THRESHOLD SINGULARITIES OF THE SPECTRAL SHIFT FUNCTION FOR GEOMETRIC PERTURBATIONS OF MAGNETIC HAMILTONIANS

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Abstract. We consider the 3D Schrödinger operator \( H_0 \) with constant magnetic field \( B \) of scalar intensity \( b > 0 \), and its perturbations \( H_\pm \) (resp., \( H_0 \)) obtained by imposing Dirichlet (resp., Neumann) conditions on the boundary of the bounded domain \( \Omega_{\text{in}} \subset \mathbb{R}^3 \). We introduce the Krein spectral shift functions \( \xi(E; H_\pm, H_0) \), \( E \geq 0 \), for the operator pairs \( (H_\pm, H_0) \), and study their singularities at the Landau levels \( \Lambda_q := b(2q + 1) \), \( q \in \mathbb{Z}_+ \), which play the role of thresholds in the spectrum of \( H_0 \). We show that \( \xi(E; H_+, H_0) \) remains bounded as \( E \uparrow \Lambda_q \), \( q \in \mathbb{Z}_+ \), being fixed, and obtain three asymptotic terms of \( \xi(E; H_-, H_0) \) as \( E \uparrow \Lambda_q \), and of \( \xi(E; H_\pm, H_0) \) as \( E \downarrow \Lambda_q \). The first two terms are independent of the perturbation while the third one involves the logarithmic capacity of the projection of \( \Omega_{\text{in}} \) onto the plane perpendicular to \( B \).

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1. Introduction

Let

\[
B = (0, 0, b), \quad b > 0,
\]

be a vector in \( \mathbb{R}^3 \) which has the physical interpretation of a constant magnetic field. Then

\[
\begin{align*}
A(x) &:= \frac{b}{2} (-x_2, x_1, 0), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \\
\Pi(A) = (\Pi_1(A), \Pi_2(A), \Pi_3(A)) &:= -i \nabla - A
\end{align*}
\]

is a magnetic potential which generates \( B \), i.e. \( \text{curl } A = B \),

\[
\Pi(A) = (\Pi_1(A), \Pi_2(A), \Pi_3(A)) := -i \nabla - A
\]

is the magnetic gradient, and

\[
-\Delta_A := \sum_{j=1}^3 \Pi_j(A)^2 = \left(-i \frac{\partial}{\partial x_1} + \frac{bx_2}{2} \right)^2 + \left(-i \frac{\partial}{\partial x_2} + \frac{bx_1}{2} \right)^2 - \frac{\partial^2}{\partial x_3^2}
\]

is the magnetic Laplacian. In order to define the domain of an appropriate realization of \( -\Delta_A \), self-adjoint in \( L^2(\mathbb{R}^3) \), we need the following notations. Let \( \Omega \) be an open non-empty subset of \( \mathbb{R}^3 \). Introduce the magnetic Sobolev spaces

\[
H^s_A(\Omega) := \{ u \in \mathcal{D}'(\Omega) | \Pi(A)^\alpha u \in L^2(\mathbb{R}^3), \quad \alpha \in \mathbb{Z}_+^3, \quad 0 \leq |\alpha| \leq s \}, \quad s \in \mathbb{Z}_+.
\]
Denote by $H_{s,0}^a(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the norm of $H_s^a(\Omega)$ defined by

$$
\|u\|_{H_s^a(\Omega)}^2 := \sum_{\alpha \in \mathbb{Z}^3, |\alpha| \leq s} \int_\Omega |\Pi(A)^\alpha u|^2 \, dx.
$$

Then the operator $H_0 := -\Delta_A$ with domain $\mathcal{D}(H_0) := H_0^2(\mathbb{R}^3)$ is self-adjoint in $L^2(\mathbb{R}^3)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$ (see e.g. [17, Appendix]). It is well known that

$$(1.2) \quad \sigma(H_0) = \sigma_{ac}(H_0) = [b, \infty),$$

and the Landau levels

$$\Lambda_q := b(2q + 1), \quad q \in \mathbb{Z}_+: = \{0, 1, 2, \ldots\},$$

play the role of thresholds in the spectrum $\sigma(H_0)$ of $H_0$ (see e.g. [16, 22]).

Next, as usual, we define a domain in $\mathbb{R}^d$, $d \geq 1$, as an open, connected, non-empty subset of $\mathbb{R}^d$. Let $\Omega_\text{in} \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial \Omega_\text{in} \in C^\infty$. Set

$$\Gamma := \partial \Omega_\text{in}, \quad \Omega_\text{ex} := \mathbb{R}^3 \setminus \overline{\Omega_\text{in}}.$$ 

Then the operator $H_{+,j} := -\Delta_A, j = \text{ex, in}$, with domain

$$\mathcal{D}(H_{+,j}) := \{u \in H^2_0(\Omega_j) \mid u|_\Gamma = 0\},$$

is the Dirichlet realization of $-\Delta_A$ on $\Omega_j$. Similarly, if $\nu$ is the unit normal vector at $\Gamma$, outward looking with respect to $\Omega_\text{in}$, then the operator $H_{-,j} := -\Delta_A, j = \text{ex, in}$, with domain

$$\mathcal{D}(H_{-,j}) := \{u \in H^2_0(\Omega_j) \mid \nu \cdot \Pi(A)u|_\Gamma = 0\},$$

is the Neumann realization of $-\Delta_A$ on $\Omega_j$. The operators $H_{\pm,j}, j = \text{ex, in}$, are self-adjoint in $L^2(\Omega_j)$. Moreover, $H_{+,j}$ (resp., $H_{-,j}$) corresponds to the closed quadratic form

$$(1.3) \quad \int_{\Omega_j} |\Pi(A) u|^2 \, dx$$

with domain $H_{+,0}^1(\Omega_j)$ (resp., $H_{+,0}^1(\Omega_j)$).

Using the orthogonal decomposition $L^2(\mathbb{R}^3) = L^2(\Omega_\text{in}) \oplus L^2(\Omega_\text{ex})$, set

$$H_{\pm} := H_{+,\text{in}} \oplus H_{+,\text{ex}}.$$ 

The aim of the article is to study the asymptotic behavior of the spectral shift functions $\xi(E; H_{\pm}, H_0)$ defined in the next section, as the energy $E$ approaches a given Landau level $\Lambda_q, q \in \mathbb{Z}_+$. 

The article is organized as follows. In Section 2 we introduce the spectral shift functions $\xi(E; H_{\pm}, H_0)$ and describe their main properties. In Section 3 we state our main result, Theorem 3.1, and briefly comment on it. In Section 4 we prove several important auxiliary results, Propositions 2.1, 2.2, and 2.3, while the proof of Theorem 3.1 can be found in Section 5. Finally, the Appendix contains the details concerning some technical results used in the main text of the article.
2. The spectral shift function

Let $X$ be a separable Hilbert space. Denote by $\mathfrak{B}(X)$ (resp., $\mathfrak{S}_\infty(X)$) the class of linear bounded (resp., compact) operators acting in $X$, and by $\mathfrak{S}_p(X)$, $p \in [1, \infty)$, the $p$th Schatten-von Neumann space of operators $T \in \mathfrak{S}_\infty(X)$ for which the norm

$$
\|T\|_p := \left(\text{Tr}(T^*T)^{p/2}\right)^{1/p}
$$

is finite. In particular, $\mathfrak{S}_1(X)$ is the trace class, and $\mathfrak{S}_2(X)$ is the Hilbert-Schmidt class over $X$. If $X = L^2(\mathbb{R}^3)$, we omit $X$ in the notations $\mathfrak{B}(X)$ and $\mathfrak{S}_p(X)$, $p \in [1, \infty]$.

By the Dirichlet-Neumann bracketing and the non-negativeness of the quadratic form (1.3), we have

$$
H_+ \geq H_0 \geq H_- \geq 0.
$$

By (1.2), and $b > 0$, we find that the operators $H_0$, and hence $H_+$, are invertible. It is not difficult to see that $H_-$ is invertible as well. To this end, arguing as in the proof of Proposition 2.1 below, we find that

$$(H_- + I)^{-1} - (H_0 + I)^{-1} \in \mathfrak{S}_2 \subset \mathfrak{S}_\infty.
$$

Therefore, the Weyl theorem on the invariance of the essential spectrum under relatively compact perturbations yields

$$
\sigma_{\text{ess}}(H_-) = \sigma_{\text{ess}}(H_0) = [b, \infty).
$$

Hence, if $0 \in \sigma(H_-)$, then the zero should be a discrete eigenvalue of $H_-$. Let $u \in \mathcal{D}(H_-)$ such that $H_- u = 0$. By (1.3), we have

$$
\Pi(A)u_{|_{\Omega_{\text{in}}}} = 0, \quad \Pi(A)u_{|_{\Omega_{\text{ex}}}} = 0.
$$

Taking into account the explicit expression (1.1) for $A$, we find that the only element $u \in \mathcal{D}(H_-)$ which satisfies (2.2), is $u = 0$, and hence $0 \notin \sigma(H_-)$.

Further, (2.1) implies

$$
H_-^{-1} \geq H_0^{-1} \geq H_+^{-1}.
$$

Set

$$
V_+ := H_0^{-1} - H_+^{-1}, \quad V_- := H_-^{-1} - H_0^{-1}.
$$

Then, (2.3) yields $V_\pm \geq 0$.

**Proposition 2.1.** We have

$$
V_\pm \in \mathfrak{S}_2.
$$

Moreover,

$$
H_\pm^{-2} - H_0^{-2} \in \mathfrak{S}_1.
$$
The proof of Proposition 2.1 can be found in Section 4.1.

Remark: In [2, 4, 5], the authors consider second-order elliptic differential operators in \( \mathbb{R}^d, d \geq 2 \), equip them with Dirichlet or Neumann boundary conditions on appropriate hypersurfaces, and obtain results closely related to our Proposition 2.1. Although, formally, our operator \( H_0 \) is not in the classes of the operators considered in [2, 4, 5], the methods applied there may improve relations (2.4) and (2.5) which, nonetheless, are sufficient for the purposes of this article.

Using (2.5), we define the spectral shift function (SSF) \( \xi(E; H_\pm, H_0) \) as

\[
\xi(E; H_\pm, H_0) := \begin{cases} 
-\xi(E^{-2}; H_\pm^{-2}, H_0^{-2}) & \text{if } E > \inf \sigma(H_\pm), \\
0 & \text{if } E < \inf \sigma(H_\pm),
\end{cases}
\]

where, for almost every \( E > 0 \),

\[
(2.6) \quad \xi(E^{-2}; H_\pm^{-2}, H_0^{-2}) := \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \arg \det \left( (H_\pm^{-2} - E^{-2} - i\varepsilon) (H_0^{-2} - E^{-2} - i\varepsilon)^{-1} \right),
\]

the branch of the argument being fixed by the condition

\[
\lim_{\im z \to \infty} \arg \det \left( (H_\pm^{-2} - z) (H_0^{-2} - z)^{-1} \right) = 0
\]

(see the original work [21] or [34, Chapter 8]). The SSF \( \xi(\cdot; H_\pm, H_0) \) is the unique element of \( L^1_{\text{loc}}(\mathbb{R}) \) which satisfies the Lifshits-Krein identity

\[
\text{Tr} \left( f(H_\pm) - f(H_0) \right) = \int_{\mathbb{R}} f'(E) \xi(E; H_\pm, H_0) \, dE, \quad f \in C^\infty_0(\mathbb{R}),
\]

and the normalization condition

\[
\xi(E; H_\pm, H_0) = 0, \quad E < \inf \sigma(H_\pm).
\]

Since \( \inf \sigma(H_\pm) > 0 \), so that \( \xi(E; H_\pm, H_0) = 0 \) for \( E \in (-\infty, 0] \), in the sequel we will consider \( \xi(E; H_\pm, H_0) \) only for \( E > 0 \).

For almost every \( E \in [b, \infty) = \sigma_{\text{ac}}(H_0) \), the Birman-Krein formula implies

\[
\det S(E; H_\pm, H_0) = e^{-2\pi i \xi(E; H_\pm, H_0)}
\]

where \( S(E; H_\pm, H_0) \) is the scattering matrix for the operator pair \((H_\pm, H_0)\) (see [3] or [34, Chapter 8]). On the other hand, for almost every \( E \in (0, b) \) we have

\[
(2.7) \quad \xi(E; H_-, H_0) = -\text{Tr} \mathbf{1}_{(-\infty, E)}(H_-).
\]

Here and in the sequel \( \mathbf{1}_S \) denotes the characteristic function of the set \( S \). Thus, \( \mathbf{1}_S(T) \) is the spectral projection of \( T \) corresponding to the Borel set \( S \subset \mathbb{R} \), and by (2.7) \(-\xi(E; H_-, H_0)\) is equal to the number of the eigenvalues of \( H_- \) less than \( E \) and counted with the multiplicities.

A priori, the SSF \( \xi(E; H_\pm, H_0) \) is defined only for almost every \( E \in \mathbb{R} \). Our next goal is to introduce a canonic representative of the class of equivalence \( \xi(\cdot; H_\pm, H_0) \) following
the main ideas of [27] (see below Proposition 2.3). Let $\mathbb{C}_\pm := \{ z \in \mathbb{C} \mid \pm \text{Im } z > 0 \}$. For $z \in \mathbb{C}_-$ set

$$T^\pm(z) := V_\pm^\frac{1}{2}(H_0^{-1} - z^{-1})^{-1}V_\pm^\frac{1}{2}.$$ 

**Proposition 2.2.** Let $E \in (0, \infty) \setminus b(2\mathbb{Z}_+ + 1)$. Then there exists a norm limit

$$T^\pm(E) := n - \lim_{C_- \ni z \to E} T^\pm(z) \in \phi_2,$$

and

$$\text{Im } T^\pm(E) \in \phi_1.$$ 

Moreover, $\text{Re } T^\pm(E)$ (resp., $\text{Im } T^\pm(E)$) depends continuously in $\phi_2$ (resp., in $\phi_1$) on $E \in (0, \infty) \setminus b(2\mathbb{Z}_+ + 1)$.

The proof of Proposition 2.2 can be found in Subsection 4.2.

Let $T = T^*$ be a compact operator in a Hilbert space. For $s > 0$ set

$$n_\pm(s; T) = \text{Tr } \mathbb{1}_{(s, \infty)}(\pm T).$$

Thus $n_+(s, T)$ (resp., $n_-(s, T)$) is just the number of the eigenvalues of $T$ counted with the multiplicities, greater than $s > 0$ (resp., less than $-s < 0$).

For $E \in (0, \infty) \setminus b(2\mathbb{Z}_+ + 1)$ set

$$\hat{\xi}(E; H_\pm, H_0) := \pm \frac{1}{\pi} \int \mathbb{R} n_\pm(1; \text{Re } T^\pm(E) + t \text{Im } T^\pm(E)) \frac{dt}{1 + t^2}.$$ 

**Proposition 2.3.** The function $\hat{\xi}(\cdot; H_\pm, H_0)$ is well defined on $(0, \infty) \setminus b(2\mathbb{Z}_+ + 1)$, bounded on every compact subset of $(0, \infty) \setminus b(2\mathbb{Z}_+ + 1)$, and continuous on $(0, \infty) \setminus (\sigma_p(H_\pm) \cup b(2\mathbb{Z}_+ + 1))$ where $\sigma_p(H_\pm)$ denotes the set of the eigenvalues of $H_\pm$.

Moreover, for almost every $E \in (0, \infty)$ we have

$$\xi(E; H_\pm, H_0) = \hat{\xi}(E; H_\pm, H_0).$$

The proof of Proposition 2.3 can be found in Subsection 4.3.

**Remark:** In view of Proposition 2.3 we identify in the sequel the SSF $\xi(E; H_\pm, H_0)$ with $\hat{\xi}(E; H_\pm, H_0)$, and assume that it is defined for every $E \in (0, \infty) \setminus b(2\mathbb{Z}_+ + 1)$.

### 3. Main Results

Let $\mathcal{E} \subset \mathbb{R}^2$ be a Borel set, and $\mathcal{M}(\mathcal{E})$ denote the set of compactly supported probability measures on $\mathcal{E}$. Then the *logarithmic capacity* of $\mathcal{E}$ is defined as $\text{Cap}(\mathcal{E}) := e^{-\mathcal{I}(\mathcal{E})}$ where

$$\mathcal{I}(\mathcal{E}) := \inf_{\mu \in \mathcal{M}(\mathcal{E})} \int_{\mathcal{E}} \int_{\mathcal{E}} \ln |x - y|^{-1} d\mu(x) d\mu(y).$$

The properties of $\text{Cap}(\mathcal{E})$ we need, are summarized in Subsection 5.5. A systematic exposition of the theory of the logarithmic capacity can be found, for example, in [30, Chapter 5] and [23, Chapter II, Section 4].

Let $\mathcal{E} \subset \mathbb{R}^2$ be a Borel set such that $\text{Cap}(\mathcal{E}) \in (0, \infty)$. Set

$$(3.1) \quad \mathcal{E}(\mathcal{E}) := 1 + \ln \left(b \cdot \text{Cap}(\mathcal{E})^2\right).$$
Note that if $E$ is a bounded domain, then $\text{Cap}(E) \in (0, \infty)$.

For $x \in \mathbb{R}^3$, we write $x = (x_\perp, x_\parallel)$ where $x_\perp = (x_1, x_2) \in \mathbb{R}^2$ are the variables on the plane perpendicular to the magnetic field $B$ while $x_\parallel = x_3 \in \mathbb{R}$ is the variable along $B$.

For $x = (x_\perp, x_\parallel) \in \mathbb{R}^3$ define the projections $\pi_\perp(x) := x_\perp$, $\pi_\parallel(x) := x_\parallel$. Note that if $\Omega \subset \mathbb{R}^3$ is a (bounded) domain, then $\pi_\perp(\Omega) \subset \mathbb{R}^2$ is a (bounded) domain as well. Set $O_{\text{in}} := \pi_\perp(\Omega_{\text{in}})$.

Thus, $O_{\text{in}}$ is the projection of the obstacle $\Omega_{\text{in}}$ onto the plane perpendicular to the magnetic field $B$.

For $\lambda > 0$ small enough, and $C \in \mathbb{R}$ set

$$\ln_2(\lambda) := \ln | \ln \lambda |, \quad \ln_3(\lambda) := \ln \ln_2(\lambda),$$

and

$$\Phi_0(\lambda) := \frac{| \ln \lambda |}{\ln_2(\lambda)}, \quad \Phi_1(\lambda; C) := \Phi_0(\lambda) \left( 1 + \frac{\ln_3(\lambda)}{\ln_2(\lambda)} + \frac{C}{\ln_2(\lambda)} \right).$$

**Theorem 3.1.** Let $\Omega_{\text{in}}$ be a bounded domain with $\partial \Omega_{\text{in}} \in C^\infty$. Fix $q \in \mathbb{Z}_+$. Then,

$$\xi(\Lambda_q - \lambda; H_+, H_0) = O(1),$$

(3.2)

$$\xi(\Lambda_q - \lambda; H_-, H_0) = -\frac{1}{2} \Phi_1(\lambda; \mathcal{C}(O_{\text{in}})) + o \left( \frac{| \ln \lambda |}{\ln_2(\lambda)^2} \right),$$

(3.3)

$$\xi(\Lambda_q + \lambda; H_\pm, H_0) = \pm \frac{1}{4} \Phi_1(\lambda; \mathcal{C}(O_{\text{in}})) + o \left( \frac{| \ln \lambda |}{\ln_2(\lambda)^2} \right),$$

(3.4)

as $\lambda \downarrow 0$.

**Remarks:** (i) Evidently, (3.2) and (3.4) with sign “+” imply

$$\lim_{\lambda \downarrow 0} \xi(\Lambda_q - \lambda; H_+, H_0) = 0,$$

(3.5)

while (3.3) and (3.4) with sign “-” imply

$$\lim_{\lambda \downarrow 0} \xi(\Lambda_q - \lambda; H_-, H_0) = 2.$$  

(3.6)

In a certain sense, relations (3.5) and (3.6) can be considered as generalizations of the classical Levinson theorem (see the original work [24] or the survey article [33]), which relates the (finite) number of the negative eigenvalues of the non-magnetic Schrödinger operator $-\Delta + V$ with electric potential $V$ which decays fast enough at infinity, and the limit $\lim_{E \uparrow 0} \xi(E; -\Delta + V, -\Delta)$ where $\xi(E; -\Delta + V, -\Delta)$ is the SSF for the operator pair $(-\Delta + V, -\Delta)$.

(ii) By the so-called telescopic property of the SSF, we have

$$\xi(E; H_+, H_-) = \xi(E; H_+H_0) - \xi(E; H_-, H_0), \quad E \in (0, \infty) \setminus b(2\mathbb{Z}_+ + 1).$$
Therefore, (3.2) - (3.3) imply
\[ \xi(\Lambda_q - \lambda; H_+, H_-) = \frac{1}{2} \Phi_1(\lambda; \mathcal{C}(O_{in})) + o\left( \frac{|\ln \lambda|}{\ln^2(\lambda)^2} \right), \quad \lambda \downarrow 0, \]
while (3.4) implies
\[ \xi(\Lambda_q + \lambda; H_+, H_-) = \frac{1}{2} \Phi_1(\lambda; \mathcal{C}(O_{in})) + o\left( \frac{|\ln \lambda|}{\ln^2(\lambda)^2} \right), \quad \lambda \downarrow 0. \]
In particular, similarly to (3.5)-(3.6), we have
\[ \lim_{\lambda \downarrow 0} \frac{\xi(\Lambda_q - \lambda; H_+, H_-)}{\xi(\Lambda_q + \lambda; H_+, H_-)} = 1. \]

(iii) According to (2.7), we have
\[ (3.7) \]
\[ \xi(\Lambda_0 - \lambda; H_-, H_0) = -\text{Tr} \mathbf{1}_{(-\infty, \lambda_0 - \lambda)}(H_-) = -\text{Tr} \mathbf{1}_{(-\infty, \lambda_0 - \lambda)}(H_{-,ex}) - \text{Tr} \mathbf{1}_{(-\infty, \lambda_0 - \lambda)}(H_{-,in}), \quad \lambda > 0. \]
Since the operator \( H_{-,in} \) is a second-order elliptic partial differential operator acting in a bounded domain with smooth boundary, its spectrum \( \sigma(H_{-,in}) \) is discrete, and
\[ \text{Tr} \mathbf{1}_{(-\infty, \lambda_0 - \lambda)}(H_{-,in}) = O(1), \quad \lambda \downarrow 0. \]
Then, (3.3) with \( q = 0 \) implies
\[ \text{Tr} \mathbf{1}_{(-\infty, \lambda_0 - \lambda)}(H_{-,ex}) = \frac{1}{2} \Phi_1(\lambda; \mathcal{C}(O_{in})) + o\left( \frac{|\ln \lambda|}{\ln^2(\lambda)^2} \right), \quad \lambda \downarrow 0, \]
which describes the accumulation of the discrete spectrum of the exterior Neumann magnetic Laplacian \( H_{-,ex} \) at \( \Lambda_0 = \inf \sigma_{ess}(H_{-,ex}) \).

Let us compare Theorem 3.1 with similar results available in the literature. The threshold singularities of the SSF for the operator pair \((H_0 + V, H_0)\) where \( V \) is a real-valued fast decaying electric potential, were considered in [14]. The cases of \( \lambda \) of power-like decay, exponential decay, and compact support were handled. Formally, our Theorem 3.1 resembles the results of [14] on compactly supported \( V \), which however are less precise than (3.3) and (3.4): the right-hand side of the analogue of (3.3) (resp., of (3.4)) in [14] is \(-\frac{1}{2} \Phi_0(\lambda)(1 + o(1)) \) (resp., \( \pm \frac{1}{4} \Phi_0(\lambda)(1 + o(1)) \)).

A problem closely related to the analysis of the SSF \( \xi(\cdot; H_0 + V, H_0) \) as \( E \to \Lambda_q \) for a given \( q \in \mathbb{Z}_+ \), is the investigation of accumulation of resonances of \( H_0 + V \) at \( \Lambda_q \) performed in [8, 9, 10]. The asymptotic distribution of resonances near the Landau levels for the operators \( H_\pm \) considered in this article, is studied in [12].

Let us mention also some 2D results related to Theorem 3.1. It is well known that in the 2D case the spectrum of the Landau Hamiltonian is purely point and consists of the Landau levels which are eigenvalues of infinite multiplicity (see (4.6) – (4.7) below). Hence, the problem of the singularities of the SSF for the 2D analogue of the operator pair \((H_\pm, H_0)\) reduces to the study of the accumulation of the discrete eigenvalues of the 2D analogues of \( H_\pm \) at the Landau levels. Such a study was undertaken in [28] for
the Dirichlet case, in [26, 18] for the Neumann case, and in [18] for Robin boundary conditions.

4. Proofs of the auxiliary results

4.1. Proof of Proposition 2.1. We start with the following key Lemma

Lemma 4.1. Let \( \omega \in C_0^\infty(\mathbb{R}^3; [0, 1]) \) such that \( \omega = 1 \) in a vicinity of \( \Gamma \). Then we have

\[
V_{\pm}^{\frac{1}{2}} = V_{\pm}^{\frac{1}{2}} H_0 \omega H_0^{-1}.
\]

Proof. Let \( P_{\pm} \) be the orthogonal projection onto \( (\text{Ker} \, V_{\pm})^\perp \). Then, \( V_{\pm}^{\frac{1}{2}} = V_{\pm}^{\frac{1}{2}} P_{\pm} \). Set \( \tilde{\omega} := 1 - \omega \). Note that \( \tilde{\omega} \) vanishes in vicinity of \( \Gamma \). We have

\[
V_{\pm}^{\frac{1}{2}} = V_{\pm}^{\frac{1}{2}} P_{\pm} H_0 (\omega + \tilde{\omega}) H_0^{-1}.
\]

Therefore, in order to prove (4.1), it suffices to show that

\[
P_{\pm} H_0 \tilde{\omega} H_0^{-1} = 0.
\]

Define the operator \( H_{00} := -\Delta_A \) with domain

\[
\mathcal{D}(H_{00}) := \{ u \in H^2_A(\mathbb{R}^3) \mid u|_\Gamma = \nu \cdot \Pi(A) u|_\Gamma = 0 \}.
\]

Thus the operators \( H_0, H_+ \), and \( H_- \) are extensions of the operator \( H_{00} \). If \( u \in L^2(\mathbb{R}^3) \), then \( \tilde{\omega} H_0^{-1} u \in \mathcal{D}(H_{00}) \) and \( H_j \tilde{\omega} H_0^{-1} u = H_{00} \tilde{\omega} H_0^{-1} u, j = 0, +, - \). Therefore, \( V_{\pm} H_0 \tilde{\omega} H_0^{-1} u = 0 \), i.e. \( H_0 \tilde{\omega} H_0^{-1} u \in \ker V_{\pm} \) which implies that (4.2) holds true.

Further, we note that

\[
H_0 \omega H_0^{-1} = \omega + [H_0, \omega] H_0^{-1}
\]

and obtain a convenient representation of the commutator \([H_0, \omega]\).

To this end, we introduce the Landau Hamiltonian \( H_{0,\perp} \), i.e. the 2D Schrödinger operator with constant scalar magnetic field \( b > 0 \),

\[
H_{0,\perp} = \left( -i \frac{\partial}{\partial x_1} + \frac{bx_2}{2} \right)^2 + \left( -i \frac{\partial}{\partial x_2} - \frac{bx_1}{2} \right)^2, \quad x_\perp = (x_1, x_2) \in \mathbb{R}^2,
\]

essentially self-adjoint on \( C_0^\infty(\mathbb{R}^2) \), and self-adjoint in \( L^2(\mathbb{R}^2) \). We have

\[
H_{0,\perp} = a^* a + b
\]

where

\[
a^* = -2ie^\phi \frac{\partial}{\partial \zeta} e^{-\phi} = -2i \left( \frac{\partial}{\partial \zeta} - \frac{\partial \phi}{\partial \zeta} \right), \quad \zeta = x_1 + ix_2,
\]

is the magnetic creation operator,

\[
a = -2ie^{-\phi} \frac{\partial}{\partial \bar{\zeta}} e^\phi = -2i \left( \frac{\partial}{\partial \bar{\zeta}} + \frac{\partial \phi}{\partial \bar{\zeta}} \right), \quad \bar{\zeta} = x_1 - ix_2,
\]
is the magnetic annihilation operator, and \( \phi(x_\perp) := \frac{b|x_\perp|^2}{4}, \ x_\perp \in \mathbb{R}^2 \), so that \( \Delta \phi = b \).

The operators \( a \) and \( a^* \) are closed on their common domain \( \mathcal{D}(a) = \mathcal{D}(a^*) = \mathcal{D}(H_{0,\perp}^{1/2}) \), they are mutually adjoint in \( L^2(\mathbb{R}^2) \), and satisfy
\[
[a, a^*] = 2b.
\]

It is well known that
\[
(4.6) \quad \sigma(H_{0,\perp}) = \bigcup_{j \in \mathbb{Z}_+} \{ \Lambda_j \},
\]

\[
\text{Ker}(H_{0,\perp} - \Lambda_j) = (a^*)^j \text{Ker}a, \quad j \in \mathbb{Z}_+,
\]

\[
\text{Ker}a := \left\{ u \in L^2(\mathbb{R}^2) \mid u = ge^{-\phi}, \ \frac{\partial g}{\partial \zeta} = 0 \right\},
\]

and, accordingly,
\[
(4.7) \quad \text{dim Ker}(H_{0,\perp} - \Lambda_j) = \infty, \quad j \in \mathbb{Z}_+.
\]

Denote by \( p_j \) the orthogonal projection onto \( \text{Ker}(H_{0,\perp} - \Lambda_j), \ j \in \mathbb{Z}_+ \). Next, set
\[
H_{0,\|} := -\frac{d^2}{dx_\|^2}, \quad \mathcal{D}(H_{0,\|}) = H^2(\mathbb{R}).
\]

Then we have
\[
H_0 = H_{0,\perp} \otimes I_\| + I_\perp \otimes H_{0,\|}
\]

where \( I_\perp \) and \( I_\| \) are the identities in \( L^2(\mathbb{R}^2_{x_\perp}) \) and \( L^2(\mathbb{R}^2_{x_\|}) \) respectively, and a simple calculation implies the following

**Lemma 4.2.** Let \( \omega \in C^\infty_0(\mathbb{R}^2; \mathbb{R}) \). Then we have
\[
(4.8) \quad K(\omega) := [H_0, \omega] = -\Delta \omega + \sum_{j=1}^3 \omega_j G_j = \Delta \omega + \sum_{j=1}^3 G_j \omega_j = -K(\omega)^*,
\]

where
\[
\omega_1 := -2i \frac{\partial \omega}{\partial \zeta}, \quad \omega_2 := -2i \frac{\partial \omega}{\partial \xi}, \quad \omega_3 := -2 \frac{\partial \omega}{\partial x_\|},
\]

\[
G_1 := a^* \otimes I_\|, \quad G_2 := a \otimes I_\|, \quad G_3 := I_\perp \otimes \partial,
\]

and
\[
\partial := \frac{d}{dx_\|}, \quad \mathcal{D}(\partial) := H^1(\mathbb{R}).
\]

Note that \( \text{supp} \Delta \omega \subset \text{supp} \omega \) and \( \text{supp} \omega_j \subset \text{supp} \omega, \ j = 1, 2, 3 \). Moreover, the operators \( K(\omega)H_0^{-1} \) and, hence, \( H_0^{-1}K(\omega)^* \) are compact in \( L^2(\mathbb{R}^3) \).
Lemma 4.3. Let $\eta \in C_c^\infty(\mathbb{R}^3)$, $j = 0, +, -$.

(i) We have

$$\eta H_j^{-1/2} \in \mathcal{S}_4,$$

and hence

$$\eta H_j^{-1} \eta \in \mathcal{S}_2.$$ 

(ii) Moreover,

$$\eta H_j^{-1} \in \mathcal{S}_2.$$ 

Proof. The validity of (4.9) and (4.11) follows easily from the diamagnetic inequality (see e.g. [1] and [20]), and the results of [6] concerning the spectral properties of elliptic non-magnetic differential operators. \(\square\)

Now we are in position to prove Proposition 2.1. As above, let $\omega \in C_c^\infty([0, 1])$ satisfy $\omega = 1$ in a vicinity of $\Gamma$, and let $\eta \in C_c^\infty(\mathbb{R}^3; [0, 1])$ satisfy $\eta = 1$ in a vicinity of $\text{supp } \omega$. By Lemma 4.1 and (4.3), we have

$$V_\pm = (H_0^{-1}K^* + \omega) \eta V_\pm \eta (\omega + KH_0^{-1}).$$

Since $\eta V_\pm \eta \in \mathcal{S}_2$ by (4.10), and the operators $H_0^{-1}K^* + \omega$ and $\omega + KH_0^{-1}$ are bounded, we obtain (2.4). Let us now prove (2.5). Write

$$H_\pm^{-2} - H_0^{-2} = \mp V_\pm (\omega + K(\omega)H_0^{-1})H_0^{-1} \mp H_\pm^{-1}(H_0^{-1}K^*(\omega) + \omega)V_\pm.$$ 

Let us show that

$$V_\pm (\omega + K(\omega)H_0^{-1})H_0^{-1} \in \mathcal{S}_1.$$

By (2.4), we have $V_\pm \in \mathcal{S}_2$, (4.11) implies $\omega H_0^{-1} \in \mathcal{S}_1$, and therefore

$$V_\pm \omega H_0^{-1} \in \mathcal{S}_1.$$ 

Further, let $\theta \in C_c^\infty(\mathbb{R}^3; [0, 1])$ satisfy $\theta = 1$ on $\text{supp } \omega$. Then, by (4.8), we have

$$V_\pm K(\omega)H_0^{-2} = V_\pm K(\omega)\theta H_0^{-2} = V_\pm K(\omega)H_0^{-1}\theta H_0^{-1} + V_\pm K(\omega)H_0^{-1}K(\theta)H_0^{-2}.$$ 

Since $V_\pm \theta H_0^{-1} \in \mathcal{S}_2$, and $K(\omega)H_0^{-1}$ is bounded, we get

$$V_\pm K(\omega)H_0^{-1}\theta H_0^{-1} \in \mathcal{S}_1.$$ 

Further, by (4.8), we have

$$V_\pm K(\omega)H_0^{-1}\theta H_0^{-1} = V_\pm K(\omega)H_0^{-1}\left(\Delta \theta + \sum_{j=1}^{3} G_j \theta_j \right)H_0^{-1}.$$ 

Since $V_\pm \Delta \theta H_0^{-1}, \theta_j H_0^{-1} \in \mathcal{S}_2$, while the operators $K(\omega)H_0^{-1}, K(\omega)H_0^{-1}G_j$ are bounded, we find that (4.17) yields $V_\pm K(\omega)H_0^{-1}K(\theta)H_0^{-2} \in \mathcal{S}_1$, which combined with (4.14), (4.11), and (4.16) implies (4.13). In a similar manner we prove that

$$H_\pm^{-1}(H_0^{-1}K^*(\omega) + \omega)V_\pm \in \mathcal{S}_1.$$ 

Putting together (4.12), (4.13), and (4.18), we obtain (2.5).
4.2. Proof of Proposition 2.2. Let \( z \in \mathbb{C}_- \). Combining (1.1) and (1.3) with (4.8), we find that

\[
T(z) = V^\pm_\pm(\omega + K(\omega)H_0^{-1})(H_0^{-1} - z^{-1})(\omega + H_0^{-1}K(\omega)^*)V^\pm_\pm.
\]

Evidently,

\[
(H_0^{-1} - z^{-1})^{-1} = -z^2(H_0 - z)^{-1} - z,
\]

\[
H_0^{-1}(H_0 - z)^{-1} = (H_0 - z)^{-1}H_0^{-1} = \frac{1}{z}(H_0 - z)^{-1} - \frac{1}{z}H_0^{-1},
\]

\[
H_0^{-1}(H_0 - z)^{-1}H_0^{-1} = \frac{1}{z^2}(H_0 - z)^{-1} - \frac{1}{z^2}H_0^{-1} - \frac{1}{z}H_0^{-2}.
\]

Combining (1.19) with (4.20) - (4.22), and taking into account that \( V^\pm_\pm(\omega + KH_0^{-1}) = V^\pm_\pm \), we get

\[
T(z) = M^+_1(z) + R^+_1(z)
\]

where the main term is

\[
M^+_1(z) := V^\pm_\pm(z\omega + K)(H_0 - z)^{-1}(z\omega + K^*)V^\pm_\pm,
\]

while the rest is

\[
R^+_1(z) := zV^\pm_\pm(\omega H_0^{-1}K^* + KH_0^{-1}\omega + KH_0^{-2}K^* - I)V^\pm_\pm + V^\pm_\pm KH_0^{-1}K^*V^\pm_\pm.
\]

Since \( R^+_1 \) extends to an affine function form \( \mathbb{C} \) to \( \mathcal{G}_2 \), we obtain the following elementary

**Proposition 4.4.** For every \( E \in \mathbb{R} \) there exists \( R^+_1(E) = R^+_1(E)^* \in \mathcal{G}_2 \) such that

\[
\lim_{\mathbb{C}_- \ni \omega \to E} \|R^+_1(z) - R^+_1(E)\|_2 = 0,
\]

\( R^+_1(E) \) depends continuously in \( \mathcal{G}_2 \) on \( E \), and

\[
\|R^+_1(E)\|_2 = O(|E| + 1), \quad E \in \mathbb{R}.
\]

Set \( P_j := p_j \oplus I, j \in \mathbb{Z}_+ \). For a given \( q \in \mathbb{Z}_+ \) put

\[
P^\leq_q := \sum_{j \leq q} P_j, \quad P^\geq_q := \sum_{j > q} P_j.
\]

Thus, \( P^\leq_q \) and \( P^\geq_q \) are orthogonal projections in \( L^2(\mathbb{R}^3) \), and \( P^\leq_q + P^\geq_q = I \). Taking into account (4.21), we find that

\[
M^+_1(z) = M^+_2(z) + R^+_2(z)
\]

where

\[
M^+_2(z) = M^+_2(z; q) := -V^\pm_\pm(z\omega + K)P^\leq_q(H_0 - z)^{-1}(z\omega + K^*)V^\pm_\pm,
\]

\[
R^+_2(z) = R^+_2(z; q) := -V^\pm_\pm(z\omega + K)P^\geq_q(H_0 - z)^{-1}(z\omega + K^*)V^\pm_\pm.
\]
Proposition 4.5. Fix \( q \in \mathbb{Z}_+ \). For every \( E \in (-\infty, \Lambda_{q+1}) \) there exists \( R_2^+(E) = R_2^+(E)^* \in \mathcal{S}_2 \) such that
\[
\lim_{z \to E} \|R_2^+(z) - R_2^+(E)\|_2 = 0,
\]
\( R_2^+(E) \) depends continuously in \( \mathcal{S}_2 \) on \( E \), and
\[
\|R_2^+(E)\|_2 = O \left( (E^2 + 1) \left( 1 + |E| (\Lambda_{q+1} - E)^{-1} \right) \right), \quad E \in (-\infty, \Lambda_{q+1}).
\]

Proof. We have
\[
R_2^+(z) = -V_\pm^\frac{1}{2} (z \omega + K) H_0^{-1/2} (P_q^+ (I + z(H_0 - z)^{-1})) H_0^{-1/2} (z \omega + K^*) V_\pm^\frac{1}{2}.
\]

Now the claims of the proposition follow from the facts that by (2.4) we have \( V_\pm \in \mathcal{S}_2 \), the operators \( \omega H_0^{-1/2}, \ K H_0^{-1/2}, \) and \( P_q^+ \), are bounded,
\[
n - \lim_{\delta \to 0} P_q^+(H_0 - E + i \delta))^{-1} = \sum_{j > q} p_j \otimes (H_{0,\|} + \Lambda_j - E)^{-1},
\]
the operator \( \sum_{j > q} p_j \otimes (H_{0,\|} + \Lambda_j - E)^{-1} \) depends continuously in \( \mathfrak{B} \) on \( E \in (-\infty, \Lambda_{q+1}), \) and
\[
\left\| \sum_{j > q} p_j \otimes (H_{0,\|} + \Lambda_j - E)^{-1} \right\| = (\Lambda_{q+1} - E)^{-1}.
\]

Further,
\[
M_2^+(z; q) = \sum_{j \leq q} M_{2,j}^+(z)
\]
where
\[
M_{2,j}^+(z) := -V_\pm^\frac{1}{2} (z \omega + K) P_j (H_0 - z)^{-1} (z \omega + K^*) V_\pm^\frac{1}{2}, \quad z \in \mathcal{C}_-, \quad j \in \mathbb{Z}_+.
\]
Let \( \omega_4 \in C_0^\infty(\mathbb{R}; [0, 1]) \) be such a function that \( \omega_4(x_\|) = 1 \) if \( x_\| \in \pi_\| (\text{supp} \omega) \). Then,
\[
M_{2,j}^+(z) := -V_\pm^\frac{1}{2} (z \omega + K) P_j \left( p_j \otimes (\omega_4(H_{0,\|} + \Lambda_j - z)^{-1} \omega_4) \right) P_j (z \omega + K^*) V_\pm^\frac{1}{2}, \quad j \leq q.
\]
Define the operator
\[
L_j(z) := (z \omega - \Delta \omega + \omega_1 G_1 + \omega_2 G_2) P_j, \quad z \in \mathcal{C}, \quad j \in \mathbb{Z}_+,
\]
so that \((z \omega + K) P_j = L_j(z) + \omega_3 G_3 P_j\). Set
\[
\mathcal{R}(z) = \omega_4(H_{0,\|} - z)^{-1} \omega_4, \quad \mathcal{R}(z) = \omega_4 \partial (H_{0,\|} - z)^{-1} \omega_4, \quad z \in \mathcal{C}_-.
\]
Then we have
\[
M_{2,j}^+(z) =
\]
\[
- V_\pm^\frac{1}{2} \left( L_j(z) p_j \otimes \mathcal{R}(z - \Lambda_j) L_j(\bar{z})^* - \omega_3 p_j \otimes (I_\| - (\Lambda_j - z) \mathcal{R}(z - \Lambda_j)) \omega_3 \right) V_\pm^\frac{1}{2}
+ V_\pm^\frac{1}{2} \left( L_j(z) p_j \otimes \mathcal{R}(z - \Lambda_j) \omega_3 + \omega_3 p_j \otimes \bar{\mathcal{R}}(z - \Lambda_j) L_j(\bar{z})^* \right) V_\pm^\frac{1}{2}.
\]
Lemma 4.6. Let $E \in \mathbb{R} \setminus \{0\}$. Then there exist operators $\mathcal{R}(E), \widetilde{\mathcal{R}}(E) \in \mathcal{S}_2(L^2(\mathbb{R}))$ such that

$$n - \lim_{C \to z \to E} \|\mathcal{R}(z) - \mathcal{R}(E)\|_2, \quad n - \lim_{C \to z \to E} \|\widetilde{\mathcal{R}}(z) - \widetilde{\mathcal{R}}(E)\|_2.$$ 

Moreover, the operator $\mathcal{R}(E)$ admits the integral kernel

$$(4.27) \quad \mathcal{K}_E(x\|, x') = \begin{cases} \frac{1}{2\sqrt{|E|}} w_4(x\|) e^{-|E||x| - |x'|} w_4(x'), & E < 0, \\ -\frac{1}{2\sqrt{E}} w_4(x\|) e^{-i|E||x| - |x'|} w_4(x'), & E > 0, \end{cases} x\|, x' \in \mathbb{R},$$

while the operator $\widetilde{\mathcal{R}}(E)$ admits the integral kernel

$$\widetilde{\mathcal{K}}_E(x\|, x') = \begin{cases} -\text{sign}(x\| - x') w_4(x\|) e^{-|E||x| - |x'|} w_4(x'), & E < 0, \\ -\text{sign}(x - x') w_4(x) e^{-i|E||x| - |x'|} w_4(x'), & E > 0, \end{cases} x\|, x' \in \mathbb{R},$$

so that $\mathcal{R}(E)$ and $\widetilde{\mathcal{R}}(E)$ depend continuously in $\mathcal{S}_2(L^2(\mathbb{R}))$ on $E \in \mathbb{R} \setminus \{0\}$, and

$$\|\mathcal{R}(E)\|_2 \leq (2\sqrt{|E|})^{-1} \|\omega_4\|_{L^2(\mathbb{R})}^2, \quad \|\widetilde{\mathcal{R}}(E)\|_2 \leq 2^{-1} \|\omega_4\|_{L^2(\mathbb{R})}^2, \quad E \in \mathbb{R} \setminus \{0\}.$$ 

We omit the proof based on elementary facts from complex and functional analysis.

Remark: In fact, $\mathcal{R}(E) \in \mathcal{S}_1(L^2(\mathbb{R}))$ (see [11, Eq. (4.4)]) but we will not use this in the article.

For $j \in \mathbb{Z}_+$ and $E \in \mathbb{R} \setminus \{\Lambda_j\}$ set

$$M_{2,j}^\pm (E) = -V_{2,j}^\pm \left(L_j(E) p_j \otimes \mathcal{R}(E - \Lambda_j) L_j(E)^* - \omega_3 p_j \otimes (I\| - (\Lambda_j - E) \mathcal{R}(E - \Lambda_j)) \omega_3\right) V_{2,j}^\pm + V_{2,j}^\pm \left(L_j(E) p_j \otimes \widetilde{\mathcal{R}}(E - \Lambda_j) \omega_3 + \omega_3 p_j \otimes \widetilde{\mathcal{R}}(E - \Lambda_j) L_j(E)^*\right) V_{2,j}^\pm.$$ 

(4.28)

Proposition 4.7. Let $j \in \mathbb{Z}_+$ and $E \in \mathbb{R} \setminus \{\Lambda_j\}$. Then we have

$$\text{Re} \ M_{2,j}^\pm (E) \in \mathcal{S}_2, \quad \text{Im} \ M_{2,j}^\pm (E) \in \mathcal{S}_1,$$

$$\lim_{C \to z \to E} \|M_{2,j}^\pm (z) - M_{2,j}^\pm (E)\|_2 = 0,$$

the operator $\text{Re} \ M_{2,j}^\pm (E)$ (resp., $\text{Im} \ M_{2,j}^\pm (E)$) depends continuously in $\mathcal{S}_2$ (resp., in $\mathcal{S}_1$) on $E$, and

$$\|\text{Re} \ M_{2,j}^\pm (E)\|_2, \quad \|\text{Im} \ M_{2,j}^\pm (E)\|_1 = O \left((E^2 + 1) \left|E - \Lambda_j\right|^{-1/2}\right), \quad E \in \mathbb{R} \setminus \{\Lambda_j\}.$$ 

Proof. Set

$$F_{1,j}(E) := L_j(E) p_j \otimes \mathcal{R}(E - \Lambda_j) L_j(E)^*, \quad F_{2,j}(E) := -(\Lambda_j - E) \omega_3 p_j \otimes \mathcal{R}(E - \Lambda_j) \omega_3,$$

$$F_{3,j}(E) := -L_j(E) p_j \otimes \widetilde{\mathcal{R}}(E - \Lambda_j) \omega_3, \quad F_{4,j}(E) := -\omega_3 p_j \otimes \widetilde{\mathcal{R}}(E - \Lambda_j) L_j(E)^*.$$
Then,
\begin{equation}
M_{2,j}^\pm(E) = -\sum_{\ell=1}^4 V_\pm^{\frac{1}{2}} F_{\ell,j}(E)V_\pm^{\frac{1}{2}} - V_\pm^{\frac{1}{2}} \omega_3 P_j \omega_3 V_\pm^{\frac{1}{2}},
\end{equation}
so that
\begin{equation}
\text{Re } M_{2,j}^\pm(E) = -\sum_{\ell=1}^4 \text{Re} \left(V_\pm^{\frac{1}{2}} F_{\ell,j}(E)V_\pm^{\frac{1}{2}}\right) - V_\pm^{\frac{1}{2}} \omega_3 P_j \omega_3 V_\pm^{\frac{1}{2}},
\end{equation}
\begin{equation}
\text{Im } M_{2,j}^\pm(E) = -\sum_{\ell=1}^4 \text{Im} \left(V_\pm^{\frac{1}{2}} F_{\ell,j}(E)V_\pm^{\frac{1}{2}}\right).
\end{equation}

Taking into account Lemma 4.6 and the facts that the orthogonal projection $p_j$ has an integral kernel in $C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ while the functions $\omega$ and $\omega_k$, $k = 1, 2, 3$, are in $C^\infty_0(\mathbb{R}^3)$, we find that $F_{\ell,j}$, $\ell = 1, \ldots, 4$, are continuous functions from $\mathbb{R} \setminus \{\Lambda_j\}$ to $\mathcal{S}_2$, and
\begin{equation}
\|F_{1,j}(E)\|_2 = O \left( (E^2 + 1) |E - \Lambda_j|^{-1/2} \right),
\end{equation}
\begin{equation}
\|F_{2,j}(E)\|_2 = O \left( (|E| + 1) |E - \Lambda_j|^{1/2} \right),
\end{equation}
\begin{equation}
\|F_{\ell,j}(E)\|_2 = O (|E| + 1), \quad \ell = 3, 4.
\end{equation}
Since, by (2.4), we have $V_\pm \in \mathcal{S}_2$, we find that $V_\pm^{\frac{1}{2}} F_{\ell,j}(E)V_\pm^{\frac{1}{2}} \in \mathcal{S}_1$. Moreover, the continuity of $F_{\ell,j}$ in $\mathcal{S}_2$ implies the continuity of $V_\pm^{\frac{1}{2}} F_{\ell,j}(E)V_\pm^{\frac{1}{2}}$ in $\mathcal{S}_1$, and
\begin{equation}
\|V_\pm^{\frac{1}{2}} F_{\ell,j}(E)V_\pm^{\frac{1}{2}}\|_1 \leq \|V_\pm\|_2 \|F_{\ell,j}\|_2, \quad \ell = 1, \ldots, 4.
\end{equation}
Finally, by (2.4), we have $V_\pm^{\frac{1}{2}} \omega_3 P_j \omega_3 V_\pm^{\frac{1}{2}} \in \mathcal{S}_2$. Therefore, the claims of the proposition follow from representation (4.29) and the properties of $F_{\ell,j}$ established above.

Now Proposition 2.2 follows from Propositions 4.5, 4.4, and 4.7.

### 4.3. Proof of Proposition 2.3
As above, we denote by $X$ a separable Hilbert space.

**Lemma 4.8.** [27, Lemma 2.1] Let $T_1 = T_1^* \in \mathcal{S}_\infty(X)$, $T_2 = T_2^* \in \mathcal{S}_1(X)$. Then for any $s > 0$ we have
\begin{equation}
\frac{1}{\pi} \int_{\mathbb{R}} n_\pm(s; T_1 + tT_2) dt \frac{dt}{1 + t^2} \leq n_\pm(s/2, T_1) + \frac{2}{\pi s} \|T_2\|_1.
\end{equation}

Our next lemma contains an elementary Chebyshev-type estimate for the eigenvalue counting functions of compact operators.

**Lemma 4.9.** Let $T = T^* \in \mathcal{S}_p(X)$, $p \in [1, \infty)$. Then for any $s > 0$ we have
\begin{equation}
n_+(s; T) := n_+(s; T) + n_-(s; T) \leq s^{-p} \|T\|_p^p.
\end{equation}
By Lemma 4.8 with \( s = 1 \), and Lemma 4.9 with \( s = 1/2 \) and \( p = 2 \), we obtain
\[
|\tilde{\xi}(E; H_\pm, H_0)| \leq 4\|\text{Re} T^\pm(E)\|_2^2 + \frac{2}{\pi}\|\text{Im} T^\pm(E)\|_1.
\]
Combining (4.32) with Proposition 2.2 we find that \( \tilde{\xi}(E; H_\pm, H_0) \) is well defined for any \( E \in (0, \infty) \setminus b(2\mathbb{Z}_+ + 1) \), and \( \tilde{\xi}(\cdot; H_\pm, H_0) \) is bounded on every compact subset of \( (0, \infty) \setminus b(2\mathbb{Z}_+ + 1) \).

Let us now prove the continuity of \( \tilde{\xi}(\cdot; H_\pm, H_0) \) following the main ideas of the proof of the continuity part of [11] Proposition 2.5. Let \( E_0 \in (0, \infty) \setminus b(2\mathbb{Z}_+ + 1) \). Assume that
\[
\lim_{E \to E_0} \|\text{Re} T^\pm(E) - \text{Re} T^\pm(E_0)\|_1 = \lim_{E \to E_0} \|\text{Im} T^\pm(E) - \text{Re} T^\pm(E_0)\|_1 = 0,
\]
(4.34) \[ \pm 1 \not\in \sigma(T^\pm(E_0)). \]
Then, by [27] Lemma 2.5 we have
\[
\lim_{E \to E_0} \tilde{\xi}(E; H_\pm, H_0) = \tilde{\xi}(E_0; H_\pm, H_0).
\]
Proposition 2.2 implies (4.33) for any \( E_0 \in (0, \infty) \setminus b(2\mathbb{Z}_+ + 1) \). Moreover, (4.34) with \( E_0 \in (0, \infty) \setminus (\sigma_p(H_\pm) b(2\mathbb{Z}_+ + 1)) \) will follow from

**Lemma 4.10.** Let \( E \in (0, \infty) \setminus b(2\mathbb{Z}_+ + 1) \). Assume that
\[
\pm 1 \in \sigma_p(T^\pm(E)).
\]
Then
\[
E \in \sigma_p(H_\pm).
\]

**Proof.** If \( E \in (0, b) \) then \( E \in \rho(H_0) \) so that in the Neumann case the lemma follows from the Birman-Schwinger principle. In the Dirichlet case, (4.36) and, hence, (4.35) cannot hold true with \( E \in (0, b) \). That is why we assume that \( E > b \), and will follow the general lines of the proof of [32] Section XIII.8, Lemma 8].

Let \( 0 \not\in \varphi = \varphi^\pm \in L^2(\mathbb{R}^3) \) satisfy
\[
T^\pm(E) \varphi = \pm \varphi.
\]
(4.37) \[ T^\pm(E) \varphi = \pm \varphi. \]
Set \( \phi := V_{\pm}^{\frac{1}{2}} \varphi \), and
\[
w_s(t) := (1 + t^2)^{s/2}, \quad t \in \mathbb{R}, \quad s \in \mathbb{R}.
\]
As usual, we denote the multiplier by \( w_s \) acting in \( L^2(\mathbb{R}) \) by the same symbol \( w_s \). Moreover, for \( s \in \mathbb{R} \), set
\[
(W_s u)(x_\perp, x_\parallel) := w_s(x_\parallel) u(x_\perp, x_\parallel), \quad (x_\perp, x_\parallel) \in \mathbb{R}^3, \quad u \in \mathcal{D}(W_s) \subset L^2(\mathbb{R}^3).
\]
Writing \( \phi = H_0^{-1} \omega H_0 \phi \) with \( \omega \in C^\infty_0(\mathbb{R}^3; [0, 1]) \) such that \( \omega = 1 \) in a neighborhood of \( \Gamma \) (see Lemma 4.11), and commuting \( W_s \) with \( H_0^{-1} \) appropriately many times, we easily find that
\[
W_s \phi \in L^2(\mathbb{R}^3), \quad s \in \mathbb{R}.
\]
Let \( \{ \varphi_{k,q} \}_{k \in \mathbb{Z}_+} \) be an orthogonal basis of Ran \( p_q, q \in \mathbb{Z}_+ \), for example the canonic basis defined in (A.1) – (A.2) below. Set
\[
\phi_{k,q}(x) := \int_{\mathbb{R}^2} \phi(x_\perp, x_\parallel) \varphi_{k,q}(x_\perp) \, dx_\perp, \quad x_\parallel \in \mathbb{R}, \quad k, q \in \mathbb{Z}_+.
\]
Evidently,
\[
\big\| \phi \big\|^2_{L^2(\mathbb{R}^3)} = \sum_{(k,q) \in \mathbb{Z}^2_+} \| \varphi_{k,q} \|^2_{L^2(\mathbb{R})}.
\]
Moreover, by (4.38), we find that
\[
(4.39) \quad w_s \varphi_{k,q} \in L^2(\mathbb{R}), \quad s \in \mathbb{R}, \quad k, q \in \mathbb{Z}_+.
\]
Further, for any \( z \in \rho(H_0) \) we have
\[
(4.40) \quad (H_0 - z)^{-1} = \sum_{q \in \mathbb{Z}_+} p_q \otimes (H_0 + \Lambda_q - z)^{-1}.
\]
Let \( q_0 \) be the largest integer satisfying \( q_0 < \frac{E - b}{2b} \). Since \( E > b \), we have \( q_0 \geq 0 \).

By (4.37), (4.20), and (4.40), we get
\[
0 = \lim_{\varepsilon \downarrow 0} \text{Im} \langle (H_0^{-1} - (E + i\varepsilon)^{-1})^{-1} \phi, \phi \rangle = -E^2 \lim_{\varepsilon \downarrow 0} \text{Im} \langle (H_0 - E - i\varepsilon)^{-1} \phi, \phi \rangle = -\pi E^2 \sum_{q=0}^{q_0} \sum_{k \in \mathbb{Z}_+} (E - \Lambda_q)^{-1/2} \left( \left| \hat{\varphi}_{k,q} \left( -\sqrt{E - \Lambda_q} \right) \right|^2 + \left| \hat{\varphi}_{k,q} \left( \sqrt{E - \Lambda_q} \right) \right|^2 \right),
\]
that is
\[
(4.41) \quad \hat{\varphi}_{k,q} \left( \pm \sqrt{E - \Lambda_q} \right) = 0, \quad k \in \mathbb{Z}_+, \quad q = 0, \ldots, q_0,
\]
where \( \hat{\varphi}_{k,q} \) is the Fourier transform of \( \varphi_{k,q} \). Then, by [31, Section IX.9, Lemma 3,], relations (4.39) and (4.41) imply the existence of a function \( \beta_{k,q} \in L^2(\mathbb{R}) \) such that
\[
\beta_{k,q} = \lim_{\varepsilon \downarrow 0} (H_0 + \Lambda_q - E - i\varepsilon)^{-1} \phi_{k,q}; \quad k \in \mathbb{Z}_+, \quad q = 0, \ldots, q_0.
\]

Set
\[
\beta_{k,q} := (H_0 + \Lambda_q - E)^{-1} \phi_{k,q}, \quad k \in \mathbb{Z}_+, \quad q > q_0,
\]
and
\[
\beta(x_\perp, x_\parallel) := \sum_{(k,q) \in \mathbb{Z}_+^2} \beta_{k,q}(x_\parallel) \varphi_{k,q}(x_\perp), \quad (x_\perp, x_\parallel) \in \mathbb{R}^3.
\]
Then \( \beta \in L^2(\mathbb{R}^3) \), and
\[
(4.42) \quad \beta = \lim_{\varepsilon \downarrow 0} (H_0 - E - i\varepsilon)^{-1} \phi.
\]
Set \( \psi := -E^2 \beta - E \phi \). Then, by (4.20) and (4.42), we have
\[
(4.43) \quad \psi = \lim_{\varepsilon \downarrow 0} (H_0^{-1} - (E + i\varepsilon)^{-1})^{-1} \phi \in L^2(\mathbb{R}^3).
\]
Moreover, (4.37) implies
\[
(4.44) \quad V_\pm \psi = \pm \phi.
\]
By (4.43) and (4.44), we easily find that
\[ H_0^{-1} \psi = \pm V_\pm \psi + E^{-1} \psi \]
which is equivalent to \( H_\pm \psi = E \psi, \psi \in \mathcal{D}(H_\pm) \). Since \( \psi \neq 0 \), we arrive at (4.36).

Finally, we prove (2.11), following the general ideas of [27]. By the invariance principle, (4.45)
\[ \xi(E; H_\pm) = -\xi(E^{-1}; H_\pm, H_0^{-1}), \quad E \in (0, \infty), \]
where \( \xi(E^{-1}; H_\pm, H_0^{-1}) := \xi(E^{-2}; H_\pm^2, H_0^{-2}) \), and \( \xi(E^{-2}; H_\pm^2, H_0^{-2}) \) is defined by (2.6).

Let \( \{ \lambda_j^\pm \}_{j \in \mathbb{N}} \) be the non-increasing sequence of the non-zero eigenvalues of \( V_\pm \), and \( \{ f_j^\pm \}_{j \in \mathbb{N}} \) be the corresponding orthonormal eigenfunctions, so that
\[ V_\pm = \sum_{j \in \mathbb{N}} \lambda_j^\pm \langle \cdot, f_j^\pm \rangle f_j^\pm. \]

For \( \ell \in \mathbb{N} \) set
\[ V_{\pm, \ell} := \sum_{j=1}^\ell \lambda_j^\pm \langle \cdot, f_j^\pm \rangle f_j^\pm, \quad S_{\pm, \ell} := H_0^{-1} + V_{\pm, \ell}, \]
\[ T_\ell^\pm(E) := n - \lim_{C \to 0} V_{\pm, \ell} \langle \cdot, f_j^\pm \rangle f_j^\pm, \quad E \in (0, \infty) \setminus b(2\mathbb{Z}_+ + 1). \]

It is easy to check that \( \text{Re} T_\ell^\pm(E) \in \mathcal{S}_2, \text{Im} T_\ell^\pm(E) \in \mathcal{S}_1 \), and
\begin{equation}
(4.46) \quad \lim_{\ell \to \infty} \| \text{Re} T_\ell^\pm(E) - \text{Re} T^\pm(E) \|_2 = \lim_{\ell \to \infty} \| \text{Im} T_\ell^\pm(E) - \text{Im} T^\pm(E) \|_1 = 0.
\end{equation}

Since the ranks of the operators \( V_{\pm, \ell}, \ell \in \mathbb{N} \) are finite, and, hence \( V_{\pm, \ell} \in \mathcal{S}_1 \), we find that [27, Theorem 1.1] implies
\begin{equation}
(4.47) \quad \xi(E^{-1}; S_{\pm, \ell}, H_0^{-1}) = \mp \frac{1}{\pi} \int_{\mathbb{R}} n_{\pm}(1; \text{Re} T_\ell^\pm(E) + t\text{Im} T_\ell^\pm(E)) \frac{dt}{1 + t^2}
\end{equation}
for almost every \( E \in (0, \infty) \). It remains to pass to the limit as \( \ell \to \infty \) at both hand sides of (4.47).

We have
\begin{equation}
(4.48) \quad \xi(E^{-1}; S_{\pm, \ell}, H_0^{-1}) = \xi(E^{-1}; S_{\pm, \ell}, H_\pm^{-1}) + \xi(E^{-1}; H_\pm^{-1}, H_0^{-1}).
\end{equation}

Bearing in mind (2.5), we apply [27, Lemma 4.2], and obtain
\begin{equation}
(4.49) \quad \lim_{\ell \to \infty} \xi(E^{-1}; S_{\pm, \ell}, H_\pm^{-1}) = 0
\end{equation}
for almost every \( E \in (0, \infty) \).

Next, combining (4.47), (4.46), and Lemma 4.10, we find that [27, Lemma 2.5] implies
\begin{equation}
(4.50) \quad \lim_{\ell \to \infty} \int_{\mathbb{R}} n_{\pm}(1; \text{Re} T_\ell^\pm(E) + t\text{Im} T_\ell^\pm(E)) \frac{dt}{1 + t^2} = \mp \xi(E; H_\pm, H_0),
\end{equation}
for every \( E \in (0, \infty) \setminus (\sigma_p(H_\pm) \cup b(2\mathbb{Z}_+ + 1)) \).

Putting together (4.45), (4.47), and (4.48)-(4.50), we obtain (2.11).

5. Proof of Theorem 3.1

Throughout the section the parameter \( q \in \mathbb{Z}_+ \) is fixed as in Theorem 3.1.
5.1. The effective Hamiltonians. Define the rank-one operator
\[ p := \langle \cdot, \omega_4 \rangle \omega_4, \]
self-adjoint in \( L^2(\mathbb{R}) \). For \( \lambda \in (-b, 0) \cup (0, b) \), set
\[ M_{3,q}^\pm(\lambda) := -\frac{i(\lambda)}{2\sqrt{|\lambda|}} M_q^\pm \]
where
\[ i(\lambda) := \begin{cases} 1 & \text{if } \lambda < 0, \\ -i & \text{if } \lambda > 0, \end{cases} \]
\[ M_q^\pm := V^\pm_+ T_q p_q \otimes p_T^* V^\pm_-, \]
and
\[ (5.1) \quad T_q := L_q(\Lambda_q), \]
the operator \( L_q(z) \) being defined in \((4.25)\). Note that the operators \( M_q^\pm \) are self-adjoint and non-negative so that the operators \( M_{3,q}^\pm(\lambda) \) are self-adjoint and non-positive if \( \lambda < 0 \) and purely imaginary if \( \lambda > 0 \).

Proposition 5.1. Let \( q \in \mathbb{Z}_+ \), \( \epsilon \in (0, 1) \). Then we have
\[ (5.2) \quad \xi(\Lambda_q - \lambda; H_+, H_0) = O(1), \]
\[ (5.3) \quad -n_+((1 - \epsilon)2\sqrt{\lambda}; M_q^-) + O(1) \leq \xi(\Lambda_q - \lambda; H_-, H_0) \leq -n_+((1 + \epsilon)2\sqrt{\lambda}; M_q^-) + O(1), \]
\[ (5.4) \quad \frac{1}{\pi} \text{Tr} \arctan \left( \frac{M_q^+}{(1 + \epsilon)2\sqrt{\lambda}} \right) + O(1) \leq \xi(\Lambda_q + \lambda; H_+, H_0) \leq \frac{1}{\pi} \text{Tr} \arctan \left( \frac{M_q^+}{(1 - \epsilon)2\sqrt{\lambda}} \right) + O(1), \]
\[ (5.5) \quad -\frac{1}{\pi} \text{Tr} \arctan \left( \frac{M_q^-}{(1 - \epsilon)2\sqrt{\lambda}} \right) + O(1) \leq \xi(\Lambda_q + \lambda; H_-, H_0) \leq -\frac{1}{\pi} \text{Tr} \arctan \left( \frac{M_q^-}{(1 + \epsilon)2\sqrt{\lambda}} \right) + O(1), \]
as \( \lambda \downarrow 0 \).

Remark: According to Proposition 5.1, the operators \( M_q^\pm \) play the role of effective Hamiltonians in the asymptotic analysis of the SSF \( \xi(E; H\pm, H_0) \) as the energy \( E \) approaches the Landau level \( \Lambda_q \), \( q \in \mathbb{Z}_+ \).

For the proof of Proposition 5.1 we need the well known Weyl inequalities for the eigenvalues of compact operators, described in the following
Lemma 5.2. [7] Theorem 9, Section 9.2] Let $X$ be a separable Hilbert space, and $T_j = T_j^* \in \mathcal{S}_\infty(X)$, $j = 1, 2$. Then for any $s_j > 0$ we have
\[ n_\pm(s_1 + s_2; T_1 + T_2) \leq n_\pm(s_1, T_1) + n_\pm(s_2, T_2). \]

Proof of Proposition 5.1. Set
\[ R_{3,q}^\pm(\lambda) := T^\pm(\Lambda_q + \lambda) - M_{3,q}^\pm(\lambda). \]

By Propositions 4.5, 4.4, and 1.7, estimates (4.30)-(4.31), and the explicit form (4.27) of the integral kernel of the operator $\mathcal{R}(\lambda)$, we have
\[ \|\text{Re } R_{3,q}^\pm(\lambda)\|_2 = O(1), \quad \|\text{Im } R_{3,q}^\pm(\lambda)\|_1 = O(1), \quad \lambda \downarrow 0. \]

Applying Lemma 5.2, we get
\[ \frac{1}{\pi} \int \frac{n_\pm(1 + \epsilon; \text{Re } M_{3,q}^\pm(\lambda) + t\text{Im } M_{3,q}^\pm(\lambda))}{1 + t^2} dt. \]

Using Lemmas 4.8 and 4.9 with $s > 0$ and $p = 2$, we obtain
\[ \frac{1}{\pi} \int n_\epsilon(s; \text{Re } R_{3,q}^\pm(\lambda) + t\text{Im } R_{3,q}^\pm(\lambda)) \frac{dt}{1 + t^2} \leq \frac{4}{s^2} \|\text{Re } R_{3,q}^\pm(\lambda)\|_2^2 + \frac{2}{\pi s} \|\text{Im } R_{3,q}^\pm(\lambda)\|_1. \]

Putting together (2.10), (5.7), (5.8), and (5.6), we get
\[ \frac{1}{\pi} \int n_\pm(1 + \epsilon; \text{Re } M_{3,q}^\pm(\lambda) + t\text{Im } M_{3,q}^\pm(\lambda)) \frac{dt}{1 + t^2} + O(1) \leq \pm \xi(\Lambda_q + \lambda; H_\pm, H_0) \leq \]
\[ \frac{1}{\pi} \int n_\pm(1 - \epsilon; \text{Re } M_{3,q}^\pm(\lambda) + t\text{Im } M_{3,q}^\pm(\lambda)) \frac{dt}{1 + t^2} + O(1) \]
as $\lambda \to 0$. Simple calculations show that for $s > 0$ we have
\[ \frac{1}{\pi} \int n_\epsilon(s; \text{Re } M_{3,q}^\pm(\lambda) + t\text{Im } M_{3,q}^\pm(\lambda)) \frac{dt}{1 + t^2} = 0, \]
\[ \frac{1}{\pi} \int n_\epsilon(s; \text{Re } M_{3,q}^\pm(\lambda) + t\text{Im } M_{3,q}^\pm(\lambda)) \frac{dt}{1 + t^2} = n_\pm(2s\sqrt{\lambda}; \mathcal{M}_q^\pm), \]
if \( \lambda < 0 \), and
\[
\frac{1}{\pi} \int_{\mathbb{R}} n_{\pm}(s; \text{Re} M_{3; q}^\pm(\lambda) + t \text{Im} M_{3; q}^\pm(\lambda)) \frac{dt}{1 + t^2} = \frac{1}{\pi} \text{Tr arctan} \left( \frac{M_{3; q}^\pm}{2s\sqrt{-\lambda}} \right),
\]
if \( \lambda > 0 \). Now the claims of the proposition follow from estimates (5.9) and identities (5.10) - (5.12).

Note that (5.2) is identical with (3.2), so that in order to complete the proof of Theorem 3.4 it remains to prove (3.3) - (3.4) using respectively (5.3) and (5.4) - (5.5). Here we state two more lemmas needed for the estimates of \( n_+(s; M_q^\pm) \).

**Lemma 5.3.** Let \( X_j, j = 1, 2, \) be Hilbert spaces, and \( J : X_1 \to X_2 \) be a linear compact operator. Then we have
\[
n_+(s; J^* J) = n_+(s; J J^*), \quad s > 0.
\]

**Proof.** The claim follows immediately form [7, Chapter 8, Section 1, Theorem 4]. \( \square \)

**Lemma 5.4.** Let \( X_j, j = 1, 2, \) be Hilbert spaces, \( J : X_1 \to X_2 \) be a linear compact operator, and \( T \in \mathcal{S}_\infty(X_2) \). Then we have
\[
n_+(s; J^* (I - T) J) \geq n_+(s; (1 - \varepsilon) J^* J - \text{Tr} 1_{[\varepsilon, \infty)}(T)), \quad s > 0, \quad \varepsilon \in (0, 1).
\]

**Proof.** We have
\[
J^* (I - T) J = J^* ( (1 - \varepsilon) I + 1_{(-\infty, \varepsilon)}(T)(\varepsilon I - T) + 1_{[\varepsilon, \infty)}(T)(\varepsilon I - T) ) J.
\]
Evidently,
\[
1_{(-\infty, \varepsilon)}(T)(\varepsilon I - T) \geq 0, \quad \text{rank} J^* 1_{[\varepsilon, \infty)}(T)(\varepsilon I - T) J \leq \text{Tr} 1_{[\varepsilon, \infty)}(T).
\]
By the mini-max principle and [7, Chapter 9, Section 3, Theorem 3], now (5.13) follows from (5.14) and (5.15). \( \square \)

For further references, set
\[
M_{4; q}^\pm := (p_q \otimes p) T_q^* V_\pm T_q(p_q \otimes p).
\]
The operator \( M_{4; q}^\pm \) will be considered as a compact self-adjoint operator in the Hilbert space \((p_q \otimes p)L^2(\mathbb{R}^3)\). By Lemma 5.3, we have
\[
n_+(s; M_{4; q}^\pm) = n_+(s; M_{4; q}^\pm), \quad s > 0.
\]

5.2. **Lower bounds of \( n_+(s; M_{4; q}^\pm) \) in the Dirichlet case.** In this and in the following subsection we assume \( \omega = 1 \) in a neighborhood of \( \overline{\Omega_{\text{in}}} \), where \( \omega \) is the function which participates in the definition of the operator \( L_q(z) \) (see (4.23)), and hence in that of \( T_q \) (see (5.11)), and of \( M_{4; q}^\pm \) (see (5.16)).

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain. Note that for any \( x_\perp \in \mathbb{R} \) the function
\[
\mathbb{R} \ni x_\parallel \mapsto 1_{\Omega}(x_\perp, x_\parallel) \in \{0, 1\}
\]
is Lebesgue measurable and has a bounded support. Set
\[
w_\Omega(x_\perp) := \int_{\mathbb{R}} 1_{\Omega}(x_\perp, x_\parallel) dx_\parallel, \quad x_\perp \in \mathbb{R}^2.
\]
Evidently, \( w_\Omega(x_\perp) \geq 0 \) for every \( x_\perp \in \mathbb{R}^2 \), and \( w_\Omega(x_\perp) > 0 \) if and only if \( x_\perp \in \pi_\perp(\Omega) \).

**Proposition 5.5.** Let the domain \( \Omega_\subset \subset \mathbb{R}^3 \) satisfy \( \overline{\Omega_\subset} \subset \Omega_{in} \). Then we have

\[
(5.19) \quad n_+(s; M^+_{4,q}) \geq n_+ \left( 4s \| \omega_1^2 \|_{L^2(\mathbb{R})}^2; p_q w_{\Omega_\subset} p_q \right) + O(1), \quad s > 0.
\]

**Proof.** By definition of \( T_q \) (see (5.11)), we have

\[
T_q = L_q(\Lambda_q) = (\Lambda_q \omega + K) P_q - \omega_3 P_q G_3 = (\Lambda_q \omega - \omega H_0) P_q + H_0 \omega P_q - \omega_3 P_q G_3.
\]

Using that \( \partial_{x_3} \omega_4 = 0 \) on the support of \( \omega \) and hence of \( \omega_3 \), we obtain

\[
\omega_3 G_3(p_q \otimes p) = 0, \quad \omega H_0 P_q(p_q \otimes p) = \Lambda_q \omega P_q(p_q \otimes p), \quad T_q(p_q \otimes p) = H_0 \omega P_q(p_q \otimes p),
\]

and, hence,

\[
(5.20) \quad M^+_{4,q} = (p_q \otimes p) P_q \omega H_0 V_+ H_0 \omega P_q(p_q \otimes p).
\]

On the other hand,

\[
V_+ = H_0^{-1} - H_{+,in}^{-1} \oplus H_{+,ex}^{-1} = V_{+,0} - R_{in},
\]

with

\[
V_{+,0} := H_0^{-1} - 0 \oplus H_{+,ex}^{-1}, \quad R_{in} := H_{+,in}^{-1} \oplus 0.
\]

Obviously,

\[
-R_{in}^\frac{1}{2} = \Lambda_q R_{in}^\frac{1}{2} \left( 0 \oplus H_{+,ex}^{-1} - \Lambda_q^{-1} \right) = \left( 0 \oplus H_{+,ex}^{-1} - \Lambda_q^{-1} \right) \Lambda_q R_{in}^\frac{1}{2},
\]

and since \( 0 \oplus H_{+,ex}^{-1} = H_0^{-1} - V_{+,0} \), we have

\[
R_{in} = \Lambda_q^2 \left( H_0^{-1} - \Lambda_q^{-1} - V_{+,0} \right) R_{in} \left( H_0^{-1} - \Lambda_q^{-1} - V_{+,0} \right).
\]

Moreover, from the above relations and the fact that \( \omega \) is equal to 1 on \( \overline{\Omega_{in}} \), we obtain

\[
R_{in}(H_0^{-1} - \Lambda_q^{-1}) H_0 \omega P_q(p_q \otimes p) = R_{in}(\omega - \Lambda_q^{-1} \omega H_0) P_q(p_q \otimes p) + R_{in}[\omega, H_0] P_q(p_q \otimes p) = 0.
\]

Using this relation and the dual one, we deduce

\[
M^+_{4,q} = (p_q \otimes p) P_q \omega H_0(V_{+,0} - R_{in}) H_0 \omega P_q(p_q \otimes p)
\]

\[
= (p_q \otimes p) P_q \omega H_0(V_{+,0} - \Lambda_q^2 V_{+,0} R_{in} V_{+,0}) H_0 \omega P_q(p_q \otimes p).
\]

Since the operator \( V_{+,0}^\frac{1}{2} R_{in} V_{+,0}^\frac{1}{2} \) is compact, Lemma [5.4] implies

\[
(5.21) \quad n_+(2s; (p_q \otimes p) P_q \omega H_0 V_{+,0} H_0 \omega P_q(p_q \otimes p)) - \text{Tr} \mathbb{1}_{\{1/2, \infty\}}(V_{+,0}^\frac{1}{2} R_{in} V_{+,0}^\frac{1}{2}), \quad s > 0.
\]

Further,

\[
(5.22) \quad V_{+,0} = \left( H_0^{-1} - (H_0 + \mathbb{1}_{\Omega_\subset})^{-1} \right) + \left( (H_0 + \mathbb{1}_{\Omega_\subset})^{-1} - 0 \oplus H_{+,ex}^{-1} \right).
\]
By \( \Omega_\subset \cap \Omega_{\text{ex}} = \emptyset \), the restriction to \( \mathcal{D}(H_{+,\text{ex}}^{1/2}) \) of the quadratic form of \( H_0 + \mathbb{1}_{\Omega_\subset} \) coincides with the one of \( H_{+,\text{ex}} \). Hence, by Proposition 2.1 (i), the second term on the r.h.s. of (5.22) is non-negative. Next, the resolvent identity yields

\[
H_0^{-1} - (H_0 + \mathbb{1}_{\Omega_\subset})^{-1} = H_0^{-1} \mathbb{1}_{\Omega_\subset} \left( I - \mathbb{1}_{\Omega_\subset}(H_0 + \mathbb{1}_{\Omega_\subset})^{-1} \mathbb{1}_{\Omega_\subset} \right) \mathbb{1}_{\Omega_\subset} H_0^{-1}.
\]

Thus, the mini-max principle implies

\[
(5.23) \quad n_+ (s; (p_q \otimes p) P_q \omega H_0 V_{+,0} H_0 \omega P_q (p_q \otimes p)) \geq n_+ \left( s; (p_q \otimes p) P_q \omega \mathbb{1}_{\Omega_\subset} \left( I - \mathbb{1}_{\Omega_\subset}(H_0 + \mathbb{1}_{\Omega_\subset})^{-1} \mathbb{1}_{\Omega_\subset} \right) \mathbb{1}_{\Omega_\subset} \omega P_q (p_q \otimes p) \right), \quad s > 0.
\]

Applying Lemma 5.4 and taking into account that

\[
(5.24) \quad (p_q \otimes p) P_q \omega \mathbb{1}_{\Omega_\subset} \omega P_q (p_q \otimes p) = (p_q \otimes p) \mathbb{1}_{\Omega_\subset} (p_q \otimes p),
\]

we obtain

\[
(5.25) \quad n_+ \left( s; (p_q \otimes p) P_q \omega \mathbb{1}_{\Omega_\subset} \left( I - \mathbb{1}_{\Omega_\subset}(H_0 + \mathbb{1}_{\Omega_\subset})^{-1} \mathbb{1}_{\Omega_\subset} \right) \mathbb{1}_{\Omega_\subset} \omega P_q (p_q \otimes p) \right) \geq n_+ \left( s; (p_q \otimes p) \mathbb{1}_{\Omega_\subset} (p_q \otimes p) \right) \quad \text{Tr} \mathbb{1}_{\{1/2,\infty\}} (H_0 + \mathbb{1}_{\Omega_\subset})^{-1} \mathbb{1}_{\Omega_\subset}), \quad s > 0.
\]

Finally, the operator \( (p_q \otimes p) \mathbb{1}_{\Omega_\subset} (p_q \otimes p) \) with domain \( (p_q \otimes p)L^2(\mathbb{R}^3) \) is unitarily equivalent to the operator \( \|\omega_q\|_{L^2(\mathbb{R}^3)} p_q w_{\Omega_\subset} p_q \) with domain \( p_q L^2(\mathbb{R}^2) \), where \( w_{\Omega_\subset} \) is the function defined in (5.18). Therefore,

\[
(5.26) \quad n_+ \left( s; (p_q \otimes p) \mathbb{1}_{\Omega_\subset} (p_q \otimes p) \right) = n_+ \left( \|\omega_q\|_{L^2(\mathbb{R}^3)}^2 s; p_q w_{\Omega_\subset} p_q \right), \quad s > 0.
\]

Now (5.19) follows from (5.22), (5.23), (5.25), and (5.26).

\[
5.3. \quad \text{Lower bounds of } n_+ (s; M_{4,q}^{-}) \text{ in the Neumann case.}
\]

\[
\textbf{Proposition 5.6. Let the domain } \Omega_\subset \subset \mathbb{R}^3 \text{ satisfy } \overline{\Omega_\subset} \subset \Omega_{\text{in}}. \text{ Then there exists a constant } c > 0 \text{ such that}
\]

\[
(5.27) \quad n_+ (s; M_{4,q}^{-}) \geq n_+ \left( cs\|\omega_q\|_2^2; p_q w_{\Omega_\subset} p_q \right) + O(1), \quad s > 0.
\]

\[
\textbf{Proof. By analogy with (5.20), we obtain}
\]

\[
M_{4,q}^{-} = (p_q \otimes p) P_q \omega H_0 V_{-} H_0 \omega P_q (p_q \otimes p).
\]

On the other hand, for any \( \delta > 0 \), we have

\[
V_{-} := H_{-,\text{in}}^{-1} \oplus H_{-,\text{ex}}^{-1} - H_0^{-1} = V_{-,0} - R_{\text{in}},
\]

with

\[
V_{-,0} := (1 + \delta)H_{-,\text{in}}^{-1} \oplus H_{-,\text{ex}}^{-1} - H_0^{-1}, \quad R_{\text{in}} := \delta H_{-,\text{in}}^{-1} \oplus 0.
\]

For \( \delta > 0 \) such that \( \Lambda_q (1 + \delta) \) is not an eigenvalue of \( H_{-,\text{in}} \), set

\[
r_{\text{in}} := \delta^2 H_{-,\text{in}}^{1/2} \left( I - \Lambda_q (1 + \delta) H_{-,\text{in}}^{-1} \right)^{-1} \oplus 0.
\]
Obviously,
\[-R^+_{in} = \Lambda_q r_{in} \left( (1 + \delta)H^{-1}_{-in} \oplus H^{-1}_{-ex} - \Lambda_q^{-1} \right) = \left( (1 + \delta)H^{-1}_{-in} \oplus H^{-1}_{-ex} - \Lambda_q^{-1} \right) \Lambda_q r_{in},\]
and since \((1 + \delta)H^{-1}_{-in} \oplus H^{-1}_{-ex} = V_{-0} + H_0^{-1}\), we have
\[R_{in} = \Lambda_q^2 \left( H_0^{-1} - \Lambda_q^{-1} + V_{-0} \right) r_{in}^2 \left( H_0^{-1} - \Lambda_q^{-1} + V_{-0} \right).
\]
Moreover,
\[R_{in}(H_0^{-1} - \Lambda_q^{-1})H_0 \omega P_q(p_q \otimes p) = R_{in}(\omega - \Lambda_q^{-1} \omega H_0)P_q(p_q \otimes p) + R_{in}[\omega, H_0]P_q(p_q \otimes p) = 0.
\]
Using this relation and the dual one, we deduce
\[M_{4,q} = (p_q \otimes p)P_q \omega H_0(V_{-0} - R_{in})H_0 \omega P_q(p_q \otimes p)
= (p_q \otimes p)P_q \omega H_0 \left( V_{-0} - \Lambda_q^2 V_{-0} r_{in}^2 V_{-0} \right) H_0 \omega P_q(p_q \otimes p).
\]
By Lemma 5.4
\[(5.29)\]
\[n_+ (s; M_{4,q}) \geq n_+ (2s; (p_q \otimes p)P_q \omega H_0 V_{-0} H_0 \omega P_q(p_q \otimes p) - \text{Tr} \mathbb{1}_{[1/2, \infty)}(\Lambda_q^2 V_{-0} r_{in}^2 V_{-0}).
\]
Pick \(\kappa \in (0, b)\), and write
\[(5.30)\]
\[V_{-0} = \left( (1 + \delta)H^{-1}_{-in} \oplus H^{-1}_{-ex} - (H_0 - \kappa \mathbb{1}_{\Omega <})^{-1} \right) + \left( (H_0 - \kappa \mathbb{1}_{\Omega <})^{-1} - H_0^{-1} \right),
\]
the operator \(V_{-0}\) being defined in \((5.28)\). Now choose \(\kappa\) sufficiently small so that \(\kappa(1 + \delta)^{-1}\) is smaller than the ground state of \(H_{-\infty}\). Then on \(\mathcal{D}(\mathbb{H}_{-\infty} - \kappa \mathbb{1}_{\Omega <})\frac{1}{2} = H^A_4(\mathbb{R}^3)\) the quadratic form of \(\frac{1}{1 + \delta}H_{-\infty} \oplus H_{-\infty}\) is dominated by the one of \(H_0 - \kappa \mathbb{1}_{\Omega <}\). More precisely, for any \(u \in H^A_4(\mathbb{R}^3)\) we have
\[\|\Pi(A)u\|_{L^2(\Omega_{ex})}^2 + \|\Pi(A)u\|_{L^2(\Omega_{in})}^2 - \kappa \|u\|_{L^2(\Omega_{in})}^2 \geq \|\Pi(A)u\|_{L^2(\Omega_{in})}^2 + \frac{1}{1 + \delta} \|\Pi(A)u\|_{L^2(\Omega_{in})}^2.
\]
By [28], Proposition 2.1 (i)), this shows that the first term of \((5.30)\) is a non-negative operator. Moreover, the resolved identity implies
\[\left( (H_0 - \kappa \mathbb{1}_{\Omega <})^{-1} - H_0^{-1} \right) = \kappa H_0^{-1} \mathbb{1}_{\Omega <} \left( I + \kappa \mathbb{1}_{\Omega <} (H_0 - \kappa \mathbb{1}_{\Omega <})^{-1} \mathbb{1}_{\Omega <} \right) \mathbb{1}_{\Omega <} H_0^{-1} \geq \kappa H_0^{-1} \mathbb{1}_{\Omega <} H_0^{-1}.
\]
Taking into account \((5.30)\), we get
\[V_{-0} \geq \left( (H_0 - \kappa \mathbb{1}_{\Omega <})^{-1} - H_0^{-1} \right) \geq \kappa H_0^{-1} \mathbb{1}_{\Omega <} H_0^{-1},
\]
and the mini-max principle implies
\[(5.31)\]
\[n_+ (s; (p_q \otimes p)P_q \omega H_0 V_{-0} H_0 \omega P_q(p_q \otimes p) \geq n_+ (s; \kappa (p_q \otimes p)P_q \omega \mathbb{1}_{\Omega <} \omega P_q(p_q \otimes p)), \quad s > 0.
\]
Finally, taking into account \((5.24)\), by analogy with \((5.26)\), we obtain
\[(5.32)\]
\[n_+ (s; (p_q \otimes p)P_q \omega \mathbb{1}_{\Omega <} \omega P_q(p_q \otimes p)) = n_+ \left( \|\omega_4\|_{L^2(\mathbb{R})}^2; s; p_q w < p_q \right), \quad s > 0.
\]
Now \((5.27)\) follows from \((5.29)\), \((5.31)\), and \((5.32)\).
5.4. Upper bounds of \( n_+(s; M_+^q) \).

**Proposition 5.7.** Let \( q \in \mathbb{Z}_+ \). Then there exist constants \( C_q^\pm > 0 \) such that
\[
(5.33) \quad n_+(s; M_+^{\pm q}) \leq n_+\left(C_q^\pm s; p_0 \mathbf{1}_{\pi\perp(supp \omega)} p_0\right) + O(1), \quad s > 0.
\]

**Proof.** Evidently,
\[
M_+^{\pm q} \leq \|V_\pm\| M_{5,q}
\]
where
\[
M_{5,q} := (p_q \otimes p) T_q^* T_q (p_q \otimes p).
\]
Similarly to \( M_+^{\pm q} \), the operator \( M_{5,q} \) will be considered as a compact self-adjoint operator in the Hilbert space \((p_q \otimes p)L^2(\mathbb{R}^3)\). Then,
\[
(5.34) \quad n_+(s; M_+^{\pm q}) \leq n_+(s; \|V_\pm\| M_{5,q}), \quad s > 0.
\]

Set \( \omega_0 := \Lambda_q \omega - \Delta \omega \). Define the operator
\[
M_{6,q} := p_q \left( \sum_{j,k=0}^{2} g_q^j w_{jk} g_k \right) p_q
\]
where
\[
g_0 = I_\perp, \quad g_1 = a^*, \quad g_2 = a,
\]
and
\[
w_{jk}(x_\perp) := \int_{\mathbb{R}} \omega^j(x_\perp, x_\parallel) \omega_k(x_\perp, x_\parallel) dx_\parallel, \quad x_\perp \in \mathbb{R}^2, \quad j, k = 0, 1, 2.
\]

Then the operator \( M_{5,q} \) with domain \((p_q \otimes p)L^2(\mathbb{R}^3)\) is unitarily equivalent to the operator \( \|\omega_4\|^2_{L^2(\mathbb{R})} M_{6,q} \) with domain \( p_q L^2(\mathbb{R}^2) \). Therefore,
\[
(5.35) \quad n_+(s; M_{5,q}) = n_+ \left( s \|\omega_4\|^2_{L^2(\mathbb{R})}; M_{6,q} \right), \quad s > 0.
\]

In the Appendix A we show that the operator \( M_{6,q} \) is unitarily equivalent to
\[
(5.36) \quad M_{7,q} := p_0 v_q p_0,
\]
where \( v_q : \mathbb{R}^2 \to \mathbb{R} \) is an appropriate bounded multiplier so that \( M_{7,q} \) is self-adjoint on its domain \( p_0 L^2(\mathbb{R}^2) \). More precisely, if \( q \geq 1 \), we have
\[
(5.37) \quad v_q := L_q \left( -\frac{\Delta}{2b} \right) w_{00} + 2b(q+1)L_{q+1} \left( -\frac{\Delta}{2b} \right) w_{11} + 2bq L_{q-1} \left( -\frac{\Delta}{2b} \right) w_{22}
\]
\[
- 8\text{Re} L_{q-1}^{(2)} \left( -\frac{\Delta}{2b} \right) \frac{\partial^2 w_{21}}{\partial \zeta^2} - 4\text{Im} L_q^{(1)} \left( -\frac{\Delta}{2b} \right) \frac{\partial w_{01}}{\partial \zeta} - 4\text{Im} L_{q-1}^{(1)} \left( -\frac{\Delta}{2b} \right) \frac{\partial w_{20}}{\partial \zeta},
\]
where
\[
L_q^{(m)}(t) := \sum_{j=0}^{q} \frac{q + m}{q - j} \frac{(-t)^j}{j!}, \quad t \in \mathbb{R}, \quad q \in \mathbb{Z}_+, \quad m \in \mathbb{Z}_+.
\]
are the Laguerre polynomials; as usual, we write $L_q^{(0)} = L_q$. If $q = 0$, then

$$v_0 := w_{00} + 2bL_1 \left( -\frac{\Delta}{2b} \right) w_{11} - 4\text{Im} \frac{\partial w_{01}}{\partial \zeta}.$$

Therefore,

$$n_+(s; M_6,q) = n_+(s; M_7,q), \quad s > 0.$$

Note that $v_q \in C^\infty_0(\mathbb{R}^2; \mathbb{R})$ and we have

$$\mu_q := \max_{x_\bot \in \mathbb{R}^2} v_q(x_\bot) \in (0, \infty).$$

Indeed, if $v_q \leq 0$, then (5.17), (5.34), and (5.39) would imply that $M_4^+q \leq 0$ which is impossible by Propositions 5.5 and 5.6. Moreover, it is easy to check that

$$\text{supp } v_q \subset \pi_\bot(\text{supp } \omega).$$

Therefore,

$$M_7,q \leq \mu_q p_0 \mathbb{1}_{\pi_\bot(\text{supp } \omega)} p_0,$$

and, hence,

$$n_+(s; M_7,q) \leq n_+(s; \mu_q p_0 \mathbb{1}_{\pi_\bot(\text{supp } \omega)} p_0), \quad s > 0.$$

Now (5.13) follows from (5.34), (5.35), (5.39), and (5.40).

5.5. **Properties of the logarithmic capacity.** First, we list several elementary properties of the logarithmic capacity (see e.g. [30, Chapter 5]):

(i) Let $E_1, E_2$ be Borel subsets of $\mathbb{R}^2$ such that $E_1 \subset E_2$. Then, evidently,

$$\text{Cap}(E_1) \leq \text{Cap}(E_2).$$

(ii) Let $K \subset \mathbb{R}^2$ be a compact set. For $\delta > 0$, put

$$K_\delta := \{x_\bot \in \mathbb{R}^2 \mid \text{dist} (x_\bot, K) \leq \delta\}.$$  

Then we have

$$\lim_{\delta \downarrow 0} \text{Cap}(K_\delta) = \text{Cap}(K).$$

Next, we formulate a result which allows to approximate the logarithmic capacity of a bounded plane domain by the logarithmic capacities of curves contained in the domain. Let $\gamma \subset \mathbb{R}^2$ be a Jordan curve, i.e. a simple closed curve. We will say that $\gamma$ is $C^2$-smooth if there exists a $C^2$-smooth diffeomorphism $x : S^1 \to \gamma$.

**Proposition 5.8.** [13, Proposition 5.6] Let $D \subset \mathbb{R}^2$ be a bounded domain. Then there exists a sequence $\{\gamma_j\}_{j \in \mathbb{N}}$ of $C^2$-smooth Jordan curves such that $\gamma_j \subset D$ and

$$\lim_{j \to \infty} \text{Cap}(\gamma_j) = \text{Cap}(\overline{D}).$$

**Corollary 5.9.** Let $D \subset \mathbb{R}^2$ be a bounded domain. Then

$$\text{Cap}(D) = \text{Cap}(\overline{D}).$$
Proof. Let $\{\gamma_j\}_{j \in \mathbb{N}}$ be the sequence of curves introduced in Proposition 5.8. Then (5.43) follows immediately from $\gamma_j \subset D \subset \overline{D}$ and (5.42). \qed

5.6. Eigenvalue asymptotics for the Toeplitz operators $p_q \mathbb{1}_O p_q$. Let $O \subset \mathbb{R}^2$ be a bounded domain. Fix $q \in \mathbb{Z}_+$. In the Hilbert space $p_q L^2(\mathbb{R}^2)$, consider the operator $p_q \mathbb{1}_O p_q$ which is self-adjoint, compact, and non-negative. Moreover, the results of [29, Subsection 4.3] imply that all the eigenvalues of rank $p_q \mathbb{1}_O p_q$ are strictly positive. Denote by $\{\nu_{k,q}(O)\}_{k \in \mathbb{Z}_+}$ the non-decreasing sequence of the eigenvalues of $p_q \mathbb{1}_O p_q$.

Proposition 5.10. [15, Lemma 2] Let $O \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary, and $q \in \mathbb{Z}_+$. Then we have

$$\lim_{k \to \infty} \left( k! \nu_{k,q}(O) \right)^{1/k} = \frac{b \text{Cap}(O)^2}{2},$$

or, equivalently,

$$(5.44) \quad \ln \nu_{k,q}(O) = -k \ln k + (\mathcal{C}(O) - \ln 2)k + o(k), \quad k \to \infty,$$

where $\mathcal{C}(O)$ is the constant defined in (3.1).

Corollary 5.11. Under the hypotheses of Proposition 5.10 for any constant $c > 0$ we have

$$(5.45) \quad n_+(c\sqrt{\lambda}; p_q \mathbb{1}_O p_q) = \frac{1}{2} \Phi_1(\lambda; \mathcal{C}(O)) + o \left( \frac{\ln \lambda}{\ln_2(\lambda)^2} \right), \quad \lambda \downarrow 0.$$

The proof of the corollary can be found in Subsection A.2 of the Appendix.

Corollary 5.12. Under the hypotheses of Proposition 5.10 for any constant $c > 0$ we have

$$(5.46) \quad \frac{1}{\pi} \text{Tr} \arctan \left( \frac{p_q \mathbb{1}_O p_q}{c \sqrt{\lambda}} \right) = \frac{1}{4} \Phi_1(\lambda; \mathcal{C}(O)) + o \left( \frac{\ln \lambda}{\ln_2(\lambda)^2} \right), \quad \lambda \downarrow 0.$$

The proof of the corollary is contained in Subsection A.3 of the Appendix.

5.7. Proof of (3.3) - (3.4). For $\lambda > 0$ small enough, and $q \in \mathbb{Z}_+$, set

$$\Xi_{q,1}(\lambda) := -\xi(\Lambda_q - \lambda; H_-, H_0) - 2^{-1} \Phi_1(\lambda; 1 + \ln b),$$

$$\Xi_{q,2}(\lambda) := \pm \xi(\Lambda_q + \lambda; H_+, H_0) - 2^{-2} \Phi_1(\lambda; 1 + \ln b).$$

Then (3.3) is equivalent to

$$\lim_{\lambda \downarrow 0} \Xi_{q,1}(\lambda) = \ln \text{Cap}(O_\text{in})^2,$$

while (3.4) is equivalent to

$$\lim_{\lambda \downarrow 0} \Xi_{q,2}(\lambda) = \ln \text{Cap}(O_\text{in})^2.$$
Let us first prove (5.47), starting with the corresponding lower asymptotic bound. Combining (5.3) and (5.17) with (5.27), we find that for each domain $\Omega_<$ such that $\Omega_< \subset \Omega_{in}$ there exists a constant $c > 0$ such that
\begin{equation}
-\xi(\Lambda_q - \lambda; H_-, H_0) \geq n_+(c\sqrt{\lambda}; p_q w_{\Omega_<} p_q) + O(1), \quad \lambda \downarrow 0,
\end{equation}
where $w_{\Omega_<}$ is the function defined in (5.18). Let us construct a suitable sequence of domains compactly embedded in $\Omega_{in}$. Let
\begin{equation}
\gamma_j^\prec := \{x_j(s) | s \in S^1\}, \quad j \in \mathbb{N},
\end{equation}
be a sequence of $C^2$-smooth Jordan curves such that $\gamma_j^\prec \subset \Omega_{in}$ and
\begin{equation}
\lim_{j \to \infty} \text{Cap}(\gamma_j^\prec) = \text{Cap}(\Omega_{in}) = \text{Cap}(\Omega_{in}),
\end{equation}
whose existence is guaranteed by Proposition 5.8. Let
\begin{equation}
\mathbf{n}_j(s) := \frac{(x'_{2,j}(s), -x'_{1,j}(s))}{|x'_j(s)|}, \quad s \in S^1,
\end{equation}
be the normal unit at $\gamma_j^\prec$, $j \in \mathbb{N}$. For $\tau \in (0, 1]$ set
\begin{equation}
\Omega_{j,\tau}^\prec := \{x_j(s) + t\mathbf{n}_j(s) | s \in S^1, \ |t| < \tau \varepsilon_j\}
\end{equation}
where $\varepsilon_j > 0$, $j \in \mathbb{N}$, is chosen so small that $\overline{\Omega_{j,\tau}^\prec} \subset \Omega_{in}$ and the boundary $\partial \Omega_{j,\tau}^\prec$ is Lipschitz. Then, evidently, the domains $\Omega_{j,\tau}^\prec$ with $\tau \in (0, 1)$ have the same properties.

Set
\begin{equation}
\Omega_{j,\tau} := \{(x_\perp, x_\parallel) | x_\perp \in \Omega_{in} | x_\perp \in \Omega_{j,\tau}^\prec\}, \quad j \in \mathbb{N}, \quad \tau \in (0, 1].
\end{equation}
Evidently, $\Omega_{j,\tau}$ is a domain and $\overline{\Omega_{j,\tau}} \subset \Omega_{in}$. Since $w_{\Omega_{j,\tau}}(x_\perp) \geq 0$ for every $x_\perp \in \mathbb{R}^2$ and $w_{\Omega_{j,\tau}}(x_\perp) > 0$ if and only if $x_\perp \in \Omega_{j,\tau}^\prec$, we have
\begin{equation}
w_{\Omega_{j,\tau}} \geq \mathbf{1}_{\Omega_{j,\tau}^\prec} w_{\Omega_{j,\tau}} \geq c_1 \mathbf{1}_{\Omega_{j,\tau}^\prec}
\end{equation}
where
\begin{equation}
c_1 := \inf_{x_\perp \in \Omega_{j,\tau}^\prec} w_{\Omega_{j,\tau}}(x_\perp).
\end{equation}
Since
\begin{equation}
w_{\Omega_{j,\tau}}(x_\perp) = \begin{cases} w_{\Omega_{in}}(x_\perp) & \text{if } x_\perp \in \Omega_{j,\tau}^\prec, \\
0 & \text{if } x_\perp \in \mathbb{R}^2 \setminus \Omega_{j,\tau}^\prec,
\end{cases}
\end{equation}
we have
\begin{equation}
c_1 = \inf_{x_\perp \in \Omega_{j,\tau}^\prec} w_{\Omega_{in}}(x_\perp).
\end{equation}
Let us now show that if $K \subset \Omega_{in}$ is a compact set, then
\begin{equation}
\inf_{x_\perp \in K} w_{\Omega_{in}}(x_\perp) > 0.
\end{equation}
Let $y_\perp \in \Omega_{in}$. Then there exists $y_\parallel \in \mathbb{R}$ such that $y := (y_\perp, y_\parallel) \in \Omega_{in}$. Since $\Omega_{in}$ is open, there exists $r = r(y_\perp) > 0$ such that $B_r(y_\perp) \times (y_\parallel - r, y_\parallel + r) \subset \Omega_{in}$ where $B_r(y_\perp) :=$
\( \{ x_{\perp} \in \mathbb{R}^2 \mid |x_{\perp} - y_{\perp}| < r \} \). Therefore, for every \( y_{\perp} \in \mathcal{O}_{\text{in}} \) there exists \( r = r(y_{\perp}) > 0 \) such that for each \( x_{\perp} \in \mathcal{B}_r(y_{\perp}) \) we have

\[
w_{\mathcal{O}_{\text{in}}}(x_{\perp}) = \int_{\mathbb{R}} \mathbb{1}_{\mathcal{O}_{\text{in}}}(x_{\perp}, t) dt \geq \int_{y_{\perp} - r}^{y_{\perp} + r} dt = 2r.
\]

Since \( \mathcal{K} \) is a compact subset of \( \mathcal{O}_{\text{in}} \), there exists a finite set \( \{ y_{\perp,j} \}_{j=1}^{J} \subset \mathcal{O}_{\text{in}} \) with \( J \in \mathbb{N} \) such that \( \mathcal{K} \subset \bigcup_{j=1}^{J} \mathcal{B}_r(y_{\perp,j}) \). Set \( \rho := \min_{j=1,\ldots,J} r(y_{\perp,j}) \). Then we have

\[
w_{\mathcal{O}_{\text{in}}}(x_{\perp}) \geq 2\rho > 0, \quad x_{\perp} \in \mathcal{K},
\]

which implies (5.53). By (5.52) and (5.53), we obtain \( c_1 > 0 \). Therefore, (5.54) and the mini-max principle yield

\[
n_+(s; p_q \mathbb{1}_{\mathcal{O}_{\text{in}}}, p_q) \geq n_+(s; c_1 p_q \mathbb{1}_{\mathcal{O}_{\text{in}}}^{\mathcal{K},r/2} p_q) = n_+(c_1^{-1} s; p_q \mathbb{1}_{\mathcal{O}_{\text{in}}^{\mathcal{K},r/2}} p_q), \quad s > 0.
\]

Now, (5.49), (5.54), and (5.45) imply

\[
\lim inf_{\lambda \downarrow 0} \Xi^{-}_{q,1}(\lambda) \geq \ln \text{Cap}(\mathcal{O}_{j,\tau/2}^{\mathcal{K}}).
\]

By \( \gamma_j^{\mathcal{K}} \subset \mathcal{O}_{j,\tau/2}^{\mathcal{K}} \subset \overline{\mathcal{O}_{\text{in}}}^{\mathcal{K}} \) and (5.50), we have

\[
\lim_{j \to \infty} \text{Cap}(\mathcal{O}_{j,\tau/2}^{\mathcal{K}}) = \text{Cap}(\overline{\mathcal{O}_{\text{in}}}) = \text{Cap}(\mathcal{O}_{\text{in}}),
\]

which combined with (5.55) yields

\[
\lim inf_{\lambda \downarrow 0} \Xi^{-}_{q,1}(\lambda) \geq \ln \text{Cap}(\mathcal{O}_{\text{in}})^2.
\]

Let us now estimate \( \Xi^{-}_{q,1}(\lambda) \) from above. Combining (5.3) and (5.17) with (5.33), we find that for each \( \omega \in C_{0}^{\infty}(\mathbb{R}^3; \mathbb{R}) \) satisfying \( \omega = 1 \) on \( \overline{\mathcal{O}_{\text{in}}} \), there exists a constant \( c > 0 \) such that

\[
- \xi(A_q - \lambda; H_{-}, H_0) \leq n_+(c \sqrt{\lambda}; p_0 \mathbb{1}_{\mathcal{O}_{\text{in}}}^{\mathcal{K},r/2} p_0) + O(1), \quad \lambda \downarrow 0.
\]

For \( \delta > 0 \) small enough set

\[
\Omega_{\delta} := \left\{ x \in \mathbb{R}^3 \mid \text{dist}(x, \mathcal{O}_{\text{in}}) \leq \delta \right\}, \quad \mathcal{O}_{\delta} := \left\{ x_{\perp} \in \mathbb{R}^2 \mid \text{dist}(x_{\perp}, \mathcal{O}_{\text{in}}) \leq \delta \right\},
\]

and choose \( \omega \) so that \( \text{supp} \omega = \Omega_{\delta} \). Then we have

\[
\pi_{\perp}(\text{supp} \omega) = \pi_{\perp}(\Omega_{\delta}) \subset \mathcal{O}_{\delta}.
\]

In order to check the above inclusion, assume that \( x_{\perp} \in \pi_{\perp}(\Omega_{\delta}) \). Then there exists \( x \in \Omega_{\delta} \) such that \( \pi_{\perp}(x) = x_{\perp} \) and \( y \in \mathcal{O}_{\text{in}} \) satisfying

\[
\text{dist}(x, \mathcal{O}_{\text{in}}) = \text{dist}(x, \overline{\mathcal{O}_{\text{in}}}) = |x - y| \leq \delta.
\]

Let \( y_{\perp} := \pi_{\perp}(y) \in \pi_{\perp}(\mathcal{O}_{\text{in}}) = \overline{\mathcal{O}_{\text{in}}} \). We have

\[
|x_{\perp} - y_{\perp}| \leq |x - y| \leq \delta.
\]

Therefore,

\[
\text{dist}(x_{\perp}, \mathcal{O}_{\text{in}}) = \text{dist}(x_{\perp}, \overline{\mathcal{O}_{\text{in}}}) \leq |x_{\perp} - y_{\perp}| \leq \delta,
\]
and, hence, \( x_\perp \in \mathcal{O}_\delta \).

Since \( \mathcal{O}_\delta \) is compact, there exist finite coverings of \( \mathcal{O}_\delta \) by squares with sides parallel to the coordinate axes, of arbitrarily small size. Hence, there exists a domain \( \mathcal{O}_{2\delta}^\perp \subset \mathbb{R}^2 \) with Lipschitz boundary such that

\[
(5.59) \quad \overline{\mathcal{O}_m} \subset \mathcal{O}_\delta \subset \mathcal{O}_{\delta}^\perp, \quad \overline{\mathcal{O}_{2\delta}^\perp} \subset \mathcal{O}_{2\delta}.
\]

Then (5.58) and the mini-max principle imply

\[
-\xi(\Lambda_q - \lambda; H_-, H_0) \leq n_\pm(c\sqrt{\lambda}; p_0 \mathbb{1}_{\mathcal{O}_{\delta}^\perp}p_0) + O(1), \quad \lambda \downarrow 0,
\]

which combined with (5.45) yields

\[
(5.60) \quad \limsup_{\lambda \downarrow 0} \Xi_{q,1}^\pm(\lambda) \leq \ln \text{Cap}(\mathcal{O}_\delta^\perp)^2.
\]

By (5.59), (5.41), and (5.43), we have

\[
(5.61) \quad \lim_{\delta \downarrow 0} \text{Cap}(\mathcal{O}_{\delta}^\perp) = \text{Cap}(\mathcal{O}_m).
\]

Therefore, (5.60) and (5.61) imply

\[
(5.62) \quad \limsup_{\lambda \downarrow 0} \Xi_{q,1}^\pm(\lambda) \leq \ln \text{Cap}(\mathcal{O}_m)^2.
\]

Putting together (5.57) and (5.62), we obtain (5.47).

The proof of (5.48) is quite similar. Note that for any trace-class operator \( T = T^* \geq 0 \), we have

\[
(5.63) \quad \text{Tr} \arctan T = \int_0^\infty n_\pm(s; T) \frac{ds}{1 + s^2}.
\]

Combining (5.4)-(5.5) and (5.17) with (5.19) or (5.27), and (5.33), and bearing in mind (5.63) and the mini-max principle, we find that there exist constants \( c_\leq \geq c_\geq > 0 \) such that

\[
\frac{1}{\pi} \text{Tr} \arctan \left( \frac{c_\leq p_0 \mathbb{1}_{\mathcal{O}_{j,\tau/2}^\perp} p_0}{c_\geq \sqrt{\lambda}} \right) + O(1) \leq \pm \xi(\Lambda_q + \lambda; H_\pm, H_0) \leq \frac{1}{\pi} \text{Tr} \arctan \left( \frac{p_0 \mathbb{1}_{\tau}(\sup \omega) p_0}{c_\geq \sqrt{\lambda}} \right) + O(1), \quad \lambda \downarrow 0.
\]

(5.64)

Putting together (5.64) and (5.46), we get

\[
\ln \text{Cap}(\mathcal{O}_{j,\tau/2}^\perp)^2 \leq \liminf_{\lambda \downarrow 0} \Xi_{q,2}^\pm(\lambda) \leq \limsup_{\lambda \downarrow 0} \Xi_{q,2}^\pm(\lambda) \leq \ln \text{Cap}(\mathcal{O}_\delta^\perp)^2,
\]

which together with (5.56) and (5.61), implies (5.48).
Appendix A.

A.1. Unitary equivalence of $M_{6,q}$ and $M_{7,q}$. In our proof of the unitary equivalence of the operators $M_{6,q}$ and $M_{7,q}$ (see Subsection 5.4) we will follow closely the argument of the proof of [25, Proposition 4.1]. Set

$$\varphi_{k,0}(x_\perp) := \sqrt{\frac{b}{2\pi k!}} \left( \frac{b}{2} \right)^{k/2} \zeta^k e^{-b|x|^2/4}, \quad x_\perp \in \mathbb{R}^2, \; k \in \mathbb{Z}_+, \tag{A.1}$$

$$\varphi_{k,q}(x) := \sqrt{\frac{1}{(2b)^q q!}} (a*)^q \varphi_{k,0}(x), \quad x_\perp \in \mathbb{R}^2, \; k \in \mathbb{Z}_+, \; q \in \mathbb{N}. \tag{A.2}$$

Then $\{\varphi_{k,q}\}_{k \in \mathbb{Z}_+}$ is an orthonormal basis of $p_qL^2(\mathbb{R}^2)$ called sometimes the angular momentum basis (see e.g. [29] or [11, Subsection 9.1]). Evidently, for $k \in \mathbb{Z}_+$ we have

$$a^* \varphi_{k,q} = \sqrt{2b(q+1)} \varphi_{k,q+1}, \quad q \in \mathbb{Z}_+, \quad a \varphi_{k,q} = \begin{cases} \sqrt{2bq} \varphi_{k,q-1}, & q \geq 1, \\ 0, & q = 0. \end{cases} \tag{A.3}$$

Define the unitary operator $\mathcal{W} : p_qL^2(\mathbb{R}^2) \to p_0L^2(\mathbb{R}^2)$ by $\mathcal{W} : u \mapsto v$ where

$$u = \sum_{k \in \mathbb{Z}_+} c_k \varphi_{k,q}, \quad v = \sum_{k \in \mathbb{Z}_+} c_k \varphi_{0,k}, \quad \{c_k\}_{k \in \mathbb{Z}_+} \in \ell^2(\mathbb{Z}_+).$$

We will show that

$$M_{6,q} = \mathcal{W}^* M_{7,q} \mathcal{W}. \tag{A.4}$$

For $V \in C_0^\infty(\mathbb{R}^2)$, $m, s \in \mathbb{Z}_+$, and $k, \ell \in \mathbb{Z}_+$, set

$$\Upsilon_{m,s}(V; k, \ell) := \langle V \varphi_{k,m}, \varphi_{k,s} \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{R}^2)$. Taking into account (A.3), we easily find that

$$\langle M_{6,q} u, u \rangle = \sum_{k \in \mathbb{Z}_+} \sum_{\ell \in \mathbb{Z}_+} (\Upsilon_{q,q}(w_{00}; k, \ell) + 2b(q+1) \Upsilon_{q+1,q+1}(w_{11}; k, \ell) + 2bq \Upsilon_{q-1,q-1}(w_{22}; k, \ell)) c_k \overline{c_\ell} + 2\text{Re} \sum_{k \in \mathbb{Z}_+} \sum_{\ell \in \mathbb{Z}_+} \left( 2b\sqrt{q(q+1)} \Upsilon_{q+1,q-1}(w_{21}; k, \ell) \right) c_k \overline{c_\ell} + 2\text{Re} \sum_{k \in \mathbb{Z}_+} \sum_{\ell \in \mathbb{Z}_+} \left( \sqrt{2b(q+1)} \Upsilon_{q+1,q}(w_{01}; k, \ell) + \sqrt{2bq} \Upsilon_{q,q-1}(w_{20}; k, \ell) \right) c_k \overline{c_\ell},$$

if $q \geq 1$, and

$$\langle M_{6,0} u, u \rangle = \sum_{k \in \mathbb{Z}_+} \sum_{\ell \in \mathbb{Z}_+} (\Upsilon_{0,0}(w_{00}; k, \ell) + 2b \Upsilon_{1,1}(w_{11}; k, \ell)) c_k \overline{c_\ell} + 2\sqrt{2b} \text{Re} \sum_{k \in \mathbb{Z}_+} \sum_{\ell \in \mathbb{Z}_+} \Upsilon_{1,0}(w_{01}; k, \ell) c_k \overline{c_\ell}.$$
Moreover,
\begin{equation}
\langle M_7, \mathcal{W} u, \mathcal{W} u \rangle = \sum_{k \in \mathbb{Z}_+} \sum_{\ell \in \mathbb{Z}_+} \Upsilon_{0,0}(v_q; k, \ell)c_k c_\ell, \quad q \in \mathbb{Z}_+.
\end{equation}

In [11, Lemma 9.2] (see also the remark after Eq.(2.2) in [10]), it was shown that
\begin{equation}
\Upsilon_{m,m}(V; k, \ell) = \Upsilon_{0,0}\left( L_m \left( -\frac{\Delta}{2b} \right) V; k, \ell \right), \quad m \in \mathbb{Z}_+,
\end{equation}
for any \( V \in C_0^\infty(\mathbb{R}^2) \). Moreover, by [25, Eq. (4.27)], we have
\begin{equation}
2b \sqrt{q(q + 1)} \Upsilon_{q+1,q-1}(V; k, \ell) = \Upsilon_{0,0}\left( -4L_{q-1}^{(2)} \left( -\Delta \frac{2b}{q}\right) \frac{\partial^2 V}{\partial z^2}; k, \ell \right).
\end{equation}

It remains to handle the quantity \( \Upsilon_{q+1,q}(V; k, \ell) \) with \( q \in \mathbb{Z}_+ \). We have
\begin{equation}
\Upsilon_{q+1,q}(V; k, \ell) = \sqrt{2b(q + 1)} \Upsilon_{q,q}(\mathbb{C}^2 V, a^*; k, \ell) + \frac{q}{q + 1} \Upsilon_{q,q+1}(V; k, \ell).
\end{equation}

Taking into account (A.8), as well as the facts that
\[ [V, a^*] = 2i \frac{\partial V}{\partial \zeta}, \]
by (A.9), and that
\[ \sum_{j=0}^q L_j^{(m)}(t) = L_q^{(m+1)}(t), \quad t \in \mathbb{R}, \quad q \in \mathbb{Z}_+, \quad m \in \mathbb{Z}_+, \]
by [19, Eq. 8.974.3], we find that (A.11) implies
\begin{equation}
\sqrt{2b(q + 1)} \Upsilon_{q+1,q}(V; k, \ell) = 2i \Upsilon_{0,0}\left( L_q^{(1)} \left( -\frac{\Delta}{2b} \right) \frac{\partial V}{\partial \zeta}; k, \ell \right).
\end{equation}

By (A.9), (A.12), and the definition (5.37) – (5.38) of \( v_q \), we find that (A.5), (A.6), and (A.7) imply (A.4).

A.2. Proof of Corollary 5.11. First of all, we note that elementary calculations show that for any constants \( c > 0 \) and \( C \in \mathbb{R} \) we have
\begin{equation}
\Phi_1(c\lambda; C) = \Phi_1(\lambda; C) + o(1),
\end{equation}
and
\begin{equation}
\Phi_1(\sqrt{\lambda}; C) = \frac{1}{2} \Phi_1(\lambda; C + \ln 2) + o\left( \frac{|\ln \lambda|}{\ln^2(\lambda)} \right),
\end{equation}
as \( \lambda \downarrow 0 \). Further, by definition,
\[ n_+(\lambda; p_q \mathbf{1}_D p_q) = \# \left\{ k \in \mathbb{Z}_+ \mid v_{k,q} > \lambda \right\}, \quad \lambda > 0. \]
Therefore,
\begin{equation}
(A.15) \quad n_+ (\lambda; p_q \mathbb{1}_O p_q) = \# \{ k \in \mathbb{Z}_+ \mid \lfloor \ln \nu_{k,q} \rfloor < \lfloor \ln \lambda \rfloor \} + O(1), \quad \lambda \downarrow 0.
\end{equation}
Combining (5.44) and (A.15), we find that for every \( \varepsilon > 0 \) we have
\begin{equation}
\# \{ k \in \mathbb{Z}_+ \mid k \ln k - (\mathcal{C}(O) - \ln 2 - \varepsilon)k < \lfloor \ln \lambda \rfloor \} + O(1) \leq n_+ (\lambda; p_q \mathbb{1}_O p_q) \leq \# \{ k \in \mathbb{Z}_+ \mid k \ln k - (\mathcal{C}(O) - \ln 2)k < \lfloor \ln \lambda \rfloor \} + O(1), \quad \lambda \downarrow 0.
\end{equation}

We have
\begin{equation}
(A.16) \quad F_C^{-1}(x) = x \ln x - Cx, \quad x > 0.
\end{equation}
Note that \( F'_C(x) > 0 \) if \( x > e^{C-1} \). Hence,
\begin{equation}
(A.17) \quad \# \{ k \in \mathbb{Z}_+ \mid k \ln k - Ck < \lfloor \ln \lambda \rfloor \} = F_C^{-1}(\lfloor \ln \lambda \rfloor) + O(1), \quad \lambda \downarrow 0.
\end{equation}
We have
\begin{equation}
(A.18) \quad F_C^{-1}(y) = \frac{y}{\ln y} + \frac{y \ln \ln y}{(\ln y)^2} + \frac{Cy}{(\ln y)^2} + o \left( \frac{y}{(\ln y)^2} \right), \quad y \to \infty.
\end{equation}
To see this, set \( u = u(y) := \frac{\ln y}{y} F_C^{-1}(y) - 1 \) for \( y > 0 \) large enough. Then \( u \) satisfies the equation
\begin{equation*}
u = \frac{\ln \ln y}{\ln y} + \frac{C}{\ln y} + \left( \frac{\ln \ln y}{\ln y} + \frac{C}{\ln y} \right) u - (1 + u) \frac{\ln (1 + u)}{\ln y}.
\end{equation*}
Applying a suitable version of the contraction mapping principle, we find that
\begin{equation*}
u(y) = \frac{\ln \ln y}{\ln y} + \frac{C}{\ln y} + o \left( \frac{1}{\ln y} \right), \quad y \to \infty,
\end{equation*}
which implies (A.18). Putting together (A.16), (A.17), and (A.18), we get
\begin{equation}
(A.19) \quad n_+ (\lambda; p_q \mathbb{1}_O p_q) = \Phi_1 (\lambda; \mathcal{C}(O) - \ln 2) + o \left( \frac{\lfloor \ln \lambda \rfloor}{\ln^2 (\lambda)^2} \right), \quad \lambda \downarrow 0.
\end{equation}
Bearing in mind (A.13) and (A.14), we conclude that (A.19) implies (5.45).

A.3. Proof of Corollary 5.12 First of all, we note that similarly to (A.13) for any constant \( C \in \mathbb{R} \) we have
\begin{equation}
(A.20) \quad \Phi_1 (\lambda \lfloor \ln \lambda \rfloor; C) = \Phi_1 (\lambda; C) + O(1),
\end{equation}
\begin{equation}
(A.21) \quad \Phi_1 (\lambda \lfloor \ln \lambda \rfloor^{-1}; C) = \Phi_1 (\lambda; C) + O(1),
\end{equation}
as \( \lambda \downarrow 0 \). Further, (5.63) yields
\begin{equation}
(A.22) \quad \text{Tr} \ \text{arctan} \left( \lambda^{-1} p_q \mathbb{1}_O p_q \right) = \int_{\mathbb{R}} \frac{n_+ (\lambda t; p_q \mathbb{1}_O p_q)}{1 + t^2} dt, \quad \lambda > 0.
\end{equation}
Let us estimate from above the integral $\int_{\mathbb{R}} \frac{n_+(\lambda t; p_q \mathbb{1}_n p_q)}{1 + t^2} dt$ with $\lambda > 0$ small enough. Taking into account (A.19), (A.21), and the fact that the function $n_+(\cdot; p_q \mathbb{1}_n p_q)$ is non-increasing, we find that for any $\varepsilon > 0$ we have

$$\int_{\mathbb{R}} \frac{n_+(\lambda t; p_q \mathbb{1}_n p_q)}{1 + t^2} dt = \int_{|\ln \lambda|^{-1}}^{\infty} \frac{n_+(\lambda t; p_q \mathbb{1}_n p_q)}{1 + t^2} dt + \int_{0}^{|\ln \lambda|^{-1}} \frac{n_+(\lambda t; p_q \mathbb{1}_n p_q)}{1 + t^2} dt \leq$$

$$n_+(\lambda |\ln \lambda|^{-1}; p_q \mathbb{1}_n p_q) \int_{|\ln \lambda|^{-1}}^{\infty} \frac{dt}{1 + t^2} + \int_{0}^{|\ln \lambda|^{-1}} \Phi_1(\lambda t; \mathcal{C}(\mathcal{O}) - \ln 2 + \varepsilon) dt =$$

(A.23) $$\frac{\pi}{2} \Phi_1(\lambda |\ln \lambda|^{-1}; \mathcal{C}(\mathcal{O}) - \ln 2) + o \left( \frac{|\ln \lambda|}{\ln^2(\lambda)^2} \right), \quad \lambda \downarrow 0,$$

where at the last line we have used that

$$\int_{0}^{|\ln \lambda|^{-1}} \Phi_1(\lambda t; \mathcal{C}(\mathcal{O}) - \ln 2 + \varepsilon) dt = o(1), \quad n_+(\lambda |\ln \lambda|^{-1}; p_q \mathbb{1}_n p_q) \arctan(|\ln \lambda|^{-1}) = o(1),$$

as $\lambda \downarrow 0$. Let us now estimate from below $\int_{\mathbb{R}} \frac{n_+(\lambda t; p_q \mathbb{1}_n p_q)}{1 + t^2} dt$ with small $\lambda > 0$. By (A.19) and (A.21), we easily obtain

$$\int_{\mathbb{R}} \frac{n_+(\lambda t; p_q \mathbb{1}_n p_q)}{1 + t^2} dt \geq \int_{0}^{|\ln \lambda|} \frac{n_+(\lambda t; p_q \mathbb{1}_n p_q)}{1 + t^2} dt \geq$$

(A.24) $$n_+(\lambda |\ln \lambda|; p_q \mathbb{1}_n p_q) \int_{0}^{|\ln \lambda|} \frac{dt}{1 + t^2} = \frac{\pi}{2} \Phi_1(\lambda; \mathcal{C}(\mathcal{O}) - \ln 2) + o \left( \frac{|\ln \lambda|}{\ln^2(\lambda)^2} \right), \quad \lambda \downarrow 0.$$

Now, (A.22), (A.23), and (A.24) imply

$$\frac{1}{\pi} \text{Tr} \arctan \left( \lambda^{-1} p_q \mathbb{1}_n p_q \right) = \frac{1}{2} \Phi_1(\lambda; \mathcal{C}(\mathcal{O}) - \ln 2) + o \left( \frac{|\ln \lambda|}{\ln^2(\lambda)^2} \right), \quad \lambda \downarrow 0,$$

which combined with (A.13) and (A.14) yields (5.46).

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