POLYNOMIAL APPROXIMATION IN SOBOLEV SPACES ON THE UNIT SPHERE AND THE UNIT BALL

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Abstract. This work is a continuation of the recent study by the authors on approximation theory over the sphere and the ball. The main results define new Sobolev spaces on these domains and study polynomial approximations for functions in these spaces, including simultaneous approximation by polynomials and relation between best approximation to a function and to its derivatives.

1. Introduction

In a recent work [3], the authors defined new moduli of smoothness and K-functionals on the unit sphere \( S^{d-1} = \{ x \in \mathbb{R}^d : \| x \| = 1 \} \) and the unit ball \( B^d = \{ x \in \mathbb{R}^d : \| x \| \leq 1 \} \) of \( \mathbb{R}^d \), where \( \| x \| \) denotes the usual Euclidean norm, and used them to characterize the best approximation by polynomials. This work is a continuation of [3] and studies polynomial approximation in Sobolev spaces.

The new modulus of smoothness on the sphere is defined in terms of forward differences in the angle \( \theta_{i,j} \) of the polar coordinates on the \((x_i, x_j)\) planes, and it is essentially the maximum over all possible angles of the moduli of smoothness of one variable in \( \theta_{i,j} \). There are \( \binom{d}{2} \) such angles, which are clearly redundant as a coordinate system. Nevertheless, our new definition effectively reduces a large part of problems in approximation theory on \( S^d \) to problems on \( S^1 \), which allows us to tap into the rich resources of trigonometric approximation theory for ideas and tools and adopt them for problems on the sphere. These same angles also become indispensable for our new definition of moduli of smoothness on the unit ball \( B^d \).

In fact, our moduli on \( B^d \) are defined as the maximum of moduli of smoothness of one variable in these angles and of one additional term that takes care of the boundary behavior. We had two ways to define the additional term, the first one is deduced from the results on the sphere and the second one is the direct extension of the Ditzian-Totik modulus of smoothness on \([-1, 1]\), both of which capture the boundary behavior of the unit ball and permit both direct and inverse theorem for the best approximation. For \( d = 1 \), approximation by polynomials on \( B^1 = [-1, 1] \) is often deduced from approximation by trigonometric polynomials on the circle \( S^1 \) by projecting even functions and their approximations on \( S^1 \) to \([-1, 1]\). This procedure can be adopted to higher dimension by projection functions on \( S^d \) onto \( B^d \), and this is how our first modulus of smoothness on \( B^d \) was defined. It should be mentioned that our new moduli of smoothness on the sphere and on the ball are

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computable; in fact, the computation is not much harder than what is needed for computing classical modulus of smoothness of one variable. A number of examples were given in [3].

In the present paper we continue the work in this direction, study best approximation of functions and their derivatives. On the sphere, our result will be given in terms of differential operators $D_{i,j} = x_i \partial_j - x_j \partial_i$, which can be identified as partial derivatives with respect to $\theta_{i,j}$. We shall define Sobolev spaces and Lipschitz spaces in terms of $D_{i,j}$ on these two domains and study approximation by polynomials in these spaces, including simultaneous approximation of functions and their derivatives. The study is motivated by a question of Kendall Atkinson (cf. [1]) in numerical solution of Poisson equation.

The paper is organized as follows. The main results in [3] on the unit sphere will be recalled in Section 2 and the new results on the sphere will be developed in Section 3. The results in [3] on the unit ball will be recollected and further clarified in Section 4. Finally, the new results on the ball are developed in Section 5.

Throughout this paper we denote by $c, c_1, c_2, \ldots$ generic constants that may depend on fixed parameters and their values may vary from line to line. We write $A \lesssim B$ if $A \leq cB$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2. Polynomial approximation on the sphere: Recent Progress

In this section we recall recent progress on polynomial approximation on the sphere as developed in [3].

Let $L^p(S^{d-1})$ be the $L^p$-space with respect to the usual Lebesgue measure $d\sigma$ on $S^{d-1}$ with norm denoted by $\| \cdot \|_p := \| \cdot \|_{L^p(S^{d-1})}$ for $1 \leq p \leq \infty$, where for $p = \infty$, we replace $L^\infty$ by $C(S^{d-1})$, the space of continuous functions on $S^{d-1}$ with the uniform norm.

2.1. Polynomial spaces and spherical harmonics. We denote by $\Pi^d_n$ the space of polynomials of total degree $n$ in $d$ variables, and by $\Pi^d_n(S^{d-1}) := \Pi^d_n|_{S^{d-1}}$ the space of all polynomials in $\Pi^d_n$ restricted on $S^{d-1}$. In the following we shall write $\Pi^d_n$ for $\Pi^d_n(S^{d-1})$ whenever it causes no confusing. The quantity of best approximation is then defined by

$$ E_n(f)_p := \inf_{g \in \Pi^d_{n-1}} \| f - g \|_p, \quad 1 \leq p \leq \infty. $$

Let $\mathcal{H}^d_n$ denote the space of spherical harmonics of degree $n$ on $S^{d-1}$, which are the restrictions of homogeneous harmonic polynomials to $S^{d-1}$. Let $\Delta_0$ be the Laplace-Beltrami operator on the sphere, defined by

$$ \Delta_0 f(x) := \Delta \left[ f \left( \frac{y}{|y|} \right) \right](x), \quad x \in S^{d-1}, $$

where $\Delta := \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$ is the usual Laplace operator and it acts on the variables $y$. Then the spherical harmonics are the eigenfunctions of $\Delta_0$,

$$ \Delta_0 Y = -n(n + d - 2)Y, \quad Y \in \mathcal{H}^d_n. $$

The reproducing kernel of the space $\mathcal{H}^d_n$ in $L^2(S^{d-1})$ is given by the zonal harmonic

$$ Z_{n,d}(x, y) := \frac{n + \lambda}{\lambda} C_n^\lambda((x, y)), \quad \lambda = \frac{d - 2}{2}, $$
where \( \langle x, y \rangle \) denotes the Euclidean dot product of \( x, y \in \mathbb{R}^d \). Let \( C_n^\lambda \) be the Gegenbauer polynomial with index \( \lambda \), normalized by \( C_n^\lambda(1) = \binom{n + 2\lambda - 1}{n} \). Let \( \eta \) be a \( C^\infty \)-function on \([0, \infty)\) with the properties that \( \eta(x) = 1 \) for \( 0 \leq x \leq 1 \) and \( \eta(x) = 0 \) for \( x \geq 2 \). We define
\[
(2.5) \quad V_n f(x) := \int_{\mathbb{S}^{d-1}} f(y)K_n(\langle x, y \rangle) \, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \quad n = 1, 2, \ldots ,
\]
where
\[
K_n(t) := \sum_{k=0}^{2n} \eta \left( \frac{k}{n} \right) \frac{k + \lambda}{\lambda} C_k^\lambda(t), \quad t \in [-1, 1].
\]
Then \( V_n f \in \Pi_{2n}^d \), \( V_n f = f \) for all \( f \in \Pi_n^d \), and for \( f \in L^p(\mathbb{S}^{d-1}) \) or \( C(\mathbb{S}^{d-1}) \),
\[
(2.6) \quad ||V_n f - f||_p \leq cE_n(f)_p, \quad 1 \leq p \leq \infty.
\]

2.2. A class of differential operators on \( \mathbb{S}^{d-1} \). One of the main tools in our study is a class of differential operators \( D_{i,j} \), \( 1 \leq i \neq j \leq d \), that commute with the Laplace-Beltrami operator. Let \( SO(d) \) denote the group of rotations on \( \mathbb{R}^d \), and let \( e_1, \ldots, e_d \) denote the standard orthogonal basis in \( \mathbb{R}^d \). For \( 1 \leq i \neq j \leq d \) and \( t \in \mathbb{R} \), we denote by \( Q_{i,j,t} \) the rotation by the angle \( t \) in the \((x_i, x_j)\)-plane, oriented such that the rotation from the vector \( e_i \) to the vector \( e_j \) is assumed to be positive. For example, the action of the rotation \( Q_{1,2,t} \in SO(d) \) is given by
\[
(2.7) \quad Q_{1,2,t}(x_1, \ldots, x_d) = (x_1 \cos t - x_2 \sin t, x_1 \sin t + x_2 \cos t, x_3, \ldots, x_d)
\]
\[
= (s \cos(\phi + t), s \sin(\phi + t), x_3, \ldots, x_d),
\]
where \( (x_1, x_2) = s(\cos \phi, \sin \phi) \), and other \( Q_{i,j,t} \) are defined likewise.

To each \( Q \in SO(d) \) corresponds an operator \( L(Q) \) in the space \( L^2(\mathbb{S}^{d-1}) \), defined by \( L(Q)f(x) := f(Q^{-1}x) \) for \( x \in \mathbb{S}^{d-1} \), which is a group representation of \( SO(d) \). The infinitesimal operator of \( L(Q_{i,j,t}) \) has the form
\[
(2.8) \quad D_{i,j} := \frac{\partial}{\partial t} [L(Q_{i,j,t})]\bigg|_{t=0} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}, \quad 1 \leq i < j \leq d.
\]
In particular, it is easy to verify that, taking \((i, j) = (1, 2)\) as an example,
\[
(2.9) \quad D_{1,2}^r f(x) = \left(-\frac{\partial}{\partial \phi}\right)^r f(s \cos \phi, s \sin \phi, x_3, \ldots, x_d).
\]
The following useful observation, which asserts that \( D_{i,j}^r f \) is independent of smooth extensions of \( f \), is a simple consequence of (2.9):

**Proposition 2.1.** Let \( x_0 \in \mathbb{S}^{d-1} \), and let \( F \) and \( G \) be two smooth functions on an open neighborhood \( U \subset \mathbb{R}^d \) of \( x_0 \) which coincide on \( U \cap \mathbb{S}^{d-1} \). Then \( D_{i,j}^r F(x_0) = D_{i,j}^r G(x_0) \).

The operators \( D_{i,j} \) are connected to the usual tangential partial derivatives according to the following formula \([3, 3.15]\):
\[
(2.10) \quad \frac{\partial}{\partial x_j} \left[f \left( \frac{x}{\|x\|} \right) \right]_{\|x\|=1} = \partial_j f - x_j \sum_{i=1}^{d} x_i \partial_i f = \sum_{\{i:1 \leq i \neq j \leq d\}} x_i D_{i,j} f.
\]
The operators $D_{i,j}$ are also closely related to the Laplace-Beltrami operator $\Delta_0$. In fact, $\Delta_0$ satisfies the following decomposition \[ \Delta_0 = \sum_{1 \leq i < j \leq d} D^2_{i,j}, \] (2.11)

Furthermore, each operator $D_{i,j}$ in this decomposition commutes with $\Delta_0$. In particular, by (2.3), this implies that the spaces of spherical harmonics on $S^{d-1}$ are invariant under $D_{i,j}$.

### 2.3. Moduli of smoothness and K-functionals on $S^{d-1}$

For $1 \leq i < j \leq d$ and $\theta \in [-\pi, \pi]$, we define the $r$th difference operator $\Delta^{r}_{i,j,\theta}$ by

$$\Delta^{r}_{i,j,\theta} := (I - T_{Q_{i,j,\theta}})^{r} = \sum_{k=0}^{r} (-1)^{k} \binom{r}{k} T_{Q_{i,j,\theta}}^{k},$$

where $T_{Q} f(x) := f(Qx)$ for $Q \in SO(d)$. This differential operator can be expressed in terms of the usual forward difference as, for example, for $(i,j) = 1,(2)$,

$$\Delta^{r}_{1,2,\theta} f(x) = \Delta^{r}_{\theta} f(x_{1}\cos(\cdot) - x_{2}\sin(\cdot), x_{1}\sin(\cdot) + x_{2}\cos(\cdot), x_{3}, \ldots, x_{d}),$$

where $\Delta^{r}_{\theta}$ is acted on the variable $(\cdot)$, and is evaluated at $t = 0$. The following new modulus of smoothness was recently introduced in [3, Definition 2.2]:

**Definition 2.2.** For $r \in \mathbb{N}$, $t > 0$, and $f \in L^{p}(S^{d-1})$, $1 \leq p < \infty$, or $f \in C(S^{d-1})$ for $p = \infty$, define

$$\omega_{r}(f,t)_{p} := \sup_{|\theta| \leq t} \max_{1 \leq i < j \leq d} \| \Delta^{r}_{i,j,\theta} f \|_{p}. \tag{2.13}$$

This modulus of smoothness enjoys most of the properties of classical moduli of smoothness ([3, Proposition 2.7]), and it permits both direct and inverse theorems ([3, Theorem 3.4]):

**Theorem 2.3.** For $f \in L^{p}(S^{d-1})$ if $1 \leq p < \infty$ and $f \in C(S^{d-1})$ if $p = \infty$,

$$E_{n}(f)_{p} \leq c \omega_{r}(f,n^{-1})_{p}, \quad 1 \leq p \leq \infty. \tag{2.14}$$

On the other hand,

$$\omega_{r}(f,n^{-1})_{p} \leq c n^{-r} \sum_{k=1}^{n} k^{r-1} E_{k-1}(f)_{p}, \quad 1 \leq p \leq \infty. \tag{2.15}$$

A new K-functional on $S^{d-1}$ was defined in terms of the differential operators $D_{i,j}$ in [3, Definition 2.4]:

**Definition 2.4.** For $r \in \mathbb{N}_{0}$ and $t \geq 0$,

$$K_{r}(f,t)_{p} := \inf_{g \in C^{r}(S^{d-1})} \left\{ \| f - g \|_{p} + t^{r} \max_{1 \leq i < j \leq d} \| D^{r}_{i,j} g \|_{p} \right\}. \tag{2.16}$$

As in the classical setting, these two gadgets are equivalent ([3, Theorem 3.6]).

**Theorem 2.5.** Let $r \in \mathbb{N}$ and let $f \in L^{p}(S^{d-1})$ if $1 \leq p < \infty$ and $f \in C(S^{d-1})$ if $p = \infty$. For $0 < t < 1$,

$$\omega_{r}(f,t)_{p} \sim K_{r}(f,t)_{p}, \quad 1 \leq p \leq \infty.$$
Finally, we point out that there are several well studied moduli of smoothness on \( S^{d-1} \) (see, for instance, [5] [8]). One of the advantages of our new modulus is that it reduces many problems in approximation on \( S^{d-1} \) to the corresponding problems of trigonometric approximation of one variable, the latter is classical and well studied. Another advantage is that our modulus is relatively easier to compute, as demonstrated in Part 3 of [3].

3. Sobolev spaces and simultaneous approximation on \( S^{d-1} \)

The classical Sobolev space \( W^r_p \) on \( S^{d-1} \) is defined via the fractional order Laplace-Beltrami operator (see, for example, [2] [7] [8]):

\[
W^r_p := \left\{ f \in L^p(S^{d-1}) : \|f\|_{W^r_p} := \|f\|_p + \|(-\Delta)^{r/2} f\|_p < \infty \right\}.
\]

We shall introduce a new Sobolev type space on \( S^{d-1} \) in this section and then study approximation by polynomials for functions in this new space. Our new Sobolev space is defined via the differential operators \( D_{i,j} \), \( 1 \leq i < j \leq d \), which, by (2.11), are more primitive than \( \Delta \). First, however, we need a lemma.

**Lemma 3.1.** For \( f, g \in C^1(S^{d-1}) \) and \( 1 \leq i \neq j \leq d \),

\[
\int_{S^{d-1}} f(x)D_{i,j}g(x)d\sigma(x) = -\int_{S^{d-1}} D_{i,j}f(x)g(x)d\sigma(x).
\]

**Proof.** By the rotation invariance of the Lebesgue measure \( d\sigma \), we obtain, for any \( \theta \in [-\pi, \pi] \),

\[
\int_{S^{d-1}} f(x)g(Q_{i,j,-\theta}x)d\sigma(x) = \int_{S^{d-1}} f(Q_{i,j,\theta}x)g(x)d\sigma(x).
\]

Differentiating both sides of this identity with respect to \( \theta \) and evaluating the resulted equation at \( \theta = 0 \) lead to the desired equation (3.2). \( \square \)

The equation (3.2) allows us to define distributional derivatives \( D^r_{i,j} \) on \( S^{d-1} \) for \( r \in \mathbb{N} \) via the identity,

\[
\int_{S^{d-1}} D^r_{i,j}f(x)g(x)d\sigma(x) = (-1)^r \int_{S^{d-1}} f(x)D^r_{i,j}g(x)d\sigma(x), \quad g \in C^\infty(S^{d-1}).
\]

We can now define our new Sobolev space on the sphere.

**Definition 3.2.** For \( r \in \mathbb{N} \) and \( 1 \leq p \leq \infty \), we define the Sobolev space \( W^r_p(S^{d-1}) \) to be the space of functions \( f \in L^p(S^{d-1}) \) whose distributional derivatives \( D^r_{i,j}f \), \( 1 \leq i < j \leq d \), all belong to \( L^p(S^{d-1}) \), with norm

\[
\|f\|_{W^r_p(S^{d-1})} := \|f\|_p + \sum_{1 \leq i < j \leq d} \|D^r_{i,j}f\|_p,
\]

where \( L^p(S^{d-1}) \) is replaced by \( C(S^{d-1}) \) when \( p = \infty \).

The following proposition compares the new Sobolev space with the classical one defined in (3.1):

**Proposition 3.3.** For \( 1 < p < \infty \) and \( r = 1 \) or \( 2 \), one has \( W^r_p = W^r_p \) and \( \|f\|_{W^r_p} \sim \|f\|_{W^r_p} \).

In general, for \( r \geq 3 \) and \( 1 < p < \infty \),

\[
W^r_p \subset W^r_p \quad \text{and} \quad \|f\|_{W^r_p} \lesssim \|f\|_{W^r_p}.
\]
Definition 3.5. For \( r \in \mathbb{N} \), \( 1 \leq p \leq \infty \), and \( \alpha \in [0, 1) \), we define the Lipschitz space \( W_p^{r,\alpha} \equiv W_p^{r,\alpha}(S^{d-1}) \) to be the space of all functions \( f \in W_p^r \) with

\[
\|f\|_{W_p^{r,\alpha}(S^{d-1})} := \|f\|_p + \max_{1 \leq i < j \leq d} \sup_{0 < |\theta| \leq 1} \frac{\|\Delta_{i,j}^{r,\alpha}(D_{i,j}^r f)\|_p}{|\theta|^\alpha} < \infty,
\]

where \( \ell \) is a fixed positive integer, for example, \( \ell = 1 \).

Our next theorem gives an equivalent characterization of the space \( W_p^{r,\alpha} \). For the same set of parameters as in the definition of \( W_p^{r,\alpha} \), we define the space

\[
H_p^{r,+:\alpha} := \left\{ f \in L^p(S^{d-1}) : \|f\|_{H_p^{r,+:\alpha}} := \|f\|_p + \sup_{0 < t \leq 1} \frac{\omega_{r+\ell}(f, t)_{p}}{t^{r+\alpha}} < \infty \right\},
\]

Theorem 3.6. If \( r \in \mathbb{N} \), \( 1 \leq p \leq \infty \), and \( \alpha \in [0, 1) \), then

\[
W_p^{r,\alpha} = H_p^{r,+:\alpha} \quad \text{and} \quad \|f\|_{W_p^{r,\alpha}} \sim \|f\|_{H_p^{r,+:\alpha}}.
\]
Thus, by the definition of the distributional derivative $\mathfrak{D} \in g$, the proof of Lemma 2.6 of [3]. Consequently, it follows that
\begin{equation}
\|\Delta^{r+\ell}_{i,j,\theta} f\|_p \leq c |\theta|^r \|\Delta^{\ell}_{i,j,\theta}(D^r_{i,j} f)\|_p.
\end{equation}
Using Lemma 2.6 (ii) in [3], we have
\begin{equation}
\|\Delta^{r+\ell}_{i,j,\theta} f\|_p = \|\Delta^{\ell}_{i,j,\theta}(\Delta^{r}_{i,j,\theta} f)\|_p \leq c |\theta|^r \|\Delta^{\ell}_{i,j,\theta}(D^r_{i,j} f)\|_p.
\end{equation}
However, from (2.12) and (2.9), a quick computation shows that $\Delta_{i,j,\theta} D_{i,j} = D_{i,j} \Delta_{i,j,\theta}$, hence, by iteration, $\Delta^{\ell}_{i,j,\theta} D_{i,j} = D^{\ell}_{i,j} \Delta^{r}_{i,j,\theta}$. As a result,
\begin{equation}
\|D^{\ell}_{i,j} \Delta^{r}_{i,j,\theta} f\|_p = \|\Delta^{\ell}_{i,j,\theta} D^r_{i,j} f\|_p.
\end{equation}
Together, these two displayed equations yield (3.5).

Conversely, assume $f \in H^{r+\alpha}_p$. We first show that $D^r_{i,j} f \in L^p(\mathbb{S}^{d-1})$. For $g \in C^\infty(\mathbb{S}^{d-1})$ and $G_x(t) := g(Q_{i,j,t} x)$, we can write [3, (4.8)]
\begin{equation}
\Delta^r_{i,j,\theta}(g)(x) = \int_0^\theta \cdot \cdot \cdot \int_0^\theta G^{(r)}_x(t_1 + \cdots + t_r) dt_1 \cdots dt_r,
\end{equation}
which implies, in particular, that
\begin{equation}
\lim_{\theta \to 0} \frac{\Delta^r_{i,j,\theta}(g)(x)}{\theta} = G^{(r)}_x(0) = D^r_{i,j} g(x).
\end{equation}
Thus, by the definition of the distributional derivative $D^r_{i,j} f$, it follows that, for $g \in C^\infty(\mathbb{S}^{d-1})$,
\begin{equation}
\int_{\mathbb{S}^{d-1}} [D^r_{i,j} f(x)] g(x) d\sigma = (-1)^r \int_{\mathbb{S}^{d-1}} f(x) D^r_{i,j} g(x) d\sigma = (-1)^r \lim_{\theta \to 0} \int_{\mathbb{S}^{d-1}} f(x) \frac{\Delta^r_{i,j,\theta} g(x)}{\theta} d\sigma = (-1)^r \lim_{\theta \to 0} \int_{\mathbb{S}^{d-1}} \frac{\Delta^r_{i,j,\theta} f(x)}{\theta} g(x) d\sigma,
\end{equation}
where the last step uses the rotation invariance of the Lebesgue measure $d\sigma$. However, by the Marchaud inequality (Proposition 2.7 of [3]), for $f \in H^{r+\alpha}_p$,
\begin{equation}
\begin{split}
\omega_x(f, t)_p \leq c t^r \int_t^1 \frac{\omega_x + t f(x, u)}{u^{r+1}} du \leq c t^r \|f\|_{H^{r+\alpha}_p}.
\end{split}
\end{equation}
Hence, by Hölder’s inequality, we deduce with $\frac{1}{p'} + \frac{1}{p'} = 1$ that
\begin{equation}
\left| \int_{\mathbb{S}^{d-1}} [D^r_{i,j} f(x)] g(x) d\sigma(x) \right| \leq c \|f\|_{H^{r+\alpha}_p} \|g\|_{p'},
\end{equation}
which implies, upon taking supreme over all $g$ with $\|g\|_{p'} \leq 1$ that $D^r_{i,j} f \in L^p(\mathbb{S}^{d-1})$. Next we note that for $\ell \in \mathbb{N}$,
\begin{equation}
\|\Delta^{\ell}_{i,j,\theta}(D^r_{i,j} f)\|_p \leq c \int_0^\theta \|\Delta^{r+\ell}_{i,j,\theta} f\|_{p'} \frac{du}{u^{r+1}},
\end{equation}
which follows from the analogue result for trigonometric functions [4, (7.1)] as in the proof of Lemma 2.6 of [3]. Consequently, it follows that
\begin{equation}
\|f\|_{W^{r,\alpha}_p} \leq \|f\|_p + c \max_{1 \leq i < j \leq d} \sup_{0 \leq |\theta| \leq 1} \frac{1}{|\theta|^\alpha} \int_0^\theta \|\Delta^{r+\ell}_{i,j,\theta} f\|_{p'} \frac{du}{u^{r+1}} \leq c \|f\|_{H^{r+\alpha}_p}
\end{equation}
since $0 < \alpha < 1$. This completes the proof.

Theorem 3.6 together with the Jackson theorem implies the following:

\[\square\]
Corollary 3.7. If \( r \in \mathbb{N}, \alpha \in [0,1), f \in W_p^{r,\alpha}, \) and \( 1 \leq p \leq \infty \), then
\[
E_n(f)_p \leq cn^{-r-\alpha}\|f\|_{W_p^{r,\alpha}}.
\]

4. Polynomial approximation on \( \mathbb{B}^d \): Recent Progress

In this section, we recall recent progress on polynomial approximation on \( \mathbb{B}^d \) as developed in \[9\]. For \( \mu \geq 0 \), let \( W_\mu \) denote the weight function on \( \mathbb{B}^d \) defined by
\[
W_\mu(x) := (1 - \|x\|^2)^{\mu-1/2}.
\]
For \( 1 \leq p < \infty \) we denote by \( \|f\|_{p,\mu} \) the norm for the weighted \( L^p \) space \( L^p(\mathbb{B}^d, W_\mu) \),
\[
\|f\|_{p,\mu} := \left( \int_{\mathbb{B}^d} |f(x)|^p W_\mu(x) dx \right)^{1/p},
\]
and \( \|f\|_{\infty,\mu} := \|f\|_{\infty} \) for \( f \in C(\mathbb{B}^d) \). When we need to emphasize that the norm is taken over \( \mathbb{B}^d \), we write \( \|f\|_{p,\mu} = \|f\|_{L^p(\mathbb{B}^d, W_\mu)} \). For \( f \in L^p(\mathbb{B}^d, W_\mu), 1 \leq p < \infty \), or \( f \in C(\mathbb{B}^d), p = \infty \), the best approximation by polynomials is defined by
\[
E_n(f)_{p,\mu} := \inf_{p \in \Pi_n^d} \|f - p\|_{p,\mu}.
\]

4.1. Weighted orthogonal polynomial expansions on \( \mathbb{B}^d \). Let \( \mathcal{V}_n^d(W_\mu) \) denote the space of orthogonal polynomials of degree \( n \) with respect to the weight function \( W_\mu \) on \( \mathbb{B}^d \). We denote by \( P_n^\mu(x, y) \) the reproducing kernel of \( \mathcal{V}_n^d(W_\mu) \) in \( L^2(\mathbb{B}^d, W_\mu) \). It is shown in \[9\] Theorem 2.6] that
\[
P_n^\mu(x, y) = \int_{S^{d-1}} Z_{n,d+m}(\langle x, y \rangle + \sqrt{1 - \|y\|^2} \langle x', \xi \rangle) d\sigma(\xi)
\]
for any \( x, y \in \mathbb{B}^d \) and \( (x, x') \in S^{d+m-1} \), where \( Z_{n,d}(t) \) is the zonal harmonic defined in \[2.1\] and \( \mu = \frac{n-1}{2d} \). For \( \eta \) being a \( C^\infty \)-function on \( [0, \infty) \) that satisfies the properties as defined in Section 1.1, we define an operator
\[
V_n^\mu f(x) := a_\mu \int_{\mathbb{B}^d} f(y) K_n^\mu(x, y) W_\mu(y) dy, \quad x \in \mathbb{B}^d,
\]
where \( a_\mu \) is the normalization constant of \( W_\mu \) and
\[
K_n^\mu(x, y) := \sum_{k=0}^{2n} \eta \left( \frac{k}{n} \right) P_k^\mu(x, y).
\]
This operator plays the same role as \( V_n f \) in the study on \( S^{d-1} \). In particular, \( V_n^\mu f \in \Pi_{2n}^d \) and \( \|f - V_n^\mu f\|_{p,\mu} \leq c E_n(f)_{p,\mu}, 1 \leq p \leq \infty \).

The spaces \( \mathcal{V}_n^d(W_\mu) \) of orthogonal polynomials are also the eigenspaces of the following second order differential operator:
\[
\mathcal{D}_\mu := \sum_{i=1}^{d} (1 - x_i^2) \partial_i^2 - 2 \sum_{1 \leq i < j \leq d} x_i x_j \partial_i \partial_j - (d + 2\mu) \sum_{i=1}^{d} x_i \partial_i.
\]
Indeed, elements of \( \mathcal{V}_n^d(W_\mu) \) satisfy (cf. \[9\] p. 38)
\[
\mathcal{D}_\mu P = -n(n + d + 2\mu - 1) P \quad \text{for all} \quad P \in \mathcal{V}_n^d(W_\mu).
\]
It was shown in [3, Proposition 7.1] that the differential operator $D_\mu$ can be decomposed as a sum of second order differential operators:

$$D_\mu = \sum_{i=1}^d D_{i,i}^2 + \sum_{1 \leq i < j \leq d} D_{i,j}^2 = \sum_{1 \leq i \leq j \leq d} D_{i,j}^2,$$

where the operators $D_{i,j}^2$ for $1 \leq i < j \leq d$ are defined as in (2.8), and

$$D_{i,i}^2 = D_{\mu,i,i}^2 := \left[ W_\mu(x) \right]^{-1} \partial_i \left[ (1 - \|x\|^2) W_\mu(x) \right] \partial_i, \quad 1 \leq i \leq d.$$

For the rest of the paper, we will always set $\varphi(x) = \sqrt{1 - \|x\|^2}$. We have the following useful estimates:

**Proposition 4.1.** If $1 < p < \infty$, and $g \in C^2(B^d)$ then

$$\|D_\mu g\|_{p,\mu} \sim \sum_{1 \leq i \leq d} \|D_{i,j}^2 g\|_{p,\mu}.$$  

Furthermore, if $r \in \mathbb{N}$, $1 < p < \infty$, and $g \in C^{2r}(B^d)$ then

$$c_1 \|\varphi_r \partial_r^2 g\|_{p,\mu} \leq \|D_{i,j}^2 g\|_{p,\mu} \leq c_2 \|\varphi_r \partial_r^2 g\|_{p,\mu} + c_2 \|g\|_{p,\mu}, \quad 1 \leq i \leq d.$$

**Proof.** (4.10) was proved in [3, Theorem 7.3]. In the case when $r = 1$, (4.11) was shown in [3, Theorem 7.4], and the proof there works equally well for $r \geq 1$. □

### 4.2. Moduli of smoothness and K-functionals.

Two moduli of smoothness and their equivalent K-functionals on $B^d$ were introduced in [3]. Our first modulus on $B^d$ was defined via an extension $\tilde{f}$ of a function $f : B^d \to \mathbb{R}$ to $B^{d+1}$, defined by

$$\tilde{f}(x, x_{d+1}) := f(x), \quad (x, x_{d+1}) \in B^{d+1}, \quad x \in B^d.$$  

More precisely, it is defined as follows ([3, Definition 5.3]).

**Definition 4.2.** For $r \in \mathbb{N}$, $t > 0$, and $f \in L^p(B^d, W_\mu)$, $1 \leq p < \infty$, or $f \in C(B^d)$ for $p = \infty$, define

$$\omega_r(f, t)_{p,\mu} := \sup_{|\theta| \leq t} \left\{ \max_{1 \leq i < j \leq d} \|\Delta_{i,j,\theta} f\|_{L^p(B^d, W_\mu)} \right\},$$

where for $m = 1$, $\|\Delta_{i,d+1,\theta} f\|_{L^p(B^{d+1}, W_{\mu-1/2})}$ is replaced by $\|\Delta_{i,d+1,\theta} f\|_{L^p(B^{d+1})}$.

In the case when $\mu = \frac{n-1}{2}$ and $m \in \mathbb{N}$, we have established in [3, Theorem 5.5] the direct theorem, that is, the Jackson inequality

$$E_n(f)_{p,\mu} \leq c \omega_r(f, n^{-1})_{p,\mu}, \quad 1 \leq p \leq \infty,$$

and the corresponding inverse theorem,

$$\omega_r(f, n^{-1})_{p,\mu} \leq c n^{-r} \sum_{k=1}^n k^{r-1} E_k(f)_{p,\mu},$$

in terms of this new modulus of smoothness. Moreover, it was also shown in [3, Theorem 5.8] that the modulus of smoothness $\omega_r(f, t)_{p,\mu}$ is equivalent to the following
\begin{equation}
\hat{C}_r(f, t)_{p, \mu} := \inf_{g \in C^r(\mathbb{B}^d)} \left\{ \| f - g \|_{L^p(\mathbb{B}^d, W_\mu)} + t^r \max_{1 \leq i < j \leq d} \| D_{r,ij}^t g \|_{L^p(\mathbb{B}^d, W_\mu)} \right\},
\end{equation}

where if \( m = 1 \), then \( \| D_{r,ij}^t g \|_{L^p(\mathbb{B}^d, W_\mu)} \) is replaced by \( \| D_{r,ij+1}^t g \|_{L^p(\mathbb{B}^d, W_\mu)} \).

Our second new modulus in [3] can be considered as a higher-dimensional analogue of the classical Ditzian-Totik modulus on the interval \([-1, 1]\). In the unweighted case, this modulus is defined as follows [3 Definition 6.7]:

**Definition 4.3.** For \( r \in \mathbb{N}, t > 0, \) and \( f \in L^p(\mathbb{B}^d, W_\mu), 1 \leq p < \infty, \) or \( f \in C(\mathbb{B}^d) \) for \( p = \infty, \) define

\begin{equation}
\hat{\omega}_r(f, t)_p := \sup_{0 < |h| \leq t} \left\{ \max_{1 \leq i < j \leq d} \| \Delta_{r,i,j,h} f \|_p, \max_{1 \leq i \leq d} \| \hat{\Delta}_{r,i,\varphi e_i} f \|_p \right\},
\end{equation}

where \( \hat{\Delta}_{r,i,\varphi e_i} \) denotes the central difference and

\( \hat{\Delta}_{r,i,\varphi e_i} f(x) := \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (\xi - k) \varphi e_i) \),

and we assume \( \hat{\Delta}_{r,i,\varphi e_i} = 0 \) if either of the points \( x \pm r \frac{\Delta \varphi(x)}{2} e_i \) does not belong to \( \mathbb{B}^d \).

The weighted version \( \hat{\omega}_r(f, t)_{p, \mu} \) of the above modulus can also be defined, but is more complicated. Both direct and inverse theorems were established in terms of \( \hat{\omega}_r(f, t)_p \) in [3 Theorem 6.13]. Furthermore, it was shown in [3 Theorem 6.10] that the modulus of smoothness \( \hat{\omega}_r(f, t)_p \) is equivalent to the following K-functional

\begin{equation}
\hat{K}_r(f, t)_{p, \mu} := \inf_{g \in C^r(\mathbb{B}^d)} \left\{ \| f - g \|_{p, \mu} + t^r \max_{1 \leq i < j \leq d} \| D_{r,ij}^t g \|_{p, \mu} + t^r \max_{1 \leq i \leq d} \| \varphi \partial_i g \|_{p, \mu} \right\}
\end{equation}

in the sense that, for the equivalence between \( \hat{K}_r(f, t)_p \) and \( \omega_r(f, t)_p \) in the unweighted case,

\[ c^{-1} \hat{\omega}_r(f, t)_p \leq \hat{K}_r(f, t)_p \leq c \hat{\omega}_r(f, t)_p + c t^r \| f \|_p. \]

The two K-functionals, hence their equivalent moduli of smoothness, are connected as shown in [3 Theorem 6.2].

**Theorem 4.4.** Let \( \mu = \frac{m-1}{2} \) and \( m \in \mathbb{N} \). Let \( f \in L^p(\mathbb{B}^d, W_\mu) \) if \( 1 \leq p < \infty, \) and \( f \in C(\mathbb{B}^d) \) if \( p = \infty. \) We further assume that \( r \) is odd when \( p = \infty. \) Then

\begin{equation}
\hat{K}_1(f, t)_{p, \mu} \sim K_1(f, t)_{p, \mu},
\end{equation}

and for \( r \geq 1 \), there is a \( t_r > 0 \) such that

\begin{equation}
K_r(f, t)_{p, \mu} \leq c \hat{K}_r(f, t)_{p, \mu} + c t^r \| f \|_{p, \mu}, \quad 0 < t < t_r.
\end{equation}

Finally, we point out that it was shown in [3] that both moduli \( \hat{\omega}_r(f, t)_{p, \mu} \) and \( \omega_r(f, t)_{p, \mu} \) enjoy most of the properties of classical moduli of smoothness and they are computable as demonstrated in Part 3 of [3]. In comparison, the only other modulus of smoothness [10] on the unit ball that is strong enough to characterize the best approximation is hardly computable.
4.3. Representation of the term $D^r_{i,d+1} \tilde{f}$. The term $D^r_{i,d+1} \tilde{g}$ appears in the definition of our first K-functional $K_r(f,t)_{p,\mu}$ in (4.10) on the ball, where $\tilde{g}(x,x_{d+1}) = g(x)$ as in (4.12). Notice that $\tilde{f}$ is a function in $x \in \mathbb{R}^d$, but the operator $D^r_{i,d+1} = x_i \partial_{d+1} - x_{d+1} \partial_i$ involves $x_{d+1}$, so that $D^r_{i,d+1} \tilde{f}$ is indeed a function of $(x,x_{d+1})$ in $\mathbb{B}^{d+1}$. The following lemma gives an explicit formula of this term in terms of $f$.

**Lemma 4.5.** Assume that $(y,y_{d+1}) = s(x,x_{d+1}) \in \mathbb{B}^{d+1}$ with $s = \|(y,y_{d+1})\| > 0$, $x \in \mathbb{B}^d$ and $x_{d+1} = \varphi(x) \geq 0$. If $f \in C^r(\mathbb{B}^d)$, then

$$(D^r_{i,d+1} \tilde{f})(y,y_{d+1}) = \left( -\varphi(x) \frac{\partial}{\partial x_i} \right)^r [f(sx)], \quad 1 \leq i \leq d.$$ 

**Proof.** The proof uses induction. For $r = 1$, we have

$$D^1_{i,d+1} \tilde{f}(y,y_{d+1}) = (y_i \partial_{d+1} - y_{d+1} \partial_i)f(y) = -y_{d+1} \partial_i f(y).$$

Hence, using the fact that $\frac{\partial}{\partial x_i}[f(sx)] = s(\partial_i f)(sx)$ we have

$$(D^1_{i,d+1} \tilde{f})(sx, s_{d+1}) = -s x_{d+1} (\partial_i f)(sx) = -s \varphi(x)(\partial_i f)(sx) = -\varphi(x) \frac{\partial}{\partial x_i}[f(sx)].$$

Let $F_r(x,x_{d+1}) = D^r_{i,d+1} \tilde{f}(x,x_{d+1})$. Assume that the result has been established for $r$. Then

$$F_r(sx, s_{d+1}) = (D^r_{i,d+1} F_r)(sx, s_{d+1})$$

$$= s x_i (\partial_{d+1} F_r)(sx, s_{d+1}) - s x_{d+1} (\partial_i F_r)(sx, s_{d+1}).$$

On the other hand, taking derivative by chain rule shows that

$$\left( -\varphi(x) \frac{\partial}{\partial x_i} \right)^{r+1} [f(sx)] = \left( -\varphi(x) \frac{\partial}{\partial x_i} \right) [F_r(sx, s \varphi(x))]$$

$$= -s \varphi(x)(\partial_i F_r)(sx, s \varphi(x)) + s x_i (\partial_{d+1} F_r)(sx, s \varphi(x)),$$

which is the same as the right hand side of (4.20) with $x_{d+1} = \varphi(x)$. \hfill \square

**Lemma 4.6.** The function $D^r_{i,d+1} \tilde{f}(x,x_{d+1})$ is even in $x_{d+1}$ if $r$ is even, and odd in $x_{d+1}$ if $r$ is odd.

**Proof.** For $r = 1$, $D^1_{i,d+1} \tilde{f}(x,x_{d+1}) = -x_{d+1} \partial_i f(x)$ is clearly odd in $x_{d+1}$. And

$$D^2_{i,d+1} \tilde{f}(x,x_{d+1}) = -x_i \partial_i f(x) + x_{d+1}^2 \partial_i^2 f(x)$$

is even in $x_{d+1}$. The general case follows from induction upon using (4.20). \hfill \square

Recall that our $K$-functional $K_r(f,t)_{p,\mu}$ with $\mu = \frac{m-1}{2}$ in (4.16) is defined when $m = 1$, with $\|D^r_{i,d+1} \tilde{g}\|_{L^p(\mathbb{B}^{d+1})}$ in place of $\|D^r_{i,d+1} \tilde{g}\|_{L^p(\mathbb{B}^{d+1},W_{\mu-1/2})}$. Hence, as a consequence of the above lemmas, we conclude the following:

**Proposition 4.7.** For $g \in C^r(\mathbb{B}^d)$ and the Chebyshev weight $W_0$ on $\mathbb{B}^d$, we have

$$\|D^r_{i,d+1} \tilde{g}\|_{L^p(\mathbb{B}^d)} = \|((\varphi \partial_i)^r g\|_{L^p(\mathbb{B}^d, W_0)}.$$
Proposition 4.8. If $\mu \geq 0$ when dealing with $p$, whereas for $\parallel (5.3)$

Proof. Let $S_+^d = \{ x \in S^d : x_{d+1} \geq 0 \}$. By Lemma 4.6, we only need to consider $S_+^d$ when dealing with $D_{i,d+1}^r \tilde{g}$. By Lemma 4.5 with $s = 1$, we then obtain

$$\int_{S_+^d} |D_{i,d+1}^r \tilde{g}(x, x_{d+1})|^p \, d\sigma(x, x_{d+1}) = 2 \int_{S_+^d} |(\varphi(x) \partial_i)^r g(x)|^p \, d\sigma(x, x_{d+1})$$

$$= \int_{S_+^d} |(\varphi(x) \partial_i)^r g(x)|^p \frac{dx}{\sqrt{1 - \|x\|^2}},$$

which is what we want to prove. \qed

In general, using polar coordinates, Lemma 4.5 and Lemma 4.6, we can deduce

**Proposition 4.8.** If $g \in C^r(\mathbb{B}^d)$, $\mu = \frac{m-1}{2}$ and $m > 1$, then for $1 \leq p < \infty$,

$$\|D_{i,d+1}^r \tilde{g}\|_{L^p(S^d, W_{\mu-1/2}^p)} = \int_0^1 s^d (1 - s^2)^{\mu-1} \int_{S_+^d} |(\varphi(x) \partial_i)^r g(sx)|^p \frac{dx}{\sqrt{1 - \|x\|^2}} \, ds;$$

whereas for $p = \infty$, we have

$$\max_{y \in S^d} |D_{i,d+1}^r \tilde{g}(y)| = \max_{x \in \mathbb{B}^d, 0 \leq s \leq 1} \left( \varphi(x) \frac{\partial}{\partial x_i} \right)^r |g(sx)|.$$

5. **Sobolev Spaces and Simultaneous Approximation on $\mathbb{B}^d$**

We start with the definition of a Sobolev space on $\mathbb{B}^d$.

**Definition 5.1.** For $1 \leq p \leq \infty$, $f \in C^r(\mathbb{B}^d)$, and $r \in \mathbb{N}$, we define

$$\|f\|_{W_p^r(\mathbb{B}^d, W_\mu)} := \|f\|_{p,\mu} + \sum_{1 \leq i < j \leq d} \|D_{i,j}^r f\|_{p,\mu} + \sum_{i=1}^d \|\varphi^i \partial_i^r f\|_{p,\mu},$$

and define $W_p^r(\mathbb{B}^d, W_\mu)$ to be the completion of $C^r(\mathbb{B}^d)$ with respect to the norm $\| \cdot \|_{W_p^r(\mathbb{B}^d, W_\mu)}$.

**Remark 5.1.** Since convergence in the norm $\| \cdot \|_{W_p^r(\mathbb{B}^d, W_\mu)}$ implies convergence in the weighted $L^p$-norm $\| \cdot \|_{p, \mu}$, we may assume that $W_p^r(\mathbb{B}^d, W_\mu) \subset L^p(\mathbb{B}^d, W_\mu)$ when $p < \infty$, and $W_p^r(\mathbb{B}^d, W_\mu) \subset C(\mathbb{B}^d)$ when $p = \infty$. As a consequence, we can also extend the definitions of the operators $D_{i,j}^r f$ and $\varphi^i \partial_i^r f$ to the whole space $W_p^r(\mathbb{B}^d, W_\mu)$.

The following proposition follows readily from (6.15) and (6.16) of \[3\] and Proposition 4.1.

**Proposition 5.2.** If $f \in C^r(\mathbb{B}^d)$, $\mu \geq 0$ and $1 \leq p \leq \infty$, then

$$\sum_{1 \leq i \leq d} \|D_{i,d+1}^r f\|_{L^p(S^d, W_{\mu-1/2}^p)} \leq c \|f\|_{W_p^r(\mathbb{B}^d, W_\mu)}.$$

Furthermore, if $f \in C^{2r}(\mathbb{B}^d)$ and $1 < p < \infty$ then

$$\|f\|_{W_p^{2r}(\mathbb{B}^d, W_\mu)} \sim \sum_{1 \leq i \leq j \leq d} \|D_{i,j}^{2r} f\|_{p,\mu}.$$
Theorem 5.3. Let \( \mu = \frac{m+1}{2} \) with \( m \in \mathbb{N} \). For \( f \in W^r_p(\mathbb{B}^d, W_\mu) \), \( 1 \leq p \leq \infty \),

\begin{equation}
E_2n(f)_{p, \mu} \leq c n^{-r} \max_{1 \leq i,j \leq d} E_n(D_{i,j}^r f)_{p, \mu} + \max_{1 \leq i \leq d} E_n(D_{i,d+1}^r \tilde{f})_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})}.
\end{equation}

Furthermore, \( V_n^\mu f \), defined by \( (1.44) \), provides the near best simultaneous approximation for all \( D_{i,j}^r f \), \( 1 \leq i < j \leq d+1 \) in the sense that

\begin{align*}
\|D_{i,j}^r (f - V_n^\mu f)\|_{L^p} &\leq c E_n(D_{i,j}^r f)_{p, \mu}, \quad 1 \leq i < j \leq d, \\
\|D_{i,d+1}^r (\tilde{f} - V_n^\mu f)\|_{L^p} &\leq c E_n(D_{i,d+1}^r \tilde{f})_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})}, \quad 1 \leq i \leq d.
\end{align*}

Proof. For \( f \) defined on \( \mathbb{B}^d \), we define \( F(x, x') := f(x), x \in \mathbb{B}^d \), \( (x, x') \in \mathbb{S}^{d+m-1} \).

By [3, Lemma 5.2], \( (V_n F)(x, x') = V_n^\mu f(x) \). Furthermore, by [3, Lemma 5.7],

\[ K_r(f, n^{-1})_{L^p(\mathbb{B}^{d+m})} \sim K_r(F, n^{-1})_{L^p(\mathbb{S}^{d+m})} \]

Hence, it follows that

\begin{align*}
E_2n(f)_{L^p(\mathbb{B}^d)} &\leq c K_r(f - V_n^\mu f, n^{-1})_{L^p(\mathbb{B}^{d+m})} \\
&\leq c K_r(F - V_n F, n^{-1})_{L^p(\mathbb{B}^{d+m})} \\
&\leq c n^{-r} \max_{1 \leq i, j \leq d+m} \|D_{i,j}^r (F - V_n F)\|_{L^p(\mathbb{S}^{d+m})} \\
&= c n^{-r} \max_{1 \leq i, j \leq d+1} \|D_{i,j}^r F - (D_{i,j}^r V_n F)\|_{L^p(\mathbb{S}^{d+m})},
\end{align*}

where the last step follows from the fact that \( V_n(D_{i,j}^r g) \) depends on \( x_j, 1 \leq j \leq d, \)
and \( x_{d+k} \), which implies that we only need to consider \( 1 \leq i < j \leq d + 1 \).

Denote by \( V_n^{\mu,d} \) the operator \( (1.44) \) associated with \( W_\mu \) on \( \mathbb{B}^d \) and \( \tilde{f}(x, x_{d+1}) = f(x) \). By Lemma 5.2 of [3],

\begin{equation}
V_n F(x, x') = V_n^{\mu-1/2,d+1} \tilde{f}(x, x_{d+1}) = V_n^{\mu,d} f(x).
\end{equation}

Since \( D_{i,j}^r V_n = V_n D_{i,j}^r \) on the sphere, that for \( 1 \leq i, j \leq d \),

\[ D_{i,j}^r V_n^{\mu,d} f(x) = D_{i,j}^r V_n F(x, x') = V_n D_{i,j}^r F(x, x') = V_n^{\mu,d} D_{i,j}^r f(x). \]

Consequently, it follows from [3, (5.8)] that

\[ \|D_{i,j}^r F - D_{i,j}^r V_n F\|_{L^p(\mathbb{S}^{d+m})} = c \|D_{i,j}^r f - D_{i,j}^r V_n^{\mu,d} f(x)\|_{L^p(\mathbb{B}^{d+m})} \]

\[ = c \|D_{i,j}^r f - V_n^{\mu,d} D_{i,j}^r f(x)\|_{L^p(\mathbb{B}^{d+m})} \]

\[ \leq c E_n(D_{i,j}^r f)_{L^p(\mathbb{B}^{d+m})}. \]

Whereas for \( D_{i,d+1}^r F \) term, we have for \( 1 \leq i \leq d \),

\[ D_{i,d+1}^r V_n^{\mu-1/2} \tilde{f}(x, x_{d+1}) = D_{i,d+1}^r (V_n F)(x, x') \]

\[ = V_n(D_{i,d+1}^r F)(x, x') = V_n^{\mu-1/2} D_{i,d+1}^r \tilde{f}(x, x_{d+1}). \]

Consequently,

\[ \|D_{i,d+1}^r F - D_{i,d+1}^r V_n F\|_{L^p(\mathbb{B}^{d+m})} = c \|D_{i,d+1}^r f - V_n^{\mu-1/2} D_{i,d+1}^r f(x)\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \]

\[ \leq c E_n(D_{i,d+1}^r \tilde{f})_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})}. \]

This proves \((5.4)\). The conclusion that \( V_n^\mu f \) is the near best simultaneous approximation follows from the above proof and \((5.5)\). \( \square \)
Corollary 5.4. Let $\mu = \frac{m-1}{2}$ with $m \in \mathbb{N}$. If $f \in \mathcal{W}_p^r(\mathbb{B}^d, W_\mu)$, $1 \leq p \leq \infty$, then
\[
E_n(f)_{p, \mu} \leq c n^{-r} \|f\|_{\mathcal{W}_p^r(\mathbb{B}^d, W_\mu)}.
\]

Proof. This follows immediately from (5.1), (6.2) and Theorem 5.3.

In the next corollary, we replace $D_{i,d+1}^r f$ term in (5.4) by ordinary derivatives of $f$. First we consider the Chebyshev weight $W_0$ on $\mathbb{B}^d$ (with $\mu = 0$).

Corollary 5.5. For $f \in \mathcal{W}_p^r(\mathbb{B}^d, W_0)$, $r \in \mathbb{N}$ and $1 \leq p \leq \infty$,
\[
E_n(f)_{p,0} \leq c n^{-r} \max_{1 \leq i < j \leq d} E_n(D_{i,j}^r f)_{p,0} + c n^{-r} \max_{1 \leq i \leq d} E_n((\varphi \partial_i)^r f)_{p,0}.
\]

Proof. By (4.21), for all $f \in C^r(\mathbb{B}^d)$,
\[
(\varphi(x)\partial_i)^r f(x) = D_{i,d+1}^r f(x, x_{d+1}),
\]
where $x \in \mathbb{B}^d$ and $x_{d+1} = \varphi(x)$. The desired conclusion then follows.

For $\mu > 0$, including the case $\mu = 1/2$ (the constant weight function), however, the best that we can do is the following:

Corollary 5.6. Let $\mu = \frac{m-1}{2}$ and $m \in \mathbb{N}$. For $f \in \mathcal{W}_p^r(\mathbb{B}^d, W_\mu)$, $1 \leq p \leq \infty$,
\[
E_n(f)_{p, \mu} \leq c n^{-r} \max_{1 \leq i < j \leq d} E_n(D_{i,j}^r f)_{p, \mu} + c n^{-r} \max_{1 \leq i \leq d} E_n((\varphi \partial_i)^r f)_{p, \mu} + \max_{1 \leq i \leq d} E_n((\varphi \partial_i + \varphi^2 \partial_i^{d+1})^r f)_{p, \mu}.
\]

Proof. It was shown in Lemma 6.4 of [3] that
\[
D_{i,d+1}^r f(x, x_{d+1}) = \sum_{j=1}^r p_{j,r}(x_i, x_{d+1}) \partial_i^j f(x), \quad x \in \mathbb{B}^d, \ (x, x_{d+1}) \in \mathbb{B}^{d+1},
\]
where $p_{j,r}$ is a polynomial of degree $\leq j$. Since $E_n(f)$ is subadditive, it follows that
\[
E_n(D_{i,d+1}^r f)_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \leq \sum_{j=1}^r \inf_{g \in \Pi_n^{j,r}} \left\| p_{j,r}(\partial_i^j \bar{f} - \bar{g}) \right\|_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})}.
\]

However, using (6.11) and (6.12) of [3], we have, for $(x, x_{d+1}) \in \mathbb{B}^{d+1}$,
\[
|p_{j,r}(x, x_{d+1})| \leq \begin{cases} c, & \text{if } 1 \leq j < \frac{m-1}{2}, \\ c|x_{d+1}|^{2j-r}, & \text{if } \frac{m-1}{2} \leq j \leq r. \end{cases}
\]
Thus, by (5.6), we deduce
\[
E_n(D_{i,d+1}^r f)_{L^p(\mathbb{B}^{d+1}, W_{\mu-1/2})} \leq c \max_{1 \leq j < \frac{m-1}{2}} \inf_{g \in \Pi_n^{j,r}} \| \partial_i^j f - g \|_{p, \mu} + c \max_{\frac{m-1}{2} \leq j \leq r} \inf_{g \in \Pi_n^{j,r}} \| \partial_i^j f - g \|_{p, \mu+(j-\frac{1}{2})p}.
\]

The desired conclusion then follows from Theorem 5.3.

It remains to be seen if $D_{i,d+1}^r f$ term in (5.4) can be bounded by a term that involves only $(\varphi \partial)^r f$ in the case of $\mu > 0$. Similar to the case of $\mathbb{S}^{d-1}$, we can also define a Lipschitz space on the ball.
Definition 5.7. For \( r \in \mathbb{N} \), \( \alpha \in (0, 1) \), and \( 1 \leq p \leq \infty \), we define \( W^r_{p,\alpha}(\mathbb{B}^d, W_\mu) \) to be the space of all functions \( f : \mathbb{B}^d \to \mathbb{R} \) with finite norm

\[
\|f\|_{W^r_{p,\alpha}(\mathbb{B}^d, W_\mu)} := \|f\|_{p,\mu} + \max_{1 \leq i < j \leq d} \sup_{0 < |\theta| \leq 1} |\theta|^{-\alpha} \|\Delta_{i,j,\theta}^r f\|_{p,\mu} + \max_{1 \leq i \leq d} \sup_{0 < |\theta| \leq 1} |\theta|^{-\alpha} \|\Delta_{i,d+1,\theta}^r f\|_{L^p(\mathbb{B}^d, W_{\mu-1/2})}
\]

with the usual change when \( p = \infty \), where \( \ell \) is a fixed positive integer, say \( \ell = 1 \).

We can also give an equivalent characterization of the space \( W^r_{p,\alpha}(\mathbb{B}^d, W_\mu) \) in terms of our modulus of smoothness. For the same set of parameters as in the definition of \( W^r_{p,\alpha}(\mathbb{B}^d, W_\mu) \), we define a space

\[
H^r_{p,\alpha}(\mathbb{B}^d, W_\mu) := \left\{ f \in L^p(\mathbb{B}^d, W_\mu) : \sup_{0 < t \leq 1} \frac{\omega_{r+\ell}\left(f, t\right)_{p,\mu}}{t^{r+\alpha}} < \infty \right\}.
\]

Theorem 5.8. Let \( \mu = \frac{m-1}{2} \). If \( r \in \mathbb{N} \), \( 1 \leq p \leq \infty \), and \( \alpha \in (0, 1) \), then

\[
W^r_{p,\alpha}(\mathbb{B}^d, W_\mu) = H^r_{p,\alpha}(\mathbb{B}^d, W_\mu) \quad \text{and} \quad \|f\|_{W^r_{p,\alpha}(\mathbb{B}^d, W_\mu)} \sim \|f\|_{H^r_{p,\alpha}(\mathbb{B}^d, W_\mu)}.
\]

Proof. This follows from Theorem 8.9 since for \( F(x, x') := f(x) \), \( (x, x') \in \mathbb{B}^{d+m-1} \) and \( x \in \mathbb{B}^d \), we have \( \omega(F, t)_{L^p(\mathbb{B}^{d+m-1})} \sim \omega_r(f, t)_{p,\mu} \) by Lemma 5.4 of [3]. \( \square \)

We could also define a Lipschitz space that uses central differences of \( \partial_{\alpha} f \) in place of \( D_{\alpha+d+1,\theta}^r f \) in \( W^r_{p,\alpha}(\mathbb{B}^d, W_\mu) \), so that it is equivalent to an analogue of \( H^r_{p,\alpha}(\mathbb{B}^d, W_\mu) \) with \( \omega_{r+\ell}(f, t)_{p,\mu} \) in place of \( \omega_{r+\ell}(f, t)_{p,\mu} \).

As a consequence of the last theorem and the Jackson estimate, we have

Corollary 5.9. Let \( \mu = \frac{m-1}{2} \) and \( m \in \mathbb{N} \). If \( r \in \mathbb{N} \), \( \alpha \in (0, 1) \), \( f \in W^r_{p,\alpha}(\mathbb{B}^d, W_\mu) \), and \( 1 \leq p \leq \infty \), then

\[
E_\alpha(f)_{p,\mu} \leq C n^{-r-\alpha} \|f\|_{W^r_{p,\alpha}(\mathbb{B}^d, W_\mu)}.
\]

Let us point out that for \( f \in C^r(\mathbb{B}^d) \) the traditional definition of the Lipschitz continuity takes the form, for \( 0 < \alpha < 1 \),

\[
|\partial_{\alpha} f(x) - \partial_{\alpha} f(y)| \leq c \|x - y\|^\alpha, \quad \beta \in \mathbb{N}^d, \quad |\beta| = r
\]

for all \( x, y \in \mathbb{B}^d \). Let us denote by \( \text{Lip}_{r,\alpha} \) the space of all \( C^r(\mathbb{B}^d) \) functions that satisfy (5.8). From the definition of \( D_{i,j} \) it follows readily that

\[
\text{Lip}_{r,\alpha} \subset W^r_{\infty,\alpha}.
\]

Hence, the estimate (5.7) holds for the functions in \( \text{Lip}_{r,\alpha} \). On the other hand, our definition of \( W^r_{p,\alpha} \) is more general than \( \text{Lip}_{r,\alpha} \) as the following example shows.

Example. Let \( f_\alpha(x) = (1 - \|x\|^2 + \|x - x_0\|^2)\alpha \) on \( \mathbb{B}^d \) with a fixed \( x_0 \in \mathbb{S}^{d-1} \). Assume \( 1/2 < \alpha < 1 \). Then by [3, Example 10.1], \( \omega_r(f_\alpha, t) \sim t^{2\alpha} \), so that by Theorem 5.8, \( f_\alpha \in W^r_{\infty,2\alpha-1}(\mathbb{B}^d) \). On the other hand, setting \( x_0 = (1, 0, 0, \ldots, 0) \) shows that \( f_\alpha(x) = (1 - x_1^2 + (1 - x_1)^2)^\alpha = 2^\alpha(1 - x_1)^\alpha \), whose first partial derivative is unbounded on \( \mathbb{B}^d \) so that it is not an \( \text{Lip}_{1,\alpha} \) function. We note that \( D_{i,j} f_\alpha \in C(\mathbb{B}^{d+1}) \) for all \( 1 \leq i < j \leq d + 1 \).
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