THE REGULARITY TRANSFORMATION EQUATIONS:
HOW TO SMOOTH A CRINKLED MAP OF SPACETIME

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Abstract. We derive a system of nonlinear elliptic equations with
matrix valued differential forms as unknowns, the Regularity Trans-
formation equations (or RT-equations), and prove that they determine
whether metrics in General Relativity (GR) can be smoothed to optimal
regularity (i.e., two full derivatives smoother than the curvature tensor)
by coordinate transformation; that is, they determine whether a coordi-
nate transformation is sufficient to un-wrinkle a wrinkled map of space-
time. Shock-wave solutions constructed by the Glimm scheme in GR
with metric in \( C^{0,1} \) are non-optimal, and it is an open problem whether
there always exist coordinate transformations which smooth such met-
rics to optimal regularity \( C^{1,1} \), or whether regularity singularities exist.
In this paper we pose the problem at the general level of conne-
tions \( \Gamma \in W^{m,p} \), \( \text{Riem}(\Gamma) \in W^{m,p} \), and prove existence of solutions of the
RT-equations when \( m \geq 1, p > n \), by applying the linear theory of ellip-
tic regularity in \( L^p \) spaces to prove convergence of an iteration scheme.
The iteration scheme then provides an explicit numerical algorithm for
smoothing metrics to optimal regularity in GR, demonstrating that no
regularity singularities exist when \( m \geq 1, p > n \). The starting point
for the derivation of the RT-equations is the Riemann-flat condition, a
geometric condition for metric smoothing introduced previously by the
authors. For the case of GR shock waves (\( m = 0, p = \infty \)), the RT-
equations establish a connection between regularity singularities in GR
and classical Calderon-Zygmund singularities in elliptic PDE theory.

1. Introduction

Although the Einstein equations of General Relativity (GR) are covariant,
solutions are constructed in coordinate systems in which the PDE’s take on
a solvable form. A very first question in GR is then, which properties of
the spacetime represent the true geometry, and which are merely anomalies
of the coordinate system? In particular, does a solution of the Einstein
equations exhibit its optimal regularity in the coordinate system in which
it is constructed? Since coordinate systems define the local property of
spacetime, the coordinates in which the metric is most regular determine

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the degree to which the physics in curved spacetime corresponds locally to the physics of Special Relativity, (Einstein’s Correspondence Principle [7]).

A particularly intriguing case is GR shock waves [13, 23, 11, 17, 26, 2]. In [11], shock wave solutions of the Einstein equations generated by the Glimm scheme could only be constructed in coordinate systems in which the metric is only Lipschitz continuous ($C^{0,1}$) at shocks, even though both the connection and curvature tensor of such solutions stay bounded in $L^\infty$.

The question as to whether a $C^{0,1}$ metric can always be smoothed one order to $C^{1,1}$ by coordinate transformation is intimately related to the existence of locally inertial coordinate systems, and to the local correspondence of GR with the physics of Special Relativity. In the RSPA publication [17] authors began the discussion of regularity singularities by conjecturing that if such coordinate systems do not always exist, then shock wave interactions create a new kind of mild singularity which the authors termed regularity singularity, (see also [16, 18]). We begin here by generalizing this notion as follows:

**Definition 1.1.** Let $\Gamma$ be a connection given in a coordinate system $x$ such that (each component of) its Riemann curvature tensor $\text{Riem}(\Gamma)$ is in $W^{m,p}$ for some $m \geq 0$, $p \geq 1$, but is no smoother in the sense that $\text{Riem}(\Gamma)$ is not in $W^{m',p}$ for any $m' > m$. Then we say $\Gamma$ has optimal regularity in $x$-coordinates if $\Gamma \in W^{m+1,p}$, i.e., $\Gamma$ is one order smoother than $\text{Riem}(\Gamma)$. And we say $\Gamma$ has a regularity singularity at a point $q$ if $\Gamma$ fails to transform to optimal regularity under any $W^{m+2,p}$ coordinate transformation $x \to y$ defined in a neighborhood of $q$.

In this paper we derive the Regularity Transformation equations (RT-equations), a system of nonlinear elliptic equations with matrix valued differential forms as unknowns, and show that they determine whether or not non-optimal metrics in GR can be smoothed to optimal regularity by coordinate transformation. (The complete theory is developed in [20].) We then resolve the problem of regularity singularities for spacetimes above a threshold level of smoothness, (essentially one order smoother than the shock wave case), by giving an existence theory for the RT-equations when the connection $\Gamma$ and its Riemann curvature tensor $\text{Riem}(\Gamma)$ are in $W^{m,p}$, $m \geq 1$, $p > n$. (Detailed proofs are presented in [21].) This proves that such gravitational metrics (and in fact any such connection) can always be smoothed by one order, thereby establishing that determining optimal regularity by the RT-equations works. Moreover, the iteration scheme introduced for the existence theory converges without the need to take a subsequence, and thus provides an explicit numerical algorithm for constructing the coordinates in

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1We recover the notion of regularity singularity for GR shock waves when $p = \infty$, $m = 0$, c.f. [19]. Note also that a metric is always one derivative smoother than the connection by Christoffel’s formula, and a connection can never be more than one derivative smoother than the curvature by Koszul’s formula. [27]. Thus, for example, a connection of optimal regularity is always exactly one order smoother than its curvature tensor.
which the metric and connection exhibit optimal regularity. In words, a crinkled map of spacetime can always be smoothed by coordinate transformation, and we have a numerical algorithm for doing it, above a threshold level of smoothness.

In Section 3 we show the Einstein equations in Standard Schwarzschild Coordinates provide an example where solutions are generically non-optimal at every level of smoothness, and we conjecture that, without taking account of non-optimal solutions, the theory of the initial value problem for the Einstein equations is, in general, incomplete. Remarkable to us is that the RT-equations reduce the question of regularity singularities in Lorentzian spacetimes to an existence problem for a system of elliptic Poisson equations. So the metric signature is of no relevance to the question of regularity singularities. The RT-equations establish an elliptic regularity theory for the Einstein equations. Whether regularity singularities exist at the lowest regularity of GR shock waves, \( (\Gamma, \text{Riem}(\Gamma) \in L^\infty) \), is still an open problem.

The question as to the existence of such smoothing transformations is surprisingly subtle. In particular, the Riemann normal construction is not sufficient to smooth a connection to optimal regularity, and the construction itself is problematic for \( L^\infty \) connections. At smooth, non-interacting shock surfaces, the coordinate transformation to Gaussian normal coordinates at the surface suffices to smooth an \( L^\infty \) gravitational connection with \( L^\infty \) Riemann curvature, by one order to \( C^{0,1}_0 \) at shocks, by a now classical result of Israel in 1966, [13]. But for more general shock wave interactions, the only result we have is due to Reintjes [16], who proved that the gravitational metric can always be smoothed one order to \( C^{1,1}_1 \) in a neighborhood of the interaction of two shock waves from different characteristic families, in spherically symmetric spacetimes. This result stands as the only step forward since Israel on the problem of optimal metric regularity at GR shock waves [17]. Reintjes’ procedure for finding the local coordinate systems of optimal smoothness is orders of magnitude more complicated than the Riemann normal, or Gaussian normal construction process. The coordinate systems of optimal \( C^{1,1}_1 \) regularity are constructed in [16] by solving a complicated non-local PDE highly tuned to the structure of the interaction. Trying to guess the coordinate system of optimal smoothness apriori, for example harmonic or Gaussian normal coordinates [3], didn’t work. In Reintjes’ construction, several apparent miracles happen in which the Rankine-Hugoniot jump conditions come in to make seemingly over-determined equations consistent, but at this stage, the principle behind what PDE’s must be solved to smooth the metric in general, or when this is possible, appears entirely mysterious.

The authors’ new point of view on the question of regularity singularities began with the formulation of the Riemann-flat condition in [19], a necessary and sufficient condition for the existence of a coordinate transformation which smooths a connection in \( L^\infty \) to \( C^{0,1}_0 \). The Riemann-flat condition is the condition that there should exist a symmetric \((1,2)\)-tensor \( \tilde{\Gamma} \), one order
smoother than the connection \( \Gamma \), such that \( \text{Riem}(\Gamma - \tilde{\Gamma}) = 0 \), remarkable because it is a geometric condition on \( \tilde{\Gamma} \) alone, independent of the coordinate transformation that smooths the metric. Since \( \Gamma \) and \( \Gamma - \tilde{\Gamma} \) have the same singular set (shock set), at first we thought the Riemann-flat condition was telling us that to smooth an \( L^{\infty} \) shock wave connection one needed to extend the singular shock set to a flat connection by some sort of Nash embedding theorem. Our point of view changed again with the successful idea that we might derive a system of elliptic equations equivalent to the Riemann-flat condition, which resulted in the Regularity Transformation equations.

Our results, stated in Theorems 2.1 and 2.3 below, resolve the problem of regularity singularities at the levels of smoothness \( \Gamma \) and \( \text{Riem}(\Gamma) \) in \( W^{m,p} \), \( m \geq 1, p > n \), by establishing an existence theory for the RT-equations. The problem of solving the RT-equations at the threshold low regularity of \( L^{\infty} \) connections, the setting of GR shock waves, or for \( m < 1 \), is problematic. This is due mainly to the existence of Calderón-Zygmund singularities, by which we mean counterexamples demonstrating that solutions of the linear Poisson equation are not always in \( C^{1,1} \), i.e., not two derivatives above the sources, when the sources are in \( L^{\infty} \), \cite{14, 8}. The problem of regularity singularities at shock waves in GR is thus reduced to the problem of using the gauge freedom in the RT-equations to rule out classical Calderón-Zygmund singularities. We conclude that, for GR shock waves, the RT-equations make the unexpected connection between singularities from two apparently different subjects, and thereby place the open problem of regularity singularities within the well-studied framework of elliptic regularity theory. This is the topic of authors’ current research.

2. Statement of Results

Our first theorem establishes the equivalence of the Riemann-flat condition with the solvability of the RT-equations. Our second theorem gives an existence theorem for the RT-equations in the case \( \Gamma, d\Gamma \in W^{m,p} \), (and hence \( \text{Riem}(\Gamma) \in W^{m,p} \)), for \( m \geq 1, p > n \). By this we mean the component functions of \( \Gamma \) and \( d\Gamma \) are in \( W^{m,p} \) in some given, but otherwise arbitrary, coordinate system \( x \). Combining the two theorems, we conclude that any such connection can be mapped locally to optimal regularity by a coordinate transformation, demonstrating that regularity singularities do not exist when \( \Gamma \) and \( d\Gamma \) are in \( W^{m,p} \), \( m \geq 1, p > n \).

To state the first theorem, view \( \Gamma \equiv \Gamma_{\mu}^{\nu} dx^k \) as a matrix valued 1-form. The unknowns in the RT-equations are \( \tilde{\Gamma}, J, A \) also taken to be matrix valued differential forms as follows: Let \( J \equiv J_{\mu}^{\nu} \) denote the Jacobian of the sought after coordinate transformation which smooths the connection, viewed as a matrix-valued 0-form; let \( \tilde{\Gamma} \equiv \tilde{\Gamma}_{\mu}^{\nu} dx^k \) be a unknown tensor one order smoother than \( \Gamma \) (as required for the Riemann-flat condition \( \text{Riem}(\Gamma - \tilde{\Gamma}) = 0 \)) viewed as a matrix-valued 1-form; and \( A \equiv A_{\mu}^{\nu} \) is an auxiliary matrix.
valued 0-form introduced to impose \( \text{Curl}(J) = 0 \), the integrability condition for the Jacobian.

**Theorem 2.1.** Assume \( \Gamma \) is defined in a fixed coordinate system \( x \) on \( \Omega \), for \( \Omega \subset \mathbb{R}^n \) bounded and open with smooth boundary. Assume that \( \Gamma \in W^{m,p}(\Omega) \) and \( d\Gamma \in W^{m,p}(\Omega) \) for \( m \geq 1 \), \( p > n \). Then the following equivalence holds:

Assume there exists \( J \in W^{m+1,p}(\Omega) \) invertible, \( \tilde{\Gamma} \in W^{m+1,p}(\Omega) \) and \( A \in W^{m,p}(\Omega) \) which solve the elliptic system

\[
\begin{align*}
\Delta \tilde{\Gamma} &= \delta d\Gamma - \delta(d(J^{-1}) \wedge dJ) + d(J^{-1}A), \\
\Delta J &= \delta(J \Gamma) - \langle dJ; \tilde{\Gamma} \rangle - A, \\
d\vec{A} &= \text{div}(dJ \wedge \Gamma) + \text{div}(J d\Gamma) - d(\langle dJ; \tilde{\Gamma} \rangle), \\
\delta \vec{A} &= v,
\end{align*}
\]

with boundary data

\[
\text{Curl}(J) \equiv \partial_j J^\mu_i - \partial_i J^\mu_j = 0 \quad \text{on} \quad \partial \Omega,
\]

where \( v \in W^{m-1,p}(\Omega) \) is some vector valued 0-form free to be chosen. Then for each \( q \in \Omega \), there exists a neighborhood \( \Omega' \subset \Omega \) of \( q \) such that \( J \) is the Jacobian of a coordinate transformation \( x \mapsto y \) on \( \Omega' \), and the components of \( \Gamma \) in \( y \)-coordinates are in \( W^{m+1,p}(\Omega') \).

Conversely, if there exists a coordinate transformation \( x \mapsto y \) with Jacobian \( J = \frac{\partial y}{\partial x} \in W^{m+1,p}(\Omega) \) such that the components of \( \Gamma \) in \( y \)-coordinates are in \( W^{m+1,p}(\Omega) \), then there exists \( \tilde{\Gamma} \in W^{m+1,p}(\Omega) \) and \( A \in W^{m,p}(\Omega) \) such that \((J, \tilde{\Gamma}, A)\) solve \((2.1)\) - \((2.5)\) in \( \Omega \) for some \( v \in W^{m-1,p}(\Omega) \).

Equations \((2.1)\) - \((2.4)\) are the RT-equations. To derive the RT-equations we develop an Euclidian Cartan algebra associated with the Riemann-flat condition, summarized in Section 4, where we also introduce the operations \( \vec{\cdot}, \text{div} \) and \( \langle \cdot \cdot \rangle \). Here \( \vec{A} \), the vectorization of \( A \), is the vector valued 1-form defined by \( \vec{A} \equiv A^\mu_i dx^i \), so \( d\vec{A} = \text{Curl}(A) \). Equation \((2.3)\) is obtained by setting \( d \) of the vectorized right hand side of \((2.2)\) equal to zero, thus the identity \( d\vec{J} = \text{Curl}(J) \) implies that \((2.3)\) is equivalent to the integrability condition \( \text{Curl}(J) = 0 \) for the Jacobian, c.f. [20]. The first two terms on the right hand side of \((2.3)\) result from identity \((4.12)\). By this identity we are able to re-express seemingly uncontrolled terms involving \( \delta \Gamma \) in terms of the more regular \( d\Gamma \), resulting in a fortuitous gain of one derivative required for the whole theory to work.

The derivation of the RT-equations in Section 6 shows that given \( \tilde{\Gamma} \) satisfying the Riemann-flat condition, there exists \( J \) and \( A \) such that \((J, \tilde{\Gamma}, A)\) solve the RT-equations with the regularities required in Theorem 2.1. The converse is more subtle. The following lemma is the main step in the proof of the converse of Theorem 2.1 that existence for the RT-equations implies
existence of local coordinate transformations which smooth the connection \( \Gamma \) to optimal regularity:

**Lemma 2.2.** Assume \((J, \tilde{\Gamma}, A)\) solves the RT-equations, then

\[
\tilde{\Gamma}' \equiv -J^{-1}dJ + \Gamma
\]  

(2.6)

solves the Riemann-flat condition \( \text{Riem}(\Gamma - \tilde{\Gamma}') = 0 \), and \( \tilde{\Gamma}' \) has the regularity of \( \tilde{\Gamma} \).

The RT-equations produce the correct Jacobian \( J \), but not the correct \( \tilde{\Gamma} \) which solves the Riemann-flat condition. The miracle then is that the RT-equations boost the regularity of the correct \( \tilde{\Gamma}' \) (which does solve the Riemann-flat condition) to one level more regular than it should be based on its definition in (2.6), due to cancellations between \( J^{-1}dJ \) and \( \Gamma \), which are both one level below the required regularity. The mapping from \( \tilde{\Gamma} \) to \( \tilde{\Gamma}' \) is a gauge transformation in the sense that \( \tilde{\Gamma}' \) again solves the RT-equations, but for different matrix valued 0-form \( A' \) in place of \( A \). The gauge freedom, by which we mean the freedom to assign \( v \), and the freedom to assign boundary conditions for \( \tilde{\Gamma} \) and \( A \), is a propitious feature of the RT-equations. In particular, equation (2.1) was obtained by replacing the first order Riemann-flat condition by a second order Poisson equation for \( \tilde{\Gamma} \). But in Lemma 2.2, we recover the Riemann-flat condition from the RT-equations without having to impose the nonlinear boundary data required to recover a solution of the first order Cauchy-Riemann equations from a solution of the second order Poisson equation, c.f. Section 3.1 in [20].

Our second main theorem is the following existence result for the RT-equations.

**Theorem 2.3.** Assume the components of \( \Gamma, d\Gamma \in W^{m,p}(\Omega) \) for \( m \geq 1 \), \( p > n \geq 2 \) in some coordinate system \( x \). Then for each \( q \in \Omega \) there exists a solution \((\tilde{\Gamma}, J, A)\) of the RT-equations (2.1) - (2.4) with boundary data (2.5) defined in a neighborhood \( \Omega_q \) of \( q \) such that \( \tilde{\Gamma} \in W^{m+1,p}(\Omega_q) \), \( J \in W^{m+1,p}(\Omega_q) \), \( A \in W^{m,p}(\Omega_q) \).

As an immediate corollary of Theorems 2.1 and 2.3 we deduce the main result of this paper, which states that non-optimal connections can always be smoothed by one order, above a threshold level of regularity:

**Theorem 2.4.** Assume the components of \( \Gamma, d\Gamma \in W^{m,p}(\Omega) \) for \( m \geq 1 \), \( p > n \geq 2 \) in some coordinate system \( x \). Then for each \( q \in \Omega \) there exists a coordinate transformation \( x \mapsto y \) defined in a neighborhood of \( q \), such that the components of \( \Gamma \) in \( y \)-coordinates are in \( W^{m+1,p} \).

We outline the proof of Theorem 2.3 below, and refer to [21] for further details. For the proof we introduce an iteration scheme designed to apply the linear theory of elliptic regularity in \( L^p \) spaces. A key insight for the proof was to augment the RT-equations by ancillary elliptic equations in order to convert the non-standard boundary condition \( \text{Curl}(J) = 0 \), which is of
neither Neumann nor Dirichlet type, into Dirichlet data for \( J \) at each stage of the iteration, c.f. [21]. By this, the iteration scheme can be defined and bounds sufficient to imply convergence in the requisite spaces can be proven, by applying standard existence theorems regarding elliptic regularity in \( L^p \) spaces for the linear Poisson equation, to each iterate, [9]. The regularity \( \Gamma, Riem(\Gamma) \in W^{m,p}, m \geq 1, p > n \), is natural threshold, because this is the lowest regularity that implies \( \Gamma, Riem(\Gamma) \) are Hölder continuous by Morrey’s inequality. Since \( L^p \) is not closed under multiplication, this is currently required in the proof of convergence of the iteration scheme to control the nonlinear products on the right hand side of the RT-equations.

3. The RT-equations and the Initial Value Problem in GR

The Einstein equations \( G = \kappa T \) of General Relativity are covariant tensorial equations defined independent of coordinates. The unknowns in the equations are the metric tensor \( g \), coupled to the variables which determine the sources in \( T \). The existence of solutions to the Einstein equations are established by PDE methods in coordinate systems in which the Einstein equations take on a solvable form. The coordinate systems are typically specified by an ansatz for the metric, for example, SSC coordinates for spherically symmetric spacetimes, or harmonic coordinates, wave-gauge coordinates, etc., for the general initial value problem in four dimensions. For example, in a given coordinate system, assuming a perfect fluid, \( T^{ij} = (\rho + p)u^i u^j + pg^{ij}, \) the unknowns are \( g_{ij}, \rho, p, u_i, \) [3, 12, 27]. The question we ask here is–how do we know the gravitational metric, which is the solution of the Einstein equations in a given coordinate system, exhibits its optimal smoothness in the coordinates in which it is constructed?

Indeed, assume one were to construct a solution to the Einstein equations \( G = \kappa T \) in a given coordinate system \( x \) in which the equations produce unique solutions (locally) within a given smoothness class, starting from initial data. For example, assume the equations produce solutions of optimal smoothness with metric \( g \in W^{m+2,p} \), connection \( \Gamma \in W^{m+1,p} \), and \( Riem(\Gamma) \in W^{m,p} \). Then application of a transformation \( x \rightarrow y \) with Jacobian \( J \in W^{m+1,p} \) will in general lower the regularity of the solution space, lowering the regularity of the metric and its connection \( \Gamma \) by one order, but the transformation will preserve the regularity of the curvature tensor \( Riem(\Gamma) \in W^{m,p} \), because the connection involves derivatives of the Jacobian of the coordinate transformation, but the metric and Riemann curvature tensor, being tensors, involve only the undifferentiated Jacobian.

Therefore, if one were to then express the Einstein equations in the transformed coordinates \( y \) in which the metric is one order less smooth than optimal, the resulting existence theory posed in \( y \)-coordinates, by construction, would produce the unique transformed solution \( g \in W^{m+1,p}, \Gamma \in W^{m,p}, \) and \( Riem(\Gamma) \in W^{m,p} \). Therefore, and this is the main point, if we were to construct our solutions in the \( y \)-coordinates in the first place, then we would
not know that our unique solution was one order below optimal smoothness without knowing about the existence of the inverse transformation $y \to x$. It is precisely the existence of this transformation from $y$ back to $x$ that is guaranteed by Theorems 2.1 and 2.3 because its existence follows from existence for the RT-equations when $\Gamma \in W^{m,p}$, $d\Gamma \in W^{m,p}$, $m \geq 1$, $p > n$. Theorems 2.1 and 2.3 tell us that it is sufficient to solve the Einstein equations in a weaker sense than optimal, by stating that it is sufficient to solve a version of the Einstein equations which only produce metrics and connections one order less smooth than optimal. If $\Gamma$ and $d\Gamma$ are in $L^\infty$, then this is the difference between weak and strong solutions in the true sense of the theory of distributions.

The fact that the Einstein equations admit coordinate systems in which the metric is one degree less smooth than optimal, leads one to anticipate that the Einstein equations might be easier to solve at this lower level of smoothness—because in coordinates where the metric is one order less smooth, the equations need impose fewer constraints. Standard Schwarzschild Coordinates (SSC) provides such an example, the case when the metric takes the form

$$ds^2 = -B(t,r)dt^2 + \frac{dr^2}{A(t,r)} + r^2d\Omega^2,$$

and this represents the coordinates in which the Einstein equations for a spherically symmetric spacetime (arguably) take their simplest form. Since the first three Einstein equations in SSC are

$$-rA_r + (1 - A) = \kappa BT^{00}r^2,$$  \hspace{1cm} (3.2)  

$$A_t = \kappa BT^{10}r,$$  \hspace{1cm} (3.3)  

$$\frac{rB_r}{B} - \frac{1 - A}{A} = \frac{\kappa}{A^2}T^{11}r^2,$$  \hspace{1cm} (3.4)  

we find that the metric is generically only one level more regular than the curvature tensor, at every level of smoothness [11], in agreement with the assumptions of Theorems 2.1 and 2.3. As an application of Theorem 2.4, we thus have the following result.

**Corollary 3.1.** Assume $T \in W^{m,p}$, $m \geq 1$, $p > 4$, and let $g \equiv (A,B)$ be a solution of the Einstein equations in SSC satisfying $g \in W^{m+1,p}$, $\Gamma \in W^{m,p}$ and $d\Gamma \in W^{m,p}$ in an open set $\Omega$. Then for each $q \in \Omega$ there exists a coordinate transformation $x \to y$ defined in a neighborhood of $q$, such that, in $y$-coordinates, $g \in W^{m+2,p}$, $\Gamma \in W^{m+1,p}$, $\text{Riem}(\Gamma) \in W^{m,p}$.

Consider now the problem of trying to prove that a non-optimal metric is smoothed by one order via the more standard 3 + 1 framework for studying solutions of the Einstein equations in GR. The 3 + 1 framework is based on foliating spacetime into spatial slices parameterized by a time variable. We now argue that the 3 + 1 framework replaces the problem of smoothing metrics in spacetime, to the (seemingly formidable) problem of smoothing
the restrictions of the metric by one order on space-like hypersurfaces [3]. To make the point, recall that in wave coordinates, the spacetime metric evolves from initial data surfaces, and inherits the regularity of the initial data, via evolution by a semi-linear wave equation [25]. Thus to obtain optimal regularity in wave coordinates, the induced metric on the initial data surface must be one order more regular than the spacetime metric in the original coordinates. But the induced metric obtained by restricting the spacetime metric to a smooth hypersurface in the original coordinates in which the spacetime metric is non-optimal, would, in general, be no more regular than the spacetime metric itself, because, generically, the induced curvature on the surface would loose an order of regularity by undoing spacetime cancellations in \( \partial \Gamma \) which make the spacetime curvature one order more regular than \( \partial \Gamma \). Since the metric can only be two orders more regular than its curvature, this loss of one derivative in the curvature implies that the restricted metric would generically be no smoother than the spacetime metric. Since the curvature transforms as a tensor, it follows that a spacetime metric one order smoother in wave coordinates would require the curvature of the \textit{pre-image} of initial data hypersurfaces in wave coordinates, to be at least as smooth as the curvature of spacetime in the original coordinates, meaning no such loss of derivative in the curvature on hypersurfaces in the original coordinates. Thus to prove optimal regularity in wave coordinates, or any other \((3 + 1)\)-formulation of the initial value problem for the Einstein equations, one has to find such a non-generic initial data surface for every non-optimal metric. Establishing the existence of such surfaces without additional assumptions, appears to be a formidable problem for Lorentzian metrics [3]. Since the initial data surface would have to be tuned to each non-optimal metric, we conjecture that accounting for non-optimal regularity by the RT-equations is required to complete the solution space of the Einstein equations obtained by solving the initial value problem starting from a fixed initial data surface, at any given level of smoothness.

Results on the regularity of Lorentzian metrics in GR have been achieved from the \(3 + 1\) formulation of the Einstein equations, starting from certain additional assumptions on the geometry of the underlying spacetime, c.f. [1-4, 15]. Our approach here is entirely independent, and at this stage it is unclear to the authors how these results and underlying assumptions in [1-4, 15] fit into the framework of the RT-equations. In [1], \(W^{2,p}\)-bounds on Lorentzian metrics in terms of their Riemann curvature were derived using harmonic coordinates on spatial slices, starting from a geodesic ball assumption. For GR vacuum solutions the results in [1] were improved in [4] by means of an alternative coordinate ansatz and foliation. The work in [15] establishes consistency of the vacuum Einstein equations for initial data

\[\text{This is not a problem for Riemannian metrics which do exhibit optimal regularity in harmonic coordinates [5], because the regularity for the Laplacian comes from the source terms, not the boundary data. The elliptic RT-equations, surprisingly, accomplish this for Lorentzian metrics.}\]
with curvature bounded in $L^2$. Neither the results in \cite{1,4,6,15} nor our result in \cite{21} address GR shock waves, when the matter sources are non-zero and bounded in $L^\infty$.

The modern take on the Einstein equations is to move away from the four dimensional geometrical framework of spacetime, in favor of the 3 + 1 framework of classical physics. The effectiveness of Einstein’s original four dimensional framework as expressed through the RT-equations and the Riemann-flat condition, provides an example in which the geometry of four dimensional spacetime is fundamental.

4. EUCLIDEAN CARTAN CALCULUS FOR MATRIX VALUED DIFFERENTIAL FORMS

Our motivation in \cite{20} for introducing matrix valued differential forms begins by expressing the Riemann curvature tensor as matrix valued 2-form,

$$\text{Riem}(\Gamma) = d\Gamma + \Gamma \wedge \Gamma,$$  \hspace{1cm} (4.1)

interpreting the connection $\Gamma$ as the matrix valued 1-form $\Gamma^\mu_\nu dx^i$. By a matrix valued differential $k$-form $A$ we mean an $(n \times n)$-matrix whose components are $k$-forms over the spacetime region $\Omega \subset \mathbb{R}^n$, and we write

$$A \equiv \sum_{i_1 < \ldots < i_k} A_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$  \hspace{1cm} (4.2)

for $(n \times n)$-matrices $A_{i_1 \ldots i_k}$, assuming total anti-symmetry in the indices $i_1, \ldots, i_k \in \{1, \ldots, n\}$. The wedge product of $A$ with a matrix valued $l$-form $B = B_{j_1 \ldots j_l} dx^{j_1} \wedge \ldots \wedge dx^{j_l}$ is then defined by

$$A \wedge B \equiv \frac{1}{l!k!} A_{i_1 \ldots i_k} B_{j_1 \ldots j_l} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l},$$  \hspace{1cm} (4.3)

where “$\cdot$” denotes standard matrix multiplication. So $\Gamma \wedge \Gamma$ in (4.1) is non-zero, unless all component matrices mutually commute. We introduce the matrix valued inner product

$$\langle A ; B \rangle^\mu_\nu \equiv \sum_{i_1 < \ldots < i_k} A^\mu_{\sigma i_1 \ldots i_k} B^\sigma_{\nu i_1 \ldots i_k},$$  \hspace{1cm} (4.4)

which is a matrix valued version of the Euclidean inner product of $k$-forms, and the Hodge star operator $\ast$ by

$$A \wedge (\ast B) \equiv \langle A ; B \rangle dx^1 \wedge \ldots \wedge dx^n.$$  \hspace{1cm} (4.5)

The exterior derivative is defined as

$$dA \equiv \partial_l A_{[i_1 \ldots i_k]} dx^l \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$  \hspace{1cm} (4.6)

the co-derivative as the $(k - 1)$-form

$$\delta A \equiv (-1)^{(k+1)(n-k)} \ast (d(\ast A))$$  \hspace{1cm} (4.7)

and the Laplace operator as

$$\Delta \equiv \delta d + d\delta.$$  \hspace{1cm} (4.8)
The derivative operations (4.6), (4.7) and (4.8) act on matrix components separately and behave like the analogous operations on scalar valued differential forms. In particular, c.f. Theorem 3.7 in [5], \( \Delta \) acts component-wise as the Euclidean Laplacian,

\[
(\Delta A)_{\nu_1...i_k}^{\mu} = \Delta (A_{\nu_1...i_k}^{\mu}) = \sum_{j=1}^{n} \partial_j \partial_j (A_{\nu_1...i_k}^{\mu}).
\]  

(4.9)

We convert matrix valued differential forms to vector valued forms as follows: We let an arrow over a matrix valued 0-form \( A \) denote the conversion of \( A \) into its equivalent vector valued 1-form, i.e.,

\[
\vec{A} \equiv A^\alpha_i dx^i,
\]  

(4.10)

where \( \alpha \) labels the components of the vector. By this, we express the integrability condition for the Jacobian \( J \) as \( \text{d}\vec{J} = 0 \), since

\[
\text{Curl}(J) \equiv \frac{1}{2} (J^\alpha_{i,j} - J^\alpha_{j,i}) dx^j \otimes dx^i = J^\alpha_{i,j} dx^j \wedge dx^i = \text{d}(J^\alpha_i dx^i) \equiv d\vec{J}^\alpha.
\]

Moreover, for a matrix valued \( k \)-form \( A \), we define the operation

\[
\overrightarrow{\text{div}}(A)^\alpha \equiv \sum_{l=1}^{n} \partial_l ((A^\alpha)_{i_1...i_k}) dx^{i_1} \wedge ... \wedge dx^{i_k},
\]  

(4.11)

which creates a vector valued \( k \)-form. The operations (4.10) - (4.11) are meaningful when the dimension of the matrices equals the dimension of the physical space. For the proof of Theorems 2.1 and 2.3 we extend in [20] various identities of classical Cartan Calculus to the setting of matrix valued differential forms. The key identity required to close the RT-equations within the appropriate regularity classes is the identity

\[
d(\delta(J\Gamma)) = \overrightarrow{\text{div}}(dJ \wedge \Gamma) + \overrightarrow{\text{div}}(J \cdot d\Gamma),
\]  

(4.12)

which applies to matrix valued 1-forms \( \Gamma \) and matrix valued 0-forms \( J \), c.f. [20] for proofs. This identity has no analogue for classical scalar valued differential forms.

5. The Riemann-flat condition

To begin, consider the transformation law for a connection

\[
(J^{-1})^k_{ij} (\partial_j J^\alpha_i + J^\beta_i J^\gamma_j \Gamma^\alpha_{\beta\gamma}) = \Gamma^k_{ij},
\]  

(5.1)

where \( \Gamma^k_{ij} \) denotes the components of the connection in \( x^i \)-coordinates, \( \Gamma^\alpha_{\gamma\beta} \) denotes its components in \( y^\alpha \)-coordinates and \( J^\alpha_i \equiv \frac{\partial y^\alpha}{\partial x^i} \). Assume now that \( \Gamma^k_{ij} \in W^{m,p}(\Omega), \Gamma^\alpha_{\gamma\beta} \in W^{m+1,p}(\Omega) \) and \( J^\alpha_i \in W^{m+1,p}(\Omega) \), for \( m \geq 1 \). In other words, assume the Jacobian \( J \) smooths the connection \( \Gamma^k_{ij} \) by one order. For these given coordinates \( x \) and \( y \), we introduce

\[
\tilde{\Gamma}^k_{ij} \equiv (J^{-1})^k_{ij} (\partial_j J^\alpha_i + J^\beta_i J^\gamma_j \Gamma^\alpha_{\beta\gamma}),
\]  

(5.2)
which defines a field in $x$-coordinates. By imposing that $\tilde{\Gamma}_{ij}^k$ should transform as a $(1,2)$-tensor, (5.2) defines a tensor $\tilde{\Gamma}$. Now, (5.1) can be written equivalently as

$$(J^{-1})^k_\alpha \partial_j J^\alpha_i = (\Gamma - \tilde{\Gamma})^k_{ij}, \quad (5.3)$$

which we interpret as a condition on the fields $J$ and $\tilde{\Gamma}$ in $x$-coordinates. To obtain the Riemann-flat condition from (5.3), observe that adding a tensor to a connection yields another connection, so (5.3) is the condition that $J$ transforms the connection $\Gamma - \tilde{\Gamma}$ to zero. This implies that $\Gamma - \tilde{\Gamma}$ is a Riemann-flat connection, $\text{Riem}(\Gamma - \tilde{\Gamma}) = 0$. We conclude, that the existence of a coordinate transformation $x \mapsto y$ which lifts the connection regularity by one order implies the Riemann-flat condition, that is, the condition that there exists a symmetric $(1,2)$-tensor $\tilde{\Gamma}$ one order more regular than $\Gamma$ such that $\text{Riem}(\Gamma - \tilde{\Gamma}) = 0$. The following theorem records several further equivalences which, in particular, imply that the inverse implication is also true.

**Theorem 5.1.** Let $\Gamma^k_{ij}$ be a symmetric connection in $W^{m,p}(\Omega)$ for $m \geq 1$ and $p > n$ (in coordinates $x^i$). Then the following statements are equivalent:

(i) There exists a coordinate transformation $x^i \mapsto y^\alpha$ with Jacobian $J \in W^{m+1,p}(\Omega)$ such that $\Gamma^\alpha_{\beta\gamma} \in W^{m+1,p}(\Omega)$ in $y$-coordinates.

(ii) There exists a symmetric $(1,2)$-tensor $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ and a matrix field $J \in W^{m+1,p}(\Omega)$ which solve

$$J^{-1} dJ = \Gamma - \tilde{\Gamma}, \quad (5.4)$$

$$\text{Curl}(J)_{ij}^\alpha \equiv J_{i,j}^\alpha - J_{j,i}^\alpha = 0. \quad (5.5)$$

(iii) There exists a symmetric $(1,2)$ tensor $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ such that $\Gamma - \tilde{\Gamma}$ is Riemann-flat,

$$\text{Riem}(\Gamma - \tilde{\Gamma}) = 0. \quad (5.6)$$

(iv) There exists a symmetric $(1,2)$ tensor $\tilde{\Gamma} \in W^{m+1,p}(\Omega)$ which, when viewed as a matrix valued 1-form in $x$-coordinates, solves

$$d\tilde{\Gamma} = d\Gamma + (\Gamma - \tilde{\Gamma}) \wedge (\Gamma - \tilde{\Gamma}). \quad (5.7)$$

**Proof.** Note that (5.4) is a restatement of (5.3) in the formalism of matrix valued differential forms, and (5.5) is the condition that $J$ is integrable to define a coordinate system, (c.f. Frobenius Theorem in [21]). This shows that (i) and (ii) are equivalent and that (ii) implies (iii). The equivalence of (iii) and (iv) follows from the expression of the Riemann tensor as a matrix valued 2-form in (4.1). Finally, the implication (iii) to (i) is proved in [19] when $\Gamma \in L^\infty$ and $\tilde{\Gamma}, J \in C^{0,1}$, and the more regular case $\Gamma \in W^{m,p}$, $\tilde{\Gamma}, J \in W^{m+1,p}$ here follows by essentially the same argument, without the need of a mollification. □
By Theorems 5.1 and 2.1, existence for the RT-equations is equivalent to the Riemann-flat condition. Thus, as an immediate application of Theorems 2.1 and 2.3, we obtain the following analog of a Nash-type embedding theorem for connections with discontinuities in the \( m \)'th derivatives:

**Corollary 5.2.** If \( \Gamma, \text{Riem}(\Gamma) \in W^{m,\infty}(\Omega) \), (so \( \Gamma \in W^{m,p}_{\text{loc}} \) for \( p > n \), and \( \Gamma \) has at worst jump discontinuities in the \( m \)'th derivatives), then in a neighborhood of each point in \( \Omega \) there exists a Riemann-flat connection \( \hat{\Gamma} \), (namely, \( \Gamma = \Gamma + \hat{\Gamma} \)), which contains discontinuities in the \( m \)'th derivatives at the same locations as \( \Gamma \), and has the same jumps in the \( m \)'th derivatives across those discontinuities.

Corollary 5.1 implies that to prove optimal regularity it suffices to construct a Nash-type extension of the singular set of \( \Gamma \) to a Riemann-flat connection. However, establishing optimal regularity by the RT-equations turns out to be more feasible.

6. **The Proof of Theorem 2.1**

We begin by outlining the ideas and steps in the derivation of the RT-equations set out in detail in [20]. The idea is that by Theorem 5.1 the Riemann-flat condition \( \text{Riem}(\Gamma - \hat{\Gamma}) = 0 \) gives the equation (5.7), namely, \( d\hat{\Gamma} = d\Gamma + (\Gamma - \hat{\Gamma}) \wedge (\Gamma - \hat{\Gamma}) \), which we view as an equation for \( \hat{\Gamma} \). This can be augmented to a first order system of Cauchy-Riemann equations by addition of an equation for \( \delta \hat{\Gamma} \) with arbitrary right hand side. But to obtain a solvable system, we couple this Cauchy-Riemann system in the unknown \( \hat{\Gamma} \), to equation (5.4), namely \( J^{-1}dJ = \Gamma - \hat{\Gamma} \), for the unknown Jacobian \( J \). But equations (5.7) and (5.4) are not independent, since both are equivalent to the Riemann-flat condition. To obtain two independent equations, we employ the identity \( d\delta + \delta d = \Delta \) to derive two semi-linear elliptic Poisson equations, one for \( \Delta \hat{\Gamma} \) and one for \( \Delta J \). This results in the two second order equations (2.1) - (2.2), which closes in \((J, \hat{\Gamma})\) for fixed \( A \) upon setting \( \delta \hat{\Gamma} = J^{-1}A \). The equations are formally correct at the levels of regularity sufficient for \( J \) and \( \hat{\Gamma} \) to be one order smoother than \( \Gamma \), consistent with known results on elliptic smoothing by the Poisson equation in \( L^p \)-spaces, \[5, 8, 10, 9\].

To impose the integrability condition for \( J \), we use the freedom in \( \delta \hat{\Gamma} \) to interpret \( A \) as a variable on the right hand side of (2.1) and (2.2), and impose \( \text{Curl}(J) = 0 \) by asking that \( A \) solve the equation obtained by requiring \( d \) of the vectorized right hand side of the \( J \) equation (2.2) to equal zero. When taking \( d \) of the right hand side of (2.2), we encounter the term \( d(\delta(J \Gamma)) \) which seems to involve uncontrolled derivatives on \( \Gamma \), hence one derivative too low to get the required regularity \( A \in W^{m,p} \). But, surprisingly, this term can be re-expressed in terms of \( d\Gamma \) by the fortuitous identity (4.12), so

\[\text{(4.12)}\]

Note, \( A \in W^{m,p} \) is needed for (2.1) - (2.2) to imply the required regularity for \((J, \hat{\Gamma})\).
this term is in fact one order smoother than it initially appears to be. (This confirms that our assumptions need only control $d\Gamma$ in $W^{m,p}$, but not the complementary derivatives $\delta\Gamma$, which, by $(7.20)$, measure the derivatives not controlled by $d\Gamma$.) This gives $(2.3)$. The final form of the RT-equations is then obtained by augmenting $(2.1)$ - $(2.3)$ by equation $(2.4)$. This represents the “gauge freedom” to impose $\delta A = v$. This completes the derivation of the RT-equations and establishes the backward implication in Theorem 2.1.

We now outline the proof of the forward implication in Theorem 2.1, namely, that a solution of the RT-equations produces a Jacobian $J$ which lifts $\Gamma$ to optimal regularity. So assume $(J, \tilde{\Gamma}, A)$ solves the RT-equations. We first show that $J$ is integrable to coordinates. If $J$ is a solution of $(2.2)$ and $A$ solves $(2.3) - (2.4)$ with boundary data $(2.5)$, then, as shown in [20], $\Delta(d\vec{J}) = 0$ in $\Omega$. Thus, since $d\vec{J}$ is assumed to vanish on $\partial\Omega$ by $(2.5)$, it follows that the harmonic form $d\vec{J}$ is zero everywhere in $\Omega$, so $J$ is integrable to coordinates. To complete the forward implication, note that $\Gamma$ need not satisfy the Riemann-flat condition because the RT-equations have a larger solution space than the first order equations from which they are derived. So we define $\tilde{\Gamma}' = \Gamma - J^{-1}dJ$, which meets the Riemann-flat condition by $(5.4)$. But an additional argument is required to show that $\tilde{\Gamma}'$, like $\tilde{\Gamma}$, is indeed one level smoother than $\Gamma$, as stated in Lemma 2.2. For this, we use $(2.1)$ - $(2.2)$ to show that $\Delta\tilde{\Gamma}' \in W^{m-1,p}(\Omega)$, so that standard estimates of elliptic regularity theory imply the desired smoothness $\tilde{\Gamma}' \in W^{m+1,p}(\Omega')$ on any compactly contained subset $\Omega'$ of $\Omega$, (c.f. [20] for details). This establishes the forward implication in Theorem 2.1.

In summary, we start with two equivalent first order equations, one for $d\tilde{\Gamma}$ and one for $dJ$, both equivalent to the Riemann-flat condition. Out of these we create two independent nonlinear Poisson equations in $\tilde{\Gamma}$ and $J$ which have a larger solution space. The resulting system has the freedom to impose an auxiliary solution $A$ through the gauge freedom to impose $\delta\tilde{\Gamma}$. Since the solution space is larger, not all solutions of the RT-equations provide a $\tilde{\Gamma}$ which solves the Riemann-flat condition, but given any solution $(\tilde{\Gamma}, J, A)$ of $(2.1)$ - $(2.3)$, we show that there is enough freedom in $A$ so that there always exists $A'$ such that $(\tilde{\Gamma}', J, A')$, solves the RT-equations with $\tilde{\Gamma}' = \Gamma - J^{-1}dJ$. Then $\tilde{\Gamma}'$ meets the Riemann-flat condition by construction, and $J$ is the Jacobian of a coordinates transformation which takes $\Gamma$ to coordinates of optimal connection regularity.

One might wonder why we were not able to obtain an equation for the coordinate transformation $y$ directly, so that the simpler $dy = J$ would replace the integrability condition $Curl(J) = 0$. This is because, starting with the Riemann-flat condition $Riem(\Gamma - \tilde{\Gamma}) = 0$, the gauge freedom enters through the freedom to impose $\delta\tilde{\Gamma}$, and this expresses itself in the additional variable $A$ on the right hand side of equation $(2.1)$. To close the system, we then need a differential equation for $A$, which naturally comes from $Curl(J) = d\vec{J} = 0$ by setting $d$ of the vectorized right hand side of $(2.2)$ equal to zero, leading to the equation $(2.2)$ for $A$. Thus to obtain a closed solvable system, we are essentially forced to impose the integrability condition on $J$ in the form $Curl\vec{J} = 0$. 

4One might wonder why we were not able to obtain an equation for the coordinate transformation $y$ directly, so that the simpler $dy = J$ would replace the integrability condition $\text{Curl}(J) = 0$. This is because, starting with the Riemann-flat condition $\text{Riem}(\Gamma - \tilde{\Gamma}) = 0$, the gauge freedom enters through the freedom to impose $\delta\tilde{\Gamma}$, and this expresses itself in the additional variable $A$ on the right hand side of equation $(2.1)$. To close the system, we then need a differential equation for $A$, which naturally comes from $\text{Curl}(J) = d\vec{J} = 0$ by setting $d$ of the vectorized right hand side of $(2.2)$ equal to zero, leading to the equation $(2.2)$ for $A$. Thus to obtain a closed solvable system, we are essentially forced to impose the integrability condition on $J$ in the form $\text{Curl}\vec{J} = 0$. 


7. The Proof of Theorem 2.3

The biggest challenge of this research program was to discover a system of nonlinear equations for optimal regularity, the RT-equations, and formulate them so that existence of solutions to the nonlinear equations could be deduced from known theorems of elliptic regularity theory. The existence proof in [21], which we outline in this section, demonstrates the success of this program, i.e., that obtaining optimal regularity by the RT-equations really works. The strategy of the proof is to deduce convergence of an iteration scheme for approximating the nonlinear equations, from two standard theorems on the Dirichlet problem, stated below, taken from the linear theory of elliptic regularity in $L^p$ spaces, [5, 9, 8]. To begin, we rewrite the RT-equations (2.1) - (2.4) in the following compact form

\[
\begin{align*}
\Delta \tilde{\Gamma} &= \tilde{F}(\tilde{\Gamma}, J, A), \\
\Delta J &= F(\tilde{\Gamma}, J) - A, \\
d\tilde{A} &= d\tilde{F}(\tilde{\Gamma}, J), \\
\delta \tilde{A} &= v,
\end{align*}
\]

where

\[
\begin{align*}
\tilde{F}(\tilde{\Gamma}, J, A) &\equiv \delta d\Gamma - \delta (d(J^{-1}) \wedge dJ) + d(J^{-1}A) \\
F(\tilde{\Gamma}, J) &\equiv \delta (J \cdot \Gamma) - \langle dJ, \tilde{\Gamma} \rangle.
\end{align*}
\]

Here $\tilde{F}(\tilde{\Gamma}, J)$ is the vectorized version of $F(\tilde{\Gamma}, J)$, so that $d\tilde{F}(\tilde{\Gamma}, J)$ is identical to the right hand side of (2.3), c.f. the derivation leading to equation (3.40) in [20]. Note (7.3) - (7.4) take the Cauchy-Riemann form $d\tilde{A} = f$, $\delta \tilde{A} = g$. The consistency conditions $df = 0$, $\delta g = 0$ are met, since the right hand side of (2.3) is exact and since $\delta v = 0$ holds as an identity for 0-forms.

To handle the nonlinearities in (7.1) - (7.4), we introduce a small parameter $\epsilon > 0$ below by using the freedom to restrict to small neighborhoods, and then apply linear elliptic estimates in $L^p$ spaces to establish convergence at the sought after levels of regularity for sufficiently small $\epsilon > 0$. But we still have the problem of how to handle the non-standard boundary condition (2.5), which is neither standard Neumann nor Dirichlet data for the PDE (7.2) which determines $J$. We now introduce an equivalent formulation of the boundary condition (2.5) for (7.2), which has the advantage that it reduces to standard Dirichlet data for $J$ at each stage of the iteration scheme below. For this, observe that (7.3) implies the consistency condition $d(F(\tilde{\Gamma}, J) - \tilde{A}) = 0$, so that we can solve

\[
\begin{align*}
\begin{cases}
d\Psi = \tilde{F}(\tilde{\Gamma}, J) - \tilde{A}, \\
\delta \Psi = 0,
\end{cases} \\
\end{align*}
\]

for a vector valued 0-form $\Psi$, (c.f. Theorem 7.4 in [5]). Next, let $y$ be any solution of

\[
\Delta y = \Psi.
\]
We now claim that in place of the Poisson equation (2.2) for $J$ with the boundary condition (2.5), it suffices to solve (2.2) with boundary data $\vec{J} = d\Psi$ on $\partial\Omega$. (7.7)

To see this, write $\Delta dy = d\Delta y = d\Psi = \vec{F} - \vec{A} = \Delta \vec{J}$, which uses that, after taking $\text{vec}$ on both sides of the $J$-equation (2.2), the operation $\text{vec}$ commutes with $\Delta$ on the left hand side of (2.2) because the Laplacian acts component-wise. Thus, $\Delta(\vec{J} - dy) = 0$ in $\Omega$ and $\vec{J} - dy = 0$ on $\partial\Omega$, which implies by uniqueness of solutions of the Laplace equation that $\vec{J} = dy$ in $\Omega$. Since second derivatives commute, we conclude that $d\vec{J} = \text{Curl}(J) = 0$ in $\Omega$, on solutions of (2.2) with boundary data (7.7), as claimed. The point of using (7.7) in place of (2.5) is that (7.7) is standard Dirichlet data for $J$ in the following iteration scheme.

We now discuss the iteration scheme introduced in [21] for approximating solutions of the RT-equations (7.1) - (7.4). To start, assume a given connection $\Gamma \in W^{m,p}$ defined in $x$-coordinates on a bounded and open set $\Omega \subset \mathbb{R}^n$ with smooth boundary. We take $v = 0$ in (2.4) to fix the freedom to choose $v \in W^{m-1,p}(\Omega)$. For the existence proof, we define a sequence of differential forms $(A_k, \tilde{\Gamma}_k, J_k)$ in $\Omega$, and prove convergence to a solution $(A, \tilde{\Gamma}, J)$ of (7.1) - (7.4) with boundary data (2.5) in the limit $k \to \infty$. Define the iterates $(A_k, \tilde{\Gamma}_k, J_k)$ by induction as follows: To start, take $J_0$ to be the identity matrix and set $\tilde{\Gamma}_0 = 0$. Assume then $\tilde{\Gamma}_k$ and $J_k$ are given for some $k \geq 0$. Define $A_{k+1}$ as the solution of

$$
\begin{align*}
\begin{cases}
\delta \vec{A}_{k+1} = 0, \\
\delta \vec{A}_{k+1} = 0,
\end{cases}
\end{align*}
$$

for $A_{k+1} \cdot N = 0$ on $\partial\Omega$, where $N$ is the unit normal vector of $\partial\Omega$ which is multiplied by the matrix $A_{k+1}$. To introduce the Dirichlet data for $J_{k+1}$, we first define the auxiliary variables $\psi_{k+1}$ and $y_{k+1}$, as the solutions of

$$
\begin{align*}
\begin{cases}
\delta \psi_{k+1} = 0, \\
\delta \psi_{k+1} = 0,
\end{cases}
\end{align*}
$$

with boundary data $\psi_{k+1} \cdot N = 0$ on $\partial\Omega$ and

$$
\Delta y_{k+1} = \psi_{k+1}
$$

with boundary data $y_{k+1}(x) = x$ on $\partial\Omega$. (Note, the definitions of $A_{k+1}$, $\psi_{k+1}$ and $y_{k+1}$ do not require the previous iterates $A_k$, $\psi_k$ and $y_k$.) Now define $J_{k+1}$ to be the solution of the following standard Dirichlet boundary value problem,

$$
\begin{align*}
\Delta J_{k+1} &= F(\tilde{\Gamma}_k, J_k) - \vec{A}_{k+1}, \\
J_{k+1} &= dy_{k+1} \text{ on } \partial\Omega,
\end{align*}
$$

(7.11) (7.12)
and define $\tilde{\Gamma}_{k+1}$ as the solution of
\[ \Delta \tilde{\Gamma}_{k+1} = \tilde{F}(\tilde{\Gamma}_k, J_k, A_{k+1}), \] (7.13)
with boundary data $\tilde{\Gamma}_{k+1} = 0$ on $\partial \Omega$.

To prove that there exists a well-defined sequence of iterates $(J_k, \tilde{\Gamma}_k, A_k)_{k \in \mathbb{N}}$ and establish convergence, we introduce a small parameter $\epsilon > 0$. For this, let $\Gamma^*$ be a connection in $x$-coordinates satisfying
\[ \|\Gamma^*\|_{W^{m,p}(\Omega)} + \|d\Gamma^*\|_{W^{m,p}(\Omega)} < C_0, \] (7.14)
for $m \geq 1$ and $C_0 > 0$ a fixed constant. To introduce the small parameter assume that $\Gamma$ scales with $\epsilon > 0$ according to
\[ \Gamma = \epsilon \Gamma^*. \] (7.15)
Note that assumptions (7.14) and (7.15) can be made without loss of generality regarding the local problem of optimal metric regularity. To see this assume that $\Omega$ is the ball of radius 1. Then given any connection $\Gamma'(y) \in W^{m,p}(\Omega)$ with $d\Gamma'$ bounded in $W^{m,p}(\Omega)$, we can define $\Gamma^*(x)$ as the restriction of $\Gamma'$ to the ball of radius $\epsilon$ with its components transformed as scalars to the ball of radius 1 by the transformation $y = \epsilon x$. We then define $\Gamma(x)$ as the connection resulting from transforming $\Gamma'(y)$ as a connection under the coordinate transformation $y = \epsilon x$. We conclude that, given any connection $\Gamma'$, local existence of a solution of the RT-equations with $\Gamma = \Gamma'$ follows from the existence of a solution of the RT-equations with $\Gamma = \epsilon \Gamma^*$ for some $\epsilon > 0$. Thus, without loss of generality, we assume (7.15), c.f. [21] for details.

To incorporate $\epsilon$ into the RT-equations, we assume the scaling ansatz
\[ J_k = I + \epsilon J_k^*, \quad \tilde{\Gamma}_k = \epsilon \tilde{\Gamma}_k^*, \quad A_k = \epsilon A_k^*, \quad u_k \equiv (J_k^*, \tilde{\Gamma}_k^*), \quad a_k \equiv A_k^*. \] (7.16)
Substitute (7.15) and (7.16) into the RT-equations (7.1) - (7.4) for $v \equiv 0$ and dividing by $\epsilon > 0$, we obtain the following equivalent set of equations:
\[ \Delta u = F_u(u, a), \quad \delta \bar{a} = F_a(u), \] (7.17)
where
\[ F_u(u, a) \equiv \left( \frac{\delta d\Gamma^* - \delta d(J^{-1}dJ^*) + d(J^{-1}a)}{\delta \Gamma^* + \epsilon \delta (J^*; \Gamma^*) - \epsilon \langle dJ^*; \Gamma^* \rangle - a} \right), \] (7.18)
\[ F_a(u) \equiv \frac{\delta \bar{a}}{\delta \Gamma^*} \left( d\Gamma^* + \epsilon \frac{\delta \bar{a}}{\delta \Gamma^*} \right) + \epsilon \frac{\delta \bar{a}}{\delta \Gamma^*} \left( dJ^* \wedge \Gamma^* \right) - \epsilon \left( \langle dJ^*; \tilde{\Gamma}^* \rangle \right). \] (7.19)
Under assumption (7.15), the iterates defined by (7.18) - (7.19) generate corresponding iterates $u_k, a_k$ which successively solve (7.17), as well as iterates $\Psi_k^* = \frac{1}{\epsilon} \Psi_k$ and $y_k^* = \frac{1}{\epsilon} y_k$. It remains to prove that $(u_k, a_k)$ and $(\Psi_k, y_k)$ are

\[ \text{Note: since we only need to construct a particular solution, any boundary condition could be chosen for solving (7.5), (7.10} \text{ and (7.13} \text{ in the iteration scheme, as long as the resulting } J \text{ is invertible.} \]
well defined and converge for \( \epsilon \) sufficiently small. We state the results in two theorems:

**Theorem 7.1.** Assume \((u_k, a_k) \in W^{m+1,p}(\Omega) \times W^m,p(\Omega)\). Then \((u_{k+1}, a_{k+1})\) is well-defined and bounded in the same Sobolev space for \( \epsilon > 0 \) sufficiently small.

**Proof.** This is implied by the following two well known theorems from linear elliptic PDE theory\(^6\) which both extend component-wise to matrix and vector valued differential forms. (The possibility that we might reduce the existence theory to these two theorems was the guiding principle in the formulation of the RT-equations.)

**Theorem: (Cauchy-Riemann)** Let \( f \in W^{m,p}(\Omega) \) be a 2-form with \( df = 0 \) and let \( g \in W^{m,p}(\Omega) \) be a 0-form with \( \delta g = 0 \), where \( m \geq 0 \), \( n \geq 2 \). Then there exists a 1-form \( u = u_i dx^i \in W^{m+1,p}(\Omega) \) which solves \( du = f \) and \( \delta u = g \) in \( \Omega \) with boundary data \( u \cdot N = 0 \) on \( \partial \Omega \). Moreover, there exists a constant \( C_{\epsilon} > 0 \) depending only on \( \Omega, m, n, p \), such that

\[
\|u\|_{W^{m+1,p}(\Omega)} \leq C_{\epsilon} \left( \|f\|_{W^{m,p}(\Omega)} + \|g\|_{W^{m,p}(\Omega)} + \|u_0\|_{W^{m+1,p}(\partial \Omega)} \right). \tag{7.20}
\]

**Theorem: (Poisson)** Let \( f \in W^{m-1,p}(\Omega) \) and \( u_0 \in W^{m+\frac{p-1}{p},p}(\partial \Omega) \) both be scalar functions, and \( m \geq 1 \). Then there exists \( u \in W^{m+1,p}(\Omega) \) which solves the Poisson equation \( \Delta u = f \) with Dirichlet data \( u|_{\partial \Omega} = u_0 \). Moreover, there exists a constant \( C_{\epsilon} > 0 \) depending only on \( \Omega, m, n, p \) such that

\[
\|u\|_{W^{m+1,p}(\Omega)} \leq C_{\epsilon} \left( \|f\|_{W^{m-1,p}(\Omega)} + \|u_0\|_{W^{m+\frac{p-1}{p},p}(\partial \Omega)} \right). \tag{7.21}
\]

Namely, putting \((u_k, a_k)\) into the right hand side of (7.17), using Morrey’s inequality to estimate quadratic terms by the supnorm times appropriate Sobolev bounds, we obtain bounds of \( F_a(u) \) and \( F_u(u, a) \) in suitable Sobolev norms. Combining these bounds with the above elliptic estimates (7.20) and (7.21), generates estimates for \( a_{k+1} \in W^{m,p}(\Omega) \), \( \Psi_{k+1} \in W^{m,p}(\Omega) \), \( y_{k+1} \in W^{m+2,p}(\Omega) \) and then \( u_{k+1} \in W^{m+1,p}(\Omega) \), in terms of \((u_k, a_k)\), for \( \epsilon > 0 \) sufficiently small. For details see [21].

Our second theorem establishes convergence:

**Theorem 7.2.** There exists \((u, a)\) such that the sequence \((u_k, a_k)_{k \in \mathbb{N}}\) converges to \((u, a)\) in \( W^{m+1,p}(\Omega) \times W^m,p(\Omega) \) as \( k \to \infty \), and \((u, a)\) solves (7.17).

**Proof.** In order to establish convergence of the sequence of iterates \((u_k, a_k)_{k \in \mathbb{N}}\) in \( W^{m+1,p}(\Omega) \times W^m,p(\Omega) \), we require estimates on the differences \( a_k - a_{k-1} \) and \( u_k \equiv u_k - u_{k-1} \), in terms of the corresponding differences of source terms, \( F_u(u_k, a_{k+1}) \equiv F_u(u_k, a_{k+1}) - F_u(u_{k-1}, a_k) \) and \( F_a(u_k) \equiv F_a(u_k) - F_a(u_{k-1}) \). Combining the elliptic estimate (7.20) and (7.21) with

\(^6\)See Theorem 7.4 in [9] and Theorems 9.19 in [9] respectively.
source estimates, (for which we use that \( W^{m,p} \) is closed under multiplication when \( m \geq 1 \) and \( p > n \), by Morrey’s inequality), the main estimate proven in [21] is the following Sobolev space estimate which holds for \( \epsilon > 0 \) sufficiently small:

**Lemma 7.3.** Assume \( \epsilon \leq \min \left( \epsilon(k), \epsilon(k - 1) \right) \), where \( \epsilon(k) \equiv \frac{1}{4C_M \| u_k \|_{W^{m+1,p}}} \) and \( C_M > 0 \) is the constant from Morrey’s inequality, which only depends on \( n, p, \Omega \).

Then

\[
\| u_{k+1} \|_{W^{m+1,p}} \leq C_c C_u(k) \left( \epsilon \| u_k \|_{W^{m+1,p}} + \| u_{k+1} \|_{W^{m,p}} \right),
\]

\[
\| u_{k+1} \|_{W^{m,p}} \leq \epsilon C_c C_a(k) \| u_k \|_{W^{m+1,p}},
\]

where \( C_c > 0 \) is the constant resulting from applying (7.20) and (7.21) and

\[
C_u(k) \equiv C_s \left( 1 + \| u_k \|_{W^{m+1,p}} + \| u_{k-1} \|_{W^{m+1,p}} + \| a_{k+1} \|_{W^{m,p}} \right),
\]

\[
C_a(k) \equiv C_s \left( 1 + \| u_k \|_{W^{m+1,p}} + \| u_{k-1} \|_{W^{m+1,p}} \right),
\]

for some constant \( C_s > 0 \) only depending on \( m, n, p, \Omega \) and \( C_0 \).

The next lemma establishes the induction hypothesis sufficient to control the growth of the iterates allowed by (7.22) - (7.23) due to nonlinearities, and bound the iterates in the appropriate Sobolev spaces.

**Lemma 7.4.** Assume the induction hypothesis

\[
\| u_k \|_{W^{m+1,p}(\Omega)} \leq 4C_0 C_e^2,
\]

for some \( k \in \mathbb{N} \) and let \( C_e > 1 \). Then, if

\[
\epsilon \leq \epsilon_1 \equiv \min \left( \frac{1}{4C_s C_s (1 + 2C_c C_0 + 4C_s^2 C_0)}, \frac{1}{16C_M C_0 C_e^2} \right),
\]

the iterates satisfy for each \( l \in \mathbb{N} \)

\[
\| a_{k+l} \|_{W^{m,p}} \leq 2C_0 C_e \quad \text{and} \quad \| u_{k+l} \|_{W^{m+1,p}} \leq 4C_0 C_e^2
\]

and \( \epsilon_1 \leq \epsilon(k + l) \) for each \( l \in \mathbb{N} \).

Combining the induction assumption (7.26) with our estimates (7.22) - (7.23) to control the nonlinearities, we prove in [21] that for \( \epsilon \leq \epsilon_1 \) the estimate

\[
\| u_{k+1} \|_{W^{m+1,p}} + \| a_{k+1} \|_{W^{m,p}} \leq \epsilon C \| u_k \|_{W^{m+1,p}},
\]

holds for some constant \( C > 0 \) which depends only on \( m, n, p, \Omega \) and \( C_0 \). In the final step, assuming \( \epsilon \leq \min(\epsilon_1, \frac{1}{C}) \), we use a geometric sequence argument to show that \( (u_k, a_k)_{k \in \mathbb{N}} \) is a Cauchy sequence in the Banach space \( W^{m+1,p}(\Omega) \times W^{m,p}(\Omega) \) which then implies convergence to a solution to \( (u, a) \) of (7.17). See [21] for detailed proofs.

\[\text{Note: convergence of } \psi_k^* \text{ and } y_k^* \text{ follows directly from the convergence of } a_k \text{ and } u_k, \text{ because the auxiliary iterates } \psi_k^* \text{ and } y_k^* \text{ are only coupled to the equations for } a_k \text{ and } u_k \text{ through the boundary data (7.12), which we estimate using the "Trace Theorem" together with elliptic estimates and bounds on the nonlinear sources in terms of } u_k, \text{ c.f. [21].}\]
As a final remark, note that the iteration scheme converges without the need to restrict to a subsequence, so the iteration scheme supplies a numerical algorithm for constructing coordinate systems of optimal connection regularity.

CONCLUSION

Albert Einstein presented his theory of General Relativity in spacetime, a four dimensional geometric framework in which time was given the same status as space. Even though modern analysts have returned to the classical 3 + 1 framework in order to apply theorems from classical PDE theory, here we see that Einstein’s original four dimensional geometrical framework is fundamental for the derivation of an elliptic system of equations, the RT-equations, which tell how to construct coordinate systems of optimal metric regularity. The iteration scheme introduced to prove existence of solutions to the RT-equations provides an explicit numerical algorithm for constructing coordinate systems of optimal metric regularity above a threshold level of smoothness.

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