On the Influence of Uniform Singularity Distributions

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Abstract

Exact expressions for three-dimensional potential and force field due to uniform singularity distributed on a finite flat rectangular surface have been presented. The expressions, valid throughout the physical domain, have been found to be consistent with other expressions available as special cases of the same problem. Very accurate estimates of the capacitance of a unit square plate and a unit cube have been made using them.

Key words:
inverse square law, singularity, potential, field, boundary element method, capacitance

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1 Introduction

The importance of inverse laws has been acknowledged in various branches of science and technology for a long time. Much of the contemporary physics is also dominated by the effects of inverse laws in various guises. Whenever, a particular physical phenomenon is modeled using sources or sinks, the inverse laws come into play. These laws are found to be crucially important in gravitation, electromagnetics, ideal fluid dynamics, Stoke’s flow, acoustics, thermodynamics and many other fields. In fact, a large part of the classical physics, when assumed non-dissipative, can be described by some form of the inverse laws such as the Laplace’s and Poisson’s equations. These two linear second order partial differential equations have been considered to be among the most important differential equations in the whole of classical physics. As a result, estimation of the effects of the inverse laws has been known to be extremely important in many branches of science and technology [1].

While the effect of point sources and sinks can be easily computed, it has not been possible to obtain closed form expressions for computing the effects of distributed sources, except for very simple cases. But, since in many of the real-life problems the singularities are found to be distributed on surfaces of various shapes and sizes, it has been customary to represent them using the simple shapes for which closed form expressions are known, or simply by assuming the surface to be composed of a large number of point sources. These approximations, besides being computationally rather expensive, turn out to be significantly restricted and inaccurate.

In this work, we have presented the closed form expressions of potential and field due to a uniform distribution of source on a flat surface. Using these expressions it has been possible to find the potential and field accurately in the complete physical domain, including the critical near-field domain. Especially, the sharp changes and discontinuities which characterize the near-field domain have been easily reproduced. As a result of the numerical experiments carried out to establish the accuracy of the proposed method, it has been possible to estimate the amount of discretization required to predict the field properties up to a desired level of accuracy.

Although in the present work we have concentrated only on source distribution on flat surfaces, surfaces of other shapes, including curved ones, can be similarly handled through the use of proper geometric transformations. Similar expressions may also be used in dynamic situations where the assumption of quasi-static holds. Since the expressions are analytic and valid for the complete physical domain, and no approximations regarding the size or shape of the singular surface have been made during their derivation, their application is not limited by the proximity of other singular surfaces or their
curvature. In fact, it has been well-known that most of the difficulties in the
earlier methods arose because of nodal concentration of singularities which led
to various mathematical difficulties and to the infamous numerical boundary
layers [2,3]. Through the use of the expressions presented in this work, it is
possible to truly model the effect of distributed singularities precisely. As a
result, the problem of mathematical singularities does not arise at all, and
the real, physical singularities can be dealt with in a straight-forward manner.
Moreover, the requirement of developing special formulations (despite their
mathematical elegance and efficiency) in order to cope with the singularities
in various guises does not arise.

As an illustration of the use of the presented expressions, we have developed
a boundary element method solver, namely, the Nearly Exact Boundary El-
lement Method (NEBEM) solver. Using this solver, we have computed the
capacitances of a unit square plate and a unit cube to very high accuracy.
These problems have been considered to be two major unsolved problems of
electrostatic theory [4,5,6,7,8,9,10,11,12,13,14,15] and no analytical solu-
tions for these problems have been obtained so far. The capacitance values estimated
by the present method have been compared with very accurate results avail-
able in the literature (using boundary element methods (BEM) and others).
The comparison testifies to the accuracy of the NEBEM solver and, hence,
the usefulness of the presented expressions.

2 Theory

The expression for potential at a point \((X, Y, Z)\) in free space due to uniform
source distributed on a rectangular flat surface having corners situated at
\((x_1, z_1)\) and \((x_2, z_2)\) as shown in Fig.1 is known to be a multiple of

\[
\phi(X, Y, Z) = \frac{dx\, dz}{\sqrt{(X - x)^2 + Y^2 + (Z - z)^2}}
\]  

(1)

where the value of the multiple depends upon the strength of the source and
other physical considerations. Here, the surface under consideration, as well
as the origin of the coordinate system is on the \(XZ\) plane. The denominator
within the integrals can be easily interpreted to be the distance between the
point \((X, Y, Z)\) at which the potential is being evaluated and an infinitesimal
element on the surface. The closed form expression for \(\phi(X, Y, Z)\) is as follows:

\[
\phi(X, Y, Z) =
\]
Fig. 1. A rectangular surface with uniform distributed source

\[(X - x_1) \ln \left( \frac{D_{12} - (Z - z_2)}{D_{11} - (Z - z_1)} \right) + (X - x_2) \ln \left( \frac{D_{21} - (Z - z_1)}{D_{22} - (Z - z_2)} \right)\]
\[+ (Z - z_1) \ln \left( \frac{D_{21} - (X - x_2)}{D_{11} - (X - x_1)} \right) + (Z - z_2) \ln \left( \frac{D_{12} - (X - x_1)}{D_{22} - (X - x_2)} \right)\]
\[+ \frac{i |Y|}{2}\]
\[+ S_1 \left( \tanh^{-1} \left( \frac{R_1 + i I_1}{D_{11} |Z - z_1|} \right) - \tanh^{-1} \left( \frac{R_1 - i I_1}{D_{11} |Z - z_1|} \right) \right)\]
\[+ \tanh^{-1} \left( \frac{R_1 - i I_2}{D_{21} |Z - z_1|} \right) + \tanh^{-1} \left( \frac{R_1 + i I_2}{D_{21} |Z - z_1|} \right)\]
\[+ S_2 \left( \tanh^{-1} \left( \frac{R_2 + i I_2}{D_{22} |Z - z_2|} \right) - \tanh^{-1} \left( \frac{R_2 - i I_2}{D_{22} |Z - z_2|} \right) \right)\]
\[+ \tanh^{-1} \left( \frac{R_2 + i I_1}{D_{21} |Z - z_2|} \right) - \tanh^{-1} \left( \frac{R_2 - i I_1}{D_{21} |Z - z_2|} \right) \right)\]
\[-2 \pi Y\]

(2)

where

\[D_{11} = \sqrt{(X - x_1)^2 + Y^2 + (Z - z_1)^2} ; D_{12} = \sqrt{(X - x_1)^2 + Y^2 + (Z - z_2)^2}\]
\[D_{21} = \sqrt{(X - x_2)^2 + Y^2 + (Z - z_1)^2} ; D_{22} = \sqrt{(X - x_2)^2 + Y^2 + (Z - z_2)^2}\]
\[R_1 = Y^2 + (Z - z_1)^2 ; R_2 = Y^2 + (Z - z_2)^2\]
\[I_1 = (X - x_1) |Y| ; I_2 = (X - x_2) |Y| ; S_1 = \text{sign}(z_1 - Z) ; S_2 = \text{sign}(z_2 - Z)\]

Similarly, the force field for the above problem is given as a multiple of

\[\vec{F}(X, Y, Z) = \int \int \frac{\hat{r} dx dz}{r^2}\]

(3)
where $\mathbf{r}$ is the displacement vector from a small surface element to the $(X, Y, Z)$ point where the force field is being evaluated. Eqn.(3) has also been integrated in order to get exact expressions to estimate the force fields in the $X$, $Y$ and $Z$ directions. These expressions, valid for the complete physical domain, are as follows:

$$F_x(X, Y, Z) = \ln \left( \frac{D_{11} - (Z - z_1)}{D_{12} - (Z - z_2)} \right) + \ln \left( \frac{D_{22} - (Z - z_2)}{D_{21} - (Z - z_1)} \right) \tag{4}$$

$$F_y(X, Y, Z) = -\frac{i}{2} \operatorname{Sign}(Y) \left( S_1 \left( \tanh^{-1} \left( \frac{R_1 + i I_1}{D_{11}|Z - z_1|} \right) - \tanh^{-1} \left( \frac{R_1 - i I_1}{D_{11}|Z - z_1|} \right) \right) + \tanh^{-1} \left( \frac{R_1 - i I_2}{D_{21}|Z - z_1|} \right) + \tanh^{-1} \left( \frac{R_1 + i I_2}{D_{21}|Z - z_1|} \right) \right)$$

$$+ S_2 \left( \tanh^{-1} \left( \frac{R_2 + i I_2}{D_{22}|Z - z_2|} \right) - \tanh^{-1} \left( \frac{R_2 - i I_2}{D_{22}|Z - z_2|} \right) \right) + \tanh^{-1} \left( \frac{R_2 + i I_1}{D_{21}|Z - z_2|} \right) - \tanh^{-1} \left( \frac{R_2 - i I_1}{D_{21}|Z - z_2|} \right) \right) + C \tag{5}$$

$$F_z(X, Y, Z) = \ln \left( \frac{D_{11} - (X - x_1)}{D_{12} - (X - x_2)} \right) + \ln \left( \frac{D_{22} - (X - x_2)}{D_{21} - (X - x_1)} \right) \tag{6}$$

In Eqn.(5), $C$ is a constant of integration as follows:

$$C = \begin{cases} 0 & \text{if outside the extent of the flat surface} \\ 2\pi & \text{if inside the extent of the surface and } Y > 0 \\ -2\pi & \text{if inside the extent of the surface and } Y < 0 \end{cases}$$

The above closed-form integrations can be useful in the mathematical modeling of physical processes governed by the inverse square laws as designated by Eqns.(1) and (3). Eqns. (2) and (4)-(6), being exact and valid throughout the physical domain, can be used to formulate and solve multi-scale problems involving Dirichlet, Neumann or Robin boundary conditions.
It is well known that the potential at the centroid of a flat rectangular surface of same dimensions and having source uniformly distributed on it is as follows:

$$\phi(0, 0, 0) = 2(a \log\left(\frac{\sqrt{a^2 + b^2} + b}{a}\right) + b \log\left(\frac{\sqrt{a^2 + b^2} + a}{b}\right))$$

(7)

where $a$ and $b$ are the sides of the rectangular surface. Considering the coordinate origin at the centroid of the element shown in Fig.1 and $x_2 - x_1 = a$, $z_2 - z_1 = b$, by simple algebraic manipulation we can easily show that the expression for potential given by Eqn.(2) reduces to Eqn.(7). In addition, if the point $P(X, Y, Z)$ is infinitely far away from the surface on which the singularity is distributed, the following relations can be easily shown to be true:

$$X = X - x_1 = X - x_2$$
$$Z = Z - z_1 = Z - z_2$$
$$I_1 = I_2 = X|Y|$$
$$R_1 = R_2 = Y^2 + Z^2$$
$$D_{11} = D_{12} = D_{21} = D_{22} = \sqrt{X^2 + Y^2 + Z^2}$$

Substituting the above in Eqn.(2), we find that at a point infinitely far away from the influencing element, the potential $\phi$ is zero. Thus, the presented expressions have been found to be consistent with results in the near-field, as well as the far-field.

As discussed in the Introduction, the above expressions (with suitable constants) have been used to compute the capacitances of a unit square plate and a unit cube in the framework of BEM. The resulting BEM solver has been christened as the Nearly Exact BEM (NEBEM) solver. The details of the solver have been presented in several recent communications, e.g., [16] and we will refrain from elaborating on it here.

3 Results

In order to establish the accuracy of the expressions in Eqns.(2) and (4)-(6), we have computed the potential and field distributions of a unit flat square ($1\text{unit} \times 1\text{unit}$) conducting plate carrying uniform unit singularity density ($1\text{unit}/m^2$). Results computed using the exact expressions have been compared with those computed using a conventional zero-th order piecewise constant BEM with varying amount of discretization ($1 \times 1$, $10 \times 10$, $100 \times 100$ and $1000 \times 1000$ elements). For the line plots below, we have presented computations either along a diagonal which originates at $(-1.5, -1.5, -1.5)$ and ends at $(1.5, 1.5, 1.5)$ (please refer to Fig.1), or one which runs parallel to X-
or Z-axis very close to the edge of the element, just 10 nanounits away from it.

- Potential along the diagonals and edges: From the Figs. 2 and 3, it is clear that the usual BEM solver produces acceptable results only after the plate is discretized into $100 \times 100$ elements. The observation is true along the diagonal line, as well as the line along the edge. For the latter, an oscillation in the usual BEM results is observed for all the discretizations while the exact expressions produce quite smooth results.

Fig. 2. Comparison of potential distribution along diagonal.

- Force components along the diagonals and edges: The $X$ and $Y$ force components for the above two cases have been presented in Figs. 4, 5 and 6. Comments made above are equally applicable in these cases, only in a more significant manner. This is expected because forces are obtained as gradients of potential and, as a result, the short-comings in the computation of potential are amplified in the estimation of force components. As a result, even a $1000 \times 1000$ discretization fails to yield satisfactory results. The results from the exact expressions are excellent throughout.

- Surface plot of potential: To help visualization of the potential field, we have presented a surface plot of the potential on the conducting plate in Fig. 7. Please note that these are values on the plate itself.

Fig. 3. Comparison of potential distribution along edge.
• Surface plot of force components: Force surfaces in the $X$ and $Y$ directions are presented in the following Figs. 8 and 9. These force components have been computed at a distance of only 10 nanounits from the surface of the square plate. The sharp changes in the magnitude of these force components are found to be accurately estimated by the new expressions.

• Error plot along diagonals for potential and force components: In Figs. 10 and 11, we have presented the normalized error, defined as \((\text{Approximate} - \text{Exact})/\text{Exact}\) as a function of distance. One important fact that is immedi-
Fig. 7. Potential distribution on the plate.

Fig. 8. $F_x$ distribution at $Y = 10\text{nanounits}$

Fig. 9. $F_y$ distribution at $Y = 10\text{nanounits}$

ately apparent is that the error in force field computation is larger by almost an order of magnitude than that in the computation of potential. Moreover, it is clear that the usual BEM approximation ($1 \times 1$) is unacceptable as soon as the distance of the evaluation point is of the same order of magnitude as the size of the element. For potential calculations, the error exceeds 1%, while for force field it exceeds 10% even when the distance is twice the size of the plate. At lesser distances, the error increases rapidly. Only after the unit plate is further segmented into $100 \times 100$ elements, the computed results
are acceptable (error less than 1%) throughout the domain. But, under the framework of usual BEM, this translates into 10000 elements in place of 1!

Use of Gaussian quadrature can alleviate the problem to a certain extent, but is unlikely to provide a complete cure.

![Graph](image1.png)

**Fig. 10.** Variation of Error in computing potential along a diagonal.

![Graph](image2.png)

**Fig. 11.** Variation of Error in computing force in Y along a diagonal.

The above results prove the precision of the closed-form integrals given in Eqn.(2) and Eqns.(4)-(6) which the NEBEM solver uses as its foundation expressions. As a further application of them, we have computed the capacitance of a unit square plate and a unit square cube using the NEBEM solver. In Table1, we have presented a comparison of the values of capacitances as calculated by [4,5,6,7,8,9,10,11,12,13,14] and our estimations. In [8,10,11], the authors calculated the capacitances of the plate and cube (among many other things) using numerical path integration method, while in [12,13,14], the authors used the "random walk on the boundary" method. In [4,5,6,7,9], various forms of the BEM were used to solve the problem. In [7,9], the authors improved upon the BEM quite substantially by proposing new and more accurate expressions for the evaluation of the elements in the influence matrix. However, these expressions are not valid for the entire physical domain and thus, necessitated the use of several expressions for evaluating potential and field.
on the charged surfaces and other field points. It was also necessary to subdivide the segments on the boundary to satisfy certain approximations used to deduce the expressions [17]. Despite these limitations, it must be said that the improvement was significant through the use of these expressions and their results belong to the most accurate ones available for the present problem. In order to enhance the accuracy of the results [7,9,13] also used extrapolation techniques. Finally, in [15], new upper and lower bounds of the capacitance of an isolated cube have been presented using random walk methods. New estimates for the two capacitances (preliminary for the plate) have also been presented here. In addition, some remarks on the usual BEM have been made which, although true in general, does not apply to the NEBEM solver. It may be noted here that our results, as presented in Table 1 have been obtained through a straight-forward application of the NEBEM solver, without taking recourse to any extrapolation technique. In order to model the sharp change in the charge density near the corners and edges, finer discretization in these regions have been used. Although it is difficult to comment regarding which is the best result among the published ones, it is clear from the table that the new expressions indeed lead to very accurate results which are well within the acceptable range.

**Table 1**
Comparison of capacitance values

| Reference | Method                                      | Plate          | Cube            |
|-----------|---------------------------------------------|----------------|-----------------|
| [4]       | Surface Charge                              | 0.3607         | -               |
| [5]       | Surface Charge                              | 0.362          | 0.6555          |
| [6]       | Surface Charge                              | 0.367          | -               |
| [7]       | Refined Surface Charge and Extrapolation    | 0.3667892 ± 1.1 × 10⁻⁶ | 0.6606747 ± 5 × 10⁻⁷ |
| [8]       | Brownian Dynamics                           | -              | 0.663           |
| [9]       | Refined Boundary Element and Extrapolation  | 0.3667874 ± 1 × 10⁻⁷ | 0.6606785 ± 6 × 10⁻⁷ |
| [10]      | Refined Brownian Dynamics                   | -              | 0.660675 ± 1 × 10⁻⁵ |
| [11]      | Numerical Path Integration                  | 0.36684        | 0.66069         |
| [12]      | Walk on Spheres                             | -              | 0.660683 ± 5 × 10⁻⁶ |
| [13]      | Modified Walk on Spheres                    | -              | 0.6606867 ± 1.2 × 10⁻⁶ |
| [14]      | Random Walk on the Boundary                 | -              | 0.6606780 ± 2.7 × 10⁻⁷ |
| [15]      | Random Walk                                 | 0.36 ± 0.01    | 0.6606 ± 0.0001 |
| This work | NEBEM                                        | 0.3667869      | 0.6606746       |
Besides obtaining extremely accurate results, there are several other advantages of using the presented expressions for computation of potential and field, in general. For example, for the NEBEM solver, the boundary condition on a particular element need not be located only at the centroid of the element. It can be anywhere on the element, except probably on the edges, or the point can also be situated very close to the singular surface, while not being exactly on it. These capabilities can be of significant advantages in specific situations. In addition, since the singularities are now truly distributed on the surface, and there is no concept of nodal point in the formulation, the aspect ratio (ratio of length to breadth) of the elements can vary as wildly as necessary. The question of mathematical singularities, and the resulting numerical boundary layer, does not arise at all. As a result, it should be straight-forward to deal with real physical singularities such as close proximity of two singular surfaces, or those due to degeneracy of singular surfaces. All these advantages, along with the fact that the inverse square is an almost ubiquitous feature of the natural world, makes the expressions presented in this work particularly suitable for multi-physics multi-scale problems.

4 Conclusions

Exact expressions for potential and force field due to uniform singularity distribution on a flat surface has been presented. The expressions have been found to yield very accurate results in the complete physical domain. Of special importance is their ability to reproduce the complicated field structure in the near-field region. The errors inducted in assuming discrete point sources to represent a continuous distribution have been illustrated. Accurate estimates of the capacitance of a unit square plate and that of a unit cube have been made using the Nearly Exact BEM (NEBEM) solver which uses Eqns. (2), (4)-(6) as its foundation expressions. Comparison of the obtained results with very accurate results available in the literature has confirmed the accuracy of the closed-form integrals.

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