ADJOINTS OF IDEALS IN REGULAR LOCAL RINGS

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Introduction. Several existing results labeled “Briançon-Skoda theorem” concern an ideal \( I \) in a regular local ring \( R \). Of these, the weakest states that if \( I \) is generated by \( \ell \) elements, then \( \overline{I^{n+\ell-1}} \subseteq I^n \) (\( n > 0 \)), where “\( \overline{\cdot} \)” denotes “integral closure.” In this paper, we associate to \( I \) an integrally closed ideal \( \tilde{I} \supseteq \overline{I} \), the \textit{adjoint} of \( I \), and indicate how it can be used in place of \( \overline{I} \) to improve such results. At first, in Theorem (1.4.1), this just involves a recycling of methods from [LS]. (Even that is not without benefit, see Corollary (1.4)). But there’s more. It’s not hard to show that there is an \( n_0 \) such that \( \tilde{I}^{n+1} = \overline{I^n} \) for all \( n \geq n_0 \). The basic conjecture (1.6)—which, as we’ll see, quickly implies a number of recently proved Briançon-Skoda-type theorems—says that \( n_0 \) can be taken to be \( \ell(I) - 1 \), where \( \ell(I) \) is the analytic spread of \( I \). This conjecture does hold when \( R \) is essentially of finite type over a field of characteristic zero, or when \( \dim R = 2 \).

Section 2 deals with a conjecture, related to Grauert-Riemenschneider vanishing, about certain cohomology groups being zero. Suppose there exists a proper birational map \( f : Y \to \text{Spec}(R) \) such that \( IO_Y \) is invertible and \( Y \) is \textit{nonsingular}, i.e., locally regular. (The existence of such a \( Y \) in all characteristics is not yet certain, but it is needed in the vanishing conjecture.) Let \( \omega_Y \) be a dualizing sheaf for \( f \), chosen to be canonical in the sense that its restriction to the open set \( U \) where \( f \) is an isomorphism is \( \mathcal{O}_U \). While the definition (1.1) of \( \tilde{I} \) uses neither \( Y \) nor any duality theory, Proposition (1.3.1) states that

\[
\tilde{I} := H^0(Y, I \omega_Y);
\]

and the vanishing conjecture states that

\[
H^i(Y, I \omega_Y) = 0 \quad (i > 0).
\]

The point is that this vanishing conjecture implies conjecture (1.6). In fact it is thanks to Cutkosky’s transcendental proof of the vanishing conjecture [C] that we know (1.6) holds in characteristic zero.

Section 3 elaborates on the two-dimensional case (where the vanishing conjecture is known to hold, see Remark (2.2.1)(b)). A geometrically motivated treatment of the adjoint of a simple complete ideal \( I \) is given in [L4]; close connections with the multiplicity sequence and the conductor ideal of the local ring \( \mathfrak{a} \) of the “generic curve through \( I \)” are brought out. Roughly speaking, \( \mathfrak{a} \) is the local ring at the generic point of the exceptional divisor (a \( \mathbb{P}^1 \)) on the blowup \( Y_0 \) of any 2-generated reduction \( I_0 \) of \( I \). Propositions (3.1.1) and (3.1.2) below explore such connections for an arbitrary integrally closed \( I \). If \( Y \) is the normalization of \( Y_0 \), then \( I \omega_Y \) is just the conductor \( \mathcal{E} := \mathcal{O}_{Y_0} : \mathcal{O}_Y \) (so \( \mathcal{E} \) is independent of the choice of \( I_0 \)), and it is generated by its global sections \( \tilde{I} \). We also find in Proposition (3.3) that \( \tilde{I} = I_0 : I \).

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Furthermore, in [HS] Huneke and Swanson have shown that \( \widetilde{I} \) is just the second Fitting ideal \( \mathfrak{F}_2(I) \); and more generally (since \( I = \mathfrak{F}_1(I) \)), that for all \( n > 0 \), 
\[ \mathfrak{F}_n(I) = \mathfrak{F}_{n+1}(I). \]

Let me mention in closing that though the material in [L4] dates back to 1966, the results in this paper all came out of an effort to analyze the Briançon-Skoda theorem (3.3) in [AH1].

1. Adjoint and Briançon-Skoda theorems. Let \( R \) be a regular noetherian domain with fraction field \( K \), let \( v \) be a valuation of \( K \) whose valuation ring \( R_v \) (with maximal ideal \( m_v \)) contains \( R \), and let \( h \) be the height of the prime ideal \( p := m_v \cap R \).
We say that \( v \) is a prime divisor of \( R \) if \( R_v / m_v \) has transcendence degree \( h - 1 \) over its subfield \( R_p / pR_p \). It is equivalent that \( R_v \) be essentially of finite type over \( R \), or that \( v \) be a Rees valuation of some \( R \)-ideal \( I \), i.e., that \( R_v \) be \( R \)-isomorphic to the local ring of a point on the normalized blowup \( \overline{Y}_I := \text{Proj}(\oplus_{n \geq 0} \overline{I}^n) \), where \( \overline{I}^n \) is the integral closure of \( I^n \). Such a \( v \) is a discrete rank-one valuation. (See [A, p. 300, Thm. 1 (4) and p. 336, Prop. 3]. Note also that \( R \), being universally catenary, satisfies the “dimension formula” [EGA III, (5.6.4) and (5.6.1) (c)]; and that \( \overline{Y}_I \) is of finite type over \( R \) [R, p. 27, Thm. 1.5].)

Definition (1.1). The adjoint of an \( R \)-ideal \( I \) is the ideal
\[ \widetilde{I} := \bigcap_v \{ r \in K \mid v(r) \geq v(I) - v(J_{R_v/R}) \} \]
where the intersection is taken over all prime divisors \( v \) of \( R \), and for any essentially finite-type \( R \)-subalgebra \( S \) of \( K \), the Jacobian ideal \( J_{S/R} \) is the 0-th Fitting ideal of the \( S \)-module of Kähler differentials \( \Omega^1_{S/R} \).

Remarks (1.2). (a) \( \widetilde{I} \subset R \) because \( R \) is the intersection of its localizations at height one primes, and each such localization is the valuation ring of a \( v \) for which \( v(J_{R_v/R}) = 0 \). Hence
\[ \widetilde{I} = \bigcap_v \{ r \in R \mid v(r) \geq v(I) - v(J_{R_v/R}) \} \]
where the intersection is taken over all prime divisors \( v \) such that \( v(I) > 0 \).

(b) Being an intersection of valuation ideals, \( \widetilde{I} \) is integrally closed; and if \( \bar{I} \) is the integral closure of \( I \) then
\[ I \subset \bar{I} \subset \widetilde{I} = \widetilde{\bar{I}}. \]

(c) For any \( x \in R \), we have \( \widetilde{xI} = x \widetilde{I} \). In particular, \( x \widetilde{R} = xR \).
(d) For any two \( R \)-ideals \( I, J \), we have \( \widetilde{JI} : I = \widetilde{J} \). In particular,
\[ \widetilde{I^{n+1}} : I = \widetilde{I^n} \quad (n \geq 0). \]

(1.3). For any finite-type birational map \( f : Y \to \text{Spec}(R) \), we may—and will—identify \( \mathcal{O}_Y \) with a subsheaf of the constant sheaf \( K \) on \( Y \), so that the

\[\text{We consider two valuations with the same valuation ring to be identical.}\]
stalks \( \mathcal{O}_{Y,y} \) \((y \in Y)\) are all \( R \)-subalgebras of \( K \). If \( g: Z \to \text{Spec}(R) \) is another such map which factors via \( f \), then \( g \) is uniquely determined by \( Z \) and \( Y \), and we say that \( Z \) dominates \( Y \). The relative Jacobian \( \mathcal{J}_f \) (or, less precisely, \( \mathcal{J}_Y \)) is the coherent \( \mathcal{O}_Y \)-module whose sections over any affine open \( \text{Spec}(A) \subset Y \) are given by

\[
H^0(\text{Spec}(A), \mathcal{J}_Y) = \mathcal{J}_{A/R}.
\]

We set

\[
\omega_Y := \mathcal{O}_Y: \mathcal{J}_Y \cong \mathcal{H}om_Y(\mathcal{J}_Y, \mathcal{O}_Y).
\]

If \( Y \) is normal, \( \omega_Y \) is a canonical relative dualizing sheaf for \( f \) [LS, p. 206, (2.3)].

For any proper birational \( f: Y \to \text{Spec}(R) \) with \( Y \) normal and \( IO_Y \) invertible, we set

\[
\widetilde{I}_Y := H^0(Y, I\omega_Y),
\]

the ideal obtained by restricting the intersection in Definition (1.1) to those \( v \) such that \( R_v \) is \( \mathcal{O}_{Y,y} \) for some \( y \in Y \). So \( \widetilde{I} \subset \widetilde{I}_Y \), and \( \widetilde{I}_Y \) is a “decreasing” function of \( Y \) in the sense that for any proper birational \( g: Z \to Y \) with \( Z \) normal, we have \( \widetilde{I}_Z \subset \widetilde{I}_Y \). For any prime divisor \( w \), \( R_{w} \) is the local ring of a point on some such \( Z \); so the intersection of all \( \widetilde{I}_Z \) is just \( \widetilde{I} \).

**Proposition (1.3.1).** For any \( Y \) as above and having pseudo-rational singularities (for example, \( Y \) regular), \( \widetilde{I}_Y = \widetilde{I} \). If such a \( Y \) exists then for any multiplicative system \( M \) in \( R \), \( \widetilde{I}R_M = \widetilde{I}_R \).

**Proof.** The pseudo-rationality assumption forces \( g_*(I\omega_Z) = I\omega_Y \) for all \( g: Z \to Y \) as above (by [LT, p. 107, Corollary], and since \( IO_Y \) is invertible), whence

\[
\widetilde{I}_Z = H^0(Z, I\omega_Z) = H^0(Y, g_*I\omega_Z) = \widetilde{I}_Y,
\]

and \( \widetilde{I}_Y = \cap Z \widetilde{I}_Z = \widetilde{I} \). The rest follows from compatibility of \( H^0(Y, I\omega_Y) \) with localization on \( R \). \( \square \)

**Remarks (1.3.2).** (a) That a regular \( Y \) with \( IO_Y \) invertible always exists has been announced by Spivakovsky, but details have not appeared at the time of this writing. For the equicharacteristic zero case, see [H].

(b) In dimension 2, every normal \( Y \) birationally dominating \( \text{Spec}(R) \) has pseudo-rational singularities, [L1, p. 212, §9], [LT, p. 103, Example (a)]. So in Prop. (1.3.1), we could take \( Y \) to be the normalized blowup of \( I \).

(c) Another case in which we could take \( Y \) in (1.3.1) to be the blowup of \( I \) is when \( I = (x_1, \ldots, x_r, y) \) where \((x_1, \ldots, x_r)\) is a regular sequence such that \( R/(x_1, \ldots, x_r)R \) is still regular [LS, p. 219, Proposition, (iii)]. Here \( \widetilde{I}^n_Y \) \((n \geq 0)\) is easily calculated: indeed, if \( L \) is any \( R \)-ideal generated by a regular sequence of length \( \ell \), and such that all the powers of \( L \) are integrally closed, then the blowup \( X \) of \( L \) is normal and \( \mathcal{J}_X = (L^{\ell-1} \mathcal{O}_X)^2 \) so that

\[
\widetilde{I}^n_X = R \quad \text{for} \quad (n < \ell)
\]

\[
= L^{n-\ell+1} \quad \text{for} \quad (n \geq \ell).
\]

\( ^2 \)Because for a regular sequence \((a_1, \ldots, a_\ell)\), we have, e.g.,

\[
R_{\ell} := R[a_1 a_\ell, \ldots, a_{\ell-1} a_\ell] \cong R[t_1, \ldots, t_{\ell-1}]/(a_\ell t_1 - a_1, \ldots, a_\ell t_{\ell-1} - a_{\ell-1}),
\]

and so \( \mathcal{O}_{X,a} \) is generated by the differentials \( d(a_i/a_\ell) \) subject to \( d(a_\ell/a_\ell) = 0 \) \((1 \leq i < \ell-1)\).
For $L = I$, we have $\ell = r + 1$ or $r$.

(1.4). The following is clearly related to the Briançon-Skoda theorem in [LS, p. 204, Thm. 1”]. (Recall the above-given inclusion $\bar{I} \subset \tilde{I}$, where $\bar{I}$ and $\tilde{I}$ are the integral closure and adjoint, respectively, of the $R$-ideal $I$.)

We say that $I$ is $\ell$-generated ($\ell \geq 0$) if $I$ is generated by $\ell$ elements.

**Theorem (1.4.1).** For any $\ell$-generated ideal $I$ in a regular noetherian domain $R$:

(i) $I^{n+\ell-1} \subset I^n$ for all $n \gg 0$.

(ii) If the graded ring $\text{gr}_I R := \oplus_{n \geq 0} I^n/I^{n+1}$ contains a homogeneous regular element of positive degree, then (i) holds for all $n \geq 1$.

(iii) $\tilde{I}^{n+\ell} \subset \tilde{I}^n$ for all $n \geq 0$.

**Proof.** If $Y_0 := \text{Proj}(\oplus_{n \geq 0} I^n)$ is the blowup of $I$, and $Y$ is its normalization, then as in [LS, p. 200, Thm. 2 and proof of Corollary], we have $I^{\ell-1}\omega_Y \subset \mathcal{O}_{Y_0}$ (all inside the constant sheaf $K$ on $Y_0$). Hence

$$\tilde{I}^{n+\ell-1} \subset H^0(Y, I^{n+\ell-1}\omega_Y) \subset H^0(Y_0, I^n\mathcal{O}_{Y_0}) = \bigcup_{j \geq 0} I^{n+j} : I^j.$$  

For $n \gg 0$, $H^0(Y_0, I^n\mathcal{O}_{Y_0}) = I^n$ (by e.g., [EGA III, (2.3.1)]),$^3$ proving (i).

If $\text{gr}_I R$ has a homogeneous regular element of positive degree, then $I^{n+j} : I^j = I^n$, proving (ii).

In (i), the restriction of $n$ to sufficiently large values is annoying, and may well be unnecessary (see Conjecture (1.6) below). If so, then (iii)—and the following ungainly argument—would be superfluous.

The polynomial ring $R[t]$ is still regular. An immediate consequence of the following Lemma is that for any $R$-ideal $L$, $\tilde{L}R[t] \subset LR[t]$. (The adjoints are taken in $R$ and $R[t]$ respectively.) With $I' := (I, t)R[t]$, we have that $\text{gr}_{I'} R[t] \cong (\text{gr}_I R)[t]$ has a regular element (namely $t$) of degree 1, and we can apply (ii) to get $I^{n+\ell} \subset I^n$ for all $n \geq 2$; and since $I^{n+1}R[t] \subset (I^{n+1}R[t])^\sim \subset I^{n+\ell}$ and $I^{n+1} \cap R = I^n$, therefore (iii) results.

**Lemma (1.4.2).** Let $w$ be a prime divisor of the polynomial ring $R[t]$ and let $v$ be the restriction of $w$ to $K$, the fraction field of $R$. Then $v$ is a prime divisor of $R$, and for any $R$-ideal $L$,

$$v(L) - v(J_{R_v/R}) \geq w(L) - w(J_{R_w/R}[t]).$$

**Proof.** Let $(R_w, m_w)$ and $(R_v, m_v)$ be the (discrete) valuation rings of $w$ and $v$ respectively. Set $q := m_w \cap R_v[t]$. There are two cases to consider.

(1) $q = m_v R_v[t]$. Then the localization $R_v[t]/q$ is a discrete valuation ring contained in, and hence equal to, $R_v$. Thus $R_w/m_w = (R_w/m_w)(t)$ has transcendence degree (t.d.) 1 over $R_w/m_w$.

(2) $q \supsetneq m_v R_v[t]$, whence $q$ is maximal, of height 2, and $R_v[t]/q$ is algebraic over $R_v/m_v$. Since $R_w$ is essentially of finite type over $R[t]$, hence over $R_v[t]$, therefore $R_w$ is a prime divisor of $R_v[t]$; and so $R_w/m_w$ has t.d. 1 over $R_v[t]/q$. Thus, again, $R_w/m_w$ has t.d. 1 over $R_v/m_v$.

Now set $p' := m_w \cap R[t]$ and $p := p' \cap R = m_v \cap R$, so that $R_w/m_w$ has t.d. height $(p') - 1$ over $R[t]/p'$, and, by the preceding remarks, the t.d. of $R_v/m_v$ over $R/p$ is height $(p') - 2 +$ the t.d. of $R[t]/p'$ over $R/p$. It follows then from [ZS, p. 323, Prop. 1A] that $R_v/m_v$ has t.d. height $(p) - 1$ over $R/p$, and so $v$ is indeed a prime divisor of $R$.

$^3$For a more elementary proof, apply [ZS, pp. 154–155, Lemmas 4 and 5] to the ideal $\mathfrak{B} := (0)$ in $\text{gr}_I R$ to find an integer $q$ such that for any $n$ and any $x \in H^0(Y_0, I^n\mathcal{O}_{Y_0}) \setminus I^n$, $x \notin I^q$ (because the leading form of $x$ annihilates all homogeneous elements of large degree ...). Such an $x$ must lie in $\mathfrak{B}_{\tilde{I}}$. But there exists $p$ such that $\mathfrak{B}_{\tilde{I}}^{n+p} \subset \tilde{I}^n$, and therefore $p \leq q$. 


The last assertion follows from the relation

\[ J_{R_w/R[t]} = J_{R_w/R} J_{R_w[t]/R[t]} = J_{R_w/R} J_{R_w/R}. \]

(See [LS, p. 201, (1.1)] for the first equality.) \( \square \)

Suppose now that \( R \) is local, with maximal ideal \( m \). For an \( R \)-ideal \( I \), the analytic spread \( \ell(I) \) is the dimension of the ring \( \oplus_{n \geq 0} I^n/mI^n \). When \( R/m \) is infinite, \( I \) has an \( \ell(I) \)-generated reduction \( I_0 \subset I \), i.e., \( I_0I^n = I^{n+1} \) for some \( n \geq 0 \).

**Corollary (1.4.3).** For \( R \) local, assertions (i) and (iii) in Theorem (1.4.1) hold with \( \ell \) the analytic spread of \( I \). And if \( I \) has an \( \ell \)-generated reduction \( I_0 \) such that \( gr_{I_0}R \) contains a homogeneous regular element of positive degree, then (i) holds for all \( n \geq 0 \).

**Proof.** By arguing as in the proof of (1.4.1)(iii), with \( R[t] \) replaced by its localization \( S := R[t]/mR[t] \), and \( I' := IS \), we reduce to the case where \( R/m \) is infinite. Then we can apply (1.4.1) to an \( \ell \)-generated reduction \( I_0 \), noting that for any valuation \( v \) such that \( R_v \) contains \( R \) we have \( v(I_0) = v(I) \), whence \( \widetilde{I}_p = \widetilde{I}^p \) for all \( p \geq 0 \). \( \square \)

The following statement was conjectured by Huneke.

**Corollary (1.4.4).** If \( (R, m) \) is a \( d \)-dimensional regular local ring and \( I \) is an \( m \)-primary ideal, then for all \( n \geq 1 \),

\[ \widetilde{I}^{n+d-1} : m^{d-1} \subset \widetilde{I}^n. \]

**Proof.** Replacing \( (R, I) \) by \( (S := R[t]/mR[t], IS) \) if necessary, we may assume that \( R/m \) is infinite. Then \( I \) has a \( d \)-generated reduction \( I_0 \) such that \( gr_{I_0}R \) is a polynomial ring in \( d \) variables over \( R/I_0 \); so Corollary (1.4.3) gives \( \widetilde{I}^{n+d-1} : \subset \widetilde{I}^n. \) Thus it suffices to show that \( \widetilde{I}^{n+d-1} : m^{d-1} \subset \widetilde{I}^{n+d-1} \), for which it’s clearly enough (see (1.2) (a)) that for any prime divisor \( v \) of \( R \) such that \( m_v \cap R = m \),

\[ v(J_{R_v/R}) \geq v(m^{d-1}). \]

But \( R_v \) contains \( R' := R[x_2/x_1, \ldots, x_d/x_1] \) for some generating set \( (x_1, x_2, \ldots, x_d) \) of \( m \), and then

\[ J_{R_v/R} = J_{R_v/R'} J_{R'/R} = m^{d-1} J_{R_v/R'} \]

(see [LS, p. 201, (1.1) and top of p. 202]), which gives the desired result. \( \square \)

**Lemma (1.5).** Let \( R \) be a regular noetherian domain, let \( I \) be an \( R \)-ideal, and set \( G := \oplus_{n \geq 0} I^n, \widetilde{G} := \oplus_{n \geq 0} \widetilde{I}^n. \) Then \( \widetilde{G} \) is a finitely generated graded \( G \)-module, and hence there is an \( n_0 \) such that

\[ \widetilde{I}^{n+1} = \widetilde{I}^n \quad \text{for all} \quad n \geq n_0. \]

**Proof.** \( \widetilde{G} \) is a graded \( G \)-module because, clearly, \( I^p \widetilde{I}^q \subset \widetilde{I}^{p+q} \) \( (p, q \geq 0) \). Now just observe, with \( Y \) the normalized blowup of \( I \), that by (1.3), \( \widetilde{G} \) is a submodule of \( \oplus_{n \geq 0} H^0(Y, I^n \omega_Y) \), which is finitely generated over \( G \) [EGA III, (3.3.2)]. \( \square \)

As we’ll see in (2.3) below, the following refinement of Lemma (1.5) holds true when \( R \) is essentially of finite type over a characteristic-zero field, or when \( \dim R = 2 \). (The 2-dimensional case also results from Prop. (3.1.2); see also Prop. (4.2) of [HS]. For another example, see Remark (1.3.2)(a)).
Conjecture (1.6). Let $R$ be a regular local ring, and let $I$ be an $R$-ideal of analytic spread $\ell$. Then

$$\widehat{I^{n+1}} = I\widehat{I^n} \quad \text{for all } n \geq \ell - 1.$$  

We illustrate the usefulness of this conjecture (when it holds) by indicating how it implies some Briançon-Skoda-type theorems recently proved for equicharacteristic regular local rings by Aberbach and Huneke. These theorems are all of the form

$$I_{n+\ell-1} \subset I_n \quad (n > 0),$$

where the “coefficient ideal” $A$ depends only on $I$. Under the assumption that (1.6) holds, we need only show that $\widehat{I_{\ell-1}} \subset A$ in order to get the stronger assertion

$$\widehat{I_{n+\ell-1}} = I_n \widehat{I_{\ell-1}} \subset I_n A.$$  

(1.6.1). In [AH2] $A$ is taken to be the sum of all ideals $A'$ such that $IA' = \widehat{IA'}$. By (1.6) and (1.2)(b), $I\widehat{I_{\ell-1}} = I_{\ell} = \widehat{I_{\ell-1}}$, so that $\widehat{I_{\ell-1}} \subset A$.  

(1.6.2). In [AH1, p. 350, Thm. 3.3], $A$ is taken to be the intersection of the primary components of $I_{\ell-h}$ belonging to the minimal primes $p_1, \ldots, p_e$ of $I$, where $h := \max_i h_i := \max_i \text{height}(p_i)$. (To check that $\ell \geq h$, just localize at each $p_i$.)

To show that $\widehat{I_{\ell-1}} \subset A$, localize at $p = p_i (1 \leq i \leq e)$, and note that

$$\widehat{I_{\ell-1}} R_p \subset I\widehat{I_{\ell-1}} R_p \subset I_{\ell-h_i} \subset I_{\ell-h},$$

where the first inclusion is elementary, and the second is given by (1.4.1)(ii).

Moreover, if (1.6) holds, then, with $I_p := IR_p$, we have

$$\widehat{I_{\ell-1}} = I_{\ell-h} \widehat{I_{p_{i=1}h_i}} = I_{\ell-h} \widehat{I_{p_{i=1}h_i-1}};$$

and hence if $\widehat{I_{\ell-1}} R_p = \widehat{I_{\ell-1}}$ for all $p_i$ (see Prop. (1.3.1)), then $\widehat{I_{\ell-1}}$ is contained in the intersection of the primary components of $I_{\ell-h} \widehat{I_{h-1}}$ belonging to the $p_i$.

(1.6.3). In [AHT, Thm. 7.6], the above-mentioned Theorem 3.3 of [AH1] is strengthened. Here the inductive description of $A$ is somewhat complicated. So suffice it to say that the inclusion $\widehat{I_{\ell-1}} \subset A$ can be established by alternately localizing at suitable associated primes of height $i$ and applying (1.6), as $i$ goes, one step at a time, from $\ell - 1$ down to the height of $I$.

2. A vanishing conjecture. Again, let $I$ be an ideal in a regular local ring $(R, m)$. Throughout this section we make the following assumption—which is satisfied at least over varieties in characteristic zero [H, p. 143, Cor. 1], or whenever $\dim R = 2$, as follows e.g., from the Hoskin-Deligne formula, see [L3, p. 223, (3.1.1)].

Assumption (2.1). There exists a map $f: Y \to \text{Spec}(R)$ which factors as a sequence of blowups with nonsingular centers, such that $I\mathcal{O}_Y$ is invertible.

The basic conjecture (1.6) will be deduced from the following vanishing conjecture.
Vanishing Conjecture (2.2). With $I$ and $f: Y \to \text{Spec}(R)$ as above,

$$H^i(Y, I\omega_Y) = 0 \text{ for all } i > 0.$$ 

Remarks(2.2.1). (a) Cutkosky has proved the vanishing conjecture for local rings essentially of finite type over a field of characteristic zero, see [C]. He uses Kodaira vanishing, which fails, in general, in positive characteristic—but that does not preclude the conjecture holding for special maps such as $f$.

(b) It was noted in (1.3) that $\omega_Y$ is a dualizing sheaf for $f$. By duality [L2, p. 188], the conjecture is equivalent to the vanishing of $H^i_E(Y, (IO_Y)^{-1})$ for all $i < \dim R$, where $E := f^{-1}\{m\}$ is the closed fiber. For $d = 2$, this dual assertion is proved in [L2, p. 177, Thm. 2.4].

(c) For $I = R$, the conjecture is a form of Grauert-Riemenschneider vanishing, and is readily proved by induction on the number of blowups making up the map $f$. For arbitrary $I$, the conjecture is equivalent to the vanishing of $H^i(Y, Q)$ for all $i > 0$ and every invertible quotient $Q$ of a finite direct sum of copies of $\omega_Y$ (because $IO_Y$ is a quotient of a direct sum of copies of $O_Y$ ...)

Moreover, if $g: Z \to \text{Spec}(R)$ is the normalized blowup of $I$ and $h: Y \to Z$ is the domination map, then using the Leray spectral sequence for $f = gh$, and ampleness of $IO_Z$, one shows that the vanishing of $R^ih_*\omega_Y$ ($i > 0$) is equivalent to the vanishing of $H^i(Y, I^n\omega_Y)$ for all $n \gg 0$. In other words, Conjecture (2.1) is somewhat stronger than Grauert-Riemenschneider vanishing for “sandwiched singularities.”

(d) Theorem (4.1) of [L6, p. 153] shows that there is an $R$-ideal $L$ such that $Y$ in (2.1) is the blowup of $L$, i.e., the Proj of the Rees ring $R[Lt]$ (a indeterminate), and such that furthermore $R[Lt]$ is Cohen-Macaulay (CM). This leads to another conjecture which can be shown to imply the vanishing one:

CM Conjecture. Let $L = II'$, with $L, I$ and $I'$ integrally closed $R$-ideals, and assume that $R[Lt]$ is CM and normal. Then for some $e > 0$, the ideal $IR[L^{et}]$ (which is divisorial) is CM as an $R[L^{et}]$-module.

(2.3). We show next that Conjecture (1.6) follows from the vanishing conjecture.5 Thus (1.6) does hold for local rings of smooth points of algebraic varieties in characteristic zero, or when $\dim R = 2$. (See the preceding remarks (a) and (b).)

We first reduce to the case where $R/m$ is infinite by passing, as usual, to $S := R[t]_{m,R[t]}$. We have already seen, in proving (1.4.1)(iii), that for any $R$-ideal $L$, $\widetilde{LS} \subset \widetilde{LS}$; but now we need equality, which we get by applying Prop.(1.3.1) to $Y \otimes_R S$, with $Y$ as in (2.1). (I don’t know a more elementary way!)

Now let $I_0 = (a_1, \ldots, a_t)R$ be a reduction of $I$, so that $I_0O_Y = IO_Y$. Let $F$ be the direct sum of $\ell$ copies of $(I_0O_Y)^{-1}$, and let $\sigma: F \to O_Y$ be the $O_Y$-homomorphism defined by the sequence $(a_1, \ldots, a_t)$. Then we have a Koszul complex

$$K(F, \sigma): 0 \to \Lambda F \to \Lambda^{\ell-1}F \to \cdots \to \Lambda^1F \xrightarrow{\sigma} O_Y \to 0$$

(see [LT, p. 111]) which is locally split, so that $K(F, \sigma) \otimes I^{n+1}\omega_Y$ ($n \geq \ell - 1$) is exact. By (2.2), and with $H^i(-) := H^i(Y, -)$,

$$H^1(I^{n-1}\omega_Y) = H^2(I^{n-2}\omega_Y) = \cdots = H^{\ell-1}(I^{n+1-\ell}\omega_Y) = 0.$$
Hence, as in [LT, p. 112, Lemma (5.1)] we can conclude that

$$H^0(I^{n+1}\omega_Y) = IH^0(I^n\omega_Y),$$

i.e., by Proposition (1.3.1), $\tilde{I}^{n+1} = \tilde{I}^n$. □

3. Dimension 2. Except in Lemma (3.2.1), $(R, m)$ will be a two-dimensional regular local ring and $I$ will be an $m$-primary $R$-ideal. The purpose of this section is to give a number of alternative descriptions of $\tilde{I}$.

(3.1). It is pointed out in the footnote on p. 235 of [L4] that when $I$ is a simple integrally closed ideal, the definition of the adjoint of $I$ given in [L4, p. 229] and [L5, p. 299] agrees with the one in this paper (see Proposition (1.3.1)). Let us extend this result—more specifically, the not-quite-correctly stated Corollary (4.1) of [L4, p. 233]—to arbitrary $I$.

The point basis of $I$ is the family of integers $\text{ord}_S(I^S)_{S \supset R}$ where $S$ runs through all two-dimensional regular local rings between $R$ and its fraction field, and $I^S := (\gcd(IS))^{-1}IS$, the $S$-transform of $I$. There are only finitely many $S$ for which $\text{ord}_S(I^S) \neq 0$; these are called the base points of $I$ [L4, p. 225]. Two $m$-primary ideals $I$ and $I'$ have the same point basis iff their integral closures coincide [L3, p. 209, (1.10)]

Consider a sequence of regular schemes

$$\text{Spec}(R) = X_{0} \leftarrow X_{1} \leftarrow \cdots \leftarrow X_{n+1} = X$$

where $f_i: X_{i+1} \to X_i$ is obtained by blowing up a point on $X_i$ whose local ring $(S_i, m_i)$ is a base point of $I$, and where $I\mathcal{O}_X$ is invertible. Denote by $m_i\mathcal{O}_X$ the invertible $\mathcal{O}_X$-ideal whose stalk at $x \in X$ is $m_i\mathcal{O}_{X,i}$ if $\mathcal{O}_{X,i} \supset S_i$, and $\mathcal{O}_{X,i}$ otherwise. Then

$$\omega_X^{-1} = \prod_{i=0}^{n} m_i\mathcal{O}_X,$$

see the end of the proof of Corollary (1.4.4), or the footnote in [L4, p. 235]. Let $E_i \cong \mathbb{P}^1_{S_i/m_i}$ be the curve on $X$ corresponding to the $m_i$-adic valuation; and let $[S_i:R]$ be the degree of the field extension $S_i/(m_i)/(R/m)$. The intersection number $(E_i \cdot E_i)$ is $-d_i[S_i:R]$ for some positive integer $d_i$, and $d_i = 1$ iff $I^{S_i}$ generates an invertible ideal on $X_{i+1}$, i.e., iff $I^{S_i}$ is of the form $m_i^d$ ($d > 0$), in which case $(I\mathcal{O}_X \cdot E_i) = d[S_i:R]$. Moreover, as in [L4, p. 235],

$$(\omega_X \cdot E_i) = -(E_i \cdot E_i) - 2[S_i:R].$$

It follows that $(I\omega_X \cdot E_i) \geq 0$ for all $i$, and hence, by [L1, p. 220, Thm. (12.1)(ii)], $I\omega_X$ is generated by its global sections, i.e., by $\tilde{I}$, see Prop. (1.3.1). Thus:

(3.1.1) $$I\omega_X = \tilde{I}\mathcal{O}_X \quad \text{i.e.,} \quad \tilde{I}\mathcal{O}_X = \tilde{I} \prod_{i=0}^{n} m_i\mathcal{O}_X.$$ 

For any $S \supset R$, we have then

$$\text{ord}_S(I) - \text{ord}_S(\tilde{I}) = \sum \text{ord}_S(m_i).$$
On the other hand, setting, for any two-dimensional regular local \( T \) between \( R \) and its fraction field, \( r_T := \text{ord}_T(I^T), \tilde{r}_T := \text{ord}_T(\tilde{I}^T) \), we have

\[
\text{ord}_S(I) - \text{ord}_S(\tilde{I}) = \sum_{T \subset S} \text{ord}_S(m_i)(r_T - \tilde{r}_T),
\]

see [L4, p. 301, Remark (1)]. By induction on the length of the unique sequence of quadratic transforms \( R := R_0 \subset R_1 \subset \cdots \subset S \) (see [A, p. 343, Thm. 3]), we deduce that

\[
r_S - \tilde{r}_S = 1 \quad (S = S_1, S_2, \ldots, S_n)
= 0 \quad \text{otherwise}.
\]

But since \( S_1, S_2, \ldots, S_n \) are precisely the base points of \( I \), i.e., those \( S \) such that \( r_S > 0 \), what this amounts to is that \( \tilde{r}_S = (\max(0, r_S - 1)) \). Thus:

**Proposition (3.1.2).** \( \tilde{I} \) is the unique integrally closed ideal whose point basis is

\[
(\max(0, \text{ord}_S(I^S) - 1))_{S \supset R}.
\]

For any two-dimensional regular local \( T \) between \( R \) and its fraction field, the point basis of the transform \( I^T \) is obtained from that of \( I \) by restriction to those \( S \) which contain \( T \). Moreover, a theorem of Zariski states that \( I^T \) is integrally closed if \( I \) is (see e.g., [L5, p. 300]). We have then the following generalization of [L4, p. 231, Thm. (3.1)]:

**Corollary (3.1.3).** Adjoint commutes with transform: for all \( T \), \( \tilde{I}^T = \tilde{I}^T \).

(3.2). For the next result, let \( I \) be any non-zero integrally closed ideal in a \( d \)-dimensional regular local ring \( R \), such that \( I \) has a reduction \( I_0 \) generated by a regular sequence \( (a_1, a_2, \ldots, a_d) \). Let \( Y_0 \) be the blowup up of \( I_0 \), let \( \pi: Y \to Y_0 \) be the normalization map, and let \( \mathcal{C} \) be the conductor of \( Y \) in \( Y_0 \). Then \( \mathcal{C} \) is independent of \( I_0 \):

**Lemma (3.2.1).** With the preceding notation, we have \( \mathcal{C} = I^{d-1}\omega_Y \).

*Proof.* Noting that \( Y_0 \to \text{Spec}(R) \) is a local complete intersection map (see footnote under (1.3.2)(c)), and arguing as on pp. 205–207 of [LS], we find that

\[
\pi_*\omega_Y = \mathcal{H}\text{om}(\pi_*\mathcal{O}_Y, \omega_{Y_0}) = \mathcal{H}\text{om}(\pi_*\mathcal{O}_Y, (I_0\mathcal{O}_{Y_0})^{1-d}),
\]

so that

\[
\pi_*I^{d-1}\omega_Y = \pi_*I_0^{d-1}\omega_Y = I_0^{d-1}\pi_*\omega_Y = \mathcal{H}\text{om}(\pi_*\mathcal{O}_Y, \mathcal{O}_{Y_0}) = \pi_*\mathcal{C},
\]

whence the assertion. \( \square \)

More can be said in the two-dimensional case.
**Proposition (3.2.2).** With the preceding notation, when \( d = \dim R = 2 \), \( \mathfrak{C} \) is generated by its global sections \( H^0(Y, \mathcal{C}) = H^0(Y, I \omega_Y) = \tilde{I} \), i.e., \( \mathfrak{C} = \tilde{I} \mathcal{O}_Y \).

**Proof.** Choose \( X \) as in (3.1), and let \( g : X \to Y \) be the domination map (which exists because \( \tilde{I} \mathcal{O}_X \) is invertible). As in the proof of Prop. (1.3.1), \( g_*(I \omega_X) = I \omega_Y = \mathfrak{C} \), the last equality by Lemma (3.2.1). Also, by [L1, p. 209, Prop. (6.5)], the \( \mathcal{O}_Y \)-ideal \( \tilde{I} \mathcal{O}_Y \) is integrally closed. Hence, and by (3.1.1),

\[
\tilde{I} \mathcal{O}_Y = g_*(\tilde{I} \mathcal{O}_X) = g_*(I \omega_X) = I \omega_Y = \mathfrak{C},
\]

whence the assertion. □

Here is another characterization of \( \tilde{I} \).

**Proposition (3.3).** Let \( I \) be an \( m \)-primary integrally closed ideal in a regular local ring \( R \) of dimension 2, let \( f : Y \to \text{Spec}(R) \) be the normalized blowup of \( I \), and let \( I_0 = (a, b)R \) be a reduction of \( I \). Then with \( D \) an injective hull of \( R/m \), we have a duality isomorphism

\[
R/\tilde{I} \cong \text{Hom}_R(I/I_0, D).
\]

Hence the \( R \)-module \( I/I_0 \) depends only on \( I \), and its annihilator \( I_0 : I \) is just \( \tilde{I} \).

**Proof.** Recall that \( H^1(Y, \mathcal{O}_Y) = 0 \) [L1, p. 199, Prop. (1.2)]. With \( \mathcal{I} := I \mathcal{O}_Y = (a, b)\mathcal{O}_Y \), we have the exact Koszul complex

\[
0 \to \mathcal{I}^{-1} \xrightarrow{-(b:a)} \mathcal{O}_Y \oplus \mathcal{O}_Y \xrightarrow{(a,b)} \mathcal{I} \to 0,
\]

whence an exact homology sequence, with \( H^\bullet(-) := H^\bullet(Y, -) \),

\[
R \oplus R = H^0(\mathcal{O}_Y \oplus \mathcal{O}_Y) \xrightarrow{(a,b)} H^0(\mathcal{I}) = I \to H^1(\mathcal{I}^{-1}) \to H^1(\mathcal{O}_Y \oplus \mathcal{O}_Y) = 0,
\]

yielding

\[
I/I_0 \cong H^1(\mathcal{I}^{-1}).
\]

We already noted that \( H^1(\mathcal{O}_Y) = 0 \), and since \( f \) has fibers of dimension < 2 therefore \( H^2(\mathcal{O}_Y) = 0 \); so

\[
H^1(\mathcal{I}^{-1}) \cong H^1(\mathcal{I}^{-1}/\mathcal{O}_Y).
\]

Further, with \( E := Y \otimes_R (R/m) \) the closed fiber, we have that \( \mathcal{I}^{-1}/\mathcal{O}_Y \) vanishes on \( U := Y \setminus E \cong \text{Spec}(R) \setminus \{m\} \). We conclude that

\[
H^1_E(\mathcal{I}^{-1}/\mathcal{O}_Y) \cong H^1(\mathcal{I}^{-1}/\mathcal{O}_Y) \cong I/I_0.
\]

Denoting the dualizing functor \( \text{Hom}_R(-, D) \) by \( -' \), we have, by [L2, p. 188],

\[
H^2_E(\mathcal{I}^{-1}) \cong \text{Ext}^0(\mathcal{I}^{-1}, \omega_Y)' \cong H^0(\mathcal{I} \otimes \omega_Y)' = (\tilde{I})',
\]

and similarly

\[
H^2_E(\mathcal{O}_Y) \cong \text{Ext}^0(\mathcal{O}_Y, \omega_Y)' \cong H^0(\omega_Y)' = R'.
\]

Recall from (2.2.1)(b) that \( H^2_E(\mathcal{I}^{-1}) = 0 \). So there is an exact sequence

\[
0 \to H^1_E(\mathcal{I}^{-1}/\mathcal{O}_Y) \to H^2_E(\mathcal{O}_Y) \to H^2_E(\mathcal{I}^{-1}) \to 0
\]

whose dual is an exact sequence

\[
0 \to \tilde{I} \to R \to \text{Hom}_R(I/I_0, D) \to 0
\]

which gives the desired conclusion. □
ADJOINTS OF IDEALS IN REGULAR LOCAL RINGS

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