Solving the At-Most-Once Problem with Nearly Optimal Effectiveness

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Abstract

We present and analyze a wait-free deterministic algorithm for solving the at-most-once problem: how \( m \) shared-memory fail-prone processes perform asynchronously \( n \) tasks at most once. Our algorithmic strategy provides for the first time nearly optimal effectiveness, which is a measure that expresses the total number of tasks completed in the worst case. The effectiveness of our algorithm equals \( n - 2m + 2 \). This is up to an additive factor of \( m \) close to the known effectiveness upper bound \( n - m + 1 \) over all possible algorithms and improves on the previously best known deterministic solutions that have effectiveness only \( n - \log m \cdot o(n) \). We also present an iterated version of our algorithm that for any \( m = O\left(\frac{n^{3+\epsilon}}{\log n}\right) \) is both effectiveness-optimal and work-optimal, for any constant \( \epsilon > 0 \). We then employ this algorithm to provide a new explicit algorithmic solution for the Write-All problem which is work optimal for any \( m = O\left(\frac{n^{3+\epsilon}}{\log n}\right) \) improving the previously best known result of \( m = O\left(\frac{n^{4}}{\log n}\right) \).

Keywords: At-most-once problem, Write-All, I/O automata, asynchronous shared memory.

1 Introduction

The at-most-once problem for asynchronous shared memory systems was introduced by Kentros et al. \[15\] as the problem of performing a set of \( n \) jobs by \( m \) fail-prone processes while maintaining at-most-once semantics.

The at-most-once semantic for object invocation ensures that an operation accessing and altering the state of an object is performed no more than once. This semantic is among the standard semantics for remote procedure calls (RPC) and method invocations and it provides important means for reasoning about the safety of critical applications. Uniprocessor systems may trivially provide solutions for at-most-once semantics by implementing a central schedule for operations. The problem becomes very challenging for autonomous processes in a system with concurrent invocations on multiple objects.

Perhaps the most important question in this area is devising algorithms for the at-most-once problem with good effectiveness. The complexity measure of effectiveness \[15\] describes the number of jobs completed (at-most-once) by an implementation, as a function of the overall number of jobs \( n \), the number of processes \( m \), and the number of crashes \( f \). The only deterministic solutions known exhibit very low effectiveness \( \left(\frac{1}{n^{\log m}} - 1\right)^{\log m} \) (see \[15\]) which for most choices of the parameters is very far from optimal (unless \( m = O(1) \)). Contrary to this, the present work presents the first deterministic algorithm for the at-most-once problem which is optimal up to additive factors of \( m \). Specifically our effectiveness is...
The at-most-once problem has also many similarities with the Write-All problem for the shared memory model \([1, 10, 14, 16, 24]\). First presented by Kanellakis and Shvartsman \([14]\), the Write-All problem is concerned with performing each task \textit{at-least-once}. Most of the solutions for the Write-All problem, exhibit super-linear work when \(m \ll n\). Malewicz \([24]\) was the first to present a solution for the Write-All problem that has linear work for a non-trivial number of processors. The algorithm presented by Malewicz \([24]\) has work \(O(n + m^3 \log n)\) and uses test-and-set operations. Later Kowalski and Shvartsman \([16]\) presented a solution for the Write-All problem that for any constant \(\epsilon > 0\) has work complexity \(O(n + m^{3+\epsilon} \log n)\). Their algorithm uses a collection of \(q\) permutations with contention \(O(q \log q)\) for a properly chosen constant \(q\).

We note that the at-most-once problem becomes much simpler when shared-memory is supplemented by some type of read-modify-write operations. For example, one can associate a \textit{test-and-set} bit with each task, ensuring that the task is assigned to the only process that successfully sets the shared bit. An effectiveness optimal implementation can then be easily obtained from any Write-All solution. Thus, in this paper we deal only with the more challenging setting where algorithms use atomic read/write registers.

\textbf{Contributions:} In this paper we present and analyze the algorithm \(\text{KK}_\beta\) that solves the at-most-once problem. The algorithm is parametrized by \(\beta \geq m\) and has effectiveness \(n - \beta - m + 2\). If \(\beta < m\) the correctness of the algorithm is still guaranteed, but the termination of the algorithm cannot be guaranteed.
For $\beta = m$ the algorithm has optimal effectiveness of $n - 2m + 2$ up to an additive factor of $m$. Note that the upper bound for the effectiveness of any algorithm is $n - f$ [15], where $f \leq m - 1$ is the number of failures in the system. We further prove that for $\beta \geq 3m^2$ the algorithm has work complexity $O(nm \log n \log m)$. We use algorithm KK$_3$ with $\beta = 3m^2$, in order to construct an iterated version of our algorithm which for any constant $\epsilon > 0$, has effectiveness of $n - O(m^2 \log n \log m)$ and work complexity $O(n + m^{3+\epsilon} \log n)$. This is both effectiveness-optimal and work-optimal for any $m = O(3\sqrt[3]{n}/\log n)$. We note that our solutions are deterministic and assume worst-case behavior. In the probabilistic setting Hillel [13] shows that optimal effectiveness can be achieved with expected work complexity $O(nm^2 \log m)$.

We then demonstrate how to use the iterated version of our algorithm in order to solve the Write-All problem with work complexity $O(n + m^{3+\epsilon} \log n)$ for any constant $\epsilon > 0$. Our solution improves on the algorithm of Malewicz [24], which is the best known explicit result, in two ways. Firstly our solution is work optimal for a wider range of $m$, namely for any $m = O(3\sqrt[3]{n}/\log n)$ compared to the $m = O(\sqrt[3]{n}/\log n)$ of Malewicz. Secondly our solution does not assume the test-and-set primitive used by Malewicz [24], and relies only on atomic read/write memory. Note that there is a Write-All algorithm due to Kowalski and Shvartsman [16], which is work optimal for a wider range of processors $m$ than our algorithm, specifically for $m = O(2\sqrt{n})$. However, their algorithm uses a collection of $q$ permutations with contention $O(q \log q)$, while it is not known to date how to construct such permutations in polynomial time. Thus their result is so far existential, while ours is explicit.

2 Model, Definitions, and Efficiency

We define our model, the at-most-once problem, and measures of efficiency.

2.1 Model and Adversary

We model a multi-processor as $m$ asynchronous, crash-prone processes with unique identifiers from some set $P$. Shared memory is modeled as a collection of atomic read/write memory cells, where the number of bits in each cell is explicitly defined. We use the Input/Output Automata formalism [22, 23] to specify and reason about algorithms; specifically, we use the asynchronous shared memory automaton formalization [9, 23]. Each process $p$ is defined in terms of its states $\text{states}_p$ and its actions $\text{acts}_p$, where each action is of the type input, output, or internal. A subset $\text{start}_p \subseteq \text{states}_p$ contains all the start states of $p$. Each shared variable $x$ takes values from a set $V_x$, among which there is $\text{init}_x$, the initial value of $x$.

We model an algorithm $A$ as a composition of the automata for each process $p$. Automaton $A$ consists of a set of states $\text{states}(A)$, where each state $s$ contains a state $s_p \in \text{states}_p$ for each $p$, and a value $v \in V_x$ for each shared variable $x$. Start states $\text{start}(A)$ is a subset of $\text{states}(A)$, where each state contains a $\text{start}_p$ for each $p$ and an $\text{init}_x$ for each $x$. The actions of $A$, $\text{acts}(A)$ consists of actions $\pi \in \text{acts}_p$ for each process $p$. A transition is the modification of the state as a result of an action and is represented by a triple $(s, \pi, s')$, where $s, s' \in \text{states}(A)$ and $\pi \in \text{acts}(A)$. The set of all transitions is denoted by $\text{trans}(A)$. Each action in $\text{acts}(A)$ is performed by a process, thus for any transition $(s, \pi, s')$, $s$ and $s'$ may differ only with respect to the state $s_p$ of process $p$ that invoked $\pi$ and potentially the value of the shared variable that $p$ interacts with during $\pi$. We also use triples $(\{\text{vars}_s\}, \pi, \{\text{vars}_{s'}\})$, where $\text{vars}_s$ and $\text{vars}_{s'}$ are subsets of variables in $s$ and $s'$ respectively, as a shorthand to describe transitions without having to specify $s$ and $s'$ completely; here $\text{vars}_s$ and $\text{vars}_{s'}$ contain only the variables whose value changes as the result of $\pi$, plus possibly some other variables of interest.

An execution fragment of $A$ is either a finite sequence, $s_0, \pi_1, s_1, \ldots, \pi_r, s_r$, or an infinite sequence, $s_0, \pi_1, s_1, \ldots, \pi_r, s_r, \ldots$, of alternating states and actions, where $(s_k, \pi_{k+1}, s_{k+1}) \in \text{trans}(A)$ for any $k \geq 0$. If $s_0 \in \text{start}(A)$, then the sequence is called an execution. The set of executions of $A$ is $\text{execs}(A)$. We say that execution $\alpha$ is fair, if $\alpha$ is finite and its last state is a state of $A$ where no locally controlled action is enabled, or $\alpha$ is infinite and every locally controlled action $\pi \in \text{acts}(A)$ is performed infinitely many times or there are infinitely many states in $\alpha$ where $\pi$ is disabled. The set of fair executions of $A$ is $\text{fairexecs}(A)$.
An execution fragment $\alpha'$ extends a finite execution fragment $\alpha$ of $A$, if $\alpha'$ begins with the last state of $\alpha$. We let $\alpha \cdot \alpha'$ stand for the execution fragment resulting from concatenating $\alpha$ and $\alpha'$ and removing the (duplicated) first state of $\alpha'$.

For two states $s$ and $s'$ of an execution fragment $\alpha$, we say that state $s$ precedes state $s'$ and we write $s < s'$ if $s$ appears before $s'$ in $\alpha$. Moreover we write $s \leq s'$ if state $s$ either precedes state $s'$ in $\alpha$ or the states $s$ and $s'$ are the same state of $\alpha$. We use the term precedes and the symbols $<$ and $\leq$ in a same way for the actions of an execution fragment. We use the term precedes and the symbol $<$ if an action $\pi$ appears before a state $s$ in an execution fragment $\alpha$ or if a state $s$ appears before an action $\pi$ in $\alpha$. Finally for a set of states $S$ of an execution fragment $\alpha$, we define as $s_{\text{max}} = \max S$ the state $s_{\text{max}} \in S$, s.t. $\forall s \in S, s \leq s_{\text{max}}$ in $\alpha$.

We model process crashes by action $\text{stop}_p$ in $\text{acts}(A)$ for each process $p$. If $\text{stop}_p$ appears in an execution $\alpha$ then no actions $\pi \in \text{acts}_p$ appear in $\alpha$ thereafter. We then say that process $p$ crashed. Actions $\text{stop}_p$ arrive from some unspecified external environment, called adversary. In this work we consider an omniscient, online adversary [14] that has complete knowledge of the algorithm executed by the processes. The adversary controls asynchrony and crashes. We allow up to $f < m$ crashes. We denote by $\text{fairexecs}_f(A)$ all fair executions of $A$ with at most $f$ crashes. Note that since the processes can only communicate through atomic read/write operations in the shared memory, all the asynchronous executions are linearizable. This means that concurrent actions can be mapped to an equivalent sequence of state transitions, where only one process performs an action in each transition, and thus the model presented above is appropriate for the analysis of a multi-process asynchronous atomic read/write shared memory system.

2.2 At-Most-Once Problem, Effectiveness and Complexity

We consider algorithms that perform a set of tasks, called jobs. Let $A$ be an algorithm specified for $m$ processes with ids from set $\mathcal{P} = \{1\ldots m\}$, and for $n$ jobs with unique ids from set $\mathcal{J} = \{1\ldots n\}$. We assume that there are at least as many jobs as there are processes, i.e., $n \geq m$. We model the performance of job $j$ by process $p$ by means of action $\text{do}_{p,j}$. For a sequence $c$, we let $\text{len}(c)$ denote its length, and we let $c|\pi$ denote the sequence of elements $\pi$ occurring in $c$. Then for an execution $\alpha$, $\text{len}(\alpha|\text{do}_{p,j})$ is the number of times process $p$ performs job $j$. Finally we denote by $F_\alpha = \{p|\text{stop}_p \text{ occurs in } \alpha\}$ the set of crashed processes in execution $\alpha$. Now we define the number of jobs performed in an execution. Note here that we are borrowing most definitions from Kentros et al. [15].

**Definition 2.1** For execution $\alpha$ we denote by $\mathcal{J}_\alpha = \{j \in \mathcal{J}|\text{do}_{p,j} \text{ occurs in } \alpha \text{ for some } p \in \mathcal{P}\}$. The total number of jobs performed in $\alpha$ is defined to be $\text{Do}(\alpha) = |\mathcal{J}_\alpha|$.

We next define the at-most-once problem.

**Definition 2.2** Algorithm $A$ solves the at-most-once problem if for each execution $\alpha$ of $A$ we have $\forall j \in \mathcal{J} : \sum_{p \in \mathcal{P}} \text{len}(\alpha|\text{do}_{p,j}) \leq 1$.

**Definition 2.3** Let $S$ be a set of elements with unique identifiers. We define as the rank of element $x \in S$ and we write $[x]_S$, the rank of $x$ if we sort in ascending order the elements of $S$ according to their identifiers.

**Measures of Efficiency.** We analyze our algorithms in terms of two complexity measures: effectiveness and work. Effectiveness counts the number of jobs performed by an algorithm in the worst case.

**Definition 2.4** The effectiveness of algorithm $A$ is: $E_A(n, m, f) = \min_{\alpha \in \text{fairexecs}_f(A)}(\text{Do}(\alpha))$, where $m$ is the number of processes, $n$ is the number of jobs, and $f$ is the number of crashes.
A trivial algorithm can solve the at-most-once problem by splitting the $n$ jobs in groups of size $\frac{n}{m}$ and assigning one group to each process. Such a solution has effectiveness $E(n, m, f) = (m - f) \cdot \frac{n}{m}$ (consider an execution where $f$ processes fail at the beginning of the execution).

Work complexity measures the total number of basic operations (comparisons, additions, multiplications, shared memory reads and writes) performed by an algorithm. We assume that each internal or shared memory cell has size $O(\log n)$ bits and performing operations involving a constant number of memory cell costs $O(1)$. This is consistent with the way work complexity is measured in previous related work [14, 16, 24].

**Definition 2.5** The work of algorithm $A$, denoted by $W_A$, is the worst case total number of basic operations performed by all the processes of algorithm $A$.

Finally we repeat here as a Theorem, Corollary 1 from Kentros et al. [15], that gives an upper bound on the effectiveness for any algorithm solving the at-most-once problem.

**Theorem 2.1** from Kentros et al. [15]

For all algorithms $A$ that solve the at-most-once problem with $m$ processes and $n \geq m$ jobs in the presence of $f < m$ crashes it holds that $E_A(n, m, f) \leq n - f$.

## 3 Algorithm $KK_\beta$

Here we present algorithm $KK_\beta$, that solves the at-most-once problem. Parameter $\beta \in \mathbb{N}$ is the termination parameter of the algorithm. Algorithm $KK_\beta$ is defined for all $\beta \geq m$. If $\beta = m$, algorithm $KK_\beta$ has optimal up to an additive factor of $m$ effectiveness. Note that although $\beta \geq m$ is not necessary in order to prove the correctness of the algorithm, if $\beta < m$ we cannot guarantee termination of algorithm $KK_\beta$.

The idea behind the algorithm $KK_\beta$ (see Fig. 1) is quite intuitive and is based on an algorithm for renaming processes presented by Attiya et al. [2]. Each process $p$, picks a job $i$ to perform, announces (by writing in shared memory) that it is about to perform the job and then checks if it is safe to perform it (by reading the announcements other processes made in the shared memory, and the jobs other processes announced they have performed). If it is safe to perform the job $i$, process $p$ will proceed with the $\text{do}_{p,i}$ action and then mark the job completed. If it is not safe to perform $i$, $p$ will release the job. In either case, $p$ picks a new job to perform. In order to pick a new job, $p$ reads from the shared memory and gathers information on which jobs are safe to perform, by reading the announcements that other processes made in the shared memory about the jobs they are about to perform, and the jobs other processes announced they have already performed. Assuming that those jobs are ordered, $p$ splits the set of “free” jobs in $m$ intervals and picks the first job of the interval with rank equal to $p$’s rank. Note that since the information needed in order to decide whether it is safe to perform a specific job and in order to pick the next job to perform is the same, these steps are combined in the algorithm. In Figure 1 we use function $\text{rank}(\text{SET}_1, \text{SET}_2, i)$, that returns the element of set $\text{SET}_1 \setminus \text{SET}_2$ that has rank $i$. If $\text{SET}_1$ and $\text{SET}_2$ have $O(n)$ elements and are stored in some tree structure like red-black tree or some variant of $B$-tree, the operation $\text{rank}(\text{SET}_1, \text{SET}_2, i)$, costs $O(|\text{SET}_2| \log n)$ assuming that $\text{SET}_2 \subseteq \text{SET}_1$. We will prove that the algorithm has effectiveness $n - (\beta + m - 2)$. For $\beta = O(m)$ this effectiveness is asymptotically optimal for any $m = o(n)$. Note that by Theorem 2.1 the upper bound on effectiveness of the at-most-once problem is $n - f$, where $f$ is the number of failed processes in the system. Next we present algorithm $KK_\beta$ in more detail.

**Shared Variables.** $\text{next}$ is an array with $m$ elements. In the cell $\text{next}_q$ of the array process $q$ announces the job it is about to perform. From the structure of algorithm $KK_\beta$, only process $q$ writes in cell $\text{next}_q$. On the other hand any process may read cell $\text{next}_q$. 

5
Shared Variables:
\[ \text{next} = \{ \text{next}_1, \ldots, \text{next}_m \}, \text{next}_q \in \{0, \ldots, n\} \text{ initially 0} \]
\[ \text{done} = \{ \text{done}_1, \ldots, \text{done}_m, \text{done}_n \}, \text{done}_{n,i} \in \{0, \ldots, n\} \text{ initially 0} \]

Signature:
Input:
stop_{p,j}, p \in \mathcal{P}

Output:
do_{p,j}, p \in \mathcal{P}, j \in \mathcal{J}

State:
\[ \text{STATUS}_p \in \{ \text{compNext}, \text{setNext}, \text{gatherTry}, \text{gatherDone}, \text{check}, \text{do}, \text{done}, \text{end}, \text{stop} \}, \]
\[ \text{FREE}_p, \text{DONE}_p, \text{TRY}_p \subseteq \mathcal{J}, \text{ initially } \text{FREE}_p = \mathcal{J} \text{ and } \text{DONE}_p = \text{TRY}_p = \emptyset \]
\[ \text{POS}_p = \{ \text{POS}_p(1), \ldots, \text{POS}_p(m) \}; \text{ where } \text{POS}_p(i) \in \{1, \ldots, n\}, \text{ initially } \text{POS}_p(i) = 1 \]
\[ \text{NEXT}_p \in \{1, \ldots, n+1\}, \text{ initially undefined} \]
\[ \text{TMP}_p \in \{1, \ldots, n\}, \text{ initially undefined} \]

Transitions of process \( p \):

Input \( \text{stop}_p \):

Effect:
\[ \text{STATUS}_p \leftarrow \text{stop} \]

Internal \( \text{compNext}_p \):

Precondition:
\[ \text{STATUS}_p = \text{compNext} \]

Effect:
\[ \text{if } |\text{FREE}_p| \geq \beta \text{ then } \]
\[ \text{TMP}_p \leftarrow \frac{|\text{FREE}_p|}{|\text{FREE}_p| - (m-1)} \]
\[ \text{if } \text{TMP}_p \geq 1 \text{ then } \]
\[ \text{TMP}_p \leftarrow \left(\frac{1}{\text{TMP}_p} \right) \cdot \text{TMP}_p + 1 \]
\[ \text{NEXT}_p \leftarrow \text{rank}(\text{FREE}_p, \text{TRY}_p, \text{TMP}_p) \]
\[ \text{else } \]
\[ \text{NEXT}_p \leftarrow \text{rank}(\text{FREE}_p, \text{TRY}_p, p) \]
\[ \text{end} \]
\[ \text{Q}_p \leftarrow 1 \]
\[ \text{TRY}_p \leftarrow \emptyset \]
\[ \text{STATUS}_p \leftarrow \text{setNext} \]
\[ \text{end} \]

Internal \( \text{check}_p \):

Precondition:
\[ \text{STATUS}_p = \text{check} \]

Effect:
\[ \text{if } \text{NEXT}_p \notin \text{TRY}_p \]
\[ \text{AND } \text{NEXT}_p \notin \text{DONE}_p \]
\[ \text{then } \text{STATUS}_p \leftarrow \text{do} \]
\[ \text{else } \]
\[ \text{STATUS}_p \leftarrow \text{compNext} \]
\[ \text{end} \]

Internal Read \( \text{gatherDone}_p \):

Precondition:
\[ \text{STATUS}_p = \text{gatherDone} \]

Effect:
\[ \text{if } \text{Q}_p \neq p \text{ then } \]
\[ \text{TMP}_p \leftarrow \text{done}_p, \text{POS}_p(\text{Q}_p) \]
\[ \text{if } \text{POS}_p(\text{Q}_p) < n \text{ AND } \text{TMP}_p > 0 \text{ then } \]
\[ \text{DONE}_p \leftarrow \text{DONE}_p \cup \{ \text{TMP}_p \} \]
\[ \text{FREE}_p \leftarrow \text{FREE}_p \setminus \{ \text{TMP}_p \} \]
\[ \text{POS}_p(\text{Q}_p) \leftarrow \text{POS}_p(\text{Q}_p) + 1 \]
\[ \text{else } \]
\[ \text{Q}_p \leftarrow \text{Q}_p + 1 \]
\[ \text{end} \]
\[ \text{if } \text{Q}_p > m \text{ then } \]
\[ \text{Q}_p \leftarrow 1 \]
\[ \text{STATUS}_p \leftarrow \text{check} \]
\[ \text{end} \]

Internal Write \( \text{done}_p \):

Precondition:
\[ \text{STATUS}_p = \text{done} \]

Effect:
\[ \text{done}_p, \text{POS}_p(\text{Q}_p) \leftarrow \text{NEXT}_p \]
\[ \text{DONE}_p \leftarrow \text{DONE}_p \cup \{ \text{NEXT}_p \} \]
\[ \text{FREE}_p \leftarrow \text{FREE}_p \setminus \{ \text{NEXT}_p \} \]
\[ \text{POS}_p(\text{Q}_p) \leftarrow \text{POS}_p(\text{Q}_p) + 1 \]
\[ \text{STATUS}_p \leftarrow \text{compNext} \]

Figure 1: Algorithm KK\( \beta \): Shared Variables, Signature, States and Transitions

\( \text{done} \) is an \( m \times n \) matrix. In line \( q \) of the matrix, process \( q \) announces the jobs it has performed. Each cell of line \( q \) contains the identifier of exactly one job that has been performed by process \( q \). Only process \( q \) writes in the cells of line \( q \) but any process may read them. Moreover, process \( q \) updates line \( q \) by adding entries at the end of it.

Internal Variables of process \( p \). The variable \( \text{STATUS}_p \in \{ \text{compNext}, \text{setNext}, \text{gatherTry}, \text{gatherDone}, \text{check}, \text{do}, \text{done}, \text{end}, \text{stop} \} \) records the status of process \( p \) and defines its next action as follows: \( \text{compNext} - p \) is ready to compute the next job to perform (this is the initial status of \( p \)), \( \text{setNext} - p \) computed the next job to perform and is ready to announce it, \( \text{gatherTry} - p \) reads the array \( \text{next} \) in shared memory in order to compute the \( \text{TRY} \) set, \( \text{gatherDone} - p \) reads the matrix \( \text{done} \) in
shared memory in order to update the DONE and FREE sets, check - p has to check whether it is safe to perform its current job, do - p can safely perform its current job, done - p performed its current job, end - p terminated, stop - p crashed.

FREE_p, DONE_p, TRY_p ⊆ J are three sets that are used by process p in order to compute the next job to perform and whether it is safe to perform it. We use some tree structure like red-black tree or some variant of B-tree \cite{11} for the sets FREE_p, DONE_p and TRY_p, in order to be able to add, remove and search elements in them in O(\log n). FREE_p is initially set to \emptyset and contains an estimate of the jobs that are still available. DONE_p is initially empty and contains an estimate of the jobs that have been performed. No job is removed from DONE_p or added to FREE_p during the execution of algorithm KK_p. TRY_p is initially empty and contains an estimate of the jobs that other processes are about to perform. It holds that |TRY_p| < m, since there are m − 1 processes apart from process p that may be attempting to perform a job.

POS_p is an array of m elements. Position POS_p(q) of the array contains a pointer in the line q of the shared matrix done. POS_p(q) is the element of line q that process p will read from. In the special case where q = p, POS_p(p) is the element of line p that process p will write into after performing a new job. The elements of the shared matrix done are read when process p is updating the DONE_p set.

NEXT_p contains the job process p is attempting to perform.

TM_p is a temporary storage for values read from the shared memory.

Q_p ∈ \{1, \ldots, m\} is used as indexing for looping through process identifiers.

**Actions of process p.** We visit them one by one below.

- **compNext_p:** Process p computes the set FREE_p \ \TRY_p and if it has more or equal elements to \beta, were \beta is the termination parameter of the algorithm, process p computes its next candidate job, by splitting the FREE_p \ \TRY_p set in m parts and picking the first element of the p-th part. In order to do that it uses the function rank(SET_1, SET_2, i), which returns the element of set SET_1 \ \setminus SET_2 with rank i. Finally process p sets the TRY_p set to the empty set, the Q_p internal variable to 1 and its status to set_next in order to update the shared memory with its new candidate job. If the FREE_p \ \TRY_p set has less than \beta elements process p terminates.

- **setNext_p:** Process p announces its new candidate job by writing the contents of its NEXT_p internal variable in the p-th position of the next array. Remember that the next array is stored in the shared memory. Process p changes its status to gather_try, in order to start collecting the TRY_p set from the next array.

- **gatherTry_p:** With this action process p implements a loop, which reads from the shared memory all the positions of the array next and updates the TRY_p set. In each execution of the action, process p checks if Q_p is equal with p. If it is not equal, p reads the Q_p-th position of the array next, checks if the value read is less than n + 1 and if it is, adds the value it read in the TRY_p set. If Q_p is equal with p, p just skips the step described above. Then p checks if the value of Q_p + 1 is less than m + 1. If it is, then p increases Q_p by 1 and leaves its status gather_fry, otherwise p has finished updating the TRY_p set and thus sets Q_p to 1 and changes its status to gather_done, in order to update the DONE_p and FREE_p sets from the contents of the done matrix.

- **gatherDone_p:** With this action process p implements a loop, which updates the DONE_p and FREE_p sets with values read from the matrix done, which is stored in shared memory. In each execution of the action, process p checks if Q_p is equal with p. If it is not equal, p uses the internal variable POS_p(Q_p), in order to read fresh values from the line Q_p of the done matrix. In detail, p reads the shared variable done_{Q_p, POS_p(Q_p)}, checks if POS_p(Q_p) is less than n + 1 and if the value read is greater than 0. If both conditions hold, p adds the value read at the DONE_p set, removes the value read from the FREE_p set and increases \POS_p(Q_p) by one. Otherwise, it means that either process Q_p has terminated (by performing all the n jobs) or the line Q_p does not contain any new completed jobs. In either case p increases the value of Q_p by 1. The value of Q_p is increased by 1 also if Q_p was equal with p. Finally p checks whether Q_p is greater than m; if it is, p has completed the loop and thus changes its status to check.
check\(_p\): Process \(p\) checks if it is safe to perform its current job. This is done by checking if \(\text{NEXT}_p\) belongs to the set \(\text{TRY}_p\) or to the set \(\text{DONE}_p\). If it does not, then it is safe to perform the job \(\text{NEXT}_p\) and \(p\) changes its status to \(\text{do}\). Otherwise it is not safe, and thus \(p\) changes its status to \(\text{comp\_next}\), in order to find a new job that may be safe to perform.

dop\(_{p,i}\): Process \(p\) performs job \(i\). Note that \(\text{NEXT}_p = j\) is part of the preconditions for the action to be enabled in a state. Then \(p\) changes its status to \(\text{done}\).

done\(_p\): Process \(p\) writes in the \(\text{done}_{p,\text{pos}_{p}(p)}\) position of the shared memory the value of \(\text{NEXT}_p\), letting other processes know that it performed job \(\text{NEXT}_p\). Also \(p\) adds \(\text{NEXT}_p\) to its \(\text{DONE}_p\) set, removes \(\text{NEXT}_p\) from its \(\text{FREE}_p\) set, increases \(\text{pos}_{p}(p)\) by 1 and changes its status to \(\text{comp\_next}\).

stop\(_p\): Process \(p\) crashes by setting its status to \(\text{stop}\).

### 4 Correctness and Effectiveness Analysis

Next we begin the analysis of algorithm KK\(_\beta\), by proving that KK\(_\beta\) solves the at-most-once problem. That is, there exists no execution of KK\(_\beta\) in which 2 distinct actions dop\(_{p,i}\) and dop\(_{q,i}\) appear for some \(i \in \mathcal{J}\) and \(p,q \in \mathcal{P}\). In the proofs, for a state \(s\) and a process \(p\) we denote by \(s.\text{FREE}_p\), \(s.\text{DONE}_p\), \(s.\text{TRY}_p\), the values of the internal variables \(\text{FREE}, \text{DONE}\) and \(\text{TRY}\) of process \(p\) in state \(s\). Moreover with \(s.\text{next}\) and \(s.\text{done}\) we denote the contents of the array \(\text{next}\) and the matrix \(\text{done}\) in state \(s\). Remember that \(\text{next}\) and \(\text{done}\), are stored in shared memory.

**Lemma 4.1** There exists no execution \(\alpha\) of algorithm KK\(_\beta\), such that \(\exists i \in \mathcal{J}\) and \(\exists p,q \in \mathcal{P}\) for which dop\(_{p,i}\), dop\(_{q,i}\) \(\in \alpha\).

**Proof.** Let us for the sake of contradiction assume that there exists an execution \(\alpha \in \text{execs}(\text{KK}\_\beta)\) and \(i \in \mathcal{J}\) and \(p,q \in \mathcal{P}\) such that dop\(_{p,i}\), dop\(_{q,i}\) \(\in \alpha\). We examine two cases.

**Case 1** \(p = q\): Let states \(s_1, s'_1, s_2, s'_2 \in \alpha\), such that the transitions \(s_1, \text{dop}_{p,i}, s'_1\), \(s_2, \text{dop}_{p,i}, s'_2\) \(\in \alpha\) and without loss of generality assume \(s'_1 \leq s_2\) in \(\alpha\). From Figure 1 we have that \(s_1.\text{NEXT}_p = i, s'_1.\text{STATUS}_p = \text{done}\) and \(s_2.\text{NEXT}_p = i, s_2.\text{STATUS}_p = \text{do}\). From algorithm KK\(_\beta\), state \(s_2\) must be preceded in \(\alpha\) by transition \((s_3, \text{check}_p, s'_3)\), such that \(s_3.\text{NEXT}_p = i\) and \(s'_3.\text{NEXT}_p = i, s'_3.\text{STATUS}_p = \text{do}\), where \(s'_1\) precedes \(s_3\) in \(\alpha\). Finally \(s_3\) must be preceded in \(\alpha\) by transition \((s_4, \text{done}_p, s'_4)\), where \(s'_1\) precedes \(s_4\), such that \(s_4.\text{NEXT}_p = i\) and \(i \in s'_4.\text{DONE}_p\). Since \(s'_4\) precedes \(s_3\) and during the execution of KK\(_\beta\) no elements are removed from \(\text{DONE}_p\), we have that \(i \in s_3.\text{DONE}_p\). This is a contradiction, since the transition \((\{\text{NEXT}_p = i, i \in \text{DONE}_p\}, \text{check}_p, \{\text{NEXT}_p = i, \text{STATUS}_p = \text{do}\}) \notin \text{trans}(\text{KK}\_\beta)\).

**Case 2** \(p \neq q\): Given the transition \((s_1, \text{dop}_{p,i}, s'_1)\) in \(\alpha\), we deduce from Figure 1 that there exist in \(\alpha\) transitions \((s_2, \text{set\_next}_p, s'_2)\), \((s_3, \text{gather\_try}_p, s'_3)\), \((s_4, \text{check}_p, s'_4)\), where \(s_2.\text{next}_p = s'_2.\text{NEXT}_p = i, s_3.\text{NEXT}_p = i, s_3.\text{Q}_p = q, s_4.\text{NEXT}_p = i, s'_4.\text{NEXT}_p = i, s'_4.\text{STATUS}_p = \text{do}\), such that \(s_2 < s_3 < s_4 < s_1\) and there exists no action \(\pi = \text{comp\_next}_p\) in \(\alpha\) between states \(s_2\) and \(s'_1\). This essentially means that in the execution fragment \(\alpha' \in \alpha\) starting from state \(s_2\) and ending with \(s'_1\) there exists only a single \(\text{check}_p\) action - the one in transition \((s_4, \text{check}_p, s'_4)\) - that leads in the performance of job \(i\). Similarly for transition \((t_1, \text{dop}_{q,i}, t'_1)\) there exist in \(\alpha\) transitions \((t_2, \text{set\_next}_q, t'_2)\), \((t_3, \text{gather\_try}_q, t'_3)\), \((t_4, \text{check}_q, t'_4)\), where \(t'_2.\text{next}_q = t'_2.\text{NEXT}_q = i, t_3.\text{next}_q = t_3.\text{NEXT}_q = i, t_3.\text{Q}_q = p, t_4.\text{NEXT}_q = i, t_4.\text{NEXT}_q = i, t'_4.\text{STATUS}_q = \text{do}\), such that \(t_2 < t_3 < t_4 < t_1\) and there is no action \(\pi' = \text{comp\_next}_q\) occurring in \(\alpha\) between states \(t_2\) and \(t'_1\).
In the execution $\alpha$, either state $s_2 < s_3$ or $s_3 < s_2$ which implies $t_2 < s_3$. We will show that if $s_2 < s_3$ then $\text{do}_{j_{\alpha,i}}$ cannot take place, leading to a contradiction. The case were $t_2 < s_3$ is symmetric and will be omitted. So let us assume that $s_2$ precedes $t_3$ in $\alpha$. We have two cases, either $t_3.next_p = i$ or $t_3.next_p \neq i$.

In the first case $i \in t_3.\text{TRY}_q$. From Figure 1 the only action in which entries are removed from the $\text{TRY}_q$ set, is the $\text{compNext}_q$ where the $\text{TRY}_q$ set is reset to $\emptyset$. This means that $i \in t_4.\text{TRY}_q$ since $\exists \pi' = \text{compNext}_q \in \alpha$, such that $t_2 < \pi' < t_1$. This is a contradiction since $(t_4, \text{check}_{q}, t_4') \notin \text{trans}(\text{KK}\beta), \text{if } i \in t_4.\text{TRY}_q, t_4.\text{NEXT}_q = i \text{ and } t_4'.\text{STATUS}_q = \text{do}$.

If $t_3.\text{next}_p \neq i$, since $(s_2, \text{setNext}_p, s_2) \in \alpha \text{ and } s_2' < t_3$ there exists action $\pi_1 = \text{setNext}_p \in \alpha$, such that $s_2' < \pi_1 < t_3$. Moreover from Figure 1 there exists action $\pi_2 = \text{compNext}_p \in \alpha$, such that $s_2 < \pi_1 < s_1$. Since $\exists \pi = \text{compNext}_p \in \alpha$, such that $s_2 < \pi < s_1$, it holds that $s_1 < s_\pi < s_\pi< s_3$. Furthermore, from Figure 1 there exists transition $(s_5, \text{done}_{q,p}, s_5') \in \alpha$ and $j \in \{1, \ldots, n\}$, such that $s_5.\text{POS}_{p}(p) = j, s_5.\text{done}_{p,j} = 0, s_3.\text{NEXT}_p = i, s_5.\text{done}_{p,j} = i$ and $s_5' < s_5 < s_\pi < s_3$. It must be the case that $i \notin t_2.\text{DONE}_q$, since $t_2.\text{NEXT}_q = i$. From that and from Figure 1 we have that there exists transition $(t_6, \text{gatherDone}_{q}, t_6')$ in $\alpha$, such that $t_6.\text{Q}_q = p, t_6.\text{POS}_{p}(p) = j$ and $t_3 < t_6 < t_4$. Since $s_5' < s_3$ and $\text{done}_{p,j}$ from Figure 1 cannot be changed in execution $\alpha$, we have that $t_6.\text{done}_{p,j} = i$ and as a result $i \in t_6.\text{DONE}_q$. Moreover during the execution of algorithm KK$\beta$ entries in set DONE are only added and never removed, thus we have that $i \in t_4.\text{DONE}_q$. This is a contradiction since $(t_4, \text{check}_{q}, t_4') \notin \text{trans}(\text{KK}\beta), \text{if } i \in t_4.\text{DONE}_q, t_4.\text{NEXT}_q = i \text{ and } t_4'.\text{STATUS}_q = \text{do}$. This completes the proof.

Next we examine the effectiveness of the algorithm.

Lemma 4.2 For any $\beta \geq m$, $f \leq m - 1$ and for any finite execution $\alpha \in \text{execs}(\text{KK}\beta)$ with $\text{Do}(\alpha) \leq n - (\beta + m - 1)$, there exists a (non-empty) execution fragment $\alpha'$ such that $\alpha \cdot \alpha' \in \text{execs}(\text{KK}\beta)$.

Proof. From the algorithm KK$\beta$, we have that for any process $p$ and any state $s \in \alpha$, $|s.\text{FREE}_p| \geq n - \text{Do}(\alpha)$ and $|s.\text{TRY}_p| \leq m - 1$. The first inequality holds since the $s.\text{FREE}_p$ set is estimated by $p$ by examining the done matrix which is stored in shared memory. From Figure 1 a job $j$ is only inserted in line $q$ of the matrix done, if a $\text{do}_{j_{\alpha,i}}$ action has already been performed by process $q$. The second inequality is obvious. Thus was have that $\forall p \in \mathcal{P}$ and $\forall s \in \alpha$, $|s.\text{FREE}_p \setminus s.\text{TRY}_p| \geq n - (\text{Do}(\alpha) + m - 1)$. If $\text{Do}(\alpha) \leq n - (\beta + m - 1)$, $\forall p \in \mathcal{P}$ and $\forall s \in \alpha$ we have that $|s.\text{FREE}_p \setminus s.\text{TRY}_p| \geq \beta$. Since there can be $f \leq m - 1$ failed processes in our system, at the final state $s'$ of $\alpha$ there exists at least one process $p \in \mathcal{P}$ that has not failed. This process has not terminated, since from Figure 1a process $p$ can only terminate if in the enabling state $s$ of action $\text{compNext}_{p}, |s.\text{FREE}_p \setminus s.\text{TRY}_p| < \beta$. This process can continue executing steps and thus there exits (non-empty) execution fragment $\alpha'$ such that $\alpha \cdot \alpha' \in \text{execs}(\text{KK}\beta)$. This means that if the KK$\beta$ algorithm has effectiveness less then or equal to $n - (\beta + m - 1)$, there should be some infinite fair execution $\alpha$ of the algorithm with $\text{Do}(\alpha) \leq n - (\beta + m - 1)$ (since no finite execution of algorithm could terminate). Next we prove that the algorithm KK$\beta$ is wait-free (the algorithm has no infinite fair executions) and thus there exists no such execution $\alpha \in \text{execs}(\text{KK}\beta)$.

Lemma 4.3 For any $\beta \geq m$, $f \leq m - 1$ there exists no infinite fair execution $\alpha \in \text{execs}(\text{KK}\beta)$.

Proof. We will prove this by contradiction. Let $\beta \geq m$ and $\alpha \in \text{execs}(\text{KK}\beta)$ an infinite fair execution with $f \leq m - 1$ failures, and let $\text{Do}(\alpha)$ be the jobs executed by execution $\alpha$ according to Definition 27. Since $\alpha \in \text{execs}(\text{KK}\beta)$ and from Lemma 4.1 KK$\beta$ solves the at-most-once problem, $\text{Do}(\alpha)$ is finite. Clearly there exists at least one process in $\alpha$ that has not crashed and does not terminate(some process need to take steps in $\alpha$ in order for it to be infinite). Since $\text{Do}(\alpha)$ and $f$ are finite, there exists a state $s_0$ in $\alpha$ such that after $s_0$ no process crashes, no process terminates, no do action takes place in $\alpha$ and no process adds new
entries in the done matrix in shared memory. The later holds since the execution is infinite and fair, the Do(α) is also finite, consequently any non failed process q that has not terminated will eventually update the q line of the done matrix to be in agreement with the doq,s actions it has performed. Moreover any process q that has terminated, has already updated the q line of done matrix with the latest do action it performed, before it terminated, since in order to terminate it must have reached a compNext action that has set its status to end.

We define the following sets of processes and jobs according to state $s_0$. $J_\alpha$ are jobs that have been performed in $\alpha$ according to Definition 2.1 $P_\alpha$ are processes that do not crash and do not terminate in $\alpha$. By the way we defined state $s_0$ only processes in $P_\alpha$ take steps in $\alpha$ after state $s_0$. STUCK$\alpha = \{i \in J \setminus J_\alpha |$ failed process $p : s_0.next_p = i\}$, i.e., STUCK$\alpha$ expresses the set of jobs that are held by failed processes. DONE$\alpha = \{i \in J_\alpha | \exists p \in P$ and $j \in \{1, \ldots, n\} : s_0.done_p(j) = i\}$, i.e., DONE$\alpha$ expresses the set of jobs that have been performed before state $s_0$ and the processes that performed them managed to update the shared memory. Finally we define POOL$\alpha = J \setminus (J_\alpha \cup$ STUCK$\alpha$).

After state $s_0$, all processes in $P_\alpha$ will keep executing. This means that whenever such process $p \in P_\alpha$ takes action compNext$_p$ in $\alpha$, the first if statement is true. Specifically it holds that for $\forall p \in P_\alpha$ and for all the enabling states $s \geq s_0$ of actions compNext$_p$ in $\alpha$, $|FREE_p \setminus TRY_p| \geq \beta$.

From Figure 1 we have that for any $p \in P_\alpha$, $\exists s_p$ after state $s_0$ in $\alpha$ such that $\forall$ states $s \geq s_p, s.DONE_p = DONE_\alpha, s.FREE_p = J \setminus DONE_\alpha$ and $s.FREE_p \setminus s.TRY_p \subseteq POOL_\alpha$. Let $s_0' = \max_{p \in P_\alpha}[s_p]$. From the above we have: $|[J \setminus DONE_\alpha| \geq \beta \geq m$ and $|POOL_\alpha| \geq \beta \geq m$, since $\forall p \in P_\alpha$ we have that for all the enabling states $s \geq s_0'$ of actions compNext$_p$ in $\alpha$, $|FREE_p \setminus TRY_p| \geq \beta$ and $\forall s' \geq s_0'$ we have that $s'.FREE_p = J \setminus DONE_\alpha$ and $s'.FREE_p \setminus s'.TRY_p \subseteq POOL_\alpha$.

Let $p_0$ be the process with the smallest process identifier in $P_\alpha$. We examine 2 cases according to the size of $\alpha \setminus DONE_\alpha$.

**Case A** $|[J \setminus DONE_\alpha| \geq 2m - 1$: Let $x_0 \in POOL_\alpha$ be the job such that $[x_0]_{POOL_\alpha} = \left\lceil (p_0 - 1) \cdot \frac{|J \setminus DONE_\alpha| - (m - 1)}{m} \right\rceil + 1$. Such $x_0$ exists since $\forall p \in P_\alpha \land \forall s \geq s_0'$ it holds $s.DONE_p \setminus s.TRY_p \subseteq POOL_\alpha$, s.FREE_p = J \setminus DONE_\alpha$ from which we have that $|POOL_\alpha| \geq |J \setminus DONE_\alpha| - |s.TRY_p| \geq |J \setminus DONE_\alpha| - (m - 1)$.

It follows that any $p \in P_\alpha$ that executes action compNext$_p$ after state $s_0'$, will have its NEXT$_p$ variable pointing in a job $x$ with $[x]_{POOL_\alpha} \geq \left\lceil (p - 1) \cdot \frac{|J \setminus DONE_\alpha| - (m - 1)}{m} \right\rceil + 1$. Thus $\forall p \in P_\alpha, \exists s' \geq s_0'$ in $\alpha$ such that $\forall$ states $s \geq s', [s.next_p]_{POOL_\alpha} \geq \left\lceil (p - 1) \cdot \frac{|J \setminus DONE_\alpha| - (m - 1)}{m} \right\rceil + 1$. Let $s_0'' = \max_{p \in P_\alpha}[s'']$, we have to study 2 cases for $p_0$:

**Case A.1** After $s_0''$, process $p_0$ executes action compNext$_{p_0}$ and the transition leads in state $s_1 > s_0''$ such that $s_1.NEXT_{p_0} = x_0$. Since $[x_0]_{POOL_\alpha} = \left\lceil (p_0 - 1) \cdot \frac{|J \setminus DONE_\alpha| - (m - 1)}{m} \right\rceil + 1$ and $p_0 = \min_{p \in P_\alpha}[p]$, from the previous discussion we have that $\forall s \geq s_1$ and $\forall p \in P \setminus \{p_0\}, s.next_p \neq x_0$. Thus when $p_0$ executes action check$_p$ of Figure 1 for the first time after state $s_1$, the condition will be true, so in some subsequent transition $p_0$ will have to execute action d$p_{p_0,x_0}$, performing job $x_0$, which is a contradiction, since after state $s_0$ no jobs are executed.

**Case A.2** After $s_0''$, process $p_0$ executes action compNext$_{p_0}$ and the transition leads in state $s_1 > s_0''$ such that $s_1.NEXT_{p_0} = x_0$. Since $p_0 = \min_{p \in P_\alpha}[p]$, it holds that $\forall x \in POOL_\alpha$ such that $[x]_{POOL_\alpha} \leq \left\lceil (p_0 - 1) \cdot \frac{|J \setminus DONE_\alpha| - (m - 1)}{m} \right\rceil + 1$, $\exists p \in P_\alpha$ such that $s_1.next_p = x$. Let the transition $(s_2,compNext_{p_0},s_2') \in \alpha$, where $s_2 > s_1$, be the first time that action compNext$_{p_0}$ is executed after state $s_1$. We have that $\forall x \in POOL_\alpha$ such that $[x]_{POOL_\alpha} \leq \left\lceil (p_0 - 1) \cdot \frac{|J \setminus DONE_\alpha| - (m - 1)}{m} \right\rceil + 1$, $x \notin s_2.DONE_{p_0} \cup s_2.TRY_{p_0}$, since from the discussion above we have that $\forall s \geq s_1$ and $\forall p \in P_\alpha \setminus \{p_0\}$, $[s.next_p]_{POOL_\alpha} \geq \left\lceil (p - 1) \cdot \frac{|J \setminus DONE_\alpha| - (m - 1)}{m} \right\rceil + 1$. Thus $[x_0]_{s_2.DONE_{p_0} \setminus s_2.TRY_{p_0}} = [x_0]_{POOL_\alpha} =}$
\[ \left( p_0 - 1 \right) \cdot \left( \left( s_0 \text{DONE}_{p_0} \right) - (m-1) \right) \] + 1. As a result, \( s_2' \cdot \text{NEXT}_{p_0} = x_0 \). With similar arguments like in case A.1, we can see that job \( x_0 \) will be performed by process \( p_0 \), which is a contradiction, since after state \( s_0 \) no jobs are executed.

**Case B** \(| s \ \text{DONE}_{p} | < 2m - 1\): Let \( x_0 \in \text{POOL}_{p} \) be the job such that \( [x_0]_{\text{POOL}_{p}} = p_0 \). Such \( x_0 \) exists since \( \beta \geq m \) and \( \text{POOL}_{p} \geq \beta \). It follows that any \( p \in \mathcal{P}_p \) that executes action \( \text{compNext}_{p} \) after state \( s' \), will have its \( \text{NEXT}_{p} \) variable pointing in a job \( x \) with \( [x]_{\text{POOL}_{p}} \geq p \). Thus \( \forall p \in \mathcal{P}, \exists s' \geq s' \) in \( \alpha \) such that \( \forall s \geq s' \), \( s, \text{next}_{p} \text{POO}_{l} \geq p \). Let \( s'' = \max_{p \in \mathcal{P}} [s'_p] \), we have to study 2 cases for \( p_0 \):

**Case B.1** After \( s''_0 \), process \( p_0 \) executes action \( \text{compNext}_{p_0} \) and the transition leads in state \( s_1 > s''_0 \) such that \( s_1, \text{NEXT}_{p_0} = x_0 \). Since \( [x_0]_{\text{POOL}_{p}} = p_0 \) and \( p_0 = \min_{p \in \mathcal{P}} [p] \), from the previous discussion we have that \( \forall s \geq s_1 \) and \( \forall p \in \mathcal{P} \setminus \{p_0\}, s, \text{next}_{p} \neq x_0 \). Thus when \( p_0 \) executes action \( \text{compNext}_{p} \) of Figure \( [1] \) for the first time after state \( s_1 \), the condition will be true, so in some subsequent transition \( p_0 \) will have to execute action \( \text{DO}_{p_0, x_0} \), performing job \( x_0 \), which is a contradiction, since after state \( s_0 \) no jobs are executed.

**Case B.2** After \( s''_0 \), process \( p_0 \) executes action \( \text{compNext}_{p_0} \) and the transition leads in state \( s_1 > s''_0 \) such that \( s_1, \text{NEXT}_{p_0} > x_0 \). Since \( p_0 = \min_{p \in \mathcal{P}} [p] \), it holds that \( \forall x \in \text{POOL}_{p} \) such that \( [x]_{\text{POOL}_{p}} \leq p_0 \), \( \exists p \in \mathcal{P} \) such that \( s_1, \text{next}_{p} = x \). Let the transition \( \left( s_2, \text{compNext}_{p_0}, s_2' \right) \in \alpha \), where \( s_2 > s_1 \), be the first time that action \( \text{compNext}_{p_0} \) is executed after state \( s_1 \). We have that \( \forall x \in \text{POOL}_{p_0} \) such that \( [x]_{\text{POOL}_{p}} \leq p_0 \), \( x \notin s_2, \text{DONE}_{p_0} \cup s_2, \text{TRY}_{p_0} \), from the discussion above we have that \( \forall s \geq s_1 \) and \( \forall p \in \mathcal{P} \setminus \{p_0\}, [s, \text{next}_{p}]_{\text{POOL}_{p}} \geq p \). Thus \( [x_0]_{s_0, \text{FREE}_{p_0} \setminus s_2, \text{TRY}_{p_0}} = [x_0]_{\text{POOL}_{p}} = p_0 \). As a result, \( s_2' \cdot \text{NEXT}_{p_0} = x_0 \). With similar arguments like in case B.1, we can see that job \( x_0 \) will be performed by process \( p_0 \), which is a contradiction, since after state \( s_0 \) no jobs are executed.

Using the last two lemmas we can find the effectiveness of algorithm KK\( \beta \).

**Theorem 4.4** For any \( \beta \geq m, f \leq m - 1 \) algorithm KK\( \beta \) has effectiveness \( E_{KK\beta}(n, m, f) = n - (\beta + m - 2) \).

**Proof.** From Lemma 4.2 we have that any finite execution \( \alpha \in \text{execs}(KK\beta) \) with \( D\alpha \leq n - (\beta + m - 1) \) can be extended, essentially proving that in such an execution no process has terminated. Moreover from Lemma 4.3 we have that KK\( \beta \) is wait free, and thus there exists no infinite fair execution \( \alpha \in \text{execs}(KK\beta) \), such that \( D\alpha \leq n - (\beta + m - 1) \). Since finite fair executions are executions were all non-failed processes have terminated, from the above we have that \( E_{KK\beta}(n, m, f) \geq n - (\beta + m - 2) \).

If all processes but the process with id \( m \) fail in an execution \( \alpha \) in such a way that \( J_{\alpha} \cap \text{STUCK}_{\alpha} = \emptyset \) and \( |\text{STUCK}_{\alpha}| = m - 1 \) (where STUCK\( \alpha \) is defined as in the proof of lemma 4.3), then there exists adversarial strategy, that can result in \( \beta + m - 2 \) jobs not having been performed when process \( m \) terminates. Such an execution will be a finite fair execution where \( n - (\beta + m - 2) \) jobs are performed. From this and the previous claims we have that \( E_{KK\beta}(n, m, f) = n - (\beta + m - 2) \).

## 5 Work Complexity Analysis

In this section we are going to prove that for \( \beta \geq 3m^2 \) algorithm KK\( \beta \) has work complexity \( O(nm \log n \log m) \).

The main idea of the proof, is to demonstrate that under the assumption \( \beta \geq 3m^2 \), process collisions on a job cannot accrue without making progress in the algorithm. In order to prove that, we first demonstrate that if two different processes \( p, q \) set their \( \text{NEXT}_{p}, \text{NEXT}_{q} \) internal variables to the same job \( i \) in some \( \text{compNext} \) actions, then at the enabling states of those actions the \( \text{DONE}_{p} \) and \( \text{DONE}_{q} \) sets of the processes, have at least \( |q - p|/m \) different elements, given that \( \beta \geq 3m^2 \). Next we prove that if two processes \( p, q \) collide three consecutive times, while trying to perform some jobs, then the size of the set \( \text{DONE}_{p} \cup \text{DONE}_{q} \) that
processes $p$ and $q$ know has increased by at least $|q - p| \cdot m$ elements. This essentially tells us that every three collisions between the same two processes a significant number of jobs has been performed, and thus enough progress has been made. In order to prove the above statement, we need to formally define what we mean by collision, and tie such a collision with some specific state, so that we have a fixed “point” in the execution for which to reason. Finally we use the argument about the progress made if three consecutive collisions happen between two processes $p$, $q$, in order to prove that a process $p$ cannot collide with a process $q$ more than $2 \left\lceil \frac{n}{m|q - p|} \right\rceil$ times in any execution. This is proved by contradiction, proving that if process $p$ collides with process $q$ more than $2 \left\lceil \frac{n}{m|q - p|} \right\rceil$ times, there exist states for which the set $|\text{DONE}_p \cup \text{DONE}_q|$ has more than $n$ elements which is impossible. The last statement is used in order to prove the main theorem on the work complexity of algorithm KK$_\beta$ for $\beta \geq 3m^2$. We obtain the main theorem on the work complexity by counting the total number of collisions and the cost of each collision.

We start by proving that if two processes $p$, $q$ decide, with some compNext actions, to perform the same job $i$, then their DONE sets at the enabling states of those compNext actions, differ in at-least $|q - p| \cdot m$ elements.

**Lemma 5.1** If $\beta \geq 3m^2$ and in an execution $\alpha \in \text{execs(KK}_\beta)$ there exist states $s_1$, $t_1$ and processes $p$, $q \in \mathcal{P}$ with $p < q$ such that $s_1.\text{NEXT}_p = t_1.\text{NEXT}_q = i \in \mathcal{J}$, then there exist transitions $(s_2, \text{compNext}_p, s_2')$, $(t_2, \text{compNext}_q, t_2')$, where $s_2.\text{NEXT}_p = t_2'.\text{NEXT}_q = i$, $s_2.\text{STATUS}_p = t_2'.\text{STATUS}_q = \text{set\_next}$, such that there exist no action $\pi_1 = \text{compNext}_p$ with $s_2' < \pi_1 < s_1$ and no action $\pi_2 = \text{compNext}_q$ with $t_2' < \pi_2 < t_1$ and $|s_2.\text{DONE}_p \cap t_2.\text{DONE}_q| > (q - p) \cdot m$ or $|s_2.\text{DONE}_p \cap t_2.\text{DONE}_q| > (q - p) \cdot m$.

**Proof.** We will prove this by contradiction. From algorithm KK$_\beta$ there must exist transitions $(s_2, \text{compNext}_p, s_2')$, $(t_2, \text{compNext}_q, t_2')$ where $s_2'.\text{NEXT}_p = i$ and $t_2'.\text{NEXT}_q = i$, such that there exist no actions $\pi_1 = \text{compNext}_p$, $\pi_2 = \text{compNext}_q$ with $s_2' < \pi_1 < s_1$ and $t_2' < \pi_2 < t_1$, if there exist $s_1, t_1 \in \alpha$ and $p, q \in \mathcal{P}$ with $p < q$ such that $s_1.\text{NEXT}_p = t_1.\text{NEXT}_q = i \in \mathcal{J}$, since those are the transitions that set NEXT$_p$ and NEXT$_q$ to $i$. So in order to get a contradiction we must assume that $|s_2.\text{DONE}_p \cap t_2.\text{DONE}_q| \leq (q - p) \cdot m$ and $|s_2.\text{DONE}_p \cap t_2.\text{DONE}_q| \leq (q - p) \cdot m$.

We will prove that if this is the case $s_2'.\text{NEXT}_p \neq t_2'.\text{NEXT}_q$.

Let $A = \mathcal{J} \setminus s_2.\text{DONE}_p = s_2.\text{FREE}_p$ and $B = \mathcal{J} \setminus t_2.\text{DONE}_q = t_2.\text{FREE}_q$, thus from the contradiction assumption we have that: $|A \cap B| \leq (q - p) \cdot m$ and $|A \cap B| \leq (q - p) \cdot m$.

It could either be that $|A| < |B|$ or $|A| \geq |B|$.

**Case 1** $|A| < |B|$: From the contradiction assumption we have that $|A \cap B| \leq (q - p) \cdot m$. Thus we have that:

$$|t_2.\text{FREE}_q \setminus t_2.\text{TRY}_q \cap s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p| \leq m(q - p) + m - 1 \quad (1)$$

Since $s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p$ can have up to $m - 1$ less elements than $A$ - the elements of set $s_2.\text{TRY}_p$ - and it can be the case that $s_2.\text{TRY}_p \cap t_2.\text{TRY}_q = \emptyset$.

Moreover, since $s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p \subseteq A$ and $|s_2.\text{FREE}_p \setminus s_2.\text{TRY}_p| \geq \beta \geq 3m^2$, $|A| \geq 3m^2$. Similarly $|B| \geq 3m^2$. We have:

$$\frac{(q - 1)|B|}{m} = \frac{p - 1)|B|}{m} + \frac{(q - p)|B|}{m} > \frac{(p - 1)|A|}{m} + \frac{(q - p)|B|}{m} \Rightarrow (2)$$

$$\Rightarrow (q - 1)|B| > (p - 1)|A| + 3m(q - p) \Rightarrow (3)$$

$$\Rightarrow \left(\frac{(q - 1)|B| - (m - 1)}{m}\right) + 1 \geq \left(\frac{(p - 1)|A| - (m - 1)}{m}\right) + 1 + 3m(q - p) \Rightarrow (4)$$

12
Since \(s_2'.\text{NEXT}_{p} = t_2'.\text{NEXT}_{q} = i\), it must be the case that \([i]_{s_2.\text{FREE}_{p}\setminus s_2.\text{TRY}_{p}} = \left( (p - 1)\frac{|A|-(m-1)}{m} + 1\right)\) and \([i]_{t_2.\text{FREE}_{q}\setminus t_2.\text{TRY}_{q}} = \left( (q - 1)\frac{|B|-(m-1)}{m} + 1\right)\). Equation 4 gives that \([i]_{s_2.\text{FREE}_{p}\setminus s_2.\text{TRY}_{p}} \geq [i]_{s_2.\text{FREE}_{p}\setminus s_2.\text{TRY}_{p}} + 3m(q - p)\). This means that \(t_2.\text{FREE}_{q} \setminus t_2.\text{TRY}_{q}\) must have at least \(3m(q - p)\) more elements with rank less than the rank of \(i\), than set \(s_2.\text{FREE}_{p} \setminus s_2.\text{TRY}_{p}\). This is a contradiction since from \(7\) we have that \(|t_2.\text{FREE}_{q} \setminus t_2.\text{TRY}_{q} \cap s_2.\text{FREE}_{p} \setminus s_2.\text{TRY}_{p}| \leq m(q - p) + m - 1\)

**Case 2** \(|B| \leq |A|\): We have that \(|A \cap B| \leq (q - p) \cdot m\) and \(|A \cap B| \leq (q - p) \cdot m\) from the contradiction assumption. Thus we have that:

\[
|t_2.\text{FREE}_{q} \setminus t_2.\text{TRY}_{q} \cap s_2.\text{FREE}_{p} \setminus s_2.\text{TRY}_{p}| \leq m(q - p) + m - 1
\]

Since \(s_2.\text{FREE}_{p} \setminus s_2.\text{TRY}_{p}\) can have up to \(m - 1\) less elements than \(A\) - the elements of set \(s_2.\text{TRY}_{p}\) - and it can be the case that \(s_2.\text{TRY}_{p} \cap t_2.\text{TRY}_{p} = \emptyset\).

From the contradiction assumption and the case 2 assumption we have that \(|B| \leq |A| \leq |B| + (q - p)m\).

Moreover \(|A| \geq \beta \geq 3m^2\) and \(|B| \geq \beta \geq 3m^2\). We have:

\[
(q - 1)\frac{|B| + (q - p)m}{m} = (p - 1)\frac{|B| + (q - p)m}{m} + (q - p)\frac{|B| + (q - p)m}{m} \geq
\]

\[
\geq (p - 1)\frac{|A|}{m} + (q - p)\frac{|B|}{m} + 3m(q - p) + (q - p)^2 \Rightarrow
\]

\[
\Rightarrow (q - 1)\frac{|B|}{m} \geq (p - 1)\frac{|A|}{m} + 3m(q - p) + (q - p)^2 - (q - 1)(q - p) \Rightarrow
\]

\[
\Rightarrow (q - 1)\frac{|B|}{m} \geq (p - 1)\frac{|A|}{m} + (3m - p - 1)(q - p) \Rightarrow
\]

\[
\Rightarrow (q - 1)\frac{|B|}{m} \geq (p - 1)\frac{|A|}{m} + 2m(q - p) \Rightarrow
\]

\[
\Rightarrow (q - 1)\frac{|B| - (m - 1)}{m} + 1 \geq (p - 1)\frac{|A| - (m - 1)}{m} + 1 + 2m(q - p)
\]

Since \(s'_2.\text{NEXT}_{p} = t'_2.\text{NEXT}_{q} = i\), it must be the case that \([i]_{s_2.\text{FREE}_{p}\setminus s_2.\text{TRY}_{p}} = \left( (p - 1)\frac{|A|-(m-1)}{m} + 1\right)\) and \([i]_{t_2.\text{FREE}_{q}\setminus t_2.\text{TRY}_{q}} = \left( (q - 1)\frac{|B|-(m-1)}{m} + 1\right)\). Equation 6 gives that \([i]_{s_2.\text{FREE}_{p}\setminus s_2.\text{TRY}_{p}} \geq [i]_{s_2.\text{FREE}_{p}\setminus s_2.\text{TRY}_{p}} + 2m(q - p)\). This means that \(t_2.\text{FREE}_{q} \setminus t_2.\text{TRY}_{q}\) must have at least \(2m(q - p)\) more elements with rank less than the rank of \(i\) than set \(s_2.\text{FREE}_{p} \setminus s_2.\text{TRY}_{p}\). This is a contradiction since from \(7\) we have that \(|t_2.\text{FREE}_{q} \setminus t_2.\text{TRY}_{q} \cap s_2.\text{FREE}_{p} \setminus s_2.\text{TRY}_{p}| \leq m(q - p) + m - 1\).

Next we are going to prove that if 2 processes \(p, q \in \mathcal{P}\) with \(p < q\) “collide” three times, their DONE sets at the third collision will contain at least \(m(q - p)\) more jobs than they did at the first collision. This will allow us to find an upper bound on the collisions a process may participate in. It is possible that both processes become aware of a collision or only one of them does while the other one successfully completes the job. At the proofs that follow, for a state \(s\) in execution \(\alpha\) we define as \(s.\text{DONE}\) the following set: \(s.\text{DONE} = \{i \in J \mid \exists p \in \mathcal{P} \land j \in \{1, \ldots, n\} : s.\text{done}_{p}(j) = i\}\). We also need the following definitions.

**Definition 5.1** In an execution \(\alpha \in \text{execs}(\mathcal{K}\mathcal{K}_\beta)\), we say that process \(p\) collided with process \(q\) in job \(i\) at state \(s\), if (i) there exist in \(\alpha\) transitions \((s_1, \text{compNext}_{p}, s'_1)\), \((t_1, \text{compNext}_{q}, t'_1)\) and \((s_2, \text{check}_{p}, s'_2)\), with \(s_1 < s_2\) and \(t_1 < s_2\), \(s'_1.\text{NEXT}_{p} = t'_2.\text{NEXT}_{q} = s_2.\text{NEXT}_{p} = i\), \(s'_1.\text{STATUS}_{p} = t'_1.\text{STATUS}_{q} = \text{set}_{\text{next}}\), \(s'_2.\text{STATUS}_{p} = \text{comp}_{\text{next}}\), (ii) let \(\alpha'\) be the execution fragment that begins with state \(s'_1\) and ends with state \(s_2\), there exists no action \(\pi_1 = \text{comp}_{\text{next}} p \in \alpha'\) and either there exists in \(\alpha'\) transition
\( (s, \text{gatherTry}_p, s') \) such that \( s.Q_p = q, s.next_q = i \), or transition \( (s, \text{gatherDone}_p, s') \) and \( j \in \{1, \ldots, n\} \) such that \( s.Q_p = q, s.POS_p(q) = j, s.done_{q,j} = i \) and \( i \notin s.TRY_p \).

According to Def. 5.1, process \( p \) collided with process \( q \) in job \( i \) at state \( s \), if process \( p \) attempted to preform job \( i \), but was not able to, because it detected in state \( s \) that either process \( q \) was trying to perform job \( i \) or process \( q \) has already performed job \( i \).

**Definition 5.2** In an execution \( \alpha \in \text{execs}(\text{KK}_\beta) \), we say that processes \( p, q \) collide in job \( i \) at state \( s \), if process \( p \) collided with process \( q \) or process \( q \) collided with process \( p \) in job \( i \) at state \( s \), according to Definition 5.1.

**Lemma 5.2** In an execution \( \alpha \in \text{execs}(\text{KK}_\beta) \) for any \( \beta \geq m \) if there exist processes \( p, q \), jobs \( i_1, i_2 \in \mathcal{J} \) and states \( s_1 < s_2 \) such that process \( p \) collided with process \( q \) in job \( i_1 \) at state \( s_1 \) and in job \( i_2 \) at state \( s_2 \) according to Definition 5.1 then there exist transitions \( (s_1, \text{compNext}_p, s_1'), (s_2, \text{compNext}_p, s_2') \), \( (t_1, \text{compNext}_q, t_1'), (t_2, \text{compNext}_q, t_2') \) where \( s_1'.\text{NEXT}_p = t_1'.\text{NEXT}_q = i_1, s_2'.\text{NEXT}_p = t_2'.\text{NEXT}_q = i_2, s_1'.\text{STATUS}_p = s_2'.\text{STATUS}_p = t_1'.\text{STATUS}_q = t_2'.\text{STATUS}_q = \text{set}_\text{next} \) and there exists no action \( \pi_1 = \text{compNext}_p \) for which \( s_1 < \pi_1 < s_1, s_2 < \pi_1 < s_2 \) such that: \( s_1 < s_2 \) and \( t_1 < t_2 \).

**Proof.** From Definition 5.1 we have that there exist transitions \( (s_1, \text{compNext}_p, s_1'), (s_2, \text{compNext}_p, s_2') \) with \( s_1'.\text{NEXT}_p = i_1, s_2'.\text{NEXT}_p = i_2, s_1'.\text{STATUS}_p = s_2'.\text{STATUS}_p = \text{set}_\text{next} \), and there exists no action \( \pi_1 = \text{compNext}_p \) for which \( s_1 < \pi_1 < s_1 \) or \( s_2 < \pi_1 < s_2 \). From the later and the fact that \( s_1 < s_2 \), it must be the case that \( s_1 < s_1' < s_2 < s_2' \). Furthermore from Definition 5.1 we have that there exist transitions \( (t_1, \text{compNext}_q, t_1'), (t_2, \text{compNext}_q, t_2') \) with \( t_1'.\text{NEXT}_q = i_1, t_2'.\text{NEXT}_q = i_2, t_1'.\text{STATUS}_q = t_2'.\text{STATUS}_q = \text{set}_\text{next} \), such that \( t_1' < t_1 \) and \( t_2' < t_2 \). We can pick those transitions in \( \alpha \) in such a way that there exists no other transition between \( t_1' \) and \( s_1 \) that sets \( \text{next}_q \) to \( i_1 \) and similarly there exists no other transition between \( t_2' \) and \( s_2 \) that sets \( \text{next}_q \) to \( i_2 \). We need to prove now that \( t_1 < t_2 \). We will prove this by contradiction.

Let \( t_2 < t_1 \). Since \( t_1' < s_1 \), we have that \( t_2 < t_1 < t_1' < s_1 < s_2 < s_2' \). Since from Definition 5.1 either \( s_1.\text{next}_p = i_1 \) or there exists \( j \in \{1, \ldots, n\} \) such that \( s_1.\text{done}_{q,j} = i_1 \), it must be the case that \( s_2.\text{STATUS}_p = \text{gather}_\text{done} \), \( s_2.Q_p = q \) and there exists \( j' \in \{1, \ldots, n\} \) such that \( s_2.\text{done}_{p,j'} = i_2 \). This means that there exists transition \( (t_3, \text{done}_q, t_3') \) and \( j' \in \{1, \ldots, n\} \) such that \( t_3'.\text{done}_{p,j'} = i_2 \) and \( t_2 < t_3' < t_1 < t_1' < s_1 < s_2 < s_2' \).

If \( s_1.\text{STATUS}_p = \text{gather}_\text{try} \) then from algorithm \( \text{KK}_\beta \) we have that \( s_1.\text{DONE} \subseteq s_2.\text{DONE}_p \) and as a result \( i_2 \in s_2.\text{DONE}_p \), which is a contradiction since \( (s_2, \text{compNext}_p, s_2') \notin \text{trans}(\text{KK}_\beta) \) if \( i_2 \in s_2.\text{DONE}_p \) and \( s_2'.\text{STATUS}_p = \text{set}_\text{next} \).

If \( s_1.\text{STATUS}_p = \text{gather}_\text{done} \) then from algorithm \( \text{KK}_\beta \) we have that \( s_1.Q_p = q \) and there exists \( j \in \{1, \ldots, n\} \) such that \( s_1.\text{POS}_p(q) = j \) and \( s_1.\text{done}_{q,j} = i_1 \). Since \( t_2 < t_3' < t_1 < t_1' < s_1 < s_2 < s_2' \) it must be the case that \( j' < j \) and as a result \( i_2 \in s_1.\text{DONE}_p \). Clearly \( s_1.\text{DONE} \subseteq s_2.\text{DONE}_p \), which is a contradiction since \( (s_2, \text{compNext}_p, s_2') \notin \text{trans}(\text{KK}_\beta) \) if \( i_2 \in s_2.\text{DONE}_p \) and \( s_2'.\text{NEXT}_p = i_2, s_2'.\text{STATUS}_p = \text{set}_\text{next} \).

**Lemma 5.3** In an execution \( \alpha \in \text{execs}(\text{KK}_\beta) \) for any \( \beta \geq m \) if there exist processes \( p, q \), jobs \( i_1, i_2 \in \mathcal{J} \) and states \( s_1 < s_2 \) such that process \( p \) collided with process \( q \) in job \( i_1 \) at state \( s_1 \) and process \( q \) collided with process \( p \) in job \( i_2 \) at state \( s_2 \) according to Definition 5.1 then there exist transitions \( (s_1, \text{compNext}_p, s_1'), (s_2, \text{compNext}_p, s_2') \).
\[(s_2, \text{compNext}_p, s_2'), (t_1, \text{compNext}_q, t_1'), (t_2, \text{compNext}_q, t_2') \text{ where } s_1' = \text{compNext}_p = t_1', \text{compNext}_q = i_1,\]
\[s_2' = \text{compNext}_p = t_2', \text{compNext}_q = i_2, s_1'.\text{STATUS}_p = s_2'.\text{STATUS}_p = \text{set}_\text{next}, \text{ and there exists no actions } \pi_1 = \text{compNext}_p, \pi_2 = \text{compNext}_q \text{ for which } s_1 < \pi_1 < \tilde{s}_1, t_2 < \pi_2 < \tilde{s}_2 \text{ such that}:\]

\[s_1 < s_2 \text{ and } t_1 < t_2.\]

**Proof.** From Definition [5.1] we have that there exist transitions \((s_1, \text{compNext}_p, s_1')\), \((s_2, \text{compNext}_p, s_2')\) with \(s_1'.\text{NEXT}_p = i_1, s_2'.\text{NEXT}_p = i_2, s_1'.\text{STATUS}_p = s_2'.\text{STATUS}_p = \text{set}_\text{next}, \text{ and there exists no action } \pi_1 = \text{compNext}_p \text{ for which } s_1 < \pi_1 < \tilde{s}_1. \text{ Furthermore from Definition } [5.1] \text{ we have that there exist transitions } \((t_1, \text{compNext}_q, t_1'), (t_2, \text{compNext}_q, t_2')\) \text{ with } t_1'.\text{NEXT}_q = i_1, t_2'.\text{NEXT}_q = i_2, t_1'.\text{STATUS}_q = t_2'.\text{STATUS}_q = \text{set}_\text{next, and there exists no action } \pi_2 = \text{compNext}_q \text{ for which } t_2 < \pi_2 < \tilde{s}_2. \text{ From the later and the fact that } \tilde{s}_1 < \tilde{s}_2, \text{ it must be the case that } t_1 < \tilde{s}_1 < t_2 < \tilde{s}_2. \text{ We can pick the transitions that are enabled by states } t_1 \text{ and } s_2 \text{ in } \alpha \text{ in such a way that there exists no other transition between } t_1' \text{ and } \tilde{s}_1 \text{ that sets } \text{NEXT}_q \text{ to } i_1 \text{ and similarly there no other transition between } s_2' \text{ and } \tilde{s}_2 \text{ that sets } \text{NEXT}_p \text{ to } i_2. \text{ We need to prove now that } s_1 < s_2. \text{ We will prove this by contraction.}\]

Let \(s_2 < s_1. \text{ From algorithm KK}_\beta \text{ there exist transitions } (s_3, \text{setNext}_p, s_3'), (s_4, \text{done}_p, s_4') \) and \((t_3, \text{setNext}_q, t_3'), \text{ where } s_3'.\text{next}_p = i_2, s_4'.\text{next}_p = i_2, t_3'.\text{next}_q = i_1 \text{ and } s_2 < s_3' < s_4 < s_1, t_1 < t_3 < t_2. \text{ There are 2 cases, either } s_3' < t_3' \text{ or } t_3' < s_3'.\]

**Case 1** \(s_3' < t_3': \text{ We have that } s_3' < t_3' < t_2 \text{ and } (t_2, \text{compNext}_q, t_2'), \text{ where } t_2'.\text{NEXT}_q = i_2 \text{ and } t_2'.\text{STATUS}_q = \text{set}_\text{next \ which means that } i_2 \notin t_2'.\text{TRY}_q \cup t_2'.\text{DONE}_q. \text{ This is a contradiction since the } t_2'.\text{TRY}_q \text{ and } t_2'.\text{DONE}_q \text{ actions are computed by gatherTry}_q \text{ and gatherDone}_q \text{ actions that are preceded by state } s_3'. \text{ So either } i_2 \notin t_2'.\text{TRY}_q \text{ or } i_2 \notin t_2'.\text{DONE}_q, \text{ since a new setNext}_p \text{ action may take place only after state } s_4'.\]

**Case 2** \(t_3' < s_3': \text{ We have that } t_3' < s_3' < s_1 \text{ and } (s_1, \text{compNext}_p, s_1'), \text{ where } s_1'.\text{NEXT}_p = i_1 \text{ and } s_1'.\text{STATUS}_p = \text{set}_\text{next \ which means that } i_1 \notin s_1'.\text{TRY}_p \cup s_1'.\text{DONE}_p. \text{ This is a contradiction since the } s_1'.\text{TRY}_p \text{ and } s_1'.\text{DONE}_p \text{ sets are computed by gatherTry}_p \text{ and gatherDone}_p \text{ actions that are preceded by state } t_3'. \text{ There exists transition } (s_4, \text{gatherTry}_p, s_4') \text{ in } \alpha \text{ with } s_4'.Q_p = q \text{ such that there exists no } \pi_1 = \text{compNext}_p \text{ where } s_4' < \pi_1 < s_1. \text{ If } s_4'.\text{next}_q = i_1 \text{ we have a contradiction since } i_1 \notin s_1'.\text{TRY}_p. \text{ If } s_4'.\text{next}_q \neq i_1 \text{ there exists an action } \pi_2 = \text{setNext}_q \text{ in } \alpha, \text{ such that } t_3 < \pi_2 < s_4. \text{ If this } \pi_2 = \text{setNext}_q \text{ is preceded by transition } (t_4, \text{done}_q, t_4') \text{ with } t_4'.\text{NEXT}_q = i_1, \text{ we have a contradiction since } i_1 \notin t_4'.\text{DONE}_q \text{ and } s_1'.\text{DONE}_p \text{ is computed by gatherDone}_p \text{ actions that are preceded by state } t_4', \text{ which results in } i_1 \notin s_1'.\text{DONE}_p. \text{ If there exists no such transition we have again a contradiction since state } \tilde{s}_1 \text{ as defined by Definition } [5.1] \text{ could not belong in } \alpha.\]

**Lemma 5.4** If \(\beta \geq m \text{ and in an execution } \alpha \in \text{execs(KK}_\beta) \text{ there exist processes } p \neq q, \text{ jobs } i_1, i_2, i_3 \in J \text{ and states } \tilde{s}_1 < \tilde{s}_2 < \tilde{s}_3 \text{ such that } p, q \text{ collide in job } i_1 \text{ at state } \tilde{s}_1, \text{ in job } i_2 \text{ at state } \tilde{s}_2 \text{ and in job } i_3 \text{ at state } \tilde{s}_3 \text{ according to Definition } [5.2] \text{ then there exist states } s_1 < s_3 \text{ and } t_1 < t_3 \text{ such that}\]

\[s_1.\text{DONE}_p \cup t_1.\text{DONE}_q \subseteq s_3.\text{DONE}_p \cap t_3.\text{DONE}_q\]

\[|s_3.\text{DONE}_p \cup t_3.\text{DONE}_q| - |s_1.\text{DONE}_p \cup t_1.\text{DONE}_q| \geq m \cdot |q - p|\]

15
Proof. From Definitions 5.1 5.2 we have that there exist transitions \((s_1, \text{compNext}_p, s_1'), (s_2, \text{compNext}_p, s_2'), (s_3, \text{compNext}_p, s_3')\) and \((t_1, \text{compNext}_q, t_1'), (t_2, \text{compNext}_q, t_2'), (t_3, \text{compNext}_q, t_3')\), where \(s_1\text{.next}_p = t_1\text{.next}_q = i_1, s_2\text{.next}_p = t_2\text{.next}_q = i_2, s_3\text{.next}_p = t_3\text{.next}_q = i_3, s_1\text{.status}_p = s_2\text{.status}_p = s_3\text{.status}_p = t_1\text{.status}_q = t_2\text{.status}_q = t_3\text{.status}_q = \text{set\_next}\text{ and } s_1 < s_1, t_1 < s_1, s_2 < s_2, t_2 < s_2, \text{ and } s_3 < s_3, t_3 < s_3.\) We pick from \(\alpha\) the transitions \((s_1, \text{compNext}_p, s_1'), (t_1, \text{compNext}_q, t_1')\), in such a way that there exists no other \text{compNext}_p, \text{compNext}_q\ between states \(s_1, s_1\) respectively \(t_1, t_1\) that sets \text{next}_p, \text{next}_q\ to \(i_1\). We can pick in a similar manner the transitions for jobs \(i_2, i_3\). From Lemmas 5.2 5.3 and Definitions 5.1 5.2 we have that \(s_1 < s_2 < s_3\) and \(t_1 < t_2 < t_3.\) We will first prove that:

\[
s_1\text{.DONE}_p \cup t_1\text{.DONE}_q \subseteq s_3\text{.DONE}_p \cap t_3\text{.DONE}_q
\]

From algorithm KK_\(\beta\) we have that there exists in \(\alpha\) transitions \((s_4, \text{setNext}_p, s_4'), (t_4, \text{setNext}_q, t_4')\) with \(s_4\text{.next}_p = i_2, t_4\text{.next}_q = i_2\) and there exist no action \(\pi_1 = \text{compNext}_p\), such that \(s_2 < \pi_1 < s_4\), and no action \(\pi_2 = \text{compNext}_q\), such that \(t_2 < \pi_2 < t_4\). We will prove that \(t_1 < s_4\) and \(s_1 < t_4\). We start by proving that \(t_1 < s_4\). In order to get a contradiction we assume that \(s_4 < s_1\). From algorithm KK_\(\beta\) we have that there exists in \(\alpha\) transition \((t_4, \text{gatherTry}_q, t_4')\), with \(t_4\text{.Q}_q = p\), and there exists no action \(\pi_2 = \text{compNext}_q\), such that \(t_4' < \pi_2 < t_2\). We have that \(s_4 < t_1 < t_4' < t_2\) and \(i_2 \notin t_2\text{.TRY}_q \cup t_2\text{.DONE}_q.\) If \(t_4\text{.next}_p = i_2\) we have a contradiction since \(i_2 \notin s_2\text{.TRY}_q.\) If \(t_4\text{.next}_q \neq i_2\) there exists an action \(\pi_3 = \text{setNext}_p\) in \(\alpha\), such that \(s_4 < \pi_3 < t_4.\) If this \(\pi_3 = \text{setNext}_p\) is preceded by transition \((s_5, \text{done}_p, s_5')\) with \(s_5\text{.next}_p = i_2\) we have a contradiction since \(i_2 \notin t_4\text{.DONE} \cup t_2\text{.DONE}_q\) is computed by gatherDone actions that are preceded by state \(t_4\), which results in \(i_2 \notin t_2\text{.DONE}_q.\) If there exists no such transition we have again a contradiction state \(s_2\) as defined by Definition 5.2 could not belong in \(\alpha\).

From the discussion above we have that \(t_1 < s_4\). Thus \(t_1\text{.DONE}_q < s_4\text{.DONE}_p\), moreover \(s_3\text{.DONE}_p\) is computed by gatherDone actions that are preceded by state \(s_4\), from which we have that \(t_1\text{.DONE}_q \subseteq s_3\text{.DONE}_p.\) It is easy to see that \(s_1\text{.DONE}_p \subseteq s_3\text{.DONE}_p\) holds, thus we have that \(s_1\text{.DONE}_p \cup t_1\text{.DONE}_q \subseteq s_3\text{.DONE}_p.\) With similar arguments as before, we can prove that \(s_1\text{.DONE}_p \cup t_1\text{.DONE}_q \subseteq t_3\text{.DONE}_q\), which gives us that \(s_1\text{.DONE}_p \cup t_1\text{.DONE}_q \subseteq s_3\text{.DONE}_p \cap t_3\text{.DONE}_q.\)

Now it only remains to prove that:

\[
|s_3\text{.DONE}_p \cup t_3\text{.DONE}_q| - |s_1\text{.DONE}_p \cup t_1\text{.DONE}_q| > m \cdot |q - p|
\]

If \(p < q\) from Lemma 5.1 we have that \(|s_3\text{.DONE}_p \cap t_3\text{.DONE}_q| > (q - p)m\) or \(|s_3\text{.DONE}_p \cap t_3\text{.DONE}_q| > (q - p)m\). Since \(s_1\text{.DONE}_p \cup t_1\text{.DONE}_q \subseteq s_3\text{.DONE}_p \cap t_3\text{.DONE}_q,\) we have that:

\[
|s_3\text{.DONE}_p \cup t_3\text{.DONE}_q| - |s_1\text{.DONE}_p \cup t_1\text{.DONE}_q| > (q - p) \cdot m
\]

If \(q < p\) with similar arguments we have that:

\[
|s_3\text{.DONE}_p \cup t_3\text{.DONE}_q| - |s_1\text{.DONE}_p \cup t_1\text{.DONE}_q| > (p - q) \cdot m
\]

Combining the above we have:

\[
|s_3\text{.DONE}_p \cup t_3\text{.DONE}_q| - |s_1\text{.DONE}_p \cup t_1\text{.DONE}_q| > m \cdot |q - p|
\]

\(\square\)
Lemma 5.5 If $\beta \geq 3m^2$ there exists no execution $\alpha \in \text{execs(KK}_\beta)$ at which process $p$ collided with process $q$ in more than $2 \left[\frac{n}{m|q-p|}\right] + 1$ states according to Definition 5.7.

Proof. Let execution $\alpha \in \text{execs(KK}_\beta)$ be an execution at which process $p$ collided with process $q$ in at least $2 \left[\frac{n}{m|q-p|}\right] + 1$ states. Let us examine the first $2 \left[\frac{n}{m|q-p|}\right] + 1$ such states. Let those states be $\bar{s}_1 < \bar{s}_2 < \ldots < \bar{s}_2 \left[\frac{n}{m|q-p|}\right] + 1$. From Lemma 5.2 we have that there exists states $s_1 < s_2 < \ldots < s_2 \left[\frac{n}{m|q-p|}\right] + 1$ that enable the $\text{compNext}_p$ actions and states $t_1 < t_2 < \ldots < t_2 \left[\frac{n}{m|q-p|}\right] + 1$ that enable the $\text{compNext}_q$ actions that lead to the collisions in states $\bar{s}_1 < \bar{s}_2 < \ldots < \bar{s}_2 \left[\frac{n}{m|q-p|}\right] + 1$. Then from Lemma 5.4 we have that $\forall i \in \{1, \ldots, \left[\frac{n}{m|q-p|}\right]\}$:

\[
|s_{2i+1}.\text{DONE}_p \cup t_{2i+1}.\text{DONE}_q| - |s_{2i-1}.\text{DONE}_p \cup t_{2i-1}.\text{DONE}_q| > m|q-p| \tag{7}
\]

\[
|s_{2i+1}.\text{DONE}_p \cup t_{2i+1}.\text{DONE}_q| - |s_1.\text{DONE}_p \cup t_1.\text{DONE}_q| > im|q-p| \tag{8}
\]

\[
|s_{2i+1}.\text{DONE}_p \cup t_{2i+1}.\text{DONE}_q| > im|q-p| \tag{9}
\]

From (9) we have that:

\[
|s_2 \left[\frac{n}{m|q-p|}\right] + 1.\text{DONE}_p \cup t_2 \left[\frac{n}{m|q-p|}\right] + 1.\text{DONE}_q| > m|q-p| \left[\frac{n}{m|q-p|}\right] \geq n \tag{10}
\]

Equation (10) leads to a contradiction since $s_2 \left[\frac{n}{m|q-p|}\right] + 1.\text{DONE}_p \cup t_2 \left[\frac{n}{m|q-p|}\right] + 1.\text{DONE}_q \subseteq \mathcal{J}$ and $|\mathcal{J}| = n$. □

Theorem 5.6 If $\beta \geq 3m^2$ algorithm KK$_\beta$ has work complexity $W_{KK}_\beta = O(nm \log n \log m)$.

Proof. We start with the observation that in any execution $\alpha$ of algorithm KK$_\beta$, if there exists process $p$, job $i$, transition $(s_1, \text{done}_p, s'_1)$ and $j \in \{1, \ldots, n\}$ such that $s_1.\text{POS}_p(p) = j$, $s_1.\text{NEXT}_p = i$, for any process $q \neq p$ there exists at most one transition $(t_1, \text{gatherDone}_q, t_1')$ in $\alpha$, with $t_1.\text{Q}_q = p$, $t_1.\text{POS}_q(p) = j$ and $t_1 \geq s_1$. Such a transition corresponds exactly one read operation from the shared memory, one insertion at the set $\text{DONE}_q$ and one removal from the set $\text{FREE}_q$, thus such a transition costs $O(\log n)$ work. Clearly there exist at most $m - 1$ such transitions for each $\text{done}_p$. From Lemma 4.1 for all processes there can be at most $n$ actions $\text{done}_p$ in any execution $\alpha$ of algorithm KK$_\beta$. Each $\text{done}_p$ action performs one write operation in shared memory, one insertion at the set $\text{DONE}_q$ and one removal from the set $\text{FREE}_q$, thus such an action has cost $O(\log n)$ work. Furthermore any $\text{done}_p$ is preceded by $m - 1$ $\text{gatherTry}_p$ read actions that read the next array and each add at most one element to the set $\text{TRY}_p$ with cost $O(\log n)$ and $m - 1$ $\text{gatherDone}_p$ read actions that do not add elements in the $\text{DONE}_p$ set. Note that we have already counted the $\text{gatherDone}_p$ read actions that result in adding jobs at the $\text{DONE}_p$ set. Finally any $\text{done}_p$ action is preceded by one $\text{compNext}_p$ action. This action is dominated by the cost of $\text{rank}(\text{FREE}_p, \text{TRY}_p, i)$ function that has cost $O(m \log n)$, if the sets $\text{FREE}_p$, $\text{TRY}_p$ are represented with some efficient tree structure that allows insertion, deletion and search of an element in $O(\log n)$. We discussed at Section 3 what such tree structures could be. That gives us a total of bound of $O(n m \log n)$ work associated with the $\text{done}_p$ actions.

If a process $p$ collided with a process $q$ in job $i$ at state $s$, we have extra an extra $\text{compNext}_p$ action, $m - 1$ extra $\text{gatherTry}_p$ read actions and insertions in the $\text{TRY}_p$ set and $m - 1$ $\text{gatherDone}_p$ read actions that do not add elements in the $\text{DONE}_p$ set. Thus each collision costs $O(m \log n)$ work. Since $\beta \geq 3m^2$ from Lemma 5.5 for two distinct processes $p, q$ we have that in any execution $\alpha$ of algorithm KK$_\beta$ there
exist less than \(2 \left[ \frac{n}{m(q-p)} \right] \) collisions. For process \(p\) if we count all such collisions with any other process \(q\) we get:

\[
\sum_{q \in \mathcal{P} - \{p\}} 2 \left[ \frac{n}{m(q-p)} \right] \leq 2(m-1) + \frac{2n}{m} \sum_{q \in \mathcal{P} - \{p\}} \frac{1}{q-p} \leq 2(m-1) + \frac{4n}{m} \sum_{i=1}^{\frac{m}{n}} \frac{1}{i} \leq 2(m-1) + \frac{4n}{m} \log m
\]

(11)

If we count the total number of collisions for all the \(m\) processes we get that if \(\beta \geq 3m^2\) in any execution of algorithm \(\text{KK}_\beta\) there can be at most \(2m^2 + 4m \log m < 4(n+1) \log m\) collisions (since \(n > \beta\)). Thus collisions cost \(O(nm \log n \log m)\) work. Finally any process \(p\) that fails may add in the work complexity less than \(O(m \log n)\) work from its \(\text{compNext}_p\) action and from reads (if the process fails without performing a \(\text{done}_p\) action after its latest \(\text{compNext}_p\) action). So for the work complexity of algorithm \(\text{KK}_\beta\) if \(\beta \geq 3m^2\) we have that \(W_{\text{KK}_\beta} = O(nm \log n \log m)\).

\[
\frac{W_{\text{KK}_\beta}}{W_{\text{Comp}}(\beta)} = \frac{O(nm \log n \log m)}{O(n^2 \log n \log n)} = \frac{O(n \log n)}{O(n)} = \frac{\log n}{n}
\]

\[\frac{\log n}{n} \leq \frac{1}{\sqrt{n}} \quad \text{for} \quad n \geq 2
\]

\[\frac{1}{\sqrt{n}} < \frac{1}{n} < \frac{1}{n^2}
\]

\[\therefore \quad \frac{W_{\text{KK}_\beta}}{W_{\text{Comp}}(\beta)} < \frac{1}{n^2}
\]

\[\therefore \quad \text{KK}_\beta \text{ is asymptotically better than Comp}
\]

6 An Asymptotically Work Optimal Algorithm

Here we demonstrate how to use algorithm \(\text{KK}_\beta\) with \(\beta = 3m^2\) if \(m = O(\sqrt{n})\), in order to solve the at-most-once problem with effectiveness \(n - O(m^2 \log n \log m)\) and work complexity \(O(n + m(3+\epsilon) \log n)\), for any constant \(\epsilon > 0\), such that \(1/\epsilon\) is a positive integer. We construct algorithm \(\text{IterativeKK}(\epsilon)\) Fig. 2 that performs iterative calls to a variation of \(\text{KK}_\beta\), which we call \(\text{IterStepKK}\). \(\text{IterativeKK}(\epsilon)\) has \(3 + 1/\epsilon\) distinct \(\text{done}\) matrices in shared memory, with different granularities. One \(\text{done}\) matrix, stores the regular jobs performed, while the remaining \(2 + 1/\epsilon\) matrices store \(\text{super-jobs}\). \(\text{Super-jobs}\) are groups of consecutive jobs. From them, one stores super-jobs of size \(m \log n \log m\), while the remaining \(1 + 1/\epsilon\) matrices, store super-jobs of size \(m^{1-\epsilon} \log n \log m^{1+\epsilon}\) for \(i \in \{1, \ldots, 1/\epsilon\}\).

| IterativeKK(\(\epsilon\)) for process \(p\): |
|-----------------------------|
| 00  size\(_{p,1}\) \(\leftarrow 1\) |
| 01  size\(_{p,2}\) \(\leftarrow m \log n \log m\) |
| 02  FREE\(_{p}\) \(\leftarrow \text{map}(\mathcal{J},\text{size}_{p,1},\text{size}_{p,2})\) |
| 03  FREE\(_{p}\) \(\leftarrow \text{IterStepKK}(\text{FREE}_{p},\text{size}_{p,2})\) |
| 04  for(\(i \leftarrow 1, i \leq \lfloor 1/\epsilon \rfloor, i + +\) |
| 05  size\(_{p,1}\) \(\leftarrow \text{size}_{p,2}\) |
| 06  size\(_{p,2}\) \(\leftarrow m^{1-\epsilon} \log n \log m^{1+\epsilon}\) |
| 07  FREE\(_{p}\) \(\leftarrow \text{map}(\text{FREE}_{p},\text{size}_{p,1},\text{size}_{p,2})\) |
| 08  FREE\(_{p}\) \(\leftarrow \text{IterStepKK}(\text{FREE}_{p},\text{size}_{p,2})\) |
| 09  endfor |
| 10  size\(_{p,1}\) \(\leftarrow \text{size}_{p,2}\) |
| 11  size\(_{p,2}\) \(\leftarrow 1\) |
| 12  FREE\(_{p}\) \(\leftarrow \text{map}(\text{FREE}_{p},\text{size}_{p,1},\text{size}_{p,2})\) |
| 13  FREE\(_{p}\) \(\leftarrow \text{IterStepKK}(\text{FREE}_{p},\text{size}_{p,2})\) |

Figure 2: Algorithm \(\text{IterativeKK}(\epsilon)\): pseudocode

The algorithm \(\text{IterStepKK}\) is different from \(\text{KK}_\beta\) in three ways. First, all instances of \(\text{IterStepKK}\) work for \(\beta = 3m^2\). Moreover \(\text{IterStepKK}\) has a termination flag in shared memory. This termination flag is initially 0 and is set to 1 by any process that decides to terminate. Any process that discovers that \(|\text{FREE}_{p} \setminus \text{TRY}_{p}| < 3m^2\) in its \(\text{compNext}_p\) action, sets the termination flag to 1, computes new \(\text{FREE}_{p}\) and \(\text{TRY}_{p}\) set, returns the set \(\text{FREE}_{p} \setminus \text{TRY}_{p}\) and terminates the current iteration. Any process \(p\) that checks if it is safe to perform a job, checks the termination flag first and if the flag is 1, the process instead of performing
the job, computes new FREE_p and TRY_p set, returns the set FREE_p \ TRY_p and terminates the current iteration. Finally, IterStepKK takes as inputs the variable size and a set SET_1, such that |SET_1| > 3m^2, and returns the set SET_2 as output. SET_1 contains super-jobs of size size. In IterStepKK, with an action do_p,j process p performs all the jobs of super-job j. IterStepKK performs as many super-jobs as it can and returns in SET_2 the super-jobs, which it can verify that no process will perform upon the termination of the algorithm IterStepKK. In IterativeKK (\epsilon) we use also the function SET_2 = map (SET_1, size_1, size_2), that takes the set of super-jobs SET_1, with super-jobs of size size_1 and maps it to a set of super-jobs SET_2 with size size_2.

**Theorem 6.1** Algorithm IterativeKK (\epsilon) has work complexity W_{IterativeKK(\epsilon)} = O(n + m^{3+\epsilon} \log n) and and effectiveness E_{IterativeKK(\epsilon)}(n, m, f) = n - O(m^2 \log n \log m).

**Proof.** In order to determine the effectiveness and work complexity of algorithm IterativeKK (\epsilon), we compute the jobs preformed by and the work spend in each invocation of IterStepKK. Moreover we compute the work that the invocations to the map () function add. The first invocation to function map () in line 02 can be completed by process p with work O(m n log n log m log n), since process p needs to construct a tree with m log n log m elements. This contributes for all processes O(m n log n) work. From Theorem 5.6 we have that IterStepKK in line 03 has total work O(n + m n log n log m log n log m) = O(n), where the first n comes from do actions and the second term from the work complexity of Theorem 5.6. Note that we count O(1) work for each normal job executed by a do action on a super-job. That means that in the invocation of IterStepKK in line 03, do actions cost m log n log m work. Moreover from Theorem 4.4 we have effectiveness m log n log m (3m^2 + m - 2) on the super-jobs of size m log n log m. From the super-jobs not completed, up to m − 1 may be contained in the TRY_p sets upon termination in line 03. Since those super-jobs are not added (and thus are ignored) in the output FREE_p set in line 03, up to (m − 1)m log n log m jobs may not be performed by IterativeKK (\epsilon). The set FREE_p returned by algorithm IterStepKK in line 03 has no more than 3m^2 + m − 2 super-jobs of size m log n log m.

In each repetition of the loop in lines 04 − 09, the map () function in line 07 constructs a FREE_p set with at most O(m^{2+\epsilon} / log m) elements, which costs O(m^{2+\epsilon}) per process p for a total of O(m^{3+\epsilon}) work for all processes. Moreover each invocation of IterStepKK in line 08 costs from Theorem 5.6 O(3m^3 \log n log m n + m^{3+\epsilon} \log m) \leq O(m^{3+\epsilon} \log n) work, where the term 3m^3 \log n log m log n is an upper bound on the work needed for the do actions on the super-jobs. From Theorem 4.4 we have that each output FREE_p set in line 08 has at most 3m^2 + m − 2 super-jobs. Moreover from each invocation of IterStepKK in line 08 at most m − 1 super-jobs are lost in TRY sets. Those account for less than (m − 1)m log n log m jobs in each iteration, since the size of the super-jobs in the iterations of the loop in lines 04 − 09 is strictly less than m log n log m.

When we leave the loop in lines 04 − 09, we have a FREE_p set with at most 3m^2 + m − 2 super-jobs of size log n log^{1+1/\epsilon} m, which means that in line 12 function map () will return a set FREE_p with less than (3m^2 + m − 2)(log n log^{1+1/\epsilon} m) elements that correspond to jobs and not super-jobs. This costs for all processes a total of O(m^3 \log n log m log n log log m) < O(m^{3+\epsilon} \log n), since \epsilon is a constant. Finally we have that IterStepKK in line 13 has from Theorem 5.6 work O(m^3 \log^2 m \log n log log m) < O(m^{3+\epsilon} \log n), also from Theorem 4.4 it has effectiveness (3m^2 + m − 2)(log n log^{1+1/\epsilon} m) − (3m^2 + m − 2)

If we add up all the work we have that W_{IterativeKK(\epsilon)} = O(n + m^{3+\epsilon} \log n) since the loop in lines 04 − 09 repeats 1 + 1/\epsilon times and \epsilon is a constant. Moreover for the effectiveness, we have that less that or equal to (m − 1)m log n log m jobs will be lost in the TRY set at line 03. After that strictly less than (m − 1)m log n log m jobs will be lost in the TRY sets of the iterations of the loop in lines 04 − 09 and less than 3m^2 + m − 2 jobs will be lost from the effectiveness of the last invocation of IterStepKK in line 13. Thus we have that E_{IterativeKK(\epsilon)}(n, m, f) = n - O(m^2 \log n \log m). \hfill \square

For any \epsilon = O(3+\epsilon/\sqrt{n/\log n}), algorithm IterativeKK (\epsilon) is work optimal and asymptotically effectiveness optimal.
6.1 An Asymptotically Optimal Work Complexity Algorithm for the Write-All Problem

\texttt{WA\_IterativeKK (} \epsilon \texttt{) for process } p:\nn00 \texttt{size}_{p,1} \leftarrow 1\nn01 \texttt{size}_{p,2} \leftarrow m \log n \log m\nn02 \texttt{FREE}_p \leftarrow \texttt{map (} J, \texttt{size}_{p,1}, \texttt{size}_{p,2}\texttt{)}\nn03 \texttt{FREE}_p \leftarrow \texttt{WA\_IterStepKK (} \texttt{FREE}_p, \texttt{size}_{p,2}\texttt{)}\nn04 \texttt{for} (i \leftarrow 1, i \leq 1/\epsilon, i + +)\nn05 \texttt{size}_{p,1} \leftarrow \texttt{size}_{p,2}\nn06 \texttt{size}_{p,2} \leftarrow m^{1-i\epsilon} \log n \log 1+i m\nn07 \texttt{FREE}_p \leftarrow \texttt{map (} \texttt{FREE}_p, \texttt{size}_{p,1}, \texttt{size}_{p,2}\texttt{)}\nn08 \texttt{FREE}_p \leftarrow \texttt{WA\_IterStepKK (} \texttt{FREE}_p, \texttt{size}_{p,2}\texttt{)}\nn09 \texttt{endfor}\nn10 \texttt{size}_{p,1} \leftarrow \texttt{size}_{p,2}\nn11 \texttt{size}_{p,2} \leftarrow 1\nn12 \texttt{FREE}_p \leftarrow \texttt{map (} \texttt{FREE}_p, \texttt{size}_{p,1}, \texttt{size}_{p,2}\texttt{)}\nn13 \texttt{FREE}_p \leftarrow \texttt{WA\_IterStepKK (} \texttt{FREE}_p, \texttt{size}_{p,2}\texttt{)}\nn14 \texttt{for} (i \in \texttt{FREE}_p)\n15 \quad \texttt{do}_{p,i}\nn16 \texttt{endfor}\n
Figure 3: Algorithm \texttt{WA\_IterativeKK (} \epsilon \texttt{)}: pseudocode

Based on \texttt{IterativeKK (} \epsilon \texttt{)} we construct algorithm \texttt{WA\_IterativeKK (} \epsilon \texttt{)} Fig. 3 that solves the Write-All problem \cite{14} with work complexity \( O(n + m^{3+\epsilon} \log n) \), for any constant \( \epsilon > 0 \), such that \( 1/\epsilon \) is a positive integer. From Kanellakis and Shvartsman \cite{14} the Write-All problem for the shared memory model, consists of: “Using \( m \) processors write 1’s to all locations of an array of size \( n \)” Algorithm \texttt{WA\_IterativeKK (} \epsilon \texttt{)} is different from \texttt{IterativeKK (} \epsilon \texttt{)} in two ways. It uses a modified version of \texttt{IterStepKK}, that instead of returning the \( \texttt{FREE}_p \setminus \texttt{TRY}_p \) set upon termination returns the set \( \texttt{FREE}_p \) instead. Let us name this modified version \texttt{WA\_IterStepKK}. Moreover in \texttt{WA\_IterativeKK (} \epsilon \texttt{)} after line 13 process \( p \) instead of terminating, executes all tasks in the set \( \texttt{FREE}_p \). Note that since we are interested in the Write-All problem, when process \( p \) performs a job \( i \) with action \( \texttt{do}_{p,i} \), process \( p \) just writes 1, in the \( i \)-th position of the Write All array \( wa[1, \ldots, n] \) in shared memory.

**Theorem 6.2** Algorithm \texttt{WA\_IterativeKK (} \epsilon \texttt{)} solves the Write-All problem with work complexity \( W_{\texttt{WA\_IterativeKK (} \epsilon \texttt{)}} = O(n + m^{3+\epsilon} \log n) \).

**Proof.** (of Theorem 6.2) We prove this with similar arguments as in the proof of Theorem 6.1. As in the proof of Theorem 6.1 after each invocation of \texttt{WA\_IterStepKK} the output set \( \texttt{FREE}_p \) has less than \( 3m^2 + m - 1 \) super-jobs, from Theorem 4.4. The difference is that now we don’t leave jobs in the \( \texttt{TRY}_p \) sets, since we are not interested in maintaining the at-most-once property between successive invocations of the \texttt{WA\_IterStepKK} algorithm. Since after each invocation of \texttt{WA\_IterStepKK} the output set \( \texttt{FREE}_p \) has the same upper bound on super-jobs as in \texttt{IterativeKK (} \epsilon \texttt{)}, with similar arguments as in the proof of Theorem 6.1 we have that at line 13 the total work performed by all processes is \( O(n + m^{3+\epsilon} \log n) \). Moreover from Theorem 4.4 the output \( \texttt{FREE}_p \) set in line \( p \) has less \( 3m^2 + m - 2 \) jobs. This gives us for all processes a total work of \( O(m^{3+\epsilon}) \) for lines the loop in lines 14 – 16. After the loop in lines 14 – 16 all jobs have been performed, since we left no \( \texttt{TRY} \) sets behind, thus algorithm \texttt{WA\_IterativeKK (} \epsilon \texttt{)} solves the Write-All problem with work complexity \( W_{\texttt{WA\_IterativeKK (} \epsilon \texttt{)}} = O(n + m^{3+\epsilon} \log n) \).

For any \( m = O(\sqrt[3]{n/\log n}) \), algorithm \texttt{WA\_IterativeKK (} \epsilon \texttt{)} is work optimal.
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