ARE 1+1 AND 2+2 EXCEPTIONAL SIGNATURES?

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Abstract

We prove that 1+1 and 2+2 target ‘spacetimes’ of a 0-brane are exceptional signatures. Our proof is based on the requirement of \( SL(2, R) \) and ‘Lorentz’ symmetries of a first order lagrangian. Using a special kind of 0-brane called ‘quatl’, we also show that the exceptional signatures 1 + 1 and 2 + 2 are closely related. Moreover, we argue that the 2 + 2 target ‘spacetime’ can be understood either as 2 + 2 worldvolume ‘spacetime’ or as ‘1 + 1-matrix-brane’. The possibility that the exceptional 2 + 2−signature implies an exceptional chirotope is briefly outlined.

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1.- Introduction

In both mathematics and physics there are exceptional cases. The five regular polyhedra, five exceptional groups, four division algebras, among others, offer beautiful exceptions in mathematics [1]. On the other hand, the four fundamental forces of nature and the five superstring theories [2] provide with two examples of exceptional objects in physics. The dream in this framework is to find a general theory of exceptions. For instance, $M$-theory [3]-[5] is one of the latest dreams in this quest in connection with the five superstring theories.

One of the most important open problems in fundamental physics is to find what the exceptional signatures of the space time are. Traditionally, in the basic formulation of gauge field theories it is introduced by hand the signature of one timelike dimension and three spacelike dimensions, that is 1+3 signature. In Kaluza-Klein theory the situation is similar since one chooses as starting point $T$-time, with $T = 1$, and $S$-spacelike dimensions. In contrast supersymmetry offers an excellent mechanism to find exceptional signatures. For instance, only with $T = 1, S = 9, T = 9, S = 1$ or $T = 5, S = 5$ supersymmetry makes sense in superstring theory. Moreover, supersymmetry application in $p$-brane theory leads to exceptions in both time and space coordinates [6]. However these exceptions are too general in the sense that are modulo some specific value. In this scenario it may seem surprising to realize that one does not even need supersymmetry to find exceptional signatures. Recent studies in two time physics [7] observed that the symmetry $SL(2, R)$ in a first order classical lagrangian associated with a 0-brane is accomplished only if one considers a scenario with two times and arbitrary spacelike coordinates. In this work, we analyze this possibility more carefully founding that the $SL(2, R)$ symmetry, together with a global Lorentz symmetry, leads to the interesting result that 1 + 1 and 2 + 2 dimensions are exceptional signatures.

At ‘human scale’ our World appears to have 1 + 3 signature. But why does Nature make this signature an exceptional case? Recently some authors [8]-[9] have proposed a possible explanation for this question. Their proposals are based essentially in spin considerations [8] and group-theoretical argument a la Wigner [9]. From the point of view of $p$-brane theory, however, one has two type of signatures: the background target spacetime signature, and the worldvolume signature. Thus, the considerations in references [8] and [9] may be applied to the background target spacetime signature but not necessarily to the whole scenario of $p$-branes.

Through the years there has been some wisdom that such two types of
signatures are connected. For instance, in superstring theory one starts with 1 + 1 signature associated with the worldsheet and supersymmetry and quantum consistency of the theory requires 1 + 9, 5 + 5 or 9 + 1 background target spacetime signatures. Similarly, if one starts with 1 + 2 worldvolume signature it is expected that quantum consistency requires 1 + 10 background target spacetime signature. Hull [10] has shown that if one considers closed time-like dimensions in the worldsheet of string theory or worldvolume p-branes, then duality implies other signatures for the target spacetime. In the case of type IIA string theory, Hull suggested that besides the 1 + 9 background target spacetime the signatures 0 + 10, 2 + 8, 5 + 5, 4 + 6 are also allowed. On the other hand, a 2 + 2-brane signature seems to require 2 + 10 background target spacetime signature as it has been shown by several authors [11]-[14]. Thus, according to these developments our result that 1 + 1 and 2 + 2 dimensions are exceptional signatures must refer to worldvolume signature of a 1 + 1-brane and 2 + 2-brane respectively, rather than to the background target spacetime signature associated with our World.

The key idea of our proof is based on the observation that a target spacetime of a 0-brane may become after first quantization in the worldvolume of a kind of a p-brane [15]-[16]. Thus, 1 + 1 and 2 + 2 target spacetime of a 0-brane can be seen as the 1 + 1 and 2 + 2 worldvolume spacetime of a 1 + 1-brane and 2 + 2-brane respectively. The idea turns out to be similar to the worldsheets for worldsheets proposed by Green [17], in which the string worldsheet itself emerges as the embedding space of a two dimensional string theory, and the 0-branes condensation suggested by Townsend [18] in which a 0-branes condensate leading to a description of a membrane.

Presumably, $M$-theory should have the property to predict not only the dimensionality of our World, but also its signature. Further, $M$-theory should be an exceptional theory and consequently one should expect that predicts also exceptional $t + s$-branes. In this work we show that the values $t = 1, s = 1$ and $t = 2, s = 2$ are exceptional and therefore one should expect that 1 + 1-brane and 2 + 2-brane are exceptional $t + s$-branes in agreement with our results. Moreover, in section 3, using a special kind of 0-brane called quat [15] we show that the signatures 1 + 1 and 2 + 2 are closely related. In fact, we show that after quantization a quat living in 2 + 2 target ‘spacetime’ leads to two equivalent branes, namely ‘1 + 1-matrix-brane’ and 2 + 2-brane. This means that, in principle, one can think either in 1 + 1-matrix-brane or 2 + 2-brane as a candidate for the $M$-theory. At this respect, it is important to mention that the case of 2 + 2-brane is precisely what Ketov [19] has suggested as a candidate for the $M$-theory.

It is worth mentioning that the relevance of the 2 + 2 signature has emerged
in a different context. In mathematics, the signature 2 + 2 is known as Atiyah-Ward signature as a recognition of the work of Atiyah and Ward [20] (see also [21] and Refs. [22]-[24]) in which they conjecture the possible classification of lower-dimensional integrable models by means of self-dual Yang-Mills connection in the 2 + 2−signature. Physically, the 2 + 2−signature emerges in a consistent \( N = 2 \) superstring theory as discussed by Ooguri and Vafa [25]. These authors proved that \( N = 2 \) strings provide a consistent quantum theory of self-dual gravity \( a \ la \) Plebanski in 2 + 2 dimensions (see [26]-[27] and Refs. therein). The \( N = 2 \) string concept has been extended to \( N = (2, 1) \) heterotic string [28]-[30] and to \( N = (2, 2) \) open and closed strings [19]. Both of these cases suggested to see \( M \)−theory as a 2 + 2−brane embedded in 2 + 10 dimensions [11]-[14]. Siegel [31] has observed that spin-independent coupling and ghost nature of \( SO(2, 2) \) make \( N=4 \) super Yang-Mills in 2 + 2 dimensions a topological-like theory. Finally, it has been emphasized [32]-[33] that Majorana-Weyl spinor exists in spacetime of 2 + 2−signature.

In section 4, we make some comments about the importance of the ‘1 + 1−matrix-brane’ or 2 + 2-brane for future developments of \( M \)−theory. In particular, we argue that the present work may suggest that there should be an exceptional chirotope connected with the 2 + 2−signature. (Chirotope concept is part of the oriented matroid theory [34], which, as it has been suggested [35]-[39], may provide the underlying mathematical framework for \( M \)−theory, and it may be applied to different scenarios of physics [40] via the angular momentum concept and bundle matroid theory [41]-[44].) The idea is essentially suggested because in general, chirotopes and \( p \)-branes appear to be connected [39]. From the results of reference [39] it is straightforward to see that the \( t + s \)-branes and chirotopes should be also linked. Therefore, the exceptional feature of the 2 + 2-brane should lead necessarily to an exceptional chirotope.

### 2.- A 0-brane and the signature of the ‘spacetime’

Usually, the motion of a 0−brane is described by the position coordinates \( x^\mu(\tau) \) where \( \tau \) is an arbitrary parameter and the index \( \mu \) goes from 0 to 3. The corresponding lagrangian is

\[
L = -m(-\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu})^{1/2}.
\]

Here, \( m \) is the inertial mass and \( \eta_{\mu\nu} = diag(-1, 1, 1, 1) \) is the Minkowski metric. The signature of the target spacetime determined by \( \eta_{\mu\nu} \) is in this case 1 + 3. It can be proved that the first order lagrangian

\[4\]
\[ L = \dot{x}^\mu p_\mu - \frac{\lambda}{2} (p^\mu p_\mu + m^2), \]  

(2)

where \( p_\mu \) is the canonical momentum associated with \( x^\mu \), is classical equivalent to the lagrangian (1). The advantage of the first order lagrangian (2) over the lagrangian (1) is that allows the possibility to consider the massless case

\[ L = \dot{x}^\mu p_\mu - \frac{\lambda}{2} (p^\mu p_\mu). \]  

(3)

Observe that the signature given by \( \eta_{\mu\nu} \) is an assumption of the above problem.

Our main idea is to discuss the above problem from other perspective. First of all, we shall assume a general signature with \( t \)–time and \( s \)–space coordinates, with \( t > 0 \) and \( s > 0 \), instead of the usual signature 1 + 3. Secondly, in order to emphasize that the physical system is moving in a more general target spacetime, we shall change the notation writing \( \xi^i(\tau) \) rather than \( x^\mu(\tau) \) for describing the position of the system, with \( i, j \) running over all values of \( t + s \).

Let us rewrite in terms of \( \xi^i(\tau) \) the corresponding lagrangian (3),

\[ \mathcal{L} = \dot{\xi}^i p_i - \frac{\lambda}{2} (p^i p_i). \]  

(4)

Up to total derivative this lagrangian can be written as

\[ \mathcal{L} = \frac{1}{2} (\dot{\xi}^i p_i - \xi^i \dot{p}_i) - \frac{\lambda}{2} (p^i p_i). \]  

(5)

Let us now introduce the notation

\[ \xi^i_1 \equiv \xi^i \]  

(6)

and

\[ \xi^i_2 \equiv p^i. \]  

(7)

Writing \( \xi^i_1 \) and \( \xi^i_2 \) in the compact form \( \xi^a \), with the index \( a = 1, 2 \), we see that (5) becomes

\[ \mathcal{L} = \frac{1}{2} \varepsilon^{ab} \dot{\xi}_a \xi_b - \frac{\lambda}{2} (\xi_2^i \xi_2^i), \]  

(8)

where \( \varepsilon^{ab} \) is the completely antisymmetric tensor with \( \varepsilon^{12} = 1 \). It is evident from (8) that the first term is manifestly \( SL(2, R) \) invariant, while the second one is not. Thus, in order to make the full lagrangian \( \mathcal{L} \) manifest \( SL(2, R) \) invariant it is necessary to add new constraints besides the \( p^i p_i = 0 \) constraint. The resultant lagrangian is (see [7] and Refs. therein)
\[ \mathcal{L}_0 = \frac{1}{2} \varepsilon^{ab} \xi^i_a \xi^i_b - \frac{\lambda^{ab}}{2} (\xi^i_a \xi^i_b), \]

(9)

where \( \lambda^{ab} = \lambda^{ba} \) denotes three different Lagrange multipliers.

The constraints derived from (8) are

\[ \xi^i_a \xi^i_b = 0, \]

(10)

which mean

\[ \xi^i_1 \xi^i_1 = 0, \]

(11)

\[ \xi^i_1 \xi^i_2 = 0, \]

(12)

and

\[ \xi^i_2 \xi^i_2 = 0. \]

(13)

According to the notation (6) and (7) this is equivalent to

\[ \xi^i \xi^j \eta_{ij} = 0, \]

(14)

\[ \xi^i p^j \eta_{ij} = 0, \]

(15)

and

\[ p^i p^j \eta_{ij} = 0, \]

(16)

respectively (see Ref. [45] for a discussion of these constraints). On the other hand, applying Noether's procedure to (8) one learns that the angular momentum

\[ L^{ij} = \xi^i p^j - \xi^j p^i \]

(17)

or

\[ L^{ij} = \varepsilon^{ab} \xi^i_a \xi^j_b \]

(18)

is a conserved dynamic variable obeying the Lorentz group algebra.

Our goal is now to determine signatures of \( \eta_{ij} \) for which the formulae (14)-(17) make sense. For this purpose, let us separate from (14)-(16) one time variable in the form

\[ - (\xi^i)^2 + \xi^i \xi^j \eta_{ij} = 0, \]

(19)
- $\xi^1 p^1 + \xi^p p^j \eta_{ij} = 0$,  

and

$$-(p^1) + p^j p^{j'} \eta_{ij} = 0,$$

where the indices $i', j'$, etc. run from 2 to $t + s$. The formula (20) leads to

$$(\xi^1)^2(p^1)^2 - \xi^p p^j \eta_{ij} \xi^k p^{j'} \eta_{k'} = 0.$$  

Using (14) and (16) we find that (22) becomes

$$\xi^i \xi^j \eta_{ij} p^k p^{j'} \eta_{k'} - \xi^i \xi^j \eta_{ij} \xi^k p^{j'} \eta_{k'} = 0,$$

which can also be written as

$$(\delta^i_{j'} \delta_{k'}^{j'} - \delta^i_{j'} \delta_{k'}^{j'}) \xi^i \xi^j p^k p^{j'} = 0.$$  

Let us now introduce the completely antisymmetric symbol

$$\epsilon_{i_2 \ldots i_{t+s}}.$$  

This is a rank $t + s - 1$ tensor which values are $+1$ or $-1$ depending on even or odd permutations of

$$\epsilon_{2 \ldots t+s},$$  

respectively. Moreover, $\epsilon_{i_2 \ldots i_{t+s}}$ takes the value 0, unless $i_2 \ldots i_{t+s}$ are all different.

The formula (24) can be written in terms of $\epsilon_{i_2 \ldots i_{t+s}}$ in the form

$$\epsilon^{i' j' \ldots i_{t+s}} \epsilon^{k k' \ldots i_{t+s}} \xi^i \xi^j p^k p^{j'} = 0,$$

where we dropped the nonzero factor $\frac{1}{(t+s-2)!}$. Moreover, the formula (27) can be rewritten as

$$\epsilon^{i' j' \ldots i_{t+s}} \epsilon^{k k' \ldots i_{t+s}} L^{i' k'} L^{j' p} = 0.$$  

Here, we used (17) and dropped some numerical factors. Observe that $L^{i' k'} \ldots i_{t+s} = \frac{1}{2} \epsilon^{i' j' k' \ldots i_{t+s}} L^{i' k'}$ is the dual of $L^{i' k'}$.

From (27) we first observe that for the values $t = 1$ and zero otherwise the expression (28) is an euclidean relation and, therefore, implies $L^{j' p} = 0$. This is true for all values of $s$ except for the case $s = 1$, which cannot be considered from (28). Indeed, for the case $1 + 1$ we need to go back to (14)-(16) and write,
\[-(\xi_1)^2 + (\xi_2)^2 = 0, \quad (29)\]

\[-\xi_1^2 p_1 + \xi_2^2 p_2 = 0, \quad (30)\]

and

\[-(p_1)^2 + (p_2)^2 = 0. \quad (31)\]

One can check, using (29) and (31) that (30) is an identity. Thus (29) and (31) do not lead to any link between \(\xi\) and \(p\) and therefore in this case the angular momentum (17) is well defined. Therefore, we have discovered that the symmetry \(SL(2, R)\) is not consistent with the Lorentz symmetry for all cases \(1+s\), except \(s = 1\). This means that for the case of one time the signature \(1+1\) is exceptional.

Another consequence of our analysis is that the minimal dimensionality in which (28) makes sense is \(2+1\), since the other case \(1+2\) we have already discovered that implies \(L_{j'l'} = 0\). Since the formula (28) is true up to sign we find that the analysis of the cases \(t+1\) is exactly the same that \(1+s\) for \(t \neq 1\) and \(s \neq 1\). Thus, the case \(2+1\) also implies \(L_{j'l'} = 0\). Hence, this proves that with two times the minimal case in which the symmetry \(SL(2, R)\) is perfectly consistent with the Lorentz symmetry, corresponds to the \(2+2\) signature. In principle we may continue with this procedure founding that \(3+3\), and so on are consistent possibilities. But, considering that (19)-(21) are only three constraints we see that there are not enough constraints to eliminate all additional degrees of freedom in all possible cases with \(t \geq 3\) and \(s \geq 3\). In fact, one should expect that this will lead to unpleasant possibilities at the quantum level [7].

Summarizing by imposing the symmetry \(SL(2, R)\) and the Lorentz symmetry in the lagrangian (9) we have shown that the signatures \(1+1\) and \(2+2\) are distinguished for any other and therefore they are exceptional.

3.- Relation between the signatures \(1+1\) and \(2+2\): the quatl theory

In this section, we shall show that a particular kind of 0–brane called the quatl [15] provides a link between the signatures \(1+1\) and \(2+2\), and also allows the possibility to transfer the target spacetime properties of the signatures \(1+1\) and \(2+2\) to a worldvolume properties of \(1+1\)–brane and
Consider the line element
\[ ds^2 = d\xi^\mu d\xi^\nu \eta_{\mu\nu}. \] (32)
Here, we shall assume that the indices \( \mu, \nu \in \{1, 2, 3, 4\} \) and that \( \eta_{\mu\nu} = \text{diag}(-1, -1, 1, 1) \) determine the 2 + 2 signature. By defining
\[ \zeta^{11} \equiv \xi^3, \quad \zeta^{22} \equiv \xi^4, \quad \zeta^{12} \equiv \xi^1, \quad \zeta^{21} \equiv \xi^2, \] (33)
it is not difficult to show that (23) can also be written as
\[ ds^2 = d\zeta^{am} d\zeta^{bn} \eta_{ab} \eta_{mn}, \] (34)
where \( a, b, m, n \in \{1, 2\} \) and \( \eta_{ab} = \text{diag}(-1, 1) \) and \( \eta_{mn} = \text{diag}(-1, 1) \). We have that while \( \xi^\mu \) are coordinates associated to a target spacetime of signature 2 + 2 the coordinates \( \zeta^{am} \) are associated with a target space time of signature 1 + 1. Thus, the equivalence between (32) and (34) determines an interesting connection between the signatures 1 + 1 and 2 + 2.

Now, consider the transition \( \eta_{\mu\nu} \to G_{\mu\nu}(\xi^a) \) in (32), where \( G_{\mu\nu} = G_{\nu\mu} \) is a curved symmetric metric. A natural question is to see which the corresponding transition in (34) is. One can check that the transitions \( \eta_{ab} \to \varphi_1 g_{ab}(\zeta^{am}) \) and \( \eta_{mn} \to \varphi_2 \gamma_{mn}(\zeta^{am}) \), where \( g_{ab} \) and \( \gamma_{mn} \) are two different nonsymmetric metrics and \( \varphi_1 \) and \( \varphi_2 \) are two conformal factors, provide a possible identification between \( G_{\mu\nu} \) and the two nonsymmetric metrics \( g_{ab}(\zeta^{am}) \) and \( \gamma_{mn}(\zeta^{am}) \). In fact, this can be seen by just observing that the two metrics \( g_{ab} \) and \( \gamma_{mn} \) together with the conformal factors \( \varphi_1 \) and \( \varphi_2 \) lead exactly to the same number of degrees of freedom associated with the symmetric metric \( G_{\mu\nu} \).

The next step is to make dynamic the above kinematic relation between the signatures 1 + 1 and 2 + 2. For this purpose let us first consider the \((t + s)\)-brane action
\[ S = \frac{1}{2} \int d\zeta^{t+s} \sqrt{-G} \left[ G^{AB} \frac{\partial y^A}{\partial \zeta^B} \frac{\partial y^B}{\partial \zeta^A} - (\zeta^{(t+s)} - (t + s) - 2) \right], \] (35)
where we shall assume that \( \gamma_{\rho\sigma} \) is a background metric associated with some \( \sigma \)-model in \( T + S \) dimensions. From (35) we obtain the constraint
\[ \frac{\partial y^\nu}{\partial \zeta^A} \frac{\partial y^\sigma}{\partial \zeta^B} \gamma_{\nu\sigma} - \frac{G_{AB}}{2} \left[ \frac{G^{CD} \partial y^\nu}{\partial \zeta^C} \frac{\partial y^\sigma}{\partial \zeta^D} \gamma_{\nu\sigma} - ((t + s) - 2) \right] = 0. \]  

(36)

From (36) we find the expression

\[ G^{CD} \frac{\partial y^\nu}{\partial \zeta^C} \frac{\partial y^\sigma}{\partial \zeta^D} \gamma_{\nu\sigma} = t + s. \]

(37)

Now, we ask ourselves what the analogue of (36) for a 0–brane will be. Presumably, we shall have

\[ P_A P_B - \frac{G_{AB}}{2} \left[ G^{CD} P_C P_D - \frac{(t + s) - 2}{T + S} \right] = 0 \]

(38)

and the analogue of (37) will be

\[ G^{CD} P_C P_D = \frac{t + s}{T + S}. \]

(39)

In principle, (38) and (39) can be obtained from (36) and (37) by assuming that

\[ \gamma_{\mu\nu} = x^{\hat{M}}(A) \frac{\partial x^\mu}{\partial y^\nu} x^{R(C)} \frac{\partial x^S}{\partial y^\nu} \eta_{\hat{M}\hat{R}} \eta_{\hat{N}\hat{S}}. \]

(40)

Here, \( x^{\hat{N}}(A) \) is eigenstate of \( P_C = -i \frac{\partial}{\partial \zeta^C} \) and is an element of \( SO(T, S) \), where \( T \) and \( S \) determine the signature of the flat metric \( \eta_{\hat{M}\hat{R}} \).

The corresponding first order lagrangian from which (38) and (39) can be derived is

\[ S = \int d\tau \left\{ \dot{\zeta}^A P_A - \sqrt{-G} \left[ G^{AB} P_A P_B - \frac{(t + s) - 2}{T + S} \right] \right\}. \]

(41)

Let us now focus in the case of \( 2 + 2 \) dimensions. We have \( t = 2 \) and \( s = 2 \) and therefore in this case (35) becomes

\[ S = \frac{1}{2} \int d\zeta^{2+2} \sqrt{-G} \left[ G^{AB} \frac{\partial y^\nu}{\partial \zeta^A} \frac{\partial y^\sigma}{\partial \zeta^B} \gamma_{\nu\sigma} - 2 \right], \]

(42)

which corresponds to the bosonic \( 2+2 \)–brane action. While the corresponding 0–brane action (41) is reduced to

\[ S = \frac{1}{2} \int d\tau \left\{ \dot{\zeta}^A P_A - \sqrt{-G} \left[ G^{AB} P_A P_B - \frac{2}{T + S} \right] \right\}. \]

(43)
For the particular case of $\varphi_1 = \varphi_2 = 1$, it is straightforward to show that this action can be written in terms of $\zeta^{am}$ and $p_{am}$ as follows:

$$S = \int d\tau \left[ \dot{\zeta}^{am} p_{am} - \frac{1}{2} \sqrt{-g} \sqrt{-\gamma} \left( g^{ab} \gamma^{mn} p_{am} p_{bn} - \frac{2}{T+S} \right) \right], \quad (44)$$

where $g_{ab}$ and $\gamma_{mn}$ are the two auxiliary metrics mentioned above. It turns out that the coordinates $\zeta^{am}(\tau)$ describe the position of a quatl [15]. The states $|x^N_{(A)}\rangle$ which allow the connection between the actions (42) and (43) are the so called ketzal [15]. It is particularly interesting to observe that the formalism that connects (43) and (44) only works for the signatures $2+2$ and $1+1$. In fact, assume that $\zeta^A$ is a vector living in a target spacetime of signature $q+q$, in other words, we consider the same number of time and space coordinates. Now, consider the possibility to write the coordinates $\zeta^A$ as square matrix $\zeta^{am}$ where the maximum value of the indices $a$ and $m$ is $q$. It is not difficult to see that this is possible if $q$ satisfies the formula $q^2 = q+q$, which has the unique solution $q = 2$. Thus, a connection of the form (43) and (44) is only possible for a vector $\zeta^A$ in a target spacetime of $2+2$--signature and a matrix $\zeta^{am}$ associated with a $1+1$--signature. It is worth mentioning that the group theory provides us with an alternative explanation about the connection between the coordinates $\zeta^A$ and $\zeta^{am}$. In fact, the natural global continuous symmetry associated with the signature $2+2$ is $SO(2,2)$. But due to the isomorphism $SO(2,2) \cong SL(2,R)' \otimes SL(2,R)$ one finds that a $2+2$ dimensional vector $\zeta^A$ may be written as $\zeta^{am}$, where the index $a$ refers to $SL(2,R)'$ and the index $m$ to $SL(2,R)$ (see Ref. [19]).

From the above observations it becomes evident that (42) may also be written as

$$S = \frac{1}{2} \int d\zeta^{2+2} \sqrt{-g} \sqrt{-\gamma} \left[ g^{ab} \gamma^{mn} \frac{\partial y^\nu}{\partial \zeta^{am}} \frac{\partial y^\sigma}{\partial \zeta^{an}} \gamma_{\nu\sigma} - 2 \right], \quad (45)$$

which presumably it will correspond to the first quantization of the $0$--brane called quatl described by the action (44). It seems that action (45) has not been considered in the literature. For distinguishing from the $2+2$--brane, we shall call the physical system described by (45) the '1+1--matrix-brane'. The action (45) appears interesting because at the quantum level may allow to follow similar technics to the one used in string theory. In particular, (45) may allow to determine the critical dimensions of the $1+1$--matrix-brane. In principle, one may think on $s+t$--matrix-brane, but its correspondence with the $t+s$--brane should require more degrees of freedom for connecting the coordinates $\zeta^A$ and the square matrix $\zeta^{am}$. 
4.- Some final thoughts about the 1 + 1 and 2 + 2 signatures

From the above discussion it is evident that in general, one may consider three different kinds of ‘spacetimes’: the target spacetime ($T$), the scenario where a 0–brane moves, the worldvolume ($W$) associated with a $t + s$–brane, and the $T + S$ background target ($BT$) spacetime where the $t + s$–brane evolves. We have shown that, in the case of the signatures $1 + 1$ and $2 + 2$, and only in this case, there is a clear relation between $T$ and $W$. In fact, in this case, after quantizing the system, the $T$ of a quatl becomes the $W$ spacetime of $2 + 2$–brane. This proves that the $2 + 2$–brane can be understood as a first quantization of a quatl, a kind of $0$–brane moving in a spacetime of $1+1$–signature [15]. Traditionally, the $W$ spacetime of $1+1$–brane determines the $BT$–spacetime via quantum consistency and supersymmetrizability. Thus, in the case of $2 + 2$ one should expect a connection of the form

$$T \leftrightarrow W \leftrightarrow BT.$$  

Supersymmetrizability established that for a $2 + 2$–brane, the signature of $BT$ should be $2 + 10$, but a quantum analysis as in the case of string theory, as far as we know is lacking. Looking the action (45) one wonders if the signature of $BT$ for a supersymmetric $2 + 2$–brane should be $2 + 18$ rather than $2 + 10$. The reason for this is that the action contains two independent metrics in $1+1$ dimensions and one should expect that each metric leads to $1 + 9$ dimensions.

The $2 + 2$–brane in a $BT$ of $2 + 10$–spacetime has been proposed as a candidate of the $M$–theory. But one of the dreams is that $M$–theory is an exceptional theory. Thus our proof that $2 + 2$ is an exceptional signature coincides with this expectation regarding $M$–theory.

Hull [10] has shown that it makes sense to consider compactifications not only in the spacelike coordinates, but also in the timelike coordinates. From this point of view one may think in a double dimensional compactification procedure in order of reducing $2 + 2$ to $1 + 1$ dimensions and simultaneously $2 + 10$ going to $1 + 9$. And this is another way to see the close relation between the signatures $2 + 2$ and $1 + 1$. Another possibility for a link between the signature $2 + 2$ and $1 + 1$ is to apply the Carlini-Greensite [46] procedure for transforming a spacelike coordinate into timelike coordinates and vice versa.

Cosmology may also provide the possibility to see the exceptionality of the signature $2 + 2$. In fact, recently it has been shown [47] that radius-duality in a Kaluza-Klein cosmological models applies to all possible dimensions $T + S$ in the spacetime of $BT$ with the only exception of $2 + S$. One is tempting
to speculate that similar technics applied to the spacetime $W$ will lead to the
exceptionality of the $2+2-$signature. This is expected since $2+2-$brane moves
naturally $2+10-$spacetime which according to duality should be exceptional.
Therefore, one should expect that the cosmological exceptionality of the $2+10$
spacetime may be translated to the $2+2-$brane new world scenario.

It has been proposed [35]-[40] the so called oriented matroid theory [34]
as the underlaying mathematical structure for $M-$theory. Since $2+2-$brane
is also a candidate for the $M-$theory one should expect a relation between
matroid theory and $2+2-$brane theory. The proof that the signature $2+2$
is exceptional relied on the combination of both group symmetries $SL(2,R)$
and $SO(S,T)$. Specifically one assumes that the $SL(2,R)$-symmetry implies
a nonvanishing angular momentum $L^{i\ell'}$ associated with the group $SO(S,T)$.
But it has been shown [39] that any nonvanishing angular momentum can
be identified with the chirotope concept of the oriented matroid theory [34].
Hence, it is correct to assure that the $SL(2,R)-$symmetry implies a chirotope
structure for two times physics. In particular, according to the result of the
present work concerning the exceptionality of the $2+2-$signature one must
have an exceptional chirotope associated to the $2+2-$brane. In other words,
the $SL(2,R)-$symmetry should imply an exceptional chirotope structure for
$M-$theory.

There are a number of possible developments of the present work. By con-
sidering supersymmetry restrictions of the values $T,S,t$ and $s$ in connection
with a superconformal group, Batrachenko, Duff and Lu [48] have reexamined
the $2-$brane case at the end of the De Sitter universe. It may be interesting to
understand the present development from the point of view of supersymmetry
and superconformal group. There has been an attempt to explain why the
$BT-$spacetime has $1+3-$signature using fractal technics [49]. It seems interest-
ing to apply fractal technics to the case $W-$spacetime in order to find an
alternative explanation of the exceptionality of the $2+2-$signature. Finally,
self-duality seems to be deeply connected with the $2+2-$signature [50]. Thus
it seems also interesting for further research to investigate self-duality concept
at the level of constraints of the ‘$1+1-$matrix-brane’ system.

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