On the Sum of Distance Laplacian Eigenvalues of Graphs

Shariefuddin Pirzada and Saleem Khan

Abstract. Let $G$ be a connected graph with $n$ vertices, $m$ edges and having diameter $d$. The distance Laplacian matrix $D^L$ is defined as $D^L = \text{Diag}(Tr) - D$, where $\text{Diag}(Tr)$ is the diagonal matrix of vertex transmissions and $D$ is the distance matrix of $G$. The distance Laplacian eigenvalues of $G$ are the eigenvalues of $D^L$ and are denoted by $\delta_1, \delta_1, \ldots, \delta_n$. In this paper, we obtain (a) the upper bounds for the sum of $k$ largest and (b) the lower bounds for the sum of $k$ smallest non-zero, distance Laplacian eigenvalues of $G$ in terms of order $n$, diameter $d$ and Wiener index $W$ of $G$. We characterize the extremal cases of these bounds. Also, we obtain the bounds for the sum of the powers of the distance Laplacian eigenvalues of $G$. Finally, we obtain a sharp lower bound for the sum of the $\beta$th powers of the distance Laplacian eigenvalues, where $\beta \neq 0, 1$.

1 Introduction

A graph $G = (V, E)$ consists of the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G)$. We assume all the graphs under consideration are simple and connected. Further, $|V(G)| = n$ is the order and $|E(G)| = m$ is the size of $G$. The degree of $v$, denoted by $d_G(v)$ (we simply write $d_v$) is the number of edges incident on the vertex $v$. As usual, $K_n$ is a complete graph with $n$ vertices and $K_{1,n-1}$ is a star graph with $n$ vertices. Also, $K_{a,b}$ is a complete bipartite graph with two partite sets $V_1$ and $V_2$ of cardinalities $a$ and $b$, respectively, such that each vertex of $V_1$ is adjacent to every vertex of $V_2$. For other standard definitions, we refer [12].

The adjacency matrix $A = (a_{ij})$ of $G$ is an $n \times n$ matrix whose $(i, j)$-entry is equal to 1, if $v_i$ is adjacent to $v_j$ and equal to 0, otherwise. Let $\text{Deg}(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees $d_i = d_{v_i}, i = 1, 2, \ldots, n$ of $G$. The positive semi-definite matrix $L(G) = \text{Deg}(G) - A(G)$ is the Laplacian matrix of $G$. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues of $G$. Let $S_k(G)$ be the sum of the $k$ largest Laplacian eigenvalues of $G$. Several researchers have been investigating the parameter $S_k(G)$ because of its importance in
dealing with many problems in the theory, for instance, Brouwer’s conjecture, Laplacian energy. More recent work on \( S_k(G) \) can be seen in [5, 6, 7, 8, 13].

In \( G \), the distance between two vertices \( u, v \in V(G) \), denoted by \( d_{uv} \), is defined as the length of a shortest path between \( u \) and \( v \). The diameter of \( G \) is the maximum distance between any two vertices of \( G \). The distance matrix of \( G \) is denoted by \( D(G) \) and is defined as \( D(G) = (d_{uv})_{u,v \in V(G)} \). The vertex transmission \( Tr_G(v) \) of a vertex \( v \) is defined as the sum of the distances from \( v \) to all other vertices in \( G \), that is, \( Tr_G(v) = \sum_{u \in V(G)} d_{uv} \). A graph \( G \) is said to be \( k \)-transmission regular if \( Tr_G(v) = k \), for each \( v \in V(G) \). The Wiener index (also called transmission) of a graph \( G \), denoted by \( W(G) \), is the sum of distances between all unordered pairs of vertices in \( G \). Clearly, \( W(G) = \frac{1}{2} \sum_{v \in V(G)} Tr_G(v) \). For any vertex \( v_i \in V(G) \), the vertex transmission \( Tr_G(v_i) \) is also called the transmission degree of \( v_i \).

Let \( Tr(G) = \text{diag}(Tr_1, Tr_2, \ldots, Tr_n) \) be the diagonal matrix of vertex transmissions of \( G \). Aouchiche and Hansen [1] introduced the Laplacian for the distance matrix of a connected graph. The matrix \( D^L(G) = Tr(G) - D(G) \) (or simply \( D^L \)) is called the distance Laplacian matrix of \( G \). The eigenvalues of \( D^L \) are called the distance Laplacian eigenvalues of \( G \). Since \( D^L(G) \) is a real symmetric positive semi-definite matrix, we denote its eigenvalues by \( \delta_i \)'s and order them as \( 0 = \delta_n \leq \delta_{n-1} \leq \cdots \leq \delta_1 \). The largest distance Laplacian eigenvalue \( \delta_1 \) is called the distance Laplacian spectral radius of \( G \). More work on distance Laplacian eigenvalues can be found in [2, 9, 11].

Motivated by the parameter \( S_k(G) \) of the Laplacian matrix, we introduce the following. For \( 1 \leq k \leq n - 1 \), let \( U_k \) denote the sum of the \( k \) largest distance Laplacian eigenvalues and \( L_k \) denote the sum of \( k \) smallest positive distance Laplacian eigenvalues of the graph \( G \), that is,

\[
U_k = \sum_{i=1}^{k} \delta_i \quad \text{and} \quad L_k = \sum_{i=1}^{k} \delta_{n-i}.
\]

In Section 2, we obtain the upper bounds for \( U_k \) in terms of order \( n \), diameter \( d \) and Wiener index \( W \) of \( G \). Also, we find the lower bounds for \( L_k \) in terms of the same parameters. In particular, we obtain the bounds for \( U_k \) and \( L_k \) when \( G \) is a bipartite graph. We characterize the extremal cases of these bounds. We derive the bounds for the sum of the powers of the distance Laplacian eigenvalues of \( G \). Also, we obtain a sharp lower bound for the sum of the \( \beta \)th powers of the distance Laplacian eigenvalues, where \( \beta \neq 0, 1 \).

## 2 On the sum of the distance Laplacian eigenvalues of graphs

We begin with the following observation due to Caen [3].
Lemma 2.1. [3] Let \([n] = \{1, 2, \ldots, n\}\) be the canonical \(n\)-element set and let \([n]^{(2)}\) denote the set of 2-element subsets of \([n]\), that is, the edge set of \(K_n\). To each entry \(\{i, j\} = ij\) in \([n]^{(2)}\), associate a real variable \(x_{ij}\), then for \(n \geq 2\), and for all \(x'_{ij}\)'s, we have
\[
\left(\sum_{ij} x_{ij}^2\right)^2 + \left(\frac{n-1}{2}\right) \left(\sum_{ij} x_{ij}^2 - \frac{n-1}{2} \sum_i \left(\sum_{j \neq i} x_{ij}\right)^2\right) \geq 0.
\]

The following result gives the upper bound for the sum of the squares of the vertex transmissions in terms of the Wiener index \(W\), the diameter \(d\) and the order \(n\) of the graph \(G\).

Lemma 2.2. Let \(G\) be a connected graph with \(n\) vertices having diameter \(d\). Then
\[
\sum_i Tr^2(i) \leq \frac{2W^2}{n-1} + \frac{n(n-1)(n-2)d^2}{2}
\]
with equality if and only if \(G \cong K_n\).

Proof. Substituting \(d_{ij}\) for \(x_{ij}\) in Lemma 2.1 and noting that each \(d_{ij} \leq d\), we have
\[
\left(\sum_{ij} d_{ij}^2\right)^2 + \left(\frac{n-1}{2}\right) \left(\sum_{ij} d_{ij}^2 - \frac{n-1}{2} \sum_i \left(\sum_{j \neq i} d_{ij}\right)^2\right) \geq 0
\]
or
\[
W^2 + \left(\frac{n-1}{2}\right) \sum_{ij} d_{ij}^2 - \frac{n-1}{2} \sum_i Tr^2(i) \geq 0.
\]

Then
\[
\sum_i Tr^2(i) \leq \frac{2W^2}{n-1} + \frac{2}{n-1} \left(\frac{n-1}{2}\right) \left(\frac{n(n-1)}{2}d^2\right),
\]
or
\[
\sum_i Tr^2(i) \leq \frac{2W^2}{n-1} + \frac{n(n-1)(n-2)d^2}{2},
\]
proving the inequality.

Assume that the equality holds in the above inequalities. Then clearly each \(d_{ij} = d\). Since \(G\) is connected, there is at least one \(d_{ij} = 1\) and thus \(d = 1\) which clearly shows that \(G \cong K_n\).

Conversely, if \(G \cong K_n\), then it is easy to see that \(d = 1\), \(W = \frac{n(n-1)}{2}\) and \(\sum_i Tr^2(i) = n(n-1)^2\). Substituting these values in the above inequalities, we observe that the equality holds. \(\Box\)

Now, we obtain an upper bound for \(U_k\) in terms of the Wiener index, the diameter and the order of the graph \(G\). The proof follows by using similar techniques as in Zhou [14].

Theorem 2.1. Let \(G\) be a connected graph with \(n\) vertices having diameter \(d\). For \(1 \leq k \leq n-2\), we have
\[
U_k \leq \frac{2Wk}{n-1} + \frac{\sqrt{k(n-k-1)(dn(n-1) - 2W)(dn(n-1) + 2W)}}{\sqrt{2(n-1)}}
\]
with equality if and only if \(G \cong K_n\). Equality always holds for \(k = n-1\).
**Proof.** It is easy to see that
\[
\sum_{i=1}^{n-1} \delta_i = \sum_{i \in V} \text{Tr}(i) = 2W \quad \text{and} \quad \sum_{i=1}^{n-1} \delta_i^2 = \sum_{i} \text{Tr}^2(i) + 2 \sum_{1 \leq i < j \leq n} d_{ij}^2.
\]
For \(1 \leq k \leq n-2\), using Cauchy-Schwarz inequality, we have
\[
(2W - U_k)^2 = (\delta_{k+1} + \cdots + \delta_{n-1})^2 \leq (n-k-1)(\delta_{k+1}^2 + \cdots + \delta_{n-1}^2)
\]
\[
= (n-k-1)\left(\sum_i \text{Tr}^2(i) + 2 \sum_{1 \leq i < j \leq n} d_{ij}^2 - (\delta_1^2 + \cdots + \delta_k^2)\right)
\]
\[
\leq (n-k-1)\left(\sum_i \text{Tr}^2(i) + 2 \sum_{1 \leq i < j \leq n} d_{ij}^2 - \frac{U_k^2}{k}\right),
\]
which implies that
\[
U_k^2 - \frac{4kWU_k}{n-1} + \frac{4kW^2}{n-1} - \frac{k(n-k-1)}{n-1} \left(\sum_i \text{Tr}^2(i) + 2 \sum_{1 \leq i < j \leq n} d_{ij}^2\right) \leq 0.
\]
Thus,
\[
U_k \leq \frac{2Wk + \sqrt{k(n-k-1)\left[(n-1)\left(\sum_i \text{Tr}^2(i) + 2 \sum_{1 \leq i < j \leq n} d_{ij}^2\right) - 4W^2\right]}}{n-1}.
\]
Using Lemma 2.2, we get
\[
U_k \leq \frac{2Wk + \sqrt{k(n-k-1)\left[\frac{2W^2}{n-1} + \frac{n(n-1)(n-2)d^2}{2} + d^2n(n-1)\right] - 4W^2}}{n-1}.
\]
On further simplifications, we have
\[
U_k \leq \frac{2Wk + \sqrt{k(n-k-1)\left(d^2n^2(n-1)^2-4W^2\right)}}{n-1},
\]
or,
\[
U_k \leq \frac{2Wk}{n-1} + \frac{\sqrt{k(n-k-1)(dn(n-1) - 2W)(dn(n-1) + 2W)}}{\sqrt{2}(n-1)},
\]
proving the inequalities.

Assume that the equality hold in above inequalities. Then all the above inequalities have to be equalities and after using Cauchy-Schwarz theorem and Lemma 2.2, we observe that \(G \cong K_n\).

Conversely, let \(G \cong K_n\). Taking the characteristic polynomial of \(K_n\) into consideration, it
is quite easy to check that $U_k = nk$, $W = \frac{n(n-1)}{2}$ and $d = 1$. Using these values in the main inequality, we observe that the equality holds.

Using the fact that trace of a matrix is equal to sum of its eigenvalues and noting that $2W = U_{n-1}$, we see that equality always holds in main inequality when $k = n - 1$.

From Theorem 2.1, we obtain the following upper bound for the spectral radius $\delta_1$ of the distance Laplacian matrix of a graph.

**Theorem 2.2.** Let $G$ be a connected graph on $n$ vertices having diameter $d$. Then

$$\delta_1 \leq \frac{2W}{n-1} + \frac{\sqrt{(n-2)(dn(n-1) - 2W)(dn(n-1) + 2W)}}{\sqrt{2}(n-1)}$$

with equality if and only if $G \cong K_n$.

Using arguments same as in Theorem 2.1, we have the following lower bound for $L_k$.

**Theorem 2.3.** Let $G$ be a connected graph with $n$ vertices having diameter $d$. Then, for $1 \leq k \leq n - 2$, we have

$$L_k \geq \frac{2Wk}{n-1} - \frac{\sqrt{k(n-k-1)(dn(n-1) - 2W)(dn(n-1) + 2W)}}{\sqrt{2}(n-1)}$$

with equality if and only if $G \cong K_n$. Equality always holds for $k = n - 1$.

As a consequence of Theorem 2.3, we get the following upper bound for the smallest non-zero distance Laplacian eigenvalue $\delta_{n-1}$ of $G$.

$$\delta_{n-1} \geq \frac{2W}{n-1} - \frac{\sqrt{(n-2)(dn(n-1) - 2W)(dn(n-1) + 2W)}}{\sqrt{2}(n-1)}$$

with equality if and only if $G \cong K_n$.

Now, we have the following observation about the sum of the squares of the distances in a graph.

**Lemma 2.3.** Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges having diameter $d$. Then

$$\sum_{1 \leq i < j \leq n} d_{ij}^2 \leq \frac{2m(1 - d^2) + d^2n(n-1)}{2}$$

with equality if and only if $d \leq 2$. 

Proof. Since $G$ contains $m$ edges, therefore there are exactly $m$ distances equal to 1 and the remaining distances (if there are any) are greater or equal to 2. As each $d_{ij} \leq d$, we have

$$
\sum_{1 \leq i < j \leq n} d_{ij}^2 \leq m + d^2 \left( \frac{n(n-1)}{2} - m \right)
= \frac{2m(1-d^2) + d^2n(n-1)}{2},
$$

proving the inequality.

Assume that the equality hold in above inequalities. If there are no non-adjacent pair of vertices, then clearly $d = 1$, so that $G$ is a complete graph. If there are some non-adjacent pair of vertices, then from the above proof we observe that the distance between every non-adjacent pair of vertices is same and equals to $d$. Since $G$ is connected, at least one such distance equals to 2, so that $d = 2$.

Conversely, if $d = 1$, then $G$ is a complete graph and it is easy to check that the equality holds in this case. Similarly, we can easily see that the equality holds when $d = 2$. This proves the result.

From Lemma 2.3, we obtain the following corollary for the class of connected bipartite graphs.

**Corollary 2.4.** Let $G$ be a connected bipartite graph with $n \geq 3$ vertices, $m$ edges and having diameter $d$. Then

$$
\sum_{1 \leq i < j \leq n} d_{ij}^2 \leq \frac{2m(1-d^2) + d^2n(n-1)}{2}
$$

with equality if and only if $G$ is a complete bipartite graph.

We require the following lemma due to Zhou [14].

**Lemma 2.4.** [14] Let $G$ be a connected bipartite graph on $n$ vertices and $m$ edges and let $d_v$ be the degree of any vertex $v$ of $G$. Then

$$
\sum_{v \in V(G)} d_v^2 \leq mn
$$

with equality if and only if $G$ is a complete bipartite graph.

We have the following observation for the sum of the squares of the vertex transmissions of a bipartite graph.

**Lemma 2.5.** Let $G$ be a connected bipartite graph having $n \geq 3$ vertices, $m$ edges and diameter $d$ with bi-partition of vertex set as $\{V_1, V_2\}$ with $|V_1| = a$, $|V_2| = b$, $a \geq b \geq 1$ and $a + b = n$. Then

$$
\sum_i Tr^2(i) \leq mn(1-d)^2 + nd^2(n-1)^2 + 4md(1-d)(n-1)
$$

with equality if and only if $G$ is a complete bipartite graph.
Proof. Without loss of generality, let \( i \in V_1 \). Since \( G \) is bipartite we have the following inequality

\[
\text{Tr}(i) \leq d_i + d(b - d_i) + d(a - 1),
\]

or,

\[
\text{Tr}(i) \leq d_i(1 - d) + d(n - 1)
\]

so that

\[
\text{Tr}^2(i) \leq d_i^2(1 - d)^2 + (d(n - 1))^2 + 2dd_i(1 - d)(n - 1).
\]

Taking the summation over all vertices in \( V \), we have

\[
\sum_i \text{Tr}^2(i) \leq \sum_i d_i^2(1 - d)^2 + \sum_i (d(n - 1))^2 + \sum_i 2dd_i(1 - d)(n - 1).
\]

Using Lemma 2.4 and noting that \( \sum_i d_i = 2m \), from the above inequality, we have

\[
\sum_i \text{Tr}^2(i) \leq mn(1 - d)^2 + nd^2(n - 1)^2 + 4md(1 - d)(n - 1).
\]

Assume that the equality hold in above inequality. Then we observe that the distance between every non-adjacent pair of vertices is the same and equals to 2, which is possible only if \( G \) is complete bipartite.

Conversely, it is easy to check that the equality holds for complete bipartite graphs. \( \Box \)

We note the following observation.

**Lemma 2.6.** [1] The distance Laplacian characteristic polynomial of a complete bipartite graph \( K_{a,b} \) is

\[
P^L_{K_{a,b}}(x) = x(x - n)(x - (2a + b))^{a-1}(x - (2b + a))^{b-1}.
\]

Now, we obtain an upper bound for \( U_k \) when \( G \) is a connected bipartite graph.

**Theorem 2.5.** Let \( G \) be a connected bipartite graph with \( n \geq 3 \) vertices, \( m \) edges and having diameter \( d \). Then, for \( 1 \leq k \leq n - 2 \), we have

\[
U_k \leq \frac{2Wk + \sqrt{k(n - k - 1)[(n - 1)\left(m(1 - d)(n + 3dn - 2d + 2) + d^2n^2(n - 1)\right) - 4W^2]}}{n - 1}
\]

with equality if and only if \( k = n - 2 \) and \( G \cong K_{1,n-1} \) or \( G \cong K_{n-2,n-2} \). Equality always holds for \( k = n - 1 \).
Proof. Let $1 \leq k \leq n - 2$. From the proof of Theorem 2.1 and using Corollary 2.4 and Lemma 2.5, we get the required inequality.

Let the equality hold in above inequality. Then we observe that
\[ \delta_1 = \delta_2 = \cdots = \delta_k \quad \text{and} \quad \delta_{k+1} = \delta_{k+2} = \cdots = \delta_{n-1} \]
and $G$ is a complete bipartite graph. Taking Lemma 2.6 into consideration, we have $k = n - 2$ and $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Conversely, if $k = n - 2$ and $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, then we can easily check that equality follows in both cases.

Using same argument as in Theorem 2.1, we see that equality holds when $k = n - 1$. \qed

Some results for $\delta_1$ in bipartite graphs can be seen in [10]. Using Theorem 2.5, we get the following strict upper bound for distance Laplacian spectral radius for the class of connected bipartite graphs with $n \geq 4$.

**Theorem 2.6.** Let $G$ be a connected bipartite graph with $n \geq 4$ vertices, $m$ edges and having diameter $d$. Then
\[
\delta_1 < \frac{2W + \sqrt{(n-2)\left[(n-1)(m(1-d)(n+3dn-2d+2)+d^2n^2(n-1))-4W^2\right]}}{n-1}.
\]

With the help of Theorem 2.5, we can easily obtain the following upper bound for $L_k$.

**Theorem 2.7.** Let $G$ be a connected bipartite graph with $n \geq 3$ vertices, $m$ edges and having diameter $d$. Then, for $1 \leq k \leq n - 2$, we have
\[
L_k \geq \frac{2Wk - \sqrt{k(n-k-1)\left[(n-1)(m(1-d)(n+3dn-2d+2)+d^2n^2(n-1))-4W^2\right]}}{n-1}
\]
with equality if and only if $k = n - 2$ and $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$. Equality always holds for $k = n - 1$.

The following lemma will be used in the sequel.

**Lemma 2.7.** [1] Let $G$ be a connected graph on $n$ vertices and $m \geq n$ edges. Let $G^*$ be the connected graph obtained from $G$ by the deletion of an edge. Let $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n-1} \geq \delta_n = 0$ and $\delta_1^* \geq \delta_2^* \geq \cdots \geq \delta_{n-1}^* \geq \delta_n^* = 0$ denote the distance Laplacian spectra of $G$ and $G^*$, respectively. Then, $\delta_i^* \geq \delta_i$, for all $i = 1, 2, \ldots, n$.

Now, let $\beta$ be a real number such that $\beta \neq 0, 1$ and let $S_\beta(G) = \sum_{i=1}^{n-1} \delta_i^\beta$, that is, the sum of the $\beta$th powers of distance Laplacian eigenvalues of $G$. 
Lemma 2.8. Let $G$ be a connected graph with $n$ vertices and $m \geq n$ edges. Let $G^*$ be the connected graph obtained from $G$ by the deletion of an edge. Let $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n-1} \geq \delta_n = 0$ and $\delta_1^* \geq \delta_2^* \geq \cdots \geq \delta_{n-1}^* \geq \delta_n^* = 0$ be the distance Laplacian eigenvalues of $G$ and $G^*$, respectively. Then

(i) $S_\beta(G^*) > S_\beta(G)$ for $\beta > 0$ and $S_\beta(G^*) < S_\beta(G)$ for $\beta < 0$.

(ii) For any connected graph $G$ with $n$ vertices $S_\beta(G) \geq (n-1)n^\beta$ if $\beta > 0$, and $S_\beta(G) \leq (n-1)n^\beta$ if $\beta < 0$, with either of the equality if and only if $G \cong K_n$.

Proof. Using Lemma 2.7, we have $\sum_{i=1}^{n-1} \delta_i^* - \sum_{i=1}^{n-1} \delta_i \geq 2$. Hence (i) follows directly from Lemma 2.7. We know that the eigenvalues of $K_n$ are $n, n, \ldots, n, 0$, so that (ii) follows from (i). \qed

The following lemma will be used in the proof of Theorem 2.8.

Lemma 2.9. \cite{4} Let $G$ be a connected graph such that $D^L$ has an eigenvalue with multiplicity $n-2$. Let $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n-1} \geq \delta_n = 0$ be the eigenvalues of $D^L$. Then exactly one of the following condition holds.

(i) $m(\delta_1) = n-2$ and $G \cong K_{1,n-1}$ or $G \cong K_{p,p}$.

(ii) $m(\delta_{n-1}) = n-2$ and $G \cong K_{n-2} \lor K_2$.

Now, we have the following result.

Theorem 2.8. Let $\beta$ be a real number with $\beta \neq 0,1$ and let $G$ be a connected graph with $n \geq 3$ vertices. Let $R = \prod_{i=1}^{n-1} \delta_i$. Then

$$S_\beta(G) \geq \delta_1^\beta + (n-2)\left(\frac{R}{\delta_1}\right)^{\frac{\beta}{n-2}}$$

with equality if and only if $G \cong K_n$ or $G \cong K_{n-2} \lor K_2$.

Proof. By the arithmetic-geometric mean inequality, we have

$$S_\beta(G) = \delta_1^\beta + \sum_{i=2}^{n-1} \delta_i^\beta \geq \delta_1^\beta + (n-2)\left(\prod_{i=2}^{n-1} \delta_i^\beta\right)^{\frac{1}{n-2}} = \delta_1^\beta + (n-2)\left(\frac{\prod_{i=1}^{n-1} \delta_i}{\delta_1}\right)^{\frac{\beta}{n-2}}$$

with equality if and only if $\delta_2 = \delta_3 = \cdots = \delta_{n-1}$.

Now, if $\delta_1 = \delta_2$, then $\delta_1 = \delta_2 = \delta_3 = \cdots = \delta_{n-1}$, so that $G \cong K_n$. If $\delta_1 \neq \delta_2$, by Lemma 2.9, we have $G \cong K_{n-2} \lor K_2$. \qed
Conclusions. The parameters $U_k$ and $L_k$ introduced in this paper will be of great importance in the investigation of the distribution of distance Laplacian eigenvalues of a graph. In particular, it will throw more light in the study of the distance Laplacian spectral radius and distance Laplacian energy of a graph. Although the bounds for $U_k$ and $L_k$ are obtained in terms of the order, Wiener index and diameter of the graph, there is enough scope to obtain more bounds for $U_k$ and $L_k$ in terms of several other parameters, which will lead to interesting discussion on the problem.

Acknowledgments

We are highly thankful to the anonymous referees for their valuable suggestions. The research of S. Pirzada is supported by the SERB-DST research project number CRG/2020/000109. The research of Saleem Khan is supported by MANUU.

References

[1] M. Aouchiche and P. Hansen, Two Laplacians for the distance matrix of a graph, Linear Algebra Appl., 439 (2013), 21–33.

[2] M. Aouchiche and P. Hansen, Some properties of the distance Laplacian eigenvalues of a graph, Czechoslovak Mathematical Journal, 64(139) (2014), 751–761.

[3] D. D. Caen An upper bound on the sum of squares of degrees in a graph, Discrete Math., 185 (1998), 245–248.

[4] R. Fernandes, M. A. A. de Freitas, C.M. da Silva Jr and R. R. Del-Vecchio, Multiplicities of distance Laplacian eigenvalues and forbidden subgraphs, Linear Algebra Appl., 541 (2018), 81–93.

[5] H. A. Ganie, S. Pirzada and Vilmar Trevisan, On the sum of $k$ largest Laplacian eigenvalues of a graph and clique number, Mediterranean J. Math., 18 (2021), Artcle No. 15.

[6] H. A. Ganie, S. Pirzada, Bilal A. Rather and Rezwan Ul Shaban, On Laplacian eigenvalues of graphs and Brouwer’s conjecture, J. Ramanujan Math. Soc., 36(1) (2021), 13–21.

[7] H. A. Ganie, S. Pirzada, Bilal A. Rather and Vilmar Trevisan, Further developments on Brouwer’s conjecture for the sum of Laplacian eigenvalues of graphs, Linear Algebra Appl., 588 (2020), 1–18.
[8] H. A. Ganie, S. Pirzada, Rezwan Ul Shaban and X. Li, Upper bounds for the sum of Laplacian eigenvalues of a graph and Brouwer's conjecture, Discrete Math. Algorithms Appl., 11(2) (2019), 195008 (15 pages).

[9] M. Nath and S. Paul, On the distance Laplacian spectra of graphs, Linear Algebra Appl., 460 (2014), 97–110.

[10] A. Niu, D. Fan and G. Wang, On the distance Laplacian spectral radius of bipartite graphs, Discrete Appl. Math., 186 (2015), 207–213.

[11] S. Pirzada and Saleem Khan, On distance Laplacian spectral radius and chromatic number of graphs, Linear Algebra Appl., 625 (2021), 44–54.

[12] S. Pirzada, An Introduction to Graph Theory, 1st edition, Universities Press, Orient Black-swan, Hyderabad, 2012.

[13] S. Pirzada and H. A. Ganie, On the Laplacian eigenvalues of a graph and Laplacian energy, Linear Algebra Appl., 486 (2015), 454–468.

[14] B. Zhou, On Laplacian eigenvalues of a graph, Z. Naturforsch, 59a (2004), 181–184.

Shariefuddin Pirzada  Department of Mathematics, University of Kashmir, Srinagar, Kashmir, India
E-mail: pirzadasd@kashmiruniversity.ac.in

Saleem Khan  Department of Mathematics, University of Kashmir, Srinagar, Kashmir, India
E-mail: khansaleem1727@gmail.com