Thermal conductivity for a bidimensional dilute gas within the Chapman-Enskog approximation

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Abstract

In this work we explicitly calculate the thermal conductivity for a bidimensional dilute gas of neutral molecules by solving Boltzmann’s equation. Chapman-Enskog’s method is used in order to analytically obtain the transport coefficient to first approximation. The result is expressed in terms of a collision integral for an unspecified molecular interaction model. The particular case of a hard disks model is addressed yielding a $T^{1/2}$ dependence with the temperature which is consistent with the one obtained by J. V. Sengers [1] and widely used in the literature as the low density limit in the Enskog expansion. The corresponding value for bidimensional Maxwellian molecules is also obtained.

1 Introduction

Two-dimensional fluid systems have been vastly studied experimentally as well as numerically [1, 2, 3, 4, 5, 6, 7]. These systems have attracted scientists’ attention due primarily to the wide variety of their applications, for example in biological membranes and fluxes in surfaces as graphite. Additionally, computer simulations for two-dimensional systems serve as prototypes for testing theoretical results through molecular dynamics or Monte Carlo simulations [4, 8, 9]. The analytical value of transport coefficients in 2D fluids are also required in order to verify the non-relativistic limit for relativistic dilute fluids [10, 11]. Since experimental data for neutral relativistic gases is not available, numerical results are of particular relevance in such high temperature systems, amongst which the two dimensional case is much more simple to address [9]. However, most of the theoretical works on this field are limited to the case of dense gases. Moreover, the question of whether a density expansion is appropriate and the origin of the divergence of the corresponding transport coefficients for low density limits have been a topic of intense research [2, 10, 11].

To the author’s knowledge, the only work in which transport coefficients for the dilute bidimensional gas are calculated can be found in Ref. [1] where only the particular case of a hard disks gas is presented as a side calculation in order to address the triple collision contribution in such a system. In the present work an explicit calculation of the thermal conductivity for a dilute bidimensional gas are shown without specifying a particular interaction potential. The heat flux and corresponding conductivity are calculated for two particular cases, the Maxwellian molecules and the hard disk model which yields results consistent with Ref. [1].

The rest of this work is organized as follows. Section two establishes the basic formalism for calculating the heat flux from the linearized Boltzmann equation within the Chapman-Enskog approximation. The general expressions for the heat flux and thermal conductivity are obtained in section three in terms of a collision integral calculated in detail in the Appendix. Section four addresses the particular cases of hard disks and Maxwellian molecules. The final section includes a brief discussion and some final remarks.

2 Boltzmann equation and Chapman-Enskog approximation

The evolution equation for the single particle distribution function corresponding to a dilute bidimensional gas of neutral, non-interacting molecules is given by the Boltzmann equation [12]

$$\frac{df}{dt} = \int \left[ f' f' - f f \right] g \sigma d\chi d^2v_1. \quad (1)$$

Here $f$, the distribution function, is the density of particles in the four dimensional phase space such that $f (\vec{r}, \vec{v}, t) d^2r d^2v$ yields the number of particles at time $t$ with a position vector between $\vec{r}$ and $\vec{r} + d\vec{r}$ and velocity between $\vec{v}$ and $\vec{v} + d\vec{v}$. The right hand side represents the change in the distribution function due to collisions where interactions between the particle in question and particles denoted with a 1 subindex are considered. Non-primed and primed quantities denote values before and after the collision, respectively. The relative velocity is denoted by $\vec{g}$, with $g$ being its magnitude. Also $\chi$ and $\sigma$ are the scattering angle and the impact parameter respectively, which are the 2D counterparts to the solid angle and the scattering cross section in the 3D case.

The solution of the homogeneous Boltzmann equation for a bidimensional system can be shown to be given by

$$f^{(0)} = \frac{nm}{2\pi kT} e^{-\frac{m\vec{g}^2}{2kT}}, \quad (2)$$
by means of which one can establish the corresponding expressions for the state variables, in this case particle density

\[ n = \int f^{(0)}d^2v, \quad (3) \]

hydrodynamic velocity

\[ \vec{u} = \int f^{(0)} \vec{v} d^2v, \quad (4) \]

and internal energy

\[ \varepsilon = \int f^{(0)} \left( \frac{1}{2}mv^2 \right) d^2v. \quad (5) \]

The relationship between the molecular \( \vec{v} \), hydrodynamic \( \vec{u} \) and chaotic, or peculiar, \( \vec{c} \) velocities appearing in the previous expressions is given by \( \vec{v} = \vec{u} + \vec{c} \) being \( \vec{v} \) the velocity of a molecule as measured by an arbitrary observer while \( \vec{c} \) is the one seen from a system comoving with the fluid, that is an observer moving with velocity \( \vec{u} \).

At this point it is convenient to introduce the Chapman-Enskog’s hypothesis which allows one to express the complete solution to Eq. (1) as an infinite series of terms ordered in the Knudsen parameter, which quantifies the deviation from the equilibrium one due to the gradients present in the system [12]. To first order in the gradients, corresponding to the Navier-Stokes regime, one has

\[ f = f^{(0)}(1 + \phi), \quad (6) \]

where \( \phi \) is of first order in the Knudsen parameter and thus in the gradients. In this context, as is well known, the transport equations which represent balance equations for the set of variables given in Eqs. (3)-(5), can be readily obtained by multiplying Eq. (1) by a collisional invariant and integrating over velocity space. Such a set requires closure, or constitutive, relations expressing the dissipative fluxes in terms of gradients of the state variables. In particular, for the heat flux one obtains the expression

\[ \vec{q} = \int f^{(0)} \phi \left( \frac{1}{2}mc^2 \right) \vec{c} d^2c, \quad (7) \]

which will be used in this work to obtain an analytic expression for the thermal conductivity \( \kappa \) from the Fourier relation \( \vec{q} = -\kappa \nabla T \).

The linearized Boltzmann equation to first order in Knudsen’s parameter is given by [12]

\[ \frac{df^{(0)}}{dt} = \mathcal{I}(\phi), \quad (8) \]

where

\[ \mathcal{I}(\phi) = \int f^{(0)} f^{(0)} [\phi' + \phi' - \phi_1 - \phi] g \sigma d\chi dv_1^2, \quad (9) \]

and in the absence of external forces the left hand is expressed as

\[ \frac{\partial f^{(0)}}{\partial \vec{u}} \cdot \frac{\partial \vec{u}}{\partial t} + \frac{\partial f^{(0)}}{\partial T} \frac{\partial T}{\partial t} + f^{(0)} \frac{\partial f^{(0)}}{\partial n} + \vec{v} \cdot \left( \frac{\partial f^{(0)}}{\partial \vec{u}} \cdot \frac{\partial \vec{u}}{\partial \vec{r}} + \frac{\partial f^{(0)}}{\partial T} \frac{\partial T}{\partial \vec{r}} + \frac{\partial f^{(0)}}{\partial n} \frac{\partial n}{\partial \vec{r}} \right), \quad (10) \]

using the well known functional hypothesis by means of which the distribution function depends on time and space only implicitly through the state variables. Introducing Eq. (2) for the equilibrium distribution function and using Euler’s equations to express the time derivatives in terms of the gradients, as required by the Chapman-Enskog method, for the existence of the solution [12], Eq. (8) can be shown to be reduced to

\[ f^{(0)} \left( \frac{mc^2}{2kT} - 2 \right) \vec{c} \cdot \nabla T = \mathcal{I}(\phi), \quad (11) \]

where we have also neglected the terms involving the gradient of the hydrodynamic velocity since by Curie’s principle, only forces and fluxes of the same tensorial rank couple and thus \( \nabla \vec{u} \) will not drive heat flux.

### 3 Solution to the integral equation

The Chapman-Enskog hypothesis together with the linearized Boltzmann equation yield the integral equation given in Eq. (11). Although much simpler to solve than the integro-differential one given by Eq. (1), its complete solution can be involved. However, as will be shown, the two-dimensional calculation is straightforward and the collisional bracket corresponding to the heat flux can be calculated directly for a hard disks collisional model.

Since Eq. (11) is an inhomogeneous integral equation, its general solution is given within this approximation by Eq. (6) where the correction \( \phi \) is expressed as

\[ \phi = \alpha + \vec{\alpha} \cdot \vec{c} + \alpha_2 c^2 + A(c) \vec{c} \cdot \frac{\nabla T}{T}. \quad (12) \]
The first three terms, a linear combination of the collisional invariants, determine the solution of the corresponding homogeneous equation. The last term is a proposed particular solution to the complete inhomogeneous equation. Here \( A(c) \) is to be determined and depends only on the magnitude of \( c \) since the dependence with its direction is fixed, due to the structure of the inhomogeneous term. In order to guarantee uniqueness of the solution, the usual subsidiary conditions are imposed which ensure that density, internal energy and hydrodynamic velocity, being state variables, are determined exclusively through the equilibrium distribution function, a hypothesis already established in Eqs. (3)-(5). These conditions are thus satisfied as long as

\[
\int f^{(0)}(0) \phi d^2c = 0
\]

\[
\int f^{(0)}(0) \phi \frac{c^2}{2} d^2c = 0
\]

\[
\int f^{(0)}(0) \phi c d^2c = 0.
\]

Substitution of Eq. (12) in the first two lines of Eq. (13) leads to

\[
\int f^{(0)}(0) (\alpha + \alpha_2 c^2) d\tilde{c} = 0,
\]

and

\[
\int f^{(0)}(0) (\alpha + \alpha_2 c^2) c^2 d\tilde{c} = 0,
\]

which can be written as an homogeneous system of equations with the only trivial solution \( \alpha = \alpha_2 = 0 \). For the third line of Eq. (13) one has

\[
\int f^{(0)}(0) \phi \tilde{c} v = \int f^{(0)}(0) \tilde{c} \cdot \left( A(c) \frac{\nabla T}{T} + \tilde{\alpha}_1 \right) d\tilde{c} = 0,
\]

which implies that \( \phi \) can be written as

\[
\phi = A(c) \tilde{c} \frac{\nabla T}{T},
\]

where now \( A(c) \) needs to be determined such that it satisfies the inhomogeneous equation (11), being still subject to the condition

\[
\frac{\nabla T}{T} \cdot \int f^{(0)}(0) A(c) \tilde{c} d\tilde{c} = \frac{1}{2} \frac{\nabla T}{T} \cdot \mathbb{I} \int f^{(0)}(0) A(c) c^2 d\tilde{c} = 0,
\]

where \( \mathbb{I} \) is the identity matrix. Introducing Eq. (17) and defining an auxiliary variable \( x = mc^2/2kT \) in Eq. (11), the integral equation for \( \phi \) can be written as

\[
f^{(0)}(x - 2) \tilde{c} \cdot \frac{\nabla T}{T} = \mathbb{I} \left( A(x) \tilde{c} \cdot \frac{\nabla T}{T} \right)
\]

where \( A(x) \) satisfies

\[
\int A(x) p(x) dx = 0,
\]

with

\[
p(x) = xe^{-x},
\]

being a weight function. Following the standard procedure, \( A(x) \) is then written as an infinite series in orthogonal polynomials. In this case, the appropriate basis are the associated Laguerre, or Sonine, polynomials \( \mathcal{L}^{(1)}_n \) since

\[
\int e^{-x} \mathcal{L}^{(1)}_n(x) \mathcal{L}^{(1)}_m(x) x dx = \frac{\Gamma(n + 2)}{n!} \delta_{mn},
\]

with

\[
\mathcal{L}^{(1)}_0(x) = 1 \quad \text{and} \quad \mathcal{L}^{(1)}_1(x) = 2 - x.
\]

Thus, we have

\[
A(x) = \sum_{n=0}^{\infty} a_n \mathcal{L}^{(1)}_n(x),
\]

and the condition in Eq. (20) is written as

\[
\sum_{n=1}^{\infty} a_n \int p(x) \mathcal{L}^{(1)}_n(x) dx = 0.
\]

Equation (25) implies \( a_0 = 0 \) since,

\[
\sum_{n=0}^{\infty} a_n \int p(x) \mathcal{L}^{(1)}_n(x) \mathcal{L}^{(1)}_0(x) dx = a_0 \int p(x) \mathcal{L}^{(1)}_0(x) \mathcal{L}^{(1)}_0(x) dx = 0,
\]
and thus, the solution to the integral equation (19) is given by Eq. (24) where the coefficients \( a_n \) are obtained from

\[
-f^{(0)} \mathcal{L}^{(1)} (x) \hat{c} = \sum_{n=1}^{\infty} a_n \mathcal{I} \left( \mathcal{L}^{(1)}_n (x) \hat{c} \right).
\]  

(27)

With the solution proposed, the heat flux defined in Eq. (7) is

\[
\hat{q} = \sum_{n=0}^{\infty} a_n \frac{\nabla T}{T} \int f^{(0)} \mathcal{L}^{(1)}_n (x) \frac{m c^2}{2} \hat{c} d^2 c,
\]

which can be written in terms of \( x \) as

\[
\hat{q} = \frac{n}{m} \frac{(kT)^2}{T} \frac{\nabla T}{T} \sum_{n=1}^{\infty} a_n \int \mathcal{L}^{(1)}_n (x) \exp (x) \, dx.
\]

(29)

Writing \( x \) in terms of the Sonine polynomials as

\[
x = 2 \mathcal{L}^{(1)}_0 (x) - \mathcal{L}^{(1)}_1 (x),
\]

(30)

and using the orthogonality condition (22) one obtains that only the first coefficient of the infinite series is needed in order to calculate the thermal conductivity namely,

\[
\hat{q} = -2 \frac{n}{m} \frac{(kT)^2}{T} \frac{\nabla T}{T} a_1.
\]

(31)

Since only \( a_1 \) is required for the purposes of this work, the scalar product of Eq. (24) with \( \mathcal{L}^{(1)}_m (x) \hat{c} \) is calculated and the resulting equation is integrated over velocity space. The left hand side yields

\[
\frac{2 k n T}{m} \int \mathcal{L}^{(1)}_1 (x) \mathcal{L}^{(1)}_m (x) p (x) \, dx = 4 \frac{n k T}{m} \delta_{1m},
\]

and thus

\[
4 \frac{n k T}{m} \delta_{1m} = - \sum_{n=1}^{\infty} a_n \int \mathcal{L}^{(1)}_n (x) \hat{c} \cdot \mathcal{I} \left( \mathcal{L}^{(1)}_n (x) \hat{c} \right) \, d^2 c.
\]

(33)

Equation (33) can now be solved for \( a_1 \) to different orders of approximation using the standard variational method, whose details can be found in Ref. [13]. To first approximation one obtains

\[
a_1 = -4 \frac{n k T}{m} \left( \int \mathcal{L}^{(1)}_1 (x) \hat{c} \cdot \mathcal{I} \left( \mathcal{L}^{(1)}_1 (x) \hat{c} \right) \, d^2 c \right)^{-1}.
\]

(34)

The integral above is usually written in terms of a collisional bracket and requires in most cases an elaborate procedure in order to be evaluated. However, the calculation can be simplified by using momentum conservation as follows

\[
\mathcal{I} \left( \mathcal{L}^{(1)}_1 (x) \hat{c} \right) = \int f^{(0)} f^{(0)} \left[ (2 - x') \hat{c}' + (2 - x') \hat{c}' - (2 - x') \hat{c} - (2 - x) \hat{c} \right] g \sigma d \chi d^2 c_1
\]

\[
= - \frac{m}{kT} \mathcal{I} \left[ \frac{c^2}{2} \right],
\]

(35)

Also, since by interchanging particles after and before the collision, and dummy indices it can be shown that

\[
\int \hat{H} \cdot \mathcal{I} \left( \hat{G} \right) \, d^2 c = \int \hat{G} \cdot \mathcal{I} \left( \hat{H} \right) \, d^2 c,
\]

(36)

one obtains

\[
\int \mathcal{L}^{(1)}_1 (x) \hat{c} \cdot \mathcal{I} \left( \mathcal{L}^{(1)}_1 (x) \hat{c} \right) \, d^2 c = \frac{m}{kT} I,
\]

(37)

where

\[
I = \int \mathcal{I} \left[ \left( \frac{m c^2}{2 k T} - 2 \right) \hat{c} \right] \frac{c^2}{2} \, d^2 c,
\]

(38)

and thus \( a_1 \) can be expressed as

\[
a_1 = -4 \frac{n k T^2}{m^2} I^{-1}.
\]

(39)

The detailed calculation of \( I \) is shown in the Appendix where the expression

\[
I = -\frac{n^2 \beta}{32 \pi} \int \exp \left( -\frac{\beta}{2} g^2 \right) g^5 \sin^2 \chi \sigma (\chi, g) \, d\chi d^2 g
\]

(40)
is obtained with $\beta = m/2kT$. It is worthwhile to point out that in the bidimensional case, the integral is in principle much simpler to calculate than in the 3D general case. The general expression for the thermal conductivity of a dilute bidimensional gas is thus given by

$$\lambda = -\frac{1}{7} \frac{8n^2k^4T^3}{ m^3}$$  \hspace{1cm} (41)

This expression for the transport coefficient is the main result of this work. In the following section, two particular models for the intermolecular interaction will be used in order to obtain explicit temperature dependence of $\lambda$.

4 The hard disks and Maxwellian molecules models

In order to perform the integration in Eq. (40) with respect to the scattering angle $\chi$, a particular model for the differential cross section needs to be introduced. For example, in the case of hard disks $\sigma (\chi, g)$ depends on the scattering angle as follows

$$\sigma = \left| \frac{d b}{d \chi} \right| = \left| \sin \left( \frac{\chi}{2} \right) \right|.$$  \hspace{1cm} (42)

which leads to the following expressions for $I$ and $a_1$ (see Appendix):

$$I = -n^2d\beta^{-5/2} \sqrt{\frac{\pi}{2}}$$  \hspace{1cm} (43)

and

$$a_1 = \frac{1}{nd} \left( \frac{m}{\pi kT} \right)^{1/2}.$$  \hspace{1cm} (44)

Using the results above, the heat flux can be written as

$$\vec{q} = -\frac{2}{d} \left( \frac{k^3T}{\pi m} \right)^{1/2} \nabla T,$$  \hspace{1cm} (45)

from which we obtain the thermal conductivity for the hard disk dilute gas:

$$\lambda = \frac{2}{d} \left( \frac{k^3T}{\pi m} \right)^{1/2}.$$  \hspace{1cm} (46)

which is consistent with results quoted in Ref. [1].

The Maxwellian disks model is a particular case of the point centers of repulsion potential and approximate the molecular interaction for the case of high temperature gases in which the repulsive forces dominate over the attractive ones. In such a case, the potential can be expressed as [13]

$$\phi (r) = \kappa r - \nu,$$  \hspace{1cm} (47)

where $\nu$ is a measure of the hardness of the molecules. Maxwellian molecules are characterized by $g\sigma (\chi, g)$ being independent of $g$, a property that in two dimensions is obtained for $\nu = 2$. In such a case one obtains a scattering angle given by

$$\chi = \pi \left( 1 - \left( 1 + \frac{\kappa}{2m g^2 b^2} \right)^{-1/2} \right),$$  \hspace{1cm} (48)

which leads to the following expression for the scattering cross section

$$\sigma (g, \chi) = \sqrt{\frac{\kappa}{2m}} \frac{\pi^2}{g} (\chi (2\pi - \chi))^{-3/2}.$$  \hspace{1cm} (49)

With this result

$$I = -\frac{\kappa}{2m} \Gamma \frac{\pi^2 n^2}{2\beta^2},$$  \hspace{1cm} (50)

where $\Gamma$ is given in ((73)) in the Appendix. Thus the heat flux and thermal conductivity are given by

$$\vec{q} = -\frac{1}{\Gamma} \sqrt{\frac{2}{mk}} \frac{k^2T}{\pi^2} \nabla T$$  \hspace{1cm} (51)

and

$$\lambda = \frac{1}{\Gamma \pi^2} \sqrt{\frac{2}{\kappa m k^2 T}}.$$  \hspace{1cm} (52)

with $\Gamma \approx 0.21$. Should de noted that for this model the thermal conductivity grows linearly with the temperature, in the same way as happens in the 3D case.
5 Discussion and final remarks

In this work, the standard procedure to compute transport coefficients has been used in order to obtain the thermal conductivity of a dilute bidimensional gas. As mentioned above, the temperature dependence $T^{1/2}$ and the proportionality constant, for the hard disks case, agrees with Ref. [1] whose result is in general quoted as the low density limit and often used as a normalizing of a dilute bidimensional gas. As mentioned above, the temperature dependence

In this work, the standard procedure to compute transport coefficients has been used in order to obtain the thermal conductivity for hard spheres and Maxwellian molecules interactions and observed that the dependence with the temperature is the same as in the three dimensional case for both models. It is of the authors’ opinion that the explicit and systematic fashion in which this work is presented will be useful for generalizations to other dimensions, potentials and transport coefficients. To the authors’ knowledge, this type of calculation has not been published in the available literature and could also be used as a baseline for the calculation of bidimensional transport coefficients for relativistic dilute gases which will be reported elsewhere.

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Appendix

To calculate the integral $I$ given in Eq. (58), we follow the steps in Ref. [14] by changing to relative and center of mass velocity variables

$$\vec{g} = \vec{c} - \vec{c}_1 \quad \vec{G} = \vec{c}_1 + \vec{c},$$

and thus

$$\vec{c} = \frac{\vec{G} + \vec{g}}{2} \quad \vec{c}_1 = \frac{\vec{G} - \vec{g}}{2},$$

with the corresponding Jacobian being $J = 2^{-2}$. By introducing these new variables in Eq. (58), $I$ takes the form

$$I = \frac{1}{32} \int f^{(0)} f_1^{(0)} \left( \frac{m}{2kT} \right) \left( \frac{G^2 + 2\vec{G} \cdot \vec{g} + g^2}{4} \right) - 2 \left( \vec{G} + \vec{g} \right) \cdot \left[ \vec{G} \cdot (\vec{g} \cdot \vec{g} - \vec{g} \vec{g}) \right] g \sigma d\Omega d^2 G d^2 g. \quad (55)$$

Remembering the parameter $\beta = m/2kT$, one has for the equilibrium distribution function

$$f^{(0)} = (\frac{n}{\pi}) \frac{1}{2} \pi e^{-\beta \vec{c}^2}, \quad (56)$$

and

$$I = \frac{1}{32} \left( \frac{n}{\pi} \right)^2 \int \exp \left[ \frac{\beta}{2} G^2 \right] \left[ \beta \left( \frac{G^2 + 2G \cdot g + g^2}{4} \right) - 2 \right] \left( \vec{G} + \vec{g} \right) \cdot \left[ \vec{G} \cdot (\vec{g} \cdot \vec{g} - \vec{g} \vec{g}) \right] g \sigma (\chi, g) d\chi d^2 G d^2 g. \quad (57)$$

The integration with respect to $\vec{G}$ can be separate in two terms as follows

$$\int \exp \left[ \frac{-\beta}{2} G^2 \right] \left[ \beta \left( \frac{G^2 + 2G \cdot g + g^2}{4} \right) - 2 \right] \left( \vec{G} + \vec{g} \right) \cdot \left[ \vec{G} \cdot (\vec{g} \cdot \vec{g} - \vec{g} \vec{g}) \right] g \sigma (\chi, g) d\chi d^2 G d^2 g =$$

$$\int \exp \left[ \frac{-\beta}{2} G^2 \right] \left[ \beta \left( \frac{G^2 + 2G \cdot g + g^2}{4} \right) - 2 \right] \vec{G} \cdot \left[ \vec{G} \cdot (\vec{g} \cdot \vec{g} - \vec{g} \vec{g}) \right] g \sigma (\chi, g) d\chi d^2 G d^2 g$$

$$+ \int \exp \left[ \frac{-\beta}{2} G^2 \right] \left[ \beta \left( \frac{G^2 + 2G \cdot g + g^2}{4} \right) - 2 \right] \vec{g} \cdot \left[ \vec{G} \cdot (\vec{g} \cdot \vec{g} - \vec{g} \vec{g}) \right] g \sigma (\chi, g) d\chi d^2 G d^2 g. \quad (58)$$

For the first term (second line in Eq. (58)), since $\vec{G} \cdot \left[ \vec{G} \cdot (\vec{g} \cdot \vec{g} - \vec{g} \vec{g}) \right] = \sum_{i,j=1}^{2} G_i G_j (g_i g_j - g_i g_j)$, one has

$$\sum_{i,j=1}^{2} (g_i g_j' - g_i g_j) \int \exp \left[ \frac{-\beta}{2} G^2 \right] \left[ \beta \left( \frac{G^2 + 2G \cdot g + g^2}{4} \right) - 2 \right] G_i G_j d^2 G \quad (59)$$

$$= \sum_{i,j=1}^{2} (g_i g_j' - g_i g_j) \int \exp \left[ \frac{-\beta}{2} G^2 \right] \left[ \frac{G^2 + g^2}{4} - 2 \right] G_i G_j d^2 G,$$
which vanishes since the integral is proportional to $\delta_{ij}$ and
\[ (g_i'g_j' - g_ig_j)\delta_{ij} = g'^2 - g^2 = 0. \] (60)

For the second term (third line in Eq. (58)), considering that $\bar{G} \cdot (\bar{g}' \bar{g}' - \bar{g} \bar{g}) = \sum_{i,j=1}^2 g_iG_j (g_i'g_j' - g_ig_j)$, the integration in $\bar{G}$ yields
\[ \sum_{i,j=1}^2 g_i (g_i'g_j' - g_ig_j) g \int \exp \left[ -\frac{\beta}{2}G^2 \right] \left[ \beta \left( \frac{G^2 + 2\bar{G} \cdot \bar{g} + g^2}{4} \right) - 2 \right] G_j d^2G = \sum_{i,j=1}^2 g_i (g_i'g_j' - g_ig_j) g_j \] (61)

and thus
\[ I = \frac{n^2\beta}{32\pi} \sum_{i,j=1}^2 \int \exp \left( -\frac{\beta}{2}g'^2 \right) (g_i'g_j' - g_ig_j) g_i g_j g \sigma (\chi, g) d\chi d^2g, \] (62)

where the integration over the scattering angle $\chi$ can be performed by aligning the $y$ axis with the initial relative velocity such that
\[ \bar{g} = g (0, 1), \] (63)

and using polar coordinates
\[ \bar{g}' = g (\sin \chi, \cos \chi). \] (64)

Since
\[ \int_0^{2\pi} \sigma (g, \chi) [g_i'g_j' - g_ig_j] g_i g_j d\chi = -g^4 \int_0^{2\pi} \sigma (g, \chi) \sin^2 \chi d\chi \] (65)

we have
\[ I = -\frac{n^2\beta}{32\pi} \int \exp \left( -\frac{\beta}{2}g'^2 \right) g^5 \sin^2 \chi \sigma (\chi, g) d\chi d^2g, \] (66)

which is precisely Eq. (40).

A specific molecular interaction model needs to be introduced in order to perform the $\chi$ integration. For hard disks $\sigma = \frac{d}{2} |\sin (\frac{\chi}{2})|$, thus
\[ I = -\frac{d}{2} \frac{n^2\beta}{32\pi} \int \exp \left( -\frac{\beta}{2}g'^2 \right) g^5 |\sin (\frac{\chi}{2})| \sin^2 \chi d\chi d^2g \] (67)

and integrating over the scattering angle $\int_0^{2\pi} |\sin (\chi/2)| \sin^2 \chi d\chi = 32/15$, one obtains
\[ I = -n^2d\beta^{-5/2} \sqrt{\frac{\pi}{2}} \] (68)

which is the expression used in the main text to obtain the thermal conductivity for this model.

On the other hand, for Maxwellian molecules, since $\chi$ is given by
\[ \chi = \pi - \int_0^{s_{\text{max}}} \frac{2ds}{\sqrt{1 - s^2 - \frac{2s}{mg^2} \Phi (\frac{s}{a})}} \] (69)

where $s = b/r$ and $s_{\text{max}}$ is the positive root of $1 - s^2 - \frac{2s}{mg^2} \Phi (b/s) = 0$, one has
\[ \chi = \pi - \int_0^{\frac{s}{a}} \frac{2ds}{\sqrt{1 - a^2s^2}} = \pi \left( 1 - \frac{1}{a} \right) \] (70)

where $a^2 = 1 + (\kappa/2m) (1/g^2b^2)$ which is the expression for $\chi$ given in Eq. (48). Solving for the impact parameter one obtains
\[ b (\chi) = \sqrt{\frac{m}{kg}} \left( \frac{\pi}{\pi - \chi} \right)^2 - 1 \right)^{-1/2} \] (71)

which leads to Eq. (49) after differentiation. Introducing $\sigma (g, \chi)$ in Eq. (40) yields
\[ I = -\frac{n^2\pi^2}{2\beta^2} \sqrt{\frac{k}{2m}} \Gamma, \] (72)

where we have stated
\[ \Gamma = \int \sin^2 \chi (\chi (2\pi - \chi))^{-3/2} d\chi. \] (73)

The integral in equation (73) can be solved numerically to obtain $\Gamma \approx 0.21$. 
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