Upper and lower bounds for the solution of a stochastic prey-predator system with foraging arena scheme

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Abstract

We investigate some probabilistic aspects of the unique global strong solution of a two dimensional system of stochastic differential equations describing a prey-predator model perturbed by Gaussian noise. We first establish, for any fixed $t > 0$, almost sure upper and lower bounds for the components $X(t)$ and $Y(t)$ of the solution vector: these explicit estimates emphasize the interplay between the various parameters of the model and agree with the asymptotic results found in the literature. Then, standing on the aforementioned bounds, we derive upper and lower estimates for the joint moments and distribution function of $(X(t), Y(t))$. Our analysis is based on a careful use of comparison theorems for stochastic differential equations and exploits several peculiar features of the noise driving the equation.

Key words and phrases: stochastic predator-prey models, Brownian motion, stochastic differential equations, comparison theorems, moments and distribution functions.

AMS 2000 classification: 60H10, 60H30, 92D30.

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1 Introduction

In theoretical ecology the system of equations
\[
\begin{align*}
\frac{dx(t)}{dt} &= x(t)(a_1 - b_1 x(t)) - c_1 h(x(t), y(t)) y(t), \quad x(0) = x; \\
\frac{dy(t)}{dt} &= y(t)(-a_2 - b_2 y(t)) + c_2 h(x(t), y(t)) y(t), \quad y(0) = y,
\end{align*}
\]
constitutes a fundamental class of models for predator-prey interaction. Here, \(x(t)\) and \(y(t)\) represent the population densities of prey and predator at time \(t \geq 0\), respectively; \(a_1\) the prey intrinsic growth rate; \(a_2\) the predator intrinsic death rate; \(a_1/b_1\) the carrying capacity of the ecosystem; \(b_2\) the predator intraspecies competition; \(h(x(t), y(t))\) the intake rate of predator; \(c_2/c_1\) the trophic efficiency. We observe that equation (1.1) encompasses the classic Lotka-Volterra model \[22,27\], which is obtained setting \(b_1 = b_2 = 0\) and \(h(x, y) = x\).

To catch the different features of specific environments, several choices for the so-called functional response \(h(x, y)\) have been suggested in the literature; we mention, among others,

- Holling II function \[15\]: \(h(x, y) = \frac{x}{\beta + \alpha} \);  
- ratio dependent functional responses \[3,4\]: \(h(x, y) = \tilde{h}(x/y)\);  
- foraging arena models \[2,28\]: \(h(x, y) = \frac{x}{\beta + \alpha x + \alpha_2 y}\);  
- Beddington-DeAngelis model \[3,11\]: \(h(x, y) = \frac{x}{\beta + \alpha_1 x + \alpha_2 y}\);  
- CrowleyMartin model \[9\]: \(h(x, y) = \frac{x}{\beta + \alpha_1 x + \alpha_2 y + \alpha_3 x y}\);  
- HassellVarley model \[26\]: \(h(x, y) = \frac{x}{\alpha_1 x + \alpha_2 y + m}\).

(\(\beta, \alpha_1, \alpha_2, \alpha_3\), are positive real numbers, \(m \in \mathbb{N}\) and \(\tilde{h}: \mathbb{R} \to \mathbb{R}\) a suitable regular function). What distinguishes the Holling II function from other models is the absence of \(y\); on this issue the paper \[25\] presents statistical evidence from 19 predator-prey systems that the BeddingtonDeAngelis, CrowleyMartin and HassellVarley models (whose functional responses depend on both prey and predator abundances) can provide better descriptions compared to those with Holling-type functions (see also \[14\]). Moreover, as remarked in \[1\], models based on ratio-dependent functional responses exhibit singular behaviours.

With the aim of introducing environmental noise in the model, different types of stochastic perturbation for the system (1.1) have been considered and studied. Among the most common, we find the Itô-type stochastic differential equation
\[
\begin{align*}
\frac{dX(t)}{dt} &= [X(t)(a_1 - b_1 X(t)) - c_1 h(X(t), Y(t)) Y(t)] dt + \sigma_1 X(t) dB_1(t), \quad X(0) = x; \\
\frac{dY(t)}{dt} &= [Y(t)(-a_2 - b_2 Y(t)) + c_2 h(X(t), Y(t)) Y(t)] dt + \sigma_2 Y(t) dB_2(t), \quad Y(0) = y,
\end{align*}
\]
where \{B_1(t), B_2(t)\}_{t \geq 0} is a standard two dimensional Brownian motion and \(\sigma_1, \sigma_2\) positive real numbers. System (1.2) tries to catch random fluctuations in the growth rate \(a_1\) and death rate \(a_2\). Some references in this stream of research are [7], in the case of foraging arena schemes, [12], [17], [21] treating the case of Beddington-DeAngelis functional response, and [24] dealing with Hassell-Varley model. It is worth mentioning that all these papers are devoted to the study of global existence, uniqueness, positivity and asymptotic properties for the specific model of type (1.2) considered.

Our investigation is focused on the system

\[
\begin{align*}
    dX(t) &= \left[X(t)(a_1 - b_1 X(t)) - c_1 \frac{X(t)Y(t)}{\beta + Y(t)}\right] dt + \sigma_1 X(t) dB_1(t), \quad X(0) = x; \\
    dY(t) &= \left[Y(t)(a_2 - b_2 Y(t)) + c_2 \frac{X(t)Y(t)}{\beta + Y(t)}\right] dt + \sigma_2 Y(t) dB_2(t), \quad Y(0) = y,
\end{align*}
\]

which is proposed and analysed in [7]. It corresponds to equation (1.2) with a foraging arena functional response. It is proved in [7] that system (1.3) possesses a unique global strong solution \{(X(t), Y(t))\}_{t \geq 0} fulfilling the condition

\[P(X(t) > 0 \text{ and } Y(t) > 0, \text{ for all } t \geq 0) = 1.\]

Moreover, the authors investigate the asymptotic behaviours of \(X(t)\) and \(Y(t)\), as \(t\) tends to infinity, and identify three different regimes:

- if \(a_1 < \frac{\sigma_1^2}{2}\), then
  \[
  \lim_{t \to +\infty} X(t) = \lim_{t \to +\infty} Y(t) = 0,
  \]
  almost surely and exponentially fast;

- if \(\frac{\sigma_1^2}{2} < a_1 < \frac{\sigma_1^2}{2} + \frac{b_1 \beta a_2}{c_2} + \frac{b_1 \beta \sigma_2^2}{2c_2} =: \phi\), then almost surely
  \[
  \lim_{t \to +\infty} Y(t) = 0, \quad \text{exponentially fast},
  \]
  and
  \[
  \lim_{t \to +\infty} \frac{1}{t} \int_0^t X(r) dr = \frac{a_1 - \sigma_1^2/2}{b_1};
  \]

- if \(a_1 > \frac{\phi}{1 - \sigma_2^2/2c_2 - a_2/c_2}\) and \(a_2 + \frac{\sigma_2^2}{2} < c_2\), then system (1.3) has a unique stationary distribution.

The case

\[\phi < a_1 < \frac{\phi}{1 - \sigma_2^2/2c_2 - a_2/c_2};\]
with \( a_2 + \frac{\sigma_2^2}{2} < c_2 \), is not investigated but the authors mention that computer simulations indicate the existence of stationary distributions for both \( X(t) \) and \( Y(t) \) also in that regime.

The goal of our work is to present a novel analysis for systems of the type (1.2), which in the current study take the form (1.3). We derive explicit upper and lower bounds for the components \( X(t) \) and \( Y(t) \) of the solution of equation (1.3) at any fixed time \( t \geq 0 \). Such almost sure estimates depend solely on the parameters describing the model under investigation and the noise driving the equation. Their derivation is based on a careful use of comparison theorems for stochastic differential equations and standard stochastic calculus' tools. The estimates we obtain reflect the intrinsic interplay between the parameters of the model and enlighten the probabilistic dependence structure of \( X(t) \) and \( Y(t) \). We also remark that our bounds, which are valid for any fixed time \( t \geq 0 \), agree in the limit as \( t \) tends to infinity with the asymptotic results proven in [7] and summarized above. We then utilize the previously mentioned bounds to get upper and lower estimates for the joint moments and distribution function of \((X(t), Y(t))\). We propose closed form expressions which rely on new estimates for a logistic-type stochastic differential equation.

It is important to remark that, while systems of the type (1.2) with Beddington-DeAngelis or Crowley-Martin or Hassell-Varley functional responses can be treated, as far as finite time analysis in concerned, with a change of measure approach, the unboundedness of \( h(x, y) = \frac{x}{x + b_2 y} \), as a function of \( x \), prevents from the use of a similar approach for (1.3). We will in fact prove in Section 3.1 below the failure of the Novikov condition for the corresponding change of measure.

The paper is organized as follows: Section 2 collects some auxiliary results on the solution of a logistic stochastic differential equation that plays a major role in our analysis; in Section 3 we state and prove our first main theorem: almost sure upper and lower bounds for \( X(t) \) and \( Y(t) \), for any \( t \geq 0 \). Here, we also comment on the impossibility of a change of measure approach and compare our findings with the asymptotic results from [7]; Section 4 contains our second main result, which proposes upper and lower estimates for the joint moments of \((X(t), Y(t))\); in Section 5 upper and lower bounds for the joint probability function of \((X(t), Y(t))\) constitutes our third and last main theorem.

### 2 Preliminary results

In this section we will prove some auxiliary results concerning the solution of the logistic stochastic differential equation

\[
dL(t) = L(t)(a - bL(t))dt + \sigma L(t)dB(t), \quad L(0) = \lambda. \tag{2.1}
\]

Here \( a, b, \sigma \) and \( \lambda \) are positive real numbers and \( \{B(t)\}_{t \geq 0} \) is a standard one dimensional Brownian motion. It is well known (see for instance formula (4.51) in [20] or formula
(2.1) in [18] for the case of time-dependent parameters) that equation (2.1) possesses a unique global positive strong solution which can be represented as

$$L(t) = \frac{\lambda e^{(a-\sigma^2/2)t+\sigma B(t)}}{1 + b \int_0^t e^{(a-\sigma^2/2)r+\sigma B(r)} dr}, \quad t \geq 0.$$ \hfill (2.2)

We start focusing on the asymptotic behaviour of the solution of equation (2.1). For notational convenience we set

$$m(t) := \inf_{r \in [0,t]} B(r) \quad \text{and} \quad M(t) := \sup_{r \in [0,t]} B(r). \hfill (2.3)$$

We also refer the reader to the paper [13] for a small time analysis of \(\{L(t)\}_{t \geq 0}\).

**Proposition 2.1.** Let \(\{L(t)\}_{t \geq 0}\) be the unique global strong solution of (2.1). Then,

$$\lim_{t \to +\infty} L(t) = \begin{cases} 0, & \text{if } a < \sigma^2/2; \\ \frac{a-\sigma^2/2}{b}, & \text{if } a > \sigma^2/2, \end{cases} \hfill (2.4)$$

almost surely.

**Proof.** If \(a < \sigma^2/2\), the numerator of (2.2) tends to zero almost surely by virtue of

$$\mathbb{P}\left(\lim_{t \to +\infty} \frac{B(t)}{t} = 0\right) = 1,$$

(see for instance Theorem 3.4 in [23]). Moreover, according to formula 1.8.4 page 612 in [6],

$$\mathbb{P}\left(\int_0^{+\infty} \lambda e^{(a-\sigma^2/2)r+\sigma B(r)} dr < +\infty\right) = 1,$$

which is equivalent to say that the denominator of (2.2) converges almost surely to a positive finite limit, as \(t\) tends to infinity. This proves the first equality in (2.4).

When \(a > \sigma^2/2\), recalling the notation (2.3) we can write for all \(t \geq 0\) that

$$\frac{\lambda e^{(a-\sigma^2/2)t+\sigma B(t)}}{1 + b \lambda e^{\sigma M(t)} \int_0^t e^{(a-\sigma^2/2)r} dr} \leq L(t) \leq \frac{\lambda e^{(a-\sigma^2/2)t+\sigma B(t)}}{1 + b \lambda e^{\sigma m(t)} \int_0^t e^{(a-\sigma^2/2)r} dr},$$

or equivalently

$$\frac{\lambda e^{(a-\sigma^2/2)t+\sigma B(t)}}{1 + b \lambda e^{\sigma M(t)} e^{(a-\sigma^2/2)r_1 - \frac{1}{2}}} \leq L(t) \leq \frac{\lambda e^{(a-\sigma^2/2)t+\sigma B(t)}}{1 + b \lambda e^{\sigma m(t)} e^{(a-\sigma^2/2)r_1 - \frac{1}{2}}}.$$ \hfill (2.5)

Since

$$\mathbb{P}\left(\lim_{t \to +\infty} \frac{M(t)}{t} = \lim_{t \to +\infty} \frac{m(t)}{t} = 0\right) = 1,$$

(see Lemma 4.1 in [7]) we deduce that both left and right hand sides in (2.5) converge to \(\frac{a-\sigma^2/2}{b}\), thus proving the second equality in (2.4). \qed
From formula (2.2) we see that, for any $t > 0$, the random variable $L(t)$ is a function of the Geometric Brownian motion $e^{(a-\sigma^2/2)t + \sigma B(t)}$ and its integral $\int_0^t e^{(a-\sigma^2/2)r + \sigma B(r)} dr$. Using the joint probability density function of the random vector 

\[
e^{(a-\sigma^2/2)t + \sigma B(t)}, \int_0^t e^{(a-\sigma^2/2)r + \sigma B(r)} dr\]

which can be found in [29], the authors of [10] write down an expression for the probability density function of $L(t)$: see formula (40) there. However, the authors mention that, due to the presence of oscillating integrals, the numerical treatment of such expression is rather tricky.

In the next two results, instead of insisting with exact formulas, we propose upper and lower estimates for the moments $E[L(t)^p]$ and distribution function $P(L(t) \leq z)$; the bounds we obtain involve integrals whose numerical approximations do not present the aforementioned difficulties. We also mention the paper [3] which uses an approach based on power series to approximate the moments of $L(t)$.

In the sequel, we will write for $t > 0$

\[
N_{0,t}(r) := \frac{1}{\sqrt{2t}} e^{-\frac{r^2}{2t}}, \quad r \in \mathbb{R},
\]

and

\[
N'_{0,t}(r) := \frac{d}{dr} N_{0,t}(r) = -\frac{r}{t} \frac{1}{\sqrt{2t}} e^{-\frac{r^2}{2t}}, \quad r \in \mathbb{R}.
\]

**Proposition 2.2.** Let $\{L(t)\}_{t \geq 0}$ be the unique global strong solution of (2.1). Then, for any $p \geq 0$, we have

\[
E[L(t)^p] \leq 2k_p(t) \int_0^{+\infty} (1 + b\lambda e^{-\sigma z} K_p(t))^{-p} N_{0,t}(z) dz,
\]

and

\[
E[L(t)^p] \geq 2k_p(t) \int_0^{+\infty} (1 + b\lambda e^{\sigma z} K_p(t))^{-p} N_{0,t}(z) dz,
\]

where

\[
k_p(t) := \lambda^p e^{p(a-\sigma^2/2)t + p^2\sigma^2 t/2}, \quad \text{and} \quad K_p(t) := \frac{\lambda e^{(a-\sigma^2/2+p\sigma^2)t} - 1}{a - \sigma^2/2 + p\sigma^2}.
\]

**Proof.** Fix $p \geq 0$; then,

\[
E[L(t)^p] = E \left[ \frac{\lambda^p e^{p(a-\sigma^2/2)t + p\sigma B(t)}}{(1 + b \int_0^t \lambda e^{(a-\sigma^2/2)r + \sigma B(r)} dr)^p} \right]
\]

6
We now observe that, according to the Girsanov’s theorem, the law of \( \{B(t)\}_{t \geq 0} \) under the equivalent probability measure
\[
dQ := e^{\rho \sigma B(t) - \frac{1}{2} \sigma^2 t} d\mathbb{P}
\]
coinsides with the one of \( \{B(t) + p\sigma t\}_{t \geq 0} \) under the measure \( \mathbb{P} \). Therefore,
\[
\mathbb{E}[L(t)^p] = k_p(t) \mathbb{E} \left[ \frac{e^{\rho \sigma B(t) - \frac{1}{2} \sigma^2 t}}{\left( 1 + b \int_{0}^{t} \lambda e^{(a-\frac{\sigma^2}{2})(1+\sigma B(r))} dr \right)^p} \right]
\]
\[
= k_p(t) \mathbb{E} \left[ \frac{1}{\left( 1 + b \int_{0}^{t} \lambda e^{(a-\frac{\sigma^2}{2})(1+\sigma B(r))} dr \right)^p} \right]
\]
\[
= k_p(t) \mathbb{E} \left[ \left( 1 + b \int_{0}^{t} \lambda e^{(a-\frac{\sigma^2}{2})(1+\sigma B(r))} dr \right)^{-p} \right].
\]
Now, adopting the notation (2.3), we can estimate as
\[
\mathbb{E}[L(t)^p] = k_p(t) \mathbb{E} \left[ \left( 1 + b \int_{0}^{t} \lambda e^{(a-\frac{\sigma^2}{2})(1+\sigma B(r))} dr \right)^{-p} \right]
\]
\[
\geq k_p(t) \mathbb{E} \left[ \left( 1 + be^{\sigma M(t)} \int_{0}^{t} \lambda e^{(a-\frac{\sigma^2}{2}+p\sigma^2)} dr \right)^{-p} \right]
\]
\[
= k_p(t) \mathbb{E} \left[ \left( 1 + be^{\sigma M(t)} K_p(t) \right)^{-p} \right],
\]
and similarly
\[
\mathbb{E}[L(t)^p] = k_p(t) \mathbb{E} \left[ \left( 1 + b \int_{0}^{t} \lambda e^{(a-\frac{\sigma^2}{2})(1+\sigma B(r))} dr \right)^{-p} \right]
\]
\[
\leq k_p(t) \mathbb{E} \left[ \left( 1 + be^{\sigma m(t)} \int_{0}^{t} \lambda e^{(a-\frac{\sigma^2}{2}+p\sigma^2)} dr \right)^{-p} \right].
\]
Moreover, recalling that, for \( A \in \mathcal{B}(\mathbb{R}) \) and \( t > 0 \), we have
\[
\mathbb{P}(m(t) \in A) = 2 \int_A \mathcal{N}_{0,t}(z) \mathbf{1}_{(-\infty,0)}(z) dz \quad \text{and} \quad \mathbb{P}(M(t) \in A) = 2 \int_A \mathcal{N}_{0,t}(z) \mathbf{1}_{[0,\infty)}(z) dz,
\]
(see formula (8.2) in Chapter 2 from [19]) we can conclude that
\[
\mathbb{E}[L(t)^p] \geq k_p(t) \mathbb{E}\left[(1 + be^\sigma M(t) K_p(t))^{-p}\right] = 2k_p(t) \int_0^{+\infty} (1 + be^\sigma K_p(t))^p \mathcal{N}_{0,t}(z) dz,
\]
and
\[
\mathbb{E}[L(t)^p] \leq k_p(t) \mathbb{E}\left[(1 + be^\sigma M(t) K_p(t))^{-p}\right] = 2k_p(t) \int_0^{+\infty} (1 + be^{-\sigma} K_p(t))^p \mathcal{N}_{0,t}(z) dz.
\]

Proposition 2.3. Let \( \{L(t)\}_{t \geq 0} \) be the unique global strong solution of (2.1). Then, for any \( z > 0 \) and \( t > 0 \), we have the bounds
\[
\mathbb{P}(L(t) \leq z) \leq -2 \int_{\frac{1 + b e^\sigma k(t)}{1 + b e^\sigma K(t)}}^{\frac{1 + b e^\sigma k(t)}{1 + b e^\sigma K(t)}} \mathcal{N}_{0,t}(2v-u) dudv, \tag{2.8}
\]
and
\[
\mathbb{P}(L(t) \leq z) \geq -2 \int_{\frac{1 + b e^\sigma k(t)}{1 + b e^\sigma K(t)}}^{\frac{1 + b e^\sigma k(t)}{1 + b e^\sigma K(t)}} \mathcal{N}_{0,t}(u-2v) dudv, \tag{2.9}
\]
with
\[
k(t) := \lambda e^{(a-\sigma^2/2)t} \quad \text{and} \quad K(t) := \frac{\lambda e^{(a-\sigma^2/2)t} - 1}{a - \sigma^2/2}.
\]

Proof. We first prove (2.9): from (2.2) we have
\[
L(t) \geq \frac{\lambda e^{(a-\sigma^2/2)t + \sigma B(t)}}{1 + b e^\sigma M(t)} \int_0^t \lambda e^{(a-\sigma^2/2)r} dr = \frac{k(t) e^{\sigma B(t)}}{1 + bK(t) e^{\sigma M(t)}}.
\]
The last member above is a function of the two dimensional random vector \((B(t), M(t))\), whose joint probability density function is given by the expression
\[
f_{B(t),M(t)}(u, v) = \begin{cases} -2\mathcal{N}_{0,t}^\prime(2v-u), & \text{if } v > 0 \text{ and } u < v, \\ 0, & \text{otherwise} \end{cases}
\]

Therefore, for any \( z > 0 \), we obtain
\[
\mathbb{P}(L(t) \leq z) \leq \mathbb{P}\left( \frac{k(t)e^{\sigma B(t)}}{1 + bK(t)e^{\sigma M(t)}} \leq z \right)
\]
\[
= -2 \int_{\{ \frac{k(t)e^{\alpha u}}{1 + bK(t)e^{\sigma v}} \leq z \} \cap \{ v > 0 \} \cap \{ u < v \}} N_{0,t}'(2v - u) dudv,
\]
completing the proof of \((2.9)\). Similarly,
\[
L(t) \leq \frac{\lambda e^{(a - \sigma^2/2)t + \sigma B(t)}}{1 + be^{\sigma m(t)}} \int_0^t \lambda e^{(a - \sigma^2/2)r} dr = \frac{k(t)e^{\sigma B(t)}}{1 + bK(t)e^{\sigma m(t)}}.
\]
The last member above is a function of the two dimensional random vector \((B(t), m(t))\), whose joint probability density function is given by the expression
\[
f_{B(t), m(t)}(u, v) = \begin{cases} -2N_{0,t}'(u - 2v), & \text{if } v < 0 \text{ and } u > v, \\ 0, & \text{otherwise}. \end{cases}
\]
Therefore, for any \( z > 0 \), we obtain
\[
\mathbb{P}(L(t) \leq z) \geq \mathbb{P}\left( \frac{k(t)e^{\sigma B(t)}}{1 + bK(t)e^{\sigma M(t)}} \leq z \right)
\]
\[
= -2 \int_{\{ \frac{k(t)e^{\alpha u}}{1 + bK(t)e^{\sigma v}} \leq z \} \cap \{ v > 0 \} \cap \{ u < v \}} N_{0,t}'(u - 2v) dudv.
\]
The proof is complete. \(\square\)

**Remark 2.4.** We observe that the inequality \( u < v \) implies
\[
\frac{k(t)e^{\sigma u}}{1 + bK(t)e^{\sigma v}} \leq \frac{k(t)e^{\sigma v}}{1 + bK(t)e^{\sigma v}} \leq \frac{k(t)}{bK(t)}.
\]
Therefore, the upper bound \((2.8)\) becomes trivial for \( z \geq \frac{k(t)}{bK(t)} \); in fact, in that case
\[
\{ u < v \} \Rightarrow \left\{ \frac{k(t)e^{\sigma u}}{1 + bK(t)e^{\sigma v}} \leq \frac{k(t)}{bK(t)} \right\} \Rightarrow \left\{ \frac{k(t)e^{\sigma u}}{1 + bK(t)e^{\sigma v}} \leq z \right\}
\]
which yields
\[
\int_{\{ \frac{k(t)e^{\sigma u}}{1 + bK(t)e^{\sigma v}} \leq z \} \cap \{ v > 0 \} \cap \{ u < v \}} -2N_{0,t}'(2v - u) dudv
\]
\[
= \int_{\{ v > 0 \} \cap \{ u < v \}} -2N_{0,t}'(2v - u) dudv = 1.
\]
3 First main theorem: almost sure bounds

Our first main theorem provides explicit almost sure upper and lower bounds for the solution of (1.3) at any given time \( t \). It is useful to introduce the following notation: let

\[
L_1(t) := \frac{G_1(t)}{1 + b_1 \int_0^t G_1(r)dr}, \quad t \geq 0, \tag{3.1}
\]

and

\[
L_2(t) := \frac{G_2(t)}{1 + b_2 \int_0^t G_2(r)dr}, \quad t \geq 0, \tag{3.2}
\]

where for \( t \geq 0 \) we set

\[
G_1(t) := xe^{(a_1-\sigma_1^2/2)t+\sigma_1 B_1(t)} \quad \text{and} \quad G_2(t) := ye^{-(a_2+\sigma_2^2/2)t+\sigma_2 B_2(t)};
\]

the parameters \( a_1, a_2, b_1, b_2, \sigma_1, \sigma_2, x, y \) are those appearing in equation (1.3). According to the previous section, the stochastic processes \( \{L_1(t)\}_{t \geq 0} \) and \( \{L_2(t)\}_{t \geq 0} \) satisfy the equations

\[
dL_1(t) = L_1(t)(a_1 - b_1 L_1(t))dt + \sigma_1 L_1(t)dB_1(t), \quad L_1(0) = x, \tag{3.3}
\]

and

\[
dL_2(t) = L_2(t)(-a_2 - b_2 L_2(t))dt + \sigma_2 L_2(t)dB_2(t), \quad L_2(0) = y, \tag{3.4}
\]

respectively. Therefore, the two dimensional process \( \{(L_1(t), L_2(t))\}_{t \geq 0} \) is the unique strong solution of system (1.3) when \( c_1 = c_2 = 0 \), i.e. when the interaction term \( \frac{X(t)Y(t)}{\beta+Y(t)} \) is not present.

3.1 Comments on the use of Girsanov theorem

We have just mentioned that, by removing the ratio \( \frac{X(t)Y(t)}{\beta+Y(t)} \) from its drift, equation (1.3) reduces to the uncoupled system

\[
\begin{cases}
  dL_1(t) = L_1(t)(a_1 - b_1 L_1(t))dt + \sigma_1 L_1(t)dB_1(t), & L_1(0) = x; \\
  dL_2(t) = L_2(t)(-a_2 - b_2 L_2(t))dt + \sigma_2 L_2(t)dB_2(t), & L_2(0) = y,
\end{cases} \tag{3.5}
\]

whose solution is explicitly represented via formulas (3.1) and (3.2). Since such drift removal can be performed with the use of Girsanov theorem, one may wonder whether the almost sure properties of (1.3) can be deduced from those of (3.5) under a suitable equivalent probability measure. Aim of the present subsection is to show that this not case: we are in fact going to prove that the Novikov condition corresponding to the just mentioned drift removal is not fulfilled.
First of all, we notice that system (3.5) can be rewritten as

\[
\begin{aligned}
    dL_1(t) &= L_1(t)(a_1 - b_1 L_1(t))dt + \sigma_1 L_1(t) \left( dB_1(t) + \frac{c_1 L_2(t)}{\sigma_1(\beta + L_2(t))} dt - \frac{c_1 L_2(t)}{\sigma_1(\beta + L_2(t))} dt \right); \\
    L_1(0) &= x; \\
    dL_2(t) &= L_2(t)(-a_2 - b_2 L_2(t))dt + \sigma_2 L_2(t) \left( dB_2(t) - \frac{c_2 L_1(t)}{\sigma_2(\beta + L_2(t))} dt + \frac{c_2 L_1(t)}{\sigma_2(\beta + L_2(t))} dt \right); \\
    L_2(0) &= y,
\end{aligned}
\]

or equivalently

\[
\begin{aligned}
    dL_1(t) &= L_1(t)(a_1 - b_1 L_1(t))dt - c_1 \frac{L_1(t)L_2(t)}{\beta + L_2(t)} dt + \sigma_1 L_1(t)\tilde{B}_1(t), \\
    L_1(0) &= x; \\
    dL_2(t) &= L_2(t)(-a_2 - b_2 L_2(t))dt + c_2 \frac{L_1(t)L_2(t)}{\beta + L_2(t)} dt + \sigma_2 L_2(t)\tilde{B}_2(t), \\
    L_2(0) &= y,
\end{aligned}
\tag{3.6}
\]

where we set

\[
\tilde{B}_1(t) := B_1(t) + \int_0^t \frac{c_1 L_2(r)}{\sigma_1(\beta + L_2(r))} dr, \quad t \geq 0,
\]

and

\[
\tilde{B}_2(t) := B_2(t) - \int_0^t \frac{c_2 L_1(r)}{\sigma_2(\beta + L_2(r))} dr, \quad t \geq 0.
\]

Now, if the Novikov condition

\[
\mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^T \left( \frac{c_1 L_2(r)}{\sigma_1(\beta + L_2(r))} \right)^2 + \left( \frac{c_2 L_1(r)}{\sigma_2(\beta + L_2(r))} \right)^2 dr \right\} \right] < +\infty \tag{3.7}
\]

is satisfied for some \( T > 0 \), then the stochastic process \( \{(\tilde{B}_1(t), \tilde{B}_1(t))\}_{t \in [0,T]} \) is according to the Girsanov theorem a standard two dimensional Brownian motion on the probability space \((\Omega, \mathcal{F}_T, \mathbb{Q})\) (here \( \{\mathcal{F}_t\}_{t \geq 0} \) denotes the augmented Brownian filtration) with

\[
d\mathbb{Q} := \exp \left\{ - \int_0^T \frac{c_1 L_2(r)}{\sigma_1(\beta + L_2(r))} dB_1(r) - \frac{1}{2} \int_0^T \left( \frac{c_1 L_2(r)}{\sigma_1(\beta + L_2(r))} \right)^2 dr \right\} \times \exp \left\{ \int_0^T \frac{c_2 L_1(r)}{\sigma_2(\beta + L_2(r))} dB_2(r) - \frac{1}{2} \int_0^T \left( \frac{c_2 L_1(r)}{\sigma_2(\beta + L_2(r))} \right)^2 dr \right\} d\mathbb{P}.
\]

Moreover, in this case equation (3.6) implies that the two dimensional process \( \{(L_1(t),L_2)\}_{t \in [0,T]} \) is a weak solution of (1.3) with respect to \((\Omega, \{\mathcal{F}_t\}_{t \in [0,T]}; \mathbb{Q}; \{(\tilde{B}_1(t), \tilde{B}_1(t))\}_{t \in [0,T]}\). We now prove that condition (3.7) cannot be true without additional assumptions on the parameters of our model. In fact,

\[
\mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^T \left( \frac{c_1 L_2(r)}{\sigma_1(\beta + L_2(r))} \right)^2 + \left( \frac{c_2 L_1(r)}{\sigma_2(\beta + L_2(r))} \right)^2 dr \right\} \right]
\]

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$$\geq \mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^T \left( \frac{c_2 L_1(r)}{\sigma_2 (\beta + L_2(r))} \right)^2 dr \right\} \right]$$

$$= \mathbb{E} \left[ \exp \left\{ \frac{c_2^2}{2\sigma_2^2} \int_0^T \frac{L_1^2(r)}{(\beta + L_2(r))^2} dr \right\} \right]$$

$$\geq \mathbb{E} \left[ \exp \left\{ \frac{c_2^2}{2\sigma_2^2 M_2} \int_0^T L_1^2(r) dr \right\} \right]$$

where we introduced the notation

$$M_2 := \sup_{r \in [0,T]} (\beta + L_2(r))^2.$$ 

We now apply Jensen’s inequality to the Lebesgue integral and use the identity

$$\int_0^T L_1(r) dr = \frac{1}{b_1} \ln \left( 1 + b_1 \int_0^T G_1(r) dr \right)$$

to get

$$\mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^T \left( \frac{c_1 L_2(r)}{\sigma_1 (\beta + L_2(r))} \right) + \left( \frac{c_2 L_1(r)}{\sigma_2 (\beta + L_2(r))} \right)^2 dr \right\} \right]$$

$$\geq \mathbb{E} \left[ \exp \left\{ \frac{c_2^2}{2\sigma_2^2 M_2} \int_0^T L_1^2(r) dr \right\} \right]$$

$$= \mathbb{E} \left[ \exp \left\{ \frac{c_2^2 T}{2\sigma_2^2 M_2 T} \int_0^T L_1^2(r) dr \right\} \right]$$

$$\geq \mathbb{E} \left[ \exp \left\{ \frac{c_2^2}{2\sigma_2^2 M_2 T} \left( \int_0^T L_1(r) dr \right)^2 \right\} \right]$$

$$= \mathbb{E} \left[ \exp \left\{ \frac{c_2^2}{2\sigma_2^2 M_2 T b_1^2} \left( \ln \left( 1 + b_1 \int_0^T G_1(r) dr \right) \right)^2 \right\} \right]$$

$$\geq \mathbb{E} \left[ \exp \left\{ \frac{c_2^2}{2\sigma_2^2 M_2 T b_1^2} \left( \ln \left( 1 + b_1 K_1(T)e^{\sigma_1 m_1(T)} \right) \right)^2 \right\} \right]$$

$$\geq \mathbb{E} \left[ \exp \left\{ \frac{c_2^2}{2\sigma_2^2 M_2 T b_1^2} \left( \ln \left( b_1 K_1(T)e^{\sigma_1 m_1(T)} \right) \right)^2 \right\} \right]$$

$$= \mathbb{E} \left[ \exp \left\{ \frac{c_2^2}{2\sigma_2^2 M_2 T b_1^2} \left( \sigma_1 m_1(T) + \ln(b_1 K_1(T)) \right)^2 \right\} \right].$$

Here, we set

$$K_1(T) = \frac{e^{(a_1 - \sigma_1^2/2)T} - 1}{a_1 - \sigma_1^2/2} \quad \text{and} \quad m_1(T) := \min_{t \in [0,T]} B_1(t).$$

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Using the independence between $B_1$ and $B_2$, we can write the last expectation as

$$
E \left[ \exp \left\{ \frac{c_1^2}{2\sigma_1^2 M_2 T b_1^2} (\sigma_1 m_1(T) + \ln(b_1 K_1(T)))^2 \right\} \right]
= \int_{\beta^2}^{+\infty} \left( \int_{-\infty}^{0} e^{\frac{C}{\sqrt{2\pi} T} (\sigma_1 u + D)} \frac{2}{\sqrt{2\pi T}} e^{-\frac{u^2}{T}} du \right) d\mu(z),
$$

where $\mu$ stands for the law of $M_2$, $C := \frac{c_1^2}{\sigma_1^2 b_1^2}$ and $D := \ln(b_1 K_1(T))$. It is now clear that the inner integral above is finite if and only if $z \geq C \sigma_1^2$. Since $z$ ranges in the interval $]\beta^2, +\infty[$, we deduce that the last condition is verified for all $z \in ]\beta^2, +\infty[$ only when $\beta^2 \geq C \sigma_1^2$, which in our notation means

$$
\beta \geq \frac{c_2 \sigma_1}{b_1 \sigma_2}.
$$

(3.8)

Therefore, if the parameters describing system (1.3) do not respect the bound (3.8), then inequality

$$
E \left[ \exp \left\{ \frac{1}{2} \int_0^T \left( \frac{c_1 L_2(r)}{\sigma_1 (\beta + L_2(r))} \right)^2 + \left( \frac{c_2 L_1(r)}{\sigma_2 (\beta + L_2(r))} \right)^2 dr \right\} \right]
\geq 2 \int_{\beta^2}^{+\infty} \left( \int_{-\infty}^{0} e^{\frac{C}{\sqrt{2\pi} T} (\sigma_1 u + D)} \frac{1}{\sqrt{2\pi T}} e^{-\frac{u^2}{T}} du \right) d\mu(z) = +\infty,
$$

which is valid for all $T > 0$, implies the failure of Novikov condition (3.7). From this point of view the almost sure properties of the solution of (1.3) cannot be deduced from those of the uncoupled system (3.5).

**Remark 3.1.** The functional response in the foraging arena model formally appears to be a particular case of the one that characterizes the Beddington-DeAngelis model (take $\alpha_1 = 0$). However, referring to the change of measure technique mentioned above, we see that the Novikov condition corresponding to the Beddington-DeAngelis model would amount at the finiteness of

$$
E \left[ \exp \left\{ \frac{1}{2} \int_0^T \left( \frac{c_1 L_2(r)}{\sigma_1 (\beta + \alpha_1 L_1(r) + \alpha_2 L_2(r))} \right)^2 + \left( \frac{c_2 L_1(r)}{\sigma_2 (\beta + \alpha_1 L_1(r) + \alpha_2 L_2(r))} \right)^2 dr \right\} \right].
$$

Since the two ratios in the Lebesgue integral are upper bounded almost surely by $\frac{c_1}{\sigma_1 \alpha_2}$ and $\frac{c_2}{\sigma_2 \alpha_1}$, respectively, we get immediately the finiteness, for all $T > 0$, of the expectation above. Therefore, in the Beddington-DeAngelis model one may utilize the change of measure approach to study almost sure properties of the solution on any finite interval of time $[0, T]$. The same reasoning applies also to the Crowley-Martin and Hassell-Varley functional responses.
3.2 Statement and proof of the first main theorem

Recall that, according to the discussion in Section 1, the quantity
\[ \phi := \frac{\sigma_1^2}{2} + \frac{b_1 \beta a_2}{c_2} + \frac{b_1 \beta \sigma_2^2}{2c_2} \]
is a threshold determining the asymptotic behaviour of \( X(t) \) and \( Y(t) \).

**Theorem 3.2.** Let \( \{(X(t), Y(t))\}_{t \geq 0} \) be the unique global strong solution of (1.3). Then, for all \( t \geq 0 \) the following bounds hold almost surely:

\[ L_2(t) \leq Y(t) \leq L_2(t) \left( 1 + b_1 \int_0^t G_1(r) dr \right)^{\frac{\sigma_2}{\sigma_1}} ; \]  (3.9)

if \( a_1 < \phi \), then

\[ L_1(t)e^{-\frac{\sigma_1}{\sigma_2} (1 + b_1 \int_0^t G_1(r) dr) \ln(1 + b_2 \int_0^t G_2(r) dr)} \leq X(t) \leq L_1(t); \]  (3.10)

if \( a_1 > \phi \), then

\[ L_1(t)e^{-c_1 t} \leq X(t) \leq L_1(t). \]  (3.11)

**Remark 3.3.** We assumed at the beginning of this manuscript that the Brownian motions \( \{B_1(t)\}_{t \geq 0} \) and \( \{B_2(t)\}_{t \geq 0} \), driving the two dimensional system (1.3), are independent. However, this assumption is not needed in the derivation of the almost sure bounds stated above, as long as system (1.3) possesses a positive global strong solution. Therefore, the estimates (3.9), (3.10) and (3.11) remain true in the case of correlated Brownian motions as well.

**Remark 3.4.** The bounds in Theorem 3.2 are consistent with the asymptotic results obtained in [7]. In fact:

- \( a_1 < \frac{\sigma_1^2}{2} \): taking the limit as \( t \) tends to infinity in (3.10) we get
  \[ 0 \leq \lim_{t \to +\infty} X(t) \leq \lim_{t \to +\infty} L_1(t), \]
  which, in combination with the first equation in (2.4) for \( L_1 \), gives
  \[ \lim_{t \to +\infty} X(t) = 0. \]

On the other hand, if we take the limit in (3.9) we obtain

\[ 0 \leq \lim_{t \to +\infty} Y(t) \leq \lim_{t \to +\infty} L_2(t) \left( 1 + b_1 \int_0^t G_1(r) dr \right)^{\frac{\sigma_2}{\sigma_1}}. \]

According to formula 1.8.4 page 612 in [6] the random variable \( \int_0^{+\infty} G_1(r) dr \) is finite almost surely; this fact and the first equation in (2.4) for \( L_2 \) yield

\[ \lim_{t \to +\infty} Y(t) = 0, \]

completing the proof of (1.4).
\[ \sigma_2^2 < a_1 < \phi = \frac{\sigma_2^2}{2} + \frac{b_1 \beta a_2}{\xi_2} + \frac{b_1 \beta \sigma_2^2}{2\xi_2}; \] first of all, note that
\[ L_2(t) \sim e^{-(a_2+\sigma_2^2/2)t+o(t)}, \quad \text{as } t \text{ tends to infinity.} \]

Moreover, according to Proposition 2.1, inequality \( \frac{\sigma_2^2}{2} < a_1 \) implies that
\[ \int_0^t G_1(r)dr \sim G_1(t), \quad \text{as } t \text{ tends to infinity;} \]

since \( G_1(t) \sim e^{(a_1-\sigma_1^2/2)t+o(t)} \), we deduce that
\[ \int_0^t G_1(r)dr \sim e^{(a_1-\sigma_1^2/2)t+o(t)}, \quad \text{as } t \text{ tends to infinity.} \]

Combining these observations we get
\[ L_2(t) \left( 1 + b_1 \int_0^t G_1(r)dr \right)^{\frac{\sigma_2}{b_1}} \sim e^{-(a_2+\sigma_2^2/2)t+o(t)} \left( e^{(a_1-\sigma_1^2/2)t+o(t)} \right)^{\frac{\sigma_2}{b_1}} \]
\[ = e^{(-a_2-\sigma_2^2/2+\frac{\sigma_2^2}{b_1}(a_1-\sigma_1^2/2))t+o(t)}. \quad (3.12) \]

The restriction \( a_1 < \frac{\sigma_2^2}{2} + \frac{b_1 \beta a_2}{\xi_2} + \frac{b_1 \beta \sigma_2^2}{2\xi_2} \) entails the negativity of the exponent in the last exponential and hence, passing to the limit in (3.9), we conclude
\[ \lim_{t \to +\infty} Y(t) = 0; \]
this corresponds to (1.5). In addition, from (3.10) we obtain
\[ \lim_{t \to +\infty} \frac{1}{t} \int_0^t X(r)dr \leq \lim_{t \to +\infty} \frac{1}{t} \int_0^t L_1(r)dr = \lim_{t \to +\infty} L_1(t) = \frac{a_1 - \sigma_1^2/2}{b_1}. \]

Here, we utilized de l’Hôpital’s rule and the second equation in (2.4) for \( L_1 \). This partially proves (1.6);

- \( a_1 > \phi \): from Proposition 2.1 and equality (3.12) we deduce that the lower and upper bounds in (3.11) tend to zero and \( \frac{a_1 - \sigma_1^2/2}{b_1} \), respectively, while the lower and upper bounds in (3.9) tend to zero and infinity, respectively. These bounds allow for possible oscillatory behaviours of \( X(t) \) and \( Y(t) \), which are typical in the presence of stationary distributions.

Proof. We start finding the Itô’s differential of the stochastic process \( \frac{1}{L_1(t)} \):
\[
\frac{d}{dt} \frac{1}{L_1(t)} = -\frac{1}{L_1^2(t)} dL_1(t) + \frac{1}{L_1^3(t)} d\langle L_1 \rangle_t \\
= -\frac{a_1 - b_1 L_1(t)}{L_1(t)} dt - \frac{\sigma_1}{L_1(t)} dB_1(t) + \frac{\sigma_2^2}{L_1(t)} dt 
\]
Following the previous reasoning we also find that
\[
= \frac{\sigma_1^2 - a_1 + b_1 L_1(t)}{L_1(t)} dt - \frac{\sigma_1}{L_1(t)} dB_1(t).
\]
Combining this expression with the first equation in (1.3) we get
\[
\frac{d X(t)}{L_1(t)} = X(t) \frac{1}{L_1(t)} + \frac{1}{L_1(t)} dX(t) + d \langle X, 1/L_1 \rangle (t)
\]
\[
= X(t) \left( \frac{\sigma_1^2 - a_1 + b_1 L_1(t)}{L_1(t)} dt - \frac{\sigma_1}{L_1(t)} dB_1(t) \right)
\]
\[
+ \frac{1}{L_1(t)} \left[ X(t) \left( a_1 - b_1 X(t) - \frac{c_1 Y(t)}{\beta + Y(t)} \right) dt + \sigma_1 X(t) dB_1(t) \right]
\]
\[
- \frac{\sigma_1^2}{L_1(t)} dt
\]
\[
= X(t) \left[ \frac{\sigma_1^2}{L_1(t)} - a_1 + b_1 L_1(t) + a_1 - b_1 X(t) - \frac{c_1 Y(t)}{\beta + Y(t)} \right] dt
\]
\[
= X(t) \left[ b_1 \left( L_1(t) - X(t) \right) - \frac{c_1 Y(t)}{\beta + Y(t)} \right] dt. \tag{3.13}
\]
Since \( \frac{X(0)}{L_1(0)} = 1 \), the last chain of equalities implies
\[
\frac{X(t)}{L_1(t)} = \exp \left\{ b_1 \int_0^t \left( L_1(r) - X(r) \right) dr - c_1 \int_0^t \frac{Y(r)}{\beta + Y(r)} dr \right\}. \tag{3.13}
\]
Following the previous reasoning we also find that
\[
\frac{d}{L_2(t)} = -\frac{1}{L_2(t)} dL_2(t) + \frac{1}{L_2(t)} d\langle L_2 \rangle_t
\]
\[
= -\frac{-a_2 - b_2 L_2(t)}{L_2(t)} dt - \frac{\sigma_2}{L_2(t)} dB_2(t) + \frac{\sigma_2^2}{L_2(t)} dt
\]
\[
= \frac{\sigma_2^2 + a_2 + b_2 L_2(t)}{L_2(t)} dt - \frac{\sigma_2}{L_2(t)} dB_2(t).
\]
Combining this expression with the second equation in (1.3) we get
\[
\frac{d Y(t)}{L_2(t)} = Y(t) \frac{1}{L_2(t)} + \frac{1}{L_2(t)} dY(t) + d \langle Y, 1/L_2 \rangle (t)
\]
\[
= Y(t) \left( \frac{\sigma_2^2 + a_2 + b_2 L_2(t)}{L_2(t)} dt - \frac{\sigma_2}{L_2(t)} dB_2(t) \right)
\]
\[
+ \frac{1}{L_2(t)} \left[ Y(t) \left( -a_2 - b_2 X(t) + \frac{c_2 X(t)}{\beta + Y(t)} \right) dt + \sigma_2 Y(t) dB_2(t) \right]
\]
\[
- \frac{\sigma_2^2 Y(t)}{L_2(t)} dt
\]
\[
Y(t) - \frac{L(t)^2}{L_2(t)} - \sigma^2 - a_2 - b_2 Y(t) + \frac{c_2 X(t)}{\beta + Y(t)} - \sigma_2^2 \right] dt \\
= Y(t) \left[ b_2 (L_2(t) - Y(t)) + \frac{c_2 X(t)}{\beta + Y(t)} \right] dt.
\]

Since \( Y(0) = 1 \), the last chain of equalities implies

\[
Y(t) = \exp \left\{ b_2 \int_0^t (L_2(r) - Y(r)) dr + c_2 \int_0^t \frac{X(r)}{\beta + Y(r)} dr \right\}.
\] (3.14)

We now observe that

\[ P \left( \frac{X(t)Y(t)}{\beta + Y(t)} > 0 \right) = 1, \quad \text{for any } t \geq 0 \]

(remember that \( X(t) \) and \( Y(t) \) are positive for all \( t \geq 0 \)); therefore, by means of standard comparison theorems for SDEs (see for instance Theorem 1.1 in Chapter VI from [16]) we deduce that

\[
X(t) \leq L_1(t), \quad \text{for all } t \geq 0,
\] (3.15)

and

\[
Y(t) \geq L_2(t), \quad \text{for all } t \geq 0,
\] (3.16)

where \( \{L_1(t)\}_{t \geq 0} \) and \( \{L_2(t)\}_{t \geq 0} \) solve (3.3) and (3.4), respectively. Therefore, equation (3.13) leads to

\[
\exp \left\{ -c_1 \int_0^t \frac{Y(r)}{\beta + Y(r)} dr \right\} \leq \frac{X(t)}{L_1(t)} \leq 1,
\]

or equivalently,

\[
L_1(t) \exp \left\{ -c_1 \int_0^t \frac{Y(r)}{\beta + Y(r)} dr \right\} \leq X(t) \leq L_1(t),
\] (3.17)

while equation (3.14) leads to

\[
1 \leq \frac{Y(t)}{L_2(t)} \leq \exp \left\{ c_2 \int_0^t \frac{X(r)}{\beta + Y(r)} dr \right\},
\]

or equivalently,

\[
L_2(t) \leq Y(t) \leq L_2(t) \exp \left\{ c_2 \int_0^t \frac{X(r)}{\beta + Y(r)} dr \right\}.
\] (3.18)

The lower bound in (3.17) and upper bound in (3.18) are not explicit yet since they depend on the solution itself. To solve this problem we first recall that the process
Figure 1: Upper bounds for the function $y \mapsto \frac{y}{2+y}$ (thick line) with the function $y \mapsto \frac{y}{2}$ (regular line) and the function $y \mapsto 1$ (thin line).

$\{L_2(t)\}_{t \geq 0}$ is positive and converges almost surely to zero exponentially fast, as $t$ tends to infinity. Now, by virtue of (3.15), (3.16) and the infinitesimal behaviour of $L_2$, we can upper bound the right hand side in (3.18) as

$$L_2(t) \exp \left\{ c_2 \int_0^t \frac{X(r)}{\beta + Y(r)} dr \right\} \leq L_2(t) \exp \left\{ c_2 \int_0^t \frac{L_1(r)}{\beta + L_2(r)} dr \right\} \leq L_2(t) \exp \left\{ \frac{c_2}{\beta} \int_0^t L_1(r) dr \right\},$$

In addition, since

$$L_1(t) = \frac{1}{b_1} \frac{d}{dt} \ln \left( 1 + b_1 \int_0^t G_1(r) dr \right),$$

the last member above can be rewritten as

$$L_2(t) \exp \left\{ \frac{c_2}{\beta} \int_0^t L_1(r) dr \right\} = L_2(t) \exp \left\{ \frac{c_2}{\beta b_1} \ln \left( 1 + b_1 \int_0^t G_1(r) dr \right) \right\} = L_2(t) \left( 1 + b_1 \int_0^t G_1(r) dr \right)^{\frac{c_2}{\beta b_1}}.$$

Combining this estimate with (3.18) we obtain (3.9).

For the lower bound in (3.17), we observe that the function $y \mapsto \frac{y}{\beta + y}$, for $y > 0$, can be sharply upper bounded by affine functions in two different ways: the upper bound $y \mapsto 1$ is sharp at infinity but not accurate at zero while the upper bound $y \mapsto \frac{y}{\beta}$ is sharp at zero but very bad at infinity. Therefore, according to the asymptotic results proved in [7] and mentioned in the Introduction, we now proceed distinguishing two different regimes:

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• when \( a_1 < \phi \), the process \( \{Y_t\}_{t \geq 0} \) tends to zero exponentially fast and hence we utilize the process \( \frac{Y_t}{\beta + Y_t} \) to upper bound \( \frac{Y_t}{\beta + Y_t} \). The left hand side of (3.17) is then simplified to

\[
L_1(t) \exp \left\{ -c_1 \int_0^t \frac{Y(r)}{\beta + Y(r)} dr \right\} \geq L_1(t) \exp \left\{ -\frac{c_1}{\beta} \int_0^t Y(r) dr \right\} \\
\geq L_1(t) \exp \left\{ -\frac{c_1}{\beta} \int_0^t L_2(r) \left( 1 + b_1 \int_0^r G_1(u) du \right)^{\frac{c_2}{\beta b_1}} dr \right\} \\
\geq L_1(t) \exp \left\{ -\frac{c_1}{\beta} \left( 1 + b_1 \int_0^t G_1(r) dr \right)^{\frac{c_2}{\beta b_1}} \int_0^t L_2(r) dr \right\} \\
= L_1(t) \exp \left\{ -\frac{c_1}{\beta b_2} \left( 1 + b_1 \int_0^t G_1(r) dr \right)^{\frac{c_2}{\beta b_1}} \ln \left( 1 + b_2 \int_0^t G_2(r) dr \right) \right\}. \quad (3.19)
\]

Here, in the second inequality we utilized the upper bound in (3.9) while in the last equality we employed the identity

\[
L_2(t) = \frac{1}{b_2} \frac{d}{dt} \ln \left( 1 + b_2 \int_0^t G_2(r) dr \right).
\]

Inserting (3.19) in the left hand side of (3.17), one gets (3.10);

• when \( a_1 > \phi \), the process \( \{Y_t\}_{t \geq 0} \) has a more oscillatory behaviour; therefore, we prefer to upper bound the ratio \( \frac{Y_t}{\beta + Y_t} \) with one. This gives

\[
L_1(t) \exp \left\{ -c_1 \int_0^t \frac{Y(r)}{\beta + Y(r)} dr \right\} \geq L_1(t)e^{-c_1 t},
\]

and (3.17) reduces to (3.11).

\[\Box\]

**Remark 3.5.** It is important to emphasize that both the lower bounds in (3.10) and (3.11) remain valid without restrictions on the parameters: this is clear from the proof of Theorem 3.2 and in particular from the use of the comparison principle we made. In fact, one may combine the two lower estimates as

\[
L_1(t) \max \left\{ e^{-\frac{c_1}{\beta b_2} \left( 1 + b_1 \int_0^t G_1(r) dr \right)^{\frac{c_2}{\beta b_1}} \ln \left( 1 + b_2 \int_0^t G_2(r) dr \right)} \right\} \leq X(t) \leq L_1(t),
\]

and argue on the different values attained by the maximum above. However, such analysis would necessarily involve the non directly observable quantities \( \int_0^t G_1(r) dr \), \( \int_0^t G_2(r) dr \) and their probabilities. That is why we preferred to suggest which lower bound is better suited for the given set of parameters.
4 Second main theorem: bounds for the moments

The next theorem presents upper and lower estimates for the joint moments of $X(t)$ and $Y(t)$ at any given time $t$. These bounds, which rely on the almost sure inequalities (3.9), (3.10) and (3.11) are represented through closed form expressions involving Lebesgue integrals; such integrals can be evaluated via numerical approximations or Monte Carlo simulations.

We also mention that in [7] the authors prove an asymptotic upper bound for the moments $E [(X(t)^2 + Y(t)^2)_{\theta/2}]$ with $\theta$ being a positive real number.

**Theorem 4.1.** Let $\{(X(t), Y(t))\}_{t \geq 0}$ be the unique global strong solution of (1.3). For all $t \geq 0$ we have the following estimates:

1. if $p, q \geq 0$ with $\frac{qc_2}{\beta b_1} - p \geq 1$, then

$$E[X(t)^p Y(t)^q] \leq 2k_{1,p}(t)k_{2,q}(t) \left(1 + b_1 \frac{e^{(\frac{qc_2}{\beta b_1} + p - 1)\frac{\sigma^2}{2}}} {a_1 + (\frac{qc_2}{\beta b_1} + p - 1)\frac{\sigma^2}{2}} - 1\right)^{-\frac{qc_2}{\beta b_1} - p} \int_{0}^{+\infty} \left(1 + b_2 e^{-\sigma z K_2(q)(t)}\right)^{-q} N_{0,t}(z) dz. \quad (4.1)$$

2. if $p, q \geq 0$ and $a_1 > \phi$, then

$$E[X(t)^p Y(t)^q] \geq 4e^{-p c_1 t} k_{1,p}(t)k_{2,q}(t) \int_{0}^{+\infty} \left(1 + b_1 x e^{\sigma z K_1(p)(t)}\right)^{-p} N_{0,t}(z) dz$$

$$\times \int_{0}^{+\infty} \left(1 + b_2 y e^{\sigma z K_2(q)(t)}\right)^{-q} N_{0,t}(z) dz. \quad (4.2)$$

3. if $p, q \geq 0$ and $a_1 < \phi$, then

$$E[X(t)^p] \geq -4k_1(t)^p \int_{A} \frac{e^{(\sigma_1 v_1 - \frac{\sigma_2}{2}(1+b_1 K_1(t)e^{\sigma_1 v_1}))\frac{\sigma_2}{2}(1+b_1 K_2(t)e^{\sigma_2 v_2}) \ln(1+b_2 K_2(t)e^{\sigma_2 v_2})}}{(1+b_1 K_1(t)e^{\sigma_1 v_1})^p} \times N_{0,t}^\prime(2v_1 - u_1)N_{0,t}(v_2) du_1 dv_1 dv_2, \quad (4.3)$$

where

$$A := \{(u_1, v_1, v_2) \in \mathbb{R}^3 : v_1 > 0, u_1 < v_1, v_2 > 0\},$$

while

$$E[Y(t)^q] \geq 2k_{2,q}(t) \int_{0}^{+\infty} \left(1 + b_2 y e^{\sigma z K_2(q)(t)}\right)^{-q} N_{0,t}(z) dz. \quad (4.4)$$
Here,
\[ k_1(t) := xe^{(a_1 - \sigma_1^2/2)t}, \quad K_1(t) := x \frac{e^{(a_1 - \sigma_1^2/2)t} - 1}{a_1 - \sigma_1^2/2}, \quad K_2(t) := y \frac{e^{(a_2 - \sigma_2^2/2)t} - 1}{a_2 - \sigma_2^2/2}, \]
\[ k_{1,p}(t) := x^p e^{p(a_1 - \sigma_1^2/2)t + p^2 \sigma_1^2 t/2}, \quad K_{1,p}(t) := x \frac{e^{(a_1 - \sigma_1^2/2 + p\sigma_1^2)t} - 1}{a_1 - \sigma_1^2/2 + p\sigma_1^2}, \]
\[ k_{2,p}(t) := y^p e^{p(a_2 - \sigma_2^2/2)t + p^2 \sigma_2^2 t/2}, \quad K_{2,p}(t) := y \frac{e^{(a_2 - \sigma_2^2/2 + p\sigma_2^2)t} - 1}{a_2 - \sigma_2^2/2 + p\sigma_2^2}. \]

Proof. 1. Using (3.9) and (3.10) (or (3.11)), we can write
\[
\mathbb{E} [X(t)^p Y(t)^q] \leq \mathbb{E} \left[ L_1(t)^p L_2(t)^q \left( 1 + b_1 \int_0^t G_1(r) dr \right)^{\frac{qc_2}{p\rho_1}} \right] \\
= \mathbb{E} \left[ L_1(t)^p \left( 1 + b_1 \int_0^t G_1(r) dr \right)^{\frac{qc_2}{p\rho_1}} \right] \mathbb{E} [L_2(t)^q] \\
= \mathbb{E} \left[ \frac{G_1(t)^p}{\left( 1 + b_1 \int_0^t G_1(r) dr \right)^{p}} \left( 1 + b_1 \int_0^t G_1(r) dr \right)^{\frac{qc_2}{p\rho_1} - p} \right] \mathbb{E} [L_2(t)^q] \\
= \mathbb{E} \left[ G_1(t)^p \left( 1 + b_1 \int_0^t G_1(r) dr \right)^{\frac{qc_2}{p\rho_1} - p} \right] \mathbb{E} [L_2(t)^q] \\
= \mathcal{I}_1 \mathcal{I}_2,
\]

where we set
\[ \mathcal{I}_1 := \mathbb{E} \left[ G_1(t)^p \left( 1 + b_1 \int_0^t G_1(r) dr \right)^{\frac{qc_2}{p\rho_1} - p} \right] \quad \text{and} \quad \mathcal{I}_2 := \mathbb{E} [L_2(t)^q]. \]

From (2.6) we get immediately that
\[ \mathcal{I}_2 \leq 2k_{2,q}(t) \int_0^{+\infty} \left( 1 + b_2 ye^{-\sigma_2 z} K_{2,q}(t) \right)^{-q} N_0(t)(z) dz. \]

Now, mimicking the proof of Proposition 2.2 we can write
\[
\mathcal{I}_1 = k_{1,p}(t) \mathbb{E} \left[ e^{\rho_1 B_1(t) - p^2 \sigma_1^2 t/2} \left( 1 + b_1 \int_0^t G_1(r) dr \right)^{\frac{qc_2}{p\rho_1} - p} \right] \\
= k_{1,p}(t) \mathbb{E} \left[ \left( 1 + b_1 \int_0^t G_1(r) e^{\rho_1 r} dr \right)^{\frac{qc_2}{p\rho_1} - p} \right] \\
= k_{1,p}(t) \left\| 1 + b_1 \int_0^t G_1(r) e^{\rho_1 r} dr \right\|_{\frac{qc_2}{p\rho_1} - p}(\Omega).
\]

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Observe that the condition $\frac{qc_2}{b_1} - p \geq 1$ allows for the use of triangle and Minkowski’s inequalities for the norm of the space $L^{\frac{qc_2}{b_1} - p} (\Omega)$; therefore, we obtain

\[
\mathcal{I}_1 = k_{1,p}(t) \left\| 1 + b_1 \int_0^t G_1(r) e^{p_2 t^r} dr \right\|_{L^{\frac{qc_2}{b_1} - p} (\Omega)}^{\frac{qc_2}{b_1} - p} \\
\leq k_{1,p}(t) \left( 1 + b_1 \left\| \int_0^t G_1(r) e^{p_2 t^r} dr \right\|_{L^{\frac{qc_2}{b_1} - p} (\Omega)}^{\frac{qc_2}{b_1} - p} \right)^{\frac{qc_2}{b_1} - p} \\
\leq k_{1,p}(t) \left( 1 + b_1 \int_0^t \| G_1(r) \|_{L^{\frac{qc_2}{b_1} - p} (\Omega)}^{\frac{qc_2}{b_1} - p} e^{p_2 t^r} dr \right)^{\frac{qc_2}{b_1} - p} \\
= k_{1,p}(t) \left( 1 + b_1 x e^{\left( a_1 + \frac{qc_2}{b_1} + p - 1 \right)} \left( \frac{qc_2}{b_1} - p \right) \right).
\]

Combining the estimates for $\mathcal{I}_1$ and $\mathcal{I}_2$ we obtain

\[
\mathbb{E} [X(t)^p Y(t)^q] \leq 2 k_{1,p}(t) k_{2,q}(t) \left( 1 + b_1 x e^{\left( a_1 + \frac{qc_2}{b_1} + p - 1 \right)} \left( \frac{qc_2}{b_1} - p \right) \right) \\
\times \int_0^{+\infty} (1 + b_2 ye^{-\sigma z K_2(t)})^{-q} N_{\alpha,2}(z)dz.
\]

2. From (3.9) and (3.11) we can write

\[
\mathbb{E} [X(t)^p Y(t)^q] \geq e^{-p_1 t} \mathbb{E} [L_1(t)^p L_2(t)^q] \\
= e^{-p_1 t} \mathbb{E} [L_1(t)^p] \mathbb{E} [L_2(t)^q].
\]

Inequality (2.7) completes the proof of (4.2).

3. The lower bound (4.1) is obtained setting $p = 0$ in (4.2); to prove the lower bound (4.3) we observe that

\[
X(t) \geq L_1(t) e^{-\frac{\sigma_1}{\sigma_2} \int_0^t G_1(v) dv} e^{\frac{\sigma_2}{\sigma_1} \ln(1 + b_2 f_0^t G_2(r) dr)} \\
= \frac{G_1(t) e^{-\frac{\sigma_1}{\sigma_2} \int_0^t G_1(v) dv}}{1 + b_1 f_0^t G_1(r) dr} \\
\geq \frac{G_1(t) e^{-\frac{\sigma_1}{\sigma_2} \int_0^t G_1(v) dv}}{1 + b_1 K_1(t) e^{\sigma_1 M_1(t)} \ln(1 + b_2 K_2(t) e^{\sigma_2 M_2(t)})} \\
= \frac{k_1(t) e^{-\frac{\sigma_1}{\sigma_2} \int_0^t G_1(v) dv}}{1 + b_1 K_1(t) e^{\sigma_1 M_1(t)}}.
\]

(4.5)
The last member above is a function of the three dimensional random vector $(B_1(t), M_1(t), M_2(t))$ whose joint probability density function is given by

\[
\int_{B_1(t), M_1(t), M_2(t)} f(u_1, v_1, v_2) = \begin{cases} 
-4 \mathcal{N}_0, t(2v_1 - u_1) \mathcal{N}_0, t(v_2), & \text{if } v_1 > 0, u_1 < v_1 \text{ and } v_2 > 0, \\
0, & \text{otherwise.}
\end{cases}
\]

Therefore, for any $p \geq 0$ we get

\[
\mathbb{E}[X(t)^p] \geq \mathbb{E} \left[ k_1(t) e^{\sigma_1 B_1(t)} \frac{(1+b_1 K_1(t) e^{\sigma_1 M_1(t)}) e^{\sigma_2 M_2(t)}}{1 + b_1 K_1(t) e^{\sigma_1 M_1(t)}} \right]^p
\]

\[
= -4 k_1(t)^p \int_A \frac{(1 + b_1 K_1(t) e^{\sigma_1 u_1})^p}{(1+b_1 K_1(t) e^{\sigma_1 u_1})^p} \mathcal{N}_0, t(2v_1 - u_1) \mathcal{N}_0, t(v_2) du_1 dv_1 dv_2,
\]

where

\[
A := \{ (u_1, v_1, v_2) \in \mathbb{R}^3 : v_1 > 0, u_1 < v_1, v_2 > 0 \}.
\]

This proves (4.3).

\[ \square \]

**Remark 4.2.** Due to the complexity of the left hand side in (3.16) we were not able to obtain a lower bound for the joint moments $\mathbb{E}[X(t)^p Y(t)^q]$ in the regime $a_1 < \phi$. However, according to the argument of Remark 3.3 inequality (3.3) can be utilize also in that regime.

## 5 Third main theorem: bounds for the distribution functions

The last main theorem of this paper concerns with upper and lower estimates for the distribution functions of $X(t)$ and $Y(t)$.

**Theorem 5.1.** Let $\{(X(t), Y(t))\}_{t \geq 0}$ be the unique global strong solution of (1.3). Then, for all $t \geq 0$ and $z_1, z_2 > 0$ we have the following bounds:

1. \[
\mathbb{P}(X(t) \leq z_1) \geq -2 \int_{\mathcal{N}_0, t(2v - u) du dv}, \quad (5.1)
\]

and

\[
\mathbb{P}(Y(t) \leq z_2) \geq -\frac{4 \beta b_1}{\sigma_1 c_2} \int_0^{z_2/(1+b_1 K_1(t))} \left( \int_{\mathcal{N}_0, t(u - 2v) du} \right)
\]

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\[ \times \mathcal{N}_{0,t} \left( \frac{1}{\sigma_1} \ln \left( \frac{\left( \frac{b_1 \sigma_1}{\sqrt{2}} \right) \frac{e^{b_1 \sigma_1}}{b_1 K_1(t)}}{b_1 K_1(t)} \right) \right) \frac{1}{\left( \frac{\frac{b_1 \sigma_1}{\sqrt{2}}}{\sqrt{2}} \right) - 1} d\zeta; \quad (5.2) \]

2. if \( a_1 > \phi \), then
\[
\mathbb{P}(X(t) \leq z_1, Y(t) \leq z_2) \leq 4 \int \frac{1}{1 + b_1 K_1(t) e^{\sigma_1 v_1}} \mathcal{N}'_{0,t}(2v-u) du dv \\
\times \int \frac{1}{1 + b_2 K_2(t) e^{\sigma_2 v_2}} \mathcal{N}'_{0,t}(2v-u) du dv;
\]
(5.3)

3. if \( a_1 < \phi \), then
\[
\mathbb{P}(X(t) \leq z_1) \leq -4 \int_{A_{z_1} \cap \{v_1 > 0, u_1 < v_1, v_2 > 0\}} \mathcal{N}'_{0,t}(2v-u_1) \mathcal{N}_0(t)(v_2) du_1 dv_1 dv_2, \quad (5.4)
\]

where
\[
A_{z_1} := \left\{ (u_1, v_1, v_2) \in \mathbb{R}^3 : \frac{k_1(t) e^{\sigma_1 u_1 - \frac{c_1}{\sqrt{2}} (1 + b_1 K_1(t) e^{\sigma_1 v_1}) \frac{e^{\frac{\sigma_1}{\sqrt{2}} \ln(1 + b_2 K_2(t) e^{\sigma_2 v_2})}}{1 + b_1 K_1(t) e^{\sigma_1 v_1}} \leq z_1 \right\},
\]

and
\[
\mathbb{P}(Y(t) \leq z_2) \leq -2 \int \frac{1}{1 + b_2 K_2(t) e^{\sigma_2 v_2}} \mathcal{N}'_{0,t}(2v-u) du dv. \quad (5.5)
\]

Here,
\[
k_1(t) := xe^{(a_1 - \sigma_1^2/2)t}, \quad \text{and} \quad K_1(t) := xe^{(a_1 - \sigma_1^2/2)t} - 1,
\]
while
\[
k_2(t) := ye^{(a_2 - \sigma_2^2/2)t}, \quad \text{and} \quad K_2(t) := ye^{(a_2 - \sigma_2^2/2)t} - 1.
\]

Proof. 1. The upper bound in (3.10) (or (3.11)) yields
\[
\mathbb{P}(X(t) \leq z_1) \geq \mathbb{P}(L_1(t) \leq z_1)
\]
which in combination with (2.9) gives (5.1). We now prove (5.2); the estimate
\[
\int_0^t G_1(r) dr \leq K_1(t) e^{\sigma_1 M_1(t)},
\]

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together with the upper estimate in (3.9), entails

\[
P(Y(t) \leq z_2) \geq \mathbb{P}\left( L_2(t) \left( 1 + b_1 \int_0^t G_1(r)dr \right)^{\frac{\beta_2}{\beta b_1}} \leq z_2 \right)
\]

\[
= \mathbb{E}\left[ \mathbb{P}\left( L_2(t) \left( 1 + b_1 \int_0^t G_1(r)dr \right)^{\frac{\beta_2}{\beta b_1}} \leq \frac{z_2}{L_2(t)} \right) | \mathcal{F}_t^2 \right]
\]

\[
= \mathbb{E}\left[ \mathbb{P}\left( \int_0^t G_1(r)dr \leq \left( \frac{z_2}{L_2(t)} \right)^{\frac{\beta b_1}{\beta_2}} - 1 \right) / b_1 | \mathcal{F}_t^2 \right]
\]

\[
\geq \mathbb{E}\left[ \mathbb{P}\left( K_1(t)e^{\sigma_1 M_1(t)} \leq \left( \frac{z_2}{L_2(t)} \right)^{\frac{\beta b_1}{\beta_2}} - 1 \right) / b_1 | \mathcal{F}_t^2 \right]
\]

\[
= \mathbb{E}\left[ \mathbb{P}\left( M_1(t) \leq \frac{1}{\sigma_1} \ln \left( \frac{\left( \frac{z_2}{L_2(t)} \right)^{\frac{\beta b_1}{\beta_2}} - 1}{b_1 K_1(t)} \right) | \mathcal{F}_t^2 \right) \right].
\]

Here \( \{\mathcal{F}_t^2\}_{t \geq 0} \) denotes the natural augmented filtration of the Brownian motion \( \{B_2(t)\}_{t \geq 0} \). Note that the almost sure positivity of the random variable \( M_1(t) \) implies that the probability in the last member above is different from zero if and only if

\[
\frac{\left( \frac{z_2}{L_2(t)} \right)^{\frac{\beta b_1}{\beta_2}} - 1}{b_1 K_1(t)} > 1
\]

which is equivalent to say that

\[
L_2(t) \leq \frac{z_2}{(1 + b_1 K_1(t))^{\frac{\beta_2}{\beta b_1}}}
\]

Therefore,

\[
P(Y(t) \leq z_2) \geq \mathbb{E}\left[ \mathbb{P}\left( M_1(t) \leq \frac{1}{\sigma_1} \ln \left( \frac{\left( \frac{z_2}{L_2(t)} \right)^{\frac{\beta b_1}{\beta_2}} - 1}{b_1 K_1(t)} \right) | \mathcal{F}_t^2 \right) \right]
\]

\[
= \int_0^{z_2/(1 + b_1 K_1(t))^{\frac{\beta_2}{\beta b_1}}} \mathbb{P}\left( M_1(t) \leq \frac{1}{\sigma_1} \ln \left( \frac{\left( \frac{z_2}{L_2(t)} \right)^{\frac{\beta b_1}{\beta_2}} - 1}{b_1 K_1(t)} \right) \right) dF_2(\zeta),
\]

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where $F_2$ denotes the distribution function of the random variable $L_2(t)$. We now integrate by parts and notice that

$$
P(Y(t) \leq z_2) \geq \int_0^{z_2/(1+b_1K_1(t))} \left( \frac{1}{\sigma_1} \ln \left( \frac{\beta b_1}{b_1K_1(t)} \right) \right) dF_2(\zeta)
$$

Moreover, since from (2.9) we know that

$$
F_2(\zeta) = \mathbb{P}(L_2(t) \leq \zeta) \geq -2 \int_{\{y \geq \delta_{k_2(t)c_2N_1(t)} \}} N_{0,t}(u-2v) dudv,
$$

we can conclude that

$$
P(Y(t) \leq z_2) \geq \frac{2\beta b_1}{\sigma_1 c_2} \int_0^{z_2/(1+b_1K_1(t))} F_2(\zeta) N_{0,t}\left( \frac{1}{\sigma_1} \ln \left( \frac{\beta b_1}{b_1K_1(t)} \right) \right) \frac{1}{\zeta^2} d\zeta.
$$

2. Using the lower bounds in (3.9) and (3.11) we obtain

$$
\mathbb{P}(X(t) \leq z_1, Y(t) \leq z_2) \leq \mathbb{P} \left( (L_1(t)e^{-c_1t} \leq z_1, L_2(t) \leq z_2) \right)
$$

$$
= \mathbb{P} \left( (L_1(t)e^{-c_1t} \leq z_1) \mathbb{P}(L_2(t) \leq z_2) \right)
$$

$$
= \mathbb{P} \left( (L_1(t) \leq z_1e^{c_1t}) \mathbb{P}(L_2(t) \leq z_2) \right).
$$

With the help of (2.8) we conclude that

$$
P(X(t) \leq z_1, Y(t) \leq z_2) \leq 4 \int_{\{y \geq \delta_{k_2(t)c_2N_1(t)} \}} N_{0,t}(2v-u) dudv
$$

$$
\times \int_{\{y \geq \delta_{k_2(t)c_2N_1(t)} \}} N_{0,t}(2v-u) dudv.
$$
3. We now prove (5.4); we know from (3.10) and (4.5) that

\[ X(t) \geq L_1(t)e^{-\frac{c_2}{b_2}(1+b_1K_1(t)e^{\sigma_1B_1(t)})} \frac{\int_0^t G_1(v)dv}{1+b_1K_1(t)e^{\sigma_1B_1(t)}} \leq c_2^{\beta}b_1^{\beta} \ln(1+b_2K_2(t)e^{\sigma_2B_2(t)}) \]

Hence, we can write

\[ \mathbb{P}(X(t) \leq z_1) \leq \mathbb{P}\left( \frac{k_1(t)e^{\sigma_1B_1(t)}-\frac{c_2}{b_2}(1+b_1K_1(t)e^{\sigma_1B_1(t)})}{1+b_1K_1(t)e^{\sigma_1B_1(t)}} \leq z_1 \right) \]

\[ = -4 \int_{A_{z_1}(v_1)>0, u_1<v_1, v_2>0} \mathcal{N}_0(t)du_1dv_1dv_2, \]

where

\[ A_{z_1} := \left\{ (u_1, v_1, v_2) \in \mathbb{R}^3 : \frac{k_1(t)e^{\sigma_1u_1}-\frac{c_2}{b_2}(1+b_1K_1(t)e^{\sigma_1u_1})}{1+b_1K_1(t)e^{\sigma_1u_1}} \leq z_1 \right\} , \]

This coincides with (5.4). Moreover, from the lower estimate in (3.9) we get

\[ \mathbb{P}(Y(t) \leq z_2) \leq \mathbb{P}(L_2(t) \leq z_2); \]

inequality (2.8) completes the proof of (5.5). 

\[ \square \]

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