The conforming virtual element method for polyharmonic problems

P. F. Antonietti\textsuperscript{a}, G. Manzini\textsuperscript{b}, and M. Verani\textsuperscript{c}

\textsuperscript{a} MOX, Dipartimento di Matematica, Politecnico di Milano, Italy; e-mail: paola.antonietti@polimi.it
\textsuperscript{b} Group T-5, Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM, USA; e-mail: gmanzini@lanl.gov
\textsuperscript{c} MOX, Dipartimento di Matematica, Politecnico di Milano, Italy; e-mail: marco.verani@polimi.it

Abstract

In this work, we exploit the capability of virtual element methods in accommodating approximation spaces featuring high-order continuity to numerically approximate differential problems of the form $\Delta^p u = f$, $p \geq 1$. More specifically, we develop and analyze the conforming virtual element method for the numerical approximation of polyharmonic boundary value problems, and prove an abstract result that states the convergence of the method in the energy norm.

Key words: virtual element method, polytopal mesh, polyharmonic problem, high-order methods

1. Introduction

In the recent years, there has been a tremendous interest to numerical methods that approximate partial differential equations (PDEs) on computational meshes with arbitrarily-shaped polygonal/polyhedral (polytopic, for short) elements. A nonexhaustive list of such methods include the Mimetic Finite Difference method (see e.g., [22,20,4,16,8]), the Polygonal Finite Element Method (see e.g., [43]), the polygonal Discontinuous Galerkin Finite Element Methods (see e.g., [5,26,24,7,11]), the Hybridizable Discontinuous Galerkin and Hybrid High-Order Methods (see e.g., [31,32]), the Gradient Discretization method (see e.g., [34],[33]). An alternative approach that is also proved to be very successful is provided by the Virtual Element method (VEM), which was originally proposed in [14] for the numerical treatment of second-order elliptic problems [29,28], and readily extended to Cahn-Hilliard equation [3], Stokes equations [2], Laplace-Beltrami equation [35], Darcy-Brinkam equation [44], discrete topology optimization problems [6], fracture networks problems [17], eigenvalue problems [38]. The mixed virtual element formulation was proposed in [21]; the nonconforming Virtual element formulations was proposed for second-order elliptic problems in [12], and later extended to general advection-reaction-diffusion problems, Stokes equation, the biharmonic problems, the eigenvalue problems, and the Schrodinger equation in [27,30,48,37,9]. Efficient multigrid methods for the resulting linear system of equations in [10]. A posteriori error estimates can be found in [25].

In this work, we propose the conforming VEM for the numerical approximation of polyharmonic problems. A peculiar feature of VEM is the possibility of designing approximation spaces characterized by high-order continuity properties [15]. This turns out to be crucial when differential operators of order higher than two have to be considered, as, for example, in the numerical treatment of biharmonic problems (see, e.g.,
the plate bending problem or the Cahn-Hilliard equation) and polyharmonic problems. The numerical approximation of polyharmonic problems has been first addressed in the eighties by [18] and, more recently, in [13,40,45,42,36]. It is worth mentioning an increasing interest in the numerical approximation of models involving high-order differential operators, e.g., [46,47,41] in the context of sixth order Cahn-Hilliard equations. To the best of our knowledge, the conforming VEM proposed in this article is the first work addressing the approximation of arbitrary-order polyharmonic problems on polygonal meshes.

The outline of the paper is as follows. In Section 2, we introduce the continuous polyharmonic problem involving the differential operator $\Delta^p$ for any integer $p \geq 1$. In Section 3, we introduce the conforming VEM approximation of arbitrary order. In this case, the global VEM space is made of $p$ involving the differential operator $\Delta^r$ the approximation of arbitrary-order polyharmonic problems on polygonal meshes.

In Section 4, we derive the error estimates in different norms. Finally, in Section 5, we offer our final comments and conclusions.

Notation and technicalities. Throughout the paper, we consider the usual multi-index notation. In particular, if $v$ is a sufficiently regular bivariate function and $\nu = (\nu_1, \nu_2)$ a multi-index with $\nu_1, \nu_2$ nonnegative integer numbers, the function $D^\nu v = \partial^{(\nu_1)} v / \partial x_1^{\nu_1} \partial x_2^{\nu_2}$ is the partial derivative of $v$ of order $|\nu| = \nu_1 + \nu_2 > 0$. For $\nu = (0,0)$, we adopt the convention that $D^\nu v$ coincides with $v$. Also, for the sake of exposition, we may use the shortcut notation $\partial_x v, \partial_y v, \partial_{xx} v, \partial_{xy} v, \partial_{yy} v$, to denote the first- and second-order partial derivatives along the coordinate directions $x$ and $y$; $\partial_n v, \partial_{nn} v, \partial_{nt} v, \partial_{tt} v$ to denote the first- and second-order normal and tangential derivatives of order one and two along a given mesh edge; and $\partial_n^m v$ and $\partial_t^m v$ to denote the normal and tangential derivative of $v$ of order $m$ along a given mesh edge. Finally, let $\mathbf{n} = (n_x, n_y)$ and $\mathbf{t} = (t_x, t_y)$ be the unit normal and tangential vectors to a given edge $e$ of an arbitrary polygon $K$. We recall the following relations between the first derivatives of $v$:

$$\partial_n v = n_x \partial_x v + n_y \partial_y v, \quad \partial_t v = t_x \partial_x v + t_y \partial_y v,$$

and the second derivatives of $v$:

$$\partial_{nn} v = \mathbf{n}^T H(v) \mathbf{n}, \quad \partial_{nt} v = \mathbf{n}^T H(v) \mathbf{t}, \quad \partial_{tt} v = \mathbf{t}^T H(v) \mathbf{t},$$

where matrix $H(v)$ is the Hessian of $v$, i.e., $H_{11}(v) = \partial_{xx} v, H_{12}(v) = H_{21}(v) = \partial_{xy} v, H_{22}(v) = \partial_{yy} v$.

2. The continuous polyharmonic problem

Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain with boundary $\Gamma$. For an integer $p \geq 1$, we are interested in developing the conforming Virtual Element method for the numerical approximation of the following problem:

$$\Delta^p u = f \quad \text{in } \Omega,$$

$$\partial_n^j u = 0 \quad \text{for } j = 0, \ldots, p-1 \text{ on } \Gamma,$$

(recall the conventional notation $\partial_n^0 u = u$). Let

$$V \equiv H_0^p(\Omega) = \{ v \in H^p(\Omega) : \partial_n^j v = 0 \text{ on } \Gamma, j = 0, \ldots, p-1 \}.$$

We denote the duality pairing between $V$ and its dual $V^*$ by $\langle \cdot, \cdot \rangle$. The variational formulation of (3) reads as: Find $u \in V$ such that

$$a(u,v) = \langle f, v \rangle \quad \forall v \in V,$$

where, for any nonnegative integer $\ell$, the bilinear form on the left is given by:
\[
\begin{aligned}
a(u, v) &= \begin{cases}
\int_{\Omega} \nabla^\ell u \cdot \nabla^\ell v \, dx & \text{for } p = 2\ell + 1, \\
\int_{\Omega} \Delta^\ell u \Delta^\ell v \, dx & \text{for } p = 2\ell.
\end{cases}
\end{aligned}
\]

Whenever \( f \in L^2(\Omega) \), we may consider the duality pairing between \( L^2(\Omega) \) and itself given by the \( L^2 \)-inner product:

\[
\langle f, v \rangle = (f, v) = \int_{\Omega} f v \, dx.
\]

The existence and uniqueness of the solution to (4) follows from the Lax-Milgram lemma because of the continuity and coercivity of the bilinear form \( a(\cdot, \cdot) \) with respect to \( \| \cdot \|_V := \| \cdot \|_{p, \Omega} \) which is a norm on \( H^p_0(\Omega) \). Moreover, since \( \Omega \) is a convex polygon, from [39] we know that \( u \in H^{2p-\omega}(\Omega) \cap H^p_0(\Omega) \) if \( f \in H^{m-\omega}(\Omega) \), \( m \leq p \) and it holds that \( \| u \|_{2p-\omega} \leq C \| f \|_{-\omega} \). In the following, we denote the coercivity and continuity constants of \( a(\cdot, \cdot) \) by \( \alpha \) and \( M \), respectively.

3. The conforming Virtual Element approximation

3.1. Abstract framework

Let \( \{ \Omega_h \}_h \) be a sequence of decompositions of \( \Omega \) where each mesh \( \Omega_h \) is a collection of nonoverlapping polygonal elements \( K \) with boundary \( \partial K \), and let \( \mathcal{E}_h \) be the set of edges \( e \) of \( \Omega_h \). Each mesh is labeled by \( h \), the diameter of the mesh, defined as usual by \( h = \max_{K \in \Omega_h} h_K \), where \( h_K = \sup_{x,y \in K} |x - y| \). We denote the set of vertices in \( \mathcal{T}_h \) by \( V_h = V_h^\text{int} \cup V_h^\text{bd} \), where \( V_h^\text{int} \) and \( V_h^\text{bd} \) are the subsets of interior and boundary vertices, respectively. Accordingly, \( V_h^K \) is the set of vertices of \( K \). The symbol \( h_v \) denotes the average of the diameters of the polygons sharing the vertex \( v \). For functions in \( \Pi_K \in \Omega_h H^p(K) \), we define the seminorm \( \| v \|_h^K = \sum_{K \in \Omega_h} a^K(\cdot, \cdot) \), being \( a^K(\cdot, \cdot) \) the restriction of \( a(\cdot, \cdot) \) to \( K \).

The formulation of the Virtual Element method for solving problem (4) only requires three mathematical objects: the finite dimensional conforming Virtual Element space \( V_{h,r}^p \subset V \), the bilinear form \( a_h(\cdot, \cdot) \), and the linear functional \( \langle f_h, \cdot \rangle \). Their definition is the topic of this section. Using such objects, we formulate the VEM as: Find \( u_h \in V_{h,r}^p \) such that

\[
a_h(u_h, v_h) = \langle f_h, v_h \rangle \quad \forall v_h \in V_{h,r}^p.
\]

The well-posedness of the VEM given in (7), which implies existence and uniqueness of the solution \( u_h \), is a consequence of the Lax-Milgram lemma. An abstract convergence result is available, which depends only on the following assumptions:

(H1) for each \( h \) and an assigned integer number \( r \geq 2p - 1 \) we are given:

(i) the global Virtual Element space \( V_{h,r}^p \) with the following properties:
   - \( V_{h,r}^p \) is a finite dimensional subspace of \( H^p_0(\Omega) \);
   - its restriction \( V_{h,r}^p(K) \) to any element \( K \) of a given mesh \( \Omega_h \), called the local Virtual Element space, is a finite dimensional subspace of \( H^p(K) \);
   - \( P_r(K) \subset V_{h,r}^p(K) \) where \( P_r(K) \) is the space polynomials of degree up to \( r \geq 1 \) defined on \( K \);

(ii) the symmetric and coercive bilinear form \( a_h : V_{h,r}^p \times V_{h,r}^p \rightarrow \mathbb{R} \) admitting the decomposition

\[
a_h(u_h, v_h) = \sum_{K \in \Omega_h} a^K_h(u_h, v_h) \quad \forall u_h, v_h \in V_{h,r}^p,
\]

where each local summation term \( a^K_h(\cdot, \cdot) \) is also a symmetric and coercive bilinear form;

(iii) an element \( f_h \) of the dual space \( V_{h,r}^* \), which allows us to define the continuous linear functional \( \langle f_h, \cdot \rangle \).
\( \textbf{(H2)} \) for each \( h \) and each mesh element \( K \in \Omega_h \), the local symmetric bilinear form \( a^K_h(\cdot, \cdot) \) possesses the two following properties:

(i) \textbf{r-Consistency}: for every polynomial \( q \in \mathbb{P}_r(K) \) and function \( V^p_{h,r}(K) \) we have:

\[
a^K_h(v_h, q) = a^K(v_h, q); \tag{8}
\]

(ii) \textbf{Stability}: there exist two positive constants \( \alpha_*, \alpha^* \) independent of \( h \) and \( K \) such that for every \( v_h \in V^p_{h,r}(K) \) it holds:

\[
\alpha_* a^K(v_h, v_h) \leq a^K_h(v_h, v_h) \leq \alpha^* a^K(v_h, v_h). \tag{9}
\]

To apply the Lax-Milgram lemma we need \( a^K_h(\cdot, \cdot) \) to be coercive and continue. The coercivity of \( a^K_h(\cdot, \cdot) \) follows from the coercivity of \( a(\cdot, \cdot) \) and the stability property \( \textbf{(H2)} \) (with coercivity constant \( \alpha^* \alpha \)). The continuity of \( a^K_h(\cdot, \cdot) \) follows from its symmetry, assumption \( \textbf{(H2)} \) and the continuity of \( a(\cdot, \cdot) \) (with continuity constant \( \alpha^* M \)). Denoting by \( \mathbb{P}_r(\Omega_h) \) the space of piecewise (possibly discontinuous) polynomials defined over the mesh \( \Omega_h \), the following abstract convergence result hold.

**Theorem 3.1** Let \( u \) be the solution of the variational problem \( (4) \). Then, for every Virtual Element approximation \( u_I \) in \( V^p_{h,r} \) and any piecewise polynomial approximation \( u_\pi \in \mathbb{P}_r(\Omega_h) \) of \( u \) we have:

\[
\|u - u_h\|_V \leq C \left( \|u - u_I\|_V + \|u - u_\pi\|_V + \|f_h - f\|_{V^p_{h,r}} \right), \tag{10}
\]

where \( C \) is a constant independent of \( h \) that may depend on \( \alpha, \alpha_*, \alpha^*, M, \) and \( r \), and,

\[
\|f - f_h\|_{V^p_{h,r}} = \sup_{v_h \in V^p_{h,r} \setminus \{0\}} \frac{\langle f - f_h, v_h \rangle}{\|v_h\|_V} \tag{11}
\]

is the approximation error of the right-hand side given in the norm of the dual space \( V^p_{h,r} \).

**Proof.** The proof of this theorem is similar to the proofs of the convergence theorem in the energy norm for the Virtual Element approximation of lower-order elliptic problems [14,23]. We briefly sketch how the proof works for completeness of exposition. First, an application of the triangular inequality implies that:

\[
\|u - u_h\|_V \leq \|u - u_I\|_V + \|u_I - u_\pi\|_V. \tag{12}
\]

Let \( \delta_h = u_h - u_I \). Starting from the definition of \( \|\cdot\|_V \), we find that:

\[
\alpha_* \|\delta_h\|_V^2 = \alpha_* a(\delta_h, \delta_h) \tag{use (9)}
\]

\[
\leq a_h(\delta_h, \delta_h) \tag{use \( \delta_h = u_h - u_I \)}
\]

\[
\leq a_h(\delta_h, u_\pi) - a_h(\delta_h, u_I) \tag{use (7)}
\]

\[
\leq \langle f_h, \delta_h \rangle - \sum_{K \in \Omega_h} a^K_h(\delta_h, u_I) \tag{add \( \pm u_\pi \)}
\]

\[
\leq \langle f_h, \delta_h \rangle - \sum_{K \in \Omega_h} \left( a^K_h(\delta_h, u_I) + a^K(\delta_h, u_\pi) \right) \tag{use (8)}
\]

\[
\leq \langle f_h, \delta_h \rangle - \sum_{K \in \Omega_h} \left( a^K(\delta_h, u_I - u_\pi) + a^K(\delta_h, u_\pi) \right) \tag{add \( \pm u \)}
\]

\[
\leq \langle f_h, \delta_h \rangle - \sum_{K \in \Omega_h} \left( a^K(\delta_h, u_I - u_\pi) + a^K(\delta_h, u_\pi - u) + a^K(\delta_h, u) \right) \tag{use (4)}
\]

\[
= \langle f_h - f, \delta_h \rangle - \sum_{K \in \Omega_h} \left( a^K(\delta_h, u_I - u_\pi) + a^K(\delta_h, u_\pi - u) \right). \tag{13}
\]

Then, we use (9), add and subtract \( u \), use the continuity of \( a^K \), sum over all the elements \( K \), divide by \( \|\delta_h\|_V \), take the supremum of the right-hand side error term on \( V^p_{h,r} \setminus \{0\} \), and obtain

\[
\alpha_* \|\delta_h\|_V \leq \sup_{v_h \in V^p_{h,r} \setminus \{0\}} \frac{|\langle f_h - f, v_h \rangle|}{\|v_h\|_V} + M (\alpha^* \|u_I - u\|_V + (1 + \alpha^*) \|u - u_\pi\|_h). \tag{13}
\]
The assertion of the theorem follows by using (13) in (12) and suitably defining constant the $C$.

Let $K \subset \mathbb{R}^2$ be a polygonal element and set

$$a^K(u, v) = \begin{cases} \int_K \nabla \Delta^\ell u \cdot \nabla \Delta^\ell v \, dx & \text{for } p = 2\ell + 1, \\ \int_K \Delta^\ell u \Delta^\ell v \, dx & \text{for } p = 2\ell. \end{cases}$$

For an odd $p$, i.e., $p = 2\ell + 1$, a repeated application of the integration by parts formula yields

$$a^K(u, v) = - \int_K \Delta^p u \, v \, dx + \int_{\partial K} \partial_n (\Delta^\ell u) \Delta^\ell v \, ds$$

$$+ \sum_{i=1}^{\ell} \left( \int_{\partial K} \partial_n (\Delta^{p-i} u) \Delta^{i-1} v \, ds - \int_{\partial K} \Delta^{p-i} u \partial_n (\Delta^{i-1} v) \, ds \right), \quad (14)$$

while, for an even $p$, i.e., $p = 2\ell$, we have

$$a^K(u, v) = - \int_K \Delta^p u \, v \, dx + \sum_{i=1}^{\ell} \left( \int_{\partial K} \partial_n (\Delta^{p-i} u) \Delta^{i-1} v \, ds - \int_{\partial K} \Delta^{p-i} u \partial_n (\Delta^{i-1} v) \, ds \right). \quad (15)$$

3.2. Virtual element spaces

For $p \geq 1$ and $r \geq 2p - 1$, the local Virtual Element space on element $K$ is defined by

$$V^p_{h,r}(K) = \left\{ v_h \in H^p(K) : \Delta^p v_h \in \mathbb{P}_{r-2p}(K), \, D^p v_h \in C^0(\partial K), \, |\nu| \leq p - 1, \right. \left. v_h \in \mathbb{P}_r(e), \, \partial_n v_h \in \mathbb{P}_{r-i}(e), \, i = 1, \ldots, p - 1 \, \forall e \in \partial K \right\}, \quad (16)$$

with the conventional notation that $\mathbb{P}_{-1}(K) = \{0\}$. The Virtual Element space $V^p_{h,r}(K)$ contains the space of polynomials $\mathbb{P}_r(K)$, for $r \geq 2p - 1$. Moreover, for $p = 1$, it coincides with the conforming Virtual Element space for the biharmonic equation [23]; for $p = 2$, it coincides with the conforming Virtual Element space for the biharmonic equation.

We characterize the functions in $V^p_{h,r}(K)$ through the following degrees of freedom:

1. For any vertex $v$ of $K$:
   $$(D1) \ h^{\nu}_{e} D^\nu v_h(v), \ |\nu| \leq p - 1$$

2. For any edge $e$ of $\partial K$:
   $$(D2) \ h^{-1}_{e} \int_e q v_h \, ds \text{ for any } q \in \mathbb{P}_{r-2p}(e)$$

3. For any $e$ of $\partial K$:
   $$(D3) \ h^{-1+j}_{e} \int_e q \partial_n^j v_h \, ds \text{ for any } q \in \mathbb{P}_{r-2p+j}(e), \, j = 1, \ldots, p - 1$$

4. For any $q \in \mathbb{P}_{r-2p}(K)$:
   $$(D4) \ h^{-2}_{K} \int_K q v_h \, ds$$

Here, as usual, we assume that $\mathbb{P}_{-n}(\cdot) = \{0\}$ for $n \geq 1$. In $(D3)$, the index $j$ starts from 1 instead of 0 since for $j = 0$ we would find the degrees of freedom that are already listed in $(D2)$ (recall that $\partial_n^j v_h = v_h$ for $j = 0$). We note that for any sufficiently regular two-dimensional domain $\Omega$ we have the embedding

$$C^m(\Omega) \subset H^p(\Omega) \text{ if } m \leq p - 1.$$
Therefore, \( v \) interpolation of the traces in the right-hand side of (14). Then, we are left to prove that a function \( v \) equal to the dimension of \( V \) \( v(\cdot) \) is constant in \( K \). Using the Divergence Theorem we find that:

\[
\Delta v_h |K = \int_K \Delta v_h \, dx = \int_{\partial K} \partial_n v_h \, ds = 0.
\]

Therefore, \( v_h \) is the solution of the boundary value problem \( \Delta v_h = 0 \) in \( K \) with boundary conditions \( v_h = 0 \) on \( \partial K \), and, thus, \( v_h = 0 \) in \( K \).

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**Fig. 1.** Triharmonic problem: edge degrees of freedom of the Virtual Element space \( V_{h,r}(K) \). Here, \( p \) is the order of the partial differential operator and \( p = 3 \) corresponds to the triharmonic case; \( r = 5, 6 \) are the integer parameters that specify the maximum degree of the polynomial subspace \( P_r(\cdot) \) of the VEM space \( V_{h,r}(K) \). The (green) dots at the vertices represent the vertex values and each (red) vertex circle represents an order of derivation. The (black) dot on the edge represents the moment of \( v_{h|e} \); the arrows represent the moments of \( \partial_n v_{h|e} \); the double arrows represent the moments of \( \partial_{nn} v_{h|e} \).

- the degrees of freedom \((D1)\) provides \( v_A, v_B, \partial^k_x v_A \) and \( \partial^k_x v_B \) for \( k = 1, \ldots, p - 1 \), i.e., 2\( p \) degrees of freedom, which are enough to interpolate \( v_{h|e} \) in \( P_{r}(e) \) if \( r = 2p - 1 \). When \( r > 2p - 1 \) the remaining \((r + 1) - 2p \) conditions required to interpolate \( v_{h|e} \) in \( P_{r}(e) \) are provided by the degrees of freedom \((D2)\).
- The tangential derivatives \( \partial^k_x v_{h|e} \) in \( P_{r-k}(e) \) for \( k = 1, \ldots, r - 1 \) can be obtained by deriving \( k \) times the interpolated polynomial \( v_{h|e} \) along \( e \);
- similarly, for each \( j = 1, \ldots, p - 1 \), the degrees of freedom \((D1)\) provides \( 2(p - j) \) conditions, i.e., \( \partial^k_x \partial^h_j v_A \) and \( \partial^k_x \partial^h_j v_B \), for \( k = j, \ldots, p - 1 \). The remaining \( r + 1 - 2(p - j) \) conditions to interpolate \( \partial^h_j v_h \) in \( P_{r-j}(e) \) are provided by the \((r - 2p + j + 1) \) degrees of freedom \((D3)\). The tangential derivatives \( \partial^k_x \partial^h_j v_{h|e} \) in \( P_{r-j-k}(e) \) for \( k = 1, \ldots, r - j - 1 \) can be obtained by deriving \( k \) times the interpolated polynomial \( \partial^h_j v_h \) along \( e \).

Figure 1 illustrates the degrees of freedom on a given edge \( e \) for \( p = 3 \) (triharmonic case) and \( r = 5, 6 \).

Finally, we note that the internal degrees of freedom \((D4)\) make it possible to define the orthogonal polynomial projection of \( v_{h} \) onto the space of polynomial of degree \( r - 2p \).

The dimension of \( V_{h,r}^p(K) \) is:

\[
\dim V_{h,r}^p(K) = \frac{p(p+1)}{2} \mathcal{N}_K + \mathcal{N}_K \sum_{j=0}^{p-1} \dim P_{r-2p+j}(e) + \dim P_{r-2p}(K)
\]

\[
= \frac{p(p+1)}{2} \mathcal{N}_K + \mathcal{N}_K \sum_{j=0}^{p-1} (r-2p+j+1) + \frac{(r-2p+1)(r-2p+2)}{2},
\]

where \( \mathcal{N}_K \) is the number of vertices, which equals the number of edges, of \( K \).

The following lemma ensures that the above choice of degrees of freedom is unisolvent in \( V_{h,r}^p(K) \).

**Lemma 3.2** The degrees of freedom \((D1)-(D4)\) are unisolvent for \( V_{h,r}^p(K) \).

**Proof.** To ease the presentation, we first consider the lowest order space \((r = 5)\) for the triharmonic problem \((p = 3)\). A counting argument implies that the cardinality of the set of degrees of freedom \((D1) - (D3)\) is equal to the dimension of \( V_{h,5}^3(K) \) (note that in this specific case \((D4)\) is empty as there is no volumetric integral in the right-hand side of (14)). Then, we are left to prove that a function \( v_h \) in \( V_{h,5}^3(K) \) is zero if its degrees of freedom are zero. From the previous discussion on the degrees of freedom, we know that the edge polynomial interpolation of the traces of \( v_h \), \( \partial_n v_h \), \( \partial_{nn} v_h \) and \( \partial_{nnn} v_h \), and, hence, \( \Delta v_h = \partial_{nn} v_h + \partial_{nnn} v_h \), must be zero if the degrees of freedom of \( v_h \) are zero. Hence, (14) with \( \ell = 1 \), we find that \( \| \nabla(\Delta v_h) \|_{L^2(K)} = 0 \) inside \( K \), which implies that \( \Delta v_h \) is constant in \( K \). Using the Divergence Theorem we find that:

\[
\Delta v_h |K = \int_K \Delta v_h \, dx = \int_{\partial K} \partial_n v_h \, ds = 0.
\]

Therefore, \( v_h \) is the solution of the boundary value problem \( \Delta v_h = 0 \) in \( K \) with boundary conditions \( v_h = 0 \) on \( \partial K \), and, thus, \( v_h = 0 \) in \( K \).
The case of a generic $p$ can be treated analogously by properly employing relations (14) (odd $p$) and (15) (even $p$) in combination with the following observations:

(a) the polynomial trace $\Delta^p v_{ih}|_{\partial K} = \partial^p_n \partial^\ell v_{ih}$ for every integers $\alpha$, $\beta$, and $\nu$ (with $\nu \geq 1$) such that $\alpha + \beta = 2\nu$ must be zero if the degrees of freedom $(D1) - (D3)$ of $v_h$ are zero;

(b) the volumetric integrals in (14) and (15) are zero if the degrees of freedom $(D4)$ are zero;

(c) $a^K(v_h, v_h)$ is a norm on $H^0_\delta(K)$.

To define the elliptic projection $\Pi^r_{\nu,K} : V^p_{h,r}(K) \rightarrow \mathbb{P}_r(K)$, we first need to introduce the vertex average projector $\hat{\Pi}^K : V^p_{h,r}(K) \rightarrow \mathbb{P}_0(K)$, which projects any smooth enough function defined on $K$ onto the space of constant polynomials. Let $\psi$ be a continuous function defined on $K$. The vertex average projection of $\psi$ onto the constant polynomial space is defined as:

$$\hat{\Pi}^K \psi = \frac{1}{N_K} \sum_{v \in \partial K} \psi(x_v),$$

where $x_v$ is the position of vertex $v$. Finally, we define the elliptic projection $\Pi^r_{\nu,K} : V^p_{h,r}(K) \rightarrow \mathbb{P}_r(K)$ as the solution of the finite dimensional variational problem

$$a^K(\Pi^r_{\nu,K} v_h, q) = a^K(v_h, q) \quad \forall q \in \mathbb{P}_r(K),$$

$$\hat{\Pi}^K D^\nu \Pi^r_{\nu,K} v_h = \hat{\Pi}^K D^\nu v_h \quad |\nu| \leq p - 1.$$

Such operator has two important properties:

(i) it is a polynomial-preserving operator in the sense that $\Pi^r_{\nu,K} q = q$ for every $q \in \mathbb{P}_r(K)$.

(ii) $\Pi^r_{\nu,K} v_h$ is computable using only the degrees of freedom of $v_h$. In fact, in view of the integration by parts formulas (14) and (15), the right-hand side of (18) takes the form (depending on the parity of $p$):

$$a^K(v_h, q) = - \int_K \Delta^p q v_h \, dx + \int_{\partial K} \partial_n(\Delta^\ell q) \Delta^\ell v_h \, ds$$

$$+ \sum_{i=1}^\ell \left\{ \int_{\partial K} \partial_n(\Delta^{p-i} q) \Delta^{i-1} v_h \, ds - \int_{\partial K} \Delta^{p-i} q \, \partial_n(\Delta^{i-1} v_h) \, ds \right\},$$

or

$$a^K(v_h, q) = - \int_K \Delta^p q v_h \, dx + \sum_{i=1}^\ell \left\{ \int_{\partial K} \partial_n(\Delta^{p-i} q) \Delta^{i-1} v_h \, ds - \int_{\partial K} \Delta^{p-i} q \, \partial_n(\Delta^{i-1} v_h) \, ds \right\}. \quad (21)$$

In (20) and (21), $\Delta^{p-i} q$, and $\Delta^p q$ are easily computable from $q$. The volumetric integral on $K$ can be expressed using the degrees of freedom $(D5)$ since it is the moment of $v_h$ against $\Delta^p q$, which is a polynomial of degree $r - 2p$. The edge traces of $\Delta^\ell v_h$, $\partial_n(\Delta^{i-1} v_h)$, and $\Delta^{i-1} v_h$ are computable from the degrees of freedom $(D1) - (D4)$ of $v_h$ by solving suitable polynomial interpolation problems.

Building upon the local spaces $V^p_{h,r}(K)$ for all $K \in \Omega_h$, the global conforming Virtual Element space $V^p_{h,r}$ is defined on $\Omega$ as

$$V^p_{h,r} = \left\{ v_h \in H^0_\delta(\Omega) : v_h|_K \in V^p_{h,r}(K) \ \forall K \in \Omega_h \right\}. \quad (22)$$

We remark that the associated global space is made of $C^{p-1}$ functions. Indeed, the restriction of a Virtual Element function $v_h$ to each element $K$ belongs to $H^0(K)$ and glue with $C^{p-1}$-regularity across the internal mesh faces. The global degrees of freedom induced by the local degrees of freedom are listed as follows:

- $h^{|\nu|} D^\nu v_h(v)$, $|\nu| \leq p - 1$ for every interior vertex $v$ of $\Omega_h$,
Remark 3.3

For \( h^{-1} \int q v_h \, ds \) for any \( q \in \mathbb{P}_{r-2p}(e) \) and every interior edge \( e \in \mathcal{E}_h \);

- \( \int q \partial_n v_h \, ds \) for every \( q \in \mathbb{P}_{r-5}(e) \) and every interior edge \( e \in \mathcal{E}_h \);

- \( h^{-1+j} \int q \partial_i^j v_h \, ds \) for any \( q \in \mathbb{P}_{r-2p+j}(e) \) \( i = 1, \ldots, p - 1 \) and every interior edge \( e \in \mathcal{E}_h \);

- \( h^{-2} \int q v_h \, ds \) for any \( q \in \mathbb{P}_{r-2p}(K) \) and every \( K \in \Omega_h \).

**Remark 3.3** For \( p = 3 \) (triharmonic case) we can also consider the following modified lowest order space

\[
\tilde{V}_{h,5}(K) = \left\{ v_h \in H^3(K) : \Delta^3 v_h = 0, v_h, \partial_n v_h, \partial_{nn} v_h \in C^0(\partial K), \quad v_h \in \mathbb{P}_5(e), \partial_n v_h \in \mathbb{P}_3(e), \partial_{nn} v_h \in \mathbb{P}_2(e) \quad \forall e \in \partial K \right\}
\]

with associated dofs

\[ (D1') \quad h^{-1} D^i v_h(\nu), \quad |\nu| \leq 2 \text{ for any vertex } \nu \text{ of } \partial K; \]

\[ (D2') \quad h \int q \partial_n v_h \, ds \text{ for any edge } e \text{ of } \partial K. \]

Using the same argument of the proof of Lemma 3.2, we can still prove that (i) the degrees of freedom \((D1')\) and \((D2')\) are unisolvent in \( \tilde{V}_{h,5}(K) \); (ii) the space of polynomials of degree 4 are a subspace of \( \tilde{V}_{h,5}(K) \); (iii) the elliptic projection of \( v_h \) is still computable from this choice of degrees of freedom; (iv) the associated global space

\[
\tilde{V}_{h,5} = \left\{ v_h \in H^3_0(\Omega) : \left. v_h \right|_{K} \in \tilde{V}_{h,5}(K) \quad \forall K \in \Omega_h \right\},
\]

which is obtained by gluing together all the elemental spaces \( \tilde{V}_{h,5}(K) \), is still made of \( C^2 \) functions. Analogously, in the general case one can build the following modified lowest order spaces (containing the space of polynomials of degree \( 2p - 2 \))

\[
\tilde{V}_{h,2p-1}^p(2p-1)(K) = \left\{ v_h \in H^p(K) : \Delta^p v_h = 0, D^\nu v_h \in C^0(\partial K), |\nu| \leq p - 1, \quad v_h \in \mathbb{P}_{2p-1}(e), \partial^\nu v_h \in \mathbb{P}_{2p-2-i}(e), \quad i = 1, \ldots, p - 1 \quad \forall e \in \partial K \right\},
\]

with associated dofs

\[ (D1') \quad h^{-1} D^i v_h(\nu), \quad |\nu| \leq p - 1 \text{ for any vertex } \nu \text{ of } \partial K; \]

\[ (D2') \quad h^{-1+j} \int q \partial_i v_h \, ds \text{ for any } q \in \mathbb{P}_{i-2}(e) \text{ and any edge } e \text{ of } \partial K, \quad i = 1, \ldots, p - 1. \]

3.3. Construction of the bilinear form

We write the symmetric bilinear form \( a_h : \mathbb{V}_{h,r}^p \times \mathbb{V}_{h,r}^p \to \mathbb{R} \) as the sum of local terms

\[
a_h(u_h, v_h) = \sum_{K \in \Omega_h} a^K_h(u_h, v_h),
\]

where each local term \( a^K_h : \mathbb{V}_{h,r}^p(K) \times \mathbb{V}_{h,r}^p(K) \to \mathbb{R} \) is a symmetric bilinear form. We set

\[
a^K_h(u_h, v_h) = a^K(\Pi^r, \Pi^r, K u_h, \Pi^r, K v_h) + S^K(u_h - \Pi^r, K u_h, v_h - \Pi^r, K v_h),
\]

where \( S^K : \mathbb{V}_{h,r}^p(K) \times \mathbb{V}_{h,r}^p(K) \to \mathbb{R} \) is a symmetric positive definite bilinear form such that

\[
\sigma_* a^K(u_h, v_h) \leq S^K(u_h, v_h) \leq \sigma^* a^K(u_h, v_h) \quad \forall v_h \in \mathbb{V}_{h,r}^p(K) \text{ with } \Pi^r, K v_h = 0,
\]

for two some positive constants \( \sigma_* \), \( \sigma^* \) independent of \( h \) and \( K \). The bilinear form \( a^K_h(\cdot, \cdot) \) has the two fundamental properties of consistency and stability stated by the following lemma.
Remark 3.5 The right-hand side of (33) is computable by using the degrees of freedom \((D\ell)\). Using the definition of the
\[ a_h^K(u_h, v_h) = \left( a_h^K(u_h, u_h) \right)^{1/2} \left( a_h^K(v_h, v_h) \right)^{1/2} \leq \alpha^* M \left\| u_h \right\|_{V,K} \left\| v_h \right\|_{V,K} \forall u_h, v_h \in V_h^r(K). \]

Collecting together the local terms, we can formulate the global V-ellipticity and continuity properties as follows:
\[ \alpha_a v_h, v_h \leq a_h(v_h, v_h) \leq \alpha^* a^K(v_h, v_h) \forall v_h \in V_h^p \]
\[ a_h(u_h, v_h) \leq \alpha^* M \left\| u_h \right\|_V \left\| v_h \right\|_{V,K} \forall u_h, v_h \in V_h^p. \]

3.4. Construction of the load term

We denote by \( f_h \) the piecewise polynomial approximation of \( f \) on \( \Omega_h \) given by
\[ f_{h|K} = \Pi^{0,K}_{r,p} f, \]
for \( r \geq 2p - 1 \) and \( K \in \Omega_h \). Then, we set
\[ \langle f_h, v_h \rangle = \sum_{K \in \Omega_h} \int_K f_h v_h \, dx. \]
Using the definition of the \( L^2 \)-orthogonal projection we find that
\[ \langle f_h, v_h \rangle = \sum_{K \in \Omega_h} \int_K \Pi^{0,K}_{r,p} f v_h \, dx = \sum_{K \in \Omega_h} \int_K \Pi^{0,K}_{r,p} f \Pi^{0,K}_{r-p} v_h \, dx = \sum_{K \in \Omega_h} \int_K f \Pi^{0,K}_{r-p} v_h \, dx. \]

The right-hand side of (33) is computable by using the degrees of freedom \((D1) - (D5)\) and the enhanced approach \([1]\).

Remark 3.5 An alternative formulation that does not require the enhancement is given by taking \( \tau = \max(0, r - 2p) \) for \( r \geq 2p - 1 \) and \( f_h = \Pi^{0,K}_{r} f \). The resulting approximation is suboptimal.

3.5. VEM approximation of polyharmonic problems with basis functions of arbitrary degree of continuity

In this last section we briefly sketch the construction of global Virtual Element spaces with higher order of continuity. More precisely, let us consider the local Virtual Element space defined as before, for \( r \geq 2p - 1 \):
\[ V^p_{h,r}(K) = \left\{ v_h \in H^p(K) : \Delta^p v_h \in \mathbb{P}_{r-2p}(K), D^r v_h \in C^0(\partial K), |v| \leq p - 1, \right\} \]
\[ v_h \in \mathbb{P}_{e}(e), \right\| \partial^j v_h \right\|_{H^0(e)}, j = 1, \ldots, p - 1 \forall e \in \partial K \].
\[(D1)\ h^{|\nu|}D^\nu v_h(v), \ |\nu| \leq p - 1 \text{ for any vertex } v \text{ of } K;\]
\[(D2)\ h_e^{-1} \int_e v_h q \, ds \text{ for any } q \in \mathbb{P}_{r-2p}(e), \text{ for any edge } e \in \partial K;\]
\[(D3)\ h_e^{-1+j} \int_e \partial^n v_h q \, ds \text{ for any } q \in \mathbb{P}_{r-2p+j}(e), j = 1, \ldots, p-1 \text{ for any edge } e \in \partial K;\]
\[(D4')\ h_K^{-2} \int_K q v_h \, ds \text{ for any } q \in \mathbb{P}_{r-2(p-1)}(K);\]
where as usual we assume \(\mathbb{P}_{-n}(\cdot) = \{0\} \) for \(n = 1, 2, 3, \ldots\).

In view of the above choice of the degrees of freedom, the following properties hold true:
(i) the dofs are unisolvent. Indeed, proceeding as before, it is enough to use (14) or (15) and observe that \(\Delta^tv_h|_e = \partial^n_\alpha \partial^n_\beta v_h\) with \(\alpha + \beta = 2i\) is a polynomial uniquely identified by the values of the dofs;
(ii) \(\mathbb{P}_r(K) \subset V_{h,r}^2\), for \(r \geq 2p - 1\);
(iii) the choice \((D4')\) instead of \((D4)\) still guarantees that the associated global space is made of \(C^{p-1}\) functions, but now \((D1)-(D4')\) can be employed to solve a differential problem involving the \(\Delta^{p-t}\) operator by employing \(C^{p-1}(\Omega)\) basis functions. For instance:
(a) Choosing \(p\) and \(t\) such that \(p - t = 1\) we obtain \(C^{p-1}\) conforming VEM for the solution of the Laplacian problem. For example, for \(p = 3, t = 2\) and \(r = 5\), the local space \(V_{h,5}^2(K)\) endowed with the corresponding degrees of freedom \((D1) - (D4')\) can be employed to build a global space made of \(C^2\) functions for the approximation of the Laplace problem. It is worth mentioning that the new choice \((D4')\) (differently from the original choice \((D4)\)) is essential for the computability of the elliptic projection (see (18)-(19)) with respect to the bilinear form \(a^K(\cdot, \cdot) = \int_K \nabla(\cdot)\nabla(\cdot)\,dx\).
(b) Choosing \(p\) and \(t\) such that \(p - t = 2\) we get \(C^{p-1}\) conforming VEM for the solution of the Biharmonic problem. For example, for \(p = 3, t = 1\) and \(r = 5\), similarly to the previous case, the space \(V_{h,5}^4(K)\) together with \((D1) - (D4')\) gives rise to a global space made of \(C^2\) functions that can be employed for the solution of the biharmonic problem. Again, the particular choice \((D4')\) makes possible the computability of the elliptic projection with respect to the bilinear form \(a^K(\cdot, \cdot) = \int_K \Delta(\cdot)\Delta(\cdot)\,dx\).

4. Convergence analysis

4.1. Mesh regularity and polynomial interpolation error estimates

We consider the following mesh regularity assumptions:

\textbf{(M)} There exists a positive constant \(\gamma\) independent of \(h\) (and \(K\)) such that \(\{\Omega_h\}\):

(i) \(K\) is star-shaped with respect to every point of a ball of radius \(\gamma h_K\), where \(h_K\) is the diameter of \(K\);
(ii) for every edge \(e\) of the cell boundary \(\partial K\) of every cell \(K\) of \(\Omega_h\), it holds that \(h_e \leq \gamma h_K\), where \(h_e\) denotes the length of \(e\).

We refer to \(\gamma\) as the mesh regularity constant.

In view of assumptions \textbf{M(i)}-\textbf{M(ii)} on \(\Omega_h\), we define, for every smooth enough function \(w\) the Virtual Element interpolant \(w^I\), which is the function in \(V_{h,r}\) uniquely identified by the same degrees of freedom of \(w\). More precisely, if \(\chi_i(w)\) denotes the \(i\)-th global degree of freedom of \(w\), there exists a unique Virtual Element function \(w^I \in V_{h,r}\) such that \(\chi_i(w - w^I) = 0\). Combining the Bramble-Hilbert Lemma and scaling arguments as in the finite element framework (see, e.g., [14]and [19]) we can prove that for every \(K \in \Omega_h\) and every function \(w \in H^2(K)\), it holds
\[
\|w - w^I\|_{s,K} \leq Ch_K^{\min(\beta,r+1)-s}\|w\|_{\beta,K} \quad s = 0, 1, \ldots, p
\] (35)
for some positive constant \(C\) independent of \(h\).
Under the same assumptions and using similar techniques, we can also prove that the existence of a piecewise polynomial approximation \( w_\pi \in \mathcal{P}_\ell(\Omega_h) \) such that the local estimate holds

\[
\|w - w_\pi\|_{s,K} \leq C h_{K}^{\min(\beta,f+1) - s} \|w\|_{\beta,K} \quad s = 0, 1, \ldots, p, \; 1 \leq \beta \leq \ell + 1,
\]

for some positive constant \( C \) independent of \( h \) and every mesh element \( K \). The elliptic projection \( \Pi_{h}^{\nabla,K} w \) and the \( L^2 \)-orthogonal projection \( \Pi_{h}^{0,K} w \) of \( w \) are both instances of \( w_\pi \) for which estimate (36) holds.

### 4.2. Convergence in the energy norm

By using standard estimates for the interpolation error, we can derive the convergence rate of the approximation error in the energy norm. First, we need a technical lemma that estimates the approximation error of the load term.

**Lemma 4.1** Consider a function \( f \in H^{r-(p-1)}(\Omega) \) and its \( L^2 \)-orthogonal projection on the space of polynomials of degree \( r-p \), denoted by \( f_h = \Pi_{r-p}^{0,K} f \). Then, there exists a positive constant \( C \), which is independent of \( h \), such that

\[
\langle f - f_h, v_h \rangle \leq C h^{r+1} |f|_{r-(p-1)} |v_h|_p \quad \forall v_h \in V_{h,r}^p.
\]

**Proof.** First, we note that \( (I - \Pi_{r-p}^{0,K} f) \) is orthogonal to the polynomials of degree \( r-p \) (recall that \( r \geq 2p-1 \)) and that \( v_h \) belongs to \( H_{h}^{r,p}(\Omega) \). We employ the Cauchy-Schwarz inequality (twice) and use (35) (with \( w = f \) and \( w_\pi = \Pi_{r-p}^{0,K} f \) ) to obtain the estimate:

\[
\langle f - f_h, v_h \rangle = \sum_{K \in \Omega_h} \int_{K} (I - \Pi_{r-p}^{0,K} f) (I - \Pi_{r-p}^{0,K} v_h) \, dx \leq \sum_{K \in \Omega_h} \left\| (I - \Pi_{r-p}^{0,K} f) \right\|_{0,K} \left\| (I - \Pi_{r-p}^{0,K} v_h) \right\|_{0,K}
\]

\[
\leq C \sum_{K \in \Omega_h} h_{K}^{r-(p-1)} |f|_{r-(p-1),K} h_{K}^{p} |v_h|_{p,K} \leq C h^{r+1} |f|_{r-(p-1)} |v_h|_p,
\]

where \( C \) denotes a positive constant independent of \( h \). \( \square \)

**Theorem 4.2** Let \( f \in H^{r+p+1}(\Omega) \) and let \( u \) be the solution of the variational problem (4) and \( u_h \in V_{h,r}^p \) be the solution of the Virtual Element problem (7). Under the mesh regularity assumption (M), we find that

\[
\|u - u_h\|_V \leq C h^{r-(p-1)} (|u|_{r+p+1} + |f|_{r-p+1}).
\]

**Proof.** The assertion of the theorem follows by estimating each term of the right-hand side of (10) separately and using interpolation estimate (35) and Lemma 4.1, cf. inequality (37). \( \square \)

### 4.3. Convergence in lower order norms

We first prove a technical lemma that will be useful in the error analysis of the next subsections. **Lemma 4.3** Let \( f \in H^{r-p+1}(\Omega) \) and let \( u \) be the solution of the variational problem (4) and \( u_h \in V_{h,r}^p \) be the solution of the Virtual Element problem (7). Then, for any function \( \psi \in H^\beta(\Omega) \cap H_0^1(\Omega) \) (\( \beta > p \)) it holds that

\[
a(\psi, u - u_h) \leq C h^{(r-p-1) + \min(\beta,r+1) - p} (|u|_{r+p+1} + |f|_{r-p+1}) \|\psi\|_\beta,
\]

for some positive constant \( C \) independent of \( h \).

**Proof.** To derive (39), we add and subtract the Virtual Element interpolant of \( \psi \) denoted by \( \psi_I \) to the left-hand side of (39), and, then, use (3) and (7), and obtain:

\[
a(\psi, u - u_h) = a(u - u_h, \psi - \psi_I) + a(u - u_h, \psi_I)
\]

\[
= a(u - u_h, \psi - \psi_I) + \langle f - f_h, \psi_I \rangle + a_h(u_h, \psi_I) - a(u_h, \psi_I)
\]

\[
= T_1 + T_2 + T_3.
\]

11
To estimate term $T_1$, we use the continuity of $a(\cdot,\cdot)$ with respect to the norm $|\cdot|_V = |\cdot|_p$, the estimate in the energy norm (38), and interpolation error estimate (35) ($s = p$)

$$T_1 \leq \|u - u_h\|_V \|\psi - \psi_I\|_V \leq C h^{r-(p-1)} \left( |u|_{r+p+1} + |f|_{r-(p-1)} \right) |\psi - \psi_I|_p \tag{41}$$

$$\leq C h^{r-(p-1)+\min(\beta,r+1)-p} \left( |u|_{r+p+1} + |f|_{r-(p-1)} \right) |\psi|_\beta. \tag{42}$$

To estimate term $T_2$, we first note that $(I - \Pi_{r-\frac{1}{2}}^0)f$ is orthogonal to the polynomials of global degree up to $r - p$ defined on $K$ and that $\psi_I \in H^p(\Omega)$. Then, we apply the Cauchy-Schwarz inequality (twice) and obtain:

$$(f - f_h, \psi_I) = \sum_{K \in \mathcal{K}} (I - \Pi_{r-\frac{1}{2}}^0)f (I - \Pi_{p-1}^0)\psi_I dx \leq \sum_{K \in \mathcal{K}} \left\| (I - \Pi_{r-\frac{1}{2}}^0)f \right\|_{0,K} \left\| (I - \Pi_{p-1}^0)\psi_I \right\|_{0,K}$$

$$\leq \left( \sum_{K \in \mathcal{K}} \left\| (I - \Pi_{r-\frac{1}{2}}^0)f \right\|_{0,K}^2 \right)^{\frac{1}{2}} \left( \sum_{K \in \mathcal{K}} \left\| (I - \Pi_{p-1}^0)\psi_I \right\|_{0,K}^2 \right)^{\frac{1}{2}}.$$

The first term on the right is bounded by using local estimate (36):

$$\left\| (I - \Pi_{r-\frac{1}{2}}^0)f \right\|_{0,K} \leq C h^{r-(p-1)} |f|_{r-(p-1),K}. $$

The second term on the right is transformed by applying estimate (36) (with $w = \psi_I$ and $\psi = \Pi_{p-1}^0\psi_I$), adding and subtracting $\psi$ and applying estimate (35):

$$\left\| (I - \Pi_{p-1}^0)\psi_I \right\|_{0,K} \leq C h^p |\psi_I|_{p,K} \leq C h^p (|\psi_I|_{p,K} + |\psi - \psi_I|_{p,K}) \leq C h^p (|\psi_I|_{p,K} + h^{\min(\beta,r+1)-p} |\psi|_{\beta,K}).$$

Collecting all the local terms, using the Cauchy-Schwarz inequality, and the assumption that $\beta \geq p$ (so $h^p \geq h^{\min(\beta,r+1)}$) yields:

$$T_2 \leq C h^{r-(p-1)} |f|_{r-(p-1)} \left(h^p |\psi|_{p} + h^{p+\min(\beta,r+1)-p} |\psi|_{\beta} \right) \leq C h^{r+1} |f|_{r-(p-1)} |\psi|_{\beta}. \tag{43}$$

To bound $T_3$, we first split it in the summation of local terms. Then, we use the $r$-consistency and stability property of $a_h$, and the continuity property of $a$ and $a_h$, and we obtain

$$T_3 = \sum_{K \in \mathcal{K}_h} \left( a_h^K(u_h, \psi_I) - a^K(u_h, \psi_I) \right)$$

$$= \sum_{K \in \mathcal{K}_h} \left( a_h^K(u_h - u_\pi, \psi_I) - a^K(u_h - u_\pi, \psi_I) \right)$$

$$= \sum_{K \in \mathcal{K}_h} \left( a_h^K(u_h - u_\pi, \psi_I - \psi_\pi) - a^K(u_h - u_\pi, \psi_I - \psi_\pi) \right)$$

$$\leq C |u_h - u_\pi|_V \|\psi_I - \psi_\pi\|_V. \tag{44}$$

Adding and subtracting $u$ and using the estimate in the energy norm (38) and the estimate for the polynomial interpolation (36), we find that

$$\|u_h - u_\pi\|_V \leq \|u_h - u\|_V + \|u - u_\pi\|_V \leq C h^{r-(p-1)}. \tag{45}$$

Adding and subtracting $\psi_\pi$ and, then, using estimates (35) and (36) we find that

$$\|\psi_I - \psi_\pi\|_V \leq \|\psi_I - \psi\|_V + \|\psi - \psi_\pi\|_V \leq C h^{\min(\beta,r+1)-p} |\psi|_{\beta}. \tag{46}$$

The bound on $T_3$ following by using (45) and (46) in (44):

$$T_3 \leq C h^{(r-(p-1)) + \min(\beta,r+1)-p} |\psi|_{\beta}. \tag{47}$$

The assertion of the lemma follows by substituting (42), (43), and (47), in (40).
In view of this lemma, we can readily state and prove the convergence theorems for the four possible combinations of even and odd \( p \) and even and odd norm indices.

**Theorem 4.4 (Even \( p \), even norms)** Let \( f \in H^{r-p+1}(\Omega) \) and let \( u \) be the solution of the variational problem (4) with \( p = 2\ell \) and \( u_h \) be the solution of the Virtual Element method (7). Then, there exists a positive constant \( C \) independent of \( h \) such that

\[
|u - u_h|_{2i} \leq Ch^{r+1-2i} \left( |u|_{r+p+1} + |f|_{r-(p-1)} \right),
\]

for every integer \( i = 0, \ldots, \ell - 1 \).

**Proof.** For \( i = 0, \ldots, \ell - 1 \), let \( \psi \in H^{2\ell} \cap H^p \) be the solution of the problem

\[
\Delta^{p-i} \psi = \Delta^i(u - u_h) \in L^2(\Omega),
\]

with the stability property

\[
\|\psi\|_{2(p-i)} \leq C |u - u_h|_{2i},
\]

We use (49) and integrate by parts to obtain:

\[
|u - u_h|_{2i}^2 = \|\Delta^i(u - u_h)\|^2_0 = (\Delta^i(u - u_h), \Delta^i(u - u_h)) = (\Delta^{p-i} \psi, \Delta^i(u - u_h))
\]

where we employed the fact that \( |u|_{2i} = \|\Delta^i v\|_0 \) for any \( v \in H^p \). The assertion of the theorem follows from an application of Lemma 4.3 (use \( \beta = 2(p-i) \) together with \( r \geq 2p - 1 \) and the stability property (50). \( \square \)

**Theorem 4.5 (Even \( p \), odd norms)** Let \( f \in H^{r-p+1}(\Omega) \) and let \( u \) be the solution of the variational problem (4) with \( p = 2\ell \) and \( u_h \) be the solution of the Virtual Element method (7). Then, there exists a positive constant \( C \) independent of \( h \) such that

\[
|u - u_h|_{2i+1} \leq Ch^{r+1-(2i+1)} \left( |u|_{r+p+1} + |f|_{r-(p-1)} \right),
\]

for every integer \( i = 0, \ldots, \ell - 1 \).

**Proof.** For \( i = 0, \ldots, \ell - 1 \), let \( \psi \in H^{2\ell} \cap H^p \) be the solution of the problem:

\[
-\Delta^{p-i} \psi = \Delta^{i+1}(u - u_h) \in H^{-1}(\Omega),
\]

with the stability property

\[
\|\psi\|_{2(p-i)-1} \leq C |u - u_h|_{2i+1}.
\]

We use (52) and integrate by parts to obtain:

\[
|u - u_h|_{2i+1}^2 = \|\nabla \Delta^i(u - u_h)\|^2_0 = (\nabla \Delta^i(u - u_h), \nabla \Delta^i(u - u_h))
\]

where we employed the fact that \( |v|_{2i+1} = \|\nabla \Delta^i v\|_0 \) for any \( v \in H^p \). The assertion of the theorem follows from an application of Lemma 4.3 (use \( \beta = 2(p-i) \) together with \( r \geq 2p - 1 \) and the stability property (53). \( \square \)

**Theorem 4.6 (Odd \( p \), even norms)** Let \( u \) be the solution of the variational problem (4) and let \( u_h \) be the solution of the Virtual Element method (7). Then, there exists a positive constant \( C \) independent of \( h \) such that

\[
|u - u_h|_{2i} \leq Ch^{r+1-2i} \left( |u|_{r+p+1} + |f|_{r-(p-1)} \right),
\]

for every integer \( i = 0, \ldots, \ell - 1 \).
Proof. For $i = 0, \ldots, \ell$, let $\psi \in H^{2(p-i)}(\Omega) \cap H_0^{p-i}(\Omega)$ be the solution of the problem
\begin{equation}
-\Delta^{p-i}\psi = \Delta^i(u - u_h) \in L^2(\Omega)
\end{equation}
with the stability property
\begin{equation}
\|\psi\|_{2(p-i)} \leq C |u - u_h|_{2i}.
\end{equation}
We use (58) and integrate by parts to obtain:
\begin{equation}
|u - u_h|_{2i}^2 = \|\Delta^i(u - u_h)\|_0^2 = (\Delta^i(u - u_h), \Delta^i(u - u_h))
\end{equation}
\begin{equation}
= (-\Delta^{p-i}\psi, \Delta^i(u - u_h)) = (-\Delta^{i+1}\psi, \Delta^i(u - u_h))
\end{equation}
\begin{equation}
= (\nabla \Delta^i\psi, \nabla \Delta^i(u - u_h)) = a(\psi, u - u_h).
\end{equation}
The assertion of the theorem follows from an application of Lemma 4.3 (use $\beta = 2(p - i)$ together with $r \geq 2p - 1$) and the stability property (56).

\begin{thm}[Odd $p$, odd norms] Let $u$ be the solution of the variational problem (4) and let $u_h$ be the solution of the Virtual Element method (7). Then, there exists a positive constant $C$ independent of $h$ such that
\begin{equation}
|u - u_h|_{2i+1} \leq C h^{(r+1)-(2i+1)} \left( |u|_{r+p+1} + |f|_{r-(p-1)} \right),
\end{equation}
for every integer $i = 0, \ldots, \ell - 1$.
\end{thm}

Proof. For $i = 0, \ldots, \ell$, let $\psi \in H^{2(p-i)-1}(\Omega) \cap H_0^p(\Omega)$ be the solution of the problem:
\begin{equation}
-\Delta^{p-i}\psi = \Delta^{i+1}(u - u_h) \in H^{-1}(\Omega),
\end{equation}
with the stability property
\begin{equation}
\|\psi\|_{2(p-i)-1} \leq C |u - u_h|_{2i+1}.
\end{equation}
We use (58) and integrate by parts to obtain:
\begin{equation}
|u - u_h|_{2i+1}^2 = \|\nabla \Delta^i(u - u_h)\|_0^2 = (\nabla \Delta^i(u - u_h), \nabla \Delta^i(u - u_h))
\end{equation}
\begin{equation}
= (\nabla \Delta^{i+1}(u - u_h), \nabla \Delta^i(u - u_h)) = (-\Delta^{p-i}\psi, \Delta^i(u - u_h)) = (-\Delta^{i+1}\psi, \Delta^i(u - u_h))
\end{equation}
\begin{equation}
= (\nabla \Delta^i\psi, \nabla \Delta^i(u - u_h)) = a(\psi, u - u_h)
\end{equation}
where again we employed the fact that $|v|_{2i+1} = \|\nabla \Delta^i v\|_0$ for any $v \in H_0^p(\Omega)$. The assertion of the theorem follows from an application of Lemma 4.3 (use $\beta = 2(p - i) - 1$ together with $r \geq 2p - 1$) and the stability property (59).

5. Conclusions

In this paper, we developed the conforming Virtual Element discretization of arbitrary order for polyharmonic problems, which requires the discretization of operator like $\Delta^p u$ for integer $p \geq 1$. To this end, we introduced local and global Virtual Element approximation spaces together with suitable discrete bilinear forms for odd and even $p$. The convergence of the method has been proved and optimal error estimates derived in suitable norms. The numerical implementation of the current method deserves a careful study because of the severe ill-conditioning of the polyharmonic differential operator and the need of high order polynomials, whose degree should be at least 5 in the simplest case $p = 3$. For this reason, the implementation of this conforming VEM is under investigation and will be addressed in a forthcoming publication.
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References

[1] B. Ahmad, A. Alsaedi, F. Brezzi, L. Marini, and A. Russo. Equivalent projectors for Virtual Element Method. *Computers & Mathematics with Applications*, 66(3):376–391, 2013.

[2] P. F. Antonietti, L. Beirão Da Veiga, D. Mora, and M. Verani. A stream Virtual Element formulation of the Stokes problem on polygonal meshes. *SIAM J. Numer. Anal.*, 52(1):386–404, 2014.

[3] P. F. Antonietti, L. Beirão Da Veiga, S. Scacchi, and M. Verani. A $C^1$ Virtual Element Method for the Cahn-Hilliard equation with polygonal meshes. *SIAM J. Numer. Anal.*, 54(1):34–56, 2016.

[4] P. F. Antonietti, N. Bigoni, and M. Verani. Mimetic discretizations of elliptic control problems. *Journal of Scientific Computing*, 56(1):14–27, 2013.

[5] P. F. Antonietti, F. Brezzi, and L. D. Marini. Bubble stabilization of Discontinuous Galerkin Methods. *Comput. Methods Appl. Mech. Engrgy.*, 198(21-26):1651–1659, 2009.

[6] P. F. Antonietti, M. Bruggi, S. Scacchi, and M. Verani. On the Virtual Element Method for topology optimization on polygonal meshes: A numerical study. *Computers and Mathematics with Applications*, 74(5):1091–1109, 2017.

[7] P. F. Antonietti, A. Cangiani, J. Collis, Z. Dong, E. H. Georgoulis, S. Giani, and P. Houston. Review of Discontinuous Galerkin finite element methods for partial differential equations on complicated domains. G.R. Barrenechea et al. (eds.), Building Bridges: Connections and Challenges in Modern Approaches to Numerical Partial Differential Equations, Lecture Notes in Computational Science and Engineering 114:279–307, 2016.

[8] P. F. Antonietti, L. Formaggia, A. Scotti, M. Verani, and N. Verzotti. Mimetic finite difference approximation of flows in fractured porous media. *M2AN Math. Model. Numer. Anal.*, 50(3):809–832, 2016.

[9] P. F. Antonietti, G. Manzini, and M. Verani. The fully nonconforming Virtual Element method for biharmonic problems. *Mathematical Models & Methods in Applied Sciences*, 28(2), 2018.

[10] P. F. Antonietti, L. Mascotto, and M. Verani. A multigrid algorithm for the $p$-version of the Virtual Element method. *M2AN Math. Model. Numer. Anal.*, 52(1):337–364, 2018.

[11] P. F. Antonietti, G. Pennesi, and P. Houston. Fast numerical integration on polytopic meshes with applications to discontinuous Galerkin finite element methods. *J. Sci. Comput.*, 2018.

[12] B. Ayuso de Dios, K. Lipnikov, and G. Manzini. The nonconforming Virtual Element Method. *ESAIM Math. Model. Numer. Anal.*, 50(3):879–904, 2016.

[13] J. W. Barrett, S. Langdon, and R. Nürnberg. Finite element approximation of a sixth order nonlinear degenerate parabolic equation. *Numer. Math.*, 96(3):401–434, 2004.

[14] L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. D. Marini, and A. Russo. Basic principles of virtual element methods. *Mathematical Models & Methods in Applied Sciences*, 23(1):119–214, 2013.

[15] L. Beirão da Veiga and G. Manzini. A virtual element method with arbitrary regularity. *IMA Journal on Numerical Analysis*, 34(2):782–799, 2014. DOI: 10.1093/imanum/drt018, (first published online 2013).

[16] L. Beirão da Veiga, K. Lipnikov, and G. Manzini. The Mimetic Finite Difference Method for elliptic problems, volume 11 of MS&A. Modeling, Simulation and Applications. Springer, Cham, 2014.

[17] M. F. Benedetto, S. Berrone, S. Pieraccini, and S. Scialò. The Virtual Element Method for discrete fracture network simulations. *Comput. Meth. Appl. Mech. Engrgy.*, 280:135–156, 2014.

[18] J. H. Bramble and R. S. Falk. A mixed-Lagrange multiplier finite element method for the polyharmonic equation. *RAIRO Modél. Math. Anal. Numér.*, 19(4):519–557, 1985.

[19] S. Brenner and L. Scott. *The Mathematical Theory of Finite Element Methods*. Springer-Verlag, Berlin/Heidelberg, 1994.

[20] F. Brezzi, A. Buffa, and K. Lipnikov. Mimetic finite differences for elliptic problems. *ESAIM: Mathematical Modelling and Numerical Analysis*, 43(2):277–295, 2009.

[21] F. Brezzi, R. S. Falk, and L. D. Marini. Basic principles of mixed Virtual Element Methods. *ESAIM Math. Model. Numer. Anal.*, 48(4):1227–1240, 2014.

[22] F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of the mimetic finite difference method for diffusion problems on polyhedral meshes. *SIAM J. Numer. Anal.*, 43(5):1872–1896 (electronic), 2005.
[23] F. Brezzi and L. D. Marini. Virtual Element Methods for plate bending problems. *Comput. Methods Appl. Mech. Engrg.*, 253:455–462, 2013.

[24] A. Cangiani, Z. Dong, E. Georgoulis, and P. Houston. hp-version Discontinuous Galerkin Methods for advection-diffusion-reaction problems on polytopic meshes. *M2AN Math. Model. Numer. Anal.*, 50(3):699–725, 2016.

[25] A. Cangiani, E. Georgoulis, T. Pryer, and O. Sutton. A posteriori error estimates for the virtual element method. *Numerische Mathematik*, 137(4):857–893, 2017.

[26] A. Cangiani, E. H. Georgoulis, and P. Houston. hp-Version discontinuous Galerkin methods on polygonal and polyhedral meshes. *Math. Models Methods Appl. Sci.*, 24(10):2009–2041, 2014.

[27] A. Cangiani, V. Gyrya, and G. Manzini. The nonconforming Virtual Element Method for the Stokes equations. *SIAM J. Numer. Anal.*, 54(6):3411–3435, 2016.

[28] A. Cangiani, V. Gyrya, G. Manzini, and O. Sutton. Chapter 14: Virtual element methods for elliptic problems on polygonal meshes. In K. Hormann and N. Sukumar, editors, *Generalized Barycentric Coordinates in Computer Graphics and Computational Mechanics*, pages 1–20. CRC Press, Taylor & Francis Group, 2017.

[29] A. Cangiani, G. Manzini, A. Russo, and N. Sukumar. Hourglass stabilization of the virtual element method. *International Journal on Numerical Methods in Engineering*, 102(3-4):404–436, 2015.

[30] A. Cangiani, G. Manzini, and O. Sutton. Conforming and nonconforming virtual element methods for elliptic problems. *IMA Journal of Numerical Analysis*, 37(3):1317–1354, 2017.

[31] B. Cockburn, B. Dong, and J. Guzmán. A superconvergent LDG-Hybridizable Galerkin Method for second-order elliptic problems. *Math. Comp.*, 77(264):1887–1916, 2008.

[32] D. A. Di Pietro, A. Ern, and S. Lemaire. An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators. *Comput. Methods Appl. Math.*, 14(4):461–472, 2014.

[33] J. Droniou. Finite volume schemes for diffusion equations: Introduction to and review of modern methods. *Mathematical Models and Methods in Applied Sciences*, 24(08):1575–1619, 2014.

[34] R. Eymard, C. Guichard, and R. Herbin. Small-stencil 3D schemes for diffusive flows in porous media. *ESAIM Math. Model. Numer. Anal.*, 46(2):265–290, 2012.

[35] M. Frittelli and I. Sgura. Virtual Element Method for the Laplace Beltrami equation on surfaces, 2018. DOI=10.1051/m2an/2017040 (arXiv:1612.02369v1).

[36] D. Gallistl. Stable splitting of polyharmonic operators by generalized Stokes systems. *Math. Comp.*, 86(308):2555–2577, 2017.

[37] F. Gardini, G. Manzini, and G. Vacca. The nonconforming virtual element method for eigenvalue problems, 2018. arXiv:1802.02942v1.

[38] F. Gardini and G. Vacca. Virtual element method for second order elliptic eigenvalue problems. *IMA Journal of Numerical Analysis*, 2017. Preprint, arXiv:1610.03675.

[39] F. Gazzola, H.-C. Grunau, and G. Sweers. *Polyharmonic boundary value problems*, volume 1991 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2010. Positivity preserving and nonlinear higher order elliptic equations in bounded domains.

[40] T. Gudi and M. Neilan. An interior penalty method for a sixth-order elliptic equation. *IMA J. Numer. Anal.*, 31(4):1734–1753, 2011.

[41] A. Miranville. Sixth-order Cahn-Hilliard systems with dynamic boundary conditions. *Math. Methods Appl. Sci.*, 38(6):1127–1145, 2015.

[42] M. Schedensack. A new discretization for m-th-Laplace equations with arbitrary polynomial degrees. *SIAM J. Numer. Anal.*, 54(4):2138–2162, 2016.

[43] N. Sukumar and A. Tabarraei. Conforming Polygonal Finite Elements. *Internat. J. Numer. Methods Engrg.*, 61(12):2045–2066, 2004.

[44] G. Vacca. An $H^1$-conforming Virtual Element for Darcy and Brinkman equations. *Math. Models Methods Appl. Sci.*, 28(1):159–194, 2018.

[45] M. Wang and J. Xu. Minimal finite element spaces for 2m-th-order partial differential equations in $\mathbb{R}^n$. *Math. Comp.*, 82(281):25–43, 2013.

[46] S. Wise, J. S. Lowengrub, J. S. Kim, K. Thornton, P. W. Voorhees, and W. C. Johnson. Quantum dot formation on a strain-patterned epitaxial thin film. *Applied Physics Letters*, 87(13):133102, 2005.

[47] J. Zhao, S. Chen, and B. Zhang. The nonconforming Virtual Element Method for plate bending problems. *Math. Models Methods Appl. Sci.*, 26(9):1671–1687, 2016.