Non-CM elliptic curves with infinitely many almost prime Frobenius traces

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Abstract. Let $E$ be an elliptic curve defined over $\mathbb{Q}$ and without complex multiplication. For a prime $p$ of good reduction for $E$, we write #$_{E_p}(\mathbb{F}_p) = p + 1 - a_p(E)$ for the number of $\mathbb{F}_p$-rational points of the reduction $E_p$ of $E$ modulo $p$. Under the Generalized Riemann Hypothesis (GRH), we study the primes $p$ for which the integer $|a_p(E)|$ is a prime. In particular, we prove the following results: (i) the number of primes $p < x$ for which $|a_p(E)|$ is a prime is bounded from above by $C_1(E) x (\log x)^2$ for some constant $C_1(E)$; (ii) the number of primes $p < x$ for which $|a_p(E)|$ is the product of at most 4 distinct primes, counted without multiplicity, is bounded from below by $C_2(E) x (\log x)^2$ for some constant $C_2(E)$; (iii) the number of primes $p < x$ for which $|a_p(E)|$ is the product of at most 5 distinct primes, counted with multiplicity, is bounded from below by $C_3(E) x (\log x)^2$ for some positive constant $C_3(E)$ > 0. Under GRH, we also prove the convergence of the sum of the reciprocals of the primes $p$ for which $|a_p(E)|$ is a prime. Furthermore, under GRH, together with Artin’s Holomorphy Conjecture and a Pair Correlation Conjecture for Artin L-functions, we prove that the number of primes $p < x$ for which $|a_p(E)|$ is the product of at most 2 distinct primes, counted with multiplicity, is bounded from below by $C_4(E) x (\log x)^2$ for some constant $C_4(E)$. The constants $C_i(E)$, 1 ≤ i ≤ 4, are defined explicitly in terms of $E$ and are factors of another explicit constant $C(E)$ that appears in the conjecture that #$_{\{p < x : |a_p(E)| \text{ is prime}\}} \sim C(E) x (\log x)^2$.

1. Introduction

In the mid 1600s, Fermat stated a characterization in terms of modular arithmetic for the primes that may be written as $m^2 + Dn^2$ for some integers $m, n$, for each $D \in \{1, 2, 3\}$. His statements sparked the development of significant branches of contemporary number theory, which, in turn, provided the tools for proving a characterization of the primes $p$ represented by a given arbitrary positive definite, primitive, integral, binary quadratic form (see [Co89] for a beautiful account of this proof and its history). An outcome of this characterization is that, as $x \to \infty$,

$$\# \{p \leq x : p = Q(m, n) \text{ for some integers } m, n\} \sim C_0(Q) \frac{x}{\log x},$$

Key words and phrases: elliptic curves, primes, sieve methods

2010 Mathematics Subject Classification: 11G05, 11G25, 11A41, 11N05, 11N36

A.C.C. was partially supported by a Collaboration Grant for Mathematicians from the Simons Foundation under Award No. 709008.
where $C_0(Q)$ is some explicit positive constant that depends on the binary quadratic form $Q$.

In 1997, Fouvry and Iwaniec [FoIw97] pursued the study of primes $p$ that may be written as $m^2 + n^2$ for some integers $m, n$ such that $m$ is a prime. In particular, they proved the striking result that, as $x \to \infty$,

$$\# \{p \leq x : p = m^2 + n^2 \text{ for some integers } m, n \text{ such that } m \text{ is prime} \} \sim C \frac{x}{(\log x)^2},$$

where $C$ is some explicit positive constant. Moreover, in 2010, Friedlander and Iwaniec [Friw10] Thm. 18.6] provided a simplified proof of (2), while in 2020, Lam, Schindler, and Xiao [LaScXi20] proved a generalization of (2) which applies to any arbitrary positive definite, primitive, integral, binary quadratic form $Q(m, n)$, and not only to the form $Q(m, n) = m^2 + n^2$. In particular, they proved that, as $x \to \infty$,

$$\# \{p \leq x : p = Q(m, n) \text{ for some integers } m, n \text{ such that } m \text{ is prime} \} \sim C(Q) \frac{x}{(\log x)^2},$$

where $C(Q)$ is some explicit constant that depends on the binary quadratic form $Q$. These results may be viewed under the unifying theme of equidistribution and primes envisioned by Sarnak [Sa08].

Primes of the forms $m^2 + n^2$ for some integers $m, n$ such that $m$ is a prime also appear naturally in the setting of elliptic curves. Indeed, for the elliptic curve $E : y^2 = x^3 - x$, any odd prime $p$ for which the Frobenius trace $a_p(E)$ is the double of a prime $\ell$ gives rise to the representation $p = \ell^2 + n^2$ for some non-zero integer $n$ (see below for the definition of $a_p(E)$). This is not an isolated example, but rather a particular case of a general phenomenon related to a pair $(E, p)$, where $E$ is an elliptic curve defined over $\mathbb{Q}$ and with complex multiplication, and $p$ is a prime of good reduction for $E$ having the property that the Frobenius trace $a_p(E)$ is a prime or twice a prime.

With this motivation in mind, our purpose in this paper is to investigate the primality of the Frobenius traces of an elliptic curve defined over $\mathbb{Q}$ and without complex multiplication, as explained in what follows.

Let $E$ be an elliptic curve defined over $\mathbb{Q}$, of conductor $N_E$, and without complex multiplication. For each rational prime $p \nmid N_E$, we denote by $E_p$ the reduction of $E$ modulo $p$. We recall that the reduced curve $E_p$ is itself an elliptic curve over the finite field $\mathbb{F}_p$ with $p$ elements and that its group of $\mathbb{F}_p$-rational points, $E_p(\mathbb{F}_p)$, has size $\#E_p(\mathbb{F}_p) = p + 1 - a_p(E)$ for some integer $a_p(E)$ which satisfies the bound

$$|a_p(E)| < 2\sqrt{p}.$$

For a positive integer $k$, we denote by $\omega(k)$ the number of prime factors of $k$, counted without multiplicity, and by $\Omega(k)$ the number of prime factors of $k$, counted with multiplicity. A positive integer $n$ is called a $Q_k$-integer if $\omega(n) \leq k$, and a $P_k$-integer if $\Omega(n) \leq k$. With this notation and terminology, for an arbitrary $x > 0$ and an arbitrary integer $k \geq 1$, we set

$$\pi_{E, \text{prime trace}}(x) := \# \{p \leq x : p \nmid N_E, |a_p(E)| \text{ is prime} \},$$

$$\pi_{E, Q_k, \text{trace}}(x) := \# \{p \leq x : p \nmid N_E, a_p(E) \notin \{0, \pm 1\}, \omega(|a_p(E)|) \leq k \},$$

$$\pi_{E, P_k, \text{trace}}(x) := \# \{p \leq x : p \nmid N_E, a_p(E) \notin \{0, \pm 1\}, \Omega(|a_p(E)|) \leq k \}.$$
In [CoJo22], Cojocaru and Jones make the following prediction:

**Conjecture**

Let $E$ be an elliptic curve defined over $\mathbb{Q}$, of conductor $N_E$, and without complex multiplication. There exists a constant $C(E)$ such that, as $x \to \infty$,

$$\pi_{E, \text{prime trace}}(x) \sim C(E) \frac{x}{(\log x)^2}.$$  

The constant is explicitly defined as

$$C(E) := 2 \cdot \frac{m_E}{\phi(m_E)} \cdot \frac{\# \{ M \in \text{Gal}(\mathbb{Q}(E[m_E])/\mathbb{Q}) : \text{tr} M \in (\mathbb{Z}/m_E\mathbb{Z})^\times \} }{\# \text{Gal}(\mathbb{Q}(E[m_E])/\mathbb{Q})} \cdot \prod_{\ell \mid m_E} \left(1 - \frac{1}{\ell^3 - \ell^2 - \ell + 1}\right),$$

where $m_E$ is the torsion conductor of $E/\mathbb{Q}$ (see Section 2.2 for a definition), $\text{Gal}(\mathbb{Q}(E[m_E])/\mathbb{Q})$ is the Galois group of the $m_E$-division field of $E$, viewed as a subgroup of the matrix group $\text{GL}_2(\mathbb{Z}/m_E\mathbb{Z})$, and $\phi(m_E)$ is the Euler function of $m_E$.

Let us note that, in [CoJo22], the authors show that the constant $C(E)$ is positive for infinitely many elliptic curves $E$.

A weaker form of this conjecture, with $C(E)$ not given explicitly, was proposed by Cojocaru as a topic of investigation in the Bachelor’s thesis of Lane [La05]. Therein, it was proven that, under the Generalized Riemann Hypothesis for Dedekind zeta functions, we have

$$\pi_{E, \text{prime trace}}(x) \ll_E \frac{x}{(\log x)^2},$$  

(10)

$$\pi_{E, \text{Q2-trace}}(x) \gg_E \frac{x}{(\log x)^2},$$  

(11)

and

$$\pi_{E, \text{P2-trace}}(x) \gg_E \frac{x}{(\log x)^2},$$  

(12)

where the constants implied in the $\ll_E$ and $\gg_E$ notation depend on $E$ in an unspecified way.

Our goal in the paper is to improve upon the above results in several aspects, as follows.

**Theorem 1.** Let $E$ be an elliptic curve defined over $\mathbb{Q}$, of conductor $N_E$, and without complex multiplication. Assume that there exists some $\frac{1}{2} \leq \theta < 1$ such that the $\theta$-quasi Generalized Riemann Hypothesis holds for Dedekind zeta functions. Then, for all $x > e$,

$$\pi_{E, \text{prime trace}}(x) \leq \left(\frac{3}{1 - \theta} + o(1)\right) C(E) \frac{x}{(\log x)^2},$$  

(13)

where $C(E)$ is the explicit constant introduced in [4]. In particular, when $\theta = \frac{1}{2}$, (13) becomes

$$\pi_{E, \text{prime trace}}(x) \leq (6 + o(1)) C(E) \frac{x}{(\log x)^2}.$$  

(14)
As a corollary to Theorem 1, we obtain the convergence of the sum of $1/p$, where $p$ runs over primes for which $a_p(E)$ is a prime, a result reminiscent of a famous theorem of Brun about the convergence of the sum of $1/p$, where $p$ runs over primes for which $p + 2$ is also a prime.

**Corollary 2.** Let $E$ be an elliptic curve defined over $\mathbb{Q}$, of conductor $N_E$, and without complex multiplication. Assume that there exists some $\frac{1}{2} \leq \theta < 1$ such that the $\theta$-quasi Generalized Riemann Hypothesis holds for Dedekind zeta functions. Then

$$
\sum_{p \mid N_E, \text{prime}} \frac{1}{p} < \infty.
$$

More precisely, for any $\varepsilon > 0$, there exists $x_0 = x_0(E, \theta, \varepsilon)$ such that

$$
\sum_{p \geq x_0, \text{prime}} \frac{1}{p} \leq \left( \frac{3}{1 - \theta} + \varepsilon \right) C(E) \frac{1}{\log x_0},
$$

where $C(E)$ is the explicit constant introduced in [7].

Our second main result is reminiscent of Chen’s lower bound [Ch73] about almost twin primes.

**Theorem 3.** Let $E$ be an elliptic curve defined over $\mathbb{Q}$, of conductor $N_E$, and without complex multiplication.

(i) Assume that there exists some $\frac{1}{2} \leq \theta < 1$ such that the $\theta$-quasi Generalized Riemann Hypothesis holds for Dedekind zeta functions. Denote by $r_1, r_2$ the positive integers defined by

$$
r_1 = r_1(\theta) := 1 + \left[ \frac{1}{0.83} \left( \frac{3}{2(1 - \theta)} - \frac{1}{6} \right) \right],
$$

$$
r_2 = r_2(\theta) := 1 + \left[ \frac{5}{2(1 - \theta)} - \frac{5}{12} \right].
$$

Then, for all $x > e$,

$$
\pi_{E, Q_{r_1}, \text{trace}}(x) \geq \frac{3}{1 - \theta} (0.00692 + o(1)) C(E) \frac{x}{(\log x)^2}
$$

and

$$
\pi_{E, P_{r_2}, \text{trace}}(x) \geq \frac{3}{1 - \theta} (0.3162 + o(1)) C(E) \frac{x}{(\log x)^2}.
$$

In particular, when $\theta = \frac{1}{2}$, the upper bounds (15) and (16) become

$$
\pi_{E, Q_{44}, \text{trace}}(x) \geq (0.0415 + o(1)) C(E) \frac{x}{(\log x)^2}
$$

and

$$
\pi_{E, P_{5}, \text{trace}}(x) \geq (1.8972 + o(1)) C(E) \frac{x}{(\log x)^2}.
$$
(ii) Assume that the Generalized Riemann Hypothesis holds for Dedekind zeta functions, and that Artin’s Holomorphy Conjecture and a Pair Correlation Conjecture hold for Artin L-functions. Then, for all $x > e$,

$$
\pi_{E, P_2}^{\text{trace}}(x) \geq (1 + o(1)) C(E) \frac{x}{(\log x)^2}.
$$

Our approach to the study of the primality of $a_p(E)$ takes inspiration from classical studies of the primality of $p + a$, for some fixed even integer $a$. In particular, it draws on the analogy between the interpretation of the divisibility $m \mid a_p(E)$ for some non-zero integer $m$ as a Chebotarev condition at $p$ in the division field $\mathbb{Q}(E[m])$ of $E$ and the interpretation of the divisibility $m \mid (p + a)$ for some non-zero integer $m$ as a Chebotarev condition at $p$ in the cyclotomic field $\mathbb{Q}(\zeta_m)$. For example, in light of this analogy, analytic methods, such as Turán’s normal order method and Titchmarsh’ conditional approach to the divisor problem for $p + a$, have already been used in the study of the divisors of $a_p(E)$ in works such as [GuMu14], [MuMu84], and [Po16].

In this paper, we pursue the analogy between the arithmetic of $a_p(E)$ and that of $p + a$ by employing sieve methods in the spirit of the study of the primality of $p + 1 - a_p(E)$ in analogy to that of $p + a$ pursued in [Co05], [DaWu12], [IwJu10], [Ju08], [MiMu01], and [StWe05]. In particular, after setting an elliptic curve sieve problem in Section 3, we use a version of the Selberg Upper Bound Sieve from [HaRi74] to prove Theorem 4 in Section 4 and we use a version of Greaves Lower Bound Sieve from [HaRi85] to prove Theorem 5 in Section 5. The key ingredients in these proofs are Theorems 9 and 11 from Section 2 based on conditional effective versions of the Chebotarev Density Theorem. The special feature of these density theorems is their number field setting, which is derived not from the division field $\mathbb{Q}(E[m])$ of $E$, but rather from the splitting field $J_{E,m}$ of the modular polynomial $\Phi_m(X, j(E))$, where $j(E)$ is the $j$-invariant of the elliptic curve $E$ (see [Co89] and d[Second setting] in Section 2 for this number field setting, and [Co89] Ch. 3] for modular polynomials). This choice of number field setting leads to improvements from $Q_5$ to $Q_4$, and from $P_7$ to $P_5$, in Theorem 8 over [11] - [12], under the Generalized Riemann Hypothesis, and to the $P_2$ result of Theorem 2 under the Generalized Riemann Hypothesis, Artin’s Holomorphy Conjecture, and a Pair Correlation Conjecture. The improvements in the explicit constants of $E$ emerging in all of our main results come from the specific versions of upper bound and lower bound sieves that we use.

**Notation.** Throughout the paper, we use the following notation.

- Given a finite set $S$, we denote its cardinality by $\# S$.
- Given suitably defined real functions $h_1, h_2$, we say that $h_1 = o(h_2)$ if $\lim_{x \to \infty} \frac{h_1(x)}{h_2(x)} = 0$; we say that $h_1 = O(h_2)$ or, equivalently, that $h_1 \ll h_2$, if $h_2$ is positive valued and there exists a positive constant $c$ such that $|h_1(x)| \leq c h_2(x)$ for all $x$ in the common domain of $h_1$ and $h_2$; we say that $h_1 \asymp h_2$ if $h_1, h_2$ are
positive valued and $h_1 \ll h_2 \ll h_1$; we say that $h_1 = O_D(h_2)$ or, equivalently, that $h_1 \ll_D h_2$, if $h_1 = O(h_2)$ and the implied O-constant $c$ depends on priorly given data $D$; we say that $h_1 \sim h_2$ if $\lim_{x \to \infty} \frac{h_1(x)}{h_2(x)} = 1.$

- Given integers $m \geq 2$ and $n \geq 2$, we write $m \mid n^\infty$ to mean that every prime dividing $m$ also divides $n$.
- Given an integer $n \geq 1$, we write $\phi(n)$ for the Euler function of $n$, $\omega(n)$ for the number of prime factors of $n$, counted without multiplicity, and $\Omega(n)$ for the number of prime factors of $n$, counted with multiplicity.
- We use the letters $p$ and $\ell$ to denote positive rational primes. We denote by $\pi(x)$ the number of primes $p \leq x$ and we recall that, by the Prime Number Theorem, $\pi(x) \sim \frac{x}{\log x}$.
- Given an odd prime $\ell$, we use the notation $(\ell\mod{c})$ for the Legendre symbol.
- Given an integer $m \geq 1$, we denote by $\mathbb{Z}/m\mathbb{Z}$ the ring of integers modulo $m$. When $m$ is a prime $\ell$, we denote $\mathbb{Z}/\ell\mathbb{Z}$ by $\mathbb{F}_\ell$ to emphasize its field structure. For an integer $a$, we denote by $a \pmod{\ell}$ its residue class modulo $\ell$.
- Given a prime $\ell$, we denote by $\mathbb{Z}_\ell$ the ring of $\ell$-adic integers. We set $\hat{\mathbb{Z}} := \varprojlim \mathbb{Z}/m\mathbb{Z}$ and recall that there exists a ring isomorphism $\hat{\mathbb{Z}} \simeq \prod_{\ell} \mathbb{Z}_\ell$.
- Given a number field $K$, we denote by $\mathcal{O}_K$ its ring of integers, by $\sum_K$ the set of non-zero prime ideals of $\mathcal{O}_K$, by $[K : \mathbb{Q}]$ the degree of $K$ over $\mathbb{Q}$, by $d_K \in \mathbb{Z}\setminus\{0\}$ the discriminant of an integral basis of $\mathcal{O}_K$, and by $\text{disc}(K/\mathbb{Q}) = \mathbb{Z}d_K \subseteq \mathbb{Z}$ the discriminant ideal of $K/\mathbb{Q}$. For a prime ideal $\wp \in \sum_K$, we denote by $N_{K/\mathbb{Q}}(\wp)$ its norm in $K/\mathbb{Q}$. We say that $K$ satisfies the Generalized Riemann Hypothesis (GRH) if the Dedekind zeta function $\zeta_K$ of $K$ has the property that, for any $\rho \in \mathbb{C}$ with $0 \leq \Re \rho \leq 1$ and $\zeta_K(\rho) = 0$, we have $\Re(\rho) = \frac{1}{2}$. For $\frac{1}{2} \leq \theta < 1$, we say that $K$ satisfies the $\theta$-quasi Generalized Riemann Hypothesis ($\theta$-quasi GRH) if the Dedekind zeta function $\zeta_K$ of $K$ has the property that, for any $\rho \in \mathbb{C}$ with $0 \leq \Re \rho \leq 1$ and $\zeta_K(\rho) = 0$, we have $\Re(\rho) \geq \theta$. Note that $\frac{1}{2}$-quasi GRH is the same as GRH.
- Given a Galois extension $L/K$ of number fields and given an irreducible character $\chi$ of the Galois group of $L/K$, we denote by $f(\chi) \subseteq \mathcal{O}_K$ the global Artin conductor of $\chi$, by $A_{\chi} := |d_L|\chi(1)N_{K/\mathbb{Q}}(f(\chi)) \in \mathbb{Z}$ the conductor of $\chi$, and by $A_{\chi}(T)$ the function of a positive real variable $T > 3$ defined by the relation

$$\log A_{\chi}(T) = \log A_{\chi} + \chi(1)[K : \mathbb{Q}]\log T.$$
function of \( L(s, \chi, L/K) \) by

\[
P_T(X, \chi) := \sum_{-T \leq \gamma_1 \leq T} \sum_{-T \leq \gamma_2 \leq T} w(\gamma_1 - \gamma_2)e((\gamma_1 - \gamma_2)X),
\]

where \( \gamma_1 \) and \( \gamma_2 \) range over all the imaginary parts of the non-trivial zeroes \( \rho = \frac{1}{2} + i\gamma \) of \( L(s, \chi, L/K) \), counted with multiplicity, and where, for an arbitrary real number \( u \), \( e(u) := \exp(2\pi i u) \) and \( w(u) := \frac{4}{1 + u^2} \). We say that the extension \( L/K \) satisfies the Pair Correlation Conjecture (PCC) if, for any irreducible character \( \chi \) of the Galois group of \( L/K \) and for any \( A > 0 \) and \( T > 3 \), provided \( 0 \leq Y \leq A\chi(1)[K : \mathbb{Q}] \log T \), we have

\[
P_T(Y, \chi) \ll A\chi(1)^{-1}T \log A\chi(T).
\]

- Given a field \( F \), we denote by \( \overline{F} \) a fixed algebraic closure of \( F \).
- Given a non-zero unitary commutative ring \( R \), we denote by \( R^\times \) its group of multiplicative units.
- Given a non-zero unitary commutative ring \( R \) and an integer \( n \geq 1 \), we denote by \( M_n(R) \) the ring of \( n \times n \) matrices with entries in \( R \) and by \( I_n \) the identity matrix in \( M_n(R) \). For an arbitrary matrix \( M \in M_n(R) \), we denote by \( \text{tr} M \) and \( \det M \) its trace and determinant, and by \( M^t \) its transpose. We define the general linear group \( \text{GL}_n(R) \) as the collection of \( M \in M_n(R) \) with \( \det M \in R^\times \). We define the projective general linear group \( \text{PGL}_n(R) \) as the quotient group \( \text{GL}_n(R)/\{aI_n : a \in R^\times \} \).

Acknowledgements. We thank Damaris Schindler for useful remarks related to the asymptotic formula (3) and Nathan Jones for helpful conversations related to the constants (9) and (104).

2. Preliminaries about elliptic curves

2.1. Generalities about Galois representations associated to elliptic curves. Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \) and of conductor \( N_E \). For a fixed arbitrary positive integer \( m \), we denote by \( E[m] \) the group of \( \overline{\mathbb{Q}} \)-rational points of \( E \) of order dividing \( m \) and by \( \mathbb{Q}(E[m]) \) the field obtained by adjoining to \( \mathbb{Q} \) the \( x \) and \( y \) coordinates of the points of \( E[m] \). We recall from the theory of elliptic curves that the group \( E[m] \) is isomorphic to \( (\mathbb{Z}/m\mathbb{Z})^2 \), that the field extension \( \mathbb{Q}(E[m])/\mathbb{Q} \) is finite and Galois, and that the rational primes that ramify in \( \mathbb{Q}(E[m]) \) are among the prime factors of \( mN_E \). By fixing a \( \mathbb{Z}/m\mathbb{Z} \)-basis of \( E[m] \), we obtain the residual modulo \( m \) Galois representation of \( E/\mathbb{Q} \),

\[
\overline{\rho}_{E,m} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_2(\mathbb{Z}/m\mathbb{Z}),
\]

which has the property that

\[
\mathbb{Q}(E[m]) = \overline{\mathbb{Q}}^\text{Ker}\overline{\rho}_{E,m}.
\]

Taking the inverse limit over \( m \) of the representations \( \overline{\rho}_{E,m} \), we obtain a continuous Galois representation, the adelic Galois representation of \( E/\mathbb{Q} \),

\[
\rho_E : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GL}_2\left( \mathbb{Z} \right).
\]
Setting $m$ to be powers $\ell^k$ of a fixed prime $\ell$ and taking the inverse limit over $k$ of the representations $\overline{\rho}_{E,\ell^k}$, we obtain another continuous representation, the $\ell$-adic Galois representation of $E/\mathbb{Q}$,

$$\rho_{E,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Z}_\ell) .$$

It is known that, for each prime $p \nmid mN_E$, the $p$-Weil polynomial

$$P_{E,p}(X) := X^2 - a_p(E)X + p \in \mathbb{Z}[X]$$

satisfies the congruence

$$P_{E,p}(X) \equiv \det(XI_2 - \overline{\rho}_{E,m}(\text{Frob}_p)) \mod m,$$

where $\text{Frob}_p \in \text{Gal}(\mathbb{Q}(E[m])/\mathbb{Q})$ denotes the Frobenius element at an arbitrary non-zero prime ideal $p$ of $\mathbb{Q}(E[m])$, lying above $p$. Thus, we always have the congruence

$$(19) \quad \text{tr}\overline{\rho}_{E,m}(\text{Frob}_p) \equiv a_p(E) \mod m.$$ 

This congruence suggests that the field extension $\mathbb{Q}(E[m])/\mathbb{Q}$ plays a crucial role in the study of the arithmetic properties of $a_p(E)$, an observation which we will put to use in our proofs.

2.2. Division fields of elliptic curves. As in Subsection §2.1, let $E$ be an elliptic curve defined over $\mathbb{Q}$ and of conductor $N_E$, and let $m$ be an arbitrary positive integer. Thanks to [18], the Galois group $\text{Gal}(\mathbb{Q}(E[m])/\mathbb{Q})$, which we denote by

$$G_E(m) := \text{Gal}(\mathbb{Q}(E[m])/\mathbb{Q}),$$

may be identified with a subgroup of $\text{GL}_2(\mathbb{Z}/m\mathbb{Z})$. As a consequence, the degree of the extension $\mathbb{Q}(E[m])/\mathbb{Q}$ has the natural upper bound

$$(20) \quad [\mathbb{Q}(E[m]) : \mathbb{Q}] \leq \# \text{GL}_2(\mathbb{Z}/m\mathbb{Z}) = m^4 \prod_{\ell|m} \left(1 - \frac{1}{\ell}\right) \left(1 - \frac{1}{\ell^2}\right) \leq m^4.$$

If $E/\mathbb{Q}$ is without complex multiplication, then Serre’s Open Image Theorem for elliptic curves, proven in [Se72], implies the existence of a smallest positive integer $m_E$ having the property that, upon writing the priorly fixed arbitrary integer $m$ uniquely as

$$(21) \quad m = m_1m_2$$

for some positive integers $m_1, m_2$ such that

$$m_1 \mid m_E^\infty \quad \text{and} \quad \gcd(m_2, m_E) = 1,$$

there exists a subgroup $H_{E,m_1} \leq \text{GL}_2(\mathbb{Z}/m_1\mathbb{Z})$ satisfying a group isomorphism

$$(22) \quad G_E(m) \simeq H_{E,m_1} \times \text{GL}_2(\mathbb{Z}/m_2\mathbb{Z}).$$

For future purposes, we recall that $m_E$ is an even positive integer (see [Jo10]), which we refer to as the torsion conductor of $E/\mathbb{Q}$.
As a consequence of (22), if \(E/Q\) is without complex multiplication, then the degree of \(Q(E[m])/Q\) is the product of the function of \(m_1\) defined by \([H_{E,m_1} : Q]\) and the explicit multiplicative function of \(m_2\) defined by \(#\text{GL}_2(\mathbb{Z}/m_2\mathbb{Z})\). Consequently, the degree of \(Q(E[m])/Q\) obeys the lower bound

\[
m_2^4 \prod_{\ell | m_2} \left(1 - \frac{1}{\ell}\right) \left(1 - \frac{1}{\ell^2}\right) = #\text{GL}_2(\mathbb{Z}/m_2\mathbb{Z}) \leq [Q(E[m]) : Q].
\]

Our approach to the study of the prime factors of \(a_p(E)\) will rely mostly on the properties of a particular subfield of the division field \(Q(E[m])\), defined as follows. Upon identifying \(\mathbb{G}_E(m)\) with its image under \(\rho_{E,m}\) in \(\text{GL}_2(\mathbb{Z}/m\mathbb{Z})\), we set \(J_{E,m}\) to be the subfield of \(Q(E[m])\) fixed by the scalar subgroup \(\text{Scal}_{\mathbb{G}_E(m)}\) of \(\mathbb{G}_E(m)\), that is,

\[
J_{E,m} := Q(E[m])^{\text{Scal}_{\mathbb{G}_E(m)}},
\]

where

\[
\text{Scal}_{\mathbb{G}_E(m)} := \mathbb{G}_E(m) \cap \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/m\mathbb{Z}) : a \in (\mathbb{Z}/m\mathbb{Z})^\times \right\}.
\]

The subfield \(J_{E,m}\) is, in fact, the splitting field of the modular polynomial \(\Phi_m(X, j(E))\) over \(Q\), but we will not use this property in our proofs.

We observe that \(\text{Scal}_{\mathbb{G}_E(m)} \leq \mathbb{G}_E(m)\) and deduce that \(J_{E,m}/Q\) is a finite Galois extension, whose Galois group we denote by

\[
\hat{\mathbb{G}}_E(m) := \text{Gal}(J_{E,m}/Q).
\]

Moreover, we observe that \(J_{E,m} = \mathbb{Q}^{\text{Ker} \hat{\rho}_{E,m}}\), where

\[
\hat{\rho}_{E,m} : \text{Gal}(\mathbb{Q}/Q) \rightarrow \text{PGL}_2(\mathbb{Z}/m\mathbb{Z})
\]

is the Galois representation obtained by composing the natural projection \(\text{GL}_2(\mathbb{Z}/m\mathbb{Z}) \rightarrow \text{PGL}_2(\mathbb{Z}/m\mathbb{Z})\) with \(\mathbb{P}_{E,m}\). As a consequence, we obtain that the degree of \(J_{E,m}/Q\) satisfies the upper bounds

\[
[J_{E,m} : Q] \leq #\text{PGL}_2(\mathbb{Z}/m\mathbb{Z}) = m^3 \prod_{\ell | m} \left(1 - \frac{1}{\ell^2}\right) \leq m^3.
\]

If \(E/Q\) is without complex multiplication, then, using factorization (21) of \(m\) and invoking Serre’s Open Image Theorem as before, we deduce that

\[
\hat{\mathbb{G}}_E(m) \simeq \frac{\mathbb{G}_E(m)}{\text{Scal}_{\mathbb{G}_E(m)}} \simeq \frac{H_{E,m_1}}{\text{Scal}_{H_{E,m_1}}} \times \text{PGL}_2(\mathbb{Z}/m_2\mathbb{Z}).
\]

Consequently, the degree of \(J_{E,m}/Q\) is the product of the function of \(m_1\) defined by \]\frac{#H_{E,m_1}}{#\text{Scal}_{H_{E,m_1}}}\] and the explicit multiplicative function of \(m_2\) defined by \(#\text{PGL}_2(\mathbb{Z}/m_2\mathbb{Z})\).
2.3. Applications of the Chebotarev Density Theorem for division fields of elliptic curves.

As in Subsections §2.1 and §2.2, let $E$ be an elliptic curve defined over $\mathbb{Q}$, of conductor $N_E$, and let $m$ be an arbitrary positive integer. Throughout this subsection, we assume that $E/\mathbb{Q}$ is without complex multiplication and use the notation $m_E$ for the torsion conductor of $E/\mathbb{Q}$, the even integer whose existence is ensured by Serre’s Open Image Theorem for elliptic curves, as mentioned in Subsection §2.2. Similarly to the previous two subsections, we use factorization (21) for $m$ and we appeal to the group isomorphism (22) whenever needed.

Crucial to our analytic study of the primality of the Frobenius traces $a_p(E)$ of $E$ are applications in the settings $\mathbb{Q}(E[m])/\mathbb{Q}$ and $J_{E,m}/\mathbb{Q}$ of an effective version of the Chebotarev Density Theorem, which we now recall.

Let $L/K$ be a Galois extension of number fields, with $G := \text{Gal}(L/K)$, and let $\emptyset \neq \mathcal{C} \subseteq G$ be a union of conjugacy classes of $G$. We denote by disc$(L/K) \subseteq \mathcal{O}_K$ the discriminant ideal of $L/K$. We set

$$\pi_\mathcal{C}(x, L/K) := \sum_{\substack{p \in \Sigma_K \setminus \text{disc}(L/K) \subseteq \mathcal{O}_K \setminus (\mathcal{O}_K/\mathcal{P}) \leq x}} \delta_\mathcal{C} \left( \left( \frac{L/K}{p} \right) \right),$$

where $\delta_\mathcal{C}(\cdot)$ is the characteristic function of $\mathcal{C}$, the sum is over non-zero prime ideals $p$ of $\mathcal{O}_K$ which are unramified in $L/K$ and have norm $N_{K/\mathbb{Q}}(p) \leq x$, and $\left( \frac{L/K}{p} \right) \subseteq G$ is the Artin symbol at $p$ in $L/K$.

The Chebotarev Density Theorem asserts that, as $x \to \infty$,

$$\pi_\mathcal{C}(x, L/K) \sim \frac{\# \mathcal{C}}{\# G} \pi(x).$$

In studies such as ours, the above asymptotic formula is needed in a formulation which highlights the dependence of the growth of the error term $|\pi_\mathcal{C}(x, L/K) - \frac{\# \mathcal{C}}{\# G} \pi(x)|$ on the extension $L/K$ and on the set $\mathcal{C}$.

In order to state effective versions of (26), we introduce the notation

$$P(L/K) := \{ p : \exists p \in \Sigma_K \text{ such that } p | p \text{ and } p | \text{disc}(L/K) \}$$

and

$$M(L/K) := 2[L : K] |d_K| \prod_{p \in P(L/K)} p,$$

and we recall from [Se81] Prop. 5, p. 129] that

$$\log |N_{K/\mathbb{Q}}(\text{disc}(L/K))| \leq ([L : \mathbb{Q}] - [K : \mathbb{Q}]) \left( \sum_{p \in P(L/K)} \log p \right) + [L : \mathbb{Q}] \log[L : K].$$

We are now ready to state the effective versions of (26) needed in the proofs of our main results.

**Theorem 4.** Let $L/K$ be a Galois extension of number fields, with $G := \text{Gal}(L/K)$, and let $\emptyset \neq \mathcal{C} \subseteq G$ be a union of conjugacy classes of $G$. Assume that, for some $\frac{1}{2} \leq \theta < 1$, the $\theta$-quasi-GRH holds for the Dedekind zeta function of $L$. Then

$$\pi_\mathcal{C}(x, L/K) = \frac{\# \mathcal{C}}{[L : K]} \pi(x) + O \left( (\# \mathcal{C}) x^\theta [K : \mathbb{Q}] \left( \frac{\log |d_L|}{[L : \mathbb{Q}]} + \log x \right) \right).$$
Proof. The original reference is [LaOd77]. For the above formulation with $\theta = \frac{1}{2}$, see [Se81] Thm. 4, p. 133. The same proof goes through for $\frac{1}{2} < \theta < 1$. □

Theorem 5. Let $L/K$ be a Galois extension of number fields, with $G := \text{Gal}(L/K)$, and let $\emptyset \neq C \subseteq G$ be a union of conjugacy classes of $G$. Denote by $\text{Gal}(L/K)^\#$ the set of conjugacy classes of $\text{Gal}(L/K)$. Assume that GRH holds for the Dedekind zeta function of $L$, and that AHC and PCC hold for the extension $L/K$. Then

$$\pi_C(x, L/K) = \frac{\#C}{[L : K]} \pi(x) + O \left( \left( \frac{\# \text{Gal}(L/K)^\#}{[L : K]} \right)^{\frac{1}{2}} x^{\frac{1}{2}} [K : \mathbb{Q}]^{\frac{3}{2}} \log(M(L/K)x) \right).$$

Proof. This is [MuMuWo18] Theorem 1.2, p. 402]. □

We now highlight two particular elliptic curve settings for Theorems 4-5 which will be relevant to our study.

(First setting) $L = \mathbb{Q}(E[m]), K = \mathbb{Q}, C = C_E(m, \alpha)$

for a fixed $\alpha \in \mathbb{Z}$, where

$$C_E(m, \alpha) := \{ M \in G_E(m) : \text{tr } M = \alpha \text{ (mod } m) \};$$

(Second setting) $L = J_{E,m}, K = \mathbb{Q}, C = \hat{C}_E(m, 0),$

where

$$\hat{C}_E(m, 0) := \left\{ \hat{M} \in \hat{G}_E(m) : \text{tr } M = 0 \text{ (mod } m) \right\},$$

with $M \in \text{GL}_2(\mathbb{Z}/m\mathbb{Z})$ denoting an arbitrary representative of a given coset $\hat{M} \in \text{PGL}_2(\mathbb{Z}/m\mathbb{Z})$.

Observe that the group isomorphism (22) gives rise to the bijection

$$C_E(m, \alpha) \rightarrow C_E(m_1, \alpha) \times C(m_2, \alpha)$$

$$M \mapsto (M_1, M_2),$$

where

$$C_E(m_1, \alpha) := \{ M_1 \in H_{E,m_1} : \text{tr } M_1 = \alpha \text{ (mod } m_1) \},$$

$$C(m_2, \alpha) := \{ M_2 \in \text{GL}_2(\mathbb{Z}/m_2\mathbb{Z}) : \text{tr } M_2 = \alpha \text{ (mod } m_2) \}.$$

Similarly, observe that the group isomorphism (25) gives rise to the bijection

$$\hat{C}_E(m, 0) \rightarrow \hat{C}_E(m_1, 0) \times \hat{C}(m_2, 0)$$

$$\hat{M} \mapsto \left( \hat{M}_1, \hat{M}_2 \right),$$

where

$$\hat{C}_E(m_1, 0) := \left\{ \hat{M}_1 \in H_{E,m_1}/\text{Scal}_{H_{E,m_1}} : \text{tr } M_1 = 0 \text{ (mod } m_1) \right\},$$

$$\hat{C}(m_2, 0) := \left\{ \hat{M}_2 \in \text{PGL}_2(\mathbb{Z}/m_2\mathbb{Z}) : \text{tr } M_2 = 0 \text{ (mod } m_2) \right\}.$$
with $M_1 \in H_{E,m_1}$ an arbitrary representative of a given coset $\overline{M}_1 \in H_{E,m_1}/\text{Scal}_{H_{E,m_1}}$, and with $M_2 \in \text{GL}_2(\mathbb{Z}/m_2 \mathbb{Z})$ an arbitrary representative of a given coset $\overline{M}_2 \in \text{PGL}_2(\mathbb{Z}/m_2 \mathbb{Z})$.

Using the above two number field settings and the above observations, in the next two propositions we obtain immediate applications of Theorems 4 - 5.

**Proposition 6.** Let $E$ be an elliptic curve defined over $\mathbb{Q}$, of conductor $N_E$, without complex multiplication, and of torsion conductor $m_E$. Let $m = m_1m_2$ be a positive integer such that $m_1 \mid m_E^\infty$ and $\gcd(m_2, m_E) = 1$.

(i) Let $\alpha \in \mathbb{Z}$. Assume that, for some $\frac{1}{4} \leq \theta < 1$, the $\theta$-quasi-GRH holds for the Dedekind zeta function of $\mathbb{Q}(E[m])$. Then

$$\# \{ p \leq x : p \nmid mN_E, a_p(E) \equiv \alpha \mod{m}\} = \frac{\#C_E(m_1, \alpha) \cdot \#C(m_2, \alpha)}{\#H_{E,m_1} \cdot \#\text{GL}_2(\mathbb{Z}/m_2 \mathbb{Z})} \pi(x) + O_E \left( \#C(m_2, \alpha) x^\theta \log(mx) \right).$$

(ii) Assume that, for some $\frac{1}{4} \leq \theta < 1$, the $\theta$-quasi-GRH holds for the Dedekind zeta function of $J_{E,m}$. Then

$$\# \{ p \leq x : p \nmid mN_E, a_p(E) \equiv 0 \mod{m}\} = \frac{\# \hat{C}_E(m_1, 0) \cdot \# \text{Scal}_{H_{E,m_1}} \cdot \# \hat{C}(m_2, 0)}{\#H_{E,m_1} \cdot \#\text{PGL}_2(\mathbb{Z}/m_2 \mathbb{Z})} \pi(x) + O_E \left( \#\hat{C}(m_2, 0) x^\theta \log(mx) \right).$$

**Proof.** Recalling (27) and that the ramified primes of $\mathbb{Q}(E[m])/\mathbb{Q}$, hence of $J_{E,m}/\mathbb{Q}$, are among the prime factors of $mN_E$, by applying (20), respectively (24), we deduce that

$$\frac{\log |d_{\mathbb{Q}(E[m])}|}{[\mathbb{Q}(E[m]): \mathbb{Q}]} \leq \sum_{p \in \mathcal{P}(\mathbb{Q}(E[m])/\mathbb{Q})} \log p + \log[\mathbb{Q}(E[m]): \mathbb{Q}] \ll \log(mN_E)$$

and

$$\frac{\log |d_{J_{E,m}}|}{[J_{E,m}: \mathbb{Q}]} \leq \sum_{p \in \mathcal{P}(J_{E,m}/\mathbb{Q})} \log p + \log[J_{E,m}: \mathbb{Q}] \ll \log(mN_E).$$

The asymptotic formulae claimed in the statement of the theorem now follow from Theorem 4 by using these estimates, along with (28) and (29). \qed

**Proposition 7.** Let $E$ be an elliptic curve defined over $\mathbb{Q}$, of conductor $N_E$, without complex multiplication, and of torsion conductor $m_E$. Let $m = m_1m_2$ be a positive integer such that $m_1 \mid m_E^\infty$ and $\gcd(m_2, m_E) = 1$. Assume that GRH holds for the Dedekind zeta function of $J_{E,m}$, and that AHC and PCC hold for the extension $J_{E,m}/\mathbb{Q}$. Then

$$\# \{ p \leq x : p \nmid mN_E, a_p(E) \equiv 0 \mod{m}\} = \frac{\# \hat{C}_E(m_1, 0) \cdot \# \text{Scal}_{H_{E,m_1}} \cdot \# \hat{C}(m_2, 0)}{\#H_{E,m_1} \cdot \#\text{PGL}_2(\mathbb{Z}/m_2 \mathbb{Z})} \pi(x) + O_E \left( \left( \#\hat{C}(m_2, 0) \right)^{\frac{1}{2}} \left( \#\text{PGL}_2(\mathbb{Z}/m_2 \mathbb{Z}) \right)^{\frac{1}{2}} x^{\frac{1}{2}} \log(mx) \right).$$
Proof. The proof is similar to that of part (ii) of Proposition 6, the only difference being that of applying Theorem \ref{thm:5} instead of Theorem \ref{thm:4}. \hfill \Box

To put these propositions to use, we need a satisfactory understanding of the size of the unions of conjugacy classes that occur in the main term and the error term of these three asymptotic formulae. We record such counts below.

Lemma 8. Let \( \ell \) be an odd prime and let \( \alpha \in \mathbb{Z} \). Then

\begin{equation}
\# \mathcal{C}(\ell, \alpha) = \begin{cases} 
\ell^3 - \ell^2 - \ell & \text{if } \alpha \not\equiv 0 \pmod{\ell}, \\
\ell^3 - \ell^2 & \text{if } \alpha \equiv 0 \pmod{\ell};
\end{cases}
\end{equation}

\begin{equation}
\# \mathcal{C}(\ell^2, \alpha) = \ell^6 - \ell^5 & \text{if } \alpha \equiv 0 \pmod{\ell};
\end{equation}

\begin{equation}
\# \hat{\mathcal{C}}(\ell, 0) = \ell^2;
\end{equation}

\begin{equation}
\# \hat{\mathcal{C}}(\ell^2, 0) = \ell^4.
\end{equation}

Proof. Let us focus on proving formula (30) for \( \# \mathcal{C}(\ell, \alpha) \) in the case \( \alpha \equiv 0 \pmod{\ell} \). First, we see easily that there are \( \ell^3 \) matrices with zero trace in \( M_2(\mathbb{Z}/\ell\mathbb{Z}) \). Next, we determine how many of these matrices have zero determinant. Any matrix \( M \in M_2(\mathbb{Z}/\ell\mathbb{Z}) \) with zero trace can be written in the form

\[ M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \]

for some cosets \( a \pmod{\ell}, b \pmod{\ell}, c \pmod{\ell} \in \mathbb{Z}/\ell\mathbb{Z} \). For any fixed pair \( (b \pmod{\ell}, c \pmod{\ell}) \in (\mathbb{Z}/\ell\mathbb{Z})^2 \), there are \( 1 + \left[ \frac{-bc}{\ell} \right] \) possible cosets \( a \pmod{\ell} \) such that \( \det M = 0 \pmod{\ell} \). Thus, the number of matrices \( M \in M_2(\mathbb{Z}/\ell\mathbb{Z}) \) with \( \text{tr} M = \det M = 0 \pmod{\ell} \) equals

\[ \sum_{b \pmod{\ell}, c \pmod{\ell} \in \mathbb{Z}/\ell\mathbb{Z}} \left( 1 + \left[ \frac{-bc}{\ell} \right] \right) = \ell^2 + \sum_{b \pmod{\ell}, c \pmod{\ell} \in (\mathbb{Z}/\ell\mathbb{Z})^2} \left( \frac{-bc}{\ell} \right) = \ell^2. \]

We deduce that

\begin{equation}
\# \{ M \in \text{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : \text{tr} M = 0 \pmod{\ell} \} = \ell^3 - \ell^2 = \ell^2 \phi(\ell),
\end{equation}

establishing formula (30) for \( \# \mathcal{C}(\ell, \alpha) \) in the case \( \alpha \equiv 0 \pmod{\ell} \).

Now, let us focus on proving formula (30) for \( \# \mathcal{C}(\ell, \alpha) \) in the case \( \alpha \not\equiv 0 \pmod{\ell} \). Note that, for any \( \alpha_1 \pmod{\ell}, \alpha_2 \pmod{\ell} \in (\mathbb{Z}/\ell\mathbb{Z})^2 \), by choosing \( \beta \in \mathbb{Z} \) such that \( \beta \equiv \alpha_2 \alpha_1^{-1} \pmod{\ell} \), the map

\[ \text{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) \]

\[ M \mapsto \beta \pmod{\ell} \cdot M \]
induces a bijection
\[
\{ M \in \text{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : \text{tr} M = \alpha_1(\text{mod } \ell) \} \rightarrow \{ M \in \text{GL}_2(\mathbb{Z}/\ell\mathbb{Z}) : \text{tr} M = \alpha_2(\text{mod } \ell) \}.
\]

This observation leads to formula (30) for \#C(\ell, 0) in the case \( \alpha \not\equiv 0 \text{ (mod } \ell) \).

Next, observe that we may write any \( M \in \text{M}_2(\mathbb{Z}/\ell^2\mathbb{Z}) \) with \( \text{tr} M \equiv 0 \text{ (mod } \ell) \) uniquely in the form
\[
M = \begin{pmatrix} \hat{a}_0 + \hat{a}_1 \ell & \hat{b}_0 + \hat{b}_1 \ell \\ \hat{c}_0 + \hat{c}_1 \ell & -\hat{a}_0 - \hat{a}_1 \ell \end{pmatrix}
\]
for some \( \hat{a}_0, \hat{a}_1, \hat{b}_0, \hat{b}_1, \hat{c}_0, \hat{c}_1 \in \mathbb{Z}/\ell\mathbb{Z} \). From here, the calculation is identical to the one for formula (30) for \#C(\ell, 0), except for taking into account that we have three free variables \( \hat{a}_1, \hat{b}_1, \hat{c}_1 \), so both the number of matrices with zero trace and the number of matrices with zero trace and zero determinant increase by a factor of \( \ell^3 \). We deduce that
\[
\# \{ M \in \text{GL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) : \text{tr} M = 0 \text{ (mod } \ell^2) \} = \ell^6 - \ell^5 = \ell^4 \phi(\ell^2).
\]

From (34) and (35), respectively, we conclude that
\[
\#\hat{C}(\ell, 0) = \#C(\ell, 0) \frac{\phi(\ell)}{\phi(\ell^2)} = \ell^2
\]
and
\[
\#\hat{C}(\ell^2, 0) = \#C(\ell^2, 0) \frac{\phi(\ell^2)}{\phi(\ell^4)} = \ell^4,
\]
which completes the proof of the lemma.

Of primary interest to us are the following applications of part (ii) of Propositions [6] - [7].

**Theorem 9.** Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \), of conductor \( N_E \), without complex multiplication, and of torsion conductor \( m_E \).

(i) Let \( d \) be a squarefree positive integer such that \( \gcd(d, m_E) = 1 \). Assume that there exists some \( \frac{1}{2} \leq \theta < 1 \) such that the \( \theta \)-quasi-GRH holds for the Dedekind zeta function of \( J_{E,dk} \) for all positive squarefree integers \( k \) with \( k \mid m_E \). Then
\[
\# \{ a_p(E) : p \leq x, p \mid N_E, \gcd(a_p(E), m_E) = 1, a_p(E) \equiv 0 \text{ (mod } d) \} = \frac{1}{d} \left( \prod_{\ell \mid d} \left( 1 - \frac{1}{\ell^2} \right)^{-1} \right) C_1(E) \pi(x) + O_E \left( d^2 x^6 \log(dx) \right),
\]
where
\[
C_1(E) := \frac{\# \{ M \in G_E(m_E) : \gcd(\text{tr} M, m_E) = 1 \}}{\#G_E(m_E)}.
\]
(ii) Let \( \ell \) be a prime such that \( \ell \nmid m_E \). Assume that there exists some \( \frac{1}{2} \leq \theta < 1 \) such that the \( \theta \)-quasi-GRH holds for the Dedekind zeta function of \( J_{E,\ell^2} \). Then

\[
\# \{ a_p(E) : p \leq x, p \nmid N_E, \gcd(a_p(E), m_E) = 1, a_p(E) \equiv 0 \pmod{\ell^2} \} = \frac{1}{\ell^2 - 1} C_1(E) \pi(x) + O_E \left( \ell^4 x^{\theta} \log(\ell x) \right),
\]

with \( C_1(E) \) defined in (36).

**Proof.** Let \( m \) be a positive integer with \( \gcd(m, m_E) = 1 \). Note that, in the notation \( m = m_1 m_2 \) of Proposition 6, we have \( m_1 = 1 \) and \( m_2 = m \). We want to estimate the cardinality of the set

\[
(37) \quad A_{E,m} := \{ a_p(E) : p \leq x, p \nmid N_E, \gcd(a_p(E), m_E) = 1, a_p(E) \equiv 0 \pmod{m} \}
\]

when \( m \) is one of the following two types: either \( m \) is an odd squarefree positive integer such that, for some \( \frac{1}{2} \leq \theta < 1 \), the \( \theta \)-quasi-GRH holds for the Dedekind zeta function of \( J_{E, m_k} \) for all positive squarefree integers \( k \) with \( k \mid m_E \); or \( m = \ell^2 \) for some odd prime \( \ell \) such that, for some \( \frac{1}{2} \leq \theta < 1 \), the \( \theta \)-quasi-GRH holds for the Dedekind zeta function of \( J_{E, \ell^2} \).

Before making these particular choices of \( m \), let us observe that

\[
(38) \quad \# A_{E,m} = \# \{ a_p(E) : p \leq x, p \nmid N_E, \gcd(a_p(E), m_E) = 1, a_p(E) \equiv 0 \pmod{m} \}
\]

\[
= \# \{ a_p(E) : p \leq x, p \nmid mN_E, \gcd(a_p(E), m_E) = 1, a_p(E) \equiv 0 \pmod{m} \} + O(\log m)
\]

\[
= \left( \sum_{p \leq x} \sum_{\substack{p \mid mN_E \atop p \nmid \gcd(a_p(E), m_E)}} \mu(k) \right) + O(\log m)
\]

\[
= \left( \sum_{k \mid m_E} \mu(k) \sum_{\substack{p \nmid \gcd(a_p(E), m_E) \atop p \leq x : p \mid mN_E}} \right) + O(\log m)
\]

\[
= \left( \sum_{k \mid m_E} \mu(k) \# \{ p \leq x : p \nmid mkN_E, a_p(E) \equiv 0 \pmod{mk} \} + O(\log k) \right) + O(\log m)
\]

\[
= \left( \sum_{k \mid m_E} \mu(k) \# \{ p \leq x : p \mid mkN_E, a_p(E) \equiv 0 \pmod{mk} \} \right) + O(\log x).
\]

To pass to the second and fourth lines above, we used that for any positive integer \( n \), \( \omega(n) \leq 2 \log n \). To pass to the fifth line above, we used that \( \gcd(m, k) = 1 \), since \( k \mid m_E \) and \( \gcd(m, m_E) = 1 \).
By invoking part (ii) of Proposition 6 under the assumption of a $\theta$-quasi-GRH for $J_{E,mk}$ for all positive squarefree integers $k$ with $k \mid m_E$, we see that \((38)\) gives

\[
\#A_{E,m} = \frac{\#\hat{C}(m,0)}{\#\text{PGL}_2(\mathbb{Z}/m\mathbb{Z})} \left( \sum_{\substack{k \geq 1 \\ k \mid m_E}} \mu(k) \frac{\#\hat{C}_E(k,0) \cdot \#\text{Scal}_{H_{E,k}}}{\#H_{E,k}} \right) \pi(x) + O_E \left( \#\hat{C}(m,0) x^\theta \log(mN_E x) \right). \tag{39}
\]

Let us analyze the summation over $k \mid m_E$ occurring in \((39)\). We observe that, for each $k \mid m_E$, we have

\[
\#\hat{C}_E(k,0) \cdot \#\text{Scal}_{H_{E,k}} = \# \{ M \in G_E(k) : \text{tr} M = 0 \pmod{k} \}.
\]

Furthermore, we observe that

\[
\# \{ M \in G_E(k) : \text{tr} M = 0 \pmod{k} \} = \frac{\# \{ M \in G_E(k) : \text{tr} M = 0 \pmod{k} \}}{\#G_E(k)} = \frac{\# \{ M \in G_E(k) : \text{tr} M = 0 \pmod{k} \}}{\#G_E(m_E)} = C_1(E).
\]

By plugging \((40)\) in \((39)\), we obtain that, under the assumption of a $\theta$-quasi-GRH for $J_{E,mk}$ for all positive squarefree integers $k$ with $k \mid m_E$, we have

\[
\#A_{E,m} = \frac{\#\hat{C}(m,0)}{\#\text{PGL}_2(\mathbb{Z}/m\mathbb{Z})} C_1(E) \pi(x) + O_E \left( \#\hat{C}(m,0) x^\theta \log(mN_E x) \right). \tag{41}
\]

Finally, we specialize \((41)\) to our two desired types of $m$.

(i) In \((41)\), we take $m = d$ for some odd squarefree positive integer $d$ coprime to $m_E$. The claimed estimate for $\#A_{E,d}$ follows by invoking the Chinese Remainder Theorem and by recalling that, from Lemma 8, for any odd prime $\ell$, we have $\#\hat{C}(\ell,0) = \ell^2$.

(ii) In \((41)\), we take $m = \ell^2$ for some odd prime $\ell \nmid m_E$. The claimed estimate for $\#A_{E,\ell^2}$ follows by recalling that, from Lemma 8, $\#\hat{C}(\ell^2,0) = \ell^4$.

\[\square\]

**Theorem 10.** Let $E$ be an elliptic curve defined over $\mathbb{Q}$, of conductor $N_E$, without complex multiplication, and of torsion conductor $m_E$. 

\[\square\]
(i) Let \( d \) be a squarefree positive integer such that \( \gcd(d, m_E) = 1 \). Assume that GRH holds for the Dedekind zeta function of \( J_{E,dk} \), and that AHC and PCC hold for the extension \( J_{E,dk}/\mathbb{Q} \), for all positive squarefree integers \( k \) with \( k \mid m_E \). Then

\[
\# \{ a_p(E) : p \leq x, p \mid N_E, \gcd(a_p(E), m_E) = 1, a_p(E) \equiv 0 \pmod{d} \} = \frac{1}{d} \left( \prod_{\ell \mid d} \left( 1 - \frac{1}{\ell^2} \right) \right) C_1(E) \pi(x)
+ O_E \left( x^{\frac{1}{2}} \log(dx) \right),
\]

where \( C_1(E) \) is defined in (36).

(ii) Let \( \ell \) be a prime such that \( \ell \nmid m_E \). Assume that GRH holds for the Dedekind zeta function of \( J_{E,\ell^2} \), and that AHC and PCC hold for the extension \( J_{E,\ell^2}/\mathbb{Q} \). Then

\[
\# \{ a_p(E) : p \leq x, p \mid N_E, \gcd(a_p(E), m_E) = 1, a_p(E) \equiv 0 \pmod{\ell^2} \} = \frac{1}{\ell^2 - 1} C_1(E) \pi(x)
+ O_E \left( x^{\frac{1}{2}} \log(\ell x) \right),
\]

with \( C_1(E) \) defined in (36).

Proof. We proceed similarly to the proof of Theorem 9, but invoke Proposition 7 instead of Proposition 6. Moreover, for any squarefree positive integer \( d \) such that \( \gcd(d, m_E) = 1 \) and for any prime \( \ell \) such that \( \ell \nmid m_E \), we appeal to the upper bounds

\[
\# \text{PGL}_2(\mathbb{Z}/d\mathbb{Z}) \leq \frac{d}{d^2} = \frac{1}{d^2}
\]

and

\[
\# \text{PGL}_2(\mathbb{Z}/\ell^2\mathbb{Z}) \leq \frac{\ell^2}{\ell^6} = \frac{1}{\ell^4}.
\]

Proof.

We end this section with an immediate application of part (i) of Theorem 4, which we will need in the proof of our two main theorems.

Lemma 11. Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \), without complex multiplication, of conductor \( N_E \), and of torsion conductor \( m_E \). Assume that there exists some \( \frac{1}{2} \leq \theta < 1 \) such that the \( \theta \)-quasi-GRH holds for the Dedekind zeta functions of the division fields of \( E \). Then, for any \( \alpha \in \mathbb{Z} \) with \( \alpha \neq 0 \),

\[
\# \{ p \leq x : p \mid N_E, a_p(E) = \alpha \} \ll_E \frac{x^{1 - \frac{1}{k}}} {\log x}.
\]

and

\[
\# \{ p \leq x : p \mid N_E, a_p(E) = 0 \} \ll_E \frac{x^{1 - \frac{1}{k}}} {\log x}.
\]
Proof. Let $\ell$ be a prime such that $\ell \nmid m_E$. Then

$$\# \{p \leq x : p \mid N_E, a_p(E) = \alpha\} \leq \# \{p \leq x : p \mid N_E, a_p(E) \equiv \alpha \pmod{\ell}\}.$$  

For the left hand side of the above inequality, we invoke part (i) of Proposition 6 together with (30) of Lemma 8, and deduce that

$$\# \{p \leq x : p \mid N_E, a_p(E) \equiv \alpha \pmod{\ell}\} \ll E \frac{x}{\ell \log x} + \ell^3 x^\theta \log(\ell x).$$  

We choose $\ell \asymp \frac{1 - \theta}{(\log x)^2}$ and obtain the desired upper bound for $\# \{p \leq x : p \mid N_E, a_p(E) = \alpha\}$.  

When $\alpha = 0$, the result can be strengthened by invoking part (ii) of Proposition 6 and (32) of Lemma 8, leading to the upper bounds

$$\# \{p \leq x : p \mid N_E, a_p(E) = 0\} \ll E \frac{x}{\ell \log x} + \ell^2 x^\theta \log(\ell x).$$  

In this case, we choose $\ell \asymp \frac{-1 + \theta}{(\log x)^2}$ and obtain the desired upper bound for $\# \{p \leq x : p \mid N_E, a_p(E) = 0\}$.  

Remark 12. Much better conditional upper bounds for $\# \{p \leq x : p \mid N_E, a_p(E) = \alpha\}$ are known for a non-zero integer $\alpha$ and the much stronger unconditional result $\# \{p \leq x : p \mid N_E, a_p(E) = 0\} \ll E x^{3/4}$ is also known. For the purpose of Theorems 1 and 3, the weaker upper bound (42) of Lemma 11 under the assumption of a $\theta$-quasi-GRH and not of the full GRH, suffices. We recorded this weaker upper bound to be able to refer to it in the upcoming sections.

3. Sieve setting

3.1. General sieve setting. Inspired by classical approaches towards the twin prime conjecture, we study the primality of the Frobenius traces associated to an elliptic curve using sieve methods. The general sieve setting that we will be using is as follows:

- $\mathcal{A}$ is a finite sequence of integers;
- $\mathcal{P}$ is a set of primes;
- for each prime $\ell \in \mathcal{P}$, $\mathcal{A}_\ell$ and $\mathcal{A}_{\ell^2}$ are defined by
  \[ \mathcal{A}_\ell := \{a \in \mathcal{A} : a \equiv 0 \pmod{\ell}\} \]
  \[ \text{and} \]
  \[ \mathcal{A}_{\ell^2} := \{a \in \mathcal{A} : a \equiv 0 \pmod{\ell^2}\}; \]
- for each positive squarefree integer $d$ composed of primes of $\mathcal{P}$, $\mathcal{A}_d$ is defined by
  \[ \mathcal{A}_d := \bigcap_{\ell \mid d} \mathcal{A}_\ell = \{a \in \mathcal{A} : a \equiv 0 \pmod{d}\}; \]
- $z, z_1, z_2 > 0$ are positive real numbers;
• $P(z)$ is the positive integer defined by

$$P(z) := \prod_{\ell \in P \atop \ell < z} \ell;$$

• $S(A, P, z)$ is the cardinality of the sifted set $A\setminus \left(\cup_{d \mid P(z)} A_d\right)$, that is,

$$S(A, P, z) := \# \{a \in A : \gcd(a, P(z)) = 1\}.$$

The general sieve problem is that of estimating the cardinality $S(A, P, z)$ under additional assumptions. In particular, for the two sieves that we will be using, the Selberg Upper Bound Sieve and the Greaves Lower Bound Sieve (see Section 4 and Section 5, respectively), we will make the assumption that there exist a real number $X > 0$ and a non-zero multiplicative function $w : \mathbb{N}\setminus\{0\} \to \mathbb{R}$ such that, for any positive squarefree integer $d$ composed of primes of $P$, we have

$$\#A_d = \frac{w(d)}{d} X + R_d \text{ for some } R_d \in \mathbb{R}. \quad (44)$$

For the Greaves Lower Bound Sieve, in addition to (44), we will make the assumption that, for any prime $\ell \in P$, we have

$$\#A_{\ell^2} = \frac{w(\ell^2)}{\ell^2} X + R_{\ell^2} \text{ for some } R_{\ell^2} \in \mathbb{R}. \quad (45)$$

Moreover, for both sieves, we will make the assumptions that there exists $\varepsilon > 0$ such that, for any prime $\ell \in P$, we have

$$0 \leq \frac{w(\ell)}{\ell} \leq 1 - \varepsilon, \quad (46)$$

and that there exist $L, A \geq 1$ such that, for any $z_1, z_2$ with $2 \leq z_1 < z_2$, we have

$$-L \leq \sum_{z_1 \leq \ell \leq z_2 \atop \ell \in P} \frac{w(\ell)}{\ell} \log \ell - \log \frac{z_2}{z_1} \leq A. \quad (47)$$

We note that, for each of the two sieves, the estimate for the cardinality of the sifted set will be formulated in terms of the function

$$W(z) := \prod_{\ell \mid P(z)} \left(1 - \frac{w(\ell)}{\ell}\right). \quad (48)$$

### 3.2. Elliptic curve sieve setting.

To prove Theorems 1 and 3, we will use the Selberg Upper Bound Sieve and the Greaves Lower Bound Sieve, respectively, in the following setting.

We fix an elliptic curve $E$ defined over $\mathbb{Q}$, without complex multiplication, of conductor $N_E$, and of torsion conductor $m_E$, and we assume that there exists some $\frac{1}{2} \leq \theta < 1$ such that the $\theta$-quasi-GRH holds for
the Dedekind zeta functions of $\mathbb{Q}(E[m])$ and $J_{E,m}$ for all positive integers $m$. We fix $x > 2$ (to be thought of as going to infinity) and set

$$A = A_E := \{a_p(E) : p \leq x, p \nmid N_E, \gcd(a_p(E), m_E) = 1\},$$

$$\mathcal{P} = \mathcal{P}_E := \{\ell : \ell \nmid m_E\}.$$

Recalling that $m_E$ is even, we see that all the primes $\ell \in \mathcal{P}$ are odd. Consequently, all the squarefree integers composed of primes of $\mathcal{P}$ are also odd.

With these definitions, we see that, for each positive squarefree $d$ with $\gcd(d, m_E) = 1$ and for each prime $\ell \nmid m_E$, we have

$$A_d = A_{E,d} = \{a_p(E) : p \leq x, p \nmid N_E, \gcd(a_p(E), m_E) = 1, a_p(E) \equiv 0 \pmod{d}\},$$

$$A_{\ell^2} = A_{E,\ell^2} = \{a_p(E) : p \leq x, p \nmid N_E, \gcd(a_p(E), m_E) = 1, a_p(E) \equiv 0 \pmod{\ell^2}\}.$$

In order to check the sieve assumptions mentioned in Subsection 3.1, it remains to identify $X$ and $w(\cdot)$, and to bound $|R_d|$ and $|R_{\ell^2}|$ from above, which is what we do next.

Noting that $\mathcal{A}_d$ and $\mathcal{A}_{\ell^2}$ are the sets introduced in the proof of Theorem 9, we deduce that

$$\#\mathcal{A}_d = \frac{1}{d} \left(\prod_{\ell \mid d} \left(1 - \frac{1}{\ell^2}\right)^{-1}\right) C_1(E)\pi(x) + O_E \left(d^2 x^\theta \log(dx)\right),$$

$$\#\mathcal{A}_{\ell^2} = \frac{1}{\ell^2 - 1} C_1(E)\pi(x) + O_E \left(\ell^4 x^\theta \log(\ell x)\right),$$

with $C_1(E)$ defined in (36). We conclude that, in our particular sieve setting, we may take

$$X := C_1(E)\pi(x)$$

and

$$w(d) := \prod_{\ell \mid d} \left(1 - \frac{1}{\ell^2}\right)^{-1},$$

in which case

$$|R_d| \ll_{E} d^2 x^\theta \log(dx)$$

and

$$|R_{\ell^2}| \ll_{E} \ell^4 x^\theta \log(\ell x).$$

Let us point out that, in (52) and (53), the exponent $\theta$ reflects the assumption of the $\theta$-quasi-GRH.

If we assume that $\theta = \frac{1}{2}$ and if, in addition, we assume that AHC and PCC hold for the Artin L-functions of the extensions $J_{E,m}/\mathbb{Q}$ for all positive integers $m$, then, using Theorem 10, we obtain that

$$|R_d| \ll_{E} x^\frac{7}{4} \log(dx)$$
and

\[(55) \quad |R_{E\ell}| \ll_E x^{\frac{3}{12}} \log(x).\]

3.3. Sieve assumptions for the elliptic curve sieve setting. We will verify that the sieve assumptions (46) and (47) mentioned previously are satisfied.

It is easy to see that the function \(w(\cdot)\) is decreasing on prime values. Hence, for any prime \(\ell\), we have \(0 \leq \frac{w(\ell)}{\ell} \leq \frac{2}{3}\). As such, the first assumption (46) is satisfied.

For the second assumption, fix \(z_1, z_2\) with \(2 \leq z_1 < z_2\). Then

\[
\sum_{z_1 \leq \ell < z_2} \frac{w(\ell)}{\ell} \log(\ell) = \sum_{z_1 \leq \ell < z_2} \left(1 - \frac{1}{\ell^2}\right) \log(\ell) = \sum_{z_1 \leq \ell < z_2} \frac{\ell}{\ell^2 - 1} \log(\ell).
\]

Using Mertens’ First Theorem,

\[(57) \quad \left| \sum_{\ell \leq y} \frac{\log(\ell)}{\ell} - \log y \right| \leq 2 \forall y > e,
\]

we see that the first sum in line (56) differs from \(\log \frac{z_2}{z_1}\) by at most 4. Extending the range of the second sum to all primes \(\ell \geq 2\) yields a series that converges to a value less than 1. As such,

\[(58) \quad \left| \sum_{z_1 \leq \ell < z_2} \frac{w(\ell)}{\ell} \log(\ell) - \log \frac{z_2}{z_1} \right| < 5,
\]

which confirms that the second assumption (47) holds.

3.4. Main term estimate for the elliptic curve sieve setting. We will now estimate the function \(W(z)\) introduced in (48). For this, we need Mertens’ Third Theorem,

\[(59) \quad \lim_{y \to \infty} (\log y) \prod_{\ell \leq y} \left(1 - \frac{1}{\ell}\right) = e^{-\gamma},
\]

where \(\gamma\) is Euler’s constant. We also need the following property of convergent products.

**Lemma 13.** Suppose a series \(\sum_{n \geq 1} a_n\) converges absolutely. Define \(F(y) := \sum_{n \geq y} |a_n|\). Then, for large enough \(y\),

\[
\prod_{n \geq y} (1 + a_n) = 1 + O(F(y)).
\]
PROOF. We start by taking the logarithm of the product. Then, provided $y$ is large enough, we are guaranteed to have $|a_n| < 1$ and we may rewrite $\log(1 + a_n)$ as a power series. We obtain

$$\log \prod_{n \geq y} (1 + a_n) = \sum_{n \geq y} \log(1 + a_n) = \sum_{n \geq y} \sum_{k \geq 1} \frac{(-a_n)^k}{k} \leq \sum_{n \geq y} \sum_{k \geq 1} |a_n|^k \ll F(y).$$

Note that $F(y) = o(1)$ since we assumed that $\sum_{n \geq 1} a_n$ converges absolutely. Then, putting everything together, we obtain that

$$\prod_{n \geq y} (1 + a_n) = e^{O(F(y))} = 1 + O(F(y)). \quad \square$$

We are now ready to analyze the function $W(z)$ defined in (48). Using (51), for any $z > m_E$ we obtain that

$$W(z) = \prod_{\ell < z} \left(1 - \ell^{-1} \left(1 - \frac{1}{\ell^2}\right)^{-1}\right)$$

$$= \left(\prod_{\ell < z} \left(1 - \frac{1}{\ell}\right)\right) \cdot \left(\prod_{\ell \geq z} \left(1 - \frac{1}{\ell^2 - \ell} + 1\right)\right)$$

$$= \frac{m_E}{\phi(m_E)} \cdot \frac{\phi(m_E)}{m_E} \cdot \left(\prod_{\ell \geq z} \left(1 + \frac{1}{\ell^2 - \ell}\right)\right)$$

To pass from the fourth line to the fifth, we used Lemma 13. To pass from the fifth line to the sixth, we used Mertens’ Third Theorem (59).

For later purposes, we record the above calculation as

$$W(z) = C_2(E) \cdot \left(\frac{e^{-\gamma}}{\log z} + o\left(\frac{1}{\log z}\right)\right),$$

where

$$C_2(E) := \frac{m_E}{\phi(m_E)} \cdot \prod_{\ell \mid m_E} \left(1 - \frac{1}{\ell^3 - \ell^2 - \ell + 1}\right).$$

4. Proof of Theorem 1 and Corollary 2

To prove Theorem 1 we will use a simplified version of the Selberg Upper Bound Sieve, as presented in [HaRi74] Thm. 8.3, p. 231].
Theorem 14. Let $\mathcal{A}$ be a finite sequence of integers and let $\mathcal{P}$ be a set of primes. Use the notation $\mathcal{A}_t$, $\mathcal{A}_d$, $P(z)$, $S(\mathcal{A}, \mathcal{P}, z)$, and $W(z)$ introduced in Subsection 3.1. Then, under assumptions (44), (46) and (47), there exists $B > 0$ such that, for any $z > 0$,

(62) \[ S(\mathcal{A}, \mathcal{P}, z) \leq XW(z) \left( e^\gamma + \frac{BL}{(\log z)^{1/14}} \right) + \sum_{d \leq z^2, d \mid P(z)} 3^{\omega(d)} |R_d|. \]

When applying the above theorem to our particular elliptic curve setting, the following estimate will be helpful.

Lemma 15. Let $a \in (-1, \infty)$ and $k \in \mathbb{N} \setminus \{0\}$. For each $y > e$, we have

(63) \[ \sum_{n \leq y} n^a k^{\omega(n)} \ll_{a, k} y^{a+1} (\log y)^{k-1}. \]

Proof. We will first prove the estimate for $a = 0$ by proceeding by induction on $k$. The base case $k = 1$ is clear. Note that, for any $k \in \mathbb{N} \setminus \{0\}$,

(64) \[ k^{\omega(n)} \leq \sum_{d_1d_2...d_k=n} 1. \]

Now, assume that (63) holds for $a = 0$ and some fixed $k$. Then we see from (64) that

\[
\sum_{n \leq y} (k+1)^{\omega(n)} \leq \sum_{n \leq y} \sum_{d_1...d_{k+1}=n} 1 \\
= \sum_{d_{k+1} \leq y} \sum_{n \leq y} \frac{n}{d_{k+1}} \sum_{d_1...d_k=\frac{n}{d_{k+1}}} 1 \\
= \sum_{d_{k+1} \leq y} \sum_{m \leq \frac{n}{d_{k+1}}} \sum_{d_1...d_k=m} 1 \\
\ll_k \sum_{d_{k+1} \leq y} \frac{y}{d_{k+1}} \left( \log \frac{y}{d_{k+1}} \right)^{k-1} \\
\ll_k y (\log y)^k.
\]

This completes the induction for $a = 0$. To prove the estimate for $a \neq 0$, we start by fixing $a > -1$ and $k \in \mathbb{N} \setminus \{0\}$. Then, using partial summation, we obtain

\[
\sum_{n \leq y} n^a k^{\omega(n)} = y^a \sum_{n \leq y} k^{\omega(n)} - a \int_1^y t^{a-1} \sum_{n \leq t} k^{\omega(n)} dt \\
\ll_a y^{a+1} (\log y)^{k-1} + \int_1^y t^a (\log t)^{k-1} dt.
\]

The integral above can be evaluated through repeated uses of integration by parts, leading to the upper bound \( \ll_{a, k} y^{a+1} (\log y)^{k-1}. \)
Remark 16. Before we begin proving Theorem 1, it is worth remarking on an interesting wrinkle that arises from the set of Frobenius traces \(a_p(E)\) being a multiset, i.e. displaying the feature that certain values of \(a_p(E)\) repeat for different values of \(p\). Theorem 14 detects each \(a \in A\) whose only prime factors are large and thus provides an upper bound for the number of large primes appearing in \(A\). In particular, it gives no information about the small primes appearing in \(A\). When \(A\) is a set (that is, has no repeated elements), this is not a problem, since we may write, for any \(z > 0\),

\[
\#\{a \in A : a \text{ prime}\} = \#\{a \in A : a \text{ prime, } |a| < z\} + \#\{a \in A : a \text{ prime, } |a| \geq z\} \leq 2z + S(A, P, z).
\]

In this case, choosing \(z\) of negligible size, the sieve on its own leads to an upper bound for the number of primes appearing in \(A\). In contrast, when \(A\) is a multiset, since we cannot bound \(\#\{a \in A : a \text{ prime, } |a| < z\}\) by \(2z\) (for \(A\) may contain some small prime infinitely many times), the sieve itself is not enough to bound the number of primes in \(A\) without additional information about \(A\). For our particular elliptic curve setting, this additional information is derived from Lemma 11.

Proof of Theorem 1. We fix an elliptic curve \(E\) defined over \(\mathbb{Q}\), without complex multiplication, of conductor \(N_E\), and of torsion conductor \(m_E\). We assume that there exists some \(\frac{1}{2} \leq \theta < 1\) such that the \(\theta\)-quasi-GRH holds for the Dedekind zeta functions of \(\mathbb{Q}(E[m])\) and \(J_{E,m}\) for all positive integers \(m\). We fix \(x > 0\), to be thought of as going to infinity. As in Subsection 3.2, we define

\[
A := \{a_p(E) : p \leq x, p \nmid N_E, \gcd(a_p(E), m_E) = 1\},
\]

\[
P := \{\ell \text{ prime : } \ell \nmid m_E\}.
\]

We recall that \(m_E\) is even, which implies that all primes \(\ell \in P\) are odd.

With these choices, we showed in (49), (50), (51), (52), and (60) that

\[
\#A_d = \frac{1}{d} \left( \prod_{\ell | d} \left( 1 - \frac{1}{\ell^2} \right)^{-1} \right) C_1(E) \pi(x) + O_{E} \left( d^2 x^\theta \log(dx) \right),
\]

\[
X = C_1(E) \pi(x),
\]

\[
w(d) = \prod_{\ell | d} \left( 1 - \frac{1}{\ell^2} \right)^{-1},
\]

\[
|R_d| \ll_{E} d^2 x^\theta \log(dx),
\]

and that, for any \(z > m_E\),

\[
W(z) = C_2(E) \cdot \left( \frac{e^{-\gamma}}{\log z} + o \left( \frac{1}{\log z} \right) \right).
\]

Furthermore, we showed that \(w(\cdot)\) satisfies assumptions (46) and (47), which are needed in order to apply Theorem 14.
Now let \( z = z(x) > m_E \) be a parameter of \( x \), to be chosen optimally later. Recalling the definition of \( S(A, \mathcal{P}, z) \) and invoking Lemma 11 and Theorem 14, we obtain that

\[
\pi_{E, \text{prime trace}}(x) = \#\{a_p(E) : p \leq x, p \nmid N_E, a_p(E) \text{ prime}\}
\]

\[
= \#\{a_p(E) : p \leq x, p \nmid N_E, a_p(E) \text{ prime}, |a_p(E)| \geq z\}
\]

\[
+ \#\{a_p(E) : p \leq x, p \nmid N_E, a_p(E) \text{ prime}, |a_p(E)| < z\}
\]

\[
\leq S(A, \mathcal{P}, z) + O_E \left( \frac{x^{1 - \frac{1}{4} \theta}}{(\log x)^{\frac{1}{2}}} \right)
\]

\[
\leq XW(z) \left( e^\gamma + \frac{5B}{(\log z)^{1/14}} \right) + \sum_{d \leq x^2, \gcd(d, m_E) = 1} 3^{\omega(d)} |R_d| + O_E \left( \frac{x^{1 - \frac{1}{4} \theta}}{(\log x)^{\frac{1}{2}}} \right)
\]

(65)

for some absolute constant \( B > 0 \).

In order for the last inequality to be meaningful, we want to ensure that its last two terms are \( o \left( \frac{x}{(\log x)^2} \right) \).

We claim that this is the case if we choose

\[
z := \frac{x^{1 - \theta}}{(\log x)^2}.
\]

Using bound (52) for \( R_d \) and Lemma 15, we obtain that

\[
\sum_{d \leq x^2, \gcd(d, m_E) = 1} 3^{\omega(d)} |R_d| \ll E \sum_{d \leq x^2} d^{2\omega(d)} x^\theta \log x \ll x^\theta x^6 \log x (\log z)^2 \ll \frac{x}{(\log x)^9}.
\]

Then, using (66), we see that

\[
\frac{x^{1 - \frac{1}{4} \theta}}{(\log x)^{\frac{1}{2}}} \ll \frac{x^{1 - \frac{1}{4} \theta}}{(\log x)^{5/2}}
\]

As such, choice (66) of \( z \) makes the last two terms in (65) be \( o \left( \frac{x}{(\log x)^2} \right) \).

It remains to examine the first term in (65). Recalling (50) and (60), we see that

\[
XW(z) e^\gamma = C_1(E)C_2(E)\pi(x) \left( \frac{1}{\log z} + o \left( \frac{1}{\log z} \right) \right) = \left( \frac{3}{1 - \theta} + o(1) \right) C(E) \frac{x}{(\log x)^2},
\]

where \( C(E) \) is as in (9).

Altogether, by using (67) in (65) and by gathering all the error terms into the little o-notation, we obtain that

\[
\pi_{E, \text{prime trace}}(x) \leq \left( \frac{3}{1 - \theta} + o(1) \right) C(E) \frac{x}{(\log x)^2}.
\]

This completes the proof of Theorem 1. \( \square \)

From Theorem 1 we can derive the convergence of the sum of the reciprocals of the primes \( p \) with the property that the Frobenius trace \( a_p(E) \) is also a prime.
Proof of Corollary 2. Fix $\varepsilon_0 > 0$. By Theorem 1 there exists $x_0 = x_0(E, \theta, \varepsilon_0)$ such that, for all $x \geq x_0$,

$$\pi_{E, \text{prime trace}}(x) \leq \left( \frac{3}{1 - \theta} + \varepsilon_0 \right) C(E) \frac{x}{(\log x)^2}.$$ 

Then, by using partial summation and the above inequality, we deduce that

$$\sum_{p \geq x_0 \atop \text{a}_p \text{ prime}} \frac{1}{p} = \frac{\pi_{E, \text{prime trace}}(t)}{t} \bigg|_{x_0}^{\infty} + \int_{x_0}^{\infty} \frac{\pi_{E, \text{prime trace}}(t)}{t^2} dt \leq \left( \frac{3}{1 - \theta} + \varepsilon_0 \right) C(E) \frac{1}{\log x_0} \pi_{E, \text{prime trace}}(x_0) \leq \left( \frac{3}{1 - \theta} + \varepsilon_0 \right) C(E) \frac{1}{\log x_0}.$$

□

5. Proof of Theorem 3

Our main tool in the proof of Theorem 3 is the weighted Greaves Lower Bound Sieve of dimension one, proven in [HaRi85] and recalled below.

In the general sieve setting of Subsection 3.1, we assume (44), (45), (46), and (47), along with

$$0 \leq w(\ell) < \ell \quad \forall \ell \in \mathcal{P},$$

(68)

We set

$$V_0 := 0.074368...$$

as in [HaRi85] (6.16), p. 205] and fix positive real numbers $U$ and $V$ such that

$$V_0 < V < U,$$

(69)

$$V \leq \frac{1}{4},$$

(70)

$$U \geq \frac{1}{2},$$

(71)

$$U + 3V \geq 1.$$ 

(72)

We fix

$$z > 0$$

and consider an arithmetic function

$$g(\cdot) : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{R}$$

having the property that, on primes $\ell \in \mathcal{P}$,

$$g(\ell) := \begin{cases} \frac{1}{U + zV} \left( \frac{\log \ell}{\log z} - V \right) & \text{if } zV \leq \ell < z^U, \\ 0 & \text{otherwise.} \end{cases}$$

(73)
Note that
\begin{equation}
0 \leq g(\ell) \leq 1 \quad \forall \ell \in \mathcal{P}
\end{equation}
and
\begin{equation}
g(\ell) = 0 \quad \forall \ell \in \mathcal{P} \text{ with } \ell < z^V.
\end{equation}
Using the notation
\begin{equation}
\{u\}^+ := \max\{u, 0\}
\end{equation}
for any given \(u \in \mathbb{R}\), we define
\[
\mathcal{G}(\cdot) : \mathbb{N}\setminus\{0\} \rightarrow \mathbb{R},
\]
\[
\mathcal{G}(n) := \left\{ 1 - \sum_{\substack{1 \leq \ell \leq n \leq \mathcal{P} \\text{ with } \ell < z^V}} (1 - g(\ell)) \right\}^+.
\]
Finally, we define the weighted function
\begin{equation}
H(A, \mathcal{P}; z^V, z^U) := \sum_{a \in A} \mathcal{G}\left(\gcd\left(a, P\left(z^U\right)\right)\right).
\end{equation}

The weighted Grieves Lower Bound Sieve provides a lower bound for \(H(A, \mathcal{P}; z^V, z^U)\). We will show in Lemma [19] that, under additional hypotheses and with particular choices of \(U\) and \(V\), the growth of \(H(A, \mathcal{P}; z^V, z^U)\), as a function of \(z\), may be used to estimate, from below, the number of almost-primes in \(A\).

To state the sieve, we require the following additional notation. For a positive integer \(r\) and a positive real number \(t\), we set
\[
h_{2r}(t) := \int \ldots \int_{3t_{2r} + \ldots + t_{1} \geq 1; \forall 1 \leq i \leq r - 1} \ldots \int_{t_{2r} < 1 - t - t_1 - \ldots - t_{2r}} \frac{1}{1 - t - t_1 - \ldots - t_{2r}} \frac{1}{t_1 \ldots t_{2r}} dt_1 \ldots dt_{2r}
\]
and
\[
h(t) := \sum_{r \geq 1} h_{2r}(t).
\]
For a positive real number \(t\) with \(t \leq \frac{1}{4}\), we set
\[
\psi(t) := \frac{1}{1 - t} - h(t).
\]
We recall from [HaKi85, p. 205]) that, for any \(t \geq V_0\),
\begin{equation}
\psi(t) \geq 0.
\end{equation}
Finally, for \(V\) satisfying (69) - (72), we set
\[
\alpha(V) := \int_{V}^{\frac{1}{27}} \psi(t) \, dt
\]
and
\[
\beta(V) := \int_{V}^{1} \psi(t) \frac{dt}{t}.
\]

The following is the weighted Gries Lower Bound Sieve [HaRi85 Thm. A, p. 206].

**Theorem 17.** Let \( \mathcal{A} \) be a finite sequence of integers and let \( \mathcal{P} \) be a set of primes. Use the notation \( \mathcal{A}_c, \mathcal{A}_d, P(z), S(\mathcal{A}, \mathcal{P}, z), \) and \( W(z) \) introduced in Subsection 3.1. Let \( U < V \) be such that (69) - (72) hold. With the above notation and under assumptions (44), (45), (46), (47), and (68), there exist sequences of real numbers \((a_m), (b_n)\) such that \(|a_m| \leq 1 \forall m, |b_n| \leq 1 \forall n, \) and such that, for any real numbers \( M, N \) satisfying

\[
M > z^U, \quad N > 1, \quad MN = z,
\]

provided that \( z \to \infty, \) we have

\[
H(\mathcal{A}, \mathcal{P}, z^V, z^U) \geq XW(z) \left( \frac{2e^\gamma}{U - V} \left( U \log \frac{1}{U} + (1 - U) \log \frac{1}{1 - U} \right) \right.
\]
\[
- \left( \log \frac{4}{3} - \alpha(V) \right) - V \log 3 - V \beta(V) + O_{A,U} \left( \log \log \log z \right) \bigg) \bigg)
\]
\[
- (\log z)^\frac{1}{2} \left| \sum_{m < M} \sum_{n < N, n \in P(z^U)} a_m b_n R_{mn} \right|.
\]

Note that the lower bound (80) may be rewritten as

\[
H(\mathcal{A}, \mathcal{P}, z^V, z^U) \geq 2e^\gamma XW(z) \left( J(U, V) + O_{A,U} \left( \log \log \log z \right) \right) \bigg) \bigg) \bigg) \bigg) \bigg) \bigg)
\]
\[
- (\log z)^\frac{1}{2} \left| \sum_{m < M} \sum_{n < N, n \in P(z^U)} a_m b_n R_{mn} \right|,
\]

where

\[
J(U, V) := \frac{1}{U - V} \left( U \log \frac{1}{U} + (1 - U) \log \frac{1}{1 - U} - \log \frac{4}{3} + \alpha(V) - V \log 3 - V \beta(V) \right).
\]

With this notation, we remark that, in order for (81) to be meaningful, we need to choose parameters \( U, V \) such that \( J(U, V) > 0. \)

**Remark 18.** We recall from [DaWu12 p. 115] that, for any \( \frac{1}{6} \leq V \leq \frac{1}{4}, \) \( J(U, V) \) may be studied using the simplified integral formulae

\[
\alpha(V) = \log \frac{4(1 - V)}{3} - \int_{\frac{1}{4}}^{\frac{1}{2}} \log(2 - uV) \log(\frac{1 - \frac{1}{V}}{1 - V}) du,
\]
\[
\beta(V) = \log \frac{1 - V}{3V} - \int_{\frac{1}{4}}^{\frac{1}{2}} \log(2 - uV) \log(\frac{1 - \frac{1}{V}}{1 - V}) du.
\]
Numerical computations using these formulae suggest that $J(U, V)$ is close to zero if $|1 - U - V| < 0.0005$. In our application, we will make choices of $U$ and $V$ which ensure both the positivity of $J(U, V)$ and a balance between the magnitudes of the terms occurring on the right-hand side of (81).

Now let us relate $H(A, P, z^V, z^U)$ to the number of almost-primes in $A$.

**Lemma 19.** Let $A$ be a finite sequence of integers and let $P$ be a set of primes. Let $U < V$ be such that (69) - (72) hold. Assume that, for each $a \in A$, if a prime $\ell$ satisfies $\ell | a$, then $\ell \in P$. Additionally, assume that

$$\exists r \in \mathbb{N}\setminus\{0\} \text{ such that } \max \{|a| : a \in A\} \leq z^{rU+V}. \tag{85}$$

Then

$$\# \{a \in A : \omega(a) \leq r\} \geq H(A, P, z^V, z^U)$$

and

$$\# \{a \in A : \Omega(a) \leq r\} \geq H(A, P, z^V, z^U) + O \left( \sum_{z^V \leq \ell < z^U} |A_{\ell}| \right).$$

**Proof.** We start by establishing the following two properties of $G(n)$:

$$0 \leq G(n) \leq 1 \forall n \tag{86}$$

and

$$G(n) = 0 \forall n \text{ such that } \gcd(n, P(z^V)) > 1. \tag{87}$$

To prove (86), let $n$ be a non-zero natural number. We will consider several cases of $n$ according to its possible prime factors. If $n$ is not divisible by any $\ell \in P$, then, from the definition of $G$, we get $G(n) = 1$. If $n$ is divisible by some $\ell \in P$ outside the range $z^V \leq \ell \leq z^U$, then, again from the definition of $G$, we get $G(n) = 0$. Now, fix $\ell \in P$ with $z^V \leq \ell < z^U$ and observe that $0 \leq \frac{1}{U-V} \left( \frac{\log \ell}{\log z} - V \right) < 1$, that is, $0 < 1 - g(\ell) \leq 1$. Then, for any $n$ such that $\ell \mid n$, we have $0 \leq 1 - \sum_{\ell \mid n} \left(1 - g(\ell)\right) < 1$, from which we get that $0 \leq G(n) < 1$.

To prove (87), let $n$ be a non-zero natural number such that $\gcd(n, P(z^V)) > 1$. Fix a prime $\ell \in P$ such that $\ell \mid n$ and $\ell < z^V$. Observe that $g(\ell) = 0$, which implies that $\sum_{\ell \mid n} \left(1 - g(\ell)\right) \geq 1$, since we have shown that each summand is non-negative. Then $G(n) = 0$, as claimed.

From (86), we obtain that

$$\sum_{a \in A} 1 \geq \sum_{a \in A} G(\gcd(a, P(z^U))) = H(A, P, z^V, z^U). \tag{88}$$
Note that, so far, we have not used assumption (85). We claim that, under assumption (85), each integer $a$ counted on the left hand sum above satisfies $\omega(a) \leq r$.

To justify this claim, we fix $a \in A$ such that $\mathcal{G} \left( \gcd \left( a, P \left( z^U \right) \right) \right) > 0$ and deduce from (87) that $\gcd \left( a, P \left( z^V \right) \right) = 1$. Then

$$0 < 1 - \sum_{\ell | a \atop \ell \leq z^U} \left( 1 - \frac{1}{U - V} \left( \frac{\log \ell}{\log z} - V \right) \right)$$

$$= 1 - \frac{1}{U - V} \sum_{\ell | a \atop \ell \leq z^U} \left( U - \frac{\log \ell}{\log z} \right)$$

$$\leq 1 - \frac{1}{U - V} \sum_{\ell | a \atop \ell \leq z^U} \left( U - \frac{\log \ell}{\log z} \right) - \frac{1}{U - V} \sum_{\ell^k | a \atop \ell \geq z^U \atop k \geq 2} \left( U - \frac{\log \ell}{\log z} \right)$$

$$\leq 1 - \frac{U}{U - V} \omega \left( a; z^U \right) + \frac{1}{U - V} \frac{\log a}{\log z},$$

where we used the notation

$$\omega(n; y) := \sum_{\ell | n} 1 + \sum_{\ell^k | n \atop \ell \geq y \atop k \geq 2} 1.$$

Following easy algebraic manipulations and invoking assumption (85), we obtain that

$$U \cdot \omega \left( a; z^U \right) < U - V + \frac{\log a}{\log z} < U - V + (rU - V) = U(r + 1),$$

which gives that $\omega (a; z^U) < r + 1$. Since $\omega(a) \leq \omega(a; z^U)$, we infer that $\omega(a) \leq r$, as desired.

We conclude that, under assumption (85),

$$\# \{ a \in A : \omega(a) \leq r \} \geq \sum_{a \in A} 1 \geq \mathcal{H} \left( A, P, z^V, z^U \right),$$

completing the first part of the lemma.

For the second part of the lemma, we start by rewriting the left hand sum of (88) as

$$\sum_{a \in A} 1 = \sum_{a \in A \atop \mathcal{G} \left( \gcd \left( a, P \left( z^U \right) \right) \right) > 0} 1 + \sum_{a \in A \atop \Omega(a) = \omega \left( a, z^U \right) \atop \mathcal{G} \left( \gcd \left( a, P \left( z^U \right) \right) \right) > 0} 1 + \sum_{a \in A \atop \Omega(a) > \omega \left( a, z^U \right) \atop \mathcal{G} \left( \gcd \left( a, P \left( z^U \right) \right) \right) > 0} 1.$$
Therefore,

\[ \sum_{a \in A} \Omega(a) \geq \omega(a, z^U) \]

for all \( a \in A \) with \( z^V \leq \ell < z^U \) and \( \ell^2 \mid a \). By combining (88), (89), and (90), we deduce that

\[ \# \{ a \in A : \Omega(a) \leq r \} \geq H(A, P, z^V, z^U) + O \left( \sum_{z^V \leq \ell < z^U} |A_{\ell^2}| \right), \]

completing the second part of the lemma. \( \square \)

We are now ready to apply Theorem 17 to the study of the almost-prime Frobenius traces associated to an elliptic curve. Our particular sieve setting is the same as that used in the proof of Theorem 1, hence the similarity between the first two paragraphs below and those in the beginning of the proof of our first main theorem.

**Proof of Theorem 3.** (i) We fix an elliptic curve \( E \) defined over \( \mathbb{Q} \), without complex multiplication, of conductor \( N_E \), and of torsion conductor \( m_E \). We assume that there exists some \( \theta < 1 \) such that the \( \theta \)-quasi-GRH holds for the Dedekind zeta functions of \( \mathbb{Q}(E[m]) \) and \( J_{E,m} \) for all positive integers \( m \). We fix \( x > 2 \), to be thought of as going to infinity. As in Subsection 3.2, we define

\[ A := \{ a_p(E) : p \leq x, \gcd(a_p(E), m_E) = 1 \}, \]

\[ P := \{ \ell \text{ prime} : \ell \mid m_E \}. \]

We recall that \( m_E \) is even, which implies that all primes \( \ell \in P \) are odd.

We showed in Subsection 3.2 that

\[ \#A_d = \frac{w(d)}{d} X + R_d, \]

\[ \#A_{\ell^2} = \frac{w(\ell^2)}{\ell^2} X + R_{\ell^2}, \]

where

\[ X = C_1(E) \pi(x), \]

\[ w(d) = \prod_{\ell \mid d} \left( 1 - \frac{1}{\ell^2} \right)^{-1}, \]

\[ |R_d| \ll_E d^2 x^\theta \log(dx), \]

\[ |R_{\ell^2}| \ll_E \ell^4 x^\theta \log(\ell x). \]

We also showed that, for any \( z > m_E \),

\[ W(z) = C_2(E) \cdot \left( \frac{e^{-\gamma}}{\log z} + o \left( \frac{1}{\log z} \right) \right). \]
Additionally, we showed that $\kappa(\cdot)$ satisfies assumptions (14), (15), (16), and (17). Observing that, for any prime $\ell \in P$, we have

$$0 < \kappa(\ell) = \left(1 - \frac{1}{\ell}\right)^{-1} = \frac{\ell^2}{\ell^2 - 1} < \ell,$$

we see that $\kappa(\cdot)$ also satisfies (68). Therefore, we may apply Theorem 17 which we do below.

We fix an arbitrary positive real number $z > m_E$, to be thought of as going to infinity, we fix $U, V$ satisfying (92) - (72), and we fix $M, N$ satisfying (79). Using (81), we obtain that

$$H(A, P, z^V, z^U) \geq C(E)\frac{\pi(x)}{\log z} (J(U, V) + o(1)) - \log z)^{\frac{1}{2}} \left| \sum_{m < M} \sum_{n < N} a_m b_n R_{mn} \right|,$$

where $C(E) = 2C_1(E)C_2(E)$ and where $J(U, V)$ is as in (82).

Remembering that $MN = z$ (from (71)) and that $|a_m| \leq 1 \forall m$, $|b_n| \leq 1 \forall n$ (from the setting of Theorem 17), we deduce that

$$\left| \sum_{m < M} \sum_{n < N} a_m b_n R_{mn} \right| \leq \sum_{d \leq z} 2^{\omega(d)} |R_d| \ll_E \sum_{d \leq z} 2^{\omega(d)} d^2 x^\theta \log(dx) \leq x^\theta z^3 (\log x)(\log z).$$

Our goal is to make choices of $z, U$, and $V$ that satisfy the following constraints: they ensure that $J(U, V) > 0$; they balance the growth of $\frac{\pi(x)}{\log z} J(U, V)$, as a function of $x$, with the growth of $\frac{x}{(\log x)^x}$; and they ensure that $x^\theta z^3 (\log x)(\log z)^{1 + \frac{1}{2}} = o \left( \frac{x}{(\log x)^x} \right)$. Further restrictions on these parameters will emerge upon ensuring that we are allowed to apply Lemma 19 and upon seeking to minimize the number of prime factors of the integer $\kappa_p(E)$.

To move towards a choice of $z$, we fix a positive real number $0 < \xi < 1$, which will be specified later, and set

$$z := \frac{x^\xi}{(\log x)^2}.$$

Then (91) becomes

$$H(A, P, \frac{x^\xi V}{(\log x)^2 V}, \frac{x^\xi U}{(\log x)^2 U}) \geq C(E)\frac{\pi(x)}{(\xi \log x - 2 \log \log x)} (J(U, V) + o(1)) \left| \sum_{m < M} \sum_{n < N} a_m b_n R_{mn} \right|,$$

while (72) gives

$$(\xi \log x - 2 \log \log x)^{\frac{1}{2}} \left| \sum_{m < M} \sum_{n < N} a_m b_n R_{mn} \right| \ll_E \frac{x^{3\xi + \theta}}{(\log x)^3} (\xi \log x - 2 \log \log x)^{\frac{1}{2}}.$$ 

Note that the function on the right hand side of the above inequality is $o \left( \frac{x}{(\log x)^x} \right)$ if

$$\xi \leq \frac{1 - \theta}{32},$$

and set

$$z := \frac{x^\xi}{(\log x)^2}.$$
To ensure hypothesis (85) of Lemma 19, we choose a positive integer $r$, to be specified later, we recall that $|a_p(E)| < 2\sqrt{p} \leq 2\sqrt{x}$, and we note that assumption (85) is satisfied if

$$2\sqrt{x} \leq \left(\frac{x^\xi}{(\log x)^2}\right)^{rU+V}.$$ 

By examining the exponents of $x$ on each side, we see that this inequality holds if $\frac{1}{2} < \xi(rU + V)$, i.e., if

$$r > \frac{1}{U} \left(\frac{1}{2\xi} - V\right).$$

From this last inequality, we see that if any two of the three parameters $U$, $V$, $\xi$ are held constant, then $r$ is minimized when the third parameter takes its largest possible value.

We choose $\xi$ as the largest possible, that is,

$$\xi := 1 - \frac{\theta}{3}.$$

Then the main term in (94) is $\frac{3J(U,V)}{1-\theta}C(E)\frac{\pi(x)}{\log x}$. Since we are seeking a main term as close as possible to $C(E)\frac{\pi(x)}{\log x}$, we want $J(U,V)$ positive and as close as possible to $1 - \frac{\theta}{3}$.

Recalling that $\frac{1}{2} \leq \theta < 1$, which is equivalent to $0 < 1 - \frac{\theta}{3} \leq \frac{1}{6}$, we choose

$$U := 0.83, \ V := \frac{1}{6}.$$ 

In this case, $J(0.83, \frac{1}{6}) = 0.00692...$ and the minimal value of $r$ that we can choose is

$$r_1 := 1 + \left[\frac{1}{0.83} \left(\frac{3}{2(1-\theta)} - \frac{1}{6}\right)\right].$$

Invoking Lemma 19, we conclude that

$$\#\{a_p(E) : p \leq x, p \nmid N_E, \gcd(a_p(E), m_E) = 1, \omega(a_p(E)) \leq r_1\} \geq \frac{3}{1-\theta}(0.00692 + o(1))C(E)\frac{\pi(x)}{\log x}.$$ 

Note that removing the gcd condition $\gcd(a_p(E), m_E) = 1$ will only make the set of primes $p$ larger. As such, we deduce that

$$\#\{a_p(E) : p \leq x, p \nmid N_E, \omega(a_p(E)) \leq r_1\} \geq \frac{3}{1-\theta}(0.00692 + o(1))C(E)\frac{\pi(x)}{\log x}.$$ 

Now we focus on proving a similar result about $\Omega(a_p(E))$. Using part (ii) of Proposition 9, we deduce that

$$\sum_{z^V \leq \ell < z^U} \sum_{\ell \in \mathcal{P}} \left(\frac{\pi(x)}{\ell^2} + \ell^4x^\theta \log x\right) \ll \frac{x^{1-\xi V}}{(\log x)^{1-2V}} + \frac{x^{5\xi U + \theta}}{(\log x)^{10U-1}}.$$ 

Since $\xi$ and $V$ are both positive, the first term on the right hand side inequality above is $o\left(\frac{x}{(\log x)^r}\right)$. In order for the second term to also be $o\left(\frac{x}{(\log x)^r}\right)$, we need

$$5\xi U + \theta \leq 1.$$
\[ U \leq \frac{1 - \theta}{5\xi}. \]

Thus, we may set
\[ \xi := \frac{1 - \theta}{3}, \]
\[ U := \frac{3}{5}. \]

In this case, the right hand side of (96) is minimized when we choose the largest possible \( V \), i.e.,
\[ V := \frac{1}{4}. \]

Then assumption (85) from Lemma 19 is satisfied for
\[ r_2 := 1 + \left[ \frac{5}{2(1 - \theta)} \right] - \frac{5}{12}. \]

Once again, with these choices, we obtain that the error term in (91) is negligible and that \( J(\frac{3}{5}, \frac{1}{4}) = 0.3162... > 0. \)

Altogether, from the second part of Lemma 19 we deduce that
\[ (101) \quad \# \{ a_p(E) : p \leq x, p \nmid N_E, \gcd(a_p(E), m_E) = 1, \Omega(a_p(E)) \leq r_2 \} \geq \frac{3}{1 - \theta} (0.3162 + o(1)) C(E) \frac{x}{(\log x)^2}. \]

Since removing the gcd condition only makes the set larger, we obtain that
\[ (102) \quad \# \{ a_p(E) : p \leq x, p \nmid N_E, \Omega(a_p(E)) \leq r_2 \} \geq \frac{3}{1 - \theta} (0.3162 + o(1)) C(E) \frac{x}{(\log x)^2}. \]

We make one final remark. One may worry that the statements (99) and (102) are misleading, since they seem to offer lower bounds for the number of primes \( p \) such that the integer \( a_p(E) \) is almost prime, while the primes \( p \) being counted would also include those for which \( a_p(E) = \pm 1 \). However, this inclusion does not impact the final result, since, by Lemma 11, \( \# \{ p \leq x : a_p(E) = \pm 1 \} \ll_E x^{1 - \frac{1}{12}} = o \left( \frac{x}{(\log x)^2} \right) \).

Thus, (99) and (102) give
\[ \# \{ a_p(E) : p \leq x, p \nmid N_E, \gcd(a_p(E), m_E) = 1, a_p(E) \neq \pm 1, \omega(a_p(E)) \leq r_1 \} \geq \frac{3}{1 - \theta} (0.00692 + o(1)) C(E) \frac{x}{(\log x)^2} \]
and
\[ \# \{ a_p(E) : p \leq x, p \nmid N_E, \gcd(a_p(E), m_E) = 1, a_p(E) \neq \pm 1, \Omega(a_p(E)) \leq r_2 \} \geq \frac{3}{1 - \theta} (0.3162 + o(1)) C(E) \frac{x}{(\log x)^2}. \]

The proof of part (i) of Theorem 3 is now complete.

(ii) We proceed as in part (i), with the exception of making the assumptions of GRH, AHC, and PCC. The effect of this change is in the estimates for \( |R_d| \) and \( |R_{\ell^2}| \), which, according to (54) and (55), are
\[ |R_d| \ll_E x^{\frac{1}{2}} \log(dx), \]
\[ |R_{\ell^2}| \ll_E x^{\frac{1}{6}} \log(\ell x). \]
In this case, the sum estimated in (92) becomes
\[ \left| \sum_{m \leq M} \sum_{n \leq N \atop m|P(z)} a_m b_n R_{mn} \right| \leq \sum_{d \leq z \atop d|P(z)} 2^{\omega(d)} |R_d| \ll_E x^{\frac{1}{2}} z (\log x) (\log z) \]
and the sum estimated in (100) becomes
\[ \sum_{z^{V} \leq \ell < z^{U}} \#A_{\ell^2} \ll_E \sum_{z^{V} \leq \ell < z^{U}} \left( \frac{\pi(x)}{\ell^2} + x^{\frac{1}{2}} \log x \right) \ll \frac{\pi(x)}{\sqrt{z}} + \frac{x^{\frac{1}{2}} z^{U} \log x}{U}. \]

We choose
\[ z := \frac{x^{\frac{1}{2}}}{(\log x)^{5}}, \]
\[ U := 0.5111286..., V := \frac{1}{4}, \]
i.e. $U$ is chosen to be the solution to $J(U, \frac{1}{4}) = \frac{1}{2}$. In this way, we deduce that
\[ H(A_0, P, z^{V}, z^{U}) \geq C(E) \frac{\pi(x)}{\log x} (1 + o(1)) + o \left( \frac{x}{(\log x)^{2}} \right). \]
The above choices of $z, U, V$ also imply that
\[ \sum_{z^{V} \leq \ell < z^{U}} \#A_{\ell^2} = o \left( \frac{x}{(\log x)^{2}} \right). \]

Moreover, hypothesis (85) of Lemma 19 is ensured if
\[ r = 2. \]

Therefore
\[ \# \{ p \leq x, p \nmid N_{E}, \gcd(a_p(E), m_E) = 1, a_p(E) \neq \pm 1, \Omega(a_p(E)) \leq 2 \} \geq (1 + o(1)) C(E) \frac{x}{(\log x)^{2}} \]
The proof of part (ii) of Theorem 3 is now complete. \qed

6. Concluding remarks and further questions

In the setting of an elliptic curve $E$ defined over $\mathbb{Q}$, of conductor $N_{E}$, without complex multiplication, and of torsion conductor $m_{E}$, from the Chebotarev density theorem we know that a positive set of primes $p$ have the property that $a_p(E)$ is even. For these primes, instead of studying the primality of $a_p(E)$, we can study the primality of $\frac{a_p(E)}{2}$. For example, it is conjectured in [CoJo22] that, similarly to [5] and [10], we have
\begin{align*}
\# \{ p \leq x : p \nmid N_{E}, |a_p(E)|/2 \text{ is prime} \} &\sim C'(E) \frac{x}{(\log x)^{2}}, \\
\end{align*}
where
\begin{align*}
C'(E) := \frac{m_{E}}{\phi(m_{E})} \cdot \frac{\# \{ M \in \text{Gal}(\mathbb{Q}(E[2m_{E}])/\mathbb{Q}) : \text{tr} M \in 2(\mathbb{Z}/2m_{E}\mathbb{Z})^{\times} \} \prod_{\ell \nmid m_{E}} \left( 1 - \frac{1}{\ell^3 - \ell^2 - \ell + 1} \right)}. \\
\end{align*}
The proofs of Theorems 1, Corollary 2, and Theorem 3 can be easily adapted to also prove results about the (almost) primality of the integers \( \frac{a_p(E)}{2} \). With more work, the methods of proofs for these results can also be adapted to obtain results about the (almost) primality of the Frobenius traces associated to an arbitrary product of non-isogenous elliptic curves defined over \( \mathbb{Q} \) and without complex multiplication, and to an arbitrary abelian variety defined over \( \mathbb{Q} \), of dimension \( g \), whose adelic Galois representation has open image in the symplectic group \( \text{GSp}_{2g}(\hat{\mathbb{Z}}) \). This more complex adaptation is relegated to a future project.

In the setting of an elliptic curve \( E \) defined over \( \mathbb{Q} \), of conductor \( N_E \), and with complex multiplication by some order \( \mathcal{O} \) in an imaginary quadratic field \( K \), all primes \( p \nmid N_E \) of ordinary reduction for \( E \) have the property that \( \mathbb{Q} \left( \sqrt{a_p(E)^2 - 4p} \right) \simeq K \). This special property of \( E \) leads to a tight connection between the study of the primality of \( a_p(E) \) (or of \( \frac{a_p(E)}{2} \)) and the study of the primes \( p \) for which there exist a prime \( \ell \) and an integer \( n \) such that \( p = Q_{E,\mathcal{O}}(\ell, n) \), where \( Q_{E,\mathcal{O}}(\ell, n) \) is a particular positive definite, primitive, integral, binary quadratic form, defined by \( E \) and by the imaginary quadratic order \( \mathcal{O} \). As explained in the upcoming notes \([CoJo22]\) and \([Jo22]\), the analogues of the asymptotic formulae (8) and (103) for an elliptic curve \( E \) with complex multiplication agree with the asymptotic formula (8) for \( Q_{E,\mathcal{O}} \).

In light of the above remark, an ambitious research project is that of adapting some of the ideas and methods of \([FoIw97]\), \([FrIw10]\ Thm. 18.6), and \([LaScXi20]\) to study conjectures (8) and (103) in the case of an elliptic curve \( E \) without complex multiplication.

We conclude by posing the following question.

**Question**

Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \), of conductor \( N_E \), and without complex multiplication. Let \( K \) be an imaginary quadratic field, of discriminant \( d_K \). Define \( t_K \) to be 1 if \( 2 \nmid d_K \) and to be 2 if \( 2 \mid d_K \). Are there infinitely many primes \( p \nmid N_E \) such that \( \mathbb{Q} \left( \sqrt{a_p(E)^2 - 4p} \right) \simeq K \) and \( \frac{|a_p(E)|}{t_K} \) is prime?

### References

- [Ch73] J.R. Chen, *On the representation of a larger even integer as the sum of a prime and the product of at most two primes*, Sci. Sinica 16, 1973, pp. 157–176.
- [Co05] A.C Cojocaru, *Reductions of an elliptic curve with almost prime orders*, Acta Arithmetica 119, No. 3, 2005, pp. 265–289.
- [CoJo22] A.C. Cojocaru and N. Jones, *Questions about elliptic curves with infinitely many prime Frobenius traces*, in preparation.
- [Co89] D.A. Cox, *Primes of the form \( x^2 + ny^2 \). Fermat, class field theory and complex multiplication*, John Wiley & Sons, Inc., New York, 1989.
- [DaJu10] C. David and J. Jiménez Urroz, *Square-free discriminants of Frobenius rings*, International Journal of Number Theory 6, 2010, pp. 1391-1412.
- [DaWu12] C. David and J. Wu, *Almost prime values of the order of elliptic curves over finite fields*, Forum Math. 24, No. 1, 2012, pp. 99–119.
- [FoIw97] É. Fouvry and H. Iwaniec, *Gaussian primes*, Acta Arithmetica 79, No. 3, 1997, pp. 249–287.
J. Friedlander and H. Iwaniec, *Opera de Cribo*, American Mathematical Society Colloquium Publications, Vol. 57, AMS, Providence RI, 2010.

S. Gun and M.R. Murty, *Divisors of Fourier coefficients of modular forms*, New York Journal of Mathematics 20, 2014, pp. 229–239.

H. Halberstam and H.-E. Richert, *Sieve methods*, London Mathematical Society Monographs, Academic Press Inc., New York, 1974.

H. Halberstam and H.-E. Richert, *A weighted sieve of Greaves’ type, II*, Elementary and Analytic Theory of Numbers, pp. 183-215, Banach Center Publication 171, 1985.

H. Iwaniec and J. Jiménez-Urroz, *Orders of CM elliptic curves modulo p with at most two primes*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 9, No. 4, 2010, pp. 815–832.

J. Jiménez-Urroz, *Almost prime orders of CM elliptic curves modulo p*, in “Algorithmic number theory,” Lecture Notes in Comput. Sci., 5011, Springer, Berlin, 2008, pp. 74–87.

N. Jones, *Almost all elliptic curves are Serre curves*, Trans. Amer. Math. Soc. 362, No. 3, 2010, pp. 1547–1570.

B.F. Jones, *Primes, elliptic curves, and binary quadratic forms*, in preparation.

N. Koblitz, *Primality of the number of points on an elliptic curve over a finite field*, Pacific J. Math. 131, 1988, pp. 157–165.

J. Lagarias and A. Odlyzko, *Effective versions of the Chebotarev density theorem*, in: A. Fröhlich (Ed.), Algebraic Number Fields, Academic Press, New York, 1977, pp. 409–464.

S. Lang and H. Trotter, *Frobenius distributions in GL₂-extensions*, Lecture Notes in Mathematics 504, Springer Verlag, Berlin - New York, 1976.

P.C-H. Lam, D. Schindler, S.Y. Xiao, *On prime values of binary quadratic forms with a thin variable*, Journal of the London Mathematical Society 102 (2), 2020, pp. 749–772.

M. Lane, *Elliptic curve analogues of the twin prime conjecture*, Bachelor’s Thesis, Princeton University, 2005, 115 pages.

S.A. Miri and V.K. Murty, *An application of sieve methods to elliptic curves*, in: Progress in Cryptology - Indocrypt 2001 (Chennai), Lecture Notes in Comput. Sci. 2247, Springer, Berlin, 2001, pp. 91–98.

M.R. Murty and V.K. Murty, *Prime divisors of Fourier coefficients of modular forms*, Duke Mathematics Journal 51, 1984, No. 1, pp. 57–76.

M.R. Murty, V.K. Murty, and P-J. Wong, *The Chebotarev density theorem and the pair correlation conjecture*, J. Ramanujan Math. Soc. 33, No. 4, 2018, pp. 399–426.

P. Pollack, *A Titchmarsh divisor problem for elliptic curves*, Math. Proc. Cambridge Philos. Soc. 160, No. 1, 2016, pp. 167–189.

P. Sarnak, *Équidistribution and primes*, Géométrie différentielle, physique mathématique, mathématiques et société II, Astérisque No. 322, 2008, pp. 225–240.

J-P. Serre, *Propriétés galoisiennes des points d’ordre fini des courbes elliptiques*, Inventiones Mathematicae 15, 1972, pp. 259–331.

J-P. Serre, *Quelques applications du théorème de densité de Chebotarev*, Publ. Math. I. H. E. S., No. 54, 1981, pp. 123–201.

J. Steuding and A. Weng, *On the number of prime divisors of the order of elliptic curves modulo p*, Acta Arith. 117, 2005, pp. 341–352.
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