A FLOER FUNDAMENTAL GROUP

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Abstract. The main purpose of this paper is to provide a description of the fundamental group of a symplectic manifold in terms of Floer theoretic objects.

As an application, we show that when counted with a suitable notion of multiplicity, non degenerate Hamiltonian diffeomorphisms have enough fixed points to generate the fundamental group.

In many ways, the topology of a space influences its geometry. This is particularly true in symplectic geometry, and having a symplectic interpretation of a topological invariant is a good tool to explore this relationship. The celebrated Floer Homology ([7][8]) is of course a strong illustration of this phenomenon. Introduced to prove the homological version of the Arnold conjecture ([1]), it quickly became one of the most powerful tools in symplectic geometry.

However, all the techniques derived from the original Floer construction are homological (or at least chain complex based) in nature (although the use of local coefficients allows to involve some homotopical invariants). The notion of cobordism (among moduli spaces) is even at the root of the original ideas of M. Gromov [9] of using pseudo-holomorphic curves in symplectic geometry. As a consequence, purely homotopical tools are still missing, and it is the goal of this paper to provide a symplectic interpretation of the fundamental group.

Although all the objects this construction is based on are classical Floer theoretic objects, the essential non abelianity the homology has to forget is here caught by a careful attention to “augmentations” and 1-dimensional moduli spaces.

This construction applies to both the absolute (i.e. Hamiltonian fixed points problem) and the relative (i.e. intersections of a Lagrangian submanifold with its deformation under Hamiltonian isotopies problem) settings. Although the latter can be expected to hold the most interesting applications, we choose to focus on the former here for the sake of simplicity and to better highlight the main ideas: the generalisation to the latter entails exactly the same issues as for the homology and involves no new idea.

2000 Mathematics Subject Classification. 57R17; 53D40, 14F35.

Key words and phrases. Symplectic topology, Fundamental group, Floer theory, Morse theory, Homotopy, Arnold conjecture.
The most immediate outcome of this construction regards estimates on the number of fixed points of Hamiltonian diffeomorphisms, but interestingly enough not in the most expected way.

More precisely, let \((M,\omega)\) be a symplectic manifold. Recall that given a Hamiltonian function \(H : M \times S^1 \to \mathbb{R}\), its differential induces a (time dependent) vector field \(X_H\) via the relation \(dH = \omega(X_H, \cdot)\), and that a Hamiltonian diffeomorphism is the time one map of the flow of such a vector field.

The (non degenerate) Arnold conjecture claims that if they are all non degenerate, the number of fixed points of a Hamiltonian diffeomorphism is at least the minimal number of critical points of a Morse function. A closely related question regards the number of critical points of a stable Morse function, i.e. a Morse function on \(M \times \mathbb{R}^N\) (for arbitrary \(N\)) that is quadratic at infinity.

The algebraic study of chain complexes done by V. V. Sharko [13], implies several constraints for stable-Morse functions, and in [4], M. Damian shows that all the algebraic operations on the complex can be realized geometrically in this setting. In particular, stable-Morse functions may indeed have strictly less critical points than regular Morse functions.

In a recent work [11], K. Ono and A. Pajitnov use the Floer complex with local coefficients to extend these constraints to the Hamiltonian setting. In particular, they show the following

**Theorem 0.1** (K. Ono, A. Pajitnov). Suppose \(M\) is a weakly monotone symplectic manifold and let \(H\) be an Hamiltonian function on it. Then, if they are all non degenerate, the number \(p(H)\) of fixed points of the associated Hamiltonian diffeomorphism satisfies

\[
p(H) \geq \delta(\pi_1(M))
\]

where \(\delta(\pi_1(M))\) is the minimal number of generators of the kernel of the augmentation \(\mathbb{Z}[\pi_1(M)] \to \mathbb{Z}\).

Notice that in general, \(\delta(\pi_1(M))\) may be strictly smaller than the minimal number of generators of the fundamental group itself.

The interpretation of the fundamental group proposed in this paper suggests a new point of view on this kind of question: since we consider “augmented” fixed points rather than real geometric ones, it naturally leads to a notion of multiplicity (which depends on auxiliary data like a metric or an almost complex structure), for which we recover the minimal number of generators of the fundamental group as a lower bound (see [18]).

It should be noticed here that both the construction of the fundamental group and its corollary regarding counting of critical points with multiplicity have an obvious analog in the Morse stable setting, and have their own interest. They would definitely deserve a dedicated study, but it is our choice here to focus on the Hamiltonian case, and the stable-Morse case,
which is strictly parallel, will only be roughly sketched, mainly as a finite dimensional illustration of the main construction (see section 4.5).

In the first section of the paper, the main definitions, statements and applications are presented. In the second section, farther required notations and tools are introduced. The third section is dedicated to the comparison of Morse and Floer loops. The fourth section is devoted to the description of the relations, and the fifth to the proof of the applications.

This work would not exist without the crucial help of a few people. I am particularly thankful to O. Cornea, whose deep topological insight and generosity nourished me for years, to J.-Y. Welschinger and B. Chantraine to whom I am indebted for the keystone of this paper, which is the notion of augmentation, to A. Oancea who served as a compass to me and M. Damian who also owns a large part of this work. Finally, I’m particularly grateful to A.V. Duffrène who indirectly but deeply influenced the birth of this paper.

1. Introduction and main statements.

1.1. Preliminaries. Let \((M, \omega)\) be a compact symplectic manifold without boundary. For technical reasons, \(M\) will be supposed to be either

- aspherical, which means \(c_1(TM) = \omega = 0\) on the image of the Hurewitz morphism \(\pi_2(M) \to H_2(M)\), or
- monotone, which means there is a positive constant \(\kappa\) such that \(c_1(TM) = \kappa[\omega]\) in \(H_2(M, \mathbb{R})\).

These assumptions will allow us to easily

- avoid the transversality issues related to the multiply covered negative curves,
- avoid bubbles on 0 and 1 dimensional moduli spaces,
- ensure finiteness of the number of (lifted) orbits of given Conley-Zehnder index.

Given a Hamiltonian function \(H : M \times \mathbb{S}^1 \to \mathbb{R}\), we let \(X_H\) be the associated Hamiltonian vector field, \(\phi_H^t\) its flow, and \(\mathcal{P}(H)\) the set of its contractible 1-periodic orbits.

To handle the index computation when \(c_1(TM)\) does not necessarily vanish on \(\pi_2(M)\), we consider the covering \(\tilde{\mathcal{P}}(H)\) associated to the group \(\pi_2(M)/\ker c_1\). It is obtained from \(\mathcal{P}(H)\) by adjoining a class of capping to the orbit in the following way:

\[
(1) \quad \tilde{\mathcal{P}}(H) = \{ (\gamma, \bar{\gamma}), \gamma \in \mathcal{P}(H), \bar{\gamma} : D \to M, \bar{\gamma}|_{\partial D} = \gamma \}/\sim
\]

where \((\gamma, \bar{\gamma}) \sim (\gamma', \bar{\gamma}')\) if \(\gamma = \gamma'\) and \(\mu_{CZ}(\bar{\gamma}) = \mu_{CZ}(\bar{\gamma}')\) (recall that in the aspherical case, all the cappings define the same index, and the covering is in fact trivial, while in the monotone case \(\mu_{CZ}(\bar{\gamma}) = \mu_{CZ}(\bar{\gamma}')\) if and only if \(\omega(\bar{\gamma}) = \omega(\bar{\gamma}')\)). In the sequel, \(\tilde{\mathcal{P}}(H)\) will completely replace \(\mathcal{P}(H)\) and no explicit reference to the covering will be made anymore. In particular, what
we call an Hamiltonian orbit from now on will in fact be a lift of such an orbit to $\tilde{P}(H)$.

Each element $x$ in $\tilde{P}(H)$ then has a well defined Conley-Zehnder index $\mu_{CZ}$. For convenience, we shift the Conley-Zehnder index by $n$ and let

$$|x| = \mu_{CZ}(x) + n.$$ 

The set of orbits $\tilde{P}(H)$ splits according to this index, and we let

$$\tilde{P}_k(H) = \{x \in \tilde{P}(H), |x| = k\}.$$ 

Given an $\omega$ compatible almost complex structure $J$, we are interested in the Floer moduli spaces and some classical variants of such. Recall the Floer equation for a map $u: \mathbb{R} \times S^1 \to M$ is the following:

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = J(u)X_H(u)$$ 

Moreover, we fix once for all a smooth function $\beta: \mathbb{R} \to [0, 1]$ such that

$$\begin{cases} 
\beta(s) = 1 & \text{if } s \leq 0 \\
\beta(s) = 0 & \text{if } s \geq 1
\end{cases}$$

and use it to cutoff the Hamiltonian term of the Floer equation on one or both ends of the cylinder by considering the equation

$$(F_i) \quad \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = \chi_i(s) J(u)X_H(u)$$

for different functions $\chi_i: \mathbb{R} \to [0, 1]$ derived from $\beta$, namely

1. $\chi_1 \equiv 1$ defines the usual Floer equation,
2. $\chi_2(s) = \beta(s)$ defines the lower capping equation,
3. $\chi_3(s) = \beta(-s)$ defines the upper capping equation,
4. $\chi_4,R(s) = \beta(R+s)\beta(R-s)$ defines “$R$-perturbed” sphere equation.

Solutions of finite energy of this equation have converging ends, either to a point by the classical removal of singularities arguments if the Hamiltonian term is cut off on this end, or to an Hamiltonian orbit if not. In the former case, considering the end as a neighborhood of 0 in $\mathbb{C} \setminus \{0\}$, the map $u$ extends holomorphically through 0. The limit value will both be referred to as the center of the capping, and as $u(+\infty)$ or $u(-\infty)$. We abusively but conveniently say that such a trajectory ends at the $\emptyset$ symbol to describe the fact that this limit point is not constrained.

We are interested in the moduli spaces described below and depicted on figure 1. Let $\star$ be a point in $M$, and $\mathcal{U}$ be the space of (smooth, if $J$ is smooth) maps $u: \mathbb{R} \times S^1 \to M$ that have finite energy i.e. such that $\iint \|\frac{\partial u}{\partial s}\|^2 ds dt < +\infty$. If $a$ is an oriented disc, let $\bar{a}$ denote the disc with opposite orientation, and if $b$ is another disc or tube sharing boundary with
a with compatible orientation, let \(a^\# b\) denote the gluing of the two.

\[
\begin{align*}
\mathcal{M}(y, x) &= \{ u \in \mathcal{U}, (F_1), \lim_{s \to \pm \infty} u(s, \cdot) = \frac{x}{y}, \text{ and } [y^\# u^\# x] = 0 \}/\mathbb{R} \\
\mathcal{M}(y, \emptyset) &= \{ u \in \mathcal{U}, (F_2), \lim_{s \to -\infty} u(s, \cdot) = y, \text{ and } [y^\# u] = 0 \} \\
\mathcal{M}(\star, x) &= \{ u \in \mathcal{U}, (F_3), \lim_{s \to \pm \infty} u(s, \cdot) = \frac{\star}{x}, \text{ and } [u^\# \star] = 0 \} \\
\mathcal{M}(\star, \emptyset) &= \{(R, u) \in \mathbb{R} \times \mathcal{U}, (F_{4, R}), \lim_{s \to +\infty} u(s, \cdot) = \star \text{ and } [u] = 0 \}
\end{align*}
\]

where the brackets denote homotopy classes in \(\pi_2(M)\), and their vanishing express the compatibility of the trajectory \(u\) with the prescribed lifts of its ends to the covering space \(\tilde{P}(H)\).

Notice that in the last case, the \(R\) parameter is allowed to vary.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{floer_moduli_spaces.png}
\caption{Floer moduli spaces.}
\end{figure}

Remark 1. The elements of \(\mathcal{M}(x, \emptyset)\) for \(x \in \tilde{P}_0(H)\) are generally used to define an augmentation on the Floer homology, and we will often refer to a capping \(\alpha \in \mathcal{M}(x, \emptyset)\) as an “augmentation” of \(x\), and to a couple \((x, \alpha)\) as an augmented Hamiltonian orbit.

It is well known (\cite{8}, \cite{10}) that for a generic triple \((H, J, \star)\), all these moduli spaces are smooth manifolds of dimensions

\[
\begin{align*}
\dim \mathcal{M}(y, x) &= |y| - |x| - 1 \\
\dim \mathcal{M}(y, \emptyset) &= |y| \\
\dim \mathcal{M}(\star, x) &= -|x| \\
\dim \mathcal{M}(\star, \emptyset) &= 1
\end{align*}
\]

From now on, \((H, J, \star)\) will be supposed to be chosen so that all these moduli spaces are indeed defined transversely.

Moreover, all these moduli spaces are compact up to breaks, bubbling off, or vanishing of the Hamiltonian strip in the last case (i.e. \(R \leq -1\)) and we
let
\[ \mathcal{M}(x, y) = \mathcal{M}(x, y), \quad \mathcal{M}(x, \emptyset) = \mathcal{M}(x, \emptyset) \]
\[ \mathcal{M}(\ast, y) = \mathcal{M}(\ast, y), \quad \mathcal{M}(\ast, \emptyset) = \mathcal{M}(\ast, \emptyset) \]
be the Gromov-Floer compactifications of the previous moduli spaces.

In all this paper, only 0 and 1 dimensional moduli spaces will be considered, and, since \( M \) is supposed to be aspherical or monotone, no bubbling of sphere can occur on such moduli spaces. This means they will all be compact up to breaks, and to vanishing of the Hamiltonian strip in the last case.

In particular, each 0 dimensional moduli space \( \mathcal{M}(y, x) \) is compact, and hence finite, and we let
\[ \sharp_{\text{abs}}(\mathcal{M}(y, x)) = \sum_{\gamma \in \mathcal{M}(x, y)} (+1) \]
denote the (absolute) number of elements in \( \mathcal{M}(y, x) \).

Remark 2. It is usual, when working with pseudo-holomorphic curves or Floer trajectories, to consider the algebraic number \( \sharp_{\text{alg}} \mathcal{M}(x, y) \) of elements in a 0-dimensional moduli space, i.e. to take signs coming from some orientation of the moduli space into account. We stress however that this definition refers to the absolute number, i.e. the sum where each element counts for +1.

Finally, all the moduli spaces are given coherent orientations (see for instance [3] for a detailed discussion of the orientations). In particular, the orientation of an element \( \gamma \) in a 0 dimensional moduli space is a sign \( \epsilon(\gamma) \in \{\pm 1\} \). Although they do not appear in the counting process of definition 1.7 nor in the statements of the main theorems, the orientations will be used in the proofs.

1.2. Floer steps and loops. Given a broken configuration \( (\beta, \alpha) \in \mathcal{M}(y, x) \times \mathcal{M}(x, \emptyset) \), where \( x \in \mathcal{P}_0(H) \) and \( y \in \mathcal{P}_1(H) \cup \{\ast\} \), the gluing construction ([8], [10]) gives rise to a one dimensional family of trajectories starting with \( (\beta, \alpha) \) and ending at some other broken trajectory \( (\beta', \alpha') \in \mathcal{M}(y, x') \times \mathcal{M}(x', \emptyset) \).

This relation between \( (\alpha, \beta) \) and \( (\alpha', \beta') \) will be denoted by
\[ (\alpha, \beta) \overset{x}{\leftrightarrow} (\alpha', \beta'). \]

Remark 3. The main property of the orientations we will be using is that “the gluing reverses the orientation”, i.e. for trajectories in 0 dimensional spaces, we have
\[ (\beta, \alpha) \overset{x}{\leftrightarrow} (\beta', \alpha') \Rightarrow \epsilon(\beta)\epsilon(\alpha) = -\epsilon(\beta')\epsilon(\alpha') \]

This “gluing” relation gives rise to the main definition of this paper which is the notion of Floer step depicted on figure 2. There will be two kinds of such, and we begin with “pure” Floer steps.
Definition 1.1. For $y \in \mathring{\mathcal{P}}_1(H)$, let a (pure) Floer step through $y$ be a quadruple $(\alpha, \beta, \beta', \alpha')$ with $\alpha \in \mathcal{M}(x, \emptyset)$, $\beta \in \mathcal{M}(y, x)$, $\beta' \in \mathcal{M}(y, x')$, and $\alpha' \in \mathcal{M}(x', \emptyset)$ for some $x, x' \in \mathring{\mathcal{P}}_0(H)$ such that

$$(\beta, \alpha) \overset{\Sigma}{\leftrightarrow} (\beta', \alpha').$$

Remark 4. In other words, a Floer step through $y \in \mathring{\mathcal{P}}_1(H)$ is a connected component of $\mathcal{M}(y, \emptyset)$ with non-empty boundary.

This first definition deserves some comments.

An enlightening point of view is the Morse setting. There, any loop in the manifold can be deformed into a sequence of “Morse steps” which are just travels along the unstable manifolds of index 1 critical points. A way to understand what such a travel is from the moduli space point of view, is to regard a point $p$ in the unstable manifold of a critical point $y$ as the Morse trajectory from $y$ to $p$ interrupted at $p$ (see [2] for instance), hence as an element of $\mathcal{M}(y, \emptyset)$ (where the cutoff function $\beta$ instantly goes from 1 to 0 instead of smoothly, but this plays no essential role here).

The compactification of this moduli space is obtained by adding the two non-interrupted Morse trajectories rooted at $y$. They should be thought of as broken at the (trivially augmented) index 0 critical points, and a travel through the unstable manifold of $y$ is hence the exact analog of a step in the sens given in [1].

This interpretation suggests that the analog of an index 0 Morse critical point is not an index 0 Hamiltonian orbit per se, but an augmented one, i.e. a couple $(x, \alpha)$ where $x \in \mathring{\mathcal{P}}_0(H)$ and $\alpha \in \mathcal{M}(x, \emptyset)$. Similarly, the analog of index 1 Morse critical points should be an index 1 Hamiltonian orbits endowed with some extra data, which is precisely that of a (pure) Floer step.

More generally, the Floer unstable manifold of any orbit $x$ could be defined as the moduli space $\mathcal{M}(x, \emptyset)$: more precisely, each component of this space can be thought of as a different copy (augmentation) of $x$, and the points of the component as the points of its unstable manifold.
Remark 5. The major difference however between the Morse and Floer settings is that in the former case, the unstable manifolds are all balls, while in the latter, there is no control a priori on this topology. This strongly limits the use of unstable manifolds.

Remark 6. An other difference between the Floer and Morse settings regarding steps is that in the latter, there is exactly one step per index 1 critical point (with its inverse), while in the former, there might be several steps through the same periodic orbit (see section 4.5 and figures 4.5.2 and 11 for a finite dimensional illustration).

![Figure 3. The Floer base point associated to \( \star \).](image)

We now turn to the definition of the second type of steps.

Starting with some \((\beta, \alpha) \in \mathcal{M}(\star, x) \times \mathcal{M}(x, \emptyset)\) for some \(x \in \tilde{\mathcal{P}}_0(H)\), gluing at \(x\) gives rise to a one dimensional family of trajectories in \(\mathcal{M}(\star, \emptyset)\). However, the other end of this family may not be a break, but a vanishing of the Hamiltonian strip. In this case, the limit sphere attached at \(\star\) is \(J\)-holomorphic, and null homotopic (recall the compatibility conditions with the lifting of the orbits in the definition of the moduli spaces). It hence has to be the constant at \(\star\). This implies that all the couples \((\beta, \alpha)\) in \(\bigcup_{x \in \tilde{\mathcal{P}}_0(H)} \mathcal{M}(\star, x) \times \mathcal{M}(x, \emptyset)\) come by pairs but one, denoted by \((\beta_*, \alpha_*)\) that is related to \(\star\) (seen as a constant sphere):

\[
\exists x_* \in \tilde{\mathcal{P}}_0(H), \exists (\beta_*, \alpha_*) \in \mathcal{M}(\star, x_*) \times \mathcal{M}(x_*, \emptyset), \quad (\beta_*, \alpha_*) \leftrightarrow \star
\]

Remark 7. This remark is already present in the construction of the PSS morphisms \([12]\) to prove that the composition of the two PSS morphisms induces the identity in homology.

This couple \((\beta_*, \alpha_*)\) will play a crucial role in all the constructions below, and the capping \(\alpha_*\) can be considered as our Floer base point. Consequently, the center of \(\alpha_*\) will be of particular interest to us, and we let

\[
\star = \alpha_*(+\infty).
\]
**Definition 1.2.** Let a Floer step through \( \star \) be a quadruple \( (\alpha, \beta, \beta', \alpha') \) with \( \alpha \in \mathcal{M}(x, \emptyset) \), \( \beta \in \mathcal{M}(\star, x) \), \( \beta' \in \mathcal{M}(\star, x') \), and \( \alpha' \in \mathcal{M}(x', \emptyset) \) for some \( x, x' \in \tilde{P}_0(H) \) such that \( (\beta, \alpha) \sim (\beta', \alpha') \).

**Remark 8.** This kind of step has no Morse analog since an index 0 critical point always has one and only one augmentation which is the trivial one in this context. In the Floer context however, such steps might exist, and there might be several of them (see section 4.5 and figure 4.5.2 for a finite dimensional illustration).

**Remark 9.** Notice there are only finitely many Floer steps of both kinds: there are finitely many periodic orbits, and because of the monotonicity assumption, finitely many lifts of each can have index 0 or 1, and finely, each 0 dimensional moduli space is compact and hence finite.

Notice Floer steps are oriented: with the above notations, \( \alpha \) is the start of the step and \( \alpha' \) its end.

**Figure 4.** A Floer free path (it may also contain steps through \( \star \)).

Define a **Floer free path** as a sequence of consecutive steps:

**Definition 1.3.** A Floer free path is a sequence of Floer steps

\[(\alpha_1, \beta_1, \beta'_1, \alpha'_1), (\alpha_2, \beta_2, \beta'_2, \alpha'_2), \ldots, (\alpha_N, \beta_N, \beta'_N, \alpha'_N)\]

such that

\[\alpha'_1 = \alpha_2, \ldots, \alpha'_{i-1} = \alpha_{i+1}, \ldots, \alpha'_{N-1} = \alpha_N\]

With the above notations, \( \alpha_1 \) is the start of the path and \( \alpha_N \) its end.

**Definition 1.4.** A Floer free loop is a Floer free path that has the same start and end. A Floer based loop is a Floer free path that starts and ends at \( \alpha_\ast \).
The main result of this paper is that $\pi_1(M)$ can be described in terms of Floer based loops.

Let $\mathcal{L}_*(H)$ be the set of all Floer based loops. It carries obvious concatenation and simplification rules that turn it into a group. Considering a step as a one parameter family of disc, the evaluation at the center of the discs defines a map

$$\mathcal{L}_*(H) \xrightarrow{\text{ev}} \Omega(M, \ast).$$

The main result of this paper is then the following:

**Theorem 1.5.** With the above notations, the evaluation map induces a surjective morphism

$$\mathcal{L}_*(H) \twoheadrightarrow \pi_1(M, \ast).$$

It is then natural to ask for a description of the relations. Although they do not depend on any extra data, we need some to present explicit generators: picking a generic Morse function $f$ that has a single minimum at $\ast$ and a Riemannian metric $g$, one can explicitly build a family $R(H, f)$ out of Floer and Piunikhin-Schwarz-Salamon moduli spaces (see section 4), such that the generated normal subgroup $\mathcal{R}_*(H)$ satisfies the following theorem.

**Theorem 1.6.** With the above notations, we have

$$\mathcal{L}_*(H) / \mathcal{R}_*(H) \sim \pi_1(M, \ast).$$

**Remark 10.** In particular, the generated subgroup $\mathcal{R}_*(H)$ does not depend on the Morse function used to define it. It would be very interesting to have purely Hamiltonian generators of the relations. There are natural candidates since we have Floer analogs of Morse 2-cells, but the lack of control on the topology of these “Floer unstable manifolds” prevents from using them efficiently. More fundamentally, the Floer situations can undergo more bifurcations than the Morse one, and the study of the Floer specific ones (birth/death of orbits of index $-1$ and $0$, creation/cancellation of pairs of Floer trajectories of opposite signs) seem to indicate that Floer like 2-cells cannot be enough.

In the same way, it would be very interesting to investigate if a “direct” comparison morphisms for fundamental groups associated to different Hamiltonians or almost complex structures, i.e. a comparison morphism that does not factor through the comparison to the Morse situation can be defined. Here again, the Floer specific bifurcations seem to already require this factorisation. If so, it would be interesting to better understand how deep this impossibility is.

1.3. **Application.**
Definition 1.7. Define the multiplicity of an Hamiltonian orbit $y \in \tilde{\mathcal{P}}_1(H)$ as the number of steps through it, i.e.

$$\nu_J(y) = \frac{1}{2} \sum_{x \in \tilde{\mathcal{P}}_0(H)} \sharp_{\text{abs}} \mathcal{M}(y, x) \cdot \sharp_{\text{abs}} \mathcal{M}(x, \emptyset)$$

Define the multiplicity of the point $\star$ as the number

$$\nu_J(\star) = \frac{1}{2} \left( \sum_{x \in \tilde{\mathcal{P}}_0(H)} \sharp_{\text{abs}} \mathcal{M}(\star, x) \cdot \sharp_{\text{abs}} \mathcal{M}(x, \emptyset) \right) - \frac{1}{2}$$

Notice the counting here is not algebraic but geometric: it is not hard to see that the algebraic count would always be 0.

Although they may seem to be $\frac{1}{2} \mathbb{Z}$ valued, these numbers are in fact integer valued: as already observed, the gluing construction groups the broken trajectory $(\beta, \alpha)$ from some $y \in \tilde{\mathcal{P}}_1(H)$ to $\emptyset$ by pairs, so there is an even number of such, and the same holds for broken trajectories from $\star$ to $\emptyset$ but for $(\beta_\star, \alpha_\star)$, which proves there is an odd number of such in this case.

The following statement is a direct corollary of our construction, and will be proven in section 5.1.

Theorem 1.8. Let $\rho(\pi_1(M))$ be the minimal number of elements in a generating family of $\pi_1(M)$. Then

$$\nu_J(\star) + \sum_{y \in \tilde{\mathcal{P}}_1(H)} \nu_J(y) \geq \rho(\pi_1(M)).$$

In other words, counted with multiplicities, $\{\star\} \cup \tilde{\mathcal{P}}_1(H)$ contains sufficiently many elements to generate $\pi_1(M)$.

Remark 11. In the Morse setting, all the index 1 critical points have multiplicity 1 and $\nu_J(\star) = 0$, so that the inequality (10) in this context becomes the usual lower estimate of the number of index 1 critical points of a Morse function by the minimal number of generators of the fundamental group $\pi_1(M)$.

Remark 12. Although the natural interpretation of the $\nu_J(\star)$ term is a multiplicity for $\star$, it gathers contributions of index 0 orbits (namely $\frac{1}{2} \sharp_{\text{abs}} \mathcal{M}(\star, x) \cdot \sharp_{\text{abs}} \mathcal{M}(x, \emptyset)$ for each $x \in \tilde{\mathcal{P}}_0(H)$), and inequality (10) means that if counted suitably, there are enough of index 0 and 1 Hamiltonian periodic orbits to generate the fundamental group.

Remark 13. This inequality is obviously different in nature from the Morse inequalities derived from the Floer homology, since $\rho(\pi_1(M))$ is an homotopical invariant. It is also different from the results derived by K. Ono and A. Pajitnov (11) from the algebraic study of the Floer complex with local coefficient, since it leads to a larger lower bound that is known to be false in the stable Morse setting where A. Pajitnov and K. Ono technique also apply, and for which some kind of notion of multiplicity is necessarily required.
Remark 14. The $\nu_J(\ast)$ term may be unexpected, since it automatically vanish in the Morse setting. It is a very natural question to ask how essential it is. The construction seems to indicate it plays a crucial role, but on the other hand, the results of K. Ono and A. Pajitnov prove there are indeed many orbits, and hence limit its contribution.

The weaker theorem 1.9 below (proven in section 5.2) ensures the existence of at least one orbit of index 1 when $\pi_1(M) \neq \{1\}$. Although it is far from optimal in any respect, we do not resist to mention it here, because we like its simple and geometric proof. We stress however that it is not an application of the construction of the fundamental group.

**Theorem 1.9.** Suppose $\pi_1(M) \neq \{1\}$. Then every non degenerate Hamiltonian $H$ has to have at least one contractible 1-periodic orbit of Conley Zehnder index $1 - n$.

2. More notations and preliminaries.

2.1. **Moduli spaces.** In addition to the already introduced moduli spaces we will need hybrid Morse-Floer moduli spaces, depicted in figure 5 and defined below.

![Figure 5. Hybrid moduli spaces.](image)

Let $f$ be a Morse functions and $g$ a Riemannian metric on $M$. Let $\text{Crit}_k(f)$ be the set of index $k$ critical points, and suppose $\text{Crit}_0(f) = \{\ast\}$. For $y \in \mathcal{P}_1(H)$ and $x \in \text{Crit}_0(f)$, we let

$$\mathcal{M}^0(y, x) = \{u \in \mathcal{M}(y, \emptyset), u(+\infty) \in W^s(x)\}$$

where $W^s(x)$ is the stable manifold of $x$.

Similarly, for $y \in \text{Crit}_1(f)$ and $x \in \mathcal{P}_0(H)$, we let

$$\mathcal{M}^0(y, x) = \{u \in \mathcal{M}(\emptyset, x), u(-\infty) \in W^u(y)\}$$

where $W^u(y)$ is the unstable manifold of $y$.

These spaces are compact up to bubbling of sphere or breaks, either at an intermediate Hamiltonian orbit or at an intermediate Morse critical point (see [12]), and the compactification is denoted by $\mathcal{M}(y, x)$. Moreover, $(f, g)$
is supposed to be chosen generically, so that all these spaces are cut out transversely. In particular, they have the expected dimensions:
\[
\dim \mathcal{M}(y, x) = |y| - |x|
\]
(where the Morse index is also denoted by $| \cdot |$) and there is a gluing construction proving every broken configuration does indeed appear on the boundary of a bigger moduli space.

2.2. Crocodile walk. We now introduce the main technical tool of all the subsequent constructions.

Consider a Hamiltonian orbit $z$ of index 2. Let $B(z)$ be the space of twice broken trajectories from $z$ to $\varnothing$:
\[
B(z) = \bigcup_{|y|=1, |x|=0} \mathcal{M}(z, y) \times \mathcal{M}(y, x) \times \mathcal{M}(x, \varnothing).
\]
For each such trajectory, the gluing construction can take place either at the upper or lower break. Gluing at the upper break defines an involution
\[
\sharp^* B(z) \rightarrow B(z) \quad (\gamma, \beta, \alpha) \mapsto (\gamma', \beta', \alpha)
\]
where $(\gamma', \beta')$ is such that $(\gamma, \beta) \sharp^* (\gamma', \beta')$. Similarly, gluing at the lower break, defines another involution
\[
\sharp B(t, x) \rightarrow B(t, x) \quad (\gamma, \beta, \alpha) \mapsto (\gamma, \beta', \alpha')
\]
We will refer to the iteration of alternatively $\sharp^*$ and $\sharp$ as running a “crocodile walk” on the set $B(z)$ of twice broken trajectories from $z$ to $\varnothing$.

Remark 15. The crocodile walk can in fact be defined much more generally on any 0 dimensional space of twice broken trajectories or cappings or other objects, as long as breaks are the only compacity failures (like the space of twice broken Floer trajectories between orbits of relative index 3 for instance).

The crocodile walk is the iteration of a one to one map on a finite set, so the orbits all have to be cyclic.

Recall all the moduli spaces are given coherent orientations. In particular, a trajectory $\gamma$ in a 0 dimensional moduli space is endowed with a sign $\epsilon(\gamma)$. The maps $\sharp^*$ and $\sharp$ then change the product of the signs associated to the 3 components of the broken trajectory. In particular, the crocodile walk necessarily loops after an even number of steps, i.e. after iterating $\sharp \circ \sharp^*$ a certain number of times.

To an orbit of the crocodile walk is not only associated a sequence of twice broken trajectories, but also an abstract polyhedron representing the way the trajectories in the different interpolation moduli spaces fit together in the following way.
Figure 6. An orbit of the crocodile walk on the space of twice broken trajectories from $z \in \tilde{\mathcal{P}}_2(H)$ to $\varnothing$.

An orbit $W$ of the crocodile walk is a sequence 

$$((\gamma_1, \beta_1, \alpha_1), (\gamma_2, \beta'_1, \alpha_1), (\gamma_2, \beta_2, \alpha_2), \ldots, (\gamma_N, \beta_N, \alpha_N))$$

such that 

$$(\gamma_k, \beta_k) \leftrightarrow (\gamma_{k+1}, \beta'_k) \quad \text{and} \quad (\beta'_k, \alpha_k) \leftrightarrow (\beta_{k+1}, \alpha_{k+1}).$$

Let $\mathcal{M}^\bullet_k$ (resp. $\mathcal{M}^\bullet_{-k}$) be an abstract copy of the component of the moduli space relating $(\gamma_k, \beta_k)$ to $(\gamma_{k+1}, \beta'_k)$ (resp. $(\beta'_k, \alpha_k)$ to $(\beta_{k+1}, \alpha_{k+1})$). Let $\Sigma \mathcal{M}^\bullet_k$ be its suspension: it is the suspension of a segment and hence can be identified with the standard diamond.

Recall that before compactification, the evaluation along the real line defines a map

$$\mathcal{M}^\circ_k \times \mathbb{R} \xrightarrow{\text{ev}} \mathcal{M}.$$ 

Since the action is strictly decreasing along the Floer trajectories, it can be used for a parametrization of the trajectories, and after a suitable normalisation, we get a continuous map

$$\mathcal{M}^\circ_k \times [-1, 1] \xrightarrow{\text{ev}} \mathcal{M}.$$ 

that extends continuously to the compactification, and descends to the suspension

$$\Sigma \mathcal{M}^\bullet_k \xrightarrow{\text{ev}} \mathcal{M}.$$ 

We think of $\Sigma \mathcal{M}^\bullet_k$ as a diamond, and on the four sides, the evaluation map is the action-normalized evaluation along the broken trajectories $(\gamma_k, \beta_k)$ on the left and $(\gamma_{k+1}, \beta'_k)$ on the right.

A similar construction can also be achieved for the $\mathcal{M}^\bullet_{-k}$ spaces. The lower end of the trajectories is not constrained however, and the suspension should be replaced by the half suspension $\Sigma' \mathcal{M}^\bullet_{-k} = \mathcal{M}^\bullet_{-k} \times [-1, 1] / \mathcal{M}^\bullet_{-k} \times \{1\}$. We think of this as a truncated diamond, or a pentagon. It is endowed with an evaluation map whose restriction

- to the upper left side (i.e. $[0, 1] \times \{(\beta'_k, \alpha_k)\}$) is $\beta'_k$.
• to the lower left side (i.e. \([-1,0] \times \{(\beta'_k, \alpha_k)\}\) is \(\alpha_k\)
• to the upper right side (i.e. \([0,1] \times \{(\beta_{k+1}, \alpha_{k+1})\}\) is \(\beta_{k+1}\)
• to the lower right side (i.e. \([-1,0] \times \{(\beta_{k+1}, \alpha_{k+1})\}\) is \(\alpha_{k+1}\)
• to the bottom side (i.e. \([-1] \times M_{\bullet}^k\) is the evaluation at the center of the augmentations \(ev(u) = u(+\infty)\)

We identify all these diamonds and pentagons along their shared sides in the order of the gluings appearing in the orbit \(W\): formally, we let

\[
P_2(W) = \left( \bigcup_{k=1}^{N} \Sigma M_k^k \cup \Sigma M_{\bullet}^k \right) \bigg/ \sim
\]

where \(\sim\) is the identification, for each \(k\) of

• the upper right side of \(\Sigma M_k^k\) with the upper left side of \(\Sigma M_{k+1}^k\),
• the lower right side of \(\Sigma M_k^k\) with the upper left side of \(\Sigma M_{k+1}^k\),
• the lower left side of \(\Sigma M_k^k\) with the upper right side of \(\Sigma M_{k+1}^k\).

The resulting 2 dimensional polyhedron \(P_2(W)\) is a disc. Moreover, since it is compatible with all the identifications, the evaluation map descends to \(P_2(W)\) and defines a continuous map

\[
P_2(W) \xrightarrow{ev} M.
\]

**Remark** 16. Here, the lower pentagons \(\Sigma M_{\bullet}^k\) don’t play any essential role and could be omitted without affecting the topology of \(P_2(W)\). In general however, if the lower end of the trajectories were constrained (like in the twice broken trajectories between orbits of relative index 3 for instance), full diamonds would be used as the lower polygons, and the resulting polyhedron \(P_2(W)\) would a sphere.

2.2.1. *Hybrid walks.* As already observed, the “crocodile walk” can in fact be run on many kind of moduli spaces. In particular, the top or intermediate orbits could be critical points of our Morse function \(f\).

What should be kept in mind however, is that as already observed, the space \(\mathcal{M}(\bullet, \emptyset)\) has one (and only one) boundary component which is not a break but the vanishing of the Hamiltonian stripe. This means that reaching any configuration of the form \((\gamma, \beta_\bullet, \alpha_\bullet)\) stops the crocodile.

In particular, let \(b \in \text{Crit}_1(f)\) and let

\[
B(b) = \bigcup_{\substack{y \in P_1(H) \cup \{\star\} \atop x \in P_0(H)}} \mathcal{M}(b, y) \times \mathcal{M}(y, x) \times \mathcal{M}(x, \emptyset)
\]

be the space of twice broken trajectories from \(b\) to \(\emptyset\).

Let \(\{\gamma_-, \gamma_+\} = \mathcal{M}(b, \star)\) be the two Morse trajectories rooted at \(b\) (recall \(\text{Crit}_0(f) = \{\star\}\)).

**Proposition 2.1.** With the above notations, exactly one orbit of the crocodile walk on \(B(b)\) is not cyclic. It starts at \((\gamma_-, \beta_\bullet, \alpha_\bullet)\) and ends at \((\gamma_+, \beta_\bullet, \alpha_\bullet)\).
Proof. The only possible stops for the Crocodile walk are \((\gamma_-, \beta_*, \alpha_*)\) and \((\gamma_+, \beta_*, \alpha_*)\). Starting from \((\gamma_-, \beta_*, \alpha_*)\), the crocodile walk can only move in one direction (i.e. upper gluing first), and will stop when reaching a configuration of the form \((\gamma', \beta_*, \alpha_*)\) where \(\gamma' \in \mathcal{M}(y, *) = \{\gamma_-, \gamma_+\}\). Notice the last configuration has to be reached after an upper gluing, and hence after an odd number of gluings from start. For orientation reasons, this implies that \(\epsilon(\gamma') = -\epsilon(\gamma_-)\), and hence \(\gamma' = \gamma^+\).

\[\text{Figure 7. The non cyclic orbit of the crocodile walk on a space of twice broken hybrid trajectories.}\]

The construction of the polyhedron \(P_2(W)\) still makes sense for this non-cyclic orbit but results in a half disc now (a half disc is topologically the same as a disc, but it is a way to emphasize that the boundary splits into two distinct parts: a diameter and a half circle). It is still endowed with a continuous evaluation map to \(M\)

\[
P_2(W) \xrightarrow{ev} M,
\]
given by the trajectories \(\gamma_-\) and \(\gamma_+\) on the “diameter”, and a Floer loop conjugated by a path \(\lambda_{**}\) on the half circle, where \(\lambda_{**}\) is the path from \(*\) to \(\ast\) given by the evaluation along the real lines of \(\beta_*\) and \(\alpha_*\).

3. Homotopies

Similarly to \[\square\] define a Morse step as a path traveling once along the unstable manifold of an index 1 Morse critical point \(y \in \text{Crit}_1(f)\), and represent it algebraically by \(y^\pm\). Notice that since \(f\) has a single minimum, all the steps are necessarily consecutive and every sequence of steps defines a Morse based loop. Let \(L_*(f)\) be the corresponding group, i.e. the free group generated by \(\text{Crit}_1(f)\).
3.1. From $L_*(H)$ to $L_*(f)$. Pushing a generic topological loop down by the flow of the Morse function $f$ deforms it into a Morse loop, i.e. a word in the index 1 critical points. Here generic means that the loop avoids the stable manifolds of all the index $k \geq 2$ Morse critical points. Our genericity assumptions imply that the Floer loops are indeed generic in this sense, and we get a well defined map

$$L_*(H) \xrightarrow{\phi} L_*(f)$$

that is a group morphism. Let $\lambda_{**}$ be the piece of flow line from $\ast$ to $\ast$ and consider the associated change of base point:

$$\Omega(M, \ast) \xrightarrow{\phi_\Omega} \Omega(M, \ast)$$

These maps fit into the following diagram

(15)

where the first square only commutes up to homotopy (i.e. $\pi \circ \text{ev} \circ \phi = \pi \circ \phi_\Omega \circ \text{ev}$). This last fact is emphasized in the following obvious proposition for later use:

**Proposition 3.1.** For $w \in L_*(H)$, $w$ and $\phi(w)$ are homotopic in the following sense:

$$\text{ev}(\phi(w)) \sim \lambda_{**}^{-1} \text{ev}(w) \lambda_{**} \quad \text{in } \pi_1(M, \ast)$$

(where $\lambda_{**}$ is the piece of Morse flow line from $\ast$ to $\ast$).

Since the second row of (14) is onto, theorem 1.5 comes down to proving that any Morse loop can be deformed into a Floer loop. Unfortunately, this deformation cannot be obtained like $\phi$ by pushing a loop by a flow, since there is no such thing as a Floer flow on the loop space.

However, the definition of the map $\phi$ can be reinterpreted in terms of moduli spaces and crocodile walk in the following way, which makes sense in the Floer setting. Consider that $\ast$ as endowed with a trivial augmentation. Recall that given some index 1 critical point $y$ of $f$, the one parameter family obtained by gluing this augmentation to a Morse trajectory $\gamma \in M(y, \ast) = \{\gamma, \gamma'\}$ consists in stopping the trajectory $\gamma$ before reaching $\ast$ at an higher and higher level, and once the level of $y$ is reached, stopping $\gamma'$ at an lower and lower level until $\ast$ is reached again.

Given a Floer step $(\alpha, \beta, \beta', \alpha')$, let $\lambda_{\alpha}$ (resp. $\lambda_{\alpha'}$) be the piece of Morse trajectory running from the center of $\alpha$ (resp. $\alpha'$) down to $\ast$ (notice that by our genericity assumption, all the Morse trajectories through the center
of such augmentations do indeed end at $\star$). A crocodile walk can then be started at the broken trajectory $(\beta, \alpha, \lambda)$. 

Along this walk, the lower break is always $\star$, while the upper one could be either a Morse or a Hamiltonian break. The Morse breaks do not stop the crocodile walk, and by definition of a Floer step, the first Hamiltonian break is reached at $(\beta', \alpha', \lambda_{\alpha'})$, and this stops the crocodile walk. Keeping only the lower parts of the corresponding sequence of broken trajectories, we get a Morse loop, and the concatenation of all these loops for all the steps in a Floer loop $\gamma$ is the loop $\phi(\gamma)$.

The next section uses this interpretation of the homotopy to turn Morse loops into Floer ones.

3.2. From $L_*(f)$ to $L_*(H)$. Let $b$ be an index 1 critical point of $f$ and $(\gamma^+_b, \gamma^-_b)$ be the two Morse trajectories from $b$ to $\star$.

Recall from 2.2.1 that exactly one orbit of the crocodile walk on the set

$$B_b = \bigcup_{y \in \hat{P}_1(H) \cup \{\star\}} \mathcal{M}(b, y) \times \mathcal{M}(y, x) \times \mathcal{M}(x, \varnothing)$$

of hybrid twice broken trajectories from $b$ to $\varnothing$ is not periodic: it starts at $(\gamma^-_b, \beta_{\star}, \alpha_{\star})$ and ends at $(\gamma^+_b, \beta_{\star}, \alpha_{\star})$ (see figure 7).

Let $(\gamma^-_b, \beta_{\star}, \alpha_{\star}), (\gamma_1, \beta_1, \alpha_1), \ldots, (\gamma_N, \beta_N, \alpha_N), (\gamma^+_b, \beta_{\star}, \alpha_{\star})$ be this crocodile orbit. We have

- $\gamma_i \in \mathcal{M}(b, y)$ for some $y \in \{\star\} \cup \hat{P}_1(H)$,
- $\beta_i \in \mathcal{M}(y, x)$ for some $y \in \{\star\} \cup \hat{P}_1(H)$ and $x \in \hat{P}_0(H)$,
- $\alpha_i \in \mathcal{M}(x, \varnothing)$ for some $x \in \hat{P}_0(H)$,

and, for all $i$ with $0 \leq i < N/2$ :

$$\gamma_{2i} = \gamma_{2i+1}, \beta_{2i} = \beta_{2i+1}, \alpha_{2i} = \alpha_{2i+1}$$

As a consequence, the sequence of steps $(\alpha_1, \beta_1, \alpha_2), (\alpha_3, \beta_3, \alpha_4), \ldots, (\alpha_{N-1}, \beta_{N-1}, \beta_N, \alpha_N)$ is a Floer based loop. Denoting it by $\psi(b)$ (see figure 7), we get a map

$$L_*(f) \xrightarrow{\psi} L_*(H).$$

Proposition 3.2. For $\gamma \in L_*(f)$, we have

$$\text{ev}(\psi(\gamma)) \sim \lambda_{\star}^{-1} \text{ev}(\gamma) \lambda_{\star} \text{ in } \pi_1(M, \star),$$

where $\lambda_{\star}$ is the path from $\star$ to $\star$ obtained by evaluating $\beta_{\star}$ and $\alpha_{\star}$ along their real ray.

Proof. For each Morse step $\tau$, $\psi(\tau)$ is constructed from an orbit $W$ of crocodile walk. To each such an orbit is associated a polyhedron $P_2(W)$ (see section 2.2) which is a half disc endowed with an evaluation map to $M$. Looking at the boundary of this disc gives the desired relation. \qed
3.3. **Proof of theorem 1.5.** Let $\lambda_{**} = \lambda_{**} \cdot \lambda_{**}$ be the loop from $\star$ to itself defined by the evaluation along the real line of $\beta_{**}$ and $\alpha_{**}$ and the Morse flow line from $\star = \alpha_{**}(+\infty)$ to $\star$.

The following proposition gathers propositions 3.1 and 3.2:

**Proposition 3.3.** For $\gamma \in L_{**}(f)$, we have

$$\text{ev}(\phi \circ \psi(\gamma)) \sim \lambda_{***}^{-1}\text{ev}(\gamma)\lambda_{***}$$

in $\pi_1(M, \star)$.

**Proof of theorem 1.5.** From (14), the theorem reduces to proving that

$$L_{**}(H) \xrightarrow{\text{ev}} \pi_1(M, \star)$$

is onto. Let $g \in \pi_1(M)$. Pick a Morse representation $\gamma$ of $\lambda_{***}g\lambda_{***}^{-1}$. Then from proposition 3.3, $g = \pi \circ \text{ev} \circ \phi(\psi(\gamma))$ in $\pi_1(M)$, which ends the proof. □

4. Relations

We just proved the Floer loops form a generating family for $\pi_1(M, \star)$. The subgroup of relations is then nothing but

$$R_{**}(H) = \ker (L_{**}(H) \xrightarrow{\text{ev}} \pi_1(M, \star)).$$

In other word, the fundamental group can be seen as the group of Floer loops where two Floer loops are considered equivalent if they have homotopic evaluations in $M$:

$$L_{**}(H)/R_{**}(H) \sim \pi_1(M, \star).$$

It is natural to ask for a Floer theoretic interpretation of the relations. It is the object of this section to provide a family of generators of $R_{**}(H)$ that can be expressed in terms of crocodile walks on Floer and PSS moduli spaces. Notice the proposed generators will depend on the choice of a Morse function, but the generated group does not.

4.1. **Relations associated to the homotopy $\psi \circ \phi$.** Recall $\star$ is seen as the base point in the Morse setting, and $\alpha_*$ as the base point in the Floer setting, and that the homotopies $\psi$ and $\phi$ act by conjugation by $\lambda_{**}$ and $\lambda_{**}$. Recall the diagram (15) and the surjectivity of the evaluation:

$$L_{**}(H) \xrightarrow{\text{ev}} \pi_1(M, \star).$$

In particular, consider the loop $\lambda_{**} = \lambda_{**} \cdot \lambda_{**}$ obtained by concatenating the piece of Morse flow line from $\star = \alpha_{**}(+\infty)$ to $\star$, and the evaluation along the real line on $\beta_{**}$ and $\alpha_{**}$.

Let $\Lambda$ be a Floer realisation of $\lambda_{**}$ :

$$\exists \Lambda \in L_{**}(H), \text{ev}(\Lambda) \sim \lambda_{**} \text{ in } \pi_1(M, \star)$$

Given a Floer loop $\gamma \in L_{**}(H)$, we already observed that $\psi(\phi(\gamma))$ is homotopic to $\lambda_{**}^{-1}\gamma\lambda_{**}^{-1}$, and this proves that

$$\gamma^{-1} \Lambda \psi(\phi(\gamma)) \Lambda^{-1} \sim 1 \text{ in } \pi_1(M, \star)$$
Figure 8. Composed homotopies and conjugation.

**Definition 4.1.** Define the set of "\(\psi \circ \phi\) homotopy relations" as
\[
R_{\psi \phi} = \{ \gamma^{-1}\Lambda\psi(\phi(\gamma))\Lambda^{-1}, \ \gamma \in \mathcal{L}_\ast(H) \}
\]

**Remark 17.** Such relations can be characterized (up to the computation of \(\Lambda\)) in terms of crocodile walks: appending \(\lambda_{***}\) to \(\alpha_\ast\) at the beginning of the first step of \(\gamma\) gives a trajectory from some \(y \in \tilde{P}_1(H) \cup \{\ast\}\) to \(\emptyset\) that is broken three times. A variant of the crocodile walk can be started there, that consists in

1. running the usual crocodile walk as far as possible on the two lower level breaks,
2. when stopped, perform one gluing at the top level, and restart a crocodile walk on the two lower ones.

The Floer loop \(\psi(\phi(\gamma))\) can then be recovered by repeating this process for each step in \(\gamma\).

**4.2. Relations associated to Morse 2-cells.** Let \(c\) be an index 2 Morse critical point, and \(\rho_c\) the associated relation in \(\mathcal{L}_\ast(f)\). The image under \(\psi\) of this relation in \(\mathcal{L}_\ast(H)\) is still homotopically trivial (since \(\psi\) acts by conjugation at the homotopy level). We call \(\psi(\rho_c)\) the Floer relation induced by \(c\), and let
\[
R_M = \{ \psi(\rho_c), \ c \in \text{Crit}_2(f) \}
\]

**Remark 18.** These relations can again be described in terms of crocodile walks. Namely, consider the set
\[
B(c) = \bigcup_{b \in \text{Crit}_1(f)} \mathcal{M}(c,b) \times \mathcal{M}(b,y) \times \mathcal{M}(y,x) \times \mathcal{M}(x,\emptyset)
\]
of 3 times broken trajectories from \(c\) to \(\emptyset\) that are rigid. The same variant of the crocodile walk as earlier can be run on this space following the algorithm:
Figure 9. A Floer relation associated to a Morse 2-cell.

(i.) run the crocodile walk as far as possible on the two lower breaks (i.e. keeping the first one fixed)
(ii.) when it stopped, perform one upper gluing, and repeat (i.)
It is just a reformulation of all the construction above that the bottom Floer loop obtained in this way is \( \psi(\rho) \).

4.3. The Floer fundamental group. We can finally define the subgroup of relations:

**Definition 4.2.** Let \( R^\ast(\mathcal{H}) \) be the normal subgroup of \( L^\ast(\mathcal{H}) \) generated by the relations induced by all the “\( \psi \circ \phi \)” homotopies and all the Morse 2-cells:
\[
R^\ast(\mathcal{H}) = \langle R_{\psi \phi}, R_M \rangle.
\]

**Definition 4.3.** The Floer fundamental group associated to \( (\mathcal{H}, J, \ast, f, g) \) is defined as the group
\[
\pi_1(\mathcal{H}) = L^\ast(\mathcal{H}) / R^\ast(\mathcal{H}).
\]

**Remark 19.** The group should be denoted as \( \pi_1(\mathcal{H}, J, \ast, f, g) \) to emphasise the dependency on all the parameters. However, we will see that \( R^\ast(\mathcal{H}) \) does not depend on \( (f, g) \), so that \( \pi_1(\mathcal{H}) \) does only depend on \( \mathcal{H}, J, \) and \( \ast \). We still keep the dependency on \( J \) and \( \ast \) implicit to reduce notations.

In the same way, we let \( R^\ast(f) \) be normal subgroup of \( L^\ast(f) \) generated by the “boundary of Morse 2-cells” and \( \pi_1(f) := L^\ast(f) / R^\ast(f) \) and recall as it is well known that \( \pi_1(f) \simeq \pi_1(M, \ast) \).

**Theorem 4.4.** With the above notations
\[
R^\ast(\mathcal{H}) = \ker(L^\ast(\mathcal{H}) \xrightarrow{ev} \pi_1(M, \ast)).
\]
In particular, the following maps are group isomorphisms:
\[
\pi_1(\mathcal{H}) \xrightarrow{ev} \pi_1(M, \ast) \quad \text{and} \quad \pi_1(\mathcal{H}) \xrightarrow{\phi} \pi_1(f).
\]
The proof of this statement is the object of the next section

4.4. Morphisms.

4.4.1. Compatibility of \( \phi \) with the relations and surjectivity.

**Lemma 4.5.** The morphism \( \phi \) sends relations to relations :
\[
\phi(\mathcal{R}_*(H)) \subseteq \mathcal{R}_*(f).
\]

**Proof.** It was already observed that the evaluation of a relation in \( R_{\psi \phi}(H) \) or \( R_M(H) \) is trivial in \( \pi_1(M, \ast) \).

As a consequence, every Floer loop \( w \) in \( \mathcal{R}_*(H) \) evaluates as a trivial loop in \( \pi_1(M, \ast) \), and since \( \phi \) acts by conjugation by a fixed path, \( \phi(w) \) is also trivial and hence belongs to \( \mathcal{R}_*(f) \).

\[ \square \]

**Lemma 4.6.** The induced map \( \pi_1(H, \ast) \xrightarrow{\phi} \pi_1(f, \ast) \) is surjective.

**Proof.** We already proved \( \mathcal{L}_*(H) \xrightarrow{\phi} \mathcal{L}_*(f) \xrightarrow{\pi} \pi_1(f) \) is onto.

4.4.2. Compatibility of \( \psi \) with relations and injectivity.

**Lemma 4.7.** The morphism \( \psi \) sends relations to relations :
\[
\psi(\mathcal{R}_*(f)) \subseteq \mathcal{R}_*(H).
\]

**Proof.** Since \( \mathcal{R}_*(f) \) is generated by the “boundaries of 2-cells” associated to index 2 critical points, \( \psi(\mathcal{R}_*(f)) \) is contained in the normal subgroup generated by \( R_M(H) \), and is hence clearly a subgroup of \( \mathcal{R}_*(H) \).

\[ \square \]

**Corollary 4.8.** The induced map \( \pi_1(f, \ast) \xrightarrow{\psi} \pi_1(H, \ast) \) is injective. The composition \( \pi_1(f) \xrightarrow{\psi} \pi_1(H) \xrightarrow{\phi} \pi_1(f) \) is the conjugation by \( \lambda_{**} \).

**Proof.** The last part of the statement implies the first and is already contained in proposition 3.3.

\[ \square \]

4.4.3. Kernel of \( \phi \).

**Proof of proposition 4.4.** We already proved the map \( \pi_1(H) \xrightarrow{\phi} \pi_1(f) \) to be surjective.

We now want to prove it is injective.

Let \( \gamma \in \mathcal{L}_*(H) \) such that \( \phi(\gamma) \in \mathcal{R}_*(f) \). Then \( \psi(\phi(\gamma)) \in \psi(\mathcal{R}_*(f)) \subseteq \mathcal{R}_*(H) \).

This means \( \psi \circ \phi(\gamma) = 1 \) modulo \( \mathcal{R}_*(H) \). But \( \gamma \sim \Lambda \psi \circ \phi(\gamma) \Lambda^{-1} \) modulo \( \mathcal{R}_*(H) \), so \( \gamma = 1 \) modulo \( \mathcal{R}_*(H) \), and \( \phi \) is injective on \( \pi_1(H) \).

This ends the proof that ker(\( \mathcal{L}_*(H) \xrightarrow{\phi} \mathcal{L}_*(f) \)) = \( \mathcal{R}_*(H) \), and hence of proposition 4.4.

\[ \square \]

**Remark 20.** As already observed, this shows in particular that \( \mathcal{R}_*(H) = \text{ker}(\mathcal{L}_*(H) \xrightarrow{\psi} \pi_1(M, \ast)) \) does not depend on the Morse function used to define it.
4.5. A finite dimensional model. Recall a stable Morse function on $M$ is a Morse function on $M \times \mathbb{R}^N$ that is quadratic at infinity.

To some extent, a stable Morse function can be considered as a simplified finite dimensional model for the Action functional on the free loop space. As a side comment on the construction of the Floer fundamental group, this section is devoted to a quick sketch of its analog in the stable-Morse setting. Although it would deserve a dedicated discussion, it is only addressed here to shed some light on the phenomena encountered along the construction that do not appear in the usual Morse setting, namely the existence of several steps through the same critical point or of steps through $\star$. Therefore, we limit ourselves to the ground definition of the relevant moduli spaces, and leave all proofs and checks to the reader, who could just as well and safely skip this section.

4.5.1. Setting. Let $M$ be a smooth closed manifold of dimension $n$, $N_\pm$ be two integers, $N = N_+ + N_-$ and $H$ be a Morse function on $M \times \mathbb{R}^N$ that is quadratic at infinity with signature $(N_+, N_-)$. Namely, we suppose there is a compact $K$ such that $\forall (m, u, v) \in (M \times \mathbb{R}^{N_+} \times \mathbb{R}^{N_-}) \setminus K$, $H(m, u, v) = u^2 - v^2$.

For convenience, the Morse index will be shifted by $N_-$ and we let, for a critical point $p$ of $H$:

$$|p| = \mu(p) - N_-$$

where $\mu$ denotes the usual Morse index.

We also pick a Riemannian metric $g$ on $M \times \mathbb{R}^N$ and denote by $\phi^t$ the associated negative gradient flow of $H$. For $x, y \in \text{Crit}(H)$ we define

$$\mathcal{M}(y, x) = (W^u(y) \cap W^s(x)) / \mathbb{R}$$

$$\mathcal{M}(y, \emptyset) = W^u(y) \cap (M \times \mathbb{R}^{N_-})$$

We also pick a preferred point $\star \in M$ and consider :

$$\mathcal{M}(\star, x) = (\{\star\} \times \mathbb{R}^{N_-}) \cap W^s(x)$$

$$\mathcal{M}(\star, \emptyset) = \{(p, R) \in (\{\star\} \times \mathbb{R}^{N_+}) \times [0, +\infty], \phi^R(p) \in M \times \mathbb{R}^{N_-}\}$$

The triple $(H, g, \star)$ is supposed to be chosen generically so that all the considered moduli spaces are cutout transversely (which is more than Morse-Smale in general). In this situation, they are all smooth manifolds with dimension :

$$\dim \mathcal{M}(y, x) = |y| - |x| - 1$$

$$\dim \mathcal{M}(y, \emptyset) = |y|$$

$$\dim \mathcal{M}(\star, x) = -|x|$$

$$\dim \mathcal{M}(\star, \emptyset) = 1$$

Moreover, all these moduli spaces are compact up to the usual break of Morse trajectories phenomenon, and the gluing construction also makes sens in this setting.
With these notations, the definitions given in the Floer setting literally make sense and give rise to suited notions of base point, steps, and loops in the Morse stable setting.

4.5.2. **Base point.** A trajectory in $\mathcal{M}(\star, \emptyset)$ is a piece of Morse flow line from $\{\star\} \times \mathbb{R}^{N_-}$ to $M \times \mathbb{R}^{N_+}$. It has a transit time $R$ which provides a proper projection $\mathcal{M}(\star, \emptyset) \to (0, +\infty)$. All the Morse breaks happen over $R = +\infty$, while over $R = 0$, the trajectories are reduced to a single point in $\{\star\} \times \mathbb{R}^{N_-} \cap M \times \mathbb{R}^{N_+} = \{\star\}$. As a consequence, there is only one trajectory above $R = 0$, and hence, all the (broken) trajectories above $R = +\infty$ are grouped by pairs but one, denoted by $(\alpha_*, \beta_*)$ which is connected to $\star$ through $\mathcal{M}(\star, \emptyset)$. We consider $\alpha_*$ as our stable-Morse base point.

![Figure 10](image_url)

**Figure 10.** A situation with a non trivial “step through $\star$”:
$(\alpha_*, \beta_*)$ is the base point while $(\alpha, \beta_*) \leftrightarrow (\alpha', \beta_*)$ is a step.

4.5.3. **Steps and loops.** Similarly, definitions 1.1 and 1.2 in the Morse stable setting define a step as a quadruple $(\alpha, \beta, \beta', \alpha')$ such that
- $\alpha \in \mathcal{M}(x, \emptyset)$, where $x \in \text{Crit}(H)$ with $|x| = 0$
- $\beta \in \mathcal{M}(y, x)$, where $y = \star$ or $y \in \text{Crit}(H)$ with $|y| = 1$
- $\beta' \in \mathcal{M}(y, x')$, for some $x' \in \text{Crit}(H)$ with $|x'| = 0$
- $\alpha' \in \mathcal{M}(x', \emptyset)$,

and

$$(\alpha, \beta) \leftrightarrow (\alpha', \beta').$$
Fundamental group. Defining a based loop like in 1.4 as a sequence of steps starting and ending at the base point, and $L_*(H)$ as the set of all based loops, we have an “evaluation” map induced by the projection $M \times \mathbb{R}^N \to M$

$$L_*(H) \xrightarrow{ev} \Omega(M, \ast).$$

**Theorem 4.9.** The composition $L_*(H) \xrightarrow{ev} \Omega(M, \ast) \xrightarrow{\pi} \pi_1(M, \ast)$ is onto.

### 5. Application.

#### 5.1. Generating $\pi_1(M)$ with steps. Theorem 1.8 is a direct consequence of a weaker version of theorem 1.5 where Floer loops are replaced by Floer steps.

**Proof of theorem 1.8.** Fix a generic set of data $(H, J, \ast, f, g)$. Let $S_*(H)$ be the free group generated by the Floer steps. Then $L_*(H)$ is a subgroup of $S_*(H)$ and the map $\phi$ extends to $S_*(H)$, and factors through the inclusion:

$$
\begin{tikzcd}
L_*(H) \ar{r} & S_*(H) \ar{r}{\phi} & L_*(f) \ar{r} & \pi_1(M, \ast).
\end{tikzcd}
$$

This proves that the induced morphism $S_*(H) \to \pi_1(M, \ast)$ is onto, and hence that $S_*(H)$ has at least $\rho(\pi_1(M))$ generators.

On the other hand, $S_*(H)$ is generated by all the Floer steps, and there are $\sum_{y \in \mathcal{P}_1(H)} \nu_J(y)$ of them through an index 1 orbit, and $\nu_J(\ast)$ of them through $\ast$. \qed
Notice statement 1.8 also holds in the Morse stable setting with a strictly parallel proof.

5.2. **Proof of theorem 1.9** In this section, we want to prove theorem 1.9, namely that if \( \pi_1(M) \neq \{1\} \), then every non-degenerate Hamiltonian \( H \) should have at least one contractible 1 periodic orbit of index 1 (i.e. Conley-Zehnder index \( 1 - n \)).

This is not a consequence of the above construction, but uses similar ideas arranged slightly differently.

Suppose \( H \) has no index 1 orbit, and pick a generic triple \((J, f, g)\) where \( J \) is an almost complex structure compatible with \( \omega \), \( f \) a Morse function with a single minimum \( \star \) and \( g \) a Riemannian metric.

Let \( b \) be an index 1 Morse critical point, such that the unstable manifold of \( b \) defines a non trivial loop \( \gamma \) in \( M \), and let \( \gamma_- \) and \( \gamma_+ \) be the two Morse flow lines rooted at \( b \). For convenience, we consider \( \gamma \) as based at \( b \) and let :

\[
\gamma = \gamma_+ \gamma_-^{-1}.
\]

For \( x \in \hat{\mathcal{P}}_0(H) \), consider the space

\[
B(b) = \{\gamma_-, \gamma_+\} \times \mathcal{M}(\star, x).
\]

Since \( H \) has no index 1 orbit, \( B(b) \) is the set of all broken trajectories from \( b \) to \( x \).

In particular, gluing \( \gamma_\pm \) with a trajectory \( \beta \in \mathcal{M}(\star, x) \) defines a 1 dimensional family of trajectories from \( b \) to \( x \) whose other end has to be of the same form. This defines a one to one correspondence :

\[
B(b) \xrightarrow{\sigma} B(b) \quad \begin{array}{c}
(\gamma_\epsilon, \beta) \\
\mapsto \quad (\gamma_{\epsilon'}, \beta')
\end{array} \quad \text{such that} \quad (\gamma_\epsilon, \beta) \xrightarrow{\hat{\gamma}} (\gamma_{\epsilon'}, \beta').
\]

Permuting \( \gamma_- \) and \( \gamma_+ \) defines another one to one correspondence

\[
B(b) \xrightarrow{\tau} B(b) \quad \begin{array}{c}
(\gamma_{\pm}, \beta) \\
\mapsto \quad (\gamma_{\mp}, \beta)
\end{array}.
\]

Notice both \( \sigma \) and \( \tau \) reverse the orientation.

Consider now an orbit of \( \rho = \tau \circ \sigma \). It has to be cyclic, and is a sequence

\[
(\gamma_{\epsilon_1}, \beta_1), \ldots, (\gamma_{\epsilon_k}, \beta_k)
\]

such that \((\gamma_{\epsilon_1}, \beta_1) \xrightarrow{\hat{\gamma}} (\gamma_{-\epsilon_{i+1}}, \beta_{i+1})\) (with the convention that \((\gamma_{\epsilon_{k+1}}, \beta_{k+1}) = (\gamma_{\epsilon_1}, \beta_1))

To each gluing, is associated a one dimensional space, and we let \( \Sigma_i \) be its suspension. It is a diamond, endowed with an evaluation map to \( M \) that coincides with

- \( \gamma_{\epsilon_i} \) on the upper left edge,
- \( \beta_i \) on the lower left edge,
- \( \gamma_{-\epsilon_{i+1}} \) on the upper right edge,
- \( \beta_{i+1} \) on the lower right edge.
Gluing all these diamonds side by side along the lower edges, we obtain a disc, endowed with a continuous evaluation map to $M$, whose restriction to the boundary is
\[
\gamma_{\epsilon_1}^{-1} \gamma_{-\epsilon_2} \gamma_{\epsilon_2}^{-1} \cdots \gamma_{-\epsilon_k} \gamma_{\epsilon_k}^{-1} \gamma_{-\epsilon_1}.
\]
and this loop is therefore trivial.

But $\gamma_{-\epsilon_i} \gamma_{\epsilon_i}^{-1} = \gamma^{-\epsilon_i}$ so $\gamma \sum \epsilon_i = 1$. Moreover, the orientation of the couple $(\gamma_{\epsilon_i}, \beta_i)$ is constant with respect to $i$ because one moves from one to the next by two gluings, and it can be supposed to be positive without loss of generality. This means that $\epsilon_i = \epsilon(\beta_i)$ for all $i$ and hence $\sum \epsilon_i = \sum \epsilon(\beta_i)$ (where $\epsilon(\beta_i)$ is the orientation of $\beta_i$). As a consequence, we get
\[
\gamma \sum \epsilon(\beta_i) \sim 1 \text{ in } \pi_1(M, \star).
\]

Observe now that the orbits of $\rho$ induce a partition of $M(\star, x)$, so repeating this for all the orbits $O_1, \ldots, O_N$ of $\rho$, we derive
\[
\gamma \sum O_1 \epsilon(\beta_i) \cdots \gamma \sum O_N \epsilon(\beta_i) = \gamma \sum_{\beta \in M(\star, x)} \epsilon(\beta) = \gamma^{n_x} \sim 1 \text{ in } \pi_1(M, \star),
\]
where $n_x = \sharp \mathcal{M}(\star, x)$ is the algebraic number of elements in $M(\star, x)$. Recall this number is the component along $x$ of the image of $\star$ under the PSS morphism:
\[
PSS(\star) = \sum_{x \in \tilde{\mathcal{P}}_0(H)} n_x x.
\]
Because the PSS morphism is an isomorphism, the image of $\star$ has to be primitive, so that
\[
\gcd(n_x, x \in M(\star, x)) = 1.
\]

As a corollary, there are integers $\{u_x, x \in \tilde{\mathcal{P}}_0(H)\}$ such that $\sum_{x \in \tilde{\mathcal{P}}_0(H)} u_x n_x = 1$ and hence
\[
\gamma = \gamma \sum u_x n_x = 1 \text{ in } \pi_1(M, \star).
\]
This is a contradiction, since we supposed $\gamma$ was non trivial. This ends the proof of theorem 1.9.

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