A SOLUTION OF THE 3D REFLECTION EQUATION FROM QUANTIZED ALGEBRA OF FUNCTIONS OF TYPE B

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Abstract. Let $A_q(g)$ be the quantized algebra of functions associated with simple Lie algebra $g$ defined by generators obeying the so called RTT relations. We describe the embedding $A_q(B_2) \hookrightarrow A_q(C_2)$ explicitly. As an application, a new solution of the Isaev-Kulish 3D reflection equation is constructed by combining the embedding with the previous solution for $A_q(C_2)$ by the authors.

1. Introduction

The reflection equation [1, 2] is a companion system of the Yang-Baxter equation [3] in 2 dimensional (2D) integrable systems with boundaries. A 3D analogue of such a system is known by Isaev and Kulish [4] as the 3D reflection equation which accompanies the Zamolodchikov tetrahedron equation [5]. In [6], the first solution of the 3D reflection equation was constructed by invoking the representation theory [7] of the quantized algebra of functions $A_q(g)$ [8, 9] for $g$ of type $C$. It was done by succeeding the approach in [10], where the solution of the tetrahedron equation [11] was obtained (up to misprint) as the intertwiner for $A_q(g)$-modules with $g$ of type $A$.

In this paper we extend such results to $g$ of type $B$. The algebra $A_q(B_n)$ is formulated following [9], and its fundamental representations $\pi_i (i = 1, 2, 3)$ are presented for $n = 3$. According to the general theory [7], one has the equivalence of the tensor products $\pi_{121} \simeq \pi_{212}$ and $\pi_{323} \simeq \pi_{232}$ ($\pi_{i_1 \cdots i_r} := \pi_{i_1} \otimes \cdots \otimes \pi_{i_r}$) according to the Coxeter relations $s_1 s_2 s_1 = s_2 s_1 s_2$ and $s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2$ among the simple reflections in the Weyl group $W(B_3)$. We determine their intertwiners $S$ and $J$ explicitly and show that they yield a new solution to the 3D reflection equation, confirming a conjecture in [6]. In fact, our strategy is to attribute the analysis to the $A_q(C_3)$ case [6]. The key to this is the embedding $A_q(B_2) \hookrightarrow A_q(C_2)$ in Theorem 2.1. It enable us to bypass the complicated intertwinning relation (11) for type $B$ and to utilize the result on the type $C$ case. The embedding originates in a formulation of $A_q(g)$ by the RTT relations [9] and the fact that the vector representation of $C_2$ is the spinor one for $B_2$.

The paper is organized as follows. In Section 2 we recall the Hopf algebra $A_q(B_n)$ following [9]. The embedding $A_q(B_2) \hookrightarrow A_q(C_2)$ in Theorem 2.1 will be the key to the proof of our main Theorem 1.2. In Section 3 we present the fundamental representations $\pi_i (i = 1, 2, 3)$ of $A_q(B_3)$. The $\pi_2$ and $\pi_3$ corresponding to the subalgebra $A_q(B_2)$ actually come from the fundamental representations of $A_q(C_2)$ through the embedding (Remark 3.2). In Section 4 we formulate the intertwiners along the scheme parallel to $A_q(C_3)$ [6] and determine them. In Section 5 we explain that these intertwiners yield a new solution to the 3D reflection equation, which is distinct from the type $C$ case in [6].
2. Algebra $A_q(B_n)$

Let $N = 2n + 1$ with $n \in \mathbb{Z}_{>2}$. We define $A_q(B_n)$ following [9], where it was denoted by $\text{Fun}(SO_q^2(N))$. $A_q(B_n)$ is a Hopf algebra generated by $t_{ij}$ $(1 \leq i, j \leq N)$ with the relations

$$R_{ij, mp} t_{im} t_{pl} = t_{jp} t_{lm} R_{mp, kl}, \quad C_{jk} C_{lm} t_{ij} t_{lk} = C_{ij} C_{kl} t_{kj} t_{lm} = \delta_{lm}, \quad (1)$$

where the repeated indices are summed over $\{1, 2, \ldots, N\}$. The structure constants are specified by $C_{ij} = \delta_{i, N+1-j} q^{\rho_{ij}}$ with $(q_1, \ldots, q_N) = (2n-1, \ldots, 3, 1, 0, -1, -3, \ldots, -2n+1)$, and $\sum_{i,j,m,l} R_{ij, ml} E_{im} \otimes E_{jl} = 2^2 \lim_{x \to \infty} x^{-2} R(x)_{k=q^{-2}}$, where $R(x)$ is the quantum $R$ matrix [12, 13] for the vector representation of $U_q(B_0^{(1)})$ given in [13] eq.(3.6). The former one in (1) is the so called RTT relation on the generators $T = (t_{ij})$. The coproduct is given by $\Delta(t_{ij}) = \sum_{k} t_{ik} \otimes t_{kj}$. We omit the antipode and counit as they are not necessary in this paper.

Reflecting the equivalence $U_q(B_2) \simeq U_q(C_2)$, the simplest case $A_q(B_2)$ is related to $A_q(C_2)$. The latter was introduced in [9] and detailed in [6, sec. 3.1], where it was denoted by $A_q(Sp_4)$. The $A_q(C_2)$ is the Hopf algebra defined similarly to $A_q(B_2)$ by replacing the structure constants $R_{ij, mp}$ and $C_{jk}$ by those associated with the quantum $R$ matrix of the vector representation of $U_q(C_2^{(1)})$. The generators $t_{ij}$ $(1 \leq i, j \leq 4)$ of $A_q(C_2)$ ([6 sec. 3.1]) will be denoted by $s_{ij}$ here for distinction. The vector representation of $U_q(C_2)$ is the spinor one in terms of $U_q(B_2)$. This fact is reflected in the following.

Theorem 2.1. There is an embedding of the Hopf algebra $\iota : A_q(B_2) \hookrightarrow A_q(C_2)$ which maps the generators $\{t_{ij} \mid 1 \leq i, j \leq 5\}$ as follows:

$$t_{ij} \mapsto (-1)^{\delta_{i,1} + \delta_{j,5}} (\sqrt{r})^{\delta_{i,3}} (s_{i,6} s_{i,7} s_{n,0} - q s_{i,7} s_{n,6} s_{i,6}) \quad (j \neq 3),$$

$$t_{33} \mapsto s_{22} s_{33} - q s_{21} s_{34} + q s_{24} s_{31} - q^2 s_{23} s_{32},$$

where $r = 1 + q^2$ and $\xi_i, \eta_k$ are given by $(\xi_i \xi_j \eta_\ell \eta_k / m_n n_{n'} n'_{n'}) = (1^{11223} 23344)$. We have checked that under $\iota$, all the relations on $\{t_{ij}\}$ in $A_q(B_2)$ are guaranteed by those on $\{s_{ij}\}$ in $A_q(C_2)$ by tedious but direct calculations. We note that in the different formulation of $A_q(g)$ of [14], one has the isomorphism $A_q(B_2) \simeq A_q(C_2)$ by definition rather than the embedding.

3. Fundamental representations of $A_q(B_3)$

Let $\text{Osc}_q = (1, a^+, a^-, k)$ be the $q$-oscillator algebra, i.e. an associative algebra with the center 1 and the relations

$$k a^+ = q a^+ k, \quad k a^- = q^{-1} a^- k, \quad a^- a^+ = 1 - q^2 k^2, \quad a^+ a^- = 1 - k^2.$$

It has a representation on the Fock space $\mathcal{F}_q = \oplus_{m \geq 0} \mathbb{C}(q)|m\rangle$:

$$|1\rangle = |m\rangle, \quad |k| |m\rangle = q^{mn} |m\rangle, \quad a^+ |m\rangle = |m + 1\rangle, \quad a^- |m\rangle = (1 - q^{2m}) |m - 1\rangle.$$

We shall also use $\text{Osc}_{q^2}$ with $q$ replaced by $q^2$. For distinction it is denoted by $\text{Osc}_{q^2} = (1, A^+, A^-, K)$, which acts on $\mathcal{F}_{q^2}$.

From now on we consider $A_q(B_3)$ which includes $A_q(B_2)$ as a subalgebra. Set $(q_1, q_2, q_3) = (q^2, q^2, q)$, which reflects the squared root length of the simple roots of
B_3. Consider the maps \( \pi_i (i = 1, 2, 3) : A_q(B_3) \to \text{Osc}_q \) that send the generators \((t_{ij})_{1 \leq i, j \leq 7}\) to the following:

\[
\begin{align*}
\pi_1 : & \quad \begin{pmatrix}
\mu_1 A^+ & \alpha_1 K \\
\beta_1 K & \nu_1 A^+
\end{pmatrix}
\begin{pmatrix}
\kappa_1 1 \\
\sigma_1 1
\end{pmatrix}, \\
\pi_2 : & \quad \begin{pmatrix}
\mu_2 A^- & \alpha_2 K \\
\beta_2 K & \nu_2 A^+
\end{pmatrix}
\begin{pmatrix}
\kappa_2 1 \\
\sigma_2 1
\end{pmatrix}, \\
\pi_3 : & \quad \begin{pmatrix}
\mu_3 (a^-)^2 & \alpha_3 k a^- \\
-r\alpha_3 \mu_3 a^- k & a^+ a^- - k^2 \\
-r(q\alpha_3^{-1}) \mu_3 k^2 & r\alpha_3^{-1} k a^+ \\
& \mu_3 (a^+)^2
\end{pmatrix}
\begin{pmatrix}
\kappa_3 1 \\
\kappa_3 1
\end{pmatrix},
\end{align*}
\]

where \( r \) is defined in Theorem 2.1 and blanks mean 0. The symbols \( \alpha_i, \beta_i, \mu_i, \nu_i, \sigma_i, \kappa_i, \kappa_{31}, \kappa_{32} \) are parameters. They obey the constraint

\( \alpha_i \beta_i = -q^2 \mu_i \nu_i, \quad \sigma_i = \pm 1 \quad (i = 1, 2). \)  

\( \) 

**Proposition 3.1.** The maps \( \pi_i (i = 1, 2, 3) \) are naturally extended to the algebra homomorphisms. The resulting representations of \( A_q(B_3) \) on \( F_q \) are irreducible.

We call \( \pi_i (i = 1, 2, 3) \) the fundamental representations of \( A_q(B_3) \). It is easy to infer the fundamental representation \( \pi_i \) of \( A_q(B_n) \) for general \( n \).

**Remark 3.2.** The symbols \((t_{ij})_{2 \leq i, j \leq 6}\) generate \( A_q(B_2) \). The 5x5 submatrices of \([4] \) without the first and the last rows and columns provide the two fundamental representations of \( A_q(B_2) \). Denote them by \( \pi_i^{B_2} \) and \( \pi_2^{B_2} \), respectively. Similarly one can extract the fundamental representations of \( A_q(C_2) \) from the 4x4 submatrices of \([6] \) eq.\((3.4), (3.5)\). Denote them by \( \pi_i^{C_2} \) and \( \pi_2^{C_2} \), respectively. Then \( \pi_2^{B_2} \) coincides with the representation induced from \( \pi_3^{C_2} \) via Theorem 2.1 with a suitable adjustment of the parameters, i.e. \( \pi_i^{B_2} (f) = \pi_3^{C_2} (\kappa (f)) \) for any \( f \in A_q(B_2) \).

4. **INTERTWINERS**

The Weyl group \( W(B_3) = \langle s_1, s_2, s_3 \rangle \) is generated by the simple reflections with the relations \( s_1^2 = s_2^2 = s_3^2 = 1, s_1 s_3 = s_3 s_1, s_1 s_2 s_1 = s_2 s_1 s_2 \) and \( s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2 \).

According to \([7] \) one should have the equivalence

\[ \pi_{13} \simeq \pi_{31}, \quad \pi_{121} \simeq \pi_{212}, \quad \pi_{2321} \simeq \pi_{3232} \]  

\( \)
under an appropriate tuning of the parameters. Here and in what follows we use the shorthand \( \pi_{i_1, \ldots, i_r} = \pi_{i_1} \otimes \cdots \otimes \pi_{i_r} \). It is easy to see that \( \pi_{13} \simeq \pi_{31} \) holds if and only if
\[
\kappa_1 = \sigma_1, \quad \kappa_{31} = \kappa_{32}
\]
and the intertwiner is just the transposition \( P(x \otimes y) = y \otimes x \). The equivalence \( \pi_{121} \simeq \pi_{212} \) holds if and only if
\[
\kappa_1 = \kappa_2 = \sigma_2, \quad \alpha_1 \beta_1 = \alpha_2 \beta_2
\]
are further satisfied. Assuming them we introduce the intertwiner \( \Phi \in \text{End}(F_{q^2} \otimes F_{q^2} \otimes F_{q^2}) \) characterized by
\[
\pi_{212}(\Delta(f)) \circ \Phi = \Phi \circ \pi_{121}(\Delta(f)) \quad (\forall f \in A_q(B_3))
\]
and the normalization \( \Phi([0] \otimes [0] \otimes [0]) = [0] \otimes [0] \otimes [0] \). Set \( S = \Phi P_{13} \), where \( P_{13}(x \otimes y \otimes z) = z \otimes y \otimes x \). The \( S \) is regarded as a matrix \( S = (S_{ijk}^{abc}) \) whose elements are specified by \( S([i] \otimes [j] \otimes [k]) = \sum_{a,b,c \in \mathbb{Z}_{\geq 0}} S_{ijk}^{abc} [a] \otimes [b] \otimes [c] \). We use the notation \( \delta_i^0 = 1(a = i) \), \( \delta_i^0 = 0(a \neq i) \) and
\[
(q)_i = \prod_{j=1}^i (1 - q^j), \quad \left[ i_1, \ldots, i_r \right] = \left( \prod_{i=1}^r (q)_{i_k} \prod_{k=1}^r (q)_{j_k} \right) = 0 \quad (\forall i_k, j_k \in \mathbb{Z}_{\geq 0}, \text{otherwise}).
\]

The following result confirms the conjecture stated in [6, eq.(4.2)].

**Theorem 4.1.** Under [5], [7] and [8], the following formulas are valid:
\[
S_{ijk}^{abc} = (-\alpha_1 \beta_1 q^{-2})_j \mu_1^{a+j} \mu_2^{b-a-k} \delta_1^{a+b+c} S_{ijk}^{abc},
\]
\[
S_{ijk}^{abc} = \delta_{i+j}^0 \delta_{j+k}^0 \sum_{\lambda + \mu = b} (-1)^{\lambda} q^{2i(\lambda+j+2(k+1)+2(\mu-k))} \left[ i, j, c + \mu, \lambda, i - \mu, j - \lambda, \mu \right]_{q^a}.
\]

This \( S_{ijk}^{abc} \) is equal to \( R_{ijk}^{abc} \) in [6, eq.(2.20)] with \( q \) replaced by \( q^2 \). Our notation here is taken consistently with [6, eq.(4.2)]. This \( S \) coincides with the 3D R matrix [11] which follows as the intertwiner of \( A_{q^2}(SL_3) \)-modules [11]. See [6, sec.2] for an exposition. It satisfies the tetrahedron equation
\[
S_{156} S_{246} S_{145} S_{123} = S_{123} S_{145} S_{246} S_{356}
\]
and \( S = S^{-1} \). Let us turn to \( \pi_{2323} \simeq \pi_{3232} \) in [6]. It holds if and only if
\[
\kappa_{31} = \kappa_{32} = \pm 1, \quad \alpha_1 \beta_2 = \pm q^2.
\]

The constraints [6], [7], [8] and [10] are summarized as
\[
\sigma = \kappa_i = \sigma_i, \quad \rho = \kappa_{3i}, \quad \alpha_i \beta_i = -\varepsilon q^2, \quad \mu_i \nu_i = \varepsilon \quad (i = 1, 2)
\]
in terms of the three independent sign factors \( \sigma, \rho, \varepsilon \in \{1, -1\} \).

Let \( \Xi : F_{q^2} \otimes F_{q^2} \otimes F_{q^2} \to F_{q^2} \otimes F_{q^2} \otimes F_{q^2} \) be the intertwiner characterized by
\[
\pi_{2323}(\Delta(f)) \circ \Xi = \Xi \circ \pi_{2323}(\Delta(f)) \quad (\forall f \in A_q(B_3))
\]
and the normalization \( \Xi([0] \otimes [0] \otimes [0] \otimes [0]) = [0] \otimes [0] \otimes [0] \otimes [0] \). Introduce further \( J = \Xi P_{14} P_{23} \in \text{End}(F_{q^2} \otimes F_{q^2} \otimes F_{q^2} \otimes F_{q^2}) \), where \( P_{14} P_{23}(x \otimes y \otimes z \otimes w) = w \otimes z \otimes y \otimes x \). The \( J \) is regarded as a matrix \( J = (J_{ijkl}^{abcd}) \) whose elements are specified by
\[
J([i] \otimes [j] \otimes [k] \otimes [l]) = \sum_{a,b,c,d \in \mathbb{Z}_{\geq 0}} J_{ijkl}^{abcd} [a] \otimes [b] \otimes [c] \otimes [d].
\]
Setting $J_{abcd} = \varepsilon^{b+c+1} \varepsilon^{a+c} (\sigma \mu \nu \rho \tau \lambda)_{ijkl}$, it can easily be checked that $J_{ijkl}$ depends only on $q$. In this sense we call $J = (J_{ijkl})$ the parameter-free part of the intertwiner for $A_q(B_3)$.

Let $\mathcal{X} = (\mathcal{X}_{ijkl})$ be the parameter-free part in the similar sense of the intertwiner for the quantized function algebra $A_q(C_3)$. The elements $\mathcal{X}_{ijkl}$ are polynomials in $q$ and given explicitly in [6, Th.3.4]. The $J$ and $\mathcal{X}$ will simply be referred to as type $B$ and type $C$, respectively. Now we present the main result of the paper, which confirms the conjecture in [6, eq.(4.1)].

**Theorem 4.2.** The type $B$ and $C$ intertwiners are related by the transposition of the components as $J = P_{14}P_{23}K_{14}P_{23}$, i.e. $J_{ijkl} = \mathcal{X}_{kjil}$ holds.

**Proof.** The $J$ and $\mathcal{X}$ are characterized as the intertwiners of the representations of the subalgebras $A_q(B_2)$ and $A_q(C_2)$, respectively. Thus the assertion follows from Theorem 3.1 and Remark 3.2. □

As a corollary of [6, Th.3.5], $J_{ijkl}$ is a polynomial in $q^2$ vanishing unless $(a + 2b + c + d + e + f = (i + j + k + l)$. Moreover $J$ has the birational and combinatorial counterparts as shown in [6, Table 1].

**Example 4.3.** The following is the list of all the nonzero $J_{ijkl}$:

\[
\begin{align*}
J_{0003}^{1102} &= q^8(1 - q^6)/(1 - q^{12}), & J_{0201}^{1102} &= q^4(1 - q^6)/(1 - q^4 + q^6 - q^8 - q^{10}), \\
J_{1021}^{1102} &= -q^6(1 - q^6), & J_{0003}^{1012} &= -q^2(1 - q^6), \\
J_{1102}^{1102} &= q^4(1 - q^8 + q^{14}), & J_{0111}^{1012} &= 1 - q^4 + q^{10}.
\end{align*}
\]

5. 3D REFLECTION EQUATION

The intertwiners $S$ and $J$ yield a solution of the 3D reflection equation proposed by Isaev and Kulish [4]. Since its derivation is the same as the type $C$ case in [6, sec.3.6], we shall only describe the outline.

Let $w_0$ be the longest element of the Weyl group $W(B_3)$ and consider the two reduced expressions $w_0 = s_1s_2s_3s_2s_1s_2s_3s_2s_3s_2s_1 = s_3s_2s_3s_2s_1s_2s_3s_2s_1$. According to [7, Th. 5.7, Cor. 1], we have the equivalence $\pi_{12321323} \cong \pi_{32321323}$ of the representations of $A_q(B_3)$. The intertwiner for this can be constructed by composing the ‘basic ones’ $P, \Phi$ and $\Xi$ for $\mathfrak{g}$. Moreover there are two ways to do so, the results of which must coincide since $\pi_{12321323} \cong \pi_{32321323}$ is irreducible. This postulate leads to the equality

$$S_{456}S_{489}S_{3579}S_{269}S_{258}S_{1678}S_{1234} = J_{1234}J_{1678}J_{258}S_{269}J_{3579}S_{456}S_{489}$$

in $\text{End}(F_3 \otimes F_2 \otimes F_1 \otimes F_0)$, where the indices signify the positions of the tensor components on which the intertwiners act non trivially. The equation (12) is the 3D reflection (or “tetrahedron reflection”) equation introduced in [11], which is not a spectral parameter. In view of Theorem 1.2, one may substitute $S_{ijkl} = \mathcal{X}_{ijkl}$. The resulting relation confirms the conjecture stated in [6, eq.(4.3)].

A pair $(S, J)$ is called a solution to the 3D reflection equation if it satisfies (12) and $S$ satisfies the tetrahedron equation (10) by itself. Thus we have established a new solution of the 3D reflection equation distinct from the previous one $(R, X)$ in [6]. The two solutions $(S, J)$ and $(R, X)$ are associated with $A_q(B_3)$ and $A_q(C_3)$, respectively. Although they are simply related as $(S, J) = (R_{q \to q^2}, P_{14}P_{23}K_{P_{14}P_{23}})$, the fact that these replacements do keep (12) is highly nontrivial. The main achievement of this paper is to have established it by exploiting the embedding in Theorem 2.1.
Let us finish by giving a diagram for the 3D reflection equation

\[
S_{489}J_{3579}S_{269}S_{258}J_{1678}J_{1234}S_{654} = S_{654}J_{1234}J_{1678}S_{258}S_{269}J_{3579}S_{489}.
\] (13)

This is $S^{-1}_{456} \times \text{eq.}(12) \times S_{456}$ with the properties mentioned after (9) applied. It is depicted in Fig.1.

**Figure 1.** The 2D projection diagram of the 3D reflection equation (13).

To draw the LHS, first put three lines 8, 5 and 2 that intersect at one point, the center. Take generic points A, B and C on every second half-infinite line emanating from the center along 8, 5 and 2. The points P, Q and R are the crossings of the triangle ABC with the lines 8, 5 and 2. The broken lines are formed by connecting P, Q and R. The RHS is obtained by changing A, B and C in such a way that the lines 6, 9 and 4 are shifted and the triangle ABC is reversed.

There is a natural geometric interpretation of Fig.1. Let us call the plane containing all the lines the *boundary plane*. Fig. 1 shows the projection of the three planes reflected by the boundary plane. A physical interpretation is the world sheets of three straight strings in 3D exhibiting boundary reflections.

The intersections of the three and four arrows correspond to $S$ and $J$ in (13), respectively. The indices 1, 3 and 7 are attached only to $J$, hence represent the boundary degrees of freedom. In Fig.1 they are depicted by the broken lines. The other solid lines are the projection of the configurations (or events) in 3D explained in what follows onto the boundary plane.

To each broken line, associate a pair of half-planes in 3D sharing the broken line as the common boundary. They represent the world sheet of a straight string moving in 3D and reflected by the boundary plane exactly at the broken line. The six in total half-planes should be located on the same side of the boundary plane. A broken line is like a spine of an open (by a generic angle) book and the half-planes are like the front and back of the book. The books should be “upright” on the boundary plane since the incident and reflection angles coincide. There are three such books with spines 1, 3 and 7. At the intersection R of the spines 1 and 3, the fronts and backs of the two books generate the four intersecting half-lines on them. Their projections are the solid lines 2 and 4. The crossing of these four lines 1, 2, 3, 4 yields $J_{1234}$. The other $J$ can be understood similarly. The intersections of three planes without a spine take place off the boundary plane and correspond to three string scatterings encoded by $S$. By an elementary geometry one can prove that the 2D projection of the 3D configuration explained so far exactly forms the pattern depicted in Fig.1. Finally the orientations of the lines indicated by the arrows signify the time ordering of the
scattering and reflection events. They specify the order of the products of various $S$ and $J$ in \[13\].

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