Simple Moufang loops and Galois extensions

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Abstract

The paper establishes an one-to-one correspondence between simple Moufang loops and Paige loops constructed over Galois extension over prime field in its algebraic closure. Using this connection it describes fully the family of nonassociative finite simple Moufang loops. It describes the generators, the loop structure of subloops, the automorphism group of nonassociative simple Moufang loops.

Keywords: simple nonassociative Moufang loop, matrix Cayley-Dickson algebra, Paige loop, Galois extension, splitting field, Galois group, generators, automorphism group, lattice, Krull topology, profinite group.

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Historically, the first examples of simple Moufang loops were constructed by Paige in [1]. Namely, given a field $F$, write $M^*(F)$ for the set of all elements of unit norm in matrix Cayley-Dickson algebra $C(F)$ over $F$ (also called split octonion algebra in the literature), and let $M(F)$ be the quotient of $M^*(F)$ by its centre $Z(M^*(F)) = \{\pm 1\}$ (more detailed definitions will be presented below). Paige proved that $M(F)$ is a nonassociative simple Moufang loop. These loops are also known as Paige loops. Liebeck [2] used the classification of finite simple groups to conclude that there are no other nonassociative simple Moufang loops besides $M(F)$, $F$ finite.

Further, [3] (see, also [4]) indicates concrete elements $a, b, c \in M(F)$, that generate the loop $M(F)$, for $F = GF(p)$. Elements $a, b, c$ contain only 0 and 1 in their structure. This result is generalized in [5], using the classical results on generators of unimodular groups. It is proved that every Paige loop $M(F)$, $F = GF(p^n)$, is 3-generated. The core of the proof consists in considering a concrete triplet of generators of $M(F)$. The main result of [6], proved in a quite cumbersome manner, is that the automorphism group of Paige loop $M(GF(2))$ is isomorphic to Chevalley group $G_2(GF(2))$. In [7] it is proved that the automorphism group of the Paige loop $M(F)$, where $F$ is a perfect field, is isomorphic to the semidirect product $G_2(F) \rtimes Aut F$, where $G_2(F)$ is the Chevalley group.

The aforementioned results from [3–7] can be proved by different methods. But the crucial moment of this proofs relies heavily on the results of Doro [8], that relate Moufang loops to groups with triality. [4] rewrites
many proofs using the geometric loop theory. It also uses the connections between composition algebras, simple Moufang loops, simple Moufang 3-nets, S-simple groups and groups with triality. This paper contains a new approach to investigation of simple Moufang loops - using the one-to-one connection between simple Moufang loops (equivalent to Paige loops) and Galois extension over prime field in its algebraic closure and also normal subgroups of automorphism group of algebraic closure of prime field. It describes the generators, the automorphism group, the subloop structure of simple Moufang loops. The aforementioned results regarding the simple Moufang loops become a particular case of the corresponding results from this paper.

For basic definitions and properties of Moufang loops, alternative algebras, fields see [9], [10], [11] respectively. Often the needed results from this books will be used without reference. It is also worth mentioning that this paper is a continuation of paper [12] and, for purposes of completeness, the content of the latter is included in this paper.

Remind that for an alternative algebra $A$ with unity the set $U(A)$ of all invertible elements of $A$ forms a Moufang loop with respect to multiplication [13]. It is proved by analogy to Lemma 1 from [14, 15].

**Lemma 1.** Let $A$ be an alternative algebra with unity and let $Q$ be a subloop of $U(A)$. Then the restriction of any homomorphism of algebra $A$ upon $Q$ will be a homomorphism on the loop. More concretely, any ideal $J$ of $A$ induces a normal subloop $Q \cap (1 + J)$ of $Q$.

Let $L$ be a free Moufang loop, let $F$ be a field and let $FL$ be a loop algebra of loop $L$ over field $F$. We remind that $FL$ is a free module with basis $\{g | g \in L\}$ and the multiplication of basis elements is defined by their multiplication in loop $L$. Let $(u,v,w) = uv \cdot w - u \cdot vw$ denote the associator of elements $u,v,w$ of algebra $FL$. We denote by $I$ the ideal of loop algebra $FL$, generated by the set

$$\{(a,b,c) + (b,a,c), (a,b,c) + (a,c,b) | \forall a,b,c \in L\}.$$ 

It is shown in [13, 14] that algebra $FL/I$ is alternative and loop $L$ is embedded (isomorphically) in the loop $U(FL/I)$. Further we identify the loop $L$ with its isomorphic image in $U(FL/I)$. Hence the free loop $L$ is a subloop of loop $U(FL/I)$. Without causing any misunderstandings we will denote by $FL$ the quotient algebra $FL/I$ and call it alternative loop algebra (in [14, 15] it is called "loop algebra"). Sums $\sum_{g \in L} \alpha_g g$, are elements of algebra $FL$, where $\alpha_g \in F$. Further, we will identify the field $F$ with subalgebra $F1$ of algebra $FL$, where 1 is the unit of loop $L$.

Let now $Q$ be an arbitrary Moufang loop. Then $Q$ has a representation as a quotient loop $L/H$ of the free Moufang loop $L$ by the normal subloop $H$. We denote by $\omega H$ the ideal of alternative loop algebra $FL$, generated
by the elements $1 - h$ ($h \in H$). By Lemma 1, $\omega H$ induces a normal subloop $K = L \cap (1 + \omega H)$ of loop $L$ and $F(L/K) = FL/\omega H$.

We denote $L/K = \overline{Q}$, thus $FL/\omega H = F\overline{Q}$. As every element in $FL$ is a finite sum $\sum_{g \in L} \alpha_g q$, where $\alpha_g \in F$, $q \in L$, then the finite sum $\sum_{q \in \overline{Q}} \alpha_q q$, where $\alpha_q \in F$, $q \in \overline{Q}$ will be elements of algebra $F\overline{Q}$. Let us determine the homomorphism of $F$-algebras $\varphi : FL \to F(L/H)$ by the rule $\varphi(\sum \lambda_q q) = \sum \lambda_q Hq$. The mapping $\varphi$ is $F$-linear, then for $h \in H$, $q \in L$ we have $\varphi((1 - h)q) = Hq - H(q) = Hq - Hq = 0$. Hence $\omega H \subseteq \ker \varphi$. The loop $\overline{Q}$ is a subloop of loop $U(F\overline{Q})$ and as $\omega H \subseteq \ker \varphi$, then the homomorphisms $FL \to FL/\omega H = F\overline{Q}$ and $FL \to FL/\ker \varphi = F(L/H) = FQ$ induces a homomorphism $\pi$ of loop $\overline{Q}$ upon loop $Q$. Hence we have.

**Lemma 2.** Let $Q$ be an arbitrary Moufang loop. Then the loop $\overline{Q}$ is embedded in loop of invertible elements $U(F\overline{Q})$ of alternative algebra $F\overline{Q}$ and the homomorphism $FL \to FL/\omega H$ of alternative loop algebra $FL$ induces a homomorphism $\pi : \overline{Q} \to Q$ of loops.

**Remark.** The Lemma 2 answers positively to Question 1’ from [16]: is it true that any Moufang loop can be imbedded into a homomorphic image of a loop of type $U(A)$ for a suitable unital alternative algebra?

Let now $Q$ be a simple Moufang loop. Then $\ker \pi$ will be a proper maximal normal subloop of $\overline{Q}$. Let $J_1$, $J_2$ be proper ideals of algebra $F\overline{Q}$. We prove that the sum $J_1 + J_2$ is also a proper ideal of $F\overline{Q}$. Indeed, by Lemma 1 $K_1 = \overline{Q} \cap (1 + J_1)$, $K_2 = \overline{Q} \cap (1 + J_2)$ will be normal subloops of loop $\overline{Q}$. We have that $K_1 \subseteq \ker \pi$, $K_2 \subseteq \ker \pi$. Then product $K_1 K_2 \subseteq \ker \pi$, as well. But $K_1 K_2 = (\overline{Q} \cap (1 + J_1))(\overline{Q} \cap (1 + J_2)) = \overline{Q} \cap (1 + J_1)(1 + J_2) = \overline{Q} \cap (1 + J_1 + J_2 + J_1 J_2) = \overline{Q} \cap (1 + J_1 + J_2) \subseteq \ker \pi$, i.e. $\overline{Q} \cap (1 + J_1 + J_2)$ is a proper normal subloop of $\overline{Q}$. Then from Lemma 1 it follows that $J_1 + J_2$ is a proper ideal of algebra $F\overline{Q}$, as required.

We denote by $S$ the ideal of algebra $F\overline{Q}$, generated by all proper ideals $J_i$ ($i \in I$) of $F\overline{Q}$. Let us show that $S$ is also a proper ideal of $F\overline{Q}$. If $I$ is a finite set, then the statement follows from the first case. Let us now consider the second possible case. The algebra $F\overline{Q}$ is generated as a $F$-module by elements $x \in \overline{Q}$. Let there be such ideals $J_1, \ldots, J_k$ that for element $1 \neq a \in \overline{Q}$, $a \in \sum J_i$ and let us suppose that for element $b \in \overline{Q}$, $b \notin \sum J_i$. We denote by $T$ the set of all ideals of algebra $F\overline{Q}$, containing the element $a$, but not containing the element $b$. By Zorn’s Lemma there is a maximal ideal $I_1$ in $T$. We denote by $I_2$ the ideal of algebra $F\overline{Q}$, generated by all proper ideals of $F\overline{Q}$ that don’t belong to ideal $I_1$. Then $S = I_1 + I_2$. $I_1, I_2$ are proper ideals of $F\overline{Q}$ and by the first case $S$ is also proper ideal of $F\overline{Q}$. By Lemma 1 $K = \overline{Q} \cap (1 + S)$ is a normal subloop of $\overline{Q}$. We denote $\overline{Q} = \overline{Q}/K$. Then $F\overline{Q} = F\overline{Q}/S$ is a simple algebra. As $K \subseteq \ker \pi$ then $\pi$ induce a homomorphism $\rho : \overline{Q} \to Q$. Hence we prove.

**Lemma 3.** Let $Q$ be a simple nonassociative Moufang loop. Then the
loop $Q$ is embedded in loop of invertible elements $U(FQ)$ of alternative algebra $FQ = F(Q)$ and the homomorphism $FL \to FL/\omega H$ of alternative loop algebra $FL$ induces a homomorphism $\rho: Q \to Q$ of loops.

Let $F$ be an arbitrary field. Let us consider a classical matrix Cayley-Dickson algebra $C(F)$. It consists of matrices of form $(\begin{array}{cc} \alpha_1 & \alpha_{12} \\ \alpha_{21} & \alpha_2 \end{array})$, where $\alpha_1, \alpha_2 \in F$, $\alpha_{12}, \alpha_{21} \in F^3$. The addition and multiplication by scalar of elements of algebra $C(F)$ is represented by ordinary addition and multiplication by scalar of matrices, and the multiplication of elements of algebra $C(F)$ is defined by the rule

$$\left( \begin{array}{cc} \alpha_1 & \alpha_{12} \\ \alpha_{21} & \alpha_2 \end{array} \right) \left( \begin{array}{cc} \beta_1 & \beta_{12} \\ \beta_{21} & \beta_2 \end{array} \right) = \left( \begin{array}{cc} \alpha_1\beta_1 + (\alpha_{12}, \beta_{21}) & \alpha_1\beta_{12} + \beta_{21} \alpha_{12} - \alpha_{21} \times \beta_{21} \\ \beta_1\alpha_{21} + \alpha_{21}\beta_{21} + \alpha_{12} \times \beta_{12} & \alpha_2\beta_2 + (\alpha_{21}, \beta_{21}) \end{array} \right),$$

(1)

where for vectors $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, $\delta = (\delta_1, \delta_2, \delta_3) \in A^3 (\gamma, \delta) = \gamma_1\delta_1 + \gamma_2\delta_2 + \gamma_3\delta_3$ denotes their scalar product and $\gamma \times \delta = (\gamma_2\delta_3 - \gamma_3\delta_2, \gamma_3\delta_1 - \gamma_1\delta_3, \gamma_1\delta_2 - \gamma_2\delta_1)$ denotes the vector product. Algebra $C(F)$ is alternative. For $a \in C(F)$ the norm $n(a)$ is defined by the equality $n(a) = \alpha_1\alpha_2 - (\alpha_{12}, \alpha_{21})$. If $\alpha_{ij} = (\alpha_{ij}^{(1)}, \alpha_{ij}^{(2)}, \alpha_{ij}^{(3)})$, then

$$n(a) = \alpha_1\alpha_2 - \alpha_{12}^{(1)}\alpha_{21}^{(1)} - \alpha_{12}^{(2)}\alpha_{21}^{(2)} - \alpha_{12}^{(3)}\alpha_{21}^{(3)}.$$  

(2)

The algebra $C(F)$ is split. Then from [1, Theorem] it follows.

**Lemma 4.** Let $F$ be an arbitrary field. Then the Moufang loop $M(F) = M^*(F)/ < -1 >$ of the matrix Cayley-Dickson algebra $C(F)$ is simple and the loop $< -1 >$ coincides with the center of loop $M^*(F)$.

The loops $M(F)$ of Lemma 4 are sometimes called Paige loops.

By Lemma 4 the center $Z$ of loop $M^*(F)$ coincides with subloop $< -1 >$. As $M^*(F)/Z$ is a simple loop, a question appears. Is the center $Z$ of loop $M^*(F)$ emphasized by the direct factor? The answer is negative. Let field $F$ consist of 5 elements and let $H$ be a direct completion of center $Z$. If $\alpha$ were the generator of the multiplicative group of field $F$, then one of the elements $\pm \left( \begin{array}{cc} 1 \alpha \\ 0 \alpha \end{array} \right)$ would lie in $H$. The square of this element is equal to $-1$, i.e., it lies in the intersection $H \cap Z$, which is impossible. Therefore center $Z$ cannot have a direct factor in $M^*(F)$.

Let now $P$ be an algebraically closed field and let $Q$ be a simple nonassociative Moufang loop. By Lemma 3 the loop $\overline{Q}$ is embedded in loop of invertible elements of simple alternative algebra $P\overline{Q}$. We denote $\overline{Q} = G$.

If $a \in G$, then it follows from the equality $aa^{-1} = 1$ that $n(a)n(a)^{-1} = 1$, i.e. $n(a) \neq 0$. The associator $(a, b, c)$ of elements $a, b, c$ of an arbitrary loop
is defined by the equality \( ab \cdot c = (a \cdot bc)(a, b, c) \). Identity \((xy)^{-1} = y^{-1}x^{-1}\) holds in Moufang loops. Therefore, if \( a, b, c \) are elements of Moufang loop \( G \), then \( u = (a, b, c) = (a \cdot bc)^{-1}(ab \cdot c) = (c^{-1}b^{-1} \cdot a^{-1})(ab \cdot c) \), \( n(u) = n(c^{-1})n(b^{-1})n(a^{-1})n(a) \cdot n(b)n(c) = n(c)^{-1}n(b)^{-1}n(a)^{-1}n(a)n(b)n(c) = 1 \), i.e. \( u \in M^*(P) \). We denote by \( G' \) the subloop generated by all associators of Moufang loop \( G \). If \( G' = G \), then \( G \subseteq M^*(P) \), i.e. the loop \( G \) is embedded in \( M^*(P) \). Now we suppose that \( G' \neq G \). It is shown in [17, 18] that the subloop \( G' \) is normal in \( G \). The finite sum \( \sum_{g \in G} \alpha_g g \), where \( \alpha_g \in P \), \( g \in G \) are elements of algebra \( PG \). Let \( \eta : PG \to P(G/G') \) be a homomorphism of \( P \)-algebras determined by the rule \( \eta(\sum \alpha_g g) = \sum \alpha_g gG' \) (\( g \in G \)) and let \( P(G/G') = PG/\ker \eta \). As the quotient loop \( P(G/G') \) is non-trivial, then \( PG/\ker \eta \neq PG \). Hence \( \ker \eta \) is a proper ideal of \( PG \). The algebra \( PG \) is simple. Then the ideal \( \ker \eta \) cannot be the proper ideal of \( PG \). Hence the case \( G' \neq G \) is impossible and, consequently, the loop \( G \) is embedded in loop \( M^*(P) \).

The alternative algebra \( PG \) is simple. By Kleinfeld Theorem [10] it is a Cayley-Dickson algebra over their center. Field \( P \) is algebraically closed. Then algebra \( PG \) is split. The matrix Cayley-Dickson algebra \( C(P) \) is also split. But any two split nonassociative composition algebras over an algebraically closed field are isomorphic. Therefore algebra \( PG = P\overline{Q} \) is isomorphic to the matrix Cayley-Dickson algebra \( C(P) \). Further we identify \( P\overline{Q} \) with \( C(P) \). It is proved.

**Lemma 5.** Let \( P \) be an algebraically closed field and let \( Q \) be a simple Moufang loop. Then loop \( \overline{Q} \) is embedded in loop \( M^*(P) \) of matrix Cayley-Dickson algebra \( P\overline{Q} = C(P) \).

It is known that for any field \( K \) there exists an algebraic closure \( \overline{K} \), containing \( K \) as subfield. Let \( S \) be the set of all roots of all polynomials of degree \( \geq 1 \) from polynomial ring \( K[X] \). Then the field \( \overline{K} \) is the adding of the set \( S \) to field \( K \). The elements of \( \overline{K} \) are polynomials of elements from \( S \) with coefficients from \( K \). Further we will consider that \( \Delta \) is a prime field and denote by \( P \) (or \( \overline{\Delta} \)) its algebraic closure. Every field contains as subfield an unique prime field. Then every subfield of \( P \) can be presented as an extension of prime field \( \Delta \).

Let \( \varphi : \overline{Q} \to M^*(P) \) be the embedding considered in Lemma 5. Further we identify \( \overline{Q} \) with \( \varphi(\overline{Q}) \). The subfield over \( \Delta \) of field \( P \), generated by matrix elements \( \alpha, \alpha^{(k)}_{ij} \), where \( \alpha_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}) \), of all matrices \( \left( \begin{array}{cc} \alpha_1 & \alpha_{12} \\ \alpha_{21} & \alpha_2 \end{array} \right) \) \( \in \overline{Q} \) will be denoted by \( P_Q \) and will be called subfield over \( \Delta \) of field \( P \), corresponding to matrices loop \( \overline{Q} \).

**Lemma 6.** Let \( Q \) be a simple Moufang loop and let \( P_Q = F \) be the subfield over \( \Delta \) of algebraic closure \( P \) corresponding to matrix loop \( \overline{Q} \). Then
\[ M^*(P_Q) = \overline{Q}. \]

**Proof.** Let \( L \) be such a free Moufang loop that \( Q = L/H \) for simple Moufang loop \( Q \). Denote by \( FL \) the loop algebra and by \( FL \) the alternative loop algebra of loop \( L \) over field \( F \). Let

\[
FL \rightarrow FL/1 = FL \rightarrow FL/\omega H = F\overline{Q} \rightarrow F\overline{Q}/S = F\overline{Q}
\]

be the homomorphisms of algebras constructed in Lemmas 2, 3. These homomorphisms of algebras induce the homomorphisms of loops

\[
L \rightarrow L/B = Q \rightarrow \overline{Q}/K = \overline{Q}.
\]

Let now \( R = M(F) \) be the simple Moufang loop considered in Lemma 4 and let \( L_1 \) be such a free Moufang loop that \( R = L_1/H_1 \). As in the case of loop \( Q \) we consider the homomorphisms

\[
FL_1 \rightarrow FL_1/1 = FL_1 \rightarrow FL_1/\omega H_1 = F\overline{R} \rightarrow F\overline{R}/S_1 = F\overline{R},
\]

\[
L_1 \rightarrow L_1/B_1 = R \rightarrow R/K_1 = \overline{R},
\]

of algebras and loops respectively. These homomorphisms induce the homomorphisms \( \varphi : FL_1 \rightarrow F\overline{R} \) and \( \psi : L_1 \rightarrow \overline{R} \) of algebras and loops such that \( \varphi u = \psi u \) for all \( u \in L_1 \).

By Lemma 5, \( \overline{Q} \subseteq M^*(P_Q) = \overline{R} \) and \( F\overline{Q} = C(F) (F = P_Q) \), where \( C(F) \) is matrix Cayley-Dickson algebra. Hence \( F\overline{Q} = F\overline{R} \). Let us suppose that \( \overline{Q} \subset \overline{R} \). Let \( a \in \overline{R}\overline{Q} \). It follows from equality \( F\overline{Q} = F\overline{R} \) that \( a = \sum \alpha_i a_i \), where \( \alpha_i \in F \), \( a_i \in \overline{Q} \), and the sum \( \sum \alpha_i a_i \) is non-trivial, i.e. \( i > 1 \). Let \( \psi u = a \), \( \psi u_i = a_i \) for some \( u, u_1 \in L_1 \). Then the inverse image of equality \( a = \sum \alpha_i a_i \) under homomorphism \( \varphi : FL \rightarrow F\overline{R} \) has a form \( u = \sum \alpha_i u_i + \sum \beta_j v_j \) for some \( \beta_j \in F \), \( v_j \in L_1 \). The sum \( \sum \alpha_i \) in equality \( a = \sum \alpha_i \) is non-trivial. Then, obviously, the sum \( \sum \alpha_i u_i + \sum \beta_j v_j \) in equality \( u = \sum \alpha_i u_i + \sum \beta_j v_j \) is also non-trivial. Hence in loop algebra \( FL_1 \) the element \( u \in L_1 \) is linearly expressed through the elements of loop \( L_1 \). But this contradicts the definition of loop algebra \( FL_1 \). Consequently, \( \overline{Q} = \overline{R} = M^*(F) \). This completes the proof of Lemma 6.

Let \( Q \) be a simple Moufang loop and let \( P_Q \) be the subfield over \( \Delta \) of algebraic closure \( P \) corresponding to matrix loop \( \overline{Q} \). From Lemma 6 it follows that \( \overline{Q} \) consists of elements of form \( a = \begin{pmatrix} \alpha_1 & (\alpha_2, \alpha_3, \alpha_4) \\ (\alpha_5, \alpha_6, \alpha_7) & \alpha_8 \end{pmatrix} \) with norms one, where \( \alpha_1, \ldots, \alpha_8 \in P_Q \). Denote \( T_i = \{ \alpha_i \in P_Q \mid \forall a \in \overline{Q} \} \), \( i = 1, 2, \ldots, 8 \). The elements

\[
\begin{pmatrix} \alpha & (\alpha, \alpha, \alpha) \\ (0, 0, 0) & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \alpha & (0, 0, 0) \\ (\alpha, \alpha, \alpha) & \alpha^{-1} \end{pmatrix}, \begin{pmatrix} \alpha^{-1} & (0, 0, 0) \\ (\alpha, \alpha, \alpha) & \alpha^{-1} \end{pmatrix}
\]

on
are norm one. From here it follows that \( \alpha \in T_i \) implies \( \alpha \in T_j, \ i, j = 1, \ldots, 8 \), or \( T_1 = T_2 = \ldots = T_8 \). Let us denote \( BP_Q = T_i \). Then only the elements from \( BP_Q \) are as components of matrices elements from \( \overline{Q} \). The norms of elements \( \left( \begin{array}{cc} \alpha & (1,0,0) \\ (0,0,0) & \alpha^{-1} \end{array} \right), \left( \begin{array}{cc} 1 & (0,0,1) \\ (0,0,0) & 1 \end{array} \right) \) are one for \( \alpha \in BP_Q \). By (1)

\[
\left( \begin{array}{cc} \alpha & (1,0,0) \\ (0,0,0) & \alpha^{-1} \end{array} \right) \left( \begin{array}{cc} \beta & (1,0,0) \\ (0,0,0) & \beta^{-1} \end{array} \right) = \left( \begin{array}{cc} \alpha \beta & (1,0,0) \\ (0,0,0) & \alpha^{-1} \beta^{-1} \end{array} \right),
\]

\[
\left( \begin{array}{cc} 1 & (0,0,1) \\ (0,0,0) & 1 \end{array} \right) \left( \begin{array}{cc} 1 & (0,0,1) \\ (0,0,0) & 1 \end{array} \right) = \left( \begin{array}{cc} 1 + \alpha \beta & (0,0,1) \\ (\alpha - \beta, 0, 0) & 1 \end{array} \right).
\]

From these equalities it follows that \( \alpha, \beta \in BP_Q \) implies \( \alpha^{-1}, \alpha \beta, \alpha + \beta, \alpha - \beta \in BP_Q \). Hence we proved.

**Lemma 7.** Let \( Q \) be a simple Moufang loop, let \( \Delta \) be a prime field and let \( P \) be its algebraic closure. Then the set \( BP_Q \) of \( P \) is a subfield over \( \Delta \) and \( BP_Q = P_Q \).

Every field contains an unique simple subfield. Then \( \Delta \subseteq P_Q \) and \( P_Q \) is an extension of field \( \Delta \), \( \Delta \subseteq P_Q \). The field \( P \) is algebraically closed. Then \( \Delta \subseteq P_Q \) is an algebraically extension.

We will show that the extension \( \Delta \subseteq P_Q \) is normal. Really, let \( \{\alpha, \alpha_1, \alpha_2, \ldots\} \) be a basis of vector space \( P_Q \) over \( \Delta \). Then \( P_Q = \Delta(\alpha, \alpha_1, \alpha_2, \ldots) \). Let \( f_\alpha(X) \) be an irreducible polynomial of polynomial ring \( \Delta[X] \) corresponding to element \( \alpha \), \( f_\alpha(\alpha) = 0 \). We consider that \( \text{deg} f_\alpha(X) > 1 \). Let \( \beta \neq \alpha \) be a root of \( f_\alpha(X) \). To prove the normality of extension \( \Delta \subseteq P_Q \) it is sufficient to prove that \( \beta \in P_Q \). Let us assume the contrary, that \( \beta \notin P_Q \). We denote \( \Delta_\beta = \Delta(\alpha_1, \alpha_2, \ldots) \). Over field \( \Delta_\alpha \) the polynomial \( f_\alpha(X) \) takes the form \( f(X) = (x - \theta_1)(x - \theta_2)g_\alpha(X) \), where \( \theta_1, \ldots, \theta_r \in \Delta_\alpha \) and the polynomial \( g_\alpha \) is an irreducible polynomial over \( \Delta_\alpha \). As \( \beta \notin P_Q \) then \( \text{deg} g_\alpha > 1 \). If the degree \( |\Delta_\alpha(\alpha) : \Delta_\alpha| = n > 1 \) then by [11, Prop. 3, cap. VII] and \( |\Delta_\alpha(\beta) : \Delta_\alpha| = n \). In these cases the elements in \( \Delta_\alpha(\alpha) \) have the form \( \sum_{k=0}^{n-1} \theta_k \alpha^k \) and the elements in \( \Delta_\alpha(\beta) \) are form \( \sum_{k=0}^{n-1} \theta_k \beta^k \), where \( \theta_k \in \Delta_\alpha \). Then the mapping \( \varphi: \Delta_\alpha(\alpha) \to \Delta_\alpha(\beta) : \sum_{k=0}^{n-1} \theta_k \alpha^k \to \sum_{k=0}^{n-1} \theta_k \beta^k \) is an isomorphism over \( \Delta_\alpha \).

We have \( P_Q = \Delta_\alpha(\alpha) \) and \( \Delta_\alpha \neq P_Q \). Then \( M(\Delta_\alpha) \neq M(P_Q) = Q \). From here it follows that for \( \alpha \) there exist such elements \( \omega_1, \omega_2, \ldots, \omega_7 \in \Delta_\alpha \) that \( \alpha, \omega_1, \ldots, \omega_7 \) are the components of some matrix element \( u \) of Moufang loop \( M^*(P_Q) \). The norm \( n(u) = 1 \) or by (2) \( \alpha \omega_1 - \omega_2 \omega_3 - \omega_4 \omega_5 - \omega_6 \omega_7 = 1 \). Then \( \varphi(\alpha \omega_1 - \omega_2 \omega_3 - \omega_4 \omega_5 - \omega_6 \omega_7) = \varphi(1) \), \( \varphi(\alpha) \omega_1 - \omega_2 \omega_3 - \omega_4 \omega_5 - \omega_6 \omega_7 = 1 \), \( \beta \omega_1 = \omega_2 \omega_3 + \omega_4 \omega_5 + \omega_6 \omega_7 + 1 \), \( \beta \omega_1 \in P_Q \). By Lemma 7 \( \beta \in P_Q \). We get a contradiction as \( \beta \notin P_Q \). Hence \( \beta \in P_Q \). Consequently, any root of
polynomial $f_\alpha(X)$ lies in $P_Q$. Then $f_\alpha(X)$ factors over $\Delta$ in $P_Q$ as a product of linear factors. Consequently, $P_Q$ is a splitting field of the set of irreducible polynomials $\{f_\alpha(X)|\alpha = \alpha_1, \alpha_2,\ldots\}$ over $\Delta$. By [11, Theorem 4, cap. VII] the extension $\Delta \subset P_Q$ is normal.

The prime field $\Delta$ can be only the field of rational numbers $R$ or the Galois field $GF(p)$. In the first case $\text{char } P = 0$ and by [19] the extension $\Delta \subset P_Q$ is perfect. Recall, a field $F$ is called perfect if any irreducible polynomial of polynomial ring $F[X]$ is separable. Let us consider the second case. By [19] the field $GF(p)$ is perfect and any algebraical extension of perfect field is perfect. Hence the field $P_Q$ over $\Delta$ is perfect. In particular, the extension $\Delta \subset P_Q$ is separable. The algebraically separable and normal extension is called Galois. Consequently, the extension $\Delta \subset P_Q$ is Galois. Hence we proved.

**Lemma 8.** The extension $\Delta \subset P_Q$ is Galois and $P_Q$ is a perfect field.

**Theorem 1.** Let $\Delta$ be a prime field and let $P$ be its algebraic closure. Only and only the Paige loops $M(F)$, where $F$ is a Galois extension over $\Delta$ in $P$, are with precise till isomorphism nonassociative simple Moufang loops.

**Proof.** Let $Q$ be a simple Moufang loop. By Lemma 7 the subfield $P_Q = BP_Q$ over $\Delta$ corresponds to it. By Lemma 8 the extension $\Delta \subset P_Q$ is Galois. If $Q, G$ are simple Moufang loops and $Q \neq G$, then $P_Q = BP_Q \neq BP_G = P_G$.

Conversely, let $F$ be a Galois extension over $\Delta$ of $P$ and let $P_Q$ be the subfield of $P$ corresponding to Paige loop $Q = M(F)$. By Lemma 8 $P_Q$ is a normal extension over $\Delta$. It is clearly that $P_Q \subset F$. We suppose that $1 \neq a \in F \setminus P_Q$. Let $f(X)$ be an irreducible polynomial of polynomial ring $\Delta[X]$ of element $a$, $f(a) = 0$. Let $a, a_1,\ldots,a_n$ be the roots of $f(X)$ and let $S$ be the splitting field of polynomial $f(X)$ over $\Delta$. By [11, Theorem 3, cap. VII] the normality of some extension $K$ over $\Delta$ equals to the fact, that every irreducible polynomial of $\Delta[X]$, with a root in $K$, factorizes into linear factors in $K$. The extensions $F, P_Q$ over $\Delta$ are normal. Then $\{a,a_1,\ldots,a_n\} \subset F$, $\{a,a_1,\ldots,a_n\} \not\subset P_Q$, $M(S) \subset M(F) - Q$ and $\{a_1,a_2,\ldots,a_n\} \not\subset P_Q$, $M(S) \not\subset M(P_Q) = Q$. We get a contradiction. Hence $F = P_Q$. This completes the proof of Theorem 1.

**Theorem 2.** Let $\Delta$ be a prime field, let $P$ be its algebraic closure and let $q = p^n$, where $p = p(Q)$ is a prime, $n = n(Q)$ is an integer. Then for a nonassociative Moufang loop $Q$ the following statements are equivalent:

1) $Q$ is a finite simple loop;

2) $Q$ is isomorphic to one of the Paige loop $M(F)$, where $F \subset P$ is a Galois field $GF(q)$ over $\Delta$;

3) $Q$ is isomorphic to Paige loop $M(F)$, where $F \subset P$ is a splitting field over $\Delta$ of $(q - 1)$-th roots of unity;
4) $Q$ is isomorphic to Paige loop $M(F)$, where $F$ is the finite field $GF(q)$.

**Proof.** It is clear that the loop $Q$ is finite then and only then the subfield $BP_Q$ over $\Delta$ from Lemma 7 is finite. Hence $BP_Q = GF(q)$, i.e. 1) $\Rightarrow$ 2).

The implication 2) $\Leftrightarrow$ 3) follows from Theorem 1.

2) $\Leftrightarrow$ 3). The Galois field $GF(q)$ is defined in a unique manner in algebraic closure $P$ as a splitting field of polynomial $x^q - x$ and its elements are the roots of this polynomial. But the set of all roots of polynomial $x^q - x$ consists from $\{0\}$ and the set of roots of polynomial $x^{q-1} - 1$. Hence 2) $\Leftrightarrow$ 3).

Finally, the prime field $\Delta$ is fixed in the conditions of the theorem, but arbitrary. Let $\text{char} \ \Delta = p$ and let $GF(p^n)$ be a Galois field over $\Delta$. Then $\Delta \subseteq GF(p^n)$. In such a case the equivalence of 1) and 4) derives from the equivalence of 1) and 2). This completes the proof of Theorem 2.

Let $\Delta$ be a prime field and let $P$ be its algebraic closure. By Theorem 1 only and only the Paige loops $M(F)$, where $F$ is a Galois extension over $\Delta$ in $P$, are with precise till isomorphism nonassociative simple Moufang loops. By Theorem 2 with precise till isomorphism the only nonassociative finite simple Moufang loops are the loops $M(F)$, where $F = GF(p^n)$ is a Galois field over $\Delta$ in $P$. Next let us consider the simple Moufang loops in form of $M(F)$, where $F \subseteq P$. It is clear that for such loops the inclusion relation $\subseteq$ is meaningful.

**Corollary 1.** 1. Only the minimal nonassociative finite simple Moufang loops are the Paige loops $M(GF(p))$, $p$ prime. Only the minimal nonassociative infinite simple Moufang loops are the Paige loops $M(Q)$, where $Q$ is the field of rational numbers.

2. Only the simple Moufang loops $M(\Delta)$, where $\Delta$ is a prime field, do not contain proper nonassociative subloops.

3. The loop $M(GF(q^m))$ is contained into loop $M(GF(p^n))$ when and only when $q = p$ and $n$ is divisible by $m$.

4. Only the maximal nonassociative simple Moufang loops is the Paige loops $M(P)$, where $P$ is an algebraic closure of some prime field $\Delta$. For various prime fields $\Delta$ the corresponding loops $M(P)$ are isomorphic.

**The proof** follows immediately from Theorems 1, 2 and well known fact from the field theory: $GF(p^n) \subseteq GF(p^m)$ when and only when $m$ is divisible by $n$.

**Corollary 2.** If three elements $a, b, c$ of simple Moufang loop $M(F)$ are not connected through associative law, for example, $ab \cdot c \neq a \cdot bc$, and the matrix components of these elements $a, b, c$ generate the field $F$, in particular, at least one component is a primitive of field $F$, then these elements $a, b, c$ generate the loop $M(F)$.

Any element $\neq 0$ is a primitive element in prime field $GF(p)$. Then from Corollary 1 it follows.

**Corollary 3.** Let $\Delta = GF(p)$. Then any three elements $a, b, c \in M(\Delta)$,
not connected through the associative law, generate the loop $M(\Delta)$.

Now using the Theorem 1 and Lemma 8 we generalize the main result of [7], stated at the beginning of this article.

**Proposition 1.** The automorphism group of arbitrary Paige loop $M(F)$ is isomorphic to the semidirect product $G_2(F) \rtimes G(F/\Delta) = G_2(F) \rtimes Aut F$, where $G_2(F)$ is the Chevalley group, $Aut F = G(F/\Delta)$ is the Galois group of Galois extension of a field $F$ over a prime field $\Delta$ in algebraic closure $\Delta$.

**Proof.** It is known that the automorphism groups of prime fields $\Delta$ coincide with the identical automorphism, $Aut \Delta = \{id\}$. Then $G(F/\Delta) = Aut F$. From Proposition 1 it follows.

**Corollary 4.** The automorphism group of Paige group $M(F)$ is isomorphic to the Chevalley group $G_2(F)$ if and only if the field $F$ is prime.

The automorphism group of Galois field $GF(p^n)$ is the cyclic group of order $n$ and is generated by automorphism $\varphi : a \to a^p$. If by item 3) of Corollary 1 $GF(p^m) \subseteq GF(p^{nk})$ then the set of all automorphisms of $Aut GF(p^{nk})$, which induces the identical automorphism on $GF(p^m)$ forms a cyclic group generated by automorphism $\varphi^k$. Then from Proposition 1 it follows.

**Corollary 5.** The automorphism group of Paige loop $M(GF(p^n))$ is isomorphic to the semidirect product $G_2(F) \rtimes Aut GF(p^n)$. The group $Aut GF(p^n)$ is described above.

Let $K$ be a field, let $G = Aut K$ and let $H$ be a subgroup of group $G$. The fixed field of group $G$ is the subfield $K^H$ of $K$ defined by $K^H = \{x \in K|\varphi x = x \forall \varphi \in H\}$. The Galois group $G(K/E)$ of normal extension $K$ over $E$ is the set of all field automorphisms of $K$ which keep all elements of $E$ fixed. Let $F$ be a subfield, $E \subset F \subset K$. In [11, Theorem 3, cap. VIII] it is proved that the extension $F$ over $E$ is normal when and only when the subgroup $G(K/F)$ is normal in $G$. In such a case $G(F/E), \cong G/G(K/F)$. This isomorphism induces an one-to-one mapping between the set of the normal extension $S$ over $E$ in $K$ and the set of all normal subgroups $H = G(K/F)$ of $G$, get by formula $S = K^H$.

We consider now a normal extension $\Delta \subset K$, where $\Delta$ is a prime field. If $\Delta$ is the field of rational numbers then the extension $\Delta \subset K$ is separable [11, 19] and hence $\Delta \subset K$ is a Galois extension. Let now $\Delta$ be the Galois field $GF(p)$. We have $Aut K = G(K/\Delta) = G$ and $K^G = \Delta$ as $Aut F = \{id\}$ if and only if $F$ is a prime field. Then from [11, Proposition 11, cap. V; 19] it follows that the normal extension $\Delta \subset K$ is separable. Hence and in case $\Delta = GF(p)$ the normal extension $\Delta \subset K$ is Galois.

The Theorem 1 identifies all nonassociative simple Moufang loops with Paige loops $M(F)$ constructed over all Galois extensions $F$ over prime field $\Delta$ in it algebraic closure $\Delta$. By Lemma 7 the field $F$ determines the loop
$M(F)$ in a unique manner.

**Theorem 3.** Let $\Delta$ be a prime field, let $\overline{\Delta}$ be its algebraic closure and let $P$ be a Galois extension of $\Delta$ in $\overline{\Delta}$. We denote by $\Sigma$ the set of all Galois extensions of $\Delta$ in $P$. Then between the set $\Sigma_L$ of all Paige loops $M(F)$, where $F \in \Sigma$, and the set $\Sigma_G$ of all normal subgroups $H = G(P/F)$ of automorphism group $G = \text{Aut} \ P$ there exists an one-to-one mapping defined by $F = P^H$. Concretely:

1) each Paige loop $M(F), F \in \Sigma$, has a corresponding normal subgroup $H = G(P/F) \in \Sigma_G$ induced by isomorphism $\text{Aut} \ F \cong \text{Aut} \ P/G(P/H)$;

2) the field $F$ is defined in a unique manner $F = P^H$;

3) for each normal subgroup $H \in \Sigma_G$ we can find a field $F \in \Sigma$ that has the above described relationship with the normal subgroup $H$;

4) let $M(F), H$ and $M(F'), H'$ be the pairs described in item 1. Then $M(F) \subset M(F')$ (resp. $M(F) \neq M(F')$ or $M(F) \supset M(F')$) if and only if $H \supset H'$ (resp. $H \neq H'$ or $H \subset H'$).

Let $\Delta$ be a prime field and let $\overline{\Delta}$ be their algebraic closure. We denote by $\mathcal{M}(\overline{\Delta})$ the set of all subloops of Paige loop $M(\overline{\Delta})$, which are nonassociative simple loops (i.e. Paige loops $M(F), F \subset \overline{\Delta}$), by $\mathcal{R}(\overline{\Delta})$ we denote the set of all Galois extensions of subfield $F$ of $\overline{\Delta}$ over $\Delta$ and by $\mathcal{G}(\overline{\Delta})$ we denote the set of all normal subgroups of automorphism group $\text{Aut} \ \overline{\Delta}$. The sets $\mathcal{M}(\overline{\Delta}), \mathcal{R}(\overline{\Delta}), \mathcal{G}(\overline{\Delta})$ form respectively the lattices $L(\mathcal{M}(\overline{\Delta})), L(\mathcal{R}(\overline{\Delta})), L(\mathcal{G}(\overline{\Delta}))$ with respect to intersection $A \cap B$ as intersection of sets $A, B$ and union $A \cup B$ as the least algebra that contains subalgebras $A, B$. This lattices are full with zero $M(\Delta)$, $\Delta$, unitary group and with unity $M(\overline{\Delta}), \overline{\Delta}, \text{Aut} \ \overline{\Delta}$ respectively (more detail see Theorem 2).

Let $\Delta_1, \Delta_2$ be a prime fields and let $Q$ be a simple Moufang loop. By Theorem 1 $Q \cong M(F_1)$ and $Q \cong M(F_2)$ for some Galois extensions $\Delta_1 \subset F_1 \subset \overline{\Delta_1}$ and $\Delta_2 \subset F_2 \subset \overline{\Delta_2}$. Further, from Theorem 3 it follows that the intersection of two normal subgroups of group $\text{Aut} \ \overline{\Delta_1}$ corresponds to the union of fields, corresponding to these subgroups and union of normal subgroups corresponds to the intersection of fields. Then from here and definition of Paige loop it follows.

**Proposition 2.** Let $\Delta_1, \Delta_2$ be a prime fields. Then the lattices $L(\mathcal{M}(\overline{\Delta_1})), L(\mathcal{M}(\overline{\Delta_2})), L(\mathcal{R}(\overline{\Delta_1})), L(\mathcal{R}(\overline{\Delta_2}))$ are isomorphic among themselves and are inverse isomorphic with isomorphically lattices $L(\mathcal{G}(\overline{\Delta_1})), L(\mathcal{G}(\overline{\Delta_2}))$.

It is known [20] that the lattice of all normal subgroups of an arbitrary group is Dedekind’s (modular), i.e. satisfies the identity $x \cap (y \cup z) = y \cup (x \cap z)$. The dedekind property is maintained under inverse isomorphisms. Then from Proposition 1 it follows.

**Corollary 6.** The lattices considered in Proposition 2 are Dedekind’s.

Let $Q, G$ be isomorphic loops, let $\sigma : Q \to G$ be an isomorphism an let $\sigma^{-1} : G \to Q$ be the inverse isomorphism. Denote by $\text{Iso} \ (Q, G)$ the set
of all isomorphisms from $Q$ on $G$. If $\varphi \in Aut \, Q$ then $\sigma \varphi \in Iso \, (Q,G)$. Analogically, if $\psi \in Aut \, B$ then $\psi \sigma \in Iso \, (G,Q)$. If $\tau \in Iso \, (Q,G)$ is another isomorphism then $\tau^{-1} \sigma \in Aut \, Q$, $\tau \sigma^{-1} \in Aut \, G$. Hence two isomorphisms differ to automorphism. Moreover, the automorphism groups $Aut \, Q$, $Aut \, G$ are isomorphic with respect to one-to-one mappings $\varphi \to \sigma \varphi \sigma^{-1}$, $\psi \to \sigma^{-1} \psi \sigma$. Consequently, the group $Aut \, Q$ is defined uniquely with an accuracy to mapping, analogically to conjugate.

According to Theorem 1 and to the aforementioned we will consider that any simple Moufang loop $Q$ is a Paige loop $M(F)$, where $F$ is a Galois extension over a prime field $\Delta$ in algebraic closure $\overline{\Delta}$, for described the group $Aut \, Q$. In this form $M(F)$ the automorphism group $Aut \, Q$ is described in Proposition 1. For the finite loop $Q$ the group $Aut \, Q$ is described in Corollary 5 more detailed.

Now we pass to infinite loop $Q$. Many properties of finite Galois groups are proved using the calculation methods, but for infinite Galois groups it is used, as usually the topological methods (see, for example, [21]). To recall this. Let $E \subset K$ be a Galois extension. If Galois group $G(K/E) = G$ is defined the Krull topology, taking as the fundamental system of neighborhoods of unity the set of subgroups $G(S/E)$, where $S$ ranges over all intermediate fields, $E \subset S \subset K$, such that $E \subset S$ is a finite Galois extension. We prove that the family of normal subgroups $G(E/k)$, such that the extension $k \subset E$ is finite, defines the same topology. From here it follows that the group $G$ can be presented in the form of inverse limit

$$G = \lim_{\leftarrow} G(S_i/E),$$

where $S_i$ ranges over all intermediate fields such that $E \subset S_i$ is a finite Galois extension. Hence $G$ is a profinite group. We remind that that a topological group is called profinite if is isomorphic to the inverse limit of an inverse system of discrete finite groups. The class of all profinite groups and the class of all compact and totally disconnected topological groups coincide. From here it follows that the group $G$ is compact and totally disconnected.

Finite groups are profinite in discrete topology and for a prime field $\Delta$ $G(F/\Delta) = Aut \, F$. Then from the aforementioned, Propositions 2, 3 it follows.

**Theorem 4.** Let $Q$ be a simple Moufang loop. Then $Aut \, Q \cong G_2(F) \times Aut \, F$, where $G_2(F)$ is the Chevalley group, $Aut \, F = G(F/\Delta)$, $G(F/\Delta)$ is the Galois group of Galois extension of a field $F$ over a prime field $\Delta$ in algebraic closure of $\Delta$. The group $Aut \, F$ is profinite and if the field $F$ is infinite then the group $Aut \, F$ is compact and totally disconnected in Krull topology, and is isomorphic to the inverse limit of an inverse system of finite normal subgroups of group $Aut \, F$. 

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