CONVERGENCE IN LAW FOR COMPLEX GAUSSIAN MULTIPLICATIVE CHAOS IN PHASE III

HUBERT LACOIN

Abstract. Gaussian Multiplicative Chaos (GMC) is informally defined as a random measure $e^{\gamma X} d\mu$ where $X$ is Gaussian field on $\mathbb{R}^d$ (or an open subset of it) whose correlation function is of the form $K(x, y) = \log \frac{1}{|x-y|} + L(x, y)$, where $L$ is a continuous function $x$ and $y$ and $\gamma = \alpha + i\beta$ is a complex parameter. In the present paper, we consider the case $\gamma \in \mathcal{P}_{III}$ where

$$\mathcal{P}_{III} := \{\alpha + i\beta : \alpha, \gamma \in \mathbb{R}, |\alpha| < \sqrt{d/2}, \alpha^2 + \beta^2 \geq d\}.$$  

We prove that if $X$ is replaced by the approximation $X_\varepsilon$ obtained by convolution with a smooth kernel, then $e^{\gamma X_\varepsilon} d\mu$, when properly rescaled, has an explicit non-trivial limit in distribution when $\varepsilon$ goes to zero. This limit does not depend on the specific convolution kernel which is used to define $X_\varepsilon$ and can be described as a complex Gaussian white noise with a random intensity given by a real GMC associated with parameter $2\alpha$.

2010 Mathematics Subject Classification: 60F99, 60G15, 82B99.
Keywords: Random distributions, log-correlated fields, Gaussian Multiplicative.

1. Model and results

1.1. The exponential of a log-correlated field via convolution approximation. Given an open domain $D \subset \mathbb{R}^d$, we consider $K : D^2 \to \mathbb{R}$ to be a positive definite kernel on $D$ which admits a decomposition in the following form

$$K(x, y) := L(x, y) + \log \frac{1}{|x-y|}, \quad (1.1)$$

where $L$ is a continuous function. A kernel $K$ is positive definite if for any bounded continuous function $\rho$ with compact support on $D$

$$\int_{D^2} K(x, y) \rho(x) \rho(y) dx dy \geq 0. \quad (1.2)$$

We want to consider a Gaussian field $X$ with covariance $K$ and find a way to make sense of the distribution $e^{\gamma X(x)} dx$ when $\gamma = \alpha + i\beta$ is a complex parameter. Such an exponentiation a log-correlated field is what is called Gaussian Multiplicative Chaos (GMC) and has been the focus of a large amount of mathematical work first in the case $\gamma \in \mathbb{R}$ (see [20] for a review and references) and more recently $\gamma \in \mathbb{C}$ (see [9, 11, 10, 14, 16, 15] and references therein). The standard way to define GMC is to consider $X$ taking value in the field of distribution and then use an approximation of $X$ and a passage to the limit to define its exponential.

Since $K$ is infinite on the diagonal, it is not possible to define directly a Gaussian field indexed by $D$ with covariance function $K$. We consider thus a distributional Gaussian field, that is, a field indexed by a set of signed measure. Like in [1], we define $\mathcal{M}_K^+$ to be
the set of positive Borel measures on \( D \) such that
\[
\int_{\mathbb{D}^2} |K(x,y)|\mu(dx)\mu(dy) < \infty
\]
and let \( \mathcal{M}_K \) be the space of signed measure spanned by \( \mathcal{M}_K^+ \)
\[
\mathcal{M}_K := \{ \mu_+ - \mu_- : \mu_+, \mu_- \in \mathcal{M}_K^+ \}.
\]
We define \( K \) as the following quadratic form on \( \mathcal{M}_K \)
\[
K(\mu, \mu') = \int_{\mathbb{D}^2} K(x,y)\mu(dx)\mu'(dy),
\]
and finally define \( X = (\langle X, \mu \rangle)_{\mu \in \mathcal{M}_K} \) as the random field indexed by \( \mathcal{M}_K \) with covariance kernel given by \( K \).

The distributional field \( X \) can be approximated by a sequence of functional fields - fields indexed by a subset of \( D \) - by the mean of convolution with smooth kernels (we have to consider a strict subset of \( D \) to avoid boundary effects). Consider \( \theta \) a non-negative \( C^\infty \) function whose compact support is included in \( B_{\mathbb{R}^d}(0,1) \) the \( d \)-dimensional Euclidean ball of radius one, \( \int_{B(0,1)} \theta(x)dx = 1 \). We define for \( \varepsilon > 0 \), \( \theta_\varepsilon := \varepsilon^{-d}\theta(\varepsilon^{-1} \cdot) \) and consider the convoluted version of \( X \) on the set
\[
D_\varepsilon := \{ x \in D : \min_{y \in \mathbb{R}^d \cap \overline{D}} |x - y| \geq 2\varepsilon \}
\]
(or \( D_\varepsilon = \mathbb{R}^d \) convention if \( D = \mathbb{R}^d \)). It is defined by
\[
X_\varepsilon(x) := \langle X, \theta_\varepsilon(x - \cdot) \rangle
\]
where the function \( \theta_\varepsilon(x - \cdot) \) is identified with the measure \( \theta_\varepsilon(y - x)dy \) on \( D \). With this definition one can check that \( X_\varepsilon(x) \) has covariance
\[
K_\varepsilon(x,y) := \mathbb{E}[X_\varepsilon(x)X_\varepsilon(y)] = \int_{\mathbb{R}^{2d}} \theta_\varepsilon(x - z_1)\theta_\varepsilon(y - z_2)K(z_1, z_2)dz_1dz_2.
\]
We simply write \( K_\varepsilon(x) \) when \( x = y \). Note that since \( K_\varepsilon \) is infinitely differentiable, by Kolmogorov’s criterion (see e.g. [17, Theorem 2.9]), there exists a version of \( X_\varepsilon \) which is continuous in \( x \). Considering this version of the field, we can make sense of integrals of measurable functionals of \( X_\varepsilon(\cdot) \). This allows to define, for any \( f \in C_c(D) \) (continuous with a compact support), the quantity
\[
M_\varepsilon^{(\gamma)}(f) := \int_{\mathbb{R}^d} f(x)e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2} K_\varepsilon(x)}1_{D_\varepsilon}(x)dx.
\]
The restriction to \( D_\varepsilon \) ensures that \( K_\varepsilon \) is well defined and uniformly bounded, which is convenient. It disappears when \( \varepsilon \) goes to zero, since for \( \varepsilon \) sufficiently small the support of \( f \) is included in \( D_\varepsilon \). The question of focus in the present paper is the existence of a nontrivial limit of the distribution \( M_\varepsilon^{(\gamma)}(\cdot) \) when \( \varepsilon \) tend to zero (possibly with a rescaling by a factor depending on \( \varepsilon \)). Such a limit gives natural interpretation for the formal distribution \( e^{\gamma X}dx \).
1.2. The case of real GMC. The case when the parameter in the exponentiation is real (in that case we write it as $\alpha$ instead of $\gamma$) has been extensively studied, starting with the work of Kahane [13] (see for instance [1, 5, 21], we refer to the introduction in [1] for a detailed chronological account). These works established that when $\alpha \in (-\sqrt{2d}, \sqrt{2d})$ then $M_\varepsilon^{(\alpha)}$ converges to a non trivial limit. As this result, together its variant Theorem [6] presented in the next section, play a pivotal role in our proof so we state it in full details.

**Theorem A.** [1, Theorem 1.1] For any $\alpha \in \mathbb{R}$ with $|\alpha| < \sqrt{2d}$, then there exists a random distribution $M_0^{(\alpha)}$ such that for every $\theta$ and every continuous $f$ with compact support in $D$, we have the following convergence in $L_1$

$$\lim_{\varepsilon \to 0} M_\varepsilon^{(\alpha)}(f) = M_0^{(\alpha)}(f).$$

The distribution $M_0^{(\alpha)}$ is a locally finite measure whose support is dense in $D$. The limit does not depend on the convolution kernel $\theta$ used in the definition of $M_\varepsilon^{(\alpha)}$.

The condition $|\alpha| < \sqrt{2d}$ is optimal: when $|\alpha| \geq \sqrt{2d}$, then $\lim_{\varepsilon \to 0} M_\varepsilon^{(\alpha)}(f) = 0$ for all $f$. When $\alpha = \pm \sqrt{2d}$, one can still obtain a non-trivial limit by adding a scaling factor: the measure $\sqrt{\log(1/\varepsilon)} M_\varepsilon^{(\pm \sqrt{2d})}(\cdot)$ converges in probability to a positive measure referred to as the critical multiplicative chaos (see [3] for a first derivation [8] for uniqueness of the limit and [19] for an up-to-date review). When $|\alpha| > \sqrt{2d}$, one should still obtain a convergence after an adequate rescaling but the convergence is of a different nature since it only holds in distribution (see [18] for such a result with $X_\varepsilon$ replaced by a martingale sequence of approximation similar to the one considered in Section 2 of the present paper).

1.3. Our main result. Our main theorem concerns the convergence of $M_\varepsilon^{(\gamma)}$ for complex values $\gamma$. More precisely we consider $\gamma$ in the following range of parameters

$$\mathcal{P}_III := \{\alpha + i\beta : \alpha, \beta \in \mathbb{R}, |\alpha| < \sqrt{d/2}, \alpha^2 + \beta^2 \geq d\}. \quad (1.10)$$

We require an assumption on the regularity of the function $L$ present in (1.1) (a condition which is also present for papers investigating the subcritical complex chaos [10, 14] for a similar reasons). Let us recall the definition for the Sobolev space with index $s \in \mathbb{R}$ on $\mathbb{R}^k$ which is the Hilbert space of complex valued function associated with the norm

$$\|\varphi\|_{H^s(\mathbb{R}^k)} := \left(\int_{\mathbb{R}^k} (1 + |\xi|^2)^s |\hat{\varphi}(\xi)|^2d\xi\right)^{1/2}, \quad (1.11)$$

where $\hat{\varphi}(\xi)$ is the Fourier transform of $\varphi$ defined for $\varphi \in C_c^\infty(\mathbb{R}^k)$ by

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^k} e^{i\xi \cdot x} \varphi(x)dx.$$

Now for $U \subset \mathbb{R}^k$ open, the local Sobolev space $H^s_{loc}(U)$ denotes the function which belongs to $H^s(U)$ after multiplication by an arbitrary smooth function with compact support

$$H^s_{loc}(U) := \left\{\varphi : U \to \mathbb{R} \mid \rho \varphi \in H^s(\mathbb{R}^d) \text{ for all } \rho \in C_c^\infty(U)\right\}, \quad (1.12)$$

where above $\rho \varphi$ is identified with its extension by zero on $\mathbb{R}^k$. We are going to assume that the covariance kernel $K$ is of the form (1.1) with $L \in H^s_{loc}(D^2)$ for some exponent $s > d$. 
Before stating the result we need to introduce some notation. Let us define the function \( \ell_\theta \) on \( \mathbb{R}^d \), obtained by convoluting \( z \mapsto \log 1/|z| \) twice with \( \theta \), that is
\[
\ell_\theta(z) := \int_{\mathbb{R}^d} \log \left( \frac{1}{|z + z_1 - z_2|} \right) \theta(z_1)\theta(z_2)dz_1dz_2.
\] (1.13)
and set
\[
v(\epsilon, \theta, \gamma) := \begin{cases} \epsilon^{-|\gamma|^2-2d} \left( \frac{1}{2} \int_{\mathbb{R}^d} e^{\gamma^2|z|^2}dz \right)^{-1/2} & \text{if } |\gamma| > \sqrt{d}, \\
\left( \frac{\epsilon^{-d}}{1/(d/2)} \log \frac{1}{\epsilon} \right)^{-1/2} & \text{if } |\gamma| = \sqrt{d},
\end{cases}
\] (1.14)
(the quantity \( \frac{\epsilon^{-d}}{1/(d/2)} \) corresponds to half of the volume of the \((d-1)\)-dimensional sphere).

Our result establishes that for \( \gamma \in \mathcal{P}_{\text{III}} \) when \( \epsilon \) goes to zero \( v(\epsilon, \theta, \gamma)M_\epsilon(\gamma) \) converges in distribution towards a complex Gaussian white noise, whose intensity is the random positive measure \( M_0^{(2\alpha)}(e^{\gamma^2L}) \), defined by
\[
M_0^{(2\alpha)}(e^{\gamma^2L}g) := \int_{\mathbb{R}^d} e^{\gamma^2L(x)}g(x)M_0^{(2\alpha)}(dx),
\] for \( g \in C_c(D) \). Above, we have used the short-hand notation \( L(x) := L(x,x) \). This convergence holds in \( H_{u}^{-u}(D) \) for any \( u > d/2 \).

**Theorem 1.1.** Let \( X \) be a Gaussian random field on \( D \) whose covariance kernel is of the form \( \mathcal{L} \) with \( L \in H_{u}^{s}(D^2) \) for some \( s > d \). For \( \gamma \in \mathcal{P}_{\text{III}} \), \( u > d/2 \), the distribution \( v(\epsilon, \theta, \gamma)M_\epsilon(\gamma) \) converges in law, for the \( H_{u}^{-u}(D) \) topology, towards a non-trivial limit. The limiting random distribution \( M_0^{(\gamma)} \) is a complex white noise with random intensity given by \( M_0^{(2\alpha)}(e^{L_\cdot}) \).

Let us precise the exact meaning of the expression

“\(A complex white noise with random intensity given by \( M_0^{(2\alpha)}(e^{L_\cdot})\).”\n
As a complex valued distribution, the law of \( M_0^{(\gamma)} \) is entirely characterized entirely by that of \( \Re[e^{-i\omega}M_0^{(\gamma)}(f)] \) for real valued \( f \in C_c^\infty(D) \) and \( \omega \in (0,2\pi] \) (here and in the remainder of the paper \( \Re \) is used to denote the real part of a complex number). The complex white noise of intensity \( M_0^{(2\alpha)}(e^{L_\cdot}) \), is defined as the unique random distribution which satisfies
\[
\forall (f, \omega) \in C_c^\infty(D) \times [0,2\pi), \quad \mathbb{E}[e^{i\omega \Re[e^{-i\omega}M_0^{(\gamma)}(f)]]} = \mathbb{E}[e^{-\frac{\omega^2}{2}}M_0^{(2\alpha)}(e^{Lf^2})].
\] (1.15)

To prove Theorem 1.1 we prove separately the tightness of \( v(\epsilon, \theta, \gamma)M_\epsilon(\gamma) \) in \( H_{u}^{-u}(D) \) if \( u > d/2 \) and the convergence of the finite dimensional marginals. The proof of the tightness result below, while a bit technical, follows a standard approach and for this reason is given in Appendix B.

**Proposition 1.2.** Under the assumptions of Theorem 1.1, given \( \rho \in C_c^\infty(D) \). The random sequence \( (v(\epsilon, \theta, \gamma)M_\epsilon(\gamma)(\rho \cdot))_{\epsilon \in (0,1)} \) is tight in \( H_{u}^{-u}(\mathbb{R}^d) \) for any \( u > d/2 \).

As \( M_\epsilon^{(\gamma)}(\cdot) \) is a distribution, that is, a linear form on a functional space, it is sufficient to check that the law of \( v(\epsilon, \theta, \gamma)M_\epsilon^{(\gamma)}(f) \) converges for every \( f \in C_c^\infty(D) \). In order to use
Lévy’s Theorem, we can consider the Fourier transform of real valued random variables, so that we need to check for the limiting law of \(M_\varepsilon^{(\gamma)}(f, \omega)\) for every \(\omega \in [0, 2\pi]\) where
\[
M_\varepsilon^{(\gamma)}(f, \omega) := \Re \left(e^{-i\omega M_\varepsilon^{(\gamma)}(f)}\right) = \int_{\mathcal{D}_\varepsilon} f(x)e^{\alpha X_\varepsilon(x) + \beta^2 - \alpha^2 K_\varepsilon(x)} \cos(\beta(X_\varepsilon(x) - 2\alpha\beta K_\varepsilon(x) - \omega))dx.
\]
(1.16)

The main technical achievement of the paper is the proof of the following convergence

**Theorem 1.3.** Under the assumption of Theorem 1.1, given \(f \in C_c(D)\) and \(\omega \in [0, 2\pi]\), we have
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[e^{iv(\varepsilon, \theta, \gamma)M_\varepsilon^{(\gamma)}(f, \omega)}\right] = \mathbb{E} \left[e^{-\frac{1}{2}M_0^{(2\alpha)}(e^{L}f^2)}\right].
\]
(1.17)

**Remark 1.4.** Note that as in Theorem A or its extension to the complex case [14] Theorem 2.1 the obtained limit does not depend on the convolution kernel used to define \(X_\varepsilon\) (the scaling factor \(v(\varepsilon, \theta, \gamma)\) does, although not when \(|\gamma| = \sqrt{d})\).

**Remark 1.5.** An important observation is that, in contrast with Theorem A, \(M_\varepsilon^{(\gamma)}\) here does not converge in probability. The convergence in law in Theorem 1.1 is not a limitation of the result. The absence of a limit in probability is illustrated by Remark 2.4.

1.4. A review of results on complex GMC. Let us now try to give some perspective on our results by relating it to the existing literature on complex GMC. The set \(P_{\text{III}}\) in (1.10) corresponds in fact - up to boundary - to one of the phases in a diagram which contains three. Let us introduce these three phases for the needs of the discussion
\[
P_{\text{sub}} := \{\alpha + i\beta : \alpha^2 + \beta^2 < d\} \cup \{\alpha + i\beta : \alpha \in (\sqrt{d}/2, \sqrt{2d}) ; |\alpha| + |\beta| < \sqrt{2d}\},
\]
\[
P_{\text{II}} := \{\alpha + i\beta : |\alpha| + |\beta| > \sqrt{2d} ; |\alpha| > \sqrt{d}/2\},
\]
\[
P_{\text{III}} := \{\alpha + i\beta : \alpha^2 + \beta^2 > d ; |\alpha| < \sqrt{d}/2\}.
\]
(1.18)

This phase diagram appears in in [16] in the context of the study of complex GMC is defined by considering
\[
M_\varepsilon^{(\alpha, \beta)}(dx) := e^{\alpha X_\varepsilon + i\beta Y_\varepsilon - \frac{\alpha^2 - \beta^2}{2} K_\varepsilon(x)} dx
\]
(1.19)

where \(X\) and \(Y\) are independent log-correlated fields of covariance \(K\) and also for related models such as complex Multiplicative Cascades [3] complex Random Energy Model [12] or complex branching Brownian Motion [6, 7]. Each of the phases in (1.18) is conjectured to correspond to a different scaling regime for \(M_\varepsilon^{(\gamma)}\).

The subcritical phase \(P_{\text{sub}}\). When \(\gamma \in P_{\text{sub}}\) is has been proved that \(M_\varepsilon^{(\gamma)}\) converges to a random distribution. More precisely has been proved in [10] (with the assumption \(L \in H^s_{\text{loc}}(D^2)\), for \(s > d\)) that the random distribution \(M_0^{(\alpha)}\) of Theorem A has a unique analytic continuation on the domain \(P_{\text{sub}}\). In [14], it was proved (under the same assumption) that this analytic continuation is the limit of \(M_\varepsilon^{(\gamma)}\) for any choice of convolution kernel \(\theta\), extending Theorem A to the full region \(P_{\text{sub}}\). Convergence of \(M_\varepsilon^{(\alpha, \beta)}\), recall (1.19), for

\[\text{To be completely accurate, the field } X_\varepsilon \text{ considered in [16] is not a convolution but rather a martingale approximation of the type considered in Section 2. This difference is not relevant for the present discussion.}\]
by an appropriate factor. More precisely we have renormalization to consider in that case should be
\[ p \] in the limit. In this case the right scaling should be \( (\log 1/\varepsilon)^{\frac{3d}{2}} \varepsilon^{-d} M_{\varepsilon}^{(\gamma)}(dx) \). This phenomenon is called freezing and has been proved in \([6, 18]\) for the complex exponential of Branching Brownian Motion, but it remains a challenging conjecture for complex GMC (both for \( M_{\varepsilon}^{(\alpha,\beta)} \) and \( M_{\varepsilon}^{(\gamma)} \)).

The glassy phase \( \mathcal{P}_{II} \). When \( \gamma \in \mathcal{P}_{II} \), it is conjectured that in the limit when \( \varepsilon \to 0 \) the distribution of \( M_{\varepsilon}^{(\gamma)} \) is supported by small neighborhood of the points where \( X_{\varepsilon} \) is close to be maximized, yielding an atomic distribution (a countable sum of weighted Dirac masses) in the limit. In this case the right scaling should be \( (\log 1/\varepsilon)^{\frac{3d}{2}} \varepsilon^{\frac{d}{2}} M_{\varepsilon}^{(\gamma)}(dx) \). The third phase \( \mathcal{P}_{III} \). When \( \gamma \in \mathcal{P}_{III} \) the fluctuations of \( X_{\varepsilon} \) makes the phases of \( e^{i\beta X_{\varepsilon}(x)} \) decorrelate even on small scale, and this accounts for the appearance of a white noise appear in the limit. The intensity of the corresponding white noise has to be given by the limit of the square of the modulus of the local variations, that is \( e^{2\gamma X_{\varepsilon}dx} \). The convergence of \( \varepsilon^{|\gamma|^2 - d} M_{\varepsilon}^{(\alpha,\beta)} \) (recall \([1,11]\)) was established in \([16]\). The proof relied on the computation of all the conditional moments of \( M_{\varepsilon}^{(\alpha,\beta)} \) when conditioning w.r.t. to the field \( X \). This approach is heavily relying on the independence of \( X \) and \( Y \) and cannot be adapted to the present context.

In this paper, partly inspired the techniques used in \([15]\) to study the Sine-Gordon model (which corresponds to the case \( \alpha = 0 \)) we take a completely different approach which relies on convergence of martingale brackets after using a martingale decomposition.

1.5. Open questions. Note that our result Theorem \([1,1]\) does not only compute the scaling limit of \( M_{\varepsilon}^{(\gamma)} \) in \( \mathcal{P}_{III} \) but also on the frontier between \( \mathcal{P}_{III} \) and \( \mathcal{P}_{sub} \) (recall the definition of \( \mathcal{P}_{III} \) \([1,10]\)). Let us discuss here shortly what should occur on the rest of the frontier between \( \mathcal{P}_{III} \) and other phases.

To formulate this conjecture, let us introduce the critical real multiplicative chaos, which corresponds to the point \( |\alpha| = \sqrt{2d} \) and \( \beta = 0 \). It has been proved \([4,8]\) that while \( M_{\varepsilon}^{(\pm \sqrt{2d})}(dx) \) converges to zero, we obtain a non trivial limit in probability after rescaling by an appropriate factor. More precisely we have
\[
\lim_{\varepsilon \to 0} (\log 1/\varepsilon)^{1/2} M_{\varepsilon}^{(\pm \sqrt{2d})}(dx) =: M^{(\pm \sqrt{2d})}(dx).
\]

The measure \( M^{(\pm \sqrt{2d})} \) is referred to as the critical multiplicative chaos.

The frontier \( \mathcal{P}_{II}/\mathcal{P}_{III} \). When \( |\alpha| = \sqrt{d/2} \), \( |\beta| > \sqrt{d/2} \), it is natural to conjecture that \( M_{\varepsilon}^{(\gamma)} \) properly renormalized should converges to a white noise whose intensity is given by \( M^{(2\alpha)} \). Such a result has been proved in \([16]\) for the chaos given in \([1,19]\). The proper renormalization to consider in that case should be \( (\log 1/\varepsilon)^{\frac{1}{2}} \varepsilon^{-d/2} M_{\varepsilon}^{(\gamma)} \).

The triple point \( |\alpha| = |\beta| = \sqrt{d/2} \). This should be similar to the \( \mathcal{P}_{II}/\mathcal{P}_{III} \) frontier though possibly more technical to handle. In that case \( (\log 1/\varepsilon)^{-\frac{1}{2}} M_{\varepsilon}^{(\gamma)} \) should converge to a complex white noise with intensity given by \( M^{(2\alpha)} \).
First hints on the organization of the paper. Proposition 1.2 is proved in Appendix B. The proof of Theorem 1.3 which is the main technical achievement of the paper rely on a martingale decomposition of $M^{(\gamma)}$ which is introduced in Section 2. Additional details on this decomposition are needed to describe the remainder of the organization of the paper, a more detailed picture is given in Section 2.6.

**Notation.** Let us list here a few convention adopted in the paper. If $G$ is a generic function of two variables we will write $G(x)$ for $G(x, x)$. If $(J_s)_{s \geq 0}$ is a continuous function or a random process indexed by $s$ we set

$$J_{[a,b]} := J_b - J_a$$

(1.20)

The letter $C$ is used to denote generic positive constants used in the computation. The value of $C$ is allowed to change from one equation to another within the same proof.

2. Decomposition of the proof of Theorem 1.3

2.1. Star-scale invariant kernels. We are going to say that the kernel $K$ has a star-scale invariant part (with kernel $\kappa$) if it can be written in the form

$$K(x, y) = K_0(x, y) + \int_0^\infty \kappa(e^t|x - y|)dt$$

(2.1)

where $K_0(x, y)$ is a bounded Hölder continuous positive definite kernel on $\mathcal{D}$ (recall 1.2) and the function $\kappa : \mathbb{R}_+ \to \mathbb{R}$, satisfies the following assumptions:

1. $\kappa$ is Lipschitz-continuous and non-negative,
2. $\kappa(0) = 1$, $\kappa(r) = 0$ for $r \geq 1$.
3. $(x, y) \mapsto \kappa(|x - y|)$ defines a positive definite function on $\mathbb{R}^d \times \mathbb{R}^d$.

Note that if $K$ satisfies (2.1) then

$$L(x, y) := K(x, y) + \log |x - y|,$$

(2.2)

can be extended to a continuous function on $\mathcal{D}^2$, so that $K$ having a star-scale invariant part implies that it can be written in the form (1.1).

The converse is not true and there are positive definite kernel $K$ of the type given in Equation (1.1) that cannot be decomposed as in (2.1) for any choice of $\kappa$. However if $L \in H^s_{\text{loc}}(\mathcal{D}^2)$ for some $s > d$, then $K$ can be approximated very well by a kernel of the type (2.1). This is the content of the following result (proved in Appendix A).

**Lemma 2.1.** Given $K$ a covariance kernel on $\mathcal{D}$ of the form (1.1) with $L \in H^s_{\text{loc}}(\mathcal{D}^2)$ for $s > d$, $\mathcal{D}'$ a bounded open set whose closure satisfies $\overline{\mathcal{D}'} \subset \mathcal{D}$ and $\delta > 0$, then there exists a kernel $K^{(\delta)}$ of the form (2.1) on $\mathcal{D}'$ such that

(A) For all $x, y \in \mathcal{D}'$, $|K^{(\delta)}(x, y) - K(x, y)| \leq \delta$.

(B) $\Delta^{(\delta)}(x, y) = K^{(\delta)}(x, y) - K(x, y)$ is a positive definite kernel on $\mathcal{D}'$.

In order to prove Theorem 1.3, the most important step is to prove the convergence of $M^{(\gamma)}(f, \omega)$ for a field whose covariance kernel satisfies the assumption in (2.1) and use our approximation Lemma 2.1 to conclude.

**Proposition 2.2.** Given a Gaussian field $X$ defined on $\mathcal{D}$ whose covariance kernel satisfies (2.1), a convolution kernel $\theta$, and $\gamma \in \mathcal{P}_+$, we have for every $f \in C_c(\mathcal{D})$ and $\omega \in [0, 2\pi)$

$$\lim_{\varepsilon \to 0} \mathbb{E} \left[ e^{i\varepsilon (x, \delta, \gamma) M^{(\gamma)}(f, \omega)} \right] = \mathbb{E} \left[ e^{-\frac{1}{2}M^{(2\alpha)}(e^{\gamma}|t|^2f^2)} \right].$$

(2.3)
The proof of Proposition [2.2] is technical and require several steps. We provide a detailed roadmap at the end of this section. Let us first explain how we deduce our main result from it.

2.2. Proving Theorem [1.3] from Proposition [2.2]. Let us fix \( f \in C_c(D) \). We consider a bounded open set \( D' \) which includes the support of \( f \) and which is such that its topological closure satisfies \( \overline{D'} \subset D \). Given \( \delta > 0 \) we let \( K^{(\delta)} \) be a kernel satisfying the assumptions (A) and (B) of Lemma 2.1. We can construct two Gaussian fields \( X \) and \( X^{(\delta)} \) indexed by \( D' \) on the same probability space in such a way that \( Z := X^{(\delta)} - X \) is independent of \( X \) and has covariance \( \Delta^{(\delta)}(x,y) \). Letting \( M^{(\gamma,\delta)}_\varepsilon \) denote the exponential of the convoluted field \( X^{(\delta)}_\varepsilon \) we have from Proposition [2.2]

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ e^{iv(\varepsilon,\theta,\gamma)M^{(\gamma,\delta)}_\varepsilon(f,\omega)} \right] = \mathbb{E} \left[ e^{-M^{(2\alpha,\delta)}_0(\varepsilon|\gamma|^2L^2f^2)} \right].
\]

(2.4)

Now in order to obtain a conclusion for the exponential of the original field \( X \) we are going to show that there exists a constant \( C \) (allowed to depend on \( f, \gamma, \) and on the covariance kernel \( K \)) such that for all \( \varepsilon \) sufficiently small

\[
\left| \mathbb{E} \left[ e^{iv(\varepsilon,\theta,\gamma)M^{(\gamma,\delta)}_\varepsilon(f,\omega)} \right] - \mathbb{E} \left[ e^{iv(\varepsilon,\theta,\gamma)M^{(\gamma)}_\varepsilon(f,\omega)} \right] \right| \leq C\sqrt{\delta},
\]

\[
\left| \mathbb{E} \left[ e^{-M^{(2\alpha,\delta)}_0(\varepsilon|\gamma|^2L^2f^2)} \right] - \mathbb{E} \left[ e^{-M^{(2\alpha,\delta)}_0(\varepsilon|\gamma|^2L^2f^2)} \right] \right| \leq C\sqrt{\delta}
\]

(2.5)

The combination of (2.4) and (2.5) yields

\[
\limsup_{\varepsilon \to 0} \left| \mathbb{E} \left[ e^{iv(\varepsilon,\theta,\gamma)M^{(\gamma)}_\varepsilon(f,\omega)} \right] - \mathbb{E} \left[ e^{-M^{(2\alpha,\delta)}_0(\varepsilon|\gamma|^2L^2f^2)} \right] \right| \leq 2C\sqrt{\delta},
\]

(2.6)

which is sufficient to conclude since \( \delta \) may be chosen arbitrarily small.

Let us now prove (2.5). Using the fact that \( u \mapsto e^{iu} \) and \( u \mapsto e^{-u} \) are Lipshitz function on \( \mathbb{R} \) and \( \mathbb{R}_+ \) respectively (for the first line we also use that \( |\mathbb{R}e(e^{-i\omega z})| \leq |z| \)) we have

\[
\left| \mathbb{E} \left[ e^{iv(\varepsilon,\theta,\gamma)M^{(\gamma,\delta)}_\varepsilon(f,\omega)} - e^{iv(\varepsilon,\theta,\gamma)M^{(\gamma)}_\varepsilon(f,\omega)} \right] \right| \leq v(\varepsilon, \theta, \gamma)\mathbb{E} \left[ \left| M^{(\gamma,\delta)}_\varepsilon(f) - M^{(\gamma)}_\varepsilon(f) \right| \right],
\]

\[
\left| \mathbb{E} \left[ e^{-M^{(2\alpha,\delta)}_0(\varepsilonL^2f^2)} - e^{-M^{(2\alpha,\delta)}_0(\varepsilonL^2f^2)} \right] \right| \leq \mathbb{E} \left[ M^{(2\alpha,\delta)}_0(\varepsilonL^2f^2) - M^{(2\alpha,\delta)}_0(\varepsilonL^2f^2) \right].
\]

(2.7)

To bound the r.h.s. of the first line in (2.7), we rely on Cauchy-Schwarz and evaluate the second moment. Using the independence of \( X \) and \( Z \) and the assumption (B) on \( \Delta^{(\delta)} \) setting

\[
\Delta^{(\delta)}_\varepsilon(x,y) := \int_{(D')^2} \Delta^{(\delta)}(z_1, z_2)\theta_\varepsilon(x-z_1)\theta_\varepsilon(y-z_2)dz_1dz_2.
\]

(2.8)

we obtain that - here we assume that \( \varepsilon \) is sufficiently small so that the support of \( f \) is included in \( D'_\varepsilon \) (recall (1.6))

\[
\mathbb{E} \left[ \left| M^{(\gamma,\delta)}_\varepsilon(f) - M^{(\gamma)}_\varepsilon(f) \right|^2 \right] = \int_{(D')^2} f(x)f(y) \left( \varepsilon \right)^2 \Delta^{(\delta)}_\varepsilon(x,y) - 1 \right) e^{\varepsilon |\gamma|^2K_\varepsilon(x,y)}dxdy
\]

\[
\leq (e^{\varepsilon |\gamma|^2} - 1) \int_{(D')^2} f(x)f(y)e^{\varepsilon |\gamma|^2K_\varepsilon(x,y)}dxdy.
\]

(2.9)
In the second line above we simply used that $\Delta_{\varepsilon}^{(\delta)}(x, y) \leq \delta$ which follows immediately from the assumption (A) in Lemma 2.1. Using the estimate (3.4) for $K_{\varepsilon}$ we obtain that

$$\int_{D^2} f(x)f(y)e^{\gamma^2K_{\varepsilon}(x,y)}dx\,dy \leq C\int_{\mathbb{R}^d} f(x)f(y)(|x-y|\vee \varepsilon)^{-|\gamma|^2}dx\,dy$$

$$\leq e^{C|\gamma|^2}\|f\|^2\int_{\mathbb{R}^d} (|z| \vee \varepsilon)^{-|\gamma|^2}dz. \tag{2.10}$$

The right-hand side is of order $\varepsilon^{-|\gamma|^2}$ if $|\gamma| > \sqrt{d}$ and of order $\log(1/\varepsilon)$ if $|\gamma| = \sqrt{d}$. This allows to conclude from (2.9) that for a constant $C$ which may depend on all parameters but $\delta$ and $\varepsilon$, we have

$$v(\varepsilon, \theta, \gamma)^2E\left[|M_{\varepsilon}^{(\gamma,\delta)}(f) - M_{\varepsilon}^{(\gamma)}(f)|^2\right] \leq C\delta. \tag{2.11}$$

Let us now evaluate the r.h.s. of the second line in (2.7). Since, by Theorem A, $M_{\varepsilon}^{(2\alpha,\delta)}(e^L f^2)$ and $M_{\varepsilon}^{(2\alpha)}(e^L f^2)$ both converge in $L_1$, it is sufficient to prove a bound on

$$E\left[M_{\varepsilon}^{(2\alpha,\delta)}(e^{|\gamma|^2L} f^2) - M_{\varepsilon}^{(2\alpha)}(e^{|\gamma|^2L} f^2)\right]$$

which is uniform in $\varepsilon$. Assuming that the support of $f$ is included in $D_{\varepsilon}$, letting $Z_{\varepsilon}$ denote the field $Z$ convoluted with $\theta_{\varepsilon}$ we have

$$E\left[M_{\varepsilon}^{(2\alpha,\delta)}(e^{|\gamma|^2L} f^2) - M_{\varepsilon}^{(2\alpha)}(e^{|\gamma|^2L} f^2)\right]$$

$$\leq E\left[\int_{D_{\varepsilon}} e^{2\alpha Z_{\varepsilon}(x)-2\alpha^2\Delta_{\varepsilon}^{(\delta)}(x)} - 1 \right] e^{\gamma^2L(x)} f^2(x) M_{\varepsilon}^{(2\alpha)}(dx) \tag{2.12}.$$}

Averaging with respect to $X$, and using the fact that $X$ and $Z$ are independent, we obtain that the l.h.s. in (2.12) is equal to

$$\int_D E\left[\left(e^{2\alpha Z_{\varepsilon}(x)-2\alpha^2\Delta_{\varepsilon}^{(\delta)}(x)} - 1 \right] e^{\gamma^2L(x)} f^2(x)dx \leq \sqrt{e^{4\alpha^2\delta}} - 1 \int_D e^{\gamma^2L(x)} f^2(x)dx \tag{2.13}.$$}

where for the inequality we only used Cauchy-Schwarz together with

$$E\left[\left(e^{2\alpha Z_{\varepsilon}(x)-2\alpha^2\Delta_{\varepsilon}^{(\delta)}(x)} - 1 \right]^2\right] = e^{4\alpha^2\Delta_{\varepsilon}^{(\delta)}(x)} - 1 \leq e^{4\alpha^2\delta} - 1. \tag{2.14}$$

\[ \Box \]

2.3. The martingale approximation of the GMC. Using the assumption (2.1), we can introduce another functional approximation of the field $(X_t)_{t \geq 0}$ besides the convolution approximation. Introducing the notation $Q_t(x,y) := \kappa(\varepsilon^t(x-y))$, we define $(X_t(x))_{x \in D, t \geq 0}$ as the Gaussian field parametrized by $D \times \mathbb{R}_+$ with covariance function

$$E[X_s(x)X_t(y)] = K_0(x,y) + \int_0^{s\wedge t} Q_u(x,y)du =: K_{s\wedge t}(x,y). \tag{2.15}$$

By construction $X_t(\cdot)$ is a martingale for the canonical filtration $(\mathcal{F}_t)$ defined by

$$\mathcal{F}_t := \sigma(X_s, s \in [0,t)). \tag{2.16}$$

Since, from our assumptions on $K_0$ and $\kappa$, $K_t(x,y)$ is Hölder continuous (in space and time) then by Kolmogorov-Čensov Theorem $(X_t(x))_{x \in D, t \geq 0}$ admits a modification which
is space time continuous. On the same probability space, one can define a distributional
field $X$ indexed by $\mathcal{M}_K$ by setting

$$\langle X, \mu \rangle := \lim_{t \to \infty} \int X_t(x) d\mu(x), \quad (2.17)$$

(the convergence holds in $L_2$). The field $X$ has covariance $K$ and $(X_t)$ is thus a sequence
of approximation for $X$. We then consider the following distribution valued martingale by setting for $f \in C_c(\mathcal{D})$

$$M_t^{(\gamma)}(f) := \int_{\mathbb{R}^d} f(x) e^{\gamma X_t(x)} - \frac{2K_t(x)}{f} \, dx. \quad (2.18)$$

The notation $K_t$, $X_t$ and $M_t^{(\gamma)}$ conflicts with $K_\varepsilon$, $X_\varepsilon$ and $M_\varepsilon^{(\gamma)}$ introduced earlier, but we believe this abuse of notation to be harmless for all our purposes: the variable $\varepsilon$ will always that be used for convolution approximation while $t$ and a other latin letters will be used for martingale approximations.

The following result illustrates the fact that the martingale approximation has an effect
which is similar to the convolution approximation. It can be deduced from the proof in [1] and is also a particular case of [14, Remark 2.4].

**Theorem B.** For any $\alpha \in \mathbb{R}$ with $|\alpha| < \sqrt{2d}$, then there exists a random distribution $M_0^{(\alpha)}$

such that for every $\theta$ and every contious $f \in C_c(\mathcal{D})$, we have the following convergence in $L_1$

$$\lim_{t \to \infty} M_t^{(\alpha)}(f) = M_0^{(\alpha)}(f).$$

where the distribution $M_0^{(\alpha)}$ is the same as in Theorem A.

Before proving Proposition 2.2, we are going to prove that $M_t^{(\gamma)}$, once renormalized
also converges to a white noise with intensity $M_0^{(2\alpha)}$. We only prove convergence of finite
dimensional distribution but, as the reader can check, the proof of tightness (Proposition 1.2) needs no adaptation in this case. While this result is not necessary to prove Proposition 2.2, it provides a step of intermediate difficulty and thus serves a didactical purpose. Furthermore Theorem 2.3 presents an interest in itself since it provides further indication
that the scaling limit is universal since it is the same for convolution approximation and
for martingale approximation.

As for the convolution case, we define for $\omega \in [0, 2\pi)$$M_t^{(\gamma)}(f, \omega) := \Re(e^{-i\omega M_t^{(\gamma)}(f)}). \quad (2.19)$$

and set

$$\Psi(t, \kappa, \gamma) := \begin{cases} e^{\frac{d-|\gamma|^2}{2}t} \left( \frac{1}{2} \int_{\mathbb{R}^d} e^{-|\gamma|^2 \ell_\kappa(|z|)} \, dz \right)^{-1/2} & \text{if } |\gamma| > \sqrt{d}, \\
\left( \frac{\delta_t}{(d/2)} \right)^{-1/2} & \text{if } |\gamma| = \sqrt{d}, \end{cases} \quad (2.20)$$

where

$$\ell_\kappa(r) := \int_0^\infty \kappa(e^{-t}r) - \kappa(e^{-t}) \, dt. \quad (2.21)$$

**Theorem 2.3.** Let $X$ a Gaussian field on $\mathcal{D}$ whose covariance kernel satisfies (2.1), and

$\gamma \in \mathcal{P}_III$. We have for every $f \in C_c(\mathcal{D})$ and $\omega \in [0, 2\pi)$

$$\lim_{t \to \infty} \mathbb{E} \left[ e^{i\Psi(t, \kappa, \gamma) M_t^{(\gamma)}(f, \omega)} \right] = \mathbb{E} \left[ e^{-\frac{1}{2} M_0^{(2\alpha)}(e^{i\gamma^2 L_f^2})} \right]. \quad (2.22)$$
The function $L$ appearing in the theorem above is simply defined by (1.1). Prefer to use $L$ instead of $K_0$ in the result since the latter depends on the specific choice which is made for $\kappa$ while $L$ is entirely determined by the kernel $K$. Since we have

$$L(x,y) := K_0(x,y) + \int_0^\infty \kappa(e^t|x-y|)dt - \log \frac{1}{|x-y|},$$

looking at the value of the difference of the two last terms near the diagonal, we obtain that $L(x)$ and $K_0(x)$ only differ by a constant

$$L(x) = K_0(x) - \int_0^\infty (1 - \kappa(e^{-t})) dt.$$  

We introduce the notation

$$j_\kappa := \int_0^\infty (1 - \kappa(e^{-t})) dt.$$  

The proof of Theorem 2.3 relies on a very simple strategy. We need to prove that the quadratic variation of $pM^{(\gamma)}_t$ (recall that this is a continuous martingale with respect to the filtration $(\mathcal{F}_t)$) satisfies the law of large number given in the following proposition, and then use a martingale Central Limit Theorem which we introduce in the next section.

**Proposition 2.4.** Under the assumption of Theorem 2.3 setting $N_t := M^{(\gamma)}_t(f, \omega)$ we have

$$\lim_{t \to \infty} \mathbb{E} \left[ e^{i\xi \langle N \rangle_t} \right] = \mathbb{E} \left[ e^{-\frac{1}{2} \xi^2 Z} \right].$$

**2.4. A martingale CLT with a random variance.** One of the key ideas our proof is the use of the following Central Limit Theorem with a random variance. While this is a quite natural result, we could not find it stated or proved in the existing literature. We present it here in a more general framework than required for our proof of Theorem 2.3 since it could have applications in other contexts.

We consider $(\mathcal{F}_t)$ to be a continuous filtration, $(N_t)$ be a continuous local martingale for the filtration $(\mathcal{F}_t)$, $Z$ a non-negative random variable and $v : \mathbb{R}_+ \to \mathbb{R}$ be a non-increasing positive continuous function such that $\lim_{t \to \infty} v(t) = 0$.

**Theorem 2.5.** If the quadratic variation of $N$ satisfies the following convergence in probability

$$\lim_{t \to \infty} v(t)^2 \langle N \rangle_t = Z$$

then we have for every $\xi \in \mathbb{R}$

$$\lim_{t \to \infty} \mathbb{E} \left[ e^{i\xi \langle N \rangle_t} / v(t)^2 \right] = \mathbb{E} \left[ e^{-\frac{1}{2} \xi^2 Z} \right].$$

In other words we have the following convergence in distribution

$$N_t / v(t) \xrightarrow{t \to \infty} \sqrt{Z} \times N$$

where $N$ is a standard normal variable independent of $Z$.

**Remark 2.6.** Our proof only uses marginally the fact that $N$ is a continuous process and the result can most likely be extended to the case of càdlàg local martingales with an adequate assumption on the size of the jumps.

**Proof of Theorem 2.5.** It follows directly from the combination of Proposition 2.4 and Theorem 2.5. □
Remark 2.7. In fact Theorem 2.3 could also be extended to the proof of some invariance principle. Let us consider some particular cases to simplify the exposition. When \( v(t) = 1/\sqrt{t} \) we have

\[
\left( \frac{N_{st}}{\sqrt{t}} \right)_{s \geq 0} \Rightarrow \sqrt{Z}(B_s)_{s \geq 0}
\]  

(2.30)

where \( B \) is a standard Brownian motion independent of \( Z \). In the case where \( v(t) = e^{-\alpha t/2} \) \((\alpha > 0)\) we have

\[
\left( e^{-\alpha t/2} N_{t+s} \right)_{s \geq 0} \Rightarrow \sqrt{Z}(X_s)_{s \geq 0}
\]  

(2.31)

converges to an Ornstein Uhlenbeck process of covariance \( \mathbb{E}[X_sX_t] = e^{-\alpha |t-s|} \), independent of \( Z \). These two examples imply that under the assumption of Theorem 2.3, \( M_t^{(\gamma)}(f) \) does not converge in probability.

2.5. Roadmap to prove Proposition 2.2. To prove Proposition 2.2, we are going to follow some of the same ideas used the proof of Theorem 2.3. We consider a martingale indexed by \( t \) whose limit is given by \( M_t^{(\gamma)} \). For this we consider \( (X_{t,\varepsilon})_{t \in D} \) (recall (1.6)) defined by

\[
X_{t,\varepsilon}(x) = \int_{\mathbb{R}^d} \theta_\varepsilon(x - z)X_t(z)dz.
\]  

(2.32)

We let \( K_{t,\varepsilon} \) denote the covariance of \( X_{t,\varepsilon} \)

\[
K_{t,\varepsilon}(x,y) := \int_{(\mathbb{R}^d)^2} \theta_\varepsilon(x - z_1)\theta_\varepsilon(y - z_2)K_t(z_1, z_2)dz_1dz_2
\]  

(2.33)

Note that if \( X \) is the field defined by (2.17) then we have

\[
\lim_{t \to \infty} X_{t,\varepsilon}(x) = X_\varepsilon(x) \quad \text{and} \quad \mathbb{E}[X_\varepsilon(x) \mid \mathcal{F}_t] = X_\varepsilon(x).
\]

Given \( \gamma \in \mathcal{P}_\text{III} \), \( f \in C_c(\mathbb{D}) \) and \( \omega \in [0,2\pi) \), we define

\[
N_t^{(\varepsilon)} = \int_{\mathbb{D}} f(x)e^{\alpha X_{t,\varepsilon}(x)} \cos(\beta X_{t,\varepsilon}(x) - \alpha \beta K_{t,\varepsilon}(x) - \omega)dx.
\]  

(2.34)

Note that

\[
N_t^{(\varepsilon)} = \mathbb{E}[M_t^{(\gamma)}(f, \omega) \mid \mathcal{F}_t] \quad \text{and} \quad \lim_{t \to \infty} N_t^{(\varepsilon)} = M_t^{(\gamma)}(f, \omega).
\]  

(2.35)

Instead of the asymptotic when of the quadratic variation when \( t \) tends to infinity like in Proposition 2.4, we must prove a similar statement about \( \langle N_t^{(\varepsilon)} \rangle \) when \( \varepsilon \) tends to 0.

Proposition 2.8. Under the assumption of Theorem 2.3 and with \( N_t^{(\varepsilon)} \)

\[
\lim_{\varepsilon \to 0} v(\varepsilon, \theta, \gamma)^2 \langle N_t^{(\varepsilon)} \rangle_{\infty} = M_0^{(2\alpha)}(e^{-|\gamma|^2}Lf^2).
\]  

(2.36)

We cannot deduce Theorem 2.3 from Proposition 2.8 using Theorem 2.5 but we will be using similar ideas to conclude

2.6. Organization of the paper.

- In Section 3 we introduce a couple of classical tools and technical estimates that are used throughout the paper.
- Sections 4 and 5 are devoted to the proof of Theorem 2.3. More precisely Theorem 2.3 is proved in Section 4 and then in Section 5 we prove Proposition 2.4.
• Sections 6 and 7 are devoted to the proof of Proposition 2.2. The proof are a bit more technical than those concerning Theorem 2.3 but follow the same main ideas. In Section 6, we show how Proposition 2.2 can be deduced from Proposition 2.8 while in Section 7 we prove Proposition 2.8.

• Two technical results are proved in appendices. Lemma 2.1 is proved in Appendix A, while Proposition 1.2 is proved in Appendix B.

3. Technical preliminaries

3.1. Gaussian tools. Before starting the proof, let us mention one inequality and one identity that will be used repeatedly. First, for any \( \sigma > 0 \) and \( t \geq 0 \), we have

\[
\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2}} du \leq e^{-\frac{t^2}{2\sigma}}. \tag{3.1}
\]

We refer to this inequality as the Gaussian tail bound. Our second tool is the Cameron-Martin formula. Let \((Y(z))_{z \in Z}\) be an arbitrary centered Gaussian field indexed by an arbitrary set \(Z\). We let \(\mathbf{H}\) denote its covariance and \(\mathbb{P}\) denote its law. For any \(z \in Z\) let us define \(\mathbb{P}_z\) the measure tilted by \(Y(z)\).

\[
\frac{d\mathbb{P}_z}{d\mathbb{P}} := e^{Y(z) - \frac{1}{2}H(z,z)}. \tag{3.2}
\]

**Proposition 3.1.** Under the probability law \(\mathbb{P}_z\), \(Y\) is a Gaussian field with covariance \(\mathbf{H}\), with mean value equal to

\[
\mathbb{E}_z[Y(z')] = H(z, z'). \tag{3.3}
\]

3.2. Estimates for the covariance kernels. We are going to list a collection of useful estimates for the kernel \(K_\varepsilon, K_t, K_{t,\varepsilon}\) defined in (1.8), (2.15) and (2.33) (under the most general assumption (1.1) for the \(K_\varepsilon\) and (2.1) for the others).

First note that if we fix a compact in \(K \subset D\), then there exists a constant \(C\) (depending on \(K, \mathcal{K}, \theta\) and \(\kappa\)) such that for every \(t \geq 0\) and every \(\varepsilon\) such that \(D_\varepsilon \subset \mathcal{K}\), every \(x, y\) in \(K\)

\[
\begin{align*}
|K_t(x, y) - \log \left( \frac{1}{|x - y|} \vee e^{-t} \right) | & \leq C, \\
|K_\varepsilon(x, y) - \log \left( \frac{1}{|x - y|} \right) | & \leq C, \\
|K_{t,\varepsilon}(x, y) - \log \left( \frac{1}{|x - y| \vee \varepsilon} \vee e^{-t} \right) | & \leq C.
\end{align*} \tag{3.4}
\]

The estimates above can be proved by hand using the definitions and are left to the reader. Secondly, let us give some estimate concerning the local regularity of the Kernel near the diagonal.

**Lemma 3.2.** There exists a function \(\eta : \mathbb{R}_+ \to \mathbb{R}_+\) such that \(\lim_{a \to 0} \eta(a) = 0\) and a constant \(C\) which is such that for every \(x, y \in K\), \(t \geq 0\) and every \(\varepsilon > 0\) such that \(\mathcal{K} \subset D_\varepsilon\),

\[
\begin{align*}
|K_t(x, y) - K_t(x, x)| & \leq \eta(|x - y|) + Ce^t|x - y|, \\
|K_{t,\varepsilon}(x, y) - K_{t,\varepsilon}(x, x)| & \leq \eta(|x - y|) + Ce^t|x - y|.
\end{align*} \tag{3.5}
\]
Proof. We let \( \eta \) be the modulus of continuity of \( K_0 \) restricted to \( K \). Then we have by definition
\[
|K_0(x, y) - K_0(x, x)| \leq \eta(|x - y|) \quad \text{and} \quad |K_{0,\varepsilon}(x, y) - K_{0,\varepsilon}(x, x)| \leq \eta(|x - y|)
\]
We can thus assume that \( K_0 \equiv 0 \). In that case the inequality
\[
|K_t(x, y) - K_t(x, x)| \leq C\varepsilon|x - y|
\]
follows from the fact that \( \kappa \) is Lipshitz, and this also holds after convolution.

\[\square\]

4. PROOF OF THEOREM 2.5

Without loss of generality, we assume to simplify notation that \( \xi = 1 \). Given \( t \) we fix \( u(t) \in [0, t] \) in such a fashion that asymptotically \( v(t) \ll v(u) \ll 1 \). To fix ideas we choose
\[
u(t) := \inf \{ s : v(s) = \sqrt{v(t)} \}.
\]
We introduce the stopping time
\[
T = T(t, A) := \inf \{ s > 0 : v(t)^2 \langle N \rangle_s \geq A \}.
\]
We also assume that \( Z > 0 \) with positive probability (the proof of the case \( Z \equiv 0 \) is only simpler). We set \( Z_t^A := E[Z \wedge A \mid \mathcal{F}_u] \). Note that by definition \( Z \) (cf. (2.27)) is measurable with respect to \( \mathcal{F}_u \) so that the following convergence holds in \( L_1 \),
\[
\lim_{t \to \infty} Z_t^A = Z \wedge A.
\]
We have (recall the notation (1.20))
\[
\begin{align*}
&\left| E \left[ e^{iv(t) N_t} - e^{-\frac{1}{2} Z} \right] \right| \\
&\leq \left| E \left[ e^{iv(t) N_t} - e^{iv(t) N_{[u,T]}^A} \right] \right| + \left| E \left[ e^{iv(t) N_{[u,T]}^A} - e^{-\frac{1}{2} Z^A} \right] \right| + \left| E \left[ e^{\frac{1}{2} Z^A} - e^{-\frac{1}{2} Z} \right] \right| \\
&= E_1(t, A) + E_2(t, A) + E_3(t, A).
\end{align*}
\]
We are going to show that for \( i \in \{ 1, 2, 3 \} \)
\[
\lim_{A \to \infty} \limsup_{t \to \infty} E_i(t, A) = 0.
\]
The case of \( E_3 \) is the simplest, by dominated convergence we have
\[
\lim_{t \to \infty} E \left[ e^{-\frac{1}{2} Z^A_{[u,T]}^A} \right] = E \left[ e^{-\frac{1}{2} (Z \wedge A)} \right]
\]
and, also by dominated convergence, the r.h.s. tends to zero when \( A \) tends to infinity. Let us move to the case of \( E_1 \). We have
\[
E \left[ e^{iv(t) N_t} - e^{iv(t) N_{[u,T]}} \right] \leq 2P[T < t] + E \left[ e^{iv(t) N_{[u,T]}} - 1 \indic{T \geq t} \right].
\]
We have \( P[T < t] \leq P[v(t)^2 \langle N \rangle_t \geq A] \) and thus, as by assumption \( v(t)^2 \langle N \rangle_t \) tends to \( Z \) we have
\[
\limsup_{t \to \infty} P[T(t, A) < t] \leq P[Z \geq A]
\]
To see that when \( t \) goes to infinity, the second term in the r.h.s. of (4.7) tends to zero, we only need to check that \( v(t) N_{u,T} \) tends to zero in probability. We have
\[
E \left[ v(t)^2 (N_{u,T} - N_0)^2 \right] = E \left[ v(t)^2 \langle N \rangle_{u,T} \right].
\]
The quantity inside the expectation is bounded above by $A$ and tends to zero in probability (here we are using that $v(u) \ll v(t)$) so that by dominated convergence
\[
\lim_{t \to \infty} \mathbb{E} \left[ v(t)^2 (N_u - N_0)^2 \right] = 0.
\]
This concludes the proof of (4.5) for $i = 1$. Let us finally control $E_2$. Using the martingale exponentiation - $(N_{[u \wedge T, t \wedge T]} | t \geq u)$ is a martingale with quadratic variation bounded by $A v(t)^{-2}$ - we have
\[
\mathbb{E} \left[ e^{iv(t)N_{[u \wedge T, t \wedge T]}} \right] = 1
\]
and thus
\[
\mathbb{E} \left[ e^{-\frac{1}{2} Z_u^A} \right] = \mathbb{E} \left[ e^{iv(t)N_{[u \wedge T, t \wedge T]}} \right].
\]
and thus we have from Jensen’s inequality
\[
\left| \mathbb{E} \left[ e^{iv(t)N_{[u \wedge T, t \wedge T]}} - e^{-\frac{1}{2} Z_u^A} \right] \right| = \left| \mathbb{E} \left[ e^{iv(t)N_{[u \wedge T, t \wedge T]}} \left( 1 - e^{\frac{1}{2} v(t)^2 (N_{[u \wedge T, t \wedge T]} - Z_u^A) \right) \right] \right|
\]
\[
\leq \mathbb{E} \left[ \left| 1 - e^{\frac{1}{2} v(t)^2 (N_{[u \wedge T, t \wedge T]} - Z_u^A) \right| \right].
\]
Now combining the assumption that $v(t)^2 \langle N \rangle_t$ converges to $Z$ with our definition of $u$ and $T$ and (4.3) we have the following convergence in $L_1$
\[
\lim_{t \to \infty} v(t)^2 \langle N \rangle_{[u \wedge T, t \wedge T]} - Z_u^A = 0.
\]
Taking the limit (using dominated convergence) in the inequality (4.12), we obtain
\[
\lim_{t \to \infty} E_2(t, A) = 0.
\]
which concludes the proof of (4.5) for $i = 2$, and thus that of the theorem. □

5. PROOF OF PROPOSITION 2.4

Our proof is simply based on an explicit computation of the quadratic variation using Itô calculus. This computation is more convenient (for notation) when considering the complex valued martingale
\[
W_t := M_t^{(\gamma)} (f) = \int_D e^{\gamma \xi_t(x)} - \frac{\gamma^2}{2} K_t(x) f(x) dx.
\]
The bracket of $M_t^{(\gamma)} (f, \omega)$ can then easily be expressed in terms of $\langle W, \overline{W} \rangle_t$ and $\langle W, W \rangle_t$. Informally we are going to prove that for large $t$
\[
d\langle W, \overline{W} \rangle_t \approx C e^{\frac{1}{2} t^2 - d M_0^{(2a)}} e^{\frac{1}{2} t^2 f^2} dt
\]
and that $d\langle W, W \rangle_t$ is of a much smaller order. We then simply integrate that inequality in $t$ to obtain the required asymptotics.
5.1. **Computing the quadratic variation.** We fix $\gamma \in \mathcal{P}_\text{III}$ and $f \in C_c(\mathbb{R}^d)$ be fixed. We define the complex martingale $W_t$ by

$$W_t := M_t^{(\gamma)}(f) = \int_D e^{\gamma X_t(x)} - \frac{\gamma^2}{2} K_t(x) f(x) \, dx$$

Now note that for $x \in D$, $(X_t(x))_{t \geq 0}$ is a continuous time martingale. Using Itô calculus we have

$$dW_t := \gamma \int_D e^{\gamma X_t(x)} - \frac{\gamma^2}{2} K_t(x) f(x) \, dx \, dx.$$  \hspace{1cm} (5.1)

Now by construction we have $d\langle X(x), X(y) \rangle_t = Q_t(x, y)$ and we obtain the following expressions for the brackets $\langle W, W \rangle_t$ and $\langle W, W \rangle_t$

$$\langle W, W \rangle_t = |\gamma|^2 \int_0^t A_s \, ds \quad \text{and} \quad \langle W, W \rangle_t = \gamma^2 \int_0^t B_s \, ds,$$  \hspace{1cm} (5.2)

where $A_s$ and $B_s$ are defined by

$$A_s := \int_D f(x)f(y)Q_s(x, y)e^{\gamma X_s(x) + \gamma X_s(y)}(\beta^2 - \alpha^2)K_s(x) \, dx \, dy,$$

$$B_s := \int_D f(x)f(y)Q_s(x, y)e^{\gamma X_s(x) + \gamma X_s(y)} - \gamma^2 K_s(x) \, dx \, dy.$$  \hspace{1cm} (5.3)

The core of our proof is to show that $A_s$ and $B_s$ properly rescaled converge to $M_0^{(2\alpha)}$ and 0 respectively. Let us define $\hat{K}_s = K_s - K_0$ or

$$\hat{K}_s(x, y) := \int_0^s \kappa(e^t|x-y|) \, dt$$  \hspace{1cm} (5.4)

and set

$$a(s, \kappa, \gamma) := \int_{\mathbb{R}^2} Q_s(0, z)e^{\gamma^2 \hat{K}_s(0, z)} \, dz.$$  \hspace{1cm} (5.5)

**Proposition 5.1.** We have

$$\lim_{t \to \infty} a(t, \kappa, \gamma)^{-1} A_t = M_0^{(2\alpha)}(e^{\gamma^2|K_0|^2}),$$

$$\lim_{t \to \infty} a(t, \kappa, \gamma)^{-1} B_t = 0.$$  \hspace{1cm} (5.6)

The proof of Proposition 5.1 is detailed in the next subsection and requires several technical steps. Let us show first that it implies Proposition 2.4.

**Proof of Proposition 2.4 using Proposition 5.1.** First let us show that

$$\lim_{t \to \infty} \pi(t, \kappa, \gamma)^2 \gamma^2 \int_0^t a(s, \kappa, \gamma) \, ds = 2e^{-|\gamma|^2}\kappa.$$  \hspace{1cm} (5.7)

We have

$$\int_0^t |\gamma|^2a(s, \kappa, \gamma) \, ds = \int_{|z| \leq 1} \left(e^{\gamma^2 \hat{K}_s(0, z)} - 1\right) \, dz$$

$$= e^{(|\gamma|^2 - d)t} \int_{|y| \leq \epsilon t} \left(e^{-|\gamma|^2 \int_0^t (1 - \kappa(e^{-s}y)) \, ds} - e^{-|\gamma|^2t}\right) \, dy.$$
where the last inequality is obtained using the change of variable $y = e^t z$. When $|\gamma| > \sqrt{d}$ the $e^{-|\gamma|^2 t}$ term, once integrated tend to zero. Now we have
\[
\lim_{t \to \infty} \int_0^t (1 - \kappa(e^{-s}y))ds = \ell_\kappa(y) + j_\kappa, \tag{5.8}
\]
and when $|y| \leq e^t$ we have
\[
\int_0^t (1 - \kappa(e^{-s}y))ds \geq (\log |y| - C_\gamma). \tag{5.9}
\]
With (5.8) and (5.9) we use dominated convergence and obtain
\[
\lim_{t \to \infty} \int_{\mathbb{R}^d} e^{-|\gamma|^2 t} (1 - \kappa(e^{-s}y))ds \mathbb{1}_{\{|y| \leq e^t\}}dy = e^{-|\gamma|^2 t} \ell_\kappa(y)dy, \tag{5.10}
\]
so that (5.6) holds. When $|\gamma| = \sqrt{d}$, considering the first equality in (5.7), we disregard the $-1$ in the integral which only yields a contribution of constant order. Observe that we have $\tilde{K}_t(0, z) \leq t$ for all $z$, and that whenever $|z| \in [e^{-t}, 1]$
\[
\tilde{K}_t(0, z) = \log \frac{1}{|z|} + \int_0^{\log 1/|z|} (\kappa(e^t|z|) - 1)dz = \log \frac{1}{|z|} - j_\kappa \tag{5.11}
\]
This yields
\[
\int_{|z| \leq 1} e^{d\tilde{K}_t(0, z)}dz = e^{-j_\kappa t} \int_{|z| \in [e^{-t}, 1]} |z|^{-d}dz = \int_{|z| \leq e^{-t}} e^{d\tilde{K}_t(0, z)}dz. \tag{5.12}
\]
The first term is exactly $e^{-j_\kappa t} 2a_d^{\frac{d+2}{2}}$ (the last factor being the volume of the $d - 1$ dimensional sphere) while the other term is of order one which yields (5.6) in that case too.

The quadratic variation of $N_t = M_t(\gamma)(f, \omega)$ is a linear combination of $\langle W, \overline{W} \rangle_t$ and $\langle W, W \rangle_t$ we have
\[
\langle N \rangle_t = \frac{1}{2} \langle W, \overline{W} \rangle_t + \frac{1}{2} \Re(e^{-2i\omega} \langle W, \overline{W} \rangle_t) \tag{5.13}
\]
In view of (5.6)-(5.13) and (2.24), we only need to prove the following convergence in $L_1$
\[
\lim_{t \to \infty} \int_0^t \langle W, \overline{W} \rangle_t \|\gamma^2 a(s, \kappa, \gamma)\|ds = M_0(2\alpha) (\gamma^2 K_0 f^2) \quad \text{and} \quad \lim_{t \to \infty} \int_0^t \langle W, W \rangle_t \|\gamma^2 a(s, \kappa, \gamma)\|ds = 0, \tag{5.14}
\]
As a direct consequence of (5.2) and (5.5) we have
\[
\lim_{t \to \infty} \int_0^t \langle W, \overline{W} \rangle_t \|\gamma^2 a(s, \kappa, \gamma)\|ds = M_0(2\alpha) (\gamma^2 K_0 f^2) \quad \text{and} \quad \lim_{t \to \infty} \int_0^t \langle W, W \rangle_t \|\gamma^2 a(s, \kappa, \gamma)\|ds = 0. \tag{5.15}
\]
To conclude we need only to check that both $\int_0^t \|\gamma^2 a(s, \kappa, \gamma)\|ds$ and $\langle W, \overline{W} \rangle_t \sqrt{t}$ are negligible with respect to $\int_0^t \|\gamma^2 a(s, \kappa, \gamma)\|ds$. For the first quantity, it is sufficient to observe that from definition and (3.4), $a(s, \kappa, \gamma)$ is of order $e^{|\gamma|^2 - d}s$. As for the second, we have
\[
\mathbb{E} \left[ \langle W, \overline{W} \rangle_u \right] = \mathbb{E} \left[ \|W_u - W_0\|^2 \right] \leq \int_{\mathbb{D}^2} f(x)f(y)e^{|\gamma|^2 K_0(x,y)}dxdy \leq e^{|\gamma|^2 K_0} \|f\|^2 \int_{\mathbb{R}^d} e^{|\gamma|^2 K_0(0, z)}dz, \tag{5.16}
\]
Now $\int_{\mathbb{R}^d} e^{|\gamma|^2 \tilde{K}_u(0, z)}dz$ is either of order $u$ (if $|\gamma|^2 = d$) or $e^{|\gamma|^2 - d}u$ (if $|\gamma|^2 > d$) and with $u = \sqrt{t}$ this is negligible w.r.t. $\int_0^t \|\gamma^2 a(s, \kappa, \gamma)\|ds$ in all cases.
5.2. Proof of Proposition 5.1 For the sake of simplicity we are going to assume (recall (2.21)) that $K_0 \equiv 0$. This does not alter the proof at all but provides some welcome simplification in the notation (for instance we have $K_t(x) := K_t(x, x) = t$ for every $x$). Since in that case $K$ is translation invariant, we can extend it to a kernel $\mathbb{R}^d \times \mathbb{R}^d$, thus without loss of generality we are going to assume that $D = \mathbb{R}^d$. In this case note that $L(x, x) = -j_x$ for all $x \in \mathbb{R}^d$. To alleviate further the notation we write $a(t)$ for $a(t, \kappa, \gamma)$. We set for this proof $u := u_t = t - \log t$. We are going to show the result using intermediate steps. Recalling the notation (1.20) we set

$$A_t^{(1)} := \int_{\mathbb{R}^{2d}} f^2(x) Q_t(x, y) e^{\gamma X_t(x) + \gamma X_t(y) + (\beta^2 - \alpha^2)t} dxdy,$$

$$A_t^{(2)} := \int_{\mathbb{R}^{2d}} f^2(x) Q_t(x, y) e^{\gamma X_u(x) + \gamma X_u(y) + \gamma^2 K_u(x, y) + (\beta^2 - \alpha^2)u} dxdy$$

Using the triangle inequality we have

$$|A_t - a(t) M_0^{(2\alpha)} (f^2)| \leq |A_t - A_t^{(1)}| + |A_t^{(1)} - A_t^{(2)}| + |A_t^{(2)} - a(t) M_0^{(2\alpha)} (f^2)| + a(t) |M_0^{(2\alpha)} (f^2) - M_0^{(2\alpha)} (f^2)|. \quad (5.18)$$

We are going to show that the expectation of each of the terms of the right-hand side are $o(e(\gamma^2 - \beta^2 \kappa t))$ (which is the same as $o(a(t))$). The fourth term is controlled by applying Theorem [3]. The first three terms require more work.

Step 1: Bounding $\mathbb{E}[|A_t - A_t^{(1)}|]$. We let $w_f(u) := \max_{|x_1 - x_2| \leq u} f(x_1 - x_2)$ denote the modulus of continuity of $f$. We have

$$|A_t - A_t^{(1)}| \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x)||f(y) - f(x)| Q_t(x, y) e^{\alpha (X_t(x) + X_t(y)) + (\beta^2 - \alpha^2) t} dxdy \quad (5.19)$$

and hence

$$\mathbb{E} \left[ |A_t - A_t^{(1)}| \right] \leq w_f(e^{-t}) \int_{\mathbb{R}^d} |f(x)| \mathbf{1}_{\{|y - x| \leq e^{-t}\}} e^{\gamma |x|^2} dxdy$$

$$= \frac{\pi^{d/2}}{\Gamma \left( \frac{d}{2} + 1 \right)} e^{(\gamma^2 - \beta^2) t} w_f(e^{-t}) \int_{\mathbb{R}^d} |f(x)| dx. \quad (5.20)$$

This implies that

$$\lim_{t \to \infty} e^{(\gamma^2 - \beta^2) t} \mathbb{E} \left[ |A_t - A_t^{(1)}| \right] = 0. \quad (5.21)$$

Step 1: Bounding $\mathbb{E}[|A_t^{(1)} - A_t^{(2)}|]$. Now let us consider the second term $A_t^{(2)} - A_t^{(1)}$. This is the most delicate step. Let us set

$$\xi_t(x, y) := e^{\gamma X_t(x) + \gamma X_t(y) + (\beta^2 - \alpha^2)t} - \mathbb{E} [e^{\gamma X_t(x) + \gamma X_t(y) + (\beta^2 - \alpha^2)t} | \mathcal{F}_u],$$

$$\zeta_t(x, y) := Q_t(x, y) f(x)^2 \xi_t(x, y)$$

we have

$$A_t^{(1)} - A_t^{(2)} = \int_{\mathbb{R}^{2d}} \zeta_t(x, y) dxdy \quad (5.23)$$
Our method to bound $A_t^{(1)} - A_t^{(2)}$ depends on whether $\alpha \in [0, \sqrt{d}/2)$ or $\alpha \in [\sqrt{d}/2, \sqrt{d}/2)$ (by symmetry we can assume without loss of generality that $\alpha \geq 0$). When $\alpha < \sqrt{d}/2$ it is sufficient to compute the second moment of $A_t^{(2)} - A_t^{(1)}$. We have

$$\mathbb{E} \left[ |A_t^{(1)} - A_t^{(2)}|^2 \right] = \int_{\mathbb{R}^d} \mathbb{E} \left[ \xi_t(x_1, y_1) \xi_t(x_2, y_2) \right] dx_1 dx_2 dy_1 dy_2.$$  \hspace{1cm} (5.24)

We have for any $x_1, x_2, y_1, y_2$

$$| \mathbb{E} \left[ \xi_t(x_1, y_1) \xi_t(x_2, y_2) \right] | \leq \frac{1}{2} \left( \mathbb{E} \left[ |\xi_t(x_1, y_1)|^2 \right] + \mathbb{E} \left[ |\xi_t(x_2, y_2)|^2 \right] \right) \leq e^{(6\alpha^2 + 2\beta^2)t}. $$ \hspace{1cm} (5.25)

Now by construction, whenever $|x_1 - x_2| \geq e^{-u} + 2e^{-t}$, we have

$$\mathbb{E} [\xi_t(x_1, y_1) \xi_t(x_2, y_2) | \mathcal{F}_u] = 0,$$ \hspace{1cm} (5.26)

since at distance $e^{-u}$ the correlation of the field $X_{[u,t]}$ vanish and the prefactor $Q_t(x_1, y_1)Q_t(x_2, y_2)$ is zero if $|x_1 - y_1| \lor |x_2 - y_2| \geq e^{-t}$. Hence we have

$$\mathbb{E} \left[ |A_t^{(2)} - A_t^{(1)}|^2 \right] \leq \int_{\mathbb{R}^d} e^{(6\alpha^2 + 2\beta^2)t} f(x_1)^2 f(x_2)^2 Q_t(x_1, y_1)Q_t(x_2, y_2)1_{|x_1 - x_2| \leq e^{-u}} dx_1 dx_2 dy_1 dy_2$$

$$\leq C e^{2(\gamma^2 - d)t} e^{4\alpha^2 - t} \| f \|_4^4. $ \hspace{1cm} (5.27)

If $4\alpha^2 < d$ then the second exponential factor goes to zero (recall that $u = t - \log t$) and we can deduce that

$$\lim_{t \to \infty} e^{(d - |\gamma|^2 + 4\alpha^2)\gamma} \mathbb{E} \left[ |A_t^{(2)} - A_t^{(1)}|^2 \right] \leq \lim_{t \to \infty} e^{(d - |\gamma|^2)\gamma} \mathbb{E} \left[ |A_t^{(2)} - A_t^{(1)}|^2 \right]^{1/2} = 0. $$ \hspace{1cm} (5.28)

When $\alpha \in [\sqrt{d}/2, \sqrt{d}/2)$ we need to combine the above argument with a truncation procedure. Let us fix some parameter $\lambda$ that satisfies the following assumptions

$$\lambda \in (2\alpha, 4\alpha) \quad \text{and} \quad \frac{1}{2} (4\alpha - \lambda) > 4\alpha^2 - d. $$ \hspace{1cm} (5.29)

Such a value of $\lambda$ exists when $\alpha \in [\sqrt{d}/2, \sqrt{d}/2)$. We introduce the event

$$A_t(x) := \{ X_u(x) \leq \lambda u \} $$ \hspace{1cm} (5.30)

Now recalling (5.23) we have

$$A_t^{(2)} - A_t^{(1)} = \int_{\mathbb{R}^d} \xi_t(x, y) 1_{A_t^{(2)}(x)} dx dy + \int_{\mathbb{R}^d} \xi_t(x, y) 1_{A_t^{(1)}(x)} dx dy$$

$$=: W_t^{(1)} + W_t^{(2)}. $$ \hspace{1cm} (5.31)

We are going to compute the first moment of $W_t^{(1)}$ and the second moment of $W_t^{(2)}$. We have

$$\mathbb{E} [\xi_t(x, y) 1_{A_t^{(2)}(x)}] \leq Q_t(x, y) f(x)^2 \left[ e^{(\beta^2 - \alpha^2)t} \mathbb{E} \left[ e^{\alpha(X_t(x) + X_t(y))} 1_{A_t^{(2)}(x)} \right] \right. $$

$$+ e^{(\beta^2 - \alpha^2)u + |\gamma|^2 K_{[u,t]}(x,y)} \mathbb{E} \left[ e^{\alpha(X_u(x) + X_u(y))} 1_{A_t^{(2)}(x)} \right] \right]. $$ \hspace{1cm} (5.32)
Using the Cameron-Martin formula (Proposition 3.1) to compute the expectations we find

$$
E\left[e^{\alpha(X_t(x)+X_t(y))}1_{A_t^c}(x)\right] = e^{\alpha^2(t+K_t(x,y))}P\left[X_u(x) > \lambda u - \alpha(u + K_u(x,y))\right],
$$

$$
E\left[e^{\alpha(X_u(x)+X_u(y))}1_{A_t^c}(x)\right] = e^{\alpha^2(u+K_u(x,y))}P\left[X_u(x) > \lambda u - \alpha(u + K_u(x,y))\right].
$$

(5.33)

Using $u$ and $t$ as upper bounds for $K_t$ and $K_u$ we obtain that

$$
E\left[|\zeta_t(x,y)1_{A_t^c}(x)|\right] \leq 2Q_t(x,y)|f(x)|^2e^{\gamma^2}\int P[X_u(x) \geq (\lambda - 2\alpha)u],
$$

(5.34)

Recalling that $\lambda > 2\alpha$, using Gaussian tail estimates (3.1), we bound the above probability by $e^{-\frac{(\lambda-2\alpha)^2\gamma}{2\lambda}}$, and obtain

$$
E\left[|W_t^{(1)}|\right] \leq 2\|f\|^2e^{\gamma^2}\frac{(\lambda-2\alpha)^2\gamma}{2\lambda} \int P[X_u(x) \geq (\lambda - 2\alpha)u].
$$

(5.35)

Let us now control the second moment of $W_t^{(2)}$. We have

$$
E\left[|W_t^{(2)}|^2\right] = \int \int E\left[|\zeta_t(x_1,y_1)\zeta_t(x_2,y_2)1_{A_t(x_1)\cap A_t(x_2)}|\right] dx_1dx_2dy_1dy_2
$$

(5.36)

From (5.26), as $A_t(x_1) \cap A_t(x_2)$ is $F_u$ measurable, since the correlations of the increments of $X_{[u,t]}$ have range at most $e^{-u}$, and we have

$$
E\left[|\zeta_t(x_1,y_1)\zeta_t(x_2,y_2)1_{A_t(x_1)\cap A_t(x_2)}|\right] = 0 \quad \text{if} \quad |x_1 - x_2| \leq 3e^{-u}
$$

(5.37)

When $|x_1 - x_2| < 3e^{-u}$ we have

$$
|E\left[|\zeta_t(x_1,y_1)\zeta_t(x_2,y_2)1_{A_t(x_1)\cap A_t(x_2)}|\right]|
$$

$$
\leq |f(x_1)f(x_2)|^2Q_t(x_1,y_1)Q_t(x_2,y_2)\frac{1}{2}E\left[|\zeta_t(x_1,y_1)|^21_{A_t(x_1)} + |\zeta_t(x_2,y_2)|^21_{A_t(x_2)}\right],
$$

(5.38)

so that

$$
E\left[|W_t^{(2)}|^2\right] \leq \int \int |f(x_1)f(x_2)|^2Q_t(x_1,y_1)Q_t(x_2,y_2)1_{|x_1 - x_2| \leq 3e^{-u}} dx_1dx_2dy_1dy_2
$$

$$
\times \sup_{|x-y| \leq e^{-t}} E\left[|\zeta_t(x,y)|^21_{A_t(x)}\right].
$$

(5.39)

The integral in the first line is smaller than $C\|f\|^4e^{-d(2t+u)}$. Finally, using the Cameron-Martin formula (Proposition 3.1), we have

$$
E\left[|\zeta_t(x,y)|^21_{A_t(x)}\right] \leq E\left[e^{\gamma X_t(x)+\gamma X_t(y)+(\beta^2-\alpha^2)t}1_{A_t(x)}\right]
$$

$$
= e^{4\alpha^2K_t(x,y)+2\gamma^2}\int P\left[X_u(x) \leq \lambda u - 2\alpha(u + K_u(x,y))\right],
$$

(5.40)

Using the fact that $K_u(x,y) \geq u - 1$ if $|x-y| \leq e^{-t}$, and the fact that $\lambda < 4\alpha$, (5.29), we have from Gaussian tails estimates (3.1)

$$
P\left[X_u(x) \leq \lambda u - 2\alpha(u + K_u(x,y))\right] \leq Ce^{-\frac{4\alpha-\lambda}{2}u}
$$

(5.41)

Combining (5.39)-(5.40)-(5.41) yields

$$
E\left[|W_t^{(2)}|^2\right] \leq C\|f\|^4e^{\frac{4\alpha^2}{2}[2(\gamma^2-d)-d]} \leq \|f\|^4e^{2(\gamma^2-d-d)t}
$$

(5.42)
where, using the fact that $|t - u| \ll t$, the last line is valid for $t$ sufficiently large with (cf. (5.29))

$$\delta := \frac{1}{4} \left[ (4\alpha - \lambda)^2 - 4\alpha^2 + d \right] > 0. \quad (5.43)$$

Recalling (5.31), combining (5.35) and (5.42) we obtain that

$$\lim_{t \to \infty} e^{(d-|\gamma|^2)t} \mathbb{E} \left[ |A^{(2)} - A^{(1)}| \right] = 0. \quad (5.44)$$

**Step 3: Bounding $\mathbb{E} \left[ |A^{(2)} - a(t)M_0^{(2\alpha)}(f^2)| \right]$.** This is easier and just comes down to a simple first moment estimate. We have

$$a(t)M_u^{(2\alpha)}(f^2) := \int_{\mathbb{R}^{2d}} f(x)^2 Q_t(x, y) e^{\gamma^2 K_t(x, y)} e^{2\alpha X_u(x) - 2\alpha^2 u} \, dx \, dy \quad (5.45)$$

so that from Jensen’s inequality

$$\mathbb{E} \left[ a(t)M_u^{2\alpha}(f^2) - A^1_t \right] \leqslant \int_{\mathbb{R}^{2d}} f(x)^2 Q_t(x, y) e^{\gamma^2 K_t(x, y)} \times \mathbb{E} \left[ e^{2\alpha X_u(x) - 2\alpha^2 u} - e^{\gamma^2 K_t(x, y)} e^{2\alpha X_u(x) + 2\alpha^2 u} \right] \, dx \, dy. \quad (5.46)$$

Using the Cameron-Martin formula (Proposition 3.1) the expectation in the integral is equal to

$$\mathbb{E} \left[ e^{2\alpha X_u(x) - 2\alpha^2 u} \left( 1 - e^{\gamma^2 K_t(x, y) + |\gamma|^2 (u - K_t(x, y))} \right) \right]$$

$$= \mathbb{E} \left[ 1 - e^{\gamma^2 (u - K_t(x, y))} \right]. \quad (5.47)$$

We can explicitly compute the second moment in the last line and obtain

$$\mathbb{E} \left[ 1 - e^{\gamma^2 (u - K_t(x, y))} \right]^2 = e^{2\gamma^2 (u - K_t(x, y))} - 1, \quad (5.48)$$

Now when $|x - y| \leqslant e^{-t}$, using Lemma 3.2 (note that $K_t(x, x) = u$)

$$e^{2\gamma^2 (u - K_t(x, y))} - 1 \leqslant e^{|\gamma|^2 (Ce^{-t} + \eta(e^{-t}))} - 1 \leqslant \delta(t). \quad (5.49)$$

where $\lim_{t \to \infty} \delta(t) = 0$ Going back to (5.46), combining (5.47) and (5.49) yields

$$\mathbb{E} \left[ a(t)M_u^{2\alpha}(f^2) - A^1_t \right] \leqslant C \sqrt{\delta(t)} \int_{\mathbb{R}^{2d}} f(x)^2 Q_t(x, y) e^{\gamma^2 K_t(x, y)} \, dx \, dy = C \sqrt{\delta(t)} \|f\|^2 a(t), \quad (5.50)$$

and finishes the proof for the convergence of $A_t$. We are left with that of $B_t$.

---

2The term $\eta(e^{-t})$ is not needed when $K_0 \equiv 0$, cf. the proof of Lemma 3.2 but adding it make the proof easier to adapt to the general case.
Final step: Bounding $\mathbb{E}[|B_t|]$. For $B_t$ we use the same ideas as for $A_t$, but require only one intermediate step. We define

$$B_t^{(1)} = \mathbb{E}[B_t | F_u] = \int_{\mathbb{R}^d} f(x) f(y) Q_s(x, y) e^{\gamma(X_u(x) + X_u(y)) - \gamma^2 (u-K_{[w, t]}(x, y))} \, dx \, dy,$$  

(5.51)

a give a separate bound for $\mathbb{E}
[|B_t^{(1)}|]$ and $\mathbb{E}
[|B_t - B_t^{(1)}|]$. We have

$$\mathbb{E}
[|B_t^{(1)}|] \leq \int_{\mathbb{R}^d} f(x) f(y) Q_s(x, y) e^{\gamma(X_u(x) + X_u(y)) + (\beta^2 - \alpha^2)(u-K_{[w, t]}(x, y))} \, dx \, dy$$

$$\leq \int_{\mathbb{R}^d} f(x) f(y) Q_s(x, y) e^{\sigma^2 K_t(x, y) + \beta^2 (u-K_{[w, t]}(x, y))} \, dx \, dy$$

(5.52)

$$\leq C \|f\|^2 e^{\frac{(\gamma^2-d)(t-2\beta^2(t-u))}{2}}.$$

and hence $\lim_{t \to \infty} e^{d-|\gamma|^2} \mathbb{E}
[|B_t^{(1)}|] = 0$. We can then bound $\mathbb{E}[|B_t - B_t^{(1)}|]$. The reasoning is identical to that used for $A_t^{(1)} - A_t^{(2)}$ so we provide details only the more delicate case $\alpha \in [\sqrt{d}/2, \sqrt{d}/2]$. We define $A_t(x)$ as in (5.30) and in analogy with (5.22) set

$$\xi_t(x, y) := e^{\gamma(X_t(x)+X_t(y)) - \gamma^2 t - \frac{1}{2} \mathbb{E}[e^{\gamma X_t(x)+\gamma X_t(y)-\gamma^2 t} | F_u]},$$

(5.53)

$$\zeta_t(x, y) := Q_t(x, y) f(x) f(y) \xi_t(x, y).$$

We have

$$|B_t^{(1)} - B_t| \leq \int_{\mathbb{R}^d} \zeta_t(x, y) 1_{A_t}(x) \, dx \, dy + \int_{\mathbb{R}^d} \zeta_t(x, y) 1_{A_t}(x) \, dx \, dy$$

(5.54)

We prove similarly to (5.32) that

$$\mathbb{E}
[\zeta_t(x, y) 1_{A_t}(x)] \leq Q_t(x, y) f(x) f(y) e^{\frac{|\gamma|^2 (t-2\beta^2 t)}{2}}$$

(5.55)

which implies that

$$\mathbb{E}
\left[\int_{\mathbb{R}^d} \zeta_t(x, y) 1_{A_t}(x) \, dx \, dy\right] \leq C \|f\|^2 e^{\frac{(\gamma^2-d) (t-2\beta^2 t)}{2}}.$$

(5.56)

Finally we control the second moment of the first term in (5.54) like that of $W_t^{(2)}$ in (5.39). Repeating the same steps we prove that

$$\mathbb{E}
\left[\left|\int_{\mathbb{R}^d} \zeta_t(x, y) 1_{A_t}(x) \, dx \, dy\right|^2\right] \leq C \|f\|^4 e^{\frac{4\alpha^2 (2|\gamma|^2-d) (t-2\beta^2 t)}{2}}$$

(5.57)

and conclude in the same manner.

6. Deducing Proposition 2.2 from Proposition 2.8

The present section and the next are dedicated to the proof of Proposition 2.2 on which the main result of the paper, Theorem 1.3 relies. While with the combination of martingale filtration and convolution, the notation become more cumbersome and the computations a bit more delicate, the proof rely almost exclusively on the ideas developped in Section 4 and 5.

In this section, we adapt the proof of Theorem 2.5 to show that Proposition 2.8 implies Proposition 2.2. Recall that $N_t^{(\epsilon)} = \mathbb{E}(M^{(\gamma)}_\epsilon(f) | F_t)$. We set for the proof $Z :=$
$M^{(2\alpha)}(e^L f^2)$ and $Z_t^A := \mathbb{E}[Z \wedge A \mid \mathcal{F}_t]$. Since $N_{\infty}^{(e)} = M_\varepsilon^{(\gamma)}(f, \omega)$ we only need to show that

$$
\lim_{\varepsilon \to 0} \mathbb{E} \left[ e^{iv(\varepsilon)N_{\infty}^{(e)}} - e^{-\frac{1}{2}Z_t^A} \right] = 0.
$$

(6.1)

We write simply $v(\varepsilon)$ for $v(\varepsilon, \theta, \gamma)$. We set $t_\varepsilon := \sqrt{\log 1/\varepsilon}$, and

$$
T(A, \varepsilon) := \inf\{s \geq 0 : \langle N_{(e)}^{(e)} \rangle_t \geq v(\varepsilon)^{-2} A\}.
$$

(6.2)

Then we decompose the expectation we want to bound as we did in (4.4)

$$
\left| \mathbb{E} \left[ e^{iv(\varepsilon)N_{\infty}^{(e)}} - e^{-\frac{1}{2}Z_t^A} \right] \right|
\leq \left| \mathbb{E} \left[ e^{iv(\varepsilon)N_{\infty}^{(e)}} - e^{iv(\varepsilon)N_{(e)}^{(e)}} \right] \right| + \left| \mathbb{E} \left[ e^{iv(\varepsilon)N_{(e)}^{(e)}} - e^{-\frac{1}{2}Z_t^A} \right] \right| + \left| \mathbb{E} \left[ e^{-\frac{1}{2}Z_t^A} - e^{-\frac{1}{2}Z_t^A} \right] \right|
\leq E_1(\varepsilon, A) + E_2(\varepsilon, A) + E_3(\varepsilon, A).
$$

(6.3)

To prove (6.1) we need to show that

$$
\forall i \in \{1, 2, 3\}, \lim_{A \to \infty} \lim_{\varepsilon \to 0} \sup E_i(\varepsilon, A) = 0.
$$

(6.4)

For $E_3$ we can use dominated convergence twice

$$
\lim_{A \to \infty} \lim_{\varepsilon \to 0} E_3(\varepsilon, A) = \lim_{A \to \infty} \mathbb{E} \left[ e^{-\frac{1}{2}Z_t^A} - e^{-\frac{1}{2}Z_t^A} \right] = 0.
$$

For $E_1$ we have

$$
\mathbb{E} \left[ \left| e^{iv(\varepsilon)N_{\infty}^{(e)}} - e^{iv(\varepsilon)N_{(e)}^{(e)}} \right| \right] \leq 2\mathbb{P}[T < \infty] + \mathbb{E} \left[ \left| 1 - e^{-iv(\varepsilon)N_1^{(e)}} \right| \right]
$$

(6.5)

Combining Proposition [2.4] and Portmanteau Theorem we have

$$
\lim_{\varepsilon \to 0} \sup \mathbb{P}[T(A, \varepsilon) < \infty] \leq \mathbb{P}[Z \geq A].
$$

(6.6)

To bound the second term in (6.5) we only need to show that $v(\varepsilon)N_1^{(e)}$ tend to zero in probability. The reader can check that with our choice for $t_{\varepsilon}$

$$
\mathbb{E} \left[ (N_1^{(e)})^2 \right] \leq \begin{cases} 
C_\varepsilon |\gamma|^2 - d & \text{if } |\gamma|^2 > d, \\
C \varepsilon & \text{if } |\gamma|^2 = d,
\end{cases}
$$

(6.7)

and the r.h.s. in (6.7) much smaller than $v(\varepsilon)^{-2}$ in both cases. Finally for $i = 2$, using an exponential martingale we have, like for (4.10)

$$
\mathbb{E} \left[ \left| e^{iv(\varepsilon)N_{(e)}^{(e)}} \right| \left| 2^{-N_{(e)}} \langle N_{(e)} \rangle_{[\tau, T]} \right| \mid \mathcal{F}_t \right] = 1.
$$

(6.8)

As a consequence we have

$$
\mathbb{E} \left[ e^{iv(\varepsilon)N_{(e)}^{(e)}} + \frac{1}{2} \langle N_{(e)} \rangle_{[\tau, T]} - Z_t^A - e^{-\frac{1}{2}Z_t^A} \right] = 0
$$

(6.9)

and thus

$$
E_2(\varepsilon, A) \leq \mathbb{E} \left[ \left| 1 - e^{\frac{1}{2}v(\varepsilon)^2 \langle N_{(e)} \rangle_{[\tau, T]} - Z_t^A} \right| \right].
$$

(6.10)
To conclude, we simply observe that both \( v(\varepsilon)^2 \langle N^{(\varepsilon)} \rangle_{[t,T]} \) and \( Z^4_t \) are bounded and both converge in probability to \( Z \wedge A \) when \( \varepsilon \) goes to zero so that by dominated convergence
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ 1 - e^{\frac{1}{2} (v(\varepsilon)^2 \langle N^{(\varepsilon)} \rangle_{[t,T]} - Z^4_t)} \right] = 0. \tag{6.11}
\]
This concludes the proof of (6.1). \( \square \)

7. Proof of Proposition 2.8

As for the proof of Proposition 2.8 we are going to control the derivative of the brackets associated with a complex valued martingale. We set
\[
W^{(\varepsilon)}_t := \int_{\mathbb{R}^d} f(x)e^{\gamma X_{t,\varepsilon}(x) - \gamma^2 K_{t,\varepsilon}(x,y)} \mathbf{1}_{D_\varepsilon} \, dx.
\]
Assuming that \( \varepsilon \) is sufficiently small the support of \( f \) is included in \( D_\varepsilon \) so we do not need to worry about the indicator function. Using that \( (X_{t,\varepsilon}(x))_{t \geq 0} \) is a continuous martingale, we have from Itô calculus
\[
dW^{(\varepsilon)}_t := \gamma \int_{\mathbb{R}^d} f(x)e^{\gamma X_{t,\varepsilon}(x) - \gamma^2 K_{t,\varepsilon}(x,y)} \, dX_{t,\varepsilon}(x) \, ds. \tag{7.1}
\]
Since by construction we have \( d\langle X_{t,\varepsilon}(x), X_{t,\varepsilon}(y) \rangle_t = Q_{t,\varepsilon}(x,y) \) where
\[
Q_{t,\varepsilon}(x,y) := \int_{\mathbb{R}^d} \theta_\varepsilon(x-z_1) \theta_\varepsilon(y-z_2) \kappa(t|z_1 - z_2|) \, dz_1 \, dz_2, \tag{7.2}
\]
this yields
\[
\langle W^{(\varepsilon)}, W^{(\varepsilon)} \rangle_\infty = |\gamma|^2 \int_0^\infty A_{t,\varepsilon} \, dt \quad \text{and} \quad \langle W^{(\varepsilon)}, W^{(\varepsilon)} \rangle_\infty = \gamma^2 \int_0^\infty B_{t,\varepsilon} \, dt. \tag{7.3}
\]
We set
\[
A_{t,\varepsilon} := \int_{\mathbb{R}^{2d}} f(x)f(y)Q_{t,\varepsilon}(x,y)e^{\gamma X_{t,\varepsilon}(x) + \gamma X_{t,\varepsilon}(y) + \frac{\kappa^2}{2}(K_{t,\varepsilon}(x) + K_{t,\varepsilon}(y))} \, dx \, dy,
\]
\[
B_{t,\varepsilon} := \int_{\mathbb{R}^{2d}} f(x)f(y)Q_{t,\varepsilon}(x,y)e^{\gamma X_{t,\varepsilon}(x) + \gamma X_{t,\varepsilon}(y) + \frac{\kappa^2}{2}(K_{t,\varepsilon}(x) + K_{t,\varepsilon}(y))} \, dx \, dy. \tag{7.4}
\]
Recalling the definition of \( \hat{K}_s \) (5.4) let us define
\[
\hat{K}_{s,\varepsilon}(x,y) := \int_{\mathbb{R}^d} \hat{K}(z_1, z_2) \theta_\varepsilon(x-z_1) \theta_\varepsilon(y-z_2) \, dz_1 \, dz_2 = \int_0^s Q_{t,\varepsilon}(x,y) \, dt \tag{7.5}
\]
\[
a(t, \kappa, \gamma, \varepsilon) := \int_{\mathbb{R}^d} Q_{t,\varepsilon}(z)e^{\kappa^2 \hat{K}_{s,\varepsilon}(0,z)} \, dz.
\]
We write simply \( \hat{K}_\varepsilon \) when \( t = \infty \). The key estimate to prove Proposition 2.8 is the following

**Proposition 7.1.** We have the following convergences in \( L_1 \)
\[
\lim_{\varepsilon \to \infty} a(t, \kappa, \gamma, \varepsilon)^{-1} A_{t,\varepsilon} = M_0^{(2\alpha)}(e^{K_0 f^2}),
\]
\[
\lim_{\varepsilon \to \infty} a(t, \kappa, \gamma, \varepsilon)^{-1} B_{t,\varepsilon} = 0. \tag{7.6}
\]
Proof of Proposition 2.8. Let us prove first that we have

$$
\lim_{\varepsilon \to 0} |\gamma|^2 v(\varepsilon, \theta)^2 \int_0^\infty a(t, \kappa, \gamma, \varepsilon) dt = 2e^{-|\gamma|^2 j}\kappa. \tag{7.7}
$$

When $|\gamma|^2 > d$ we have

$$
|\gamma|^2 \int_0^\infty a(t, \kappa, \gamma, \varepsilon) dt = \int_{\mathbb{R}^d} \left( e^{|\gamma|^2 \hat{K}_\varepsilon(0,z)} - 1 \right) \, dz = e^{d-|\gamma|^2} \int_{|y| \leq \varepsilon^{-1}+2} \left( e^{|\gamma|^2 (\hat{K}_\varepsilon(0,\varepsilon y)-\log(1/\varepsilon))} - e^{|\gamma|^2} \right) \, dy. \tag{7.8}
$$

Now integrating $e^{|\gamma|^2}$ yield something tending to zero so we just need to worry about the first term in the integral. To compute the other term, we consider the following decomposition

$$
\hat{K}(x, y) = \log \left( \frac{1}{|x-y| \sqrt{2}} \right) + \left( \int_0^\infty \kappa(e^t|x-y|) dt - \log \left( \frac{1}{|x-y| \sqrt{2}} \right) \right).
$$

Let us call $U(x, y)$ the second term, it is uniformly bounded and equal to $-j\kappa$ on the diagonal (recall (2.25)). When $|y| \leq 2 + \varepsilon^{-1}$ and $\varepsilon \leq 1/4$ we have

$$
\hat{K}_\varepsilon(0, \varepsilon y) = \int_{\mathbb{R}^d} \theta_\varepsilon(z_1) \theta_\varepsilon(z_2) \left( \log \frac{1}{|z_1 + z_2 - \varepsilon y|} + U(\varepsilon y - z_1 + z_2) \right) \, dz_1 dz_2 \tag{7.9}
$$

Performing a change of variable and recalling (1.13) we have

$$
\int_{\mathbb{R}^d} \theta_\varepsilon(z_1) \theta_\varepsilon(z_2) \log \frac{1}{|z_1 + z_2 - \varepsilon y|} \, dz_1 dz_2 = \ell_\theta(y) + \log(1/\varepsilon), \tag{7.10}
$$

while

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \theta_\varepsilon(z_1) \theta_\varepsilon(z_2) U(\varepsilon y - z_1 + z_2) = -j\kappa. \tag{7.11}
$$

Has a conclusion we obtain that

$$
\lim_{\varepsilon \to 0} \left( \hat{K}_\varepsilon(0, \varepsilon y) - \log(1/\varepsilon) \right) = \ell_\theta(y) \tag{7.12}
$$

By dominated convergence, this proves that

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} e^{|y|^2 (\hat{K}_\varepsilon(0,\varepsilon y)-\log(1/\varepsilon))} \, dy = e^{-|\gamma|^2 j}\kappa \int_{\mathbb{R}^d} e^{-|\gamma|^2 \ell_\theta(y)} \, dy. \tag{7.13}
$$

and hence recalling (7.8) that (7.7) holds. Now when $|\gamma| = \sqrt{d}$ we split the integral

$$
\int_{\mathbb{R}^d} e^{d\hat{K}_\varepsilon(0,z)} \, dz \leq \int_{|z| \leq 1+2\varepsilon} e^{d\hat{K}_\varepsilon(0,\varepsilon y)} \, dz \tag{7.14}
$$

and hence recalling (7.9) (replacing $\varepsilon y$ by $z$) we see that $\hat{K}_\varepsilon(0, \varepsilon y) \leq \log(3)(1/\varepsilon) + C$ when $|z| \geq \log \log(1/\varepsilon)$. This yields

$$
\int_{\log \log(1/\varepsilon) \leq |z| \leq 1+2\varepsilon} e^{d\hat{K}_\varepsilon(0,z)} \, dz \leq C \log \log(1/\varepsilon)^d. \tag{7.15}
$$
Finally, using (7.9) \((U \text{ is a Lipshitz function})\) we obtain that
\[
\{|z| \in (\epsilon \log(1/\epsilon), \log(1/\epsilon)^{-1})\} = \{|\tilde{K}_{\varepsilon}(0, z) - \log |z| + j_n| \leq C \log \log(1/\epsilon)^{-1}\}
\]
so that
\[
\lim_{\varepsilon \to 0} \frac{e^{d_\varepsilon} \int_{|z| \in (\epsilon \log(1/\epsilon), \log(1/\epsilon)^{-1})} e^{d\tilde{K}_{\varepsilon}(0, z)} dz}{\int_{|z| \in (\epsilon \log(1/\epsilon), \log(1/\epsilon)^{-1})} |z|^{-d} dz} = 1 \quad (7.16)
\]
The denominator is asymptotically equivalent to \(\frac{d!}{(d/2)!} \log(1/\epsilon)\) (the prefactor is simply the volume of the \(d - 1\) dimensional sphere of radius one). Combined with (7.11) and (7.15), this yields (7.7) also in that case.

Now recalling that \(N^{(\epsilon)} = 2 \Re W^{(\epsilon)} e^{-i\omega}\) we have
\[
\langle N^{(\epsilon)} \rangle_\infty = \frac{1}{2} \langle W^{(\epsilon)}, \overline{W^{(\epsilon)}} \rangle_\infty + \frac{1}{2} 2 \Re (e^{-2i\omega} \langle W^{(\epsilon)}, W^{(\epsilon)} \rangle_\infty) \quad (7.17)
\]
In view of (7.7), it is sufficient to prove that
\[
\lim_{\varepsilon \to 0} \frac{\langle W^{(\epsilon)}, \overline{W^{(\epsilon)}} \rangle_\infty}{|\gamma|^2 \int_0^\infty a(t, \kappa, \gamma, \varepsilon) dt} = M_0^{(2a)}(e^{K_0} f^2) \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\langle W^{(\epsilon)}, W^{(\epsilon)} \rangle_\infty}{\int_0^\infty a(t, \kappa, \gamma, \varepsilon) dt} = 0. \quad (7.18)
\]
Let us set \(t_\varepsilon := \sqrt{\log(1/\varepsilon)}\) (we need \(1 < t_\varepsilon < \log(1/\varepsilon)\)). From Proposition 7.1 and (7.3) we have
\[
\lim_{\varepsilon \to 0} \frac{\langle W^{(\epsilon)}, \overline{W^{(\epsilon)}} \rangle_{[t_\varepsilon, \infty)}}{|\gamma|^2 \int_0^{t_\varepsilon} a(s, \kappa, \gamma, \varepsilon) ds} \quad \text{and} \quad \frac{\langle W^{(\epsilon)}, W^{(\epsilon)} \rangle_{[t_\varepsilon, \infty)}}{\int_0^{t_\varepsilon} a(s, \kappa, \gamma, \varepsilon) ds} = 0. \quad (7.19)
\]
To conclude we just need to show that \(\mathbb{E} \left[\langle W^{(\epsilon)}, \overline{W^{(\epsilon)}} \rangle_{t_\varepsilon}\right]\) and \(\int_0^{t_\varepsilon} a(s, \kappa, \gamma, \varepsilon) ds\) are of an order of magnitude \(e^{(|\gamma|^2 - d) t_\varepsilon}\) (if \(|\gamma| > \sqrt{d}\)) or \(t_\varepsilon\) (if \(|\gamma| = \sqrt{d}\), details are left to the reader.

\[\square\]

### 7.1. Proof of Proposition 7.1

We proceed as in the case without convolution (Proposition (5.1)) in several steps. The case of \(\Lambda_{t, \varepsilon}\) is the more delicate so we treat it in full details and then sketch briefly the proof for \(B_{t, \varepsilon}\). We assume that \(K_0 = 0\) for the sake of simplifying notation. For the need of the computation we need to fix \(u < t\) such that \(X_{u, \varepsilon}(x) - X_{u, \varepsilon}(y)\) is small whenever \(Q_{t, \varepsilon}(x, y) > 0\). Keeping (3.1) and Lemma 3.2 in mind, a convenient choice is
\[
u := u(t, \varepsilon) = \left(t \wedge \log \frac{1}{\varepsilon}\right) - \log \left(t \wedge \log \frac{1}{\varepsilon}\right). \quad (7.20)
\]
We set
\[
k(t, \varepsilon) := K_{t, \varepsilon}(x, x)
\](since \(K_0 = 0\), the field is translation invariant so it does not depend on \(x\)). Note that from (3.1) we have
\[
\left|K(t, \varepsilon) - \left(t \wedge \log \frac{1}{\varepsilon}\right)\right| \leq C, \quad (7.21)
\]
so that we have in particular in the \(\varepsilon \to 0\) and \(t \to \infty\) regime
\[
1 \ll k(t, \varepsilon) - k(u, \varepsilon) \ll k(t, \varepsilon) \quad \text{and} \quad |k(u, \varepsilon) - u| \leq C \quad (7.22)
\]
We define, using (1.20)
\[ A_{t,\varepsilon}^{(1)} := \int_{\mathbb{R}^d} f(x)^2 Q_{t,\varepsilon}(x, y) e^{\gamma X_{t,\varepsilon}(x) + \tau X_{t,\varepsilon}(y) + (\beta^2 - \alpha^2) k(t, \varepsilon)} \, dx \, dy, \]
\[ A_{t,\varepsilon}^{(2)} := \int_{\mathbb{R}^d} f(x)^2 Q_{t,\varepsilon}(x, y) e^{\gamma X_{u,\varepsilon}(x) + \tau X_{u,\varepsilon}(y) + (\beta^2 - \alpha^2) k(u, \varepsilon) + |\gamma|^2 K_{u,\varepsilon}(x, y)} \, dx \, dy, \]
(7.23)

Let us also introduce
\[ M_{t,\varepsilon}^{(2\alpha)}(f^2) := \int_{\mathbb{R}^d} f(x)^2 e^{2\alpha X_{u,\varepsilon}(x) - 2\alpha^2 k(t, \varepsilon)} \, dx \, dy \]  
(7.24)

Writing \( a(t, \varepsilon) \) for \( a(t, \kappa, \gamma, \varepsilon) \) we have
\[ |A_{t,\varepsilon} - a(t, \varepsilon)M_{0}^{(2\alpha)}(f^2)| \leq |A_{t,\varepsilon} - A_{t,\varepsilon}^{(1)}| + |A_{t,\varepsilon}^{(1)} - A_{t,\varepsilon}^{(2)}| \\
\quad + |A_{t,\varepsilon}^{(2)} - a(t, \kappa, \gamma, \varepsilon)M_{t,\varepsilon}^{(2\alpha)}| + a(t, \varepsilon)|M_{t,\varepsilon}^{(2\alpha)}(f^2) - M_{0}^{(2\alpha)}(f^2)| \]  
(7.25)

and we need to show that all of the four terms in the r.h.s. rescaled by \( a(t, \kappa, \gamma, \varepsilon) \) tend to zero. This is done in four separate steps. The argument for the first three terms are essentially the same as for the proof of Proposition 5.1 in Section 5.

**Step 1: Bounding** \( E\left[|A_{t,\varepsilon} - A_{t,\varepsilon}^{(1)}|\right] \). Let us prove first that
\[ \lim_{\varepsilon \to 0} a(t, \varepsilon)^{-1} E\left[|A_{t,\varepsilon} - A_{t,\varepsilon}^{(1)}|\right] = 0 \]  
(7.26)

Now since \( Q_{t,\varepsilon}(x, y) = 0 \) when \( |x - y| > e^{-t} + 2\varepsilon \), letting \( w_f \) denote the modulus of continuity of \( f \) we have
\[ E\left[|A_{t} - A_{t}^{(1)}|\right] \leq w_f(e^{-t} + 2\varepsilon) \int_{\mathbb{R}^d} |f(x)|Q_{t,\varepsilon}(x, y)e^{\gamma|2 K_{t,\varepsilon}(x, y)|} \, dx \, dy \]
\[ = w_f(e^{-t} + 2\varepsilon)\|f\|_1 a(t, \varepsilon). \]  
(7.27)

We conclude the proof by simply observing that \( w_f(e^{-t} + 2\varepsilon) \) tends to zero.

**Step 2: Bounding** \( E\left[|A_{t,\varepsilon} - A_{t,\varepsilon}^{(2)}|\right] \). Let us now prove that
\[ \lim_{\varepsilon \to 0} a(t, \varepsilon)^{-1} E\left[A_{t,\varepsilon}^{(1)} - A_{t,\varepsilon}^{(2)}\right] = 0 \]  
(7.28)

We provide details only on the more delicate case \( \alpha \in [\sqrt{d}/2, \sqrt{d}/2] \) (the reader can repeat the procedure in Section 5 for the case \( \alpha \in (0, \sqrt{d}/2) \)). Let us set
\[ \xi_{t,\varepsilon}(x, y) := e^{\gamma X_{t,\varepsilon}(x) + \tau X_{t,\varepsilon}(y) + (\beta^2 - \alpha^2) k(t, \varepsilon)} - E\left[e^{\gamma X_{t,\varepsilon}(x) + \tau X_{t,\varepsilon}(y) + (\beta^2 - \alpha^2) k(t, \varepsilon)} \mid F_u\right], \]  
(7.29)

\[ \zeta_{t,\varepsilon}(x, y) := Q_{t,\varepsilon}(x, y)f(x)^2 \xi_{t,\varepsilon}(x, y). \]

We fix \( \lambda \) satisfying (5.29) and set similarly to (5.30)
\[ A_{t,\varepsilon}(x) = \{X_{t,\varepsilon}(x) \leq \lambda k(u, \varepsilon)\} \]
(7.30)

(recall that now \( u \) depends also on \( \varepsilon \)). We have
\[ A_{t,\varepsilon}^{(1)} - A_{t,\varepsilon}^{(2)} = \int_{\mathbb{R}^d} \zeta_{t,\varepsilon}(x, y) 1_{A_{t,\varepsilon}(x)} \, dx \, dy + \int_{\mathbb{R}^d} \zeta_{t,\varepsilon}(x, y) 1_{A_{t,\varepsilon}(x)} \, dx \, dy =: W_{t,\varepsilon}^{(1)} + W_{t,\varepsilon}^{(2)} \]  
(7.31)
To prove (7.28) we are going to bound the $L_1$ norm of the first term and the $L_2$ norm of the second term. We have used the Cameron-Martin formula (Proposition 3.1)

$$
\mathbb{E} \left[ |\xi_t(x,y)\|_{\mathcal{A}^c_t(x)} \right] \leq 2\mathbb{E} \left[ e^{\alpha(X_{t,t}(x)+X_{t,t}(y))-\alpha^2 k(t,\varepsilon)} \mathbf{1}_{\mathcal{A}^c_t(x)} \right] \\
= 2e^{\beta k(t,\varepsilon)+\alpha^2 K_{t,\varepsilon}(x,y)} \mathbb{P} \left[ X_{u,\varepsilon} > (\lambda - \alpha)k(u,\varepsilon) - \alpha K_{u,\varepsilon}(x,y) \right] \\
\leq 2e^{\gamma^2 k(t,\varepsilon)} \mathbb{P} \left[ X_{u,\varepsilon}(x) > (\lambda - 2\alpha)k(u,\varepsilon) \right].
$$

(7.32)

Using the Gaussian tail bound (3.1), this entails that

$$
\mathbb{E}[|W_{t,\varepsilon}^{(1)}|] \leq \mathbb{E} \left[ \int_{\mathbb{R}^d} |\zeta_{t,\varepsilon}(x,y)| \mathbf{1}_{\mathcal{A}^c_t(x)} \, dx \, dy \right] \\
\leq 2e^{\gamma^2 k(t,\varepsilon)-\frac{(\lambda-2\alpha)^2 k(u,\varepsilon)}{2}} \int_{\mathbb{R}^d} f(x)^2 Q_{t,\varepsilon}(x,y) \, dx \, dy = a(t,\varepsilon)e^{-\frac{(\lambda-2\alpha)^2 k(u,\varepsilon)}{2}} \| f \|^2. 
$$

(7.33)

For the second moment computation $W_{t,\varepsilon}^{(2)}$, note that the range of correlation of $X_{u,\varepsilon}$ is smaller than $e^{-u}+2\varepsilon$ and that $Q_{t,\varepsilon}(x,y) = 0$ when $|x-y| \geq e^{-t} + 2\varepsilon$ so that

$$
|x_1 - x_2| \geq 6\varepsilon + 2e^{-t} + e^{-u} \implies \mathbb{E} [\zeta_{t,\varepsilon}(x_1,y_1) \bar{\zeta}_{t,\varepsilon}(x_2,y_2) \mid \mathcal{F}_u] = 0.
$$

(7.34)

Hence taking $\varepsilon$ sufficiently small and $t$ sufficiently large, we have, whenever $|x_1 - x_2| \leq 2e^{-u}$

$$
\mathbb{E} [\zeta_{t,\varepsilon}(x_1,y_1) \bar{\zeta}_{t,\varepsilon}(x_2,y_2) \mathbf{1}_{\mathcal{A}_{t,\varepsilon}(x_1) \cap \mathcal{A}_{t,\varepsilon}(x_2)}] = 0.
$$

(7.35)

When $x_1$ and $x_2$ are closer to each other, we bound the covariance by the mean of the variances. We have

$$
\mathbb{E} [\xi_{t,\varepsilon}(x,y) \|_{\mathcal{A}_{t,\varepsilon}(x)}^2] \leq \mathbb{E} [e^{2\alpha(X_{t,t}(x)+X_{t,t}(y))} - 2(\beta^2 - \alpha^2)k(t,\varepsilon)] \\
= e^{2\gamma^2 k(t,\varepsilon)+4\alpha^2 K_{t,\varepsilon}(x,y)} \mathbb{P} \left[ X_{u,\varepsilon} \leq (\lambda - 2\alpha)k(u,\varepsilon) - 2\alpha K_{u,\varepsilon}(x,y) \right].
$$

(7.36)

Our definition of $u$ and Lemma 3.2 implies that there exists some positive constant $C$ satisfying

$$
Q_{t,\varepsilon}(x,y) > 0 \quad \implies \quad K_{u,\varepsilon}(x,y) \geq k(u,\varepsilon) - C
$$

Thus using Gaussian tail bounds (3.1) we have, whenever $Q_{t,\varepsilon}(x,y) > 0$

$$
\mathbb{P} [X_{u,\varepsilon} \leq (\lambda - 2\alpha)k(u,\varepsilon) - 2\alpha K_{u,\varepsilon}(x,y)] \leq C e^{-\frac{(\lambda-4\alpha)^2 k(u,\varepsilon)}{2}}.
$$

(7.37)

Thus we obtain that

$$
\mathbb{E} \left[ |W_{t,\varepsilon}^{(2)}|^2 \right] \leq C e^{(2\gamma)^2+4\alpha^2 k(t,\varepsilon)-\frac{(\lambda-4\alpha)^2 k(u,\varepsilon)}{2}} \times \int_{\mathbb{R}^d} f(x_1)^2 f(x_2)^2 Q_{t,\varepsilon}(x_1,y_1)Q_{t,\varepsilon}(x_2,y_2) \mathbf{1}_{|x_1 - x_2| \leq 2e^{-u}} \, dx_1 \, dx_2 \, dy_1 \, dy_2.
$$

(7.38)

Now the integral above can be bounded by

$$
C \| f \|^2 \| \varepsilon^{du} \left( \int_{\mathbb{R}^d} Q_{t,\varepsilon}(0,z) \, dz \right)^2.
$$

(7.39)

Thus we obtain that

$$
\mathbb{E} \left[ |W_{t,\varepsilon}^{(2)}|^2 \right] \leq C \| f \|^2 \| a(t,\varepsilon)^2 e^{4\alpha^2 k(t,\varepsilon)-du - \frac{(\lambda-4\alpha)^2 k(u,\varepsilon)}{2}}.
$$

(7.40)

It only remains to prove that the exponential terms goes to zero. This is simply a consequence of (5.29) and (7.22) (which essentially allows to replace $u$ and $k(u,\varepsilon)$ by $k(t,\varepsilon)$).
Step 3: Bounding $\mathbb{E} \left[ |A_{t,\varepsilon}^{(2)} - M_{u,\varepsilon}^{(2\alpha)}| \right]$. We prove now that
\[
\lim_{\varepsilon \to 0} a(t, \varepsilon)^{-1} \mathbb{E} \left[ A_{t,\varepsilon}^{(1)} - A_{t,\varepsilon}^{(2)} \right] = 0. \tag{7.41}
\]
This much easier, we have from Jensen’s inequality
\[
\mathbb{E} \left[ |A_{t,\varepsilon}^{(2)} - a(t, \varepsilon) M_{u,\varepsilon}^{(2\alpha)}| \right] \leq \int_{\mathbb{R}^2} f(x)^2 Q_t(x, y) e^{\gamma |t|^2 K_{t,\varepsilon}(x, y)}
\times \mathbb{E} \left[ e^{2\alpha X_{u,\varepsilon}(x) - 2\alpha^2 k(u, \varepsilon)} \left| \mathcal{E} \left( X_{u,\varepsilon}(y) - X_{u,\varepsilon}(x) \right) + |\gamma|^2 (k(u, \varepsilon) - K_{u,\varepsilon}(x, y)) - 1 \right| \right] \, dx \, dy \tag{7.42}
\]
Using the Cameron-Martin formula (Proposition 3.1) we have
\[
\mathbb{E} \left[ e^{2\alpha X_{u,\varepsilon}(x) - 2\alpha^2 k(u, \varepsilon)} \left| \mathcal{E} \left( X_{u,\varepsilon}(y) - X_{u,\varepsilon}(x) \right) + |\gamma|^2 (k(u, \varepsilon) - K_{u,\varepsilon}(x, y)) - 1 \right| \right] = \mathbb{E} \left[ e^{ \mathcal{E}(Y_{u,\varepsilon}(y) - Y_{u,\varepsilon}(x)) - \mathcal{E}^2(k(u, \varepsilon) - K_{u,\varepsilon}(x, y)) - 1} \right] \tag{7.43}
\]
and finally
\[
\mathbb{E} \left[ e^{ \mathcal{E}(Y_{u,\varepsilon}(y) - Y_{u,\varepsilon}(x)) - \mathcal{E}^2(k(u, \varepsilon) - K_{u,\varepsilon}(x, y)) - 1} \right] \leq \mathbb{E} \left[ e^{ \mathcal{E}^2(k(u, \varepsilon) - K_{u,\varepsilon}(x, y)) - 1} \right] = e^{2\gamma|t|^2 (k(u, \varepsilon) - K_{u,\varepsilon}(x, y)) - 1}. \tag{7.44}
\]
Now, using Lemma 3.2 and our definition of $u$, we have whenever $|x - y| \leq e^t + 2\varepsilon$
\[
k(u, \varepsilon) - K_{u,\varepsilon}(x, y) \leq C e^{-u} |x - y| + \eta(|x - y|) \leq \delta(t, \varepsilon) \tag{7.45}
\]
where $\delta(t, \varepsilon)$ when $t$ goes to infinity and $\varepsilon$ does to 0. Altogether we obtain that
\[
\mathbb{E} \left[ |A_{t,\varepsilon}^{(2)} - a(t, \varepsilon) M_{u,\varepsilon}^{(2\alpha)}| \right] \leq C \|f\|_2^2 a(t, \varepsilon) \sqrt{\delta(t, \varepsilon)}, \tag{7.46}
\]
concluding the proof of (7.41).

Step 4: Bounding $\mathbb{E} \left[ |M_{t,\varepsilon}^{(2\alpha)} - M_0^{(2\alpha)}| \right]$. Let us finally discuss the fourth term. We want to prove the following
\[
\lim_{\varepsilon \to 0} t \to \infty \mathbb{E} \left[ |M_{t,\varepsilon}^{(2\alpha)}(f^2) - M_0^{(2\alpha)}(f^2)| \right] = 0. \tag{7.47}
\]
Omitting $f^2$ for readability we have
\[
\mathbb{E} \left[ |M_{t,\varepsilon}^{(2\alpha)} - M_0^{(2\alpha)}| \right] = \mathbb{E} \left[ |M_{t,\varepsilon}^{(2\alpha)} - M_{t}^{(2\alpha)}| \right] + \mathbb{E} \left[ |M_{t}^{(2\alpha)} - M_0^{(2\alpha)}| \right]. \tag{7.48}
\]
The second term tends to zero when $t \to \infty$ thanks to Theorem 3. As for the first one, note that we have
\[
M_{t,\varepsilon}^{(2\alpha)} - M_{t}^{(2\alpha)} = \mathbb{E} \left[ M_{t}^{(2\alpha)} - M_0^{(2\alpha)} \mid \mathcal{F}_t \right]
\]
so that by Jensen inequality for conditional expectation we have
\[
\mathbb{E} \left[ |M_{t,\varepsilon}^{(2\alpha)} - M_{t}^{(2\alpha)}| \right] \leq \mathbb{E} \left[ |M_{t}^{(2\alpha)} - M_0^{(2\alpha)}| \right] \tag{7.49}
\]
and by Theorem A the right-hand side tends to zero when $\varepsilon$ goes to zero, yielding (7.47).
Final step: Bounding $B_{t,\varepsilon}$. Finally to bound $B_{t,\varepsilon}$ we proceed as we did for $B_t$. We show that

$$\lim_{\varepsilon \to 0} a(t,\varepsilon)^{-1}\mathbb{E}[|B_{t,\varepsilon} - \mathbb{E}[B_{t,\varepsilon} | \mathcal{F}_u]|] = 0,$$

(7.50)

For the first line we repeat the argument of Step 3 above while for the second line we use the same proof as in Step 2.

Acknowledgement: The author is thankful to J.F. Le Gall, Rémi Rhodes and Vincent Vargas for enlightening discussions. This work was realized during H.L. extended stay in Aix-Marseille University funded by the European Union’s Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 837793.

Appendix A. Proof of Lemma 2.1

Let $D'$ and $\delta$ be fixed. We consider $\kappa_0$ a $C^\infty$ kernel satisfying the assumptions listed below (2.1). Setting $\eta := (s - d)/2$ we consider $K^\delta$ defined by

$$K^\delta(x,y) := K(x,y) + \eta\delta \int_0^\infty e^{-\eta t} \kappa_0(e^t |x-y|)dt. \tag{A.1}$$

Note that $K'$ satisfies assumptions (A)-(B) since by construction

$$(K^\delta - K)(x,y) = \eta\delta \int_0^\infty e^{-\eta t} \kappa_0(e^t |x-y|)dt \tag{A.2}$$

is a positive definite kernel and is equal to $\delta$ on the diagonal. To prove that $K^\delta$ can be written in the form (2.1) we need to prove the following claim

Lemma A.1. If $t_0$ is sufficiently large then

$$K^\delta(x,y) - \int_{t_0}^\infty \kappa_0(e^t |x-y|)dt = K^\delta(x,y) - \int_0^{t_0} \kappa_0(e^{t_0+u}|x-y|)du.$$

is positive definite and Hölder continuous.

The lemma implies that (2.1) is satisfied for $K^\delta(x,y)$ with $\kappa(u) = \kappa_0(e^{t_0}u)$. It is a consequence of the two following estimates.

Lemma A.2. Given $s > d$, there exists a constant $C(s,\kappa_0,K)$ such that for any $\varphi \in C^\infty_c(D')$ we have

$$\int_{\mathbb{R}^{2d}} \left( K(x,y) - \int_{t_0}^\infty \kappa_0(e^t |x-y|)dt \right) \varphi(x)\varphi(y)dxdy \geq -Ce^{-\frac{(d-s)t_0}{2}}\|\varphi\|_{H^{-s/2}(\mathbb{R}^d)}^2 \tag{A.3}$$

Lemma A.3. For any $\varphi \in C^\infty_c(D')$ we have

$$\int \left( \eta\delta \int_0^\infty e^{-\eta t} \kappa_0(e^t |x-y|)dt \right) \varphi(x)\varphi(y)dxdy \geq \delta C(\kappa_0,s)\|\varphi\|_{H^{-s/2}(\mathbb{R}^d)}^2 \tag{A.4}$$

Let us first deduce Lemma A.1 from Lemma A.2 and Lemma A.3.
Proof of Lemma A.1. First note that
\[
K_0^\varepsilon(x, y) = \int_0^\infty \kappa_0(e^t|x - y|)dt
\]
\[
= L(x, y) + \eta \delta \int_0^\infty e^{-\eta t} \kappa_0(e^t|x - y|)dt + \left( \log \frac{1}{|x - y|} - \int_0^\infty \kappa_0(e^t|x - y|)dt \right). \quad (A.5)
\]
Each of the three terms are Hölder continuous on \(D' \times D'\) (\(L(x, y)\) because of its Sobolev regularity combined with Morrey’s inequality, the other two can be checked by hand). Now combining (A.3) and (A.4) we have
\[
\int_{\mathbb{R}^d} \left( K_0^\varepsilon(x, y) - \int_0^\infty \kappa_0(e^t|x - y|)dt \right) \varphi(x) \varphi(y)dxdy
\]
\[
\geq \left( \delta c(k_0, s) - C e^{-\frac{(d-1)s}{2}t_0} \right) \| \varphi \|_{H^{-s/2}(\mathbb{R}^d)}^2 \quad (A.6)
\]
and the r.h.s. is positive if \(t_0\) is chosen to be sufficiently large.

\[
\square
\]

Proof of Lemma A.2. First we notice that we have
\[
K(x, y) = \left( L(x, y) + \log \frac{1}{|x - y|} - \int_0^\infty \kappa(e^t|x - y|)dt \right) + \int_0^\infty \kappa(e^t|x - y|)dt
\]
\[
= \tilde{L}(x, y) + \tilde{K}(x, y), \quad (A.7)
\]
where \(\tilde{L} \in H^s_{\text{loc}}(D' \times D')\) (this is simply because \(\log \frac{1}{|x - y|} - \int_0^\infty \kappa(e^t|x - y|)dt\) is a \(C^\infty\) function). Now we consider \(\varepsilon > 0\) sufficiently small so that \(D' \subset D_\varepsilon\) (1.6). Recalling (1.8), we are going to use the \(\varepsilon\) subscript notation for convolution with \(\theta_\varepsilon\) on both coordinates for \(\tilde{L}\) also. Since \(K_\varepsilon\) is definite positive, it is sufficient to show that the inequality (A.3) holds for the following kernel
\[
K(x, y) - K_\varepsilon(x, y) - \int_0^\infty \kappa_0(e^t|x - y|)dt
\]
\[
= (\tilde{L} - \tilde{L}_\varepsilon)(x, y) + \int_0^{t_0} \kappa_0(e^t|x - y|)dt - \tilde{K}_\varepsilon(x, y) \quad (A.8)
\]
Now using [10] Lemma 4.6, we have for any \(\varphi \in C^\infty_c(D')\)
\[
\left| \int \varphi(x) \varphi(y)(\tilde{L} - \tilde{L}_\varepsilon)(x, y)dxdy \right| \leq C_{D'} \| \tilde{L} - \tilde{L}_\varepsilon \|_{H^s(\mathbb{R}^{2d})} \| \varphi \|_{H^{-s/2}(\mathbb{R}^{2d})}^2. \quad (A.9)
\]
To control the remaining part, let us set
\[
\psi(x) := \int \left( \tilde{K}_\varepsilon(x, y) - \int_0^{t_0} \kappa_0(e^t|x - y|)dt \right) \varphi(y)dy. \quad (A.10)
\]
We have using Plancherel Theorem
\[
\int_{\mathbb{R}^d} \varphi(x) \varphi(y) \left( \int_0^{t_0} \kappa_0(e^t|x - y|)dt - \tilde{K}_\varepsilon(x, y) \right) dxdy
\]
\[
= - \int_{\mathbb{R}^d} \varphi(x) \psi(x)dx = -(2\pi)^{-d} \int_{\mathbb{R}^d} \hat{\psi}(\xi) \hat{\varphi}(\xi)d\xi. \quad (A.11)
\]
Now using the formula for Fourier transform of convolution and rescaled functions, we have
\[ \hat{\psi}(\xi) = \left( \int_{0}^{\infty} (|\hat{\theta}(\varepsilon \xi)|^2 - 1)_{t \leq t_0} e^{-d\tau_{0}(e^{-t} \xi)} dt \right) \hat{\varphi}(\xi) =: T(\xi) \hat{\varphi}(\xi) \]  
(A.12)

To conclude we only need an upper bound on \( T(\xi) \). Note that since \( |\hat{\theta}(\xi)| \leq 1 \) for every \( \xi \) and, since by Bochner’s Theorem \( \hat{\varphi} \) is pointwise real and non-negative, we have
\[ T(\xi) \leq |\hat{\theta}(\varepsilon \xi)|^2 \int_{t_0}^{\infty} e^{-d\tau_{0}(e^{-t} \xi)} dt \leq \frac{1}{d} |\hat{\theta}(\varepsilon \xi)|^2 e^{-dt_0}. \]  
(A.13)

We can now set \( \varepsilon = e^{-\frac{(s+d)t_0}{2s}} \). Now since \( \theta \) is \( C^\infty \), \( \hat{\theta} \) decays faster than any polynomial, and hence we can find a constant \( C \) (depending on \( \theta \) and \( s \)) which is such that
\[ T(\xi) \leq Ce^{-dt_0}(1 + |\xi|^2)^{-s/2} \leq Ce^{-(d-s)t_0/2}(1 + |\xi|^2)^{-s}, \]  
(A.14)

which is sufficient to conclude.

Proof of Lemma A.3. Following the same computation as in (A.11)-(A.12) we obtain that
\[ \int \left( \eta \delta \int_{0}^{\infty} e^{-\eta \tau_{0}}(e^t|x-y|) dt \right) \varphi(x) \varphi(y) dx dy = \eta \delta \int \left( \int_{0}^{\infty} e^{-(\eta+d)t} \hat{\tau}_{0}(e^{-t} \xi) dt \right) \hat{\varphi}(\xi)^2 d\xi. \]  
(A.15)

Now as \( \hat{\tau}_{0} \) is non-negative (and positive around 0) we have
\[ \eta \int_{0}^{\infty} e^{-(\eta+d)t} \hat{\tau}_{0}(e^{-t} \xi) dt \geq c_\eta (1 + |\xi|^2)^{-(d+\eta)/2}, \]  
(A.16)

which is sufficient to conclude.

\[ \square \]

Appendix B. Proof of Proposition 1.2

In order to check the tightness we work with the Fourier transform \( \hat{M}_\varepsilon(\gamma; \rho)(\xi) \) of \( M_\varepsilon(\gamma; \rho) \) which is almost surely finite. Most of the time we will omit the dependence in \( \gamma, \rho \) for better readability, and simply write \( v(\varepsilon) \) for \( v(\varepsilon, \theta, \gamma) \).

We need to show that \( (v(\varepsilon)M_\varepsilon(\gamma; \rho)) \) is tight in \( H^{-u}(\mathbb{R}^d) \), which, by isometry, is equivalent to showing that \( v(\varepsilon)\hat{M}_\varepsilon \) is tight in the space \( \hat{H}^{-u}(\mathbb{R}^d) := L^2(\mathbb{R}^d, (1 + |\xi|^2)^{-u}d\xi) \).

To prove the later statement we are going to use the following variant of the Frechet-Kolmogorov compactness criterion (see e.g. [2, Theorem 4.26]).

Proposition B.1. A subset of \( K \) of \( \hat{H}^{-u}(\mathbb{R}^d) \) is relatively compact if and only if it satisfies the two following conditions
\[ \begin{align*} 
&\text{(i) } \lim_{R \to \infty} \sup_{\varphi \in K} \int_{|\xi| > R} |\varphi(\xi)|^2 (1 + |\xi|^2)^{-u}d\xi = 0, \\
&\text{(ii) } \lim_{a \to 0} \sup_{\varphi \in K} \int_{\mathbb{R}^d} |\varphi(\xi + a) - \varphi(\xi)|^2 (1 + |\xi|^2)^{-u}d\xi = 0. 
\end{align*} \]

Now the tightness of \( \hat{M}_\varepsilon \) will be proved using the following simple estimates.

Lemma B.2. We have
\[ \mathbb{E}[v(\varepsilon)^2 |\hat{M}_\varepsilon(\xi)|^2] \leq C(\rho) \]
\[ \mathbb{E}[v(\varepsilon)^2 |\hat{M}_\varepsilon(\xi + a) - \hat{M}_\varepsilon(\xi)|^2] \leq |a|^2. \]  
(B.1)
Proof. The proof of both bounds follows from direct computation. We have
\[
\mathbb{E}[|\widehat{M}_\varepsilon(\xi)|^2] = \int_{\mathbb{R}^d} \rho(x)\rho(y)e^{i\xi \cdot (x-y)}\mathbb{E}\left[e^{\gamma X_\varepsilon(x)+\gamma X_\varepsilon(y)} - \frac{1}{\gamma}K_\varepsilon(x) - \frac{1}{\gamma}K_\varepsilon(y)\right] dx dy
\]
\[
= \int_{\mathbb{R}^d} \rho(x)\rho(y)e^{i\xi \cdot (x-y)}e^{\gamma |x|}K_\varepsilon(x,y) dx dy \leq \int_{\mathbb{R}^d} \rho(x)\rho(y)e^{\gamma |x|}K_\varepsilon(x,y) dx dy. \quad (B.2)
\]
The last integral is of order \(v(\varepsilon)^2\). In the same fashion we have
\[
\mathbb{E}[|\widehat{M}_\varepsilon(\xi + a) - \widehat{M}_\varepsilon(\xi)|^2]
\]
\[
= \int_{\mathbb{R}^d} \rho(x)\rho(y) (e^{i\xi \cdot x} - e^{i(\xi + a) \cdot x}) (e^{-i\xi \cdot y} - e^{-i(\xi + a) \cdot y}) e^{\gamma |x|}K_\varepsilon(x,y) dx dy
\]
\[
\leq |a|^2 \int_{\mathbb{R}^d} \rho(x)\rho(y)|x||y|e^{\gamma |x|}K_\varepsilon(x,y) dx dy. \quad (B.3)
\]
Since the support of \(\rho\) is compact and hence bounded, the last integral is also of order \(v(\varepsilon)^2\).

Now given \(A\) and \(d/2 < u' < u\) we define \(K_A := K_A^{(1)} \cap K_A^{(2)}\) with
\[
K_A^{(1)} := \left\{ \varphi : \int_{\mathbb{R}^d} |\varphi(\xi)|^2 (1 + |\xi|^2)^{-u'} d\xi \leq A \right\}
\]
\[
K_A^{(2)} := \left\{ \varphi : \forall |a| \leq 1, \|\varphi(\cdot + a) - \varphi\|_{H^{-u}(\mathbb{R}^d)} \leq A \sqrt{|a|} \right\} \quad (B.4)
\]
It is immediate to check from Lemma \([B.1]\) that \(K_A^{(1)} \cap K_A^{(2)}\) is relatively compact. To conclude the proof of Proposition \([1.2]\) we only have to check the following

Lemma B.3. We have
\[
\lim_{A \to \infty} \sup_{\varepsilon \in (0,\varepsilon_0)} \mathbb{P}\left[v(\varepsilon)\widehat{M}_\varepsilon \notin K_A\right] = 0. \quad (B.5)
\]

Proof. To show that
\[
\lim_{A \to \infty} \sup_{\varepsilon \in (0,\varepsilon_0)} \mathbb{P}\left[v(\varepsilon)\widehat{M}_\varepsilon \notin K_A^{(1)}\right] = 0, \quad (B.6)
\]
it is sufficient to observe that from Proposition \([B.2]\) and the fact that \(s' > d/2\), we have
\[
\mathbb{E}\left[\int_{\mathbb{R}^d} v(\varepsilon)^2 |\widehat{M}_\varepsilon(\xi)|^2 (1 + |\xi|^2)^{-u'} d\xi\right] < C'(\rho). \quad (B.7)
\]
Then \((B.6)\) simply follows from Markov inequality. Let us now prove
\[
\lim_{A \to \infty} \sup_{\varepsilon \in (0,\varepsilon_0)} \mathbb{P}\left[v(\varepsilon)\widehat{M}_\varepsilon \notin K_A^{(2)}\right] = 0. \quad (B.8)
\]
We introduce
\[
\Phi(u) := \max_{\xi \in \mathbb{R}^d} \left(\frac{1 + |\xi - a|^2}{1 + |\xi|^2}\right)^{u/2} \quad (B.9)
\]
Using again Lemma \([B.2]\) and Markov inequality we have given \(b \in \mathbb{R}^d\)
\[
\mathbb{P}\left[\int_{\mathbb{R}^d} v(\xi)^2 |\widehat{M}_\varepsilon(\xi + b) - \widehat{M}_\varepsilon(\xi)|^2 (1 + |\xi|^2)^u d\xi \geq u\right] \leq \frac{C|b|^2}{u}. \quad (B.10)
\]
We apply it to $b_{k,i} = 2^{-k}e_i$ for $k \geq 1$ and $i \in [1,d]$ where $e_i$ are the unit coordinate vectors. Hence setting

$$K_A^{(3)} := \left\{ \varphi : \forall (k, i) \in \mathbb{N} \times [1, d], \| \varphi(\cdot + b_{k,i}) - \varphi\|_{\tilde{H}^{-u}(\mathbb{R}^d)} \geq \frac{A 2^{-k/2} (\sqrt{2} - 1)}{\sqrt{2} d \Phi(u)} \right\}. \quad (B.11)$$

We obtain after a union bound over $i$ and $k$ that

$$\mathbb{P} \left[ \nu(\xi) \hat{M}_\varepsilon \notin K_A^{(3)} \right] \leq \frac{C}{A^2} \quad (B.12)$$

Then (B.8) follows from the inclusion $K_A^{(3)} \subset K_A^{(2)}$, which we prove now. Consider $\varphi \in K_A^{(3)}$. Now note that for any $\phi$

$$\max_{|a| \leq 1} \frac{\| \phi(\cdot + a) \|_{\tilde{H}^{-u}(\mathbb{R}^d)}}{\| \phi \|_{\tilde{H}^{-u}(\mathbb{R}^d)}} \leq \Phi(s). \quad (B.13)$$

Applying this to $\phi = \varphi(\cdot + b_{i,k}) - \varphi$, we obtain that for all $(k, i) \in \mathbb{N} \times [1, d]$, and $|a| \leq 1$

$$\| \varphi(\cdot + a + b_{i,k}) - \varphi(\cdot + a) \|_{\tilde{H}^{-u}(\mathbb{R}^d)} \leq \frac{A 2^{-k/2} (\sqrt{2} - 1)}{\sqrt{2} d} \quad (B.14)$$

Given $a$ with $2^{-k_0} \leq |a| \leq 2^{1-k_0}$, assuming without loss of generality that all coordinates are positive we can write $a$ in the following form

$$a := \sum_{i=1}^{d} \sum_{k \geq k_0} \chi(k, i, a) b_{k,i},$$

with $\chi(k, i, a) \in \{0, 1\}$ (the decomposition is not necessarily unique). Write

$$a_{k,i} := \sum_{j=1}^{d} \sum_{m \geq k_0} \chi(m, i, a) b_{m,j} (1 \{ m \leq k-1 \} + 1 \{ m = k, j \leq i \}).$$

Then using (B.14) and the triangle inequality we obtain that for every $\phi \in K_A^{(3)}$, $k \geq k_0$ and $i \in [1,d]$

$$\| \varphi(\cdot + a_{i,k}) - \varphi \|_{\tilde{H}^{-u}(\mathbb{R}^d)} \leq A \left(1 - \frac{1}{\sqrt{2}}\right) \sum_{m=k_0}^{k} 2^{-k/2} \leq A 2^{-k_0/2} \leq A \sqrt{|a|}. \quad (B.15)$$

Passing to the limit we obtain that $\| \varphi(\cdot + a) - \varphi \|_{\tilde{H}^{-u}(\mathbb{R}^d)} \leq A \sqrt{|a|}$, and thus that $\varphi \in K_A^{(2)}$, which concludes the proof.

\[\square\]

References

[1] Nathanaël Berestycki. An elementary approach to Gaussian multiplicative chaos. *Electron. Commun. Probab.*, 22:Paper No. 27, 12, 2017.

[2] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[3] B. Derrida, M. R. Evans, and E. R. Speer. Mean field theory of directed polymers with random complex weights. *Comm. Math. Phys.*, 156(2):221–244, 1993.

[4] Bertrand Duplantier, Rémi Rhodes, Scott Sheffield, and Vincent Vargas. Critical Gaussian multiplicative chaos: convergence of the derivative martingale. *Ann. Probab.*, 42(5):1769–1808, 2014.

[5] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185(2):333–393, 2011.

[6] Lisa Hartung and Anton Klimovsky. The glassy phase of the complex branching Brownian motion energy model. *Electron. Commun. Probab.*, 20:no. 78, 15, 2015.
[7] Lisa Hartung and Anton Klimovsky. The phase diagram of the complex branching Brownian motion energy model. *Electron. J. Probab.*, 23:Paper No. 127, 27, 2018.

[8] Janne Junnila and Eero Saksman. Uniqueness of critical gaussian chaos. *Electron. J. Probab.*, 22:31 pp., 2017.

[9] Janne Junnila, Eero Saksman, and Lauri Viitasaari. On the regularity of complex multiplicative chaos. *arXiv e-prints*, page arXiv:1905.12027, May 2019.

[10] Janne Junnila, Eero Saksman, and Christian Webb. Decompositions of log-correlated fields with applications. *Ann. Appl. Probab.*, 29(6):3786–3820, 2019.

[11] Janne Junnila, Eero Saksman, and Christian Webb. Imaginary multiplicative chaos: moments, regularity and connections to the Ising model. *Ann. Appl. Probab.*, 30(5):2099–2164, 2020.

[12] Zakhar Kabluchko and Anton Klimovsky. Complex random energy model: zeros and fluctuations. *Probab. Theory Related Fields*, 158(1-2):159–196, 2014.

[13] Jean-Pierre Kahane. Sur le chaos multiplicatif. (On multiplicative chaos). *Ann. Sci. Math. Qué.*, 9:105–150, 1985.

[14] Hubert Lacoin. A universality result for subcritical Complex Gaussian Multiplicative Chaos. *arXiv e-prints*, page arXiv:2003.14024, March 2020.

[15] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. A probabilistic approach of ultraviolet renormalisation in the boundary Sine-Gordon model. *to appear in Probab. Theory Related Fields*.

[16] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. Complex Gaussian multiplicative chaos. *Comm. Math. Phys.*, 337(2):569–632, 2015.

[17] Jean-François Le Gall. *Brownian motion, martingales, and stochastic calculus*, volume 274 of *Graduate Texts in Mathematics*. Springer, 2016.

[18] Thomas Madaule, Rémi Rhodes, and Vincent Vargas. Glassy phase and freezing of log-correlated gaussian potentials. *Ann. Appl. Probab.*, 26(2):643–690, 04 2016.

[19] Ellen Powell. Critical Gaussian multiplicative chaos: a review. *arXiv e-prints*, page arXiv:2006.13767, June 2020.

[20] Rémi Rhodes and Vincent Vargas. Gaussian multiplicative chaos and applications: a review. *Probab. Surv.*, 11:315–392, 2014.

[21] Raoul Robert and Vincent Vargas. Gaussian multiplicative chaos revisited. *Ann. Probab.*, 38(2):605–631, 2010.

IMPA, Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110 Rio de Janeiro, CEP-22460-320, Brasil.