Two-dimensional $O(3)$ $\sigma$-model up to correlation length $10^5$

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We carry out a high-precision Monte Carlo simulation of the two-dimensional $O(3)$-invariant $\sigma$-model at correlation lengths $\xi$ up to $10^5$. Our work employs a new and powerful method for extrapolating finite-volume Monte Carlo data to infinite volume, based on finite-size-scaling theory. We compare the extrapolated data to the renormalization-group predictions. The deviation from asymptotic scaling, which is $\approx 25\%$ at $\xi \sim 10^5$, decreases to $\approx 4\%$ at $\xi \sim 10^5$.

We study the lattice $\sigma$-model taking values in the unit sphere $S^{N-1} \subset \mathbb{R}^N$, with nearest-neighbor action $\mathcal{H}(\sigma) = -\beta \sum_{\langle x,y \rangle} \sigma_x \cdot \sigma_y$. Perturbative renormalization-group computations through three loops [1,2] predict that the exponential correlation length (inverse mass gap) $\xi^{(exp)}$, the second-moment correlation length $\xi^{(2)}$, and the susceptibility $\chi$ behave for $N = 3$ (in infinite volume) as

$$\xi^#(\beta) = C_{\xi^#} \frac{e^{2\pi \beta}}{2\pi\beta} \left[ 1 - \frac{0.091}{\beta} + \cdots \right]$$

$$\chi(\beta) = C_\chi \frac{e^{4\pi \beta}}{(2\pi\beta)^4} \left[ 1 - \frac{0.001}{\beta} + \cdots \right]$$

as $\beta \to \infty$. The nonperturbative constant $C_{\xi^{(exp)}}$ has been computed recently using the thermodynamic Bethe Ansatz [3]:

$$C_{\xi^{(exp)}} = 2^{-5/2} \left( \frac{\log \frac{1}{\beta}}{8} \right).$$

The remaining nonperturbative constants are known analytically only at large $N$ [4]:

$$C_{\xi^{(0)}/C_{\xi^{(exp)}}} = 1 - \frac{0.003225}{N} + O(1/N^2)$$

$$C_\chi = \frac{\pi}{16} \left[ 1 - \frac{4.267}{N} + O(1/N^2) \right]$$

A high-precision Monte Carlo study [5] yields $C_{\xi^{(0)}/C_{\xi^{(exp)}}} = 0.9993 \pm 0.0006$ for $N = 3$, in good agreement with (4). Previous studies up to $\xi \sim 100$ agree with these predictions to within about $20-25\%$ [6].

In order to extrapolate finite-volume Monte Carlo data to infinite volume, we used a novel method [7] based on the finite-size-scaling Ansatz

$$\frac{\mathcal{O}(\beta, sL)}{\mathcal{O}(\beta, L)} = F_\mathcal{O} \left( \xi(\beta, L)/L; s \right),$$

which is correct up to terms of order $\xi^{-\omega}$ and $L^{-\omega}$; here $\mathcal{O}$ is any long-distance observable, $s$ is a fixed scale factor (usually $s = 2$), $L$ is the linear lattice size, $F_\mathcal{O}$ is a universal function, and $\omega$ is a correction-to-scaling exponent. For similar extrapolation methods, see [8,9].

Details of our simulation and of the extrapolation process can be found in Sokal’s talk at this conference and in [10]. Our preferred fit is shown in Figure 1, where we compare also with the perturbative prediction

$$F_\xi (x; s) = s \left[ 1 - \frac{\log s}{8\pi x^2} - \left( \frac{\log s}{128\pi^2} + \frac{\log s}{16\pi^2} \right) \frac{1}{x^4} \right]$$

valid for $x \gg 1$. 

*Speaker at the conference.*
The extrapolated values $\xi^{(2)}$ from different lattice sizes at the same $\beta$ are consistent within statistical errors: only one of the 24 $\beta$ values has a $\chi^2$ too large at the 5% level; and summing all $\beta$ values we have $\chi^2 = 86.56$ (106 DF, level = 92%).

In Figure 2 (points + and $\times$) we plot $\xi^{(2)}$;estimate divided by the two-loop and three-loop predictions (1)-(4). The discrepancy from three-loop asymptotic scaling, which is $\approx 16\%$ at $\beta = 2.0$ ($\xi \approx 200$), decreases to $\approx 4\%$ at $\beta = 3.0$ ($\xi \approx 10\%$). This is roughly consistent with the expected $1/\beta^2$ corrections. Notice that the points in the curve fluctuate less than what one would expect on the basis of the reported error bars; this is due to a strong statistical correlation of the estimates for the points with higher values of $\beta$. Probably also the slight bump at $2.3 \leq \beta \leq 2.6$ is spurious, arising from correlated statistical or systematic errors.

We can also try an “improved expansion parameter” [11,2,13] based on the energy $E = \langle \sigma_0 \cdot \sigma \rangle$. First we invert the perturbative expansion

$$E(\beta) = 1 - \frac{1}{2\beta} - \frac{1}{16\beta^2} - \frac{0.038512}{\beta^3} + O(1/\beta^4) \quad (8)$$

and substitute into (1); this gives a prediction for $\xi$ as a function of $1 - E$. For $E$ we use the value measured on the largest lattice; the statistical errors and finite-size corrections on $E$ are less than $5 \times 10^{-5}$, and therefore induce a negligible error (less than 0.5%) on the predicted $\xi$. The corresponding observed/predicted ratios are also shown in Figure 2 (points $\Box$ and $\Diamond$). The “improved” 3-loop prediction is in excellent agreement with the data.

In Figure 3 we report the ratio

$$R_\chi(\beta) = \frac{C_\chi}{C_{\xi(2)}} \frac{\chi_{\infty,\text{estimate}}(\xi^{(2)};\text{estimate})^2}{\chi_{\infty,\text{theor}}(\xi^{(2)};\text{theor})^2} \quad (9)$$

where by the suffix theor we denote the one-, two- or three-loop prediction either for the standard perturbation theory in $1/\beta$ or for the “improved” one in $1 - E$. The curves appear to become flat for increasing $\beta$ and to converge to the same value $\approx 10.8$, thus providing an estimate for the universal ratio $C_\chi/C_{\xi(2)}$. 

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**Figure 1.** $\xi(\beta,2L)/\xi(\beta,L)$ versus $\xi(\beta,L)/L$. Different symbols indicate different lattice sizes $L$. Dashed curve is the perturbative prediction (7).

**Figure 2.** $\xi^{(2)}_{\infty,\text{estimate}}/\xi^{(2)}_{\infty,\text{theor}}$ versus $\beta$. Error bars are one standard deviation (statistical error only). There are four versions of $\xi^{(2)}$: standard perturbation theory in $1/\beta$ gives points $+$ (2-loop) and $\times$ (3-loop); “improved” perturbation theory in $1 - E$ gives points $\Box$ (2-loop) and $\Diamond$ (3-loop).
Let us summarize the conceptual basis of our analysis. The main assumption is that if the Ansatz (6) with a given function \( F_\xi \) is well satisfied by our data for \( L_{\text{min}} \leq L \leq 256 \) and \( 1.65 \leq \beta \leq 3 \), then it will continue to be well satisfied for \( L > 256 \) and for \( \beta > 3 \). Obviously this assumption could fail, e.g., if [14] at some large correlation length \(( \gtrsim 10^3 \) the model crosses over to a new universality class associated with a finite-\( \beta \) critical point. In this respect our work is subject to the same caveats as any other Monte Carlo work on a finite lattice. However, it should be emphasized that our approach does not assume asymptotic scaling [eq. (1)], as \( \beta \) plays no role in our extrapolation method. Thus, we can make an unbiased test of asymptotic scaling. The fact that we confirm (1) with the correct nonperturbative constant \( (3)/(4) \) is, we believe, good evidence in favor of the asymptotic-freedom picture.

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