Proposed symmetry relations, e.g., quark-lepton complementarity (QLC) or tribimaximal mixing (TBM), need to be imposed at a high scale $\sim 10^{12}$ GeV characterising the large masses of right-handed neutrinos required to implement the seesaw mechanism. RG evolution down to the laboratory scale $\lambda \sim 10^3$ GeV, generically prone to spoil these relations and their predicted neutrino mixing patterns, can be made to preserve them by appropriately constraining the Majorana phases $\alpha_{2,3}$. This is explicitly demonstrated in the MSSM for two versions of QLC and two versions of TBM. A preference for $\alpha_2 \simeq \pi$ (i.e. $m_1 \simeq -m_2$) emerges in each case. Discrimination among the four cases is shown to be possible by future measurements of $\theta_{13}$.

Preliminaries

The unitary neutrino mixing matrix $U^\nu$ acts between mass eigenstates $|i \rangle (i = 1, 2, 3)$ and flavour eigenstates $|a \rangle (a = e, \mu, \tau)$ by $\text{Eq. 1}$

$$|a \rangle = (U^\nu)_{ai} |i \rangle .$$

The SM fermion mass term in the chiral flavour basis is $\text{Eq. 2}$

$$\mathcal{L}_m = -\bar{f}_a M^f f_b + \text{h.c.}$$

Let the unitary transformations of $f_{La}$ and $f_{Rb}$ to the mass basis be $\text{Eq. 3a, 3b}$

$$f_{La} = U^f_{ai} f_{Li},$$

$$f_{Rb} = W^f_{bj} f_{Rj}.$$  

Thus, in the mass basis, $\text{Eq. 4}$

$$\mathcal{L}_m = -\bar{f}_{Li} M^{f(D)}_{ij} f_{Rj} + \text{h.c.},$$

where the diagonal mass matrix $M^{f(D)}$ is given by the biunitarily transformed $M^f$:

$$M^{f(D)} = U^f M^f W^f,$$

The charged current weak interaction for quarks is $\text{Eq. 6a, 6b}$

$$\mathcal{L}_{cc} = \bar{u}_{La} \gamma^\mu d_{La} W^\mu_+ + \text{h.c.} = \bar{u}_{Li} \gamma^\mu V^{CKM}_{ij} d_{Lj} W^\mu_+ + \text{h.c.},$$

$$V^{CKM} = U^u U^d.$$
The corresponding interaction in the lepton sector can be written as
\[ \mathcal{L}_{cc}^\ell = \bar{\ell}_L \gamma^\mu \nu_L W^-_\mu + \text{h.c.} = \bar{\ell}_L \gamma^\mu U_{ij}^{PMNS} \nu_{Lj} W^-_\mu + \text{h.c.}, \quad (7a) \]
\[ U^{PMNS} = U^\ell U^\nu. \quad (7b) \]

The above similarity leads one to attribute a CKM-like form \([1]\) to \(U^{PMNS}\) in terms of three angles \(\theta_{12}, \theta_{23}, \theta_{13}\) and a Dirac phase \(\delta_\ell\):
\[ U^{CKM-form} = \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_\ell} \\
  -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_\ell} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_\ell} & s_{23}c_{13} \\
  s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_\ell} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_\ell} & c_{23}c_{13}
\end{pmatrix}, \quad (8) \]
with \(c_{ij} = \cos \theta_{ij}\) and \(s_{ij} = \sin \theta_{ij}\). However, for Majorana neutrinos, there is an additional diagonal matrix factor containing two more phases (since one can be absorbed in the overall neutrino phase):
\[ U^{PMNS} = U^{CKM-form} \text{diag.}(1, e^{-i\alpha_2}, e^{-i\alpha_3}). \quad (9) \]
Eq. (9) means that, for Majorana neutrinos with a mass term
\[ \mathcal{L}_m = -\frac{1}{2} \bar{\nu} \mathcal{M}^D \nu + \text{h.c.}, \quad (10) \]
(5) in fact becomes
\[ \mathcal{M}^D = U^{\nu} \mathcal{M}^{\nu} U^{\nu*} = \text{diag.}(m_1, m_2, m_3). \quad (11) \]
Given (9), one can take \(m_1 = |m_1|, m_2 = |m_2|e^{i\alpha_2}, m_3 = |m_3|e^{i\alpha_3}\) in (11).

**Neutrino factfile**

We now know that at least two of the three light neutrinos are massive. In the notation of (11), one can already make statements \([2]\) on three scales respectively, namely solar, atmospheric and cosmological:
\[ \sqrt{\delta m^2_S} \equiv |m_2|^2 - |m_1|^2|^{1/2} \sim 0.009 \text{ eV}, \quad (12a) \]
\[ \sqrt{\delta m^2_A} \equiv |m_3|^2 - |m_2|^2|^{1/2} \sim 0.05 \text{ eV}, \quad (12b) \]
\[ \sum_{i=1}^{3} |m_i| < \mathcal{O}(1) \text{ eV}. \quad (12c) \]

Furthermore, while the value of the Dirac phase \(\delta_\ell\) in (8) is unknown, we do know what two of the angles in \(U^{PMNS}\) are and have an upper bound on the third:
\[ \theta_{12} = 33.8^\circ \pm 2.4^\circ, \quad \theta_{23} = 45^\circ \pm 4^\circ, \quad \theta_{13} < 13^\circ, \delta_\ell = ? \quad (13) \]
The magnitudes of the elements of the matrices \(V^{CKM}\) and \(U^{PMNS}\) are now roughly known to be
\[ |V^{CKM}| = \begin{pmatrix}
  0.97 & 0.22 & 0.003 \\
  0.22 & 0.97 & 0.04 \\
  0.01 & 0.04 & 0.99
\end{pmatrix}, \quad |U^{PMNS}| \sim \begin{pmatrix}
  0.8 & 0.5 & < 0.14 \\
  0.4 & 0.6 & 0.7 \\
  0.4 & 0.6 & 0.7
\end{pmatrix}. \quad (14) \]
and present a striking contrast between small and large deviations from the unit matrix in the two cases respectively.

We don’t yet know if the ordering of the neutrino masses is ‘normal’ ($|m_3| > |m_2| > |m_1|$) or ‘inverted’ ($|m_2| > |m_1| > |m_3|$). So three mass patterns for the three neutrinos are still possible:

1. Normal hierarchical (NH): $|m_1| \ll |m_2| \sim 0.009\,\text{eV} \ll |m_3| \sim 0.05\,\text{eV}$,
2. Inverted hierarchical (IH): $|m_3| \ll |m_1| \lesssim |m_2| \sim 0.05\,\text{eV}$,
3. Quasi-degenerate (QD): $0.05\,\text{eV} < |m_1| \sim |m_2| \sim |m_3| \lesssim \mathcal{O}(0.33)\,\text{eV}$.

Within the QD pattern, the mass ordering could be either normal or inverted. We generically club the IH and QD cases under the heading ‘nonhierarchical’ (NH):

$$NH \equiv \{IH, QD\}.$$

**Mass parametrization with Majorana phases**

We introduce three real parameters, two of them dimensionless ($\rho_A$ and $\epsilon_S$) and one dimensional ($m_0$), such that

\begin{align*}
|m_1| &= m_0(1 - \rho_A)(1 - \epsilon_S), \\
|m_2| &= m_0(1 - \rho_A)(1 + \epsilon_S), \\
|m_3| &= m_0(1 + \rho_A),
\end{align*}

with $0 < \epsilon_S \leq 1$, $-1 \leq \rho_A \leq 1$, $0 < m_0 < \mathcal{O}(0.33)\,\text{eV}$. The sign of $\rho_A$ is positive (negative) for a normal (inverted) mass ordering. The solar as well as atmospheric mass scales and the sum of the neutrino masses are given respectively by

\begin{align*}
\delta m^2_S &= 4m_0^2(1 - \rho_A)^2\epsilon_S, \\
|\delta m^2_A| &= 4m_0^2|\rho_A|, \\
\sum_i |m_i| &= 3m_0 \left(1 - \frac{\rho_A}{3}\right). 
\end{align*}

It is convenient to define a derived dimensionless parameter $\Gamma$ by

$$\Gamma \equiv \rho_A^{-1} - \rho_A$$

which is positive (negative) for a normal (inverted) mass ordering and is allowed by the present data to be anywhere between zero and $\pm 182$. Sample values of these quantities are given in Table 1 for the three mass patterns.
Running neutrino masses and mixing angles

Loop divergences and corresponding renormalization procedures turn coupling strengths \( g_i \) into functions of the evolution variable \( t = (16\pi^2)^{-1}\ln Q/\wedge \), where \( Q \) is the running energy and \( \wedge \) some fixed (high) scale. In particular, this is true of the fermionic Yukawa couplings relevant to neutrino masses and mixing angles. As a result, the latter become functions of \( t \):

\[
\begin{align*}
\langle m_i(t)\rangle &\rightarrow \langle m_i(t)\rangle(t), \\
\langle \theta_{ij}(t)\rangle &\rightarrow \langle \theta_{ij}(t)\rangle(t), \\
\langle \delta(t)\rangle &\rightarrow \langle \delta(t)\rangle(t), \\
\langle \alpha_2,3\rangle &\rightarrow \langle \alpha_2,3\rangle(t).
\end{align*}
\]

Our basic idea \cite{3} is to consider certain neutrino symmetries, which fix the neutrino mixing pattern, to be operative at a high scale \( Q = \wedge \). We choose \( \wedge \sim 10^{12} \) GeV characterising the mass scale of heavy right handed neutrinos responsible for the seesaw origin of tiny neutrino masses. We then see the effects of RG evolution \cite{4} down to a laboratory scale \( \lambda \sim 1 \) TeV on that pattern. We do so within the Minimal Supersymmetric Standard Model (MSSM) \cite{5} which is why we have chosen \( \lambda \) to be of the order of the expected scale of soft supersymmetry breaking. While one can debate the precise values of \( \wedge \) and \( \lambda \) that have been chosen, the effects that we are concerned with are only logarithmically sensitive to \( \wedge/\lambda \). Moreover, the RG effects are controlled by factors such as \(|m_i + m_j|^2(|m_i|^2 - |m_j|^2)^{-1}\Delta_\tau\) where the dimensionless parameter \( \Delta_\tau \lesssim O(10^{-2}) \). For neutrinos with a normal hierarchy, this is negligible. Only for nonhierarchical neutrinos are these effects significant.

The neutrino Majorana mass matrix originates at the scale \( \wedge \) from a dimension 5 operator

\[
\mathcal{O} = c_{\alpha\beta} \frac{(\ell_\alpha \cdot H)(\ell_\beta \cdot H)}{\wedge}.
\]

In (18), \( \ell_\alpha \) and \( H \) are the \( SU(2) \) doublet lepton and Higgs fields respectively, with \( \alpha \) being a generation index and \( c_{\alpha\beta} \) being dimensionless coefficients that run with the energy scale. Then

\[
(M_{\nu,\wedge})_{\alpha\beta} \sim c_{\alpha\beta} \frac{v^2}{\wedge},
\]

with \( v = 246 \) GeV and \( \wedge \sim M_{MAJ} \), the Majorana mass characterising the set of heavy SU(2)-singlet Majorana neutrinos \{N\}. The coefficients \( c_{\alpha\beta} \) are the ones which evolve from \( Q = \wedge \) to \( Q = \lambda \).

One-loop contributions to the evolution of \( M_{\nu} \) from gauge bosons, gauginos and sfermions of the MSSM lead to the relation \cite{6}

\[
M_{\nu,\lambda} = I_k I_\kappa^T M_{\nu,\wedge} I_\kappa
\]

with

\[
\begin{align*}
I_k &= \exp \left[-\int_0^{t(\lambda)} d\tau \{6g_2^2(\tau) + 2g_Y^2(\tau) - 6Tr(Y_\ell Y_\ell^T)(\tau)\}\right], \\
I_\kappa &= \exp \left[-\int_0^{t(\lambda)} d\tau (Y_\ell Y_\ell^T)(\tau)\right].
\end{align*}
\]
In (21), $g_{2,Y}$ are the SU(2), U(1) gauge couplings and $Y_{u,\ell}$ are the Yukawa coupling matrices in family space for up-quarks, charged leptons. Let us define a dimensionless quantity

$$\Delta_r \equiv m_r^2 (\tan^2 \beta + 1) (8\pi^2 v^2)^{-1} \ln \frac{\Lambda}{\chi},$$

(22)

where $\tan \beta = v_u/v_d$, $v_{u,d}$ being the VEV of the neutral Higgs coupling to up, down type of fermions. This $\Delta_r \lesssim 10^{-2}$ for $\tan \beta \lesssim 30$.

In the basis with $U^\ell = I$, i.e., $\mathcal{M}^\ell = \text{diag.}(m_e, m_\mu, m_\tau)$, to linear order in $\Delta_r$ one has $^{[3]}$

$$I_k \simeq \text{diag.}(1, 1, 1 - \Delta_r) + \mathcal{O}(\Delta_r^2).$$

(23)

Now (20) can be rewritten as

$$\mathcal{M}_{\nu,\lambda} = I_k \text{diag.}(1, 1, 1 - \Delta_r) \mathcal{M}_{\nu,\lambda} \text{diag.}(1, 1, 1 - \Delta_r) + \mathcal{O}(\Delta_r^2).$$

(24)

Since in this basis the unitary matrix $U^\nu$ diagonalising $\mathcal{M}^\nu$ equals $U_{PMNS}^{\lambda}$ and $U_{PMNS}^{\nu}$, and consequently $m_{\nu,\iota,\jmath}^\lambda$ to $m_{\lambda,\iota,\jmath}^\nu$, respectively. Though we make these approximations to facilitate the use of analytically transparent expressions, our final results, shown in later figures, are based on the numerical integration of the full equations of Antusch et al. $^{[4]}$

The high scale symmetries considered by us dictate $\theta_{13}^\wedge$ to be a small parameter which is negligible. Moreover, $\theta_{13}^\wedge \Delta_r \lesssim 10$ and so $\mathcal{O}(\theta_{13}^\wedge \Delta_r)$ terms can also be neglected. Then the evolution of all the above parameters can be computed analytically in a simple manner. We have

$$\theta_{12}^\wedge = \theta_{12}^\wedge + k_{12} \Delta_r + \mathcal{O}(\theta_{13}^\wedge \Delta_r, \Delta_r^2),$$

(25a)

$$\theta_{23}^\wedge = \theta_{23}^\wedge + k_{23} \Delta_r + \mathcal{O}(\theta_{13}^\wedge \Delta_r, \Delta_r^2),$$

(25b)

$$\theta_{13}^\wedge = k_{13} \Delta_r + \mathcal{O}(\theta_{13}^\wedge \Delta_r, \Delta_r^2),$$

(25c)

$$a_{23}^\wedge = a_{23}^\wedge + 2 a_{23} \Delta_r + \mathcal{O}(\theta_{13}^\wedge \Delta_r, \Delta_r^2),$$

(25d)

$$|m_\lambda^\Delta| = I_k |m_\lambda^\Delta| \left[ 1 + \mu_1 \Delta_r + \mathcal{O}(\theta_{13}^\wedge \Delta_r, \Delta_r^2) \right],$$

(25e)

$$\delta^\ell = d_\ell \Delta_r + \mathcal{O}(\theta_{13}^\wedge \Delta_r, \Delta_r^2).$$

(25f)

The values of $k_{ij}$ are

$$k_{12} = \frac{1}{2} \sin 2\theta_{12}^\wedge \sin^2 \theta_{23}^\wedge \left| \frac{m_1^\wedge + m_2^\wedge}{m_3^\wedge} \right|^2 \left( m_3^\wedge - |m_2^\wedge|^2 \right),$$

$$k_{23} = \frac{1}{2} \sin 2\theta_{23}^\wedge \left( \cos^2 \theta_{12}^\wedge \left| \frac{m_1^\wedge + m_3^\wedge}{m_2^\wedge} \right|^2 + \sin^2 \theta_{12}^\wedge \left| \frac{m_1^\wedge + m_2^\wedge}{m_2^\wedge} \right|^2 \right),$$

$$k_{13} = \frac{1}{4} \sin 2\theta_{23}^\wedge \left[ 1 + \cos^2 \theta_{12}^\wedge \cos (\alpha_2^\wedge - \alpha_3^\wedge) + \sin^2 \theta_{12}^\wedge \cos \alpha_3^\wedge \right] + \mathcal{O}(\theta_{13}^\wedge, \epsilon_3^\wedge).$$

(26a, 26b)
The sign of $k_{12}$ is always positive, as is clear from (26a).

**High scale neutrino symmetries**

We consider four cases in this category: QLC 1 [7], QLC 2 [7, 8], TBM 1 [9] and TBM 2 [10]. None of these determines the ordering of the neutrino masses which can be normal or inverted in each case. Recall first the respective forms of $U$ for bimaximal and tribimaxial mixing.

\[
U_{BM}^{BM} = \begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
-1/2 & 1/2 & 1/\sqrt{2} \\
1/2 & -1/2 & 1/\sqrt{2}
\end{pmatrix}, \quad \theta_{23}^\text{BM} = \pi/4 = \theta_{12}^\text{BM}, \quad \theta_{13}^\text{BM} = 0,
\]

\[
U_{TBM}^{TBM} = \frac{1}{\sqrt{6}} \begin{pmatrix}
2 & \sqrt{2} & 0 \\
-1 & \sqrt{2} & \sqrt{3} \\
-1 & -\sqrt{2} & \sqrt{3}
\end{pmatrix}, \quad \theta_{23}^\text{TBM} = \pi/4, \quad \theta_{12}^\text{TBM} = \sin^{-1} \frac{1}{\sqrt{3}} \approx 35.26^\circ, \quad \theta_{13}^\text{TBM} = 0.
\]

In (27) we have also listed the corresponding values of the mixing angles at the scale $\wedge$ where these symmetries are implemented. The content of that implementation in each case is summarised in Table 2.

| QLC 1 | QLC 2 | TBM 1 | TBM 2 |
|-------|-------|-------|-------|
| $U^{PMNS} = V^{CKM} U^{\nu, BM}$ | $U^{PMNS} = V^{TBM} V^{CKM}$ | $U^{PMNS} = U^{\nu, TBM}$ | $U^{PMNS} = \tilde{V}^{CKM} U^{\nu, TBM}$ |
| $U^u = I$ basis $\rightarrow U^d = U^e$ | $U^d = I$ basis $\rightarrow U^u = U^e$ | Family $A_4$ or $S_3$ | $\tilde{V}^{CKM} \approx \begin{pmatrix}
1 & \theta_{C/3} & 0 \\
\theta_{C/3} & 1 & -|V_{cb}| \\
0 & |V_{cb}| & 1
\end{pmatrix} + O(\theta_{C}^3)$ |
| $\theta_{12} \approx \frac{\pi}{4} - \theta_C/\sqrt{2} \approx 35.4^\circ$ | $\theta_{12} \approx \frac{\pi}{4} - \theta_C \approx 32.4^\circ$ | $\theta_{13} \approx \sin^{-1} \frac{1}{\sqrt{3}} \approx 35.3^\circ$ | $\theta_{12} \approx \sin^{-1} \frac{1}{\sqrt{3}} - \theta_{C/2} \approx 32.3^\circ$ |
| $\theta_{23} \approx \frac{\pi}{4} - |V_{cb}| - \theta_{C/2}^2 \approx 42.1^\circ$ | $\theta_{23} \approx \frac{\pi}{4} - \frac{V_{cb}}{\sqrt{2}} \approx 43.4^\circ$ | $\theta_{23} \approx 45^\circ$ | $\theta_{23} \approx \frac{\pi}{4} - |V_{cb}| \approx 42.7^\circ$ |
| $\theta_{13} \approx \frac{\theta_{C}}{\sqrt{2}} \approx 8.9^\circ$ | $\theta_{13} \approx \frac{V_{cb}}{\sqrt{3}} \approx 1.6^\circ$ | $\theta_{13} \approx 0$ | $\theta_{13} \approx \frac{\theta_{C}}{\sqrt{2}} \approx 3.1^\circ$ |

| Table 2: Statement and consequence of each high scale symmetry |

**Correlated constraints**

The $3\sigma$ allowed ranges [1] for neutrino mass and mixing parameters are tabulated below.

| $7 \times 10^{-5} eV^2 < \delta m_2^2 < 9.1 \times 10^{-3} eV^2$ | $1.7 \times 10^{-3} eV^2 < |\delta m_1^2| < 3.3 \times 10^{-3} eV^2$ |
| $30^\circ < \theta_{12} < 39.2^\circ$ | $35.5^\circ < \theta_{23} < 55.5^\circ$ |
| $\theta_{13} < 12^\circ$ |

| Table 3: $3\sigma$ allowed values of neutrino mass and mixing parameters |
The tightest constraints come from $\theta_{12}$. Figure 1 shows exclusion regions in the $m_0^\lambda \tan \beta - \alpha_2^\lambda$ plane for each high scale symmetry considered. The peak at $\alpha_2^\lambda \simeq \pi$ shows a preference in all these models for the approximate result $m_1 \simeq -m_2$ which is also a desired result for leptogenesis [11] with nonhierarchical neutrinos. The positivity of $k_{12}$ dictates that the measured value of $\theta_{12}$ should exceed $35.4^\circ$, $32.4^\circ$, $35.26^\circ$ and $35.3^\circ$ for the QLC 1, QLC 2, TBM 1 and TBM 2 cases respectively. In particular, reduced errors on $\theta_{12}$, possibly from SNO 3, can really put QLC 1 and TBM 2 out of commission.

Turning to the other mixing angles, the measured value of $\theta_{23}$ has to exceed (be less than) $42.5^\circ$, $42.7^\circ$, $45^\circ$, $42.5^\circ$ for the QLC 1, QLC 2, TBM 1 and TBM 2 cases respectively for a normal (inverted) ordering of neutrino masses. On the other hand, $\Delta \theta_{13} = \theta_{13}^\lambda - \theta_{13}^\lambda$ depends on $m_0 \tan \beta$. The allowed regions in the $\theta_{13}^\lambda - m_0^\lambda \tan \beta$ plane for the four cases are shown in Fig. 2. In particular, a measured value of $\theta_{13} < 6^\circ$ will exclude QLC 1. Also, if $m_0^\lambda \tan \beta < 2$ eV (i.e. $m_0^\lambda \tan \beta < 1.4$ eV), TBM 2 will be distinguishable from QLC 2 and TBM 1. Finally, values of $m_0^\lambda \tan \beta > 4.4$ eV (i.e. $m_0^\lambda \tan \beta > 3.1$ eV) are disallowed for all four cases.

Figure 1: Exclusion regions (above the curves) in the $m_0^\lambda \tan \beta - \alpha_2^\lambda$ plane.

Figure 2: Allowed regions in the $\theta_{13} - m_0^\lambda \tan \beta$ plane for all four cases.
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References

[1] Particle Data Group: W.-M. Yao et al., J. Phys. G33, 1 (2006).

[2] R.N. Mohapatra et al., arXiv: hep-ph/050213. R.N. Mohapatra and A.Y. Smirnov, Ann. Rev. Nucl. Part. Sci. 56, 569 (2006).

[3] A. Dighe, S. Goswami and P. Roy, arXiv: 0704.3735 [hep-ph].

[4] e.g. S. Antusch et al. Nucl. Phys. B674, 401 (2003).

[5] M. Drees, R.M. Godbole and P. Roy, Theory and Phenomenology of Sparticles, Hackensack, USA: World Scientific (2004).

[6] P.H. Chankowski and S. Pokorski, Int. J. Mod. Phys. A17, 575 (2002).

[7] H. Minakata and A.Y. Smirnov, Phys. Rev. D70 073009 (2004).

[8] M. Raidal, Phys. Rev. Lett. 93, 161801 (2004).

[9] P.F. Harrison, D.H. Perkins and W.G. Scott, Phys. Lett. B458, 79 (1999); ibid B530, 167 (2002).

[10] S.F. King, JHEP 0508, 105 (2005).

[11] W. Buchmüller, P. di Bari and M. Plumacher, Nucl. Phys. B665, 445 (2003).