On the Hyperbolic Gluing Equations and Representations of Fundamental Groups of Closed 3-Manifolds

Tian Yang

April 23, 2010

Abstract

We show that for a representation of the fundamental group of a triangulated closed 3-manifold (not necessarily hyperbolic) into $\text{PSL}_2(\mathbb{C})$ so that any edge loop has non-trivial image under the representation, there exist uncountably many solutions to the hyperbolic gluing equation whose associated representations are conjugate to the given representation, and whose volumes are equal to the volume of the given representation.

1 Introduction

In [10], Thurston introduced a system of algebraic equations—called the hyperbolic gluing equations—for constructing hyperbolic metrics on orientable 3-manifolds with torus cusps. He used solutions to the hyperbolic gluing equations to produce a complete hyperbolic metric on the figure-eight knot complement in the early stages of formulating his geometrization conjecture. On a closed, oriented, triangulated 3-manifold $M$, the hyperbolic gluing equations can be defined in the same way: We assign to each edge of each oriented tetrahedron in the triangulation a shape parameter $z \in \mathbb{C} \setminus \{0, 1\}$ such that

(a) opposite edges of each tetrahedron have the same shape parameter;
(b) the three shape parameters assigned to three pairs of opposite edges in each tetrahedron are $z$, $\frac{1}{z}$, and $1 - \frac{1}{z}$ subject to an orientation convention; and
(c) for each edge $e$ in $M$, if $z_1, \ldots, z_k$ are shape parameters assigned to the edges sharing $e$ as an edge, then we have

$$\prod_{i=1}^{k} z_i = 1. \quad (1)$$

The equations (1) are termed the hyperbolic gluing equations, and the set of all solutions is the parameter space $\mathcal{P}(M)$. The space $\mathcal{P}(M)$ depends on the triangulation of $M$. Given any $Z \in \mathcal{P}(M)$, the associated representation, denoted $\rho_Z$, is defined by Yoshida in [11]; and the volume of $Z$, denoted $\text{Vol}(Z)$, is well-defined using the Lobachevsky-Milnor formula.

In our joint work with F.Luo and S.Tillmann [5], we were able to show the hyperbolic structure on a closed, oriented, triangulated hyperbolic 3-manifold can be constructed from a solution to the hyperbolic gluing equation using any triangulation with essential edges. An edge in $T$ is termed essential if it is not a null homotopic loop in $M$. This is clear the case if it has distinct end-point, but we allow the triangulation of $M$ to be semi-simplicial (or singular), so that some or all edges may be loops in $M$. It is well known that a closed 3-manifold $M$ is hyperbolic if and only if there exists a discrete and faithful representation $\rho: \pi_1(M) \to \text{PSL}_2(\mathbb{C})$ of the fundamental group of $M$ into $\text{PSL}_2(\mathbb{C})$, and the main
Our main observation in the present paper is that the constraint that the 3-manifold \(M\) is hyperbolic, or equivalently the representation \(\rho: \pi_1(M) \to PSL_2(\mathbb{C})\) is discrete and faithful, could be removed, and we have

**Theorem 1.1** Let \(M\) be an oriented, closed 3-manifold, \(\mathcal{T}\) be a triangulation of \(M\) so that each edge \(e\) in \(\mathcal{T}\) is essential, and \(\rho: \pi_1(M) \to PSL_2(\mathbb{C})\) be a representation of the fundamental group of \(M\) into \(PSL_2(\mathbb{C})\) so that \(\rho([e]) \neq 1\), for all loop \(e\) in \(\mathcal{T}\). (2)

Then

1. there exist uncountably many solutions \(Z_\rho\) to the hyperbolic gluing equation;
2. the associated representation \(\rho_{Z_\rho}\) is conjugate to \(\rho\); and
3. \(Vol(Z_\rho) = Vol(\rho)\).

We note that, in the special case that \(\mathcal{T}\) is simplicial, i.e., every 3-simplex in \(\mathcal{T}\) has distinct vertices, every representation \(\rho: \pi_1(M) \to PSL_2(\mathbb{C})\) satisfies condition (2); and in the case that \(\rho: \pi_1(M) \to PSL_2(\mathbb{C})\) is discrete and faithful, Theorem 1.1 implies Theorem 1.1 of [5] as a special case.

The present paper is organized as follows. In section 2, some basic definitions on hyperbolic geometry are reviewed. Theorem 1.1 is proven in section 3 using the spinning construction summarized in [5] and a theorem of Luo on the continuous extension of the volume function [3].

### 2 The parameter space

#### 2.1 The hyperbolic gluing equation and the volume of solutions

If \(\sigma\) is an oriented 3-simplex with edges from one vertex labeled by \(e_1, e_2\) and \(e_3\) so that the opposite edges have the same labeling, then the cyclic order of \(e_1, e_2\) and \(e_3\) viewed from each vertex depends only on the orientation of the tetrahedron, i.e. independent of the choice of the vertices. Note that each pair of opposite edges \(e_i\) corresponds to a normal isotopy class of quadrilateral (normal quadrilateral for short) \(q_i\) in \(\sigma\) so that \(q_1 \to q_2 \to q_3 \to q_1\) is the cyclic order induced by the cyclic order on the edges from a vertex. To define hyperbolic gluing equation, we need the following notation. Let \(e\) be an edge in \(\mathcal{T}\), and \(q\) be a normal quadrilateral in \(\sigma\). The index \(i(q,e)\) is the integer 0, 1 or 2 defined as follows. \(i(q,e) = 0\) if \(e\) is not an edge of \(\sigma\). \(i(q,e) = 1\) if \(e\) is the only edge in \(\sigma\) facing \(q\) and \(i(q,e) = 2\) if \(e\) are the two edges in \(\sigma\) facing \(q\). Let \(Q\) be the set of normal quadrilaterals in \(\mathcal{T}\), we have

**Definition 2.1** Suppose \((M, \mathcal{T})\) is a triangulated oriented close 3-manifold. The hyperbolic gluing equation is defined for \(Z = (z_q) \in (\mathbb{C} \setminus \{0, 1\})^Q\) so that

(a) for each edge \(e\) in \(\mathcal{T}\),

\[
\prod_{q \in Q} z_q^{i(q,e)} = 1,
\]

and
(b) if $\sigma$ is a 3-simplex in $T$, and $q_1 \to q_2 \to q_3 \to q_1$ is the cyclic order of normal quadrilaterals in $\sigma$, then

$$z_{q_{i+1}} = \frac{1}{1 - z_{q_i}},$$

where $q_{3+1}$ is understood to be $q_1$.

The set of all solutions to the hyperbolic gluing equation is called the parameter space, and is denoted by $\mathcal{P}(M)$.

Let $z_\sigma = (z_{q_1}, z_{q_2}, z_{q_3})$ be the complex numbers assigned to $q_i$, $i \in \{1, 2, 3\}$, then we have

**Definition 2.2** The volume of $z_\sigma$ is defined to be the sum of the Lobachevsky functions

$$\text{Vol}(z_\sigma) = \sum_{i=1}^{3} \Lambda(\arg(z_{q_i}))$$

$$= \sum_{i=1}^{3} \left( -\int_{0}^{\arg(z_{q_i})} \ln |2 \sin t| dt \right);$$

and the volume of $Z = (z_\sigma) \in \mathcal{P}(M)$ is defined by

$$\text{Vol}(Z) = \sum_{\sigma \in T} \text{Vol}(z_\sigma).$$

### 2.2 The shape parameters of an ideal tetrahedron

Let $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup S^2_\infty$ be the compactification of $\mathbb{H}^3$, where $S^2_\infty$ is the sphere at infinity. We have

**Definition 2.3** Let $\sigma$ be an ideal tetrahedron in $\overline{\mathbb{H}^3}$ with vertices $\{v_i\} \subset S^2_\infty$, $i \in \{1, \ldots, 4\}$, and $e_{ij}$ be the edge from $v_i$ to $v_j$. Identifying $S^2_\infty$ with $\mathbb{C} \cup \{\infty\}$, the shape parameter of $\sigma$ at $e_{ij}$ is defined by the following cross-ratio

$$z_{ij} = (v_i, v_j; v_k, v_l) = \frac{v_i - v_k}{v_k - v_l} \cdot \frac{v_j - v_l}{v_j - v_k}$$

where $(i, j, k, l)$ is an even permutation of $(1, 2, 3, 4)$.

A direct cross-ratio calculation shows the following well known

**Proposition 2.4**

1. For all $\{i, j\}, \{k, l\} \subset \{1, \ldots, 4\}$, $i, j \neq k, l$,

$$z_{ij} = z_{kl},$$

i.e., opposite edges share the same shape parameter, so we can denote the shape parameter of $\sigma$ at $e_{ij}$ and $e_{kl}$ by $z_q$, where $q$ is the normal quadrilateral facing $e_{ij}$ and $e_{kl}$, and

2. if $q_1 \to q_2 \to q_3 \to q_1$ is the cyclic order of normal quadrilaterals in $\sigma$, then

$$z_{q_{i+1}} = \frac{1}{1 - z_{q_i}},$$

where $q_{3+1}$ is understood to be $q_1$. 

3
For an ideal tetrahedron $\sigma$ with shape parameters $z_{q_1}, z_{q_2}$ and $z_{q_3}$, the hyperbolic volume is calculated by Milnor as

$$Vol_{\H^3}(\sigma) = \sum_{i=1}^{3} \Lambda(\arg(z_{q_i})).$$

We call an ideal tetrahedron $\sigma \subset \H^3$ flat if it lies in a totally geodesic plan. When $\sigma$ is flat, we have that $\{z_{q_i}\}$ are real numbers and $Vol(\sigma) = 0$.

### 2.3 The associated representation

Given a solution $Z \in \mathcal{P}(M)$ to the hyperbolic gluing equation for a triangulated cusped 3-manifold $(M, T)$ with essential edges, the associated representation $\rho_Z : \pi_1(M) \to PSL_2(\C)$ was described by Yoshida \[11\] via constructing the pseudo developing maps. For a closed triangulated 3-manifold with essential edges, the construction of the associated representation is essentially the same as Yoshida’s. Namely, for each $Z \in \mathcal{P}(M)$, there is a continuous map $D_Z : \tilde{M} \to \H^3$ taking 3-simplices in $T$ to ideal straight simplices, and a representation $\rho_Z$ which makes it invariant. This construction is described in details in Section 4.5, \[3\].

### 3 The existence of solutions to the hyperbolic equation

#### 3.1 The proof of 1. of Theorem \[11\]

**Proof** Let $\pi : \tilde{M} \to M$ be the universal cover of $M$, and $\tilde{T}$ the triangulation of $\tilde{M}$ induced from $T$. Let $V$ be the set of vertices of $\tilde{T}$.

We take any $\rho$-equivariant map $F : \pi^{-1}(V) \to S^2_\infty$ so that for any 3-simplex in $\tilde{T}$, the four points $\{F(v) \mid v \text{ is a vertex of } \sigma\}$ are distinct. The existence of such $F$ is guaranteed by condition \[2\]. Indeed, let $D \subset \tilde{M}$ be a fundamental domain of $M$ which is a union of $3$-simplices of $\tilde{T}$. Let $V = \{v_1, \ldots, v_{|V|}\}$ and $V_i = \pi^{-1}(v_i) \cap D$. We take a point $u_i \in V_i$ for each $i \in \{1, \ldots, |V|\}$. Then by the $\rho$-equivariance, $F(V_i)$ should be determined by $F(u_i)$. Namely, if $u_i' \in V_i$ with $u_i' = \gamma \cdot u_i$ for some $\gamma \in \pi_1(M)$, then $F(u_i') = \rho(\gamma)F(u_i)$. Let $e$ be an edge in $D$ such that both of its vertices $w_1, w_2$ are in $V_i$ for some $i \in \{1, \ldots, |V|\}$, i.e., $w_j = \gamma_j \cdot u_i$ for some $\gamma_j \in \pi_1(M), j \in \{1, 2\}$, we see that since by condition \[2\], $\rho([e]) \neq 1$, there are at most two $y \in S^2_\infty$ such that $\rho([e]) \cdot y = y$. Therefore, for a generic choice of $z \in S^2_\infty$,

$$\rho(\gamma_2) \cdot z = \rho([e]) \cdot (\rho(\gamma_1) \cdot z) \neq \rho(\gamma_1) \cdot z.$$

Since there are in total finitely many edges in $D$, for a generic choice of $(z_1, \ldots, z_{|V|}) \in (S^2_\infty)^{|V|}$, the map defined by

$$F(u_i) = z_i, \quad i \in \{1, \ldots, |V|\}$$

satisfies the property that for each edge $e$ of $D$ with vertices $w_1$ and $w_2$ such that $w_1, w_2 \in V_i$ for some $i \in \{1, \ldots, |V|\}$,

$$F(w_2) = \rho([e]) \cdot F(w_1) \neq F(w_1).$$

Furthermore, since there are only finitely many vertices in $D$, for a generic choice of $(z_1, \ldots, z_{|V|}) \in (S^2_\infty)^{|V|}$, the map $F$ satisfies that for each $e$ in $D$ with vertices $w_1$ and $w_2$,
\[ F(w_2) \neq F(w_1). \]

Therefore, for each 3-simplex \( \sigma \) in \( T \), the four points \( \{ F(v) \mid v \text{ is a vertex of } \sigma \} \) are distinct.

For any 3-simplex \( \sigma \) of \( T \), let \( \tilde{\sigma} \) in \( \tilde{T} \) be a lift of \( \sigma \). Then the four distinct points \( \{ F(v) \mid v \text{ is a vertex of } \sigma \} \) determine an ideal tetrahedron \( \sigma_\infty \) with them as vertices. We assign the shape parameters of \( \sigma_\infty \) to the corresponding normal quadrilaterals of \( \sigma \), and get an assignment \( Z_\rho \subset (\mathbb{C} \setminus \{0, 1\})^2 \) of complex numbers to the set of normal quadrilaterals in \( T \). We claim that \( Z_\rho \) is a solution to the hyperbolic gluing equation.

By Proposition 2.4, we see that \( Z_\rho \) satisfies (b) of Definition 2.1. The verification of (a) is a cross-ratio calculation which is exactly the same as in [5]. We include it here for the readers’ convenience. Let \( e \in E \), and \( \tilde{e} \) in \( \tilde{T} \) a lift of \( e \) with end points \( v \) and \( w \). Let \( \sigma_1, ..., \sigma_k \) be tetrahedra in \( \tilde{T} \) sharing \( \tilde{e} \) as an edge in a cyclic order, and \( q_i \subset \sigma_i \) be the normal quadrilateral facing \( \tilde{e} \). Let \( u_i \) and \( u_{i+1} \) be the other two vertices of \( \sigma_i \) so that \( u_i \in \sigma_{i-1} \cap \sigma_i \). We make a convention that \( u_{k+1} = u_1 \). Let \( \sigma_{i, \infty} \) be the ideal tetrahedron determined by vertices \( F(v) \), \( F(w) \), \( F(u_i) \) and \( F(u_{i+1}) \), and \( l \) be the geodesic connecting \( F(v) \) and \( F(w) \), then \( \{ \sigma_{i, \infty} \}_{i=1}^k \) share \( l \) as an edge in a cyclic order. Without loss of generality, we can assume that \( F(v) = 0 \) and \( F(w) = \infty \in S^2_\infty = \mathbb{C} \cup \{\infty\} \). Suppose \( z_i \) is the complex number assigned to \( q_i \), i.e., the shape parameter of \( \sigma_{i, \infty} \) at \( l \), then

\[
\prod_{q \in Q} \frac{e^{i(q,e)}}{z_q} = \prod_{i=1}^k z_i
\]
\[
= \prod_{i=1}^k (0, \infty; F(u_i), F(u_{i+1}))
\]
\[
= \prod_{i=1}^k \frac{F(u_i)}{F(u_{i+1})}
\]
\[
= 1,
\]

which verifies (a).

From the arbitrariness of \( F \), we see that there are uncountably many choices of \( Z_\rho \).

We call the solutions \( Z_\rho \in \mathcal{P}(M) \) constructed above the solutions from the spinning construction.

After we obtained the result, it was brought to our attention that similar construction had also appeared a little earlier in the work of Kasheve, Korepanov and Martyushev [6].

### 3.2 Spinning construction and the proof of 2. of Theorem 1.1

According to Thurston’s notes, Section 6.1, [10], any \( k+1 \) points \( v_0, ..., v_k, 1 \leq k \leq 3 \), in \( \mathbb{H}^3 \) determine a straightening map (or straight \( k \)-simplex) \( \sigma_{v_0, ..., v_k} : \Delta^k \to \mathbb{H}^3 \), whose image is the convex hull of \( v_0, ..., v_k \). Similarly, any \( k+1 \) points \( v_0, ..., v_k, 1 \leq k \leq 3 \), in \( S^2_\infty \) determine an ideal straightening map (or ideal straight \( k \)-simplex), see Section 2.2, [5] for details. The (ideal) straightening map is natural in the following sense (see also [7] for the proof).

**Proposition 3.1** 1. If \( \Delta' \) is an \( m \)-face of \( \Delta^k \) so that \( \sigma_{v_0, ..., v_k}(\Delta') \) has vertices \( v_{i_0}, ..., v_{i_m} \), then
\[
\sigma_{v_0, ..., v_k}|_{\Delta'} = \sigma_{v_{i_0}, ..., v_{i_m}}.
\]
2. If \( g \in Iso(\mathbb{H}^n) \), the group of isometries of \( \mathbb{H}^3 \), then

\[
g \circ \sigma_{v_0, \ldots, v_k} = \sigma_{g \cdot v_0, \ldots, g \cdot v_k}.
\]

To prove 2. of Theorem 1.1 we need the following technical Lemma whose proof is contained in [3].

**Lemma 3.2** Let \( \{ \sigma_i : \Delta^k \to \mathbb{H}^3 \ t \in \mathbb{R}_{\geq 0} \} \) be a family of straight \( k \)-simplices so that the \( i \)-th vertex \( v_{i,t} \) of \( \sigma_i \) lies in a geodesic ray \( l_i \), and \( v_{i,t} \) moves toward the end point \( v_i^* \) of \( l_i \) at unit speed, i.e., \( d(v_{i,t}, v_{i,t}) = t \). If \( v_{i,0}^*, \ldots, v_k^* \) are pairwise distinct, then as \( t \) tends to \( \infty \) the family \( \{ \sigma_i \} \) converges pointwise to an ideal straight \( k \)-simplex \( \sigma_{\infty} : \Delta^k \to \overline{\mathbb{H}^3} \) whose vertices are \( v_{0,\infty}^*, \ldots, v_k^* \).

**Proof of 2. of Theorem 1.1**

We take an arbitrary \( \rho \)-equivariant map \( f : \tilde{M} \to \mathbb{H}^3 \) and apply the following spinning construction. Let \( D \subset M \) be a fundamental domain of \( M \) which is a union of 3-simplices of \( \tilde{T} \) with some 0-, 1- and 2-faces removed so that \( \pi|_D : D \to M \) is one-to-one and onto, and \( T_D \) be the triangulation of \( D \) restricted from \( \tilde{T} \). Let \( V_D \) be the set of vertices of \( T_D \). For each \( v \in V_D \), let \( l_v \) be a geodesic in \( \mathbb{H}^3 \) passing through \( f(v) \) \( F(v) \in S^2_\infty \) as one of its end-points. We parameterize \( l_v : (-\infty, \infty) \to \mathbb{H}^3 \) so that \( l_v(0) = f(v) \), \( \|F'(t)\|_{\mathbb{H}^3} = 1 \), \( \forall t \in (-\infty, \infty) \), and \( l_v(t) \to F(v) \) as \( t \to +\infty \), and define a family of piecewise smooth \( \rho \)-equivariant maps \( f_t : \tilde{M} \to \mathbb{H}^3, t \in [0, \infty) \) as follows. Define

\[
f_t(v) = \exp(t \cdot l'_v(0)), \ \forall v \in V_D,
\]

and

\[
f_t(\gamma \cdot v) = \rho(\gamma) \cdot f_t(v), \ \forall \gamma \in \pi_1(M), v \in V_D.
\]

Extend \( f_t \) to the 1-, 2- and 3-simplices of \( \tilde{T} \) by straightening maps. By 1. of Proposition 3.1 \( f_t \) is well defined, and by 2. of Proposition 3.1 \( f_t \) is \( \rho \)-equivariant. From the definition of \( f_t \), we see that for each vertex \( \tilde{v} \) of \( \tilde{T} \), \( f_t(\tilde{v}) \) approaches to \( F(\tilde{v}) \in S^2_\infty \), and for each face \( \tilde{\sigma} \) of \( \tilde{T} \), \( f_t(\tilde{\sigma}) \) lies in a totally geodesic plane.

By Lemma 3.2, \( f_t : \tilde{M} \to \mathbb{H}^3 \) pointwise converges to a piecewise smooth \( \rho \)-equivariant map \( f_\infty : \tilde{M} \to \mathbb{H}^3 \) such that

1. \( \forall \tilde{v} \in \pi^{-1}(I), \ f_\infty(\tilde{v}) = F(\tilde{v}) \); and
2. \( f_\infty(\tilde{M} \setminus \pi^{-1}(V)) \subset \mathbb{H}^3 \).

Given the solution \( Z_\rho \) to the hyperbolic gluing equation, \( f_\infty \) can be regarded as the pseudo developing map described in Section 2.3 which gives rise to the associated representation. Tautologically the map \( f_\infty : \tilde{M} \setminus \pi^{-1}(V) \to \mathbb{H}^3 \) is \( \rho_{Z_\rho} \)-equivariant. Therefore, \( f_\infty \) is both \( \rho \) and \( \rho Z_\rho \)-equivariant, and for all \( \gamma \in \pi_1(M) \) and \( x \in \tilde{M} \setminus \pi^{-1}(V) \); and we have

\[
\rho_{Z_\rho}(\gamma) \cdot f_\infty(x) = f_\infty(\gamma \cdot x)
\]

i.e., \( \rho_{Z_\rho}(\gamma)| f_\infty(\tilde{M} \setminus \pi^{-1}(V)) = \rho(\gamma)| f_\infty(\tilde{M} \setminus \pi^{-1}(V)) \). It is clear that \( f_\infty(\tilde{M} \setminus \pi^{-1}(V)) \) contains more than four points. Indeed, for and 3-simplex \( \tilde{\sigma} \) in \( \tilde{T} \), the interior of the ideal tetrahedron \( f_\infty(\tilde{\sigma}) \) contains an open subset of a totally geodesic plane (generically, the interior of the ideal tetrahedron \( f_\infty(\tilde{\sigma}) \) is itself open in \( \mathbb{H}^3 \), and the only “bad” extremal case happens only if that for any \( \tilde{\sigma} \) in \( \tilde{T} \), \( f_\infty(\tilde{\sigma}) \) is flat). Therefore,

\[
\rho_{Z_\rho}(\gamma) = \rho(\gamma) \in \text{PSL}_2(\mathbb{C}), \ \forall \gamma \in \pi_1(M),
\]
i.e., \( \rho_{Z^\rho} = \rho : \pi_1(M) \to PSL_2(\mathbb{C}). \)

Let \( \mathcal{R}(M) \) be the set of representations \( \rho : \pi_1(M) \to PSL_2(\mathbb{C}) \). As we pointed out in the introduction, in the case that the triangulation \( \mathcal{T} \) is simplicial, every \( \rho \in \mathcal{R}(M) \) satisfies condition (2), and we have

**Theorem 3.3** If \( M \) is a closed, oriented 3-manifold, and \( \mathcal{T} \) is a simplicial triangulation of \( M \), then the map \( Y : \mathcal{P}(M) \to \mathcal{R}(M) \) defined by \( Y(Z) = \rho_{Z^\rho} \) is surjective.

By 2. of Theorem 1.1, we see that our construction is the inverse construction of Yoshida’s as described in Section 2.3, and we have

**Theorem 3.4** If \((M, T, \rho)\) satisfies the condition of Theorem 1.1, then all the solutions \( Z^\rho \) of the hyperbolic gluing equation such that \( \rho_{Z^\rho} \) is conjugate to \( \rho \) are from the spinning construction.

### 3.3 Continuous extension of the volume function and the proof of 3. of Theorem 1.1

Given a hyperbolic 3-simplex \( \sigma \) with vertices \( v_1, \ldots, v_4 \), the \( i \)-th face is defined to be the 2-simplex facing \( v_i \). The dihedral angle between the \( i \)-th and \( j \)-th faces is denoted by \( a_{ij}(\sigma) \). As a convention, we define \( a_{ii}(\sigma) = \pi \), and call the symmetric matrix \([a_{ij}(\sigma)]_{6 \times 6}\) the angle matrix of \( \sigma \). It is well known that the angle matrix \([a_{ij}(\sigma)]_{6 \times 6}\) determines \( \sigma \) up to isometry.

To prove 3. of Theorem 1.1 we need the following theorem of Luo \[3\].

**Theorem 3.5 (Luo)** Let \( X \subset \mathbb{R}^{6 \times 6} \) be the space of angle matrices of all hyperbolic 3-simplices. The volume function \( V : X \to \mathbb{R} \) can be extended continuously to the closure of \( X \) in \( \mathbb{R}^{6 \times 6} \).

We point out that Luo’s original result, Theorem 1.1 in \[3\], is more general than Theorem 3.5. It covers the cases of simplices in arbitrary dimensions, and in both hyperbolic and spherical geometry. See \[3\] for details.

**Proof of 3. of Theorem 1.1**

Let \( Z^\rho \) be a solution to the hyperbolic gluing equation constructed in Section 3.1. By Definition 2.2,

\[
Vol(Z^\rho) = \sum_{\sigma \in T} Vol(z^\rho, \sigma) = \sum_{\sigma \in T_0} Vol_{\mathbb{H}^3}(f_\infty(\sigma))
\]

Since \( f_t : \tilde{M} \to \mathbb{H}^3 \) constructed in Section 3.2 is \( \rho \)-equivariant, \( \forall t \in [0, +\infty) \), as defined by Dunfield in Section 2.5 of \[1\],

\[
Vol(\rho) = \int_D f_t^*(dVol_{\mathbb{H}^3}) = \sum_{\sigma \in T_0} \int_{\sigma} f_t^*(dVol_{\mathbb{H}^3}) = \sum_{\sigma \in T_0} Vol_{\mathbb{H}^3}(f_t(\sigma)), \quad \forall t \in [0, +\infty).
\]
For any 3-simplex $\sigma \in T_D$, since $f_t|_\sigma$ pointwise converges to $f_\infty|_\sigma$, the angle matrices $[a_{ij}(f_t(\sigma))]_{6 \times 6}$ converges to $[a_{ij}(f_\infty(\sigma))]_{6 \times 6} \in \mathbb{X}$. By Theorem 3.5,

$$Vol_{H^3}(f_\infty(\sigma)) = \lim_{t \to +\infty} Vol_{H^3}(f_t(\sigma)).$$

Therefore,

$$Vol(Z_\rho) = \sum_{\sigma \in T_D} Vol_{H^3}(f_\infty(\sigma))$$
$$= \sum_{\sigma \in T_D} \lim_{t \to +\infty} Vol_{H^3}(f_t(\sigma))$$
$$= \lim_{t \to +\infty} \sum_{\sigma \in T_D} Vol_{H^3}(f_t(\sigma))$$
$$= \lim_{t \to +\infty} Vol(\rho)$$
$$= Vol(\rho)$$

Acknowledgment

Research of the author is partially supported by the NSF. The author would like to thank Feng Luo and Stephan Tillmann for useful discussions, Feng Luo for encouraging him to write up this result, and Stephan Tillmann for pointing out an error in an earlier draft, sharing important ideas and providing instructive suggestions generously.

References

[1] N. Dunfield, *Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds*, Invent. Math. 136 (1999), no. 3, 623–657.

[2] S. Francaviglia, *Hyperbolic volume of representations of fundamental groups of cusped 3-manifolds*, Int. Math. Res. Not. (2004), no. 9, 425–459.

[3] F. Luo, *Continuity of the volume of simplices in classical geometry*, Commun. Contemp. Math. 8 (2006), no. 3, 411–431.

[4] F. Luo, *Triangulated 3-manifolds: from Haken’s normal surfaces to Thurston’s algebraic equation*, arXiv:1003.4413v1.

[5] F. Luo, S. Tillmann and T. Yang, *Thurston’s spinning construction and the existence of solutions to Thurston’s hyperbolic gluing equations for closed hyperbolic 3-manifolds*, arXiv:math/1004.2992 v1

[6] R. Kashaev, I. Korepanov and E. Martyushev *A finite-dimensional TQFT for three-manifolds based on group PSL(2, C) and cross-ratios*, arXiv:0809.4239v1

[7] H. Segerman, *On spun-normal and twisted squares surfaces*, Proc. Amer. Math. Soc. 137 (2009), no.12, 4259–4273.

[8] S. Tillmann, *Degenerations of ideal hyperbolic triangulations* arXiv:math/0508295

[9] W.P. Thurston, *Hyperbolic structures on 3-manifolds I: Deformation of acylindrical manifolds*, Ann. of Math. (2) 124 (1986), no.2,203–246.

[10] W.P. Thurston, *Three-dimensional geometry and topology*, 1979-1981, http://www.msri.org/publications/books/gt3m/
[11] T. Yoshida, *On ideal points of deformation curves of hyperbolic 3-manifolds with one cusp*, Topology 30 (1991), no. 2, 155–170.

Tian Yang  
Department of Mathematics, Rutgers University  
New Brunswick, NJ 08854, USA  
(tianyang@math.rutgers.edu)