The dependence of relative dispersion on turbulence scales in Lagrangian Stochastic Models

A. Maurizi\(^1\), G. Pagnini\(^{1,2}\) and F. Tampieri\(^1\)

\(^1\)ISAC-CNR, via Gobetti 101, I-40129 Bologna, Italy
\(^2\)Facoltà di Scienze Ambientali, Università di Urbino, Campus Scientifico Sogesta, I-61029 Urbino, Italy

October 29, 2018

Abstract

The aim of the article is to investigate the relative dispersion properties of the Well Mixed class of Lagrangian Stochastic Models. Dimensional analysis shows that given a model in the class, its properties depend solely on a non-dimensional parameter, which measures the relative weight of Lagrangian-to-Eulerian scales. This parameter is formulated in terms of Kolmogorov constants, and model properties are then studied by modifying its value in a range that contains the experimental variability. Large variations are found for the quantity \( g^* = 2gC_\alpha^{-1} \), where \( g \) is the Richardson constant, and for the duration of the \( t^3 \) regime. Asymptotic analysis of model behaviour clarifies some inconsistencies in the literature and excludes the Ornstein-Uhlenbeck process from being considered a reliable model for relative dispersion.

1 Introduction

Relative dispersion is a process that depends on the combination of the Eulerian and Lagrangian properties of turbulence. If particle separation falls in the inertial subrange, the Eulerian spatial structure affects the dispersion, which can be regarded as a Lagrangian these property (Monin and Yaglom, 1975). The combination of properties requires that both descriptions be considered (see e.g., Boffetta et al., 1999).

Lagrangian Stochastic Modelling (LSM) is one turbulence representation that naturally combines Eulerian spatial structure and Lagrangian temporal correlation. In fact, as formulated by Thomson (1990) using the Well Mixed Condition (WMC), Lagrangian and Eulerian statistics are accounted for through the second order Lagrangian structure function and the probability density function (pdf) of Eulerian velocity. Several studies prove that this approach leads to the qualitative reproduction of the main properties, as expected from the Richardson theory (see Thomson, 1990; Reynolds, 1995; Sawford, 2001, among others). Furthermore, recent experimental studies seem to confirm the validity of
the Markovianity assumption for the velocity (Porta et al., 2001; Renner et al., 2001; Mordant et al., 2001).

However, the intrinsic non-uniqueness of the WMC formulation (see, e.g., Sawford, 1999) and the indeterminacy of the Kolmogorov constants (see, e.g., Sreenivasan, 1995; Anfossi et al., 2000, for reviews) do not allow for a completely reliable representation of the process. In particular, the value of the Richardson constant predicted by previous studies is not uniquely determined (see, among others, Thomson, 1990; Kurbanmuradov, 1997; Borgas and Sawford, 1994; Reynolds, 1999). Whether this indeterminism is a result of the different formulation of models, or of the different values of the parameters adopted is still unclear, and no systematic studies have been performed so far.

It is worth noting that, even focusing attention only on the dependence on the model constants produces significant variability. As an example, Borgas and Sawford (1994) present the variation of the Richardson constant value with the Lagrangian Kolmogorov constant $C_0$.

The aim of this article is to investigate the general properties of models based on the WMC with regard to inertial subrange relative dispersion features. In Section 2 the properties of the WMC are evidenced through a dimensional analysis, while the limit for vanishing spatial correlation is studied in Section 3. Subsequently a model formulation is discussed in Section 4 and results analysed in Section 5.

2 The non-dimensional form of the well mixed condition

Following the logical development of Thomson (1987), Thomson (1990) (hereinafter T90) extended the method for the selection of single particle Lagrangian Stochastic Models to models for the evolution of particle pair statistics. In the latter models, the state of a particle pair is represented by the joint vector of position and velocity $(\mathbf{x}, \mathbf{u}) \equiv (x^{(1)}, x^{(2)}, u^{(1)}, u^{(2)})$, where the upper index denotes the particle, whose evolution is given by the set of Langevin type equations (LE) (with implied summation over repeated indices):

$$\begin{cases}
    \frac{dx_i}{dt} = u_i \, dt \\
    \frac{du_i}{dt} = a_i(x, u, t) \, dt + b_{ij}(x, t) \, dW_j(t),
\end{cases}$$

where $i, j = 1..6$. The coefficients $a$ and $b$ are determined, as usual, through the well known Well Mixed Condition (Thomson, 1987) and the consistency with the inertial subrange scaling, respectively. Further details are not given here, in that they are well established and widely used in the literature (see, e.g., Sawford, 2001, for a review). The only remark we would make is that, although Thomson (1987) himself studied this alternative, the tensor $b_{ij}$ cannot be dependent on $u$ in order to allow Eq. (1) to describe a physically meaningful process. In fact, as shown, for instance, by van Kampen (1981), the Itô and Stratonovich calculus give different results when $b_{ij} = b_{ij}(u)$. In particular, the WMC would not have a unique definition. Thus, from now on, $b_{ij} = \sqrt{C_{0i} \varepsilon \delta_{ij}}$, $i, j = 1..6$ will be used according to the usual scaling of Lagrangian structure function (Thomson, 1987), where $\varepsilon$ is the mean dissipation rate of turbulent kinetic energy.
It should be remembered here that the WMC is satisfied by constraining the Fokker-Planck equation associated to Eq. (1) (see, e.g., Gardiner 1990) to be consistent with the Eulerian probability density function of the flow. In the case of particle pairs the considered pdf is the one-time, two-point joint pdf of $x^{(i)}$ and $u^{(i)}$, $i = 1, 2$, accounting for the spatial structure of the turbulent flow considered. The open question about the non-uniqueness of the solution in more than one dimension (see, e.g., Sawford 1999) is not addressed here. However, the following analysis will show that the problem studied is independent of the particular solution selected.

In order to highlight the effect of turbulence features on the model formulation, characteristic scales for particle pair motion must be identified. Because the process of relative dispersion has to deal with both Eulerian and Lagrangian properties (see, e.g., Monin and Yaglom 1975, p. 540), such scales can be defined by considering the second order Eulerian and Lagrangian structure functions, i.e.,

$$
\langle \Delta v^2 \rangle \sim C_K (\varepsilon \Delta r)^{2/3}
$$

(2)

for Eulerian velocity $v$ for a separation $\Delta r = ||\Delta r||$, according to the standard Kolmogorov (1941) theory (hereinafter K41), and

$$
\langle \Delta u^2 \rangle \sim C_0 (\varepsilon t)
$$

(3)

for Lagrangian velocity $u$ (see, e.g., Monin and Yaglom 1975), where $\Delta v = ||v(r + \Delta r) - v(r)||$ and $\Delta u = ||u(t + dt) - u(t)||$. A length scale $\lambda$ can be defined in the Eulerian frame, so that in the inertial subrange (namely, for $\eta \ll r \ll \lambda$ where $\eta$ is the Kolmogorov microscale) the structure function for each component may be written as

$$
\langle \Delta v_i^2 \rangle = 2 \sigma^2 (\Delta r / \lambda)^{2/3}
$$

(4)

where $\sigma = \sqrt{\langle |v|^2 \rangle / 3}$, which together with Eq. (2) provides a definition for $\lambda$.

A Lagrangian time scale $\tau$ can be defined in a similar way using Eq. (3) and the Lagrangian version of Eq. (1). Thus, for $\tau_\eta \ll t \ll \tau$, one has

$$
\langle \Delta u_i^2 \rangle = 2 \sigma^2 t / \tau
$$

(5)

from which one can retrieve the known relationship

$$
\varepsilon = \frac{2 \sigma^2}{C_0 \tau}
$$

(6)

suggested by Tennekes (1982). It should be observed that scales for the inertial subrange, at variance with their integral version, can be defined independently of non-homogeneity or unsteadiness, provided that the scales of such variations are sufficiently large to allow an inertial subrange to be identified. As far as the velocity is concerned, $\sigma$ can be recognised as the appropriate scale of turbulent fluctuations in both descriptions.

The quantities $\sigma$, $\lambda$ and $\tau$ can then be used respectively to make velocity $u_i$, position $x_i$ and time $t$ non-dimensional. They also form a non-dimensional parameter

$$
\beta = \frac{\sigma \tau}{\lambda} = \frac{C_K^{3/2}}{\sqrt{2} C_0},
$$

(7)
the last equality being based on the combination of Eqs. (2) and (3) with (4) and (5). The parameter $\beta$ can be recognised as a version of the well known Lagrangian-to-Eulerian scale ratio. The approach adopted here evidences its connection to fundamental constants of the K41 theory.

In non-dimensional form, Eq. (1) reads
\begin{align}
\frac{dx_i}{dt} &= \beta u_i dt \\
\frac{du_i}{dt} &= a_i dt + \sqrt{2} dW_i(t).
\end{align}
(8)

where, with a change of notation with respect to Eq. (1), all the quantities involved are without physical dimensions.

The associated Fokker-Planck equation is
\begin{equation}
\frac{\partial p_L}{\partial t} + \beta u_i \frac{\partial p_L}{\partial x_i} + \frac{\partial a_i p_L}{\partial u_i} = \frac{\partial^2 p_L}{\partial u_i \partial u_i},
\end{equation}
(9)

where $p_L$ is the pdf of the Lagrangian process described by Eq. (8) for some initial conditions. Using the WMC, $a$ can be written as
\begin{equation}
a_i = \frac{\partial \ln p_E}{\partial u_i} + \phi_i
\end{equation}
(10)

where
\begin{equation}
\frac{\partial \phi_i p_E}{\partial u_i} = -\frac{\partial p_E}{\partial t} - \beta u_i \frac{\partial p_E}{\partial x_i}
\end{equation}
(11)

and $p_E$ is the Eulerian one-time, two-point joint pdf of $x$ and $u$.

An advantage of this choice of scales emerges clearly in Eq. (9). It shows that, given a Eulerian pdf, once the non-uniqueness problem is solved by selecting a suitable solution to Eq. (10), or applying a further physical constraint to Eq. (11) [Sawford (1999)], any solution of Eq. (9) will depend on one parameter only, namely on the Lagrangian-to-Eulerian scale ratio. It can also be observed that this dependence is completely accounted for by the non-homogeneity term, which is an intrinsic property of the particle pair dispersion process in spatially structured velocity fields.

In looking for the universal properties of pair-dispersion in the inertial subrange, it is useful to rewrite the Richardson $t^3$ law in non-dimensional form, i.e., $\Delta x^2 = g^* \beta^2 t^3$ where $g^* = 2g/C_0$. In this form, the numerical value of the “normalised” Richardson constant $g^*$ depends on $\beta$ only. This dependence is investigated in the following Sections to highlight the intrinsic properties of the LSM.

3 The spatial decorrelation limit

In the limit $\beta \to \infty$, corresponding to a vanishing Eulerian correlation scale, the non-dimensionalisation defined in the previous section fails to apply. However, in this limit, the WMC solution can be proven to reduce to an homogeneous process (see Appendix). In particular, selecting a Gaussian pdf will give the Ornstein-Uhlenbeck (OU) process. It is worth noting that the OU process has sometimes been used to describe Lagrangian velocity in turbulent flows, for instance by Gifford [1982], who pioneered the stochastic approach to atmospheric
dispersion. The Novikov (1963) model and the NGLS model (Thomson, 1990, p. 124) are simple applications of this concept.

Adopting the choices made in the previous Section, but using the spatial scale defined by \( \tau \sigma \) rather than the vanishing \( \lambda \) as a length scale, the OU process equivalent to Eq. (8) is described by the non-dimensional set of linear LE

\[
\begin{align*}
\frac{dx_i}{dt} &= u_i dt \\
\frac{du_i}{dt} &= -u_i dt + \sqrt{2} dW_i
\end{align*}
\] (12)

where \( i = 1..6 \). The equations for the relative quantities \( (\Delta u_i, \Delta x_i) \) can be obtained from the difference between quantities relative to the first \( (i = 1, 2, 3) \) and second \( (i = 4, 5, 6) \) particles. The resulting set of equations reads

\[
\begin{align*}
\frac{d\Delta x_i}{dt} &= \Delta u_i dt \\
\frac{d\Delta u_i}{dt} &= -\Delta u_i dt + 2 dW_i
\end{align*}
\] (13)

where \( i = 1..3 \).

Equation (13) can be solved analytically for correlation functions and variances (see e.g., Gardiner, 1990). Some basic results are summarised below (see also Gifford, 1982).

The second order moment of velocity difference turns out to be an exponential function dependent on the time interval only

\[
\langle (\Delta u_i - \Delta u_{0i})^2 \rangle = \langle \Delta u_{0i}^2 \rangle \exp(-t) .
\] (14)

By integrating Eq. (14), the displacement variance for a single component is

\[
\langle (\Delta x_i - \Delta x_{0i})^2 \rangle = (\langle \Delta u_{0i}^2 \rangle - 2)(1 - \exp(-t)) + 4t - 4(1 - \exp(-t)) .
\] (15)

For short times (but expanding Eq. (15) to the third power of \( t \)), it turns out that

\[
\langle (\Delta x_i - \Delta x_{0i})^2 \rangle \simeq \langle \Delta u_{0i}^2 \rangle t^2 + \left(\frac{4}{3} - \langle \Delta u_{0i}^2 \rangle \right) t^3.
\] (16)

From Eq. (16) it can be observed that, when initial relative velocity \( \Delta u_{0i} \) is distributed in equilibrium with Eulerian turbulence (i.e., \( \langle \Delta u_{0i}^2 \rangle = 2 \)), a \( t^2 \) regime takes place with a negative \( t^3 \) correction (Hunt, 1985). On the other hand, if \( \langle \Delta u_{0i}^2 \rangle = 0 \) the ballistic regime displays a \( t^3 \) growth with a coefficient 4, i.e., \( 2C_0 \) for the dimensional version (Novikov, 1963; Monin and Yaglom, 1975; Borgas and Sawford, 1991).

4 Model formulation and numerical simulations

In order to proceed with the analysis of the dependence of model features on parameter \( \beta \), we select as a possible solution to Eq. (10), the expression given by T90 (his eq. 18) for Gaussian pdf. The spatial structure is accounted for using the Durbin (1980) formula for longitudinal velocity correlation, which is compatible with the 2/3 scaling law in the inertial subrange. Although this form is known not to satisfy completely the inertial subrange requirements (it prescribes a Gaussian distribution for Eulerian velocity differences, while inertial subrange requires a non-zero skewness), it has been successfully used in basic studies (Borgas and Sawford, 1994) and applications (Reynolds, 1999), and provides a useful test case for studying the results shown above.
The stochastic model is formulated for the variable \((x,u)\) rather than for the variable \((\Delta x/\sqrt{2}, \Delta u/\sqrt{2})\) as in Thomson’s original formulation. In the present case, assuming homogeneous and isotropic turbulence, the covariance matrix \(\mathcal{V}(x)\) of the Eulerian pdf is expressed by

\[
\mathcal{V} = \begin{pmatrix}
    I & \mathcal{R}^{(1,2)}(x)
    \\
    \mathcal{R}^{(2,1)}(x) & I
\end{pmatrix}
\]

(17)

where \(I\) is the identity matrix and

\[
\mathcal{R}^{(p_1,p_2)}(x) = \langle u_i^{(p_1)} u_j^{(p_2)} \rangle
\]

(18)

where \(p_1, p_2 = 1, 2 \ (p_1 \neq p_2)\) are the particles indices. The quantity \(\langle u_i^{(p_1)} u_j^{(p_2)} \rangle \equiv \langle u_i(x^{(p_1)}) u_j(x^{(p_2)}) \rangle\) is the two-point covariance matrix, which is expressed in terms of longitudinal and transverse functions \(F\) and \(G\) (see, e.g., Batchelor, 1953) as

\[
\mathcal{R}_{ij} = F(\Delta x) \Delta x_i \Delta x_j + G(\Delta x) \delta_{ij}
\]

(19)

where \(\Delta x = ||x^{(1)} - x^{(2)}||\),

\[
F = -\frac{1}{2\Delta x} \frac{\partial f}{\partial \Delta x}
\]

(20)

and

\[
G = f + \frac{\Delta x}{2} \frac{\partial f}{\partial \Delta x}
\]

(21)

It goes without saying that \(\mathcal{R}_{ij}^{(p_1,p_2)} = \mathcal{R}_{ij}^{(p_2,p_1)} = \mathcal{R}_{ji}^{(p_1,p_2)}\). As in Durbin (1980), \(F\) and \(G\) are computed from the parallel velocity correlation

\[
f(\Delta x) = 1 - \left(\frac{\Delta x^2}{\Delta x^2 + 1}\right)^{1/3}
\]

(22)

which is K41 compliant for \(\Delta x \ll 1\).

Using the above formulation, Eqs. (8) were solved numerically for a number of trajectories large enough to provide reliable statistics for the relevant quantities. Particular attention was paid to the time-step–independence of the solution (details are not reported here). It was found that the time step strongly depends on \(\beta\) because large values of the parameter increase non-homogeneity, which requires greater accuracy. Despite the widespread use of variable–time-step algorithms (see, e.g., Thomson, 1990; Schwere et al., 2002) based, in particular, on spatial derivatives, here a fixed time step short enough for time-step independence of the solution was used throughout the computation.

Simulations were performed for two different initial conditions for velocity difference: i) the distributed case, where velocity differences are given according to the second-order Eulerian structure function and ii) the delta case \((\Delta u_{ij}^2)_0 = 0\), where both particles of a pair are released with the same velocity, which is normally distributed with variance 1. The two cases correspond to the limiting cases considered in Section 3. The former describes “real” fluid particles, i.e., particles distributed like fluid at all times, while the latter represents, from the point of view of relative dispersion, marked particles leaving a “forced” source, where they were completely correlated (as for a jet). The initial condition for the spatial variable was \(\Delta x_0 = 10^{-5}\beta\) for all simulations. It can be noted
that this corresponds to different positions in the inertial subrange for different simulations ($\Delta x_0$ differs from case to case). However, the dimensional $\lambda \Delta x_0$ is chosen small enough to provide at least three decades of inertial subrange.

The $\beta$ parameter was varied in the range $[10^{-2} : 10^2]$, well beyond physically meaningful values. In fact, values reported in the literature range from $O(10^{-1})$ to $O(10^1)$ (Hinze, 1955; Hanna, 1981; Sato and Yamamoto, 1987; Koeltzsch, 1994) with $\beta = O(1)$ taken as a reference (Corrsin, 1963). This choice was made in order to infer asymptotic properties of the model. Note that, from a numerical point of view, different values of $\beta$ were obtained by varying the length scale $\lambda$, keeping $\sigma$, $\tau$ and $C_0$ fixed. In other words, with reference to Eq. (7), the variation of $\beta$ was obtained by varying $C_K$.

5 Results and discussion

Figures 1 (from a to i) show the results of simulations for the two initial conditions and for different values of $\beta$. The non-dimensional quantity $\langle \Delta x^2 \rangle \beta^{-2}$ is plotted against the non-dimensional time $t$. The OU analytical solutions ($\beta = \infty$) are reported for reference. The general behaviour qualitatively fulfills the expectations of Taylor (1921) and Richardson (1926). It presents an initial ballistic regime which differs for the two cases: the distributed case shows a $t^2$, while the delta case presents a “false” $t^3$ according to Eq. (16). After the ballistic regime there is a transition to an inertial range $t^3$ regime, which then becomes well established until a pure diffusive regime takes place.

This generically correct behaviour merits further consideration. A “true” $t^3$ is observed, which depends on the spatial flow structure and influences dispersion properties. In particular, increasing $\beta$ causes an increase in the normalised Richardson coefficient $g^*$ (Fig. 2). It is worth noting that the “false” $t^3$ regime, according to the findings reported in Sect. 3, is not dependent on the structure, and therefore does not vary with $\beta$. In fact, as pointed out by Sawford (2001), there should be a range where $\langle \Delta x^2 \rangle \beta^{-2} = 4t^3$ for $t \ll t_0$, $t_0$ being the time at which memory of the initial conditions is lost (see also Borgas and Sawford, 1991). It is clear now that this regime does not originate from any spatial structure and is intrinsic to the solution with the delta initial condition, as explained by Eq. (16).

According to Monin and Yaglom (1975, p. 541), the “true” $t^3$ regime should be independent of the initial conditions. Thus, the starting point of this regime can be selected at the point where the solutions for the two cases coincide, as clearly occurs in Figs. 1a to f. Therefore, the temporal extension of the $t^3$ regime is probably shorter than the one that could be estimated using intersections with the idealised ballistic regime, on the one hand, and with the diffusive regime, on the other. Note, however, that the extension of the inertial regime remains a decreasing function of $\beta$, which asymptotically converges to zero.

Another point of interest evident in Figs. 1 and 2 is that the present results do not agree with the theoretical findings of Borgas and Sawford (1991) (hereinafter BS91), although they compare well with the numerical results of Borgas and Sawford (1994) (hereinafter BS94). In fact, BS94 (their last figure) showed the results obtained by varying $C_0$ in their models. However, as shown

\[1\] In the sense that it is only a correction to the ballistic $t^2$ regime, which depends on the initial conditions and not on spatial structure.
Figure 1: Mean square separation normalised with $\beta$ as a function of time for different values of $\beta$. Thick lines represent results of present simulations, while thin lines are the analytical Ornstein-Uhlenbeck solutions (continuous: distributed case; dotted: delta case). a) $\beta = 0.01$, b) $\beta = 0.1$, c) $\beta = 0.2$. 
Figure 1: (continued) d) $\beta = 0.5$, e) $\beta = 1$, f) $\beta = 2$. 
Figure 1: (continued) g) $\beta = 5$, h) $\beta = 10$, i) $\beta = 100$. 
in Sect. 2, β is the only parameter on which the model depends. Because of the constancy of ε in BS94, the variation of C_0 corresponds to a variation of τ, and hence β. The results of BS94 for the implementation of the T90 model are reported in Fig. 2, and show a complete agreement with the present results. The values indicated do not satisfy the kinematic constraint $g = 2C_0 - \gamma$, where γ is a positive quantity, proposed by BS91, based on a double asymptotic expansion. It should be observed, however, that γ is derived from kinematic features and depends on integrals of correlation functions. For vanishing correlation, one obtains γ → 0, suggesting that in BS91 the ballistic part of the OU process is an upper limit for dispersion in the Richardson regime.

Nevertheless, this discrepancy can be explained as follows. From Eq. (16) it is clear that, at any time $t < t_{\text{diff}}$, where $t_{\text{diff}}$ is the time when pure diffusion takes place, the displacement variance for the OU process in the distributed case is always larger than the displacement variance for the OU process in the delta case. The two cases represent the limit for any process based on the WMC for $\beta \to \infty$, as shown in Sect. 3. In particular, focusing attention on the time range between the ballistic and diffusive regimes, it can be observed that $\lim_{\beta \to \infty} \text{T90}(\text{distributed}) = \text{OU}(\text{distributed})$ and, furthermore, $\text{T90}(\text{distributed}) < \text{OU}(\text{distributed})$ for any finite β. It can be concluded that a $\beta'$ must exist for which $\text{T90}(\text{distributed}) = \text{OU}(\text{delta})$, and $\text{T90}(\text{distributed}) > \text{OU}(\text{delta})$ for $\beta > \beta'$, in disagreement with BS91. Recalling that the OU process is the limit for any WMC process with Gaussian $p_E$, this result can be considered to be applicable to more general kinematic properties, which should therefore depend on the ratio between Eulerian and Lagrangian scales. Thus, the limitation to $g^*$ in the BS91 derivation possibly derives from an implicit assumption concerning the spatial structure and/or the value of β, which defines a range of applicability of the result.

Proceeding further with the analysis, it can also be said that, because of the
existence of a time $t_0$ after which the solution is not dependent on the initial conditions, it might be expected that $T_{90}(\text{distributed}) = T_{90}(\text{delta})$ for $t > t_0$. However, as the Lagrangian time increases with respect to $\sigma^{-1}\lambda$, an increasing number of particle pairs reaches the end of the inertial range ($\Delta x \gg 1$) still remembering their initial conditions. This results in a range of $\beta > 1$ in which the $\text{delta}$ solution never reaches the $\text{distributed}$ solution before the onset of the diffusive regime. Therefore, it is not possible to define any $g^\ast$. Nevertheless, for $\beta > \beta'$ there exists a range of $t$ where $T_{90}(\text{delta}) > \text{OU}(\text{delta})$, which shows that $T_{90}(\text{delta})$ converges to $\text{OU}(\text{delta})$ in a non-monotonic way.

When, for $\beta \lesssim 1$, the expected independence on the initial conditions is recovered, it can be noted that $t_0$ itself is a function of $\beta$. Thus, the duration of the $t^3$ regime depends also (and mainly) on the starting time of the diffusive regime. It is observed that decreasing $\beta$ increases the time at which the diffusion regime becomes fully developed.

### 6 Conclusions

The dimensional analysis of the WMC, through the non-dimensionalisation of the Fokker-Planck equation has shown that only one parameter plays a role in the determination of two particle dispersion properties. This parameter is the Lagrangian-to-Eulerian scale ratio $\beta$, which can be reliably defined in terms of inertial subrange constants. The dimensional analysis leads to the definition of a normalised Richardson constant $g^\ast$ whose scale is identified with $C_0$, as suggested by the comparison of Lagrangian and Eulerian properties. Given a particular model, the numerical value of $g^\ast$ depends solely on the value of $\beta$ adopted. This also applies to the duration of the $t^3$ regime.

Using the T90 formulation, it has been shown that the results of Novikov (1963) are recovered for $\beta \to \infty$, which means that in the model the spatial structure is negligible with respect to the Lagrangian time correlation. This limit corresponds to the OU process, whose general properties highlight that the observed $t^3$ growth is actually a correction to the ballistic regime $t^2$. Moreover, because of the absence of any genuine $t^3$ regime, it is not possible to define any Richardson coefficient. This means that $2C_0$ cannot be considered in general as the upper limit for $g$. Therefore there is no inconsistency in models that produce $g > 2C_0$, as occurs in the present study and in BS94.

### Acknowledgements

G. Pagnini is supported by the CNR-fellowship n. 126.226.BO.2.

### References

Anfossi, D., G. Degrazia, E. Ferrero, S. E. Gryning, M. G. Morselli, and S. T. Castelli, 2000: Estimation of the Lagrangian structure function constant $C_0$ from surface-layer wind data. Boundary-Layer Meteorol., 95, 249–270.

Batchelor, G., 1953: The theory of homogeneous turbulence, 1970th ed., Cambridge University Press.
Boffetta, G., A. Celani, A. Crisanti, and A. Vulpiani, 1999: Pair dispersion in synthetic fully developed turbulence. Phys. Rev. E, 60, 6734–6741.

Borgas, M. S. and B. L. Sawford, 1991: The small-scale structure of acceleration correlations and its role in the statistical theory of turbulent dispersion. J. Fluid Mech., 228, 295–320.

Borgas, M. S. and B. L. Sawford, 1994: A family of stochastic models for two-particle dispersion in isotropic homogeneous stationary turbulence. J. Fluid Mech., 279, 69–99.

Corrsin, S., 1963: Estimates of the relations between Eulerian and Lagrangian scales in large Reynolds number turbulence. J. Atmos. Sci., 20, 115–119.

Durbin, P. A., 1980: A stochastic model for two-particle dispersion and concentration fluctuations in homogeneous turbulence. J. Fluid Mech., 100, 279–302.

Gardiner, C. W., 1990: Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences, 2nd ed., Springer-Verlag.

Gifford, F. A., 1982: Horizontal diffusion in the atmosphere: a Lagrangian-dynamical theory. Atmos. Environ., 15, 505–512.

Hanna, S., 1981: Lagrangian and Eulerian time-scale relations in the daytime boundary layer. J. Appl. Meteorol., 20, 242–249.

Hinze, J. O., 1959: Turbulence, Mc Graw-Hill, New York.

Hunt, J. C. R., 1985: Turbulent diffusion from sources in complex flows. Ann. Rev. Fluid Mech., 17, 447–485.

Koeltzsch, K., 1999: On the relationship between the Lagrangian and Eulerian time scale. Atmos. Environ., 33, 117–128.

Kolmogorov, A. N., 1941: The local structure of turbulence in incompressible viscous fluid for very large reynolds numbers. Dokl. Akad. Nauk SSSR, 30, 301.

Kurbanmuradov, O. A., 1997: Stochastic Lagrangian models for two-particle relative dispersion in high-Reynolds number turbulence. Monte Carlo Methods and Appl., 3, 37–52.

Monin, A. S. and A. M. Yaglom, 1975: Statistical fluid mechanics, vol. II, MIT Press, Cambridge, 874 pp.

Mordant, N., P. Metz, O. Michel, and J.-F. Pinton, 2001: Measurement of Lagrangian velocity in fully developed turbulence. Phys. Rev. Letters, 87, 214501/1–214501/4.

Novikov, E. A., 1963: Random force method in turbulence theory. Sov. Phys. JETP, 17, 1449–1454.

Porta, A. L., G. A. Voth, A. M. Crawford, J. Alexander, and E. Bodenschatz, 2001: Fluid particle acceleration in fully developed turbulence. Nature (London), 409, 1017–1019.
Renner, C., J. Peinke, and R. Friedrich, 2001: Experimental indications for Markov properties of small-scale turbulence. *J. Fluid Mech.*, **433**, 383–409.

Reynolds, A. M., 1999: The relative dispersion of particle pairs in stationary homogeneous turbulence. *J. Appl. Meteorol.*, **38**, 1384–1390.

Richardson, L. F., 1926: Atmospheric diffusion shown on a distance-neighbor graph. *Proc. R. Soc. London Ser. A*, **110**, 709–737.

Sato, Y. and K. Yamamoto, 1987: Lagrangian measurements of fluid-particle motion in an isotropic turbulent field. *J. Fluid Mech.*, **175**, 183–199.

Sawford, B. L., 1999: Rotation of trajectories in Lagrangian stochastic models of turbulent dispersion. *Boundary-Layer Meteorol.*, **93**, 411–424.

Sawford, B. L., 2001: Turbulent relative dispersion. *Ann. Rev. Fluid Mech.*, **33**, 289–317.

Schwere, S., A. Stohl, and M. W. Rotach, 2002: Practical considerations to speed up Lagrangian stochastic particle models. *Computer & Geosciences*, **28**, 143–154.

Sreenivasan, K. R., 1995: On the universality of the Kolmogorov constant. *Phys. of Fluids*, **7**, 2778–2784.

Taylor, G. I., 1921: Diffusion by continuous movements. *Proc. London Math. Soc.*, **20**, 196–211.

Tennekes, H., 1982: Similarity relations, scaling laws and spectral dynamics, *Atmospheric turbulence and air pollution modeling*, F. T. M. Nieuwstadt and H. van Dop, eds., Reidel, pp. 37–68.

Thomson, D. J., 1987: Criteria for the selection of stochastic models of particle trajectories in turbulent flows. *J. Fluid Mech.*, **180**, 529–556.

Thomson, D. J., 1990: A stochastic model for the motion of particle pairs in isotropic high-Reynolds-number turbulence, and its application to the problem of concentration variance. *J. Fluid Mech.*, **210**, 113–153.

van Kampen, N. G., 1981: *Stochastic Processes in Physics and Chemistry*, North-Holland, Amsterdam.

**Appendix**

The stationary structure function of the second order, Eq. (4), can be generalized to an arbitrary integer order $n$, in non-dimensional terms as

$$
\langle \Delta u^n \rangle = \langle \Delta u^n \rangle_e \Delta r^h n,
$$

(A-1)

where $\langle \cdot \rangle_e$ denotes Eulerian equilibrium statistics and, when $n = 2$, $\langle \Delta u^2 \rangle_e = 2$. The inertial subrange and spatial decorrelation limit are recovered for $h = 1/3$ and $h = 0$, respectively.
Considering the characteristic function \( \hat{p}_E(\Delta w; \Delta r) \) of the stationary Eulerian pdf of velocity differences \( p_E(\Delta u; \Delta r) \) and using Eq. (A-1), it turns out that

\[
\hat{p}_E(\Delta w; \Delta r) = \sum_{n=0}^{\infty} (i \Delta r^h \Delta w)^n (\Delta u^n)^{-1} = \hat{f}(\Delta r^h \Delta w), \quad (A-2)
\]

with \( i = \sqrt{-1} \). From Eq. (A-2) it follows that

\[
p_E(\Delta u; \Delta r) = \frac{1}{\Delta r^h} f \left( \frac{\Delta u}{\Delta r^h} \right), \quad (A-3)
\]

where the factor \( \Delta r^{-h} \) conserves the normalization and, for the constant values \( h = 1/3, 0 \), Eq. (A-3) defines the self similar regimes of the inertial subrange and the spatial decorrelation limit, respectively.

Using the dimensional quantities \( \Delta r' = \lambda \Delta r \) and \( \Delta u' = \sigma \Delta u \) for the particle separation and the velocity differences, respectively, for any finite Lagrangian correlation time \( \tau \) and particle separation \( \Delta r' \), the following identity holds

\[
\lim_{\beta \to \infty} \varphi(\Delta r) \equiv \lim_{\lambda \to 0} \varphi(\Delta r'/\lambda), \quad (A-4)
\]

where \( \varphi \) is a generic continuous bounded function. Since continuity is required in the transition from the inertial subrange regime to the equilibrium, the scaling exponent \( h \) is assumed to be a monotonic decreasing function of \( \Delta r'/\lambda \). Thus

\[
\lim_{\lambda \to 0} \lambda^h = 1. \quad (A-5)
\]

As observed in Section 2, the only term affected by variations of \( \beta \) in Eq. (9) is the non-homogeneous one. Therefore for any finite \( \Delta r' \) using Eq. (A-3) and Eq. (A-5), it turns out that

\[
\lim_{\beta \to \infty} \beta \frac{\partial p_E}{\partial r} \sim \lim_{\lambda \to 0} \left\{ \lambda^h \frac{h}{\Delta r'^{h+1}} f \left( \frac{\Delta u' \sigma^{-1}}{(\Delta r' \lambda^{-1})^h} \right) + \lambda^{2h} \frac{h}{\Delta r'^{2h+1}} \frac{\Delta u' \sigma^{-1}}{\sigma} f' \left( \frac{\Delta u' \sigma^{-1}}{(\Delta r' \lambda^{-1})^h} \right) \right\} \to 0 \quad (A-6)
\]

which shows that the non-homogeneous term vanishes in this limit.