Analytic study of clustering in shaken granular material using zero-range processes

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We show that models used to described granular clustering due to vertical shaking belong to the class of zero-range processes. This correspondence allows us to derive exactly in a very easy and straightforward manner a number of properties of the models like particle distribution functions, phase diagram, and characteristic time of clusterization.

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I. INTRODUCTION

Non-equilibrium phase transitions were observed in many simple systems [1, 2, 3]. Recently, it was also found that shaken granular material exhibits clustering depending on the shaking strength [4]. In his paper Jens Eggers suggested a model for the description of the clustering of vertically shaken granular material. Originally, it was introduced for the two box setup. Later it and its modified version were used to describe experiments having more boxes [5, 6, 7, 8, 9, 10, 11].

The analytic studies of the above mentioned papers were difficult and in many cases did not give general results. In the case of exclusion models, it was proved [12] that the correspondence to an already solved model, namely the zero-range process [11] can be of great help, as many results can be obtained directly.

The aim of this paper to derive analytically for the general case the probability distribution, phase diagram and the coarsening time using the above correspondence. We note that specific cases with small number of boxes are already solved [3, 7, 9, 10, 11].

II. THE MODEL

The experimental setup looks as follows. The system consists of a container separated into boxes by walls which are open upwards. The whole system is then vertically shaken and particles can hop above the walls from one compartment to the other. The model of Eggers [4] defines particle fluxes between the boxes based upon the physical properties of the system like shaking strength and local particle density, etc.. Two different steady states were found, both in the experiments and in the model: a homogeneous where the containers hold roughly equal number of particles, and a condensed steady state where one compartment contains nearly all particles.

We first define the zero-range process following [12]: We consider a one-dimensional finite lattice of sites and periodic boundary conditions. The total particle number is denoted by N.

The dynamics of the system is given by rates u(n), at which a particle leaves a site. The hopping rates u(n) depend only on the number of particles on the site of departure and external parameters but independent of the properties of the target site. Here, we consider only the symmetric case when the hopping to the left or to the right is equally probable.

The important attribute of the zero-range process is that it yields a steady state described by a product measure. By this it is meant that the steady state probability \( P(\{n_\mu\}) \) of finding the system in configuration \( \{n_1, n_2, \ldots, n_L\} \) is given by a product of factors \( f(n_\mu) \) that are called marginals

\[
P(\{n_\mu\}) = \frac{1}{Z(L, N)} \prod_{\mu=1}^{L} f(n_\mu),
\]

where \( Z(L, N) \) is the normalization factor. For the zero-range process \( f(n_\mu) \) is given by

\[
f(n) = \begin{cases} 
\frac{1}{m} & \text{for } n \geq 1 \\
1 & \text{for } n = 0
\end{cases}
\]

The marginals are defined up to a multiplicative factor.

It is important to note that the steady state particle distribution can be formally read off generally for all \( N \) and \( L \). This will allow us to calculate the general phase diagrams.

The Eggers model [4] has a quadratic temperature dependence while Lipowski and Droz [5] uses a linear form. For all quantities we use an index \( E \) for Eggers model and \( LD \) for the Lipowski-Droz model. The fluxes are defined in the following way

\[
\begin{align*}
   u_E(n) &= A n^2 e^{-B n^2} \\
   u_{LD}(n) &= \frac{n}{N} \exp\left(-1/[T_0 + \Delta(1 - n/N)]\right)
\end{align*}
\]

where \( A, B, T_0, \) and \( \Delta \) contain physical quantities. It is of importance that \( B \) is proportional to the inverse shaking strength while \( \Delta \) increases with the strength of the agitation. The \( u(n) \) functions are compared on Fig. 1.

Our main purpose in this paper is to study the clustering process of this model. Furthermore, as cases with \( L = 2, 3 \) [5, 6, 7, 8, 9, 10, 11] are already solved we focus on the general case with at least moderate \( L \) and large \( N \gg 1 \) particle number.
III. PARTICLE DISTRIBUTION

Before calculating the general particle distribution function we remark that in both models only two different steady states may be stable: (i) The symmetric, where all sites contain the same amount of particles, and (ii) The asymmetric, or condensed steady state, where one site contains more particles than the others which have equal number of particles.

The reason why only those two are stable is that in the steady state all sites must have the same flux. The hopping probability curve has a maximum thus sites may be on both sides of the maximum at the positions with equal flux. However, when the flux has negative derivate only one site may be at a given position because more would be unstable against fluctuations.

Thus we calculate only the probability distribution function for the case where \( L - 1 \) sites contain the same number of particles \( a \) the remaining site thus holds \( b = N - (L - 1)a \) particles. The marginals take the form

\[
\begin{align*}
  f_E(n) &= \frac{\exp\left(\frac{E}{n+1}(2n+1)\right)}{A(n)^2} \\
  f_{LD}(n) &= \frac{N!}{a!} \left[ T_0 + \Delta \left( 1 - \frac{n}{N} \right) \right]^{-N/\Delta}.
\end{align*}
\]

We are lucky that for both models the unnormalized probability distribution can be obtained in a closed form:

\[
\begin{align*}
  P_E(a, \ldots, a, b) &= \frac{\exp[B(\frac{a}{b-1}a^3 + \frac{b}{b-1}a^2 + \frac{b}{b-1}a)]}{A^2 a!^2 b!^2} 	imes \exp[B(b^3/3 + b^2/2 + b/6)] \\
  P_{LD}(a, \ldots, a, b) &= \frac{N!}{a!(L-1)!} \left[ T_0 + \Delta \left( 1 - \frac{a}{N} \right) \right]^{-N(L-1)/\Delta} 	imes \left[ T_0 + \Delta \left( 1 - \frac{b}{N} \right) \right]^{-N/\Delta}.
\end{align*}
\]

IV. PHASE DIAGRAM

We already discussed that both models have two possible stable steady states which give rise to four different cases on the phase diagram: I) only the symmetric state is stable, II) only the asymmetric state is stable, III) both states are stable but asymmetric is more probable, IV) both states are stable but symmetric is more probable. In this section we determine the three lines separating these states.

On line III-IV the symmetric solution loses its stability. In the symmetric steady state all sites contain \( N/L \) particles. Only a flux curve with positive derivate may be stable for many sites thus \( N/L \) must be less than the position of the maximum of the flux curve. This gives the following two relations

\[
\begin{align*}
  (E) &\quad B = L^2/N^2 \\
  (LD) &\quad T_0 = \sqrt{\Delta/L} - \Delta(1-1/L).
\end{align*}
\]

The other two line can only be determined through the probability distribution function.

A. The LD model

Let us define the particle ratios: \( x = a/N, \beta = b/N \). The following equation describes the extremal points of the probability distribution function

\[
\ln \frac{\beta}{x} = \frac{1}{T_0 + \Delta (1 - \beta)} - \frac{1}{T_0 + \Delta (1 - x)}.
\]

It is obvious to see that \( \beta = x \) is always a solution and that the above equation is \( N \) independent. The exact position of the clusterized state can only be determined numerically.

An important question is whether the transition from symmetric to asymmetric steady state is continuous or not. The third order Taylor expansion of Eq. 7 around the symmetric solution of \( x = 1/L \) shows that if \( L > 2 \) no continuous transition is possible as there is no solution of the equation. While for \( L = 2 \) continuous solution is possible if \( \Delta < 2/3 \). This result was also shown with a general argument in [10]. It is important to reiterate because in the following calculations are valid for \( x \ll 1 \) which can only be used in case of discontinuous transitions.

The line I-IV, the limit of the stability of the asymmetric steady state can be derived from Eq. 7. Assuming that \( x \ll 1 \) and the fact that at the point where other roots except from \( \beta = x \) appear the value and the derivate of both sides of the equation should be equal. This results in the following implicit equation

\[
0 = \ln \left( \frac{\Delta(L - 1)}{T_0^2} + \frac{\Delta}{(T_0 + \Delta)^2} \right) + \frac{T_0^2 + \Delta(T_0 - 1)}{T_0(T_0 + \Delta)}.
\]
Line I-IV can be determined from the above results
\[ BN^2 = 8 \ln L + \mathcal{O}(1). \]  
(11)

Line III-IV can be determined from the equality of the probability distribution functions as
\[ BN^2 = 6 \ln L + \mathcal{O}(1). \]  
(12)

The above two equations suffer from strong corrections to the leading order \( \ln L \) term. However, this accuracy is enough to get a clear picture of the phase diagram which is shown on Fig. 2b.

V. COARSENING TIME

The coarsening process in general drives the system to have bigger but a smaller number of macroscopic clusters until only one prevails. As the clusters become bigger the coarsening process slows down enormously because the size of the clusters to be dissolved is bigger and the relative flux is much smaller. Thus, in general it is enough to study the competition of the last two big clusters to get the asymptotic behaviour of the coarsening time \( \tau \).

The above picture applies only if the if the homogeneous steady state is not stable. In the opposite case, where the symmetric steady state is also stable a fluctuation large enough is needed to drive the system out of the potential well for which the time needed is in general proportional to \( \exp(-N^2) \). Here we study only the case when the system is in section II of the phase space.

For obvious reasons, the two big clusters are generally at \( \sim L/2 \) distance. If a big cluster looses a particle to its neighbour the particle starts a random walk. If the distance between the two clusters is \( L/2 \) then the probability of reaching the other is \( 2/L \) and it takes \( L^2/4 \) timesteps. Thus the coarsening time can be written as
\[ \tau = \frac{L^3}{8} \int_{1}^{N/2-x_1} \frac{dn}{u(N/2-n) - u(N/2+n)} \]  
(13)

The coarsening time is to be dominated by the state when both clusters have almost the same number of particles. Thus, we use the first order Taylor series around this point:
\[ \tau_{LD} \approx L^3 \exp(2/(2T_0 + \Delta))(2T_0 + \Delta)^2 N \ln(N/2) \]
\[ \tau_E \approx \frac{L^3 \exp(BN^2/4)}{AN^3B} \ln(N/2) \]  
(14)

For the \( E \) model we have to identify the \( N \) dependence of \( A \). This parameter sets the speed of the hopping but in all previous calculations it was unimportant. We show that in the transition regime it should scale as \( 1/N^2 \) exactly as \( B \). One reason could be the physical derivation of Eggers where one can look up the correspondence.
of $A$ and $B$, and find that they have the same $N$ dependence. The other reason is that we want to have the same order of fluxes in different systems. The value of the maximum of $u(n)$ is $AN^2$ so it is obvious to have $A \propto 1/N^2$ in the transition regime. Finally, we can get the same $N$ scaling as for the $LD$ model namely

$$\tau \propto N \ln(N/2),$$

which is shown on Fig. 3b.

The $L$ dependence of the coarsening time can also be read off. For the $E$ model in region II of the phase space $B$ has a $L$ dependence faster than $L^2$. It is easy to see that if $B \propto L^\gamma$ then the coarsening time grows as $\tau_E \propto \exp(L^\gamma)$. For the $LD$ model the dominating term will also be the exponential as in region II of the phase space both $\Delta$ and $T_0$ should decrease at least as $1/L$ for which $\tau_{LD} \propto \exp(L)$. The results are tested against numerical simulations on Fig. 3b.

As we already mentioned at the model definition $B$ is inversely proportional to the shaking strength, while $\Delta$ is proportional to the agitation. From physical point of view the $L$ scaling of the coarsening time is also the same for both models in spite of all differences.

VI. CONCLUSION

In this paper we showed that models proposed to described clustering of shaken granular material belongs to the group of zero-range processes which is very efficient in describing non-equilibrium phase transitions.

The probability distribution which describes the steady state of the system can be read off in a straightforward manner. This allows us to determine the phase diagram analytically for both models in the general case, for $L, N \ll 1$.

The coarsening time is also calculated for both models and tested against numerical simulations. The two models have qualitatively different phase diagram but surprisingly similar coarsening time scaling. The particle number scaling was found to be the same linear, the box number dependence was found to be exponential.

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