Noise-resilient phase transitions and limit-cycles in coupled Kerr oscillators

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Keywords: open quantum system, limit-cycle, quantum synchronization, Keldysh formalism, Bose–Hubbard model, cumulant expansion

Abstract

In recent years, there has been considerable focus on exploring driven-dissipative quantum systems, as they exhibit distinctive dissipation-stabilized phases. Among them dissipative time crystal is a unique phase emerging as a shift from disorder or stationary states to periodic behaviors. However, understanding the resilience of these non-equilibrium phases against quantum fluctuations remains unclear. This study addresses this query within a canonical parametric quantum optical system, specifically, a multi-mode cavity with self- and cross-Kerr non-linearity. Using mean-field (MF) theory we obtain the phase diagram and delimit the parameter ranges that stabilize a non-stationary limit-cycle phase. Leveraging the Keldysh formalism, we study the unique spectral features of each phase. Further, we extend our analyses beyond the MF theory by explicitly accounting for higher-order correlations through cumulant expansions. Our findings unveil insights into the modifications of the open quantum systems phases, underscoring the significance of quantum correlations in non-equilibrium steady states. Importantly, our results conclusively demonstrate the resilience of the non-stationary phase against quantum fluctuations, rendering it a dissipation-induced genuine quantum synchronous phase.

1. Introduction

Understanding the quantum phase transitions of many-body systems and their temporal evolution has been one of the important objectives in contemporary physics. Traditionally, phase transitions have been explored within the thermodynamic limits of closed systems, where they reach thermal equilibrium in a steady state. However, in recent years, the investigation of phase transitions and critical phenomena in driven-dissipative many-body quantum systems has emerged as a significant area of study. Various experimental platforms, such as cavity arrays, superconducting circuits, and exciton-polaritons, have provided versatile setups to analyze the interplay between (in)coherent drive, dissipation, and interaction within the non-equilibrium steady state (NESS). These include phenomena like multi-stability and crystallization in driven-dissipative nonlinear resonator arrays [11–14], spins [15], and synchronized switching in arrays of coupled Josephson junctions [16].

Within this realm, the limit-cycle (LC) stands out as one of the most prevalent collective behaviors observed in synchronized systems where all constituents’ internal dynamics become coordinated [17–19]. The study of synchronization in coupled oscillators has a diverse history in nonlinear dynamics, spanning from the synchronized flashing of fireflies to the coordinated ticking of coupled clocks and the phase patterns in oscillatory neural circuits [20].

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To date, mean-field (MF) LC-phases have been anticipated in various open quantum systems, including optomechanical resonators [21–25], Rydberg lattices [26], Fermi–Hubbard and Bose–Hubbard arrays [27–31], Heisenberg lattices [32], spin arrays [33], Dicke model [34, 35], and non-linear photonic crystal cavities [36]. However, due to its inherent assumptions, the MF treatment does not encompass the effects of quantum fluctuations. Only a few recent studies have delved into the concept of dissipation-stabilized quantum synchronization, also known as dissipative time crystals, within the context of the super-operator spectrum and dynamical symmetries [30, 37–41]. Furthermore, only a few recent experimental works have reported the observation of a dissipative time crystal [42, 43].

In this article, our focus centers on experimentally feasible nonlinear parametric oscillators capable of exhibiting complexity, especially in the realm of limit cycles akin to those recently investigated in [44–46]. The physical system is a dissipative multi-mode cavity as depicted in figure 1, which is driven on one mode and coupled to others through cross-Kerr nonlinearity. Under varying parameters, a multifaceted phase diagram emerges, featuring diverse steady states, multi-stability, and limit cycles. For comparison, this system is juxtaposed with its closed counterpart lacking dissipation, revealing a much simpler phase diagram composed of only two types of steady states.

While the MF approach often yields accurate phase transition features in equilibrium scenarios, its adequacy is limited when applied to out-of-equilibrium open quantum systems. It serves as a useful starting point, offering insights into nuanced dynamics and phases, but the reliability of the results is not assured [47, 48]. To reinforce and refine the findings, we employ two distinct methods: (i) utilizing the Keldysh formalism, enabling a comprehensive study ranging from quantum to thermodynamic limits on an equal footing, and (ii) incorporating higher-order correlations up to the 2nd order. Both approaches corroborate the phase transition features and furnish quantitative adjustments. The results and methodologies introduced herein empower the exploration of non-equilibrium steady-state (NESS) quantum properties beyond the conventional MF approximation, highlighting dissipative phase transitions, such as dissipative time crystals, in open quantum systems.

The paper’s structure is as follows: section 2 introduces our model of a driven three-mode cavity with self- and cross-kerr nonlinearity. Section 3 presents numerical results about attractive interaction, delineating the MF phase diagram, elucidating uniform and multi-stable regions, and delineating the emergence of the LC phase. A comparison is drawn between our results and the closed system’s physics in its thermalized ground state to underscore the profound impact of dissipation on newly emerging phases. Furthermore, we delve into the spectral characteristics of various phases derived from Keldysh Green’s functions, highlighting dissipation-induced crossovers in the open system. The section 3.3 within section 3 elaborates on the beyond-MF results incorporating two-point correlations through the 2nd-cumulant approach. Here, we contrast the Gaussian approximation of cavity states with their respective MF outcomes, detailing the implications of quantum correlations on LC and its robustness. Finally, section 4 concludes our results and presents prospects for further research endeavors.

2. The model

We consider the dynamics of a three-mode lossy cavity with self- and cross-Kerr non-linearity subject to a single-photon drive and a single-photon loss [31, 49–51]. Such nonlinearities, as well as the parametric four-wave mixing, can be realized either via \( \chi^{(3)} \) optical nonlinearities with rotating wave approximation, as previously studied in the large-photon limit [52–55], or parametric processes in superconducting cavities [56–60]. It is worth mentioning that in the weak nonlinearity limit of customary media classical treatments suffice and quantum features are mainly inaccessible. In the rotating frame at the frequency of the drive, the Hamiltonian reads as \( \hbar = 1 \)
where $\Omega_2$ is the drive rate at the frequency of $\omega_2$, $\hat{a}_m, \hat{a}_m^\dagger$ is the annihilation and creation operators of the $m$th-cavity mode, respectively, $\Delta_m = \omega_m - \omega_2$ is the detuning of the $m$th cavity mode from the drive, and $U_0$ is the interaction rate.

For $U_0 \gg 0$ the system energy increases with increasing the particle number hence, a repulsive interaction. Similarly, when $U_0 \ll 0$ the system energy decreases with increasing the particle number and the interaction is attractive. In the presence of the single-photon loss from the cavity at the rate of $2\gamma$, the Markov–Born approximation leads to the following Lindblad dissipator for the cavity density operator

$$D(\hat{\rho}) = \sum_{m=1}^{3} \gamma_m \left( 2\hat{a}_m^\dagger \hat{a}_m \hat{\rho} - \left\{ \hat{a}_m^\dagger \hat{a}_m, \hat{\rho} \right\} \right).$$

The time evolution of the multi-mode cavity density operator $\dot{\hat{\rho}}(t)$ is determined via the following master equation

$$\frac{d}{dt} \hat{\rho}(t) = -i [\hat{H}, \hat{\rho}] + D(\hat{\rho}).$$

In this work, we are interested in the long-time solution of this density matrix, where all the transient dynamics are over. This is obtained by numerically integrating the coupled equations of motion (EoM) (cf appendix A) implemented by the 4th-order Runge–Kutta method.

When the drive detuning is swept from the red to the blue, the nonlinear dynamics sets in as a parametric amplification process, and photons are created in the side modes as correlated pairs. As shown in appendix A, for an attractive interaction, a multi-stability region exists for the red-detuned drive satisfying $\Delta_2 \gg \sqrt{3}\gamma_2$. Similarly, for repulsive interactions, a multi-stability phase exists for the blue-detuned coherent drives when $\Delta_2 \ll -\sqrt{3}\gamma_2$. For some drive rates and detuning, the amplification moves into a regime of self-sustained oscillations, aka LC phase, due to the nonlinear coupling. In the next sections, we detail the phase diagrams of the aforementioned system assuming an attractive interaction.

3. Open system

As shown recently, the phase diagram of the open quantum system can be noticeably different due to the emergence of dissipation-stabilized phases and phenomena [4]. For the open system, we determine the phase diagram by evolving the system for many randomized initial conditions and examining the dynamical stability of the stagnation points using the Bogoliubov matrix spectrum (cf appendix A) [61].

In addition to multi-stable phases, it is also possible to find regions of parameter space where no stationary fixed point exists. In such cases, the system may be attracted to time-dependent solutions such as sustainable oscillations, as found in other coupled nonlinear systems. In particular, we search for the complete set of stable attractors of the long-time dynamics, including the fixed points, the multi-stable coexistence phases, and time-dependent trajectories. The possible steady-states for the evolution of the system under equation (3) are phases where either the only populated mode is the driven mode, $(\alpha_n = 2 \neq 0, \alpha_n \neq 2 = 0)$, called the uniform phase, or phases where the side modes, $a_1, a_3$, are also populated and host a LC.

In figure 2 we plot the three-mode cavity open phase diagram. It comprises four different regions: (I) the white region that presents only one uniform stable steady-state solution which we identify as the low population (LP) phase, (II) the dark blue region of tri-stability between two uniform phases namely, the LP and the high-population (HP), and a non-uniform solution where the non-driven modes showcase a LC behavior, (III) the blue region of bi-stability between the LP and the HP uniform phases, and finally, (IV) the light blue region that presents a HP uniform phase. At the boundary between region (I) and (IV), denoted as the striped region, an exceptional point appears and the LP and HP are connected, smoothly modifying the phase transition to a crossover.

When compared to the phase diagram of the closed system (cf figure 8 of appendix B), it becomes clear that the dissipation has a marked impact. It lifts the boundary of the closed-system phase diagram (red dashed line), stabilizes an excited state of the closed system, the HP, in the region (IV), renders the LP as the only attractor of the dynamics in (I), and leads to limit cycles.
3.1. Steady states

Here, we focus on different regions of the phase diagram to clarify the nature of the underlying attractors including their stability. As discussed in appendix A, the possible fixed points of the EoM are determined as \( \frac{d}{dt} \langle \hat{a}_m \rangle = 0 \) in equation (A2). Among the possible fixed points in the long-time limit, the semiclassical dynamics eventually evolve towards some stable fixed points for all initial conditions, known as the steady-states of the system.

We use the MF approximation, i.e. we factorize higher order moments as \( \langle \hat{a}_m \hat{a}_n \rangle \approx \langle \hat{a}_m \rangle \langle \hat{a}_n \rangle \), and obtain the EoM for the MF order parameters \( \alpha_m \), with \( m = 1, 2, 3 \). When \( U_0 \Delta_2 \geq 0 \), the EoM has one stable solution only as \( \alpha_{m=2} \neq 0 \), whereas when \( U_0 \Delta_2 \leq 0 \), there might exist several stable steady-state solutions (cf appendix A). We self-consistently determine the steady state of the system which, in general, may allow LC solutions. For these solutions, the long-time limit of the steady-state has the general form of \( e^{i \Delta \omega t} \).

Physically, the LC solutions and their associated frequencies \( \omega_{LC} \) can be thought of as the frequency of the parametrically generated pair, i.e. the non-driven, via the parametric Kerr process [31].

To draw a more comprehensive picture of different phases, in figures 3(a)–(c) we show the MF occupation of each cavity mode, \( n_m = |\alpha_m|^2 \), versus the coherent drive rate \( \Omega_2 \) at various detunings, \( \Delta_2 = -5, 0, +5 \) (vertical dashed lines in figure 2), for an attractive interaction of \( U_0 = -1 \), and bare cavity spacing as \( \Delta_1 = \Delta_2 + 1 \) (More discussion on the effect of the cavity spectrum can be found in [31]). The red lines show the behavior of the 2nd-mode while the blue lines correspond to the 1st- and 3rd-mode populations. The solid and dashed lines signify the stable and unstable solutions, respectively. For \( U_0 \Delta_2 \geq 0 \), panels (a) and (b), there is only one stable steady-state, i.e. the uniform phase HP, but when \( U_0 \Delta_2 < \sqrt{3} \) we observe the regions of multi-stability and transitions to the non-uniform phases, figure 3(c).

In figures 3(d)–(f) we show three exemplary time evolution, obtained from the direct integration of MF equations, for several randomized initial conditions corresponding to various phases at points \( T_{1,2,3} \), i.e. \( \Omega_2 = 3 \), in figure 2. For each detuning, the upper row shows the temporal behavior of \( n_2 = |\alpha_2|^2 \) while the lower panel shows the real part of the 1st-mode order parameter, i.e. \( \text{Re}(\alpha_1) \).

For \( T_1 \) and \( T_2 \), corresponding to \( \Delta_2 = -5, 0 \) respectively, there is only a uniform phase where \( \alpha_{1,3} = 0 \) and \( \alpha_2 \neq 0 \). Accordingly, in figures 3(d) and (e) we see that all time traces converge to only one non-zero value (HP) for the driven mode and a zero value for the side modes, independent of the initial conditions.

On the other hand, for \( T_3 \) corresponding to \( \Delta_2 = +5 \), where a multi-stability is predicted, one can see that the time traces in the upper panel of figure 3(f), converge to three different values for \( n_2 \) at LP (red traces), LC (orange traces), and HP (brown traces). From those three phases, only the LC corresponds to non-zero values for the 1st, 3rd modes (orange line in upper panel of figure 3(f)), where their order parameter \( \text{Re}(\alpha_1) \) shows a periodic long-time behavior (light blue in the lower panel of figure 3(f)).
Figure 3. Cavity mode occupation vs. coherent drive rate ($\Omega_2$) for $\Delta_2 = -5, 0, +5$ in (a)–(c), respectively (cf orange dashed cut-lines in figure 2). In each panel, the red lines show the driven mode population, $n_2$, and the blue lines show the non-driven modes occupation, $n_1 = n_3$. Solid lines show the stable solutions while the dashed lines correspond to unstable branches. In (a) and (b), there is only a uniform stable steady-state (HP) whereas in (c) multiple stable steady-states are possible (LP, LC, HP). Panels (d)–(f) show the time evolution of the MF equations of motion for the corresponding detuning of (a)–(c) for 100 randomized initial conditions. In each panel, the upper row shows the population of the driven mode ($n_2 = |\alpha_2|^2$), and the lower row shows Real ($\alpha_1$). In (d) and (e) the system always evolves towards a uniform steady state with only the driven mode populated (HP). In (f) there are three possible steady-states, two uniform phases LP, HP (red and brown), and a LC (orange). In the LC case, the side modes show an oscillatory behavior at the frequency $\omega_{LC}$, as depicted with light blue traces in the bottom panel (f). All modes have the same decay rate $\gamma_0$ to which other rate parameters are normalized. The time is in units of $1/\gamma_0$. $U_0 = -1, \Omega_2 = 3$ in all temporal calculations. The side modes are equally-spaced around the driven mode as $\omega_1, \omega_3 = \omega_2 \mp 1$.

Figure 4. (a), (b) The order parameter, $|\alpha_2|^2$, and (c), (d) real (dashed) and imaginary (solid) part of the excitation spectrum, $\omega$, on top of the LP (light green) and HP (dark green), along the vertical ($\Delta_2 = +5$) and horizontal ($\Omega_2 = 0.5$) cut lines in figure 2.

3.2. Spectral functions

To understand the unique properties of different phases better it is instructive to study their corresponding spectrum. Figures 4(a), (c) and (b), (d), shows the population of the 2nd mode, $|\alpha_2|^2$, (top) and the fluctuation eigenvalues (bottom) vs coherent driving rate at $\Delta_2 = +5$ and detuning at $\Omega_2 = 0.5$, respectively. In both cases, $|\alpha_1|^2$ behavior highlights a smooth crossover from the HP to the LP between regions (I) and (IV). As the latter transition traverses an exceptional point region, the imaginary part of the eigenvalues coalesce to zero while their real parts split, hosting over- and under-damped fluctuations.

Using a Keldysh action approach we readily get access to the Green’s function of the system and its associated dynamical observable. In figure 5, we compare the spectral functions, $A(\omega) = -2 \text{Im}[G_R(\omega)]$, of the three-modes in the LP (a) and HP (b) phases for different detuning and at fixed coherent driving rate $\Omega_2 = 0.5$, points $P_{LP, HP}$ in figure 2, two points in the region (I) and (IV) of the open phase diagram, i.e. where only one stable attractor exists.

As can be seen, the LP phase has the response of a ground-state, i.e. a positive (negative) peak at the positive (negative) frequencies (cf appendix B). We note that peaks at negative frequencies are unresolved due to the scale resolution. On the other hand, the HP phase features a peak swap with a positive (negative) peak at the negative (positive) frequencies, a hallmark of a stabilized excited state (cf appendix B for more information, and see figure 10 from the appendix for more data on the co-existence region). The spectral
Figure 5. Spectral function of the three-mode harmonic cavity uniform phase, for the points $P_{LP} = (\Delta_2 = 5, \Omega_2 = 0.5)$, (a), and $P_{HP} = (\Delta_2 = -5, \Omega_2 = 0.5)$, (b), (cf figure 2). The spectral function of $P_{HP}$ features a peak swap compared to $P_{LP}$. This signals a transition between particle- to hole-like physics from region (I) to (IV) of the open phase diagram.

functions highlight a more profound difference between the two phases rather than just a different mode occupation. Interestingly, the peak swap and hence the change in the behavior, is also present in the response of the not-populated side modes which signifies a normal to an excited phase transition, signaling a particle-to-hole-like physics transition, when going from region (I) to region (IV) of the open phase diagram [5, 6, 62].

3.3. Beyond the MF and the phase robustness

So far, we studied the MF-phase diagram of the open system and investigated the dissipation-stabilized phases in contrast to the closed system. Further, we employed the Keldysh approach to study the spectral signatures of each phase and delimit the onset of a PT, beyond the mean field. For example, we showed how transitions from the ground state to the excited state accompanied by a change in the behavior from particle-like to hole-like can be obtained from the spectral functions.

As can be inferred from the Heisenberg EoM in equation (A1) however, the quartic interaction leads to an infinite hierarchy of moments. Therefore, any semi-analytic or numerical calculations require a truncation of this hierarchy. The MF treatment ignores the correlation via the factorization approximation and hence truncates the cumulant expansions to the 1st-order. Recent studies, however, show that including higher-order quantum correlations in open quantum systems leads to marked deviations from the MF results, especially close to the phase transition points [47, 48]. Therefore, it is quite natural to examine the validity of MF-predicated phase boundaries and delineate the truncation artifacts from the true phase transitions. So far, there have been several approaches to include the effect of higher-order correlation including exact diagonalization, diagrammatic expansions, functional renormalization group, numerical or density matrix renormalization group analysis, and phase space methods [63–72].

In this section, we extend the EoM to include the moments dynamics up to the two-point correlation, while assuming a vanishing 3rd-order, to find the Gaussian approximation of the NESS with an emphasis on the robustness of the LC-phase in the presence of the quantum fluctuations. Further details and the comparison between the results of this approach with the MF and the full density matrix calculations for a single-mode Kerr cavity can be found in appendix D.

Figures 6(a)–(c) shows the population of the side modes ($\langle \hat{a}_{1,3}^{\dagger} \hat{a}_{1,3} \rangle$ in blue), the driven mode ($\langle \hat{a}_{2}^{\dagger} \hat{a}_{2} \rangle$ in red), and the correlation between the generated pairs in the side modes ($\langle \hat{a}_{1} \hat{a}_{3} \rangle$ in brown), as a function of the coherent drive rate at $\Delta_2 = -5, 0, +5$, respectively.

For panels (a) and (b) corresponding to the MF uniform phase, i.e. one solution for the driven mode and no occupation of the side modes, the Gaussian approximation results are in good agreement with the MFs depicted in figures 3(a) and (b). For the non-driven modes, however, the MF predicts zero population while the 2nd-cumulant indicates a finite but low occupation. This can be understood in terms of the quantum fluctuations ignored in the MF and partially resumed in the Gaussian approximation. Besides, the weak correlation between the side modes, i.e. $g_{13}$, reinforces that interpretation and justifies the resemblance of the Gaussian approximation and the MF results in this phase.

For $\Delta_2 = +5$ however, the MF (figure 3(c)) and the 2nd cumulant approach (figure 6(c)) show marked differences. While the MF results in a multi-stable behavior, a 1st-order phase transitions, and the emergence of an LC phase for the non-driven modes, the cumulant approximation results are unique and continuous for all modes. For the non-driven modes and before the MF LC-phase, the trend is quite similar to the ones depicted in panels (a) and (b), i.e. a finite but low population. Unlike the MF case however, the transition to
the LC-phase is not a discontinuous 1st-order as suggested by the MF, but instead it is a large but continuous change of the order parameter, i.e. $\langle \hat{a}_1^{\dagger} \hat{a}_1 \rangle$. This transition is accompanied by a maximum in the non-driven modes correlations, i.e. $\langle \hat{a}_1^{\dagger} \hat{a}_3 \rangle$ the solid brown line, a quantity that signifies the correlated photon-pair generation within this phase. It is interesting to note that this correlation is low before the LC phase and drops but remains finite after the LC. Moreover, while the MF predicts a uniform phase after the LC without any populations in the non-driven modes, the Gaussian approximation predicts a finite non-zero population. The fact that the side-mode population (solid blue line) follows the correlation (solid brown line) delineates that this is indeed a correlation effect, and describes its absence in the MF results.

Figure 6(d)–(f) present the temporal evolution of the driven mode for the corresponding detuning in (a)–(c) at a fixed coherent drive rate of $\Omega_2 = 3$ starting from the vacuum state at $t = 0$, i.e. $\langle \hat{a}_m^{\dagger} \hat{a}_m \rangle = 0$. For the uniform cases shown in (a) and (b) the order parameter ($n_2$) reaches a stationary value at the long-time limit after some transient behavior, a value which is very close to the corresponding MF one. On the contrary, the order parameter of the LC phase (figure 6(f)) shows sustainable oscillations in time, after a short transient period.

It is interesting to note that, unlike the MF-predicted LC however, where the time-periodic behavior was only observable in the side-modes, here the coupled correlations lead to oscillatory behaviors in all modes. Unlike the MF multi-stable results depicted in figure 3(f), this is the only plausible solution in this regime. Therefore, these dissipation-stabilized oscillatory behaviors which are susceptible to quantum noise, signify the emergence of a dissipative time crystal [37].

To further examine the impact of added two-point correlations on spectral and Green's functions, in figures 7(a) and (b), we compare the spectral functions obtained from the Keldysh approach, based on the 2nd-cumulant results (solid lines), with the ones determined from the MFs (dashed lines) for LP and HP.
phases, respectively. There is no qualitative change in the response and the peak swap between the LP and HP is still present. The 2nd-cumulant approximation however leads to corrections to the eigenfrequencies of the system highlighted as a frequency pulling towards lower values. The effect is more pronounced in the HP, in agreement with stronger corrections to the MF results where the Gaussian approximation predicts a non-vanishing population for the non-driven modes in contrast to zero MF values. The slight modifications of the spectral responses combined with the results depicted in figures 6(a) and (b) a clear evidence for the robustness of LP and HP phases against the correlation noise.

4. Conclusion and outlook

There has been speculation for a considerable time that the limit cycle, a recurring phenomenon in various nonlinear classical systems, might qualify as a dissipative time crystal. Although the MF treatment of several many-body systems suggests the emergence of periodic long-time behavior, the endurance of these periodic dynamics against quantum fluctuations has remained an open question. In this study, we present beyond-MF results for a canonical quantum optical system, a multi-mode parametrically-coupled bosonic system, by comprehensively exploring its phase diagram within and beyond the MF approximation. Our focus particularly centers on an MF-limit cycle (LC) phase. Utilizing the Keldysh formalism and extending MF results through a higher-order cumulant expansion, we confirmed the robustness of the LC phase and the overall phase diagram topology. Additionally, we identified corrections to the MF outcomes, including changes in correlation spectra, non-zero populations of non-driven modes, oscillating cross-correlations, and the absence of MF multi-stability far from the thermodynamic limit.

The methods and findings showcased in this study can be applied to a diverse range of many-body systems to comprehend the impact of higher-order correlations and explore the resilience of dissipation-stabilized phases predicted by MF. Extending this work to investigate the dynamic behavior of such systems under a parametric drive with modulated amplitude or exploring system transitions from stationary dynamics to dissipative time crystals and chaotic phases represents a straightforward extension. Another avenue involves changing the drive, e.g. subjecting the system to a two-photon parametric drive that could preserve the $Z_2$ symmetry of the modes, potentially leading to new phase transitions due to the spontaneous symmetry breaking [73–78]. Investigating whether this symmetry breaking can accommodate a dissipative time crystal within certain parameter ranges would be interesting.

Finally, exploring these dissipative time crystals for potential quantum computations and metrology are intriguing directions [79, 80]. Since the early days of quantum mechanics, exploring quantum corrections to classical phenomena has been a prevalent area of interest. Thus, revisiting synchronization from a quantum perspective, as an emergent phenomenon among coupled bodies, holds significance beyond its curiosity-driven aspects.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Acknowledgments

M S would like to thank O Zilberberg for fruitful discussions and acknowledge financial support from the Swiss National Science Foundation through Grant No. PP00P2_163818. H A would like to thank B Buca for several stimulating discussions and the helpful comments on the revised version, and acknowledge financial support from Baden-Württemberg Stiftung Eliteprogram award, the Purdue University Startup fund, the Industry-University Cooperative Research Center Program at NSF under Grant No. 2224960, and the Air Force Office of Scientific Research under Award Number FA9550-23-1-0489. S F Y would like to thank the NSF through the CUA PFC grant and via PHY-2207972 the AFOSR via FA9550-19-1-0233.
Appendix A. The mean-field EoM, dynamical stability, and covariance matrix

Directly from the master equation of equation (3) we obtain the Heisenberg equations of motion for the operators as

$$\begin{align*}
\frac{d}{dt} \hat{a}_1 &= -i(\Delta_1 - i\gamma_1) \hat{a}_1 - iU_0 \left[ \left( \hat{a}_1^\dagger \hat{a}_1 + 2\hat{a}_2^\dagger \hat{a}_2 + 2\hat{a}_3^\dagger \hat{a}_3 \right) \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \right] + \sqrt{2\gamma_1} \hat{\xi}_m(t), \\
\frac{d}{dt} \hat{a}_2 &= -i(\Delta_2 - i\gamma_2) \hat{a}_2 - iU_0 \left[ \left( \hat{a}_2^\dagger \hat{a}_2 + 2\hat{a}_3^\dagger \hat{a}_3 \right) \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_3 \right] - i\Omega_2 + \sqrt{2\gamma_2} \hat{\xi}_2(t), \\
\frac{d}{dt} \hat{a}_3 &= -i(\Delta_3 - i\gamma_3) \hat{a}_3 - iU_0 \left[ \left( \hat{a}_3^\dagger \hat{a}_3 + 2\hat{a}_3^\dagger \hat{a}_3 \right) \hat{a}_3 + \hat{a}_3^\dagger \hat{a}_3 \right] + \sqrt{2\gamma_3} \hat{\xi}_3(t),
\end{align*}$$

(A1)

where $\hat{\xi}_m(t)$ is a zero-mean noise operator with the correlations of $\langle \hat{\xi}_m(t_1) \hat{\xi}_m^\dagger(t_2) \rangle = \delta_{m,0} \delta(t_1 - t_2)$ and $\langle \hat{\xi}_m^\dagger(t_1) \hat{\xi}_m(t_2) \rangle = 0$.

Ignoring the quantum correlations in the aforementioned EoM and employing the factorization assumption, one can get a set of coupled non-linear equations for describing the MF of $\langle \hat{a}_m \rangle = \alpha_m e^{i\phi_m(t)}$ as

$$\begin{align*}
\frac{d}{dt} \langle \hat{a}_1 \rangle &= -i(\Delta_1 - i\gamma_1) \langle \hat{a}_1 \rangle - iU_0 \left[ |\langle \hat{a}_1 \rangle|^2 + 2|\langle \hat{a}_2 \rangle|^2 \right] \langle \hat{a}_1 \rangle + \langle \hat{a}_2 \rangle^2 \langle \hat{a}_3 \rangle^* \\
\frac{d}{dt} \langle \hat{a}_2 \rangle &= -i(\Delta_2 - i\gamma_2) \langle \hat{a}_2 \rangle - iU_0 \left[ |\langle \hat{a}_2 \rangle|^2 + 2|\langle \hat{a}_3 \rangle|^2 \right] \langle \hat{a}_2 \rangle + \langle \hat{a}_2 \rangle^* \langle \hat{a}_3 \rangle - i\Omega_2, \\
\frac{d}{dt} \langle \hat{a}_3 \rangle &= -i(\Delta_3 - i\gamma_3) \langle \hat{a}_3 \rangle - iU_0 \left[ |\langle \hat{a}_3 \rangle|^2 + 2|\langle \hat{a}_2 \rangle|^2 \right] \langle \hat{a}_3 \rangle + \langle \hat{a}_2 \rangle^2 \langle \hat{a}_3 \rangle^*.
\end{align*}$$

(A2)

As can be seen, while the coherent drive restricts $(2\phi_2 - \phi_1 - \phi_3)$, hence breaking the $U(1)$-symmetry of this field, there is no additional constraint on the phase of the non-driven modes. This phase of freedom leads to the emergence of the LC phase. The stagnation points of the aforementioned EoM are determined as $LHS = 0$. When dynamics are contractive at a particular stagnation point, field operators can be linearized around that MF with a fluctuation vector $[\delta \Phi]$ as

$$\delta \Phi = \begin{bmatrix} \delta \hat{a}_1 & \delta \hat{a}_2 & \delta \hat{a}_3 & \delta \hat{a}_1^* & \delta \hat{a}_2^* & \delta \hat{a}_3^* \end{bmatrix}^T,$n

(A3)

where the superscript $T$ means the matrix transpose.

To evaluate the stability of these solutions we employ the dynamical stability analysis. $\mathcal{M}$, i.e. the Bogoliubov matrix of the small excitation, has the following structure

$$\mathcal{M} = \begin{bmatrix} R & S \end{bmatrix},$$

(A4)

where

$$R = -i \begin{bmatrix}
\Delta_1 - i\gamma_1 + 2U_0N & 2U_0 \langle \hat{a}_1^* \rangle \langle \hat{a}_2 \rangle^* + 2U_0 \langle \hat{a}_2 \rangle \langle \hat{a}_3 \rangle^* & 2U_0 \langle \hat{a}_1 \rangle \langle \hat{a}_3 \rangle^* \\
2U_0 \langle \hat{a}_1 \rangle^* \langle \hat{a}_2 \rangle + 2U_0 \langle \hat{a}_2 \rangle \langle \hat{a}_3 \rangle & \Delta_2 - i\gamma_2 + 2U_0N & 2U_0 \langle \hat{a}_1 \rangle \langle \hat{a}_3 \rangle^* \\
2U_0 \langle \hat{a}_1 \rangle^* \langle \hat{a}_3 \rangle & 2U_0 \langle \hat{a}_1 \rangle \langle \hat{a}_3 \rangle^* + 2U_0 \langle \hat{a}_3 \rangle & \Delta_3 - i\gamma_3 + 2U_0N
\end{bmatrix},$$

and

$$S = -iU_0 \begin{bmatrix}
\langle \hat{a}_1 \rangle^2 & \langle \hat{a}_2 \rangle^2 & \langle \hat{a}_3 \rangle^2 \\
\langle \hat{a}_2 \rangle^2 + 2\langle \hat{a}_1 \rangle \langle \hat{a}_3 \rangle & \langle \hat{a}_2 \rangle^2 + 2\langle \hat{a}_1 \rangle \langle \hat{a}_3 \rangle & \langle \hat{a}_3 \rangle^2 \\
\langle \hat{a}_2 \rangle^2 + 2\langle \hat{a}_1 \rangle \langle \hat{a}_3 \rangle & 2\langle \hat{a}_2 \rangle \langle \hat{a}_3 \rangle & \langle \hat{a}_3 \rangle^2
\end{bmatrix},$$

(A5)

for $N = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$ being the mean value of the total number of the photons in the cavity.

When MF is dynamically-stable the Bogoliubov matrix $\mathcal{M}$ is negative-definite. For the stationary state of the stable solutions, we can define the covariance matrix as $\Gamma_n(\omega) = \langle [\delta \Phi(\omega)]^\dagger [\delta \Phi(\omega)] \rangle$, with $\Gamma_{nm}(\omega)$ determined as

$$\mathcal{F} \left\{ \lim_{t \to \infty} \langle \delta \Phi_m^\dagger(t + \tau) \delta \Phi_n(t) \rangle \right\}_\tau.$$

(A6)

The diagonal entries of the covariance matrix are related to the cavity transmission spectrum (auto-correlations), and the off-diagonal entries signify the cross-correlations between the modes hence, are related to the entanglement between the modes.
A.1. Uniform phase

Within this phase, only the 2nd-mode has a non-zero MF and \( \alpha_{1,3} = 0 \), making \( R, S \) diagonal and anti-diagonal matrices, respectively. The eigenvalues of the dynamical stability matrix \( \mathcal{M} \) for identical loss rates of the cavity modes \( \gamma_0 \), are

\[
\lambda_{1,2} = -\gamma_0 \pm i \sqrt{3U_0^2n_2^2 + 4U_0\Delta_2n_2 + \Delta_2^2} \\
\lambda_{3,4,5,6} = -\gamma_0 \pm i \left( \frac{\Delta_1 - \Delta_3}{2} \pm \sqrt{3U_0^2n_2^2 + 4U_0(\Delta_2 - \delta_D)n_2 + (\Delta_2 - \delta_D)^2} \right),
\]

where the dispersion parameter \( \delta_D \) is defined as \( 2\delta_D = 2\Delta_2 - (\Delta_1 + \Delta_3) = 2\omega_2 - \omega_1 - \omega_3 \).

As can be seen, for \( U_0\Delta_2 \gg 0 \) and \( U_0(\Delta_2 - \delta_D) \gg 0 \), matrix \( \mathcal{M} \) is negative definite in the uniform phase hence, stagnation points are dynamically stable.

The multi-mode system is within the uniform phase when \( \Omega_2 \) is either small or large. If the former, the cross-interaction compared to the self-interaction is so small that the pair generation process cannot start. In the latter, the number of particles in the driven mode is so large that the self-interaction shifts the driven mode out of resonance by several \( \gamma_0 \) such that the inter-modal scattering ceases. In other words, in the extreme of a strong coherent drive, the large self-interaction dominates all interactions hence, pushing the system to the single-body dynamics again.

On the single-mode branch only, one gets

\[
\frac{dn_2}{d\Omega_2} = \frac{2\Omega_2}{3U_0^2n_2^2 + 4U_0\Delta_2n_2 + (\Delta_2^2 + \gamma_2^2)}. \tag{A8}
\]

When \( U_0\Delta_2 \ll 0 \), there might be points where the slope first diverges and later changes the sign. These turning points in the MF dynamics signify the existence of multiple MF attractors. More specifically, the boundaries of the uniform phase can be determined as

\[
n_2 = -2U_0\Delta_2 \pm \sqrt{U_0^2(\Delta_2^2 - 3\gamma_2^2)} \tag{A9}.
\]

The above equation determines that for \( \delta_D = 0 \) the multi-stability exists when \( \Delta_2 \gtrsim 2\sqrt{3} \).

A similar argument for a dispersive cavity, i.e. \( \delta_D \neq 0 \), shows that the multi-stability can exist for \( U_0(\Delta_2 - \delta_D) \ll 0 \) and \(|\Delta_2 - \delta_D| \gtrsim 2\sqrt{3} \).

The covariance matrix within this uniform phase has the following general form

\[
\Gamma_{\psi}(\omega) = \begin{bmatrix}
\langle \hat{a}_1(\omega)\hat{a}_1^\dagger(\omega) \rangle & 0 & 0 & 0 & 0 & \langle \hat{a}_1(\omega)\hat{a}_3(\omega) \rangle \\
0 & \langle \hat{a}_2(\omega)\hat{a}_2^\dagger(\omega) \rangle & 0 & 0 & \langle \hat{a}_1(\omega)\hat{a}_2(\omega) \rangle & 0 \\
0 & 0 & \langle \hat{a}_3(\omega)\hat{a}_3^\dagger(\omega) \rangle & \langle \hat{a}_3(\omega)\hat{a}_1(\omega) \rangle & 0 & 0 \\
0 & 0 & 0 & \langle \hat{a}_1(\omega)\hat{a}_3(\omega) \rangle & 0 & 0 \\
0 & \langle \hat{a}_1^\dagger(\omega)\hat{a}_2(\omega) \rangle & 0 & 0 & \langle \hat{a}_3^\dagger(\omega)\hat{a}_3(\omega) \rangle & 0 \\
\langle \hat{a}_1^\dagger(\omega)\hat{a}_3^\dagger(\omega) \rangle & 0 & 0 & 0 & 0 & \langle \hat{a}_3^\dagger(\omega)\hat{a}_1(\omega) \rangle
\end{bmatrix}.
\]

The structure of this matrix implies that the 2nd-mode does not correlate with the side modes. That physically is consistent with the picture of these non-driven modes not being populated through the parametric process via the 2nd-mode.

Consequently, the covariance matrix for the quadratures gets the following block diagonal form

\[
\Gamma_{\text{sym}}(\omega) = \begin{bmatrix}
\Gamma_2(\omega) & 0_{4 \times 4} \\
0_{2 \times 2} & \Gamma_{\pm}(\omega)
\end{bmatrix} \tag{A10}.
\]

The above form emphasizes again that the driven mode does not correlate with the other two side modes.

A.2. LC phase

As discussed before, while the U(1)-symmetry of the un-driven system is broken by a coherent drive, the non-driven modes still have some phase freedom since \( \Phi_0 = 2\Phi_2 - \Phi_1 - \Phi_3 \), is the only constraint imposed by the coherent drive.

Within this phase, there is no stationary state, and the long-time limit of the 1st, 3rd MF has an oscillatory behavior as \( e^{i\theta_{1,3}t} \). In other words, due to the aforementioned phase constraint, if \( \dot{a}_1 \) oscillates as \( e^{i\theta_{1,3}t} \), \( \dot{a}_3 \) should vary as \( e^{-i\theta_{1,3}t} \). Going back to the equation (A1), one can see that this oscillation can be interpreted in terms of a re-normalized detuning of the parametrically-populated modes as \( \Delta_{1,3} = \Delta_{1,3} \pm \omega_{LC} \).
Appendix B. Closed system

To draw a preliminary understanding of the system we first describe the closed system. In general, the Hamiltonian describes the dynamics of any bosonic ensemble with two-body interaction thermalizing to the Bose–Einstein condensate at the thermodynamic limit.

In the presence of the coherent drive of the 2nd-mode, the system features a phase transition as a function of the coherent drive rate \( \Omega_2 \). We study the energy potential landscape, \( \bar{H}_3 \), for a zero occupation of the 1st and 3rd-mode

\[
\bar{H}_3 = \Delta_2 |\alpha_2|^2 + \frac{U_0}{2} |\alpha_2|^4 + \Omega_2 \alpha_2^* \alpha_2, \tag{B1}
\]

where \( \alpha_2 \) is the order parameter. Without loss of generality, we assume \( \Omega_2 \) to be real and find the potential extrema to be given by

\[
U_0 \text{Re}(\alpha_2)^3 + \Delta_2 \text{Re}(\alpha_2) + \Omega_2 = 0, \tag{B2}
\]

\[
\text{Im}(\alpha_2) = 0. \tag{B3}
\]

This leaves us with a cubic equation in \( \text{Re}(\alpha_2) \). We study the discriminant and find the boundaries between the regions with only one and three possible solutions

\[
\Omega_2 = \frac{2}{3\sqrt{3}} \sqrt{\frac{\Delta_2^3}{|U_0|}}, \tag{B4}
\]

where we explicitly assumed \( U_0 < 0 \) in our case.

In the parameter regime where three solutions are allowed, we find one low population (LP) and one high population (HP) phase accompanied by an unphysical one, i.e. imaginary eigenfrequencies of excitations.

To understand the nature of the extrema of the potential, we find the excitation spectrum associated with the uniform case Hamiltonian \( \bar{H}_3 \) [6]. We obtain the eigenfrequencies

\[
\omega_3^\pm = \pm \sqrt{(n_2 U_0 + \Delta_2) (3n_2 U_0 + \Delta_2)}, \tag{B5}
\]

with eigenvectors

\[
v_\pm = \left( \frac{U_0 \alpha_2^3}{\omega_3^\pm - \Delta_2 - 2U_0 n_2}, 1 \right), \tag{B6}
\]

and their associated symplectic norms

\[
ds_3^2 = \left| \frac{U_0 |\alpha_2|^2}{\omega_3^\pm - \Delta_2 - 2U_0 n_2} \right|^2 - 1, \tag{B7}
\]

where \( n_2 = |\alpha_2|^2 \).

The symplectic norm describes the nature of a state of the system. Whenever all positive (negative) frequency eigenmodes have a positive (negative) symplectic norm it describes the ground state. On the other hand, if a physical state of the system, i.e. a state with real excitation eigenmodes, has at least one excitation mode with a negative (positive) norm at positive (negative) excitation frequency, then the state is an excited state of the closed system. Additionally, a positive symplectic norm is associated with particle-like processes where an external excitation is absorbed by the system whereas a negative norm underpins a hole-like process where an excitation in the system is destroyed [6]. Finally, the symplectic norm helps identify the so-called negative-mass instabilities [81].

In figure 8, we plot the mean-field energy potential. Due to the single-photon drive of the 2nd-mode, the system features a phase transition as a function of the coherent drive rate \( \Omega_2 \) between an LP and an HP phase. The phase diagram comprises two qualitatively distinct regions in parameter space with (I), the energy functional has only one clear extremum, the HP phase; II, three extrema exist including a ‘proper’ LP phase ground-state, a saddle point, and an HP phase that reveals itself as a maximum, i.e. an excited-state.

In figure 9, we plot the order parameter \( |\alpha_2|^2 \) and the excitation spectra on top of the possible solutions along the vertical orange cut (a), (c) and the horizontal brown cut (b), (d) of figure 8. In both cases the order parameter \( |\alpha_2|^2 \) has a finite value within the entire range but in the region (II), two distinct phases with different populations are possible. Besides, there is a third phase whose spectrum is fully imaginary and therefore not a physical state (gray lines in figures 9(a) and (b)). The HP exists throughout the phase diagram.
Figure 8. Closed-system phase diagram as a function of the driven mode detuning $\Delta_2$ and the coherent driving rate $\Omega_2$. Two distinct regions are indicated by their respective mean-field energy potential ($\bar{H}_3$) landscape as a function of the real part of the cavity field, $\text{Re}(\alpha_2)$. The dark-blue region indicates the parameter regime where the LP phase is the ground state of the system and the HP represents a physically-allowed excited state. The LP phase disappears in the light-blue region where the HP is the only physical state. All rates are normalized to $|U_0|$.

Figure 9. (a), (b) The order parameter, $|\alpha_2|$, and (c), (d) the excitation spectrum, $\omega$, on top of the LP (light hues) and HP (dark hues), along the orange and brown cut lines of figure 8, as a function of the coherent drive rate, $\Omega_2$, and detuning $\Delta_2$, respectively. Real (solid) and imaginary (dashed) values with green [blue] hues encode the particle- [hole-]excitations, i.e. $d\sigma^2_s > 0$ [$d\sigma^2_s < 0$]. In (a), (c), at the $\Omega_2$ boundary between the light blue and dark blue regions in figure 8, the LP ceases to exist and the only possible state is the excited-state HP. In (b), (d), at the $\Delta_2$ boundary, the LP appears as a possible state of the system. The HP always features a norm swap with the negative norm for the positive eigenmode. The gray lines indicate the third unphysical solution.

but always presents a negative (positive) symplectic norm with a positive (negative) eigenfrequency. Therefore, we confirm that it is indeed an excited state of the closed system. Due to the presence of the negative interactions a 'true ground-state' of the unbounded Hamiltonian is only possible in a limited region of the parameter space (see figure 8) and is identified with the LP phase, accompanied by a positive (negative) symplectic norm with a positive (negative) eigenfrequency [5, 6].

Appendix C. Keldysh formalism

The exact diagonalization of the Liouvillian super-operator suffers from the finite size effects, i.e. the truncation of the Fock space, while the MF approach ignores quantum fluctuations. To include the correlation effects while approaching the thermodynamic limit (the typical validity range of an MF treatment), we employ Keldysh formalism [82].

We readily write the rotated Keldysh action, associated with the Hamiltonian (1) and subjected to the dissipation (2), in the quantum and classical fields $a_{m,q}$, for $m = 1, 2, 3$ as

$$S_k(\vec{\alpha}_c, \eta) = \sum_{m=1}^{3} S^m_0 + S_{\text{int}} + S_{\text{drive}} + S_\gamma,$$  (C1)
where \( S^m_\text{int}, S_\text{drive}, S_\gamma \) are defined in appendix C.1. Here, we only highlight that the symmetry \( a_1 \leftrightarrow a_3 \) is still present in the action (C1).

We perform the saddle point approximation via \( \partial S_k / \partial \alpha_{i,q} = 0 \), set the quantum fields to zero \( \alpha_{m_q} = 0 \), and obtain the EoM for the classical fields \( \alpha_{m_c} \) (cf appendix C.1). These equations coincide with the mean-field ones upon rotating back to physical fields, i.e. \( \alpha_{m_c} = \sqrt{2} \omega_m \).

To go beyond mean-field, we study the fluctuations around the MF stationary-states as \( \alpha_m = \alpha_{m_c} + \delta \alpha_m \).

We expand the Keldysh action in equation (C1) and retain terms up to second order in fluctuations, i.e. only the Gaussian parts. Therefore, the Gaussian action can be written in the normal form as

\[
S_G = \int_\omega \delta \Phi^\dagger(\omega) M(\omega) \delta \Phi(\omega),
\]

where we went to Fourier space with the three-mode, 12 component, nambu-spinor \( \delta \Phi(\omega) \), and the matrix \( M(\omega) \) is given by

\[
M_{12 \times 12} = \frac{1}{2} \left[ \begin{array}{c|c} 0_{6 \times 6} & [G^A(\omega)]^{-1}_{6 \times 6} \end{array} \right],
\]

where \( [G^A(\omega)]^{-1}, [G^P(\omega)]^{-1}, P^K(\omega) \) are the inverse of the advanced, retarded Green’s functions, and the Keldysh component, respectively (cf appendix C.3).

Using Gaussian integration we determine the single-mode action and the associated Green’s functions. We here report solely the uniform phase case, \( \alpha_2 \neq 0, \alpha_{1,3} = 0 \) hence, the single-mode Green’s functions have simple analytical expressions

\[
G_R^m = \frac{4(\omega + \Delta_2 + i \gamma_2 + U_0 n_2)}{4(\omega + \Delta_2 + i \gamma_2) (-\omega + \Delta_2 - i \gamma_2) + 8U_0 \Delta_2 n_2 + 3U_0^2 n_2^2}
\]

\[
G_R^m = \frac{4(\omega + \Delta_3 + i \gamma_3 + U_0 m_2)}{4(\omega + \Delta_3 + i \gamma_3) (-\omega + \Delta_3 - i \gamma_3) + 8U_0 \Delta_3 m_2 + 3U_0^2 m_2^2}
\]

where \( n_2 = |\alpha_2|^2 \) and \( (m, \bar{m}) = (1, 3), (3, 1) \).

From the poles of Green’s function, we obtain the eigenvalues

\[
\omega^2 = \pm \frac{1}{2} \sqrt{4\Delta^2_2 + 8\Delta_2 U_0 n_2 + 3U_0^2 n_2^2 + i \gamma_2}
\]

\[
\omega^2 = \pm \frac{1}{2} \sqrt{(\Delta_m + \Delta_3 + i \gamma_3 - i \gamma_3) + i \gamma_3} + 4U_0 n_2 (\Delta_m + \Delta_3 + i \gamma_3 - i \gamma_3) + 3U_0^2 n_2^2
\]

where \( \omega^i \) is associated with the \( i \)th cavity mode and \( i = 1, 2, 3 \).

First, we notice that the driven mode behaves as a single-driven Kerr-oscillator whereas the symmetric modes \( \alpha_{1,3}, \) even though empty and not driven, present a non-trivial Green’s function, and their Green’s functions are ‘coupled’. Second, the eigenvalues we obtained from Green’s functions coincide, upon the substitutions \( \alpha_m = \sqrt{2} \omega_m, \gamma_0 = \gamma_1 = \gamma_2 = \gamma_3, \Delta_1 + \Delta_3 = 2(\Delta_2 - \delta_D) \), with those obtained from the stability matrix (cf appendix A).

C.1. Rotated Keldysh action

In this appendix, we give the explicit expressions for the rotated Keldysh actions used in the equation (C1).

The Keldysh action of the \( m \)-th mode with the self-interaction reads as

\[
S^m_i = \alpha^*_{m_c} \left( i \frac{\partial}{\partial t} - \Delta_m \right) \alpha_{m_c} + \alpha^*_{m_c} \left( i \frac{\partial}{\partial t} - \Delta_m \right) \alpha_{m_c} - \frac{U_0}{2} (|\alpha_{m_c}|^2 + |\alpha_{m_c}|^2) (\alpha_{m_c} \alpha^*_{m_c} + \alpha^*_{m_c} \alpha_{m_c}).
\]

The action due to the cross-term interactions between the modes reads as

\[
S_{int} = -U_0 \left[ (|\alpha_{1_1}|^2 + |\alpha_{1_3}|^2) \left( \alpha_{2} \alpha_{2}^* + \alpha_{2}^* \alpha_{2} + \alpha_{3} \alpha_{3}^* + \alpha_{3}^* \alpha_{3} \right) \right] \]

\[
- \alpha_{2} \alpha_{2}^* \left( \alpha_{3} \alpha_{3}^* + \alpha_{3}^* \alpha_{3} \right) \left( \alpha_{1} \alpha_{1}^* + \alpha_{1}^* \alpha_{1} + \alpha_{3} \alpha_{3}^* + \alpha_{3}^* \alpha_{3} \right) \]

\[
- \alpha_{2} \alpha_{2}^* \left( \alpha_{3} \alpha_{3}^* + \alpha_{3}^* \alpha_{3} \right) \left( \alpha_{1} \alpha_{1}^* + \alpha_{1}^* \alpha_{1} + \alpha_{3} \alpha_{3}^* + \alpha_{3}^* \alpha_{3} \right) \]

\[
- \frac{U_0}{2} \left[ (|\alpha_{2}^2|^2 + |\alpha_{2}^2|^2) \left( \alpha_{1} \alpha_{1}^* + \alpha_{1}^* \alpha_{1} + \alpha_{3} \alpha_{3}^* + \alpha_{3}^* \alpha_{3} \right) + c.c. \right].
\]
The coherent drive action is

\[ S_{\text{drive}} = -\Omega_2 \sqrt{2} \left( \alpha_{2_1} + \alpha_{2_1}^* \right). \]  

And finally, the action due to the Lindblad dissipator reads as

\[ S_\gamma = i \sum_{m=1}^{3} \gamma_m \left( 2|\alpha_{m}|^2 + \alpha_{m}^* \alpha_{m} - \alpha_{m} \alpha_{m}^* \right). \]

The action is stationary at the saddle point, determined via \( \partial S_\gamma / \partial \alpha_{m} = 0 \), which leads to \( \alpha_{m} = 0 \) and the following equations of motion for \( \alpha_{m} \):

\[ \frac{d}{dt} \alpha_{1} = -i (\Delta_1 - i\gamma_1) \alpha_{1} - i \frac{U_0}{2} \left[ (|\alpha_{1}|^2 + 2|\alpha_{2}|^2 + 2|\alpha_{3}|^2) \alpha_{1} + \alpha_{2}^* \alpha_{3}^* \right], \]

\[ \frac{d}{dt} \alpha_{2} = -i (\Delta_2 - i\gamma_2) \alpha_{2} - i \frac{U_0}{2} \left[ (|\alpha_{1}|^2 + 2|\alpha_{2}|^2 + 2|\alpha_{3}|^2) \alpha_{2} + 2\alpha_{3}^* \alpha_{1} \alpha_{3} \right] - i \Omega_2 \sqrt{2}, \]

\[ \frac{d}{dt} \alpha_{3} = -i (\Delta_3 - i\gamma_3) \alpha_{3} - i \frac{U_0}{2} \left[ (|\alpha_{1}|^2 + 2|\alpha_{2}|^2 + 2|\alpha_{3}|^2) \alpha_{3} + \alpha_{2}^* \alpha_{1}^* \right]. \]

Comparing the EoM for \( \alpha_{m} \) of the action saddle points with equation (A1) for the MFs, one can see that \( \alpha_{m} = \sqrt{2}(d_{m}) \), which further clarifies the meaning of the classical fields in terms of the mean-fields.

C.2. Approximated Gaussian action

As mentioned in the text, the effects of quantum fluctuations \( \delta \alpha_{m-1} \), can be included by expending the Keldysh action around the saddle points. Since the Hamiltonian of equation (1) is quartic, the expansion has terms linear—quartic in fluctuations, in general. Here, we report the explicit form of the Keldysh action up to the second order in fluctuations, i.e. only the Gaussian parts

\[ S_k^{(2)} = \sum_{m=1}^{3} \delta \alpha_{m}^* \left[ \left( i \frac{\partial}{\partial \alpha} - \Delta_m \right) - U_0 \left( |\alpha_{1}|^2 + |\alpha_{2}|^2 + |\alpha_{3}|^2 \right) - i \gamma_m \right] \delta \alpha_{m} + i 2 \gamma_m \delta \alpha_{m}^* \delta \alpha_{m}. \]

C.3. M matrix coefficients

In this appendix, we give the explicit form of the entries of the matrix \( M \) in equation (C3)

\[ [G^4(\omega)]]^{-1} \text{ reads as} \]

\[
\begin{array}{cccccccc}
\omega - \Delta_1 - U_0 N - i\gamma_1 & -\frac{U_0}{2} \alpha_{1} & -\frac{U_0}{2} \alpha_{2} & -\frac{U_0}{2} \alpha_{3} & -\frac{U_0}{2} (\alpha_{1}^2 + 2\alpha_{1} \alpha_{2}) & -\frac{U_0}{2} (\alpha_{2}^2 + 2\alpha_{2} \alpha_{3}) & -\frac{U_0}{2} (\alpha_{3}^2 + 2\alpha_{3} \alpha_{1}) \\
\omega - \Delta_2 - U_0 N + i\gamma_2 & -\frac{U_0}{2} \alpha_{1} & -\frac{U_0}{2} \alpha_{2} & -\frac{U_0}{2} \alpha_{3} & -\frac{U_0}{2} (\alpha_{1}^2 + 2\alpha_{1} \alpha_{2}) & -\frac{U_0}{2} (\alpha_{2}^2 + 2\alpha_{2} \alpha_{3}) & -\frac{U_0}{2} (\alpha_{3}^2 + 2\alpha_{3} \alpha_{1}) \\
\omega - \Delta_3 - U_0 N + i\gamma_3 & -\frac{U_0}{2} \alpha_{1} & -\frac{U_0}{2} \alpha_{2} & -\frac{U_0}{2} \alpha_{3} & -\frac{U_0}{2} (\alpha_{1}^2 + 2\alpha_{1} \alpha_{2}) & -\frac{U_0}{2} (\alpha_{2}^2 + 2\alpha_{2} \alpha_{3}) & -\frac{U_0}{2} (\alpha_{3}^2 + 2\alpha_{3} \alpha_{1})
\end{array}
\]
\[ [G^R(\omega)]^{-1} \] reads as

\[
\frac{(\omega - \Delta_1 - U_0 N + i\gamma_1)}{4(\omega + \Delta_2 - i\gamma_2)} - \frac{\omega}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)} - \frac{\omega}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)} - \frac{\omega}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)} - \frac{\omega}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)}
\]

\[
- \frac{\omega}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)} - \frac{\omega}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)} - \frac{\omega}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)} - \frac{\omega}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)} - \frac{\omega}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)}
\]

where in the above equations \( N = |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 \) is the MF total number of particles in the cavity.

And finally, we have \( P^k \) as

\[
\begin{pmatrix}
  i2\gamma_1 [I]_{2 \times 2} & 0 & 0 \\
  0 & i2\gamma_2 [I]_{2 \times 2} & 0 \\
  0 & 0 & i2\gamma_3 [I]_{2 \times 2}
\end{pmatrix}
\]

for \([I]_{2 \times 2}\) being an identity matrix so rank 2.

C.4. Single-mode action

We can determine the single-mode actions of each mode by Gaussian integration. Considering the steady state with empty mode \( a_{1,3} \), figures 3(a) and (b), simplifies the analytical expressions for the single mode Green’s function to

\[
G^A_1 = \frac{4(\omega + \Delta_2 - i\gamma_2 + U_0 |\alpha_2|^2)}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)} + \frac{4U_0^2 |\alpha_2|^4}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)}
\]

\[
G^B_1 = \frac{4(\omega + \Delta_2 - i\gamma_2 + U_0 |\alpha_2|^2)}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)} + \frac{4U_0^2 |\alpha_2|^4}{4(\omega + \Delta_2 - i\gamma_2)(\omega - \Delta_1 - U_0 N + i\gamma_1)}
\]

where \((m, \bar{m}) = (1, 3), (3, 1)\). From the poles of Green’s function, we obtain the eigenvalues

\[
\epsilon^2 = \pm \frac{1}{2} \sqrt{4\Delta_1^2 + 8\Delta_2U_0 |\alpha_2|^2 + 3U_0^2 |\alpha_2|^4 + i\gamma_2}
\]

\[
\epsilon^m = \frac{1}{2} \left( \pm \sqrt{(\Delta_m + \Delta_m + i\gamma_m - i\gamma_m)^2 + 4U_0 |\alpha_2|^2 (\Delta_m + \Delta_m + i\gamma_m - i\gamma_m)^2 + 3U_0^2 |\alpha_2|^4 + (\Delta_m - \Delta_m + i\gamma_m + i\gamma_m)} \right)
\]

Interestingly, the driven mode behaves as a single-driven Kerr-oscillator whereas the symmetric modes \( a_{1,3} \), even though empty, present a non-trivial Green’s function.

C.5. Spectral functions

Using a Keldysh action approach we readily get access to the Green’s function of the system and associated dynamical observable. In figure 10, we compare the spectral functions, \( A(\omega) = -2Im[G^R(\omega)] \), of the three-modes in the LP (top) and HP (bottom) phases for different coherent driving rate at fixed detuning \( \Delta_2 = +5 \), points \( S_{1,2,3} \) in figure 2. From the top panel, we see that the LP phase has the response of a ‘ground state’, i.e. a positive (negative) peak at positive (negative) frequencies, see appendix B. We note that peaks at negative frequencies are unresolved due to the scale resolution. On the other hand, the HP phase features a peak swap with a positive (negative) peak at negative (positive) frequencies, features of a stabilized excited state (cf appendix B). We note that the two uniform phases have different occupations of the 2nd-mode, see figure 3 LP and HP, nevertheless, their spectral functions highlight a more profound difference between them than just a different mode occupation. Interestingly, the peak swap is also present in the response of the empty side modes and we can think of it as being in the presence of a normal phase to excited normal phase transition.
Appendix D. Beyond MF and 2nd-order cumulants in a single-mode cavity

Following the approach of employing cumulant expansion for exploring the dynamics and phase transitions as detailed in [48, 83] we perform an extension to the second order, i.e. writing the three-body correlations appearing in EoM of equation \((A1)\) as multiplications of the two-body and single-body moments. In what follows we detail the approach for the single-mode Kerr cavity and compare the results with ones obtained from the MF and the full density matrix solution. The extension to the 3-mode case directly follows the approach here to derive the EoM for 15 independent first- and second-order correlations.

For the single-mode driven cavity, we have
\[
\frac{d}{dt} \langle \hat{a} \rangle = -i (\Delta - i \gamma) \langle \hat{a} \rangle - i U_0 \langle \hat{a} \hat{a}^\dagger \rangle - i \Omega_2 \tag{D1}
\]

where we assumed a vanishing 3rd-order cumulant hence replacing the three-body correlation of \(\langle \hat{A} \hat{B} \hat{C} \rangle\) in terms of the two-body correlation and the single-body expectation values as follow
\[
\langle \hat{A} \hat{B} \hat{C} \rangle = \langle \hat{A} \rangle \langle \hat{C} \rangle + \langle \hat{A} \hat{C} \rangle \langle \hat{B} \rangle + \langle \hat{B} \hat{C} \rangle \langle \hat{A} \rangle - 2 \langle \hat{A} \rangle \langle \hat{B} \rangle \langle \hat{C} \rangle
\]

Note that the above relation is the same as Wick’s theorem results for the Gaussian states.

Therefore, we have
\[
\langle \hat{a}^\dagger \hat{a}^\dagger \rangle = 2 \left( \langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle \right) + \langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle. \tag{D2}
\]

As can be seen from equation \((D1)\), the two-body correlations are needed to describe the dynamics of \(\langle \hat{a} \rangle\).

\[
\frac{d}{dt} \langle \hat{a}^\dagger \hat{a} \rangle = -2 \gamma \langle \hat{a}^\dagger \hat{a} \rangle + i \Omega_2 \left( \langle \hat{a} \rangle - \langle \hat{a}^\dagger \rangle \right),
\]

\[
\frac{d}{dt} \langle \hat{a}^\dagger \rangle = -i 2 \Omega_2 \langle \hat{a} \rangle - i 2 \Omega_2 \langle \hat{a} \rangle - i 2 \Omega_2 \langle \hat{a} \rangle - i 2 U_0 \langle \hat{a} \hat{a}^\dagger \rangle. \tag{D3}
\]

The last term of the second equation involves the four-body correlation of \(\langle \hat{a}^3 \hat{a}^\dagger \rangle\), rendering the sets of EoM to a non-closed set.

Finally, the desired 4-body correlation of the single-mode Kerr cavity reads as
\[
\langle \hat{a}^3 \hat{a}^\dagger \rangle = 3 \langle \hat{a} \hat{a} \rangle \langle \hat{a}^3 \rangle - 2 \langle \hat{a} \rangle \langle \hat{a} \rangle \langle \hat{a}^3 \rangle. \tag{D4}
\]

As can be seen, the last equation is in terms of the single-body expectation values, i.e. \(\langle \hat{a} \rangle, \langle \hat{a} \rangle^\dagger\), and two-body correlations as \(\langle \hat{a} \hat{a} \rangle, \langle \hat{a} \hat{a}^\dagger \rangle\), only.
Finally, the EoM for the Gaussian approximation of the solution reads as

\[
\frac{d}{dt} \langle a \rangle = -i \left( \Delta - i\gamma + 2U_0 \langle a \dagger a \rangle - 2U_0 \langle a \dagger \rangle \langle a \rangle \right) \langle a \rangle - iU_0 \langle a \dagger \rangle \langle a \rangle - i\Omega_2, \\
\frac{d}{dt} \langle a \dagger a \rangle = -2\gamma \langle a \dagger a \rangle + i\Omega_2 \left( \langle a \rangle - \langle a \dagger \rangle \right), \\
\frac{d}{dt} \langle a^2 \rangle = -i2\Omega_2 \langle a \rangle - i2 \left( \Delta - i\gamma + 0.5U_0 + 3U_0 \langle a \dagger a \rangle \right) \langle a \rangle + i4U_0 \langle a \rangle^3 \langle a \dagger \rangle. \tag{D5}
\]

Figure 11 compares the results of the MF theory (blue lines) with the full-density matrix solution (red line) and the one obtained from the Gaussian approximation (green line). As can be seen, the Gaussian approximation not only matches with the full-density matrix solutions far away from the phase transition point but also captures the transition behavior from the LP to the HP. Specifically, the MF bistability region and the associated hysteresis behavior disappear and the two branches are connected via a rapid but continuous change. The remained discrepancy between the 2nd-cumulant results and the density matrix within the transition region is related to the non-Gaussian nature of the cavity state due to the quartic interaction. We note that the multi-stable behavior does not appear in the full quantum mechanical determination of NESS and the quantum solution is unique, as well as the Gaussian approximation, while the MF approach gives multiple dynamically stable solutions.

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