CONSTRUCTION OF TYPE II BLOW-UP SOLUTIONS FOR THE ENERGY-CRITICAL WAVE EQUATION IN DIMENSION 5

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Abstract. We consider the semilinear wave equation with focusing energy-critical nonlinearity in space dimension $N = 5$:

$$\partial_{tt}u = \Delta u + |u|^{4/3}u,$$

with radial data. It is known [7] that a solution $(u, \partial_t u)$ which blows up at $t = 0$ in a neighborhood (in the energy norm) of the family of solitons $W_\lambda$, decomposes in the energy space as

$$(u(t), \partial_t u(t)) = (W_\lambda(t) + u_0^*, u_1^*) + o(1),$$

where $\lim_{t \to 0} \lambda(t)/t = 0$ and $(u_0^*, u_1^*) \in \dot{H}^1 \times L^2$. We construct a blow-up solution of this type such that the asymptotic profile $(u_0^*, u_1^*)$ is any pair of sufficiently regular functions with $u_0^*(0) > 0$. For these solutions the concentration rate is $\lambda(t) \sim t^4$. We also provide examples of solutions with concentration rate $\lambda(t) \sim t^{\nu+1}$ for $\nu > 8$, related to the behaviour of the asymptotic profile near the origin.

1. Introduction

1.1. General setting. We are interested in the problem of constructing type II blow-up solutions for the energy-critical wave equation in space dimension $N = 5$:

$$\partial_{tt}u = \Delta u + |u|^{4/3}u,$$

$(t, x) \in \mathbb{R} \times \mathbb{R}^5$.

Denote $f(u) := |u|^{4/3}u$. It will be convenient to write the wave equation as a first-order in time system:

$$\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \left( \frac{u}{\partial_t u} \right) = \left( \frac{\partial_t u}{\Delta u + f(u)} \right), \\
\left( \frac{u(t_0)}{\partial_t u(t_0)} \right) = \left( \frac{u_0}{u_1} \right) \in \dot{H}^1 \times L^2.
\end{array} \right.$$  

(NLW)

This equation is locally well-posed in the energy space $\dot{H}^1 \times L^2$ (see for example [12] and the references therein). In particular, for any initial data $(u_0, u_1)$ there exists a maximal interval of existence $(T_-, T_+)$, $-\infty \leq T_- < t_0 < T_+ \leq +\infty$, and a unique solution $(u, \partial_t u) \in C((T_-, T_+); \dot{H}^1 \times L^2)$. This solution conserves the energy:

$$E(u(t), \partial_t u(t)) = \frac{1}{2} \int |\partial_t u|^2 \, dx + \frac{1}{2} \int |\nabla u|^2 \, dx - \int F(u) \, dx = E(u_0, u_1),$$

where $F(u) = \int f(u) \, du = \frac{3}{10}|u|^{10/3}$ (notice that $\int F(u) \, dx$ is finite by the Sobolev imbedding).

For a function $v : \mathbb{R}^5 \to \mathbb{R}$ and $\lambda > 0$, we denote

$$v_\lambda(x) := \frac{1}{\lambda^{3/2}} v \left( \frac{x}{\lambda} \right), \quad v_\lambda(x) := \frac{1}{\lambda^{5/2}} v \left( \frac{x}{\lambda} \right).$$

A change of variables shows that

$$E((u_0)_\lambda, (u_1)_\lambda) = E(u_0, u_1).$$
Equation $\text{(NLW)}$ is invariant under the same scaling. If $(u, \partial_t u)$ is a solution of $\text{(NLW)}$ and $\lambda > 0$, then
\[
t \mapsto \left( u\left( \frac{t-t_0}{\lambda} \right), \partial_t u\left( \frac{t-t_0}{\lambda} \right) \right)
\]
is also a solution with initial data $\left( (u_0)_\lambda, (u_1)_\lambda \right)$ at time $t = 0$. This is why equation $\text{(NLW)}$ is called energy-critical.

We introduce also the infinitesimal generators of scale change:
\[
\Lambda v := -\frac{d}{d\lambda}v_{|\lambda=1} = \left( \frac{3}{2} + x \cdot \nabla \right)v,
\]
\[
\Lambda_0 v := -\frac{d}{d\lambda}v_{|\lambda=1} = \left( \frac{5}{2} + x \cdot \nabla \right)v.
\]

A fundamental object in the study of $\text{(NLW)}$ is the family of solutions $(u, \partial_t u) = (W_\lambda, 0)$, where $$W(x) = \left( 1 + \frac{|x|^2}{15} \right)^{-3/2}.$$ The functions $W_\lambda$ are called ground states. In this paper we are interested in radial solutions $(u, \partial_t u)$ of $\text{(NLW)}$ such that $\inf_x \| (u - W_\lambda, \partial_t u) \|_{\dot{H}^1 \times L^2}$ remains small for $T_- < t \leq t_0$. In the case $N = 3$ it was proved by Krieger, Nakanishi and Schlag \cite{KNS} that such solutions form a codimension one manifold in a neighbourhood of the family $\{W_\lambda\}$. This is expected to hold also for $N = 5$. The asymptotic behaviour of such (not necessarily radial) solutions as $t \to T_-$ was described by Duyckaerts, Kenig and Merle in \cite{DKM1}, both in the case $T_- = -\infty$ and $T_- > -\infty$. In the second case, which is relevant for us, they obtain the following result.

**Theorem.** \cite{DKM1} Theorem 2] Let $(u, \partial_t u)$ be a solution of $\text{(NLW)}$ such that $T_- = 0$ and $\inf_x \| (u - W_\lambda, \partial_t u) \|_{\dot{H}^1 \times L^2}$ remains small for $T_- < t \leq T_0$. Then there exists a $C^0$ function $\lambda(t) : (0, T_0) \to (0, +\infty)$, such that
\begin{equation}
\lim_{t \to 0^+} \left( u(t) - W_{\lambda(t)}, \partial_t u(t) \right) = (u_0^*, u_1^*) \in \dot{H}^1 \times L^2,
\end{equation}
and the convergence is strong in $\dot{H}^1 \times L^2$. In addition, $\lambda(t) \ll t$ as $t \to 0^+$.

In this context, $W_\lambda$ is called the bubble of energy and $(u_0^*, u_1^*)$ is called the asymptotic profile.

Solutions of this type were first constructed by Krieger, Schlag and Tataru \cite{KST1} in space dimension $N = 3$, where it is shown that for any $\nu > 0$ there exists a solution such that the concentration speed is $\lambda(t) \sim t^{1+\nu}$. Similar results where obtained for energy-critical wave maps by the same authors \cite{KST2}, for energy-critical NLS in dimension $N = 3$ by Ortola and Perelman \cite{OP} and for energy-critical Schrödinger maps by Perelman \cite{Per}. Using a different approach, Hillairet and Raphaël \cite{HR} obtained $C^\infty$ blow-up solutions for energy-critical wave equation in dimension $N = 4$ with blow-up rate $\lambda(t) = t \exp \left( - \sqrt{-\log t} (1 + o(1)) \right)$. Collot \cite{Col} obtained a related result for supercritical wave equation in large dimension.

It follows from the classification of solutions with energy $E(W)$ by Duyckaerts and Merle \cite{DM1} that necessarily $(u_0^*, u_1^*) \neq 0$. In other words, we have non-existence of minimal energy blow-up solutions. Analogous result is true also for energy-critical wave maps, energy-critical Schrödinger maps and energy-critical NLS.

This is in contrast with the $L^2$-critical NLS where the conformal invariance produces explicit solutions concentrating a bubble of mass and tending weakly to 0 at blow-up. Existence of blow-up solutions with a non-zero smooth asymptotic profile was first observed by Bourgain and Wang \cite{BW}. Blow-up solutions close to the ground state in the case of $L^2$-critical NLS were extensively studied in a series of papers by Merle and Raphaël. They examined in particular the relationship between regularity of the asymptotic profile and the blow-up speed. One can consult a survey \cite{Mer} for an
account of these results in a proper perspective and a presentation of recent developments in the case of $L^2$-critical gKdV.

1.2. Main results. The aim of this paper is to construct solutions which blow up by concentration of one bubble of energy in space dimension $N = 5$. Our approach differs substantially from [15] in that it produces a blow-up solution with a given asymptotic profile. This profile is seen as a source term which permits concentration of the bubble. This point of view is close to a recent construction by Martel, Merle and Raphael [17] in the case of $L^2$-critical gKdV.

Denote $X^* := H^{s+1} \cap H^1$. We prove the following two results.

**Theorem 1.1.** Let $(u_0^*, u_1^*) \in X^4 \times H^4$ be any radial functions with $u_0^*(0) > 0$. Let $(u^*(t), \partial_t u^*(t))$ be the solution of (NLW) for the initial data $(u^*(0), \partial_t u^*(0)) = (u_0^*, u_1^*)$. There exists a solution $(u, \partial_t u)$ of (NLW) defined on a time interval $(0, T_0)$ and a $C^1$ function $\lambda(t) : (0, T_0) \to (0, +\infty)$ such that

\[
\lim_{t \to 0^+} \| (u(t) - W_{\lambda(t)} - u^*(t), \partial_t u(t) + \lambda(t)(\Lambda W)_{\lambda(t)} - \partial_t u^*(t)) \|_{H^1 \times L^2} = O(t^{9/2})
\]

as $t \to 0^+$, and $\lambda(t) = \left(\frac{32}{315}\right)^2 (u^*(0, 0))^2 t^4 + o(t^4)$.

**Theorem 1.2.** Let $\nu > 8$. There exists a solution $(u, \partial_t u)$ of (NLW) defined on the time interval $(0, T_0)$ such that

\[
\lim_{t \to 0^+} \| (u(t) - W_{\lambda(t)} - u_0^*, \partial_t u(t) - u_1^*) \|_{H^1 \times L^2} = 0,
\]

where $\lambda(t) = t^{\nu+1}$, and $(u_0^*, u_1^*)$ is an explicit radial $C^2$ function.

We will refer to the situation of Theorem 1.1 as the non-degenerate case and to the situation of Theorem 1.2 as the degenerate case. Note that in Theorem 1.1 we allow any regular $(u_0^*, u_1^*)$ with $u_0^*(0) > 0$. Our result might be seen as a first step in a possible classification of all blow-up solutions with a non-degenerate asymptotic profile. Theorem 1.2 demonstrates how the asymptotic behaviour of $(u_0^*, u_1^*)$ at $x = 0$ influences the blow-up speed. The condition $\nu > 8$ is imposed by our method. It could be improved at the cost of some technical details, but we are far from obtaining the whole range $\nu > 0$ as in [15] for $N = 3$.

Let us mention that radiality is only a simplifying assumption. All the estimates used here are true also in the non-radial situation.

In Theorem 1.2 the function $u_0^*$ is given explicitly by (1.1) and $u_1^* = 0$. It follows from our proof that there exists a $C^1$ function $\lambda(t) : (0, T_0) \to (0, +\infty)$ such that $\tilde{\lambda}(t) = t^{\nu+1} + o(t^{\nu+1})$ and the solution $(u, \partial_t u)$ satisfies

\[
\lim_{t \to 0^+} \| (u(t) - W_{\tilde{\lambda}(t)} - u^*(t), \partial_t u(t) + \tilde{\lambda}(t)(\Lambda W)_{\tilde{\lambda}(t)} - \partial_t u^*(t)) \|_{H^1 \times L^2} = O(t^{7\nu-\frac{8}{3}}).
\]

1.3. Structure of the proof. In Section 2 we present a formal computation which explains the relation between the asymptotic behaviour of $(u_0^*, u_1^*)$ and the blow-up speed, as well as the relevance of the condition $u_0^*(0) > 0$.

In Section 3 we specify an ansatz $(\varphi_0(t), \varphi_1(t))$ in the non-degenerate case and prove appropriate bounds on the error of this approximate solution.

In Section 4 we choose $(u_0^*(0), u_1^*(0))$ such that the same procedure leads to an approximate solution with $\lambda(t) \sim t^{1+\nu}$, and we prove appropriate bounds on the error in this situation.

Section 5 covers both the non-degenerate and the degenerate case. We use a well-known compactness argument introduced by Merle [19] and used by several authors starting with the work of Martel [10] for constructions of multi-solitons. We take a decreasing sequence $t_n \to 0^+$ and we define $(u_n, \partial_t u_n)$ as the solution of (NLW) such that $(u_n(t_n), \partial_t u_n(t_n))$ is close to the approximate solution at time $t = t_n$. The heart of the analysis is to obtain uniform energy bounds for this
sequence. That is to say, there exists $T_0 > 0$ such that $(u_n(t), \partial_t u_n(t))$ stays close to $(\varphi_0(t), \varphi_1(t))$ for $t_n \leq t \leq T_0$, with bounds independent of $n$. Note that the exponential instability of $W_\lambda$ causes an additional difficulty in the argument. We use the shooting method to eliminate the unstable mode. The blow-up solution $(u, \partial_t u)$ is obtained as a weak limit of a subsequence of $(u_n, \partial_t u_n)$. To obtain the crucial uniform energy bounds, we use a mixed energy-virial functional. This method was introduced by Raphaël and Szeftel [22] for a construction of minimal mass blow-up solutions for NLS.

In Appendix A we prove sequential weak continuity of the dynamical system (NLW) under some natural (non-optimal) condition, which is an adaptation of an analogous result of Bahouri and Gérard in the defocusing case [2, Corollary 1]. This result is required in order to extract a weak limit of the sequence $(u_n, \partial_t u_n)$.

In Appendix B we provide for reader’s convenience some well-known estimates of the $X^1 \times H^1$ norm of solutions of (NLW). The persistence of $X^1 \times H^1$ regularity is used in Section 5. The energy estimates are used in Section 4. They are non-optimal, but sufficient for our purposes. We prove also propagation of regularity in a neighbourhood of the origin in the non-degenerate case, which is used in Section 3.

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1.5. Notation. For $v, w \in L^2$ we denote

$$\langle v, w \rangle := \int_{\mathbb{R}^5} v \cdot w \, dx.$$  

We use the same notation for the duality pairing when $v \in \dot{H}^{-s}$ and $w \in \dot{H}^s$.

Linearizing $-\Delta V - f(V)$ around $V = W_\lambda$ we obtain a self-adjoint operator

$$L_\lambda h := -\Delta h - f'(W_\lambda) h.$$  

Differentiating $-\Delta W_\lambda - f(W_\lambda) = 0$ with respect to $\lambda$ we find

$$L_\lambda (\Lambda W)_\lambda = 0.$$  

We denote $L := L_1 = -\Delta - f'(W)$.

We will also use the notation $v(t) := (v(t), \partial_t v(t))$.

We denote $Z$ a fixed radial $C_0^\infty$ function such that $\langle AW, Z \rangle > 0$.

Finally, $\chi$ is a fixed standard $C^\infty$ cut-off function ($\chi(r) = 1$ for $r \leq 1$, $\chi(r) = 0$ for $r \geq 2$, $\chi'(r) \leq 0$).

2. Formal picture and construction of blow up profiles

2.1. Inverting the operator $L$. We define

$$\kappa := \frac{\langle AW, f'(W) \rangle}{\langle AW, AW \rangle} = \frac{128}{105\pi}.$$  

Proposition 2.1. There exist radial functions $A, B \in C^\infty(\mathbb{R}^5)$ such that

$$LA = \kappa AW + f'(W), \quad LB = -\Lambda_0 AW.$$  

In addition, $A(r) \sim r^{-1}, A'(r) \sim r^{-2}, A''(r) \sim r^{-3} \text{ and } B(r) \sim r^{-1}, B'(r) \sim r^{-2}, B''(r) \sim r^{-3}$ as $r \to +\infty$. 

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Proof. In the proof we will use some standard facts from the theory of Sturm-Liouville equations, see for example [23, Chapter 5].

Solving equation (2.22) is equivalent to solving the following ODE:

\begin{equation}
-(p(r)y')' + q(r)y = g(r),
\end{equation}

with \( r \in (0, +\infty) \), \( p(r) = r^4 \), \( q(r) = -r^4 f'(W) \) and \( g(r) = g_A(r) = r^4 (\kappa AW(r) + f'(W(r))) \) or \( g(r) = g_B(r) = -r^4 A_0 \Lambda W(r) \). Notice that \( |g(r)| \lesssim r^4 \) for small \( r \).

We know that \( \Lambda W(r) \) is a solution of (2.3) with \( g(r) = 0 \). Let \( \Gamma(r) \) be a second solution normalized in such a way that

\begin{equation}
\mathcal{W}(\Lambda W, \Gamma) = r^4 (\Lambda W \cdot \Gamma' - (\Lambda W)' \cdot \Gamma) = 1
\end{equation}

(\( \mathcal{W} \) is the modified wronskian, in particular its value is independent of \( r \)).

Take \( r_1 < \sqrt{15} \), \( r_2 > \sqrt{15} \) (recall that \( r = \sqrt{15} \) is the unique point where \( \Lambda W \) vanishes) and define

\begin{align*}
y_1(r) &:= \Lambda W(r) : \int_{r_1}^{r} \frac{ds}{s^4 (\Lambda W(s))^2}, \quad \text{for } r < \sqrt{15}, \\
y_2(r) &:= \Lambda W(r) : \int_{r_2}^{r} \frac{ds}{s^4 (\Lambda W(s))^2}, \quad \text{for } r > \sqrt{15}. 
\end{align*}

It can be easily checked that \( y_1 \) and \( y_2 \) are solutions of the homogeneous equation and verify \( \mathcal{W}(\Lambda W, y_1) = \mathcal{W}(\Lambda W, y_2) = 1 \). Hence, we have \( y_j = a_j \Lambda W + \Gamma \) for some scalar coefficients \( a_1, a_2 \). Directly from the formulas defining \( y_1 \) and \( y_2 \) we obtain the asymptotic behaviour of \( y_1 \) as \( r \to 0^+ \) and of \( y_2 \) as \( r \to \infty \):

\begin{align*}
y_1(r) &\sim -\int_{r}^{r_1} \frac{ds}{s^4} \sim -\frac{1}{r^3}, \quad r \to 0^+, \\
y_2(r) &\sim -\frac{1}{r^3} \int_{r_2}^{r} \frac{ds}{s^4} \sim -1.
\end{align*}

As adding a constant multiple of \( \Lambda W \) does not change these asymptotics, we obtain that \( \Gamma(r) \sim -r^{-3} \) as \( r \to 0^+ \) and \( \Gamma(r) \sim -1 \) as \( r \to +\infty \). From the relation \( \mathcal{W}(\Lambda W, \Gamma) = 1 \) we get

\[ \Gamma' = \frac{r^{-4} + (\Lambda W)' \cdot \Gamma}{\Lambda W}, \]

which immediately gives \( \Gamma'(r) \sim r^{-4} \) as \( r \to 0 \) and \( \Gamma'(r) \sim \pm r^{-1} \) as \( r \to +\infty \) (it can be checked that the sign is " + ", but we will not use this fact).

For \( r_0, r \in (0, +\infty) \) we define

\begin{equation}
s(r, r_0) := \Lambda W(r_0) \Gamma(r) - \Gamma(r_0) \Lambda W(r).
\end{equation}

We see that \( s(r_0, r_0) = 0 \) and \( r_0^4 \frac{\partial}{\partial r_0} s(r, r_0)|_{r=r_0} = 1 \), which means that \( s(r, r_0) \) is the second fundamental solution of (2.3). Now using the Duhamel formula we obtain a solution of the non-homogeneous equation (2.3):

\begin{align*}
A(r) &= \int_{0}^{r} s(r, r') g_A(r') \, dr', \\
B(r) &= \int_{0}^{r} s(r, r') g_B(r') \, dr'.
\end{align*}
Fix $r > 0$ and let $|h| \leq \frac{1}{2} r$. In the estimates which follow, all the constants may depend on $r$. We have
\[
\frac{A(r + h) - A(r)}{h} - \int_0^r \frac{d}{dr} s(r, r') g_A(r') \, dr' \leq \int_0^r \left| \frac{s(r + h, r') - s(r, r')}{h} - \frac{d}{dr} s(r, r') \right| |g_A(r')| \, dr' + \frac{1}{h} \int_r^{r+h} |s(r + h, r')| \cdot |g_A(r')| \, dr'.
\]

Formula (2.5) implies that $|s(\bar{r}, r_0)| \lesssim h$ when $|\bar{r} - r| \leq h$ and $|r - r_0| \leq h$. Hence, the second term above converges to 0 as $h \to 0$. For $0 \leq r_0 \leq r$ and $|\bar{r} - r| \leq \frac{1}{2} r$ we have the bound $\left| \frac{d^2}{dr^2}s(\bar{r}, r_0) \right| \lesssim r_0^{-3}$. This implies
\[
\left| \frac{s(r + h, r') - s(r, r')}{h} - \frac{d}{dr} s(r, r') \right| \leq \frac{1}{2} \sup_{|\bar{r} - r| \leq h} \left| \frac{d^2}{dr^2}s(\bar{r}, r') \right| |h| \lesssim (r')^{-3} \cdot |h|,
\]
so the first term above also converges to 0 as $h \to 0$. This shows that $A(r)$ (and similarly $B(r)$) is continuously differentiable and
\[
\begin{align*}
A'(r) &= \int_0^r \frac{d}{dr} s(r, r') g_A(r') \, dr', \\
B'(r) &= \int_0^r \frac{d}{dr} s(r, r') g_B(r') \, dr'.
\end{align*}
\]

It is clear from these formulas that $\lim_{r \to 0^+} A'(r) = \lim_{r \to 0^+} B'(r) = 0$.

It follows from above considerations that $A$ and $B$, seen as functions on $\mathbb{R}^5$, are $C^1$, so they are $C^\infty$ by elliptic regularity.

Now we consider the behaviour of $A(r)$ and $B(r)$ as $r \to +\infty$. From the crucial orthogonality relation $\int_0^{+\infty} \Lambda W(r') A(r') \, dr' = 0$ we deduce that
\[
\left| \int_0^r \Lambda W'(r') g(r') \, dr' \right| \leq \left| \int_r^{+\infty} \Lambda W(r') g(r') \, dr' \right| \lesssim r^{-1}.
\]

From this and the asymptotics of $\Gamma$ and $g_A$ it follows that $|A(r)| \lesssim r^{-1}$ and similarly $|B(r)| \lesssim r^{-1}$. Using the asymptotics of $\Gamma'$ we obtain also $|A'(r)| \lesssim r^{-2}$ and $|B'(r)| \lesssim r^{-2}$. The fact that $|A''(r)| \lesssim r^{-3}$ and $|B''(r)| \lesssim r^{-3}$ follows from the differential equation.

We define $A$ and $B$ as the solutions of (2.2) satisfying the orthogonality condition
\[
\int A Z \, dx = \int B Z \, dx = 0.
\]

2.2. Determination of blow-up speeds. Let $u^*(t, x)$ be the solution of (NLW) for initial data $(u^*(0), \partial_t u^*(0)) = (u_0, u_1)$. At a formal level, while computing the interaction of $u^*$ with the soliton, we will treat $u^*$ as a function constant in space and $C^2$ in time, $u^*(t, x) \simeq v^*(t)$. (In the non-degenerate case we will take $v^*(t) = u^*(t, 0)$ and in the degenerate case $v^*(t) = qt^\beta$, where $q$ and $\beta$ are appropriate constants.) We will construct a solution which blows up at $t = 0$ and is defined for small positive $t$. This means that in our situation the characteristic length $\lambda$ will increase in time. The usual method of performing a formal analysis of blow-up solutions in the case of the wave equation consists in defining $b := \lambda t$ and searching a solution in the form of a power series in $b$. Following this scheme, we write
\[
\begin{align*}
\left\{ \begin{array}{l}
\quad u = W_\lambda + u^*(t) + b^2 T_\lambda + \text{lot} \\
\quad \partial_t u = -b(\Lambda W)_\lambda + \partial_t u^* + \text{lot}
\end{array} \right.
\end{align*}
\]
Here, the profile $T$ is undetermined, and we search a convenient blow up speed. Neglecting irrelevant terms and replacing $\lambda t := \frac{dt}{dr} \lambda(t)$ by $b$, we compute
\[
\partial_t u = -b_t (\Lambda W)_{\Delta} + \frac{b^2}{\lambda} (\Lambda_0 \Lambda W)_{\Delta} + \partial_t u^* + \text{lot}.
\]
On the other hand,
\[
\Delta u + f(u) = -\frac{1}{\lambda} b^2 (LT)_{\Delta} + f'(W_\lambda)v^* + \Delta u^* + f(u^*) + \text{lot}.
\]
We discover that, formally at least, we should have
\[
(2.10) \quad LT = -\Lambda_0 \Lambda W + \frac{\lambda}{b^2} [b_t \Lambda W + v^*(t)\sqrt{t} f'(W)].
\]
Proposition 2.1 shows that if
\[
(2.11) \quad b_t = \kappa v^*(t) \lambda^{1/2},
\]
then equation (2.10) has a decaying regular solution $T = B + \frac{v^*(t)\lambda^{3/2}}{b^2} A$. We call equation (2.11) together with the equation $\lambda_t = b$ formal parameter equations. In the non-degenerate case $v^*(t) = v^*(t,0)$ is close to $v^*(0,0)$, so we expect that there exists a solution of the formal parameter equations which is close to
\[
(2.12) \quad (\lambda(t), b(t)) = \left( \frac{v^2 u^*(0,0)^2}{144} t^4, \frac{\kappa^2 u^*(0,0)^2}{36} \nu^3 \right).
\]
This is indeed the case, as follows from our analysis in Section 5.

In the degenerate case we have $v^*(t) = q t^\beta$, and the formal parameter equations have a solution
\[
(2.13) \quad (\lambda(t), b(t)) = (t^{1+\nu}, (1+\nu) t^\nu)
\]
if we choose $q = \frac{\nu(1+\nu)}{\kappa}$ and $\beta = \frac{\nu - 3}{2}$.

3. Approximate solution in the non-degenerate case

3.1. Bounds on the profile ($P_0, P_1$). The functions $A$ and $B$ from the previous section do not belong to the space $\mathcal{H}^1$. We will place a cut-off at the light cone, that is at distance $t$ from the center. Given modulation parameters $(\lambda(t), b(t))$, we define:
\[
(3.1) \quad P_0(t) := \chi(t) (\lambda(t)^{3/2} v^*(t) A(t) + b(t)^2 B(t)).
\]
Recall that in the non-degenerate case $v^*(t) = u^*(t,0) \in C^2$ by Proposition 15.6 and Schauder estimates.

Remark 3.1. Because of the finite speed of propagation, without loss of generality we can replace $(u_0^*, u_1^*)$ by $(\chi(t) u_0^*, \chi(t) u_1^*)$, where $\rho$ is a strictly positive constant to be chosen later. Thus, without loss of generality we can assume that the support of $(u_0^*, u_1^*)$ is contained in a small ball and that $\|(u_0^*, u_1^*)\|_{\mathcal{H}^1}$ is small.

Remark 3.2. The fact that the profile $(P_0, P_1)$ is cut at $r = t = t^1$ can be considered as a coincidence. The power of $t$ has been chosen in order to optimize the estimates. This is the only power for which we can obtain the estimate of the error term which has asymptotically the same size as the profile $P_0$. Also, for this choice, $\|P_1\|_{L^2}$ (the forth term of the asymptotic expansion which will be defined in a moment) is asymptotically the same as $\|P_0\|_{\mathcal{H}^1}$. However, the angle of the cone has no significance for us.
Remark 3.3. Notice that the orthogonality condition which we choose to define \( A \) and \( B \) has little significance due to a relatively fast decay of \( \Delta W \). We will use the same orthogonality condition as for the error term, as this choice simplifies slightly the computation. Observe that the fact that \( Z \) has compact support implies that if \( \lambda(t) \ll t \), then \( \int P_0(t)Z_{\lambda}\,dx = 0 \) for small \( t \).

In the error estimates which will follow, on the right hand side we will always replace \( \lambda(t) \) by \( t^4 \) and \( b(t) \) by \( t^3 \), as this is the regime that we are going to consider later in the bootstrap argument. In this section, all the constants may depend on \( u^* \).

**Lemma 3.4.** Assume that \( \lambda(t) \sim t^4 \) and \( b(t) \sim t^3 \). Then

\[
\|P_0(t)\|_{H^1} \lesssim t^{9/2}.
\]

**Proof.** It is sufficient to show that \( \|\chi(t)A_\lambda\|_{H^1} \lesssim t^{-3} \) (the computation for \( B_\lambda \) is the same). We have

\[
\|\chi(t)A_\lambda\|_{H^1}^2 = \int_0^{\infty} \left( (\chi(t)A_\lambda(r))^2 \right) r^4 \,dr = \int_0^{\infty} \left( (\chi(t)A_\lambda(r))' \right)^2 r^4 \,dr
\]

\[
\lesssim \int_0^{\infty} \chi(t)A_\lambda(r) r^4 \,dr + \int_0^{\infty} \left( \frac{\lambda}{t}\chi(t)A_\lambda(r) \right)^2 r^4 \,dr
\]

\[
\lesssim \int_0^{2\sqrt{t}} r^4 \frac{1}{r^4} \,dr + \frac{\lambda^2}{t^2} \int_{2\sqrt{t}}^{\infty} r^4 \frac{1}{r^2} \,dr \lesssim \frac{t}{\lambda} \sim t^{-3}.
\]

□

**Lemma 3.5.** Assume that \( \lambda(t) \sim t^4 \) and \( b(t) \sim t^3 \). Then

\[
\|L_\lambda P_0 - \lambda^{3/2}v^*(t)L_\lambda A_\lambda - b^2 L_\lambda B_\lambda\|_{L^2} \lesssim t^{7/2}.
\]

**Proof.** We will do the computation only for the terms with \( A \). The terms with \( B \) are asymptotically the same. We need to check that

\[
\|(1 - \chi(t)) f'(W_\lambda) A_\lambda\|_{L^2} + \|\Delta((1 - \chi(t)) A_\lambda)\|_{L^2} \lesssim t^{-5/2}
\]

For the first term we have even some margin since

\[
\|(1 - \chi(t)) f'(W_\lambda) A_\lambda\|_{L^2} = \frac{1}{\lambda} \|(1 - \chi(t)) f'(W) A\|_{L^2}
\]

\[
\lesssim \frac{1}{\lambda} \left( \int_{t/\lambda}^{\infty} (r^{-4}r^{-1})r^4 \,dr \right)^{1/2} \sim \frac{1}{\lambda} \cdot \left( \frac{\lambda}{t} \right)^{5/2} \sim t^{7/2}.
\]

For the second term, we have a few possibilities. Recall that \( \Delta = \partial_{rr} + \frac{2r}{\lambda} \). Either the laplacian hits directly \( A \):

\[
\|\Delta(1 - \chi(t)) A_\lambda\|_{L^2} = \frac{1}{\lambda} \|(1 - \chi(t)) \Delta A\|_{L^2} \lesssim \frac{1}{\lambda} \left( \int_{t/\lambda}^{\infty} (r^{-3})r^4 \,dr \right)^{1/2} \sim \frac{1}{\lambda} \cdot \sqrt{\frac{\lambda}{t}} \sim t^{-5/2},
\]

either one derivative hits \( \chi \):

\[
\frac{1}{t} \|\chi(t)\frac{d}{dr}(A_\lambda)\|_{L^2} = \frac{1}{t} \|\chi(t)A_\lambda(r)\|_{L^2} \lesssim t^{-1} \left( \int_{t/\lambda}^{2\sqrt{t}} (r^{-2})r^4 \,dr \right)^{1/2} \sim t^{-1} \cdot \sqrt{\frac{t}{\lambda}} \sim t^{-5/2},
\]

and analogously the term \( \frac{1}{t} \|\chi(t)\frac{d}{dr}(A_\lambda)\|_{L^2} \), or two derivatives hit \( \chi \), and we get

\[
\frac{1}{t^2} \|\chi(t)A_\lambda\|_{L^2} = \frac{\lambda}{t^2} \|\chi(t)A_\lambda\|_{L^2} \lesssim \frac{\lambda}{t^2} \left( \int_{t/\lambda}^{2\sqrt{t}} (r^{-1})r^4 \,dr \right)^{1/2} \sim \frac{\lambda}{t^2} \cdot t^{3/2} \sim t^{-5/2}.
\]

□
We define $P_1(t)$ as a formal time derivative of $P_0(t)$, which means that we replace $\lambda$ by $b$ and $b_t$ by $\kappa v^*(t)\lambda^{1/2}$, see (2.11), and we do not differentiate the cut-off function. Explicitly, set

$$
(3.4) \quad P_1(t) = \chi(t)\frac{3}{2} \lambda^{3/2} b A \lambda - \lambda^{3/2} b(\Lambda A)\lambda 
$$

$$
+ \lambda^{5/2} \partial_t v^*(t) A \lambda + 2\lambda^{3/2} b B \lambda - b^3(\Lambda B)\lambda
.$$  

Notice that in the regime (2.12) the coefficient $\lambda^{5/2}$ is smaller than the other coefficients (all of which are, asymptotically, of the same size). However, we prefer to keep the corresponding term in the definition of $P_1$.

**Lemma 3.6.** Assume that $\lambda(t) \sim t^4$ and $b(t) \sim t^3$. Then

$$
(3.5) \quad \|P_1(t)\|_{L^2} \lesssim t^{9/2}
$$

**Proof.** All the terms except for the one mentioned above have the same asymptotics, so we will do the computation only for the first one. It is sufficient to show that $\|\chi(t) A\|_{L^2} \lesssim t^{-9}$. We have

$$
\|\chi(t) A\|_{L^2}^2 \sim \|\chi(t) A(r)\|_{L^2(r^4 dr)}^2 
$$

$$
\lesssim \int_0^{2t/\lambda} (r^{-1})^2 r^4 dr \lesssim \left(\frac{t}{\lambda}\right)^3 \sim t^{-9}.
$$

\[\square\]

Our ansatz $\varphi(t) = (\varphi_0(t), \varphi_1(t))$ is defined as follows:

$$
(3.6) \quad \begin{cases} 
\varphi_0(t) = W_{\lambda(t)} + P_0(t) + u^*(t), \\
\varphi_1(t) = -b(t)(\Lambda W)_{\lambda(t)} + P_1(t) + \partial_t u^*(t),
\end{cases}
$$

where $P_0$ and $P_1$ are given by (3.1) and (3.4).

The error term $\varepsilon(t) = (\varepsilon_0(t), \varepsilon_1(t))$ is defined by the formula:

$$
(3.7) \quad \begin{cases} 
u(t) = \varphi_0(t) + \varepsilon_0(t), \\
\partial_t \nu(t) = \varphi_1(t) + \varepsilon_1(t).
\end{cases}
$$

We shall impose the orthogonality condition

$$
\int \varepsilon_0 Z \lambda dx = 0.
$$

**Lemma 3.7.** If $\lambda \sim t^4$, $b \sim t^3$ and $t$ is small enough, then

$$
(3.8) \quad |\lambda_t - b| \leq \|\varepsilon\|_{H^1 \times L^2}.
$$

**Proof.** To find the formula for $\lambda_t$, first we write

$$
- b(\Lambda W)_{\lambda} + \partial_t u^* + P_1(t) + \varepsilon_1(t) = \partial_t u = -\lambda_t(\Lambda W)_{\lambda} + \partial_t u^* + \partial_t P_0(t) + \partial_t \varepsilon_0 \Rightarrow \partial_t \varepsilon_0 = (\lambda_t - b)(\Lambda W)_{\lambda} + (P_1 - \partial_t P_0) + \varepsilon_1.
$$

Notice that for small $t$ and $\lambda \sim t^4$ we have

$$
\int (P_1(t) - \partial_t P_0(t)) Z \lambda dx = (\lambda_t - b)[\lambda^{3/2} v^*(t) \langle \Lambda A, Z \rangle_{L^2} + b^2 \langle \Lambda B, Z \rangle_{L^2}].
$$
This follows from (2.8) and the fact that \( \text{supp}(Z_\lambda) \) is contained in the light cone for small \( t \). This gives
\[
0 = \frac{d}{dt} \int_0^t \varepsilon_0 Z_\lambda \, dx = \int_0^t \partial_t \varepsilon_0 Z_\lambda \, dx - \lambda_t \int_0^t \frac{1}{\lambda} (\Lambda_0 Z)_\lambda \, dx \\
= \int (\lambda_t - b) \langle \Lambda W, Z \rangle + (\lambda_t - b) \langle \lambda^{3/2} v^*(t) \langle \Lambda A, Z \rangle + b^2 \langle \Lambda B, Z \rangle \rangle_{L^2} \\
+ \langle \varepsilon, Z_\lambda \rangle_{L^2} - \lambda_t \int_0^t \frac{1}{\lambda} (\Lambda_0 Z)_\lambda \, dx,
\]
and we obtain
\[
(\lambda_t - b) \langle \Lambda W, Z \rangle + \lambda^{3/2} v^*(t) \langle \Lambda A, Z \rangle + b^2 \langle \Lambda B, Z \rangle = -\langle \varepsilon, Z_\lambda \rangle + \lambda_t \langle \varepsilon, \frac{1}{\lambda} (\Lambda_0 Z)_\lambda \rangle.
\]
Rearranging the terms we get
\[
\lambda_t = \left(1 - \frac{\langle \varepsilon, \frac{1}{\lambda} (\Lambda_0 Z)_\lambda \rangle}{\langle \Lambda W, Z \rangle_{L^2} + \lambda^{3/2} v^*(t) \langle \Lambda A, Z \rangle + b^2 \langle \Lambda B, Z \rangle} \right)^{-1} \cdot \left(b - \frac{\langle \varepsilon, Z_\lambda \rangle}{\langle \Lambda W, Z \rangle_{L^2} + \lambda^{3/2} v^*(t) \langle \Lambda A, Z \rangle + b^2 \langle \Lambda B, Z \rangle} \right).
\]
(3.9)

For \( t \) small enough, (3.8) follows.

Remark 3.8. To be precise, our rigorous argument goes the other way round – we use (3.9) and (2.11) to define the local evolution of the modulation parameters, and then by doing exactly the same computation as above, but in the opposite direction, we find that the orthogonality condition \( \langle \varepsilon_0, \frac{1}{\lambda} Z_\lambda \rangle_{L^2} = 0 \) is preserved if it is verified at the initial time (which will be the case). Notice also that using (2.8) we obtain
\[
\langle u - W_\lambda - u^*, Z_\lambda \rangle = 0.
\]
Differentiating this condition we find
\[
\lambda_t \langle \Lambda W, Z \rangle + \langle \varepsilon, \frac{1}{\lambda} (\Lambda_0 Z)_\lambda \rangle = -\langle \partial_t u - \partial_t u^*, Z_\lambda \rangle.
\]

We need to estimate the error between the formal and the actual time derivative of \( P_0 \):

Lemma 3.9. Assume that \( \lambda(t) \sim t^4 \) and \( b(t) \sim t^3 \). Then
\[
\| \partial_t P_0 - P_1 \|_{H^1} \lesssim \sqrt{t} (t^3 + \| \varepsilon \|_{H^1 \times L^2}).
\]

Proof. The error has two parts – one comes from differentiating in time the cut off function and the other one from \( |\lambda_t - b| \).
\[
\partial_t P_0 - P_1 = -\frac{r}{t^2} \chi'(\frac{r}{t}) (\lambda^{3/2} v^*(t) A_\lambda + b^2 B_\lambda) \\
+ \chi(\frac{r}{t}) (\lambda_t - b) [v^*(t) (\frac{3}{2} \lambda^{3/2} A_\lambda - \lambda^{3/2} (\Lambda A)_\lambda) - b^2 (\Lambda B)_\lambda].
\]

Using Proposition 2.11 we can write:
\[
\| \frac{r}{t^2} \chi'(\frac{r}{t}) A_\lambda \|_{H^1} = \| r \chi'(\frac{r}{t}) A \|_{H^1} \lesssim \| \chi'(\lambda t) \cdot \frac{1}{t} \|_{L^2} + \lambda \| r \chi'(\lambda t) \cdot \frac{1}{t} \|_{L^2} + \| r \chi'(\lambda t) \cdot \frac{1}{t^2} \|_{L^2} \\
\lesssim \| \chi'(\lambda t) \cdot \frac{1}{t} \|_{L^2} + \frac{\lambda}{t} \| \chi'(\lambda t) \|_{L^2} \sim (\frac{t}{\lambda})^{3/2} \sim t^{-9/2}.
\]

The same computation is valid also for \( A \) replaced by \( B \). Now we have
\[
\| \frac{r}{t^2} \chi'(\frac{r}{t}) \lambda^{3/2} A_\lambda \|_{H^1} \lesssim \frac{\lambda}{t^2} \lambda^{3/2} \cdot t^{-9/2} \sim t^2 t^6 t^{-9/2} = t^{7/2},
\]

\[10\]
Lemma 3.10. Assume that \( \lambda(t) \sim t^4 \) and \( b(t) \sim t^3 \). Then
\[
\| \partial_t P_1 \|_{L^2} \lesssim \sqrt{t}(t^3 + \| \varepsilon \|_{H^1 \times L^2}).
\]

Proof. Consider first the terms coming from differentiating the cut-off function. Like in the proof of the previous lemma, we have
\[
\| \frac{r}{\lambda} \chi'(\frac{r}{t}) A_\lambda \|_{L^2} \lesssim \| \chi'(\frac{r}{t}) \|_{L^2} \sim \left( \frac{t}{\lambda} \right)^{5/2},
\]
which gives
\[
\| \frac{r}{\lambda} \chi'(\frac{r}{t}) v^*(t) \lambda^{3/2} b A_\lambda \|_{L^2} \lesssim \frac{\lambda^{3/2} b}{t^2} \left( \frac{t}{\lambda} \right)^{5/2} \sim t^{7/2}.
\]
The term \( \frac{r}{\lambda} \chi'(\frac{r}{t}) \lambda^{5/2} \partial_t v^*(t) A_\lambda \|_{L^2} \) is even smaller.

Consider now the other terms. They are of one of the following six types:
\[
\begin{align*}
&\chi(\frac{r}{t}) \lambda_t \lambda^{1/2} b T_\lambda, \\
&\chi(\frac{r}{t}) \lambda_t b T_\lambda, \\
&\chi(\frac{r}{t}) b_t \lambda^{3/2} T_\lambda, \\
&\chi(\frac{r}{t}) b_t b T_\lambda, \\
&\chi(\frac{r}{t}) \lambda_t \lambda^{3/2} d_t v^*(t) T_\lambda, \\
&\chi(\frac{r}{t}) \lambda t^{5/2} d_t v^*(t) T_\lambda,
\end{align*}
\]
where \( T \in \{ A, B, \Lambda A, \Lambda B, \Lambda_0 A, \Lambda_0 B, \Lambda_0 \Lambda A, \Lambda_0 \Lambda B \} \). In all the situations \( T \) is regular and decays like \( r^{-1} \) (see Proposition 2.1), so we can write
\[
\| \chi(\frac{r}{t}) T_\lambda \|_{L^2} \lesssim \left( \int_{t/\lambda}^{2t/\lambda} \left( \frac{1}{r} \right)^2 r^4 \, dr \right)^{1/2} \lesssim \left( \frac{t}{\lambda} \right)^{3/2} \sim t^{-9/2}.
\]
Using the fact that \( \lambda \sim t^4 \), \( b \sim t^3 \), \( \lambda_t \lesssim b + \| \varepsilon \| \), \( b_t \lesssim \sqrt{\lambda} \) and that \( v^*(t) \) is \( C^2 \) we obtain
\[
\lambda_t \lambda^{1/2} b + \lambda \left( \frac{b^3}{\lambda} \right) + b_t \lambda^{3/2} + b_t b^2 + \lambda_t \lambda^{3/2} |d_t v^*| + \lambda^{5/2} |d_{tt} v^*| \lesssim t^5(t^3 + \| \varepsilon \|),
\]
which finishes the proof.

The last lemma shows that \( \varphi \) is “almost constant” after rescaling.

Lemma 3.11. Let \( c_1 > 0 \). If \( T_0 \) is sufficiently small, then for \( t \in (0, T_0] \) there holds
\[
\| \partial_t (\varphi_0)_{1/\lambda} \|_{H^1} \leq \frac{c_1}{t}.
\]
Proof. By the definition of \( \varphi_0 \) and \( P_0 \) we get

\[
(\varphi_0)_{1/\lambda} = W + \chi(\frac{\lambda}{t})[\lambda^{3/2}A + b^2B] + (u^*)_{1/\lambda}.
\]

The terms with \( A \) and \( B \) are similar, so we only consider the first one. We observe that \( |\frac{\lambda}{t}| \leq \frac{1}{t} \) for small \( t \), with an explicit numerical constant. Now

\[
\partial_t(\chi(\frac{\lambda}{t})\lambda^{3/2}A) = \frac{3}{2} \frac{\lambda}{t} \chi(\frac{\lambda}{t})\lambda^{3/2}A - \frac{\lambda}{t} \chi'(\frac{\lambda}{t})\lambda^{3/2}A.
\]

The size of the first term is acceptable by Lemma 3.13. For the second one, it is sufficient to notice that \( \frac{\lambda}{t^2} \leq \frac{3}{t} \) on the support of \( \chi \). The conclusion follows again from Lemma 3.13. (Notice that we have a large margin for these two terms.)

Next, we have

\[
\|\partial_t(u^*)_{1/\lambda}\|_{H^1} \leq \|\partial_t u^*\|_{H^1} + \frac{\lambda}{\lambda} \|\Lambda u^*\|_{H^1}.
\]

By Proposition 3.2, the first term is bounded for small \( t \). Choosing \( \rho \) small enough (see Remark 3.1), we can guarantee that \( \|\Lambda u^*(t)\|_{H^1} \) will stay small for small \( t \), which is exactly what we need. \( \square \)

3.2. Error of the ansatz. Our next objective is to estimate the error of the approximate solution, defined as

\[
\psi(t) = \begin{pmatrix} \psi_0(t) \\ \psi_1(t) \end{pmatrix} := \begin{pmatrix} \partial_t \varphi_0(t) \\ \partial_t \varphi_1(t) \end{pmatrix} - \begin{pmatrix} \varphi_1(t) \\ \Delta \varphi_0(t) + f(\varphi_0(t)) \end{pmatrix}.
\]

In order to do this we first need to extract the principal terms of the nonlinear term, which is based on the following pointwise estimate:

**Lemma 3.12.**

\[
(3.15) \quad |f(k + l + m) - [f(k) + f(m) + f'(k)l + f'(m)m]| \lesssim |f(l)| + f'(l)|k| + f'(m)|k| + f'(m)|l|.
\]

*Proof.* The inequality is homogeneous, so we can suppose that \( k^2 + l^2 + m^2 = 1 \). The right hand side vanishes only for \( (k, l, m) \in \{ (\pm 1, 0, 0), (0, 0, \pm 1) \} \), so it suffices to prove the inequality in a neighborhood of these 4 points, where it is an easy consequence of the Taylor expansion of \( f \). \( \square \)

**Lemma 3.13.** If \( \lambda(t) \sim t^4, b \sim t^3 \) and \( t \) is small, then

\[
(3.16) \quad \|f(\varphi_0(t)) - [f(W_{\lambda(t)}) + f(u^*) + f'(W_{\lambda(t)})P_0(t) + f'(W_{\lambda(t)})u^*(t)]\|_{L^2} \lesssim t^4.
\]

*Proof.* We put in the preceding lemma \( k = W_{\lambda(t)}l = P_0(t), m = u^*(t) \), and we estimate the \( L^2 \) norm of the 4 terms on the right hand side of (3.15). When \( P_0(t) \) appears, we split it into two parts. We sometimes forget \( \chi \), as its presence here can only help (there are no derivatives).

Term \( \|f(l)\|_{L^2} \):

\[
(\chi(\frac{\lambda}{t})\lambda^{3/2}A)^{7/3} \lesssim \chi(\frac{\lambda}{t})^{7/3},
\]

and \( r^{-1/3} \) is integrable near 0, so \( \|((\chi(\frac{\lambda}{t})\lambda^{3/2}A)^{7/3}\|_{L^2} \lesssim \chi(\frac{\lambda}{t})^{7/3}\|\chi(\frac{\lambda}{t})^{7/3}\|_{L^2} \ll t^4 \). In a similar way, \( \|((\chi(\frac{\lambda}{t})b^2B_{\lambda})^{7/3}\|_{L^2} \ll t^4 \).

Term \( \|f'(l)|k|\|_{L^2} \):

By a change of variables we get

\[
\|((\lambda^{3/2}A)^{4/3}W_{\lambda})\|_{L^2} = \lambda^{4/3}W_{\lambda} \sim t^4
\]

(exponent of \( \lambda \) on the left = \( (3/2 - 3/2) \cdot (4/3) - 3/2 = -3/2 \), and the \( L^2 \) scaling is \(-5/2\)).

In a similar way,

\[
\|((b^2B_{\lambda})^{4/3}W_{\lambda})\|_{L^2} = \lambda^{-1}b^{8/3}W_{\lambda} \sim t^4
\]

Term \( \|f'(m)|k|\|_{L^2} \):

We use once again the \( L^\infty \) bound of \( u^* \) and the fact that \( \|W_{\lambda}\|_{L^2} \sim \lambda \).
Term “$f'(m)|l|”$: Using (3.2) and the fact that $u^*(t)$ is bounded in $L^{20/3}$ for small $t$ (by Proposition 3.2), we have

$$\|f'(u^*)P_0\|_{L^2} \leq \|f'(u^*)\|_{L^2} \cdot \|P_0\|_{L^{10/3}} \lesssim t^{9/2}. \qed$$

We can now estimate $\psi(t)$.

**Proposition 3.14.** Assume that $\lambda(t) \sim t^4$ and $b(t) \sim t^3$. Then

\begin{align*}
(3.17) \quad &\|\psi_0(t) + (\lambda_t - b)\frac{1}{\lambda}(\Delta W)\lambda\|_{H^1} \lesssim \sqrt{t}\|\varepsilon(t)\|_{H^1 \times L^2} + t^3), \\
(3.18) \quad &\|\psi_1(t) - (\lambda_t - b)\frac{b}{\lambda}(\Lambda_0 AW)\lambda\|_{L^2} \lesssim \sqrt{t}\|\varepsilon(t)\|_{H^1 \times L^2} + t^3).
\end{align*}

**Proof.** The first inequality is just a reformulation of Lemma 3.9.

For the second inequality, we divide the error into several parts:

$$\psi_1 = \frac{d}{dt}\varphi_1 - (\Delta \varphi_0 + f(\varphi_0)) = (\frac{b}{\lambda}(\Lambda_0 AW)\lambda + \frac{d}{dt}P_1 + \partial_t u^*) - (\Delta W_\lambda + \Delta P_0 + \Delta u^*) - (f(W_\lambda) + f(u^*) + f'(W_\lambda)P_0(t) + f'(W_\lambda)u^*).$$

Now we can use Lemma 3.13 to replace $f(\varphi_0)$ by the sum of its principal terms. Rearranging the terms and using (2.11), we can rewrite the sum above as follows:

$$\psi_1 = (\lambda_t - b)\frac{b}{\lambda}(\Lambda_0 AW)\lambda - (\Delta W_\lambda + f(W_\lambda)) + (\partial_t u^* - \Delta u^* - f(u^*)) - v^*(t)\sqrt{\lambda}(LA + \kappa AW + f'(W))\lambda + \frac{b^2}{\lambda}(LB + \Lambda_0 AW)\lambda + (-\Delta P_0 - f'(W_\lambda)P_0(t) - v^*(t)\sqrt{\lambda}(LA)\lambda + \frac{b^2}{\lambda}(LB)\lambda + (v^*(t) - u^*(t))\sqrt{\lambda}(f'(W))\lambda + \partial_t P_1 + O(t^{7/2}).$$

Now we proceed line by line.

Line 1. This is the correction that we substract in (3.18).

Line 2. Both terms equal 0.

Line 3. Both terms equal 0 by the definition of $A$ and $B$.

Line 4. This error is due to the presence of the cut-off function in (3.1), and Lemma 3.5 tells us that it is acceptable.

Line 5. This error is due to the fact that we replace the interaction with $u^*(t)$ by the interaction with the constant in space function $v^*(t)$. It follows from Proposition 3.6 that $|v^*(t) - u^*(t, r)| \lesssim r$ uniformly in time when $r \leq t$ and $t$ is small. Hence,

$$\|(v^*(t) - u^*(t, r))f'(W_\lambda)\|_{L^2(r \leq t)} \lesssim \sqrt{r}\sqrt{(f'(W))\lambda}\|_{L^2} \sim \lambda^{3/2} \sim t^0.$$

(We have used the fact that $rf'(W) \in L^2$.) In the zone $r \geq t$ first we use the fact that $v^*$ is bounded and

$$\|f'(W_\lambda)\|_{L^2(r \geq t)} = \sqrt{\lambda}\|f'(W)\|_{L^2(r \geq t)} \lesssim \sqrt{\lambda}(\lambda/t)^{3/2} \sim t^{13/2}.$$
As for $u^*$, we know from Proposition 3.2 that it is bounded in $L^{10}$. By Hölder $\|u^* \cdot f'(W_\lambda)\|_{L^2(r \geq t)} \leq \|u^*\|_{L^{10}} \cdot \|f'(W_\lambda)\|_{L^{5/2}(r \geq t)}$, and a routine computation shows that the last term is bounded by $(\lambda/t)^2 \sim t^6$.

Line 6. This error is small by Lemma 3.10.

\[\square\]

4. Approximate solution in the degenerate case

4.1. Bounds on the profile $(P_0, P_1)$. This section is very similar to the previous one. Formula (3.1) is still valid, but recall that in the present case we take $v^*(t) = qt^\beta$ where $q = \frac{\nu(1+\nu)}{\kappa}$ and $\beta = \frac{\nu-3}{2}$. The function $u_0^*$ is defined as follows:

\[u_0^*(x) := \chi\left(\frac{t}{\rho}\right) \cdot p|x|^\beta, \quad p = \frac{3q}{(\beta + 1)(\beta + 3)}, \quad \rho > 0 \text{ small.}\]

(by the finite speed of propagation the cut-off does not affect the behaviour at zero for small times, cf. Remark 3.1). We take $u_1^* = 0$.

In the error estimates which will follow, on the right hand side we will always replace $\lambda(t)$ by $t^{1+\nu}$ and $b(t)$ by $t^\nu$, since this is the regime considered later in the bootstrap argument.

Lemma 4.1. Assume that $\lambda(t) \sim t^{1+\nu}$ and $b(t) \sim t^\nu$. Then

\[\|P_0(t)\|_{H^1} \lesssim t^{3\nu/2}.\]

Proof. Recall that $v^*(t) \sim t^{\beta} = t^{(\nu-3)/2}$, so $\lambda^{3/2}v^*(t) \sim b^2 \sim t^{2\nu}$. Hence, it is sufficient to show that $\|\chi(\frac{\tau}{t})A_\lambda\|_{H^1}^2 + \|\chi(\frac{\tau}{t})B_\lambda\|_{H^1}^2 \lesssim t^{-\nu}$. The computation in the proof of Lemma 3.4 gives

\[\|\chi(\frac{\tau}{t})A_\lambda\|_{H^1}^2 \lesssim \frac{t}{\lambda} \sim t^{-\nu},\]

and similarly for the second term.

\[\square\]

Lemma 4.2. Assume that $\lambda(t) \sim t^{1+\nu}$ and $b(t) \sim t^\nu$. Then

\[\|L_\lambda P_0 - \lambda^{3/2}v^*(t)L_\lambda A_\lambda - b^2L_\lambda B_\lambda\|_{L^2} \lesssim t^{3\nu/2-1}.\]

Proof. We will do the computation only for the terms with $A$. The terms with $B$ are asymptotically the same. We need to check that

\[\|(1 - \chi(\frac{\tau}{t}))f'(W_\lambda)A_\lambda\|_{L^2} + \|\Delta((1 - \chi(\frac{\tau}{t}))A_\lambda)\|_{L^2} \lesssim t^{-\nu/2-1}\]

The computations in the proof of Lemma 3.5 imply that the first term is bounded by $\frac{1}{\lambda}(\frac{\tau}{t})^{5/2} \sim t^{3\nu/2-1}$, and the second by

\[1 \cdot \sqrt{\frac{\lambda}{t}} + \frac{\lambda}{t} \cdot \sqrt{\frac{\tau}{\lambda}} + \frac{\lambda}{t^2} \cdot (\frac{\tau}{\lambda})^{3/2} \sim (t \cdot \lambda)^{-1/2} \sim t^{-\nu/2-1}.\]

\[\square\]

In the degenerate case the profile $P_1(t)$ is defined by the same formula (3.4).

Lemma 4.3. Assume that $\lambda(t) \sim t^{1+\nu}$ and $b(t) \sim t^\nu$. Then

\[\|P_1(t)\|_{L^2} \lesssim t^{3\nu/2}.\]

Proof. Notice that $\frac{d}{dt}v^*(t) \sim t^{\beta-1} \sim t^{(\nu-5)/2}$. This implies that

\[v^*(t) \cdot \lambda^{3/2}b \sim \frac{d}{dt}v^*(t) \cdot \lambda^{5/2} \sim b^3 \sim t^{3\nu}.\]
so all the terms in the definition of $P_1(t)$ have asymptotically the same size and it suffices to show that $\|\chi(t)A_\Delta\|_{L^2}^2 \lesssim t^{-3\nu}$ (the other terms are similar). The computation in the proof of Lemma 3.6 gives
\[
\|\chi(t)A_\Delta\|_{L^2}^2 \lesssim \left(\frac{t}{\lambda}\right)^3 \sim t^{-3\nu}.
\]

□

Estimate (3.8) and its proof are valid in the degenerate case.

Lemma 4.4. Assume that $\lambda(t) \sim t^{1+\nu}$ and $b(t) \sim t^\nu$. Then
\[
\|\partial_t P_0 - P_1\|_{H^1} \lesssim t^{\nu/2-1}(t^\nu + \|\varepsilon\|_{H^1 \times L^2}).
\]
Proof. As in the proof of Lemma 3.9, we write
\[
\partial_t P_0 - P_1 = -\frac{r}{t^2} \chi(t) (\lambda^{3/2}v^*(t)A_\lambda + b^2 B_\lambda)
+ \chi(t) (\lambda - b)[v^*(t)] \left(\frac{3}{2} \lambda^{3/2}A_\lambda - \lambda^{3/2}(\Lambda A)_\lambda - b^2(\Lambda B)_\lambda\right).
\]
The computation in the proof of Lemma 3.9 implies
\[
\|\chi(t)A_\lambda\|_{H^1} \lesssim \left(\frac{t}{\lambda}\right)^{3/2} \sim t^{-3\nu/2}.
\]
Multiplying by $\frac{r}{t^2} \lambda^{3/2}v^*(t) \sim t^{3\nu-1}$ we obtain the required bound on the first term. The second term of the first line is similar.

The second line is bounded exactly as in the proof of Lemma 3.9.

□

Lemma 4.5. Assume that $\lambda(t) \sim t^{1+\nu}$ and $b(t) \sim t^\nu$. Then
\[
\|\partial_t P_1\|_{L^2} \lesssim t^{\nu/2-1}(t^\nu + \|\varepsilon\|_{H^1 \times L^2}).
\]
Proof. We indicate only the modifications with respect to the proof of Lemma 3.10. The term coming from differentiating the cut-off function is estimated as before by
\[
\frac{\lambda}{t^2} v^*(t) \lambda^{3/2} b \cdot \left(\frac{t}{\lambda}\right)^{5/2} \sim t^{3\nu/2-1}.
\]
For the other terms, we get
\[
\|\chi(t)T_\Delta\|_{L^2} \lesssim \left(\frac{t}{\lambda}\right)^{3/2} \sim t^{-3\nu/2}.
\]

□

4.2. Error of the ansatz. This subsection differs from the non-degenerate case, because we work here only with $X^1$ regularity and some more effort is required in order to estimate the terms involving $u^*$.

Lemma 4.6. If $\lambda(t) \sim t^{1+\nu}$, $b \sim t^\nu$, $\nu > 8$ and $t$ is small, then
\[
\|f(\varphi_0(t)) - [f(W_{\lambda(t)}) + f(u^*) + f'(W_{\lambda(t)})P_0(t) + f'(W_{\lambda(t)})u^*(t)]\|_{L^2} \ll \frac{t^{5\nu-7}}{t^{1/2}}.
\]
Proof. As in the proof of Lemma 3.13 we use Lemma 3.12 with $k = W_{\lambda(t)}$, $l = P_0(t)$ and $m = u^*(t)$. We obtain that the $L^2$ norm of the term \( \|f(l)\| \) is bounded by
\[
\left((v^* \cdot \lambda)^{7/3} + b^{14/3}\right)\|\chi(t)T_\Delta\|_{L^2},
\]
which is better than required. For the term \( \|f'(l)\| \) we obtain the bound $(v^*)^{4/3} : \lambda + b^{4/3} \lambda^{-1} \sim t^{5\nu/3-1}$, which is again better than required.
Term “$f'(m)|k|$": Let $(u_t^*, \partial_t u_t^*)$ be the solution of the free wave equation for the initial data $(u_t^*(0), \partial_t u_t^*(0)) = (u_0^*, u_t^*)$. We write

$$
\|f'(u_t^*) \cdot W_\lambda\|_{L^2} \lesssim \|f'(u_t^*) \cdot W_\lambda\|_{L^2(|x| \leq \frac{\lambda}{\sqrt{2}} t)} + \|f'(u_t^*) \cdot W_\lambda\|_{L^2(|x| \geq \frac{\lambda}{\sqrt{2}} t)} + \|f'(u_t^*) \cdot W_\lambda\|_{L^2(|x| \geq \frac{\lambda}{\sqrt{2}} t)}
$$

and we examine separately the three terms on the right hand side. It follows from Proposition 4.7 that for $|x| \leq \frac{\lambda}{\sqrt{2}} t$ we have the bound $|u_t^*(t, x)| \lesssim t^\beta = t^{\frac{\beta}{\beta - 1}}$, which implies $\|f'(u_t^*)\|_{L^\infty} \lesssim t^{\frac{\beta}{\beta - 1}}$, hence

$$
\|f'(u_t^*) \cdot W_\lambda\|_{L^2(|x| \leq \frac{\lambda}{\sqrt{2}} t)} \lesssim t^{\frac{\beta}{\beta - 1}} \|W_\lambda\|_{L^2} \sim t^{\frac{\beta}{\beta - 1}} t^{\nu - 1} \ll t^{\frac{\beta}{\beta - 1}}.
$$

From Proposition 4.8 we infer

$$
\|u^* - u_t^*\|_{L^{20/3}(|x| \leq \frac{\lambda}{\sqrt{2}} t)} \lesssim t^{\frac{\beta}{\beta - 1}} t^{\frac{\nu}{\nu - 2}},
$$

hence

$$
\|f'(u^* - u_t^*)\|_{L^{5}(|x| \leq \frac{\lambda}{\sqrt{2}} t)} \lesssim t^{\frac{\beta}{\beta - 1} \nu - \frac{2\nu}{\nu - 2}},
$$

which leads to

$$
\|f'(u^* - u_t^*) \cdot W_\lambda\|_{L^2(|x| \leq \frac{\lambda}{\sqrt{2}} t)} \leq \|f'(u^* - u_t^*)\|_{L^{5}(|x| \leq \frac{\lambda}{\sqrt{2}} t)} \cdot \|W_\lambda\|_{L^{10/3}(|x| \leq \frac{\lambda}{\sqrt{2}} t)} \lesssim t^{\frac{\beta}{\beta - 1} \nu - \frac{2\nu}{\nu - 2}},
$$

which is more than sufficient for $\nu > 8$.

For $|x| \geq \frac{\lambda}{\sqrt{2}} t$, we know from Proposition 4.2 that $\|f'(u^*)\|_{L^5}$ is bounded for small $t$. By a change of variables we obtain

$$
\|W_\lambda\|_{L^{10/3}(|x| \geq \frac{\lambda}{\sqrt{2}} t)} \lesssim \left( \int_{\lambda/2 \lambda}^{+\infty} (r^{-3})^{10/3} r^4 \, dr \right) \sim (\lambda / t)^{3/2} \sim t^{\frac{\beta}{\beta - 1}} t^\nu \ll t^{\frac{\beta}{\beta - 1}}.
$$

Term “$f'(m)|\ell|$": Using (4.2) we have

$$
\|f'(u^*) \cdot P_0\|_{L^2} \leq \|f'(u^*)\|_{L^5} \cdot \|P_0\|_{H^1} \lesssim t^{3/2} \ll t^{\frac{\beta}{\beta - 1}} t^\nu.
$$

□

We can now estimate $\psi(t)$.

**Proposition 4.7.** Assume that $\lambda(t) \sim t^{1+\nu}$ and $b(t) \sim t^\nu$. Then

$$
\|\psi_0(t) + (\lambda - b) \frac{1}{\lambda} (\Lambda W)_\lambda\|_{H^1} \lesssim t^{\frac{3\nu}{\nu - 2}} + t^{\nu + 1}\|\epsilon(t)\|_{H^1 \times L^2},
$$

(4.8)

$$
\|\psi_1(t) - (\lambda - b) \frac{b}{\lambda} (\Lambda_0 W)_\lambda\|_{L^2} \lesssim t^{\frac{3\nu}{\nu - 2}} + t^{\nu - 1}\|\epsilon(t)\|_{H^1 \times L^2}.
$$

(4.9)

**Proof.** The first inequality follows from Lemma 4.1.

For the second inequality, as in the proof of Proposition 3.14 using Lemma 4.6 and rearranging the terms, we get:

$$
\psi_1 = (\lambda - b) \frac{b}{\lambda} (\Lambda_0 W)_\lambda - (\Delta W_\lambda + f(W_\lambda)) + (\partial_t u^* - \Delta u^* - f(u^*)) - v^*(t) \sqrt{\lambda} (\Lambda W)_\lambda + (\Lambda_0 \Lambda W)_\lambda + (\Delta P_0 - f'(W_\lambda) P_0 - v^*(t) \sqrt{\lambda} (\Lambda W)_\lambda - \frac{b^2}{\lambda} (L B + \Lambda_0 W)_\lambda + (v^*(t) - u^*(t)) \sqrt{\lambda} (f'(W)_\lambda) + \partial_t P_1 + O(t^{\frac{3\nu}{\nu - 2}}).
$$
We have
\begin{align*}
&\| (v^* - u^*) f'(W_\lambda) \|_{L^2} \lesssim \| (v^* - u_{\lambda,0}^*) f'(W_\lambda)\|_{L^2(\|x\| \leq \frac{1}{2} t)} \\
&\quad + \| (u_{\lambda,0}^* - u^*) f'(W_\lambda)\|_{L^2(\|x\| \leq \frac{1}{2} t)} \\
&\quad + \| v^* \cdot f'(W_\lambda) \|_{L^2(\|x\| \geq \frac{1}{2} t)} \\
&\quad + \| u^* \cdot f'(W_\lambda) \|_{L^2(\|x\| \geq \frac{1}{2} t)}.
\end{align*}
From Proposition B.7 it follows in particular that \(|v^*(t) - u_{\lambda,0}^*(t, r)| \lesssim r\) when \(r \leq \frac{1}{2} t\), hence the proof of Proposition 3.14 gives the bound
\[
\| (v^* - u_{\lambda,0}^*) \cdot f'(W_\lambda) \|_{L^3(\|x\| \leq \frac{1}{2} t)} \lesssim \lambda^{3/2} \ll t^{\frac{7}{2} - \frac{3}{4}}.
\]
From Proposition B.8 and the fact that \(\| f'(W_\lambda) \|_{L^{5/2}} = \| f'(W) \|_{L^{5/2}}\) we get
\[
\| (u^* - u_{\lambda,0}^*) \cdot f'(W_\lambda) \|_{L^2(\|x\| \geq \frac{1}{2} t)} \lesssim t^{\frac{7}{6} + \frac{2}{5}} = t^{\frac{1}{5} - \frac{3}{4}}.
\]
We have
\[
\| f'(W_\lambda) \|_{L^2(\|x\| \geq \frac{1}{2} t)} \lesssim \frac{\lambda^2}{\|x\|^3} \lesssim \lambda^2 t^{-3/2}
\]
and
\[
\| f'(W_\lambda) \|_{L^{5/2}(\|x\| \geq \frac{1}{2} t)} \lesssim \frac{\lambda^2}{\|x\|^2} \lesssim \lambda^2 t^{-2}.
\]
Using boundedness of \(v^*\) in \(L^\infty\), boundedness of \(u^*\) in \(L^1\) and Hölder inequality we obtain the required bounds, which terminates the proof.

Lemma 3.11 is still valid in the degenerate case, as well as its proof (we use Lemma 4.1 instead of Lemma 3.4).

5. EVOLUTION OF THE ERROR TERM

The evolution of the error term \(\varepsilon\) is governed by the following system of differential equations:
\[
\frac{d}{dt} \begin{pmatrix} \varepsilon_0 \\ \varepsilon_1 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 - \psi_0 \\ \Delta \varepsilon_0 + [f(\varphi_0 + \varepsilon_0) - f(\varphi_0)] - \psi_1 \end{pmatrix},
\]
coupled with the equations (3.3) and (2.11) for the modulation parameters \(\text{Mod} := (\lambda, b)\). We denote \((T_-, T_+)\) the maximal interval of existence of \(u\).

We introduce the energy functional adapted to our ansatz:
\[
I(t) := \int \frac{1}{2} |\varepsilon_1|^2 + \frac{1}{2} |\nabla \varepsilon_0|^2 - \left[ F(\varphi_0 + \varepsilon_0) - F(\varphi_0) - f(\varphi_0) \varepsilon_0 \right] dx.
\]
Essentially we will perform a bootstrap argument in order to control this functional just by integrating in time its time derivative. We need a virial correction term which is defined as follows:
\[
J(t) := b \int \varepsilon_1 \cdot \left( \frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_\lambda + (\nabla a)_\lambda \cdot \nabla \right) \varepsilon_0 dx,
\]
where \(a_\lambda(r) = a(\frac{r}{\lambda})\), \((\nabla a)_\lambda(r) = \nabla a(\frac{r}{\lambda})\), \((\Delta a)_\lambda(r) = \Delta a(\frac{r}{\lambda})\) and
\[
a(r) := \begin{cases} 
\frac{1}{2} r^2 & |r| \leq R \\
\frac{15}{8} R r - \frac{5}{2} R^2 + \frac{5}{4} R^3 r^{-1} - \frac{1}{8} R^5 r^{-3} & |r| \geq R
\end{cases}
\]
\((R)\) is a big radius to be chosen later, see Proposition 5.24.
Lemma 5.1. The function \( a(r) \) defined above, viewed as a function on \( \mathbb{R}^5 \), has the following properties:

- \( a \in C^{3,1} \),
- \( a \) is strictly convex,
- \( |a(r)| \leq r, \ |a'(r)| \leq 1, \ |a''(r)| \leq r^{-1} \) when \( r \to +\infty \) (the constant depends on \( R \)),
- \( -\frac{1}{r^2} \leq \Delta^2 a(r) \leq 0 \).

Proof. It is apparent from the formula defining \( a \) that \( a \) is regular except for \( r = R \). A computation shows that \( a(r), a'(r), a''(r) \) and \( a'''(r) \) are Lipschitz near \( r = R \). For \( r \geq R \) we have \( a''(r) = \frac{5}{2}(\frac{r}{R})^3 - \frac{3}{2}(\frac{r}{R})^5 > 0 \), which proves strict convexity. For \( r > R \) one can compute \( \Delta^2 a(r) = -\frac{15}{r^2} \) (where \( \Delta = \partial_r + \frac{4}{r} \partial_r \) is the laplacian in dimension \( N = 5 \)). \( \Box \)

We define the mixed energy-virial functional:

\[
H(t) = I(t) + J(t).
\]

The proof of the following result, which will occupy most of this section, is valid both in the non-degenerate and the degenerate case. The non-degenerate case is obtained for \( \nu = 3 \). We denote also:

\[
\gamma := \begin{cases} \frac{7}{2} & \text{in the non-degenerate case,} \\ \frac{7}{2} \nu - \frac{7}{3} & \text{in the degenerate case,} \end{cases}
\]

which is the exponent of \( t \) in the error estimates in Proposition B.4 and Proposition B.7 respectively.

We will use the notation:

\[
\|\nabla a_{\lambda \varepsilon_0}\|_{L^2}^2 := \int \sum_{i,j} (\partial_i j a)_{\lambda \varepsilon_0} \partial_i \varepsilon_0 \partial_j \varepsilon_0 \, dx.
\]

Proposition 5.2. Let \( \nu = 3 \) or \( \nu > 8 \). Suppose that \( \lambda \sim t^{1+\nu}, b \sim t^\nu \) and let \( c > 0 \). If \( R \) is chosen large enough, then there exist strictly positive constants \( T_0 \) and \( C_1 \) such that for \( [T_1, T_2] \subset (0, T_0] \cap (T_-, T_+) \) there holds

\[
H(T_2) \leq H(T_1) + \int_{T_1}^{T_2} \left( -\frac{b}{\lambda} (\|\nabla a_{\lambda \varepsilon_0}\|_{L^2}^2 - \int (f(\varphi_0 + \varepsilon_0) - f(\varphi_0)) \varepsilon_0 \, dx) \right.
\]

\[
+ \left. \left( \frac{C}{t} \|\varepsilon\|^2_{H^1 \times L^2} + C_1 t^{\gamma} \cdot \|\varepsilon\|_{H^1 \times L^2} \right) \right) \, dt.
\]

The proof of this result is going to be an algebraic computation which is not justified in the space \( H^1 \times L^2 \). However, we do not need any uniform control of the regularity or the decay, so we can use the following density argument. We can approximate a given \( \varepsilon \) in \( H^1 \times L^2 \) in such a way that the initial data \( (u(T_1), \partial_t u(T_1)) \) will be in \( X^1 \times H^1 \) and of compact support. Then locally the evolution will have the same proprieties by Proposition B.6 and will be close to the original one in \( H^1 \times L^2 \) for all \( t \in [T_1, T_2] \) by local well-posedness in \( H^1 \times L^2 \). The \( \varepsilon \) has sufficient regularity and decay to justify all the computations. Since the estimate (5.5) depends continuously (in \( H^1 \times L^2 \)) on \( \varepsilon \), we are done.

We shall split the proof of Proposition 5.2 into several Lemmas. We always work under the hypotheses of Proposition 5.2, that is \( \lambda \sim t^{1+\nu}, b \sim t^\nu \) and \( \|\varepsilon\|_{H^1 \times L^2} \leq t^{\gamma+1} \). Notice that \( \gamma + 1 > \nu \). In the non-degenerate case \( \gamma + 1 = \frac{9}{2} > 3 = \nu \) and in the degenerate case \( \gamma + 1 = \frac{7}{6} \nu - \frac{4}{3} > \nu \) because \( \nu > 8 \). This means that \( \|\varepsilon\|_{H^1 \times L^2} \ll b \) and \( \|\varepsilon\|_{H^1 \times L^2} \ll \frac{1}{t} \) for small \( t \). In what follows \( c \) stands for any small strictly positive constant.

We use the method introduced in [22], which consists in differentiating the nonlinear term in self-similar variables. The resulting error will be corrected by the virial term \( J \). Concretely, we
have:

\[
\frac{d}{dt} \int [F(\varphi_0 + \varepsilon_0) - F(\varphi_0) - f(\varphi_0)\varepsilon_0] \, dx \\
= \frac{d}{dt} \int [F((\varphi_0)_{1/\lambda} + (\varepsilon_0)_{1/\lambda}) - F((\varphi_0)_{1/\lambda}) - f((\varphi_0)_{1/\lambda})(\varepsilon_0)_{1/\lambda}] \, dx \\
= \int [f((\varphi_0)_{1/\lambda} + (\varepsilon_0)_{1/\lambda}) - f'((\varphi_0)_{1/\lambda})(\varepsilon_0)_{1/\lambda}] \partial_t ((\varphi_0)_{1/\lambda}) \, dx \\
+ \int [f((\varphi_0)_{1/\lambda} + (\varepsilon_0)_{1/\lambda}) - f((\varphi_0)_{1/\lambda})][(\varepsilon_0)_{1/\lambda} + \frac{\lambda t}{\lambda} (\Lambda \varepsilon_0)_{1/\lambda}] \, dx.
\]

The first term can be neglected, as shown by Lemma 3.11. Scalling back the second term we obtain

\[
\frac{d}{dt} \int [F(\varphi_0 + \varepsilon_0) - F(\varphi_0) - f(\varphi_0)\varepsilon_0] \, dx \simeq \int [f(\varphi_0 + \varepsilon_0) - f(\varphi_0)]((\varepsilon_0)_{1/\lambda} + \frac{\lambda t}{\lambda} (\Lambda \varepsilon_0)_{1/\lambda}) \, dx.
\]

Here and later the sign \(\simeq\) means that the difference of the two sides has size at most \(\frac{C}{\varepsilon}\|\varepsilon\|_{H^1 \times L^2}^2 + C_1 t^{\gamma} \cdot \|\varepsilon\|_{H^1 \times L^2}^2\). Also, when we say that a term is “negligible”, it always means that its absolute value is bounded by \(\frac{C}{\varepsilon}\|\varepsilon\|_{H^1 \times L^2}^2 + C_1 t^{\gamma} \cdot \|\varepsilon\|_{H^1 \times L^2}^2\).

Using the equations (5.1), (5.7) and integrating by parts, we obtain standard cancellations:

\[
\frac{d}{dt} I(t) \simeq \int \varepsilon_1 \varphi_1 \, dx - \int [\Delta \varepsilon_0 + f(\varphi_0 + \varepsilon_0) - f(\varphi_0)]\varepsilon_0 \, dx \\
- \frac{\lambda t}{\lambda} \int [f(\varphi_0 + \varepsilon_0) - f(\varphi_0)]\varepsilon_0 \, dx \\
\simeq -\int \varepsilon_1 \varphi_1 \, dx + \int [\Delta \varepsilon_0 + f(\varphi_0 + \varepsilon_0) - f(\varphi_0)]\varphi_0 \, dx \\
- \frac{\lambda t}{\lambda} \int [f(\varphi_0 + \varepsilon_0) - f(\varphi_0)]\varphi_0 \, dx.
\]

Consider now the virial term \(J(t)\).

**Lemma 5.3.**

\[
\frac{d}{dt} J(t) \leq \int \varepsilon_1 \varphi_1 \, dx - \frac{b}{\lambda} \|\nabla a(\varepsilon_0)\|_{L^2}^2 + \frac{\lambda t}{\lambda} \int [f(\varphi_0 + \varepsilon_0) - f(\varphi_0)]\Lambda_0 \varepsilon_0 \, dx \\
+ \frac{c}{t} \|\varepsilon\|_{H^1 \times L^2}^2 + C_1 t^{\gamma} \cdot \|\varepsilon\|_{H^1 \times L^2}^2.
\]

Notice the cancellation of \(\int \varepsilon_1 \varphi_1 \, dx\) in (5.8) and (5.9). This is important because the bound on \(\|\varphi_1\|\) given by Proposition 3.14 and Proposition 4.7 is only \(\frac{1}{t} \|\varepsilon\|\), which is borderline but not sufficient to close the bootstrap. Moreover, \(\Lambda_0 - \Lambda = \text{Id}\), so \(J\) eliminates the unbounded part of the operator \(\Lambda\) acting on \(\varepsilon_0\).

**Proof of Lemma 5.3.** We compute

\[
\frac{d}{dt} J(t) = b_1 \int \varepsilon_1 \cdot \left(\frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_{\lambda} + (\nabla a)_{\lambda} \cdot \nabla\right)\varepsilon_0 \, dx \\
- \frac{b \lambda t}{\lambda} \int \varepsilon_1 \cdot \left(\frac{1}{\lambda} \cdot \frac{1}{2} (\Lambda_{3/2} \nabla a)_{\lambda} + (\Lambda_{5/2} \nabla a)_{\lambda} \cdot \nabla\right)\varepsilon_0 \, dx \\
+ b \int \varepsilon_1 \cdot \left(\frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_{\lambda} + (\nabla a)_{\lambda} \cdot \nabla\right)\varepsilon_0 \, dx \\
+ b \int \varepsilon_1 \cdot \left(\frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_{\lambda} + (\nabla a)_{\lambda} \cdot \nabla\right)\varepsilon_0 \, dx.
\]
Consider the first two lines. From Lemma 5.1 and Hardy inequality it follows that
\[
\frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_{\lambda} + (\nabla a)_{\lambda} \cdot \nabla
\]
and
\[
\frac{1}{\lambda} \cdot \frac{1}{2} (\Lambda_{3/2} a)_{\lambda} - (\Lambda_{5/2} \nabla a)_{\lambda} \cdot \nabla
\]
are uniformly bounded as operators $\dot{H}^1 \to L^2$ (the bound depends on $R$). Moreover, it is clear that $|b_t| + \frac{1}{\lambda} |\Lambda_{\lambda}| \ll t^{-1}$. Hence, the first two lines are negligible.

Using again (5.1) we get
\[
\int b \varepsilon_{1t} \cdot \left( \frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_{\lambda} + (\nabla a)_{\lambda} \cdot \nabla \right) \varepsilon_0 \, dx + b \int \varepsilon_1 \cdot \left( \frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_{\lambda} + (\nabla a)_{\lambda} \cdot \nabla \right) \varepsilon_0 \, dx
\]
\[
= b \int (\Delta \varepsilon_0 + f(\varphi_0 + \varepsilon_0) - f(\varphi_0)) \cdot \left( \frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_{\lambda} + (\nabla a)_{\lambda} \cdot \nabla \right) \varepsilon_0 \, dx
\]
\[
+ b \int \varepsilon_1 \cdot \left( \frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_{\lambda} + (\nabla a)_{\lambda} \cdot \nabla \right) \varepsilon_1 \, dx
\]
\[
- b \int \psi_1 \cdot \left( \frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_{\lambda} + (\nabla a)_{\lambda} \cdot \nabla \right) \varepsilon_0 \, dx - b \int \varepsilon_1 \cdot \left( \frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_{\lambda} + (\nabla a)_{\lambda} \cdot \nabla \right) \psi_0 \, dx.
\]

Proposition 3.14 and Proposition 4.7 imply that $\|\varepsilon_0\|_{L^2} \leq \frac{1}{\lambda} \|\varepsilon\|_{H^1 \times L^2} + t^\gamma$. Using once again uniform boundedness of the operator (5.10), we obtain that the first term of the last line is negligible. Consider now the second term. We will show that this is true and recapitulate in order to finish the proof of Lemma 5.4.

It follows from Proposition 3.14 and Proposition 4.7 that in (5.12) $\psi_0$ can be replaced by $- (\lambda_t - b) \frac{1}{\lambda} (\Lambda W)_{\lambda}$ and $\psi_1$ by $(\lambda_t - b) \frac{1}{\lambda} (\Lambda_0 W)_{\lambda}$. Hence, using (5.8), it suffices to prove that $\|\Lambda_0 W - \left[ \frac{1}{2} \Delta a + \nabla a \cdot \nabla \right] \Lambda_0 W \|_{L^2}$ is arbitrarily small when $R$ is large enough. But this is clear, since $\left[ \frac{1}{2} \Delta a + \nabla a \cdot \nabla \right] \Lambda_0 W (r) = \Lambda_0 W (r) - r^{-3}$ for all $r$, with a constant independent of $R$.

The second line of (5.11) is 0 by integration by parts and we are left with the first line. The term with $\Delta \varepsilon_0$ is computed via a classical Pohozaev identity:
\[
\int \left( \frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_{\lambda} + (\nabla a)_{\lambda} \cdot \nabla \right) \varepsilon_0 \Delta \varepsilon_0 \, dx = - \frac{1}{\lambda} \|\nabla a, \lambda \varepsilon_0\|_{L^2}^2 + \frac{1}{\lambda^3} \int (\Delta^2 a)_{\lambda} \varepsilon_0^2 \, dx.
\]

By Lemma 5.1, the last term is finite and $\leq 0$.

The nonlinear part is calculated in the following lemma.

**Lemma 5.4.**
\[
\left| b \int \left[ \frac{1}{\lambda} \cdot \frac{1}{2} (\Delta a)_{\lambda} + (\nabla a)_{\lambda} \cdot \nabla \right] \varepsilon_0 \cdot [f(\varphi_0 + \varepsilon_0) - f(\varphi_0)] \, dx
\]
\[
- \frac{\lambda_t}{\lambda} \int [f(\varphi_0 + \varepsilon_0) - f(\varphi_0)] \Lambda_0 \varepsilon_0 \, dx \right| \leq \frac{c}{\lambda} \|\varepsilon\|_{H^1 \times L^2}^2.
\]

We will admit for a moment that this is true and recapitulate in order to finish the proof of Lemma 5.4. Identity (5.13) implies that the term with $\Delta \varepsilon_0$ in the first line of (5.11) is smaller than $- \frac{\lambda_t}{\lambda} \|\nabla a, \lambda \varepsilon_0\|_{L^2}^2$. Lemma 5.4 implies that the difference between the other term of the first line of (5.11) and $\frac{\lambda_t}{\lambda} \int [f(\varphi_0 + \varepsilon_0) - f(\varphi_0)] \Lambda_0 \varepsilon_0 \, dx$ is negligible. The second line of (5.11) is 0, and the difference between the last line and $\int \varepsilon_1 \psi_1 \, dx$ is negligible, as follows from the computation above. This proves (5.9).

In order to prove Lemma 5.4, we need two auxiliary facts:
Lemma 5.5.
\[ |f(k+l) - f(k) - f'(k)l - \frac{1}{2} f''(k)l^2| \lesssim |f(l)|, \]
\[ |F(k+l) - F(k) - f(k)l - \frac{1}{2} f''(k)l^2| \lesssim |F(l)| + |f''(k)|l^3. \]

Proof. For \(|l| \leq \frac{1}{2}|k|\) this follows from the Taylor expansion and for \(|l| \geq \frac{1}{2}|k|\) this is obvious by the triangle inequality. \(\square\)

Lemma 5.6. There exists a constant \(C_2\) independent of \(R\) such that for small \(t\),
\begin{align}
\label{eq:5.15} ||x| \cdot |\nabla \phi_0||_{L^{10/3}} & \leq C_2, \\
\label{eq:5.16} ||\lambda(\nabla a)_\lambda \cdot |\nabla \phi_0||_{L^{10/3}} & \leq C_2.
\end{align}
Moreover,
\begin{equation}
\label{eq:5.17} \| (x - \lambda(\nabla a)_\lambda) \cdot \nabla \phi_0 \|_{L^{10/3}} \leq c \tag{if \(R\) is large enough and \(\rho\) small enough.}
\end{equation}

Proof. Recall that \(\varphi_0(t) = W_{\lambda(t)} + P_0(t) + u^*(t)\), and we can estimate the three terms separately. The third one gives \(||x| \cdot |\nabla u^*||_{L^{10/3}}\), which is bounded by Proposition 1.2 and the fact that \(u^*\) has compact support. It is easy to check that \(||x| \cdot |\nabla (W_\lambda)||_{L^{10/3}} = ||x| \cdot |\nabla W||_{L^{10/3}},\) which gives the boundedness of the first term. Finally, we compute
\[ \nabla \left[ \chi \left( \frac{x}{\lambda} \right) \lambda^{3/2} A_\lambda(x) \right] = \nabla \left[ \chi \left( \frac{x}{\lambda} \right) A(\frac{x}{\lambda}) \right] = \frac{1}{\lambda} \left( \nabla \chi \right) \left( \frac{x}{\lambda} \right) A(\frac{x}{\lambda}) + \frac{1}{\lambda} \chi \left( \frac{x}{\lambda} \right) \nabla A(\frac{x}{\lambda}), \]
and it is sufficient to use the inequalities \(|A(x/\lambda)| \lesssim |x|/|\lambda|\) and \(|\nabla A(x/\lambda)| \lesssim \lambda^2/|x|^2\). The second term of \(P_0\) is bounded in the same way. Notice that we obtain in fact that \(||x| \cdot |\nabla \phi_0(t)||_{L^{10/3}}\) is small when \(t\) is small.

Clearly \(|\lambda(\nabla a)_\lambda| \lesssim |x|\) uniformly in \(R\), so (5.16) follows from (5.15).

The proof of (5.17) is similar. The terms \(||x| \cdot |\nabla u^*||_{L^{10/3}}\) and \(|\lambda(\nabla a)_\lambda \cdot |\nabla u^*||_{L^{10/3}}\) are small when \(\rho\) is small. By rescaling we get
\[ \| (x - \lambda(\nabla a)_\lambda) \cdot |\nabla W_\lambda\|_{L^{10/3}} = \| (x - \nabla a) \cdot |\nabla W\|_{L^{10/3}}. \]

By definition \(\nabla a = x\) for \(|x| \leq R\), so
\[ \| (x - \nabla a) \cdot |\nabla W\|_{L^{10/3}} \lesssim \| |x| \cdot |\nabla W|\|_{L^{10/3}} \rightarrow 0 \quad \text{when} \quad R \rightarrow +\infty. \]
Smallness of \(||(x - \lambda(\nabla a)_\lambda)) \cdot |\nabla P_0(t)||_{L^{10/3}}\) for small \(t\) follows from smallness of \(||x| \cdot |\nabla P_0(t)||_{L^{10/3}}\). \(\square\)

Proof of Lemma 5.4. First, as for the linear terms, using integration by parts we transform the integral so that the unbounded operator \(\Lambda_0\) (and its approximation \(\frac{1}{2} \Delta a + \nabla a \cdot \nabla\)) no longer acts on \(\varepsilon_0\):
\begin{equation}
\label{eq:5.18} \int \frac{1}{\lambda} x \cdot \nabla \varepsilon_0 f(\varphi_0 + \varepsilon_0) \, dx = \int \frac{1}{\lambda} x \cdot \nabla (\varphi_0 + \varepsilon_0) f(\varphi_0 + \varepsilon_0) \, dx - \int \frac{1}{\lambda} x \cdot \nabla \varphi_0 f(\varphi_0 + \varepsilon_0) \, dx
\end{equation}
\[ = -5 \int \frac{1}{\lambda} F(\varphi_0 + \varepsilon_0) \, dx - \int \frac{1}{\lambda} x \cdot \nabla \varphi_0 f(\varphi_0 + \varepsilon_0) \, dx \]
and analogously
\begin{equation}
\label{eq:5.19} \int (\nabla a)_\lambda \cdot |\nabla \varepsilon_0 f(\varphi_0 + \varepsilon_0)\| \, dx = - \int \frac{1}{\lambda} (\Delta a)_\lambda F(\varphi_0 + \varepsilon_0) \, dx - \int (\nabla a)_\lambda \cdot |\nabla \varphi_0 f(\varphi_0 + \varepsilon_0)\| \, dx. \end{equation}
Using Lemma 5.5 we see that
\[ \int |F(\varphi + \varepsilon_0) - (F(\varphi_0) + f(\varphi_0)\varepsilon_0 + \frac{1}{2} f'(\varphi_0)\varepsilon_0^2)| \, dx \lesssim \|\varepsilon_0\|_{H^1 \times L^2}^3 \leq f(\|\varepsilon\|_{H^1 \times L^2}) \]
Similarly, from Lemma 5.5 and Lemma 5.6 we get
\[ \int \left| x \cdot \nabla \varphi_0 f(\varphi + \varepsilon_0) - x \cdot \nabla \varphi_0 (f(\varphi_0) + f'(\varphi_0)\varepsilon_0 + \frac{1}{2} f''(\varphi_0)\varepsilon_0^2) \right| \, dx \lesssim f(\|\varepsilon\|_{H^1 \times L^2}). \]
Notice that \( \frac{\lambda_t}{\lambda} f(\|\varepsilon\|_{H^1 \times L^2}) \lesssim \frac{1}{\lambda} \|\varepsilon\|_{H^1 \times L^2}^2 \), so the above two inequalities together with (5.18) imply that
\[ \frac{\lambda_t}{\lambda} \int x \cdot \nabla \varepsilon_0 f(\varphi + \varepsilon_0) \, dx \simeq -\frac{5\lambda_t}{\lambda} \int (F(\varphi_0) + f(\varphi_0)\varepsilon_0 + \frac{1}{2} f'(\varphi_0)\varepsilon_0^2) \, dx \]
\[ - \frac{\lambda_t}{\lambda} \int x \cdot \nabla \varphi_0 (f(\varphi_0) + f'(\varphi_0)\varepsilon_0 + \frac{1}{2} f''(\varphi_0)\varepsilon_0^2) \, dx. \]
Integrating by parts we find
\[ \int x \cdot \nabla \varphi_0 f(\varphi_0) \, dx = \int x \cdot \nabla F(\varphi_0) \, dx = -5 \int F(\varphi_0) \, dx \]
and
\[ \int x \cdot \nabla \varphi_0 f'(\varphi_0)\varepsilon_0 \, dx = \int x \cdot \nabla f(\varphi_0)\varepsilon_0 \, dx = -5 \int f(\varphi_0)\varepsilon_0 \, dx - \int x \cdot \nabla \varepsilon_0 f(\varphi_0) \, dx. \]
Thus, (5.20) simplifies to
\[ \frac{\lambda_t}{\lambda} \int x \cdot \nabla \varepsilon_0 (f(\varphi_0 + \varepsilon_0) - f(\varphi_0)) \, dx \simeq \frac{\lambda_t}{\lambda} \int \left( -\frac{5}{2} f'(\varphi_0)\varepsilon_0^2 - \frac{1}{2} x \cdot \nabla \varphi_0 f''(\varphi_0)\varepsilon_0^2 \right) \, dx. \]
Using just a pointwise estimate and Hölder we obtain
\[ \frac{\lambda_t}{\lambda} \int \varepsilon_0 [f(\varphi_0 + \varepsilon_0) - f(\varphi_0)] \, dx \simeq \frac{\lambda_t}{\lambda} \int f'(\varphi_0)\varepsilon_0^2 \, dx. \]
Combining with (5.21) we have
\[ \frac{\lambda_t}{\lambda} \int \Lambda_0 \varepsilon_0 (f(\varphi_0 + \varepsilon_0) - f(\varphi_0)) \, dx \simeq -\frac{\lambda_t}{2\lambda} \int x \cdot \nabla \varphi_0 f''(\varphi_0)\varepsilon_0^2 \, dx \simeq -\frac{b}{2\lambda} \int x \cdot \nabla \varphi_0 f''(\varphi_0)\varepsilon_0^2 \, dx, \]
where the last almost-equality follows from the fact that \( |\lambda_t - b| \lesssim \|\varepsilon\|_{H^1 \times L^2} \).
Analogously, we obtain
\[ b \int \left[ \frac{1}{2} (\Delta a) + (\nabla a) \cdot \nabla \right] \varepsilon_0 (f(\varphi_0 + \varepsilon_0) - f(\varphi_0)) \, dx \simeq -\frac{b}{2} \int (\nabla a) \cdot \nabla \varphi_0 f''(\varphi_0)\varepsilon_0^2 \, dx. \]
Comparing (5.22) and (5.23), we see that in order to finish the proof, we need to check that
\[ \int |(x - \lambda (\nabla a) \lambda) \cdot \nabla \varphi_0 f''(\varphi_0)\varepsilon_0^2| \, dx \leq c \|\varepsilon\|_{H^1 \times L^2}^2 \]
when \( R \) is sufficiently large. Using Sobolev and Hölder inequalities this boils down to
\[ \|(x - \lambda (\nabla a) \lambda) \cdot \nabla \varphi_0 f''(\varphi_0)\|_{L^{3/2}} \leq c, \]
and this follows from (5.17) and boundedness of \( f''(\varphi_0) \) in \( L^1 \).
Proof of Proposition 5.2. From (5.8), (5.9) and the fact that \( \Lambda_0 - \Lambda = \text{Id} \), we have
\[
\frac{d}{dt} H = \frac{d}{dt} I + \frac{d}{dt} J \leq \int (\Delta \varepsilon_0 + f(\varphi_0 + \varepsilon_0) - f(\varphi_0)) \psi_0 \, dx
- \frac{b}{\lambda} \| \nabla_{a,\lambda} \varepsilon_0 \|_{L^2}^2 + \frac{\lambda t}{\lambda} \int (f(\varphi_0 + \varepsilon_0) - f(\varphi_0)) \varepsilon_0 \, dx + \frac{c}{t} \| \varepsilon \|_{H^1 \times L^2}^2 + C_1 t^\gamma \| \varepsilon \|_{H^1 \times L^2}.
\]
Notice that
\[
\| f(\varphi_0 + \varepsilon_0) - f(\varphi_0) \|_{H^{-1}} \lesssim \| \varepsilon_0 \|_{H^1}.
\]
This follows from the inequality \(|f(k + l) - f(k)| \lesssim |l| + |f(l)|\) and the fact that \( \varphi_0 \) is bounded in \( H^1 \). If we recall that \( \frac{\lambda t - b}{\lambda} \lesssim \| \varepsilon \|_{H^1 \times L^2} \ll \frac{1}{\alpha} \), we see that in the second line we can replace \( \lambda t \) by \( b \), hence to finish the proof we only have to prove that
\[
\int (\Delta \varepsilon_0 + f(\varphi_0 + \varepsilon_0) - f(\varphi_0)) \psi_0 \, dx \leq \frac{c}{t} \| \varepsilon \|_{H^1 \times L^2}^2 + C_1 t^\gamma \| \varepsilon \|_{H^1 \times L^2}.
\]
Inequalities (3.17) and (4.8) show that it is sufficient to check that
\[
\left| \int (\Delta \varepsilon_0 + f(\varphi_0 + \varepsilon_0) - f(\varphi_0)) (\Lambda W)_\lambda \, dx \right| \leq \frac{c \lambda}{t} \| \varepsilon \|_{H^1 \times L^2}^2 + C_1 t^\gamma \| \varepsilon \|_{H^1 \times L^2},
\]
which in turn will follow from (5.8) and
\[
\left| \int (\Delta \varepsilon_0 + f(\varphi_0 + \varepsilon_0) - f(\varphi_0)) (\Lambda W)_\lambda \, dx \right| \leq \frac{c \lambda}{t} \| \varepsilon \|_{H^1 \times L^2} + C_1 \lambda t^\gamma.
\]
From pointwise bounds (for example the first inequality in Lemma 5.3), one deduces
\[
\| f(\varphi_0 + \varepsilon_0) - f(\varphi_0) - f'(\varphi_0) \varepsilon_0 \|_{H^{-1}} \lesssim \| \varepsilon \|_{H^1}^2 \ll \frac{\lambda}{t} \| \varepsilon \|_{H^1 \times L^2},
\]
hence it suffices to show that
\[
\left| \int (\Delta \varepsilon_0 + f'(\varphi_0) \varepsilon_0) (\Lambda W)_\lambda \, dx \right| = \left| \int \varepsilon_0 \cdot [\Delta + f'(\varphi_0)] (\Lambda W)_\lambda \, dx \right| \leq \frac{c \lambda}{t} \| \varepsilon \|_{H^1 \times L^2} + C_1 \lambda t^\gamma.
\]
Observe that \([\Delta + f'(W_\lambda)](\Lambda W)_\lambda = 0\), so we are left with proving that
\[
\left| \int \varepsilon_0 \cdot [f'(\varphi_0) - f'(W_\lambda)](\Lambda W)_\lambda \, dx \right| \leq \frac{c \lambda}{t} \| \varepsilon \|_{H^1 \times L^2} + C_1 \lambda t^\gamma.
\]
By Hölder inequality it suffices to show that
\[
(5.24) \quad \| (f'(\varphi_0) - f'(W_\lambda))(\Lambda W)_\lambda \|_{L^{10/3}} \leq \frac{c \lambda}{t}.
\]
The inequality \(|f'(k + l) - f'(l)| \lesssim |f'(l)| + |f''(l)| \cdot |l|\) for \( k = W_\lambda \) and \( l = u^*(t) + P_0(t) \) reduces (5.24) to checking that
\[
(5.25) \quad \| f'(u^*)(\Lambda W)_\lambda \|_{L^{10/7}} \leq \frac{c \lambda}{t},
\]
\[
(5.26) \quad \| f'(P_0)(\Lambda W)_\lambda \|_{L^{10/7}} \leq \frac{c \lambda}{t},
\]
\[
(5.27) \quad \| u^* \cdot f'((\Lambda W)_\lambda) \|_{L^{10/7}} \leq \frac{c \lambda}{t},
\]
\[
(5.28) \quad \| P_0 \cdot f'((\Lambda W)_\lambda) \|_{L^{10/7}} \leq \frac{c \lambda}{t}.
\]
Again using Hölder we get \( \| P_0 \cdot f'((\Lambda W)_\lambda) \|_{L^{10/7}} \leq \| P_0 \|_{L^{10/3}} \| f'((\Lambda W)_\lambda) \|_{L^{5/2}} \lesssim \| P_0 \|_{H^1} \). From Lemma 3.3 (or the degenerate version Lemma 1.1) we have \( \| P_0 \|_{H^1} \lesssim t^\gamma + 1 \ll \frac{1}{\alpha} \). This proves (5.28) and (5.26) is very similar.
From Proposition 5.2, we know that \( \|u^*\|_{L,10} \) is bounded. Hence \( \|u^* \cdot f'(\Lambda W)\|_{L,10} \lesssim \|u^*\|_{L,10} \cdot \|f'(\Lambda W)\|_{L,5/3} \lesssim \lambda \). This proves (5.27) and (5.25) is similar.

6. Construction of a uniformly controlled sequence and conclusion

In this section we will analyse finite dimensional phenomena of our dynamical system – modulation equations and eigendirections of the linearized operator \( L \). We will also define precisely the bootstrap assumptions and finish the proof of the main theorems.

It is known that the operator \( L = -\Delta - f'(W) \) has a unique simple strictly negative eigenvalue \(-c_0^2\) (by convention \( e_0 > 0 \)), with a unique positive eigenfunction \( Y \) such that \( \|Y\|_{L,2} = 1 \). This function \( Y \) is radial, smooth and decays exponentially. This follows from classical results of spectral theory and theory of elliptic equations, see \([8, Proposition 5.5]\), where it is also shown that there exists a constant \( c_1 > 0 \) such that

\[
(6.1) \quad g \in \tilde{H}^1_{rad}, \quad \langle g, Y \rangle = \langle \nabla g, \nabla Y \rangle = 0 \quad \Rightarrow \quad \langle g, Lg \rangle \geq c_1 \|\nabla g\|_{L,2}^2.
\]

We need here a slight modification of this coercivity lemma.

Lemma 6.1. For any \( c > 0 \) there exists \( c_L, C > 0 \) such that

\[
(6.2) \quad \langle g, Lg \rangle \geq c_L \|\nabla g\|_{L,2}^2 - C \langle g, Y \rangle^2 - c \langle g, Z \rangle^2.
\]

Proof. We first show that

\[
(6.3) \quad g \in \tilde{H}^1_{rad}, \quad \langle g, Y \rangle = \langle g, Z \rangle = 0 \quad \Rightarrow \quad \langle g, Lg \rangle \geq c_2 \|\nabla g\|_{L,2}^2.
\]

To prove (6.3), decompose \( g = a\Lambda W + h, \langle h, \Delta \Lambda W \rangle = 0 \). Notice that \( \langle \Lambda W, Y \rangle = 0 \), thus \( \langle h, Y \rangle = 0 \) and (6.1) implies

\[
\langle g, Lg \rangle = \langle h + a\Lambda W, Lh \rangle = \langle h, Lh \rangle \geq c_1 \|\nabla h\|_{L,2}^2.
\]

Let \( \Lambda \tilde{W} \) be the orthogonal projection of \( \Delta \Lambda W \) on \( Z^+ \) in \( \tilde{H}^{-1} \). We have

\[
\|\nabla h\|_{L,2}^2 = \|\nabla g - a\nabla \Lambda W\|_{L,2}^2 = \|\nabla g\|_{L,2}^2 - 2a\langle \nabla g, \nabla \Lambda W \rangle + a^2 \|\nabla \Lambda W\|_{L,2}^2
\]

\[
= \|\nabla g\|_{L,2}^2 + 2a \langle g, \Lambda \tilde{W} \rangle + a^2 \|\nabla \Lambda W\|_{L,2}^2.
\]

The functions \( \Delta \Lambda W \) and \( Z \) are not perpendicular in \( \tilde{H}^{-1} \), so \( \|\Lambda \tilde{W}\|_{H^{-1}} < \|\nabla \Lambda W\|_{L,2} \), and (6.3) follows from Cauchy-Schwarz inequality.

In order to prove (6.2), we decompose

\[
(6.4) \quad g = aY + b\Lambda W + \bar{g}, \quad \langle \bar{g}, Y \rangle = \langle \bar{g}, Z \rangle = 0.
\]

Projecting (6.4) on \( Y \) and \( Z \) we have

\[
(6.5) \quad a^2 \lesssim \langle g, Y \rangle^2, \quad b^2 \lesssim \langle g, Z \rangle^2 + a^2 \langle g, Y \rangle^2 \lesssim \langle g, Z \rangle^2 + \langle g, Y \rangle^2.
\]

From (6.3) we obtain

\[
\langle \bar{g}, L\bar{f} \rangle \geq c_2 \|\nabla \bar{g}\|_{L,2}^2,
\]

thus

\[
\langle g, Lg \rangle = \langle aY + b\Lambda W + \bar{g}, -c_0^2 aY + L\bar{g} \rangle = -c_0^2 a^2 + \langle \bar{g}, L\bar{g} \rangle \geq c_2 \|\nabla \bar{g}\|_{L,2}^2 - c_0^2 a^2.
\]

From the inequality \((x - y)^2 \geq \frac{1}{2} x^2 - y^2 \) we have

\[
\|\nabla \bar{g}\|_{L,2}^2 \geq \frac{1}{2} \|\nabla g - b\nabla \Lambda W\|_{L,2}^2 - a^2 \|\nabla Y\|_{L,2}^2.
\]

From the inequality \((x - y)^2 \geq \frac{c}{1 + c} x^2 - cy^2 \) we have

\[
\|\nabla g - b\nabla \Lambda W\|_{L,2} \geq \frac{c}{1 + c} \|\nabla g\|_{L,2} - cb^2 \|\nabla \Lambda W\|_{L,2}.
\]
If we choose $c$ small enough and put everything together using (6.5), we obtain (6.2).

From now on we will denote

$$\alpha(g) := \langle g, \mathcal{Y} \rangle, \quad \alpha_\lambda(g) := \left\langle g, \frac{1}{\lambda} \mathcal{Y}_\lambda \right\rangle.$$ 

We prove a version of the coercivity lemma with a localized gradient term.

**Lemma 6.2.** Let $c > 0$. If $R$ is large enough, then there exists a constant $C$ such that

\[
(6.6) \quad \int_{|x| \leq R} |\nabla g|^2 \, dx - \int_{\mathbb{R}^5} f'(W) g^2 \, dx \geq -c\|\nabla g\|^2_{L^2} - C|\alpha(g)|^2.
\]

In the proof we assume that $g$ is radial, which is justified because later we use it for $g = \varepsilon_0$. Notice however that the non-radial case follows by considering the radial rearrangement.

**Proof.** Define the projection $\Psi_R : H^1 \to \dot{H}^1$ by the formula:

\[
(6.7) \quad \Psi_R g(r) = \begin{cases} g(r) - g(R) & \text{if } r \leq R, \\ 0 & \text{if } r \geq R. \end{cases}
\]

By (6.2) applied to $\Psi_R g$ we have

\[
(1 + \frac{c}{2}) \int_{|x| \leq R} |\nabla \varepsilon_0|^2 \, dx = (1 + \frac{c}{2}) \int_{\mathbb{R}^5} |\nabla (\Psi_R \varepsilon_0)|^2 \, dx \\
\geq (1 + \frac{c}{2}) \int f'(W)|\Psi_R \varepsilon_0|^2 \, dx - C(\Psi_R \varepsilon_0, \mathcal{Y})^2.
\]

Recall that in dimension $N = 5$ for a radial function $g$ we have $g(R) \leq R^{-\frac{4}{5}}\|\nabla g\|_{L^2}$ (this is sometimes called the Strauss Lemma; it follows from the Cauchy-Schwarz inequality), so we have a pointwise estimate

$$|g|^2 \leq (1 + \frac{2}{c}) R^{-3}\|\nabla g\|_{L^2}^2 + (1 + \frac{c}{2})|\Psi_R g|^2.$$

Now we notice that

$$\int_{|x| \leq R} f'(W) \, dx \sim R,$$

so for any $\delta > 0$ the first term above gives a small contribution to the quadratic form for $R$ large. Similarly,

$$|\langle g - \Psi_R g, \mathcal{Y} \rangle| \lesssim R^{-\frac{4}{5}}\|\nabla g\|_{L^2} + \int_{|x| \geq R} |g| \mathcal{Y} \, dx \leq \left( R^{-\frac{4}{5}} + \|\mathcal{Y}\|_{L^{10/7}(|x| \geq R)} \right)\|\nabla g\|_{L^2},$$

which is small when $R$ is large. As $|\langle \Psi_R g, \mathcal{Y} \rangle|^2 \leq 2\langle g, \mathcal{Y} \rangle^2 + 2(g - \Psi_R g, \mathcal{Y})^2$, the proof is finished.

We are ready to state coercivity properties of the functional $H$ from the previous section.

**Proposition 6.3.** Under the assumptions of Proposition 5.2 there exist $T_0, c_H, \alpha_0, C_2 > 0$ such that for $t \in (0, T_0] \cap (T_-, T_+)$ there holds

\[
(6.8) \quad |\alpha_\lambda(\varepsilon_0)| \leq \alpha_0 \parallel \varepsilon_0 \parallel_{H^1} \quad \Rightarrow \quad H(t) \geq c_H \parallel \varepsilon \parallel^2_{H^1 \times L^2}.
\]

If $[T_1, T_2] \subset (0, T_0] \cap (T_-, T_+)$ and $|\alpha_\lambda(\varepsilon_0)| \leq \frac{1}{c_0} t^{\gamma + 1}$ for all $t \in [T_1, T_2]$, then

\[
(6.9) \quad H(T_2) \leq H(T_1) + \frac{c_H}{10} \int_{T_1}^{T_2} \frac{1}{t} \parallel \varepsilon \parallel^2_{H^1 \times L^2} \, dt + C_2 t^{2\gamma + 2}.
\]

The constants $\frac{c_H}{10}$ and $\frac{1}{c_0}$ have no special signification, but this formulation will be convenient later.
Proof. Let
\[ I_L(t) := \int \frac{1}{2} \| \varepsilon_1 \|^2 + \frac{1}{2} \nabla \varepsilon_0 \|^2 - \frac{1}{2} f'(W_\lambda) \varepsilon_0^2 \, dx. \]
Recall that \( \langle \varepsilon_0, Z \rangle = 0 \). Lemma 5.1 implies (after rescaling) that if we take \( \alpha_0 \) small enough, then there exists a constant \( c > 0 \) such that \( I_L(t) \geq c \| \varepsilon \|^2_{H^1 \times L^2} \).

We can assume that \( \| u^* \|_{X^1 \times H^1} \) is as small as we like, so by pointwise estimates we get \( |I(t) - I_L(t)| \leq \frac{1}{3} c \| \varepsilon \|^2_{H^1 \times L^2} \). Moreover, it is clear from the definition of \( J \) that for small \( t \) we have \( |J(t)| \leq \frac{1}{3} c \| \varepsilon \|^2_{H^1 \times L^2} \). This proves the result with \( c_H = \frac{1}{4} c \).

In order to prove (6.9), notice first that, by pointwise estimates and smallness of \( \| \varphi_0 - W_\lambda \|_{H^1} \), in (5.5) we can replace \( \int (f(\varphi_0 + \varepsilon_0) - f(\varphi_0)) \varepsilon_0 \, dx \) by \( \int f'(W_\lambda) \varepsilon_0^2 \, dx \). Convexity of \( a \) (see Lemma 5.1) implies that
\[ \| \nabla \alpha \lambda \varepsilon \|^2 \geq \int_{|x| \leq R_\lambda} \| \nabla \varepsilon_0 \|^2 \, dx, \]
so from (5.5) and Lemma 6.2 (after rescaling) we obtain
\[ H(T_2) \leq H(T_1) + \int_{T_1}^{T_2} \frac{c_H}{2t} \| \varepsilon \|^2_{H^1 \times L^2} + C_4 t^\gamma \cdot \| \varepsilon \|^2_{H^1 \times L^2} + C \| \varepsilon_0 \|^2 \, dt \leq H(T_1) + \frac{c_H}{10} \int_{T_1}^{T_2} \frac{1}{t} \| \varepsilon \|^2_{H^1 \times L^2} \, dt + C_2 t^{2\gamma + 2}, \]
where \( C_2 \) is a constant.

In order to close the bootstrap, it is necessary to control the stable and unstable directions. More precisely, it is necessary to eliminate the unstable mode.

Define
\[ \alpha^- : = \int \mathcal{Y}_\lambda \varepsilon_1 - \frac{c_0}{\lambda} \mathcal{Y}_\lambda \varepsilon_0 \, dx \]
and
\[ \alpha^+ : = \int \mathcal{Y}_\lambda \varepsilon_1 + \frac{c_0}{\lambda} \mathcal{Y}_\lambda \varepsilon_0 \, dx. \]
Notice that \( -\frac{c_0^2}{\lambda^2} \) is the unique strictly negative eigenvalue of \( L_\lambda \).

We will define an auxiliary function \( l(t) \) which measures the distance of the modulation parameters from the approximate trajectory (2.12) or (2.13). This function has a slightly different form in the non-degenerate and degenerate cases. In the non-degenerate case we define
\[ l(t) = \frac{1}{2} \left( \frac{b}{t^3} + \frac{2\lambda}{t^4} \right) + \frac{\kappa^2 u^*(0,0)^2}{24} + \frac{1}{2} \left( \frac{b}{t^3} - \frac{3\lambda}{t^4} - \frac{\kappa^2 u^*(0,0)^2}{144} \right)^2, \]
and in the degenerate case
\[ l(t) = \frac{1}{2} \left( \frac{b}{t^3} + \frac{\nu}{t^4} \right) + \frac{\lambda}{t^4 + 1} - (\nu + \nu + 1)^2 + \frac{1}{2} \left( \frac{b}{t^3} - \frac{\lambda}{t^4} \right) \left( \nu + 1 \right) - (\nu - \nu)^2, \]
where \( \nu := -\frac{1}{2} + \frac{1}{2} \sqrt{\nu^2 + (\nu + 1)^2} \).

We will write \( \alpha^+(t) \) and \( \alpha^-(t) \) instead of \( \alpha^-_\lambda(\varepsilon) \) and \( \alpha^+_\lambda(\varepsilon) \). In the next few propositions we describe the evolution of Mod(t) := (\lambda(t), b(t), \alpha^-(t), \alpha^+(t)) in the “modulation cylinder” defined as:
\[ \mathcal{C}(t) := \{ (\lambda, b, \alpha^-, \alpha^+) : \ l(t) \leq t^{\gamma + 1 - \nu} \ and \ -t^{\gamma + 1} \leq \alpha^- + \alpha^+ \leq t^{\gamma + 1} \}. \]

In the non-degenerate case we denote
\[ \lambda_{app}(t) = \frac{\kappa^2 u^*(0,0)^2}{144} t^4, \quad b_{app}(t) = \frac{\kappa^2 u^*(0,0)^2}{36} t^3, \]
and in the degenerate case
\[ \lambda_{app}(t) := t^{\nu+1}, \quad b_{app}(t) := (\nu + 1)t^\nu. \]

Solving a 2 × 2 linear system we check easily that
\[
(6.13) \quad l(t) \leq t^{\gamma+1-\nu} \Rightarrow |\frac{\lambda}{\lambda_{app}} - 1| \leq t^{\frac{1}{2}(\gamma+1-\nu)}, \quad |\frac{b}{b_{app}} - 1| \leq t^{\frac{1}{2}(\gamma+1-\nu)},
\]
with constants which depend only on \( \nu \).

We have \( a_{\lambda}(\varepsilon_0) = \frac{1}{2\varepsilon_0}(\alpha^+ - \alpha^-) \), so
\[
(6.14) \quad \text{Mod}(t) \in \mathcal{C}(t) \Rightarrow |a_{\lambda}(\varepsilon_0)| \leq \frac{1}{c_0}l^{\gamma+1}.
\]

**Remark 6.4.** The formula for \( l \) is found by linearizing the parameter equations near \( (\lambda_{app}, b_{app}) \) and diagonalizing the resulting system.

We can finally state a result on uniform in time energy bounds.

**Proposition 6.5.** Let \( C_4 > 0 \) be a fixed constant. There exist \( C_0 > 0 \) and \( T_0 > 0 \) having the following property. Let \( 0 < T_1 < T_0 \). Suppose that
\[
(6.15) \quad \| \varepsilon(T_1) \|_{H^1 \times L^2} \leq C_4 T_1^{\gamma+1}, \quad \text{Mod}(T_1) \in \text{Int}(\mathcal{C}(T_1)).
\]

Then, either there exists a time \( t, T_1 \leq t \leq T_0 \), such that \( \text{Mod}(t) \notin \partial \mathcal{C}(t) \), or the solution exists on \([T_1, T_0]\) and for all \( t \in [T_1, T_0] \) there holds
\[
(6.16) \quad \| \varepsilon \|_{H^1 \times L^2} \leq C_0 t^{\gamma+1}.
\]

**Proof.** Let \( T_0 \) be the time provided by Proposition 6.3. Let \( T_3 \) be the maximal time of existence of the solution and let \( T_2 := \min(T_0, T_3) \). Suppose that \( \text{Mod}(t) \notin \partial \mathcal{C}(t) \) for \( T_1 \leq t \leq T_2 \). By continuity of \( \text{Mod}(t) \) this means that \( \text{Mod}(t) \in \text{Int}(\mathcal{C}(t)) \) for \( T_1 \leq t \leq T_2 \). We will show first that if \( C_0 \) is large enough, then (6.16) holds for \( t \in [T_1, T_2] \). Argue by contradiction, assuming that there exists \( T_3 < T_2 \) such that \( \| \varepsilon(T_3) \|_{H^1 \times L^2} = C_0 T_3^{\gamma+1} \). At \( t = T_3 \) (6.12) gives \( |a_{\lambda}(\varepsilon_0)| \leq \frac{1}{c_0}l^{\gamma+1} \). In particular, if \( C_0 \) is large, we will have \( |a_{\lambda}(\varepsilon_0)| \leq C_0 \| \varepsilon_0 \|_{H^1} \), so by Proposition 6.3 we obtain
\[
(6.17) \quad H(T_3) \geq c_H C_0^2 T_3^{2\gamma+2}.
\]

On the other hand, for \( t \in [T_1, T_3] \) we have \( \| \varepsilon \|_{H^1 \times L^2}^2 \leq C_0^2 t^{2\gamma+2} \) and \( |a_{\lambda}(\varepsilon_0)| \leq \frac{1}{c_0}l^{\gamma+1} \), so from (6.9) we deduce that
\[
H(T_3) \leq H(T_1) + \frac{c_H C_0^2}{10(2\gamma + 2)} T_3^{2\gamma+2} + C_2 T_3^{2\gamma+2}.
\]

Notice that \( H(T_1) \leq \| \varepsilon(T_1) \|_{H^1 \times L^2}^2 \leq C_4^2 T_1^{2\gamma+2} \leq C_4^2 T_3^{2\gamma+2} \). Returning to (6.17) we deduce
\[
c_H C_0^2 \leq C_4^2 + \frac{c_H C_0^2}{10(2\gamma + 2)} + C_2,
\]
which is impossible if \( C_0 \) is large enough.

Hence, \( T_3 = T_2 \). To prove that \( T_2 = T_0 \), notice that by the Cauchy theory in the critical space there exists \( \delta > 0 \) such that
\[
\|(u_0 - W, u_1)\|_{H^1 \times L^2} \leq \delta \Rightarrow \text{the solution } u(t) \text{ with } u(0) = (u_0, u_1) \text{ exists at least for } t \in (-1, 1).
\]
Lemma 6.7. Suppose, for the sake of contradiction, that for example
\[(6.19)\]

\[\|u(t) - W_\lambda, u_1\|_{\mathcal{H}^1 \times L^2} \leq \delta \Rightarrow \text{the solution } u(t) \text{ with } u(0) = (u_0, u_1) \]
exists at least for \( t \in (-\lambda, \lambda) \).

\[\text{If } \|u\|_{\mathcal{H}^1 \times L^2} \text{ is sufficiently small and } T_0 \text{ is chosen sufficiently small, \[3.28\] and \[3.29\] show that our solution verifies the sufficient condition in \[6.18\] for any } t < T_2 \text{ with } \lambda = \lambda(t). \text{ Taking } t \text{ close to } T_2 \text{ we obtain that the solution cannot blow up at } T_2, \text{ hence } T_2 = T_0. \]

The crucial element of the preceding result is that the constant \( C_0 \) is independent of \( T_1 \). From now, \( C_0 \) has a fixed value given by Proposition\[6.5\] and the constants which appear later are allowed to depend on \( C_0 \). In particular, when we use the notation \( \lesssim \) or \( O \), the constant may depend on \( C_0 \).

We examine now the evolution of the eigenvectors \( \alpha^- \) and \( \alpha^+ \).

**Lemma 6.6.** If \( \|\varepsilon\|_{\mathcal{H}^1 \times L^2} \lesssim t^{\gamma+1} \), \( \lambda \sim t^{\mu+1} \) and \( b \sim t^{\nu} \), then
\[\|\varepsilon\|_{\mathcal{H}^1 \times L^2} \lesssim t^{\gamma+1} \]

\[(6.19)\]

\[\frac{d}{dt} \alpha_\lambda^+ \approx (\lambda_0' \cdot (-L_\lambda \varepsilon_0) + \frac{\varepsilon_0}{\lambda} \lambda_\lambda \varepsilon_1) \lesssim t^{\gamma}, \]

\[(6.20)\]

\[\frac{d}{dt} \alpha_\lambda^- \approx (\lambda_0' \cdot \varepsilon_0 + \frac{\varepsilon_0}{\lambda} \lambda_\lambda \varepsilon_1) \lesssim t^{\gamma}. \]

**Proof.** We will do the computation for \(6.19\), because the one for \(6.20\) is exactly the same.

\[\frac{d}{dt} \alpha_\lambda^+ = \int (\lambda_\lambda' \cdot (-L_\lambda \varepsilon_0) + \frac{\varepsilon_0}{\lambda} \lambda_\lambda \varepsilon_1) \, dx\]

\[+ \int \frac{-\lambda_0' \cdot (\lambda_0' \lambda_\lambda \varepsilon_0 + \frac{\varepsilon_0}{\lambda} \lambda_\lambda \varepsilon_0) \, dx\]

\[+ \int \lambda_\lambda' \cdot (f(\varphi_0 + \varepsilon_0) - f(\varphi_0) - f'(\varphi) \varepsilon_0) \, dx\]

\[+ \int \lambda_\lambda' \cdot (f'(\varphi_0) - f'(W_\lambda)) \varepsilon_0 \, dx\]

\[+ \int \lambda_\lambda' \cdot (-\psi_0) + \frac{\varepsilon_0}{\lambda} \lambda_\lambda' \cdot (-\psi_0) \, dx. \]

The first line is \( \frac{\varepsilon_0}{\lambda} \lambda_\lambda \varepsilon_1 \) and it suffices to estimate the remaining ones. For the last line we use Proposition\[3.14\] and \( L^2 \)-orthogonality of \( \Lambda \) and \( \lambda \). Using \( \lambda_t \sim t^{\nu}, \lambda \sim t^{\mu+1} \) and \( \|\varepsilon\|_{\mathcal{H}^1 \times L^2} \lesssim C t^{\gamma+1} \), the second line is seen to be bounded by \( C t^{\gamma} \). The proof of \[3.28\] shows that \( \|\lambda_\lambda' \cdot (f'(\varphi) - f'(W_\lambda)) \|_{L^1 / \lambda} \lesssim \frac{C t^{\gamma + 1}}{t^{\gamma+1}} \), so using \( \|\varepsilon_0\|_{L^1 / \lambda} \lesssim t^{\gamma+1} \) we obtain the required bound for the fourth line. Finally, \( \|f(\varphi_0 + \varepsilon_0) - f(\varphi_0) - f'(\varphi) \varepsilon_0\|_{\mathcal{H}^1} \lesssim C t^{2\gamma+2} \) and \( \|\lambda_\lambda'\|_{\mathcal{H}^1} \lesssim \frac{C}{t^{\nu+1}} \sim t^{-\nu-1} \), so by Cauchy-Schwarz the third line is bounded by \( C t^{2\gamma-\nu+1} \lesssim t^{\gamma} \).

We know from Proposition\[6.5\] that if we start at \( t = T_1 \) with \( \varepsilon \) small enough, then \( \varepsilon \) is controlled in \( \mathcal{H}^1 \times L^2 \) unless \( \text{Mod} \) leaves the cylinder \( \mathcal{C} \). It turns out that this can happen only because of \( \alpha^+ \). The other parameters are trapped in the cylinder for small times:

**Lemma 6.7.** Under the assumptions of Proposition\[6.3\], suppose that \( \text{Mod}(t) \) leaves \( \text{Int}(\mathcal{C}(t)) \) before \( t = T_0 \). If \( T_0 \leq T_0 \) is the first time for which \( \text{Mod}(T_2) \in \partial \mathcal{C}(T_2) \), then \( |\alpha^+(T_2)| = T_2^{\gamma+1} \). In addition, suppose that at time \( T_3, T_1 \leq T_3 < T_2 \), we have \( \alpha^+(T_3) > \frac{1}{2} T_3^{\gamma+1} \). Then \( \alpha^+(T_2) = T_2^{\gamma+1} \). Analogously, if \( \alpha^+(T_3) < -\frac{1}{2} T_3^{\gamma+1} \), then \( \alpha^+(T_2) = -T_2^{\gamma+1} \).

**Proof.** Suppose, for the sake of contradiction, that for example \( l(T_2) = T_2^{\gamma+1} \). In particular this implies \( \frac{d}{dt} l(t_1) \geq 0 \), and we will show that it is impossible.
We start with the degenerate case. Using \( \text{(6.13)} \) and \( \sqrt{x} = \frac{1 + x}{2} + O(|1 - x|^2) \) we obtain
\[
\sqrt{\lambda} = \frac{1}{2}(t^{\frac{\nu}{2}(\nu + 1)} + \lambda t^{-\frac{1}{2}(\nu + 1)}) + O(t^{\gamma + \frac{1}{2} - \frac{1}{2} \nu}).
\]
Recall that \( b_t = (\nu + 1)\nu t^{\frac{1}{2}(\nu - 3)} \sqrt{\lambda} \), so we get
\[
(6.21) \quad \frac{b_t}{t^{\nu - 1}} = \frac{(\nu + 1)\nu}{2}(1 + \frac{\lambda}{t^{\nu + 1}}) + O(t^{\gamma + 1 - \nu}).
\]
From Lemma 3.7 and \( \text{(6.13)} \) we have
\[
(6.22) \quad \frac{\lambda t}{t^{\nu}} = \frac{b_t}{t^{\nu}} + O(t^{\gamma + 1 - \nu}).
\]
Using \( \text{(6.21)} \) and \( \text{(6.22)} \) we can compute \( \frac{d}{dt} l(t) \):
\[
\frac{d}{dt} l(t) = \left( \frac{b_t}{t^{\nu}} + \nu \frac{\lambda}{t^{\nu + 1}} - (\nu + \nu + 1) \right) \left( \frac{b_t}{t^{\nu}} + \nu \frac{\lambda}{t^{\nu + 1}} - \nu \frac{b_t}{t^{\nu + 1}} - \nu (\nu + 1) \frac{\lambda}{t^{\nu + 2}} \right) + \left( \frac{b_t}{t^{\nu}} - (\nu + 1) \frac{\lambda}{t^{\nu + 1}} - (\nu - \nu) \right) \left( \frac{b_t}{t^{\nu}} - (\nu + 1) \frac{\lambda}{t^{\nu + 1}} - \nu \frac{b_t}{t^{\nu + 1}} - (\nu + 1)(\nu + 1) \frac{\lambda}{t^{\nu + 2}} \right) + \frac{1}{t} O(\sqrt{l(t)} t^{\gamma + 1 - \nu}).
\]
If we use the definition of \( \nu \), this simplifies to
\[
\frac{d}{dt} l(t) = - (\nu - \nu) \left( \frac{b_t}{t^{\nu}} + \nu \frac{\lambda}{t^{\nu + 1}} - (\nu + \nu + 1) \right)^2 - (\nu + \nu + 1) \left( \frac{b_t}{t^{\nu}} - (\nu + 1) \frac{\lambda}{t^{\nu + 1}} - (\nu - \nu) \right)^2 + \frac{1}{t} O(\sqrt{l(t)} t^{\gamma + 1 - \nu}) = \frac{1}{t} (- (\nu - \nu) l(t) + O(t^{\frac{1}{2}(\gamma + 1 - \nu)})).
\]
At time \( t = T_2 \) by assumption \( l(T_2) = T_2^{1 - \nu} \), so for \( T_2 \) small enough the formula above implies \( \frac{d}{dt} l(T_2) < 0 \), which is impossible.

In the non-degenerate case the computation is similar, but we must take into account that in this case
\[
b_t = \kappa u^*(t, 0) \sqrt{\lambda} = \kappa u^*(0, 0) \sqrt{\lambda}(1 + O(t)),
\]
which leads to
\[
b_t = \kappa u^*(0, 0) \cdot \frac{1}{2} \left( \frac{\kappa u^*(0, 0)}{12} t^2 + \frac{12 \kappa u^*(0, 0)}{\lambda t^{-2}} + O(t^3) \right).
\]
Then, the computation is the same as before:
\[
\frac{d}{dt} l(t) = \left( \frac{b_t}{t^{\nu}} + \frac{2 \lambda t}{t^{\nu}} - \frac{\kappa^2 u^*(0, 0)^2}{24} \right) \left( \frac{b_t}{t^{\nu}} + \frac{2 \lambda t}{t^{\nu}} - \frac{3 b_t}{t^4} - \frac{8 \lambda}{t^3} \right) + \left( \frac{b_t}{t^{\nu}} + \frac{2 \lambda t}{t^{\nu}} - \frac{\kappa^2 u^*(0, 0)^2}{24} \right) \left( \frac{b_t}{t^{\nu}} + \frac{2 \lambda t}{t^4} - \frac{3 b_t}{t^4} - \frac{8 \lambda}{t^3} \right) \leq \frac{1}{t} \left( \frac{b_t}{t^{\nu}} + \frac{2 \lambda t}{t^{-2}} \right) + \frac{1}{t} \left( \frac{b_t}{t^{\nu}} + \frac{3 \lambda t}{t^4} \right) \leq \frac{1}{t} \left( 2 l(t) + O(t^{7/4}) \right).
\]
Since \( \gamma + 1 - \nu = \frac{3}{2} - \frac{7}{4} < 0 \), we are done.
Now suppose that \( |\alpha^{-}(T_2)| = T_2^{\gamma+1} \). As \( t^{1+|\gamma|} \sim t^{\gamma+\nu} \gg t^\gamma \), (6.20) implies that \( \frac{d}{dt} \alpha_\nu^- \) and \( \alpha_\nu^- \) have opposite signs, which is impossible.

Again by contradiction, suppose that \( \alpha_\nu^+(T_3) > \frac{1}{2}T_3^{\gamma+1} \) and \( \alpha_\nu^+(T_2) = -T_2^{\gamma+1} \). By continuity, there exists the smallest \( T > T_3 \) such that \( \alpha_\nu^+(T_4) = \frac{1}{2}T_4^{\gamma+1} \). Necessarily \( \frac{d}{dt} \alpha_\nu^+(T_4) \leq \frac{2\gamma+1}{2}T_4^{\gamma+1} \), which is in contradiction with (6.19).

**Proposition 6.8.** There exist strictly positive constants \( C_0 \) and \( T_0 \) such that for all \( T_1 \in (0, T_0) \) there exists a solution \( u \) defined on \([T_1, T_0]\) which for all \( t \in [T_1, T_0] \) verifies
\[
(6.23) \quad \|(u - W_\lambda - u^*, \partial_t u + \lambda_t(\Lambda W)_\Delta - \partial_t u^*)\|_{H^1 \times L^2} \leq C_0 t^{\gamma+1},
\]
\[
(6.24) \quad \left| \frac{\lambda}{\lambda_{\text{app}}} - 1 \right| \leq C_0 t^{\frac{1}{2} (\gamma + 1 - \nu)}.
\]

**Proof.** We consider the degenerate case. The proof in the non-degenerate case is similar.

Let \( \lambda = \lambda_{\text{app}}(T_1), b = b_{\text{app}}(T_1) \). For \( a \in [-\frac{2}{3} T_1^{\gamma+1}, \frac{2}{3} T_1^{\gamma+1}] \), let \( \varepsilon_a(T_1) = a \frac{(\Lambda^+ - \Omega_{\text{app}})(Z_\Delta, 0)}{(\gamma_{\text{app}} - \frac{2}{3})(\gamma_{\text{app}} - \frac{2}{3})} \), and consider the corresponding evolution. Of course (6.15) is verified for a universal constant \( C_4 \).

Let \( C_0 \) be the constant provided by Proposition 6.5. We will show that there exists a parameter \( a \) for which the solution exists until \( t = T_0 \) and satisfies (6.16). Suppose this is not the case. Let \( A^+ = \{ a : \alpha_\nu^+(T_2) = T_2^{\gamma+1} \} \) and \( A^- = \{ a : \alpha_\nu^+(T_2) = -T_2^{\gamma+1} \} \), where \( T_2 \) is the exit time given by Lemma 6.7. By the second part of Lemma 6.7, we know that \( -\frac{2}{3} T_1^{\gamma+1} \in A^- , \ \frac{2}{3} T_1^{\gamma+1} \in A^+ \), and that \( A^- \) and \( A^+ \) are open sets. Indeed, let \( a \in A^+ \). This means in particular that for \( T_1 \leq t \leq T_2 \) we have \( \alpha_\nu^+(t) \geq -\frac{1}{2}t^{\gamma+1} \) and \( \alpha_\nu^+(T_2) = T_2^{\gamma+1} \). By continuity of the flow, for close enough initial data we will still have \( \alpha_\nu^+(t) > -t^{\gamma+1} \) for \( T_1 \leq t \leq T_2 \) and \( \alpha_\nu^+(T_2) \geq \frac{1}{2} T_2^{\gamma+1} \). By Lemma 6.7 the corresponding solutions escape from the cylinder by positive values of \( \alpha^+ \). Thus \( A^+ \cup A^- \) would be a partition of \( [-\frac{2}{3} T_1^{\gamma+1}, \frac{2}{3} T_1^{\gamma+1}] \) into two disjoint open sets, which is impossible.

Using (6.16), (4.2), (4.4) and (3.3) we obtain (6.23).

Estimate (6.24) follows from (6.13) and the fact that \( \text{Mod}(t) \in \mathcal{C}(t) \). \( \square \)

**Proof of Theorem 1.1.** Let \( t_n \) be a decreasing sequence such that \( t_n > 0 \) and \( t_n \to 0 \). Let \( u_n \) be the solution given by Proposition 6.8 for \( T_1 = t_n \) and let \( \lambda_n : [t_n, T_0] \to (0, +\infty) \) be the corresponding modulation parameter. The sequence \( u_n(T_0) \) is bounded in \( H^1 \times L^2 \). After extracting a subsequence, it converges weakly to some function \( (u_0, u_1) \). Let \( u(t) \) be the solution of (NL W) for the Cauchy data \( u(T_0) = (u_0, u_1) \). We will show that \( u \) satisfies (1.2).

Let \( 0 < T_1 < T_0 \) and \( T_1 \leq t \leq T_0 \). Using (6.23), (6.24) and \( |\lambda_t| \lesssim t^3 \) we get
\[
\|(u_n - W_{\lambda_{\text{app}}} - u^*, \partial_t u_n - \partial_t u^*)\|_{H^1 \times L^2} \leq C_0 t^2.
\]
This shows that if \( T_0 \) is sufficiently small, then the sequence \( u_n \) satisfies the conditions of Proposition A.1 on the time interval \([T_1, T_0]\), hence
\[
u(u_n(T_1) \rightharpoonup u(T_1)).
\]

Weak lower semi-continuity of the norm implies that at time \( t = T_1 \) we have
\[
\|(u - W_{\lambda_{\text{app}}} - u^*, \partial_t u - \partial_t u^*)\|_{H^1 \times L^2} \leq C_0 T_1^{3/4}.
\]
This bound holds for all \( T_1 \) such that \( 0 < T_1 < T_0 \). In particular, the orthogonality condition:
\[
(6.25) \quad \langle u - W_\lambda - u^*, Z_\Delta \rangle = 0.
\]
defines uniquely a continuous function \( \lambda(T_1) : (0, T_0) \to (0, +\infty) \). We will prove that \( \lambda_n(T_1) \to \lambda(T_1) \).

Using (3.10) for the solution \( u_n \) at time \( T_1 \) and passing to a limit \( n \to \infty \) we obtain that all the accumulation points of \( \lambda_n(T_1) \) verify the orthogonality condition (6.25). Hence \( \lambda_n(T_1) \to \lambda(T_1) \).
Passing to a limit in (3.11) we get \( \frac{d}{dt} \lambda_n(T_1) \to \frac{d}{dt} \lambda(T_1) \). Passing to a limit in (6.23) and (6.24) finishes the proof.

The proof of Theorem 1.2 follows the same lines, so we will skip it.

APPENDIX A. WEAK CONTINUITY OF THE FLOW NEAR A FIXED PATH

**Proposition A.1.** Let \( v : [0, 1] \to \dot{H}^1 \times L^2 \) be a continuous path in the energy space. There exist a constant \( \delta > 0 \) with the following property. Let \( u_n \) be a sequence of radial solutions of (NLW) defined on the interval \([0, 1]\), such that

\[
\sup_{t \in [0, 1]} \| u_n - v \|_{\dot{H}^1 \times L^2} \leq \delta.
\]

Suppose that \( u_n(0) \to (u_0, u_1) \) in \( \dot{H}^1 \times L^2 \) and let \( u \) be the solution of (NLW) for the initial data \( u(0) = (u_0, u_1) \). Then \( u \) is defined on \([0, 1]\) and for all \( t \in [0, 1] \) we have

\[
u_n(t) \to u(t) \quad \text{in } \dot{H}^1 \times L^2.
\]

**Remark A.2.** Notice that without the assumption (A.1) the result is false. More generally, existence of type II blow-up solutions in some space excludes weak continuity of the flow in this space, and existence of type II blowup solutions in our case follows from Theorems 1.1 and 1.2. One might search weaker conditions than (A.1); we have chosen a simple condition which is sufficient for our needs.

**Proof.**

**Step 1.** Suppose that \( u \) is not defined on \([0, 1]\) and let \( T_+ \leq 1 \) be its final time of existence. In Step 2, we will prove (A.2) for all \( t < T_+ \). In particular, by the lower weak semi-continuity of the norm, this shows that

\[
\sup_{t \in (0, T_+)} \| u_n - v \|_{\dot{H}^1 \times L^2} \leq \delta.
\]

By local well-posedness in the energy space and compactness of \( \{ v(t) : t \in [0, 1] \} \), if \( \delta > 0 \) is small enough, there exists \( \tau > 0 \) such that the solution corresponding to the initial data \( u_n(t) \) is defined at least on the interval \( (-\tau, \tau) \). This means that \( u \) cannot blow up at \( T_+ \), and so it is defined for \( t \in [0, 1] \).

If \( \delta \) is chosen small enough, depending on \( v(1) \), then by the Cauchy theory the solutions \( u_n \) exist on an interval \((1 - t', 1 + t')\) for some \( t' > 0 \). By eventually choosing \( t' \) smaller, we can assume that \( u \) also exists on \((1 - t', 1 + t')\). Hence, by repeating the same procedure we obtain weak convergence also for \( t = 1 \).

**Step 2.** Let \( t < T_+ \). In order to prove (A.2), it is sufficient to show that any subsequence of \( u_n \) (which we will still denote \( u_n \)) admits a subsequence such that the required convergence takes place. By the result of Bahouri-Gérard a subsequence of \( u_n(0) \) admits a profile decomposition such that the first profile is \( U_n^1(t) = S(t)(u_0, u_1) \) (corresponding to parameters \( t_{1,n} = 0, \lambda_{1,n} = 1 \)). By the triangle inequality \( \| u_n - (u_0, u_1) \|_{\dot{H}^1 \times L^2} \leq 2\delta \), so all the other profiles are small, in particular they are global and disperse. By definition of \( T_+ \) the assumptions of Proposition 2.8 in [6] (which is a version of [2] Main Theorem for the focusing equation) are satisfied for \( \theta_n = t \), in particular formula (2.22) from [6] yields:

\[
u_n(t) = u(t) + \sum_{j=2}^J U_n^j(t) + w_n^J(t) + r_n^J(t).
\]

Here, \( w_n^J(t) = S(t)w_n^J(0) \to 0 \) as \( n \to +\infty \) (indeed, \( w_n^J(0) \to 0 \) for \( J > 1 \) by definition of the profiles, and \( S(t) \) is a bounded linear operator). By Lemma A.3 below also \( U_n^J(t) \to 0 \) when \( J > 1 \), which finishes the proof.

\[\square\]
Lemma A.3. Let $U$ be a solution of equation (NLW) such that $\|U\|_{H^1 \times L^2}$ is small. Let $t_n, \lambda_n$ be a sequence of parameters such that one of the following holds:

1. $t_n = 0$ and $\lambda_n \to 0$,
2. $t_n = 0$ and $\lambda_n \to +\infty$,
3. $t_n/\lambda_n \to +\infty$,
4. $t_n/\lambda_n \to -\infty$.

Fix $t \in \mathbb{R}$ and define

$$U_n(x) = \left( \frac{1}{\lambda_n^{3/2}} U^j \left( \frac{t - t_n}{\lambda_n}, \frac{x}{\lambda_n} \right), \frac{1}{\lambda_n^{5/2}} \partial_t U^j \left( \frac{t - t_n}{\lambda_n} \right) \right).$$

Then $U_n \to 0$ in $\dot{H}^1 \times L^2$.

Proof. Again it is sufficient to show this for a subsequence of any subsequence. Thus we can assume that $\frac{t - t_n}{\lambda_n} \to t_0 \in [-\infty, +\infty]$.

Suppose first that $t_0$ is a finite number. Extracting again a subsequence we can assume that $\lambda_n \to \lambda_0 \in [0, +\infty[$. If $\lambda_0$ was a strictly positive finite number, we would obtain that also $t_n$ has a finite limit, which is impossible. Thus $\lambda_n \to 0$ or $\lambda_n \to +\infty$, and in both cases we get our conclusion.

In the case $\frac{t - t_n}{\lambda_n} \to \pm \infty$ we have dispersion, so $\|U_n - (S(\tau_n)V)_{\lambda_n}\|_{H^1 \times L^2} \to 0$, and it is well known that $(S(\tau_n)V)_{\lambda_n} \to 0$ when $\tau_n \to \pm \infty$ and $\lambda_n$ is any sequence (in the case of space dimension $N = 5$ this follows for example from the strong Huyghens principle).

APPENDIX B. LOCAL THEORY IN HIGHER REGULARITY

In this section we use the energy method to prove two results about preservation of regularity.

B.1. Energy estimates in $X^1 \times H^1$. Recall that we denote $X^s := \dot{H}^{s+1} \cap \dot{H}^1$. We have classical energy estimates for the linear wave equation:

Lemma B.1. Let $s \in \mathbb{N}$. Let $I = [0, T_0]$ be a time interval, $g \in C(I, H^s)$ and $(u_0, u_1) \in X^s \times H^s$. Then the Cauchy problem

$$\begin{cases}
\partial_t u - \Delta u = g, \\
(u(0), \partial_t u(0)) = (u_0, u_1)
\end{cases}$$

has a unique solution $(u, \partial_t u) \in C(I, X^s \times H^s)$ and for all $t \in I$ there holds

$$\|(u, \partial_t u)\|_{X^s \times H^s} \leq \|(u_0, u_1)\|_{X^s \times H^s} + \int_0^t \|g(\tau)\|_{H^s} \, d\tau. \quad (B.1)$$

For a proof of a more general result one can consult for example [1, Theorem 4.4]. Using finite speed of propagation and Sobolev Extension Theorem on each time slice we get a localised version of the energy estimate:

$$\|(u, \partial_t u)\|_{X^s \times H^s(B(0, \rho))} \lesssim \|(u_0, u_1)\|_{X^s \times H^s(B(0, \rho+t))} + \int_0^t \|g(\tau)\|_{H^s(B(0, \rho+t))} \, d\tau \quad (B.2)$$

Now we use the case $s = 1$ to prove energy estimates in $X^1 \times H^1$ for (NLW).

Proposition B.2. For all $M_0 > 0$ there exists $T_0 = T_0(M_0) > 0$ such that the following is true. Let $(u_0, u_1) \in X^1 \times H^1$ with $\|(u_0, u_1)\|_{X^1 \times H^1} \leq M_0$. Then the Cauchy problem:

$$\begin{cases}
\partial_t u - \Delta u = f(u), \\
(u(0), \partial_t u(0)) = (u_0, u_1)
\end{cases}$$

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has a unique solution \((u, \partial_t u) \in C([0, T_0], X^1 \times H^1)\) and this solution verifies

\[
(B.3) \quad \sup_{t \in [0, T_0]} \left\| (u(t), \partial_t u(t)) \right\|_{X^1 \times H^1} \leq 2\left\| (u_0, u_1) \right\|_{X^1 \times H^1}.
\]

Moreover, let \(u_\ell\) denote the solution of the free wave equation for the same initial data \((u_0, u_1)\). Then

\[
(B.4) \quad \sup_{t \in [0, T_0]} \left\| (u(t), \partial_t u(t)) - (u_\ell(t), \partial_t u_\ell(t)) \right\|_{X^1 \times H^1} \lesssim f\left(\left\| (u_0, u_1) \right\|_{X^1 \times H^1}\right).
\]

This will follow easily from the following lemma.

**Lemma B.3.** Let \(u, v \in X^1\). Then

\[
(B.5) \quad \left\| f(u) \right\|_{H^1} \leq C f\left(\left\| u \right\|_{X^1}\right),
\]

\[
(B.6) \quad \left\| f(u) - f(v) \right\|_{H^1} \leq C \left\| u - v \right\|_{X^1} \cdot (f'(\left\| u \right\|_{X^1}) + f'(\left\| v \right\|_{X^1})).
\]

**Proof.** We have \(\| f(u) \|_{L^2} = \| u \|_{L^{7/3}}^{7/3} \lesssim f(\| u \|_{X^1})\) from the Sobolev imbedding. By Hölder inequality,

\[
\| \nabla f(u) \| = \| \nabla u \cdot f'(u) \|_{L^2} \lesssim \| \nabla u \|_{L^{10/3}} \cdot \| f'(u) \|_{L^5} \lesssim \| u \|_{X^1} \cdot \| u \|_{L^{4/3}} \lesssim f(\| u \|_{X^1}),
\]

again by Sobolev imbedding. This proves (B.5).

To prove (B.6), we write \(f(u) - f(v) \lesssim \| u - v \| (f'(u) + f'(v))\), hence

\[
\left\| f(u) - f(v) \right\|_{L^2} \lesssim \| u - v \|_{L^{14/3}} \cdot \| f'(u) + f'(v) \|_{L^{7/2}} \lesssim \| u - v \|_{L^{14/3}} \cdot (\| u \|_{L^{4/3}} + \| v \|_{L^{4/3}})
\]

\[
\lesssim \| u - v \|_{X^1} \cdot (f'(\| u \|_{X^1}) + f'(\| v \|_{X^1})).
\]

Finally,

\[
\| \nabla f(u) - \nabla f(v) \| \lesssim \| \nabla(u - v)(f'(u) + f'(v)) \| + \| u - v \| (\| \nabla u \| + \| \nabla v \|)(|f''(u)| + |f''(v)|),
\]

and it suffices to notice that

\[
\left\| \nabla u - \nabla v \right\| (f'(u) + f'(v)) \|_{L^2} \lesssim \| \nabla u - \nabla v \|_{L^{10/3}} \cdot \| f'(u) + f'(v) \|_{L^5}
\]

\[
\lesssim \| u - v \|_{X^1} \cdot (f'(\| u \|_{X^1}) + f'(\| v \|_{X^1})),
\]

and

\[
\left\| u - v \right\|_{L^{10}} \cdot (\| \nabla u \|_{L^{10/3}} + \| \nabla v \|_{L^{10/3}}) \cdot (\| f''(u) \|_{L^{10}} + \| f''(v) \|_{L^{10}})
\]

\[
\lesssim \| u - v \|_{X^1} \cdot (f'(\| u \|_{X^1}) + f'(\| v \|_{X^1})).
\]


**Proof of Proposition B.2.** Let \(B\) denote the ball of centre 0 and radius \(2\| (u_0, u_1) \|_{X^1 \times H^1}\) in the space \(X^1 \times H^1\). Given \((u, \partial_t u) \in C([0, T], B)\), let \(\bar{u} = \Phi(u)\) denote the solution of the Cauchy problem

\[
\begin{align*}
\partial_t \bar{u} - \Delta \bar{u} &= f(u), \\
(\bar{u}(0), \partial_t \bar{u}(0)) &= (u_0, u_1)
\end{align*}
\]

It follows from Lemma (B.5) and (B.1) that if \(T \leq \frac{M}{C f(T_0)}\), then \((\bar{u}, \partial_t \bar{u}) \in C([0, T], B)\). It follows from (B.6) and (B.1) that if \(T \leq \frac{M}{4C f(T_0)}\), then \(\Phi\) is a contraction, so it has a unique fixed point, which is the desired solution.

The function \(v := u - u_\ell\) solves the Cauchy problem

\[
\begin{align*}
\partial_t v - \Delta v &= f(u), \\
(v(0), \partial_t v(0)) &= 0,
\end{align*}
\]

so (B.4) follows from (B.1). \(\square\)
B.2. Persistence of $X^1 \times H^1$ regularity. We recall the classical Strichartz inequality:

**Lemma B.4.** \[\textbf{[10]}\] Let $I = [0, T_0]$ be a time interval, $g \in C(I, L^2)$ and $(u_0, u_1) \in \dot{H}^1 \times L^2$. Let $u$ be the solution of the Cauchy problem

\[
\begin{cases}
\partial_t u - \Delta u = g, \\
(u(0), \partial_t u(0)) = (u_0, u_1).
\end{cases}
\]

Then

\[
\|u\|_{L^{7/3}(I; L^{14/3})} \leq \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|g\|_{L^1(I; L^2)},
\]

with a constant independent of $I$.

From the local theory of \([\text{NLW}]\) in the critical space we know that if $u \in C((T_-, T_+); \dot{H}^1 \times L^2)$ is a solution of \([\text{NLW}]\) and $I = [T_1, T_2] \subset (T_-, T_+)$, then

(B.7) \[\|u\|_{L^{7/3}(I; L^{14/3})} < +\infty.\]

**Proposition B.5.** Suppose that $0 \in I = [T_1, T_2] \subset (T_-, T_+)$ and that $(u_0, u_1) \in X^1 \times H^1$. Then $u \in C(I, X^1 \times H^1)$.

**Proof.** The proof is classical, see for example \[\textbf{[4]}\] Chapter 5] for more general results in the case of NLS.

We consider positive times. The proof for negative times is the same. Let $T_*$ be the maximal time of existence of $u$ in $X^1 \times H^1$. Suppose that $T_* < T_+$. From Proposition B.2 it follows that

(B.8) \[\lim_{t \to T_*} \|u\|_{X^1 \times H^1} = +\infty.\]

Consider the time interval $I = [T_* - \tau, T_*]$. Deriving \([\text{NLW}]\) once and using Lemma B.4 we get

(B.9) \[\|\nabla u\|_{L^{7/3}(I; L^{14/3})} \leq C\|(u(T_* - \tau), \partial_t u(T_* - \tau))\|_{X^1 \times H^1} + C\|\nabla (f(u))\|_{L^1(I; L^2)},\]

with $C$ independent of $\tau$. From Hölder inequality we have

\[
\|\nabla (f(u))\|_{L^1(I; L^2)} \leq \|\nabla u\|_{L^{7/3}(I; L^{14/3})} \cdot f'(\|u\|_{L^{7/3}(I; L^{14/3})}).
\]

By (B.7), the last term is arbitrarily small when $\tau \to 0^+$, so for $\tau$ small enough the second term on the right hand side of (B.9) can be absorbed by the left hand side, which implies $\|\nabla u\|_{L^{7/3}(I; L^{14/3})} < +\infty$ and $\|\nabla (f(u))\|_{L^1(I; L^2)} < +\infty$. This is in contradiction with (B.8), because of the energy estimate (B.1). \[\square\]

B.3. Propagation of regularity around a non-degenerate point.

**Proposition B.6.** Let $(u_0, u_1) \in X^4 \times H^4$ such that $u_0(0) > 0$. Let $(u, \partial_t u) \in C([0, T_0]; X^1 \times H^1)$ be the solution of the Cauchy problem:

\[
\begin{cases}
\partial_t u - \Delta u = f(u), \\
(u(0), \partial_t u(0)) = (u_0, u_1),
\end{cases}
\]

cunstructed in Proposition B.2. There exists $\tau, \rho > 0$ such that $(u, \partial_t u)$ satisfies:

(B.10) \[\chi(\frac{v}{\rho}) u, \chi(\frac{v}{\rho}) \partial_t u \in C([0, \tau]; X^4 \times H^4)\]

(where $\chi$ is a standard regular cut-off function).
Proof. Denote \( v_0 := u_0(0) > 0 \) and introduce an auxiliary function \( \tilde{f} \in C^\infty, \tilde{f}(u) = f(u) \) when \( u \geq v_0/2, f(u) = 0 \) when \( u \leq 0 \). Using Faà di Bruno formula one can prove an analog of Lemma B.1

\[
\|f(u)\|_{H^4} \leq C(\|u\|_{X^4}),
\]

\[
\|f(u) - \tilde{f}(v)\|_{H^4} \leq \|u - v\|_{X^4} \cdot C(\|u\|_{X^4} + \|v\|_{X^4}),
\]

where \( C : \mathbb{R}_+ \to \mathbb{R}_+ \) is a continuous function. The same procedure as in the proof of Proposition B.2 leads to the conclusion that there exists \( \tau > 0 \) such that the Cauchy problem:

\[
\left\{ \begin{array}{l}
\partial_t \tilde{u} - \Delta \tilde{u} = \tilde{f}(\tilde{u}), \\
(\tilde{u}(0), \partial_t \tilde{u}(0)) = (u_0, u_1)
\end{array} \right.
\]

has a solution \((\tilde{u}, \partial_t \tilde{u}) \in C([0, \tau], X^4 \times H^4)\). By continuity and Schauder estimates, if we take \( \tau \) and \( \rho \) sufficiently small, we have \( \tilde{u}(t, x) > \frac{1}{2} v_0 \) for \( |x| \leq 4 \rho \) and \( 0 \leq t \leq \tau \). We may assume that \( \tau \leq 2 \rho \). Consider \( v = u - \tilde{u} \). We will prove that \( v = 0 \) when \( 0 \leq t \leq \tau \) and \( |x| \leq 2 \rho \), which will finish the proof. The function \( v \) solves the Cauchy problem:

\[
\left\{ \begin{array}{l}
\partial_t v - \Delta v = f(u) - \tilde{f}(\tilde{u}), \\
(v(0), \partial_t v(0)) = 0.
\end{array} \right.
\]

We run the localized energy estimate (B.2) for \( |x| \leq 2 \rho + |t - \tau| \). We suppose that \( \tau \leq 2 \rho \), so \( |x| \leq 4 \rho \), which means that \( \|f(u) - f(u)\|_{H^1} = \|f(u) - f(u)\|_{H^1} \leq \|u - \tilde{u}\|_{X^1} \) (the norm is taken in the ball \( B(0, 2 \rho + |t - \tau|) \)). From (B.2) and Gronwall inequality we deduce that \( u = \tilde{u} \) when \( |x| \leq 2 \rho + |t - \tau| \), in particular when \( |x| \leq 2 \rho \). \( \square 

B.4. Short-time asymptotics in the case \((u_0, u_1) = (p|x|^\beta, 0)\). Let \((u, \partial_t u)\) denote the solution of (NLW) corresponding to the initial data

\[
(u_0, u_1) = (\chi(\frac{x}{\rho}) p|x|^\beta, 0),
\]

where \( p, \rho > 0 \) and \( \beta > \frac{5}{2} \) are constants and \( \chi \) is a standard cut-off function. Let \((u_L, \partial_t u_L)\) denote the solution of the free wave equation corresponding to the same initial data.

Proposition B.7. Let \( q = \frac{(\beta+1)(\beta+3)}{3} p \). There exist \( T_0 > 0 \) and a constant \( C > 0 \) such that for \( 0 \leq t \leq T_0 \) and \( |x| \leq \frac{1}{2} t \) there holds

\[
|u_L(t, x) - qt^\beta| \leq Ct^\beta - 2|x|^2.
\]

Proof. Define

\[
w(y) := \int_{\partial B(0,1)} p|\omega + ye_1|^\beta \, d\sigma(\omega), \quad -\frac{1}{\sqrt{2}} < y < \frac{1}{\sqrt{2}},
\]

where \( B(0,1) \) denote the unit ball in \( \mathbb{R}^5 \), \( d\sigma \) is the surface measure on the unit sphere and \( e_1 = (1,0,0,0,0) \). Notice that

\[
|\omega + ye_1|^\beta = (1 - \omega_1^2 + (y + \omega_1)^2)^{\beta/2} = (1 + \omega_1^2)^{\beta/2} \cdot (1 + y + \frac{2\omega_1}{1 + \omega_1^2})^{\beta/2}
\]

can be developed in a power series in \( y \) which converges uniformly for \( -\frac{1}{\sqrt{2}} < y < \frac{1}{\sqrt{2}} \). Hence, \( w \) is an analytic function. It is also symmetric, so it is in fact analytic in \( y^2 \).

\[
w(y) = \bar{w}(y^2), \quad \bar{w}(z) \text{ analytic for } |z| < \frac{1}{2}
\]

We have \( \bar{w}(0) = w(0) = p \).
The representation formula for solutions of the free wave equation, see for example [9, p. 77], yields
\[ u_L(t, x) = \frac{1}{3} \left( \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial t} \right) \left( t^3 \int_{\partial B(x, t)} p|y|^\beta \, d\sigma(y) \right). \]

A change of variables shows that for \(|x| < \frac{1}{\sqrt{2}} t\) and \(t\) sufficiently small we have
\[ u_L(t, x) = \frac{1}{3} \left( \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial t} \right) \left( t^3 \cdot t^\beta \tilde{w}(\frac{|x|^2}{t^2}) \right) = t^\beta \tilde{w}(\frac{|x|^2}{t^2}), \]
where \(\tilde{w}_1(z)\) is analytic for \(|z| < \frac{1}{2}\). It is easily seen that \(\tilde{w}_1(0) = \frac{(\beta+1)(\beta+3)}{3} p = q\) (all the terms coming from differentiating \(\tilde{w}\) vanish at \(z = 0\)). Hence, there exists a constant \(C\) such that \(|\tilde{w}_1(z) - q| \leq C|z|\) for \(|z| \leq \frac{1}{4}\), and the conclusion follows.

**Proposition B.8.** For \(t\) small enough there holds
\[ \|u - u_L\|_{X^1(|x| \leq \frac{3}{4} t)} \lesssim t^{2\beta + \frac{\beta}{4}}. \]

**Proof.** From (B.4) and finite speed of propagation we obtain
\[ \|u - u_L\|_{X^1(|x| \leq \frac{3}{4} t)} \lesssim f(\|u_0, u_1\|)_{X^1 \times H^1(|x| \leq \frac{3}{4} t)}. \]
We have
\[ \|u_0, u_1\|_{X^1 \times H^1(|x| \leq \frac{3}{4} t)}^2 \sim \int_0^{\frac{3}{4} t} (r^{\beta-2}) r^4 \, dr \sim t^{2\beta + 1}, \]
and the conclusion follows.

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