Braiding and entanglement in spin networks:
a combinatorial approach to topological phases

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Abstract

The spin network quantum simulator relies on the $su(2)$ representation ring (or its $q$–deformed counterpart at $q = \text{root of unity}$) and its basic features naturally include (multipartite) entanglement and braiding. In particular, $q$–deformed spin network automata are able to perform efficiently approximate calculations of topological invariants of knots and 3–manifolds. The same algebraic background is shared by 2D lattice models supporting topological phases of matter that have recently gained much interest in condensed matter physics. These developments are motivated by the possibility to store quantum information fault– tolerant in a physical system supporting fractional statistics since a part of the associated Hilbert space is insensitive to local perturbations. Most of currently addressed approaches are framed within a ‘double’ quantum Chern–Simons field theory, whose quantum amplitudes represent evolution histories of local lattice degrees of freedom.

We propose here a novel combinatorial approach based on ‘state sum’ models of the Turaev–Viro type associated with $SU(2)_q$–colored triangulations of the ambient 3–manifolds. We argue that boundary 2D lattice models (as well as observables in the form of colored graphs satisfying braiding relations) could be consistently addressed. This is supported by the proof that the Hamiltonian of the Levin–Wen condensed string net model in a surface $\Sigma$ coincides with the corresponding Turaev–Viro amplitude on $\Sigma \times [0, 1]$ presented in the last section.

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1 Braiding and entanglement in spin networks

By spin networks we mean here ‘computational’ graphs the nodes and edges of which are labeled by dimensions of SU(2) irreducible representations (irreps) and by SU(2) recoupling transformations, respectively. For this reason spin networks can be thought of as a generalized quantum computational framework for dealing with unitary transformations among (entangled) many–angular momenta states. Actually, the computational space of the spin network simulator[1] encodes in its very definition the representation ring of the Lie group SU(2) given by finite–dimensional Hilbert spaces supporting irreps of SU(2) endowed with two operations, tensor product \( \otimes \) and direct sum \( \oplus \) (providing a ring structure over the field \( \mathbb{C} \)) as well as unitary operators relating (multiple tensor products of) such spaces.

Unlike the usual standard circuit model[2], here it is possible to handle eigenstates of \( N \) (pairwise coupled) angular momenta labeled by integers and half–integers \( j_1, j_2, \ldots, j_N \) (in \( h \) units) and not simply \( N \)–qubit states for which \( j_1 = j_2 = \ldots = j_N = 1/2 \). The (re)coupling theory of \( N \) SU(2) angular momenta provides the whole class of unitary transformations that can be performed on many–body quantum systems described by this kind of states[3]. Any such general transformation –expressed in terms of a \( 3n j \)–coefficient (\( N = n + 1 \)) when a choice of basis sets is made explicit– can be split into a finite sequence of two basic unitaries, called twist (or ‘trivial braiding’) and associator in the language of tensor categories[4]. The former acts on the (ordered) tensor product of two Hilbert spaces \( V, W \) supporting irreps of SU(2) by swapping them, namely

\[
T_{V,W} : V \otimes W \to W \otimes V \quad \text{with} \quad T_{W,V} \circ T_{V,W} = \text{Id}_{V \otimes W} \tag{1}
\]

and its action on a binary coupled state amounts to a trivial phase transform. The associator \( F \) relates different binary bracketing structures in the triple tensor product of irreps \( V, U, W \) (intertwiner)

\[
F : (V \otimes U) \otimes W \to V \otimes (U \otimes W) \tag{2}
\]

and is implemented on any binary coupled state as a transformation involving one Racah–Wigner \( 6j \) symbol. Notice that both (1) and (2) are isomorphisms but the associator reflects a physically measurable modification of the way in which intertwiner spaces are coupled. More complicated multiple tensor products can be related by various combinations of braidings and associators, so that the basic isomorphisms must satisfy compatibility conditions, a so–called pentagon identity and two hexagon identities[4] (in SU(2) angular momentum theory they correspond to the Biedenharn–Elliott identity and the Racah identity, respectively[3]).

The most effective way of dealing with non–trivial braiding operators –to be used in connection with the study of both braid group representations and braid statistics (typically occurring in anyonic systems)– is to move to the representation ring \( \mathcal{R}(SU(2)_q) \) (modular tensor category) of the \( q \)–deformed Hopf algebra of the Lie group SU(2), \( SU(2)_q \) (\( q \) a root of unity). Then the operation \( T \) in (1) turns out to be substituted by a non–trivial braiding

\[
R_{V,W} : V \otimes W \to W \otimes V \quad \text{with} \quad R_{W,V} \circ R_{V,W} \neq \text{Id}_{V \otimes W} \tag{3}
\]

The \( q \)–deformed spin network model of computation[5] is modelled on the \( q \)–representation ring

\[
( \mathcal{R}(SU(2)_q) ; R ; F ), \tag{4}
\]
where $F$ denotes from now on the $q$–counterpart of the associator (2). Once suitable basis sets are chosen, the isomorphisms $R$ and $F$ can be made explicit. In particular, $F$ contains the $q$–deformed counterpart of the $6j$–symbol and, regarding it as a unitary matrix, it is also referred to as ‘duality’ (or ‘fusion’) matrix in conformal field theories[6].

The framework outlined above was exploited in a series of papers[5, 7], where families of $q$–deformed spin network automata were implemented for processing efficiently classes of computationally–hard problems in geometric topology –in particular, approximate calculations of topological invariants of links (collections of knots) and 3–manifolds[9]. A prominent role was played there by ‘universal’ unitary braiding operators associated with representations of the braid group in $\mathfrak{g}(SU(2)_q)$. Suitable traces of matrices of these representations provide polynomial invariants of $SU(2)_q$–colored links, while weighted sums of the latter give topological invariants of 3–manifolds presented as complements of knots in the 3–sphere. These invariants are in turn recognized as partition functions and vacuum expectation values of physical observables in 3–dimensional Chern–Simons–Witten (CSW) Topological Quantum Field Theory (TQFT)[10]. The CSW environment actually provides not only a physical interpretation of such quantities but it is universal in the sense that includes, besides the quantum group interpretation quoted above, monodromy representations of the braid group arising in a variety of (boundary) conformal field theories[6] (where point–like excitations confined in 2–dimensional regions evolve along braided worldlines).

It is worth mentioning that the $q$–spin network approach represents a naturally ‘discretized’ version of the topological framework for quantum computation proposed a few years ago and further improved recently[11]. In the light of some basic questions raised by that paper, we are going to outline a novel approach to the whole matter of topological phases (section 2) providing in particular a transparent combinatorial description of condensed strings nets (section 3).

### 2 \(SU(2)_q\)-colored triangulations: 3D invariant partition functions and induced 2D lattice models

The issue of connections between (topological) gauge theories in 3 spacetime dimensions and 2D (integrable) lattice models[12] has been intensively investigated over the years in a variety of different contexts. The renewed interest driven by the search for a fault–tolerant quantum computer based on manipulations of Non–Abelian quantum Hall states[11, 13] represents a major challenge on both experimental and theoretical grounds. Most of the currently addressed approaches are framed within a CSW environment, but an \textit{ab initio} discretized framework could reveal much more effective, as briefly outlined in the following.

The Turaev–Viro (TV) ‘state sum’ model[14] provides a well–defined (i.e. finite) topological invariant for any closed 3–manifolds $M^3$. Given a triangulation $T^3$ of $M^3$, namely a dissection into tetrahedra, an $SU(2)_q$–coloring is assigned to it according to a set of admissible initial data (not made explicit here). A state functional for $T^3$ is built, where the $N_0$ vertices, $N_1$ edges and $N_3$ tetrahedra are suitably weighted. Finally, a summation over all admissible colored triangulations of $M^3$ is performed, the resulting functional being invariant under a set of combinatorial moves ensuring that its value
depends only on the topological type of the manifold $M^3$. The TV state sum reads

$$Z_{TV}[M^3; q] = \sum_{\{j\}} w^{-N_0} \prod_{A=1}^{N_3} w_A \prod_{B=1}^{N_3} \left\{ j_1 \ j_2 \ j_3 \right\}_B,$$

(5)

where the summation is over all colourings $\{j\}$ labeling irreps of $SU(2)_q$ ($q = \exp\{2\pi i/r\}$, with $\{j = 0, 1/2, 1, \ldots, r-1\}$); $w_A = (-1)^{2jA}[2jA+1]_q$ where $[\cdot]_q$ denote the quantum dimension of the irrep; $w = 2r/(q-q^{-1})^2$ and $\{\ldots\}_B$ represents the $q^{-6j}$ symbol whose entries are associated with the six edges of tetrahedron $B$.

The invariant (5) equals the square modulus of the Reshetikhin–Turaev invariant [9], which in turn represents the Chern–Simons partition function $Z_{CS}$ for an oriented 3–manifold $M^3$, namely $Z_{TV}[M^3; q] \leftrightarrow |Z_{CS}[M^3; k]|^2$, where $k = 2(r-1)$ is the level of Chern–Simons functional. This feature is crucial in view of applications to topological phases, where a ‘doubling’ of the basic CS setting is needed [11].

The TV model can be further extended to deal with both 3–manifolds with 2D boundary component(s) and ‘observables’ related to embedded links and (ribbon) graphs. The extension to a pair $(M^3, \Sigma)$ with $\Sigma \equiv \partial M^3$ can be carried out in two different ways, depending on whether the boundary surface $\Sigma$ inherits a fixed triangulation [15] or a ‘fluctuating’ one [16]. Observables in the form of colored graphs, satisfying braiding relations, can be consistently introduced [15, 17] in any oriented triangulated compact manifold $(M^3, \Sigma)$.

On the basis of such results we are currently addressing a combinatorial reformulation of theoretical foundations of topological phases [18]. A first result is discussed in the last section.

### 3 Topological spin liquids: a combinatorial description

We will now briefly introduce the condensed string nets of Levin and Wen [19] in $(2+1)$ dimension and show that the projector to the ground state of the boundary model can be obtained from the corresponding Turaev–Viro partition function $Z_{TV}(\Sigma \times [0, 1], X)$ for the cylinder $M^3 = \Sigma \times [0, 1]$ ($\Sigma$ is oriented and $X$ denotes a fixed identical triangulation on both $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$).

Consider the string net model defined on the honeycomb lattice [13] with microscopic degrees of freedom associated to the edges. The degrees of freedom are called string types and are endowed with $N + 1$ labels and orientations. The change of orientation of an edge is equivalent to the change of the label $i$ to its dual $i^*$ and the 0 label is self–dual. The Hamiltonian is a sum of potential and kinetic energy. When the relative coefficient of the kinetic piece is large, the ground state is a condensate of a dense string net. The universal long distance behaviour of these nets is characterized by the so–called fixed–point wave functions on the space of all configurations. The fixed–point wave functions $\Phi$ are in one–to–one correspondence with modular tensor categories [20] and obey the following rules (written by using the original notation [19],...
although slightly different from the conventions in \([5]\):

\[
\Phi \left( \begin{array}{c}
\text{i} \\
\end{array} \right) = \Phi \left( \begin{array}{c}
\text{j} \\
\end{array} \right)
\]

(6)

\[
\Phi \left( \begin{array}{c}
\text{i} \\
\text{j} \\
\end{array} \right) = d_i \Phi \left( \begin{array}{c}
\text{0} \\
\end{array} \right)
\]

(7)

\[
\Phi \left( \begin{array}{c}
\text{i} \\
\text{j} \\
\text{k} \\
\end{array} \right) = \delta_{ij} \Phi \left( \begin{array}{c}
\text{j} \\
\text{k} \\
\end{array} \right)
\]

(8)

\[
\Phi \left( \begin{array}{c}
\text{i} \\
\text{j} \\
\text{k} \\
\text{l} \\
\end{array} \right) = \sum_n F_{\text{klm}}^{\text{ijm}} \Phi \left( \begin{array}{c}
\text{j} \\
\text{i} \\
\text{l} \\
\text{k} \\
\end{array} \right)
\]

(9)

where the \(F\) matrix and the quantum dimensions \(d_i\) satisfy consistency conditions given by

\[
F_{ij}^{*k} = v_{k}^{-1} v_{i} v_{j} \delta_{ijk}
\]

\[
F_{klmn}^{*} = F_{lkmn}^{*} = F_{klnm}^{*} = F_{kmjn}^{*} = F_{ijm}^{*} v_{m} v_{n} v_{j} v_{i}
\]

\[
\sum_{n=0}^{N} F_{kp}^{*s} F_{mns}^{*} F_{kl}^{*s} F_{qr}^{*s} = F_{ijp}^{*s} F_{mls}^{*s}
\]

(10)

with \(v_i = \sqrt{\text{TV}_i}\) (= \(w_i\) in TV notation) and \(\delta_{ijk} = 0, 1\). The latter is the branching rule: the triple \(\{ijk\}\) is admissible at a vertex when \(\delta_{ijk} = 1\). These rules coincide with those of Turaev and Viro\([14]\) for the evaluation of closed manifold invariants in three dimensions and the (first two lines of the) consistency conditions encode the standard properties of the \(q - 6j\) symbol suitable normalized as

\[
d_{n} \left\{ \begin{array}{c}
i \\
j \\
k \\
l \\
\end{array} \right\} = F_{klmn}^{ijm}
\]

(11)

while the last relation in (10) is the Biedenharn–Elliott (pentagon) identity (cfr. section 1).

The Hamiltonian for the string–net model on the honeycomb lattice reads

\[
H = - \sum_I Q_I - \sum_p B_p,
\]

(12)

where \(I\) runs over vertices and \(p\) over plaquettes. \(Q_I = \delta_{ijk}\) with \(i, j, k\) being the colours of the edges incident to the vertex \(I\). The ‘magnetic constraints’ are given by the sum \(\sum_{s=0}^{N} a_s B_p^s\) with \(B_p^s\) to be described below. We consider the case when the coefficients are

\[
a_s = \frac{d_s}{\sum_{i=0}^{N} d_i^2}.
\]

(13)

This normalization prescription corresponds to the existence of smooth continuum limit of the lattice model in one hand and gives the precise weight of the Turaev–Viro evaluation on the other, as proved below.
Figure 1: a) shows a part of the honeycomb lattice and its dual graph. In b) the ‘plane’ of the triangle $abc$ is the ‘plane’ of the honeycomb lattice and there is another one with triangle $a''b''c''$. The matrix element of the projector is interpreted as a three dimensional extrapolation between the bra and the ket string net state given by a triangulation. The prism in the figure correspond to a part of the projector $P$ where there is a tetrahedron for every associated $F$ symbol.

The magnetic constraints

There is a nice intuitive way of determining the matrix elements of the magnetic constraints described in appendix C[19]. In short, for each hexagon of the lattice, one draws a smaller concentric hexagon with labels $s$ at every edge in hexagon $p$, then (i) connects the hexagons $p$ and $s$ with new edges and labels them with 0 (this does not change $\Phi$) and (ii) use the rules (9) repeatedly until one reaches the single hexagon again. The resulting coefficients define the matrix elements of the constraint given explicitly by

$$B_p^a = a_i b_j c_k d_l e_m f_n \sum F^{bg'h'} F^{ch'i'} F^{dji'} F^{ej'k'} F^{fjk'l'} F^{alg'}$$

(14)

The operators $B_p$ with the coefficients given by (13) are projectors (as the $Q_{I'}$’s) and all these constraint operators commute with each other.

The image of the projector $P = \prod_q Q_q \prod_p B_p$ is the ground state of the model. We are going to show explicitly that

$$P = Z_{TV}(\Sigma \times [0, 1], X),$$

(15)

where $\Sigma$ is the (oriented) surface on which the honeycomb lattice is defined and $X$ is the dual triangulation of $\Sigma \times \{0\} \simeq \Sigma \times \{1\} \simeq \Sigma$. To build the dual graph one places a vertex in the center of each face and connects the new vertices in neighbouring faces. In particular, the dual of the trivalent honeycomb lattice is a triangular graph as shown in Fig. 1b).

Let us start constructing $\prod_p B_p$. Each operator $B_p^a$ contains six $F$ symbols corresponding to the six vertices of a hexagon. We collect the three $F$ symbols from three
different $B^*_p$ operators corresponding to a given vertex. Their order does not matter since the $\bar{B}$ operators commute. Choosing the order $B^{s_1}_p B^{s_2}_p B^{s_3}_p$ (cfr. Fig. 1a) we get the contribution $F_{s_1 b' c'}^{a b' c'} F_{s_2 c' a'}^{b' a' c'} F_{s_3 a' b'}^{c' b' a'}$. Note also that we have omitted denoting the orientation for the sake of simplicity, but it is not difficult to incorporate it consistently. Comparing this expression with Fig. 1b), we find that it corresponds to the triangulation of the prism with boundary triangles $abc$ and $a'' b'' c''$. In this identification we have associated a $q - 6j$ or $F$ symbol to every tetrahedron, which is the prescription of the Turaev–Viro state sum (5).

The second step consists in checking whether the gluing along the triangulated boundary quadrilaterals can be made consistent with the algebraic pattern. Using the freedom of multiplying the $F$ symbols at a given vertex in arbitrary order (due to the commuting $B_p$ operators that they are part of), the consistency can be achieved. It is clear now that we have a triangulation given by the dual graph with edge labels $a'' b'' c'' d'' \ldots$ and an identical one with labels $abcd \ldots$ and these bound a 3–dimensional triangulation given by triangular prisms suitably glued together.

We now check whether the weights associated to edges and vertices match that of the TV prescription. Taking into account the normalisation (11) we find that there is a factor of $d_i$ associated to each internal edge denoted by $a'' b'' c'' d'' \ldots$. Because of the coefficient (13) it is also true for the internal edges ‘perpendicular’ to $\Sigma$ labelled by $s_i$ in the figures (this weight comes from the numerator of $a_s$, the denominator will be associated to vertices bounding these edges). To the boundary edges with labels $a'' b'' c'' d'' \ldots$ there is also a $d_{a''}, d_{b''}$ etc. associated, whereas to those with label $abcd \ldots$ there is no non–trivial weight. This is consistent with the TV prescription.

The last type of simplexes we have and not discussed yet are the boundary vertices. We associate the square root of the denominator $\left( \sum_i d_i^2 \right)^{-1/2} = D^{-1} = w^{-1}$ in TV notation) of the coefficient $a_s$ to the two boundary vertices connected by the edge labeled by $s$. This way we covered both boundaries and found agreement with the TV prescription.

The last step is the sum over internal colouring, which has to be performed to get the partition function. This corresponds to the summation over the labels $s_i$ and $a'' b'' c'' d'' \ldots$, which come from $\sum_i a_s B^*_p$ and the matrix multiplication in $\prod_p B_p$. Finally, note, that the electric constraints $\prod Q_q$ are taken care of by the product $\prod B_p$ as had non–admissible labels met at a vertex, a corresponding $q - 6j$ symbol gives zero. This concludes the proof of (15).

1For example we can introduce a black and white colour on boundaries and glue only black to white boundary, that is one with $d_i$ weights to one with no weights of vertices.
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