The discovery of the D-brane in the string theory has altered the notion of extra-dimensions dramatically. The gauge fields are confined on the brane and only the gravity can propagate in the whole higher dimensional spacetime. Inspired by this possibility, Randall and Sundrum constructed a model where the size of the extra-dimension can be adjusted. In this model, gauge fields are confined on the brane and only the gravity can propagate in the whole higher dimensional spacetime. The theory on the D-brane described by the Born-Infeld action is not like Einstein-Maxwell theory in the lower order of the gradient expansion, i.e., the Maxwell field does not appear in the theory. Thus the careful analysis and statement for cosmology on self-gravitating D-brane should be demanded in realistic models.

I. INTRODUCTION

We consider a D-brane coupled with gravity in type IIB supergravity on $S^5$ and derive the effective theory on the D-brane in two different ways, that is, holographic and geometrical projection methods. We find that the effective equations on the brane obtained by these methods coincide. The theory on the D-brane described by the Born-Infeld action is not like Einstein-Maxwell theory in the lower order of the gradient expansion, i.e., the Maxwell field does not appear in the theory. Thus the careful analysis and statement for cosmology on self-gravitating D-brane should be demanded in realistic models.
bulk matter and gravitational fields. After that we can derive an effective theory for a gravitating brane. We expect that almost the same result is obtained in the holographic approach.

We will work in type IIB supergravity on $S^5$ because the AdS/CFT correspondence was originally formulated between super Yang-Mills theory and type IIB supergravity aided by D-branes \cite{19}. For simplicity, however, we will turn off several fields in the course of calculation. The rest of this paper is composed of two main parts. In Sec. II, we will adopt the holographic method. We first describe the strategy and then obtain the solution to the Hamilton-Jacobi equation. Therein we will compare the results obtained in each methods and present their interpretations.

II. HOLOGRAPHIC APPROACH

We will derive the effective action for gravitating D-brane using the AdS/CFT correspondence. See Ref. \cite{10} for the study of holography on probe D-branes. This section is organised as follows. We begin with the Hamilton-Jacobi equation in the Sec. II A and give its solution in the Sec. II B. Then we derive the effective theory on the D-brane discusssions. Therein we will compare the results obtained in each methods and present their interpretations.

A. Type IIB Supergravity on $S^5$ and Hamilton-Jacobi equation

We begin with the action for type IIB supergravity on $S^5$:

$$S = \frac{1}{2\kappa^2} \int d^5x\sqrt{-g} \left\{ e^{-2\phi + \frac{4}{5}\rho} \left[ (5)R + 4(\nabla \phi)^2 + \frac{5}{4}(\nabla \rho)^2 - 5\nabla \phi \nabla \rho - \frac{1}{2}|H|^2 \right] - \frac{1}{2}e^{\frac{2}{5}\rho} \left[ (\nabla \chi)^2 + |F|^2 + |G|^2 \right] + e^{-2\phi + \frac{4}{5}\rho} R(S^5) \right\},$$

where $H_{MNK} = \frac{1}{2} \partial_{[M} B_{NK]}, \; F_{MN} = \frac{1}{2} \partial_{[M} C_{NK]}, \; G_{K_1K_2K_3K_4K_5} = \frac{1}{5} \partial_{[K_1} D_{K_2K_3K_4K_5]}, \; \tilde{F} = F + \chi H$ and $\tilde{G} = G + C \wedge H$. $|A_4|^2 = \frac{1}{9} A_{K_1\cdots K_4} A^{K_1\cdots K_4}$. $M, N = 0, 1, 2, 3, 4$ and hereafter we set $2\kappa^2 = 1$. For example, see Ref. \cite{3} for the derivation.

Recently Sato and Tsuchiya derived the Born-Infeld action for a probe D-brane as a solution to the Hamilton-Jacobi equation \cite{5}. Since the effective action is obtained via the transition amplitude from the vacuum to the boundary state representing the probe D3-brane, it could be classical counter-terms. The solutions to Hamilton-Jacobi equation is raised in the classical limit of Wheeler-De Witt equation.

In this paper we will consider the self-gravitating D3-brane, not probe one. Our purpose is to get the action for the gravitating D-brane where we can discuss the cosmology correctly. For this purpose we first write down the full expression of the Hamilton-Jacobi equation:

$$-e^{2\phi + \frac{4}{5}\rho} \left[ \left( \delta S \right)_{\delta \phi \rho} + \frac{1}{2} \left( \delta S \right)^2 - \frac{1}{2} q_{\mu \nu} \delta S_{\delta \phi} + \frac{4}{5} \left( \delta S \right)^2 + \frac{\delta S}{\delta \phi} + \left( \frac{\delta S}{\delta \rho} - \chi \frac{\delta S}{\delta C_{\mu \nu}} - 6 C_{\alpha \beta} \frac{\delta S}{\delta D_{\mu \nu \alpha \beta}} \right) \right]$$

$$-e^{-2\phi + \frac{4}{5}\rho} \left[ (4)R + 4D^2 \phi - \frac{5}{2} D^2 \rho - 4(D\phi)^2 - 16(D\rho)^2 + 53 D\phi D\rho - \frac{11}{12} H_{\mu \nu \rho} H^{\mu \nu \rho} \right] - e^{-2\phi + \frac{4}{5}\rho} R(S^5)$$

$$-e^{\frac{2}{5}\rho} \left[ \frac{1}{12}(D\chi)^2 - \frac{1}{12} \tilde{F}_{\mu \nu \alpha} \tilde{F}^{\mu \nu \alpha} \right] - e^{-\frac{2}{5}\rho} \left[ \frac{1}{2} \frac{(\delta S)}{\delta \chi} + \frac{(\delta S)}{\delta C_{\mu \nu}} + \left( \frac{\delta S}{\delta D_{\mu \nu \alpha \beta}} \right)^2 \right] = 0,$$

where $q_{\mu \nu}$ and $D_{\mu}$ are the induced metric on the D3-brane and its covariant derivative. $\mu, \nu = 0, 1, 2, 3$.

In Ref. \cite{3}, all fields were supposed to be constant and then it was shown that the Born-Infeld action with the Wess-Zumino terms is a solution up to full orders of $\alpha'$:

$$S_{BI} = \alpha \int d^4x \sqrt{-g} e^{-2\phi + \rho} + \beta \int d^4x e^{-\phi} \sqrt{-\det(q_{\mu \nu} + B_{\mu \nu}) + \gamma \left( \int D + \int C \wedge B + \frac{1}{2} \int \chi B \wedge B \right)}.$$

where $\alpha^2 = 5R(S^5)$ and $\beta^2 = \gamma^2$. In our paper, on the other hand, we will not assume these fields are constant. To solve the Hamilton-Jacobi equation, we will employ the gradient expansion scheme in proceeding sections.
B. Solution to Hamilton-Jacobi equation

Let us solve the Hamilton-Jacobi equation using the gradient expansion scheme. The expansion parameter is $\epsilon = \ell^2/L^2$, where $\ell$ and $L$ are the bulk curvature scale and the typical gradient scale on the brane, respectively. The solution is expanded as

$$S = S_0 + S_1 + S_2 + \cdots.$$  \hspace{1cm} (4)

For example, $S_1$ is expected to contain a linear combination of $(4)R$, $B_{\mu\nu}B^{\mu\nu}$, $(D\phi)^2$ and so on.

1. 0th order

In the zeroth order the Hamilton-Jacobi equation becomes

$$- e^{2\phi - \frac{2}{q}} \frac{\delta S_0}{\sqrt{-q}} \left[ \frac{\delta S_0}{\delta q_{\mu\nu}} q_{\mu\nu} + \frac{1}{2} \left( \frac{\delta S_0}{\delta \phi} \right)^2 + \frac{1}{2} q_{\mu\nu} q_{\alpha\beta} \frac{\delta S_0}{\delta q_{\mu\nu}} \delta q_{\alpha\beta} + \frac{1}{2} q_{\mu\nu} \frac{\delta S_0}{\delta \phi} \delta q_{\mu\nu} \right]$$

$$- e^{-2\phi + \frac{2}{q}} R(S^2) - 12 e^{-\frac{2}{q}} \frac{\delta S_0}{\sqrt{-q} \delta D_{\mu\nu\alpha\beta}} \left( \frac{\delta S_0}{\delta D_{\mu\nu\alpha\beta}} \right)^2 = 0.$$  \hspace{1cm} (5)

It is easy to see that the solution can be written as

$$S_0 = \int d^4 x \sqrt{-q} \left[ \alpha_0 e^{-2\phi + \frac{2}{q}} + \beta_0 e^{-\phi} + \frac{\gamma_0}{24} e^{\mu\nu\alpha\beta} D_{\mu\nu\alpha\beta} \right].$$  \hspace{1cm} (6)

Substituting the above into Eq. 5, we have an equation for $\alpha_0$, $\beta_0$ and $\gamma_0$

$$\left[ \frac{1}{5} \alpha_0^2 - R(S^2) \right] e^{-2\phi + \frac{2}{q}} - \frac{1}{2} (\beta_0^2 - \gamma_0^2) e^{-\frac{2}{q}} = 0,$$

from which we find

$$\alpha_0^2 = 5R(S^2) \text{ and } \beta_0^2 = \gamma_0^2.$$  \hspace{1cm} (7)

2. 1st order

In the first order the Hamilton-Jacobi equation becomes

$$- e^{2\phi - \frac{2}{q}} \frac{\delta S_0}{\sqrt{-q}} \left[ \frac{2 \delta S_0}{\delta q_{\mu\nu}} q_{\mu\nu} + \left( \frac{\delta S_0}{\delta \phi} + \frac{1}{2} \frac{\delta S_0}{\delta \phi} q_{\mu\nu} + \frac{\delta S_0}{\delta \rho} \right) \frac{\delta S_1}{\delta \phi} \frac{\delta S_1}{\delta \rho} + \frac{1}{2} \frac{\delta S_1}{\delta \phi} \frac{\delta S_0}{\delta \rho} q_{\mu\nu} + \left( \frac{8 \delta S_0}{\delta \rho} + \frac{\delta S_0}{\delta \phi} \right) \frac{\delta S_1}{\delta \rho} \frac{\delta S_1}{\delta \phi} \right]$$

$$+ \left( \frac{\delta S_1}{\delta B_{\mu\nu}} - \chi \frac{\delta S_1}{\delta C_{\mu\nu}} - 6 C_{\alpha\beta} \frac{\delta S_0}{\delta D_{\mu\nu\alpha\beta}} \right)^2 \right) - e^{-2\phi + \frac{2}{q}} \left[ (4)R + 4D^2\phi - \frac{5}{2} D^2\rho - 4(D\phi)^2 \right]$$

$$- \frac{15}{8} (D\rho)^2 + 5D\phi D\rho - \frac{1}{12} H_{\mu\nu\alpha} H^{\mu\nu\alpha} + e^{\frac{2}{q} \phi} \left[ \frac{1}{2} (D\chi)^2 + \frac{1}{12} F_{\mu\nu\alpha} F^{\mu\nu\alpha} \right] - e^{-\frac{2}{q} \phi} \left[ \frac{1}{\sqrt{-q} \delta C_{\mu\nu}} \right] \frac{\delta S_1}{\delta C_{\mu\nu}} = 0.$$  \hspace{1cm} (9)

For simplicity we set $H_{\mu\nu\alpha} = 0$ and $F_{\mu\nu\alpha} = 0$. Thus $B_{\mu\nu}$ and $C_{\mu\nu}$ are closed, and then written by the vector potentials. We will also set $C_{\mu\nu} = 0$ at the end of calculations. Using the solution of $S_0$, Eq. 9 becomes

$$- e^{2\phi - \frac{2}{q}} \frac{\delta S_0}{\sqrt{-q}} \left[ \beta_0 e^{-\phi} \left( \frac{1}{2} q_{\mu\nu} + \frac{\delta S_1}{\delta \rho} \right) - \frac{2}{5} \alpha_0 e^{-2\phi + \frac{2}{q}} \frac{\delta S_1}{\delta \rho} + \frac{1}{\sqrt{-q}} \left( \frac{\delta S_1}{\delta B_{\mu\nu}} - \chi \frac{\delta S_1}{\delta C_{\mu\nu}} - 6 C_{\alpha\beta} \frac{\delta S_0}{\delta D_{\mu\nu\alpha\beta}} \right) \right]$$

$$- e^{-2\phi + \frac{2}{q}} \left[ (4)R + 4D^2\phi - \frac{5}{2} D^2\rho - 4(D\phi)^2 - \frac{15}{8} (D\rho)^2 + 5D\phi D\rho \right] + e^{\frac{2}{q} \phi} \left( D\chi \right)^2 - e^{-\frac{2}{q} \phi} \left( \frac{1}{\sqrt{-q} \delta C_{\mu\nu}} \right)^2 = 0.$$  \hspace{1cm} (10)

Here remember that the AdS/CFT correspondence will hold in the limit of

$$\alpha_0 \to 0,$$  \hspace{1cm} (11)
that is, the AdS and $S^5$ curvature radii are much longer than the string length. In this limit we can see that the solution for $S_1$ is given by

$$
S_1 = \frac{1}{\beta_0} \int d^4x \sqrt{-q} e^{-\phi + \frac{2}{5} \rho} \left[ \frac{1}{2} (4)R + 4(D\phi)^2 + \frac{35}{16} (D\rho)^2 - \frac{25}{4} D\phi D\rho \right] - \frac{1}{4\beta_0} \int d^4x \sqrt{-q} e^{-\phi + \frac{2}{5} \rho} (D\chi)^2 + \frac{\gamma_0}{4} \int d^4x \sqrt{-q} e^{-\phi} B_{\mu\nu} B^{\mu\nu} + \frac{\gamma_0}{2} \int d^4x \sqrt{-q} e^{\mu\nu\alpha\beta} \left[ B_{\mu\nu} C_{\alpha\beta} + \frac{\chi}{2} B_{\mu\nu} B_{\alpha\beta} \right].
$$

(12)

Hereafter we will consider the limit of $\alpha_0 = 0$ and $R(\phi^1) = 0$.

3. 2nd order

Next we consider the second order. The Hamilton-Jacobi equation is

$$
\frac{1}{\sqrt{-q}} \left[ \frac{1}{2} q_{\mu\nu} \delta S_2 - \frac{e^{-\phi}}{\beta_0 (\sqrt{-q})^2} \left( \frac{4}{\beta_0} R_{\mu\nu} (4) R_{\mu\nu} + \frac{1}{2} (4) R^2 \right) \right]_{\mu\nu} - \frac{3\beta_0 e^{-\phi}}{8} \frac{1}{4} \left( \frac{1}{4} \right)^2 
$$

where $(B^2)_{\mu\nu} = B_{\mu\alpha} B_{\nu\alpha}$ and $\text{Tr}(B^2) = B_{\mu\nu} B^{\mu\nu}$.

As seen soon, $(4) R_{\mu\nu} = O(B^2)$ and $(4) R = O(B^4)$ will be held. Bearing this in mind, $S_2$ can be evaluated as

$$
S_2 = \frac{1}{20\beta_0} \int d^4x \sqrt{-q} e^{-\phi + \frac{5}{2} \rho} (4) R_{\mu\nu} (4) R^{\mu\nu} + \frac{1}{2\beta_0} \int d^4x \sqrt{-q} e^{-\phi + \frac{2}{5} \rho} (4) R_{\mu\nu} (1) T_{\mu\nu}
$$

(14)

where we set $\beta_0 = \gamma_0$ so that the Born-Infeld action is realised for the flat D-brane with the constant field $B_{\mu\nu}$. We also defined

$$
(1) T_{\mu\nu} = -(B^2)_{\mu\nu} + \frac{1}{4} q_{\mu\nu} \text{Tr}(B^2).
$$

(15)

4. Summary

The total solution to the Hamilton-Jacobi equation is summarised by

$$
S_{\text{tot}} = -(S_0 + S_1 + S_2 + \cdots) = -(\hat{S}_{BI} + \hat{S}_{EH} + \hat{S}_{WZ} + \hat{S}_2),
$$

(16)

where

$$
\hat{S}_{BI} = \beta_0 \int d^4x \sqrt{-q} e^{-\phi} \left[ 1 + \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{8} \left( \text{Tr}(B^4) - \frac{1}{4} \text{Tr}(B^2)^2 \right) \right],
$$

(17)

$$
\hat{S}_{EH} = \frac{1}{\beta_0} \int d^4x \sqrt{-q} \left[ e^{-\phi + \frac{2}{5} \rho} \left( \frac{1}{2} (4) R + 4(D\phi)^2 + \frac{35}{16} (D\rho)^2 - \frac{25}{4} D\phi D\rho \right) - \frac{1}{4} e^{-\phi + \frac{2}{5} \rho} (D\chi)^2 \right],
$$

(18)

$$
\hat{S}_{WZ} = \beta_0 \int d^4x \sqrt{-q} e^{\mu\nu\alpha\beta} \left( \frac{1}{24} \left( 2 \right)_{\alpha\beta} + \frac{1}{4} B_{\mu\nu} C_{\alpha\beta} + \frac{\chi}{8} B_{\mu\nu} B_{\alpha\beta} \right),
$$

(19)

and

$$
\hat{S}_2 = \frac{1}{20\beta_0} \int d^4x \sqrt{-q} e^{-\phi + \frac{5}{2} \rho} (4) R_{\mu\nu} (4) R^{\mu\nu} + \frac{1}{2\beta_0} \int d^4x \sqrt{-q} e^{-\phi + \frac{2}{5} \rho} (4) R_{\mu\nu} (1) T_{\mu\nu} + \cdots.
$$

(20)

Note that in the flat and constant field limit $\hat{S}_{BI} = S_{BI}$ up to the current order. It is also noted that there is non-trivial coupling $(4) R^{\mu\nu} (1) T_{\mu\nu}$ and so on.
C. Effective equation on D-brane

1. Strategy

In the braneworld the AdS/CFT correspondence may be formulated by the following partition functional argument \[^7\]:

\[
Z = \int \mathcal{D}g e^{iS_{\text{bulk}}(g) + i\frac{\lambda}{2} \mathcal{S}_{\text{D-brane}}(q) + iS_{\text{GH}}(g)}
= \int \mathcal{D}g e^{i\frac{\lambda}{2} \mathcal{S}_{\text{D-brane}}(q) + i\mathcal{S}_{\text{ct}} + i \int d^4 x q_{\mu\nu} \mathcal{T}^{\mu\nu}_{\text{CFT}}},
\]

(21)

where \(S_{\text{ct}}\) represents the counter-terms which make the action finite and is given by the on-shell solution of the Hamilton-Jacobi equation, \(S_{\text{ct}} = -(S_0 + S_1 + S_2 + \cdots)\). The variational principle implies

\[
\frac{1}{\sqrt{-q}} \frac{\delta S_{\text{D-brane}}}{\delta q_{\mu\nu}} + 2 \frac{1}{\sqrt{-q}} \left( \frac{\delta S_{\text{ct}}}{\delta q_{\mu\nu}} + \frac{\delta \mathcal{S}_{\text{CFT}}}{\delta q_{\mu\nu}} \right) = 0.
\]

(22)

To fix the first term we must specify the D-brane (cut-off brane) action. In this paper we consider two types of branes.

2. Born-Infeld membrane

First let us examine the case where the brane action is the Born-Infeld type:

\[
S_{\text{D-brane}} = \beta \int d^4 x e^{-\phi} \sqrt{-\det(q_{\mu\nu} + B_{\mu\nu})}.
\]

(23)

The energy-momentum tensor of this brane becomes

\[
\mathcal{T}^{\mu\nu}_{\text{BI}} = e^{-\phi} q_{\mu\nu} - e^{-\phi} T^{(1)}_{\mu\nu} + (^{(2)} \mathcal{T}^{\mu\nu}_{\text{BI}}),
\]

(24)

and

\[
(^{(2)} \mathcal{T}^{\mu\nu}_{\text{BI}}) = e^{-\phi} \left\{ -\frac{1}{4} \text{Tr}(B^2) \left[ (B^2)_{\mu\nu} - \frac{1}{8} q_{\mu\nu} \text{Tr}(B^4) \right] + (B^4)_{\mu\nu} - \frac{1}{8} q_{\mu\nu} \text{Tr}(B^4) \right\}.
\]

(25)

Substituting Eq. (23) into Eq. (22), we can obtain the effective gravitational equation on the brane. At that time we set

\[
\beta = 2\beta_0,
\]

(26)

so that the brane geometry could be four dimensional Minkowski spacetime.

In the first order the effective equation becomes just vacuum one:

\[
(4) G_{\mu\nu} = \beta_0 e^{3\phi - \frac{5}{2} \phi} \left( -\frac{1}{2} T^{(2)}_{\text{BI} \mu\nu} + 2 \frac{1}{\sqrt{-q}} \frac{\delta S_2}{\delta q_{\alpha\beta}} q_{\alpha\rho} q_{\beta\nu} \right) + T^{\text{CFT}}_{\mu\nu}
= -3(D_\mu D_\nu - q_{\mu\nu} D^2) \phi + \frac{5}{2} (D_\mu D_\nu - q_{\mu\nu} D^2) \rho + \left[ D_\mu \phi D_\nu \phi - 5 q_{\mu\nu} (D\phi)^2 \right] + \frac{15}{8} \left[ D_\mu \rho D_\nu \rho - \frac{13}{6} q_{\mu\nu} (D\rho)^2 \right]
+ \frac{1}{2} e^{2\phi} \left[ D_\mu \chi D_\nu \chi - \frac{1}{2} q_{\mu\nu} (D\chi)^2 \right] - \frac{5}{4} \left[ D_\mu \rho D_\nu \phi + D_\mu \phi D_\nu \rho - 7 q_{\mu\nu} D\phi D\rho \right] + T^{\text{CFT}}_{\mu\nu} + \cdots.
\]

(27)

We may naively expect that Einstein-Maxwell theory governs the physics on the D-brane described by the Born-Infeld action. However, the result is not the case. Since \(S_{\text{BI}}\) is same as \(S_{\text{BI}}\) up to the order of \((B^4)_{\mu\nu}\), the first order Einstein equation does not have the source of the Maxwell field while the contribution from holographic CFT exists.

As a result, the gravitational equation up to the second order is given by

\[
(4) G_{\mu\nu} = T^{\text{CFT}}_{\mu\nu} - 3(D_\mu D_\nu - q_{\mu\nu} D^2) \phi + \frac{5}{2} (D_\mu D_\nu - q_{\mu\nu} D^2) \rho + \left[ D_\mu \phi D_\nu \phi - 5 q_{\mu\nu} (D\phi)^2 \right]
\]
previous sections. and then obtain the effective theory in the low-energy limit. We will use the slightly different notations from the geometrical method\cite{12, 15} in this section. To do so we will solve the bulk spacetime in the long wave approximation to understand why we obtain such consequences, we will re-derive the gravitational equation on the D-brane using the Born-Infeld action originated from the fact that the Born-Infeld action is a solution to the Hamilton-Jacobi equation. In order to this result.

In the above we have dropped the second order terms which couple to the scalar fields to keep the form compact.

3. Nambu-Goto membrane

For the comparison, it might be worth considering the brane described by the Nambu-Goto action

\[ S_{NG} = 2\beta_0 \int d^4x \sqrt{-q} e^{-\phi}. \]  

(29)

At the first order, the effective equation becomes

\[ G_{\mu\nu} = \beta_0 e^{2\phi - \frac{2}{\beta_0}} T^{(1)}_{\mu\nu} - 3(D_\mu D_\nu - q_{\mu\nu} D^2) \phi + \frac{5}{2} (D_\mu D_\nu - q_{\mu\nu} D^2) \rho + \frac{15}{8} (D_\mu D_\nu - q_{\mu\nu} D^2) \phi^2 + \frac{15}{6} (D_\mu D_\nu - q_{\mu\nu} D^2) \rho^2 - 5 q_{\mu\nu} D_\alpha \phi D_\rho^\alpha \rho + T^{\text{CFT}}_{\mu\nu} + \cdots. \]  

(30)

The cancellation does not occur and Einstein-Maxwell-scalar theory is realised on the brane. This is also an unexpected result.

III. GEOMETRICAL APPROACH

In the previous section, we saw unexpected results for the effective theory on the brane. It seems that they are originated from the fact that the Born-Infeld action is a solution to the Hamilton-Jacobi equation. In order to understand why we obtain such consequences, we will re-derive the gravitational equation on the D-brane using the geometrical method\cite{12, 13} in this section. To do so we will solve the bulk spacetime in the long wave approximation and then obtain the effective theory in the low-energy limit. We will use the slightly different notations from the previous sections.

The rest of this section is organised as follows. In the Subsec. III.A we give a formulation of the geometrical approach and stress that we must solve the bulk fields and gravity somehow. Then we solve them in long wave approximation up to leading order for the gravitational theory on the brane. Finally we derive the gravitational equation on the D-branes described by Born-Infeld action in Subsec. III.B.

A. formulation

The full metric is written as

\[ ds^2 = e^{2\phi(x)} dy^2 + q_{\mu\nu}(y, x) dx^\mu dx^\nu. \]  

(31)

The induced metric on the brane is \( h_{\mu\nu}(x) = q_{\mu\nu}(y_0, x) \), where we suppose that the brane is located at \( y = y_0 \).

In the geometrical approach, the gravitational equation on the brane is given by

\[ (G_{\mu\nu}(h) = \frac{2}{3} \left[ T_{\mu\nu} + h_{\mu\nu} \left(T_{yy} - \frac{1}{4} T \right) \right] + KK_{\mu\nu} - K_\mu^\alpha K_{\nu\alpha} - \frac{1}{2} (K^2 - K_{\alpha\beta}K)_{\alpha\beta} h_{\mu\nu} - E_{\mu\nu}. \]  

(32)
where

\[ T_{MN} = -2(\nabla_M \nabla_N - g_{MN} \nabla^2)\phi + \frac{5}{4}(\nabla_M \nabla_N - g_{MN} \nabla^2)\rho + \frac{1}{2} e^{2\phi} \left[ \nabla_M \nabla_N \chi - \frac{1}{2} g_{MN} (\nabla \chi)^2 \right] \]
\[ + \frac{5}{16} \left[ \nabla_M \nabla_N \rho - 3 g_{MN} (\nabla \rho)^2 \right] - 2 g_{MN} (\nabla \phi)^2 + \frac{5}{2} g_{MN} \nabla_K \phi \nabla^K \rho + \frac{1}{4} (H_{MKL} H_{NKL} - g_{MN} |H|^2) \]
\[ + \frac{1}{4} e^{2\phi} (\hat{F}_{MKL} \hat{F}_{NKL} - g_{MN} |\hat{F}|^2) + \frac{1}{96} e^{2\phi} \tilde{G}_{MK} K_{23} \tilde{G}_{N} K_{12} K_{34}, \]
\[ \text{(33)} \]

and then

\[ T_{\mu
u} + h_{\mu\nu} \left( T_{yy} - \frac{1}{4} T \right) = -2(D_{\mu} D_{\nu} \phi - h_{\mu\nu} D^2 \phi) + \frac{5}{4} (D_{\mu} D_{\nu} \rho - h_{\mu\nu} D^2 \rho) + \frac{1}{2} e^{2\phi} \left[ D_{\mu} \nabla_D \chi - \frac{5}{8} \eta_{\mu\nu} (\nabla \chi)^2 \right] \]
\[ + \frac{5}{16} \left[ D_{\mu} D_{\nu} \rho - \frac{5}{2} \eta_{\mu\nu} (D \rho)^2 \right] - \frac{3}{2} h_{\mu\nu} (D \phi)^2 + \frac{15}{8} h_{\mu\nu} D_{\alpha} \phi D^\alpha \rho \]
\[ - 2(K_{\mu\nu} - h_{\mu\nu} K) \nabla_{\alpha} \phi + \frac{5}{4} (K_{\mu\nu} - h_{\mu\nu} K) \nabla_{\alpha} \rho \]
\[ + \frac{3}{16} e^{2\phi} h_{\mu\nu} (\nabla \phi)^2 - \frac{15}{32} h_{\mu\nu} (\nabla \rho)^2 - \frac{3}{2} h_{\mu\nu} (\nabla \phi)^2 + \frac{15}{8} h_{\mu\nu} \nabla_{\alpha} \phi \nabla_{\beta} \rho \]
\[ + \frac{1}{2} \left( H_{\mu
u\alpha} H_{\gamma\alpha} - \frac{1}{16} h_{\mu\nu} H_{\alpha\beta} H_{\gamma\alpha\beta} \right) + \frac{1}{2} e^{2\phi} \left( \hat{F}_{\mu\nu} \hat{F}_{\gamma\alpha} - \frac{1}{16} h_{\mu\nu} \hat{F}_{\alpha\beta} \hat{F}_{\gamma\alpha\beta} \right) \]
\[ + \frac{1}{24} e^{2\phi} \left( \tilde{G}_{\mu\nu\alpha\beta} \tilde{G}_{\gamma\alpha} \alpha_1 \alpha_2 \alpha_3 - \frac{1}{16} h_{\mu\nu} \tilde{G}_{\alpha\beta} \tilde{G}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \right). \]
\[ \text{(34)} \]

\( E_{\mu\nu} \) is the projected five-dimensional Weyl tensor defined by \( E_{\mu\nu} = (^5)C_{\mu M N} n^M n^N \). It is obvious that the above equation is not closed in four dimensions. Moreover, when bulk fields exist, \( E_{\mu\nu} \) is not negligible in low energy limits\[1, 17, 18\]. Since the Born-Infeld action appears as a solution to the Hamilton-Jacobi equation, we guess that \( E_{\mu\nu} \) contains a part of the Born-Infeld energy-momentum tensor.

As the previous section, for simplicity, we turn off almost fields except scalar fields, \( B_{\mu\nu} \) and \( \tilde{G}_{\mu\nu\alpha\beta} \).

To obtain the background solution which is consistent with the junction condition, we assume that the action for the brane is given by

\[ S_{\text{brane}} = 2 \beta \int d^4 x e^{-\phi} \sqrt{-\det(h + B)} + 2 \beta \int d^4 x \sqrt{-h} \bar{e}^{\mu\nu\alpha\beta} \left[ \frac{1}{4} B_{\mu\nu} C_{\alpha\beta} + \frac{\chi}{8} B_{\mu\nu} B_{\alpha\beta} + \frac{1}{24} D_{\mu\nu\alpha\beta} \right]. \]
\[ \text{(35)} \]

The boundary conditions at the brane are brought by the junction conditions:

\[ (K_{\mu\nu} - h_{\mu\nu} K) e^\phi + \left( 2 \partial_{\mu} \phi - \frac{5}{4} \partial_{\mu} \rho \right) h_{\mu\nu} \bigg|_{(y_0, x)} = -\frac{1}{4} e^{\phi + \frac{3}{2} \phi} T_{\mu\nu}^{BI}, \]
\[ \text{(36)} \]

\[ 4K e^\phi - 8 \partial_{\mu} \phi + 5 \partial_{\mu} \rho \bigg|_{(y_0, x)} = \beta e^{\phi + \frac{3}{2} \phi} \left[ 1 - \frac{1}{4} \text{Tr}(B^2) + \frac{1}{32} (\text{Tr}(B^2))^2 - \frac{1}{8} \text{Tr}(B^4) + O(B^6) \right], \]
\[ \text{(37)} \]

\[ -K e^\phi - \partial_{\mu} \rho + 2 \partial_{\mu} \phi \bigg|_{(y_0, x)} = 0, \]
\[ \text{(38)} \]

\[ \partial_{\mu} \chi \bigg|_{(y_0, x)} = -\frac{1}{8} \beta e^{\phi + \frac{3}{2} \phi} \epsilon^{\mu\nu\alpha\beta} B_{\mu\nu} B_{\alpha\beta}, \]
\[ \text{(39)} \]

\[ H_{\mu\nu\alpha} \bigg|_{(y_0, x)} = -\beta e^{\phi + \frac{3}{2} \phi} \left[ B_{\mu\nu} - \frac{1}{4} \text{Tr}(B^2) B_{\mu\nu} + (B^3)_{\mu\nu} + O(B^5) \right], \]
\[ \text{(40)} \]

\[ \hat{F}_{\mu\nu} \bigg|_{(y_0, x)} = -\frac{3}{2} e^{\phi + \frac{3}{2} \phi} \epsilon_{\mu\nu\alpha\beta} B^{\alpha\beta}, \]
\[ \text{(41)} \]
and
\[ \hat{G}_{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3} (y_0, x) = -\beta e^{-\frac{2}{\phi} \rho} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4}, \] (42)

where
\[ T_\mu^{\mathrm{BI}} = 2\beta e^{-\phi} \left\{ h_{\mu \nu} - \frac{4}{3} \frac{\mathrm{Tr}(B^2)}{4} \left[ (B^2)_{\mu \nu} - \frac{1}{8} h_{\mu \nu} \mathrm{Tr}(B^2) \right] + \frac{1}{8} h_{\mu \nu} \mathrm{Tr}(B^3) + O(B^4) \right\}. \] (43)

Using these junction conditions, the Eq. (32) becomes
\[ G_{\mu \nu} (h) = \frac{5}{6} \beta^2 e^{2\phi - \frac{2}{\phi} \rho} T_{\mu \nu} - \frac{4}{3} \left( D_{\mu} D_{\nu} - h_{\mu \nu} D^2 \right) \phi + \frac{5}{6} \left( D_{\mu} D_{\nu} - h_{\mu \nu} D^2 \right) \rho 
+ \frac{1}{3} e^{2\phi} \left[ D_{\mu} \chi D_{\nu} \chi - h_{\mu \nu} (D\chi)^2 \right] + \frac{5}{24} \left[ D_{\mu} D_{\nu} \chi + \frac{3}{2} h_{\mu \nu} (D\phi)^2 \right] - h_{\mu \nu} (D\phi)^2 + \frac{5}{4} h_{\mu \nu} D_{\alpha} \phi D^\alpha \rho 
- E_{\mu \nu} + O(B^4). \]

The bulk “evolutional” equations are
\[ L^\alpha_{\mu} = \frac{4}{3} R - \left( T_{\mu}^{\alpha} - \frac{4}{3} T^\alpha \right) - K^2 - D^2 \phi - (D\varphi)^2, \] (45)
\[ L_{\alpha} K_{\nu}^{\mu} = \left( \frac{4}{3} \hat{R}_{\nu}^{\mu} - (D^\mu D_{\nu} \varphi + D^\nu \varphi D_{\nu} \varphi) \right)_{\text{traceless}} - \left( T_{\nu}^{\mu} - \frac{1}{4} \delta_{\nu}^{\mu} T^{\alpha}_{\alpha} \right) - K \hat{K}_{\nu}^{\mu}, \] (46)
\[ 4D_{\mu}^2 \phi - \frac{5}{2} \frac{\partial_{\mu} \rho}{\partial_{\mu} \rho} - 8 (\partial_{\mu} \phi)^2 - \frac{25}{8} \left( \partial_{\mu} \phi \right)^2 + 10 \partial_{\mu} \phi \partial_{\mu} \phi - \frac{3}{2} e^{2\phi} (\partial_{\mu} \chi)^2 + e^\chi K \left( 4 \partial_{\mu} \phi \right) - \frac{5}{2} \rho \right) 
+ \left[ \frac{5}{2} e^{2\phi} \hat{G}^2 - D_{\mu} \varphi D^\mu \left( 4 \phi - \frac{5}{3} \rho \right) + 4D^2 \phi - \frac{5}{2} D^2 \rho - 8 (D\phi)^2 - \frac{25}{8} (D\rho)^2 \right] 
+ 10 D_{\alpha} \phi D^\alpha \rho - \frac{3}{2} e^{2\phi} (D\chi)^2 + 2 |H|^2 + \frac{1}{2} e^{2\rho} |\tilde{F}|^2 \right] e^{2\varphi} = 0, \] (47)
\[ \partial_{\mu} \left( e^{2\phi} \partial_{\mu} \chi \right) + D_{\alpha} \left( e^{2\phi} D_{\alpha} \chi \right) + e^{2\phi} K \partial_{\alpha} \chi - e^{2\phi} e^{2\rho} D_{\mu} \chi D^\mu \varphi = 0, \] (49)
\[ \partial_{\mu} \left( e^{2\phi} H^{\mu \nu} + e^{2\rho} \chi \tilde{F}^{\nu \mu} \right) + e^{2\phi} K \left( e^{-2\phi} \chi \frac{H^{\mu \nu}}{\phi} + e^{2\rho} \chi \tilde{F}^{\nu \mu} \right) + \frac{1}{2} e^{2\rho} F_{\alpha \beta} \tilde{G}^{\mu \nu \alpha \beta} = 0, \] (50)
\[ \partial_{\mu} \left( e^{2\phi} \tilde{F}^{\nu \mu} \right) + e^{2\phi} K \tilde{F}^{\nu \mu} + \frac{1}{2} e^{2\rho} H_{\alpha \beta} \tilde{G}^{\mu \nu \alpha \beta} = 0, \] (51)
and
\[ \partial_{\mu} \left( e^\phi \tilde{G}_{\mu_1 \nu_1 \mu_2 \nu_2} \right) = K e^\phi \tilde{G}_{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3} \] (52)

The Hamiltonian and momentum constraints are
\[ -\frac{1}{2} \left\{ (4) R - \frac{3}{4} K^2 + \hat{K}^\alpha_{\beta} \hat{K}^\beta_{\alpha} \right\} = T_{\mu \nu} e^{-2\phi}, \] (53)
and
\[ D_\nu K^\nu - D_\mu K = T_{\mu \nu} e^{-\varphi}, \] (54)
respectively. The constraints for \( H_{\mu \nu}, \tilde{F}_{\mu \nu} \) and \( \tilde{G}_{\mu_1 \mu_2 \mu_3 \mu_4} \) are
\[ D^\alpha (e^{-\varphi - \frac{2}{\phi} \tilde{\rho}} H_{\mu \nu} + e^{-\varphi - \frac{2}{\phi} \tilde{\rho}} \tilde{F}_{\mu \nu}) = 0, \] (55)
\[ D^\alpha (e^{\varphi + \frac{2}{\phi} \tilde{\rho}} \tilde{F}_{\mu \nu}) = 0, \] (56)
and
\[ D^\alpha (e^{-\varphi + \frac{2}{\phi} \tilde{\rho}} \tilde{G}_{\mu_1 \mu_2 \mu_3 \mu_4}) = 0. \] (57)

B. Solving of the bulk and effective theory

Let us solve the bulk equations (40)-(52) with the junction conditions (36)-(42) in the long wave approximation. The infinitesimal parameter of the expansion is \( \epsilon = (\ell/L)^2 \), where \( \ell \) is the bulk curvature scale and \( L \) is the typical scale on the brane.

1. 0th-order

In the 0th order the evolutional equations are
\[ e^{-\varphi} \partial_y \tilde{K}^{(0)}_{\nu} = - \tilde{K} \tilde{K}^{(0)}_{\nu}, \] (58)
\[ e^{-\varphi} \partial_y \tilde{K} = - \left( T^{(0)}_{\mu} - \frac{4}{3} \tilde{G}^{(0)} - K^2, \right) \] (59)
\[ 4\phi_0 - \frac{5}{2} \rho_0 + e^{\varphi} \tilde{K}^{(0)} \left( 4\partial_y \phi_0 - \frac{5}{2} \partial_y \rho_0 \right) - 8(\partial_y \phi_0)^2 - \frac{25}{8}(\partial_y \rho_0)^2 + 10\partial_y \phi_0 \partial_y \rho_0 + \frac{5}{2} e^{2\varphi + 2\phi_0} |\tilde{G}|^2 = 0, \] (60)
\[ - \partial_y^2 \phi_0 + \partial_y^2 \rho_0 + e^{\varphi} \tilde{K} \left( -\partial_y \phi_0 + \partial_y \rho_0 \right) + 2(\partial_y \phi_0)^2 + \frac{5}{4}(\partial_y \rho_0)^2 - \frac{13}{4} \partial_y \phi_0 \partial_y \rho_0 - e^{2\varphi + 2\phi_0} |\tilde{G}|^2 = 0, \] (61)
and
\[ \partial_y \left( e^{\frac{2}{\phi} \phi_0} \tilde{G}^{(0)}_{\mu_1 \mu_2 \mu_3 \mu_4} \right) = \tilde{K} e^{\frac{2}{\phi} \phi_0} \tilde{G}^{(0)}_{\mu_1 \mu_2 \mu_3 \mu_4}. \] (62)

The constraint equations are
\[ D^{(0)}_{\mu_1} \left( e^{-\varphi + \frac{2}{\phi} \phi_0} \tilde{G}^{(0)}_{\mu_1 \mu_2 \mu_3 \mu_4} \right) = 0. \] (63)

The junction conditions are
\[ \tilde{K}^{(0)}_{\nu} (y_0, x) = 0, \quad \tilde{K}^{(0)} (y_0, x) = - \beta e^{\phi_0 + \frac{2}{\phi} \rho_0}, \quad \partial_y \phi_0 (y_0, x) = 0, \quad \partial_y \rho_0 (y_0, x) = \beta e^{\varphi - \phi_0} - \frac{5}{4} \rho_0, \] (64)
and
\[ \tilde{G}^{(0)}_{\mu_1 \mu_2 \mu_3 \mu_4} (y_0, x) = - \beta e^{\varphi - \frac{2}{\phi} \rho_0} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4}. \] (65)

The background solution is easily found as
\[ \phi_0 = \phi_0 (x), \quad \rho_0 = \frac{4}{5} \log(y/y_0) + \sigma_0 (x), \quad \chi_0 = \chi_0 (x), \quad \tilde{C}_{\mu \nu} = 0, \quad \tilde{G}^{(0)}_{\mu_1 \mu_2 \mu_3 \mu_4} = - \beta e^{\varphi - \frac{2}{\phi} \rho_0} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4}, \] (66)
where
\[ \varphi(x) = -\phi_0(x) + \frac{5}{4}\sigma_0(x) + \log \left(\frac{4}{5y_0}\beta\right). \] (67)

The extrinsic curvature is given by
\[ ^{(0)}K^\mu_\nu = -\frac{1}{5y}e^{-\varphi(x)}\delta^\mu_\nu, \] (68)
and then the metric becomes
\[ ^{(0)}g_{\mu\nu} = a^2(y)h_{\mu\nu}(x), \] (69)
where
\[ a(y) = \left(\frac{y}{y_0}\right)^{-\frac{1}{2}}. \] (70)

The behavior of the background metric brings us a serious problem, that is, the four-dimensional gravity cannot be recovered on the brane. The minimum way to see this is the dimensional reduction from five to four dimensions
\[ \int d^5x\sqrt{-g}(5)R \sim \int dy a^2(y) \int d^4x\sqrt{-h}(4)R(h). \] (71)

In the above \( \int dy a^2(y) = \infty \) when we consider the infinite extra dimensions, which implies that the four dimensional gravity cannot be recovered. This problem might be regarded as a sort of no-go theorem proposed by Maldacena\[20\]. The simple resolution to this problem is the compactification and/or introduction of the another brane. There may be another possibility that the bulk or brane action is modified via some quantum effects. We leave this issue for future study.

Note that adding a Wess-Zumino term \( \int D \) in the brane action is essential to obtain solutions. There is no solution with similar form when brane is supposed to be described only by Nambu-Goto action. This fact is consistent with that the Nambu-Goto action alone cannot satisfy the Hamilton-Jacobi equation.

2. 1st order

Next we turn to the 1st order equations. The junction conditions become
\[ ^{(1)}K(y_0, x) = -\frac{1}{4}\beta e^{\phi_0 - \frac{5}{4}\rho_0}\text{Tr}(B^2), \] (72)
\[ ^{(1)}K^\mu_\nu(y_0, x) = \frac{1}{2}\beta e^{\phi_0 - \frac{5}{4}\rho_0} T^\mu_\nu, \] (73)
\[ \partial_y \phi_1(y_0, x) = -\frac{1}{4}\beta e^{\varphi + \phi_0 - \frac{5}{4}\rho_0}\text{Tr}(B^2), \] (74)
and
\[ \partial_y \rho_1(y_0, x) = -\frac{1}{4}\beta e^{\varphi + \phi_0 - \frac{5}{4}\rho_0}\text{Tr}(B^2). \] (75)

For the gravitational equation on the brane, the key equations are the evolutional equation for the traceless part of the extrinsic curvature and the Hamiltonian constraint:
\[ \partial_y ^{(1)}K^\nu_\nu = (4)^{(1)}\bar{K}^\mu_\nu - (D^\mu D_\nu \varphi + D^\nu \varphi D_\mu \varphi)_{\text{traceless}} - \left( T^\mu_\nu - \frac{1}{4}\delta^\mu_\nu T^\alpha_\alpha \right) - (K)^{(1)}\bar{K}^\mu_\nu, \] (76)
and
\[ -\frac{1}{2}(4)^{(1)}R + \frac{3}{4}(0)^{(1)}(1)^{(1)}(1)^{(1)} - \frac{1}{4}K \bar{K} = T^\mu_\nu e^{-2\varphi}. \] (77)
Since the right-hand side in Eq. (76) contains $\tilde{F}_{\mu\nu}$ and $H_{\mu\nu}$, we also need to solve their bulk equations

$$\partial_{\nu}X^{\mu\nu} + e^{\phi} K^{\nu} X^{\mu\nu} + \frac{1}{2} e^{\tilde{\phi}_{\rho}} F_{\rho\alpha\beta} \tilde{G}^{\alpha\beta\mu\nu} = 0,$$

(78)

and

$$\partial_{\nu}(e^{\tilde{\phi}_{\rho}} \tilde{F}^{\mu\nu}) + e^{\varphi + \tilde{\phi}_{\rho}} K^{\nu} \tilde{F}^{\mu\nu} - \frac{1}{2} e^{\tilde{\phi}_{\rho}} H_{\rho\alpha\beta} \tilde{G}^{\alpha\beta\mu\nu} = 0,$$

(79)

where $X^{\mu\nu} = e^{-2\phi_{0}} + e^{\tilde{\phi}_{\rho}} H^{\mu\nu} + e^{\tilde{\phi}_{\rho}} \chi \tilde{F}^{\mu\nu}$ and $(T_{\mu}^{\nu} - \frac{1}{4} \delta_{\nu}^{\mu} T^{\alpha})^{(1)}$ is the first order part of $(T_{\mu}^{\nu} - \frac{1}{4} \delta_{\nu}^{\mu} T^{\alpha})$. The solutions are easily found

$$H_{\mu\nu}(y, x) = -a(y)^{\beta} e^{\phi_{0} - \frac{1}{2} \phi_{0}} B_{\mu\nu}(x),$$

(80)

and

$$\tilde{F}_{\mu\nu}(y, x) = -a(y)^{\beta}_{\frac{1}{2}} e^{\phi_{0} - \frac{1}{2} \phi_{0}} \epsilon_{\mu\nu\rho\sigma}(h) B_{\rho\sigma}(h),$$

(81)

Let us derive the gravitational equation on the brane. From the Hamiltonian constraint on the brane, we first obtain

$$(4)^{th} \hat{R}(h) = -2 \hat{D}^{2} \phi_{0} + \frac{6}{5} \hat{D} \phi_{0})^{2} + \frac{1}{2} e^{2\phi_{0}} (\hat{D} \chi_{0})^{2},$$

(82)

where $\hat{D}_{\mu}$ is the covariant derivative with respect to $h_{\mu\nu}$.

Substituting the above solutions into Eq. (76) and integrating over $y$, we obtain

$$\frac{1}{y_{0}} K^{\mu}_{\nu}(y, x) = \left[ \frac{5}{8} a^{-7} e^{\varphi} \left[ (4)^{th} \hat{R}_{\mu}^{\nu}(h) + 2 \hat{D}^{\mu} \hat{D}_{\nu} \phi_{0} - \frac{5}{4} \hat{D}^{\mu} \hat{D}_{\mu} \phi_{0} - \frac{5}{16} \hat{D}^{\mu} \rho_{0} \hat{D}_{\nu} \rho_{0} - \frac{5}{4} \hat{D}^{\mu} \rho_{0} \hat{D}_{\nu} \rho_{0} \right] \right. \left. - \frac{5}{4} \hat{D}^{\mu} \phi_{0} \hat{D}_{\nu} \rho_{0} - \frac{1}{2} e^{2\phi_{0}} \hat{D}_{\mu} \chi_{0} \hat{D}_{\nu} \chi_{0} - \hat{D}^{\mu} \hat{D}_{\nu} \phi - \hat{D}^{\mu} \phi \hat{D}_{\nu} \phi - a^{14} \beta^{2} e^{2\phi_{0} - \frac{1}{2} \phi_{0}} \hat{D}_{\sigma} \hat{D}_{\nu} \phi + \alpha \chi^{\mu}(x) \right]$$

(83)

where $\chi_{\mu}(x)$ is the constant of integration. Together with the junction condition for $K^{\mu}_{\nu}$ and the Hamiltonian constraint, we finally obtain the effective equation on the brane:

$$(4)^{th} G_{\mu\nu}(h) = - \left( \hat{D}_{\mu} \hat{D}_{\nu} + \frac{3}{4} h_{\mu\nu} \hat{D}^{2} \right) \left( 2 \phi_{0} - \frac{5}{4} \rho_{0} \right) + \frac{16}{5} \left( \hat{D}_{\mu} \rho_{0} \hat{D}_{\nu} \rho_{0} - \frac{7}{4} h_{\mu\nu} \hat{D} \rho_{0}^{2} \right)$$

$$+ \frac{1}{2} e^{2\phi_{0}} \left[ \hat{D}_{\mu} \chi_{0} \hat{D}_{\nu} \chi_{0} - \frac{1}{4} h_{\mu\nu} \hat{D} \chi_{0}^{2} \right] - (\hat{D} \phi_{0})^{2} h_{\mu\nu} + \frac{5}{4} \hat{D}_{\sigma} \phi_{0} \hat{D}^{\sigma} \rho_{0} h_{\mu\nu}$$

$$+ \hat{D}_{\mu} \hat{D}_{\nu} \phi - \frac{1}{4} h_{\mu\nu} (\hat{D} \phi)^{2} + \hat{D}_{\mu} \phi \hat{D}_{\nu} \phi - \frac{1}{4} h_{\mu\nu} (\hat{D} \phi)^{2} + \chi_{\mu\nu}(x).$$

(84)

This is main result in this section. Although we can write $\phi$ in terms of $\phi_{0}$ and $\rho_{0}$, we leave it from a pedagogical point of view. As in the previous section, it turns out again that the gravitational equation on the brane is not like Einstein-Maxwell theory. $(4)^{th}$ is exactly canceled out!

Comparing Eq. (84) with (44), we find that the relation between $E_{\mu\nu}$ and $\chi_{\mu\nu}$ is

$$- E_{\mu\nu} = \tilde{\chi}_{\mu\nu} - \frac{5}{6} \beta^{2} e^{2\phi_{0} - \frac{1}{2} \phi_{0}} T_{\mu\nu}^{(1)} - \frac{5}{3} \left[ \hat{D}_{\mu} \hat{D}_{\nu} \phi_{0} - \frac{1}{4} h_{\mu\nu} \hat{D} \phi_{0}^{2} \right] + \frac{5}{3} \left[ \hat{D}_{\mu} \hat{D}_{\nu} \rho_{0} - \frac{1}{4} h_{\mu\nu} \hat{D} \rho_{0}^{2} \right]$$

$$+ \frac{1}{6} e^{2\phi_{0}} \left[ \hat{D}_{\mu} \chi_{0} \hat{D}_{\nu} \chi_{0} - \frac{1}{4} h_{\mu\nu} \hat{D} \chi_{0}^{2} \right] + \frac{5}{3} \left[ \hat{D}_{\mu} \rho_{0} \hat{D}_{\nu} \rho_{0} - \frac{1}{4} h_{\mu\nu} \hat{D} \rho_{0}^{2} \right] + \hat{D}_{\mu} \phi_{0} \hat{D}_{\nu} \phi_{0} - \frac{1}{2} h_{\mu\nu} \hat{D}_{\mu} \phi_{0} \hat{D}_{\nu} \phi_{0}$$

(85)

We should again notice that the form of the brane action Eq. (84) is essential to have consistent solutions. Solutions for $H_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ in the bulk are automatically consistent with boundary conditions derived from Eq. (84). We cannot find a consistent solution except for a trivial solution $B_{\mu\nu} = 0$, if one choose the brane action without Born-Infeld term.
IV. DISCUSSION

In this paper we derived the effective theory on the D-brane described by Born-Infeld action in type IIB supergravity. To bring out the essence we focused on the gravity, the U(1) gauge field and the scalars. We considered two different derivations by holographic and geometrical approaches. In the both, it turns out that the effective theory is four-dimensional Einstein+scalar+holographic CFT (or integration of constant) and that the Maxwell fields do not appear at the leading order. This is a bad news for the D-braneworld scenario. Now we have a caution: careful considerations will be demanded in a realistic model.

Usually “the integration of constant” $\tilde{\chi}_{\mu\nu}$ in Eq. (84) is expected to correspond to the holographic CFT energy-momentum tensor $T_{\mu\nu}^{\text{CFT}}$. Comparing Eq. (27) with Eq. (84), we can easily confirm that this is the case, that is, $\tilde{\chi}_{\mu\nu}$ is related to $T_{\mu\nu}^{\text{CFT}}$ at the leading order. More precisely,

$$T_{\mu\nu}^{\text{CFT}} = \tilde{\chi}_{\mu\nu} + \frac{5}{4} h_{\mu\nu} \left[ \nabla^2 (-\phi_0 + \rho_0) + 3(\nabla\phi_0)^2 - \frac{11}{2} \nabla\phi_0 \nabla\rho_0 + \frac{5}{2} (\nabla\rho_0)^2 \right]$$

$$= \tilde{\chi}_{\mu\nu} - \frac{1}{4} h_{\mu\nu} J_{\text{CFT}}^{(0)},$$

where

$$J_{\text{CFT}}^{(0)} = \frac{1}{\sqrt{-h}} \frac{\delta \Gamma_{\text{CFT}}}{\delta \rho}.$$  

In the above we have used the equations for the scalar fields which can be obtained through the variational principle of the action $S_{\text{D-brane}} + S_{\text{ct}} + \Gamma_{\text{CFT}}$ in the previous section

$$\nabla^2 (\phi_0 - \rho_0) - 3(\nabla\phi_0)^2 - \frac{5}{2} (\nabla\rho_0)^2 + \frac{11}{2} \nabla_{\mu}\phi_0 \nabla^{\mu} \rho_0 = -\frac{1}{5} J_{\text{CFT}}^{(0)}.$$  

Thus we can confirm the desirable result here.

As stressed in the Sec. III B 1, the current background solution is not like AdS spacetime and the gravity cannot be confined on the brane at low energy without compactification. If we compactify the extra dimension, we must introduce another brane. In this case, the integration of constant $\tilde{\chi}_{\mu\nu}$ is not the holographic CFT energy-momentum tensor but just the energy-momentum tensor on the brane. If the brane is vacuum, $\tilde{\chi}_{\mu\nu} = 0$. Then the effective theory is not like Einstein-Maxwell. But, if the Maxwell field lives on the another brane, we can see that the field also appears on the D-brane.

In this paper we saw the drastic changes from the probe D-brane case when we take into account the self-gravity of the brane. Compared to the probe brane, the new ingredients are junction conditions. The consistent solutions are extremely limited. In type IIB supergravity, indeed, we obtain the consistent bulk solution for the Born-Infeld action, but we do not for the Nambu-Goto one. This fact implies that one should be careful in connecting an effective action derived from AdS/CFT like correspondence to an effective action on a self-gravitating brane.

There are several remaining studies. The first is the higher order corrections and its meaning. In the holographic approach, we obtained the coupling between the curvature and the stress tensor of the gauge fields. The systematic analysis will be interesting. The second is about the localization of fermion fields on the D-brane. We hope that these issues will be addressed in near future.

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[1] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999); ibid, 4690 (1999).
[2] K. Maeda and M. Sasaki(eds.), Brane world: New perspective in cosmology, Prog. Theor. Phys. Supplement, 148(2002).
[3] T. Shiromizu, T. Torii and T. Uesugi, hep-th/0302223; M. Sami, N. Dadhich, and T. Shiromizu, hep-th/0304187.
[4] For the review, A. A. Tseytlin, hep-th/9908105.
[5] M. Sato and A. Tsuchiya, Prog. Theor. Phys. 109, 687 (2003).
[6] S. S. Gubser, Phys. Rev. D63, 084017 (2001); L. Anchordoqui, C. Nunez and L. Olsen, JHEP 10, 050 (2000).
[7] S. B. Giddings, E. Katz, and L. Randall, JHEP 0003, 023 (2000).
[8] T. Shiromizu and D. Ida, Phys. Rev. D64, 044015 (2001); S. de Haro, K. Skenderis, and S. N. Solodukhin, hep-th/0011230.
[9] T. Shiromizu, T. Torii, and D. Ida, JHEP 0203, 007 (2002).
[10] A. Hashimoto, Phys. Rev. D60, 127902 (1999); U. H. Danielsson and M. Kruczenski, JHEP 05, 028 (2000); L. Rastelli and M. V. Raamsdonk, JHEP 12, 005 (2000).
[11] S. Nojiri, S.D. Odintsov, S. Zerbini, Phys. Rev. D62, 064006 (2000); S. Nojiri, S.D. Odintsov, Phys. Lett. B484, 119 (2000); S. W. Hawking, T. Hertog and H. S. Reall, Phys. Rev. D62, 043501 (2000); L. Anchordoqui, C. Nunez and K. Olsen, JHEP 10, 050 (2000); K. Koyama and J. Soda, JHEP 05, 027 (2001).
[12] T. Shiromizu, K. Maeda and M. Sasaki, Phys. Rev. D62, 024012 (2000); M. Sasaki, T. Shiromizu and K. Maeda, Phys. Rev. D62, 024008 (2000).
[13] R. Maartens, Phys. Rev. D62, 084023 (2000).
[14] K. Maeda and D. Wands, Phys. Rev. D62, 124009 (2000); A. Mennim and R. A. Battye, Class. Quant. Grav. 18, 2171 (2001); C. Barcelo and M. Visser, JHEP 10, 019 (2002).
[15] S. Kanno and J. Soda, Phys. Rev. D66, 043526 (2002); ibid, 083506, (2002); S. Kobayashi and K. Koyama, JHEP 12, 056 (2002); T. Shiromizu and K. Koyama, Phys. Rev. D67, 084022 (2003);
[16] S. Kanno and J. Soda, hep-th/0303203.
[17] Y. Himemoto and T. Tanaka, Phys. Rev. D67, 084014 (2003).
[18] D. Langlois and M. Sasaki, hep-th/0302009.
[19] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998); S. S. Gubser, I. R. Klevanov and A. M. Polyakov, Phys. Lett. B428, 105 (1998); E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998); O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rep. 323, 183 (2000).
[20] J. M. Maldacena, Int. J. Mod. Phys. A16, 822 (2001).
[21] There could be another solution which cannot be written in an analytic form. But, we cannot derive the effective equation in an analytic form at leading order.