One-dimensional quantum walks with a general perturbation of the radius 1

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Abstract
We give a complete description of the discrete spectrum of one-dimensional Hamiltonian with a general perturbation of the radius 1.

1 Introduction
We consider the one particle continuous time one-dimensional quantum walk with an Hamiltonian $H$ acting in the Hilbert space $l_2(Z)$, where $Z$ is the one-dimensional lattice and $l_2(Z)$ is the space of complex valued square summable sequences $f = \{f_n \in C, n \in Z\}$.

Let $\{e_n\}_{n \in Z}$ be the standard basis of $l_2(Z)$. If $f \in l_2(Z)$ then we have $f = \sum_{n \in Z} f_n e_n$.

The Hamiltonian $H = H_0 + H_1, l_2(Z) \rightarrow l_2(Z)$ is defined as follows:

$$
\begin{align*}
H_0 e_n &= -\lambda (e_{n+1} - 2e_n + e_{n-1}), n \in Z \\
H_1 e_0 &= -\lambda_1 (e_1 - 2e_0 + e_{-1}) + \mu e_0 \\
H_1 e_1 &= -\lambda_1 e_0 + \mu_1 e_1 \\
H_1 e_{-1} &= -\lambda_1 e_0 + \mu_1 e_{-1} \\
H_1 e_n &= 0, \text{if } |n| > 1,
\end{align*}
$$

(1)

where $\lambda, \lambda_1, \mu, \mu_1 \in R$. Without loss of generality we can assume that $\lambda > 0$. Our aim is to describe the discrete spectrum of the Hamiltonian $H$.

The continuous time quantum walk evolution is defined as

$$
f(t) = e^{-itH} f(0) \in l_2(Z)
$$

where $f(0)$ is the initial state.

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We generalize the result of [1], where authors investigated the discrete spectrum of the Hamiltonian which is defined by two parameters \( \mu \) and \( \mu_1 \), 
\[
H = H_0 + H_1 : l^2_2(\mathbb{Z}) \to l^2_2(\mathbb{Z})
\]

\[
H_0 e_n = -\frac{1}{2}e_{n+1} + e_n - \frac{1}{2}e_{n-1}
\]

\[
H_1 e_0 = \mu e_0
\]

\[
H_1 e_1 = \mu_1 e_1
\]

\[
H_1 e_{-1} = \mu_1 e_{-1}
\]

\[
H_1 e_n = 0, \text{ if } |n| > 1
\]

where \( l^2_2(\mathbb{Z}) \subset l^2(\mathbb{Z}) \) is the subspace of symmetric sequences: \( f_n = f_{-n} \) for all \( n \in \mathbb{Z} \).

2 Main results

Notation It will be convenient to introduce new parameters

\[
\alpha = \frac{\lambda_1}{\lambda}, \quad \delta = \frac{\mu_1}{\lambda}, \quad \sigma = \frac{2\lambda_1 + \mu}{2\lambda}
\]

Define the functions of parameters \( \alpha, \delta, \sigma \):

\[
c_1 = c_1(\alpha, \delta, \sigma) = (\delta - 1)(\sigma - 1) - (\alpha + 1)^2
\]

\[
c_2 = c_2(\alpha, \delta, \sigma) = (\delta + 1)(\sigma + 1) - (\alpha + 1)^2
\]

where \((\alpha, \delta, \sigma) \in \mathbb{R}^3\).

Let us introduce the following partition of \( \mathbb{R}^3 \):

\[
U_1(+, +) = \{(\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 > 0, c_2 > 0, \delta > 1, \sigma > 1\}
\]

\[
U_0(+, +) = \{(\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 \geq 0, c_2 \geq 0, |\delta| \leq 1, |\sigma| \leq 1\}
\]

\[
U_{-1}(+, +) = \{(\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 > 0, c_2 > 0, \delta < -1, \sigma < -1\}
\]

\[
U(-, +) = \{(\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 < 0, c_2 \geq 0\}
\]

\[
\cup \{(\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 = 0, c_2 > 0, \delta > 1, \sigma > 1\}
\]

\[
U(-, -) = \{(\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 < 0, c_2 < 0\}
\]

\[
U(+, -) = \{(\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 \geq 0, c_2 < 0\}
\]

\[
\cup \{(\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 > 0, c_2 = 0, \delta < -1, \sigma < -1\}
\]

Figures 1-3 (see Appendix) show sections of the subsets from partition (4) by planes \( \alpha = c \) for various constants \( c \neq -1 \).
Finally we put
\[ \kappa = \frac{\lambda(\delta + 1)^2}{\delta} = \frac{(\mu_1 + \lambda)^2}{\mu_1} \] (5)
for \( \mu_1 \neq 0 \).

**Results** It follows from Weyl’s theorem [3] on essential spectrum that the essential spectrum of the Hamiltonian \( H \) is the same as for the homogeneous Hamiltonian \( H_0 \) and coincides with the segment \([0, 4\lambda]\).

Denote by \( s_d(H) \) the discrete spectrum [2] of \( H \). We give a complete description of \( s_d(H) \). We prove that the discrete spectrum consists of at most three eigenvalues. One of them we find explicitly. For the other two we define the intervals of the real axis containing these eigenvalues.

**Theorem 2.1** Let \( \lambda > 0 \) and \( \kappa \) be defined by (5). For \( \alpha = \frac{\lambda}{\mu_1} \neq -1 \) we have
a) if \( (\alpha, \delta, \sigma) \in U_1(+, +) \), then \( s_d(H) = \{\nu_1, \nu_2, \nu_3\} \), where \( \nu_1 = \kappa, \nu_1, \nu_2 \in (4\lambda, \infty) \) and, moreover,
\[ 4\lambda < \nu_1 < 2\lambda \left( \min \left\{ \sigma, \frac{\delta^2 + 1}{2\delta} \right\} + 1 \right) \leq 2\lambda \left( \max \left\{ \sigma, \frac{\delta^2 + 1}{2\delta} \right\} + 1 \right) < \nu_2; \]
b) if \( (\alpha, \delta, \sigma) \in U_0(+, +) \), then \( s_d(H) = \emptyset \);
c) if \( (\alpha, \delta, \sigma) \in U_{-1}(+, +) \) then \( s_d(H) = \{\nu_1, \nu_2, \nu_3\} \), where \( \nu_1 = \kappa, \nu_1, \nu_2 \in (-\infty, 0) \) and, moreover,
\[ \nu_1 < 2\lambda \left( \min \left\{ \sigma, \frac{\delta^2 + 1}{2\delta} \right\} + 1 \right) \leq 2\lambda \left( \max \left\{ \sigma, \frac{\delta^2 + 1}{2\delta} \right\} + 1 \right) < \nu_2 < 0; \]
d) if \( (\alpha, \delta, \sigma) \in U_1(-, +) \), then
\[ d1) \text{ for } \delta \leq 1 \text{, } s_d(H) = \{\nu_1\} \text{, where } \nu_1 \in (4\lambda, \infty), \]
\[ d2) \text{ for } \delta > 1 \text{, } s_d(H) = \{\nu_1, \nu_2\} \text{, where } \nu_1 \in (4\lambda, \infty), \nu_2 = \kappa; \]
e) if \( (\alpha, \delta, \sigma) \in U_{-1}(-, +) \), then
\[ e1) \text{ for } \delta \leq 1 \text{, } s_d(H) = \{\nu_1, \nu_2\} \text{, where } \nu_1 \in (-\infty, 0), \nu_2 \in (4\lambda, \infty), \]
\[ e2) \text{ for } \delta > 1 \text{, } s_d(H) = \{\nu_1, \nu_2, \nu_3\} \text{, where } \nu_1 \in (-\infty, 0), \nu_2 \in (4\lambda, \infty), \nu_3 = \kappa; \]
f) if \( (\alpha, \delta, \sigma) \in U_1(+, -) \), then
\[ f1) \text{ for } \delta \leq 1 \text{, } s_d(H) = \{\nu_1\} \text{, where } \nu_1 \in (-\infty, 0) \text{, and} \]
\[ f2) \text{ for } \delta > 1 \text{, } s_d(H) = \{\nu_1, \nu_2\} \text{, where } \nu_1 \in (-\infty, 0), \nu_2 = \kappa; \]
g) geometric multiplicity of all eigenvalues is 1; the eigenvector \( f = \{f_n\} \in l_2(\mathbb{Z}) \) corresponding to the eigenvalue \( \kappa \) is an odd function \( f_n = -f_{-n} \) and the eigenvectors \( f = \{f_n\} \in l_2(\mathbb{Z}) \) corresponding to the eigenvalues \( \nu_1, \nu_2 \) are even functions \( f_n = f_{-n} \).

For \( \alpha = \lambda_1/\lambda = -1 \) one can find eigenvalues in an explicit form.

**Theorem 2.2** Let \( \lambda > 0 \) and \( \kappa \) be defined by (5). For \( \alpha = \lambda_1/\lambda = -1 \)
• if $|\delta| > 1$, $|\sigma| > 1$, $\mu \neq \kappa$, then $s_d(H) = \{\mu, \kappa\}$
• if $|\delta| \leq 1$, $|\sigma| > 1$, then $s_d(H) = \{\mu\}$
• if $|\delta| > 1$, $|\sigma| \leq 1$, then $s_d(H) = \{\kappa\}$
• if $|\delta| \leq 1$, $|\sigma| \leq 1$, then $s_d(H) = \{\emptyset\}$

• the eigenvalue $\kappa$ is of the geometrical multiplicity 2 and has two linearly independent eigenvectors one of which is an odd function and the other is an even function.

• if $\mu = \kappa$, $|\delta| > 1$, $|\sigma| > 1$, there is only one eigenvalue of the geometrical multiplicity 2 with the same eigenvectors as in the previous item.

Note that for $\alpha = \lambda_1/\lambda = -1$ there exists the eigenvalue $\mu$ with the eigenvector $e_0$. By definition (1), one can obtain

$$He_0 = H_0e_0 + H_1e_0 =$$
$$= -\lambda(e_1 - 2e_0 + e_{-1}) - \lambda_1(e_1 - 2e_0 + e_{-1}) + \mu e_0 = \mu e_0$$

So, if $\mu \in [0, 4\lambda]$, then there exists the eigenvalue $\nu = \mu$ belonging to the essential spectrum of Hamiltonian $H$.

3 Proof of main results

For $f \in l_2(\mathbb{Z})$ we have

$$Hf = \sum_{n \in \mathbb{Z}} \lambda (-f_{n+1} + 2f_n - f_{n-1}) e_n + (\mu_1 f_{-1} - \lambda_1 f_0) e_{-1} +$$
$$+ ((2\lambda_1 + \mu)f_0 - \lambda_1 f_1 - \lambda_1 f_{-1}) e_0 + (\mu_1 f_1 - \lambda_1 f_0) e_1 \quad (6)$$

Let us consider the Hilbert space $L_2(S)$ of square integrable functions defined on the unit circle $S$. The elements of $L_2(S)$ we will denote by $\hat{f}$. It is known that any two separable Hilbert spaces are isomorphic, so we have $l_2(\mathbb{Z}) \cong L_2(S)$.

Consider an isomorphism $U : l_2(\mathbb{Z}) \rightarrow L_2(S)$ such that $e_n \mapsto \frac{1}{\sqrt{2\pi}} e^{i\pi n}$. Then $U$ is an unitary operator and for $f = \sum_{n \in \mathbb{Z}} f_n e_n$ we have

$$Uf = \sum_{n \in \mathbb{Z}} f_n \frac{1}{\sqrt{2\pi}} e^{i\pi n}$$

Introduce operator $\hat{H} = UHU^{-1} : L_2(S) \rightarrow L_2(S)$ which is unitarily equivalent to $H$. Using (6) we get
\[
\hat{H}\hat{f} = \sum_{n \in \mathbb{Z}} \lambda (-f_{n+1} + 2f_n - f_{n-1}) \frac{1}{\sqrt{2\pi}} e^{i\varphi n} + (\mu_1 f_1 - \lambda_1 f_0) \frac{1}{\sqrt{2\pi}} e^{-i\varphi} + \\
+ ((2\lambda_1 + \mu) f_0 - \lambda_1 f_1 - \lambda_1 f_{-1}) \frac{1}{\sqrt{2\pi}} + (\mu_1 f_1 - \lambda_1 f_0) \frac{1}{\sqrt{2\pi}} e^{i\varphi} = \\
\sum_{n \in \mathbb{Z}} \lambda \left( -f_{n+1} \frac{1}{\sqrt{2\pi}} e^{i\varphi(n+1)} e^{-i\varphi} + 2f_n \frac{1}{\sqrt{2\pi}} e^{i\varphi n} - f_{n-1} \frac{1}{\sqrt{2\pi}} e^{i\varphi(n-1)} e^{i\varphi} \right) - \\
- \frac{\lambda_1}{\sqrt{2\pi}} f_0(e^{-i\varphi} + e^{i\varphi} - 2) - \frac{\lambda_1}{\sqrt{2\pi}} (f_{-1} + f_1) + \frac{\mu}{\sqrt{2\pi}} f_0 + \frac{\mu_1}{\sqrt{2\pi}} (f_{-1} e^{-i\varphi} + f_1 e^{i\varphi}) = \\
-\lambda (e^{-i\varphi} + e^{i\varphi} - 2) \hat{f} - \frac{\lambda_1}{\sqrt{2\pi}} \left( f_{-1} + f_1 + f_0(e^{-i\varphi} + e^{i\varphi} - 2) \right) + \\
+ \frac{\mu}{\sqrt{2\pi}} f_0 + \frac{\mu_1}{\sqrt{2\pi}} (f_{-1} e^{-i\varphi} + f_1 e^{i\varphi}) \\
(7)
\]

Thus,
\[
\hat{H}\hat{f} = -2\lambda (\cos \varphi - 1) \hat{f} - \frac{\lambda_1}{\sqrt{2\pi}} \left( f_{-1} + f_1 + 2f_0(\cos \varphi - 1) \right) + \\
+ \frac{\mu}{\sqrt{2\pi}} f_0 + \frac{\mu_1}{\sqrt{2\pi}} (f_{-1} e^{-i\varphi} + f_1 e^{i\varphi}) \\
(8)
\]

Because of unitarily equivalence of operators \( \hat{H} \) and \( H \), their point spectra are identical.

If \( \hat{f} \) is an eigenfunction with eigenvalue \( \nu \), i.e. \( \hat{H}\hat{f} = \nu \hat{f} \), then by (8) we have
\[
\nu \hat{f} = -2\lambda (\cos \varphi - 1) \hat{f} - \frac{\lambda_1}{\sqrt{2\pi}} \left( f_{-1} + f_1 + 2f_0(\cos \varphi - 1) \right) + \\
+ \frac{\mu}{\sqrt{2\pi}} f_0 + \frac{\mu_1}{\sqrt{2\pi}} (f_{-1} e^{-i\varphi} + f_1 e^{i\varphi})
\]
and, hence,
\[
\hat{f} = \frac{1}{\sqrt{2\pi}} \frac{-\frac{\lambda_1}{\sqrt{2\pi}} \left( f_{-1} + f_1 + 2f_0(\cos \varphi - 1) \right) + \frac{\mu}{\sqrt{2\pi}} f_0 + \frac{\mu_1}{\sqrt{2\pi}} (f_{-1} e^{-i\varphi} + f_1 e^{i\varphi})}{\cos \varphi + \frac{\mu}{2\pi} f_0 + \frac{\mu_1}{2\pi} (f_{-1} e^{-i\varphi} + f_1 e^{i\varphi})} = \\
= \frac{1}{\sqrt{2\pi}} \frac{-\frac{\lambda_1}{\sqrt{2\pi}} \left( f_{-1} + f_1 \right) + (\sigma - \alpha \cos \varphi) f_0 + \frac{\mu}{2\pi} (f_{-1} e^{-i\varphi} + f_1 e^{i\varphi})}{\cos \varphi + \frac{\mu}{2\pi} f_0 + \frac{\mu_1}{2\pi} (f_{-1} e^{-i\varphi} + f_1 e^{i\varphi})}
\]
Since
\[
f_{-1} e^{-i\varphi} + f_1 e^{i\varphi} = (f_{-1} + f_1) \cos \varphi + i (-f_{-1} + f_1) \sin \varphi
\]
we get
\[
\hat{f} = \frac{1}{\sqrt{2\pi}} \left( f_0 + f_1 \right) + \frac{\alpha}{\sqrt{2\pi}} \left( f_0 + f_1 \right) \cos \varphi + \frac{\nu^{2\lambda}}{2\pi} - 1 \]

\[
+ \frac{1}{\sqrt{2\pi}} \left( (f_1 + f_0) \cos \varphi + i (f_1 - f_0) \sin \varphi \right) \cos \varphi + \frac{\nu^{2\lambda}}{2\pi} - 1 \}
\]

\[
= \frac{1}{\sqrt{2\pi}} \left( -\alpha + \delta \cos \varphi \right) + (\sigma - \alpha \cos \varphi) f_0 + i\delta \frac{f_1 - f_{-1}}{2} \sin \varphi \cos \varphi + \frac{\nu^{2\lambda}}{2\pi} - 1 \}
\]

Put
\[
\gamma = \nu \frac{2\lambda}{2\pi} - 1 \}
\]

It is evident that \( \gamma \in [-1, 1] \iff \nu \in [0, 4\lambda] \). Then

\[
\hat{f} = \frac{1}{\sqrt{2\pi}} \left( f_0 + f_1 + f_{-1} \right) \cos \varphi + \gamma \cos \varphi + \gamma \]

If \( \gamma \notin [-1, 1] \), then \( \hat{f} \in L_2(S) \).

Applying the inverse Fourier transform we get

\[
f_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f} e^{-ik\varphi} d\varphi = \]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{f_0 + f_1 + f_{-1}}{2} \right) (\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0 + i\delta \frac{f_1 - f_{-1}}{2} \sin \varphi \cos \varphi + \gamma e^{-ik\varphi} d\varphi \]

for all \( k \in \mathbb{Z} \). It follows, that the coordinates \( f_0, f_1, f_{-1} \) of eigenvector \( f = \{f_k\} \) satisfy the following homogeneous linear system:

\[
f_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{f_0 + f_1 + f_{-1}}{2} \right) (\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0 + i\delta \frac{f_1 - f_{-1}}{2} \sin \varphi \cos \varphi + \gamma e^{-ik\varphi} d\varphi \]

\[
f_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{f_0 + f_1 + f_{-1}}{2} \right) (\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0 + i\delta \frac{f_1 - f_{-1}}{2} \sin \varphi \cos \varphi + \gamma e^{-ik\varphi} d\varphi \]

\[
f_{-1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{f_0 + f_1 + f_{-1}}{2} \right) (\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0 + i\delta \frac{f_1 - f_{-1}}{2} \sin \varphi \cos \varphi + \gamma e^{-ik\varphi} d\varphi \]

Thus, we proved the following lemma:

**Lemma 3.1** Real \( \nu \notin [0, 4\lambda] \) is an eigenvalue of the Hamiltonian \( H \) iff linear system \( (12)-(14) \) has a nontrivial solution for \( \gamma = \nu \frac{2\lambda}{2\pi} - 1 \). The geometric multiplicity of eigenvalue \( \nu \) is equal to \( 3 - r \), where \( r \) is the rank of linear system \( (12)-(14) \).
Now we simplify integrals in equations (12)–(14). Let us denote
\[ f' = f - f_1 + f_2, \quad f'_{-1} = f_1 - f_{-1} \]

We have from equation (12)
\[
f_0 = \int_{-\pi}^{\pi} f'_1 (\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0 + i \delta f'_{-1} \sin \varphi \, d\varphi = \int_{-\pi}^{\pi} f'_1 (\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0 \, d\varphi
\]
since
\[
\int_{-\pi}^{\pi} i \frac{\delta}{2} (-f'_{-1} + f'_1) \sin \varphi \cos \varphi + i \gamma d\varphi = 0
\]
because of the integrand is an odd function.

Further, from equation (13) we get
\[
f_1 = \int_{-\pi}^{\pi} f'_1 (\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0 \cos \varphi + \gamma i \sin \varphi \, d\varphi
\plus\int_{-\pi}^{\pi} i \delta f'_{-1} \sin \varphi \cos \varphi + \gamma \cos \varphi \, d\varphi
\]
Because of the integrands are odd functions we have
\[
\int_{-\pi}^{\pi} f'_1 (\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0 \sin \varphi \, d\varphi = 0
\]
\[
\int_{-\pi}^{\pi} \sin \varphi \cos \varphi d\varphi = 0
\]
So
\[
f_1 = \int_{-\pi}^{\pi} f'_1 (\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0 \cos \varphi d\varphi + \int_{-\pi}^{\pi} \delta f'_{-1} \sin^2 \varphi \, d\varphi
\]
Similarly, one can show that
\[
f_{-1} = \int_{-\pi}^{\pi} f'_1 (\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0 \cos \varphi d\varphi - \int_{-\pi}^{\pi} \delta f'_{-1} \sin^2 \varphi \, d\varphi
\]
Hence, system (12)–(14) can be rewritten as follows
\[
f_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'_1 (\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0 \, d\varphi
\]
\[ f_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_1'(\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0}{\cos \varphi + \gamma} \cos \varphi d\varphi + \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta f_{-1}' \sin^2 \varphi d\varphi \] (16)

\[ f_{-1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_1'(\delta \cos \varphi - \alpha) + (\sigma - \alpha \cos \varphi) f_0}{\cos \varphi + \gamma} \cos \varphi d\varphi - \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta f_{-1}' \sin^2 \varphi d\varphi \] (17)

Adding equations (16) and (17), subtracting equation (17) from (16) and dividing the resulting equations by 2 we come to the linear system with respect to variables \( f_0, f_1', f_{-1}' \):

\[ f_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\sigma - \alpha \cos \varphi) f_0 \cos \varphi d\varphi + (\delta \cos \varphi - \alpha) f_1'}{\cos \varphi + \gamma} \] 
\[ f_1' = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\sigma - \alpha \cos \varphi) \cos \varphi f_0 + (\delta \cos \varphi - \alpha) \cos \varphi f_1'}{\cos \varphi + \gamma} \cos \varphi d\varphi \] 
\[ f_{-1}' = f_{-1}' \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta \sin^2 \varphi d\varphi \]

Finally, we get the system which is equivalent to the original system (12)–(14):

\[ f_0 \left( 1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sigma - \alpha \cos \varphi}{\cos \varphi + \gamma} d\varphi \right) + f_1' \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\alpha - \delta \cos \varphi}{\cos \varphi + \gamma} d\varphi = 0 \] (18)

\[ -\frac{f_0}{2\pi} \int_{-\pi}^{\pi} \frac{(\sigma - \alpha \cos \varphi) \cos \varphi}{\cos \varphi + \gamma} d\varphi + f_1' \left( 1 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\alpha - \delta \cos \varphi}{\cos \varphi + \gamma} \cos \varphi d\varphi \right) = 0 \] (19)

\[ f_{-1}' \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\delta (1 - \cos^2 \varphi)}{\cos \varphi + \gamma} d\varphi - 1 \right) = 0 \] (20)

Define the following functions

\[ g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\varphi}{\cos \varphi + x} \]
\[ h(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos \varphi d\varphi}{\cos \varphi + x} \]
\[ v(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos^2 \varphi d\varphi}{\cos \varphi + x} \]
where $|x| > 1$. By lemma A.2,

$$g(x) = \frac{1}{x\sqrt{1 - x^2}}$$

$$h(x) = xg(x) - 1, \quad v(x) = xh(x),$$

$$g(x)v(x) = h^2(x) + h(x)$$

Let us express coefficients of linear system (18)–(20) in terms of functions $g, h, v$:

$$f_0(1 - a\gamma - \sigma g(\gamma)) + f_1'(a\gamma + \delta h(\gamma)) = 0,$$

$$f_0(\sigma h(\gamma) + \alpha v(\gamma)) + f_1'(1 - a\gamma - \delta v(\gamma)) = 0$$

$$f'_{-1}(\delta (g(\gamma) - v(\gamma)) - 1) = 0$$

Coefficients of this system are defined only for $|\gamma| > 1$. We need to find such $\gamma$ that system (24)–(26) has nontrivial solution. A homogeneous linear system has non trivial solution if its determinant equals zero. Denote by $D'(\alpha, \delta, \sigma, \gamma)$ the determinant of this system and let $D(\alpha, \delta, \sigma, \gamma)$ be the determinant of system (24)–(25), consisting of the first two equations. We have

$$D'(\alpha, \delta, \sigma, \gamma) = D(\alpha, \delta, \sigma, \gamma) (\delta (g(\gamma) - v(\gamma)) - 1)$$

So in order to determine eigenvalues we need to find roots of the equation $D'(\alpha, \delta, \sigma, \gamma) = 0$. Simple algebra gives

$$D(\alpha, \delta, \sigma, \gamma) = (1 - a\gamma - \sigma g(\gamma))(1 - a\gamma - \delta v(\gamma)) -$$

$$- (a\gamma + \delta h(\gamma))(\sigma h(\gamma) + \alpha v(\gamma)) =$$

$$= (1 - a\gamma)\gamma - \delta v(\gamma) + a\gamma\delta v(\gamma) -$$

$$- \sigma g(\gamma)(1 - \delta v(\gamma)) + \sigma g(\gamma)a\gamma(\gamma) -$$

$$- \sigma g(\gamma)\sigma h(\gamma) - \alpha^2 g(\gamma)v(\gamma) - \delta v(\gamma)\gamma - \delta h(\gamma)v(\gamma) =$$

$$= (1 - a\gamma)\gamma - \delta v(\gamma) - \sigma g(\gamma)(1 - \delta v(\gamma)) -$$

$$- \alpha^2 g(\gamma)v(\gamma) - \delta v(\gamma) =$$

$$= (1 - a\gamma)\gamma - \delta \sigma h^2(\gamma) - \sigma g(\gamma) - (\alpha^2 - \alpha\delta) g(\gamma)v(\gamma) - \delta v(\gamma)$$

By (25) we have $g(\gamma)v(\gamma) = h(\gamma) + h(\gamma)$ and, hence,

$$D(\alpha, \delta, \sigma, \gamma) = (1 - a\gamma)\gamma - \delta \sigma h^2(\gamma) - \sigma g(\gamma) - (\alpha^2 - \alpha\delta) h^2(\gamma) -$$

$$- (\alpha^2 - \alpha\delta) h(\gamma) - \delta v(\gamma) =$$

$$1 - 2ah(\gamma) + \alpha^2 h^2(\gamma) - \delta \sigma h^2(\gamma) - \sigma g(\gamma) - \alpha^2 h^2(\gamma) +$$

$$+ \alpha\delta h^2(\gamma) - \alpha^2 h(\gamma) + \sigma \delta h(\gamma) - \delta v(\gamma) =$$
Remark that \( h(\gamma) \neq 0 \) for \( |\gamma| > 1 \). Let us divide \( D(\alpha, \delta, \sigma, \gamma) \) by \( h(\gamma) \). Using (22), we obtain

\[
D(\alpha, \delta, \sigma, \gamma) = h(\gamma) \left( \frac{1}{h(\gamma)} - \frac{g(\gamma)}{h(\gamma)} - \alpha^2 - 2\alpha + \sigma\delta - \delta \gamma \right)
\]

Put

\[
l(\gamma) = \gamma \left( 1 + \sqrt{1 - \gamma^{-2}} \right) = \begin{cases} 
\gamma + \sqrt{\gamma^2 - 1} & \gamma \geq 1 \\
\gamma - \sqrt{\gamma^2 - 1} & \gamma \leq -1
\end{cases}
\]

By (21), we have

\[
\frac{1}{h(\gamma)} = \frac{\sqrt{1 - \gamma^{-2}}}{1 - \sqrt{1 - \gamma^{-2}}} = \frac{1 - \sqrt{1 - \gamma^{-2}}}{1 - \sqrt{1 - \gamma^{-2}}} = -1 + \frac{1}{1 - \sqrt{1 - \gamma^{-2}}}
\]

and

\[
g(\gamma) = \frac{1}{\gamma(1 - \sqrt{1 - \gamma^{-2}})}
\]

Since

\[
\frac{1}{1 - \sqrt{1 - \gamma^{-2}}} = \frac{1 + \sqrt{1 - \gamma^{-2}}}{(1 - \sqrt{1 - \gamma^{-2}})(1 + \sqrt{1 - \gamma^{-2}})} = \gamma^2(1 + \sqrt{1 - \gamma^{-2}}) = \gamma l(\gamma)
\]

we get

\[
\frac{1}{h(\gamma)} = -1 + \gamma l(\gamma), \quad \frac{g(\gamma)}{h(\gamma)} = l(\gamma)
\]

and, hence,

\[
D(\alpha, \delta, \sigma, \gamma) = h(\gamma) \left( \gamma l(\gamma) - \sigma l(\gamma) - \delta (\gamma - \sigma) - (\alpha + 1)^2 \right) = h^{-1}(\gamma) \left( (\gamma - \sigma) (l(\gamma) - \delta) - (\alpha + 1)^2 \right)
\]

Consider the second factor in (27). By (22) we have

\[
g(\gamma) - v(\gamma) = g(\gamma) - \gamma h(\gamma) = g(\gamma) - \gamma (g(\gamma) - 1) = -g(\gamma)\gamma^2 (1 - \gamma^{-2}) + \gamma
\]
Now using (21) we get

\[ g(\gamma) - v(\gamma) = \gamma \left( 1 - \sqrt{1 - \gamma^{-2}} \right) = \frac{\gamma (1 - \sqrt{1 - \gamma^{-2}})}{1 + \sqrt{1 - \gamma^{-2}}} = \frac{1}{\gamma \left( 1 + \sqrt{1 - \gamma^{-2}} \right)} = \frac{1}{l(\gamma)} \quad (30) \]

and the second factor in (27) is

\[ \delta (g(\gamma) - v(\gamma)) - 1 = \frac{\delta}{l(\gamma)} - 1 = -\frac{l(\gamma) + \delta}{l(\gamma)} \]

where \( l(\gamma) \) is defined by (28). According to (27)

\[ D'(\alpha, \delta, \sigma, \gamma) = -h(\gamma) l^{-1}(\gamma) \left( (\gamma - \sigma) (l(\gamma) - \delta) - (\alpha + 1)^2 \right) (l(\gamma) - \delta) \quad (31) \]

Thus \( D'(\alpha, \delta, \sigma, \gamma) = 0 \) iff or \( l(\gamma) - \delta = 0 \) or \( (\gamma - \sigma) (l(\gamma) - \delta) - (\alpha + 1)^2 = 0 \).

Note that \( |l(\gamma)| \geq 1 \) and the equation \( l(\gamma) - \delta = 0 \) has a unique root \( \gamma_\delta = \frac{\delta^2 + 1}{2\delta} \) iff \( |\delta| \geq 1 \) (i.e. \( |\gamma| = 1 \) iff \( |\delta| = 1 \)). Hence, in case of \( |\delta| > 1 \) we have eigenvalue

\[ \kappa = 2\lambda \left( \frac{\delta^2 + 1}{2\delta} + 1 \right) = \lambda \left( \frac{\delta + 1}{\delta} \right)^2 = \left( \mu_1 + \lambda \right)^2 \mu_1 \]

Let us find an eigenvector corresponding to this eigenvalue. To satisfy linear system (24)–(26) one can put

\[ f'_1 = f_1 + f_{-1} = 0, \quad f_0 = 0, \quad f'_{-1} = f_1 - f_{-1} = 2f_1 \]

and for all \( |k| > 1 \) by (11) we have

\[ f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\delta}{2} \left( f_{-1} e^{-i\varphi} + f_1 e^{i\varphi} \right) e^{-ik\varphi} d\varphi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\delta \bar{f}_1}{2} \left( -e^{-i\varphi} + e^{i\varphi} \right) e^{-ik\varphi} d\varphi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta \bar{f}_1 \sin \varphi \cos \varphi + \gamma \sin k\varphi d\varphi = \frac{\delta \bar{f}_1}{2\pi} \int_{-\pi}^{\pi} \sin \varphi \sin k\varphi \cos \varphi + \gamma d\varphi = \frac{\delta \bar{f}_1}{4\pi} \int_{-\pi}^{\pi} \cos \varphi \left( (k - 1)\varphi - (k + 1)\varphi \right) d\varphi \quad (32) \]

We see that \( f_k \) is an odd function of \( k : f_k = -f_{-k} \).
Let $\alpha \neq -1$. Define the function

$$F(\gamma) = (\gamma - \sigma)(l(\gamma) - \delta)$$

(33)

for $|\gamma| \geq 1$. As $\gamma \to \pm \infty$ $F(\gamma) \to \infty$.

Consider the equation

$$F(\gamma) = (\alpha + 1)^2$$

(34)

One can see (since $l(\pm 1) = \pm 1$)

$$c_1 = c_1(\alpha, \delta, \sigma) = (\delta - 1)(\sigma - 1) - (\alpha + 1)^2 = F(1) - (\alpha + 1)^2$$

$$c_2 = c_2(\alpha, \delta, \sigma) = (\delta + 1)(\sigma + 1) - (\alpha + 1)^2 = F(-1) - (\alpha + 1)^2$$

The equation $l(\gamma) = \delta$ has a unique root

$$\gamma_\delta = \frac{\delta^2 + 1}{2\delta} = \frac{1}{2} \left( \delta + \frac{1}{\delta} \right)$$

iff $|\delta| \geq 1$.

a) Let $(\alpha, \delta, \sigma) \in U_1(+, +) = \{ (\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 > 0, c_2 > 0, \sigma > 1, \delta > 1 \}$

The equation $F(\gamma) = 0$ has two roots $\sigma$ and $\gamma_\delta$ which are greater than 1. Put

$$\gamma_{min} = \min(\sigma, \gamma_\delta), \gamma_{max} = \max(\sigma, \gamma_\delta)$$

Function $F(\gamma)$ is strictly decreasing and positive on the interval $(1, \gamma_{min})$ and takes all values in the interval $(0, F(1))$; $F(\gamma)$ is strictly increasing and positive on the interval $(\gamma_{max}, \infty)$ and takes all positive real values. For $\gamma \in (\gamma_{min}, \gamma_{max})$ we have $F(\gamma) < 0$. The condition $c_1 > 0$ is equivalent to $F(1) > (\alpha + 1)^2$. So equation $F(\gamma) = (\alpha + 1)^2$ has exactly two solutions $\gamma_1 \in (1, \gamma_{min}), \gamma_2 \in (\gamma_{max}, \infty)$.

For $\gamma < -1$ there is no solution as $F(-1) \not> F(1) > (\alpha + 1)^2$ and $F(\gamma)$ is strictly decreasing on the interval $(-\infty, -1)$.

b) Let

$$(\alpha, \delta, \sigma) \in U_0(+, +) = \{ (\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 \geq 0, c_2 \geq 0, |\sigma| \leq 1, |\delta| \leq 1 \}$$

In this case the function $F(\gamma)$ is strictly decreasing on the interval $(1, \gamma_{min})$ and takes all values in the interval $(0, F(1))$; for $\gamma \in (1, \infty)$ $F(\gamma)$ is strictly increasing and takes all values in the interval $(F(1), \infty)$:

It follows from $c_1 \geq 0, c_2 \geq 0$ that $F(-1) \geq (\alpha + 1)^2$ and $F(1) \geq (\alpha + 1)^2$.

So the equation $F(\gamma) = (\alpha + 1)^2$ has no solutions satisfying condition $|\gamma| > 1$.

The proof of c) is similar to the proof of a).

d) Let $(\alpha, \delta, \sigma) \in U(-, +) = \{ (\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 < 0, c_2 \geq 0 \} \cup$
∪ \{ (\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 = 0, c_2 > 0, \sigma > 1, \delta > 1 \}. If \ c_1 < 0, c_2 \geq 0 then we have
\[ F(-1) \geq (\alpha + 1)^2 > F(1) \]

The condition \( c_1 < 0 \) implies that equation \( F(\gamma) = (\alpha + 1)^2 \) has a unique solution satisfying condition \( \gamma > 1 \). Indeed, two cases are possible. In the first one for \( \gamma \in (1, \infty) \) \( F(\gamma) \) is strictly increasing taking all values in the interval \((F(1), \infty)\) and, since \((\alpha + 1)^2 > F(1)\) the equation \((\ref{eq:F})\) has a unique solution \(\gamma > 1\). The second case is similar to item a) except for now the condition \((\alpha + 1)^2 > F(1)\) holds and, hence, the equation \((\ref{eq:F})\) has no solution belonging to the interval \((1, \gamma_{\text{min}})\), but exactly one solution \(\gamma > \gamma_{\text{max}}\).

For \( \gamma < -1 \) there is no solution since \( F(-1) \geq (\alpha + 1)^2 \) and \( F(\gamma) \) is strictly decreasing for \( \gamma < -1 \).

So there exists a unique solution \( \gamma \in (1, +\infty) \) satisfying \((\ref{eq:F})\).

The case \( \{(\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 = 0, c_2 > 0, \sigma > 1, \delta > 1\} \) is similar to item a) except for now the condition \((\alpha + 1)^2 = F(1)\) holds and, hence, the equation \((\ref{eq:F})\) has a unique solution \(\gamma \) satisfying condition \(\gamma > 1\).

e) Let \((\alpha, \delta, \sigma) \in U(-, -) = \{(\alpha, \delta, \sigma) \in \mathbb{R}^3 : c_1 < 0, c_2 < 0\} \). As well as for the item d) one can show that conditions \( c_1 < 0, c_2 < 0 \) imply the existence of exactly two solutions of equation \( F(\gamma) = (\alpha + 1)^2 \) one of which is strictly greater than 1 and the other is strictly less than \(-1\).

The proof of the item f) is similar to the proof of the item e).

to prove the item g) note, that the eigenvector corresponding to \( \kappa \), defined by \((\ref{eq:kappa})\), is an odd function.

Let us find an eigenvector corresponding to the eigenvalue \( \nu \), where \( \nu = 2\lambda (\gamma + 1) \) and \( \gamma \) is a root of equation \((\ref{eq:gamma})\). Note, that if \( \alpha \neq -1 \) then \( \ell(\gamma) \neq \delta \).

It follows from \((\ref{eq:gamma})\) and \((\ref{eq:lambda})\), that \( f'_{-1} = -\frac{f_{-1} + f_1}{2} = 0 \) and, hence, \( f_{-1} + f_1 = \frac{f_1}{k} \). By \((\ref{eq:nu})\) we have
\[ f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_1(-\alpha + \delta \cos \varphi + (\sigma - \alpha \cos \varphi)f_0}{\cos \varphi + \gamma} e^{-ik\varphi} d\varphi = \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_1(-\alpha + \delta \cos \varphi + (\sigma - \alpha \cos \varphi)f_0}{\cos \varphi + \gamma} (\cos k\varphi - i \sin k\varphi) d\varphi \]

Because of the integrand is an odd function we have
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_1(-\alpha + \delta \cos \varphi + (\sigma - \alpha \cos \varphi)f_0}{\cos \varphi + \gamma} i \sin k\varphi d\varphi = 0 \]

and, hence
\[ f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_1(-\alpha + \delta \cos \varphi + (\sigma - \alpha \cos \varphi)f_0}{\cos \varphi + \gamma} \cos k\varphi d\varphi \quad (\ref{eq:f_k}) \]
It follows that \( f_k \) is an even function of \( k \): \( f_k = f_{-k} \).

This completes the proof of Theorem 2.1.

In order to prove theorem 2.2 consider the case of \( \alpha = -1 \). According to (31) we have

\[
D'(\alpha, \delta, \sigma, \gamma) = -h(\gamma)l^{-1}(\gamma)(\gamma - \sigma)(l(\gamma) - \delta)^2
\]

and the equation for eigenvalues will take the simple form

\[
(\gamma - \sigma)(l(\gamma) - \delta)^2 = 0
\]

We are looking for roots satisfying the condition \( |\gamma| > 1 \). So

- if \( |\delta| > 1, |\sigma| > 1 \), then there are two roots \( \gamma = \sigma \) and \( \gamma = \frac{\delta^2 + 1}{2\delta} \)
- if \( |\delta| \leq 1, |\sigma| > 1 \), then there is one root \( \gamma = \sigma \)
- if \( |\delta| > 1, |\sigma| \leq 1 \), then there is one root \( \gamma = \frac{\delta^2 + 1}{2\delta} \)
- if \( |\delta| \leq 1, |\sigma| \leq 1 \), then there are no roots

Remind that \( \gamma = \frac{\nu}{2\lambda} - 1 \) and \( \sigma = \frac{2\lambda + \mu}{2\lambda} \). If \( \alpha = \frac{\lambda}{\lambda} = -1 \) then \( \sigma = \frac{\mu}{2\lambda} - 1 \) and equality \( \gamma = \sigma \) is equivalent to \( \nu = \mu \).

According to (33) equality \( \gamma = \frac{\delta^2 + 1}{2\delta} \) is equivalent to \( \nu = \kappa \). By lemma 3.1 this eigenvalue is of multiplicity 2 since the rank of the system (18)–(20) equals 1.

If \( \mu = \kappa \), then there is only one eigenvalue of multiplicity 2 because of the rank of the system (18)–(20) equals 1.

The theorem is proved.

Appendix

A.1 Partition of parameter space

Here we give an illustration of partition (4). We consider sections of the parameter space \( \mathbb{R}^3 \) by planes \( \alpha = c \) for various constants \( c \).

A.2 Lemma A.2

Consider the functions

\[
g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\varphi}{\cos \varphi + x}, \quad (36)
\]
Figure 1: Case of $|c + 1| < 1$

$$h(x) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos \varphi d\varphi$$

$$v(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2 \varphi d\varphi$$

where $|x| > 1$.

**Lemma A.2** For $|x| > 1$

$$g(x) = \frac{1}{x\sqrt{1-x^2}}.$$  \hspace{1cm} (39)

$$h(x) = xg(x) - 1, v(x) = xh(x),$$  \hspace{1cm} (40)

$$g(x)v(x) = h^2(x) + h(x)$$  \hspace{1cm} (41)

**Proof** Let us make change of variable $z = e^{i\varphi}$ in integral \([36]\). We get

$$g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\varphi}{\cos \varphi + x} = \frac{1}{2\pi i} \int_{\Gamma} \frac{dz}{\left(z + \frac{x - 1}{2} + x\right)z} =$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \frac{2dz}{z^2 + 2xz + 1} = \frac{1}{2\pi i} \int_{\Gamma} \frac{2dz}{(z-a_1)(z-a_2)}$$

where $\Gamma = \{ z : |z| = 1 \}$ is the unit circle and $a_1 = -(x - \sqrt{x^2 - 1})$, $a_2 = -(x + \sqrt{x^2 - 1})$ are roots of square equation $z^2 + 2xz + 1 = 0$. For $|x| > 1$
only one of these roots is strictly less than 1 by module. If \( x \neq 1 \), then \(|a_1| = |x - \sqrt{x^2 - 1}| < 1\) and
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{2dz}{(z-a_1)(z-a_2)} = \text{Res}_{a_1} \frac{2}{(z-a_1)(z-a_2)} = \frac{1}{\sqrt{x^2-1}^2} = \frac{1}{x\sqrt{1-x^{-2}}}
\]

If \( x \neq 1 \), then \(|a_2| = |x + \sqrt{x^2 - 1}| < 1\) and
\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{2dz}{(z-a_1)(z-a_2)} = \]
\[ \text{Res}_{a}^{2} \frac{2}{(z - a_1)(z - a_2)} = \frac{1}{-\sqrt{x^2 - 1}} = \frac{1}{x\sqrt{1 - x^{-2}}} \]

For integral (37), we have

\[ h(x) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos \varphi d\varphi}{\cos \varphi + x} = \frac{x}{2\pi} \int_{-\pi}^{\pi} \frac{d\varphi}{\cos \varphi + x} - \frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi = xg(x) - 1. \]

Further, we get

\[ v(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos^2 \varphi + 2x \cos \varphi + x^2 - (2x \cos \varphi + x^2)}{\cos \varphi + x} d\varphi = \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos \varphi (\cos \varphi + x) - x \cos \varphi}{\cos \varphi + x} d\varphi = \]

\[ = \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} \frac{\cos \varphi d\varphi}{\cos \varphi + x} - \int_{-\pi}^{\pi} \cos \varphi d\varphi \right) = \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos \varphi d\varphi}{\cos \varphi + x} = xh(x) \]

Finally,

\[ g(x)v(x) = g(x)xh(x) = (h(x) + 1)h(x) = h^2(x) + h(x) \]

This completes the proof of the lemma.

References

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