Pure braids, a new subgroup of the mapping class group and finite-type invariants

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Abstract

In the study of the relation between the mapping class group $\Gamma_g$ of a surface $\Sigma_g$ of genus $g$ and the theory of finite-type invariants of homology 3-spheres, three subgroups of $\Gamma_g$ play a large role. They are the Torelli group $\mathcal{T}_g$, the Johnson subgroup $\mathcal{K}_g$ and a new subgroup $\mathcal{L}_g^L$, which contains $\mathcal{K}_g$, defined by a choice of a Lagrangian subgroup $L \subseteq H_1(\Sigma_g)$. In this work we determine the quotient $\mathcal{L}_g^L/\mathcal{K}_g \subseteq \Gamma_g/\mathcal{K}_g$, in terms of the precise description of $\Gamma_g/\mathcal{K}_g$ given by Johnson and Morita. We also study the lower central series of $\mathcal{L}_g^L$ and $\mathcal{K}_g$, using some natural imbeddings of the pure braid group in $\mathcal{L}_g^L$ and the theory of finite-type invariants.

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1 Introduction

Let $\Gamma_g$ denote the group of isotopy classes of orientation-preserving diffeomorphisms of the bounded surface $\Sigma_g$ of genus $g$ with one boundary component which are the identity on the boundary. In [7] and [8] the relation between the mapping class group and the theory of finite-type invariants of homology spheres is investigated. It turns out that the three different notions of finite-type which are discussed in [7] require one to consider three different subgroups of the mapping class group. Two of these subgroups are familiar from previous work of D. Johnson, S. Morita and others on the structure of the mapping class group (see [15] and [22] for surveys). These are the Torelli group $T_g$ (maps which induce the identity automorphism on $H_1(\Sigma_g)$) and the Johnson subgroup $K_g$ (generated by Dehn twists on bounding closed curves). But the original notion of finite-type introduced by Ohtsuki in [25] requires one to consider a new subgroup defined as follows. Choose a Lagrangian subspace $L \subseteq H = H_1(\Sigma_g)$, i.e. a summand of rank $g$ on which the intersection form vanishes. Define $L^g$ to be the subgroup of $\Gamma_g$ generated by Dehn twists on closed curves whose homology class lies in $L$. Of course $L^g$ depends on $L$ but its conjugacy class is independent of the choice of $L$. It is also clear that $K_g \subseteq L^g$. In fact, the relation between $L^g$ and $K_g$ is comparable to the relation between $K_g$ and $T_g$.

In this work we will give a precise determination of how $L^g$ sits in $\Gamma_g$, or, more precisely, how $L^g/K_g$ sits in $\Gamma_g/K_g$, using the description given by Johnson [12] and [13]. The analogous result for a closed surface is obtained as a corollary and is, in fact, somewhat simpler to state than the bounded case.

A second purpose of this paper is to study the lower central series of the groups $L^g$ and $K_g$. We show that the lower central series of $L^g$ is dominated by a refinement of the relative weight filtration of $\Gamma_g$. We also make use of a natural imbedding of the pure braid group $P_g$ on $g$ strands into $L^g$ (first used by Oda, in an unpublished paper [24] to give lower bounds for the ranks of the associated gradeds of the relative weight filtration of $\Gamma_g$), and of $P_{2g}$ into $K_g$, to give lower bounds on the ranks of the lower central series quotients of these groups. These estimates augment the lower bounds given in [8] using the theory of finite-type invariants of homology 3-spheres.

Of course the definitive results on the lower central series of $T_g$ are those of Hain [10].

2 Statement of Results

2.1 The group $L^g$

Let $\overline{L}_g$ denote the subgroup of $\Gamma_g$ consisting of maps whose induced automorphism of $H$ is the identity on $L$. Then $L^g \subseteq \overline{L}_g$ and $T_g \subseteq \overline{L}_g$. Under the obvious isomorphism $\Gamma_g/T_g \cong Sp(\mathbb{Z}, 2g)$, the symplectic group over $\mathbb{Z}$ of genus $g$, $\overline{L}_g/T_g$ corresponds to the subgroup $B_g$ consisting of symplectic automorphisms of $H$ which are the identity on $L$. $B_g$ is isomorphic to $S_2(L)$, the additive subgroup of symmetric elements of $L \otimes L$. If $f \in B_g$, then $\phi - 1$ induces an element of $\text{Hom}(H/L, L) \cong \text{Hom}(L^*, L) \cong L \otimes L$ (using duality via the intersection pairing

\footnote{The author thanks S. Morita for informing him of the existence of this paper and providing a copy.}
on \( H \). Alternatively if we choose a symplectic basis \( \{ x_i, y_i \} \) for \( H \), where \( \{ x_i \} \) is a basis of \( L \), then \( \phi \) is represented by a matrix in the form \( \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \) and \( \phi \rightarrow A \) defines an isomorphism of \( B_g \) with the additive group of symmetric integral matrices. See [4] for more details.

Recall the Johnson homomorphism \( J : T_g \rightarrow \Lambda^3 H \), which is onto and whose kernel is the Johnson subgroup \( K_g \) (see [12] and [13]). If \( f \in T_g \) and \( \alpha \in F = \pi_1(\Sigma_g) \), the free group on \( 2g \) generators, then we can write

\[
 f_*(\alpha) = \alpha \Phi_f(\alpha) \mod F_3
\]

where, for any group \( G \), \( G_q \) denotes the \( q \)-th lower central series subgroup: \( G_1 = G \), \( G_{q+1} = [G, G_q] \). \( \Phi_f(\alpha) \) can be regarded, therefore, as an element in \( F_2/F_3 \cong \Lambda^2 H \) and so \( \Phi_f \) defines a homomorphism \( H \rightarrow \Lambda^2 H \) or, dually, an element of \( H \otimes \Lambda^2 H \). Johnson shows in [12] that \( \Phi_f \) lies in the subgroup \( \Lambda^3 H \subseteq H \otimes \Lambda^2 H \), where this imbedding is defined by

\[
 h_1 \land h_2 \land h_3 \rightarrow h_1 \otimes (h_2 \land h_3) + h_2 \otimes (h_3 \land h_1) + h_3 \otimes (h_1 \land h_2)
\]

and, in [13], that the correspondence \( f \rightarrow \Phi_f \) defines an isomorphism \( T_g/K_g \cong \Lambda^3 H \). From this we can, therefore, regard \( L_g/L_g \) as an extension of \( \Lambda^3 H \) by \( B_g \).

**Theorem 1.** There exists a homomorphism

\[
 J : \mathcal{L}_g \rightarrow \Lambda^3(H/L) \oplus H/L
\]

extending the Johnson homomorphism, which satisfies the following:

(a) \( J \) is onto,

(b) \( \ker J = \mathcal{L}_g^L \)

(c) There is a commutative diagram

\[
\begin{array}{cccccc}
 1 & \rightarrow & \mathcal{L}_g^L & \rightarrow & \mathcal{L}_g^L & \rightarrow & \Lambda^3(H/L) \oplus H/L & \rightarrow & 1 \\
 1 & \uparrow & \downarrow J & & \downarrow p \oplus c & & \downarrow & \downarrow & \downarrow \\
 1 & \rightarrow & \mathcal{L}_g & \rightarrow & \Lambda^3 H & \rightarrow & 1
\end{array}
\]

(2)

where \( p \) is the projection \( \Lambda^3 H \rightarrow \Lambda^3(H/L) \) and \( c \) is the composition of the projection \( H \rightarrow H/L \) with the contraction map \( \Lambda^3 H \rightarrow H \) defined by

\[
 h_1 \land h_2 \land h_3 \rightarrow (h_1 \cdot h_2)h_3 + (h_2 \cdot h_3)h_1 + (h_3 \cdot h_1)h_2
\]
If \( \Gamma_{g,0} \) denotes the mapping class group of a closed surface \( \Sigma_{g,0} \) of genus \( g \), i.e. the group of isotopy classes of orientation-preserving diffeomorphisms of \( \Sigma_{g,0} \), then we can define the analogous subgroups \( \mathcal{L}_{g,0}^L, \mathcal{L}_{g,0}^T \subseteq \Gamma_{g,0} \). We also have the corresponding Torelli group \( \mathcal{T}_{g,0} \) and Johnson subgroup \( \mathcal{K}_{g,0} \). The homomorphism \( J : \mathcal{T}_g \to \Lambda^3 H \) is shown, in \([12]\), to induce a homomorphism \( J' : \mathcal{T}_{g,0} \to \Lambda^3 H/H \) whose kernel is \( \mathcal{K}_{g,0} \).

**Corollary 1.** There exists an epimorphism \( J_0 : \mathcal{L}_{g,0}^L \to \Lambda^3 (H/L) \) whose kernel is \( \mathcal{L}_{g,0}^L \) and which satisfies the commutative diagram:

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathcal{L}_{g,0}^L & \longrightarrow & \mathcal{L}_{g,0}^L & \longrightarrow & \Lambda^3 (H/L) & \longrightarrow & 1 \\
& & \uparrow & & \uparrow & \downarrow & \rho' & & \\
1 & \longrightarrow & \mathcal{K}_{g,0} & \longrightarrow & \mathcal{T}_{g,0} & \longrightarrow & \Lambda^3 H/H & \longrightarrow & 1
\end{array}
\]

where \( \rho' \) is induced by \( p \).

We can also characterize \( \mathcal{L}_{g,0}^L \subseteq \Gamma_{g,0} \) in terms of the induced outer automorphisms of \( \pi = \pi_1(\Sigma_{g,0}) \). Let \( \pi_L \subseteq \pi \) be the subgroup of elements whose homology class belongs to \( L \). If \( f \in \mathcal{L}_{g,0}^L \), then \( f \cdot |\pi_L \equiv \text{id} \mod \pi_2 \).

**Corollary 2.** If \( f \in \Gamma_{g,0} \), then \( f \in \mathcal{L}_{g,0}^L \) if and only if \( f \cdot |\pi_L \equiv \text{id} \mod [\pi, \pi_L] \).

Note that this condition on \( f \cdot | \) depends only on its outer automorphism class.

We also identify the lower central series terms of \( \mathcal{L}_g^L \), modulo \( \mathcal{K}_g \). Let \( K = \ker(p \oplus e) \).

By Theorem \([1]\), we have an isomorphism:

\[
\mathcal{J}' : \mathcal{L}_{g,0}^L \cap \mathcal{T}_g / \mathcal{K}_g \cong K
\]

We now define a filtration of \( \Lambda^3 H \):

\[
\Lambda^3 H = K_0 \supset K_1 \supset K_2 \supset K_3 = \Lambda^3 L \supset K_4 = 0
\]

by letting \( K_m \) be the subgroup of \( \Lambda^3 H \) generated by elements \( h_1 \wedge h_2 \wedge h_3 \), where at least \( m \) of the \( h_i \) lie in \( L \). Note that \( K_1 \supset K \supset K_2 \).

**Theorem 2.** Under the isomorphism \( \mathcal{J}' \) of \([4]\) the lower central series terms of \( \mathcal{L}_g^L \) satisfy:

\[
((\mathcal{L}_g^L)_m \cdot \mathcal{K}_g) / \mathcal{K}_g \cong K_m \text{ for } m \geq 2
\]

In particular \( (\mathcal{L}_g^L)_4 \subseteq \mathcal{K}_g \), but \( (\mathcal{L}_g^L)_3 \not\subseteq \mathcal{K}_g \).

Furthermore \( \mathcal{L}_g^L \cap \mathcal{T}_g = \mathcal{K}_g : [\mathcal{L}_g^L, \mathcal{T}_g] \).

**Question 1.** Does \([\mathcal{L}_g^L, \mathcal{L}_g^L] \) contain \( \mathcal{K}_g \)?
We obtain some further information on the lower central series of $\mathcal{L}_g^L$ inside $\mathcal{K}_g$. Recall the filtration on $\Gamma_g$ defined by:

$$\Gamma_g[n] = \{ f \in \Gamma_g | f_* : F \to F \text{ is the identity mod } F_{n+1} \}$$

This is called the relative weight filtration in [23]. Thus $\Gamma_g[1] = \mathcal{T}_g$ and $\Gamma_g[2] = \mathcal{K}_g$.

**Theorem 3.** $(\mathcal{L}_g^L)_q \subseteq \Gamma_g[n]$ if $q \geq \binom{n+3}{2} - 6$

Thus $\mathcal{L}_g^L$ is residually nilpotent.

In fact we have a more delicate description of the relation between the lower central series of $\mathcal{L}_g^L$ and the relative weight filtration which requires us to define a more refined filtration. We first recall the extended Johnson homomorphisms from [15] as formulated by Morita [20]. Let $\mathbb{L}(H)$ be the graded free Lie algebra (over $\mathbb{Z}$) on $H$. $\mathbb{L}_n(H)$ is generated by the brackets of length $n$. Then

$$J_n : \Gamma_g[n] \to H \otimes \mathbb{L}_{n+1}(H)$$

is defined by $f_*(\alpha) = \alpha J_n(\alpha) \mod F_{n+2}$, where $J_n(\alpha) \in F_{n+1}/F_{n+2} \cong \mathbb{L}_{n+1}(H)$. Then $\Gamma_g[n + 1] = \ker J_n$ and

$$\text{Im } J_n \subseteq \ker \{ b : H \otimes \mathbb{L}_{n+1}(H) \to \mathbb{L}_{n+2}(H) \} \quad (5)$$

where $b(h \otimes \alpha) = [h, \alpha]$ (see [21]). Thus $J_1((\mathcal{L}_g^L)_m) = K_m$ (by Theorem 2).

We generalize the filtration $\{ K_m \}$ as follows:

$$\mathbb{L}_m(H) = \mathbb{L}_m^0(H) \supseteq \mathbb{L}_m^1(H) \supseteq \cdots \supseteq \mathbb{L}_m^m(H) = \mathbb{L}_m(L) \supseteq \mathbb{L}_m^{m+1}(H) = 0$$

where $\mathbb{L}_m^r(H)$ is the subgroup generated by brackets of length $m$ of which at least $r$ of the entries belong to $L$. We can then define a filtration of $H \otimes \mathbb{L}_m(H)$ by $\mathcal{F}_m^r = (L \otimes \mathbb{L}_m^{r-1}(H)) \oplus (H \otimes \mathbb{L}_m^r(H))$. Thus $\mathcal{F}_m^{m+1} = L \otimes \mathbb{L}_m(L)$ and $\mathcal{F}_m^{m+2} = 0$.

**Theorem 4.** $J_n((\mathcal{L}_g^L)_{q+r}) \subseteq \mathcal{F}_{n+1}^r$ if $q \geq \binom{n+3}{2} - 6$.

### 2.2 Imbedding the pure braid group in the mapping class group

It has been pointed out several times in the literature that one can define interesting maps from the group of framed pure braids to the mapping class group. The framed pure braid group on $h$ strands, $\mathcal{P}_h$, which is canonically isomorphic to the Cartesian product of the usual pure braid group $\mathcal{P}_h^c$ with the free abelian group of rank $h$, can be represented as the group of isotopy classes of diffeomorphisms of the disk with $h$ holes, $D_h$, which are the identity on the boundary. Any imbedding of $D_h$ into $\Sigma_g$ defines, therefore, a homomorphism $\theta : \mathcal{P}_h \to \Gamma_g$. For example one can imbed $D_{2g}$ into $\Sigma_g$ as in Figure [1]. This map was considered by Hatcher-Thurston [11] and, with a formulation in terms of tangles, Matveev-Polyak [18] in a slightly more general setting of what Matveev-Polyak call admissible braids. It is not hard to see that $\theta(\mathcal{P}_{2g}) \subseteq \mathcal{L}_g^L$, where $L$ is the subspace spanned by the meridians of the handles. We
will be more interested, though, in a smaller version of this map. Consider the imbedding $D_g \subseteq \Sigma_g$ indicated in Figure 2. The resulting map $\psi : \mathcal{P}_g \to \Gamma_g$ was considered in [24] Oda who showed that $\psi$ induces imbeddings of the lower-central series quotients of $\mathcal{P}_g$ into the associated graded quotients of $\Gamma_g$ using the relative weight filtration: $\psi_n : (\mathcal{P}_g)/((\mathcal{P}_g)_{n+1} \to \Gamma_g[n-1]/\Gamma_g[n]$. The well-known split exact sequence $1 \to F^{q-1} \to \mathcal{P}_g \to \mathcal{P}_g/1 \to 1$ gives an isomorphism:

$$\mathcal{P}_g/((\mathcal{P}_g)_{n+1} \cong (\mathcal{P}_g)_{n}/((\mathcal{P}_g)_{n+1} \oplus F^{q-1}_n/F^{q-1}_{n+1}$$

(6)

See e.g. [2]. This enables Oda to use $\psi_n$ to give some explicit lower bounds for the rank of $\Gamma_g[n-1]/\Gamma_g[n]$. We observe easily that $\psi$ actually maps into $\mathcal{L}_g^L$, for the correct choice of $L$, and describe how the image fits into the refined filtration $F_m^L$ of $H \otimes \mathbb{L}(H)$.

**Theorem 5.** $\psi(\mathcal{P}_g) \subseteq \mathcal{L}_g^L$ and $\text{Im}(J_{n-1}\psi_n) \subseteq F^{n+1}_n = L \otimes \mathbb{L}_n(L)$.

Thus $(\mathcal{P}_g)/((\mathcal{P}_g)_{n+1} \subseteq \Gamma_g[n-1]/\Gamma_g[n]$ lies in the bottom stage of this filtration.

**Remark 2.1.** It is not hard to prove, by a similar argument, that the conclusions of Theorem 4 are also true for the Hatcher-Thurston map.

As in Equation (5) $\text{Im}(J_{n-1}\psi_n) \subseteq \ker\{b : L \otimes \mathbb{L}_n(L) \to \mathbb{L}_{n+1}(L)\}$, but for $n \geq 3$, this inclusion is definitely proper. One sees this by computing the difference between the rank of $\ker b$ and the rank of $(\mathcal{P}_g)/((\mathcal{P}_g)_{n+1}$, using Equation (4). For example, for $n = 3$ this difference is $\frac{1}{6}(g^3 - g)$ and for $n = 4$ it is $\frac{1}{8}(g^3 - g)(g - 2)$. For $n \leq 2$ see Theorem 4 below.

The non-triviality of $J_{n-1}\psi_n$ shows that $\mathcal{L}_g^L$ is not nilpotent. More precisely, combining this with Theorem 4 gives a lower bound on the rate of descent of the lower central series of $\mathcal{L}_g^L$. We denote the rank of $(\mathcal{P}_g)/((\mathcal{P}_g)_{n+1}$ by $r(g,n)$. This can be computed explicitly from Equation (5).

**Corollary 3.** The image of the map $(\mathcal{P}_g)/((\mathcal{P}_g)_q \to (\mathcal{L}_g^L)/((\mathcal{L}_g^L)_q has rank \geq r(g,n)$ if $q \geq \binom{n+3}{2} - 6$. 

![Figure 1. The Hatcher-Thurston map](image-url)
This estimate is independent of the estimate on the rank of the lower central series of $L^L_g$ given in [8]. Recall the map $\phi_n^L : (L^L_g)^{3n}/(L^L_g)^{3n+1} \otimes \mathbb{Q} \to A_n^{\text{conn}}(\emptyset)$ defined in [8] and shown there (Theorem 7) to be onto if $g \geq 5n + 1$, where $A_n^{\text{conn}}(\emptyset)$ is a vector space defined by trivalent graphs with $2n$ vertices and $3n$ edges. Now it is easy to see that the composition $(P_g)^{3n} \to (L^L_g)^{3n} \to A_n^{\text{conn}}(\emptyset)$ is zero. If $h \in \text{Im} \psi$ then $S^1_h = S^1$ since $h$ extends to a diffeomorphism of the handlebody bounded by $\Sigma_g$. Thus combining Corollary 3 and [8, Theorem 7] we have:

**Corollary 4.** $\text{rank}(L^L_g)^{3n}/(L^L_g)^{3n+1} \geq r(g, 3n) + \dim A_n^{\text{conn}}(\emptyset)$ if $q \geq (3n+3)/2 - 6$ and $g \geq 5n + 1$.

For the special cases $n = 1, 2$ we have:

**Theorem 6.** (a) $\psi_1$ induces an isomorphism $\mathcal{P}_g/(\mathcal{P}_g)_2 \cong L^L_g/(L^L_g \cap \mathcal{T}_g) \cong B_g$.

(b) $J_1 \psi_2$ induces an isomorphism $(\mathcal{P}_g)_2/(\mathcal{P}_g)_3 \otimes \mathbb{Q} \cong ((L^L_g)_3 \cdot \mathcal{K}_g)/\mathcal{K}_g \otimes \mathbb{Q} \cong \Lambda^3 L \otimes \mathbb{Q}$.

Thus $\psi^{-1}(\mathcal{K}_g) = (\mathcal{P}_g)_3$.

**Question 2.** Can (b) be improved to say $(\mathcal{P}_g)_2/(\mathcal{P}_g)_3 \cong \Lambda^3 L$?

**Remark 2.2.** For the Hatcher-Thurston map $\theta$ one has $J_1 \theta((\mathcal{P}_g)_2) \subseteq K_3 = \Lambda^3 L$, according to Remark [2], but it is also not hard to show that $J_1(\theta((\mathcal{P}_g) \cap \mathcal{K}_g)) \subseteq K_2$. A consequence of this is that $\theta((\mathcal{P}_g)_2) \subsetneq L^L_g$.

We will consider one more mapping of the pure braid group into the mapping class group. Consider the imbedding $D_g \subseteq \Sigma_g$ indicated in Figure 3. Denote the resulting map by $\kappa : \mathcal{P}_g \to \Gamma_g$. We will prove:
Theorem 7. \( \kappa(P_g) \subseteq K_g \) and \( \kappa((P_g)_n) \subseteq \Gamma_g[2n] \). The induced map \( \kappa_n : (P_g)_n/(P_g)_{n+1} \rightarrow \Gamma_g[2n]/\Gamma_g[2n+1] \) is a monomorphism. Thus the induced map \( (P_g)_n/(P_g)_{n+1} \rightarrow (K_g)_n/(K_g)_{n+1} \) is also a monomorphism and so \( \text{rank}(K_g)_n/(K_g)_{n+1} \geq r(g,n) \).

As above we have, from [8], a map \( \phi^K_n : (K_g)_n/(K_g)_{n+1} \rightarrow A^n(\emptyset) \) which is onto if \( g \geq 5n+1 \). The composition \( (P_g)_n/(P_g)_{n+1} \rightarrow (K_g)_n/(K_g)_{n+1} \rightarrow A^n(\emptyset) \) is zero since, if \( h \in \text{Im} \phi^K_n \) then \( S^3_h = S^3 \). Thus we have, from Theorem 7 and [8]:

Corollary 5. \( \text{rank}(K_g)_n/(K_g)_{n+1} \geq r(g,n) + \dim A^n(\emptyset) \) if \( g \geq 5n+1 \).

2.3 Applications of the theory of finite-type invariants

The fact that there are useful connections between the theory of finite-type invariants of homology spheres and the structure of subgroups of the mapping class group was established in [7] and [8]. In particular, as we have mentioned above, Theorem 6 of [8] gives lower bounds for the dimensions of the graded quotients of the lower central series of the subgroups \( L^L_g, T_g \) and \( K_g \) in terms of certain vector spaces \( A^n(\emptyset) \) defined by connected trivalent graphs.

As a further application of this result we have the following relation between the lower central series of these groups.

Theorem 8. Let \( g \geq 5n+1 \). Then

1. \( (K_g)_n \not\subseteq (T_g)_{2n+1} \cup (L^L_g)_{3n+1} \),
2. \( (T_g)_{2n} \not\subseteq (K_g)_{n+1} \cup (L^L_g)_{3n+1} \),
3. \( (L^L_g)_{3n} \not\subseteq (T_g)_{2n+1} \cup (K_g)_{n+1} \).

We also examine the map \( \psi : P_g \rightarrow L^L_g \), defined in the previous section, and its relationship to the theory of finite-type invariants of knots and homology spheres. Let \( \mathcal{V} \) denote
the $\mathbb{Q}$-vector space generated by isotopy classes of oriented smooth knots in $S^3$ and let $\mathcal{M}$ denote the $\mathbb{Q}$-vector space generated by diffeomorphism classes of smooth oriented homology spheres. In [3] Garoufalidis defines a natural linear map $\Psi : \mathcal{V} \to \mathcal{M}$ by $\Psi(K) = S^3_K$, where, for any knot $K$ in a homology sphere $M$, $M_K$ denotes the homology sphere obtained by doing a +1-surgery on $K \subseteq M$. Filtrations are defined on $\mathcal{V}$ and $\mathcal{M}$ by the following devices. For any positive integer $q$, a singular $q$-knot is an immersion of $S^1$ into $S^3$ with exactly $q$ transverse (ordered) double points. If $K$ is a singular $q$-knot, the resolution $\hat{K}$ of $K$ is the element of $\mathcal{V}$ defined by $\hat{K} = \sum \epsilon(-1)^{\epsilon}K_\epsilon$, where $\epsilon$ ranges over all sequences $(\epsilon_1, \ldots, \epsilon_q)$, $\epsilon_i = 0$ or 1, $|\epsilon| = \sum \epsilon_i$ and $K_\epsilon$ denotes the knot obtained from $K$ by resolving each double point to give a positive, resp. negative crossing according to whether $\epsilon_i$ is, resp. 0 or 1. Now define $\mathcal{F}_q(\mathcal{V})$ to be the subspace of $\mathcal{V}$ generated by the resolutions of all $q$-singular knots. This gives a decreasing filtration of $\mathcal{V}$ (see [1]). To deal with $\mathcal{M}$ define, for any $q$-component link $J$ in a homology sphere, an element $[M; J] \in \mathcal{M}$ by the formula $[M; J] = \sum \epsilon(-1)^{\epsilon}M_\epsilon$, where $\epsilon$ is as above and $J_\epsilon$ is the sublink of $J$ consisting of exactly those components for which $\epsilon_i = 1$. Now $\mathcal{F}_q(\mathcal{M})$ is defined to be the subspace of $\mathcal{M}$ generated by all $[M; J]$ for which $J$ has $q$ components. This is a decreasing filtration of $\mathcal{M}$ (see [25]). It is shown in [25] and [1] that $\mathcal{F}_{3q-2}(\mathcal{M}) = \mathcal{F}_{3q}(\mathcal{M})$, for all $q$.

The associated graded quotients

$$\mathcal{G}_q(\mathcal{V}) = \mathcal{F}_q(\mathcal{V})/\mathcal{F}_{q+1}(\mathcal{V}) \text{ and } \mathcal{G}_q(\mathcal{M}) = \mathcal{F}_q(\mathcal{M})/\mathcal{F}_{q+1}(\mathcal{M})$$

are described in terms of spaces of trivalent graphs $\mathcal{A}(S^1), \mathcal{A}(\emptyset)$, where $\mathcal{A}(X)$, for any 1-manifold $X$, is a certain $\mathbb{Q}$-vector space defined from trivalent graphs which contain $X$ as a prescribed subgraph (see [1], [25] and [10]). It was conjectured by Garoufalidis in [3] and proved by Habegger [3] that $\Psi(\mathcal{F}_q(\mathcal{V})) \subseteq \mathcal{F}_q(\mathcal{M})$ and so we have induced maps $\Psi_q : \mathcal{G}_q(\mathcal{V}) \to \mathcal{G}_q(\mathcal{M})$. This map is studied in [1], using the work of [10], and shown to be non-trivial for all $q$.

The map $\psi : \mathcal{P}_q \to \mathcal{L}^L_q$ and $\Psi : \mathcal{V} \to \mathcal{M}$ are related by means of ‘actions’ of $\mathcal{P}_q$ on $\mathcal{V}$ and $\mathcal{L}^L_q$ on $\mathcal{M}$. Let $K$ be an oriented knot in $S^3$ and $D \subseteq S^3$ an imbedded disk which meets $K$ transversely in $q$ points so that, at each intersection point, $K$ points in the same normal direction to $D$. If $b$ is any $g$-strand pure braid then we can cut $S^3$ along $D$ and insert $b$ so that the orientations agree, to form a new knot $K_b$. This action was defined by Stanford in [20] who showed that if $b \in (\mathcal{P}_q)_m$, then $K_b - K \in \mathcal{F}_m(\mathcal{V})$. Note also that if $b$ is a framed pure braid and $K$ is a framed knot, then $K_b$ is also framed in a natural way. If $s$ is the self-linking number of $K$ and $\{s_i\}$ are the self-linking numbers of the strands of $b$, then the self-linking number of $K_b$ is $s + \sum s_i$.

We now describe the action of $\mathcal{L}^L_q$ on $\mathcal{M}$. Suppose $M$ is a homology sphere and $i : \Sigma_g \subseteq M$ is an imbedding. If $L_i = L \subseteq H = H_1(\Sigma_g)$ is the kernel of the inclusion into one of the complementary summands, then, for any $f \in \mathcal{L}_g$, we can cut $M$ open along $i(\Sigma_g)$ and reglue, using $f$, to define a new homology sphere $M_f$ (see [8]). It is shown in [8] that if $f \in (\mathcal{L}_g)_m$ then $M_f - M \in \mathcal{F}_m(\mathcal{M})$.

Now suppose we are given $K, D$ as above and $b \in \mathcal{P}_q$. Choose an imbedding $i : \Sigma_g \to S^3$ whose image is the boundary of a regular neighborhood of $K \cup D$. Then $i$ will also define an imbedding $j$ of $\Sigma_g$ into $S^3_K$. Let $b$ denote also the framed braid whose self-linking numbers are determined by the equations $\sum l_{ij} = 0$, where $l_{ij}$ denotes the linking number of the $i$-th
and $j$-th strand, if $i \neq j$, and the self-linking number of the $i$-th strand if $i = j$. Then for every $i = 1, \cdots, g$, then we have:

**Theorem 9.** $\Psi(K_b) = M_{\psi(b)}$, where $M = S^3_K$.

### 3 Proof of Theorem 1

As mentioned already, different choices for $L$ result in conjugate subgroups $L \oplus Lg$. Therefore it will suffice to prove Theorems 1-4 for any convenient choice of $L$. (The other theorems will require certain choices for $L$). In particular we will choose $L$ to be the subgroup of $H$ generated by meridians of the handles for some representation of $\Sigma_g$ as the boundary of a handle-body. Thus we can choose a basis $\{x_i, y_i\}$ for $F = \pi_1(\Sigma_g)$ to satisfy:

- The induced basis of $H$ (which we will also denote by $x_i, y_i$) is symplectic, i.e. under the intersection pairing on $H$, we have $x_i \cdot x_j = y_i \cdot y_j = 0$ and $x_i \cdot y_j = \delta_{ij}$,
- $\{x_i\}$ is a basis for $L$,
- The element $[x_1, y_1] \cdots [x_g, y_g] \in F$ represents the boundary curve of $\Sigma_g$.

We will call such a basis *admissible for $L$*, We will also use this term for the induced basis of $H$.

#### 3.1 Construction of $J$

We begin by defining a crossed homomorphism on $L \oplus Lg$. Let $F_L \subseteq F$ be the set of all elements of $F$ which map into $L \subseteq H$ under abelianization. Let $\hat{L}$ be the abelianization of $F_L$. We define

$$\hat{J} : L \oplus Lg \to \text{Hom}(\hat{L}, \Lambda^2 H)$$

by the formula $\hat{J}(f) \beta \equiv \alpha^{-1} f_i(\alpha) \mod F_3$, under the identification $F_2/F_3 \cong \Lambda^2 H$, where $\alpha \in F_L$ is any lift of $\beta \in \hat{L}$.

We then have the following crossed-homomorphism formula

$$\hat{J}(fg) = \hat{J}(g) + \hat{J}(f) \circ g_* \quad (7)$$

where $g_*$ is the automorphism of $\hat{L}$ defined by $g$.

Composing $\hat{J}$ with the projection $\Lambda^2 H \to \Lambda^2(H/L)$ defines

$$\bar{J} : L \oplus Lg \to \text{Hom}(\hat{L}, \Lambda^2(H/L))$$

Composing $\hat{J}$ with the contraction map $c : \Lambda^2 H \to \mathbb{Z}$ defines

$$J_\omega : L \oplus Lg \to \text{Hom}(\hat{L}, \mathbb{Z})$$

$c$ is defined, using the intersection pairing on $H$, by $h_1 \wedge h_2 \to h_1 \cdot h_2$.

We point out two properties of $\bar{J}$ and $J_\omega$. 
Lemma 3.1.  
(a) \( \bar{J}(f)|F_2 = J_\omega(f)|F_2 = 0 \), for any \( f \in \overline{L}_g^I \).

(b) \( \bar{J} \) and \( J_\omega \) are homomorphisms.

Thus we have homomorphisms:
\[
\bar{J} : \overline{L}_g^I \to \text{Hom}(L, \Lambda^2(H/L)), \quad J_\omega : \overline{L}_g^I \to \text{Hom}(L, \mathbb{Z})
\]

Proof of Lemma 3.1. To prove (a) consider a generating element \([h_1, h_2] \in F_2\). If \( f \in \overline{L}_g^I \), then we can write \( f_*([h_i]) = h_il_i \), for some elements \( l_i \in F_L \) and so we have:
\[
f_*([h_1, h_2]) = [h_1, h_2][l_1, h_2][l_1, l_2][l_1, l_2] \mod F_3
\]
Thus \( \bar{J}(f)[h_1, h_2] = l_1 \wedge h_2 + h_1 \wedge l_2 + l_1 \wedge l_2 \). Since this element projects to zero in \( \Lambda^2(H/L) \), (a) follows for \( \bar{J} \). For \( J_\omega \) we note that \( l_1 \cdot l_2 = 0 \), since the intersection form is zero on \( L \) and \( h_1 \cdot l_2 + l_1 \cdot h_2 = 0 \) since \( f_* \) is an isometry.

To prove (b) we need only note that \( g_* \) is the identity on \( L \) and apply (a) and Equation (8).
\[ \square \]

Now we follow Johnson and reformulate \( \bar{J} \) first as a homomorphism
\[
\bar{J} : \overline{L}_g^I \to H/L \otimes \Lambda^2(H/L)
\]
using the duality isomorphism \( \text{Hom}(L, \mathbb{Z}) \cong H/L \) defined by the intersection form on \( H \). If \( \{x_i, y_i\} \) is an admissible basis of \( F \) for \( L \), then \( \bar{J}(f) = \sum_i y_i \otimes \bar{J}(f) \cdot x_i \).

We have, for any free-abelian group \( V \), a short exact sequence:
\[
0 \longrightarrow \Lambda^3V \xrightarrow{\eta} V \otimes \Lambda^2V \xrightarrow{\theta} \mathbb{L}_3(V) \longrightarrow 0
\]
where \( \eta(v_1 \wedge v_2 \wedge v_3) = v_1 \otimes (v_2 \wedge v_3) + v_2 \otimes (v_3 \wedge v_1) + v_3 \otimes (v_1 \wedge v_2) \) and \( \theta(v_1 \otimes (v_2 \wedge v_3)) = [v_1, [v_2, v_3]] \). We leave the proof as an exercise for the reader.

We can use this sequence to show that \( \bar{J}(\overline{L}_g^I) \subseteq \Lambda^3(H/L) \). If \( f \in \overline{L}_g^I \), then we can write \( f_*(x_i) = x_iC_i \), for some \( C_i \in F_2 \) which represents \( \bar{J}(f) \cdot x_i \), and \( f_*(y_i) = y_il_i \) for some \( l_i \in F_L \). By distributivity of brackets we have:
\[
f_*[x_i, y_i] = [x_i, y_i][C_i, y_i][x_i, l_i] \alpha_i \mod F_4 \tag{8}
\]
where \( \alpha_i \) is a product of elements of \( F_3/F_4 \cong \mathbb{L}_3(H) \) which vanish when projected into \( \mathbb{L}_3(H/L) \). Since any element of \( \Gamma_g \) leaves the boundary curve of \( \Sigma_g \) fixed, we have that, up to conjugacy, \( f_*([\prod_i[x_i, y_i]]) = \prod_i[x_i, y_i] \). Our first deduction from Equation (8) is that \( \prod_i[x_i, l_i] \in F_3 \). Now if we write out \( l_i = (\prod_j x_{ij})C_i' \), for some \( C_i' \in F_2 \), we can conclude that \( e_{ij} = e_{ji} \) and so \( \prod_i[x_i, l_i] \mod F_4 \) reduces to a product of triple brackets, each of which has some \( x_i \) as one of its entries, and so vanishes in \( \mathbb{L}_3(H/L) \). Thus we can conclude, from Equation (8), that \( \sum_i[C_i, y_i] = 0 \) in \( \mathbb{L}_3(H/L) \). But this means that \( \theta \bar{J}(f) = 0 \) and so \( \bar{J} \) actually gives us a homomorphism \( \overline{L}_g^I \to \Lambda^3(H/L) \).

We can now set \( J = \bar{J} \oplus J_\omega \), where \( J_\omega \) is reformulated as a homomorphism \( J_\omega : \overline{L}_g^I \to H/L \), using the duality isomorphism \( \text{Hom}(L, \mathbb{Z}) \cong H/L \).
3.2 Proof of (a)

Since the Johnson homomorphism \( J \) is onto \( \Lambda^3 H \), it is obvious that \( J \) is onto. To deal with \( J_{\omega} \) we first point out that \( J_{\omega}|T_g \) can be alternatively defined by the formula: \( J_{\omega}(f) = \sigma J(f) \), where \( \sigma : \Lambda^3 H \to H \to H/L \) is defined by contraction:

\[
\sigma(h_1 \wedge h_2 \wedge h_3) = (h_1 \cdot h_2) h_3 + (h_2 \cdot h_3) h_1 + (h_3 \cdot h_1) h_2
\]

Let \( \{x_i, y_i\} \) be an admissible basis of \( H \). Now choose \( f \in T_g \) so that \( J(f) = x_i \wedge y_i \wedge y_j \), for a prescribed \( i \neq j \). Then \( J_{\omega}(f) = y_j \) while \( J(f) = 0 \) in \( \Lambda^3(H/L) \). From this it follows that \( J \) is onto.

\[ \square \]

3.3 Proof of (b)

It is essentially proved in [7] that \( J(\mathcal{L}_g^L) = 0 \) but we take a different approach to give a unified proof that \( J(\mathcal{L}_g^L) = 0 \).

Suppose that \( f \) is a Dehn twist along the curve \( \gamma \) in \( \Sigma_g \). If \( \gamma \) bounds in \( \Sigma_g \), then \( f \in K_g \) and there is nothing to prove. If \( \gamma \) does not bound, then we may assume that \( \gamma \) is a meridian curve of a handle in some representation of \( \Sigma_g \) as the boundary of a handle-body. Let \( \{x_i, y_i\} \) be the standard basis of \( F \), where \( x_i \) represents the meridian of the \( i \)-th handle and \( y_i \) represents the longitude. Note that this is not necessarily an admissible basis. We may assume that \( \gamma \) represents \( x_1 \). Then we see easily that

\[
f_*(x_i) = x_i \ (1 \leq i \leq g), \quad f_*(y_i) = \begin{cases} y_i & \text{if } 1 \leq i \leq g, \\ y_1 x_1 & \text{if } i = 1 \end{cases}
\]

All we know about \( L \), though, is that \( x_1 \in F_L \) and, as a consequence, an element \( w \in F \) lies in \( F_L \) only if the exponent sum of \( y_1 \) is 0. From this it follows that for any \( w \in F_L \), \( J(f) \cdot w = a \wedge x_1 \), where \( a \in H \) is a linear combination of \( x_1, \ldots, x_g, y_2, \ldots, y_g \). But then it is clear that \( a \wedge x_1 \) maps to 0 in \( \Lambda^2(H/L) \) and \( c(a \wedge x_1) = 0 \). Thus \( \mathcal{L}_g^L \subseteq \ker J \).

Now suppose \( f \in \mathcal{L}_g^L \) and \( J(f) = 0 \). We want to show that \( f \in \mathcal{L}_g^L \). We first recall that the images of \( \mathcal{L}_g^L \) and of \( \mathcal{L}_g^L \) under the canonical map \( \Gamma_g \to Sp(Z,g) \) coincide and are equal to the subgroup \( B_g \subseteq Sp(Z,g) \) defined in Section 2.4. This is proved in [4]. Since there exists some \( g \in \mathcal{L}_g^L \) so that \( f = g \cdot H \), then, by replacing \( f \) by \( f^{-1} \), we may assume that \( f \in T_g \). Since \( K_g \subseteq \mathcal{L}_g^L \), it will suffice to find \( g \in \mathcal{L}_g^L \cap T_g \) so that \( J(g) = J(f) \). In fact we will find such a \( g \in [\mathcal{L}_g^L, T_g] \subseteq \mathcal{L}_g^L \cap T_g \) (since \( \mathcal{L}_g^L \) is invariant under conjugation by \( T_g \)), which will, in addition, show that \( \mathcal{L}_g^L \cap T_g = [\mathcal{L}_g^L, T_g] \cdot K_g \) (part of the conclusion of Theorem 2).

Let \( K = \ker \{ p \oplus c : \Lambda^3 H \to \Lambda^3(H/L) \oplus H/L \} \). We need to show that, for any \( a \in K \), there exists \( g \in [\mathcal{L}_g^L, T_g] \) such that \( J(g) = a \). Some examples of elements of \( K \) are, in terms of an admissible basis of \( H \):

1. \( a = x_i \wedge x_j \wedge x_k \) for any \( i, j, k \),
2. \( a = x_i \wedge x_j \wedge y_k \) for any \( i, j, k \),
3. \( a = x_i \wedge y_j \wedge y_k \) for \( i, j, k \) distinct and
4. \( a = y_i \land x_i \land y_k + x_j \land y_j \land y_k \) for \( i, j, k \) distinct.

In fact we will leave it as an exercise for the reader to prove that any element of \( K \) is a linear combination of elements of these four types.

To compute \( J \) on \( [\mathcal{L}_g^L, \mathcal{T}_g] \) we need the following observation. Let \( f \in \Gamma_g \) and \( h \in \mathcal{T}_g \).

Then we have, for the classical Johnson homomorphism \( J : \mathcal{T}_g \to \Lambda^3H \):

\[
J([f, h]) = J(fh^{-1}) - J(h) = (f_\ast - 1)J(h)
\]

Since \( J \) is onto and \( f_\ast \) can be any element of \( B_g \), for \( f \in \mathcal{L}_g^L \), this formula determines \( J([\mathcal{L}_g^L, \mathcal{T}_g]) \). We construct some examples.

1. Choose \( h \) so that \( J(h) = x_i \land x_j \land y_k \) for \( i, j \) distinct, and \( f \) so that \( f_\ast(y_k) = y_k + x_k \) and \( f_\ast \) is the identity on every other basis element. Then it is straightforward to compute, from Equation (8), that \( J([f, h]) = x_i \land x_j \land x_k \).

2. Choose \( h \) so that \( J(h) = y_i \land x_j \land y_k \) for \( i, k \) distinct, and \( f \) so that \( f_\ast(y_i) = y_i + x_i \) and \( f_\ast \) is the identity on every other basis element. Then \( J([f, h]) = x_i \land x_j \land y_k \).

3. Choose \( h \) so that \( J(h) = y_i \land y_j \land y_k \) for \( i, j, k \) distinct, and \( f \) so that \( f_\ast(y_i) = y_i + x_i \) and \( f_\ast \) is the identity on every other basis element. Then \( J([f, h]) = y_i \land y_j \land y_k \).

4. Choose \( h \) so that \( J(h) = y_i \land y_j \land y_k \) for \( i, j, k \) distinct, and \( f \) so that \( f_\ast(y_i) = y_i + x_j \), \( f_\ast(y_j) = y_j + x_i \) and \( f_\ast \) is the identity on every other basis element. Then

\[
J([f, h]) = y_i \land x_i \land y_k + x_j \land y_j \land y_k + x_j \land x_i \land y_k
\]

Now it is clear that the examples given here cover all the cases above.

This completes the proof of (b). In fact these arguments also prove (c) and so the proof of Theorem [1] is complete.

\[ \square \]

### 3.4 Proof of Corollary [1]

Recall the exact sequence (see e.g. [19])

\[
1 \longrightarrow \pi_1(T) \longrightarrow \eta \longrightarrow \pi_1(T) \longrightarrow 1
\]

where \( T \) is the tangent circle bundle of \( \Sigma_{g,0} \). The exact homotopy sequence of the bundle \( T \to \Sigma_{g,0} \) gives a central extension (since \( \Sigma_{g,0} \) is orientable).

\[
1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(T) \longrightarrow \pi_1(\Sigma_{g,0}) \longrightarrow 1
\]

where \( \pi = \pi_1(\Sigma_{g,0}) \). \( \eta \) satisfies the following properties:

1. \( \text{Im} \eta \subseteq \mathcal{T}_g \)

2. \( \eta|\mathbb{Z} \) is defined by \( \eta(1) = \text{Dehn twist along a circle bounding the puncture} \), and \( \eta|\pi \) is characterized by the property that the automorphism of \( \pi \) induced by \( \eta(\alpha) \) is conjugation by \( \alpha \).
3. There is a commutative diagram

\[ \begin{array}{ccc}
\pi_1(T) & \xrightarrow{\eta} & \pi_g \\
\rho \downarrow & & \downarrow \xi \\
\pi & \xrightarrow{\eta'} & \text{Aut} \pi
\end{array} \]

where Aut \( \pi \) is the group of automorphisms of \( \pi \), \( \eta'(g) = \text{conjugation by} \ g \), and \( \xi(f) \) is the automorphism of the fundamental group of \( \Sigma_{g,0} \) induced by \( f \).

**Claim.**

(a) \( \bar{J} \circ \eta = 0 \)

(b) If \( \rho(\alpha) \in \pi \) has homology class \( h \in H \), then \( J_\omega \eta(\alpha) \equiv h \mod L \).

Assuming this, we then see, from (a), that \( \bar{J} \) induces the homomorphism \( J_0 \) (since \( \bar{L}_{g,0} = e(\bar{L}_g) \)) and, since by (b), \( J_\omega \circ \eta \) is onto (and \( L_{g,0} = e(\bar{L}_g) \)), \( \ker J_0 = L_{g,0} \).

**Proof of Claim.** It follows from (3) above that \( \bar{J}\eta(\alpha) \cdot x = h \wedge x \), for any \( x \in L \), where \( h \in H \) is the homology class of \( \rho(\alpha) \). (In fact, since \( \text{Im} \eta \subseteq T_{g,0}, \bar{J}(\eta(\alpha)) \) is given by the standard Johnson homomorphism and this formula is true for any \( x \in H \).) If \( x \in L \), then \( h \wedge x \rightarrow 0 \in \Lambda^2(H/L) \)– this proves (a). Similarly \( J_\omega \eta(\alpha) \in \text{Hom}(L, \mathbb{Z}) \) is the functional \( x \rightarrow h \cdot x \) which, under the duality \( \text{Hom}(L, \mathbb{Z}) \cong H/L \), corresponds to the reduction of \( h \). This proves (b).

Finally note that \( p(H) = 0 \) since \( p(h) \) is defined to be \( \sum_i x_i \wedge y_i \wedge h \), for any symplectic basis \( \{x_i, y_i\} \) of \( H \) and, since we can choose this basis so that \( x_i \in L \), each term in this sum goes to \( 0 \in \Lambda^3(H/L) \). This completes the proof of Corollary \( \Box \).

3.5 Proof of Corollary \( \Box \)

**Claim.** Suppose \( f' \in \Gamma_g \) induces \( f \in \Gamma_{g,0} \). Then \( f'_* \) (the induced automorphism of \( F \)) satisfies

\[ f'_*|_{F_L} \equiv id \mod [F, F_L] \tag{10} \]

if and only if \( f_*|_{\pi_L} \equiv id \mod [\pi, \pi_L] \).

This follows from the fact that \( \ker \{ F \rightarrow \pi \} \subseteq [F, F_L] \).

Now (10) is clearly equivalent to \( f' \in \bar{L}_g \) and \( \bar{J}(f') = 0 \), and this, by definition, is the same as \( J_0(f) = 0 \). So Corollary \( \Box \) is an immediate consequence of Corollary \( \Box \).

4 Proof of Theorem \( \Box \)

As pointed out above, we have already proved the last assertion of Theorem \( \Box \) and so we address the first part. Note that this is equivalent to the assertion that \( J((\bar{L}_g)_m) = K_m \) for \( m \geq 2 \).
4. PROOF OF THEOREM ??

4.1 The case \( m = 2 \)

We first show that \( J([\mathcal{L}_g^L]_2) \subseteq K_2 \). This is equivalent to the two inclusions:

1. \( \Phi_f(H) \subseteq \ker\{\Lambda^2 H \to \Lambda^2 (H/L)\} \) and

2. \( \Phi_f(L) \subseteq \Lambda^2 L \)

where \( \Phi_f \) is defined in Section 2.1. (1) is already proved in [7], so we address (2). According to Theorem 1, \( \hat{J} \) induces a function \( \mathcal{L}_g^L \to \text{Hom}(\hat{L}, X) \), where \( X = \ker\{\Lambda^2 H \to \Lambda^2 (H/L)\} \). Now \( \Lambda^2 L \subseteq X \) and the induced function \( J' : \mathcal{L}_g^L \to \text{Hom}(\hat{L}, X/L^2 L) \) is a homomorphism. This follows from an alternative version of the crossed-homomorphism formula for \( \hat{J} \):

\[
\hat{J}(fg) = \hat{J}(f) + f \circ \hat{J}(g)
\]

where \( f_* : \Lambda^2 H \to \Lambda^2 H \) is defined by \( f_*(h_1 \wedge h_2) = f_*(h_1) \wedge f_*(h_2) \). If \( f \in \mathcal{L}_g^L \) then \( f_* \) induces the identity map on \( L \) and on \( H/L \). From this it follows that \( f_*|X \equiv 0 \mod \Lambda^2 L \) and so \( J' \) is a homomorphism. But then we can conclude that \( J(([\mathcal{L}_g^L]_2) = 0 \), which means that \( \Phi_f(\hat{L}) \subseteq \Lambda^2 L \) for any \( f \in ([\mathcal{L}_g^L]_2 \). This proves (2).

We now need to show that \( K_2 \subseteq J(([\mathcal{L}_g^L]_2) \). In fact we will show that \( K_2 \subseteq J([\mathcal{L}_g^L \cap H/L^2 L, \mathcal{L}_g^L]) \). Let \( \{x_i, y_k\} \) be an admissible basis of \( H \). Then \( K_2 \) is generated by the following elements:

(a) \( x_i \wedge x_j \wedge x_k \), for \( i, j, k \) distinct, and

(b) \( x_i \wedge x_j \wedge y_k \), for \( i \neq j \) and \( j \neq k \).

Choose \( f \in \mathcal{L}_g^L \) so that \( f_*(y_i) = y_i + x_i \) and \( f_* \) is the identity on all other basis elements. By Section 3.3 we can choose \( h \in \mathcal{L}_g^L \cap H/L^2 L \) so that \( J(h) = y_i \wedge x_j \wedge x_k \), if \( i, j, k \) are distinct. Then, by Equation (3), \( J([f, h]) = x_i \wedge x_j \wedge x_k \). If \( i \neq j \) and \( j \neq k \), then, by Section 3.3, we can choose \( h_2 \in \mathcal{L}_g^L \cap H/L^2 L \) so that \( J(h_2) = y_i \wedge x_j \wedge y_k \). Then by Equation (3), \( J([f, h_2]) = x_i \wedge x_j \wedge y_k \).

This completes the proof of Theorem 2 for the case \( m = 2 \).

4.2 The cases \( m \geq 3 \)

We first show that \( J([\mathcal{L}_g^L]_m) \subseteq K_m \). We have already proved this for \( m = 2 \) in the previous section. We now argue by induction on \( m \) using Equation (4) and the fact that \( (f_* - 1)K_m \subseteq K_{m+1} \). This latter inclusion follows from the expansion:

\[
(f_* - 1)(a_1 \wedge a_2 \wedge a_3) = (f_* - 1)a_1 \wedge f_*a_2 \wedge f_*a_3 + a_1 \wedge (f_* - 1)a_2 \wedge f_*a_3 + a_1 \wedge a_2 \wedge (f_* - 1)a_3
\]

and the following easy fact:

If \( f \in \mathcal{L}_g^L \), then \( (f_* - 1)a \in L \) for any \( a \in H \), and \( (f_* - 1)a = 0 \) if \( a \in L \) \hspace{1cm} (11)

All that remains to prove is that \( K_3 \subseteq J([\mathcal{L}_g^L]_3) \). But \( K_3 = \Lambda^3 L \) is obviously generated by the elements \( x_i \wedge x_j \wedge x_k \) and if we choose \( h \in ([\mathcal{L}_g^L]_2) \) so that \( J(h) = x_i \wedge x_j \wedge y_k \), which we can do by the previous section, and choose \( f \in \mathcal{L}_g^L \) so that \( f_*(y_k) = y_k + x_k \) and \( f_* \) is the identity on all other basis elements, then \( J([f, h]) = x_i \wedge x_j \wedge x_k \).

The proof of Theorem 2 is now complete. \( \square \)
5 Proof of Theorems 3 and 4

The argument is a generalization of that in Section 4.2. Both these theorems will follow from:

Lemma 5.1. 1. If \( f \in \Gamma_g \) and \( g \in \Gamma_g \), then \( J_n([f, g]) = (f - 1)J_n(g) \).

2. If \( f \in \overline{L}^r \), then \( (f - 1)F \subseteq F^{r+1} \).

Proof of (1). This just follows from the easy formula

\[
J_n(fg - 1) = fJ_n(g)
\]

and the fact that \( J_n \) is a homomorphism.

Proof of (2). First we show that \( (f - 1)L_n(H) \subseteq L_n^{r+1} \). \( L_n \) is generated by \( n \)-fold brackets

\[
[a_1, \cdots, a_i \cdots, a_n]
\]

where at least \( r \) of the \( a_i \) belong to \( L \). We can expand in the following manner:

\[
(f - 1)[a_1, \cdots, a_i \cdots, a_n] = \sum_{i=1}^{n} [a_1, \cdots, a_{i-1}, (f - 1)a_i, f_n a_{i+1} \cdots, f_n a_n] \quad (12)
\]

Now, it follows from (11) that each of the terms on the right side of (12) lies in \( L_n^{r+1} \).

Now suppose \( a \otimes \alpha \in F_r \). Then we can expand

\[
(f - 1)(a \otimes \alpha) = (f - 1)a \otimes f_n \alpha + a \otimes (f - 1)\alpha \quad (13)
\]

If \( a \in L \) and \( \alpha \in L_n^{r-1} \), then the first term on the right side of (13) vanishes and the second lies in \( L \otimes L_n^{r} \subseteq F_n^{r+1} \). If \( \alpha \in L_n^{r} \), then the first term lies in \( L \otimes L_n^{r} \subseteq F_n^{r+1} \) and the second lies in \( H \otimes L_n^{r+1} \subseteq F_n^{r+1} \).

This completes the proof of the Lemma and, therefore, of Theorems 3 and 4.

6 Proof of Theorems 5, 6 and 7

6.1 Proof of Theorems 5 and 6

The framed pure braid group \( \mathcal{P}_g \) on \( g \) strands is defined to be the group, under stacking, of pure braids, where each strand is equipped with a normal framing which is standard on the boundary. This is easily seen to be canonically isomorphic to \( \mathcal{P}_g^0 \times \mathbb{Z}^g \), where \( \mathcal{P}_g^0 \) is the usual pure braid group, since there is a well-defined self-linking number for a normal framing. the imbedding \( \psi : \mathcal{P}_g \rightarrow \Gamma_g \) is defined in (14) as follows. Choose an imbedding of \( D_g \), the disk with \( g \) holes, into \( \Sigma_g \) so that the internal boundary circles map to \( g \) independent meridians and the external circle maps to a bounding closed curve (see Figure 3).

According to the arguments of Artin, \( \mathcal{P}_g \) can be regarded as the group of isotopy classes of diffeomorphisms of \( D_g \) which are the identity on the boundary. This representation of \( \mathcal{P}_g \), together with the chosen imbedding \( D_g \subseteq \Sigma_g \), defines \( \psi \). Of course \( \psi \) depends on the choice of imbedding. An algebraic formulation of \( \psi \) can be given as follows. Let \( \{x_i, y_i\} \) be a basis of \( F = \pi_1(\Sigma_g) \) which satisfies:
1. The homology classes of \( \{ x_i, y_i \} \) form a symplectic basis of \( H \).

2. The boundary circle of \( \Sigma_g \) defines the element

\[
(x_1 \cdots x_g)^{-1} (y_1 x_1 y_1^{-1} \cdots y_g x_g y_g^{-1}) \in F
\]

See Figure 4. An element \( \alpha \in \mathcal{P}_g \) is determined by a sequence \( \lambda_1, \cdots, \lambda_g \in F' \), the free group.

\[
\lambda_1 x_1 \lambda_1^{-1} \cdots \lambda_g x_g \lambda_g^{-1} = x_1 \cdots x_g
\]

(14)

Then \( \psi(\alpha) \in \Gamma_g \) is the element which defines the following automorphism of \( F \):

\[
x_i \mapsto \lambda_i x_i \lambda_i^{-1}, \quad y_i \mapsto y_i \lambda_i^{-1}
\]

(15)

Obviously \( \psi \) is an imbedding. It is not hard to prove the following Proposition (see [23]).

**Proposition 6.1.** \( \psi \) induces an imbedding of the quotients:

\[
\psi_n : (\mathcal{P}_g)_n / (\mathcal{P}_g)_{n+1} = \mathcal{P}_g[n] / \mathcal{P}_g[n+1] \to \Gamma_g[n-1] / \Gamma_g[n]
\]

for every \( n \geq 1 \).

\( \{ \mathcal{P}_g[n] \} \) is the weight filtration of \( \mathcal{P}_g \), i.e. \( \alpha \in \mathcal{P}_g[n] \) if and only if the longitudes \( \lambda_i \in F'_n \).

In fact \( (\mathcal{P}_g)_n = \mathcal{P}_g[n] \).
6 PROOF OF THEOREMS ??, ?? AND ??

Proof of Theorem 3. Since any diffeomorphism of $D_g$ is a composition of Dehn twists along closed curves in $D_g$, then $\psi(P_g) \subseteq \mathcal{L}_g^L$ if we set $L = \operatorname{Im}\{H_1(D_g) \rightarrow H_1(\Sigma_g)\}$. Thus $L$ is generated by \{x_i\}. Now suppose $\lambda_i \in \mathcal{F}_g'$ are the lengths of $\alpha \in (P_g)_n$. Then $J_{n-1}\psi_n(\alpha) \in \operatorname{Hom}(H, \mathbb{Q})$ is given by $x_i \mapsto [\lambda_i, x_i] = 0 \in \mathbb{Q}$, $y_i \mapsto -\lambda_i$. The dual element of $H \otimes \mathbb{Q}$ is, therefore, $-\sum x_i \otimes \lambda_i$, which belongs to $L \otimes \mathbb{Q}$.

Proof of Theorem 4. If $\alpha \in P_g$ with lengths $\lambda_i \in F$, then we can write their reductions $l_i \in L = F'_2/F'_2$ in the form $l_i = \sum_j a_{ij}x_j$. It is clear that $a_{ij}$ is the linking number of the $i$-th and $j$-th strands of (the closure of) $\alpha$, if $i \neq j$, and the self-linking number of the $i$-th strand if $i = j$. Thus $A = (a_{ij})$ is a symmetric matrix and it is clear that any symmetric matrix can be realized by some pure braid. The definition of $\psi(\alpha)$ given in [13] shows that the symplectic automorphism determined by $\psi(\alpha)$ is represented by the matrix $\left( \begin{smallmatrix} I & A \\ 0 & I \end{smallmatrix} \right)$. This proves (a).

To prove (b) we will construct an analogue of the Johnson homomorphism $J_b : (P_g)_2 = P_g[2] \rightarrow \Lambda^3(L)$ whose kernel is $(P_g)_3 = P_g[3]$ such that $J_b$ induces an isomorphism $(P_g)_2/(P_g)_3 \otimes \mathbb{Q} \cong \Lambda^3L \otimes \mathbb{Q}$. Suppose $\alpha \in (P_g)_2$ has lengths $\lambda_i \in F'_2$ and, under the isomorphism $F'_2/F'_2 \cong \Lambda^2L$, $\lambda_i \mapsto l_i$. We define $J_b(\alpha) = \sum x_i \otimes l_i \in L \otimes \Lambda^2L$. It is clear that the kernel of $J_b$ is $(P_g)_3$. We see that $\operatorname{Im} J_b \subseteq \Lambda^3L$ by a copy of Johnson’s argument. Consider the exact sequence:

$$0 \rightarrow \Lambda^3L \rightarrow L \otimes \Lambda^2L \rightarrow \mathbb{L}_3(L) \rightarrow 0$$

(16)

where $\sigma(a \otimes b) = [a, b]$. Equation (14), when read in $F'_2/F'_2$, becomes $\prod_i[x_i, \lambda_i] = 1$ or, equivalently, $\sigma J_b(\alpha) = 0$.

Now the desired fact that $\operatorname{Im} J_b \otimes \mathbb{Q} = \Lambda^3L \otimes \mathbb{Q}$ will follow from the fact that $d_g = \dim((P_g)_2/(P_g)_3 \otimes \mathbb{Q})$ is equal to $\dim\Lambda^3L \otimes \mathbb{Q}$. To prove this we use the split-exact sequence which computes the lower central series of the pure braid groups (see [4]):

$$1 \rightarrow F'_2/F'_2 \rightarrow (P_g)_2/(P_g)_3 \rightarrow (P_g-1)_2/(P_g-1)_3 \rightarrow 1$$

From this sequence we get the recursive formula $d_g = d_{g-1} + (\binom{g-1}{2})$. Solving this ($d_1 = 0$) gives $d_g = \binom{g}{2}$, the dimension of $\Lambda^3L$.

6.2 Proof of Theorem 7

First of all it is clear that any simple closed curve in $D_g$ is a separating curve in $\Sigma_g$ under the imbedding $D_g \subseteq \Sigma_g$ which defines $\kappa$. Therefore the corresponding Dehn twist of $\Sigma_g$ defines an element of $K_g$. For the remainder of the proof we need an algebraic description of the map $\kappa : P_g \rightarrow \Gamma_g$. To do this we will find it convenient to introduce a map $\delta : F' \rightarrow F$, where $F'$ is the free group on generators $x_1, \ldots, x_g$ and $F$ is the free group on $x_1, \ldots, x_g, y_1, \ldots, y_g$, defined by $\delta(x_i) = [x_i, y_i]$. Now suppose \{\lambda_i\} are the longitudinal elements in $F'$ which define the element $\alpha \in P_g$. Then $\kappa(\alpha)$ is the element of $\Gamma_g$ which corresponds to the automorphism of $F$ given by

$$\begin{align*}
x_i & \mapsto \delta(\lambda_i)x_i\delta(\lambda_i)^{-1} \\
y_i & \mapsto \delta(\lambda_i)y_i\delta(\lambda_i)^{-1}
\end{align*}$$

(17)
If $\lambda_i \in F'_n$, then $\delta(\lambda_i) \in F_{2n}$ and so $\kappa(\alpha) \in \Gamma_g[2n]$. But it is proved in [24] that $\alpha \in (P_g)_n$ if and only if $\lambda_i \in F'_n$ for every $i$. This shows that $\kappa((P_g)_n) \subseteq \Gamma_g[2n]$.

For the proof of injectivity we will need the following:

**Lemma 6.2.** $\delta$ induces an injection $F'_n/F'_{n+1} \rightarrow F_{2n}/F_{2n+1}$.

**Proof.** $F'_n/F'_{n+1}$ is the free abelian group generated by the standard basic Lie elements $\{z_i\}$ on the symbols $x_1, \ldots, x_g$ (with that ordering) of degree $n$, as defined in [17, p. 334]. It will suffice, therefore, to observe that $\{\delta(z_i)\}$ are distinct standard basic Lie elements on the symbols $x_1, \ldots, x_g, y_1, \ldots, y_g$ (with that ordering) of degree $2n$. But it follows, by a straightforward inductive argument on the degree that these elements are basic and, in the standard ordering, as defined in [17, p. 334], $z_\nu < z_\mu$ implies $\delta(z_\nu) < \delta(z_\mu)$. \hfill \Box

To prove that $\kappa_n$ is injective we use the Johnson-Morita map $J_{2n}: \Gamma_g[2n] \rightarrow H \otimes \mathbb{L}_n(H)$ whose kernel is $\Gamma_g[2n+1]$. If $\alpha \in (P_g)_n$, then, from (17), we have

$$J_{2n}(\alpha) = \sum_i (x_i \otimes [\delta(\lambda_i), y_i] - y_i \otimes [\delta(\lambda_i), x_i])$$

It follows from this that if $J_{2n}(\alpha) = 0$, then $\delta(\lambda_i) \in F_{2n+1}$ for all $i$. From Lemma 6.2 we conclude that $\lambda_i \in F'_{n+1}$ and so $\alpha \in (P_g)_{n+1}$.

Finally, to see that the induced map $(P_g)_n/(P_g)_{n+1} \rightarrow (K_g)_n/(K_g)_{n+1}$ is injective it suffices to observe that $(K_g)_n \subseteq \Gamma_g[2n]$, for every $n$ (see [23, Cor. 3.3]), and so the injection $\kappa_n$ can be factored

$$(P_g)_n/(P_g)_{n+1} \rightarrow (K_g)_n/(K_g)_{n+1} \rightarrow \Gamma_g[2n]/\Gamma_g[2n+1]$$

\hfill \Box

7 Proof of Theorem 8 and Theorem 9

7.1 Proof of Theorem 8

Since $K_g$ and $T_g$ are normal and any two choices of $L$ give conjugate subgroups $L^L_g$ the truth of the assertions for any single choice of $L$ implies the truth for any other choice. Recall from 8 the following material.

Let $i : \Sigma_{g,0} \rightarrow S^3$ be any Heegard imbedding, and $L \subseteq H$ a suitable Lagrangian subspace. Then the association $f \mapsto S^f_L$, for any $f \in L^L_g$, defines a linear map $\mathbb{Q}L^L_g \rightarrow \mathcal{M}$, where $\mathbb{Q}G$, for any group $G$, denotes the group algebra over $\mathbb{Q}$. This map induces the following three maps on the associated graded lower central series quotients:

$$\phi_n^L : G_3(I_g^L) \otimes \mathbb{Q} \rightarrow A_n^{\text{conn}}(\emptyset)$$

$$\phi_n^T : G_2(I_g^T) \otimes \mathbb{Q} \rightarrow A_n^{\text{conn}}(\emptyset)$$

$$\phi_n^K : G_n(I_g^K) \otimes \mathbb{Q} \rightarrow A_n^{\text{conn}}(\emptyset)$$

where $A_n^{\text{conn}}(\emptyset)$ is the subspace of $A_n(\emptyset)$ spanned by connected trivalent graphs and $G_n(G)$ is the lower central series quotient $G_n/G_{n+1}$, for any group $G$. 
It is proved in [8] that these maps are onto if \( g \geq 5n + 1 \). Thus we can, for example, choose \( f \in (K_g)_n \) so that \( \phi^K_n(f \otimes 1) \neq 0 \). If \( f \in (L_g^L)_{3n+1} \) then we would have \( \phi^K_n(f) \) defined and zero. But \( \phi^K_n \) and \( \phi^L_n \) take the same value on any element in both their domains, since they are defined by the same construction. Thus \( f \notin (L_g^L)_{3n+1} \). Similarly \( f \notin (T_g)_{2n+1} \).

The proofs of the other two non-inclusions are the same.

### 7.2 Proof of Theorem 9

Let \( b, D \) and \( K \subseteq S^3 \) be as indicated on page 8. Let \( N_0 \approx I \times D \) be a regular neighborhood of \( D \) in \( S^3 \) and \( N \) a regular neighborhood of \( K \cup D \). Let \( i : \Sigma_g \to \partial N \) be a homeomorphism. Then \( K_b \subseteq S^3 \) is defined by replacing \( K \cap N_0 \), which consists of \( g \) straight line segments, by the pure braid \( b \). Thus we have \( K_b \subseteq N \subseteq S^3 \). Let \( D_g = D \times 1 \cap \Sigma_g \), which determines the homomorphism \( \Psi \). See Figure 5.

![Figure 5. The action of a pure braid on a knot](image)

The main point here is the following:

**Lemma 7.1.** There is a homeomorphism \( h \) of \( N \) onto itself which satisfies:

1. \( h(K) = K_b \),
2. \( h|\partial N = i\psi(b)i^{-1} \).

**Proof of Lemma.** Recall the correspondence between a pure braid \( b \) and a homeomorphism \( h' \) of \( D_g \) onto itself. If \( b \) is considered as a path in the configuration space of \( g \) points in the 2-disk, then this path extends to a diffeotopy of the 2-disk. The end point of this diffeotopy fixes \( \delta D \) and the \( g \) points and \( h' \) is just the restriction to \( D_g \). The isotopy class, rel \( \delta D \), of \( h' \) is uniquely determined. For framed pure braids, the same construction determines the isotopy class, rel \( \delta D_g \), of \( h' \) is the identity on \( \delta D_g \). The diffeotopy then defines a diffeomorphism \( \bar{h} \) of \( I \times D_g \) onto itself which is the identity on \( 0 \times D_g \), \( h' \) on \( 1 \times D_g \) and, if
$d$ denotes the union of the $g$ interior disks whose complement is $D_g$, $\bar{h}(I \times d) = b$. Here we think of a framed pure braid as a coordinatized tubular neighborhood of the actual braid. We can identify $N_0$ with $I \times D_g$ so that a tubular neighborhood of $N_0 \cap K$ is identified with $I \times d$. Then $\bar{h}$ becomes a diffeomorphism of $N_0$ and we can extend it over $N$ by declaring it to be the identity on $N - N_0$. This defines the desired $h$. (1) is clear from the construction and (2) follows from the definition of $\psi$. \hfill \square

We now return to the proof of Theorem 9. Let $N'$ be the closure of the complement of $N$ in $S^3$. Then $S^3_{\psi(b)}$ is, by definition, $N \cup_f N'$, where $f = i\psi(b)i^{-1}$. But we can use the diffeomorphism $h$ of Lemma 7.1 to construct a diffeomorphism $\tilde{h} : S^3_{\psi(b)} \to S^3$ as follows. Regarding $S^3$ as $N \cup_{id} N'$ we define $\tilde{h}|N = h$ and $\tilde{h}|N' = \text{identity}$. By Lemma 7.1, $\tilde{h}(K) = K_b$. The effect of $\tilde{h}$ on the framing of $K$ is that an $n$-framing on $K$ is mapped to a $(n + \sum_{ij} l_{ij})$-framing on $K_b$ and so, for our choice of framing on $b$, a $+1$-surgery on $K \subseteq S^3_{\psi(b)}$ induces, via $\tilde{h}$, a $+1$-surgery on $K_b \subseteq S^3$.

Now $(S^3_{\psi(b)})_K = (S^3_K)_{\psi(b)}$ and when we show that $\psi(b) \in \Lambda_g^{L_j}$, for the given imbedding $j : \Sigma_g \subseteq S^3_K$, the proof of the Theorem will be complete. Clearly $\text{Im } \psi \subseteq \Lambda_g^{L_i}$ where $i\Sigma_g \subseteq S^3$ is defined from $K$ above. But $L_i \neq L_j$. In fact if $\{x_i, y_i\}$ is the symplectic basis defined by the meridians and longitudes of $\Sigma_g = \delta N$ so that $L_i$ is spanned by $\{x_i\}$-- then $L_j$ is spanned by $\{x_i - y\}$, where $y = \sum_i y_i$. Thus to show $\psi(b) \in \Lambda_g^{L_j}$ we need to show that $\psi(b)(y) = y$. But if $\{l_{ij}\}$ is the linking matrix of the framed link $b$, then $\psi(b)(y_i) = y_i + \sum_j l_{ij} x_j$ and so $\psi(b)(y) = y + \sum_j (\sum_i l_{ij}) x_j$. Since we have required $\sum_i l_{ij} = 0$ the proof is complete. \hfill \square

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