A TOPOLOGICAL GROUP OF EXTENSIONS OF \( \mathbb{Q} \) BY \( \mathbb{Z} \)

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Abstract. The group of extensions (as in the title), endowed with something like a connection at Archimedean infinity, is isomorphic to the adèle-class group of \( \mathbb{Q} \): which is a topological group with interesting Haar measure.

For Mike Boardman and Takashi Ono: dear friends and colleagues

1.1 An extension of one abelian group \( A \) (ie, a \( \mathbb{Z} \)-module), by another \( (B) \) is an exact sequence

\[
\mathcal{E} : 0 \to B \xrightarrow{i} E \xrightarrow{j} A \to 0;
\]

the exact functor

\[
C \mapsto C \otimes \mathbb{Z} \mathbb{R} := C_R
\]

associates to \( \mathcal{E} \) an extension

\[
\mathcal{E}_R : 0 \to B_R \xrightarrow{i_R} E_R \xrightarrow{j_R} A_R \to 0
\]

of real vector spaces, which necessarily splits. This note is concerned with extensions \( \mathcal{E} \) as above, which have been rigidified by the choice

\[
s_\mathcal{E} : A_R \to E_R
\]

of a splitting of \( \mathcal{E}_R \), ie a homomorphism such that \( j_R \circ s_\mathcal{E} = 1_{A_R} \). I will refer to the pair \( \tilde{\mathcal{E}} := (\mathcal{E}, s_\mathcal{E}) \) as an extension of \( \mathbb{Z}_0 \)-modules.

A congruence \( \alpha \) of two extensions \( \mathcal{E}, \mathcal{E}' \) of \( A \) by \( B \) is a commutative diagram

\[
\begin{array}{ccc}
0 & \to & B & \xrightarrow{i} & E & \xrightarrow{j} & A & \to & 0 \\
\downarrow{1_B} & & \downarrow{\alpha} & & \downarrow{1_A} & & \downarrow{1_B} & & \downarrow{1_A} \\
0 & \to & B' & \xrightarrow{i'} & E' & \xrightarrow{j'} & A & \to & 0
\end{array}
\]

cf eg [6 III §1]. Let us say that a congruence \( \alpha : \tilde{\mathcal{E}} \equiv \tilde{\mathcal{E}}' \) of rigidified extensions is a congruence \( \alpha : \mathcal{E} \equiv \mathcal{E}' \) of their underlying extensions of

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\[ \begin{array}{ccc}
E_R & \xrightarrow{s_{\varepsilon}} & A_R \\
\downarrow{\alpha_R} & & \downarrow{1_{AR}} \\
E'_{\mathbb{R}} & \xrightarrow{s_{\varepsilon'}} & A_R
\end{array} \]

Congruence classes of such rigidified extensions define an abelian group-valued bifunctor \( \text{Ext}^{\mathbb{Z}}(A, B) \), with a straightforward generalization of Baer sum (as we shall check below) as composition. Forgetting the splitting data defines an epimorphism

\[ \text{Ext}^{\mathbb{Z}}(A, B) \to \text{Ext}_{\mathbb{Z}}(A, B). \]

1.2.1 Proposition This forgetful homomorphism fits in an exact sequence

\[ 0 \to \text{Hom}_R(A_R, B_R) \to \text{Ext}^{\mathbb{Z}}_0(A, B) \to \text{Ext}_{\mathbb{Z}}(A, B) \to 0; \]

in particular, the exact sequence

\[ 0 \to \mathbb{R} \to \text{Ext}^{\mathbb{Z}}_0(\mathbb{Q}, \mathbb{Z}) \to \text{Ext}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \to 0 \]

is isomorphic to the sequence

\[ 0 \to \mathbb{R} := \Sigma_0 \to \Sigma \to (\hat{\mathbb{Z}} \otimes \mathbb{Q})/\mathbb{Q} \to 0 \]

defined by the inclusion of the path-component \( \Sigma_0 \) of the identity in the solenoid

\[ o \to \mathbb{Q} \to A_\mathbb{Q} := \mathbb{R} \times (\hat{\mathbb{Z}} \otimes \mathbb{Q}) \to \Sigma \cong \text{Hom}(\mathbb{Q}, \mathbb{T}) \to 1 \]

(i.e. the Pontrjagin dual of the rational numbers: which is connected but not path-connected).

1.2.2 More generally, if \( o_K \) is the ring of algebraic integers in a number field \( K \), then

\[ \text{Ext}^{\mathbb{Z}}_0(\mathbb{Q}, o_K) \cong A_K/K \]

is naturally isomorphic to the adele-class group of \( K \) [3 §14, 5 §5.3]; a compact topological group (a product of \(|K : \mathbb{Q}| \) copies of \( \Sigma \)) with canonical Haar measure whose square equals the absolute value of the discriminant \( D_{K/\mathbb{Q}} \) of \( K \) over \( \mathbb{Q} \).

Remark The injective resolution

\[ 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \]

identifies

\[ \text{Ext}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \cong (\hat{\mathbb{Z}} \otimes \mathbb{Q})/\mathbb{Q} \]
with the cokernel of the map defined by tensoring the (dense) inclusion
\[ \mathbb{Z} \to \hat{\mathbb{Z}} = \prod \mathbb{Z}_p \]
(of the integers into the product of the profinite integers) with \( \mathbb{Q} \): which is superfluous, since \( \hat{\mathbb{Z}}/\mathbb{Z} \) is uniquely divisible and is thus already a \( \mathbb{Q} \)-vector space, whose natural \([\?]\) topology is then \textbf{indiscrete}. See [2 Theorem 25] for an account (which motivated this note) of this classical group of extensions.

1.3.1 \textbf{Proof:} We need first to define the Baer sum of two rigidified extensions \( \tilde{E}_0, \tilde{E}_1 \) as the quotient \( \tilde{E}_1 + \) of the pullback \( E_1 \leftarrow \bullet \to \bullet \to E_0 \to \to \bullet \to 0 \) by the image of the map
\[ b \mapsto i_\Delta(b) = (i_0(b), -i_1(b)) : B \to E_+ . \]
The resulting sum is split, after tensoring with \( \mathbb{R} \), by
\[ A \ni a \mapsto (s_0(a), s_1(a)) \in E_+ = E/(\text{image } i_\Delta) . \]
To check the first assertion of the proposition, note that if two extensions \( \tilde{E}, \tilde{E}' \) of \( \mathbb{Z}_0 \)-modules have congruent underlying extensions of \( \mathbb{Z} \)-modules, then tensoring those extensions with \( \mathbb{R} \) defines an isomorphism
\[ 0 \to B_\mathbb{R} \xrightarrow{i} E_\mathbb{R} \xrightarrow{j} A_\mathbb{R} \to 0 \]
\[ 0 \to B_\mathbb{R} \xrightarrow{i'} E'_\mathbb{R} \xrightarrow{j'} A_\mathbb{R} \to 0 \]
of extensions of real vector spaces, with splittings \( s, s' \). If
\[ \rho' := s' - s \circ \alpha : A_\mathbb{R} \to E'_\mathbb{R} \]
then \( j' \circ \rho' = 0 \), so the image of \( \rho' \) lies in the image of \( i' \), defining
\[ (i')^{-1} \circ \rho' \in \text{Hom}_\mathbb{R}(A_\mathbb{R}, B_\mathbb{R}) . \]
On the other hand such homomorphisms act freely on the \( \mathbb{Z}_0 \)-module extensions of \( A \) by \( B \): \( \text{Hom}_\mathbb{R}(A_\mathbb{R}, B_\mathbb{R}) \) is the group, under Baer sum, defined by splittings of the exact sequence
\[ 0 \to B_\mathbb{R} \to B_\mathbb{R} \oplus A_\mathbb{R} \to A_\mathbb{R} \to 0 . \]
To prove the second assertion of the proposition we construct a homomorphism

$$\overline{\Delta} : \text{Ext}_{\mathbb{Z}_0}(\mathbb{Q}, \mathbb{Z}) \to \text{Hom}(\mathbb{Q}, T)$$

as follows: given a diagram

$$
\begin{array}{c}
0 \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
\rightarrow \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
E_{\mathbb{R}} \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
\rightarrow \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
\rightarrow \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
E_{\mathbb{R}} \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
\rightarrow \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
\rightarrow \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
E_{\mathbb{R}} \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
\rightarrow \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
\rightarrow \end{array}
$$

let

$$\Delta_0(e) := e_{\mathbb{R}} - s(j(e)_{\mathbb{R}}) : E \to E_{\mathbb{R}}$$

(where $x_{\mathbb{R}} := x \otimes 1_{\mathbb{R}}$); then $j_{\mathbb{R}} \circ \Delta_0 = 0$, so the image of $\Delta_0$ lies in the image of $i_{\mathbb{R}}$, defining

$$\Delta := i_{\mathbb{R}}^{-1} \circ \Delta_0 : E \to \mathbb{R}.$$ 

But now $\Delta \circ i : \mathbb{Z} \to E \to \mathbb{R}$ is the usual inclusion, so $\Delta$ induces a homomorphism

$$[\overline{\Delta} : \mathbb{Q} = E/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}] \in \text{Hom}(\mathbb{Q}, T) \ldots$$

1.3.2 Here is an example:

$$
\begin{array}{c}
0 \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
\rightarrow \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
\mathbb{Z}(p) \oplus \mathbb{Z}[p^{-1}] \\ \downarrow \\ \mathbb{R} \oplus \mathbb{R}
\end{array}
\begin{array}{c}
\rightarrow \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
\rightarrow \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
\mathbb{Z}(p) \oplus \mathbb{Z}[p^{-1}] \\ \downarrow \\ \mathbb{R} \oplus \mathbb{R}
\end{array}
\begin{array}{c}
\rightarrow \\ \downarrow \\ \mathbb{R}
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
\rightarrow \end{array}
$$

with $i(k) = (k, k), j(u, v) = u - v$, and $\sigma(x) = ((s + 1)x, sx)$ for some $s \in \mathbb{R}$.

[Check that $j$ is onto, ie that $q \in \mathbb{Q}$ equals $u - v$ for some $u \in \mathbb{Z}(p)$ and $v \in \mathbb{Z}[p^{-1}]$:

Let $q = q_0p^{-n}/q_1$ with $p \nmid q_0, q_1$ and $n > 0$ (otherwise the claim is immediate, with $v = 0$). Then $(p^n, q_1) = 1$ implies the existence of $\alpha, \beta$ such that

$$\alpha p^n + \beta q_1 = 1,$$

so if $a = q_0\alpha, b = -q_0\beta$ then $ap^n - bq_1 = q_0$ and hence

$$q = \frac{ap^n - bq_1}{q_1p^n} = \frac{a}{q_1} - \frac{b}{p^n}.$$ 

Now we have $\Delta_0(u, v) = (u, v) - \sigma(u - v) = i(v - s(u - v))$, so

$$\Delta(u, v) = (s + 1)v - su \mod \mathbb{Z},$$

eg \overline{\Delta}(q) = sq + q_0\beta p^{-n} \in \mathbb{T}.$
1.3.3 More generally, Ext_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}^k) is isomorphic to the product $\Sigma^k \cong (\mathbb{A}_Q/\mathbb{Q})^k$. The ring $\mathfrak{o}_K$ of integers in a number field, however, has more structure, which endows its adèle-class group $\mathbb{A}_K/K \cong (\mathbb{A}_Q/\mathbb{Q})^k$ with a Haar measure normalized [8 V §4 Prop 7] as asserted in the proposition.

1.4 Remarks I suppose the proposition above has a natural reformulation in Arakelov geometry [4 §1], i.e. in terms of $\mathbb{Z}$-modules $A, B$ endowed with positive-definite inner products on $A_{\mathbb{R}}, B_{\mathbb{R}}$; but I don’t know anything about Arakelov geometry.

It also seems plausible that

$$\text{Ext}_{\mathbb{F}[t]}(\mathbb{F}(t), \mathbb{F}[t]) \cong \mathbb{A}_{\mathbb{F}(t)}/\mathbb{F}(t)$$

for finite fields $\mathbb{F}$; but its analog of Haar measure seems to be more like an invariant one-form [1].

1.5 Acknowledgements I owe A. Salch [6 §3] thanks, for bringing these matters to my attention. This note grew out of conversations with Ch. Deninger and R. Meyer at the September 2013 Oberwolfach workshop on noncommutative geometry, and I would like to thank them both for their interest and help.

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1 Indeed this note is essentially a footnote to work of Bost and Künnemann [9]; more thanks to ChD for this reference!