Novel Hierarchies & Hidden Dimensions in Integrable Field Models: Theory & Application

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Abstract. Focusing on the problem of finding the right number of infinite series of conserved charges in integrable models with infinite degrees of freedom, we explore the well known nonlinear Scrödinger (NLS) equation discovering novel sets of charges with hidden integrable hierarchies. This prompts us to construct a new integrable NLS equation in 2 + 1-dimensions. Few important applications of our result including a 2D analytic model for the ocean rogue wave are reported.

1. Introduction
Theory and application of nonlinear integrable systems has a long and rich history starting from the KdV equation and its application to shallow water waves to the NLS, sine-Gordon, derivative NLS, Toda chain etc. stretching from the discrete to the continuum models, with corresponding applications to various fields. At the same time the classical theory was developed into the quantum domain with the discovery of new tools like Yang-Baxter equation Bethe ansatz and construction of quantum integrable models like spin chains, t-J model, Hubbard model etc. [1, 2].

The notion of integrability is intimately related to the symmetries of the system, which following the Nöther’s theorem are linked to the conserved quantities. To be precise, the system to be integrable the total number of the associated conserved charges $C_n$ must coincide with the degrees of its freedom.

For mechanical systems with finite $N$ degrees of freedom the definition of integrability becomes straightforward, since such a system becomes integrable, when the total number of the associated conserved quantities: $C_n$, $n = 1, 2, .., M$, $M$ matches with $N$. If $M < N$, the system might be partially integrable, while for $M > N$ it becomes a superintegrable system. Examples of such integrable systems with finite degrees of freedom may be given by the following classic cases.

I. Kepler Problem
Degrees of freedom of this 3D problem is $N = 3$, while the total number of conserved quantities is $M = 4$, which makes this well known problem a superintegrable system.

II. Euler, Lagrange, Kovalevskaya Top: These classic top problems posses $N = 4$ degrees of freedom, while as Kovalevskaya has shown, the total number of conserved charges of the system is also $M = 4$. This confirms the integrability of the system, which has been tested by Kovalevskaya through the Painlevé
III. Toda chain  This nonlinear $N$ coupled oscillator type problem on a chain has $N$ degrees of freedom with $M = N$ number of conserved charges, making the system completely integrable.

However the problem may arise when the above concept of integrability is applied to the field theoretic models having infinite degrees of freedom with $N \to \infty$. Extending the above criterion of integrability to the systems with infinite degrees of freedom, we naturally demand the existence of infinite number of independent conserved charges $C_n$, $n = 1, 2, \ldots$, with $M \to \infty$. This very criterion of integrability is applied to all known integrable field models in 1 + 1-dimensions like KdV, NLS, derivative NLS, sine-Gordon models etc. However a big, not well answered question for the field theoretic models is that, is there any guarantee that these two infinities: $N \to \infty$ and $M \to \infty$ would coincide, that is the infinite degrees of freedom for such systems would match the number of associated conserved quantities. Nevertheless, in (1 + 1)-dimensional integrable field models like in the NLS equation

$$i q_t - (q_{xx} + 2|q|^2 q) = 0,$$

all conserved charges: $C_n$, $n = 1, 2, \ldots$ associated with the model is believed to be well known, with known integrable hierarchies, derived from these charges. On the other hand, since an infinity added to another infinity would make again an infinity, there might exist more than one sets of charges, yet undiscovered, for the true integrability of a field model. Though this might be an universal problem concerning the integrability of all models with infinite degrees of freedom, here we will focus only on the NLS field model (1).

Our aim therefore is to focus on this less discussed but extremely crucial conceptual problem regarding the integrable NLS model, asking some important questions like, can there be new sets of charges yet undiscovered for the well known NLS model, can there be hidden space-dimensions underlying the standard 1 + 1-dimensional NLS equation.

2. Generation of charges

Since the conserved charges are intimately linked to the integrability of a system, we present, for going deep into such properties, first a simple but systematic method for generating the full set of finite number of charges for the NLS equation, based on the Lax operator method, which is also an important signature of an integrable system. The space-Lax equation related to an nonlinear integrable equation can be given by an auxiliary linear system

$$\Phi_x = U_1(\lambda)\Phi,$$

where the Lax operator $U_1$ for the NLS model can be expressed in the AKNS form

$$U_1 = i(\lambda \sigma^3 + U^{(0)}), \quad U^{(0)} = \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix},$$

where $q, q^*$ are the basic NLS fields, $\lambda$ is the spectral parameter.

Scale dimension: Before proceeding further we would focus on an important concept of scale dimension (or inverse length dimension), which is usually not much emphasised in the literature. Note that all terms in any object should be of the same scale dimension with the dimension of $\partial_x$ being $L^{-1}$, which we define as scale dimension 1. Therefore looking at the Lax equation (2) we may conclude that the scale dimension of $U_1$ should be 1, with that of $q, q^*$ as well as of $\lambda$ must also be 1. We shall actively use this concept below to identify the higher scale dimensions of different objects of our interest.
By taking the Jost matrix function as $\Phi(\lambda, x) = (\phi, \bar{\phi})$, the infinite set of conserved charges: $C_n$, $n = 1, 2, \ldots$ expressed as $C_n = \int dx \rho_n$, $n = 1, 2, \ldots$ can be linked to
\[
\phi(x, \lambda) = e^{\int^x r(\lambda, x')dx'}, \quad \rho(\lambda, x) = \sum_{n=1}^{\infty} \rho_n(x)\lambda^{-n}
\]
resulting to the generation of charges through
\[
\ln \phi(x = \infty, \lambda) = \sum_{n=1}^{\infty} C_n\lambda^{-n}.
\]
Therefore constructing Riccati type equation from the Lax equation (2) one can systematically generate the well known conserved quantities for the NLS model as $C_n = \int dx \rho_n$, $n = 1, 2, \ldots$ where
\[
\rho_1 = |q|^2, \quad \rho_2 = i(q^*_x - q^* q_x), \quad \rho_3 = q^*_x q_x + |q|^4,
\]
and so on. Taking for example $H = C_3$, one can derive the well known NLS equation (1) from $i\sigma^3 = \{q, H\}$, using the PB relation
\[
\{q(x), q^*(x')\} = i\delta(x - x'),
\]
where time $t = t_2$ is of scale dimension 2 (one can easily check this using the concept of scale dimension, defined above.)

For $H = C_4$ with the same PB we get the higher NLS equation
\[
i\sigma^3 - (q_{xxx} + 6|q|^2 q_x) = 0, \quad c.c. = 0,
\]
with time $t = t_3$ having scale dimension 3. Thus one derives the integrable hierarchy of the NLS equation taking the conserved charges $C_n, n = 1, 2, \ldots$ as the respective Hamiltonian of the system: $H = C_n, n = 1, 2, \ldots$ with the corresponding time: $t \equiv t_n$.

We have noticed above an interesting fact, which has not been focused much, is that the NLS field $q(x, t_1, t_2, t_3, \cdots)$ involved in the integrable hierarchy contains multi-dimensional time having different scale dimensions (see similar observation for the Toda chain in Suris [3]). We intend to exploit this known though less discussed fact to the deepest extend, asking the intriguing question whether we can revert the concept of multi-dimensional time to construct an integrable NLS equation with multi-dimensional space $x_m, m = 1, 2, \ldots$.

Another nontrivial possibility which will be our primary concern here also seems to have been ignored in the existing literature. To understand this point, let us focus first on the fact that, the well known conserved charges of the NLS model, as we have shown above, can be derived from the Lax operator $U_1$ in the AKNS form (3) having the scale dimension 1. It is worth recalling that not only the NLS, but also all other well known integrable systems, e.g., KdV, mKdV, sine-Gordon, Toda chain etc. have their conserved charges and the integrable hierarchies constructed from the same Lax operator $U_1$ in the AKNS form of scale dimension 1 (with different conjugate field $r$).

Therefore, we raise a nontrivial question asking whether one can use a different Lax operator with higher scale dimension for generating new series of conserved quantities and hence new integrable hierarchy for the same model, e.g. NLS model. For concreteness we use a Lax operator with scale dimension 2 for the NLS equation, choosing it as $U_2 \sim V_{NLS}$ with Lax equation
\[
\Phi_y = U_2(\lambda)\Phi, \quad y \equiv x_2, \\
U_2 = i \left( 2\lambda^2 \sigma^3 + \lambda U^{(0)} - \sigma^3 U^{(0)} - i\sigma^3 U^{(0)} \right)
\]
where $U^{(0)}$ is the same scale dimension 1 object as defined in (3). It is not difficult to see that $U_2$ containing $\lambda^2$ etc. is of scale dimension 2.
3. New charges, hidden hierarchies and novel 2D NLS model

Repeating now the same procedure for finding the well known infinite set of charges from the Lax equation (2) with Lax operator $U_1$ of scale dimension 1, as shown above, for the Lax equation (8) associated with $U_2$ of scale dimension 2, we can derive a new set of infinite number of conserved charges. In explicit form they can be given as

$$C_n^{(2)} = \int dy \rho_n^{(2)}$$

where

$$\rho_1^{(2)} = i(q^*_yx - q^*q)_x,$$

$$\rho_2^{(2)} = iq^*_yq + q^*_xq_x + |q|^4 + c.c., \quad \rho_3^{(2)} = q^*_yq_x,$$

$$\rho_4^{(2)} = iq^*_yq_x + q^*_yq_y - i|q|^2(q^*_yq - q^*_yx) - 2|q|^2q^*_xq_x$$

$$+ (q^2q^2 + q^2q^2) + c.c.,$$

etc. It is important to note that, all the above densities of conserved charges $\rho_n^{(2)}(x, y)$ are dependent on both the space dimensions $x, y$ as a consequence of the fact that the basic fields $q, q^*$ of this NLS system are now dependent also on the 2-dimensional space coordinates. These hidden space dimensions are manifestation of a dual picture of the multi-dimensional time in the NLS field discussed above.

Taking these new conserved quantities as Hamiltonians one can find therefore a new form of integrable hierarchy:

$$iq_n = \{q, H\}_{(2)}, \quad H = C_n^{(2)}, \quad n = 1, 2, \ldots$$

hidden in the NLS system.

3.1. Novel PB structure

The above hierarchy (12) could be derived when we use a new kind of PB structure

$$\{q(x), q^*_y(y')\}_{(2)} = i\delta(y - y').$$

For comparison recall the standard bosonic type of PB: $\{q(x), q^*(x')\} = i\delta(x - x')$ valid for the well known NLS model. It is remarkable that this completely new type of PB (13) can be derived from the Yang-Baxter equation (YBE)

$$\{U_2 \otimes \tilde{U}_2\}_{(2)} = [r, U_2 \otimes I + I \otimes \tilde{U}_2]$$

related to the Lax operator $U_2$, but associated again with the same rational $r$ as in case of 1D NLS model. Note that the validity of the YBE confirms also the complete integrability of the 2D NLS system. We wish to elaborate on this point including its quantum generalisation elsewhere [4].

3.2. Novel 2D integrable NLS equation

If we choose as Hamiltonian $H = C_4^{(2)}$, in the above 2D NLS hierarchy (12) $iq_4 = \{q, H\}_{(2)}$, using the PB structure (13) we can arrive at a novel $(2 + 1)$-dimensional integrable NLS equation given as

$$iq_t + q_xy + 2iq(qq^*_x - q^*_x) = 0$$

Note that while the well known 2D extension of the NLS equation is a nonintegrable system with unstable soliton solution, our 2D NLS model (15) is a completely integrable system exhibiting exact soliton solutions. Some specific properties and details of this novel model can be found in our recent reports [5, 6, 7].
4. Hierarchy of hierarchies in the NLS system

Now the question remains to be addressed whether this novel integrable hierarchy of a 2D integrable NLS equation is really independent from the well known NLS hierarchy, to which we will return below.

Recall that the compatibility of the linear system (2) with the Lax pair \((U_1, V_2)\) leading to the flatness condition

\[ U_{1t} - V_{2x} + [U_1, V_2] = 0 \]

gives the 1D NLS equation (1). The NLS Hierarchy similarly is generated from the Lax pair \((U_1, V_n)\), \(n = 1, 2, 3, \ldots\), which can be presented in the \((x, t)\) plain with a vertical line with a single jump \(x_1\) along the \(x\)-direction while a jump of \(t_1, t_2, \ldots, t_n\) in the \(t\)-direction making the vertical line \((1, n)\), \(n = 1, 2, \ldots\) (See first column in Fig. 1).

Now we ask some important questions, which seem to have been never put forward, as: Is this hierarchy complete? Can there be other hidden hierarchies?

To address these questions let us take first the case of the integrable hierarchy of the 2D-NLS equation we have already discussed above having the Lax equations \(\Phi_y = U_2 \Phi\), \(\Phi_{t_n} = V_n \Phi\), with the Lax pairs \((U_2, V_n)\), \(n = 1, 2, \ldots\), yielding the hierarchy from the flatness condition

\[ U_{2t} - V_{nx} + [U_2, V_n] = 0. \]

In similarity with the well known NLS hierarchy representing the first column in the \(x, t\) plane, the present case would naturally represent the second column with a double jump \(x_2\) along the \(x\)-direction, while jumps of \(t_1, t_2, \ldots, t_n\) along the \(t\)-direction making the next vertical line \((2, n)\), \(n = 1, 2, \ldots\) (See second vertical column in Fig. 1).

Interchanging \(U_m \leftrightarrow V_n\) in the above scheme one can derive similarly dual hierarchies along the horizontal lines in the \(x, t\) plane with the generation of infinite sets of dual charges \(J^{(n)}_m\).

Therefore, exploring these hidden hierarchies we can conclude finally that, different Lax pairs \((U_m, V_n)\), \(n = 1, 2, \ldots; m = 1, 2, \ldots\)

through their flatness condition

\[ U_{nt_m} - V_{mx_n} + [U_n, V_m] = 0 \]

would generate complete hierarchies covering all vertical and horizontal lines at all 2D lattice points \((n, m)\) in the \(x, t\) plane (see Fig. 1).

As shown in the figure:

1. \((m, n) = 1, 2, \ldots\) with \(m < n\) describes the vertical lines with novel Integrable hierarchies with conserved charges \(C^{(m)}_n\).
2. The diagonal line \(n = m\) corresponds to linear equations (since nonlinear part of these equations \([U_n, V_n] = 0\) ).
3. Horizontal lines with \(m > n\) correspond to the dual hierarchies with dual charges (or currents) \(J^{(n)}_m = \int dt_j n_j^{(n)}\).

Now we address to the independence of all these hidden integrable hierarchies with likely conclusions that:

Conserved charges \(C^{(m)}_n\) (from \(U_m\)) commute within each hierarchy & hence they are independent (with fixed \(m\)).

Dual charges \(J^{(n)}_m\) (generated by \(V_n\)) commute within each hierarchy & hence they are independent (with fixed \(n\)).
However, these charges (dual charges) within a hierarchy are likely to be related to the charges belonging to other hierarchies with different $m$ (different $n$ for dual charges), due to their underlying continuity relations

$$\partial_t \rho_n^{(m)} = \partial_{x,m} \rho_n^{(m)}$$

Nevertheless the hidden hierarchies we found here yield novel equations in higher dimension with applicable interest.

5. Applications

We briefly present now various possibilities of application of our result, more elaborate account of which can be found in our recent work [6, 8].

5.1. Non-periodic models with nontrivial boundaries

The usual integrable models with conserved charges considered above are valid only for periodic or trivial boundary conditions. However if we consider a more general integrable model in the interval $[x_0, x_1]$ with nontrivial boundary conditions, then interestingly the standard conserved charges are no longer conserved, while the actual conserved charges $\tilde{C}_k$ take the form

$$\tilde{C}_k = C_k + I_k^{BC},$$

where the boundary contribution is expressed through the dual charges $J_k \equiv J_k^{(1)}$ that we have generated above through the time-Lax operator $V_2$ related to the dual hierarchy of the NLS equation in the form

$$I_k^{BC} = J_k(x_0,t) - J_k(x_1,t), \quad k = 1, 2, \ldots$$

5.2. Integrable defect contribution through dual charge

Models with defect point at $x = x_d$, preserving their integrability become popular in recent years. To find the infinite set of conserved quantities of the whole system including the defect point the defect contribution can be given as

$$\tilde{C}_k = C_k^- + C_k^+ + I_k^{def}.$$
The defect contribution, as we show, can again be expressed through the dual charges $J_k$ at the defect point as

$$I_{\text{def}}^k = J_k(x_d^+) - J_k(x_d^-), \quad x_d^\pm = (x_d \pm \epsilon)|_{\epsilon \to 0}, \quad k = 1, 2, \ldots$$

with ± denoting the left/right limits from the defect point. The details of this method with application to the Toda chain with an integrable defect site can be found in our recent work [8].

5.3. Ocean rough wave model through integrable 2D NLS

Oceanic rogue wave, one of the mysteries of nature has not been understood and modelled yet satisfactorily. The rogue waves are 2D surface waves, which can appear suddenly on a relatively calm ocean, could attain enormously large amplitude (17-30 meters) and disappear fast without a trace. Such monster wave usually comes as a single event and preceded by strange hole waves. Nonlinearity, modulational instability and ocean current are supposed to be the crucial factors in the formation of ocean rogue waves.

However strangely, the analytic rogue wave modes available so far, are only in one space dimension and known as the Peregrine breather [9] and its higher order extensions, while the ocean rogue waves being surface waves are truly two-dimensional objects. Our aim therefore is to propose an analytic 2D surface rogue wave model based on a solution of our 2D integrable NLS equation (15), controlled by the ocean current represented by a term $U_c(x, t)q_x$ (see for details our report [6]). Our proposed rogue wave model is given by a 2D dynamical lump solution

$$P(2d)(x, y, t) = e^{4iy}(-1 + \frac{1 - i4y}{\alpha x^2 + \mu t^2 + 4y^2 + c})$$

with tunable amplitude & steepness (due to the existence of free parameters $c, \alpha$, which are absent in the available rogue wave model: Peregrine breather.)

5.4. Real time behaviour of our 2D rogue wave model

The lump solution (16) constructed by us shows close similarity with the dynamical behaviour of the observed ocean rogue wave as evident from its real time dynamics as described below.

1) In distant past ($t \to -\infty$) the solution turns into a plane wave $e^{4ix}$.
2) At $t = 0$ the solution becomes a high and steep 2D lump peaked at $x = y = 0$, accompanied by hole states.
3) It disappears again into the plane wave $e^{4ix}$ in future ($t \to +\infty$)

More vivid description of the dynamics of the model is given graphically Fig. 2a-c as snap-shots of the 2D wave at different moments of time.

6. Concluding remarks

Focusing on the problem of ambiguity in determining the right number of conserved charges in integrable field models with infinite degrees of freedom, we have found novel infinite sets of charges and dual charges together with the hidden hierarchies in the integrable NLS model. We have also discovered hidden possibilities of constructing higher dimensional integrable NLS model within this framework reverting the concept of multi-dimensional time. However these equations, as in other $2 + 1$ dimensional known integrable equations are accompanied by constraint equations. As a concrete example we have constructed a novel $2 + 1$ dimensional integrable NLS equation with direct application to model ocean surface rogue wave. Other applications of our result include an easy and explicit way of constructing defect contribution.
Figure 2. Creation of 2D hole at time $t_h = -0.83$, as told in marine-lores.

Figure 3. The hole splits into two and drifted away from the centre (at $t = -0.40$).

Figure 4. The full grown surface rogue wave at $t = 0.0$.

in integrable defect models and boundary contributions in integrable models with nontrivial boundary conditions, both expressed through dual charges.
Since our result is loaded with new ideas, we hope to develop them for new applications as well as extension to cover quantum integrable models.

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