The universal conservative superalgebra

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Abstract. We introduce the class of conservative superalgebras, in particular, the superalgebra $U(V)$ of bilinear operations on a superspace $V$. Moreover, we show that each conservative superalgebra modulo its maximal Jacobian ideal is embedded into $U(V)$ for a certain superspace $V$.

Keywords: superalgebra, Kantor product, conservative algebra.

MSC2010: 17A30, 17D99

1 Introduction

I. Kantor introduced the class of conservative algebras in [4]. This class includes some well-known classes of algebras, such as associative, Jordan, Lie, Leibniz, and Zinbiel algebras [10].

To define conservative algebras we firstly introduce some notations. Let $V$ be a vector space over a field $F$, let $\varphi$ be a linear map on $V$, and let $B$ be a bilinear map on $V$ (i.e., an algebra). Then we can consider a product of $\varphi$ and $B$, which is a bilinear map $[\varphi, B]$ on $V$ given by

$$[\varphi, B](x, y) = \varphi(B(x, y)) - B(\varphi(x), y) - B(x, \varphi(y)).$$

(1)

Note that this product measures how far is $\varphi$ from being a derivation of the algebra $B$. The relation (1) may be also considered as a transformation of a bilinear operator $B$ under the action of an infinitesimal transformation $x \mapsto x + t\varphi(x)$. Indeed, the right-hand side of (1) is the coefficient at the first degree of $t$ in the series $e^{\varphi t}(B(e^{-\varphi t}(x), e^{-\varphi t}(y)))$. Thus, $\{[L_a, B] : a \in V\}$ is the set of all algebras which arise from the initial algebra $B$ by the...
action of left shifts $L_a, a \in V$. Thus, the definition of a conservative algebra given below says that this set (for every conservative algebra $B$) is transformed into itself under other actions of the left shifts $L_a, a \in V$.

An algebra $A$ with a multiplication $\cdot$ is called a (left) conservative algebra if there exists an algebra structure $\ast$ (called an associated algebra) on the underlying space of $A$ such that

$$[L_b, [L_a, \cdot]] = -[L_{a \ast b}, \cdot];$$

here, as usual, $L_a$ stands for the operator of left multiplication by $a$: $L_a(x) := a \cdot x := ax$ for all $x \in A$ (in what follows, the symbol $:= $ denotes an equality by definition; and $(\mathcal{V})$ stands for the linear span of a set $\mathcal{V}$ over the ground field). Replacing the left multiplications with the right multiplications and modifying correspondingly the above relation, we can define right conservative algebras and obtain a similar theory. The associative and Lie algebras give obvious examples of conservative algebras (see it further for the supercase).

In the theory of conservative algebras, the algebra $U(n)$, which was introduced in [8], is of great importance. Let $V_n$ be an $n$-dimensional vector space over $F$. The space of the algebra $U(n)$ is the space of all bilinear operations on $V_n$ (further we identify a bilinear operation $A \in U(n)$ with the algebra structure that it defines on $V_n$, and do the same in the supercase). To define a product $\triangle$ on $U(n)$ we fix a nonzero vector $u \in V_n$. Then for $A, B \in U(n)$ we put

$$(A \triangle_u B)(x, y) = [L_u^A, B](x, y) = A(u, B(x, y)) - B(A(u, x), y) - B(x, A(u, y)),$$

(2)

where $L_u^A : x \mapsto A(u, x)$ is the left multiplication with respect to $A$. This product is called the Kantor product of $A$ and $B$. One can easily check that the algebras obtained by different choices of nonzero $u \in V_n$ are isomorphic (see the proof for the superalgebra case below).

Note also that (1) gives a Lie action of $gl_n$ on $U(n)$, since it coincides with the natural action of $gl_n$ on $U(n) = V_n^* \otimes V_n^* \otimes V_n$.

One can verify that the algebra $U(n)$ is conservative with the associated multiplication $\nabla$ given, for example, by $A \nabla_u B(x, y) = -B(u, A(x, y))$ (there are other associated multiplications as well). In the theory of conservative algebras, the algebra $U(n)$ plays a role analogous to the role of $gl_n$ in the theory of Lie algebras, that is, every finite-dimensional conservative algebra (modulo its maximal Jacobian ideal) may be embedded into $U(n)$ for some $n$. Some properties of the algebra $U(2)$ were studied in [10, 11].

The Kantor product of a multiplication by itself is called its Kantor square. It gives a map $K$ from any variety $\mathcal{V}$ of algebras to some class of algebras $K(\mathcal{V})$. The Kantor squares of multiplications satisfying certain conditions (such as the associativity, the Jacobi identity and others) were studied in [9].

The main aim of this paper is to introduce the conservative superalgebra $U(n, m)$, which is the super-counterpart of the algebra $U(n)$, and prove that every finite-dimensional conservative superalgebra is embedded (modulo its maximal Jacobian ideal) into $U(n, m)$ for some non-negative integer $n, m$.

## 2 The Conservative Superalgebras

In this section we introduce the class of conservative superalgebras, whose definition is a complete analogue of the notion of a conservative algebra given in [4, 8], and we also prove some of their elementary properties, which are used further.
2.1 Notation and Main Definitions

As usual, all (super)spaces and (super)algebras are considered over a field $\mathbb{F}$. Algebras and superalgebras are in general assumed to be nonassociative, noncommutative, and without unity. Let $A = A_0 \oplus A_1$ be a superalgebra (i.e., it is a $\mathbb{Z}_2$-graded algebra: $A_0, A_1 \subseteq A_{1+\overline{1}}$, the elements in $A_0$ are even, and the elements in $A_1$ are odd), $(-1)^{xy} := (-1)^{p(x)p(y)}$, where $p(x)$ is the parity of $x$, that is, $p(x) = i$ if $x \in A_i$. The elements in $A_i$ are homogeneous. We use the same notation for a homogeneous operator $\varphi$: $(-1)^{\varphi} := (-1)^{p(\varphi)}$ (here $p(\varphi)$ is the parity of $\varphi$: $\varphi(A_i) \subseteq A_{1+\overline{p(\varphi)}}$), and so on. In what follows, if the parity of an element (operator) appears in a formula, then this element (operator) is assumed to be homogeneous. Sometimes we simply write “subspace” instead of “subsuperspace”, “subalgebra” instead of “subsuperalgebra”, and so on. The ideals are assumed to be homogeneous, i.e., an ideal $I$ contains with every element $x = x_0 + x_1 \in I$ its even and odd components $x_i \in I \cap A_i$, $i = 1, 2$.

Following [4, 8], we define some supercommutators. Let $V$ be a vector superspace, let $A$ be a linear operator on $V$, and let $B$ and $C$ be bilinear operators on $V$. For all $x, y, z \in V$, put

$$[A, x] = A(x), \ [B, x](y) = B(x, y),$$

$$[A, B](x, y) = A(B(x, y)) - (-1)^{B\mathcal{A}}B(A(x), y) - (-1)^{A\mathcal{B}+\mathcal{X}}B(x, A(y)), \hspace{1cm} (3)$$

$$[B, C](x, y, z) = B(C(x, y), z) + (-1)^{C\mathcal{B}}B(x, C(y, z)) + (-1)^{y(C\mathcal{X}+\mathcal{Z})}B(y, C(x, z))$$

$$-(-1)^{BC\mathcal{Z}}C(B(x, y), z) - (-1)^{B(C\mathcal{Z}+\mathcal{X})}C(x, B(y, z)) - (-1)^{CB+\mathcal{X}y\mathcal{Z}B\mathcal{Y}}C(y, B(x, z)). \hspace{1cm} (4)$$

By definition we put $[X, Y] = (-1)^{XY}[Y, X]$ for all $X, Y \in \{a, A, B\}$.

Denote by $\mathcal{U}(V)$ the space of all bilinear operations on $V$. Then one can verify that $[4]$ defines an action of $\mathfrak{gl}(V)$ on $\mathcal{U}(V)$.

Let $M(x, y) = xy$ be an even bilinear operation that defines a superalgebra structure on $V$. Let $L_a$ be as above. We say that $M$ is a conservative superalgebra if there exists an even superalgebra $M^*(x, y) = x \ast y$ (called the associated superalgebra) on the same space such that

$$[L_b, [L_a, M]] = -(-1)^{ab}[L_{ab+b}, M] \hspace{1cm} (5)$$

for all $a, b \in V$.

The above relation can be written explicitly as an identity of degree 4 with respect to the multiplications $M$ and $M^*$:

$$b(ab(xyz)) - b((ax)y) - (-1)^{ax}b(xy) = (-1)^{ab}\cdot a((bx)y) + (-1)^{ab}(a(bx))y +$$

$$(-1)^{ax}(bx)(ay) - (-1)^{a+x}\cdot a((xby)) + (-1)^{a+x}(a(xy))+(a(by)) + (-1)^{ab}\cdot x(a(by)) =$$

$$-(-1)^{ab}(a * b)(xy) + (-1)^{ab}(a * b)(xy) + (-1)^{ab}((a * b)x)y + (-1)^{x(ab+b)+ab}x((a * b)y).$$

One may also use the general approach to define the conservative superalgebras. Namely, let $\Gamma := \Gamma_0 \oplus \Gamma_1$ be the Grassmann superalgebra in generators 1, $\xi_i, \ i \in \mathbb{N}, \Gamma_0 = \{1, \xi_1, \ldots, \xi_{2k} : k \in \mathbb{N}\}$, $\Gamma_1 = \{\xi_1, \ldots, \xi_{2k-1} : k \in \mathbb{N}\}$. Let $\mathcal{A} := \mathcal{A}_0 \oplus \mathcal{A}_1$ be a superalgebra and $\cdot$ and $\ast$ be two products on $\mathcal{A}$. Consider its Grassmann enveloping $\Gamma(\mathcal{A}) := (\mathcal{A}_0 \otimes \Gamma_0) \oplus (\mathcal{A}_1 \otimes \Gamma_1)$, and extend the products $\cdot$ and $\ast$ to $\Gamma(\mathcal{A})$ as follows:

$$(a \otimes f) \cdot (b \otimes g) = (-1)^{ab}ab \otimes fg,$$

$$(a \otimes f) \ast (b \otimes g) = (-1)^{ab}a \ast b \otimes fg$$

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for all homogeneous \( a, b \in A, f, g \in \Gamma \) \((p(a) = p(f), p(b) = p(g))\). Then \((A, \cdot)\) is conservative with an associated multiplication \(*\) if and only if \((\Gamma(A), \cdot)\) is a conservative algebra with an associated multiplication \(*\).

2.2 Examples

The Lie superalgebras give obvious examples of conservative superalgebras. Indeed, let \( L \) be a Lie superalgebra with a product \( M \). Then the Jacobi identity and the anticommutativity imply that \([L_a, M] = 0\) for all \( a \in L \). Thus, the left and right-hand sides of (5) are zero for arbitrary product \( M^* \) on \( L \). As another example we have associative superalgebras. In this case

\[
[L_a, M](x, y) = -(-1)^{ax} xay; \quad [L_b, [L_a, M]](x, y) = (-1)^{b(a+x)} xaby,
\]

and (5) holds with \( x \ast y := xy \).

A linear space \( U \) with a bilinear operation \( M : U \times U \mapsto U \) is called a terminal algebra provided that

\[
[[[M, a], M], M] = 0
\]

for every \( a \in U \). Note that (6) is an identity of degree 4, therefore, the class of terminal algebras is a variety. The class of terminal algebras is vast: it includes, for example, Jordan algebras, Lie algebras and (left) Leibniz algebras. In [3, 4] it was shown that the commutative algebras satisfying (6) are Jordan algebras. Assuming that in (6) we have the supercommutators, we arrive at the definition of terminal superalgebra. Passing to the supercase we see that the commutative superalgebras satisfying (6) are Jordan superalgebras.

The following result can be obtained by a direct computation:

**Proposition 1.** Let \( \text{char} \mathbb{F} \neq 3 \). An algebra \( M \) is terminal if and only if it is conservative and the multiplication in the associated superalgebra \( M^* \) can be defined by

\[
M^*(x, y) = \frac{2}{3}xy + \frac{1}{3}yx.
\]

The following theorem provides us with different examples of conservative superalgebras.

**Theorem 2.** Let \( \mathcal{V} \) be a homogeneous variety of algebras. Assume that there exist \( \alpha, \beta \in \mathbb{F} \) such that every \( \mathcal{V} \)-algebra is conservative with the associated multiplication given by the rule \( a \ast b = \alpha ab + \beta ba \). Then every \( \mathcal{V} \)-superalgebra is conservative with the associated multiplication \( a \ast b = \alpha ab + (-1)^{ab} \beta ba \).

**Proof.** Let \( M \) be a \( \mathcal{V} \)-superalgebra. Consider the Grassmann enveloping \( \Gamma(M) \), which is an algebra in \( \mathcal{V} \). By our assumptions, \( \Gamma(M) \) is a conservative algebra, and the multiplication in the associated algebra \( \Gamma(M)^* \) can be defined by the formula \( \Gamma(M)^*(x, y) = \alpha xy + \beta yx \) for all \( x, y \in \Gamma(M) \). This multiplication is obviously induced by a multiplication \( M^* \) on the space of \( M \) which is given by \( M^*(x, y) = \alpha xy + (-1)^{xy} \beta yx \). Therefore, by the general approach above, \( M \) is a conservative superalgebra with an associated multiplication \( M^* \).

It follows that associative, quasi-associative, Jordan, terminal, Lie, Leibniz, and Zinbiel superalgebras are conservative (see [10]). In particular, a superalgebra \( M \) is terminal if and
only if it is conservative and the multiplication in the associated superalgebra $M^*$ can be given by

$$M^*(x, y) = \frac{2}{3}xy + (-1)^{xy} \frac{1}{3}yx. \quad (7)$$

As we have seen, associative and Jordan superalgebras are conservative with the associated multiplication $M = M^*$. It is natural to ask what is the subclass of conservative superalgebras with this additional restriction.

A superalgebra $U$ is called a noncommutative Jordan superalgebra if $U$ is flexible (that is, the operator identity $[R_x, L_y] = [L_x, R_y]$ holds in $U$) and its symmetrized superalgebra (the algebra on the space of $U$ with the multiplication $x \circ y = \frac{1}{2}(xy + (-1)^{xy}yx)$) is Jordan. For more information on noncommutative Jordan superalgebras see [13, 12] and references therein.

**Proposition 3.** A flexible conservative superalgebra with the product $M$ whose associated superalgebra has the same product $M^* = M$ is a noncommutative Jordan superalgebra.

**Proof.** Let $A$ be such flexible conservative superalgebra. Then $\Gamma(A)$ is a flexible conservative algebra with the extended products $M = M^*$. By [4, Proposition 1], $\Gamma(A)$ is a noncommutative Jordan algebra. Therefore, by the general definition of a superalgebra of a given variety $V$, $A$ is a noncommutative Jordan superalgebra. \qed

**Proposition 4.** A conservative superalgebra $M$ with a unity is a noncommutative Jordan superalgebra.

**Proof.** Let $A$ be a conservative superalgebra with a unity. Then $\Gamma(A)$ is a conservative algebra with a unity. By [4, Proposition 2], $\Gamma(A)$ is a noncommutative Jordan algebra. Therefore, $A$ is a noncommutative Jordan superalgebra. \qed

### 2.3 Operator Superalgebras

Let $M$ be a conservative superalgebra on an underlying vector space $V$. Considering both parts of (5) as operators acting on $y \in V$, we obtain the following operator relation:

$$[L_b, [L_a, L_x]] - [L_b, L_{ax}] - (-1)^{ab}[L_a, L_{bx}] + (-1)^{ab}L_{a(bx)} + (-1)^{ab}[L_{a+b}, L_x] - (-1)^{ab}L_{(a+b)x} = 0.$$

Therefore, $U_0(V) := \langle L_a, [L_a, L_b] : a, b \in V \rangle \leq \mathfrak{gl}(V)$. Moreover, since [4] gives an action of the Lie superalgebra $\mathfrak{gl}(V)$ on $U(V)$, we immediately get

$$[[L_b, L_a], M] = [L_{b+a-(-1)^{ab}a+b}, M],$$

which implies that $U_1(V) := \langle M, [L_a, M] : a \in V \rangle$ is a $U_0(V)$-submodule of $U(V)$. This also implies that the operators $[L_b, L_a] - (-1)^{ab}L_{b+a-(-1)^{ab}a+b}, a, b \in V$, are superderivations of $M$.

**Corollary 5.** Let $M$ be a terminal superalgebra. The linear transformations $[L_a, L_b] - \frac{1}{3}L_{[a,b]}$ are superderivations of $M$ for all $a, b \in M$. \qed

**Remark.** Let $M$ be a (super)algebra such that $U_1(M)$ is a $U_0(M)$-submodule of $U_1(M)$. Then $M$ is called rigid or quasi-conservative [7]. In [2], the simple linearly compact rigid commutative and anticommutative superalgebras over an algebraically closed field of characteristic 0 were classified, and in [1] the 2-dimensional rigid algebras were classified.
2.4 Jacobi Elements and Quasiunities

An element \( a \) in a superalgebra \( M \) is called a \textit{Jacobi element} provided that
\[
a(xy) = (ax)y + (-1)^{ax}x(ay)
\]
holds for all \( x, y \in M \).

In other words, \( a \) is a Jacobi element if \( L_a \) is a superderivation of \( M \). The relation (8) can be rewritten in the following forms:
\[
[L_a, L_x] = L_{ax} \quad \text{for every } x \in M,
\]
\[
[L_a, M] = 0.
\]

Denote by \( J \) the space of all Jacobi elements of a superalgebra \( M \). Let \( N := \{ a \in M : L_a = 0 \} \) be the \textit{left annihilator} of \( M \). Obviously, \( N \subseteq J \). An ideal \( I \) of \( M \) is called a \textit{Jacobi ideal} provided that \( I \subseteq J \).

The following statement is immediate from the definitions and (9).

\textbf{Lemma 6.} Let \( M \) be a superalgebra, and let \( J \) and \( N \) be as above. Then \( J \) is a subsuperalgebra of \( M \); \( N \) is an ideal of \( J \), and the quotient superalgebra \( J/N \) is isomorphic to a subsuperalgebra of the Lie superalgebra of derivations of \( M \). If \( M \) possesses a unity then \( J = 0 \); and if \( M \) is a Lie superalgebra then \( J = M \).

An even element \( e \in M \) is said to be a \textit{left quasiunity} if the equality
\[
e(xy) = (ex)y + x(ey) - xy
\]
holds for all \( x, y \in M \). This condition is equivalent to the relations
\[
[L_e, L_x] = L_{ex-x} \quad \text{for every } x \in M,
\]
\[
[L_e, M] = -M.
\]

Obviously, a left unity is a left quasiunity. But, in general, the converse is not true (see examples in the next section). The following theorem is proved similarly to [8, Theorem 1], so we only give an outline of the proof.

\textbf{Theorem 7.} Let \( M \) be a conservative superalgebra. The associated superalgebra \( M^* \) is defined up to an arbitrary superalgebra with values in \( J \). Moreover, the following relations hold:
\[
M^*(a, b) \equiv 0 \pmod{J}, \quad a \in J,
\]
\[
M^*(a, b) \equiv -(-1)^{ab}ba \pmod{J}, \quad b \in J.
\]

If \( M \) has a left quasiunity \( e \), then
\[
M^*(e, a) \equiv a, \quad M^*(a, e) \equiv 2a - ea \pmod{J}.
\]
Proof. The first statement and (13) are immediate by (5) and (10). To prove (14) it suffices to show that
\[
[L_b, [L_a, M]] = [L_{ba}, M]
\]
for all \( a \in M, b \in J \). It follows easily from (9) and (10) that
\[
[L_b, [L_a, M]] = [[L_b, L_a], M] + (-1)^{ab}[L_a, [L_b, M]] = [L_{ba}, M].
\]
The first of the equations (15) follows from (5) and (12). The second is proved by analogy with (14): using (11) and (12) we get
\[
[L_e, [L_a, M]] = [L_{ea-2a}, M].
\]
\[\blacksquare\]

3 The Universal Conservative Superalgebra

In this section we define the superalgebra structure on the space \( \mathcal{U}(V) \) of all bilinear operations on a superspace \( V \) and prove that it is conservative. Moreover, we show that every finite-dimensional conservative superalgebra is embedded (modulo its maximal Jacobi ideal) in \( \mathcal{U}(V) \) for a certain finite-dimensional space \( V \).

3.1 The superalgebra \( \mathcal{U}(V) \)

Let \( V \) be a superspace. The space of the superalgebra \( \mathcal{U}(V) \) is the superspace of all bilinear operations on \( V \). Fix a nonzero homogeneous \( a \in V \). Define the multiplication \( \triangle_a \) in \( \mathcal{U}(V) \) by the rule
\[
(A \triangle_a B)(x, y) = A(a, B(x, y)) - (-1)^{B(A+a)}B(A(a, x), y) - (-1)^{(A+a)(B+a)}B(x, A(a, y)).
\]

Consider the natural action of the group \( \text{gl}(V) \) of even automorphisms of \( V \) on \( \mathcal{U}(V) \) :
\[
\varphi(A)(x, y) = \varphi(A(\varphi^{-1}(x), \varphi^{-1}(y)))
\]
(we denote an automorphism and its action by the same symbol \( \varphi \)). A direct computation shows that the mapping \( A \mapsto \varphi(A) \) is an isomorphism between \( (\mathcal{U}(V), \triangle_a) \) and \( (\mathcal{U}(V), \triangle_{\varphi(a)}) \). Therefore, different nonzero even (respectively, odd) vectors \( a \) give rise to isomorphic even (respectively, odd) superalgebras, which we denote \( \mathcal{U}(V)^0 \) and \( \mathcal{U}(V)^1 \), respectively.

Moreover, consider the opposite superspace \( V^\Pi \) given by \( V_0^\Pi = V_1^\Pi = V_0 \). Then the parity-reversing isomorphism \( V \cong V^\Pi \) induces an isomorphism between \( \mathcal{U}(V^\Pi)^1 \) and the odd superalgebra obtained from \( \mathcal{U}(V)^0 \) by reversing the parity. Therefore, it suffices to consider only the superalgebras \( \mathcal{U}(V)^0 \). For the sake of simplicity, we denote them by \( \mathcal{U}(V) \).

If \( V = V_{n,m} \) is a finite-dimensional superspace with \( \dim V_0 = n \) and \( \dim V_1 = m \) (further in this case we say that \( V \) is of dimension \( n + m \)) then we denote \( \mathcal{U}(V) \) by \( \mathcal{U}(n,m) \).

Theorem 8. Let \( V \) be a superspace, and let \( a \in V_0 \). The superalgebra \( (\mathcal{U}(V), \triangle_a) \) is conservative, and the associated multiplication can be given by
\[
A \triangledown^1_a B(x, y) = -(-1)^{AB}B(a, A(x, y))
\]
or
\[ A \triangledown_a^2 B(x, y) = \frac{1}{3}(A^* \triangle_a B + (-1)^{AB} \tilde{B} \triangle_a A), \] (21)

where \( A^*(x, y) = A(x, y) + (-1)^{xy} A(y, x) \) and \( \tilde{B}(x, y) = 2(-1)^{xy} B(y, x) - B(x, y) \).

**Proof.** Let \( \triangle_a \) be the product on \( \mathcal{U}(V) \) given by (18). A straightforward computation shows that for \( W, V \in \mathcal{U}(V) \) and \( x, y \in V \) we have

\[
\begin{align*}
[L_A, \triangle_b](W, V)(x, y) &= (-1)^{(A+a)W}(W(A(a, b), V(x, y)) - \\
&~ -(-1)^{(V+W+A+a+b)}V(W(A(a, b), x), y) - (-1)^{(V+x)(W+A+a+b)}V(x, W(A(a, b), y)).
\end{align*}
\]

In other words,

\[
[L_A, \triangle_b](W, V) = (-1)^{AW} W \triangle_{A(a,b)} V. \tag{22}
\]

Now, for \( A \triangledown_a^1 B(x, y) = (-1)^{AB} B(a, A(x, y)) \) we have

\[
[L_B, [L_A, \triangle_a]](W, V) = (-1)^{(A+B)W} W \triangle_{B(A(a,a))} V = (-1)^{AB}[L_{A\triangledown_1 B}, \triangle_a](W, V).
\]

Analogously one can show that \( (A \triangledown_a^2 B)(a, a) = (-1)^{AB} B(a, A(a, a)) \), which proves that we can also take \( \triangledown_a^2 \) as the multiplication in the associated superalgebra. \( \square \)

By (10) and (22), the Jacobi subspace \( J \) of \( \mathcal{U}(V), \triangle_a \) consists precisely of those \( A(x, y) \in \mathcal{U}(V) \) for which \( A(a, a) = 0 \), so we may identify the spaces \( \mathcal{U}(V)/J \) and \( V \) by the mapping \( A \mapsto A(a, a) \). In particular, for the algebra \( \mathcal{U}(n, m) \) we have \( \text{codim}(J) = n + m \).

If the mapping \( \mathcal{U}(V) \to \mathfrak{gl}(V) \) given by \( A \mapsto L_a^A \), is surjective (which is always the case if \( V \) is of countable dimension) then every operator \( A \) such that \( L_a^A = -id \) is a left unity of \( (\mathcal{U}(V), \triangle_a) \).

**Lemma 9.** The algebra \( \mathcal{U}(V) \) has no nonzero Jacobi ideals.

**Proof.** By above, a bilinear operator \( A \) lies in the Jacobi subspace \( J \subseteq (\mathcal{U}(V), \triangle_a) \) if and only if \( A(a, a) = 0 \). Therefore, for every \( A \) in a Jacobi ideal of \( \mathcal{U}(V) \) and every \( B \in \mathcal{U}(V) \) we have

\[
0 = (A \triangle_a B)(a, a) = A(a, B(a, a)),
\]

\[
0 = (B \triangle_a A)(a, a) = (-1)^{AB} A(B(a, a), a) - (-1)^{AB} A(a, B(a, a)) = (-1)^{AB} A(B(a, a), a),
\]

whence \( A(a, V) = A(V, a) = 0 \). Now, this relation holds for \( B \triangle_a A \) for every \( B \in \mathcal{U}(V) : \)

\[
0 = (B \triangle_a A)(a, y) = (-1)^{AB} A(B(a, a), y)
\]

for all \( y \in V \), and we get \( A = 0 \). \( \square \)
3.2 Terminal Subalgebras of $\mathcal{U}(n,m)$

In this section we give some examples of terminal non-Jordan superalgebras, which are subsuperalgebras of $\mathcal{U}(n,m)$. These superalgebras are analogs of the simple terminal algebras $W_n$, $S_n$, $H_n$ introduced in [6].

**The algebra** $W_{n,m}$. The space of $W_{n,m}$ consists of all supersymmetric bilinear operations on a vector superspace $V$ of dimension $n + m$.

It is easy to see that the algebra $M := W_{n,m}$ is terminal; indeed, for supersymmetric operations $A, B$ the multiplication (21) specializes exactly to (7).

It follows from (10), (12) and (22) that the Jacobi subspace consists of the elements $A \in W_{n,m}$ such that $A(a, a) = 0$, and the left quasiunits satisfy the condition $A(a, a) = -a$.

Note also that the algebra $W_{n,m}$ has left units; these are the elements $A \in W_{n,m}$ for which $A(a, x) = -x$ for all $x$.

**The superalgebra** $S_{n,m}$ is the subsuperalgebra of $W_{n,m}$ consisting of the bilinear operators $A(x, y)$ such that the supertrace of every $T_a$ is zero for all $a$, where $T_a(x) = A(a, x)$.

**The superalgebra** $H_{n,m}$ ($n$ even) is the subsuperalgebra of $W_{n,m}$ consisting of the bilinear operators "preserving" a nondegenerate skew-symmetric bilinear consistent superform $\langle \cdot, \cdot \rangle$:

$$\langle A(x, y), z \rangle = (-1)^{yz} \langle A(x, z), y \rangle.$$  

All assertions and calculations made for $W_{n,m}$ hold also for $S_{n,m}$ and $H_{n,m}$, except that the latter two superalgebras have no left units.

3.3 The Main Theorem

Let $M$ be a conservative superalgebra on a space $V$ with the Jacobi subspace $J$. Consider the space $W$, which we define as $W = V/J$ if $M$ has a left quasunity, and $W = V/J \oplus E$ in the opposite case, where $E$ is the one-dimensional even space with a basis element $\epsilon$.

Assume that $M$ possesses a quasunity. Define the adjoint mapping $\text{ad} : M \rightarrow \mathcal{U}(W)$ as follows:

$$\text{ad}(a)(\alpha, \beta) = (-1)^{\beta(a+\alpha)}((\beta * a) * \alpha + (-1)^{\alpha a} \beta * (\alpha a) - (-1)^{\alpha a}(\beta * \alpha) * a).$$  (23)

If $M$ does not have a quasunity, we define the adjoint mapping $\text{ad} : M \rightarrow \mathcal{U}(W)$ by the equation above and the following equations:

$$\text{ad}(a)(\alpha, \epsilon) = a * \alpha + (-1)^{\alpha a} \alpha a - (-1)^{\alpha a} \alpha * a,$$  (24)

$$\text{ad}(a)(\epsilon, \beta) = (-1)^{\alpha \beta} \beta * a, \text{ ad}(a)(\epsilon, \epsilon) = a.$$  (25)

We check that this mapping is well-defined. Firstly, we prove that if $\alpha \in J$ or $\beta \in J$ then $\text{ad}(a)(\alpha, \beta) \in J$. Indeed, if $\beta \in J$, then the correctness follows easily from (13). If $\alpha \in J$, then by (14)

$$\text{ad}(a)(\alpha, \beta) \equiv (-1)^{\alpha(a+\beta)}(-\alpha(\beta * a) + (-1)^{\alpha \beta} \beta * (\alpha a) + (\alpha \beta) * a) \ (\text{mod} \ J).$$

Now, (9) and (16) imply

$$[L_\alpha, [L_\alpha, [L_\beta, M]]] = [[L_\alpha, L_\alpha], [L_\beta, M]] + (-1)^{\alpha a}[L_\alpha, [L_\alpha, [L_\beta, M]]] =$$
On the other hand, \[ [L_{\alpha}, [L_\beta, M]] + (-1)^{\alpha\alpha}[L_\alpha, [L_\alpha, M]] = -[L_{(-1)^{\beta(\alpha+a)\beta*(\alpha+a)}+(-1)^{\alpha\beta*(\alpha\beta)*a}}, M]. \]

On the other hand, \[ [L_\alpha, [L_\alpha, [L_\beta, M]]] = -(-1)^{\alpha\beta}[L_\alpha, [L_\beta*a, M]] = -(-1)^{\alpha\beta}[L_\alpha(\beta*a), M]. \]

Comparing these expressions and using (10), we infer that ad(\(\alpha, \beta\) \(\in\) J). The correctness of (24) and (25) follows from (13) and (14). Indeed, (24) follows from (15). To prove (25) we note that \(\text{ad}(\alpha, \beta) \equiv (-1)^{\alpha\beta}(\alpha\beta + \beta* \alpha) \mod J\).

Now, (11), (17), and the Jacobi identity allow us to rewrite the expression \([L_\epsilon, [L_\alpha, [L_\beta, M]]]\) in two different ways:

\[
[L_\epsilon, [L_\alpha, [L_\beta, M]]] = [[L_\epsilon, L_\alpha], [L_\beta, M]] + [L_\alpha, [L_\epsilon, [L_\beta, M]]] = \]

\[
[L_{\epsilon\alpha-a}, [L_\beta, M]] + [L_\alpha, [L_{\epsilon\beta-2\beta}, M]] = -(-1)^{\alpha\beta}[[L_{\beta*(\epsilon\alpha)+\beta*(\epsilon\beta)*a-3\beta*a}, M], \]

and

\[
[L_\epsilon, [L_\alpha, [L_\beta, M]]] = -(-1)^{\alpha\beta}[L_\epsilon, [L_\beta*a, M]] = -(-1)^{\alpha\beta}[[L_{\epsilon(\beta*a)-2(\beta*a)}, M], \]

Comparing these expressions, we obtain \(\text{ad}(\alpha, \beta) \equiv (-1)^{\alpha\beta} \beta * a\).

Moreover, it is easy to check that the mapping \(\text{ad}\) does not depend on the associated multiplication *.* Indeed, let *1 and *2 be two associated multiplications on \(M\). By (13) and the inclusion \(\alpha * 1 \beta = \alpha * 2 \beta \in J\) which holds for all \(\alpha, \beta \in M\) by (5), we have

\[
(\beta * 1 a) * 1 \alpha + (-1)^{\alpha\beta} (\beta * 1 a) * 1 \alpha = \]

\[
(\beta * 2 a) * 2 \alpha + (-1)^{\alpha\beta} (\beta * 2 a) * 2 \alpha \in J \]

for all \(\alpha, \beta \in M\). We proceed analogously with (24) and (25).

Prove that the adjoint mapping \(\text{ad} : M \rightarrow (U(W), \triangle_a)\) is a homomorphism for certain \(a \in W\). We begin with

**Lemma 10.** Let \((U_{-1}, U_0, U_1)\) be a triple of superspaces. Given some (superanticommutative) commutators

\[ [U_{-1}, U_0] \subseteq U_{-1}, \ [U_0, U_0] \subseteq U_0, \ [U_1, U_0] \subseteq U_1, \ [U_{-1}, U_1] \subseteq U_0, \]

for every \(a \in U_{\pm 1}\) we define an algebra structure \(M_a\) on \(U_{\pm 1}\) (whose parity coincides with one of \(a\)) by the rule

\[ M_a(x, y) = [[a, x], y], \ x, y \in U_{\pm 1}. \]

Assume that for the commutators above the Jacobi superidentity holds whenever defined. Then for every even \(a \in U_{-1}\) the mapping

\[ (U_1, M_a) \rightarrow (U(U_{-1}), \triangle_a), \]

\[ x \mapsto -M_x, \]

is an algebra homomorphism.
Proof. A direct computation shows that for \( b, c \in U_{-1} \) we have

\[
(-M_x)\Delta_a(-M_y)(b, c) = [[x, a], [[y, [x, a]], b]] - (-1)^{xy}[y, [[x, a], b]], c] - (-1)^{(y+b)[[y, [x, a]], b]], c],
\]

and these expressions are equal by the Jacobi identity.

Remark. It is possible to construct a (unique in a sense) \( \mathbb{Z} \)-graded Lie superalgebra \( \mathcal{L} = \sum_{i=-\infty}^{\infty} \mathcal{L}_i \) such that \( \mathcal{L}_i = U_i, \ i = -1, 0, 1, \) such that the commutators on \( \mathcal{L}_i \) coincide with the commutators on \( U_i \) (see [3]), but we will not need this superalgebra here.

Theorem 11. Let \( M \) be a conservative superalgebra on a vector space \( V \) with the Jacobi subspace \( J \). Let either \( W = V/J \) or \( W = V/J \oplus \langle e \rangle \) as above. The adjoint mapping \( \text{ad} : M \to (U(W), \triangle_e) \) is a homomorphism whose kernel is the maximal Jacobi ideal. In particular, if \( V \) is finite-dimensional and \( J \) is of codimension \( n + m \), then we have a homomorphism \( \text{ad} : M \to \mathcal{U}(k, m) \), where \( k = n \) if \( M \) has a quasiunity and \( k = n + 1 \) otherwise.

Proof. As above, take

\[
\mathcal{U}_0(M) := \langle L_a, [L_a, L_b] : a, b \in V \rangle,
\]

\[
\mathcal{U}_1(M) := \langle M, [L_a, M] : a \in V \rangle.
\]

Consider the triple of spaces \((\mathcal{U}_1(M), \mathcal{U}_0(M), V)\) and define the commutators among them by [3], [4], and the usual commutation in \( \mathfrak{gl}(V) \) (recall that \( \mathcal{U}_0(M) \) is a Lie superalgebra that acts on \( \mathcal{U}_1(M) \) and \( V \)). Then one can easily check that the commutators [3] and [4] satisfy the Jacobi super-identity whenever defined: the only case not checked above is of the double commutator of the elements \( A \in \mathfrak{gl}(V), B \in \mathcal{U}(V), x \in V \), and it can be verified directly.

Therefore, we are under the conditions of the previous lemma. Take \( M \in \mathcal{U}_1(M) \). We have a homomorphism \( M \to (\mathcal{U}(\mathcal{U}_1(M)), \Delta_M) \). Calculate the bilinear map \(-M_a\) explicitly:

\[
-M_a([L_\alpha, M], [L_\beta, M]) = -[[a, [L_\alpha, M]], [L_\beta, M]] = -[[L_\alpha, L_\alpha] + (-1)^{aa} L_{aa}, [L_\beta, M]] =
\]

\[
-[L_\alpha, [L_\alpha, L_\beta]], M] + (-1)^{aa}[L_{aa}, [L_\alpha, L_\beta], M]] - (-1)^{aa}[L_{aa}, [L_\beta, M]] =
\]

\[
[L_{(1)\beta} + (\beta), M] = [L_\alpha + \beta, [L_\alpha, M]] = [L_\alpha, [L_\beta, M]] = [L_{(-1)\beta} + (\beta), M] = [L_{\text{ad}(\alpha, \beta), M}],
\]

\[
-M_a([L_\alpha, M], [L_\beta, M]) = -[[a, M], [L_\beta, M]] = [L_\alpha, [L_\beta, M]] = [L_{(-1)^{\beta}\beta}, M] = [L_{\text{ad}(\alpha, \beta)}, M],
\]

\[
-M_a([L_\alpha, M], M) = -[[a, L_\alpha, M], M] = -[[L_\alpha, L_\alpha] + (-1)^{aa} L_{aa}, M] = [L_{\text{ad}(\alpha, \beta)}, M],
\]

\[
M_a(M, M) = [[a, M], M] = [L_\alpha, M].
\]

Consider the mapping \( \psi : V \to \mathcal{U}_1(M) \) given by \( x \mapsto [L_x, M] \). Then [10] means exactly that \( \ker(\psi) = J \). Therefore, we have an injective map \( \tilde{\psi} : V/J \to \mathcal{U}_1(M) \). Defining, if needed, \( \tilde{\psi}(e) = -M \) (if \( M \) has a left quasiunity \( e \) then \([L_e, M] = -M \) by [12]) we have an isomorphism between \( W \) and \( \mathcal{U}_1(M) \). This induces an isomorphism between \((\mathcal{U}(\mathcal{U}_1(M)), \Delta_M) \) and \((\mathcal{U}(W), \Delta_e) \) given by [19]. Now, by the formulas above this homomorphism in composition with the homomorphism which was constructed above is exactly the adjoint mapping.

Now we show that \( \ker(\text{ad}) = I \), where \( I \) is the maximal Jacobi ideal of \( M \). Indeed, if \( a \in I \), then it follows from [13] and [14] that \( \text{ad}(a) = 0 \).

Conversely, if \( \text{ad}(a) = 0 \), then \( a \in J \) by [25]. Since \( \text{ad} \) is a homomorphism, \( \ker \text{ad} \subseteq I \), and the theorem is proved.\]
Consider \( (\mathcal{U}(V), \triangle_a) \) for a fixed nonzero \( a \in V \). Check that the adjoint homomorphism, which is applied to \( \mathcal{U}(V) \), is the identity mapping. We know that \( \mathcal{U}(V) \) has a left unity, and the maximal Jacobi ideal of \( \mathcal{U}(V) \) is zero. Therefore, in our case the space \( W \) is \( \mathcal{U}(V)/J \) that we identify with \( V \) by the mapping \( A \mapsto A(a,a) \).

Now, let \( A,B,C \in \mathcal{U}(V) \), and let \( B(a,a) = u, C(a,a) = v \). Show that \( \text{ad}(A)(B,C)(a,a) = A(u,v) \), which establishes the required isomorphism. Recall that the adjoint mapping does not depend on the choice of associated multiplication. Therefore, we choose it as (20). Now, a direct computation shows the desired equality:

\[
\text{ad}(A)(B,C)(a,a) = (-1)^{C(A+B)}((C \nabla^1_a A) \nabla^1_a B + (-1)^{BAC} \nabla^1_a (B \triangle_a A)) - (-1)^{B(A+C)}(C \nabla^1_a B) \nabla^1_a A)(a,a) = \\
(-1)^{C(A+B)}((-1)^{B(A+C)+AC}B(a, A(a, C(a,a))) - (-1)^{BAC}(B(a, A(a, C(a,a)))) - \\
(-1)^{AB}A(B(a,a), C(a,a)) - (-1)^{AB}A(a, B(a, C(a,a)))) - \\
(-1)^{BAC}(B(a, C(a,a))) = A(B(a,a), C(a,a)) = A(u,v).
\]

In particular, the equality \( \text{ad} = \text{id} \) gives another proof of the fact that \( \mathcal{U}(V) \) has no nonzero Jacobi ideals.

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