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Energy-based modeling of localization and necking in plasticity

Tuncay Yalçinkaya\textsuperscript{a,b,\,*}, Giovanni Lancia\textsuperscript{c}

\textsuperscript{a}Department of Engineering Science, University of Oxford, Parks Road, Oxford OX1 3PJ, UK
\textsuperscript{b}Department of Aerospace Engineering, Middle East Technical University, 06800 Ankara, Turkey
\textsuperscript{c}Department of Civil and building Engineering, and Architecture, Polytechnic University of Marche, Via Brecce Bianche, 60131 Ancona, Italy

Abstract

In this paper two different non-local plasticity models are presented and compared to describe the necking and fracture through a non-convex energy, where fracture is regarded as the extreme localization of the plastic strain. The difference between the models arises from the evolution of plastic deformation. The first (rate-dependent) approach, proposed in Yalçinkaya et al. (2011) follows the principle of virtual work to get balance equations and the dissipation inequality, in order to obtain the plastic evolution equation. The free-energy is given by the sum of a non-convex plastic term, and two quadratic terms with respect to the elastic deformation and the plastic deformation gradient. In the second (rate-independent) model, developed in Del Piero et al. (2013a), the plastic evolution is determined by incremental minimization of an energy functional which is equal to the free-energy of the previous model. The numerical example considers a convex-concave plastic energy to address the response of a tensile steel bar, where plastic strains localize intrinsically up to fracture. The numerical results exhibit good agreement between the two models. The solutions provided by the rate-dependent model approach those of the rate-independent model, as the imposed deformation rate reduces.

Keywords: Non-convexity, necking, strain-gradient plasticity, localization, softening

1. Introduction

During simple tension or compression, ductile metals initially display macroscopically uniform deformation modes. However, after a finite amount of straining, it is common that such homogeneous deformation patterns yield to ones which are characterized by localized modes, such as localized bands and necking which is often a direct precursor to ductile rupture through profuse void formation and growth within the band (see e.g. Asaro and Rice (1977) and Asaro (1979)). There have been numerous methods developed to model the formation of the so called localization or the cohesive zone accompanied by the macroscopic softening of the material in the context of damage and plasticity theories. Recently the concept of cohesive energy diffused over the volume is considered in the modeling of brittle fracture (see e.g. Volokh (2004)) and damage (see e.g. Benallal and Marigo (2007) and Pham et al. (2011)), and in

\textsuperscript{*} Corresponding author. Tel.: +44 1865 2 83495
E-mail address: tuncay.yalcinkaya@eng.ox.ac.uk

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the variational theory of fracture (see e.g. Freddi and Royer-Carfagni (2010)). In particular, Pham and Marigo (2010) deals with a model of energy minimization, in which rupture is preceded by progressive damage. While in most of these frameworks the localization is governed by a damage variable, in the current approach it evolves as the intrinsic property of the cohesive or the free energy itself. Therefore the solution is strongly affected by the convexity-concavity properties of the energy. A convex cohesive energy produces a work-hardening response, with the deformation almost homogeneously distributed over the bar. On the other hand, a concave cohesive energy is required to capture a softening response accompanied by strain localization, where the fracture is obtained as the extreme localization on singular points. Analogous correlations between the shape of a cohesive functional and the evolution of fracture have been found in Del Piero and Truskinovsky (2009), Del Piero et al. (2013a), Lancioni (2014), Del Piero et al. (2013b) while, in the rate-dependent strain-gradient plasticity model (see Yalcinkaya et al. (2011), Yalcinkaya et al. (2012), Klusemann and Yalcinkaya (2013), Klusemann et al. (2013), Yalcinkaya (2013)), the convex-concave shape of a plastic energy determines the formation and evolution of deformation patterns at different length scales.

In this paper both rate-dependent and rate-independent formation of necking phenomena is covered through two different plasticity models based on the non-convexity of the free energy. The first framework, proposed in Yalcinkaya et al. (2011), Yalcinkaya et al. (2012) to simulate the formation of dislocation cell type microstructures, follows the principle of virtual work to get balance equations at macro (linear momentum balance) and at micro level. The dissipation inequality is exploited to get the microstress definitions, and the plastic evolution equation is obtained to satisfy the reduced dissipation inequality for thermodynamical consistency. It assumes an additive decomposition of the free-energy, which consists of a convex-concave plastic term, and two quadratic terms with respect to the elastic deformation and the plastic deformation gradient. The resulting rate-dependent model accounts for processes where the plastic deformation is partially recoverable. Both the displacement and the plastic slip are considered as primary variables. These fields are determined on a global level by solving simultaneously the linear momentum balance and the plastic slip evolution equation, which has a convex-concave term enabling the localization of the deformation field related fracture of the material. In the second model, developed in Del Piero et al. (2013b) and Del Piero et al. (2013a), the plastic evolution is determined by incremental minimization of a global energy functional which is composed of three parts as mentioned above. In this case, the plastic term is supposed to be totally dissipative. Therefore, plastic deformations are not recoverable and dissipative stress is different than the previous one. The resulting framework is rate-independent, contrary to the previous model. The identical convex-concave plastic potential is incorporated in this model as well, and the outcome is compared with experiments.

The purpose of this work is to compare the results and ability of the two rate-dependent and rate-independent plasticity models in capturing localization phenomena leading to fracture in ductile metallic materials. For the sake of thorough mechanistic understanding, the derivations and implementations are done in a 1D mathematical setting, which does not exclude its extension to multi slip strain gradient crystal plasticity setting. A convex-concave plastic potential is considered. The concavity of the function is used to simulate the localization leading to failure of a steel bar. When the amount of plastic deformation corresponds to the values in the concave region the instability starts and continues until the ultimate failure of the material.

The paper is organized as follows. First, in Section 2 the problem is shortly described. Then, in Section 3, the development of the rate dependent model is addressed. In section 4, the theory of the rate independent model is presented. Section 5, studies the numerical results from the both models. Last, in section 6 the concluding remarks are summarized.

2. Problem statement

In this paper the tensile deformation of a bar is studied, corresponding to a one-dimensional problem. Consider a one-dimensional domain \((0, l)\) of length \(l\). The displacement of a point \(x \in (0, l)\) at the time instant \(t\) is denoted by \(u = u(x,t)\). Here and in the following, the dependence on time of a certain variable is indicated by a subscript, when it is necessary. At the endpoints are assigned the boundary conditions \(u(0) = 0\) and \(u(l) = l \varepsilon_i\), where \(\varepsilon_i\) is an imposed deformation, function of time. We suppose that the deformation is decomposed additively into an elastic part \(\varepsilon^e\) and a plastic part \(\varepsilon^p = \gamma\), i.e.,

\[
\varepsilon'(x,t) = \varepsilon^e(x,t) + \varepsilon^p(x,t) = \varepsilon(x,t) = \varepsilon^e(x,t) + \gamma(x,t).
\]
3. The rate-dependent model

In this section the rate-dependent model, shortly called RD model, is summarized. The state variables are assumed to be the elastic strain $\varepsilon^e$, the plastic slip (plastic strain) $\gamma$ and the gradient of plastic slip $\gamma'$, and $\sigma = \sigma_i(x), \pi = \pi_i(x)$ and $\xi = \xi_i(x)$ are the respective conjugated stresses.

The balance equations

$$\sigma' = 0, \quad \pi - \sigma - \xi' = 0,$$

are obtained from the principle of virtual power (see Yalcinkaya et al. (2011) for details). Then, the dissipation inequality is considered in order to satisfy the thermodynamical consistency

$$D = P_1 - \dot{\psi} \geq 0,$$

where $P_1 = \sigma\dot{\varepsilon}^e + \pi\dot{\gamma} + \xi\dot{\gamma}'$ and $\psi(e^s, \gamma, \gamma') = \psi_e(e^s) + \psi_{\gamma}(\gamma) + \psi_{\gamma'}(\gamma')$ are the local internal power, and the free-energy, respectively. Assuming the constitutive assumptions

$$\sigma = \psi'_e(e^s), \quad \xi = \psi'_{\gamma'}(\gamma'),$$

the inequality (3) is reduced to

$$D = (\pi - \psi'_e)\dot{\gamma} = \sigma^d\dot{\gamma} \geq 0,$$

where the quantity within the brackets is identified as the dissipative stress $\sigma^d$. It follows that the stress conjugated to $\gamma$ is the sum of an energetic and a dissipative term

$$\pi = \psi'_e + \sigma^d.$$

Here and in the next rate-independent model we assume

$$\psi_e(e^s) = \frac{1}{2}Ee^s^2, \quad \text{and} \quad \psi_{\gamma'}(\gamma') = \frac{1}{2}A\gamma'^2,$$

and, from (4), the stresses have the expressions

$$\sigma = E\dot{e}^s, \quad \xi = A\dot{\gamma}'.$$

where $E$ is the Youngs modulus and $A$ is the non-local parameter including the internal length scale. To satisfy the second law of thermodynamics (5), we make the following general constitutive assumption

$$\sigma^d = s \left( \frac{\dot{\gamma}}{\dot{\gamma}_0} \right)^m \text{sign}(\dot{\gamma}),$$

in order to have a power law crystal plasticity type relation between the stress and the evolution of the plastic slip field. Here $s$ is the resistance to dislocation slip, $\dot{\gamma}_0$ is the reference slip rate and $m$ is the rate sensitivity exponent. In this paper we assume $m = 1$. In this context, let us define $c = s/\dot{\gamma}_0$ as a friction coefficient. Thus now $\sigma^d = c\dot{\gamma}$. Then after using (6), the balance equation (2) and (8), we obtain the evolution equation for the plastic deformation

$$E\dot{e}^s - \psi'_e + A\dot{\gamma}' - c\dot{\gamma} = 0.$$
Equations (2) and (10) are solved by implicit time integration, using $u$ and $\gamma$ as independent variables. Let $(u_t, \Delta)$ be the solution at the previous instant $t$. For a given time step $\tau$, the solution at the current instant $t+\tau$ is approximated by the first-order Taylor expansion

$$u_{t+\tau} = u_t + \tau u_t, \quad \gamma_{t+\tau} = \gamma_t + \tau \gamma_t.$$  

(11)

The rates $\dot{u_t}$ and $\dot{\gamma_t}$ are the unknowns of the following equations

$$(\sigma_t + \tau \dot{\sigma_t})' = 0, \quad \sigma_t + \tau \dot{\sigma_t} - (\psi_e'(\gamma_t) + \tau \psi_e''(\gamma_t) \dot{\gamma_t}) + Ay_t'' + \tau A y_t'' - c \gamma_t = 0,$$

(12)

which are obtained by linearizing equations (2) and (10). The last term in (12) has been approximated by the backward Euler integration $\dot{\gamma} = (\gamma_{t+\tau} - \gamma_t)/\tau$. Using (8), and the relations $\dot{\epsilon} = \dot{u_t} - \dot{\gamma_t}$ and $\sigma' = \sigma_t - \psi_e'(\gamma_t) + Ay_t''$, and being $\sigma' = 0$, the above equations turn into

$$(\dot{u_t}' - \dot{\gamma_t})' = 0, \quad E(\dot{u_t}' - \dot{\gamma_t}) - (\psi_e''(\gamma_t) + \frac{c}{\tau}) \dot{\gamma_t} + Ay_t'' = -\frac{\sigma_t'}{\tau}.$$  

(13)

The boundary conditions are

$$\dot{u_t}(0) = 0, \quad \dot{u_t}(l) = l \dot{e}; \quad \dot{\gamma_t}(0) = \dot{\gamma_t}(l) = 0 \quad \text{(hard b.c.)}, \quad \text{or} \quad \dot{\gamma_t}'(0) = \dot{\gamma_t}'(l) = 0 \quad \text{(soft b.c.)}. \quad (14)$$

4. The rate-independent model

In this section the basic ingredients of the rate-independent model proposed in Del Piero et al. (2013a) are recalled. This model will be labeled RD model in the following. Consider the body strain energy

$$E(\epsilon^e, \gamma) = \int_0^l \left( \psi_e(\epsilon^e) + \theta(\gamma) + \psi_\gamma(\gamma') \right) dx,$$

(15)

where $\psi_e$ is the elastic energy density, $\theta$ is the cohesive energy density, and $\psi_\gamma$ is the non-local energy. We assume that $\psi_e$ and $\psi_\gamma$ have the expressions (8), and that $\theta$ is monotonic increasing.

The energies $\psi_e$ and $\psi_\gamma$ are stored in the free-energy $\psi = \psi_e + \psi_\gamma$, while $\theta$ is totally dissipative. Thus it obeys the dissipation inequality

$$\dot{\theta}(\gamma) = \theta'(\gamma) \dot{\gamma} \geq 0,$$

(16)

which brings to

$$\dot{\gamma} \geq 0, \quad (17)$$

since $\theta$ is monotonic increasing. Within this framework, the dissipation inequality (3) reads $D = (\sigma - \psi_e) \dot{\epsilon} + \pi \dot{\gamma} + (\xi - \psi_\gamma) \dot{\gamma}' \geq 0$, and, with the constitutive assumptions (4), it reduces to

$$\pi \dot{\gamma} \geq 0. \quad (18)$$

Comparing (16) and (18), the internal stress conjugated to $\gamma$ is

$$\pi = \theta'(\gamma), \quad (19)$$

and it is totally dissipative.

Let $(\delta \epsilon^e, \delta \gamma)$ be a perturbation of a given configuration $(\epsilon^e, \gamma)$. Since the dissipation inequality (17) requires that $\gamma$ can only grow, $\delta \gamma$ must be non-negative. A configuration $(\epsilon^e, \gamma)$ is equilibrated if the first variation of the energy is non-negative

$$\delta E(\epsilon^e, \gamma; \delta \epsilon^e, \delta \gamma) = \int_0^l \left( \sigma \delta \epsilon^e + (\pi - \xi') \delta \gamma \right) dx + [\xi \delta \gamma]_l^0 \geq 0. \quad (20)$$

In the above inequality, integration by parts is applied to the non-local term, and the constitutive assumptions (4) and (19) are used. From it, for perturbations with $\delta \gamma = 0$, we obtain the macroscopic stress balance (2), and for arbitrary perturbations such that $\delta \epsilon^e + \delta \gamma = 0$, we get the inequalities

$$\sigma \leq \pi - \xi', \quad \xi(0) \delta \gamma(0) \leq 0, \quad \xi(l) \delta \gamma(l) \geq 0. \quad (21)$$
Inequality (21), represents a yield condition, which states that the stress \( \sigma \) cannot be greater than the yield limit \( \pi - \xi' \), and inequalities (21) are satisfied if \( \gamma = 0 \) (strong condition) or \( \xi = 0 \) (soft condition) at \( x = 0, l \).

As in the previous section, the evolution problem is formulated by considering \( u \) and \( \gamma \) as independent variables. The energy at the instant \( t \) rewrites

\[
F_t = E(u'_t - \gamma'_t, \gamma'_t) = \int_0^l \left( \psi(u'_t - \gamma'_t) + \theta(\gamma'_t) + \psi(\gamma'_t) \right) \, dx. \tag{22}
\]

We consider the linear approximations (11), and the second-order development of the energy at the instant \( t + \tau \)

\[
F_{t+\tau}(\bar{u}_t, \bar{\gamma}_t) = F_t + \tau F'_t(\bar{u}_t, \bar{\gamma}_t) + \frac{1}{2} \tau^2 F''_t(\bar{u}_t, \bar{\gamma}_t) = F_t + \tau J_t(\bar{u}_t, \bar{\gamma}_t), \tag{23}
\]

with \( J_t(\bar{u}_t, \bar{\gamma}_t) = \int_0^l \left( E(u'_t - \gamma'_t)(\bar{u}'_t - \bar{\gamma}'_t) + \theta'({\gamma}'_t){\bar{\gamma}}'_t + A\gamma'_t \right) \, dx + \frac{1}{2} \tau \int_0^l \left( E(\bar{u}'_t - \bar{\gamma}'_t)^2 + \theta''(\gamma'_t)\gamma'_t^2 + 2\theta'(\gamma'_t)\bar{\gamma}'_t \right) \, dx. \]

The incremental evolution is determined by solving the following constrained quadratic programming problem at each time step:

\[
(\bar{u}_t, \bar{\gamma}_t) = \arg\min \{ J_t(\bar{u}_t, \bar{\gamma}_t) \mid \bar{\gamma}_t \geq 0; \, \bar{u}_t(0) = 0, \, \bar{u}_t(l) = l\bar{\varepsilon}; \, \bar{\gamma}_t(0) = \bar{\gamma}_t(l) = 0 \} \tag{24}
\]

Both rate-dependent and rate-independent evolution problems are solved incrementally by means of finite element method, where the displacement and plastic slip fields are considered as the global degrees of freedom. Yalcinkaya et al. (2011) and Del Piero et al. (2013a) study the details of the numerical solution schemes.

5. Numerical modeling of necking and fracture in tensile steel bars

We consider the homogeneous bar of length \( l = 140 \) mm illustrated in Fig. 1(b), clamped at the left endpoint and subjected to the tensile displacement \( \varepsilon_l \) at the right endpoint. It schematizes the tensile test performed on the bone-shaped sample represented in Fig. 1(a), where the deformations of the enlargements at the extremities are neglected.

For the plastic energy (\( \psi_t \) in rate-dependent model and \( \theta \) in rate-independent model), we consider a piecewise cubic function. It is a convex function in the interval \( 0 \leq \gamma < 0.16 \), a concave function for \( 0.16 \leq \gamma < 0.7 \), and a constant function for \( \gamma > 0.7 \). Its analytical expression is

\[
\psi_t = \begin{cases}
0.75\gamma^2 - 1.59\gamma^3, & 0 \leq \gamma < 0.16, \\
0.013 + 0.1224(\gamma - 0.16) - 1700(\gamma - 0.16)^2 - 0.57(\gamma - 0.16)^3, & 0.16 \leq \gamma < 0.7, \\
0.2528 & \gamma \geq 0.7.
\end{cases} \tag{25}
\]

In Lancioni (2014), simple formulas are proposed, which relate the above polynomial coefficients to experimental data. For \( \gamma > 0.7 \), the bar deforms plastically without spending plastic energy, and this corresponds to complete breaking. Thus, the simulations presented in the following are interrupted when \( \gamma \) reaches the breaking value \( \gamma = 0.7 \).

The material parameters are: Young’s Modulus \( E = 210 \) GPa, the non-local parameter \( A = 2 \) kN, and the friction coefficient is identified as \( c = 0.015 \) GPa/s.

The experimental stress-elongation \( \sigma - \Delta s \) curves are compared in Fig. 1. In the horizontal axis, \( \Delta s \) represents the relative displacement between two points, indicated by dots in Fig. 1(a), and mutually distant \( \Delta x_{\text{meas}} = 80 \) mm, which is measured by an extensometer, and, in the vertical axis, \( \sigma \) is the normal stress. Three different deformation rates \( \dot{\varepsilon} = 1 \cdot 10^{-10}, 0.8 \cdot 10^{-3}, \) and \( 1 \cdot 10^{-2} \) s\(^{-1}\) are considered for the RD model. The simulations predict three phases: an initial perfectly elastic phase, which interrupts when \( \sigma \) reaches the yield value \( \sigma^c = 0.375 \) GPa, an hardening phase, and a final softening phase. The RI and RD models give practically the same hardening branches, but they differ in predicting the softening curves. The branch of the RI model is long and sloped, being very close to the experimental curve. The softening curves of the RD are shorter, and they strongly depend on the deformation rate \( \dot{\varepsilon} \) and they get longer for increasing deformation rate. The stress-strain response curves plotted in Fig. 2 differ from those of Fig. 1 for the values of \( \dot{\varepsilon} \) (imposed deformation) considered in the horizontal axis. Since \( \dot{\varepsilon} = 1 \cdot 10^{-10} \) is sufficiently small, the corresponding curve approximates the response curve of the limit static case \( \dot{\varepsilon} = 0 \). At this stage it is quite different from the curve of the RI model. However, note that the RD model gives results close to those of the RI model when small deformation rates are considered.
The evolution of $\gamma$ is described in Fig. 3. The results of the RI model and RD model with $\dot{\varepsilon} = 0.8 \cdot 10^{-3} \text{ s}^{-1}$ are compared. In both the cases, the softening phase is characterized by a progressive localization of $\gamma$ in smaller and smaller portions in the middle of the bar, up to the final fracture, occurring when $\gamma$ reaches the breaking value $\gamma = 0.7$. Thus both the RI and RD models describe fracture as the ending phase of a strain localization process. But they differ for the description outside the localization zone: for RI model, $\gamma$ maintain constant, and only the elastic deformation $\varepsilon$ reduces, being $\varepsilon = \sigma/E$ (perfectly elastic unloading); for the RD model, $\gamma$ reduces, and it is partially recovered outside the localization portion (elasto-plastic unloading). The deformation rates at fracture are reported in Fig. 4. Much higher rates are reached in the RD model case. Indeed the softening phase occurs much faster in the RD model than in the RI model.

High deformation rates are considered in Fig. 5 ($\dot{\varepsilon} = 0.1, 1, \text{ and } 10 \text{ s}^{-1}$). As the deformation rate increases, larger maximum stresses are attained, and the softening branches extend. Fig. 5(c) shows the delay in the evolution of the plastic deformation encountered when large deformation rates are considered. Note that at high deformation rates RD model produces homogeneous deformation field.
6. Summary and Conclusion

A comparison study is presented where the ability of two plasticity models on capturing the deformation localization leading to fracture of steel bars under tensile loading. The first model is a rate-dependent one deduced from classical thermodynamics, and it accounts for the partially recoverable plastic deformations. The other one is a rate-independent variational model which accounts for incremental energy minimization, and it considers totally dissipative plastic deformations. Both plasticity models are enhanced by a gradient energy and a nonconvex plastic potential contribution. The first contribution introduces a length parameter into the model allowing studies at different lengths scales and the second one introduces an instability leading to localization of the deformation field. The instability due to second contribution is stabilized by the gradient energy function. The numerical examples illustrate the agreement between the models and the experiment. At low deformation rates the results of the rate-dependent model approaches to the rate-independent case. However when the rate is increased the rate-dependent model gives homogeneous solution. The numerical examples illustrate the potential of both models to capture rate-dependent and rate-independent formation of necking.

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Fig. 5. (a) $\sigma - \varepsilon$ response curves at high deformation rates. (b) Enlargement at the onset of the hardening plastic phase. (c) Profiles of $\gamma$ at $\varepsilon = 0.0025$, for different deformation rates.

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