Second order formalism in Poincaré gauge theory

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Abstract

Changing the set of independent variables of Poincaré gauge theory and considering, in a manner similar to the second order formalism of general relativity, the Riemannian part of the Lorentz connection as function of the tetrad field, we construct theories that do not contain second or higher order derivatives in the field variables, possess a full general relativity limit in the absence of spinning matter fields, and allow for propagating torsion fields in the general case, the spin density playing the role of the source current in a Yang-Mills type equation for the torsion. The equivalence of the second order and conventional first order formalism is established and the corresponding Noether identities are discussed. Finally, a concrete Lagrangian is constructed and by means of a Yasskin type ansatz, the field equations are reduced to a conventional Einstein-Proca system. Neglecting higher order terms in the spin tensor, approximate solutions describing the exterior of a spin polarized neutron star are presented and the possibility of the experimental detection of the torsion fields is briefly discussed.

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1 Introduction

In general relativity, the procedure to consider, during the variation of the Lagrangian, the connection and the metric as independent of each other is known as Palatini, or first order, formalism. Indeed the classical Hilbert-Einstein Lagrangian contains no second derivatives of those fields. On the other hand, in the conventional metric approach of general relativity, where the only independent field is the metric, the Lagrangian contains second derivatives of the metric, although incidentally, the terms containing them can be eliminated by subtracting a boundary term. This is known as second order formalism.

In Poincaré gauge theory (PGT), the usual procedure is to consider the connection $\Gamma_{ab}^m$ and the tetrad $e^a_m$ as independent fields and consequently, the field equations are derived through variation with respect to those fields. For a detailed description of PGT, we refer to Refs. [1] and [2]. Usually, the Lagrangians are constructed from terms at most second order in curvature and torsion, and thus contain no second derivatives of the independent fields. We will therefore refer to this as the first order formalism.

However, in view of the relation $\Gamma_{ab}^m = \Gamma_{ab}^m + K_{ab}^m$, where $\Gamma_{ab}^m$ is the Christoffel connection, which can be expressed as function of the tetrad (and its derivatives) only, and $K_{ab}^m$ is the contortion tensor, one might equally well consider $K_{ab}^m$ and $e^a_m$ as independent fields and carry out the variation with respect to those fields. Then, similar to the classical approach in general relativity (GR), the Christoffel
part of the connection is considered as a function of the tetrad. We will refer to this as the second order
formalism.

What could be the use of this formalism? Well, in conventional PGT, the most general Lagrangian
is constructed under the requirement that it should not contain second and higher order derivatives of
the independent fields \( e^a_m \) and \( \Gamma^{ab}_m \) for the usual reasons (Cauchy problem). It is easy to see that, if
we require instead the Lagrangian not to contain second and higher order derivatives of \( e^a_m \) and \( K^{ab}_m \),
now seen as independent fields, then this will allow for other terms in the Lagrangian. Also, some terms
previously allowed are now forbidden, as is the case, e.g., for terms of second order in the curvature,
which will contain second derivatives of the tetrad field in this formalism. The Cauchy problem can
now be stated with respect to \( e^a_m \) and \( K^{ab}_m \). Since the determination of the latter fields allows for the
determination of \( \Gamma^{ab}_m \) too, no initial value problem will arise this way.

On the other hand, one could argue that, in view of the underlying gauge structure of the theory,
it appears quite unnatural to consider as fundamental variables a set different to the gauge fields of the
Poincaré group. It turns out, however, that both formalisms are completely equivalent. In other words,
it is simply a matter of choice and convenience which formalism we use. This equivalence ultimately
means that theories that contain higher derivatives in one formalism may be reformulated in the other
formalism such that no higher derivatives occur. The change of the fundamental variables thus does not
change anything concerning the viability or consistency of the theory, but it shows that certain theories
that are apparently in conflict with the Cauchy initial value problem need not be so in reality.

Summarizing, the second order formalism allows us to investigate different Lagrangians than those
usually considered in PGT. This, in turn, will allow us to construct a theory with a complete classical
general relativity limit, and thus a complete agreement with the experimental situation, without any
constraints on eventual parameters of the theory, but with dynamical torsion fields arising in the presence
of spinning matter. As we have pointed out in Ref. \([3]\), this is not possible in conventional PGT for the
following reasons. Since in the usual first order approach, the independent fields are the tetrad \( e^a_m \) and the
Lorentz connection \( \Gamma^{ab}_m \), in order to get propagating modes for both fields, their first derivatives should
appear quadratically in the Lagrangian. Therefore, apart from a term suitable of producing the general
relativity limit, there has to be at least one term quadratic in the curvature tensor. In the presence of
spinless matter, we wish the field equations to reduce to the Einstein equations of general relativity. If
the Cartan equation (i.e., the connection equation) in this case leads to a vanishing torsion (Riemannian
geometry), then the term quadratic in the curvature will necessarily contribute to the Einstein equation,
which will thus not be of the GR form. (And therefore, in order to be in agreement with experiments,
constraints will have to be imposed on the coupling constants of such theories.) The only exceptions to
this are very artificial Lagrangians, with terms of the form \( R^{[ik]} [l^{[ik]}] \) and \( R^{[ikm]} R^{[ik]} \), constructed from
the antisymmetric part of the Ricci tensor and the completely antisymmetric (in the last three indices)
part of the curvature tensor. Those terms will lead to contributions in the Einstein equation that vanish
together with the torsion in the spinless case (in view of the Bianchi identities in the Riemannian limit).
In the presence of spinning matter fields, those terms will eventually lead to propagating torsion, but in
general, the field equations have very few and quite strange solutions, certainly not of the form expected
for a Yang-Mills like theory. On the other hand, theories whose Cartan equation in the vanishing spin
limit leads to a vanishing curvature (teleparallel geometry) and presenting a Yang-Mills like behavior for
the Lorentz connection can easily be constructed using curvature squared terms, without problems with
the general relativity limit. Those theories have been analyzed in detail in Ref. \([3]\), where we have shown
that they all present problems with an additional symmetry arising in the classical limit. Schematically,
we can write the linear approximation of such theories in the following form of wave equations

$$\Box \Gamma^a_{\text{m}} = \sigma^a_{\text{m}} \quad \text{and} \quad \Box g_{ik} = T_{ik} + \mathcal{O}(R), \quad (1)$$

which leads, in the spinless case, to $\Gamma^a_{\text{m}} = 0$, and therefore the curvature corrections $\mathcal{O}(R)$ will vanish and we are left with the GR equation $\Box g_{ik} = T_{ik}$. The problem is that, even for $\Gamma^a_{\text{m}}$ fixed to zero, we can Lorentz rotate the tetrad field $e^a_{\text{m}} \rightarrow \Lambda^a_b e^b_{\text{m}}$, without changing the metric $g_{ik}$. Thus, the tetrad field is not uniquely fixed by the field equations, and therefore, the behavior of spinning test matter, e.g., the spin precession and the trajectory of a Dirac particle, cannot be uniquely predicted [3]. It is clear that this argumentation is valid independently of the formalism (first or second) one uses for the derivation of the field equations. It is a general problem of the set of equations $\Gamma^a_{\text{m}} = 0$ and $G_{ik} = T_{ik}$, i.e., of the teleparallel form of GR.

On the other hand, a theory with an GR limit in a Riemannian geometry (vanishing torsion), should behave as

$$\Box K^a_{\text{m}} = \sigma^a_{\text{m}} \quad \text{and} \quad \Box g_{ik} = T_{ik} + \mathcal{O}(K), \quad (2)$$

where now the spinless case leads to a vanishing contortion $K^a_{\text{m}} = 0$, and thus to a vanishing torsion, and therefore, as before, the second equation reduces to the field equations of GR. In contrast to (1), the set of equations (2) fixes the geometry completely. Indeed, if we know the contortion and the metric, then the independent field tensors, torsion and curvature, are fixed, up to a Poincaré transformation of course, which is the underlying gauge symmetry of the theory [3].

However, as we have argued above, no Lagrangian of conventional PGT leads to equations of the form (2). In order to get such field equations using the conventional first order approach, one would have to use higher order derivatives of the fields $(\Gamma^a_{\text{m}}, e^a_{\text{m}})$ in the Lagrangian, which is (apparently) forbidden in view of the initial value problem.

We will solve this problem using the second order approach and construct an explicit example of a theory with equations of type (2), containing only first order derivatives of the independent fields $(K^a_{\text{m}}, e^a_{\text{m}})$.

The article is structured as follows. In the next section, we will establish the equivalence of the first and second order formalisms as far as the field equations are concerned. In section 3, we will compare the Noether currents and their conservation equations in the two approaches. Section 4 is devoted to the construction of an exemplifying theory that, using the second order formalism, allows for a full general relativity limit in the absence of spinning matter. Dynamical torsion fields will arise if the matter possesses a non-vanishing spin density. In section 5, extending slightly the Yasskin ansatz known from Einstein-Yang-Mills theory with internal symmetry group, we simplify the system of field equations for a special class of solutions, suitable for the description of spin polarized neutron stars. Finally, in section 6, explicit solutions are discussed and in section 7, the effects of the torsion fields on the spin precession of a test body are briefly analyzed in view of an eventual experimental detection of the non-Riemannian contributions.

2 Equivalence of first and second order formalisms

Let us first give a short review of the basic concepts of Riemann-Cartan geometry and fix our notations and conventions. For a complete introduction into the subject, the reader may consult Refs. [1] and [2].

Latin letters from the beginning of the alphabet ($a, b, c, \ldots$) run from 0 to 3 and are (flat) tangent space indices. Especially, $\eta_{ab}$ is the Minkowski metric $\text{diag}(1, -1, -1, -1)$ in tangent space. Latin letters
from the middle of the alphabet \((i, j, k \ldots)\) are indices in a curved spacetime with metric \(g_{ik}\) as before. We introduce the Poincaré gauge fields, the tetrad \(e^a_m\) and the connection \(\Gamma^{ab}_m\) (antisymmetric in \(ab\)), as well as the corresponding field strengths, the curvature and torsion tensors

\[
\begin{align*}
R^{ab}_{\ m} &= \Gamma^{ab}_{m,\ l} - \Gamma^{ab}_{l\ m} + \Gamma^a_c \Gamma^{cb}_{\ m} - \Gamma^a_{cm} \Gamma^{cb}_l \\
T^a_{\ lm} &= e^a_{m,\ l} - e^a_{l\ m} + e^b_m \Gamma^{ab}_{\ l\ m} - e^b_l \Gamma^{ab}_m.
\end{align*}
\]

(3)

(4)

The spacetime connection \(\Gamma^i_{\ ml}\) and the spacetime metric \(g_{ik}\) can now be defined through

\[
\begin{align*}
e^a_{m,\ l} + \Gamma^a_{bl} e^b_m &= e^a_i \Gamma^i_{ml} \\
\delta^i_a e^b_i &= g_{ik}.
\end{align*}
\]

(5)

(6)

It is understood that there exists an inverse to the tetrad, such that \(e^a_i e^i_b = \delta^a_b\). It can easily be shown that the connection splits into two parts,

\[
\Gamma^{ab}_m = \hat{\Gamma}^{ab}_m + K^{ab}_m,
\]

(7)

such that \(\Gamma^{ab}_m\) is torsion-free and is essentially a function of \(e^a_m\). \(K^{ab}_m\) is the contortion tensor (see below). Especially, the spacetime connection \(\hat{\Gamma}^i_{\ ml}\) constructed from

\[
e^a_{m,\ l} + \hat{\Gamma}^a_{bl} e^b_m = e^a_i \hat{\Gamma}^i_{ml},
\]

(8)

is just the (symmetric) Christoffel connection of general relativity, a function of the metric only.

The gauge fields \(e^a_m\) and \(\Gamma^{ab}_m\) are covector fields with respect to the spacetime index \(m\). Under a local Lorentz transformation (more precisely, the Lorentz part of a Poincaré transformation, see Ref. [3]) in tangent space, \(\Lambda^a_b(x^m)\), they transform as

\[
e^a_m \rightarrow \Lambda^a_b e^b_m, \quad \Gamma^{ab}_m \rightarrow \Lambda^a_c \Lambda^b_d \Gamma^{cd}_m - \Lambda^a_{cm} \Lambda^b_c.
\]

(9)

The torsion and curvature are Lorentz tensors with respect to their tangent space indices as is easily shown. The contortion \(K^{ab}_m\) is also a Lorentz tensor and is related to the torsion through \(K^i_{\ lm} = \frac{1}{2} (T^i_{\ lm} + T^i_{\ ml} - T^i_{\ lm})\), with \(K^i_{\ lm} = e^a_i e^b_i K^{ab}_m\) and analogously for \(T^i_{\ lm}\). The inverse relation is \(T^{i}_{\ lm} = -2K^i_{\ l|m}\).

All quantities constructed from the torsion-free connection \(\hat{\Gamma}^{ab}_m\) or \(\hat{\Gamma}^i_{\ ml}\) will be denoted with a hat, for instance \(\hat{R}^{il}_{\ km} = e^a_i e^b_i \hat{R}^{ab}_{\ km}\) is the usual Riemann curvature tensor.

We are now ready to compare the field equations arising in first and second order formalism and to establish their equivalence.

Let \(\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_m\) be the Lagrangian density in the first order formalism and \(\check{\mathcal{L}} = \check{\mathcal{L}}_0 + \check{\mathcal{L}}_m\) the same Lagrangian in the second order formalism. The indices 0 and \(m\) refer to the gravitational (or free) part and to the matter part respectively. Thus, we can write

\[
\mathcal{L}(e^a_m, \Gamma^{ab}_m) = \check{\mathcal{L}}(e^a_m, K^{ab}_m).
\]

(10)

In concrete cases, the Lagrangians may differ by a total divergence, which does not influence the field equations and will therefore not be written out explicitly. For instance, in Ref. [3], we considered the teleparallel Lagrangian

\[
e(R - \frac{1}{4} T^{ijkl} T_{ijkl} - \frac{1}{2} T^{ikl} T_{ikl} + \frac{1}{2} T^{ik} T_{ik} T^{ml} m) = e \hat{R}.
\]

(11)
where the left hand side is the first order form, while the right hand side is the usual Einstein-Hilbert Lagrangian expressed in terms of the tetrad only. In (11), a divergence term has been suppressed. (In other words, the left and right hand sides are not really equal, but can be rendered equal by adding a physically irrelevant divergence.)

The field equations in the conventional, first order formalism are well known,

\[ \frac{\delta \mathcal{L}}{\delta \Gamma_{ab}^m} = 0 \quad \text{and} \quad \frac{\delta \mathcal{L}}{\delta e_a^m} = 0. \]  

As usual, they will be referred to as Cartan and Einstein equations, respectively. Both formalisms are related by equation (7),

\[ \Gamma_{ab}^m = \hat{\Gamma}_{ab}^m(e_a^m) + K_{ab}^m. \]

For the Cartan equation in the second order formalism, we find

\[ \frac{\delta \tilde{\mathcal{L}}}{\delta K_{ab}^m} = \frac{\delta \mathcal{L}}{\delta \Gamma_{cd}^l} \frac{\delta \Gamma_{cd}^l}{\delta K_{ab}^m} = \frac{\delta \mathcal{L}}{\delta \Gamma_{ab}^m} = 0. \]  

This differs by the second term from the corresponding equation in (12). This can be seen explicitly by carrying out the variation of this term using the explicit form of \( \hat{\Gamma}_{cd}^l \) from (8) and of the Christoffel symbols \( \hat{\Gamma}_{ik}^l \).

However, we can instead use immediately the Cartan equation \( \delta \mathcal{L}/\delta \Gamma_{cd}^l = 0 \) (which we have shown to be equivalent to the Cartan equation in the second order formalism) to eliminate the second term in (14), and therefore we have the on-shell relation \( \delta \tilde{\mathcal{L}}/\delta e_a^m = \delta \mathcal{L}/\delta e_a^m \).

To summarize, we have the following relations

\[ \frac{\delta \tilde{\mathcal{L}}}{\delta e_a^m} = \frac{\delta \mathcal{L}}{\delta e_a^m} \quad \text{and} \quad \frac{\delta \tilde{\mathcal{L}}}{\delta K_{ab}^m} = \frac{\delta \mathcal{L}}{\delta \Gamma_{ab}^m}, \]

which shows clearly the equivalence of both sets of field equations. The above derivation also explains why some Lagrangians are allowed only in the first order formalism while others can only be considered in the second order formalism. The reason is that the terms resulting from the higher order derivatives (when the wrong formalism is applied to a certain Lagrangian), can actually be eliminated using the second field equation, thereby bringing the field equations into a form that would have resulted immediately if the right formalism had been used right from the start. (By wrong and right, we simply mean the formalism where we have higher order derivatives, or we have not, respectively. Both formalisms are correct, since they are equivalent.)

Note that in order to show the equivalence of both sets of field equations, we have to start with the full Lagrangian. We can of course not conclude that for given stress-energy tensor and spin density, the left hand sides of the field equations will have the same form in both procedures. The source tensors have to be modified at the same time.
Let us define the conventional canonical stress-energy and spin density tensors in the first order formalism

\[ T^a_i = \frac{1}{2\epsilon} \frac{\delta L_m}{\delta e^i_a} \quad \text{and} \quad \sigma_{ab}^m = \frac{1}{\epsilon} \frac{\delta L_m}{\delta \Gamma^a_{bm}}, \]  

as well as their second order analogues

\[ \tilde{T}^a_i = \frac{1}{2\epsilon} \frac{\delta \tilde{L}_m}{\delta e^i_a} \quad \text{and} \quad \tilde{\sigma}_{ab}^m = \frac{1}{\epsilon} \frac{\delta \tilde{L}_m}{\delta K^a_{bm}}. \]  

Just as before (see (13) and (14)), one shows that

\[ \tilde{\sigma}_{ab}^m = \sigma_{ab}^m \quad \text{and} \quad \tilde{T}^a_i = T^a_i + \frac{1}{2} \sigma_{cd}^m \frac{\delta \tilde{\Gamma}^{cd}_m}{\delta e^i_a}. \]  

Now, use the relation (8), \( e^i_{a,k} + \Gamma^a_{bk} e^k_i = \tilde{e}^i_a \tilde{\Gamma}_i^k \), and the explicit form of the Christoffel symbols \( \tilde{\Gamma}_i^k \) to derive

\[ \tilde{\Gamma}^{cd}_m = \frac{1}{2} \left[ e^{ci}_m e^{di}_m e^c_{i,m} - e^{di}_m e^{ci}_m e^d_{i,m} + e^{di}_m e^{ck}_m e^{lb}_k e^{lb}_k - e^{ci}_m e^{dk}_m e^{lk}_k e^{lk}_k \right]. \]  

We can now evaluate \( \tilde{T}^a_i \) in (18). The result is

\[ \tilde{T}^{lm} = T^{lm} + \frac{1}{2} (\sigma^{lmk} - \sigma^{kml} - \sigma^{klm})_{,k}, \]  

where \( : \) denotes the usual covariant derivative with the Christoffel connection.

Note that (19) looks basically like the Belinfante-Rosenfeldt relation between the symmetric Hilbert (or metric) stress-energy tensor \( \tilde{T}^{lm} \) and the canonical (or Noether) tensor \( T^{lm} \) known from purely Riemannian theory, see Refs. [1] and [2], especially the discussion on relocalizations in Ref. [2]. We will see that \( \tilde{T}^{lm} \) is not yet the symmetric tensor, but it is a tensor that reduces to the symmetric Hilbert tensor for vanishing torsion.

In order to prevent any kind of misunderstandings, let us emphasize once again that we do not modify the underlying gauge structure of the theory. Especially, the relation (10) makes clear the we do not change the coupling prescriptions in the matter Lagrangians. Thus, for instance, the Dirac particle still couples minimally to the gauge connection \( \Gamma_{ab}^i \) (which, from a mathematical point of view, is the fundamental field variable). The only difference is that we express this connection in terms of \( e^i_a \) and \( K^a_{bm} \). Any change in the coupling prescription would lead to a fundamentally different theory, especially concerning the gauge symmetry.

In the next section, we will derive the conservation equations for spin density and stress energy tensor in both approaches.

3 The Noether identities

3.1 First order formalism

Let us briefly review the derivation of the Noether identities related to local, tangent space Lorentz transformations to general coordinate transformations. Under an infinitesimal Lorentz transformation (9), with \( \Lambda^a_b = \delta^a_b + \varepsilon^a_b \) (\( \varepsilon^{ab} = -\varepsilon^{ba} \)) the independent fields \( e^m_a \) and \( \Gamma_{a}^{m}_{bm} \) undergo the following change:

\[ \delta \Gamma_{a}^{m}_{bm} = -\varepsilon^{ab}_{,m} - \Gamma^{a}_{cm} e^{bh}_{m} - \Gamma^{b}_{cm} e^{ah}_{m} \quad \text{and} \quad \delta e^m_a = \varepsilon^a_c e^m_c. \]  

6
Let us note at this point that, if we define the parameters $\varepsilon^a$ and $\xi^a$ to as generalized Poincaré gauge transformations, with $\Gamma_{ab}^m$ on tangent space indices and with $\hat{\Gamma}_{ki}$ (torsion free) on spacetime indices. The requirement of Lorentz gauge invariance therefore leads to

$$2T^{[ac]} + D_m a_{cm} = 0.$$  \hspace{1cm} (22)

The first equation can be written in the short form $\delta \Gamma_{ab}^m = -D_m \varepsilon^{ab}$. The change in the matter Lagrangian therefore reads

$$\Delta L_m = \frac{\delta \mathcal{L}_m}{\delta e^a_m} \delta e^m_a + \frac{\delta \mathcal{L}_m}{\delta \Gamma_{ab}^m} \delta \Gamma_{ab}^m = \varepsilon(2T^{[ac]} + D_m a_{cm})\varepsilon_{ac},$$

\hspace{1cm} (21)

where we have omitted a total divergence. The covariant derivative operator $D_m$ is defined to act with $\Gamma_{ab}^m$ on tangent space indices and with $\hat{\Gamma}_{ki}$ (torsion free) on spacetime indices. The requirement of Lorentz gauge invariance therefore leads to

$$2T^{[ac]} + D_m a_{cm} = 0.$$  \hspace{1cm} (22)

Slightly more complicated is the case of the coordinate invariance. Under an infinitesimal coordinate transformation,

$$\hat{x}^i = x^i + \xi^i,$$

\hspace{1cm} (23)

the fields $\Gamma_{ab}^m$ and $e^m_a$ transform as spacetime covectors (or one-forms), while the inverse tetrad $e^m_a$ is a contravariant vector with respect to the index $m$. Thus,

$$\tilde{e}^m_a(\hat{x}) = e^m_a(x) + \tilde{e}^m_{a,k}(\hat{x})\xi^k, \quad \tilde{\Gamma}_{ab}^m(\hat{x}) = \Gamma_{ab}^m(x) - \xi^k \Gamma_{ab}^{m,k,k}.$$  

Since we are interested in the change of the Lagrangian under an active transformation, we have to evaluate the change of the fields at the same point $x$, and thus have to express the transformed fields in the old coordinates, i.e.,

$$\tilde{e}^m_a(x) = e^m_a(x) + \tilde{e}^m_{a,k}(x)\xi^k, \quad \tilde{\Gamma}_{ab}^m(x) = \Gamma_{ab}^m(x) - \xi^k \Gamma_{ab}^{m,k,k}.$$  

In the $\xi$-terms, we can replace $\tilde{e}^m_{a,k}(\hat{x})$ by $e^m_{a,k}(x)$ and $\tilde{\Gamma}_{ab}^{m,k}(\hat{x})$ by $\Gamma_{ab}^{m,k,k}(x)$, since the difference will be of order $\xi^2$. Finally, we find

$$\delta e^m_a = e^m_a(x) - e^m_a(x) = \xi^k e^m_{a,k} - \xi e^m_{a,k},$$

\hspace{1cm} (24)

$$\delta \Gamma^{ab}_m = \Gamma^{ab}_m(x) - \Gamma^{ab}_m(x) = -\xi^k \Gamma^{ab}_{m,k} - \xi \Gamma^{ab}_{m,k,k}.$$  \hspace{1cm} (25)

Let us note at this point that, if we define the parameters $\varepsilon^a = -e^m_a \xi^m$ and $\varepsilon^{ab} = \xi^k \Gamma_{ab}^{m,k,k}$, the above transformations can be written in the form

$$\delta e^m_a = \varepsilon^a e^m_a - D_m \varepsilon^a - \varepsilon^m T^a_{lm},$$

$$\delta \Gamma^{ab}_m = -D_m \varepsilon^{ab} - \varepsilon^l R^{ab}_{lm}.$$  

Since we are also free to perform an additional, independent, Lorentz transformation (20), we may again consider $\varepsilon^a$ and $\varepsilon^{ab}$ as independent of each other. In this form, the transformations are sometimes referred to as generalized Poincaré gauge transformations, with $\varepsilon^a$ generating the translations and $\varepsilon^{ab}$ the Lorentz rotations [1]. We will remain in the usual interpretation of active coordinate transformations and return to the form (24)-(25). The change of the Lagrangian reads

$$\delta \mathcal{L}_m = \frac{\delta \mathcal{L}_m}{\delta e^m_a} \delta e^m_a + \frac{\delta \mathcal{L}_m}{\delta \Gamma^{ab}_m} \delta \Gamma^{ab}_m$$

$$= -2(\varepsilon^m T^a_{m,a,k} \xi^m - 2\varepsilon^m T^a_{m,a,k} \xi^k + (\varepsilon \sigma^{m}_{ab} \Gamma_{ab}^{m,k}) \xi^k - \varepsilon \sigma^{m}_{ab} \Gamma_{ab}^{m,k} \xi^k).$$
Requiring \( \delta \mathcal{L}_m = 0 \) and regrouping carefully the terms, we finally get

\[
(D_m T^m_b)_{eb} + T^m_b T^b_{mk} = \frac{1}{2} R^{ab}_{mk} \sigma^{ab}_{m} + \frac{1}{2} \Gamma^{ab}_{m} (D_m \sigma^{ab}_{m} + 2 T_{[ab]}).
\]  

(26)

The last term vanishes with (22) and the first two terms can be rewritten in a different form, using the relation \( T^m_{[im]} = -2 K^i_{[lm]} \) in the second term, and the explicit form of the covariant derivative \( D_m \) in the first term, to get

\[
T^m_{k;m} - K^{i m} T^i_{m} = \frac{1}{2} R^{ab}_{mk} \sigma^{ab}_{m}.
\]  

(27)

Recall that \( K^{i m} \) is antisymmetric in \( im \), so that classical matter (with \( T^m_{[mi]} = 0 \) and \( \sigma^{ab}_{m} = 0 \)) will satisfy the general relativistic conservation law \( T^m_{k;m} = 0 \).

### 3.2 Second order formalism

In the second order formalism, we have to evaluate the changes of the fields \( e^m_a \) and \( K^{ab}_{m} \) under Lorentz and coordinate transformations. The Lorentz transformation now reads

\[
\delta e^m_a = \varepsilon^{mn} c^{m}_{a}, \quad \delta K^{ab}_{m} = \varepsilon^{ac} K^{cb}_{m} + \varepsilon^{b c} K^{ac}_{m},
\]  

(28)

which shows clearly that \( K^{ab}_{m} \) transforms as Lorentz tensor. Evaluating

\[
\delta \hat{\mathcal{L}}_m = \frac{\delta \hat{\mathcal{L}}_m}{\delta e^m_a} \delta e^m_a + \frac{\delta \hat{\mathcal{L}}_m}{\delta K^{ab}_{m}} \delta K^{ab}_{m},
\]

and requiring \( \delta \hat{\mathcal{L}}_m = 0 \) we easily derive the following relation

\[
0 = 2 \hat{T}^{[ac]} - K^{bc}_{m} \sigma^{a}_{b m} + K^{ba}_{m} \sigma^{c}_{b m}.
\]  

(29)

This does not look like a conservation equation for the spin density. We see that in the case where the torsion vanishes, the stress-energy tensor becomes symmetric. Therefore, we can see \( \hat{T}^{im} \) as a generalization of the symmetric Hilbert tensor. Let us note at this point that if we had considered as independent the fields \( K^{ik}_{lm} \) and \( e^m_a \) (instead of \( K^{ab}_{m} \) and \( e^a_m \)), we would have found a stress-energy tensor that is always symmetric. This, however, does not present any advantages (the variation of the gravitational Lagrangian would in most cases become more complicated) and so we preferred to stay a step closer to the conventional gauge approach, considering \( K^{ab}_{m} \) with its natural tangent space indices. The advantage of this approach is the linear relation between \( \Gamma^{ab}_{m} \) and \( K^{ab}_{m} \) and the resulting complete equivalence of the Cartan field equation in first and second order formalism. The alternative approach, with \( K^{i m} \) and \( e^m_a \), has been used in the framework of Einstein-Cartan-Dirac theory in the past [4]. It leads of course to a symmetric Einstein equation.

Under coordinate transformations, the fields transform exactly as in the previous section (because just as \( \Gamma^{ab}_{m} \), \( K^{ab}_{m} \) too is a spacetime covector), i.e.,

\[
\delta e^m_a = \tilde{e}^m_a (x) - e^m_a (x) = \xi^m_a e^k_a - \xi^k_a e^m_a, \\
\delta K^{ab}_{m} = \tilde{K}^{ab}_{m} (x) - K^{ab}_{m} (x) = -\xi^k_m K^{ab}_{k} + \xi^k K^{ab}_{m k}.
\]  

(30)

(31)
In the same way as in the previous section, we derive the conservation equation in the form

\[
(D_m \tilde{T}_a^m)e_k^a + \tilde{T}_a^m T^a_{mk} - \frac{1}{2} \sigma_{li}^m \ K_{lk}^m
= \frac{1}{2} \left[ K_{ab}^{ck} m - K_{ab}^{ck} m, k + \hat{\Gamma}_{cm}^b K_{ck}^a k + \hat{\Gamma}_{cm}^a K_{ac}^b k - \hat{\Gamma}_{ck}^a K_{cb}^m m + \hat{\Gamma}_{ck}^b K_{ac}^b m + K_{ab}^{ck} e_k^a \right] \sigma_{ab}^m.
\]

We used (29) to put the equation into this form. A shorter way of writing this equation is the following

\[
(D_m \tilde{T}_a^m)e_k^a + \tilde{T}_a^m T^a_{mk} - \frac{1}{2} \sigma_{li}^m \ K_{lk}^m = \frac{1}{2} (\hat{R}_{ab}^{mk} - \hat{R}_{ab}^{mk}) \sigma_{ab}^m.
\]

### 3.3 Discussion

Comparing (22) and (29), we can derive the following relation between the antisymmetric parts of the stress-energy tensors:

\[
2(\tilde{T}_{lm}^m - T_{lm}^m) = \sigma_{lm}^k.
\]

This relation is easily verified using the explicit form of \(\tilde{T}_{lm}^m\) in terms of \(T_{lm}^m\), Eq. (19).

Also, from (26) and (32), we get a relation between the divergences of the stress-energy tensors of the form

\[
(T^m_a k - \tilde{T}^m_a k)_{mk} = \frac{1}{2} \hat{\Gamma}_{ab} \sigma_{ab}^m.
\]

We used (33) to bring the relation into this form. Again, this relation can be derived directly from (19), using the symmetry properties of \(\sigma_{lm}^k\) and of the Riemann tensor \(\hat{R}_{lm}^k\).

Note that both sets of equations, (22,26) and (29,32), are valid, it is not the one or the other set. (The fact that the Lagrangian is invariant under Lorentz gauge and under general coordinate transformations does not depend on the choice of the independent field variables.) They are simply different equations for different quantities. The important thing is to use the right quantities as right hand side of the gravitational field equations (this depends on the choice of our formalism, first or second order).

Another point concerns the physical interpretation of \(T^i_{ik}\) and \(\tilde{T}^i_{ik}\). If we take the point of view that the measurable quantities are not the densities, but rather the integrated quantities, namely the momentum vector and the spin tensor \([5]\), then it is easy to see that \(T^i_{ik}\) and \(\tilde{T}^i_{ik}\) lead, in general, to different definitions of the momentum. Apart from this ambiguity, there is the problem that even from a single stress-energy tensor, one can define many inequivalent momentum vectors, integrating \(T^i_{ia}\), \(T^i_{ai}\), \(T^i_{ik}\) etc. (see Ref. [6] for an interesting discussion on this point). Usually, one would like to define the momentum in a way that is conserved. This is of course not possible in our case, since whatever momentum vector one defines, it will always be subject to gravitational forces coming from curvature and torsion, and therefore it cannot be conserved. (The other way around, whatever momentum vector one uses, it will always be conserved in the limit of vanishing curvature and torsion.) As it seems, it is rather a matter of convention how one wishes to define spin and momentum, under the only restriction that for vanishing fields, they should be conserved.

In view of the rather unusual equation (29), which does not look like a conservation equation, one might be tempted to reject the set of current densities \(\tilde{T}_{ik}^m, \sigma_{ab}^m\) and to claim that the canonical currents \(T_{ik}^m, \sigma_{ab}^m\) as they arise in the conventional, first order formalism, should be considered as the true, physical currents. However, let us recall the fact that, for instance, the canonical stress-energy tensor
of the Dirac Lagrangian contains non-axial torsion parts, which, as is well known, do not couple to the Dirac particle in the first place, and are neither contained in the Dirac equation, nor in the Lagrangian (where they cancel out with the hermitian conjugated term). We have brought this to attention in Ref. [5], to show that one has to be very careful on how to extract physical quantities from $T_m^a$. Thus, neither can the conventional, first order canonical tensors, be used without precautions to derive physically valid results.

In order to gain at least some insight into the contents of $T_{ik}$ and $\tilde{T}_{ik}$, let us take a look at the flat limit of our equations, i.e., assume a spacetime with $g_{ik} = \eta_{ik}$. In this case, we find from (19)

$$\tilde{T}_{lm} = T_{lm} + \frac{1}{2}(\sigma^{lmk} - \sigma^{kml} - \sigma^{klm}).k,$$

If we define the momentum vector as we do in special relativity,

$$P_k = \int T^i_k \, dS_i,$$

then, for the alternative stress-energy tensor, we have

$$\tilde{P}_k = \int \tilde{T}^i_k \, dS_i = P_k + \frac{1}{2} \int (\sigma^{i}_{m} - \sigma^{l}_{m} - \sigma^{l}_{i} m)_{l} \, dS_i.$$

Clearly, the last term can be converted into a 4d volume integral over the divergence of the integrand, which vanishes however, due to the antisymmetry in $il$ of the expression in brackets. Thus, we have

$$\tilde{P}_k = P_k.$$

Of course, we also have (see (34) or (35))

$$\tilde{T}^m_{k,m} = T^m_{k,m}.$$  

Usually, the relations (38) and (39) are said to show the equivalence of the two stress-energy tensors. (Same charges, same conservation law.) More generally, a change of the stress-energy tensor and the spin in the form

$$T_{lm} \rightarrow T_{lm} + X^{mkl},$$

$$\sigma_{km}^i \rightarrow \sigma_{km}^i - (X_{km}^i - X_{mk}^i)$$

with $X^{mkl} = -X^{mlk}$ is called relocalization of stress-energy and spin and is said not to affect physical quantities.

It is easily seen that (35) is a relocalisation of $T_{lm}$, without, however, the corresponding transformation of the spin density. Nevertheless, (26) and (29) coincide in flat space. The remaining relation is

$$\tilde{T}^m_{k,m} = T^m_{k,m} = \frac{1}{2}K_{k}^m(\sigma^{l}_{im},l - K_{lm}^j \sigma^{j}_{jm} - K_{jl}^m \sigma^{j}_{il})$$

$$+ \frac{1}{2}(K_{k}^{ab} - K_{ab}^{m,k} + K^{ac}_{cm} K_{k}^{bc} + K^{ab}_{cm} K_{k}^{ac}) \sigma_{ab}.$$
Unfortunately, we cannot express the equation for the spin precession (22) or (29) in a form independent of the stress-energy tensor. (Using (35) in (33) leads to an identity.) However, since the spin density is the same in both approaches, it will be subject to the same equations of motion, expressed in one or the other way.

We therefore conclude that in the flat limit, \( g_{ik} = \eta_{ik} \), the equations of motions for the current densities obtained in both pictures are identical as far as their physical contents is concerned.

In curved spacetime, this is not the case anymore. Especially, we get two different equations for the stress-energy tensor and we have to decide, based on physical arguments, how to define define the momentum vector.

Let us once again refer the reader to the extended discussion on canonical and metric stress-energy tensors and their relation via relocalizations in the general framework of metric affine theories in Hehl et al., [2].

4 Model construction

Comparing equations (27) and (32), we see that the following tensor

\[
F_{mk}^{ab} = \frac{1}{2}(R_{mk}^{ab} - \tilde{R}_{mk}^{ab})
\]

\[
= K_{k,m}^{ab} - K_{m,k}^{ab} + \Gamma_{cm}^{a}K_{k}^{cb} + \Gamma_{cm}^{b}K_{k}^{ac}
- \Gamma_{ck}^{a}K_{m}^{cb} - \Gamma_{ck}^{b}K_{m}^{ac} + K_{cm}^{a}K_{k}^{bc} + K_{cm}^{b}K_{k}^{ac}
\]

plays, in the context of the second order formalism, a similar role as the curvature tensor \( R_{lm}^{ab} \) in the conventional approach. Essentially, \( F_{mk}^{ab} \) has the structure of a Lorentz Yang-Mills tensor, but with \( \Gamma_{cm}^{a} \) replaced by \( K_{cm}^{a} \). The terms containing the Christoffel connection \( \tilde{\Gamma}_{cm}^{a} \) are needed to get the correct tensor behavior under a Lorentz transformation.

Apart from \( F_{mk}^{ab} \), which we will use to produce the dynamical, Yang-Mills like, behavior of the torsion field, we will need a term that guarantees the correct general relativity limit of the theory. In the second order formalism, this is simply the Einstein-Hilbert Lagrangian, expressed in terms of the tetrad field, \( \tilde{R} \). For reasons that will become clear later, we also include a mass term \( K_{m}^{ab}K_{l}^{ab} \) into our Lagrangian.

Thus, we start with

\[
\mathcal{L}_{0} = e(a\tilde{R} - bF_{mk}^{ab}F_{mk}^{ab} - cK_{m}^{ab}K_{l}^{ab} + \tilde{T}_{m}^{k} + \tilde{T}_{l}^{k}).
\]

This Lagrangian contains no second order derivatives of the independent fields \( e_{i}^{a} \) and \( K_{i}^{ab} \), apart from the well known higher order terms contained in \( \tilde{R} \), which can, however, be eliminated by subtracting a surface term. On the other hand, if \( \mathcal{L}_{0} \) were to be treated in the first order formalism, there would arise higher order derivatives in the tetrad fields, coming from the term \( \tilde{R}_{lm}^{ab} \tilde{R}_{lm}^{ab} \) contained in the second term of (43).

Including the matter Lagrangian \( \mathcal{L}_{m} \), we are let to the following set of field equations

\[
-4b \, D_{k}F_{mk}^{ab} + 2cK_{m}^{ab} = \sigma_{ab}^{m}
\]

\[
-a\tilde{G}_{ki} = \tau_{ki}^{(1)} + \tau_{ki}^{(2)} + \tau_{ki}^{(3)} + \tilde{T}_{ki},
\]

where

\[
\tau_{ki}^{(1)} = -2b(F_{mi}^{cb}F_{mk}^{cb} - \frac{1}{4}g_{ki}F_{mi}^{cb}F_{mk}^{cb}).
\]
\[ \tau^{(2)}_{ki} = -c(K^{ch}_{k}K_{cb} - \frac{1}{2}g_{ki}K^{cb}_{ch}) \]  
\[ \tau^{(3)}_{ki} = -2b \left[ (F_{imn}^{n}K^{lm}_{n}t)_{,i} + (F_{inm}^{l}K_{k,m})_{,i} - (F_{kmn}^{l}K^{lm}_{n}t)_{,i} - (F_{mni}^{l}K_{k,m})_{,i} \right]. \]

We recognize in (44) a Yang-Mills type equation for the field \( K^{ab}_{m} \), endowed with a mass term proportional to \( c \). (Recall that we have defined \( D \) to act with the full Lorentz connection \( \Gamma^{ab}_{m} \) on tangent space indices, and with the Christoffel symbol on spacetime indices.) The corresponding symmetric, traceless stress-energy tensor is found in (46) and the stress-energy contribution from the mass term is given by (47).

The only non-symmetric contributions in (45) are contained in the unusual term \( \tau^{(3)}_{ki} \), a term that results from the \( \hat{\Gamma}^{a}_{b}m \) couplings in \( F^{ab}_{mk} \). In this term, the Lorentz gauge structure of the theory is not obvious, since the spacetime and Lorentz indices of \( F^{ab}_{mk} \) and \( K^{ab}_{m} \) are completely mixed up. This is also the reason why we chose to write down Eq. (45) immediately in terms of spacetime tensors, as opposed to the original form \( \hat{G}^{a}_{i} = T^{a}_{ik} + \ldots \) that results from the variation with respect to \( e^{a}_{i} \). In contrast to (44), which respects clearly the underlying gauge structure, nothing useful is revealed by keeping (45) in its mixed form, and it is convenient to pull down all quantities to physical spacetime such that one has to deal only with one kind of indices. Concretely, we find for the antisymmetric part of (45)

\[ \hat{T}^{[ik]} = -2b(F^{i}_{n}K^{kn}_{m} - F^{k}_{n}K^{in}_{m})_{,t}. \]

The calculations are rather lengthy, but one can directly verify that (49), together with (44), leads to the Noether identity (29).

In the classical, spinless limit, it is immediately clear that (44) leads to the groundstate solution \( K^{ab}_{m} = 0 \), while (45) reduces to

\[ \hat{G}_{ik} = T_{ik}, \]

which are the field equations of general relativity. (We took the opportunity to fix the parameter \( a \) to \( a = -1 \) to be consistent with the conventional choice of units.) Also, in view of (19), there is no need, in the classical limit, to distinguish between \( T_{ik} \) and \( \hat{T}^{ik} \).

Thus, the theory described by (43) allows for a complete general relativity limit in the absence of spinning matter, while a source with intrinsic spin, e.g., an elementary particle or a spin polarized neutron star, will give rise to dynamical torsion fields via (44). This cannot be achieved within the conventional first order formalism if one is not willing to allow for higher order derivatives, and therefore, no such theory has yet been presented. In order to avoid misunderstandings, let us emphasize that this does not mean that there are no consistent Poincaré gauge theories based on the first order formalism without higher derivatives. What it means is that such theories do not have a full general relativity limit. Therefore, constraints on the free parameters of the theory will arise from the classical experiments, based on planetary motion and light propagation. This is rather disappointing, since it essentially means generalizing GR, but holding the changes small enough to ensure that they do not affect classical physics in a sensible way. On the other hand, in our model, classical measurements are not affected at all, independently of the coupling constants \( b, c \), and therefore, the post-general relativistic effects will only arise if spinning matter is introduced. Thus, the generalization affects only spinning matter, in view of which the theory was generalized in the first place. In short: Additional fields (torsion) for additional degrees of freedom (spin).

Finally, let us make a remark on the mass term in (43). The reason for its introduction is easily seen in context of the classical limit. Suppose \( c = 0 \) in (44)-(45). Then, every GR solution, satisfying in addition
$F_{ab} = 0$, will be solution to the field equations with $\sigma_{ab} = 0$. However, from $F_{ab} = 0$, we can not conclude to the unique solution $K_{ab} = 0$ (we will see an explicit example in the next section). However, changing $K_{ab}$ by a term that leaves $F_{ab}$ unaffected will have physical consequences (e.g., concerning the spin precession of a test particle) and therefore, this additional freedom cannot be accepted. This is not confined to the classical limit. From (43), it is obvious that, for $c = 0$, the field equations will only determine $F_{ab}$, but not $K_{ab}$, allowing therefore, in general, for a whole class of solutions for $K_{ab}$. This will become very clear in the context of the solutions we will analyze in the next section. The introduction of the mass term obviously resolves this problem.

5 Yasskin type reduction

Since the classical limit of our theory coincides completely with general relativity, the only way to test the new features related to the presence of torsion fields is to look at field configurations generated by a source with intrinsic spin. One way is for instance to study interactions between elementary particles. On the astrophysical level, one can consider massive, spin polarized bodies, especially neutron stars, and analyze, e.g., the evolution of a system of two such bodies or, which is simpler from the computational point of view, study the motion and spin precession of a test body (a smaller neutron star or elementary particles) in the vicinity of a central source.

In this section, we try to get solutions for the fields at the exterior of such a macroscopic spinning body. As discussed extensively in Ref. [5], such a matter distribution (located at the origin) is described by a spin density of the form

$$\sigma_{ab} = \sigma_{ab} \rho(x),$$

(50)

where $\rho(x)$ is a suitably normalized density function and $\sigma_{ab}$ is the integrated spin tensor. This spin density also coincides with that describing the generalized Weyssenhoff fluid in Riemann-Cartan spacetimes [7]. Then, following Yasskin [8], we look for solutions that have essentially the same structure as the current density, but in view of our rather special Lagrangian, we make the ansatz not for the Lorentz gauge potential $\Gamma_{ab m}$, but for the contortion $K_{ab m}$, i.e., we set

$$K_{ab m} = \sigma_{ab} A_m,$$

(51)

where the field $A_m$ will have to be determined by the field equations. In Ref. [8], with a similar ansatz, the Einstein-Yang-Mills field equations were reduced essentially to an Einstein-Maxwell system of equations, under the assumption that the gauge charges (in our case, $\sigma_{ab}$) are constant. In order to preserve Lorentz invariance, one cannot require $\sigma_{ab}$ to be constant, but instead has to use the constraint

$$\hat{D}_i \sigma_{ab} = 0,$$

(52)

which, with the help of (51), can also be written in the form

$$D_i \sigma_{ab} = 0.$$

(53)

In other words, the contortion fields, which in our second order approach play a role similar to the Yang-Mills potentials, drop out from the covariant derivative of $\sigma_{ab}$. This is similar to Ref. [8], where the gauge covariant derivatives coincide, for the specific ansatz, with the partial derivatives, which makes the requirement that the charges be constant a covariant one.

What does Eq. (52) mean physically? Well, in view of a physical interpretation of the spin tensor, one will have to use further constraints, the so-called spin supplementary condition (SSC), like $\sigma_{ik} u^i = 0$
or similar. Roughly, there will be a local coordinate system where \( \sigma_{ik} = \text{const} \) and one might then choose the SSC \( \sigma_i = 0 \) and define the spin vector \( \sigma^\mu = \varepsilon^{\mu\nu\lambda} \sigma_{\nu\lambda} \) \((\mu, \nu, \lambda = 1, 2, 3)\). In other words, \((52)\) is simply the covariant expression of the fact that the source is described by a constant spin vector. \(\text{Constant in the sense of a fixed vector, pointing in a fixed direction, as opposed to a radially orientated, spherically symmetric spin vector field, for instance.}\) This is just what we expect for a spin polarized body. Note, on the occasion, that \(\sigma_{ab}\sigma^{ab} = 2\sigma^2\). Therefore, let us define the spin \(s\) through

\[\sigma_{ab}\sigma^{ab} = 2s^2.\] (54)

In practice, we will apply \((52)\) only to the contortion field \((51)\), and not to \((50)\), since we will only deal with exterior solutions here. The above arguments are therefore only a simplified version of what should be the result of an analysis of the interior dynamics of the neutron star, which is beyond the scope of this article. For our purposes, the star can equally well be considered to be pointlike \((i.e., \rho(x) = \delta(x)\) in \((50)\)) as long as \((52)\) is satisfied.

With the ansatz \((50)-(52)\), the field tensor \((42)\) reduces to

\[F_{ik} = \sigma_{ab}F_{ik},\] (55)

where \(F_{ik} = A_{k,i} - A_{i,k}\), whereas the field equations \((44)\) and \((45)\) take the simple form

\[-4\beta F_{ik} = -2cA^m + \rho(x)u^m\] (56)

\[\tilde{G}_{ik} = -4bs^2(F_{mi}F_{nk} - \frac{1}{4}g_{ik}F_{lm}F_{lm}) - 2cs^2(A_kA_i - \frac{1}{2}g_{ik}A_iA^l) + \tilde{T}_{ik}.\] (57)

Thus, with a slight extension of the method presented in Ref. \[8\], we have brought our equations into an Einstein-Proca system.

Several remarks are in order at this point. First, the ansatz \((51)\) is supposed to hold at the exterior of the source. We do not know anything about the field configurations inside the source. Nevertheless, we have included the source terms in \((56)-(57)\) in order to be able to fix eventual parameters of the solutions \(\text{(like the Schwarzschild mass etc.)}\). You may consider the source terms as boundary conditions, having in mind that the above form of the field equations is valid only at the exterior. Secondly, and not unrelated to the first remark, you may have noticed that equation \((57)\) is symmetric, while one expects an asymmetric stress-energy tensor for a spinning body. This, of course, reflects again the fact that the equations do not hold inside the matter distribution. In order to ensure the continuity of the solutions at the boundary of the source, the stress-energy tensor should have a symmetric limit at that boundary. Recall however that the tensor \(\tilde{T}_{ik}\), as we have outlined at the end of section 2, is not the canonical stress-energy tensor, but a tensor that is already quite close to the symmetric Hilbert tensor. More precisely, we have shown that its antisymmetric part is given by \(\text{(see \((29)\))}\)

\[2\tilde{T}^{[ac]} = K^{bc}_{\ m}a^a_{\ b}^m - K^{ba}_{\ m}a^c_{\ b}^m.\]

Using the spin density \((50)\) and assuming that \(K^{ab}_{\ m}\) tends to the form \((51)\) as we approach the boundary, we see that \(T^{[ac]}\) vanishes without any restrictions on the stress-energy tensor. Therefore, the symmetry of \((57)\) is not a constraint on the stress-energy tensor, but simply the result of the specific form \((51)\) of the torsion.
On the other hand, if we assume that $K_{ab}^m$ has a similar form also at the interior of the source (this is to be understood as the mean value over a macroscopic region, not on an elementary particle level), then we can conclude that the stress-energy tensor is symmetric all over, and that it essentially coincides with the Hilbert tensor of general relativity. This is of course in contrast to conventional, first order, Poincaré gauge theory, where a symmetric stress-energy tensor is only possible for vanishing spin.

For the simplified equations (56)-(57), the reason for the necessity of the mass term (i.e., the $c$-terms) is especially clear. For $c = 0$, we are dealing with an Einstein-Maxwell system. Thus, the torsion will only be determined up to a gauge transformation $A_m \rightarrow A_m + f_m$. However, measurable results, like the evolution of spin and momentum of a test particle in the given field configuration [5], will depend on the full torsion tensor. Otherwise stated, the field equations for $c = 0$ do not completely determine the physical fields.

Unfortunately, the mass term, necessary on theoretical grounds, is rather disturbing from a computational point of view, since the known solutions of the Einstein-Maxwell system do not easily generalize to the Proca field.

6 Discussion and solutions of the field equations

In the last section, we have reduced the field equations of our specific Poincaré gauge theory to a purely Riemannian problem: The description of a massive vector field in curved spacetime. The discussion of the system (56)-(57) is therefore subject to the framework of general relativity, and can be found in the corresponding literature. Nevertheless, for the sake of completeness, we will briefly sketch the main features of this system and refer to the original articles for details.

It is well known that in the cases of practical interest, no exact solution of the Einstein-Proca field equations has been found to date. Especially, no generalization of the spherically symmetric Reissner-Nordstrom solution, nor of the axially symmetric Kerr-Newman solution are known for the case of a massive vector field. Moreover, it has been established [9] that there is no black hole solution other then for $A_i = 0$. This is a special case of the no-hair theorem for black holes. Here, we are interested in neutron stars. In view of the no-hair theorem, the vacuum solutions we eventually find for the exterior of the star will appear with a naked singularity at the origin. This, however, does not mean that they are unphysical. As vacuum solutions, they are restricted to the exterior of the star and will have to be completed by a suitable interior solution, thereby removing the singularity. On the other hand, the absence of black hole solutions indicates that there is a rather fundamental difference between the Maxwell and the Proca case. Especially, it seems that there are no solutions that tend to the Einstein-Maxwell black holes (Kerr-Newman or Reissner-Nordstrøm) for $c \rightarrow 0$ ($c$ is the mass parameter of the vector field in (56)-(57)), as might have been expected.

A detailed analysis of the static, spherically symmetric case has been performed and numerical solutions have been derived in Refs. [10]-[12]. The authors of the three articles essentially agree in their conclusions, and there is no need to repeat the analysis here. The main feature of the solutions is that, for large distances from the source, the metric tends to the Schwarzschild solution. This is in agreement with the expected exponential decrease of the massive vector field known from the special relativistic limit. Indeed, for $g_{ik} = \eta_{ik}$, starting with the general spherically symmetric, static field $A_i = (\varphi(r), \psi(r) \hat{\mathbf{z}})$ in the comoving system of the source, $u^i = \delta^i_0$, one easily shows from (56) that $\psi(r) = 0$, and that $\varphi(r)$ is solution to the equation

$$-4b\Delta \varphi + 2c\varphi = \rho,$$

(58)
with solution

\[ \varphi \sim e^{-\sqrt{c/2b} r}. \]

In order to fix the proportionality factor, we have to take a look at equations (50) and (51). In contrast to the usual form of the Maxwell (or Proca) equations, the charge \( \sigma_{ab} \) has essentially been removed from \( A_i \) in the ansatz for \( K^{ab}_{\text{m}} \), i.e., in the flat case, the density \( \rho(r) \) is supposed to be normalized to \( \int \rho(r) d^3x = 1 \). It is then easy to see (consider the special case \( c = 0 \) and \( \rho(r) = \delta(r) \)) that the complete solution reads

\[ \varphi(r) = \frac{1}{16\pi b} e^{-\sqrt{c/2b} r}. \] (59)

Since the systematic numerical analysis can be found in Refs. [10]-[12], we will apply here a different approach to the system (56)-(57). Namely, we will proceed in the spirit of the WKB approach of quantum mechanics, which consists in expanding the system in terms of Planck’s constant. In our case, this means expanding in terms of the spin tensor \( \sigma_{ab} \) or simply in terms of the parameter \( s \) introduced in (54). To zeroth order (i.e., for \( s = 0 \)), we simply get the Schwarzschild solution with \( A_i = 0 \). We are interested in the first order corrections to this solution. Thus, in (56)-(57), we simply neglect the terms in \( s^2 \). Again, form (57), we find the Schwarzschild solution. (Just as in Einstein-Maxwell theory, where the corrections to the metric are of second order in the electric charge, here, the corrections are of second order in the spin.) At this stage, we are dealing with a Proca field on a Schwarzschild background. Finally, in order to solve equation (56), we expand the metric in terms of the Schwarzschild mass \( m \) and neglect the term of order \( \varphi(r) \frac{m}{r} \) and higher, leaving us with the solution (59).

Summarizing, at large distances, our solution looks like the Schwarzschild solution with \( A_i = 0 \). Approaching the source, the first order corrections are given by the solution (59) for \( A_i = (\varphi,0,0,0) \). The next order of approximation consists in interaction terms of the gravitational potential \( m/r \) and the vector field \( A_i \), leading to corrections in (59). Only at the next order (\( \sim A^2 \)) will the corrections to the Schwarzschild metric become apparent.

Note that the Schwarzschild metric is not only a very good approximation because of the exponential decrease of \( \varphi \), but also because of the absolute smallness of spin effects in general. In practice, rotational effects (leading to Kerr type corrections) will by far dominate the (intrinsic) spin effects. Not only will the rotational spin of a neutron star be much larger than its intrinsic spin, but moreover, the rotational effects lead to first order corrections in the metric, in contrast to the second order spin effects. Experimentally, this means that the spin effects cannot be observed by analyzing the geodesics, but rather by measuring directly the torsion field (through its interaction with spinning test bodies, see Ref. [5]).

Let us emphasize that, although the Schwarzschild metric is subject to higher order corrections, it will retain its spherical symmetry to each order. The same holds for \( A_i \), but not for \( K^{ab}_{\text{m}} \), which is given by (51).

Unfortunately, the analysis of Refs. [10]-[12] has not been extended to the axially symmetric case. Since this is the case of practical interest (neutron stars are supposed to be subject to a rotation of very high frequency), we will extend the simple approach of the above considerations also to this case. As before, we thus neglect terms of second order in the spin. This, clearly, leads to the Kerr metric. Then, we also neglect interactions of the gravitational potential with the vector field \( A_i \).

First, let us recall the Kerr-Newman solution for the massless vector field, i.e., the axially symmetric solution of (56)-(57) for \( c = 0 \). Using spherical coordinates \( x^i = (t, r, \theta, \varphi) \), the Kerr-Newman metric...
reads
\[ ds^2 = \frac{\rho^2 - 2mr + a^2}{\rho^2} dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\vartheta^2 - \frac{(2mr - q^2)2a \sin^2 \vartheta}{\rho^2} dt d\vartheta - [a^2 + r^2 + a^2 \sin^2 \vartheta \frac{2mr - q^2}{\rho^2}] d\varphi^2, \] (60)

with \( \rho^2 = r^2 + a^2 \cos^2 \vartheta \), \( \Delta = r^2 - 2mr + q^2 + a^2 \). Here, \( m, q, a \) are constants of integration. As is easily seen taking suitable limits, \( m \) is the Schwarzschild mass, \( a \) is related to the angular momentum of the source (see any textbook on general relativity) and the charge parameter \( q \) is proportional to the intrinsic spin \( s \) (see below). The corresponding field \( A_i \) has the form
\[ A_i = \frac{q}{s \sqrt{2b}} \left( \frac{r}{\rho^2}, 0, 0, -\frac{ar \sin^2 \vartheta}{a^2 \cos^2 \vartheta + r^2} \right). \] (61)

The constant \( q \) can be evaluated by taking the Coulomb limit, i.e., take \( a = 0 \) and compare with Eq. (59) for \( c = 0 \). The result is
\[ q = \frac{s \sqrt{2}}{16\pi \sqrt{b}}, \] (62)

and the solution takes the form
\[ A_i = \frac{1}{16\pi b} \left( \frac{r}{\rho^2}, 0, 0, -\frac{ar \sin^2 \vartheta}{a^2 \cos^2 \vartheta + r^2} \right). \] (63)

This is the exact solution for the massless vector field. A difference to the spherically symmetric case arises if we consider the flat limit of this solution. In the Reissner-Nordstrøm case, the solution \( A_i \) is simply the Coulomb potential, which is a solution of \( F^i_{jk} = 0 \) in the flat case too. At first sight, it seems as if (62) is not a solution on the flat background \( ds^2 = dt^2 - dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \). This, however, is the result of a false interpretation of the coordinate system. Indeed, the flat limit of (60) is found for \( q \rightarrow 0, m \rightarrow 0 \), i.e.,
\[ ds^2 = dt^2 - \frac{r^2 + a^2 \cos^2 \vartheta}{r^2 + a^2} dr^2 - (r^2 + a^2 \cos^2 \vartheta)d\vartheta^2 - (a^2 + r^2) \sin^2 \vartheta d\varphi^2, \]

which is easily shown to be flat (\( \tilde{R}^i_{klm} = 0 \)) and related to the Minkowski metric \( \eta_{ik} = diag(1, -1, -1, -1) \) by the coordinate transformation
\[ x = \sqrt{r^2 + a^2 \sin \vartheta \cos \varphi} \]
\[ y = \sqrt{r^2 + a^2 \sin \vartheta \cos \varphi} \]
\[ z = r \cos \vartheta. \]

We now find that (62) is indeed a solution in the flat background expressed in the non-spherical coordinates \( r, \vartheta, \varphi \). Since it is rather difficult to generalize this to the massive case, we will neglect contributions that are of second order in the rotational momentum \( a \). To that order, the coordinates used in (62) coincide with conventional spherical coordinates, and indeed the field
\[ A_i = \frac{1}{16\pi b} \left( \frac{r}{r}, 0, 0, -\frac{a \sin^2 \vartheta}{r} \right), \] (63)

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which differs from (62) by terms of order $a^2$ and higher, is a solution to the Maxwell equations on a flat background (in spherical coordinates). Physically, we see that (63) contains an electric monopole and a magnetic dipole contribution, while (62) contains, as is well known, the complete series of odd electric multipoles (monopole, quadrupole...) and even magnetic multipoles (dipole, octupole...). Thus, replacing (62) by (63), we essentially neglect quadrupole and higher moments. We then make the following ansatz for the massive vector field:

$$A_i = \frac{1}{16\pi b} \left( \frac{1}{r} f(r), 0, 0, -\frac{a\sin^2\vartheta}{r} g(r) \right). \tag{64}$$

Putting this into (56) yields, on a flat background, the unique asymptotically flat solution

$$f(r) = e^{-\sqrt{c/2b} \frac{r}{r}},$$

$$g(r) = (1 + \sqrt{c/2b} \frac{r}{r})e^{-\sqrt{c/2b} \frac{r}{r}}, \tag{65}$$

where the constants of integration have been fixed by requiring the limit (63) for $c \to 0$.

Summarizing, the Kerr metric, as before the Schwarzschild metric, is a very good approximation for three reasons: (1) The spin is small compared to the angular momentum of the source. (2) Eventual corrections to the metric are of second order in $A_i$ and thus in the spin (in contrast to the angular momentum contributions which are first order). (3) The field $A_i$, and thus the metric corrections, decreases exponentially.

The first order approximation of $A_i$ is given by (64)-(65), neglecting interactions of the metric with $A_i$ (order $\sim A_i m/r$) and second order terms in the spin and in the rotational momentum.

Possibly, one can find exact solutions of (56) in Schwarzschild or Kerr backgrounds. This would yield the complete analogue of the WKB approximation, neglecting only terms of second and higher order in the spin, but including the spin-gravity couplings $\sim A_i m/r$. However, as we have argued before, it is not the scope of this article to analyze in detail the equations in the form (56)-(57), because this set of equations is well known from classical general relativity.

Finally, one could also consider the possibility of a very small $c$, too small for the exponential decrease to be sensible on astrophysical scales. (In analogy to theories with a small photon mass.) In this case, one can consider the term in $c$ as a gauge fixing term but irrelevant else. This means that we can simply take over the solutions from Einstein-Maxwell theory, i.e., the Reissner-Nordstrøm metric together with the Coulomb potential, or the Kerr-Newman metric with the field (62), but in the gauge that is consistent with $c \neq 0$, which is essentially $A_i^0 = 0$, as can be easily derived taking the divergence of (56). Note that (62) and also the Coulomb solution in its usual form are already in this gauge. Thus, the term in $c$ forces us to choose the gauge $A_i^0 = 0$, but the $c$-contributions will then be neglected in the field equations.

Does every solution of (56)-(57) tend to a solution of the Einstein-Maxwell equations in the limit $c = 0$? Our approximate solutions seem to indicate that this is indeed the case. One should, however, be cautious, because there are nevertheless severe differences between the massive and the massless vector fields. One example is the previously mentioned absence of black hole solutions in the massive case. Thus, there might be surprises in the strong field region which have not been revealed by our approximations.

Let us remind that the whole section was based on the specific ansatz (51), which was adapted to the macroscopic spin distribution (50). This does not mean that, even if we retain (50), there are no other solutions. Especially, the existence of black holes cannot be excluded a priori. On the other hand, for a different spin density, for instance the totally antisymmetric tensor of the Dirac particle, a different ansatz will be necessary, leading to entirely different solutions.
7 Spin precession in torsion field

Finally, we address the question of the detectability of the torsion fields by analyzing the behavior of test bodies in a given field configuration. We confine ourselves to the concrete solutions of the previous section.

It is well known that the equations for the spin precession and for the momentum evolution in a Riemann-Cartan geometry depend on the nature of the test body in question. Especially, elementary particles with different spin couple in a different manner to the torsion. A complete review, as well as a simple method for the derivation of the equations of motion, has been presented in Ref. [5], where the references to the original articles can be found.

For simplicity, we consider again the case where the test particle is a macroscopic, spin polarized body. Thus, the system (central source + test body) we are interested in is essentially a neutron star binary, under the restriction, for computational simplicity, that the central star has a much larger mass and can be considered to be at rest.

The deviation from geodesic motion is given by a direct coupling of the test particles’ spin to the curvature tensor. Even for the classical spin (i.e., the angular momentum in the body’s rest frame), which presents a similar coupling to the curvature (Papapetrou equations in general relativity), these corrections have not yet been measured in experiment, and the contributions from the intrinsic spin and the non-Riemannian fields will be even smaller. Therefore, the only experimentally relevant equation is the spin precession equation, which reads [5]

\[ D\sigma_{ik}^{(2)} + \hat{D}S_{ik}^{(2)} = P_i u_k - P_k u_i, \]  

(66)

where D denotes the covariant derivative with respect to proper time using the full connection \( \Gamma_{lm}^i \), and \( \hat{D} \) the corresponding derivative with the Christoffel connection, i.e., \( D\sigma_{ik}^{(2)} = d\sigma_{ik}^{(2)} / d\tau - \Gamma_{lm}^i \sigma_{ik}^{(2)} u^m - \Gamma_{km}^i \sigma_{il}^{(2)} u^m \) and \( \hat{D}S_{ik}^{(2)} = dS_{ik}^{(2)} / d\tau - \Gamma_{lm}^i S_{ik}^{(2)} u^m - \hat{\Gamma}_{km}^i S_{il}^{(2)} u^m \). We denote by \( \sigma_{ik}^{(2)} \) the (intrinsic) spin tensor of the test body and by \( S_{ik}^{(2)} \) its angular momentum in the rest frame (due to rotation). The index \( (2) \) is attached to avoid confusion with the corresponding quantities of the source. \( P_i \) is the momentum vector of the test body, which is not necessarily parallel to the velocity \( u^i \). However, the right hand side of (66) is of higher order [5] and will be neglected in the following. Finally, we have to impose some conditions on the structure of the test body. It seems reasonable, for neutron stars, to make the ansatz (strong spin-rotation coupling)

\[ S_{ik}^{(2)} = g\sigma_{ik}^{(2)}, \]  

(67)

which leaves us with

\[ \hat{D}\sigma_{ik}^{(2)} + \frac{1}{1 + g} K_{lm}^i \sigma_{ik}^{(2)} u^m + \frac{1}{1 + g} K_{km}^i \sigma_{il}^{(2)} u^m = 0, \]  

(68)

where we have separated into Riemannian and non-Riemannian contributions. Next, introduce the spin vector \( \sigma_{ik}^{(2)} = \frac{1}{2} \eta^{iklm} u_k \sigma_{lm}^{(2)} \) (\( \eta^{iklm} = |g|^{-1/2} \varepsilon^{iklm} \)) and use the spin supplementary condition \( \sigma_{ik}^{(2)} u^k = 0 \) to find

\[ \hat{D}\sigma_{ik}^{(2)} + \frac{1}{1 + g} K_{lm}^i \sigma_{ik}^{(2)} u^m = 0, \]  

(69)

where we have used the approximate validity of the geodesic equation \( \hat{D}u^i = 0 \) and higher order terms have been omitted [5]. This equation, together with the solutions of the previous section, can be used to evaluate the spin precession and especially to measure the influence of the torsion contributions.
Here, our scope is to illustrate the effects of the dynamical torsion fields. Therefore, we confine ourselves to the simple case of the non-rotating test body ($S_{ik}^{(2)} = 0$, i.e., $g = 0$) and use the spherically symmetric solution (no rotation of the source). This is not a very realistic case (as we have argued before, the rotational effects, when dealing with neutron stars, will in general be by far more important than the intrinsic spin effects), but it is a useful example to illustrate directly the post-general relativistic effects due to spin and torsion.

With $K_{ab}^i$ from (51), with $A_m = (\varphi, 0, 0, 0)$ and introducing the spin vector of the source, $\sigma_i^{(1)} = \frac{1}{2} \eta^{iklm} u_k \sigma_{lm}^{(1)}$, using only the Newtonian order of the Schwarzschild background (compare with Ref. [4]), we find to lowest order for the relevant spin precession equation

$$ \frac{d\sigma^{(2)}}{dt} = \frac{3}{2} \frac{m}{r^3} [\vec{L} \times \sigma^{(2)}] + \varphi(r) [\sigma^{(1)} \times \sigma^{(2)}], $$

(70)

where $\varphi(r)$ is given by Eq. (59). The first term, with the orbital momentum $\vec{L}$, is identical to the precession of the rotational spin in general relativity (as derived from the lowest approximation of the Papapetrou equations). This is a general result of Poincaré gauge theory: In purely Riemannian spacetimes, no distinction can be made between intrinsic and classical (rotational) spin. The torsion, however, couples only to the intrinsic spin. Therefore, the other way around, only a particle with intrinsic spin can distinguish between Riemannian and non-Riemannian geometry.

The above result is easily generalized to the more realistic case of rotating neutron stars. One can also proceed to a more complete treatment considering the neutron star binary as a two body problem. On the other hand, Eq. (70) is already enough to discuss the possibility of an experimental detection. It is clear that the spin effects are very small, even for completely spin polarized stars. Moreover, the torsion decreases exponentially, and practical results will strongly depend on the coupling constants $b$ and $c$, especially on $c$. A possible, indirect, detection could be based on gravitational waves emitted by a merging binary system, since this will allow for conclusions on the evolution of the system in the strong field regime.

8 Conclusions

Considering, in the framework of Poincaré gauge theory, the Christoffel part of the Lorentz connection as a function of the tetrad field in a formalism similar to the second order formalism used in general relativity and in metrical theories in general, we were able to present a theory that allows for dynamically propagating torsion fields, preserving nevertheless a full general relativity limit in the absence of spinning matter. This is not possible in the framework of the conventional, first order formalism, without introducing higher order derivatives of the independent field variables and apparently running into trouble with the Cauchy initial value problem. The equivalence of the second order formalism with the conventional approach has been established and therefore, ultimately, we have shown that in the conventional formalism, certain Lagrangians lead to consistent theories despite the appearance of higher derivatives. A concrete model has been presented, and for a specific class of solutions, the field equations have been reduced to an Einstein-Proca system using a slight modification of the Yasskin method known from conventional Einstein-Yang-Mills theories. Approximate solutions have been discussed for concrete cases, corresponding to the exterior of spin polarized neutron stars. In view of an eventual experimental detection of the non-Riemannian effects, the spin precession equation of a test body in such a field configuration has been briefly analyzed.
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