Demazure submodules
of level-zero extremal weight modules
and specializations of Macdonald polynomials

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Abstract

In this paper, we give a characterization of the crystal bases $B^+_x(\lambda)$, $x \in W_{af}$, of Demazure submodules $V^+_x(\lambda)$, $x \in W_{af}$, of a level-zero extremal weight module $V(\lambda)$ over a quantum affine algebra $U_q$, where $\lambda$ is an arbitrary level-zero dominant integral weight, and $W_{af}$ denotes the affine Weyl group. This characterization is given in terms of the initial directions of semi-infinite Lakshmibai-Seshadri paths, and is established under a suitably normalized isomorphism between the crystal basis $B(\lambda)$ of the level-zero extremal weight module $V(\lambda)$ and the crystal $B^+_{\infty}(\lambda)$ of semi-infinite Lakshmibai-Seshadri paths of shape $\lambda$, which is obtained in our previous work. As an application, we obtain a formula expressing the graded character of the Demazure submodule $V^+_{w_0}(\lambda)$ in terms of the specialization at $t = 0$ of the symmetric Macdonald polynomial $P_\lambda(x; q, t)$.

1 Introduction.

Extremal weight modules over quantized universal enveloping algebras of symmetrizable Kac-Moody algebras were introduced by [Kas3]. Since then, the study of level-zero extremal weight modules over quantum affine algebras especially has been the subject of a number of papers. Among them, we would like to mention [Kas6] and [BN], in which many of the crucial results on the structure of level-zero extremal weight modules and their crystal bases are obtained.

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In our previous paper [INS], for an arbitrary level-zero dominant integral weight \( \lambda \), we gave an explicit realization of the crystal basis \( \mathcal{B}(\lambda) \) of the extremal weight module \( V(\lambda) \) of extremal weight \( \lambda \), in terms of semi-infinite Lakshmibai-Seshadri paths (SiLS paths for short) of shape \( \lambda \); here, SiLS paths are analogs of Littelmann’s LS paths, which are defined by using the semi-infinite Bruhat order in place of the usual Bruhat order on the affine Weyl group \( W_{af} \), and Peterson’s coset representatives in place of the usual minimal(-length) coset representatives. Namely, we proved that the crystal basis \( \mathcal{B}(\lambda) \) is isomorphic as a crystal to the crystal \( \mathbb{B}^{\ast \lambda}(\lambda) \) of SiLS paths of shape \( \lambda \). Note that both of (the crystal graphs of) these crystals have infinitely many connected components in general, and hence an isomorphism between these crystals is not uniquely determined.

The purpose of this paper is to give a characterization of the crystal bases \( \mathcal{B}_{x}^{\pm}(\lambda) \) of Demazure(-type) submodules \( V_{x}^{\pm}(\lambda) := U_{q}^{+}S_{x}^{\text{norm}}v_{\lambda} \) of the extremal weight module \( V(\lambda) \) of extremal weight \( \lambda \), where \( x \) runs over the affine Weyl group \( W_{af} \). Here, \( v_{\lambda} \) denotes the generating extremal weight vector of \( V(\lambda) \) of weight \( \lambda \), and \( S_{x}^{\text{norm}}v_{\lambda} \in V(\lambda) \) is an extremal weight vector of weight \( x\lambda \) in the \( W_{af} \)-orbit of \( v_{\lambda} \); also, \( U_{q}^{+} \) denotes the positive part of a quantum affine algebra \( U_{q} \). This characterization is given in terms of the initial directions of SiLS paths, and is established after normalizing suitably the isomorphism \( \mathcal{B}(\lambda) \cong \mathbb{B}^{\ast \lambda}(\lambda) \) of crystals.

To be more precise, let \( \lambda = \sum_{i \in I} m_{i} \varpi_{i} \), with \( m_{i} \in \mathbb{Z}_{\geq 0} \) for \( i \in I \), be an arbitrary level-zero dominant integral weight, where the \( \varpi_{i} \), \( i \in I := I_{af} \setminus \{0\} \), are the level-zero fundamental weights. We set \( J := \{ i \in I \mid m_{i} = 0 \} \), and \( \mathbb{B}_{x \geq}(\lambda) := \{ \eta \in \mathbb{B}^{\ast \lambda}(\lambda) \mid x \geq \iota(\eta) \} \) for \( x \in (W^{J})_{af} \), where \( \iota(\eta) \in (W^{J})_{af} \) denotes the initial direction of a SiLS path \( \eta \in \mathbb{B}^{\ast \lambda}(\lambda) \); here, \( (W^{J})_{af} \) denotes the set of Peterson’s coset representatives for the cosets in \( W_{af}/(W_{J})_{af} \), with \( (W_{J})_{af} := W_{J} \ltimes Q_{J}^{\vee} \) a subgroup of \( W_{af} = W \ltimes Q^{\vee} \). The following is the main result of this paper.

**Theorem 1.** Let \( x \in (W^{J})_{af} \). Then, under a suitably normalized isomorphism

\[
\Psi_{x}^{\lambda} : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathbb{B}_{x \geq}^{\ast \lambda}(\lambda)
\]

of crystals, there holds the equality

\[
\Psi_{x}^{\lambda} (\mathcal{B}_{x}^{\pm}(\lambda)) = \mathbb{B}_{x \geq}^{\ast \lambda}(\lambda).
\]

Here we should mention that although the statement of the theorem above is of the form similar to that of Kashiwara’s result in [Kas2] (see also [Kas5, Chapitre 9]) or Littelmann’s result in [LL, §5] in the case of integrable highest weight modules, its proof is much more difficult because both of the crystals \( \mathcal{B}(\lambda) \) and \( \mathbb{B}^{\ast \lambda}(\lambda) \) may have infinitely many connected components, and because the \( \lambda \)-weight space of \( V(\lambda) \) may be infinite-dimensional even if these crystals are connected, in contrast to the case of highest weight crystals.

As an application of Theorem 1 above, we compute the graded character \( \text{gch} \, V_{w_{0}}^{\pm}(\lambda) \) of the Demazure submodule \( V_{w_{0}}^{\pm}(\lambda) \) for the longest element \( w_{0} \in W \), and obtain a formula.
expressing $gch V_{w_0}^+(\lambda)$ in terms of the specialization $P_\lambda(x; q, 0)$ of the symmetric Macdonald polynomial $P_\lambda(x; q, t)$. Namely, we prove (see Theorem 6.4.1) that

$$gch V_{w_0}^+(\lambda) = \left( \prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^r) \right)^{-1} P_\lambda(x; q, 0),$$

where $\lambda = \sum_{i \in I} m_i \omega_i$ is as above. Here the right-hand side of the equality above is called a $q$-Whittaker function in [BF], where the simply-laced cases are mainly treated; hence our result gives a new representation-theoretic interpretation of $q$-Whittaker functions for all untwisted cases.

Also, for each $w \in W^J$, we introduce a certain quotient $U_w^+(\lambda)$ of $V_w^+(\lambda)$, and give a characterization (Theorem 7.2.2) of its crystal basis as a subset of $B^+_{\infty}(\lambda)$ (or, more precisely, its connected component $B^+_0(\lambda)$) in terms of initial directions. In our forthcoming paper [LNS3], we will prove that the graded character of $U_w^+(\lambda)$ is identical to the specialization at $t = 0$ of a nonsymmetric Macdonald polynomial; this result would generalize [LNS2] Corollary 9.10], in which $w$ is the longest element $w_0$ of $W$.

This paper is organized as follows. In §2 we fix our notation for untwisted affine root data, and recall the definition of SiLS paths. Also, we briefly review our results in [INS] that we use in this paper. In §3 we first recall basic properties of extremal weight modules and their crystal bases. Then, we review some results in [BN] that we need in this paper. In §4 we introduce Demazure submodules $V_x^+(\lambda)$, $x \in (W^J)_af$, of the extremal weight module $V(\lambda)$, and their crystal bases $B_x^+(\lambda) \subset B(\lambda)$, $x \in (W^J)_af$. Also, we state our main result, i.e., Theorem 4 above. In §3 we prove some fundamental properties of the crystal bases $B_x^+(\lambda)$, $x \in (W^J)_af$, of Demazure submodules $V_x^+(\lambda)$, $x \in (W^J)_af$, and similar properties for the crystals $B_{x, af}^+ (\lambda) \subset B_{\infty}(\lambda)$, $x \in (W^J)_af$. Then, after a suitable normalization of the isomorphism $B(\lambda) \cong B_{\infty}(\lambda)$ of crystals, we finally establish our main result (= Theorem 4) stated in §4. In §6 we obtain the graded character formula above for the Demazure submodule $V_{w_0}^+(\lambda)$; in §7, we introduce the quotient $U_w^+(\lambda)$ of $V_w^+(\lambda)$ for each $w \in W^J$, and give a characterization of its crystal basis.

2 Semi-infinite Lakshmibai-Seshadri paths.

2.1 Untwisted affine root data.

Let $g_{af}$ be an untwisted affine Lie algebra over $\mathbb{C}$ with Cartan matrix $A = (a_{ij})_{i,j \in I_{af}}$. Let $h_{af} = (\bigoplus_{j \in I_{af}} \mathbb{C} \alpha_j^\vee) \oplus \mathbb{C} D$ denote the Cartan subalgebra of $g_{af}$, where $\{ \alpha_j^\vee \}_{j \in I_{af}} \subset h_{af}$ is the set of simple coroots, and $D \in h_{af}$ is the scaling element (or the degree operator). We denote by $\{ \alpha_j \}_{j \in I_{af}} \subset h_{af}$ the set of simple roots, and by $\Lambda_j \in h_{af}^*$, $j \in I_{af}$, the fundamental weights; note that $\langle D, \alpha_j \rangle = \delta_{j,0}$ and $\langle D, \Lambda_j \rangle = 0$ for $j \in I_{af}$, where $\langle \cdot, \cdot \rangle : h_{af} \times h_{af}^* \to \mathbb{C}$ denotes the canonical pairing of $h_{af}$ and $h_{af}^* := \text{Hom}(h_{af}, \mathbb{C})$. Let $\delta = \sum_{j \in I_{af}} a_{j,0} \alpha_j \in h_{af}$ and $c = \sum_{j \in I_{af}} a_{j,0} \alpha_j^\vee \in h_{af}$ denote the null root and the canonical central element of $g_{af}$,
We know from [Kac, Proposition 6.3] that
\[ W = \text{Weyl group of } g \] thought of as the (finite) Weyl group of \( g \) as a dominant integral weight for \( h \) where \( \Delta := \Delta \)
we call an element of \( P^\vee \) and \( P = \left( \bigoplus_{j \in I_\af} \mathbb{Z} \alpha_j \right) \oplus \mathbb{Z} D \subset \mathfrak{h} \) and \( P = \left( \bigoplus_{j \in I_\af} \mathbb{Z} \Lambda_j \right) \oplus \mathbb{Z} \delta \subset \mathfrak{h}^*; \) (2.1.1)
it is clear that \( P_\af \) contains \( \alpha_j \) for all \( j \in I_\af \), and that \( P_\af \cong \text{Hom}_{\mathbb{Z}}(P^\vee_\af, \mathbb{Z}) \). Also, we set
\[
Q_\af := \bigoplus_{j \in I_\af} \mathbb{Z} \alpha_j \quad \text{and} \quad Q^+_\af := \pm \sum_{j \in I_\af} \mathbb{Z}_{\geq 0} \alpha_j,
\]
we call an element of \( P^+ \) a level-zero dominant integral weight, which can be thought of as a dominant integral weight for \( g \). Also, set \( W := \langle r_j \mid j \in I \rangle \subset W_\af \), which can be thought of as the (finite) Weyl group of \( g \). For \( \xi \in Q^\vee \), let \( t_\xi \in W_\af \) denote the translation in \( \mathfrak{h}^*_\af \) with respect to \( \xi \) (see [Kac, §6.5]). Then we know from [Kac, Proposition 6.5] that \( \{ t_\xi \mid \xi \in Q^\vee \} \) forms an abelian normal subgroup of \( W_\af \), for which \( t_\xi t_\zeta = t_{\xi + \zeta}, \xi, \zeta \in Q^\vee \), and \( W_\af = W \times \{ t_\xi \mid \xi \in Q^\vee \} \) hold; remark that for \( w \in W \) and \( \xi \in Q^\vee \), we have
\[
wt_\xi \mu = w \mu - \langle \xi, \mu \rangle \delta \quad \text{if } \mu \in \mathfrak{h}^*_\af \text{ satisfies } \langle c, \mu \rangle = 0. \quad (2.1.2)
\]
We know from [Kac, Proposition 6.3] that
\[
\Delta_\af = \{ \alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z} \},
\]
\[
\Delta_\af^+ = \Delta^+ \cup \{ \alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}_{>0} \},
\]
where \( \Delta := \Delta_\af \cap Q \) is the (finite) root system for \( g \), and \( \Delta := \Delta \cap \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \). Note that if \( \beta \in \Delta_\af \) is of the form \( \beta = \alpha + n\delta \) with \( \alpha \in \Delta \) and \( n \in \mathbb{Z} \), then \( r_\beta = r_\alpha t_{n\alpha^\vee} \).
For a subset \( J \) of \( I \), we set
\[
Q_\af := \bigoplus_{j \in J} \mathbb{Z} \alpha_j, \quad Q^\vee_\af := \bigoplus_{j \in J} \mathbb{Z} \alpha_j^\vee, \quad Q^+_\af := \sum_{j \in J} \mathbb{Z}_{\geq 0} \alpha_j^\vee,
\]
\[
\Delta_\af := \Delta \cap Q_\af, \quad \Delta_\af^+ := \Delta^+ \cap \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i, \quad W_J := \langle r_j \mid j \in J \rangle.
\]
Also, denote by
\[
[\cdot] = [\cdot]_{I\setminus J} : Q^\vee \to Q^\vee_{I\setminus J}
\]
(2.1.3)
the projection from \( Q^\vee = Q^\vee_{I\setminus J} \oplus Q^\vee_J \) onto \( Q^\vee_{I\setminus J} \) with kernel \( Q^\vee_J \). Let \( W^J \) denote the set of minimal-length coset representatives for the quotient \( W/W_J \); we know from [BR §2.4] that
\[
W^J = \{ w \in W \mid w\alpha \in \Delta^+ \text{ for all } \alpha \in \Delta^+_J \}. \tag{2.1.4}
\]
For \( w \in W \), we denote by \([w] = [w]^J \in W^J\) the minimal coset representative for the coset \( wW_J \) in \( W/W_J \).

### 2.2 Peterson’s coset representatives.

Let \( J \) be a subset of \( I \). Following [P] (see also [LS §10]), we define
\[
(\Delta_J)_{af} := \{ \alpha + n\delta \mid \alpha \in \Delta_J, n \in \mathbb{Z} \} \subset \Delta_{af},
\]
(2.2.1)
\[
(\Delta_J)^+_af := (\Delta_J)_{af} \cap \Delta^+_af = \Delta_J^+ \cup \{ \alpha + n\delta \mid \alpha \in \Delta_J, n \in \mathbb{Z}_{>0} \},
\]
(2.2.2)
\[
(W_J)_{af} := W_J \ltimes \{ t_\xi \mid \xi \in Q^\vee_J \} = \langle r_\beta \mid \beta \in (\Delta_J)^+_af \rangle,
\]
(2.2.3)
\[
(W^J)_{af} := \{ x \in W_{af} \mid x\beta \in \Delta^+_af \text{ for all } \beta \in (\Delta_J)^+_af \}.
\]
(2.2.4)

Then we know the following from [P] (see also [LS Lemma 10.6]).

**Proposition 2.2.1.** For each \( x \in W_{af} \), there exist a unique \( x_1 \in (W^J)_{af} \) and a unique \( x_2 \in (W_J)_{af} \) such that \( x = x_1x_2 \).

We define a (surjective) map \( \Pi^J : W_{af} \to (W^J)_{af} \) by \( \Pi^J(x) := x_1 \) if \( x = x_1x_2 \) with \( x_1 \in (W^J)_{af} \) and \( x_2 \in (W_J)_{af} \).

An element \( \xi \in Q^\vee \) is said to be \( J \)-adjusted if \( \langle \xi, \gamma \rangle \in \{ -1, 0 \} \) for all \( \gamma \in \Delta^+_J \) (see [LNS31] Lemma 3.8]). Let \( Q^\vee,J^\text{-ad} \) denote the set of \( J \)-adjusted elements.

**Lemma 2.2.2** ([INS Lemma 2.3.5]).

1. For each \( \xi \in Q^\vee \), there exists a unique \( \phi_J(\xi) \in Q^\vee_J \) such that \( \xi + \phi_J(\xi) \in Q^\vee,J^\text{-ad} \). In particular, \( \xi \in Q^\vee,J^\text{-ad} \) if and only if \( \phi_J(\xi) = 0 \).

2. For each \( \xi \in Q^\vee \), the element \( \Pi^J(t_\xi) \in (W^J)_{af} \) is of the form \( \Pi^J(t_\xi) = z_\xi t_{\xi + \phi_J(\xi)} \) for a specific element \( z_\xi \in W_J \). Also, \( \Pi^J(w_\xi) = [w]z_\xi t_{\xi + \phi_J(\xi)} \) for every \( w \in W \) and \( \xi \in Q^\vee \).

3. We have
\[
(W^J)_{af} = \{ wz_\xi t_\xi \mid w \in W^J, \xi \in Q^\vee,J^\text{-ad} \}. \tag{2.2.5}
\]

**Lemma 2.2.3** ([INS Lemma 2.3.6]). Let \( x \in (W^J)_{af} \) and \( j \in I_{af} \). Then, \( x^{-1}\alpha_j \notin (\Delta_J)_{af} \) if and only if \( r_jx \in (W^J)_{af} \).
2.3 Parabolic semi-infinite Bruhat graphs.

**Definition 2.3.1** ([P]). Let \( x \in W_{\text{af}} \), and write it as \( x = wt_\xi \) for \( w \in W \) and \( \xi \in Q^\vee \). Then we define the semi-infinite length \( \ell^\infty(x) \) of \( x \) by \( \ell^\infty(x) := \ell(w) + 2\langle \xi, \rho \rangle \), where \( \rho := (1/2) \sum_{\alpha \in \Delta^+} \alpha \).

**Definition 2.3.2.** Let \( J \) be a subset of \( I \).

1. Define the (parabolic) semi-infinite Bruhat graph \( \text{SiB}^J \) to be the \( \Delta_{af}^+ \)-labeled, directed graph with vertex set \((W^J)_{af}\) and \( \Delta_{af}^+ \)-labeled, directed edges of the following form: \( x \xrightarrow{\beta} r_\beta x \) for \( x \in (W^J)_{af} \) and \( \beta \in \Delta_{af}^+ \), where \( r_\beta x \in (W^J)_{af} \) and \( \ell^\infty(r_\beta x) = \ell^\infty(x) + 1 \).

2. The semi-infinite Bruhat order is a partial order \( \preceq \) on \((W^J)_{af}\) defined as follows: for \( x, y \in (W^J)_{af} \), we write \( x \preceq y \) if there exists a directed path from \( x \) to \( y \) in \( \text{SiB}^J \); also, we write \( x \prec y \) if \( x \preceq y \) and \( x \neq y \). (In [INS], we used the symbol \( \preceq^x \) for the semi-infinite Bruhat order, but in the present paper, we use the symbol \( \preceq \) instead of \( \preceq^x \).)

**Remark 2.3.3** ([INS], Corollary 4.2.2]). Let \( J \) be a subset of \( I \). Let \( x \in (W^J)_{af} \) and \( \beta \in \Delta_{af}^+ \) be such that \( x \xrightarrow{\beta} r_\beta x \) in \( \text{SiB}^J \). Then, \( \beta \) is either of the following forms: \( \beta = \alpha \) with \( \alpha \in \Delta^+ \), or \( \beta = \alpha + \delta \) with \( -\alpha \in \Delta^+ \). Moreover, if \( x = wz_\xi t_\xi \) for \( w \in W^J \) and \( \xi \in Q_{\vee, J}^{\text{v, J-ad}} \) (see (2.2.3)), then \( w^{-1} \alpha \in \Delta^+ \setminus \Delta_J^+ \) in both cases above. Hence, if we write \( r_\beta x \in (W^J)_{af} \) as: \( r_\beta x = vz_\xi t_\xi \) with some \( v \in W^J \) and \( \zeta \in Q_{\vee, J}^{\text{v, J-ad}} \), then \( [\zeta - \xi] \in Q_{\vee, J}^{\text{v, J-ad}} \) in both cases above, where \([\cdot] : Q^\vee = Q_{\vee, J}^{\text{v, J-ad}} \mapsto Q_{\vee, J}^{\text{v, J-ad}} \) is the projection (see (2.1.3)).

**Lemma 2.3.4.** Let \( J \) be a subset of \( I \). Let \( w_1, w_2 \in W^J \), and fix \( \xi \in Q_{\vee, J}^{\text{v, J-ad}} \). Then, \( w_1z_\xi t_\xi \succeq w_2z_\xi t_\xi \) if and only if \( w_1 \geq w_2 \) with respect to the (ordinary) Bruhat order \( \geq \) on \( W^J \).

**Proof.** We first show the “if” part; we may assume that \( w_1 \) covers \( w_2 \), that is, \( w_1 = r_\alpha w_2 \) for some \( \alpha \in \Delta^+ \), and \( \ell(w_1) = \ell(w_2) + 1 \). Then we can easily see that \( w_2z_\xi t_\xi \xrightarrow{\alpha} w_1z_\xi t_\xi \) in the semi-infinite Bruhat graph \( \text{SiB}^J \), and hence \( w_1z_\xi t_\xi \succeq w_2z_\xi t_\xi \).

We next show the “only if” part; for simplicity, we give a proof only in the case where \( w_1z_\xi t_\xi \) covers \( w_2z_\xi t_\xi \), that is, \( w_2z_\xi t_\xi \xrightarrow{\beta} w_1z_\xi t_\xi \) for some \( \beta \in \Delta_{af}^+ \) (the proof for the general case is similar). By Remark 2.3.3, \( \beta \) is either of the following forms: \( \beta = \alpha \) with \( \alpha \in \Delta^+ \), or \( \beta = \alpha + \delta \) with \( -\alpha \in \Delta^+ \). Suppose that \( \beta = \alpha + \delta \) with \( -\alpha \in \Delta^+ \). Then,

\[
r_\beta w_2z_\xi t_\xi = r_{-\alpha} w_2z_\xi t_\xi = r_{-\alpha} w_2z_\xi t_{\xi + z_\xi^{-1}w_2^{-1}w_1}.
\]

Since \( r_\beta w_2z_\xi t_\xi = w_1z_\xi t_\xi \), but \( \xi + z_\xi^{-1}w_2^{-1}w_1 \neq \xi \), this is a contradiction. Thus, \( \beta = \alpha \) with \( \alpha \in \Delta^+ \). Then we can easily check that \( w_1 = r_\alpha w_2 \), and \( \ell(w_1) = \ell(w_2) + 1 \), which implies that \( w_2 \geq w_1 \). Thus we have proved the lemma. \( \square \)
Lemma 2.3.5 ([INS Remark 4.1.3]). Let \( \lambda \in P^+ \), and set \( J = J_\lambda := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \). For \( x \in (WJ)_{af} \) and \( j \in I_{af} \), the element \( r_jx \) is contained in \((WJ)_{af}\) if and only if \( \langle \alpha_j^\vee, x\lambda \rangle \neq 0 \) (see also Lemma 2.2.3). Moreover, in this case,

\[
\begin{aligned}
 x \prec r_jx &\iff \langle \alpha_j^\vee, x\lambda \rangle > 0, \\
 r_jx \prec x &\iff \langle \alpha_j^\vee, x\lambda \rangle < 0. 
\end{aligned}
\]  

(2.3.1)

Lemma 2.3.6. Let \( \lambda \in P^+ \), and set \( J = J_\lambda := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \). Assume that \( x, y \in W_{af} \) satisfy \( x \leq y \). Let \( j \in I_{af} \).

1. If \( \langle \alpha_j^\vee, x\lambda \rangle > 0 \) and \( \langle \alpha_j^\vee, y\lambda \rangle \leq 0 \), then \( r_jx \leq y \).

2. If \( \langle \alpha_j^\vee, x\lambda \rangle \geq 0 \) and \( \langle \alpha_j^\vee, y\lambda \rangle < 0 \), then \( x \leq r_jy \).

3. If \( \langle \alpha_j^\vee, x\lambda \rangle > 0 \) and \( \langle \alpha_j^\vee, y\lambda \rangle > 0 \), or if \( \langle \alpha_j^\vee, x\lambda \rangle < 0 \) and \( \langle \alpha_j^\vee, y\lambda \rangle < 0 \), then \( r_jx \leq r_jy \).

Proof. Parts (1) and (2) follow immediately from [INS Lemma 4.1.6]. Let us prove part (3). We give a proof only for the case that \( \langle \alpha_j^\vee, x\lambda \rangle > 0 \) and \( \langle \alpha_j^\vee, y\lambda \rangle > 0 \); the proof for the case that \( \langle \alpha_j^\vee, x\lambda \rangle < 0 \) and \( \langle \alpha_j^\vee, y\lambda \rangle < 0 \) is similar. By (2.3.1) and the assumption that \( \langle \alpha_j^\vee, y\lambda \rangle > 0 \), we have \( y \prec r_jy \), and hence \( x \prec r_jy \). Since \( \langle \alpha_j^\vee, x\lambda \rangle > 0 \) and \( \langle \alpha_j^\vee, r_jy\lambda \rangle < 0 \) by the assumption, we deduce by part (1) that \( r_jx \leq r_jy \). This proves the lemma. 

2.4 Semi-infinite Lakshmibai-Seshadri paths.

In this subsection, we fix \( \lambda \in P^+ \), and set \( J = J_\lambda := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \subset I \).

Definition 2.4.1. For a rational number \( 0 < a < 1 \), define \( \text{SiB}(\lambda; a) \) to be the subgraph of \( \text{SiB}^J \) with the same vertex set but having only the edges of the form: \( x \stackrel{\beta}{\longrightarrow} y \) with \( a\langle \beta^\vee, x\lambda \rangle \in \mathbb{Z} \).

Definition 2.4.2. A semi-infinite Lakshmibai-Seshadri path (SiLS path for short) of shape \( \lambda \) is, by definition, a pair \((x; a)\) of a (strictly) decreasing sequence \( x : x_1 \succ \cdots \succ x_s \) of elements in \((WJ)_{af}\) and an increasing sequence \( a : 0 = a_0 < a_1 < \cdots < a_s = 1 \) of rational numbers satisfying the condition that there exists a directed path from \( x_{u+1} \) to \( x_u \) in \( \text{SiB}(\lambda; a_u) \) for each \( u = 1, 2, \ldots, s - 1 \). We denote by \( \mathbb{B}^\infty(\lambda) \) the set of all SiLS paths of shape \( \lambda \).

Following [INS §3.1], we equip the set \( \mathbb{B}^\infty(\lambda) \) with a crystal structure (with weights in \( P_{af} \)) in the following way; for the definition of crystals, see [Kas4 §7.2] and [HK Definition 4.5.1] for example. For \( \eta = (x_1, \ldots, x_s; a_0, a_1, \ldots, a_s) \in \mathbb{B}^\infty(\lambda) \), we define \( \overline{\eta} : [0, 1] \to \mathbb{R} \otimes P_{af} \) to be the piecewise-linear, continuous map whose “direction vector” for the interval \([a_{u-1}, a_u]\) is equal to \( x_u\lambda \in P_{af} \) for each \( 1 \leq u \leq s \), that is,

\[
\overline{\eta}(t) := \sum_{p=1}^{u-1} (a_p - a_{p-1})x_p\lambda + (t - a_{u-1})x_u\lambda \quad \text{for} \ t \in [a_{u-1}, a_u], \ 1 \leq u \leq s; \tag{2.4.1}
\]
note that $\eta$ is a Lakshmibai-Seshadri path (LS path for short) of shape $\lambda$ by [INS, Proposition 3.1.3 together with Eq. (2.2.2)]. Then we define $\text{wt} : \mathcal{B}_+^\infty(\lambda) \to P_{af}$ by $\text{wt}(\eta) := \eta(1) \in P_{af}$ (see [L2, Lemma 4.5(a)]).

Now we define operators $e_j, f_j$, $j \in I_{af}$, which we call root operators for $\mathcal{B}_+^\infty(\lambda)$. Set

$$
\begin{cases}
H_j^n(t) := \langle \alpha_j, \eta(t) \rangle & \text{for } t \in [0, 1], \\
m_j^n := \min \{ H_j^n(t) \mid t \in [0, 1] \}.
\end{cases}
$$

(2.4.2)

Remark 2.4.3. Since $\eta$ is an LS path of shape $\lambda$, we see from [L2, Lemma 4.5(d)] that all local minima of the function $H_j^n(t), t \in [0, 1]$, are integers. In particular, the minimum $m_j^n$ is a nonpositive integer (recall that $\eta(0) = 0$, and hence $H_j^n(0) = 0$).

We define $e_j \eta$ as follows. If $m_j^n = 0$, then we set $e_j \eta := 0$, where 0 is an additional element not contained in any crystal. If $m_j^n \leq -1$, then set

$$
\begin{cases}
t_1 := \min \{ t \in [0, 1] \mid H_j^n(t) = m_j^n \}, \\
t_0 := \max \{ t \in [0, t_1] \mid H_j^n(t) = m_j^n + 1 \}.
\end{cases}
$$

(2.4.3)

from Remark [2.4.3] it follows that $H_j^n(t)$ is strictly decreasing on $[t_0, t_1]$. Let $1 \leq p \leq q \leq s$ be such that $a_{p-1} \leq t_0 < a_p$ and $t_1 = a_q$. Then we define $e_j \eta$ by

$$
e_j \eta := (x_1, \ldots, x_p, r_jx_p, r_jx_{p+1}, \ldots, r_jx_q, x_{q+1}, \ldots, x_s);
$$

$$a_0, \ldots, a_{p-1}, t_0, a_p, \ldots, a_q = t_1, \ldots, a_s);$$

if $t_0 = a_{p-1}$, then we drop $x_p$ and $a_{p-1}$, and if $r_jx_q = x_{q+1}$, then we drop $x_{q+1}$ and $a_q = t_1$.

Similarly, we define $f_j \eta$ as follows. If $H_j^n(1) - m_j^n = 0$, then we set $f_j \eta := 0$. If $H_j^n(1) - m_j^n \geq 1$, then set

$$
\begin{cases}
t_0 := \max \{ t \in [0, 1] \mid H_j^n(t) = m_j^n \}, \\
t_1 := \min \{ t \in [t_0, 1] \mid H_j^n(t) = m_j^n + 1 \}.
\end{cases}
$$

(2.4.4)

from Remark [2.4.3] it follows that $H_j^n(t)$ is strictly increasing on $[t_0, t_1]$. Let $0 \leq p \leq q \leq s-1$ be such that $t_0 = a_p$, and $a_q < t_1 \leq a_{q+1}$. Then we define $f_j \eta$ by

$$
f_j \eta := (x_1, \ldots, x_p, r_jx_p, r_jx_{p+1}, \ldots, r_jx_q, r_jx_{q+1}, x_{q+1}, \ldots, x_s);
$$

$$a_0, \ldots, a_p = t_0, \ldots, a_q, t_1, a_{q+1}, \ldots, a_s);$$

if $t_1 = a_{q+1}$, then we drop $x_{q+1}$ and $a_{q+1}$, and if $x_p = r_jx_{p+1}$, then we drop $x_p$ and $a_p = t_0$.

Set $e_j 0 = f_j 0 := 0$ for all $j \in I_{af}$.

**Theorem 2.4.4** (see [INS, Theorem 3.1.5]).

1. The set $\mathcal{B}_+^\infty(\lambda) \cup \{0\}$ is stable under the action of the root operators $e_j$ and $f_j$, $j \in I_{af}$.
(2) For each \( \eta \in \mathcal{B}(\lambda) \) and \( j \in I_{af} \), we set
\[
\begin{align*}
\varepsilon_j(\eta) &:= \max\{ k \geq 0 \mid e^k\eta \neq 0 \}, \\
\varphi_j(\eta) &:= \max\{ k \geq 0 \mid f^k\eta \neq 0 \}.
\end{align*}
\]

Then, the set \( \mathcal{B}(\lambda) \), equipped with the maps \( e_j, f_j, j \in I_{af} \), and \( \varepsilon_j, \varphi_j, j \in I_{af} \), defined above, is a crystal with weights in \( P_{af} \).

For \( \eta = (x_1, \ldots, x_s; a_0, a_1, \ldots, a_s) \in \mathcal{B}(\lambda) \), we set
\[
\iota(\eta) := x_1 \quad \text{and} \quad \kappa(\eta) := x_s. \tag{2.4.5}
\]

The following lemma will be used in the proof of Lemma 5.4.1 below.

**Lemma 2.4.5.** Let \( \eta \in \mathcal{B}(\lambda) \) and \( j \in I_{af} \). If \( \langle \alpha_j^\vee, \kappa(\eta)\lambda \rangle > 0 \), then \( f_j\eta \neq 0 \). Moreover, \( \kappa(f_j^{\max}\eta) = r_j\kappa(\eta) \) if and only if \( \langle \alpha_j^\vee, \kappa(\eta)\lambda \rangle > 0 \), where \( f_j^{\max}\eta := f_j^{\varepsilon_j(\eta)}\eta \).

**Proof.** If \( \langle \alpha_j^\vee, \kappa(\eta)\lambda \rangle > 0 \), then we see that \( H_j^\eta(1) - m_j^\eta > 0 \), and hence \( H_j^\eta(1) - m_j^\eta \geq 1 \) by Remark 2.4.3. Therefore, \( f_j\eta \neq 0 \) by the definition of the root operator \( f_j \).

Assume first that \( \langle \alpha_j^\vee, \kappa(\eta)\lambda \rangle > 0 \), and suppose, for a contradiction, that \( \kappa(f_j^{\max}\eta) = \kappa(\eta) \). Then we have \( \langle \alpha_j^\vee, \kappa(f_j^{\max}\eta)\lambda \rangle = \langle \alpha_j^\vee, \kappa(\eta)\lambda \rangle > 0 \), and hence \( f_jf_j^{\max}\eta \neq 0 \) by the assertion just shown. However, this contradicts the definition of \( f_j^{\max}\eta \). Thus we obtain \( \kappa(f_j^{\max}\eta) = r_j\kappa(\eta) \).

Assume next that \( \langle \alpha_j^\vee, \kappa(\eta)\lambda \rangle \leq 0 \); we will show by induction on \( k \) that \( \kappa(f_j^k\eta) = \kappa(\eta) \) for all \( 0 \leq k \leq \varphi_j(\eta) \). If \( k = 0 \), then the assertion is obvious. Assume that \( k > 0 \), and set \( \eta' := f_j^{k-1}\eta \); by our induction hypothesis, we have \( \kappa(\eta') = \kappa(\eta) \). Take \( t_0, t_1 \in [0,1] \) as in (2.4.4), with \( \eta' \) in place of \( \eta \). Then the function \( H_j^\eta(t) \) is strictly increasing on \([t_0, t_1] \) (see the comment following (2.4.4)). Since \( \kappa(\eta') = \kappa(\eta) \), we have \( \langle \alpha_j^\vee, \kappa(\eta')\lambda \rangle = \langle \alpha_j^\vee, \kappa(\eta)\lambda \rangle \leq 0 \), which implies that the function \( H_j^\eta(t) \) is weakly decreasing on \([1 - \epsilon, 1] \) for a sufficiently small \( \epsilon > 0 \). Therefore, we deduce that \( t_1 < 1 \), and hence \( \kappa(f_j\eta') = \kappa(\eta') \) by the definition of the root operator \( f_j \). Combining the above, we obtain \( \kappa(f_j^k\eta) = \kappa(f_j\eta') = \kappa(\eta') = \kappa(\eta) \). This proves the lemma. \( \square \)

### 2.5 SiLS paths associated with multi-partitions.

As in the previous subsection, we fix \( \lambda \in P^+ \), and set \( J = J_\lambda := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \subset I \).

We write \( \lambda \in P^+ \) as \( \lambda = \sum_{i \in I} m_i\overline{\omega}_i \) with \( m_i \in \mathbb{Z}_{\geq 0}, i \in I \), and define
\[
\begin{align*}
\overline{\text{Par}}(\lambda) &:= \{ c_0 = (\rho^{(i)})_{i \in I} \mid \rho^{(i)} \text{ is a partition of length} \leq m_i \text{ for each } i \in I \}, \tag{2.5.1} \\
\text{Par}(\lambda) &:= \{ c_0 = (\rho^{(i)})_{i \in I} \mid \rho^{(i)} \text{ is a partition of length} < m_i \text{ for each } i \in I \} \tag{2.5.2}
\end{align*}
\]
we understand a partition whose length is less than 0 to be the empty partition \( \emptyset \) (the sets \( \overline{\text{Par}}(\lambda) \) and \( \text{Par}(\lambda) \) are identical to \( N_{\mathcal{R}_0}(\lambda) \) and \( N_{\mathcal{R}_0}(\lambda)' \) in the notation of [BN] Definition 4.2.
and page 371, respectively). Note that \( \text{Par}(\lambda) \subset \overline{\text{Par}(\lambda)} \). For \( c_0 = (\rho(i))_{i \in I} \in \text{Par}(\lambda) \), we set 
\[ |c_0| := \sum_{i \in I} |\rho(i)|, \]
where for a partition \( \chi = (\chi_1 \geq \chi_2 \geq \cdots \geq \chi_m) \), we set \( |\chi| := \chi_1 + \cdots + \chi_m \).
We equip the set \( \text{Par}(\lambda) \) with a crystal structure as follows: for each \( c_0 = (\rho(i))_{i \in I} \in \text{Par}(\lambda) \), we set
\[
\begin{align*}
  e_j c_0 & = f_j c_0 := 0, & \varepsilon_j(c_0) & = \varphi_j(c_0) := -\infty & \text{for } j \in I_{af}, \\
  \text{wt}(c_0) & := -|c_0|\delta.
\end{align*}
\]

Let \( \text{Conn}(\mathbb{B}^\infty_<(\lambda)) \) denote the set of all connected components of \( \mathbb{B}^\infty_<(\lambda) \), and let \( \mathbb{B}^\infty_0(\lambda) \in \text{Conn}(\mathbb{B}^\infty_<(\lambda)) \) denote the connected component of \( \mathbb{B}^\infty_<(\lambda) \) containing \( \eta_e := (e; 0, 1) \in \mathbb{B}^\infty_<(\lambda) \).

**Proposition 2.5.1.** Keep the notation above.

1. Each connected component \( C \in \text{Conn}(\mathbb{B}^\infty_<(\lambda)) \) of \( \mathbb{B}^\infty_<(\lambda) \) contains a unique element of the form:

\[
\eta^C = (z_{\xi_1}t_{\xi_1}, z_{\xi_2}t_{\xi_2}, \ldots, z_{\xi_s}t_{\xi_s}, e; a_0, a_1, \ldots, a_s) \tag{2.5.4}
\]

for some \( s \geq 1 \) and \( \xi_1, \xi_2, \ldots, \xi_s \in Q_{J^\text{ad}}^\vee \) (see \[\text{INS}, \text{Proposition 7.1.2}\]); recall that \( e \) denotes the unit element of \( W_{af} \).

2. There exists a bijection \( \Theta : \text{Conn}(\mathbb{B}^\infty_<(\lambda)) \rightarrow \text{Par}(\lambda) \) such that \( \text{wt}(\eta^C) = \lambda - |\Theta(C)|\delta = \lambda + \text{wt}(\Theta(C)) \) (see \[\text{INS}, \text{Proposition 7.2.1 and its proof}\]).

3. Let \( C \in \text{Conn}(\mathbb{B}^\infty_<(\lambda)) \). Then, there exists an isomorphism \( C \overset{\sim}{\rightarrow} \{\Theta(C)\} \otimes \mathbb{B}^\infty_0(\lambda) \) of crystals that maps \( \eta^C \) to \( \Theta(C) \otimes \eta_e \). Consequently, \( \mathbb{B}^\infty_<(\lambda) \) is isomorphic as a crystal to \( \text{Par}(\lambda) \otimes \mathbb{B}^\infty_0(\lambda) \) (see \[\text{INS}, \text{Proposition 3.2.4 and its proof}\]).

### 2.6 Dual crystal of \( \mathbb{B}^\infty_<(\lambda) \).

As in the previous subsection, we fix \( \lambda \in P^+ \), and set \( J = J_\lambda := \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\} \subset I \).
Let \( w_0 \in W \) denote the longest element of the (finite) Weyl group \( W \), and define an involution \( \sigma : I \rightarrow I \) by \( w_0\sigma j = -\alpha_{\sigma(j)} \) for \( j \in I \); recall that \( w_0^2 = e \), and hence \( \sigma^2 \) is the identity map on \( I \). Note that \(-w_0\lambda \in P^+ \), and \( J_{-w_0\lambda} = \{i \in I \mid \langle \alpha_i^\vee, -w_0\lambda \rangle = 0\} = \sigma(J) \). Also, let \( w_{\sigma(J),0} \in W_{\sigma(J)} \) denote the longest element of the (finite) Weyl group \( W_{\sigma(J)} \); we see from \( (2.1.4) \) that the minimal coset representative \( [w_0]_{\sigma(J)} \in W_{\sigma(J)} \) is identical to \( w_0w_{\sigma(J),0} \).

Now, for \( x \in (W^J)_{af} \), we set \( x^\vee := x[w_0]_{\sigma(J)} = xw_0w_{\sigma(J),0} \). Then it follows easily from definition \( (2.2.4) \) that \( x^\vee \in (W_{\sigma(J)})_{af} \); notice that \([w_0]_{\sigma(J)}\beta = w_0w_{\sigma(J),0}\beta \in (\Delta_J)_{af}^{-} \) for all \( \beta \in (\Delta_{\sigma(J)})_{af}^{+} \). Moreover, we have (cf. \[\text{LNS}^1, \text{Proposition 4.3 (2)}\])

\[
\ell^\infty(x^\vee) = \ell(w_0) - \ell(w_{\sigma(J),0}) - \ell^\infty(x) \quad \text{for every } x \in (W^J)_{af}. \tag{2.6.1}
\]

Indeed, if \( x = wz_\xi t_\xi \) for \( w \in W^J \) and \( \xi \in Q_{J^\text{ad}}^\vee \) (see \( (2.2.5) \)), then

\[
\ell^\infty(x^\vee) = \ell^\infty(wz_\xi t_\xi w_{\sigma(J),0}) = \ell^\infty(wz_\xi w_{\sigma(J),0} t_{w_0w_{\sigma(J),0}^{-1}\xi}).
\]
\[
\ell([wz_\text{w}_{\sigma(j),0}]^{\sigma(j)}) = \ell([wz_\text{w}_{\sigma(j),0}]^{\sigma(j)}) = \ell([w_0(w_0w_0)(w_0z_\text{w}_{\sigma(j),0}]^{\sigma(j)}) = \ell([w_0(w_0w_0)]^{\sigma(j)})
\]
\[
= \ell(w_0) - \ell(w_{\sigma(j),0}) - \ell(w_0w_0) \quad \text{by } [\text{LNS}^4, \text{Proposition 4.3(2)}]
\]
\[
= \ell(w_0) - \ell(w_{\sigma(j),0}) - \ell(w).
\]
\]

Substituting (2.6.3) and (2.6.4) into (2.6.2), we obtain (2.6.1). From the definition of (parabolic) semi-infinite Bruhat graphs, by using (2.6.1), we easily obtain the next lemma.

**Lemma 2.6.1.** Let \(0 < a \leq 1\) be a rational number. Let \(x, y \in (W^J)_{af}\), and \(\beta \in \Delta_{af}^+\). Then, \(x \overset{\beta}{\rightarrow} y\) in \(\text{SiB}(\lambda; a)\) if and only if \(y^\vee \overset{\beta}{\rightarrow} x^\vee\) in \(\text{SiB}(-w_0\lambda; a)\).

For \(\eta = (x_1, \ldots, x_s; a_0, a_1, \ldots, a_s) \in \mathbb{B}^{\text{SiB}}(\lambda)\), we set
\[
\eta^\vee := (x_1^\vee, \ldots, x_s^\vee; 1 - a_s, \ldots, 1 - a_1, 1 - a_0).
\]

By Lemma 2.6.1, we see that \(\eta^\vee \in \mathbb{B}^{\text{SiB}}(-w_0\lambda)\). Also, in the same way as [\text{L2} Lemma 2.1 e)] (cf. [\text{Kas4} §7.4]), it is easily shown that for \(\eta \in \mathbb{B}^{\text{SiB}}(\lambda)\),
\[
\begin{align*}
\text{wt}(\eta^\vee) &= -\text{wt}(\eta), \quad \text{and} \\
(e_j\eta)^\vee &= f_j\eta^\vee, \quad (f_j\eta)^\vee = e_j\eta^\vee \quad \text{for all } j \in I_{af},
\end{align*}
\]
where we set \(0^\vee := 0\).

3 Extremal weight modules and their crystal bases.

3.1 Quantized universal enveloping algebras.

Let \((\cdot, \cdot)\) denote the nondegenerate, symmetric, \(\mathbb{C}\)-bilinear form on \(h_{af}\), normalized as in [\text{Kac} §6], and fix a positive integer \(d \in \mathbb{Z}_{>0}\) such that \((\alpha_j, \alpha_j)/2 \in \mathbb{Z}d^{-1}\) for all \(j \in I_{af}\). Let \(q\) be an
indeterminate, and set \( q_s := q^{1/d} \). Denote by \( U_q = U_q(\mathfrak{g}_{\text{af}}) = \langle E_j, F_j, q^h \mid j \in \mathcal{I}_{\text{af}}, h \in d^{-1}P^\vee \rangle \) the quantized universal enveloping algebra over \( \mathbb{Q}(q_s) \) associated with \( \mathfrak{g}_{\text{af}} \), where \( E_j \) and \( F_j \) denote the Chevalley generators corresponding to the simple root \( \alpha_j \) for \( j \in I \), and denote by \( U_q^+ = \langle E_j \mid j \in \mathcal{I}_{\text{af}} \rangle \) (resp., \( U_q^- = \langle F_j \mid j \in \mathcal{I}_{\text{af}} \rangle \)) the \( \mathbb{Q}(q_s) \)-subalgebra of \( U_q \) generated by \( E_j \) (resp., \( F_j \)), \( j \in \mathcal{I}_{\text{af}} \). We define a \( \mathbb{Q}(q_s) \)-algebra involutive automorphism \( \vee : U_q \to U_q \) and a \( \mathbb{Q}(q_s) \)-algebra involutive antiautomorphism \( * : U_q \to U_q \) by:

\[
E_j^\vee = F_j, \quad F_j^\vee = E_j, \quad (q^h)^\vee = q^{-h}; \tag{3.1.1}
\]

\[
E_j^* = E_j, \quad F_j^* = F_j, \quad (q^h)^* = q^{-h} \tag{3.1.2}
\]

for \( j \in \mathcal{I}_{\text{af}} \) and \( h \in d^{-1}P^\vee \). Also, we define a \( \mathbb{Q} \)-algebra involutive automorphism \( - : U_q \to U_q \) by:

\[
\overline{E}_j = E_j, \quad \overline{F}_j = F_j, \quad \overline{q}^\mu = q^{-h}, \quad \overline{q}_s = q_s^{-1} \tag{3.1.3}
\]

for \( j \in \mathcal{I}_{\text{af}} \) and \( h \in d^{-1}P^\vee \).

Let \( (\mathcal{L}(\pm\infty), \mathcal{B}(\pm\infty)) \) denote the crystal basis of \( U_q^\pm \), with \( u_{\pm\infty} \in \mathcal{B}(\pm\infty) \) the element corresponding to \( 1 \in U_q^\pm \). Recall from [Kas2, Theorem 2.1.1] and [Kas4, §8.3] that \( * : U_q \to U_q \) induces an involution on \( \mathcal{B}(\pm\infty) \), which is also denoted by \( * \); we call this involution the \( * \)-operation on \( \mathcal{B}(\pm\infty) \).

### 3.2 Extremal weight vectors and extremal elements.

Let \( M \) be an integrable \( U_q \)-module. A nonzero weight vector \( v \in M \) of weight \( \lambda \in P_{\text{af}} \) is said to be extremal (see [Kas6, §3.1] and [Kas7, §2.6]) if there exists a family \( \{v_x\}_{x \in W_{\text{af}}} \) of weight vectors in \( M \) satisfying the conditions that \( v_x = v \), and for every \( j \in \mathcal{I}_{\text{af}} \) and \( x \in W_{\text{af}} \) with \( n := \langle \alpha_j, x\lambda \rangle \geq 0 \) (resp., \( \leq 0 \)), the equalities \( E_jv_x = 0 \) and \( F_j^{(n)}v_x = v_{r_jx} \) (resp., \( F_jv_x = 0 \) and \( E_j^{(-n)}v_x = v_{r_jx} \)) hold, where \( E_j^{(k)} \) and \( F_j^{(k)} \) are the divided powers of \( E_j \) and \( F_j \) for \( k \in \mathbb{Z}_{\geq 0} \); observe that the weight of \( v_x \) is equal to \( x\lambda \). Then the Weyl group \( W_{\text{af}} \) acts on the set of extremal weight vectors in \( M \) by

\[
S_{r_j}^{\text{norm}} v := \begin{cases} 
F_j^{(n)} v & \text{if } n := \langle \alpha_j, \mu \rangle \geq 0, \\
E_j^{(-n)} v & \text{if } n := \langle \alpha_j, \mu \rangle \leq 0
\end{cases} \tag{3.2.1}
\]

for an extremal weight vector \( v \in M \) of weight \( \mu \in P_{\text{af}} \) and \( j \in \mathcal{I}_{\text{af}} \) (see [Kas7, (2.23)]); if \( \{v_x\}_{x \in W_{\text{af}}} \) is the family of weight vectors associated with an extremal weight vector \( v \), then \( v_x = S_x^{\text{norm}} v \) for all \( x \in W_{\text{af}} \).

Now, let \( \mathcal{B} \) be a regular (or normal) crystal in the sense of [Kas6, §2.2] (or [Kas3, p. 389]). By [Kas3, §7], the Weyl group \( W_{\text{af}} \) acts on \( \mathcal{B} \) by

\[
S_{r_j} b := \begin{cases} 
f_j^{\mu} b & \text{if } n := \langle \alpha_j, \mu \rangle \geq 0, \\
ed_j^{-n} b & \text{if } n := \langle \alpha_j, \mu \rangle \leq 0
\end{cases} \tag{3.2.2}
\]
for \( b \in \mathcal{B} \) and \( j \in I_{af} \), where \( e_j \) and \( f_j \), \( j \in I_{af} \), are the Kashiwara operators on \( \mathcal{B} \). An element \( b \in \mathcal{B} \) of weight \( \lambda \in P_{af} \) is said to be extremal (see \cite{Kas7} §2.6; cf. \cite{Kas6} §3.1]) if \( e_j S_x b = 0 \) (resp., \( f_j S_x b = 0 \)) for all \( x \in W_{af} \) and \( j \in I_{af} \) such that \( \langle \alpha_j^\vee, x \lambda \rangle \geq 0 \) (resp., \( \leq 0 \)).

**Lemma 3.2.1.** Let \( M \) be an integrable \( U_q \)-module, and assume that \( M \) has a crystal basis \( (\mathcal{L}, \mathcal{B}) \) with global basis \( \{ G(b) \mid b \in \mathcal{B} \} \) (see \cite{Kas6} Definitions 2.2.2 and 2.2.3 for example); note that \( \mathcal{B} \) is a regular crystal. If \( b \in \mathcal{B} \) is an extremal element of weight \( \lambda \), then the global basis element \( G(b) \) is an extremal weight vector of weight \( \lambda \). Moreover, we have \( G(S_x b) = S_x^{\text{norm}} G(b) \) for all \( x \in W_{af} \).

**Proof.** We set \( v := G(b) \), and \( v_x := G(S_x b) \) for \( x \in W_{af} \); it suffices to show that the family \( \{ v_x \}_{x \in W_{af}} \) satisfies the condition for \( v \) to be an extremal weight vector. It is obvious that \( v_x = v \). Let \( x \in W_{af} \), and \( j \in I_{af} \). Assume that \( n := \langle \alpha_j^\vee, x \lambda \rangle \geq 0 \). Then, \( e_j S_x b = 0 \) and \( f_j^{n+1} S_x b = f_j S_{r_j x} b = 0 \) by the definition of an extremal element. Hence, in exactly the same way as \cite{Kas6} Lemma 5.1.1, we obtain \( E_j v_x = E_j G(S_x b) = 0 \) and \( F_j^{(n)} v_x = F_j^{(n)} G(S_x b) = G(f_j^n S_x b) = G(S_{r_j x} b) = v_{r_j x} \). Similarly, it is shown that if \( n := \langle \alpha_j^\vee, x \lambda \rangle \leq 0 \), then \( F_j v_x = 0 \) and \( E_j^{(-n)} v_x = v_{r_j x} \). This proves the lemma.

### 3.3 Extremal weight modules.

Let \( \lambda \in P_{af} \) be an arbitrary integral weight. Let \( V(\lambda) \) denote the extremal weight module of extremal weight \( \lambda \) over \( U_q \), which is an integrable \( U_q \)-module generated by a single element \( v_\lambda \) with the defining relation that \( v_\lambda \) is an “extremal weight vector” of weight \( \lambda \) (for details, see \cite{Kas3} §8 and \cite{Kas6} §3). We know from \cite{Kas3} Proposition 8.2.2 that \( V(\lambda) \) has a crystal basis \( (\mathcal{L}(\lambda), \mathcal{B}(\lambda)) \) with global basis \( \{ G(b) \mid b \in \mathcal{B}(\lambda) \} \). Denote by \( u_\lambda \) the element of \( \mathcal{B}(\lambda) \) such that \( G(u_\lambda) = v_\lambda \in V(\lambda) \).

**Remark 3.3.1.** Let \( \lambda \in P_{af} \). The crystal basis \( \mathcal{B}(\lambda) \) is a regular crystal, and \( u_\lambda \in \mathcal{B}(\lambda) \) is an extremal element of weight \( \lambda \). Also, by Lemma 3.2.1, we have

\[
G(S_x u_\lambda) = S_x^{\text{norm}} G(u_\lambda) = S_x^{\text{norm}} v_\lambda \quad \text{for all } x \in W_{af}.
\]

For \( \mu \in P_{af} \), let \( \mathcal{T}_\mu = \{ \tau_\mu \} \) denote the crystal consisting of a single element of weight \( \mu \) (see \cite{Kas4} Example 7.3]). We see from \cite{Kas3} Theorems 2.1.1 (v) and 3.1.1] that \( \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \) is a regular crystal for each \( \mu \in P_{af} \). We define the \( \ast \)-operation on the crystal \( \tilde{\mathcal{B}} := \bigsqcup_{\mu \in P_{af}} \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \) as follows (see \cite{Kas3} Corollary 4.3.3]: for \( b_1 \otimes \tau_\mu \otimes b_2 \in \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \) with \( \mu \in P_{af} \),

\[
(b_1 \otimes \tau_\mu \otimes b_2)^\ast := b_1^\ast \otimes \tau_{-\mu-\text{wt}(b_1)-\text{wt}(b_2)} \otimes b_2^\ast.
\]

For each \( j \in I_{af} \), we define maps \( e_j^\ast \) and \( f_j^\ast \) from \( \tilde{\mathcal{B}} \sqcup \{ 0 \} \) to itself by \( e_j^\ast := \ast \circ e_j \circ \ast \) and \( f_j^\ast := \ast \circ e_j \circ \ast \), where we understand \( 0^\ast = 0 \); we know from \cite{Kas3} Theorem 5.1.1] that for every \( j \in I_{af} \), the maps \( e_j^\ast \) and \( f_j^\ast \) are strict morphisms of crystals. Also, for each \( x \in W_{af} \), the
map \( S_x^* := \ast \circ S_x \circ \ast \) is a strict automorphism of the crystal \( \tilde{B} \) that maps \( B(\infty) \otimes T_\mu \otimes B(-\infty) \) onto \( B(\infty) \otimes T_{\mu x} \otimes B(-\infty) \). We know the following from [Kas3, Proposition 8.2.2], [Kas6] §3.1, and [Kas7] §2.6.

### Proposition 3.3.2.

1. For each \( \lambda \in P_{af} \), the subset \( \{ b \in B(\infty) \otimes T_\lambda \otimes B(-\infty) \mid b^\vee \text{ is extremal} \} \) is a subcrystal of \( B(\infty) \otimes T_\lambda \otimes B(-\infty) \subset \tilde{B} \). Moreover, it is isomorphic as a crystal to the crystal basis \( B(\lambda) \); hence, we regard \( B(\lambda) \) as a subcrystal of \( B(\infty) \otimes T_\lambda \otimes B(-\infty) \subset \tilde{B} \).

2. Let \( \lambda \in P_{af} \), and \( x \in W_{af} \). If \( b \in B(\lambda) \subset B(\infty) \otimes T_\lambda \otimes B(-\infty) \), then \( S_x^* (b) \in B(x\lambda) \subset B(\infty) \otimes T_{x\lambda} \otimes B(-\infty) \). Thus, \( S_x^* \) gives an isomorphism of crystals from \( B(\lambda) \) onto \( B(x\lambda) \), that is,

\[
S_x^* : B(\lambda) \rightarrow B(x\lambda).
\]

3. Let \( \lambda \in P_{af} \), and \( x \in W_{af} \). There exists a \( U_q \)-module isomorphism

\[
V(\lambda) \sim V(x\lambda)
\]

that maps \( v_\lambda \in V(\lambda) \) to \( S_{x^{-1}}^* v_{x\lambda} \in V(x\lambda) \). Moreover, this isomorphism is compatible with the global bases; namely, for \( b \in B(\lambda) \), the global basis element \( G(b) \in V(\lambda) \) is sent to \( G(S_x^*(b)) \in V(x\lambda) \) under this isomorphism.

### 3.4 Dual crystal of \( B(\lambda) \).

It is easily seen that the \( \mathbb{Q}(q_s) \)-algebra involutive automorphism \( \vee : U_q \rightarrow U_q \) (see (3.1.1)) induces a bijection \( \vee : B(\pm \infty) \rightarrow B(\mp \infty) \); we see that for \( b \in B(\pm \infty) \),

\[
\begin{aligned}
\text{wt}(b^\vee) &= -\text{wt}(b), \quad \text{and} \\
(e_j b)^\vee &= f_j b^\vee, \quad (f_j b)^\vee = e_j b^\vee \quad \text{for all } j \in I_{af},
\end{aligned}
\]

where we set \( 0^\vee := 0 \). We define an involution \( \vee \) on the crystal \( \bigsqcup_{\mu \in P_{af}} B(\infty) \otimes T_\mu \otimes B(-\infty) \) as follows: for \( b_1 \otimes \tau_\mu \otimes b_2 \in B(\infty) \otimes T_\mu \otimes B(-\infty) \), \( \mu \in P_{af} \),

\[
(b_1 \otimes \tau_\mu \otimes b_2)^\vee := b_2^\vee \otimes \tau_{-\mu} \otimes b_1^\vee \in B(\infty) \otimes T_{-\mu} \otimes B(-\infty);
\]

we see that the same equalities as those in (3.4.1) hold for \( b \in \bigsqcup_{\mu \in P_{af}} B(\infty) \otimes T_\mu \otimes B(-\infty) \) and \( j \in I_{af} \).

**Remark 3.4.1.** We deduce that \( (S_x b)^\vee = S_x b^\vee \) for all \( x \in W_{af} \) and \( b \in \bigsqcup_{\mu \in P_{af}} B(\infty) \otimes T_\mu \otimes B(-\infty) \), which implies that if \( b \in \bigsqcup_{\mu \in P_{af}} B(\infty) \otimes T_\mu \otimes B(-\infty) \) is an extremal element, then so is \( b^\vee \).
From the definitions (3.1.1) and (3.1.2), we see easily that $\vee \circ * = \ast \circ \vee$ holds on $U_q$, and hence $(b^\ast)^\vee = (b^\vee)^\ast$ for all $b \in \mathcal{B}(\pm \infty)$. Hence, from the definitions (3.3.2) and (3.4.2), it follows that

$$\vee \circ * = \ast \circ \vee \quad \text{holds on} \quad \bigsqcup_{\mu \in \varphi^{+}} \mathcal{B}(\infty) \otimes T_{\mu} \otimes \mathcal{B}(-\infty).$$

(3.4.3)

Therefore, we deduce from Remark 3.4.1 that if $b \in \mathcal{B}(\lambda) \cong \{ b \in \mathcal{B}(\infty) \otimes T_{\lambda} \otimes \mathcal{B}(-\infty) \mid b^\ast \text{ is extremal} \}$, then $b^\vee \in \mathcal{B}(-\lambda) \cong \{ b \in \mathcal{B}(\infty) \otimes T_{-\lambda} \otimes \mathcal{B}(-\infty) \mid b^\ast \text{ is extremal} \}$. Note that the following equalities hold for $b \in \mathcal{B}(\lambda)$:

$$\begin{cases}
\text{wt}(b^\vee) = -\text{wt}(b), \\
(e_j b)^\vee = f_j b^\vee, (f_j b)^\vee = e_j b^\vee \quad \text{for all } j \in I_{\text{af}}.
\end{cases}$$

(3.4.4)

### 3.5 Isomorphism theorem.

Let $\lambda = \sum_{i \in I} m_i \varpi_i \in P^+$ be a level-zero dominant integral weight, and and $\text{Par}(\lambda)$ and $\text{Par}(\lambda)$ as defined in (2.5.1) and (2.5.2), respectively. For each $c_0 \in \text{Par}(\lambda)$, we define an element $S^{-}_c \in U_q^+$ of weight $|c_0|\delta$ as on [BN] page 352; this is a basis element of the “imaginary part” of $U_q^+$, and is identical to $B_c = L(c, 0)$ for $c = (0, c_0, 0) = c_0$ (see [BN] the paragraph including Eq. (3.11)). Also, we set $S^{-}_c = S^{-}_{c_0} \in U_q$ (see [BN] Remark 4.1); note that the weight of $S^{-}_{c_0}$ is equal to $-|c_0|\delta$. We deduce from [BN] Proposition 3.27 that

$$b(c_0) := S^{-}_{c_0} + q_{\ast} \mathcal{L}(\infty)$$

(3.5.1)

is contained in $\mathcal{B}(\infty)$.

Let $\mathcal{B}_0(\lambda)$ denote the connected component of $\mathcal{B}(\lambda)$ containing $u_{\lambda}$. We know the following from [BN] Proposition 4.3 and Theorem 4.16] (and their proofs).

**Proposition 3.5.1.** Keep the notation and setting above.

1. For each $c_0 \in \text{Par}(\lambda)$, the element $u^{c_0} := b(c_0) \otimes \tau_{\lambda} \otimes u_{-\infty}$ is an extremal element of weight $\lambda - |c_0|\delta$ contained in $\mathcal{B}(\lambda) \subset \mathcal{B}(\infty) \otimes T_{\lambda} \otimes \mathcal{B}(-\infty)$.

2. Each connected component of $\mathcal{B}(\lambda)$ contains a unique element of the form $u^{c_0} = b(c_0) \otimes \tau_{\lambda} \otimes u_{-\infty}$ with $c_0 \in \text{Par}(\lambda)$. Moreover, in this case, there exists an isomorphism of crystals from the connected component containing $u^{c_0}$ onto $\{ c_0 \} \otimes \mathcal{B}_0(\lambda)$ that maps $u^{c_0}$ to $c_0 \otimes u_{\lambda}$. Consequently, $\mathcal{B}(\lambda)$ is isomorphic as a crystal to $\text{Par}(\lambda) \otimes \mathcal{B}_0(\lambda)$.

We know from [INS] Proposition 3.2.2 that there exists an isomorphism $\mathcal{B}_0(\lambda) \sim \mathbb{B}_{0}^{\sim}(\lambda)$ of crystals that maps $u_{\lambda} \in \mathcal{B}_0(\lambda)$ to $\eta_e = (e; 0, 1) \in \mathbb{B}_{0}^{\sim}(\lambda)$. Therefore, by combining Propositions 2.5.1 and 3.5.1 we deduce that there exists an isomorphism

$$\Psi_{\lambda} : \mathcal{B}(\lambda) \sim \mathbb{B}_{0}^{\sim}(\lambda)$$

(3.5.2)
of crystals that maps $u^{c_0} \in \mathcal{B}(\lambda)$ to $\eta^{\Theta^{-1}(c_0)} \in \mathbb{B}^\times(\lambda)$ for each $c_0 \in \text{Par}(\lambda)$. Also, we define a bijection $\Psi^\vee : \mathcal{B}(\lambda) \to \mathbb{B}^\times(\lambda)$ by the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{B}(\lambda) & \xrightarrow{\Psi^\vee} & \mathbb{B}^\times(\lambda) \\
S_{w_0} \downarrow & & \downarrow \\
\mathcal{B}(-w_0 \lambda) & \xrightarrow{\Psi^{-w_0 \lambda}} & \mathbb{B}^\times(-w_0 \lambda).
\end{array}
$$

Then we see from (2.6.6) and (3.4.4) that the map $\Psi^\vee$ above is an isomorphism of crystals.

**Remark 3.5.2.** Keep the notation and setting above.

1. We know from [INS, Remark 7.2.2] that $\eta^C$ is an extremal element for every $C \in \text{Conn}(\mathbb{B}^\times(\lambda))$.

2. Let $\eta \in \mathbb{B}^\times(\lambda)$ be such that $\eta(t) \equiv t \lambda \mod \mathbb{R} as for all $t \in [0, 1]$. Then it is easily seen by using (2.2.5) that $\eta$ is of the form:

$$
\eta = (z_{\zeta_1} t_{\zeta_1}, \ldots, z_{\zeta_s} t_{\zeta_s}; a_0, a_1, \ldots, a_s)
$$

for some $s \geq 1$ and $\zeta_1, \ldots, \zeta_s \in Q^\vee, J$-ad (see also [INS, Proposition 7.1.1]). Moreover, by the same argument as for [INS, Eq. (5.1.6)], we can show that

$$
S_x \eta = (\Pi^J(xz_{\zeta_1} t_{\zeta_1}), \ldots, \Pi^J(xz_{\zeta_s} t_{\zeta_s}); a_0, a_1, \ldots, a_s)
$$

for all $x \in W_{af}$. In particular, $\eta = S_{z_{\zeta_s} t_{\zeta_s}} \eta^C$, with $C$ the connected component containing $\eta$.

### 4 Characterization of Demazure subcrystals in terms of SiLS paths.

#### 4.1 Demazure subcrystals of $\mathcal{B}(\lambda)$.

Let $\lambda \in P_{af}$. For each $x \in W_{af}$, we set

$$
V^\pm_x(\lambda) := U_q^\pm S^\text{norm}_x v_\lambda \subset V(\lambda).
$$

(4.1.1)

Under the $U_q$-module isomorphism $V(\lambda) \cong V(x \lambda)$ in (3.3.3), we have

$$
V(\lambda) \supset V^\pm_x(\lambda) = U_q^\pm S^\text{norm}_x v_\lambda \cong U_q^\pm v_\lambda = V^\pm_e(x \lambda) =: V^\pm(x \lambda) \subset V(x \lambda).
$$

We know from [Kas7, §2.8] that $V^\pm(x \lambda) = U_q^\pm v_{x \lambda}$ is compatible with the global basis of $V(x \lambda)$, that is, there exists a subset $\mathcal{B}^\pm(x \lambda)$ of the crystal basis $\mathcal{B}(x \lambda)$ such that

$$
V^\pm(x \lambda) = \bigoplus_{b \in \mathcal{B}^\pm(x \lambda)} \mathbb{Q}(q_s) G(b) \subset V(x \lambda) = \bigoplus_{b \in \mathcal{B}(x \lambda)} \mathbb{Q}(q_s) G(b).
$$

(4.1.2)
Since the $U_q$-module isomorphism $V(\lambda) \sim V(x\lambda)$ in (3.3.3) is compatible with the global bases, it follows that $V^\pm(x\lambda) = U^\pm S_{x\lambda}^{\text{norm}}v_\lambda$ is also compatible with the global basis of $V(\lambda)$; namely, if we define $B^\pm(\lambda)$ to be the inverse image of $B^\pm(x\lambda)$ under the isomorphism $S^*_x : \mathcal{B}(\lambda) \sim \mathcal{B}(x\lambda)$ of crystals, i.e., $B^\pm(\lambda) := S^{-1}_x(B^\pm(x\lambda))$, then

$$V^\pm(x\lambda) = \bigoplus_{b \in B^\pm(\lambda)} \mathbb{Q}(q_\lambda)G(b) \subset V(\lambda) = \bigoplus_{b \in B(\lambda)} \mathbb{Q}(q_\lambda)G(b). \quad (4.1.3)$$

Here, by [Kashi], p. 234, the subsets $B^\pm(x\lambda)$ in (4.1.2) can be described as follows:

$$B^+ (x\lambda) = \mathcal{B}(x\lambda) \cap (u_\infty \otimes \tau_{x\lambda} \otimes \mathcal{B}(-\infty)),$$

$$B^- (x\lambda) = \mathcal{B}(x\lambda) \cap (\mathcal{B}(\infty) \otimes \tau_{x\lambda} \otimes u_{-\infty}). \quad (4.1.4)$$

From these, we obtain

$$B^+_x(\lambda) = S^*_x \left\{ \mathcal{B}(x\lambda) \cap (u_\infty \otimes \tau_{x\lambda} \otimes \mathcal{B}(-\infty)) \right\}, \quad (4.1.6)$$

$$B^-_x(\lambda) = S^*_x \left\{ \mathcal{B}(x\lambda) \cap (\mathcal{B}(\infty) \otimes \tau_{x\lambda} \otimes u_{-\infty}) \right\}. \quad (4.1.7)$$

Remark 4.1.1. From (4.1.7) (resp., (4.1.6)), using the tensor product rule for crystals, we see that the set $B^-_x(\lambda) \cup \{0\}$ (resp., $B^-_x(\lambda) \cup \{0\}$) is stable under the action of the Kashiwara operator $f_j$ (resp., $e_j$) for all $j \in I_{af}$ (see also [Kashi] Lemma 2.6 (i)).

Lemma 4.1.2. Let $\lambda \in P^+$, and set $J = J_\lambda := \{i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0\}$. For $x, y \in W_{af}$,

$$V^\pm_x(\lambda) = V^\pm_y(\lambda) \iff B^\pm_x(\lambda) = B^\pm_y(\lambda) \iff x^{-1}y \in (W_J)_{af}. \quad (4.1.8)$$

Proof. It is obvious from the definitions that $V^\pm_x(\lambda) = V^\pm_y(\lambda)$ if and only if $B^\pm_x(\lambda) = B^\pm_y(\lambda)$. First we show that if $B^\pm_x(\lambda) = B^\pm_y(\lambda)$, then $x^{-1}y \in (W_J)_{af}$. We see from the definitions that the weights of elements in $B^\pm_x(\lambda)$ are all contained in $x\lambda \pm Q^+_{af}$. Moreover, since $V^\pm_x(\lambda)|_{x\lambda} = \mathbb{Q}(q_\lambda)S_{x\lambda}^{\text{norm}}v_\lambda = \mathbb{Q}(q_\lambda)G(S_xu_\lambda)$ by Remark 3.3.1, we deduce that $S_xu_\lambda$ is a unique element of weight $x\lambda$ in $B^\pm_x(\lambda)$. Similarly, the weights of elements in $B^\pm_y(\lambda)$ are all contained in $y\lambda \pm Q^+_{af}$, and $S_yu_\lambda$ is a unique element of weight $y\lambda$ in $B^\pm_y(\lambda)$. Since $B^\pm_x(\lambda) = B^\pm_y(\lambda)$ by the assumption, we conclude from the above that $x\lambda = y\lambda$, and hence $B^\pm_x(\lambda)|_{x\lambda} = B^\pm_y(\lambda)|_{y\lambda}$. Because $B^\pm_x(\lambda)|_{x\lambda} = \{S_xu_\lambda\}$ and $B^\pm_y(\lambda)|_{y\lambda} = \{S_yu_\lambda\}$ as seen above, it follows immediately that $S_xu_\lambda = S_yu_\lambda$, and hence $S_x^{-1}y u_\lambda = u_\lambda$. Therefore, by [INS] Proposition 5.1.1, we obtain $x^{-1}y \in (W_J)_{af}$, as desired.

Next we show that if $x^{-1}y \in (W_J)_{af}$, then $V^\pm_x(\lambda) = V^\pm_y(\lambda)$. If $x^{-1}y \in (W_J)_{af}$, then we have $S_xu_\lambda = S_yu_\lambda$ by [INS] Proposition 5.1.1, and hence

$$S_{x}^{\text{norm}}v_\lambda = G(S_xu_\lambda) = G(S_yu_\lambda) = S_{y}^{\text{norm}}v_\lambda \quad \text{by Remark 3.3.1}$$

Therefore, we obtain $V^\pm_x(\lambda) = U^\pm S_{x}^{\text{norm}}v_\lambda = U^\pm S_{y}^{\text{norm}}v_\lambda = V^\pm_y(\lambda)$, as desired. □
Remark 4.1.3. Let \( \lambda \in P^+ \), and set \( J = J_\lambda := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \).

(1) We have \( \{ x \in W_{af} \mid x \lambda = \lambda \} \supseteq (W_J)_{af} \), in which the equality holds if and only if \( \lambda \) is of the form \( \lambda = m \alpha_i \) for some \( m \in \mathbb{Z}_{\geq 0} \) and \( i \in I \).

(2) Let \( x, y \in W_{af} \). In view of Lemma 4.1.2 and part (1), the equality \( x \lambda = y \lambda \) does not necessarily imply the equality \( V_x^\pm (\lambda) = V_y^\pm (\lambda) \). However, in this case, there exists a \( U_q \)-module automorphism \( V(\lambda) \rightarrow V(\lambda) \) such that \( v_\lambda \mapsto S^\text{norm}_x v_\lambda \) (see (3.3.3)). Under this automorphism, \( V_x^\pm (\lambda) = U_q S^\text{norm}_x v_\lambda \) is mapped to \( U_q S^\text{norm}_x S^\text{norm}_y v_\lambda = U_q S^\text{norm}_y v_\lambda = V_y^\pm (\lambda) \). Thus we conclude that if \( x \lambda = y \lambda \), then \( V_x^\pm (\lambda) \) is conjugate to \( V_y^\pm (\lambda) \) under a \( U_q \)-module automorphism of \( V(\lambda) \). Similarly, if \( x \lambda = y \lambda \), then \( B_x^\pm (\lambda) \) is conjugate to \( B_y^\pm (\lambda) \) under the crystal automorphism \( S_y^* \) of \( B(\lambda) \).

4.2 Demazure subcrystals of \( B_\infty^\pm (\lambda) \).

Let \( \lambda \in P^+ \), and set \( J = J_\lambda := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \). For \( \eta \in B_\infty^\pm (\lambda) \), we define \( \iota(\eta) \in (W^J)_{af} \) and \( \kappa(\eta) \in (W^J)_{af} \) as in (2.4.3). For each \( x \in (W^J)_{af} \), we set

\[
\begin{align*}
B_{\geq x}(\lambda) &:= \{ \eta \in B_\infty^\pm (\lambda) \mid x \geq \iota(\eta) \}, \\
B_{x \geq}(\lambda) &:= \{ \eta \in B_\infty^\pm (\lambda) \mid \kappa(\eta) \geq x \}.
\end{align*}
\]

We are now ready to state the main result of this paper.

**Theorem 4.2.1.** For every \( \lambda \in P^+ \) and \( x \in (W_J)_{af} \), there hold the equalities

\[
\Psi_{\lambda}(B_x^-(\lambda)) = B_{\geq x}(\lambda) \quad \text{and} \quad \Psi_{\lambda}(B_x^+(\lambda)) = B_{x \geq}(\lambda).
\]

**Remark 4.2.2.** In view of Lemma 4.1.2 together with Proposition 2.2.1 we may assume that \( x \in (W_J)_{af} \) in Theorem 4.2.1.

Here, let us remark that the second equality in (4.2.2) follows from the first one in (4.2.2). Let \( \lambda \in P^+ \), and \( x \in (W_J)_{af} \); recall that \( \mu := -w_0 \lambda \in P^+ \), and \( x^\vee \in (W_J)_{af} \) (for the definitions, see (2.6)). Since \( S_y^* \circ \vee = \vee \circ S_y^* \) holds on \( \overline{B} = \bigsqcup_{\mu \in P_{af}} B(\infty) \otimes \tau_{\mu} \otimes B(-\infty) \) for all \( y \in W_{af} \) (see Remark 3.4.1 and (3.4.3)), we have

\[
S_{w_0}^* (B_x^-(\lambda))^\vee = S_{w_0}^* \left( S_{x^{-1}}^* \left( B(x \lambda) \cap (u_\infty \otimes \tau_{x \lambda} \otimes B(-\infty)) \right) \right)^\vee \quad \text{by (4.1.6)}
\]

\[
= S_{w_0}^* S_{x^{-1}}^* \left( B(x \lambda) \cap (u_\infty \otimes \tau_{x \lambda} \otimes B(-\infty)) \right)^\vee
\]

\[
= S_{w_0 x^{-1}}^* (B(-x \lambda) \cap (B(\infty) \otimes \tau_{-x \lambda} \otimes u_{-\infty}))
\]

\[
= S_{w_0 x^{-1}}^* (B(x w_0 \mu) \cap (B(\infty) \otimes \tau_{x w_0 \mu} \otimes u_{-\infty})
\]

\[
= B_{-x \vee}(\mu) \quad \text{by (4.1.7)}
\]

\[
= B_{x \vee}(\mu) \quad \text{by Lemma 4.1.2}
\]

Also, it is easily seen from the definitions (2.6.5) and (4.2.1), by using Lemma 2.6.1, that \( B_{x \geq}(\lambda) = (B_{\geq x}(\mu))^\vee \). Therefore, if the equality \( \Psi_{\mu}(B_{\vee}^-(\mu)) = B_{\geq x}(\mu) \) holds, then the equality \( \Psi_{\lambda}(B_x^-(\lambda)) = B_{x \geq}(\lambda) \) also holds by the definition (3.5.3) of \( \Psi_{\lambda} \). Thus, the remaining task is to prove the first equality in (4.2.2).
5 Proof of Theorem 4.2.1.

Throughout this section, we fix \( \lambda = \sum_{i \in I} m_i \omega_i \in P^+ \), with \( m_i \in \mathbb{Z}_{\geq 0} \) for \( i \in I \), and then set \( J = J_{\lambda} := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \).

5.1 Fundamental properties of \( B_x^- (\lambda) \), part 1.

Proposition 5.1.1 and Corollary 5.1.2 below are easy consequences of \([\text{Kas} 7]\ §2.8\), but we include their proofs for the convenience of the reader.

**Proposition 5.1.1.** Let \( x \in W_{af} \) and \( j \in I_{af} \) be such that \( \langle \alpha_j^\vee, x \lambda \rangle \geq 0 \). Then,

\[
B_x^- (\lambda) = \{ e_j^k b \mid b \in B_{r_j x}^- (\lambda), \ k \in \mathbb{Z}_{\geq 0} \} \setminus \{ 0 \};
\]

in particular, \( B_x^- (\lambda) \supset B_{r_j x}^- (\lambda) \). Consequently, the set \( B_x^- (\lambda) \cup \{ 0 \} \) is stable under the action of the Kashiwara operator \( e_j \) for \( j \in I_{af} \) such that \( \langle \alpha_j^\vee, x \lambda \rangle \geq 0 \) (see also Remark 4.1.1).

**Proof.** First, we show the equality

\[
U_q^- S_x^{\text{norm}} v_\lambda = U_q^{(j)} U_q^- S_{r_j x}^{\text{norm}} v_\lambda.
\]

(5.1.2)

Here, \( U_q^{(j)} \) denotes the \( \mathbb{Q}(q_{\alpha}) \)-subalgebra of \( U_q \) generated by \( E_j, F_j, \) and \( q_{\alpha}^{\alpha_j^\vee} \), which is isomorphic to the quantized universal enveloping algebra associated with \( \mathfrak{sl}_2 \); recall the triangular decomposition \( U_q^{(j)} = \langle E_j \rangle \langle q_{\alpha_j^\vee} \rangle \langle F_j \rangle \) of \( U_q^{(j)} \). We show the inclusion \( \supset \) as follows:

\[
U_q^{(j)} U_q^- S_{r_j x}^{\text{norm}} v_\lambda = U_q^{(j)} U_q^- S_{r_j x}^{\text{norm}} S_x^{\text{norm}} v_\lambda = U_q^{(j)} U_q^- F_j^{(\langle \alpha_j^\vee, x \lambda \rangle)} S_x^{\text{norm}} v_\lambda
\]

\[
\subset U_q^{(j)} U_q^- S_{r_j x}^{\text{norm}} v_\lambda = \langle E_j \rangle \langle q_{\alpha_j^\vee} \rangle \langle F_j \rangle U_q^- S_x^{\text{norm}} v_\lambda
\]

\[
= \langle E_j \rangle U_q^- S_x^{\text{norm}} v_\lambda \subset U_q^- \langle E_j \rangle S_x^{\text{norm}} v_\lambda \quad \text{since} \quad [E_j, F_l] \in \langle q_{\alpha_j^\vee} \rangle \quad \text{for all} \quad l \in I_{af}
\]

\[
= U_q^- S_x^{\text{norm}} v_\lambda \quad \text{since} \quad E_j S_x^{\text{norm}} v_\lambda = 0 \quad \text{by the assumption} \quad \langle \alpha_j^\vee, x \lambda \rangle \geq 0;
\]

the second equality in (1) follows from the fact that \( S_x^{\text{norm}} v_\lambda \) is an extremal weight vector of weight \( x \lambda \), and the assumption that \( \langle \alpha_j^\vee, x \lambda \rangle \geq 0 \). Similarly, we show the opposite inclusion \( \subset \) as follows:

\[
U_q^- S_x^{\text{norm}} v_\lambda = U_q^- S_{r_j x}^{\text{norm}} S_{r_j x}^{\text{norm}} v_\lambda = U_q^- E_j^{(\langle \alpha_j^\vee, x \lambda \rangle)} S_{r_j x}^{\text{norm}} v_\lambda
\]

\[
\subset U_q^- \langle E_j \rangle S_{r_j x}^{\text{norm}} v_\lambda \subset \langle E_j \rangle U_q^- S_{r_j x}^{\text{norm}} v_\lambda \subset U_q^{(j)} U_q^- S_{r_j x}^{\text{norm}} v_\lambda.
\]

Thus we obtain the equality (5.1.2). Since the left-hand side of (5.1.2) is identical to \( V_x^- (\lambda) \), we see that \( U_q^{(j)} U_q^- S_{r_j x}^{\text{norm}} v_\lambda \) is compatible with the global basis of \( V(\lambda) \), and has the crystal basis \( B_x^- (\lambda) \).

Now, recall from (3.3.3) that there exists a \( U_q \)-module isomorphism \( V(\lambda) \cong V(r_j x \lambda) \) that maps \( v_\lambda \) to \( S_{x-r_j x}^{\text{norm}} v_{r_j x \lambda} \). Under this isomorphism, the right-hand side \( U_q^{(j)} U_q^- S_{r_j x}^{\text{norm}} v_\lambda \subset V(\lambda) \) of (5.1.2) is mapped to \( U_q^{(j)} U_q^- v_{r_j x \lambda} = U_q^{(j)} V^- (r_j x \lambda) \subset V(r_j x \lambda) \). When we regard \( V(r_j x \lambda) \)}
as a $U_q^{(j)}$-module by restriction, $V^-(r_j x \lambda)$ is regarded as an $(F_j)$-module, which is compatible with the global basis of $V(r_j x \lambda)$. Therefore, we deduce from (the dual version of) $\text{[Kas7]}$ Eq. (2.28) and comments following it] that $U_q^{(j)} V^-(r_j x \lambda) = \langle E_j \rangle V^-(r_j x \lambda)$ is also compatible with the global basis of $V(r_j x)$, and has the crystal basis

$$\{ e_j^k b \mid b \in B^-(r_j x \lambda), k \in \mathbb{Z}_{\geq 0} \} \setminus \{ 0 \}.$$ 

Recall that the $U_q$-module isomorphism $V(\lambda) \xrightarrow{\sim} V(r_j x \lambda)$ is compatible with the global bases, and induces the crystal isomorphism $S_{r_j x}^* : B(\lambda) \xrightarrow{\sim} B(r_j x \lambda)$. Namely, we have the following correspondences of modules and their crystal bases:

$$\begin{align*}
V(\lambda) & \xrightarrow{\sim} V(r_j x \lambda) \\
U_q^{(j)}U_q^{-S_{r_j x}^\text{norm} v_\lambda} & \xrightarrow{\sim} U_q^{(j)}V^-(r_j x \lambda) \\
B_x^- (\lambda) & \xrightarrow{S_{r_j x}^*} \{ e_j^k b \mid b \in B^-(r_j x \lambda), k \in \mathbb{Z}_{\geq 0} \} \setminus \{ 0 \}.
\end{align*}$$

From this, we conclude that

$$B_x^- (\lambda) = (S_{r_j x}^*)^{-1} \{ e_j^k b \mid b \in B^-(r_j x \lambda), k \in \mathbb{Z}_{\geq 0} \} \setminus \{ 0 \}$$

$$= \{ e_j^k (S_{r_j x}^*)^{-1} (b) \mid b \in B^-(r_j x \lambda), k \in \mathbb{Z}_{\geq 0} \} \setminus \{ 0 \}$$

$$= \{ e_j^k b \mid b \in B_{r_j x}^- (\lambda), k \geq 0 \} \setminus \{ 0 \} \quad \text{by (4.1.7).}$$

This completes the proof of the proposition. \hfill \Box

**Corollary 5.1.2.** Let $x \in W_{af}$, and $j \in I_{af}$. For every $b \in B_x^- (\lambda)$, we have $f_j^{\max b} := f_j^{\varphi_j (b)} b \in B_{r_j x}^- (\lambda)$, where $\varphi_j (b) := \max \{ k \in \mathbb{Z}_{\geq 0} \mid f_j^k b \neq 0 \}$.

**Proof.** If $\langle \alpha_j^\vee, x \lambda \rangle \geq 0$, then the assertion follows from Proposition 5.1.1 and Remark 4.1.1 Assume that $\langle \alpha_j^\vee, x \lambda \rangle < 0$. Let $b \in B_x^- (\lambda)$. We see from Remark 4.1.1 that $f_j^{\max b} \in B_x^- (\lambda)$. Also, it follows from Proposition 5.1.1 that $B_x^- (\lambda) \subset B_{r_j x}^- (\lambda)$. Combining these, we obtain $f_j^{\max b} \in B_x^- (\lambda) \subset B_{r_j x}^- (\lambda)$, as desired. \hfill \Box

### 5.2 Fundamental properties of $B_x^- (\lambda)$, part 2.

Now we recall some results in $\text{[Kas6]}$ and $\text{[BN]}$. We define a $\mathbb{Q}(q_0)$-subalgebra $U_q'$ of $U_q$ by

$$U_q' := \langle E_j, F_j, q^h \mid j \in I_{af}, h \in d^{-1} (\bigoplus_{j \in I_{af}} \mathbb{Z} \alpha_j^\vee) \rangle \subset U_q,$$

which can be thought of as the quantized universal enveloping algebra $U_q (g_{af}')$ associated with the derived subalgebra $g_{af}' := [g_{af}, g_{af}]$ of $g_{af}$. We know from $\text{[Kas6]}$ p. 142 that for each $i \in I$, there exists a $U_q'$-module automorphism $z_i : V(\varpi_i) \to V(\varpi_i)$ that maps $v_{\varpi_i}$ to $v_{\varpi_i}^{[1]}$, where for $k \in \mathbb{Z}$, we denote by $u_{\varpi_i}^{[k]}$ the (unique) element of weight $\varpi_i + k \delta$ in $B(\varpi_i)$, and set $v_{\varpi_i}^{[k]} :=$
$G(u_{[k]})$ (see [Kas6] Proposition 5.8); note that $z_i$ commutes with the Kashiwara operators $e_j, f_j, j \in I_{af}$, on $V(\varpi_i)$. Moreover, we see from [Kas6] Propositions 5.12 and 5.15 that $z_i$ preserves the crystal lattice $\mathcal{L}(\varpi_i)$ of $V(\varpi_i)$, and hence induces a $\mathbb{Q}$-linear automorphism $z_i : \mathcal{L}(\varpi_i)/q_s\mathcal{L}(\varpi_i) \rightarrow \mathcal{L}(\varpi_i)/q_s\mathcal{L}(\varpi_i)$. Because $z_i$ commutes with the Kashiwara operators $e_j, f_j, j \in I_{af}$, on $\mathcal{L}(\varpi_i)/q_s\mathcal{L}(\varpi_i)$, and because $z_i(u_{\varpi_i}) = u_{i}^{[1]} \in \mathcal{B}(\varpi_i)$, it follows immediately from [Kas6] Proposition 5.12 that $z_i$ preserves the crystal basis $\mathcal{B}(\varpi_i) \subset \mathcal{L}(\varpi_i)/q_s\mathcal{L}(\varpi_i)$ of $V(\varpi_i)$ (see also [Kas6] Theorem 5.17).

Recall that $\lambda \in P^+$ is of the form $\lambda = \sum_{i \in I} m_i \varpi_i$, with $m_i \in \mathbb{Z}_{\geq 0}$ for $i \in I$. We fix an arbitrary total ordering on $I$, and then set $V(\lambda) := \bigotimes_{i \in I} V(\varpi_i)^{\otimes m_i}$. We can easily show (see also [BN], the comment preceding Eq. (4.8)) that $\tilde{v}_\lambda := \bigotimes_{i \in I} v_{\varpi_i}^{\otimes m_i} \in \tilde{V}(\lambda)$ is an extremal weight vector of weight $\lambda$. By [BN] Eq. (4.8) and Corollary 4.15, there exists a $U_q$-module embedding

$$\Phi_{\lambda} : V(\lambda) \hookrightarrow \tilde{V}(\lambda) = \bigotimes_{i \in I} V(\varpi_i)^{\otimes m_i} \quad (5.2.1)$$

that maps $v_\lambda$ to $\tilde{v}_\lambda := \bigotimes_{i \in I} v_{\varpi_i}^{\otimes m_i}$.

**Remark 5.2.1.** We can show by induction on $\ell(x)$ that for every $x \in W_{af}$,

$$S^\text{norm}_x v_\lambda = S^\text{norm}_x \left( \bigotimes_{i \in I} v_{\varpi_i}^{\otimes m_i} \right) \in \mathbb{Q}(q_s) \left( \bigotimes_{i \in I} (S^\text{norm}_x v_{\varpi_i})^{\otimes m_i} \right);$$

cf. [AK] Lemma 1.6 (1)]. Therefore, under the $U_q$-module embedding $\Phi_{\lambda} : V(\lambda) \hookrightarrow \tilde{V}(\lambda)$ in (5.2.1), $V^-_{\lambda}(x) \subset V(\lambda)$ for $x \in W_{af}$ is mapped as follows:

$$V^-_{\lambda}(x) = U_q S^\text{norm}_x v_\lambda \xrightarrow{\Phi_{\lambda}} U_q (S^\text{norm}_x v_{\varpi_i})^{\otimes m_i} \subset \bigotimes_{i \in I} (U_q S^\text{norm}_x v_{\varpi_i})^{\otimes m_i}. \quad (5.2.2)$$

Recall that $\tilde{V}(\lambda)$ has the crystal basis $(\tilde{\mathcal{L}}(\lambda), \tilde{\mathcal{B}}(\lambda))$, where

$$\tilde{\mathcal{L}}(\lambda) := \bigotimes_{i \in I} \mathcal{L}(\varpi_i)^{\otimes m_i}, \quad \tilde{\mathcal{B}}(\lambda) := \bigotimes_{i \in I} \mathcal{B}(\varpi_i)^{\otimes m_i}.$$

We see from [BN] p. 369, the 2nd line from the bottom] that $\Phi_{\lambda}(\mathcal{L}(\lambda)) \subset \tilde{\mathcal{L}}(\lambda)$, and hence $\Phi_{\lambda}$ induces a $\mathbb{Q}$-linear embedding of $\mathcal{L}(\lambda)/q_s\mathcal{L}(\lambda)$ into $\tilde{\mathcal{L}}(\lambda)/q_s\tilde{\mathcal{L}}(\lambda)$, which we denote by $\Phi_{\lambda}|_{q=0}$. Note that we have the following commutative diagram for all $j \in I_{af}$:

$$\begin{array}{ccc}
\mathcal{L}(\lambda)/q_s\mathcal{L}(\lambda) & \xrightarrow{\Phi_{\lambda}|_{q=0}} & \tilde{\mathcal{L}}(\lambda)/q_s\tilde{\mathcal{L}}(\lambda) \\
e_j, f_j & & \downarrow e_j, f_j \\
\mathcal{L}(\lambda)/q_s\mathcal{L}(\lambda) & \xrightarrow{\Phi_{\lambda}|_{q=0}} & \tilde{\mathcal{L}}(\lambda)/q_s\tilde{\mathcal{L}}(\lambda).
\end{array}$$

For each $i \in I$ and $1 \leq l \leq m_i$, we define $z_{i,l}$ to be the $U'_q$-module automorphism of $\tilde{V}(\lambda)$ which acts as $z_i$ only on the $l$-th factor of $V(\varpi_i)^{\otimes m_i}$ in $\tilde{V}(\lambda)$, and as the identity map
on the other factors of $\tilde{V}(\lambda)$; notice that $z_{i,l}$ commutes with the Kashiwara operators $e_j, f_j, j \in I_{af}$, on $\tilde{V}(\lambda)$. Since $z_{i,l}$ preserves $\tilde{\mathcal{L}}(\lambda)$ by the definition above, we deduce that $z_{i,l}$ induces a $\mathbb{Q}$-linear automorphism $z_{i,l} : \tilde{\mathcal{L}}(\lambda)/q_s\tilde{\mathcal{L}}(\lambda) \rightarrow \tilde{\mathcal{L}}(\lambda)/q_s\tilde{\mathcal{L}}(\lambda)$, which commutes with the Kashiwara operators $e_j, f_j, j \in I_{af}$, on $\tilde{\mathcal{L}}(\lambda)/q_s\tilde{\mathcal{L}}(\lambda)$. Also, notice that the $z_{i,l}$'s, $i \in I$, $1 \leq l \leq m_i$, commute with each other. For $c_0 = (\rho^{(i)}_{i \in I}) \in \operatorname{Par}(\lambda)$, we set

$$s_{c_0}(z^{-1}) := \prod_{i \in I} s_{\rho^{(i)}}(z_{i,1}^{-1}, \ldots, z_{i,m_i}^{-1}) \in \operatorname{End}_\mathbb{Q}(\tilde{\mathcal{L}}(\lambda)/q_s\tilde{\mathcal{L}}(\lambda)).$$

(5.2.3)

Here, for a partition $\rho = (\rho_1 \geq \cdots \geq \rho_m \geq 0)$ of length less than $m \in \mathbb{Z}_{\geq 1}$, $s_{\rho}(x) = s_{\rho}(x_1, \ldots, x_m)$ denotes the Schur polynomial in the variables $x_1, \ldots, x_m$ corresponding to the partition $\rho$, that is, the character of the finite-dimensional, irreducible, polynomial representation of $\operatorname{GL}(m)$ whose highest weight corresponds to $\rho$ (see [F] §8.2); note that for each $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{Z}_{\geq 0}^m$, the coefficient $c_{\nu}$ of $x^\nu = x_1^{\nu_1} \cdots x_m^{\nu_m}$ in $s_{\rho}(x)$ is equal to the dimension of the $\nu$-weight space, and in particular, $c_{\nu} = 1$ for $\nu = (\rho_1, \ldots, \rho_m, 0) \in \mathbb{Z}_{\geq 0}^m$, which is the highest weight. In particular, we have

$$s_{\rho}(x) = x^{\rho_\nu} + \sum_{\nu \in \mathbb{Z}_{\geq 0}^m, \nu \neq \nu_\rho} c_{\nu} x^\nu.$$

(5.2.4)

By [BN] Proposition 4.13, together with the fact “$\operatorname{sgn}(c, p) = 1$” shown on page 375 of [BN], the image of the crystal basis $\mathcal{B}(\lambda) \subset \mathcal{L}(\lambda)/q_s\mathcal{L}(\lambda)$ under the $\mathbb{Q}$-linear embedding $\Phi_{\lambda|q=0} : \mathcal{L}(\lambda)/q_s\mathcal{L}(\lambda) \hookrightarrow \tilde{\mathcal{L}}(\lambda)/q_s\tilde{\mathcal{L}}(\lambda)$ is identical to

$$\{ s_{c_0}(z^{-1})b \mid c_0 \in \operatorname{Par}(\lambda), b \in \tilde{B}_0(\lambda) \} \subset \tilde{\mathcal{L}}(\lambda)/q_s\tilde{\mathcal{L}}(\lambda),$$

(5.2.5)

where $\tilde{B}_0(\lambda)$ denotes the connected component of $\tilde{B}(\lambda)$ containing $\tilde{u}_\lambda := \otimes_{i \in I} u_{\infty i}^{z_{i,m_i}}$, and

$$\Phi_{\lambda|q=0}(u^{c_0}) = s_{c_0}(z^{-1})\tilde{u}_\lambda \quad \text{for each } c_0 \in \operatorname{Par}(\lambda);$$

(5.2.6)

recall from [8.2] that $u^{c_0} = b(c_0) \otimes \tau_\lambda \otimes u_{-\infty}$ is an extremal element of weight $\lambda - |c_0|\delta$ contained in $B(\lambda)$. For later use, we rewrite the right-hand side $s_{c_0}(z^{-1})\tilde{u}_\lambda$ of (5.2.6) as follows. We set

$$Z_{\geq 0}^\lambda := \{ \nu = (\nu^{(i)}_{i \in I})_{i \in I} = (\nu^{(i)}_1, \ldots, \nu^{(i)}_{m_i})_{i \in I} \mid \nu^{(i)} \in \mathbb{Z}_{\geq 0}, i \in I \},$$

and then

$$\tilde{u}_{\lambda}^{[\nu]} := \otimes_{i \in I} (u_{\omega_i}^{[\nu^{(i)}_1]} \otimes \cdots \otimes u_{\omega_i}^{[\nu^{(i)}_{m_i}]}) \in \tilde{B}(\lambda) \quad \text{for } \nu = (\nu^{(i)}_1, \ldots, \nu^{(i)}_{m_i})_{i \in I} \in Z_{\geq 0}^\lambda.$$

Also, if $c_0 = (\rho^{(i)}_{i \in I}) \in \operatorname{Par}(\lambda)$, with $\rho^{(i)} = (\rho^{(i)}_1 \geq \cdots \geq \rho^{(i)}_{m_i} \geq 0)$ for $i \in I$, then we set $\nu^{c_0} := (\nu^{(i)}_{\rho^{(i)}})_{i \in I} = (\rho^{(i)}_1, \ldots, \rho^{(i)}_{m_i-1}, 0)_{i \in I} \in Z_{\geq 0}^\lambda$. Since $z_i$ maps $u_{\omega_i} \in B(\omega_i)$ to $u_{\omega_i}^{[1]} \in B(\omega_i)$ for each $i \in I$, we deduce from (5.2.4) that

$$s_{c_0}(z^{-1})\tilde{u}_\lambda = \tilde{u}_{\lambda}^{[\nu^{c_0}]} + \sum_{\nu \in Z_{\geq 0}^\lambda, \nu \neq \nu^{c_0}} c_{\nu} \tilde{u}_{\lambda}^{[\nu]},$$

(5.2.7)

where $c_{\nu} = \prod_{i \in I} c_{\nu^{(i)}} \in \mathbb{Z}_{\geq 0}$ if $\nu = (\nu^{(i)}_{i \in I}) \in Z_{\geq 0}^\lambda$. 

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Proposition 5.2.2. Let $c_0 \in \text{Par}(\lambda)$, and $x, y \in (W^J)_{af}$. Then,

$$S_y(u^{c_0}) \in B_x(\lambda) \iff y \succeq x. \quad (5.2.8)$$

In order to show the proposition above, we need some lemmas.

Lemma 5.2.3. Let $x, y \in (W^J)_{af}$. If $y \succeq x$, then $B_x^-(\lambda) \subset B_y^-(\lambda)$.

Proof. We may assume that $x \xrightarrow{\beta} y$ in $\text{SiB}^J$ for some $\beta \in \Delta^+_\alpha$. Write $x \in (W^J)_{af}$ in the form $x = wz \xi t_{\xi}$, with $w \in W^J$ and $\xi \in Q^{V^J, \text{ad}}$ (see (2.2.5)). By Remark 2.3.3, $\beta$ is either of the following forms: $\beta = \alpha$ for some $\alpha \in \Delta^+$, or $\beta = \alpha + \delta$ for some $-\alpha \in \Delta^+$; in both cases, we have $w^{-1}\alpha \in \Delta \setminus \Delta_J$, and hence $\langle \beta^\vee, x \lambda \rangle = \langle \alpha^\vee, w \xi t_{\xi} \lambda \rangle = \langle w^{-1} \alpha^\vee, \lambda \rangle > 0$. Thus we obtain $\langle \beta^\vee, y \lambda \rangle = \langle \beta^\vee, r \lambda x \lambda \rangle < 0$. Also, by (4.1.7), we have

$$B_x^-(\lambda) = S_{y^{-1}}^* \left( B(y \lambda) \cap (B(\infty) \otimes \tau_{y \lambda} \otimes u_{-\infty}) \right),$$

$$B_x^-(\lambda) = B_{r \beta y}^-(\lambda) = S_{y^{-1}}^* \left( B(r \beta y \lambda) \cap (B(\infty) \otimes \tau_{r \beta y \lambda} \otimes u_{-\infty}) \right) = S_{y^{-1}}^* S_{r \beta}^* \left( B(r \beta y \lambda) \cap (B(\infty) \otimes \tau_{r \beta y \lambda} \otimes u_{-\infty}) \right).$$

Therefore, in order to show that $B_x^-(\lambda) \supset B_y^-(\lambda)$, it suffices to show that

$$S_{r \beta}^* \left( B(r \beta \mu) \cap (B(\infty) \otimes \tau_{r \beta \mu} \otimes u_{-\infty}) \right) \subset B(\mu) \cap (B(\infty) \otimes \tau_{\mu} \otimes u_{-\infty}), \quad (5.2.9)$$

where we set $\mu := r \beta y \lambda$. The set on the right-hand side of the inclusion $\subset$ in (5.2.9) is the crystal basis of $V^-(-\mu) = U_q^{-} v_\mu$ (see (4.1.5)), that is,

$$B(\mu) \cap (B(\infty) \otimes \tau_{\mu} \otimes u_{-\infty}) = \{ b \in B(\mu) \mid G(b) \in V^-(-\mu) \}. \quad (5.2.10)$$

The set on the left-hand side of the inclusion $\subset$ in (5.2.9) is the crystal basis of $V_{r \beta}^-(-\mu) = U_q^{-} S_{r \beta}^\text{norn} v_\mu$ by (4.1.7), that is,

$$S_{r \beta}^* \left( B(r \beta \mu) \cap (B(\infty) \otimes \tau_{r \beta \mu} \otimes u_{-\infty}) \right) = \{ b \in B(\mu) \mid G(b) \in V_{r \beta}^-(-\mu) \}. \quad (5.2.11)$$

Since $\langle \beta^\vee, \mu \rangle = -\langle \beta^\vee, y \lambda \rangle > 0$ as shown above, it follows from [Kas7, Proposition 2.8] that $S_{r \beta}^\text{norn} v_\mu \in V^-(-\mu) = U_q^{-} v_\mu$, which implies that $V^-(-\mu) \supset V_{r \beta}^-(-\mu)$. Combining this containment with (5.2.10) and (5.2.11), we conclude (5.2.9). This proves the lemma.

Lemma 5.2.4. For each $y \in W_{af}$, $S_y(u^{c_0}) \in B_y^-(\lambda)$.

Proof. We prove the assertion by induction on $\ell(y)$. If $\ell(y) = 0$, then $y = e$, and hence the assertion follows immediately from Proposition 3.5.1 (4.1.7), and the definition of $u^{c_0}$. Assume that $\ell(y) > 0$, and take $j \in I_{af}$ such that $\ell(r_j y) = \ell(y) - 1$; by our induction hypothesis, we have $S_{r_j y}(u^{c_0}) \in B_{r_j y}(\lambda)$. If $\langle \alpha_j^\vee, r_j y \lambda \rangle \geq 0$, then $S_y(u^{c_0}) = S_{r_j} S_{r_j y}(u^{c_0}) = f_{r_j}^{\text{max}} S_{r_j y}(u^{c_0})$ since $u^{c_0}$ is an extremal element. Therefore, we deduce $S_y(u^{c_0}) \in B_y^-(\lambda)$ by Corollary 5.1.2 since $S_{r_j y}(u^{c_0}) \in B_{r_j y}(\lambda)$ by our induction hypothesis. If $n := \langle \alpha_j^\vee, r_j y \lambda \rangle \leq 0$, then $S_y(u^{c_0}) = S_{r_j} S_{r_j y}(u^{c_0}) = e_{r_j}^{-} S_{r_j y}(u^{c_0})$. Since $\langle \alpha_j^\vee, y \lambda \rangle \geq 0$, and $S_{r_j y}(u^{c_0}) \in B_{r_j y}(\lambda)$ by our induction hypothesis, it follows from Proposition 5.1.1 that $S_y(u^{c_0}) \in B_y^-(\lambda)$. This proves the lemma.
Proof of Proposition 5.2.4. The “if” part follows immediately from Lemmas 5.2.3 and 5.2.4. Indeed, assume that \( y \geq x \). Then, \( B'_x(\lambda) \subset B'_x(\lambda) \) by Lemma 5.2.3. Therefore, by Lemma 5.2.4, \( S_y(u^{c_0}) \in B'_y(\lambda) \subset B'_x(\lambda) \). This proves the “if” part.

Now, we prove the “only if” part.

Claim 1. Let \( c_0 \in \text{Par}(\lambda) \), and \( \xi, \zeta \in Q^{\nu, J-u} \). If \( f_{\zeta \xi}(u^{c_0}) \in B_{\zeta \xi}(\lambda) \), then \( \zeta \xi \geq \zeta \xi \).

Proof of Claim. By Proposition 6.2.2, it suffices to show that \([\xi - \zeta] \in Q^{\nu'}_{1,J} \), where \([\cdot] : Q^{\nu'}_{1,J} \oplus Q^{\nu'}_{1,J} \to Q^{\nu'}_{1,J} \) is the projection (see (2.1.3)). By (5.2.6) and (5.2.7), we have

\[
\Phi|_{\nu = 0}(S_{\zeta \xi}(u^{c_0})) = S_{\zeta \xi}(u^{c_0}(z^{-1})u_{\lambda}) = S_{\zeta \xi}(u_{\lambda}^{[\nu c_0]}) + \sum_{\nu} c_{\nu} S_{\zeta \xi}(u_{\lambda}^{[\nu]}) ;
\]

\( \in B(\lambda) \)

notice that for \( \nu, \nu' \in \mathbb{Z}_{\geq 0} \), \( S_{\zeta \xi}(u_{\lambda}^{[\nu]}) = S_{\zeta \xi}(u_{\lambda}^{[\nu']}) \) if and only if \( \nu = \nu' \), and hence that the elements \( S_{\zeta \xi}(u_{\lambda}^{[\nu]}) \in B(\lambda), \nu \in \mathbb{Z}_{\geq 0} \), are linearly independent over \( Q \). Also, from Lemma 1.6 (1), we deduce that for every \( \nu = (\nu^{(i)}_{1}, \ldots, \nu^{(i)}_{m}) \), \( i \in I \), we have

\[
S_{\zeta \xi}(u_{\lambda}^{[\nu]}) = S_{\zeta \xi}(u^{(i)}) \otimes \cdots \otimes u^{(i)} ;
\]

\( \in B(\lambda) \)

Since the global basis element \( G(S_{\zeta \xi}(u^{c_0})) \in \mathcal{L}(\lambda) \subset V(\lambda) \) is contained in \( V^{-}_{\zeta \xi}(\lambda) = U_{q}^{\text{form}} v^{(i)} \) the assumption, and since \( \Phi(\mathcal{L}(\lambda)) \subset \tilde{\mathcal{L}}(\lambda) \) as mentioned above, it follows from Remark 5.2.1 that

\[
\Phi|_{\nu = 0}(S_{\zeta \xi}(u^{c_0})) \in \left( \bigotimes_{i \in I} V^{-}_{\zeta \xi}(\mathcal{W}) \right) \cap \tilde{\mathcal{L}}(\lambda).
\]

Because \( \Phi|_{\nu = 0} : \mathcal{L}(\lambda)/q_{\bar{s}} \mathcal{L}(\lambda) \hookrightarrow \tilde{\mathcal{L}}(\lambda)/q_{\bar{s}} \tilde{\mathcal{L}}(\lambda) \) is induced by \( \Phi \) : \( V(\lambda) \hookrightarrow \tilde{V}(\lambda) \), and because \( V^{-}_{\zeta \xi}(\mathcal{W}) \) has the global basis \( \{ G(b) \mid b \in B^{-}_{\zeta \xi}(\mathcal{W}) \} \) for each \( i \in I \), we see that

\[
\Phi|_{\nu = 0}(S_{\zeta \xi}(u^{c_0})) \in \text{Span}_{Q} \left( \bigotimes_{i \in I} B^{-}_{\zeta \xi}(\mathcal{W}) \right) \subset \tilde{\mathcal{L}}(\lambda)/q_{\bar{s}} \tilde{\mathcal{L}}(\lambda) = \text{Span}_{Q} \tilde{B}(\lambda).
\]

Here we recall that \( \tilde{B}(\lambda) = \bigotimes_{i \in I} B(\mathcal{W}) \) is a \( Q \)-basis of the \( Q \)-vector space \( \tilde{\mathcal{L}}(\lambda)/q_{\bar{s}} \tilde{\mathcal{L}}(\lambda) \), and that \( \bigotimes_{i \in I} B^{-}_{\zeta \xi}(\mathcal{W}) \) is a subset of \( \tilde{B}(\lambda) \) generating the vector space \( U \) over \( Q \) in (5.2.14). Therefore, we deduce from (5.2.12) and (5.2.14) that

\[
S_{\zeta \xi}(u_{\lambda}^{[\nu c_0]}) \in \bigotimes_{i \in I} B^{-}_{\zeta \xi}(\mathcal{W}) \cap U.
\]

If \( c_0 = (\rho^{(i)})_{i \in I} \) with \( \rho^{(i)} = (\rho^{(i)}_{1} \geq \cdots \geq \rho^{(i)}_{m_{i} - 1} \geq 0) \) for \( i \in I \), then it follows immediately from (5.2.13) that

\[
S_{\zeta \xi}(u_{\lambda}^{[\nu c_0]}) = \bigotimes_{i \in I} \left( S_{\zeta \xi}(u^{(i)}_{\mathcal{W}}) \otimes \cdots \otimes S_{\zeta \xi}(u^{(i)}_{\mathcal{W}}) \otimes \sum_{\nu} c_{\nu} S_{\zeta \xi}(u_{\mathcal{W}}^{(i)}) \right).
\]
By combining this equality with (5.2.15), we find that for every \( i \in I \setminus J \), the tensor factor \( S_{z\xi t_{\zeta}}(u_{\omega_i}) \) in the position (*) is contained in \( B_{z\xi t_{\zeta}}(\omega_i) \). Let \( i \in I \setminus J \). Since the weights of elements in \( B_{z\xi t_{\zeta}}(\omega_i) \) are contained in \( z\xi t_{\zeta} \omega_i - Q_{af}^+ \), we conclude that \( z\xi t_{\zeta} \omega_i = wt(S_{z\xi t_{\zeta}}(u_{\omega_i})) \in z\xi t_{\zeta} \omega_i - Q_{af}^+ \); since \( z\xi, z\zeta \in W_J \), and \( i \in I \setminus J \), we have \( z\xi \omega_i = z\zeta \omega_i = \omega_i \). From these, we obtain

\[
\omega_i - \langle \xi, \omega_i \rangle \delta = z\xi t_{\zeta} \omega_i - Q_{af}^+ = \omega_i - \langle \xi, \omega_i \rangle \delta - Q_{af}^+.
\]

and hence \( \langle \xi - \zeta, \omega_i \rangle \delta \in Q_{af}^+ \). Hence it follows that \( \langle \xi - \zeta, \omega_i \rangle \geq 0 \) for every \( i \in I \setminus J \), which implies that \( [\xi - \zeta] \in Q_{I \setminus J}^+ \). This proves the claim. 

**Claim 2.** Let \( c_0 \in \text{Par}(\lambda) \), and \( y \in (W^J)_{af} \), \( \zeta \in Q^\vee, J-ad \). If \( S_y(u^{c_0}) \in B_{z\xi t_{\zeta}}(\lambda) \), then \( y \succeq z\xi t_{\zeta} \).

**Proof of Claim 2.** Write \( y \in (W^J)_{af} \) in the form \( w \cdot z\xi t_{\zeta} \), with \( w \in W_J \) and \( \xi \in Q^\vee, J-ad \) (see \([2.2.5]\)). We prove the claim by induction on \( \ell(w) \). If \( \ell(w) = 0 \), then \( w = e \), and hence the claim follows immediately from Claim 1. Assume that \( \ell(w) > 0 \), and take \( j \in I \) such that \( \ell(r_j w) = \ell(w) - 1 \); in this case, we have \( -w^{-1} \alpha_j \in \Delta^+ \setminus \Delta^+_j \) (see [LNS\(^{[5]}\)] Proposition 5.10) for example), which implies that \( n := \langle \alpha_j^\vee, y \lambda \rangle = \langle \alpha_j^\vee, w \lambda \rangle < 0 \). Therefore, we obtain \( r_jy \in (W^J)_{af} \) and \( y > r_jy \) by Lemma \([2.3.5]\). Also, since \( wt(u^{c_0}) = \lambda - |c_0| \delta \), we have

\[
S_{r_jy}(u^{c_0}) = S_{r_j}S_y(u^{c_0}) = e_j^{-n}S_y(u^{c_0}).
\]

Here, since \( j \in I \), we have \( \langle \alpha_j^\vee, z\xi t_{\zeta} \lambda \rangle = \langle \alpha_j^\vee, \lambda \rangle \geq 0 \). Hence it follows from Proposition \([5.1.1]\) that the set \( B_{z\xi t_{\zeta}}(\lambda) \cup \{0\} \) is stable under the action of the root operator \( e_j \). Because \( S_y(u^{c_0}) \in B_{z\xi t_{\zeta}}(\lambda) \) by the assumption, we deduce that \( S_{r_jy}(u^{c_0}) = e_j^{-n}S_y(u^{c_0}) \in B_{z\xi t_{\zeta}}(\lambda) \). Therefore, by our induction hypothesis, we obtain \( r_jy \succeq z\xi t_{\zeta} \). Since \( y > r_jy \) as seen above, we conclude that \( y > r_jy \geq z\xi t_{\zeta} \), as desired.

Now, let \( x, y \in (W^J)_{af} \), and assume that \( S_y(u^{c_0}) \in B_{z\xi t_{\zeta}}(\lambda) \). By [AK] Lemma 1.4], there exist \( j_1, j_2, \ldots, j_p \in I_{af} \) such that

1. \( \langle \alpha_j^\vee, r_{j_{m-1}} \cdots r_{j_2} r_{j_1} x \lambda \rangle > 0 \) for all \( 1 \leq m \leq p \);
2. \( r_{j_p} r_{j_{p-1}} \cdots r_{j_2} r_{j_1} x \lambda \in \lambda + \mathbb{Z} \delta \).

By Lemma \([2.3.5]\) together with condition (1), we see that \( r_{j_m} \cdots r_{j_2} r_{j_1} x \in (W^J)_{af} \) for all \( 0 \leq m \leq p \). From this, we deduce by condition (2) that \( r_{j_p} r_{j_{p-1}} \cdots r_{j_2} r_{j_1} x = z\xi t_{\zeta} \) for some \( \zeta \in Q^\vee, J-ad \). We show by induction on the length \( p \) of the sequence above that \( y \succeq x \). If \( p = 0 \), then \( x = z\xi t_{\zeta} \), and hence the assertion follows immediately from Claim 1. Assume that \( p > 0 \).

**Case 1.** Assume that \( \langle \alpha_{j_1}^\vee, y \lambda \rangle > 0 \); note that \( r_{j_1}y \in (W^J)_{af} \) by Lemma \([2.3.5]\). Since \( u^{c_0} \in B(\lambda) \) is an extremal element of weight \( \lambda - |c_0| \delta \), we have

\[
S_{r_{j_1} y}(u^{c_0}) = S_{r_{j_1}} S_y(u^{c_0}) = f_{j_1}^{max} S_y(u^{c_0}).
\]
Since \( S_y(u^0) \in \mathcal{B}_x^-(\lambda) \) by the assumption, it follows from Corollary 5.1.2 that \( S_{r_jy}(u^0) = f_{j_1} \max S_y(u^0) \in \mathcal{B}_{r_jx}^-(\lambda) \). Therefore, by our induction hypothesis (applied to \( r_j, x \in (W^J)_{af} \)), we obtain \( r_jy \succeq r_jx \). Because \( \langle \alpha_j^\vee, r_jx \lambda \rangle < 0 \) by condition (1), and because \( \langle \alpha_j^\vee, r_jy \lambda \rangle < 0 \) by our assumption above, we conclude from Lemma 2.3.6 (3) that \( y \succeq x \).

**Case 2.** Assume that \( \langle \alpha_j^\vee, y \lambda \rangle \leq 0 \). Since \( u^0 \) is an extremal element of weight \( \lambda - |c_0| \delta \), it follows that \( f_{j_1} S_y(u^0) = 0 \), and hence \( f_{j_1} \max S_y(u^0) = f_{j_1}^0 S_y(u^0) = S_y(u^0) \). Since \( S_y(u^0) \in \mathcal{B}_x^-(\lambda) \) by the assumption, we deduce from Corollary 5.1.2 that \( S_y(u^0) = f_{j_1} \max S_y(u^0) \in \mathcal{B}_{r_jx}^-(\lambda) \). Therefore, by our induction hypothesis (applied to \( r_j, x \in (W^J)_{af} \)), we obtain \( y \succeq r_jx \). Since \( \langle \alpha_j^\vee, x \lambda \rangle > 0 \) by condition (1), we have \( r_jx \triangleright x \) by Lemma 2.3.5. Hence we conclude that \( y \succeq r_jx \triangleright x \), as desired.

This completes the proof of the proposition. □

**Corollary 5.2.5.** Let \( x, y \in (W^J)_{af} \). Then, \( \mathcal{B}_x^-(\lambda) \subset \mathcal{B}_x^-(\lambda) \) if and only if \( y \succeq x \).

**Proof.** The “if” part is already proved in Lemma 5.2.3. Let us prove the “only if” part. Since \( S_y(u_\lambda) \in \mathcal{B}_y^-(\lambda) \) by Lemma 5.2.4, we have \( S_y(u_\lambda) \in \mathcal{B}_x^-(\lambda) \). Therefore, by applying Proposition 5.2.2 to \( c_0 = (\rho^{(i)})_{i \in I} \) with \( \rho^{(i)} = 0 \) for all \( i \in I \), we obtain \( y \succeq x \) (note that in this case, \( b(c_0) = u_\infty \) and \( u^0 = u_\lambda \)). This proves the corollary. □

### 5.3 Fundamental properties of \( \mathcal{B}_x^x(\lambda) \).

**Lemma 5.3.1** (cf. Remark 4.1.1). Let \( x \in (W^J)_{af} \). The set \( \mathcal{B}_x^x(\lambda) \cup \{0\} \) (resp., \( \mathcal{B}_x^-_x(\lambda) \cup \{0\} \)) is stable under the action of the root operator \( f_j \) (resp., \( e_j \)) for all \( j \in I_{af} \).

**Proof.** We give a proof only for \( \mathcal{B}_x^x(\lambda) \); the proof for \( \mathcal{B}_x^-_x(\lambda) \) is similar. Let \( \eta \in \mathcal{B}_x^x(\lambda) \), i.e., \( \kappa(\eta) \succeq x \), and let \( j \in I_{af} \) be such that \( f_j \eta \neq 0 \). If \( \kappa(f_j \eta) = \kappa(\eta) \), then there is nothing to prove. So, assume that \( \kappa(f_j \eta) = r_j \kappa(\eta) \). Then we deduce from the comment following (2.4.4) that \( \langle \alpha_j^\vee, \kappa(\eta) \lambda \rangle > 0 \), and hence \( r_j \kappa(\eta) \triangleright \kappa(\eta) \) by Lemma 2.3.5. Since \( \kappa(\eta) \succeq x \) by our assumption, it follows that \( \kappa(f_j \eta) = r_j \kappa(\eta) \triangleright \kappa(\eta) \succeq x \), which implies that \( f_j \eta \in \mathcal{B}_x^x(\lambda) \). This proves the lemma. □

**Proposition 5.3.2** (cf. Proposition 5.1.1). Let \( x \in (W^J)_{af} \), and \( j \in I_{af} \).

1. If \( \langle \alpha_j^\vee, x \lambda \rangle > 0 \) (note that \( r_jx \in (W^J)_{af} \) by Lemma 2.3.3), then
   \[
   \mathcal{B}_x^x(\lambda) = \{ e_j^k \eta \mid \eta \in \mathcal{B}_{r_jx}^x(\lambda), k \in \mathbb{Z}_{\geq 0} \} \cup \{0\} \quad (\subset \mathcal{B}_{r_jx}^x(\lambda)).
   \] (5.3.1)

2. The set \( \mathcal{B}_x^x(\lambda) \cup \{0\} \) is stable under the action of the root operator \( e_j \) for \( j \in I_{af} \) such that \( \langle \alpha_j^\vee, x \lambda \rangle \geq 0 \).
Proof. (1) First we prove the inclusion $\subset$. Let $\eta \in B_{\geq x}^\infty(\lambda)$; note that $\kappa(\eta) \succeq x$ by the definition. Assume that $\langle \alpha_j^\vee, \kappa(\eta) \lambda \rangle \leq 0$. Since $\langle \alpha_j^\vee, x \lambda \rangle > 0$ by the assumption, we see by Lemma 2.3.6(1) that $\kappa(\eta) \succeq r_j x$, and hence $\eta \in B_{\geq r_j x}^\infty(\lambda)$. Thus, $\eta = e_j^0 \eta$ is contained in the set on the right-hand side of (5.3.1). Assume that $\langle \alpha_j^\vee, \kappa(\eta) \lambda \rangle > 0$. It follows from Lemma 2.3.5 that $\kappa(f_j^{\max} \eta) = r_j \kappa(\eta)$. Also, because $\langle \alpha_j^\vee, \kappa(\eta) \lambda \rangle > 0$ and $\langle \alpha_j^\vee, x \lambda \rangle > 0$ by the assumption, we deduce from Lemma 2.3.6(3), together with our assumption $\kappa(\eta) \succeq x$, that $\kappa(f_j^{\max} \eta) = r_j \kappa(\eta) \succeq r_j x$, which implies that $f_j^{\max} \eta \in B_{\geq r_j x}^\infty(\lambda)$. From this, we conclude that $\eta$ is contained in the set on the right-hand side of (5.3.1). This proves the inclusion $\subset$.

Next we prove the opposite inclusion $\supset$. Let $\eta \in B_{\geq r_j x}^\infty(\lambda)$, and assume that $e_j^\ell \eta \neq 0$ for some $k \in \mathbb{Z}_{\geq 0}$; note that $\kappa(\eta) \succeq r_j x$, and that $\kappa(e_j^\ell \eta)$ is equal either to $\kappa(\eta)$ or to $r_j \kappa(\eta)$. If $\kappa(e_j^\ell \eta) = \kappa(\eta)$, then we have $\kappa(e_j^\ell \eta) = \kappa(\eta) \succeq r_j x$. Since $\langle \alpha_j^\vee, x \lambda \rangle > 0$ by the assumption, it follows from Lemma 2.3.5 that $r_j x \succeq x$. Combining these, we obtain $\kappa(e_j^\ell \eta) \succeq x$, which implies that $e_j^\ell \eta \in B_{\geq x}^\infty(\lambda)$. Assume that $\kappa(e_j^\ell \eta) = r_j \kappa(\eta)$. Then we see from the definition of the root operator $e_j$ (see the comment following (2.4.3)) that $\langle \alpha_j^\vee, \kappa(\eta) \lambda \rangle < 0$. Recall that $\langle \alpha_j^\vee, r_j x \lambda \rangle < 0$ by the assumption. Since $\kappa(\eta) \succeq r_j x$ by our assumption, we deduce from Lemma 2.3.6(3) that $\kappa(e_j^\ell \eta) = r_j \kappa(\eta) \succeq x$, which implies that $e_j^\ell \eta \in B_{\geq x}^\infty(\lambda)$. This proves part (1).

(2) The assertion for $j \in I_{af}$ such that $\langle \alpha_j^\vee, x \lambda \rangle > 0$ follows immediately from (5.3.1). Let $j \in I_{af}$ be such that $\langle \alpha_j^\vee, x \lambda \rangle = 0$, and let $\eta \in B_{\geq x}^\infty(\lambda)$ be such that $e_j \eta \neq 0$; note that $\kappa(\eta) \succeq x$ by our assumption, and that $\kappa(e_j \eta)$ is equal either to $\kappa(\eta)$ or to $r_j \kappa(\eta)$. If $\kappa(e_j \eta) = \kappa(\eta)$, then it is obvious that $e_j \eta \in B_{\geq x}^\infty(\lambda)$. If $\kappa(e_j \eta) = r_j \kappa(\eta)$, then $\langle \alpha_j^\vee, \kappa(\eta) \lambda \rangle < 0$ by the same argument as in the proof of part (1). Since $\langle \alpha_j^\vee, x \lambda \rangle = 0$, and $\kappa(\eta) \succeq x$ by our assumption, it follows from Lemma 2.3.6(2) that $\kappa(e_j \eta) = r_j \kappa(\eta) \succeq x$, which implies that $e_j \eta \in B_{\geq x}^\infty(\lambda)$. This completes the proof of the proposition. \[\square\]

Corollary 5.3.3 (cf. Corollary 5.1.2). Let $x \in (W^J)_{af}$ and $j \in I_{af}$ be such that $\langle \alpha_j^\vee, x \lambda \rangle \neq 0$. (note that $r_j x \in (W^J)_{af}$ by Lemma 2.3.5). For every $\eta \in B_{\geq x}^\infty(\lambda)$, we have $f_j^{\max} \eta \in B_{\geq r_j x}^\infty(\lambda)$.

Proof. If $\langle \alpha_j^\vee, x \lambda \rangle > 0$, then the assertion follows immediately from (5.3.1) and Lemma 5.3.1. Assume that $\langle \alpha_j^\vee, x \lambda \rangle < 0$. Let $\eta \in B_{\geq x}^\infty(\lambda)$. We see from Lemma 5.3.1 that $f_j^{\max} \eta \in B_{\geq x}^\infty(\lambda)$. Also, it follows from Proposition 5.3.2(1) that $B_{\geq x}^\infty(\lambda) \subset B_{\geq r_j x}^\infty(\lambda)$. Combining these, we obtain $f_j^{\max} \eta \in B_{\geq x}^\infty(\lambda) \subset B_{\geq x}^\infty(\lambda)$, as desired. \[\square\]

Proposition 5.3.4 (cf. Corollary 5.2.5). Let $x, y \in (W^J)_{af}$. Then, $y \succeq x$ if and only if $B_{\geq y}^\infty(\lambda) \subset B_{\geq y}^\infty(\lambda)$.

Proof. The “only if” part is obvious from the definitions. Let us prove the “if” part. It is obvious from the definition that $(y; 0, 1) \in B_{\geq y}^\infty(\lambda)$ is contained in $B_{\geq y}^\infty(\lambda)$. Since $B_{\geq y}^\infty(\lambda) \subset B_{\geq x}^\infty(\lambda)$ by the assumption, we have $(y; 0, 1) \in B_{\geq x}^\infty(\lambda)$, and hence $y \succeq x$. This proves the proposition. \[\square\]
5.4 Proof of Theorem 4.2.1

We need the following technical lemma.

Lemma 5.4.1. For each \( \eta \in \mathbb{B}^\infty (\lambda) \) and \( x \in W_{af} \), there exist \( j_1, j_2, \ldots, j_p \in I_{af} \) satisfying the following conditions:

(i) \( \langle \alpha_{j_m}^\vee, r_{j_m-1} \cdots r_{j_2} r_{j_1} x \lambda \rangle \geq 0 \) for all \( 1 \leq m \leq p \);

(ii) \( f_{j_p}^{\max} f_{j_{p-1}}^{\max} \cdots f_{j_2}^{\max} f_{j_1}^{\max} \eta = S_{\delta \xi} \eta^C \) for some \( \xi \in Q^\vee, I^{\text{ad}} \) and \( C \in \text{Conn}(\mathbb{B}^\infty (\lambda)) \).

Proof. First, we see from [AK, Lemma 1.4] that there exist \( j_1, j_2, \ldots, j_a \in I_{af} \) such that

(1a) \( \langle \alpha_{j_m}^\vee, r_{j_m-1} \cdots r_{j_2} r_{j_1} x \lambda \rangle \geq 0 \) for all \( 1 \leq m \leq a \);

(2a) \( r_{j_a} r_{j_{a-1}} \cdots r_{j_2} r_{j_1} x \lambda = \lambda + \mathbb{Z} \delta \).

We deduce from condition (2a) that \( r_{j_a} r_{j_{a-1}} \cdots r_{j_2} r_{j_1} x = z t_\zeta \) for some \( z \in W_j \) and \( \zeta \in Q^\vee \).

Let \( w_0 = r_{j_b} r_{j_{b-1}} \cdots r_{j_{a+1}} r_{j_{a+1}} \) be a reduced expression of the longest element \( w_0 \in W \). Then, there hold the following:

(1b) for all \( a+1 \leq m \leq b \),

\( \langle \alpha_{j_m}^\vee, r_{j_m-1} \cdots r_{j_{a+1}} r_{j_a} \cdots r_{j_2} r_{j_1} x \lambda \rangle = \langle \alpha_{j_m}^\vee, r_{j_{a+1}} \cdots r_{j_{a+1}} \lambda \rangle \geq 0; \)

(2b) \( r_{j_b} r_{j_{b-1}} \cdots r_{j_{a+1}} r_{j_a} \cdots r_{j_2} r_{j_1} x = w_0 z t_\zeta; \)

(3b) the element

\( \eta' := f_{j_b}^{\max} f_{j_{b-1}}^{\max} \cdots f_{j_{a+1}}^{\max} f_{j_a}^{\max} \cdots f_{j_2}^{\max} f_{j_1}^{\max} \eta \)

corresponds to \( w_0 \)

\( \in \mathbb{B}^\infty (\lambda) \)

is a lowest weight element with respect to \( I \), i.e., \( f_j \eta' = 0 \) for all \( j \in I \) (this follows from [Kas5, Corollarie 9.1.4 (2)] since \( \mathbb{B}^\infty (\lambda) \) is a regular crystal).

It follows from Lemma 2.4.5 together with condition (3b), that \( \langle \alpha_j^\vee, \kappa(\eta') \lambda \rangle \leq 0 \) for all \( j \in I \), which implies that \( \kappa(\eta') \lambda \equiv w_0 \lambda \mod \mathbb{R} \delta \) since \( W \lambda \cap (-P^+) = \{w_0 \lambda\} \). We deduce from (the dual version of) [NS3, Lemma 4.3.2] that for the element \( \eta' \) in (3b), there exist \( j_{b+1}, j_{b+2}, \ldots, j_p \in I_{af} \) satisfying the following conditions:

(1c) for all \( b+1 \leq m \leq p \),

\( \langle \alpha_{j_m}, r_{j_m-1} \cdots r_{j_{b+1}} r_{j_{b+1}} w_0 \lambda \rangle = \langle \alpha_{j_m}, r_{j_{b+1}} \cdots r_{j_{b+1}} \lambda \rangle > 0; \)

(2c) \( \eta'' := f_{j_p}^{\max} f_{j_{p-1}}^{\max} \cdots f_{j_{b+2}}^{\max} f_{j_{b+1}}^{\max} \eta \) is an element of \( \mathbb{B}^\infty (\lambda) \) such that \( \eta''(t) \equiv t \lambda \mod \mathbb{R} \delta \) for all \( t \in [0, 1] \).
We see by Remark 3.5.2(2) that $\eta'' = S_{z \xi^\ell} \eta^C$ for some $\xi \in Q^\vee, J$-ad and $C \in \text{Conn}(B^\infty_\lambda(\lambda))$.

Concatenating the three sequences of elements in $I_{af}$ above, we obtain the sequence

$$
\tilde{j}_1, \tilde{j}_2, \ldots, \tilde{j}_m, \tilde{j}_{m+1}, \tilde{j}_{m+2}, \ldots, \tilde{j}_n, \tilde{j}_{n+1}, \tilde{j}_{n+2}, \ldots, \tilde{j}_{m+n} \quad \text{satisfy (1a), (2a), (3a) successively.}
$$

We show that this sequence indeed satisfies conditions (i) and (ii). We see from the definition of $\eta'$ in condition (3b) and condition (2c) that the sequence above satisfies condition (ii). For $1 \leq m \leq b$, it follows immediately from (1a) and (1b) that the sequence above satisfies condition (i). Also, for $b + 1 \leq m \leq p$, we have

$$
\langle \alpha_{j_m}, r_{j_m-1} \cdots r_{j_{b+1}} r_{j_b} \cdots r_{j_2} r_{j_1} x \lambda \rangle = \langle \alpha_{j_m}, r_{j_m-1} \cdots r_{j_{b+1}} w_0 z \xi \lambda \rangle \quad \text{by condition (2b)}
$$

$$
= \langle \alpha_{j_m}, r_{j_m-1} \cdots r_{j_{b+1}} w_0 \lambda \rangle > 0 \quad \text{by (1c)}.
$$

This completes the proof of the lemma.

**Lemma 5.4.2.** Let $b \in B(\lambda)$, and set $\eta := \Psi_{\lambda}(b) \in B^\infty_\lambda(\lambda)$. If $x \in (W^J)_{af}$ satisfies $b \in B_{\lambda}^-(\lambda)$, then $\kappa(\eta) \geq x$, and hence $\eta \in B_{\geq x}^\infty(\lambda)$.

**Proof.** We fix an arbitrary $x \in (W^J)_{af}$ such that $b \in B_{\lambda}^-(\lambda)$. Take $j_1, j_2, \ldots, j_p \in I_{af}$ satisfying conditions (i) and (ii) in Lemma 5.4.1 for $\eta = \Psi_{\lambda}(b) \in B^\infty_\lambda(\lambda)$ and $x \in (W^J)_{af} \subset W_{af}$. We prove the assertion by induction on the length $p$ of this sequence. If $p = 0$, then $\eta = S_{z \xi^\ell} \eta^C$ for some $\xi \in Q^\vee, J$-ad and $C \in \text{Conn}(B^\infty_\lambda(\lambda))$; note that in this case, $\kappa(\eta) = z \xi t_\xi$ by Remark 3.5.2(2). By the assumption, we have $S_{z \xi^\ell}(u^{(C)}) = \Psi_{\lambda}^{-1}(\eta) = b \in B_{\lambda}^-(\lambda)$. Therefore, we conclude from Proposition 5.2.2 that $\kappa(\eta) = z \xi t_\xi \geq x$, and hence $\eta \in B_{\geq x}^\infty(\lambda)$, as desired. Now, assume that $p > 0$.

**Case 1.** Assume that $\langle \alpha_{j_1}^\vee, x \lambda \rangle > 0$; note that $r_{j_1} x \in (W^J)_{af}$ by Lemma 2.3.3 and that $f_{j_1}^{\max} \in B_{r_{j_1} x}^-(\lambda)$ by Corollary 5.1.2. Observe that the sequence $j_2, j_3, \ldots, j_p \in I_{af}$ satisfies conditions (i) and (ii) in Lemma 5.4.1 for $f_{j_1}^{\max} \eta = \Psi_{\lambda}(f_{j_1}^{\max} b)$ and $r_{j_1} x$. Therefore, by our induction hypothesis, we obtain $\kappa(f_{j_1}^{\max} \eta) \geq r_{j_1} x$, and hence $f_{j_1}^{\max} \eta \in B_{\geq r_{j_1} x}^\infty(\lambda)$. Since $\langle \alpha_{j_1}^\vee, x \lambda \rangle > 0$ by our assumption, we conclude from (5.3.1) that $\eta \in B_{\geq x}^\infty(\lambda)$.

**Case 2.** Assume that $\langle \alpha_{j_1}^\vee, x \lambda \rangle = 0$; remark that $f_{j_1}^{\max} \in B_{\lambda}^-(\lambda)$ by Remark 4.1.1. Observe that the sequence $j_2, j_3, \ldots, j_p \in I_{af}$ satisfies conditions (i) and (ii) in Lemma 5.4.1 for $f_{j_1}^{\max} \eta = \Psi_{\lambda}(f_{j_1}^{\max} b)$ and $x$; notice that $r_{j_1} x \lambda = x \lambda$ since $\langle \alpha_{j_1}^\vee, x \lambda \rangle = 0$. Therefore, by our induction hypothesis, we obtain $\kappa(f_{j_1}^{\max} \eta) \geq x$, and hence $f_{j_1}^{\max} \eta \in B_{\geq x}^\infty(\lambda)$. Since $\langle \alpha_{j_1}^\vee, x \lambda \rangle = 0$ by our assumption, it follows from Proposition 5.3.2(2) that $\eta \in B_{\geq x}^\infty(\lambda)$.

This proves the lemma. \[ \square \]
Lemma 5.4.3. Let $\eta \in \mathbb{B}_{\Sigma x}^\infty(\lambda)$, and set $b := \Psi_\lambda^{-1}(\eta) \in \mathcal{B}(\lambda)$. If $x \in (W^J)_{af}$ satisfies $\eta \in \mathbb{B}_{\Sigma x}^\infty(\lambda)$, then $b \in \mathcal{B}_x^-(\lambda)$.

Proof. We fix an arbitrary $x \in (W^J)_{af}$ such that $\eta \in \mathbb{B}_{\Sigma x}^\infty(\lambda)$. Take $j_1, j_2, \ldots, j_p \in I_{af}$ satisfying conditions (i) and (ii) in Lemma 5.4.1 for $\eta$ and $x$. We prove the assertion by induction on the length $p$ of this sequence. If $p = 0$, then $\eta = S_{z_1 t_1} \eta^C$ for some $\xi \in Q^J, J_{af}$ and $C \in \text{Conn}(\mathbb{B}_{\Sigma x}^\infty(\lambda))$; note that in this case, $\kappa(\eta) = z_1 t_1$ by Remark 3.5.2 (2). Since $\eta \in \mathbb{B}_{\Sigma x}^\infty(\lambda)$ by the assumption, we have $z_1 t_1 \geq x$. Therefore, we conclude from Proposition 5.2.2 that $b = \Psi_\lambda^{-1}(\eta) = S_{z_1 t_1} (u^{\Theta(C)}) \in \mathcal{B}_x^-(\lambda)$. Now, assume that $p > 0$.

Case 1. If $\langle \alpha_{j_1}^\vee, x \lambda \rangle > 0$, then $r_{j_1} x \in (W^J)_{af}$ by Lemma 2.3.5 and $f_{j_1}^{\max} \eta \in \mathbb{B}_{\Sigma r_{j_1} x}^\infty(\lambda)$ by Corollary 5.3.3. Observe that the sequence $j_2, \ldots, j_p \in I_{af}$ satisfies conditions (i) and (ii) in Lemma 5.4.1 for $f_{j_1}^{\max} \eta$ and $r_{j_1} x$. Therefore, by our induction hypothesis, we obtain $f_{j_1}^{\max} b \in \mathcal{B}_{r_{j_1} x}^-(\lambda)$. Since $\langle \alpha_{j_1}^\vee, x \lambda \rangle > 0$ by our assumption, it follows from (5.1.1) that $b \in \mathcal{B}_{r_{j_1} x}^-(\lambda)$.

Case 2. Assume that $\langle \alpha_{j_1}^\vee, x \lambda \rangle = 0$. By Lemma 5.3.1, we have $f_{j_1}^{\max} \eta \in \mathbb{B}_{\Sigma x}^\infty(\lambda)$. Observe that the sequence $j_2, \ldots, j_p \in I_{af}$ satisfies conditions (i) and (ii) in Lemma 5.4.1 for $f_{j_1}^{\max} \eta$ and $x$; notice that $r_{j_1} x \lambda = x \lambda$ since $\langle \alpha_{j_1}^\vee, x \lambda \rangle = 0$. Therefore, by our induction hypothesis, we obtain $f_{j_1}^{\max} b \in \mathcal{B}_x^-(\lambda)$. Since $\langle \alpha_{j_1}^\vee, x \lambda \rangle = 0$ by our assumption, it follows from Proposition 5.1.1 that $b \in \mathcal{B}_x^-(\lambda)$.

This proves the lemma. 

Proof of Theorem 4.2.1. As mentioned at the end of 4.2, our remaining task is to prove the equality

$$\Psi_\lambda(\mathcal{B}_x^-(\lambda)) = \mathbb{B}_{\Sigma x}^\infty(\lambda) \quad \text{for all } x \in (W^J)_{af}. \quad (5.4.1)$$

The inclusion $\subset$ in (5.4.1) follows immediately from Lemma 5.4.2. Also, the opposite inclusion $\supset$ in (5.4.1) follows immediately from Lemma 5.4.3. This completes the proof of Theorem 4.2.1. 

6 Graded characters.

In this section, we fix $\lambda = \sum_{i \in I} m_i \varpi_i \in P_+$, and set $J = J_\lambda := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \}$.

6.1 Graded character formula for $V_e^-(\lambda)$.

The weights of the Demazure submodule $V_e^-(\lambda) = U_q^- v_\lambda$ (corresponding to $x = e$, the identity element) are all contained in $\lambda - Q^+_{af} \subset \lambda - Q_{af}$, and hence every weight space of $V_e^-(\lambda)$ is finite-dimensional. Therefore, we can define the (ordinary) character $\text{ch} V_e^-(\lambda)$ of $V_e^-(\lambda)$ to be

$$\text{ch} V_e^-(\lambda) := \sum_{\beta \in Q_{af}} \dim V_e^-(\lambda)_{\lambda - \beta} x^{\lambda - \beta}.$$
Here we recall that an element \( \beta \in Q_{af} \) can be written uniquely in the form: \( \beta = \gamma + k\delta \) for \( \gamma \in Q \) and \( k \in \mathbb{Z} \); if we set \( x^\delta := q \), then \( x^{\lambda - \beta} = x^{\lambda - \gamma} q^k \). Now we define the graded character \( gch V_e^- (\lambda) \) of \( V_e^- (\lambda) \) to be

\[
gch V_e^- (\lambda) := \sum_{\gamma \in Q, k \in \mathbb{Z}} \dim V_e^- (\lambda)_{\lambda - \gamma + k\delta} x^{\lambda - \gamma} q^k,
\]

which is obtained from the ordinary character \( ch V_e^- (\lambda) \) by replacing \( x^\delta \) with \( q \).

**Theorem 6.1.1.** Keep the notation and setting above. The graded character \( gch V_e^- (\lambda) \) of \( V_e^- (\lambda) \) can be expressed as

\[
gch V_e^- (\lambda) = \left( \prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^{-r}) \right)^{-1} P_\lambda (x; q^{-1}, 0),
\]

where \( P_\lambda (x; q, 0) \) denotes the specialization at \( t = 0 \) of the symmetric Macdonald polynomial \( P_\lambda (x; q, t) \).

### 6.2 Degree function.

Let \( \text{cl} : \mathbb{R} \otimes_{\mathbb{Z}} P_{af} \to (\mathbb{R} \otimes_{\mathbb{Z}} P_{af})/\mathbb{R} \delta \) denote the canonical projection. For an element \( \eta = (x_1, \ldots, x_s; a_0, a_1, \ldots, a_s) \in \mathbb{B}_\omega (\lambda) \), we define \( \text{cl}(\eta) \) to be the piecewise-linear, continuous map \( \text{cl}(\eta) : [0, 1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{af}/\mathbb{R} \delta \) whose “direction vector” for the interval \([a_{u-1}, a_u]\) is equal to \( \text{cl}(x_u \lambda) \) for each \( 1 \leq u \leq s \); that is,

\[
(\text{cl}(\eta))(t) := \sum_{p=1}^{u-1} (a_p - a_{p-1}) \text{cl}(x_p \lambda) + (t - a_{u-1}) \text{cl}(x_u \lambda)
\]

for \( t \in [a_{u-1}, a_u] \), \( 1 \leq u \leq s \).

Because the map \( \eta : [0, 1] \to \mathbb{R} \otimes_{\mathbb{Z}} P_{af} \) defined by (2.4.1) is an LS path of shape \( \lambda \), the map \( \text{cl}(\eta) \) above is a “projected (by cl)” LS path of shape \( \lambda \), introduced in [NS1 (3.4)] and [NS2 page 117] (see also [LNS2 §2]). By [LNS3 Theorem 3.3], the crystal, denoted by \( \mathcal{B}(\lambda)_{\text{cl}} \) in [NS1] and [NS2], of “projected” LS paths of shape \( \lambda \) is identical to the crystal, denoted by QLS(\( \lambda \)), of all quantum Lakshmibai-Seshadri paths (QLS paths for short) of shape \( \lambda \) under the identification \( \text{cl}(W_{af} \lambda) = W \text{cl}(\lambda) \cong W_J \); for the definition of QLS paths of shape \( \lambda \), see [LNS3 §3.2].

**Remark 6.2.1.** Define a surjective map \( \text{cl} : (W_J)_{af} \to W_J \) by (see (2.2.5))

\[
\text{cl}(x) = w \quad \text{if} \quad x = wz_\xi t_\xi, \quad \text{with} \quad w \in W_J \quad \text{and} \quad \xi \in Q^\vee, J-\text{ad}.
\]

The quantum LS path of shape \( \lambda \) corresponding to the map \( \text{cl}(\eta) \) defined by (6.2.1) is given as:

\[
(\text{cl}(x_1), \ldots, \text{cl}(x_s); a_0, a_1, \ldots, a_s),
\]

where, for each \( 1 \leq p < q \leq s \) such that \( \text{cl}(x_p) = \cdots = \text{cl}(x_q) \), we drop \( \text{cl}(x_p), \ldots, \text{cl}(x_{q-1}) \) and \( a_p, \ldots, a_{q-1} \).

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Thus we obtain a map \( \text{cl} : B^\pi\leftarrow(\lambda) \rightarrow \text{QLS}(\lambda) \). Recall that root operators on \( \text{QLS}(\lambda) \) are defined in exactly the same manner as those on \( B^\pi\leftarrow(\lambda) \) (see, e.g., \cite[§2.3]{LNS32}); from these definitions of root operators on \( B^\pi\leftarrow(\lambda) \) and \( \text{QLS}(\lambda) \), we see that the map \( \text{cl} : B^\pi\leftarrow(\lambda) \rightarrow \text{QLS}(\lambda) \) commutes with root operators. Because the crystal \( \text{QLS}(\lambda) \) is connected by \cite[Proposition 3.23]{NSTI}, the map \( \text{cl} \) above is surjective.

**Lemma 6.2.2.** Let \( \psi \in \text{QLS}(\lambda) \), and assume that \( \psi = X \text{cl}(\eta) \) for some monomial \( X \) in root operators. Then,

\[
\text{cl}^{-1}(\psi) = \{ Xs_{t_\zeta}\eta^C : \zeta \in \text{Conn}(B^\pi\leftarrow(\lambda)) \}
\]

for the definition of \( \eta^C \), see Proposition \[2.5.1](1).

**Proof.** First we prove the inclusion \( \supset \). Let \( C \in \text{Conn}(B^\pi\leftarrow(\lambda)) \), and \( \zeta \in Q^\vee_1, j \). Since the map \( \text{cl} : B^\pi\leftarrow(\lambda) \rightarrow \text{QLS}(\lambda) \) commutes with root operators, and since \( \text{cl}(s_{t_\zeta}\eta^C) = \text{cl}(\eta) \) (see Remark \[3.5.2](2)), we have

\[
\text{cl}(Xs_{t_\zeta}\eta^C) = X \text{cl}(s_{t_\zeta}\eta^C) = X \text{cl}(\eta) = \psi,
\]

which implies that \( Xs_{t_\zeta}\eta^C \in \text{cl}^{-1}(\psi) \).

Next we prove the opposite inclusion \( \subset \). Write \( X : X = x_{j_1}x_{j_2}\cdots x_{j_p} \), where \( x_{j_q} \) is either \( e_{j_q} \) or \( f_{j_q} \) for each \( 1 \leq q \leq p \). We prove the inclusion \( \subset \) by induction on \( p \). Assume that \( p = 0 \), i.e., \( \psi = \text{cl}(\eta) \). If \( \eta \in \text{cl}^{-1}(\psi) \), then we deduce from Remark \[3.5.2](2) that \( \eta \) is of the form:

\[
\eta = (z_{\zeta_1}t_{\zeta_1}, \ldots, z_{\zeta_s}t_{\zeta_s} ; a_0, a_1, \ldots, a_s)
\]

for some \( s \geq 1 \) and \( \zeta_1, \ldots, \zeta_s \in Q^\vee, J \), and hence that \( \eta = S_{z_{\zeta_s}t_{\zeta_s}}\eta^C \), with \( C \) the connected component containing \( \eta \). Because

\[
\langle h_j, \text{wt}(s_{t_{\zeta_s}}\eta^C) - \lambda \rangle = 0 \quad (j \in \lambda + 2\delta)
\]

for \( j \in J \), we see that \( S_{t_{j_1}}s_{t_{\zeta_s}}\eta^C = s_{t_{\zeta_s}}\eta^C \) for \( j \in J \). Therefore, we have \( s_{z_{\zeta_s}t_{\zeta_s}}\eta^C = s_{z_{\zeta_s}t_{\zeta_s}}\eta^C = S_{z_{\zeta_s}t_{\zeta_s}}\eta^C \). Also, since \( s_{t_{j_3}}\eta^C = \eta^C \) for \( \beta \in Q^\vee_1 \), it follows that

\[
\eta = S_{z_{\zeta_s}t_{\zeta_s}}\eta^C = S_{t_{\zeta_s}}\eta^C = S_{t_{\zeta_s}}\eta^C = S_{t_{\zeta_s}}\eta^C
\]

(note that \( \zeta_s - [\zeta_s] \in Q^\vee_1 \)), where \( [\cdot] : Q^\vee = Q^\vee_1 + Q^\vee_2 \rightarrow Q^\vee_1 \) denotes the projection (see \[2.1.3]). Thus, we conclude that \( \eta \) is contained in the set \( \{ s_{t_{\zeta_s}}\eta^C : \zeta \in \text{Conn}(B^\pi\leftarrow(\lambda)) \} \).

Assume now that \( p > 0 \), and set \( j := j_1, X' := x_{j_2}\cdots x_{j_p}, \psi' := X' \text{cl}(\eta) \in \text{QLS}(\lambda) \). We define \( y_j := e_j \) (resp., \( y_j := f_j \)) if \( x_j = e_j \) (resp., \( x_j = f_j \)); note that \( y_j \psi = \psi' \). Let \( \eta \in \text{cl}^{-1}(\psi) \). Since the map \( \text{cl} : B^\pi\leftarrow(\lambda) \rightarrow \text{QLS}(\lambda) \) commutes with root operators, we have \( \text{cl}(y_j\eta) = y_j \text{cl}(\eta) = y_j \psi = \psi' \), and hence \( y_j\eta \in \text{cl}^{-1}(\psi') \). By our induction hypothesis, there exists \( C \in \text{Conn}(B^\pi\leftarrow(\lambda)) \) and \( \zeta \in Q^\vee_1 \) such that \( y_j\eta = X'S_{t_{\zeta_s}}\eta^C \). Therefore, we deduce that \( \eta = X'S_{t_{\zeta_s}}\eta^C = Xs_{t_{\zeta_s}}\eta^C \). Thus we have proved the inclusion \( \subset \). This completes the proof of the lemma. \( \square \)
Lemma 6.2.3. For each $\psi \in \text{QLS}(\lambda)$, there exists a unique $\eta_\psi \in \mathbb{B}_0^\infty(\lambda)$ such that $\text{cl}(\eta_\psi) = \psi$ and $\kappa(\eta_\psi) \in W.J$.

Proof. We write $\psi = X \text{cl}(\eta_e)$ for some monomial $X$ in root operators. Note that for $C \in \text{Conn}(\mathbb{B}_0^\infty(\lambda))$, $C = \mathbb{B}_0^\infty(\lambda)$ if and only if $\eta^C = \eta_e$ (see Proposition 2.5.1(1)). Therefore, by Lemma 6.2.2

$$\text{cl}^{-1}(\psi) \cap \mathbb{B}_0^\infty(\lambda) = \{XS_t\eta_e \mid \zeta \in Q^\vee \lambda.J\}.$$

(6.2.3)

Let us write $\kappa(X\eta_e) \in (W.J)_{af}$ as: $\kappa(X\eta_e) = wz_\xi t_\xi$ for $w \in W.J$ and $\xi \in Q^\vee.J_{af}$, and write $\Pi^J(t_{-\xi}) = z_{-\xi} t_{-\xi + \phi_J(-\xi)} \in (W.J)_{af}$ (see Lemma 2.2.2(2)) as: $\Pi^J(t_{-\xi}) = t_{-\xi}y$ for some $y \in (W.J)_{af}$. Since $S_t\eta_e = \eta_e$ for all $\beta \in Q^\vee.J$, we have $S_{t_{-\xi}}\eta_e = S_{t_{-\xi}}\eta_e = S_{t_{-\xi}}\eta_e$, where $[\cdot] : Q^\vee = Q_{1,J}^\vee \oplus Q_{J}^\vee \rightarrow Q_{1,J}^\vee$. Hence it follows from (6.2.3) that $XS_{t_{-\xi}}\eta_e = XS_{t_{-\xi}}\eta_e$ is contained in $\text{cl}^{-1}(\psi) \cap \mathbb{B}_0^\infty(\lambda)$. We show that $\kappa(XS_{t_{-\xi}}\eta_e) = w \in W.J$. Note that

$$S_{t_{-\xi}}\eta_e = (\Pi^J(t_{-\xi}) ; 0, 1) = (z_{-\xi} t_{-\xi + \phi_J(-\xi)} ; 0, 1) = (t_{-\xi}y ; 0, 1)$$

(6.2.4)

by Remark 3.5.2(2). Here we remark that $\eta_e = (e ; 0, 1)$ and $S_{t_{-\xi}}\eta_e$ are both of the form as in [INS] (7.1.1)]. Because $X\eta_e$ is of the form:

$$X\eta_e = (\ldots, wz_\xi t_\xi ; 0, \ldots, 1) = (\ldots, (wz_\xi t_\xi)e ; 0, \ldots, 1)$$

by the assumption that $\kappa(X\eta) = wz_\xi t_\xi$, we deduce from [INS] Lemma 7.1.4, together with (6.2.4), that $XS_{t_{-\xi}}\eta_e$ is of the form:

$$XS_{t_{-\xi}}\eta_e = (\ldots, (wz_\xi t_\xi)(z_{-\xi} t_{-\xi + \phi_J(-\xi)} ; 0, \ldots, 1)$$

$$= (\ldots, (wz_\xi t_\xi)(t_{-\xi}y) ; 0, \ldots, 1),$$

and hence that $\kappa(XS_{t_{-\xi}}\eta_e) \in (W.J)_{af}$ is equal to $(wz_\xi t_\xi)(t_{-\xi}y)$; here, $z_\xi y$ must be equal to $e$, since $(wz_\xi t_\xi)(t_{-\xi}y) = wz_\xi y \in (W.J)_{af}$, with $w \in W.J \subset (W.J)_{af}$ and $z_\xi y \in (W.J)_{af}$. Hence we conclude that $\kappa(XS_{t_{-\xi}}\eta_e) = w$. This proves the existence of $\eta_\psi$.

It remains to prove the uniqueness of $\eta_\psi$. Assume that $\kappa(XS_{t_{\xi_1}}\eta_e), \kappa(XS_{t_{\xi_2}}\eta_e) \in W.J$ for some $\xi_1, \xi_2 \in Q^\vee_{1,J}$. Then we have $\kappa(XS_{t_{\xi_1}}\eta_e) = \kappa(XS_{t_{\xi_2}}\eta_e)$. Observe that

$$S_{t_{\xi_k}}\eta_e = (\Pi^J(t_{\xi_k}) ; 0, 1) = (z_{\xi_k} t_{\xi_k + \phi_J(\xi_k)} ; 0, 1) \quad \text{for } k = 1, 2.$$

If we write $\kappa(X\eta_e) = wz_\xi t_\xi$ as above, then we see from [INS] Lemma 7.1.4] that

$$\kappa(XS_{t_{\xi_1}}\eta_e) = (wz_\xi t_\xi)\Pi^J(t_{\xi_1}) \quad \text{and} \quad \kappa(XS_{t_{\xi_2}}\eta_e) = (wz_\xi t_\xi)\Pi^J(t_{\xi_2}).$$

Therefore, we have $(wz_\xi t_\xi)\Pi^J(t_{\xi_1}) = (wz_\xi t_\xi)\Pi^J(t_{\xi_2})$, and hence $\Pi^J(t_{\xi_1}) = \Pi^J(t_{\xi_2})$. From this, it follows that

$$S_{t_{\xi_1}}\eta_e = (\Pi^J(t_{\xi_1}) ; 0, 1) = (\Pi^J(t_{\xi_2}) ; 0, 1) = S_{t_{\xi_2}}\eta_e,$$

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which implies that $XS_{t_1} \eta_e = XS_{t_2} \eta_e$. Thus we have proved the uniqueness of $\eta_\psi$. This completes the proof of the lemma.

Now we define the (tail) degree function $\text{Deg}_{\text{tail}} : \text{QLS}(\lambda) \to \mathbb{Z}_{\leq 0}$ as follows. For $\psi \in \text{QLS}(\lambda)$, take $\eta_\psi \in \mathbb{B}^{\mathbb{Z}}_0(\lambda)$ as in Lemma 6.2.3. We write $\eta_\psi$ as:

$$
\eta_\psi = (w_1 z_{\xi_1} t_{\xi_1}, \ldots, w_{s-1} z_{\xi_{s-1}} t_{\xi_{s-1}}, w; a_0, a_1, \ldots, a_{s-1}, a_s)
$$

for $w_1, \ldots, w_{s-1}, w \in W^J$ and $\xi_1, \ldots, \xi_{s-1} \in Q^V, \text{J-ad}$. Since

$$
w_1 z_{\xi_1} t_{\xi_1} \succ \cdots \succ w_{s-1} z_{\xi_{s-1}} t_{\xi_{s-1}} \succ w
$$

with respect to the semi-infinite Bruhat order, it follows from Remark 2.3.3 (and the definition of the semi-infinite Bruhat order) that $[\xi_{s-1}] = [\xi_{s-1} - 0] \in Q^V_{J\setminus J}$, and $[\xi_u - \xi_{u+1}] \in Q^V_{J\setminus J}$ for every $1 \leq u \leq s - 2$; in particular, $[\xi_u] \in Q^V_{J\setminus J}$ for all $1 \leq u \leq s - 1$. Therefore, we have

$$
w_u z_{\xi_u} t_{\xi_u} \lambda \in w_u \lambda + \mathbb{Z}_{\leq 0} \delta \subset \lambda - Q^+ + \mathbb{Z}_{\leq 0} \delta
$$

for every $1 \leq u \leq s - 1$. Also, we have $w \lambda \in \lambda - Q^+$. From these, we deduce that

$$
\text{wt}(\eta_\psi) = \lambda - \beta + K \delta \quad \text{for some } \beta \in Q^+ \text{ and } K \in \mathbb{Z}_{\leq 0}.
\tag{6.2.5}
$$

We define $\text{Deg}_{\text{tail}}(\psi) := K \in \mathbb{Z}_{\leq 0}$; it is easily seen that the $\text{Deg}_{\text{tail}}$ thus defined agrees with the one in [LNS32, §5.2] (see [LNS32, Remark 5.2]). The next proposition follows from (the proof of) [LNS32, Proposition 9.8].

**Proposition 6.2.4.** Keep the notation and setting above. There holds the equality

$$
\sum_{\psi \in \text{QLS}(\lambda)} q^{\text{Deg}_{\text{tail}}(\psi)} x^{\text{wt}(\psi)} = P_\lambda(x; q^{-1}, 0).
\tag{6.2.6}
$$

### 6.3 Proof of the graded character formula.

We recall from §4.1 that $V^{-}(\lambda) = \bigoplus_{b \in \mathbb{B}^{-}(\lambda)} \mathbb{Q}(q_s)G(b)$. Therefore, by Theorem 4.2.1, we obtain

$$
\text{ch } V^{-}(\lambda) = \sum_{\eta \in \mathbb{B}^{-}(\lambda)} x^{\text{wt}(\eta)}.
$$

Because

$$
\mathbb{B}^{-}(\lambda) = \bigsqcup_{\psi \in \text{QLS}(\lambda)} (\text{cl}^{-1}(\psi) \cap \mathbb{B}^{-}(\lambda)),
$$

we see that

$$
\text{ch } V^{-}(\lambda) = \sum_{\psi \in \text{QLS}(\lambda)} \left( \sum_{\eta \in \text{cl}^{-1}(\psi) \cap \mathbb{B}^{-}(\lambda)} x^{\text{wt}(\eta)} \right),
\tag{6.3.1}
$$
In order to obtain a graded character formula for $V_\epsilon^-(\lambda)$, we will compute the sum $(\ast)$ above of the terms $x^{\text{wt}(\eta)}$ over all $\eta \in \text{cl}^{-1}(\psi) \cap B^\pm_{\geq e}(\lambda)$ for each $\psi \in \text{QLS}(\lambda)$. Let $\psi \in \text{QLS}(\lambda)$, and take $\eta_\psi \in B^\pm_{\geq e}(\lambda)$ as in Lemma 6.2.3. Let $X$ be a monomial in root operators such that $\eta_\psi = X \eta_e$; we see that $\psi = X \text{cl}(\eta_e)$. By Lemma 6.2.2, we have

$$\text{cl}^{-1}(\psi) = \{ X S_{t_\zeta} \eta^C | C \in \text{Conn}(B^\pm_{\geq e}(\lambda)), \zeta \in Q_{I,J}^+ \}.$$  

We claim that

$$\text{cl}^{-1}(\psi) \cap B^\pm_{\geq e}(\lambda) = \{ X S_{t_\zeta} \eta^C | C \in \text{Conn}(B^\pm_{\geq e}(\lambda)), \zeta \in Q_{I,J}^+ \}.$$  

(6.3.2)

First we show the inclusion $\subset$. Let $\eta \in \text{cl}^{-1}(\psi) \cap B^\pm_{\geq e}(\lambda)$, and write $\eta$ as: $\eta = X S_{t_\zeta} \eta^C$ for $C \in \text{Conn}(B^\pm_{\geq e}(\lambda))$ and $\zeta \in Q_{I,J}^+$. Let $w := \kappa(\eta_\psi) \in W^J$; note that $\eta_e = (e; 0, 1)$ is of the form as in [INS] (7.1.1), and $X \eta_e = \eta_\psi$ is of the form:

$$X \eta_e = (\ldots, w; 0, \ldots, 1) = (\ldots, we; 0, \ldots, 1).$$

We see from Remark 3.5.2(2) that $S_{t_\zeta} \eta^C$ is also of the form as in [INS] (7.1.1), with $\kappa(S_{t_\zeta} \eta^C) = \Pi^j(t_\zeta)$ (recall that $\kappa(\eta^C) = e$). Therefore, we deduce from [INS] Lemma 7.1.4 that $X S_{t_\zeta} \eta^C$ is of the form:

$$X S_{t_\zeta} \eta^C = (\ldots, w \Pi^j(t_\zeta); 0, \ldots, 1),$$

and hence that $\kappa(X S_{t_\zeta} \eta^C) = w \Pi^j(t_\zeta) = w z \zeta t_{\zeta + \phi_1(\zeta)}$ (see Lemma 2.2.2(2)). Since $\eta = X S_{t_\zeta} \eta^C \in B^\pm_{\geq e}(\lambda)$, we have $w z \zeta t_{\zeta + \phi_1(\zeta)} \succeq e$. Hence it follows from Remark 2.3.3 that $\zeta = [\zeta + \phi_1(\zeta)] \in Q_{I,J}^+$ is contained in $Q_{I,J}^+$. Thus, $\eta$ is contained in the set on the right-hand side of (6.3.2). Conversely, let $C \in \text{Conn}(B^\pm_{\geq e}(\lambda))$, and $\zeta \in Q_{I,J}^+$. Then, by the same argument as above, we see that $\kappa(X S_{t_\zeta} \eta^C)$ is equal to $w z \zeta t_{\zeta + \phi_1(\zeta)}$, where $w = \kappa(\eta_\psi) \in W^J$. Since $\lceil \zeta + \phi_1(\zeta) \rceil - 0 = \zeta = \zeta \in Q_{I,J}^+$, we see from [INS] Proposition 6.2.2 (with $a = 1$) that $z \zeta t_{\zeta + \phi_1(\zeta)} \succeq e$ (i.e., $t_0$). Also, we see from Lemma 2.3.4 that $w z \zeta t_{\zeta + \phi_1(\zeta)} \succeq z \zeta t_{\zeta + \phi_1(\zeta)}$, since $w \geq e$ with respect to the (ordinary) Bruhat order $\succeq$ on $W^J$. Combining these inequalities, we obtain $\kappa(X S_{t_\zeta} \eta^C) = w z \zeta t_{\zeta + \phi_1(\zeta)} \succeq e$, which implies that $X S_{t_\zeta} \eta^C \in \text{cl}^{-1}(\psi) \cap B^\pm_{\geq e}(\lambda)$. This proves the claim (6.3.2).

Let $C \in \text{Conn}(B^\pm_{\geq e}(\lambda))$, and write $\Theta(C) \in \text{Par}(\lambda)$ as: $\Theta(C) = (\rho^{(i)}_{i \in I})_{i \in I}$, with $\rho^{(i)} = (\rho^{(i)}_1 \geq \cdots \geq \rho^{(i)}_{m_i-1})$ for each $i \in I$. Also, let $\zeta \in Q_{I,J}^+$, and write it as: $\zeta = \sum_{i \in I} c_i \alpha_i^\vee$, with $c_i \in \mathbb{Z}_{\geq 0}$, $i \in I$. For each $i \in I$, we set $c_i + \rho^{(i)} := (c_i + \rho^{(i)}_1 \geq \cdots \geq c_i + \rho^{(i)}_{m_i-1} \geq c_i)$, which is a partition of length less than or equal to $m_i$. Then we set

$$(c_i)_{i \in I} + \Theta(C) := (c_i + \rho^{(i)}_{i \in I} \in \text{Par}(\lambda));$$  

(6.3.3)

for the definition of $\text{Par}(\lambda)$, see (2.5.1). We compute:

$$\text{wt}(S_{t_\zeta} \eta^C) = t_\zeta(\text{wt}(\eta^C)) = t_\zeta(\lambda - \left| \rho^{(i)}_{i \in I} \right| \delta) = \lambda - (\zeta, \lambda) \delta - \left| \rho^{(i)}_{i \in I} \right| \delta.$$
\[
= \lambda - \left( \sum_{i \in I} m_i c_i \right) \delta - \left| (\rho^{(i)})_{i \in I} \right| \delta = \text{wt}(\eta_\epsilon) - \left| (c_i + \rho^{(i)})_{i \in I} \right| \delta.
\]

From this computation, together with (6.2.5), we deduce that
\[
\text{wt}(X S_{\kappa} \eta^C) = \text{wt}(X \eta_\epsilon) - \left| (c_i + \rho^{(i)})_{i \in I} \right| \delta = \text{wt}(\eta_\psi) - \left| (c_i + \rho^{(i)})_{i \in I} \right| \delta
\]
\[
= \text{wt}(\psi) + (\text{Deg}^{\text{tail}}(\psi) - \left| (c_i + \rho^{(i)})_{i \in I} \right| \delta).
\]

Therefore, we conclude that for each \( \psi \in \text{QLS}(\lambda) \),
\[
\sum_{\eta \in \text{cl}^{-1}(\psi) \cap B_{\geq e}(\lambda)} x^{\text{wt}(\eta)} = \sum_{C \in \text{Conn}(B_{\geq e}(\lambda))} C^{\text{wt}(X S_{\kappa} \eta^C)} = x^{\text{wt}(\psi)} x^{\text{Deg}^{\text{tail}}(\psi)} \sum_{e_0 \in \text{Par}(\lambda)} \prod_{e_0 \in \text{Par}(\lambda)} x^{-|e_0| \delta}
\]
\[
= x^{\text{wt}(\psi)} q^{\text{Deg}^{\text{tail}}(\psi)} \sum_{e_0 \in \text{Par}(\lambda)} q^{-|e_0|} \quad \text{(by replacing } x^\delta \text{ with } q)\]
\[
= x^{\text{wt}(\psi)} q^{\text{Deg}^{\text{tail}}(\psi)} \prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^{-r})^{-1}.
\]

Substituting this equality into (6.3.1), we finally obtain
\[
gch V^{-}_e(\lambda) = \sum_{\psi \in \text{QLS}(\lambda)} x^{\text{wt}(\psi)} q^{\text{Deg}^{\text{tail}}(\psi)} \prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^{-r})^{-1}
\]
\[
= \left( \prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^{-r}) \right)^{-1} P_\lambda(x; q^{-1}, 0) \quad \text{by Proposition 6.2.4.}
\]

This completes the proof of Theorem 6.1.1.

Remark 6.3.1. Let \( \psi \in \text{QLS}(\lambda) \). We see from (6.3.4) that for every \( \eta \in \text{cl}^{-1}(\psi) \cap B_{\geq e}(\lambda) \),
\[
\text{wt}(\eta) - \text{wt}(\eta_\psi) \in \mathbb{Z}_{\leq 0}\delta,
\]
with \( \text{wt}(\eta) - \text{wt}(\eta_\psi) = 0 \) if and only if \( \eta = \eta_\psi \).

6.4 Graded character formula for \( V^{+}_{w_0}(\lambda) \).

We define the character \( \text{ch} V^{+}_{w_0}(\lambda) \) and the graded character \( \text{gch} V^{+}_{w_0}(\lambda) \) of the Demazure submodule \( V^{+}_{w_0}(\lambda) \) in exactly the same manner as those of \( V^{-}_e(\lambda) \) are defined in §6.1.

Recall from §2.6 the bijection \( \vee : B_{\geq e}(\lambda) \rightarrow B_{\geq e}(-w_0 \lambda) \). It follows from Lemma 2.6.1 that observe that
\[
\left( B_{\geq e}(-w_0 \lambda) \right)^{\vee} = B_{\geq e}(-w_0 \lambda).
\]

From this, together with Theorem 4.2.1, we deduce that
\[
\text{ch} V^{+}_{w_0}(\lambda) = \sum_{\eta \in B_{\geq e}(-w_0 \lambda)} x^{\text{wt}(\eta)} = \sum_{\eta \in B_{\geq e}(-w_0 \lambda)} x^{\text{wt}(\eta)^\vee} = \sum_{\eta \in B_{\geq e}(-w_0 \lambda)} x^{-\text{wt}(\eta)} \quad \text{by (2.6.6),}
\]

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which is equal to the one obtained from \( V_e^{-}(w_0\lambda) \) by replacing \( x \) with \( x^{-1} \). Therefore, the graded character \( \text{gch} \ V_{w_0}^{+}(\lambda) \) is obtained from the graded character \( \text{gch} \ V_e^{-}(w_0\lambda) \) by replacing \( x \) with \( x^{-1} \) and \( q \) with \( q^{-1} \). Also, we know from the proof of Proposition 9.8 that
\[
P_{-w_0\lambda}(x^{-1}; q, 0) = P_{\lambda}(x; q, 0).
\]
Thus, we have the following theorem.

**Theorem 6.4.1.** Keep the notation and setting above. The graded character \( \text{gch} \ V_{w_0}^{+}(\lambda) \) of \( V_{w_0}^{+}(\lambda) \) can be expressed as
\[
\text{gch} \ V_{w_0}^{+}(\lambda) = \left( \prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^r) \right)^{-1} P_{-w_0\lambda}(x^{-1}; q, 0)
\]
\[
= \left( \prod_{i \in I} \prod_{r=1}^{m_i} (1 - q^r) \right)^{-1} P_{\lambda}(x; q, 0).
\]

## 7 Certain quotients of Demazure submodules.

In this section, we fix \( \lambda = \sum_{i \in I} m_i \omega_i \in P_+ \), and set \( J = J_{\lambda} := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \).

### 7.1 Technical lemmas.

The following is an easy lemma.

**Lemma 7.1.1** (see Lemma \[6.2.3\]). The subset \( \{ \eta_\psi \mid \psi \in \text{QLS}(\lambda) \} \cup \{ 0 \} \) of \( \mathbb{B}^{\mathfrak{H}}(\lambda) \cup \{ 0 \} \) is stable under the action of the root operators \( e_j \) and \( f_j \) for \( j \in I \).

**Proof.** It suffices to show that \( x_j \eta_\psi = \eta_{x_j\psi} \), assuming that \( x_j \eta_\psi \neq 0 \) for \( \psi \in \text{QLS}(\lambda) \) and \( j \in I \), where \( x_j \) is either \( e_j \) or \( f_j \). Since the map \( \text{cl} : \mathbb{B}^{\mathfrak{H}}(\lambda) \to \text{QLS}(\lambda) \) commutes with root operators, it follows that \( \text{cl}(x_j\eta_\psi) = x_j\psi \neq 0 \). It is obvious that \( x_j\eta_\psi \in B_0(\lambda) \). Also, we deduce from the definition of root operators that \( \kappa(x_j\eta_\psi) \) is equal either to \( \kappa(\eta_\psi) \in W^J \) or to \( r_j\kappa(\eta_\psi) \in (W^J)_{af} \). In the latter case, we have \( r_j\kappa(\eta_\psi) \in W \cap (W^J)_{af} \) since \( \kappa(\eta_\psi) \in W^J \) and \( j \in I \), which implies that \( r_j\kappa(\eta_\psi) \in W^J \) since \( W \cap (W^J)_{af} = W^J \) by \[2.2.5\]. Therefore, by the uniqueness of \( \eta_{x_j\psi} \), we obtain \( x_j\eta_\psi = \eta_{x_j\psi} \), as desired.

We know from \[LNS^{\mathfrak{H}} \ S5.1\] that there exists a bijection \( * : \text{QLS}(\lambda) \to \text{QLS}(-w_0\lambda) \) such that for \( \psi \in \text{QLS}(\lambda) \),
\[
\begin{align*}
\text{wt}(\psi^*) &= -\text{wt}(\psi), \quad \text{and} \\
(e_j\psi)^* &= f_j\psi^*, (f_j\psi)^* = e_j\psi^* \quad \text{for all } j \in I_{af},
\end{align*}
\]
where we set \( 0^* := 0 \). We see easily from the definitions that the following diagram commutes:
\[
\begin{align*}
\mathbb{B}^{\mathfrak{H}}(\lambda) &\xrightarrow{\psi} \mathbb{B}^{\mathfrak{H}}(-w_0\lambda) \\
\text{cl} &\downarrow \quad \text{cl} \\
\text{QLS}(\lambda) &\xrightarrow{*} \text{QLS}(-w_0\lambda).
\end{align*}
\]
The next lemma follows immediately from Lemma 6.2.3, by using the commutative diagram (7.1.2), together with (2.6.6) and (7.1.1).

**Lemma 7.1.2** (cf. [NS3, Proposition 3.1.3]). For each \( \psi \in \text{QLS}(\lambda) \), the element \( \tilde{\eta} \psi := (\eta^* \psi)^* \) is a unique element in \( \mathbb{B}_{\lambda}^\infty \) such that \( \text{cl}(\tilde{\eta} \psi) = \psi \) and \( \iota(\eta \psi) \in W^J \).

We can prove the next lemma by an argument similar to the one for Lemma 7.1.1.

**Lemma 7.1.3.** The subset \( \{ \tilde{\eta} \psi \mid \psi \in \text{QLS}(\lambda) \} \cup \{ 0 \} \) is stable under the action of the root operators \( e_j \) and \( f_j \) for \( j \in I \).

**Remark 7.1.4.** By Lemmas 7.1.1 and 7.1.3, each of the sets \( \{ \eta \psi \mid \psi \in \text{QLS}(\lambda) \} \) and \( \{ \tilde{\eta} \psi \mid \psi \in \text{QLS}(\lambda) \} \) has a crystal structure for \( U_q(g) \), where \( g \) is the canonical finite-dimensional simple Lie subalgebra of \( g_{\text{ad}} \). Moreover, these crystals for \( U_q(g) \) are both isomorphic to \( \text{QLS}(\lambda) \), regarded as a crystal for \( U_q(g) \) by restriction.

### 7.2 Certain quotients of Demazure submodules and their crystal bases.

We define

\[
X^-_e(\lambda) := \sum_{c_0 \in \text{Par}(\lambda)} U_q^- S_{c_0}^{-} v_{\lambda};
\]

note that \( X^-_e(\lambda) \subset V^-_e(\lambda) = U_q^- v_{\lambda} \) since \( S_{c_0}^- \in U_q^- \) for all \( c_0 \in \text{Par}(\lambda) \) (see §3.3). We denote by \( \Xi^-_\lambda : V^-_e(\lambda) \to V^-_e(\lambda)/X^-_e(\lambda) \) the canonical projection, and set

\[
U_w(\lambda) := \Xi^-_\lambda(V^-_w(\lambda)) \quad \text{for each } w \in W^J;
\]  

we have \( V^-_w(\lambda) \subset V^-_e(\lambda) \) since \( w \succeq e \) (see Corollary 5.2.5).

**Theorem 7.2.1.** Keep the notation and setting above.

1. There exists a subset \( B(X^-_e(\lambda)) \) of \( B(\lambda) \) such that

\[
X^-_e(\lambda) = \bigoplus_{b \in B(X^-_e(\lambda))} \mathbb{Q}(q_s) G(b).
\]

Under the isomorphism \( \Psi_{\lambda} : B(\lambda) \to \mathbb{B}_{\lambda}^\infty \), the subset \( B(X^-_e(\lambda)) \subset B(\lambda) \) is mapped to the subset

\[
\mathbb{B}_{\Xi^-_e(\lambda)} \setminus \{ \eta \psi \mid \psi \in \text{QLS}(\lambda) \}
\]

of \( \mathbb{B}_{\lambda}^\infty \). Therefore, if we define \( B(U^-_e(\lambda)) \subset B(\lambda) \) to be the inverse image of \( \{ \eta \psi \mid \psi \in \text{QLS}(\lambda) \} \subset \mathbb{B}_{\lambda}^\infty \) under the isomorphism \( \Psi_{\lambda} \), then \( \{ \Xi^-_\lambda(G(b)) \mid b \in B(U^-_e(\lambda)) \} \) is a \( \mathbb{Q}(q_s) \)-basis of the quotient \( U^-_e(\lambda) = V^-_e(\lambda)/X^-_e(\lambda) \).
(2) For each $w \in W^J$, the quotient $U_w^-(\lambda) = \Xi^-_w(V_w^-) \subseteq V_w^- \in \Lambda$ has a $\mathbb{Q}(q_s)$-basis
\[\{\Xi^-_w(G(b)) \mid b \in B(U_w^-)\}\], where $B(U_w^-) = \text{the inverse image of the following subset of } B_0^\infty(\lambda) \text{ under the isomorphism } \Psi_\lambda:\]

\[\{\eta_\psi \mid \psi \in \text{QLS}(\lambda) \text{ such that } \kappa(\eta_\psi) \geq w\},\]

where $\kappa(\eta_\psi) \geq w$ means that $\kappa(\eta_\psi) \in W^J$ is greater than or equal to $w \in W^J$ with respect to the Bruhat order on $W^J$.

We will prove Theorem 7.2.1 in the next subsection.

Similarly, we define
\[X^+_w(\lambda) := \sum_{c_0 \in \text{Par}(\lambda)} U^+_q S_{c_0} S^\text{norm}_{w_0} v_\lambda;\]

note that $X^+_w(\lambda) \subseteq V^+_w(\lambda) = U^+_q S^\text{norm}_{w_0} v_\lambda$ since $S_{c_0} \in U^+_q$ for all $c_0 \in \text{Par}(\lambda)$. We denote by $\Xi^+_\lambda : V^+_w(\lambda) \rightarrow V^+_w(\lambda)/X^+_w(\lambda)$ the canonical projection, and set
\[U^+_w(\lambda) := \Xi^+_w(V^+_w(\lambda)) \text{ for each } w \in W^J;\]

we have $V^+_w(\lambda) \subseteq V^+_w(\lambda)$ since $[w_0]^J \geq w$.

**Theorem 7.2.2.** Keep the notation and setting above.

1. There exists a subset $B(X^+_w(\lambda))$ of $B(\lambda)$ such that
\[X^+_w(\lambda) = \bigoplus_{b \in B(X^+_w(\lambda))} \mathbb{Q}(q_s) G(b).\]

Under the isomorphism $\Psi_\lambda^- : B(\lambda) \rightarrow B^\infty_0(\lambda)$, the subset $B(X^+_w(\lambda)) \subseteq B(\lambda)$ is mapped to the subset
\[B^\infty_0(\lambda) \setminus \{\tilde{\eta}_\psi \mid \psi \in \text{QLS}(\lambda)\}\]

of $B^\infty_0(\lambda)$. Therefore, if we define $B(U^+_w(\lambda)) \subseteq B(\lambda)$ to be the inverse image of $\{\tilde{\eta}_\psi \mid \psi \in \text{QLS}(\lambda)\}$ under the isomorphism $\Psi_\lambda^-$, then $\{\Xi^+_w(G(b)) \mid b \in B(U^+_w(\lambda))\}$ is a $\mathbb{Q}(q_s)$-basis of the quotient $U^+_w(\lambda) = V^+_w(\lambda)/X^+_w(\lambda)$.

(2) For each $w \in W^J$, the quotient $U^+_w(\lambda) = \Xi^+_w(V^+_w(\lambda)) \subseteq V^+_w(\lambda)$ has a $\mathbb{Q}(q_s)$-basis
\[\{\Xi^+_w(G(b)) \mid b \in B(U^+_w(\lambda))\},\]

where $B(U^+_w(\lambda))$ is defined to be the inverse image of the following subset of $B^\infty_0(\lambda)$ under the isomorphism $\Psi_\lambda^-$:

\[\{\tilde{\eta}_\psi \mid \psi \in \text{QLS}(\lambda) \text{ such that } w \geq r(\tilde{\eta}_\psi)\}.\]

We leave the proof of Theorem 7.2.2 to the reader since it is similar to that of Theorem 7.2.1 use also [BN] Lemma 5.2.

In our forthcoming paper [LNS3], we will prove that for each $w \in W^J$, the graded character of $U^+_w(\lambda)$ is identical to the specialization at $t = 0$ of a nonsymmetric Macdonald polynomial, so is the graded character of $U^-_w(\lambda)$.
7.3 Proof of Theorem 7.2.1

Recall from 

\[ \Phi_\lambda : V(\lambda) \hookrightarrow \tilde{V}(\lambda) = \bigotimes_{i \in I} V(\pi_i)^{\otimes m_i} \]

that maps \( v_\lambda \) to \( \tilde{v}_\lambda = \bigotimes_{i \in I} v_{\pi_i}^{m_i} \). For each \( c_0 = (\rho^{(i)})_{i \in I} \in \text{Par}(\lambda) \), we define a \( U_q' \)-module homomorphism \( s_{c_0}(z^{-1}) : \tilde{V}(\lambda) \to \tilde{V}(\lambda) \) by:

\[ s_{c_0}(z^{-1}) = \prod_{i \in I} s_{\rho^{(i)}}(z_{i,1}^{-1}, \ldots, z_{i,m_i}^{-1}), \]

where for \( i \in I \) and \( 1 \leq l \leq m_i \), \( z_{i,l} : \tilde{V}(\lambda) \to \tilde{V}(\lambda) \) is the \( U_q' \)-module automorphism of \( \tilde{V}(\lambda) \), and for \( i \in I \), \( s_{\rho^{(i)}}(x_1, \ldots, x_{m_i}) \) denotes the Schur polynomial corresponding to the partition \( \rho^{(i)} \). We claim that

\[ s_{c_0}(z^{-1})(\text{Image } \Phi_\lambda) \subset \text{Image } \Phi_\lambda. \]

Indeed, since \( V(\lambda) = U_q v_\lambda = U_q' \tilde{v}_\lambda \), it follows that \( \text{Image } \Phi_\lambda = U_q \tilde{v}_\lambda = U_q' \tilde{v}_\lambda \). Therefore, we see from [BN Proposition 4.10] that

\[ s_{c_0}(z^{-1})(\text{Image } \Phi_\lambda) = s_{c_0}(z^{-1})(U_q' \tilde{v}_\lambda) = U_q' s_{c_0}(z^{-1}) \tilde{v}_\lambda = U_q' s_{c_0} \tilde{v}_\lambda \subset \text{Image } \Phi_\lambda, \]

as desired. Hence we can define a \( U_q' \)-module homomorphism \( z_{c_0} : V(\lambda) \to V(\lambda) \) in such a way that the following diagram commutes:

\[ \begin{array}{ccc}
V(\lambda) & \xrightarrow{\Phi_\lambda} & \tilde{V}(\lambda) \\
\downarrow_{\ z_{c_0}} & & \downarrow_{s_{c_0}(z^{-1})} \\
V(\lambda) & \xrightarrow{\Phi_\lambda} & \tilde{V}(\lambda).
\end{array} \quad (7.3.1) \]

Observe that \( z_{c_0} v_\lambda = S_{c_0^{-}} v_\lambda \), and that \( z_{c_0} \) commutes with root operators on \( V(\lambda) \). Because \( z_{i,l} \) preserves the crystal lattice \( \tilde{L}(\lambda) = \bigotimes_{i \in I} L(\pi_i)^{\otimes m_i} \subset \tilde{V}(\lambda) \) for all \( i \in I \) and \( 1 \leq l \leq m_i \) (see [5.2]), and because \( \Phi_\lambda(L(\lambda)) \subset \tilde{L}(\lambda) \), we deduce that \( z_{c_0}(L(\lambda)) \subset L(\lambda) \). Thus, we obtain an induced \( \mathbb{Q} \)-linear map \( z_{c_0} : L(\lambda)/q L(\lambda) \to L(\lambda)/q L(\lambda) \), for which the following diagram commutes:

\[ \begin{array}{ccc}
L(\lambda)/q L(\lambda) & \xrightarrow{\Phi_\lambda|_{q=0}} & \tilde{L}(\lambda)/q \tilde{L}(\lambda) \\
\downarrow_{z_{c_0}} & & \downarrow_{s_{c_0}(z^{-1})} \\
L(\lambda)/q L(\lambda) & \xrightarrow{\Phi_\lambda|_{q=0}} & \tilde{L}(\lambda)/q \tilde{L}(\lambda). \quad (7.3.2) \end{array} \]

It follows from [BN p. 371] (see also [5.2.5]) that

\[ B(\lambda) = \{ z_{c_0} b \mid c_0 \in \text{Par}(\lambda), b \in B_0(\lambda) \}. \quad (7.3.3) \]

Also, by [5.2.6], we have \( z_{c_0} u_\lambda = u^{c_0} \) for \( c_0 \in \text{Par}(\lambda) \) (for the definition of \( u^{c_0} \), see Proposition 3.5.1).
Remark 7.3.1. Let \( c_0 = (\rho(i))_{i \in I} \in \overline{\text{Par}(\lambda)} \). Let \( c_i \in \mathbb{Z}_{\geq 0}, i \in I \), be the number of columns of length \( m_i \) in the Young diagram corresponding to the partition \( \rho(i) \), and set \( \xi := \sum_{i \in I} c_i \alpha_i^\vee \in Q^\vee_{\lambda, J} \). Also, let \( \varrho(i), i \in I \), denote the partition corresponding to the Young diagram obtained from the Young diagram corresponding to \( \rho(i) \) by removing all columns of length \( m_i \) (i.e., the first \( c_i \)-columns), and set \( c'_0 := (\varrho(i))_{i \in I} \); note that \( c'_0 \in \text{Par}(\lambda) \). Then we deduce from [BN, Lemma 4.14 and its proof] that

\[
\zeta_{c_0} u_\lambda = S_{t_\xi} \zeta_{c'_0} u_\lambda = S_{t_\xi} u_{c'_0}.
\]  

(7.3.4)

Lemma 7.3.2. We have

\[
\mathcal{B}^-_e(\lambda) = \left\{ \zeta_{c_0} b \mid c_0 \in \text{Par}(\lambda), b \in \mathcal{B}^-_e(\lambda) \cap \mathcal{B}_0(\lambda) \right\}.
\]  

(7.3.5)

Moreover, for every \( c_0 \in \overline{\text{Par}(\lambda)} \) and \( b \in \mathcal{B}^-_e(\lambda) \cap \mathcal{B}_0(\lambda) \), the element \( \zeta_{c_0} b \) is contained in \( \mathcal{B}^-_e(\lambda) \).

Proof. First we prove the inclusion \( \supset \). Let \( b \in \mathcal{B}^-_e(\lambda) \cap \mathcal{B}_0(\lambda) \), and write it as \( b = Xu_\lambda \) for a monomial \( X \) in root operators. For \( c_0 \in \text{Par}(\lambda) \), we have \( \zeta_{c_0} b = X \zeta_{c_0} u_\lambda = Xu_{c_0} \). Set \( \eta := \Psi_\lambda(b) \) and \( \eta' := \Psi_\lambda(\zeta_{c_0} b) \), where \( \Psi_\lambda : \mathcal{B}(\lambda) \to \mathbb{B}^\widetilde{\tau}(\lambda) \) is the isomorphism of crystals. Then, we have \( \eta = X\eta_c \) and \( \eta' = X \Psi_\lambda(u_{c_0}) = X\eta_c \), with \( C := \Theta^{-1}(c_0) \in \text{Conn}(\mathbb{B}^\widetilde{\tau}(\lambda)) \). Therefore, we deduce from [INS, Lemma 7.1.4] that \( \kappa(\eta) = \kappa(\eta') \). Also, since \( b \in \mathcal{B}^-_e(\lambda) \cap \mathcal{B}_0(\lambda) \), it follows that \( \kappa(\eta) \succeq e \), and hence \( \kappa(\eta') = \kappa(\eta) \succeq e \). Thus we obtain \( \eta' \in \mathbb{B}^\widetilde{\tau}_{\geq e}(\lambda) \), which implies that \( \zeta_{c_0} b \in \mathcal{B}^-_e(\lambda) \).

Next we prove the opposite inclusion \( \subset \). Let \( b' = \zeta_{c_0} b \) for some \( c_0 \in \text{Par}(\lambda) \) and \( b \in \mathcal{B}_0(\lambda) \) (see (7.3.3)); we need to show that \( b \in \mathcal{B}^-_e(\lambda) \). Set \( \eta := \Psi_\lambda(b) \in \mathbb{B}^\widetilde{\tau}(\lambda) \), and \( \eta' := \Psi_\lambda(b') \in \mathbb{B}^\widetilde{\tau}(\lambda) \). Then, by entirely the same argument as above, we deduce that \( \kappa(\eta) = \kappa(\eta') \succeq e \). Thus we obtain \( \eta \in \mathbb{B}^\widetilde{\tau}_{\geq e}(\lambda) \), which implies that \( b \in \mathcal{B}^-_e(\lambda) \).

For the second assertion, let \( c_0 = (\rho(i))_{i \in I} \in \overline{\text{Par}(\lambda)} \), and \( b \in \mathcal{B}^-_e(\lambda) \cap \mathcal{B}_0(\lambda) \). We write \( b \) as \( b = Xu_\lambda \) for a monomial \( X \) in the Kashiwara operators. Define \( \xi = \sum_{i \in I} c_i \alpha_i^\vee \in Q^\vee_{\lambda} \), and \( c'_0 = (\varrho(i))_{i \in I} \in \text{Par}(\lambda) \) as in Remark 7.3.1 (with \( c_0 = (\rho(i))_{i \in I} \) above). Then, we have

\[
\zeta_{c_0} b = X \zeta_{c_0} u_\lambda = X S_{t_\xi} \zeta_{c'_0} u_\lambda = X S_{t_\xi} u_{c'_0}.
\]

Now we set \( \eta := \Psi_\lambda(b) \in \mathbb{B}^\widetilde{\tau}(\lambda) \), and \( \psi := \text{cl}(\eta) \in \text{QLS}(\lambda) \); note that \( \eta = Xu_\lambda \), and hence \( \psi = X \text{cl}(\eta_c) \). We see that

\[
\Psi_\lambda(\zeta_{c_0} b) = X S_{t_\xi} \Psi_\lambda(u_{c_0}) = X S_{t_\xi} \eta_c, \quad \text{with } C := \Theta^{-1}(c'_0) \in \text{Conn}(\mathbb{B}^\widetilde{\tau}(\lambda)).
\]

Since \( \xi \in Q^\vee_{\lambda, J} \), it follows from (6.3.2) that \( \Psi_\lambda(\zeta_{c_0} b) \in \mathbb{B}^\widetilde{\tau}_{\geq e}(\lambda) \), which implies that \( \zeta_{c_0} b \in \mathcal{B}^-_e(\lambda) \). This proves the lemma.

\[\square\]
Proof of Theorem 7.2.1. First, we prove that if we set

\[ \mathcal{B} := \{ z_{c_0}b \mid c_0 \in \operatorname{Par}(\lambda) \setminus (\emptyset)_{i \in I}, \ b \in B_e^-(\lambda) \cap B_0(\lambda) \} \subset \mathcal{B}(\lambda), \]

(7.3.6)

then we have

\[ X_e^-(\lambda) = \bigoplus_{b \in \mathcal{B}} \mathbb{Q}(q_s)G(b). \]

(7.3.7)

Because \( S_{c_0}v_\lambda = z_{c_0}v_\lambda \) for every \( c_0 \in \operatorname{Par}(\lambda) \), we have

\[ X_e^-(\lambda) = \sum_{c_0 \neq (\emptyset)_{i \in I}} U_q^- S_{c_0}v_\lambda = \sum_{c_0 \neq (\emptyset)_{i \in I}} U_q^- z_{c_0}v_\lambda = \sum_{c_0 \neq (\emptyset)_{i \in I}} z_{c_0}(U_q^- v_\lambda) = \sum_{c_0 \neq (\emptyset)_{i \in I}} z_{c_0}(V_e^-(\lambda)). \]

(7.3.8)

Let \( c_0 \in \operatorname{Par}(\lambda) \setminus (\emptyset)_{i \in I} \), and \( b \in B_e^-(\lambda) \cap B_0(\lambda) \). Then we deduce that \( G(z_{c_0}b) = z_{c_0}G(b) \); indeed, since \( b \in B_0(\lambda) \), we see by [BN Theorem 4.16 (ii)] that \( z_{c_0}G(b) = G(b') \) for some \( b' \in B(\lambda) \). Here,

\[ b' = G(b') + q_s \mathcal{L}(\lambda) = z_{c_0}G(b) + q_s \mathcal{L}(\lambda) = z_{c_0}(G(b) + q_s \mathcal{L}(\lambda)) = z_{c_0}b, \]

from which we get \( G(z_{c_0}b) = z_{c_0}G(b) \), as desired. Since \( G(b) \in V_e^-(\lambda) \), it follows from (7.3.8) that \( G(z_{c_0}b) = z_{c_0}G(b) \in X_e^-(\lambda) \), and hence \( X_e^-(\lambda) \supset \bigoplus_{b \in \mathcal{B}} \mathbb{Q}(q_s)G(b) \). Now we show the opposite inclusion \( \subset \) in (7.3.7). Since \( \{ G(b) \mid b \in B_e^-(\lambda) \} \) is a \( \mathbb{Q}(q_s) \)-basis of \( V_e^-(\lambda) \), we see from (7.3.8) that

\[ X_e^-(\lambda) = \operatorname{Span}_{\mathbb{Q}(q_s)} \{ z_{c_0}G(b) \mid c_0 \in \operatorname{Par}(\lambda) \setminus (\emptyset)_{i \in I}, \ b \in B_e^-(\lambda) \}. \]

(7.3.9)

Let \( c_0 \in \operatorname{Par}(\lambda) \setminus (\emptyset)_{i \in I} \), and \( b \in B_e^-(\lambda) \). By Lemma 7.3.2 we can write the \( b \) as: \( b = z_{c_0}b' \) for some \( c_0' \in \operatorname{Par}(\lambda) \) and \( b' \in B_e^-(\lambda) \cap B_0(\lambda) \). Then we have \( z_{c_0}b = z_{c_0}z_{c_0'}b' \). Because \( z_{c_0} \) and \( z_{c_0'} \) are defined by using Schur polynomials (see (5.2.3)), their product \( z_{c_0}z_{c_0'} \) can be expressed as:

\[ z_{c_0}z_{c_0'} = \sum_{c_0'' \in \operatorname{Par}(\lambda)} n_{c_0''} z_{c_0''}, \quad n_{c_0''} \in \mathbb{Z}_{\geq 0}; \]

here we remark that \( |c_0| + |c_0'| \geq 1 \) since \( c_0 \neq (\emptyset)_{i \in I} \). Therefore, we deduce that

\[ z_{c_0}G(b) = z_{c_0}G(z_{c_0'}b') = \sum_{c_0'' \in \operatorname{Par}(\lambda)} n_{c_0''} G(z_{c_0''}b') \in \bigoplus_{b \in \mathcal{B}} \mathbb{Q}(q_s)G(b). \]

From this, together with (7.3.9), we obtain \( X_e^-(\lambda) \subset \bigoplus_{b \in \mathcal{B}} \mathbb{Q}(q_s)G(b) \). Combining these, we obtain (7.3.7), as desired; we write \( \mathcal{B}(X_e^-(\lambda)) \) for the set \( \mathcal{B} \).
Next, we prove that
\[
\Psi_\lambda(B(X_e^-(\lambda))) = B_\leq_e^\infty(\lambda) \setminus \{\eta_\psi \mid \psi \in \text{QLS}(\lambda)\}.
\]

For this purpose, it suffices to show that for each \(\psi \in \text{QLS}(\lambda)\),
\[
\text{cl}^{-1}(\psi) \cap \Psi_\lambda(B(X_e^-(\lambda))) = \left(\text{cl}^{-1}(\psi) \cap B_\leq_e^\infty(\lambda)\right) \setminus \{\eta_\psi\}.
\]
(7.3.10)

Let \(\psi \in \text{QLS}(\lambda)\), and write the \(\eta_\psi \in B_\leq_e^\infty(\lambda)\) as \(\eta_\psi = X\eta_c\) for some monomial \(X\) in root operators; recall from (6.3.2) that
\[
\text{cl}^{-1}(\psi) \cap B_\leq_e^\infty(\lambda) = \{X\eta_c^C \mid C \in \text{Conn}(B_\leq_e^\infty(\lambda)), \zeta \in Q_{I,J}^\lambda+\}\.
\]

Let us show the inclusion \(\supset\) in (7.3.10). Let \(\eta\) be an element in the set on the right-hand side of (7.3.10), and write it as: \(\eta = XS_{I\lambda}\eta^C\), with \(C \in \text{Conn}(B_\leq_e^\infty(\lambda))\) and \(\zeta \in Q_{I,J}^\lambda+\). We write \(\zeta\) as \(\zeta = \sum_{i \in I} c_i\alpha_i^\lambda\), \(c_i \in \mathbb{Z}_{\geq 0}\), \(i \in I\). Then we define \(c_0 := (c_i)_{i \in I} + \Theta(C) \in \text{Par}(\lambda)\) as in (6.3.3). Here we claim that \(c_0 \neq (\emptyset)_{i \in I}\). Indeed, by the computation in (6.3.4), we have
\[
\text{wt}(\eta) = \text{wt}(XS_{I\lambda}\eta^C) = \text{wt}(\psi) + (\text{Deg}^{\text{tail}}(\psi) - |c_0|)\delta.
\]

Since \(\eta \neq \eta_\psi\) by our assumption, it follows from Remark 6.3.1 that \(\text{wt}(\eta) \neq \text{wt}(\eta_\psi) = \text{wt}(\psi) + \text{Deg}^{\text{tail}}(\psi)\delta\). Therefore, we deduce that \(|c_0| \neq 0\), which implies that \(c_0 \neq (\emptyset)_{i \in I}\).

Now, we set \(b := \Psi^{-1}_\lambda(\eta_\psi) \in B_e^-(\lambda) \cap B_0(\lambda)\); note that \(b = Xu_\lambda\). Then we see by (7.3.6) that \(z_{c_0}b \in B(X_e^-(\lambda))\). Also, since \(z_{c_0}b = z_{c_0}Xu_\lambda = Xz_{c_0}u_\lambda = XS_{I\lambda}u^{\Theta(C)}\) by Remark 7.3.1, we have \(\Psi_\lambda(z_{c_0}b) = XS_{I\lambda}\eta^C = \eta\). Hence we conclude that \(\eta \in \Psi_\lambda(B(X_e^-(\lambda)))\). Thus we have shown the inclusion \(\supset\).

Let us show the opposite inclusion \(\subset\). Since \(B(X_e^-(\lambda)) \subset B_e^-(\lambda)\), it follows immediately from Theorem 4.2.1 that
\[
\text{cl}^{-1}(\psi) \cap \Psi_\lambda(B(X_e^-(\lambda))) \subset \text{cl}^{-1}(\psi) \cap B_\leq_e^\infty(\lambda).
\]

Therefore, it suffices to show that \(\eta_\psi \notin \Psi_\lambda(B(X_e^-(\lambda)))\). Suppose, for a contradiction, that there exists \(b' \in B(X_e^-(\lambda))\) such that \(\Psi_\lambda(b') = \eta_\psi\). By (7.3.6), we can write it as: \(b' = z_{c_0}b\) for some \(c_0 \in \text{Par}(\lambda) \setminus (\emptyset)_{i \in I}\) and \(b \in B_e^-(\lambda) \cap B_0(\lambda)\). We show that \(\eta := \Psi_\lambda(b) \in \text{cl}^{-1}(\psi)\).

Let us write \(b = Yu_\lambda\) for some monomial \(Y\) in root operators (note that \(\eta = Y\eta_c\)), and define \(\zeta = \sum_{i \in I} c_i\alpha_i^\lambda \in Q_{I,J}^{\lambda+}\) and \(c_0 = (\rho(i))_{i \in I} \in \text{Par}(\lambda)\) in such a way that \(c_0 = (c_i)_{i \in I} + c'_0\) (see Remark 7.3.1). Then, by (7.3.4), we have
\[
b' = z_{c_0}b = z_{c_0}Yu_\lambda = Yz_{c_0}u_\lambda = YS_{I\lambda}u^{c'_0}.
\]

Also, we see that
\[
\eta_\psi = \Psi_\lambda(b') = \Psi_\lambda(YS_{I\lambda}u^{c'_0}) = YS_{I\lambda}\eta^C, \quad \text{with } C := \Theta^{-1}(c'_0) \in \text{Conn}(B_\leq_e^\infty(\lambda)),
\]
(7.3.11)
and hence that
\[
\psi = \text{cl}(\eta_\psi) = \text{cl}(YS_t^\kappa \eta^{C}) = Y \text{cl}(S_t^\kappa \eta^{C}) = Y \text{cl}(\eta_e) \quad (\text{see Remark } 3.5.2)
\]
\[
= \text{cl}(Y \eta_e) = \text{cl}(\Psi_\lambda(Y u_\lambda)) = \text{cl}(\Psi_\lambda(b)).
\]

Thus, we obtain \( \eta = \Psi_\lambda(b) \in \text{cl}^{-1}(\psi) \), as desired. Since \( b \in B_\varepsilon^-(\lambda) \) by our assumption, we have \( \eta = \Psi_\lambda(b) \in B_{\geq e}(\lambda) \). Hence it follows from Remark 6.3.1 that \( \text{wt}(\eta) - \text{wt}(\eta_\psi) \in \mathbb{Z}_{\leq 0} \delta \). On the other hand, by (7.3.11), we have
\[
\begin{align*}
\text{wt}(\eta_\psi) &= \psi \lambda - \text{wt}(\eta) = w_\delta \\
&= \psi \lambda - (\psi \lambda - \text{wt}(\eta)) = \psi \lambda - |c_0| \delta = \psi \lambda - |c_0| \delta,
\end{align*}
\]
and hence \( \text{wt}(\eta) - \text{wt}(\eta_\psi) = |c_0| \delta \in \mathbb{Z}_{\geq 0} \delta \). Combining these, we deduce that \( |c_0| = 0 \), which implies that \( c_0 = (\emptyset)_{i \in I} \). However, this contradicts our assumption that \( c_0 \in \text{Par}(\lambda) \setminus (\emptyset)_{i \in I} \). Thus we have shown the inclusion \( \subset \). This completes the proof of part (1) of Theorem 7.2.1.

Finally, we prove part (2) of Theorem 7.2.1. Let \( w \in W^J \). Because
\[
U_w^-(\lambda) \cong V_w^-(\lambda)/\left(V_w^-(\lambda) \cap X_w^-(\lambda)\right),
\]
we deduce that \( V_w^-(\lambda) \cap X_w^-(\lambda) \) has a \( \mathbb{C}(q) \)-basis \( \{ G(b) \mid b \in B_w^-(\lambda) \cap B(X_w^-(\lambda)) \} \). It follows immediately from part (1) that
\[
\Psi_\lambda(B_w^-(\lambda) \cap B(X_w^-(\lambda))) = B_{\geq w}^\lambda(\lambda) \cap \left( B_{w}^\lambda(\lambda) \setminus \{ \eta_\psi \mid \psi \in \text{QLS}(\lambda) \} \right)
\]
\[
= B_{\geq w}^\lambda(\lambda) \setminus \left( B_{w}^\lambda(\lambda) \cap \{ \eta_\psi \mid \psi \in \text{QLS}(\lambda) \} \right);
\]

note that \( B_w^-(\lambda) \subset B_e^-(\lambda) \), since \( w \in W^J \) and hence \( w \succeq e \) (see Lemma 2.3.4). Therefore, if we define \( B(U_w^-) \) to be the inverse image of the set
\[
B_{\geq w}^\lambda(\lambda) \cap \{ \eta_\psi \mid \psi \in \text{QLS}(\lambda) \}
\]
under the isomorphism \( \Psi_\lambda \), then the set \( \{ G(b) \mid b \in B(U_w^-(\lambda)) \} \) is a \( \mathbb{C}(q) \)-basis of \( U_w^-(\lambda) \). Here, observe that
\[
B_{\geq w}^\lambda(\lambda) \cap \{ \eta_\psi \mid \psi \in \text{QLS}(\lambda) \} = \{ \eta_\psi \mid \psi \in \text{QLS}(\lambda) \text{ such that } \kappa(\eta_\psi) \succeq w \}.
\]

Since \( \kappa(\eta_\psi) \) and \( w \) are both contained in \( W^J \), it follows from Lemma 2.3.4 that \( \kappa(\eta_\psi) \succeq w \) if and only if \( \kappa(\eta_\psi) \succeq w \) with respect to the (ordinary) Bruhat order on \( W^J \). Thus we have proved part (2). This completes the proof of Theorem 7.2.1. \( \square \)

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