Bergman kernels 
and the pseudoeffectivity of 
relative canonical bundles 
-I-

Bo BERNDTSSON

1Chalmers University of Technology  
Departement of Mathematics  
S-41296 Göteborg

Mihai PĂUN

2Université Henri Poincaré  
Institut Élie Cartan  
54000 Nancy

Abstract. The main result of the present article is a (practically optimal) criterion 
for the pseudoeffectivity of the twisted relative canonical bundles of surjective projective 
maps. Our theorem has several applications in algebraic geometry; to start with, we ob-
tain the natural analytic generalization of some semipositivity results due to E. Viehweg 
and F. Campana. As a byproduct, we give a simple and direct proof of a recent result 
due to C. Hacon–J. McKernan and S. Takayama concerning the extension of twisted 
pluricanonical forms. More applications will be offered in the sequel of this article.

§0 Introduction

In this article our primary goal is to establish some positivity results concerning the 
twisted relative canonical bundle of projective morphisms.

Let $X$ and $Y$ be non-singular projective manifolds, and let $p : X \to Y$ be a surjective 
projective map, whose relative dimension is equal to $n$. Consider also a line bundle $L$ 
over $X$, endowed with a -possibly singular- metric $h = e^{-\phi}$, such that the curvature 
current is semipositive. We denote by $I(h)$ the multiplier ideal sheaf of $h$ (see e.g. [10], 
[21], [25]). Let $X_y$ be the fiber of $p$ over a point $y \in Y$, such that $y$ is not a critical 
value of $p$. We also assume at first that the restriction of the metric $\phi$ to $X_y$ is not 
identically $-\infty$. Under these circumstances, the space of $(n, 0)$ forms $L$-valued on $X_y$ 
which belong to the multiplier ideal sheaf of the restriction of the metric $h$ is endowed 
with a natural $L^2$–metric as follows

$$\|u\|_y^2 := \int_{X_y} c_n u \wedge \overline{u} \exp(-\phi)$$

(we use the standard abuse of notation in the relation above). Let us consider an 
orthonormal basis $(u_j^{(y)})$ of the space $H^0(X_y, (K_{X_y} + L) \otimes I(h))$, endowed with the 
$L^2$ metric above. The relative canonical bundle of $p$ is denoted $K_{X/Y} := K_X - p^*K_Y$. 
Recall that the bundles $K_{X_y}$ and $K_{X/Y}|_{X_y}$ are isomorphic. Via this identification 
(which will be detailed in the paragraph 1) the sections above can be used to define
a metric on the bundle $K_{X/Y} + L$ restricted to the fiber $X_y$, called the Bergman kernel metric. This definition immediately extends also to fibers such that the metric $\phi$ is identically equal to $-\infty$ on the fiber. In this case the Bergman kernel vanishes identically on the fiber, and the Bergman kernel metric is also equal to $-\infty$ there.

Let $Y^0 \subset Y$ be the Zariski open set of points that are not critical values of $p$ in $Y$, and let $X^0 \subset X$ be the inverse image of $Y^0$ with respect to $p$. As $y$ varies in $Y^0$, the above construction defines the relative Bergman kernel metric on the $K_{X/Y} + L$ over $X^0$.

Then we have the next result, which gives a pseudoeffectivity criterion for the bundle $K_{X/Y} + L$.

0.1 Theorem. Let $p : X \to Y$ be a surjective projective map between smooth manifolds, and let $(L, h)$ be a holomorphic line bundle endowed with a metric $h$ such that:

(1) the curvature current of $(L, h)$ is semi-positive on $X$, i.e. $\Theta_h(L) \geq 0$;

(2) $H^0(X_y, (K_{X_y} + L) \otimes I(h)) \neq 0$ for some $y \in Y^0$.

Then the relative Bergman kernel metric of the bundle $K_{X/Y} + L|X^0$ is not identically $-\infty$. It has semipositive curvature current and extends across $X \setminus X^0$ to a metric with semipositive curvature current on all of $X$.

Several versions of the theorem above were established by the first author in his series of articles on the plurisubharmonic variation of the Bergman kernels (see [1], [2], [3] and also [22] for the first results in this direction). Let us point out the main improvements we have got in the present article. In the first place, we allow the metric $h$ to be singular. Secondly, the map $p$ is not supposed to be a smooth fibration—this will be crucial for the applications, as we will see in a moment. The way we are dealing with the singularities of $p$ is by a careful estimate of the local weight of the fiberwise Bergman kernel metric near the singular points: the Ohsawa-Takegoshi extension theorem shows that its local weights are uniformly bounded as we are getting close to the singular loci of $p$. These local weights are plurisubharmonic in $X_0$, so it follows from classical pluripotential theory that they extend uniquely to plurisubharmonic functions across $X \setminus X_0$. This means that the relative Bergman kernel metric extends uniquely to a metric with semipositive curvature current across $X \setminus X_0$.

A similar result has been announced by H. Tsuji in [31] and [31b]. His idea of proof is based on an interesting reduction to the case of a locally trivial fibration (and then use of the results from [2]), but the somehow sketchy argument does not seem to be quite complete.

One aspect of theorem 0.1 can be seen as a "global" version of the Ohsawa-Takegoshi theorem: indeed, if $Y$ is just a small polydisk, then under the hypothesis (1) and (2) above the theorem of Ohsawa-Takegoshi shows the existence of a section of the adjoint bundle extending the one we have on the central fiber, thus we get more than the pseudo-effectivity of the adjoint bundle. In the case under consideration, the base $Y$ is compact and simple examples show that we cannot expect such a statement, but the theorem 0.1 implies that the metric version (i.e. replacing effectivity by pseudo-effectivity) still holds.
Assume that $E$ is a pseudoeffective line bundle on a projective manifold $X$, in the sense that $E$ carries a (possibly singular) metric with semipositive curvature. Then as a consequence of the $L^2$ theory, there exists an ample line bundle $A$ on $X$ such that $H^0(X, mE + A) \neq 0$, for all $m \gg 0$ (see e.g. [10]). In this context, as a first application of the theorem 0.1 we have the next statement.

**0.2 Corollary.** Let $p : X \to Y$ be a projective surjective map between non-singular manifolds $X$ and $Y$. Let $(L, h_L)$ be a pseudo-effective line bundle over $X$. Assume that for some $y \in Y$, not a critical value of $p$, we have $H^0(X_y, (K_{X_y} + L) \otimes \mathcal{I}(h_L)) \neq 0$. Then for any ample line bundle $A \to Y$ which is positive enough we have $H^0(X, m(K_{X/Y} + L) + p^*A) \neq 0$, for all $m \geq 0$.

Of course, the content of the above statement is that the positivity we have to add to the bundle $m(K_{X/Y} + L)$ in order to make it effective comes from $Y$. A particular case of this result can be derived from the work of F. Campana (see [6]); also, it is consistent with the semi-positivity results obtained by T. Fujita, Y. Kawamata, J. Kollár and E. Viehweg (see [15], [19], [20], [37] as well as the references therein).

Let us explain in a few words how the corollary is derived from theorem 0.1. We have the decomposition

$$m(K_{X/Y} + L) + p^*A = K_X + (m - 1)(K_{X/Y} + L) + L + p^*(A - K_Y).$$

We denote by $h_B$ the metric obtained in the theorem 1; we use it to endow the bundle $(m - 1)(K_{X/Y} + L) + L + p^*(A - K_Y)$ with the metric $h_B^{m-1} \otimes h_L \otimes h_{p^*(A - K_Y)}$ where $A \to Y$ is assumed to be positive enough, to compensate for the possible negativity of the canonical bundle of $Y$ (for the precise positivity properties of $A$, see the paragraph 3). Now by hypothesis there exists $u_y \in H^0(X_y, (K_{X_y} + L) \otimes \mathcal{I}(h))$ and if we denote by $u_A$ some section of $p^*A$, then the section $u_y \otimes u_A$ satisfies the integrability properties needed in order to extend it over $X$ by using once again the Ohsawa-Takegoshi theorem. The fact that the metric $h_B$ is explicitly given over a Zariski dense open set is crucial here.

In Corollary 0.2 we prove the existence of global sections to $m(K_{X/Y} + L) + A$, assuming the existence of sections to $(K_{X/Y} + L) \otimes \mathcal{I}(h)$ over at least some fiber. It is natural to ask what happens if we only assume from the start existence of certain sections over fibers to multiples of $K_{X/Y} + L$. In section 4 we prove a version of Theorem 0.1 for the Narasimhan-Simha metric, see [26], which is an analog of the Bergman kernel metric for multiples of the canonical bundle, recently reviewed by Tsuji [32], [32a] and [32b]. This generalizes a classical result of Kawamata [18] for the nontwisted case and will be developed further in the sequel to this paper [4].

The proofs in section 4 are still based on the same positivity results for direct image bundles from [3] as the proof of Theorem 0.1, but require an additional twist. In particular, we need (weaker) versions of the results from [3] for nonsmooth metrics. These are discussed in section 3 - hopefully they are also of some independent interest.

To introduce the next application, let us recall the notion of restricted volume (see [5], [14]). Let $E \to X$ be a holomorphic line bundle. If $V \subset X$ is an irreducible
$d$-dimensional sub-manifold, let us denote by
\[ H^0(X|V, mE) := \text{Im}(H^0(X, mE) \to H^0(V, mE_V)), \]
and let $h^0(X|V, mE)$ be the dimension of this space. Then the restricted volume of $E$ to $V$ is
\[ \text{Vol}_V(X, L) := \limsup \frac{d!}{m^d} h^0(X|V, mL) \]

The definition of a maximal center is slightly more involved, and it will not be recalled here (see e.g. [16], [30] and also the paragraph 6 of this article). Let us just mention that given an effective $\mathbb{Q}$–divisor $D = \sum_j \nu_j Z_j$, the maximal centers of the pair $(X, D)$ are the higher codimensional analog of the irreducible components $Z_j$ above such that $\nu_j = 1$.

The corollary above can be used to provide a rather simple proof of the next statement due to S. Takayama ([30], see also [16] and [32]), which is crucial in the investigation of the properties of the pluricanonical series.

0.3 Theorem ([29], [30], [16]). Let $X$ be a non-singular projective manifold, such that $K_X$ is pseudo-effective. Let $L$ be a line bundle which can be written as $L = A + D$ where $A$ is $\mathbb{Q}$-ample and $D$ is a $\mathbb{Q}$-effective divisor. Assume that $V$ is an irreducible maximal center of $(X, D)$ such that there exists a section of some multiple of $K_X + L$ vanishing on some ample divisor and which is not identically zero when restricted to $V$. Then
\[ \text{Vol}_V(X, K_X + L) \geq \text{Vol}(V). \]

Even if our proof goes along the same lines as the previous ones, the exposition is substantially simplified by the use of the corollary 0.2, which allows us to bypass the use of Kawamata’s subadjunction theorem [19].

There are 2 major ingredients needed in the proof of 0.3. The first one is the technique invented by Y.-T. Siu to prove the invariance of plurigenera, see [33], [34]. The version which will be used here is due to S. Takayama, but we will offer a simpler proof, in the same spirit as in [28]. The second one is simply the corollary 0.2.

Let us explain vaguely how the two techniques combine to prove the theorem. First, if the maximal center $V$ is a hypersurface, then the theorem 0.3 is a consequence of the invariance of plurigenera techniques. If $V$ has higher codimension, one is lead to reduce to the divisor case by using a modification $\mu : Y \to X$. We consider the restriction of the map $\mu$ to a well-chosen exceptional divisor $S$ which maps onto $V$. We know how to extend twisted pluricanonical sections on $S$. The crucial result which allows us to inject the space of pluricanonical sections on $V$ to a space of twisted pluricanonical sections on $S$ is precisely Corollary 0.2 - the injection map is given by multiplication with a section of a twisted relative canonical bundle $K_{S/V} + B$, whose existence is guaranteed by Corollary 0.2.

Acknowledgements. The second named author would like to thank F. Campana, Y. Kawamata, S. Takayama and E. Viehweg for very interesting discussions concerning various aspects of this work; he would equally like to express his gratitude to his friends
and colleagues at the Institute Élie Cartan (Nancy) for the nice atmosphere during the “groupe de travail” where topics like the extension of pluricanonical forms were extensively analyzed. We also would like to thank the referees for detailed and constructive criticism of the original manuscript.

§1 Relative Bergman kernels

Let \( p : X \mapsto Y \) be a surjective holomorphic map with compact \( n \)-dimensional fibers, onto a complex manifold \( Y \). At first we will also assume that the map \( p \) defines a smooth fibration, i.e., that the differential \( dp \) is surjective at each point, so that the fibers

\[ X_y := p^{-1}(y) \]

are smooth manifolds. Let \((L, \phi)\) be a Hermitian holomorphic line bundle over the total space \( X \). For each \( y \) in \( Y \) we let

\[ E_y = H^0(X_y, L|_{X_y} + K_{X_y}) \]

be the space of global holomorphic \( L \)-valued \((n, 0)\)-forms on \( X_y \). The Hermitian metric \( \phi \) on \( L \) induces a Hilbert norm \( H_y \) on \( E_y \) by

\[ \|u\|^2_y = \int_{X_y} c_n u \wedge \overline{u} e^{-\phi}. \]

Near a point in \( Y \) we can choose local coordinates \( t = (t_1, \ldots, t_m) \), which define a local \((m, 0)\)-form, \( dt = dt_1 \wedge \ldots \wedge dt_m \), that trivializes the canonical bundle of \( Y, K_Y \). A choice of such form gives a natural map from forms, \( u \), on the fibers \( X_y \) to sections, \( \tilde{u} \), of \( K_X \) over \( X_y \) by

\[ \tilde{u} = u \wedge p^*(dt) = u \wedge dt, \]

where in the last equality we abuse notation slightly by thinking of \( t \) as a function on \( X \). When dealing with forms on \( X \) we will mostly use the notation \( dt \), but we will recur to \( p^*(dt) \) for emphasis occasionally.

Conversely, given a local section \( \tilde{u} \) of \( K_X \) we can write \( \tilde{u} = u \wedge dt \) locally. The restriction of \( u \) to fibers is then uniquely defined and thus defines a form on \( K_{X_y} \). We will call \( u \) the trace of \( \tilde{u} \) on \( K_{X_y} \). Altogether the correspondence between \( u \) and \( \tilde{u} \) gives us an identification between forms on a fiber and restrictions of sections to \( K_X \) to the fiber. This identification clearly depends on the choice of \( dt \), but the map \((u, dt) \mapsto u \wedge dt \) is an invariantly defined isomorphism

\[ K_{X_y} + p^*(K_Y)|_{X_y} \leftrightarrow K_X|_{X_y}, \]

for any \( y \). Hence we also get an isomorphism

\[ K_{X_y} \mapsto (K_X - p^*(K_Y))|_{X_y} = K_{X/Y}|_{X_y}, \]
where $K_{X/Y} = (K_X - p^*(K_Y))$ is the relative canonical bundle of $p$. If $t$ is any choice of local coordinates on $Y$, then the isomorphism is given by

$$u \mapsto u \wedge p^*(dt)/dt.$$ 

The Hilbert norm (1.1) therefore defines a norm on sections of $L + K_{X/Y}$ over $X_y$.

For any $y$ in $Y$ we now define the Bergman kernel for this norm by

$$B_y = \sum u_j \otimes \overline{u}_j,$$

for any choice of orthonormal basis $(u_j)$ of the space of sections of $(L + K_{X/Y})|_{X_y}$. It is well known that this definition is independent of the choice of orthonormal basis. In the manifold case, that we are dealing with here, the Bergman kernel is not a function, but $\log B_y$ defines a metric on $(L + K_{X/Y})|_{X_y}$ in the sense that

$$u \otimes \overline{u}/B_y$$

is a well defined function if $u$ is a section of $L + K_{X/Y}$ over $X_y$. Alternately, we could choose an arbitrary smooth metric, $\chi$, on $(L + K_{X/Y})|_{X_y}$. Then $B_y e^{-\chi} =: e^\xi$ is a well defined global function on $X_y$. Hence

$$\log B_y = \xi + \chi$$

is also a metric on $(L + K_{X/Y})|_{X_y}$.

As $y$ varies, we get a metric on the line bundle $L + K_{X/Y}$ over all of $X$. We will refer to this metric as the relative Bergman kernel metric and write $B = e^\psi$, with the understanding that $\psi$ is a metric which is represented by different local functions, given different local trivializations.

In the classical case of domains in $\mathbb{C}^n$ there is a well known extremal characterization of the Bergman kernel as

$$B(x) = \sup_{\|h\| \leq 1} |h(x)|^2.$$ 

We will use repeatedly a variant of this in our setting. Choose local coordinates $x$ near a point in a fiber $X_y$. Then $dx/dt$ is a local frame for $K_{X/Y}$. Choose also some local trivialization, $e$, of $L$ near the same point. With respect to these local frames $\psi$ is represented by a function $\psi'$, and any section $\tilde{u}/dt = u \wedge p^*(dt)/dt$ to $L + K_{X/Y}$ over $X_y$ can be written

$$\tilde{u}/dt = u'(x)dx/dt \otimes e.$$ 

Then

$$e^{\psi'(x)} = \sup_{\|u\| \leq 1} |u'(x)|^2.$$ 

The proof of this is exactly as in the classical case. It is of paramount importance that, since our local trivializations are given by $dx/dt$, the coefficient that enters in the extremal characterization is the coefficient of $\tilde{u}$ (and not any coefficient of $u$). It is the presence of the factor $p^*(dt)$ - which vanishes on the singular locus of $p$ - in $\tilde{u}$ that will allow us to control the Bergman kernel metric near the singular locus.
Our first result, Theorem 0.1, says that if $\phi$ is a, possibly singular, metric of non-negative curvature on $L$, then $\psi$ is a metric of nonnegative curvature of $L + K_{X/Y}$ - unless $B$ is identically 0. More precisely, $\psi$ is semipositive over the Zariski open set where $p$ is a smooth fibration, and extends in a unique way to a semipositive metric over the singular fibers.

To start explaining the proof of this, let us first assume that $\phi$ is smooth, and that $p$ defines a smooth fibration, i.e., that $dp$ is surjective everywhere. Theorem 0.1 is then an easy consequence of Theorem 1.2 in [3]. This theorem says that the Hilbert norms $\| \cdot \|_y$ on $E_y$ define a semipositively curved Hermitian metric on a vector bundle $E$ over $Y$, with fibers $E_y$. The complex structure on $E$ is such that a local section of $E$ $u \mapsto u_y$ is holomorphic if and only if the associated sections $y \mapsto \tilde{u}_y = u_y \wedge dt$ to $K_X$ are holomorphic, and hence define holomorphic sections $\tilde{u}/dt$ to $K_{X/Y}$. (The bundle $E$ is the direct image $p_* (L + K_{X/Y})$ of $L + K_{X/Y}$ under the map $p$.) The precise statement of Theorem 1.2 in [3] is that $E$ with the $L^2$-Hermitian metric is (weakly) Nakano positive, but the weaker property of Griffiths positivity is all we use here.

Choose local coordinates $(t, z)$ near a point in $X$, with respect to which the map $p$ is the trivial fibration $(t, z) \mapsto t$. These local coordinates give us local frames, $dt$ and $dt \wedge dz$ for $p^*(K_Y)$ and $K_X$ respectively, and hence the local frame $dt \wedge dz/dt$ for $K_{X/Y}$. With respect to this frame, $\psi$ is given by a function $\psi'$, and we need to prove that this function is plurisubharmonic. This means that

$$\psi'(t, h(t))$$

is plurisubharmonic of $t$ if $(t, h(t))$ is any local holomorphic map.

We now use the same local coordinates, and the map $h$, to define a local holomorphic section of the dual bundle of $E$, $E^*$. Let $u_y$ be an element in $E_y$, and let

$$u_y \wedge p^*(dt)/dt$$

be the associated element in $K_{X/Y} + L$. With respect to the trivializations

$$u_y \wedge p^*(dt)/dt = u'(t, z)(dz \wedge dt)/dt \otimes e.$$ 

We put

$$\xi_y(u_y) = u'(t, h(t)).$$

If $u_y$ depends holomorphically on $y$, then $u'$ is a holomorphic function of $t$ and $z$, so it follows that $y \mapsto \xi_y$ is a holomorphic section of $E^*$. By the extremal characterization of Bergman kernels

$$\|\xi_y\|^2 = \sup_{\|u\| \leq 1} |u'(t, h(t))|^2 = e^{\psi'(t, h(t))}.$$
Since the logarithm of the norm of any section of the dual of a Griffiths positive bundle is plurisubharmonic, it follows that $\psi'$ is plurisubharmonic, which is what we wanted to prove.

This proves Theorem 0.1 in the smooth case, i.e. when both the metric $\phi$ and the fibration $p$ are smooth. Let us now still assume that $\phi$ is smooth but relax the assumption on $p$, so that $p$ is a general surjective holomorphic map. Then the degeneracy locus of $p$ is an analytic subvariety $W$ of $X$ and $p(W)$ is an analytic subvariety of $Y$. By Sard’s theorem $p(W)$ has zero measure in $Y$, so it is a proper subvariety, $W'$ of $Y$. Outside of $p^{-1}(W')$, $p$ is a smooth fibration, and by the previous discussion, $\psi$ has semipositive curvature there. We want to prove that $\psi$ extends to a semipositive metric across $p^{-1}(W')$. Since $p^{-1}(W')$ is pluripolar, it suffices for this to prove that $\psi$ stays locally bounded from above. The next lemma is the crucial step.

**1.1 Lemma.** Let $D$ be a polydisk in $\mathbb{C}^{n+m}$, and let $\phi$ be a plurisubharmonic function in $D$. Let

$$p : D \mapsto V$$

be a holomorphic map from $D$ to a bounded open set in $\mathbb{C}^m$. Assume 0 is a regular value of $p$ and let $D_0 := p^{-1}(0)$. Let $u$ be a holomorphic $(n,0)$-form on $D_0$. Let

$$\tilde{u} = u \wedge p^*(dt)$$

where $t = (t_1, \ldots, t_m)$ are standard coordinates on $\mathbb{C}^m$ and write

$$\tilde{u} = u' dz,$$

where $z = (z_1, \ldots, z_{n+m})$ are standard coordinates on $\mathbb{C}^{n+m}$. Then there is a constant $C_K$ such that for any compact subset $K$ of $D$ and $z$ in $D_0 \cap K$,

$$|u'(z)|^2 \leq C_K \int_{D_0} [u,u] e^{-\phi}.$$

**Proof.** By the Ohsawa-Takegoshi extension theorem there is a holomorphic $(n+m,0)$-form $\tilde{U}$ in $D$ such that

$$\tilde{U} = \tilde{u}$$

on $D_0$ and

$$\int_D |\tilde{U}|^2 e^{-\phi} \leq C \int_{D_0} [u,u] e^{-\phi},$$

where the constant $C$ only depends on the sup-norm of $p$. (See section 5, Theorem 5.1, and the comments immediately after that theorem for an appropriate version of the Ohsawa-Takegoshi extension theorem.) Since $\phi$ is locally bounded from above it follows that

$$\int_K |\tilde{U}|^2 \leq C' \int_{D_0} [u,u] e^{-\phi}.$$

The lemma now follows from the meanvalue inequality applied to $\tilde{U}$. \qed

**1.2 Remark.** The main point in the lemma is that the constant $C_K$ does not blow up as we approach a singular fiber - the Ohsawa-Takegoshi theorem implies that the
If we choose local coordinates and local trivializations near a point on a singular fiber of \( p \) it follows immediately from the lemma, together with the extremal characterization of Bergman kernels, that the relative Bergman kernel stays bounded near a singular fiber. Hence the metric \( \psi \) extends (uniquely!) to a semipositive metric on \( L + K_{X/Y} \).

(Notice that the Bergman kernel is not identically equal to 0 since by assumption \( K_{X/Y} + L \) has some section over some fiber in \( X_0 \).)

1.3 Remark. Two examples may serve to illustrate what happens near the singular locus. Let us first look at the case when \( X \) is the blow-up of \( Y \) at a point, and, say, \( L \) is trivial. Then

\[ s := p^\ast(dt)/dt \]

defines a global holomorphic section of the relative canonical bundle \( K_{X/Y} \) (\( s \) does not depend on the choice of local frame \( dt \)). Since \( s \) vanishes to order \( (n-1) \) precisely on the exceptional divisor \( D \), we see that

\[ K_{X/Y} = (n-1)(D), \]

and that \( h_D := |s|^2 \) defines a singular pseudoeffective metric on \( K_{X/Y} \). We claim that \( h_D \) equals the relative Bergman kernel metric in this case. To see this, note that outside of the singular locus the fibers \( X_y \) are just points, and elements, \( u \), in \( E_y \) are just numbers. Then

\[ \tilde{u}/dt = u \wedge p^\ast(dt)/dt = u'dx/dt, \]

and to compute the Bergman kernel of this zero-dimensional fiber we need to take the supremum of \( |u'|^2 \) over all \( u \) of modulus 1. Clearly this equals \( |s'|^2 \) where

\[ s = s' dx/dt, \]

proving our claim. A very similar argument applies, and the same conclusion holds, in the more general situation of a map \( p \) with a finite number of preimages over a generic point in the base.

The second example illustrates what happens when the dimension of the fiber does not jump at the singular locus. The perhaps simplest such example is when \( n = m = 1 \) and \( p(z_1, z_2) = z_1^2 - z_2^3 \). Then the fiber \( D_0 = \{ z_1^2 = z_2^3 \} \) can be parametrized by

\[ \lambda \mapsto (\lambda^3, \lambda^2), \]

and any holomorphic form \( u \) on \( D_0 \) can be written

\[ u = h(\lambda)d\lambda, \]

with \( h \) holomorphic off 0. Then \( u \) has finite \( L^2 \)-norm on \( D_0 \) if and only if \( h \) extends holomorphically across 0. On the other hand

\[ dp = 2z_1 dz_1 - 3z_2^2 dz_2, \]
and
\[ d\lambda = (1/z_2)dz_1 - (z_1/z_2^2)dz_2, \]
so we get
\[ \tilde{u} = u \wedge dp = -\lambda^2 h(\lambda)dz_1 \wedge dz_2. \]
Therefore \( \tilde{u}' \) is finite at the origin if and only if \( h \) has a pole of order at most 2 at 0. Since by the extremal characterization of Bergman kernels, the Bergman kernel at the origin equals
\[ \sup |\tilde{u}'(0)|^2/\|u\|^2, \]
we see that Bergman kernels tend to be small - not big - on singular fibers.

It now remains to relax the condition that \( \phi \) be smooth to obtain the full proof of Theorem 0.1. We may again assume that \( p \) is a smooth fibration, since the general case then follows from Lemma 1.1 by the same argument as before. We then need to approximate \( \phi \) by a sequence of smooth semipositive metrics. It is here that we use the assumption that \( p : X \hookrightarrow Y \) is projective, which permits us to reduce the problem to the Stein case. This is the object of the next section.

\section{Stein and projective fibrations}

We will now discuss the proof of Theorem 0.1 in the general case. First, since \( X \) is assumed to be projective, we can remove a divisor and get a Stein submanifold. On this Stein manifold the line bundle \( L \) has a holomorphic section. Removing the zero divisor of this section we get a smaller Stein submanifold over which \( L \) is trivial. We will prove that the relative Bergman kernel metric for this Stein submanifold has nonnegative curvature. But, since divisors are removable for \( L^2 \)-holomorphic functions, the relative Bergman kernel metrics for \( X \) and for the Stein submanifold are identical, so Theorem 0.1 follows.

We first consider a smooth Stein fibration over a domain \( W \) in \( \mathbb{C}^m \). By this we mean that we are given a Stein manifold \( D \) of dimension \( n + m \), together with a holomorphic surjective map \( p \) from \( D \) to \( W \) whose derivative \( dp \) is everywhere of maximal rank. Then the fibers \( D_t = p^{-1}(t) \) are \( n \)-dimensional Stein manifolds. Let \( \phi \) be a plurisubharmonic function on the total space \( D \).

We let \( E_t \) be the space of global holomorphic \((n,0)\)-forms \( u \) on \( D_t \) that lie in \( L^2 \) in the sense that
\[ \|u\|_t^2 = \int_{D_t} c_n u \wedge \bar{u}e^{-\phi} < \infty. \]
This is now, in general, an infinite dimensional space, but we can still define its Bergman kernel
\[ B_t = \sum u_j \otimes \bar{u}_j, \]
and a metric \( B = e^\psi \) on \( K_{D/W} = K_D \) as in the previous section. If \( u \) lies in \( E_t \) we write \( \tilde{u} = u \wedge dt \) and
\[ \tilde{u} = u'dz, \]
given local coordinates on $D$. We then have, by the extremal characterization of Bergman kernels, that
\[ e^{\psi(z)} = \sup |u'(z)|^2, \]
with respect to the induced local trivializations, where the supremum is taken over all forms $u$ of norm at most 1.

2.1 Theorem. In the above situation, the relative Bergman kernel metric has nonnegative curvature, i.e.
\[ i\partial\bar{\partial}\psi \geq 0, \]
if the Bergman kernel is not identically 0.

The strategy of the proof is as follows. We first prove the theorem in a 'good' situation, where the weight function $\phi$ is smooth and the Stein manifold is of very special type. It is then clear that the relative Bergman kernel is continuous, so to prove that it is plurisubharmonic it suffices to verify subharmonicity on complex lines. This means in particular that we may assume the base is one dimensional. (This is because the complex structure on our bundle $E$ restricted to a complex line in the base coincides with the structure we get if we first restrict the fibration $X \to Y$ to the line and then construct our vector bundle from there.) Finally, the general case is reduced to the 'good' case.

Since $D$ is Stein it has a smooth strictly plurisubharmonic exhaustion, $\rho$. Let $D^c = \{ \rho < c \}$. By Sard's lemma, $D^c$ is smoothly bounded and strictly pseudoconvex for all choices of $c$ outside a closed set of measure 0.

Let $c$ be one such noncritical value and let $C_c$ be the set of all $t$ in the disk such that $c$ is critical for the restriction of $\rho$ to $D^c_t := D^c \cap p^{-1}(t)$.

2.2 Lemma. $C_c$ is a closed subset of $\Delta$ of zero measure.

Proof. Let $t = \tau + i\sigma$. Note that a point $x$ is critical for the restriction of $\rho$ to $D_t$ if and only if $d\rho, d\tau$ and $d\sigma$ are linearly dependent at $x$ (here we view $t$ as a function on $X$). If $x$ is not critical for $\rho$ on $X$, $d\rho$ is not 0 at $x$, so $d\rho|_x$ must be a linear combination of $d\tau$ and $d\sigma$. This in turn means that $d\tau$ and $d\sigma$ are linearly dependent on the kernel of $d\rho$, so that the restriction of $p$ to $\partial D^{\rho(x)}$ has a critical point at $x$. The lemma therefore follows from Sard’s lemma applied to the restriction of $p$ to $\partial D^{\rho(x)}$. $\square$

Take $t_0$ in $\Delta$. We shall prove that the fiberwise Bergman kernel for $D^c$ is log-plurisubharmonic in $D^c \cap p^{-1}(V)$, if $V$ is a sufficiently small neighbourhood of $t_0$.

2.3 Lemma. There is a holomorphic tangent vector field, $f$, to $D$ such that
\[ dp(f) = \partial/\partial t. \]

Proof. Such a field is easily found locally by choosing local coordinates with respect to which the fibration is trivial, and a nonholomorphic field $f'$ is then obtained globally by a partition of unity. Let $w = \overline{\partial} f'$. Then $w$ is a $\overline{\partial}$-closed $(0,1)$-form with values in the subbundle of the holomorphic tangent bundle of $D$ consisting vectors tangent to fibers (i.e. the kernel of $dp$). Since $D$ is Stein, $w = \overline{\partial} g$, where $g$ is a field with values in
the same subbundle. Replacing \( f' \) by \( f = f' - g \) we get a holomorphic tangent vector field on \( D \) satisfying the claim in the lemma.

Let \( \Omega := D^c_{t_0} + 1/2 \), and consider the flow of our field \( f \) from the lemma above. The flow after time \( t - t_0 \) maps \( \Omega \) to a domain in \( D_t \), which contains \( D^c_t \) and is contained in \( D^c_{t+1} \) if \( t \) lies in \( V \), a small enough neighbourhood of \( t_0 \). Let \( \hat{\Omega} \) be the image of \( \Omega \) under the flow for \( t \) in \( V \).

We now follow the arguments from [2]. Since there is a fiberpreserving biholomorphism between \( \Omega \times V \) and \( \hat{\Omega} \), it first follows that the Bergman kernel for \( \hat{\Omega} \) is log plurisubharmonic, by the argument from [2], section 2. (In [2], this is proved for the product of domains in Euclidean space, but the same arguments apply for products of a domain with a manifold, provided we consider \((n,0)\)-forms instead of functions.) We then replace our weight function \( \phi \) by \( \phi_j \), where \( \phi_j \) form a sequence of plurisubharmonic weights, all equal to \( \phi \) in \( D^c \), and tending to infinity outside the closure of \( D^c \).

To prove the log plurisubharmonicity of the Bergman kernel in \( D^c \cap p^{-1}(V) \) we shall use the following version of Lemma 3.1 from [2].

\[ \text{2.4 Lemma. Let } G_0 \text{ and } G_1 \text{ be Stein manifolds, with } G_0 \text{ compactly included in } G_1. \]
\[ \text{Let } \phi_j \text{ be a sequence of continuous weight functions in } G_1 \text{ such that } \phi_j = \phi \text{ in } G_0 \text{ and } \phi_j \text{ increases to infinity almost everywhere in } G_1 \setminus G_0. \]
\[ \text{Let } B_j \text{ be the Bergman kernel for the space } A_j \text{ of holomorphic } (n,0)\text{-forms } u \text{ in } G_1 \text{ with Hilbert norm} \]
\[ c_n \int_{G_1} u \wedge \bar{u} e^{-\phi_j}. \]
\[ \text{Then for any } z \in G_0, B_j(z,z) \text{ increases to } B(z,z), \text{ the Bergman kernel for the closure of the space } A_1 \text{ in the space } A \text{ of holomorphic } (n,0)\text{-forms } u \text{ in } G_0 \text{ with Hilbert norm} \]
\[ c_n \int_{G_0} u \wedge \bar{u} e^{-\phi}. \]
\[ \text{Proof. From the extremal characterization of Bergman kernels it is clear that } B_j \text{ increases, and it is also clear that } B_j \leq B. \]
\[ \text{To prove the opposite inequality, we take a form } u \text{ in } A_1 \text{ with norm at most 1 in } A, \text{ which almost realizes the supremum in the extremal characterization. By monotone convergence, the norm of } u \text{ in } A_j \text{ tends to the norm of } u \text{ in } A \text{ as } j \text{ tends to infinity. Hence } \lim_{j} B_j(z,z) \geq B(z,z). \]

We now apply this Lemma 2.4 to the domains \( D^c_t \) inside of \( \hat{\Omega} \), with weight functions \( \phi_j = \phi + \psi_j \), with \( \psi_j \) being a sequence of plurisubharmonic functions all equal to 0 in \( D^c_t \) and increasing to infinity in \( \hat{\Omega} \) outside the closure of \( D^c_t \) (suitable convex functions of \( \rho \) will do for \( \psi_j \)). We claim that if \( t \) is taken outside of the exceptional set \( C_c \) of Lemma 2.2, then any form on \( D^c_t \) can be approximated by forms on \( \hat{\Omega}_t \), so that the Bergman kernel \( B \) of Lemma 2.4 equals the Bergman kernel on \( D^c_t \).

This claim follows from the next lemma, which is a variant of Lemma 3.4 from [2]. The version here is however substantially simpler since we have taken care to have all the domains involved smoothly bounded, by avoiding critical values of \( \rho \). This permits us to bypass a nontrivial result of Bruna and Burgues that was used in [2].
2.5 Lemma. Let \( \rho \) be a smooth plurisubharmonic exhaustion function in an \( n \)-dimensional Stein manifold \( G \), and put \( G_c = \{ \rho < c \} \). Let \( u \) be a holomorphic form of bidegree \((n,0)\) in \( G_0 \), such that

\[
 c_n \int_{G_0} u \wedge \bar{u} e^{-\phi} < \infty,
\]

where \( \phi \) is a plurisubharmonic weight function on \( G \). Assume that 0 is not a critical value of \( \rho \), and take \( c > 0 \). Then \( u \) can be approximated in \( L^2 \)-norm on \( G_0 \) by a sequence of \((n,0)\)-forms that extend holomorphically to \( G_c \).

Proof. Since, for any small \( \epsilon > 0 \), \( G_{c_\epsilon} \) is Runge with respect to \( G_c \), it suffices to approximate \( u \) by forms holomorphic in some small neighbourhood of \( G_{0} \). For this, we first approximate \( u \) by \( v \), smooth in a neighbourhood of \( \overline{G}_{0} \), with \( \bar{\partial}v \) small, by decomposing \( u = \sum u_j \), where \( u_0 \) has compact support in \( G_0 \), and each of the other \( u_j \) is supported in a small neighbourhood of a boundary point. We may assume that these small neighbourhoods of boundary points are starshaped, and then approximate the corresponding \( u_j \) by \( v_j \) obtained from a small dilation (this is where we use that 0 is noncritical). Putting \( v_0 = u_0 \) and \( v = \sum v_j \) it follows that \( v \) approximates \( u \) in \( L^2 \) and that \( \bar{\partial}v \) is small in a neighbourhood of \( G_0 \). To obtain a holomorphic approximant we finally solve a \( \bar{\partial} \)-equation in \( G_{c_\epsilon} \). \( \square \)

Consider now the sequence of Bergman kernels on \( \tilde{\Omega}_t \) for the weights \( \phi_j \). By Lemma 2.4 they form an increasing sequence that on each \( D_{c_\epsilon}^c \) converges to the Bergman kernel on \( D_{c_\epsilon}^c \), with weight \( \phi \) for the space of holomorphic forms that can be approximated by forms holomorphic on \( \tilde{\Omega}_t \). Since the Bergman kernels on \( \tilde{\Omega}_t \) for the weights \( \phi_j \) are nonnegatively curved, the upper semicontinuous regularisation of the limit is also a nonnegatively curved metric (on \( K_D \)), that we denote \( e^{\Psi_c} \). On the other hand, when \( t \) is outside of \( C_c \), Lemma 2.5 says that any holomorphic form on \( D_{c_\epsilon}^c \) can be approximated by forms holomorphic in \( \tilde{\Omega}_t \), so outside of \( p^{-1}(C_c) \) the limit is just the fiberwise Bergman kernel on \( D_{c_\epsilon}^c \). It is not hard to check (by a normal family argument) that the fiberwise Bergman kernels are already uppersemicontinuous outside of \( p^{-1}(C_c) \), so \( e^{\Psi_c} \) is just the fiberwise Bergman kernel on \( D_{c_\epsilon}^c \) there, in particular almost everywhere.

Summing up, we have proved that, over \( V \), the relative Bergman kernel metric on \( D_c \) is nonnegatively curved outside of \( p^{-1}(C_c) \) and extends (uniquely) to a nonnegatively curved metric, \( \Psi_c \) across \( p^{-1}(C_c) \). Since \( V \) is a small neighbourhood of an arbitrary point in the disk, the same thing holds on all of \( D_c \). Let \( e^{\psi_c} \) be the fiberwise Bergman kernel for all \( t \) in \( \Delta \). A normal family argument shows that

\[
\psi_c \geq \Psi_c,
\]

and, as we know, equality holds outside \( C_c \). Moreover we claim that

\[
\psi_c + \epsilon \leq \Psi_c,
\]

for \( \epsilon > 0 \). This follows since if \( t \) lies in \( \Delta \) and \( \epsilon' < \epsilon \) is noncritical for \( \rho \) on the fiber over \( t \), then

\[
\psi_c + \epsilon \leq \psi_{c+\epsilon'} = \Psi_{c+\epsilon'} \leq \Psi_c.
\]
Both $\psi_c$ and $\Psi_c$ decrease with $c$, so they must have the same limit as $c$ tends to infinity. Since $\Psi_c$ is nonnegatively curved for all $c$ this limit is nonnegatively curved.

However, for each $t$, the limit of $e^{\psi_c}$ as $c$ tends to infinity on the fiber over $t$, equals the Bergman kernel for $D_t$. (To see this, note first that the limit must be larger than the Bergman kernel for $D_t$ since the Bergman kernel is a decreasing function of the domain. The opposite inequality follows from the extremal characterization of Bergman kernels, since the extremals for each domain $D_t$ have a subsequence that converges weakly in $L^2$ to a form of norm at most 1.)

This proves Theorem 2.1 when $\phi$ is smooth. Finally, to remove the assumption that $\phi$ be smooth we again consider regular values $c$ of the exhaustion function. A general plurisubharmonic $\phi$ can now be written as a decreasing limit of smooth strictly plurisubharmonic $\phi_\nu$ in a neighbourhood of the closure of $D^c$. Applying Theorem 2.1 to $D^c$ with the weight $\phi_\nu$, we see that the corresponding Bergman kernels are plurisubharmonic and decrease to the Bergman kernel for the weight $\phi$. Hence the limit is again plurisubharmonic, and letting $c$ tend to infinity we see that $\psi$ is plurisubharmonic or identically equal to $-\infty$ in $D$. Note that there is no need to take upper semicontinuous regularizations - it follows from the argument above that the fiberwise Bergman kernels as they are give us an upper semicontinuous metric. This is of course under the assumption in Theorem 2.1, that the fibration $p$ be smooth. We lose the explicit character of our metric when we extend across the singularities of the fibration using Lemma 1.1.

Notice also that it may well happen that the limit of the sequence of Bergman kernel on $D_t^c$ as $c$ tends to infinity is equal to 0. This happens precisely when there are no nontrivial $L^2$ holomorphic forms on the fiber $D_t$. If on some fiber there is a nontrivial $L^2$ holomorphic form, then the Bergman kernel on that fiber is not identically 0, so the relative Bergman kernel metric is a nonnegatively curved (singular) metric. 

Thus, by the discussion at the beginning of this section, the proof of the theorem 0.1 is also finished

§3 Positivity of the direct image bundle.

As noted in section 1, positivity of the relative Bergman kernel metric (i.e. plurisubharmonicity of the Bergman kernel) follows in the smooth case from the (Griffiths) semipositivity of the direct image bundle

$$E = p_*(K_{X/Y} + L).$$

A priori, the positivity of $E$ is however a stronger statement: It implies plurisubharmonicity of any function $\log |\xi|^2$, for $\xi$ any holomorphic section of the dual bundle $E^*$, whereas plurisubharmonicity of Bergman kernels corresponds to the special case of point evaluations. In this section we shall prove a variant of this stronger statement when the metric on $L$ is not necessarily smooth, and also show that it corresponds to a stronger positivity condition on the Bergman kernel.

To start with we shall discuss the perhaps not completely standard notion of a singular metric on a vector bundle (see also de Cataldo, [7]). By this we basically mean a measurable map from the base, $Y$, to the space of nonnegative hermitian forms
on the fiber. We do however need to allow the hermitian form to take the value $\infty$ for some vectors at some points in the base. We say that $|\xi|^2$ is a quadratic form with values in $[0, \infty]$ on a vector space $V$ if $|\xi|^2$ is a nonnegative quadratic form in the usual sense on a subspace $V_0$ of $V$ and $|\xi|^2 = \infty$ for any vector not in $V_0$. Given a singular metric, $h$, the norm function $|\xi|^2_h$ is a measurable function from the total space of $E$ to $[0, \infty]$, whose restriction to any fiber is a quadratic form in the sense just described.

**Definition 3.1** A singular metric $h$ is **negatively curved**, if $\log |\xi|^2_h$ is plurisubharmonic on the total space of $E$.

Hence, a negatively curved metric is upper semicontinuous, in particular locally bounded from above, so it is a quadratic form in the usual sense on any fiber. Note that the definition means that a metric is negatively curved if and only if $\log |\xi|^2(p)$ is plurisubharmonic for any choice of local holomorphic section of $E$. Since $\log(v_1 + v_2)$ is plurisubharmonic if $\log v_1$ and $\log v_2$ are plurisubharmonic, it follows that a sum of negatively curved metrics is negatively curved.

Any choice of $h$ induces a dual metric $h^*$ on the dual bundle $E^*$. Then $h^*$ is also an hermitian form on each fiber, possibly taking the value $\infty$ on some vectors.

**Definition 3.2** A singular metric $h$ is **positively curved**, if $h^*$ is negatively curved.

Note that if $h$ is smooth, then $h$ is positive if and only if

$$\Theta^h \geq 0$$

in the sense of Griffiths.

**3.1 Proposition.** Let $E$ be a trivial holomorphic vector bundle over a polydisk $U$, equipped with a singular metric denoted by $h$. If $h$ is negatively curved over $U$, then over any smaller polydisk there is a sequence of smooth, strictly negatively curved metrics, $h_\nu$, decreasing pointwise to $h$. Similarly, a positively curved metric can be approximated from below by smooth strictly positively curved metrics on smaller polydisks.

**Proof.** Assume $h$ is negatively curved. By adding a small multiple of a trivial metric (this does not destroy the property of being negatively curved) we may assume $h$ is strictly positive. Let

$$h_\nu(p) = \int \chi_\nu(p - q)h(q),$$

where $\chi_\nu$ is an approximate identity. Then

$$|\xi|^2_{h_\nu} = \int \chi_\nu(p - q)|\xi|^2_h$$

and it follows easily that $h_\nu$ is a smooth negatively curved metric. Then multiply by $e^{p^2/\nu}$ to get strict positivity. The corresponding statement for positively curved metrics follows by taking duals. $\square$

We will next give another criterion for positivity of a bundle. Since the property is local we continue to assume that $E$ is a trivial bundle over a polydisk, $U$. Given $(E, h)$ we consider the bundle of unit balls

$$\Omega = \{(p, z) \in U \times \mathbb{C}^N; |z|^2_{h(p)} \leq 1\}.$$
The next preliminary statement will be very useful in what follows.

**3.2 Proposition.** Assume $h$ is smooth and strictly positively curved, and let $(p_0, z_0)$ be a point on $\partial \Omega$. Then there is a local holomorphic section of $E$ near $p_0$, $s$, such that $s(p_0) = z_0$ and $s$ takes values in $\Omega$.

*Proof.* First choose $s$ holomorphic so that $s(p_0) = z_0$ and $D's = 0$ at $p_0$, where $D'$ is the $(1, 0)$ part of the Chern connection of $h$.

Then, if $\Theta^h$ is the curvature of $h$,

$$i\partial \bar{\partial} \log |s|^2 = -\langle \Theta^h s, s \rangle_h / |s|^2_h < 0.$$  

This means that there is a second degree holomorphic polynomial, $q(p)$, such that

$$\log |s|^2_h - 2 \Re q$$

has a local maximum at $p = p_0$. Then $se^{-q}$ satisfies the conclusion of the above proposition.

The main use of the positivity assumption for bundles $E$ in this section and the next one will be to convert plurisubharmonic functions on the total space of this bundle on plurisubharmonic functions on the base. The precise statement is the following (see also Demailly [12]):

**3.3 Proposition.** Let $V(p, z)$ be a continuous plurisubharmonic function defined in some neighbourhood of $\Omega$ intersected with the fiber $E_0$. Put

$$V^*(p) = \sup_{z \in \Omega_p} V.$$  

If $h$ is positively curved, then $V^*$ is plurisubharmonic of $p$. Conversely, if $V^*$ is plurisubharmonic of $p$ for any continuous plurisubharmonic $V$, then $h$ is positively curved.

*Proof.* We prove first the converse. Let $\xi(p)$ be a local holomorphic section of $E^*$. Put

$$V(p, z) = \log(|\xi(p) \cdot z|^2 + \epsilon).$$

By assumption $V^*$ is plurisubharmonic. Letting $\epsilon$ go to 0, we see that $\log |\xi|^2_{h^*}$ is plurisubharmonic. Hence $E^*$ is negatively curved, so $E$ is positively curved.

Now assume that $E$ is positively curved, and let $h_\nu$ be sequence of smooth strictly positively curved metrics increasing to $h$. Let $\Omega_\nu$ be the corresponding bundles of unit balls. Then $\Omega_\nu$ decreases to $\Omega$, so the corresponding functions

$$V_\nu^*(p) = \sup_{z \in (\Omega_\nu)_p} V$$

decrease to $V^*$. Hence it is enough to prove our claim for the metrics $h_\nu$; in other words we may assume that $h$ is smooth and strictly positively curved. To prove that $V^*$ is plurisubharmonic we can take the base dimension $m = 1$, and moreover $V$ to
be strictly plurisubharmonic. We first claim that $V^*$ has no local maximum. For this, assume to get a contradiction that 0 is a local maximum for $V^*$. Say

$$V^*(0) = V(0, z_0).$$

Take a local holomorphic section of $E$ near 0, satisfying the conclusion of Proposition 2. Then $V(p, s(p))$ is strictly subharmonic of $p$ and has a local maximum for $p = 0$, which is impossible.

Hence $V^*$ satisfies the maximum principle, and it follows that $V^* + \text{Re} q(p)$ also satisfies the maximum principle if $q$ is any holomorphic polynomial. Thus $V^*$ is subharmonic.

Let us now return to the special case of vector bundles, $E$, that arise as the direct image of $K_{X/Y} + L$, where $L$ is a line bundle over $X$, fibered over $Y$. Let

$$B(x, x') = B_y(x, x') = \sum u_j(x) \otimes \bar{u}(x'),$$

be the (relative) Bergman kernel defined at the beginning of section 1, but now considered also outside of the diagonal. The plurisubharmonicity of $\log B(x, x)$ is equivalent to saying that if $x(y)$ is a local holomorphic section of the fibration $X \mapsto Y$, and if $a(y)$ is a local holomorphic section of $-(K_{X/Y} + L)$, then

$$|a(y)|^2 B(x(y), x(y))$$

is plurisubharmonic in $y$. Now choose for some $N > 0$, $N$ holomorphic sections of the fibration $x_j(y)$, and $N$ holomorphic sections of $-(K_{X/Y} + L)$, $a_j(y)$, and consider the quadratic form

$$\sum a_j \bar{a}_k B(x_j, x_k) =: B(a, a).$$

3.4 Proposition. If $E$ is positively curved with respect to the $L^2$-metric (1.1), then for any $N > 0$ and any local sections as above

$$B(a, a)$$

is a plurisubharmonic function of $y$. Conversely, if this holds for $N$ equal to the rank of $E$, then the $L^2$-metric of $E$ is positively curved.

Proof: Given local sections $x_j(y)$ and $a_j(y)$ as above, let

$$\xi_y = \sum a_j(y) ev_{x_j(y)},$$

with $ev_x$ denoting the evaluation functional at $x$. Then $\xi$ is a holomorphic section of the dual bundle, and it is easily verified that $B(\xi) = |\xi|^2$. Hence $B(a, a)$ is plurisubharmonic if $E$ is positively curved. Conversely, if $N$ is the rank of $E$ and if $x_j(y)$ are chosen to be in general position for $y$ close to $y_0$, then any local section of $E^*$ can be obtained in this manner. Hence $E^*$ is negatively curved, so $E$ is positively curved. \qed
We state now the following generalization of the theorem 0.1, which will play an important role in the next part of the paper. (It also motivates the notions introduced and analyzed during this section.)

**Theorem 3.5.** Let $p : X \to Y$ be a smooth holomorphic fibration, and let $L$ be a pseudoeffective line bundle over $X$. Assume that

$$E = p_*(K_{X/Y} + L)$$

is locally free. Then $E$ is positively curved in the sense of Definition 3.2.

**Proof.** By the previous proposition we just need to check that $B(a, a)$ is plurisubharmonic for any choice of sections $x_j$ and $a_j$. By the proof of that proposition, this means that $|\xi|^2$ is plurisubharmonic. For the case of one single section $x_j$ this follows from Theorem 0.1, as explained immediately before the statement of proposition 3.4. The case of an $N$-tuple of sections is proved in the same way: all the steps in the proof for $N = 1$ work in the same way for general $N$. 

---

§4 The Narasimhan-Simha metric

The Narasimhan-Simha metric is a metric on (possibly twisted) multiples of the canonical bundle, $mK_X$, see e.g. Narasimhan-Simha [26], Kawamata [18] and Tsuji [32b] in a more general context. It is defined in a way similar to the Bergman kernel metric, but using $L^p$-spaces instead of $L^2$, with $p = 2/m$.

If $X$ is a compact complex manifold with a holomorphic hermitian line bundle $(L, \phi)$, the Narasimhan-Simha metric (we will call it NS metric in the sequel) for $mK_X + L$ is defined as the dual of the next metric on $-(mK_X + L)$:

$$\|\xi\|^2 := \sup |\sigma(x) \cdot \xi|^2$$

the sup being taken over all sections $\sigma$ to $mK_X + L$ such that

$$\int_X |\sigma \wedge \bar{\sigma}|^{1/m} e^{-\phi/m} \leq 1$$

and $\xi$ is some vector in $-(mK_X + L)_x$.

This defines $h_m$ as a metric on $mK_X + L$. The point of this definition is that the quantity $|\sigma \wedge \bar{\sigma}|^{1/m}$ transforms as a measure on $X$ (in the case of $L$ trivial), so its integral can be defined without choosing any metric on $K_X$. In the relative situation, when the manifold $X$ is fibered over $Y$, we define relative NS metrics in a similar way as in section 1, identifying $K_{X_y}$ with $K_{X/Y}$ restricted to $X_y$, and get a metric on $mK_{X/Y} + L$.

We shall now see that the methods of the previous section also give positivity results for the relative NS metric. For this we will use a family of $L^2$-metrics indexed by the sections to $mK_X + L$, defined by

$$\|\sigma\|^2_u = \int_X |\sigma|^2 / |u|^{2-2/m} e^{-\phi/m}$$
for sections \( \sigma \) to \( mK_X + L \). Thus the \( L^p \)-norm occurring in the definition of the NS metric equals the 'diagonal' metric \( \| \sigma \|^2 \). This family of metrics seems to be quite interesting in their own right and play e.g. a fundamental role in Siu’s proof of the invariance of plurigenera.

Let us now first consider an abstract model of the situation above: Take \( E \to U \) to be a trivial vector bundle of rank \( N \), with an hermitian metric \( h \), positive over

\[
U = \{(t, w); t \in \mathbb{C}, |t| < 1, w \in \mathbb{C}^N \}.
\]

Choosing a trivialization, we can take

\[
E = U \times \mathbb{C}^N = \{(t, w, z); (t, w) \in U, z \in \mathbb{C}^N \}.
\]

Let

\[
\Delta = \{(t, w, z); w = z \},
\]

and

\[
\Omega_\Delta = \Omega \cap \Delta = \{(t, z, z); \|z\|^2_{h(z)} \leq 1 \}.
\]

We state now the next result, analog to the proposition 5.3 of the preceding section; it has a major rôle in our investigation of the convexity properties of the NS metric.

4.1 Proposition. Assume \( V(t, w, z) \) is continuous and plurisubharmonic on \( E \). Put

\[
V^\circ(t) = \sup_{\Omega_{\Delta,t}} V,
\]

where

\[
\Omega_{\Delta,t} = \{(t, z, z); |z|^2_{h(z)} \leq 1 \}.
\]

Then \( V^\circ \) is subharmonic.

Proof. Again, we may assume \( h \) is smooth, strictly positively curved, \( V \) strictly plurisubharmonic, and it suffices to prove that \( V^\circ \) has no local maximum. Assume to the contrary that \( 0 \) is a local maximum point for \( V^\circ \), and say

\[
V^\circ(0) = V(0, z_0, z_0).
\]

Let \( s \) be the local holomorphic section from Proposition 5.2, and put

\[
C = \text{graf}(s) \cap \Delta.
\]

Since the graf of \( s \) has dimension \( N + 1 \) and \( \Delta \) has codimension \( N \), the dimension of \( C \) is at least 1. Moreover

\[
(0, z_0, z_0) \in C \subset \Omega_\Delta.
\]

Thus \( V \) has a local maximum on \( C \), which contradicts the strict plurisubharmonicity assumption.

As a consequence of the previous considerations, we obtain the next statement which is a "twisted version" of the theorem 1 in the paper [18] by Y. Kawamata. In this
corollary we assume as a hypothesis that the direct image of $mK_{X/Y} + L$ under $p$ is locally free, i.e., that for any fiber of $p$, any holomorphic section of $(mK_{X/Y} + L)|_{X_Y}$ over the fiber, extends holomorphically to some neighborhood of the fiber. That this hypothesis is satisfied for $L$ trivial (or $L$ having a semipositive smooth metric) is exactly the invariance of (twisted) plurigenera.

4.2 Corollary. Let $p : X \to Y$ be a smooth projective fibration and let $L$ be a holomorphic line bundle over $X$ endowed with a semipositively curved, possibly singular, metric. Assume that the direct image of $mK_{X/Y} + L$ under the map $p$ is locally free. Then the relative (and twisted) NS metric on $mK_{X/Y} + L$ has semipositive curvature current.

Proof. We will assume that in fact $X$ is fibered over the unit disk (see the comments before the proof of the theorem 2.1 concerning the general case) and let $F$ be the direct image of $mK_{X/Y} + L$. Since $F$ is locally trivial it can be thought of as $F = D \times \mathbb{C}^N$.

Now let $E$ be this bundle, pulled back over the total space of $F$ under the projection of $F$ to $D$. Then

$$E = D \times \mathbb{C}_w^N \times \mathbb{C}_z^N,$$

and we consider $E$ as a bundle over $U := D \times \mathbb{C}^N_w$. Now, a point $u = (t, w)$ in $U$ is a vector in $F$, i.e., a global section of $mK_{X_t} + L$ over $X_t$. Then $|u|^{2 - 2/m}e^{-\phi/m}$ is a metric on $(m - 1)K_{X_t} + L$ over $X_t$ and induces a hermitian metric on $E_u$. Since $u$ is holomorphic as a function of $t$ and $w$ the metric $|u|^{2 - 2/m}e^{-\phi/m}$ is pseudoeffective on $(m - 1)K_{X_t} + L$ considered as a bundle over $X \times \mathbb{C}^N_w$. Hence, by Theorem 3.5, the induced hermitian metric on $E$ is positively curved. We can then apply Proposition 4.1 to

$$V_\epsilon(t, w, z) = \log(|s(p) \cdot z|^2 + \epsilon),$$

with $s$ now equal to a section of $E^*$ giving point evaluations. By Proposition 4.1 $V_\epsilon$ is plurisubharmonic. Hence $V : = \lim V_\epsilon$ as $\epsilon$ goes to zero is also plurisubharmonic. But $V$ is precisely the NS metric.

Remark. In the fundamental article [18] of Y. Kawamata, the convexity properties of the NS metric are derived via the cyclic cover trick and subtle results in Hodge theory. In our proof we replace the cyclic covers by twisting the base with the space of all global holomorphic sections. More precisely, we have considered the product of the base $\Delta$ with the space of sections $H^0(X, mK_{X/\Delta} + L)$; next, we have used the additional parameters to define a metric on the bundle we are interested in. Moreover, one can see that given a map $p : X \to Y$ the above construction has no global meaning, but nevertheless we can use this local construction in order to derive (global!) properties of the NS metric, as indicated in the corollary above. Let us also mention that a more general version of the above statement will be given in the paper [4]: we will work in the context of the singular metrics and surjective projective maps.

Finally we mention one more consequence of the argument we just used, where for simplicity we take $L$ to be trivial. Let $X$ be one fixed projective manifold and consider
the line bundle $m_1K_X$ over $X$. Let $m_0$ be a positive integer and let $u$ be a global holomorphic section of $m_0K_X$; we use $u$ to define a metric on $(m_1 - 1)K_X$ as follows

$$|\sigma|^2_u = |\sigma|^2/|u|^{2(m_1 - 1)/m_0}$$

and corresponding $L^2$-metrics

$$\|\sigma\|^2_u := \int_X |\sigma|^2_u$$

on $H^0(X, m_1K_X)$. Let $B_u(z)$ be the corresponding Bergman kernel.

**4.3 Proposition.** Let $p : \Gamma(X, m_0K_X) \times X \to X$ be the projection on the second factor and consider the bundle $p^*(m_1K_X)$. Then the Bergman kernel $B_u(z)$ defines a pseudoeffective metric on $p^*(m_1K_X)$. (Intuitively: it depends plurisubharmonically on $u$ and $z$ together.) In particular, let

$$Z = \{(u, z) \in \Gamma(X, m_0K_X) \times X ; B_u(z) = 0\}.$$

Then either $Z$ is all of $\Gamma(X, m_0K_X) \times X$ or $Z$ is pluripolar in $\Gamma(X, m_0K_X) \times X$

**Proof.** Let $p$ be the projection map from $\Gamma(X, m_0K_X) \times X$ to $\Gamma(X, m_0K_X)$. This is a trivial fibration with fiber $X$ over $\Gamma(X, m_0K_X)$ and its relative canonical bundle is equal to the inverse image of $K_X$. The norms

$$|\sigma|^2_u = |\sigma|^2/|u|^{2(m_1 - 1)/m_0}$$

define a pseudoeffective metric on $(m_1 - 1)K_X$, so by Theorem 0.1 the Bergman kernel (metric) is either identically 0 or pseudoeffective. In the latter case it can vanish only on a pluripolar set, therefore the theorem is proved.

### §5 Proof of the corollary 0.2

Before going into the details of the proof, we would like to mention the next result of F. Campana (see [6]); it is a generalization of E. Viehweg–type semi-positivity theorems (see e.g. [36]; in connection with this, see also the paper [19] of Y. Kawamata)

**Theorem ([6]).** Let $p : X \to Y$ be a projective surjective map with connected fibers between non-singular manifolds $X$ and $Y$. Let $L$ be a line bundle on $X$, whose first Chern class contains an effective $\mathbb{Q}$–divisor, which has trivial multiplier ideal sheaf when restricted to the general fiber of $p$. Then $p^*(p(K_{X/Y} + L))$ is weakly positive, for any positive and enough divisible integer $m$.

We do not intend to explain here the notion of weak positivity introduced by E. Viehweg in [37], [38]; however, let us point out a consequence of the previous result which is stated in the paper [9].

**Theorem ([9]).** Under the above assumptions, for any $\mathbb{Q}$–ample divisor $A_Y \to Y$ some large enough multiple of the line bundle $K_{X/Y} + L + p^*A_Y$ is effective.
Therefore, our corollary 0.2 can be seen as a generalization and/or effective version of the result above (since there is no hypothesis on the multiplier sheaf in our statement and moreover we are able to control the restriction over some generic fiber of the sections produced by the theorem above).

Next, we are going to explain the proof of 0.2. As we have already mentioned, our arguments rely heavily on the theorem 0.1, and on the next version of the Ohsawa-Takegoshi theorem, proved by L. Manivel (see [13], [23], [27] and the references quoted there).

5.1 Theorem ([23]). Let $X$ be a projective or Stein $n$-dimensional manifold, and let $Z \subset X$ be the zero set of a holomorphic section $s \in H^0(X, E)$ of a vector bundle $E \to X$; the subset $Z$ is assumed to be non-singular, of codimension $r = \text{rank}(E)$. Let $(F, h)$ be a line bundle on $X$, endowed with a (possibly singular) metric $h$, such that:

1. $\Theta_h(F) + \sqrt{-1} \partial \bar{\partial} \log |s|^2 \geq 0$ as current on $X$;

2. $\Theta_h(F) + \sqrt{-1} \partial \bar{\partial} \log |s|^2 \geq 1/\alpha \frac{\langle \Theta(E)s, s \rangle}{|s|^2}$ for some $\alpha \geq 1$;

3. $|s|^2 \leq \exp(-\alpha)$ on $X$, and the restriction of the metric $h$ on $Z$ is well defined.

Then every section $u \in H^0(Z, (K_X + F|_Z) \otimes \mathcal{O}(h|_Z))$ admits an extension $\tilde{u}$ to $X$ such that

$$\int_X \tilde{u} \wedge \bar{u} \exp(-\varphi_F) \leq C_{\alpha} \int_Z |u|^2 \exp(-\varphi_F) dV_Z |\wedge^r (ds)|^2,$$

where $h = e^{-\varphi_F}$, provided the right hand side is finite.

Notice that the integrand in the integral over $Z$ here is independent of the metric chosen.

We will often use the Theorem 5.1 in the following situation: $(E, h_E)$ is a holomorphic line bundle with a smooth hermitian metric (remark that we do not require any curvature conditions for it) and $Z$ is a hypersurface defined by a section $s$ to $E$. $(F', h_{F'})$ is another holomorphic hermitian line bundle over $X$ with a possibly singular metric with semipositive curvature current, such that

$$\Theta_h(F') \geq e \Theta_h(E)$$

in the sense of currents. Then $F := E + F'$ endowed with the product metric $h_Eh_{F'}$ satisfies the curvature assumptions in Theorem 5.1 if $\alpha$ is large enough. Hence any section of

$$(K_X + F)|_Z$$

that is square integrable over $Z$ extends to a section of $K_X + F$ over all of $X$. Note also that by adjunction $(K_X + E)|_Z = K_Z$, so we are extending sections to $K_Z + F'$, and that the integrability assumption means that

$$\int_Z c_n u \wedge \bar{u} \exp(-\varphi_{F'}) < \infty.$$
Also, in the papers quoted above, the metric \( h \) is non-singular; however, since the manifold \( X \) above is projective, one can derive 5.1 from the smooth case, by a regularization argument. We refer to [34], [35] and [24] for the details concerning the regularization process. Remark also that the only additional conditions needed in order to allow singular metrics are the integrability, and the generic finiteness of the restriction to the manifold we want to extend our form.

The bundle we are interested in can be decomposed as follows:

\[
m(K_{X/Y} + L) + p^*A = K_X + (m - 1)(K_{X/Y} + L) + L + p^*(A - K_Y)
\]

(the bundle \( A \) will be chosen in a moment). Our goal is to show that it is effective by extending (a multiple of) the section \( 0 \neq u \in H^0(X_y, (K_{X_y} + L) \otimes \mathcal{I}(h)) \) whose existence is given by the hypothesis.

We denote by \( h_B \) the metric on \( K_{X/Y} + L \) given by the theorem 1; the fact which will be crucial in what follows is that for any positive integer \( k \), we have

\[
|u|^2_{h_B^k} \leq O(1)
\]

uniformly on \( X_y \), just by the construction of the metric \( h_B \).

We choose now the bundle \( A_Y \to Y \) positive enough such that:

(a) \( H^0(Y, A_Y) \neq 0 \);

(b) The point \( y \in Y \) is the common zero set of the sections \( (s_j) \) of a ample line bundle \( B \to Y \) and \( A_Y - K_Y \geq 2B \), in the sense that the difference is an ample line bundle.

The bundle \( (m - 1)(K_{X/Y} + L) + L + p^*(A_Y - K_Y) \) is endowed with the metric

\[
h_m := h_B^{\otimes (m-1)} \otimes h_L \otimes h_{A_Y - K_Y};
\]

its curvature is semi-positive on \( X \), and the restriction to \( X_y \) is well defined. The section we want to extend is \( u_m := u^\otimes m \otimes s_{p^*A_Y} \), where \( s_{p^*A_Y} \) is the pull-back of some non-zero section given by the property (a) above. By the property (b) the positivity conditions in the extension theorem of Ohsawa-Takegoshi are satisfied with the bundle \( F \) given by

\[
F = (m - 1)(K_{X/Y} + L) + L + p^*(A_Y - K_Y)
\]

since

\[
\Theta_{h_m} ((m - 1)(K_{X/Y} + L) + L + p^*(A_Y - K_Y)) + \sqrt{-1} \partial \partialbar \log |s|^2 \geq p^*\Theta (A_Y - K_Y - B).
\]

The right hand side semi-positive and it dominates the bundle \( B \); thus the requirements (1), (2) and (3) are verified.

Now the integrability condition is obviously ok, since we have

\[
\int_{X_y} |u_m|^2_{h_m} dV \leq C \int_{X_y} |u|^2_{h_L} dV < \infty
\]
(the first inequality comes from the fact that the section $u$ is pointwise bounded with respect to the fiberwise $L^2$ metric $h_B$, and the second inequality is just the hypothesis).

Therefore, by the Ohsawa-Takegoshi theorem we can extend the section $u_m$ over $X$ and the corollary 0.2 is proved. \hfill \Box

**Remark.** In fact, we have just proved more than the corollary 0.2 states. Indeed, we have shown that given $u \in H^0(X_y, (K_{X_y} + L) \otimes I(h))$, then the section $u^\otimes m \otimes s_{p^*}A_Y$ extends over all of $X$ for any positive integer $m$.

### §6 Maximal centers and asymptotic extensions

In this paragraph we give a complete proof of the theorem 0.3. We have already mentioned that this result is not new, but we feel that the arguments which will be used here (which are mainly analytic) could be very useful in other problems in algebraic geometry.

To start with, we would like to recall the notions of restricted volume and maximal center of a line bundle, respectively of a $\mathbb{Q}$-divisor.

**Definition.** Let $E \to X$ be a line bundle, and let $V \subset X$ be an $d$-dimensional subset (which could be singular, but reduced and irreducible). We define

$$H^0(X|V,mE) := \operatorname{Im}(H^0(X,mE) \to H^0(V,mE_V)),$$

and let $h^0(X|V,mE)$ be the dimension of this space. Then the restricted volume of $E$ to $V$ is

$$\operatorname{Vol}_V(E) := \limsup \frac{d!}{md} h^0(X|V,mL).$$

If the line bundle $E$ is numerically effective, then we have $\operatorname{Vol}_V(E) = E^d : V$; this is a consequence of the Riemann-Roch theorem (see [5] and also [14]). This notion turned out to be very useful to deal with the linear systems which do have base-points, and one of the most spectacular result in the theory is the theorem 0.3 which will be discussed next. For further properties and consequences, we refer to [14].

Let $D$ be an effective $\mathbb{Q}$-divisor on $X$. We denote by $I(D)$ the multiplier ideal sheaf associated to $D$ (see e.g. [11], [25]). To introduce the notion of maximal center of the pair $(X,D)$, let us assume that we have the decomposition $D = S + D'$, where $S$ is a smooth hypersurface, and $D'$ is an effective $\mathbb{Q}$-divisor which does not contain $S$ in its support. Then the adjunction formula gives $K_X + D|_S = K_S + D'|_S$, in particular the difference $K_X + D|_S - K_S$ is an effective $\mathbb{Q}$-divisor on $S$. As we will see in a moment, a maximal center is a substitute in codimension greater than 1 of the hypersurface $S$ above. We recall the following notions.

**Definition.**
1. We say that the pair \((X, D)\) is klt at \(x \in X\) if \(\mathcal{I}(D)_x = \mathcal{O}_{X,x}\); the support of the quotient \(\mathcal{O}_X/\mathcal{I}(D)\) is denoted by \(\text{Nklt}(X, D)\).

2. The log-canonical threshold of the pair \((X, D)\) at \(x\) is the rational number

\[
\text{lct}_x(X, D) := \sup \{ t \in \mathbb{Q}_+ / \text{I}(tD)_x = \mathcal{O}_{X,x} \}
\]

3. We denote by \(\mathcal{MC}(X, D)\) the set of irreducible components \(V\) of \(\text{Nklt}(X, D)\) such that the log-canonical threshold of \((X, D)\) at the generic point of \(V\) is equal to 1. The elements of this set are called maximal centers of \((X, D)\).

We refer to the references [11], [19], [21] for a more extensive presentation and some properties of the notions introduced above; here we will content ourself to recall some standard facts which will be needed in what follows.

- Given a point \(x \in X\), by a slight perturbation of the divisor \(D\) we can assume that there is at most one element of the set \(\mathcal{MC}(X, D)\) containing \(x\).
- If the maximal center \(V\) is a divisor, then we have the decomposition \(D = V + D'\), where \(D'\) is an effective \(\mathbb{Q}\)-divisor, such that \(V\) is not contained in the support of \(D'\).
- Assume now that the codimension of \(V\) is at least 2, and assume that there exist an ample \(\mathbb{Q}\)-line bundle \(A \to X\) such that \(L := A + D\) is a genuine line bundle on \(X\). In this context we have the next “concentration” lemma of Kawamata-Shokurov, see e.g. [30].

6.1 Lemma ([30]). Let \(L = A + D\) be a line bundle, where \(A\) and \(D\) are \(\mathbb{Q}\)-divisors ample and effective, respectively. We assume that \(V \in \mathcal{MC}(X, D)\) is of codimension at least 2. Then there exist (another) decomposition \(L = A + \tilde{D}\) such that:

(A) \(V \in \mathcal{MC}(X, \tilde{D})\).

(B) There exist \(\mu : Y \to X\) a sequence of blow-ups, such that we have

\[
\mu^*(K_X + L) = K_Y + S + \sum_j \nu_j E_j
\]

where \(S\) and \(E_j\) are in normal crossings, and the next properties hold:

(B.1) \(S\) is a smooth, \(\mu\)-contractible hypersurface of \(Y\), such that \(\mu(S) = V\);

(B.2) If for some index \(j\) we have \(V \subset \mu(E_j)\), then the corresponding multiplicity \(\nu_j\) is strictly less than 1.

(B.3) If the multiplicity \(\nu_j\) is a negative number, then the corresponding divisor \(E_j\) is contracted by \(\mu\).

(B.4) Some of the \(E_j\) is ample on \(Y\), and the corresponding coefficient is positive and strictly less than 1.

Remark that if we don’t need the property that a unique exceptional divisor with multiplicity 1 in the decomposition above dominates \(V\), all the other facts in the lemma are immediate consequences of the fact that \(V\) is the maximal center of \((X, D)\). However, this property will simplify the arguments of the proof of the next corollary.
We have divided the proof of 0.3 into two parts: the first one uses techniques developed in connection with the invariance of plurigenera, and the second one uses the semi-positivity of the relative canonical bundles (corollary 0.2).

§6.1 Invariance of Plurigenera techniques

The next result is due to S. Takayama in [30]; his proof follow very closely the techniques introduced by Y.-T. Siu in [34]. Rather than reproducing here his arguments, we will use a more direct approach taken from [28] (see also the papers [8], [30] and [36] for other variations and further results on this theme). The version which is best adapted for our proof of 0.3 is the following.

6.1.1 Theorem ([30], [36]). Let $Z \subset X$ be a non-singular hypersurface and let $L = A + D$ be a big line bundle, where $A$ and $D$ are $\mathbb{Q}$-line bundles ample and effective, respectively. Let $A_X$ be a line bundle on $X$, endowed with a smooth metric $h_{A_X}$ whose curvature is semi-positive. Assume that $D$ does not contain $Z$ in its support and that $\mathcal{I}(D|_Z) = \mathcal{O}_V$. Then for all $m \in \mathbb{Z}_+$, the restriction morphism

$$H^0(X, m(K_X + Z + L) + A_X) \to H^0(Z, (m(K_X + Z + L) + A_X)|_Z)$$

is surjective.

The proof of this theorem is given in the last subsection in order not to break the general argument.

§6.2 End of the proof of 0.3

Let $\mu : Y \to X$ be a log-resolution of the pair $(X, D)$, such that the properties in the lemma 6.1 are satisfied. We can re-write the relation (6.1) as follows

$$(6.2.1) \quad \mu^*(K_X + L) = K_Y + S + D_Y - E_Y$$

where $D_Y, E_Y$ are effective divisors and $E_Y$ is contracted by $\mu$.

If we denote by $D^h_Y$ the “horizontal” part of the divisor $D_Y$, (i.e. the components of $D_Y$ whose projection on $X$ contain $V$) then by the property $(B_2)$ of the lemma, we have $[D^h_Y] = 0$ (we recall that the integral part, respectively the fractional part of a divisor is obtained by considering the integral part, respectively the fractional part of its coefficients).

We restrict the above equality to $S$ and use adjunction to get

$$\mu^*(K_X + L)|_S = K_S + D_Y - E_Y.$$ 

Next we subtract the inverse image of $K_V$ and split the $\mathbb{Q}$–divisors $D_Y$ and $E_Y$ according to their integral/fractional parts; the above relation becomes

$$(6.2.2) \quad \mu^*(K_X + L - K_V)|_S - [D_Y] - [-E_Y]|_S = K_{S/V} + \{D_Y\} + \{-E_Y\}|_S.$$
We consider the $\mathbb{Q}$-bundle $G := \{D_Y\} + \{-E_Y\}|_S$. According with the notations in the lemma 6.1, we have

$$G = \sum_j (\nu_j - [\nu_j])E_j$$

and we can endow it with the canonical singular metric (observe in the first place that $G$ is a genuine line bundle, by the relation (6.2.2) above). Along the next few lines, we will highlight some of its metric properties, needed in what will follow.

$(G_1)$ The multiplier ideal sheaf of the metric is trivial. Indeed, this is an obvious consequence of the “normal crossings” condition and the fact that the coefficients $G$ are in $(0,1)$ (since the divisors $E_j$ are disjoint).

$(G_2)$ Given $v \in V$ generic, we have $H^0(S_v, K_{S_v} + G) \neq 0$. Indeed, by the relation (6.2.2) and the property $(B_2)$ of the lemma 6.1, the bundle $K_{S_v} + G|_{S_v}$ is isomorphic with $-[-E_Y]|_{S_v}$, so our claim is that this line bundle has at least a non-zero section. It is the case since $-[-E_Y]$ has a section whose restriction to $S$ is non-trivial (recall the property $(B.2)$), and so will be the further restriction of this section of some generic fiber.

$(G_3)$ The line bundle $G$ is big. To see this, we recall that the property $(B_4)$ says that some of the $E_j$ in the decomposition $(1)$ is ample on $S$ and that the corresponding coefficient $\nu_j$ belongs to $(0,1)$; therefore, the ample part is not lost when we take $\nu_j - [\nu_j]$ in the construction of $G$.

In conclusion, we have shown that we are in good position to apply our corollary 0.2: for any positive integer $m$, we get a non-trivial section $\psi_m \in H^0(S, \mu^*A_X + m(K_{S/V} + G)|_S)$, where $A_X$ denotes here some positive enough but fixed line bundle on $X$. The multiplication with $\psi_m$ will define an injective map

$$(6.2.3) \quad \Psi_m : H^0(V, mK_V) \to H^0(S, \mu^*A_X + m(K_Y + S + G)|_S)$$

as follows: $\Psi_m(\tau) = \mu^*\tau \otimes \psi_m$.

The result proved in the preceding paragraph 6.1 come now into the picture: the sections in the image of the morphism $\Psi_m$ can be extended to $Y$ thanks to the theorem 6.1.1. Indeed, the line bundle $G$ is big (see $(G_3)$ above) and the multiplier ideal of the restriction of its effective part to $S$ is trivial (by the same argument as in $(G_1)$).

Let us consider now a section $u \in H^0(V, mK_V)$. By the discussion above, there exists a section $\tilde{u} \in H^0(Y, \mu^*A_X + m(K_Y + S + G))$ such that $\tilde{u}|_S = \Psi_m(u)$. We denote by $s_m[D_Y]$ the canonical section of the bundle $m[D_Y]$, and then we have $\tilde{u} \otimes s_m[D_Y] \in H^0(Y, \mu^*A_X + m(\mu^*(K_X + L) - [-E_Y]))$ by the definition of $G$ and the equality (6.2.1).

We recall now that every hypersurface in the support of the $\mathbb{Q}$-divisor $E_Y$ is contracted by $\mu$, therefore Hartogs type arguments show that

$$(6.2.4) \quad \tilde{u} \otimes s_m[D_Y] = \mu^*(J_m(u)) \otimes s_{-m[-E_Y]}$$

for some section $J_m(u) \in H^0(X, A_X + m(K_X + L))$. In conclusion, we have defined an injective linear map

$$J_m : H^0(V, mK_V) \to H^0(X|V, A_X + m(K_X + L)).$$
By hypothesis there exists a section $u \in H^0(X, m_0(K_X + L) - A_X)$ such that $u|_V \neq 0$. Then by a multiplication with the section $u$, the above relation show the existence of a injective map

$$J'_m : H^0(V, mK_V) \to H^0(X|_V, (m + m_0)(K_X + L))$$

and this clearly implies the theorem 0.3, since the non-effective integer $m_0$ will disappear as $m \to \infty$).

**Remark.** As we have already mentioned in beginning the section 5, the version of the corollary 0.2 needed above can also be obtained by the generalization due to F. Campana of some semi-positivity results of E. Viehweg. However, we prefer to apply our result, not just because the technique needed to prove it is by far lighter, but also it provides an effective link between the canonical series of the manifolds $V$ and $X$, respectively (see the comments page 15).

§6.3 Proof of Theorem 6.1.1
We finally give the proof of Theorem 6.1.1.

Let us fix an ample line bundle $B \to X$, positive enough so that the next conditions hold true.

- For each $0 \leq p \leq m - 1$, the bundle $p(K_X + Z + L) + B$ is generated by its global sections, which we denote by $(s_j^p)$;

- Any section section of the bundle $m(K_X + Z + L) + A_X + B|_Z$ admits an extension to $X$.

Let $u \in H^0(Z, m(K_X + Z + L) + A_X|_Z)$ be the section we want to extend. We consider the following statement:

$$(P_{k,p}) : \text{The sections } u^k \otimes s_j^{(p)} \in H^0(Z, (km + p)(K_X + Z + L) + kA_X + B|_Z) \text{ extend to } X, \text{ for each } k \in \mathbb{Z}_+, 0 \leq p \leq m - 1 \text{ and } j = 1, ..., N_p.$$  

If we can show that $P_{k,p}$ is true for any $k$ and $p$, then another application of the extension theorem of Ohsawa–Takegoshi will end the proof; the argument goes as follows.

Let $\tilde{u}_j^{(km)} \in H^0(X, km(K_X + Z + L) + kA_X + B)$ be an extension of $u^k \otimes s_j^{(0)}$. We take $k \gg 0$, so that

$$\frac{m - 1}{mk} B < A,$$

in the sense that the difference is ample (we recall that $L = A + D$). Let $h_A$ be a smooth, positively curved metric on $A - \frac{m-1}{mk} B$.

We apply now the extension theorem 5.1 with $F := (m - 1)(K_X + Z + L) + L + A_X + Z$; a metric $h$ on $F$ is constructed as follows:
(1) on the factor \((m - 1)(K_X + Z + L) + L + A_X\) we take the algebraic metric given by the family of sections \(\tilde{u}_j^{(k)}\) (more precisely, we take the \(\frac{m-1}{mk}\)th root of this metric) multiplied with the canonical singular metric on \(D\) twisted the metrics \(h_A\) on \(A - \frac{m-1}{mk}B\) and \(h_A^{1/m}\) on \(1/mA_X\) respectively;

(2) we take an arbitrary, smooth metric \(h_Z\) on the bundle associated to the divisor \(Z\).

With our choice of the bundle \(F\), the section \(u\) we want to extend become a section of the adjoint bundle \(K_X + F|_Z\) and the positivity requirements (1) and (2) in the extension theorem are satisfied since we have

\[
\Theta_{h}(F) + \sqrt{-1}\partial\bar{\partial}\log \|s\|_{h_Z}^2 \geq \Theta_{h_A}(A - \frac{m-1}{mk}B) > 0
\]

(insthe notation above, \(s\) is the canonical section of the bundle \(\mathcal{O}(Z)\)). Remark that the curvature of the bundle \(\mathcal{O}(Z)\) does not affect in any way the \((1,1)\)-form above, since it cancel out by the Poincaré-Lelong identity. Also, by an appropriate choice of the constant \(\alpha\) in the extension theorem and a rescaling of the section \(s\), the conditions (2) and (3) will be satisfied as well.

Concerning the integrability of \(u\), remark that we have

\[
\int_Z \frac{|u|^2}{(\sum_j |\tilde{u}^{(km)}|^2)^{\frac{m-1}{mk}}} = \int_Z \frac{|u|^2 \exp(-\varphi_D)}{(\sum_j |u^{\otimes km} \otimes s_j^{(0)}|^2)^{\frac{m-1}{mk}}} \leq C \int_Z \exp(-\varphi_D)
\]

and this last integral converge by the hypothesis concerning the multiplier ideal sheaf of the restriction of \(D\) to \(Z\).

Thus, it is enough to check the property \(\mathcal{P}_{k,p}\). For this, we will use an inductive procedure; the first steps are as follows.

(1) For each \(j = 1, ..., N_0\), the section \(u \otimes s_j^{(0)} \in H^0(Z, m(K_X + Z + L) + A_X + B|_Z)\) admits an extension \(\tilde{u}_j^{(m)} \in H^0(X, m(K_X + Z + L) + A_X + B)\), by the property \(\bullet\).

(2) We use the sections \(\tilde{u}_j^{(m)}\) to construct a metric on \(m(K_X + Z + L) + A_X + B\); we multiply it with the (positively curved) metric of \(L\) induced by \(A\) et \(D\); thus, we obtain a metric on the bundle \(m(K_X + Z + L) + L + A_X + B\).

(3) Let \(F := m(K_X + Z + L) + L + A_X + B + Z\); for each \(j = 1, ..., N_1\) let us consider the section \(u \otimes s_j^{(1)} \in H^0(Z, (K_X + F)|_Z)\). It is integrable with respect to the metric produced at (2), since

\[
\int_Z \frac{\|u \otimes s_j^{(1)}\|^2}{\sum_q \|u \otimes s_q^{(0)}\|^2} \exp(-\varphi_D) dV < \infty
\]

again by hypothesis and \(\bullet\).

(4) We apply the extension theorem (as above, we take an arbitrary smooth metric on \(\mathcal{O}(Z)\)) and we get \(\tilde{u}_j^{(m+1)} \in H^0(X, (m+1)(K_X + Z + L) + A_X + B)\) whose restriction on \(Z\) is precisely \(u \otimes s_j^{(1)}\).
Now the assertion $P_{k,p}$ will be obtained by iterating the procedure (1)-(4) several times. Indeed, assume that the proposition $P_{k,p}$ has been checked, and consider the set of global sections

$$(\tilde{u}_{j}^{(km+p)}) \in H^0(X, (km + p)(K_X + Z + L) + kA_X + B)$$

which extend $u^k \otimes s_{j}^{(p)}$. The metric associated to them twisted with the metric of $L$ induce a metric on the bundle $(km + p)(K_X + Z + L) + L + kA_X + B$. Now if $p < m - 1$, we can consider the family of sections $u^k \otimes s_{j}^{(p+1)} \in H^0(Z, K_X + F|_Z)$, where $F := (km + p)(K_X + Z + L) + L + kA_X + B + Z$. Each of them is integrable with respect to the previous metric (by the same arguments as in (3) above) and the extension theorem 3.1 show that $P_{k,p+1}$ is verified. In the remaining case $p = m - 1$, we will consider the bundle $F$ as above twisted with $A_X$ and its corresponding metric. Over $Z$, the section we will consider are $u^{k+1} \otimes s_{j}^{(0)} \in H^0(Z, K_X + F)$; the rest of the proof goes along the same lines, thus we skip it. \qed
References

[1] Berndtsson, B.: Positivity of direct image bundles and convexity on the space of Kahler metrics; math arXiv: math.CV/0608367.
[2] Berndtsson, B.: Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains; (math arXiv:mathCV 0505469, to appear in Ann Inst Fourier).
[3] Berndtsson, B.: Curvature of Vector bundles associated to holomorphic fibrations; to appear in Ann. of Maths. (2007).
[4] Berndtsson, B., Păun, M.: Bergman kernels and the pseudo-effectivity of the relative canonical bundles –Part two–; in preparation.
[5] Boucksom, S.: On the volume of a big line bundle; Intern. J. Math. 13 (2002), 1043–1063.
[6] Campana, F.: Special varieties, orbifolds and classification theory; Ann. Inst. Fourier 54 (2004), 499–665.
[7] de Cataldo, M.A.: Singular hermitian metrics on vector bundles J. Reine Angew. Math. 502 (1998), 93-122.
[8] Claudon, B.: Invariance for multiples of the twisted canonical bundle; math.AG/0511736, to appear in Annales de l’Institut Fourier.
[9] Debarre, O.: Systèmes pluricanoniques sur les variétés de type général; séminaire Bourbaki 970, 2006-2007.
[10] Demailly, J.-P.: Singular hermitian metrics on positive line bundles; Proceedings of the Bayreuth conference Complex algebraic varieties, April 2-6, 1990, edited by K. Hulek, T. Peternell, M. Schneider, F. Schreyer, Lecture Notes in Math. 1507, Springer-Verlag, 1992.
[11] Demailly, J.-P.: A numerical criterion for very ample line bundles; J. Differential Geom. 37 (1993), 323-374.
[12] Demailly, J.-P.: Pseudoconvex-concave duality and regularization of currents; Several complex variables (Berkeley, CA, 1995–1996), 233–271, Math. Sci. Res. Inst. Publ., 37, Cambridge Univ. Press, Cambridge, 1999.
[13] Demailly, J.-P.: On the Ohsawa-Takegoshi-Manivel extension theorem; Proceedings of the Conference in honour of the 85th birthday of Pierre Lerong, Paris, September 1997, Progress in Mathematics, Birkauser, 1999.
[14] Ein, L. et al. Restricted volumes and base loci of linear series; available at math.AG/0607221.
[15] Fujita, T.: On Kähler fibre spaces over curves; J. Math. Soc. Japan 30 (1978), 779794.
[16] C. D. Hacon, J. McKernan: Boundedness of pluricanonical maps of varieties of general type; Invent. Math. Volume 166, Number 1 / October, 2006, 1-25.
[17] Kawamata, Y.: Minimal models and the Kodaira dimension of algebraic fiber spaces; Journ. Reine Angew. Math. 363 (1985) 1-46.
[18] Kawamata, Y.: Kodaira dimension of algebraic fiber spaces over curves; Invent. Math. 66 (1982) no. 1, 57–71.
[19] Kawamata, Y.: Subadjunction of log canonical divisors; Amer. J. Math. 120 (1998) 893–899.
[20] Kollár, J.: Subadditivity of the Kodaira dimension: Fibres of general type; Algebraic geometry, Sendai 1985, Advanced Studies in Pure Math. 10 (1987) 361398.
[21] Lazarsfeld, R.: Positivity in Algebraic Geometry; Springer, Ergebnisse der Mathematik und ihrer Grenzgebiete.
[22] Maitani F., Yamaguchi H.: Variation of Bergman metrics on Riemann surfaces; Mathematische Annalen Volume 330, Number 3 / November, 2004, 477-489.
[23] Manivel, L.: Un théorème de prolongement L2 de sections holomorphes d’un fibré hermitien; Math. Zeitschrift 212 (1993), 107-122.
[24] McNeal J., Varolin D.: Analytic inversion of adjunction: L2 extension theorems with gain; Ann. Inst. Fourier (Grenoble) 57 (2007), no. 3, 703–718.
[25] Nadel, A.M.: Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature; Ann. of Math. 132 (1990), 549596.
[26] Narasimhan, M. S.; Simha, R. R.: Manifolds with ample canonical class. Invent. Math. 5 (1968) 120–128.
Bergman kernels and the pseudoeffectivity of relative canonical bundles

[27] Ohsawa, T., Takegoshi, K.: On the extension of $L^2$ holomorphic functions; Math. Z., 195 (1987), 197–204.

[28] Păun, M.: Siu’s Invariance of Plurigenera: a One-Tower Proof; preprint 2005, to appear in J. Diff. Geom. .

[29] Takayama, S: On the Invariance and Lower Semi–Continuity of Plurigenera of Algebraic Varieties; J. Algebraic Geom. 16 (2007), no. 1, 1–18.

[30] Takayama, S: Pluricanonical systems on algebraic varieties of general type; Invent. Math. Volume 165, Number 3 / September, 2005, 551-587.

[31] Tsuji, H.: Variation of Bergman kernels of adjoint line bundles; math.CV/0511342.

[31b] Tsuji, H.: Dynamical construction of Kähler-Einstein metrics; math.AG/0606023.

[32] Tsuji, H.: Pluricanonical systems of projective varieties of general type I; Osaka J of Math 43-4 (2006), 967-995.

[32a] Tsuji, H.: Pluricanonical systems of projective varieties of general type II; Osaka J of Math 44-3 (2007), 723-734.

[32b] Tsuji, H.: Curvature semipositivity of relative pluricanonical systems; math.AG/0703729.

[33] Siu, Y.-T.: Invariance of Plurigenera; Inv. Math., 134 (1998), 661-673.

[34] Siu, Y.-T.: Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type; Complex geometry (Göttingen, 2000), 223–277, Springer, Berlin, 2002.

[35] Siu, Y.-T.: Multiplier ideal sheaves in complex and algebraic geometry; Sci. China Ser. A 48, 2005.

[36] Varolin, D.: A Takayama-type extension theorem; math.CV/0607323 to appear in Comp. Math.

[37] Viehweg, E.: Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces; Algebraic Varieties and Analytic Varieties, Advanced Studies in Pure Math. 1 (1983) 329353.

[38] Viehweg, E.: Quasi-Projective Moduli for Polarized Manifolds; Springer-Verlag, Berlin, Heidelberg, New York, 1995 as: Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 30.

(version of January 21, 2008, printed on March 1, 2022)

Bo Berndtsson, bob@math.chalmers.se
Mihai Păun, paun@iecn.u-nancy.fr