Weighted approximation in higher-dimensional missing digit sets

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Abstract

In this note, we use the mass transference principle for rectangles, recently obtained by Wang and Wu (Math. Ann., 2021), to study the Hausdorff dimension of sets of “weighted $\Psi$-well-approximable” points in certain self-similar sets in $\mathbb{R}^d$. Specifically, we investigate weighted $\Psi$-well-approximable points in “missing digit” sets in $\mathbb{R}^d$. The sets we consider are natural generalisations of Cantor-type sets in $\mathbb{R}$ to higher dimensions and include, for example, four corner Cantor sets (or Cantor dust) in the plane with contraction ratio $\frac{1}{n}$ with $n \in \mathbb{N}$.

1 Introduction and motivation

The work of this current paper is motivated by a question posed in a seminal paper by Mahler [36]; namely, how well can we approximate points in the middle-third Cantor set by:

(i) rational numbers contained in the Cantor set, or
(ii) rational numbers not in the Cantor set?

The first contribution to this question was arguably made by Weiss [47], who showed that almost no point in the middle-third Cantor set is very well approximable with respect to the natural probability measure on the middle-third Cantor set. Since this initial contribution, numerous authors have contributed to answering these questions, approaching them from many different perspectives. For example, Levesley, Salp, and Velani [35] considered triadic approximation in the middle-third Cantor set, different subsets of the first named author, Baker, Chow, and Yu [3, 6, 13] studied dyadic approximation in the middle-third Cantor set, Kristensen [34] considered approximation of points in the middle-third Cantor set by algebraic numbers, and Tan, Wang and Wu [42] have recently studied part (i) by introducing a new notion of the “height” of a rational number. There has also been considerable effort invested in trying to generalise some of the above results to more general self-similar sets in $\mathbb{R}$ and also to various fractal sets in higher dimensions. See, for example, [4, 10–12, 14, 18, 22,
23, 27, 31, 38, 46, 48] and references therein. The results in this paper can be thought of as a contribution to answering a natural $d$-dimensional weighted variation of part (i) of Mahler’s question. In particular, we will be interested in weighted approximation in $d$-dimensional “missing digit” sets.

Before we introduce the general framework we will consider here, we provide a very brief overview of some of the classical results on weighted Diophantine approximation in the “usual” Euclidean setting which provide further motivation for the current work. Fix $d \in \mathbb{N}$ and let $\Psi = (\psi_1, \ldots, \psi_d)$ be a $d$-tuple of approximating functions $\psi_i : \mathbb{N} \rightarrow [0, \infty)$ with $\psi_i(r) \to 0$ as $r \to \infty$ for each $1 \leq i \leq d$. The set of weighted simultaneously $\Psi$-well-approximable points in $\mathbb{R}^d$ is defined as

$$W_d(\Psi) := \left\{ x = (x_1, \ldots, x_d) \in [0, 1]^d : \left| x_i - \frac{p_i}{q}\right| < \psi_i(q), \ 1 \leq i \leq d, \text{ for i.m. } (p_1, \ldots, p_d, q) \in \mathbb{Z}^d \times \mathbb{N} \right\},$$

where i.m. denotes infinitely many. Note that the special case where each approximating function is the same, that is $\Psi = (\psi, \ldots, \psi)$, is generally the more intensively studied set. The case where each approximating function is potentially different, usually referred to as weighted simultaneous approximation, is a natural generalisation of this. Simultaneous approximation (i.e. when the approximating function is the same in each coordinate axis) can generally be seen as a metric generalisation of Dirichlet’s Theorem, whereas weighted simultaneous approximation is a metric generalisation of Minkowski’s Theorem. Weighted simultaneous approximation has earned interest in the past few decades due to Schmidt and natural connections to Littlewood’s Conjecture, see for example [7–9, 15, 41].

Motivated by classical works due to the likes of Khintchine [28, 29] and Jarník [26] which tell us, respectively, about the Lebesgue measure and Hausdorff measures of the sets of classical simultaneously $\Psi$-well-approximable points (i.e. when $\Psi = (\psi, \ldots, \psi)$), one may naturally also wonder about the “size” of sets of weighted simultaneously $\Psi$-well-approximable points in terms of Lebesgue measure, Hausdorff dimension, and Hausdorff measures. Khintchine [30] showed that if $\psi : \mathbb{N} \rightarrow [0, \infty)$ and $\Psi(q) = (\psi(q)^{\tau_1}, \ldots, \psi(q)^{\tau_d})$ for some $\tau = (\tau_1, \ldots, \tau_d) \in (0, 1)^d$ with $\tau_1 + \tau_2 + \cdots + \tau_d = 1$, then

$$\lambda_d(W_d(\Psi)) = \left\{ \begin{array}{ll} 0 & \text{ if } \sum_{q=1}^{\infty} q^d \psi(q) < \infty, \\ 1 & \text{ if } \sum_{q=1}^{\infty} q^d \psi(q) = \infty, \text{ and } q^d \psi(q) \text{ is monotonic.} \end{array} \right.$$ 

Throughout we use $\lambda_d(X)$ to denote the $d$-dimensional Lebesgue measure of a set $X \subset \mathbb{R}^d$. For more general approximating functions $\Psi(q) = (\psi_1(q), \ldots, \psi_d(q))$, with $\prod_{i=1}^{d} \psi_i(q)$ monotonically decreasing and $\psi_i(q) < q^{-1}$ for each $1 \leq i \leq d$, it has been proved, see [19, 24, 30, 40], that

$$\lambda_d(W_d(\Psi)) = \left\{ \begin{array}{ll} 0 & \text{ if } \sum_{q=1}^{\infty} q^d \psi_1(q) \cdots \psi_d(q) < \infty, \\ 1 & \text{ if } \sum_{q=1}^{\infty} q^d \psi_1(q) \cdots \psi_d(q) = \infty. \end{array} \right.$$ 

For approximating functions of the form $\Psi(q) = (\psi_1(q), \ldots, \psi_d(q))$ where

$$\psi_i(q) = q^{-t_i-1}, \text{ for some vector } t = (t_1, \ldots, t_d) \in \mathbb{R}_{\geq 0}^d,$$
Rynne [39] proved that if $\sum_{i=1}^{d} t_i \geq 1$, then

$$\dim_{\mathcal{H}} W_d(\Psi) = \min_{1 \leq k \leq d} \left\{ \frac{1}{t_k + 1} \left( d + 1 + \sum_{t_k \geq t_i} (t_k - t_i) \right) \right\}.$$ 

Throughout, we write $\dim_{\mathcal{H}} X$ to denote the Hausdorff dimension of a set $X \subset \mathbb{R}^d$, we refer the reader to [20] for definitions and properties of Hausdorff dimension and Hausdorff measures. Rynne’s result has recently been extended to a more general class of approximating functions by Wang and Wu [44, Theorem 10.2].

In recent years, there has been rapidly growing interest in whether similar statements can be proved when we intersect $W_d(\Psi)$ with natural subsets of $[0, 1]^d$, such as submanifolds or fractals. The study of such questions has been further incentivised by many remarkable works of the recent decades, such as [31, 32, 43], and applications to other areas, such as wireless communication theory [1].

## 2 $d$-dimensional missing digit sets and main results

In this paper we study weighted approximation in $d$-dimensional missing digit sets, which are natural extensions of classical missing digit sets (i.e. generalised Cantor sets) in $\mathbb{R}$ to higher dimensions. A very natural class of higher dimensional missing digit sets included within our framework are the four corner Cantor sets (or Cantor dust) in $\mathbb{R}^2$ with contraction ratio $\frac{1}{n}$ for $n \in \mathbb{N}$.

Throughout we consider $\mathbb{R}^d$ equipped with the supremum norm, which we denote by $\| \cdot \|$. For subsets $X, Y \subset \mathbb{R}^d$ we define $\text{diam}(X) = \sup\{ \| u - v \| : u, v \in X \}$ and $\text{dist}(X, Y) = \inf\{ \| x - y \| : x \in X, y \in Y \}$. We define higher-dimensional missing digit sets via iterated function systems as follows. Let $b \in \mathbb{N}$ be such that $b \geq 3$ and let $J_1, \ldots, J_d$ be proper subsets of $\{0, 1, \ldots, b-1\}$ such that for each $1 \leq i \leq d$, we have

$$N_i := \# J_i \geq 2.$$ 

Suppose $J_i = \{a_{i1}^{(i)}, \ldots, a_{iN_i}^{(i)}\}$. For each $1 \leq i \leq d$, we define the iterated function system

$$\Phi^i = \{f_j : [0, 1] \to [0, 1] \}_{j=1}^{N_i} \quad \text{where} \quad f_j(x) = \frac{x + a_{ji}^{(i)}}{b}.$$ 

Let $K_i$ be the attractor of $\Phi^i$; that is, $K_i \subset \mathbb{R}$ is the unique non-empty compact set which satisfies

$$K_i = \bigcup_{j=1}^{N_i} f_j(K_i).$$

We know that such a set exists due to work of Hutchinson [25]. Equivalently $K_i$ is the set of $x \in [0, 1]$ for which there exists a base $b$ expansion of $x$ consisting only of digits from $J_i$. In view of this, we will also use the notation $K_i(J_i)$ to denote this set. For example, in this notation, the classical middle-third Cantor set is precisely the set $K_3(\{0, 2\})$. We call the
sets $K_i(J_i)$ missing digit sets since they consist of numbers with base-$b$ expansions missing specified digits. Note that, for each $1 \leq i \leq d$, the Hausdorff dimension of $K_i$, which we will denote by $\gamma_i$, is given by

$$\gamma_i = \dim_H K_i = \frac{\log N_i}{\log b}.$$  

We will be interested in the higher-dimensional missing digit set

$$K := \prod_{i=1}^{d} K_i$$

formed by taking the Cartesian product of the sets $K_i$, $1 \leq i \leq d$. As a natural concrete example, we note that the four corner Cantor set in $\mathbb{R}^2$ with contraction ratio $\frac{1}{b}$ (with $b \geq 3$ an integer) can be written in our notation as $K_b(\{0, b-1\}) \times K_b(\{0, b-1\})$.

We note that $K$ is the attractor of the iterated function system

$$\Phi = \left\{ f_{(j_1,\ldots,j_d)} : [0,1]^d \to [0,1]^d, (j_1,\ldots, j_d) \in \prod_{i=1}^{d} J_i \right\}$$

where

$$f_{(j_1,\ldots,j_d)} \left( \begin{array}{c} x_1 \\ \vdots \\ x_d \end{array} \right) = \left( \begin{array}{c} \frac{x_1 + j_1}{b} \\ \vdots \\ \frac{x_d + j_d}{b} \end{array} \right).$$

Notice that $\Phi$ consists of

$$N := \prod_{i=1}^{d} N_i$$

maps and so, for convenience, we will write

$$\Phi = \{ g_j : [0,1]^d \to [0,1]^d \}_{j=1}^{N}$$

where the $g_j$’s are just the maps $f_{(j_1,\ldots,j_d)}$ from above written in some order. The Hausdorff dimension of $K$, which we denote by $\gamma$, is

$$\gamma = \dim_H K = \frac{\log N}{\log b}.$$  

We will write

$$\Lambda = \{1,2,\ldots,N\} \quad \text{and} \quad \Lambda^* = \bigcup_{n=0}^{\infty} \Lambda^n.$$  

We write $i$ to denote a word in $\Lambda$ or $\Lambda^*$ and we write $|i|$ to denote the length of $i$. For $i \in \Lambda^*$ we will also use the shorthand notation

$$g_i = g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_{|i|}}.$$  

We adopt the convention that $g_{\emptyset}(x) = x$.  

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Let $\Psi : \Lambda^* \to [0, \infty)$ be an approximating function. For each $x \in K$, we define the set

$$W(x, \Psi) = \{y \in K : \|y - g_i(x)\| < \Psi(i) \text{ for infinitely many } i \in \Lambda^* \}.$$ 

The following theorem is a special case of [4, Theorem 1.1].

**Theorem 1.** Let $\Phi$ and $K$ be as defined above. Let $x \in K$ and let $\varphi : \mathbb{N} \to [0, \infty)$ be a monotonically decreasing function. Let $\Psi(i) = \text{diam}(g_i(K))\varphi(|i|)$. Then, for $s > 0$,

$$\mathcal{H}^s(W(x, \Psi)) = \begin{cases} 0 & \text{if } \sum_{i \in \Lambda^*} \Psi(i)^s < \infty, \\ \mathcal{H}^s(K) & \text{if } \sum_{i \in \Lambda^*} \Psi(i)^s = \infty. \end{cases}$$

Of particular interest to us here is the following easy corollary.

**Corollary 1.** Let $\Phi$ and $K$ be as above and suppose that $\text{diam}(K) = 1$. Let $\psi : \mathbb{N} \to [0, \infty)$ be such that $b^n\psi(b^n)$ is monotonically decreasing and define $\varphi : \mathbb{N} \to [0, \infty)$ by $\varphi(n) = b^n\psi(b^n)$. Let $\Psi(i) = \text{diam}(g_i(K))\varphi(|i|)$. Recall that $\gamma = \dim_h K$. Then, for $x \in K$, we have

$$\mathcal{H}^\gamma(W(x, \Psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} (b^n\psi(b^n))^\gamma < \infty, \\ \mathcal{H}^\gamma(K) & \text{if } \sum_{n=1}^{\infty} (b^n\psi(b^n))^\gamma = \infty. \end{cases}$$

**Proof.** It follows from Theorem 1 that

$$\mathcal{H}^\gamma(W(x, \Psi)) = \begin{cases} 0 & \text{if } \sum_{i \in \Lambda^*} \Psi(i)^\gamma < \infty, \\ \mathcal{H}^\gamma(K) & \text{if } \sum_{i \in \Lambda^*} \Psi(i)^\gamma = \infty. \end{cases}$$

However, in this case, by the definition of $\varphi$ and our assumption that $\text{diam}(K) = 1$, we have

$$\sum_{i \in \Lambda^*} \Psi(i)^\gamma = \sum_{n=1}^{\infty} \sum_{i \in \Lambda^*} |i| = n (\text{diam}(g_i(K))\varphi(|i|))^\gamma = \sum_{n=1}^{\infty} \sum_{i \in \Lambda^*} |i| = n (b^n\psi(b^n))^\gamma = \sum_{n=1}^{\infty} N^n \psi(b^n)^\gamma = \sum_{n=1}^{\infty} (b^n\psi(b^n))^\gamma.$$ 

\[ \square \]

For an approximating function $\psi : \mathbb{N} \to [0, \infty)$, define

$$W(x, \psi) = \{y \in K : \|y - g_i(x)\| < \psi(|i|) \text{ for infinitely many } i \in \Lambda^* \}.$$ 

In essence, $W(x, \psi)$ is a set of “simultaneously $\psi$-well-approximable” points in $K$. The following statement regarding these sets can be deduced immediately from Corollary 1.

**Corollary 2.** Let $\Phi$ and $K$ be defined as above and let $\psi : \mathbb{N} \to [0, \infty)$ be such that $b^n\psi(b^n)$ is monotonically decreasing. Suppose further that $\text{diam}(K) = 1$. Then,

$$\mathcal{H}^\gamma(W(x, \psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} (b^n\psi(b^n))^\gamma < \infty, \\ \mathcal{H}^\gamma(K) & \text{if } \sum_{n=1}^{\infty} (b^n\psi(b^n))^\gamma = \infty. \end{cases}$$
Here we will be interested in weighted versions of the sets $W(x, \psi)$. More specifically, for $t = (t_1, \ldots, t_d) \in \mathbb{R}^d_{\geq 0}$ and for $x \in K$, we define the weighted approximation set

$$W(x, \psi, t) = \{ y = (y_1, \ldots, y_d) \in K : |y_j - g_i(x)_j| < \psi(b^n_i)^{1+t_i}, 1 \leq j \leq d, \text{ for i.m. } i \in \Lambda^* \}.$$ 

Here we are using the notation $g_i(x) = (g_i(x)_1, \ldots, g_i(x)_d)$. Our main results relating to the Hausdorff dimension of sets of the form $W(x, \psi, t)$ are as follows.

**Theorem 2.** Let $\Phi$ and $K$ be defined as above. Recall that $\gamma = \dim_H K$ and $\gamma_i = \dim_H K_i$ for each $1 \leq i \leq d$. Let $\psi : \mathbb{N} \to [0, \infty)$ be such that $b^n \psi(b^n)$ is monotonically decreasing. Further suppose that $\text{diam}(K) = 1$ and

$$\sum_{n=1}^{\infty} (b^n \psi(b^n))^\gamma = \infty.$$ 

Then, for $t = (t_1, \ldots, t_d) \in \mathbb{R}^d_{\geq 0}$, we have

$$\dim_H W(x, \psi, t) \geq \min_{1 \leq k \leq d} \left\{ \frac{1}{1 + t_k} \left( \gamma + \sum_{j : t_j \leq t_k} (t_k - t_j) \gamma_j \right) \right\}.$$ 

If $\psi$ satisfies more stringent divergence conditions, then we an show that the lower bound given in Theorem 2 in fact gives an exact formula for the Hausdorff dimension of $W(x, \psi, t)$. More precisely, we are able to show the following.

**Theorem 3.** Let $\Phi$ and $K$ be as defined above. Let $x \in K$ and let $\psi : \mathbb{N} \to [0, \infty)$ be such that:

(i) $b^n \psi(b^n)$ is monotonically decreasing,

(ii) $\sum_{n=1}^{\infty} (b^n \psi(b^n))^\gamma = \infty$, and

(iii) $\sum_{n=1}^{\infty} (b^n \psi(b^n))^{\gamma + \epsilon} < \infty$ for every $\epsilon > 0$.

Then, for $t = (t_1, \ldots, t_d) \in \mathbb{R}^d_{\geq 0}$, we have

$$\dim_H W(x, \psi, t) = \min_{1 \leq k \leq d} \left\{ \frac{1}{1 + t_k} \left( \gamma + \sum_{j : t_j \leq t_k} (t_k - t_j) \gamma_j \right) \right\}.$$ 

As an example of an approximating function which satisifies conditions (i) – (iii), one can think of $\psi(q) = (q(\log_6 q)^{1/\gamma})^{-1}$. This function naturally appears when one considers analogues of Dirichlet’s theorem in missing digit sets (see [18, 22]). As a corollary to Theorem 3 we deduce the following statement which can be interpreted as a higher-dimensional
weighted generalisation of [35, Theorem 4]. In [35, Theorem 4], Levesley, Salp, and Velani establish the Hausdorff measure of the set of points in a one-dimensional base-\( b \) missing digit set (i.e. of the form \( K_b(J) \) in our present notation) which can be well-approximated by rationals with denominators which are powers of \( b \). Before we state our corollary, we fix one more piece of notation. Given an approximating function \( \psi : \mathbb{N} \to [0, \infty) \), an infinite subset \( \mathcal{B} \subset \mathbb{N} \), and \( t = (t_1, \ldots, t_d) \in \mathbb{R}_{\geq 0}^d \), we define

\[
W_{\mathcal{B}}(\psi, t) = \left\{ x \in K : \left| x_i - p_i \right| < \psi(q)^{1+t_i}, 1 \leq i \leq d \right\}.
\]

**Corollary 3.** Fix \( b \in \mathbb{N} \) with \( b \geq 3 \) and let \( \mathcal{B} = \{ b^n : n = 0, 1, 2, \ldots \} \). Let \( K \) be a higher dimensional missing digit set as defined above (with base \( b \)) and write \( \gamma = \dim_H K \). Furthermore, suppose that \( \{0, b-1\} \subset J_i \) for every \( 1 \leq i \leq d \). In particular, this also means that \( \dim K = 1 \). Let \( \psi : \mathbb{N} \to [0, \infty) \) be an approximating function such that

(i) \( b^n \psi(b^n) \) is monotonically decreasing with \( b^n \psi(b^n) \to 0 \) as \( n \to \infty \),

(ii) \( \sum_{n=1}^{\infty} (b^n \psi(b^n))^{\gamma} = \infty \), and

(iii) \( \sum_{n=1}^{\infty} (b^n \psi(b^n))^{1+\varepsilon} < \infty \) for every \( \varepsilon > 0 \).

Then

\[
\dim_H W_{\mathcal{B}}(\psi, t) = \min_{1 \leq k \leq d} \left\{ \frac{1}{1+t_k} \left( \gamma + \sum_{j: t_j \leq t_k}(t_k - t_j) \gamma_j \right) \right\}.
\]

**Proof.** Observe that the conditions imposed in the statement of Corollary 3 guarantee that Theorem 3 is applicable. Furthermore, by our assumption that \( b^n \psi(b^n) \to 0 \) as \( n \to \infty \), we may assume without loss of generality that \( \psi(b^n) < b^{-n} \) for all \( n \in \mathbb{N} \).

Next, we note that if \( p = (p_1, \ldots, p_d) \in \mathbb{Z}^d \) and \( \frac{p}{b^n} = \left( \frac{p_1}{b^n}, \ldots, \frac{p_d}{b^n} \right) \notin K \), then we must have

\[
\text{dist} \left( \frac{p}{b^n}, K \right) \geq b^{-n}, \quad \text{where} \quad \text{dist}(x, K) = \inf \{ \|x - y\| : y \in K \}.
\]

(Recall that we use \( \| \cdot \| \) to denote the supremum norm in \( \mathbb{R}^d \).) Thus we need only concern ourselves with pairs \( (p, q) \in \mathbb{Z}^d \times \mathcal{B} \) for which \( \frac{p}{q} \in K \).

Let \( G = \{ x = (x_1, \ldots, x_d) \in \{0, 1\}^d \} \) and note that \( G \subset K \) by the assumption that \( \{0, b-1\} \subset J_i \) for each \( 1 \leq i \leq d \). For any \( x \in G \) and any \( j \in \Lambda^n \) it is possible to write \( g_j(x) = \frac{p}{b^n} \) for some \( p \in (\mathbb{N} \cup \{0\})^d \). Hence

\[
W(x, \psi, t) \subset W_{\mathcal{B}}(\psi, t).
\]

Furthermore, the set of all rational points of the form \( \frac{p}{b^n} \) contained in \( K \) is

\[
\bigcup_{x \in G} \bigcup_{j \in \Lambda^n} g_j(x).
\]
Hence
\[ W_b(\psi, t) \subset \bigcup_{x \in G} W(x, \psi, t). \]

By the finite stability of Hausdorff dimension (see [20]), Corollary 3 now follows from Theorem 3.

Notice that in Theorem 2, Theorem 3, and Corollary 3, we insist on the same underlying base \( b \) in each coordinate direction. This is somewhat unsatisfactory and one might hope to be able to obtain results where we can have different bases \( b_i \) in each coordinate direction. The first steps towards proving results relating to weighted approximation in this setting can be seen in [44, Section 12]. Proving more general results with different bases in different coordinate directions is likely to be a very challenging problem since such sets are self-affine and, generally speaking, self-affine sets are more difficult to deal with than self-similar or self-conformal sets. Indeed, very little is currently known even regarding non-weighted approximation in self-affine sets.

**Structure of the paper:** The remainder of the paper will be arranged as follows. In Section 3 we will present some measure theoretic preliminaries which will be required for the proofs of our main results. The key tool required for proving Theorem 2 is a mass transference principle for rectangles proved recently by Wang and Wu [44]. We introduce this in Section 4. In Section 5 we present our proof of Theorem 2 and we conclude in Section 6 with the proof of Theorem 3.

### 3 Some Measure Theoretic Preliminaries

Recall that \( \gamma = \text{dim}_H K \) and that \( \gamma_i = \text{dim}_H K_i \) for \( 1 \leq i \leq d \), where \( K \) and \( K_i \) are as defined above. Furthermore, note that \( 0 < \mathcal{H}^\gamma(K) < \infty \) and \( 0 < \mathcal{H}^{\gamma_i}(K_i) < \infty \) for each \( 1 \leq i \leq d \), see for example [20, Theorem 9.3]. Let us define the measures

\[
\mu := \frac{\mathcal{H}^\gamma|_K}{\mathcal{H}^\gamma(K)} \quad \text{and} \quad \mu_i := \frac{\mathcal{H}^{\gamma_i}|_{K_i}}{\mathcal{H}^{\gamma_i}(K_i)} \quad \text{for each} \ 1 \leq i \leq d.
\]

So, for \( X \subset \mathbb{R}^d \), we have

\[
\mu(X) = \frac{\mathcal{H}^\gamma(X \cap K)}{\mathcal{H}^\gamma(K)}.
\]

Similarly, for \( X \subset \mathbb{R} \), for each \( 1 \leq i \leq d \) we have

\[
\mu_i(X) = \frac{\mathcal{H}^{\gamma_i}(X \cap K_i)}{\mathcal{H}^{\gamma_i}(K_i)}.
\]

Note that \( \mu \) defines a probability measure supported on \( K \) and, for each \( 1 \leq i \leq d \), \( \mu_i \) defines a probability measure supported on \( K_i \). Note also that the measure \( \mu \) is \( \delta \)-Ahlfors regular with \( \delta = \gamma \) and, for each \( 1 \leq i \leq d \), the measure \( \mu_i \) is \( \delta \)-Ahlfors regular with \( \delta = \gamma_i \) (see, for example, [37, Theorem 4.14]).
We will also be interested in the product measure

\[ M := \prod_{i=1}^{d} \mu_i. \]

We note that \( M \) is \( \delta \)-Ahlfors regular with \( \delta = \gamma \). This fact follows straightforwardly from the Ahlfors regularity of each of the \( \mu_i \)'s.

**Lemma 1.** The product measure \( M = \prod_{i=1}^{d} \mu_i \) on \( \mathbb{R}^d \) is \( \delta \)-Ahlfors regular with \( \delta = \gamma \).

**Proof.** Let \( B = \prod_{i=1}^{d} B(x_i, r) \), \( r > 0 \), be an arbitrary ball in \( \mathbb{R}^d \). The aim is to show that \( M(B) \asymp r^\gamma \). Recall that for each \( 1 \leq i \leq d \), the measure \( \mu_i \) is \( \delta \)-Ahlfors regular with \( \delta = \gamma_i = \dim_H K_i = \frac{\log N_i}{\log b} \). Also recall that \( N = \prod_{i=1}^{d} N_i \) and \( \gamma = \dim_H K = \frac{\log N}{\log b} \). Thus, we have

\[ M(B) = \prod_{i=1}^{d} \mu_i(B(x_i, r)) \asymp \prod_{i=1}^{d} r^{\gamma_i} = r^{\sum_{i=1}^{d} \gamma_i}. \]

Note that

\[ \sum_{i=1}^{d} \gamma_i = \sum_{i=1}^{d} \frac{\log N_i}{\log b} = \frac{\log(\prod_{i=1}^{d} N_i)}{\log b} = \frac{\log N}{\log b} = \gamma. \]

Hence, \( M(B) \asymp r^\gamma \) as claimed. \( \square \)

We also note that, up to a constant factor, the product measure \( M \) is equivalent to the measure \( \mu = \frac{\mathcal{H}^\gamma|_K}{\mathcal{H}^\gamma(K)} \).

**Lemma 2.** Let \( M = \prod_{i=1}^{d} \mu_i \). Then, up to a constant factor, \( M \) is equivalent to \( \mu \); i.e. for any Borel set \( F \subset \mathbb{R}^d \), we have \( M(F) \asymp \mu(F) \).

Lemma 2 follows immediately upon combining Lemma 1 with \([21, \text{Proposition 2.2 (a) + (b)})\].

In our present setting, where \( K \) is a self-similar set with well-separated components, we can actually show the stronger statement that \( \mu = M \).

**Proposition 1.** The measures \( \mu \) and \( M \) are equal, i.e. for every Borel set \( F \subset \mathbb{R}^d \), we have \( \mu(F) = M(F) \).

**Proof.** For each \( 1 \leq i \leq d \), there exists a unique Borel probability measure (see, for example, \([21, \text{Theorem 2.8})\]) \( m_i \) satisfying

\[ m_i = \sum_{j=1}^{N_i} \frac{1}{N_i} m_i \circ f_j^{-1}. \]  \quad (2)

Likewise, there exists a unique Borel probability measure \( m \) satisfying

\[ m = \sum_{j=1}^{N} \frac{1}{N} m \circ g_j^{-1}. \]  \quad (3)
We begin by showing that $\mu_i$ satisfies (2) for each $1 \leq i \leq d$. Note that $H^\gamma_i(f_{j_1}(K_i) \cap f_{j_2}(K_i)) = 0$ for any $1 \leq j_1, j_2 \leq N_i$ with $j_1 \neq j_2$. Thus, for any Borel set $X \subset \mathbb{R}^d$, we have

$$
\mu_i(X) = \frac{1}{H^\gamma_i(K_i)} H^\gamma_i(X \cap K_i)
= \frac{1}{H^\gamma_i(K_i)} \sum_{j=1}^{N_i} H^\gamma_i(X \cap f_j(K_i))
= \frac{1}{H^\gamma_i(K_i)} \sum_{j=1}^{N_i} H^\gamma_i(f_j^{-1}(X) \cap K_i)
= \frac{1}{H^\gamma_i(K_i)} \sum_{j=1}^{N_i} \left( \frac{1}{N_i} \right) H^\gamma_i(f_j^{-1}(X) \cap K_i)
= \frac{1}{H^\gamma_i(K_i)} \sum_{j=1}^{N_i} \frac{1}{N_i} H^\gamma_i(f_j^{-1}(X) \cap K_i)
= \sum_{j=1}^{N_i} \frac{1}{N_i} \mu_i \circ f_j^{-1}(X).
$$

By an almost identical argument, it can be shown that $\mu$ satisfies (3).

Finally, we show that $M$ also satisfies (3) and, hence, by the uniqueness of solutions to (3), we conclude that $M$ must be equal to $\mu$. Since $\mu_i$ satisfies (2) for each $1 \leq i \leq d$, we have

$$
M = \prod_{i=1}^{d} \mu_i
= \prod_{i=1}^{d} \left( \sum_{j=1}^{N_i} \frac{1}{N_i} \mu_i \circ f_j^{-1} \right)
= \sum_{j=(j_1,...,j_d) \in \prod_{i=1}^{d} \{1,...,N_i\}} \frac{1}{N} \prod_{i=1}^{d} \mu_i \circ f_j^{-1}
= \sum_{j=1}^{N} \frac{1}{N} M \circ g_j^{-1}.
$$

\[\square\]

4 Mass transference principle for rectangles

To prove Theorem 2, we will use the mass transference principle for rectangles established recently by Wang and Wu in [44]. The work of Wang and Wu generalises the famous Mass Transference Principle originally proved by Beresnevich and Velani [16]. Since its initial discovery in [16], the Mass Transference Principle has found many applications, especially in Diophantine Approximation, and has by now been extended in numerous directions. See
and references therein for further information. Here we shall state the general “full measure” mass transference principle from rectangles to rectangles established by Wang and Wu in [44, Theorem 3.4].

Fix an integer \( d \geq 1 \). For each \( 1 \leq i \leq d \), let \((X, | \cdot |_i, m_i)\) be a bounded locally compact metric space equipped with a \( \delta_i \)-Ahlfors regular probability measure \( m_i \). We consider the product space \((X, | \cdot |, m)\) where

\[
X = \prod_{i=1}^{d} X_i, \quad | \cdot | = \max_{1 \leq i \leq d} | \cdot |_i, \quad \text{and} \quad m = \prod_{i=1}^{d} m_i.
\]

Note that a ball \( B(x, r) \) in \( X \) is the product of balls in \( \{X_i\}_{1 \leq i \leq d} \);

\[
B(x, r) = \prod_{i=1}^{d} B(x_i, r) \quad \text{for} \quad x = (x_1, \ldots, x_d).
\]

Let \( J \) be an infinite countable index set and let \( \beta : J \to \mathbb{R}_{\geq 0} : \alpha \mapsto \beta_\alpha \) be a positive function such that for any \( M > 1 \), the set

\[
\{ \alpha \in J : \beta_\alpha < M \}
\]

is finite. Let \( \rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) be a non-increasing function such that \( \rho(u) \to 0 \) as \( u \to \infty \).

For each \( 1 \leq i \leq d \), let \( \{R_{\alpha,i} : \alpha \in J\} \) be a sequence of subsets of \( X_i \). Then, the resonant sets in \( X \) that we will be concerned with are

\[
\left\{ R_\alpha = \prod_{i=1}^{d} R_{\alpha,i} : \alpha \in J \right\}.
\]

For a vector \( a = (a_1, \ldots, a_d) \in \mathbb{R}_{\geq 0}^d \), write

\[
\Delta(R_\alpha, \rho(\beta_\alpha)^a) = \prod_{i=1}^{d} \Delta(R_{\alpha,i}, \rho(\beta_\alpha)^{a_i}),
\]

where \( \Delta(R_{\alpha,i}, \rho(\beta_\alpha)^{a_i}) \) appearing on the right-hand side denotes the \( \rho(\beta_\alpha)^a \)-neighbourhood of \( R_{\alpha,i} \) in \( X_i \). We call \( \Delta(R_{\alpha,i}, \rho(\beta_\alpha)^{a_i}) \) the part of \( \Delta(R_\alpha, \rho(\beta_\alpha)^a) \) in the \( i \)th direction.

Fix \( a = (a_1, \ldots, a_d) \in \mathbb{R}_{\geq 0}^d \) and suppose \( t = (t_1, \ldots, t_d) \in \mathbb{R}_{\geq 0}^d \). We are interested in the set

\[
W_a(t) = \{ x \in X : x \in \Delta(R_\alpha, \rho(\beta_\alpha)^{a+t}) \quad \text{for i.m.} \quad \alpha \in J \}.
\]

We can think of \( \Delta(R_\alpha, \rho(\beta_\alpha)^{a+t}) \) as a smaller “rectangle” obtained by shrinking the “rectangle” \( \Delta(R_\alpha, \rho(\beta_\alpha)^a) \).

Finally, we require that the resonant sets satisfy a certain \( \kappa \)-scaling property, which in essence ensures that locally our sets behave like affine subspaces.
Definition 1. Let $0 \leq \kappa < 1$. For each $1 \leq i \leq d$, we say that $\{R_{\alpha,i}\}_{\alpha \in J}$ has the $\kappa$-scaling property if for any $\alpha \in J$ and any ball $B(x, r)$ in $X_i$ with centre $x_i \in R_{\alpha,i}$ and radius $r > 0$, for any $0 < \varepsilon < r$, we have

$$c_1r^{\delta_i\kappa}\varepsilon^{\delta_i(1-\kappa)} \leq m_i(B(x_i, r) \cap \Delta(R_{\alpha,i}, \varepsilon)) \leq c_2r^{\delta_i\kappa}\varepsilon^{\delta_i(1-\kappa)}$$

for some absolute constants $c_1, c_2 > 0$.

In our case $\kappa = 0$ since our resonant sets are points. For justification of this, and calculations of $\kappa$ for other resonant sets, see [2]. Wang and Wu established the following mass transference principle for rectangles in [44].

Theorem 4 (Wang – Wu, [44]). Assume that for each $1 \leq i \leq d$, the measure $m_i$ is $\delta_i$-Ahlfors regular and that the resonant set $R_{\alpha,i}$ has the $\kappa$-scaling property for $\alpha \in J$. Suppose

$$m\left(\limsup_{\beta \to \infty} \Delta(R_{\alpha}, \rho(\beta_{\alpha})^a)\right) = m(X).$$

Then we have

$$\dim_H W_a(t) \geq s(t) := \min_{A \in \mathcal{A}} \left\{ \sum_{k \in \mathcal{K}_1} \delta_k + \sum_{k \in \mathcal{K}_2} \delta_k + \kappa \sum_{k \in \mathcal{K}_3} \delta_k + (1 - \kappa) \sum_{k \in \mathcal{K}_2} a_k \delta_k - \sum_{k \in \mathcal{K}_2} t_k \delta_k \right\},$$

where

$$\mathcal{A} = \{a_i, a_i + t_i : 1 \leq i \leq d\}$$

and for each $A \in \mathcal{A}$, the sets $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ are defined as

$$\mathcal{K}_1 = \{k : a_k \geq A\}, \quad \mathcal{K}_2 = \{k : a_k + t_k \leq A\} \setminus \mathcal{K}_1, \quad \mathcal{K}_3 = \{1, \ldots, d\} \setminus (\mathcal{K}_1 \cup \mathcal{K}_2)$$

and thus give a partition of $\{1, \ldots, d\}$.

5 Proof of Theorem 2

To prove Theorem 2, we will apply Theorem 4 with $X_i = K_i$, $m_i = \mu_i$ and $|\cdot| = |\cdot|$ (absolute value in $\mathbb{R}$) for each $1 \leq i \leq d$. Then, in our setting, we will be interested in the product space $(X, \|\cdot\|, M)$ where

$$X = \prod_{i=1}^d K_i = K, \quad M = \prod_{i=1}^d \mu_i,$$

and $\|\cdot\|$ denotes the supremum norm in $\mathbb{R}^d$. Recall that for each $1 \leq i \leq d$, the measure $\mu_i$ is $\delta_i$-Ahlfors regular with

$$\delta_i = \gamma_i = \dim_H K_i$$

and the measure $M$ is $\delta$-Ahlfors regular with

$$\delta = \gamma = \dim_H K.$$
For us, the appropriate indexing set is
\[ J = \{ i \in \Lambda^* \}. \]

We define our weight function \( \beta : \Lambda^* \to \mathbb{R}_{\geq 0} \) by
\[ \beta_{|i|} = \beta(i) = |i|. \]
Note that \( \beta \) satisfies the requirement that for any real number \( M > 1 \) the set \( \{ i \in \Lambda^* : \beta_i < M \} \) is finite. Next we define \( \rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) by
\[ \rho(u) = \psi(b^u). \]
Since \( b^n \psi(b^n) \) is monotonically decreasing by assumption, it follows that \( \psi(b^n) \) is monotonically decreasing and \( \psi(b^n) \to 0 \) as \( n \to \infty \).

For a fixed \( x = (x_1, \ldots, x_d) \in K \), we define the resonant sets of interest as follows. For each \( i \in \Lambda^* \), take
\[ R^x_i = g_i(x). \]
Correspondingly, for each \( 1 \leq j \leq d \),
\[ R^x_{i,j} = g_i(x)_j, \]
where \( g_i(x) = (g_i(x)_1, \ldots, g_i(x)_d) \). So, \( R^x_{i,j} \) is the coordinate of \( g_i(x) \) in the \( j \)th direction. In each coordinate direction, the \( \kappa \)-scaling property is satisfied with \( \kappa = 0 \), since our resonant sets are points.

Let us fix \( a = (1, 1, \ldots, 1) \in \mathbb{R}^d_{\geq 0} \). Then, in this case, we note that
\[ \limsup_{\alpha \in J} \Delta(R^x_{\alpha}, \rho(\beta_{\alpha})^a) = \limsup_{i \in \Lambda^*} \Delta(g_i(x), \psi(b^{|i|})^a) = W(x, \psi), \]
where \( W(x, \psi) \) is as defined in (1). Moreover, it follows from Corollary 2 and Proposition 1 that \( M(W(x, \psi)) = M(K) \), since we assumed that \( \sum_{n=1}^{\infty} (b^n \psi(b^n))^\gamma = \infty \).

Now suppose that \( t = (t_1, \ldots, t_d) \in \mathbb{R}^d_{\geq 0} \). Then, in our case,
\[ W_a(t) = W(x, \psi, t), \]
which is the set we are interested in. So, recalling that \( \kappa = 0 \) in our setting, we may now apply Theorem 4 directly to conclude that
\[ \dim_H W(x, \psi, t) \geq \min_{A \in A} \left\{ \sum_{k \in \mathcal{K}_1} \delta_k + \sum_{k \in \mathcal{K}_2} \delta_k + \sum_{k \in \mathcal{K}_3} \delta_k - \sum_{k \in \mathcal{K}_2} t_k \delta_k \right\} =: s(t), \]
where
\[ A = \{ 1 \} \cup \{ 1 + t_i : 1 \leq i \leq d \} \]
and for each \( A \in A \) the sets \( \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \) are defined as follows:
\[ \mathcal{K}_1 = \{ k : 1 \geq A \}, \quad \mathcal{K}_2 = \{ k : 1 + t_k \leq A \} \setminus \mathcal{K}_1, \quad \text{and} \quad \mathcal{K}_3 = \{ 1, \ldots, d \} \setminus (\mathcal{K}_1 \cup \mathcal{K}_2). \]
Note that \( K_1, K_2, K_3 \) give a partition of \( \{1, \ldots, d\} \).

To obtain a neater expression for \( s(t) \), as given in the statement of Theorem 2, we consider the possible cases which may arise. To this end, let us suppose, without loss of generality, that

\[
0 < t_{i_1} \leq t_{i_2} \leq \cdots \leq t_{i_d}.
\]

**Case 1: \( A = 1 \)**

If \( A = 1 \), then \( K_1 = \{1, \ldots, d\} \), \( K_2 = \emptyset \), and \( K_3 = \emptyset \). In this case, the “dimension number” simplifies to

\[
\sum_{j=1}^{d} \delta_j = \sum_{j=1}^{d} \dim H_{K_j} = \sum_{j=1}^{d} \log N_j \log b = \log \left( \prod_{j=1}^{d} N_j \right) \log b = \dim H K.
\]

**Case 2: \( A = 1 + t_{i_k} \) with \( t_{i_k} > 0 \)**

Suppose \( A = 1 + t_{i_k} \) for some \( 1 \leq k \leq d \) and that \( t_{i_k} > 0 \) (otherwise we are in Case 1). Suppose \( k \leq k' \leq d \) is the maximal index such that \( t_{i_k} = t_{i_{k'}} \). In this case,

\[
K_1 = \emptyset, \quad K_2 = \{i_1, \ldots, i_{k'}\}, \quad \text{and} \quad K_3 = \{i_{k'+1}, \ldots, i_d\}
\]

and the “dimension number” is

\[
\sum_{j=1}^{k'} \delta_{i_j} + \sum_{j=k'+1}^{d} \delta_{i_j} - \sum_{j=1}^{k'} \delta_{i_j} \cdot \frac{t_{i_j}}{1 + t_{i_k}} = \frac{1}{1 + t_{i_k}} \left( \left( 1 + t_{i_k} \right) \sum_{j=1}^{k'} \delta_{i_j} + \sum_{j=k'+1}^{d} \delta_{i_j} - \sum_{j=1}^{k'} \delta_{i_j} \cdot t_{i_k} \delta_{i_j} \right)
\]

\[
= \frac{1}{1 + t_{i_k}} \left( \sum_{j=1}^{d} \delta_{i_j} + \sum_{j=1}^{k'} \delta_{i_j} \left( t_{i_k} - t_{i_j} \right) \right)
\]

\[
= \frac{1}{1 + t_{i_k}} \left( \dim H K + \sum_{j=1}^{k'} \left( t_{i_k} - t_{i_j} \right) \dim H K_{i_j} \right).
\]

Putting the two cases together, we conclude that

\[
\dim H W(x, \psi, t) \geq \min_{1 \leq k \leq d} \left\{ \frac{1}{1 + t_k} \left( \gamma + \sum_{j : t_j \leq t_k} (t_k - t_j) \gamma_j \right) \right\},
\]

as claimed. This completes the proof of Theorem 2.

### 6 Proof of Theorem 3

Let

\[
\mathcal{A}_n(x, \psi, t) := \bigcup_{i \in \Lambda^n} \Delta (R_i^x, \psi(b^n)^{1+t}) = \bigcup_{i \in \Lambda^n} \prod_{j=1}^{d} B \left( R_i^x, \psi(b^n)^{1+t} \right).
\]
Then

\[ W(x, \psi, t) = \limsup_{n \to \infty} A_n(x, \psi, t). \]

For any \( m \in \mathbb{N} \) we have that

\[ W(x, \psi, t) \subset \bigcup_{n \geq m} A_n(x, \psi, t). \tag{4} \]

Observe that \( A_n(x, \psi, t) \) is a collection of \( N^n = (b^n)^\gamma \) rectangles with sidelengths \( 2\psi(b^n)^{1+t_j} \) in each \( j \)th coordinate axis.

Fix some \( 1 \leq k \leq d \). Throughout suppose that \( n \) is sufficiently large such that \( \psi(b^n) < 1 \). Condition (i) of Theorem 3 implies that \( \psi(b^n)^{1+t_k} \leq \psi(b^n) \to 0 \) as \( n \to \infty \), and so for any \( \rho > 0 \) there exists a sufficiently large positive integer \( n_0(\rho) \) such that

\[ \psi(b^n)^{1+t_k} \leq \rho \quad \text{for all } n \geq n_0(\rho). \]

Suppose \( n \geq n_0(\rho) \) and that for each \( 1 \leq j \leq d \) we can construct an efficient finite \( \psi(b^n)^{1+t_k} \)-cover \( B_j(i, k, \rho) \) for \( B \left( R_{i,j}^x, \psi(b^n)^{1+t_j} \right) \) with cardinality \( \#B_j(i, k, \rho) \) for each \( i \in \Lambda^n \). Then we can construct a \( \psi(b^n)^{1+t_k} \)-cover of \( \Delta \left( R_{i}^x, \psi(b^n)^{1+t} \right) \) for each \( i \in \Lambda^n \) with cardinality \( \prod_{j=1}^d \#B_j(i, k, \rho) \) by considering the Cartesian product of the individual covers \( B_j(i, k, \rho) \) for each \( 1 \leq j \leq d \). By (4)

\[ \bigcup_{n \geq n_0(\rho)} A_n(x, \psi, t) \tag{5} \]

is a cover of \( W(x, \psi, t) \). So, supposing that we can find such covers \( B_j(i, k, \rho) \), we have that

\[ \bigcup_{n \geq n_0(\rho)} \bigcup_{i \in \Lambda^n} \prod_{j=1}^d B_j(i, k, \rho) \]

is a \( \psi(b^n)^{1+t_k} \)-cover of \( W(x, \psi, t) \).

To calculate the values \( \#B_j(i, k, \rho) \) we consider two possible cases depending on the fixed \( 1 \leq k \leq d \). Without loss of generality suppose that \( 0 < t_1 \leq t_2 \leq \cdots \leq t_d \). Then, since we are assuming that \( \psi(b^n) < 1 \), we have that \( \psi(b^n)^{1+t_1} \geq \cdots \geq \psi(b^n)^{1+t_d} \).

**Case 1: \( t_j \geq t_k \)**

In this case, \( \psi(b^n)^{1+t_k} \geq \psi(b^n)^{1+t_j} \) and so, for any \( i \in \Lambda^n \), we have

\[ B \left( R_{i,j}^x, \psi(b^n)^{1+t_k} \right) \supset B \left( R_{i,j}^x, \psi(b^n)^{1+t_j} \right). \]

Hence, we may take our covers to be \( B(i, k, \rho) = B \left( R_{i,j}^x, \psi(b^n)^{1+t_k} \right) \), and so \( \#B_j(i, k, \rho) = 1 \).

**Case 2: \( t_j < t_k \)**

In this case, \( \psi(b^n)^{1+t_k} < 2\psi(b^n)^{1+t_j} \). Let \( u \in \mathbb{N} \) be the unique integer such that

\[ b^{-u} \leq 2\psi(b^n)^{1+t_j} < b^{-u+1}, \tag{6} \]
and observe that, for any \( i \in \Lambda^n \), we have

\[
B \left( R_{i,j}^x, \psi(b^{|i|}1^{+t_j}) \right) \subset \bigcup_{a=(a_1,\ldots,a_{u-1}) \in \Lambda^{u-1}_j, f_a \in \Phi^i, 1 \leq i \leq u-1} f_a([0,1]),
\]

where \( \Lambda_j = \{1, \ldots, N_j\} \). Let \( A \) denote the set of \( a \in \Lambda^{u-1}_j \) such that

\[
f_a([0,1]) \cap B \left( R_{i,j}^x, \psi(b^n)^{1+t_j} \right) \neq \emptyset.
\]

Note by the definition of \( u \), and the fact that the mappings \( f_a \) of the same length are pairwise disjoint up to possibly a single point of intersection, that \( \#A \leq 2 \) since

\[
diam \left( f_a([0,1]) \right) = b^{-u-1} > diam \left( B \left( R_{i,j}^x, \psi(b^n)^{1+t_j} \right) \right).
\]

Observe that \( f_b([0,1]) \subset f_a([0,1]) \) if and only if \( b = ac \) for \( c \in \Lambda^*_j := \bigcup_{n=0}^{\infty} \Lambda^n_j \), where we write \( ac \) to denote the concatenation of the two words \( a \) and \( c \). Let \( v \geq 0 \) be the unique integer such that

\[
b^{-u-v} \leq \psi(b^n)^{1+t_k} < b^{-u-v+1}.
\]

Note that \( v \) is well defined since \( \psi(b^n)^{1+t_k} < 2\psi(b^n)^{1+t_j} < b^{-u+1} \), and so \( v \geq 0 \). Then

\[
\bigcup_{a \in A, c \in \Lambda^n_j} f_{ac}([0,1]) \supset B \left( R_{i,j}^x, \psi(b^{|i|}1^{+t_j}) \right).
\]

Notice that the left-hand side above gives rise to a \( \psi(b^n)^{1+t_k} \)-cover for the right-hand side and let us denote this cover by \( B_j(i,k,\rho) \). By the above arguments an easy upper bound on \( \#B_j(i,k,\rho) \) is seen to be \( 2N^n_j \). Furthermore, by (6) and (7) we have that

\[
\#B_j(i,k,\rho) \leq 2N^n_j = 2(b^n)^{\gamma_j} \leq 2(1+b^{-u}a\psi(b^n)^{1-t_k})^{\gamma_j} \leq 2^{1+\gamma_j}b^{\gamma_j}a\psi(b^n)^{t_j-t_k}\gamma_j.
\]

Summing over \( 1 \leq j \leq d \) and \( i \in \Lambda^n \) for each \( n \geq n_0(\rho) \) we see that

\[
H^\rho_{\rho}(W(x,\psi,t)) \ll \sum_{n \geq n_0(\rho)} \left( (\psi(b^n)^{1+t_k})^s \times \sum_{i \in \Lambda^n} \#B_j(i,k,\rho) \right)
\]

\[
\ll \sum_{n \geq n_0(\rho)} \left( (\psi(b^n)^{1+t_k})^s \times N^n \prod_{j:t_j < t_k} b^{\gamma_j} \psi(b^n)^{(t_j-t_k)\gamma_j} \right)
\]

\[
\ll \sum_{n \geq n_0(\rho)} \psi(b^n)^{s(1+t_k) + \sum_{j:t_j < k} (t_j-t_k)\gamma_j - \gamma} \psi(b^n)^{b^n} \gamma^\gamma.
\]

Thus, it follows from condition (iii) in Theorem 3 that for any

\[
s \geq s_0 = \frac{\gamma + \sum_{j:t_j < k} (t_k - t_j)\gamma_j + \delta \gamma}{1 + t_k}
\]

with \( \delta > 0 \),
we have
\[ \mathcal{H}^s_\rho(W(x, \psi, t)) \to 0 \quad \text{as } \rho \to 0. \]
This implies that \( \dim_H W(x, \psi, t) \leq s_0 \). The above argument holds for any initial choice of \( k \), and so we conclude that
\[
\dim_H W(x, \psi, t) \leq \min_{1 \leq k \leq d} \left\{ \frac{1}{1 + t_k} \left( \gamma + \sum_{j: t_j < t_k} (t_k - t_j) \gamma_j \right) \right\}.
\]
Combining this upper bound with the lower bound result from Theorem 2 completes the proof of Theorem 3.

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