Risk Bounds for Quantile Trend Filtering

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Abstract

We study quantile trend filtering, a recently proposed method for nonparametric quantile regression with the goal of generalizing existing risk bounds known for the usual trend filtering estimators which perform mean regression. We study both the penalized and the constrained version (of order $r \geq 1$) of univariate quantile trend filtering. Our results show that both the constrained and the penalized version (of order $r \geq 1$) attain the minimax rate up to log factors, when the $(r-1)$th discrete derivative of the true vector of quantiles belongs to the class of bounded variation signals. Moreover we also show that if the true vector of quantiles is a discrete spline with a few polynomial pieces then both versions attain a near parametric rate of convergence. Corresponding results for the usual trend filtering estimators are known to hold only when the errors are sub-Gaussian. In contrast, our risk bounds are shown to hold under minimal assumptions on the error variables. In particular, no moment assumptions are needed and our results hold under heavy-tailed errors. Our proof techniques are general and thus can potentially be used to study other nonparametric quantile regression methods. To illustrate this generality we also employ our proof techniques to obtain new results for multivariate quantile total variation denoising and high dimensional quantile linear regression.

Keywords: Total variation, nonparametric quantile regression, local adaptivity, fused lasso.

1 Introduction

1.1 Introduction

In this paper we focus on the problem of nonparametric quantile regression for the quantile sequence model. Specifically, let $y \in \mathbb{R}^n$ be a vector of independent random variables and for a given quantile level $\tau \in (0, 1)$, let $\theta^*$, a vector of $\tau$-quantiles of $y$, be given by

$$
\theta^*_i = \arg \min_{a \in \mathbb{R}} E \{ \rho_\tau(y_i - a) \}
$$

where $\rho_\tau(x) = \max\{\tau x, (\tau - 1)x\}$ is the usual check function used for quantile regression. Upon observing $y$, the problem is to estimate the vector of quantiles $\theta^*$. We call this the quantile sequence model. This generalizes the usual Gaussian sequence model where the quantile $\tau$ is taken to be 0.5 and the distribution of $y$ is taken to be multivariate normal with the covariance matrix a multiple of identity.

Our main focus in this paper is on signals (quantile sequences) that have bounded $r$th order total variation. For a vector $\theta \in \mathbb{R}^n$, let us define $D^{(0)}(\theta) = \theta$, $D^{(1)}(\theta) = (\theta_2 - \theta_1, \ldots, \theta_n - \theta_{n-1})^\top$ and $D^{(r)}(\theta)$, for $r \geq 2$, is recursively defined as $D^{(r)}(\theta) = D^{(1)}(D^{(r-1)}(\theta))$. Note that $D^{(r)}(\theta) \in \mathbb{R}^{n-r}$. For simplicity, we
denote the operator $D^{(1)}$ by $D$. For any positive integer $r \geq 1$, let us now define the $r$th order total variation of a vector $\theta$ as follows:

$$TV^{(r)}(\theta) = n^{r-1} \| D^{(r)}(\theta) \|_1$$

where $\| . \|_1$ denotes the usual $\ell_1$ norm of a vector.

**Remark 1.** The $n^{r-1}$ term in the above definition is a normalizing factor and is written following the convention adopted in the trend filtering literature; see for instance Guntuboyina et al. (2020). If we think of $\theta$ as evaluations of a $r$ times differentiable function $f : [0, 1] \rightarrow R$ on the grid $(1/n, 2/n, \ldots, n/n)$ then the Riemann approximation to the integral $\int_{[0,1]} |f^{(r)}(t)| dt$ is precisely equal to $TV^{(r)}(\theta)$. Here $f^{(r)}$ denotes the $r$th derivative of $f$. Thus, for natural instances of $\theta$, the reader can imagine that $TV^{(r)}(\theta) = O(1)$.

Let us now define the constrained quantile trend filtering (CQTF) estimator which is one of the main objects of study in this paper, and it is given as

$$\hat{\theta}^{(r)}_V = \arg \min_{\theta \in \mathbb{R}^n : TV^{(r)}(\theta) \leq V} \sum_{i=1}^n \rho_r(y_i - \theta_i).$$

Here $V$ is a tuning parameter.

The other estimator we focus on in this paper is the penalized quantile trend filtering estimator (PQTF) defined as follows:

$$\hat{\theta}^{(r)}_\lambda = \arg \min_{\theta \in \mathbb{R}^n} \left\{ \sum_{i=1}^n \rho_r(y_i - \theta_i) + \lambda TV^{(r)}(\theta) \right\},$$

for a tuning parameter $\lambda > 0$. This is the quantile regression version of the standard trend filtering estimator proposed first by Kim et al. (2009).

The PQTF estimator has already been proposed in the literature. The PQTF estimator with $r = 1$ appeared in Li and Zhu (2007). When $r = 1$, we refer to the PQTF estimator as quantile fused lasso. More recently, Brantley et al. (2020) proposed the general quantile trend filtering estimator (PQTF) of order $r \geq 1$. However, to the best of our knowledge, not much is known about the theoretical properties (such as risk bounds) of the PQTF and the CQTF estimators.

Due to $\ell_1$ penalization both the CQTF and PQTF estimators enforce $D^{(r)}(\hat{\theta}^{(r)}_V)$ and $D^{(r)}(\hat{\theta}^{(r)}_\lambda)$ respectively to be sparse. It is known that for $\theta \in \mathbb{R}^n$, $D^{(r)}(\theta)$ has $k$ nonzero entries if and only if $\theta = (f(1/n), f(2/n), \ldots, f(n/n))^\top$ for a discrete spline function $f$, consisting of $(k+1)$ polynomials of degree $r-1$ (see Proposition D.3 in Guntuboyina et al. (2020)). For this reason, just like the usual trend filtering estimators, the CQTF and the PQTF estimators both fit discrete splines. For the precise definition of a discrete spline see Section 2 in Mangasarian and Schumaker (1971).

### 1.2 Notation

Let $\{a_n\}$ and $\{b_n\} \subset R$ be two positive sequences. We write $a_n = O(b_n)$ if there exists constants $C > 0$ and $n_0 > 0$ such that $n \geq n_0$ implies that $a_n \leq C b_n$. We also use the notation $a_n = \tilde{O}(b_n)$ to indicate that $a_n \leq C b_n g(\log n)$ for $n \geq n_0$ where $g(\cdot)$ is a polynomial function. Furthermore, if $a_n = O(b_n)$ and $b_n = O(a_n)$ then we write $a_n = \Theta(b_n)$ or $a_n \asymp b_n$. For a sequence of random variables $X_n$ and a positive sequence $a_n$ we write $X_n = O_{\text{pr}}(a_n)$ if for every $\epsilon > 0$ there exists $M > 0$ such that $\text{pr} \left( |X_n| \geq M a_n \right) < \epsilon$ for all $n$. For any positive integer $n$, we denote the set of positive integers from $1$ to $n$ by $[n]$. For any vector $v \in \mathbb{R}^m$ we denote its usual Euclidean or $\ell_2$ norm by $\|v\|$. Furthermore, for a vector $v \in \mathbb{R}^m$, we define $\|v\|_0 = |\{j \in \{1, \ldots, m\} : v_j \neq 0\}|$. Finally, for a set $K \subset \mathbb{R}^n$ we define its Rademacher width (or
complexity) as

\[ RW(K) = E \left( \sup_{v \in K} \sum_{i=1}^{n} \xi_i v_i \right), \]

where \( \xi_1, \ldots, \xi_n \) are independent Rademacher random variables.

### 1.3 Summary of Our Results

The usual (mean regression) univariate trend filtering estimators are defined similarly to the PQTF and the CQTF estimators with the \( \rho_\tau \) function replaced by the \( x \to x^2 \) function. These estimators were independently introduced by Steidl et al. (2006) and Kim et al. (2009).

A continuous version of these trend filtering estimators, where discrete derivatives are replaced by continuous derivatives, was proposed much earlier in the statistics literature by Mammen and van de Geer (1997) under the name locally adaptive regression splines. By now, there exists a body of literature studying the risk properties of trend filtering under squared error loss. There exists two strands of risk bounds for trend filtering in the literature focussing on two different aspects.

Firstly, for a given constant \( V > 0 \) and \( r \geq 1 \), \( \Theta\{n^{-2r/(2r+1)}\} \) rate is known to be the minimax rate of estimation over the space \( BV_n^{(r)}(V) \); (see e.g, Donoho and Johnstone (1994)) where for any integer \( r \geq 1 \),

\[ BV_n^{(r)}(V) = \{ \theta \in \mathbb{R}^n : TV^{(r)}(\theta) \leq V \}. \]

A standard terminology in this field terms this \( \Theta(n^{-2r/(2r+1)}) \) rate as the slow rate. It is also known that a well tuned trend filtering estimator is minimax rate optimal over the parameter space \( BV_n^{(r)}(V) \) and thus attains the slow rate. This result has been shown in Tibshirani (2014) and Wang et al. (2014) building on earlier results by Mammen and van de Geer (1997).

Secondly, it is also known that an ideally tuned trend filtering (of order \( r \)) estimator can adapt to \( \|D^r(\theta)\|_0 \), the number of non zero elements in the \( r \)th order differences, under some assumptions on \( \theta^* \). Such a result has been shown in Guntuboyina et al. (2020) and Ortelli and van de Geer (2019a). In this case, the Trend Filtering estimator of order \( r \) attains the \( \tilde{O}\{\|D^{(r)}(\theta)\|_0/n\} \) rate which can be much faster than the \( n^{-2r/(2r+1)} \) rate. Standard terminology in this field terms this as the fast rate.

Our goal in this paper is to extend these two types of results for quantile trend filtering estimators under minimal assumptions on the distribution of the components of the data vector \( y \). We are able to do this to a large extent with two main differences from the existing results. To the best of our knowledge, the results for usual trend filtering all hold under sub-Gaussian noise and under mean squared error loss. Our results for quantile trend filtering estimators hold under an extremely mild assumption on the growth of the CDF’s of the components of \( y \) around the true quantiles; see Section 2 where this assumption is stated. In particular, our results hold even when the distribution of \( y \) is heavy-tailed (with no moments existing) such as the Cauchy distribution. In this sense, our results are stronger than the existing results for the usual trend filtering estimators. On the other hand, our results hold under a Huber type loss which is in general smaller than the mean squared error loss.

Our loss function is given by the function \( \Delta_n^2 : \mathbb{R}^n \to \mathbb{R} \) defined as

\[ \Delta_n^2(v) = \frac{1}{n} \sum_{i=1}^{n} \min \{ |v_i|, v_i^2 \} \]  

which, up to constants, is a Huber loss, see Huber (1964). We also write \( \Delta^2(v) = n \Delta_n^2(v) \). The main reason why our bounds are for the Huber loss is that this loss naturally appears as a lower bound to the quantile population loss; see (10) and Section 3 for a more detailed explanation.
In our first result in Theorem 1, we show that the CQTF estimator satisfies
\[
\Delta_n^2(\hat{\theta}_n^{(r)}) - \theta^* = O_{pr} \left\{ n^{-2r/(2r+1)} \right\},
\]
where the notation \( O_{pr}(\cdot) \) is defined in Section D.1. Therefore, the CQTF estimator attains the minimax rate for estimating signals in \( BV_n^r(V) \). See Section 5 where we state precisely in what sense this is minimax rate optimal. Additionally, in Theorem 3 we show that a similar result is satisfied by the PQTF estimator. These results generalize the slow rate results for trend filtering to the quantile setting.

Now let us consider the case when \( \|D^{(r)}\theta^*\|_0 = s \) and the elements of \( \{j : (D^{(r)}\theta^*)_j \neq 0\} \) satisfy a minimal spacing condition. In Theorem 2 we prove that, with an ideal tuning parameter \( V = TV^{(r)}(\theta^*) = V^* \), the CQTF estimator satisfies
\[
\Delta_n^2(\hat{\theta}_n^{(r)} - \theta^*) = O_{pr} \left\{ \frac{(s+1)}{n} \log \left( \frac{en}{s+1} \right) \right\}.
\]

Our result generalizes the fast rate results of Guntuboyina et al. (2020) to the quantile setting. We also show in Theorem 4 that the PQTF estimator of orders \( r \in \{1, 2, 3, 4\} \) when the tuning parameter \( \lambda \) is chosen appropriately, attains the above fast rate. This result generalizes the fast rate results of Ortelli and van de Geer (2019a) to the quantile setting.

In this paper we actually formulate a general quantile sequence problem under convex constraints. The setup is that we have a vector \( y \in \mathbb{R}^n \) of independent random variables and \( \theta^* \in \mathbb{R}^n \) is a corresponding vector of \( \tau \in (0, 1) \) quantiles of \( y \). Suppose it is known that \( \theta^* \in K \subset \mathbb{R}^n \) where \( K \) enforces a constraint on the vector \( \theta^* \). This is a generalization of the Gaussian sequence model with constraints on the mean vector. We call this the constrained quantile sequence problem. A natural estimator for this problem is the following:
\[
\hat{\theta}_K = \arg\min_{\theta \in K} \sum_{i=1}^n \rho_{\tau}(y_i - \theta_i)
\]

We call this estimator the constrained quantile sequence estimator or the CQSE estimator. For example, if \( K = \{\theta \in \mathbb{R}^n : TV^{(r)}(\theta) \leq V\} \) for some integer \( r \geq 1 \) and some \( V > 0 \) then the above estimator is the CQTF estimator of order \( r \) with tuning parameter \( V \) as defined in (2).

We prove a general result bounding the risk of \( \hat{\theta}_K \) in terms of the Rademacher width of \( K \); see Theorem 7 in Section 3. Therefore our proof technique can potentially be used for other CQSE estimators with different constraint sets \( K \). In this context, we also consider two other related quantile estimation problems with different constraint sets \( K \) and prove results for the corresponding CQSE estimators that appear to be new. The first problem we consider is two dimensional quantile total variation denoising which is the quantile version of the ubiquitous total variation denoising estimator (see Rudin et al. (1992)) used in image processing. Here, in Theorem 5 we generalize existing results of Hutter and Rigollet (2016); Chatterjee and Goswami (2019b) to the quantile setting. To the best of our knowledge, quantile total variation denoising has not been proposed and studied before in the literature. Another setting we consider is high-dimensional quantile regression. We study the quantile version of lasso and in Theorem 6 we prove a slow rate for quantile lasso under the fixed design setup which holds without any assumptions on the design matrix. Previous results in this problem show a fast rate but with restricted eigenvalue conditions imposed on the design matrix as in Belloni and Chernozhukov (2011); Fan et al. (2014).

1.4 Some Related Literature

In this section we mention some other existing works in the literature which are closely related to our work. Since its introduction by Koenker and Bassett Jr (1978), quantile regression has become a prominent
tool in statistics. The attractiveness of quantile regression is due to its flexibility for modelling conditional distributions, construction of predictive models, and even outlier detection applications. The problem of one-dimensional nonparametric quantile regression goes back at least to Utreras (1981); Cox (1983); Eubank (1988) who focused on median regression. Koenker et al. (1994) introduced quantile smoothing splines in one dimension. These are defined as the solution to problems of the form

$$
\minimize_{g \in \mathcal{C}} \left[ \sum_{i=1}^{n} \rho_{\tau}(y_i - g(x_i)) + \lambda \left\{ \int_{0}^{1} |g''(x)|^{p} \, dx \right\}^{1/p} \right],
$$

assuming that $0 < x_1 < \ldots < x_n < 1$, where $\lambda > 0$ is a tuning parameter, $p \geq 1$, and $\mathcal{C}$ a suitable class of functions. When $p = 1$ this is related to the quantile version of locally adaptive regression splines of order 2 which appeared later in Mammen and van de Geer (1997). The theoretical properties of quantile smoothing splines were studied in He and Shi (1994). Specifically, the authors in He and Shi (1994) demonstrated that quantile smoothing splines attain the rate $n^{-2r/(2r+1)}$, for estimating quantile functions in the class of Hölder functions of exponent $r$.

It is natural to believe that the connections between quantile version of adaptive regression splines proposed in Mammen and van de Geer (1997) and quantile trend filtering would be similar to the connections between the mean regression counterparts. It is known that both attain similar rates over appropriate bounded variation function classes but trend filtering is computationally more efficient; see Tibshirani (2014).

In the context of median regression in one dimension, the authors in Brown et al. (2008) showed that a wavelet-based quantile regression approach attains minimax rates for estimating the median function, when the latter belongs to Besov spaces which is related to the sort of bounded variation classes considered in this paper. However, our focus in this paper is not on wavelet methods. Despite the optimality of wavelet methods, it is also known that total variation based methods can outperform wavelet methods in practice, see Tibshirani (2014); Wang et al. (2016).

A precursor of trend filtering can be traced back in the machine learning literature to Rudin et al. (1992) who proposed a two dimensional total variation penalized method for image denoising applications. To the best of our knowledge, the quantile version of this estimator has not been considered before in the literature. Due to the ubiquity of this image denoising method, we study the quantile 2D total variation denoising estimator in this paper; see Section 2.4.

On the computational front, it is known that the usual trend filtering estimator with $r = 1$ can be solved in $O(n)$ time, see for instance Johnson (2013). More recently, Hochbaum and Lu (2017) showed that the corresponding quantile fused lasso estimator (PQTF) with $r = 1$, can be computed in $O(n \log n)$ time. For $r > 1$, Brantley et al. (2020) proposed an alternating direction method of multipliers (ADMM) based algorithm for computing PQTF estimators of order $r$.

## 2 Main Results

### 2.1 Assumption

For all our theorems, unless stated otherwise, we consider any fixed quantile level $\tau \in (0, 1)$, and any fixed integer $r \in \{1, 2, \ldots\}$. The quantities $\epsilon_i = y_i - \theta_i^*$, $i = 1, \ldots, n$ which are unobservable are referred to as the errors. We also generically write $V^* = TV^{(r)}(\theta^*)$ where $\theta^*$ is the true signal.

We state all of our results under the following assumption on the distribution of the components of $y$.

**Assumption A:** There exist constants $L > 0$ and $f > 0$ such that for any positive integer $n$ and any $\delta \in \mathbb{R}^n$ satisfying $\|\delta\|_{\infty} \leq L$ we have for all $i = 1, \ldots, n$,

$$
|F_{y_i}(\theta_i^* + \delta_i) - F_{y_i}(\theta_i^*)| \geq f|\delta_i|,
$$
where $F_{y_i}$ is the CDF of $y_i$.

If the cumulative distribution functions $F_{y_i}$ have probability density functions $f_{y_i}$ with respect to Lebesgue measure then Assumption A is a weaker assumption than requiring that for any positive integer $n$,

$$\inf_{\|\delta\|_{\infty} \leq L} \min_{i=1, \ldots, n} f_{y_i}(\theta_i^r + \delta_i) \geq f_r,$$

which appeared as Condition 2 in He and Shi (1994), and is related to condition D.1 in Belloni and Chernozhukov (2011). Such an assumption ensures that the quantile of $y_i$ is uniquely defined and there is a uniformly linear growth of the CDF around a neighbourhood of the quantile. An assumption of such a flavor (making the quantile uniquely defined) is clearly going to be necessary. We think this is a mild assumption otherwise set $\eta = 0$.

Let $\Delta^2_n(\hat{\theta}_V^{(r)} - \theta^*) = O_{pr} \left[ n^{-2r/(2r+1)} V^{2/(2r+1)} \max \left\{ 1, \left( \frac{V}{n^{r-1}} \right)^{(2r-1)/(2r+1)} \right\} \right]$.

**Remark 2.** The above theorem holds for any $r \in (0, 1)$. The role of $\tau$ is not made explicit on the right hand side in Theorem 1. The proof of Theorem 1 reveals that the closer $\tau$ is to $\{0, 1\}$, the larger the constants are in the upper bound in Theorem 1.

**Remark 3.** Theorem 1 can be thought of as generalizing Theorem 2.1 from Guntuboyina et al. (2020) to the quantile regression setting. Aside from the different loss $\Delta_n(\cdot)$ and our result being a $O_{pr}$ statement, our result also differs from Theorem 2.1 in Guntuboyina et al. (2020) in that our upper bound has an extra term.

This is the factor

$$\max \left\{ 1, \left( \frac{V}{n^{r-1}} \right)^{(2r-1)/(2r+1)} \right\}$$

which can go to infinity if $V$ grows faster than $n^{r-1}$. However, under the natural scaling $V^* = O(1)$ one can choose $V = O(1)$ as well and thus the above term is also $O(1)$.

We now state our first result which is the slow rate result for the CQTF estimator.

**Theorem 2.** Let $\{y_i\}_{i=1}^n$ be any sequence of independent random variables which satisfies Assumption A and $\theta^*_i$ be the sequence of $\tau$ quantiles of $y_i$. If $V$ is chosen such that $V \geq V^* = TV^{(r)}(\theta^*)$ then

$$\Delta^2_n(\hat{\theta}_V^{(r)} - \theta^*) = O_{pr} \left[ n^{-2r/(2r+1)} V^{2/(2r+1)} \max \left\{ 1, \left( \frac{V}{n^{r-1}} \right)^{(2r-1)/(2r+1)} \right\} \right].$$

We now state our fast rate result for the CQTF estimator.

**Theorem 2.** Let $\{y_i\}_{i=1}^n$ be any sequence of independent random variables which satisfies Assumption A and $\theta^*_i$ be the sequence of $\tau$ quantiles of $y_i$. Let $s = \|D^{(r)}\theta^*_s\|_0$ and $S = \{ j : (D^{(r)}\theta^*_j)_j \neq 0 \}$. Let $j_0 < j_1 < \ldots < j_{s+1}$ be such that $j_0 = 1$, $j_{s+1} = n - r$ and $j_1, \ldots, j_s$ are the elements of $S$. With this notation define $\eta_j = \eta_{j+1} = 0$. Then for $j \in S$ define $\eta_j$ to be 1 if $(D^{(r-1)}\theta^*)_j < (D^{(r-1)}\theta^*)_{j+1}$, otherwise set $\eta_j = -1$. Suppose that $\theta^*$ satisfies the following minimum length assumption

$$\min_{l \in [s], \eta_{j_l} \neq \eta_{j_{l+1}}} (j_{l+1} - j_l) \geq \frac{cn}{s + 1}$$

for some constant $c$ satisfying $0 \leq c \leq 1$. Then we have that

$$\Delta^2_n(\hat{\theta}_V^{(r)} - \theta^*) = O_{pr} \left[ \max \left\{ \frac{V^*}{n^{r-1}}, 1 \right\} \left( \frac{s + 1}{n} \log \left( \frac{en}{s + 1} \right) \right) \right].$$
Remark 4. Theorem 2 shows that the constrained quantile trend filtering estimator attains, off by a logarithmic factor, the rate attained by an oracle estimator that knows the set $S$. Thus, Theorem 2 can be thought of as generalizing Theorem 2.2 of Guntuboyina et al. (2020) to the quantile setting. Our minimum length assumption is identical to the one assumed by Guntuboyina et al. (2020). In particular it requires that when two consecutive change points correspond to two opposite changes in trend, then the two points should be sufficiently separated.

Remark 5. Notice that Theorem 2 provides an upper bound that depends on $V^*$. This was not the case in Theorem 2.2 from Guntuboyina et al. (2020) which gave an upper bound that is independent of $V^*$. Nevertheless, in the case $V^* = O(n^{-r})$ (which covers the canonical regime) we do obtain the same rate from Theorem 2.2 in Guntuboyina et al. (2020).

2.3 Results for PQTF Estimator

From a computational point of view the penalized quantile trend filtering seems to present a more appealing method than its constrained counterpart. The optimization problems corresponding to the CQTF and the PQTF estimators are both linear programs that can be solved using any generic linear programming software. However, the PQTF optimization problem has special structures that enable more efficient computation. Existing works (e.g Hochbaum and Lu (2017); Brantley et al. (2020)) have studied different types of algorithms that can efficiently solve the penalized quantile trend filtering problem. This is in contrast to the CQTF optimization problem that has not received similar attention from a computational perspective perhaps due to its inherent difficulty. This makes it important to also study the risk properties of the PQTF estimator. We now present our slow rate result for the PQTF estimator.

Theorem 3. Let $\{y_i\}_{i=1}^n$ be any sequence of independent random variables which satisfies Assumption A and $\theta_i^*$ be the sequence of $\tau$ quantiles of $y_i$. Suppose that $V^* = \Theta(1)$. Given any $\epsilon \in (0, 1)$ there exists a positive constant $c_{1,\epsilon}$ only depending on $\epsilon$ and $V^*$ such that if $\lambda$ is chosen to be

$$\lambda = c n^{1/(2r+1)} (\log n)^{1/(2r+1)},$$

for a constant $c$ satisfying $c > c_{1,\epsilon}$ then

$$\Delta_2^2(\hat{\theta}^{(r)}_{\lambda} - \theta^*) \leq c_{2,\epsilon} n^{-2r/(2r+1)} (\log n)^{1/(2r+1)}$$

with probability at least $1 - \epsilon$. Here, $c_{2,\epsilon} > 0$ is a constant that only depends on $c, \epsilon$ and $V^*$.

Remark 6. Apart from an extra log factor, the bound in Theorem 3 gives the same rate as the bound in Theorem 1. As we mention above both the choice of $\lambda$ and our upper bound in Theorem 3 depend on $V^*$ and it is possible to track down the dependence on $V^*$ by following our proof. However, this dependence on $V^*$ is not simple to state and thus for clarity of presentation we state the above theorem only under the natural scaling $V^* = \Theta(1)$.

We now present our fast rate result for the PQTF estimator.

Theorem 4. Fix any $r \in \{1, 2, 3, 4\}$. Let $\{y_i\}_{i=1}^n$ be any sequence of independent random variables which satisfies Assumption A and $\theta_i^*$ be the sequence of $\tau$ quantiles of $y_i$. Consider the same notations as in Theorem 2 and the same minimum length assumption as in (7). In addition, suppose that $\theta^*$ satisfies the following two conditions:

- $V^* = O(1)$. 

• \( \frac{(s+1)}{n} \log \left( \frac{en}{s+1} \right) (\log n) \log(s+1) = O(1) \) where we recall that \( s = \|D^{(r)}\theta^*\|_0 \).

Then given any \( \epsilon \in (0, 1) \) there exists a constant \( c_\epsilon > 0 \) only depending on \( \epsilon \) and \( V^* \) such that if \( \lambda \) is chosen to be

\[
\lambda = c \max \left\{ \frac{n^{-1/2}(s+1) \log n \log(s+1) \log \frac{n}{s+1}}{V^*}, n^{-1/2} \left( \frac{1}{s+1} \right)^{r-1/2} (\log n)^{1/2} \right\}
\]

for a constant \( c \) satisfying \( c > c_{1,\epsilon} \) then

\[
\Delta_n^2(\hat{\theta}_\lambda^{(r)} - \theta^*) \leq c_{2,\epsilon} \frac{(s+1)}{n} \log \left( \frac{en}{s+1} \right) (\log n)\{\log(s+1)\},
\]

with probability at least \( 1 - \epsilon \). Here, \( c_{2,\epsilon} > 0 \) is another constant that depends on \( c, \epsilon \) and \( V^* \).

**Remark 7.** Theorem 4 shows that the PQTF estimator with appropriate tuning parameter, up to log factors, attains the same fast rate result that CQTF attains in Theorem 2. Theorem 4 can also be thought of as an extension of Corollary 1.2 from Ortelli and van de Geer (2019a) to the quantile setting. We rely on the proof machinery developed in Ortelli and van de Geer (2019a) which explains why we can only prove the above extension of Corollary 1.2 from Ortelli and van de Geer (2019a) to the quantile setting. We rely on the proof machinery developed in Ortelli and van de Geer (2019a) which explains why we can only prove the above theorem for \( r \leq 4 \).

**Remark 8.** Both Theorem 3 and Theorem 4 give fast rate results under a particular choice of the tuning parameter. In Theorem 2 we need to set \( V = V^* \) which is hard to achieve in practice. Even for the mean regression case with gaussian noise, the best available result (see Corollary 2.3 in Guntuboyina et al. (2020)) says that the tuning parameter \( V \) should be such that \( (V - V^*)^2 \) scales like \( O\left( \frac{\log n}{n} \right) \) in order to achieve the fast rate. In Theorem 4, we have more margin of error to choose \( \lambda \) as we see from its proof, as long as \( \lambda \) is chosen larger than the given threshold, doubling \( \lambda \) will at most double the MSE. In this sense, for attaining the fast rates the PQTF estimator seems to be more robust to the choice of tuning parameter. For our simulations, we have found that the BIC based approach suggested in Brantley et al. (2020) to choose the tuning parameter in a data driven way works well.

### 2.4 Result for Quantile Total Variation Denoising

Total variation Denoising (TVD) in 2 dimensions was proposed by Rudin et al. (1992) which subsequently has become a standard image denoising method. In this subsection, we propose the quantile version of the TVD estimator and study its risk properties in general dimensions.

Fix a dimension \( d \geq 2 \). Let us denote the \( d \) dimensional lattice with \( n \) points by \( L_{d,n} := \{1, \ldots, m\}^d \) where \( n = m^d \). We can also think of \( L_{d,n} \) as the \( d \) dimensional regular lattice graph with edges and vertices. Then, thinking of \( \theta \in R^n \) as a function on \( L_{d,n} \) we define

\[
TV(\theta) := \frac{1}{n^{d-1}} \sum_{(u,v) \in E_{d,n}} |\theta_u - \theta_v| \tag{8}
\]

where \( E_{d,n} \) is the edge set of the graph \( L_{d,n} \). One way to motivate the above definition is as follows. If we think \( \theta_{i_1, \ldots, i_n} = f(\frac{i_1}{n}, \ldots, \frac{i_n}{n}) \) for a differentiable function \( f : [0, 1]^d \to R \) then the above definition is precisely the Reimann approximation for \( \int_{[0,1]^d} \|\nabla f\|_1 \). Of course, the definition in (8) applies to arbitrary arrays, not just for evaluations of a differentiable function on the grid. See Sadhanala et al. (2016) who calls this scaling the canonical scaling.

We now define the Quantile Total Variation Denoising estimator (QTVD) as follows:

\[
\hat{\theta}_V = \arg\min_{\theta \in R^n : TV(\theta) \leq V} \sum_{v \in L_{d,n}} \rho_r(y_v - \theta_v)
\]
where $V$ is a tuning parameter.

For $d \geq 2$, this is the quantile version of the usual constrained TVD estimator where again the $\rho$ function is replaced by the $x \rightarrow x^2$ function. The risk properties of the usual constrained TVD estimator have been thoroughly studied in Chatterjee and Goswami (2019b). The corresponding penalized version of the TVD estimator has also been studied in Hutter and Rigollet (2016), Sadhanala et al. (2016). These works show that a well tuned TVD estimator is nearly (up to log factors) minimax rate optimal over the class $\{\theta \in \mathbb{R}^n : TV(\theta) \leq V\}$ of bounded variation signals in any dimension. The following theorem extends this result to the quantile setting.

**Theorem 5.** Suppose that Assumption A holds. If $V$ is chosen to satisfy $V \geq V^* := TV(\theta^*)$ and $V^* = O(1)$, then

$$
\Delta_n^2(\hat{\theta}_V - \theta^*) = O_{\text{pr}} \left( \frac{V \log n}{n^{1/d}} \right),
$$

for $d = 2$, and

$$
\Delta_n^2(\hat{\theta}_V - \theta^*) = O_{\text{pr}} \left( \frac{V \log n}{n^{1/d}} \right),
$$

for $d > 2$.

**Remark 9.** Theorem 5 shows that the QTVD estimator is minimax rate optimal over the class $\{\theta \in \mathbb{R}^n : TV(\theta) \leq V\}$ of bounded variation signals in any dimension $d \geq 2$, see discussion in Section 5. This result can be thought of as generalizing Theorem 2.1 from Chatterjee and Goswami (2019b) to the quantile regression setting.

### 2.5 Result for High-dimensional Quantile Linear Regression

Now we consider high-dimensional linear quantile regression. We study the constrained version of the $\ell_1$-QR estimator defined in Knight and Fu (2000) and studied in Belloni and Chernozhukov (2011). $\ell_1$-QR is commonly used as a robust tool for variable selection and prediction with high-dimensional covariates and is the quantile version of the constrained lasso estimator proposed in Tibshirani (1996).

Suppose that we are given $\{(x_i, y_i)\}_{i=1}^n \subset \mathbb{R}^p \times \mathbb{R}$ with the $\{x_i\}_{i=1}^n$ fixed, and with $y_1, \ldots, y_n$ independent random variables. Let $X \in \mathbb{R}^{n \times p}$, whose $i$th row is $x_i^\top$. We now consider the estimator

$$
\hat{\theta} = \arg \min_{\theta \in \mathbb{R}^n : \theta = X\beta, \|\beta\|_1 \leq V, \beta \in \mathbb{R}^p} \left\{ \sum_{i=1}^n \rho_\tau(y_i - \theta_i) \right\},
$$

where $V$ is a tuning parameter.

With the notation from above, we now present our next result.

**Theorem 6.** Suppose that Assumption A holds and $\theta^* = X\beta^*$ for some $\beta^* \in \mathbb{R}^p$. If $V$ is chosen such that $V \geq V^* := \|\beta^*\|_1$ then there exists a constant $C > 0$ such that

$$
E \left\{ \Delta_n^2(\hat{\theta} - \theta^*) \right\} \leq C V (\log p)^{1/2} \max_{j=1, \ldots, p} \|X_{.,j}\|,
$$

where $X_{.,j}$ is the $j$th column of $X$ and $\hat{\theta}$ is the estimator defined in (9).

**Remark 10.** Theorem 6 implies that in the case that the columns of $X$ are normalized, which is the following standard assumption in high dimensional regression,

$$
\max_{j=1, \ldots, p} \|X_{.,j}\| \leq n^{1/2},
$$


then we attain a slow rate bound scaling like $\{\log p/n\}^{1/2}$. It is well known that such a bound holds for the usual lasso without any assumptions on the design matrix $X$; see for instance Chatterjee (2013). To the best of our knowledge, this slow rate bound for quantile lasso has not appeared in the literature before. Previous works (Belloni and Chernozhukov, 2011; Fan et al., 2014; Sun et al., 2019) make restricted eigenvalue type assumptions on the design matrix and attain fast rates of convergence.

3 Proof Ideas

3.1 General Ideas

In this section, we provide an overview of the main ideas underlying our proofs. Full proofs (along with proof outlines for the major theorems) are given in the Appendix. We first prove a general result about the CQSE estimator (defined in (6)) when the constraint set $K$ is convex.

Theorem 7. Let $K \subset \mathbb{R}^n$ be a convex set. Let us define a function $R : [0, \infty) \rightarrow \mathbb{R}$ as follows:

$$R(t) = RW(K \cap \{\theta : \Delta^2(\theta - \theta^*) \leq t^2\}).$$

Suppose the distributions of $y_1, \ldots, y_n$ obey Assumption A. Then the following inequality is true for any $t > 0$,

$$\Pr(\Delta^2(\hat{\theta}_K - \theta^*) > t^2) \leq \frac{C R(t)}{t^2},$$

where $C$ is a constant that only depends on the distributions of $y_1, \ldots, y_n$.

As a consequence of Theorem 7 we obtain two corollaries in Sections A.3.1 and A.3.2 that can be used to obtain asymptotic rates of convergence for CQSE estimators.

To prove Theorem 7 we view $\hat{\theta}_K$ as an M estimator as we now explain. We define $\hat{M} : \mathbb{R} \rightarrow \mathbb{R}$ and $\hat{M}_i : \mathbb{R} \rightarrow \mathbb{R}$ for each $i \in [n]$ satisfying

$$\hat{M}(\theta) = \sum_{i=1}^n \hat{M}_i(\theta_i),$$

where

$$\hat{M}_i(\theta_i) = \rho_\tau(y_i - \theta_i) - \rho_\tau(y_i - \theta_i^*).$$

Also define the expected versions $M : \mathbb{R} \rightarrow \mathbb{R}$ and $M_i : \mathbb{R} \rightarrow \mathbb{R}$ for each $i \in [n]$ satisfying

$$M(\theta) = \sum_{i=1}^n M_i(\theta_i),$$

where $M_i(\theta_i) = E\{\hat{M}_i(\theta_i)\}$. With this notation, the CQSE estimator can also be written as

$$\hat{\theta}_K = \arg\min_{\theta \in K} \hat{M}(\theta)$$

and a true quantile sequence $\theta^* = \arg\min_{\theta \in K} M(\theta)$. Therefore, the CQSE estimator is an M estimator or an instance of Empirical Risk Minimization.

Remark 11. Note that $|\hat{M}_i(\theta_i)| \leq |\theta_i - \theta_i^*|$ for all $i \in [n]$. Therefore, $E\hat{M}_i(\theta_i)$ is always well defined even if $y_i$ does not have any moments.
Since we are viewing the CQSE estimator as an M estimator, the natural loss function to measure its performance would be $M(\hat{\theta}_K)$ and show that $M(\hat{\theta}_K)$ goes to 0 as $n \to \infty$. Using the M estimation viewpoint, we first prove the following result.

**Proposition 1.** Let $K \subset \mathbb{R}^n$ be a convex set. Let us define a function $\mathcal{M} : [0, \infty) \to \mathbb{R}$ as follows:

$$\mathcal{M}(t) = RW(K \cap \{\theta : M(\theta) \leq t^2\}).$$

Then the following inequality is true for any $t > 0$,

$$\Pr(M(\hat{\theta}_K) > t^2) \leq \frac{2M(t)}{t^2}.$$

The above proposition is very similar to Theorem 7, the only difference being that the loss function $\Delta^2(\hat{\theta} - \theta^*)$ is replaced with the function $M(\hat{\theta})$. This proposition is shown by first reducing the task of bounding $\Pr(M(\hat{\theta}_K) > t^2)$ to bounding

$$E \sup_{\theta \in K \cap \{v : M(v) \leq t^2\}} [M(\theta) - \hat{M}(\theta)].$$

Here the numerator in the bound is an expectation of suprema of a mean zero process. We then further bound this expected suprema by using symmetrization and contraction results commonly employed in empirical process theory, see Section 2.3 in Van Der Vaart and Wellner (1996) and Theorem 4.12 in Ledoux and Talagrand (2013).

However, handling $M$ in concrete problems such as quantile trend filtering is not convenient as it depends on the distribution of $y$. Here, a particular property of $M$ comes in handy for us as one can show that if Assumption A holds then for all $\delta \in \mathbb{R}^n$, we have for a constant $c_0 > 0$ that

$$M(\theta^* + \delta) \geq c_0\Delta^2(\delta). \quad (10)$$

This is the content of Lemma 13 in the Appendix. This makes it possible for us to convert the result in Proposition 1 to Theorem 7. Lemma 13 is the reason why we use $\Delta_n^2(\cdot)$ as the loss function throughout this paper.

The estimator $\hat{\theta}_K$ can be thought of as the quantile version of constrained least squares in the Gaussian sequence model. The study of convex constrained least squares in the Gaussian sequence model has a long history and is, by now, well established (see e.g., Van de Geer (1990); Van Der Vaart and Wellner (1996); Hjort and Pollard (2011); Chatterjee et al. (2015)). The general theory says that risk bounds (under the squared error loss) for the convex constrained least squares estimator can be deduced from the localized Gaussian width term

$$G_2(t) = GW(K \cap \{\theta : \|\theta - \theta^*\| \leq t\}). \quad (11)$$

Theorem 3.1 should be thought of as a quantile version of such a result. In our case, the localized Rademacher width $RW(K \cap \{\theta : \Delta^2(\theta - \theta^*) \leq t^2\})$ determines an upper bound on the loss function $\Delta^2(\hat{\theta} - \theta^*)$. Since Rademacher width is upper bounded by a constant times Gaussian width; see Lemma 8 in the Appendix, the main difference in our result versus results for convex constrained least squares is that the $l_2$ norm is replaced by the loss function $\Delta$.

### 3.2 Theorems 1, 2, and 5

Theorems 1, 2, and 5 are all bounding the risk for a particular instance of the CQSE estimator. For example, in Theorem 1 the constraint set $K = \{\theta \in \mathbb{R}^n : TV^{(r)}(\theta) \leq V\}$ where $V \geq V^* = TV^{(r)}(\theta^*)$ and in Theorem 2 we consider the same $K$ with $V = V^*$. 

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The starting point for proving Theorems 1, 2, 5 is Theorem 7 which behooves us to bound the local Rademacher width term \( R(t) \) for any \( t \geq 0 \). Since Rademacher width is upper bounded by Gaussian width, it suffices to bound the local Gaussian width term \( G_1(t) = GW(K \cap \{ \theta : \Delta^2(\theta - \theta^*) \leq t^2 \}) \). Now, tight bounds for the related local Gaussian width term \( G_2 \) (defined in (11)) exists in the literature and in particular we use Lemmas B.1–B.3 from Guntuboyina et al. (2020). However, to bound \( G_1(t) \) by \( G_2(t) \) one needs to bound convert the \( \ell_2 \) norm in \( G_2(t) \) to the \( \Delta \) function in \( G_1(t) \). The majority of our proof executes this conversion which constitutes one of the technical contributions of this work. We have written more detailed proof outlines for Theorems 1, 2 in the Appendix.

For Theorem 5, the relevant constraint set is \( K = \{ \theta \in \mathbb{R}^n : TV(\theta) \leq V \} \). Here also, we bound the local Gaussian width term \( G_1(t) \) by reducing the problem to bounding \( G_2(t) \) which then can be further bounded by using existing results from Hutter and Rigollet (2016).

### 3.3 Theorem 6

In this case, the constraint set is

\[
K = \{ \theta \in \mathbb{R}^n : \theta = X\beta, \|\beta\|_1 \leq L, \beta \in \mathbb{R}^p \}
\]

for a given fixed design matrix \( X \in \mathbb{R}^{n \times p} \). Since this is a compact set, we directly use Corollary 15 to get an expectation bound. This means that we need to simply bound \( RW(K) \) which can be done using standard existing results.

### 3.4 Theorem 3 and Theorem 4

The PQTF estimator is not an instance of the CQSE estimator. Therefore, the proof here is necessarily different. Here, we still continue to use the general idea of viewing the PQTF estimator as a penalized M estimator and using appropriately modified versions of the symmetrization and contraction results.

However, due to the presence of the penalty term, the proofs of Theorem 3 and Theorem 4 are longer and contain additional preliminary localization arguments compared to the proofs of Theorem 1 and Theorem 2 respectively. In Theorem 4, we have extensively used the recent ideas developed in Ortelli and van de Geer (2019a) and adapted their argument to our quantile setting.

For the convenience of the reader, we have included a proof outline (before the formal proof) for each of our Theorems 1–4 in the Appendix. We hope that these proof outlines convey the main ideas of our proofs and make reading our proofs easier.

### 4 Experiments

We now proceed to illustrate with simulations the empirical performance of quantile trend filtering. As benchmark methods, we consider the usual (mean regression) trend filtering estimator of order \( r = 1 \) and \( r = 2 \) denoted as TF1 and TF2 respectively, and quantile smoothing splines (QS) (introduced in Koenker et al. (1994)) which we implement using the R package “fields”. Notice that TF1 and TF2 only provide estimates for \( \tau = 0.5 \). As for quantile trend filtering, we consider the penalized estimator (3) with orders \( r = 1 \) and \( r = 2 \) which we denote as PQTF1 and PQTF2 respectively. These are implemented in R via ADMM, similarly to Brantley et al. (2020), using the R package “glmgen”.

For the different trend filtering based methods we consider values of \( \lambda \) such that \( \log_{10}(\lambda n^{r-1}) \) is in a grid of 300 evenly spaced points between 1 and 4.5 and we choose their corresponding penalty parameter to be the value that minimizes the average mean squared error over 100 Monte Carlo replicates. Here, for each instance of an estimator \( \hat{\theta} \) we consider the mean squared error \( \frac{1}{n} \sum_{i=1}^{n} (\hat{\theta}_i - \theta_i^*)^2 \) as a measure of its
Table 1: Average mean squared error times 10, $\frac{10}{n} \sum_{i=1}^{n} (\theta^*_i - \hat{\theta}_i)^2$, averaging over 100 Monte carlo simulations for the different methods considered. Captions are described in the text.

| $n$   | Scenario   | $\tau$ | PQTF1 | PQTF2 | QS   | TF1 | TF2 |
|-------|------------|--------|-------|-------|------|-----|-----|
| 10000 | 1          | 0.5    | 0.023 | 0.08  | 0.21 | 0.016 | 0.4 |
| 5000  | 1          | 0.5    | 0.046 | 0.12  | 0.23 | 0.034 | 0.65|
| 1000  | 1          | 0.5    | 0.18  | 0.29  | 0.32 | 0.12  | 0.94|
| 10000 | 2          | 0.5    | 0.037 | 0.11  | 0.13 | 4917385.2 | 5743.119|
| 5000  | 2          | 0.5    | 0.066 | 0.15  | 0.17 | 25215.87 | 286.45|
| 1000  | 2          | 0.5    | 0.29  | 0.43  | 0.45 | 354693.6 | 11522.6|
| 10000 | 3          | 0.5    | 0.015 | 0.063 | 0.17 | 2.26  | 0.95|
| 5000  | 3          | 0.5    | 0.029 | 0.092 | 0.18 | 0.14  | 0.65|
| 1000  | 3          | 0.5    | 0.13  | 0.24  | 0.26 | 2.23  | 1.04|
| 10000 | 4          | 0.5    | 0.045 | 0.009 | 0.015| 0.065 | 0.016|
| 5000  | 4          | 0.5    | 0.075 | 0.019 | 0.027| 0.24  | 0.031|
| 1000  | 4          | 0.5    | 0.30  | 0.082 | 0.098| 0.29  | 0.31|
| 10000 | 5          | 0.5    | 0.13  | 0.056 | 0.041| 61625.82 | 134.80|
| 5000  | 5          | 0.5    | 0.24  | 0.099 | 0.086| 1063110.0 | 877.85|
| 1000  | 5          | 0.5    | 1.92  | 0.35  | 0.35 | 1443060.0 | 11531.79|
| 10000 | 6          | 0.9    | 0.18  | 0.070 | 0.075* | * | * |
| 5000  | 6          | 0.9    | 0.29  | 0.13  | 0.14 | * | * |
| 1000  | 6          | 0.9    | 1.19  | 0.39  | 0.41 | * | * |
| 10000 | 6          | 0.1    | 0.16  | 0.065 | 0.070 | * | * |
| 5000  | 6          | 0.1    | 0.31  | 0.13  | 0.14 | * | * |
| 1000  | 6          | 0.1    | 1.27  | 0.46  | 0.47 | * | * |

performance with $\theta^*$ the true vector of quantiles. Additional simulation results reporting the $\Delta^2_n(\theta^* - \hat{\theta})$ values instead of the MSE are presented in Section F in the Appendix.

Next we describe the generative models or scenarios. For each scenario we generate 100 data sets for different values of $n$ in the set $\{1000, 5000, 10000\}$. We then report the average mean squared error, based on optimal tuning, of the different competing methods. In each scenario the data are generated as $y_i = \theta^*_i + \epsilon_i$, $i = 1, \ldots, n$, (12)

where $\theta^* \in \mathbb{R}^n$, and the errors $\{\epsilon_i\}_{i=1}^{n}$ are independent with $\epsilon_i \sim F_i$ for some distributions $F_i$ with $i \in [n]$. We now explain the different choices of $\theta^*$ and $F_i$’s that we consider.

Scenario 1(Piecewise Constant Quantiles, Normal Errors) In this case we take $\theta^*$ to satisfy $\theta^*_i = 1$ for $i \in \{1, \ldots, [n/3]\} \cup \{n - 2[n/3] + 1, \ldots, n\}$ and $\theta^*_i = 0$ otherwise. We take the $F_i$’s to be $N(0, 1)$. Since the errors are normal and the true signal is piecewise constant, it is natural to expect that TF1 will be the best method. This is verified in Table 2. However, we also see that PQTF1 is a close competitor.

Scenario 2(Piecewise Constant Quantiles, Cauchy Errors) This is the same as Scenario 1, where we replace $N(0, 1)$ with Cauchy$(0, 1)$ errors. In this situation the errors have no mean. As a result, TF1 and TF2 completely breakdown as shown in Table 2. In contrast, as expected, the quantile methods are robust and can still provide reasonable estimates. In fact, we see that PQTF1 is the best method. This is reasonable since the true median curve is piecewise constant. The second best method is PQTF2.

Scenario 3(Piecewise Constant Quantiles, Heteroscedastic t Errors) Once again, we take $\theta^*$ as in Scenario 1. With regards to the $F_i$’s, we set $\epsilon_i = t^{1/2} / n^{1/2} v_i$, where the $v_i$’s are independent draws from $t(2)$. Here $t(2)$ denotes the t-distribution with 2 degrees of freedom. The empirical performances here are similar to that of Scenario 2. Table 2 suggests that PQTF1 is the best method followed by PQTF2. Interestingly, TF1 and TF2 are not so unreasonable but their behavior seems erratic as the MSE does not decrease with $n$. 

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Figure 1: The top left panel shows $\theta^*$, the true median, for Scenarios 1, 2, and 3. The next three panels in the top row correspond to data generated according to Scenarios 1, 2 and 3. Similarly, the middle panels show the true median curve and instances of data for Scenarios 4 and 5. Finally, the bottom row shows the true quantile curve for Scenario 6 associated with $\tau = 0.9$, and an instance of data generated according to Scenario 6.

A possible explanation for this is that the errors have mean but do not have variance.

Scenario 4 (Piecewise Linear Quantiles, t Errors) We set $\theta^*_i = 3(i/n)$ for $i \in \{1, \ldots, \lfloor n/2 \rfloor \}$, and $\theta^*_i = 3(1 - i/n)$ for $\{\lfloor n/2 \rfloor + 1, \ldots, n\}$. The errors are then independent draws from $t(3)$. Since the true median curve is piecewise linear, Scenario 4 offers a model that seems more amenable for PQTF2. This intuition is confirmed in Table 2 where PQTF2 outperforms the competitors followed by QS.

Scenario 5 (Sinusoidal Quantiles, Cauchy Errors) The signal is taken as $\theta^*_i = \cos(6\pi i/n)$ for $i \in \{1, \ldots, n\}$. We then generate $\epsilon_i \sim \text{ind Cauchy}(0, 1)$ for $i = 1, \ldots, n$. Here the true median curve is infinitely differentiable. Table 2 shows that the best performance is given by QS and PQTF2 is a close second. As with the other scenarios that have Cauchy errors, TF1 and TF2 provide poor estimates.

Scenario 6 (Piecewise smooth quantiles, Heteroscedastic Errors) For our last scenario we generate data as $y$ as

$$y_i = \begin{cases} \frac{v_i(0.25\sqrt{(i/n)+1.375})}{3} & \text{if } i \in \{1, \ldots, \lfloor n/2 \rfloor \} \\ \frac{v_i(7\sqrt{(i/n)-2})}{3} & \text{if } i \in \{\lfloor n/2 \rfloor + 1, \ldots, n\} \end{cases},$$

where the $v_i$'s are independent draws from $t(2)$. Unlike the previous scenarios, Scenario 6 presents a case
where the median is constant but the other quantiles change. For instance, as illustrated in Figure 1, the 0.9th quantile is piecewise smooth but continuous. By the nature of Scenario 6, one would expect PQTF2 to be the best method as the pieces of the 0.1 and 0.9th quantile curves can be well approximated by linear functions. This is indeed what we find in Table 2.

Finally, Figure 1 illustrates the true signals and one data set example for each of the different scenarios that we consider. Overall, we see that the PQTF estimator performs well across different scenarios and under the presence of heavy tailed errors thereby supporting our theoretical findings.

5 Discussion

To summarize, in this paper we have studied quantile trend filtering and some other quantile regression methods. Our risk adaptive bounds generalize previous work to quantile setting. The main advantage of our results is that they hold under very general conditions without requiring moment conditions and allowing for heavy-tailed distributions. We now discuss some issues related to our work in this paper.

Unlike trend filtering with sub-Gaussian errors, our risk bounds are based on \( \Delta(\cdot) \) instead of the squared error loss function \( \| \cdot \| \). In general, it is the case that the former is smaller. It is a natural question whether our results also hold under squared error loss. One thing we can say is that when the set \( K \) in the constrained quantile sequence model is contained in an \( \ell_\infty \) ball whose radius does not grow with \( n \), then our convergence rates based on \( \Delta(\cdot) \) also hold under \( \| \cdot \| \). This is because \( \| \cdot \| \) and \( \Delta(\cdot) \) are equivalent up to constants when evaluated in a compact set.

We have often stated that our convergence rates are minimax rate optimal. To clarify on this let us consider the case of Theorem 1. As Nussbaum (1985) showed (see the discussion in Tibshirani (2014)), there exists a constant \( c > 0 \) such that

\[
\inf_{\hat{\theta}} \sup_{\theta^*: TV^{(r)}(\theta^*) \leq V, \|\theta^*\|_{\infty} \leq 1} E \left( \frac{1}{n} \|\hat{\theta} - \theta^*\|^2 \right) \geq c \left( \frac{V}{n} \right)^{2r/(2r+1)} \tag{13}
\]

where the infimum is taken over all estimators and the \( y_i \)'s are independent draws from \( N(\theta^*_i, \sigma^2) \) for a known \( \sigma \). Since the parameter space is within the \( \ell_\infty \) ball of radius 1, therefore the left hand side in (13) equals the following up to a constant

\[
\inf_{\hat{\theta}} \sup_{\theta^*: TV^{(r)}(\theta^*) \leq V, \|\theta^*\|_{\infty} \leq 1} E \left\{ \frac{\Delta^2_n (\hat{\theta} - \theta^*)}{\Delta_n (\hat{\theta} - \theta^*)} \right\}.
\]

It now follows that the rates in Theorem 1 and Theorem 3 (up to log factors) are minimax in the sense that they match the rate in (13). Similarly, it can be seen that the rate in Theorem 5 is minimax up to log factors in that sense that it matches the minimax rates of mean estimation with sub-Gaussian noise in the class of 2D bounded variation signals, see Hutter and Rigollet (2016); Chatterjee and Goswami (2019b).

One natural extension of our work is to consider estimation of multiple quantiles with trend filtering. This can be formulated as follows. Let \( \Lambda \subset (0, 1) \) be a finite set and consider the estimator

\[
\{\hat{\theta}^{(r)}_i\} = \arg \min_{\{\theta(\tau)\}_{\tau \in \Lambda} \subset \mathbb{R}^n} \sum_{\tau \in \Lambda} \sum_{i=1}^n \rho_r \{ y_i - \theta_i(\tau) \},
\]

subject to

\[
TV^{(r)}(\theta(\tau)) \leq V(\tau), \forall \tau \in \Lambda
\]

\[
\theta(\tau) \leq \theta(\tau'), \forall \tau < \tau', \tau, \tau' \in \Lambda,
\]

where \( \{V(\tau)\} \) are tuning parameters for \( \tau \in \Lambda \). Let \( \theta^*_i(\tau) \) be a true \( \tau \)th quantile sequence for each \( \tau \in \Lambda \). If Assumption A holds for each \( \theta^*(\tau) \) instead of \( \theta^* \), then one can show using similar arguments as in the
proof of Theorem 1 that
\[ \sum_{\tau \in \Lambda} \Delta_n^2 \left\{ \theta^*(\tau) - \hat{\theta}_r^{(r)} \right\} = O_{pr} \left\{ n^{-2r/(2r+1)} \right\}, \]
provided that \( V(\tau) \geq TV^{(r)}(\theta(\tau)^*) = O(1) \) and \( V(\tau) = O(1) \). This is an extension of the upper bound in Theorem 1 to the case where we are estimating finitely many quantiles simultaneously. However, it might be of interest to consider the case when the number of quantiles to be estimated is allowed to grow with \( n \). We leave this for future investigation.

With regards to our results on both the CQTF and PQTF estimators, all of our bounds are \( O_{pr}(\cdot) \) statements. It would be interesting to attempt to translate these results to expectation or high probability bounds on the estimation error measured with \( \Delta_n^2(\cdot) \). Our guess is that if we allow heavy tailed errors such as the Cauchy distribution then the expectation of \( \Delta_n^2(\hat{\theta} - \theta^*) \) may not even exist. More investigations need to be done on how heavy the errors can be while ensuring in expectation or high probability bounds for \( \Delta_n^2(\hat{\theta} - \theta^*) \).

Since we give a general bound for the convex constrained quantile sequence estimation problem it would also be interesting to investigate whether our proof technique can be used in shape constrained quantile problems such as isotonic regression (see Chatterjee et al. (2015)) and convex regression (see Guntuboyina and Sen (2015)).

It is worthwhile to mention that trend filtering can be generalized for general graphs as was proposed by Wang et al. (2016) which included theoretical and computational developments. In the particular case of the fused lasso on general graphs, several recent works Padilla et al. (2018, 2020); Ortelli and van de Geer (2019b) have studied its risk properties. It will be interesting to investigate whether these types of results can be extended to the quantile setting.

Finally, the Dyadic CART estimator; orginally proposed in Donoho (1997), has been shown to enjoy computational and certain statistical advantages over trend filtering while nearly maintaining all its known theoretical guarantees; see Chatterjee and Goswami (2019a). It would be also be interesting to develop quantile versions of Dyadic CART as an alternative to quantile trend filtering.

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A Proof of Proposition 1

A.1 Lemmas Required for Proof of Proposition 1

We first recall the following well known fact bounding Rademacher Width by Gaussian Width; e.g see Page 132 in Wainwright (2019).

Lemma 8. We have
\[ RW(K) \leq \left( \frac{\pi}{2} \right)^{1/2} GW(K) \] (15)
where
\[ GW(K) = E \left( \sup_{v \in K} \sum_{i=1}^n z_i v_i \right), \]
for \( z_1, \ldots, z_n \) independent standard normal random variables.

We now recall some definitions.
Definition 1. The function $\Delta^2 : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\Delta^2(\delta) := \sum_{i=1}^{n} \min\{|\delta_i|, \delta^2_i\}.$$ 

We also write $\Delta(\delta) = \{\Delta^2(\delta)\}^{1/2}$.

Definition 2. We define the empirical loss function

$$\hat{M}(\theta) = \sum_{i=1}^{n} \hat{M}_i(\theta_i),$$

where

$$\hat{M}_i(\theta_i) = \rho_{\tau}(y_i - \theta_i) - \rho_{\tau}(y_i - \theta^*_i).$$

Setting $M_i(\theta_i) = E\{\rho_{\tau}(y_i - \theta_i) - \rho_{\tau}(y_i - \theta^*_i)\}$, the population version of $\hat{M}$ becomes

$$M(\theta) = \sum_{i=1}^{n} M_i(\theta_i).$$

Notice that in the previous definition the functions $M$ and $\hat{M}$ depend on $n$ and $\theta^*$ but we omit making this dependence explicit for simplicity.

With the notation from Definition 2, we consider the $M$-estimator

$$\hat{\theta} = \arg \min_{\theta \in \mathbb{R}^n} \hat{M}(\theta) \quad \text{subject to} \quad \theta \in K,$$

and $\theta^* \in \arg \min_{\theta \in \mathbb{R}^n} M(\theta)$. Throughout, we assume that $\theta^* \in K \subset \mathbb{R}^n$ where $K$ denotes a general constraint set throughout this section.

Lemma 9. With the notation from before,

$$M(\hat{\theta}) \leq \sup_{v \in K} \left\{ M(v) - \hat{M}(v) \right\}.$$ \hspace{1cm} (17)

Proof.

$$M(\hat{\theta}) = M(\hat{\theta}) - \hat{M}(\hat{\theta}) + \hat{M}(\hat{\theta})$$
$$\leq M(\hat{\theta}) - \hat{M}(\hat{\theta})$$
$$\leq \sup_{v \in K} \left\{ M(v) - \hat{M}(v) \right\},$$

where the first inequality follows since $\hat{M}(\hat{\theta}) \leq 0$. 

Next, we proceed to bound the right hand side of Equation 17 by the standard technique of symmetrization.

Lemma 10. (Symmetrization). It holds that

$$E \left[ \sup_{v \in K} \left\{ M(v) - \hat{M}(v) \right\} \right] \leq 2 E \left\{ \sup_{v \in K} \sum_{i=1}^{n} \xi_i \hat{M}_i(v_i) \right\},$$

where $\xi_1, \ldots, \xi_n$ are independent Rademacher variables independent of $\{y_i\}_{i=1}^{n}$. 

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Proof. Let \( \bar{y}_1, \ldots, \bar{y}_n \) be an independent and identically distributed copy of \( y_1, \ldots, y_n \), and let \( \bar{M}_i \) the version of \( M_i \) corresponding to \( \bar{y}_1, \ldots, \bar{y}_n \). Then,

\[
E \left( \sup_{v \in K} \sum_{i=1}^n \left[ E\{\bar{M}_i(v_i)\} - \bar{M}_i(v_i) \right] \right) = E \left( \sup_{v \in K} \sum_{i=1}^n \left[ E\{\bar{M}_i(v_i)\} - \bar{M}_i(v_i) \right] \right).
\]

Condition on \( y_1, \ldots, y_n \) and let

\[
X_v = \sum_{i=1}^n \left\{ \bar{M}_i(v_i) - \bar{M}_i(v_i) \right\}.
\]

Then

\[
\sup_{v \in K} E_{\bar{y}_1, \ldots, \bar{y}_n | y_1, \ldots, y_n} X_v \leq E_{\bar{y}_1, \ldots, \bar{y}_n | y_1, \ldots, y_n} \sup_{v \in K} X_v.
\]

We can take the expected value with respect to \( y_1, \ldots, y_n \) to get

\[
E \left( \sup_{v \in K} \sum_{i=1}^n \left\{ M_i(v_i) - \hat{M}_i(v_i) \right\} \right) \leq E \left( \sup_{v \in K} \sum_{i=1}^n \left\{ \hat{M}_i(v_i) - \hat{M}_i(v_i) \right\} \right)
\]

\[
= E \left( \sup_{v \in K} \sum_{i=1}^n \left\{ \xi_i \hat{M}_i(v_i) \right\} \right)
\]

\[
\leq E \left\{ \sup_{v \in K} \sum_{i=1}^n \xi_i \hat{M}_i(v_i) \right\} + E \left\{ \sup_{v \in K} \sum_{i=1}^n -\xi_i \hat{M}_i(v_i) \right\}
\]

\[
= 2E \left\{ \sup_{v \in K} \sum_{i=1}^n \xi_i \hat{M}_i(v_i) \right\},
\]

where the first equality follows because \((\xi_1(\hat{M}_{n,1}(v_1) - \hat{M}_{n,1}(v_1)), \ldots, \xi_n(\hat{M}_{n,n}(v_n) - \hat{M}_{n,n}(v_n)))\) and \((\hat{M}_{n,1}(v_1) - \hat{M}_{n,1}(v_1), \ldots, \hat{M}_{n,n}(v_n) - \hat{M}_{n,n}(v_n))\) have the same distribution. The second equality follows because \(-\xi_1, \ldots, -\xi_n\) are also independent Rademacher variables.

\[ \square \]

Lemma 11. (Contraction principle). With the notation from before we have that

\[
E \left\{ \sup_{v \in K} \sum_{i=1}^n \xi_i \hat{M}_i(v_i) \right\} \leq 2RW (K - \theta^*) = 2RW (K).
\]

Proof. Recall that \( \hat{M}_i(v_i) = \rho_r(y_i - v_i) - \rho_r(y_i - \theta_i^*) \). Clearly, these are 1-Lipschitz continuous functions. Therefore,

\[
E \left\{ \sup_{v \in K} \sum_{i=1}^n \xi_i \hat{M}_i(v_i) \right\} = E \left( E \left\{ \sup_{v \in K} \sum_{i=1}^n \xi_i \hat{M}_i(v_i) \mid y \right\} \right)
\]

\[
\leq E \left( E \left\{ \sup_{v \in K} \sum_{i=1}^n \xi_i v_i \mid y \right\} \right)
\]

\[
= E \left\{ \sup_{v \in K} \sum_{i=1}^n \xi_i (v_i - \theta_i^*) \right\} + E \left( \sum_{i=1}^n \xi_i \theta_i^* \right)
\]

\[
= E \left\{ \sup_{v \in K} \sum_{i=1}^n \xi_i (v_i - \theta_i^*) \right\},
\]

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where the inequality follows from, the contraction principle for Rademacher complexity, see Theorem 4.12 in Ledoux and Talagrand (2013).

The following corollary can be used for proving upper bounds for general constraint estimators as in (16) when the set \( K \) is compact.

**Corollary 12.** With the notation from before,

\[
E \left\{ M(\hat{\theta}) \right\} \leq 2 \operatorname{RW}(K),
\]

where the right most inequality holds for a general set \( K \).

**Proof.** This follows from Lemmas 9–11. \( \square \)

### A.2 Proof of Proposition 1

We now prove Proposition 1 whose statement we now recall here.

**Proposition 1.** Let \( K \subset \mathbb{R}^n \) be a convex set. Let us define a function \( M : [0, \infty) \to \mathbb{R} \) as follows:

\[
M(t) = \operatorname{RW}(K \cap \{ \theta : M(\theta) \leq t^2 \}).
\]

Then the following inequality is true for any \( t > 0 \),

\[
\operatorname{pr}\{ M(\hat{\theta}) > t^2 \} \leq \frac{2M(t)}{t^2}.
\]

**Proof.** Suppose that \( M(\hat{\theta}) > t^2 \). (18)

First, notice that \( M \) is continuous. To see this, let \( \theta, \tilde{\theta} \in \mathbb{R}^n \). Then

\[
|M(\theta) - M(\tilde{\theta})| = \left| n \sum_{i=1}^{n} E \left\{ \rho_r(y_i - \theta_i) - \rho_r(y_i - \tilde{\theta}_i) \right\} \right|
\leq \sum_{i=1}^{n} E \left\{ |\rho_r(y_i - \theta_i) - \rho_r(y_i - \tilde{\theta}_i)| \right\}
\leq \sum_{i=1}^{n} |\theta_i - \tilde{\theta}_i|,
\]

where the second inequality follows by the fact that \( \rho_r(\cdot) \) is a 1-Lipschitz function. Hence, \( M \) is continuous.

Next, let \( q^2 := M(\hat{\theta}) \). Then define \( g : [0, 1] \to \mathbb{R} \) as \( g(u) = M\{ (1 - u)\theta^* + u\hat{\theta} \} \). Clearly, \( g \) is a continuous function with \( g(0) = 0 \), and \( g(1) = q^2 \). Therefore, there exists \( u_{\hat{\theta}} \in [0, 1] \) such that \( g(u_{\hat{\theta}}) = t^2 \). Hence, letting \( \tilde{\theta} = (1 - u_{\hat{\theta}})\theta^* + u_{\hat{\theta}}\hat{\theta} \) we observe that by the convexity of \( \hat{M} \) and the basic inequality

\[
\hat{M}(\tilde{\theta}) = \hat{M}\{ (1 - u_{\hat{\theta}})\theta^* + u_{\hat{\theta}}\hat{\theta} \} \leq (1 - u_{\hat{\theta}})\hat{M}(\theta^*) + u_{\hat{\theta}}\hat{M}(\hat{\theta}) \leq 0.
\]

Furthermore, \( \tilde{\theta} \in K \) by convexity of \( K \), and \( M(\tilde{\theta}) = t^2 \) by construction. This implies that

\[
\sup_{v \in K : M(v) \leq t^2} M(v) - M(v) \geq M(\tilde{\theta}) - \hat{M}(\tilde{\theta}) \geq M(\tilde{\theta}).
\]
Therefore,
\[
\Pr\left\{ M(\hat{\theta}) > t^2 \right\} \leq \Pr\left\{ \sup_{v \in K : M(v) \leq t^2} M(v) - \hat{M}(v) \geq t^2 \right\} \\
\leq \frac{1}{t^2} \mathbb{E}\left\{ \sup_{v \in K : M(v) \leq t^2} M(v) - \hat{M}(v) \right\} \\
\leq \frac{2}{t^2} \mathcal{R}W \left\{ \{ v \in K : M(v) \leq t^2 \} \right\} \\
= \frac{2}{t^2} \mathcal{M}(t)/t^2,
\]
where the second inequality follows from Markov’s inequality, and the third as in Lemmas 10 and 11. This completes the proof. 

\[
\square
\]

### A.3 Proof of Theorem 7 and Associated Corollaries

We start by recalling Assumption A.

**Assumption A:** There exists a constant \( L > 0 \) and \( \ell > 0 \) such that for any positive integer \( n \) and any \( \delta \in \mathbb{R}^n \) satisfying \( \|\delta\|_\infty \leq L \) we have that
\[
|F_{y_i}(\theta_i^* + \delta_i) - F_{y_i}(\theta_i^*)| \geq \ell |\delta_i|,
\]
for all \( i = 1, \ldots, n \), where we recall that \( F_{y_i} \) is the CDF of \( y_i \).

Theorem 7 follows with the same argument in the proof of Proposition 1 combined with the following lemma.

**Lemma 13.** Suppose that Assumption A holds. Then there exists a constant \( c_0 \) such that for all \( \delta \in \mathbb{R}^n \), we have
\[
M(\theta^* + \delta) \geq c_0 \Delta^2(\delta).
\]

**Proof.** First, we notice that by Equation B.3 in Belloni and Chernozhukov (2011), we have that
\[
M_i(\theta_i^* + \delta_i) - M_i(\theta_i^*) = \int_0^{\delta_i} \{ F_{y_i}(\theta_i^* + z) - F_{y_i}(\theta_i^*) \} \, dz,
\]
for all \( i = 1, \ldots, n \). Hence, supposing that \( |\delta_i| \leq L \), we obtain from Assumption A that
\[
M_i(\theta_i^* + \delta_i) - M_i(\theta_i^*) \geq \frac{\delta_i^2 \ell}{2}.
\]

Suppose now that \( \delta_i > L \). Then by (19), we obtain
\[
M_i(\theta_i^* + \delta_i) - M_i(\theta_i^*) \geq \int_{\delta_i - L/2}^{\delta_i} \{ F_{y_i}(\theta_i^* + z) - F_{y_i}(\theta_i^*) \} \, dz
\]
\[
\geq \int_{-L/2}^{L/2} \{ F_{y_i}(\theta_i^* + L/2) - F_{y_i}(\theta_i^*) \} \, dz
\]
\[
= \left( \delta_i - \frac{L}{2} \right) \{ F_{y_i}(\theta_i^* + L/2) - F_{y_i}(\theta_i^*) \}
\]
\[
\geq \frac{\delta_i^2 \ell}{2} \frac{L}{2} = \frac{\delta_i \ell}{2} c_0,
\]
where the first two inequalities follow because \( F_{y_i} \) is monotone, and the third inequality by Assumption A.

The case \( \delta_i < -L \) can be handled similarly. The conclusion follows combining the three different cases. 

\[
\square
\]
A.3.1 Corollary 14

**Corollary 14.** Consider the notation from Theorem 7. If \( \{r_n\} \) is a sequence such that

\[
\lim_{t \to \infty} \sup_{n \geq 1} \frac{\mathcal{R}(tr_n r_n^{1/2})}{t^2 r_n^2} = 0, \tag{20}
\]

then

\[
\frac{1}{n} \Delta^2(\hat{\theta}_K - \theta^*) = O_{\text{pr}}(r_n^2).
\]

**Proof.** Let \( \epsilon > 0 \) be given. Notice that for any \( c_1 > 0 \) we have that

\[
\Pr \left\{ \frac{1}{n} \Delta^2(\hat{\theta}_K - \theta^*) > c_1 r^2_n \right\} \leq C \frac{\mathcal{R}(c_1 r_n r_n^{1/2})}{c_1^2 r_n^2} < \epsilon,
\]

where the first inequality holds by Theorem 7 and the last by choosing \( c_1 \) large enough exploiting (20).

A.3.2 Corollary 15

**Corollary 15.** Let \( K \subset \mathbb{R}^n \) be a convex set. Suppose the distributions of \( y_1, \ldots, y_n \) obey Assumption A. Then the following expectation bound holds:

\[
E \{ \Delta^2(\hat{\theta}_K - \theta^*) \} \leq C \mathcal{R}(K)
\]

where \( C \) is a constant that only depends on the distributions of \( y_1, \ldots, y_n \).

**Proof.** This follows by combining Theorem 12 with Lemma 13.

B Proof of Theorem 1 and Theorem 2

We first provide a sketch of our proofs for the sake of convenience of the reader. This sketch is meant to convey the overall proof structure.

B.1 Proof sketch of Theorem 1 and Theorem 2

As discussed in Section 3, we must upper bound the quantity

\[
\mathcal{R}(K_V - \theta^* \cap \{\delta : \Delta^2(\delta) \leq t^2\})
\]

for \( t > 0 \). Here, \( K_V = \{ \theta : \|D^{(r)} \theta\|_1 \leq V n^{1-r} \} \), where \( V^* = n^{r-1} \|D^{(r)} \theta^*\|_1 \) and \( V \geq V^* \) when we are proving Theorem 1 and \( V = V^* \) when we are proving Theorem 2. This differs from the usual least squares setting where the quantity of interest is

\[
\mathcal{R}(K_V - \theta^* \cap \{\delta : \|\delta\|^2 \leq t^2\}).
\]

To proceed in the proof of Theorems 1 and 2, we start by writing

\[
\mathcal{R}(K_V - \theta^* \cap \{\delta : \Delta^2(\delta) \leq t^2\}) \leq T_1 + T_2,
\]

where

\[
T_1 := E \left\{ \sup_{\delta \in K_V - \theta^* : \Delta^2(\delta) \leq t^2} \xi^\top P_{\mathcal{R}^\perp} \delta \right\}
\]
and

\[ T_2 := E \left\{ \sup_{\delta \in K_{V^*} - \theta^* : \Delta^2(\delta) \leq \tilde{t}^2} \xi^\top P_{R^\perp} \delta \right\}. \]

where \( R \) is the subspace spanned by the rows of \( D^{(r)} \) (recall its definition on Page 2 in the paper), \( R^\perp \) is the orthogonal complement of \( R \) and \( P_R, P_{R^\perp} \) are the orthogonal projection matrices for the corresponding subspaces.

Next we consider different steps.

**Step 1:** Bounding \( T_1 \). We attain this by writing

\[ P_{R^\perp} \delta = \sum_{j=1}^r (\delta^\top v_j) v_j, \]

where \( \{v_1, \ldots, v_r\} \) form an orthonormal basis of \( R^\perp \). We then upper bound \(|\xi^\top P_{R^\perp} \delta|\) using Lemmas 20 and 21, exploiting the fact that \( \delta \in K - \theta^* \) and \( \Delta^2(\delta) \leq t^2 \) as in the definition of \( T_1 \).

**Step 2:** Bounding \( T_2 \). The bound for \( T_2 \) is going to be the leading order term. The key observation we use here is Lemma 17 which states that

\[ \|\delta\|^2 \leq \max\{\|\delta\|_\infty, 1\} \Delta^2(\delta), \tag{21} \]

for all \( \delta \in \mathbb{R}^n \). Then Lemmas 21 and 22 provide upper bounds on \( \|P_R\delta\|_\infty \) and \( \Delta^2(P_R\delta) \) respectively. This together with (21) leads to

\[ T_2 \leq E \left\{ \sup_{\delta \in K_{V^*} - \theta^*: \|\delta\| \leq \tilde{t}} \xi^\top P_{R^\perp} \delta \right\} \]

for some \( \tilde{t} \) that is of the same order of magnitude as \( t \).

**Step 3:**

We obtain that the bound on \( T_1 \) is a lower order term. Hence, the main task is to get good bounds on \( T_2 \).

- **Proof of Theorem 1:** The bound for \( T_2 \) given in the last display is exactly the local Gaussian Width of the set \( K_{V^*} - \theta^* \) for \( V \geq V^* \) and a bound for this local Gaussian width is available in Lemma B.1 from Guntuboyina et al. (2020).

- **Proof of Theorem 2:** The main difference with Theorem 1 is in the way we handle \( T_2 \). Similarly as in Step 2, we show that for \( V = V^* \)

\[ T_2 \leq E \left\{ \sup_{\delta : K - P_{R^\perp} \theta^* : \|\delta\| \leq \tilde{t}} \xi^\top \delta \right\} \leq E \left\{ \sup_{\delta : T_{K_{V^*}}(P_{R^\perp} \theta^*), \|\delta\| \leq \tilde{t}} \xi^\top \delta \right\} \]

for some \( \tilde{t} \) that is of the same order of magnitude as \( t \) and \( T_{K_{V^*}}(P_{R^\perp} \theta^*) \) is the tangent cone at \( P_{R^\perp} \theta^* \) with respect to the convex set \( K_{V^*} \); see (32) for the precise definition. The Gaussian width of such a tangent cone is again available in Appendix B.2 in Guntuboyina et al. (2020) and we directly employ this result to finish the proof.

### B.2 Proofs of Theorem 1 and Theorem 2

We now start our formal proofs. We first state some lemmas that we will require.
B.2.1 Intermediate results required for Proofs of Theorem 1 and Theorem 2

Lemma 16. It holds that

\[ \mathcal{R}^\perp = \Pi := \text{Span} \{ v \in \mathbb{R}^n : v_i = p(i/n), \text{ for } i = 1, \ldots, n, \text{ and } p(\cdot) \text{ a polynomial of degree at most } r - 1 \} . \]

Proof. First, notice that \( \Pi \) equals to the column space of the matrix

\[
\begin{pmatrix}
1 & \frac{1}{n} & \cdots & \left(\frac{1}{n}\right)^{r-1} \\
1 & \frac{2}{n} & \cdots & \left(\frac{2}{n}\right)^{r-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{pmatrix} \in \mathbb{R}^{n \times r},
\]

which is a Vandermonde matrix of rank \( r \), if \( r < n \). Furthermore, \( v \in \Pi \) implies that \( D^r(v) = 0 \), which holds by an iterative application of the mean value theorem and the fact that the \( r \)th derivative of a polynomial of is constant and equals to 0. Therefore, \( \Pi \subset \mathcal{R}^\perp \) and \( \dim(\mathcal{R}^\perp) = \dim(\Pi) \). Hence, the claim follows. \( \square \)

Lemma 17. Let \( \delta \in \mathbb{R}^n \). Then

\[ \| \delta \|^2 \leq \max\{ \| \delta \|_\infty, 1 \} \Delta^2(\delta). \] (22)

Proof. We notice that

\[
\| \delta \|^2 = \sum_{i : |\delta_i| \leq 1} |\delta_i|^2 + \sum_{i : |\delta_i| > 1} |\delta_i|^2 \\
\leq \sum_{i : |\delta_i| \leq 1} |\delta_i|^2 + \| \delta \|_\infty \sum_{i : |\delta_i| > 1} |\delta_i| \\
\leq \max\{ \| \delta \|_\infty, 1 \} \left( \sum_{i : |\delta_i| \leq 1} |\delta_i|^2 + \sum_{i : |\delta_i| > 1} |\delta_i| \right) \\
= \max\{ \| \delta \|_\infty, 1 \} \Delta^2(\delta).
\]

\( \square \)

Lemma 18. Let \( v \in \mathcal{R}^\perp \) such that \( \| v \| = 1 \). Then

\[ \| v \|_\infty \leq \frac{b_r}{n^{1/2}} , \]

for a positive constant \( b_r \) that only depends on \( r \).

Proof. Let \( \{ q_m \}_{m=0}^{r-1} \) be the normalized Legendre polynomials of degree at most \( r - 1 \) which have domain in \([-1, 1]\) and satisfy

\[ \int_{-1}^{1} q_m(x) q_{m'}(x) \, dx = 1_{\{ m = m' \}} . \]

Next notice that, by Lemma 16, \( v \) can be written as

\[ v_i = \sum_{j=0}^{r-1} a_j q_j(x_i), \]

where \( x_i = -1 + 2i/n \) for \( i = 1, \ldots, n \), and where \( a_0, \ldots, a_{r-1} \in \mathbb{R} \). Let \( g : \mathbb{R} \to \mathbb{R} \) be defined as

\[ g(x) = \sum_{j=0}^{r-1} a_j q_j(x) . \]

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The notice that for \( A_1 = (-1, x_1) \), and \( A_i = (x_{i-1}, x_i) \) for all \( i > 1 \), we have that
\[
\left| \sum_{j=0}^{r-1} a_j^2 - \frac{2}{n} \right| = \left| \int_{-1}^{1} [g(x)]^2 dx - \frac{2}{n} \sum_{i=1}^{n} [g(x_i)]^2 \right|
\leq \frac{1}{n} \sum_{i=1}^{n} \int_{A_i} |[g(x)]^2 - [g(x_i)]^2| \, dx
\leq \frac{1}{n} \sum_{i=1}^{n} \int_{A_i} \| [g^2]' \|_\infty |x - x_i| \, dx
\leq \frac{1}{n} \| [g^2]' \|_\infty.
\]

However,
\[
\{ [g(x)]^2 \}' = \left\{ \sum_{j=0}^{r-1} a_j^2 q_j(x)^2 + \sum_{j \neq j'} a_j a_{j'} q_j(x) q_{j'}(x) \right\}'.
\]

Therefore,
\[
\left| \sum_{j=0}^{r-1} a_j^2 - \frac{2}{n} \right| \leq c_r \| a \|_\infty^2 \leq \frac{c_r}{n} \sum_{j=0}^{r-1} a_j^2,
\]
for some constant \( c_r > 0 \) that only depends on \( r \). Hence, for large enough \( n \),
\[
\sum_{j=0}^{r-1} a_j^2 \leq \frac{\tilde{c}_r}{n},
\]
for a constant \( \tilde{c}_r > 0 \) that depends on \( r \). As a result
\[
\| v \|_\infty \leq \max_{i=1,\ldots,n} \sum_{j=0}^{r-1} |q_j(x_i)||a_j| \leq \| a \|_\infty \max_{x \in [-1,1]} \sum_{j=0}^{r-1} |q_j(x)| \leq \frac{c_r \| a \|_\infty^{1/2}}{n^{1/2}} \max_{x \in [-1,1]} \sum_{j=0}^{r-1} |q_j(x)|,
\]
and the claim follows. \( \square \)

**Lemma 19.** If \( \delta \in \mathbb{R}^n \) and \( TV^{(r)}(\delta) \leq V \), then
\[
\| P_R \delta \|_\infty \leq \tilde{C}_r \frac{V}{n^{r-1}},
\]
for a constant \( \tilde{C}_r > 1 \) that depends on \( r \).

**Proof.** Let \( M := \{ D^{(r)} \}^+ \in \mathbb{R}^{n \times (n-r)} \) be the Moore–Penrose inverse of \( D^{(r)} \). First, we notice that by Lemma 13 in Wang et al. (2016), we have that \( M = P_R H_2/(r-1)! \) where \( H_2 \) consists of the last \( n-r \) columns of the \( (r-1) \)th order falling factorial basis matrix. Here, as in Wang et al. (2014), we have that for \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, n-r\} \),
\[
(H_2)_{i,j} = h_j(i/n),
\]
where
\[
h_j(x) = \prod_{l=1}^{r-1} \left( x - \frac{j+l}{n} \right) 1_{\{x \geq \frac{i+r-1}{n}\}}.
\]
Then for $e_i$ an element of the canonical basis in $\mathbb{R}^{n-r}$ we have that

$$
\|e_i^T M\|_\infty \leq \|P_R e_i\|_1 \|H_2\|_\infty / (r-1)!
\leq (\|e_i\|_1 + \|P_R e_i\|_1) \|H_2\|_\infty / (r-1)!
\leq [1 + \|P_R e_i\|_1] / (r-1)!
$$

where the first inequality follows from Hölder’s inequality, the second from the triangle inequality and the last by the definition of $H_2$.

Next let $v_1, \ldots, v_r$ be an orthonormal basis of $P_R \perp$. Then

$$
\|P_R e_i\|_1 \leq \|\sum_{j=1}^{r} (e_i^T v_j) v_j\|_1 \leq \sum_{j=1}^{r} \|e_i^T v_j\| \|v_j\|_1 \leq \sum_{j=1}^{r} \|v_j\|_\infty \|v_j\|_1 \leq \sum_{j=1}^{r} \|v_j\|_\infty n^{1/2}.
$$

Hence, from Lemma 18 we obtain that

$$
\|M\|_\infty = \max_{i=1,\ldots,n, j=1,\ldots,n-r} |M_{i,j}| = O(1).
$$

(23)

Finally, if $\delta \in \mathbb{R}^n$ and $TV^{(r)}(\delta) \leq V$, then

$$
\|P_R \delta\|_\infty = \|\{D^{(r)}\} + D^{(r)} \delta\|_\infty \leq \|M\|_\infty \|D^{(r)} \delta\|_1 = O\left\{\|D^{(r)} \delta\|_1\right\}
$$

where the first inequality follows from Hölder’s inequality, and the last from (23). The claim follows.

**Lemma 20.** Let $\delta \in \mathbb{R}^n$ and $v \in \mathcal{R} \perp$ such that $\|v\| = 1$. Then

$$
|\delta^T v| \leq \frac{b_r}{n^{1/2}} \Delta^2(\delta) + \Delta(\delta),
$$

where $b_r > 0$ is the constant from Lemma 18.

**Proof.** Notice that

$$
|\delta^T v| \leq \sum_{i=1}^{n} |\delta_i| |v_i|
= \sum_{i=1}^{n} |\delta_i| |v_i| 1\{|\delta_i| > 1\} + \sum_{i=1}^{n} |\delta_i| |v_i| 1\{|\delta_i| \leq 1\}
\leq \|v\|_\infty \sum_{i=1}^{n} |\delta_i| 1\{|\delta_i| > 1\} + \|v\|_\infty \left(\sum_{i=1}^{n} \delta_i^2 1\{|\delta_i| \leq 1\}\right)^{1/2}
\leq \frac{b_r}{n^{1/2}} \Delta^2(\delta) + \Delta(\delta),
$$

where the first inequality follows from the triangle inequality, the second from Hölder and Cauchy–Schwarz inequalities, and the last by the definition of $\Delta^2(\cdot)$ and Lemma 18. The claim follows.

**Lemma 21.** Let $\delta \in \mathbb{R}^n$ with $\Delta^2(\delta) \leq t^2$. Then

$$
\|P_R \perp \delta\|_\infty \leq \gamma(t, n) := \tilde{b}_r \left(\frac{t}{n^{1/2}} + \frac{t^2}{n}\right),
$$

where $\tilde{b}_r > 0$ depends on $r$ only.
Proof. Let $v_1, \ldots, v_r$ an orthonormal basis of $\mathcal{R}^\perp$. Then

$$P_{\mathcal{R}^\perp} \delta = \sum_{j=1}^r (\delta^T v_j) v_j.$$  

Hence,

$$|(P_{\mathcal{R}^\perp} \delta)_{i}| \leq r \left( \max_{j=1,\ldots,r} \|v_j\|_\infty \right) \left( \max_{j=1,\ldots,r} |\delta^T v_j| \right) \leq r \frac{b_r}{n^{1/2}} \left( \max_{j=1,\ldots,r} |\delta^T v_j| \right),$$  

(24) where the last inequality follows from Lemma 18. Now, for $j \in \{1, \ldots, r\}$, we have by Lemma 20 that

$$|\delta^T v_j| \leq \frac{b_r}{n^{1/2}} t^2 + t.$$  

(25)

The claim follows combining (24) with (25).

\[\blacksquare\]

**Proposition 2.** Under Assumption A we have that

$$RW \left[ \left\{ \delta : TV^{(r)}(\delta) \leq 2V, \Delta^2(\delta) \leq t^2 \right\} \right] \leq C_r \{m(t, n)\}^{1-1/(2r)} (n^{1/2}V)^{1/(2r)} + a(t, n)$$

where $a(t, n)$ is a lower order term defined as

$$a(t, n) := C_r \left\{ \frac{t^2}{n^{1/2}} + t \right\} + C_r m(t, n) \{\log(en)\}^{1/2},$$

and

$$m(t, n) := \tilde{c}_r \max \left\{ (V/n^{r-1})^{1/2}, 1 \right\} \left[ 1 + \{\gamma(t, n)\}^{1/2} \right] t + \frac{t^2}{n^{1/2}},$$

for some positive constants $C_r, \tilde{c}_r$.

**Remark 12.** For the choice of $t$ that we make within the proof of Theorem 12, the term $m(t, n)$ is $\Theta(t)$ and hence the reader can safely think of $m(t, n)$ in the right hand side above as $t$.

Proof. First, we observe that

$$E \left\{ \sup_{v \in K : \Delta^2(\theta - \theta^*) \leq t^2} \sum_{i=1}^n \xi_i (v_i - \theta_i^*) \right\} \leq E \left\{ \sup_{\delta : TV^{(r)}(\delta) \leq 2V, \Delta^2(\delta) \leq t^2} \xi^T P_{\mathcal{R}^\perp} \delta \right\} + E \left\{ \sup_{\delta : TV^{(r)}(\delta) \leq 2V, \Delta^2(\delta) \leq t^2} \xi^T P_{\mathcal{R}\delta} \right\}$$

(26)

= $T_1 + T_2$.

Hence, we proceed to bound $T_1$ and $T_2$.

**Bounding $T_1$.**

Let $v_1, \ldots, v_r$ an orthonormal basis of $\mathcal{R}^\perp$. Then by Lemma 18, it holds that $\|v_j\|_\infty \leq b_r/n^{1/2}$, for $j = 1, \ldots, r$. Hence, for any $\delta \in \mathbb{R}^n$ with $\Delta^2(\delta) \leq t^2$,

$$\xi^T P_{\mathcal{R}^\perp} \delta \leq \left| \sum_{j=1}^r \delta^T v_j \cdot \xi^T v_j \right|$$

$$\leq \sum_{j=1}^r |\delta^T v_j| \cdot |\xi^T v_j|$$

$$\leq r \left( \max_{j=1,\ldots,r} |\xi^T v_j| \right) \left( \max_{j=1,\ldots,r} |\delta^T v_j| \right)$$

$$\leq r \left( \max_{j=1,\ldots,r} |\xi^T v_j| \right) \left( \frac{b_r t^2}{n^{1/2}} + t \right)$$

(27)
where the last inequality follows from Lemma 20. Therefore,
\[
T_1 \leq r \left( \frac{b_r t^2}{n^{1/2}} + t \right) \sum_{j=1}^{r} E \left( |\xi^\top v_j| \right) \leq C_r \left( \frac{t^2}{n^{1/2}} + t \right),
\] (28)
for some positive constant \(C_r > 0\), and where the last inequality follows since \(\xi^\top v_j\) are sub-Gaussian random variables with variance 1.

**Bounding \(T_2\).**

We now proceed to bound \(T_2\). Towards that end we first prove a lemma.

**Lemma 22.** Let \(\delta \in \mathbb{R}^n\) with \(\Delta^2(\delta) \leq t^2\). Then,
\[
\Delta^2(P_R \delta) \leq h(t, n) := c_r \left( t^2 + t^2 \gamma(t, n) + n \left( \frac{t^2}{n} + \frac{t^4}{n^2} \right) \right),
\]
with \(\gamma(t, n)\) as in Lemma 21, and for some constant \(c_r > 0\).

**Proof.** Set \(\tilde{\delta} = P_R \delta\). By Lemma 21 we have that \(\|\tilde{\delta} - \delta\|_\infty \leq \gamma(t, n)\). Also,
\[
\Delta^2(\tilde{\delta}) = \sum_{i=1}^{n} \min\{ |\tilde{\delta}_i|, \delta_i^2 \} \\
\leq \sum_{i=1}^{n} |\tilde{\delta}_i| 1_{\{ |\tilde{\delta}_i| > 1 \}} + \sum_{i=1}^{n} \tilde{\delta}_i^2 1_{\{ |\tilde{\delta}_i| \leq 1 \}},
\]
and so
\[
\Delta^2(\tilde{\delta}) \leq \sum_{i=1}^{n} \{ |\tilde{\delta}_i| + \gamma(t, n) \} 1_{\{ |\tilde{\delta}_i| > 1 \}} + \sum_{i=1}^{n} \left[ 2\tilde{\delta}_i^2 + 2\gamma(t, n) \right] 1_{\{ |\tilde{\delta}_i| \leq 1 \}} \\
\leq \left[ 2t^2 + 2t^2 \gamma(t, n) + 2n \gamma(t, n) \right] 1_{\{ |\tilde{\delta}_i| > 1 \}} \\
\leq 2 \left\{ t^2 + 2t^2 \gamma(t, n) + 4m \left( \frac{t^2}{n} + \frac{t^4}{n^2} \right) \right\},
\]
where the second inequality follows form the fact that
\[
| \left\{ i : |\tilde{\delta}_i| > 1 \right\} | \leq t^2.
\]
\(\Box\)

Next, let \(\delta \in \mathbb{R}^n\), \(\tilde{\delta} = P_R \delta\) and suppose that \(\Delta^2(\tilde{\delta}) \leq t^2\), and \(\text{TV}^{(r)}(\delta) \leq 2V\). Then from Lemmas 17, 19 and 22, we obtain that
\[
\|\tilde{\delta}\| \leq m(t, n) := c_r \max \left\{ \left( V/n^{r-1} \right)^{1/2}, 1 \right\} \left( \left[ 1 + \{ \gamma(t, n) \}^{1/2} \right] t + \frac{t^2}{n^{1/2}} \right), \tag{29}
\]
for a positive constant \(c_r\) that depends on \(r\). As a result from (26), we obtain
\[
T_2 \leq E \left\{ \sup_{\delta : \text{TV}(\delta) \leq 2V, \|\delta\| \leq m(t, n)} \xi^\top \delta \right\}. \tag{30}
\]
Therefore, by Lemma B.1 from Guntuboyina et al. (2020) and Lemma 8,
\[
T_2 \leq C_r m(t, n) \left\{ \frac{n^{1/2} V}{m(t, n)} \right\}^{1/(2r)} + C_r m(t, n) \{ \log(en) \}^{1/2}. \tag{31}
\]
for a positive constant \(C_r\) that depends on \(r\). The conclusion follows. \(\Box\)
B.2.2 Proof of Theorem 1

Finally, we present the proof of Theorem 1.

Proof. This follows immediately from Proposition 2 and Corollary 14 by setting

\[ r_n \asymp n^{-r/(2r+1)} V^{1/(2r+1)} \max \left\{ 1, \left( \frac{V}{n^{r-1}} \right)^{(2r-1)/(4r+2)} \right\}. \]

\[ \square \]

B.2.3 Proof of Theorem 2

We now present our proof of Theorem 2. Throughout we write

\[ K = \left\{ \theta \in \mathbb{R}^n : \|D^{(r)} \theta\|_1 \leq \frac{V^*}{n^{r-1}} \right\}. \]

Notice that to arrive at the conclusion of Theorem 2, by Theorem 7, it is enough to bound

\[ RW \left( \{ \delta \in K - \theta^* : \Delta^2(\delta) \leq t^2 \} \right), \]

for \( t < cn^{1/2} \) where \( c > 0 \) is a constant.

However,

\[ RW \left( \{ \delta \in K - \theta^* : \Delta^2(\delta) \leq t^2 \} \right) \leq E \left\{ \sup_{\delta \in K - \theta^* : \Delta^2(\delta) \leq t^2} \xi^\top P_{R^\perp} \delta \right\} \]

\[ + E \left\{ \sup_{\delta \in K - \theta^* : \Delta^2(\delta) \leq t^2} \xi^\top P_R \delta \right\} =: T_1 + T_2. \]

Then, \( T_1 \) can be bounded with the same argument that \( T_1 \) was bounded in the proof of Proposition 2. To control \( T_2 \), we define the tangent cone of \( \theta \in K \) as

\[ T_K(\theta) = \text{Closure} \left\{ \delta \in \mathbb{R}^n : \delta = a(v - \theta), \ v \in K, \ a \geq 0 \right\}, \quad (32) \]

and notice that as in Equation (30),

\[ T_2 \leq E \left\{ \sup_{\delta : K - P_R \theta^*, \|\delta\| \leq m(t,n)} \xi^\top \delta \right\}. \]

\[ \leq E \left\{ \sup_{\delta \in T_K(P_R \theta^*), \|\delta\| \leq \max\{1,(V^*/n^{r-1})^{1/2}c(t,r)\}} \xi^\top \delta \right\} \]

\[ = \max \left\{ 1, \left( \frac{V^*}{n^{r-1}} \right)^{1/2} tc(r) \right\} E \left\{ \sup_{\delta \in T_K(P_R \theta^*), \|\delta\| \leq 1} \xi^\top \delta \right\}, \quad (33) \]

\[ \leq \max \left\{ 1, \left( \frac{V^*}{n^{r-1}} \right)^{1/2} tc(r) c_r \right\} \left( s + 1 \right) \log \left( \frac{en}{s+1} \right)^{1/2} \]

for some positive constant \( c_r \), where the last inequality holds by Appendix B.2 in Guntuboyina et al. (2020) and Lemma 8.
C Proof of Theorem 3

Throughout this section we will use $C$ as a generic positive constant that can change from line to line. Furthermore, for an appropriate $\lambda$ to be chosen later we write

$$\hat{\theta} = \arg \min_{\theta \in \mathbb{R}^n} \sum_{i=1}^n \rho_r(y_i - \theta_i) + \lambda \|D^{(r)}\theta\|_1,$$

C.1 Proof outline

We now provide a high level overview of the proof of Theorem 3.

Step 1. We show in Proposition 3 that for any given $\epsilon > 0$ if $\hat{\theta}$ is in the line segment between $\hat{\theta}$ and $\theta^*$, then $\hat{\theta} - \theta^*$ belongs to a restricted set $A$ (depending on $\epsilon$) with probability at least $1 - \epsilon/4$. We call this event $\Omega_1$ and this restricted set is of the form

$$A = \{\delta : \text{TV}^{(r)}(\delta) \leq CV^* + \text{some extra terms}\}$$

for some positive constant $C$, see the precise definition in (34). Then we show, using the convexity of $\hat{M}$, the optimality of $\hat{\theta}$, and Lemma 13 that for any $t > 0$ it holds that

$$\{\Delta^2(\hat{\delta}) \geq t^2\} \subset \left\{ \sup_{\delta \in A, \Delta^2(\delta) \leq t^2} \left[ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) + \lambda \|D^{(r)}\theta^*\|_1 - \lambda \|D^{(r)}(\theta^* + \delta)\| \right] \geq c_0 t^2 \right\},$$

where $c_0 > 0$ is as in Lemma 13. This step uses ideas very similar to the proof of Theorem 7.

Step 2. We define another high probability event $\Omega_2$ as in (43). Then based on Proposition 3 and Lemma 23, we obtain that $\Omega_1 \cap \Omega_2$ happens with probability at least $1 - \epsilon/2$. Hence, we do our analysis conditioning on $\Omega_1 \cap \Omega_2$. We start with also assuming that $\Delta^2(\hat{\delta}) \geq t^2$ for some $t > 0$ (whose value is to be specified later).

Step 3. We show that if $\delta \in A$, $\Delta^2(\delta) \leq t^2$ and $\Omega_1 \cap \Omega_2$ holds then

$$\text{TV}^{(r)}(\delta) \leq C$$

for some $C > 0$. It then follows from Steps 1 and 2 above that we can reduce our focus to upper bounding the probability of the event

$$\left\{ \sup_{\delta \in K} \left[ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) + \lambda \|D^{(r)}\theta^*\|_1 - \lambda \|D^{(r)}(\theta^* + \delta)\| \right] \geq c_0 t^2 \right\}$$

where

$$K := \left\{ \delta : \text{TV}^{(r)}(\delta) \leq C, \Delta^2(\delta) \leq t^2 \right\}.$$

Step 4. Next we observe that

$$\sup_{\delta \in K} \{ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) + \lambda \|D^{(r)}\theta^*\|_1 - \lambda \|D^{(r)}(\theta^* + \delta)\| \} \leq \sup_{\delta \in K} \{ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) \} + \lambda \|D^{(r)}\theta^*\|_1.$$

Hence, to show that $\text{pr}(\Delta^2(\hat{\delta}) \geq t) \leq \epsilon$, from Step 3 and an application of Markov’s inequality, it suffices to show that

$$\frac{1}{c_0 t^2} \mathbb{E} \left[ \sup_{\delta \in K} \{ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) \} \right] + \frac{\lambda \|D^{(r)}\theta^*\|_1}{c_0 t^2} \leq \epsilon$$

Step 5. Setting

$$t \asymp n^{1/(4r+2)} (\log n)^{1/(4r+2)}$$
we now proceed to show that for some positive constant \( c \)
\[
\frac{1}{c_0 t^2} E \left[ \sup_{\delta \in K} \left\{ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) \right\} \right] \leq c \epsilon
\]

exactly similarly as in the proof of Theorem 1. Now by setting \( \lambda \) to satisfy
\[
\lambda \asymp n^{(2r-1)/(2r+1)} (\log n)^{1/(2r+1)} \| D^{(r)} \theta^* \|_1^{-(2r-1)/(2r+1)},
\]

see (44), we can also verify that
\[
\frac{\lambda \| D^{(r)} \theta^* \|_1}{c_0 t^2} \leq c \epsilon
\]

and conclude the proof.

C.2 Restricted set for Proof of Theorem 3 (Step 1)

**Proposition 3.** Let \( \epsilon \in (0, 1) \) and \( u_i = \tau - 1 \{ y_i \leq \theta_i^* \} \) for \( i = 1, \ldots, n \). Then there exists positive constants \( C_\epsilon, \tilde{C}_\epsilon, a_\epsilon \) only depending on \( \epsilon \) such that if we set
\[
\lambda = C_\epsilon n^{(2r-1)/(2r+1)} (\log n)^{1/(2r+1)} \| D^{(r)} \theta^* \|_1^{-(2r-1)/(2r+1)},
\]

then with probability at least \( 1 - \epsilon/4 \),
\[
\kappa(\hat{\theta} - \theta^*) \in A, \quad \forall \kappa \in [0, 1],
\]

where
\[
A := \left\{ \delta : \| D^{(r)} \delta \|_1 \leq \tilde{C}_\epsilon \max \left\{ \frac{V^*}{\gamma}, \gamma \Delta^2(P_\theta \delta), \gamma^{1/2} \Delta(P_\theta \delta), \frac{V^*}{\gamma}, A^{-1} u^\top P_\theta \delta \right\} \right\}, \quad (34)
\]

where
\[
\gamma := \frac{B^2}{A^2},
\]

\( B = a_\epsilon \tilde{B} \), with \( A \) given by
\[
A = Cn^{(2r-1)/(2r+1)} (\log n)^{1/(2r+1)} \| D^{(r)} \theta^* \|_1^{-(2r-1)/(2r+1)},
\]

for any fixed large enough constant \( C > 1 \), and
\[
\tilde{B} = n^{(2r-1)/(4r+2)} (\log n)^{1/(4r+2)} \| D^{(r)} \theta^* \|_1^{2/(4r+2)}.
\]

C.2.1 Auxiliary lemmas for proof of Proposition 3

First we state a result which was proven in the proof of Corollary 7 from Wang et al. (2016).

**Lemma 23.** *Corollary 7 in Wang et al., 2016*. There exists \( A \) satisfying
\[
A = Cn^{(2r-1)/(2r+1)} (\log n)^{1/(2r+1)} \| D^{(r)} \theta^* \|_1^{-(2r-1)/(2r+1)},
\]

for any constant \( C > 1 \) large enough such that
\[
\sup_{x \in \text{row}\{D^{(r)}\} : \| D^{(r)} x \|_1 \leq 1} \frac{u^\top x - A}{\| x \|} = O_p \left( \tilde{B} \right),
\]

where \( u = (u_1, \ldots, u_n)^\top \) is a vector with independent coordinates satisfying \( u_i \sim \text{subGaussian}(\sigma^2) \) for \( i = 1, \ldots, n \), with \( \sigma > 0 \) a constant and
\[
\tilde{B} = n^{(2r-1)/(4r+2)} (\log n)^{1/(4r+2)} \| D^{(r)} \theta^* \|_1^{2/(4r+2)}.
\]
Lemma 24. With the notation from Lemma 23, we have that
\[
\sup_{x \in \text{row} \{D(r)\} : \|D(r)x\|_1 \leq 1} \frac{u^\top x - A}{\Delta(x)} = O_{\Pr} \left( \tilde{B} \right),
\]
where \( u = (u_1, \ldots, u_n)^\top \) is a vector with independent coordinates satisfying
\[
\Pr(u_i = \tau) = 1 - \tau, \quad \Pr(u_i = \tau - 1) = \tau, \quad \text{for } i = 1, \ldots, n. \tag{35}
\]

Proof. Let \( x \) be such that \( x \in \text{row} \{D(r)\} \) and \( \|D(r)x\|_1 \leq 1 \), then by Lemmas 17 and 19 there exists a constant \( \tilde{C}_r > 0 \) independent of \( x \) such that
\[
\Delta(x) \geq \tilde{C}_r^{-1/2} \|x\|.
\]
Hence,
\[
\sup_{x \in \text{row} \{D(r)\} : \|D(r)x\|_1 \leq 1} \frac{u^\top x - A}{\Delta(x)} \leq \tilde{C}_r^{1/2} \sup_{x \in \text{row} \{D(r)\} : \|D(r)x\|_1 \leq 1} \frac{u^\top x - A}{\|x\|}
\]
and the claim follows by Lemma 23.

Lemma 25. Let \( u = (u_1, \ldots, u_n)^\top \) is a vector with independent coordinates satisfying (35). Recall that \( R = \text{row} \{D(r)\} \) and \( R^\perp \) denote its orthogonal complement. Then
\[
\sup_{x \in \mathbb{R}^n} \frac{u^\top P_{R^\perp} x}{\frac{\Delta^2(x)}{n} + \Delta(x)} = O_{\Pr}(1),
\]
where \( P_{R^\perp} \) denotes the orthogonal projection onto \( R^\perp \).

Proof. Let \( v_1, \ldots, v_r \) an orthonormal basis of \( R^\perp \). Then proceeding as in Equation (27),
\[
\begin{align*}
    u^\top P_{R^\perp} \delta &\leq r \left( \max_{j=1,\ldots,r} |u^\top v_j| \right) \left( \max_{j=1,\ldots,r} |\delta^\top v_j| \right) \\
    &\leq r \left( \max_{j=1,\ldots,r} |u^\top v_j| \right) \left\{ \frac{b_r}{n^{1/2}} \Delta^2(\delta) + \Delta(\delta) \right\} \tag{36}
\end{align*}
\]
where the second inequality follows from Lemma 20. The claim follows since
\[
E \left\{ \left( \max_{j=1,\ldots,r} |u^\top v_j| \right) \right\} = O(1).
\]

C.3 Proof of Proposition 3

Proof. Let \( B = a_{\epsilon} \tilde{B}, \ a_{\epsilon} > 0 \), with \( \tilde{B} \) as in Lemma 24, and such that
\[
\sup_{x \in \text{row} \{D(r)\} : \|D(r)x\|_1 \leq 1} \frac{u^\top x - A}{\Delta(x)} \leq B, \tag{37}
\]
happens with probability at least \( 1 - \epsilon/4 \). From here on, we suppose that (37) holds.
Now pick $\kappa \in [0, 1]$ fixed, and let $\bar{\delta} = \kappa(\hat{\theta} - \theta^*)$. Then by the optimality of $\hat{\theta}$ and convexity of the quantile loss, we have that

$$
\sum_{i=1}^n \rho_r(y_i - \tilde{\theta}_i) + \lambda \|D^{(r)}\hat{\theta}\|_1 \leq \sum_{i=1}^n \rho_r(y_i - \theta^*_i) + \lambda \|D^{(r)}\theta^*\|_1,
$$

where $\tilde{\theta} = \theta^* + \bar{\delta}$. Then as in the proof of Lemma 3 from Belloni and Chernozhukov (2011),

$$
0 \leq \lambda \left[\|D^{(r)}\theta^*\|_1 - \|D^{(r)}\hat{\theta}\|_1\right] + (\tilde{\theta} - \theta^*)^\top u. \tag{38}
$$

Next, notice that

$$
(\tilde{\theta} - \theta^*)^\top u = u^\top P_R(\tilde{\theta} - \theta^*) + u^\top P_{R^\perp}(\tilde{\theta} - \theta^*) = (u^\top x)\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 + u^\top P_{R^\perp}(\tilde{\theta} - \theta^*), \tag{39}
$$

where

$$
x := \frac{1}{\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1} P_R(\tilde{\theta} - \theta^*).
$$

Hence,

$$
\|D^{(r)}x\|_1 \leq 1,
$$

which combined with (39) and Lemma 24 implies

$$
(\tilde{\theta} - \theta^*)^\top u = \begin{cases} B\Delta(x) + A \|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 + u^\top P_{R^\perp}(\tilde{\theta} - \theta^*) \\ \leq B \max\{\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1^{1/2}, 1\} \Delta \left\{ P_R(\tilde{\theta} - \theta^*) \right\} + A \|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 + u^\top P_{R^\perp}(\tilde{\theta} - \theta^*), \tag{40} \end{cases}
$$

where the inequality follows from the fact that $\Delta(tv) \leq \max\{t, \sqrt{t}\} \Delta(v)$ for $v \in \mathbb{R}^n$ and $t \geq 0$.

Suppose now that $\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 \geq 1$. If

$$
A\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 < B\|\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1^{1/2} \Delta \left\{ P_R(\tilde{\theta} - \theta^*) \right\},
$$

then

$$
\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 \leq \frac{B^2 \Delta^2 \left\{ P_R(\tilde{\theta} - \theta^*) \right\}}{A^2} \tag{41}
$$

If

$$
A\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 \geq B\|\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1^{1/2} \Delta \left\{ P_R(\tilde{\theta} - \theta^*) \right\},
$$

then

$$
(\tilde{\theta} - \theta^*)^\top u \leq 2A\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 + u^\top P_{R^\perp}(\tilde{\theta} - \theta^*). \tag{42}
$$

Hence, choosing $\lambda = 3A$, and combining (38) with (42),

$$
A\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 \leq \lambda \left\{\|D^{(r)}\theta^*\|_1 - \|D^{(r)}\hat{\theta}\|_1\right\} + 3A\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 + u^\top P_{R^\perp}(\tilde{\theta} - \theta^*) \leq 6A\|D^{(r)}\theta^*\|_1 + u^\top P_{R^\perp}(\tilde{\theta} - \theta^*),
$$

\[32]
with the second inequality follows by the triangle inequality. Therefore,

$$
\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 \leq \max \left\{ \frac{V^*}{n^{r-1}}, \frac{6V^*}{n^{r-1}}, A^{-1}u^\top P_{R\perp}(\tilde{\theta} - \theta^*), \frac{B^2 \Delta^2 \{ P_R(\tilde{\theta} - \theta^*) \}}{A^2} \right\}.
$$

Next suppose that $$\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 < 1$$. If

$$A\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 < B\Delta \left\{ P_R(\tilde{\theta} - \theta^*) \right\},$$

then

$$\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 < \frac{B}{A} \Delta \left\{ P_R(\tilde{\theta} - \theta^*) \right\}.$$ If

$$A\|D^{(r)}(\tilde{\theta} - \theta^*)\|_1 \geq B\Delta \left\{ P_R(\tilde{\theta} - \theta^*) \right\},$$

we proceed as before. The claim follows.

\[\square\]

C.4 Proof of Theorem 3

Proof. Steps 1–2 in proof outline.

Let $$\epsilon \in (0, 1)$$. By Proposition 3 and Lemmas 23–25 we can suppose that the following events

$$\Omega_1 = \left\{ \kappa(\hat{\theta} - \theta^*) \in A, \ \forall \kappa \in [0, 1] \right\},$$

$$\Omega_2 = \left\{ \sup_{x \in R^n} \frac{u^\top P_{R\perp} \frac{\Delta^2(x)}{n} + \Delta(x)}{E} \leq E \right\},$$

happen with probability at least $$1 - \epsilon/2$$ for some constant $$E$$, and with $$A$$ as in Lemma 3. Furthermore, we set

$$\lambda = 3c_\epsilon n^{(2r-1)/(2r+1)} (\log n)^{1/(2r+1)} \|D^{(r)}(\theta^*)\|^2_1 \leq (2r-1)/(2r+1)$$

and $$A = \lambda/3$$ in Lemma 23, where $$c_\epsilon > 1$$.

Then, for a choice of $$t > 0$$ to be specified later, we have

$$\text{pr} \left\{ \Delta^2(\hat{\delta}) > t^2 \right\} \leq \text{pr} \left[ \left\{ \Delta^2(\hat{\delta}) > t^2 \right\} \cap \Omega_1 \cap \Omega_2 \right] + \frac{\epsilon}{2}.$$}

Next suppose that the event

$$\left\{ \Delta^2(\hat{\delta}) > t^2 \right\} \cap \Omega_1 \cap \Omega_2$$

holds. Then, proceeding as in the proof of Proposition 1 there exists $$\hat{\delta} = u_\delta \hat{\delta}$$ with $$u_\delta \in [0, 1]$$ such that $$\hat{\delta} \in A$$ and $$\Delta^2(\hat{\delta}) = t^2$$. Hence, by the basic inequality,

$$\tilde{M}(\theta^* + \hat{\delta}) + \lambda \left[ \|D^{(r)}(\theta^* + \hat{\delta})\|_1 - \|D^{(r)}(\theta^*)\|_1 \right] \leq 0.$$ Therefore,

$$\sup_{\tilde{\delta} \in A, \Delta^2(\tilde{\delta}) \leq t^2} \left[ M(\theta^* + \delta) - \tilde{M}(\theta^* + \tilde{\delta}) + \lambda \left\{ \|D^{(r)}(\theta^*)\|_1 - \|D^{(r)}(\theta^* + \tilde{\delta})\|_1 \right\} \right] \geq M(\theta^* + \hat{\delta}) \geq c_0 t^2,$$
where the second inequality follows from Lemma 9. Therefore,

\[
\Pr \left \{ \Delta^2 (\delta) > t^2 \right \} \cap \Omega_1 \cap \Omega_2 \leq \Pr \left \{ \sup_{\delta \in A, \Delta^2 (\delta) \leq t^2} \left[ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) + \lambda \left( \|D^{(r)}\theta^*\|_1 - \|D^{(r)}(\theta^* + \delta)\|_1 \right) \right] \geq c_0 t^2 \right \} \cap \Omega_1 \cap \Omega_2
\]

\[
\leq \frac{1}{c_0 t^2} E \left \{ \sup_{\delta \in A, \Delta^2 (\delta) \leq t^2} \left[ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) + \lambda \left( \|D^{(r)}\theta^*\|_1 - \|D^{(r)}(\theta^* + \delta)\|_1 \right) \right] \right \}
\]

\[
\leq \frac{1}{c_0 t^2} E \left \{ \sup_{\delta \in A, \Delta^2 (\delta) \leq t^2} \|D^{(r)}\delta\|_1 \right \},
\]

where the second inequality follows from Markov’s inequality, and the last from the triangle inequality.

**Step 3 in proof outline.**

Next, define

\[
t := c_r n^{1/(4r+2)} \left( \log n \right)^{1/(4r+2)}
\]

and notice that for \( \delta \in A \) with \( \Delta(\delta) \leq t \) it holds, by Lemma 22, that

\[
\Delta(P_R \delta) \leq \tilde{C}_r t,
\]

for some positive constant \( \tilde{C}_r \). Hence, if in addition \( \Omega_1 \cap \Omega_2 \) holds then for a constant \( \tilde{C} \) independent of \( c_r \),

\[
\|D^{(r)}\delta\|_1 \leq C_0 \max \left \{ \frac{V^*}{\sqrt{n}}, \gamma \Delta^2 (P_R \delta), \gamma^{1/2} \Delta(P_R \delta), \frac{V^*}{\sqrt{n}}, A^{-1} u^\top P_R \delta \right \}
\]

\[
\leq C_0 \max \left \{ \frac{V^*}{\sqrt{n}}, \gamma \Delta^2 (P_R \delta), \gamma^{1/2} \Delta(P_R \delta), \frac{V^*}{\sqrt{n}}, A^{-1} \left( \frac{\Delta^2 (\delta)}{n} + \Delta(\delta) \right) \right \}
\]

\[
\leq C_0 \max \left \{ \frac{V^*}{\sqrt{n}}, \tilde{C}_r^2 B^2 t^2, \tilde{C}_r B t, \frac{V^*}{\sqrt{n}}, A^{-1} \left( \frac{t^2}{n} + t \right) \right \}
\]

\[
\leq \tilde{C} n^{1-r} \max \left \{ V^*, (V^*)^{4/(2r+1)} \right \},
\]

where the first inequality follows from the definition of \( \Omega_1 \), the second because we are assuming that \( \Omega_2 \) holds, the third since \( \Delta(\delta) \leq t \), and the fourth by definition of \( A, B \) and \( t \) as simple algebra shows that

\[
\tilde{C}_r^2 B^2 t^2 \leq \frac{C (V^*)^{4r/(2r+1)}}{c_r^2 n^{r-1}},
\]

\[
\tilde{C}_r B t \leq \frac{C (V^*)^{4/(4r+2)}}{n^{r-1}},
\]

and

\[
A^{-1} \left( \frac{t^2}{n} + t \right) \leq \frac{C}{c_r} \left \{ \frac{(V^*)^{(2r-1)/(2r+1)}}{n^{r-1}} \right \} \frac{1}{(n \log n)^{1/(4r+2)}}.
\]

**Steps 4–5 in proof outline.**

Let us now define

\[
\mathcal{H}(t) = \left \{ \delta : \|D^{(r)}\delta\|_1 \leq \tilde{C} n^{1-r} \max \left \{ V^*, (V^*)^{4/(2r+1)} \right \} \text{, and } \Delta(\delta) \leq t \right \}.
\]
Then for some constant $a > 0$ that depends on $V^*$ but independent of $c_\epsilon$, (45) and (47) imply
\begin{align*}
\Pr\left[ \{ \Delta^2(\hat{\delta}) > t^2 \} \cap \Omega_1 \cap \Omega_2 \right] & \leq \frac{1}{c_0 t^2} E \left[ \sup_{\delta \in \mathcal{H}(t)} \left\{ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) \right\} \right] \\
& \quad + \frac{\lambda}{c_0 t^2} \sup_{\delta \in \mathcal{H}(t)} \| D^{(r)} \delta \|_1, \\
& \leq \frac{2}{c_0 t^2} E \left[ \sup_{\delta \in \mathcal{H}(t)} \sum_{i=1}^n \xi_i \delta_i \right] \\
& \quad + \frac{\lambda}{c_0 t^2} \left[ \tilde{C} n^{1-r} \max\{V^*, (V^*)^{4r/(4r+2)}\} \right] \\
& \leq \left[ a \left\{ c_\epsilon (\log n)^{1/(4r+2)} \right\}^{1-1/(2r)} n^{1/(2r+1)} \right. \\
& \quad + \left. \tilde{C} n^{1-r} \max\{V^*, (V^*)^{4r/(4r+2)}\} \right] \frac{1}{c_0 t^2}
\end{align*}
with $\xi_1, \ldots, \xi_n$ are independent Rademacher variables, where the second inequality follows as in the proof of Lemmas 10–11, and third by Proposition 2. Hence, given our choice of $\lambda$,
\begin{align*}
\Pr\left[ \{ \Delta^2(\hat{\delta}) > t^2 \} \cap \Omega_1 \cap \Omega_2 \right] & \leq \left[ a \left\{ c_\epsilon (\log n)^{1/(4r+2)} \right\}^{1-1/(2r)} n^{1/(2r+1)} \right. \\
& \quad + \left. \tilde{C} n^{1-r} \max\{V^*, (V^*)^{4r/(4r+2)}\} \right] \frac{1}{c_0 t^2} \\
& \leq \frac{\epsilon}{2}
\end{align*}
provided that $c_\epsilon$ is large enough. \hfill \Box

### D Theorem 4

Since we rely on proof machinery developed in Ortelli and van de Geer (2019a), we start by introducing some relevant notation from Ortelli and van de Geer (2019a).

#### D.1 Notation

Throughout this section we will use $C$ as a generic positive constant that can change from line to line. Let $m = n - r$ be the number of rows of $D^{(r)}$. For a vector $b \in \mathbb{R}^m$ and a set $S \subset D := \{1, \ldots, m\}$ we denote by $b_S$ the vector $b_S = (b_j)_{j \in S}$ and we write $b_{-S} = (b_j)_{j \in \{1, \ldots, m\} \setminus S}$.

Following Ortelli and van de Geer (2019a), we take $S$ a subset of $\{1, \ldots, m\}$ with $s = |S|$. We also denote by $t_1, \ldots, t_s$ the elements of $S$ and assume that $r + 1 < t_1 < t_2 < \ldots < t_s \leq n$, and let $t_0 = r$ and $t_{s+1} = n$. Then we denote $n_i = t_i - t_{i-1}$ for $i \in \{1, \ldots, s + 1\}$.

In our entire proof we take $S$ to be the same as in Ortelli and van de Geer (2019a) which satisfies $\{j : (D^{(r)} \theta^*_j) \not= 0\} \subset S$, and $s := |S| \geq |\{j : (D^{(r)} \theta^*_j) \not= 0\}|$. Furthermore, we write $N_{-S} = \{ \theta \in \mathbb{R}^n : (D^{(r)} \theta)_{-S} = 0 \}$ and denote $r_S = \dim(N_{-S})$. Also, the matrix $D_{-S}^{(r)}$ denotes the matrix obtained after removing from $D^{(r)}$ the rows indexed by $S$, and we set $\Psi^{-S} = (D_{-S}^{(r)})^\top (D_{-S}^{(r)} (D_{-S}^{(r)})^\top)^{-1}$. The $j$th
column of \( \Psi^{-S} \) is denoted as \( \psi^{-S}_j \). Furthermore, we denote the orthogonal projections onto \( \mathcal{N}_S \) and \( \mathcal{N}_S^\perp \) as \( P_{\mathcal{N}_S} \) and \( P_{\mathcal{N}_S^\perp} \) respectively.

For a vector \( w_S \) such that \( 0 \leq w_j \leq 1 \) we write \( (1 - w_S)(D^{(r)}\theta)_S = \{(1 - w_j)(D^{(r)}\theta)_j\}_{j \in \mathcal{D}\setminus S} \). We then study the estimator

\[
\hat{\theta} = \arg\min_{\theta \in \mathbb{R}^n} \sum_{i=1}^n \rho_r(y_i - \theta_i) + \lambda \|D^{(r)}\theta\|_1,
\]

for some \( \lambda > 0 \).

We also let \( A \) be such that

\[
A \geq \max_{j \in \mathcal{D}\setminus S} \|\psi^{-S}_j\|(\log n)^{1/2}, \tag{50}
\]

and by Section 3.1 in Ortelli and van de Geer (2019a), we have that (50) holds if

\[
A \geq A^*: = C^* n^{r} \left( \frac{1}{s+1} \right)^{r-1/2} \left( \frac{\log n}{n} \right)^{1/2}, \tag{51}
\]

for some constant \( C^* > 0 \).

With the notation from above, we also borrow the following definition from Ortelli and van de Geer (2019a).

**Definition 3.** For any sign vector \( q_S \in \{-1, 1\}^s \) its noiseless effective sparsity is

\[
\Gamma^2(q_S) = \left( \max \left\{ q_S^\top (D^{(r)}\theta)_S - \|(D^{(r)}\theta)_S\|_1 : \theta \in \mathbb{R}^n \right\} \right)^2.
\]

Its noisy effective sparsity is defined as

\[
\Gamma^2(q_S, w_S) = \left( \max \left\{ q_S^\top (D^{(r)}\theta)_S - \|(1 - w_S)(D^{(r)}\theta)_S\|_1 : \theta \in \mathbb{R}^n \right\} \right)^2,
\]

with

\[
w_j = \frac{\|\psi^{-S}_j\|(\log n)^{1/2}}{A^*}
\]

for \( j \in \mathcal{D}\setminus S \).

### D.2 Proof outline

We now provide a high level overview of the proof of Theorem 4. The first three steps in this proof are very similar to the first three steps in the proof outline of Theorem 3.

**Step 1.**
We show in Proposition 4 that for any given \( \epsilon > 0 \) if \( \tilde{\theta} \) is in the line segment between \( \hat{\theta} \) and \( \theta^* \) then \( \tilde{\delta} = \tilde{\theta} - \theta^* \) belongs to a restricted set \( \mathcal{A} \) (depending on \( \epsilon \)) with probability at least \( 1 - \epsilon/4 \). We call this event \( \Omega_1 \) and this restricted set is of the form

\[
\mathcal{A} = \{ \delta : TV^{(r)}(\delta) \leq CV^* + \text{some extra terms} \}
\]

for some positive constant \( C \), see the precise definition in (34). Here, the additional extra terms appearing in (52) are different to the corresponding ones in Step 1 of the proof of Theorem 3. Next we obtain, using the convexity of \( M \), the optimality of \( \hat{\theta} \), and Lemma 13 that for any \( t > 0 \) it holds that

\[
\{ \Delta^2(\delta) \geq t^2 \} \subset \left\{ \sup_{\delta \in \mathcal{A}, \Delta^2(\delta) \leq t^2} \left[ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) + \lambda \|D^{(r)}\theta^*\|_1 - \|D^{(r)}(\theta^* + \delta)\|_1 \right] \geq ct^2 \right\},
\]
where \( c_0 > 0 \) is as in Lemma 13. Again, this step uses ideas very similar to the proof of Theorem 7.

**Step 2.**
We define another high probability event \( \Omega_2 \) as in (55). Then based on Proposition 4 and Lemma 23, we obtain that \( \Omega_1 \cap \Omega_2 \) happens with probability at least \( 1 - \epsilon/2 \). Hence, we do our analysis conditioning on \( \Omega_1 \cap \Omega_2 \). We start with also assuming that \( \Delta^2(\hat{\delta}) \geq t^2 \) for a large enough \( t > 0 \) (whose value is to be specified later).

**Step 3.**
We show that if \( \delta \in \mathcal{A}, \ \Delta^2(\delta) \leq t^2 \) and \( \Omega_1 \cap \Omega_2 \) holds then

\[
TV^{(r)}(\delta) \leq C
\]

for some \( C > 0 \). Here the details of the calculations are different to the corresponding ones in Step 3 of the proof of Theorem 3 but it leads us to obtaining a similar conclusion. It then follows from Steps 1 and 2 above that we can reduce our focus to upper bounding the probability of the event

\[
\left\{ \sup_{\delta \in K} \left[ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) + \lambda \|D^{(r)}\theta^*\|_1 - \lambda \|D^{(r)}(\theta^* + \delta)\|_1 \right] \geq c_0 t^2 \right\}
\]

where

\[
K := \left\{ \delta : TV^{(r)}(\delta) \leq C, \ \Delta^2(\delta) \leq t^2 \right\}.
\]

**Step 4.**
Using Markov’s inequality and Step 3, it follows that \( \text{pr}(\Delta^2(\hat{\delta}) \geq t) \leq \epsilon \) holds if

\[
\frac{1}{c_0 t^2} E \left[ \sup_{\delta \in K} \left\{ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) + \lambda \|D^{(r)}\theta^*\|_1 - \lambda \|D^{(r)}(\theta^* + \delta)\|_1 \right\} \right] \leq \epsilon.
\]

Then, using symmetrization and contraction results from Empirical Process Theory; see Lemmas 28 and 29, it reduces our task to show that

\[
U := \frac{4}{c_0 t^2} E \left[ \sup_{\delta \in K} \left\{ \sum_{i=1}^n \xi_i \delta_i + \frac{\lambda}{2} \|D^{(r)}\theta^*\|_1 - \frac{\lambda}{2} \|D^{(r)}(\theta^* + \delta)\|_1 \right\} \right] \leq \epsilon,
\]

for \( \xi_1, \ldots, \xi_n \) independent Rademacher variables.

**Step 5.**
We now write

\[
U \leq T_1 + T_2 + T_3
\]

with

\[
T_1 := \frac{4}{c_0 t^2} E \left\{ \sup_{\delta \in K} \xi^\top P_{R} \delta \right\},
\]

\[
T_2 := \frac{4}{c_0 t^2} E \left\{ \sup_{\delta \in K} \xi^\top P_{N-s} P_{R} \delta \right\},
\]

and

\[
T_3 := \frac{4}{c_0 t^2} E \left[ \sup_{\delta \in K} \left\{ \xi^\top P_{N-s} \delta + \frac{\lambda}{2} \|D^{(r)}\theta^*\|_1 - \frac{\lambda}{2} \|D^{(r)}(\theta^* + \delta)\|_1 \right\} \right]
\]

where \( P_{R}^\perp, P_{R}, P_{N-s}^\perp \) and \( P_{N-s} \) are defined in Section D.1. In the subsequent proof we set \( t \) and \( \lambda \) to satisfy:

\[
t \asymp (s + 1)^{1/2} (\log^{1/2} n) \left\{ \log \left( \frac{n}{s + 1} \right) \right\}^{1/2} \log^{1/2} (s + 1),
\]

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and
\[ \lambda \asymp \max \left\{ \frac{n^{r-1}(s+1)\log n \log(s+1)\log \frac{n}{s+1}}{V^*}, n^{r-1/2} \left( \frac{1}{s+1} \right)^{r-1/2} (\log n)^{1/2} \right\}. \]

As we see in the proof of Theorem 4, \( T_1 \) and \( T_2 \) turn out to be lower order terms as compared to \( T_3 \). Therefore, from here our goal is to show that there exists a positive constant \( c \) such that \( \max\{T_1, T_2, T_3\} \leq c\epsilon \).

**Step 6.**
Bounding \( T_1 \). This is done exactly similarly as in bounding the corresponding term \( T_1 \) inside the proof of Theorem 1.

**Step 7.**
Bounding \( T_2 \). This is handled using Lemmas 17, 19 and 22.

**Step 8.**
We define an event \( \Omega_3(b) \) for a constant \( b > 5 \), see (67), and using a standard concentration inequality for maxima of subgaussian random variables we show that \( \Omega_3(b)^c \) happens with high probability. Next we write
\[ T_3 = T_{3,1} + T_{3,2}, \]
where
\[ T_{3,1} := \frac{4}{c_0 t^2} E \left[ \sup_{\delta \in K} \left\{ \xi^\top P_{N,S} P_R \delta + \frac{\lambda}{2} \|D^{(r)}\theta^*\|_1 - \frac{\lambda}{2} \|D^{(r)}(\theta^* + \delta)\|_1 \right\} \mid \Omega_3(b) \right] \Pr\{\Omega_3(b)\} \]
and
\[ T_{3,2} := \frac{4}{c_0 t^2} E \left[ \sup_{\delta \in K} \left\{ \xi^\top P_{N,S} P_R \delta + \frac{\lambda}{2} \|D^{(r)}\theta^*\|_1 - \frac{\lambda}{2} \|D^{(r)}(\theta^* + \delta)\|_1 \right\} \mid \Omega_3(b)^c \right] \Pr\{\Omega_3(b)^c\}. \]

**Step 9.**
Bounding \( T_{3,1} \). This a lower order term that can be upper bounded exploiting the definition of \( \Omega_3(b) \).

**Step 10.**
Bounding \( T_{3,2} \). This is done following the ideas for proving fast rates for trend filtering as laid out in Section 3.3 of Ortelli and van de Geer (2019a).

**D.3 Restricted set for Proof of Theorem 4 (Step 1)**
The following result is obtained similarly to Proposition 3.

**Proposition 4.** Let \( \epsilon \in (0,1) \) then there exists \( A \) satisfying (51) such that for
\[ \lambda = 3A, \]
with probability at least \( 1 - \epsilon/8 \),
\[ \kappa(\hat{\theta} - \theta^*) \in \mathcal{A}, \quad \kappa \in [0,1], \]
with
\[ \mathcal{A} := \left\{ \delta : \|D^{(r)}\delta\|_1 \leq \tilde{C}_\epsilon \max \left\{ \frac{V^*}{n^{r-1}}, \gamma \Delta^2(P_R \delta), \gamma^{1/2} \Delta(P_R \delta), \frac{V^*}{n^{r-1}} + A^{-1}u^\top P_{R,S} \delta \right\} \right\}, \]
where \( \tilde{C}_\epsilon > 0 \) is a constant that depends on \( \epsilon \),
\[ \gamma := \frac{B^2}{A^2}, \]
\[ B = a_\epsilon \tilde{B}, \]
for some constant \( a_\epsilon > 0 \) that depends , with \( u_i = \tau - 1\{y_i \leq \theta^*_i\} \) for \( i = 1, \ldots, n \). Here,
\[ \tilde{B} = (s+1)^{1/2} \log^{1/2}(s+1). \]
D.3.1 Auxiliary lemmas for proof of Proposition 4

Lemma 26. (Lemma A.2 in Ortelli and van de Geer, 2019a). With $A$ as in (51), it holds that

$$\sup_{x \in \mathbb{R}^n} \frac{u^\top x - A \|w_{-S}(D(r)x)_{-S}\|_1}{\|x\|} = O_{\text{pr}}(\tilde{B}),$$

where $u = (u_1, \ldots, u_n)^\top$ is a vector with independent coordinates satisfying $u_i \sim \text{SubGaussian}(\sigma^2)$ for $i = 1, \ldots, n$, with $\sigma$ a constant and $\tilde{B} = (s + 1)^{1/2} \log^{1/2}(s + 1)$.

Proof. The proof is almost identical to that of Lemma A.2 in Ortelli and van de Geer (2019a). We start by noticing that for any $x \in \mathbb{R}^n$, we have that

$$u^\top P_{N_{-S}} x = u^\top P_{N_{-S}} x + u^\top P_{N_{-S}^\perp} x.$$

Next let $v_1, \ldots, v_{r_S}$ be an orthonormal basis of $N_{-S}$ and notice that

$$|u^\top P_{N_{-S}} x|^2 \leq \|x\|^2 \left\| P_{N_{-S}} u \right\|^2 = \|x\|^2 \left( \sum_{j=1}^{r_S} |u^\top v_j|^2 \right) = \|x\|^2 \left( \sum_{j=1}^{r_S} |u^\top v_j|^2 \right) \leq \|x\|^2 r_S \max_{j=1, \ldots, r_S} |u^\top v_j|^2$$

where $P_{N_{-S}}$ and $P_{N_{-S}^\perp}$ are the orthogonal projections onto $N_{-S}$ and $N_{-S}^\perp$ respectively. Since by the sub-Gaussian tail inequality,

$$\max_{j=1, \ldots, r_S} |u^\top v_j| = O_{\text{pr}} \left( \log^{1/2}(s + 1) \right),$$

we obtain that

$$\sup_{x \in \mathbb{R}^n} \frac{u^\top P_{N_{-S}} x}{\|x\|} = O_{\text{pr}} \left[ (s + 1)^{1/2} \log^{1/2}(s + 1) \right].$$

The rest of the proof concludes by proceeding as in the proof of Lemma A.2 in Ortelli and van de Geer (2019a). \qed

As Lemma 24 we obtain the following result.

Lemma 27. With the notation from Lemma 26, we have that

$$\sup_{x \in \text{row}\{D(r)\}: \|D(r)x\|_1 \leq 1} \frac{u^\top x - A}{\Delta(x)} = O_{\text{pr}} \left( \tilde{B} \right),$$

where $u = (u_1, \ldots, u_n)^\top$ is a vector with independent coordinates satisfying

$$\text{pr}(u_i = \tau) = 1 - \tau, \quad \text{pr}(u_i = \tau - 1) = \tau, \quad \text{for} \quad i = 1, \ldots, n.$$  

(54)
D.4 Symmetrization and Contraction Lemmas for proof of Theorem 4

Lemma 28. (Symmetrization). For any set $K$ and any $\lambda > 0$ it holds that

$$E \left[ \sup_{v \in K} \left\{ M(v) - \tilde{M}(v) + \lambda\|D^{(r)}\theta^*\|_1 - \lambda\|D^{(r)}v\|_1 \right\} \right]$$

$$\leq 2E \left[ \sup_{v \in K} \left\{ \sum_{i=1}^{n} \xi_i \tilde{M}_i(v_i) + \frac{\lambda}{2}\|D^{(r)}\theta^*\|_1 - \frac{\lambda}{2}\|D^{(r)}v\|_1 \right\} \right],$$

where $\xi_1, \ldots, \xi_n$ are independent Rademacher variables independent of $\{y_i\}_{i=1}^n$.

Remark 13. The above lemma (and its proof) is almost the same as the statement of Lemma 10 except that the term inside the supremum has an additional term involving the $r$th order total variation.

Proof. We proceed using the notation argument from the proof of Lemma 10. Then for $\xi_1, \ldots, \xi_n$ independent Rademacher variables, independent of $y$ and $\tilde{y}$ we have that

$$\sup_{v \in K} \left\{ M(v) - \tilde{M}(v) + \lambda\|D^{(r)}\theta^*\|_1 - \lambda\|D^{(r)}v\|_1 \right\}$$

$$\leq E \left[ \sup_{v \in K} \left\{ \tilde{M}(v) - \tilde{M}(v) + \lambda\|D^{(r)}\theta^*\|_1 - \lambda\|D^{(r)}v\|_1 \right\} \right].$$

Hence,

$$E \left[ \sup_{v \in K} \left\{ M(v) - \tilde{M}(v) + \lambda\|D^{(r)}\theta^*\|_1 - \lambda\|D^{(r)}v\|_1 \right\} \right]$$

$$\leq E \left[ \sup_{v \in K} \left\{ \sum_{i=1}^{n} \xi_i \tilde{M}_i(v_i) + \lambda\|D^{(r)}\theta^*\|_1 - \lambda\|D^{(r)}v\|_1 \right\} \right]$$

$$= E \left[ \sup_{v \in K} \left\{ \sum_{i=1}^{n} \xi_i \tilde{M}_i(v_i) + \frac{\lambda}{2}\|D^{(r)}\theta^*\|_1 - \frac{\lambda}{2}\|D^{(r)}v\|_1 \right\} \right]$$

$$+ E \left[ \sup_{v \in K} \left\{ \sum_{i=1}^{n} \xi_i \tilde{M}_i(v_i) + \frac{\lambda}{2}\|D^{(r)}\theta^*\|_1 - \frac{\lambda}{2}\|D^{(r)}v\|_1 \right\} \right]$$

$$= 2E \left[ \sup_{v \in K} \left\{ \sum_{i=1}^{n} \xi_i \tilde{M}_i(v_i) + \frac{\lambda}{2}\|D^{(r)}\theta^*\|_1 - \frac{\lambda}{2}\|D^{(r)}v\|_1 \right\} \right].$$

Lemma 29. (Contraction principle). Let $h_1, \ldots, h_n : R \to R$ $\eta$-Lipschitz functions for some $\eta > 0$. Then for any compact set $K$ and for $\xi_1, \ldots, \xi_n$ independent Rademacher variables we have that

$$E \left[ \sup_{v \in K} \left\{ \sum_{i=1}^{n} \xi_i h_i(v_i) + \frac{\lambda}{2}\|D^{(r)}\theta^*\|_1 - \frac{\lambda}{2}\|D^{(r)}v\|_1 \right\} \right]$$

$$\leq E \left[ \sup_{v \in K} \left\{ \eta \sum_{i=1}^{n} \xi_i v_i + \frac{\lambda}{2}\|D^{(r)}\theta^*\|_1 - \frac{\lambda}{2}\|D^{(r)}v\|_1 \right\} \right]$$

for any $\lambda > 0.$
Remark 14. The above lemma (and its proof) is a version of the standard contraction result (Theorem 4.12 in Ledoux and Talagrand, 2013) except that the term inside the supremum has an additional term involving the $r$th order total variation. This lemma can be proved by following the standard proof argument of the original result. We provide this proof here for the sake of completeness.

Proof. We begin by defining the function
\[ g_{n-1}(v) = \sum_{i=1}^{n-1} \xi_i h_i(v_i) + \frac{\lambda}{2} \|D^{(r)} \theta^*\|_1 - \frac{\lambda}{2} \|D^{(r)} v\|_1. \]

Let $v^+, v^- \in K$ such that
\[ v^+ \in \arg \max_{v \in K} \{ g_{n-1}(v) + h_n(v_n) \}, \]
and
\[ v^- \in \arg \max_{v \in K} \{ g_{n-1}(v) - h_n(v_n) \}. \]

Next, letting $a = \text{sign}(v_n^+ - v_n^-)$, we notice that
\[
E \left[ \sup_{v \in K} \{ g_{n-1}(v) + \xi_n h_n(v_n) \} \middle| \xi_1, \ldots, \xi_{n-1} \right] = \frac{1}{2} \sup_{v \in K} \{ g_{n-1}(v) + h_n(v_n) \} + \frac{1}{2} \sup_{v \in K} \{ g_{n-1}(v) - h_n(v_n) \}
\]
\[
\leq \frac{1}{2} \left\{ g_{n-1}(v^+) + h_n(v_n^+) \right\} + \frac{1}{2} \eta a \{ v_n^+ - v_n^- \}
\]
\[
\leq \frac{1}{2} \sup_{v \in K} \{ g_{n-1}(v) + \eta a v_n \} + \frac{1}{2} \sup_{v \in K} \{ g_{n-1}(v) - \eta a v_n \}
\]
\[
= E \left[ \sup_{v \in K} \{ g_{n-1}(v) + \eta \xi_n v_n \} \middle| \xi_1, \ldots, \xi_{n-1} \right]
\]
and the proof concludes by proceeding with a similar argument for the other $i$'s, $i \neq n.$ \hfill \Box

D.5 Proof of Theorem 4

Proof. Steps 1–2 in proof outline.

Let $\epsilon \in (0, 1).$ By Lemmas 25 and Proposition 4 we can suppose that the following events
\[
\begin{align*}
\Omega_1 &= \left\{ \kappa(\hat{\theta} - \theta^*) \in A, \quad \forall \kappa \in [0, 1] \right\}, \\
\Omega_2 &= \left\{ \sup_{x \in \mathbb{R}^n} \frac{u^T P_{R_k} x}{\Delta(x) + \Delta(x)} \leq E_1 \right\},
\end{align*}
\]
(55)
happen with probability at least $1 - \epsilon/2$ for some constant $E_1 > 0$, and with $A$ as in Lemma 4.

Following Ortelli and van de Geer (2019a), we take $S$ to be such that $\{(j : (D^{(r)} \theta^*)_j \neq 0) \subset S$, and $s := |S| \succ |\{j : (D^{(r)} \theta^*)_j \neq 0\}|.$ Then for $\epsilon > 0$ we set
\[
A = c_\epsilon \max \left\{ \frac{n^{r-1}(s + 1) \log n \log(s + 1) \log \frac{n}{s + 1}}{V^*}, n^{r-1/2} \left( \frac{1}{s + 1} \right)^{r-1/2} (\log n)^{1/2} \right\}
\]
(56)
for a large enough constant $c_\epsilon$ such that the events in (55) happen with probability at least $1 - \epsilon/2.$ We also set $\lambda = 3A.$

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Step 3 in proof outline.

Next, define
\[ t := c \epsilon (s + 1)^{1/2} (\log^{1/2} n) \left\{ \log \left( \frac{n}{s + 1} \right) \right\}^{1/2} \log^{1/2} (s + 1), \]
and notice that for \( \delta \in A \) with \( \Delta(\delta) \leq t \) it holds, by Lemma 22, that
\[
\Delta^2(P_R \delta) \leq \tilde{C}_r \left\{ t^2 + t^2 \gamma(t, n) + \frac{t^4}{n} \right\} \leq C t^2,
\]
for some positive constants \( \tilde{C}_r \) and \( C \), where we have used the fact that \( t/n^{1/2} = O(1) \).

If in addition \( \Omega_1 \cap \Omega_2 \) holds then
\[
\| D^{(r)} \delta \|_1 \leq C_0 \max \left\{ \frac{V^*}{n^{r-1}}, \frac{B^2}{A^2} \Delta^2(P_R \delta), \frac{B}{A} \Delta(P_R \delta), \frac{V^*}{n^{r-1}} + A^{-1} \left( \frac{t^2}{n} + t \right) \right\}. \tag{57}
\]

Next, notice that
\[
\frac{B^2 t^2}{A^2} \leq \frac{V^*}{n^{r-1} \log n} \frac{\epsilon^2}{n^{r-1} \log \left( \frac{n}{s + 1} \right)} \frac{V^*}{n^{r-1}} < C \frac{V^*}{n^{r-1}},
\]
for large enough \( n \). Furthermore,
\[
\frac{B t}{A} \leq \frac{V^*}{n^{r-1} (\log n)^{1/2}},
\]
and
\[
\frac{t^2}{A n} + \frac{t}{A} < \frac{c \epsilon}{n} \cdot \frac{V^*}{n^{r-1}} \cdot \frac{V^*}{n^{r-1}} + \frac{V^*}{n^{r-1}} < 2 \frac{V^*}{n^{r-1}}.
\]
Hence, for a constant \( \tilde{C} > 0 \) we have that
\[
\| D^{(r)} \delta \|_1 \leq \tilde{C} \frac{V^*}{n^{r-1}}. \tag{58}
\]

Steps 4–5 in proof outline.

Furthermore, denoting \( \bar{c} = \tilde{c} - \theta^* \), we notice that
\[
\Pr \left\{ \Delta^2(\bar{c}) > t^2 \right\} \leq \Pr \left\{ \left\{ \Delta^2(\bar{c}) > t^2 \right\} \cap \Omega_1 \cap \Omega_2 \right\} + \frac{\xi}{2}. \tag{59}
\]

Then proceeding as in the proof of Theorem 3,
\[
\Pr \left\{ \left\{ \Delta^2(\bar{c}) > t^2 \right\} \cap \Omega_1 \cap \Omega_2 \right\} \leq \Pr \left\{ \sup_{\delta \in A, \Delta^2(\bar{c}) \leq t^2} \left[ M(\theta^* + \delta) - \tilde{M}(\theta^* + \delta) + \lambda \| D^{(r)} \theta^* \|_1 - \lambda \| D^{(r)}(\theta^* + \delta) \|_1 \right] > c_0 t^2 \right\} \tag{60}
\]
\[
\cap \Omega_1 \cap \Omega_2 \right\}.
\]
And so, from (58) and by Markov’s inequality

$$\Pr \left[ \left\{ \Delta^2(\hat{\delta}) > t^2 \right\} \cap \Omega_1 \cap \Omega_2 \right] \leq \Pr \left( \sup_{\delta : \|D(\gamma)\delta\|_1 \leq \tilde{C}V^*/n^{r-1}, \Delta^2(\delta) \leq t^2} \left[ M(\theta^* + \delta) - \hat{M}(\theta^* + \delta) + \lambda \|D(\gamma)\theta^*\|_1 - \lambda \|D(\gamma)(\theta^* + \delta)\|_1 \right] > c_0 t^2 \right) \cap \Omega_1 \cap \Omega_2$$

where

$$K := \left\{ \delta \in \mathbb{R}^n : \|D(\gamma)\delta\|_1 \leq \tilde{C}V^*/n^{r-1}, \Delta^2(\delta) \leq t^2 \right\}.$$  (61)

Therefore, by Lemmas 28–29, we obtain that for $\xi_1, \ldots, \xi_n$ independent Rademacher variables independent of $y$, it holds that

$$\Pr \left[ \left\{ \Delta^2(\hat{\delta}) > t^2 \right\} \cap \Omega_1 \cap \Omega_2 \right] \leq \frac{4}{c_0 t^2} E \left[ \sup_{\delta \in K} \left\{ \sum_{i=1}^n \xi_i \delta_i + \frac{\lambda}{2} \|D(\gamma)\theta^*\|_1 - \frac{\lambda}{2} \|D(\gamma)(\theta^* + \delta)\|_1 \right\} \right] + \frac{\epsilon}{2}. \quad (62)$$

which combined with (59) leads to

$$\Pr \left[ \left\{ \Delta^2(\hat{\delta}) > t^2 \right\} \right] \leq \frac{4}{c_0 t^2} E \left[ \sup_{\delta \in K} \left\{ \sum_{i=1}^n \xi_i \delta_i + \frac{\lambda}{2} \|D(\gamma)\theta^*\|_1 - \frac{\lambda}{2} \|D(\gamma)(\theta^* + \delta)\|_1 \right\} \right] + \frac{\epsilon}{2}. \quad (63)$$

Then, we must give an upper bound to

$$U := \frac{4}{c_0 t^2} E \left[ \sup_{\delta \in K} \left\{ \sum_{i=1}^n \xi_i \delta_i + \frac{\lambda}{2} \|D(\gamma)\theta^*\|_1 - \frac{\lambda}{2} \|D(\gamma)(\theta^* + \delta)\|_1 \right\} \right], \quad (64)$$

where $\xi_1, \ldots, \xi_n$ are independent Rademacher variables independent of $y$ with $K$ as in (61). Towards that end, notice that

$$U \leq T_1 + T_2 + T_3$$

where

$$T_1 := \frac{4}{c_0 t^2} E \left\{ \sup_{\delta \in K} \xi^\top P_R \delta \right\},$$

$$T_2 := \frac{4}{c_0 t^2} E \left\{ \sup_{\delta \in K} \xi^\top P_{N-s} P_R \delta \right\},$$

and

$$T_3 := \frac{4}{c_0 t^2} E \left[ \sup_{\delta \in K} \left\{ \xi^\top P_{N-s} P_R \delta + \frac{\lambda}{2} \|D(\gamma)\theta^*\|_1 - \frac{\lambda}{2} \|D(\gamma)(\theta^* + \delta)\|_1 \right\} \right]$$

with $\mathcal{R}$ as defined in Section B.1.

**Step 6 in proof outline.**

Next we proceed to bound $T_1, T_2$ and $T_3$. First, we notice that as in (28) it follows that $T_1 = \mathcal{O}(\frac{1}{t^2})$.

**Step 7 in proof outline.**
To bound $T_2$ notice that for a positive constant $C > 0$,

$$
T_2 \leq \frac{4}{c_0 t^2} E \left\{ \sup_{\delta \in K : \|P_N \delta\|^2 \leq h(t, n)} \max\{\tilde{C}_r \tilde{C} \frac{V^*}{n-1}, 1\} \left( E^T P_N \delta \right) \right\}
$$

$$
\leq \frac{4}{c_0 t^2} E \left( h(t, n) \max\{\tilde{C}_r \tilde{C} \frac{V^*}{n-1}, 1\} \left( E \max_{1 \leq j \leq s} |\varepsilon^T v_j|^2 \right) \right)^{1/2}
$$

$$
\leq \frac{C}{c_0 t^2} \{h(t, n)\}^{1/2}(s + 1)^{1/2} \left( E \max_{1 \leq j \leq s} |\varepsilon^T v_j|^2 \right)
$$

$$
\leq \frac{2C}{c_0 t^2} \{h(t, n)\}^{1/2}(s + 1)^{1/2}(\log n)^{1/2}
$$

where the first inequality follows from Lemmas 17, 19 and 22, the second as (53), and the fourth by the expected value of maxima of subGaussian random variables inequality.

Therefore for a universal constant $C$ independent of $c_e$ we obtain,

$$
T_2 \leq \frac{C}{c_e} \leq \frac{\epsilon}{3},
$$

(65)

where the first inequality follows from our choice of $t$ and the second inequality in (65) follows by choosing $c_e$ large enough.

**Step 8 in proof outline.**

Next we proceed to bound $T_3$. Based on (56), suppose that

$$
\lambda = 3c_e n^{r-1/2} \left( \frac{1}{s+1} \right)^{r-1/2} (\log n)^{1/2}.
$$

Then, for $\delta \in K$ we have by Hölder’s inequality and choosing $c_e$ large enough that

$$
\|\xi^T P_N \delta\|_1 - \lambda \|D^{(r)}(\theta^* + \delta)\|_1 \leq \|w_{\theta}(D^{(r)}(\theta^* + \delta))\|_1 \leq \frac{A}{5(\log n)^{1/2}} \max_{j \in D \setminus S} \|\xi^T \psi_j^{-S}\|
$$

$$
+ \frac{\lambda}{2} \|D^{(r)}(\theta^* + \delta)\|_1 \leq \frac{\lambda}{2} \|D^{(r)}(\theta^* + \delta)\|_1.
$$

(66)

Next let

$$
\Omega_3(b) = \left\{ \max_{j \in D \setminus S} \|\psi_j^{-S}\| \geq b \sqrt{\log n} \right\},
$$

(67)

for $b > 0$. Then

$$
\text{pr}\{\Omega_3(b)\} \leq \exp\{-b^2(\log n)/2 + \log n\} \leq n^{1-b^2/2}.
$$

(68)

by the tail inequality for Rademacher variables and by union bound. Furthermore, from Subsections 3.3.2, 3.3.3, 3.3.4 and 3.3.5 in Ortelli and van de Geer (2019a)

$$
\max_{j \in D \setminus S} w_j \leq 1.
$$

(69)

Hence, combining (66)–(69) we obtain that

$$
\frac{4}{c_0 t^2} E \left\{ \sup_{\delta \in K} \left\{ \xi^T P_N \delta \|D^{(r)}(\theta^* + \delta)\|_1 - \lambda \|D^{(r)}(\theta^* + \delta)\|_1 \right\} \right\} \text{pr}\{\Omega_3(b)\}
$$

$$
\leq \frac{4}{c_0 t^2} \left\{ n^{1/2} \{h(t, n)\}^{1/2} + 3c_e n^{1/2} \left( \frac{1}{s+1} \right)^{r-1/2} (\log n)^{1/2} \right\} n^{1-b^2/2}
$$

(70)

$$
\rightarrow n \rightarrow 0,
$$

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where the inequality holds by Lemma 22 and Cauchy–Schwarz inequality, and the limit by the definition of \( t \) and choosing \( b = 5 \).

**Step 9 in proof outline.**

Additionally, given that \( \Omega_3(b)^c \) holds then (66) and (69) imply that for large enough \( c_{*} \), defining \( q_S := \text{sign}\{(D^{(r)}\theta^*)_S\} \), it holds that

\[
\sup_{\delta \in K} \left\{ \xi^T P_{N_{+}^L} P_{R} \delta + \frac{\lambda}{2} \| D^{(r)} \theta^* \|_1 - \frac{\lambda}{2} \| D^{(r)} (\theta^* + \delta) \|_1 \right\} \\
\leq \frac{\lambda}{2} \sup_{\delta \in K} \left[ \| w^- S \{D^{(r)} \delta\}_S \|_1 + \| D^{(r)} \theta^* \|_1 - \| D^{(r)} (\theta^* + \delta) \|_1 \right] \\
= \frac{\lambda}{2} \sup_{\delta \in K} \left[ \| w^- S \{D^{(r)} \delta\}_S \|_1 + \| \{D^{(r)} \theta^*\}_S \|_1 - \| D^{(r)} (\theta^* + \delta) \|_1 \right] \\
= \frac{\lambda}{2} \sup_{\delta \in K} \left[ \| \{D^{(r)} \theta^*\}_S \|_1 - \| \{D^{(r)} (\theta^* + \delta)\}_S \|_1 - \| (1 - w^- S) \{D^{(r)} (\theta^* + \delta)\}_S \|_1 \right] \\
\leq \frac{\lambda}{2} \sup_{\delta \in K} \left\{ \frac{\Gamma(q_S, w^- S)}{n^{1/2}} \| P_{R} \delta \|, \right\} \\
\leq \frac{\lambda \Gamma(q_S, w^- S) \{h(t, n)\}^{1/2}}{2n^{1/2}} \left( \max \left\{ C, C \right\} \right)^{1/2},
\]

where the second to last inequality follows from Lemma A.3 in Ortelli and van de Geer (2019a), and the last from Lemmas 17, 19 and 22.

However, by Section 3.3 in Ortelli and van de Geer (2019a), it holds that for some \( C_2 > 0 \),

\[
\Gamma(q_S, w^- S) \leq C_2 [\log \{n/(s + 1)\}]^{1/2} \{n(s + 1)\}^{1/2} \left( \frac{s + 1}{n} \right)^{r-1/2}.
\]

Hence,

\[
\sup_{\delta \in K} \left\{ \xi^T P_{N_{+}^L} P_{R} \delta + \frac{\lambda}{2} \| D^{(r)} \theta^* \|_1 - \frac{\lambda}{2} \| D^{(r)} (\theta^* + \delta) \|_1 \right\} \\
\leq C \left[ (s + 1) \log n \log \{n/(s + 1)\}\right]^{1/2} \{h(t, n)\}^{1/2}.
\]

Therefore, for large enough \( c_{*} \),

\[
\frac{4}{c_0 l^2} E \left[ \sup_{\delta \in K} \left\{ \xi^T P_{N_{+}^L} P_{R} \delta + \frac{\lambda}{2} \| D^{(r)} \theta^* \|_1 - \frac{\lambda}{2} \| D^{(r)} (\theta^* + \delta) \|_1 \right\} \right\} \Omega_3(b)^c \right| \Pr\{\Omega_3(b)^c\}
\leq \frac{4C \{h(t, n)\}^{1/2} [(s + 1) \log n \log \{n/(s + 1)\}]^{1/2}}{c_0 l^2}
\leq \frac{\epsilon}{6},
\]

which together with (70) implies that \( T_3 < \epsilon/3 \).

Next assume that

\[
\lambda = 3c_{*} n^{r-1}(s + 1) \log n \log(s + 1) \log \frac{n}{s+1}.
\]

Then, letting

\[
\tilde{\lambda} = 3c_{*} n^{r-1/2} \left( \frac{1}{s + 1} \right)^{r-1/2} (\log n)^{1/2},
\]

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we have that $\lambda \geq \tilde{\lambda}$ by the definition of $\lambda$, which implies that

$$\sup_{\delta \in K} \left\{ \xi^T P_{\mathcal{N}^\perp_S} P_R \delta + \frac{\lambda}{2} \|D^{(r)} \theta^*\|_1 - \frac{\lambda}{2} \|D^{(r)} (\theta^* + \delta)\|_1 \right\}$$

$$\leq \sup_{\delta \in K} \left\{ \xi^T P_{\mathcal{N}^\perp_S} P_R \delta + \frac{\tilde{\lambda}}{2} \|D^{(r)} \theta^*\|_1 - \frac{\tilde{\lambda}}{2} \|D^{(r)} (\theta^* + \delta)\|_1 \right\} + \frac{1}{2} \sup_{\delta \in K} \left\{ (\lambda - \tilde{\lambda}) \|D^{(r)} \theta^*\|_1 - (\lambda - \tilde{\lambda}) \|D^{(r)} (\theta^* + \delta)\|_1 \right\}$$

$$\leq \sup_{\delta \in K} \left\{ \xi^T P_{\mathcal{N}^\perp_S} P_R \delta + \frac{\tilde{\lambda}}{2} \|D^{(r)} \theta^*\|_1 - \frac{\tilde{\lambda}}{2} \|D^{(r)} (\theta^* + \delta)\|_1 \right\} + \frac{\lambda}{2} \|D^{(r)} \theta^*\|_1$$

and we notice that

$$\lambda \|D^{(r)} \theta^*\|_1 = 3c_\epsilon (s + 1) \log n \log(s + 1) \log \frac{n}{s + 1},$$

and in this case we also obtain that $T_3 \leq \epsilon/3$ by proceeding as before. The proof follows.

\[\square\]

## E Proof Theorem 5

### E.0.1 Controlling the Rademacher width

**Proposition 5.** Let $t > 0$ and

$$K = \{ \theta : \text{TV} (\theta) \leq V \}.$$  

Then

$$RW(\{ \delta \in K - \theta^* : \Delta^2(\delta) \leq t^2 \}) \leq C \left\{ Vm^{d-1} c(d, n) + \frac{t^2}{n^{1/2}} + t \right\},$$

where

$$c(d, n) \leq \begin{cases} \tilde{C} \log n & \text{if } d = 2, \\ \tilde{C}(\log n)^{1/2} & \text{if } d > 2, \end{cases}$$

for some positive constant $\tilde{C}$ that depends on $d$.

**Proof.** Let $\nabla$ be an incidence matrix of $L_{d,n}$, $N$ the number of rows of $\nabla$, and $\Pi$ the orthogonal projection onto the span of $(1, \ldots, 1)^T \in \mathbb{R}^n$. Notice that for $\xi_1, \ldots, \xi_n$ independent Rademacher variables we have that

$$RW(\{ \delta \in K - \theta^* : \Delta^2(\delta) \leq t^2 \}) \leq E \left\{ \sup_{\delta : \Delta^2(\delta) \leq t^2, \|\nabla \delta\|_1 \leq 2Vm^{d-1}} \xi^T \nabla^+ \nabla \delta \right\} + E \left\{ \sup_{\delta : \Delta^2(\delta) \leq t^2} \xi^T \Pi \delta \right\}$$

$$\leq 2Vm^{d-1} E \left\{ \| \nabla^+ \xi \|_\infty \right\} + \left( \frac{bt^2}{n} + \frac{t}{n^{1/2}} \right) E \left( \left| \sum_{i=1}^n \xi_i \right| \right)$$

$$\leq C \left\{ Vm^{d-1} \max_{j=1, \ldots, N} \| \nabla^+_{\cdot, j} \|_\infty \log n \right\}^{1/2} + \frac{t^2}{n^{1/2}} + t,$$

for some positive constant $C$, where the second inequality follows by Hölder’s inequality and Lemma 20, and the last by the Sub-Gaussian maximal inequality.

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Next, we recall from Propositions 4 and 6 from Hutter and Rigollet (2016), that

$$\max_{j=1, \ldots, N} \| \nabla^+_{\cdot, j} \| \leq \begin{cases} \tilde{C} (\log n)^{1/2} & \text{if } d = 2, \\ \tilde{C} & \text{if } d > 2, \end{cases}$$  

(73)

for some positive constant $\tilde{C}$ that depends on $d$. Hence, the claim follows combining (72)–(73).

\section*{E.0.2 Proof of Theorem 5}

Theorem 5 follows immediately from Theorem 7 and Proposition 5, by choosing

$$t \propto \begin{cases} (V m^{d-1} \log^2 n)^{1/2} & \text{if } d = 2, \\ \{V m^{d-1} \log n\}^{1/2} & \text{if } d > 2. \end{cases}$$

\section*{E.1 Theorem 6}

\textbf{Proof.} Let

$$K_V := \{ \beta \in \mathbb{R}^p : \| \beta \|_1 \leq V \}.$$

Also, let $\{\xi_i\}_{i=1}^n$ be independent Rademacher random variables independent of $\{(y_i)\}_{i=1}^n$. By Corollary 15, there exists a constant $C > 0$ such that

$$\frac{1}{n} E \left\{ \Delta^2 (\hat{\theta} - \theta^*) \right\} \leq \frac{CV}{n} E \left( \sup_{v \in K_1} \sum_{i=1}^n \xi_i x_i^\top v \right) = \frac{CV}{n} E \left( \sup_{v \in XK_1} \xi^\top v \right),$$

where

$$K_1 = \{ v : \| v \|_1 \leq 1 \}.$$ 

By the proof of Theorem 2.4 in Rigollet and Hütter (2015), there exists a constant $C_2 > 0$ such that

$$E \left( \sup_{v \in XK_1} \xi^\top v \right) \leq \frac{C_2 (\log p)^{1/2} \max_{j \in [p]} \| X_{\cdot, j} \|}{n}.$$

The claim of the theorem then follows. \qed
### Additional experiments

Table 2: Average $\Delta^2_n$ distance times $10, 10 \cdot \Delta^2_n(\theta^* - \hat{\theta})$, averaging over 100 Monte carlo simulations for the different methods considered. Captions are described in the main paper.

| $n$ | Scenario | $\tau$ | PQTF1 | PQTF2 | QS  | TF1 | TF2 |
|-----|----------|--------|-------|-------|-----|-----|-----|
| 10000 | 1        | 0.5    | 0.023 | 0.08  | 0.21| 0.016| 0.4 |
|  5000 | 1        | 0.5    | 0.046 | 0.12  | 0.23| 0.034| 0.65|
|  1000 | 1        | 0.5    | 0.18  | 0.29  | 0.32| 0.12 | 0.94|
| 10000 | 2        | 0.5    | 0.037 | 0.11  | 0.13| 6.67 | 6.33|
|  5000 | 2        | 0.5    | 0.066 | 0.15  | 0.17| 2.45 | 2.80|
|  1000 | 2        | 0.5    | 0.29  | 0.43  | 0.45| 8.08 | 9.41|
| 10000 | 3        | 0.5    | 0.015 | 0.063 | 0.17| 0.18 | 0.54|
|  5000 | 3        | 0.5    | 0.029 | 0.092 | 0.18| 0.13 | 0.65|
|  1000 | 3        | 0.5    | 0.13  | 0.24  | 0.26| 0.38 | 1.04|
| 10000 | 4        | 0.5    | 0.045 | 0.009 | 0.015| 0.063| 0.016|
|  5000 | 4        | 0.5    | 0.075 | 0.019 | 0.027| 0.10 | 0.031|
|  1000 | 4        | 0.5    | 0.30  | 0.082 | 0.098| 0.28 | 0.31|
| 10000 | 5        | 0.5    | 0.13  | 0.056 | 0.041| 1.55 | 1.91|
|  5000 | 5        | 0.5    | 0.24  | 0.099 | 0.085| 3.24 | 3.8 |
|  1000 | 5        | 0.5    | 1.91  | 0.35  | 0.35 | 5.38 | 6.00|
| 10000 | 6        | 0.9    | 0.18  | 0.070 | 0.075| *   | *  |
|  5000 | 6        | 0.9    | 0.29  | 0.13  | 0.14| *   | *  |
|  1000 | 6        | 0.9    | 1.19  | 0.39  | 0.40| *   | *  |
| 10000 | 6        | 0.1    | 0.16  | 0.065 | 0.070| *   | *  |
|  5000 | 6        | 0.1    | 0.31  | 0.13  | 0.14| *   | *  |
|  1000 | 6        | 0.1    | 1.27  | 0.46  | 0.47| *   | *  |
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