Some remarks on \( q \)-deformed multiple polylogarithms

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Abstract

We introduce general \( q \)-deformed multiple polylogarithms which even in the dilogarithm case differ slightly from the deformation usually discussed in the literature. The merit of the deformation as suggested, here, is that \( q \)-deformed multiple polylogarithms define an algebra, then (as in the undeformed case). For the special case of \( q \)-deformed multiple \( \zeta \)-values, we show that there exists even a noncommutative and noncocommutative Hopf algebra structure which is a deformation of the commutative Hopf algebra structure which one has in the classical case. Finally, we discuss the possible correspondence between \( q \)-deformed multiple polylogarithms and a noncommutative and noncocommutative self-dual Hopf algebra recently introduced by the author as a quantum analog of the Grothendieck-Teichmüller group.

1 Motivation

In [CL] a family of noncommutative deformations \( S^{4,\theta} \) of the four dimensional sphere has been introduced from the instanton algebra. Concretely, the algebra of functions on \( S^{4,\theta} \) is given by the generators \( t, \alpha, \alpha^*, \beta, \beta^* \) and relations

\[
\alpha \alpha^* = \alpha^* \alpha, \quad \beta \beta^* = \beta^* \beta
\]
\[
\alpha \beta = \lambda \beta \alpha, \quad \alpha^* \beta = \overline{\lambda} \beta \alpha^*
\]

\[
0 = \alpha \alpha^* + \beta \beta^* + t^2 - t
\]

and the requirement that \( t \) commutes with all the other generators. With \( \varphi, \psi \) angles,

\[
0 \leq \varphi \leq \frac{\pi}{2}, \quad -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}
\]

and \( u, v \) the unitary generators of the algebra of smooth functions on the noncommutative 2-torus, one has

\[
\alpha = \frac{u}{2} \cos \varphi \cos \psi
\]

\[
\beta = \frac{v}{2} \sin \varphi \cos \psi
\]

\[
t = \frac{1}{2} + \frac{1}{2} \sin \psi
\]

and the Dirac operator corresponding to the round metric on the sphere is given as (see [CL])

\[
D = (\cos \varphi \cos \psi)^{-1} \delta_1 \gamma_1 + (\sin \varphi \cos \psi)^{-1} \delta_2 \gamma_2
\]

\[
+ \frac{1}{\cos \psi} \sqrt{-1} \left( \frac{\partial}{\partial \varphi} + \frac{1}{2} \cot \varphi - \frac{1}{2} \tan \varphi \right) \gamma_3 + \sqrt{-1} \left( \frac{\partial}{\partial \psi} - \frac{3}{2} \tan \varphi \right) \gamma_4
\]

where \( \delta_1, \delta_2 \) are the derivations on the noncommutative torus and \( \gamma_i \) the Dirac matrices. Using an Ansatz

\[
f = \sum_{n,m} a_{nm} (\varphi, \psi) u^n v^m
\]

for the Dirac equation

\[
Df = 0
\]

we get a \( q \)-deformation of a classical partial differential equation in \( \varphi \) and \( \psi \) for \( a_{nm} \). So, solving the Dirac equation on \( S^4,\theta \) translates into a problem of solving a differential equation for certain \( q \)-special functions.

From a slightly different perspective, one can also understand this from the following consideration: It is one of the basic properties of quantum field theory that one can represent it in terms of infinitely many coupled harmonic oscillators. This property is not independent of the fact that the symmetry group of flat three dimensional space is related to \( SU(2) \) which in turn links
to the oscillator algebra. There is a similar situation for the case of the $S^{4,\theta}$. The algebras $S^{4,\theta}$ contain the suspension of a noncommutative 3-sphere given by an analytic continuation of the quantum group $SU_q(2)$ to the case where $q$ is a root of unity. Now, the quantum group $SU_q(2)$ relates through the Jordan-Schwinger isomorphism between the (roughly speaking dual) quantum algebra $\hat{U}_q(sl_2)$ and the extended algebra of two $q$-oscillators $A_q^{ext,2}$ to $q$-oscillators (see [KS]). By the $SU_q(2)$ symmetry of the suspended noncommutative 3-sphere, it follows that a projective module over $S^{4,\theta}$ carries always a coaction of basically $SU_q(2)$ on it (taking the coaction of $SU_q(2)$ on the suspended noncommutative 3-sphere and the trivial coaction on the generator $t$ of $S^{4,\theta}$). Remember that a projective module is the noncommutative analog of a vector bundle. Therefore, the quantum state space of a noncommutative gauge theory over $S^{4,\theta}$ should always carry a corepresentation of $SU_q(2)$. But then, by the results cited above, such a noncommutative quantum field theory can always be interpreted as a system of coupled $q$-oscillators. Now, $q$-oscillators relate to $q$-Hermite polynomials as $q$-special functions (see [KS]). So, at least if we treat the quantum field theories on $S^{4,\theta}$ perturbatively, starting from a system of free $q$-oscillators, we should, again, be led to expressions in terms of $q$-special functions.

In conclusion, both arguments suggest that perturbative quantum field theory on $S^{4,\theta}$ should have a formulation in terms of $q$-special functions. Since in usual quantum gauge theories on commutative space-time, the weights of Feynman graphs can be calculated as values of so called multiple polylogarithms (see [BK], [Kre]) and this fact plays a considerable role for the structural properties of quantum field theory (see [CK 1998], [Kon], [KoSo]), a first step toward perturbative quantum field theory on $S^{4,\theta}$ should be to look for $q$-deformations of multiple polylogarithms. For the logarithm and dilogarithm cases, $q$-deformations have been studied (see e.g. [K11], [Koo] and the literature cited therein) but even for the dilogarithm case, we use a slightly different form for the $q$-deformation, here. The main advantage of the form which we give, here, is that the $q$-deformed multiple polylogarithms form an algebra, then, and for the restriction to the case of $q$-deformed multiple $\zeta$-values we even get a noncommutative deformation of the classical Hopf algebra structure of multiple polylogarithms (see [Gon 2001]). Since precisely these structural features of multiple polylogarithms give the above mentioned link to quantum field theory, we consider this to be an important property for the purpose we have in mind.
For the reader who is not acquainted with the notations and conventions of $q$-calculus and $q$-special functions, we refer e.g. to [KS] for a short introduction.

2  $q$-deformed multiple polylogarithms

Let

$$Li_{n_1,\ldots,n_m}(z_1,\ldots,z_m; q) = \sum_{0<k_1<\ldots<k_m} \frac{z_1^{k_1}\ldots z_m^{k_m}}{(1-q^{k_1})^{n_1}\ldots(1-q^{k_m})^{n_m}}$$

be the $q$-deformed multiple polylogarithms. We have

$$(1-q)^{n_1+\ldots+n_m} Li_{n_1,\ldots,n_m}(z_1,\ldots,z_m; q) \xrightarrow{q\to 1} Li_{n_1,\ldots,n_m}(z_1,\ldots,z_m)$$

where

$$Li_{n_1,\ldots,n_m}(z_1,\ldots,z_m) = \sum_{0<k_1<\ldots<k_m} \frac{z_1^{k_1}\ldots z_m^{k_m}}{k_1^{n_1}\ldots k_m^{n_m}}$$

are the classical multiple polylogarithms. With the $q$-derivative defined as (see e.g. [KS])

$$D_{z,q}f(z_0) = \frac{f(qz_0) - f(z_0)}{(q-1)z_0}$$

we have the following lemma:

**Lemma 1** We have

$$D_{z,q}Li_{n_1,\ldots,n_m}(z_1,\ldots,z_m; q) = \frac{1}{(1-q)z_j} Li_{n_1,\ldots,n_j-1,\ldots,n_m}(z_1,\ldots,z_m; q)$$

Especially

$$D_{z,q}Li_n(z, q) = \frac{1}{(1-q)z} \cdot \frac{1}{1-z}$$

**Proof.** By calculation. ■

In contrast to the $q$-deformations introduced for the logarithm and the dilogarithm case in [Kir] and [Koo], we have chosen a version for the $q$-deformation which has the following important property:
Lemma 2 The rational linear span of the $Li_{n_1,\ldots,n_m}(z_1,\ldots,z_m;q)$ forms an associative algebra over $\mathbb{Q}$.

Proof. Follows from the definition. ■

A special case of this is the fact that the so called symmetry relations (see [Gon 1997]) hold also in the $q$-deformed case, as is seen from the following corollary:

Corollary 3 We have

$$Li_1(x,q)Li_1(y,q) = Li_{1,1}(x,y,q) + Li(y,x,q) + Li_2(xy,q)$$

Also there is a $q$-deformed version of the so called distribution relations (see [Gon 1997] for the classical case. again), now:

Lemma 4 We have

$$Li_2(x,q^n) = \frac{1}{n} \sum_{y^n=x} Li_2(y,q)$$

and

$$Li_{1,1}(x_1,x_2,q^n) = \frac{1}{n} \sum_{y_i^n=x_i} Li_{1,1}(y_1,y_2,q)$$

Proof. Direct calculation. ■

An important consequence of the property

$$\frac{d}{dz_j} Li_{n_1,\ldots,n_m}(z_1,\ldots,z_m) = \frac{1}{z_j} Li_{n_1,\ldots,n_j-1,\ldots,n_m}(z_1,\ldots,z_m)$$

of the classical multiple polylogarithms is the integral formula

$$Li_{n_1,\ldots,n_m}(z_1,\ldots,z_m) = \int_0^{z_j} \frac{1}{w_j} Li_{n_1,\ldots,n_j-1,\ldots,n_m}(z_1,\ldots,w_j,\ldots,z_m)$$
which leads as a special case to the integral formula

$$
\zeta(n_1, ..., n_m) = \sum_{0 < k_1 < ... < k_m} \frac{1}{k_1^{n_1} ... k_m^{n_m}}
= \int_0^1 \frac{dt}{1-t} \circ \frac{dt}{t} \circ ... \circ \frac{dt}{t} \circ ... \circ \frac{dt}{1-t} \circ \frac{dt}{t} \circ ... \circ \frac{dt}{t} \circ ... \circ \frac{dt}{t}
\tag{1}
$$

noted by Kontsevich (see e.g. [Gon 2001] for the definition of the right hand integral) for the multiple \(\zeta\)-values (i.e. the multiple polylogarithms with all variables identical to unity and \(n_m > 1\)). E.g.

$$
\zeta(2) = \int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dt_1}{1-t_1} \wedge \frac{dt_2}{t_2}
$$

As a consequence of Lemma 1, we have a similar integral formula for the case of \(q\)-deformed multiple \(\zeta\)-values with the integral replaced by the \(q\)-integral (see e.g. [KS]) and the appropriate powers of \((1 - q)\) inserted. E.g.

$$
\zeta_q(2) = \frac{1}{1-q} \int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dq t_1}{1-t_1} \wedge \frac{dq t_2}{t_2}
$$

Up to now, we have considered the case where the multiple polylogarithms become \(q\)-deformed but the variables \(z_j\) remain classical commutative numbers from \(\mathbb{C}\), only. In the next section, we are going to consider variables which are \(q\)-commutative.

## 3 \(q\)-deformed multiple polylogarithms with \(q\)-commuting variables

Let \(z_j, \ j \in \mathbb{N}\) be \(q\)-commuting variables, i.e.

$$
z_i z_j = q \ z_j z_i, \ i < j
$$

We now have to give an ordering for the variables \(z_j\) in the definition of the \(q\)-deformed multiple polylogarithms as given above, in order to make this well defined in the case of \(q\)-commuting variables. We do this by requiring
that we order the variables from left to right according to increasing index, 
e.g.
\[ Li_{1,1} (z_1, z_2; q) = \sum_{0 < j < k} \frac{z_j z_k}{z_1 z_2} \left( 1 - q^j \right) \left( 1 - q^k \right) \]

but
\[ Li_{1,1} (z_2, z_1; q) = \sum_{0 < j < k} \frac{z_k z_j}{z_1 z_2} \left( 1 - q^j \right) \left( 1 - q^k \right) \]

With this definition, we have, again, the following property:

**Lemma 5** For the ordered definition of the \( q \)-deformed multiple polylogarithms with \( q \)-commuting variables as given above, the rational span of the \( q \)-deformed multiple polylogarithms forms an algebra over \( Q \). Especially, the symmetry relations hold also for the \( q \)-commuting variables, then.

**Proof.** Direct calculation, again. \( \blacksquare \)

Especially, this means that the commutator of two \( q \)-deformed multiple polylogarithms can be expressed as a sum of such.

Since the differential equation for \( q \)-deformed multiple polylogarithms remains formally valid in the case of \( q \)-commuting variables, too, we can next ask for an integral formula for the \( q \)-deformed multiple \( \zeta \)-values in this case. In trying to derive such a formula, one makes a strange observation: On the one hand, we have

\[ \zeta_q (n_1, \ldots, n_m) = \sum_{0 < k_1 < \ldots < k_m} \frac{1}{(1 - q^{k_1})^{n_1} \cdots (1 - q^{k_m})^{n_m}} \]

i.e. the \( \zeta_q (n_1, \ldots, n_m) \) do not depend on the variables \( z_j \) at all and therefore remain commuting with each other. On the other hand, consider e.g. the case of \( \zeta_q (2) \) where the integral formula reads

\[ \zeta_q (2) = \frac{1}{1 - q} \int_{0 \leq t_1 \leq t_2 \leq 1} \frac{d_q t_1}{1 - t_1} \land \frac{d_q t_2}{t_2} \]

Now, in a formal sense one would, in the case of \( q \)-commuting variables, consider the integrals not to satisfy the Fubini theorem if one multiplies to
expressions of this kind but one would expect the differentials formally to exchange with a factor \( q^{-1} \). So, formally this would lead to an exchange rule

\[
\zeta(n_1, \ldots, n_m; q) \zeta(j_1, \ldots, j_l; q) = q^{(\sum n_i)(\sum j_i)} \zeta(j_1, \ldots, j_l; q) \zeta(n_1, \ldots, n_m; q)
\]

for the \( q \)-deformed multiple \( \zeta \)-values for \( n_1 < \ldots < n_m < j_1 < \ldots < j_l \) in the ordering for the \( q \)-commutativity relations. This means that we should allow the arguments of the \( q \)-deformed multiple \( \zeta \)-values - i.e. the indices of the \( q \)-deformed multiple polylogarithms - to become noncommuting variables, too. In the sequel, we will assume this to be the case and the above relation for the \( q \)-deformed multiple \( \zeta \)-values to hold true.

We next come to the question of Hopf algebra structure. On the algebra of classical multiple polylogarithms there is a coproduct given (see [Gon 2001] for the definition). Since the coproduct \( \Delta \) preserves the subalgebra of classical multiple \( \zeta \)-values, we get a bialgebra \( \mathcal{Z} \) of classical multiple \( \zeta \)-values in this way. For the algebra \( \mathcal{Z}_q \) of \( q \)-deformed multiple \( \zeta \)-values as just defined, \( \Delta \) is no longer compatible as a coproduct. The reason for this is that the number of arguments - which determines the power of \( q \) in the above \( q \)-commutation relation - varies between the different terms of the coproduct (see the definition in [Gon 2001]). But this observation immediately suggests also the solution: One has to formally insert an appropriate number of \( q \)-commuting variables into the different terms of the coproduct in order to make it an algebra morphism of the deformed product. In conclusion, we have observed the following result, therefore:

**Lemma 6** There is a \( q \)-deformation \( \Delta_q \) of the coproduct \( \Delta \) of \( \mathcal{Z} \) which turns \( \mathcal{Z}_q \) into a bialgebra. Since there is a full Hopf algebra structure for \( \mathcal{Z} \) (see [Gon 2001]), we can by a general result in Hopf algebra cohomology (see e.g. [CP]) assume without loss of generality that \( \mathcal{Z}_q \) carries a full Hopf algebra structure, too.

**Remark 1** The commutators for the full algebra of \( q \)-deformed multiple polylogarithms - with \( q \)-commutation relations for both, the variables and the indices - are much too complicated to allow at present for an answer to the question if there exists a deformation of the full Hopf algebra of classical
multiple polylogarithms. But since the Hopf structure is intimately tied to the motivic origin of these periods (see [Gon 2001]), we expect that as a consequence of ultrarigidity (see [Sch 2001a], [Sch 2001b]), there exists no such deformation (because the multiple $\zeta$-values are supposed to give precisely those periods which are linked to the Grothendieck-Teichmüller group, see [Ko 1]).

**Remark 2** The Hopf algebra of Connes and Kreimer is supposed to be related to the Hopf algebra of classical multiple $\zeta$-values (see [CK 1998], [CK 1999]). So, we expect $\mathbb{Z}_q$ to be related to a $q$-deformation of the Connes-Kreimer algebra (see also [GS] for the question of deformations of the Connes-Kreimer algebra).

**Conjecture 7** By the arguments given in the introductory section and by the connection between the Hopf algebra structure of multiple $\zeta$-values and the Connes-Kreimer algebra mentioned in the previous remark, one expects $\mathbb{Z}_q$ to act on quantum field theories on $S^{4,0}$. On the other hand, we know that the Hopf algebra $\tilde{H}_{GT}$, introduced in [Sch 2001a] as a quantum analog of the Grothendieck-Teichmüller group, acts on quantum field theories on $S^{4,0}$ (see [Sch 2001b]). We conjecture that the corresponding representations of $\mathbb{Z}_q$ and $\tilde{H}_{GT}$, gained in this way, are isomorphic.

In conclusion, we have seen that it is possible to introduce a special $q$-deformation of multiple polylogarithms which allows for quantum counterparts of several of the algebraic features of periods which have become increasingly important in the study of quantum field theories in recent years.

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