Mass Inflation in Quantum Gravity

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ABSTRACT

Using the canonical formalism for spherically symmetric black hole inside the apparent horizon we investigate the mass inflation in the Reissner-Nordström black hole in the framework of quantum gravity. It is shown that like in classical gravity the combination of the effects of the influx coming from the past null infinity and the outflux backscattered by the black hole’s curvature causes the mass inflation even in quantum gravity. The results indicate that the effects of quantum gravity neither alter the classical picture of the mass inflation nor prevent the formation of the mass inflation singularity.

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1. Introduction

It is well known that a spherically symmetric solution of the Einstein equation with an electric charge is Reissner and Nordström geometry [1]. Since we expect that any macroscopic body in astronomical universe does not possess a net electric charge by the neutralization process with the surrounding, a consideration of the Reissner-Nordström black hole may seem to be outside the realm of reality. Nevertheless, a study of the Reissner-Nordström black hole leads us to a useful understanding of the structure of the spacetime. This is because in spite of its simple geometrical form the Reissner-Nordström black hole has two horizons, an external event horizon and an internal Cauchy horizon, whose natures are also shared by a more realistic Kerr black hole.

The existence of a region beyond the Cauchy horizon is rather disturbing since one cannot predict a future evolution of events in this region since the informations which cannot be determined by the initial data come from the singularity owing to the timelike character in an uncontrollable way. Thus the Cauchy horizon would give us a possibility of the violation of the strong cosmic censorship hypothesis of Penrose [2].

The Cauchy horizon in both the Reissner-Nordström black hole and the Kerr one is unstable against small perturbations in the initial data [3]. Namely, the energy-momentum tensor associated with the matter fields diverges at the Cauchy horizon since this null surface is that of infinite blueshift. It is thus widely believed that the Cauchy horizon would become a curvature singularity if the back reaction of the matter fields on the geometry is taken account of in a self-consistent manner although we do not have a satisfying proof of this conjecture yet [3].

A few years ago, Poisson and Israel have proposed an interesting model which examines the physical behavior of the Cauchy horizon of the charged black hole under the back reaction of the matter fields [4]. They have shown that the curvature singularity is indeed formed along the Cauchy horizon but this singularity is rather
mild in that the metric tensor is regular while the derivatives of it diverge on the Cauchy horizon. Moreover it was shown that this singularity is characterized by the exponential divergence of the local mass function with an advanced time, what is called, mass inflation. This interesting phenomenon have been subsequently investigated in more detail [5, 6, 7, 8].

Recently we have constructed the canonical formalism of a system with a spherically symmetric black hole inside the apparent horizon [9]. Since in this region the Killing vector field \( \frac{\partial}{\partial t} \) becomes spacelike while it does timelike in the exterior region, one must foliate the interior region of a black hole by a family of spacelike hypersurfaces, for example, \( r = \text{const} \). As one of applications of this canonical formalism, we have investigated the black hole radiation and shown that the mass-loss rate by the black hole radiation is exactly equal to that evaluated by Hawking in the semiclassical approximation [10].

In this article, based on the canonical formalism constructed in the previous work [9] we would like to study how quantum gravity would modify the classical picture of the mass inflation. Since singular behavior of the Cauchy horizon suggests that the classical theory of general relativity would be broken down and quantum gravity would play a dominant role there, it is very natural to study the mass inflation in the framework of quantum gravity.

The article is organized as follows. In section 2, we review a construction of the canonical formalism inside the apparent horizon. In section 3, by using the canonical formalism reviewed in section 2 we will make a formalism of quantum gravity holding in the vicinity of the apparent horizons. In section 4, we apply this formalism for the mass inflation. The last section is devoted to conclusion.

2. Review of canonical formalism

We begin by a review of our previous work [9] of constructing a canonical formalism of a spherically symmetric system with a black hole inside the horizon
(See the references [11, 12] for the canonical formalism outside the horizon). The important point is that the radial coordinate plays a role of time inside the horizon in the spherically symmetric coordinate system. Thus we must take a choice of $x^1 = \text{const}$ hypersurfaces to slice the spacetime. Later we will take the simplest choice $x^1 = r$.

The four dimensional action which we consider in this section is of the form

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi} (4) R - \frac{1}{4\pi} (4) g^{\mu\nu} (D_\mu \Phi)^\dagger D_\nu \Phi - \frac{1}{16\pi e^2} F_{\mu\nu} F^{\mu\nu} \right], \quad (1)$$

where $\Phi$ is a complex scalar field, $A_\mu$ the electromagnetic field, $F_{\mu\nu}$ the corresponding field strength, and $e$ is the electric charge of $\Phi$. To clarify the four dimensional meaning we put the suffix (4) in front of the metric tensor and the curvature scalar. We follow the conventions adopted in the MTW textbook [13] and use the natural units $G = \hbar = c = 1$. The Greek indices $\mu, \nu, \ldots$ take the values 0, 1, 2, and 3, on the other hand, the Latin indices $a, b, \ldots$ take the values 0 and 1.

The most general spherically symmetric assumption for the metric is

$$ds^2 = (4) g_{\mu\nu} dx^\mu dx^\nu,$$

$$= g_{ab} dx^a dx^b + \phi^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2)$$

where the two dimensional metric $g_{ab}$ and the radial function $\phi$ are the functions of only the two dimensional coordinates $x^a$. The substitution of (2) into (1) and then integration over the angular coordinates $(\theta, \varphi)$ leads to the following two dimensional effective action

$$S = \frac{1}{2} \int d^2x \sqrt{-g} \left[ 1 + g^{ab} \partial_a \phi \partial_b \phi + \frac{1}{2} R \phi^2 \right]$$

$$- \int d^2x \sqrt{-g} \phi^2 g^{ab} (D_a \Phi)^\dagger D_b \Phi - \frac{1}{4} \int d^2x \sqrt{-g} \phi^2 F_{ab} F^{ab}, \quad (3)$$

† We replace the coefficient of the second term $\frac{1}{4\pi}$ with $\frac{1}{8\pi}$ in case of the neutral scalar field as will be treated in later sections.
where
\[ D_a \Phi = \partial_a \Phi + ieA_a \Phi. \] (4)

Next let us rewrite the action (3) into the first-order ADM form. As mentioned before, we shall take the $x^1$ coordinate as time to cover the internal region of a black hole by spacelike hypersurfaces. Then the appropriate ADM splitting of (1+1)-dimensional spacetime is given by
\[ g_{ab} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \frac{\alpha^2}{\gamma} - \beta^2 \end{pmatrix}, \] (5)
and the normal unit vector $n^a$ which is orthogonal to the hypersurface $x^1 = \text{const}$ reads
\[ n^a = \left( \frac{\alpha}{\beta \gamma}, -\frac{1}{\beta} \right). \] (6)

After using the various formulae derived in the Ref.[9], the action (3) can be written as
\[ S = \int d^2x L = \int d^2x \left[ \frac{1}{2} \beta \sqrt{\gamma} \{ 1 - (n^a \partial_a \phi)^2 + \frac{1}{\gamma} \dot{\phi}^2 - Kn^a \partial_a (\phi^2) \right. \\
+ \left. \frac{\dot{\beta}}{\beta \gamma} \partial_0 (\phi^2) \} + \beta \sqrt{\gamma} \dot{\phi}^2 \{ |n^a D_a \Phi|^2 - \frac{1}{\gamma} |D_0 \Phi|^2 \} + \frac{1}{2} \beta \sqrt{\gamma} \dot{\phi}^2 E^2 \right] \\
+ \int d^2x \left[ \frac{1}{2} \partial_a (\beta \sqrt{\gamma} Kn^a \phi^2) - \frac{1}{2} \partial_0 (\frac{\dot{\beta}}{\sqrt{\gamma} \dot{\phi}^2}) \right] \right], \] (7)

where
\[ E = \frac{1}{\sqrt{-g}} F_{01} = \frac{1}{\beta \sqrt{\gamma}} (\dot{A}_1 - A'_0), \] (8)
the trace of the extrinsic curvature $K = g^{ab} K_{ab}$ is expressed by
\[ K = -\frac{\gamma'}{2 \beta \gamma} + \frac{\dot{\alpha}}{\beta \gamma} - \frac{\alpha}{2 \beta \gamma^2} \dot{\gamma}, \] (9)
and $\frac{\partial}{\partial x^0} = \partial_0$ and $\frac{\partial}{\partial x^1} = \partial_1$ are also denoted by an overdot and a prime, respectively.
By taking the variation of the action (7) with respect to the \( x^1 \) derivative of
the canonical variables \( \Phi(\Phi^\dagger), \phi, \gamma \) and \( A_0 \) we have the corresponding conjugate
momenta \( p_\Phi(p_{\Phi^\dagger}), p_\phi, p_\gamma \) and \( p_A \)

\[
p_\Phi = -\sqrt{\gamma} \phi^2 (n^a D_0 \Phi)^\dagger, \quad (10)
\]

\[
p_\phi = \sqrt{\gamma} n^a \partial_a \phi + \sqrt{\gamma} K \phi, \quad (11)
\]

\[
p_\gamma = \frac{1}{4\sqrt{\gamma}} n^a \partial_a (\phi^2), \quad (12)
\]

\[
p_A = -\phi^2 E. \quad (13)
\]

The Hamiltonian \( H \) is expressed by a linear combination of constraints as expected:

\[
H = \int dx^0 (\alpha H_0 + \beta H_1 + A_1 H_2), \quad (14)
\]

where

\[
H_0 = \frac{1}{\gamma} [p_\Phi D_0 \Phi + p_{\Phi^\dagger} (D_0 \Phi)^\dagger] + \frac{1}{\gamma} p_\phi \dot{\phi} - 2 \dot{p}_\gamma - \frac{1}{\gamma} p_\gamma \dot{\gamma}, \quad (15)
\]

\[
H_1 = \frac{1}{\sqrt{\gamma} \phi^2} p_\Phi p_{\Phi^\dagger} - \frac{\sqrt{\gamma}}{2} - \frac{\dot{\phi}^2}{2\sqrt{\gamma}} + \partial_0 \left( \frac{\partial_0 (\phi^2)}{2\sqrt{\gamma}} \right) + \frac{\dot{\phi}^2}{\sqrt{\gamma}} |D_0 \Phi|^2 - 2\sqrt{\gamma} p_\phi p_\gamma + \frac{2\gamma \sqrt{\gamma}}{\phi^2} p_\gamma^2 + \frac{\sqrt{\gamma}}{2\phi^2} p_A^2, \quad (16)
\]

\[
H_2 = -ie (p_\Phi \Phi - p_{\Phi^\dagger} \Phi^\dagger) - \hat{p}_A. \quad (17)
\]

Note that \( \alpha, \beta \) and \( A_1 \) are non-dynamical Lagrange multiplier fields.
The action can be cast into the first-order ADM canonical form by the dual Legendre transformation

\[ S = \int dx^1 \left[ \int dx^0 (p_\Phi \Phi' + p_\Phi^\dagger \Phi'^\dagger + p_\phi \phi' + p_\gamma \gamma' + p_A A'_0) - H \right]. \] 

(18)

As Regge and Teitelboim pointed out [14], in order to have the correct Hamiltonian which produces the Einstein equations through the Hamilton equations, one has to supplement the surface terms to the Hamiltonian (14). In the present formalism, since we take the variation of all the fields to be zero at both the singularity and the apparent horizon we do not have to add any surface terms to the Hamiltonian. This is a big difference between the present formalism considering the region inside the horizon and the previous formalism [11, 12] outside the horizon where a surface term at the spatial infinity needs to be added.

3. Quantum gravity between outer and inner apparent horizons

In this section, for the purpose of applying the canonical formalism constructed in the previous section for understanding the mass inflation [4] in the Reissner-Nordström black hole in quantum gravity, we will make a quantum gravity holding in the vicinity of apparent horizons where the complicated constraints have a tractable form while retaining the important features of the black hole physics. For simplicity, let us restrict ourselves to the neutral scalar field as the matter field, for which the $U(1)$ charge $Q$ has a fixed constant value and $p_A$ in Eq.(13) becomes $Q^2$. Thus under this situation the constraint $H_2$ generating the $U(1)$ gauge transformation identically vanishes. Moreover, we shall consider the charged Vaidya metric since the Reissner-Nordström geometry is converted into the charged Vaidya form with a spherically symmetric stream of the massless matters.

First of all, we shall consider the ingoing charged Vaidya metric. In order to
do so we will take the two dimensional coordinates \( x^a \) by

\[
x^a = (x^0, x^1) = (w - r, r),
\]

where \( w \) is the standard external advanced time coordinate which is infinite both on the future null infinity and the Cauchy horizon \([4, 7]\). Note that we have chosen \( x^1 = r \) as mentioned earlier. And we fix the gauge freedoms corresponding to the two dimensional reparametrization invariances by the gauge conditions

\[
g_{ab} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \frac{\alpha^2}{\gamma} - \beta^2 \end{pmatrix},
\]

\[
= \begin{pmatrix} -(1 - \frac{2M}{r} + \frac{Q^2}{r^2}) & \frac{2M}{r} - \frac{Q^2}{r^2} \\ \frac{2M}{r} - \frac{Q^2}{r^2} & 1 + \frac{2M}{r} - \frac{Q^2}{r^2} \end{pmatrix},
\]

where \( M \) is the mass function that depends only on \( w \). From these equations the two dimensional line element takes a form of the ingoing charged Vaidya metric

\[
ds^2 = g_{ab}dx^adx^b,
\]

\[
= -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dw^2 + 2dwdr.
\]

Similarly, we can construct the outgoing charged Vaidya metric directly from the above ingoing one in the replacement of the advanced time coordinate by the retarded one.

For later convenience let us introduce the two dimensional metric in radial double-null coordinates \((u, v)\), which has the form

\[
ds^2 = -2e^{-\lambda}dudv,
\]

where \( \lambda = \lambda(u, v) \), and \( u \) and \( v \) are a retarded time and an advanced time, respectively.
For a dynamical black hole, it is sometimes useful to consider the local definition of horizon, i.e., the apparent horizon, rather than the global one, the event horizon. In case of the Reissner-Nordström black hole, the apparent horizon consists of two horizons. One is the outer apparent horizon given by $r_+ = M + \sqrt{M^2 - Q^2}$ and the other the inner apparent horizon $r_- = M - \sqrt{M^2 - Q^2}$. In this article we confine ourselves to be $M > Q$.

In the literature [7], the classical analysis of the mass inflation has been achieved only on the two intersecting null surfaces $S^+$ and $S^-$ where the surfaces $S^+$ is radial rightgoing (↗) null geodesics, and the surface $S^-$ radial leftgoing (↘) null one. We fix the system of coordinates such that $u = 0$ is the event horizon and $v = 0$ the Cauchy horizon. Then we choose $S^+$ to be parallel to and near the event horizon while we take $S^-$ to coincide with the Cauchy horizon.

Now we would like to consider the physics along $S^+$. According to a similar procedure to our previous work [9] which took advantage of the idea in the reference [15], let us attempt to solve the Hamiltonian and momentum constraints only in the vicinity of the outer apparent horizon. Here note that in the Poisson-Israel model [4], the outer apparent horizon approaches the event horizon asymptotically around the Cauchy horizon under a flow of infalling lightlike flux. Thus it is reasonable to assume that $S^+$ is located in the vicinity of the outer apparent horizon since we are interested in the region $S = S^+ \cap S^-$. Near the outer apparent horizon, we make an approximation

$$\Phi \approx \Phi(w), M \approx M(w), \phi \approx r. \quad (23)$$

From now on we shall use $\approx$ to indicate the equalities which hold approximately near the apparent horizons. Indeed one can prove the above assumptions (23) to be consistent with the field equations [9].
Near the outer apparent horizon $r_+$, Eq.(20) yields

$$\alpha \approx +1, \gamma = \frac{1}{\beta^2} \approx 0, \quad (24)$$

and the canonical conjugate momenta (10)-(12) are given approximately as

$$p_\Phi \approx -\phi^2 \partial_w \Phi,$$
$$p_\phi \approx -\frac{1}{\gamma} \partial_w M,$$
$$p_\gamma \approx -\frac{1}{2} \phi. \quad (25)$$

Then it is easy to check the fact that the two constraints are proportional to each other

$$-\gamma H_0 \approx \sqrt{\gamma} H_1,$$
$$\approx \frac{1}{\phi^2} p_\Phi^2 + \gamma p_\phi. \quad (26)$$

As in the previous work [9], just at the apparent horizon $\gamma$ becomes zero so that the various equalities approximately hold when we restrict our attention to the interior region near but not at the apparent horizon. Thus we should assume that $\gamma$ takes a small but finite value within the present approximation level.

An imposition of the constraint (26) as an operator equation on the state produces the Wheeler-DeWitt equation

$$i \frac{\partial \Psi}{\partial \phi} = -\frac{1}{\gamma \phi^2} \frac{\partial^2}{\partial \Phi^2} \Psi. \quad (27)$$

This Wheeler-DeWitt equation is regarded as the Schrödinger equation in the superspace with the Hamiltonian $H_S = p_\Phi^2$ and the time $T_S = \frac{1}{\gamma \phi}$. Note that this superspace time $T_S$ stops on the horizon owing to the effect of an infinite gravitational time dilation.
Now it is easy to find a special solution of this Wheeler-DeWitt equation by the method of separation of variables. The result is

\[ \Psi = (B_w e^{\sqrt{A_w} \Phi(w)} + C_w e^{-\sqrt{A_w} \Phi(w)}) e^{-i \frac{A_w}{\phi^2}}, \]  

(28)

where \( A_w, B_w, \) and \( C_w \) are integration constants.

The formalism explained so far is easily extendible to the quantum gravity near the inner apparent horizon. The result is nothing but the replacement of the advanced time variable \( w \) by the retarded one \( z \). For later convenience, we will write down the conjugate momenta and the physical state in the below:

\[
\begin{align*}
p_{\Phi} &\approx -\phi^2 \partial_z \Phi, \\
p_\phi &\approx -\frac{1}{\gamma} \partial_z M, \\
p_\gamma &\approx -\frac{1}{2} \phi, \\
\Psi &= (B_z e^{\sqrt{A_z} \Phi(z)} + C_z e^{-\sqrt{A_z} \Phi(z)}) e^{-i \frac{A_z}{\phi^2}},
\end{align*}
\]

(29)

where \( A_z, B_z, \) and \( C_z \) are integration constants, and the matter field \( \Phi \) and the mass function \( M \) were assumed to be the function of only the standard external retarded time coordinate \( z \) near the inner apparent horizon.

At this stage, if one defines an expectation value \( < \mathcal{O} > \) of an operator \( \mathcal{O} \) rather naively as

\[
< \mathcal{O} > = \frac{1}{\int d\Phi |\Psi|^2} \int d\Phi \Psi^* \mathcal{O} \Psi,
\]

(30)

one can calculate \( < \partial_w M > \) by using either (25) or (26)

\[
< \partial_w M > = -\frac{A_w}{<\phi^2>}.
\]

(31)

This equation shows the black hole radiation when one chooses the constant \( A_w \).
to be a positive constant $k_w^2$. Then the mass-loss rate becomes [9]

$$< \partial_w M > = -\frac{k^2}{< r_+^2 >}.$$  \hspace{1cm} (32)

If we consider the Schwarzschild black hole, this result exactly corresponds to that calculated by Hawking in the semiclassical approach [10] and by Tomimatsu in the exterior region of the apparent horizon in quantum gravity [15]. Thus a black hole completely evaporates within a finite time. However, we are now thinking of the outer apparent horizon of the Reissner-Nordström black hole, for which if we apply the Stefan-Boltzmann law of the blackbody radiation by assuming the radiation area to be that of the outer horizon and the temperature to be the Hawking temperature [16] in the Reissner-Nordström black hole, we meet an inconformity, for which we will not pay attention in this article.

4. Mass inflation in quantum gravity

The classical physics behind the mass inflation is not so difficult to understand and arises from two key observations. One is that the influx is tremendously blueshifted along the Cauchy horizon so that the energy density of the influx inflates exponentially. The other is a separation between the Cauchy horizon, a surface of infinite blueshift, and inner apparent horizon, a surface of infinite redshift, owing to some transverse outflux. As a consequence, the mass function near the Cauchy horizon diverges exponentially with respect to an advanced time.

Now let us proceed with the study of the mass inflation in quantum gravity. According to the literature [7], let us analyse this phenomenon only on the two intersecting null surfaces $S^+$ and $S^-$. As mentioned before, we will choose $S^-$ to coincide with the Cauchy horizon, and $S^+$ to be parallel to as well as near the event horizon.
As a preparation, let us introduce the extrinsic fields [7]

\[ \theta := \phi^{-2} \partial_v(\phi^2), \]
\[ \tilde{\theta} := \phi^{-2} \partial_u(\phi^2), \]

where \( \theta \) and \( \tilde{\theta} \) denote the expansion rate of the ingoing and the outgoing fluxes, respectively. Along \( S^+ \) the radius satisfies the equation

\[ \frac{dr}{dw} = \frac{1}{2} \left[ 1 - \frac{2M(w)}{r} + \frac{Q^2}{r^2} \right]. \]  

(34)

Then we can make a choice of \( \lambda = \log \phi \) along \( S^+ \) by means of the freedom of the rescaling of the null coordinates. By solving a focusing equation [7] and Eq.(34), it is relatively straightforward to find an explicit relation between \( v \) and \( w \) near the Cauchy horizon:

\[ v \approx -e^{-\kappa_0 w}, \]

(35)

where \( \kappa_0 \) is the surface gravity of the inner horizon. Then since the expansion rate of the ingoing flux in Eq.(33) along \( S^+ \) near the Cauchy horizon \( v \to 0 \ (w \to \infty) \) can be expressed as

\[ \theta : = \phi^{-2} \partial_v(\phi^2) \approx r^{-2} \partial_v(r^2) \approx -\frac{2}{\kappa_0 v \sqrt{M^2 - Q^2}} \partial_w M \approx \frac{2\gamma}{\kappa_0 v \sqrt{M^2 - Q^2}} p^\phi, \]

(36)

we can easily evaluate the expectation value of which

\[ < \theta >_{S^+} \approx \frac{2A_w}{\kappa_0 v \sqrt{M^2 - Q^2}} < r^2_+ >. \]

(37)

In the present physical setting, the influx of the massless neutral matter \( \Phi(w) \) flows into the black hole through \( r = r_+ \) so that \( < \partial_w M > \) must be positive, which
requires the constant $A_w$ in Eq.(31) to be a certain negative constant, e.g., $-k_w^2$, which should be contrasted with the case of the Hawking radiation considered in the previous section, where the constant $A_w$ was selected to be a positive constant. Consequently, Eq.(37) becomes

\[
< \theta >_{S^+} \approx -\frac{2k_w^2}{\kappa_0 v \sqrt{M^2 - Q^2}} < r_+^2 > , \tag{38}
\]

as $v \to 0$. This result indicates that, while the radius of the two-sphere $S = S^+ \cap S^-$ is finite, $< \theta >_{S^+} \to -\infty$ at the Cauchy horizon, which is a signal of the mass inflation. It is worth remarking here that the divergent behavior of the expansion rate comes from a huge blueshift factor $\frac{1}{v}$. This situation is completely analogous to the classical result [7] although the way of deriving those results are very much different in both methods.

To show the mass inflation more clearly, let us consider the invariant definition of the black hole mass in a spherically symmetric geometry

\[
1 - \frac{2M(x^a)}{r} + \frac{Q^2}{r^2} := g^{ab} \partial_a r \partial_b r \approx -\frac{r^2}{2} e^\lambda \theta \tilde{\theta} , \tag{39}
\]

where

\[
\theta := \phi^{-2} \partial_v (\phi^2) \approx r^{-2} \partial_v (r^2) , \\
\tilde{\theta} := \phi^{-2} \partial_u (\phi^2) \approx r^{-2} \partial_u (r^2) . \tag{40}
\]

Now on the two-sphere $S = S^+ \cap S^-$ we are interested in calculating an expectation value of Eq.(39). In order to do so, one needs to calculate $< \tilde{\theta} >$ and $< \partial_u M >$ on $S^-$. In the model considered at present, the outflux is switched on at some retarded time $u = u_0 \approx 0$. Since near $u = u_0$ the inner apparent horizon almost coincides with the Cauchy horizon, we can make use of the formalism holding the inner apparent horizon (29). Thus, similar calculations to those of $< \theta >_{S^+}$ and
< ∂_u M >_{S^+} \text{ yields }

< \dot{\theta} >_{S^-} \approx \frac{2A_z}{\kappa_0 u \sqrt{M^2 - Q^2} < r_-^2 >}, \tag{41}

< \partial_u M >_{S^-} \approx -\frac{A_z}{< r_-^2 >}. \tag{42}

We can interpret that the outflux of lightlike radiation through \( r = r_- \) from the \( r_- < r < r_+ \) region to \( 0 < r < r_- \) region comes from the surface of the collapsing star [4], for which we must take the constant \( A_z \) to be a negative constant, e.g., \(-k_z^2\). As a result, after some manipulation, we obtain

\[
< 1 - \frac{2M(x^\alpha)}{r} + \frac{Q^2}{r^2} >_{S^+ \cap S^-} \approx -\frac{2k_w^2k_z^2}{\kappa_0^2 vu(M^2 - Q^2)\sqrt{< r^2 >}}. \tag{43}
\]

Note that the right-hand side of this equation is actually a negative infinity near the Cauchy horizon, which implies that \(< M > \to \infty \), that is, \emph{mass inflation}. The important point here is that the mechanism responsible for the mass inflation needs the combination of the effects of both the outflux and the influx. Incidentally we need not fear that the right-hand side of Eq.(43) also becomes divergent at the event horizon \( u = 0 \) since there is no outflux before \( u = u_0 > 0 \). This situation is also perfectly analogous to the classical result [4].

Finally, the square of the Weyl tensor on the \( S = S^+ \cap S^- \) can be calculated to be

\[
< C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} >_{S} = 12 \left( \frac{1}{r^2} - \frac{Q^2}{r^4} + \frac{r}{2} \theta \bar{\theta} \right)^2
\]

\[
\propto < \theta^2 >_{S^+} < \bar{\theta}^2 >_{S^-} \propto + \frac{1}{v^2}, \tag{44}
\]

from which we can imagine that the square of the Weyl tensor on the Cauchy horizon diverges rather strongly, thus a quantum continuation beyond the Cauchy
horizon seems to be unlikely although we need a more detailed analysis to confirm this statement.

5. Conclusion

In this article, we have explored the mass inflation which is the phenomenon that the local mass function reaches an incredibly large value in the interior of the charged black hole, by using the canonical quantization formalisms of gravity developed in our previous work [9]. As pointed out in the pioneering work by Poisson and Israel [4], the mass inflation depends on the general feature, that is, the presence of a highly blueshifted influx and a small outflux. It is remarkable to notice that this feature in the classical theory of general relativity is inherited to the case of quantum gravity.

In the sec. V in the reference [4], Poisson and Israel have speculated that the effects of quantum gravity might not alter the classical picture of the mass inflation. At least our present results seem to support their speculations. It would be very interesting to show in more general context that the quantum solutions should be still singular if the solutions to the classical equation all exhibit a singular behavior [4]. (However, see the recent work which seems to be against this conjecture [17].) It might be not quantum gravity but the combined effects of quantum gravity and supersymmetry that prevent an appearance of the classical singular solutions, which was shown at least in the context of the two dimensional dilaton gravity [18].

Kuchar has recently constructed an interesting canonical quantization formalism based on the Kruskal extension of the Schwarzschild black hole [19]. In his work, the hypersurfaces which foliate the spacetime extend from the left spatial infinity to the right one in the Kruskal diagram, thus it would be very interesting to investigate the mass inflation by his formalism.

In future works, we would like to investigate further other recently developing
problems such as the smearing of the black hole singularities [20, 21] and the black hole thermodynamics [22, 23] by using the canonical formalism holding the interior region of black hole.

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