Vibrations of a pair microparticles suspended in a plasma sheath

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Abstract. Interactions of two identical horizontally and vertically aligned microparticles suspended in the sheath region of a discharge plasma are studied. The role of the charge gradient and ion-focusing effect on the stability of vertical and horizontal vibrations and their coupling is investigated.

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1. Introduction

Self-organization processes in a complex ‘dusty’ plasma with the micron-sized grains are strongly influenced by a large electric charge accumulated by the dust particles (up to a few thousand or ten thousand electron charges) \cite{1,2}. The charge-induced self-organization of the plasma–dust system leads to formation of strongly coupled dust particle structures \cite{1}–\cite{5}. Stability of such complex plasma structures is now a subject of growing interest \cite{6}–\cite{11}.

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In ground-based complex plasma experiments, the dust particle formations are usually observed in the sheath region of a plasma discharge where the balance between gravitational and electrostatic forces occurs. There are a few characteristic features of such systems. First, the vertical ion flow in the (pre)sheath region redistributes the ions inside the Debye sphere leading to the formation of a region of enhanced ion density downstream of the suspended microparticles, the so-called ‘ion wake’ [3, 10], [12]–[17]. The wake effect produces an anisotropy in the electrostatic dust–dust interactions and can cause a ‘pairing’ instability at very low gas pressures [9]. Furthermore, the plasma sheath is an inhomogeneous structure and as a result the charge of the microparticles, caused by electron and ion currents on to the grain surfaces, strongly depends on the vertical particle position in the sheath [8], [18]–[20]. It is then reasonable to assume that a change in the particle charge is necessarily accompanied by a variation in the wake potential. This introduces another anisotropy in the particle interactions related to the vertical charge gradient. In this paper, we study stability and oscillations of the simplest particle configurations (two-particle formations horizontally or vertically aligned) taking into account the particle–wake interactions along with the vertical charge inhomogeneity.

Considering particle dynamics in the sheath region, we first need to model the dust–dust interactions. The usual assumption of the pure symmetric screened Coulomb (or Debye) dust–dust interaction is inadequate for a plasma with an ion flow, which produces the ion-focusing effect. We use here the reasonable analytical approximation often invoked for description of the particle–wake interaction (e.g. in [9, 21, 22]) and treat the ion wake as a point-like effective positive charge $q_n$ located at a distance $l$ beneath the particle. In this case, the electrostatic potential of the microparticle, $\phi_n$, can be presented as a combination of the screened Debye potentials of the particle charge itself and its effective wake charge, such as

$$\phi_n(r) = Q_n \left[ \frac{\exp(-|r - r_n|/\lambda_D)}{|r - r_n|} - \frac{\tilde{q}_n \exp(-|r - r_{nq}|/\lambda_D)}{|r - r_{nq}|} \right].$$

(1)

Here, the parameter $\tilde{q}_n = q_n / Q_n$ measures the efficiency of the ion focus (the wake charge), and $|r - r_n|$ and $|r - r_{nq}|$ stand for the distances from the $n$th particle and from the effective wake charge, respectively. Note that this model of the ion wake, on one hand provides a reasonable analytical approximation and, on the other hand, has the advantage of highlighting qualitative features of the physical processes involved.

Furthermore, we assume that when a particle $n$ acquires a small vertical displacement around its equilibrium position $\delta z_n$, the particle charge can be linearly approximated as $Q \simeq Q_0(1 + \epsilon \delta z_n)$ with $\epsilon = Q_0' = (dQ_0/dz)_0/Q_0$. The subscript ‘0’ means that the derivative is taken at the equilibrium level $z = 0$. It is reasonable to suggest that the vertical displacements affect the values of the wake charge, $q_i$ in the same way, i.e. $q \simeq q_0(1 + \epsilon \delta z_n)$.

2. Binary interactions of the horizontally arranged particles

Consider vibrations of two identical microparticles of mass $M$ and charge $Q$, separated horizontally by a distance $\Delta$ (figure 1(a)). The balance of forces in the horizontal direction determines the particle equilibrium due to the confining force and the binary repulsion of two charges. Assuming that there is a harmonic potential well in the $x$-direction with the
eigenfrequency $\Omega_x$ (related to the confinement potential via $U_{\text{conf, } x} = -M\Omega_x^2 q_x^2/2$), we obtain

$$\frac{2Q_x^2}{M\Delta^2}(1 + \kappa) \exp(-\kappa)(1 - \tilde{q}) = \Omega_x^2 \Delta. \quad (2)$$

Introducing the characteristic frequency $\Omega_0^2 = Q_0^2 \exp(-\kappa)/(M\Delta^3)$, we rewrite (2) as

$$2\Omega_0^2(1 + \kappa)(1 - \tilde{q}) = \Omega_x^2. \quad (3)$$

We have used here the standard notation for the normalized interparticle distance $\kappa = \Delta / \lambda_D$. The confining frequency $\Omega_x$ is specified by the horizontal electric field $E_x(x)$ and the equilibrium particle charge $Q_0$ through $\Omega_x^2 = M^{-1} Q_0 (\partial E_x/\partial x)_0$ (the subscript '0' refers to the derivative taken at the equilibrium position $x = \pm \Delta/2, z = 0$).

Introducing small horizontal, $\delta x_n$, and vertical, $\delta z_n$, displacements around the equilibrium position ($\delta x_n, \delta z_n \ll l, \Delta$, where $n = 1, 2$) and linearly expanding the interaction forces yields

$$\delta\ddot{x}_{1(2)} + 2\gamma \delta\dot{x}_{1(2)} = \mp \frac{\varepsilon \Delta}{2} \Omega_0^2 \delta z_{2(1)} - \Omega_x^2 \delta x_{1(2)} + \Omega_x^2 \delta x_{2(1)} - \delta x_{1(2)}) - \Omega_x^2 (\delta z_{1(2)} - \delta z_{2(1)}), \quad (4)$$

$$\delta\ddot{z}_{1(2)} + 2\gamma \delta\dot{z}_{1(2)} = -\Omega_x^2 \delta z_{1(2)} + \Omega_0^2 (1 + \kappa)(1 - \tilde{q})(\delta z_{1(2)} - \delta z_{2(1)}) + \Omega_x^2 \delta x_{2(1)} - \delta x_{1(2)}) - \varepsilon \Delta \Omega_0^2 \tilde{q} \Phi(\kappa)(\delta z_{1(2)} + \delta z_{2(1)}). \quad (5)$$

The friction by the ambient neutral gas is given by the Epstein friction coefficient $\gamma$ [23]. The squared frequencies are

$$\Omega_{xx}^2 = (1 - \tilde{q}) f(\kappa) \Omega_0^2, \quad (6)$$

$$\Omega_{xz}^2 = \tilde{q} \Phi(\kappa) \Omega_0^2. \quad (7)$$

with $f(\kappa) = (2 + 2\kappa + \kappa^2)$, and $\Phi(\kappa) = (3 + 3\kappa + \kappa^2)$. To simplify the bulky expressions for $\Omega_{ij}^2$, we have assumed that $\tilde{l} = l/\Delta \simeq \kappa^{-1} < 1$ (this is usually relevant for strongly coupled complex plasmas) and neglected the term $O(\tilde{l}^3)$.

**Figure 1.** Sketch of the particle configurations.
Frequency $\Omega_z$ in equation (5) describes vertical oscillations of a single particle immersed into a sheath electric field. It is specified by the vertical electric field profile $E(z)$ and the vertical gradient of the equilibrium particle charge $Q(z)$ through

$$\Omega_z^2 = -\frac{Q_0}{M} (\varepsilon |E_0| / \Delta + |E_0'|). \quad (8)$$

The value $\Omega_z^2$ can be either positive or negative depending on specific experimental conditions. To insure that the vertical vibrations of individual particles are stable, we assume from the outset that $\Omega_z^2 > 0$.

Harmonic perturbations $x_n, z_n \propto \exp(-i\omega t)$ lead to the dispersion relation

\[
[\ddot{\omega}^2 + 2i\gamma \ddot{\omega} - 2(1 + \kappa)(1 - \tilde{q})][\ddot{\omega}^2 + 2i\gamma \ddot{\omega} - \Omega_v^2 - 2\varepsilon \Delta \tilde{q}l(1 + \kappa)][\ddot{\omega}^2 + 2i\gamma \ddot{\omega} - 2\Phi (1 - \tilde{q})]
\]

\[
\times [\ddot{\omega}^2 + 2i\gamma \ddot{\omega} - \Omega_v^2 + 2(1 - \tilde{q})(1 + \kappa) - 4\tilde{q}^2 \tilde{\Phi}^2] = 0. \quad (9)
\]

Here, the wave frequency, the damping rate and the confining frequencies are normalized by $\Omega_0^2$, i.e., $\omega^2 / \Omega_0^2 \rightarrow \tilde{\omega}^2$, $\gamma / \Omega_0^2 \rightarrow \tilde{\gamma}$ and $\Omega_z^2 / \Omega_0^2 \rightarrow \Omega_v^2$.

In general, longitudinal (along the $x$-axis) and transverse (along the $z$-axis) oscillations of the pair of horizontally arranged particles cannot develop independently and are not symmetric. There exists only one possibility for decoupled vibrations—when the particles oscillate horizontally in phase (with equal amplitude, $A_{1x} = A_{2x}$) without any vertical displacements ($A_{1z} = A_{2z} = 0$). The mode frequency is then given by

$$\omega_{xx,1} = -i\gamma + \sqrt{\Omega_0^2 + 2\varepsilon \Delta \tilde{q}l(1 + \kappa)} \quad (10).$$

Invoking the equilibrium condition (3), one can rewrite this more explicitly as

$$\omega_{xx,1} = -i\gamma + \Omega_z. \quad (11)$$

To avoid fully damped oscillations, we assume here and below that the friction is weak, $(\gamma^2 \ll \Omega_0^2, \Omega_v^2)$. For the particle–wake model (1), the frequency (10) depends on the effective particle charge $\tilde{q}$. The value of $\Omega_z^2$ (and therefore that of $\omega_{xx,1}$) can be considerably decreased via the factor $(1 - \tilde{q})$ in comparison with the pure Coulomb interactions of the particle charges (the case $\tilde{q} = 0$). Obviously, the vibrational mode (11) is unaffected by the vertical dependence $Q(z)$, since the particle motion occurs only along the horizontal axis. Note that this type of vibrations are always stable.

The vertical oscillations in phase are characterized by the frequency

$$\omega_{xz,1} = -i\gamma + \sqrt{\Omega_0^2 + 2\varepsilon \Delta \tilde{q}l(1 + \kappa)}, \quad (12)$$

or, more explicitly,

$$\omega_{xz,1} = -i\gamma + \sqrt{\Omega_z^2 + 2\varepsilon \Delta \tilde{q}l(1 + \kappa)\Omega_0^2}. \quad (13)$$

Note that frequency (12), contrary to that of (11), depends on the charge variations and can reveal instability due to the combination of the charge gradient effect and the particle–wake interactions. This requires significant charge gradients, namely, $|\varepsilon \Delta| \gtrsim \Omega_z^2 / (2\tilde{q}\Omega_0^2)$. 

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However, even a weak-charge gradient \( (\varepsilon \Delta \ll 1) \) influences mode amplitudes of the in-phase vertical vibrations, leading to the asymmetry in the particle vibrations along the \( z \)-axis. In particular, the amplitude ratio becomes now dependent on the sign of the charge gradient

\[
\frac{A_{1z}}{A_{2z}} \simeq 1 - 2\varepsilon \Delta \frac{\Omega_x^2}{\Omega^2 + 2\Omega_{xx}^2 - \Omega_z^2}.
\]

(14)

Moreover, vertical oscillations are now accompanied by small, almost counter-phase, horizontal vibrations with the amplitude

\[
A_{1x} \simeq -A_{2x} \simeq \frac{\varepsilon \Delta \Omega_z^2}{2 \Omega_x^2 + 2\Omega_{xx}^2 - \Omega_z^2},
\]

(15)

which arise due to the vertical inhomogeneity in the equilibrium particle charge. It should be noted that observations of the coupled nonsymmetric vibrations can be very useful for estimating the values of the charge gradient in the sheath region.

Other two vibrational modes are described by equation

\[
[\tilde{\omega}^2 + 2i\tilde{\gamma}\tilde{\omega} - 2\Phi(1 - \bar{q})][\tilde{\omega}^2 + 2i\tilde{\gamma}\tilde{\omega} - \Omega_x^2 + 2(1 - \bar{q})(1 + \kappa)] - 4\bar{q}^2\tilde{l}^2\Phi^2 = 0,
\]

(16)

they do not reveal any dependence on the charge gradient. There is clear coupling between the components of \( \delta x_n \) and \( \delta z_n \) due to the particle–wake interactions. When dealing with a weak ion-focusing effect \( (\bar{q} \ll 1) \), the horizontal and vertical vibrations develop almost independently. Indeed, by solving dispersion relation (16) in the linear approximation in \( \bar{q} \), we result in two types of the counter-phase oscillations \( (A_{1x,z} + A_{2x,z} \neq 0) \). The horizontal vibration frequency is

\[
\tilde{\omega}_{xx,2} \simeq -i\tilde{\gamma} + \sqrt{2\Phi(\kappa)} \left(1 - \frac{\bar{q}}{2}\right),
\]

(17)

while the frequency of the vertical oscillations is

\[
\tilde{\omega}_{xz,2} \simeq -i\tilde{\gamma} + \sqrt{\Omega_x^2 - 2(1 + \kappa)} + \frac{(1 + \kappa)}{\sqrt{\Omega_x^2 - 2(1 + \kappa)}\bar{q}}.
\]

(18)

Longitudinal oscillations (17) developing with the characteristic frequency \( \omega_{xx,2} = \sqrt{\Omega_x^2 + 2f(\kappa)(1 - \bar{q}/2)\Omega_0^2} \) are always stable (see figure 2(a)). The weak particle–wake interactions do not significantly affect these particle vibrations: the squared frequency differs only by the factor \( (1 - \bar{q}) \) from that corresponding to the pure Coulomb repulsion of the particle charges [6]. Vibrations along the \( z \)-axis (18) arising with the frequency \( \omega_{xz,2} = \sqrt{\Omega_z^2 - 2(\kappa + 1)(1 - \bar{q})}\Omega_0^2 = \sqrt{\Omega_x^2 - \Omega_z^2} \) can be unstable if the horizontal confining potential prevails over the vertical one. In this case, the particle–wake interaction reduces the value \( \Omega_z^2 \propto (1 - \bar{q}) \), while the frequency \( \Omega_x^2 \) remains unaffected. This can be essential for the horizontal pairing of the grains: the particle configuration may remain stable at a given interparticle distance just due to the (even weak) wake effect (e.g., the region \( 3.5 < \kappa < 5 \) for \( \bar{q} = 0.3 \) in figure 2(b)).
Figure 2. Two types of solutions of equation (16) when $\gamma = 0$ and $\Omega_v = 3$: (a) $\omega_{xx,2}^2$, the branch analogues to pure horizontal oscillations in the absence of the wake effect; (b) $\omega_{xz,2}^2$, the branch analogues to pure vertical vibrations. The curves are plotted versus the normalized interparticle distance $\kappa = \Delta / \lambda_D$. The dashed line corresponds to $\tilde{q} = 0$, the solid lines denote the exact solutions: $\tilde{q} = 0.3$ (the bold solid line) and $\tilde{q} = 0.6$ (the thin solid line). The linear in $\tilde{q}$ approximations, (17) and (18), are indicated by dots ($\tilde{q} = 0.6$) and the dot-dashed line ($\tilde{q} = 0.3$), respectively.

The coupling between $\delta x_n$ and $\delta z_n$ becomes essential with the increasing effective wake charge, and the condition $\tilde{q} \lesssim 1$ requires exact solutions of equation (16) which are given by

$$\tilde{\omega}_{xx,2, xz, 2} \simeq -i\bar{\gamma} + \left[ f(1 - \tilde{q}) + \frac{1}{2} \Omega_v^2 \pm \frac{1}{2} \left( (2\Psi - \Omega_v^2)^2 - 4\Psi(2\Psi - \Omega_v^2)\tilde{q} + 4\tilde{q}^2(\Psi^2 + \Phi_1^2l^2) \right) \right]^{1/2},$$

where $\Psi = (1 + \kappa + \Phi)$. Figure 2 shows these exact solutions for the coupled vibrations as functions of the particle separation. For comparison, simplified solutions (17) and (18) are also presented. One can see reasonable agreement between approximations (17) and (18) and exact solutions (19) for $\tilde{q} \lesssim 0.3$, especially for the branch analogous to horizontal oscillations (figure 2(a)). For larger wake charges, $\tilde{q} > 0.3$, it is necessary to use exact solutions (19) for both types of solutions. It should be noted that the coupling between transverse (vertical) and longitudinal (horizontal) oscillations itself does not lead to an instability. Figure 3 shows behaviour of $\tilde{\omega}^2 = \omega^2 / \Omega_v^2$ defined by equation (16) as a function of the interparticle distance $\kappa$ for $\tilde{q} = 0.5$, and $\Omega_v = 3$. The vertical and horizontal oscillations, corresponding to the decoupled solutions (17) and (18), are indicated by dashed lines. As can be seen, the upper curve is rather a modification of the transverse branch of oscillations at small distances $\kappa < \kappa_0$ but is much closer to the longitudinal branch at larger distances $\kappa > \kappa_0$. For the lower curve, the situation is reversed. This type of reconnection between two curves describing different types of vibrations in the vicinity of the point, where the decoupled curves intersect, is in some sense analogous to the linear mode conversion considered for the dust–lattice modes in a plasma crystal [22].

Finally, we indicate a possibility for the vibrations to be unstable due to the strong particle–wake interactions. While the vibrations described by the ‘plus’ sign under the square root in (19) remain always stable (see figure 2(a)), the ‘minus’ sign can lead to the negative value under the square root. Therefore, due to the strong wake effect, solution (19) for $\omega_{xz,2}$ includes exponentially growing oscillations arising in the domain of small interparticle distances (e.g., if $\tilde{q} = 0.6$ the...
Figure 3. The ‘conversion’ of the vibrations in the case $\gamma = 0$, $\tilde{q} = 0.5$, and the confining frequency $\Omega_v = 3.5$. The uncoupled modes (17) and (18) are shown by the dashed lines.

Figure 4. The critical value of the effective wake charge $\tilde{q}_{cr}$ which determines the onset of instability of the vertical vibrations of the horizontally oriented configuration, as a function of the normalized interparticle distance $\kappa = \Delta/\lambda_D$. The dashed line corresponds to the confining frequency $\Omega_v = 2.8$, the bold solid line corresponds to $\Omega_v = 3$, and the thin solid line is for $\Omega_v = 3.5$.

Instability occurs for $\kappa < 1.75$—see thin solid line in figure 2(b)). One easily finds the critical value of $\tilde{q}$ for the onset of the unstable vibrations which follows from (19) as

$$\tilde{q}_{cr} = \frac{1}{4((1 + \kappa) + \tilde{l}^2\Phi)} \left[ (4(1 + \kappa) - \Omega_{v}^2) + \sqrt{(\Omega_{v}^4 - 8\tilde{l}^2\Phi(2(\kappa + 1) - \Omega_{v}^2))} \right].$$

Using various confining frequencies $\Omega_v$, the value $\tilde{q}_{cr}$ is plotted as a function of the parameter $\kappa$ in figure 4. The transition to the oscillatory regime can be externally controlled either by reducing the neutral gas pressure or decreasing the discharge power. A reduction of the pressure entails an increase of the screening length (and hence reduction of $\kappa$ for a fixed interparticle distance $\Delta$) and a decrease of the friction coefficient $\gamma$, while a decrease in the discharge power is usually accompanied by a decrease of the vertical confinement $\Omega_v$ thus furthering the instability. The predicted instability could be related to the experimentally observed strong growth of the amplitude of vertical vibrations (in comparison with horizontal oscillations) at low gas pressures ($p \sim 4–5$ Pa) and low input power reported in [24].
3. Binary interactions of the vertically aligned particles

Consider now vibrations of two particles of the same mass, \( M \), separated vertically by the distance \( d \) (along the \( z \)-axis), see figure 1(b). An important effect of the particle interactions in the vertical direction is the asymmetry of the forces between them: the potential acting on the lower particle (located at \( z_{01} \)) due to the upper grain (located at \( z_{02} \)) is given by equation (1), while the potential acting on the upper particle due to the lower grain can be approximated by a simple Debye repulsive potential \( (q = 0 \text{ in equation (1)}) \). Furthermore, the balance of forces in the vertical direction, in addition to the electrostatic interactions of the particle and wake charges includes the gravitational force \( F_g = Mg \) as well as the sheath electrostatic force \( F_E = Q(z)E(z) \). In equilibrium, we assume that the interparticle distance \( d \) is small compared with the distance between the lower particle and the electrode and compared with the width of the sheath region \( L \). This means that the sheath electric field changes slightly on the length scales \( z \sim d \) and one can safely put \( E(z) \propto z \partial E/\partial z \) where \( \partial E/\partial z \simeq \text{const} \). Moreover, it is reasonable to consider that \( |\partial E/\partial z|d/|E(z)| \sim d/L \ll 1 \). Also, we assume once again that the particle charge is a slow function of the vertical coordinate that satisfies the condition \( \varepsilon d \ll 1 \) (we therefore exclude an instability of the particle vibrations due strong charge gradients \( \varepsilon d > 1 \) considered earlier in [25]). One can then linearly approximate the product \( Q(z)E(z) \) in the range of distances near the particle equilibrium, so that \( F_E - Mg = -\gamma_z(z - d) \), where \( \gamma_z \propto \partial(Q(z)E(z))/\partial z \simeq \text{const} \), and \( z_0 \) denotes the equilibrium position due to the balance of the electrostatic \( F_E \) and the gravity \( F_g \) forces only (in figure 1(b), \( z_0 = 0 \)). The actual equilibrium positions of the vertically aligned grains \( (z_{01} \text{ and } z_{02} = z_{01} + d) \) become now dependent on the effective wake parameters, \( q \) and \( l \). Indeed, the force balance in the vertical direction on the lower and upper particles yields

\[
\gamma_z z_{01} = -Q_1 Q_2 \frac{(1 + \kappa) \exp(-\kappa)}{d^2} \left[ 1 - \bar{q} \left( 1 + \frac{f(\kappa)}{1 + \kappa} \right) \right], \tag{21}
\]

\[
\gamma_z z_{02} = Q_1 Q_2 \frac{(1 + \kappa) \exp(-\kappa)}{d^2}, \tag{22}
\]

where \( \kappa \) once again denotes the normalized interparticle distance \( \kappa = d/\lambda_D \), \( Q_1 \) and \( Q_2 \) refer to the equilibrium particle charges at the (different) equilibrium positions \( z_{01} \) and \( z_{02} = z_{01} + d \), respectively, and \( \bar{q} = q_2/Q_2 \simeq q_0/Q_0 \). In general, the value of \( z_{01} \) can be either negative or positive depending on specific \( \bar{q} \) for given \( \kappa \). Introducing the vertical confinement frequency through \( \Omega_z^2 = \gamma_z/M \) as well as \( \Omega^2 = Q_1 Q_2 \exp(-\kappa)/Md^3 \), we reduce (21) and (22) to the form, similar to (3), i.e.,

\[
\Omega^2(1 + \kappa) \left[ 2 - \bar{q} \left( 1 + \frac{f(\kappa)}{1 + \kappa} \right) \right] = \Omega_z^2. \tag{23}
\]

To investigate the particle stability, we exclude the regime corresponding to the unstable initial equilibrium configuration and thus focus on the case when \( \Omega_z^2 > 0 \).

The horizontal and vertical displacements of each particle can now develop independently. Horizontal perturbations \( \delta x_{1,2} \) are governed by

\[
\delta x_1 + 2\gamma \delta x_1 = -\Omega_{x1}^2 \delta x_1 + \Omega^2((1 + \kappa)(1 - \bar{q}) - \bar{q}f(\kappa))((\delta x_1 - \delta x_2), \tag{24}
\]

\[
\delta x_2 + 2\gamma \delta x_2 = -\Omega_{x2}^2 \delta x_2 + \Omega^2((1 + \kappa)(\delta x_2 - \delta x_1)), \tag{25}
\]
where the confining frequencies $\Omega^2_{x1,2}$ are related to the different particles charges via $\Omega^2_{x1,2} = \Omega^2_x(1 + \varepsilon z_{01,2})$ with $\Omega^2_x = Q_0 M^{-1} (\partial E_x / \partial x)_{z=0}$.

The vertical displacements obey

$$\ddot{\delta z}_1 + 2\gamma \dot{\delta z}_1 = -\Omega^2_x \delta z_1 - \Omega^2 \hat{q}(1 - \tilde{q}) (\delta z_1 - \delta z_2) + \Omega^2 \hat{q} \Psi^*(\kappa) (\delta z_1 - \delta z_2) - \varepsilon d \Omega^2 (\delta z_1 + \delta z_2)$$

$$\ddot{\delta z}_2 + 2\gamma \dot{\delta z}_2 = -\Omega^2_x \delta z_2 - \Omega^2 \hat{q}(1 - \tilde{q}) (\delta z_2 - \delta z_1) + \varepsilon d \Omega^2 (1 + \kappa) (\delta z_1 + \delta z_2),$$

with $\Psi^*(\kappa) = 6 + 6 + 3\kappa^2 + \kappa^3$. It should be noted that the term $\Omega^2 \propto Q_1 Q_2$ and thus implicitly contains the particle charge gradient.

We start the stability analysis with horizontal displacements of two vertically aligned microparticles and consider some limiting cases:

(i) $\varepsilon = 0, q \neq 0$. The zero charge gradient in (24) and (25) implies only modifications due to the asymmetry in the particle–wake interactions. For the two existing oscillatory modes, the frequency of the first one is not affected by the particle–wake interactions; it coincides with expression (11) and corresponds to stable in-phase horizontal oscillations with equal amplitudes ($A_{1x} = A_{2x}$). The frequency of the second vibrational mode is given by

$$\omega \simeq -i\gamma + \left[ \Omega^2_x - \Omega^2_z \left( 1 - \frac{\tilde{q}(1 + \kappa)}{2\kappa(1 + \kappa)(1 - \tilde{q}) - \tilde{q}(\kappa + 2)} \right) \right]^{1/2},$$

(here, the equilibrium condition (23) is invoked). Stability of these vibrations depends on the ratio $\Omega^2_x / \Omega^2_z$. Unstable regime appears due to weak horizontal confinement, which results in the negative term under the square root. It is important that condition for this instability, which in the linear order in $\tilde{q}$ takes the form $\Omega^2_x < \Omega^2_z (1 - \tilde{q}/2\kappa)$, is somewhat opposite to the instability condition for the vertical vibrations of two horizontally aligned particles given by equation (18). However, for the vertically arranged pair, just as in the case of the horizontal configuration, the particle–wake interactions play a stabilizing role by decreasing the instability domain.

The amplitudes of the particle vibrations (28) are now related by

$$\frac{A_{1x}}{A_{2x}} = -1 + \tilde{q} \left( 1 + \frac{\Phi(\kappa)}{1 + \kappa} \right).$$

One finds the counter-phase oscillations with unequal amplitudes, when the term $\propto \tilde{q}$ is small (for the pure symmetric interaction potential, $\tilde{q} = 0$, the transition to the counter-phase oscillations $A_{1x} = -A_{2x}$ occurs, which corroborates earlier results [7]). However, when the wake effect is significant, the vibrations for certain interparticle distances $\kappa$ can become in phase, with unequal amplitudes. The latter is illustrated in figure 5: the transition form the counter-phase to the in-phase oscillations takes place for $\tilde{q} \sim 0.4$ and $\kappa \sim 4$.

This new type of vibrations (28) provides a good diagnostic tool to estimate the effective wake parameters (e.g., by measuring the amplitudes $A_{1x}$ and $A_{2x}$ and comparing them with ratio (29) at different gas pressures).
(ii) $\varepsilon \neq 0$, $\tilde{q} = 0$. It turns out that the charge gradient effect itself leads to quadratic corrections to the both mode frequencies. In particular, the analogue of the in-phase horizontal vibrations now becomes

$$\omega_{zx,1} = -i\gamma + \Omega_x\left(1 + \frac{\varepsilon^2 d^2}{4} \frac{\Omega_x^2}{\Omega_z^2}\right)^{1/2},$$

with the amplitude ratio given by

$$\frac{A_{1x}}{A_{2x}} \simeq \left(1 - \varepsilon d \frac{\Omega_x^2}{\Omega_z^2}\right)^{-1}. \quad (31)$$

Similarly, the counter-phase vibrations develop with the frequency

$$\omega_{zx,2} = -i\gamma + \left[\Omega_x^2 \left(1 - \frac{\varepsilon^2 d^2}{4} \frac{\Omega_x^2}{\Omega_z^2}\right) - \Omega_z^2\right]^{1/2},$$

and reveal the amplitude ratio

$$\frac{A_{1x}}{A_{2x}} \simeq -1 - \varepsilon d \frac{\Omega_x^2}{\Omega_z^2}.$$

Although small charge gradient does not change significantly the mode frequencies of both types of oscillations, it influences the character of the grain vibrations: e.g., the term $|\varepsilon d \Omega_x^2 / \Omega_z^2| \gtrsim 1$ can turn in-phase vibrations (31) into counter-phase ones, and vice versa, counter-phase vibrations (33) can be transformed into in-phase oscillations. Note that useful information about the charge gradient can be obtained by observing this type of transition. In particular, one can expect that a decrease in the input power reducing the value $\Omega_z$ (while the horizontal confinement $\Omega_x$ does not change significantly) could stimulate the transition from one polarization to another and thus reveal the role of the charge gradient effect.

(iii) $\varepsilon \neq 0$, $q \neq 0$. The combined effect of the charge gradient and the ion wake modifies both vibrational modes. Solutions for the mode frequencies corresponding to set (24) and (25) are
Figure 6. The squared mode frequency $\omega_{zx,1}^2/\Omega_z^2$ (determined by equation (34) for $\gamma = 0$) and corresponding to the in-phase horizontal vibrations of the vertically arranged particles is plotted versus the normalized interparticle distance $\kappa = \Delta/\lambda_D$. The dashed line corresponds to $\tilde{q}$, $\varepsilon = 0$, the solid lines stand for the exact solutions when $\varepsilon d = 0.2$: the bold solid line denotes the case $\tilde{q} = 0.3$ and the thin solid line is for $\tilde{q} = 0.5$. The dot-dashed line indicates the case $\tilde{q} = 0.3$, $\varepsilon d = −0.1$.

now determined by bulky expressions even in the linear approximation in $\varepsilon d$. In particular, for in-phase horizontal vibrations of the vertically arranged pair, we have

$$\omega_{zx,1} \simeq -i\gamma + \Omega_z \left[ 1 - \frac{(1 + \kappa)^2\kappa}{(\bar{q}(3\kappa + 2\kappa^2 + 2) - 2\kappa(1 + \kappa))(\bar{q}(2\kappa^2 + 4\kappa + 3) - 2\kappa(1 + \kappa))} \bar{q}\varepsilon d \right]^{1/2}. $$

(34)

As can be seen, modification of the mode frequency is determined by the product of $\tilde{q}$ and $\varepsilon d$. Depending on the sign of the charge gradient, the term in brackets either increases the effective confining frequency ($\varepsilon < 0$) or decreases $\Omega_z^2$ if $\varepsilon > 0$, as indicated in figure 6. The amplitude ratio of this mode in the linear approximation implies an asymmetry due to the contribution of the term $\propto \tilde{q}\varepsilon d$, namely,

$$\frac{A_{1x}}{A_{2x}} \simeq 1 - \frac{\Omega_z^2}{\Omega_z^2} \left( 1 + \frac{\tilde{q}}{2\kappa} \right) \varepsilon d. $$

(35)

We note that, similar to the case (ii), these in-phase vibrations can turn in the counter-phase ones when $\Omega_z^2(1 + q/2\kappa)e d/\Omega_z^2 > 1$.

Influence of the combined effect of the charge gradient and the ion wake can also be significant for the counter-phase vibrational mode as well. Frequency of these oscillations $\omega_{zx,2}$ obeys equation (28) where $\Omega_z^2$ has to be replaced by

$$\Omega_z^2 \left[ 1 + \frac{\kappa(1 + \kappa)(4\kappa^2 + 7\kappa + 5) - \bar{q}(2\kappa^2 + 4\kappa + 3)(3\kappa + 2\kappa^2 + 2)}{(\bar{q}(3\kappa + 2\kappa^2 + 2) - 2\kappa(1 + \kappa))(\bar{q}(2\kappa^2 + 4\kappa + 3) - 2\kappa(1 + \kappa))} \bar{q}\varepsilon d \right]. $$

(36)

Contrary to equation (34), the term containing the charge gradient $\tilde{q}\varepsilon d$ now works in the opposite direction: the case $\varepsilon > 0$ increases the effective confining frequency and thus plays a stabilizing role, while the case $\varepsilon < 0$ decreases the effective value of $\Omega_z^2$ thus destabilizing vibrations. Note that variations in the mode frequency described by (36) are a few times larger than modifications
for the in-phase mode shown in figure 6 and thus can be significant even for small gradient and wake-charge values. The mode amplitudes are not equal in the magnitude and reveal a strong asymmetry

$$\frac{A_{1x}}{A_{2x}} \simeq -1 + \tilde{q} \left( 1 + \frac{\Phi(\kappa)}{1 + \kappa} \right) - \frac{\Omega_z^2}{\Omega_z^2} \epsilon d + \frac{\Omega_z^2}{\Omega_z^2} \left( 4 + \frac{9}{2\kappa} - \frac{2}{1 + \kappa} \right) \tilde{q} \epsilon d. \quad (37)$$

Therefore, one can again expect the possibility of transition between the counter-phase and in-phase oscillations by varying complex plasma parameters. Experimentally, identification of this phenomenon would allow us to estimate the role of the ion wake effect or evaluate the charge gradient.

Finally, consider vertical vibrations of two vertically aligned particles, described by equations (26) and (27). When the particle interactions are symmetric ($\tilde{q} = 0$ and $\epsilon = 0$), one immediately finds two modes (with equal amplitudes): the in-phase oscillations $\omega_{zz,1} = -i\gamma + \Omega_z$ and the counter-phase vibrations ($A_{1z} = -A_{2z}$) with $\omega_{zz,2} = -i\gamma + \sqrt{\Omega_z^2 + 2f(\kappa)\Omega_0^2}$. To explore the combined effect of the particle–wake interactions and the charge gradient in the asymmetric case, we start with the linear (in $\epsilon d$) term of the in-phase mode

$$\omega_{zz,1}(\kappa) \simeq -i\gamma + \left[ \frac{\Omega_z^2 + 2\tilde{q} \epsilon d}{2f(\kappa) - (5\kappa^2 + 2\kappa^3 + 8\kappa + 6}\tilde{q}} \right]^{1/2}. \quad (38)$$

Once again, the effect on this mode frequency is determined by the product $\tilde{q} \epsilon d$. However, the asymmetry in the amplitude ratio is mainly due the particle charge gradient, namely,

$$\frac{A_{1z}}{A_{2z}} \simeq 1 - 2 \frac{1 + \kappa}{f(\kappa)} \epsilon d. \quad (39)$$

As can be seen, the small gradient corrections could hardly change polarization of these vibrations.

In figure 7, we plot dependencies of the normalized squared frequency $\omega_{zz,1}^2 / \Omega_0^2$ following from (38) when $\gamma = 0$ versus $\kappa$ for the typical characteristic frequency $\Omega_z^2 / \Omega_0^2 = 2$. We see that both factors which introduce asymmetry in the vertical direction (the ion focusing and the charge gradient) can significantly affect the mode frequency (38), but the vibrations remain always stable. It turns out that (38) is a good approximation within the typical range $\kappa \sim 2–5$. The obtained linear solution fails in the vicinity of the points where the denominator of the second term under the square root goes to zero thus artificially increasing the role of the gradient term in (38). For comparison, we also give an exact solution for $\omega_{zz,1}^2 / \Omega_0^2$ in the special case where the linear approximation (38) fails (see figure 7(b)).

Next, we turn to the counter-phase vertical oscillations. In contrast to the in-phase vibrations (for which $\omega_{zz,1}$ is specified by both terms $\tilde{q}$ and $\epsilon d$ to the same extent), the mode frequency for the counter-phase vibrations

$$\omega_{zz,2}(\kappa) \simeq -i\gamma + \left[ \frac{\Omega_0^2(2f(\kappa) - (2\kappa^3 + 8\kappa + 5\kappa^2 + 6)\tilde{q} + (2\kappa^2 + 3\kappa + 4)\tilde{q} \epsilon d) + \Omega_z^2}{\Omega_z^2} \right]^{1/2}, \quad (40)$$

can be influenced by the particle–wake interactions even when $\epsilon = 0$, while the charge gradient term enters only in the combination $\tilde{q} \epsilon d$. Influence of various effective wake charges and dust-charge gradients on this vibrational mode is shown in figure 8. We note that these mode amplitudes
Figure 7. The squared mode frequency $\omega_{zz,1}^2/\Omega_1^2$ for the in-phase vertical vibrations of the vertically aligned pair in the case $\gamma = 0$ as a function of the interparticle distance for the typical $\Omega_2^2/\Omega_0^2 = 2$. (a) The dashed line corresponds to $\tilde{q}$, $\varepsilon = 0$, the solid lines denote the exact solutions: $\tilde{q} = 0.3$, $\varepsilon \delta = 0.2$ (the bold solid line), $\tilde{q} = 0.4$, $\varepsilon \delta = -0.3$ (the thin solid line), and $\tilde{q} = 0.4$, $\varepsilon \delta = 0.2$ (the dot-dashed line). (b) Comparison of the linear approximation given by (38) (the thin solid line) and the exact solution (the bold solid line) for $\tilde{q} = 0.5$, $\varepsilon \delta = -0.3$.

Figure 8. The squared mode frequency $\omega_{zz,2}^2/\Omega_1^2$ corresponding to the counter-phase vertical vibrations of the vertically aligned pair in the case $\gamma = 0$ plotted versus the normalized interparticle distance $\kappa = \Delta/\lambda_D$. The dashed line corresponds to $\tilde{q}$, $\varepsilon = 0$, the solid lines denote the exact solutions: $\tilde{q} = 0.4$ and $\varepsilon \delta = \pm 0.3$ (the thin solid lines), $\varepsilon \delta = 0$ (the bold solid line).

are not equal in magnitude and are related by

$$\frac{A_{zz}}{A_{zz}} \simeq -1 + 2\tilde{q} \left[ 1 + \frac{3}{2\kappa} - \frac{\kappa + 1}{f(\kappa)} + \frac{1}{2} \varepsilon \delta \left( \frac{1}{2\kappa} - \frac{1}{(1 + \kappa)} + \frac{1}{2} \frac{\kappa}{f(\kappa)} \right) \right]. \quad (41)$$

One thus finds that the particle–wake interactions can mainly result in transition from the counter-phase vibrational mode into the in-phase oscillations. Finally, we note that within the presented model of the particle–wake interactions and small charge gradients ($\varepsilon \delta \ll 1$), vertical vibrations of a pair of vertically arranged grains remain stable. Adding a third grain in the vertical structure will probably lead to the onset of the instability of the vertical displacements due to the asymmetry in the particle–wake interactions akin to the dipole instability predicted for a vertical particle chain [26].

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4. Discussion and conclusion

Here, we studied stability and vibrations of two-particle configurations trapped in the plasma sheath which is typical for complex plasma experiments in a gas discharge. In particular, influence of the ion-focusing effect and the spatial equilibrium charge gradient, mostly pronounced in the vertical direction, have been explored. We have demonstrated that inclusion of these anisotropic factors can crucially modify vibrations of the pair configurations. The main conclusion is that even small effective wake charges and charge gradients can affect the polarization properties of the vibrational modes and their stability.

We have investigated the vertical and horizontal particle arrangements relevant for current experiments. In both particle configurations, the particle equilibrium conditions involve dependencies on the wake parameters (see equations (3), (21) and (22)). Furthermore, in the case of horizontally arranged grains, we found that: (i) the longitudinal and transverse oscillations are coupled due to the particle–wake interaction and the charge inhomogeneity. Note that the earlier considered model of the ion wake [6, 7] (applicable only on the line behind the particles) does not lead to this effect. Experimental observations of the mode coupling can provide a verification of the considered ion-focusing model; (ii) observations of the predicted asymmetry in in-phase vertical oscillations (14) and counter-phase horizontal vibrations (15) give a useful tool for the sheath diagnostics (e.g., for estimating the charge gradient effect); (iii) vertical oscillations are predicted to be unstable for a significant charge gradient (the in-phase vibrational mode), while the counter-phase oscillations can be unstable due to the strong ion-focusing effect (figure 4). The predicted instability could be responsible for the experimentally observed self-excited vertical vibrations observed at low gas pressures and small input power [24]. The transition to the oscillatory regime of the vertical vibrations can be externally controlled by changing the discharge parameters, thus allowing us to estimate the role of the ion-focusing effect in the stability of horizontally arrangements.

In contrast to the horizontal configuration, the vertical particle alignment does not reveal any mode coupling, and the longitudinal and transverse displacements develop independently. Although the combination of the ion wake and the charge gradient modifies the eigenfrequencies of the particle vibrations significantly (figures 6–8), nevertheless, the oscillations remain stable within the considered model of the weak-charge gradient. The main effect lies in the asymmetry of the amplitudes of particle oscillations. It turns out that a weak effective wake charge can result in strong asymmetry of the amplitudes of grain oscillations and even cause the transition form one polarization to another. Figure 5 illustrates such a transition from the counter-phase to the in-phase mode; it requires relatively small value of the effective wake charge \( \sim 0.4 \). Note that the earlier considered model of the ion wake [6, 7] does not permit such mode transition. At the same time, experimental observations of the mode transition, on one hand, can provide a good diagnostic tool to estimate the effective wake charge (measuring amplitudes of the oscillations and comparing them with the predicted ratio (29)) and, on the other hand, can be a verification of the presented wake model.

The obtained results on the stability of the pair grain structures inside the sheath region can be useful for interpretation of experiments on the particle pairing in the plasma sheath, particle interactions in cluster structures (where the ion-focusing effect could also be of importance) [27, 28], as well as for diagnostics of complex plasmas and setting up future laboratory experiments. Note that in considering stability of the particle structures we have used several simplifications, which could be of importance when compare the obtained theoretical
results with experiments. First of all, we employed the model of the particle–wake interactions, in which the charged particle and the distributed ion-density enhancement beneath the grain is treated as two point-like charges separated by some characteristic distance. In the considered approximation, this reasonable assumption catches the essential topology and physics quite good. Secondly, for the vertical particle arrangement, we considered strongly asymmetric system, introducing the effective wake charge only for the upper grain. The degree of the anisotropy in the particle–wake interactions is a big issue demanding further research. Another difficulty concerns modelling complex plasma parameters in the sheath region; here, we introduced reasonable assumptions that the sheath electric field slowly depends on the vertical coordinate and the equilibrium particle charge profile remains a smooth function within the levitation heights thus referring our results to the case of weak gradients of the complex plasma parameters. Future experiments can elucidate the role of the above assumptions under various conditions.

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