure we compare the evolution obtained with the rotating code with symmetric lapse, antisymmetric lapse conditions, and the non-rotating code of Ref. [9,10] with symmetric lapse. When the same lapse conditions are used, both codes show excellent agreement. Note also that even with an antisymmetric lapse (though the early evolution is not shown) the runs will be noticeably but minimally different at late times. This is the first time these data sets have been evolved with this antisymmetric slicing condition.

We have compared many other aspects of results of our code with those obtained with the code described in Ref. [9,10] and find excellent agreement for metric functions, extracted waveforms, horizon masses, etc. The comparisons have been performed for a wide range of data sets. A detailed study of horizons and waveforms will be presented in Paper II.

C. Rotating spacetimes

Rotating spacetimes differ only slightly from the picture discussed above. Constant \( \eta \) observers along the equator are, in general, rotating about the black hole. Because of this, their fall through the horizon is slower and grid stretching is less near the equator. The peak in the metric function \( A \) is, therefore, lesser in magnitude here. This gives the data for a rotating black hole spacetime an obvious angular dependence not present in distorted non-rotating metric data.

The pure stationary Kerr spacetime is, in general, much less stable numerically than the Schwarzschild, or even the distorted Kerr, and special care is required to evolve it. We have two methods which we use to carry out the evolution. The first is to use a lapse which is symmetric across the throat. We begin our discussion of rotating black hole evolutions by showing a surface plot of metric variable \( A = g_{\eta\eta}/\Psi^4 \) for a symmetrically-sliced high resolution \((300 \times 30)\) \( J = 5 \) (\( a/m = .68 \)) Kerr spacetime in Fig. 10. The error in \( J \) as computed by Eq. (3b) reached a maximum error of 3.6% over the grid at this late time, and occurs over the peak in \( g_{\eta\eta} \). This is an important and nontrivial test, as \( J \) must be conserved during the evolution even though the different metric and extrinsic curvature components entering Eq. (3b) evolve dramatically.

Next we consider this same spacetime with an antisymmetric lapse. The antisymmetric, maximal sliced Kerr spacetime is actually a time independent analytic solution, but it is numerically unstable. Any slight numerical error will destroy the perfect balance between the Ricci terms and the derivatives of the lapse in the extrinsic curvature evolution equations.
Then the solution will evolve rapidly, behaving much like the symmetrically sliced Kerr spacetime. In order to evolve this system stably for long times, we force metric variable $F = g_{\theta \phi}/\Psi^4$ to be zero at all times (in this stationary spacetime this must be true). Extending the grid to larger radii is also helpful. When both of these things are done we are able to evolve past $50M$ before serious problems develop. The $J = 5$ Kerr evolution with an antisymmetric lapse at the throat begins to look like the symmetric lapse run at late times. A peak forms in the radial metric function $A$, although a much smaller one than is seen with the symmetric lapse. This is illustrated in Fig. 11. Because the lapse is collapsed in the inner region $\alpha$ is virtually zero over a number of grid zones and this satisfies both the symmetric and antisymmetric conditions to a good approximation. Note that even the Schwarzschild spacetime is difficult to evolve with an antisymmetric lapse since the axis instability becomes quite serious at about $60M$. The error in $J$ as computed by Eq. (8b) reached a maximum error of 1.6% over the grid.

Finally, we present a distorted rotating black hole. Calculation labeled run $r_4$ is specified by $(Q_0, \eta_0, \sigma, J, n) = (1.0, 1.0, 1.0, 10.0, 2)$. These labels correspond to the same simulations discussed in detail in Paper II, where many physical properties of the spacetimes are analyzed. On a $300 \times 48$ grid, the maximum error in the momentum integral was 1.6%. Again, at this high angular resolution the axis instability develops rapidly after about $t = 70M$, causing the code to crash. Lower resolution runs can be carried out past $t = 100M$ with similar results.

In Fig. 12 we show a surface plot of the radial metric function $A$ for the rotating hole at time $t = 60M$. Although the familiar peak develops in this function, it does develop the expected angular dependence. This is typical of all our rotating black hole evolutions.

Another typical feature of these spacetimes is "slice-wrapping," discussed in section IV A 1. This name refers to the fact that slices inside the horizon approach a limit surface with roughly constant $r$ value. This is connected to the discussion of the limit surfaces above, but here we use this feature to illustrate the full 2D behavior of the metric functions to show that they evolve as expected. As a result of slice-wrapping, in our spacetimes the value of $\Psi^2\sqrt{D}$ (which is the circumferential radius when evaluated at $\theta = \pi/2$) becomes constant over larger regions of the spacetime as the evolution continues. In a non-rotating spacetime with a lapse that is symmetric across throat this value goes to $3M/2$. These spacetimes have a different limiting value, as discussed in section IV A. In Fig. 13 we show an example of this effect for the spacetime labeled $r_4$. The slice-wrapping effect is clear inside the radius of about $\eta \approx 2.7$ where the slice moves out away from the limit surface, out across the horizon. From studies of horizons of black holes like those in Ref. [30], we know the horizon is located near $\eta = 2.7$.

The strong internal structure evident in $D = g_{\phi \phi}/\Psi^4$ well inside the black hole, near the throat ($\eta = 0$), is a remnant of the initial Brill wave, indicating that this black hole was initially quite distorted. This structure developed early on as gravitational waves propagated into the hole, but was "frozen in" as the lapse collapsed rapidly in this region of the spacetime.

A similar behavior is observed in the metric variable $B = g_{\theta \theta}/\Psi^4$ as in $D$, except that while $D = 1$ initially, $B = e^{2(q - q_0)}$ and so the antisymmetric lapse "freezes in" these different functions at the throat. Because the spacetime can evolve even a short distance away from there a sharp peak develops at the throat with this slicing condition. Fortunately, the lapse
is always small in this region and the code does not suffer as a result. In Fig. 14 we show
the behavior of $B$, and Fig. 15 we show the behavior of $D$. In both plots we are looking
at run labeled $r4$ at time $60M$. At high resolution ($300 \times 48$) the axis instability becomes
serious at about $70M$, but with lower resolutions it can go past $100M$.

VI. CONCLUSIONS

The study of rotating spacetimes presents a new level of complexity to the distorted
black hole, and we have developed a new code to evolve such spacetimes by building on
previous non-rotating work [10]. Although the rotation requires the introduction of new
metric and shift variables, we have been able to bring many of the same numerical methods
to bear on the problem that have been used before. We have shown that the new code
is able to reproduce results of Ref. [5,9,10] for spherical and highly distorted non-rotating
holes, including the behavior of both the metric functions and derived quantities. We have
also shown the effect of rotation on the metric functions and how they behave differently for
rotating black hole spacetimes. In addition to reproducing previous results, the code has also
passed other tests, such as convergence tests and the conservation of angular momentum,
and it confirms the theoretical predictions for the relation between the limit slice for a
maximally sliced black hole and its rotation parameter.

We have also introduced a new family of distorted black hole data sets, including distorted
rotating black holes and odd-parity distorted non-rotating black holes. The next paper in
this series will discuss results obtained from applying this code to these new data sets.
Specifically it will show how to calculate the location of the apparent horizon and how to
extract the waveforms for the various $\ell$ modes radiating from the black hole. It will also
analyze the behavior of these aspects of the spacetimes for a series of evolutions forming
a sequence of black holes with increasing rotation, and also for odd-parity distorted black
holes.

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FIGURES

FIG. 1. This plot shows $r_c$, the circumferential radius of the maximally sliced black hole as a function of radial coordinate $\eta$, for run labeled $o1$ at time $t = 70M$. The theoretical limit surface for a non-rotating hole is $r = 1.5M$, but our boundary condition on the lapse prevents us from reaching that point at the throat ($\eta = 0$). Nevertheless, away from the throat, the predicted limit surface is reached.

FIG. 2. In this figure the circumferential radius $r_c$ is plotted at time $t = 60M$. For Kerr black holes the limit surface is not $r_c = 1.5M$ but a higher value that depends on the rotation parameter $a/m$, as predicted by Duncan. The dashed line is the theoretically predicted $r = \text{constant}$ limit. In regions away from the throat (where the lapse is zero and the spacetime may not evolve) we see that this limit surface is reached.

FIG. 3. We show a 2D plot of the shift $\beta^\phi$ as computed by the rotating code for a Kerr black hole with $J = 5$. It is virtually identical to the analytic shift function one would obtain from the exact Kerr solution.

FIG. 4. In both plots that follow, all of the angular zones are displayed. (a) This 2D plot shows the contents of metric variable $B = g_{\theta \theta}/\Psi^4$ at a resolution of $150 \times 24$ for run $r4$, a highly distorted rotating black hole. Using this resolution, we can evolve the spacetime to $100M$ before numerical instabilities develop. (b) This 2D plot shows the metric variable $B = g_{\theta \theta}/\Psi^4$ (where $\Psi$ is a conformal factor) in run $r4$. It has a higher spatial resolution ($300 \times 48$) and it develops trouble at an earlier time. The ridges near the throat are a result of the axis instability and the radial integration we use to compute the $\beta^\phi$ component of the shift.

FIG. 5. This figure plots the convergence exponent $\sigma$ as a function of radial coordinate $\eta$ for the radial metric variable $A = g_{\eta \eta}/\Psi^4$ (where $\Psi$ is a conformal factor) of a Kerr black hole spacetime with $J = 5$, as measured along $\theta = \pi/4$, evolved to $t = 4.8M$. As discussed in the text, $\sigma$ was measured by evolving the spacetime at 3 different solutions. As our methods are second order, we expect to see roughly a horizontal line at $\sigma = 2$ ($\sigma$ is the order of convergence). The points labeled “crossing point” are places at which the direction of the convergence is changing, and we do not expect to be able to measure the convergence well at these points.

FIG. 6. This plot compares a 1D code to our 2D code for a Schwarzschild spacetime. Each solid (dashed) line represents the metric variable $A = g_{\eta \eta}/\Psi^4$ (where $\eta$ is a logarithmic radial coordinate and $\Psi$ is a conformal factor) from the 2D (1D) code at a time interval of $10M$, beginning with $10M$ and continuing to $100M$.

FIG. 7. In this plot we show the maximum violation in the Hamiltonian constraint function $\alpha|\rho|$ on the grid. The error is dominated by the effects of the axis instability and is highly localized there, and evolution becomes impossible shortly after this error exceeds unity. The error is measured in units of $M_{\text{ADM}}^{-2}$. 

FIG. 8. In this plot we show the average of the Hamiltonian constraint function $|\rho|$ (weighted by the lapse function). The errors for distorted black holes is noticeably larger since these functions are initially less smooth. The error is measured in units of $M_{ADM}^{-2}$.

FIG. 9. In this figure we compare several calculations of the radial metric function $A$ at time $t = 60M$ for run $r0$ (a distorted black hole with no angular momentum nor odd-parity distortion). Three results are shown: both codes were run with a symmetric lapse across the throat, and the present rotating code was also run with an antisymmetric lapse across the throat. We see that the rotating and non-rotating code produce practically identical results for the symmetric lapse. We also see that the antisymmetric lapse produces similar results, except near the throat where it must be different.

FIG. 10. This surface plot shows the radial metric function $A = g_{\eta\eta}/\Psi^4$ (where $\eta$ is a logarithmic radial coordinate and $\Psi$ is a conformal factor) at time $t = 80M$ for an antisymmetrically sliced Kerr spacetime with $J = 5$ ($a/m = .68$). The characteristic peak in $A$ is well-developed and its angular dependence, explained in the text, is clearly visible.

FIG. 11. This surface plot shows the radial metric function $A = g_{\eta\eta}/\Psi^4$ (where $\eta$ is a logarithmic radial coordinate and $\Psi$ is a conformal factor) at time $t = 50M$ for an antisymmetrically sliced Kerr spacetime with $J = 5$ ($a/m = .68$). At this high resolution ($300 \times 30$), the axis instability sets in shortly after this time.

FIG. 12. This is a surface plot of metric function $A = g_{\eta\eta}/\Psi^4$ (where $\eta$ is a logarithmic radial coordinate and $\Psi$ is a conformal factor) at time $t = 60M$ for the distorted rotating black hole run labeled $r4$. The presence of a $\phi$ shift in the evolution means $\eta$ = constant observers near the equator feel a “centripetal force” and fall in more slowly. This results in less grid-stretching, and a lower peak there.

FIG. 13. This surface plot is of the function $R = \Psi^2 D^{1/2}$ for the run labeled $r4$. Along the equator this value is the equatorial circumferential radius. This plot shows that the “slice-wrapping” effect affects a large portion of the grid.

FIG. 14. This surface plot depicts metric variable $B = g_{\theta\theta}/\Psi^4$ (where $\Psi$ is a conformal factor) at time $70M$ for the run labeled $r4$. Note that the angular shape of the Brill wave on the initial data slice is preserved at the throat, but outside this region it becomes very close in value to metric variable $D = g_{\phi\phi}/\Psi^4$.

FIG. 15. This surface plot depicts metric variable $D = g_{\phi\phi}/\Psi^4$ (where $\Psi$ is a conformal factor) at time $60M$ for the run labeled $r4$. Note that at the throat $D = g_{\phi\phi}/\Psi^4$ has value 1, as it does in the initial data. Farther from the throat it has the same value as $B = g_{\theta\theta}/\Psi^4$. 