On the Convergence of Orthogonal/Vector AMP: Long-Memory Message-Passing Strategy

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Abstract—This paper proves the convergence of Bayes-optimal orthogonal/vector approximate message-passing (AMP) to a fixed point in the large system limit. The proof is based on Bayes-optimal long-memory (LM) message-passing (MP) that is guaranteed to converge systematically. The dynamics of Bayes-optimal LM-AMP is analyzed via an existing state evolution framework. The obtained state evolution recursions are proved to converge. The convergence of Bayes-optimal orthogonal/vector AMP is proved by confirming an exact reduction of the state evolution recursions to those for Bayes-optimal orthogonal/vector AMP.

I. INTRODUCTION

Consider the problem of reconstructing an $N$-dimensional sparse signal vector $x \in \mathbb{R}^N$ from compressed, noisy, and linear measurements $y \in \mathbb{R}^M$ [1], [2] with $M \leq N$, given by

$$y = Ax + w.$$ (1)

In (1), $w \sim \mathcal{N}(0, \sigma^2 I_M)$ denotes an additive white Gaussian noise (AWGN) vector with variance $\sigma^2 > 0$. The matrix $A \in \mathbb{R}^{M \times N}$ represents a known sensing matrix. The signal vector $x$ has zero-mean independent and identically distributed (i.i.d.) elements with unit variance. The triple $\{A, x, w\}$ is assumed to be independent random variables.

A promising approach to efficient reconstruction is message passing (MP), such as approximate MP (AMP) [3], [4] and orthogonal/vector AMP (OAMP/VAMP) [5], [6]. When the sensing matrix has zero-mean i.i.d. sub-Gaussian elements, AMP was proved to be Bayes-optimal in the large system limit [7], [8], where both $M$ and $N$ tend to infinity with the compression ratio $\delta = M/N \in (0, 1]$ kept constant. However, AMP fails to converge for non-i.i.d. cases, such as the non-zero mean case [9] and the ill-conditioned case [10].

OAMP [5] or equivalently VAMP [6] is a powerful MP algorithm to solve this convergence issue for AMP. In this paper, these MP algorithms are referred to as OAMP. When the sensing matrix is right-orthogonally invariant, OAMP was proved to be Bayes-optimal in the large system limit [6], [11].

Strictly speaking, the Bayes-optimality of OAMP requires an implicit assumption under which state evolution recursions for OAMP converge to a fixed point after an infinite number of iterations [6], [11]. Thus, this assumption needs to be confirmed for individual problems [12]–[14]. The purpose of this paper is to prove this assumption for OAMP using the Bayes-optimal denoiser—called Bayes-optimal OAMP.

The proof is based on a Bayes-optimal long-memory (LM) MP algorithm that is guaranteed to converge systematically. LM-MP uses messages in all preceding iterations to update the current message while conventional MP utilizes messages only in the latest iteration. LM-MP was originally proposed via non-rigorous dynamical functional theory [15], [16] and formulated via rigorous state evolution [17]. A unified framework in [17] was used to propose convolutional AMP [17], [18], memory AMP [19], and VAMP with warm-started conjugate gradient (WS-CG) [20]. See [21], [22] for another state evolution approach.

A first step in the proof is a formulation of Bayes-optimal LM-OAMP, in which a message in the latest iteration is regarded as an additional measurement that depends on all preceding messages. Thus, use of the additional measurement never degrades the reconstruction performance if statistical properties of the dependent messages are grasped completely and if the current message is updated in a Bayes-optimal manner.

A second step is an application of the unified framework in [17] to state evolution analysis for LM-OAMP. It is sufficient to confirm that a general error model in [17] contains an error model for LM-OAMP. The obtained state evolution recursions represent asymptotic correlation structures for all messages in LM-OAMP. Furthermore, asymptotic Gaussianity for estimation errors [17] implies that the correlation structures provide full information on the asymptotic distributions of the estimation errors. As a result, it is possible to update the current message in a Bayes-optimal manner.

A third step is to prove that state evolution recursions for Bayes-optimal LM-OAMP converge to a fixed point under mild assumptions. While the convergence is intuitively expected from the formulation of Bayes-optimal LM-OAMP, a rigorous proof is non-trivial and based on a novel statistical interpretation for optimized LM damping in [19].

The last step is an exact reduction of state evolution recursions for Bayes-optimal LM-OAMP to those for conventional Bayes-optimal OAMP [5]. Thus, the convergence of Bayes-optimal LM-OAMP implies the convergence of conventional Bayes-optimal OAMP to a fixed point. As a by-product, the LM-MP proof strategy in this paper claims that conventional Bayes-optimal OAMP is the best in terms of convergence speed among all possible LM-MP algorithms included into a unified framework in [17].

The remainder of this paper is organized as follows: Sec-
tion reviews Bayes-optimal estimation based on dependent Gaussian measurements. The obtained results reveal a statistical interpretation for optimized LM damping, which is utilized to formulate LM-OAMP in Section III Section LV presents state evolution analysis of LM-OAMP via a unified framework in [17]. Two-dimensional (2D) discrete systems—called state evolution recursions—are derived to describe the asymptotic dynamics of LM-OAMP. Furthermore, we prove the convergence and reduction of the state evolution recursions. See [23] for the details of the proof.

Finally, see a recent paper [23] for an application of the LM-MP strategy in this paper.

II. CORRELATED AWGN MEASUREMENTS

This section presents a background to define the Bayes-optimal denoiser in LM-OAMP. We review Bayesian estimation of a scalar signal $X \in \mathbb{R}$ from $t + 1$ correlated AWGN measurements $Y_t = (Y_0, \ldots, Y_t) \in \mathbb{R}^{1 \times (t+1)}$, given by

$$Y_t = X^T + W_t.$$  

In (2), $1$ denotes a column vector whose elements are all one. The signal $X$ follows the same distribution as that of each element in the i.i.d. signal vector $x$. The AWGN vector $W_t \sim \mathcal{N}(0, \Sigma_t)$ is a zero-mean Gaussian row vector with covariance $\Sigma_t$ and independent of $X$. The covariance matrix $\Sigma_t$ is assumed to be positive definite.

This paper uses a two-step approach in computing the posterior mean estimator of $X$ given $Y_t$. A first step is computation of a sufficient statistic $S_t \in \mathbb{R}$ for estimation of $X$ given $Y_t$. The second step is evaluation of the posterior mean estimator of $X$ given the sufficient statistic $S_t$. This two-step approach is useful in proving Lemma II while it is equivalent to direct computation of the posterior mean estimator of $X$ given $Y_t$, i.e. $\mathbb{E}[X|S_t] = \mathbb{E}[X|Y_t]$. As shown in [23] Section II, a sufficient statistic for estimation of $X$ is given by

$$S_t = \frac{Y_t^T \Sigma_t^{-1} \tau}{1^T \Sigma_t^{-1} 1} = X + \tilde{W}_t,$$

where $\{\tilde{W}_\tau\}$ are zero-mean Gaussian random variables with covariance

$$\mathbb{E}[\tilde{W}_\tau \tilde{W}_\tau^T] = \frac{1^T \Sigma_t^{-1} \Sigma_t^{-1}}{1^T \Sigma_t^{-1} 1}.$$  

for all $t' \leq t$. An important observation is that the covariance (4) is independent of $t'$ as long as $t'$ is smaller than or equal to $t$. This is a key property in proving the reduction of Bayes-optimal LM-OAMP to conventional Bayes-optimal OAMP.

The Bayes-optimal estimator is defined as the posterior mean $\hat{f}_{opt}(S_t; \mathbb{E}[\tilde{W}_t^2]) = \mathbb{E}[X|S_t]$ of $X$ given the sufficient statistic $S_t$. The posterior covariance of $X$ given $S_t$ and $S_t'$ is given by

$$C(S'_t, S_t; \mathbb{E}[\tilde{W}_t^2], \mathbb{E}[\tilde{W}_t'^2]) = \mathbb{E}\left\{X - \hat{f}_{opt}(S'_t; \mathbb{E}[\tilde{W}_t'^2]) \mid S'_t, S_t\right\}.  \quad (5)$$

Note that the posterior covariance depends on the noise covariance $\mathbb{E}[W_t, \tilde{W}_t']$, which is not presented explicitly, because of $\mathbb{E}[W_t, \tilde{W}_t'] = \mathbb{E}[\tilde{W}_t'^2]$ for $t' \leq t$. These definitions are used to define the Bayes-optimal denoiser.

We prove key technical results to prove the convergence of state evolution recursions for Bayes-optimal LM-OAMP and its reduction to those for Bayes-optimal OAMP.

**Lemma 1:**

- $\mathbb{E}[\tilde{W}_t^2] \geq \mathbb{E}[\tilde{W}_t'^2]$ and $\mathbb{E}\{X - \hat{f}_{opt}(S'_t; \mathbb{E}[\tilde{W}_t'^2])\}^2 \geq \mathbb{E}\{X - \hat{f}_{opt}(S_t; \mathbb{E}[\tilde{W}_t^2])\}^2$ hold for all $t' < t$.
- $C(S'_t, S_t; \mathbb{E}[\tilde{W}_t^2], \mathbb{E}[\tilde{W}_t'^2]) = C(S'_t, S_t; \mathbb{E}[\tilde{W}_t'^2], \mathbb{E}[\tilde{W}_t^2])$ holds for all $t' < t$.
- If $[\Sigma_t]_{\tau, \tau'} = [\Sigma_t]_{\tau', \tau}$ holds for all $\tau' < \tau$, then $\mathbb{E}[\tilde{W}_t^2] = [\Sigma_t]_{\tau, \tau}$ holds.

**Proof:** We prove the first property. Since $\{Y_t\}_{t=0}^T$ holds for all $t' < t$, the optimality of the posterior mean estimator implies $\mathbb{E}\{X - \hat{f}_{opt}(S'_t; \mathbb{E}[\tilde{W}_t'^2])\}^2 \geq \mathbb{E}\{X - \hat{f}_{opt}(S_t; \mathbb{E}[\tilde{W}_t^2])\}^2$. The other monotonicity $\mathbb{E}[\tilde{W}_t^2] \geq \mathbb{E}[\tilde{W}_t'^2]$ follows from the monotonicity of the MSE with respect to the variance.

We next prove the second property. Since $\Delta_S \hat{S}_t \leq \Delta_S \hat{S}_t'$, the sufficient statistic $S_t'$ can be represented as

$$S'_t = S_t + Z_t', \quad Z_t' \sim \mathcal{N}(0, \mathbb{E}[\tilde{W}_t^2] - \mathbb{E}[\tilde{W}_t'^2]).$$

It is straightforward to confirm $\mathbb{E}[(S'_t - X)^2] = \mathbb{E}[\tilde{W}_t'^2]$ and $\mathbb{E}[(S_t - X)(S_t - X)] = \mathbb{E}[\tilde{W}_t^2]$. This representation implies that $S_t$ is a sufficient statistic for estimation of $X$ based on both $S_t'$ and $S_t$. Thus, we have $\mathbb{E}[X|S_t', S_t] = \mathbb{E}[X|S_t]$. Using this identity, we find that (5) reduces to

$$C(S'_t, S_t; \mathbb{E}[\tilde{W}_t^2], \mathbb{E}[\tilde{W}_t'^2]) = \mathbb{E}[X|S_t'] - \mathbb{E}[X|S_t'] \{\mathbb{E}[X|S_t', S_t] - \mathbb{E}[X|S_t]\} = 0. \quad (7)$$

Thus, the second property holds.

Before proving the last property, we prove the monotonicity $\Delta_S \hat{S}_{t, \tau} = [\Sigma_t]_{\tau, \tau} - [\Sigma_t]_{\tau+1, \tau+1} > 0$ for all $\tau$. For that purpose, we evaluate the determinant $\det \Sigma_t$. Subtracting the $(\tau + 1)$th column in $\Sigma_t$ from the $\tau$th column for $\tau = 0, \ldots, t - 1$, we use the assumptions $[\Sigma_t]_{\tau, \tau} = [\Sigma_t]_{\tau, \tau'} = [\Sigma_t]_{\tau, \tau}$ to have

$$\det \Sigma_t = \prod_{\tau=0}^{t-1} \Delta_S \hat{S}_{t, \tau}. \quad (8)$$
Since $\Sigma_t$ has been assumed to be positive definite, the determinants of all square upper-left submatrices in $\Sigma_t$ have to be positive. From (8) we arrive at $\Delta \Sigma_{\tau,t} > 0$ for all $\tau \in \{0, \ldots, t-1\}$.

Finally, we prove the last property. Using $\Delta \Sigma_{\tau,t} > 0$, the AWGN measurements $\{Y_{\tau}\}_{\tau=0}^t$ can be represented as

$$Y_t = X + V_t, \quad Y_{\tau-1} = Y_{\tau} + V_{\tau-1}$$

for $\tau \in \{1, \ldots, t\}$, where $\{V_{\tau}\}$ are independent zero-mean Gaussian random variables with variance $E[V_{\tau}^2] = \Sigma_{\tau,t}$, and $E[V_{\tau-1}^2] = \Delta \Sigma_{\tau-1,t} > 0$. This representation implies that $Y_t$ is a sufficient statistic for estimation of $X$ based on the AWGN measurements $\{Y_{\tau}\}_{\tau=0}^t$. Thus, we have the identity $E[(X - f_{\text{opt}}(S_t; E[W_t^2]))^2] = E[(X - f_{\text{opt}}(Y_t; \Sigma_{t,t}))^2]$, which is equivalent to $E[W_t^2] = \Sigma_{t,t}$.

The first property in Lemma 1 is used to prove that the mean-square error (MSE) for Bayes-optimal LM-OAMP is monotonically non-increasing. This property was utilized in convergence analysis for memory AMP [19]. The remaining two properties are used to prove the reduction of Bayes-optimal LM-OAMP to Bayes-optimal OAMP.

III. LONG-MEMORY OAMP

A. Long-Memory Processing

LM-OAMP is composed of two modules—called modules A and B. Module A uses a linear filter to mitigate multiuser interference while module B utilizes an element-wise nonlinear denoiser for signal reconstruction. An estimator of the signal vector $x$ is computed via MP between the two modules.

Each module employs LM processing, in which messages in all preceding iterations are used to update the current message, while messages only in the latest iteration are utilized in conventional MP. Let $x_{A \rightarrow B,t} \in \mathbb{R}^N \text{ and } \{v_{A \rightarrow B,t,t'} \in \mathbb{R} \}_{t'=0}^t$ denote messages that are passed from module A to module B in iteration $t$. The former $x_{A \rightarrow B,t}$ is an estimator of $x$ while the latter $v_{A \rightarrow B,t,t'}$ corresponds to an estimator for the error covariance $N^{-1}E[(x_{A \rightarrow B,t} - x)^T(x_{A \rightarrow B,t} - x)]$. The messages used in iteration $t$ are written as $X_{A \rightarrow B,t} = (x_{A \rightarrow B,0}, \ldots, x_{A \rightarrow B,t}) \in \mathbb{R}^{N \times (t+1)}$ and symmetric $V_{A \rightarrow B,t,t'} \in \mathbb{R}^{(t+1) \times (t+1)}$ with $\|V_{A \rightarrow B,t,t'}\| = v_{A \rightarrow B,t,t'}$ for all $t' \leq \tau$.

Similarly, we define the corresponding messages passed from module B to module A in iteration $t$ as $x_{B \rightarrow A,t} \in \mathbb{R}^N$ and $\{v_{B \rightarrow A,t,t'} \in \mathbb{R} \}_{t'=0}^t$. They are compactly written as $X_{B \rightarrow A,t} = (x_{B \rightarrow A,0}, \ldots, x_{B \rightarrow A,t}) \in \mathbb{R}^{N \times (t+1)}$ and symmetric $V_{B \rightarrow A,t,t'} \in \mathbb{R}^{(t+1) \times (t+1)}$ with $\|V_{B \rightarrow A,t,t'}\| = v_{B \rightarrow A,t,t'}$ for all $t' \leq \tau$.

Asymptotic Gaussianity for estimation errors is postulated in formulating LM-OAMP. While asymptotic Gaussianity is defined and proved shortly, a rough interpretation is that estimation errors are jointly Gaussian-distributed in the large system limit, i.e. $(x_{A \rightarrow B,t,t'} - x)(x_{A \rightarrow B,t,t'} - x)^T \sim \mathcal{N}(0, v_{A \rightarrow B,t,t'} I_N)$ and $(x_{B \rightarrow A,t,t'} - x)(x_{B \rightarrow A,t,t'} - x)^T \sim \mathcal{N}(0, v_{B \rightarrow A,t,t'} I_N)$. This rough interpretation is too strong to justify. Nonetheless, it helps us understand update rules in LM-OAMP.

B. Module A (Linear Estimation)

Module A utilizes $X_{B \rightarrow A,t}$ and $V_{B \rightarrow A,t}$ provided by module B to compute the mean and covariance messages $x_{A \rightarrow B,t}$ and $\{v_{A \rightarrow B,t,t'} \}_{t'=0}^t$ in iteration $t$. A first step is computation of a sufficient statistic for estimation of $x$. According to (3) and (4), we define a sufficient statistic $x_{A \rightarrow B,t}$ and the corresponding covariance $v_{A \rightarrow B,t,t'}$ as

$$x_{A \rightarrow B,t} = \frac{X_{B \rightarrow A,t} V_{B \rightarrow A,t}^1}{1^T V_{B \rightarrow A,t}^1 1},$$

$$v_{A \rightarrow B,t,t'} = \frac{1}{1^T V_{B \rightarrow A,t}^1 1}$$

for all $t' \leq t$.

In the initial iteration $t=0$, the initial values $x_{A \rightarrow B,0} = 0$ and $v_{A \rightarrow B,0,0} = E[\|x\|^2]/N$ are used to compute (10) and (11).

The sufficient statistic (10) is equivalent to optimized LM damping of all preceding messages $\{x_{A \rightarrow B,t'}\}_{t'=0}^t$ in [19], which was obtained as a solution to an optimization problem based on state evolution results. However, this statistical interpretation is a key technical tool in proving the main theorem.

A second step is computation of posterior mean $x_{A,t} \in \mathbb{R}^N$ and covariance $\{v_{A,t,t'} \}_{t'=0}^t$. A linear filter $W_t \in \mathbb{R}^{M \times N}$ is used to obtain

$$x_{A,t} = x_{A \rightarrow B,t} + W_t^T (y - A x_{A \rightarrow B,t}),$$

$$v_{A,t,t'} = \gamma_t v_{A \rightarrow B,t,t'} + \frac{\sigma^2}{N} \text{Tr} \left( W_t^T W_t^T \right),$$

with

$$\gamma_t = \frac{1}{N} \text{Tr} \left( (I_N - W_t^T A)^T (I_N - W_t^T A) \right).$$

In this paper, we focus on the linear minimum mean-square error (LMMSE) filter

$$W_t = v_{A \rightarrow B,t,t} \left( \sigma^2 I_M + \frac{v_{A \rightarrow B,t,t}}{v_{A \rightarrow B,t,t'}} A A^T \right)^{-1} A.$$ (15)

The LMMSE filter minimizes the posterior variance $v_{A,t,t'}$ among all possible linear filters.

The last step is computation of extrinsic messages $x_{A \rightarrow B,t} \in \mathbb{R}^N$ and $\{v_{A \rightarrow B,t,t'} \}_{t'=0}^t$ to realize asymptotic Gaussianity in module B. Let

$$\xi_{A,t,t'} = \frac{e_{t'} v_{B \rightarrow A,t,t'}}{1^T V_{B \rightarrow A,t}^1 1},$$

where $e_t$ is the $t$th column of $I$, with

$$\xi_{A,t} = \frac{1}{N} \text{Tr} \left( I_N - W_t^T A \right).$$

The extrinsic mean $x_{A \rightarrow B,t}$ and covariance $\{v_{A \rightarrow B,t,t'} \}_{t'=0}^t$ are computed as

$$x_{A \rightarrow B,t} = \frac{x_{A \rightarrow B,t} - \sum_{t'=0}^t \xi_{A,t,t'} x_{B \rightarrow A,t,t'}}{1 - \xi_{A,t}},$$

(18)
The numerator in (13) is the so-called Onsager correction of the posterior mean $\mathbf{x}_{A,t}^\text{post}$ to realize asymptotic Gaussianity. The denominator can be set to an arbitrary constant. In this paper, we set the denominator so as to minimize the extrinsic variance $\nu_{A \rightarrow B,t,t}$ for the LMMSE filter [15]. See Section II for the details.

C. Module B (Nonlinear Estimation)

Module B uses $\mathbf{X}_{A \rightarrow B,t}$ and $\mathbf{V}_{A \rightarrow B,t}$ to compute the messages $x_{t}^{\text{post}}$ and $\{v_{B,t}^{\text{post}}\}^{t+1}_{t=0}$ in the same manner as in module A. A sufficient statistic $\mathbf{x}_{A,t}^\text{post} \in \mathbb{R}^{N}$ and the corresponding covariance $\{\nu_{A \rightarrow B,t,t} \in \mathbb{R}\}^{t=0}$ are computed as

$$
\nu_{A \rightarrow B,t,t} = \frac{\mathbf{X}_{A \rightarrow B,t} \mathbf{V}_{A \rightarrow B,t}^{-1} \mathbf{1}}{1^{T} \mathbf{V}_{A \rightarrow B,t}^{-1} \mathbf{1}},
$$

where the right-hand side (RHS) of (22) means the element-wise application of $f_{\text{opt}}(\cdot)$ and covariance $\gamma$ for the correlated AWGN measurements,

$$
\mathbf{x}_{B,t+1}^{\text{post}} = f_{\text{opt}}(\mathbf{x}_{A,t}^{\text{post}}; v_{A \rightarrow B,t,t}),
$$

$$
v_{B,t+1,t+1} = \frac{1}{N} \sum_{n=1}^{N} C(\nu_{A,t}^{\text{post}}, \nu_{A,t}^{\text{post}}; \mathbf{x}_{A,t}^{\text{post}}, \mathbf{x}_{A,t}^{\text{post}}),
$$

where the right-hand side (RHS) of (22) means the element-wise application of $f_{\text{opt}}$ to $x_{A,t}^{\text{post}}$. The posterior mean $\mathbf{x}_{B,t+1}$ is used as an estimator of $\mathbf{x}_{B,t+1}$. To realize asymptotic Gaussianity in module A, the extrinsic mean $\mathbf{x}_{B,t+1}$ and covariance $\{v_{B,t+1,t+1}\}^{t=0}$ are fed back to module A. Let

$$
\mathbf{v}_{B,t+1} = \mathbf{v}_{B,t} + \mathbf{e}_{B,t}^T \mathbf{V}_{A \rightarrow B,t}^{-1} \mathbf{1},
$$

with

$$
\mathbf{v}_{B,t} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{f}_{\text{opt}}(\mathbf{x}_{A,t}^{\text{post}}, \nu_{A,t}^{\text{post}}; \mathbf{x}_{A,t}^{\text{post}}, \nu_{A,t}^{\text{post}}),
$$

where the derivative is taken with respect to the first variable. The extrinsic messages are computed as

$$
\nu_{B,t+1,t+1} = \frac{\mathbf{v}_{B,t+1,t+1} - \mathbf{v}_{B,t} - \mathbf{e}_{B,t}^T \mathbf{V}_{A \rightarrow B,t}^{-1} \mathbf{1}}{(1 - \mathbf{v}_{B,t})^2},
$$

$$
\nu_{B,t+1,t+1} = \frac{\mathbf{v}_{B,t+1,t+1} - \mathbf{v}_{B,t} - \mathbf{e}_{B,t}^T \mathbf{V}_{A \rightarrow B,t}^{-1} \mathbf{1}}{(1 - \mathbf{v}_{B,t})^2},
$$

for $t' \in \{0, \ldots, t\}$.

IV. MAIN RESULTS

A. State Evolution

The dynamics of LM-OAMP is analyzed via state evolution [17] in the large system limit. Asymptotic Gaussianity has been proved for a general error model proposed in [17]. Thus, the main part in state evolution analysis is to prove the inclusion of the error model for LM-OAMP into the general error model.

Before presenting state evolution results, we first summarize technical assumptions.

Assumption 1: The signal vector $x$ has i.i.d. elements with zero mean, unit variance, and bounded $(2 + \epsilon)$th moment for some $\epsilon > 0$.

Assumption 1 is a simplifying assumption. To relax Assumption 1 we need non-separable denoisers [25–27].

Assumption 2: The sensing matrix $A$ is right-orthogonally invariant: For any orthogonal matrix $\Phi$ independent of $A$, the equivalence in distribution $A \Phi \sim A$ holds. More precisely, in the singular-value decomposition (SVD) $A = U \Sigma V^T$ the orthogonal matrix $V$ is independent of $U \Sigma$ and Haar-distributed [28, 29]. Furthermore, the empirical eigenvalue distribution of $A^H A$ converges almost surely to a compactly supported deterministic distribution with unit first moment in the large system limit.

The right-orthogonal invariance is a key assumption in state evolution analysis. The unit-first-moment assumption implies the almost sure convergence $N^{-1} \text{Tr}(A^T A) \xrightarrow{a.s.} 1$.

Assumption 3: The Bayes-optimal denoiser $f_{\text{opt}}$ in module B is nonlinear and Lipschitz-continuous.

The nonlinearity is required to guarantee the asymptotic positive definiteness of $\mathbf{V}_{B\rightarrow A,t}$ and $\mathbf{V}_{B\rightarrow A,t}$. It is an interesting open question whether any Bayes-optimal denoiser is Lipschitz-continuous under Assumption 1.

We next define state evolution recursions for LM-OAMP, which are 2D discrete systems with respect to two positive-definite symmetric matrices $\mathbf{V}_{A \rightarrow B,t} \in \mathbb{R}^{(t+1) \times (t+1)}$ and $\mathbf{V}_{B \rightarrow A,t} \in \mathbb{R}^{(t+1) \times (t+1)}$. We write the $(t', t)$ elements of $\mathbf{V}_{A \rightarrow B,t} \in \mathbb{R}^{(t+1) \times (t+1)}$ and $\mathbf{V}_{B \rightarrow A,t} \in \mathbb{R}^{(t+1) \times (t+1)}$ as $\nu_{A,t+1,t+1}$ and $\nu_{B,t+1,t+1}$ for $t', t \in \{0, \ldots, t\}$, respectively. Consider the initial condition $\nu_{B,0,0} = 1$. State evolution recursions for module A are given by

$$
\nu_{A,t+1} = \frac{1}{1^{T} \mathbf{V}_{B \rightarrow A,t}^{-1} \mathbf{1}} \nu_{A,t+1},
$$

for $t' \leq t$, (28)

$$
\nu_{A,t+1} = \lim_{M \to N, \infty} \gamma_{t', t} \nu_{B,t+1,t+1} + \sigma^2 \nu_{A,t+1},
$$

where

$$
\nu_{B,t+1,t+1} = \frac{\nu_{A,t+1} - \mathbf{e}_{B,t}^T \mathbf{V}_{A \rightarrow B,t}^{-1} \mathbf{1}}{(1 - \mathbf{v}_{B,t})^2},
$$

for $t' \leq t$.

State evolution recursions for module B are given by

$$
\nu_{B,t+1,t+1} = \frac{1}{1^{T} \mathbf{V}_{A \rightarrow B,t,t+1}^{-1} \mathbf{1}} \nu_{B,t+1,t+1},
$$

for $t' \leq t$, (31)
Bayes-optimal OAMP [5], [6], [11].

We next prove that the state evolution recursions (28)–(33) for Bayes-optimal LM-OAMP are equivalent to those (34)–(37) for Bayes-optimal OAMP, i.e. \( \bar{v}_{B,t,t}^{\text{post}} = \bar{v}_{B,t,t}^{\text{post}} \) holds for any \( t \). Furthermore, \( \bar{v}_{B,t,t}^{\text{post}} \) converges to a constant as \( t \) tends to infinity for all \( t' \leq t \).

**Proof:** We first evaluate the RHS of (29). Let

\[
\Xi_t = \bar{v}_{B,t,t}^{\text{post}} \left( \sigma^2 I_M + \bar{v}_{B,t,t}^{\text{post}} A A^T \right)^{-1}. 
\]

Using \( \bar{v}_{B,t,t}^{\text{post}} = \bar{v}_{B,t,t}^{\text{post}} \) for (33) with \( \bar{v}_{B,t,t}^{\text{post}} \) replaced by \( \bar{v}_{B,t,t}^{\text{post}} \). In the derivation of the second equality we have used (38).

We next evaluate the RHSs of (30) and (33). Applying (38). Note that (39) implies \( \bar{v}_{A,t,t}^{\text{post}} = \bar{v}_{A,t,t}^{\text{post}} \) for all \( t' \leq t \).

We next evaluate the RHSs of (30) and (33). Applying (32), obtained from the second property in Lemma 1 to find that the RHS of (33) reduces to

\[
\bar{v}_{B,t,t}^{\text{post}} = \left( \frac{1}{\bar{v}_{B,t,t}^{\text{post}}} - \frac{1}{\bar{v}_{B,t,t}^{\text{post}}} \right)^{-1}.
\]

Similarly, we use \( \bar{v}_{B,t,t}^{\text{post}} = \bar{v}_{B,t,t}^{\text{post}} \) for (32), obtained from the first property in Lemma 1 to find that the RHS of (33) reduces to

\[
\bar{v}_{B,t,t}^{\text{post}} = \left( \frac{1}{\bar{v}_{B,t,t}^{\text{post}}} - \frac{1}{\bar{v}_{B,t,t}^{\text{post}}} \right)^{-1}.
\]

To prove \( \bar{v}_{B,t,t}^{\text{post}} = \bar{v}_{B,t,t}^{\text{post}} \) for any \( t \), it is sufficient to show \( \bar{v}_{A-B,t,t}^{\text{post}} = \bar{v}_{A-B,t,t}^{\text{post}} \) and \( \bar{v}_{A-B,t,t}^{\text{post}} = \bar{v}_{A-B,t,t}^{\text{post}} \). These identities follow immediately from the last property in Lemma 1.

Thus, the state evolution recursions for Bayes-optimal LM-OAMP are equivalent to those for Bayes-optimal OAMP.

Finally, we prove the convergence of the state evolution recursions for Bayes-optimal LM-OAMP. The first property in Lemma 1 implies that \( \{ \bar{v}_{B,t,t}^{\text{post}} \geq 0 \} \) is a monotonically non-increasing sequence as \( t \) grows. Thus, \( \bar{v}_{B,t,t}^{\text{post}} \) converges to a constant \( \bar{v}_{B,t,t}^{\text{post}} \) as \( t \) tends to infinity. Since \( \bar{v}_{B,t,t}^{\text{post}} = \bar{v}_{B,t,t}^{\text{post}} \) holds for all \( t' \leq t \), the convergence of the diagonal element \( \bar{v}_{B,t,t}^{\text{post}} \) implies that of the non-diagonal elements \( \{ \bar{v}_{B,t,t}^{\text{post}} \}. \)

**Theorem 2** implies that the state evolution recursions (34)–(37) for Bayes-optimal OAMP converge to a fixed point as \( t \) tends to infinity. Furthermore, the LM-MP proof strategy developed in this paper claims the optimality of Bayes-optimal OAMP in terms of the convergence speed among all possible LM-MP algorithms included into a unified framework in [17].
