A model of quantum reduction with decoherence

Roland Omnès
Laboratoire de Physique Théorique∗†
Université de Paris XI, Bâtiment 210, F-91405 Orsay Cedex, France

Abstract

The problem of reduction (wave packet reduction) is reexamined under two simple conditions: Reduction is a last step completing decoherence. It acts in commonplace circumstances and should be therefore compatible with the mathematical frame of quantum field theory and the standard model.

These conditions lead to an essentially unique model for reduction. Consistency with renormalization and time-reversal violation suggest however a primary action in the vicinity of Planck’s length. The inclusion of quantum gravity and the uniqueness of space-time point moreover to generalized quantum theory, first proposed by Gell-Mann and Hartle, as a convenient framework for developing this model into a more complete theory.

PCAS codes : 03.65.Ta, 03.65.Yz, 03.70.+k, 04.60.-m

LPT Orsay 04-105
October 2004

∗Unité Mixte de Recherche (CNRS) UMR 8627
†e-mail: Roland.Omnes@th.u-psud.fr
1 Introduction

Several theories or models have attempted to describe the reduction of a wave function as a physical effect [1-6]. On the other hand, Bohm’s version of quantum mechanics in terms of “real” particles was introduced to avoid reduction [7]. There is however no universal agreement on the existence of a reduction process. Gell-Mann and Hartle [8], and Griffiths [9], considered that, since quantum mechanics is fundamentally a probabilistic theory, any mechanism insuring the uniqueness of physical reality should necessarily stand outside its framework. As a matter of fact, Griffiths, and the present author [10] showed explicitly that the problem of reduction never occurs in the framework of consistent histories, decoherence being sufficient for all logical and practical purposes.

Two proposals of a more “philosophical” nature can also be mentioned. As well known, Everett assumed that every quantum possibility is actualized in some branch of a multi-valued universe [11]. An opposite viewpoint stressed that every theory –even a classical one– can only deal with potentialities and never with actuality, so that the problem of reduction might well be outside the reach of theoretical physics [12]. Many people nevertheless consider that the problem of reduction belongs properly to physics and should even be considered among the most significant ones in understanding the foundations of science.

The discovery of decoherence [13] and its experimental confirmation [14] have modified significantly the problem. Decoherence explains cleanly the suppression of macroscopic quantum interferences (except in a few well-understood cases). It is generally followed by a classical behavior of macroscopic bodies and the various possible results of a quantum measurement appear thereafter as a set of well-defined possibilities obeying standard probability calculus.

The understanding of decoherence began with models [15-17] and was extended to more general theories relying either on coarse graining [18-19], predictability sieves [20] or the quantum theory of irreversible processes [21]. Some significant problems remain however and will be encountered in the present work. They are as follow:

(A). Decoherence relies on the emergence of some “relevant” (or “collective”, or coarse-grained) observables, as opposed to the bulk of all the observables entering in a more or less well-defined “environment”. This splitting of a macroscopic (or mesoscopic) system into two interacting subsystems implies many drastic consequences
for the decoherent subsystem involving the relevant observables. Its dynamic is no
more unitary, it shows no quantum interferences, and it often develops rapidly a
classical behavior. The relevant observables are thus obviously important, but their
exact meaning is still rather obscure. One can easily guess empirically what they
are, in most practical cases, but there is yet no mathematical criterion allowing their
direct construction from the basic principles of quantum theory.

(B). Decoherence is often described as an approximate diagonalization of a re-
duced density matrix describing the relevant observables. But then a problem was
pointed out by Zurek [22]: is this property general and, if then, what is the “pointer
basis” in which the diagonal form occurs? In the present paper, diagonalization will
not be supposed universally valid (and it is not, as a matter of fact [21]).

(C). Another problem is concerned with very small probabilities. When, for
instance, decoherence amounts to diagonalization, the non-diagonal elements of the
reduced density matrix do not vanish completely. They only decrease exponentially
with time and, in some sense, macroscopic interferences never fully disappear. If
the initial state of the whole system is a pure state, it remains so mathematically,
and this survival of quantum superposition, although through very small quantities,
led some critics to consider decoherence theory as purely phenomenological and
not really fundamental [23, 24]. Some cogent arguments have been raised however
against this point of view [10]. Anyway, it is clear that the meaning of very small
probabilities must be better understood.

(D). Finally, decoherence does not of course explain or try to explain the unique-
ness of physical reality. It only implies that one can assert consistently uniqueness
according to the logic of decoherent histories [9, 20].

The present research originated as a critique of the idea of reduction and started
from two simple conditions, which might be expected from any theory making sense
of reduction:

1. If decoherence and reduction both exist, it would be very strange if they were
not strongly related. During a measurement, decoherence appears as an initial step
destroying the quantum superposition of different results, and reduction elects only
finally a unique datum among them. Reduction should be treated accordingly as a
dynamical process, like decoherence itself.

2. The uniqueness of physical reality is obvious in the case of an ordinary physical
system involving familiar laboratory devices. One understands quite well such an
experiment, at least as far down as the level of particles and fields in the standard model. Reduction, when it acts at this commonplace level of physics, should be therefore expressible in the mathematical formalism of well-known physics, at least in a phenomenological way. Its action should agree with the consistency conditions of the standard model, including relativistic invariance and renormalization.

The present models of reduction disagree with these reasonable conditions. They are always concerned for instance with the wave functions of particles, whereas the standard model tells us that the basic objects are quantum fields. As for Bohmian mechanics, in spite of its elegant beginnings, it was never able to satisfy plainly relativistic invariance and to account for quantum fields. Conversely, the present criteria can be used in principle for a search of possible reduction models, or for a critique of the reduction idea.

Some background in decoherence and precision in the corresponding language will be useful for this delicate topic. Let one assume for definiteness that the macroscopic system $S$ under consideration can be split objectively into a relevant subsystem $R$ and an environment $E$ (accordingly, one assumes that a solution exists for Problem A). The relevant (or reduced) density matrix (or state operator) is defined by

$$\rho_R = \text{Tr}_E(\rho_S) . \tag{1.1}$$

The coupling between $R$ and $E$ is supposed to define a set of projection operators $\{P_j\}$ in the Hilbert space of $R$ and decoherence leads to a block-diagonal form of $\rho_R$:

$$\rho_R = \sum_j P_j \rho_R P_j \tag{1.2}$$

Since the Hilbert space of $R$ is a subspace of the Hilbert space of $S$, the operators $P_j$ are well defined as observables of $S$ (when written as $P_j \otimes I_E$, where $I_E$ is the identity operator for $E$ and $P_j$ is a projection operator acting in the $R$ Hilbert space). One can then write

$$\rho_S = \rho_0 + \rho_1 , \tag{1.3}$$

$$\rho_0 = \sum_j P_j \rho_S P_j \tag{1.4}$$
A few remarks will help understanding the meaning of these equations. The matrix $\rho_0$ is positive with unit trace. One usually calls “density” a quantity of the same type as a density matrix (namely a trace-class linear functional on observables, not necessarily positive or having a unit trace), and the quantity $\rho_1$ is such a density (with zero trace). The state of a particle or a bunch of particles not interacting with the relevant subsystem enters in every element $P_j \rho_S P_j$ of $\rho_0$ as a common tensor factor (these objects could be measured later on). The projection operators $P_j$ and the densities are time-dependent (think for instance of a moving part of an apparatus). Very often, they are microlocal (semi-classical) projection operators, whose definition has been given elsewhere [12, 25, 26]. The quantity $\text{Tr}(P_j \rho_S P_j)$ is a constant, equal to the standard probability $p_j$ for observing at any time $t$ after decoherence the property $j$ with projection operator $P_j(t)$. The density $\rho_1$ is still poorly understood. It is clearly not zero in the case of an overall pure state though one knows that it decreases in the trace norm $\text{Tr}|\rho_1|$ when an increasing number of relevant observables is considered. Its physical meaning is related to problem A and, in the coarse-graining approach, it represents intricate long-distance time-varying phase correlation in the environment. Whether or not it takes part in reduction will not be considered in this paper.

Another important point is the non-linearity of reduction, since the Copenhagen theory of measurement [27], or its derivation from first principles [12], shows that, when the result associated with the projection index $j$ is observed, the density matrix becomes

$$\rho_S \rightarrow \rho'_S = \frac{P_j \rho_S P_j}{\text{Tr}(P_j \rho_S P_j)}.$$ (1.5)

The discussion of reduction in the present paper will proceed as follows. In Section 2, a simple model is proposed, in which the sharp transition (1.5) is replaced by a continuous (dynamical) random process, occurring after (or during) decoherence (significant analogies between this model and a previous one by Pearle [2] should be mentioned in that respect). Section 3 contains the core of the paper, a theorem showing that the mathematics of reduction is essentially unique. Quantum field theory is then introduced in Section 4, in the simple case of a non-relativistic system of particles entering in a measuring device. Related projection operators are introduced in Section 4. A discussion of infinitesimal reduction process in Section 5 shows that it must take place primarily at Planck’s scale or nearby in order to
preserve renormalization and insure a definite time direction. Finally, some suggestions in Section 6 are concerned with the theoretical framework in which reduction could enter consistently. Generalized quantum theory, first introduced by Gell-Mann and Hartle [8, 28], looks particularly promising, when using later development by Isham and coworkers [29] together with a recent proposal by Kuchar on a foliation of space-time in quantized general relativity [30-31].

2 A simple model

One is looking for a dynamical reduction process, which must therefore proceed through infinitesimal steps. It is supposed to agree with decoherence, so that its simplest expression is easily found as follows. Let one write

\[ \rho_j = P_j \rho_S P_j , \]  

(2.1)

Where as before \( P_j \) is understood as \( P_R \otimes I_E \) and time is not written explicitly. At some time \( t \), an infinitesimal reduction of one state \( j \) transforms the projection operator \( P_j \) into \( (1 + \varepsilon_j)P_j \). The quantum probability \( p_j = Tr \rho_j \) is then replaced by \( p_j + \delta p_j \), with \( \delta p_j = (1 + 2\varepsilon_j)p_j \). Complete reduction consists ultimately in a random process, so that \( \delta p_j \) is a random quantity. It is classically random, at least as far as one can say presently. The sum of all probabilities must remain however equal to 1, so that reduction cannot affect a unique state, but in principle all of them with the condition

\[ \sum_k \delta p_k = 0 \]  

(2.2)

It may be noticed that this change in the probabilities can be represented by a change in the evolution operator \( U(t) = \exp(-iHt) \). It amounts to the replacement

\[ -iHdt \rightarrow -iHdt + \sum_j P_j(t)d\varepsilon_j(t) . \]  

(2.3)

Since this change amounts formally to the use of a non-Hermitian random Hamiltonian, it is clearly not time-reversal invariant, as one could expect.

A series of infinitesimal random processes is essentially a general Brownian variation of the coordinates \( p_j \) and a few convenient definitions will be useful. One considers only the case when the number \( n \) of coordinates is finite. The various
quantities $p_k$ change continuously in a random way and it is convenient to represent them as the coordinates of a moving point $M(t)$ with coordinates $p_k(t)$ in an $n$-dimensional Euclidean space $E$. Since they always sum up to 1, one can introduce the $(n-1)$-dimensional subspace $E'$, with equation $\sum p_k = 1$ in $E$. Let us call “vertex $k$”, or $V_k$, the point in $E$ having all its coordinates equal to 0, except for the $k$-th coordinate, equal to 1. The space $E'$ contains all the vertices $V_k$, which are the vertices of a $(n-1)$-dimensional regular “tetrahedron”, also often called a simplex and denoted by $S$. The case $n = 3$ is particularly easy to visualize since the simplex is then an equilateral triangle. Choosing an arbitrary fixed origin of coordinates $O$ in $E'$, one has the vector relation

$$\overrightarrow{OM}(t) = \sum_k p_k(t) \overrightarrow{OV}_k.$$  

(2.4)

In algebraic geometry, the quantities $p_k(t)$ are called barycentric coordinates of the point $M(t)$ and they can be thought of as masses located at the vertices, $M(t)$ being their center of gravity. They can also be considered as the distance of $M(t)$ to the $(n-2)$-dimensional face of the simplex opposite to the vertex $V_k$.

Let $\{\xi_j\}$ ($j = 1$ to $n-1$) denote a set of orthogonal Cartesian coordinates in $E'$. Their relation with the quantities $\{p_k\}$ is simply obtained after introducing another coordinate $\xi_0$ in $E$ along an axis normal to $E'$, so that one has $\xi_0 = \sum p_k = 1$ in $E'$. The relation between the coordinates $\{\xi_j, \xi_0\}$ and $\{p_k\}$ is an $n \times n$ orthogonal transformation so that, conversely, each quantity $p_k$ is given by a first-degree polynomial in the quantities $\{\xi_j\}$.

One can draw two important consequences from this property. The fact that the second derivatives of $p_k$ with respect to the $\xi_j$’s vanish will play an essential role in next section. Furthermore, one can discuss equivalently the random quantities $dp_k(t)$ or $d\xi_j(t)$. Their random motion will be called non-directional if the average values $<d\xi_j(t)>$ are zero.

Consider then the correlation matrix $C$ whose coefficients are defined by

$$<d\xi_j d\xi_m> = C_{jm}(\xi)dt.$$  

(2.5)

The Brownian motion will be said homogeneous if $C$ does not depend on $\xi$, and isotropic if it is proportional to the unit matrix. One may notice that a condition for isotropy is invariance under the symmetries of the simplex or, equivalently, to permutations of the quantities $dp_k$. Because of Eq. (2.2), one then has
formula (2.6)

The Brownian motion of the point \( M(t) \), starting from the point \( M = M(0) \), must bring it after some time on a face of the simplex, in which one of the coordinates \( p_k \) vanishes. Thereafter, it will never go back into the simplex interior because the corresponding component \( \rho_k \) of the density matrix has vanished. If one now makes explicit the number \( n \) in the dimension \( n - 1 \) the initial simplex by writing it \( S_n \), its boundary consists of \( n \) simplexes \( S_{n-1} \) with dimension \( n - 2 \). After reaching such a face, the reduction mechanism becomes a Brownian motion inside it until \( M(t) \) reaches its boundary, and the same process goes on until \( M(t) \) reaches some vertex \( V_k \) in a one-dimensional simplex (which is an interval \([0, 1]\)). Its \( k \)-th coordinate is then equal to 1 and the density matrix has become \( P_k \rho S P_k / \text{Tr}(P_k \rho S P_k) \), if one can neglect the density \( \rho_1 \) in Eq. (1.3). Reduction is complete. The mechanism resulting from these simple assumptions looks therefore able, in principle, to “explain” the uniqueness of physical reality. To make sense, however, it must satisfy a very stringent condition, which is that the probability for \( M(t) \) to reach ultimately the vertex \( V_k \) should be equal to the quantum probability, \( p_k(0) \), which is the \( k \)-th barycentric coordinate of the initial point \( M = M(0) \) from which the Brownian started.

3 A uniqueness theorem

The main problem is therefore to compute the probability \( P_1 \) for the moving point \( M(t) \), starting from the position \( M = M(0) \) with coordinates \((p_1(0), p_2(0), \ldots, p_n(0))\) to reach finally a definite vertex, say \( V_1 \). It may be noticed that it does not matter whether the matrix \( C \) in Eq. (2.5) is time-dependent or not as long as one does not ask how much time the motion will take. The essential question is to find whether there are cases yielding the quantum prediction

\[
P_1 = p_1(0)
\]  

Pearle first raised this question in a different theoretical framework and obtained insightful results [1]. The following statement gives a complete answer:
**Theorem**: A necessary and sufficient condition for the validity of Eq. (3.1) is that the Brownian motion be non-directional, isotropic and homogeneous.

This theorem will be proved through a series of lemmas, some of them extending previously known results [1-2, 32].

**Lemma 1**: Eq. (3.1) is valid for a non-directional, isotropic and homogeneous Brownian motion.

**Proof**: Let one first show that the function $P_1(p_1, \cdots, p_N)$ is harmonic in the $\xi$ variables, the initial value of time (0) characterizing these quantities being omitted, just like $M(0)$ was denoted by $M$. One knows already that this function is harmonic if Eq. (3.1) holds, since $p_1(0)$ has vanishing second derivatives in the $\xi$ variables. Conversely, a function is harmonic if and only if its value at any point $M$ is the average of its values over an arbitrary $(n - 2)$-dimensional sphere centered at $M$. Let one then consider a sphere $\Sigma$ (with element of area $d\Omega$), centered at $M$ and inside the open simplex $S_n$. Denoting by $Q$ an arbitrary point on $\Sigma$, the harmonic character of $P_1(p_1, \cdots, p_N)$ will be established if one has

$$P_1(M) = \Omega^{-1} \int_{\Sigma} P_1(Q)d\Omega.$$  

(3.2)

This is easily shown as follows: The moving point $M(t)$ starting from $M$ must cross the sphere $\Sigma$ before reaching finally vertex 1. Let $\Pi(M, Q)d\Omega$ denote the probability for reaching $\Sigma$ for the first time in an infinitesimal region $d\Omega$ containing $Q$. Using composite probabilities, one has:

$$P_1(M) = \int_{\Sigma} \Pi(M, Q)P_1(Q)d\Omega.$$  

(3.3)

But for an isotropic Brownian motion, one has $\Pi(M, Q) = \Omega^{-1}$, so that Eq. (3.2) is true.

The next step consists in showing that $P_1(p_1, \cdots, p_N) = p_1$. This is obvious when $N = 2$, when the simplex reduces to an interval (for instance the interval $V_1V_2$) and there is only one coordinate $\xi_1$. $P_1$ depends then only on the variable $p_1$, which is a first-degree polynomial in $\xi_1$. Being harmonic, $P_1$ is a function $Ap_1 + B$, which is furthermore equal to 1 when the starting point is the vertex $V_1$ (where $p_1 = 1$) and equal to zero at $V_2$ (where $p_1 = 0$). Therefore $P_1 = p_1$. 

9
When $n = 3$, the simplex $S_3$ is an equilateral triangle. The case $n = 2$ has already shown that the boundary value of the harmonic function $P_1(p_1, p_2, p_3)$ is equal to $p_1$ on both sides $V_1V_2$ and $V_1V_3$ of the triangle and it obviously vanishes on the side $V_2V_3$ (where $p_1 = 0$). The expected result follows then from the uniqueness of the solution of the Dirichlet problem. The extension to any value of $n$ by means of recurrence is obvious.

**Lemma 2**: Eq. (3.1) does not hold for an anisotropic homogeneous Brownian motion.

**Proof**: The matrix $C$ is then a symmetric positive matrix, with some different eigenvalues. One can choose the axes of coordinates along its eigenvectors and perform a change of scale $\xi_j \rightarrow \eta_j = \lambda_j \xi_j$ bringing $C$ proportional to the unit matrix. The simplex is no more regular in the $\eta$ coordinates, but the previous reasoning implying the harmonic character of $P_1$ (in $\eta$) remains valid. The previous result also holds true for $n = 2$, since a Brownian motion is always isotropic in one dimension. But if one tries to descend from a simplex $S_n \cdot (n \geq 2)$ to its boundary, the boundary value of $P_1$, when expressed as a function of the $\eta$ variables of a boundary simplex will be harmonic only if the variables of type $\eta$ agree on both simplexes $S_n$ and $S_{n-1}$. But this property holds only when the boundary simplex $S_{n-1}$ is spanned by $n - 2$ eigenvectors of $C$ and it cannot hold for all the boundary simplexes. In other words, the boundary value of $P_1$ cannot be harmonic everywhere on the boundary, forbidding the descent process. Another way of getting the same answer consists in looking at the transfer from $n = 2$ to $n = 3$. The values of $P_1$ resulting from the one-dimensional Brownian motions on the three sides of the equilateral triangle have not the same algebraic expression in the $\eta$ variables.

**Lemma 3**: Eq. (3.1) does not hold for an inhomogeneous Brownian motion.

**Proof**: In view of Lemma 2, one may assume the motion isotropic. It will be enough to establish Lemma 3 in the case $n = 2$, since the property (3.1) cannot hold for an arbitrary value of $n$ if it does not hold for the final one-dimensional step for which $n = 2$. Let one then choose the coordinate $\xi = p_1$, defined in the interval $J = [0, 1]$. One can use the diffusion approximation for an inhomogeneous Brownian motion, where the probability distribution of the moving point $M(t)$ is a function
\( \rho(\xi, t) \) satisfying the following equation

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \xi} \left\{ D(\xi) \frac{\partial \rho}{\partial \xi} \right\}, \tag{3.4}
\]

where the diffusion coefficient \( D(\xi) \) is positive.

The boundary conditions for absorbing boundaries are given by

\[
\rho(0, t) = \rho(1, t) = 0. \tag{3.5}
\]

The initial value is taken as

\[
\rho(\xi, 0) = \delta(\xi - \alpha), \tag{3.6}
\]

where \( \alpha \) is the initial position \( \xi(0) \) of the moving point.

Denoting by \( L \) the operator \( \partial/\partial \xi(D \partial/\partial \xi) \), one introduces orthonormal eigenfunctions and eigenvalues satisfying the differential equations

\[
L \psi_n = -\lambda_n \psi_n, \tag{3.7}
\]

with Dirichlet boundary conditions \( \psi_n(0) = \psi_n(1) = 0 \). The quantities \( \lambda_n \) are strictly positive (zero is not an eigenvalue).

The delta function in Eq. (3.6) is written as

\[
\delta(\xi - \alpha) = \sum_n \psi_n(\xi) \psi_n(\alpha), \tag{3.8}
\]

so that

\[
\rho(\xi, t) = \sum_n \psi_n(\alpha) \psi_n(\xi) \exp(-\lambda_n t). \tag{3.9}
\]

The probability that the point \( M(t) \) ends up at the boundary \( \xi = 1 \) is given by the time integral of the flux

\[
P_1 = -D(1) \int_0^\infty \frac{\partial \rho(1, t)}{\partial \xi} dt = -D(1) \psi_n(\alpha) \psi'_n(1)/\lambda_n. \tag{3.10}
\]

The right-hand side of this expression has a simple interpretation after introducing the Green function (with complex \( z \)).

\[
G(z, \xi, \eta) = \langle \xi | (L - zI)^{-1} | \eta \rangle, \tag{3.11}
\]
i.e. the kernel of the operator \( z I - L \) with Dirichlet boundary conditions. One has

\[
P_1 = D(1) \frac{\partial}{\partial \xi} G(0, \alpha, \xi) \bigg|_{\xi=1} .
\]  (3.12)

The limit \( \xi = 1 \) in the derivative is meaningful, because \( G(0, \alpha, \xi) \) is an infinitely differentiable function of \( \alpha \) and \( \xi \) (except for \( \alpha = \xi \)) in the Cartesian product \( J \times J \). Considering \( P_1 \) as a function of \( \alpha \) and letting the operator \( L \) act on it, one gets

\[
D(\alpha) \frac{d^2 P_1}{d\alpha^2} + D'(\alpha) \frac{dP_1}{d\alpha} = D(1)\delta'(\alpha - 1) .
\]  (3.13)

(Note : A rigorous use of the delta function derivative at the point \( \xi = 1 \) can be justified through a limiting process \( \xi \to 1 \), but this is trivial because the only property we need is \( \delta'(\alpha - 1) = 0 \), when \( \alpha < 1 \). One thus gets

\[
\frac{d}{d\alpha} \left( D(\alpha) \frac{dP_1}{d\alpha} \right) = 0 , \text{or } \frac{dP_1}{d\alpha} = \frac{C}{D(\alpha)} ,
\]  (3.14)

where \( C \) is a constant. Eq. (3.1), which would suppose \( P_1(\alpha) = \alpha \), holds therefore only when \( D(\alpha) \) is a constant, i.e. only in the case of homogeneous Brownian motion.

**Lemma 4 :** Eq. (3.1) does not hold for a directional Brownian motion.

**Proof :** Consider the simplest case \( n = 2 \), with \( <d\xi/dt> = \nu \). In the diffusion approximation, the probability distribution satisfies the Fokker-Planck equation

\[
\partial \rho / \partial t = \frac{\partial}{\partial \xi} \left( D \frac{\partial \rho}{\partial \xi} - \nu \rho \right) .
\]  (3.15)

One can integrate it in time from zero to infinity, using the initial condition (3.6), to get

\[
-\delta(\xi - \alpha) = D d^2 y/d\xi^2 - \nu dy/d\xi , \text{with } y(\xi) = \int_{0}^{\infty} \rho(\xi;t)dt .
\]  (3.16)

This elementary differential equation determines explicitly the function \( y \) in view of the boundary conditions (3.5) :

\[
y(\xi) = \frac{D}{\nu} \left( e^{\nu/2} - e^{\nu/2} \right) \vartheta(\alpha-\xi) - \frac{D}{\nu} \frac{1 - e^{-\nu/2}}{e^{\nu/2} - 1} \left( e^{\nu/2} - e^{\nu/2} \right) \vartheta(\xi-\alpha) ,
\]
where \( \vartheta \) denotes the unit step-function. Noticing that \( P_1 = -Dy'(1) \), one thus gets the exact expression

\[
P_1 = \frac{(1 - \exp(-\nu \alpha/D))}{\exp(\nu/D) - 1}, \tag{3.17}
\]

which satisfies condition (3.1) when and only when \( \nu = 0 \).

Finally, one can estimate roughly the time during which a complete reduction occurs when starting from \( n \) competing states. A unique scale of time \( \tau \) will be assumed to enter in the average displacement of the moving point \( M(t) \), whatever the value of \( n \):

\[
\Delta \xi^2 = t/\tau. \tag{3.18}
\]

At every step, or more properly during every step of the descending scale in \( n \), the point \( M(t) \) has moved over a distance \( \Delta \xi \) of the order of unity (the size of the simplex) when it reaches a boundary. Accordingly, the duration of a complete reduction is of the order of \( n \tau \). Estimating the value of \( \tau \) remains however an open problem.

### 4 Non-relativistic systems and quantum field theory

It was assumed in the introduction that the action of reduction should be expressible in the framework of quantum field theory, at least in a phenomenological way. Actually, the uniqueness of physical reality is usually observed in commonplace occasions, under non-relativistic conditions. One may therefore begin by considering a non-relativistic macroscopic system. Elementary quantum mechanics considers it as made of particles, though one knows that quantum fields are more fundamental. As a first task, one must therefore cast the description of a system of non-relativistic particles into the framework of quantum field theory.

One considers for convenience a simple model of a macroscopic system where all the particles belong to the same species and are described by a scalar relativistic field \( \phi(r, t) \). In the reference frame where the system is at rest, one can introduce a field \( \Pi(r, t) \), canonically conjugate to \( \phi(r, t) \). The evolution amplitudes \( \langle \phi_{2t_2} | \phi_{1t_1} \rangle \) are given by Feynman sums.
\[
\int [d\phi][d\Pi] \exp \left\{ i \int dt \int (dr) [L + \phi(x)\rho(x) + \Pi(x)\sigma(x)] \right\}.
\]

(4.1)

The notation for Feynman sums and ordinary integrals, which is borrowed from Brown's book [33], distinguishes the space integration in the Lagrangian, denoted by \((dr)\) from a Feynman summation, denoted with square parentheses \([d\phi]\). The four-dimensional variable \(x\) stands for \((r,t)\). The Lagrangian density \(L\) is given by \(\Pi\partial_0 \phi - H(\phi, \Pi)\), and includes the Hamiltonian \(H\). Sources \(\rho\) and \(\sigma\) have been introduced for later convenience. The two fields \(\phi\) and \(\Pi\) satisfy the canonical commutation relations

\[
[\phi(r, t), \Pi(r', t)] = i\delta^{(3)}(r - r') ,
\]

(4.2)

and the same notation is used for the quantum fields and the classical fields on which Feynman sums are performed. Free fields are characterized by the Hamiltonian

\[
H = \frac{1}{2} \Pi^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}m^2\phi^2 .
\]

(4.3)

A non-relativistic version of the field formalism is obtained when mass terms \(e^{\pm \imag m t}\) are separated from the phase of the fields, thereby introducing a pair of non-relativistic fields as

\[
\psi(r, t) = e^{\imag m t} \left[ \sqrt{m/2} \phi(r, t) + i/\sqrt{2m} \Pi(r, t) \right] ,
\]

\[
\psi^\dagger(r, t) = e^{-\imag m t} \left[ \sqrt{m/2} \phi(r, t) - i/\sqrt{2m} \Pi(r, t) \right] .
\]

(4.4)

The commutation relations of these fields are those of local creation and annihilation operators, when the quantities \(e^{\pm \imag m t}\) are considered as rapidly varying and averaged out:

\[
[\psi(r, t), \psi^\dagger(r', t)] = \delta^{(3)}(r - r') .
\]

(4.5)

Neglecting similar rapidly changing terms in the action, the Feynman sum (4.1) for a free field becomes in the non-relativistic limit:

\[
\int [d\psi][d\psi^*] \exp \left\{ i \int dt \int (dr) \left[ \psi^* i \frac{\partial \psi}{\partial t} - \psi^* \left( -\nabla^2 \right) \psi + \psi^* f + \psi g \right] \right\} .
\]

(4.6)
The sources \( f \) and \( g \) are linear functions of \((\rho, \sigma)\). The effect of an external potential \( V_1(r) \) and of interaction potentials \( V_2(r, r') \) results from adding the following quantity to the Lagrangian:

\[
\int (dr)\psi^*(r, t)V_1(r)\psi(r, t) + \frac{1}{2}\int (dr)(dr')\psi^*(r, t)\psi^*(r', t)V_2(r, r')\psi(r, t)\psi(r', t) .
\] (4.7)

Finally, the relation with particle wave functions is obtained from introducing a vacuum state \( |0> \), so that a pure state of the system with wave function \( u \) is given in the field formalism by

\[
|u> = \int dr_1 \cdots dr_n u(r_1, \cdots r_n, t)\psi^\dagger(r_1, t) \cdots \psi^\dagger(r_n, t)|0> .
\] (4.8)

5 Infinitesimal reduction

One now comes to the central problem, which is the compatibility of a reduction mechanism with known physics and, particularly, relativistic quantum field theory as it is used in the standard model. The difficulty underlying this problem is emphasized by the fact that no previous model of reduction was able to solve it. The root of the difficulty, in my opinion, does not lie really in the possibility of dealing with a macroscopic system in an arbitrary reference frame. There is no reason to expect this problem of relativistic invariance to be more difficult than in standard quantum field theory. The real difficulty may well come from the projection operators entering in reduction, whether continuous or sudden.

Since the introduction of reduction by Bohr, and later in the models attempting to formalize it, it was always assumed that a specific part \( u_\lambda \) of the wave function \( u \) (representing a definite measurement result \( \lambda \)) was preserved, while the rest of the wave function vanished. It will be instructive to look at the projection operator \( P_\lambda \) realizing this effect, either completely or infinitesimally.

In the non-relativistic quantum field formalism of the previous section, this projection operator can be expressed in terms of fields by

\[
\int (dr_1) \cdots (dr_n) \cdot dr'_1 \cdots dr'_n \psi^\dagger(r_1) \cdots \psi^\dagger(r_n) \cdot \psi(r'_1) \cdots \psi(r'_n)
\times u_\lambda(r'_1, \cdots, r'_n)u^*_\lambda(r_1, \cdots, r_n) .
\] (5.1)
This expression cannot be extended however to a less trivial field theory. In quantum electrodynamics for instance, the Hilbert space involves photons, particles and antiparticles. There is no wave function describing the particles as such, but a state involving the values of the various fields at every point of space. The operator \( P_\lambda \) could be written in the form (5.1) because the \((\psi, \psi^\dagger)\) fields destroy and create particles, whereas no operator can destroy or create the fields themselves. It seems therefore that one cannot express in quantum field theory a reduction process that would act on every detail of a wave function. I am inclined to believe that this simple remark is the main reason why the GRW mechanism or Bohm’s model could not be extended to field theory.

The situation is quite different when decoherence has acted first. It was shown already that the projection operators entering in reduction are much simpler and this point can be made clear on a simple example. Consider a case where the relevant subsystem \( R \) consists of a unique massive particle with position \( r_0 \) and field \( \psi_0(r_0, t) \). The environment \( E \) consists of a large number of identical particles, as in Section 4 (this model being quite similar to Joos-Zeh’s model [17], if the massive “particle” is supposed to have a spatial extension). Let one assume that decoherence has split the state into mutually exclusive events \( \lambda \), each one of them corresponding to the location of the massive particles in a different region of space \( D_\lambda \). The corresponding projection operator (which, \( E_E \to I_E \) was denoted earlier by \( P_{AR} \otimes E_E \)) is simply given by

\[
\int_{D_\lambda} dr \psi_0^\dagger(r) \psi_0(r). \tag{5.2}
\]

Although I will not discuss the extension of this formula to a less trivial field theory, one does not expect any specific difficulty, since this operator belongs to the field algebra. In any case, the present considerations are only preliminary and they are only meant as a first encounter with past difficulties and their expected removal.

Finally, one must consider more significant consequences of the consistency of reduction with the standard model. Reduction violates time-reversal invariance and there are strong indications from the standard model that interactions violating time reversal become sizable at the scale of Planck’s length. Another remark points also to a fundamental action of reduction at this level: When a projection operator is integrated over time with a random factor \( d\varepsilon(t) \), it behaves formally as a source term analogous to the ones in Eq. (4.1). It involves however generally a somewhat
complicated function of the fields. Such source terms will obviously break renormalization, except if $d\varepsilon(t)$ varies rapidly. It is generally agreed that this situation can happen only when the time scale of variation of the source is of the order of Planck’s time.

6 Perspectives

Having to deal together with a source of reduction at the scale of Planck’s length and the macroscopic uniqueness of space-time raise immediately the questions of quantum gravity and cosmology. I shall restrict however the remarks on this topic to the perspective of generalized quantum theory, which was introduced by Gell-Mann and Hartle for this special purpose, as an extension of consistent decohering histories [8, 28]. Isham and coworkers devised for it a convenient mathematical formalism. Rather than using a unique Hilbert space, they introduced a tensor product of as many copies of the Hilbert space as there are instants of time in a family of consistent histories [29]. The procedure relies however on a foliation of space-time by space-like surfaces and the existence of such a foliation looked questionable in the Wheeler-De Witt theory of quantized gravity, in which there is no time variable. A recent proposal by Kuchar might open new possibilities for this approach and will be described here briefly [31, 32].

In the canonical formalism of quantum mechanics, a unique Hilbert space provides a representation of the commutation relations between canonically conjugate operators (for particles or fields) $[x, p] = 1$. Isham’s formalism involves a set of time-indexed operators satisfying history-dependent commutation relations, which can be written essentially as $[x(t), p(t')] = i\delta(t - t')$. Kuchar’s proposal consists in extending the space-time phase space by introducing a time operator $T$ together with its conjugate momentum. In the case of vacuum gravity, the corresponding “phase space” involves then a set of operators

$$\left( G_{AB}(y), \Pi^{AB}(y); T(y), \Pi_T(y) \right)$$

(6.1)

and a key point is that these space-time fields can be freely varied in the space-time classical action. Projection operators, as they enter in consistent histories, can also be defined.

Kuchar considered the case of pure gravity, but a very interesting problem will occur when the time foliation will be coupled to decoherence. The consistency of
decohering histories will strongly constrain the time foliation and reduction might
give it an objective local meaning. This is of course only a speculative perspective,
but it looks encouraging for further investigations.

Also encouraging is the convergence of older approaches to the problem of re-
duction with the present one. Although it was described here in a constructive way,
the present approach was initially meant as a critical analysis of the previous ones,
according to the two basic conditions that were stated in the introduction. It looks
remarkable that a critical approach brought thus back rather naturally new variants
of spontaneous reduction [3, 4], stochastic continuity [1, 2] and of the relevance of
quantum gravity in the uniqueness of space-time [5, 6].

Acknowledgements

I thank Bernard d’Espagnat for useful remarks and Karel Kuchar for an illuminating
discussion on related topics. I also thank the organizers of the Second DICE meeting
in Piombino, and some speakers in whose lectures I found an inspiration for some
of the ideas occurring in this work.
References

[1] P. Pearle, Phys. Rev. D 13, 857 (1976).
[2] P. Pearle, Phys. Rev. A 39, 2277 (1984).
[3] G-C. Ghirardi, A. Rimini, T. Weber, Phys. Rev. D34, 470 (1986).
[4] L. Diosi, Phys. Lett. A, 120, 377 (1987); 129, 419 (1988), Phys. Rev. A 40, 1165 (1989).
[5] R. Penrose, Gen. Rel. Grav., 28, 581 (1996).
[6] F. Karolyhazy, Nuovo Cimento, A 42, 390 (1966).
[7] D. Bohm, Phys. Rev. 85, 166 (1966), D. Bohm, D., J. Bub, Rev. Mod. Phys. 38, 453 (1966).
[8] M. Gell-Mann, M., J. B. Hartle, in Complexity, Entropy, and the Physics of Information, W. H. Zurek, edit. Addison-Wesley, Redwood City, CA (1991).
[9] R.G. Griffiths, Found. Phys. 23, 1601 (1993); Consistent Quantum Mechanics, Cambridge (G. B.) Cambridge University Press (2002).
[10] R. Omnès, Understanding Quantum Mechanics, Princeton University Press (1999).
[11] H. Everett, Rev. Mod. Phys., 29, 454 (1957).
[12] R. Omnès, Rev. Mod. Phys. 64, 339 (1992); The interpretation of quantum mechanics, Princeton University Press (1994).
[13] H. D. Zeh, Found. Phys. 1, 69 (1970).
[14] M. Brune, M., E. Hagley, J. Dreyer, X. Maître, A. Maali, C. Wunderlich, J. M. Raimond, S. Haroche, Phys. Rev. Letters, 77, 4887 (1996).
[15] K. Hepp, E. H. Lieb, Helv. Phys. Acta, 46, 573 (1974).
[16] A. O. Caldeira, A. J. Leggett, Physica A 121, 587 (1983).
[17] E. Joos, H. D. Zeh, Z. Phys. B 59, 229 (1985).
[18] M. Gell-Mann, J. B. Hartle, Phys. Rev. D 47, 3345 (1993).

[19] J. J. Halliwell, Phys. Rev. Lett. 83, 2481 (1999); Phys. Rev. D 58, 105015 (1998).

[20] W. H. Zurek, Rev. Mod. Phys. 75, 715 (2003).

[21] R. Omnès, Phys. Rev. A 65, 052119 (2002).

[22] W. H. Zurek, W. H., Phys. Rev. D24, 1516 (1981); Phys. Rev. D26, 1862 (1982).

[23] J. S. Bell, Hel. Phys. Acta, 48, 93 (1975).

[24] B. d'Espagnat, Veiled Reality, Addison-Wesley, Reading (1995).

[25] L. Hörmander, Ark. Mat. (Sweden), 17, 297 (1979).

[26] R. Omnès, J. Math. Phys. 38, 697 (1997).

[27] G. Lüders, Ann. Phys. 8, 322 (1951).

[28] J. B. Hartle, in Les Houches Summer School Proceedings, vol. 57, Gravitation and quantizations, B. Julia and J. Zinn-Justin, eds. North-Holland, Amsterdam (1995).

[29] C. J. Isham, N. Linden, J. Math. Phys. 35, 5452 (1994); C. J. Isham, N. Linden, S. Schrekenberg, J. Math. Phys. 35, 6360 (1994); C. J. Isham, N. Linden, J. Math. Phys. 36, 5392 (1995).

[30] I. Kouletsis, K. Kuchar, Phys. Rev. D (3) 65, 125026 (2002).

[31] K. Kuchar, communication at the 2nd.DICE meeting, Piombino (It), September 2004, unpublished.

[32] R. Omnès, Phys. Lett. A 187, 26 (1994).

[33] L. S. Brown, Quantum Field Theory, Cambridge University Press (1992).