A bootstrap-based method to estimate directional extreme risk regions at high levels

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Abstract

In multivariate extreme value analysis, the focus is on the multivariate analysis outside of the observable sampling zone, which implies that a region of interest is associated to high risk levels. This work provides an out-sample estimation method for the recently introduced Directional Multivariate Quantiles ($DMQ$) and a characterization of these quantiles at high levels using multivariate extreme value theory. The asymptotic normality of the proposed estimator is derived. We introduce a multivariate bootstrap procedure to carry out the estimation of the $DMQ$. The methodology is illustrated with a multivariate t-distribution for which the theoretical $DMQ$ are known. A real application to a financial case is also performed.

1 Introduction

The estimation of extreme level curves is important for identifying extreme events and for characterizing the joint tails of multidimensional distributions. They are usually considered as quantiles at high levels; that is, they are linked with a probability $\alpha$ of occurrence of certain event, where $\alpha$ is a very small number. This proposal considers values of $\alpha$ lower or equal than $1/n$, where $n$ denotes the sample size, which implies that the number of data points that fall beyond the quantile curve is small and can even be zero; thus we are outside of the observable region, or in other words, in the framework of out-sample estimation. This lack of relevant data points does the estimation difficult, making it necessary to introduce tools from the multivariate extreme value theory.

The main purpose of this paper is to provide an out-sample estimation method for the directional multivariate quantiles ($DMQ$) recently introduced in [Torres et al. (2015)] and [Torres et al. (2016)]. In these papers, the directional setting refers to the inclusion of a parameter of direction $u$ that allows the analysis of data by looking at the cloud of observations from different perspectives. Accurate assessments of these quantiles are sought in a diversity of applications from financial risk management (e.g. [Torres et al. (2015)]) to environmental impact assessment (e.g. [Torres et al. (2016)]). A non-parametric estimation method was developed in [Torres et al. (2015)] to estimate the directional quantile based on the empirical probability law, which is
valid just for the \textit{in-sample} scenario; that is $\alpha > 1/n$.

Both scenarios, \textit{in-sample} and \textit{out-sample}, have been widely studied in the univariate setting and recently the literature has focused on the extension to the multivariate context. Some relevant references in this area can be grouped into three categories as follows. Firstly, estimation under optimization processes; for instance, those proposals with estimation methods based on linear quantile regression (see, e.g., [Chaudhuri (1996), Serfling (2002), Hallin et al. (2010), Mukhopadhyay and Chatterjee (2011), Kong and Mizera (2012), Girard and Stupfler (2015), He and Einmahl (2017)]). This category contains estimation methods \textit{in-sample} such as the proposal by [Chaudhuri (1996)] for his notion of geometric quantiles. Recently [Girard and Stupfler (2015)] has also proposed an \textit{out-sample} estimation method for these geometric quantiles.

A second category contains methods determining level curves of joint density functions in such a way that the set of points outside those contours has a probability equal to a given level $\alpha$. These methods easily describe inner and outer regions at the given level and inherently cover the infinite set of directions through those contours (e.g., [Cai et al. (2011), Einmahl et al. (2013)]). The estimators proposed in this category have been developed mainly for the \textit{out-sample} framework. For instance, [Cai et al. (2011)] have provided estimation of bivariate contour levels for some joint densities with elliptical and non-elliptical distributions, considering cases with asymptotic dependence and asymptotic independence. Other methodologies in this category are based on the trimming through depth functions. [Serfling (2002)] have described \textit{in-sample} methods considering different depth functions and [He and Einmahl (2017)] presented an \textit{out-sample} contour estimation based on Tukey depth.

Last category considers level curve estimations using either joint distribution or survival functions (e.g. [De Haan and Huang (1995), Fernández-Ponce and Suárez-Llorens (2002), Belzunce et al. (2007), Chebana and Ouarda (2009), Di Bernardino et al. (2011)]). Works based on copulas are also classified in this group (e.g. [Chebana and Ouarda (2011), Durante and Salvadori (2010), Salvadori et al. (2011), Binois et al. (2015)]). These works have introduced proposals in both contexts \textit{in-sample} and \textit{outsample}, but most of them present the theory or have applications only in the bivariate case. Since the proposal developed in this work is somehow based on distributions, it belongs to the third category presented above.

As we have mentioned before, the methodology developed in this work includes a directional notion and one can find in the literature a few references dealing with this notion. [Chaudhuri (1996)] was one of the first works that deals with directions, but this multivariate aspect starts to take importance just in the past decade, where an accurate assessment of risk regions arises in a diversity of applications. For instance, [Embrechts and Puccetti (2006)] studied bounds for multivariate financial risks, highlighting the utility of the analysis considering two particular directions. [Belzunce et al. (2007)] presented a bivariate quantile application to air quality where the directions are related to the four classical orthants.

Other examples that highlight the importance of the directions can be seen in [Hallin et al. (2010)], where it was proposed directional projections to show a relationship between their quantile trimming and the trimming obtained through Tukey depth. [Kong and Mizera (2012)] used a similar idea to build multivariate growth chart application. [Fraiman and Pateiro-López (2012)] provided a directional projection-based definition for infinite-dimensional multivariate quantiles in Hilbert spaces. In financial risk management [Torres
et al. (2015)] showed the advantage of using the portfolio weights as the direction of analysis to provide an upper bound for the maximum loss. In [Torres et al. (2016)] was performed an application to environmental impact assessment, where it can be seen the improvement of identifying extremes by using the first principal component as a direction of analysis.

Therefore, inspired in the work of [De Haan and Huang (1995)] where an out-sample estimator for bivariate level curves of a distribution function $F$ was established, the contributions of this paper are three fold: 1) to include the directional framework given in [Torres et al. (2015), Torres et al. (2016)], 2) to provide the expression of the estimator for those directional high level sets in a general dimension $d$ and 3) to introduce a non-parametric extreme estimator based on bootstrap for these high level directional quantiles.

The paper is organized as follows. In Section 2 we summarize the main definitions and results related to the directional multivariate framework used in the paper. Section 3 introduces definitions from the multivariate extreme theory to fix conditions over the random vector $X$ to ensure the results of the paper. In Section 4, the characterization of the elements of the DMQ at high levels based on the heuristic ideas in [De Haan and Huang (1995)] are described. In Section 5 are developed statistical tools to perform an out-sample estimation of the DMQ. Section 5.1 presents the estimator, a bootstrap-based estimation method is described in Section 5.3, and also the asymptotic normality of the estimator is presented in Section 5.2. Section 6 illustrates the multivariate estimation procedure comparing both theoretical and estimated results using a multivariate $t-$distribution. Section 7 presents a directional analysis over daily filtered returns of three different international indices. Finally, in Section 8 some conclusions and perspectives are provided. Proofs and auxiliary results are postponed to Appendix A, Figures to Appendix B.

## 2 Directional multivariate quantiles (DMQ)

This section introduces the preliminary definitions and notation necessary to understand the contributions of the paper. The directional multivariate setting that we use in this paper is based on the work by [Laniado et al. (2012)], where the main contributions were based on the notion of oriented orthant.

**Definition 2.1.** An oriented orthant in $\mathbb{R}^d$ with vertex $x$ in direction $u$ is defined by,

$$
C_x^R_u = \{ z \in \mathbb{R}^d : R_u(z-x) \geq 0 \}.
$$

where $u \in \{ z \in \mathbb{R}^d : ||z|| = 1 \}$ and $R_u$ is an orthogonal matrix such that $R_u u = e$, with $e = \frac{1}{\sqrt{d}}(1,\ldots,1)'$.

Note that an oriented orthant is a translation and a rotation of the non-negative Euclidean orthant toward a vertex in the point $x$ and a direction $u$. In [Torres et al. (2015)] has been pointed out that $R_u$ is not unique for $d \geq 3$. Hence, in order to guarantee uniqueness, they state the following. Let $u$ be a unit vector with non-null components and let $M_u$ and $M_e$ be matrices defined as,

$$
M_u = [u, \ sgn(u_2)e_2, \ \cdots, \ sgn(u_d)e_d] \quad \text{and} \quad M_e = [e, \ e_2, \ \cdots, \ e_d],
$$

where $u_i, i = 1,\ldots,d$ is the $i-$th component of $u$, $sgn(\cdot)$ is the scalar sign function and $e_i$ is the $i-$th column of the $d \times d$ identity matrix. Then $M_u$ and $M_e$ have full rank and unique QR decompositions,

$$
M_u = Q_u T_u \quad \text{and} \quad M_e = Q_e T_e,
$$

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such that $T_u$, $T_e$ are triangular matrices with positive diagonal elements and $Q_u$, $Q_e$ are the correspondent orthogonal matrices (see [Horn and Johnson (2013)][Theorem 2.1.14, p.g. 89]). Therefore, [Torres et al. (2015)] have defined,

**Definition 2.2.** The QR oriented orthant with vertex $x$ in direction $u$, denoted as $C_x^u$, is the oriented orthant satisfying $R_u = Q_eQ_u^t$. 

We remark that the consideration of directions with non-null components is not restrictive, because if a vector of direction $u$ has a null component then the variable associated to the null component can be analyzed separately. Finally, quantiles at certain level $\alpha$ in direction $u$ have been defined in [Torres et al. (2015)] as,

**Definition 2.3.** Let $X$ be a random vector with associated probability distribution $P$. Then the directional multivariate quantile at level $\alpha$ in direction $u$ is defined as

$$Q_X(\alpha, u) := \partial \{ x \in \mathbb{R}^d : P(C^{-u}x) \geq 1 - \alpha \},$$

(2.3)

where $\partial$ denotes the boundary of the subset considered into brackets and $0 \leq \alpha \leq 1$.

In the univariate setting, extremes are analyzed considering the two possibilities of exceeding from either distribution or survival functions and most of the extensions of these analyses to the multivariate setting have also been concentrated on these two types of exceeding. For instance, in the bivariate case [Fernández-Ponce and Suárez-Llorens (2002), Shiau (2003), Salvadori (2004), Embrechts and Puccetti (2006)], and generalized multivariate versions are presented in [Gupta and Manohar (2005), Fraiman and Pateiro-López (2012), Cousin and Di Bernardino (2013), Di Bernardino et al. (2015)].

However, the multivariate setting offers infinite possibilities of exceeding to be considered and the directional framework explores these alternatives. First, note that the possibilities based on distributions and survival functions are provided through the directions $-e$, $e$ respectively. Hereafter we will call these classical directions.

Second, we point out that there are other interesting directions to be taken into consideration. For instance in portfolio optimization, the direction given by the portfolio weights of investments is of particular interest because it takes into account the losses due to the composition of the investment in the portfolio (see [Laniado et al. (2012), Torres et al. (2015)]). In environmental phenomena, the directional approach has also been applied to detect extreme events by considering the direction of maximum variability of the data (see [Torres et al. (2016)]). In summary, different contexts or phenomena could suggest different particular directions that consider external information in order to capture the overall behavior of the data and to improve visualization of the results.

### 3 Probabilistic assumptions

In this section, we introduce conditions over $X$ that must be satisfied in order to introduce an estimator properly defined of $Q_X(\alpha, u)$ when $\alpha \leq 1/n$. We also include some constraints to obtain asymptotic properties of the estimator that we propose. Those constraints involve notions of first and second order multivariate regular variation. We refer to [Resnick (1987), De Haan and Ferreira (2006), Resnick (2007)] for a throughout review.
Assumption A1. The random vector $X$ must be absolutely continuous with increasing marginal distribution functions and such that $\mathbb{E}[X] < \infty$.

A1 was also considered in [Torres et al. (2015)] to define a multivariate value at risk, where the existence of the first moment of the random vector $X$ is important to fix a center of location as a reference point. Hereafter, we can suppose for simplicity that $X$ is a random vector with zero mean.

Assumption A2. Given $u$, $R_u X$ possesses positive upper-end points of the marginal distributions.

This assumption was introduced in [De Haan and Huang (1995)] for the marginals distributions of $X$, but A2 is more general and establishes the condition for the correspondent rotation associated to the direction $u$. We now characterize the tail behavior of $X$.

Definition 3.1. A random vector $X$ has first order multivariate regular variation with tail index $\gamma$, if there exists a real-value function $\phi(t) > 0$ that is regularly varying at infinity\(^1\) with exponent $1/\gamma$, denoted by $RV_{1/\gamma}$, and a non-zero measure $\mu(\cdot)$ on the Borel $\sigma$–field $[-\infty, \infty)^d \setminus \{0\}$, such that,

$$t \mathbb{P}[(\phi(t))^{-1} X \in \cdot] \xrightarrow{\text{v}} \mu(\cdot), \quad (3.1)$$

where $\xrightarrow{\text{v}}$ means vague convergence and $t \to \infty$ (see, e.g., [Jessen and Mikosh (2006), Resnick (1987)]).

If $X$ verifies Definition 3.1 with $\gamma > 0$, then it is called a heavy tailed random vector and the measure of convergence $\mu(\cdot)$ in (3.1) has the homogeneity property of order $\gamma$,

$$\mu(c B) = c^{-\gamma} \mu(B), \quad (3.2)$$

for all $c > 0$ and every Borel set $B$.

Assumption A3. $X$ has first order multivariate regular variation with tail index $\gamma > 0$.

The following result establishes A3 for any orthogonal transformations $Q$ over $X$.

Proposition 3.2. Let $Q$ be an orthogonal transformation. If $X$ has first order multivariate regular variation with tail index $\gamma$, then the random vector $QX$ has first order multivariate regular variation with tail index $\gamma$.

Proof of Proposition 3.2 is given in Appendix A.

Corollary 3.3. If $X$ has first order multivariate regular variation with tail index $\gamma$. Then, there exist $\gamma_i^Q$, $i = 1, \ldots, d$ such that the marginals of $QX$ are regularly varying with corresponding tail indexes $\gamma_i^Q$.

Proof of Corollary 3.3 is a straightforward consequence of the marginal implications of Proposition 3.2 (see [De Haan and Ferreira (2006)][Chapter 6]). Note that Proposition 3.2 states that first order multivariate regular variation is a property preserved under rotations where the associated multivariate tail index $\gamma$ remains invariant. On the other hand, Corollary 3.3 ensures the univariate first order regularly varying property for the marginals of $QX$, but the tail index for each rotation and for each marginal could be different.

\(^1\)A function $\phi(\cdot) \in RV_{1/\gamma}$, if it holds that $\lim_{t \to \infty} \frac{\phi(tx)}{\phi(t)} = t^{\frac{1}{\gamma}}$, for all $t > 0$. 

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Definition 3.4. A random vector $X$ has second order multivariate regular variation with indexes $(\gamma, \pi)$, if there exist functions $\phi(\cdot) \in RV_{1/\gamma}$ and $\Lambda(t) \to 0$, such that $|\Lambda| \in RV_{\pi}$, $\pi \leq 0$; satisfying locally uniformly that,

$$\frac{t \mathbb{P} \left[ (\phi(t))^{-1}X \in B \right] - \mu(B)}{\Lambda(\phi(t))} \to \psi(B) < \infty,$$

(3.3)

where $\psi(\cdot)$ is not identically zero. (see [Resnick (2002)]).

Definition 3.4 was given in [Resnick (2002)][Section 3] in terms of relatively compact rectangles in $[0, \infty)^d \setminus \{0\}$, but we have rewritten it for all relatively compact rectangles in $[-\infty, \infty)^d \setminus \{0\}$. Moreover, it is necessary to describe a strong version of the second order regular variation, because this condition will be needed further over the marginals.

Definition 3.5. A random vector $X$ has strong second order multivariate regular variation with indexes $(\gamma, \pi)$ if the convergence in Definition 3.4 holds and

$$\frac{t \mathbb{P} \left[ (\phi(t))^{-1}X \in B \right] - \mu(B)}{\Lambda(\phi(t))\psi(B)} \to 1.$$

(3.4)

Assumption A4. $X$ has strong second order multivariate regular variation.

Proposition 3.6. If $X$ has (strong) second order multivariate regular variation with indexes $(\gamma, \pi)$, then the random vector $QX$ has (strong) second order regular variation for any orthogonal transformation $Q$.

Proof of Proposition 3.6 is given in Appendix A.

Corollary 3.7. If $X$ has (strong) second order multivariate regular variation with indexes $(\gamma, \pi)$. Then, there exist $(\gamma^Q_i, \pi^Q_i)$, $i = 1, \ldots, d$ such that the marginals of $QX$ are regularly varying with corresponding indexes $(\gamma^Q_i, \pi^Q_i)$.

The proof of Corollary 3.7 is provided by Theorem 2 in [Resnick (2002)]. A similar reasoning to the one after Corollary 3.3 works here for $\pi$ from Proposition 3.6 and $(\gamma^Q_i, \pi^Q_i)$ from Corollary 3.7. Hereafter, we consider that a random vector $X$ satisfies A1-A4 and we use the notation $\gamma_u$, $\gamma_{u,i}$, $\pi_u$ and $\pi_{u,i}$ to refer the multidimensional and the marginal indexes associated to $R_uX$.

4 Characterization of the DMQ at high levels

The objective of this section is to characterize the points of $Q_X(\alpha, u)$ defined in (2.3) at high levels (i.e., small values of the $\alpha$ level). Our proposal is based on: (1) the relationship provided in [Torres et al. (2015)][Property 3.8],

$$Q_X(\alpha, u) = R_u^t Q_{R_uX}(\alpha, e),$$

(4.1)

and (2) the heuristic ideas of the bivariate quantile parametrization given in [De Haan and Huang (1995)] extended to a general multivariate context. Therefore, we assume that the random vector $X$ verifies A1–A4. Thus, Proposition 3.2 implies that the distribution function $F_u$ of the random vector $R_uX$ belongs to the
domain of attraction of a non-degenerate multivariate extreme value distribution $G_u$. Moreover, there exist two sequences $a_u([t]), b_u([t])$ such that,

$$
\lim_{t \to \infty} t \left( 1 - F_u(a_u(j)) x_u(j) + b_u(j) \right) = -\ln(G_u(x_u)), \quad (4.2)
$$

where $\lfloor \cdot \rfloor$ is the floor function. In addition, a direct consequence of (4.2) is that each marginal of $G_u$ has the form $\exp\left\{-\left(1 + \gamma_{u,j} x_{u,j}\right)^{-1/\gamma_{u,j}}\right\}$, $j = 1, \ldots, d$, for $\gamma_{u,j} \neq 0$, and $\gamma_{u,j}$ is called the tail index of the $j$–marginal (see [De Haan and Ferreira (2006)][Chapter 6]). Then it is possible to write,

$$
\lim_{t \to \infty} t \left( 1 - F_u(j)(x_{u,j} + b_u(j))\right) = (1 + \gamma_{u,j} x_{u,j})^{-1/\gamma_{u,j}}, \quad (4.3)
$$

where $F_u$ is the $j$–marginal of $F_u$. Thus, (4.3) implies that at high levels the $(1 - \alpha)$–quantile related to $F_{u,j}$ verifies the following relationship,

$$
x_{u,j}(\alpha) \approx a_{u,j}(t) \left( \frac{1}{t^{\alpha}} \right)^{\gamma_{u,j}} - 1 + b_{u,j}(t), \quad \text{for all } j = 1, \ldots, d. \quad (4.4)
$$

However, we want to estimate $Q_{R_\alpha X}(\alpha, e)$ which according to Definition 2.3 is the set of points such that $1 - \alpha = F_u(x)$. To this end, we generalize the heuristic idea developed for the bivariate case by [De Haan and Huang (1995)] of introducing a parametrization of the quantile.

Recall that any point $x \in \mathbb{R}^d$ can be written alternatively in polar coordinates as $x = ||x|| (x/||x||) = \rho(\theta) \theta$, where $\rho(\theta) \in \mathbb{R}^+$ and $\theta$ belonging to the unit $d$–dimensional ball (for a further discussion see [Driver (2003)][pg. 217]). Note also that A2 in the $\mathbb{R}^d$ polar coordinates is equivalent to the assumption of upper-end points when $\theta = (\theta_1, \ldots, \theta_d)$ is such that $0 \leq \theta_i \leq 1$, for all $i = 1, \ldots, d$.

From now on, any point of $Q_{R_\alpha X}(\alpha, e)$ under the new parametrization will be denoted by $x_{u}(\alpha, \theta)$, $\theta \in [0, 1]^d$. Then, we provide the following characterization of the elements of $Q_{R_\alpha X}(\alpha, e)$,

$$
x_{u,j}(\alpha, \theta) = a_{u,j}(t) \left( \frac{\rho_u(\theta \theta_j / t^{\alpha})}{\gamma_{u,j}} \right) - 1 + b_{u,j}(t), \quad \text{for all } j = 1, \ldots, d. \quad (4.5)
$$

Note the difference between $x_{u,j}(\alpha)$ given in (4.4) and $x_{u,j}(\alpha, \theta)$ in (4.5). The former is related to the univariate quantile of the marginal $(R_uX)_j$ and the latter is the $j$–component of an element in $Q_{R_\alpha X}(\alpha, e)$. Therefore, given that all the elements in (4.5) are known or can be estimated, except $\rho_u(\theta)$, the problem of estimating $x_{u,j}(\alpha, \theta)$ turns into the problem of finding a solution for the scalar function $\rho_u(\theta)$ and its estimation. Then, from (4.2) and (4.5), we obtain that,

$$
\alpha = 1 - F_u(x_{u}(\alpha, \theta)) \approx t^{-1} \left\{ -\ln \left( G_u \left( \frac{x_{u,j}(\alpha, \theta) - b_{u,j}(t)}{a_{u,j}(t)} \right); \quad j = 1, \ldots, d \right) \right\}
$$

$$
= t^{-1} \left\{ -\ln \left( G_u \left( \frac{\rho_u(\theta) \theta_j / t^{\alpha}}{\gamma_{u,j}} - 1 \right); \quad j = 1, \ldots, d \right) \right\} \quad (4.6)
$$

$$
= -\frac{\alpha}{\rho_u(\theta)} \ln \left( G_u \left( \frac{\theta_j^{\gamma_{u,j}} - 1}{\gamma_{u,j}} \right); \quad j = 1, \ldots, d \right),
$$

Last equality in (4.6) is due to the homogeneity property of $G_u$ (see [De Haan and Ferreira (2006)] [Theorem 6.1.9]). Hence, from (4.6), we achieve a solution of $\rho_u(\theta)$ by approximation. This solution will be denoted as,

$$
\tilde{\rho}_u(\theta) := -\ln \left( G_u \left( \frac{\theta_j^{\gamma_{u,j}} - 1}{\gamma_{u,j}} \right); \quad j = 1, \ldots, d \right), \quad (4.7)
$$
which implies the approximation of \( x_{u,j}(\alpha, \theta) \) by,
\[
\tilde{x}_{u,j}(\alpha, \theta) := a_{u,j}(t) \left( \frac{\tilde{p}_u(\theta) \theta_j / t \alpha}{\gamma_{u,j}} - 1 \right) + b_{u,j}(t), \quad \text{for all } j = 1, \ldots, d.
\] (4.8)

Thus, \( Q_X(\alpha, u) \) is approximated at high levels by the parametrization,
\[
\tilde{Q}_X(\alpha, u, \theta) := R'_u \tilde{Q}_{RaX}(\alpha, e, \theta),
\] (4.9)

where \( \tilde{Q}_{RaX}(\alpha, e, \theta) := \{ \tilde{x}_u(\alpha, \theta), \theta \in [0,1]^d \} \). By using this characterization one can get an out-sample estimator for \( Q_X(\alpha, u) \), which is the objective of next section.

**Remark 1.** Note that it is also possible to define DMQ based on joint survival functions, for which, the following definition has to be used,
\[
Q_X(\alpha, u) := \partial \{ x \in \mathbb{R}^d : P(C_u x) \leq \alpha \}. \] (4.10)

This implies that any point \( x \) in \( Q_{RaX}(\alpha, e) \) should satisfy the equation \( \alpha = \bar{F}_u(x) \). Then, we can characterize (4.10) by adapting all results presented in this section.

## 5 Inference for DMQ at high-levels

Our interest in this section is to find directional extreme sets based on a random sample. Then, given a direction \( u \), let \( R_u X_1, \ldots, R_u X_n \) be i.i.d. random vectors distributed as \( R_u X \) and denote by \( \{(R_u X)_j : n\}_{j=1}^d, i = 1, \ldots, n \), the collection of the corresponding \( n \)-th order statistics for each marginal.

Marginal order statistics are important since they allow (4.2) (see [De Haan and Ferreira (2006)][Section 7.2]) to be written in terms of a subsample that provides significant information about the tail behavior of the joint distribution \( F_u \). This subsample is related with a tuning intermediate sequence \( k := k(n) \to \infty, k/n \to 0 \) as \( n \to \infty \). This crucial sequence leads to the break point from which the information of an ordered sample starts to be considered in the tail of the distribution. Then, we obtain,
\[
\lim_{n \to \infty} \frac{n}{k} \left( 1 - F_u(a_{u,j}(n/k) x_{u,j} + b_{u,j}(n/k); \ j = 1, \ldots, d) \right) = - \ln \left( G_u(x_u) \right).
\] (5.1)

In extreme value theory it is well-known that asymptotic results such as (5.1) provide estimators in terms of a tuning parameters as \( k \) here. Therefore, we introduce an estimator of \( Q_X(\alpha, u) \) in this way. We also provide its asymptotic normality and we describe a bootstrap based methodology to find an optimal solution of \( k \) in our multivariate approach.

### 5.1 Directional multivariate quantile estimator \( \hat{Q}_X(\alpha, u) \)

Note that if one has estimators for all the elements in (4.8), an estimator of (4.9) can be provided. For \( \hat{\gamma}_{u,j} \), \( j = 1, \ldots, d \), we consider the moments estimators given in [Dekkers et al. (1989)],
\[
\hat{\gamma}_{u,j} := M_{k,j}^{(1)} + 1 - \frac{1}{2} \left\{ 1 - \left( M_{k,j}^{(1)} \right)^2 / M_{k,j}^{(2)} \right\}^{-1}.
\] (5.2)
where \( M_{k,j}^{(r)} := \frac{1}{k} \sum_{i=0}^{k-1} \{ \ln([R_{u}X]_{j,n-i:n}) - \ln([R_{u}X]_{j,n-k:n}) \}^{r} \), \( r = 1, 2 \). As noted, \( \hat{\gamma}_{u,j} \) depends on the sample size \( n \) and the tuning parameter \( k \). This is holds also for the rest of estimators introduced in this section, but to avoid cumbersome notation these dependencies are not included in all the expressions. Later in Section 5.3, we discuss how to find an optimal \( k \) based on a joint estimation of the marginal tail indexes.

The estimators for the components of the sequences \( a_{u}(n/k), b_{u}(n/k) \) can be defined as in [De Haan and Huang (1995)],

\[
\hat{a}_{u,j}(n/k) := \lfloor R_{u}X \rfloor_{j,n-k:n} M_{k,j}^{(1)} \max(1,1 - \hat{\gamma}_{u,j}). \\
(5.3)
\]

\[
\hat{b}_{u,j}(n/k) := \lfloor R_{u}X \rfloor_{j,n-k:n}. \\
(5.4)
\]

The estimator of the scalar function \( \rho_{u} \) can be defined by

\[
\hat{\rho}_{u}^{(\theta)} := -\ln \left( \hat{G}_{u} \left( \frac{\theta^{j_{u,j}}}{\hat{\gamma}_{u,j}} ; \ j = 1, \ldots, d \right) \right), \text{ with} \\
- \ln \left( \hat{G}_{u}(x) \right) = \frac{1}{K} \sum_{i=1}^{n} 1 \left\{ d \bigcup_{j=1}^{d} \lfloor R_{u}X \rfloor_{j} > \hat{a}_{u,j}(n/k)x_{u,j} + \hat{b}_{u,j}(n/k) \right\}. \\
(5.5)
\]

Hence, by using (4.8) and Equations (5.2) to (5.5) one can get an estimator for elements of \( \tilde{Q}_{R_{u}X}(\alpha, e, \theta) \), i.e.,

\[
\hat{x}_{u,j}(\alpha, \theta, n/k) := \hat{a}_{u,j}(n/k) \left\{ \frac{\hat{\rho}_{u}^{(\theta)}}{n^{\alpha}} \hat{\gamma}_{u,j} - 1 \right\} + \hat{b}_{u,j}(n/k), \text{ for all } j = 1, \ldots, d. \\
(5.6)
\]

Finally, we get

\[
\hat{Q}_{X}(\alpha, u, \theta, n/k) = R_{u}^{*} \hat{Q}_{R_{u}X}(\alpha, e, \theta, n/k). \\
(5.7)
\]

### 5.2 Asymptotic normality for \( \hat{Q}_{X}(\alpha, u, \theta, n/k) \)

The aim now is to prove normality of \( \hat{Q}_{X}(\alpha, u, \theta, n/k) \) in (5.7) when \( n\alpha \to 0 \), as \( n \to \infty \). An important condition to complete this objective is that \( X \) verifies \( A_{4} \); that is, \( X \) has strong second order multivariate regular variation, which allows a general multivariate version of the Theorem 2.1 in [De Haan and Huang (1995)] and also adapted to the directional framework.

Note that equations (1.8) and (1.9) in [De Haan and Huang (1995)] can be written in the directional framework as follows,

\[
\sqrt{k} \left( \frac{\hat{a}_{u,j}(n/k)}{a_{u,j}(n/k)} - 1 \right) \xrightarrow{d} A_{u,j} \\
\sqrt{k} \left( \frac{\hat{b}_{u,j}(n/k) - b_{u,j}(n/k)}{a_{u,j}(n/k)} - 1 \right) \xrightarrow{d} B_{u,j} \\
\sqrt{k} (\hat{\gamma}_{u,j} - \gamma_{u,j}) \xrightarrow{d} \Gamma_{u,j}, \\
(5.8)
\]

9
Then, \( \Lambda \) converges in distribution to 
\[ \psi(\theta \cdot x) = \psi(\theta \cdot x) \] 
for all \( j \).

Proof of Theorem 5.1 is given in Appendix A. Finally, in Corollary 5.2 below, the asymptotic normality of \( \hat{\theta}_k \) is derived using that orthogonal transformations preserve the result in Theorem 5.1.

**Theorem 5.1.** Let \( s_n := \frac{k}{n^\alpha} \). Suppose that A4 holds, \( \lim_{n \to \infty} \frac{\sqrt{u}}{n^\alpha} \) converges to zero and

\[
\lim_{n \to \infty} s_n \sqrt{k} \cdot \Lambda \left( b_{u_1}(n/k) \right) \psi(u) \left( \frac{s_{n,1}^{u_1} - 1}{\gamma_{u_1}}, \ldots, \frac{s_{n,d}^{u_1} - 1}{\gamma_{u_1}} \right) = 0,
\]

where \( \Lambda(\cdot) \) and \( \psi(\cdot) := \psi(R_u) \) are the functions in Definition 3.5 that ensure A4.

Then,

\[
\sqrt{k} \left( \frac{\hat{\theta}_u \cdot x - x_{u,j}(\theta)}{\hat{\theta}_u \cdot x_{u,j}(\theta)} \right)_{j=1,\ldots,d}
\]

converges in distribution to

\[
(\rho_u(\theta) \theta_j)^{\gamma_{u,j}} \Gamma_{u,j} - (0 \wedge \gamma_{u,j}) A_{u,j} + (0 \wedge \gamma_{u,j})^2 B_{u,j},
\]

for all \( j = 1, \ldots, d \).

Proof of Theorem 5.1 is given in Appendix A. Finally, in Corollary 5.2 below, the asymptotic normality of \( \hat{\theta}_u \) is derived using that orthogonal transformations preserve the result in Theorem 5.1.

**Corollary 5.2.** The asymptotic normality property of the estimator \( \hat{\theta}_u \) is preserved under orthogonal transformations. Therefore (5.7) implies the asymptotic normality of \( \hat{\theta}_u \).

### 5.3 Bootstrap method to estimate the tuning parameter \( k = k(n) \)

From Equations (5.2)-(5.5), it can be viewed the key role of sequence \( k = k(n) \). However, it is not an easy task to establish optimal tuning parameter \( k \) for a given sample size \( n \). This tuning parameter is complicated to tackle in practice and methods to provide optimality are still a matter of research and discussion.

In the recent literature, one can find only heuristic guidelines adapted to each application (e.g. [Cai et al. (2011), Cai et al. (2015), Di Bernardino and Palacios-Rodríguez (2016)]), where the selection of parameters such as \( k \) are mostly based on the identification of a common region of stability across the estimation of particular marginal elements such as marginal tail indexes. For instance \( k \) is selected trough a graphical visualization of the common range provided by the flat behavior around the marginal estimations. On the
other hand, some sophisticated methodologies based on bootstrap have been presented in the univariate case to provide optimality on the choice of \( k \) (e.g. [Draisma et al. (1999), Danielsson et al. (2001), Ferreira et al. (2003), Qi (2008)]). Therefore, how to select an optimal value of \( k \) to perform estimation in the multivariate context? To achieve this goal, it is necessary to overcome the lack of a total order in \( \mathbb{R}^d \), for \( d \geq 2 \). In this sense, we will use the orthant order in [Torres et al. (2015)], which is a partial order in \( \mathbb{R}^d \) based on Definition 2.1. For a fixed direction \( u \),

\[
x \preceq_u y, \quad \text{if and only if,} \quad C^u_x \supseteq C^u_y,
\]

where \( x, y \in \mathbb{R}^d \) and \( C^u_x \) is as in Definition 2.2. Equivalently,

\[
x \preceq_u y, \quad \text{if and only if,} \quad R^u_x \leq R^u_y,
\]

where the inequality on the right side is component-wise. Our proposal is based on a multivariate adaptation of the univariate method introduced in [Danielsson et al. (2001)] and it is described in the following pseudo-algorithm.

**Step 1.** Pre-rotate the sample to generate \( \{R^u x_1, \ldots, R^u x_n\} \) and center that with respect to its mean.

**Step 2.** Set \( m_1 = \lfloor n^{1-\epsilon} \rfloor \) for some \( \epsilon \in (0, 1/2) \), where \( \lfloor \cdot \rfloor \) denotes the integer part function. Draw a large number \( B_1 \) of bootstrap samples of size \( m_1 \) and order each of them according to (5.10), dropping the observations with non-positive components. Denote by \( Err_j(m_1, b_1, k_j) \) the error obtained in each marginal \( j = 1, \ldots, d \), where \( k_j \) varies from 1 to \( m_1 - 1 \),

\[
Err_j(m_1, b_1, k_j) := \left( M^{(2)}_{k_j,j} - 2 \left( M^{(1)}_{k_j,j} \right)^2 \right)^2, \quad b_1 = 1, \ldots, B_1.
\]

Then, determine the value \( k_j(m_1) \) that minimizes

\[
\frac{1}{B_1} \sum_{b_1=1}^{B_1} Err_j(m_1, b_1, k_j).
\]

**Step 3.** Set \( m_2 = \lfloor m_1^2/n \rfloor \), and repeat Step 2 to obtain \( k_j(m_2) \).

**Step 4.** Estimate the associated marginal second order tail index \( \pi_j \) in Corollary 3.3 by

\[
\hat{\pi}_j = \log \left( \frac{k_j(m_1)}{-2 \log(m_1) + 2 \log(k_j(m_1))} \right),
\]

which is a consistent estimator, (see [Qi (2008)]).

**Step 5.** The optimal selection for \( k = k(n) \) is given by,

\[
\hat{k}(n) := \frac{1}{d} \sum_{j=1}^{d} \frac{k_j(m_1)}{k_j(m_2)} \left( 1 - \frac{1}{\pi_j} \right)^{1/(2\pi_j-1)}.
\]

Remark 2 in [Qi (2008)] should be applied for \( n < 2000/2^d \).
6 Illustrative example

Now, we illustrate the estimation methodology introduced in Section 5 by using a well-known distribution: the \( d \)-dimensional \( t \)-distribution with d.f. \( \nu \), which satisfies A1-A4. The multivariate (and marginal) regular variation indexes are \((\gamma, \pi) = (\gamma_{u,j}, \pi_{u,j}) = (1/\nu, -2/\nu)\) (see [Hult and Lindskog (2002)], [Hua and Joe (2011)]). To derive the theoretical DMQ for any direction \( u \), it is necessary to recall Lemma 3.1 in [Hult and Lindskog (2002)] for elliptical distributions (see Lemma A.5 in Appendix A).

Lemma A.5 in Appendix A establishes that \( R_u X \) is again a multivariate \( t \)-distribution with the same d.f. \( \nu \), but with location and scale given by \( \mu_u = R_u \mu, \Sigma_u = R_u \Sigma R_u' \). Thus, tail indexes remain invariant for all \( u \) and it makes this model suitable to perform simulation analysis in any direction and/or in different dimensions. Some results in 2D and 3D scenarios are derived using the following \( t \)-distributions,

\[
\begin{align*}
\mu &= [0, 0]'^t, & \Sigma &= \begin{bmatrix} 5 & 0.1 \\ 0.1 & 1 \end{bmatrix}, & \nu &= 3. \\
\mu &= [0, 0, 0]'^t, & \Sigma &= \begin{bmatrix} 5 & 2.44 & -1.88 \\ 2.44 & 2.12 & 0.04 \\ -1.88 & 0.04 & 2.36 \end{bmatrix}, & \nu &= 4.
\end{align*}
\] (6.1)

The main goal of this section is to illustrate the differences and the importance of the directions in our methodology, as well as, to show the performance of the estimation method for \( Q_X(\alpha, u) \). Therefore, the initial analysis over \( X \) is performed in direction \( e \). Later, it is considered the direction given by the main axis of the elliptical random vector, which is equivalent to the vector characterizing the first principal component (FPC). We focus the initial part of the study to the bivariate case and after we present the results for the case \( d = 3 \).

Figure 1 shows in black the theoretical curves \( Q_X(\alpha, e) \equiv \{ x \mid F_e(x) = 1 - \alpha \} \) for three different \( \alpha \) values (0.7, 0.4, 0.1). It is also displayed in red the theoretical curves \( Q_X(\alpha, FPC) \equiv R_{FPC}' \{ x \mid F_{FPC}(x) = 1 - \alpha \} \) for the same \( \alpha \)'s, (in this case the theoretical FPC is (0.9997 , 0.025)). These non-high level curves show visual improvements of the extreme detection through directional analysis, since FPC takes into account the shape of the data, showing better visual results than the classical direction \( e \).

Now, we proceed to describe step by step all the necessary elements for the estimation of the DMQ at high levels. Our simulations study has been done for different sample sizes, but we present two important cases: (1) \( n = 500 \) (“small”), and (2) \( n = 5000 \) (“large”). Firstly, we present the procedure in the classical direction \( e \), when \( \alpha = 1/n \) and after we show the results for the FPC direction.

1. **Tuning parameter, \( k(n) \):** We implement the methodology described in Section 5 by considering 1000 bootstrap samples to perform the selection of this parameter. To describe the distribution of this selection, Figure 2 summarizes the estimations \( \hat{k}(n) \) obtained after 100 iterations, i.e., by generating 100 different samples of the \( t \)-model each sample size \( n \in \{500, 5000\} \). It is possible to visualize the behavior \( k \to \infty \) and \( k/n \to 0 \) as \( n \to \infty \).

2. **Marginal tail index estimation, \( \gamma_{u,j} \) and sequences of normalization, \( a_{u,j}, b_{u,j} \):** The marginal tail indexes are estimated through the moments estimator in (5.2) and the optimal \( k \) provided in (5.12).
Figure 3 displays the results obtained for the ratio of $\hat{\gamma}_{e,1}/\gamma_{e,1}$ when the estimation is iterated 100 times for each sample size, i.e., 100 samples from the $t-$model are drawn, then the bootstrap selection of $k$ is performed and its corresponding $\hat{\gamma}_{e,j}$, $j = 1, 2$ are selected. We present the results for the first marginal given that it possesses more variance. Results for the second marginal are similar.

Then, the bootstrap procedure improves the estimation of $\hat{\gamma}_{e,j}$, $j = 1, \ldots, d$ and the tractability of $k$ for real applications. Also, once $\gamma_{u,j}$, $j = 1, \ldots, d$ are estimated, it is possible to complete the estimation of the sequences of normalization $a_{u,j}$ and $b_{u,j}$ through (5.3) and (5.4).

3. The scalar function $\rho_u(\theta)$: Note that (4.7) uses the function $-\ln(G_u(\cdot))$, which is the tail function of the multivariate extreme value distribution $G_u$. The theoretical tail function for a multivariate $t-$distribution is provided by using Theorem 2.3 [Nikoloulopoulos et al. (2009)]. For sake of clarity this result is recalled in Theorem A.6 in Appendix A.

Therefore for any direction $u$, we can calculate $\rho_u(\theta)$ and $\hat{\rho}_u(\theta)$ by using Lemma A.5, Theorem A.6 and Equation (5.5). Figure 4 shows the theoretical curves (magenta) and the estimated ones (blue) of $\rho_u(\theta)$. We can appreciate a good performance of the estimation for both sample sizes. Furthermore, as expected, the larger the sample size, the better the performance of the estimator.

Remark 2. Figure 4 and Figure 10 plotted $\rho(\theta)$ with the argument $\theta$ described in terms of the $(d - 1)$ angles of its polar representation.

4. The directional quantile curve $Q_X(1/n, e)$: Having all the previous ingredients, both theoretical and estimated results for $Q_X(1/n, e)$ can be calculated. We use the $t-$model to simulate 100 samples and we apply items 1 to 3 to present more than a single estimation and to construct confidence bands for the performance of the procedure in Section 5. The results are displayed in Figure 5.

Then, for both sample sizes $n \in \{500, 5000\}$, Figure 5 shows the theoretical quantiles plotted in black\(^2\), theoretical approximations through the tail function are in magenta, medians point by point of the estimated curves are in blue and confidence regions from 15% to 85% are shaded in green. We can appreciate the accuracy of the estimations of $Q_X(1/n, e)$. It is also possible to see from the confidence regions that uncertainty grows as the extreme level $\alpha = 1/n$ approaches to zero.

Now, for the $FPC$ direction, we have that the theoretical $FPC$ is (0.9997, 0.025) and the theoretical parameters to evaluate the performance of the directional estimation are,

$$\mu_{FPC} = [0, 0]', \quad \Sigma_{FPC} = \begin{bmatrix} 3.0001 & 2.0025 \\ 2.0025 & 2.9999 \end{bmatrix}.$$  

Thereby, we calculate theoretical and estimated solutions for $Q_{R_{FPC}X}(1/n, e)$, where the estimation is done by repeating the procedure described in items 1 to 4. We have considered the same sample sizes and Figure 6 presents the results in the same colors as before. Finally, applying the inverse rotation indicated in (5.7), i.e., $R_{\text{FPC}}'$, Figure 7 presents the results for $Q_X(1/n, (0.9997, 0.025))$. We can appreciate the good performance of the estimators and the improvements on the visualization of extremes, which are more in consonance with the shape of the data.

Similar results can be shown for the trivariate case presented in (6.1). Firstly, Figure 8 displays the theoretical quantile surfaces at level $\alpha = 0.1$: The black quantile is performed in the classical direction $e$ and the red

---

\(^2\)Theoretical quantiles are the computational $1 - \alpha$ isocurves and isosurfaces of the multivariate $t-$distribution.
quantile is in the *FPC* direction. These surfaces also shown more concordance with the shape of the data when the *FPC* direction is considered.

Then, we proceed to illustrate the results through the bootstrap procedure from Figure 9 to Figure 11, but just in the *FPC* direction, \((u = (0.8417, 0.4202, -0.3392))\), and without the construction of the confidence bands, i.e., an output after a single simulation due to computational difficulties building the bands in the trivariate case. In this case the considered sample sizes are \(n = 500\) and \(n = 50000\). Figure 9 is the only figure including replications of the process to describe the distribution of the tuning parameter \(k(n)\) selected by bootstrap.

Figure 10 displays the behavior of \(\rho_u(\cdot)\). The estimation is accurate in general, but specially in the central part of the parametric domain of \(\theta\), which is very important to describe properly the behavior of the directional quantile because it represents the region with curvature in the quantile surface.

Figure 11 displays the final estimation of \(Q_X(1/n, FPC)\). The visual performance is quite accurate, but a measurement to assess the quality of these results is necessary. For this reason, we have calculated the relative error in the point of maximum convexity between estimated and asymptotic theoretical quantiles, which using (4.8) and (5.6) can be written as,

\[
\frac{||\hat{x}_u(\alpha, \theta) - \tilde{x}_u(\alpha, \theta)||}{||\tilde{x}_u(\alpha, \theta)||},
\]

(6.2)

where \(\theta = (1/\sqrt{d}, \ldots, 1/\sqrt{d})\). It can be seen the good quality of the procedure in Figure 12.

7 Real case of study

Finance is one of the fields where risk analysis is very important. Specifically, when the object of analysis is a portfolio, the relevance of the analysis in the direction of the vector of weights of investment was pointed out in [Torres et al. (2015)]. Also, this field requires both estimation approaches, *in-sample* and *out-sample*.

In this section, the aim is to highlight that the methodology presented in this paper offers an alternative for decision making when investment allocation and particular management criteria are considered. Therefore, we summarize a real case analyzed in [He and Einmahl (2017)], describing the main differences between a general analysis of risks through close trimming contours and the specific directional analysis provided in this paper.

[He and Einmahl (2017)] have analyzed the daily market return of a portfolio composed by three international indices from July 2nd, 2001 to June 29th, 2007. The indices are the S&P 500 index from USA, the FTSE 100 index from UK and the Nikkei 225 index from Japan. The data contains 1564 observation and it is well-known from the financial literature that stocks returns usually reject the serial independence. Hence, one cannot work with the raw data since the assumption of independent and identically distributed observations may be inappropriate. However, [He and Einmahl (2017)] have filtered the data to solve this issue. Each time series of market returns was modeled by an exponential generalized auto-regressive conditional heteroscedasticity GARCH(1,1) and fitted the parameters by maximizing the quasi-likelihood to obtain the filtered returns, also called *innovations*, which were modeled by a \(t\)−distribution.

As we pointed out in Section 1, methods based on depth and density contours inherently consider the whole set of directions, which provides an overall analysis. However, an analysis considering particular criteria or
manager preferences is outside of the aim of those methods. [He and Einmahl (2017)] used the Tukey depth to build out-sample trimming contours for the innovations of the three leading indices (S&P 500, FTSE 100, Nikkei 225) and they suggest to consider the big loss in the US market on February 27th, 2007 as an outlier by considering its innovation far enough based on the high level contour with $\alpha = 1/10000$. Figure 13 displays the results in spherical coordinates to support their claim.

Thus, the region above the surface in Figure 13 accumulates an approximated probability of $1/10000$ in its global analysis. However, the directional approach concentrates the $\alpha$—level of probability in the set $\{x \in \mathbb{R}^d : \mathbb{P}(c_x^u) \geq 1 - \alpha\}$, where $u$ allows to incorporate the manager preferences. Then, the QR orthant concept given in Definition 2.2 provides a naive directional outlier identification analog to the rule in [He and Einmahl (2017)], because the QR orthant in direction $u$ divides the $\mathbb{R}^d$ space in $2^d$ parts which leads to the value of $\alpha = 8/10000 = 1/1250$.

For instance, if the criteria of analysis is the portfolio weights of investment and considering the latter $\alpha$—level, two examples can be chosen to highlight differences: 1) the naive diversification of the portfolio, i.e., $u = e$ and 2) an investment with large participation in the U.S. market, regular in the U.K. market and small in the Japanese market $u = (0.6, 0.35, 0.05)$. The directional analysis is carried out over the filtered losses, i.e., the negative of the innovations. As it was made in [He and Einmahl (2017)], the filtered losses can be fitted by a multivariate $t$—Student distribution, which allows us to perform the directional approach in twofold:

1) A semi-parametric method; that is, the parameters estimation of the $t$—distributed negative innovations and the calculation of the theoretical directional iso-surfaces by using the tools presented in Section 6.

2) The full non-parametric bootstrap-based method presented in Section 5.

Figure 14 focused in the direction $e$ and it can be seen that the big U.S. lost has not been identified as an outlier point because is not contained in the critical region, which suggests a leverage effect that cannot be underestimated for this particular investment. Meanwhile, Figure 15 shows that the so called big U.S. lost is indeed above the critical layer for the investment weights $u = (0.6, 0.35, 0.05)$, which leads to a similar interpretation of outlier to the one provided by [He and Einmahl (2017)].

[He and Einmahl (2017)] commented that “Neglecting the joint behaviour can lead to an overestimated diversifiability of risks across international markets and, therefore, underestimation of systematic risk”. In this sense, we add that the directional approach allows to include external information or manager criteria providing a joint local analysis that could lead to different conclusions that those in an overall joint behavior.

Also, [He and Einmahl (2017)] note that “an outlier in a high dimensional space is not necessarily an outlier in its subspaces with reduced dimensions”. We also pointed out that an outlier in one direction is not necessarily an outlier in other directions.

8 Conclusions

This paper presented an out-sample characterization of the $Q_X(\alpha, u)$, recently introduced in [Torres et al. (2015), Torres et al. (2016)]. Necessary conditions over $X$ to ensure a feasible estimation of the $DMQ$ at high levels was also presented and the proposed estimator integrates different asymptotic results from the
univariate and the multivariate extreme value theory through a heuristic parametrization in polar coordinates in \( \mathbb{R}^d \).

A usual implication from the asymptotic results that consider the extreme value theory is the participation of tuning parameters that play a key role on the estimation. Therefore, we introduced a bootstrap-based method to find an optimal solution for the tuning parameter in the multivariate framework of this work, joint to a non-parametric method to complete the estimation of the \( DMQ \). Finally, the asymptotic normality of the estimator was derived.

Based on the multivariate \( t \)–distribution, illustrations of the estimation procedure in dimensions 2 and 3 were shown. This family of distributions possesses properties such as heavy tails and closure under rotations, which provides a good example for comparing theoretical and estimated solutions. In the examples described, it was appreciated the good performance of the estimation process. Finally, a real case study to identify directional outliers in the filtered losses in a portfolio was performed. This example suggests that joint local analysis could lead to different conclusions than overall joint behavior, which provides a wider vision to the fact that neglecting the joint behaviour can lead to an overestimated diversifiability of risks across international markets.

A future interesting work is to study a multivariate risk measure in the \textit{out-sample} framework based on \( DMQ \) to analyze risks in different real case scenarios. For instance in environmental problems, where it is demanding high level quantiles in the multivariate framework.

**Appendices**

**A Auxiliary results and proofs**

This section is devoted to the proofs of main results of this paper. Furthermore, different necessary results are introduced below.

**Proof of Proposition 3.2.** An orthogonal transformation is a measurable function. Hence for each Borel set \( B \), we have that \( QB := \{QX|X \in B\} \) is a Borel set. On the other hand, we denote \( \mathbb{P} \) and \( \tilde{\mathbb{P}} \) as the probability measures of \( X \) and \( QX \), respectively. For a Borel set \( B \), we have that,

\[
\mathbb{P}[B] = \tilde{\mathbb{P}}[QB] \quad \text{or analogously} \quad \mathbb{P}[Q'B] = \tilde{\mathbb{P}}[B].
\]

Therefore, we obtain that the random vector \( QX \) is also regularly varying with tail index \( \gamma \) since

\[
t\tilde{\mathbb{P}}[(\phi(t))^{-1}QX \in \cdot] \xrightarrow{\text{w}} \tilde{\mu}(\cdot) := \mu(Q'\cdot).
\]

\( \square \)
Proof of Proposition 3.6. As before, we denote $\mathbb{P}$ and $\hat{\mathbb{P}}$ as the probability measures of $X$ and $QX$, respectively. Then, we get for any Borel set $B$,

$$
\frac{t \hat{\mathbb{P}} \left[ (\phi(t))^{-1} QX \in B \right] - \hat{\mu}(B)}{\Lambda(\phi(t))} = \frac{t \mathbb{P} \left[ (\phi(t))^{-1} X \in Q'B \right] - \mu(Q'B)}{\Lambda(\phi(t))}.
$$

Hence,

$$
\frac{t \hat{\mathbb{P}} \left[ (\phi(t))^{-1} QX \in B \right] - \hat{\mu}(B)}{\Lambda(\phi(t))} \to \hat{\psi}(B) := \psi(Q'B).
$$

The proof is analogous for the strong condition. \qed

To prove Theorem 5.1, we need to introduce in the following the directional multivariate versions of the four Lemmas 2.1-2.4 in [De Haan and Huang (1995)].

Lemma A.1. If $-\ln G_u$ has continuous first-order derivatives $(-\ln G_u)_i$, $i = 1, \ldots, d$, then

$$
\sqrt{k} (\hat{\rho}_u(\theta) - \tilde{\rho}_u(\theta))
$$

converges to

$$
V_u \left( \frac{\hat{\theta}_{u,j} - 1}{\gamma_{u,j}} ; j = 1, \ldots, d \right) + \sum_{i=1}^{d} \left( -\ln G_u \right)_i \left( \frac{\hat{\theta}_{u,j} - 1}{\gamma_{u,j}} ; j = 1, \ldots, d \right) \left[ \int_1^{\theta_i} (\ln t) t^{\gamma_{u,i}-1} dt \right] \Gamma_{u,i}.
$$

Lemma A.2. Under the conditions of Theorem 5.1,

$$
\sqrt{k} \left( \frac{\hat{Q}_u X(\alpha, e, \theta, n/k) - \hat{x}_{u,j}(\alpha, \theta)}{\hat{a}_{u,j}(n/k) \int_1^{s_n} t^{\gamma_{u,j}-1} (\log t) dt} \right)
$$

converges in distribution to

$$
(\rho_u(\theta) \theta_j)^{\gamma_{u,j}} \Gamma_{u,j} - (0 \wedge \gamma_{u,j}) A_{u,j} + (0 \wedge \gamma_{u,j})^2 B_{u,j},
$$

for all $j = 1, \ldots, d$.

Lemma A.3. Under the conditions of Theorem 5.1,

$$
\lim_{n \to \infty} \sqrt{k} (\rho_u(\theta) - \tilde{\rho}_u(\theta)) = 0,
$$

locally uniformly.

Lemma A.4. Under the conditions of Theorem 5.1,

$$
\lim_{n \to \infty} \frac{\sqrt{k} (\hat{x}_{u,j}(\alpha, \theta) - x_{u,j}(\alpha, \theta))}{\hat{a}_{u,j}(n/k) s_n^{\gamma_{u,j}+1} \psi \left( \left( \frac{\hat{\gamma}_{u,j} - 1}{\gamma_{u,j}} \right) / \hat{\gamma}_{u,j} \right)} = 0,
$$

locally uniformly, for all $j = 1, \ldots, d$. 17
The proofs of these lemmas work in a similar way as in [De Haan and Huang (1995)] but considering the arrangements due to the directional multivariate framework, then they are omitted here. Now, by using Lemmas A.1-A.4, it can be proved the main Theorem 5.1.

**Proof of Theorem 5.1.** Lemma A.1 proves the asymptotic normality of the standardized difference $\hat{\rho}_u(\theta) - \hat{\rho}_u(\theta)$. This implies the asymptotic normality of the standardized difference $Q_{R_\alpha X}(\alpha, \theta, n/k) - \bar{x}_{u,j}(\alpha, \theta)$ in Lemma A.2.

Also, Lemma A.3 proves the convergence to zero of the standardized difference $\rho_u(\theta) - \hat{\rho}_u(\theta)$, which helps to prove Lemma A.4 where the convergence to zero of the standardized difference $\bar{x}_{u,j}(\alpha, \theta) - x_{u,j}(\alpha, \theta)$ is given. Thus, Lemma A.2 and Lemma A.4 complete the result in Theorem 5.1. □

We recall below an useful result for elliptical distribution (see Lemma 3.1 in [Hult and Lindskog (2002)]).

**Lemma A.5.** If $X$ has an elliptical distribution and decomposition given by

$$X \overset{d}{=} \mu + \Sigma^{1/2} r Z,$$

where $r$ is a random variable independent from the random vector $Z$, which is uniformly distributed in the unit circle of dimension $d$, $\mu$ a location parameter and $\Sigma$ a matrix indicating scale. Then, for any orthogonal matrix $Q$, $QX$ has an elliptical distribution with associated decomposition given by

$$QX \overset{d}{=} Q\mu + Q\Sigma^{1/2} r Z.$$

Moreover, its marginals are the associated univariate elliptical distributions with parameters of location and scale given by $(Q\mu)_j$ and $(Q\Sigma^{1/2} r Z)_j$, for $j = 1, \ldots, d$.

In the following result the theoretical tail function for a multivariate $t$–distribution is obtained.

**Theorem A.6 ([Nikoloulopoulos et al. (2009)] Theorem 2.3).** The theoretical tail function of $T_{0,\Sigma,\nu}^d(\cdot)$, a $d$–dimensional $t$–distribution with d.f. $\nu$, location parameter $\mu = 0$ and scale parameter $\Sigma$, is given by,

$$- \ln \left( G \left( \frac{Z^2 - 1}{\gamma} \right) \right) = \sum_{j=1}^{d} z_j^{-1} T_{0,Q_j,\nu+1}^{-1} \left( \sqrt{\frac{\nu + 1}{\nu}} \left[ \left( \frac{z_j}{z_j} \right)^{1/\nu} - r_{i,j} \right] ; i \neq j \right),$$

where $r_{i,j}$ are the correlations between the components $i, j$. $T_{0,Q_j,\nu+1}^{-1}(\cdot)$ is a $t$–distribution in dimension $d - 1$ (removing the $j$–component), with d.f. $\nu + 1$, location parameter $\mu = 0$ and scale parameter given by,

$$Q_j = \begin{bmatrix}
1 & \cdots & r_{1,j-1,j} & r_{1,j+1,j} & \cdots & r_{1,d,j} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
r_{j-1,j,j} & 1 & r_{j-1,j+1,j} & \cdots & r_{j-1,d,j} \\
r_{j+1,j,j} & r_{j+1,j-1,j} & 1 & \cdots & r_{j+1,d,j} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
r_{d-1,j,j} & r_{d-1,j-1,j} & r_{d-1,j+1,j} & \cdots & 1
\end{bmatrix},$$

where $r_{i,l,j} = \frac{r_{i,l}-r_{i,j}r_{l,j}}{\sqrt{1-r_{i,j}^2} \sqrt{1-r_{l,j}^2}}$ for $i, l \neq j$ (see also [Opitz (2013)]).
Figure 1: Classical and $FPC$ directional quantiles for $\alpha = 0.7, 0.4$ and 0.1.
Figure 2: Distribution of the bootstrap estimation of $k$

(A) $n = 500$  
(B) $n = 5000$

Figure 3: Ratio $\hat{\gamma}_{e,1}/\gamma_{e,1}$

(A) $n = 500$  
(B) $n = 5000$
Figure 4: Theoretical and estimated curves $\rho_e(\theta)$

Figure 5: Estimations for $Q_X(1/n, e)$
Figure 6: Estimations for \( Q_{R_uX}(1/n, e) \), \( u = FPC \)

(A) \( n = 500 \)  
(B) \( n = 5000 \)

Figure 7: Estimations for \( Q_{X}(1/n, FPC) \)

(A) \( n = 500 \)  
(B) \( n = 5000 \)
Figure 8: Classical and FPC directional quantiles for $\alpha = 0.1$

Figure 9: Distribution of the bootstrap estimation of $k$

(A) $n = 500$

(B) $n = 50000$
Figure 10: Theoretical and estimated curves $\rho_u(\theta), u = FPC = (0.8417, 0.4202, -0.3392)$

Figure 11: Estimations for $Q_X(1/n, FPC)$
Figure 12: Relative error made in $Q_X(1/n, FPC)$ in the point $\theta = (1/\sqrt{d}, \ldots, 1/\sqrt{d})$

Figure 13: Outlier criteria through Tukey depth trimming for $\alpha = 1/10000$
Figure 14: Directional portfolio criteria, \( u = e \) and \( \alpha = 1/1250 \).

Figure 15: Directional portfolio criteria, \( u = (0.6, 0.35, 0.05) \) and \( \alpha = 1/1250 \).
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