Boundary Term in the Gravitational Action is the Heat Content of the Null surfaces

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Abstract

The Einstein-Hilbert Lagrangian has no well-defined variational derivative with respect to the metric. This issue has to be tackled by adding a suitable surface term to the action, which is a peculiar feature of gravity. We also know that null surfaces in spacetime exhibit (observer-dependent) thermodynamic features. This suggests a possible thermodynamic interpretation of the boundary term when the boundary is a null surface. For timelike/spacelike surfaces it is easy to construct the boundary term but there are some subtleties in the case of the null surface. The correct form of boundary term for null surfaces was obtained recently from first principles. We show that this surface term, as well as its variation, have a direct thermodynamic interpretation in terms of a heat density of null surfaces. The implications of the result are discussed.

1 Introduction

The standard action principle in general relativity, based on the Lagrangian density $R \sqrt{-g}$ does not lead to a well-defined variational principle unless we take care of the boundary contributions in some suitable manner [1–6]. There are two possible ways of handling this situation.

The first is to separate the action into a bulk term — which is quadratic in the first derivatives of the metric — and a boundary term arising from the second derivatives of the metric. Once this is done, one can simply discard the boundary term and work with the quadratic action. While the separation is foliation dependent the resulting field equations are not and everything works out satisfactorily.

The second procedure is to add a suitable boundary term to the Einstein-Hilbert action and arrange matters such that the variation of the boundary term cancels the unwanted surface variation terms coming from the Einstein-Hilbert action. The resulting action principle will lead to the standard field equations if the 3-metric is fixed at the boundary.

In either approach the nature of the boundary term — arising from the Einstein-Hilbert action or due to the addition of the extra term — depends on the nature of the boundary. For example, in the case

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of a space-like or time-like boundaries, the additional term can be built from the trace of the extrinsic curvature $\gamma_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$. In the case of a null boundary, or in those parts of the boundary which are null, one cannot define the extrinsic curvature in a natural manner. It can, however, be shown that in the case of null boundaries the surface term is essentially the sum $\Theta + \kappa$ where $\Theta$ is the expansion and $\kappa$ is the surface gravity of the null congruence defining the boundary surface [7–13].

The existence of the null surfaces is a unique feature of gravitational theory. Acting as one way membranes, the null surfaces can restrict access to spacetime events for a class of observers confined to one side of the surface. Familiar examples include event horizons in black hole spacetimes, de Sitter spacetime and the case of uniformly accelerated observers in flat spacetime. In all these cases, observers who perceive a horizon attribute to it a temperature $T$. Further, the lack of accessibility of spacetime events beyond the horizon suggests attributing an entropy density $s$ to the null surface. Together, this allows us to associate a heat density $T s$ to any patch of null surface [14–20].

Such an association of thermodynamic variables with spacetime finds a natural backdrop in the emergent gravity paradigm in which one attempts to describe gravity in a manner analogous to, say, fluid dynamics or elasticity. In this approach gravitational dynamics is not described in terms of geometrical concepts but instead is phrased completely in thermodynamic language.¹ The evolution of geometry in a bulk region of space, for e.g., is described in terms of heating/cooling of the thermodynamic degrees of freedom driven by the disparity between the degrees of freedom on the surface and bulk regions of spacetime [21–26].

The existence of such an interpretation suggests that even the conventional approach to gravity should exhibit traces of its underlying thermodynamic character. In particular, we would expect any peculiar feature of gravitational action principle — not usually found in other field theories — to be connected with the fact that gravitational dynamics has an underlying thermodynamic interpretation. As we noted before, a striking feature of gravitational dynamics is the existence of a surface term in the action principle. This fact, coupled with the second peculiar feature of gravity viz. the existence of null surfaces which can act as one way membrane, suggests that there must exist a direct connection between the surface term in the action principle and the thermodynamics of null surfaces [27, 28]. The purpose of this paper is to establish this result. (Such a connection was established in some of the previous works (see e.g. [20]), treating the null surface as a limiting case of timelike surfaces. Recently, we provided a first principle derivation of the boundary term for the null surface, without using any limiting procedure [7, 8]. Here we will not use any limiting procedures but instead will use this more rigorous approach. Further, unlike previous works, our result will be completely general and we will not require any special assumptions regarding the nature of the spacetime or other approximations.)

We will show that there exists a simple but at the same time very general interpretation of the surface term in the Einstein-Hilbert action evaluated on a null surface and the heat density of the null surface. When the null surface is described using Gaussian null coordinates, the boundary term in the action reduces to the heat density $T s$ of the null surface. Moreover, the variation of the surface term arising from a “flow” along the null congruence induced by the transformation $x^a \to x^a + \ell^a$ can be expressed in the form of the variation $T \delta s$, once again providing a thermodynamic condition. These results provide yet another link between the gravitational dynamics and spacetime thermodynamics in the context of null surfaces.

In what follows we will use the following mostly positive signature convention for the metric and shall set the fundamental constants $c$ and $G$ to unity. The Roman indices $a, b, \ldots$ will stand for four dimensional

¹Note that this is conceptually very different from attempts to derive gravitational field equations from thermodynamic considerations, finally obtaining $G_{ab} = 8\pi T_{ab}$ with the left hand side still being treated as having a geometric interpretation. In contrast, the emergent gravity paradigm interprets the field equation entirely in thermodynamic language.
spacetime coordinates and Greek indices $\mu, \nu, \ldots$ will stand for spatial coordinates.

The rest of the paper is organized as follows: We start in Section 2 by computing the boundary contribution of the total gravitational action associated with null surfaces and exploring its thermodynamic properties. Subsequently in Section 3 we have discussed a certain variation of the null boundary term having a nice thermodynamic interpretation. Finally we conclude with a discussion on our results. Relevant calculations are presented in Appendix A, Appendix C and Appendix D respectively.

2 Boundary term in the action is the Heat content

Our first task will be to show that the boundary term which arises in the Einstein-Hilbert action, when separated into a bulk and a boundary term, has a natural thermodynamic interpretation as the heat density of the null boundaries. For this purpose, to uncover the thermodynamic nature of gravitational dynamics, using the boundary term it is convenient to work with the following variables

\[ f^{ab} = \sqrt{-g}g^{ab}; \quad N^c_{ab} = -\Gamma^c_{ab} + (1/2)(\delta^c_a \Gamma^m_{mb} + \delta^c_b \Gamma^m_{ma}) \]

In terms of these variables, the Einstein-Hilbert action separates neatly into a bulk and a boundary term:

\[ \sqrt{-g}R = \frac{1}{2}N^c_{ab}\partial_c f^{ab} + \partial_c (-f^{ab}N^c_{ab}) \equiv \sqrt{-g}L_{\text{bulk}} + \sqrt{-g}L_{\text{sur}} \]

Thus one can immediately identify the term $\partial_c (-f^{ab}N^c_{ab})$ as the one leading to the boundary term of the Einstein-Hilbert action, which when integrated over a boundary surface $\phi = \text{constant}$, yields the boundary term, $-s_c f^{ab} N^c_{ab}$ with normal $s_c = \nabla_c \phi$. In what follows we will concentrate on the structure of this boundary term, in the context of null surfaces (For a brief discussion on the corresponding situation for spacelike/timelike surfaces we refer the reader to Appendix A).

Thus our job is to start from the boundary term $-s_c f^{ab} N^c_{ab}$ of the Einstein-Hilbert action, integrated over the boundary surface and then considering the thermodynamic interpretation as the boundary surface becomes null. In this spirit, we shall first rewrite the boundary term in a different manner, valid for an arbitrary surface and then we will specialize to the null surface. For clarity, we present below the full structure of the boundary term,

\[ S = -\frac{1}{16\pi G} \int_{\partial \mathcal{V}} d^3x \ s_c f^{ac} N^c_{ab}, \]

where, as mentioned earlier $\partial \mathcal{V}$ is the $\phi = \text{constant}$ surface and $s_c = \nabla_c \phi$ is the normal to the desired boundary surface, which we keep arbitrary for the moment. By expanding out the Christoffel symbol, the integrand can be rewritten as:

\[ s_c f^{ac} N^c_{ab} = f^{ab} \left[ -s_c \Gamma^c_{ab} + \frac{1}{2} s_a \Gamma^d_{bd} + \frac{1}{2} s_b \Gamma^d_{ad} \right] \]

The product of the normal vector and the Christoffel symbols can be expressed in terms of covariant derivatives of the normal vector, in particular, we can use the following results,

\[ \nabla_a s_b = \partial_a s_b - \Gamma^c_{ab} s_c; \quad \nabla_a s^a = \partial_a s^a + \Gamma^a_{ac} s^c, \]

which along with the definition $f^{ab} = \sqrt{-g}g^{ab}$ can be used to express the boundary term as,

\[ s_c f^{ab} N^c_{ab} = \sqrt{-g}g^{ab} \left( \nabla_a s_b - \partial_a s_b \right) + \sqrt{-g} \left( \nabla_a s^a - \partial_a s^a \right) \]
The Null Coordinate system, which can be constructed for any arbitrary null surface, see \[ \text{boundary, i.e.,} \]

metric components vanish in the null limit (as is evident from the GNC parametrization, see \[ \text{Note that the boundary term is not covariant, since it explicitly depends on the Christoffel symbol.} \]

We will now rewrite the first term in the integrand, namely

\[
\int d^3x \left( 2\nabla_a s^a - g^{ab} \partial_a s_b - \partial_a s^a \right). \tag{6}
\]

The above expression is another manifestation of the same, since it depends on the partial derivatives of the normal vector. To cast it into the desired form, it is advantageous to introduce the following projector

\[
\Pi^a_b = \delta^a_b + t^a s_b \tag{7}[8], \text{ where } t_b \text{ is an auxiliary vector satisfying } s_a t^a = -1, \text{ such that } s_a \Pi^a_b = 0. \text{ Use of this projector helps us to write down the boundary term in the following manner,}
\]

\[
s_c f^{ab} N^c_{ab} = 2\sqrt{g} \left\{ \Pi^a_b \nabla_a s^b \right\} + \sqrt{g} \left( -g^{ab} \partial_a s_b - \partial_a s^a \right) - 2\sqrt{g} s_b \nabla_a s^b. \tag{7}
\]

The discussion so far is applicable to any boundary, irrespective of whether it is null or spacelike/timelike. Subsequently, we will specialize to the case of a null boundary for which we can take \( s_a = \ell_a \) and \( t_a = k_a \), with the following properties: \( \ell_a \ell^a = 0 = k_a k^a \) and \( \ell_a k^a = -1 \). Thus for null boundaries, the surface term of the Einstein-Hilbert action becomes,

\[
s_c f^{ab} N^c_{ab} \big|_{\text{null}} = 2\sqrt{q} \left\{ \Pi^a_b \nabla_a \ell^b \right\} + \sqrt{q} \left( -g^{ab} \partial_a \ell_b - \partial_a \ell^a \right) - 2\sqrt{q} k^a \ell_b \nabla_a \ell^b. \tag{8}
\]

By simplifying the above expression and to attribute a thermodynamic meaning to this, it is convenient to go to a coordinate system where \( \Phi \) itself is a coordinate. This is very much similar to the Gaussian Null Coordinate system, which can be constructed for any arbitrary null surface, see [7]. In this system of coordinates with \( \ell_a = \nabla_a \Phi \) we have, \( k^a \ell_b \nabla_a \ell^b = -\kappa \), as well as \( \Pi^a_b \nabla_a \ell^b = \Theta + \kappa \). Further since \( \Phi \) itself is a coordinate, we immediately have \( \partial_a \ell_b = 0 \). Thus the surface term becomes,

\[
s_c f^{ab} N^c_{ab} \big|_{\text{null}} = 2\sqrt{q} (\Theta + \kappa) + 2\kappa \sqrt{q} - \sqrt{q} \partial_a \ell^a. \tag{9}
\]

In order to evaluate the last term in this coordinate system, we start with an alternative definition for \( \kappa \), which is

\[
\kappa = -\frac{1}{2} k^a \nabla_a \ell^2 = \frac{1}{2} \partial_s g^{\Phi \Phi}, \tag{10}
\]

where we have chosen \( k^\Phi = -1 \), in order to satisfy the relation \( \ell_a k^a = -1 \). Further, \( \partial_a \ell^a = \partial_s g^{\Phi \Phi} \) whose only non-zero component, on the null surface becomes, \( \partial_s g^{\Phi \Phi} \). This is because derivative of all the other metric components vanish in the null limit (as is evident from the GNC parametrization, see Appendix B) we obtain the surface term on a null surface to be:

\[
S = -\frac{1}{16\pi G} \int_{\partial V} d^3x \, s_c f^{ab} N^c_{ab} \big|_{\text{null}} = \frac{1}{16\pi G} \int_{\partial V} d^3x \, 2\sqrt{q} \left( \Theta + \kappa \right). \tag{11}
\]

We will now rewrite the first term in the integrand, namely \( \sqrt{q} \Theta \) in the form \( (d\sqrt{q}/d\lambda) \), since \( \Theta = (d \ln \sqrt{q}/d\lambda) \). Thus integrating this expression we will obtain a boundary contribution on the null surface which, as usual, can be ignored. Further, noting that the temperature associated with the null surface generated by the null vector \( \ell_a \) is \( T = (\kappa/2\pi) \) and defining the entropy density as one quarter of the area measure \( s = (\sqrt{q}/4G) \), we find that the surface term is indeed equal to the heat density of the null boundary, i.e.,

\[
S = \int_{\partial V} d^3x \, Ts, \tag{12}
\]

\(^2\text{This is a feature and not a bug. We expect null surfaces in flat spacetime to exhibit thermal properties; so the action cannot vanish in flat spacetime if we use non-inertial coordinates. The correct action for gravity — unlike the Einstein-Hilbert action — satisfies this criterion.}


where, we have neglected the boundary term originating from the expansion of the null generators, namely the $\sqrt{g}\Theta$ term. Since the total derivative term contributes only at the corners of the three dimensional surface, if the boundary null surface is assumed to be smooth, then the total derivative term does not contribute. Thus we have the result that the boundary term of the Einstein-Hilbert action is the heat content of the null boundary $\partial \mathcal{V}$.

### 3 Variation of the null boundary term: Thermodynamic interpretation

It turns out that not only the boundary term, but certain variation of the same has thermodynamic interpretation. In this section we will discuss such a variation and the associated interesting thermodynamic interpretation it presents. We would like to start by briefly mentioning the variation of the integrand of the Einstein-Hilbert action which yields, for arbitrary variations of the metric \[ \delta \left( \sqrt{-g} R \right) = -\partial_c \left( f^{ab} \delta N^c_{ab} \right) + R_{ab} \delta f^{ab}. \] (13)

Integrating over a spacetime volume the above variation of the action can be rewritten as:

\[
\int_{\mathcal{V}} d^4x \delta \left( \sqrt{-g} R \right) = \int_{\mathcal{V}} d^4x R_{ab} \delta f^{ab} - \int_{\partial \mathcal{V}} d^3x s_c f^{ab} \delta N^c_{ab} ,
\]

where the boundary surface $\partial \mathcal{V}$ is taken to be $\phi = \text{constant}$ with an unnormalised normal $s_c = \nabla_c \phi$. On the other hand, we can rewrite the Einstein-Hilbert Lagrangian density as, $\sqrt{-g} R = R_{ab} f^{ab}$, such that its variation becomes, $\delta (\sqrt{-g} R) = R_{ab} \delta f^{ab} + f^{ab} \delta R_{ab}$. Hence Eq. (14) suggests the following identity, $f^{ab} \delta R_{ab} = -\partial_c \left( f^{ab} \delta N^c_{ab} \right)$, which can also be derived starting from first principle computation [24].

As emphasized earlier, the boundary term in the variation of the gravitational action has the structure $s_c f^{ab} \delta N^c_{ab}$, where $s_c = \nabla_c \phi$ is the normal to the boundary hypersurface $\phi = \text{constant}$. We will now demonstrate that this variation also has a simple thermodynamic interpretation for variations of the metric arising out of the ‘flow’ along the null congruence, such that $g^{ab} \rightarrow g^{ab} + (1/2)(\nabla^a \ell^b + \nabla^b \ell^a)$.

Arbitrary variations of the boundary term on a null surface can be computed in a straightforward manner, whose technical details have been delegated to Appendix C and quote here the final result:

\[
\delta \left( \frac{1}{8\pi G} \int_{\partial \mathcal{V}} d^3x \sqrt{-g} (\Theta + \kappa) \right) = -\frac{1}{16\pi G} \int_{\mathcal{V}} d^4x \ f^{ab} \delta R_{ab} + \frac{1}{16\pi G} \int_{\partial \mathcal{V}} d^3x \ s_c \ f^{ab} \delta N^c_{ab} + \frac{1}{16\pi G} \int_{\partial \mathcal{V}} d^3x \ s_c \ f^{ab} \delta \ell^b + \frac{1}{16\pi G} \int_{\partial \mathcal{V}} d^3x \ s_c \ f^{ab} \delta \ell^a .
\]

In the above expression, $\Theta_{ab}$ is the extrinsic curvature associated with the null generator $\ell^a$ and $\Theta$ is the trace of the extrinsic curvature. Further, the quantity $P_a$ stands for a tensor density with the following expression, $P_a = \{ 2k_a (\Theta + \kappa) - k^b (\nabla_a \ell_b + \nabla_b \ell_a) \}$, where $\kappa$ is the non-affinity parameter associated with the null generators and $k^a$ is the auxiliary null vector with $\ell_a k^a = -1$.

The above expression provided a general variation of the null boundary term for arbitrary variations. Our aim is to provide a thermodynamic interpretation for this variation and for this purpose it will be advantageous to consider variations associated with displacements along the null surface, leading to $\delta g^{ab} = (1/2)(\nabla^a \ell^b + \nabla^b \ell^a)$, such that $\delta g_{ab} = -\nabla_a \ell_b$. For this variation induced by the flow along the
null congruence we have:  \( f^{ab} \delta R_{ab} = -\sqrt{-g} \nabla_a (R^b_a e^b) \), \( \delta q^{ab} = \Theta^{ab} + \text{terms proportional to}(\ell^a, k^a) \) and \( \delta \ell^a = \kappa \ell^a \). Thus from Eq. (15) we obtain,

\[
\delta \left\{ \frac{1}{8\pi G} \int_{\partial V} d^2x d\lambda \sqrt{-g} (\Theta + \kappa) \right\} = \frac{1}{16\pi G} \int_V d^2x d\lambda q R_{ab} \ell^a \ell^b + \frac{1}{16\pi G} \int_{\partial V} d^3x \partial_a \left( \sqrt{q} \Pi^a_b \delta \ell^b \right) \\
+ \frac{1}{16\pi G} \int_{\partial V} d^2x d\lambda \sqrt{q} \left\{ \Theta_{ab} \Theta^{ab} - (\Theta + \kappa) \Theta \right\} + \kappa P_{a} \ell^a .
\]

(16)

In the above expression, for the variation considered earlier, we have \( \Pi^a_b \delta \ell^b \) and hence the we obtain, \( \partial_a (\sqrt{q} \Pi^a_b \delta \ell^b) = (1/\sqrt{q}) (d/d\lambda)(\kappa \sqrt{q}) = (d\kappa/d\lambda) + \kappa \Theta \). The \( R_{ab} \ell^a \ell^b \) term appearing in the above variation can be transformed to various geometric quantities associated with the null surface by using the Raychaudhuri equation, which reads

\[
R_{ab} \ell^a \ell^b = \Theta - \left( \Theta_{ab} \Theta^{ab} - \Theta^2 \right) - (1/\sqrt{q}) (d/d\lambda)(\sqrt{q}) \Theta .
\]

(17)

Substitution of the above expression for \( R_{ab} \ell^a \ell^b \) along with the expression for \( \partial_a (\sqrt{q} \Pi^a_b \delta \ell^b) \) and the result that \( \ell_a P^a = -2\Theta \kappa \) leads to the following expression for the variation of the null boundary term associated with the Einstein-Hilbert action,

\[
-\delta \left\{ \frac{1}{8\pi G} \int_{\partial V} d^2x d\lambda \sqrt{-g} (\Theta + \kappa) \right\} = -\frac{1}{16\pi G} \int_V d^2x d\lambda \sqrt{q} \left\{ \kappa \Theta - (\Theta_{ab} \Theta^{ab} - \Theta^2) - \frac{1}{\sqrt{q}} \frac{d\sqrt{q}\Theta}{d\lambda} \right\} \\
- \frac{1}{16\pi G} \int_{\partial V} d^2x d\lambda \sqrt{q} \left\{ \Theta_{ab} \Theta^{ab} - (\Theta + \kappa) \Theta \right\} + \kappa P_{a} \ell^a + \frac{1}{\sqrt{q}} \frac{d}{d\lambda} (\kappa \sqrt{q}) \right\} \\
= \frac{1}{8\pi G} \int_{\partial V} d^2x d\lambda \sqrt{q} \Theta + \frac{1}{8\pi G} \int d^2x \sqrt{q} (\Theta - \kappa) \right\} \left. \lambda_2 \right|_{\lambda_1} .
\]

(18)

Thus, neglecting the boundary term, we obtain,

\[
-\delta \left\{ \frac{1}{8\pi G} \int_{\partial V} d^2x d\lambda \sqrt{q} (\Theta + \kappa) \right\} = \int_{\partial V} d^2x d\lambda \left( \frac{\sqrt{q}}{4} \right) = \int_{\partial V} d^2x T d\lambda
\]

(19)

where \( ds \) corresponds to the rate of change of entropy along the null generator, i.e., \( ds = (ds/d\lambda)d\lambda \). We thus see that not only the boundary term but also its variation has a simple thermodynamic interpretation. Interestingly, the above result can also be arrived at from the first principle, which we have presented in Appendix D for completeness.

4 Conclusions

Three peculiar features of gravitational theories which distinguishes them from other field theories are the following:

(i) Gravity affects the propagation of light rays and hence the causal connection between events in spacetime; no other interaction is capable of doing this. A collective manifestation of this phenomena is exhibited by the null surfaces which can act as one way membranes for a particular class of observers. The observers who perceive the null surface as a horizon, limiting their vision, attributes to it a heat density \( T_s \).
(ii) The most natural action principle for gravity contain second derivative of the dynamical variable. Integrating out these second derivatives lead to a surface term in the action and one needs to do something special to handle this surface term in order to obtain a sensible variational principle. The existence of such a surface term is yet another peculiar feature of gravity and is not prevalent in other field theories.

(iii) Gravitational field equations can be interpreted entirely in a purely thermodynamic language. The heat density of null surfaces plays a crucial role in such a formulation.

In this paper we have provided a simple synthesis of the three peculiar features of gravity listed above. We consider the action principle defined in a region with a boundary \( \partial V \) which could be either completely or partially a null surface. We then evaluate the boundary term on this null surface and show that it has a simple physical interpretation as the heat density of the null surface. We also show that the variations induced by the flow along the null surface can be interpreted as thermodynamic variations, viz., \( T \delta s \).

These results reinforce the already well established idea that gravitational dynamics should be thought of as an emergent phenomena like fluid mechanics or elasticity.

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A Connection with spacelike/timelike surfaces

We have explicitly demonstrated that the term \( s_c N_{ab} f^{ab} \) is related to \( 2\sqrt{-q} (\Theta + \kappa) \) for null surfaces. Thus one may ask what happens for non-null surfaces, i.e. can we relate this to the boundary term of non-null surfaces, namely \( 2K \sqrt{h} \). We will demonstrate that we indeed can. To see this let us start with the boundary term in the gravitational action (no variation is involved), which reads,

\[
s_c N_{ab} f^{ab} = 2\sqrt{-g} \Pi^a \nabla_a s^b + \sqrt{-g} \left( g^{ab} \partial_a s_b - \partial_a s^a \right) - 2\sqrt{-g} t^c s_b \nabla_a s^b \tag{20}\]

If we consider a spacelike surface (e.g., a \( t = \text{constant} \) surface) then we have, \( s_c = -n_c / N \), where \( n_c \) is the normal and \( N \) is the normalization factor. Thus we also have \( t_c = -N n_c \), such that \( s_c t_c = -1 \), since \( n_c n^c = -1 \). Thus the above expression can be expressed entirely in terms of the normal \( n_c \), which yields,

\[
s_c N_{ab} f^{ab} = 2\sqrt{-g} (\delta^a_b + n^a n_b) \nabla_a \left( -\frac{1}{N} n^b \right) + \sqrt{-g} \left[ g^{ab} \partial_a \left( \frac{1}{N} n^b \right) + \partial_b \left( \frac{1}{N} n^b \right) \right] + 2\sqrt{-g} n^a n_b \nabla_a \left( \frac{1}{N} n^b \right)\]

\[
= -2\sqrt{h} n^a \nabla_a \left( \frac{1}{N} n^b \right) + \sqrt{-g} \left[ g^{ab} \partial_a \left( \frac{1}{N} n^b \right) + \partial_b \left( \frac{1}{N} n^b \right) \right] + 2\sqrt{h} n^a n_b \nabla_a \left( \frac{1}{N} n^b \right)\]

\[
= -2K \sqrt{h} + \sqrt{h} \left[ g^{ab} \partial_a (n_b) + \partial_b (n^a) \right] - 2\sqrt{h} n^a \partial_a \ln N + 2\sqrt{h} n^a \partial_b \ln N
\]

\[
= -2K \sqrt{h} + \sqrt{h} \left[ g^{ab} \partial_a (n_b) + \partial_b (n^a) \right] \tag{21}\]
where we have used the fact that \( u^a n^b \nabla_a n_b = 0 \), since the normal \( n_a \) is unit normalized. Let us evaluate
the second term in the above expression, which reads,
\[
g^{ab} \partial_a n_b + \partial_b n^b = g^{tt} \partial_t n_t + g^{\alpha t} \partial_\alpha n_t = -\frac{1}{N^2} \partial_t (-N) + \frac{N^\alpha}{N^2} \partial_\alpha (-N)
\]
Thus if one moves to the synchronous frame such that \( N = 1 \) and \( N^\alpha = 0 \), then it immediately follows
that the above term will vanish and hence the quantity \( s \epsilon N^\epsilon f^{ab} \) becomes \(-2K \sqrt{H}\), the boundary term
associated with the spacelike surface.

B Brief introduction to Gaussian Null Coordinates

In this section we will briefly describe the Gaussian Null Coordinate system (often referred to as GNC),
since many of our results fits in naturally with this coordinate system. In flat spacetime, the null planes
\( X = \pm T \) are viewed as the horizon by comoving observers in Rindler coordinates. The GNC generalizes
the notion of Rindler coordinate system to an arbitrary null surface. The line element in this system
of coordinates takes the following form:
\[
ds^2 = -2ra du^2 + 2udu - 2r \beta_A du dx^A + q_{AB} dx^A dx^B.
\]
The above metric is characterized by six unknown functions, \( \alpha, \beta_A \) and \( q_{AB} \) respectively, all of which are
functions of \((u, r, x^A)\). The surfaces \( u = \text{constant} \) as well as the surface \( r = 0 \) are null surfaces, while all the
other \( r = \text{constant} \) surfaces are spacelike. The Rindler metric, central to flat spacetime thermodynamics,
is just a special case of the above metric, with \( \alpha = \text{constant} \), \( \beta_A = 0 \) and \( q_{AB} = \delta_{AB} \). Thus any result
presented using GNC will hold for any class of null surfaces, which includes the Rindler and black hole
horizons as special cases.

The surface \( r = 0 \) is the one of special interest, having the following normal vector \( \ell_a = \nabla_a r \), such that
\( \ell_a \ell^a|_{r=0} = 0 \). For this null vector we have \( \alpha(u, r, x^A) \) as the non-affinity parameter, since \( \ell^a \nabla_a \ell^b|_{r=0} = \alpha \ell^b \). In addition we need an auxiliary null vector to uniquely characterize the null surface, which we
have denoted as \( k^a \) and has the form \(- (\partial / \partial r)^a\) in the GNC. In this system of coordinates we have
\( \ell^a = (1, 2r\alpha + r^2 \beta^2, r\beta^A) \) and hence \( \partial_a \ell^a|_{r=0} = 2\alpha \), proportional to the non-affinity parameter associated
with the null surface. This is one of the results used in the main text.

The intrinsic geometry associated with the \( r = 0 \) null surface is characterized by the two-metric \( q_{AB} \)
and the null normal \( \ell^a \); being null, it is also tangential to the null surface. On the null surface one can
express, \( \ell^a = (\partial / \partial u)^a \) and hence the null surface can be parameterized by the coordinates \((u, x^A)\). The extrinsic curvature associated with the null surface is given by \( \Theta_{AB} = (1/2) \partial_u q_{AB} \) and the expansion,
i.e., the trace of the extrinsic curvature is give by \( \Theta = \partial_u \ln \sqrt{q} \). Thus integral of \( \Theta \) on the null surface
will contribute only at the \( u = \text{constant} \) boundaries of the null surface, which can be ignored in bulk
integrations by suitable boundary conditions. These are some of the results we have used in the main text.

C Variation of the null boundary term

In this appendix we will present the structure of the variation of the boundary term. First of all, from the
discussion around Eq. (14) we have the following expression for the boundary term,
\[
\int d^3 x \ s \epsilon f^{ab} \delta N^\epsilon_{ab} \ = \ - \int_\mathcal{V} d^4 x f^{ab} \delta R_{ab}.
\]
Concentrating on the integrand of the boundary term appearing on the left hand side of the above expression we obtain,

\[ s_c f^{ab} \delta N^c_{ab} = \sqrt{-g} s_c \nabla_d (\delta g^{cd} - g^{cd} g_{ik} \delta g^{ik}) = \sqrt{-g} \nabla_d (s_c \delta g^{cd}) - \sqrt{-g} (\nabla_d s_c) \delta g^{cd} - \sqrt{-g} s^c \nabla_c (g_{ik} \delta g^{ik}) = \sqrt{-g} \nabla_d (\delta s^d) - \sqrt{-g} (\nabla_c s_c) \delta g^{cd} + 2\sqrt{-g} s^c \nabla_c \delta \ln \sqrt{-g} , \]  

(25)

where, we have assumed that under variation the boundary surface \( \phi = \text{constant} \) does not change, such that \( \delta s_c = 0 \). We can now use the following relation, \( \delta (\nabla_c s^c) = \nabla_c \delta s^c + s^c \nabla_c \delta \ln \sqrt{-g} \) to rewrite Eq. (25) as,

\[ s_c f^{ab} \delta N^c_{ab} = - \sqrt{-g} \nabla_d (\delta s^d) - \sqrt{-g} \nabla_c s_c \delta g^{cd} + 2\sqrt{-g} \delta (\nabla_c s^c) . \]  

(26)

Hence the boundary term in the variation of the Einstein-Hilbert action becomes,

\[ B \equiv - \int_{\partial \mathcal{V}} d^3 x \ s_c f^{ab} \delta N^c_{ab} = \int_{\partial \mathcal{V}} d^3 x \left[ - \sqrt{-g} \nabla_d (\delta s^d) + \sqrt{-g} \nabla_c s_c \delta g^{cd} - 2\sqrt{-g} \delta (\nabla_c s^c) \right] . \]  

(27)

This expression holds true for spacelike/timelike as well as for null surfaces. For spacelike/timelike surfaces, we can choose, the normal to be \( n_a = -N s_a \), where \( n_a n^a = \epsilon \) and \( \epsilon = \mp 1 \) denotes spacelike/timelike surfaces. Then the above expression for the boundary term in the variation of the gravitational action yields [8],

\[ B = \int_{\partial \mathcal{V}} d^3 x \left[ \sqrt{h} (D_a \delta n^a) - \delta \left( 2 K \sqrt{h} \right) + \sqrt{h} (K_{ab} - K h_{ab}) \delta h^{ab} \right] . \]  

(28)

However, our main interest is in the context of null hypersurfaces, where the null normal is denoted by \( \ell_a = \nabla_a \phi \), with the variation satisfying the following conditions, \( \delta \ell_a = 0 \), \( \delta (\ell_a \ell^a) = 0 = \delta (\ell_a k^a) \). Thus from the general result, presented in Eq. (27), it immediately follows that the boundary term in the variation of the Einstein-Hilbert action for null surfaces read,

\[ B = \int_{\partial \mathcal{V}} d^3 x \left[ \partial_a \left( \sqrt{\Pi} \delta \ell^a \right) - 2\partial_a \left( \sqrt{\Pi} (\Theta + \kappa) \right) + \sqrt{\Pi} \left( \Theta_{ab} - (\Theta + \kappa) q_{ab} \right) \delta q^{ab} + \sqrt{\Pi} \left( 2 k_a (\Theta + \kappa) - k^b (\nabla_a \ell_b + \nabla_b \ell_a) \right) \delta \ell^a \right] . \]  

(29)

Here the symbols have their usual meaning. Finally using Eq. (29) and Eq. (24) one can obtain an expression for the variation of the null boundary term, \( 2\sqrt{-g} (\Theta + \kappa) \) in terms of \( \delta R_{ab} \), \( \delta q^{ab} \) and \( \delta \ell^a \), which is used in the main text.

As an aside, we make the following clarification about the variations of the kind \( \delta g^{ab} = (1/2)(\nabla^a \ell^b + \nabla^b \ell^a) \) arising from the flow \( x^a \rightarrow x^a + (1/2) \ell^a \). In such variations one should think of \( \ell^a \) as \( \epsilon \ell^a \) and one takes the \( \epsilon \rightarrow 0 \) limit at an appropriate juncture. In addition to making the infinitesimal nature of variations well-defined, the change \( \ell^a \rightarrow \epsilon \ell^a \) is also needed for dimensional reasons — a fact which is sometimes not appreciated. In general, metric coefficients can have any dimension depending on the coordinate choice. If we choose \( \ell_a = \nabla_a \phi \), it is dimensionless since \( \phi \) can be treated as a coordinate with dimension of length. So the \( \epsilon \) is required to match dimensions in both sides of the equation \( \delta g^{ab} = (\epsilon/2)(\nabla^a \ell^b + \nabla^b \ell^a) \). We do not exhibit \( \epsilon \) explicitly since we get the same result by that route as well.
D  Another perspective in the variation of null boundary term

It is also possible to provide another perspective to the variation of the null boundary term. In this we will assume that \( \delta \ell_a = 0 \), leading to \( \delta \ell^a = \kappa \ell^a \) as \( \delta g^{ab} = (1/2)(\nabla^a \ell^b + \nabla^b \ell^a) \). This ensures that \( \delta \ell^2 = 0 \), as desired. Similarly, we need to impose the condition \( \delta (\ell^a k_a) = 0 \), which yields, \( \ell^a \delta k_a = \kappa \). Thus we should not assume \( \delta k_a = 0 \).

We have derived the result that \( \delta \{2 \sqrt{-q} (\Theta + \kappa)\} \) for the above variation of the metric equals to \( T \text{d}s \), using the variation of the gravitational action. However, the above result is also derivable from direct variation of the null boundary term as well. For this purpose we need to compute the variation of the surface gravity, which can be achieved by considering variation of the equation defining surface gravity. This yields,

\[
\ell_b \delta \kappa = \delta (\ell^a \nabla_a \ell_b) = \nabla_a \ell_b \delta \ell^a + \ell^a \delta (\nabla_a \ell_b) = \kappa \ell^a \nabla_a \ell_b + \ell^a \ell_b (-\delta \Gamma^c_{ab}) = \kappa^2 \ell_b - \frac{1}{2} \ell^c (-\nabla_c \delta g_{ab} + \nabla_a \delta g_{bc} + \nabla_b \delta g_{ac}) .
\]

Thus taking inner product of this equation with \( k^b \) we finally arrived at,

\[
\delta \kappa = \kappa^2 + \frac{1}{2} k^b \ell^a \ell^c (-\nabla_c \delta g_{ab} + \nabla_a \delta g_{bc} + \nabla_b \delta g_{ac}) = \kappa^2 + \frac{1}{2} k^b \ell^a \ell^c \nabla_b \delta g_{ac}
\]

\[
= \kappa^2 - \frac{1}{2} k^b \ell^a \ell^c \nabla_b \nabla_a \ell_c = \kappa^2 - \frac{1}{2} k^b \nabla_b (\ell^a \ell^c \nabla_a \ell_b) + \frac{1}{2} k^b \ell^a \ell^c \nabla_b \ell^a \ell^c
\]

\[
= \kappa^2 - \frac{1}{2} k^b \nabla_b \left( \frac{1}{2} \ell^a \nabla_a \ell^2 \right) - \kappa^2 .
\]

Even though it is tempting to set \( (1/2)\ell^a \nabla_a \ell^2 = 0 \), one should note that this holds only on the null surface, while the above expression involves off the null surface derivative as well. Specializing to the GNC construction \([7]\) we see,

\[
\delta \kappa = -\frac{1}{4} \ell^a \partial_r \ell^a (\ell^a \ell_r \ell^2 + \ell^r \partial_r \ell^2) = \frac{1}{4} \partial_r [\delta \ell_a (2r \kappa) + 2r \kappa \partial_r (2r \kappa)]
\]

\[
= \frac{1}{4} \partial_r [2r \partial_a \kappa + 4r \kappa^2] = \kappa^2 + \frac{1}{2} \ell^a \nabla_a \kappa .
\]

One can also start from the alternative expression for \( \kappa \), which is \( -k^a \ell^b \nabla_b \ell_a \) and vary it. Since \( \delta (\ell^a k_a) = 0 \), this will lead to an identical expression as above. At this stage, one point should be emphasized, namely the variation itself carries a dimension. This is because, the metric is assumed to be dimensionless, so is \( \ell_a \). Thus the covariant derivative of \( \ell^a \) and hence \( \delta g^{ab} \) must have dimension of inverse length. This is why variation of \( \kappa \) involves a \( \kappa^2 \) term.

As the second part of the variation of the null boundary term for Einstein-Hilbert action let us compute the variation of the expansion scalar associated with the null generator, yielding.

\[
\delta \Theta = \delta (q^{ab} \nabla_a \ell_b) = \delta q^{ab} (\nabla_a \ell_b) + q^{ab} \delta (\nabla_a \ell_b)
\]

\[
= (\nabla_a \ell_b) (\delta g^{ab} + k^b \delta \ell^a + \ell^a \delta k^b + \ell^b \delta k^a + k^a \delta \ell^b) - \frac{1}{2} q^{ab} \ell^c (\nabla_c \delta g_{ab} + \nabla_a \delta g_{bc} + \nabla_b \delta g_{ac})
\]

\[
= \nabla_a \ell_b \nabla^a \ell^b + 2 \kappa \ell_a \delta k^a - 2 \kappa^2 - \frac{1}{2} q^{ab} \ell^c \nabla_c \nabla_a \ell_b + q^{ab} \ell^c \nabla_a \ell_b
\]
\[ \begin{align*}
= & \Theta_{ab} \Theta^{ab} - \frac{1}{2} g^{ab} e^c [\nabla_c, \nabla_a] e_b - \frac{1}{2} g^{ab} e^c \nabla_a e_c e_b + q^{ab} e^c \nabla_a e_b c \\
= & \Theta_{ab} \Theta^{ab} + \frac{1}{2} g^{ab} e^c R^p_{bc a} e_p - \frac{1}{2} g^{ab} \nabla_a (e^c \nabla_e e_b) + q^{ab} \nabla_a (e^c \nabla_b e_c) - \frac{1}{2} g^{ab} (\nabla_a e^c) (\nabla_b e_c) \\
= & \Theta_{ab} \Theta^{ab} + \frac{1}{2} R_{ab} e^a e_b + \frac{1}{2} \kappa \Theta - \frac{1}{2} \Theta_{ab} \Theta^{ab} \\
= & \frac{1}{2} R_{ab} e^a e_b + \frac{1}{2} \kappa \Theta + \frac{1}{2} \Theta_{ab} \Theta^{ab} .
\end{align*} \]

Thus combining the variation of the non-affinity parameter and the expansion scalar we finally obtain the following expression for the variation of the null boundary term,

\[
\delta \left[ 2 \sqrt{-g} (\Theta + \kappa) \right] = 2 \sqrt{-g} \left( \delta \Theta + \delta \kappa \right) + 2 (\Theta + \kappa) \delta \sqrt{-g} = 2 \sqrt{-g} \left( \frac{1}{2} R_{ab} e^a e_b + \frac{1}{2} \kappa \Theta + \frac{1}{2} \Theta_{ab} \Theta^{ab} + \kappa^2 + \frac{1}{2} \ell^a \nabla_a \kappa \right) + 2 (\Theta + \kappa) \left( -\frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab} \right) \\
= 2 \sqrt{-g} \left( \frac{1}{2} R_{ab} e^a e_b + \frac{1}{2} \kappa \Theta + \frac{1}{2} \Theta_{ab} \Theta^{ab} + \kappa^2 + \frac{1}{2} \ell^a \nabla_a \kappa \right) - \sqrt{-g} (\Theta + \kappa) \nabla_a \ell^a \\
= \sqrt{-g} \left( R_{ab} e^a e_b + \kappa \Theta + \Theta_{ab} \Theta^{ab} + 2 \kappa^2 + \ell^a \nabla_a \kappa \right) - \sqrt{-g} (\Theta + \kappa) (\Theta + 2 \kappa) \\
= \sqrt{-g} \left( R_{ab} e^a e_b - 3 \kappa \Theta + \Theta_{ab} \Theta^{ab} - \Theta^2 + \ell^a \nabla_a \kappa \right) \\
\tag{33}
\end{align*}
\]

Using the Raychaudhuri equation presented in Eq. (17), we obtain,

\[
-\delta \left[ 2 \sqrt{-g} (\Theta + \kappa) \right] = \sqrt{-g} \left( 2 \kappa \Theta \right) - \left( \kappa \Theta + \ell^a \nabla_a \kappa \right) + \frac{1}{\sqrt{q}} \frac{d \sqrt{q} \Theta}{d \lambda} \\
= \sqrt{q} \left( 2 \kappa \Theta \right) + \frac{1}{\sqrt{q}} \frac{d \sqrt{q} \Theta}{d \lambda} \left( \sqrt{q} (\Theta - \kappa) \right) \tag{35}
\]

Thus dividing the above expression by $16\pi G$ we again get back the desired expression presented in Eq. (18).

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