Sine wave solution
for plane waves in loop quantum gravity

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Abstract

This paper constructs an approximate sinusoidal wave packet solution to the equations of loop quantum gravity (LQG). The equations are solved in a semiclassical, small sine approximation. Eigenvalues of the volume operator are assumed to be large enough that the [volume, holonomy] commutators may be replaced by their quantum field theory limits; SU(2) holonomies are expanded in sines and cosines, sines are assumed small, and terms up to quadratic in sines are kept. The wave is unidirectional and linearly polarized. The states are coherent states tailored to the symmetry of the plane wave case. Fixing the spatial diffeomorphisms is equivalent to fixing the spatial interval between vertices of the loop quantum gravity lattice. In the classical limit, this spacing can be chosen such that the eigenvalues of the triad operators are large, as required by a semiclassical treatment. Exact continuity of variables at boundaries is not reasonable in LQG, a fundamentally discrete theory. I propose equating averages taken over vertices on opposite sides of the boundary.

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I Introduction

This paper is a continuation of a previous paper which quantized plane gravitational waves using semiclassical LQG techniques [1]. The waves are unidirectional and singly polarized. The semiclassical approximation retains terms up to order $(\sin \theta)^2$ in the Hamiltonian (sine the sine of a holonomy). Also, The fractional change in a dynamical quantity $f$, from one vertex to the next, is assumed small: $\delta f/f \ll 1$.

Section II reviews the classical background. Section III constructs a set of triads which are sinusoidal and obey the constraints approximately. Section IV sketches the construction of coherent states, to be used as basis for a Hilbert space. These states depend upon a number of angle and angular momentum parameters. Section V determines parameter values such that the expectation values of the triads reproduce the constraint solution constructed in section II.

Section II describes some known, exact solutions to the equations of classical general relativity (CGR). These solutions in themselves were not terribly helpful in constructing the sinusoidal solution presented in section III. However, the classical work motivates a choice of diffeomorphism gauge. When combined with the unidirectional assumption, the gauge greatly simplifies the scalar constraint and allows this constraint to be solved. Also, because of the simplicity of the scalar constraint, the classical authors can postulate a desired curvature and simultaneously solve the (vector and scalar) constraints; then work backwards from the curvature to compute the physical degrees of freedom.

$$\text{curvature} \rightarrow \text{constraints} \rightarrow \text{physical}.$$  

This is perhaps a reversal of the usual logic (postulate desired physical degrees of freedom; solve the constraints for the non-physical degrees of freedom; compute curvature). I have copied the classical approach when constructing the sinusoidal solution.

The sections on coherent states involve several angular parameters; I have repeated their definitions at several points in the discussion, perhaps more times than needed. However, the basic structure of these states should not be surprising to anyone familiar with the standard example, coherent states for the free particle.

If there are any surprises, they come in section III. In that section, fixing the diffeomorphism gauge fixes the distance $\Delta Z$ between
vertices, where $\Delta Z$ is measured in the local free-fall frame. That section also suggests a prescription for imposing boundary conditions, in a theory where jump discontinuities from one vertex to the next are the rule rather than the exception.

II Classical solutions

Classical solutions for both unidirectional and colliding plane waves are reviewed in the monograph by Griffiths [2]. The unidirectional waves discussed here were discovered by Brinkmann [3], rediscovered by Peres [4].

Classical treatments typically use a conformal gauge

$$ds^2 = +2 du dv g_{uv} + a^2 dx^2 + c^2 dy^2$$

with basis vectors

$$\hat{\omega}^u = g_{uv};$$
$$\hat{\omega}^v = dv;$$
$$\hat{\omega}^x = a dx;$$
$$\hat{\omega}^y = c dy;$$

For a single polarization, unidirectional wave (all metric components depend only on $u$), the non-zero components of the curvature tensor are

$$R^x_{\ u} = \left[ \ddot{a}/a + (\dot{a}/a)(\dddot{g}_{uv}/g_{uv}) \right] \times \hat{\omega}^u \wedge \hat{\omega}^x/(g_{uv})^2;$$
$$R^y_{\ u} = \left[ \ddot{c}/c + (\dot{c}/c)(\dddot{g}_{uv}/g_{uv}) \right] \times \hat{\omega}^u \wedge \hat{\omega}^y/(g_{uv})^2.$$  

Dots denote derivatives with respect to $u$. Therefore the non-zero components of the Weyl and Einstein tensors are

$$C^x_{\ uuu} = -C^y_{\ uuu};$$
$$G_{uu} = -\left[ \ddot{c}/c - (\dot{c}/c)(\dddot{g}_{uv}/g_{uv}) \right]/4(g_{uv})^2 + (c\leftrightarrow a).$$

At eq. (1) I have defined the variable $u$ to make conversions from $u$ to $z$ easier. An increase in $u$ corresponds to an increase in $z$.

$$du = (dz - cdt)/\sqrt{2}.$$
From eqs. (3) and (4), classical treatments favor the gauge
\[ g_{uv} = g_{zz} = 1; \]
\[ e_z^z = sgn(z) = \pm 1, \] (6)
where sgn(z) is the sign of \( e_z^z \) and \( E^z \). In this gauge the equations for the metric components and \( \tilde{E} \) simplify.

\[ E_X^x = | e | / e_x^X = sgn sgn(z) e_y^Y = c; \]
\[ \tilde{E}_X^x / E_X^x = \tilde{c} / c. \] (7)
sgn is the sign of the 3x3 triad determinant e. Similarly,
\[ E_Y^y = sgn sgn(z) e_x^X = a; \]
\[ \tilde{E}_Y^y / E_Y^y = \tilde{a} / a. \] (8)

In terms of triads, the Weyl and Einstein tensors become

\[ 0 = \tilde{E}_X^x / E_X^x + \tilde{E}_Y^y / E_Y^y; \]
\[ C_{uxu}^{x} = \tilde{E}_X^x / E_X^x - \tilde{E}_Y^y / E_Y^y. \] (9)

Einstein implies \( \tilde{c} / c = -\tilde{a} / a \). Weyl is simply \( 2 \tilde{c} / c \).

One can obtain a classical solution by making a choice for a and c obeying \( \tilde{c} / c = -\tilde{a} / a \). Setting the \( \tilde{E} / E \) equal to delta functions yields an impulsive curvature.

\[ E_X^x = sgn sgn(z)(1 + c_i u \Theta(-u)); \]
\[ E_Y^y = sgn sgn(z)(1 - c_i u \Theta(-u)); \]
\[ C_{uxu}^{x} = -c_i \delta(u)/4. \] (10)
c_i (i for impulse) is a constant. Theta is the Heaviside step function.

Setting the \( \tilde{E} / E \) equal to step functions produces a step wave.

\[ E_X^x = sgn sgn(z) \cos(c_s u) \Theta(-u); \quad (u < 0) \]
\[ E_Y^y = sgn sgn(z) \cosh(c_s u) \Theta(-u); \quad (u < 0) \]
\[ E_X^x = E_Y^y = sgn sgn(z); \quad (u > 0) \]
\[ C_{uxu}^{x} = -\Theta(-u)(c_s)^2 / 4. \] (11)
c_s is a constant.
Extending these solutions to the quantum domain is not straightforward. One problem is more apparent than real. Both solutions have \( \tilde{E} \) which diverge for large negative \( u \) (large negative \( z \)). These divergences are a consequence of the planar symmetry, which prevents gravitational flux lines from spreading. Even the simplest example from Newtonian gravity shows such a divergence. The static gravitational potential due to a planar sheet of matter increases linearly with \( z \). In CGR the \( \tilde{E} \) become the potentials, therefore may be expected to diverge. Fortunately, geodesics and curvature are more physical. They depend on ratios (derivative of \( \tilde{E} \))/\( \tilde{E} \), which are finite.

A possibly more serious problem with these \( \tilde{E} \) is that they are not big enough. The classical limit assumes large \( \tilde{E} \), but the \( \tilde{E} \) in eqs. (10) and (11) contain no large parameter. The classical limit requires slow variation (\( f/f \) small), therefore the parameters \( c_i \) and \( c_s \) must be small. Also, the solution must obey matching conditions at \( u = 0 \), where the solution connects to flat space. The classical matching conditions across characteristics are

\[
g_{\mu \nu}, \quad g^{ui} g_{ij}, \quad g^{ij} \partial_u g_{ij} \quad \text{continuous}
\]

in a gauge where \( g_{uu} = 0 \) and \( i,j = x,y,v \). These boundary conditions require the \( \tilde{E} \) to reduce to unity at \( u = 0 \) (where the triads match onto flat space) and forbid multiplying the \( \tilde{E} \) by large constants.

Indeed, both solutions have the ultimate ”too small” problem: at least one \( \tilde{E} \) has a zero for large enough \( |u| \). The problems with too small \( \tilde{E} \) disappear when we go to the quantum case in section III.

III A LQG solution

A Choice of gauge

The quantum \( \tilde{E} \) operators differ from their classical analogs.

\[
\frac{2}{\kappa \gamma} E_i^j (\text{classical}) \rightarrow \frac{2}{\kappa \gamma} E_i^j (\text{quantum}) \rightarrow (2/\kappa \gamma) \Delta x^j \wedge \Delta x^k E_i^j (\text{classical}).
\]

The quantum operator is multiplied by an area (in Planck lengths squared, because of the \( \kappa \gamma \) factor). Suppose the \( \Delta x^i \) are taken to be \( 10^2 \) Planck lengths (an extremely tiny length, by classical
standards). The classical triad may be order unity; yet the quantum
eigenvalue will be order $10^4$. Therefore, the typical angular momenta
in the wavefunction can be order $10^4$, far from order unity, even
though classical values are order unity. This fact goes a long way
toward solving the problem of ”no large parameter” discussed in section II.

Since the quantum operators contain the classical fields, the re-
sults of a classical gauge fixing are still relevant. I follow classical
practice, eq. (6), and choose a gauge implying $g_{uv} = g_{zz} = 1$. In the
notation of paper I, this gauge has parameter $p = 1/2$.

$$\tilde{E} (cl) = (C_{cl} E^z_Z)^{p+1/2} = C_{cl} E^z_Z; \text{ or}$$

$$(e^Z_z)^2 = \text{sgn} C_{cl}. \quad \text{(14)}$$

On the last line I have expanded the $\tilde{E}$ in triads. Since $e^Z_z$ must
match to flat space at the front of the packet,

$$C_{cl} = \text{sgn};$$

$$e^Z_z = \pm 1. \quad \text{(15)}$$

In loop quantum gravity, the spatial diffeomorphism gauge must
be chosen such that, when factors of $\Delta x^i$ are stripped out, one re-
covers the classical gauge fixing.

$$\tilde{E} (qu) = C_{qu} E^z_Z(qu);$$

$$C_{qu} = (\Delta Z)^2 C_{cl} \quad \text{(16)}$$

Each $E^z$ in $\tilde{E} (qu)$ will have an area factor $\Delta y \Delta z$; each $E^y$ in $\tilde{E}$
will have a factor $\Delta x \Delta z$. The factors of $\Delta x, \Delta y$ are also present in
$E^z$, but not the $\Delta z$. Therefore the missing $\Delta z$ factors turn up in
$C_{qu}$. $C_{cl}$ is still the classical value, sgn. When we pick $C_{qu}$, we are
picking a value for $\Delta z$ in Planck units.

More precisely, as indicated in eq. (16), we are picking $\Delta Z$. (Caps
$\Delta X^I$ indicate Local Lorentz coordinates; lower case $\Delta x^i$ indicate
coordinates on the global manifold.) The quantum $\tilde{E}$ are related to
Lorentz frame areas:

$$\tilde{E}^i_I (qu) = \text{sgn} (e/e^I_I)(cl) \Delta x^j \Delta x^k$$

$$= \text{sgn} e^I_j e^K_k \Delta x^j \Delta x^k$$

$$= \text{sgn} \Delta X^I \Delta X^K. \quad \text{(17)}$$
If these results are inserted into the first line of eq. (16), then the second line follows.

This calculation differs from most LQG calculations in that it starts from a classical result, and the classical result must be used to determine the quantum operator. This means that quantities $\Delta x^i$, $\Delta X^i$ occur in intermediate steps. Two types of area occur, and they are connected by

$$\tilde{E}^i_{\text{(qu)}} = \tilde{E}^i_{\text{(cl)}} \Delta x^j \Delta x^k$$

$$= \text{sgn } \Delta X^J \Delta X^K.$$ (18)

$\tilde{E}$ (cl) will have $n$ dependence; does this mean that the $\Delta x^j \Delta x^k$ vary, or the $\Delta X^J \Delta X^K$, or both? I assume the arbitrary labels $x$ are held fixed; the variation is in the Lorentz lengths $\Delta X$. Equivalently, I assume classical and quantum $\tilde{E}$ have the same variation with $n$, since $\tilde{E}$ (cl) and $\tilde{E}$ (qu) differ only by factors of $\Delta x^j$, which are held fixed.

Support for this assumption comes from a later result in the sections on coherent state parameters. The coherent states are approximate eigenstates of the quantum fields in eq. (17), with eigenvalues equal to an angular momentum, or $Z$ coordinate of angular momentum.

$$\tilde{E}^a_A_{\text{(qu)}} | \text{coh} \rangle = (\kappa \gamma /2) L^a_A | \text{coh} \rangle; \quad a = x, y;$$

$$\tilde{E}^z_Z_{\text{(qu)}} | \text{coh} \rangle = (\kappa \gamma) m_z | \text{coh} \rangle.$$ (19)

When the first line of eq. (18) acts on a coherent state, one gets

$$\kappa \gamma (L^a_a /2 \text{ or } m_z) = \text{sgn } \Delta X^J \Delta X^K.$$ (20)

If the Lorentz lengths $\Delta X^i$ are taken as fixed, then the canonical coordinates cannot vary in the presence of a gravitational wave, a reduction to the absurd. Also, this is a gauge fixed theory; the $\Delta x^i$ are not allowed to vary.

The quantum curvature will be linear in a small amplitude $a$; and the variations in the $\Delta X^i$ will turn out to be linear or higher in $a$. I.e. the variations are a curvature effect. The equivalence principle is not violated. If the Local Lorentz frame is visualized as an elevator in free fall, then the elevator must be large enough that an observer can detect curvature.
Because of diffeomorphism invariance, the classical triad \( e^X_x \) and the corresponding \( dx \) in the following relation scale in opposite senses.
\[
d X = e^X_x \, dx.
\]
Scaling also occurs in the present context, although not because of diffeomorphism invariance. Since an area is proportional to a canonical \( L \) or \( m \), the area and the canonical coordinate scale together.

In the present gauge, eq. (15), \( \Delta Z = \pm \Delta z \). Therefore only the \( \Delta X, \Delta Y \) vary; \( \Delta Z \) is a constant. Note \( \Delta Z \) may be expressed in terms of coherent state parameters, by using the gauge choice for \( z \) diffeomorphisms, eq. (16).

The \( x, y \) diffeomorphisms (and \( X, Y \) Gauss rotations) are fixed by setting canonical pairs equal to zero, which puts no constraint on \( \Delta x, \Delta y \). Nevertheless we will be able to express those quantities in terms of coherent state parameters, just as for \( \Delta z \). For the moment I leave \( \Delta x, \Delta y \) undetermined but respect the symmetry by setting \( \Delta x = \Delta y \). Relaxing this condition forces \( x \) and \( y \) angular momenta to have different magnitudes, which seems a pointless complication. Of course the upside to the \( \Delta x^i \) dependence is the ability to scale the angular momenta to large values.

When quantizing plane waves in QFT, using ADM variables, one often renormalizes constraints by dividing out a factor \( \Delta x \Delta y \). The QFT expressions then contain only integrals over \( z \) and integrations over \( dz \). Such a renormalization is not possible in LQG, because not every term contains an overall factor of \( \Delta x \Delta y \). Some \( \Delta x, \Delta y \) are hidden in holonomies and do not cancel out.

In QFT the integrals over transverse directions are infinite, and a renormalization is mandatory. In LQG the transverse integrals are finite. They range over the circumferences of the \( x \) and \( y \) circles (over the finite area \( \Delta x \Delta y \)).

### B  Triad zeros

The gauge choice eq. (15) forbids zeros of \( e^Z_z \); but \( ^{(2)}E \) and \( E^z \) could conceivably pass through zero simultaneously, since both contain one power of \( ^{(2)}e \), the determinant of the transverse \( e^A_a \).

As for \( N \) and \( N = e^T_T \), they cannot vanish. In most contexts, gauge consistency equations fix \( N \), and one chooses \( N > 0 \) so that
dt and dT "run" in the same direction.

\[ \begin{align*}
\text{d}T &= e_i^T \text{d}t = N \text{d}t; \\
e_i^T &= 0; \ i = xyz. \quad (21)
\end{align*} \]

The second line is the usual gauge choice which reduces full Lorentz symmetry to SU(2).

For plane waves, the consistency conditions associated with diffeomorphism gauge fixing determine the quantity \( \overline{N} \), rather than \( N \); however, an appropriate choice for \( N \) allows \( N \) to be fixed at unity.

\[ \overline{N} \text{ (cl)} := N(\mid e \mid / E_z)_{\text{cl}} = N/e_z^* = N \text{sgn}(z). \quad (22) \]

If \( \overline{N} \) is now chosen appropriately, \( N \) becomes unity:

\[ \overline{N} \text{ (cl)} = \text{sgn}(z). \quad (23) \]

Neither \( \overline{N} \) nor \( N \) can vanish.

The quantum \( \overline{N} \) is constructed from the definition eq. \( (24) \) except use quantum \( \tilde{E} \).

\[ \overline{N} \text{ (qu)} = \overline{N} \text{ (cl)}/\Delta Z. \quad (25) \]

\( \overline{N} \) is a contravariant rank one tensor, therefore needs a \( 1/\Delta Z \) to make it diffeomorphism invariant.

If I wish the light cone variable \( du \) to equal the inertial frame \( dU \),

\[ du = (e_z^* \text{d}Z - e_T^* \text{d}T)/\sqrt{2} = (\text{sgn}(z) \text{d}Z - \text{d}T)/\sqrt{2}, \quad (26) \]

I must choose

\[ e_z^* = +1 = \overline{N} \text{ (cl)}. \quad (27) \]

C The quantum scalar constraint

Our final formula for the scalar constraint \( \tilde{H} \) in paper I was

\[ \tilde{H} = \sum_n (1/\kappa) \{(1/2)(\delta(c) E_y^x/E_X^y - \delta(c) E_y^x/E_X^y)^2 E_z^x \\
- p(\delta(c) E_z^x/E_Z^z)^2 E_z^z + \delta(c) (\delta(c) E_z^z)\} = 0. \quad (28) \]
The gauge choice eq. (16) implies
\[
\delta_c \frac{(2)\tilde{E}}{(2)\tilde{E}} = \frac{\delta_c E^z_z}{E^z_z} = \frac{\delta_c E^x_X}{E^x_X} + \frac{\delta_c E^y_Y}{E^y_Y}.
\] (29)

In the constraint, eq. (28), I set \(p = 1/2\), divide through by \(E^z_z\), and use eq. (29) to eliminate \(\delta_c E^z_z\). I rewrite the double difference using
\[
\delta_c \{ \frac{\delta_c E^z_z}{E^z_z} \} = \delta_c \{ [\delta_c E^x_x/E^x_x + \delta_c E^y_y/E^y_y]E^z_z / E^z_z
\]
\[
= \delta_c (\delta_c E^x_x/E^x_x + \delta_c (\delta_c E^y_y/E^y_y) - [\delta_c E^x_x/E^x_x]^2 - [\delta_c E^y_y/E^y_y]^2
\]
\[
+ [\delta_c E^x_x/E^x_x + \delta_c E^y_y/E^y_y] \delta_c (\delta_c E^x_x/E^x_x / E^x_x. \) (30)
\]

The constraint simplifies to
\[
0 = \delta_c (\delta_c E^x_x/E^x_x) + \delta_c (\delta_c E^y_y/E^y_y), \quad (31)
\]
which is just the classical constraint, eq. (9), with derivatives replaced by differences.

D Quantum boundary conditions

If I wish to construct a sinusoidal packet, rather than a step wave, I must in effect glue together many step waves. What boundary conditions should hold at the junctions? The classical boundary conditions for the metric, eq. (12) imply corresponding boundary conditions for the classical triads. In the present gauge, \(g_{uv} = \) constant, \(g_{uu} = g_{vv} = 0\), and single polarization, the classical boundary conditions imply
\[
E^a_a, \quad \partial_u E^x_X/E^x_X, \quad \partial_u E^y_Y/E^y_Y \quad \text{continuous}, \quad (32)
\]
for \(a = x, y\).

The classical boundary conditions cannot hold exactly in LQG, because LQG is fundamentally discrete. One expects jump discontinuities in basic variables from one vertex to the next. I propose the following procedure for implementing boundary conditions. Compute an average slope and value for each variable, on each side of the boundary, by averaging over two vertex subsets located on opposite sides of the boundary. The subsets should be large enough
to approximate a classical length. Allow these subsets to approach the boundary vertex; and demand that values and slopes match. This procedure allows for jump discontinuities, while preserving a classical limit.

E A quantum gravity wave

The classical step curvature solution, eq. (11), is exact but unphysical. It extends to minus infinity; and it has metric zeros which are not true singularities. This section constructs a periodic wave packet solution which is inexact, but has a beginning and end and no metric zeros.

I do this in two steps. In the first step (this section) the solution is periodic but undamped. The following section adds the damping. The undamped solution is

\[ E_X^x(qu; n) = (\Delta Z)^2 \{1 - a \sin [(2\pi n/4q)]/2! - (a^2/32)[\cos(\pi n/q) + (\pi/q)^2(n)^2/2] \} \]  

(33)
a is a small, dimensionless, constant amplitude. q is a constant. When n changes by q (q for "quarter wave"), the phase of the sine changes by one quarter of 2 \( \pi \). The expression for \( E_Y^y \) is identical to eq. (33), except replace a by -a.

With a slight abuse of a standard notation, I can define a \( k \) vector in \( n \) space, i.e. a vector which gives the change in phase per unit change in \( n \).

\[ (2\pi n/4q) := kn; \]

\[ kn = (k/\Delta Z)(n\Delta Z) = (2\pi/(\text{wavelength}))(Z) \]  

(34)
The second line gives the connection to the usual \( k \), the change in phase per unit change in length.

The expression eq. (33) may not look much like the step solution. However, rewrite the step as

\[ E_X^x(cl; u) = 1 + \Theta(-u)[-1 + \cos(c_s u)] \]
\[ = 1 + \Theta(-u)[-\cos(u)^2/2! + (c_s u)^4/4! + \cdots] \]  

(35)
where \( c_s^2 \) is the curvature. In the quantum solution, eq. (33), the constant curvature \( c_s^2 \) becomes the sinusoidal curvature \( a \sin(2\pi n/4q) \).
The $a^2$ term in eq. (33) is not just the square of the linear term, essentially because derivatives have been replaced by differences. The solution is approximate because an exact solution would require an infinite series, whereas the quantum solution of eq. (33) stops at order $a^2$.

To check the constraint and compute curvature, I must compute $\delta^{(2)}E/E$. The second difference of the linear-in-$a$ term, eq. (33), is

$$- (a/2)\{ \sum_{\pm} \sin[(\pi/2q)(n \pm 1)] - 2\sin(n\pi/2q) \} (\Delta Z)^2$$

where I have expanded the first sine using $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$. To estimate the size of $q$, I use the connection between $q$ and the classical wavelength, eq. (34). Since that wavelength is macroscopic whereas $\Delta Z$, the change in $z$ per unit change in $n$, is of order a few hundred Planck lengths, $q$ must be astronomically large, and $1/q$ must be negligible, except when multiplied by $n$. I can therefore expand the cosine in eq. (36), which becomes

$$- (a/2)(\pi/2q)^2 \sin(\pi n/4q)(\Delta Z)^2.$$  

The term quadratic in $a$, eq. (33), is handled similarly. Trigonometric identities are used to expand functions of $n \pm 1$. Functions of $1/q$ are power-series expanded. The total second difference is

$$\delta^{(2)}E_x^x (qu; n) = (a/2)(\pi/2q)^2 \sin(n\pi/2q)[1 - (a/2) \sin(n\pi/2q)](\Delta Z)^2.$$  

Then

$$\delta^{(2)}E_x^x (qu; n)/E_x^x = (a/2)(\pi/2q)^2 \sin(2\pi n/4q),$$

correct to order $a$. The corresponding ratio involving $E_y^y$ will have equal magnitude but opposite sign ($a \rightarrow -a$). From eq. (37), this guarantees $H = 0$.

In deriving eq. (38) I have assumed that the curvature is linear in amplitude $a$. All order $a^2$ and higher corrections to eq. (38) vanish. To see how this happens in order $a^2$ write the expression for curvature in the abbreviated form

$$\delta^{(2)}E^x/E^x = (\ddot{B}_1 + \ddot{B}_2 + \cdots)/(1 + B_1 + \cdots)$$

$$= \ddot{B}_1 + \ddot{B}_2 - B_1\ddot{B}_1 + \cdots,$$  

(39)
where $B_p$ is order $a^p$; and $\cdots$ indicate terms which contribute cubic and higher terms in $a$. I have chosen $B_2$ such that

$$(\ddot{B}_1 + \ddot{B}_2 + \cdots)/(1 + B_1 + \cdots) = \ddot{B}_1 (1 + B_1 + \cdots)/(1 + B_1 + \cdots),$$

Equivalently, I have chosen

$$\frac{\ddot{B}_2}{B_1} = \frac{\ddot{B}_1}{1 + \cdots}.$$  \hspace{1cm} (40)

Then the order $a^2$ contributions to curvature cancel.

One can generalize eq. (40) to higher orders in $a$. Given $B_1, B_2, \cdots, B_{p-1}$, determine $B_p$ by solving the equation

$$\frac{\ddot{B}_p}{B_p} = \frac{\ddot{B}_1}{1 + \cdots}.$$  \hspace{1cm} (41)

Then

$$(\ddot{B}_1 + \cdots + \ddot{B}_p)/(1 + B_1 + \cdots + B_{p-1}) = \frac{\ddot{B}_1 (1 + \cdots + B_{p-1})}{(1 + \cdots + B_{p-1})} = \ddot{B}_1.$$  \hspace{1cm} (42)

The curvature is order $a$, to all orders.

To make contact with the classical curvature, eq. (9), I must divide the second difference by $(\Delta U)^2$, in order to convert differences to derivatives with respect to $U$. From eq. (5),

$$\Delta U = \Delta Z/\sqrt{2},$$

in a formalism where $T$ is held constant.

$$C_{\text{uxu}}^x (cl) = \{\delta^{(2)} E_X^x (qu; n)/[E_X^x (qu; n) - (x \rightarrow y)]\}/(\Delta U)^2 = 2(a/2)(2\pi/4q)^2 \sin(2\pi n/4q)/(\Delta U)^2 = a(2\pi \Delta Z/\lambda)^2 \sin(2\pi n/4q)(\Delta Z)^2 = 2a(2\pi/\lambda)^2 \sin(2\pi n/4q),$$  \hspace{1cm} (43)

where $\lambda$ is the classical wavelength. The initial 2 on the second line is the contribution from $(x \rightarrow y)$.

The constant curvature, characteristic of the classical step curvature, has been replaced by a sinusoidal curvature. In essence there is now a different solution for each $n$. They ”match at each boundary” because the sines and cosines change smoothly from $n$ to $n \pm 1$. 

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In this section I have chosen $E_A \,(cl.)$ to start off with leading term +1. This choice, together with the gauge choice $e^Z_\pm = +1$, implies $\text{sgn} = +1$. To obtain the opposite choice, $\text{sgn} = -1$, change one $E_B^b$ to $-E_B^b$. The new solution leaves the Weyl tensor unchanged and continues to satisfy the $\tilde{H} = 0$ constraint.

**F inclusion of damping**

The solution eq. (43) is infinite in length. To construct a packet, I must include damping factors.

$$E^\pi_X (qu; n) = (\Delta Z)^2 \{ 1 - (a/2) \exp(\mp \rho n) \sin[\mp k n + \phi]$$

$$+ (-a^2/32) \{ \exp(\mp 2 \rho n) \cos(2 k n \mp 2 \phi)$$

$$+ ( \exp[\mp 2 \rho n] \pm 2 \rho n - 1) (f/\rho^2) \cos \phi] \};$$

$$f := (k^2 + \rho^2).$$

(44)

Upper (lower) sign refers to $n > 0$ ($n < 0$). For simplicity in what follows, I will consider only the case $n > 0$ (upper sign); the $n < 0$ follows by changing $\rho \rightarrow -\rho; \; \phi \rightarrow -\phi.$ (45)

The exponential damping factors have discontinuities in derivative at $n = 0$; and from eq. (45) the angle $\phi$ is undefined at $n = 0$. A discontinuity by itself is not a problem because the damping function is defined only at discrete points. The problems at $n = 0$ will turn out to be minor; nevertheless, the value $n = 0$ will need separate consideration (given in section G).

The quantity $\phi$ is a constant phase. When one solves the differential equation $F = ma$ for the damped oscillator, one finds that each derivative shifts the phase by more than the usual $-\pi/2$.

$$(d/dt)[\exp(-\rho t) \sin(\omega t - \phi)] = \sqrt{\omega^2 + \rho^2} \exp(-\rho t) \cos(\omega t - \phi + \psi);$$

$$\cos \psi = \omega/\sqrt{\omega^2 + \rho^2}.$$ (46)

Exactly the same phenomenon occurs in the difference case. I choose a non-zero phase $\phi$ for $\tilde{E}$, so that the curvature becomes a sine wave with zero phase.

The expression eq. (44) contains undamped terms involving

$$2 \rho \mid n \mid -1.$$
The quantum triads, like the classical triads, diverge. The Christoffel symbols are constants at infinity (constant force; compare the constant electric field between parallel plates); the curvature is zero at infinity.

Initially, I included these divergent terms to eliminate a $1/\rho^2$ singularity in the limit $\rho \to 0$. With these terms included, the damped form reduces correctly to the undamped form, eq. (33).

However, these terms also have a fundamental significance. Because the rest of eq. (44) is damped, these are the only terms which survive at large $|n|$, therefore the only terms which contribute to the surface term in the Hamiltonian, the term which gives the total energy of the wave. For the details, see the section on ADM energy below.

Computation of the damped second difference is straightforward. As before, sinusoidal functions of $n \pm 1$ are expanded using trigonometric identities. As before, $k$ is assumed small and functions $\sin k$, $\cos k$ are power series expanded. A new feature: I assume $\rho$ small and power-series expand functions $\exp(-\rho)$. The second difference of the term linear in $a$ is

$$
\frac{a}{2} \exp(-\rho n) \left\{ \sin(k n - \phi) [k^2 - \rho^2] 
+ 2k \rho \cos(k n - \phi) \right\} (\Delta Z)^2 \left[ 1 + \text{order } k^2, \rho^2, k \rho \right]; n \neq 0. \tag{47}
$$

I define

$$
\begin{align*}
\cos \phi &= (k^2 - \rho^2)/f;
\sin \phi &= 2k \rho / f;
\quad f &= k^2 + \rho^2.
\end{align*}
$$

The second difference of the linear-in-$a$ term then reduces to

$$
(\frac{a}{2}) \exp(-\rho n) f \sin(k n) (\Delta Z)^2.
$$

The term quadratic in $a$, eq. (44), requires one extra trigonometric identity. After the usual expansions, that second difference becomes

$$
- \frac{a^2}{8} \exp(-2\rho n) \left\{ -(k^2 - \rho^2) \cos(2k n - 2\phi) 
+ (2k \rho k) \sin(2k n - 2\phi) + f \cos \phi \right\} (\Delta Z)^2 
= -(\frac{a^2}{8}) f \exp(-2\rho n) \left\{ -\cos(2n k - \phi) + \cos \phi \right\} (\Delta Z)^2 
= -(\frac{a^2}{4}) f \exp(-2\rho n) \left\{ \sin(k n - \phi) \sin(k n) \right\} (\Delta Z)^2. \tag{49}
$$
The last line uses the identity
\[ 2 \sin A \sin B = \cos(A - B) - \cos(A + B). \]

One can now factor out
\[ E_X^x = (\Delta Z)^2[1 - (a/2) \exp(-\rho n) \sin(k n - \phi)] + \text{order } a^2 \]
from the total second difference. The final curvature contribution is then
\[ \delta^{(2)} E_X^x (qu; n) / E_X^x = (a/2) f \exp(-\rho n) \sin(k n); \ n \neq 0. \quad (50) \]

Again, there are no order \( a^2 \) corrections.

G Curvature at \( n = 0 \)

Although notions of continuity are not always relevant, big jumps from one vertex to the next are not a good idea. I have assumed the damping factor \( \rho \) is small. This makes both discontinuities at \( n = 0 \) small: the discontinuity in the slope of the exponent \( \exp(-\rho |n|) \), and the discontinuity in the phase \( \phi \). Define
\[ \rho/k := 1/r << 1, \quad (51) \]

From eq. (51) the quantity \( 2 r \) gives an estimate of the number of wavelengths in the central, not strongly damped part of the packet. Since \( \rho \) should be small, \( r \) must be large, and the packet must contain many wavelengths.

\( k \) is very small (\( 2 \pi \) divided by the number of vertices in one wavelength); from eq. (51), \( \rho \) is even smaller. Eqs. (48) and (51) also guarantee small \( \phi \).

\[ \phi \cong +2/r \quad (n > 0); \]
\[ \cong -2/r \quad (n < 0). \quad (52) \]

The relative magnitudes are
\[ \rho << k << 1/r \sim \phi. \quad (53) \]

Because of the discontinuity in \( \phi \), \( E_X^x \) at \( n = 0 \) is undefined. I parameterize it as
\[ E_X^x(n = 0) = (a_0 + a_1 + a_2)(\Delta Z)^2, \quad (54) \]
where $a_p$ is of order $a^p$ in the small amplitude $a$. To determine the $a_i$, I must require the order $a^2$ corrections to curvature to vanish, as at eq. (39).

$E^x_X(n = 0)$ contributes to curvatures at $n = \pm 1$ and $n = 0$. Consider $n \pm 1$ first. From eq. (44),

$$\{E^x_X(\pm 2) \cong E^x_X(\pm 1)\} = \{1 \pm a(\sin \phi)/2 - (a^2/32) \cos(2\phi)\}(\Delta Z)^2.$$  

(55)

I have used the orders of magnitude eq. (53) to expand the $E^x_X$, keeping only leading order in $\rho$ and $k$, and all orders in $\phi$. The $E^x_x$ contribution to curvature is

$$\{E^x_X(\pm 2) - 2E^x_X(\pm 1) + E^x_X(0)\}/E^x_X(\pm 1)
= a_0 - 1 + [\mp a(\sin \phi)a_0/2 + a_1]
+ a_0 a^2[\cos(2\phi)/32 + (1/4) \sin^2 \phi] + a_2 \mp a_1 (a/2) \sin \phi,$$

(56)
to order $a^2$. Setting $a^2$ curvature terms to zero, last line, gives

$$a_1 = 0;$$

$$a_2 = -a^2 a_0/32.$$  

(57)

At eq. (57) I have expanded in $\phi$ and kept zeroth order. The surviving contribution at $n = \pm 1$ is now

$$\delta^{(2)} E^x_X/E^x_X(\pm 1) = a_0 - 1 \mp a(\sin \phi)a_0/2.$$  

(58)

Now consider curvature at $n = 0$.

$$\{E^x_X(+1) - 2E^x_X(0) + E^x_X(-1)\}/E^x_X(0)
= 2/a_0 - 2,$$

(59)
to order $a^2$. Even keeping all orders in $\phi$, there are no order $a^2$ terms, therefore no new constraints on the $a_i$. However, the curvature away from zero varies as sine. This suggests a zero of curvature at $n = 0$. From eq. (59), the $n = 0$ curvature vanishes if

$$a_0 = 1.$$  

(60)

The discontinuities are now minimized. Compare

$$E^x_X(0) = 1 - a^2/32;$$

$$E^x_X(\pm 1) \cong 1 \mp (a/2) \sin(k - \phi) - (a^2/32);$$

$$\delta^{(2)} E^x_X/E^x_X(0) = 0;$$

$$\delta^{(2)} E^x_X/E^x_X(\pm 1) = \pm (a/2) \sin \phi.$$  

(61)
H The ADM energy

The Hamiltonian is the sum of a constraint plus a surface term. The latter is the true Hamiltonian and gives the energy. From \([1]\), the surface term is given by

\[
-\mathcal{N} \delta(c) E_z X \Big/ E_x X \bigg|_{\sim \infty}.
\]

Because of the gauge choice, \(\mathcal{N}(qu) = 1/\Delta Z\).

Only undamped terms in \(\tilde{E}\) of order \(a^2\) survive to infinity. Since the \(\delta(c) \tilde{E}\) in eq. (63) are already order \(a^2\), and the \(\tilde{E}\) are accurate only to order \(a^2\), denominators may be set equal to unity. Then

\[
\text{ADM Energy} = \left(1/\Delta Z\right) |\Delta X \Delta Y| \left(a^2/4\right) \rho (f/\rho^2) \cos \phi \\
\approx \left(1/\Delta Z\right) |\Delta X \Delta Y| \left(a^2/4\right) \rho (k/\rho)^2 (\cos \phi \approx 1) \\
= |\Delta X \Delta Y| \left(a^2/4\right)[kr/\Delta Z].
\]

\((k/\Delta Z)\) is the usual wave vector, \(2\pi\) over wavelength. The area comes from

\[
E_z^z(qu) = E_z^z(cl) \Delta x \Delta y \\
= \text{sgn}(z) \text{sgn}(x) \text{sgn}(y) \Delta X \Delta Y \\
= |\Delta X \Delta Y| ,
\]

if we take the initial area \(\Delta x \Delta y\) to be positive.

Eq. (63) is physically plausible. The energy in weak field approximation is of order (define \(w := k/\Delta Z; \sigma := \rho/\Delta Z\))

\[
\int (\partial_z \tilde{E})^2 \sim (\text{area}) (aw)^2 \left[\sin(wz) \exp(-\sigma z)\right]^2 \\
= (\text{area}) (aw)^2 [(w)^2/(2\sigma)] \left[1/((w)^2 - \sigma^2)\right] \\
\approx (\text{area}) (aw)^2(1/2\sigma = r/2w),
\]

where \(w/\sigma = k/\rho = r\). This back-of-the-envelope estimate contains the same factors as eq. (63).

IV Coherent states

Each vertex has, not one \(x\) and one \(y\) holonomy, but rather a superposition of \(x\) and \(y\) holonomies which form coherent states. In
the free particle case, coherent states are approximate, simultane-
ous eigenfunctions of two canonically conjugate coordinates, x and p. In the Loop Quantum Gravity case, the coherent states are ap-
proximate, simultaneous eigenfunctions of both holonomy and triad.
The canonically conjugate pair is angle (the holonomy) and angular
momentum (the triads).

In section A I discuss a hidden O(3) symmetry which should be
built into the coherent states. In section B I introduce notation and
justify the form of the coherent states, using arguments which are
qualitative, but I believe intuitively convincing. Full details of the
construction are given in reference [6].

A The planar Hilbert space

Call the direction of propagation the z direction. (Lower case indices
x, y, z, a, b, · · · denote global coordinates; upper case indices X, Y, Z,
A, B, · · · denote indices rotated by local SU(2).) The spin network
in the z direction has the expected topology, a series of vertices
connected by edges in the z direction. Holonomies on the z axis
look like holonomies in the full theory. Each holonomy is integrated
from one vertex to the next.

Each vertex also has an infinite number of vertices stretching in
the x and y directions, but because of the symmetry, the holonomy
stretching from vertex n to vertex n+1 is identical to the holonomy
stretching from n-1 to n. Rather than an infinite number of vertices,
one can bend the n to n+1 holonomy around in a circle and associate
both ends of this holonomy with the same vertex n. I.e., the x and
y edges may be given the topology of a circle.

I work in a connection representation for the wave function. The
wave function at each vertex is a product of four holonomies: the two
x and y holonomies on circular edges, plus one incoming z holonomy
and one outgoing z holonomy.

The holonomies may be simplified by gauge fixing the Ė and
connection fields [8]. The off-diagonal elements E_{AZ} and E_{AZ}^z, with
a = x,y and A = X,Y, can be gauged to zero; similarly, A_{AZ} and
A_{AZ} may be set to zero. This means that the holonomies along the
longitudinal z direction are quite simple, involving only the A_{AZ}^z and
the rotation generator S_{AZ} for U(1) rotations around Z.

Coherent states for the case of U(1) symmetry are well under-
stood; see for example Thiemann and Winkler \[11\]. The basis holonomies along \(z\) are

\[
h_z = \exp[i \int M_z A_z^Z \, dz], \tag{66}
\]

where \(M_z\), an eigenvalue of the diagonal generator \(S_z\), is integer or half-integer.

Now consider the \(x\) and \(y\) holonomies. Since \(A_x^Z\) and \(A_y^Z\) have been set to zero, these involve generators \(S_x, S_y\) and are rotations in the \(X,Y\) plane. Each transverse holonomy \(h^{(1/2)}\) therefore has an axis of rotation with no \(Z\) component.

\[
h^{(1/2)} = \exp[i \tilde{m} \cdot \vec{\sigma} \theta/2]; \tag{67}
\]

\[
\tilde{m} = (\cos \phi, \sin \phi, 0),
\]

for some angle \(\phi\). (More precisely, there is one holonomy for each transverse direction \(x,y\); and one \(\phi\) for each transverse direction, \(\phi_x\) and \(\phi_y\). Since the two directions are treated equally, I discuss only the \(x\) holonomies, and suppress the subscript \(x\) for now.) When expanded out, the spin 1/2 holonomy \(h^{(1/2)}\), eq. (67), becomes

\[
h^{(1/2)} = \begin{bmatrix} \cos(\theta/2) & i \exp(-i\phi) \sin(\theta/2) \\ i \exp(+i\phi) \sin(\theta/2) & \cos(\theta/2) \end{bmatrix} \tag{68}
\]

The usual Euler angle decomposition for this rotation is

\[
h^{(1/2)} = \exp[-i\sigma_Z(\phi - \pi/2)/2] \exp(i\sigma_Y \theta/2) \exp[+i\sigma_Z(\phi - \pi/2)/2] = h^{(1/2)}(-\phi + \pi/2, \theta, \phi - \pi/2). \tag{69}
\]

The natural basis for the transverse Hilbert space might seem to be the generalization of \(h^{(1/2)}\) from 1/2 to general \(j\), the set of rotation matrices

\[
h^{(j)}(-\phi + \pi/2, \theta, \phi - \pi/2),
\]

where \(j\) is the highest weight obtained by multiplying together 2\(j\) \(h^{(1/2)}\) matrices. However, this basis is not convenient because it has complicated behavior under the action of the \(\tilde{E}\) and the volume operator. For example, for \(j = 1/2\), \(\tilde{E}\) acts as a functional derivative with respect to \(A\), and produces an anticommutator.
\[ E_A^x h^{(1/2)} = E_A^x \exp[i \int A_B^B S_B dx] = \left( \gamma \kappa / 2 \right) [\sigma_A/2, h^{(1/2)}]^+, \tag{70} \]

This anticommutator shuffles the matrix elements of \( h^{(1/2)} \) in a complicated way. The anticommutator arises because the transverse holonomy is supported by an edge with the topology of a loop. The holonomy both begins and ends at the same vertex, and \( \tilde{E} \) ”grasps” the holonomy on both sides.

In eq. \( (70) \) \( \kappa = 8\pi G; \gamma = \text{Immirsi parameter}; \) and the \( 1/2 \) comes about because the \( \tilde{E} \) grasps the \( \int A \cdot S \) argument of the holonomy at endpoints, resulting in half a delta function. The delta functions are always canceled by the area and line integrals associated with \( E_A^x \) and \( A_B^B \). I have suppressed the area integration associated with each \( \tilde{E} \).

Fortunately, the \( \tilde{E} \) reshuffle the elements of \( h \) in a relatively simple way. Introduce the operators \( E_A^x \), where as usual

\[ f_{\pm} := (f_x \pm if_y)/\sqrt{2}. \tag{71} \]

The operators \( E_A^x \) reshuffle the components of \( h \) in the same way that the familiar angular momentum operators \( L_{\pm} \) reshuffle the \( L = 1 \) Legendre polynomials \( Y_1^M \). For example, I write out the action of the anticommutator in eq. \( (70) \), for index \( A = + \).

\[ [\sigma_+/2, h^{(1/2)}]^+ = \sqrt{1/2} \begin{pmatrix} i \exp(-i\phi)(\sin \theta/2) & 2 \cos \theta/2 \\ 0 & i \exp(+i\phi) \sin \theta/2 \end{pmatrix} \tag{72} \]

Compare this matrix to the original matrix, eq. \( (68) \). \( E_A^x \) has reshuffled the matrix elements as follows

\[ \begin{align*}
(i/\sqrt{2}) \exp(-i\phi) \sin \theta/2 \to & \cos \theta/2, \\
\cos \theta/2 \to & (i\sqrt{2}) \exp(+i\phi) \sin \theta/2, \\
(i/\sqrt{2}) \exp(+i\phi) \sin \theta/2 \to & 0. \tag{73} 
\end{align*} \]

This is isomorphic to the action of the operator \( L_+ \) on the \( L = 1 \)
Legendre polynomials \cite{12,13}. The isomorphism is

\begin{align*}
L_\pm & \leftrightarrow 2 E^\pm_\pm / \gamma \kappa; \\
L_0 & \leftrightarrow 2 E^0_0 / \gamma \kappa; \\
Y^\pm_1(\theta, \phi) & \leftrightarrow N Y^\pm_1(\theta/2, \phi - \pi/2) \\
& = \mp N \sin(\theta/2) \exp[\pm(i\phi - i\pi/2)]/\sqrt{2}; \\
Y^0_1(\theta, \phi) & \leftrightarrow Y^0_1(\theta/2, \phi - \pi/2) \\
& = N \cos(\theta/2). \quad (74)
\end{align*}

Because of the half angles, normalization of the Y’s requires integrating θ from 0 to 2π. N = \sqrt{4\pi/3}.

Because the \(Y^M_1(\theta/2, \phi - \pi/2)\) transform more simply than matrix elements of \(h^{(1/2)}\) under the action of \(\tilde{E}\), one obtains a more convenient basis by using O(3) 3J coefficients and products of \(Y_1^1\)’s, rather than SU(2) coefficients and products of \(h^{(1/2)}\)’s. The resultant basis is just the set of spherical harmonics \(Y^M_L(\theta/2, \phi - \pi/2)\) for O(3).

The operator \(E^x_0\) is not the triad \(E^x_Z\), which has been gauged to zero. \(E^x_0\) is a third operator constructed to complete the trio of generators and act on states in a manner isomorphic to \(L_0\).

\[
2 E^x_0 / \gamma \kappa = [h, \sigma_3/2]_-
\] (75)

Because the three independent elements of \(h^{(1/2)}\) can be expressed in terms of the \(Y^M_1(\theta/2, \phi - \pi/2)\), the Y’s are as "complete" a set as the elements of \(h^{1/2}\). The relation between h and the \(Y^M_1\) is

\[
Nh^{(1/2)} = 1Y^0_1 + iY^+_1 S_- + iY^-_1 S_+,
\] (76)

where boldface denotes a 2x2 matrix. The equations eq. (76) may be used to replace the \(h^{(1/2)}\) in the Hamiltonian by \(Y^M_1\)’s.

One can take into account the y edges as well as the x edges, by constructing two bases, \(Y^M_{Lx}\) and \(Y^M_{Ly}\) for holonomies along the x and y directions respectively. These harmonics transform simply under the action of the \(\tilde{E}\) :

\[
(\gamma \kappa / 2)^{-1} E^p_L Y^M_L = \Sigma_N Y_{LN} \langle L, N \mid S_P \mid L, M \rangle,
\] (77)

where \(Y_{LM} = Y_{LM}(\theta/2, \phi - \pi/2)\). The unconventional half-angle reminds us of the origin of these objects in a holonomy \(h^{(1/2)}\) depending on half-angles.
The $Y$’s are known to be proportional to matrix elements of rotations,

$$\sqrt{4\pi/(2L+1)}Y^M_L = D^{(L)}_{0M}(-\phi + \pi/2, \theta/2, \phi - \pi/2).$$  \hspace{1cm} (78)

Therefore eq. (77) is also correct if $D$’s are substituted for $Y$’s. I prefer $D$’s to $Y$’s in what follows, because use of $D$’s (will require awkward factors of $\sqrt{4\pi/(2L+1)}$ in initial formulas, but) ultimately will result in fewer factors $\sqrt{4\pi/(2L+1)}$.

The transverse coherent states constructed here will not have unique values for $M_x$ and $M_y$. These states will be superpositions of $D^{(La)}_{0Ma}$ matrices ($a = x,y$); and the superpositions will contain a range of values $M_a$. (Similarly, coherent states in the longitudinal direction will not have definite $M_z$.) The superpositions are sharply peaked at central values of the $M$’s, however, so that $M$-values which violate $U(1)$ are suppressed.

## B States

Thiemann and Winkler [9, 10, 11] have constructed coherent states for the general case, full local $SU(2)$ symmetry. These coherent states may be understood intuitively as generalizations of the minimal uncertainty states for the free particle. Once this intuitive approach is understood, it is straightforward to use the same recipe to generate a set of coherent states displaying the $O(3)$ symmetry of the planar case.

The recipe for constructing a coherent state for the free particle starts from a wave function which is a delta function.

$$\delta(x - x_0) = \int \exp[ik(x - x_0)] dk/2\pi.$$  \hspace{1cm}

This wave function is certainly strongly peaked, but it is not normalizable. Also, it is peaked in position, but it should be peaked in both momentum and position. To make the packet normalizable, insert a Gaussian operator $\exp(-p^2/2\sigma^2)$. (Choosing the Gaussian form is a ”cheat”, because we know the answer; but for future reference note that all the eigenvalues $k^2$ of $p^2$ must be positive, so that the Gaussian damps for all $k$.) To produce a peak in momentum, complexify the peak position: $x_0 \rightarrow x_0 + ip_0/\sigma^2$. With these changes, the packet becomes
\begin{align*}
N \int \exp[-p^2/(2\sigma^2)] \exp[ik(x-x_0) + kp_0/\sigma^2] \frac{dk}{2\pi} \\
= \frac{N}{\sqrt{2\pi}} \cdot \exp[p_0^2/(2\sigma^2)] \cdot \exp[ip_0(x-x_0) - (x-x_0)^2/2].
\end{align*}

(79)

The last line follows after completing the square on the exponential, and exhibits the characteristic coherent state form.

There is not just one coherent state, but a family of coherent states, characterized by the parameter \(\sigma\). The shape of the wave function is highly sensitive to \(\sigma\); but the peak values \(x_0, p_0\) are independent of \(\sigma\), as is the minimal uncertainty relation \(\Delta x \Delta p = \hbar/2\). The coherent states constructed below contain a parameter \(t\) which is analogous to \(\sigma\).

Now apply the above recipe to the planar case. Position \(x\) and momentum \(p\) become the pair of angular variables \((\theta, \phi)\) on the group manifold and a pair of angular momentum variables \((L, M)\). The complete set of plane waves becomes a complete set of spherical harmonics.

To construct a delta function in angles,

\[\delta(\theta/2 - \alpha/2)\delta(\phi - \beta)/\sin(\alpha/2)\]

I introduce spherical harmonics \(D^{(L)}(u)\) depending on angles \((\alpha, \beta)\) in the same way that the \(D^{(L)}(h)\) depend on \((\theta, \phi)\). In particular the axis of rotation for \(u\) is also in the xy plane, like the axis of \(h\).

\[
D^{(L)}(u)_{0M} = D^{(L)}(-\beta + \pi/2, \alpha/2, \beta - \pi/2)_{0M} \\
= \sqrt{4\pi/(2L + 1)} Y_{LM}(\alpha/2, \beta - \pi/2).
\]

(80)

Compare eq. (80) to eq. (78). I can now write the delta function in angle as a sum over spherical harmonics.

\[
\delta(\theta/2 - \alpha/2)\delta(\phi - \beta)/\sin(\alpha/2) \\
= \sum_{L,M} ((2L + 1)/4\pi) D^{(L)}(h)_{0M} D^{(L)}(u)^*_{0M}.
\]

(81)
As discussed in the last section, it is more convenient to use D matrices rather than YLM’s, but the reader who wishes to exhibit the latter can use eqs. (78) and (80) to replace D’s by Y’s in eq. (81). The sum then takes on a form which may be more familiar, just $\sum YY^\ast$.

Continue with the recipe: dampen the sum using a Gaussian

$$\exp[-t L(L + 1)/2].$$

Complexify by extending the angles in $u$ to complex values, replacing $u$ by a matrix $g$ in the complex extension of O(3). The coherent state has the general form

$$|u, \vec{p}\rangle = N \sum_{L,M} ((2L + 1)/4\pi) \exp[-tL(L + 1)/2] D(L)(h)_{0M} D(L)(g)^\ast_{0M}$$

$$= N \sum_{L,M} \cdots D(L)(h)_{0M} D(L)(g^\dagger)_{M0}.$$ (82)

Every matrix in SL(2,C), the complex extension of SU(2), can be decomposed into a product of a Hermitean matrix times a unitary matrix (“polar decomposition”; see for example [14]). E. g. for the fundamental representation,

$$g = H u = \exp(\vec{\sigma} \cdot \vec{p}/2) u.$$ (83)

The vector $\vec{p}$ gives the matrix $H$ an axis $\hat{p}$, analogous to $\hat{m}$ and $\hat{n}$ for matrices $h$ and $u$. From this result, it follows that every matrix in O(3) also has a polar decomposition, obtained by restricting the representations of SU(2) to representations with integer spin.

$$g^{(L)} = \exp[\vec{S} \cdot \vec{p}] u^{(L)}$$

$$:= H^{(L)} u^{(L)}.$$ (83)

In the limit that the complex part of the angle, $\vec{p}$ goes to zero, eq. (82) reduces to eq. (81).

$H^{(L)}$ is expected to diverge as $\exp(p L)$ for large $L$, because of its $\exp[\vec{S} \cdot \vec{p}]$ form. Combine this with the damping factor:

$$\exp[-t L(L + 1)] \exp[p L].$$

The state is seen to have a peak $<L>$ at $p/t$. 25
All three axes of rotation are assumed to lie in the xy plane:  
\( \hat{p}, \hat{m}, \) and \( \hat{n} \) for H, h, and u respectively.

\[
\begin{align*}
\hat{m} & = (\cos \phi, \sin \phi, 0); \\
\hat{n} & = (\cos \beta, \sin \beta, 0); \\
\hat{p} & = (\cos(\beta + \mu, \sin(\beta + \mu), 0). \tag{84}
\end{align*}
\]

C Main results

\[
\begin{align*}
D^{(1)}_{0A}(h) | u, \vec{p} \rangle & = D^{(1)}_{0A}(u) | u, \vec{p} \rangle \\
& \quad + \text{SC;}
(2/\gamma \kappa)E^x_A | u, \vec{p} \rangle & = < L > \hat{p}_B D^{(1)}(u)_{BA} | u, \vec{p} \rangle \\
& \quad + \text{SC;}
< L > & = p/t; \\
D^{(1)}_{0A}(u) & := \hat{D}_A(\sqrt{4\pi/3}Y^A_1); \\
\hat{p}_B D^{(1)}(u)_{BA} & = \hat{n}_A \cos \mu - (\hat{n} \times \hat{D})_A \sin \mu. \tag{85}
\end{align*}
\]

\( \mu \) is the angle between the holonomic axis of rotation \( \hat{n} \) and \( \hat{p} \). SC denotes small correction states, down by order \( 1/\sqrt{< L >} \).

The peak value \( D^{(1)}(u) \) may be translated into a peak value of \( \theta^{1/2} \), using eqs. (76) and (78). More generally,

\[
\begin{align*}
\theta^{(j)}(h) | u, \vec{p} \rangle & = \theta^{(j)}(u) | u, \vec{p} \rangle,
\end{align*}
\]

where \( \theta^{(j)} \) is a representation of SU(2). Despite the use of an O(3) basis, we do not lose information about SU(2).

The direction of \( < L > \) is given by \( \hat{p} \) rotated by \( D(u) \). The magnitude of \( L \) is not given by \( p \), but by \( p/t \), \( t \) the damping factor. From reference [6] \( t \) should be order \( 1/L \), in order for certain SC terms to be suppressed; and \( p \) must not be too close to unity, because correction terms in \( D(H) \) are suppressed by \( \exp(-p) \). A reasonable compromise is \( p \approx 5, t \approx 5/L \).

V Determining the coherent state parameters

Usually the constraint \( H = 0 \) is considered the most difficult to solve; but here we begin with a set of transverse \( \hat{E} \) (constructed in section [III]) which satisfy this constraint. However, a solution must
obey nine additional constraints: five single polarization constraints, which constrain the four off-diagonal transverse $\tilde{E}$ and transverse $K$, as well as $K_Z^z$; two unidirectional constraints; and the two diffeomorphism constraints. A tenth constraint, the Gauss constraint

$$0 = K_A^a E_B^b \epsilon_{AB},$$

is automatically satisfied because it is linear in off-diagonal tensors. The states in the Hilbert space depend on parameters which must be adjusted so that these constraints are satisfied.

"Satisfied" in the context of coherent states means that expectation values satisfy the constraint. The state is usually not an eigenfunction of the constraint.

## A Basic variables

This section express the basic $(K, \tilde{E})$ variables in terms of coherent state parameters. The spin connection is needed in the current gauge, since $K$ is a combination of holonomy and spin connection:

$$\gamma^I = A^I - \Gamma^I \text{ (QFT)}$$

$$\rightarrow -2i \tilde{h}^I - \Gamma^I \text{ (SS)}$$

$$\hat{n} = (\cos \beta, \sin \beta, 0).$$

$\hat{h}$ (second line) acting on the coherent state, gives the peak value of $\hat{n}$ (third line), with axis of rotation $\hat{n}$ and angle of rotation $\alpha$. From paper I the products $\Gamma \cdot E$, in single polarization limit, are given by

$$\Gamma^y_x E^x_X + \Gamma^x_y E^y_Y = [\delta(c) E^y_Y/E^y_Y - \delta(c) E^x_X/E^x_X]E^z_Z;$$

$$\Gamma^y_x E^y_Y - \Gamma^x_y E^x_X = \delta(c) E^z_Z.$$  \hspace{1cm} (87)

In the present gauge we may use eq. (29) and replace $\delta(c) E^z_Z$ on the last line by

$$[\delta(c) E^x_X/E^x_X + \delta(c) E^y_Y/E^y_Y]E^z_Z.$$  \hspace{1cm} (88)

We solve for the individual $\Gamma \cdot E$.

$$\Gamma^y_Y E^y_Y = -\delta(c) (E^x_X) E^y_Y / C(qu);$$

$$\Gamma^y_X E^y_X = +\delta(c) (E^y_Y) E^x_X / C(qu);$$

$$C(qu) = \text{sgn} (\Delta Z)^2.$$
Eq. \((86)\) expresses the K’s in terms of coherent state parameters \((\alpha, \beta)\) plus the \(\Gamma\); we must now express the \(\tilde{E}\), and therefore \(\Gamma\), in terms of coherent state parameters.

\[
E_A^a(qu) = (\kappa \gamma/2) L_a \tilde{p}^{(1)}(u_a)_{BA};
\]

\[
L_a = p_a/t;
\]

\[
\tilde{p}^{(1)}(u)_{BA} = \cos \mu_a \hat{n}_A - \sin \mu_a (\hat{n}^a \times \hat{D})_A;
\]

\[
\hat{n}^a \times \hat{D}^a = (\cos(\alpha_a/2) \sin \beta_a, - \cos(\alpha_a/2) \cos \beta_a, - \sin(\alpha_a/2));
\]

\[
\hat{n}^a = (\cos \beta_a, \sin \beta_a, 0).
\]  

\(\mu\) is the angle between \(\hat{p}\) and the axis of rotation \(\hat{n}\). \(\hat{D}\) is essentially the expectation value of the holonomy; see eq. \((85)\). All basic variables \((K, \tilde{E})\) are now expressed in terms of coherent state parameters.

\section*{B Evaluation of the \(\beta_a\)}

I form combinations \(U_1 \pm U_3\) of the unidirectional constraints from paper I, then use eq. \((86)\) to eliminate the K’s.

\[
0 = \left[ \gamma K^X_y E^y_Y + E^z_Z \delta_{(c)} (E^x_X)/E^x_X \right]/\sqrt{E^z_Z}
\]

\[
= \left\{ \frac{2 \sin \beta_y \sin(\alpha_y/2)}{\gamma + \delta_{(c)} (E^x_X)/C(qu)} \right\} E^y_Y/\sqrt{E^z_Z};
\]

\[
0 = \left[ \gamma K^X_y E^x_X + E^z_Z \delta_{(c)} (E^y_Y)/E^y_Y \right]/\sqrt{E^z_Z}
\]

\[
= \left[ \frac{2 \cos \beta_x \sin(\alpha_x/2)}{\gamma + \delta_{(c)} (E^y_Y)/C(qu)} \right] E^x_X/\sqrt{E^z_Z}. \tag{90}
\]

Because all on-diagonal \(\Gamma^I\) vanish, the \(\tilde{E}\) contributions come entirely from the explicit \(\tilde{E}\) in the constraints. The single polarization constraints, on the other hand, have no explicit \(\tilde{E}\) dependence. They acquire \(\tilde{E}\) dependence from eq. \((88)\).

\[
0 = \gamma K^X_y
\]

\[
= \frac{2 \cos \beta_y \sin(\alpha_y/2) + \delta_{(c)} (E^x_X)/C(qu)}{C(qu)};
\]

\[
0 = \gamma K^Y_x
\]

\[
= \frac{2 \sin \beta_x \sin(\alpha_x/2) - \delta_{(c)} (E^y_Y)/C(qu)}{C(qu)}. \tag{91}
\]

The unidirectional constraints have an additional \(\tilde{E}/\sqrt{E^z_Z}\) on the right. However, this additional factor merely produces a constant,
when acting on a coherent state. Therefore this factor may be com-
muted to the left. The two sets of constraints, unidirectional and
single polarization, agree only if

\[ \begin{align*}
(\cos \beta_x)/\gamma &= -\sin \beta_x; \\
\cos \beta_y &= +(\sin \beta_y)/\gamma; \\
\cos \beta_x &= \eta_x\gamma/\sqrt{1 + \gamma^2}; \\
\sin \beta_x &= -\eta_x/\sqrt{1 + \gamma^2}; \\
\cos \beta_y &= \eta_y/\sqrt{1 + \gamma^2}; \\
\sin \beta_y &= \eta_y\gamma/\sqrt{1 + \gamma^2},
\end{align*} \]

(92)

where \( \eta_a = \pm 1 \), and \( \beta_a \) is the angle the holonomic rotation axis
makes with the X axis. A corollary: the two rotation axes, \( \hat{n}_x \) and
\( \hat{n}_y \), are 90 degrees apart.

The unidirectional and single polarization constraints are now
equivalent. I can drop the unidirectional constraints and focus on
the single polarization constraints; the number of independent con-
straints has dropped to seven.

C Evaluation of the \( \mu_a \)

The single polarization constraints also require vanishing of off-
diagonal angular momentum components.

\[ E^X_y = E^y_X = 0. \]

Translated into coherent state language, this means the off-diagonal
components of \( \hat{p} \) must vanish.

\[ \begin{align*}
\hat{p}^a(\alpha_a = 0) &= (\cos(\beta_a + \mu_a), \sin(\beta_a + \mu_a), 0) \\
&= \cos \mu_a \hat{n}_a - \sin \mu_a \hat{n}_a \times \hat{Z}; \\
\hat{p}_A^a(\alpha) &= \cos \mu_a (\cos \beta_a, \sin \beta_a, 0) \\
&= -\sin \mu_a (\cos(\alpha_a/2) \sin \beta_a, -\cos(\alpha_a/2) \cos \beta_a, -\sin(\alpha_a/2)); \\
\hat{p}_X^a(\alpha) &= \cos \mu_x / \cos \beta_x; \\
\hat{p}_Y^a(\alpha) &= \cos \mu_y / \sin \beta_y.
\end{align*} \]

(93)

There are two \( \hat{p} \) variables in this problem: the unrotated \( \hat{p}(\alpha = 0) \);
and the rotated \( \hat{p}(\alpha) \).

\[ \hat{p}(\alpha)_A = \hat{p}_B(\alpha = 0) D^{(1)}(u)_{BA}. \]
\( \hat{p}(\alpha = 0) \) lies in the xy plane and has components along both \( \hat{n}^a \) and \( \hat{n} \times \hat{Z} \) (first two lines). After \( \hat{p}(\alpha = 0) \) and \( \hat{Z} \) are rotated through \( \alpha/2 \) around axis \( \hat{n} \), \( \hat{Z} \) becomes \( \hat{D} \). The rotated \( \hat{p}(\alpha) \) has an unchanged component \( \cos \mu_a \) along \( \hat{n} \), and a component \( \sin \mu_a \) along the image of \( \hat{n} \times \hat{Z} \) under the rotation, \( \hat{n} \times \hat{D} \) (next two lines). If one sets off-diagonal components \( \hat{p}_{B}^a = 0, a \neq B \), then the on-diagonal components simplify greatly (last two lines).

Since \( \sin(\alpha/2) \) is small (small excitations in the classical limit), \( \alpha/2 \) must be near zero or \( \pi \); I choose the former possibility: I assume flat space is \( \alpha/2 = 0 \), and look for a solution which connects smoothly to zero. Because \( \alpha \) is small, \( \hat{p}_x \) at \( \alpha_x = 0 \) must be close to the +X axis, and \( \hat{p}_y \) close to the +Y axis (positive X and Y axes because the \( \hat{E}_A^a \) start off with value +1).

\[
\begin{align*}
\mu_x + \beta_x &= \epsilon_x; \\
\mu_y + \beta_y &= \pi/2 + \epsilon_y,
\end{align*}
\]

(94)

where \( \mu_a + \beta_a \) is the angle \( \hat{p}_a(\alpha_a = 0) \) makes with the X axis, and \( \epsilon_a \) is the small angle \( \hat{p}_a \) makes with the +X or +Y axis. If one of the \( \hat{E}_A^a \) start off with value -1 (sgn = -1) it is necessary to reverse the direction of \( \hat{p} \) by adding \( \pi \) to the corresponding \( \epsilon_a \).

To determine the \( \epsilon_a \), I write out the expressions for \( \hat{p}_B^a = 0, B \neq a, \) and insert eqs. (92) and (94).

\[
\tan \epsilon_a = \gamma \left[ 1 - \cos(\alpha_a/2) \right]/\left[ 1 + \gamma^2 \cos(\alpha_a/2) \right];
\]

\[
\epsilon_a \approx \gamma \left[ \sin^2(\alpha_a/2)/2 \right]/\left[ 1 + \gamma^2 \right].
\]

(95)

Eqs. (94) and (95) determine \( \mu \) once \( \sin(\alpha/2) \) is known. Eq. (95) is correct for either sign of \( \eta_a \), eq. (92), and both values of sgn.

As a check, the formulas for \( \hat{p}_B^a \), eq. (93), predict \( \hat{p}_A^a \) close to unity (not quite unity because there is a component \( \hat{p}_Z^a \)). From eqs. (92) and (94) for \( \beta_a \) and \( \mu_a \),

\[
\begin{align*}
\hat{p}_X &= \cos \mu_x / \cos \beta_x \\
&\approx 1 - \epsilon_x / \gamma \\
&(\approx 1 - (\hat{p}_Z^2)/2); \\
\hat{p}_Y &= \cos \mu_y / \sin \beta_y \\
&\approx 1 - \epsilon_y / \gamma \\
&(\approx 1 - (\hat{p}_Z^2)/2).
\end{align*}
\]

(96)
If $E^a_A$ starts off with -1, the corresponding $p^a$ will acquire an overall minus sign.

We have now satisfied both unidirectional constraints and two single polarization constraints. We are down to five constraints.

D Determination of $\sin(\alpha/2)$

We have a set of transverse $\tilde{E}$ which satisfy $H = 0$. We can insert them into the single polarization constraints eq. (91), and determine $\sin(\alpha/2)$, where $\alpha_a$ is the peak angle of rotation.

From eq. (44),

$$\delta(c) E^x_A(cl; n) = -(a/2)f \exp(-\rho n) \cos(kn - \phi/2)$$

$$+(-a^2/16) \{ f \exp(-2\rho n) \cos(2kn - 3\phi/2)$$

$$+[-\rho \exp(-2\rho n) + \rho] (f/\rho^2) \cos \phi \}. \quad (98)$$

I insert eqs. (98) and (92) into eq. (97). To order $a^2$,

$$2\eta_x \sin(\alpha_x/2)/\sqrt{1 + \gamma^2} = -\delta(c) E^y_A(cl)(\Delta x/\Delta Z) sng$$

$$= -(a/2)f \exp(-\rho n) \cos(kn - \phi/2)(\Delta x/\Delta Z) sgn + O a^2;$$

$$2\eta_y \sin(\alpha_y/2)/\sqrt{1 + \gamma^2} = -\delta(c) E^x_A(cl)(\Delta x/\Delta Z) sgn$$

$$= +(a/2)f \exp(-\rho n) \cos(kn - \phi/2)(\Delta x/\Delta Z) sgn + O a^2. \quad (99)$$

It may seem surprising that the single polarization constraints relate $K_x$ to $E^y$, and vice-versa. However, in the present gauge,

$$E^y = |e|/(e^Y_x) = sgn \ e^x_X. \quad (100)$$

The single polarization (and unidirectional) constraints relate $K_x$ to $e^X_x$.

A second way to relate $\tilde{E}$ to coherent state parameters: set $\tilde{E}$ equal to its expression in terms of $\tilde{p}$.

$$E^a_A(qu; n) = (\kappa \gamma/2) L_a \tilde{p}^a_A(\alpha_a); \quad \text{or}$$

$$E^a_A(cl; n)(\Delta Z \Delta x^b) = (\kappa \gamma/2) L_a \tilde{p}^a_A(\alpha_a)$$

$$= sgn (\Delta Z \Delta X^b), \quad b \neq a. \quad (101)$$
I have used eqs. (13) and (17). $\Delta z, \Delta Z, \text{and } \Delta x^b$ are constants, so that the n dependence of $\hat{E}$ (cl) is inherited by $\hat{p}$, the p in $L = p/t$, and $\Delta X^b$. From eqs. (95) and (96),

$$\hat{p}_A^a = 1 - \sin^2(\alpha_a/2)/2 (1 + \gamma^2)$$

$$\simeq 1 - (\hat{p}_Z^a)^2/2, \ a = A. \quad (102)$$

The variation in $\hat{p}_A^a$ is order $a^2$. Therefore p and $\Delta X^b$ inherit the order a variation of $\hat{E}$ (cl). Rewrite the first line of eq. (101) using the explicit expression for $E^x$ (cl), eq. (44).

$$(\Delta x \Delta Z) \{1 - (a/2) \exp(\mp \rho n) \sin[k n \mp \phi] + O a^2 \}
= (\kappa \gamma/2t)(p(\text{avg})_x + \Delta p_x)\hat{p}_X^x
:= (\kappa \gamma/2)(L(\text{avg})_x + \Delta L_x)\hat{p}_X^x. \quad (103)$$

Then

$$(\kappa \gamma/2)L(\text{avg})_x = \Delta x \Delta Z;
\Delta L_x = L(\text{avg}) \{(-a/2) \exp(\mp \rho n) \sin[k n \mp \phi] + O a^2 \}. \quad (104)$$

For $\Delta L_y$, change a to minus a. If we had chosen $\Delta x \neq \Delta y$, then we would have found $L(\text{avg})_x \neq L(\text{avg})_y$, a not especially helpful result.

The oscillations in Lorentz coordinates follow from eq. (101).

$$\text{sgn } \Delta X = \Delta x \{1 + (\mp a/2) \exp(\mp \rho n) \sin[k n \mp \phi] + O a^2 \}. \quad (105)$$

For $\Delta Y$, change the sign of a.

We now know the angles $\mu_a + \beta_a$ (= the directions of $\hat{p}_a$) from eq. (11). We know the $\beta_a$ up to sign; but we do not know $(\mu_a, \beta_a)$ separately.

However, we have some limited information about the sign of $\beta_a$. The leading contributions to $\delta(c) E^x_X$ and $\delta(c) E^y_Y$ have opposite sign. From eq. (97), this implies

$$\cos \beta_x \sin(\alpha_x/2)/\sin \beta_y \sin(\alpha_y/2) < 0; \ or \ \eta_x \sin(\alpha_x/2)/\eta_y \sin(\alpha_y/2) < 0. \quad (106)$$

This determines the relative sign of the $\beta_a$, but only if we know the relative sign of the $\alpha_a$!
It turns out only the signs of the products \((\cos \beta a \text{ or } \sin \beta a)\) times \(\sin(\alpha_a/2)\) are significant. The basic holonomy is

\[
h_a = \cos(\alpha_a/2) + i \sigma \cdot \vec{n}_a \sin(\alpha_a/2)
\]

\[
= \cos(\alpha_a/2) + i \sigma \cdot (\cos \beta_a, \sin \beta_a, 0) \sin(\alpha_a/2).
\] (107)

This expression is invariant under simultaneous sign change of both \(\vec{n}\) and \(\alpha\). From eq. (92), the two solutions for \(\beta\) (two possible signs for \(\eta\)) correspond to opposite signs for \(\vec{n}\). If a solution exists for one sign of \(\vec{n}\) and \(\alpha\), then an identical solution exists for the opposite sign of \(\vec{n}\), provided we simultaneously change the sign of \(\alpha\). The signs of the \(\alpha_a\) and \(\beta_a\) have little physical significance; they are constrained only by eq. (106).

Four out of five single polarization constraints are now satisfied. Three constraints remain: the single polarization constraint \(K_z = 0\) and two diffeomorphism constraints.

### E  \(K^Z_x, E^z_Z\) and Gauss

The single polarization constraints demand

\[
0 = \gamma K^Z_x(n)
\]

\[
= -2i \left[ \hat{h}_z^Z(n, n + 1) + \hat{h}_z^Z(n - 1, n) \right]/2
\]

\[
= 2 \left[ \sin(\theta_z/2)(n, n + 1) + \sin(\theta_z/2)(n - 1, n) \right]/2.
\] (108)

Either all \(\theta\) are zero, or \(\theta\) alternates between two values having opposite sign. If the latter, the two values must be small, because of the slow variation assumption.

Since \(E^z_Z\) grasps on both sides of the vertex, its expectation value depends on \(m_f + m_i\), the expectation values of \(S_x\) on the ingoing and outgoing sides of the vertex.

\[
E^z_Z(qu) = (\kappa \gamma/2)(m_f + m_i)
\]

\[
= E^z_Z(cl)(\Delta x)^2
\]

\[
= \text{sgn} (E^x_X E^y_Y)(cl)(\Delta x)^2.
\] (109)

On the last line I have invoked the classical form of the diffeomorphism gauge fixing, eq. (14). The classical \(\hat{E}\) go as

\[
E^x_X(cl) = 1 - A_1 + A_2;
\]

\[
E^y_Y(cl) = 1 + A_1 + A_2.
\]
where \( A_p = O \ a^p \). Therefore
\[
E^z_Z(\text{cl}) = sgn \left[ 1 - A_1^2 + 2A_2 + O \ a^3 \right];
\]
\[
(\kappa \gamma / 2)(m_f + m_i) = sgn(\Delta x)^2(1 + O \ a^2). \tag{110}
\]

\( E^z_Z(\text{cl}) \) and \( (m_f + m_i) \) are close to constants.

The quantity \( m_f - m_i \) occurs in Gauss’ Law. The Z components of the x and y holonomies are given by the Z components \( \hat{p}_Z^i \).

\[
0 = L_x \sin \mu_x \sin(\alpha_x/2) + L_y \sin \mu_y \sin(\alpha_y/2) + (m_f - m_i)
\]
\[
= L_x \sin(\epsilon_x - \beta_x) \sin(\alpha_x/2) + L_y \cos(\epsilon_y - \beta_y) \sin(\alpha_y/2) + (m_f - m_i)
\]
\[
\approx L_x (\epsilon_x \gamma + 1) \eta_x \sin(\alpha_x/2)
\]
\[
+ L_y (\epsilon_y \gamma + 1) \eta_y \sin(\alpha_y/2) \sqrt{1 + \gamma^2 + (m_f - m_i)}
\]
\[
= -[L_x \delta(c) E^x_X(\text{cl}) + L_y \delta(c) E^y_Y(\text{cl})](\Delta x/2 \Delta z) + (\kappa \gamma / 2)(m_f - m_i). \tag{111}
\]

The second line follows from eq. (94); the next two from eq. (92) keeping up to terms of order \( \epsilon_a = a^2 \); the last from eq. (91). Since the square bracket on the last line is even under \( a \leftrightarrow -a \), the bracket is 1 minus order \( a^2 \). The oscillations of \( (m_f \pm m_i) \) are therefore both order \( a^2 \). Then to order \( a \),

\[
m_f = m_i = sgn(\Delta x)^2 / \kappa \gamma. \tag{112}
\]

As for the diffeomorphism constraints, the single polarization constraint \( K_z = 0 \) guarantees that the diffeomorphism constraint \( D_2 \) is satisfied.

\[
D_2 = K_z E^z = 0.
\]

The constraint \( D_1 \) was used at eq. (109) to determine \( E^z_Z \).  

**F Checking the \( \Delta x^i \)**

We now have
\[
\Delta x \Delta Z = (\kappa \gamma / 2)L(\text{avg});
\]
\[
(\Delta x)^2 = sgn \kappa \gamma (m_f \text{ or } m_i). \tag{113}
\]

In principle the diffeomorphism gauge condition and the area operator can be used to check these assignments. In practice matters are not always simple.
Consider first the gauge condition.

\[ (\Delta Z)^2 E_Z^z(qu) = sgn(\bar{E}_Z (qu)). \]

Acting on a coherent state, this becomes

\[ (\Delta Z)^2 sgn(\kappa \gamma/2)(m_f + m_i) = sgn(\kappa \gamma/2)^2 L_x L_y \hat{p}^x \times \hat{p}^y \cdot \hat{Z} \]

\[ = (\kappa /2)^2 L_x L_y [\cos^2 \mu + \sin^2 \mu \cos(\alpha_x/2) \cos(\alpha_y/2) ] \]

\[ \cong (\Delta x \Delta Z)^2[1 + O a^2]. \] (114)

The second line is even under a ↔ -a, leading to the third line. This result is consistent with eq. (113); order \( a^2 \) terms were dropped when computing eq. (113).

Now consider a check using the area operator. The standard, three-dimensional area squared operator contains two \( \bar{E} \), therefore brings down two factors of \( S \cdot S \) when acting on a holonomy \[ [15] \]. If the holonomy has spin \( j \),

\[ area^2 = (\kappa \gamma)^2 j(j + 1). \]

There is no factor 1/2; the area operator acts away from vertices.

The area operator for the \( z \) direction reduces to a product of two \( E_Z^z \), because it acts on a holonomy containing only \( A^Z S_z \), therefore brings down \( S_z^2 \).

\[ (xy \text{ area})^2 = (\kappa \gamma)^2 m_z^2 \]

\[ = \{ \Delta x \Delta y E_Z^z(cl) \}^2 \]

\[ = \{ \Delta x \Delta y sgn(\bar{E}_Z c) \}^2 \]

\[ = \{ \Delta X \Delta Y \}^2. \] (115)

\( m_z \) can be either \( m_f \) or \( m_i \). Eq. (115) is consistent with eq. (113).

Now consider transverse areas \( \Delta x^a \Delta z \). The Hilbert space is based on \( O(3) \) holonomies, which are not well suited to an area calculation. \( O(3) \) holonomies are simple when grasped at a vertex; the area operator requires grasps away from the vertex.

However, suppose we grasp away from vertices anyway. Two grasps away from a vertex give

\[ D^{(L)}(\theta/2) \rightarrow D^{(L)}(c_1 \theta/2)[(S_x)^2 + (S_y)^2 ] D^{(L)}(c_2 \theta/2), \]

\( \phi \) dependence suppressed; \( c_1 + c_2 = 1 \). This square bracket does not commute with the D matrices. No matter how this expression is manipulated, there is \( \alpha \) and \( \theta \) dependence.
Compare this to the standard three dimensional case, where the square bracket is $S \cdot S$ which commutes with the rotation matrices. The standard area operator has no $\alpha$ or $\theta$ dependence. Also, in the planar case, when $E$ grasps an $O(3)$ holonomy at a vertex, there is no $\alpha$ or $\theta$ dependence (which was the motivation for introducing the $O(3)$ formalism in the first place).

Although an exact calculation of area is difficult, it is not too hard to show that transverse areas are given by $(\kappa \gamma / 2) L$ to leading order in sine. Strictly speaking the area operator is defined by its action on the underlying $SU(2)$ holonomies $D^{(j)}_{MN}$, not by its action on the $O(3)$ holonomies $D^{(L)}_{0M}$. However, one can estimate the area by noting

\[
\begin{align*}
D^{(j)}_{MN} &\sim \exp[iA \cdot S(j)]; \\
D^{(L)}_{0M} &\sim \exp[iA \cdot S(L)/2].
\end{align*}
\]

(116)

The $L$ rotation has the same axis as the $j$ rotation, but the rotation angle has half the magnitude, which explains the $1/2$.

Now grasp the $D^{(L)}$ with the area operator. This brings down factors of $1/2$, from eq. (116).

\[
\text{area}^2 \sim (\kappa \gamma)^2 (S_x/2)^2 + (S_y/2)^2.
\]

(117)

There is no $S_z$ because of the gauge fixing. There is no factor $1/2$ in $\kappa \gamma$, because the grasp is away from endpoints. This gives a first rough estimate of the area as

\[
\kappa \gamma \sqrt{L_a(L_a + 1)/4 - M_a^2/4} \approx \kappa \gamma (L_a/2),
\]

I have neglected $M_a$, which is higher order in sine:

\[
< \frac{M_a}{L} > = \hat{p}_Z^a \sim \sin(\alpha_a/2).
\]

Now imagine the coherent state is expanded in a complete set of $SU(2)$ harmonics. The area squared operator gives eq. (117) without the factors of $1/2$, therefore

\[
\text{area} = \kappa \gamma \sqrt{j(j+1)} - M^2 \approx \kappa \gamma j.
\]

This estimate gives the same value as before, because the peak value of $2j$ is $L$. $D^{(j)}$ is constructed from $2j$ copies of $h^{1/2}$; $D^{(L)}$ is constructed from $L$ copies of $h^{1/2}$ (rearranged to form $L$ $Y^{11}$s); peak $2j$ equals peak $L$.  

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This explains the factors of 1/2 in eq. (113). The area is given by the underlying SU(2) harmonics.

Eq. (113) can be used to eliminate the $\Delta x^i$ from eq. (99).

$$
\Delta x / \Delta Z = (\Delta x)^2 / (\Delta x / \Delta Z) = \text{sgn} m_f / (L(\text{avg})/2).
$$

VI Discussion

The $E_A^a(\text{cl})$ determine the traditional metric components $g_{ab}$.

$$
g_{xx} = e^2 / (E_X^x)^2 = \text{sgn} E_Z^y E_Y^y / E_X^x = (1 + \text{order } a^2) (1 - a \sin \cdots) / (1 + a \sin \cdots)
$$

$$
\cong 1 - a \sin(k n - \phi) \exp (-\rho n).
$$

(118)

$g_{yy}$ is identical, except for $a \rightarrow -a$. That sign change implies the usual picture of the gravitational wave as an ellipse with major and minor axes that fluctuate 180 degrees out of phase.

Turning from geometrodynamics to LQG: the presence of area elements in the LQG triads means that fixing the diffeomorphism gauge fixes $\Delta Z / \Delta n$, the metric spacing between vertices. Classical variables can be near unity, yet quantum variables can be far from unity.

In the presence of a gravitational wave, the transverse $\tilde{E}$ do not oscillate in direction, to order $a$. Their direction is given by the $\hat{p}_a$, which remain fixed.

However, the magnitude of each $\tilde{E}$ changes. To order $a$ the two $\tilde{E}$ oscillate around the same average magnitude $L(\text{avg})$, but 180 degrees out of phase.

As for the transverse holonomies, they are parameterized by axes of rotation plus angles of rotation. The axes are fixed by the Immirzi parameter, up to a reflection through the origin.

However, the angles of rotation oscillate. Those angles may be either in phase or 180 degrees out of phase. The two possibilities are physically equivalent, since the phase of either angle may be changed by reflecting the corresponding axis.

This picture is reasonable. The $\tilde{E}$ and angles represent conjugate p’s and q’s. One would expect oscillations of both.
Although this paper used $O(3)$ harmonics $Y_L$ rather than $SU(2)$ harmonics, the two have identical angular behavior, with

$$<L> = <2j>.$$  

The calculations in this paper assumed all $E^i_{cl}$ near $+1$, and $\text{sgn} = +1$. The calculations can be repeated for (say) $E^x_{cl}$ near $-1$, $\text{sgn} = -1$. This changes one formula.

$$\mu_x + \beta_x \rightarrow \pi + \epsilon_x.$$  

The expressions for $\epsilon_a$ and $L_a$ are unchanged. Also, the foregoing qualitative discussion of holonomic and triad oscillations goes through unchanged.

In this paper I am able to set $N = N = 1$. This is possible only because of the unidirectional assumption. In the general case, $N$ and $\overline{N}$ vary with $n$; and they may pass through zero.

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