Cosmological magnetic fields

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Abstract. Observations indicate that magnetic fields are associated with most structures in the universe. Upto now the origin of large scale primordial magnetic fields is an open problem. Two mechanisms in particular, taking place during inflation in the very early universe are reviewed.

1. Introduction
Observations indicate that magnetic fields are associated with most structures in the universe. Magnetic fields are observed to be of the order of $10^{13}$ G in neutron stars, $10^{3}$ G in solar type stars and of order a $\mu$G on galactic scale. Magnetic fields of order a few $\mu$G also have been detected in radio galaxies at redshifts $z > 2$. The structure of the magnetic field in galaxies depends on the type of galaxy. In spiral galaxies there is typically a large scale component whose coherence length is of order of the size of the visible disk. In addition to this there are small-scale tangled fields. In contrast, magnetic fields in elliptical galaxies seem to be random with a coherence length much smaller than the galactic scale [1]. Basically there are no direct observations of magnetic fields not associated with some kind of structure. However, limits on the strength of truly cosmological magnetic fields can be derived indirectly using for example big bang nucleosynthesis [2] or the anisotropies of the cosmic microwave background [3, 4]. Since magnetic fields enter directly into the initial conditions as well as the evolution equations or, in the case of big bang nucleosynthesis, in the interaction rates, this leads to upper bounds on the magnetic field strength. In order to explain the galactic magnetic field the standard approach is to assume some kind of dynamo mechanism which amplifies an initial "seed" magnetic field. Similarly, magnetic fields in stars are explained by the formation of protostars out of condensed interstellar matter which was pervaded by a preexisting large scale magnetic field [5].

One of the open problems is to explain the origin of this initial magnetic field. There is a multitude of proposals. However, none of them seems to entail the most "natural" model. The proposals can be classified broadly into two groups. On the one hand mechanisms which rely on processes on small scales such as vortical perturbations and phase transitions and on the other hand those which use the potential amplification of perturbations in the electromagnetic field during inflation in the very early universe, see for example, [6, 7]. Inflation provides a mechanism to amplify initially small perturbation amplitudes to cosmologically relevant values. In flat backgrounds in order to reach interesting amplitudes one has to break the conformal invariance of the background. In four dimensions the Maxwell Lagrangian describing linear electrodynamics is conformally invariant. There has been a variety of proposals to break the
conformal invariance such as, coupling to a scalar field [8], breaking Lorentz invariance [9], adding extra dimensions [10, 11], coupling to curvature terms [12] or nonlinear electrodynamics [13, 14].

In curved backgrounds with open spatial sections there is an interesting coupling between the geometry and electromagnetism that can lead to the superadiabatic amplification of the field amplitude just within the framework of standard electromagnetism [15].

In particular, here two proposals will be considered in more detail. On the one hand, the generation of magnetic seed fields from extra dimensions will be considered. On the other hand, the way how nonlinear electrodynamics can produce primordial magnetic fields of cosmologically interesting field strengths will be discussed.

2. Primordial magnetic fields from extra dimensions

Space-times with more than four dimensions have been considered since a long time starting with the work of Kaluza and Klein [16]. In this approach the extra dimensions are small enough so that our world is effectively described by physics in four dimensions.

The possibility of having large extra dimensions came about when solutions of this type were found within string/M-theory. Here the observable four dimensional universe is located on a brane which is a four dimensional hypersurface embedded in a higher dimensional space-time. In higher-dimensional gravity the four dimensional Planck scale $M_4$ is no longer fundamental, but rather it is the higher dimensional Planck scale $M_D$. Assuming that there are $n$ extra dimensions with a characteristic size $R$, using Gauss' law the four dimensional and $D$ dimensional Planck masses are related by,

$$ M_4^2 = R^n M_D^{n+2}. $$

A lower bound on the ratio $M_D/M_4$ can be found by experiments of Newtonian gravity. Experiments indicate that Newtonian gravity is valid at least down to length scales of 1 mm [17].

The background space-time is assumed to be homogeneous and anisotropic with a line element,

$$ ds^2 = a^2(\eta) \left[ d\eta^2 - \delta_{ij} dx^i dx^j \right] - b^2(\eta) \delta_{AB} dy^A dy^B, $$

where $i, j = 1, ..., 3$ and $A, B = 4, ..., 3 + n, n \geq 1$. Furthermore $a(\eta)$ is the scale factor of the "external", 3-dimensional space and $b(\eta)$ is the scale factor of the internal, $n$-dimensional space. We assume that an early $(n + 4)$-dimensional vacuum phase of the universe is matched to a standard four dimensional radiation dominated phase. Thus whereas in the higher dimensional phase both scale factors are functions of time, in the succeeding four dimensional phase only the scale factor of the three dimensional space $a(\eta)$ is dynamical. In this stage the extra dimensions are frozen at a constant value of the scale factor $b(\eta) = \text{const}$. The matching of these two phases of the universe, that is the higher-dimensional and the four dimensional stage, takes place at some time $\eta = -\eta_1$. The solutions for the scale factors during these two epochs are given by,

$$ a(\eta) = a_1 \left( \frac{-\eta}{\eta_1} \right)^\sigma, \quad b(\eta) = b_1 \left( \frac{-\eta}{\eta_1} \right)^\lambda, \quad \text{for} \quad \eta < -\eta_1 $$

$$ a(\eta) = a_1 \left( \frac{\eta + 2\eta_1}{\eta_1} \right), \quad b(\eta) = b_1, \quad \text{for} \quad \eta \geq -\eta_1 $$

In the following we set $a_1 = 1 = b_1$. Clearly, for $\eta < -\eta_1$ the solution is given by the vacuum Kasner metric, which is determined by the exponents $\sigma$ and $\lambda$ which are functions of the number
of extra dimensions $n$. These are related to the Kasner exponents $\alpha_E$ and $\alpha_I$, which satisfy the Kasner conditions $3\alpha_E + n\alpha_I = 1$ and $3\alpha_E^2 + n\alpha_I^2 = 1$, by

$$\sigma = \frac{\alpha_E}{1 - \alpha_E}, \quad \lambda = \frac{\alpha_I}{1 - \alpha_E}. \quad (5)$$

Furthermore, in the case of an expanding, external space and a contracting, internal space the exponents $\sigma$ and $\lambda$ are of the form $[10]$, 

$$\sigma = -\frac{1}{2} \left( \sqrt{\frac{3n}{n+2}} - 1 \right), \quad \lambda = \sqrt{\frac{3}{n(n+2)}}. \quad (6)$$

Starting with Maxwell’s equations in $D$ dimensions, $\nabla A_F = 0$, where $A, B = 0, \ldots, n+3$ equations for the gauge potential $A_i$ in 3+1 dimensions can be derived. Imposing the radiation gauge simplifies these equations. In order to proceed from there the canonical field $\Psi_I = b^F A_i$ is introduced and Fourier expanded. This finally leads to a mode equation, which during the early stage of dynamical extra dimensions is given by, for $\eta < -\eta_1$, $[11]$

$$\Psi''_I + \left[ k^2 + \left( -\frac{\eta}{\eta_1} \right)^{2\beta} q^2 - \frac{N}{\eta^2} \right] \Psi_I = 0, \quad (7)$$

where $' \equiv \frac{\partial}{\partial \eta}$ and $N \equiv \frac{1}{4}(n\lambda - 1)^2 - \frac{1}{4}$. Furthermore, $\beta \equiv \sigma - \lambda$. During the later radiation dominated epoch where the extra dimensions are frozen to a particular size and only the three spatial dimensions of the observable universe are dynamical, the mode equation takes the form, for $\eta \geq -\eta_1$

$$\Psi''_I + \left[ k^2 + \left( \frac{\eta + 2\eta_1}{\eta_1} \right)^2 q^2 \right] \Psi_I = 0. \quad (8)$$

The next step is to solve the mode equations in the two regimes, using the proper matching conditions at the transition time, $\eta = -\eta_1$. This leads to an expression for the Bogoliubov coefficient determining the number of created photons which can be used to calculate the total magnetic energy density in four dimensions. As the universe had a very high conductivity during most of its history, the magnetic field evolves such that its flux is conserved and thus the dimensionless ratio $r \equiv \rho_B/\rho_\gamma$ is approximately constant $[7]$, where $\rho_B$ is the magnetic field energy density and $\rho_\gamma$ the energy density of the background radiation. $r$ calculated at the galactic scale of order 1 Mpc determines the magnetic field strength at the time of galaxy formation. Assuming that a galactic dynamo has been operating since the time of galaxy formation implies a lower limit on the primordial seed field strength of $B_s \sim 10^{-20}$G $[5]$ corresponding to $r \sim 10^{-37}$. This number can be lowered significantly if the non-vanishing cosmological constant is taken into account. In that case the minimal required magnetic seed field strength is $B_s \sim 10^{-30}$G and $r \sim 10^{-57}$. Imposing in addition that $r(\omega) < 1$, the bound on $M_D/M_4$ from Newtonian gravity and that standard big bang nucleosynthesis takes place, leads to bounds on the parameters of the model. It is found that if the momenta lying along the extra dimensions are taken into account magnetic fields strong enough to seed the galactic magnetic field can be created. For details see reference $[11]$.

3. Primordial magnetic fields from nonlinear electrodynamics

Nonlinear electrodynamics naturally breaks the conformal invariance of electrodynamics in four dimensions. Thus it seems to be a good candidate to amplify perturbations of the electrodynamic
field during inflation in the early universe. Nonlinear electrodynamics can be understood as taking into account the self interaction of the electromagnetic field. Born and later Infeld were the first to treat a particular model of nonlinear electrodynamics in the search for a classical, singularity free theory of the electron [18]. Another class of models of nonlinear electrodynamics is given by the Heisenberg-Euler Lagrangian which describes effectively the propagation of a photon in an external electromagnetic field [19].

The question that arises in the context of generation of primordial magnetic fields is if models of nonlinear electrodynamics can generate primordial magnetic fields of cosmologically interesting strength. As we will see below this is actually the case. The cosmological model to be considered consists of a stage of de Sitter inflation followed by reheating and a standard radiation dominated stage. During inflation quantum fluctuations in the electromagnetic field are excited within the horizon. Being stretched beyond the horizon they become classical perturbations. It is assumed that during inflation electrodynamics is nonlinear which could be motivated by the presence of possible quantum corrections to quantum electrodynamics at high energies. Furthermore, it is assumed that once inflation ends electrodynamics becomes linear so that the subsequent evolution is described by the standard model of cosmology.

In general, electrodynamics coupled minimally to gravity can be written as, [20]

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R + \frac{1}{4\pi} \int d^4x \sqrt{-g} L(X, Y),$$

(9)

where $L(X, Y)$ is the Lagrangian of nonlinear electrodynamics. Furthermore, the invariants are denoted by $X \equiv \frac{1}{4} F_{\mu \nu} F^{\mu \nu}$ and $Y \equiv \frac{1}{4} F_{\mu \nu}^{*} F^{\mu \nu}$, where $F^{* \mu \nu}$ is the dual bi-vector given by $F^{* \mu \nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu \nu \alpha \beta} F_{\alpha \beta}$, and $\epsilon^{\mu \nu \alpha \beta}$ the Levi-Civita tensor with $\epsilon_{0123} = +1$.

The equations of motion are given by

$$\nabla_\mu P^{\mu \nu} = 0$$

(10)

where $P_{\mu \nu} = -(L X F_{\mu \nu} + L Y * F_{\mu \nu})$, furthermore $L_A$ denotes $L_A = \partial L / \partial A$, and

$$\nabla_\mu * F^{\mu \nu} = 0,$$

(11)

which implies that $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ where $\mu, \nu = 0\ldots3$. Moreover, the electromagnetic field is treated as a perturbation so that the vacuum Einstein equations apply to the background cosmology. The background metric is chosen to be of the form

$$ds^2 = a^2(\eta) \left[ -d\eta^2 + dx^2 \right].$$

(12)

Writing the Maxwell tensor in terms of the electric and magnetic field leads to a set of highly nonlinear equations coupled in these two fields [13]. Clearly, a Fourier expansion of the electric and magnetic field leads to mode coupling. Therefore, we used the long wave length approximation which is based on the observation that if variations over a characteristic comoving length scale $L$ much larger than the horizon $aH$ are considered then the spatial derivatives of a quantity can be neglected with respect to its time derivatives. In this approximation the equation for the magnetic field is given by

$$\ddot{B}_k^\mu + \frac{L_X}{L_X} \ddot{E}_k^\mu + \frac{L_Y}{L_X} \dddot{E}_k^\mu + \frac{L_Y}{L_X} \left[ \frac{L_X}{L_X} \dddot{E}_k^\mu - \frac{L_Y}{L_X} \dddot{B}_k \right] = 0,$$

(13)

where $\ddot{B}_k \equiv \frac{d^2 B_k}{d\eta^2}$, $\dddot{E}_k \equiv \frac{d^3 E_k}{d\eta^3}$ and a prime denotes the derivative with respect to conformal time $\eta$, that is $' \equiv \frac{d}{d\eta}$. Now in order to make progress one has to choose a particular theory
of nonlinear electrodynamics $L(X, Y)$. Our aim is to see if in principle it is possible to amplify sufficiently perturbations in the electromagnetic field during inflation. Thus for simplicity the Lagrangian is chosen to be of the form,

$$L = -\left(\frac{X^2}{\Lambda^8}\right)^{\frac{1}{2}}X, \quad (14)$$

where $\delta$ is a dimensionless parameter and $\Lambda$ a dimensional constant. This Lagrangian describes the abelian Pagels-Tomboulis model [21]. Originally, the nonabelian theory was proposed as an effective model of low energy QCD [22]. It is obvious that linear electrodynamics is recovered for the choice $\delta = 1$. As emphasized before the Lagrangian (14) is chosen in such a way since it leads to a simplification of the equations, but still allows to study the effects of a strongly nonlinear theory of electrodynamics on the generation of primordial magnetic fields. Using the Lagrangian (14) in the equations of motion in the long wave length approximation leads to an equation for the evolution of the energy density of the magnetic field which can be approximately solved in three different regimes depending on $\vec{B}_k^2/\vec{E}_k^2$ (for details see [13]). Basically the cases $\vec{B}_k^2/\vec{E}_k^2$ greater, equal or smaller than one are considered. The key quantity in order to decide if the amplification of the initial perturbation in the electromagnetic field during inflation is cosmologically relevant is as before the ratio of magnetic energy density to radiation energy density, $r = \rho_B/\rho_r$. Requiring that $r$ has to be larger than $10^{-37}$ which corresponds to a seed magnetic field of $B_s \sim 10^{-20}$G it has been shown that depending on the approximation and the value of the Pagels-Tomboulis parameter $\delta$ primordial magnetic fields can be generated that are strong enough to seed the galactic magnetic field. For a detailed discussion see [13].

4. Conclusions
We have presented two particular mechanisms to generate primordial magnetic fields. Both of them take place in flat backgrounds and so rely on breaking the conformal invariance of Maxwell’s equations in four dimensions. In particular, the first mechanism presented involves a stage with dynamical extra dimensions during inflation. The second mechanism assumes that due to quantum corrections electrodynamics effectively becomes non linear. In both cases it has been found that there is a range of parameters where primordial magnetic fields of cosmologically relevant strength can be created, so as to seed for example the galactic dynamo resulting in the galactic magnetic field of $10^{-6}$G today.

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