ON HECKE $L$-FUNCTIONS ATTACHED TO CUSP FORMS OF HALF-INTEGRAL WEIGHT

Winfried Kohnen

(Received 26 February 2020 and revised 13 May 2020)

Abstract. We give a characterization of cusp forms of half-integral weight of level four in the plus space in terms of a functional equation of attached $L$-series.

1. Introduction

A classical theorem of Hecke asserts that a Fourier series in $q = e^{2\pi iz}$ ($z \in \mathcal{H} =$ complex upper half-plane) indexed by positive integers and with coefficients of polynomial growth is a cusp form of integral weight $k$ for $\Gamma_1 = \text{SL}_2(\mathbb{Z})$ if and only if the associated Hecke $L$-function completed with a $\Gamma$-factor has good analytic properties and in particular satisfies a functional equation under $s \mapsto k - s$. Later on this result was generalized by Weil [8] for cusp forms for the Hecke congruence subgroups $\Gamma_0(N) \subset \Gamma_1$ of level $N$ where now one has to require good analytic properties of $L$-functions twisted by sufficiently many primitive Dirichlet characters of conductor prime to $N$.

Various generalizations later have been given by several authors where the number and type of twists necessary differ. For example, in [6] one only needs twists by the finitely many (not necessarily primitive) characters modulo $Nc$, where $1 \leq c \leq N$, and in [1] one twists by modified Ramanujan sums. The basic idea underlying the proofs of all the above results is the theory of Mellin transforms.

On the other hand, there is Shimura’s theory of modular forms of half-integral weight [7] which is important, for example, since modular forms of half-integral weight $k + \frac{1}{2}$ and level $4N$ explicitly relate to modular forms of even integral weight $2k$ and level $2N$ [4, 7] (Shimura correspondence). It is likely and maybe even obvious that most of the above indicated characterizations of cusp forms of integral weight by twisting mutatis mutandis generalize to the case of half-integral weight; cf. e.g. p. 481 in [7].

Now let $N$ be odd and squarefree and denote by $S_{k + \frac{1}{2}}^+(4N)$ the subspace of cusp forms of weight $k + \frac{1}{2}$ and level $4N$ whose Fourier coefficients of index $n$ vanish unless $(-1)^kn \equiv 0, 1 \pmod{4}$ [2, 3]. This space is canonically Hecke isomorphic to the space of cusp forms of weight $2k$ and level $N$. It is natural to ask (maybe a bit vaguely) in this context if good analytic properties of twisted $L$-functions can be used to guarantee that a Fourier series is modular and in fact lies in the plus subspace. In this note we treat the case $N = 1$ and give an affirmative answer; see the Theorem in Section 2. In fact, as in Hecke’s case, only one

2010 Mathematics Subject Classification: Primary 11F37, 11F66.

Keywords: modular forms of half-integral weight; $L$-functions.

© 2021 Faculty of Mathematics, Kyushu University
functional equation and no twists are necessary here. The proof is simple and mainly uses
the fact that the plus space in level four can be characterized as an eigenspace of a certain
quadratic operator involving the Fricke involution [2]. It is not quite obvious if and in what
direction our result can be generalized to \( N > 1 \) and we leave this as an open problem to the
reader. Note that also for \( N > 1 \) the plus subspace is an eigenspace of a quadratic operator
which, however, is more complicated to describe [3].

2. Statement of result and proof

We start with recalling some elementary definitions from [7]. The standard operation of
\( \text{GL}_2^+(\mathbb{R}) \) on \( \mathcal{H} \) is denoted by \( (A, z) \mapsto Az \). We let \( \mathcal{G} \) be the group of all pairs \( (A, \phi(z)) \) where
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(\mathbb{R})
\]
and \( \phi : \mathcal{H} \to \mathbb{C} \) is a holomorphic function such that \( \phi(z)^2 = t (\det A)^{-1/2} (cz + d) \) with \( t \in \mathbb{C} \),
\(|t| = 1\). The group law is given by
\[
(A, \phi(z))(B, \psi(z)) = (AB, \phi(Bz)\psi(z)).
\]

Let \( k \) be an integer. Then \( \mathcal{G} \) operates on functions \( f : \mathcal{H} \to \mathbb{C} \) in weight \( k + \frac{1}{2} \) by
\[
(f|_{k+\frac{1}{2}}(A, \phi))(z) = \phi(z)^{-2k-1} f(Az).
\]
We simply write \(| \) instead of \(|_{k+\frac{1}{2}} \) if there is no confusion possible.

One has an embedding \( \Gamma_0(4) \hookrightarrow \mathcal{G} \), \( A \mapsto A^* := (A, j(A, z)) \),
where
\[
j(A, z) := \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} -4 \\ d \end{pmatrix} (cz + d)^{1/2}
\]
(we always choose the argument by \(-\pi/2 < \arg(z^{1/2}) \leq \pi/2 \) \( (z \in \mathbb{C}^*) \)).

Recall that a holomorphic function \( f : \mathcal{H} \to \mathbb{C} \) is called a cusp form of weight \( k + \frac{1}{2} \)
and level four if \( f|_{k+\frac{1}{2}} A^* = f \) for all \( A \in \Gamma_0(4) \) and \( f \) is holomorphic and zero at all
cusps. We denote the space of such functions by \( S_{k+\frac{1}{2}}(4) \). The group \( \Gamma_0(4) \) has three cusps,
represented by \( \infty \), 0 and \( \frac{1}{2} \). The latter is \((k + \frac{1}{2})\)-irregular and therefore one needs to check
the cuspidal conditions to be zero only at \( \infty \) and 0. Each \( f \in S_{k+\frac{1}{2}}(4) \) has a Fourier expansion
\[
f(z) = \sum_{n\geq 1} a(n)q^n \ (q = e^{2\pi iz}, \ z \in \mathcal{H}) \text{ at } \infty.
\]

We let
\[
W_4 := \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}, \ (-2iz)^{1/2}
\]
Then \( W_4^2 \) acts as the identity and \( W_4 \) operates on \( S_{k+\frac{1}{2}}(4) \) as an involution (Fricke involution).

For a power series \( \sum_{n\geq 0} a(n)q^n \) we set
\[
\sum_{n\geq 0} a(n)q^n |U_4 := \sum_{n\geq 0} a(4n)q^n.
\]
Recall that \( U_4 \) maps \( S_{k+\frac{1}{2}}(4) \) to itself.
We let $S_{k+\frac{1}{2}}^+(4)$ be the subspace of $S_{k+\frac{1}{2}}(4)$ consisting of forms $f$ whose Fourier coefficients $a(n)$ ($n \geq 1$) vanish unless $(-1)^kn \equiv 0, 1 \pmod{4}$ [2, 3].

Finally, if $\sum_{n \geq 1} a(n)q^n$ is a power series with constant term zero and $a(n) \ll n^c$ for some $c > 0$, we denote by $L(f, s) = \sum_{n \geq 1} a(n)n^{-s}$ $(\sigma := \Re(s) > c + 1)$ the associated $L$-series. We set $\Lambda(f, s) := \pi^{-s}\Gamma(s)L(f, s)$ $(\sigma > c + 1)$.

**Theorem.** Let

$$f(z) = \sum_{n \geq 1} a(n)q^n \quad (q = e^{2\pi i z}, z \in \mathcal{H}),$$

where $a(n) \in \mathbb{C}$ and $a(n) \ll n^c$ for some $c > 0$. Then the following are equivalent:

(i) $f \in S_{k+\frac{1}{2}}^+(4)$;

(ii) the functions $\Lambda(f, s)$ and $\Lambda(f|U_4, s)$ have holomorphic continuation to $\mathbb{C}$, are bounded in every vertical strip and satisfy the functional equation

$$\Lambda(f|U_4, k + \frac{1}{2} - s) = \left(\frac{2}{2k + 1}\right)^2 \Lambda(f, s).$$

**Proof.** (i) $\Rightarrow$ (ii) Since $f \in S_{k+\frac{1}{2}}^+(4)$, by the classical Hecke argument using Mellin transforms we see that $\Lambda(f, s)$ has holomorphic continuation to an entire function, is bounded in every vertical strip and satisfies the functional equation

$$\Lambda(f|W_4, k + \frac{1}{2} - s) = \Lambda(f, s).$$

Since $f|U_4 \in S_{k+\frac{1}{2}}(4)$, the same *mutatis mutandis* with $f$ replaced by $f|U_4$ holds. In particular, replacing $s$ by $k + \frac{1}{2} - s$ we find

$$\Lambda(f|U_4|W_4, s) = \Lambda(f|U_4, k + \frac{1}{2} - s).$$

Since $f$ is in the plus subspace, one has [2]

$$f|U_4|W_4 = \left(\frac{2}{2k + 1}\right)^2 f. \quad (1)$$

Hence (ii) holds.

(ii) $\Rightarrow$ (i) From the regularity conditions and the functional equation we find in the usual way using inverse Mellin transforms and holomorphic continuation that (1) holds.

As is well known (cf. e.g. [6, Proposition 3]), the group $\Gamma_0(4)$ is generated by the matrices

$$-E, \quad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}.$$}

Certainly

$$f|(-E)^s = f, \quad f\left|\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right|^s = f.$$}

Furthermore, by direct computation one sees that

$$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}^* = W_4 \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^* W_4^{-1}, \quad (2)$$
and from (1) and (2) it follows that

\[ f \left| \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \right|^* = f. \]

Therefore, \( f \) satisfies the required transformation formulas with respect to \( \Gamma_0(4) \).

Also, since \( a(n) \ll n^c \), it follows that \( f(x + iy) \ll y^{-c-1} \) as \( y \to 0 \), uniformly in all real \( x \) [5, I-3, Proposition 1]. The latter implies that \( f \) is holomorphic at all cusps, see [5, V-14, Lemma]. Hence \( f \) is a modular form of weight \( k + \frac{1}{2} \) and level four. Furthermore \( f \) vanishes at \( \infty \) by assumption, and vanishes at 0 because by (1) we have

\[ \left( \frac{2}{2k + 1} \right)^2 f|W_4 = f|U_4. \]

Finally, by (1) \( f \) is in the plus subspace. This shows that (ii) implies (i). \( \square \)

REFERENCES

[1] S. Bettin, J. W. Bober, A. R. Booker, B. Conrey, Min Lee, G. Moltani, T. Oliver, D. J. Platt and R. S. Steiner. A conjectural extension of Hecke’s converse theorem. Ramanujan J. 47(3) (2018), 659–684.

[2] W. Kohnen. Modular forms of half-integral weight on \( \Gamma_0(4) \). Math. Ann. 248 (1980), 249–266.

[3] W. Kohnen. Newforms of half-integral weight. J. Reine Angew. Math. 333 (1982), 32–72.

[4] S. Niwa. Modular forms of half-integral weight and the integral of certain theta-functions. Nagoya Math. J. 56 (1975), 147–161.

[5] A. Ogg. Modular Forms and Dirichlet Series. Benjamin, New York, 1969.

[6] M. Razar. Modular forms for \( \Gamma_0(N) \) and Dirichlet series. Trans. Amer. Math. Soc. 231(2) (1977), 489–495.

[7] G. Shimura. On modular forms of half-integral weight. Ann. of Math. 97 (1973), 440–481.

[8] A. Weil. Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. Math. Ann. 168 (1967), 149–156.

Winfried Kohnen
Universität Heidelberg
Mathematisches Institut
INF 205
69120 Heidelberg
Germany
(E-mail: winfried@mathi.uni-heidelberg.de)