Exact Solution of the general Non Intersecting String Model

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We present a thorough analysis of the Non Intersecting String (NIS) model and its exact solution. This is an integrable $q$-states vertex model describing configurations of non-intersecting polygons on the lattice. The exact eigenvalues of the transfer matrix are found by analytic Bethe Ansatz. The Bethe Ansatz equations thus found are shown to be equivalent to those for a mixed spin model involving both $1/2$ and infinite spin. This indicates that the NIS model provides a representation of the quantum group $SU(2)_q$ ($|q| \neq 1$) corresponding to spins $s = 1/2$ and $s = \infty$. The partition function and the excitations in the thermodynamic limit are computed.

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1. Introduction

The Yang-Baxter (YB) equations assure integrability in two-dimensional lattice models and quantum field theories. The YB equations have the form

\[ R_{12}(\theta' - \theta)R_{13}(\theta)R_{23}(\theta') = R_{23}(\theta')R_{13}(\theta)R_{12}(\theta' - \theta) , \quad (1.1) \]

where the \( R \)-matrix elements \( R_{ab}^{cd}(\theta) \) (with \( 1 \leq a, b, c, d \leq q, \ q \geq 2 \)) define the statistical (or Boltzman) weights for a vertex model in two dimensions. The general Non Intersecting String (NIS) model is a \( q(2q-1) \) vertex model introduced in [1] as a solution of the Yang-Baxter equations. It generalizes the \( q = 2 \) ferroelectric models and the \( q = 3 \) model of Stroganov [2].

The non-zero Boltzmann weights of the model are depicted in Fig.1. We distinguish between indices of two different kinds that we shall call “red” and “black”, corresponding to dashed and solid lines respectively in the figure. The red indices, denoted by \( \alpha, \beta, \ldots \), take on \( r \) values, whereas the black ones, denoted by \( \mu, \nu, \ldots \), take on the remaining \( t \) values, with \( r + t = q > 2 \). Latin indices \( a, b, \ldots \) run over all the possible values \( 1, \ldots, q \). The values of the \( R \)-matrix elements in Fig.1 are:

\[
\begin{align*}
R_{\alpha\alpha}^{\alpha\alpha}(\theta) &= \sinh \theta + \sinh(\eta - \theta) , \\
R_{\mu\mu}^{\mu\mu}(\theta) &= \sinh \eta , \\
R_{\alpha\beta}^{\alpha\beta}(\theta) &= \sinh(\eta - \theta) , \quad \alpha \neq \beta , \\
R_{\mu\nu}^{\mu\nu}(\theta) &= e^\theta \sinh(\eta - \theta) , \quad \mu \neq \nu , \\
R_{\alpha\mu}^{\alpha\mu}(\theta) &= e^{\theta/2} \sinh(\eta - \theta) , \\
R_{cd}^{ab}(\theta) &= R_{ca}^{db}(\eta - \theta) .
\end{align*}
\]

(1.2)

The Boltzmann weights of the NIS model, like the six-vertex ones, depend on the spectral parameter \( \theta \) and on the anisotropy parameter \( \eta \) but, differently from the six-vertex case, \( \eta \) is not continuous but takes on discrete values, solutions of the
second order equation [1]

\[(q - r - 1)e^{2\eta} + re^{\eta} - 1 = 0 \quad . \tag{1.3}\]

We have then the two solutions $\eta$ and $\eta' + i\pi$ given by:

\[e^{\eta} = \frac{\sqrt{r^2 + 4(q - r - 1)} - r}{2(q - r - 1)} < 1 \tag{1.4}\]

with the limiting case $e^{\eta} = 1/r$ for $q = r + 1$ and

\[e^{\eta'} = \frac{\sqrt{r^2 + 4(q - r - 1)} + r}{2(q - r - 1)}. \tag{1.5}\]

We see that $\eta' \geq 0$ for $q \leq 2(r + 1)$ and $\eta' < 0$ for $q > 2(r + 1)$. It must be noticed that although the $R$-matrix becomes complex for the solution $\eta' + i\pi$, the imaginary factors cancel out in the partition function for any finite size. For simplicity, in the following we shall consider the solution (1.4) which corresponds to real negative $\eta$ and real $R$-matrix.

By definition, (see last equation in (1.2)) the model is crossing invariant and $\eta$ is the crossing parameter. As it appears clear from Fig.1, the NIS model describes configurations of non-intersecting patterns of polygons on the square lattice. For an $N$-sites system, the spin Hamiltonian associated to the NIS model can be written as

\[
H = \sum_{j=1}^{N} \left[ \sum_{a,b=1}^{q} \exp \left( \frac{n(a,b)}{2} \right) e^{ab}_j \otimes e^{ab}_{j+1} + \sum_{a,b=1}^{q} \left( \frac{n(a,b)}{2} \sinh \eta - \cosh \eta \right) e^{aa}_j \otimes e^{bb}_{j+1} \right. \\
\left. - \sinh \eta \sum_{a=1}^{q} n(a,a) e^{aa}_j \otimes e^{aa}_{j+1} \right],
\]

where $n(a,b)$ is the number of black indices among $a$ and $b$ ($n(a,b) = 0, 1$ or $2$) and $(e^{ab})_{cd} = \delta_{ac}\delta_{bd}$.
Several papers dealing with the special case $t = 0$, also called the “separable vertex model”, already appeared in the literature [1-5]. In particular, $r = q = 3$ corresponds to the model considered by Stroganov [2]. Equivalence between the separable NIS model with a $q^2$-Potts model has been established in [4]. Analytical continuation to $-2 < q < 2$ of the separable case corresponds to the scaling limit of the $O(n)$ vector model with $-2 < n < 2$ away from the critical point [5]. As it is well known, the partition function of integrable vertex models can be used for computing knot invariants [6]. The separable NIS model leads then to the Jones polynomial [7].

In this paper we find the exact solution of the NIS model in its full generality ($t \neq 0$) by means of analytic Bethe Ansatz (BA). We show that with periodic boundary conditions on a $N$-sites lattice ( $N$ even), the eigenvalues of the transfer matrix $\tau(\theta)$ of the general NIS model have the form

$$\Lambda(\theta, \vec{\lambda}) = F(\theta) \sinh^N(\eta - \theta) e^{\theta B} + \overline{F(\eta - \theta)} \sinh^N\theta e^{(n-\theta)B} ,$$

where $B$ is an integer ($0 \leq B \leq N$), $\vec{\lambda} = (\lambda_1, \ldots, \lambda_m)$ and

$$F(\theta) = e^{i\omega} \prod_{j=1}^m \frac{\sin(\lambda_j - i\theta - i\eta/2)}{\sin(\lambda_j - i\theta + i\eta/2)} .$$

The numbers $\lambda_j$ in $F(\theta)$ are obtained by solving the BA equations which read, for the NIS model, (see Sect.3):

$$\left( \frac{\sin(\lambda_k + i\eta/2)}{\sin(\lambda_k - i\eta/2)} \right)^N e^{2i(\lambda_k B - \omega)} = \prod_{i=1, i \neq k}^m \frac{\sin(\lambda_k - \lambda_i + i\eta)}{\sin(\lambda_k - \lambda_i - i\eta)} , \quad 1 \leq k \leq m . \quad (1.6)$$

Here $\omega$ is a real phase. We conjecture (see Secs.3 and 6 for more details and a proof in the limit $N \to \infty$) that $e^{i\omega}$ is in fact a root of unity i.e.

$$e^{i\omega l} = 1$$

for some integer $l$. The eigenvectors of $\tau(\theta)$ are thus labelled by $B$, $\omega$ and $m$.

The antiferromagnetic ground state is obtained for $B = 0$, $m = N/2$ and $\omega = 0$. The
The corresponding eigenvalue of $\tau(\theta)$ coincides exactly with the antiferromagnetic ground state of the six-vertex model with the identification $\gamma = -\eta$. Notice that the BA equations (1.6) are identical to those of the six-vertex model if $\gamma = -\eta$, except for the phase factor $e^{2i(\lambda_k B - \omega)}$.

We recall that for a spin-$s$ model the L.H.S. of the BA equations is given by

$$
\left(\frac{\sin(\lambda_k + is\gamma)}{\sin(\lambda_k - is\gamma)}\right)^{N_s},
$$

whose $s \rightarrow \infty$ limit is $\exp(-2i\lambda_k N_s)$. Moreover, as shown in ref. [9], integrable models with mixed spins can be constructed and solved by BA. This evidence suggested us that a deep link exists between the NIS model and a model containing both spins 1/2 and infinite spins. We are in fact able to prove that the BA equations and the eigenvalues of the NIS model coincide with those of a model containing $N$ spins 1/2 and $N_s$ spins $s$ in the limit in which $s$ is infinite. The proof of this equivalence is given in Sec. 6 in the limit $N, N_s \rightarrow \infty$. As a byproduct we find that the phases $e^{i\omega}$ are roots of unity. In this context, the discrete parameter $B$ turns out to be the contribution of the $\infty$-spins to the total $z$-component of the spin (normalized to one) for each BA state.

It is known that it is possible to associate a family of solutions of the YB algebra (1.1) to each Lie algebra $G$ [8]. Trigonometric (hyperbolic) solutions usually provide highest weight or cyclic representations of the corresponding quantum group $G_\hat{q}$. Here $\hat{q} = e^{i\gamma} (e^\gamma)$ where $\gamma$ is the anisotropy parameter in the $R$-matrix. As the discussion above indicates, the NIS model should provide a new representation of the quantum group $SU(2)_\hat{q}$ with $\hat{q} = e^\eta$ (notice that $|\hat{q}| \neq 1$). This representation is absent in the “classical” case $\hat{q} = 1$ and is probably indecomposable containing $s = 1/2$ and $s = \infty$.

The plan of the paper is as follows. In Sec. 2 we summarize the general properties of the NIS $R$-matrix and transfer matrix. In Sec. 3 we construct the BA solutions of the NIS model through analytic BA. Sec. 4 contains checks for
large values of the spectral parameter $\theta$ of the results obtained by BA. In Sec. 5 we find the free energy and the excitations in the thermodynamic limit. Finally, in Sec. 6, the connection between the NIS model and a spin 1/2-spin $\infty$ model is derived.

2. General properties of the NIS model

For the general NIS model we find, by direct computation, the following properties for the $R$ matrix

a) Regularity at $\theta = 0$

$$R(0) = \sinh \eta \mathbb{I} \ . \quad (2.1)$$

b) Unitarity

$$R(\theta)R(-\theta) = \rho(\theta) \mathbb{I} \ , \quad (2.2)$$

where $\rho(\theta) = \sinh(\eta - \theta) \sinh(\eta + \theta)$

c) Quasi periodicity

$$R(\theta + i\pi) = (u \otimes 1)R(\theta)(u^{-1} \otimes h) \ , \quad (2.3)$$

where $u$ and $h$ are the diagonal matrices $u_{ab} = u_a \delta_{ab}$ and $h_{ab} = h_a \delta_{ab}$ with

$$u_\alpha = 1 \ , \quad u_\mu = i \ , \quad \begin{cases} 1 \leq \alpha \leq r \\ r + 1 \leq \mu \leq q \end{cases}$$

$$h_\alpha = -1 \ , \quad h_\mu = 1 \ .$$

d) Invariances. A Yang-Baxter algebra is said to be invariant under a transformation group $G$ if the $R$-matrix satisfies [11]

$$[g \otimes g , R(\theta)] = 0 \ , \quad \forall \theta \in \mathbb{C} , \quad \forall g \in G .$$

In the separable case, $r = q$, from eqs.(1.2) it follows

$$gg^T = 1 \ ,$$

that is we have invariance under the orthogonal group.
In the other limiting case in which all the indices are of the black kind \((r = 0)\), we find
\[
(g g^T)_{\mu\mu} = 1, \quad \forall \mu
\]
and
\[
e^{-(\theta - \eta)} \sum_{\sigma \neq \tau} g_{\mu\sigma} g_{\nu\sigma} + e^{\theta - \eta} g_{\mu\tau} g_{\nu\tau} = 0, \quad \mu \neq \nu
\]
which, for the linear independence of the factors \(e^{\pm(\theta - \eta)}\) yields
\[
g_{\mu\rho} g_{\nu\rho} = 0, \quad \mu \neq \nu.
\]
Thus \(\mathcal{G}\) is formed by matrices \(g\) with only one element different from zero, equal to \(\pm 1\), in each row and column. The symmetry group is in this case \(S_q \times (\mathbb{Z}_2)^q\), the direct product of \(S_q\), the symmetric group of \(q\) elements, and \((\mathbb{Z}_2)^q\).

In the general case \((r \neq 0, q)\), there is no symmetry mixing the black and red sectors of the theory. Therefore the symmetry group is simply given by the direct product
\[
\mathcal{G} = O(r) \times S_t \times (\mathbb{Z}_2)^t,
\]
where \(t = q - r\) is the number of black indices.

We now turn our attention to some properties of the transfer matrix \(\tau(\theta)\) which, for a \(N\)-sites system, is defined as
\[
\tau(\theta) = \text{Tr}_A (R_{A1}(\theta) R_{A2}(\theta) \ldots R_{AN}(\theta)).
\]
From property (2.3) of quasiperiodicity one gets for the transfer matrix
\[
\tau(\theta + i\pi) = \text{Tr}_A \left( u_A R_{A1}(\theta) u_A^{-1} h_1 u_A R_{A2}(\theta) u_A^{-1} h_2 \ldots u_A R_{AN}(\theta) u_A^{-1} h_N \right)
\]
\[
= \tau(\theta) \mathcal{H},
\]
with \(\mathcal{H} = \bigotimes_{j=1}^N h_j\). Next we note that the matrix \(h\) is a symmetry of \(R(\theta)\)
(property d) and that \((hu)^{-1} = u^{-1}h\) then

\[ R_{12}(\theta + i\pi) = (hu \otimes h)R_{12}(\theta)(u^{-1}h \otimes 1) . \]

So that

\[
\tau(\theta + i\pi) = \text{Tr}_A \left[ (hu)_A h_1 R_{A1}(\theta)(u^{-1}h)_A (hu)_A h_2 R_{A2}(\theta)(u^{-1}h)_A \ldots \\
\times (hu)_A h_N R_{AN}(\theta)(u^{-1}h)_A \right] = \mathcal{H} \tau(\theta) .
\]

Thus,

\[
\tau^2(\theta + i\pi) = \tau(\theta)\mathcal{H}^2\tau(\theta) = \tau^2(\theta)
\]

since \(\mathcal{H}^2 = 1\). This shows that configurations on the square lattice with periodic boundary conditions of the NIS model are invariant under the change \(\theta \to \theta + i\pi\).

The key property for the applicability of the analytic BA technique to the general NIS model is the inversion relation satisfied by the transfer matrix, namely

\[
\tau(\theta + \eta) \tau(\theta) = [\sinh(\eta + \theta) \sinh(\eta - \theta)]^N + \sinh^N \theta \tilde{\tau}(\theta) ,
\]

where \(\tilde{\tau}(\theta)\) is a matrix regular at \(\theta = 0\). In order to prove (2.5) it is convenient to switch to the \(S\)-matrix, which is related to the \(R\)-matrix by \(S_{ab}(\theta) = R_{ab}(\theta)\).

We notice that, from eq. (2.1) and crossing invariance we have

\[
S_{ab}(\eta) = \sinh \eta \delta_b^a \delta_d^c ;
\]

so \(S(\eta)\) is proportional to

\[
\mathcal{P} = \frac{1}{q} \sum_{a,b=1}^{q} \langle aa \rangle \langle bb \rangle
\]

which is a projector onto a one-dimensional subspace. The Yang-Baxter equation
for the $S$-matrix reads

$$S_{12}(\theta' - \theta)S_{13}(\theta)S_{23}(\theta') = S_{23}(\theta')S_{13}(\theta)S_{12}(\theta' - \theta) .$$  \tag{2.6}$$

Setting $\theta' - \theta = \eta$ we get

$$\mathcal{P}_{12}S_{13}(\theta)S_{23}(\eta + \theta) = S_{23}(\eta + \theta)S_{13}(\theta)\mathcal{P}_{12}$$

which, left multiplied by $\mathcal{P}_{12}^\perp = 1 - \mathcal{P}_{12}$, yields

$$\mathcal{P}_{12}S_{13}(\theta)S_{23}(\eta + \theta)\mathcal{P}_{12}^\perp = 0 .$$

Therefore, the matrix $S_{13}(\theta)S_{23}(\eta + \theta)$ takes the block triangular form

$$S_{13}(\theta)S_{23}(\eta + \theta) = \begin{pmatrix} A_3(\theta) & 0 \\ * & B_3(\theta) \end{pmatrix}$$  \tag{2.7}$$

in the basis where $\mathcal{P}_{12}$ has the canonical form

$$\mathcal{P}_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$  \tag{2.8}$$

The matrix notation in eqs. (2.7) and (2.8) refers to the tensor product $\mathcal{V}_1 \times \mathcal{V}_2$ of the vector spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ so that the matrix elements are matrices in the vector space $\mathcal{V}_3$. From eq. (2.7) we can determine $A(\theta)$ which is given by

$$\mathcal{P}_{12}S_{13}(\theta)S_{23}(\eta + \theta)\mathcal{P}_{12} = A_3(\theta)\mathcal{P}_{12} .$$

The $S$-matrix defined by the $R$-matrix elements in eqs. (1.2) is time-reversal invariant

$$S(\theta) = S(\theta)^T .$$

By using time reversal, unitarity [eq.(2.2)], crossing invariance $[S_{12}(\theta)T_2 = S_{12}(\eta - \theta)]$ of $S(\theta)$ and from the explicit form of $\mathcal{P}$ we get

$$A(\theta) = \rho(\theta) \mathbb{1} .$$

This shows that the matrix $A(\theta)$ is diagonal.
Analogously, we can show that the matrix $B(\theta)$ is zero at $\theta = 0$. From eq. (2.7) we have
\[ P_{12}^\perp S_{13}(0)S_{23}(\eta)P_{12}^\perp = B_3(0)P_{12}^\perp . \tag{2.9} \]
As consequence of crossing and regularity, it is
\[ S_{13}(0)S_{23}(\eta) \propto P_{13}P_{23} , \]
where $P_{13}$ is the exchange operator in the spaces 1 and 3:
\[ P_{13} = \sum_{a,b=1}^q |ab\rangle \langle ba| . \]

It is then a matter of straightforward algebra to show that the LHS, and therefore $B(0)$, in (2.9) is zero. With no loss of generality, we can then rewrite eq. (2.7) in the form
\[ S_{13}(\theta)S_{23}(\eta + \theta) = \begin{pmatrix} \rho(\theta) & 0 \\ * & \sinh \theta \tilde{S}_3(\theta) \end{pmatrix} , \tag{2.10} \]
where $\tilde{S}_3(\theta)$ is a $(q^2 - 1) \times (q^2 - 1)$ dimensional matrix in $\mathcal{V}_1 \times \mathcal{V}_2$ regular at $\theta = 0$ whose elements are matrices in $\mathcal{V}_3$.

Consequently
\[ \tau(\theta)\tau(\eta + \theta) = \text{Tr}_{AB} \left( S_{A_1}(\theta)S_{B_1}(\eta + \theta)S_{A_2}(\theta)S_{B_2}(\eta + \theta) \ldots S_{A_N}(\theta)S_{B_N}(\eta + \theta) \right) \\
= \rho(\theta)^N + \sinh^N \theta \text{Tr}_{AB} \left( \tilde{S}_1(\theta) \ldots \tilde{S}_N(\theta) \right) , \]
with $\rho(\theta)$ given in eq. (2.2). This proves eq. (2.5).
3. Analytic Bethe Ansatz (BA) for the general NIS model

We start by constructing a family of eigenstates for the transfer matrix that we shall call reference states. Although these states provide only a small portion of the spectrum of $\tau(\theta)$, their knowledge is an essential ingredient for the formulation of the analytic BA which, in turn, will generate the whole spectrum of $\tau(\theta)$.

The action of $\tau(\theta)$ on a general vector $|a_1, \ldots, a_N\rangle$, is given by

$$
\tau(\theta)|a_1, \ldots, a_N\rangle = \sum_{\{c\},\{b\}=1}^{q} R_{b_1c_2}^{c_1a_1}(\theta) R_{b_2c_3}^{c_2a_2}(\theta) \ldots R_{b_Nc_1}^{c_Na_N}(\theta) |b_1, \ldots, b_N\rangle .
$$

If we restrict our attention to those states such that $a_{i+1} \neq a_i$ for $i = 1, \ldots, N$, then eq.(3.1) takes a particularly simple form. Indeed, in (3.1) let us consider the following two possibilities: (i) $c_1 \neq a_1$ and (ii) $c_1 \neq a_N$. In the case (i), $R_{b_1c_2}^{c_1a_1}(\theta)$ is non-zero only if $b_1 = c_1$ and $c_2 = a_1$. This implies that in the next factor $R_{b_2c_3}^{c_2a_2}(\theta)$, since $a_1 \neq a_2$, we must have $b_2 = a_1$ and $c_3 = a_2$ and so on, hence the sum in (3.1) reduces to just one term:

$$
\prod_{i=1}^{N} R_{a_{i}a_{i+1}}^{a_{i+1}a_{i}}(\theta) |a_N, a_1, a_2, \ldots, a_{N-1}\rangle .
$$

Here and in the following we set $a_{N+1} = a_i$.

If instead $c_1 \neq a_N$, then $c_N = a_N$ and $c_1 = b_N$ in $R_{b_Nc_1}^{c_Na_N}(\theta)$. In a way similar to case (i), this time moving backwards in the product of weight factors in eq.(3.1), we get

$$
\prod_{i=1}^{N} R_{a_{i+1}a_{i+1}}^{a_{i}a_{i+1}}(\theta) |a_2, \ldots, a_N, a_1\rangle .
$$

As we have just seen, case (ii) implies $a_1 = c_1$. Thus, the two cases considered
above exhaust all the possibilities. The total result is then
\[
\tau(\theta) |a_1, \ldots, a_N\rangle = \prod_{i=1}^{N} R_{a_i a_{i+1}}^{a_i a_{i+1}}(\theta) |a_N, a_1, a_2, \ldots, a_{N-1}\rangle \\
+ \prod_{i=1}^{N} R_{a_{i+1} a_{i+1}}^{a_i a_{i+1}}(\theta) |a_2, \ldots, a_N, a_1\rangle .
\] (3.2)

Now, the value of the weights \( R_{ab}^{ab}(\theta) = R_{bb}^{aa}(\eta - \theta) \) depends on whether \( a \) and \( b \) are black or red indices. From eqs.(1.2) we find
\[
R_{ab}^{ab}(\theta) = \sinh(\eta - \theta) e^{\frac{\theta}{2} n(a,b)},
\]
where \( n(a,b) \) is the number of black indices among \( a \) and \( b \) (\( n(a,b) = 0, 1 \) or 2). Thus, by taking into account that each index appears twice in the products, we obtain
\[
\prod_{i=1}^{N} R_{a_i a_{i+1}}^{a_i a_{i+1}}(\theta) = \sinh(N (\eta - \theta)) e^{\theta B}, \quad \prod_{i=1}^{N} R_{a_{i+1} a_{i+1}}^{a_i a_{i+1}}(\theta) = \sinh N \theta e^{(\eta - \theta)B}
\]
where \( B \) is the number of black indices in the state \( |a_1, \ldots, a_N\rangle \). Since the effect of applying \( \tau(\theta) \) amounts, apart from an overall factor, to shifting the indices \( a_i \) to the left or to the right, eigenstates of \( \tau(\theta) \) can be obtained by constructing “plane waves” of the form
\[
\Psi(\varepsilon^j_L; a_1, \ldots, a_N) = \sum_{k=0}^{L-1} \varepsilon^j_k |a_{1+k}, \ldots, a_{N+k}\rangle, \quad a_{i+1} \neq a_i \quad a_{N+i} = a_i \quad (3.3)
\]
with \( \varepsilon_L = e^{\frac{2\pi i}{L}} \). In eq.(3.3) \( L \) is the smallest integer such that \( |a_{L+1}, \ldots, a_{L+N}\rangle = |a_1, \ldots, a_N\rangle \) and \( 1 \leq j < L \). Depending on the choice of the indices \( a_1, \ldots a_N \), \( L \) takes the values \( 2 \leq L \leq N \), with \( N/L \) integer. These states are eigenvectors
of $\tau(\theta)$ with eigenvalue

$$
\Lambda(\varepsilon^j_L, \theta) = \varepsilon^j_L \sinh^N(\eta - \theta) e^{\theta B} + \varepsilon^{-j}_L \sinh^N \theta e^{(\eta - \theta) B} \tag{3.4}
$$

and they provide a basis in the sector of the Hilbert space spanned by vectors $|a_1, \ldots, a_N\rangle$ with $a_{i+1} \neq a_i$. Indeed

$$
|a_1, \ldots, a_N\rangle = \frac{1}{L} \sum_{j=1}^L \Psi(\varepsilon^j_L; a_1, \ldots, a_N)
$$

owing to $\sum_{j=1}^L \varepsilon^j_L = 0$. Notice that the eigenvalues (3.4) are highly degenerate since they do not depend on the particular choice of $|a_1, \ldots, a_N\rangle$ as long as $B$ is left unchanged.

Clearly, $B \equiv 0$ in the separable case. In the case $r = 0$, it is still possible to build reference states for which the eigenvalues are of the form (3.4) with $B \neq N$ by forming suitable linear combinations of states with one or more couples of equal indices. For example starting from states of the form

$$
|\mu \rho_3 \ldots \rho_N\rangle , \tag{3.5}
$$

with $\rho_i \neq \rho_{i+1}$ and $\mu \neq \rho_3, \rho_N$, as before we form the plane waves

$$
|\varepsilon^j_N; \mu\rangle = \sum_{k=0}^{N-1} \varepsilon^{jk}_N |\rho_{1+k}, \ldots, \rho_{N+k}\rangle , \tag{3.6}
$$

$$
\rho_i \neq \rho_{i+1} \quad \text{if} \quad i \neq 1 , \quad \rho_1 = \rho_2 = \mu , \quad \rho_{N+i} = \rho_i .
$$

Then, states of the kind

$$
\Psi(\varepsilon^j_N; \mu, \nu) = |\varepsilon^j_N; \mu\rangle - |\varepsilon^j_N; \nu\rangle
$$

are eigenstates of the transfer matrix with eigenvalues of the form (3.4) with $B = N - 2$. By taking appropriate linear combinations of plane waves involving states
with two and more couples of equal neighboring indices, we span all the possible even values of $B$ from $N$ to 0 ($N$ is supposed to be even). This also shows that, in the case where only black indices are allowed, $B$ is bound to take on even values.

We are now in the position to formulate the analytic BA for the general NIS model. Eqs. (2.5), (2.4) and crossing invariance of $\tau(\theta)$ imply the following properties for the eigenvalues $\Lambda(\theta)$ of $\tau(\theta)$:

\[ \Lambda(\theta + \eta) \Lambda(\theta) = \sinh^N(\eta + \theta) \sinh^N(\eta - \theta) + \sinh^N \theta \tilde{\Lambda}(\theta) , \quad (3.7) \]

\[ \Lambda(\theta + i\pi)^2 = \Lambda(\theta)^2 , \quad (3.8) \]

\[ \overline{\Lambda(\theta)} = \Lambda(\eta - \theta) , \quad (3.9) \]

where $\tilde{\Lambda}(\theta)$ is a regular function at $\theta = 0$.

Starting from expression (3.4) of the reference eigenvalues, we propose for the $\Lambda(\theta)$ the general analytical form

\[ \Lambda(\theta) = F(\theta) \sinh^N(\eta - \theta) e^{\theta B} + G(\theta) \sinh^N \theta e^{(\eta - \theta)B} , \quad (3.10) \]

where $B$ is an integer ($0 \leq B \leq N$) and $F(\theta)$ and $G(\theta)$ are functions to be determined. As previously discussed, $B \equiv 0$ in the separable case and $B$ even if $r = 0$. From the explicit expression of the $R$-matrix elements and eq.(3.8) it follows that $\Lambda(\theta)$ must be a polynomial in $e^\theta$ of degree at most $2N$ and of definite parity under the change $\theta \to \theta + i\pi$. Therefore with no loss of generality we can take $F(\theta)$ and $G(\theta)$ to be rational functions of $e^{2\theta}$ with the same number of zeros and poles. Moreover, in order to obtain $\Lambda(\theta)$ entire in $e^\theta$, the poles of $F(\theta)$ and $G(\theta)$ must have the same location, so that the corresponding singular contributions may be made to cancel. This last requirement is what will give us
the Bethe-Ansatz equations. Summarizing we look for $F(\theta)$ and $G(\theta)$ of the form

$$F(\theta) = e^{\phi} \prod_{i=1}^{m} \frac{e^{2\theta} - z_i}{e^{2\theta} - \xi_i}, \quad (3.11)$$

$$G(\theta) = e^{\psi} \prod_{i=1}^{m} \frac{e^{2\theta} - \zeta_i}{e^{2\theta} - \xi_i}, \quad (3.12)$$

where $\phi, \psi, z_i, \zeta_i$ and $\xi_i$ are complex parameters to be determined. The expression (3.10) and the form of $F(\theta)$ and $G(\theta)$ shows that the parity of $\Lambda(\theta)$ under the change $\theta \to \theta + i\pi$ is entirely determined by the choice of $B$.

From eqs. (3.7), (3.8) and (3.9) it follows

$$F(\theta) G(\theta - \eta) = 1, \quad G(\theta) = F(\eta - \theta) \quad (3.13)$$

or

$$F(\theta) F(-\theta) = 1. \quad (3.14)$$

By substituting the expression (3.11) into eq. (3.14) we obtain

$$e^{2\text{Re}\phi} \prod_{i=1}^{m} \frac{e^{2\theta} - z_i}{e^{2\theta} - \xi_i} = \prod_{i=1}^{m} \frac{1 - e^{2\theta} \bar{\xi}_i}{1 - e^{2\theta} z_i}. \quad (3.15)$$

Zeros and poles on both sides of eq. (3.15) coincide if $\bar{\xi}_i z_i = 1$. Equating the residues at the poles of both members of eq. (3.15) imposes

$$e^{2\text{Re}\phi} = \prod_{i=1}^{m} |\xi_i|^2.$$

On the other hand eqs. (3.13) give

$$e^{\psi} \prod_{i=1}^{m} \frac{e^{2\theta} - \zeta_i}{e^{2\theta} - \xi_i} = e^{-\phi} \prod_{i=1}^{m} \frac{e^{2\theta - 2\eta} - \zeta_i}{e^{2\theta - 2\eta} - \xi_i^{-1}},$$

that is: $\psi = -\phi$, $\zeta_i = e^{2\eta} \xi_i$ and $\xi_i = e^{2\eta} \bar{\xi}_i^{-1}$. It is then natural to define
\( \mu_i = e^{-\eta \xi_i} \) so that
\[
\zeta_i = e^{3\eta \mu_i}, \quad \Re(\phi) = m\eta \quad \text{with} \quad \{\mu_i\} = \left\{ \mu_j^{-1} \right\}.
\]
The final result is
\[
F(\theta) = e^{i\omega} \prod_{i=1}^{m} \frac{e^{2\theta + \eta} - \mu_i}{e^{2\theta} - e^{\eta \mu_i}}, \quad (3.16)
\]
\[
G(\theta) = e^{-i\omega} \prod_{i=1}^{m} \frac{e^{2\theta - \eta} - e^{2\eta \mu_i}}{e^{2\theta} - e^{\eta \mu_i}}, \quad (3.17)
\]
with \( \omega = \Im \phi \).

Reinserting eqs. (3.17) and (3.16) back in (3.10), and imposing the vanishing of the residues at the poles of \( \Lambda(\theta) \), so to actually obtain a polynomial in \( e^\theta \), we get the Bethe-Ansatz (BA) equations. The singularities of \( F(\theta) \) and \( G(\theta) \) are at those values \( \theta_k \) of \( \theta \) such that \( e^{2\theta_k} = e^{\eta \mu_k} \). The BA equations are thus:
\[
\left( \frac{\sinh \theta_k}{\sinh(\eta - \theta_k)} \right)^N e^{(\eta - 2\theta_k)B} = e^{2i\omega} \prod_{i=1}^{m} \frac{\sinh(\theta_k - \theta_i + \eta)}{\sinh(\theta_k - \theta_i - \eta)}
\]
which, after the change of variable \( \theta_k = \eta/2 - i\lambda_k \), read (assuming \( N \) even)
\[
\left( \frac{\sin(\lambda_k + i\eta/2)}{\sin(\lambda_k - i\eta/2)} \right)^N e^{2i(\lambda_k B - \omega)} = \prod_{i=1}^{m} \frac{\sin(\lambda_k - \lambda_i + i\eta)}{\sin(\lambda_k - \lambda_i - i\eta)}.
\]
(3.18)
The eigenvalues of the transfer matrix have thus the form
\[
\Lambda(\theta, \vec{\lambda}) = \left( \prod_{j=1}^{m} \frac{\sin(\lambda_j - i\theta - i\eta/2)}{\sin(\lambda_j - i\theta + i\eta/2)} \right) \sinh^N(\eta - \theta) e^{\theta B + i\omega}
\]
\[
+ \left( \prod_{j=1}^{m} \frac{\sin(\lambda_j - i\theta + 3i\eta/2)}{\sin(\lambda_j - i\theta + i\eta/2)} \right) \sinh^N \theta e^{(\eta - \theta)B - i\omega},
\]
(3.19)
with \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_m) \) solution of (3.18). We see that the BA equations of the general NIS model are very similar but not identical to those of the six-vertex
model. First, in the six-vertex with periodic boundary conditions the phase factor \( e^{i\omega} \) is absent; second and more important, a \( \lambda \)-dependent phase factor \( e^{2i\lambda_k B} \) never appears in the six-vertex model. It is usually possible to determine the phase \( \omega \) by studying the asymptotic behaviour for \( \theta \to \infty \) of the transfer matrix \[12\]. For the general NIS model the diagonalization of the transfer matrix in this limit is not simpler than the diagonalization for finite values of \( \theta \). An analysis of the large \( \theta \) asymptotics of \( \tau(\theta) \) will be the subject of the the next section. In the separable case \( e^{i\omega} \) is known to be a root of unity \[1\]. Computations on small size examples and the analysis of the next section led us to conjecture that the same result holds true in the general case; in Sec. 6 we provide a proof of this result in the thermodynamic limit.

To analyze the structure of the BA equations, we take as usual the logarithm of eq. (3.18) which reads \((\gamma = -\eta > 0)\)

\[
N \Phi(\lambda_k, \gamma/2) + 2\lambda_k B - 2\omega - \sum_{\substack{i=1
\sum_{i \neq k}}^{m} \Phi(\lambda_k - \lambda_i, \gamma) = 2\pi I_k \tag{3.20}
\]

where

\[
\Phi(\lambda, \gamma) \equiv i \log \left[ \frac{\sin(i\gamma + \lambda)}{\sin(i\gamma - \lambda)} \right] \quad \tag{3.21}
\]

and the \( I_k \) are half integers. We choose the branches of the logarithm in (3.21) in such a way that the function \( \Phi(x, \gamma) \) for real \( x \) is continuous, odd and monotonically increasing for \(-\pi/2 \leq \lambda \leq \pi/2\). In particular

\[
\Phi \left( \pm \frac{\pi}{2}, \gamma \right) = \pm \pi \quad , \tag{3.22}
\]

\[
\Phi(\lambda + \pi, \gamma) - \Phi(\lambda, \gamma) = 2\pi \quad . \tag{3.23}
\]

As customary we introduce the so called counting function \( Z_N(\lambda) \)[11] that in
the present case takes the form

\[ Z_N(\lambda) = \frac{1}{2\pi} \left[ \Phi(\lambda, \gamma/2) + 2 \left( b\lambda - \frac{\omega}{N} \right) - \frac{1}{N} \sum_{i=1}^{m} \Phi(\lambda - \lambda_i, \gamma) \right] \]  

(3.24)

where \( b = B/N \) is in the range \( 0 \leq b \leq 1 \). By construction, at each real real root \( \lambda_k \), \( Z_N(\lambda)N \) takes the half-integer value (cfr. eqs. (3.20) and (3.24)) \( I_k \):

\[ Z_N(\lambda_k) = \frac{I_k}{N} \]  

(3.25)

Let us solve the BA equations (3.20) in the thermodynamic limit. For the ground state we assume as usual the regular filling

\[ I_{k+1} - I_k = 1 \]

Then, after introducing the continuous density of roots

\[ \sigma_b(\lambda_k) = \lim_{N \to \infty} \frac{1}{N(\lambda_{k+1} - \lambda_k)} \]

in the \( N \to \infty \) limit in which the BA roots are closely spaced, we have from eq. (3.25) [13]

\[ \frac{dZ_\infty}{d\lambda} = \sigma_b(\lambda) \]  

Thus in the limit \( N \to \infty \) eq. (3.24) reads

\[ \sigma_b(\lambda) = \frac{1}{2\pi} \Phi'(\lambda, \gamma/2) + \frac{b}{\pi} - \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} d\mu \Phi'(\lambda - \mu, \gamma) \sigma_b(\mu) \]  

(3.26)

with \( \Phi'(\lambda, \gamma) = d\Phi(\lambda, \gamma)/d\lambda \). Eq. (3.26) is identical to the corresponding equation for the six-vertex model, except for the constant term \( b/\pi \) [13]. This linear
equation can be solved by Fourier series. Starting with

\[ \sigma_b(\lambda) = \sum_{k=-\infty}^{+\infty} e^{2ik\lambda} \sigma_k \]

\[ \Phi(\lambda, \gamma) = 2\lambda + 2 \sum_{k=1}^{\infty} \frac{\sin(2k\lambda)}{k} e^{-2k\gamma}, \]

we find

\[ \sigma_b(\lambda) = \frac{1 + b}{2\pi} + \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{e^{2ik\lambda}}{\cosh(k\gamma)}. \] (3.27)

This density results to be equal to the six-vertex ground state density of roots plus the constant \( b/(2\pi) \). This implies that the number of roots

\[ N \int_{-\pi/2}^{\pi/2} d\lambda \sigma_b(\lambda) = \frac{N + B}{2} \]

is \( B/2 \) larger than in the six-vertex model.

The free energy per site at fixed \( b \) is defined as

\[ f_b(\theta, \gamma) = -\lim_{N \to \infty} \frac{1}{N} \log \Lambda(\theta) \]

which, by using eq. (3.19), takes the form

\[ f_b(\theta, \gamma) = -\log |\sinh(\gamma + \theta)| - \theta b + i \int_{-\pi/2}^{\pi/2} d\lambda \Phi(\lambda - i\theta, \gamma/2) \sigma_b(\lambda). \] (3.28)

Inserting eq. (3.27) in (3.28) yields the \( b \)-independent result

\[ f_b(\theta, \gamma) = f_0(\theta, \gamma) \]

in the thermodynamic limit. This shows that all the ground states of the various \( B \)-dependent sectors of the Hilbert space are degenerate in this limit.
The excited states follow by allowing holes in the half-integers $I_k$:

$$I_{k+1} - I_k = 1 + \sum_{h=1}^{N_h} \delta_{ii_h}.$$ 

The corresponding root density and eigenvalues are identical to those of the six-vertex model since they are $B$-independent.

4. The large $\theta$ behaviour of the NIS model and of its Bethe Ansatz solution.

It is always instructive to investigate the properties of integrable models in the limit in which the spectral parameter $\theta$ takes large values. In particular the quantum group structure usually emerges in such a limit [13]. We see from eq.(1.2) that the leading contribution for large $\theta$ of the various Boltzmann weights is $e^{d\theta}$ with $d = 2, \frac{3}{2}, 1, \frac{1}{2}, 0$. This behaviour makes the diagonalization of the transfer matrix in the large $\theta$ limit as difficult as for finite values of $\theta$. This situation differs from the one of the six-vertex model where $d = 1, 0$ and the transfer matrix in the limit $\theta \to \infty$ is easily diagonalizable and non-singular.

Clearly, the purely black configurations $R_{\mu \nu}^{\lambda \sigma}(\theta) = R_{\nu \mu}^{\eta}(\eta - \theta)$ [see eqs.(1.2)] dominate in the $\theta \to +\infty$ limit. More precisely

$$R_{\lambda \sigma}^{\mu \nu}(\theta) \bigg|_{\theta \to +\infty} = \frac{e^{-\eta}}{2} e^{2\theta \delta_{\mu \lambda} \delta_{\nu \sigma}} (1 - \delta_{\mu \sigma}).$$

One then gets for the $N$-sites transfer matrix in the same limit

$$\tau(\theta) \bigg|_{\theta \to +\infty} = \left( -\frac{e^{-\eta}}{2} \right)^N e^{2N\theta} M_{2N} \left( 1 + o(e^{-\theta}) \right),$$

where

$$M_{2N} = \sum_{\{\mu\}} |\mu_1 \mu_2 \ldots \mu_N\rangle \langle \mu_2 \mu_3 \ldots \mu_N \mu_1|, \quad \mu_i \neq \mu_{i+1} \quad (4.1)$$

is a left-shift operator in the purely black sector with the condition that all the neighboring indices are different. Notice that the operator $M_{2N}$ has a non-zero
kernel since it annihilates all states having either a couple of equal neighboring black indices or at least one red index. Thus the diagonalization of $M_{2N}$ does not yield any information on the asymptotic behaviour at large $\theta$ of the states in its kernel. We then move to the next to leading contribution in $\tau(\theta)$. It is obtained by replacing in two neighboring sites the weights $R_{\mu\nu}(\theta)$ with $R_{\mu\alpha}(\theta)$ and $R_{\alpha\mu}(\theta)$ in all possible ways. From

$$R_{\mu\alpha}(\theta) = R_{\alpha\mu}(\theta) = \left(\frac{e^{-\eta}}{2}\right)e^{\frac{2\theta}{2}}$$

we find

$$\tau(\theta)_{\theta \rightarrow +\infty} = \left(\frac{e^{-\eta}}{2}\right)^N \left(e^{2\theta}M_{2N} + e^{(2N-1)\theta}M_{2N-1}\right) \left(1 + o(e^{-\theta})\right), \quad (4.2)$$

where the operator $M_{2N-1}$ is again a left-shift operator of the form (4.1) with all black indices but one which is red. Just like $M_{2N}$, the operator $M_{2N-1}$ has a non-zero kernel.

It is useful to match eq. (4.2) with the large $\theta$ behaviour of the eigenvalues (3.19) obtained from the BA equations. We have

$$A(\theta)_{\theta \rightarrow +\infty} = \left(\frac{1}{2}\right)^Ne^{(N+B)\theta}e^{i\omega-(N-m)\eta} \left[1 + e^{-2\theta} \left(2\sinh\eta \sum_{j=1}^{m} e^{-2i\lambda_j} - Ne^{2\eta}\right) \right] + \left(\frac{1}{2}\right)^Ne^{(N-B)\theta}e^{-i\omega+(B-m)\eta} \left[1 - e^{-2\theta} \left(2e^{2\eta}\sinh\eta \sum_{j=1}^{m} e^{-2i\lambda_j} + N\right) \right] + \text{lower orders in } e^\theta. \quad (4.3)$$

Comparison between eqs.(4.3) and (4.2) leads to the conclusion that the highest degree term $e^{2N\theta}$ in eq. (4.2) corresponds to an eigenvector $\psi_{2N}$ of $M_{2N}$ whose eigenvalue in the large $\theta$ limit is given by (4.3) with $B = N$. The eigenvectors of
$M_{2n}$ are reference states of the form (3.3) with all black indices. Moreover

$$M_{2N}\psi_{2N} = \varepsilon_l^j \psi_N$$

since $M_{2N}$ is a shift operator. Thus from eq. (4.3) $e^{i\omega}$ must be a $L$-th root of unity. The operator $M_{2N}$ annihilates all vectors outside the purely black sector of the Hilbert space. States with only one red index are instead captured by the operator $M_{2N-1}$ which, being a shift operator, has eigenstates of the form (3.3) and roots of unity as eigenvalues. Thus also in this case $e^{i\omega}$ is a root of unity. Again there are no BA roots if $B = N - 1$.

In order to find roots of the BA equations we must move to the next to sub-leading term, $e^{(2N-2)\theta}$, corresponding to $B = N - 2$ in the asymptotic expansion of $\tau(\theta)$ for large $\theta$. The transfer matrix in this limit is

$$
\tau(\theta) \bigg|_{\theta \to +\infty} = \left( -\frac{e^{-\eta}}{2} \right)^N \left[ e^{2N\theta} M_{2N} + e^{(2N-1)\theta} M_{2N-1} + e^{(2N-2)\theta} M_{2N-2} \right] + \left( -\frac{e^{2\theta-\eta}}{2} \right)^{N-1} \left[ \frac{\eta}{2} M_{2N} + M \right].
$$

The operator $M_{2N-2}$ is a shift operator of the form (4.1) involving 2 red indices. Its eigenstates are then of the kind (3.3) and its eigenvalues are roots of unity. The operator $M$ is given by

$$M = \sinh \eta M_{2N-2}^{(1)} + \frac{\eta}{2} M_{2N-2}^{(2)} + \frac{\eta/2}{2} M_{2N-2}^{(3)} + \frac{1 - e^{-\eta/2}}{2} M_{2N-2}^{(4)} + \frac{1}{2} M_{2N-2}^{(5)} + \frac{\eta/2}{2} M_{2N-2}^{(6)}$$

where the operators $M_{2N-2}^{(i)}$ ($i = 1, \ldots, 6$), as the diagrammatic representation of in Fig.2 shows, involve a couple of equal neighboring indices. Eigenvalues of the transfer matrix with asymptotic behaviour $e^{(2N-2)\theta}$ not corresponding to reference states can then be obtained by diagonalizing the operator $M$. Obviously (see Fig.2) the reference states (3.3) belong to the kernel of $M$ and conversely the eigenstates of $M$ are in the kernel of $M_{2N}$, $M_{2N-1}$ and $M_{2N-2}$. Let us start by
considering the plane waves $|\varepsilon_j^N; \mu\rangle$ of eq.(3.6). Calling $\nu_L$ and $\nu_R$ the (black) indices on the left and on the right of the couple $\mu\mu$ in the states $|\varepsilon_j^N; \mu\rangle$, the state

$$\Psi(\varepsilon_j^N) = e^{\eta/2} \sum_{\mu=1}^{t} |\varepsilon_j^N; \mu\rangle + 2e^{-\eta/2} \sinh \eta |\varepsilon_j^N; \nu_R \rangle + \sum_{\alpha=1}^{r} |\varepsilon_j^N; \alpha\rangle$$

satisfies

$$M\Psi(\varepsilon_j^N) = -\frac{e\eta}{2} \Psi(\varepsilon_j^N),$$

where eq. (1.3) has been taken into account. Therefore we have

$$\tau(\theta) \Psi(\varepsilon_j^N) = \frac{\rho(\theta)^N}{2} \Psi(\varepsilon_j^N).$$

This corresponds to a solution of the BA equations with $B = N - 2$, $m = 2$ and $e^{i\omega} = \varepsilon_j^N$ which is a root of unity.

In conclusion, we have verified for $B = N, N - 1, N - 2$ that $e^{i\omega}$ is a root of unity. We conjecture that this is the case for all values of $B$.

5. The partition function and the excitation spectrum in the thermodynamic limit

In this section we collect some results about the determination of the free energy per site in the thermodynamic limit and make the comparison between the result obtained by means of inversion techniques [2,14-16] and the one we get from the BA equations (3.18).

We start by observing that eq.(2.5) can be rewritten as

$$\tau(\theta + \eta) \tau(\theta) = \rho(\theta)^N \mathbb{I} + o(e^{-N}),$$

where now we have made explicit the fact that the second term on the RHS is vanishing in the limit $N \to \infty$. Correspondingly, for the eigenvalues $\Lambda(\theta)$ of $\tau(\theta)$
one gets

$$\lim_{N \to \infty} \Lambda(\theta) \Lambda(\eta + \theta) = \rho(\theta)^N .$$  (5.2)

For the ground state $\Lambda_0(\theta)$ we define

$$L_0(\theta) = \frac{1}{\sinh \eta} \lim_{N \to \infty} \Lambda_0(\theta)^{1/N}$$

which, because of (3.9) and (5.2), satisfies ( $\Lambda_0(\theta)$ is real)

$$L_0(\theta) L_0(-\theta) = \frac{\sinh(\eta + \theta) \sinh(\eta - \theta)}{\sinh^2 \eta} \equiv \varphi(\theta) \varphi(-\theta) ,$$  (5.3)

where property (3.8) has been taken into account and a suitable function $\varphi(\theta)$ has been introduced. The explicit form of $\varphi(\theta)$ has to be chosen in such a way that the following properties for $L_0(\theta)$ are satisfied: i) analyticity and absence of zeroes in the physical strip $-\eta \leq \text{Re} \theta \leq \eta$, ii) $i\pi$ periodicity, which follows from eq. (3.8) under the hypothesis that the ground state belongs to the even $B$ sector as discussed in the Sec. 3.

Eq.(5.3) can be solved iteratively. Starting from $L_0(\theta) = \varphi(\theta)$ and imposing alternatively crossing invariance (3.9) and eq.(5.3) we end up with the solution in terms of an infinite product

$$L_0(\theta) = \prod_{k=0}^{\infty} \frac{\varphi(2k\eta + \theta)}{\varphi((2k + 1)\eta + \theta)} \frac{\varphi((2k + 1)\eta - \theta)}{\varphi(2(k + 1)\eta - \theta)} \frac{\varphi(2(k + 1)\eta)}{\varphi(2k\eta)} ,$$

where the last ratio in the product has been introduced so as to make $L_0(0) = L_0(\eta) = 1$. The obvious identification is then

$$\varphi(\theta) = \frac{\sinh(\eta + \theta)}{\sinh \eta}$$

so that, in the thermodinamic limit,

$$\Lambda_0(\theta)^{1/N} = -\sinh \eta \prod_{k=0}^{\infty} \frac{\sinh((2k + 1)\eta + \theta) \sinh((2k + 1)\eta - \theta) \sinh((2k + 3)\eta)}{\sinh(2(k + 1)\eta + \theta) \sinh((2k + 3)\eta - \theta) \sinh((2k + 1)\eta)} .$$  (5.4)

This solution has the required analyticity and periodicity properties. By taking
the logarithm of $\Lambda_0(\theta)$ we obtain the free energy per site $f_{\text{NIS}}(\theta, \eta)$:

$$f_{\text{NIS}}(\theta, \eta) = - \log \Lambda_0(\theta)^{1/N}.$$ 

The expression (5.4) for the ground eigenvalue could have been obtained equally well from the BA equations (3.18) with $B = 0$. From eq.(3.28) we get for the free energy

$$f_{\text{NIS}}(\theta, \eta) = \theta + \sum_{k=1}^\infty \frac{e^{k\eta}}{k} \frac{\sinh(2k\theta)}{\cosh(k\eta)} - \log |\sinh(\eta - \theta)|. \quad (5.5)$$

The sum on the RHS can be recast in the following form

$$\sum_{k=1}^\infty \frac{e^{k\eta}}{k} \frac{\sinh(2k\theta)}{\cosh(k\eta)} = \sum_{k=1}^\infty (-1)^k \log \frac{1 - e^{2(\theta + k\eta)}}{1 - e^{-2(\theta - k\eta)}} = - \log \prod_{k=1}^\infty \frac{1 - e^{-2\theta + 4k\eta}}{1 - e^{-2\theta + 2(2k - 1)\eta}} \frac{1 - e^{2\theta + 2(2k - 1)\eta}}{1 - e^{2\theta + 4k\eta}},$$

so that (see also [16])

$$\Lambda_0(\theta)^{1/N} = e^{-f(\theta, \eta)}$$

$$= \frac{1}{2} e^{-\eta} \prod_{k=0}^\infty \frac{1 - e^{-2\theta + 4(k+1)\eta}}{1 - e^{-2\theta + 2(2k+3)\eta}} \frac{1 - e^{2\theta + 2(2k+1)\eta}}{1 - e^{2\theta + 4(k+1)\eta}},$$

which is completely equivalent to (5.4) owing to

$$\prod_{k=0}^\infty e^{2\eta} \frac{\sinh((2k + 3)\eta)}{\sinh((2k + 1)\eta)} = - \frac{e^\eta}{2 \sinh \eta}.$$

The derivation above shows the relation between the free energies of the NIS and six vertex models is

$$f_{\text{NIS}}(\theta, \eta) = f_{\text{6-V}}(-\theta, \gamma), \quad \gamma = -\eta.$$

Following ref. [15], the inversion method can also be used for determining all
those eigenvalues $\Lambda(\theta)$ such that the limit

$$l(\theta) = \lim_{N \to \infty} \frac{\Lambda(\theta)}{\Lambda_0(\theta)}$$  \hspace{1cm} (5.6)$$

is finite and non-zero. From the definition (5.6) and eq.(5.2), $l(\theta)$ satisfies the inversion relation

$$l(\theta)l(\eta + \theta) = 1 \quad \text{or} \quad l(\theta)l(-\theta) = 1,$$  \hspace{1cm} (5.7)$$

where in the last equation crossing invariance has been taken into account. From crossing invariance and eq. (5.7) it is immediate to show that $l(\theta)$, in addition to being $i\pi$ periodic owing to property (3.8), is also $2\eta$ periodic:

$$l(\theta) = \overline{l(\eta - \theta)} = \frac{1}{l(\theta - \eta)} = \frac{1}{l(2\eta - \theta)} = l(\theta - 2\eta).$$

The excitations, just like the ground state, ought to be analytic in the physical strip $-\eta \leq \text{Re} \theta \leq \eta$ but zeros are now allowed. The typical ansatz is thus written as a product of Jacobi elliptic functions of modulus $k$ as follows

$$l(\theta) = \prod_{j=1}^{n} C_j \text{sn}(D(\theta_j - i\theta)),$$

with $C_j$ and $D$ coefficients to be determined. In the following we shall assume $n$ to be even, as in the six-vertex model. The parameters $\theta_j$ specify the location of the zeroes. Periodicity $2\eta$ and $i\pi$ require

$$D = \frac{K'}{\eta} \quad \text{and} \quad \frac{K'}{K} = -\frac{2\eta}{\pi}.$$  \hspace{1cm} (5.8)$$

By making use of [17]

$$\text{sn}(z - iK') \text{sn}(z) = \frac{1}{k},$$  \hspace{1cm} (5.9)$$

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eqs. (5.7) are satisfied if

\[ C_j = \pm \sqrt{k}, \quad \text{and} \quad \{\theta_j\} = \{\theta_i - i\eta\}. \]

The simplest solution, corresponding to elementary excitations, is given by

\[ \theta_j = \alpha_j + i\eta/2, \quad 1 \leq j \leq n \quad (5.10) \]

where \( \alpha_k \) real numbers (other more general solutions would describe bound states). For the \( \alpha_k \) given by eq. (5.10) we get

\[ l(\theta) = \pm \prod_{j=1}^{n} \sqrt{k} \text{sn} \left( \frac{K'}{\eta} (\alpha_j - i\theta) + \frac{iK'}{2} \right). \quad (5.11) \]

The energy per site of the ground state is given by

\[ \epsilon_0 \equiv \frac{d}{d\theta} f_{NIS}(\theta) \bigg|_{\theta=0} = 2 \sum_{k=1}^{\infty} \frac{e^{k\eta}}{\cosh(k\eta)} - \frac{e^{-\eta}}{\sinh \eta}. \]

The momenta and energies of the first excitations can be obtained from the expression of \( l(\theta) \). We have

\[ p - p_0 = -i \ln l(0) = \sum_{j=1}^{n} p_j, \quad (5.12) \]

\[ \epsilon - \epsilon_0 = -\frac{d}{d\theta} l(\theta) \bigg|_{\theta=0} = \sum_{j=1}^{n} \epsilon(p_j), \]

which give

\[ p_j = -i\sqrt{k} \text{sn} \left( \frac{K'}{\eta} \alpha_j + \frac{iK'}{2} \right). \quad (5.13) \]
and, with the help of [17],

\[
\frac{d\text{sn}u}{du} = \sqrt{(1 - \text{sn}^2u)(1 - k^2\text{sn}^2u)},
\]

one gets

\[
\epsilon(p_j) = \frac{K'}{\eta} \sqrt{(1 - k)^2 + 4k \sin^2 p_j}.
\]

(5.14)

Notice that \( p_j \) is real owing to eq. (5.9) as it must be. From eq. (5.14) the energy gap is obtained by setting \( p_j = 0 \) with the minimal value \( n = 2 \) in eqs.(5.12):

\[
\text{gap} = 2 \frac{K'}{\eta}(1 - k).
\]

(5.15)

In summary, the free energy (5.5) and the excited states (5.11) of the general NIS model in the thermodynamic limit are completely equivalent to those of the six-vertex ones. There is one branch of excitations with non-zero gap, given by eq. (5.15), and obeying the dispersion relation (5.14).

Finite size corrections can be computed here with the method proposed in [18]. Since the model is gapful, they are exponential in the size. More precisely they are typically of order

\[
k^{N/2},
\]

where \( k < 1 \) is defined by eq. (5.8).
6. The NIS model as a mixed spin 1/2 and $\infty$-spin model

In this section we show that the BA equations and the transfer matrix eigenvalues of the NIS model derived in Section 3 can be identified with those of a spin system describing a mixture of spin 1/2 and spin $s$ particles in the limit in which $s \to \infty$. In ref. [10] it has been shown that the ground state of a pure spin $s$ model is formed by a Dirac sea of solutions of the BA equation arranged in strings of 2s roots (called 2s-strings). This situation generalizes the usual situation encountered in spin 1/2 systems where the sea is formed by real roots (1-strings). It is also known [9] that the ground state of a system with mixed spin 1/2 and $s = 1$ is just the superposition of the two seas. Since the form of the BA equations found in [9] is immediately generalizable to an arbitrary value $s$, we can conclude that the ground state of a system with mixed 1/2 and $s$ spin is formed by a sea of 2s-strings plus a sea of real roots. Namely, we have $M$ real roots $\lambda_j$ ($1 \leq j \leq M$) and $\nu_s$ 2s-strings:

$$\lambda_{\alpha,l} = \sigma_\alpha + i \left(l + \frac{1}{2}\right) \gamma + \varepsilon_{N_{1/2}N_s}$$

with $1 \leq \alpha \leq \nu_s$ and $-s \leq l \leq s - 1$. Here the real numbers $\sigma_\alpha$ are the strings’ centers and $\varepsilon_{N_{1/2}N_s}$ are quantities vanishingly small when $N_{1/2}, N_s \to \infty$.

The BA equations (with periodic boundary conditions) in presence of string solutions are then

$$\left( \frac{\sin(\lambda_k + i\gamma/2)}{\sin(\lambda_k - i\gamma/2)} \right)^{N_{1/2}} \left( \frac{\sin(\lambda_k + is\gamma)}{\sin(\lambda_k - is\gamma)} \right)^{N_s} = -\prod_{i=1}^{M} \frac{\sin(\lambda_k - \lambda_i + i\gamma)}{\sin(\lambda_k - \lambda_i - i\gamma)}$$

$$\times \prod_{\alpha=1}^{\nu_s} \prod_{j=-s}^{s-1} \frac{\sin(\lambda_k - \lambda_{\alpha,j} + i\gamma)}{\sin(\lambda_k - \lambda_{\alpha,j} - i\gamma)}$$

(6.1)
for a real root $\lambda_k$ and

$$
\left( \frac{\sin(\lambda_{a,l} + i\gamma/2)}{\sin(\lambda_{a,l} - i\gamma/2)} \right)^{N_{1/2}} \cdot \left( \frac{\sin(\lambda_{a,l} + is\gamma)}{\sin(\lambda_{a,l} - is\gamma)} \right)^{N_s} = -\prod_{i=1}^M \frac{\sin(\lambda_{a,l} - \lambda_i + i\gamma)}{\sin(\lambda_{a,l} - \lambda_i - i\gamma)} 
\times \prod_{\beta=1}^{\nu_s} \prod_{j=-s}^{s-1} \frac{\sin(\lambda_{a,l} - \lambda_{\beta,j} + i\gamma)}{\sin(\lambda_{a,l} - \lambda_{\beta,j} - i\gamma)}
$$

for a string solution $\lambda_{a,l}$. The integers $N_{1/2}$ and $N_s$ denote the number of sites where particles with spin $1/2$ and $s$ respectively sit. Both $N_{1/2}$ and $N_s$ are assumed to be even integers.

The transfer matrix eigenvalue of this mixed spin system is given by

$$
\Lambda(u, \vec{\lambda}) = \Lambda_A(u, \vec{\lambda}) + \Lambda_D(u, \vec{\lambda}) ,
$$

with

$$
\Lambda_A(u, \vec{\lambda}) = \sinh^{N_{1/2}}(\gamma - u) \left( \frac{\sinh[(s + 1/2)\gamma - u]}{\sinh[(s + 1/2)\gamma]} \right)^{N_s} \prod_{j=1}^M \frac{\sin(\lambda_j + iu + i\gamma/2)}{\sin(\lambda_j + iu - i\gamma/2)} 
\times \prod_{\alpha=1}^{\nu_s} \prod_{l=-s}^{s-1} \frac{\sin(\lambda_{a,l} + iu + i\gamma/2)}{\sin(\lambda_{a,l} + iu - i\gamma/2)} ,
$$

where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_M, \lambda_1, -s, \ldots, \lambda_{\nu_s}, s-1)$ are solutions of (6.1) and (6.2) and $\Lambda_D(u, \vec{\lambda}) = \Lambda_A(\gamma - u, \vec{\lambda})$.

The infinite $s$ limit of eq. (6.1) gives then

$$
\left( \frac{\sin(\lambda_k + i\gamma/2)}{\sin(\lambda_k - i\gamma/2)} \right)^{N_{1/2}} e^{-2i(N_s - 2\nu_s)\lambda_k} = -\exp \left( 4i \sum_{\alpha=1}^{\nu_s} \sigma_\alpha \right) \prod_{i=1}^M \frac{\sin(\lambda_k - \lambda_i + i\gamma)}{\sin(\lambda_k - \lambda_i - i\gamma)}
$$

which coincide with eqs.(3.18) if we set $\gamma = -\eta$, $N_{1/2} = N$, $B = N_s - 2\nu_s$ and $e^{i\omega} = \exp(-2i\sum_{\alpha=1}^{\nu_s} \sigma_\alpha)$. In the same way and with the same identification of
parameters the large \( s \) limit of eq.(6.4) is

\[
\Lambda_A(u, \bar{N}) = \sinh^{N_1/2}(\gamma - u)e^{-(N_s-2\nu_s)u}\exp\left(-2i\sum_{\alpha=1}^{\nu_s} \sigma_\alpha \right) \prod_{j=1}^{M} \frac{\sin(\lambda_j + iu + i\gamma/2)}{\sin(\lambda_j + iu - i\gamma/2)}
\]

which inserted back in (6.3) reproduces exactly the eigenvalue (3.19) of the NIS model if \( u = -\theta \).

One can further explore this correspondence and give a proof, at least in the limit \( N \to \infty \), to our conjecture that the phase factor \( e^{i\omega} \) of the NIS model is in fact a root of unity. To this purpose we multiply all the BA equations (6.2) corresponding to a string with center say \( \sigma_\alpha \) and take the logarithm [19]. After some algebra we obtain in the limit \( N_1/2, N_s \to \infty \)

\[
N_1/2 \Phi(\sigma_\alpha, s\gamma) + N_s \sum_{k=1}^{2s-1} \Phi(\sigma_\alpha, (k - \frac{1}{2})\gamma) = 2\pi I_\alpha + \sum_{\beta=1}^{\nu_s} \left[ \Phi(\sigma_\alpha - \sigma_\beta, 2s\gamma) + 2\sum_{k=1}^{2s-1} \Phi(\sigma_\alpha - \sigma_\beta, k\gamma) \right] + \sum_{j=1}^{M} \left[ \Phi(\sigma_\alpha - \lambda_j, (s + \frac{1}{2})\gamma) + \Phi(\sigma_\alpha - \lambda_j, (s - \frac{1}{2})\gamma) \right]
\]

(6.6)

where, for real \( \lambda \),

\[
\Phi(\lambda, \gamma) = 2\tan^{-1}[\tan \lambda \coth \gamma].
\]

As usual \( I_\alpha \) denotes a half-integer. Eqs. (6.6) are a set of equations for the real roots and the centers of the strings. We can easily solve for the latter in the infinite \( s \) limit in which (6.6) reduces to

\[
(N_1/2 + 4sN_s)\sigma_\alpha = 2\pi I_\alpha + 8s \sum_{\beta=1}^{\nu_s} (\sigma_\alpha - \sigma_\beta) + 2\sum_{j=1}^{M} (\sigma_\alpha - \lambda_j) + o(s^0)
\]

Under the hypothesis of regular filling for the ground state

\[
M = \frac{N_1/2}{2} + o(1), \quad \sum_{j=1}^{M} \lambda_j = o(1)
\]

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we get
\[ \sigma_\alpha = \frac{\pi}{2 s B} \left( I_\alpha - \frac{2}{N_s} \sum_{\beta=1}^{\nu_s} I_\beta \right). \]

This result shows that the quantity
\[ \exp \left( 2i \sum_{\beta=1}^{\nu_s} \sigma_\alpha \right) = \exp \left( \frac{i\pi}{s N_s} \sum_{\beta=1}^{\nu_s} I_\alpha \right) \]
appearing in (6.5) is indeed a root of unity and the correspondence with the eigenvalues of the NIS model holds.

Recall that the eigenvalue of \( \hat{S}_z \) on a state determined by eqs. (6.1)-(6.4) is given by
\[ S_z = s(N_s - 2\nu_s) + \left( \frac{N_{1/2}}{2} - M \right). \]

The first term is the contribution of the \( s \) spins and the second one of the spins \( 1/2 \). Then if we define
\[ \hat{B} = \lim_{s \to \infty} \frac{\hat{S}_z}{s}. \]
Hence \( B \) can be interpreted as a properly normalized eigenvalue of \( \hat{S}_z \) for \( s = \infty \).

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FIGURE CAPTIONS

1) Boltzmann weights for the general NIS model. Dashed and solid lines represent red and black indices. The remaining non-zero weights can be obtained by crossing invariance.

2) Contribution of order $e^{(2N-2)\theta}$ to the transfer matrix in the limit $\theta \to \infty$. 