A criterion for uniform finiteness in the imaginary sorts

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Abstract

Let $T$ be a theory. If $T$ eliminates $\exists^\infty$, it need not follow that $T^{eq}$ eliminates $\exists^\infty$, as shown by the example of the $p$-adics. We give a criterion to determine whether $T^{eq}$ eliminates $\exists^\infty$. Specifically, we show that $T^{eq}$ eliminates $\exists^\infty$ if and only if $\exists^\infty$ is eliminated on all interpretable sets of “unary imaginaries.” This criterion can be applied in cases where a full description of $T^{eq}$ is unknown. As an application, we show that $T^{eq}$ eliminates $\exists^\infty$ when $T$ is a $C$-minimal expansion of ACVF.

1 Conventions

Definition 1.1. Let $X$ be a definable or interpretable set in an $\aleph_0$-saturated structure. Say that $\exists^\infty$ is eliminated on $X$ if for every definable family $\{D_a\}_{a \in Y}$ of subsets of $X$, the following (equivalent) conditions hold:

1. The set $\{a \in Y : |D_a| = \infty\}$ is definable.
2. There is an $n \in \mathbb{N}$ such that for all $a \in Y$,

$$|D_a| = \infty \iff |D_a| > n.$$ 

In a non-saturated structure $M$, we use Condition $\exists^\infty$, which is invariant under elementary extensions, and stronger than Condition $\exists$. In other words, we say that “$\exists^\infty$ is eliminated on $X$” if this holds in an $\aleph_0$-saturated elementary extension $M^* \supseteq M$. This is a slight abuse of terminology.

Definition 1.2. A theory $T$ has uniform finiteness if $\exists^\infty$ is eliminated on every definable set. We also say that $T$ eliminates $\exists^\infty$.

In a 1-sorted theory, uniform finiteness is equivalent to elimination of $\exists^\infty$ on the home sort, by the following observation.

\begin{itemize}
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\end{itemize}
Observation 1.3. If $\exists^\infty$ is eliminated on $X$ and $Y$, it is eliminated on $X \times Y$. In fact, $S \subseteq X \times Y$ is finite if and only if both of the projections $S \to X$ and $S \to Y$ have finite image.

Example 1.4. If $(M, \leq, +)$ is a dense o-minimal structure, then $M$ eliminates $\exists^\infty$. Indeed, a definable set $X \subseteq M$ is infinite if and only if $X$ has non-empty interior.

Example 1.5. If $(K, +, \cdot)$ is a $p$-adically closed field, such as $\mathbb{Q}_p$, then $K$ eliminates $\exists^\infty$. In fact, a definable set $X \subseteq K$ is infinite if and only if it has interior, by work of Macintyre [4].

Example 1.6. The ordered abelian group $(\mathbb{Z}, \leq, +)$ does not eliminate $\exists^\infty$, because there is no uniform bound on the size of the finite intervals $[1, n]$.

2 When does $T^{\text{eq}}$ eliminate $\exists^\infty$?

Uniform finiteness does not pass from $T$ to $T^{\text{eq}}$. In other words, $\exists^\infty$ can be eliminated on definable sets without being eliminated on interpretable sets. This happens in $\mathbb{Q}_p$, which interprets $(\mathbb{Z}, \leq, +)$ as the value group.

In many theories, it is difficult to fully characterize interpretable sets. For example, in the theory of algebraically closed valued fields (ACVF), the classification of interpretable sets is rather complicated [1]. Moreover, this classification fails to generalize to C-minimal expansions of ACVF [2].

In Theorem 2.3, we will give a relatively simple criterion which can be used to show that $T^{\text{eq}}$ eliminates $\exists^\infty$ without first characterizing interpretable sets. As an application, we will show that $T^{\text{eq}}$ eliminates $\exists^\infty$ when $T$ is a C-minimal expansion of ACVF.

Assume henceforth that $T$ is one-sorted.

Definition 2.1. In a model $M \models T$, a unary definable set is a definable subset of $M = M^1$.

Definition 2.2. An interpretable set $X$ is a set of unary imaginaries if there is a definable relation $R \subseteq X \times M$ such that the following map is an injection:

$$x \mapsto R_x := \{m \in M : (x, m) \in R\}.$$ 

In other words, $X$ is a set of unary imaginaries if the elements of $X$ are codes for unary definable sets, in some uniform way.

Theorem 2.3. Suppose that $\exists^\infty$ is eliminated on every set of unary imaginaries. Then $T^{\text{eq}}$ eliminates $\exists^\infty$.

Proof. Let $M_0 \models T$ be a small model. Let $N_0$ be the expansion of $M_0^{\text{eq}}$ by a new sort $\mathbb{N} \cup \{\infty\}$ and functions

$$Y \to \mathbb{N} \cup \{\infty\}$$

$$a \mapsto |D_a|$$

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for every definable family \( \{D_a\}_{a \in Y} \) in \( M_0^{eq} \). Let \( N = (M^{eq}, \mathbb{N}^* \cup \{\infty\}) \) be an \( \aleph_0 \)-saturated elementary extension of \( N_0 \).

Then \( \mathbb{N}^* \) is an \( \aleph_0 \)-saturated elementary extension of \( \mathbb{N} \), \( M \) is an \( \aleph_0 \)-saturated model of \( T \), and every interpretable set \( X \) in \( M \) has a non-standard “size”

\[
|X| \in \mathbb{N}^* \cup \{\infty\}.
\]

Say that \( X \) is \textit{pseudofinite} if \(|X|\) is less than the symbol \( \infty \). (In particular, finite sets are pseudofinite.) It suffices to show that every pseudofinite interpretable set is finite, because of the \( \aleph_0 \)-saturation of \( N \).

Say that an interpretable set \( X \) in \( M \) is \textit{wild} if there is an infinite pseudofinite definable family of subsets of \( X \). Otherwise, say \( X \) is \textit{tame}. By assumption, \( \exists^\infty \) is eliminated on sets of unary imaginaries. Therefore, every pseudofinite set of unary imaginaries is finite. Equivalently, \( M^1 \) is tame.

\textit{Claim 2.4.} If \( X \) is tame, so is any definable subset of \( X \). If \( X \) and \( Y \) are tame, then so is \( X \cup Y \).

\textit{Proof.} The first statement is trivial. For the second statement, let \( \mathcal{D} \) be a pseudofinite definable family of subsets of \( X \cup Y \). Note that \( \{D \cap X : D \in \mathcal{D}\} \) is

- \textit{pseudofinite}, because \( \mathcal{D} \) is pseudofinite, and
- \textit{finite}, because \( X \) is tame

Similarly, \( \{D \cap Y : D \in \mathcal{D}\} \) is finite. Finally, the map

\[
D \mapsto (D \cap X, D \cap Y)
\]

yields an injection from \( \mathcal{D} \) into a product of two finite sets. Thus \( \mathcal{D} \) is finite. \( \square \textit{Claim} \)

\textit{Claim 2.5.} Let \( \pi : X \to Y \) be a definable map with finite fibers. If \( Y \) is tame, then so is \( X \).

\textit{Proof.} By saturation, there is a uniform upper bound \( k \) on the size of the fibers. We proceed by induction on \( k \). The base case \( k = 1 \) is trivial. Suppose \( k > 1 \). Let \( \mathcal{D} \) be a pseudofinite definable family of subsets of \( X \). Let

\[
\mathcal{E} = \{\pi(D) : D \in \mathcal{D}\}
\]

and

\[
\mathcal{F} = \{\pi(X \setminus D) : D \in \mathcal{D}\}
\]

Then \( \mathcal{E} \) and \( \mathcal{F} \) are both pseudofinite definable families of subsets of \( Y \). By tameness of \( Y \), they are both finite.

It remains to show that the fibers of \( \mathcal{D} \to \mathcal{E} \times \mathcal{F} \) are finite. Replacing \( \mathcal{D} \) with such a fiber, we may assume that \( \pi(D) \) and \( \pi(X \setminus D) \) are independent of \( D \), as \( D \) ranges over \( \mathcal{D} \). Let \( U = \pi(D) \) and \( V = \pi(X \setminus D) \) for any/every \( D \in \mathcal{D} \). Let \( Y' = U \cap V \) and \( X' = \pi^{-1}(Y') \). Then the map \( D \mapsto D \cap X' \) is injective on \( \mathcal{D} \), because every element \( D \) of \( \mathcal{D} \) contains \( \pi^{-1}(U \setminus V) \).
and is disjoint from $\pi^{-1}(V \setminus U)$. So it suffices to show that $X'$ is tame. Let $D$ be some arbitrary element of $\mathcal{D}$. Then $X' \cap D$ and $X' \setminus D$ each intersect every fiber of $X' \to Y'$, by choice of $X'$. In particular, the two maps

$$X' \cap D \to Y'$$

$$X' \setminus D \to Y'$$

have finite fibers of size less than $k$. By Claim 2.4, $Y'$ is tame, and by induction, $X' \cap D$ and $X' \setminus D$ are tame. By Claim 2.4, $X'$ is tame. □

**Claim 2.6.** Suppose that $\pi : X \to Y$ is a definable surjection with finite fibers. Suppose that $Y$ is tame. Let $F$ be a definable family of sections of $\pi$. If $F$ is pseudofinite, then $F$ is finite.

**Proof.** A section is determined by its image. □

**Claim 2.7.** Suppose $X$ and $Y$ are tame. Then so is $X \times Y$.

**Proof.** Let $\mathcal{D}$ be a pseudofinite definable family of subsets of $X \times Y$. For each $a \in X$, the set $Y_a := \{a\} \times Y \subseteq X \times Y$ is tame, so the collection

$$\mathcal{E}_a := \{D \cap Y_a : D \in \mathcal{D}\}$$

is finite. Then

$$\pi : \bigsqcup_{a \in X} \mathcal{E}_a \to X$$

is a definable surjection with finite fibers. Each element $D \in \mathcal{D}$ induces a section of $\pi$, namely, the map $\sigma_D$ sending a point $a \in X$ to (the code for) $D \cap Y_a$. This gives a definable injection from $\mathcal{D}$ to sections of $\pi$. By Claim 2.6 and the fact that $X$ is tame, it follows that $\mathcal{D}$ is finite. □

It follows that $M^n$ is tame for all $n \geq 1$. Now if $Y$ is any interpretable set, then $Y$ is a set of codes of subsets of $M^n$, for some $n$. By tameness of $M^n$, it follows that if $Y$ is pseudofinite, then $Y$ is finite. This completes the proof of Theorem 2.3. □

### 3 C-minimal expansions of ACVF

As an example, we apply Theorem 2.3 to C-minimal expansions of ACVF.  

1 Let $T$ be a C-minimal expansion of ACVF, and $K$ be a sufficiently saturated model of $T$. As in the proof of Theorem 2.3, work in a setting with nonstandard counting functions.

**Observation 3.1.** Let $B_1, \ldots, B_n$ be pairwise disjoint balls in $K$. Then the union $\bigcup_{i=1}^n B_i$ cannot be written as a boolean combination of fewer than $n$ balls.

1 See [5] for the definition of C-minimality. The theory ACVF is C-minimal by Theorem 4.11 in [5]. Certain expansions of ACVF by analytic functions are shown to be C-minimal in [3].
This follows from uniqueness of the swiss-cheese decomposition, and the fact that the residue field is infinite.

**Lemma 3.2.** There is no pseudofinite infinite set of pairwise disjoint balls.

*Proof.* Let \( S \) be such a set. By compactness, there must be some sequence \( S_1, S_2, \ldots \) such that each \( S_i \) is a finite set of pairwise disjoint balls, the \( S_i \) are uniformly interpretable (bounded in complexity), and \( \lim_{i \to \infty} |S_i| = \infty \).

The unions \( U_i = \bigcup S_i \subseteq K \) are uniformly definable (bounded in complexity), so there is some absolute bound on the number of balls needed to express \( U_i \). But Observation 3.1 says that this number is at least \( |S_i| \), a contradiction.

C-minimality implies that the value group \( \Gamma \) is densely o-minimal. Therefore \( \exists^\infty \) is eliminated in \( \Gamma \), and there are no pseudofinite infinite subsets of \( \Gamma \).

**Lemma 3.3.** There is no pseudofinite infinite set of balls.

*Proof.* Let \( S \) be such a set. Let \( S_0 \) be the set of minimal elements of \( S \). For each \( B \in S_0 \), let \( S_B \) denote the elements of \( S \) containing \( B \). In a pseudofinite poset, every element is greater than or equal to a minimal element, so

\[
S = \bigcup_{B \in S_0} S_B.
\]

The set \( S_0 \) is pseudofinite, hence finite by Lemma 3.2. Therefore, \( S_B \) is infinite for some \( B \).

Now \( S_B \) is a chain of balls. Let \( \rho : S_B \to \Gamma \) be the map sending a ball to its radius. This map is nearly injective; the fibers have size at most 2. The range of \( \rho \) is pseudofinite, hence finite. Therefore, the domain \( S_B \) is finite, a contradiction.

Finally, suppose that \( \exists^\infty \) is not eliminated on some set \( X_0 \) of unary imaginaries. Then there is a pseudofinite infinite set \( A \subseteq X_0 \). Let \( D_a \) be the unary set associated to \( a \in A \). Note that \( a \mapsto D_a \) is injective.

For each \( a \), there is a unique minimal set of balls \( B_a \) such that \( D_a \) can be written as a boolean combination of \( B_a \). The correspondence \( a \mapsto B_a \) is a definable finite-to-finite correspondence from \( A \) to the set \( B \) of balls. Let \( I \) denote the “image” of this correspondence:

\[
I := \bigcup_{a \in A} B_a.
\]

The set \( I \subseteq B \) is pseudofinite, hence finite by Lemma 3.3. The boolean algebra generated by \( I \) is finite, and contains every \( D_a \). By injectivity of \( a \mapsto D_a \), the set \( A \) is finite, a contradiction.

By Theorem 2.3, we have proven the following:

**Proposition 3.4.** \( T_{eq} \) eliminates \( \exists^\infty \) when \( T \) is a C-minimal expansion of ACVF.
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References

[1] Deirdre Haskell, Ehud Hrushovski, and Dugald Macpherson. Definable sets in algebraically closed valued fields: elimination of imaginaries. *J. reine angew. Math.*, 597:175–236, 2006.

[2] Deirdre Haskell, Ehud Hrushovski, and Dugald Macpherson. Unexpected imaginaries in valued fields with analytic structure. *Journal of Symbolic Logic*, 78(2):523–542, June 2013.

[3] L. Lipshitz and Z. Robinson. One-dimensional fibers of rigid subanalytic sets. *Journal of Symbolic Logic*, 63(1):83–88, March 1998.

[4] Angus Macintyre. On definable subsets of p-adic fields. *Journal of Symbolic Logic*, 41(3):605–610, September 1976.

[5] Dugald Macpherson and Charles Steinhorn. On variants of o-minimality. *Annals of Pure and Applied Logic*, 79:165–209, 1996.