**D-dimensional three-body bound-state problem with zero range interactions**

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We solved analytically the three-body mass-imbalanced problem embedded in $D$ dimensions for zero-range resonantly interacting particles. We derived the negative energy eigenstates of the three-body Schrödinger equation by imposing the Bethe-Peierls boundary conditions in $D$-dimensions for zero-energy two-body bound states. The solution retrieves the Efimov-like discrete scaling factor dependence with dimension. The analytical form of the mass-imbalanced three-body bound state wave function can be used to probe the effective dimension of asymmetric cold atomic traps for Feshbach resonances tuned close to the Efimov limit.

**I. INTRODUCTION**

Magnetically tunable Feshbach resonances in ultracold atomic gases open up several possibilities to explore few- and many-body physics [1]. The access to the universal regime, wherein the scattering length exceeds in magnitude all other length scales of the system, was a significant breakthrough in cold atom physics [2–4]. Not only the interactions and energies can be freely tuned in ultracold atomic traps, but also the geometry of the system. The ability of squeezing the shape of the atomic cloud opens new opportunities for studies of few-body effects in such engineered systems.

In the context of few-body problems, one intriguing phenomenon is the Efimov effect [5–7]. It consists of an infinite series of weakly bound three-body states following a universal geometrical scaling law close to the two or three-body threshold. Several ultracold atomic experiments have by now observed the Efimov effect in homonuclear [8,11] and heteronuclear systems [12–14]. In dilute gases, weakly bound Efimov trimer states mediate inelastic collisions giving rise to a rich spectrum of atom loss resonances as a function of the tunable scattering length. The universal aspects of Efimov physics, first proposed in the nuclear physics context, appear over an incredible variety of systems covering a wide range of physical scales: atomic gases [15], Bose polarons [16,17], dipolar molecules [18] and strongly interacting photons [19], to name a few examples.

Despite of many advances in theory and experiments, which allow the continuous changing of the geometry of the system and the effective dimension of the trap, associating the Efimov geometrical scaling with the squeezed system remain an important property yet to be observed. Historically, it is known theoretically the determinant role of the dimension to establish the Efimov effect - as predicted in the early 1980’s [20,21], the Efimov effect exists in three dimensions but is absent in two. The possibility to experimentally observe this prediction only appeared after the construction of Bose-Einstein condensates in traps with one [22] and two [23] dimensions. This experimental advance brought together technologies which allowed modifying continuously the geometry of the cloud.

The independent and continuous change of one spatial dimension of the trap, allied to the careful control of the scattering length, could potentially lead to the observation of a change in the Efimov geometrical ratio associated with an effective dimension. The separation of successive peaks in the three-body recombination loss [11], or even by measuring directly the binding energies of the trimers [24] are one of the possible observables to probe and study the vicinity of vanishing the Efimov effect. However, in a trapped system the successive ratios between trimer states may not be necessarily the same, but could depend non-trivially on the properties of Feshbach resonances [25]. Therefore, one should be cautious when associating the separation between recombination peaks or Efimov states to the non-integer dimension in squeezed traps.

Even still lacking clear experimental evidence, studies of three-body systems in reduced geometries have been the subject of interest in recent years. Different approaches were employed to study dimensional effects in three-body systems close to the Efimov limit. This limit is achieved when the dimer binding energy vanishes or, equivalently, the scattering length is driven to infinity, which is also known as unitary limit. In such studies, the system is embedded in fractional dimension $D$ [26–30], in mixed dimensions [31–33], in which atoms move in different spatial dimensions, or it is squeezed to lower dimensions by changing the shape of an external potential [34–39]. An approximate relation between the non-integer dimension $D$ used in this work to the squeezing in one direction by an external potential was already derived in [40] $b_{ho}^2/r_{2D}^2 = 3(D−2)/(3−D)(D−1)$, where $b_{ho}$ is the harmonic oscillator parameter and is represented in units of the rms radius of the three-body system in two dimensions $r_{2D}$.

In this work, we provide the first analytical solution for the bound-state wave function with finite binding energy for the resonant three-body mass-imbalanced problem in $D$-dimensions. The calculation uses the Bethe-Peierls (BP) boundary condition approach [41], for each pair of resonant particles in the three-body system, extended to arbitrary dimensions. For $D = 3$, this was the method originally used by Efimov to solve the three-boson prob-
lem leading to the discovery of the geometrical ratio of the binding energies \cite{5, 7}.

The method adopted here follows closely Efimov’s solution in coordinate-space using hyperspherical coordinates, which is now applied to three-distinct particles in \(D\) dimensions. In this case of a zero-range interaction, each Faddeev component of the wave function is an eigenstate of the free Schrödinger eigenvalue equation for a given binding energy. The BP boundary conditions are imposed on the full wave function, obtained by summing the three Faddeev components, to account for the zero-range interaction.

The Efimov scaling parameter is obtained from the solution of a transcendental equation in \(D\) dimensions, which comes from the Faddeev components of the wave function for three different particles. The Efimov parameter appears naturally in each Faddeev component as a direct consequence of a system of homogeneous linear equations. For the particular case of two identical bosonic particles and a distinct one, we reproduce the previous results of Ref. \cite{27} obtained with the momentum space representation.

The analytical solution of the eigenvalue equation for the three-body bound state wave function opens the possibility of future explorations of different observables, such as, the three-body radius \cite{32} and the momentum densities \cite{43, 44}, uncovering analytically the scaling laws of these quantities with the binding energy and dimension. Such scaling laws, in correspondence with limit of these quantities with the binding energy and dimension, evidences the crucial importance of the effective dimension on the Efimov physics and points out the direction for experimental investigations.

II. BETHE-PEIERLS BOUNDARY CONDITION IN \(D\)-DIMENSIONS

We derive the BP boundary condition considering a system of two non-relativistic spinless particles in \(D\)-dimensions with a short-range s-wave interaction. For relative distances beyond a finite range, two particles are non interacting and the radial wave function of the pair, working in units of \(\hbar = 1\), is known to be \cite{45}:

\[
R(r) = \sqrt{\frac{2}{\pi}} \frac{1}{r^{D/2}} \left[ \cot \delta_D(p) J_{D/2-1}(pr) - Y_{D/2-1}(pr) \right],
\]

where \(p\) is the relative momentum, \(J_{D/2-1}\) and \(Y_{D/2-1}\) are the Bessel functions of the first and second kind, and the s-wave phase-shift \(\delta_D(p)\) is given in terms of the scattering length \(a\) as:

\[
\cot \delta_D(p) = \frac{Y_{D/2-1}(pa)}{J_{D/2-1}(pa)}.
\]

The Bethe-Peierls boundary condition at zero-energy for the contact interaction can now be obtained by taking the limit to the origin of the logarithmic derivative of the reduced wave function \(u(r) = r^{D-1/2} R(r)\):

\[
\left[ \frac{d}{dr} \log u(r) \right]_{r \to 0} = \left[ \frac{D-1}{2r} - \frac{D-2}{r - r(r/a)^D/2} \right]_{r \to 0},
\]

which reproduces the well known results:

\[
\left[ \frac{d}{dr} \log u(r) \right]_{r \to 0} = \begin{cases} \frac{1}{2r}, & \text{for } D = 3; \\ \frac{1}{2r} - \frac{1}{\log(r/a)}, & \text{for } D = 2. \end{cases}
\]

We use Eq. (3) to obtain the solution of the three-body Schrödinger equation in the unitary limit.

III. THREE-BODY MASS IMBALANCED PROBLEM

We consider three different bosons with masses \(m_i\), \(m_j\), \(m_k\), and coordinates \(x_i\), \(x_j\), and \(x_k\). One can eliminate the center of mass coordinate and describe the system in terms of two relative Jacobi coordinates. One can identify three sets of such coordinates:

\[
r_i = x_j - x_k \quad \text{and} \quad \rho_i = x_i - \frac{m_j x_j + m_k x_k}{m_j + m_k}.
\]

where \((i, j, k)\) are taken cyclically among \((1, 2, 3)\). One can choose any of such sets of coordinates to solve the three-body Schrödinger equation. The Faddeev decomposition of the three-body wave function \(\Psi(x_1, x_2, x_3)\) amounts to writing it as a sum of three two-body wave functions: \(\Psi(x_1, x_2, x_3) = \psi^{(1)}(r_1, \rho_1) + \psi^{(2)}(r_2, \rho_2) + \psi^{(3)}(r_3, \rho_3)\), where we omitted the center of mass plane wave. Each component satisfies the free Schrödinger eigenvalue equation:

\[
\left[ \frac{1}{2\eta_i} \nabla_i^2 + \frac{1}{2\mu_i} \nabla_i^2 \right] \psi^{(i)}(r_i, \rho_i) = 0,
\]

where \(E\) is the system energy and the reduced masses are given by \(\eta_i = m_j m_k /(m_j + m_k)\) and \(\mu_i = m_i(m_j + m_k)/(m_i + m_j + m_k)\). The BP boundary condition applies to the total wave function; when applied to the chosen coordinates pair \((r_i, \rho_i)\), it reads, in the unitary limit \(a \to \infty\):

\[
\left[ \frac{\partial}{\partial r_i} \nabla_i^{D-1} \Psi(r_i, \rho_i) \right]_{r_i \to 0} = \frac{3 - D}{2} \left[ \frac{\Psi(r_i, \rho_i)}{r_i^{D/2}} \right]_{r_i \to 0}.
\]

This solution strategy was applied to different particles and spins in Ref. \cite{46} and we adapt it to \(D\)-dimensions following closely Efimov’s original derivation \cite{5, 7}.

For convenience, we can simplify the form of the kinetic energies by introducing the new coordinates:

\[
r_i' = \sqrt{\eta_i} r_i \quad \text{and} \quad \rho_i' = \sqrt{\mu_i} \rho_i.
\]

The three sets of primed coordinates are related to each other by the orthogonal transformations:

\[
r_i' = -r_k' \cos \theta_i + \rho_k' \sin \theta_i,
\]

\[
\rho_i' = -r_k' \sin \theta_i + \rho_k' \cos \theta_i,
\]

where \(r_i' = \sqrt{\eta_i} r_i\) and \(\rho_i' = \sqrt{\mu_i} \rho_i\).
where \( \tan \theta_i = [m_i M/(m_j m_k)]^{1/2} \), with \( M = m_1 + m_2 + m_3 \). For bosons in the partial-wave channel with vanishing total angular momentum, one can define the reduced Faddeev component as

\[
\chi^{(i)}_0(r'_i, \rho'_i) = (r'_i \rho'_i)^{\frac{\alpha_i}{2} - 1} \psi^{(i)}(r'_i, \rho'_i),
\]

The corresponding Schrödinger equation for \( \chi^{(i)}_0 \) is separable in the hyper-spherical coordinates \( r'_i = R \sin \alpha_i \) and \( \rho'_i = R \cos \alpha_i \), so that one can write:

\[
\chi^{(i)}_0(R, \alpha_i) = C^{(i)} F(R) G^{(i)}(\alpha_i),
\]

where \( R^2 = r'_i^2 + \rho'_i^2 \) and \( \alpha_i = \arctan(r'_i/\rho'_i) \), with \( F(R) \) and \( G^{(i)}(\alpha_i) \) satisfying the following equations:

\[
\begin{align*}
\frac{\partial^2}{\partial R^2} + \frac{s_n^2 - 1/4}{R^2} + \kappa_0^2 \sqrt{RF(R)} &= 0, \quad (12) \\
\frac{\partial^2}{\partial \alpha_i^2} - \frac{s_n^2 + (D - 1)(D - 3)}{\sin^2 2\alpha_i} G^{(i)}(\alpha_i) &= 0, \quad (13)
\end{align*}
\]

where \( -\kappa_0^2 = 2E \), and \( s_n \) is the Efimov parameter, to be determined by the BP boundary condition.

The definitions \( z = \cos 2\alpha_i \) and \( G^{(i)} = (1 - z^2)^{1/4} g^{(i)} \) turn Eq. \( (13) \) into the form of the associated Legendre differential equation \( (17) \) with the known analytical solutions:

\[
G^{(i)}(\alpha_i) = \sqrt{\sin 2\alpha_i} \left[ P_{s_n/2 - 1/2}^{D/2 - 1}(\cos 2\alpha_i) - \frac{2}{\pi} \tan \left( \frac{\pi(s_n - 1)/2}{2} \right) Q_{s_n/2 - 1/2}^{D/2 - 1}(\cos 2\alpha_i) \right].
\]

where \( P_n^m(z) \) and \( Q_n^m(z) \) are the associated Legendre functions. We have imposed the boundary condition that guarantees a finite value for the Faddeev component \( \psi^{(i)} \) at \( \rho_i = 0 \), which leads the reduced wave function to satisfy \( \chi^{(i)}_0(r'_i, \rho'_i = 0) = 0 \). In terms of the hyper-spherical coordinates, it leads to \( G^{(i)}(\alpha_i = \pi/2) = 0 \), since \( \rho'_i = R \cos \alpha_i \).

Therefore, the solution for \( \psi^{(i)}(r'_i, \rho'_i) \) is given by:

\[
\psi^{(i)}(r'_i, \rho'_i) = C^{(i)} K_{s_n} \left( \frac{\kappa_0 \sqrt{r'_i^2 + \rho'_i^2}}{r'_i^2 + \rho'_i^2} \right)^{D/2 - 1/2} \frac{\sqrt{\sin \left( 2 \arctan \left( \frac{r'_i}{\rho'_i} \right) \right)}}{\left( \cos \left( \arctan \left( \frac{r'_i}{\rho'_i} \right) \right) \sin \left( \arctan \left( \frac{r'_i}{\rho'_i} \right) \right) \right)^{D/2 - 1/2}} \times \left[ P_{s_n/2 - 1/2}^{D/2 - 1} \left( \cos \left( 2 \arctan \left( \frac{r'_i}{\rho'_i} \right) \right) \right) - \frac{2}{\pi} \tan \left( \frac{\pi(s_n - 1)/2}{2} \right) Q_{s_n/2 - 1/2}^{D/2 - 1} \left( \cos \left( 2 \arctan \left( \frac{r'_i}{\rho'_i} \right) \right) \right) \right],
\]

where \( K_{s_n} \) is the modified Bessel function of the second kind. One obtains the Efimov parameter \( s_n \) by considering that all three pairs of particles are resonant. Then, the BP boundary condition, Eq. \( (7) \), should be satisfied by the three-body wave function when each relative distance between two of the particles tends to zero, namely \( r_i = R \sin \alpha_i \to 0 \), implying that \( \alpha_i \to 0 \) for finite hyper-radius \( R \). The hyper-radial part of the wave function factorizes in the BP boundary condition for each \( r_i \), which depends only on the hyper-angular part of each Faddeev component \( (14) \). The resulting homogeneous linear system for the coefficients \( C^{(i)} \) reads:

\[
\frac{C^{(i)}}{2} \left( \cot \alpha_i \frac{\partial}{\partial \alpha_i} \right)^{\frac{\alpha_i}{2} - 1} \left( \sin 2\alpha_i \frac{\partial}{\partial \alpha_i} + D - 3 \right) G^{(i)}(\alpha_i) \right|_{\alpha_i \to 0} + (D - 2) \left[ \frac{C^{(j)} G^{(j)}(\theta_k)}{\sin \theta_k \cos \theta_k} \frac{\partial}{\partial \theta_k} + \frac{C^{(k)} G^{(k)}(\theta_j)}{\sin \theta_j \cos \theta_j} \frac{\partial}{\partial \theta_j} \right] = 0, \quad (16)
\]

for \( i \neq j \neq k \). Taking the three cyclic permutations of \( \{i, j, k\} \) one has a homogeneous system of three linear equations, from which one obtains the Efimov parameter \( s_n \) by solving the characteristic transcendental equation.

We remark that the key point of this work is the analytical solution, for finite energies, of each Faddeev component for bound-state systems of the three-distinct particles - this situation is more complex than our previous work given in Ref. \( 27 \). The use of the BP boundary condition results in Eq. \( (15) \) and, in order to fully define the wave function, Eq. \( (16) \) should be solved to determine the Efimov parameter \( s_n \) and the relative weights \( C^{(i)} \) of the Faddeev components of the wave function.

In the case of a purely imaginary value for the \( s_n \) parameter, the effective potential in Eq. \( (12) \) is attractive, giving rise to the well known pathological \( 1/R^2 \) interaction. This potential admits a solution at any energy with a spectrum “unbounded from below” - a phenomenon discovered long ago by Thomas \( 48 \) and referred to as the “Thomas collapse”. In particular, the transcendent-
The present method applies to the bound state of three distinct particles for dimensions $D > 2$, where the homogeneous linear system in Eq. (16) admits nontrivial solutions with purely imaginary values $s_0 \to i s_0$, i.e. in the Efimov region. That happens only for a given range of dimensions $2 < D < 4$ constrained by the condition that $s_0(D) \to 0$, which also depends on the mass imbalance in the system. Here we discuss some novel examples of trinuclear systems composed by $^6\text{Li}$, $^{23}\text{Na}$, $^{87}\text{Rb}$ and $^{133}\text{Cs}$ in $D$-dimensions from the solution of Eq. (16), which besides $s_0$ provides the relative weights $C^{(s)}$ of the Faddeev components, Eq. (15), and allows to obtain the configuration space wave function that will be explored in one example in what follows.

In Table I for some choices of mass imbalanced systems, we show the range of $D$ values, $D_c < D < D_c'$, and the critical value of the trap parameter for which the Efimov effect is present. The results in the table reveal that as the mass imbalance increases to heavy-heavy-light, the range of $D$ values for the existence of the Efimov effect widens. For two infinitely heavy masses, the lowest critical dimension tends to $D = 2$ from above, i.e. $D_c \to 2$, while the trap parameter tends to zero. The maximum critical dimension in that case tends to $D = 4$ from below, i.e. $D_c' \to 4 \ldots$, and the trap length parameter to infinity. Such subtle behaviour is clearly seen in Table I following the pattern from $^6\text{Li}^{23}\text{Na}_2$ to $^6\text{Li}^{133}\text{Cs}_2$ passing through a fully mass balanced system. When one of the $^{133}\text{Cs}$ is substituted by a lighter atom, as in $^6\text{Li}^{23}\text{Na}^{133}\text{Cs}$, the region for Efimov states shrinks. We observe that among the examples we have discussed, the smallest range of dimensions for the existence of the Efimov effect is found for three identical atoms. In this case the squeezing length in units of the rms radius in two dimensions is given by $b_{ho}/r_{2D} = 0.988$, in this trap configuration the Efimov effect vanish for three identical atoms.

Figure 1 displays the geometrical ratio between two successive Efimov states as a function of $D$ for the systems given in Table I. Noteworthy in the figure, the ratio of the energies of two successive Efimov states varies up to $\pm \infty$, while large mass asymmetries favor ratios smaller than those for $D = 3$.

![Figure 1](image-url)
We presented an analytical solution of the mass-imbalanced three-body problem in \( D \) dimensions in the Efimov limit. Use of the Bethe-Peierls boundary condition allowed us to formulate this problem and in particular show how to compute the Efimov parameter for a wide range of mass ratios and dimensions. The importance to have a relatively simple, analytical way to compute the wave function for a finite three-body energy opens up the possibility to probe the Efimov physics in ultracold atomic systems through radiofrequency spectroscopy [43]. Such a technique has been used in Ref. [49] to measure Tan’s contact parameters [50], which can be associated with thermodynamic properties of the system. Quite recently, the two-body contact was measured across the superfluid transition of a planar Bose gas [51].

Within the perspective of our work, two and three-body contacts can be computed for mass-imbalanced systems in \( D \)-dimensions using Eq. (12) by generalizing other known techniques [44], which were applied to three-body contacts can be computed for mass-imbalanced systems in \( D \)-dimensions using Eq. (12) by generalizing other known techniques [44], which were applied to three-body systems in two dimensions. We leave the study of such a possibility for a future work.

VI. ACKNOWLEDGMENTS

This work was partially supported by: Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) (grant nos. 2019/00153-8 (M.T.Y.), 2017/05660-0 (T.F.), 2020/00560-0 (D.S.R.), 2018/25225-9 (G.K.)); Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) (grant nos. 308486/2015-3 (T.F.), 303579/2019-6 (M.T.Y.), and 309262/2019-4 (G.K.)).
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