WEIGHTED ESTIMATES FOR COMMUTATORS OF MULTILINEAR HAUSDORFF OPERATORS ON VARIABLE EXPONENT MORREY-HERZ TYPE SPACES

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Abstract. In this paper, we establish the boundedness of the commutators of multilinear Hausdorff operators on the product of some weighted Morrey-Herz type spaces with variable exponent with their symbols belong to both Lipschitz space and central BMO space. By these, we generalize and strengthen some previous known results.

1. Introduction

Given \( \Phi \) be a locally integrable function on \( \mathbb{R}^n \). The \( n \)-dimensional Hausdorff operator \( H_{\Phi, A} \) is defined by

\[
H_{\Phi, A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} f(A(t)x) dt, \quad x \in \mathbb{R}^n,
\]

where \( A(t) \) is an \( n \times n \) invertible matrix for almost everywhere \( t \) in the support of \( \Phi \). It is well known that if the function \( \Phi \) and the matrix \( A \) are taken appropriately, then the Hausdorff operator \( H_{\Phi, A} \) reduces to many classical operators in analysis, for example, the Hardy operator, the Cesàro operator, the Hardy-Littlewood average operator and the Riemann-Liouville fractional integral operator. Some of their results have been significantly seen in [3], [8], [9], [17], [30], [35], [36] and references therein.

In addition, it is natural to extend the study on the linear operator to multilinear operator, which is actually necessary. Thus, the authors of this paper in [7] have recently investigated the multilinear operators of Hausdorff type \( H_{\Phi, \vec{A}} \) given as follows:

\[
H_{\Phi, \vec{A}}(\vec{f})(x) = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} f_i(A_i(t)x) dt, \quad x \in \mathbb{R}^n,
\]

where \( \Phi : \mathbb{R}^n \to [0, \infty) \) and \( A_i(t) \) (for \( i = 1, \ldots, m \)) are \( n \times n \) invertible matrices for almost everywhere \( t \) in the support of \( \Phi \), and \( f_1, f_2, \ldots, f_m : \mathbb{R}^n \to \mathbb{C} \) are...
measurable functions and $\vec{f} = (f_1, \ldots, f_m)$ and $\vec{A} = (A_1, \ldots, A_m)$. It is useful to remark that the weighted multilinear Hardy operators [16] and weighted multilinear Hardy-Cesàro operators [9] are two special cases of the multilinear Hausdorff operators $H_{\Phi, \vec{A}}$.

**Definition 1.1.** Let $\Phi, \vec{A}, \vec{f}$ be as above. The Coifman-Rochberg-Weiss type commutator of multilinear Hausdorff operator is defined by

$$H_{\vec{b}, \Phi, \vec{A}}(\vec{f})(x) = \int_{\mathbb{R}^n} \Phi(t) \prod_{i=1}^{m} f_i(A_i(t)x) \prod_{i=1}^{m} (b_i(x) - b_i(A_i(t)x)) dt, \quad x \in \mathbb{R}^n, \quad (1.3)$$

where $\vec{b} = (b_1, \ldots, b_m)$ and $b_i$ are locally integrable functions on $\mathbb{R}^n$ for all $i = 1, \ldots, m$.

Moreover, if we now take $m = n \geq 2$, $\Phi(t) = |t|^m \omega(t) \chi_{[0,1]^m}(t)$ and $A_i(t) = t_i I_m$ ($I_m$ is an identity matrix), for $t = (t_1, t_2, \ldots, t_m)$, where $\omega : [0,1]^m \to [0,\infty)$ is a measurable function, then $H_{\vec{b}, \Phi, \vec{A}}$ reduces to the commutator of weighted multilinear Hardy operator due to Fu et al. [16] defined as the following

$$H_{\omega, \vec{b}, \Phi, \vec{A}}(\vec{f})(x) = \int_{[0,1]^m} \prod_{i=1}^{m} f_i(t_i x) \prod_{i=1}^{m} (b_i(x) - b_i(t_i x)) \omega(t) dt, \quad x \in \mathbb{R}^m. \quad (1.4)$$

Also, by $\Phi(t) = |t|^n \psi(t) \chi_{[0,1]^n}(t)$ and $A_i(t) = s_i(t) I_n$, where $\psi : [0,1]^n \to [0,\infty), s_1, \ldots, s_m : [0,1]^n \to \mathbb{R}$ are measurable functions, it is clear to see that $H_{\omega, \vec{b}, \Phi, \vec{A}}$ reduces to the commutator of multilinear Hardy-Cesàro operator $U_{\vec{s}, \omega, \vec{b}}$ introduced by Hung and Ky [21] as follows

$$U_{\vec{s}, \omega, \vec{b}}(\vec{f})(x) = \int_{[0,1]^n} \prod_{i=1}^{m} f_i(s_i(t)x) \prod_{i=1}^{m} (b_i(x) - b_i(s_i(t)x)) \psi(t) dt, \quad x \in \mathbb{R}^n. \quad (1.5)$$

In recent years, the theory of function spaces with variable exponents has attracted much more interest from many mathematicians (see, e.g., [2], [4], [7], [13], [18], [24], [32] and others). It is interesting to see that this theory has had some important applications to the electronic fluid mechanics, elasticity, fluid dynamics, recovery of graphics, harmonic analysis and partial differential equations (see [1], [6], [10], [11], [14], [22], [31]).

Let $b \in BMO(\mathbb{R}^n)$ and $T$ be a Calderón-Zygmund singular integral operator with rough kernels. From classical result of Coifman, Rochberg, and Weiss [13], Karlovich and Lerner [26] developed the boundedness of commutator $[b, T]$ to generalized $L^p$ spaces with variable exponent. Also, in order to generalize the result of Chanillo [12], Izuki [23] established the boundedness of the higher order commutator on Herz spaces with variable exponent.
More recently, Wu [33] considered the $m$th-order commutator for the fractional integral as follows

$$I_{\beta,b}^m(f)(x) = \int_{\mathbb{R}^n} \frac{f(y) (b(x) - b(y))^m}{|x - y|^{n-\beta}} dy,$$

where $\beta \in (0,n), b \in BMO(\mathbb{R}^n), m \in \mathbb{N}$. Then the author established the boundedness for commutators of fractional integrals on Herz-Morrey spaces with variable exponent.

Motivated by above mentioned results, the goal of this paper is to establish the boundedness for commutators of multilinear Hausdorff operators on the product of weighted Lebesgue, central Morrey, Herz, and Morrey-Herz spaces with variable exponent with their symbols belong to both Lipschitz spaces and central BMO spaces.

Our paper is organized as follows. In Section 2, we give necessary preliminaries on weighted Lebesgue spaces, central Morrey spaces, Herz spaces, Morrey-Herz spaces with variable exponent and Lipschitz spaces, central BMO spaces. In Section 3, our main theorems are given. Finally, the results of this paper are proved in Section 4.

2. Preliminaries

In this section, let us recall some basic facts and notations which will be used throughout this paper. The letter $C$ denotes a positive constant which is independent of the main parameters, but may be different from line to line. Given a measurable set $\Omega$, let us denote by $\chi_\Omega$ its characteristic function, by $|\Omega|$ its Lebesgue measure. For any $a \in \mathbb{R}^n$ and $r > 0$, we denote by $B(a, r)$ the ball centered at $a$ with radius $r$.

Next, we write $a \lesssim b$ to mean that there is a positive constant $C$, independent of the main parameters, such that $a \leq Cb$. Besides that, we denote $\chi_k = \chi_{C_k}, C_k = B_k \setminus B_{k-1}$ and $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, for all $k \in \mathbb{Z}$.

Now, we present the definition of the Lebesgue space with variable exponent. For further readings on its deep applications in harmonic analysis, the interested reader may find in the works [11], [14] and [15].

**Definition 2.1.** Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all measurable functions $p(\cdot): \mathbb{R} \to [1, \infty]$. For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of all complex-valued measurable functions $f$ defined on $\mathbb{R}^n$ such that there exists constant $\eta > 0$ satisfying

$$F_p(f/\eta) := \int_{\mathbb{R}^n \setminus \Omega_\infty} \left( \frac{|f(x)|}{\eta} \right)^{p(x)} dx + \|f/\eta\|_{L^{\infty}(\Omega_\infty)} < \infty,$$
where $\Omega_\infty = \{ x \in \mathbb{R}^n : p(x) = \infty \}$. When $|\Omega_\infty| = 0$, it is straightforward

$$F_p(f/\eta) := \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx < \infty.$$ 

The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ then becomes a norm space equipped with a norm as follows

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \eta > 0 : F_p \left( \frac{f}{\eta} \right) \leq 1 \right\}.$$ 

Let us denote by $P_b(\mathbb{R}^n)$ the class of exponents $q(\cdot) \in P(\mathbb{R}^n)$ such that

$$1 < q_- \leq q(x) \leq q_+ < \infty,$$

for all $x \in \mathbb{R}^n$, where $q_- = \text{ess inf}_{x \in \mathbb{R}^n} q(x)$ and $q_+ = \text{ess sup}_{x \in \mathbb{R}^n} q(x)$. For $p \in P_b(\mathbb{R}^n)$, it is useful to remark that we have the following inequalities which are usually used in the sequel.

- If $F_p(f) \leq C$, then $\|f\|_{L^{p(\cdot)}} \leq \max\{ C^{\frac{1}{q_-}}, C^{\frac{1}{q_+}} \}$, for all $f \in L^{p(\cdot)}$.
- If $F_p(f) \geq C$, then $\|f\|_{L^{p(\cdot)}} \geq \min\{ C^{\frac{1}{q_-}}, C^{\frac{1}{q_+}} \}$, for all $f \in L^{p(\cdot)}$.

The space $\mathcal{P}_\infty(\mathbb{R}^n)$ is defined by the set of all measurable functions $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and there exists a constant $q_\infty$ such that

$$q_\infty = \lim_{|x| \to \infty} q(x).$$

For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the weighted variable exponent Lebesgue space $L^{p(\cdot)}_\omega(\mathbb{R}^n)$ is the set of all complex-valued measurable functions $f$ such that $f \omega$ belongs the $L^{p(\cdot)}(\mathbb{R}^n)$ space and $f$ has norm

$$\|f\|_{L^{p(\cdot)}_\omega} = \|f \omega\|_{L^{p(\cdot)}}.$$ 

Let $C^\log_0(\mathbb{R}^n)$ denote the set of all log-Hölder continuous functions $\alpha(\cdot)$ satisfying at the origin

$$|\alpha(x) - \alpha(0)| \leq \frac{C^\log_0}{\log \left( e + \frac{1}{|x|} \right)}, \text{ for all } x \in \mathbb{R}^n.$$ 

Denote by $C^\log_\infty(\mathbb{R}^n)$ the set of all log-Hölder continuous functions $\alpha(\cdot)$ satisfying at infinity

$$|\alpha(x) - \alpha_\infty| \leq \frac{C^\log_\infty}{\log \left( e + |x| \right)}, \text{ for all } x \in \mathbb{R}^n.$$ 

Next, we give the definition of variable exponent weighted Herz spaces $K^{\alpha(\cdot)}_{q(\cdot),p}$ and variable exponent weighted Morrey-Herz spaces $M K^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}$ (see [29], [32] for more details).
Definition 2.2. Let $0 < p < \infty, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The variable exponent weighted Herz space $K^{\alpha(\cdot),p}_{q(\cdot),\omega}$ is defined by

$$K^{\alpha(\cdot),p}_{q(\cdot),\omega} = \left\{ f \in L^q(\mathbb{R}^n \setminus \{0\}) : \| f \|_{K^{\alpha(\cdot),p}_{q(\cdot),\omega}} < \infty \right\},$$

where $\| f \|_{K^{\alpha(\cdot),p}_{q(\cdot),\omega}} = \left( \sum_{k=-\infty}^{\infty} \| 2^{k\alpha(\cdot)} f \chi_k \|_{L^p_q}^p \right)^{\frac{1}{p}}$.

Definition 2.3. Assume that $0 \leq \lambda < \infty, 0 < p < \infty, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The variable exponent weighted Morrey-Herz space $M K^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}$ is defined by

$$M K^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega} = \left\{ f \in L^q(\mathbb{R}^n \setminus \{0\}) : \| f \|_{M K^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} < \infty \right\},$$

where $\| f \|_{M K^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{\infty} \| 2^{k\alpha(\cdot)} f \chi_k \|_{L^p_q}^p \right)^{\frac{1}{p}}$.

It is easy to see that $M K^{\alpha(\cdot),0}_{p,q(\cdot)}(\mathbb{R}^n) = K^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$. Consequently, the Herz space with variable exponent is a special case of Morrey-Herz space with variable exponent.

Let us next state the following corollary which is used in the sequel. The proof is trivial and may be found in [22].

Lemma 2.4. Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n), q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n), p \in (0, \infty)$ and $\lambda \in [0, \infty)$. If $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, then

$$\| f \chi_j \|_{L^q(\cdot)} \leq C 2^{j(\lambda - \alpha(0))} \| f \|_{M K^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}, \text{ for all } j \in \mathbb{Z}^-,$$

and

$$\| f \chi_j \|_{L^q(\cdot)} \leq C 2^{j(\lambda - \alpha(\infty))} \| f \|_{M K^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}}, \text{ for all } j \in \mathbb{N}.$$
Next, the following theorem is stated as the embedding result on Lebesgue spaces with variable exponent (see [7]).

**Theorem 2.6.** Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $q(x) \leq p(x)$ almost everywhere $x \in \mathbb{R}^n$, and

$$\frac{1}{r(\cdot)} := \frac{1}{q(\cdot)} - \frac{1}{p(\cdot)} \quad \text{and} \quad \|1\|_{L^{r(\cdot)}} < \infty.$$  

Then there exists a constant $K$ such that

$$\|f\|_{L^{q(\cdot)}_{\omega}} \leq K\|1\|_{L^{r(\cdot)}}\|f\|_{L^{p(\cdot)}_{\omega}}.$$  

Let us recall to define Lipschitz space and central BMO space (see, for example, [25], [28], [34] for more details).

**Definition 2.7.** Let $0 < \beta \leq 1$. The Lipschitz space $\text{Lip}^\beta(\mathbb{R}^n)$ is defined as the set of all functions $f : \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\|f\|_{\text{Lip}^\beta(\mathbb{R}^n)} := \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\beta}.$$  

**Definition 2.8.** The space $\text{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions $f : \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\|f\|_{\text{BMO}(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|dx < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

**Definition 2.9.** Let $1 \leq q < \infty$ and $\omega$ be a weight function. The central bounded mean oscillation space $\text{CMO}^q(\omega)$ is defined as the set of all functions $f \in L^{q}_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\text{CMO}^q(\omega)} := \sup_{R > 0} \left( \frac{1}{\omega(B(0,R))} \int_{B(0,R)} |f(x) - f_{\omega,B(0,R)}|^q \omega(x)dx \right)^{\frac{1}{q}},$$

where

$$\omega(B(0,R)) = \int_{B(0,R)} \omega(x)dx$$

and

$$f_{\omega,B(0,R)} = \frac{1}{\omega(B(0,R))} \int_{B(0,R)} f(x)\omega(x)dx.$$

Remark that, Fefferman [34] obtain the famous result that the space $\text{BMO}(\mathbb{R}^n)$ is the dual space of Hardy space $H^1(\mathbb{R}^n)$. When $\omega = 1$, we write simply $\text{CMO}^q(\mathbb{R}^n) := \text{CMO}^q(\omega)$. The space $\text{CMO}(\mathbb{R}^n)$ can be seen as a local version of $\text{BMO}(\mathbb{R}^n)$ at the origin. Moreover, $\text{BMO}(\mathbb{R}^n) \subsetneq \text{CMO}^q(\mathbb{R}^n)$, where $1 \leq q < \infty$, and the John-Nirenberg inequality is not true in $\text{CMO}^q(\mathbb{R}^n)$. 

3. Statement of the results

Before stating our main results, we introduce some notations which will be used throughout this section. Let $\gamma_1, \ldots, \gamma_m \in \mathbb{R}, \lambda_1, \ldots, \lambda_m \geq 0, p_1, \ldots, p_m, p \in (0, \infty), 0 < \beta_1, \ldots, \beta_m \leq 1$, $q_i \in \mathcal{P}_b(\mathbb{R}^n), r_i \in \mathcal{P}_\infty(\mathbb{R}^n)$ for $i = 1, \ldots, m$ and $\alpha_1, \ldots, \alpha_m \in L^\infty(\mathbb{R}^n) \cap C_0^{\log}(\mathbb{R}^n) \cap C_\infty^{\log}(\mathbb{R}^n)$. The functions $\alpha^\ast(\cdot), q(\cdot), \gamma(\cdot)$ and numbers $\beta, \lambda$ are defined as follows

$$\beta_1 + \cdots + \beta_m = \beta,$$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_m = \lambda,$$

$$\gamma_1 + \cdots + \gamma_m + \frac{\gamma_1}{r_1(\cdot)} + \cdots + \frac{\gamma_m}{r_m(\cdot)} = \gamma(\cdot),$$

$$\frac{1}{q_1(\cdot)} + \cdots + \frac{1}{q_m(\cdot)} + \frac{1}{r_1(\cdot)} + \cdots + \frac{1}{r_m(\cdot)} = \frac{1}{q(\cdot)},$$

$$\alpha_1(\cdot) + \cdots + \alpha_m(\cdot) - \beta_1 - \cdots - \beta_m - \frac{\gamma_1 + n}{r_1(\cdot)} - \cdots - \frac{\gamma_m + n}{r_m(\cdot)} = \alpha^\ast(\cdot).$$

For a matrix $A = (a_{ij})_{n \times n}$, we define the norm of $A$ as follow

$$\|A\| = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2}. \quad (3.1)$$

As above we conclude that $|Ax| \leq \|A\| |x|$ for any vector $x \in \mathbb{R}^n$. In particular, if $A$ is invertible, then we have

$$\|A\|^{-n} \leq |\det(A^{-1})| \leq \|A^{-1}\|^n. \quad (3.2)$$

Now, we are ready to state the main results in this paper.

**Theorem 3.1.** Let $\zeta > 0, \omega_1(x) = |x|^{\gamma_1}, \ldots, \omega_m(x) = |x|^{\gamma_m}, \omega(x) = |x|^{\gamma(x)}, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n), \alpha^\ast \in L^\infty(\mathbb{R}^n) \cap C_0^{\log}(\mathbb{R}^n) \cap C_\infty^{\log}(\mathbb{R}^n), b_i \in \text{Lip}^{\beta}, \Lambda_1, \ldots, \Lambda_m > 0$ and the following conditions are true:

$$q_i(A_i^{-1}(t)\cdot) \leq \zeta.q_i(\cdot)$$

and $\|1\|_{L^{\omega_i(\cdot)}} < \infty$, a.e. $t \in \text{supp}(\Phi)$, for all $i = 1, \ldots, m$.

$$\alpha_i(0) - \alpha_i\infty \geq 0, \text{ for all } i = 1, \ldots, m, \quad (3.3)$$

either

$$\gamma_1, \ldots, \gamma_m > -n, r_1(0) = r_{1+}, r_{1\infty} = r_{1-}, \ldots, r_m(0) = r_{m+}, r_{m\infty} = r_{m-}$$

or

$$\gamma_1, \ldots, \gamma_m < -n, r_1(0) = r_{1-}, r_{1\infty} = r_{1+}, \ldots, r_m(0) = r_{m-}, r_{m\infty} = r_{m+},$$

or

$$\gamma_1 = \cdots = \gamma_m = -n. \quad (3.5)$$
Then, if
\[ C_1 = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} c_{A_i,q_i,\gamma_i}(t) \| I_n - A_i(t) \|^{\beta_i} \| 1 \|_{L^{p_i}(\cdot, \cdot)} \phi_{A_i,\lambda_i}(t) \, dt < \infty, \] (3.6)
where
\[ \phi_{A_i,\lambda_i}(t) = \max \left\{ \| A_i(t) \|^{\lambda_i-\alpha_i(0)}, \| A_i(t) \|^{\lambda_i-\alpha_i(\infty)} \right\} \times \max \left\{ \sum_{r=\Theta_n^*+1}^{\Theta_n^*-1} 2^r(\lambda_i-\alpha_i(0)), \sum_{r=\Theta_n^*-1}^{\Theta_n^*-1} 2^r(\lambda_i-\alpha_i(\infty)) \right\}, \] (3.7)
with \( \Theta_n^* = \Theta_n^*(t) \) is the greatest integer number satisfying
\[ \max_{1 \leq i \leq m} \left\{ \| A_i(t) \|, \| A_i^{-1}(t) \| \right\} < 2^{-\Theta_n^*}, \quad \text{for a.e. } t \in \mathbb{R}^n, \]
\[ c_{A_i,q_i,\gamma_i}(t) = \max \left\{ \| A_i(t) \|^{-\gamma_i}, \| A_i^{-1}(t) \|^{-\gamma_i} \right\} \max \left\{ \| \det A_i^{-1}(t) \|^{\frac{1}{\gamma_i+1}}, \| \det A_i^{-1}(t) \|^{\frac{1}{\gamma_i-1}} \right\}, \]
\[ \frac{1}{\nu_i(t, \cdot)} = \frac{1}{q_i(A_i^{-1}(t))} - \frac{1}{\zeta q_i(t)}, \quad \text{for all } i = 1, \ldots, m, \]
we have \( H^b_{\Phi,A} \) is a bounded operator from \( M_{K_{\alpha_1(\cdot),p_1}^{\alpha_1(\cdot),\lambda_1} \cdots \times M_{K_{\alpha_m(\cdot),\omega_m}^{\alpha_m(\cdot),\lambda_m}} \) to \( M_{K_{\alpha(\cdot),p}^{\alpha(\cdot),\omega}} \).

**Theorem 3.2.** Suppose that we have the given supposition of Theorem 3.1. Let \( 1 \leq p_1, \ldots, p_m < \infty, \lambda_i = 0 \) and \( \alpha_i(0) = \alpha_i(\infty), \) for all \( i = 1, \ldots, m. \) At the same time, let
\[ \frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}, \] (3.8)
\[ C_2 = \int_{\mathbb{R}^n} \left( 2 - \Theta_n^* \right)^{m-1} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} c_{A_i,q_i,\gamma_i}(t) \| I_n - A_i(t) \|^{\beta_i} \| 1 \|_{L^{p_i}(\cdot, \cdot)} \phi_{A_i,0}(t) \, dt < \infty, \] (3.9)
Then, \( H^b_{\Phi,A} \) is a bounded operator from \( K_{\gamma_1(\cdot),\omega_1}^{\gamma_1(\cdot),\omega_1} \times \cdots \times K_{\gamma_m(\cdot),\omega_m}^{\gamma_m(\cdot),\omega_m} \) to \( K_{\gamma(\cdot),\omega}^{\gamma(\cdot),\omega} \).

By using the ideas in the proof of Theorem 3.1, we give the analogous result for the Lebesgue spaces with variable exponent as follows.

**Theorem 3.3.** Let \( \xi > 0, \gamma_1, \ldots, \gamma_m < 0, \omega_1(x) = |x|^{\gamma_1}, \ldots, \omega_m(x) = |x|^{\gamma_m}, \omega(x) = |x|^{\gamma(x)}, q(\cdot) \in P(b(\mathbb{R}^n)), b_1, \ldots, b_m \in \text{Lip}^{\beta_i}, \) and let the hypothesis (3.3) in Theorem 3.1 hold. Thus, if the following conditions are true:
\[ \| \cdot \|_{L^{q_i}(\cdot)}^{\beta_i+\gamma_i(x)} < \infty, \quad \text{for all } i = 1, \ldots, m, \] (3.10)
\[ C_3 = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} c_{A_i,q_i,\gamma_i}(t) \| I_n - A_i(t) \|^{\beta_i} \| 1 \|_{L^{q_i}(\cdot, \cdot)} \, dt < \infty, \] (3.11)
then we have
\[ \| H_{K,\Phi}^\gamma(\tilde{f}) \|_{L^q(\cdot)} \lesssim C^3 \mathcal{B}_{\Lip} \prod_{i=1}^m \| f_i \|_{L^q_{\omega_i}(\cdot)}. \]

Next, we consider that all of \( r_1(\cdot), ..., r_m(\cdot) \) are constant and the following conditions hold:

(H1) \( \alpha_1(\cdot) + \cdots + \alpha_m(\cdot) - \frac{\gamma_1+n}{r_1} - \cdots - \frac{\gamma_m+n}{r_m} = \alpha^{**}(\cdot), \)

(H2) \( A_i(t) = s_i(t) a_i(t) \) for all \( i = 1, ..., m \), where \( s_i : \text{supp}(\Phi) \to \mathbb{R} \) is a measurable function such that \( s_i(t) \neq 0 \) for a.e \( t \in \text{supp}(\Phi) \) and \( a_i(t) \) is an \( n \times n \) rotation matrix for a.e \( t \in \text{supp}(\Phi) \).

Then, we also obtain the following some interesting results.

**Theorem 3.4.** Let \( \zeta > 0, \lambda_1, ..., \lambda_m > 0, \gamma_1, ..., \gamma_m > -n, \omega_1(x) = |x|^{\gamma_1}, ..., \omega_m(x) = |x|^{\gamma_m}, \omega(x) = |x|^{\gamma}, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n), b_1 \in CMO^{r_1}(\omega_1), ..., b_m \in CMO^{r_m}(\omega_m), \) the hypothesis \((3.3)\) and \((3.4)\) in Theorem \(3.1\) hold. Then, if
\[
C_4 = \int_{\mathbb{R}^n} \Phi(t) \prod_{i=1}^m c_{A_i, q_i, \gamma_i}(t) \left\| 1 \right\|_{L^{q_i}(\cdot)} \phi_{A_i, \gamma_i}(t) \left( 1 + \psi_{A_i, \gamma_i}^{1/2} \right) |s_i(t)| \left| \frac{\gamma_i+n}{r_i} + \varphi_{A_i}(t) \right| dt < \infty,
\]

where
\[
\psi_{A_i, \gamma_i}(t) = |\det A_i^{-1}(t)| \max\{ \| A_i^{-1}(t) \|^{\gamma_i}, \| A_i(t) \|^{-\gamma_i} \},
\]
\[
\varphi_{A_i}(t) = \max\{ \log(4|s_i(t)|), \log \left\| s_i(t) \right\| \},
\]
we have \( H_{K,\Phi}^\gamma \) is a bounded operator from \( MK^{\alpha_1(\cdot), \lambda_1}_{p_1, \zeta q_1(\cdot), \omega_1} \times \cdots \times MK^{\alpha_m(\cdot), \lambda_m}_{p_m, \zeta q_m(\cdot), \omega_m} \) to \( MK^{\alpha^{**}(\cdot), \lambda}_{p, \zeta q(\cdot), \omega} \).

**Theorem 3.5.** Let \( 1 \leq p, p_1, ..., p_m < \infty, \lambda_i = 0, \alpha_i(0) = \alpha_i(\infty) \), for all \( i = 1, ..., m \). Also, both the assumptions of Theorem \(3.4\) and the hypothesis \((3.5)\) in Theorem \(3.2\) hold. In addition, the following condition holds:
\[
C_5 = \int_{\mathbb{R}^n} (2 - \Theta_n^*)^{m-1} \Phi(t) \prod_{i=1}^m c_{A_i, q_i, \gamma_i}(t) \left\| 1 \right\|_{L^{q_i}(\cdot)} \phi_{A_i, \gamma_i}(t) \times \left( 1 + \psi_{A_i, \gamma_i}^{1/2} \right) |s_i(t)| \left| \frac{\gamma_i+n}{r_i} + \varphi_{A_i}(t) \right| dt < \infty.
\]

Then, we have
\[
\| H_{K,\Phi}^\gamma(\tilde{f}) \|_{K^{\alpha^{**}(\cdot), p}_{q(\cdot), \omega}} \lesssim C_5 \prod_{i=1}^m \| f_i \|_{K^{\alpha_i(\cdot), p_i}_{q_i(\cdot), \omega_i}}.
\]
Let us now assume that \( q(\cdot) \) and \( q_i(\cdot) \, (i = 1, \ldots, m) \in \mathcal{P}_\infty(\mathbb{R}^n) \), \( \lambda, \alpha, \gamma, \beta_i, r_i, \lambda_i, \alpha_i, \gamma_i, \beta, \alpha \) are real numbers such that \( r_i \in (0, \infty), \lambda_i \in \left( \frac{1}{m}, 0 \right], \alpha_i, \gamma_i \in (-n, \infty), \beta_i \in (0, 1], i = 1, 2, \ldots, m \) and denote

\[
\beta_1 + \cdots + \beta_m = \beta,
\]

\[
\alpha_1 + \cdots + \alpha_m + \frac{\alpha}{r_1} + \cdots + \frac{\alpha}{r_m} = \alpha,
\]

\[
\frac{1}{q_1(\cdot)} + \cdots + \frac{1}{q_m(\cdot)} + \frac{1}{r_1} + \cdots + \frac{1}{r_m} = \frac{1}{q(\cdot)}.
\]

We are also interested in the commutators of multilinear Hausdorff operators on the product of weighted \( \lambda \)-central Morrey spaces with variable exponent. More precisely, we have the following useful result.

**Theorem 3.6.** Let \( \omega_i(x) = |x|^{\gamma_i}, v_i(x) = |x|^{\alpha_i}, b_i \in \text{Lip}^{\beta_i} \) for all \( i = 1, \ldots, m \), \( \omega(x) = |x|^{\gamma}, v(x) = |x|^{\alpha} \) and the following conditions are true:

\[
q_i(A_i^{-1}(t) \cdot) \leq q_i(\cdot) \text{ and } \|1\|_{L^{q_i(t)}(\cdot)} < \infty, \text{ a.e. } t \in \text{supp}(\Phi), \text{ for all } i = 1, \ldots, m, \tag{3.14}
\]

\[
\beta + \alpha - \frac{\gamma}{q_\infty} + \sum_{i=1}^{m} (\gamma_i + n) \lambda_i - \alpha_i + \frac{\gamma_i}{q_i(\cdot)} = (\gamma + n) \lambda, \tag{3.15}
\]

\[
C_6 = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \|A_i(t)\|^{(n+\gamma_i)(\frac{1}{q_i(\cdot)}+\lambda_i)} c_{A_i,q_i,\alpha_i}(t) \times
\]

\[
\times \|1\|_{L^{q_i(t)}(\cdot)} \|I_n - A_i(t)\|^\beta dt < +\infty, \tag{3.16}
\]

where

\[
\frac{1}{q_i(t)} = \frac{1}{q_i(A_i^{-1}(t) \cdot)} - \frac{1}{q_i(\cdot)} \text{ for all } i = 1, \ldots, m.
\]

Then, we have \( H^6_{\Phi, A, \lambda} \) is bounded from \( B^x_{\omega_1,v_1} \times \cdots \times B^x_{\omega_m,v_m} \) to \( B^x_{\omega,v} \).

**Theorem 3.7.** Given \( \omega_i(x) = |x|^{\gamma_i}, v_i(x) = |x|^{\alpha_i}, b_i \in \text{CMO}^{r_i}(\omega_i) \) for all \( i = 1, \ldots, m \), \( \omega(x) = |x|^{\gamma}, v(x) = |x|^{\alpha} \), the hypothesis \( (3.14) \) in Theorem 3.6 and the condition \( (H_2) \) hold. In addition, the following statements are true:

\[
\alpha - \frac{\gamma}{q_\infty} + \sum_{i=1}^{m} (\gamma_i + n) \lambda_i - \alpha_i + \frac{\gamma_i}{q_i(\cdot)} = (\gamma + n) \lambda, \tag{3.17}
\]

\[
C_7 = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \|A_i(t)\|^{(n+\gamma_i)(\frac{1}{q_i(\cdot)}+\lambda_i)} c_{A_i,q_i,\alpha_i}(t) \|1\|_{L^{q_i(t)}(\cdot)} \times
\]

\[
\times (1 + \psi_{A_i,\alpha_i} |s_i(t)|^{\alpha_i + r_i} + \varphi_{A_i}(t)) dt < +\infty. \tag{3.18}
\]

Then, we conclude that \( H^6_{\Phi, A, \lambda} \) is bounded from \( B^x_{\omega_1,v_1} \times \cdots \times B^x_{\omega_m,v_m} \) to \( B^x_{\omega,v} \).
4. Proofs of the theorems

Fristly, for simplicity of notation, we denote
\[
\mathcal{B}_{\text{Lip}} = \prod_{i=1}^{m} \| b_i \|_{\text{Lip}}^{\beta_i}, \quad \mathcal{B}_{\text{CMO}, \omega} = \prod_{i=1}^{m} \| b_i \|_{\text{CMO}}^{r_i(\omega_i)} \quad \text{and} \quad \mathcal{F} = \prod_{i=1}^{m} \| f_i \|_{\mathcal{CK}^a_{\gamma_i, \zeta_i}(\omega_i)}.
\]

4.1. Proofs of Theorem 3.1 and Theorem 3.2. By using the versions of the Minkowski inequality for variable Lebesgue spaces from Corollary 2.38 in [11], we have
\[
\left\| H_{\Phi, A}^{\beta}(\mathcal{F}) \chi_k \right\|_{L_{t}^{q_i}(\omega_i)} \lesssim \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} f(A_i(t) \cdot) (b_i(t) - b_i(A_i(t))) \chi_k \|_{L_{t}^{q_i}(\omega_i)} dt. \tag{4.1}
\]

On the other hand, since \( b_i \in \text{Lip}^{\beta_i} \), we get
\[
|f(A_i(t)x)(b_i(x) - b_i(A_i(t)x))\chi_k(x)| \leq |f(A_i(t)x)| \| b_i \|_{\text{Lip}^{\beta_i}} I_n - A_i(t) \| \beta_i 2^{\beta_i k} \chi_k(x).
\]

Thus, by applying the Hölder inequality for variable Lebesgue spaces (see also Corollary 2.30 in [11]), we find
\[
\left\| \prod_{i=1}^{m} f(A_i(t) \cdot) (b_i(t) - b_i(A_i(t) \cdot)) \chi_k \right\|_{L_{t}^{q_i}(\omega_i)} \leq 2^{k\beta} \mathcal{B}_{\text{Lip}} \prod_{i=1}^{m} \| I_n - A_i(t) \|^{\beta_i} \prod_{i=1}^{m} \| f_i(A_i(t) \cdot) \chi_k \|_{L_{t}^{q_i}(\omega_i)} \| \cdot \left| \gamma_i \right|^{\gamma_i} \chi_k \|_{L_{t}^{r_i}(\omega_i)}. \tag{4.2}
\]

We observe that
\[
F_{r_i}(|\cdot|^{\gamma_i} \chi_k) = \int_{\mathcal{C}_k} |x|^{\gamma_i} dx = \int_{2^{k-1}}^{2^k} \int_{\mathbb{S}^{n-1}} r^{\gamma_i + n - 1} d\sigma(x') dr \lesssim 2^{k(\gamma_i + n)}.
\]

Case 1: \( k < 0 \). Denote by
\[
\sigma_i = \begin{cases} 
\frac{1}{r_i}, & \text{if } (\gamma_i + n) > 0, \\
\frac{1}{r_i}, & \text{otherwise}.
\end{cases}
\]

Case 2: \( k \geq 0 \). Denote by
\[
\sigma_i = \begin{cases} 
\frac{1}{r_i}, & \text{if } (\gamma_i + n) > 0, \\
\frac{1}{r_i}, & \text{otherwise}.
\end{cases}
\]

From this, by (2.1), we have
\[
\left\| \cdot \left| \cdot \right|^{\gamma_i} \chi_k \right\|_{L_{t}^{r_i}(\omega_i)} \lesssim 2^{k(\gamma_i + n)\sigma_i}, \tag{4.3}
\]
Therefore, from (4.1)-(4.3), we see that
\[
\left\| H_{\Phi, A}^\beta(f) \chi_k \right\|_{L^{q_i}(\cdot)} \lesssim 2^{k(\beta + \sum_{i=1}^m (\gamma_i + n)\sigma_i)} B_{\text{Lip}} \times 
\times \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m \left\| I_i - A_i(t) \right\|^{\beta_i} \left\| f_i(A_i(t) \cdot) \chi_k \right\|_{L^{q_i}(\cdot)} dt.
\]

Let us now fix \( i \in \{1, 2, \ldots, m\} \). Since \( \| A_i(t) \| \neq 0 \), there exists an integer number \( \ell_i = \ell_i(t) \) such that \( 2^{\ell_i} < \| A_i(t) \| \leq 2^{\ell_i+1} \). By writing \( \rho_A^*(t) \) as
\[
\rho_A^*(t) = \max_{i=1, \ldots, m} \left\{ \| A_i(t) \|, \| A_i^{-1}(t) \| \right\}.
\]
Hence, by letting \( y = A_i(t)z \) with \( z \in C_k \), we arrive at
\[
|y| \geq \| A_i^{-1}(t) \|^{-1} |z| \geq \frac{2^{\ell_i+k} \rho_A^*}{\rho_A^*} > 2^{k+\ell_i+1+\Theta_A^*},
\]
and
\[
|y| \leq \| A_i(t) \| \cdot |z| \leq 2^{\ell_i+k}.
\]
These estimations can be used to imply that
\[
A_i(t) C_k \subset \{ z \in \mathbb{R}^n : 2^{k+\ell_i+1+\Theta_A^*} < |z| \leq 2^{k+\ell_i} \}.
\]

Now, we will prove the following inequality
\[
\left\| f_i(A_i(t) \cdot) \chi_k \right\|_{L^{q_i}(\cdot)} \lesssim c_{A_i, \gamma_i, \gamma_i} \| 1 \|_{L^{r_1}(\cdot)} \sum_{r=\Theta_A^*-1}^{0} \left\| f_i \chi_{k+\ell_i+r} \right\|_{L^{q_i}(\cdot)}.
\]

Indeed, for \( \eta > 0 \), by (4.5), we get
\[
\int_{\mathbb{R}^n} \left( \left| f_i(A_i(t)x) \chi_k(x) \omega_i(x) \right| \right)^{\eta(x)} dx 
\leq \int_{A_i(t) C_k} \left( \left| f_i(z) \max_{\gamma_i} \left\{ \| A_i^{-1}(t) \|^{\gamma_i}, \| A_i(t) \|^{-\gamma_i} \right\} \omega_i(z) \right) \left( \det A_i^{-1}(t) \right)^{\eta(z)} |z| dz 
\leq \int_{\mathbb{R}^n} \left( \frac{c_{A_i, \gamma_i, \gamma_i} \left( \sum_{r=\Theta_A^*-1}^{0} f_i(z) \chi_{k+\ell_i+r}(z) \omega_i(z) \right)^{\eta(A_i^{-1}(t), z)}}{\eta} \right)^{q_i(A_i^{-1}(t), z)} dz.
\]

From this, by the definition of Lebesgue space with variable exponent, we find
\[
\left\| f_i(A_i(t) \cdot) \chi_k \right\|_{L^{q_i}(\cdot)} \leq c_{A_i, \gamma_i, \gamma_i} \left( \sum_{r=\Theta_A^*-1}^{0} \left\| f_i \chi_{k+\ell_i+r} \right\|_{L^{q_i}(A_i^{-1}(t), \cdot)} \right).
\]
In view of (3.5) and Theorem 2.6 we deduce
\[ \| f \|_{L^{q_i}(A_1^{(-1)}t))} \lesssim \| 1 \|_{L^{q_i}(t))} \| f \|_{L^{q_i}(t)).} \]

This completes the proof of the inequalities (4.6). Now, combining (4.4) and (4.6), it is easy to see that
\[ \| H_{\Phi,A}(f) \chi_k \|_{L^2} \lesssim 2^{k+\sum_{i=1}^m (\gamma_i+n+1)} \mathcal{B}_{L^p} \left( \int |t|^m \prod_{i=1}^m c_{A_i, q_i, \gamma_i}(t) \right) \| 1 \|_{L^{q_i}(t))} \times \]
\[ \| I_n - A_i(t) \|_{L^2} \sum_{\beta=0}^m \sum_{\gamma=0}^{\gamma_i-1} \| f_i \chi_{k+\ell_i+r} \|_{L^{q_i}(t))} dt. \] (4.7)

Thus, by applying Lemma 2.4 in Section 2, we have
\[ \| H_{\Phi,A}(f) \chi_k \|_{L^p} \lesssim \mathcal{B}_{L^p, \Phi} \left( \int |t|^m \prod_{i=1}^m c_{A_i, q_i, \gamma_i}(t) \right) \| 1 \|_{L^{q_i}(t))} \| I_n - A_i(t) \|_{L^{q_i}(t))} \] (4.8)

Here
\[ \mathcal{U}(t) = 2^{k+\sum_{i=1}^m (\gamma_i+n+1)} \prod_{i=1}^m \left( 2^{2k+\ell_i(\lambda_i-\alpha_i(0))} \sum_{\beta=0}^m \sum_{\gamma=0}^{\gamma_i-1} 2^{r(\lambda_i-\alpha_i(0))} \right) \]
\[ + 2^{(k+\ell_i)(\lambda_i-\alpha_i(\infty))} \sum_{\beta=0}^m \sum_{\gamma=0}^{\gamma_i-1} 2^{r(\lambda_i-\alpha_i(\infty))} \).

Since \( 2^{\ell_i-1} \| A_i(t) \| \leq 2^{\ell_i} \), for all \( i = 1, \ldots, m \), it implies that
\[ 2^{\ell_i(\lambda_i-\alpha_i(0))} + 2^{\ell_i(\lambda_i-\alpha_i(\infty))} \lesssim \max \left\{ \| A_i(t) \|_{L^{q_i}(t))}, \| A_i(t) \|_{L^{q_i}(t))} \right\}. \]

From this, we can estimate \( \mathcal{U} \) as follows
\[ \mathcal{U}(t) \lesssim 2^{k+\sum_{i=1}^m (\gamma_i+n+1)} \prod_{i=1}^m \max \left\{ \| A_i(t) \|_{L^{q_i}(t))}, \| A_i(t) \|_{L^{q_i}(t))} \right\} \times \]
\[ \times \left\{ 2^{k(\lambda_i-\alpha_i(0))} \sum_{\beta=0}^m \sum_{\gamma=0}^{\gamma_i-1} 2^{r(\lambda_i-\alpha_i(0))} + 2^{k(\lambda_i-\alpha_i(\infty))} \sum_{\beta=0}^m \sum_{\gamma=0}^{\gamma_i-1} 2^{r(\lambda_i-\alpha_i(\infty))} \right\} \]
\[ \lesssim 2^{k+\sum_{i=1}^m (\gamma_i+n+1)} \prod_{i=1}^m \max \left\{ \| A_i(t) \|_{L^{q_i}(t))}, \| A_i(t) \|_{L^{q_i}(t))} \right\} \times \]
\[ \times \max \left\{ \sum_{\beta=0}^m \sum_{\gamma=0}^{\gamma_i-1} 2^{r(\lambda_i-\alpha_i(0))}, \sum_{\beta=0}^m \sum_{\gamma=0}^{\gamma_i-1} 2^{r(\lambda_i-\alpha_i(\infty))} \right\} \left\{ 2^{k(\lambda_i-\alpha_i(0))} + 2^{k(\lambda_i-\alpha_i(\infty))} \right\}. \]
This implies that
\[ U(t) \lesssim 2^{k(\beta + \sum_{i=1}^{m} (\gamma_i + n)\sigma_i)} \prod_{i=1}^{m} \left( 2^{k\lambda_i - \alpha_i(0)} + 2^{k\lambda_i - \alpha_i(\infty)} \right) \phi_{\lambda_i}(t). \]

Thus, by (4.8), it is not difficult to show that
\[ \| H^{\frac{\beta}{2}}_{\Phi, A}(\vec{f}^k) \chi_k \|_{L^p(\cdot)} \lesssim C_1 B_{\text{Lip}, F} 2^{k(\beta + \sum_{i=1}^{m} (\gamma_i + n)\sigma_i)} \prod_{i=1}^{m} \left( 2^{k\lambda_i - \alpha_i(0)} + 2^{k\lambda_i - \alpha_i(\infty)} \right). \]

Next, using Proposition 2.5 in [29], we have
\[ \| H^{\frac{\beta}{2}}_{\Phi, A}(\vec{f}) \|_{M^{n^{\star}(\cdot), \lambda}_{K_{p,q}(\cdot), \infty}} \lesssim \max \left\{ \sup_{k_0 < 0, k_0 \in \mathbb{Z}} E_1, \sup_{k_0 \geq 0, k_0 \in \mathbb{Z}} (E_2 + E_3) \right\}, \]

where
\[ E_1 = 2^{-k_0\lambda} \left( \sum_{k=\infty}^{k_0} 2^{k\alpha(0) + \sum_{i=1}^{m} (\gamma_i + n)\sigma_i} \prod_{i=1}^{m} \left( 2^{k\lambda_i - \alpha_i(0)p} + 2^{k\lambda_i - \alpha_i(\infty)p} \right) \right)^{\frac{1}{p}}, \]
\[ E_2 = 2^{-k_0\lambda} \left( \sum_{k=\infty}^{-1} 2^{k\alpha(0) + \sum_{i=1}^{m} (\gamma_i + n)\sigma_i} \prod_{i=1}^{m} \left( 2^{k\lambda_i - \alpha_i(0)p} + 2^{k\lambda_i - \alpha_i(\infty)p} \right) \right)^{\frac{1}{p}}, \]
\[ E_3 = 2^{-k_0\lambda} \left( \sum_{k=0}^{k_0} 2^{k\alpha(0) + \sum_{i=1}^{m} (\gamma_i + n)\sigma_i} \prod_{i=1}^{m} \left( 2^{k\lambda_i - \alpha_i(0)p} + 2^{k\lambda_i - \alpha_i(\infty)p} \right) \right)^{\frac{1}{p}}. \]

Now, we need to estimate the upper bounds for \( E_1, E_2 \) and \( E_3 \). Note that, using (4.9), \( E_1 \) is dominated by
\[ E_1 \lesssim C_1 B_{\text{Lip}, F} 2^{-k_0\lambda} \left( \sum_{k=\infty}^{k_0} 2^{k(\alpha(0) + \sum_{i=1}^{m} (\gamma_i + n)\sigma_i)} \prod_{i=1}^{m} \left( 2^{k\lambda_i - \alpha_i(0)p} + 2^{k\lambda_i - \alpha_i(\infty)p} \right) \right)^{\frac{1}{p}} := C_1 B_{\text{Lip}, F} T_0. \]

In view of \( \alpha_i \), we have
\[ T_0 = 2^{-k_0\lambda} \left( \sum_{k=\infty}^{k_0} 2^{k(\sum_{i=1}^{m} \alpha_i(0) + \sum_{i=1}^{m} (\gamma_i + n)\sigma_i - \lambda_i(0))} \prod_{i=1}^{m} \left( 2^{k\lambda_i - \alpha_i(0)p} + 2^{k\lambda_i - \alpha_i(\infty)p} \right) \right)^{\frac{1}{p}}. \]

Note that, by defining \( \sigma_i \) and (3.5), it is clear to see that
\[ (\gamma_i + n)(\sigma_i - \frac{1}{r_i(0)}) = 0, \text{ for all } i = 1, \ldots, m. \]
So, we get
\[
\mathcal{T}_0 = 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k} \left( \sum_{i=1}^{m} \alpha_i(0) \right)^p \prod_{i=1}^{m} \left( 2^{k(\lambda_i-\alpha_i(0))p} + 2^{k(\lambda_i-\alpha_i)\infty p} \right) \right)^{\frac{1}{p}}
\]
\[
= 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{k_0} \prod_{i=1}^{m} \left( 2^{k\lambda_i p} + 2^{k(\lambda_i-\alpha_i\infty + \alpha_i(0))p} \right) \right)^{\frac{1}{p}}
\]
\[
\lesssim \left( \prod_{i=1}^{m} 2^{-k_0\lambda_i p} \left( \sum_{k=-\infty}^{k_0} 2^{k\lambda_i p} + \sum_{k=-\infty}^{k_0} 2^{k(\lambda_i-\alpha_i\infty + \alpha_i(0))p} \right) \right)^{\frac{1}{p}}.
\]
Because of assuming that \(\lambda_i > 0\), for all \(i = 1, \ldots, m\) and (3.4), we obtain
\[
\mathcal{T}_0 \lesssim \left( \prod_{i=1}^{m} 2^{-k_0\lambda_i p} \left\{ \frac{2^{k_0\lambda_i p}}{1 - 2^{-\lambda_i p}} + \frac{2^{k_0(\lambda_i - \alpha_i\infty + \alpha_i(0)) p}}{1 - 2^{-(\lambda_i - \alpha_i\infty + \alpha_i(0)) p}} \right\} \right)^{\frac{1}{p}}
\]
\[
\lesssim \prod_{i=1}^{m} \left\{ \frac{1}{1 - 2^{-\lambda_i p}} + \frac{2^{k_0(\alpha_i(0) - \alpha_i\infty)}}{1 - 2^{-(\lambda_i - \alpha_i\infty + \alpha_i(0)) p}} \right\} \lesssim \prod_{i=1}^{m} \left( 1 + 2^{k_0(\alpha_i(0) - \alpha_i\infty)} \right).
\]
Consequently, from (4.11), we conclude
\[
E_1 \lesssim C_1 B_{\text{Lip}, F} \prod_{i=1}^{m} \left( 1 + 2^{k_0(\alpha_i(0) - \alpha_i\infty)} \right). \quad (4.13)
\]
A similar argument as \(E_1\), we also get
\[
E_2 \lesssim C_1 B_{\text{Lip}, F} 2^{-k_0\lambda}. \quad (4.14)
\]
For \(i = 1, \ldots, m\), we define
\[
L_i = \begin{cases} 
2^{k_0(\alpha_i\infty - \alpha_i(0))} + \left| 2^{\lambda_i p} - 1 \right|^{\frac{1}{p}} + 2^{-k_0\lambda_i}, & \text{if } \lambda_i + \alpha_i\infty - \alpha_i(0) \neq 0, \\
2^{-k_0\lambda_i} (k_0 + 1)^{\frac{1}{p}} + \left| 2^{\lambda_i p} - 1 \right|^{\frac{1}{p}}, & \text{otherwise}.
\end{cases}
\]
Thus, we see that
\[
E_3 \lesssim C_1 B_{\text{Lip}, F} \mathcal{T}_\infty, \quad (4.15)
\]
where
\[
\mathcal{T}_\infty = 2^{-k_0\lambda} \left( \sum_{k=0}^{k_0} 2^{k} \left( \sum_{i=1}^{m} (\gamma_i + n) (\sigma_i - \frac{1}{r_i\infty}) \right)^p \prod_{i=1}^{m} \left( 2^{k(\lambda_i-\alpha_i(0))p} + 2^{k(\lambda_i-\alpha_i\infty)p} \right) \right)^{\frac{1}{p}}.
\]
Remark that, by defining \(\sigma_i\) and (3.5), we deduce
\[
(\gamma_i + n) (\sigma_i - \frac{1}{r_i\infty}) = 0, \text{ for all } i = 1, \ldots, m. \quad (4.16)
\]
Thus, by estimating in the same way as \(\mathcal{T}_0\), we also have
\[
\mathcal{T}_\infty \lesssim \prod_{i=1}^{m} 2^{-k_0\lambda_i} \left( \sum_{k=0}^{k_0} 2^{k\lambda_i p} + \sum_{k=0}^{k_0} 2^{k(\lambda_i + \alpha_i\infty - \alpha_i(0))p} \right)^{\frac{1}{p}} := \prod_{i=1}^{m} \mathcal{T}_{i, \infty}.
\]
In the case \( \lambda_i + \alpha_{i\infty} - \alpha_i(0) = 0 \), \( T_{i,\infty} \) is dominated by
\[
T_{i,\infty} \leq 2^{-k_0\lambda_i} \left( \frac{2^{k_0\lambda_i p} - 1}{2^{\lambda_i p} - 1} + (k_0 + 1) \right)^{\frac{1}{p}} \lesssim 2^{-k_0\lambda_i} (k_0 + 1)^{\frac{1}{p}} + |2^{\lambda_i p} - 1|^{\frac{1}{p}}.
\]
Otherwise, we get
\[
T_{i,\infty} \leq 2^{-k_0\lambda_i} \left( \frac{2^{k_0\lambda_i p} - 1}{2^{\lambda_i p} - 1} + \frac{2^{k_0(\lambda_i + \alpha_{i\infty} - \alpha_i(0)) p} - 1}{2^{\lambda_i p} - 1} \right)^{\frac{1}{p}} \lesssim 2^{k_0(\alpha_{i\infty} - \alpha_i(0))} + |2^{\lambda_i p} - 1|^{-1/p} + 2^{-k_0\lambda_i},
\]
which implies \( T_{\infty} \lesssim \prod_{i=1}^m L_i \). From this, by (4.15), we obtain
\[
E_3 \lesssim C_1 B_{\text{Lip}} F \prod_{i=1}^m L_i. \tag{4.17}
\]

By (4.10), (4.13), (4.14) and (4.17), the proof of Theorem 3.1 is finished.

Next, let us give the proof for Theorem 3.2. From Proposition 3.8 in [2], it is easy to see that
\[
\| H_{\Phi,A}^\beta (\vec{f}) \|_{K^{\alpha(\cdot),p}_{\Phi(\cdot),\infty}} \lesssim \left( \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p} \| H_{\Phi,A}^\beta (\vec{f}) \chi_k \|_{L^p(\cdot)}^p \right)^{\frac{1}{p}} + \left( \sum_{k=0}^\infty 2^{k\alpha(0)p} \| H_{\Phi,A}^\beta (\vec{f}) \chi_k \|_{L^p(\cdot)}^p \right)^{\frac{1}{p}} =: \mathcal{H}_0 + \mathcal{H}_1. \tag{4.18}
\]

Next, we need to estimate the upper bound of \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \). In view of (4.7) and (4.12), by using the Minkowski inequality, we find
\[
\mathcal{H}_0 \lesssim B_{\text{Lip}} \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m C_{A_i, q_i, a_i}(t) \| I_n - A_i(t) \|_{L^p(\cdot)}^{\delta_i} \times \left\{ \left( \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p} \prod_{i=1}^m \left( \sum_{r=\Theta_i - 1}^{0} \| f_i \chi_{k+\ell_i + r} \|_{L^{\zeta'](\cdot)}^{p_i} \right) \right)^{\frac{1}{p}} dt. \tag{4.19}
\]

Using (3.8) and the Hölder inequality, it follows that
\[
\left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p} \prod_{i=1}^m \left( \sum_{r=\Theta_i - 1}^{0} \| f_i \chi_{k+\ell_i + r} \|_{L^{\zeta'(\cdot)}^{p_i}} \right)^{p_i} \right\}^{\frac{1}{p}} \leq \prod_{i=1}^m \left\{ \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_i} \left( \sum_{r=\Theta_i - 1}^{0} \| f_i \chi_{k+\ell_i + r} \|_{L^{\zeta'(\cdot)}^{p_i}} \right)^{p_i} \right\}^{-\frac{1}{p_i}}. \tag{4.20}
\]
By $p_i \geq 1$, for all $i = 1, ..., m$, we have
\[
\left( \sum_{r=\Theta_n^*}^{0} \| f_i \chi_{k+r, \ell_i} \|_{L_{z_i}^{q_i}(\cdot)} \right)^{p_i} \leq (2 - \Theta_n^*)^{p_i-1} \sum_{r=\Theta_n^*}^{0} \| f_i \chi_{k+r, \ell_i} \|_{L_{z_i}^{q_i}(\cdot)}^{p_i}.
\]

Thus, combining (4.19) and (4.20), we deduce
\[
\mathcal{H}_0 \lesssim B_{\text{Lip}} \int_{\mathbb{R}^n} (2 - \Theta_n^*)^{m-\frac{1}{p}} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \epsilon_{A_i, q_i, \gamma_i(t)} \|1\|_{L^{q_i}(\cdot)} \| I_n - A_i(t) \|^{\beta_i} \mathcal{H}_{0,i} dt.
\]

(4.21)

Here $\mathcal{H}_{0,i} = \sum_{r=\Theta_n^*}^{0} \left( \sum_{k=-\infty}^{-1} 2^{k\alpha_i(0)} \| f_i \chi_{k+r, \ell_i} \|_{L_{z_i}^{q_i}(\cdot)}^{p_i} \right)^{\frac{1}{p_i}}$ for all $i = 1, 2, ..., m$. Hence, we estimate
\[
\mathcal{H}_{0,i} \lesssim \sum_{r=\Theta_n^*}^{0} 2^{-(\ell_i+r)\alpha_i(0)} \left( \sum_{t=-\infty}^{\infty} 2^{t\alpha_i(0)} \| f_i \chi_t \|_{L_{z_i}^{q_i}(\cdot)}^{p_i} \right)^{\frac{1}{p_i}}.
\]

By $\alpha_i(0) = \alpha_i(\infty)$ and Proposition 3.8 in [2], we get
\[
\mathcal{H}_{0,i} \lesssim \sum_{r=\Theta_n^*}^{0} 2^{-(\ell_i+r)\alpha_i(0)} \| f_i \|_{K_{\zeta_i(\cdot), \omega_i}^{\alpha_i(\cdot), p_i}} = 2^{-\ell_i\alpha_i(0)} \sum_{r=\Theta_n^*}^{0} 2^{-r\alpha_i(0)} \| f_i \|_{K_{\zeta_i(\cdot), \omega_i}^{\alpha_i(\cdot), p_i}}.
\]

(4.22)

Since $2^{\ell_i-1} < \| A_i(t) \| \leq 2^{\ell_i}$, we deduce that $2^{-\ell_i\alpha_i(0)} \lesssim \| A_i(t) \|^{-\alpha_i(0)}$. Hence, by (4.22), we have
\[
\mathcal{H}_{0,i} \lesssim \phi_{A_i, 0}(t) \cdot \| f_i \|_{K_{\zeta_i(\cdot), \omega_i}^{\alpha_i(\cdot), p_i}}.
\]

As above, by (4.21), we make
\[
\mathcal{H}_0 \lesssim C_2 B_{\text{Lip}} \prod_{i=1}^{m} \| f_i \|_{K_{\zeta_i(\cdot), \omega_i}^{\alpha_i(\cdot), p_i}}.
\]

By estimating as $\mathcal{H}_0$, we also make
\[
\mathcal{H}_1 \lesssim C_2 B_{\text{Lip}} \prod_{i=1}^{m} \| f_i \|_{K_{\zeta_i(\cdot), \omega_i}^{\alpha_i(\cdot), p_i}}.
\]

There, by (4.18), we finishes desired conclusion.
4.2. **Proofs of Theorem 3.4 and Theorem 3.5.** Applying the Minkowski inequality and the Hölder inequality for variable Lebesgue spaces, we get

\[
\| H_{\Phi,A}^b(f)\chi_k \|_{L^{r_i}(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^m} \prod_{i=1}^m \left( \| b_i(\cdot) - b_i(A_i(t)\cdot) \|_{L^{r_i}(\omega_i,B_k)} \right) \left\| f_i(A_i(t)\cdot) \chi_k \right\|_{L^{q_i}(\mathbb{R}^n)} dt.
\]

By (4.6), we deduce

\[
\| H_{\Phi,A}^b(f)\chi_k \|_{L^{r_i}(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^m} \prod_{i=1}^m c_{A_i,q_i,\gamma_i}(t) \left\| b_i(\cdot) - b_i(A_i(t)\cdot) \right\|_{L^{r_i}(\omega_i,B_k)} \times \left\| f_i \chi_k + \ell_i + r \right\|_{L^{q_i}(\mathbb{R}^n)} dt.
\]

(4.23)

On the other hand, we need to prove that

\[
\| b_i(\cdot) - b_i(A_i(t)\cdot) \|_{L^{r_i}(\omega_i,B_k)} \leq 2 \frac{k(\gamma_i + n)}{r_i} \left( 1 + \psi A_i,\gamma_i |s_i(t)| \right)^{\frac{\gamma_i + n}{r_i}} + \phi A_i(t) \| b_i \|_{CMO^{r_i}(\omega_i)}.
\]

(4.24)

In fact, we put \( a_{1,i}(\cdot) = b_i(\cdot) - b_{i,\omega_i,B_k}, a_{2,i}(\cdot) = b_i(A_i(t)\cdot) - b_{i,\omega_i,A_i(t)B_k} \) and \( a_{3,i}(\cdot) = b_{i,\omega_i,B_k} - b_{i,\omega_i,A_i(t)B_k} \). Here

\[
b_{i,\omega_i,U} = \frac{1}{\omega_i(U)} \int_U b_i(x) \omega_i(x) dx.
\]

Then, we have

\[
\| b_i(\cdot) - b_i(A_i(t)\cdot) \|_{L^{r_i}(\omega_i,B_k)} \leq \| a_{1,i} \|_{L^{r_i}(\omega_i,B_k)} + \| a_{2,i} \|_{L^{r_i}(\omega_i,B_k)} + \| a_{3,i} \|_{L^{r_i}(\omega_i,B_k)}.
\]

(4.25)

From defining the space \( CMO^{r_i}(\omega_i) \), we immediately have

\[
\| a_{1,i} \|_{L^{r_i}(\omega_i,B_k)} \leq (\omega_i(B_k))^{\frac{1}{r_i}} \| b_i \|_{CMO^{r_i}(\omega_i)} \leq 2 \frac{k(\gamma_i + n)}{r_i} \| b_i \|_{CMO^{r_i}(\omega_i)}.
\]

(4.26)

By making the formula for change of variables, we obtain

\[
\| a_{2,i} \|_{L^{r_i}(\omega_i,B_k)} \leq \int_{B_k} |b_i(A_i(t)x) - b_{i,\omega_i,A_i(t)B_k} |^r \omega_i(x) dx \leq \psi A_i,\gamma_i(t). \int_{A_i(t)B_k} |b_i(z) - b_{i,\omega_i,A_i(t)B_k} |^r \omega_i(z) dx.
\]

Because of assuming \( A_i(t) = s_i(t), a_i(t) \), we deduce

\[
\| a_{2,i} \|_{L^{r_i}(\omega_i,B_k)} \leq \psi A_i,\gamma_i s_i(t) \| b_i \|_{CMO^{r_i}(\omega_i)}.
\]

(4.27)

Next, we observe that

\[
\| a_{3,i} \|_{L^{r_i}(\omega_i,B_k)} \leq (\omega_i(B_k))^{\frac{1}{r_i}} |b_{i,\omega_i,B_k} - b_{i,\omega_i,A_i(t)B_k}|.
\]

(4.28)
By having \( s_i(t) \neq 0 \), there exists an integer number \( \theta_i = \theta_i(t) \) satisfying \( 2^{\theta_i-1} < |s_i(t)| \leq 2^{\theta_i} \). Thus, we define

\[
\sigma(\theta_i) = \begin{cases} 
\theta_i - 1, & \text{if } \theta_i \geq 1, \\
\theta_i, & \text{otherwise}, 
\end{cases}
\]

and

\[
S(\theta_i) = \begin{cases} 
\{ j \in \mathbb{Z} : 1 \leq j \leq \theta_i - 1 \}, & \text{if } \theta_i \geq 1, \\
\{ j \in \mathbb{Z} : \theta_i + 1 \leq j \leq 0 \}, & \text{otherwise}.
\end{cases}
\]

At this point, we give the estimation as below

\[
\left| b_{i, \omega, 2^{\theta_i-1}B_k} - b_{i, \omega, 2^{\theta_i}B_k} \right| \leq \sum_{j \in S(\theta_i)} \left| b_{i, \omega, 2^{\theta_i-1}B_k} - b_{i, \omega, 2^{\theta_i}B_k} \right| + \left| b_{i, \omega, 2^{\theta_i}(\theta_i)B_k} - b_{i, \omega, A_i(t)B_k} \right|. 
\]

(4.29)

When \( S(\theta_i) \) is empty set, we should understand that

\[
\sum_{j \in S(\theta_i)} \left| b_{i, \omega, 2^{\theta_i-1}B_k} - b_{i, \omega, 2^{\theta_i}B_k} \right| := 0.
\]

It is not difficult to show that

\[
\left| b_{i, \omega, 2^{\theta_i-1}B_k} - b_{i, \omega, 2^{\theta_i}B_k} \right| \lesssim \| b_i \|_{C^{\sigma^i}(\omega_i)}.
\]

In the case \( \theta_i \geq 1 \), by defining \( \sigma_i \), it follows that

\[
\left| b_{i, \omega, 2^{\theta_i}(\theta_i)B_k} - b_{i, \omega, A_i(t)B_k} \right| \leq \frac{1}{\omega_i(2^{\theta_i-1}B_k)} \int_{2^{\theta_i-1}B_k} \left| b_i(x) - b_{i, \omega, A_i(t)B_k} \right| \omega_i(x) dt
\]

\[
\leq \left( \frac{\omega_i(A_i(t)B_k)}{\omega_i(2^{\theta_i-1}B_k)} \right)^{\frac{1}{r_i}} \left( \int_{A_i(t)B_k} \left| b_i(x) - b_{i, \omega, A_i(t)B_k} \right| \omega_i(x) dt \right)^{\frac{1}{r_i}}
\]

\[
\leq \left( \frac{\omega_i(A_i(t)B_k)}{\omega_i(2^{\theta_i-1}B_k)} \right)^{\frac{1}{r_i}} \| b_i \|_{C^{\sigma^i}(\omega_i)}.
\]

Note that \( A_i(t) = s_i(t)a_i(t) \). So, we compute

\[
\left( \frac{\omega_i(A_i(t)B_k)}{\omega_i(2^{\theta_i-1}B_k)} \right) \lesssim \left( \frac{|s_i(t)| 2^{k \gamma_i}}{2^{\theta_i-1}2^{k \gamma_i}} \right)^{\frac{n+\gamma_i}{n+\gamma_i}} \leq \left( \frac{2^{\theta_i}2^{k \gamma_i}}{2^{\theta_i-1}2^{k \gamma_i}} \right)^{\frac{n+\gamma_i}{n+\gamma_i}} \lesssim 1.
\]

(4.30)

Consequently, we have

\[
\left| b_{i, \omega, 2^{\theta_i}(\theta_i)B_k} - b_{i, \omega, A_i(t)B_k} \right| \lesssim \| b_i \|_{C^{\sigma^i}(\omega_i)}.
\]
Otherwise, for $\theta_i \leq 0$, by estimating as (4.30), we deduce
\[
|b_{i,\omega_i,2^{n_i}A_i(t)B_k} - b_{i,\omega_i,A_i(t)B_k}| \leq \frac{1}{\omega_i(A_i(t)B_k)} \int_{A_i(t)B_k} |b_i(x) - b_{i,\omega_i,2^{n_i}A_i(t)B_k}| \omega_i(x) dt
\]
\[
\leq \left(\frac{\omega_i(2^{n_i}A_i(t)B_k)}{\omega_i(A_i(t)B_k)}\right)^{\frac{1}{r_i}} \left(\int_{2^{n_i}A_i(t)B_k} |b_i(x) - b_{i,\omega_i,2^{n_i}A_i(t)B_k}|^r \omega_i(x) dt\right)^{\frac{1}{r}}
\]
\[
\leq \frac{\omega_i(2^{n_i}A_i(t)B_k)}{\omega_i(A_i(t)B_k)} \|b_i\|_{CMO^{r_i}(\omega_i)} \lesssim \|b_i\|_{CMO^{r_i}(\omega_i)}.
\]
Because of $2^{\theta_i - 1} < |s_i(t)| \leq 2^{\theta_i}$, we have
\[
|\theta_i| + 1 \lesssim \left\{ \begin{array}{ll}
\log(4|s_i(t)|), & \text{if } \theta_i \geq 0, \\
\log\frac{2}{|s_i(t)|}, & \text{otherwise}.
\end{array} \right.
\]
Therefore, by having (4.29), it follows that
\[
|b_{i,\omega_i,B_k} - b_{i,\omega_i,A_i(t)B_k}| \lesssim (|\theta_i| + 1)\|b_i\|_{CMO^{r_i}(\omega_i)} \lesssim \varphi_{A_i}(t)\|b_i\|_{CMO^{r_i}(\omega_i)}.
\]
As above, by (4.28), we get
\[
\|a_{3,i}\|_{L^{r_i}(\omega_i,B_k)} \lesssim 2^{\frac{k(n+\gamma)}{r_i}} \varphi_{A_i}(t)\|b_i\|_{CMO^{r_i}(\omega_i)}.
\]
From this, by (4.26), (4.27), we finish the proof of the inequality (4.24).

Using (4.23) and (4.24), we have
\[
\|H_{\Phi,A}^b(\tilde{f})\chi_k\|_{L^{p_i}(\omega_i)} \lesssim B_{CMO^{\infty},2} \left( \sum_{i=1}^{m} \frac{\gamma_i}{r_i} \right)^k \left( \int_{\mathbb{R}^n} \Phi(t) \prod_{i=1}^{m} c_{A_i,\Phi_i,\gamma_i}(t) |t| \right)^{\frac{1}{r_i}} \times
\]
\[
\times \left( 1 + \psi_{A_i,\gamma_i}^\downarrow |s_i(t)|^{-\gamma_i/r_i} + \varphi_{A_i}(t) \right) \prod_{i=1}^{m} \int_{0}^{\Theta_{\omega_i} - 1} \int_{E_{i+k_{\ell_i}}^i} |t|^{\gamma_i/k_{\ell_i}} dt.
\]

At this point, by making Lemma 2.4 in Section 2 again, we have a similar results to (4.23) as follow
\[
\|H_{\Phi,A}^b(\tilde{f})\chi_k\|_{L^{p_i}(\omega_i)} \lesssim C_4 B_{CMO^{\infty},2} \left( \sum_{i=1}^{m} \frac{\gamma_i}{r_i} \right)^k \prod_{i=1}^{m} \left( 2^{k(\lambda_i - \alpha_i(0))} + 2^k(\lambda_i - \alpha_i(\infty)) \right).
\]

By using Proposition 2.5 in 29 again, we get
\[
\|H_{\Phi,A}^b(\tilde{f})\|_{MK_{p,q}(\omega_\infty)} \lesssim \max \left\{ \sup_{k_0 < 0, k_0 \in \mathbb{Z}} \tilde{E}_1, \sup_{k_0 \geq 0, k_0 \in \mathbb{Z}} (\tilde{E}_2 + \tilde{E}_3) \right\},
\]
(4.32)
where

\[
\tilde{E}_1 = 2^{-k_0}λ \left( \sum_{k=-\infty}^{k_0} 2^{kα^*(0)p} \left\| H_{Φ, A}(\tilde{f}) \chi_k \right\|_{L^q(\cdot)}^p \right) \frac{1}{p},
\]

\[
\tilde{E}_2 = 2^{-k_0}λ \left( \sum_{k=-\infty}^{-1} 2^{kα^*(0)p} \left\| H_{Φ, A}(\tilde{f}) \chi_k \right\|_{L^q(\cdot)}^p \right) \frac{1}{p},
\]

\[
\tilde{E}_3 = 2^{-k_0}λ \left( \sum_{k=0}^{k_0} 2^{kα_1^∞p} \left\| H_{Φ, A}(\tilde{f}) \chi_k \right\|_{L^q(\cdot)}^p \right) \frac{1}{p}.
\]

In view of (4.32), by defining \( α^* \) and estimating as (4.13), (4.14), (4.17), we also have

\[
\tilde{E}_1 \lesssim C_4 B_{CMO, \vec{ω}} \prod_{i=1}^{m} \left( 1 + 2^{k_0} \left( α_i(0) - α_i^∞ \right) \right),
\]

\[
\tilde{E}_2 \lesssim C_4 B_{CMO, \vec{ω}} \cdot 2^{-k_0}λ,
\]

\[
\tilde{E}_3 \lesssim C_4 B_{CMO, \vec{ω}} \cdot 2^{-k_0}λ \prod_{i=1}^{m} L_i.
\]

Therefore, the proof of Theorem 3.4 is completed.

Now, let us give the proof for Theorem 3.5. By making Proposition 3.8 in [2] again, we obtain

\[
\left\| H_Φ^b(\vec{f}) \right\|_{K^{α^*(0)p}_{q(\cdot), \vec{ω}}} \lesssim \left( \sum_{k=-\infty}^{-1} 2^{kα^*(0)p} \left\| H_Φ^b(\vec{f}) \chi_k \right\|_{L^q(\cdot)}^p \right)^{\frac{1}{p}}
\]

\[
+ \left( \sum_{k=0}^{∞} 2^{kα_1^∞p} \left\| H_Φ^b(\vec{f}) \chi_k \right\|_{L^q(\cdot)}^p \right)^{\frac{1}{p}}
\]

\[
\quad := G_0 + G_1. \tag{4.33}
\]

Using the Minkowski inequality, by employing (4.31), we find

\[
G_0 \lesssim B_{CMO, \vec{ω}} \int_{\mathbb{R}^n} \frac{Φ(t)}{|t|^n} \prod_{i=1}^{m} c_{A_i, g_i, \gamma_i}(t) \left\| 1 \right\|_{L^{q_i}(\cdot)}^1 \left( 1 + \psi_{A_i, \gamma_i} |s_i(t)|^{-\gamma_i + n} + |φ_{A_i}(t)| \right) \times
\]

\[
\quad \times \left\{ \sum_{k=-\infty}^{-1} 2^{k(\sum_{i=1}^{m} α_i(0)p)} \prod_{i=1}^{m} \left( \sum_{r=0}^{\gamma_i - 1} \left\| f_i \chi_k + t_i r \right\|_{L^{q_i}(\cdot)}^p \right) \right\}^{\frac{1}{p}} dt,
\]
By applying the Minkowski inequality for the variable Lebesgue space, we have
\[
G \lesssim \mathbb{B}_{CMO, \omega} \int_{\mathbb{R}^n} \Phi(t) \frac{1}{|t|} \prod_{i=1}^{m} c_{A_i, q_i, \gamma_i}(t) \cdot \|1\|_{L^{q_i}_t(t, \cdot)} \varphi_{A_i, 0}(t) \left(1 + \psi_{A_i, \gamma_i}^\alpha |s_i(t)|^\frac{\alpha+n}{\gamma_i} + \varphi_{A_i}(t)\right) \times
\]
\[
\times \left\{ \sum_{k=-\infty}^{-1} 2^k \sum_{i=0}^{m} \prod_{i=1}^{m} \left( \sum_{r=0}^{m} \|f_i \chi_{k+t_i+r} \|_{L^{\infty}_t(t, \cdot)} \right)^p \right\}^\frac{1}{p} dt.
\]

We observe that the other estimations can be done by similar arguments as Theorem 3.2. Thus, \( G_0 \) and \( G_1 \) are dominated by \( \mathcal{C} \mathcal{C}_3 \mathcal{B}_{CMO, \omega} \prod_{i=1}^{m} \|f_i\|_{K^{\alpha_i(t), p_i}} \).

This proves the assertion.

4.3. Proofs of Theorem 3.6 and Theorem 3.7. For \( R > 0 \), we write \( B := B(0, R) \) and \( \Delta_R \) as
\[
\Delta_R = \frac{1}{\omega(B)} \frac{1}{\|H_{\Phi, \alpha}(f)\|_{L^{\infty}_{(B)}}}.
\]
By applying the Minkowski inequality for the variable Lebesgue space, we have
\[
\Delta_R \lesssim \int_{\mathbb{R}^n} \frac{1}{\omega(B)} \frac{\Phi(t)}{|t|} \prod_{i=1}^{m} \|f_i(A_i(t) \cdot (b_i(\cdot) - b_i(A_i(t) \cdot)) \|_{L^{q_i}_t(t, \cdot)} dt. \tag{4.34}
\]
By estimating as (4.2) above, we get
\[
\|\prod_{i=1}^{m} f_i(A_i(t) \cdot (b_i(\cdot) - b_i(A_i(t) \cdot)) \|_{L^{q_i}_t(t, \cdot)}
\]
\[
\lesssim R^{\beta} \mathcal{B}_{\text{Lip}} \prod_{i=1}^{m} \|I_n - A_i(t)\|^{\beta_i} \prod_{i=1}^{m} \|f_i(A_i(t) \cdot) \|_{L^{q_i}_t(t, \cdot)} \|\cdot \|_{L^{q_i}_t(t, \cdot)}.
\]
\[
\lesssim R^{\beta + \sum_{i=1}^{m} \frac{\alpha_i+n}{\gamma_i}} \mathcal{B}_{\text{Lip}} \prod_{i=1}^{m} \|I_n - A_i(t)\|^{\beta_i} \prod_{i=1}^{m} \|f_i(A_i(t) \cdot) \|_{L^{q_i}_t(t, \cdot)} \tag{4.35}
\]
By (3.14) and the Theorem 2.6 we find
\[
\|f_i(A_i(t) \cdot) \|_{L^{q_i}_t(t, \cdot)} \lesssim c_{A_i, q_i, \alpha_i}(t) \cdot \|1\|_{L^{\gamma_i}_t(t, \cdot)} \cdot \|f_i\|_{L^{q_i}_t(t, \cdot)}(B(0, R ||A_i(t)||)). \tag{4.36}
\]
By the condition (3.15), we estimate
\[
\frac{R^{\beta + \sum_{i=1}^{m} \frac{\alpha_i+n}{\gamma_i}} \omega(B)^{\frac{1}{\gamma_i} + \lambda_i}}{\omega(B)^{\frac{1}{\gamma_i} + \lambda_i}} \leq \prod_{i=1}^{m} \frac{\|A_i(t)\|^{(\gamma_i+n)\left(\frac{1}{\gamma_i} + \lambda_i\right)}}{\omega_i(B(0, R ||A_i(t)||)^{\frac{1}{\gamma_i} + \lambda_i}). \tag{4.37}
\]
Thus, by having (4.34)-(4.37) and defining central Morrey spaces with variable exponent, it follows that

$$\Delta_R \lesssim C_6 B_{\text{Lip}}, \prod_{i=1}^m \| f_i \|_{B_{\omega_i v_i}^{q_i}, \lambda_i}.$$ 

Therefore, we conclude that

$$\| H_{\Phi, A}^\delta (\vec{f}) \|_{B^{q_i, \lambda}_1} \lesssim C_6 B_{\text{Lip}}, \prod_{i=1}^m \| f_i \|_{B_{\omega_i v_i}^{q_i}, \lambda_i}.$$ 

Next, we will prove Theorem 3.7. Indeed, by using the Minkowski inequality and the Hölder inequality for variable Lebesgue spaces again, it is obvious to show that

$$\Delta_R \lesssim \int_{\mathbb{R}^n} \frac{1}{\omega(B)^{\frac{1}{m} + \lambda}} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m \left| b_i(t) - b_i(A_i(t)) \right| \| f_i(A_i(t)) \|_{L^{q_i}(B(t))} dt.$$ 

By (4.24) above, we deduce

$$\Delta_R \lesssim R_i \sum_{R_i=1}^m \frac{\alpha_i + n}{\gamma_i} B_{\text{CMO, v}} \int_{\mathbb{R}^n} \frac{1}{\omega(B)^{\frac{1}{m} + \lambda}} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m \left( 1 + \psi_{A_i, \alpha_i} s_i(t) \right)^\frac{\alpha_i + n}{\gamma_i} \varphi_{A_i}(t) \times \prod_{i=1}^m \| f_i(A_i(t)) \|_{L^{q_i}(B(t))} dt.$$ 

For this, by (4.36), we get

$$\Delta_R \lesssim R_i \sum_{R_i=1}^m \frac{\alpha_i + n}{\gamma_i} B_{\text{CMO, v}} \left( \int_{\mathbb{R}^n} \frac{1}{\omega(B)^{\frac{1}{m} + \lambda}} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m \left( 1 + \psi_{A_i, \alpha_i} s_i(t) \right)^\frac{\alpha_i + n}{\gamma_i} \varphi_{A_i}(t) \times \right. \times c_{A_i, q_i, \alpha_i}(t) \left\| f_i \right\|_{L^{q_i}(B(t))} dt \right).$$

On the other hand, by (3.17), it follows that

$$\sum_{R_i=1}^m \frac{\alpha_i + n}{\gamma_i} \lesssim \prod_{i=1}^m \left( \frac{A_i(t)}{\omega_i(B(t))^{\frac{1}{m} + \lambda_i}} \right)^{\frac{1}{\gamma_i}}.$$ 

Consequently, by having (4.38), we immediately obtain

$$\| H_{\Phi, A}^\delta (\vec{f}) \|_{B^{q_i, \lambda}, \lambda_i} \lesssim C_7 B_{\text{CMO, v}} \prod_{i=1}^m \| f_i \|_{B_{\omega_i v_i}^{q_i}, \lambda_i},$$

which completes the proof.

**Acknowledgments.** This paper is supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2014.51.
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