Foliations on smooth algebraic surfaces over positive characteristic

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Abstract. We investigate the notion of the $p$-divisor for foliations on a smooth algebraic surface defined over a field of positive characteristic $p$ and we study some of their properties. We present a structure theorem for the $p$-divisor of foliations in the projective plane and the Hirzebruch surfaces where we show that, under certain conditions, such $p$-divisors are reduced.

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1 Introduction

In his monograph [7] Jouanolou showed that a very generic foliation of degree $d \geq 2$ in the projective plane has no algebraic solutions. The crucial point of his argument consists in constructing examples, for each degree $d \geq 2$, of foliations with no algebraic solutions. The term very generic means that there exists a countable union $F$ of closed sets in the space of holomorphic foliations of degree $d$ such that for any foliations $\mathcal{F}$ lying outside $F$ has no algebraic solutions.

In order to prove his theorem, Jouanolou showed the following result.

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Theorem (Jouanolou). For every \( d \in \mathbb{Z}_{\geq 2} \) the foliation in \( \mathbb{P}^2_k \) defined by the vector field
\[
v_d = (xy^d - 1) \frac{\partial}{\partial x} - (x^d - y^{d+1}) \frac{\partial}{\partial y}
\]
has no algebraic solutions.

Recently, a similar problem was considered in [11] for Hirzebruch surfaces. By assuming certain conditions in the normal bundle it is shown that a foliation in the Hirzebruch surfaces has no algebraic solutions (see [11, Theorem C]).

Interestingly, the analogue of the Jouanolou’s theorem over a field of positive characteristic is completely false (see [15]). The results established in [15] imply the following proposition.

Proposition A. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \( F \) be a foliation on \( \mathbb{P}^2_k \) and suppose that \( \deg(F) < p - 2 \). Then \( F \) has an invariant algebraic curve.

In positive characteristic \( p \) there are two classes of foliations: the foliations that are \( p \)-closed and the foliations that are not. Given a foliation \( F \) in a smooth algebraic surface \( X \) defined over a field \( k \) of characteristic \( p > 0 \), we say that it is \( p \)-closed if their tangent sheaf \( T_F \) is closed by \( p \)-powers. This is equivalent to say that the \( \mathcal{O}_X \)-morphism
\[
\psi_F : F^*T_F \longrightarrow N_F \quad v \mapsto v^p \mod T_F
\]
is the zero morphism where \( F_X \) is the absolute Frobenius morphism and \( N_F \) is the normal bundle of \( F \). The morphism \( \psi_F \) is called the \( p \)-curvature of the foliation \( F \).

When the foliation is \( p \)-closed it follows from [2, Théorème 1] that there are infinitely many algebraic solutions. On the other hand, if the foliation \( F \) is not \( p \)-closed then there is a divisor \( \Delta_F \), the \( p \)-divisor, which is defined as the degeneracy locus of the \( p \)-curvature morphism \( \psi_F \). An interesting property of the \( p \)-divisor is that every irreducible algebraic solution of the foliations is contained in the support of the \( p \)-divisor.

Since the set of foliations that are not \( p \)-closed is open we can ask about the structure of the \( p \)-divisor for generic foliations.

Problem 1. Let \( X \) be a smooth algebraic surface defined over a field \( k \) of characteristic \( p > 0 \). What can we say about the \( p \)-divisor \( \Delta_F \) of a generic foliation \( F \)?

We present results towards the solution of this problem in the case where \( X \) is the projective plane or a Hirzebruch surface. More precisely, we show the following theorems.

Theorem A. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). A generic foliation in the projective plane \( \mathbb{P}^2_k \) of degree \( d \geq 1 \) with \( p \parallel d - 1 \) has reduced \( p \)-divisor.

Theorem B. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \( \Sigma_d \) be the \( d \)-Hirzebruch surface defined over \( k \) and let \( d_1, d_2 \in \mathbb{Z}_{\geq 0} \) such that \( p \parallel d_i \), if \( d_i \neq 0 \). Let \( F \) be a fiber of the natural projection \( \pi : \Sigma_d \longrightarrow \mathbb{P}^1_k \) and \( M_d \) be a section which satisfies \( F \cdot M_d = 1 \) and \( M_d^2 = d \). Then a generic foliation in \( \Sigma_d \) with normal bundle \( N \) which is numerically equivalent to \((d_1 - d + 2)F + (d_2 + 2)M_d\) has reduced \( p \)-divisor.
1.1 Organization of the paper

In Section 2 we fix the notations that will be used in the paper. In Section 3, we introduce the definition of $p$-divisor of foliations on algebraic surfaces defined over a field of characteristic $p > 0$. We recall the global representation of foliations in the projective plane and in the Hirzebruch surface. We present the definition of the $p$-divisor of a foliation and we investigate some of their properties, in particular, some applications to algebraicity of foliations on complex projective plane. In Section 4 we consider the problem of the structure of the $p$-divisor for foliations in the projective plane and in the Hirzebruch surface of type $d > 0$. In Section 5 we study the problem for $\mathbb{P}^1_k \times \mathbb{P}^1_k$. In Section 6 we finalize by considering the Hirzebruch surface of type $d > 0$.

2 Notation

- $k$ = algebraically closed field of characteristic $p > 0$.
- $X$ = smooth algebraic surface defined over $k$.
- Curve in $X$ = effective divisor in $X$.
- $d$-Hirzebruch surface over $k$ ($d \geq 0$) = $\Sigma_d = \mathbb{P}(O_{\mathbb{P}^1_k} \oplus O_{\mathbb{P}^1_k}(d))$.
- $M$ = the curve in $\Sigma_d$ ($d > 0$) such that $M^2 = -d$.
- $\equiv$ = numerical equivalence in $\text{Div}(X)$.
- $\text{Num}_{\mathbb{Q}}(X) = (\text{Div}(X)/ \equiv) \otimes \mathbb{Q}$.
- If $(X, H)$ is a polarized surface and $D$ is a divisor in $X$, $\deg(D) = D \cdot H$.
- $U(X, H, N), V(X, H, N)$ = open sets of Lemma 3.12.
- $m_Q(C) = \text{algebraic multiplicity of the curve } C \subset X \text{ at } Q \in X$.
- If $\pi_Q: \text{Bl}_Q(X) \to X$ is the blow up at $Q$, $\mathcal{F}$ is a foliation in $X$, $\mathcal{G} = \pi_Q^* \mathcal{F}$ and $E$ is the exceptional divisor, $l(Q) = \text{ord}_E(N_G^* - \pi_Q^* N^*_X)$.
- $O(2)$ = terms of the order at least 2.

3 Foliations on surfaces and the $p$-divisor

Let $k$ be an algebraically closed field and $X$ be a smooth algebraic surface defined over $k$. A foliation $\mathcal{F}$ in $X$ is a coherent subsheaf $T_{\mathcal{F}} \subset T_X$ of rank one which satisfies the following properties:

- The sheaf $T_{\mathcal{F}}$ is closed by Lie brackets;
- The quotient $T_X/T_{\mathcal{F}}$ is torsion free, that is, $T_{\mathcal{F}}$ is saturated in $T_X$. 

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We can define a foliation in $X$ in more explicitly terms. In this terms, a foliation in $X$ consists of a system $\{(U_i, \omega_i, v_i)\}_{i \in I}$ such that:

- The collection $\{U_i\}_{i \in I}$ is an open cover of $X$;
- For each $i \in I$ we have $v_i \in T_X(U_i)$, $\omega_i \in \Omega^1_X(U_i)$ such that $i_*\omega_i = 0$;
- In $U_i \cap U_j$ we have $\omega_i = f_{ij}\omega_j$ and $v_i = g_{ij}v_j$ for some functions $f_{ij}, g_{ij} \in \mathcal{O}_X(U_{ij})^*$;
- For each $i \in I$ we have $\text{codim} \text{sing}(\omega_i) \geq 2$ and $\text{codim} \text{sing}(v_i) \geq 2$.

Note that the second definition is an alternative version of the first. Indeed, given a foliation $\{(U_i, \omega_i, v_i)\}_{i \in I}$ in $X$ we can construct a saturated subsheaf of $T_X$ in the following way: For each open $U$ of $X$ we define $T_\mathcal{F}(U)$ by

$$T_\mathcal{F}(U) = \{ v \in T_X(U) \mid i_*\omega_i|_{U_i} = 0 \text{ in } U \cap U_i \text{ for all } i \in I \}. $$

We ensure that $T_\mathcal{F}$ is a saturated subsheaf of $T_X$ by using the condition imposed in the singular set of $\omega_i$. Reciprocally, let $T_\mathcal{F}$ be a foliation in $X$ and consider the global section $\omega \in H^0(X, \Omega^1_X \otimes N_\mathcal{F})$ induced by the morphism $T_X \to N_\mathcal{F}$. Then, by definition we obtains a open cover $\{U_i\}_{i \in I}$ of $X$, 1-forms $\omega_i \in \Omega^1_X(U_i)$ and functions $\{f_{ij}\}_{i,j}$ with $f_{ij} \in \mathcal{O}_X(U_i \cap U_j)$ representing $N_\mathcal{F}$ such that $\omega_i = f_{ij}\omega_j$. Since $T_\mathcal{F}$ is a saturated subsheaf of $T_X$ we ensure that $\text{codim} \text{sing}(\omega_i) \geq 2$ for every $i$. Note that the vector fields $v_i$ are obtained in a similar way by considering a global section $v \in H^0(X, T_X \otimes T_\mathcal{F})$ induced by the inclusion $T_\mathcal{F} \subset T_X$. By construction, we obtain vector fields $\{v_i\}_{i \in I}$ in $\{U_i\}_{i \in I}$ such that for every $i$ the vector field $v_i$ is tangent to the 1-form $\omega_i$ that defines $\mathcal{F}$ in the open set $U_i$.

Let $\{(U_i, \omega_i, v_i)\}_{i \in I}$ be a foliation in $X$. The collection $\{f_{ij}^{-1}\}, \{g_{ij}\}$ determines elements of $H^0(X, \mathcal{O}_X^*) = \text{Pic}(X)$ and the line bundles associated are the conormal bundle $\Omega^1_{X/\mathcal{F}}$ and the cotangent bundle $\Omega^1_X$ associated to $\mathcal{F}$. Any divisor in the correspondent linear classes to $\Omega^1_X$ and $(\Omega^1_{X/\mathcal{F}})^*$ will be called the canonical divisor and the normal divisor associated to $\mathcal{F}$ and will be denoted by $K_\mathcal{F}$ and $N_\mathcal{F}$.

### 3.1 The $p$-divisor for foliations on smooth algebraic surfaces

We start this section recalling the following basic lemma about derivations over fields of positive characteristic.

**Lemma 3.1.** Let $k$ be an field of characteristic $p > 0$ and $R$ be a $k$-domain. Let $D \in \text{Der}_k(R)$ be a $k$-derivation. Then $D^p \in \text{Der}_k(R)$. If $f \in R$ then

$$(fD)^p = f^pD^p - fD^{p-1}(f^{p-1})D.$$ 

**Proof.** [S, Proposition (5.3)].

Let $X$ be an algebraic surface defined over a algebraically closed field $k$ of characteristic $p > 0$. Let $\mathcal{F} = \{(U_i, \omega_i, v_i)\}$ be a foliation on $X$. 

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Definition 3.2. We say that $\mathcal{F}$ is $p$-closed if for some $i$ we have $v_i \wedge v_i^p = 0$.

Remark 3.3. By Lemma 3.1 we conclude that $v_i \wedge v_i^p = 0$ for some $i$ if and only if $v_j \wedge v_j^p = 0$ for every $j$.

Suppose that $\mathcal{F}$ is not $p$-closed. In the open $U_{ij} = U_i \cap U_j$ we have $\omega_i = f_{ij} \omega_j$ and $v_i = g_{ij} v_j$. Since we are assuming that $\mathcal{F}$ is not $p$-closed we have for each $i, j \in I$

\[
0 \neq i_{v_i^p} \omega_i = i_{(g_{ij} v_j)} f_{ij} \omega_j = i_{(g_{ij} v_j^p + g_{ij} v_j^{p-1}(g_{ij} v_j))} f_{ij} \omega_j = g_{ij}^{p} f_{ij} i_{v_j^p} \omega_j \neq 0.
\]

So, the collection $\{i_{v_i^p} \omega_i\}_{i \in I}$ determines a global section $0 \neq s_\mathcal{F} \in H^0(X, (\Omega^1_X)^{\otimes p} \otimes N_\mathcal{F})$.

Definition 3.4. Let $\mathcal{F}$ be a foliation in $X$ that is not $p$-closed. The $p$-divisor associated to $\mathcal{F}$ is the zero divisor of the section $s_\mathcal{F}$, that is, $\Delta_\mathcal{F} = (s_\mathcal{F})_0 \in \text{Div}(X)$.

Remark 3.5. Let $\mathcal{F}$ be a foliation in $\mathbb{P}^2_k$ of degree $d > 0$. Then, we have the formulas:

$\Omega^1_{\mathbb{P}^2_k} = \mathcal{O}_{\mathbb{P}^2_k}(d - 1)$ and $N_\mathcal{F} = \mathcal{O}_{\mathbb{P}^2_k}(d + 2)$. In particular, if $\mathcal{F}$ is not $p$-closed then the $p$-divisor is a divisor of degree $\deg(\Delta_\mathcal{F}) = p(d - 1) + d + 2$.

Example 3.6. Let $X = \mathbb{A}^2_k$ and $\alpha \in k$. Let $\mathcal{F}$ be a foliation defined by the vector field $v = x \partial_x + \alpha y \partial_y$. Then, $\mathcal{F}$ is $p$-closed if and only if $\alpha \in \mathbb{F}_p$ and if $\alpha \notin \mathbb{F}_p$ we have $s_\mathcal{F} = (\alpha^p - \alpha)xy$.

Proposition 3.7. Let $X$ be smooth algebraic surface defined over $k$ and $\mathcal{F}$ be a foliation in $X$ that is not $p$-closed. Let $C$ be an irreducible algebraic curve on $X$. If $C$ is $\mathcal{F}$-invariant then $\text{ord}_C(\Delta_\mathcal{F}) > 0$. Reciprocally, if $p$ does not divides $\text{ord}_C(\Delta_\mathcal{F})$ then $C$ is $\mathcal{F}$-invariant.

Proof. Suppose that $C$ is $\mathcal{F}$-invariant and let $R = \mathcal{O}_X, C$ be the ring of regular functions of $X$ along $C$. Let $U$ be an affine open set such that $T_\mathcal{F}$ if given by a regular vector field $v$ and $N_\mathcal{F}$ is given by a regular 1-form $\omega$. Let $\{f = 0\}$ be the local equation for $C$ in $U$ and note that $f$ is a uniformizer parameter to the ring $R$. We need to show that $\text{ord}_C(\Delta_\mathcal{F}) > 0$. Now, we have $v(f) = f H$ and $\omega \wedge df = f \sigma$ for some $H \in R$ and $\sigma \in \Omega^1_R/k$. Contraction with the vector field $v^p$ gives $f i_{v^p}(\sigma) = i_{v^p}(\omega \wedge df) = i_{v^p} \omega df - i_{v^p}(df) \omega$. By using the equality $v(f) = f H$ we conclude that $v^p(f) = f H_p$ for some regular function $H_p$. So, $f i_{v^p} \sigma + v^p(f) \omega = i_{v^p} \omega df$ and we conclude that $i_{v^p} \omega \in \langle f \rangle$. It follows that $\text{ord}_C(\Delta_\mathcal{F}) > 0$.

Reciprocally, suppose that $\text{ord}_C(\Delta_\mathcal{F}) = \alpha \neq 0 \mod p$ and write $\Delta_\mathcal{F} = f^\alpha g$ with $g \in R^\ast$. By [4] Theorem 6.2 we know that $d(\Delta_\mathcal{F})^\alpha = 0$ and expanding this formula we obtain $\alpha \cdot g df \wedge \omega = f(g df - dg \wedge \omega)$, which implies $f|df \wedge \omega$. So, $C$ is $\mathcal{F}$-invariant.

The Proposition 3.7 has the following consequence (compare with Proposition[A]).

Corollary 3.8. Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_k$ and suppose that $p \nmid d + 2$. Then $\mathcal{F}$ has an invariant algebraic curve.
Proof. If $\mathcal{F}$ is $p$-closed then $\mathcal{F}$ admits infinitely many invariant curves by [2, Théorème 1]. We can assume that $\mathcal{F}$ is not $p$-closed. Since $p$ does not divide $d + 2$ the degree formula: $\deg(\Delta_{\mathcal{F}}) = p(d - 1) + d + 2$ shows that $\deg(\Delta_{\mathcal{F}}) \not\equiv 0 \mod p$. In particular, there exists a prime divisor $\mathfrak{p}$ in the support of $\Delta_{\mathcal{F}}$ such that $\ord_{\mathfrak{p}}(\Delta_{\mathcal{F}}) \not\equiv 0 \mod p$. By Proposition [3.7] the divisor $\mathfrak{p}$ defines a $\mathcal{F}$-invariant irreducible curve.

Let $\mathcal{F}$ be a foliation in a smooth algebraic surface $X$ and $Q \in \text{sing}(\mathcal{F})$. We say that $Q$ is not degenerated with eigenvalue $\alpha \in \mathbb{K}$ if there exists an affine open subset $U \subset \mathbb{A}^2$ which contains $Q$ such that $\mathcal{F}|_U$ can be represent by a polynomial vector field $v = v_1 + O(2)$ with $v_1 = x\partial_x + \alpha y\partial_y$ and $\alpha \neq 0$.

**Definition 3.9.** Let $\mathcal{F}$ be a foliation on $X$ and $Q \in \text{sing}(\mathcal{F})$. We say that $Q$ is $p$-reduced if $Q$ is not degenerated and has type $\alpha(Q)$ satisfying the condition: $\alpha(Q) \not\equiv F_p$.

**Lemma 3.10.** Let $\mathcal{F}$ be a foliation in an algebraic surface $X$ and $Q \in \text{sing}(\mathcal{F})$ be a $p$-reduced singularity of $\mathcal{F}$. Then, $\mathcal{F}$ is not $p$-closed.

**Proof.** Let $U$ be an affine open set that contains $Q$ and $x, y \in O_{X,Q}$ local parameters system at $Q$ in $U$. In the open $U$ the foliation is given by a vector field $v = v_1 + \tilde{v}$ where $v_1 = x\partial_x + \alpha y\partial_y$ with $\alpha \not\equiv F_p$ and $\tilde{v}$ consists of terms that has order at least two. Then, $v^p = v_1^p + \tilde{v}_p$ where $\tilde{v}_p$ contains only homogeneous terms with order at least two. Observe that $v_1^p = x\partial_x + \alpha^p y\partial_y$ and $v_1 \wedge v_1^p$ is the homogeneous component of the smallest degree that occurs in $v \wedge v^p$. Since $\alpha \not\equiv F_p$ we ensure that $v_1 \wedge v_1^p = (\alpha^p - \alpha)\partial_x \wedge \partial_y \neq 0$ and so $v \wedge v^p \neq 0$.

**Lemma 3.11.** Let $X$ be a smooth projective surface defined over $k$ and $\mathcal{F}$ be a foliation in $X$ that is not $p$-closed. Let $Q \in \text{sing}(\mathcal{F})$ be a $p$-reduced singular point of $\mathcal{F}$. Suppose that $\Delta_{\mathcal{F}}$ is a reduced divisor and let $\pi_Q : \text{Bl}_Q(X) \rightarrow X$ the blowup with center at $Q$. Then, $\pi^*_{\mathcal{F}}$ defines a foliation in $\text{Bl}_Q(X)$ with reduced $p$-divisor.

**Proof.** Let $\mathcal{G} = \pi^*_{\mathcal{F}}\mathcal{F}$ be a foliation induced in $\text{Bl}_Q(X)$. Since $Q$ is $p$-reduced we have $K_{\mathcal{G}} - \pi^*_{\mathcal{F}}K_{\mathcal{F}} = 0$ and $N_{\mathcal{G}} - \pi^*_{\mathcal{F}}N_{\mathcal{F}} = -E$. Indeed, in an affine open set $U \subset \mathbb{A}^2$ the foliation is represented by a 1-form $\omega = \omega_1 + O(2)$ where $\omega_1 = ydx - axdy$ with $\alpha \not\equiv F_p$ and $O(2)$ containing only terms with order at least two. In a convenient coordinates system the map $\pi_Q : \text{Bl}_Q(X) \rightarrow X$ associate $(x,t) \mapsto (x,xt)$. Since $\pi^*_{\mathcal{F}}\omega$ is a local section of $N_{\pi^*_{\mathcal{F}}}$ we have $[\pi^*_{\mathcal{F}}\omega]_0 = N_{\pi^*_{\mathcal{F}}}^*$. In the other hand, since $Q$ is $p$-reduced we have

$$\pi^*_{\mathcal{F}}\omega = \pi^*_{\mathcal{F}}\omega_1 + O(2) = (xtdx - \alpha x(xtdt + tdx)) + O(2) = x(t(1 - \alpha)dx + xdt + O(2)).$$

Denote $\tilde{\omega} = t(1 - \alpha)dx + xdt + O(2)$ and note that $\tilde{\omega}$ is a local section of $\pi^*_{\mathcal{F}}N_{\mathcal{F}}$. So, $N^*_{\mathcal{G}} = N^*_{\pi^*_{\mathcal{F}}} = [\pi^*_{\mathcal{F}}\omega]_0 = [x]_0 + [\tilde{\omega}]_0 = E + \pi^*_{\mathcal{F}}N^*_{\mathcal{F}}$. The formula that compares $\pi^*_{\mathcal{F}}K_{\mathcal{F}}$ and $K_{\pi^*_{\mathcal{F}}}$ follows by the adjunction formula: $K_X = K_{\mathcal{F}} - N_{\mathcal{F}}$. So, we have

$$[\Delta_{\mathcal{G}}] = pK_{\mathcal{G}} + N_{\mathcal{G}} = p\pi^*_{\mathcal{F}}K_{\mathcal{F}} + \pi^*_{\mathcal{F}}N_{\mathcal{F}} - E = \pi^*_{\mathcal{F}}[\Delta_{\mathcal{F}}] - E = \Delta_{\mathcal{F}} + (m_Q(\Delta_{\mathcal{F}}) - 1)E$$

where $\Delta_{\mathcal{F}}$ denotes the strict transform of the divisor $\Delta_{\mathcal{F}}$ and $m_Q(\Delta_{\mathcal{F}})$ is the algebraic multiplicity of $\Delta_{\mathcal{F}}$ at $Q$. Since $Q$ is $p$-reduced, by using [13] Fact 2.8 we conclude that $m_Q(\Delta_{\mathcal{F}}) = 2$ so that $[\Delta_{\mathcal{G}}] = \Delta_{\mathcal{F}} + E$.

\[\square\]

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Lemma 3.12. Let \((X,H)\) be a polarized smooth projective surface defined over algebraically closed field \(k\) with characteristic \(p > 2\). Let \(\mathcal{N}\) be an invertible sheaf and let \(\text{Fol}_\mathcal{N}(X) = \mathbb{P}(H^0(X,\Omega_X^1 \otimes \mathcal{N}))\) the space of foliations on \(X\) that has normal bundle \(\mathcal{N}\). Suppose that that space is not empty and consider the following sets

\[
U = \{ \mathcal{F} \in \text{Fol}_\mathcal{N}(X) \mid \Delta_\mathcal{F} \text{ is reduced} \} \quad \text{and} \quad V = \{ \mathcal{F} \in \text{Fol}_\mathcal{N}(X) \mid \Delta_\mathcal{F} \text{ is prime} \}.
\]

Then, \(U\) and \(V\) are open sets.

Proof. First, note that \(\deg(\Delta_\mathcal{F})\) depends only of \(X\) and \(\mathcal{N}\). Indeed, we have the following formula:

\[
\deg(\Delta_\mathcal{F}) = pK_X \cdot H + \mathcal{N} \cdot H = p(K_X + \mathcal{N}) \cdot H + H \cdot H = pK_X \cdot H + (p+1)\mathcal{N} \cdot H.
\]

Given \(e \in \mathbb{Z}_{\geq 1}\) let \(Z_e(X)\) the space that consists of all curves in \(X\) that has degree \(e\). In the following, we will use the fact that \(Z_e(X)\) is a projective algebraic variety over \(k\) (see [9, Theorem 1.4]). Define the following sets

\[
S_e = \{ (C, \mathcal{F}) \in Z_e(X) \times \text{Fol}_\mathcal{N}(X) \mid 2C \leq \Delta_\mathcal{F} \},
\]

\[
\tilde{S}_e = \{ (C, \mathcal{F}) \in Z_e(X) \times \text{Fol}_\mathcal{N}(X) \mid C \leq \Delta_\mathcal{F} \}.
\]

Since the conditions \(2C \leq \Delta_\mathcal{F}\) and \(C \leq \Delta_\mathcal{F}\) are closed relations we have that \(S_e\) and \(\tilde{S}_e\) are closed sets in \(Z_e(X) \times \text{Fol}_\mathcal{N}(X)\). Let \(\pi_e: Z_e(X) \times \text{Fol}_\mathcal{N}(X) \to \text{Fol}_\mathcal{N}(X)\) the natural projection. Since \(\pi_e\) is a proper morphism we ensure that \(\pi_e(S_e)\) and \(\pi_e(\tilde{S}_e)\) are closed sets in \(\text{Fol}_\mathcal{N}(X)\). Denote by \(\mathcal{F} \subset \text{Fol}_\mathcal{N}(X)\) the closed set consisting of \(p\)-closed foliations and consider the following sets:

\[
T_1 = \text{Fol}_\mathcal{N}(X) - (\mathcal{F} \cup \bigcup_{j=1}^{\lfloor \deg(\Delta_\mathcal{F}) \rfloor} \pi_j(S_j)) \quad \text{and} \quad T_2 = \text{Fol}_\mathcal{N}(X) - (\mathcal{F} \cup \bigcup_{j=1}^{\deg(\Delta_\mathcal{F}) - 1} \pi_j(\tilde{S}_j)).
\]

We claim that the following identities holds: \(U = T_1\) and \(V = T_2\). Indeed, the inclusions \(U \subset T_1\) and \(V \subset T_2\) are trivial. Now, let \(\mathcal{G}_1 \in T_1\) and \(\mathcal{G}_2 \in T_2\). Suppose, by contradiction, that \(\Delta_{\mathcal{G}_1}\) is not reduced and that \(\Delta_{\mathcal{G}_2}\) is not a prime divisor. In particular, there are curves \(C_1\) and \(C_2\) in \(X\) such that \(2C_1 \leq \Delta_{\mathcal{G}_1}\) and \(C_2 \leq \Delta_{\mathcal{G}_2}\) with \(\deg(C_2) < \deg(\Delta_{\mathcal{G}_2})\).

Since \(\deg(\Delta_{\mathcal{G}_j})\) depends only on \(K_X\) and \(\mathcal{N}\) by computing the degrees we have:

\[
2\deg(C_1) \leq \deg(\Delta_{\mathcal{G}_1}) = \deg(\Delta_{\mathcal{F}}) \quad \text{and} \quad \deg(C_2) \leq \deg(\Delta_{\mathcal{G}_2}) - 1 = \deg(\Delta_{\mathcal{F}}) - 1
\]

and this implies

\[
\mathcal{G}_1 \in \mathcal{F} \cup \bigcup_{j=1}^{\deg(\Delta_{\mathcal{F}})} \pi_j(S_j) \quad \text{and} \quad \mathcal{G}_2 \in \mathcal{F} \cup \bigcup_{j=1}^{\deg(\Delta_{\mathcal{F}}) - 1} \pi_j(\tilde{S}_j)
\]

a contradiction. \(\square\)

In the following we will denote the open sets above by \(U(X,H,N) \in V(X,H,N)\).
3.2 Global equations for foliations on $\mathbb{P}_k^2$

Let $e \in \mathbb{Z}_{\geq 0}$ and $k$ be an algebraically closed field. A foliation in $\mathbb{P}_k^2$ of degree $e$ is determined by a global section of $\Omega_{\mathbb{P}_k^2}^{1}(e+2)$. By using the Euler exact sequence (see [6, Theorem 8.13]) it follows that a foliation in $\mathbb{P}_k^2$ is given, module elements of $k^*$, by a 1-form $\Omega = A dx + B dy + C dz$ where $A, B, C \in k[x, y, z]_{e+1}$ with $Z(A, B, C) \subset \mathbb{P}_k^2$ finite set and such that $i_R \Omega = 0$, where $R$ is the radial vector field: $R = x \partial_x + y \partial_y + z \partial_z$.

Suppose that $k$ has characteristic $p > 0$ and let $\mathcal{F}$ be a foliation in $\mathbb{P}_k^2$ with normal bundle $N = \mathcal{O}_{\mathbb{P}_k^2}(d + 2)$ and suppose that $p \nmid \text{deg}(N)$. Suppose that $\mathcal{F}$ is defined by the homogeneous 1-form: $\omega = Adx + Bdy + Cdz$ and put: $d\omega = (d + 2)(L dy \wedge dz - M dx \wedge dz + N dx \wedge dy)$. Let $v \in \mathfrak{X}_k(\mathbb{A}_k^3)$ be the homogeneous vector field defined by: $v_\omega = L \partial_x + M \partial_y + N \partial_z$. By the [7, Proposition 1.1.4] the association $\omega \mapsto v_\omega$ defines a bijection between the set of projective 1-forms of degree $d + 2$ and the set of homogeneous vector fields in $\mathbb{A}_k^3$ of degree $d$ that has zero divergent $\text{div}(v_\omega) = L_x + M_y + N_z = 0$. The $p$-divisor is explicitly given by $\Delta_\mathcal{F} = [i_{\nu_p} \omega] \in \operatorname{Div}(\mathbb{P}_k^2)$.

**Example 3.13.** Let $\mathcal{F}$ be a foliation of degree two in $\mathbb{P}_k^2$ defined by the projective 1-form

$$\omega = yz^2 dx - z(4yz + 2xz + 2y^2)dy + (xyz + 4y^2z + 2y^3)dz.$$ 

Given a prime number $p \in \mathbb{Z}_{\geq 3}$ consider $\mathcal{F}_p$ the foliation obtained by reducing modulo $p$ the 1-form that defines $\mathcal{F}$. Then, $\mathcal{F}_p$ is not $p$-closed and $\Delta_{\mathcal{F}_p} = 3\{y = 0\} + (p+1)\{z = 0\}$.

**Proof.** We will show first that $\mathcal{F}_p$ is not $p$-closed for every prime $p > 3$. Fix a prime number $p > 3$ and consider the foliation $\mathcal{F}_p$ defined in $\mathbb{P}_k^2$ by reduction modulo $p$ of the coefficients of the 1-form $\omega$. Since this problem is local, we can restrict the foliation to the open set $U = D_z(z) \cong \mathbb{A}_k^2$. In $U$, the foliation is given by the vector field $v = (4y + 2x + 2y^2)\partial_x + y \partial_y$. Observe that $v(x) = 4y + 2x + 2y^2$. An inductive argument shows that for every $k \geq 3$ we have

$$v^k(x) = 2^2(2^{k-2} + 2^{k-3} + \cdots + 2 + 1) + 1)y + k2^ky^2 + 2^kx.$$ 

So, we conclude $v^p(x) = 4y + 2^p x = 4y + 2x$, since $0 = 2^{p-1} - 1 = (2^{p-2} + \cdots + 1)$ in $\mathbb{F}_p$. In particular, we have

$$v^p(x)v(y) - v^p(y)v(x) = y(v^p(x) - 4y - 2y^2 - 2x) = y(-2y^2) = -2y^3 \neq 0.$$ 

So, the foliation defined by $v$ is not $p$-closed, if $p > 3$. This implies that the $p$-divisor of $\mathcal{F}_p$ is given by $\Delta_{\mathcal{F}_p} = 3\{y = 0\} + (p+1)\{z = 0\}$. \hfill \square

3.3 Applications: foliations on $\mathbb{P}_k^2$ without algebraic invariant curves

The next two propositions show that, in some cases, the irreducibility of the $p$-divisor for foliations in $\mathbb{P}_k^2$ is relate with holomorphic foliations in $\mathbb{P}_k^2$ without algebraic invariant curves.
Proposition 3.14. Let $\mathcal{F}$ be a non-dicritical foliation in $\mathbb{P}^2_C$ defined by a projective 1-form $\Omega = Adx + Bdy + Cdz$. Let $K$ be a number field and suppose that $A,B,C \in \mathcal{O}_K[x,y,z]_{d+1}$, where $\mathcal{O}_K$ is the integer ring of $K$. Let $m \in \text{Spn}(\mathcal{O}_K)$ of characteristic $p$ and suppose that $p$ does not divide $d+2$. Let $\mathcal{F}_p$ be a foliation in $\mathbb{P}^2_{k(m)}$ obtained by reduction modulo $m$ of the coefficients of $\Omega$. If $\Delta_{\mathcal{F}_p}$ is irreducible then $\mathcal{F}$ has no algebraic solutions. This can be used to give a simple proof that the Jouanolou foliations of degree two and tree has no algebraic solutions.

Proof. Suppose, by contradiction, that $\mathcal{F}$ has an algebraic solution $C$. By using Galois automorphism, we can assume that $C$ is defined by an irreducible polynomial over $\mathcal{O}_K$. In particular, $C$ is reduced as a curve in $\mathbb{P}^2_C$. Let $F \in \mathcal{O}_K[x,y,z]$ be the irreducible polynomial defining $C$. By $[3]$ theorem we know that $\deg(F) \leq d+2$. Let $F \otimes k(m)$ be the polynomial obtained by reduction modulo $m$ of $F$. Note that the reduction modulo $m$ preserves the invariance in the sense that the curve describe by $F$ is invariant by the foliation $\mathcal{F}_p$. Let $G \in k(m)[x,y,z]$ be an irreducible factor of $F$. We have that the curve $\{ G = 0 \} \subset \mathbb{P}^2_{k(m)}$ is $\mathcal{F}_p$-invariant and by the Proposition 3.7 we conclude that $\{ G = 0 \} \leq \Delta_{\mathcal{F}_p}$. Since $\Delta_{\mathcal{F}}$ is irreducible we have $\{ G = 0 \} = \Delta_{\mathcal{F}}$. But, this is a contraction by comparison of degrees, since $p(d-1)+d+2 = \deg(\Delta_{\mathcal{F}_p}) > d+2 \geq \deg(G)$. \hfill $\Box$

Let $\mathcal{F}$ be a foliation in $\mathbb{P}^2_C$ and $Q$ a reduced singularity of $\mathcal{F}$. Suppose that $Q$ is not degenerated. In this case, we know that if $\alpha$ is the eigenvalue of $Q$ then $\alpha \notin \mathbb{Q}$. By $[12]$ Appendix II we know that there is an analytic coordinate system such that the foliation is given by the 1-form $\omega = -\alpha y(1+b(x,y))dx+x(1+a(x,y))dy$ with $a,b \in \{x,y\}$. In particular, if $C$ is a reduced algebraic curve that is $\mathcal{F}$-invariant with $Q \in C$ then in the analytic coordinate system above the $C$ is given by $\{x=0\}, \{y=0\}$ or $\{xy=0\}$. So, computing the Camacho-Sad index we conclude that

$$CS(\mathcal{F},C,Q) = \begin{cases} 1/\alpha & \text{if } C = \{x=0\}, \\ \alpha & \text{if } C = \{y=0\}, \\ \alpha + \alpha^{-1} + 2 & \text{if } C = \{xy=0\}. \end{cases}$$

In particular, we have that the norm of the index is bounded by a constant which depends only on eigenvalue $\alpha$ of $Q$. More precisely, we have $\|CS(\mathcal{F},C,Q)\| \leq |\alpha| + |\alpha|^{-1} + 2$. We will use this in the following example.

Proposition 3.15. Let $\mathcal{F}$ be a non-degenerated foliation in $\mathbb{P}^2_C$ defined by a projective 1-form $\omega$ with coefficients in $\mathbb{Z}$. Suppose that $\omega$ is primitive, that is, the greatest common divisor of the coefficients of $\omega$ is equal to 1. Suppose that $\mathcal{F}$ is reduced and define

$$\alpha_{\mathcal{F}} := \sup\{ |\alpha(Q)| \mid Q \in \text{sing}(\mathcal{F}) \} \quad \text{and} \quad \alpha_{\mathcal{F}}^{\#} := \sup\{ |\alpha(Q)|^{-1} \mid Q \in \text{sing}(\mathcal{F}) \}.$$  

Let $\beta_{\mathcal{F}} = \alpha_{\mathcal{F}} + \alpha_{\mathcal{F}}^{\#} + 2$ and $p$ be a prime number such that $p > (d_{\mathcal{F}} + 1)\beta_{\mathcal{F}}^{1/2}$ and suppose that $\Delta_{\mathcal{F}_p}$ is a prime divisor. Then, $\mathcal{F}$ has no algebraic solutions.

Proof. Suppose by contradiction that $\mathcal{F}$ has an algebraic solution $C$. By using Galois automorphism we can assume that $C$ is given by an irreducible polynomial defined over
we conclude that $\Delta$ is a divisor such that $\pi D \subseteq C$.

The formula above implies that $d_C \leq (d_F + 1)\beta_F^2 < p$. So, reducing $C$ modulo $p$ we ensure that the reduction $C \otimes \mathbb{F}_p$ has no $p$-factor, that is, there is no an effective divisor, such that $pD \subseteq C \otimes \mathbb{F}_p$. Let $E$ be an irreducible factor of $C \otimes \mathbb{F}_p$. By Proposition 3.7 we know that $E$ is a $\mathcal{F}_p$-invariant irreducible curve. Since the $p$-divisor of $\mathcal{F}_p$ is irreducible we conclude that $\Delta_{\mathcal{F}_p} = E$. But, this is a contradiction by degree comparison. 

The interesting fact about Proposition 3.14 and Proposition 3.15 is that we get information about the algebraicity of foliations on $\mathbb{P}_k^2$ by using only a maximal ideal.

3.4 Global equations for foliations on $\Sigma_d$

In this subsection we recall the construction of the Hirzebruch surfaces and how to represent globally foliations in that surfaces. The reference for this section is [5].

Let $\mathbb{G}_m = k^*$ the multiplicative group of $k$ and $d \in \mathbb{Z}_{\geq 0}$. Let $\mu_d$ the action of $\mathbb{G}_m^2$ in $X = (\mathbb{A}_k^2 - 0) \times (\mathbb{A}_k^2 - 0)$ defined by the morphism:

$$
\mu_d : \mathbb{G}_m^2 \times X \longrightarrow X
$$

$$
((a, b), (x_0, x_1; y_0, y_1)) \mapsto (ax_0, ax_1, by_0, \frac{b}{a^d}y_1).
$$

The quotient $\Sigma_d = X/\mu_d$ is a smooth surface defined over $k$ and is isomorphic to the $d$-Hirzebruch surface, that is, the surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(d))$. It is a ruled surface with structural morphism to $\mathbb{P}_k^1$ defined by

$$
\pi : \Sigma_d \longrightarrow \mathbb{P}_k^1
$$

$$
(x_0, x_1; y_0, y_1) \mapsto [x_0 : x_1].
$$

Let $d_1, d_2 \in \mathbb{Z}$ and $G \in k[x_0, x_1, y_0, y_1]$. We say that $G$ is bi-homogeneous of the bi-degree $(d_1, d_2)$ if for any monomial $x_0^{a_0}x_1^{a_1}y_0^{b_0}y_1^{b_1}$ in the support of $G$ we have $d_1 = a_0 + a_1 - db_1$ and $d_2 = b_0 + b_1$. Let $F$ and $M_d$ curves in $\Sigma_d$ such that $F$ is a fiber of the structural projection $\pi : \Sigma_d \longrightarrow \mathbb{P}_k^1$ and $M_d$ is a section of $\pi$ which satisfies the conditions $M_d \cdot F = 1$ and $M_d^2 = d$. Then $\{F, M_d\}$ forms a base for the vector space $\text{Num}_Q(\Sigma)$. If $D \in \text{Div}(\Sigma_d)$ is a divisor such that $D \equiv d_1F + d_2M_d$ then the global sections of the $\mathcal{O}_{\Sigma_d}(D)$ correspond to bi-homogeneous polynomials of bi-degree $(d_1, d_2)$ in $\Sigma_d$. In the following, we will use the following description of foliations in Hirzebruch surfaces (see [5] Proposition 3.2]).

Proposition 3.16. Let $d \in \mathbb{Z}_{\geq 0}$, $d_1, d_2 \in \mathbb{Z}$ and $N = \mathcal{O}_{\Sigma_d}(d_1 - d + 2, d_2 + 2)$. Then, any foliation $\mathcal{F}$ in $\Sigma_d$ with normal bundle $N$ is uniquely determined, modulo $k^*$, by a differential 1-form of the type $\Omega = A_0dx_0 + A_1dx_1 + B_0dy_0 + B_1dy_1$ where $A_0, A_1 \in H^0(\Sigma_d, \mathcal{O}(d_1 - d + 1, d_2 + 2))$, $B_0 \in H^0(\Sigma_d, \mathcal{O}_{\Sigma_d}(d_1 - d + 2, d_2 + 1))$ and $B_1 \in H^0(\Sigma_d, \mathcal{O}_{\Sigma_d}(d_1 + 2, d_2 + 1))$ are bi-homogeneous and satisfies the following conditions: $x_0A_0 + x_1A_1 - dy_1B_1 = 0$ and $y_0B_0 + y_1B_1 = 0$. 

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4 The $p$-divisor for foliations on $\mathbb{P}^2_k$

In this section, we investigate the structure of the $p$-divisor for generic foliations in the projective plane. In the next section, $k$ will be denote an algebraically closed field that has characteristic $p > 0$. The following lemma will be important to the next sections.

**Lemma 4.1.** Let $F \in k[x_0, \ldots, x_n]$ be a reduced polynomial and $l \in \mathbb{Z}$ positive integer with $\gcd(l, p) = 1$. Suppose that $x_0 \nmid F$. Then, $F(x_0^{l}, x_1, \ldots, x_n)$ is reduced.

**Proof.** First, note that if $x_0$ does not occurs in $F$ then is nothing to prove. So, we can suppose that $x_0$ occurs in $F$. We first consider the case where $F$ is irreducible. Note that it is sufficient to consider the case where $n = 0$. Indeed, if $R = k[x_1, \ldots, x_n]$ and $K$ is their fraction field then by the Gauss Lemma (see [10, Theorem 2.1]) we know that $F \in R[x_0]$ is irreducible if and only if $F = a_0 + a_1x_0 + \cdots + a_dx_0^d \in K[x_0]$ is irreducible and has content 1, that is, the greatest common divisor of the coefficients is equal to 1.

Since we are assuming that $F$ is irreducible we have, in particular, that $F$ is irreducible over $K[x_0]$ where $K = k(x_1, \ldots, x_n)$. Let $g(x_0) = F(x_0) \in K[x_0]$. Note that the content of $g$ is equal to 1, since the coefficients of $g$ are equal to the coefficients of $F$. In the other hand, the irreducibility of $F$ implies that $g$ is reduced: indeed, to see this note that it is sufficient to proof that $\gcd(g(x_0), \frac{dg}{dx_0}) = 1$. Now, by taking derivatives we have $\frac{dg}{dx_0} = lF'(x_0)x_0^{-1} \neq 0$. Since $F$ is irreducible we ensure that $F'(x_0)$ and $F(x_0)$ are coprime. In particular, we have $\gcd(g(x_0), g'(x_0)) = 1$. So, $g(x_0)$ is reduced in $R[x_0] = k[x_0, x_1, \ldots, x_n]$.

Now we will consider the general case, where $F$ is reduced. Let $R = k[x_1, \ldots, x_n]$ and $K$ their fraction field. Note that, without loss of generality, we can assume that $F \in R[x_0]$ has content equal to 1. Let $G$ and $H$ two irreducible factors of $F$. We will show that $G = G(x_0^{l}, x_1, \ldots, x_n)$ and $H = H(x_0^{l}, x_1, \ldots, x_n)$ do not have common irreducible factors. By the Gauss Lemma (see [10, Theorem 2.1]) is it sufficient to show that $G$ and $H$ do not have common irreducible factors over $K[x_0]$, where $K = k(x_1, \ldots, x_n)$. In the other hand, since $G$ and $H$ has not common factors over $k(x_0, x_1, \ldots, x_n)$ we have, in particular, that there is no common irreducible factors over $K[x_0]$. So, there exists $A(x_0), B(x_0) \in K[x_0]$ such that $A(x_0)G + B(x_0)H = 1$. Specializing the identity above to $x_0$ we conclude that $A(x_0)G + B(x_0)H = 1$ and so $G, H$ have no common irreducible factors over $K[x_0]$. So, we consider only the case where the polynomial $F$ is irreducible by considering their decomposition in irreducible factors. \hfill \Box

**Remark 4.2.** Let $l \in \mathbb{Z}_{>0}$ with $\gcd(l, p) = 1$. In general, we do not ensure that if $f \in k[x]$ is irreducible then $g(x) = f(x^l)$ is irreducible. Indeed, let $p > 2$ and consider $f(x) = x - 1$. If $l \in 2\mathbb{Z}$ then we have $g(x) = f(x^l) = (x^l)^l - 1)(x^l + 1)$.

In the following, we present the proof of the Theorem [A]. Recall the statement.

**Theorem A.** Let $k$ be an algebraically closed field of characteristic $p > 0$. A generic foliation in $\mathbb{P}^2_k$ of degree $d \geq 1$ with $p \nmid d - 1$ has reduced $p$-divisor. More precisely, if
N is a fixed bundle then $U(\mathbb{P}^2_k, O_{\mathbb{P}^2_k}(1), N) \neq \emptyset$, if $\deg(N) = 3$ or if $\deg(N) > 3$ with $p + \deg(N) - 3$.

We will consider first the case where $d \in \{1, 2\}$ and after that we will use the case $d = 2$ to show the general case. We will divide the proof in propositions.

**Proposition 4.3.** A generic foliation in $\mathbb{P}^2_k$ of degree $d \in \{1, 2\}$ has reduced $p$-divisor. More precisely, we have $U(\mathbb{P}^2_k, O_{\mathbb{P}^2_k}(1), N) \neq \emptyset$, if $\deg(N) = \{3, 4\}$.

**Proof.** If $\deg(N) = 3$, that is, $d = 1$ then there are examples of foliations with reduced $p$-divisor. Indeed, if $\alpha \in k - \mathbb{P}_p$ consider the foliation $\mathcal{F}_\alpha$ in $\mathbb{P}^2_k$ given by the 1-form $\Omega_\alpha = -yzdx + \alpha xzdy + (1 - \alpha)xyzdz$. The condition $\alpha \notin \mathbb{P}_p$ implies that $\mathcal{F}_\alpha$ is not $p$-closed (see Lemma 3.10) with $\Delta_{\mathcal{F}_\alpha} = \{x = 0\} + \{y = 0\} + \{z = 0\}$. Now, consider the case where $\deg(N) = 4$ and let $D_+(z) = \{(x : y : z) \in \mathbb{P}^2_k | z \neq 0\}$. We will get the example by compactification via the isomorphism

$$\Phi: D_+(z) \rightarrow \mathbb{A}_k^2,$$

$$[x : y : z] \mapsto \left(\frac{x}{z}, \frac{y}{z}\right),$$

of a foliation $\mathcal{G}$ in $\mathbb{A}_k^2$ given by the 1-form: $\omega = ydx - xdy + \omega_2$ where $\omega_2 = a(x, y)dx + b(x, y)dy$ for $a, b \in k[x, y]$ generic homogeneous polynomial of degree two. Note that $\mathcal{G}$ has tree $\mathcal{G}$-invariant lines which contains the point $(0, 0)$. Indeed, the lines are given explicitly by the polynomial $i_R\omega_2 = l_1l_2l_3$, where $R$ is the radial vector field in $\mathbb{A}_k^2$: $R = x\partial_x + y\partial_y$. Since $a$ and $b$ are generic, we can assume that $l_1, l_2$ and $l_3$ has multiplicity 1 along $\Delta_\mathcal{G}$. Indeed, we choose the lines $l_1, l_2, l_3$ such that for each $i$ we ensure that $l_i \cap l_\infty = \{P_i\}$ is a $p$-reduced singularity of $\mathcal{G}$. In this case, by [13, Fact 2.8] we ensure that $l_i$ occurs with multiplicity 1 in the $p$-divisor $\Delta_\mathcal{G}$. Let $\mathcal{F}$ be a foliation obtained via compactification of $\mathcal{G}$ in $\mathbb{P}^2_k$, via $\Phi$. By Lemma 3.10 we know that $\mathcal{F}$ is not $p$-closed with four invariant lines, $l_1, l_2, l_3$ and $l_\infty = \{z = 0\}$, and so $\Delta_\mathcal{F} = l_1 + l_2 + l_3 + l_\infty + C$ for some curve $C$ of degree $p$.

We will show that a generic choice of $a, b$ implies that $C$ is irreducible and has $Q = [0 : 0 : 1]$ as a singularity of multiplicity $m_Q(C) = p - 1$. For this, let $\pi: Bl_Q(\mathbb{P}^2_k) \rightarrow \mathbb{P}^2_k$ the blowup at $Q$ fix $E$ the exceptional divisor and $F$ a fiber of the natural projection $\pi: Bl_Q(\mathbb{P}^2_k) \rightarrow \mathbb{P}^1_k$. Note that $\{F, E\}$ forms a base to the vector space $\text{Num}_Q(Bl_Q(\mathbb{P}^2_k))$ and satisfies the following conditions $E^2 = -1, F \cdot E = 1$ and $F^2 = 0$. Let $\mathcal{H} = \pi^*\mathcal{F}$ the induced foliation. Since $Q$ is a radial singularity we have (see [11, Chapter 2, Section 3]) $N^*_H = \pi^*N^*_F + 2E$ and $K_H = \pi^*K_F - E$. Let $H$ be a line containing the point $Q$. Since $\pi^*N^*_F = \pi^*(-4H) = -4\pi^*H \equiv -4(F + E)$ and $\pi^*K_F = \pi^*H \equiv F + E$, we conclude

$$\Delta_\mathcal{H} \equiv pK_H + N_H \equiv 2E + \deg(\Delta_\mathcal{F})F = (p + 4)F + 4E.$$

Let $\tilde{l}_1, \tilde{l}_2, l_3$ and $\tilde{l}_\infty$ be the strict transform of $l_1, l_2, l_3$ and $l_\infty$ respectively. Since $\pi: Bl_Q(\mathbb{P}^2_k) - E \rightarrow \mathbb{P}^2_k - \{Q\}$ is an isomorphism we verify that $\tilde{l}_1, \tilde{l}_2, \tilde{l}_3$ and $\tilde{l}_\infty$ occurs in

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\[ \Delta_H, \text{ that is } \Delta_H - \delta_1 - \delta_2 - \delta_3 - \delta_\infty \geq 0. \] Moreover, since \( a, b \) are generic we can assume that \( \text{ord}_i(\Delta_H) \geq 0 \) for \( i \in \{1, 2, 3, \infty\} \). As \( (\Delta_H - \delta_1 - \delta_2 - \delta_3 - \delta_\infty) \cdot F = 1 \) there exist an irreducible curve \( C \subset Bl_Q(\mathbb{P}^2) \) that is \( H \)-invariant and such that \( C \cdot F = 1 \). In particular, we have \( C = E + \alpha F \). Note that \( \alpha > 0 \), since \( Q \) is a radial singularity and, in particular, \( E \) is not \( H \)-invariant. Writing \( \Delta_H - \delta_1 - \delta_2 - \delta_3 - \delta_\infty = C + R \) for some divisor \( R \) we conclude (using \( C \cdot F = 1 \)) that \( R \equiv bF \), for some \( b \in \mathbb{Z}_{\geq 0} \). Now, since \( \omega_2 \) is generic we may assume that \( b = 0 \). Indeed, if \( b \neq 0 \), we see that there exist a fiber of the natural projection \( p: Bl_Q(\mathbb{P}^2) \rightarrow \mathbb{P}^1 \) that is \( H \)-invariant (see Proposition [3.7]). By projecting this fiber in \( \mathbb{P}^2 \), we obtain a line which contain \( Q \) that is \( F \)-invariant. Since \( l_1, l_2, l_3 \) and \( l_\infty \) occurs in \( \Delta_F \) with multiplicity 1 and since they are the unique invariant lines of \( F \), we obtain a contraction. So, we conclude that \( R = 0 \) and \( C \) is an irreducible curve that is \( H \)-invariant with \( C \equiv E + pF \). Projecting \( C \) via the map \( \pi \) we obtain in \( \mathbb{P}^2 \) an irreducible algebraic curve of degree \( p \) that has \( Q \) as a singularity and with multiplicity \( p - 1 \). Indeed, \( \pi_*C \) is irreducible and the degree is given by

\[
\deg(\pi_*C) = \pi_*C \cdot H = \pi^*\pi_*C \cdot \pi^*H = (E + pF) \cdot (E + F) = -1 + p + 1 = p.
\]

In other hand, the multiplicity can be computed in the following way:

\[ \pi^*\pi_*C = \tilde{C} + m_Q(C)E \equiv (E + pF) + m_Q(C)E \implies 0 = \pi^*\pi_*C \cdot E = (E + pF) \cdot E - m_Q(C) \]

and so \( 0 = -1 + p - m_Q(C) \) which implies \( m_Q(C) = p - 1 \). This concludes the proof for \( d \in \{1, 2\} \). \( \square \)

**Proposition 4.4.** A generic foliation in \( \mathbb{P}^2 \) with normal bundle \( N \) and of degree \( d \geq 3 \) with \( p \nmid d - 1 \) has reduced \( p \)-divisor. More precisely, \( U(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), N) \neq \emptyset \), if \( \deg(N) \geq 5 \) with \( p \nmid \deg(N) - 3 \).

**Proof.** We use the case \( d = 2 \) to show the general case. Let \( e \in \mathbb{Z}_{\geq 0} \) such that \( \gcd(e, p) = 1 \) and \( \mathcal{F} \) be a foliation in \( \mathbb{P}^2 \) of degree 2 that has reduced \( p \)-divisor. By the preceding proposition, we can assume that there exists a foliation that leaves invariant an irreducible algebraic curve \( C \) of degree \( p \) and four lines: \( l_1, l_2, l_3 \) and \( l_\infty \), where \( l_1 \cap l_2 = l_0 \cap l_1 = l_0 \cap l_2 = \{ Q \} \). For simplicity, we suppose that \( l_1 = \{ x = 0 \} \), \( l_2 = \{ y = 0 \} \) and \( l_3 = \{ ux + vy = 0 \} \) for some constants \( u, v \in \mathbb{k}^* \). In this case, \( \mathcal{F} \) is defined by a 1-form of the type \( \Omega_0 = yzA_0dx + xzA_1dy + xyA_2dz \) for certain \( A_0, A_1, A_2 \in \mathbb{k}[x, y, z] \) homogeneous of degree 1 and such that \( A_0 + A_1 + A_2 = 0 \). Consider the finite morphism

\[
\Phi_e: \mathbb{P}^2 \rightarrow \mathbb{P}^2 \quad [x_0 : x_1 : x_2] \mapsto [x_0^e : x_1^e : x_2^e].
\]

Let \( \mathcal{H} \) the foliation in \( \mathbb{P}^2 \) defined by the saturation of the 1-form

\[
\Omega = \Phi_e^*\Omega_0 = e(\text{xyz})^{-1}[yzA_0(x^e, y^e, z^e)dx + xzA_1(x^e, y^e, z^e)dy + xyA_2(x^e, y^e, z^e)dz].
\]

Observe that \( \mathcal{H} \) is a foliation of degree \( e + 1 \) which is not \( p \)-closed. Since \( \gcd(e, p) = 1 \), we can use the Lemma [4.1] to ensure that \( \Phi_e^*l_3 \) and \( \Phi_e^*C \) are reduced curves with irreducible
distinct components. In particular, by the Proposition 3.7 it follows that $\Phi^*_C l_3$ and $\Phi^*_C C$ are $\mathcal{H}$-invariant curves. We claim that $\mathcal{H}$ has $p$-divisor $\Delta_{\mathcal{H}} = \{ x = 0 \} + \{ y = 0 \} + \{ z = 0 \} + \Phi^*_C l_3 + \Phi^*_C C$. Indeed, note that $\Delta_{\mathcal{H}}$ is a divisor with degree given by the formula

$$\deg(\Delta_{\mathcal{H}}) = pK_{\mathcal{H}} \cdot O_{\mathbb{P}^2_k}(1) + N_{\mathcal{H}} \cdot O_{\mathbb{P}^2_k}(1) = pe + e + 3.$$ 

In the other hand, by construction we know that the curves $\{ x = 0 \}, \{ y = 0 \}, \{ z = 0 \}, \Phi^*_C l_3$ and $\Phi^*_C C$ are $\mathcal{H}$-invariant. In particular, $\Delta_{\mathcal{H}} \supseteq \{ x = 0 \} + \{ y = 0 \} + \{ z = 0 \} + \Phi^*_C l_3 + \Phi^*_C C$. By comparison of the degrees we conclude the equality $\Delta_{\mathcal{H}} = \{ x = 0 \} + \{ y = 0 \} + \{ z = 0 \} + \Phi^*_C l_3 + \Phi^*_C C$. So, it follows that $\mathcal{H}$ is a foliation of degree $e + 1$ in $\mathbb{P}^2_k$ with reduced $p$-divisor. This finishes the proof of the proposition.

**Remark 4.5.** In the proof of the Theorem A we saw that is possible to find a foliation $\mathcal{F}$ of degree 2 in $\mathbb{P}^2_k$ that is not $p$-closed with $p$-divisor in the form $\Delta_{\mathcal{F}} = l_1 + l_2 + l_3 + l_4 + C$ where $C$ is an irreducible algebraic curve of degree $p$ and $l_i \neq l_j$ if $i \neq j$. Let $P, Q \in \mathbb{P}^2_k$ and fix $\Phi : \mathbb{P}^2_k \rightarrow \mathbb{P}^2_k$ an automorphism such that $\Phi(P), \Phi(Q) \notin C$. Then, $\Phi^* \mathcal{F}$ is foliation that is not $p$-closed with $p$-divisor $\Delta_{\Phi^* \mathcal{F}} = \Phi^* l_1 + \Phi^* l_2 + \Phi^* l_3 + \Phi^* l_4 + \Phi^* C$ where $\Phi^* C$ is an irreducible algebraic curve of degree $p$ such that $P, Q \notin \Phi^* C$.

5 The $p$-divisor for foliations on $\mathbb{P}^1_k \times \mathbb{P}^1_k$

In the following, we will fix $x_0, x_1$ (resp. $y_0, y_1$) as the coordinates functions of the first factor (resp. second factor) of $\mathbb{P}^1_k \times \mathbb{P}^1_k$. Let $\pi_1$ the projection of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ over the first factor and $\pi_2$ the projection of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ over the second factor. Let $F$ and $M$ fibers of $\pi_1$ and $\pi_2$ respectively. Recall that $\{ F, M \}$ is a base for the vector space $\text{Num}_Q(\mathbb{P}^1_k \times \mathbb{P}^1_k)$ which satisfies the following numerical conditions $F^2 = 0$, $F \cdot M = 1$, and $M^2 = 0$. Observe that for any fibers $F_0 = \pi_1^{-1}(\text{pt})$ and $M_0 = \pi_2^{-1}(\text{pt})$ we have $F_0 \equiv F$ and $M_0 \equiv M$.

**Lemma 5.1.** Let $C = \mathcal{Z}(f) \subset \mathbb{A}^2_k$ be an irreducible algebraic curve of degree $d$ and $f = f_1 + f_2 + \cdots + f_d$ the decomposition of $f$ in homogeneous terms as element of $k[x, y]$. Suppose that $x^d$ and $y^d$ effectively occurs in $f_d$ and consider $\overline{C}$ the bi-projectivization of $C$ in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ via the isomorphism:

$$\Phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \longrightarrow \mathbb{A}^2_k$$

$$([x_0 : x_1], [y_0 : y_1]) \mapsto \left( \frac{x_0}{x_1}, \frac{y_0}{y_1} \right).$$

Then, $\overline{C}$ is irreducible and if $\{ F, M \}$ is a base for $\text{Num}_Q(\mathbb{P}^1_k \times \mathbb{P}^1_k)$, with $F^2 = 0$, $F \cdot M = 1$ and $M^2 = 0$ then $\overline{C} \equiv \text{deg}(C)(F + M)$.

**Proof.** The compactification $\overline{C}$ is defined by the following polynomial

$$F = (x_1 y_1)^{d-k} f_k(x_0 y_1, x_1 y_0) + \cdots + (x_1 y_1)^{d-k} f_{k+1}(x_0 y_1, x_1 y_0) + \cdots + f_d(x_0 y_1, x_1 y_0).$$

It is easy to see that $F$ is a homogeneous polynomial of bi-degree $(d, d)$, so that we need only to check their irreducibility. In the other hand, note that if exists an irreducible
factor $H$ of $F$ then necessarily we have $H \not\in \langle x_1y_1 \rangle$ since the hypothesis in $f_d$ implies that $f_d(x_0y_1, x_1y_0) \not\in \langle x_1y_1 \rangle$. But, specializing to $x_1 = 1, y_1 = 1$ we conclude that $f$ is reducible. Contradiction!

Example 5.2. Let $C$ be an affine curve in $\mathbb{A}^2_k$ defined by the equation $f = x + y + xy$. Using the map

$$\Phi : U_{11} \longrightarrow \mathbb{A}^2_k$$

$$(x_0 : x_1, [y_0 : y_1]) \mapsto (x_0, y_0 - x_1, y_1)$$

and projectivizing $C$ in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ via $\Phi$ we obtain the curve $\overline{C} = \mathcal{Z}(x_0y_1 + x_1y_0 + x_0y_0)$ of bi-degree $(1, 1) + (2, 2)$. This shows that the condition in Lemma 5.1 is necessary.

Theorem 5.3. Let $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ such that $p \not| d_i$, if $d_i \neq 0$. Then, a generic foliation in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ with normal bundle $N \equiv (d_1 + 2)F + (d_2 + 2)M$ has reduced $p$-divisor. More precisely, $U(\mathbb{P}^1_k \times \mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(F + M), N) \neq \emptyset$ if

- $N \cdot F - 2 \geq 0$ and $p \nmid N \cdot F - 2$ (if nonzero);
- $\deg(N) - N \cdot F + d - 2 \geq 0$ and $p \nmid \deg(N) - N \cdot F - 2$ (if nonzero).

The proof of the Theorem 5.3 is divided in propositions. We show that given $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ such that $p \nmid d_i$ (if $d_i \neq 0$) we can find a foliation $G$ in $\Sigma_0 = \mathbb{P}^1_k \times \mathbb{P}^1_k$ with the following properties:

(i) $K_G = d_1F_0 + d_2M_0$ where $F_0 = \{x_0 = 0\}$ and $M_0 = \{y_0 = 0\}$.
(ii) $F_0$ and $M = \{y_1 = 0\}$ are $G$-invariant curves with $\{Q\} = F_0 \cap M$ a $p$-reduced singularity of $G$.
(iii) The $p$-divisor $\Delta_G$ is reduced.

We will consider two cases: A and B.

5.1 Case A

In this subsection, we proof, in particular, that a generic foliation $G$ in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ with cotangent divisor $K_G = d_1F_0 + d_2M_0$ has reduced $p$-divisor when

$$(d_1, d_2) \in \{(0, 0)\} \cup \{(l, 0) \in \mathbb{Z}^2 \mid l > 0 \text{ and } p \nmid l\} \cup \{(0, l) \in \mathbb{Z}^2 \mid l > 0 \text{ and } p \nmid l\}.$$ 

We start with the case $d_1 = d_2 = 0$.

Proposition 5.4. A generic foliation in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ with cotangent divisor trivial $K_G \equiv 0$ satisfies $\{2\}$ and $\{0\} \in \{2\}$. 

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Proof. Let \( \omega \) be a 1-form in \( \mathbb{A}^2_k \) given by \( \omega = \alpha y dx + x dy \) with \( \alpha \notin \mathbb{F}_p \). Consider \( \mathbb{A}^2_k \) as an open subset of \( \mathbb{P}^1_k \times \mathbb{P}^1_k \) via the isomorphism

\[
\Phi: \mathbb{P}^1_k \times \mathbb{P}^1_k \rightarrow \mathbb{A}^2_k \quad \left( [x_0, x_1], [y_0, y_1] \right) \mapsto \left( \frac{x_0}{x_1}, \frac{y_0}{y_1} \right)
\]

where \( \mathbb{P}^1_k \times \mathbb{P}^1_k \) and \( \mathbb{A}^2_k \) are the projective and affine 2-spaces, respectively. Projectivizing the 1-form \( \omega \) in \( \mathbb{P}^1_k \times \mathbb{P}^1_k \) by the map \( \Phi \) we obtain:

\[
\Phi^* \omega = \frac{\alpha y_0 x_0 dx_0 - \alpha x_0 y_0 dx_1}{x_1^2 y_1} + \frac{x_0 y_0 dy_0 - y_0 x_0 dy_1}{x_1 y_1^2}
\]

and considering their saturation it follows that \( \Omega = \alpha x_1 y_0 y_1 dx_0 - \alpha x_0 y_0 y_1 dx_1 + x_0 x_1 y_1 dy_0 - x_0 x_1 y_0 dy_1 \) a projective 1-form that is bi-homogeneous which defines a foliation \( \mathcal{F}_\Omega \) with \( K_{\mathcal{F}_\Omega} \equiv 0 \). In particular, satisfies (ii). Note that \( F_0, F_1 = \{ x_1 = 0 \} \), \( M = \{ y_1 = 0 \} \) and \( M_0 \) are \( \mathcal{F}_\Omega \)-invariant curves, so that \( \Delta_{\mathcal{F}_\Omega} \geq M_0 + M + F_1 + F_2 \). In the other hand, \( \Delta_{\mathcal{F}_\Omega} \equiv pK_{\mathcal{F}_\Omega} \equiv N_{\mathcal{F}_\Omega} \equiv 2F_0 + 2M_0 \) and by degree comparison we conclude the equality \( \Delta_{\mathcal{F}_\Omega} = M_0 + M + F_1 + F_2 \). In particular, \( \mathcal{F}_\Omega \) satisfies (iii). Now, let \( U_{10} \) be the open set given by \( \{ y_0 \neq 0 \} \cap \{ x_1 \neq 0 \} \). Observe that there exists an isomorphism

\[
\psi: U_{10} \rightarrow \mathbb{A}^2_k \quad \left( [x_0: x_1], [y_0: y_1] \right) \mapsto \left( \frac{x_0}{x_1}, \frac{y_1}{y_0} \right) = (x, y).
\]

Restricting the foliation \( \mathcal{F}_\Omega \) to the open \( U_{10} \) we obtain a foliation given by the 1-form: \( \sigma = \alpha y dx - x dy \) and so \( M \cap F_0 = \{ y = 0 \} \cap \{ x = 0 \} = \{ (0,0) \} \) is a \( p \)-reduced singularity. In this way, we conclude that \( \mathcal{F}_\Omega \) satisfies (iii). This finishes the proof for \( d_2 = d_1 = 0 \).}

We pass now to the a more general case.

**Proposition 5.5.** Let \( d_1, d_2 \in \mathbb{Z}_{\geq 0} \) such that \( p \nmid d_i \) if \( d_i \neq 0 \). Suppose that

\[
(d_1, d_2) \in \{(0,0)\} \cup \{(l,0) \in \mathbb{Z}^2 \mid l > 0 \text{ and } p \nmid l \} \cup \{(0,l) \in \mathbb{Z}^2 \mid l > 0 \text{ and } p \nmid l \}.
\]

Then, a generic foliation in \( \mathbb{P}^1_k \times \mathbb{P}^1_k \) with cotangent divisor \( K_G \equiv d_1 F + d_2 M \) satisfies (i), (ii) and (iii).

**Proof.** The case \( d_1 = d_2 = 0 \) was considered in the precedent proposition. We will consider the case where \( d_1 = l \) and \( d_2 = 0 \) for \( l > 0 \) with \( p \mid l \). By the symmetry of the the problem we will automatically consider the case where \( d_1 = 0 \) and \( d_2 = l \) for \( l > 0 \) with \( p \nmid l \).

**Case** \( l = 1 \): Let \( \mathcal{F} \) be a Riccati foliation with respect the first projection \( \pi_1: \mathbb{P}^1_k \times \mathbb{P}^1_k \rightarrow \mathbb{P}^1_k \). Recall (see [1] Chapter 4, Section 1) that a Riccati foliation is defined as a foliation in \( \mathbb{P}^1_k \times \mathbb{P}^1_k \) whose the general fiber \( F \) of \( \pi_1 \) is transverse to \( \mathcal{F} \). Suppose that

(a) The foliation \( \mathcal{F} \) leaves invariant only tree fibers of the first projection \( \pi_1 \). Denote that fibers by \( F_0, F_1, F_2 \).
(b) The foliation $\mathcal{F}$ leaves invariant only one fiber of the second projection $\pi_2$. Denote that fiber by $M$.

(c) The intersections $\{Q_i\} = F_i \cap M$ are $p$-reduced singularities of $\mathcal{F}$.

We show that that foliation satisfies [i],[ii] and [iii]. Note that by [i] (see §1, Chapter 4])

$$K_{\mathcal{F}} = \pi_1^\ast K_{\mathbb{P}^1_k} + F_0 + F_1 + F_2 \equiv -2F + 3F = F.$$ 

So, $\mathcal{F}$ satisfies [i]. Note that the item (c) ensures that $\mathcal{F}$ is not $p$-closed (see Lemma 3.10) and that $\mathcal{F}$ satisfies [ii], module a change of coordinates. We will show that $\mathcal{F}$ satisfies [iii]. Observe that

$$D = \Delta_{\mathcal{F}} - F_0 - F_1 - F_3 - M \geq 0$$

and by [i] it follows that $\text{ord}_M(\Delta_{\mathcal{F}}) = \text{ord}_{F_i}(\Delta_{\mathcal{F}}) = 1$ for all $i$ (see §3, Fact 2.8]). Since

$$\Delta_{\mathcal{F}} = pK_{\mathcal{F}} + N_{\mathcal{F}} \equiv pF_0 + 3F_0 + 2M_0 \equiv (p + 2)F_0 + 2M_0,$$

the following equality holds

$$D \cdot F = \Delta_{\mathcal{F}} \cdot F - M \cdot F = 2 - 1 = 1.$$ 

So, there exists an irreducible curve $C \subseteq D$ such that $C \cdot F = 1$. Write $D = C + R$ and note that we have $C \equiv uF_0 + M_0$ and $R \equiv vF_0$, for some $u, v \in \mathbb{Z}_{\geq 0}$. We will show that $v \equiv 0 \mod p$. Suppose, by contradiction, that $v \not\equiv 0$. Then, since $R \equiv vF_0$ we conclude by the Proposition 3.7 that there exists a fiber of the first projection, $F_4 \leq R$, that is $\mathcal{F}$-invariant. In the other hand, by construction of $\mathcal{F}$, we know that $F_0, F_1$ and $F_2$ are the complete list of $\mathcal{F}$-invariant fibers. Moreover, all that fibers has multiplicity 1 along the $p$-divisor $\Delta_{\mathcal{F}}$. So, the fiber $F_4$ can not exist and we conclude that $v \equiv 0 \mod p$. In the other hand,

$$u + v = C \cdot M_0 + R \cdot M_0 = D \cdot M_0 = \Delta_{\mathcal{F}} \cdot M_0 - 3 = p + 3 - 3 = p.$$ 

So, if $v \not\equiv 0 \mod p$ implies that $v = p$ and by consequence it follows $u = 0$. But this implies $C \equiv M$ so that $C$ is a fiber of the second projection which is $\mathcal{F}$-invariant. This is a contradiction, since by construction we know that $M$ is the unique fiber of the second projection that is $\mathcal{F}$-invariant and satisfies $\text{ord}_M(\Delta_{\mathcal{F}}) = 1$. So, $v = 0$ and $C$ is an irreducible curve $\mathcal{F}$-invariant with $C \equiv pF_0 + M_0$ and $\mathcal{F}$ has $p$-divisor given by

$$\Delta_{\mathcal{F}} = F_0 + F_1 + F_2 + M + C$$

which is reduced. So, $\mathcal{F}$ satisfies [i],[ii] and [iii] and we conclude the argument to the case $(d_1, d_2) = (1, 0)$.

**Case $l > 0$ and $p \nmid l$:** We use the precedent case ($l = 1$) to study the present case. Let $\mathcal{F}$ be a foliation in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ as in the precedent case. As we had proved, a generic foliation which satisfies [i],[ii] and [iii] has $p$-divisor in the form

$$\Delta_{\mathcal{F}} = F_0 + F_1 + F_3 + M + C$$
where $C$ is an irreducible curve with $C \equiv pF_0 + M_0$, $F_0$, $F_1$ and $F_2$ are fibers of the first projection $\pi_1$ and $M$ is a fiber of the second projection $\pi_2$.

Let $\Phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k \times \mathbb{P}^1_k$ be a finite ramified map of degree $l$ with ramification divisor $R$. Suppose that $\Phi$ ramifies only along the curves $F_0$ and $F_1$. Let $\mathcal{G} = \Phi^*\mathcal{F}$ the foliation defined by the pull-back of $\mathcal{F}$ by $\Phi$. Explicitly, if $\Omega$ is the bi-homogeneous 1-form which defines $\mathcal{F}$ then $\mathcal{G}$ is the foliation defined by the saturation of $\Phi^*\Omega$. Observe that since $F_0$ and $F_1$ are $\mathcal{F}$-invariant $K_\mathcal{G} = \Phi^*K_\mathcal{F}$. So, $\Phi^*K_\mathcal{F} = \Phi^*F = lF$ and we conclude that $\mathcal{G}$ satisfies (ii) and $N_\mathcal{G} \equiv (l+2)F + 2M$ and $\Delta_\mathcal{G} \equiv pK_\mathcal{G} + N_\mathcal{G} \equiv (pl + l + 2)F + 2M$. By Lemma 4.1 we know that $\Phi^*C$ and $\Phi^*F_2$ are reduced curves that has no irreducible components in common. In particular,

\[
\Delta_\mathcal{G} \geq \Phi^*C + \Phi^*F_2 + F_1 + F_0 + M.
\]

in the other hand, $\Phi^*C \equiv \Phi^*(pF + M) \equiv plF + M$ and $\Phi^*F_2 = lF$ and so

\[
\Phi^*C + \Phi^*F_2 + F_0 + F_1 + M = (pl + l + 2)F + 2M
\]

and by comparison degree we obtain the following equality

\[
\Delta_\mathcal{G} = \Phi^*C + \Phi^*F_2 + F_1 + F_0 + M.
\]

In particular, $\Delta_\mathcal{G}$ is a reduced divisor and $\mathcal{G}$ satisfies (iii). We will show that $\mathcal{G}$ satisfies (i). For that, let $Q$ the point in $F_0 \cap M$. By the precedent case, more precisely, by (iii) we know that $Q$ is a $p$-reduced singularity. Fix $U$ an affine open set around $Q = (0,0)$ such that $\Phi|_{\Phi^{-1}(U)} : \Phi^{-1}(U) \to U$ is locally defined by $\Phi : (x, y) \mapsto (x^l, y) = (\tilde{x}, \tilde{y})$ and $\mathcal{F}$ is given by the 1-form $\omega = \alpha jdx - \tilde{x}d\tilde{y} + O(2)$. Then, $\Phi^*\mathcal{F}$ is given on $\Phi^{-1}(U)$ by the 1-form $\sigma = l\alpha ydx - xdy + O(2)$. Since we are assuming that $Q$ is a $p$-reduced singularity $\alpha \not\in \mathbb{F}_p$ and so $l\alpha \not\in \mathbb{F}_p$. We conclude that $\mathcal{G}$ has $\{Q\} = F \cap M$ as $p$-reduced singularity. In particular, satisfies (ii). This finishes the proof of the case $(d_1, d_2) = (l, 0)$ with $l > 1$ and $p \nmid l$. The cases above finish the proof of the proposition. \hfill \Box

5.2 Case B

In this subsection, we show that a generic foliation $\mathcal{G}$ in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ with cotangent divisor $K_\mathcal{G} \equiv d_1F + d_2M$ satisfies the following

(i) $K_\mathcal{G} \equiv d_1F_0 + d_2M_0$ where $F_0 = \{x_0 \equiv 0\}$ and $M_0 = \{y_0 \equiv 0\}$,

(ii) $F_0$ and $M = \{y_1 \equiv 0\}$ are $\mathcal{G}$-invariant curves with $\{Q\} = F_0 \cap M$ a $p$-reduced singularity of $\mathcal{G}$,

(iii) The $p$-divisor $\Delta_\mathcal{G}$ is reduced

when $(d_1, d_2) \in \{(l_1, l_2) \in \mathbb{Z}^2 \mid l_1, l_2 > 0 \text{ and } p \nmid l_1l_2\}$. We start with the case $d_1 = d_1 = 1$.

**Proposition 5.6.** A generic foliation in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ with cotangent divisor $K_\mathcal{G} \equiv F + M$ satisfies (i), (ii) and (iii).
Proof. Indeed, in the open set $U_{11}$ consider the foliation defined by the 1-form: $\omega = ydx - xdy + b(x,y)dx + \tilde{a}(x,y)dy$. In the proof of Theorem $\square$ we saw that for a generic choice of $\tilde{a}, \tilde{b} \in k[x,y]$ the 1-form $\omega$ defines a foliation with reduced $p$-divisor in $\mathbb{A}^2_k$ which is explicitly given by $\Delta = l_1 + l_2 + l_3 + C$ where $l_1, l_2, l_3$ are distinct lines that pass through point $(0,0)$ and $C$ is an irreducible curve that passes through point $(0,0)$ of degree $p$ and with multiplicity $p - 1$ over $(0,0)$. Moreover, by the Remark $\square$ we can assume that $C$ does not pass through points $\{(0 : 1 : 0), (1 : 0 : 0)\}$. For simplicity, we will assume that $l_1 = \{x = 0\}$, $l_2 = \{y = 0\}$ and $l_3 = \{ux + vy = 0\}$ for some non zero constants $u, v \in k$.

In this case, we have $\tilde{a}(x,y) = x\alpha(x,y)$ and $b(x,y) = y\beta(x,y)$ for some $a, b \in k[x,y]$.

Using the isomorphism $\phi: U_{11} \rightarrow \mathbb{A}^2_k$ which associates $([x_0 : x_1], [y_0 : y_1]) \rightarrow \left(\frac{x_0}{x_1}, \frac{y_0}{y_1}\right)$ we can compactify $\omega$ to a foliation in $\mathbb{P}^1_k \times \mathbb{P}^1_k$. In this case, the foliation obtained is explicitly given by the 1-form: $\Omega = A_0dx_0 + A_1dx_1 + B_0dy_0 + B_1dy_1$ where

$$A_0 = x_1y_0y_1(x_1y_1 + b(x_0y_1, x_1y_0)), \quad A_1 = -x_0y_0y_1(x_1y_1 + b(x_0y_1, x_1y_0)),$$

$$B_0 = x_0x_1y_1(-x_1y_1 + a(x_0y_1, x_1y_0)), \quad B_1 = -x_0x_1y_0(-x_1y_1 + a(x_0y_1, x_1y_0)).$$

The projective 1-form $\Omega$ defines a foliation $\mathcal{F}_\Omega$ in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ such that $K_{\mathcal{F}_{\Omega}} \equiv F + M$ and so $\mathcal{F}_{\Omega}$ satisfies $(\square)$. We claim that $\Delta_{\mathcal{F}_{\Omega}}$ is reduced. Indeed, note that

$$\Delta_{\mathcal{F}_{\Omega}} \equiv pK_{\mathcal{F}_{\Omega}} + N_{\mathcal{F}_{\Omega}} \equiv p(F + M) + (3F + 3M) = (p + 3)(F + M).$$

By the homogeneous equations of $\mathcal{F}_{\Omega}$, we see that $F_0 = \{x_0 = 0\}, F_1 = \{x_1 = 0\}, M_0 = \{y_0 = 0\}$ and $M = \{y_1 = 0\}$ are $\mathcal{F}_{\Omega}$-invariant. Moreover, bi-projectivizing the line $L$ via $\Phi$ in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ we know by the Lemma $\square$ that the obtained curve $L$, is irreducible $\mathcal{F}_{\Omega}$-invariant with $\mathbb{L} \equiv F_0 + M_0$. Since $C$ does not pass through the point $\{(0 : 1 : 0), (1 : 0 : 0)\}$ we can apply the Lemma $\square$ to conclude that the irreducible curve of degree $p$ in $\mathbb{A}^2_k$ projectivize to an irreducible curve, $\mathbb{C}$, in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ that is $\mathcal{F}_{\Omega}$-invariant and such that: $\mathbb{C} \equiv p(F_0 + M_0)$. So, we obtain $\Delta_{\mathcal{F}_{\Omega}} = F_0 + F_1 + M_0 + M + \mathbb{L} + \mathbb{C}$ and we ensure that $\Delta_{\mathcal{F}_{\Omega}}$ is a reduced divisor. This shows that $\mathcal{F}$ satisfies $(\square)$.

Observe that restricting the foliation to the open set $U_{10} = \{y_0 \neq 0\} \cap \{x_1 \neq 0\}$ the foliation in $\mathbb{A}^2_k$ is given by the 1-form $\sigma = y_1(y_1 + b(x_0y_1, 1))dx_0 - x_0(-y_1 + a(x_0y_1, 1))dy_1$. Since $a, b \in k[x,y]$ are generic, we can assume that $a(x_0y_1, 1) = a_0 + O(2)$ and $b(x_0y_1, 1) = b_0 + O(2)$ where $a_0/b_0 \in \mathbb{F}_p$, so that we obtain a foliation in $\Sigma_0$ which satisfies $(\square)$ and $(\square)$ with $d_1 = d_2 = 1$. 

Using the precedent proposition we will consider the general case.

**Proposition 5.7.** Let $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ such that $p \nmid d_1$, if $d_i \neq 0$. Suppose that

$$\langle d_1, d_2 \rangle \in \{(l_1, l_2) \in \mathbb{Z}^2 | l_1, l_2 > 0 \text{ and } p \nmid l_1l_2\}.$$

Then, a generic foliation in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ with cotangent divisor $K_G \equiv d_1F + d_2M$ satisfies $(\square), (\square)$ and $(\square)$. 

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Proof. Let $l \in \mathbb{N}$ be a positive integer coprime to $p$ and consider the finite map:

$$\Phi_l : \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k \times \mathbb{P}^1_k$$

$$([x_0, x_1], [y_0, y_1]) \mapsto (l x_0 y_0 y_1, l x_1 y_0 y_1 + b(x_0^l, x_1^l)).$$

Let $R_{\Phi_l}$ be the ramification divisor of $\Phi_l$ and $G$ the foliation in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ described by the saturation of the 1-form $\Phi_l^* \Omega$, where $\Omega$ is the projectivization of the 1-form in $\mathbb{A}^2_k$ given by $\sigma = y_1(1 + b(x_0 y_1, 1)) dx_0 - x_0(-y_1 + a(x_0 y_1, 1)) dy_1$ where $a, b \in k[x, y]$ are generic with $a(x_0 y_1, 1) = a_0 + O(2)$, $b(x_0 y_1, 1) = b_0 + O(2)$ and $a_0/b_0 \notin \mathbb{F}_p$. Recall that by Proposition 5.6 we can assume that the foliation defined by $\Omega$, $\mathcal{F}_\Omega$, is not $p$-closed with $p$-divisor given by $\Delta_{\mathcal{F}_\Omega} = F_0 + F_1 + M_0 + M + L + C$ where $F_0 = \{x_0 = 0\}$, $F_1 = \{x_1 = 0\}$, $M_0 = \{y_0 = 0\}$, $M = \{y_1 = 0\}$ and with $L, C$ irreducible curves such that $C \equiv p(F_0 + M_0)$ and $L \equiv F_0 + M_0$. We have $R_{\Phi_l} = (l - 1) F_0 + (l - 1) F_1 \equiv 2(l - 1) F_0$ and since $F_0$ and $F_1$ are $\mathcal{F}_\Omega$-invariant we conclude that $K_G = \Phi^* K_\mathcal{F} = \Phi^*(F \div M) = pF + M$. In explicit terms, the foliation $G$ is defined by the 1-form: $\gamma = A_0 dx_0 + A_1 dx_1 + B_0 dy_0 + B_1 dy_1$ where

$$A_0 = lx_0 y_0 y_1(x_1^l y_0 + b(x_0^l, x_1^l)),$$

$$B_0 = x_0 x_1 y_1(-x_1^l y_0 + a(x_0^l, x_1^l)),$$

and

$$A_1 = -lx_0 y_0 y_1(x_1^l y_0 + b(x_0^l, x_1^l)),$$

$$B_1 = -x_0 x_1 y_0(-x_1^l y_0 + a(x_0^l, x_1^l)).$$

We will show the following equality: $\Delta_G = F_0 + F_1 + M_0 + M + \Phi_l^* L + \Phi_l^* C$. Note that the curves $\Phi_l^* L$ and $\Phi_l^* C$ are $G$-invariant. Indeed, since $\gcd(l, p) = 1$ and $x_0, x_1$ do not divide the equations which define $L$ and $C$, using the Lemma 4.1 we conclude that $\Phi_l^* L$ and $\Phi_l^* C$ are reduced and have distinct irreducible components. In particular, follows that $\Phi_l^* L$ and $\Phi_l^* C$ are $G$-invariant (see Proposition 3.7). Since $F_0, F_1, M_0, M$ are $G$-invariant $\Delta_G \geq F_0 + F_1 + M_0 + M + \Phi_l^* L + \Phi_l^* C$. Note that $\Delta_G$ is a divisor that has bi-degree given by the formula:

$$pK_G + N_G \equiv p(lF_0 + M_0) + ((l + 2)F_0 + 3M_0) \equiv (pl + l + 2)F_0 + (p + 3)M_0.$$

In the other hand

$$F_0 + F_1 + M_0 + M + \Phi_l^* L + \Phi_l^* C \equiv F_0 + F_0 + M_0 + M_0 + (lF_0 + M_0) + (plF_0 + pM_0)$$

$$= (pl + l + 2)F_0 + (p + 3)M_0.$$

So, by bi-degree comparison, we conclude the following equality $\Delta_G = F_0 + F_1 + M_0 + M + \Phi_l^* L + \Phi_l^* C$. This shows that $G$ satisfies (iii).

We need to show that $F_0 \cap M = \{x_0 = 0\} \cap \{y_1 = 0\}$ is a $p$-reduced singularity of $G$. Indeed, restricting the foliation $G$ to the open set $U_{10}$ we obtain a foliation in $\mathbb{A}^2_k$ given by the 1-form $\omega = l(y + b(x^l y_1, 1)) dx - x(-y + a(x^l y_1, 1)) dy$ and we conclude that $(0, 0)$ is $p$-reduced since we are assuming that $a(x_0 y_1, 1) = a_0 + O(2)$ and $b(x_0 y_1, 1) = b_0 + O(2)$ with $a_0/b_0 \notin \mathbb{F}_p$. By symmetry, that is, replacing $F_0$ by $M_0$, follows that given $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ with $p \nmid d_1$ (if $d_1 \neq 0$) we can construct foliations in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ with the following properties:

(i) The canonical divisor $K_G$ is numerically equivalent to the divisor $d_1 F_0 + d_2 M_0$ where $F_0$ and $M_0$ are fibers of the first and second projection, respectively:
(ii) The curves $F_0$ and $M_0$ are $\mathcal{G}$-invariant and $\{Q\} = F_0 \cap M_0$ is a $p$-reduced singularity of $\mathcal{G}$;

(iii) The $p$-divisor $\Delta_{\mathcal{G}}$ is reduced.

This finishes the proof of the proposition. \hfill \Box

Considering the junction of the case A and B we obtain the complete proof of the Theorem 5.3.

6 The $p$-divisor for foliations on Hirzebruch surfaces

Let $d \in \mathbb{Z}_{>0}$ and $M_d$ be a section of the natural projection $\pi: \Sigma_d \rightarrow \mathbb{P}^1_k$ and $F$ be a fiber of $\pi$ such that $M_d^2 = \mathcal{F}$ and $F = 1$. The curves $M_d$ and $F$ form a base of the vector space $\text{Num}_{Q}(\Sigma_d)$. Note that for any other divisor $D \in \text{Div}(\Sigma_d)$ such that $(D, F) = \text{Num}_{Q}(\Sigma_d)$ and which satisfies $D^2 = d$ and $D \cdot F = 1$ we have $D = M_d$. Indeed, write $D = aM_d + bF$ for some $a, b \in \mathbb{Q}$ and observe that $1 = D \cdot F = a$ and $d = D^2 = a^2d + 2ab = d + 2b$ and so $b = 0$.

Fix $\{F, M_d\}$ as a basis for $\text{Num}_{Q}(\Sigma_d)$ and let $Q$ be the point in the intersection $F \cap M$, where $M = \{y_1 = 0\}$ is the curve in $\Sigma_d$ with negative self-intersection, that is, $M^2 = -d$. Let $Bl_Q: Bl_Q(\Sigma_d) \rightarrow \Sigma_d$ the blowup of $\Sigma_d$ at $Q$ and $c_F: Bl_Q(\Sigma_d) \rightarrow Q^{d+1}$ the map that consists in the contraction of the strict transform of $F$ in $Bl_Q(\Sigma_d)$. Let $E$ the exceptional divisor associated to $Bl_Q$ and $\Phi_{d+1}: \Sigma_{d+1} \rightarrow \Sigma_d$ the rational map which consist in the composition of birational maps:

$$\Phi_{d+1}^* = Bl_Q \circ c_F^{-1}: \Sigma_{d+1} \rightarrow Bl_Q(\Sigma_d) \rightarrow \Sigma_d.$$ 

Let $\tilde{M}_{d+1}$ the strict transform of $M_d$ in $Bl_Q(\Sigma_d)$ and consider $E = (c_F)_*\mathcal{E}$ and $M_{d+1} = (c_F)_*\tilde{M}_{d+1}$ the induced curves by the contraction. Observe that $\{E, M_{d+1}\}$ forms a basis for $\text{Num}_{Q}(\Sigma_d)$ and satisfies the following conditions: $E^2 = 0$, $E \cdot M_{d+1} = 1$ and $M_{d+1}^2 = d + 1$.

**Lemma 6.1.** If $\mathcal{F}$ is a foliation in $\Sigma_d$ with $K_{\mathcal{F}} \equiv d_1F + d_2M_d$ then $\mathcal{G} = (\Phi_{d+1}^*)^* \mathcal{F}$ is a foliation in $\Sigma_{d+1}$ with $K_{\mathcal{G}} \equiv (d_1 - l(Q) + 1)E + d_2M_{d+1}$.

**Proof.** Let $\mathcal{H} = Bl_Q^* \mathcal{F}$ be the induced foliation in $Bl_Q(\Sigma_d)$. Then, $K_{\mathcal{G}} = (c_F)_*K_{\mathcal{H}}$ and

$$K_{\mathcal{H}} = Bl_Q^*K_{\mathcal{F}} + (1 - l(Q))E \equiv Bl_Q^*(d_1F + d_2M_d) + (1 - l(Q))\mathcal{E}.$$ 

Since $Bl_Q^*F = \mathcal{F} + E$ and $Bl_Q^*M_d = \tilde{M}_{d+1}$, we conclude that $K_{\mathcal{H}} \equiv d_1\tilde{F} + d_2\tilde{M}_{d+1} + (d_1 - l(Q) + 1)\mathcal{E}$ and it follows

$$K_{\mathcal{G}} = (c_F)_*K_{\mathcal{H}} \equiv (d_1 - l(Q) + 1)(c_F)_*\mathcal{E} + d_2(c_F)_*\tilde{M}_{d+1} \equiv (d_1 - l(Q) + 1)E + d_2M_{d+1}$$

which finishes the proof of the lemma. \hfill \Box
Theorem B. Let \( \Sigma_d \) the d-Hirzebruch surface over \( k \) and \( d_1, d_2 \in \mathbb{Z}_{\geq 0} \) such that \( p \nmid d_i \), if \( d_i \neq 0 \). Then, a generic foliation in \( \Sigma_d \) with normal bundle \( N \equiv (d_1 - d + 2)F + (d_2 + 2)M_d \) has reduced \( p \)-divisor. More precisely, \( U(\Sigma_d, \mathcal{O}_\Sigma_d(F + M_d), N) \neq \emptyset \) if

- \( N \cdot F - 2 \geq 0 \) and \( p \nmid N \cdot F - 2 \) (if nonzero);
- \( \deg(N) - (1 + d)N \cdot F + d - 2 \geq 0 \) and \( p \nmid \deg(N) - (1 + d)N \cdot F + d - 2 \) (if nonzero).

Proof. The proof will be done by induction on \( d \). The case \( d = 0 \) was considerate in the Theorem 5.3. Suppose that the result is true for all \( j \)-Hirzebruch surfaces with \( j \leq d \), that is, suppose that for each \( j \in \{0, \ldots, d\} \) and for each \( d_1, d_2 \in \mathbb{Z}_{\geq 0} \) with \( p \nmid d_i \) (if \( d_i \neq 0 \)) we can find a foliation \( \mathcal{H} \) such that

(i) \( K_{d} \equiv d_1F_0 + d_2M_j \), where \( F_0 \) is a fiber of the natural projection \( \pi: \Sigma_j \rightarrow \mathbb{P}^1_k \) and \( M_j \) is a section of \( \pi \) such that \( M_j^2 = j \) and \( M_j \cdot F = 1 \);

(ii) If \( M \) is the section of negative self-intersection of \( \Sigma_d \) then \( F_0 \) and \( M \) are \( \mathcal{H} \)-invariant and \( \{Q\} = F_0 \cap M \) is a \( p \)-reduced singularity of \( \mathcal{H} \) where \( M^2 = -j \);

(iii) The \( p \)-divisor \( \Delta_m \) is reduced.

We will show that the same holds over \( \Sigma_{d+1} \). Fix \( d_1, d_2 \in \mathbb{Z}_{\geq 0} \) such that \( p \nmid d_i \), if \( d_i \neq 0 \). Let \( \mathcal{G} \) a foliation in \( \Sigma_d \) which satisfies (i), (ii) and (iii) for \( j = d \).

Let \( Q \) the \( p \)-reduced singularity in \( F_0 \cap M \) and \( \pi_Q: Bl_Q(\Sigma_d) \rightarrow \Sigma_d \) the blowup with center at \( Q \). By the Lemma 3.11, we know that \( Bl_Q(\Sigma_d) \mathcal{F} \) is a foliation in \( Bl_Q(\Sigma_d) \) with reduced \( p \)-divisor. More precisely, if \( \overline{E} \) denotes the exceptional divisor and \( \mathcal{G} = \pi_Q^*\mathcal{F} \), we know that \( \Delta_{\mathcal{G}} = \Delta_{\mathcal{F}} + \overline{E} \). We claim that since \( Q \) is \( p \)-reduced we ensure that over \( \overline{E} \) there exists at least a \( p \)-reduced singularity of \( \mathcal{G} \).

Indeed, in an open neighborhood of \( Q \) the foliation is given by a 1-form of the type \( \omega = \alpha y dx + xdy + O(2) \) and considering the chart \( \pi_Q^*\mathcal{F}(x, t) \rightarrow (x, t) \) we see that in a neighborhood of \( \overline{E} \) the foliation \( \mathcal{G} \) is given by the 1-form \( \sigma = (\alpha + 1)tdx + xdt + O(2) \). In particular, \( Q \) is \( p \)-reduced. Denote by \( c_F: Bl_Q(\Sigma_d) \rightarrow \mathcal{Y} \) the contraction of the line \( \mathcal{F} \) that is the strict transform of \( F_0 \). Since \( Q \in F_0 \), the line \( \mathcal{F} \) has self-intersection \(-1 \) so that there exists that contraction. The out surface, \( \mathcal{Y} \), is the Hirzebruch surface of type \( d + 1 \) \( (\Sigma_{d+1}) \). We will show that \( c_F(\mathcal{G}) \) is a foliation in \( \Sigma_{d+1} \) that satisfies (i), (ii) and (iii). Let \( \overline{E} \) the exceptional divisor associated with the blowup \( Bl_Q \) and \( M_{d+1} \) the strict transform of the curve \( M_d \). Denote by \( E = c_F(\overline{E}) \) and \( M_{d+1} = c_F(\overline{M_{d+1}}) \) the induced curves by the contraction \( c_F \). Observe that we have the following formulas: \( E^2 = 0, E \cdot M_{d+1} = 1 \) and \( M_{d+1}^2 = d + 1 \). If \( \{P\} = c_F(\overline{F}) \) then \( c_F(\overline{Bl_Q(\Sigma_d)} - \{\overline{F}\}) \rightarrow \Sigma_{d+1} - \{P\} \) is an isomorphism. In particular, as \( \Delta_{\mathcal{F}} \) is reduced we have \( \Delta_{c_F(\mathcal{G})} \) is reduced and so that we have (iii). The local verification done above shows that \( E \cap M = \{Q_2\} \) is a \( p \)-reduced singularity of \( c_F(\mathcal{G}) \) and the Lemma 6.1 ensures that \( K_{d} \equiv d_1E + d_2M_{d+1} \). So, we obtains (i). This finishes the proof of the theorem. \( \square \)

Remark 6.2. Let \( d \in \mathbb{Z}_{\geq 0} \) and \( F \) be a section of the natural projection \( \pi: \Sigma_d \rightarrow \mathbb{P}^1_k \) and \( M_d \) a section of \( \pi \) such that \( F \cdot M_d = 1 \) and \( M_d^2 = d \). Define

\[ S_d = \{(d_1, d_2) \in \mathbb{Z}^2 \mid \text{there exists a foliation in } \Sigma_d \text{ with canonical divisor } K \equiv d_1F + d_2M_d \}. \]
Theorem B can be formulated in the following way:

Following from [5, Proposition 3.6] that:

- \( S_0 = \{(d_1, d_2) \in \mathbb{Z}^2 \mid d_1, d_2 \geq 0\} \cup \{(-2, 0)\} \cup \{(0, -2)\}. \)
- \( If \ d > 0: \ S_d = \{(d_1, d_2) \in \mathbb{Z}^2 \mid d_1 \geq -1, d_2 \geq 0\} \cup \{(d, -2)\}. \)

Now, define

\( R_d = \{(d_1, d_2) \in \mathbb{Z}^2 \mid there \ exists \ a \ foliation \ in \ \Sigma_d \ with \ reduced \ p\text{-}\text{divisor} \ and \ K \equiv d_1F + d_2M_d\}. \)

Theorem [2] can be formulated in the following way:

- \( R_0 = S_0 - S_0(p) \cup \{(-2, 0)\} \cup \{(0, -2)\}, \)
- \( Se \ d > 0: \ R_d = S_d - S_d(p) \cup \{-1, d\} \mid d \in \mathbb{Z}_{\geq 0}\} \cup \{(d, -2)\}\)

where \( S_d(p) = \{(d_1, d_2) \in S_d \mid p \mid d_1 \ or \ p \mid d_2\} \cup \{(0, 0)\}. \)

The following proposition shows that for each point \( P \) on \( l = \{(1, c) \in \mathbb{Z}^2 \mid c \in \mathbb{Z}_{\geq 0}\}\)
we can find an example of foliation such that their \( p\text{-}\text{divisor} \ has \ a \ \text{p-factor} \).

**Proposition 6.3.** Let \( \mathcal{F} \) be a foliation that is not \( p\text{-}\text{closed} \ of \ degree \( d \) in \( \mathbb{P}_k^2 \) and \( Q \in \mathbb{P}_k^2 - \text{sing}(\mathcal{F}) \cup Z(\Delta\mathcal{F}) \). Let \( \mathcal{G} = Bl_Q\mathcal{F} \) the induced foliation in \( Bl_Q(\mathbb{P}_k^2) \). Then, \( \mathcal{G} \) is not \( p\text{-}\text{closed} \ and \( \Delta\mathcal{G} \) has a \( p\text{-}\text{factor} \. \) Moreover, \( K_\mathcal{G} \equiv -F + dM_1 \).

**Proof.** The condition \( Q \notin \text{sing}(\mathcal{F}) \) implies that \( l(Q) = 0 \). So, we have \( N_\mathcal{G} = Bl_Q N_\mathcal{F} \)
and \( K_\mathcal{G} = Bl_QK_\mathcal{F} + (1 - l(Q))E = Bl_QK_\mathcal{F} + E. \) So, \([\Delta_\mathcal{G}] = pK_\mathcal{G} + N_\mathcal{G} = Bl_Q\Delta_\mathcal{F} + pE = \Delta_\mathcal{F} + pE \) where the last equality follows from the condition \( Q \notin Z(\Delta_\mathcal{F}) \). In particular, \( \text{ord}_E(\Delta_\mathcal{G}) = p \). Now, we have \( K_\mathcal{G} = Bl_QK_\mathcal{F} + E \equiv (d - 1)(E + F) + E \equiv (d - 1)F + dE \). Write \( M_1 \equiv aF + bE \). Since \( 1 = M_1 \cdot F = b \) we have \( M_1 \equiv aF + E \) and since \( 1 = M_1^2 = (aF + E)^2 = 2a - 1 \) we obtain \( a = 1 \). So, \( M_1 = F + E \) and this implies \( E = M_1 - F \). So, \( K_\mathcal{G} \equiv (d - 1)F + dE \equiv (d - 1)F + d(M_1 - F) \equiv -F + dM_1 \) which ends the proof of the proposition.

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