Contact formalism for coupled channels

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The contact formalism, a useful tool for analyzing short-range correlations, is generalized here for systems with coupled channels, such as in nuclear physics. The relevant asymptotic form is presented and contact matrices are defined. Generally, for the case of two coupled channels, two two-body functions are included in the asymptotic form, resulting a $2 \times 2$ contact matrix. Nevertheless, it is shown that if the coupling terms of the potential are very weak or very strong, only a single two-body function is needed, resulting a single contact. This universal result is directly relevant to nuclear systems, and provides a theoretical explanation for the fact that proton-neutron short-range correlations can be described using the single bound-state deuteron wave function. It is achieved by applying an appropriate boundary condition on the two-body functions. This boundary condition can be interpreted as a mean field potential imposed on the correlated pair due to the residual system.

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Introduction – The contact formalism is a relatively new technique for analyzing short-range correlations (SRCs) in quantum systems. Originally, the contact was devised by Shina Tan to describe systems consisting of two-component fermions, fulfilling the zero-range condition \cite{1}. Under this assumption, different properties of the system were related to a single variable, the contact \cite{2}. Many of these relations were also verified experimentally in ultra cold atomic systems \cite{3–7}. The success of this theory led to concentrated attempts to generalize it to other physical systems. The first studies in this direction generalized the formalism to bosonic, and mixed fermions systems \cite{8–13}, and also to different dimensions and non-trivial geometries \cite{8, 10, 14–20}. Recently, a $p$-wave contact was defined and utilized to describe systems with a resonant zero range $p$-wave interaction \cite{21–23}.

The contact formalism was also generalized and utilized to study nuclear systems \cite{24–30}. Nuclear systems do not obey the zero range condition, and as a result few significant changes had to be made. Mainly, the contribution of all partial waves should be considered, not only the $s$-wave contribution, and the known zero-range two-body (2B) functions, which are an integral part of the contact theory, should be replaced with unknown or model dependent functions. Accordingly, the nuclear contact matrices were defined. Using these matrices, new relations between different nuclear quantities, such as the photoabsorption cross section and momentum distributions, were derived and verified \cite{24, 25, 27, 29–31}. Similar contact matrices and contributions from different partial waves were also considered recently to describe SRCs in other systems \cite{32–34}.

In this work we wish to take the contact formalism one step further, adapting it to describe systems dominated by coupled channels, situation ignored in previous studies. In order to make our goal clearer, we consider a system of two-component fermions with spin $s = 1/2$, interacting via non-central 2B force that couples different orbital angular momentum $\ell$ and spin $s$ channels. This force does not necessarily need to fulfill the zero-range condition. Looking on the 2B system, it can be characterized using the quantum numbers $(\ell, s) j m$, where $j$ and $m$ are the total angular momentum and its projection. As the potential mixes different $\ell, s$ channels, an eigenstate of the 2B system does not necessarily have a well defined $\ell, s$ values. The nuclear potential is an example for such an interaction, as, due to pion exchange, the deuteron is a bound 2B state with $j = 1$ and $s = 1$ but with both $s$-wave and $d$-wave components.

Coupled channels contact formalism – Consider a system of bosons, interacting via zero range $s$-wave potential. In such system, when two particles approach each other, the total wave function $\Psi(r_1, \ldots, r_N)$ takes the form

$$\Psi \rightarrow_{r_{ij} \to 0} \varphi(r_{ij}) A(R_{ij}, \{r_k\}_{k \neq i, j}) \quad (1)$$

where, $N$ is the number of particles, $r_{ij} = r_i - r_j$ and $R_{ij} = (r_i + r_j)/2$, $\varphi(r_{ij})$ is the universal zero-energy solution of the Schrödinger equation of the 2B system, and $A$ is a regular function describing the dynamics of all other degrees of freedom. For an $s$-wave interaction and under the zero-range assumptions, $\varphi = (1/r_{ij} - 1/a)$, where $a$ is the scattering length. This factorization is closely related to the operator product expansion \cite{35}, which was used to derive some of the known contact relations \cite{17, 20, 36}.

When considering also higher partial waves, if the different channels are not coupled, the asymptotic form becomes \cite{25, 32}

$$\Psi \rightarrow_{r_{ij} \to 0} \sum_{\alpha} \varphi_{\alpha}(r_{ij}) A_{\alpha}(R_{ij}, \{r_k\}_{k \neq i, j}) \quad (2)$$

Here, the sum over $\alpha$ indicates the different channels. In a system of two-component fermions $\alpha = (\ell, s) j m$...
as discussed above. When considering coupled channels, the quantum numbers indicated by $\alpha$ are not necessarily good quantum numbers, and we should seek a different form for the universal part of the asymptotic wavefunction.

To this end, we can continue with the example of two-component fermions with the quantum numbers $(\ell, s)j m$, and consider an asymptotic $2B$ state composed of two coupled channels, given by $|\alpha\rangle = |(\ell_\alpha, s_\alpha)j m\rangle$ and $|\beta\rangle = |(\ell_\beta, s_\beta)j m\rangle$ with $\ell_\alpha \neq \ell_\beta$ (generalization to $n$ channels is straightforward). In order to find the suitable asymptotic form, we need to understand what are the zero-energy solutions. In this case the Schrödinger equation becomes a set of two coupled equations, which results two independent solutions. These solutions have the following form

$$\varphi^a(r) = \varphi_\alpha^a(r)|\alpha\rangle + \varphi_\beta^a(r)|\beta\rangle,$$

(3)

where $a = 1, 2$ indicates the two independent solutions that differ only by the functions $\varphi_\alpha^a(r)$ and $\varphi_\beta^a(r)$. We notice that each of these solutions is a mixture of both channels, $|\alpha\rangle$ and $|\beta\rangle$.

Now it becomes clear how the asymptotic wave function should look like. If we consider only these two coupled channels, it will take the form

$$\Psi \rightarrow \sum_{a=1,2} \varphi^a(r_{ij})A^a(R_{ij}, \{r_k\}_{k \neq i,j}).$$

(4)

Notice that it is a different asymptotic form than the non-coupled case, given by Eq. (2), since Eq. (4) includes two different functions $\varphi_\alpha^a(r)$ and $\varphi_\beta^a(r)$, and each of them is generally coupled to a different $A^a$ function. For simplicity, we ignore the summation over $m$ required if $\Psi$ has a well-defined total angular momentum $J$. It is simple to make the necessary changes (see Ref. [25]), and it does not affect the conclusions of this paper.

Based on the above asymptotic form, we can define the contacts for the case of coupled channels. A matrix of contacts should be defined

$$C^{ab} = 16\pi^2 N(N-1)/2 \langle A^a | A^b \rangle$$

(5)

where $a, b = 1, 2$. We have defined here a $2 \times 2$ matrix of contacts, for the case of two coupled channels. It is important to include the non-diagonal element of the matrix, since $A^1$ and $A^2$ are not necessarily orthogonal. A matrix of contacts was already defined in previous studies [25, 33, 34], in which it was implicitly assumed that asymptotically the potential does not couple different channels. Nevertheless, the off-diagonal terms of the matrix can still be important. Here we consider the case of a potential that couples between channels.

Before analyzing the universal wave-function (3) let us recast some contact relations using the asymptotic form (4). We will focus here on the momentum and density distributions. The single-particle momentum distribution, $n(k) = \int dk \rho(k)$, describing the probability to find a particle with momentum $k$, is given asymptotically by

$$n(k) \rightarrow \sum_{a,b=1,2} (\varphi_\alpha^a(k)\varphi_\beta^b(k) + \varphi_\beta^a(k)\varphi_\beta^b(k)) 2C^{ab}/16\pi^2,$$

(6)

where $\varphi_\alpha^a(k)$ is the Fourier transform of $\varphi_\alpha^a(r)$, and $\int dk n_k N(k) = N$. Similarly, the two-particle momentum distribution $F(k) = \int dk F(k)$, which describes the probability to find a particle pair with relative momentum $k$, is given by

$$F(k) \rightarrow \sum_{a,b=1,2} (\varphi_\alpha^a(k)\varphi_\alpha^b(k) + \varphi_\alpha^a(k)\varphi_\alpha^b(k)) C^{ab}/16\pi^2.$$

(7)

The derivation of these two relations is very similar to the derivation presented in [25]. It is also very simple to derive the asymptotic probability to find a pair of particles with relative distance $r$, which is denoted by $\rho(r) = \int d^2 r \rho(r)$. In this case the relation is

$$\rho(r) \rightarrow \sum_{a,b=1,2} (\varphi_\alpha^a(r)\varphi_\alpha^b(r) + \varphi_\alpha^a(r)\varphi_\alpha^b(r)) C^{ab}/16\pi^2.$$

(8)

$\rho(r)$ and $F(k)$ are normalized to the number of pairs.

Now we can see that if we have two different functions $\varphi^a$, $a = 1, 2$, in the asymptotic form (Eq. (4)), then the last three relations are quite complicated. For example, if we will compare two different eigenstates (with the same underlining interaction), $\Psi_1$ and $\Psi_2$, and look on the ratio of the two momentum distributions, $n_1(k)/n_2(k)$, where $n_i(k)$ corresponds to $\Psi_i$, there is no reason that this ratio will obtain a constant value for high momentum. This is because the values of the different four contacts $C^{ab}$ can be different for each state, and the $k$-dependence will not generally disappear. The same argument is relevant also for $F(k)$ and $\rho(r)$. On the other hand, if there is only one asymptotic function, then these ratios must have a constant value for high momentum or small distances.

The remaining question is whether two functions are indeed required in the asymptotic form. We will first focus on nuclear systems, and then use our new insights to try and provide a general answer.

**Nuclear systems** - SRCs are known to play an important role in nuclear physics. For example, a high-momentum tail originated by SRCs was identified for $k > k_F$ $\approx 1.26$ fm$^{-1}$ [37–41]. See also [42, 43]. Nuclear systems are more complicated then the simple case discussed above, since they include both protons and neutrons, each with a spin degree of freedom. Here, we will focus on proton-neutron (pn) pairs. The main channels contributing to SRCs for pn pairs are the two coupled channels $\alpha = (\ell_\alpha = 0, s_\alpha = 1, j = 1, m)$ and $\beta = (\ell_\beta = 2, s_\beta = 1, j = 1, m)$ with isospin $T = 0$. 

In the study of $pn$ SRCs in nuclear systems there are many indications that they can be described using the single $T = 0$ deuteron function to a good approximation. The deuteron is the 2B bound state in the above mentioned $j = 1$ channels. For example, in the analysis of electron-scattering experiments, it is shown that for kinematics sensitive to SRCs, the cross section becomes approximately proportional to the deuteron cross-section \cite{40, 41, 44}. Similar picture is obtained from analysis of numerical calculations. Using available numerical data \cite{37}, it was shown in Ref. \cite{25} that the $pn$ momentum distribution of the available nuclei, is approximately a multiplication of the deuteron momentum distribution, for $k > 4 \text{ fm}^{-1}$. It was also observed for heavier nuclei using different numerical methods \cite{31}. Recently, the one-body momentum distribution $a(k)$ was reproduced using the contact formalism for $k$ larger than the Fermi momentum \cite{30}. The main contribution to $a(k)$ comes from the deuteron channel, using the single bound state wave function.

All of these indicate that for some reason only one function is needed in the asymptotic form when a proton gets close to a neutron, with in the $T = 0$ deuteron channel. We notice that there is a wide agreement in the literature on the fact that the deuteron $T = 0$ channel is the dominant channel in the description of nuclear SRCs, due to the nuclear tensor force \cite{38, 39, 45–49}. Nevertheless, it was not realized that generally two independent solutions for each positive energy. With this boundary condition we will get quantization of the positive energies, due to the nuclear tensor force \cite{50}, and solve numerically the 2B Schrödinger equation. As expected, without the external boundary condition, for any positive energy we get two independent solutions $\varphi^1(r)$ and $\varphi^2(r)$. We note that for small enough energies and small enough distances, these solutions do not depend on the energy. When including the boundary condition, we can find the allowed energies and the corresponding values of the ratio $\eta$ for each energy. In Fig. 1, we present the values of this ratio for different energies and using different values for the radius of the box $R$. We stress once again that the value of $\eta$ determines the resulting 2B function $\varphi(r)$ for small enough energies and small enough values of $r$. We can see in Fig. 1 that for energies smaller than about 30 MeV, the value of this ratio is approximately constant independent of the value of $R$. This is quite a surprising result, and it means that the spherical box somehow "chooses" the same linear combination independently of the radius of the box or the energy of the 2B state.

These results can explain why in nuclear systems we can describe the $T = 0$ deuteron-channel SRCs using a single function. Since we don’t have a reason to prefer a specific radius for the box, it is important that the chosen function does not depend on the radius. In order to make sure that this is indeed the correct boundary condition, we can check if the chosen function $\varphi(r)$ can describe the deuteron wave function. In Fig. 2 we present $\rho_{pn}(r)$ as calculated directly using the deuteron wave function for AV18 and using a single positive energy solution $\varphi(r)$ in the presence of a spherical box. We can see that indeed these two quantities coincide for small distances. Thus we conclude that the box boundary condition not only lead to a single asymptotic function, but this function is also the deuteron’s asymptotic function and is thus suit-
able to describe \( pn \) SRCs for heavier nuclei. We can also see in Fig. 2 that using arbitrary positive-energy solutions, without the box boundary condition, the deuteron density \( \rho_{pn}(r) \) cannot be reproduced. Only the specific combination determined by the box is suitable.

We note that the hard-wall boundary condition of the box can be replaced by a softer boundary condition. For example, if we include an external wide harmonic-oscillator potential, instead of the the box boundary condition, we still obtain similar results. Thus, this new boundary condition can be interpreted as the effect of the residual particles on the SRC pair, through a mean field potential imposed on this pair. The exact details of this potential are not important.

This gives an explanation for the fact that \( np \) SRCs can be described using only the deuteron wave function as leading order approximation, without a second function for these coupled channels. Theoretical explanation to this phenomena has not been presented in previous studies. Thanks to the nuclear example we have been able to identify the necessary boundary condition for the 2B functions appearing in the asymptotic form. We can now in principle apply it to different systems and study the implications of this new boundary condition.

**General potential** – To this end, we consider a simpler potential. We will use the \( pn \) nuclear example in the deuteron channels as before, but with a potential that has the following components

\[
V_{\alpha\alpha} = V_{\beta\beta} = -V_0 \exp(-\Lambda^2 r^2) \tag{10}
\]

\[
V_{\alpha\beta} = V_{\beta\alpha} = -SV_0 \exp(-\Lambda^2 r^2) \tag{11}
\]

where \( S \) represents the strength of the coupling terms. If \( S = 0 \), the two channels are actually not coupled. We can numerically solve the Schrödinger equation for this potential in a box. We want to see if again we get a single solution or if in some cases we need two different functions.

We first choose a radius \( R \) for the spherical box. Then, for a given strength \( S \), we tune \( V_0 \) to produce a constant scattering length, and calculate the value of \( \eta \) as a function of the allowed energies. The results are presented in Fig. 3. We can see that as the coupling of the channels \( S \) becomes larger, \( \eta \) obtains a constant value for small energies. Thus, in this case of large \( S \), only one asymptotic function is needed. For smaller values of \( S \), there are clearly two separated branches for \( \eta \), and we expect that two asymptotic functions are required to fully describe SRCs. For very small values of \( S \), the channels are effectively not coupled, and we go back to the known non-coupled case. Then, usually, there is one preferred channel, and again only one function is needed. The same general picture is obtained for other coupled channels or if \( V_0 \) is kept constant. The dependence of the values of \( S \), that can be considered "large", on the different parameters of the potential is studied in the supplemental materials [51].

The conclusion that only one functions is needed in the strong coupling limit holds only when considering the zero-energy solutions. If energy corrections are included, two functions might be required at some point, since two branches clearly exist when going to higher energies.

**Summary** – Summing up, in this work the contact formalism was generalized to systems with coupled channels. Focusing on the case of two coupled channels, the
relevant asymptotic form was presented and the matrices of contacts were defined. The asymptotic form generally includes two different 2B functions. Looking on the specific case of nuclear systems, it was observed that only a single 2B function and a single contact are needed to describe the pn SRCs in the deuteron channel. This result led to the understanding that a boundary condition should be imposed on the 2B functions, possibly representing the effects of the remaining particles in the system. Indeed, calculating the 2B functions for the nuclear case in a box gives a single 2B function, which does not depend on the radius of the box.

This is a clear theoretical explanation for the fact that pn SRCs can be described in leading order using the deuteron wave function. We have stressed that the known deuteron-channel dominance does not explain this observation. We note again that in this paper we do not try to explain the dominance of the deuteron channel over other possible channels (such as \( T = 1 \) channels). We focus here on the deuteron channel, and explain why with in this channel a single 2B function is sufficient.

Analyzing a simpler potential, we have shown that the collapse of the asymptotic wave functions into a single function is not restricted to nuclear physics, but it is a universal phenomena if the coupling terms of the potential are very strong or very weak. This work should be relevant to any system with SRCs which is dominated by coupled channels. It can help in understanding the importance of additional coupled channels to nuclear SRCs, especially since there are no bound states in these channels. It can also be relevant to ultra-cold atomic systems with short-range interactions.

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[51] See Supplemental Material at [URL will be inserted by publisher] for the dependence of the values of $S$, that can be considered "large", on the different parameters of the potential.