 Effective Generation of Subjectively Random Binary Sequences

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Abstract

We present an algorithm for effectively generating binary sequences which would be rated by people as highly likely to have been generated by a random process, such as flipping a fair coin.

1 Introduction

This paper is a first step in modelling mathematical objects showing “subjective randomness”, or what people believe to be random. Although there is no rigorous characterization of what subjective randomness might be, it has become clear through experimentation that is quite different from stochastic randomness. A classic example which illustrates this difference is the following: when asked which of the following sequences is most likely to have been produced by flipping a fair coin 20 times,

\[
\text{OOOOOOXXXXOOOXXOOOOO} \\
\text{OOXOXOOXXOOXOXXXOOXO}
\]

most people will answer “the second sequence” even though each sequence has been produced by a random generator.

Until now, subjective randomness has mainly been the study of psychologists, cognitive scientists and artists\(^ 1\). However, in today’s age, where computer software is an integral part of everyday life, it is a natural problem

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\(^1\)see Twyla Tharp’s “Medley” or any of Jackson Pollock’s drip paintings
to ask how one can present what people would accept as “random”. Examples of this interest is the popularity of the various “random” playlist shuffling software on the market and the appearance of “randomness” in design (screensavers, tiling, etc.). In fact, this paper grew from an attempt to create exercise drill software for students. The goal was to generate images which would be simultaneously unpredictable and yet, in some way, balanced. We hope to be able to describe a good model for subjectively random two-dimensional objects sometime in the future. For now we present one for which there already exists a substantial body of research: subjectively random binary sequences.

Most of the research on subjective randomness seeks to understand what exactly are the differences between subjective randomness and stochastic randomness and to understand why this is so. Experiments usually are of two types: “production” where subjects are asked to produce examples of what they consider to be random and “judgment” where subjects are asked to identify or rate objects based on how likely they would have been produced by some random process. From these experiments, a few traits of subjective randomness have been pinpointed.

One is local representativeness the fact that people feel that a small sample should reflect properties of the population as a whole [KT], [R1], [TK]. In terms of binary sequences, this translates to concrete restrictions on the probability of alternation and the relative number of X’s and O’s in any given subsequence.

The second trait is high subjective complexity or effort needed to encode the data [C], [F2], [L3]. In the same way that algorithmic (Kolmogorov) complexity is measured as the inability of a computer to encode the sequence in less bits than the length of the sequence [LV], so can perceived randomness be measured as the inability of the human mind to memorize the sequence in significantly less steps than there are sequence elements.

The third trait concerns symmetry recognition. People are less likely to rate a sequence as random if they recognize certain symmetries in them [F3], [LO], [S], [R3], [T]. Which symmetries are recognized depends on whether the sequence elements are presented simultaneously or one bit at a time.

In studying those sequences which are subject to the above constraints, it becomes apparent that subjectively random binary sequences are anything but random in the sense that they have properties which fall far from the norm for Bernoulli trials [LV]. This is also the reason that they could not be the output of any pseudo-random generator [K], [L1], [L2]; for one, these
sequences only have very short runs of X’s and O’s. Although the locally even distribution of X’s and O’s in subjectively random sequences resembles that of low discrepancy, or quasi-random, sequences, they are not quasi-random because the alternation rate of X’s and O’s is too high from normal [CG]. The structure of subjectively random sequences is quite rigid and these sequences constitute a very small subset of all binary sequences. In fact, this set is so small (less than 10% of all sequences of length 20 or more bits) that it makes more sense to use a creation method than some brute force method using a random generator. We use all of the above conditions in creating a simple algorithm which generates (arbitrarily long) subjectively random binary sequences. We concentrate on those sequences which would be presented one bit at a time, such as in a ticker-tape or in the case when someone is actively flipping a coin.

Our algorithm depends on a function which measures subjective randomness. This function is itself a variation on two ways of measuring subjective randomness, one developed by Falk and Konold [FK], the other developed by Griffiths and Tenenbaum [GT2] but is more efficient in measuring subjective randomness on 8-bit sequences. The sequence is created one bit at a time; at each step one chooses the element which will produce the desired randomness rating on the last 8 bits of the sequence. Producing this sequence is equivalent to moving along paths in a certain associated digraph. By traveling within a strongly connected digraph associated to sufficiently random 8-bit sequences, one can produce arbitrarily long binary sequences that consistently look “random enough”.

In the first few sections of the paper we give brief accounts of the research results which will be considered in creating our subjectively random sequences. These sections restrict themselves to describing specific results and are not surveys of the substantial body of research in subjective randomness done by psychologists and cognitive scientists. For that, we recommend articles by Bar-Hillel and Wagenaar [BW], [W], Falk and Konold [FK] and by Nickerson [N].

We thank T. Griffiths for sharing data and answering questions and the referee for many useful comments and suggestions.
2 Local Representativeness

In their studies of how people perceived randomness, Kahneman and Tversky argued that, generally, people believe that small samples should reflect the properties of the population as a whole. In probability theory, the Law of Large Numbers states that large samples tend to have the same properties as the population as a whole. Kahneman and Tversky coined the term “The Law of Small Numbers” to describe people’s belief that this holds even for small samples [KT]. When small samples don’t behave expectedly, people believe that there must be another reason behind the data. For example, gamblers attributed “luck” (good or bad) for streaks in outcomes [WK].

In the case of binary sequences the Law of Small Numbers presents itself in the following way: people believe that in even short random sequences there must be roughly the same number of X’s as there are O’s with some irregularity in the order of their appearance. In other words, neither pure runs, such as XXXXXX or OOOOOO, nor alternating runs, such as XOXOXOXO or OXOXOXO, should be too long. When considering long binary sequences, these same features should be present in short subsequences, that is, in people’s windows of observation. These windows are believed to be of variable length of approximately 7 bits [KG].

These conditions account for a high alternation rate in subjectively random binary sequences. Truly random sequences tend to have an alternation rate of 0.5, but studies consistently show that subjectively random sequences have a probability of alternation of 0.6 - 0.7 [BW], [F1], [FK]. One way of understanding why this could be so is that, since there should consistently be about as many X’s and there are O’s, if there is a run of X’s, then the run shouldn’t be too long. This forces the probability of alternation to be higher than average. On the flip side, the restriction on the length of alternating runs forces the probability of alternation to not become too high.

3 Subjective Complexity

In [FK], Falk and Konold find that people tend to view a sequence as being more random if it was harder to mentally encode it. For example, the sequence XOXOXOXOXO could be described as “five times XO”. However, the sequence XOOXOXOXXXO cannot be described so concisely: “first an X, then two O’s, then two X’s, an O, then two X’s, then two O’s”. In order to ob-
jectively measure this effort, Falk and Konold define a “difficulty predictor” dp. Let $x$ be a sequence. Then

$$dp(x) = \text{the number of pure runs} + 2 \times \text{the number of alternating runs in } x$$

The idea behind this measure is that one uses shortcuts such as “four X’s” or “XO five times” to describe the sequences. The difficulty predictor measures how many different commands one would have to give, taking into account the greater effort one needs to check the length of alternating runs over pure runs (hence the factor 2). If there are multiple ways of describing the sequence $x$ as concatenation of runs, then $dp(x)$ is defined to be the minimum of the resulting numbers.

**Example:** Consider $x = \text{XXXOXOXO}$. Then we can view $x$ as

$$\text{XXX O X O X OOO} \text{ or } \text{XXX OXOX OOO} \text{ or } \text{XXOXOXO OO}$$

The first way rates $x$’s difficulty as 6, the second two rate it as 4. So $dp(x) = 4$.

In trials where subjects were asked to memorize and copy various sequences of length 21, Falk and Konold found that their difficulty predictor was a good correlator for the randomness rating that was given by the subject and concluded that it was a good measure of the sequence’s perceived randomness [FK].

Consistent with previous studies, Falk and Konold also found that those sequences rated as most random tended to have a higher than average probability of alternation. It should be noted that this is, mathematically, not a coincidence. Requiring a maximal value for $dp$ is a stronger condition than simply requiring that the probability of alternation be 0.6-0.7 and that the imbalance (defined as $|\text{sequence length}/2 - \#X’s|$) be minimal.

In a series of papers [GT1], [GT2], [GT3], Griffiths and Tenenbaum extend the results of Falk and Konold with their own measure of subjective complexity. They propose that, when presented with a binary sequence $x$, people assess the probability that $x$ is being produced by a random process as opposed to being produced by some other (regular) process. Their measure for subjective complexity is then $P(x|\text{regular})$, (where $x$ is considered more random, the smaller $P(x|\text{regular})$ is).

In experiments where they ask subjects to order 8-bit sequences with respect to how random they believe they are, Griffiths and Tenenbaum first
show that a good model for calculating $P(x|\text{regular})$ is a finite state automaton in the form of a certain hidden Markov model (HMM) [GT2]. Conceptually, the model works as follows: As someone reads a sequence $x$, they will consider each element, $O$ or $X$, as possibly being part of one of several motifs. These motifs are of varying length and the probability that $O$ or $X$ will be considered as being part of a specific motif will be a function of the motif length and whether one is changing from one motif to another.

**Example:** In the smallest HMM model they consider [GT2], the 6 state model, there are the following states which produce an $O$ as observed occurrence: $O$ (coming from the motif $O$), $O$ (coming from the motif $OX$) and $O$ (coming from the motif $XO$). Similarly, there are 3 states which produce $X$.

The probability of remaining in a motif is denoted by $\delta$, the probability of changing to (or starting at) a motif of length $k$ is equal to $C \cdot \alpha^k$ (for some $\alpha$) where $C := (1 - \delta)/(2\alpha + 2\alpha^2)$ is a normalization constant.

Let $1$ denote the state which produces $X$ (from the length 1 motif $X$), $2$ denote the state which produces $O$ (from the motif $O$), $3$, resp. $4$, denote the state which produces $X$, resp. $O$, from the motif $XO$, and $5$, resp. $6$, denote the state which produces $X$, resp. $O$, from the motif $OX$. The transition matrix giving $P(i|j)$ of going from state $i$ to state $j$ is:

$$P(i|j) = \begin{pmatrix}
\delta & C\alpha & C\alpha^2 & 0 & 0 & C\alpha^2 \\
C\alpha & \delta & C\alpha^2 & 0 & 0 & C\alpha^2 \\
C\alpha & C\alpha & 0 & \delta & 0 & C\alpha^2 \\
C\alpha & C\alpha & \delta & 0 & 0 & C\alpha^2 \\
C\alpha & C\alpha & C\alpha^2 & 0 & 0 & \delta \\
C\alpha & C\alpha & C\alpha^2 & 0 & \delta & 0
\end{pmatrix}$$

In the same spirit as for the difficulty predictor, there are many different sequences of states which can produce the same observed sequence. For example, the sequence $x = \text{XOXX}$ could be produced by any of many state sequences: the first $X$ could come from state $1$ or $3$, the $O$ could come from states $2$, $6$ or from state $4$ if the preceding $X$ was from state $3$, the second $X$ could come from state $1$ or $3$ or from state $5$ if the preceding $O$ was from state $6$, and the third $X$ could come from state $1$ or $3$ (if the previous $X$ was not also from state $3$). For each state sequence $z$, the probability $P(x,z)$ that the observed sequence $x$ was produced from $z$ will be a function of $\delta$ and $\alpha$. For example $P(\text{XOXX}, 1211) = (C\alpha)^4 = C^4\alpha^4$ and $P(\text{XOXX}, 3431) = C\alpha^2\delta^2C\alpha = C^2\alpha^3\delta^2$. The probability $P(x|\text{regular})$ is de-
fined as the max, \( P(x, z) \) over all possible state sequences which produce \( x \).

The 22 state model is the natural extension of the 6 state model to include up to length 4 motifs which are not duplications of smaller length motifs (such as XOXO or XXXX).

Both the 6 state and the 22 state models have the advantage over the difficulty predictor of being a function of the length of a sequence. For certain values of \( \delta \) and \( \alpha \), Griffiths and Tenenbaum show that there is an equivalence of the 6 state model and Falk and Konold’s difficulty predictor. In addition, they find values of \( \delta \) and \( \alpha \) such that both HMMs modelled the trials’ subjective randomness results better than the difficulty predictor [GT2]. As with the difficulty predictor, being rated maximally random (by the 22 state model) is a stronger condition than requiring that the sequence has probability of alternation 0.6-0.7 and that the imbalance between O’s and X’s is low.

4 Symmetry recognition

Griffiths and Tenenbaum’s experiments also test the role of symmetry recognition in subjective randomness and consider the possibility that subjects recognize four types of symmetry: mirror symmetry, where the second half is produced by reflection of the first half (ex. XXOOOXX), complement symmetry, where the second half is produced by reflection of the first half and exchanging X and O (ex. XOOXOXO), and duplication, where the sequence is produced by repeating the first half once (ex. XXOXXXOX). Their “context-sensitive” model for \( P(x|\text{regular}) \) considers that sequences be generated by any of four methods. The first method is repetition, where sequences are produced by the HMM. The next three methods are the symmetry methods listed above, where the first half of the sequence is produced by the HMM and the second half by the symmetry rule. Then, \( P(x|\text{regular}) \) will depend on the models \( M \) listed above:

\[
P(x|\text{regular}) = \max_{z,M} P(x, z|M)P(M)
\]

where \( P(x, z|M) \) is obtained as above and \( P(M) \) is to be determined by analysis of hard data.
In [GT3], they conclude that when all elements of a sequence are presented simultaneously, their context-sensitive model models subjective randomness best with the following parameters: \( \delta = 0.66, \alpha = 0.12, P(\text{repetition}) = 0.748, P(\text{mirror symmetry}) = 0.208, P(\text{complement symmetry}) = 0.0005 \) and \( P(\text{duplication}) = 0.039 \). However, when elements of a sequence are presented sequentially, people do not recognize mirror or complement symmetry, so the best model has the following parameters \( \delta = 0.70, \alpha = 0.11, P(\text{repetition}) = 0.962, \) and \( P(\text{duplication}) = 0.038 \).

### 5 Measuring subjective randomness effectively

Griffiths and Tenenbaum find values for \( \alpha \) and \( \delta \) in order to fit their model to the linear ordering on the sequences which is obtained by the experimental data. However, for our problem we do not need the full structure of their model; since we are interested in the sequences rated “more random”, we only need the linear ordering of the sequences produced by this model. An analysis of the HMM model shows that we obtain the same ordering as given by the 22-motif HMM by doing the following. We consider \( \alpha \) and \( \delta \) as formal parameters satisfying \( 0 < \alpha < \delta^4 < 1 \). The normalizer \( C \) can be set to \( C = 1 \). This abstract version of the HMM will assign to each sequence an expression of the form \( \alpha^a \delta^b \). Let \( x \) and \( y \) be two sequences such that \( P(x|\text{regular}) = \alpha^{i_1} \delta^{j_1} \) and \( P(y|\text{regular}) = \alpha^{i_2} \delta^{j_2} \) (where the probability \( P(\cdot|\text{regular}) \) is determined by the HMM). Then, \( x \) is subjectively more random than \( y \) if \( i_2 - i_1 \geq 0 \) and \( j_1 - j_2 \leq 4(i_2 - i_1) \).

With regard to the repetition and duplication models, we need to set \( \alpha \delta^4 < P(\text{duplication})/P(\text{repetition}) < \alpha \delta^3 \). To obtain the same ordering as in [GT3], it suffices to set \( P(\text{repetition}) = 1 \) and \( P(\text{duplication}) = \alpha \delta^{3.5} \). (In our next paper, we will consider the simultaneous case which includes all four models).

Table 1 gives the partial ordering of all 128 sequences starting with \( O \) from most to least subjectively random. The first column gives an example sequence in full and the second column gives the list of equally rated sequences in the form of a base 10 number. The third column gives the “finite state” \( P(x|\text{regular}) \) as determined by the 22 motif HMM. The fourth column

\(^2\text{This does not give a partial ordering on all monomials in } \alpha \text{ and } \delta \text{ but simply on those produced by this model.}\)
gives the "context-sensitive" $P(x|\text{regular})$ which also takes into consideration duplication symmetry.

This abstract version of the 22-motif HMM allows us to define a more efficient variant $dp_+$, which will give the same result as the 22-motif HMM: if the HMM assigns a sequence $x$ the value $\alpha^i\delta^j$ then this predictor will assign it the value $[i, j]$, where $i$ represents the sum of the lengths of the contributing motifs and the $j$ represents the length of the sequence minus the number of contributing motifs. However, it will "read" the sequence in the same spirit as Falk and Konold's difficulty predictor. In addition to length 1 and 2 motifs, $dp_+$ considers length 3 and 4 motifs and records over how many bits the motif appears. For example, viewing the sequence $\text{XOOXOOXX}$ as two instances of $\text{XOO}$ and two instances of $\text{X}$ gives a value of $[3 + 1, 8 - 2] = [4, 6]$. Here, we allow incomplete repeats to be included into the run. For example, the sequence $\text{XOXOXOX}$ would be given a value of $[2, 6]$ because it is a run of 3 1/2 instances of the length 2 motif $\text{XO}$. The rating given to a sequence $x$ would be the smallest $[i, j]$ obtained by considering $x$ as various concatenations of motifs. The part of $dp_+$ consisting of the initial conditions and Step 1 is equivalent to the abstract 6 state HMM (with conditions $C = 1$ and $0 < a < \delta^2 < 1$). The reason that $dp_+$ is more efficient (for this problem) than the HMM is that it identifies those subsequences (and motifs) which will contribute to the value of $dp_+$. Hence, over 8-bit sequences, it is recursive with small depth. For any object $c$ described using bracket notation $[\cdots]$, we will write $c[i]$ to denote its $i$th component or entry. The algorithm for $dp_+$ is as follows:

**Algorithm to compute $dp_+$:**

**INPUT:** $x = x_1x_2\cdots x_n$.

**# Special cases:**

IF $x$ is empty THEN OUTPUT $[0, 0]$
ELSE IF $x$ is a pure run THEN OUTPUT $[1, n - 1]$
ELSE IF $x$ is an alternating run and $n \geq 3$ THEN OUTPUT $[2, n-1]$

**# consider only length 1 and 2 motifs**
ELSE SET $j := 1$

# find the largest $j \geq 1$ such that $x_1\cdots x_j$ consists of $j$ repeats of $x_1$.

WHILE $j \leq n$ AND $x_j = x_1$ DO
INC Increment $j$ BY 1
END DO
IF \( j > 1 \) THEN
\[
s_1 := x_1 x_2 \ldots x_{j-1} \quad \text{and} \quad s_2 := x_j \ldots x_n.
\]
ELSE SET \( k := 1 \)
# find the largest \( k \geq 1 \) such that \( x_1 x_2 \ldots x_{2k} \) is \( k \) repeats of \( x_1 x_2 \).
WHILE \( k \leq n \) AND \( x_{2k-1} = x_1 \) AND \( x_{2k} = x_2 \) DO
INCREMENT \( k \) BY 1
END DO
IF \( k = 1 \) THEN
\[
s_1 := x_1 \quad \text{and} \quad s_2 := x_2 \ldots x_n.
\]
ELSE \( s_1 := x_1 x_2 \ldots x_{2k} \) and \( s_2 := x_{2k+1} \ldots x_n. \)
END IF
END IF
END IF

SET \( dp^+(x) = dp_+(s_1) + dp_+(s_2). \)

# consider length 3 motifs
SET Dset = \{dp^+(x)\} \# collects possibly smaller difficulty values
IF \( dp^+(x)[1] \geq 4 \) THEN
FOR \( i \) FROM 1 TO \( n-5 \) DO
# determine the largest \( j \geq i + 4 \) such that \( x_i \ldots x_j \) is a repeat of \( x_i x_{i+1} x_{i+2} \).
SET \( j := i + 3 \)
WHILE \( j \leq n \) AND \( x_j = x_{i+(j-i) \mod 3} \) DO
INCREMENT \( j \) BY 1
END DO
IF \( j - i \geq 4 \) THEN
LET Dset := Dset UNION \{dp_+(x_1 \ldots x_{i-1}) + [3, j - i - 2] + dp_+(x_{j+1} \ldots x_n)\}
END IF
END LOOP
END IF

SET \( dp^\prime\prime(x) = \min_{v \in Dset} v \)

# consider length 4 motifs
SET Dset = \{dp^\prime\prime(x)\} \# collects possibly smaller difficulty values
IF \( dp^\prime\prime(x)[1] \geq 5 \) THEN
FOR \( i \) FROM 1 TO \( n-6 \) DO
# determine the largest \( j \geq i + 5 \) such that \( x_i \ldots x_j \) is a

repeat of \( x_i x_{i+1} x_{i+2} x_{i+3} \).

SET \( j := i + 4 \)

WHILE \( j \leq n \) AND \( x_j = x_i + ((j-i) \mod 4) \) DO

INCREMENT \( j \) BY 1

END DO

IF \( j - i \geq 7 \) THEN

LET \( Dset := Dset \ \text{UNION} \ \{ dp_+ (x_1 \cdot \cdot \cdot x_{i-1}) + [4, j-i-2] + dp_+ (x_{j+1} \cdot \cdot \cdot x_n) \} \)

END IF

END LOOP

OUTPUT: \( \min_{v \in Dset} v \)

END IF

Using MAPLE, we found that, \( dp_+ \) took significantly less time and memory than the (polynomial time) Viterbi algorithm (see [R2] for a good tutorial on the Viterbi algorithm) with the 22 motif HMM. On average, the Viterbi algorithm takes 47 times the time and 400 times the space as \( dp_+ \) to calculate the difficulty of an 8-bit sequence. We also note that this algorithm duplicates the results of the 22 motif model only for sequences of 8 bits or less. It becomes less accurate as the length of the sequence increases.

Taking into account the possibility of duplication \((x = x_1 x_2 \cdot \cdot \cdot x_{n/2} x_1 x_2 \cdot \cdot \cdot x_{n/2})\), we obtain the subjective randomness rating \( sr(x) \) for any 8-bit sequence \( x \):

\[
sr(x) = \begin{cases} 
\min(dp_+(x), dp_+(x_1 x_2 \cdot \cdot \cdot x_{n/2}) + [1, 3.5]) & \text{if } x \text{ is a perfect duplication,} \\
 dp_+(x) & \text{otherwise}
\end{cases}
\]

This gives the same partial ordering on 8-bit sequences as the "context-sensitive" model described in Section 3.

6 Sequence Creation

We can now produce arbitrarily long subjectively random binary sequences. In this paper, we concentrate on those sequences which would be presented sequentially (one bit at a time) instead of simultaneously. According to [GT3] (see above), in addition to the rating given to the sequence by the HMM, the only symmetry that needs to be considered then is duplication.
The idea behind producing this sequence is the obvious one: pick the next element in a sequence (O or X) to be the one which gives a sufficiently high rating of subjective randomness on the last 8 bits of the resulting sequence. The rationale behind this is that since people use small (6-8 bits) windows of observation when analyzing a sequence, it suffices that people feel at any one time that their window looks “random” enough.

Ideally, one would like to create sequences which would be of maximal sr over every small subsequence. However, such sequences simply do not exist. Even restricting oneself to sequences for which every 8-bit subsequence $x$ is in the top 20% ($sr(x)[1] = 5$) becomes too deterministic; there are only 2 such (arbitrarily long) sequences and each has a period of 6 bits.

For any 8-bit sequence $x = x_1 \cdots x_8$, let $xO := x_2 \cdots x_8 O$ and $xX := x_2 \cdots x_8 X$. (If $x$ is the last 8 bits of the sequence that we are creating, then $xO$ or $xX$ will be the last 8 bits of the sequence to which we’ve added one more element). To any subset $S$ of 8-bit sequences, we can associate a directed graph, or digraph, $G := G(S)$ in the following way: the vertices of $G$ are indexed by the elements of $S$. The directed edges are defined as follows: $x \rightarrow y$ if $y = xO \in S$ or $y = xX \in S$.

Recall that a digraph $G$ is connected, resp. strongly connected, if, for every two vertices $x \neq y$ in $G$, there is a nondirected, resp. directed, path from $x$ to $y$. A subgraph $C$ of $G$ is a connected, resp. strongly connected, component if it is maximal for this property. A component is trivial if it consists of a single vertex. For our problem, we are particularly interested when $G$ is strongly connected. In this case, we can form arbitrarily long sequences $x$ by choosing an equally long directed path in $G$. The strong connectivity implies that there are no sinks and thus guarantees the existence of such a path.

Set $S([m, n]) := \{x|x \text{ 8-bit }, sr(x) \geq [m, n]\}$ for $[m, n]$ any of the possible sr values $[1, 7], [2, 6], \ldots, [5, 6]$. The following theorem is proved by brute force (Maple).

**Theorem 1** 1) For $[m, n] < [5, 3]$, $G(S([m, n]))$ has exactly one non-trivial strongly connected component $C([m, n])$ and $C([5, 3])$ has exactly two non-trivial strongly connected components $C_1([5, 3]), C_2([5, 3])$. These components form a sequence of nested digraphs:

$$C([1, 7]) \supset C([2, 6]) \supset \cdots \supset C([4, 6]) \supset C_i([5, 3]), \quad i = 1, 2.$$

$G(S([m, n]))$ has no non-trivial strongly connected components for $[m, n] > [5, 3]$. 12
2) For \([m, n] \leq [4, 4.5]\), the \(C([m, n])\) are also connected components of \(G(S([m, n]))\).

Table 2 gives detailed information about the \(C([m, n])\). A highly efficient way to produce sequences would be to use an incidence matrix (or table) \(M\) for the graph \(C([m, n])\), indexed by its vertices, with entries \(M_{x,y}\) equalling the probability of traveling from \(x\) to \(y\). A sequence would then be created by randomly choosing a vertex of \(C([m, n])\) (which gives the first 8 bits of the sequence) and then moving to each consecutive vertex/bit as dictated by \(M\). This method is fast (a million bits in 347 seconds using Maple) and it produces every possible sequence. However it is rigid; if one wants to tinker with the set from which these subsequences come (by making it larger), it would be preferable to use a method that it based on built-in bounds. The success of such a method would depend on being always able to remain within a subgraph which is connected and strongly connected. The subgraph being connected allows one to use the algorithm’s simple bounds condition and still remain in the graph. The strong connectedness prevents one from travelling to a sink. The algorithm is then straightforward:

**Sequence Creation Algorithm:** Input an initial \(x = x_1 \cdots x_8\) from \(C([m, n])\) and \(N\), the desired sequence length. For \(i\) from 9 to \(N\) do:

- if \(sr(xX) \geq [m, n]\) AND \(sr(xO) < [m, n]\) then set \(x_i = X\)
- if \(sr(xO) \geq [m, n]\) AND \(sr(xX) < [m, n]\) then set \(x_i = O\)
- otherwise choose \(x_i = O\) or \(X\) at random.

From the above Theorem, we see that we can use this algorithm for paths within \(C([m, n]), [m, n] \leq [4, 4.5]\). (Otherwise, we need more complicated conditions to ensure that we remain within \(C([m, n])\).) We argue that, for very long sequences, the best choice is the 164-element set \(C([4, 4])\). This might at first seem to be too large of a set. However, any 20-bit sequence produced from \(C([4, 4])\) is in the top 9% (of subjective randomness) of all 20-bit (or more) sequences. \(C([4, 4])\) also has a large cycle basis which allows for a great variety of sequences (see [GLS] for results concerning cycle bases for digraphs). Since the proportion of \# cycles/ \# vertices is significantly larger for \(C([m, n])\) (see Table 2) than it is for any \(C([m, n])\) for \([m, n] > [4, 4]\) it makes sense to choose \(C([4, 4])\) to make these sequences. Almost any 7-bit sequence can serve as the first seven bits of an element in \(C([4, 4])\). The exceptions are 1) any sequence with a streak of length 5 or more, 2) those 7-bit sequences with \(sr[1] \leq 2\), and 3) the sequences OOOXOOO and
XXXOXXX. With such an initial sequence and bounds \([m, n] = [4, 4]\) in the above algorithm, we produce sequences always within \(C([4, 4])\).

On average, such subjectively random sequences coming from \(C([4, 4])\) have equal numbers of X’s and O’s and an alternation rate of .58. Examples of such subjectively random sequences coming from are:

which is consistently more “balanced” than what is produced by a random generator:

7 Conclusion

There are many questions that one might ask concerning the sequences presented here as subjectively random. For example, wouldn’t people notice periodicity with motifs of length 5 or more? Would a HMM with more motifs give a more accurate model for subjective randomness? Would it be better to use larger “windows” when evaluating a sequence’s subjective randomness? How do the conditions for subjective randomness change as sequences become longer? Would these sequences be, in the long run, too regular (and thereby too predictable)?

These and similar questions bring us to the main problem for this program, that is, the lack of hard results which are geared to this type of modeling. Simply put, most research done on subjective randomness seeks to answer questions which have little to do with producing subjectively random objects. The algorithm presented here is based on what information there is. However, more research would have to be done to determine how to fine-tune it, if necessary.
References

[BW] M. Bar-Hillel and W. Wagenaar, The Perception of Randomness, *Adv. in Applied Math.* **12** (1991) 428-454.

[C] N. Chater, Reconciling simplicity and likelihood principles in perceptual organization. *Psych. Rev.* (1996) 103:566-581.

[CG] F.R.K. Chung and R.L. Graham, Quasi-random subsets of $\mathbb{Z}_n$, *J. Combin. Theory Ser. A* **61**. (1992), no. 1, 6486.

[F1] R. Falk, The perception of randomness, *Proceedings, Fifth International Conference for the Psychology of Mathematical Education*, Grenoble, France (1981) 222-229.

[F2] J. Feldman, Minimization of Boolean complexity in human concept learning, *Nature*, **407**, (2000), 630-633.

[F3] J. Feldman, How surprising is a simple pattern? Quantifying “Eureka!”, *Cognition*, **93**, (2004), 199-224.

[FK] R. Falk and C. Konold, Making Sense of Randomness: Implicit Encoding as a Basis for Judgment, *Psychological Review* **104** No. 2 (1997) 301-318.

[GLS] P.M. Gleiss, J. Leydold, and P.F. Stadler, Circuit Bases of Strongly Connected Digraphs, *Disc. Math. Graph Theory*, **23 Part 2** (2003) 241-260.

[GT1] T. Griffiths and J. Tenenbaum, Randomness and Coincidences: Reconciling Intuition and Probability Theory, *23rd Annual Conference of the Cognitive Science Society* (2001) 370-375.

[GT2] T. Griffiths and J. Tenenbaum, Probability, algorithmic complexity, and subjective randomness, *Proceedings of the Twenty-Fifth Annual Conference of the Cognitive Science Society* (2003).

[GT3] T. Griffiths and J. Tenenbaum, From Algorithmic to Subjective Randomness, *Advances in Neural Information Processing Systems 16* (2004).

[K] D. E. Knuth. The Art of Computer Programming, volume 2: Seminumerical Algorithms. Addison-Wesley, Reading, MA, second edition, 1981.

[KG] M. Kubovy and D.L.Gilden, Apparent randomness is not always the complement of apparent order, in “The Perception of Structure” (G. Lockhead and J.R. Pomerantz, Eds.), Amer. Psychol. Assoc., Washington, DC, 1990.

[KT] D. Kahneman and A. Tversky, Subjective probability: A judgment of representativeness. *Cognit. Psych.*, (1972) 3:430-454.

[L1] J.C. Lagarias, Pseudorandom Numbers. *Statistical Science*, **8** (1993) 31-39.
[L2] P. L’Ecuyer, Random number generation from *Handbook of Computational Statistics*, J. E. Gentle, W. Haerdle, and Y. Mori, eds., Springer-Verlag, 2004.

[L3] E.L.L. Leeuwenberg, Quantitative specification of information in sequential patterns. *Psych. Rev.* (1969), 76:216-220.

[LO] L.L. Lopes and G.D. Oden, Distinguishing between random and nonrandom events. *J. Exp. Psych: Learning, Memory and Cognition* (1987) 13:392-400.

[V] M. Li and P. Vitanyi, *An introduction to Kolmogorov complexity and its applications*, New York: Springer (1993).

[N] R. Nickerson, The Production and Perception of Randomness, *Psych. Rev.** 109 No. 2 (2002) 330-357.

[R1] M. Rabin, Inference By Believers in the Law of Small Numbers, *Quart. J. Econ.*, 117 No. 3, (2002) 775-816.

[R2] L. R. Rabiner, A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition, *Proc. of the IEEE* 77 No. 2 (1989) 257-286.

[R3] F. Restle. Theory of serial pattern learning, *Psych. Rev.*, (1970), 77:369-382.

[S] H.A. Simon, Complexity and the representation of patterned sequences of symbols. *Psych. Rev.* 79 (1972) 369-382.

[TK] A. Tversky and D. Kahneman, Belief in the Law of Small Numbers, *Psych. Bull.*, 76 No. 2 (1971) 105-110.

[W] W. A. Wagenaar, Generation of random sequences by human subjects: A critical survey of literature *Psych. Bull*. Vol. 77, No. 1, (1972) 65-72.

[WK] W. A. Wagenaar and G. B. Keren, Chance and luck are not the same. *J. Behavioral Decision Making*, 1 (1988) 65-75.
sample sequence | all sequences with same randomness rating | finite state | context-sensitive
---|---|---|---
\[0, 1, 0, 0, 1, 0, 0, 1\] | 77 | $\alpha^5\delta^6$ | $\alpha^5\delta^6$
\[0, 1, 1, 0, 0, 0, 1\] | 105 | $\alpha^3\delta^5$ | $\alpha^3\delta^5$
\[0, 1, 1, 0, 1, 0, 1, 1\] | 41, 69, 74, 82, 89, 93, 101, 107 | $\alpha^5\delta^4$ | $\alpha^5\delta^4$
\[0, 1, 1, 0, 1, 0, 1, 0\] | 38, 44, 46, 50, 52, 66, 70, 76, 78, 98, 100, 110, 114, 116, 118 | $\alpha^3\delta^4$ | $\alpha^3\delta^4$
\[0, 1, 1, 0, 1, 0, 1, 0, 0\] | 18, 22, 37, 45, 54, 72, 75, 90, 91, 104, 108 | $\alpha^4\delta^5$ | $\alpha^4\delta^5$
\[0, 1, 1, 1, 0, 0, 0, 1\] | 20, 26, 40, 43, 53, 58, 81, 83, 86, 88, 92, 94, 106, 117, 122 | $\alpha^4\delta^5$ | $\alpha^4\delta^5$
\[0, 1, 0, 1, 0, 0, 1\] | 34, 68, 102 | $\alpha^4\delta^5$ | $\alpha^4\delta^5$
\[0, 1, 1, 1, 1, 0, 1, 1\] | 9, 11, 13, 19, 23, 25, 27, 29, 33, 35, 39, 47, 49, 55, 57, 59, 61, 65, 67, 71, 79, 97, 99, 103, 111, 113, 115, 121, 123, 125 | $\alpha^4\delta^4$ | $\alpha^4\delta^4$
\[0, 1, 1, 0, 1, 0, 1, 0\] | 36, 73, 109 | $\alpha^4\delta^4$ | $\alpha^4\delta^4$
\[0, 1, 0, 1, 1, 1, 1\] | 5, 10, 21, 42, 80, 84, 87, 95 | $\alpha^3\delta^3$ | $\alpha^3\delta^3$
\[0, 1, 1, 0, 1, 0, 1, 1\] | 17, 51, 119 | $\alpha^4\delta^4$ | $\alpha^4\delta^4$
\[0, 1, 1, 1, 0, 0, 0, 0\] | 2, 4, 6, 8, 12, 14, 16, 24, 28, 30, 32, 48, 56, 60, 62, 64, 96, 112, 120, 124, 126 | $\alpha^3\delta^3$ | $\alpha^3\delta^3$
\[0, 1, 0, 1, 0, 1, 0, 1\] | 85 | $\alpha^2\delta^4$ | $\alpha^2\delta^4$
\[0, 1, 1, 1, 1, 1, 1\] | 1, 3, 7, 15, 31, 63, 127 | $\alpha^2\delta^4$ | $\alpha^2\delta^4$
\[0, 0, 0, 0, 0, 0, 0\] | 0 | $\alpha^1\delta^7$ | $\alpha^1\delta^7$

Table 1: The linear ordering of 8-bit binary sequences given by both “finite state” and “context-sensitive” models. Sequences $x_1 \cdots x_n$ are written in base 10: $2^{n-1}x_1 + \cdots + 2x_{n-1} + x_n$
| $[m, n]$ | $C([m, n])$ | num. vertices | num. arcs | cycle basis cardinality |
|-------|-------------|---------------|-----------|-------------------------|
| [5, 3] | $C_1([5, 3]) = \{44, 89, 178, 101, 203, 150\}$ $C_2([5, 3]) = \{77, 154, 52, 105, 211, 166\}$ | 6 | 6 | 1 |
|        |             |               |           | 1                       |
| [4, 6] | $C_1([5, 3]) \cup C_2([5, 3]) \cup \{22, 38, 41, 45, 46, 54, 69, 74, 75,$ $82, 90, 93, 100, 104, 107, 108, 116, 139,$ $147, 148, 151, 155, 162, 165, 173, 180,$ $181, 186, 201, 209, 210, 214, 217, 233\}$ | 46 | 58 | 13 |
| [4, 5] | $S([4, 5]) \setminus \{18, 37, 50, 66, 72, 76, 91, 94,$ $110, 118, 122, 133, 137, 145, 161, 164, 179,$ $183, 189, 205, 218, 237\}$ | 80 | 120 | 41 |
| [4, 4, 5] | $S([4, 4, 5]) \setminus \{66, 94, 122, 133, 161, 189\}$ | 102 | 158 | 57 |
| [4, 4] | $S([4, 4]) \setminus \{190, 125, 65, 130\}$ | 164 | 280 | 117 |
| [3, 7] | $S([3, 7]) \setminus \{190, 125, 65, 130\}$ | 170 | 298 | 129 |
| [3, 6] | $S([3, 6])$ | 190 | 342 | 153 |
| [3, 5, 5] | $S([3, 5, 5])$ | 196 | 360 | 165 |
| [3, 5] | $S([3, 5])$ | 238 | 462 | 225 |
| [2, 7] | $S([2, 7])$ | 240 | 467 | 228 |
| [2, 6] | $S([2, 6])$ | 254 | 505 | 252 |
| [1, 7] | $S([1, 7]) = \text{all 8-bit sequences}$ | 256 | 512 | 254 |

Table 2: Description of the strongly connected components $C([m, n])$ for each $[m, n]$ for which $C([m, n])$ is non-trivial. (Sequences $x_1 \cdots x_n$ are written in base 10: $2^{n-1}x_1 + \cdots + 2x_{n-1} + x_n$)