Nielsen equalizer theory

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Capitol Normal University, Beijing China, June 24, 2011
Given a set of maps: $f_1, \ldots, f_k : X \to Y$, the equalizer set is

$$\text{Eq}(f_1, \ldots, f_k) = \{ x \in X \mid f_1(x) = \cdots = f_k(x) \}$$

The points where all the functions agree.
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For 2 maps, this is the coincidence set.
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**Proposition**

*When X and Y have the same dimension, given \( f_1, \ldots, f_k : X \to Y \) with \( k > 2 \), we can change the maps by homotopy to be equalizer free.*
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**Proposition**

*When $X$ and $Y$ have the same dimension, given $f_1, \ldots, f_k: X \to Y$ with $k > 2$, we can change the maps by homotopy to be equalizer free.*

Make $\text{Coin}(f_1, f_i)$ finite for each $i$, then arrange for these sets to be distinct.
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Then the coincidence sets $\text{Coin}(f_1, f_i)$ will be submanifolds of $X$, and it’s possible that their intersections would be essentially nonempty.
An example: three maps \( f, g, h : T^2 \to S^2 \) given by \((1 \times 2)\) matrices:

\[
\begin{align*}
\begin{bmatrix}
3 & 1 \\
0 & 2
\end{bmatrix}, \\
\begin{bmatrix}
-1 & -1
\end{bmatrix}
\end{align*}
\]

We can compute the coincidence sets:

\[
\text{Coin}(f, g) \text{ is points } (x, y) \text{ with } 3x + y \equiv 0 \pmod{Z^2}, \text{ which is the "line" } y \equiv 3x \pmod{Z^2}.
\]
An example: three maps \( f, g, h : T^2 \to S^2 \) given by \((1 \times 2)\) matrices:

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$\text{Coin}(f, g)$ is points $(x, y)$ with $3x + y = 0x + 2y \mod \mathbb{Z}^2$, which is the “line” $y = 3x \mod \mathbb{Z}^2$. 
\[ f = (3 1), \quad g = (0 2) \quad h = (-1 - 1) \]

We can draw the coincidence sets:

\[ \text{Coin}(f, g) \]
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\[
\begin{align*}
  f &= (31), &
  g &= (02), &
  h &= (-1\ -1)
\end{align*}
\]

We have 10 isolated equalizer points.
In fact we’ll show that in this example any maps homotopic to \( f, g, h \) must have at least 10 equalizers.
In fact we’ll show that in this example any maps homotopic to $f, g, h$ must have at least 10 equalizers.

We’ll define a Nielsen number (easy matrix formula for tori), and in this case

$$N(f, g, h) = 10.$$
On Tuesday, Peter Wong suggested I have a look at:

Dobreńko, Kucharski, On the generalization of the Nielsen number, Fundamenta Mathematicae 134:1–14, 1990.

They give a very general theory for maps \( f : X \to Y \) and a subset \( B \subset Y \), and a Nielsen theory for counting \( f^{-1}(B) \). In various special cases, in appropriate codimensional settings, this gives:

- Nielsen fixed point theory (\( B = \Delta \))
- Root theory (\( B = \text{pt} \))
- Coincidence theory of \( k \) maps (\( B = \Delta \subset Y \))
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- root theory ($B = pt$)
- coincidence theory of $k$ maps ($B = \Delta \subset Y^k$)
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My lesson learned:
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\[ \text{talk to Peter more often} \]
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With apologies to Dobreńko and Kucharski, let’s continue.
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Let \( f_1, \ldots, f_k : X \to Y \) with \( \dim X = (k - 1)n \) and \( \dim Y = n \).

Let \( F, G : X \to Y^{k-1} \) be maps (codimension 0) given by

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F(x) = (f_1(x), \ldots, f_1(x)), \quad G(x) = (f_2(x), \ldots, f_k(x)).
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$$F(x) = (f_1(x), \ldots, f_1(x)), \quad G(x) = (f_2(x), \ldots, f_k(x)).$$

Then $F, G$ are maps of manifolds of the same dimension, and

$$\text{Eq}(f_1, \ldots, f_k) = \text{Coin}(F, G).$$
\[ F(x) = (f_1(x), \ldots, f_1(x)), \quad G(x) = (f_2(x), \ldots, f_k(x)) \]

\[ \text{Eq}(f_1, \ldots, f_k) = \text{Coin}(F, G) \]

This connection is deep enough to build our whole theory, in the case where \( \dim X = (k - 1)n \) and \( \dim Y = n \).
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**Theorem**

*With these dimensions, the maps can be changed by homotopy so that \( \text{Eq}(f_1, \ldots, f_k) \) is finite.*
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**Theorem**

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\[ F = (f_1, \ldots, f_1) \quad G' = (f'_2, \ldots, f'_k) \]
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\[ F = (f_1, \ldots, f_1) \quad G' = (f_2', \ldots, f_k') \]

with \( \text{Coin}(F, G') = \text{Eq}(f_1, f_2', \ldots, f_k') \) finite.
We get an equalizer index (or semi-index, or $\mathbb{Z} \oplus \mathbb{Z}_2$ index):
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Define $\text{ind}(f_1, \ldots, f_k, U) = \text{ind}(F, G, U)$.

Easy to show that this is homotopy invariant, and has appropriate other properties.
For differentiable maps we can compute the index at a point using derivative maps:

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\text{ind}(f_1, \ldots, f_k, x) = \text{ind}(F, G, x) \\
= \text{sign det}(dF - dG) \\
= \text{sign det}\begin{pmatrix}
df_1 - df_2 \\
\vdots \\
df_1 - df_k
\end{pmatrix}
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Again we can define the equalizer classes to be the coincidence classes of $F$ and $G$. 
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Equivalently, $x, x' \in \text{Eq}(f_1, \ldots, f_k)$ are in the same class when

$$x, x' \in p \text{Eq}(\tilde{f}_1, \alpha_2 \tilde{f}_2, \ldots, \alpha_k \tilde{f}_k)$$

for $\alpha_i \in \pi_1(Y)$. 
This is equivalent to making Reidemeister classes $\pi_1(Y)^{k-1}/\sim$. 
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We say \((\alpha_2, \ldots, \alpha_k) \sim (\beta_2, \ldots, \beta_k)\) if and only if there is \(z \in \pi_1(X)\) with

\[
\beta_i = \varphi_1(z)\alpha_i\varphi_i(z)^{-1} \quad \text{for all } i
\]
Also equivalent in terms of paths:
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\( x, x' \in \text{Eq}(f_1, \ldots, f_k) \) are in the same class when there is a path \( \gamma \) from \( x \) to \( x' \) with

\[
f_i(\gamma) \simeq f_1(\gamma) \quad \text{for all } i
\]
A class is essential when its index (or semi-index) is nonzero, and the number of such classes is \( N(f_1, \ldots, f_k) \).
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We also get $R(f_1, \ldots f_k)$ and $L(f_1, \ldots, f_k)$ in the usual way.
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Also we have a “minimal equalizer number” with

$$ME(f_1, \ldots, f_k) \leq N(f_1, \ldots, f_k),$$

and these are equal when $(k - 1)n \neq 2$. 
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For more than 2 maps, this always holds except 3 maps on dimensions $2 \to 1$. 
We can get all the usual results.
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**Theorem**

*If $Y$ is a Jiang space, then all nonempty equalizer classes have the same index.*
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**Theorem**

*If $f_1, \ldots, f_k : T^{(k-1)n} \rightarrow T^n$ by matrices $A_1, \ldots, A_k$, then*

$$N(f_1, \ldots, f_k) = \text{abs det} \begin{bmatrix} A_1 - A_2 \\ \vdots \\ A_1 - A_k \end{bmatrix}$$
Our old example: \( f, g, h: T^2 \rightarrow S^1 \) by

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Then we have

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Then we have

\[
N(f, g, h) = \text{abs det} \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} = 10.
\]
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The set $\text{Eq}(f_1, \ldots, f_k)$ includes a lot of information about the coincidence sets.
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For each $i, j$ we have

$$\text{Eq}(\alpha_1 \tilde{f}_1, \alpha_2 \tilde{f}_2, \ldots, \alpha_k \tilde{f}_k) \subset \text{Coin}(\alpha_i \tilde{f}_i, \alpha_j \tilde{f}_j)$$
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So every equalizer class is a subset of a coincidence class.
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**Theorem**

*Any coincidence class containing an essential equalizer class must be geometrically essential.*
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Coin\((f, g)\)

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So \(N(f, h) = 2\) in this case.
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\[
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So \(N(f, h) = 2\) in this case. Similarly \(N(f, g) = N(g, h) = 1\).
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Start with two maps $f_1, f_2$, and a coincidence class $C$ that we want to show is essential.
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Start with two maps $f_1, f_2$, and a coincidence class $C$ that we want to show is essential.

We invent a set of maps $f_3, \ldots, f_k$ and show that $C$ contains equalizer points of nonzero index.
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This only works for maps $f_1, f_2 : X \to Y$ with $\dim Y = n$ and $\dim X = (k - 1)n$ for some $k$. ($\dim X$ must be a multiple of $\dim Y$)
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gives a method for finding $N(f_1, f_2)$ by observing that they restrict to maps $T^2 \rightarrow T^2$, and this restriction respects the Nielsen number.

So Jezierski *decreases* the domain dimension to get codimension 0.
But even when \( \dim X \) isn’t a multiple of \( \dim Y \), maybe we can still make it work.

Take two maps \( f_1, f_2 : T^7 \to T^2 \) with matrices \( A_1, A_2 \), and assume that \( A_2 - A_1 \) has rank 2.

Jezierski, *The Nielsen coincidence number of maps into tori*, Quaestiones Mathematicae, 2001 gives a method for finding \( N(f_1, f_2) \) by observing that they restrict to maps \( T^2 \to T^2 \), and this restriction respects the Nielsen number.

So Jezierski *decreases* the domain dimension to get codimension 0.

This only works because \( T^2 \subset T^7 \).
Take two maps $f_1, f_2 : T^7 \to T^2$ with matrices $A_1, A_2$, and assume that $A_2 - A_1$ has rank 2.
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We do the opposite:
Take two maps $f_1, f_2 : T^7 \to T^2$ with matrices $A_1, A_2$, and assume that $A_2 - A_1$ has rank 2.

We do the opposite:

*Increase* the domain dimension:
Take two maps $f_1, f_2 : T^7 \rightarrow T^2$ with matrices $A_1, A_2$, and assume that $A_2 - A_1$ has rank 2.

We do the opposite:

*Increase* the domain dimension: let $\bar{f}_1, \bar{f}_2 : T^8 \rightarrow T^2$ by adding columns of 0s to $A_1, A_2$. 
Take two maps \( f_1, f_2 : T^7 \to T^2 \) with matrices \( A_1, A_2 \), and assume that \( A_2 - A_1 \) has rank 2.

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Not hard to show that \( N(f_1, f_2) = N(\bar{f}_1, \bar{f}_2) \).
Take two maps \( f_1, f_2 : T^7 \to T^2 \) with matrices \( A_1, A_2 \), and assume that \( A_2 - A_1 \) has rank 2.

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Not hard to show that \( N(f_1, f_2) = N(\bar{f}_1, \bar{f}_2) \).

Let \( B_1, B_2 \) be matrices of \( \bar{f}_1, \bar{f}_2 \), and \( B_2 - B_1 \) still has rank 2.
So we can invent matrices $B_3, \ldots B_5$ with

$$\begin{bmatrix}
B_2 - B_1 \\
\vdots \\
B_5 - B_1
\end{bmatrix}$$

of full rank (8).
So we can invent matrices $B_3, \ldots B_5$ with

$$
\begin{bmatrix}
B_2 - B_1 \\
\vdots \\
B_5 - B_1
\end{bmatrix}
$$

of full rank (8).

Thus $(\overline{f}_1, \overline{f}_2, g_3, \ldots, g_5)$ has essential equalizer classes and so

$N(f_1, f_2) = N(\overline{f}_1, \overline{f}_2) \neq 0$
So we can invent matrices $B_3, \ldots B_5$ with

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of full rank (8).

Thus $(\bar{f}_1, \bar{f}_2, g_3, \ldots , g_5)$ has essential equalizer classes and so $N(f_1, f_2) = N(\bar{f}_1, \bar{f}_2) \neq 0$

Hopefully this trick can be used elsewhere when we need to prove that coincidence classes are essential.
Thank you!

Paper at arxiv: “Nielsen Equalizer Theory”, and in *Topology and its applications*