To the theory of bending and oscillations of three-layered plates with a compressible filler

Makhamatali Usarov\textsuperscript{2}, Abdulkhakim Salokhiddinov\textsuperscript{1}, D M Usarov\textsuperscript{2}, Islomjon Khazratkulov\textsuperscript{1} and Nadejda Dremova\textsuperscript{3}

\textsuperscript{1}TIIAME, Department of Theoretical & Structural Mechanics, 100000 Tashkent, Uzbekistan
\textsuperscript{2}Institute of Mechanics and Seismic Stability of Structures of Academy of Sciences of the Republic of Uzbekistan, 100187 Tashkent, Uzbekistan
\textsuperscript{3}Tashkent Institute of Textile and Light Industry, 100100 Tashkent, Uzbekistan

E-mail: theormir@mail.ru

Abstract. The paper is devoted to improving the theory of bending and vibrations of three-layer plates with transverse compressible filler and thin outer bearing layers. For the outer layers, the Kirchhoff-Love hypothesis is accepted and the motion of their points is described by the equations of the theory of thin plates relative to forces and moments. Unlike bearing layers, a filler is considered as a three-dimensional body that does not obey any simplifying hypotheses. The equations of the bimoment theory of thick plates with respect to forces, moments and bimoments, created in the framework of the three-dimensional theory of elasticity, taking into account the nonlinearity of the distribution law of displacements and stresses over the thickness, are taken as the equations of motion of the filler. Expressions of forces, moments, and bimoments in the layers, as well as boundary conditions at the edges of a three-layer plate with respect to force factors are given. In the conjugate zones of the layers, the complete contact conditions for the continuity of displacements and stresses are set. An example is considered and numerical results are obtained.

1. Introduction

Among the relevant and complex problems of the mechanics of a deformable rigid body, a special place is occupied by the development of the methods for calculating design elements of buildings and structures within the framework of the theory of plates and shells.

The calculation of thin plates considering rheological and nonlinear properties of the structure materials of great importance ceat the present stage of development of the theory of thin plates and shells.

In the field of the theory of thick plates, the methods for calculating thick plates taking into account forces, moments and bimoments are successfully developed and used in modeling seismic vibrations of multi-storey buildings and structures, in calculations of three-layer structural elements with thick filler made of relatively soft material.

Dynamic calculation of structural elements built from an isotropic material under uniformly distributed dynamic load, leading to parametric resonance are considered in the studies given in [1-4].
In [5,6], the spatial model of the building box is improved taking into account the continuity of motion and stresses in the butt joints of beam and plate elements. The equations of motion of box-type elements with boundary conditions at the base of the box and contact conditions between box elements are given. The graphs of plates and beams displacements are constructed. The problems of forced vibrations of the building box under the influence of dynamic sinusoidal load are considered. To solve the problem, the finite difference method was used. Numerical results of stresses, displacements in the hazardous areas of the box-type building are obtained.

The study in [7] was devoted to the development of methods for calculating the plate model under dynamic impacts using the grid method in the framework of the theory of bimoments [8,9].

A well-known Reissner–Mindlin theory on layered anisotropic plates was generalized in [10]. The most complete case-history review of publications was presented, which set out the basic provisions of existing theories and their elaboration.

The studies conducted in [11,12] investigated the dynamic stability of circular three-layer plates of composite materials with viscoelastic properties. It was assumed that the three-layer plate consisted of a thin multilayer fiber-reinforced composite with a filler of foam. The plate is loaded in its plane with a rapidly increasing radically compressive dynamic force and loses its stability.

The current article is devoted to the development of the theory of three-layer plates with thin bearing layers and thick fillers of relatively soft material. For thin bearing layers we apply the Kirchhoff-Love hypothesis. Consider the thick filler as a three-dimensional body and apply the bimoment theory of thick plates.

Based on the general case, the layer materials are considered elastic and orthotropic ones. For orthotropic bearing layers of elastic materials, introduce the following notation:

- \( E_1^{(+)} \), \( E_2^{(+)} \), \( E_3^{(+)} \) – are the elasticity moduli; \( G_{12}^{(+)} \), \( G_{13}^{(+)} \), \( G_{23}^{(+)} \) – are the shear moduli;
- \( \nu_{12}^{(+)} \), \( \nu_{13}^{(+)} \), \( \nu_{23}^{(+)} \) – are the Poisson's ratios of the material of the lower bearing layer;
- \( E_1^{(-)} \), \( E_2^{(-)} \), \( E_3^{(-)} \) – are the elasticity moduli; \( G_{12}^{(-)} \), \( G_{13}^{(-)} \), \( G_{23}^{(-)} \) – are the shear moduli;
- \( \nu_{12}^{(-)} \), \( \nu_{13}^{(-)} \), \( \nu_{23}^{(-)} \) – are the Poisson's ratios of the material of the upper bearing layer;
- \( E_1 \), \( E_2 \), \( E_3 \) – are the elasticity moduli; \( G_{12} \), \( G_{13} \), \( G_{23} \) – are the shear moduli; \( \nu_{12}, \nu_{13}, \nu_{23} \) – are the Poisson's ratios of the filler layer material;
- \( h = 2h \) – is the filler thickness; \( 2h_u, 2h_u \) – is the thickness of bearing layers; \( a \) and \( b \) – are the dimensions of a three-layer plate in plan.

2. Statement of the problem

The layer displacement field is described with respect to a rectangular Cartesian coordinate system \((x_1, x_2, z)\) with an origin on the middle plane of the filler. The axis \((oz)\) is directed to the normal to the first layer.

The displacements of the upper bearing layer are set according to the Kirchhoff–Love hypothesis in the form \((-h - 2h_u \leq z \leq -h)\):

\[
U_{k}^{(+)} = U_{k}^{(+)} - (z + h + h_u) \frac{\partial W_{k}^{(+)}}{\partial x_k}, \quad (k = 1,2). \tag{1}
\]

Similarly, the law of displacements distribution of the lower bearing layer can be represented as \((h \leq z \leq h + 2h_u)\):

\[
U_{k}^{(-)} = U_{k}^{(-)} - (z - h - h_u) \frac{\partial W_{k}^{(-)}}{\partial x_k}, \quad (k = 1,2). \tag{2}
\]
Here $W^{(+)}, W^{(-)}$ are the deflections of bearing layers, $U_1^{(+)}, U_2^{(+)}$ and $U_1^{(-)}, U_2^{(-)}$ are the displacements of the middle planes of the layers.

Introduce the notation for contact zones displacements between layers.

The displacement of the contact zones between the bearing layers $z = h$ and $z = -h$ and the filler is denoted by $u_1^{(+)}, u_2^{(+)}, u_3^{(+)}$ and $u_1^{(-)}, u_2^{(-)}, u_3^{(-)}$. The laws of distribution of the bearing layers (1) and (2) displacements must satisfy the continuity conditions on the joining points $z = h$ and $z = -h$:

$$u_1^{(+)} = U_1^{(+)}$$
$$u_2^{(+)} = U_2^{(+)}$$
$$u_3^{(+)} = W^{(+)}$$

(3)

$$u_1^{(-)} = U_1^{(-)}$$
$$u_2^{(-)} = U_2^{(-)}$$
$$u_3^{(-)} = W^{(-)}$$

(4)

Let a three-layer plate be loaded with distributed external surface tangential $p_1^{(+)}, p_2^{(+)}, p_1^{(-)}, p_2^{(-)}$ and normal $p_3^{(+)}, p_4^{(-)}$ forces. Due to layers strain on the surfaces $z = -h$ and $z = +h$, distributed contact stresses $q_1^{(+)}, q_1^{(+)}$, $q_3^{(-)}$ and $q_2^{(-)}, q_3^{(-)}$ arise.

The longitudinal and tangential forces of the lower bearing layer are determined through unknown functions of the displacements of the points of the middle surface of the lower bearing layer $U_1^{(+)}, U_2^{(+)}$:

$$N_{11}^{(+)} = \frac{h + h}{h} \sigma_{11} dz = B_{11}^{(+)} \frac{\partial U_1^{(+)}}{\partial x_1} + B_{12}^{(+)} \frac{\partial U_1^{(+)}}{\partial x_2},$$

$$N_{12}^{(+)} = \frac{h + h}{h} \sigma_{12} dz = S_{12}^{(+)} \left( \frac{\partial U_1^{(+)}}{\partial x_1} + \frac{\partial U_2^{(+)}}{\partial x_1} \right),$$

$$N_{22}^{(+)} = \frac{h + h}{h} \sigma_{22} dz = B_{21}^{(+)} \frac{\partial U_1^{(+)}}{\partial x_1} + B_{22}^{(+)} \frac{\partial U_2^{(+)}}{\partial x_2},$$

(5)

where $B_{11}^{(+)}, B_{22}^{(+)}, B_{12}^{(+)}, S_{12}^{(+)}$ are the cylindrical rigidituynder tension and compression of the lower bearing layer of orthotropic material.

Bending and torsional moments of the lower bearing layer are determined through unknown functions of deflection of the points of the lower bearing layer $W^{(+)}$ in the form:

$$M_{11}^{(+)} = \frac{h + h}{h} \sigma_{11} dz = -D_{11}^{(+)} \frac{\partial^2 W^{(+)}}{\partial x_1^2} - D_{12}^{(+)} \frac{\partial^2 W^{(+)}}{\partial x_2^2},$$

$$M_{22}^{(+)} = \frac{h + h}{h} \sigma_{22} dz = -D_{12}^{(+)} \frac{\partial^2 W^{(+)}}{\partial x_1^2} - D_{22}^{(+)} \frac{\partial^2 W^{(+)}}{\partial x_2^2},$$

$$M_{12}^{(+)} = \frac{h + h}{h} \sigma_{12} dz = -C_{12}^{(+)} \frac{\partial^2 W^{(+)}}{\partial x_1 \partial x_2},$$

(6)

where $D_{11}^{(+)}, D_{22}^{(+)}, D_{12}^{(+)}, C_{12}^{(+)}$ are the cylindrical rigidity of the lower supporting layer of orthotropic material under bending.

The shear forces of the lower bearing layer are determined by the formulas

$$Q_{13}^{(+)} = \frac{\partial M_{11}^{(+)}}{\partial x_1} + \frac{\partial M_{12}^{(+)}}{\partial x_2} - h_0 q_1^{(+)}$$

$$Q_{23}^{(+)} = \frac{\partial M_{21}^{(+)}}{\partial x_1} + \frac{\partial M_{22}^{(+)}}{\partial x_2} - h_0 q_2^{(+)}.$$

(7)

The longitudinal and tangential forces of the upper bearing layer $N_{kj}^{(-)}$ are determined through unknown functions of displacements of the middle surface points in the lower bearing layer $U_1^{(-)}, U_2^{(-)}$ in the form:
\[ N_{11}^{(-)} = \int_{-h}^{b} \sigma_{11} dz = B_{11}^{(-)} \frac{\partial U_{1}^{(-)}}{\partial x_1} + B_{12}^{(-)} \frac{\partial U_{2}^{(-)}}{\partial x_2}, \]
\[ N_{12}^{(-)} = \int_{-h}^{b} \sigma_{12} dz = S_{12}^{(-)} \left( \frac{\partial U_{1}^{(-)}}{\partial x_1} + \frac{\partial U_{2}^{(-)}}{\partial x_2} \right), \quad N_{22}^{(-)} = \int_{-h}^{b} \sigma_{22} dz = B_{12}^{(-)} \frac{\partial U_{1}^{(-)}}{\partial x_1} + B_{22}^{(-)} \frac{\partial U_{2}^{(-)}}{\partial x_2}. \] (8)

Here \( B_{11}^{(-)}, B_{12}^{(-)}, B_{12}^{(-)}, S_{12}^{(-)} \) are the cylindrical rigidity of the upper bearing layer of orthotropic material under tension and compression.

Bending and torsion of the upper bearing layer have the expressions

\[ M_{11}^{(-)} = \int_{-h}^{b} \sigma_{11} dz = -D_{11}^{(-)} \frac{\partial^2 W^{(-)}}{\partial x_1^2} - D_{12}^{(-)} \frac{\partial^2 W^{(-)}}{\partial x_2^2}, \]
\[ M_{22}^{(-)} = \int_{-h}^{b} \sigma_{22} dz = -D_{12}^{(-)} \frac{\partial^2 W^{(-)}}{\partial x_1^2} - D_{22}^{(-)} \frac{\partial^2 W^{(-)}}{\partial x_2^2}, \quad M_{12}^{(-)} = \int_{-h}^{b} \sigma_{12} dz = -C_{12}^{(-)} \frac{\partial^2 W^{(-)}}{\partial x_1 \partial x_2}, \] (9)

where \( D_{11}^{(-)}, D_{12}^{(-)}, D_{22}^{(-)}, C_{12}^{(-)} \) are the cylindrical rigidity of the upper bearing layer of orthotropic material under bending, \( W^{(-)} \) are the unknown functions of the displacements and deflections of the points of the middle surface of the upper bearing layer.

The shear forces of the upper bearing layer are determined by the formulas

\[ Q_{13}^{(-)} = \frac{\partial M_{11}^{(-)}}{\partial x_1} + \frac{\partial M_{12}^{(-)}}{\partial x_2} - h_u q_1^{(-)}, \quad Q_{23}^{(-)} = \frac{\partial M_{11}^{(-)}}{\partial x_1} + \frac{\partial M_{22}^{(-)}}{\partial x_2} - h_u q_2^{(-)}. \] (10)

Introduce the following notation for the displacements and internal force factors of the bearing layers of the plate.

The half-sums of internal longitudinal and tangential forces and moments are denoted in the form

\[ \vec{N}_{ij} = N_{ij}^{(+)} + N_{ij}^{(-)}, \quad \vec{M}_{ij} = M_{ij}^{(+)} + M_{ij}^{(-)}, \quad \vec{Q}_{ij} = Q_{ij}^{(+)} + Q_{ij}^{(-)}, \quad \{i, j = 1, 2\}. \] (11)

The half-differences of the internal longitudinal tangential forces determine the shear moments in the form

\[ \vec{q}_{ij} = N_{ij}^{(+)} - N_{ij}^{(-)}, \quad \{i, j = 1, 2\}. \] (12)

Introduce the following generalized contact stresses using contact stresses defined as

\[ \vec{q}_k = \frac{q_k^{(+)} - q_k^{(-)}}{2}, \quad \vec{q}_k = \frac{q_k^{(+)} + q_k^{(-)}}{2}, \quad \{k = 1, 2\}, \quad \vec{q}_3 = \frac{q_3^{(+)} + q_3^{(-)}}{2}, \quad \vec{q}_3 = \frac{q_3^{(+)} - q_3^{(-)}}{2}. \] (13)

The half-sums and half-differences of the external tangential stresses are denoted in the form

\[ \vec{p}_k = p_k^{(+)} - p_k^{(-)}, \quad \{k = 1, 2\}, \quad \vec{p}_3 = p_3^{(+)} + p_3^{(-)}, \] (14.a)
\[ \vec{p}_k = p_k^{(+)} + p_k^{(-)}, \quad \{k = 1, 2\}, \quad \vec{p}_3 = p_3^{(+)} - p_3^{(-)}. \] (14.b)

The equations of motion of the bearing layers relative to the longitudinal and shear forces are obtained in the form:

\[ \frac{\partial \vec{N}_{11}^{(+)}}{\partial x_1} + \frac{\partial \vec{N}_{12}^{(+)}}{\partial x_2} + 2\vec{q}_k - 2\vec{p}_k - 4(\rho_u h_u + \rho_a h_a) \vec{U}_k - 4(\rho_u h_u - \rho_a h_a) \vec{U}_k = 0, \quad \{k = 1, 2\}. \] (15)
Here \( \rho_u, \rho_n \) are the densities of the bearing layers.

The equations of motion of the bearing layers with respect to shear moments have the form

\[
\frac{\partial^2 M_{11}}{\partial x_1^2} + 4 \frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + 2 \frac{\partial^2 M_{22}}{\partial x_2^2} - 4(\rho_u h_u + \rho_n h_n) \dot{\ddot{W}} - 4(\rho_u h_u - \rho_n h_n) \dddot{W} + (h_u + h_n) \frac{\partial q_1}{\partial x_1} + (h_u + h_n) \frac{\partial q_2}{\partial x_2} + (h_u - h_n) \frac{\partial q_1}{\partial x_2} + (h_u - h_n) \frac{\partial q_2}{\partial x_1} - (h_u + h_n) \frac{\partial p_1}{\partial x_1} - (h_u + h_n) \frac{\partial p_2}{\partial x_2} - (h_u - h_n) \frac{\partial p_1}{\partial x_2} - (h_u - h_n) \frac{\partial p_2}{\partial x_1} = 0.
\]

(17)

The equation of motion of the bending vibrations of the bearing layers is obtained in the form:

\[
2 \frac{\partial^2 M_{11}}{\partial x_1^2} + 4 \frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + 2 \frac{\partial^2 M_{22}}{\partial x_2^2} + (h_u + h_n) \frac{\partial q_1}{\partial x_1} + (h_u + h_n) \frac{\partial q_2}{\partial x_2} + (h_u - h_n) \frac{\partial q_1}{\partial x_2} + (h_u - h_n) \frac{\partial q_2}{\partial x_1} - (h_u + h_n) \frac{\partial p_1}{\partial x_1} - (h_u + h_n) \frac{\partial p_2}{\partial x_2} - (h_u - h_n) \frac{\partial p_1}{\partial x_2} - (h_u - h_n) \frac{\partial p_2}{\partial x_1} = 0.
\]

(18)

Using the distribution law of displacements of bearing layers (1), (2) and the continuity conditions for displacements (3), (4) in the zone of layers contact \( z = +h \) end \( z = -h \), we obtain the expressions:

\[
u_k^{(+)} = U_k^{(+)} + h_n \frac{\partial W^{(+)}(k)}{\partial x_k}, \quad \nu_k^{(-)} = U_k^{(-)} - h_n \frac{\partial W^{(-)}(k)}{\partial x_k}, \quad (k = 1, 2).
\]

(19)

Consider a filler as a three-dimensional body [8,9], loaded with surface stresses \( q_1^{(+)} , q_1^{(-)} , q_3^{(+)} \) and \( q_1^{(-)} , q_1^{(-)} , q_3^{(-)} \) whose material obeys the generalized Hooke law:

\[
\sigma_{11} = E_{11}\varepsilon_{11} + E_{12}\varepsilon_{22} + E_{13}\varepsilon_{33}, \quad \sigma_{22} = E_{21}\varepsilon_{11} + E_{22}\varepsilon_{22} + E_{23}\varepsilon_{33}, \quad \sigma_{33} = E_{31}\varepsilon_{11} + E_{32}\varepsilon_{22} + E_{33}\varepsilon_{33},
\]

(20)

\[
\sigma_{12} = 2G_{12}\varepsilon_{12}, \quad \sigma_{13} = 2G_{13}\varepsilon_{13}, \quad \sigma_{23} = 2G_{23}\varepsilon_{23}
\]

where \( E_{11} , E_{12} ,..., E_{33} \) are the elastic constants determined through the Poisson’s ratios and elastic moduli, \( G_{12} , G_{13} , G_{23} \) are the shear moduli of the filler material.

In contrast to the classical theory of plates, the components of the displacement vector are defined as functions of three spatial coordinates and time. The components of the strain tensor \( u_i(x_1, x_2, z, t), u_j(x_1, x_2, z, t), u_k(x_1, x_2, z, t) \) are determined by the Cauchy relations.

Expand the components of the displacement vector in a Maclaurin series in the form [8,9]:

\[
u_k = B_0^{(k)} + \frac{B_1^{(k)}}{h} \frac{z}{h} + B_2^{(k)} \left( \frac{z}{h} \right)^2 + B_3^{(k)} \left( \frac{z}{h} \right)^3 + ... + B_m^{(k)} \left( \frac{z}{h} \right)^m + ..., \quad (k = 1, 2)
\]

(21)
u_3 = A_0 + A_1 \frac{z}{h} + A_2 \left( \frac{z}{h} \right)^2 + A_3 \left( \frac{z}{h} \right)^3 + \ldots + A_m \left( \frac{z}{h} \right)^m + \ldots, \quad (22)

where B^{(k)}_m, A_m are the unknown functions of two spatial coordinates and time.

Note that the motion of the filler points in the framework of the bimoment theory of plates [9] is described by two problems.

To describe the first problem, introduce forces and bimoments using nine unknown functions \( \bar{\psi}_1, \bar{\psi}_2, \bar{\beta}_1, \bar{\beta}_2, \bar{u}_1, \bar{u}_2, \bar{r}, \bar{\gamma}, \bar{W} \), which are determined by the following relations:

\[
\bar{u}_k = \frac{u_k^{(+)} + u_k^{(-)}}{2}, \quad (k = 1, 2), \quad \bar{W} = \frac{u_3^{(+)} - u_3^{(-)}}{2}, \quad (23, a)
\]

\[
\bar{\psi}_k = \frac{1}{2h} \int_{-h}^{h} u_k dz, \quad \bar{\beta}_k = \frac{1}{2h^3} \int_{-h}^{h} u_k z^2 dz, \quad (k = 1, 2), \quad \bar{r} = \frac{1}{2h^2} \int_{-h}^{h} u_3 zdz, \quad \bar{\gamma} = \frac{1}{2h^2} \int_{-h}^{h} u_3 z^2 dz. \quad (23, b)
\]

To describe the second problem, introduce forces, moments, and bimoments using nine unknown functions \( \tilde{\psi}_1, \tilde{\psi}_2, \tilde{u}_1, \tilde{u}_2, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{r}, \tilde{\gamma}, \tilde{W} \), which are determined by the following formulas:

\[
\tilde{u}_k = \frac{u_k^{(+)} - u_k^{(-)}}{2}, \quad (k = 1, 2), \quad \tilde{W} = \frac{u_3^{(+)} + u_3^{(-)}}{2}, \quad (24, a)
\]

\[
\tilde{\psi}_k = \frac{1}{2h^2} \int_{-h}^{h} u_k zdz, \quad \tilde{\beta}_k = \frac{1}{2h^3} \int_{-h}^{h} u_k z^3 dz, \quad (k = 1, 2), \quad \tilde{r} = \frac{1}{2h^2} \int_{-h}^{h} u_3 zdz, \quad \tilde{\gamma} = \frac{1}{2h^2} \int_{-h}^{h} u_3 z^2 dz. \quad (24, b)
\]

The forces and bimoments of the first problem are determined by the formulas

\[
N_{ij} = \int_{-h}^{h} \sigma_{ij} zdz, \quad T_{ij} = \frac{1}{h^2} \int_{-h}^{h} \sigma_{ij} z^2 dz \quad (i, j = 1, 2), \quad (25)
\]

\[
\bar{p}_{k3} = \frac{1}{2h^2} \int_{-h}^{h} \sigma_{k3} zdz, \quad \bar{r}_{k3} = \frac{1}{2h^4} \int_{-h}^{h} \sigma_{k3} z^3 dz, \quad (k = 1, 2), \quad (26)
\]

The equations of motion of the first problem of a thick plate, constructed in [9] with respect to internal forces and bimoments (25) and (26), have the form:

\[
\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + 2\bar{q}_1 = \rho H \ddot{\bar{\psi}}_1, \quad \frac{\partial N_{21}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + 2\bar{q}_2 = \rho H \ddot{\bar{\psi}}_2, \quad (27)
\]

\[
\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} - 4\bar{p}_{13} + 2\bar{q}_1 = \rho H \ddot{\bar{\beta}}_1, \quad \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} - 4\bar{p}_{23} + 2\bar{q}_2 = \rho H \ddot{\bar{\beta}}_2, \quad (27)
\]

The system of six differential equations of motion (27) contains nine unknown functions \( \bar{\psi}_1, \bar{\psi}_2, \bar{\beta}_1, \bar{\beta}_2, \bar{u}_1, \bar{u}_2, \bar{r}, \bar{\gamma}, \bar{W} \).

The second problem is described by equations for forces, moments and bimoments determined by the relations
\[ M_{ij} = \int_{-h}^{h} \sigma_{ij} z dz, \quad \sigma_{i3} = \int_{-h}^{h} \sigma_{i3} dz, \quad (i, j = 1, 2). \quad (28) \]

\[ P_{ij} = \frac{1}{h^2} \int_{-h}^{h} \sigma_{ij} z^3 dz, \quad \bar{p}_{i3} = \frac{1}{2h^3} \int_{-h}^{h} \sigma_{i3} z^2 dz, \quad (i = 1, 2), \quad \bar{r}_{33} = \frac{1}{2h^4} \int_{-h}^{h} \sigma_{33} z^3 dz. \quad (29) \]

The equations of motion of the second problem of a thick plate [9] with respect to bending, torques, shear forces, and relative to longitudinal, transverse bimoments (28), (29) are written in the form

\[ \frac{\partial Q_{i3}}{\partial x_1} + \frac{\partial Q_{23}}{\partial x_2} + 2\bar{q}_3 = \rho \bar{H} \dddot{w}, \quad H \frac{\partial^2 p_{i3}}{\partial x_1^2} + H \frac{\partial^2 p_{23}}{\partial x_2^2} - 4\bar{p}_{i3} + 2\bar{q}_3 = H \rho \dddot{w}. \quad (30,a) \]

\[ \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - Q_{i3} + H \bar{q}_1 = \frac{H^2}{2} \rho \dddot{\psi}, \quad \frac{\partial M_{21}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - Q_{23} + H \bar{q}_2 = \frac{H^2}{2} \rho \dddot{\psi}_2. \quad (30,b) \]

The system of six differential equations of motion (30) includes nine unknown functions \( \dddot{\bar{\psi}}, \dddot{\psi}, \bar{u}_1, \bar{u}_2, \bar{\beta}_1, \bar{\beta}_2, \bar{r}, \bar{\gamma}, \bar{W} \).

As a result, taking into account eight terms of the Maclaurin series (21) and (22), we obtain the contact stress expressions. For the first problem, we have equations for contact stresses through unknown generalized functions in the form \( \bar{u}_1, \bar{u}_2, \bar{\psi}_1, \bar{\psi}_2, \bar{\beta}_1, \bar{\beta}_2, \bar{r}, \bar{\gamma}, \bar{W} \).

\[ \bar{q}_1 = G_{13} \left( 15 \bar{\psi} + 20 \bar{u}_1 - 105 \bar{\beta}_1 + H \frac{\partial \bar{W}}{\partial x_1} \right), \quad \bar{q}_2 = G_{23} \left( 15 \bar{\psi} + 20 \bar{u}_2 - 105 \bar{\beta}_2 + H \frac{\partial \bar{W}}{\partial x_2} \right), \quad (31) \]

And for the second problem, we have equations for contact stresses through unknown generalized functions in the form \( \dddot{\bar{\psi}}, \dddot{\psi}, \bar{u}_1, \bar{u}_2, \bar{\psi}_1, \bar{\psi}_2, \bar{\beta}_1, \bar{\beta}_2, \bar{W} \)

\[ \bar{q}_1 = G_{13} \left( 105 \dddot{\bar{\psi}} + 30 \dddot{\bar{u}}_1 - 315 \bar{\beta}_1 + H \frac{\partial \dddot{\bar{W}}}{\partial x_1} \right), \quad \bar{q}_2 = G_{23} \left( 105 \dddot{\bar{\psi}} + 30 \dddot{\bar{u}}_2 - 315 \bar{\beta}_2 + H \frac{\partial \dddot{\bar{W}}}{\partial x_2} \right), \quad (32) \]

Taking into account the notation from (23) and (24), from the conjugation conditions of the layers (3) and (4) we obtain equations that are the conjugation conditions between the bearing layers and the filler.

\[ \bar{u}_k = \bar{U}_k + \frac{h_u + h_c}{2} \frac{\partial \bar{W}}{\partial x_k}, \quad \dddot{\bar{u}}_k = \dddot{\bar{U}}_k + \frac{h_u + h_c}{2} \frac{\partial \dddot{\bar{W}}}{\partial x_k}, \quad (k = 1, 2). \quad (33) \]

Here, generalized displacements of the bearing layers are introduced in the form

\[ \bar{U}_k = \frac{U_k^{(+) + U_k^{(-)}}}{2}, \quad \dddot{\bar{U}}_k = \frac{U_k^{(+) - U_k^{(-)}}}{2}, \quad (k = 1, 2). \quad (34) \]
Equations (33) determine the generalized contact displacements of the filler \( \vec{u}_k \) and \( \vec{u}_k \). \((k = 1, 2)\) through the displacements of the median surfaces of the outer layers.

Thus, the expressions of contact displacements and stresses (31), (32) and (33) of the symmetric and asymmetric displacement problems of the theory of three-layer plates are constructed.

The equations of motion of the bearing layers (15), (16) and (17), (18) the equations of motion of the filler (27), (30) taking into account expressions (31), (32) and (33) make up the joint system of equations of motion of the points of the layers of a three-layer plate with respect to unknown generalized displacements determined by formulas (23), (24).

The boundary conditions of the problem are set with respect to generalized displacements (23), (24), or relative to force factors (11), (12) and (25), (26), (28), (29), depending on the conditions of the tasks posed [9].

If at the edge of the plate the displacements are equal to zero, then we have:

\[
\begin{align*}
\varphi_1 &= 0, \quad \varphi_2 = 0, \quad \tilde{\beta}_1 = 0, \quad \tilde{\beta}_2 = 0, \quad \tilde{\tau} = 0, \quad \tilde{\varphi} = 0, \quad \tilde{U}_1 = 0, \quad \tilde{U}_2 = 0, \quad \tilde{W} = 0, \quad \tilde{W}_1 = 0, \\
\varphi_1 &= 0, \quad \varphi_2 = 0, \quad \tilde{\beta}_1 = 0, \quad \tilde{\beta}_2 = 0, \quad \tilde{\tau} = 0, \quad \tilde{\varphi} = 0, \quad \tilde{U}_1 = 0, \quad \tilde{U}_2 = 0, \quad \tilde{W} = 0, \quad \tilde{W}_1 = 0.
\end{align*}
\]

(35)

If the edge of the plate is free of supports, then the boundary conditions have the form:

\[
\begin{align*}
N_{11} &= 0, \quad N_{12} = 0, \quad T_{11} = 0, \quad T_{12} = 0, \quad \tilde{P}_{13} = 0, \quad \tilde{\tau}_{13} = 0, \quad \tilde{N}_{11} = 0, \quad \tilde{N}_{12} = 0, \quad \tilde{M}_{11} = 0, \quad \tilde{Q}_{13} = 0, \\
M_{11} &= 0, \quad M_{12} = 0, \quad P_{11} = 0, \quad P_{12} = 0, \quad Q_{13} = 0, \quad \tilde{P}_{13} = 0, \quad \tilde{P}_{11} = 0, \quad \tilde{P}_{12} = 0, \quad \tilde{M}_{22} = 0, \quad \tilde{Q}_{13} = 0.
\end{align*}
\]

(36)

If the edge of the plate is supported, then we have the boundary conditions in the form:

\[
\begin{align*}
N_{11} &= 0, \quad N_{12} = 0, \quad T_{11} = 0, \quad T_{12} = 0, \quad \tilde{\tau} = 0, \quad \tilde{\varphi} = 0, \quad \tilde{N}_{11} = 0, \quad \tilde{N}_{12} = 0, \quad \tilde{M}_{11} = 0, \quad \tilde{W} = 0, \\
M_{11} &= 0, \quad M_{12} = 0, \quad P_{11} = 0, \quad P_{12} = 0, \quad \tilde{\tau} = 0, \quad \tilde{\varphi} = 0, \quad \tilde{P}_{11} = 0, \quad \tilde{P}_{12} = 0, \quad \tilde{M}_{11} = 0, \quad \tilde{W} = 0.
\end{align*}
\]

(37)

If the edge of the plate is supported and there is no displacement towards the tangential contour, then we have the boundary conditions:

\[
\begin{align*}
N_{11} &= 0, \quad T_{11} = 0, \quad \varphi_2 = 0, \quad \tilde{\beta}_2 = 0, \quad \tilde{\tau} = 0, \quad \tilde{\varphi} = 0, \quad \tilde{N}_{11} = 0, \quad \tilde{N}_{12} = 0, \quad \tilde{M}_{11} = 0, \quad \tilde{W} = 0, \\
M_{11} &= 0, \quad P_{11} = 0, \quad \varphi_2 = 0, \quad \tilde{\beta}_2 = 0, \quad \tilde{\tau} = 0, \quad \tilde{\varphi} = 0, \quad \tilde{P}_{11} = 0, \quad \tilde{P}_{12} = 0, \quad \tilde{M}_{11} = 0, \quad \tilde{W} = 0.
\end{align*}
\]

(38)

Example. As an example, consider the transverse bending of a three-layer plate, under the action of normal external load applied to the front surface of the upper bearing layer:

\[
p_3(\cdot) = -q_0 \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{b}\right),
\]

(39)

here \(q_0\) is the constant parameter of external load.

Obtain the equilibrium equations for the bending of three-layer plates (bearing layers and filler) by omitting the inertial terms in the equations of motion (15) - (18), (27) and (30). If assume that the edges of a rectangular three-layer plate are supported and there is a rigid diaphragm that impedes the movement of the layers extreme points in the direction of the tangential contour, then boundary conditions in the form (38) must be satisfied. The conditions at two opposite edges of the three-layer plate \(x_1 = a\), \(x_1 = a\), are

\[
\begin{align*}
N_{11} &= 0, \quad T_{11} = 0, \quad \varphi_2 = 0, \quad \tilde{\beta}_2 = 0, \quad \tilde{\tau} = 0, \quad \tilde{\varphi} = 0, \quad \tilde{N}_{11} = 0, \quad \tilde{U}_1 = 0, \quad \tilde{W} = 0, \\
M_{11} &= 0, \quad P_{11} = 0, \quad \varphi_2 = 0, \quad \tilde{\beta}_2 = 0, \quad \tilde{\tau} = 0, \quad \tilde{\varphi} = 0, \quad \tilde{P}_{11} = 0, \quad \tilde{U}_2 = 0, \quad \tilde{W} = 0.
\end{align*}
\]

(40)

At the other two opposite edges of the three-layer plate \(x_2 = b\), \(x_2 = b\), the boundary conditions should be met.
\begin{align}
N_{22} = 0, \ T_{22} = 0, \ \bar{\nu}_1 = 0, \ \bar{\beta}_1 = 0, \ \bar{\sigma} = 0, \ \bar{\varphi} = 0, \ \bar{N}_{22} = 0, \ \bar{U}_1 = 0, \ \bar{W} = 0, \\
M_{22} = 0, \ P_{22} = 0, \ \bar{\nu}_2 = 0, \ \bar{\beta}_2 = 0, \ \bar{\sigma} = 0, \ \bar{\varphi} = 0, \ \bar{P}_{22} = 0, \ \bar{U}_1 = 0, \ \bar{W} = 0. \tag{41}
\end{align}

The maximum stresses of the lower bearing layer are determined by the following formula, which is obtained from Hooke’s law taking into account (1) at \( z = h_1 + 2h_s \), in the form:

\begin{align}
\sigma_{11}^{(+)} &= E_{11}^{(+)} \left( \frac{\partial U_1^{(+)}}{\partial x_1} + v_{12}^{(+)} \frac{\partial U_2^{(+)}}{\partial x_2} \right) - h_s E_{11}^{(+)} \left( \frac{\partial^2 W^{(+)}}{\partial x_1^2} + v_{12}^{(+)} \frac{\partial^2 W^{(+)}}{\partial x_2^2} \right), \\
\sigma_{22}^{(+)} &= E_{22}^{(+)} \left( v_{21}^{(+)} \frac{\partial U_1^{(+)}}{\partial x_1} + \frac{\partial U_2^{(+)}}{\partial x_2} \right) - h_s E_{22}^{(+)} \left( v_{21}^{(+)} \frac{\partial^2 W^{(+)}}{\partial x_1^2} + \frac{\partial^2 W^{(+)}}{\partial x_2^2} \right). \tag{42}
\end{align}

The maximum stresses of the upper bearing layer are determined by the following formula, which is obtained from Hooke’s law taking into account (2) at \( z = -h_1 - 2h_s \), in the form:

\begin{align}
\sigma_{11}^{(-)} &= E_{11}^{(-)} \left( \frac{\partial U_1^{(-)}}{\partial x_1} + v_{12}^{(-)} \frac{\partial U_2^{(-)}}{\partial x_2} \right) + h_s E_{11}^{(-)} \left( \frac{\partial^2 W^{(-)}}{\partial x_1^2} + v_{12}^{(-)} \frac{\partial^2 W^{(-)}}{\partial x_2^2} \right), \\
\sigma_{22}^{(-)} &= E_{22}^{(-)} \left( v_{21}^{(-)} \frac{\partial U_1^{(-)}}{\partial x_1} + \frac{\partial U_2^{(-)}}{\partial x_2} \right) + h_s E_{22}^{(-)} \left( v_{21}^{(-)} \frac{\partial^2 W^{(-)}}{\partial x_1^2} + \frac{\partial^2 W^{(-)}}{\partial x_2^2} \right). \tag{43}
\end{align}

The maximum stress points of the upper front surface of the filler layer \( z = -h \) is determined by the following formula

\begin{equation}
\sigma^{(+)}_{11} = \bar{\sigma}_{11} - \bar{\sigma}_{11}, \quad \sigma^{(+)}_{22} = \bar{\sigma}_{22} - \bar{\sigma}_{22}, \quad \sigma^{(+)}_{33} = \bar{\sigma}_{33} - \bar{\sigma}_{33}, \quad \sigma^{(+)}_{33} = \bar{\sigma}_{33} - \bar{\sigma}_{33}, \tag{44}
\end{equation}

The maximum stress points of the lower front surface of the filler layer \( z = +h \) is determined by the following formula

\begin{equation}
\sigma^{(-)}_{11} = \bar{\sigma}_{11} + \bar{\sigma}_{11}, \quad \sigma^{(-)}_{22} = \bar{\sigma}_{22} + \bar{\sigma}_{22}, \quad \sigma^{(-)}_{33} = \bar{\sigma}_{33} + \bar{\sigma}_{33}, \tag{45}
\end{equation}

Using the expressions of Hooke’s law, taking into account (1)-(4), (31)-(34) we obtain expressions for determining the shear stresses in the middle plane of the filler layer

\begin{equation}
\sigma_{k3}^{(0)} = G_{k3} \left( \frac{15}{4H} \left( \bar{U}_k + \frac{35}{2} \bar{\nu}_k - \frac{63}{2} \bar{\beta}_k \right) + \frac{15}{16} \left( \frac{3}{2} \frac{\partial \bar{r}}{\partial x_k} - 7 \frac{\partial \bar{\varphi}}{\partial x_k} \right) + \frac{3}{8} + \frac{15(h_h + h_s)}{8} \right) \frac{\partial \bar{W}}{\partial x_k}, \quad (k = 1, 2) \tag{46}
\end{equation}

3. Method of the solution

An analytical solution to the stated problem of three-layer plates bending, satisfying the boundary conditions at the edges of the plates (40) and (41), is sought in the form of trigonometric functions.

The solution of equation (15), (17), which describes the tension - compression of the bearing layers, is written in the form:

\begin{equation}
\bar{U}_1 = \bar{C}_1 f_1(x_1, x_2), \quad \bar{U}_2 = \bar{C}_2 f_2(x_1, x_2), \quad \bar{W} = \bar{C}_3 f_3(x_1, x_2), \tag{47}
\end{equation}

where

\begin{equation}
f_1(x_1, x_2) = \cos\left( \frac{\pi x_1}{a} \right) \sin\left( \frac{\pi x_2}{b} \right), \quad f_2(x_1, x_2) = \sin\left( \frac{\pi x_1}{a} \right) \cos\left( \frac{\pi x_2}{b} \right), \quad f_3(x_1, x_2) = \sin\left( \frac{\pi x_1}{a} \right) \sin\left( \frac{\pi x_2}{b} \right)
\end{equation}
And for the filler layer, the solution of the equation describing tension-compression taking into account the transverse compression of the plate (27) is written in the form:

\[
\bar{\psi}_1 = \bar{C}_4 f_1(x_1, x_2), \quad \bar{\psi}_2 = \bar{C}_5 f_2(x_1, x_2), \quad \bar{\beta}_1 = \bar{C}_6 f_1(x_1, x_2),
\]
\[
\bar{\beta}_2 = \bar{C}_7 f_2(x_1, x_2), \quad r = \bar{C}_8 f_3(x_1, x_2), \quad \bar{\gamma} = \bar{C}_9 f_3(x_1, x_2).
\]

(48)

Where \(\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_9\) - are the unknown constants.

The solution of equation (16), (18), which describes the transverse bending and mutual displacement of the bearing layers, is written in the form:

\[
\bar{U}_1 = \bar{C}_7 f_1(x_1, x_2), \quad \bar{U}_2 = \bar{C}_8 f_2(x_1, x_2), \quad \bar{W} = \bar{C}_9 f_3(x_1, x_2).
\]

(49)

And for the filler layer, the solution of the equation describing the transverse bending taking into account the transverse shear and compression of the plate (30) is written in the form:

\[
\bar{\psi}_1 = \bar{C}_4 f_1(x_1, x_2), \quad \bar{\psi}_2 = \bar{C}_5 f_2(x_1, x_2), \quad \bar{\beta}_1 = \bar{C}_6 f_1(x_1, x_2),
\]
\[
\bar{\beta}_2 = \bar{C}_7 f_2(x_1, x_2), \quad r = \bar{C}_8 f_3(x_1, x_2), \quad \bar{\gamma} = \bar{C}_9 f_3(x_1, x_2).
\]

(50)

Where \(\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_9\) - are the unknown constants.

Unknown constants \(\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_8, \bar{C}_9\) are defined by substituting solutions (47)-(50) into equilibrium equations (15)-(18), (27) and (30).

The aim of the study is to determine and evaluate the maximum stresses of the outer layers and the filler layer, as well as contact tangential and normal stresses.

4. Analysis of numerical results

The calculations were performed for three-layer plates of a symmetrical structure with the following numerical values of the parameters. The Poisson's ratios of the layers were taken equal to \(\nu_{21}^{(\pm)} = \nu_{31}^{(\pm)} = 0.3\). The ratio of the thickness of the bearing layers to the thickness of the filler is accepted as follows: \(h_o/h = 0.4\) and the ratios of the thickness of the three-layer plate to the size of the plate in the plan varied (here \(H_o\) is the total thickness of the three-layer plate).

An analysis of the numerical results of the problem of bending a three-layer plate is presented in tables 1, 2 and 3. They show the maximum values of dimensionless stresses \(\sigma_{11}^{(+)} / q_0\) and \(\sigma_{11}^{(-)} / q_0\) and deflections \(W^{(+)} E/q_0 H\) and \(W^{(-)} E/q_0 H\) on the upper and lower layers of a three-layer plate, respectively, for three values of the ratio of the shear moduli of the bearing layers and the filler: and for three values of the Poisson's ratio of the bearing layers.

**Table 1.** Maximum dimensionless values of stress and displacement in the bearing layers of a three-layer plate at \(G_{12}^{(+)}/G_{13} = 10\).

| \(V_{21}^{(+)}\) | \(\sigma_{11}^{(-)} / q_0\) | \(W^{(-)} E/q_0 H\) | \(\sigma_{11}^{(+)} / q_0\) | \(W^{(+)} E/q_0 H\) |
|----------------|-----------------|-----------------|-----------------|-----------------|
| 0.2            | -6.693          | 9.612           | 6.804           | 9.197           |
| 0.3            | -7.402          | 9.011           | 7.599           | 8.603           |
| 0.4            | -8.301          | 8.502           | 8.402           | 8.024           |
Table 2. Maximum dimensionless values of stress and displacement in the bearing layers of a three-layer plate at $G_{12}^{(+)} / G_{13} = 100$.

| $v_{21}^{(+)}$ | $\sigma_{11}^{(-)} / q_0$ | $W^{(-)}E / q_0H$ | $\sigma_{11}^{(+)} / q_0$ | $W^{(+)}E / q_0H$ |
|----------------|------------------------|------------------|------------------------|------------------|
| 0.2            | -15.929                | 4.576            | 15.092                 | 4.183            |
| 0.3            | -18.453                | 4.368            | 17.403                 | 4.013            |
| 0.4            | -21.472                | 4.153            | 20.114                 | 3.801            |

Table 3. Maximum dimensionless values of stress and displacement in the bearing layers of a three-layer plate at $G_{12}^{(+)} / G_{13} = 400$.

| $v_{21}^{(+)}$ | $\sigma_{11}^{(-)} / q_0$ | $W^{(-)}E / q_0H$ | $\sigma_{11}^{(+)} / q_0$ | $W^{(+)}E / q_0H$ |
|----------------|------------------------|------------------|------------------------|------------------|
| 0.2            | -30.474                | 2.583            | 26.644                 | 2.202            |
| 0.3            | -34.767                | 2.398            | 29.899                 | 2.011            |
| 0.4            | -35.596                | 2.213            | 33.503                 | 1.802            |

An analysis of the numerical results showed that the stress-strain state of the layers of a three-layer plate substantially depends on the values of the Poisson coefficient of the bearing layers. With an increase in the Poisson's ratio $v_{21}^{(+)}$, the normal stresses significantly increase, and the deflections in both bearing layers, on the contrary, decrease.

It should be noted that the stresses $\sigma_{11}$ of the upper layer are compressive, and the stresses of the lower layer are tensile and they differ in value from each other, especially when the filler shear modulus is hundreds of times smaller than the shear modulus of the bearing layers. With an increase in the ratio of shear moduli $G_{12}^{(+)} / G_{13}$, the values of the normal stress also increase and the deflections of the bearing layers decrease. The established law of change in stress and deflection is observed for all three values of the ratio of the elastic moduli of the bearing layers and filler $G_{12}^{(+)} / G_{13} = 10$, 100 and 400.

Calculations show that an increase in mechanical parameters $v_{21}^{(+)}$ and $G_{12}^{(+)} / G_{13}$ leads to a significant decrease in the deflection of the bearing layers. Deflections of the upper and lower bearing layer differ from each other up to 19-22%.

5. Conclusions

Thus, a theory and methods for calculating three-layer plates with a compressive spatial filler were created. It should be noted that the proposed theory of three-layer plates is constructed with high accuracy and compiles a method for calculating a three-layer plate under the influence of static and dynamic loads, and is also fundamental in constructing a dynamic continuous layered plate model of multi-storey buildings and is able to take into account their layered and discrete structure.

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