On the nature of incompressible magnetohydrodynamic turbulence

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A novel model of incompressible magnetohydrodynamic turbulence in the presence of a strong external magnetic field is proposed for explanation of recent numerical results. According to the proposed model, in the presence of the strong external magnetic field, incompressible magnetohydrodynamic turbulence becomes nonlocal in the sense that low frequency modes cause decorrelation of interacting high frequency modes from the inertial interval. It is shown that the obtained nonlocal spectrum of the inertial range of incompressible magnetohydrodynamic turbulence represents an anisotropic analogue of Kraichnan’s nonlocal spectrum of hydrodynamic turbulence. Based on analysis performed in the framework of the weak coupling approximation, which represents one of the equivalent formulations of the direct interaction approximation, it is shown that incompressible magnetohydrodynamic turbulence could be both local and nonlocal and therefore anisotropic analogues of both the Kolmogorov and Kraichnan spectra are realizable in incompressible magnetohydrodynamic turbulence.

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I. INTRODUCTION

Although magnetohydrodynamic (MHD) turbulence has been extensively studied for the last 40 years [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], many physical aspects of the problem still remain unclear (for recent reviews see Refs. [19, 20, 21, 22]). The first model of incompressible MHD turbulence was proposed by Iroshnikov [1] and Kraichnan [2]. The Iroshnikov-Kraichnan (IK) model of MHD turbulence is based on the so-called Alfvén effect [19] - nonlinear interaction is possible only among Alfvén waves propagating in opposite directions along the mean magnetic field. Therefore, IK model assumes that the energy cascade in MHD turbulence occurs as a result of collisions between oppositely propagating Alfvén waves. Consider isotropic excitation of Alfvén waves on some outer scale \( L_0 \) with a characteristic velocity \( v_0 \ll V_A \), where \( V_A \) is the Alfvén velocity. IK model assumes that the energy transfer is local and isotropic in the wave number space. The characteristic time scale of the Alfvén wave collision is \( t_{\text{IK}} \sim (V_A k)^{-1} \), where \( k \) is the wave number. Using the governing equations of incompressible MHD it can be shown that during one collision distortion of each wave packet \( \delta \psi \) is of order

\[
\frac{\delta \psi}{\psi} \sim \frac{v_0}{V_A} \ll 1.
\]

Because these perturbations are summed with random phases \( N \sim (\psi/\delta \psi)^2 \sim (V_A/\psi)^2 \) collisions are necessary to achieve the distortion of order unity. Therefore for the energy cascade time \( \tau_{\text{cas}}^{\text{IK}} \) we have

\[
\tau_{\text{cas}}^{\text{IK}} \sim 1 \frac{V_A}{k \psi} \psi
\]

Taking into account the relations \( \varepsilon \sim v_0^2/\tau_{\text{cas}}, v_0^2 \sim k E_k \), where \( \varepsilon \) is the energy cascade rate and \( E_k \) is the one dimensional energy spectrum, we obtain

\[
\frac{v_1}{v_0} \sim \left( \frac{l_0}{l_q} \right)^{1/4}, \quad E_k^{\text{IK}} \sim (\varepsilon V_A)^{1/2} k^{-3/2},
\]

which represents IK spectrum of incompressible MHD turbulence.

IK model of the MHD turbulence is isotropic. However, presence of a mean magnetic field has a strong effect on the turbulence properties, in contrast to a mean flow in hydrodynamic turbulence, which can be eliminated by a Galilean transformation. The anisotropy of MHD turbulence had been seen in various numerical simulations [2, 3, 6, 11, 13]. A theory of anisotropic MHD turbulence was proposed by Goldreich and Sridhar (GS) [3]. GS model implies that the dynamics of turbulence is dominated by the perpendicular cascade (i.e., by the cascade in the \( \mathbf{q} \)-space, where \( \mathbf{q} \) is the component of the wave number vector perpendicular to the mean magnetic field), whereas the parallel size of turbulent 'eddies' (wave packets) is determined by critical balance condition which implies that the characteristic time scale of wave collision \( (p V_A)^{-1} \), where \( p \) is parallel wave number, is equal to the characteristic time scale the energy cascade \( \tau_{\text{cas}}^{\text{GS}} \). GS model also assumes that the energy cascade is local in the wave number space and therefore \( \tau_{\text{cas}}^{\text{GS}} \sim (v_0 q)^{-1} \). This yields (we suppose that turbulence is generated at perpendicular scales \( l_{q_0} \sim 1/l_0 \) with a characteristic velocity \( v_0 \))

\[
\frac{v_1}{v_0} \sim \left( \frac{l_q}{l_{q_0}} \right)^{1/3}, \quad E_k^{\text{GS}} \sim \varepsilon^{2/3} q^{-5/3},
\]

which represents the anisotropic analogue of the Kolmogorov spectrum of isotropic hydrodynamic turbulence. The anisotropy of fluctuations is determined by the critical balance principle, which yields \( p \sim q^\mu \), with \( \mu = 2/3 \), and therefore as the cascade proceeds to larger \( q \), the
eddies become more elongated along the direction of the mean magnetic field.

However, GS model does not totally agree with the results of the recent numerical simulations [11, 13, 14]. It has been found [14] that in the presence of the strong external magnetic field incompressible MHD turbulence is strongly anisotropic with $\mu \approx 1/2$ and has the one dimensional spectrum similar to IK one $\varepsilon_k \sim q^{-3/2}$. As it was shown in Ref. [11] the anisotropic spectrum of incompressible MHD turbulence strongly depends on the strength of the external magnetic field. It was found that the perpendicular scaling of fluctuations changed from $\varepsilon_{\perp} \sim q^{-1} \rho v^2_0$ to $\varepsilon_{\perp} \sim q^{-3/2}$. Therefore, anisotropic IK model gives $\varepsilon_{\perp} \sim q^{-1} \rho v^2_0$, so that the perpendicular cascade timescale becomes

$$
\tau_{\text{cas}} \sim \frac{1}{k_{\text{in}} q_{\text{in}}}, \quad (5)
$$

then for the one dimensional spectrum one obtains $E_k^{\text{AIK}} \sim (c V_A)^{1/2} q^{-3/2}$. This model of MHD turbulence represents an anisotropic analogue of IK model. The characteristic parallel scale of the turbulent eddies is determined by the condition similar to GS critical balance condition $1/p \sim V_A \tau_{\text{cas}}$, which yields $p \sim q^{1/2}$.

Although anisotropic IK model properly represents scaling indices of incompressible MHD turbulence in the presence of the strong external magnetic field, it still contradicts with some other features of the turbulence observed in the numerical simulations. Indeed, according to Eq. (5), the ratio of the cascade timescale to the eddy turnover time $\tau_{\text{to}} \sim (k V_A)^{-1}$ is $\tau_{\text{cas}}/\tau_{\text{to}} \sim v_l/V_A$. For the parameters used in Ref. [9] the ratio $v_l/V_A \sim 10^{-2} - 10^{-3}$, whereas the ratio $\tau_{\text{cas}}/\tau_{\text{to}} \sim 1$ (see Sec. 6.6.3 of Ref. [9]). Therefore, anisotropic IK model gives strongly underestimated estimate for the characteristic cascade rate.

In this paper we present a novel model of incompressible MHD turbulence. The main assumption of the model is the nonlocal character of MHD turbulence in the presence of a strong external magnetic field. The model implies that low frequency modes cause decorrelation of interacting high frequency modes from the inertial interval. Proposed model represents an anisotropic analogue of Kraichnan’s nonlocal spectrum of hydrodynamic turbulence [23] and properly reproduces main features of incompressible MHD turbulence observed in the numerical simulations.

Our study of incompressible MHD turbulence is performed in the framework of Kadomtsev’s weak coupling approximation (WCA) [24], which represents one of the equivalent formulations of Kraichnan’s direct interaction approximation (DIA) [23]. Our analysis does not allow us to prove that in the presence of a strong external magnetic field incompressible MHD turbulence becomes nonlocal. However, performed analysis shows that for the systems with more than one degrees of freedom the concept of adiabatic and resonant interactions does not hold and consequently, in contrast to isotropic hydrodynamic turbulence which is local, incompressible MHD turbulence could be both local and nonlocal. Obtained local solution reproduces GS model of MHD turbulence, whereas obtained nonlocal solution represents an anisotropic analogue of Kraichnan’s nonlocal spectrum of hydrodynamic turbulence.

The paper is organized as follows. The heuristics of the proposed model is presented in Sec. II. Physics of the decorrelation mechanism is discussed in Sec. III. The WCA equations of incompressible MHD turbulence are derived in Sec. IV. Analysis of the WCA equations of isotropic hydrodynamic turbulence is given in Sec. V. Local as well as nonlocal solutions of the WCA equations of incompressible MHD turbulence are obtained in Sec. VI. Locality of incompressible MHD turbulence is discussed in Sec. VII. Conclusions are given in Sec. IX.

II. HEURISTICS OF THE MODEL

Kraichnan obtained nonlocal spectrum of hydrodynamic turbulence as a possible inertial range solution of the Eulerian DIA equations [23]. Kraichnan’s nonlocal spectrum of isotropic hydrodynamic turbulence can be derived based on the following qualitative arguments [25]: assume that the autocorrelation time scale is determined by advection of a small-scale eddies (from the inertial range) of the size $k^{-1}$ through a distance of the order of its own size by a macroscopic flow of rms velocity which is determined by the characteristic velocity of energy containing eddies $v_0$, i.e., assume that $\tau_{\text{ac}} \sim 1/(v_0 k)$. Then the energy transfer rate reduces by the factor $\tau_{\text{ac}} \tau_{\text{to}}$ compared to the Kolmogorov theory (here $\tau_{\text{to}} \sim 1/(v_0 k)$ is the characteristic eddy turnover timescale) and therefore the characteristic timescale of the turbulent cascade becomes

$$
\tau_{\text{cas}} \sim \frac{1}{k_{\text{in}} l_{\text{in}}}, \quad (6)
$$

For the velocity fluctuations and the one dimensional energy spectrum one obtains $v_l \sim l^{1/4}$ and $\varepsilon_k \sim (v_0)^{1/2} k^{-3/2}$, respectively. This represents Kraichnan’s nonlocal spectrum of isotropic hydrodynamic turbulence [23].

As it was proposed by Richardson [26] and confirmed by various experiments (see, e.g., Refs. [27, 28] and references therein) the energy cascade in isotropic hydrodynamic turbulence is local and therefore the Kolmogorov spectrum is the only one that is realized in the inertial range of isotropic hydrodynamic turbulence.

We propose that in contrast to hydrodynamic turbulence, incompressible MHD turbulence is non-local in the
presence of the strong ($\gamma \gg 1$) external magnetic field, i.e., an influence of low frequency modes leads to decorrelation of interacting high frequency modes from the inertial interval. Physics of the decorrelation mechanism is discussed in the next section. We argue that $E_q \sim q^{-3/2}$ one dimensional spectrum observed in the numerical simulations represents the anisotropic analogue of the Kraichnan nonlocal spectrum. In particular, similar to GS model, presented model implies that incompressible energy transfer is strongly anisotropic and dominated by perpendicular cascade. The later cascade is assumed to be nonlocal, and similar to Eq. (6) we have for the characteristic cascade timescale

$$\tau_{cas} \sim \frac{1}{kv_0} \frac{v_0}{v_l}.$$  

for the velocity fluctuations and the one dimensional energy spectrum our model implies

$$\frac{v_l}{v_0} \sim \left(\frac{l_q}{l_{q0}}\right)^{1/4}, \quad E_q \sim (\varepsilon v_0)^{1/2} q^{-3/2}.$$  

The relation between the parallel and perpendicular sizes of turbulent eddies is determined based on the following arguments [9]: the lifetime of a wave packet is of order $\tau_{cas}$ and therefore for the correlation lengthscale along the field line we have

$$l_p \sim 1/p \sim \tau_{cas} V_A.$$  

Using Eq. (7) this yields

$$p \sim \frac{v_0}{V_A} q_0^{1/2} q_1^{1/2}.$$  

Presented model properly reproduces main characteristics of incompressible MHD turbulence observed in the numerical simulations [9] [11].

Explicit dependence of the cascade rates given by Eqs. (6) and (7) on the characteristic velocity of low frequency energy containing modes represents important feature of Kraichnan’s nonlocal spectrum as well as considered model and underlines the nonlocal character of the turbulence.

### III. PHYSICS OF THE DECORRELATION MECHANISM

The governing equations of incompressible magnetohydrodynamics in the Elsasser variables are

$$\partial_t U = -(W \cdot \nabla) U - \nabla P + \bar{\nu} \nabla^2 U + \bar{\nu}_m \nabla^2 W,$$  

$$\partial_t W = -(U \cdot \nabla) W - \nabla P + \bar{\nu}_m \nabla^2 U + \bar{\nu} \nabla^2 W,$$  

$$\nabla \cdot W = \nabla \cdot U = 0,$$  

where $U = v + b$ and $W = v - b$ are the Elsasser variables, $v$ is the velocity field, $b \equiv B/\sqrt{4\pi \rho}$ is the magnetic field in velocity units, $P$ is the total (hydrodynamic plus magnetic) pressure normalized by the density, $\partial_t \equiv \partial / \partial t$, $\bar{\nu} \equiv (\nu + \nu_m)/2$, $\bar{\nu}_m \equiv (\nu - \nu_m)/2$, $\nu$ is the kinematic viscosity and $\nu_m$ is the magnetic diffusivity.

Consider the packet of Alfvén waves propagating along the uniform magnetic field $\bar{b}_0 \parallel z$, with the characteristic magnetic field perturbation $\bar{b}_0 \perp z$, the velocity perturbation $v_0 = -\bar{b}_0$ and the characteristic parallel and perpendicular wave numbers $p_0$ and $q_0$, respectively. On this background consider a high frequency wave packet propagating in the opposite direction with the characteristic parallel and perpendicular wave numbers $p_u \gg p_0$ and $q_u \gg q_0$, respectively. Denote the Elsasser variable associated with the high frequency wave packet by $u_u$.

Assuming $\bar{b}_0$ as approximately constant and dropping dissipation terms Eq. (11) yields

$$(\partial_t - \bar{b}_0 \cdot \nabla - \bar{b}_0 \cdot \nabla + v_0 \cdot \nabla) u_u \approx 0,$$  

Noting that $v_0 = -\bar{b}_0$, this equation gives for characteristic frequencies of the high frequency packet

$$\omega_u \approx \bar{b}_0 p_u + 2v_0 q_u.$$  

If we consider dynamics of a high frequency Alfvén wave packet propagating parallel to z-axis (with the characteristic parallel and perpendicular wave numbers $p_u \sim p_u \sim p$ and $q_u \sim q_u \sim q$ respectively), then similar consideration yields for the frequency

$$\omega_u \approx \bar{b}_0 p_u.$$  

Therefore, presence of the low frequency wave packet do not influence propagation of the high frequency packet along z-axis (Alfvén effect) whereas the packet propagating in the opposite direction is moved with the velocity $2v_0$ in the direction perpendicular to z-axis. This circumstance represents the bases for understanding, why incompressible MHD turbulence becomes nonlocal in the presence of the strong external magnetic field. Indeed, consider two interacting packets of high frequency Alfvén waves moving in opposite directions on the same background as above (i.e., in the presence of strong external magnetic field and the low frequency Alfvén wave packet). The perpendicular length scale of the packets is of order $1/q$ and consequently, due to the described above action of the low frequency packet, the interacting packets would be pulled apart during the time scale $1/(q v_0)$. Therefore, the characteristic timescale of the unit act of interaction shortens compared to the Kolmogorov autocorrelation timescale $1/(q v_0)$. As it was shown in the previous section this automatically implies that the turbulence would have the spectral characteristics given by Eqs. (7)-(11).

In the consideration above we implicitly assumed that the low frequency packet do not contribute to the mean magnetic field which acts on the interacting high frequency wave packets (i.e., it has been assumed that the
anisotropic high frequency wave packets are formed along the external magnetic field \( \mathbf{b}_0 \). If this is not the case then the situation changes entirely. Indeed, assume that the low frequency wave packet contributes to the mean field acting on high frequency packets, i.e., assume that the high frequency packets are formed along the axis \( \mathbf{b}_0 + b_0 \) instead of \( \mathbf{b}_0 \). In this case, instead of Eqs. (17)-(18) similar analysis yield

\[
\omega_u \approx -(b_0 + b_0) p_u + v_0 \cdot q_u, \\
\omega_w \approx (b_0 + b_0) p_w + v_0 \cdot q_w.
\]

The second terms in Eqs. (17)-(18) describes the Doppler shift caused by the velocity field of the low frequency wave packet and could be removed by corresponding Galilean transformation. Consequently, in this case the only effect caused by the presence of the low frequency modes is the change of the mean field, and they do not cause the decorrelation of high frequency modes. This arguments lead us to the conclusion that only the low frequency modes which do not contribute to the mean field cause decorrelation of interacting high frequency packets.

Consequently, presented model implies that in the presence of the strong external magnetic field \( b_0 \gg b_0 \) the low frequency energy containing modes do not contribute to the mean field which acts on the high frequency fluctuations in the inertial interval. As a result, decorrelation mechanism described above causes formation of the anisotropic analogue of the Kraichnan nonlocal spectrum. On the other hand, in the absence of the strong external magnetic field, the mean field is formed by the low frequency (energy containing) modes, the nonlocal decorrelation mechanism is not at work, turbulence is local and therefore is described by GS model.

IV. THE WCA EQUATIONS FOR INCOMPRESSIBLE MHD TURBULENCE

Consider incompressible MHD turbulence in the presence of the constant magnetic field \( \mathbf{b}_0 \) directed along \( z \) axis. Presenting the Elsasser variables as a sum of the mean and fluctuating parts \( \mathbf{W} = \mathbf{W}_0 + \mathbf{W}_1, \mathbf{U} = \mathbf{U}_0 + \mathbf{U}_1, \) where \( \mathbf{U}_0 = \mathbf{V}_A \) and \( \mathbf{W}_0 = -\mathbf{V}_A, \) and \( \mathbf{V}_A = \mathbf{b}_0/\sqrt{4\pi p} \) is the Alfvén velocity, and dropping dissipation terms Eqs. (11)-(13) yield

\[
\partial_t \mathbf{U}_1 - \mathbf{V}_A \partial_z \mathbf{U}_1 = -(\mathbf{W}_1 \cdot \nabla) \mathbf{U}_1 - \nabla p \\
\partial_t \mathbf{W}_1 + \mathbf{V}_A \partial_z \mathbf{W}_1 = -(\mathbf{U}_1 \cdot \nabla) \mathbf{W}_1 - \nabla p
\]

Performing the Fourier transform defined as

\[
\mathbf{u}_k = \frac{1}{(2\pi)^d} \int \exp(i\mathbf{k} \cdot \mathbf{x}) \mathbf{u}_1(x,t) d^dx dt,
\]

and eliminating pressure terms we obtain

\[
(\omega - \omega_k)\mathbf{u}_k = \int [\mathbf{u}_1 - \mathbf{k}(\mathbf{k} \cdot \mathbf{u}_1)](\mathbf{k} \cdot \mathbf{w}_2) d\mathcal{F}_{1,2}^k, \quad (22)
\]

where \( \mathbf{k} \equiv (k,\omega) \), the caret denotes the unit vector, \( \mathbf{u}_1 \) denotes \( \mathbf{u}_{k_1} \), \( \omega = V_A k_z \) is the frequency of the Alfvén wave, \( d\mathcal{F}_{1,2}^k = d^d k_1 d^d k_2 \delta(k_1 - k_1 - k_2) \) and \( \delta(k_1 - k_1 - k_2) = \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \) is the Dirac delta function.

Incompressible MHD turbulence is governed by interaction of shear Alfvén waves, whereas pseudo Alfvén waves play a passive role \[5\]. \[7\]. Therefore, in the presented paper we consider the shear Alfvénic turbulence. Defining the unit polarization vector of the shear Alfvén waves \( \mathbf{e}_k = \mathbf{k} \times \mathbf{z} \), and introducing the amplitudes of the shear Alfvén waves as

\[
\mathbf{w}_k = i\phi_k \mathbf{e}_k, \quad \mathbf{u}_k = i\psi_k \mathbf{e}_k, \quad (24)
\]

Eqs. (22)-(23) reduce to the following equations

\[
(i\omega - \omega_k)\phi_k = \int_{-\infty}^{\infty} T_{1,2} \phi_{2} \mathcal{F}_{1,2}^k, \quad (25)
\]

\[
(i\omega + \omega_k)\psi_k = \int_{-\infty}^{\infty} T_{1,2} \psi_{2} \mathcal{F}_{1,2}^k, \quad (26)
\]

where \( T_{1,2} = i(\mathbf{e}_k \cdot \mathbf{e}_{k_1})[(\mathbf{k} \cdot \mathbf{e}_k) (\mathbf{k} \cdot \mathbf{e}_{k_2})] \) is the matrix element of interaction.

To achieve any progress in analysis of Eqs. (24)-(26) some closure scheme should be used. In Ref. \[3\] the eddy damped quasi normal Markovian (EDQNM) approximation was used for the analysis of Eqs. (25)-(26). In the framework of this approximation the linear damping term is added to the equation for the third order moments of the turbulent fields and afterwards some assumptions are made regarding the eddy damping rate. But an assumption of some \( a \) \textit{priori} form of the eddy damping rate automatically fixes the property of the locality of turbulence - it assumed to be local or not. Note that locality of the turbulence is one of the main assumptions of GS model \[3\]. Here we use the WCA (DIA) for the study of locality of the incompressible MHD turbulence.

In the framework of the DIA \[22\] one differentiates direct nonlinear interactions among three Fourier modes with the complanar wave vectors \( \mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 = 0 \) and indirect interactions when the modes are interacting by means of other Fourier modes. The idea of the closure of equations for the turbulent fields is based on the assumption that the influence of the indirect interactions on the turbulence dynamics can be neglected in comparison with the direct interactions.

We use the standard WCA technique \[24\] for the derivation of the WCA equations of incompressible MHD turbulence. One of the main nonlinear effects described by the nonlinear interaction terms on the right hand sides of Eqs. (25)-(26) is the nonlinear decay of the mode \( \phi_k \).

The intensity of this process is proportional to the amplitude of the mode. The WCA implies the isolation of this part of the nonlinear interaction. For this purposes one should add to both sides of Eqs. (25)-(26) the
terms \(i\zeta_+^\pm \phi_\mathbf{k}\) and \(i\zeta_-^\pm \psi_\mathbf{k}\), respectively. Multiplying obtained equations respectively by \(\phi_\mathbf{k}^*\) and \(\psi_\mathbf{k}^*\) (here and hereafter asterisk denotes the complex conjugated value), ensemble averaging, considering different wave modes as statistically independent and introducing the following notations

\[
\langle \phi_\mathbf{k} \phi_\mathbf{k}^* \rangle = I_\mathbf{k}^\pm \delta_\mathbf{k} - \mathbf{k}', \quad \langle \psi_\mathbf{k} \psi_\mathbf{k}^* \rangle = I_\mathbf{k}^\pm \delta_\mathbf{k} + \mathbf{k}', \tag{27}
\]

we obtain

\[
(\omega - \omega_\mathbf{k} + i\zeta_+^\pm)I_\mathbf{k}^\pm \delta_\mathbf{k} - \mathbf{k}' = i\zeta_+^\pm I_\mathbf{k}^\pm \delta_\mathbf{k} - \mathbf{k}' + \int_{-\infty}^{\infty} T_{1,2}(\phi_1 \psi_2 \phi_\mathbf{k}^* + \int_{-\infty}^{\infty} T_{1,2}(\psi_1 \psi_2 \phi_\mathbf{k}^* dF_{1,2}^k, \tag{28}
\]

\[
(\omega + \omega_\mathbf{k} + i\zeta_-^\pm)I_\mathbf{k}^\pm \delta_\mathbf{k} - \mathbf{k}' = i\zeta_-^\pm I_\mathbf{k}^\pm \delta_\mathbf{k} + \mathbf{k}' - \int_{-\infty}^{\infty} T_{1,2}(\psi_1 \psi_2 \phi_\mathbf{k}^* dF_{1,2}^k. \tag{29}
\]

The second terms on the right hand sides of Eqs. \[28]-\[29\] contain contributions from both direct and indirect nonlinear interactions. Following the procedure developed in Ref. \[24\] for the elimination of the contribution of the indirect nonlinear interactions, let us represent the turbulent fields as

\[
\phi_\mathbf{k} = \phi_\mathbf{k}^{(0)} + \phi_\mathbf{k}^{(1)}, \quad \phi_\mathbf{k}^{(0)} \ll \phi_\mathbf{k}^{(1)}, \tag{30}
\]

\[
\psi_\mathbf{k} = \psi_\mathbf{k}^{(0)} + \psi_\mathbf{k}^{(1)}, \quad \psi_\mathbf{k}^{(0)} \ll \psi_\mathbf{k}^{(1)}, \tag{31}
\]

where in the zeroth approximation the fluctuations are assumed to be uncorrelated. Then the third order moments on right hand sides of Eqs. \[28]-\[29\] vanish in the zeroth order approximation. In the first order approximation we have

\[
\langle \phi_1 \psi_2 \phi_\mathbf{k}^* \rangle = \langle \phi_1 \psi_2 \phi_\mathbf{k}^{(0)*} \rangle + \langle \phi_1 \psi_2 \phi_\mathbf{k}^{(0)} \phi_\mathbf{k}^{(1)*} \rangle + \langle \phi_1 \psi_2 \phi_\mathbf{k}^{(0)} \phi_\mathbf{k}^{(1)} \phi_\mathbf{k}^{(1)*} \rangle. \tag{32}
\]

For \(\phi_\mathbf{k}^{(1)}\) Eq. \[25\] gives

\[
(\omega - \omega_\mathbf{k})\phi_\mathbf{k}^{(1)} = \int_{-\infty}^{\infty} T_{1,2}\phi_1^{(1)} \psi_2^{(0)} dF_{1,2}^k. \tag{33}
\]

To isolate the contribution of only direct nonlinear interactions we note, that the duration of the unit act of the nonlinear interaction is \(t_c^\pm \sim 1/\zeta_+^\pm\). Therefore, to hold contribution of only direct nonlinear interactions we should replace \((\omega - \omega_\mathbf{k})\) by \((\omega - \omega_\mathbf{k} + i\zeta_+^\pm)\) in Eq. \[33\]. This yields

\[
\phi_\mathbf{k}^{(1)} = \frac{1}{(\omega - \omega_\mathbf{k} + i\zeta_+^\pm)} \int_{-\infty}^{\infty} T_{1,2}\phi_1^{(1)} \psi_2^{(0)} dF_{1,2}^k. \tag{34}
\]

Similarly for \(\psi_\mathbf{k}^{(1)}\) we obtain

\[
\psi_\mathbf{k}^{(1)} = \frac{1}{(\omega + \omega_\mathbf{k} + i\zeta_-^\pm)} \int_{-\infty}^{\infty} T_{1,2}\psi_1^{(0)} \phi_2^{(0)} dF_{1,2}^k. \tag{35}
\]

Substituting Eqs. \[32\], \[34\] and \[35\] into Eqs. \[28]-\[29\], considering the fields \(\phi_\mathbf{k}\) and \(\psi_\mathbf{k}\) as uncorrelated (i.e., assuming \(\langle \phi_1 \psi_2 \rangle \approx 0\) \[3\]), which physically implies that we consider MHD turbulence with zero residual energy \[15\], \[16\] or equivalently with equal kinetic and magnetic energies), using for the forth order moments the Gaussian relations

\[
\langle \phi_3 \psi_4 \psi_2 \phi_\mathbf{k}^* \rangle = \langle \phi_3 \phi_\mathbf{k}^* \psi_4 \psi_2 \rangle = I_\mathbf{k}^\pm \delta_\mathbf{k} - \mathbf{k}_3 \delta_\mathbf{k}_2 + \mathbf{k}_4 \tag{36}
\]

\[
\langle \phi_1 \psi_3 \psi_4 \phi_\mathbf{k}^* \rangle = 0 \tag{37}
\]

and taking into account that according to the definition of \(\zeta_+^\pm\), the right hand sides of Eqs. \[28]-\[29\] should not contain the terms proportional to \(I_\mathbf{k}^\pm\) we finally arrive at the following equations

\[
i\zeta_+^\pm = - \int_{-\infty}^{\infty} \frac{T_{1,2}\mathbf{k}_-}{\omega_\mathbf{k} + i\zeta_+^\pm} dF_{1,2}^k \tag{37}
\]

\[
i\zeta_-^\pm = - \int_{-\infty}^{\infty} \frac{T_{1,2}\mathbf{k}_-}{\omega_\mathbf{k} + i\zeta_-^\pm} dF_{1,2}^k \tag{38}
\]

\[
(\omega - \omega_\mathbf{k} + i\zeta_+^\pm)^2 I_\mathbf{k}^\pm = \int_{-\infty}^{\infty} [T_{1,2}]^2 I_1^2 I_2^2 dF_{1,2}^k \tag{39}
\]

\[
(\omega + \omega_\mathbf{k} + i\zeta_-^\pm)^2 I_\mathbf{k}^\pm = \int_{-\infty}^{\infty} [T_{1,2}]^2 I_1^2 I_2^2 dF_{1,2}^k. \tag{40}
\]

Eqs. \[37]-\[40\] represent the WCA (equivalently the Eulerian DIA) equations for incompressible MHD turbulence. Noting that \(T_{1,2} = T_{k,-2}^*\) and defining

\[
\Gamma_k^\pm = \frac{i}{\omega_\mathbf{k} + i\zeta_+^\pm}, \tag{41}
\]

after straightforward manipulations Eqs. \[37]-\[40\] can be rewritten as

\[
-i(\omega - \omega_\mathbf{k})\Gamma_k^+ = 1 - \Gamma_k^+ \int_{-\infty}^{\infty} [T_{1,2}]^2 I_1^2 I_2^2 dF_{1,2}^k \tag{42}
\]

\[
-i(\omega + \omega_\mathbf{k})\Gamma_k^- = 1 - \Gamma_k^- \int_{-\infty}^{\infty} [T_{1,2}]^2 I_1^2 I_2^2 dF_{1,2}^k \tag{43}
\]

\[
-i(\omega - \omega_\mathbf{k})I_k^+ = \Gamma_k^+ \int_{-\infty}^{\infty} [T_{1,2}]^2 I_1^2 I_2^2 dF_{1,2}^k - I_1^k \int_{-\infty}^{\infty} [T_{1,2}]^2 I_1^2 I_2^2 dF_{1,2}^k \tag{44}
\]

\[
-i(\omega + \omega_\mathbf{k})I_k^- = \Gamma_k^- \int_{-\infty}^{\infty} [T_{1,2}]^2 I_1^2 I_2^2 dF_{1,2}^k - I_1^k \int_{-\infty}^{\infty} [T_{1,2}]^2 I_1^2 I_2^2 dF_{1,2}^k \tag{45}
\]

The aim of the further analysis is the study of Eqs. \[37]-\[40\], or equivalently, \[32]-\[35\] and similar equations for isotropic hydrodynamic turbulence. But before performing this analysis we shortly discuss several topics related to the nature of the obtained equations.
A. The equivalence of the WCA and the DIA

As it was mentioned above, the WCA is one of the equivalent forms of the DIA. Essentially the WCA is the DIA formulated in the frequency domain instead of the time domain [23, 29]. To perform the DIA analysis of incompressible MHD turbulence one should start from Eqs. (22)-(26) in the time domain (Eqs. (15) of Ref. [1]), add nonlinear forcing terms and define the Green functions $G^+(k, t - t')$ and $G^-(k, t - t')$ of the obtained equations. Then using standard assumptions of this approach (see, e.g., [23] and references therein) for separation of the direct and indirect interactions one should obtain the closed set of equations for functions $Q^+(k, t, t')$ and $Q^-(k, t, t')$ defined as

$$
\langle \tilde{\phi}_k(t) \tilde{\phi}_k(t') \rangle \equiv Q^+(k, t, t') \delta_{k-k'},
$$

(46)

and

$$
\langle \tilde{\phi}_k(t) \tilde{\phi}_k(t') \rangle \equiv Q^-(k, t, t') \delta_{k-k'},
$$

(47)

and the Green functions $G^\pm(k, t - t')$. In Eq. (46) $\tilde{\phi}_k(t)$ is the inverse Fourier transform of $\phi_k(t)$ with respect to $\omega$. For stationary turbulence $Q^\pm(k, t, t') = Q^\pm(k, t - t')$. Close analogy between the WCA and the DIA could be seen if one notes that according to Eqs. (27) and (46), $I_k^\pm$ are the Fourier transforms of $G^\pm(k, t - t')$ with respect to $t - t'$, whereas $\Gamma_k^\pm/2\pi$ are the Fourier transforms of $G^\pm(k, t - t')$ [24].

B. The conservation laws

It can be checked that Eqs. (42)-(45) conserves total energy of the both types of the shear Alfvén waves. Indeed, performing the inverse Fourier transform of Eqs. (44)-(45) with respect to $\omega$, setting the temporal variable equal to zero, integrating both sides of the obtained equations over the whole $k$ space and taking into account that $I^\pm_k = I^\mp_k$, it can be shown that considered nonlinear interactions conserve total energy of the both types of the shear Alfvén waves $H^+ = 1/2 \int |\phi|^2 d^4k$ and $H^- = 1/2 \int |\psi|^2 d^4k$, or equivalently, Eqs. (44)-(45) conserve both the total energy $H = H^+ + H^-$ and the cross helicity $H = H^+ - H^-$. Thus, these equations allow us to develop the method for analysis of the WCA equations for the simplest example - the inertial range of isotropic hydrodynamic turbulence. Secondly, we derive Kraichnan’s nonlocal spectrum. Comparison of the results obtained in this section with the results for inertial range of incompressible MHD turbulence obtained in the next sections allows us to show that the energy spectrum $E_q \sim q^{-3/2}$ observed in the numerical simulations of incompressible MHD turbulence in a strong external magnetic field represents the anisotropic analogue of Kraichnan’s nonlocal spectrum.

The equations analogous to (37) and (41) for the inertial range of the hydrodynamic turbulence have the form

$$
i \Gamma^h_k = \int_{-\infty}^{\infty} \frac{k^2 P_{1,2}^h}{\omega_1 + i \nu k^2 + i \zeta^h_k} dF_{1,2}^k,
$$

(48)

$$=- i (\omega + i \nu k^2) I_k^h = k^2 \Gamma^h_k \int_{-\infty}^{\infty} a_{1,2} I_{1,2}^h dF_{1,2}^k - k^2 b_{1,2} I_1^h I_2^h dF_{1,2}^k,
$$

(49)

where

$$a_{1,2} = \frac{1}{2} \left[ 1 - \frac{(k \cdot k_1)^2 (k \cdot k_2)^2}{k^2 k_1^2 k_2^2} + \frac{(k \cdot k_1)(k \cdot k_2)(k_1 \cdot k_2)}{k^2 k_1^2 k_2^2} \right],
$$

(50)

$$b_{1,2} = \frac{(k \cdot k_1)^3}{k^2 k_1^2} - \frac{(k \cdot k_2)(k_1 \cdot k_2)}{k^2 k_2^2},
$$

(51)

$\Gamma^h_k$ and $I^h_k$ are related by the relation (41) with $\omega_k = -i \nu k^2$, and $I^h_k$ is defined by $\langle \psi_{k} \cdot \psi_{k} \rangle = I^h_k \delta_{k-k'}$, where $\psi_{k}$ is the Fourier transform of the turbulent velocity field.

Eqs. (45)-(51) are useless until some assumptions are made about the frequency dependence of $\Gamma^h_k$ and $I^h_k$. Equivalently, in the framework of the DIA one should make some assumptions about the time dependence of $G^h(k, t - t')$ and $Q^h(k, t - t')$ [27], which are the corresponding inverse Fourier transforms with respect to $\omega$. One of the simplest and frequently used assumptions imply [23, 27]

$$G^h(k, t - t') = \exp \left[ -|\xi^h_k|(t - t') \right] H(t - t'),
$$

(51)

$$Q^h(k, t - t') = \exp \left[ -|\xi^h_k|(t - t') \right] E_k,
$$

(53)

where $H(t)$ is the Heaviside (step) function, and $E_k$ is the energy spectrum. Usually it is also assumed [22, 31] that the temporal autocorrelation scales of the spectral function and the Green function are equal $1/\eta_k^h = 1/\xi_k^h \equiv \eta_{ac}^h$. From Eq. (53) it follows that temporal derivative of $Q^h(k, t - t')$ has the discontinuity at $\tau = t - t' = 0$.
As it was shown in Ref. 31 this makes no problems in analysis of the turbulence dynamics if one assumes

$$\lim_{\tau \to 0} \frac{d}{d\tau} Q^h(k, \tau) = 0. \quad (54)$$

Similar to Eqs. (52, 53), in the case under consideration we assume

$$\zeta_k^h = \eta_k^h, \quad (55)$$

$$f_k = \frac{\Xi_k}{\pi} \frac{\eta_k^h}{\omega^2 + (\eta_k^h)^2} \quad (56)$$

Substituting these equations into Eq. (58) and integrating over the frequency variables we obtain

$$\eta_k^h = \int_{-\infty}^{\infty} \frac{i k^2 b_2 \Xi_k}{\omega + i (\eta_k^h)^2} d\Omega_{1,2} \quad (57)$$

where $d\Omega_{1,2} \equiv d\Omega_{1} d\Omega_{2}$. As we see the right hand side of Eq. (57) depends on $\omega$, whereas according to our assumption the left hand side does not. This means, that Eqs. (56) and (58) can not be valid for the whole $(k, \omega)$ space. The solution of this inconsistency is based on the following physical arguments: introducing the autocorrelation time scale $\tau_{ac} = 1/\eta_k^h$ one also should assume that $\tau_{ac}$ is the shortest time scale in the problem and the dynamics of turbulence is totally determined by $\tau_{ac}$. Alternatively, it can be shown that the main contribution in the energy balance equation (49) comes from the modes with $\omega \lesssim \eta_k$, and therefore when studying the scaling properties of the inertial range of the turbulence one can set $\omega = 0$ in Eq. (57). Taking also into account the isotropy of the turbulence ($\eta_k^h = \eta_k^h$ and $\Xi_k^h = \Xi_k^h$) we obtain

$$\eta_k^h = k^2 \int_{-\infty}^{\infty} \frac{\Xi_k}{\eta_k^h + \eta_k^h} d\Omega_{1,2}. \quad (58)$$

It should be noted, that the standard DIA technique (see, e.g., Ref. 27) also leads to Eq. (58) without explicit assumption that $\tau_{ac}$ is the only timescale which totally determines the dynamics of turbulence in the inertial interval. As it is shown in the Appendix, the method of derivation of Eq. (58) in the framework of the DIA also implies implicitly that the formulated assumption is hold.

For analysis of Eq. (49) we perform inverse Fourier transform with respect to $\omega$. In the right hand side of the obtained equation we can set $\nu = 0$, due to the fact that the dissipation range has negligible influence on the nonlinear energy transfer in the inertial range. Substituting Eqs. (55-58), performing integration with respect to frequencies, using Eq. (54) and taking into account the identity

$$a_{1,2} = \frac{1}{2} (b_{1,2} + b_{2,1}) \quad (59)$$

we finally obtain

$$\nu k^2 E_k^h = k^2 \int_{-\infty}^{\infty} \frac{\Xi_k}{\eta_k^h + \eta_k^h} d\Omega_{1,2}. \quad (60)$$

For further simplification of this equation we multiply it by $4\pi k^2$ and integrate from some $k$ in the inertial interval to the infinity. For isotropic hydrodynamic turbulence the effective dissipation takes place for high wave numbers and therefore

$$4\pi \nu \int_{k}^{\infty} k^4 E_k^h \, dk \approx \nu \int_{-\infty}^{\infty} k^2 E_k^h \, dk = \varepsilon_h, \quad (61)$$

where $\varepsilon_h$ is the energy dissipation rate. Then Eq. (60) reduces to

$$\varepsilon_h = 4\pi \int_{k}^{\infty} k^4 \, dk \int_{-\infty}^{\infty} \frac{b_{1,2} \Xi_k}{\eta_k^h + \eta_k^h} d\Omega_{1,2}. \quad (62)$$

To study the scaling properties of the turbulence we seek for the inertial range solution of Eqs. (58) and (60) in the form

$$\Xi_k = A_h k^m, \quad (63)$$

$$\eta_k^h = B_h k^n \quad (64)$$

Substituting these expressions into Eq. (58) and taking into account that for $k > k_2$, $b_{1,2} \approx \sin^2 \alpha_2$, where $\alpha_2$ is the angle between $k$ and $k_2$, we see that the integral on the right hand side of Eq. (65) diverges at $k_2 \to 0$ if $m < -3$ and $n > 0$. Note that for the Kolmogorov spectrum $m = -11/3$ and $n = 2/3$. Therefore, the integral in Eq. (65) is dominated by the low wave number energy containing modes for which Eq. (63) is not valid.

To calculate the contribution of the energy containing modes we substitute Eq. (63) into Eq. (58) and use delta function to integrate over $k_1$. Noting that $|k - k_2| \approx k$ this yields

$$B_h k^n \approx B_h^{-1} k^{2-n} \int \Xi_{k_2} \sin^2 \alpha_2 d^3 k_2 \sim B_h^{-1} k^{2-n} v_0^3, \quad (65)$$

where $v_0$ is the characteristic velocity of energy containing vortices. From Eq. (65) we obtain

$$n = 1, \quad B_h \sim v_0 \quad (66)$$

In contrast to Eq. (65), due to the presence of the multiplier $(\Xi_k - \Xi_{k_1})$, integrals in Eq. (62) are convergent for the small values of $k_2$ when $m > -4$. Consequently, noting that the main contribution in the integral comes from the wave numbers with $k_1 \sim k_2 \sim k$ we obtain

$$\varepsilon_h \sim A_h^2 B_h^{-1} k^{8+2m-n}, \quad (67)$$

and therefore $8+2m-n = 0$ and $A_h \sim (\varepsilon_h B_h)^{1/2}$. Using Eqs. (66) and introducing the one dimensional energy spectrum $E_k = 4\pi k^2 \Xi_k$ we obtain

$$E_k \sim (\varepsilon_h v_0)^{1/2} k^{-3/2}, \quad \eta_k^h \sim kv_0. \quad (68)$$
which represents Kraichnan’s nonlocal spectrum \[23\]. Note that in the considered model the spectral energy transfer is local [i.e., integrals in Eq. \(62\) are convergent] and the nonlocal character of the turbulence is related to effective decorrelation caused by low frequency modes from energy containing interval [divergence of the integrals in Eq. \(58\) and as a consequence shortening of the autocorrelation time scale from \(1/(kv_0)\) to \(1/(kv_\alpha)\)]

Solution of the contradiction between Eq. \(58\) and Kolmogorov spectrum led Kraichnan to the formulation of so-called Lagrangian DIA \[32\]. Another approach was developed by Kadomtsev \[24\]. He argued that analysis of Eq. \(58\) which led to Eq. \(55\) and \(56\) overestimates the contribution of nonlinear interactions. It was asserted that when studying nonlinear interactions of the modes, one should distinguish two kinds of nonlinear interactions - resonant and adiabatic. In Eq. \(58\) it is implicitly assumed that all the nonlinear interactions are resonant. In fact, interactions of the modes with very different wave numbers are adiabatic rather than resonant: the influence of the large scale fluctuations on the low frequency ones leads to adiabatic (WKB) change of the frequency \(\omega\) and the wave number \(k\) of the high frequency fluctuation, similar to the weak inhomogeneity of the mean fields.

As it was shown in Ref. \[24\], the dynamics of the turbulence in the inertial range is mainly determined by the resonant nonlinear interactions whereas influence of the adiabatic interactions can be neglected. From mathematical point of view this means, that instead of integration over the whole \(k\)-space of Eq. \(58\), one should restrict the area of integration by the area \(\sigma\) \(<|k_{1,2}|<k/\sigma\), where \(\sigma\) \(<1\), say \(\sigma=1/3\), \[24\] or \(\sigma=1/\sqrt{2}\), \[11\] \[22\]. If so, using Eqs. \(63\)-(64), instead of Eq. \(65\) we obtain

\[
Bk^1 \sim \frac{1}{A} k^{\delta + m - n}.
\]

Combining this equation with Eq. \(67\) we arrive at the Kolmogorov spectrum \(E_k \sim k^{-5/3}\). Note that for the Kolmogorov spectrum \(\tau_{\alpha c} \sim k^{-2}/k\), \(\sim 1/kv_\alpha\).

As it will be shown in the Sec. VII although the concept of resonant and adiabatic interactions yields right result for inertial range of isotropic hydrodynamic turbulence, for the systems with more then one degrees of freedom it does not hold.

\[ VI. \ \text{NONLOCAL SPECTRUM OF THE INCOMPRESSIBLE MHD TURBULENCE} \]

For further analysis of Eqs. \(67\)-\(10\), or equivalently \[42\] \[43\], similar to the hydrodynamic case we assume

\[
G^\pm(k, \tau) = \exp\left(-|\xi_k^\pm| \tau \pm i\omega_k \tau\right) H(\tau),
\]

\[
Q^\pm(k, \tau) = \exp\left(-|\xi_k^\pm| \tau \pm i\omega_k \tau\right) \mathcal{E}_k,
\]

or equivalently

\[
\xi_k^\pm = \eta_k^\pm,
\]

\[
I^\pm_k = \frac{\varepsilon_k^\pm}{\pi(\omega + \omega_k)^2 + (\xi_k^\pm)^2}.
\]

In the presented paper we consider the symmetric case

\[
\eta_k^\pm = \eta_k \equiv \eta_k, \quad \mathcal{E}_k^+ = \mathcal{E}_k^- \equiv \mathcal{E}_k,
\]

which corresponds to the turbulence with zero cross helicity \(\xi_k^+ - \xi_k^- = 0\). In this case \(I^\pm_k = I^0_k\), and it can be readily checked that Eqs. \(67\) and \(68\) coincides with Eqs. \(58\) and \(10\) respectively. Therefore, in the symmetric case there remain only two independent equations.

Here we consider strong turbulence, i.e., assume that the autocorrelation timescale \(\tau_{ac} = 1/\eta_k\) is the shortest timescale presented in the problem which totally determines the dynamics of the turbulence. Taking this into account and performing integrations with respect to frequencies in Eqs. \(67\) and \(11\) similar to hydrodynamic turbulence we obtain

\[
\eta_k = \int_{-\infty}^{\infty} |T_{1,2}|^2 \frac{\mathcal{E}_k}{\eta_k + \eta_k} \ d\xi_k,
\]

\[
\bar{\nu} k^2 \mathcal{E}_k = \int_{-\infty}^{\infty} |T_{1,2}|^2 \frac{\mathcal{E}_k(\mathcal{E}_k - \mathcal{E}_k)}{\eta_k + \eta_k + \eta_k} \ d\xi_k.
\]

For analysis of anisotropic turbulence we assume that the dynamics is dominated by cascade perpendicular with respect to the mean magnetic field and therefore we assume \(\eta_k = \eta_q\). For energy spectrum we assume \(\Xi\)

\[
\mathcal{E}_k = \frac{A}{k^{m-\mu}} f\left(\frac{p}{\Lambda q^\mu}\right),
\]

where \(\mathcal{E}(u)\) is a positive symmetric function of \(u\) which becomes negligibly small for \(u \gg 1\). For \(u \ll 1\), \(\mathcal{E}(u) \sim 1\) such that \(\int_{-\infty}^{\infty} \mathcal{E}(u) du = 1\). We also define the two dimensional energy density as

\[
\bar{\mathcal{E}}_q = \int_{-\infty}^{\infty} \mathcal{E}_k dq = Aq^m.
\]

Performing the integration with respect to \(p_1\) and \(p_2\) in Eqs. \(72\)-\(75\), multiplying both sides of Eq. \(76\) by \(2\pi q\) and integrating over \(p\) and \(q\) from some value in the inertial range to infinity and defining the energy dissipation rate

\[
\varepsilon \equiv \bar{\nu} \int_{-\infty}^{\infty} k^2 \mathcal{E}_k dq = 2\pi \bar{\nu} \int_{0}^{\infty} q^2 \bar{\mathcal{E}}_q dq.
\]

we obtain

\[
\bar{\mathcal{E}}_q = \int_{-\infty}^{\infty} |T_{1,2}|^2 \frac{\bar{\mathcal{E}}_q}{q_1 + q_2} \ dq_1 dq_2,
\]

\[
\varepsilon = 2\pi \int_{q} q dq \int_{-\infty}^{\infty} |T_{1,2}|^2 \frac{\bar{\mathcal{E}}_q (\bar{\mathcal{E}}_q - \bar{\mathcal{E}}_q)}{q + q_1 + q_2} \ dq_1 dq_2.
\]
where \( dQ_{1,2}^\xi = d^2q_1d^2q_2\delta_{q_1-q_2} \).

Similar to Eq. (64) we assume \( \eta_q = Bq^\alpha \). Taking into account that \( |T_{1,2}|^2 = q^2 \cos^2 \theta_1 \sin^2 \theta_2 \), where \( \theta_1, \theta_2 \) are the angles between \( q \) and \( q_{1,2} \), respectively, we obtain that the integral on the right hand side of Eq. (80) is divergent for \( q_2 \to 0 \) if \( m < -2 \). Further analysis is analogous to one performed in Sec. IV for hydrodynamic turbulence. Similarly we obtain:

\[
\tilde{c}_q \sim (\varepsilon v_0)^{1/2} q^{-5/2}, \quad \eta_q \sim q v_0. \tag{82}
\]

For the one dimensional spectrum \( E_q \equiv 2\pi q \tilde{c}_q \) this yields \( E_q \sim (\varepsilon v_0)^{1/2} q^{-3/2} \). For the characteristic cascade timescale we obtain:

\[
\tau_{\text{casc}} \sim \frac{1}{q v_0} v_0. \tag{83}
\]

The analysis of the anisotropic turbulence is not full until some relation between parallel and perpendicular length scales of the turbulent wave packets are specified. Technically this implies determination of \( \mu \) and \( \Lambda \). This could be done based on arguments discussed in Sec. II and lead us to Eq. (110) or equivalently

\[
\mu = 1/2, \quad \Lambda = \frac{v_0}{V_A} q_0^{1/2}. \tag{84}
\]

VII. RANDOM GALILEAN INVARIANCE AND LOCALITY OF INCOMPRESSIBLE MHD TURBULENCE

The study of the locality of incompressible MHD turbulence performed in this section is based on the methods developed in Ref. 24. Let us isolate resonant interactions in Eq. (25) and represent this equation in the following form

\[
(\omega - \omega_k) \phi_k = \int_R T_{1,2} \phi_1 \psi_2 d\mathcal{F}_{1,2}^k + \int_D T_{1,2} \phi_1 \psi_2 d\mathcal{F}_{1,2}^k + \int_{A_1} T_{1,1,2} \phi_1 \psi_1 \psi_2 d^3k_1 + \int_{A_2} T_{k-2,2} \phi k_2 \psi_2 d^3k_1 \tag{85}
\]

In this equation \( R \) denotes the resonant area where \( q_1 \sim q_2 \sim q \) (as in the previous section we also assume that there exist some relation between the perpendicular and parallel length scales of turbulent wave packets [5]), \( D \) denotes the high frequency area where \( q_1 \sim q_2 \gg q \), and \( A_{1,2} \) denote the areas \( q_{1,2} \ll q \), respectively. The analysis performed in the previous section shows that the area \( D \) has negligible contribution to the integrals in Eqs. (75)-(76) and will be neglected in the further consideration. However, it should be noted, that nonlocal interactions with small scale fluctuations could have important contributions in the dynamics of MHD turbulence with nonzero helicity [13, 19].

Performing Taylor expansion in the integrands of the third and forth terms on the right hand side of Eq. (85) with respect to the small parameter \( q_{1,2}/q \) respectively and taking into account the definition of \( T_{1,2} \) and Eq. (21) we obtain

\[
(\omega - \omega_k) \phi_k = k \cdot (V_L + B_L) \phi_k + G_{\phi\phi} + G_{\psi\psi} + \int_R T_{1,2} \phi_1 \psi_2 d\mathcal{F}_{1,2}. \tag{86}
\]

Similar manipulations of Eq. (20) yields

\[
(\omega + \omega_k) \psi_k = k \cdot (V_L - B_L) \psi_k + G_{\psi\psi} + \int_R T_{1,2} \phi_1 \psi_2 d\mathcal{F}_{1,2}. \tag{87}
\]

In Eqs. (86) and (87) \( G_{ij} \) are the first order terms of the Taylor expansion \( G_{ij} \sim \partial \phi / \partial k \) and \( G_{ij} \sim \partial \psi / \partial k \). \( V_L = \int_{k_1 \ll k} \bar{v}_k d^3k_1 \) and \( B_L \int_{k_1 \ll k} \bar{b}_k d^3k_1 \) are the velocity and the magnetic fields of the low frequency modes.

According to the principle of resonant and adiabatic interactions, first terms of the right hand sides of Eqs. (86) and (87) describe the transfer of high frequency perturbations by the low frequency modes and one should eliminate these terms before applying the closure procedure described in Sec. IV. In the case under consideration this could be done as follows: the terms proportional to \( V_L \) can be eliminated by means of the Galilean transformation (this will lead to the frequency renormalization \( \omega' = \omega - k \cdot V_L \)), whereas the terms proportional to \( B_L \) can be included to \( \omega_k \). If so, the influence of low frequency modes on the dynamics of the high frequency modes comes to slow (adiabatic) deformation of the high frequency modes, described by \( G_{ij} \) in Eqs. (86) and (87). In the zeroth order approximation the adiabatic interactions can be neglected and therefore Eqs. (86) and (87) formally coincide with Eqs. (25)-(26), except the integration now is performed only over the resonant area \( R \). Applying the WCA (DIA) closure scheme described in Sec. IV to these equations, we obtain Eqs. (42)-(45) with the integration performed only over the resonant area \( R \). Then analysis similar to the performed in the previous section would lead to the GS model of the turbulence described by Eqs. (41).

The WCA (Eulerian DIA) suggests that low frequency fluctuations always cause decorrelation of interactions of high frequency fluctuations and as it is known this result is wrong [24, 32]. In contrary, the principle of resonant and adiabatic interactions states that the influence of low frequency modes on the dynamics of high frequency modes always have adiabatic character. Although this principle yields right results for inertial range of isotropic hydrodynamic turbulence as well as for any system with one degree of freedom [i.e., when there exist one starting equation similar to Eqs. (25)-(26)], in general case this principle is not correct.

Indeed, in the case of the velocity field the necessity of elimination of the terms proportional to \( V_L \) from Eqs. (86)-(87) is caused by the requirement of the invariance of the governing equations with respect to the random Galilean transformation [32]. According to this principle the ‘right’ form of Eqs. (86)-(87) has \( V_L = 0 \). In the case
of magnetic field there exists no such a fundamental principle and consequently the necessity of the elimination of the term proportional to $B_L$ is not obvious. Physical picture is the same as it was discussed in Sec. [11]. If the low frequency energy containing modes contribute to the mean magnetic field acting on the high frequency modes, then the terms proportional to $B_L$ should be included in terms proportional to $\omega_k$ on the left hand side of Eqs. (56)-(57). In this case the DIA closure scheme presented in Sec. [11] and analysis similar to one presented in Sec. [31] leads to the GS model of the incompressible MHD turbulence. On the other hand, if the low frequency modes do not contribute to the mean magnetic field, then the same kind of analysis leads to the anisotropic analogue of Kraichnan’s nonlocal spectrum. Therefore, we conclude that in contrary to the incompressible hydrodynamic turbulence, incompressible MHD turbulence could be both local and nonlocal.

In the general case it is clear that the principle of resonant and adiabatic interactions holds for any system with one degree of freedom, since in this case the zeroth order term of the Taylor expansion describing influence of the low frequency modes on the high frequency modes always can be eliminated by corresponding Galilean transformation. In the case of the system with two or more degrees of freedom random Galilean invariance can not guarantee the elimination of all zeroth order terms, as in the considered above case of Eqs. (56)-(57) and therefore the low frequency modes can cause effective decorrelation of high frequency fluctuations.

VIII. TWO DIMENSIONAL MHD TURBULENCE

It is well known that dynamics of two dimensional hydrodynamic turbulence is basically different from three dimensional turbulence. The origin of this difference is that in two dimensional case in addition to the energy there exist another conserved quantity - enstrophy. The situation is different in magnetohydrodynamics. As it was discussed above, MHD turbulence is highly anisotropic, dominated by turbulent cascade in the direction perpendicular to the mean magnetic field. Therefore it has been argued that the local two dimensional dynamics in the plane perpendicular to the mean magnetic field describe an essential part of the three dimensional turbulence (especially in the presence of the strong mean field) and could be studied by two dimensional MHD. In this context two dimensional MHD turbulence has been extensively studied by different authors [33, 34, 35]. It seems interesting to compare presented model with the features of two dimensional MHD turbulence observed in numerical simulations. Numerical simulations performed in Refs. [32, 33] show that inertial range spectrum of two dimensional MHD turbulence in the absence of a mean magnetic field has the form

$$E_k = C_k \varepsilon^{1/2} E_M^{1/4} k^{-3/2},$$

with $C_k \approx 1.8$ and $E_M$ is the density of the magnetic energy. Noting that $E_M \sim v_A^2$ we conclude that this result coincides with the prediction of the presented model for perpendicular cascade given by Eq. (8). Although this coincidence is quite remarkable, it can not be treated as confirmation of the presented model, because it is not obvious that two dimensional MHD properly reproduces local structure of the three dimensional MHD turbulence in the place perpendicular to the mean magnetic field. It also should be noted that the results obtained in Ref. [34] for the inertial range of two dimensional MHD turbulence are closer to the Kolmogorov spectrum.

On the other hand, two dimensional MHD represents the system with two degrees of freedom (see, e.g., Eqs. (1)-(2) of Ref. [33]) which keep several important properties of MHD. Namely, linear modes have the same dispersion and Alfvén effect is also presented in two dimensional MHD. In this context theoretical as well as numerical study (including the case with strong background magnetic field) of two dimensional MHD seems to have its own value, especially for the study of the turbulence locality.

IX. CONCLUSIONS

Novel model of incompressible magnetohydrodynamic turbulence in the presence of a strong external magnetic field is proposed. It is suggested that in the presence of the strong external magnetic field incompressible MHD turbulence becomes nonlocal in the sense that the low frequency modes cause decorrelation of interacting high frequency modes from the inertial interval. Obtained nonlocal spectrum of the inertial range of incompressible MHD turbulence, given by Eqs. (8) and (11), represents the anisotropic analogue of Kraichnan’s nonlocal spectrum of isotropic hydrodynamic turbulence. Based on the analysis performed in the framework of WCA it is shown that in contrast to the isotropic hydrodynamic turbulence, incompressible MHD turbulence could be both local and nonlocal and therefore an anisotropic analogues of both Kolmogorov and Kraichnan spectra are realizable in incompressible MHD turbulence.

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APPENDIX A

DIA equation that governs the dynamics of the Green function $G^h(k,t)$ for the inertial range of the isotropic hydrodynamic turbulence has the form \[ \frac{d}{dt}G^h(k,t-t') = -\int_{-\infty}^{\infty} d^3k_1 d^3k_2 \delta_{k-k_1-k_2} b_{1,2} \times \int_{t'}^t dt'' G^h(k_1,t-t'') Q^h(k_2,t-t'') G^h(k,t''-t'). \] (A1)

The standard method of the analysis [27] implies integration of this equation with respect to $\tau = t - t'$ in the interval $(0, \infty)$. Taking also into account Eqs. (52)-(53), after the straightforward manipulations this leads to Eq. (58). Described method of derivation of Eq. (58) is not unique. Before performing integration with respect to $\tau$ we can multiply both sides of Eq. (A1) by any function of $t$. For instance, if we multiply Eq. (A1) by $\exp(-\alpha t)$ and then repeat the same procedures, we obtain Eq. (57) with $\omega$ replaced by $i\alpha$. The reason why this equation is ‘wrong’, whereas Eq. (58) is ‘right’ is the same as in WCA analysis presented in Sec. V: it is implied that $\tau_{ac}$ is the shortest timescale presented in the problem which totally determines the dynamics of the turbulence.

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