Complex Analysis of a Piece of Toda Lattice

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Abstract

We study a small piece of two dimensional Toda lattice as a complex dynamical system. In particular the Julia set, which appears when the piece is deformed, is shown analytically how it disappears as the system approaches to the integrable limit.

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1 Introduction

The two dimensional Toda lattice is one of soliton equations which has become more and more important as a key object in theoretical physics. It was first formulated by Hirota in 1981[1] as a discrete version of two continuous time Toda lattice and shown by Miwa[2] its equivalence to the KP hierarchy. When quasi periodic solutions are substituted, it is nothing but the identity known as Fay’s trisecant formula which characterizes algebraic curves.

This equation has become known in other fields of physics in the last ten years. It was shown being satisfied by the string amplitudes in particle physics[3]. More recently there appeared papers demonstrating unexpected correlation of this equation with other topics in physics. The transfer matrix of the solvable lattice model with $A_1$ symmetry, for example, was shown to satisfy this equation[4][5]. This equation has been also proven to unify discrete Painlevé equations[6]. The connection of solvable cellular automata to this equation offers another example[7].

Completely integrable nonlinear systems must play fundamental roles in various phenomena in physics. It is remarkable that many integrable systems in different fields are unified into the single equation. We are interested in clarifying ultimate notion of integrability of the systems. Investigation of such systems themselves, however, will not reveal all of features of the systems. The real meaning of integrability will be clarified only in comparison with nonintegrable systems.

An arbitrary deformation of the two dimensional Toda lattice will destroy integrability and create chaos. Since the system contains infinite number of degrees of freedom it is extremely difficult to study analytically the behaviour of transition from nonintegrable to integrable phases. It should be recalled that a very little is known about analytical properties of nonintegrable systems. The main part of the studies of complex dynamical systems were limited to simple systems with one degree of freedom.

Very recently we pointed out[8] that a set of lattice points, which form a parallelogram in the two dimensional lattice space, constitute a piece of the Toda lattice. We call it a Toda molecule[9] since it is essentially what is intended to be called by this name, but used in a bit different context in the literature. The remarkable fact is that the small pieces can be separated from other parts without loosing any properties of the original Toda lattice.

The purpose of this paper is to study in detail analytical properties of the smallest piece of Toda molecules. The smallest Toda molecule is a smallest parallelogram of four lattice points. We will call it a Toda atom for convenience. Since every Toda molecule preserves properties possessed by the Toda lattice, we can study analytical properties of the system from the knowledge of a Toda atom. In the first part of this paper we show that the time evolution of a Toda atom is described by an iterative Möbius
map. The form invariance of this map certifies integrability of this system. In the second part of this paper we will consider a deformation of this piece. Under generic deformation a chaos will be generated through the time evolution. We are especially concerned with analytical property of the Julia set as the system approaches to the integrable map. We will show how the Julia set converges to the points on the orbit of Möbius map as a parameter, which interpolates between integrable and nonintegrable.

2 Pieces of Toda Lattice

In this section we like to show that the two dimensional Toda lattice can be cut into small pieces without loosing any properties possessed by the original system. To begin with let us write down the equation which was derived by Hirota as a discrete version of the two continuous time Toda lattice\[1\]:

\[
\alpha g_{n}(l+1,m)g_{n}(l,m+1) + \beta g_{n}(l,m)g_{n-1}(l,m+1) - (\alpha + \beta) g_{n+1}(l+1,m)g_{n-1}(l,m+1) = 0, \quad \alpha, \beta \in \mathbb{C}, \quad l, m, n \in \mathbb{Z}.
\]

We called this equation Hirota bilinear difference equation\footnote{This equation is also called Hirota-Miwa equation in recent literature.} and abbreviated as HBDE. This is a nonlinear system defined on the three dimensional lattice space. Our key observation is the following. For a fixed point of the lattice \((l, m, n) = (\bar{l}, \bar{m}, \bar{n})\), we denote by \(A\) the set of points \((\bar{l}, \bar{m}, \bar{n}), (\bar{l}+1, \bar{m}, \bar{n}), (\bar{l}, \bar{m}+1, \bar{n}), (\bar{l}+1, \bar{m}+1, \bar{n}), (\bar{l}, \bar{m}+1, \bar{n}+1), (\bar{l}+1, \bar{m}, \bar{n}+1)\). Then if \(g_{n}(l,m)\) is a solution of (1),

\[
f(l,m,n) = \begin{cases} 
g_{n}(l,m), & (l, m, n) \in A, \\
0, & \text{otherwise}, 
\end{cases}
\]

is also a solution of (1). This is the smallest piece of the Toda lattice.

The proof is simple. Because \(A\) is surrounded by zero, every equation on other pieces is automatically satisfied. The result can be easily generalized to larger parallelogram prism when it is surrounded by zero. We call it a Toda molecule according to ref.\[9\]. Then it will be natural to call \(A\) a Toda atom. If there are many Toda molecules in the three dimensional lattice space separated by zeros from each other it is again a solution of (1). A slice perpendicular to the \(l\) axis of such example is presented in Fig. 1.
For an illustration let us consider the one soliton state localized on the smallest parallelogram specified by \((m, n) = (0, 0), (0, 1), (1, -1), (1, 0)\) on the \((m, n)\) lattice plane, but allowed to range all integers along \(l\). Now we recall that in the usual lattice space the one soliton solution is given by

\[
\begin{align*}
    f^{1\text{sol}}(l, m, n) &= \prod_j (1 - az_j)^{-k_j} + \prod_j (1 - bz_j)^{-k_j}.
\end{align*}
\]

(3)

Here \(a, b\) are arbitrary constants and \(\{z_j\}\) are parameters which determine velocity of the soliton. \(\{k_j\}\) are variables taking values on integers. We can choose any three among \(\{k_j\}\) to relate them to our variables \((l, m, n)\). Let \(k_1, k_2, k_3\) be such three and relate them according to

\[
\begin{align*}
    k_1 &= m + n - \frac{1}{2}, \quad k_2 = -m - \frac{1}{2}, \quad k_3 = l - n - \frac{1}{2}.
\end{align*}
\]

(4)

Writing (3) explicitly we find

\[
\begin{align*}
    f^{1\text{sol}}(l, 0, 0) &= A(1 - az_3)^{-l} + B(1 - bz_3)^{-l}, \\
    f^{1\text{sol}}(l, 1, 0) &= A \frac{1 - az_2}{1 - az_1} (1 - az_3)^{-l} + B \frac{1 - bz_2}{1 - bz_1} (1 - bz_3)^{-l}, \\
    f^{1\text{sol}}(l, 0, 1) &= A \frac{1 - az_3}{1 - az_1} (1 - az_3)^{-l} + B \frac{1 - bz_3}{1 - bz_1} (1 - bz_3)^{-l}, \\
    f^{1\text{sol}}(l, 1, -1) &= A \frac{1 - az_2}{1 - az_3} (1 - az_3)^{-l} + B \frac{1 - bz_2}{1 - bz_3} (1 - bz_3)^{-l}
\end{align*}
\]

(5)

where

\[
A := \sqrt{(1 - az_1)(1 - az_2)(1 - az_3)}, \quad B := \sqrt{(1 - bz_1)(1 - bz_2)(1 - bz_3)}.
\]

We see from (3) that all points belonging to the same piece behave similarly. The parameters are related to \(\alpha, \beta\) of (1) by

\[
\alpha = z_1(z_2 - z_3), \quad \beta = z_2(z_3 - z_1),
\]
for (3) to satisfy HBDE. If we define the amplitude $\varphi_{mn}(l)$ by

$$\varphi_{mn}(l) := \frac{f(l + 1, m, n) f(l - 1, m, n)}{f^2(l, m, n)} - 1$$

it behaves as

$$\varphi_{00}^{1sol}(l) = \frac{\sinh^2 p}{\cosh^2(pl + \chi)}, \quad p := \frac{1}{2} \ln \frac{1 - az_3}{1 - bz_3}, \quad \chi := \frac{1}{2} \ln \frac{B}{A}. \quad (7)$$

This represents a localized peak along the $l$ axis. The other amplitudes $\varphi_{mn}(l)$ behave almost the same but different by the values of the phase $\chi$.

If we consider an evolution of the system in variable $l$, a Toda atom is composed of four lattice points. Since there is only one equation of motion (1), they are not independent variables. Three of them can be chosen as we like leaving one to be determined by the equation. Let $z_l$ be $f(l, 0, 0)$. The other three could be either dependent or independent of $z_l$. If they are independent of $z_l$, the equation of motion is linear in $z_l$. On the other hand if they do depend on $z_l$ they are allowed at most linear in $z_l$, for the equation to remain Hirota bilinear form. Namely we can write

$$f(l, m, n) = A_{m,n} z_l + B_{m,n}, \quad (m, n) = (0, 0), (1, 0), (0, 1), (1, -1), \quad (8)$$

with $A_{0,0} = 1, B_{0,0} = 0$. Upon substituting them together into HBDE, it is easy to see that we obtain an equation of the form

$$z_{l+1} = \frac{A z_l + B}{C z_l + D}. \quad (9)$$

$$A = -\beta B_{1,0} + (\alpha + \beta) B_{0,1} A_{1,-1}, \quad B = (\alpha + \beta) B_{0,1} B_{1,-1},$$

$$C = (\alpha + \beta)(A_{1,0} - A_{0,1} A_{1,-1}), \quad D = AB_{1,0} - (\alpha + \beta) A_{0,1} B_{1,-1}.$$  

(3) is a Möbius map. Thercfore the map is integrable.

If we remember that HBDE is invariant under the transformation of $f(l, m, n) \to e^{al+bm+cn} f(l, m, n)$, the one soliton solution (3) offers an example of (8).

The solution of (3) can be obtained as follows. A Möbius map has three fixed points. By an appropriate transformation : $z_l \to \phi_0 \circ z_l \circ \phi^{-1}$, one of the fixed points can be transformed into 0. After the transformation the map will have the form

$$z_{l+1} = \frac{\mu z_l}{1 + \nu z_l}. \quad (10)$$

(3) is easily solved for an arbitrary initial value $z_0$ to get

$$z_l = \frac{\mu^l z_0}{1 + \nu \frac{\mu}{1 - \mu} z_0}. \quad (11)$$
Applying to this the inverse transformation: \( z_l \rightarrow \phi^{-1} \circ z_l \circ \phi \), the general solution to (9) is obtained. We call the map (10) the integrable logistic map (ILM). The meaning of this name will become clear later.

We notice that (10) corresponds to the case in which one of the lattice point is fixed constant and the other three points behave the same:

\[
\begin{align*}
\phi_{ILM}(l,0,0) &= f_{ILM}(l,0,1) = f_{ILM}(l,1,-1) = z_l, \\
f_{ILM}(l,1,0) &= \frac{\mu - 1}{\nu}, \quad \mu = -\frac{\beta}{\alpha}.
\end{align*}
\] (12)

How does the amplitude look like in this case? To see it we substitute (11) into (6) and get

\[
\varphi_{ILM} = \frac{\sinh^2 p}{\cosh^2 (pl + \chi) - \cosh^2 p}, \quad p := \frac{1}{2} \ln \mu, \quad \chi := \frac{1}{2} \ln \frac{\mu z_0}{1 - \mu + \nu z_0}.
\] (13)

The similarity of this result to the one soliton solution (7) must be apparent.

We may further simplify the equation by

\[
\begin{align*}
f_{lin}(l,0,0) &= z_l, \quad f_{lin}(l,1,-1) = f_{lin}(l,1,0) = 1 - \frac{1}{\mu}, \quad f_{lin}(l,0,1) = c (c : \text{const}).
\end{align*}
\] (14)

The map turns to be linear

\[
z_{l+1} = \mu z_l + (1 - \mu)c
\] (15)

and yields the solution

\[
z_l = \mu^l (z_0 - c) + c.
\] (16)

The corresponding amplitude is

\[
\varphi_{lin}(l) = \frac{\sinh^2 p}{\cosh^2 (pl + \chi)}, \quad p := \frac{1}{2} \ln \mu, \quad \chi = \frac{1}{2} \ln \frac{z_0 - c}{c}.
\] (17)

which is again the form of (7).

### 3 Generalized Logistic Map

As we have learned in the preceding section the smallest piece of Toda lattice already possesses useful informations of the integrable dynamical systems. In this section we study a deformation of the Toda atom. There could be many different ways of deformation, some of which preserve integrability and some others destroy it. Since we are interested in studying the transition between integrable and nonintegrable maps, we must break integrability of the Toda atom.

For this purpose we recall that the Toda molecules have a characteristic form as seen in Fig. 1. Their cross sections in the \((m, n)\) plane are parallelogram declined to the same direction. It owes to the
The property of the Toda atom defined in (2). The very reason of this asymmetry comes from the asymmetry of HBDE under the exchange of $l$ and $m$ as seen in (1). The equation in which the role of $l$ and $m$ in (1) are exchanged is also integrable. In fact we could start from it without changing none of the results.

From this argument we are tempted to consider the following deformation of HBDE.

$$\alpha g_n(l+1,m)g_n(l,m+1) + \beta g_n(l,m)g_n(l+1,m+1)$$

$$- (\alpha + \beta) [(1-\gamma)\delta g_{n+1}(l+1,m) + \gamma g_{n+1}(l,m+1)]$$

$$\times [(1-\gamma')\delta' g_{n-1}(l,m+1) + \gamma' g_{n-1}(l+1,m)] = 0.$$  (18)

We notice that this equation is integrable when $\gamma = \gamma' = 0$ and $\delta\delta' = 1$, or $\gamma = \gamma' = 1$. Integrability of other cases is not known at this point. Moreover we are not able to separate some small part of lattice independently from the rest as it was done to get a Toda atom. Nevertheless it is worthwhile to study (18) defined on a portion of the lattice shown in Fig. 2a.

In order to proceed further we have to specify the model so that we can study analytical properties of the map explicitly. We will consider, in the following discussion, the map given by (10) and its deformation. We also restrict our argument to the case of $\gamma' = 0$, $\delta = \delta' = 1 = \nu$ in (18) for simplicity and define (Fig. 2b)

$$f^{GLM}(l,0,0) = f^{GLM}(l,0,1) = f^{GLM}(l,1,-1) = f^{GLM}(l,1,1) =: z_l;$$

$$f^{GLM}(l,1,0) = \frac{\mu - 1}{\nu} (\mu: \text{const})$$  (19)

The dynamics of this model is described by the map

$$z_{l+1} = f(z_l) := \mu - \frac{z_l(1-\gamma z_l)}{1+\nu(1-\gamma z_l)}; \quad z_l \in C, \quad l \in Z.$$  (20)

We call this map a generalized logistic map (GLM). Some properties are listed below:

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3If we had chosen other set of variables, HBDE looked more symmetric and the corresponding Toda atom could be either cubic or octahedron[11]. We have used asymmetric variables such that deformations can be discussed.
1. When $\gamma = 1$, the map becomes the ordinary logistic map studied in the literature extensively.

2. GLM becomes the logistic equation for all values of the parameters $\gamma, \mu, \nu$ when the continuous limit of the variable $l$ is taken. To show it let us introduce new variables $u$ and new parameters $a$ and $h$ by

$$u(l) := \frac{\nu + \gamma\mu - \gamma\nu}{\mu - 1} z_l, \quad ah := \mu - 1.$$  \hspace{1cm} (21)

We replace $z_{l+1}$ by $z_{l+h}$ and take the limit $h \to 0$. We will find that (20) reduces to the logistic equation:

$$\frac{du}{dl} = au(1-u).$$  \hspace{1cm} (22)

3. GLM includes (10) as the special case with $\gamma = 0$. This explains the name of ILM used for (10).

4. GLM generates Julia set as long as $\gamma \neq 0$. Hence it is not integrable except for $\gamma = 0$. This will be discussed later.

The most important feature of GLM is that it interpolates nonintegrable map to integrable map in the limit of continuous deformation. This fact enables us to study analytically the transition between two phases. The problem we concern in what follows is the analytical properties of the map (20). To proceed further it is more convenient to convert the map (20) into the standard form of rational map of degree 2:

$$F(z) = \phi \circ f \circ \phi^{-1}(z) = \frac{z(z + \lambda)}{1 + \lambda' z} e^{i\theta}$$  \hspace{1cm} (23)

by the Möbius transformation

$$\phi(x) = \frac{(1 - \mu)x}{(\nu\gamma - \nu - \mu\gamma)x + (\nu\gamma - \nu - \gamma)\mu e^{-i\theta}}.$$  \hspace{1cm} (24)

where

$$\lambda = \mu e^{-i\theta}, \quad \lambda' = \frac{\nu\gamma - \nu - 2\mu\gamma + \mu^2\gamma}{(\nu\gamma - \nu - \gamma)\mu} e^{i\theta}.$$  \hspace{1cm} (25)

The corresponding integrable map turns to be the following case

$$F(z) = \mu z = \lambda e^{i\theta} z, \quad \text{if} \quad \gamma = 0 \quad \text{i.e.} \quad \lambda\lambda' = 1.$$  \hspace{1cm} (26)

The main feature of a dynamical system is determined by the nature of fixed points of the map. Namely the multiplier $\Lambda$ at fixed point $a$ of the map $\varphi(z)$ is defined by the derivative of the map at $a$:

$$\Lambda := \left. \frac{d\varphi(z)}{dz} \right|_{z=a}.$$  \hspace{1cm} (27)

The fixed point $a$ is an attractor of the map if $|\Lambda| < 1$, a repeller if $|\Lambda| > 1$, and neutral if $|\Lambda| = 1$. 

In the case of GLM, the fixed points are easily found as
\[ 0, \quad p = -\lambda - \frac{1 - \lambda\lambda'}{\lambda' - e^{i\theta}}, \quad \infty. \] (28)

The corresponding multipliers are
\[ \Lambda_0 = \lambda e^{i\theta}, \quad \Lambda_p = \frac{2 - \lambda e^{i\theta} - \lambda' e^{-i\theta}}{1 - \lambda\lambda'}, \quad \Lambda_\infty = \lambda' e^{-i\theta}. \] (29)

In the integrable limit \( \lambda\lambda' \to 1 \), we observe the following characteristic features:

1. Since
\[ |\Lambda_0\Lambda_\infty| \to 1, \] (30)
the map converges either to 0 or to \( \infty \) depending on \( |\lambda| = |\mu| < 1 \) or \( > 1 \).

2. The fixed point \( p \) approaches to \( -\lambda \) and it turns to a super repeller
\[ |\Lambda_p| \to \infty. \] (31)

4 Julia Sets

In the complex dynamical systems, chaos appears from a Julia set. Given the map \( f(z) \) on a Riemann sphere \( \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \), the Riemann sphere is divided into two parts depending on whether the orbits converge or not. A set of initial values whose orbits, together with their neighborhood, converge is called Fatou set \( F(f) \). On the other hand, a set which does not is called Julia set \( J(f) \). This definition leads to the fact that the Julia set does not contain any attractive periodic cycle. In this sense the orbit in Julia set is chaotic.

By definition, Fatou set and Julia set are invariant of the map, that is
\[ f(F) = f^{-1}(F) = F, \quad f(J) = f^{-1}(J) = J. \]

It is easy to understand that attractive fixed points belong to Fatou set. Contrary it is known that repulsive fixed points belong to Julia set [13]. Then we can compute Julia set by inversely mapping a repulsive fixed point as an initial value. We show some of their examples in Fig. 3 for the map of (23).

The Julia set does not exist if the map is completely integrable. Integrable maps converge to orbits predictable for any given initial values. Conversely if there exists an orbit not predictable for some initial values, the map is not integrable. Therefore a Julia set appears in nonintegrable maps, but not in integrable maps.

In our standard map of degree 2 given by (23), a Julia set is known to exists except for at the integrable point \( \lambda\lambda' = 1 \). We like to know how it disappears from the complex plane of the variable when
the parameters approach to the limit $\lambda\lambda' \to 1$. We have given in [12] an argument about this problem for some limited range of parameters. The purpose of this section is to present another argument which should supplement our previous one.

The inverse map of (23) is easily obtained as

$$z_l = F^{-1}(z_{l+1}) = \frac{1}{2}(\rho z_{l+1} - \lambda) \pm \frac{1}{2} \sqrt{(\rho z_{l+1} + \lambda)^2 + 4z_{l+1}e^{-i\theta}(1 - \lambda\lambda')},$$  \hspace{1cm} (32)

where we defined

$$\rho := \lambda' e^{-i\theta}. \hspace{2cm} (33)$$

From this expression it is apparent that the inverse map is not unique but double valued at every step. As we pointed out in above the inverse map generates points of the Julia set if it starts from a point on the Julia set. Substituting one value of the Julia set into (32), we get two points every time. After $n$ steps the number of points of the Julia set increases as many as $2^{n+1} - 1$. This explains the nature of the Julia set. Some of the points could be those of periodic maps. They must be subtracted from the number.

In the integrable limit $\lambda\lambda' \to 1$ the inverse map (32) is still double valued. They are

$$z_l = \begin{cases} \rho z_{l+1} \\ -\lambda \end{cases}.$$  \hspace{1cm} (34)

We notice that the second solution does not depend on $z_{l+1}$, hence is the same at every step of the map. For (34) to generate the Julia set we must start from a repulsive fixed point. When $|\lambda| > 1$, the origin is such a point. Hence we find from (34) the ‘Julia set’

$$J^{ILM} = \{-\rho^n\lambda \mid n \in \mathbb{N}\} \hspace{2cm} (35)$$

for the integrable map. The number of the ‘Julia set’ increases proportional to the number of the steps $n$. Moreover the element of $J^{ILM}$ is equal to $-\mu^{-n}\lambda$, which is nothing but the solution exactly expected from the map (26), if it started from $-\lambda$.

The next problem we concern is to explore how the Julia set of GLM turns into those points of (34) in the limit $\lambda\lambda' \to 1$. Since we are interested in the transition from a nonintegrable map to the integrable map, we are to consider small values of $|\lambda\lambda' - 1|$. The inverse map (32) can be rewritten as

$$F^{-1}(z) = \begin{cases} \rho z \\ -\lambda \end{cases} \pm E(z)$$  \hspace{1cm} (36)

where we put

$$E(z) := \frac{1}{2}(\rho z + \lambda) \left( \sqrt{1 - \frac{4z\epsilon e^{-i\theta}}{(\rho z + \lambda)^2}} - 1 \right), \quad \epsilon := \lambda\lambda' - 1. \hspace{1cm} (37)$$

This set does not possess properties expected for the ordinary Julia set. We call it ‘Julia set’ only in the sense that it is generated by the inverse map starting from a repeller.
Note that $E(z)$ vanishes for small values of $\epsilon$. To see the behaviour of $E(z)$ for small $\epsilon$ we first observe the inequality which is true for all $\epsilon$:

$$|E(z)| < 3\sqrt{|z|}, \quad \forall \epsilon \in C. \quad (38)$$

The proof of this inequality owes to the following facts.

1. If $|w| < 1$,

$$\left| \frac{1}{\sqrt{w}} (\sqrt{1 - w} - 1) \right| = \frac{1}{|\sqrt{w}|} |1 - \sqrt{1 - w}| \leq \frac{1}{|\sqrt{w}|} (1 - \sqrt{1 - |w|}) \leq \frac{1}{|\sqrt{w}|} (1 - (1 - |w|)) = |\sqrt{w}| \leq 1. \quad (39)$$

2. If $|w| > 1$,

$$\left| \frac{1}{\sqrt{w}} (\sqrt{1 - w} - 1) \right| = \left| \frac{1}{\sqrt{w}} (\sqrt{1 - w} - 1) \right| < 3. \quad (40)$$

Substituting $w := \frac{4z\epsilon e^{-i\theta}}{(\rho z + \lambda)^2}$, into $E(z)$ of (37), we can write

$$|E(z)| = \sqrt{|z\epsilon|} \left| \frac{1}{\sqrt{w}} (\sqrt{1 - w} - 1) \right|, \quad (42)$$

from which (38) follows.

We can perform the inverse map (36) iteratively. Let us denote the map (36) as

$$A(z) := \rho z + E(z), \quad B(z) := -\lambda - E(z). \quad (43)$$

Then the second map becomes

$$F^{-2}(z) = \begin{cases} A \circ A(z) \\ B \circ A(z) \end{cases}, \quad (44)$$

After $n$ steps we obtain

$$F^{-n}(z) = \{ A^{\nu_1} \circ B^{\nu_2} \circ A^{\nu_3} \circ \cdots \circ B^{\nu_n}(z) \mid \nu_1 + \nu_2 + \cdots + \nu_n = n \}. \quad (45)$$

If we had started from the repeller the above maps have produced the Julia set of GLM. In the following we consider the case $|\lambda| > 1, \ |\lambda'| < 1$, so that the origin is a repulsive fixed point and the infinity is an attractive fixed point. Since $E(0) = 0$ the origin is mapped to

$$A(0) = 0, \quad B(0) = -\lambda \quad (46)$$
by the first iteration. The second iteration yields

\[ A^2(0) = 0, \ A \circ B(0) = -\rho \lambda + E(-\lambda), \ B \circ A(0) = -\lambda, \ B^2(0) = -\lambda - E(-\lambda). \] (47)

We notice that since \( E(-\lambda) \) is the order of \( \epsilon \) from (48), all of the points after the second iteration are in the neighbourhood of \( J^{ILM} \). Proceeding similarly we obtain the Julia set as follows:

\[ J^{GLM} = \{ A^\nu_1 \circ B^\nu_2 \circ A^\nu_3 \circ \cdots \circ B^\nu_\infty(0) | \ \nu_1, \nu_2, \cdots \in \mathbb{N} \} . \] (48)

We remark some important properties which result from this expression.

1. The invariance of the Julia set under the map.

   It is obvious from (48) that

   \[ J^{GLM} = A(J^{GLM}) \cup B(J^{GLM}) = F^{-1}(J^{GLM}) . \] (49)

2. An element of the form \( B \circ X \) for any \( X \in J^{GLM} \) belongs to the neighbourhood of \( -\lambda \), as seen from

   \[ B \circ X = -\lambda - E(X) . \] (50)

3. An element of the form \( A^s \circ B \circ X \) maps \( B \circ X \) to the neighbourhood of \( -\rho^s \lambda \).

   In fact after applying \( A \)'s \( s \) times we get

   \[ A^s(B \circ X) = A^{s-1}(\rho B \circ X + E(B \circ X)) \]
   \[ = A^{s-2}(\rho^2 B \circ X + \rho E(B \circ X) + E(A \circ B \circ X)) \]
   \[ = \rho^s B \circ X + \sum_{k=0}^{s-1} \rho^k E \left( A^{s-k-1} \circ B \circ X \right) \]
   \[ = -\rho^s \lambda - \rho^s E(X) + \sum_{k=0}^{s-1} \rho^k E \left( A^{s-k-1} \circ B \circ X \right) . \] (51)

Since every element of \( J^{GLM} \), beside 0, is either the form of \( B \circ X \) or \( A^s \circ B \circ X \), we conclude that every element of \( J^{GLM} \) is in the neighbourhood of \( J^{ILM} \).

We now proceed to show that \( J^{GLM} \) approaches uniformly to \( J^{ILM} \) as \( \epsilon \) goes to 0. Since the infinity is an attractive fixed point the Julia set must be in a finite region of the complex plane. We assume that they are inside of the disc of radius \( R \), i.e., \(|z| < R, \ \forall z \in J^{GLM} \). Therefore we can bound \(|E(z)|\) by

\[ |E(z)| < 3\sqrt{R} \sqrt{|\epsilon|}, \ \ z \in J^{GLM} . \] (52)

The summation of (51) can be estimated as

\[ \sum_{k=0}^{s-1} |\rho^k E \left( A^{s-k-1} \circ B \circ X \right)| < \left( \sum_{k=0}^{s-1} |\rho|^k \right) 3\sqrt{R} \sqrt{|\epsilon|} = \frac{1 - |\rho|^s}{1 - |\rho|} 3\sqrt{R} \sqrt{|\epsilon|} , \] (53)
which vanishes as $|\epsilon|$ approaches to 0 for all integer $s$ because we assume $|\rho|^s = |\lambda'|^s < 1$. This proves that all points in $J^{GLM}$ approach to $J^{ILM}$ uniformly in the integrable limit.

In the above we considered the case of $|\lambda| > 1, |\lambda'| < 1$. The other case $|\lambda| > 1, |\lambda'| < 1$ can be also treated similarly if $z_i$ is transformed into $w_i = 1/z_i$ in (23). Since this transformation is equivalent to the exchange of the role of $\lambda$ and $\lambda'$ (and replacement of $\theta$ by $-\theta$) in (23), we can replay on the $w$-plane the same argument to the above.

We conclude this paper by showing pictures which represent the convergence of the Julia set to the points of iterative maps of the integrable system. The parameters of the map are fixed at $\lambda = 4$ and $\theta = 0.03\pi$. Under the choice of these parameters, $\lambda'$, hence $\epsilon$, can be changed freely. In the integrable limit $\epsilon = 0$, $J^{ILM} = \{-4\frac{1}{4} e^{-i0.03\pi n}, n = 0, 1, 2, \cdots\}$. It is shown in Fig. 3c. As $\epsilon$ differs from 0 the Julia set expand from these points as seen in other pictures. The real and imaginary axes are not drawn except in Fig. 3a, so that the points in the neighbourhoods of $z = -4$ and 0 are visible in other figures.
By studying analytical property of a piece of Toda lattice we attempted to clarify how a nonintegrable system approaches to the integrable one. Our argument is based on the fact that the two dimensional Toda lattice can be disjoined into small pieces, which are integrable by themselves and are called Toda molecules. A Toda molecule is composed from smaller pieces, which we called Toda atoms. Hence the two dimensional Toda lattice is a crystal consisting of Toda atoms. For such a macroscopic system being integrable every piece must be joined very carefully not to create a Julia set.

In this onnection it will be worth while recalling that a similar property is possessed commonly in other integrable models. In the solvable lattice models the partition function is factorizable into a product of Boltzmann weights. The Yang-Baxter equation is a condition imposed on the factors to be connected properly. Another example is the factorizability condition imposed on the string amplitudes which led us to the $\tau$ function of the KP hierarchy. In any case the connection rule must be such that the symmetry characterizing the unit blocks is preserved under the coupling.

We will be interested in studying analytically properties of the compound system of two GLM pieces in the forth comming paper.
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