Two-Matrix Model with $ABAB$ Interaction

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Using recently developed methods of character expansions we solve exactly in the large $N$ limit a new two-matrix model of hermitean matrices $A$ and $B$ with the action $S = \frac{1}{2} (\text{tr} A^2 + \text{tr} B^2) - \frac{a}{4} (\text{tr} A^4 + \text{tr} B^4) - \frac{\beta}{2} \text{tr}(AB)^2$. This model can be mapped onto a special case of the 8-vertex model on dynamical planar graphs. The solution is parametrized in terms of elliptic functions. A phase transition is found: the critical point is a conformal field theory with central charge $c = 1$ coupled to 2D quantum gravity.

July 1998

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1. Introduction

Matrix models a proven to be a powerful tool in the study of various mathematical and physical problems, such as random geometry and enumeration of graphs, strings and two dimensional quantum gravity, chaos and mesoscopic systems, statistical mechanics of spins on random lattices etc.

The list of solvable matrix models is not very long. Solvability is usually connected to reducing the number of relevant degrees of freedom from $\sim N^2$ to $\sim N$, for matrices of size $N \times N$. Two basic and very important examples are given by the one matrix model with the partition function:

$$Z_{1MM}(\alpha) = \int dM \exp N \left[ -\frac{1}{2} \text{tr} M^2 + \frac{\alpha}{4} \text{tr} M^4 \right] \quad (1.1)$$

and the two matrix model:

$$Z_{2MM}(\alpha, \beta) = \int\int dA dB \exp N \left[ -\frac{1}{2} (\text{tr} A^2 + \text{tr} B^2) + \frac{\alpha}{4} (\text{tr} A^4 + \text{tr} B^4) + \frac{\beta}{2} \text{tr}(AB) \right] \quad (1.2)$$

where the integrals go over the $N \times N$ hermitean matrices $M, A, B$. We consider here the frequently used quartic non-linearities in the matrix potentials, although one could have a more general potential.

The large $N$ limit of these models is perturbatively equivalent to the planar (or spherical) Feynman graph expansion and describes interesting physical systems. The first one is related to the problems of enumeration of planar graphs [1] and was proposed as a model of pure 2D gravity [2,3]; the second, solved in the planar limit in [4] and [5] describes the Ising model on random dynamical planar graphs [6]. It is clear that the exact solvability of any natural generalizations or modifications of these models could be very useful in various physical and mathematical applications.

We present in this paper a very natural solvable generalization of the two matrix model given by the following integral:

$$Z(\alpha, \beta) = \int\int dA dB \exp N \left[ -\frac{1}{2} (\text{tr} A^2 + \text{tr} B^2) + \frac{\alpha}{4} (\text{tr} A^4 + \text{tr} B^4) + \frac{\beta}{2} \text{tr}(AB)^2 \right] \quad (1.3)$$

The only difference with the conventional two-matrix model (1.2) resides in the last term in the matrix potential. This “little” modification will turn out to be very important.

First of all, this new model cannot be solved by the same method as the old matrix models: the Itzykson–Zuber–Harish Chandra [7,4] formula is not applicable here to reduce it to an eigenvalue problem. In order to avoid this problem, we shall apply a character
expansion method, which was worked out in connection with the matrix models in the papers \[8\], \[9\], \[10\] where it was successfully applied to the investigation of 2D $R^2$ gravity \[8\] and the enumeration of branched coverings \[10\]. The basic point of this method is the reduction of the degrees of freedom (from $\sim N^2$ to $\sim N$) in terms of highest weights of representations in the character expansion. It is a sort of Fourier transform for the matrix variables. The large $N$ limit allows to apply the saddle point approximation for the highest weight distribution (one looks for the most probable Young tableau). Here, following the idea of \[11\], we shall use a double saddle point on both eigenvalues and highest weights.

Second, the planar graph expansion for the model (1.3) corresponds to a different statistical mechanical system than the Ising spins on a random dynamical lattice of the conventional two matrix model (1.2). The $\phi^4$-type planar diagrams of the model consist from the intersecting and self-intersecting closed paths of 2 different colours (see fig. 1). The colouring contributes to the entropy and thus describes some spin degrees of freedom.

![Fig. 1: A typical graph in the expansion of the two matrix model with $ABAB$ interaction. The two matrices ($A$, $B$) are represented by two colours (green, red).](image)

Alternatively, this statistical model model can be mapped on a special case of the 8-vertex model on random dynamical graphs. To see it let us set $X = A + iB$; $X$ is an arbitrary complex matrix, and the model can be recast as a one-matrix model:

$$Z(b, c, d) = \int dX dX^\dagger \exp N \left[ -\frac{1}{2} \text{tr}(XX^\dagger) + b \text{tr}(X^2X^\dagger^2) + \frac{c}{2} \text{tr}(XX^\dagger)^2 + \frac{d}{4} \text{tr}(X^4 + X^\dagger^4) \right]$$

(1.4)

Since $X$ is complex, the propagators carry arrows, and the three vertices reproduce the well-known configurations of the 8-vertex model. Here, the three corresponding weights $b$,
$c, d$ are not independent, since they are related to $\alpha$ and $\beta$ by:

\begin{align*}
  b &= \frac{\alpha + \beta}{8} \\
  c &= d = \frac{\alpha - \beta}{8}
\end{align*}

On a regular lattice, the 8-vertex model is critical when the weight $d$ goes to zero, so that we are left with the 6-vertex model. In the standard parametrization of the 6-vertex model (see for example [12])

\begin{align*}
  a &= \sin \frac{\lambda}{2} (\pi - \theta) \\
  b &= \sin \frac{\lambda}{2} (\pi + \theta) \\
  c &= \sin \lambda \pi \\
  d &= 0
\end{align*}

where $a$ and $b$ are the weights of two vertices which are indistinguishable after coupling to gravity (they are related to each other by a rotation of $\pi/4$); therefore $\theta = 0$. For any value of $0 \leq \lambda \leq 1$, the model has a continuum limit which is a conformal field theory with central charge $c = 1$, and can be described by a free massless compactified boson, with radius $R = \sqrt{\frac{1-\lambda}{2}}$ (up to some duality transformations).

We expect a similar critical behaviour in our two matrix model, but modified by the presence of 2D gravity [13]. This is indeed what we shall find at the critical point of our model, which occurs for $\alpha^* = \beta^* = 1/(4\pi)$. It corresponds to the 6-vertex model at the point where the weight of $(XX^\dagger)^2$ vanishes: $c = 0$, or $\lambda = 1$. The 6-vertex model coupled to gravity has already been studied in [14] and shown to be equivalent to the standard formulation of the vortex-free compactified boson coupled to gravity [15], with a radius of compactification

$$R = \frac{1 - \lambda/2}{\sqrt{2}}$$

which is different from the one on the flat square lattice. For our critical point, $\lambda = 1$ and we find a radius $R = 1/(2\sqrt{2})$, i.e. one half of the Kosterlitz–Thouless radius. But the

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1. To have the three independent constants, one would have to introduce an additional interaction $\gamma \text{tr}(A^2B^2)$.

2. Here, we assume that going from the 6-vertex model to the 8-vertex model corresponds to perturbing the free boson with a magnetic operator of charge $\pm 4$. The Kosterlitz–Thouless phase transition occurs at $\lambda = 0$, as it should be.

3. This is a sign that the radius of compactification is a function of both the weights (i.e. of the parameter $\lambda$) and the geometry of the lattice. We thank I. Kostov for pointing out the discrepancy of radii to us.
The ABAB model is not limited to the 6-vertex model: it also displays off-critical behaviour corresponding to a particular “slice” of the full 8-vertex model.

The plan of the article is as follows: in section 2 we shall apply the character expansion method to reduce the model to the sum over highest weights of the $GL(N)$ character expansion. We then derive the saddle point equations for the density of highest weights and eigenvalues, which yield equations defining the characters and the loop averages in the planar limit.

In section 3 we solve the saddle point equations in terms of incomplete elliptic integrals and fix the parameters of the solution by matching it to the large highest weight asymptotics.

In section 4 we find the critical line (of the thermodynamical limit of large lattices) and the critical point of the phase transition. We also describe the critical behaviour around the critical point, analyse some correlation functions and find the string susceptibility.

The section 5 is devoted to conclusions.

Finally, the appendices contain a detailed study of three particular lines in our two-parameter $(\alpha, \beta)$ family of models; they are the lines given by $\alpha = 0$, $\beta = 0$, and $\alpha = \beta$.

2. The model and its representation by a character expansion.

2.1. Definition of the model.

The two-matrix model with $ABAB$ term is given by the partition function

$$Z(\alpha, \beta) = \int \int dA dB \exp N \left[ -\frac{1}{2} (\text{tr} A^2 + \text{tr} B^2) + \frac{\alpha}{4} (\text{tr} A^4 + \text{tr} B^4) + \frac{\beta}{2} \text{tr}(AB)^2 \right]$$  \hspace{1cm} (2.1)

where $A$ and $B$ are hermitean $N \times N$ matrices. Here $\alpha$ and $\beta$ are positive constants to have positive weights in the diagrammatic expansion; of course, as usual in matrix models in connection with quantum gravity, the analytic continuation from $\alpha, \beta < 0$ to $\alpha, \beta > 0$ which allows to define the integral (2.1) only makes sense at $N \to \infty$.

As already mentioned in the introduction, there is no obvious way to do the integration over the relative “angle” $\Omega$ between the two matrices $A$ and $B$, because no formula is known for the integral over the unitary group $\int d\Omega \exp[c\text{tr}(A\Omega B\Omega^\dagger)^2]$. To circumvent this problem, we use a character expansion of the term $\exp[N\frac{\beta}{2} \text{tr}(AB)^2]$ as a class-function of $AB$: representations of $GL(N)$ are parametrized by their shifted highest weights $h_i = m_i + N - i$ ($i = 1 \ldots N$), where the $m_i$ are the standard highest weights. Then one has

$$Z(\alpha, \beta) \sim \sum_{\{h\}} (N\beta/2)^{\#h/2} \frac{\Delta(h_{\text{even}}/2)}{\prod_i (h_{\text{even}}^i/2)!} \frac{\Delta((h_{\text{odd}} - 1)/2)}{\prod_i ((h_{\text{odd}}^i - 1)/2)!} \int \int dA dB \exp N \left[ -\frac{1}{2} (\text{tr} A^2 + \text{tr} B^2) + \frac{\alpha}{4} (\text{tr} A^4 + \text{tr} B^4) \right] \chi_{\{h\}}(AB)$$
where the sum is over all integers $h_i$ that satisfy $h_1 > h_2 > \ldots > h_N \geq 0$, and $\#h = \sum m_i = \sum h_i - \frac{N(N-1)}{2}$ is the number of boxes of the Young tableau. $\Delta(·)$ is the Van der Monde determinant, $\chi_{\{h\}}$ is the $GL(N)$ character associated to the set of shifted highest weights $\{h\}$, and the $h_{even/odd}^i$ are the even/odd $h_i$, which must be in equal numbers. It is now possible, using character orthogonality relations, to integrate over the relative angle between $A$ and $B$; this leads to a separation into one-matrix integrals:

$$Z(\alpha, \beta) \sim \sum_{\{h\}} (N\beta/2)^{\#h/2} c_{\{h\}}[R_{\{h\}}(\alpha)]^2$$

(2.2)

where $c_{\{h\}}$ is a coefficient:

$$c_{\{h\}} = \frac{1}{\prod_i [h_i/2]! \prod_{i,j} (h_{even}^i - h_{odd}^j)}$$

and $R_{\{h\}}(\alpha)$ is the one-matrix integral

$$R_{\{h\}}(\alpha) = \int dM \chi_{\{h\}}(M) \exp N \left[ -\frac{1}{2} \text{tr} M^2 + \frac{\alpha}{4} \text{tr} M^4 \right]$$

(2.3)

which appears squared in (2.2) because the contributions from the two matrices $A$ and $B$ are identical.

2.2. Study of $R_{\{h\}}(\alpha)$

The one-matrix integral $R_{\{h\}}$ closely resembles what was studied in [11] (see in particular appendix 2), and we use the same approach. First we go over to eigenvalues:

$$R_{\{h\}}(\alpha) = \int \prod_k d\lambda_k \Delta(\lambda) \det (\lambda_k^{h_j}) \exp N \left[ -\frac{1}{2} \sum_k \lambda_k^2 + \frac{\alpha}{4} \sum_k \lambda_k^4 \right]$$

(2.4)

where $\Delta(\lambda) = \det (\lambda_k^{N-j}) = \prod_{j<k} (\lambda_j - \lambda_k)$.

Since we now have $N$ degrees of freedom and an action of order $N^2$, we can use a saddle point method on the eigenvalues $\lambda_k$. In order to do so, we define the resolvent

$$\omega(\lambda) = \frac{\text{tr} \frac{1}{\lambda - M}}{N} = \frac{1}{N} \sum_k \frac{1}{\lambda - \lambda_k}$$

In the $N \to \infty$ limit, the $\lambda_k$ form a continuous density on some support $[-\lambda_c, \lambda_c]$, which implies that $\omega(\lambda)$ has a cut on $[-\lambda_c, \lambda_c]$. If we introduce the notation $\phi(\lambda) = \frac{1}{2} (\omega(\lambda + i0) + \omega(\lambda - i0))$, then $\phi(\lambda_k) = \frac{1}{N} \frac{\partial}{\partial \lambda_k} \log \Delta(\lambda)$. 

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Next we want to differentiate \( \det(\lambda_k^{h_j}) \); as \( \det(\lambda_k^{h_j})/\Delta(\lambda) \) is a regular function of the \( \lambda_k \) (since for example, it is a Itzykson–Zuber type integral), we can introduce another function \( h(\lambda) \) which has the same cut as \( \omega(\lambda) \) on \([-\lambda_c, +\lambda_c]\) and such that

\[
\hat{h}(\lambda_k) = \frac{1}{N} \lambda_k \frac{\partial}{\partial \lambda_k} \log \det \left( \lambda_k^{h_j} \right)
\]  

(we have introduced the factor \( \lambda_k \) in front of the \( \frac{\partial}{\partial \lambda_k} \) for convenience). The saddle point equation of (2.3) can now be written as

\[
\lambda \dot{\psi}(\lambda) + \hat{h}(\lambda) - \lambda^2 + \alpha \lambda^4 = 0 \quad \lambda \in [-\lambda_c, \lambda_c]
\]  

Here comes the key remark: as \( \omega(\lambda) \) and \( h(\lambda) \) have the same cut on \([-\lambda_c, \lambda_c]\), the slashes can be removed in (2.6):

\[
\lambda \omega(\lambda) + h(\lambda) - \lambda^2 + \alpha \lambda^4 = 0
\]  

where \( h^{\dagger} \) is \( h \) on the other side of the cut \([-\lambda_c, \lambda_c]\). Expanding in powers of \( 1/\lambda \) as \( \lambda \to \infty \), we find:

\[
h^{\dagger}(\lambda) = -\alpha \lambda^4 + \lambda^2 - 1 - \sum_{n=1}^{\infty} \frac{1}{\lambda^{2n}} \left\langle \frac{\text{tr}}{N} M^{2n} \right\rangle
\]  

2.3. Functional inversion and analytic structure of \( \lambda(h) \).

In the same way as we have considered a saddle point on the eigenvalues, we shall use a saddle point on the highest weights. Before doing so, we need to understand better the analytic structure of the functions involved.

In a very similar fashion as we have defined \( \omega(\lambda) \), we define \( H(h) \) to be

\[
H(h) = \sum_k \frac{1}{h - h_k}
\]

(here, since the \( h_k \) scale as \( N \) in the large \( N \) limit, we do not need a \( 1/N \) factor in front). After appropriate rescaling \( h \to h/N \), the \( h_k \) tend to a continuous density \( \rho(h) \) as \( N \to \infty \). We shall find out that part of the density is saturated at its maximum value 1 (the same phenomenon occurs in e.g. [16]); therefore we define the end points \( h_1 \) and \( h_2 \) such that

\[
0 < \rho(h) < 1 \quad h_1 < h < h_2
\]

Then \( H(h \pm i0) = \hat{H}(h) \pm i\pi \rho(h) \) on the cut \([0, h_2]\), with \( \hat{H}(h_k) = \frac{1}{N} \frac{\partial}{\partial h_k} \log \Delta(h) \).
Next we introduce the function $L(h)$ which has the same cut as $H(h)$, and such that

$$L(h_k) = \frac{2}{N} \frac{\partial}{\partial h_j} \log \det(\lambda^k_j)$$

(the factor of 2 is due to parity reasons). If we define $\lambda(h) = \exp(\frac{1}{2}L(h))$, then, as proven in a similar context in the appendix 1 of [11], $h(\lambda)$ and $\lambda(h)$ are functional inverses of each other as multi-valued functions. In particular, on all sheets of $\lambda(h)$ such that $\lambda(h) \to \infty$ as $h \to \infty$, one has, according to (2.8),

$$h = -\alpha \lambda^4(h) + \lambda^2(h) - 1 - \sum_{n=1}^{\infty} \frac{1}{\lambda^{2n}(h)} \left\langle \frac{\text{tr} N M^{2n}}{N} \right\rangle$$

(2.9)

A similar expansion was found for large $N$ characters in [9] (see also [17]).

Inverting the expansion (2.9) to express $\lambda^2$ (we shall from now on always use $\lambda^2$ and not $\lambda$ for parity reasons) as a function of $h$, shows that there are two sheets that satisfy $\lambda^2 \to \infty$ as $h \to \infty$: we shall call these $\lambda^2_\pm$. One of them, $\lambda_+$ is the “physical sheet” i.e. the original function $\lambda(h)$. The simplest analytic structure for $\lambda^2(h)$ is then the following: there is a semi-infinite cut $[h_3, +\infty]$ connecting $\lambda^2_+$ and $\lambda^2_-$, the finite cut $[h_1, h_2]$, and a possible pole/zero at $h = 0$. One can then show that $\lambda^2$ satisfies:

$$\lambda^2_+(h) \lambda^2_-(h) = \frac{h}{\alpha} e^{H(h)}$$

(2.10)

The figure 2 describes this analytic structure.

*Fig. 2:* Analytic structure of $\lambda^2(h)$. $\lambda^2_+(h)$ has a double zero at $h = 0$.  

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2.4. Saddle point equation on the highest weights

We finally consider the saddle point equation on \( \{h\} \) in equation (2.2). From (2.4) one infers that 
\[
2 \left( \frac{d}{dh} \log R_h \equiv \mathcal{L}(h) \right),
\]
while \( c_{\{h\}} \) has the following large \( N \) limit:
\[
\log c_{\{h\}} = - \sum_i h_i \left( \log \frac{h_i}{2} - 1 \right) - \frac{1}{2} \log \Delta(h)
\]
so that, after rescaling of the \( h_i \),
\[
\frac{d}{dh} \log \left( \frac{c_{\{h\}}}{N^{\#h/2}} \right) = \left( \mathcal{H}(h) + \log \frac{h}{2} \right)/2.
\]
Therefore the saddle point equation reads:
\[
\mathcal{L}(h) - \frac{\mathcal{H}(h)}{2} = \frac{1}{2} \log(h/\beta) \quad \forall h \in [h_1, h_2]
\quad (2.11)
\]

3. Solution of the saddle point equations in terms of elliptic functions.

In order to solve the saddle point equations, we need to remove the logarithmic cuts of the function \( 2L(h) - H(h) \) on the physical sheet. This is achieved by defining the function \( D(h) = 2L(h) - H(h) - 3 \log h + \log(h - h_1) \). \( D \) has only two (square root type) cuts: \([h_1, h_2] \) and \([h_3, +\infty] \), and it satisfies the following equations:
\[
D = \log \frac{h - h_1}{\beta h^2} \quad \forall h \in [h_1, h_2]
\]
\[
D = \log \frac{h - h_1}{\alpha h^2} \quad \forall h \in [h_3, +\infty]
\quad (3.1)
\]

The solution of (3.1) is given in terms of elliptic functions (for some useful formulae on elliptic functions, see [15]). Define \( r(h) = \sqrt{(h - h_1)(h - h_2)(h - h_3)} \) and the incomplete elliptic integral of the third kind \( \tilde{\Phi}_h \) real:
\[
\tilde{\Phi}_h(h) = r(h) \int_{h + i0}^{+\infty} \frac{dh'}{(h - h')r(h')}
\quad (3.2)
\]
Then \( D(h) \) is given by
\[
D(h) = \log \frac{h - h_1}{\beta h^2} - \tilde{\Phi}_{h_3}(h) \frac{\log(\beta/\alpha)}{i\pi} - \tilde{\Phi}_{h_1}(h) + 2\Phi_0(h)
\quad (3.3)
\]

Next define elliptic parametrizations of \( h \): \( x \) is such that
\[
\text{sn}(x, k) \equiv \text{sn} x = \sqrt{\frac{h_3 - h_1}{h_3 - h}}
\]
where \( k = \sqrt{\frac{b_3-b_2}{b_3-b_1}} \), \( x \) is chosen such that \( 0 \leq \text{Re} \, x \leq K, \) \(-iK' \leq \text{Im} \, x \leq iK' \) (where \( K \) and \( K' \) are the quarter-periods), and \( x(h) \) has a cut on \([h_1, +\infty]\). Also define the rescaled \( \Theta \) functions: \( \Theta_a(x) = \theta_a(\pi x/2K) \), where the \( \theta_a, \ a = 1 \ldots 4, \) are the usual \( \theta \) functions. Finally, let \( Z_a(x) = \left(d/dx\right) \log \Theta_a(x) \).

Then \( \Phi(h) \) is given explicitly by:

\[
\Phi(h) = \tilde{x} \left[ 2Z_1(x) - \frac{i\pi}{K} \right] + \log \frac{\Theta_1(x - \tilde{x})}{\Theta_1(x + \tilde{x})} \tag{3.4}
\]

where \( \tilde{x} = x(h = \tilde{h}) \).

Next we must adjust the behaviour of \( D(h) \) as \( h \to \infty \), which will fix the unknown constants \( h_1, h_2, h_3 \). Starting from \( h = -\alpha \lambda^4 + \lambda^2 - 1 + O(1/\lambda^2) \) (Eq. (2.9)), one proves that

\[
2L_\pm(h) = \log \frac{-h}{\alpha} \mp \frac{1}{\sqrt{-\alpha h}} + \frac{1}{h} + O(h^{-3/2}) \tag{3.5}
\]

with \( L \equiv L_+ \). As \( h \to \infty, \) \( x \sim \sqrt{h_3 - h}/\sqrt{-h}; \) when \( x \to 0, \) \( Z_1(x) \sim 1/x \), so there is a divergent part in (3.3) which must cancel out; this leads to the first condition: \( (x(h = 0) \equiv x_0, x(h = h_1) = K, x(h = h_2 + i0) = K + iK', x(h = h_3 + i0) = iK') \)

condition \# 1: \( 2x_0 = K + \frac{\log(\beta/\alpha)}{\pi} K' \)

Using (3.6) and the explicit expressions of \( \Phi_{h_1} \) and \( \Phi_{h_3}, \) \( D(h) \) can be rewritten in the simpler form

\[
D(h) = \log \frac{h - h_1}{-\alpha h^2} - \frac{\log(\beta/\alpha)}{K} x(h) + 2 \log \frac{\Theta_1(x_0 - x(h))}{\Theta_1(x_0 + x(h))} \tag{3.7}
\]

or alternatively as a function of \( y \equiv x - K): \)

\[
D(h) = \log \frac{h - h_1}{-\alpha h^2} - \frac{\log(\beta/\alpha)}{K} y(h) + 2 \log \frac{\Theta_2(x_0 - y(h))}{\Theta_2(x_0 + y(h))} \tag{3.8}
\]

By matching the \( 1/\sqrt{h} \) terms in the \( h \to \infty \) asymptotics, one obtains the second condition:

condition \# 2: \( \Omega_1 = \frac{1}{\sqrt{\alpha (h_3 - h_1)}} \)

We have defined:

\[
\Omega_a = \frac{\log(\beta/\alpha)}{K} + 4Z_a(x_0) \quad a = 1 \ldots 4
\]

From (3.8) we obtain the expression for the density of highest weights:

\[
\rho(h) = 1 + \frac{\log(\beta/\alpha)}{i\pi K} y(h) - \frac{2}{i\pi} \log \frac{\Theta_2(x_0 - y(h))}{\Theta_2(x_0 + y(h))} \tag{3.10}
\]

When \( h \in [h_1, h_2], \) \( y \) is purely imaginary, \( \text{Im} \, y \in [0, K'], \) so that \( \rho(h) \) is real.
To fix $h_1$, $h_2$ and $h_3$, we need a third condition, which is the normalisation of the density $\rho(h)$:

$$\int_{h_1}^{h_2} dh \rho(h) = 1 - h_1$$

Integrating by parts and changing variables from $h$ to $y$ yields:

$$\int_0^{iK'} dy h(y) \left[ \frac{\log(\beta/\alpha)}{K} - 2(Z_2(y-x_0) - Z_2(y+x_0)) \right] = -i\pi \quad (3.11)$$

As $Z_2(y-x_0) - Z_2(y+x_0)$ is an elliptic function:

$$\frac{1}{2} \left[ Z_2(y-x_0) - Z_2(y+x_0) \right] = -Z_4(x_0) + cn(x_0) dn(x_0) sn(x_0) \frac{dc^2 y}{1 - sn^2(x_0) dc^2 y}$$

(\(dc \equiv \frac{dn}{dh}\)). one can express (3.11) in terms of the complete elliptic integrals of the first and second kinds. The resulting equation fixes $\alpha$:

**condition # 3:**  \[ \alpha = \frac{1}{\pi \Omega_1^2} \left( E' \Omega_1 - K' \Omega_4 \frac{1}{\sn^2(x_0)} \right) \quad (3.12) \]

To summarize, given $\log(\beta/\alpha)$ and the elliptic nome $q = e^{-\pi K'/K}$, one can compute $x_0$ from (3.6), then $\alpha$ and $\beta$ from (3.12), and finally $h_1$, $h_2$, $h_3$ from (3.9).

4. The critical line and the critical point.

We now investigate the critical properties of the model. Just as in the standard eigenvalue problem, criticality is attained when a branch point collides with a critical point (i.e. a point where the derivative vanishes), so that the square root singularity degenerates into a \((h-h_0)^{3/2}\) behavior. If one starts with small coupling constants $\alpha$ and $\beta$ and then increases these constants, two phenomena can occur, which correspond to two pieces of the critical line.
4.1. Criticality of type A.

The first type of criticality appears when the singularity appears in the vicinity of $h_3$, i.e. at the start of the infinite cut $[h_3, +\infty]$. This is what happens, for example, if one fixes $\beta$ to a small value, and increases $\alpha$: for $\alpha$ smaller than a critical value $\alpha_c(\beta)$, $h_3$ is well-defined and real, but for $\alpha > \alpha_c$ $h_3$ becomes complex, which is the sign of a change of analytic structure, and therefore of criticality.

In order to find the critical line in the $(\alpha, \beta)$ plane, one looks at the behavior of $D(h)$ as $h \to h_3$, i.e. $x \to \pm iK'$. The procedure is very similar to looking at the $x \to 0$ behavior; criticality is obtained by cancelling the $\sqrt{h - h_3}$ term, that is when

$$\Omega_4 = \frac{\log(\beta/\alpha)}{K} + 4Z_4(x_0) = 0$$

(compare with (3.9)).

Using (3.9) one can simplify this condition:

$$\frac{\text{cn} x_0 \text{dn} x_0}{\text{sn} x_0} = \frac{1}{4\sqrt{\alpha(h_3 - h_1)}}$$

and reexpress it in terms of $h$:

$$\frac{h_1 h_2}{h_3} = \frac{1}{16\alpha}$$

(4.2)

4.2. Criticality of type B.

This time the singularity occurs at the level of the finite cut $[h_1, h_2]$; this is what happens for small $\alpha$ and increasing $\beta$.

Criticality is more precisely found by looking at the behaviour of $D(h)$ as $h \to h_2$, i.e. $x \to K \pm iK'$. Cancellation of the $\sqrt{h - h_2}$ term implies:

$$\Omega_3 = \frac{\log(\beta/\alpha)}{K} + 4Z_3(x_0) = 0$$

(4.3)

Combining it with (3.9) yields

$$\frac{\text{cn} x_0}{\text{sn} x_0 \text{dn} x_0} = \frac{1}{4\sqrt{\alpha(h_3 - h_1)}}$$

or equivalently

$$\frac{h_1 h_3}{h_2} = \frac{1}{16\alpha}$$

(4.4)
4.3. Double criticality: the critical point.

When one imposes both (4.2) and (4.4), one finds \( h_3 = h_2, \) \( h_1 = 1/(16\alpha) \): the critical point corresponds to the limit when the two cuts touch each other. Therefore we are in the trigonometric limit \( k \to 0 \). We also find \( \alpha = \beta \). In this case one can solve directly the equations (3.1) which combine into a single equation; the result is:

\[
D(h) = \log \frac{h - h_1}{-\alpha h^2} - 2 \log \frac{\sqrt{h_1 - h} + \sqrt{h_1}}{-\sqrt{h_1 - h} + \sqrt{h_1}} = \log \frac{h - h_1}{-\alpha} - 4 \log (\sqrt{h_1 - h} + \sqrt{h_1})
\]

\[
\rho(h) = 1 - \frac{4}{\pi} \arctan \sqrt{h/h_1 - 1}
\]

\( \rho(h) \) is zero for \( h = 2h_1 \), so one has \( h_2 = h_3 = 2h_1 \). Finally the normalisation of the density gives

\[
h_1 = \frac{\pi}{4}
\]

and therefore \( \alpha = \beta = 1/(4\pi) \).

The critical line and the critical point have been drawn numerically on figure 4. The critical line is divided in two: the left of the critical point \( (\alpha > \beta) \) is the phase A (criticality is of type A), whereas the right \( (\beta > \alpha) \) is the phase B. Note that the trigonometric limit \( q \to 0 \) is precisely the neighborhood of the critical point, and on the contrary that the hyperbolic limit \( q \to 1 \) is the limit \( \alpha \to 0 \) or \( \beta \to 0 \).

![Fig. 4: Critical line, critical point and equipotentials of \( q \) \( (q = 0(0.03)0.6) \) in the \( (\alpha, \beta) \) plane.](image-url)
To complete the picture at the critical point, we need to go back from $D(h)$ to $L(h)$, which requires to compute $H(h)$. In fact, $H(h)$ is not a standard function but $dH/dh$ is:

\[
\frac{dH}{dh} = \int_0^{h_2} \rho(h') \left( \frac{dh'}{(h-h')^2} \right)
\]

\[
= -\frac{2}{\pi h} \sqrt{\frac{h_1}{h-h_1}} \log \frac{\sqrt{h-h_1} + \sqrt{h_1}}{\sqrt{h-h_1} - \sqrt{h_1}}
\]

This expression is sufficient to expand $2L(h) = D(h) - H(h) + 3 \log h - \log(h-h_1)$ around the “critical point” $h_c \equiv h_2 = h_3$, which provides us with the loop function $h(\lambda)$ for large loops. Only $H(h)$ has a singularity at this point: $dH/dh \propto \log(h-h_c)$, and therefore

\[
H(h) \propto (h-h_c) \log(h-h_c),
\]

which we can invert:

\[
h - h_c \propto \frac{\lambda - \lambda_c}{\log(\lambda - \lambda_c)}
\]

(4.5)

We shall now study the properties of the vicinity of the critical line. In particular, we will need the exact expression for a correlation function. By functional inversion, $\lambda(h)$ provides us with all correlation functions of the type $\langle \text{tr} A^{2n} \rangle$. In practice, one can easily compute $\langle \text{tr} A^2 \rangle$ by inverting (2.9) up to order $1/\lambda^2$:

\[
2L(h) = \log \frac{-h}{\alpha} - \frac{1}{\sqrt{-\alpha h}} + \frac{1}{h} + \left( \sqrt{\alpha} \left( \frac{\text{tr} A^2}{N^2} \right) + \frac{1}{24 \alpha^{3/2}} - \frac{1}{2 \sqrt{\alpha}} \right) \frac{1}{2 (-h)^{3/2}} + O(h^{-2})
\]

(4.6)
where \( L(h) = 2 \log \lambda(h) \), and expanding the explicit expression (3.7) of \( D(h) \).

Here, we shall use another correlation function, \( \langle \text{tr}(AB)^2 \rangle \). It is given by:

\[
\langle \frac{\text{tr}}{N} (AB)^2 \rangle = \frac{\partial}{\partial \beta} Z
\]

\[
= \frac{1}{\beta} \langle \# h \rangle
\]

\[
= \frac{1}{\beta} \left[ \frac{1}{2} h^2_1 - \frac{1}{2} + \int_{h_1}^{h_2} \rho(h)h \, dh \right]
\]

\[
= \frac{1}{\beta} \left[ -\frac{1}{2} - \int_0^1 iK' \frac{dy}{dy} \frac{h^2}{2} \right]
\]

At this point, the calculation becomes similar to that of \( \alpha \). The result is:

\[
\langle \frac{\text{tr}}{N} (AB)^2 \rangle = \frac{1}{\beta} \left[ -\frac{1}{2} + h_3 + \pi \frac{E'(\Omega_4/s^2x_0 - \Omega_1^2(2 - k^2)) + K'\Omega_1^2(1 - k^2)}{(E'\Omega_1 - K'\Omega_4/s^2x_0)^2} \right]
\]

(4.7)

\[
\langle \frac{\text{tr}}{N} (AB)^2 \rangle \sim \Delta^{3/2}
\]

(4.8)

4.4. Vicinity of the critical line.

Let us suppose first that we approach the critical line at fixed slope \( s \neq 0 \); then \( q \) tends to a non-zero critical value \( q_c \), and one can show the following facts: \( \frac{d}{dq} \alpha|_{q=q_c} = 0 \), \( \frac{d}{dq} \langle \text{tr}(AB)^2 \rangle|_{q=q_c} = 0 \) and also \( \frac{d}{dq} h_k|_{q=q_c} = 0 \) with \( k = 1, 2 \) for criticality of type A, \( k = 1, 3 \) for criticality of type B. This implies in particular that

\[
\langle \frac{\text{tr}}{N} (AB)^2 \rangle \sim \Delta^{3/2}
\]

where \( \Delta \equiv \alpha_c - \alpha \) is the renormalized cosmological constant, and therefore the string susceptibility exponent is \(-\frac{1}{2}\): this means that on the whole critical line except at the critical point, the continuum theory is a \( c = 0 \) theory (pure gravity).

In fact, the analytic structure found around any point on the critical line in phase A (resp. B) is identical to that found for the endpoint \( \alpha = 1/12, \beta = 0 \) (resp. \( \beta = 2/9, \alpha = 0 \)). These two particular points have been studied in detail in appendix A. The conclusion one draws from the more explicit calculations done at these points is the following: though we have in both phases a \( c = 0 \) theory, for phase A the loop scaling function is the usual pure gravity loop function, whereas in phase B this function describes non-trivial loops (eq. (A.6)).
4.5. Vicinity of the critical point.

We now investigate the vicinity of the critical point $\alpha = \beta = 1/(4\pi)$, which is also the small $q$ region. We refer to appendix B for details on small $q$ expansions.

If we define the deviation from criticality

$$\Delta \equiv 1 - 4\pi\alpha$$

and the slope in the $(\alpha, \beta)$ plane

$$s \equiv \frac{\log(\beta/\alpha)}{4\pi}$$

then we obtain up to order $q^3$ the equation defining $q$ in terms of $\alpha$ and $\beta$:

$$\Delta = 2s + 8q^2 \log q^{-1} + 2s^2 \log^2 q^{-1} + 4q^2 - 4s^2$$

(4.9)

We can find from here the shape of the critical line near the critical point. It is given by the equation:

$$\frac{d}{dq}\Delta = 16q \log q^{-1} - \frac{4s^2}{q} \log q^{-1} = 0$$

(4.10)

or

$$q = \pm s/2$$

(4.11)

for $\beta < \alpha$ and $\beta > \alpha$, respectively. One can check (using the last two expansions of (B.3)) that it fits well the conditions (4.1) and (4.3).

In particular, this implies that the slope of this line at the critical point is:

$$\frac{d\alpha}{d\beta} = \frac{1}{1 - 2\pi}$$

(4.12)

It is possible to expand (4.8) around $q = 0$; the result is of the form

$$\left< \frac{\text{tr}}{N}(AB)^2 \right> = c_1 + c_2 s + c_3 s^2 \log^2 q^{-1} + c_4 s^2 \log q^{-1} + c_5 q^2 \log q^{-1} + c_6 s^2 + c_7 q^2$$

(4.13)

where the coefficients $c_i$ are given in appendix B.

For $s = 0$, from (4.9) $\Delta \propto q^2 \log q^{-1}$, and the singular part of (4.13) yields

$$\left< \frac{\text{tr}}{N}(AB)^2 \right>_{\text{sing}} \propto \frac{\Delta}{\log \Delta}$$

(4.14)

that is a zero string susceptibility exponent, plus logarithmic corrections. This is characteristic of a $c = 1$ model coupled to gravity [19].
Let us also note that on the critical line \( s = \pm 2q \), one can check that, as expected,

\[
\frac{d}{dq} \left( \frac{\langle \text{tr} (AB)^2 \rangle}{N} \right) = 0
\]

The free energy of the underlying statistical-mechanical model in the thermodynamical limit can be obtained by the following trick [6]. The planar free energy of the matrix model (i.e. the partition function of the statistical model on connected planar graphs) near the critical line looks as

\[
\log Z \sim \sum_n n^{\gamma_{str}-3} [\alpha/\alpha^*(s)]^n
\]

where \( \alpha^*(s) \) is the radius of convergence of the series at fixed \( s \), and \( n \) (the number of \( A^4 \) vertices) characterizes the size of the graphs. Therefore, \( \log[\alpha^*(s)] \) plays the role of the free energy per volume unit in the thermodynamical limit. By plugging (4.11) into (4.9) we find:

\[
\Delta^*(s) = 1 - 4\pi \alpha^*(s) = 2s + s^2 \log(4/s^2) + s^2 \log^2(4/s^2) - 3s^2
\]

Whereas this function and its first derivative are finite and continuous the second derivative (playing the role of the specific heat of the problem with \( s \) along the critical line being the “temperature”) develops a \( \log^2 \) type singularity:

\[
\frac{d^2}{ds^2} \Delta^*(s) \sim 2 \log^2(4/s^2) \quad \text{for} \ s \to 0
\]

giving rise to a second order phase transition.

This is an unusual result since for most of the spin models on random dynamical graphs the transition is usually of the third order. A similar situation, but with a \( \log \) type singularity occurs in the model of percolation on dynamical random graphs solved in [20].

5. Conclusion

Our results show that the possibilities to find new exact solutions of the matrix models are far from being exhausted. The character expansion method used here enlarges the class of such models with a potentially very rich spectrum of critical behaviours and thus with numerous possible applications to mesoscopic physics, enumeration of graphs, statistical-mechanical models on random dynamical graphs, etc.

For example, a rather general class of such models for \( p \) hermitean matrices can be given by the action: \( S = \text{tr}(\sum_{k=1}^p V_k(M_k) + f(\prod_{k=1}^p M_k)) \) for arbitrary functions \( V_k \) and \( F \). It is easy to see from the orthogonality of characters that the exponent of the last
term can be integrated over relative angles of the matrices. But even this is not the most general example.

One interesting question we can pose here is the following: is there any integrable structure behind the representation expansions of partition functions like (2.2)? For more conventional matrix models, we are used to the fact that they can be represented as tau-functions of some integrable hierarchies of classical differential equations. These tau-functions are usually some $N \times N$ determinants or Fredholm determinants in the grand canonical ensemble with respect to $N$.

The particular elements of this formula remind the objects known from the tau-functions: the characters or their integrals with one-matrix weights (like in (2.3)), as well as the product $\prod (h_{\text{even}} - h_{\text{odd}})^{-1}$ have determinant representations (or, in other words, some fermionic analogues). But we don’t know such a representation for the whole sum over Young tableaux. If such a representation exists it could allow us to analyse the critical behaviour for all genera of random graphs in the double scaling limit, like for the more standard matrix models [21], [22], [23].

Let us also note that the universal behaviours of the physical quantities we found, like the loop function (4.5) look different from the known patterns for the $c = 1$ matter coupled to 2D gravity. A possible explanation is that the loops we are considering differ from the usual ones (through different “boundary conditions” on the loop). If so, it would be interesting to make the connection between the different types of loops.

Also, the critical point we have in our model is only one point in the critical line of $c = 1$ CFTs. In order to describe more precisely the properties of this point, it would be interesting to investigate the possibility of a non-trivial scaling limit around the critical point $\alpha_\ast = \beta_\ast = 1/(4\pi)$. This would shed light on the perturbing operator around the fixed point, which is directly related to the radius of compactification $R$ in the equivalent bosonic formulation; it would give an independent check of the predicted radius $R = 1/(2\sqrt{2})$.

Finally, it would of course be very interesting to obtain the solution of the full 8-vertex model coupled to 2D gravity (with all couplings $b, c, d$ independent in (1.4)). This would allow to describe the most general critical properties of $c = 1$ coupled to 2D gravity. But for the moment, even character expansion methods cannot solve this more general matrix model.

**Acknowledgements**

The authors would like to thank I. Kostov for many useful discussions.
Appendix A. Study of particular cases.

A.1. $\beta = 0$: the usual one matrix model revisited.

When $\beta = 0$ the model reduces to two decoupled one-matrix models with the standard action $\frac{1}{2} \text{tr} M^2 + \frac{\alpha}{4} \text{tr} M^4$. We shall now compare our solution with the classical solution of pure gravity.

Since the expansion in characters is trivial for $\beta = 0$, the saddle point character is necessarily the trivial representation, i.e. $\rho(h) = 1$ for $h \in [0, 1]$. Therefore

$$H(h) = \log \frac{h}{h - 1}$$

which implies that $h_1 = h_2 = 1$ (the cut reduces to a pole at $h = 1$), and

$$\lambda_\pm^2(h) = \frac{1}{\sqrt{\alpha}} \frac{(\sqrt{h_3} \mp \sqrt{h_3 - h})^2}{\sqrt{h_3 - 1} \mp \sqrt{h_3 - h}}$$  \hspace{1cm} (A.1)

$h_3$ is given explicitly as a function of $\alpha$ by

$$h_3 = \frac{5/4 - 3\alpha + \sqrt{1 - 12\alpha}}{9\alpha}$$

which has a singularity at $\alpha = 1/12$, the critical point of pure gravity.

On the other hand, one can use the standard method for this model, which is to solve directly the saddle point equations for the eigenvalues. Indeed, when the character is trivial, it is clear from (2.5) that $h(\lambda) = \lambda \omega(\lambda)$; then (2.6) becomes

$$2\beta^2 - \lambda^2 + \alpha \lambda^4 = 0$$

which yields

$$h_\pm = \frac{1}{2} \left[ \lambda^2 - \alpha \lambda^2 \pm (\alpha \lambda^2 - 1 + \frac{1}{2} \alpha \lambda_+^2) \sqrt{\lambda^2 (\lambda^2 - \lambda_+^2)} \right]$$  \hspace{1cm} (A.2)

where $h(\lambda) \equiv h_+(\lambda)$ is the physical sheet, and $h^\dagger(\lambda) \equiv h_-(\lambda)$ is the other sheet connected by the cut $[-\lambda_c, +\lambda_c]$. The end point of the cut $\lambda_c$ is given by:

$$\lambda_c^2 = \frac{2}{3\alpha} (1 - \sqrt{1 - 12\alpha})$$

Note that $h^\dagger(\lambda)$ has the correct expansion (2.8) as $\lambda \to \infty$.

---

4 As a function of $\lambda^2$, the cut of $h(\lambda^2)$ is $[0, \lambda_c^2]$. 
It is easy to show that $\lambda^2(h)$ given by (A.1), and $h(\lambda^2)$ given by (A.2), are functional inverses of each other: indeed they are the solution of the equation

$$\alpha \lambda^4(h - 1) - \lambda^2(h - \alpha h_3 \lambda_c^2) + h^2 = 0$$

which is quadratic in $\lambda^2$ and in $h$.

Criticality is of course obtained when $\alpha \to \alpha_c = 1/12$. By computing correlation functions one immediately finds that the string susceptibility $\gamma = -\frac{1}{2}$, that is pure gravity.

Let us also remind the reader of the universal loop scaling function. The asymptotics of the loop average $\langle \text{tr} A^{2n} \rangle$ (which is interpreted as the summation over surfaces with a fixed boundary of length $n$), $n$ large, are dominated by the singularity of the resolvent $\omega(\lambda)$ (or of $h(\lambda)$) which is closest to $\lambda = \infty$. Here, it is the square root singularity starting at $\lambda = \lambda_c$. Equivalently, one can say that $h_c \equiv h(\lambda = \lambda_c)$ is a critical point of the function $\lambda(h)$. When $\alpha \to \alpha_c$, $h_c$ collides with the branch point $h_3$, as expected. One can now take a scaling limit where the renormalized cosmological constant $\Delta = \alpha_c - \alpha \to 0$ (so that the average area of the closed surface $\langle A \rangle \propto 1/\Delta$ diverges) and the renormalized boundary cosmological constant $\lambda - \lambda_c \to 0$ (so that the typical size of the loop $\langle l \rangle \propto 1/(\lambda - \lambda_c)$ diverges) in a correlated manner. Here, the scaling function is:

$$\frac{h - h_c}{\Delta^{3/4}} \propto \left(\text{cst} - \frac{\lambda - \lambda_c}{\Delta^{1/2}}\right) \left(\frac{\lambda - \lambda_c}{\Delta^{1/2}}\right)^{1/2}$$

\[\text{A}.4\]

A.2. $\alpha = 0$.

When $\alpha = 0$, only the vertex $\text{tr}(ABAB)$ survives, so that one has a model with two types of loops which intersect each other all over the surface. Discarding one of the two types of loops, one finds a model of dense loops, i.e. loops that cover the entire space. This looks very similar to the dense phase of the $O(1)$ model (i.e. low-temperature phase of the Ising model coupled to gravity), which has already been studied in [24]. In fact, the $ABAB$ model with $\alpha = 0$ (and its $O(n)$ generalisation) was already studied in [25] (by completely different methods from ours), where it was shown to be equivalent to a modified $O(1)$ model. Let us compare here our results with those of [24] and [25].

If $\alpha = 0$, the expansion (2.9) implies that the analytic structure looks different from the generic case $\alpha > 0$: indeed, $\lambda$ has no cut going to infinity. In other words, when $\alpha \to 0$, $h_3$ tends to infinity (more precisely, one can show that $h_3 \sim 1/(4\alpha)$). We are left with a single cut $[h_1, h_2]$; using the general solution and the additional relation $\lambda^2(h) = h \exp(H(h))$ which follows from the fact that $\alpha = 0$, we can explicitly the function $\lambda(h)$:

$$\lambda(h) = \frac{1}{2} \frac{(h + \sqrt{(h - h_1)(h - h_2)} + \sqrt{h_1 h_2})^2}{h(\sqrt{h - h_1} + \sqrt{h - h_2})}$$

\[\text{A}.5\]
$h_1$ and $h_2$ are given as solutions of a third degree equation: 
\[ h_{1,2} = \pm \beta \frac{2}{3} x^2 + \frac{1}{3} x - \frac{1}{3} \]
where
\[ x^3 - \frac{16}{3 \beta^2} x + \frac{64}{3 \beta^2} = 0 \]
The discriminant \( \Delta' \equiv \frac{4}{81 \beta^2} - 1 \) vanishes when \( \beta = \beta_c = \frac{2}{9} \) (the value of \( \beta_c \) coincides with what was found in [25]). Then \( h_1 = \frac{2}{3}, h_2 = \frac{8}{3} \), and \( \lambda_c = \lambda(h = h_2) = 3/\sqrt{2} \) satisfies \( \beta_c \lambda_c^2 = 1 \), as follows from general arguments [25].

One can easily compute the string susceptibility in the limit \( \beta \to \beta_c \); using \( x = 6(1 + \frac{1}{\sqrt{3}} \Delta'^{1/2} + \cdots) \), one finds
\[ \left\langle \frac{\text{tr} N A^2}{N} \right\rangle_{\text{sing}} \propto \Delta'^{3/2} \]
so that the string susceptibility exponent \( \gamma = -1/2 \), which indicates a \( c = 0 \) model.

However, the loop scaling function is non-trivial; in order to see this, we use the simple formula:
\[ \frac{d}{dh} L(h) = 2 \lambda^{-1} \frac{d \lambda}{dh} = \frac{h - 2 \sqrt{h_1 h_2}}{h \sqrt{(h-h_1)(h-h_2)}} \]
which shows that \( \lambda(h) \) has a critical point at \( h_c = 2 \sqrt{h_1 h_2} \). This point meets the branch point \( h_2 \) when \( \beta \to \beta_c \), as expected. In the limit \( \beta \to \beta_c \), the proper scaling ansatz is that
\[ h - h_c \propto \Delta'^{1/2}, \lambda - \lambda_c \propto \Delta'^{3/4} \]
Then the universal loop scaling function is given by:
\[ \frac{\lambda - \lambda_c}{\Delta'^{3/4}} \propto \left( \frac{h - h_c}{\Delta'^{1/2}} \right)^2 \left( \text{cst} + \frac{h - h_c}{\Delta'^{1/2}} \right)^{1/2} \]  
(A.6)

Note that the scaling \( \lambda - \lambda_c \propto \Delta'^{3/4} \) is unusual since it implies that \( \langle l \rangle \propto \langle A \rangle^{3/4} \).

In both limits \( \frac{h - h_c}{\Delta'^{1/2}} \to 0 \) and \( \frac{h - h_c}{\Delta'^{1/2}} \to \infty \), one recovers the same asymptotics as the loop function found in [24]; however, the full loop function is different, suggesting that for loops of the same size as a typical loop on the surface, the physics of these two models differs qualitatively.

A.3. \( \alpha = \beta \).

This is the line on which the critical point lies; on this line it is possible to reexpress the function \( D(h) \) using only standard functions (log, square root . . .). However, the parameters of the problem still depend on the constants \( k, K', E' \) which are all functions of the elliptic nome \( q \). Let us see how this works.
By putting $\alpha = \beta$ we obtain from (3.6) that $x_0 = K/2$ and from (3.9) that

$$h_1 = \frac{k'}{4\alpha(1 + k')},$$

$$h_2 = \frac{k'}{4\alpha(1 + k')},$$

$$h_3 = \frac{1}{4\alpha(1 + k')^2}.$$  \hspace{1cm} (A.7)

If we employ the functional relations for $\text{sn}$ and $\text{cn}$, together with the formulas $\text{sn}(K/2) = \frac{1}{\sqrt{k + 1}}$ etc, we find:

$$\rho(h) = 1 + \frac{2}{i\pi} \log \frac{-i\sqrt{(h - h_1)(h_2 - h) + \sqrt{(h_2 - h_1)(h_3 - h)}}}{i\sqrt{(h - h_1)(h_3 - h) + \sqrt{(h_3 - h_1)(h_2 - h)}}}.$$  \hspace{1cm} (A.8)

The constants $h_1$, $h_2$, $h_3$ are given by (A.7), where $\alpha$ is:

$$\alpha = \frac{E' - (1 - k')K'}{2\pi(1 + k')}.$$  \hspace{1cm} (A.9)

**Appendix B. Small $q$ expansions.**

We rewrite the condition (3.6) as

$$\frac{\pi}{2K} x_0 = \frac{\pi}{4} + s \log q^{-1}$$

where it is reminded that

$$s \equiv \frac{\log(\beta/\alpha)}{4\pi}$$

In these variables, the defining equation (3.12) for $\alpha$ becomes:

$$2\pi^2 \alpha = KE' - \frac{1}{2s + \frac{\theta''_1(\pi x_0)}{\theta'_1(\pi x_0/2K)}} - KK' \frac{2s + \frac{\theta''_1(\pi x_0)}{\theta'_1(\pi x_0/2K)}}{\left[2s + \frac{\theta''_1(\pi x_0)}{\theta'_1(\pi x_0/2K)}\right]^2} \text{sn}^2 x_0.$$  \hspace{1cm} (B.1)

We then use the $q$ expansions:

$$K = \pi/2(1 + 4q + 4q^2 + O(q^3))$$

$$K' = \frac{\log q^{-1}}{2}(1 + 4q + 4q^2 + O(q^3))$$

$$E' = 1 + 4q \log q^{-1} - 4q - 8q^2 \log q^{-1} + 12q^2 + O(q^3 \log q^{-1})$$

$$k = 4q^{1/2}(1 - 4q + 14q^2 + O(q^3))$$

$$k' = 1 - 8q + 32q^2 + O(q^3)$$

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and the double $q$, $s$ expansions:

\[
\begin{align*}
\frac{1}{\text{sn}^2(x_0)} &= 2(1 - 2s \log q^{-1} - 4q + 4s^2 \log^2 q^{-1} + 16qs \log q^{-1} + 16q^2) + \cdots \\
\frac{\theta'_1}{\theta_1} \left( \frac{\pi x_0}{2K} \right) &= 1 - 2s \log q^{-1} + 2s^2 \log^2 q^{-1} + 4q^2 + \cdots \quad \text{(B.3)} \\
\frac{\theta'_4}{\theta_4} \left( \frac{\pi x_0}{2K} \right) &= 4q + \cdots \\
\frac{\theta'_3}{\theta_3} \left( \frac{\pi x_0}{2K} \right) &= -4q + \cdots
\end{align*}
\]

where $s$ is assumed to be – at most – of order $q$: this turns out to be a correct assumption, since one has $|s| \lesssim 2q$ (the limiting values define the critical line). In (B.3) the error is of order $q^3 \log^3 q^{-1}$.

The deviation of $\alpha$ from criticality is then given by:

\[
\Delta = 2s + 8q^2 \log q^{-1} + 2s^2 \log^2 q^{-1} + 4q^2 - 4s^2 + O(q^3 \log^3 q^{-1}) \quad \text{(B.4)}
\]

The analogous expression for $\Delta' = 1 - 4\pi \beta$ is:

\[
\Delta' = 2(1 - 2\pi)s + 8q^2 \log q^{-1} + 2s^2 \log^2 q^{-1} + 4q^2 - 4s^2(1 - 2\pi + 2\pi^2) + O(q^3 \log^3 q^{-1}) \quad \text{(B.5)}
\]

On the critical line, $s \sim \pm 2q$ and therefore its slope is $\Delta/\Delta' \sim 1/(1 - 2\pi)$.

In the same way, one can expand the correlation function (4.7); the result is

\[
\left\langle \frac{\text{tr}}{N} (AB)^2 \right\rangle = \frac{2\pi}{3} \left( -3 + 2\pi + 2s(-3 + 9\pi - 4\pi^2) - 6s^2 \log^2 q^{-1} + 24\pi s^2 \log q^{-1} \\
- 24q^2 \log q^{-1} + 4\pi s^2(3 - 12\pi + 4\pi^2) + 12q^2(-1 + 4\pi) \right) + O(q^3 \log^3 q^{-1})
\]

which is of the form (4.13).
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