The square lattice Ising model on the rectangle

II: Finite-size scaling limit

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Abstract

Based on the results published recently [1], the universal finite-size contributions to the free energy of the square lattice Ising model on the $L \times M$ rectangle, with open boundary conditions in both directions, are calculated exactly in the finite-size scaling limit $L, M \to \infty$, $T \to T_c$, with fixed temperature scaling variable $x \propto (T/T_c - 1)M$ and fixed aspect ratio $\rho \propto L/M$. We derive exponentially fast converging series for the related Casimir potential and Casimir force scaling functions. At the critical point $T = T_c$ we confirm predictions from conformal field theory [2, 3].

The presence of corners and the related corner free energy has dramatic impact on the Casimir scaling functions and leads to a logarithmic divergence of the Casimir potential scaling function at criticality.
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I. INTRODUCTION

In the first part of this work \[1\], denoted I in the following, we computed the partition function $Z$ of the two-dimensional Ising model on the $L \times M$ rectangle with open boundary conditions in both directions and with anisotropic reduced couplings $K^{\leftrightarrow}$ and $K^{\updownarrow}$ in horizontal and vertical direction, in units of $k_B T$ with Boltzmann constant $k_B$, at arbitrary temperatures below and above the critical point. This second part is devoted to the finite-size scaling (FSS) behavior near criticality.

We first recall the main results of part I in terms of the size dependent reduced free energy $F(L, M) = -\log Z$. From (I.87) we get the total free energy of the considered model,

$$F(L, M) = -\frac{L}{2} \left[ M \log \left( \frac{2}{-z_-} \right) + \sum_{\mu=1}^{M} \hat{\gamma}_\mu \right]$$

$$- \frac{1}{2} \log \left[ \frac{2}{t_- z_-} \right] \frac{M^2}{d_{o,e}^2} T_2 \prod_{\mu=1}^{M} \left( \frac{t_+ z_+ - \hat{\lambda}_{\mu,+}}{M \hat{\lambda}_{\mu,-}^2 + z_+ \hat{\lambda}_{\mu,+} - t_+} \right)$$

$$- \log \det(1 + Y),$$

where $\hat{\lambda}_\mu = e^{\hat{\gamma}_\mu} > 1$ are the $M$ dominant eigenvalues of the $2M \times 2M$ transfer matrix $T_2$ (I.27), given by the positive zeroes of the characteristic polynomial $P_M(\varphi)$ (I.45). Alternatively, $\hat{\lambda}_{\mu,+} = \cosh \hat{\gamma}_\mu$ are the $M$ eigenvalues of $T_+$(I.30a). $z = \tanh K^{\leftrightarrow}$ and $t = \exp(-2K^{\updownarrow})$ parametrize the two couplings, and $a_{\pm} = \frac{1}{2}(a \pm a^{-1})$ is a handy shortcut from (I.20). For a definition of the other quantities $d_{o,e}$ (I.70b), $v_\mu$ (I.87a) and the $M/2 \times M/2$ residual matrix $Y$ (I.87b) the reader is referred to part I. In \[1\] we decomposed the leading term $F_{\text{strip}}$ from (I.92) into two parts,

$$F_{\text{strip}}(L, M) = L f_{b,s}(M) + F_{s,c}(M),$$

where $f_{b,s}(M)$ and $F_{s,c}(M)$ are the contributions from the bulk and from the two horizontal $(\leftrightarrow)$ surfaces, and $F_{s,c}(M)$ with the remaining contributions from the two vertical $(\updownarrow)$ surfaces and from the four corners, see figure \[1\]. These terms have been analyzed in great detail by R. J. Baxter recently\[2\].

\footnote{The couplings $(z, t)$ are denoted $(t^*, u^*)$ in \[4\], where $z^* = \frac{1}{1+z}$ is the dual of $z$.}
Figure 1. Sketch of the decomposition of the total free energy $F(L, M)$ (dotted line) into the different constituents. The black rectangle is the system, and the other solid lines denote infinite volume contributions solely dependent on temperature, while the dashed lines symbolize free energy parts containing residual finite-size contributions.

For a detailed discussion of the different free energy contributions we also recall the definition of the (total) residual free energy, or Casimir potential, $F_{\text{res}}$ (I.89), which is responsible for nontrivial finite-size effects such as the critical Casimir effect [5],

$$F_{\text{res}}(L, M) \equiv F(L, M) - F(\infty)(L, M),$$

(3a)

where the infinite volume contribution $F(\infty)$ can be decomposed into bulk, surface and corner contributions according to (I.90)

$$F(\infty)(L, M) \equiv LMf_b + Lf_s^{\leftrightarrow} + Mf_s^{\uparrow\downarrow} + f_c,$$

(3b)

for our rectangular geometry. The bulk free energy per spin $f_b$, the surface free energies per surface spin pair $f_s^\delta$, with direction $\delta \in \{\leftrightarrow, \uparrow\downarrow\}$, and the corner free energy $f_c$ are defined in the thermodynamic limit $L, M \to \infty$ and do not depend on $L$ or $M$. While $f_b$ and $f_s^\delta$ are known since a long time from the seminal works of Onsager [6] and McCoy.
& Wu [7], the corner free energy $f_c$ below $T_c$ was derived only recently by Baxter [4], confirming a conjectured product formula by Vernier & Jacobsen [8]. The corresponding product formula for temperatures above $T_c$ was given in (I.A7d). The logarithmic divergence $f_c \simeq \frac{1}{8} \log |1 - T/T_c| + \mathcal{O}(1)$ detailed in appendix B will lead to a considerable complication of the FSS analysis, as shown below.

Comparing (1) with (3), we first focus on the $L$ dependent terms. The free energy per row $f_{b,s}$ is the dominant outcome of one application of the transfer matrix $T_2$. It can be further decomposed,

$$f_{b,s}(M) = M f_b + f_{s}^{\leftrightarrow} + f_{b,s}^{\text{res}}(M),$$

and contains a residual contribution at finite $M$ that is equal to the leading large-$L$ contribution to $F_\infty^{\text{res}}$ from (I.94),

$$f_{b,s}^{\text{res}}(M) \equiv \lim_{L \to \infty} L^{-1} F_\infty^{\text{res}}(L, M).$$

Inserting (4) into (1),

$$F(L, M) = L[M f_b + f_s^{\leftrightarrow} + f_{b,s}^{\text{res}}(M)] + F_{s,c}(M) + F_{\text{strip}}(L, M),$$

and matching with (3), we can eliminate the leading $L$-dependent terms and find three contributions to the total residual free energy (3a),

$$F_\infty^{\text{res}}(L, M) = L f_{b,s}^{\text{res}}(M) + F_{s,c}^{\text{res}}(M) + F_{\text{strip}}^{\text{res}}(L, M),$$

where we defined the residual surface-corner contribution

$$F_{s,c}^{\text{res}}(M) \equiv F_{s,c}(M) - [M f_s^{\leftrightarrow} + f_c],$$

which is independent of $L$. Equation (7) shows that the residual free energy can be decomposed very similar to (3).

We now turn to the critical Casimir force per area $M$, that is defined as $L$-derivative of the total residual free energy (3a),

$$\mathcal{F}(L, M) \equiv -\frac{1}{M} \frac{\partial}{\partial L} F_\infty^{\text{res}}(L, M),$$

2 A constant term $- \log 2$ attributed to $f_c^<\leq$ in [1, 4] stems from the broken symmetry below $T_c$ and has to be moved from $F_\infty$ to $F_\infty^{\text{res}}$, see section VC.
and by \( \mathcal{F}(L, M) = -\frac{1}{M} F_{\text{res},s}(M) - \frac{1}{M} \frac{\partial}{\partial L} F_{\text{strip}}(L, M) \),

where the first one \( F_{\text{res},s} \) is already known from the stripe geometry \( L/M \to \infty \). Note that the surface-corner contribution \( F_{\text{res},c} \) drops out in the \( L \)-derivative as expected \([9, 10]\), which renders the analysis of the Casimir force \( F \) simpler than the analysis of the Casimir potential \( F_{\infty} \).

With these definitions, we now summarize the FSS theory for the Casimir potential and Casimir force, and introduce the corresponding universal FSS functions.

II. FINITE-SIZE SCALING THEORY

In this chapter we will formulate the finite-size scaling theory for the residual free energy, or critical Casimir potential, as well as for the critical Casimir force per area in the general case of a \( d \)-dimensional weakly anisotropic system\(^3\) with size \( V = L_{\leftrightarrow} L_{\updownarrow}^{d-1} \) and different couplings \( K_{\leftrightarrow} \) and \( K_{\updownarrow} \). We use \( \leftrightarrow \) for the direction parallel to the force and \( \updownarrow \) for all other directions, and we rewrite \( L_{\leftrightarrow} \equiv L \) and \( L_{\updownarrow} \equiv M \) in this and in the following chapter. When we later apply this theory to our model, we will let the transfer matrix \( T \) propagate parallel to the Casimir force, as the calculation of the force requires the derivative of the residual free energy with respect to \( L \).

Near criticality, the anisotropic couplings \( K_{\leftrightarrow} \) and \( K_{\updownarrow} \) lead to weakly anisotropic critical behavior, characterized by a weakly anisotropic bulk correlation length\(^4\)

\[
\xi_{\delta,\infty}(\tau) \overset{\tau \to 0}{\approx} \xi_{\pm,\infty}^\delta \tau^{-\nu},
\]

where \( \tau = T/T_c - 1 \) denotes the reduced temperature, and \( \xi_{\pm,\infty}^\delta \) denotes the correlation length amplitude in direction \( \delta \in \{\leftrightarrow, \updownarrow\} \) above criticality. The correlation length exponent is \( \nu = 1 \) for the 2d Ising model. Finite-size scaling theory predicts that near the critical point \( \tau \to 0 \) and for \( L_\delta \to \infty \) with fixed geometric aspect ratio \( r = L_{\leftrightarrow}/L_{\updownarrow} \), the residual free energy \( F_{\infty} \)\(^3\) only depends on the two length ratios \( L_{\leftrightarrow}/\xi_{\infty}^{\leftrightarrow}(\tau) \) and \( L_{\updownarrow}/\xi_{\infty}^{\updownarrow}(\tau) \)\(^1\)\(^1\)\(^1\). These two

\(^3\) For a discussion of weakly vs. strongly anisotropic critical behavior see, e.g., \([11]\).

\(^4\) “\( \approx \)” denoted “asymptotically equal”
ratios can be combined to a *reduced aspect ratio*

\[
\rho \equiv \frac{L_{\leftrightarrow}}{L_{\uparrow} \xi_{\uparrow}} \approx 0 \frac{L_{\uparrow}}{L_{\leftrightarrow} \xi_{\uparrow}} (\tau),
\]

(12a)

which does not depend on temperature and encodes both the system shape as well as the coupling anisotropy [11].

Approaching the critical point, the critical correlations are bounded by the smallest length ratio in the system. For the given geometry with arbitrary reduced aspect ratio \(\rho\) this leads to three different possible choices for the temperature scaling variable \(x\) [14, 15]: If \(0 \leq \rho \lesssim 1\) (\(1 \lesssim \rho \leq \infty\)), the correlations are limited by \(L_{\leftrightarrow}\) \((L_{\uparrow})\), while for arbitrary finite aspect ratio \(0 < \rho < \infty\) the geometric mean \(L_{o} \equiv V^{1/d}\) can be used as relevant length, avoiding the preference for one direction \(\delta\). In all three cases, the temperature scaling variable is given by

\[
x_{\delta} \equiv \tau \left( \frac{L_{\delta}}{\xi_{\delta}} \right)^{\frac{1}{\nu}} \approx 0 \left( \frac{L_{\delta}}{\xi_{\delta}^{\nu}} (\tau) \right)^{\frac{1}{\nu}}, \quad \delta \in \{\leftrightarrow, \circ, \uparrow\},
\]

(12b)

with geometric mean correlation length \(\xi_{\circ}^{\infty}\) satisfying \(\xi_{o}^{d} \equiv \xi_{\circ}^{\infty} \xi_{\uparrow}^{d-1}\), and the system is completely described by the two scaling variables \(x_{\delta}\) and \(\rho\) in the FSS limit. The three variables \(x_{\delta}\) obey the identity

\[
x_{\leftrightarrow} \rho^{\frac{1}{d} - \frac{1}{\nu}} = x_{\circ} = x_{\uparrow} \rho^{\frac{1}{d} - \frac{1}{\nu}}.
\]

(13)

Following Fisher & de Gennes [5], the residual free energy \(3a\) then fulfills the scaling ansatz \([14,15]\)

\[
F_{\infty}^{\text{res}}(L_{\leftrightarrow}, L_{\uparrow}) \approx \rho 1^{-d} \Theta_{\leftrightarrow}(x_{\leftrightarrow}, \rho) = \Theta_{o}(x_{o}, \rho) = \rho \Theta_{\uparrow}(x_{\uparrow}, \rho),
\]

(14)

with the universal Casimir potential scaling functions \(\Theta_{\delta}\). Note that both \(\Theta_{\leftrightarrow}(x_{\leftrightarrow}, \rho \to 0)\) and \(\Theta_{\uparrow}(x_{\uparrow}, \rho \to \infty)\) are finite by construction, while \(\Theta_{o}(x_{o}, \rho)\) diverges in both limits [14].

The Casimir force in \(\leftrightarrow\) direction per area \(L_{\uparrow}^{d-1}\),

\[
F(L_{\leftrightarrow}, L_{\uparrow}) \equiv -\frac{1}{L_{\uparrow}^{d-1}} \frac{\partial}{\partial L_{\leftrightarrow}} F_{\infty}^{\text{res}}(L_{\leftrightarrow}, L_{\uparrow}),
\]

(15)

satisfies the finite-size scaling form

\[
F(L_{\leftrightarrow}, L_{\uparrow}) \approx L_{\delta}^{-d} \vartheta_{\delta}(x_{\delta}, \rho)
\]

(16)

for all three cases \(\delta \in \{\leftrightarrow, \circ, \uparrow\}\), leading to the identities \([14,15]\)

\[
\rho^{-d} \vartheta_{\leftrightarrow}(x_{\leftrightarrow}, \rho) = \vartheta_{o}(x_{o}, \rho) = \rho \vartheta_{\uparrow}(x_{\uparrow}, \rho)
\]

(17)
analogous to (14). We conclude with three scaling relations between \( \Theta \delta \) and \( \vartheta \delta \)[14, 15],

\[
\begin{align*}
\vartheta_{\leftrightarrow}(x_{\leftrightarrow}, \rho) &= - \left[ 1 - d + \frac{1}{\nu} \frac{x_{\leftrightarrow}}{\partial} \frac{\partial}{\partial \rho} \right] \Theta_{\leftrightarrow}(x_{\leftrightarrow}, \rho) \\
\vartheta_{o}(x_{o}, \rho) &= - \left[ \frac{1}{\nu} \frac{x_{o}}{\partial} + \frac{\rho \partial}{\partial \rho} \right] \Theta_{o}(x_{o}, \rho) \\
\vartheta_{\uparrow}(x_{\uparrow}, \rho) &= - \left[ 1 + \frac{\rho \partial}{\partial \rho} \right] \Theta_{\uparrow}(x_{\uparrow}, \rho) = - \frac{\partial}{\partial \rho} \left[ \rho \Theta_{\uparrow}(x_{\uparrow}, \rho) \right].
\end{align*}
\]

(18)

The FSS functions defined above are universal and only depend on the bulk and surface
universality classes, the system shape and the boundary conditions. This universality was
clearly demonstrated in [16], where the Casimir force scaling function \( \vartheta(x, 0) \) of the XY
universality class with Dirichlet boundary conditions showed quantitative agreement between
experiments on liquid \(^4\)He at the \( \lambda \) transition [17] and Monte Carlo simulations of the
classical XY spin model on a simple cubic lattice [16], both in the thin film limit \( \rho \to 0 \).
Subsequent theoretical studies [18, 19] as well as experiments on binary liquid mixtures
[20–22] demonstrated the universal behavior within the framework of the Ising universality
class, for an overview see, e.g., [23].

III. SCALING FUNCTIONS FOR THE CONSIDERED GEOMETRY

In a transfer matrix (TM) formulation as utilized in part I, the system can have arbitrary
real length in propagation direction of the TM, while the other \( d - 1 \) lengths are fixed.
Therefore, we identify \( \leftrightarrow \) with the propagation direction and use the scaling variable \( x_{\uparrow} \)
and the corresponding scaling functions \( \Theta_{\uparrow}(x_{\uparrow}, \rho) \) and \( \vartheta_{\uparrow}(x_{\uparrow}, \rho) \) for the description of the FSS
behavior. In the following we will usually drop the index \( \uparrow \) from the quantities \( x_{\uparrow} \equiv x \),
\( \Theta_{\uparrow} \equiv \Theta \) and \( \vartheta_{\uparrow} \equiv \vartheta \) for simplicity.

Combining the residual free energy decomposition (7) with the scaling form (14) we now
discuss the according decomposition of the scaling functions \( \Theta(x, \rho) \) and \( \vartheta(x, \rho) \) for the
considered Ising model on the rectangle. Note that this discussion can be generalized to
higher dimensions within a TM formulation. According to (7) and (14) we decompose \( \Theta \)
into three parts,

\[
\Theta(x, \rho) = \Theta^{(oo)}(x) + \rho^{-1} \Theta_{s,c}(x) + \Psi(x, \rho),
\]

(19)

where

\[
\Theta^{(oo)}(x) \simeq L_{\uparrow} f^{\text{res}}_{b,s}(L_{\uparrow})
\]

(20)
is identified with the known Casimir potential scaling function for open boundary conditions in strip geometry $\rho \to \infty$ \cite{24,26},

$$\Theta^{(\infty)}(x) \equiv -\frac{1}{2\pi} \int_0^\infty d\omega \log \left(1 + \frac{\sqrt{x^2 + \omega^2} - x}{\sqrt{x^2 + \omega^2} + x} e^{-2\sqrt{x^2 + \omega^2}}\right),$$

(21)

which becomes independent from the BCs in $\leftrightarrow$ direction in this limit and can therefore be calculated from the known exact solution with periodic BCs in $\leftrightarrow$ direction \cite{15}. The second term in (19) contains the surface contributions from the $\uparrow\downarrow$ edges as well as the corner contributions,

$$\Theta_{s,c}(x) \approx F_{s,c}^{\text{res}}(L_{\downarrow}),$$

(22)

is independent of $\rho$ and will be discussed in chapter \textit{VC}. Finally, the third term in (19) describes the strip residual free energy contribution, which fulfills

$$\rho \Psi(x, \rho) \equiv -\log \Sigma(x, \rho) \simeq F_{\text{strip}}^{\text{res}}(L_{\leftrightarrow}, L_{\downarrow}),$$

(23)

where the scaling function $\Sigma$ of the strip residual partition function (I.83) is given by

$$\Sigma(x, \rho) \simeq Z_{\text{strip}}^{\text{res}}(L_{\leftrightarrow}, L_{\downarrow}) = e^{-F_{\text{strip}}^{\text{res}}(L_{\leftrightarrow}, L_{\downarrow})},$$

(24)

and will be computed in the next chapter. From the limit $\Psi(x, \rho \to \infty) \to 0$ \cite{11} we get the expected result

$$\lim_{\rho \to \infty} \Theta(x, \rho) = \Theta^{(\infty)}(x).$$

(25)

According to (10), the total Casimir force scaling function (16) can be decomposed to

$$\vartheta(x, \rho) = -\Theta^{(\infty)}(x) + \psi(x, \rho),$$

(26)

where the strip Casimir force from (10),

$$F_{\text{strip}}(L_{\leftrightarrow}, L_{\downarrow}) \equiv -\frac{1}{L_{\downarrow}^{d-1}} \frac{\partial}{\partial L_{\downarrow}} F_{\text{strip}}^{\text{res}}(L_{\leftrightarrow}, L_{\downarrow}),$$

(27)

corresponds to the second term,

$$\psi(x, \rho) \equiv \frac{\partial}{\partial \rho} \log \Sigma(x, \rho) \simeq L_{\downarrow}^d F_{\text{strip}}(L_{\leftrightarrow}, L_{\downarrow}),$$

(28)

while the other contribution

$$\Theta^{(\infty)}(x) \simeq -L_{\downarrow}^d F_{b,s}(L_{\downarrow}) = L_{\downarrow} \frac{\partial}{\partial L_{\downarrow}} [L_{\leftrightarrow} f_{b,s}^{\text{res}}(L_{\uparrow})] = L_{\downarrow} f_{b,s}^{\text{res}}(L_{\downarrow}),$$

(29)

is again known \cite{21} in our case. Note that the total Casimir force scaling function (26) could also have been obtained using the scaling relation (\textit{18c}). In the following, we will first calculate the FSS function $\Sigma(x, \rho)$ and determine the Casimir force scaling function $\vartheta(x, \rho)$. 
IV. FINITE-SIZE SCALING LIMIT OF THE STRIP RESIDUAL PARTITION FUNCTION

A. The characteristic polynomial

We now turn back to the 2d Ising model on the $L \times M$ rectangle discussed in I and revert the variables $L_{\leftrightarrow}$ and $L_{\uparrow}$ back to $L \equiv L_{\leftrightarrow}$ and $M \equiv L_{\uparrow}$. We are allowed to assume isotropic couplings $t^* = z$ in the FSS limit in order to keep things simple, as near criticality a coupling anisotropy $K_{\leftrightarrow} \neq K_{\uparrow}$ can be compensated by a scale transformation of the real variable $L \mapsto L\xi_{\uparrow}/\xi_{\leftrightarrow}$ in conjunction with $K_{\leftrightarrow} \mapsto K_{\uparrow}$, without changing the generalized aspect ratio $\rho$ [11]. Then, the critical point is at $z = z_c \equiv \sqrt{2} - 1$, and we can use $\tau = 1 - z/z_c$ for the reduced temperature, with corresponding isotropic correlation length amplitude $\xi_+ = 1/2$. The resulting FSS variables $x$ and $\rho$ from (12) in terms of the relevant length $M$ become

$$x = 2M \left(1 - \frac{z}{z_c}\right), \quad \rho = \frac{L}{M}. \quad (30)$$

We now perform the FSS limit of the results from I by replacing all quantities with the $M$-dependent scaling forms and then performing the limit $M \to \infty$ with fixed $x$ and $\rho$. As the strip residual partition function $Z_{\text{strip}}^{\text{res}}$ is convergent in this limit, we do not encounter
Figure 3. Universal characteristic polynomial $P(\Phi)$, Eq. (33), for different scaled temperatures $x$ in the finite-size scaling limit. The first zero is doubly degenerate at $x = -1$ and becomes imaginary below.

regularization problems as in the case of the edge contribution $F_{s,c}$ discussed later. We will use the tilde $\tilde{}$ for quantities in the FSS limit and capitals for scaling variables.

Therefore, we replace the temperature variable $z$ and the length $L$ according to

$$z \mapsto z_c \left(1 - \frac{x}{2M}\right), \quad L \mapsto \rho M, \quad (31)$$

and first turn to the characteristic polynomial $P_M(\varphi)$ (I.45). For large $M$ the relevant scaling contributions are obtained by a rescaling of the angle variable $\hat{\varphi}$ (I.86) according to

$$\varphi \mapsto \frac{\Phi}{M}, \quad (32)$$

which immediately leads to the remarkably simple universal FSS form of the characteristic polynomial (I.45),

$$P(\Phi) \equiv \cos \Phi + \frac{x}{\Phi} \sin \Phi, \quad (33)$$

with infinitely many zeroes $\Phi_\mu, \mu \in \mathbb{N}$. All zeroes $\Phi_\mu$ are real and positive except $\Phi_1$, which is zero for $x = -1$ and becomes imaginary for $x < -1$, see figure 3 and table 1.

Comparing $P(\Phi)$ with $P_M(\varphi)$ of a finite system as shown in figure 2, we notice the following: while for small $|\varphi| \lesssim \pi/2$ both curves become more and more similar for large $M$,
Table I. Location of the first few zeroes of $P(\Phi)$, for $\mu = 1, \ldots, 4$ and several values of $x$.

| $x$ | $\Phi_1$ | $\Phi_2$ | $\Phi_3$ | $\Phi_4$ |
|-----|----------|----------|----------|----------|
| -4  | 3.997302692 i | 3.916435368 | 7.355927023 | 10.63585142 |
| -3  | 2.984704585 i | 4.078149765 | 7.472192660 | 10.7277106 |
| -2  | 1.915008048 i | 4.274782271 | 7.596546020 | 10.81267333 |
| -1  | 0        | 4.493409458 | 7.725251837 | 10.90412166 |
| 0   | $\pi/2$  | $3\pi/2$  | $5\pi/2$  | $7\pi/2$  |
| 1   | 2.028757838 | 4.913180439 | 7.978665712 | 11.08553841 |
| 2   | 2.288929728 | 5.086985094 | 8.096163603 | 11.17270587 |
| 3   | 2.455643863 | 5.232938454 | 8.204531363 | 11.25604301 |
| 4   | 2.570431560 | 5.354031841 | 8.302929183 | 11.33482558 |

The deviations for $|\varphi| > \pi/2$ stem from the lattice dispersion encoded in $P_M(\varphi)$, which is different from the continuum dispersion of $P(\Phi)$. We will see below that a careful regularization of the resulting infinite products is necessary in order to overcome the UV singularities emerging in the FSS limit.

The scaling limit of the eigenvalues $\hat{\lambda}$, expressed through the Onsager-$\hat{\gamma}$, is determined from the isotropic version of (I.43),

$$\cos \hat{\varphi} = z_+^* z_+ - \cosh \hat{\gamma},$$

under the rescaling

$$\hat{\gamma} \mapsto \frac{\Gamma}{M}, \quad \hat{\lambda}^{-L} = e^{-L\hat{\gamma}} \mapsto e^{-\rho\Gamma},$$

(35)

to be

$$\Gamma = \sqrt{x^2 + \Phi^2}.$$  (36)

At the zeroes $\Phi_\mu$ we find the simple relation

$$\Phi_\mu \cot \Phi_\mu = \sigma_\mu \Gamma_\mu \cos \Phi_\mu = -x,$$

with parity (I.76)

$$\sigma_\mu \equiv (-1)^{\mu-1}.$$  (38)

The two remaining quantities entering the residual matrix $Y$ (I.87b), $p_\mu$ and $v_\mu$ from (I.74) and (I.87a), are calculated in the next section.
Figure 4. Complex zeroes of $P(\Phi)$ for three temperatures $x = \{-\frac{5}{4}, 0, \frac{5}{4}\}$ below (blue), at (black), and above (red) the critical point $T_c$, for $\mu = 4$. At $x = -1$ the first zero $\Phi_1 = 0$. The upper/lower plot shows the integration contours $\tilde{C}_\pm$ from (41c) (green lines) as well as $C_0$ from (50) and $C_\pm$ from (49) (brown lines) for the terms $\nu \leq \mu$, respectively. The log/sqrt branch cuts are shown as solid/dashed magenta lines.

**B. Contour integration**

We first calculate the regularized scaling limit $\tilde{p}_\mu$ of the product $p_\mu$ (I.74) using the rescaling

$$c_\mu - c_\nu = \cos \hat{\varphi}_\mu - \cos \hat{\varphi}_\nu \mapsto \cos \frac{\Phi_\mu}{M} - \cos \frac{\Phi_\nu}{M} = \frac{\Phi_\nu^2 - \Phi_\mu^2}{2M^2} + O(M^{-4}).$$

An important simplification stems from the fact that the residual matrix $Y$ only contains products $p_\mu \in \alpha p_\nu \in \varepsilon$, of one odd and one even factor, respectively, such that $\mu$-independent terms in $(\cdot)^{-\sigma_\mu \sigma_\nu}$ cancel in the resulting products. We can therefore drop the factor $1/2M^2$ and instead insert a regularizing denominator, that ensures the convergence of the infinite product, to get

$$\tilde{p}_\mu \equiv \lim_{N \to \infty} \frac{\prod_{\nu=1}^N (\Phi_\nu^2 - \Phi_\mu^2)^{-\sigma_\mu \sigma_\nu}}{\prod_{\nu=1}^N (\Phi_\nu^2)^{-\sigma_\mu \sigma_\nu}} = \Phi_\mu^2 \prod_{\nu=1}^\infty \left(1 - \frac{\Phi_\nu^2}{\Phi_\mu^2}\right)^{-\sigma_\mu \sigma_\nu}. \quad (40)$$
Here, $\prod'$ denotes the regularized product, with zero and infinite factors removed. This alternating product over the zeroes $\Phi_\nu$ of $P(\Phi)$ [33] can be calculated by complex contour integration using Cauchy’s residue theorem: we first rewrite the product as two sums, split into terms $\nu<\mu$ and $\nu>\mu$, respectively, as we have to avoid the zero $\Phi_\mu$. In the first sum, the argument of the log is negated to always be positive, leading to an extra overall factor $\sigma_\mu$. This circumvents the cut of the logarithm in the complex plane (solid magenta lines on the real axes in figure 4) in the resulting contour integrals. We find

$$\tilde{p}_\mu = \Phi_\mu^2 \exp \left[ -\sigma_\mu \sum_{\nu=1}^{\infty} \sigma_\nu \log \left( 1 - \frac{\Phi_\nu^2}{\Phi_\mu^2} \right) \right]$$ (41a)

$$= \sigma_\mu \Phi_\mu^2 \exp \left[ -\sigma_\mu \left\{ \sum_{\nu=1}^{\mu-1} \sigma_\nu \log \left( \frac{\Phi_\nu^2}{\Phi_\mu^2} - 1 \right) + \sum_{\nu=\mu+1}^{\infty} \sigma_\nu \log \left( 1 - \frac{\Phi_\nu^2}{\Phi_\mu^2} \right) \right\} \right]$$ (41b)

$$= \sigma_\mu \Phi_\mu^2 \exp \left[ -\sigma_\mu \sum_{\nu=1}^{\infty} \frac{1}{2\pi i} \oint_{\tilde{C}_\pm} d\Phi \log \left( \pm \frac{\Phi_\nu^2 - \Phi_\mu^2}{\Phi_\mu^2} \right) R(\Phi) \right]$$, (41c)

with alternating counting polynomial $R(\Phi)$ fulfilling

$$\text{Res}_{\Phi=\Phi_\nu} R(\Phi) = \sigma_\nu,$$ (42)

which will be constructed in the next chapter. The two contours $\tilde{C}_\pm$ enclose the respective zeroes $\Phi_\nu$ and are shown as green lines in figure 4.

**C. Construction of counting polynomial $R(\Phi)$**

The alternating counting polynomial $R(\Phi)$ satisfying (42) is constructed in the following way: we discriminate the even and odd zeroes by first rewriting

$$P(\Phi) = \cos \Phi + \frac{x}{\Phi} \sin \Phi = \Re \left[ \exp \left( \frac{x}{i\Phi} \right) \right].$$ (43)

Normalizing the modulus of $1 + x/(i\Phi)$ to one, the zeroes of the real part move to $\pm i$, and the condition for the odd (+) and even (−) zeroes becomes

$$P_{\pm}(\Phi) \equiv 1 \pm \exp \left( \frac{x}{i\Phi} \right) = 0,$$ (44)

with $\Gamma = \sqrt{x^2 + \Phi^2}$ from [36], leading to the odd and even counting polynomials

$$R_{\pm}(\Phi) \equiv Q_{\pm}'(\Phi),$$ (45)

5 The case $\mu = 1$, $x < -1$, with imaginary $\Phi_1$, has to be handled separately, but leads to the same result for $P_{\pm}$. 
fulfilling \( \text{Res}_{\Phi = \Phi_0} R_\pm(\Phi) = \delta_{\sigma_\mu, \pm 1} \), with symmetrized antiderivatives

\[
Q_\pm(\Phi) \equiv \frac{1}{2} \log[P_\pm(\Phi)P_\pm(-\Phi)] = \frac{1}{2} \log \left[ 2 \left( 1 \pm \frac{x}{\Gamma} \cos \Phi \mp \frac{\Phi}{\Gamma} \sin \Phi \right) \right].
\]  (46)

For the alternating counting polynomial \( R(\Phi) \equiv Q'(\Phi) \) we find the antiderivatives

\[
Q(\Phi) \equiv Q_+(\Phi) - Q_-(\Phi) = \text{artanh} \left( \frac{x}{\Gamma} \cos \Phi - \frac{\Phi}{\Gamma} \sin \Phi \right),
\]  (47)
leading to the result

\[
R(\Phi) = -\sqrt{\frac{\Gamma^2}{\Phi^2} + \frac{x}{\cos \Phi} - \frac{2}{\Gamma^2}} R(\Phi),
\]  (48)
where \( \mathbb{C}^+ \) denotes the complex domain \( \{ z \in \mathbb{C} | -\frac{\pi}{2} < \arg z \leq \frac{\pi}{2} \lor z = 0 \} \). The resulting analytic structure of the integrands is shown in figure 4. \( R(\Phi) \) has square root branch cuts from \( \pm ix \) to \( \pm i\infty \) as well as a simple pole at zero, with residuuum \( \text{Res}_{\Phi=0} R(\Phi) = -1 \).

Additionally, the logarithm contributes log branch cuts running either from \( -\Phi \mu \) to \( \Phi \mu \) for \( \mathcal{C}_+ \), or from \( \pm \Phi \mu \) to \( \pm \infty \) for \( \mathcal{C}_- \).

**D. Calculation of the integrals**

We now deform the contour \( \tilde{C}_- \) to \( C_0 \) at the imaginary axis and to the line \( C_- \) and move \( \tilde{C}_+ \) to \( C_+ \) as shown in figure 4. The resulting contribution from \( C_\pm \) can be calculated using (47) and reads

\[
-\sigma_\mu \sum_{\pm} \frac{1}{2\pi i} \int_{C_\pm} d\Phi \log \left( \pm \frac{\Phi^2 - \Phi_\mu^2}{2} \right) R(\Phi) = -\log \left[ \frac{\Phi_\mu}{4} \left( \frac{1}{x + \Gamma^{-2}} \right) \right].
\]  (49)

The pole at zero and the special behavior of \( \Phi_1 \) at \( x = -1 \) has to be carefully analyzed, leading to the result

\[
\tilde{p}_\mu = \frac{4\sigma_\mu \Phi_\mu}{1 + x \Gamma^{-2}} \left( \frac{x + 1}{2x} \Phi_\mu \right)^{-\sigma_\mu} \exp \left[ \frac{\sigma_\mu}{\pi i} \int_{|x|}^{i\infty} d\Phi \log \left( \frac{x + 1}{2x} - \frac{2}{\Phi_\mu^2} \right) R(\Phi) \right],
\]  (50)
where we have used the identity, c.f. (47),

\[
\frac{1}{\pi i} \int_{|x|}^{i\infty} d\Phi R(\Phi) = \frac{1}{2} \text{sign} x - 1.
\]  (51)

In (50) we can again drop \((\mu\text{-independent terms})^{\sigma_\mu}\) to get

\[
\tilde{p}_\mu = \frac{4\sigma_\mu \Phi_\mu^{1-\sigma_\mu}}{1 + x \Gamma^{-2}} \exp \left[ \frac{\sigma_\mu}{\pi i} \int_{|x|}^{i\infty} d\Phi \log \left( 1 - \frac{\Phi_\mu^2}{\Phi_\mu^2} \right) R(\Phi) \right].
\]  (52)
We can combine this result for \( \tilde{p}_\mu^\dagger \) with the scaling form of the coefficients \( \tilde{v}_\mu \) from (I.87a),

\[
\hat{v}_\mu \mapsto \tilde{v}_\mu \equiv \tilde{p}_\mu^\dagger \sigma_\mu (\Gamma_\mu - \sigma_\mu x)^\sigma_\mu
\]

to find, as \((\Gamma - x)(\Gamma + x) = \Phi^2\), the resulting scaling form of the matrix elements

\[
\tilde{v}_\mu = \frac{4}{\Gamma_\mu - x} \exp \left[ \frac{\sigma_\mu}{\pi i} \int_{|x|}^{i\infty} d\Phi \log \left( 1 - \frac{\Phi^2}{\Phi_\mu^2} \right) R(\Phi) \right].
\]

In the special case \( \mu = 1 \) at \( x \to -1 \), where \( \Phi_1 \to 0 \), the integral diverges logarithmically while the prefactor \( \Gamma_\mu - x \) goes to zero. Using the series expansion around \( x = -1 \), \( \Phi_1 = \sqrt{3(x + 1)} + \mathcal{O}(x + 1)^{3/2} \) we can nonetheless proceed and find

\[
\tilde{v}_1 |_{x = -1} = 12 \exp \left[ \frac{1}{\pi i} \int_{1}^{i\infty} d\Phi \log (-\Phi^2) R(\Phi) \right] = 6.39303337215 \ldots .
\]

The scaling form of the Cauchy matrix \( T \) from (I.70a) reads

\[
(\tilde{T})_{\mu\nu} \equiv \frac{1}{\Phi_\nu^2 - \Phi_\mu^2} = \frac{1}{\Gamma_\nu^2 - \Gamma_\mu^2},
\]

where we have moved the factor \( 2M^2 \) from the expansion around \( M = \infty \) into the Cauchy determinant (I.80). Combining (55) with the diagonal matrices

\[
(\Gamma)_{\mu\mu} \equiv \Gamma_\mu, \quad (\tilde{V})_{\mu\mu} \equiv \tilde{v}_\mu,
\]

we find the result for the residual matrix \( Y \) in the finite-size scaling limit

\[
\tilde{Y}(x, \rho) = -e^{-\rho \Gamma_e} \tilde{V}_e \tilde{T}_{e,o} e^{-\rho \Gamma_o} \tilde{V}_o \tilde{T}_{o,e},
\]

from which we can calculate the universal partition function scaling function \( \Sigma \) and the Casimir potential scaling function \( \Psi \) according to

\[
\Sigma(x, \rho) = \det [1 + \tilde{Y}(x, \rho)],
\]

\[
\Psi(x, \rho) = -\rho^{-1} \log \Sigma(x, \rho).
\]

Note that \( \Sigma \) depends on the aspect ratio \( \rho \) only via the two exponentials in \( \tilde{Y}(x, \rho) \).

**E. Representation of the partition function scaling function \( \Sigma(x, \rho) \)**

While the scaled residual matrix \( \tilde{Y} \) is infinite dimensional, its matrix elements become exponentially small for large \( \mu, \nu \) at least if \( \rho \gtrsim 1 \),

\[
(\tilde{Y})_{\nu e, \mu o} = \mathcal{O}(e^{-\rho(\Gamma_\nu + \Gamma_\mu)}),
\]
such that we can take the upper left $N \times N$ submatrix for a rapidly converging calculation of the determinant. This direct approach is however only applicable for $N \lesssim 10$ if $\rho$ is left as a free parameter, as the symbolic evaluation of a general $N \times N$ determinant is exponentially hard and requires $N!$ terms. However, we alternatively can expand the determinant according to (I.97) and directly calculate all terms $O(e^{-2\pi \rho n})$ up to $n \leq N$. Therefore we define the set $\mathcal{S}_n$ of all subsets $s$ of the natural numbers with equal number of even and odd elements $\mu$ fulfilling the condition

$$\mathcal{S}_n = \left\{ s \mid s \subset \mathbb{N} \land \sum_{\mu \in s} \sigma_\mu = 0 \land \sum_{\mu \in s} (\mu - \frac{1}{2}) = 2n \right\},$$

(60)
e.g., $\mathcal{S}_1 = \{\{1, 2\}\}$ and $\mathcal{S}_4 = \{\{1, 8\}, \{3, 6\}, \{5, 4\}, \{7, 2\}, \{1, 3, 2, 4\}\}$. We then can define the $N$-th approximant to the partition function scaling function

$$\Sigma^{(N)}(x, \rho) \equiv 1 + \sum_{n=1}^{N} \sum_{s \in \mathcal{S}_n} a_s e^{-\rho \Gamma_s},$$

(61a)

with

$$a_s \equiv \prod_{\{\mu, \nu\} \subset s} \left( \Phi^2_{\mu} - \Phi^2_{\nu} \right)^{-2\sigma_\mu \sigma_\nu} \prod_{\mu \in s} v_\mu, \quad \Gamma_s \equiv \text{Tr} \Gamma_s = \sum_{\mu \in s} \Gamma_\mu,$$

(61b)

to get an exponentially precise approximation to the scaling function (58a)

$$\Sigma(x, \rho) = \Sigma^{(N)}(x, \rho) + o(e^{-2\pi \rho N}).$$

(62)

Note that (61a) is a perturbative series, and the number of elements in $\mathcal{S}_n$ equals the famous integer partition function $P_n$ from number theory [27], $|\mathcal{S}_n| = P_n$. Consequently, the calculation of, e.g., $\Sigma^{(30)}$ requires only 28629 terms instead of the $30! = 2.65 \times 10^{32}$ terms needed for the direct evaluation of the determinant.

In table II the leading coefficients $a_s$ and $\Gamma_s$ are given for the two cases $x = \pm 1$. Already with these few terms the error is smaller than $e^{-8\pi} \approx 10^{-11}$ for all $\rho \geq 1$, while the case $\rho < 1$ can be calculated using the symmetry under exchange of the two directions $\uparrow$ and $\leftrightarrow [14] (52)$,

$$\Theta(x, \rho) = \rho^{-2} \Theta(x\rho, \rho^{-1}).$$

(63)

With these expressions we now present results for the Casimir scaling functions. We first turn to the critical point $x = 0$. 

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### V. RESULTS

#### A. Results at $x = 0$

At criticality $x = 0$ we work with the volume FSS functions $\Sigma_\circ, \Psi_\circ$ and $\psi_\circ$ in order to compare with conformal field theory (CFT) results. Remember that $\Sigma(0, \rho) = \Sigma_\circ(0, \rho), \rho \Psi(0, \rho) = \Psi_\circ(0, \rho)$ and $\psi(0, \rho) = \psi_\circ(0, \rho)$. We use the superscript $^{(0)}$ for quantities at $x = 0$. For $x = 0$ the characteristic polynomial reduces to

$$P^{(0)}(\Phi) = \cos \Phi,$$

with trivial zeroes $\Phi^{(0)}_{\mu} = (\mu - \frac{1}{2})\pi, \mu \in \mathbb{N}$. Consequently, the infinite product $\tilde{p}_\mu^+$ from (52) can be calculated exactly at $x = 0$ and can be expressed through the Euler beta function $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$, with the result

$$\tilde{p}_\mu^{(0)} = 4\sigma_\mu \Phi^{(0)}_\mu \left[ \frac{1}{\sqrt{2\pi}}B\left(\frac{\mu}{2}, \frac{1}{2}\right) \right]^{2\sigma_\mu}.$$

The resulting coefficients $\tilde{v}_\mu$ (54a) become

$$\tilde{v}_\mu^{(0)} = \tilde{p}_\mu^{(0)} \sigma_\mu (\Phi^{(0)}_\mu)^{\sigma_\mu} = 4 \left( \Phi^{(0)}_\mu \right)^{1+\sigma_\mu} \left[ \frac{1}{\sqrt{2\pi}}B\left(\frac{\mu}{2}, \frac{1}{2}\right) \right]^{2\sigma_\mu},$$

| order $n$ | set $s$ | $x = -1$ | $x = 1$ |
|-----------|--------|----------|----------|
|           |        | $a_s$     | $\Gamma_s/2\pi$ | $a_s$ | $\Gamma_s/2\pi$ |
| 1         | $\{1, 2\}$ | 0.41416034599  | 0.89179907560 | 0.15689480307  | 1.15797017264 |
| 2         | $\{3, 2\}$ | 0.58023590813  | 1.97241431063 | 0.27677728168  | 2.07776833638 |
| 2         | $\{1, 4\}$ | 0.02228040130  | 1.90188245064 | 0.01146254079  | 2.13146302530 |
| 3         | $\{3, 4\}$ | 0.48130844027  | 2.98249768567 | 0.31674195444  | 3.05126118904 |
| 3         | $\{5, 2\}$ | 0.05012321797  | 2.9769865033  | 0.02613926677  | 3.06476730850 |
| 3         | $\{1, 6\}$ | 0.00537233691  | 2.90454040035 | 0.00297985504  | 3.12373747328 |
| 4         | $\{5, 4\}$ | 0.47345042883  | 3.98708202537 | 0.34034540402  | 4.03826016115 |
| 4         | $\{3, 6\}$ | 0.04512462939  | 3.98515563538 | 0.03206462405  | 4.0435363702 |
| 4         | $\{7, 2\}$ | 0.01444309494  | 3.9784169683  | 0.00782432208  | 4.05964386240 |
| 4         | $\{1, 8\}$ | 0.00206454953  | 3.90577401397 | 0.00118447481  | 4.12008924457 |
| 4         | $\{1, 3, 2, 4\}$ | 0.31379568621 | 3.87429676127 | 0.07798039866  | 4.20923136168 |

Table II. Amplitudes $a_s$ and exponents $\Gamma_s$ from (61a) for $x = \pm 1$ and $n = 1, \ldots, 4$. 
and with
\[ \tilde{Y}(0, \rho) = -e^{-\rho \Phi(0)} \tilde{V}_e^{(0)} T_{e,o} e^{-\rho \Phi(0)} \tilde{V}_o^{(0)} T_{o,e} \]  
(67)

from (57) and (58a) we arrive at the rapidly converging series
\[ \Sigma_o(0, \rho) = 1 + \frac{1}{4} e^{-2 \pi \rho} + \frac{13}{32} e^{-4 \pi \rho} + \frac{55}{128} e^{-6 \pi \rho} + \frac{1235}{2048} e^{-8 \pi \rho} + \frac{4615}{8192} e^{-10 \pi \rho} + \mathcal{O}(e^{-12 \pi \rho}). \]  
(68)

This result is identical to the prediction from conformal field theory [3], which can be written in several ways,
\[ \Sigma_o(0, \rho) = e^{-\frac{\pi^2}{2} \eta \rho^{\frac{1}{2}} (i \rho)} = (e^{-2 \pi \rho})^{\frac{1}{4}} \Pi \left( -\frac{1}{4} \right) e^{-2 \pi \rho}, \]  
(69)

with the Dedekind eta function \( \eta \), the \( q \)-Pochhammer symbol \( (q)_\infty \), or in terms of the \( q \)-products introduced in (I.A2). Here we have taken into account the additional contribution \( e^{-\frac{\pi^2}{2} \rho} \) from the strip geometry, see below for details. While we were not able to proof this correspondence, the series coefficients agree at least for the first 30 terms and we have no doubt about the equivalency.

Observing the fact that for both even and odd \( \mu \), \( \tilde{v}_\mu^{(0)} \) is \( 2 \pi^2 \) times a squared rational number, we define the square root
\[ \tilde{w}_\mu^{(0)} \equiv \sqrt{\tilde{v}_\mu^{(0)}} = 2 \left( \Phi_\mu^{(0)} \right)^{\frac{1+\sigma_\mu}{2}} \left( \frac{1}{\sqrt{2 \pi}} B \left( \frac{\mu}{2}, \frac{1}{2} \right) \right)^{\sigma_\mu} \]  
(70a)
\[ = \sqrt{2} \sigma_\mu \left( \mu - \frac{1}{2} \right)^{\frac{1+\sigma_\mu}{2}} B \left( \frac{1}{2}, \frac{1}{2}, \mu + \frac{1 - \sigma_\mu}{2}, \frac{1 + \sigma_\mu}{2} \right) \]  
(70b)

as \( B \left( \frac{\mu}{2}, \frac{1}{2} \right)^{-1} = -\frac{1}{2 \pi} B \left( \frac{\mu+1}{2}, -\frac{1}{2} \right) \). The generating function \( W^{(0)}(\eta) \) of the coefficients \( \tilde{w}_\mu^{(0)} \) can also be given and reads
\[ W^{(0)}(\eta) = \sum_{\mu=1}^{\infty} \tilde{w}_\mu^{(0)} \eta^{\mu-1} = \frac{\pi}{\sqrt{2}} \frac{1}{1 - \eta^{\frac{1 + \eta}{1 - \eta}}}. \]  
(71)

Using the coefficients \( \tilde{w}_\mu^{(0)} \) we can write \( \tilde{Y} \) symmetrically: defining
\[ X(0, \rho) \equiv e^{-\frac{\pi}{2} \Phi(0)} \tilde{W}_e^{(0)} T_{e,o} \tilde{W}_o^{(0)} e^{-\frac{\pi}{2} \Phi(0)} \]  
(72)

we have
\[ \Sigma_o(0, \rho) = \det \left( 1 + \bar{XX} \right), \]  
(73)

where the bar denotes the transpose. As a final remark, we point out that the general residual matrix (57) can also be written symmetrically using \( \tilde{w} = \sqrt{\tilde{v}} \). However, \( \tilde{v} \) from
(54a) is not a formal square. Maybe this symmetric representation can be utilized to proof the equivalence of (68) and (69).

From (58b) and (68) the series of the strip Casimir potential scaling function \( \Psi_o(0, \rho) \) reads, with \( q \equiv e^{-2\pi \rho} \),

\[
\Psi_o(0, \rho) = -\log \Sigma_o(0, \rho) = -\frac{1}{4} \left[ q + \frac{3}{2} q^2 + \frac{4}{3} q^3 + \frac{7}{4} q^4 + \frac{6}{5} q^5 + \frac{12}{6} q^6 + \frac{8}{7} q^7 + \mathcal{O}(q^8) \right] 
\]

with \( \Sigma_o(0, \rho) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} q^n \),

\[
(74a)
\]

\[
= -\frac{1}{4} \sum_{n=1}^{\infty} \frac{\sigma(n)}{n} q^n, 
\]

\[
(74b)
\]

with the divisor sum function \( \sigma(n) = \sum_{d|n} d \) \[27\].

The strip Casimir force at \( x = 0 \) is given by

\[
\psi_o(0, \rho) = -\partial_\rho \Psi_o(0, \rho) = -\frac{\pi}{2} \sum_{n=1}^{\infty} \sigma(n) q^n = -\frac{\pi}{48} \left[ 1 + \frac{1}{\eta'(i\rho)} \right] = \frac{\pi}{48}(E_2(i\rho) - 1), 
\]

with Ramanujan’s weight two Eisenstein series \( E_2 \) \[28\], and with special value at \( \rho = 1 \),

\[
\psi_o(0, 1) = \frac{1}{16} - \frac{\pi}{48} = -0.0029498469 \ldots . 
\]

(75b)

All these results can easily be deduced from different representations of the double series

\[
4\Psi_o(0, \rho) = -\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} q^{jk} = \sum_{j=1}^{\infty} \log(1 - q^j) = \log(q)_{\infty} = \frac{\pi \rho}{12} + \log \eta(i\rho) 
\]

which can be rewritten as a double sum over \( n = jk \) and the divisors \( d \) of \( n \),

\[
4\Psi_o(0, \rho) = -\sum_{n=1}^{\infty} \sum_{d|n} \frac{d}{n} q^n = -\sum_{n=1}^{\infty} \frac{\sigma(n)}{n} q^n. 
\]

(76)

(77)

The total Casimir force \( \vartheta_o \) at \( x_o = 0 \) is even simpler than (75a) and reads

\[
\vartheta_o(0, \rho) = \frac{\pi}{48} + \psi_o(0, \rho) = -\frac{i}{4} \frac{\eta'(i\rho)}{\eta(i\rho)} = \frac{\pi}{48} E_2(i\rho), 
\]

leading to the result of Cardy & Peschel \[2\], that the amplitude of the logarithmic divergence of the Ising finite size free energy is \( \frac{1}{16} \), as

\[
\vartheta_o(0, 1) = \frac{\pi}{48} + \left( \frac{1}{16} - \frac{\pi}{48} \right) = \frac{1}{16}. 
\]

(78)

(79)

In chapter \[VD\] we will return to this point.
B. Results for general $x$ and $\rho$

Using [61], we can calculate the FSS functions for given $x$ with arbitrary precision, while the aspect ratio $\rho$ remains a free parameter in the expressions. However, we first have to estimate the surface-corner contribution $\Theta_{s,c}(x)$ for a complete picture.

C. The Casimir potential scaling function $\Theta(x,\rho)$

For the computation of the (total) Casimir potential scaling function $\Theta(x,\rho)$ we need the surface-corner contribution $\Theta_{s,c}(x)$ from (19),

$$\Theta_{s,c}(x) = -\rho \Theta^{(oo)}(x) + \log \Sigma(x,\rho) + \Theta_{o}(x,\rho) \quad \forall \rho,$$

with volume scaling variable $x_o = x \rho^{1/2}$ from [13], that unfortunately could not be calculated directly from the regularized FSS limit of $F_{s,c}^{reg}(L_T)$ yet. However, we can utilize the symmetry of the square, where $\rho = 1$, under the exchange of the two lattice directions $\leftrightarrow$ and $\uparrow\downarrow$, which implies $\partial_{\rho} \Theta_{o}(x_o,\rho)|_{\rho=1} = 0$ [14], together with (18b), (16) and the $2d$ Ising value $d\nu = 2$. We get the scaling relation

$$\partial_{\rho}(x_o,1) = -\frac{x_o}{2} \frac{\partial}{\partial x_o} \Theta_{o}(x_o,1),$$

which can be solved for $\Theta_{o}(x_o,1)$ to find, using (26),

$$\Theta_{o}(x_o,1) = 2 \int_{x_o}^{x_o \infty} d\xi \xi^{-1} \vartheta_{o}(\xi,1)$$

$$= -2 \int_{x_o}^{x_o \infty} d\xi \xi^{-1} \Theta^{(oo)}(\xi) + 2 \int_{x_o}^{x_o \infty} d\xi \xi^{-1} \psi_o(\xi,1).$$

As $\vartheta_{o}(0,1) = \frac{1}{16}$, $\Theta^{(oo)}(0) = -\frac{\pi}{48}$ and $\psi_o(0,1) = \frac{1}{16} - \frac{\pi}{48}$ are all finite, see last chapter, the integrals $I^{(1,2)}_{o}(x_o)$, and consequently $\Theta_{o}(x_o,1)$, diverge logarithmically at $x_o = 0$.

The integral $I^{(1)}_{o}(x_o)$ over $\Theta^{(oo)}$ from [21] can be evaluated analytically by exchanging the two integrals and using the formula $\int_{x}^{\infty} d\xi \log(1 + ae^{-b\xi}) = -b^{-1} \text{Li}_2(-ae^{-b\xi})$, with polylogarithm $\text{Li}_2$, with the result

$$I^{(1)}_{o}(x_o) = -\frac{1}{2\pi} \int_{|x_o|}^{\infty} d\Omega \frac{1}{\Omega^2 - x_o^2} \text{Li}_2\left(-\frac{\Omega - x_o}{\Omega + x_o} e^{-2\Omega}\right).$$

(83a)
To leading order, $I^{(1)}_o(x_o)$ diverges logarithmically and has a jump at zero from the change of the integration limits,
\[ I^{(1)}_o(x_o) \simeq -\frac{\pi}{24} \log |x_o| - C \frac{|x_o|}{x_o}, \quad (83b) \]
with Catalan’s constant $C$.

The second integral $I^{(2)}_o(x_o)$ has to be evaluated numerically, so we split off the log singularity and write
\[ I^{(2)}_o(x_o) = \psi_o(0, 1) \log(1 + x_o^{-2}) + 2 \int_{x_o}^{x_o \infty} d\xi \xi^{-1} \left[ \psi_o(\xi, 1) - \frac{1}{1 + \xi^2} \psi_o(0, 1) \right]. \quad (84a) \]
While the integrand is analytic at $x_o = 0$, the integral again develops a jump discontinuity at $x_o = 0$ from the different integration limits above and below zero,
\[ I^{(2)}_o(x_o) \simeq -\left( \frac{1}{8} - \frac{\pi}{24} \right) \log |x_o| + \left( C - \frac{3}{4} \log 2 \right) \frac{|x_o|}{x_o}. \quad (84b) \]
Via the low-temperature integration limit in (84a) we have respected the additional contribution $-\log 2$ due to the broken symmetry in the ordered phase, see [14] for details, that was inadvertently attributed to the corner free energy $f_c^\leq$ in [1, 4]. Therefore, $\Theta_{s,c}(x \to -\infty) = \Theta_o(x_o \to -\infty, \rho) = -\log 2$ in agreement with the corner-free toroidal and cylindrical case [14, 15], while $f_c \to 0$ both in the high- and low-temperature limit.

As both $\Theta^{(\infty)}(x)$ and $\log \Sigma(x, \rho)$ are analytic around $x = 0$, $\Theta_{s,c}(x)$ from (80) fulfills
\[ \Theta_{s,c}(x) = -\frac{1}{8} \log |x| - \frac{3}{4} \log 2 \frac{|x|}{x} + \text{regular terms} \quad (85) \]
for small $x$. The same asymptotic behavior holds for the Casimir potential, but with a different (in)dependency on $\rho$,
\[ \Theta_o(x_o, \rho) = -\frac{1}{8} \log |x_o| - \frac{3}{4} \log 2 \frac{|x_o|}{x_o} + \text{terms regular in } x_o. \quad (86) \]
Remember that $\Theta_o$ is a function symmetric under $\leftrightarrow / \dagger$ exchange, while $\Theta_{s,c}$ is a property of the $\leftrightarrow$ boundary. The result for $\Theta_{s,c}(x)$ is shown in figure 5 together with the Casimir potential scaling function $\Theta(x, \rho)$ for several values of $\rho \geq 1$. The behavior of both quantities is dominated by the logarithmic divergence at $x = 0$. Furthermore, both are much larger below criticality, which is expected for systems with open boundary conditions.

While the (total) Casimir potential $\Theta(x, \rho)$ diverges logarithmically for $x \to 0$ and therefore is infinite at criticality for all finite aspect ratios $0 < \rho < \infty$, we can nevertheless
Figure 5. Universal Casimir potential scaling function \( \Theta(x, \rho) \) for the Ising rectangle with different aspect ratios \( \rho \geq 1 \), together with the surface-corner contribution \( \Theta_{s,c}(x) \) and the strip residual contribution \( \Psi(x,1) \). The latter is multiplied by 8. The corresponding curves for \( \rho < 0 \) fulfill (63). Note that \( \Theta(x \to -\infty, \rho) = -\rho^{-1} \log 2 \) (dotted lines).

As consequence, we have the unusual situation that the (finite) Casimir amplitude is different from the (divergent) Casimir potential at criticality.

In summary, we observe the following logarithmic contributions to the total free energy \( F \) and its residual \( F_{\text{res}} \): the total finite-size free energy (1) has a log term at criticality, originally predicted by Cardy & Peschel [2] using conformal field theory,

\[
F(\tau=0; L, M) = LMf_b(0) + [L + M]f_s(0) - \frac{1}{16} \log(LM) + \text{regular terms},
\]

with reduced temperature \( \tau = 1 - z/z_c \), while the infinite volume contribution (3b) has log
terms and a jump originating from the corner free energy, see (B1) in the appendix,
\[ F_\infty(\tau; L, M) = LMf_b(\tau) + |L + M|f_s(\tau) + \frac{1}{8} \log |\tau| + \frac{3}{4} \log 2 \frac{|\tau|}{\tau} + \text{regular terms} \]  
(88b)
such that
\[ F_\text{res}\infty(\tau; L, M) = -\frac{1}{16} \log(\tau^2LM) - \frac{3}{4} \log 2 |\tau| + \text{regular terms}. \]  
(88c)
In the FSS limit, where \( x_\circ = 2\tau\sqrt{LM} \) and \( \rho = L/M \), both log contributions combine to (86).

We conclude that both the logarithmic divergence and the jump in the Casimir potential scaling function \( \Theta_\circ(x_\circ, \rho) \) (86) stem from the near critical behavior of the corner free energy \( f_c \).

D. The Casimir force scaling function \( \vartheta(x, \rho) \)

Finally we come to the Casimir force scaling function \( \vartheta(x, \rho) \) at arbitrary \( x \) and \( \rho \). From (26) we can easily determine the Casimir force with high precision, provided \( \rho \gtrsim 1 \),
\[ \vartheta(x, \rho) = -\Theta^{(oo)}(x) + \psi(x, \rho). \]  
(89)
For \( \rho \lesssim 1 \), however, the convergence is suboptimal, and we instead use (18a) and the \( \leftrightarrow, \updownarrow \) exchange symmetry and calculate the Casimir force in \( \updownarrow \) direction instead, to get the equivalent expression
\[ \vartheta(x, \rho) = \vartheta^{(oo)}(x) - \rho x \Theta'_s(x) - \frac{x \partial}{\partial x} \Psi(x, \rho^{-1}) - \psi(x, \rho^{-1}), \]  
(90)
with (24, 26)
\[ \vartheta^{(oo)}(x) \equiv -\frac{1}{\pi} \int_0^\infty d\omega \sqrt{x^2 + \omega^2} \left( 1 + \frac{\sqrt{x^2 + \omega^2 + x} e^{2\sqrt{x^2 + \omega^2}}}{\sqrt{x^2 + \omega^2} - x} \right)^{-1} \]  
(91)
from (18a), that does not suffer from convergence problems if \( \rho \lesssim 1 \). At \( \rho = 1 \) we can derive the expression
\[ x \Theta'_s(x) = \Theta^{(oo)}(x) + \vartheta^{(oo)}(x) - 2\psi(x, 1) - \frac{x \partial}{\partial x} \Psi(x, 1), \]  
(92)
that implies that \( -x \Theta'_s(x) \) is approximately equal to the difference \( \vartheta_\updownarrow(x_\updownarrow, \infty) - \vartheta_\leftrightarrow(x_\leftrightarrow, 0) \), that is, the distance between the dashed and the solid black curves in figure 6.

The resulting universal Casimir force scaling functions for different values of the aspect ratio \( \rho \) are displayed in figure 6. The force is attractive for small aspect ratios \( \rho \lesssim 1/4 \).
Figure 6. Universal Casimir force scaling function $\vartheta_\delta(x_\delta, \rho)$ for the Ising rectangle with different aspect ratios $\rho$. For $\rho \geq 1$ we show $\vartheta \equiv \vartheta_\uparrow$ over $x \equiv x_\uparrow$ as defined by (18c), while for $\rho \leq 1$ we show $\vartheta_\leftrightarrow$ over $x_\leftrightarrow$ from (18a). Note that the curve for $\rho = \infty$ is masked behind the curve for $\rho = 2$. Also shown are the surface-corner contribution $-x\Theta_{s,c}'(x)$ from (92) and the strip contribution $\psi(x,1)$ from (28) to the Casimir force.

and becomes repulsive for larger aspect ratios, a behavior which is very similar to the fully periodic case [14, 15]. However, the force at criticality $x = 0$ in a square system $\rho = 1$ does not vanish such as in the periodic case. This is directly related to the log divergence of the corresponding Casimir potential and is interpreted as a consequence of a long-range repulsive corner-corner interaction. At criticality, the Casimir force changes sign at $\rho_0 = 0.523521700017999266800\ldots$ and is attractive for $\rho < \rho_0$. Note that $i\rho_0$ is the only purely imaginary zero of the Eisenstein series $E_2$. 
VI. SCALING LIMIT OF EFFECTIVE SPIN MODEL

Finally, we discuss the FSS limit of the effective spin model introduced in (I.99). In the FSS limit $M \to \infty$, $T \to T_c$ with constant $x$ and $\rho$, we find the thermodynamic limit $N \to \infty$ of the scaled reduced Hamiltonian

$$
\tilde{H}_{\text{eff}}(x, \rho) = -\sum_{\mu<\nu=1}^{N} \tilde{K}_{\mu\nu}(x)s_\mu s_\nu + \rho \sum_{\mu=1}^{N} \Gamma_\mu(x)s_\mu + b\left[\sum_{\mu=1}^{N} \sigma_\mu s_\mu\right]^2, \tag{93}
$$

with scaled interaction constants

$$\tilde{K}_{\mu\nu} = -\sigma_\mu \sigma_\nu \log \frac{\tilde{v}_\mu \tilde{v}_\nu}{(\Phi_\mu^2 - \Phi_\nu^2)^2}, \tag{94}\]$$

and with $\tilde{v}_\mu$ from (54a).

For large $\mu + \nu$, with $\nu - \mu \ll \nu + \mu$, the interactions are asymptotically

$$\tilde{K}_{\mu\nu} \simeq 2\sigma_\mu \sigma_\nu \log \left[\frac{\pi}{2} |\nu - \mu|\right], \tag{95}\[
$$

while for large $\nu$ with $\mu \ll \nu$

$$\tilde{K}_{\mu\nu} = 2\sigma_\mu \sigma_\nu \log \left[\frac{\pi^{3/2} \nu^{3/2}}{2^{3/2}(\mu - \frac{1}{2})B(\frac{3}{2}, \frac{3}{2})[1 + O(\nu^{-1})]}\right]. \tag{96}\[
$$

In all cases the interaction grows logarithmically with $\mu$ and $\nu$.

The aspect ratio $\rho$ takes the role of an applied homogeneous magnetic field, which acts on the spins $s_\mu \in \{0, 1\}$ via a magnetic moment $\Gamma_\mu$ that grows linearly with $\mu$. As a consequence, the spins are asymptotically fixed to $s_\mu = 0$ for large $\mu$ if $\rho > 0$, such that the spin dynamics is mainly restricted to the first few spins.

The strip Casimir force scaling function is related to the magnetization scaling function of the effective model, c.f. (I.103),

$$\psi(x, \rho) = \frac{\partial}{\partial \rho} \log \Sigma(x, \rho) = -\left\langle \sum_{\mu=1}^{\infty} \Gamma_\mu s_\mu\right\rangle_{\text{eff}} = -\sum_{\mu=1}^{\infty} \Gamma_\mu \langle s_\mu\rangle_{\text{eff}} = -\tilde{M}_{\text{eff}}(x, \rho). \tag{97}\[
$$

To summarize, we can define a universal effective spin model (93) describing the FSS limit of the 2d Ising model on the rectangle. The couplings $\tilde{K}_{\mu\nu}$ between the spins as well as the magnetic moments $\Gamma_\mu$ depend on $x$, while the aspect ratio $\rho$ takes the role of a homogeneous magnetic field. Thermodynamic quantities of this model are directly related to universal scaling functions of the underlying Ising universality class.
VII. CONCLUSIONS

Based on the results published recently \[1\], we calculated the universal finite-size scaling functions of the Casimir potential and the Casimir force for the Ising universality class on the $L \times M$ rectangle, with open boundary conditions in both directions, and with arbitrary aspect ratio $\rho$. The calculations were done in the finite-size scaling limit $L, M \to \infty$, $T \to T_c$, with fixed temperature scaling variable $x \propto (T/T_c - 1) M$ and fixed aspect ratio $\rho \propto L/M$. We have analytically derived exponentially fast converging series for the related Casimir potential and Casimir force scaling functions. At the critical point $T = T_c$ we could confirm predictions from conformal field theory for both the size dependent critical free energy (88a) [2] as well as for the shape dependence of the Casimir amplitude $\Delta_\circ(\rho)$ (87) [3].

The presence of corners and the related corner free energy has dramatic impact on the Casimir scaling functions and leads to a logarithmic divergence of the Casimir potential scaling function at criticality. As consequence, we have the unusual situation that the (finite) Casimir amplitude is different from the (divergent) Casimir potential at criticality. This behavior was not known from other geometries and boundary conditions.

These strong influence of system corners on the critical Casimir force might give rise to new applications in the framework of colloidal suspensions confined in finite near-critical binary liquid mixtures [29, 30].

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Appendix A: Series expansion for zeroes $\Phi_\mu$

We derive a series representation of $\Phi_\mu$ in terms of the values $\Phi_\mu^{(0)}$ at $x = 0$ around $\Phi_\mu^{(0)} = \infty$: writing $\Phi_{0,\mu} \equiv \Phi_\mu^{(0)}$ and with

$$\Phi_\mu = \Phi_{0,\mu} + \Delta_\mu$$

(A1)
we have
\[ P(\Phi_\mu) = P(\Phi_{0,\mu} + \Delta_\mu) = \frac{x}{\Phi_{0,\mu} + \Delta_\mu} \cos \Delta_\mu - \sin \Delta_\mu = 0, \] (A2)
leading to the recursion relation
\[ \Delta_\mu^{(k)} \mapsto \Delta_\mu^{(k+1)} = \arctan \left( \frac{x}{\Phi_{0,\mu} + \Delta_\mu^{(k)}} \right). \] (A3)
This recursion can easily be performed analytically with a computer algebra system like Mathematica [31]: Starting with an empty series expansion \( \Delta_\mu^{(0)} = \mathcal{O}(\Phi_{0,\mu}^{-1}) \), we can simply apply (A3) \( n \) times using the command
\[ \Delta_\mu[n_] := \text{Nest}[\text{ArcTan}\left( x/\Phi_{0,\mu} + \#:\right) & \text{Series}\left[ 1/\Phi_{0,\mu},\{\Phi_{0,\mu},\infty,0\}\right],n] \] (A4)
to get the correct series expansion up to \( \mathcal{O}(\Phi_{0,\mu}^{-2n+1}) \), with the result
\[ \Phi_\mu^2 = \Phi_{0,\mu}^2 + 2x - x^2(2x + 3)/3\Phi_{0,\mu}^2 + 2x^3(2x^2 + 5x^2 + 5)/5\Phi_{0,\mu}^4 + \mathcal{O}(\Phi_{0,\mu}^{-(2n+1)}) \] (A5)
for \( \Phi_\mu^2 \).

**Appendix B: Expansion of \( q \)-products around \( q = 1 \)**

The isotropic corner free energy near \( T_c \) can be derived from the \( q \)-product representation of Vernier & Jacobsen [8] as well as from (I.A7d). Written in terms of the reduced temperature \( \tau = 1 - z/z_c \), the expansion is given by
\[ f_c(\tau) = \frac{1}{8} \log |\tau| - \frac{2}{\pi} C + \frac{9}{16} \log 2 + \frac{3}{4} \log 2 \frac{|\tau|}{\tau} + \mathcal{O}(\tau), \] (B1)
with Catalan’s constant \( C \).

The isotropic surface free energy near \( T_c \) is calculated using the result of McCoy & Wu [7, (4.35)] together with an expansion of the \( q \)-products derived in [8] and (I.A7) to be
\[ f_s(\tau) = f_s(0) + \frac{|\tau|}{2} + \left( \frac{1}{4} - \frac{3 \log 2}{2\pi} + \frac{\log |\tau| - 1}{\pi} \right) \tau + \mathcal{O}(\tau^2). \] (B2)
The critical value reads
\[ f_s(0) = -\frac{3}{4} \log z_c - 2 \left[ \zeta(1,0)(-1, 1/8) + \zeta(1,0)(-1, 3/8) - \zeta(1,0)(-1, 5/8) - \zeta(1,0)(-1, 7/8) \right] \] (B3)
\[ = 0.1817314169844 \ldots, \]
with generalized Riemann zeta function \( \zeta(s, a) = \sum_{k=0}^{\infty} (k + a)^{-s} \). Note that the exact value of the critical surface free energy \( f_s(0) \) given in [B3] was not published yet. Both [B1] and [B3] can be derived from the product representations (I.A5b) and (I.A7b), expanded around the limit \( q \to 1 \).

[1] Alfred Hucht. The square lattice Ising model on the rectangle I: finite systems. *J. Phys. A: Math. Theor.*, 50(6):065201, 2017. arXiv:1609.01963.

[2] John Cardy and Ingo Peschel. Finite-size dependence of the free energy in two-dimensional critical systems. *Nucl. Phys. B*, 300:377, 1988.

[3] P. Kleban and I. Vassileva. Free energy of rectangular domains at criticality. *J. Phys. A: Math. Gen.*, 24:3407, 1991.

[4] R. J. Baxter. The bulk, surface and corner free energies of the square lattice Ising model. *J. Phys. A: Math. Theor.*, 50(1):014001, 2017. arXiv:1606.02029.

[5] M. E. Fisher and P.-G. de Gennes. Phénomènes aux parois dans un mélange binaire critique. *C. R. Acad. Sci. Paris, Ser. B*, 287:207, 1978.

[6] L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev.*, 65:117, 1944.

[7] B. M. McCoy and T. T. Wu. *The Two-Dimensional Ising Model*. Harvard University Press, Cambridge, 1973.

[8] Eric Vernier and Jesper Lykke Jacobsen. Corner free energies and boundary effects for Ising, Potts and fully-packed loop models on the square and triangular lattices. *J. Phys. A: Math. Theor.*, 45:045003, 2012. arXiv:1110.2158.

[9] H. W. Diehl, Daniel Grüneberg, Martin Hasenbusch, Alfred Hucht, Sergei B. Rutkevich, and Felix M. Schmidt. Exact thermodynamic Casimir forces for an interacting three-dimensional model system in film geometry with free surfaces. *EPL*, 100(1):10004, Oct 2012. arXiv:1205.6613.

[10] H. W. Diehl, Daniel Grüneberg, Martin Hasenbusch, Alfred Hucht, Sergei B. Rutkevich, and Felix M. Schmidt. Large-\( n \) approach to thermodynamic Casimir effects in slabs with free surfaces. *Phys. Rev. E*, 89:062123, Jun 2014. arXiv:1402.3510.

[11] Alfred Hucht. On the symmetry of universal finite-size scaling functions in anisotropic systems.
[12] D. P. Landau and R. H. Swendsen. Monte Carlo renormalization-group study of the rectangular Ising ferromagnet: Universality and a fixed line. *Phys. Rev. B*, 30:2787, 1984.

[13] J. O. Indekeu, M. P. Nightingale, and W. V. Wang. Finite-size interaction amplitudes and their universality: Exact, mean-field, and renormalization-group results. *Phys. Rev. B*, 34(1):330–342, Jul 1986.

[14] Alfred Hucht, Daniel Grüneberg, and Felix M. Schmidt. Aspect-ratio dependence of thermodynamic Casimir forces. *Phys. Rev. E*, 83:051101, Mar 2011.

[15] Hendrik Hobrecht and Alfred Hucht. Critical Casimir force scaling functions of the two-dimensional Ising model at finite aspect ratios. *J. Stat. Mech.: Theory Exp.*, 2017:024002, Feb 2017. arXiv:1611.05622.

[16] Alfred Hucht. Thermodynamic Casimir effect in $^4$He films near $T_\lambda$: Monte Carlo results. *Phys. Rev. Lett.*, 99(18):185301, Nov 2007.

[17] R. Garcia and M. H. W. Chan. Critical fluctuation-induced thinning of $^4$He films near the superfluid transition. *Phys. Rev. Lett.*, 83:1187, 1999.

[18] O. Vasilyev, A. Gambassi, A. Maciołek, and S. Dietrich. Monte Carlo simulation results for critical Casimir forces. *EPL*, 80:60009, 2007.

[19] O. Vasilyev, A. Gambassi, A. Maciołek, and S. Dietrich. Universal scaling functions of critical Casimir forces obtained by Monte Carlo simulations. *Phys. Rev. E*, 79(4):041142, 2009.

[20] M. Fukuto, Y. F. Yano, and P. S. Pershan. Critical Casimir effect in three-dimensional Ising systems: Measurements on binary wetting films. *Phys. Rev. Lett.*, 94:135702, 2005.

[21] C. Hertlein, L. Helden, A. Gambassi, S. Dietrich, and C. Bechinger. Direct measurement of critical Casimir forces. *Nature*, 451:172, 2008.

[22] A. Gambassi, A. Maciołek, C. Hertlein, U. Nellen, L. Helden, C. Bechinger, and S. Dietrich. Critical Casimir effect in classical binary liquid mixtures. *Phys. Rev. E*, 80(6):061143, Dec 2009.

[23] Andrea Gambassi. The Casimir effect: From quantum to critical fluctuations. *Journal of Physics: Conference Series*, 161(1):012037, 2009.

[24] Helen Au-Yang and Michael E. Fisher. Wall effects in critical systems: Scaling in Ising model strips. *Phys. Rev. B*, 21:3956, 1980.

[25] R. Evans and J. Stecki. Solvation force in two-dimensional Ising strips. *Phys. Rev. B*, 49:8842–
8851, Apr 1994.

[26] J. G. Brankov, D. M. Dantchev, and N. S. Tonchev. *Theory of Critical Phenomena in Finite-Size Systems – Scaling and Quantum Effects*. World Scientific, Singapore, 2000.

[27] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford Univ. Press, Oxford, 5 edition, 1979.

[28] Eric W. Weisstein. *Eisenstein Series*. From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/EisensteinSeries.html.

[29] Hendrik Hobrecht and Alfred Hucht. Direct simulation of critical Casimir forces. *EPL*, 106(5):56005, Jun 2014. arXiv:1405.4088.

[30] Hendrik Hobrecht and Alfred Hucht. Many-body critical Casimir interactions in colloidal suspensions. *Phys. Rev. E*, 92:042315, Oct 2015.

[31] Wolfram Research, Inc. *Mathematica V11.0*. Champaign, Illinois, 2016.