Solutions and perturbation analysis of the matrix equation
\[ X - \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q \]

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Abstract
Consider the nonlinear matrix equation \( X - \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q \). This paper shows that there exists a unique positive definite solution to the equation without any restriction on \( A_i \). Three perturbation bounds for the unique solution to the equation are evaluated. A backward error of an approximate solution for the unique solution to the equation is derived. Explicit expressions of the condition number for the unique solution to the equation are obtained. The theoretical results are illustrated by numerical examples.

Keywords: nonlinear matrix equation, positive definite solution, perturbation bound, backward error, condition number

1. Introduction
In this paper the nonlinear matrix equation
\[ X - \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q \] (1.1)
is investigated, where \( A_1, A_2, \ldots, A_m \) are \( n \times n \) complex matrices, \( m \) is a positive integer and \( Q \) is a positive definite matrix. Here, \( A_i^* \) denotes the conjugate transpose of the matrix \( A_i \).

This type of nonlinear matrix equations arises in many practical applications. The equation \( X - A^* X A = Q \) which is representative of Eq. (1.1) for \( m = 1 \) comes from ladder networks, dynamic programming, control theory, stochastic filtering, statistics and so forth [1–3, 23, 24, 34]. When \( m > 1 \), Eq. (1.1) is recognized as playing an important role in solving a system of linear equations in many physical calculations.

For the equation \( X \pm A^* X^{-2} A = Q \), there were many contributions in the literature to the theory, applications and numerical solutions [4, 10, 12, 13, 15, 16, 21, 28, 31, 35, 37]. The general equations such as \( X \pm A^* X^{-2} A = Q \) [18, 19, 38, 39], \( X^t \pm A^* X^{-2} A = Q \) [5, 6, 22, 33] and \( X \pm A^* X^{-q} A = Q \) [14, 21, 32] were also investigated by many scholars. In addition, He and

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Long [17] and Duan et al. [8] have studied the similar equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$. sarhan et al. [27] discussed the existence of extremal positive definite solution of the matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$. Duan et al. [7] proved that the equation $X - \sum_{i=1}^{m} A_i^* X A_i = Q$ ($0 < |\delta| < 1$) has a unique positive definite solution. They also proposed an iterative method for obtaining the unique positive definite solution. However, to our best knowledge, there has been no perturbation analysis for Eq. (1.1) in the known literatures.

The rest of the paper is organized as follows. Section 2 gives some preliminary lemmas that will be needed to develop this work. Section 3 proves the existence of a unique positive definite solution to Eq. (1.1) without any restriction on $A_i$. Section 4 gives three perturbation bounds for the unique solution to Eq. (1.1). Section 5 derives a backward error of an approximate solution for the unique solution to Eq. (1.1). Furthermore, in Section 6, the condition number of the unique solution to Eq. (1.1) is discussed. Finally, several numerical examples are presented in Section 7.

We denote by $C^{n\times n}$ the set of $n \times n$ complex matrices, by $H^{n\times n}$ the set of $n \times n$ Hermitian matrices, by $I$ the identity matrix, by $\mathbf{i}$ the imaginary unit, by $\| \cdot \|$ the Frobenius norm and by $\| \cdot \|_F$ the spectral norm, by $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ the maximal and minimal eigenvalues of $M$, respectively. For $A = (a_{ij})$, $A \in C^{n\times n}$ and a matrix $B$, $A \otimes B = (a_{ij} B)$ is a Kronecker product, and vec$A$ is a vector defined by vec$A = (a_{11}, \ldots, a_{1n})^T$. For $X, Y \in H^{n\times n}$, we write $X \succeq Y$ (resp. $X \succ Y$) if $X - Y$ is Hermitian positive semi-definite (resp. definite).

2. Preliminaries

Lemma 2.1. [27] If $A \succeq B > 0$, then $0 < A^{-1} \preceq B^{-1}$.

Lemma 2.2. [20]. For every positive definite matrix $X \in H^{n\times n}$, if $X + \Delta X \succeq (1/\nu)X > 0$, then
\[
\|X^{-\frac{1}{2}}A^*(X + \Delta X)^{-1}X^{-\frac{1}{2}}\|_F \leq (\|X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}}\|_F + \nu\|X^{-\frac{1}{2}}\Delta XX^{-\frac{1}{2}}\|_F^2)\|X^{-\frac{1}{2}}AX^{-\frac{1}{2}}\|_F^2.
\]

Lemma 2.3. [17]. The matrix differentiation has the following properties:

1. $d(F_1 \pm F_2) = dF_1 \pm dF_2$;
2. $d(kF) = k(dF)$, where $k$ is a complex number;
3. $d(F^*) = (dF)^*$;
4. $d(F_1 F_2 F_3) = (dF_1) F_2 F_3 + F_1 (dF_2) F_3 + F_1 F_2 (dF_3)$;
5. $dF^{-1} = -F^{-1} (dF) F^{-1}$;
6. $dF = 0$, where $F$ is a constant matrix.

3. Positive definite solution of the matrix Eq.(1.1)

In this section, the existence of a unique positive definite solution of Eq. (1.1) is proved. Moreover, some properties of the unique positive definite solution of Eq. (1.1) are obtained.

Theorem 3.1. If $F(X) = Q + \sum_{i=1}^{m} A_i^* X^{-1} A_i$, then $F([Q, Q + \sum_{i=1}^{m} A_i^* Q^{-1} A_i]) \subseteq [Q, Q + \sum_{i=1}^{m} A_i^* Q^{-1} A_i]$.

Proof. Let $\Omega = [Q, Q + \sum_{i=1}^{m} A_i^* Q^{-1} A_i]$. By Lemma 2.1, we obtain $0 < X^{-1} \preceq Q^{-1}$ for every $X \in \Omega$.

Applying Eq. (1.1) yields $Q \preceq F(X) \preceq Q + \sum_{i=1}^{m} A_i^* Q^{-1} A_i$. Therefore $F(\Omega) \subseteq \Omega$. □
Theorem 3.2. There exists a unique positive definite solution $X$ to Eq. (3.1) and the iteration

$$X_0 > 0, \quad X_n = Q + \sum_{i=1}^{m} A_i^* X_{n-1}^{-1} A_i, \quad n = 1, 2, \cdots \quad (3.1)$$

converges to $X$.

To prove the above theorem, we first verify the following lemma.

Lemma 3.1. Let $F(X) = Q + \sum_{i=1}^{m} A_i^* X^{-1} A_i$. If $0 < t < 1$ and $X \in [Q, Q + \sum_{i=1}^{m} A_i^* Q^{-1} A_i]$, then

$$F^2(tX) \geq t(1 + \eta(t)) F^2(X),$$

where

$$\eta(t) = \frac{(1-t)\lambda_{\min}(Q)}{t \lambda_{\max}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\max}(A_i^* A_i)}{\lambda_{\min}(Q)}}.$$

Proof. According to Theorem 3.1 for every $X \in [Q, Q + \sum_{i=1}^{m} A_i^* Q^{-1} A_i]$ and $F^2(X) \in [Q, Q + \sum_{i=1}^{m} A_i^* Q^{-1} A_i]$. Hence we have

$$F^2(tX) - t(1 + \eta(t)) F^2(X) = (1-t)Q + t \sum_{i=1}^{m} A_i^* \left[(tQ + \sum_{i=1}^{m} A_i^* X^{-1} A_i)^{-1} - (Q + \sum_{i=1}^{m} A_i^* X^{-1} A_i)^{-1} \right] A_i$$

$$\geq (1-t)\lambda_{\min}(Q) \left[ \sum_{i=1}^{m} \lambda_{\max}(A_i^* A_i) \right] I = 0, \quad 0 < t < 1.$$

Proof of Theorem 3.2. Let $F(X) = Q + \sum_{i=1}^{m} A_i^* X^{-1} A_i$, and $\Omega = [Q, I + \sum_{i=1}^{m} A_i^* Q^{-1} A_i]$. The proof will be divided into two steps.

(1) We prove the special case of Theorem 3.2 when $X_0 = Q$.

It is easy to check that

$$Q \leq X_1 = Q + \sum_{i=1}^{m} A_i^* X_0^{-1} A_i = F(Q) = Q + \sum_{i=1}^{m} A_i^* Q^{-1} A_i,$$

$$Q \leq X_2 = Q + \sum_{i=1}^{m} A_i^* X_1^{-1} A_i = F^2(Q) \leq F(F(Q)).$$
Let
\[ F^2(Q) \leq X_3 = Q + \sum_{i=1}^{m} A_i X_i^{-1} A_i = F^3(Q) \leq F(Q), \]
\[ F^3(Q) \leq X_4 = Q + \sum_{i=1}^{m} A_i X_i^{-1} A_i = F^4(Q) \leq F^3(Q). \]

By induction, it yields that
\[ Q \leq F^{2k}(Q) \leq F^{2k+2}(Q) \leq F^{2k+1}(Q) \leq F^{2k+1}(Q) \leq Q + \sum_{i=1}^{m} A_i Q^{-1} A_i, \quad k \in \mathbb{Z}^+. \]

Hence the sequences \( \{F^{2k}(Q)\} \) and \( \{F^{2k+1}(Q)\} \) are convergent. Let \( \lim_{k \to \infty} F^{2k}(Q) = X^{(1)} \) and \( \lim_{k \to \infty} F^{2k+1}(Q) = X^{(2)} \). It is clear that \( X^{(1)} \) and \( X^{(2)} \) are positive fixed points of \( F^2(X) \).

In the following part, we first prove that \( X^{(1)} = X^{(2)} \). Suppose that \( Y_1 \) and \( Y_2 \) are two positive fixed points of \( F^2 \) in \( \Omega \). We compute
\[
Y_1 = F^2(Y_1) \geq \frac{1}{\sum_{i=1}^{m} \lambda_{\text{max}}(A_i^2 A_i)} \left( Q + \sum_{i=1}^{m} A_i Q^{-1} A_i \right),
\]
\[
Y_2 = F^2(Y_2) = t Y_1,
\]
where
\[
t = \frac{1}{\sum_{i=1}^{m} \lambda_{\text{min}}(A_i^2 A_i) + \lambda_{\text{min}}(Q)}. \]

Let \( t_0 = \sup \{t : Y_1 \geq t Y_2\} \). Then \( 1 \leq t_0 < +\infty \). On the contrary, suppose that \( 0 < t_0 < 1 \). Then \( Y_1 \geq t_0 Y_2 \). According to Lemma 3.1 and the monotonicity of \( F^2(X) \), we have
\[
Y_1 = F^2(Y_1) \geq F^2(t_0 Y_2) \geq (1 + \eta(t_0)) t_0 F^2(Y_2) = (1 + \eta(t_0)) t_0 Y_2.
\]

By the definition of \( \eta(t) \), we obtain \( (1 + \eta(t_0)) t_0 > t_0 \), which is a contradiction to the definition of \( t_0 \). Hence we have \( t_0 \geq 1 \) and \( Y_1 \geq Y_2 \). Similarly, we get \( Y_1 \leq Y_2 \). Therefore \( Y_1 = Y_2 \), i.e., the equation \( X = F^2(X) \) has only one positive definite solution. Hence \( X^{(1)} = X^{(2)} \).

Second, we prove that \( \lim_{n \to \infty} X_n \) is the unique fixed point of \( F \) in \( \Omega \). By \( X^{(1)} = X^{(2)} \), it follows that \( X^{(1)} = X^{(2)} = \lim_{n \to \infty} F^n(Q) \) is the unique fixed point of \( F^3 \). Moreover, the positive definite solution of equation \( F(X) = X \) solves \( X = F^3(X) \). Therefore \( F(X) = X \) has only one positive definite solution and \( \lim_{n \to \infty} F^n(Q) = \lim_{n \to \infty} F^n(X_n) \) is the unique fixed point of \( F \).

(2) We prove the case of Theorem 3.2 when \( X_0 > 0 \). From iteration 3.1, we obtain
\[
X_1 \geq Q,
\]
\[
Q \leq X_2 = Q + \sum_{i=1}^{m} A_i X_i^{-1} A_i \leq Q + \sum_{i=1}^{m} A_i Q^{-1} A_i = F(Q).
\]
and

\[ F^2(Q) \leq X_3 = I + \sum_{i=1}^{m} A_i^* X_i^{-1} A_i \leq F(Q). \]

By induction, we have

\[ F^{2k}(Q) \leq X_{2k+1} \leq F^{2k-1}(Q) \quad \text{and} \quad F^{2k-2}(Q) \leq X_{2k} \leq F^{2k-1}(Q). \]

Therefore

\[ \lim_{k \to \infty} X_k = \lim_{n \to \infty} F^n(Q). \]

It follows that \( \lim X_k \) is the unique positive definite solution of Eq.(1.1). □

**Theorem 3.3.** If \( X \) is a positive definite solution of Eq.(1.1), then \( Q \leq X \leq Q + \sum_{i=1}^{m} A_i^* Q^{-1} A_i \).

**Proof.** That \( X \) is a positive definite solution of Eq.(1.1) implies \( X > 0 \). Then \( X^{-1} > 0 \) and \( A_i^* X^{-1} A_i \geq 0 \). Hence \( X = Q + \sum_{i=1}^{m} A_i^* X^{-1} A_i \geq Q \). Consequently, \( X^{-1} \leq Q^{-1} \) and \( X \leq Q + \sum_{i=1}^{m} A_i^* Q^{-1} A_i \). □

**Theorem 3.4.** Every positive definite solution \( X \) of Eq.(1.1) is in \([\beta I, \alpha I]\), where \( \alpha \) and \( \beta \) are respectively the solutions of the following equations

\[ x = \lambda_{\max}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\max}(A_i^* A_i)}{\lambda_{\min}(Q) + \sum_{i=1}^{m} \lambda_{\min}(A_i^* A_i)}, \quad (3.2) \]

\[ x = \lambda_{\min}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\min}(A_i^* A_i)}{\lambda_{\max}(Q) + \sum_{i=1}^{m} \lambda_{\max}(A_i^* A_i)}. \quad (3.3) \]

Moreover,

\[ \lambda_{\min}(Q) \leq \beta \leq \alpha. \quad (3.4) \]

**Proof.** We define the sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) as follows:

\[ \beta_0 = \lambda_{\min}(Q), \quad \alpha_n = \lambda_{\max}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\max}(A_i^* A_i)}{\beta_n}, \quad \beta_{n+1} = \lambda_{\min}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\min}(A_i^* A_i)}{\alpha_n}, \quad n = 0, 1, 2, \cdots. \quad (3.5) \]
From (3.5), it follows that
\[
\begin{align*}
\beta_0 &\leq \lambda_{\text{max}}(Q) \leq \alpha_0 = \lambda_{\text{max}}(Q) + \sum_{i=1}^{m} \lambda_{\text{max}}(A_i^*A_i), \\
\lambda_{\text{min}}(Q) &= \beta_0 \leq \beta_1 = \lambda_{\text{min}}(Q) + \sum_{i=1}^{m} \lambda_{\text{min}}(A_i^*A_i) \leq \lambda_{\text{min}}(Q) + \sum_{i=1}^{m} \lambda_{\text{max}}(A_i^*A_i), \\
\beta_1 &\leq \lambda_{\text{max}}(Q) \leq \alpha_1 = \lambda_{\text{max}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{max}}(A_i^*A_i)}{\beta_1} \leq \lambda_{\text{max}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{max}}(A_i^*A_i)}{\lambda_{\text{min}}(Q)} = \alpha_0.
\end{align*}
\]

We suppose that \(\lambda_{\text{max}}(Q) \leq \alpha_k \leq \alpha_{k-1}\) and \(\lambda_{\text{min}}(Q) \leq \beta_{k-1} \leq \beta_k \leq \lambda_{\text{min}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{min}}(A_i^*A_i)}{\lambda_{\text{max}}(Q)}\). Then
\[
\begin{align*}
\lambda_{\text{min}}(Q) &\leq \beta_k = \lambda_{\text{min}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{min}}(A_i^*A_i)}{\alpha_{k-1}} \\
&\leq \beta_{k+1} = \lambda_{\text{min}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{min}}(A_i^*A_i)}{\alpha_k} \leq \lambda_{\text{min}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{min}}(A_i^*A_i)}{\lambda_{\text{max}}(Q)}, \\
\lambda_{\text{max}}(Q) &\leq \alpha_{k+1} = \lambda_{\text{max}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{max}}(A_i^*A_i)}{\beta_{k+1}} \\
&\leq \alpha_k = \lambda_{\text{max}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{max}}(A_i^*A_i)}{\beta_k}.
\end{align*}
\]

Hence, for each \(k\) we have \(\lambda_{\text{max}}(Q) \leq \alpha_{k+1} \leq \alpha_k\) and \(\lambda_{\text{min}}(Q) \leq \beta_{k+1} \leq \beta_k \leq \lambda_{\text{min}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{min}}(A_i^*A_i)}{\lambda_{\text{max}}(Q)}\), which imply that the sequences \(\{\alpha_n\}\) and \(\{\beta_n\}\) are monotonic and bounded. Therefore, they are convergent to positive numbers. Let
\[
\alpha = \lim_{n \to \infty} \alpha_n, \quad \beta = \lim_{n \to \infty} \beta_n.
\]

Taking limits in (3.5) yields
\[
\alpha = \lambda_{\text{max}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{max}}(A_i^*A_i)}{\beta}, \quad \beta = \lambda_{\text{min}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{min}}(A_i^*A_i)}{\alpha}.
\]

which imply
\[
\alpha = \lambda_{\text{max}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{max}}(A_i^*A_i)}{\lambda_{\text{min}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{min}}(A_i^*A_i)}{\alpha}}, \quad \beta = \lambda_{\text{min}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{min}}(A_i^*A_i)}{\lambda_{\text{max}}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\text{max}}(A_i^*A_i)}{\beta}}.
\]
Therefore $\alpha$ and $\beta$ satisfy (3.2) and (3.3), respectively. We will prove that $X \in [\beta I, \alpha I]$ for any positive definite solution $X$. According to Theorem 3.3 and the sequences in (3.5), we have

$$\beta_0 I \leq Q \leq X \leq (\lambda_{\max}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\max}(A_i^*A_i)}{\lambda_{\min}(Q)}) I = \alpha_0 I$$

for each positive definite solution $X$. From $X = Q + \sum_{i=1}^{m} A_i^*X^{-1}A_i$, it follows that $X = Q + \sum_{i=1}^{m} A_i^*(Q + \sum_{i=1}^{m} A_i^*X^{-1}A_i)^{-1}A_i$. Hence

$$\left( \lambda_{\min}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\min}(A_i^*A_i)}{\lambda_{\max}(Q)} \right) I \leq X \leq \left( \lambda_{\max}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\max}(A_i^*A_i)}{\lambda_{\min}(Q)} \right) I. \quad (3.7)$$

Using $\beta_0 I \leq X \leq \alpha_0 I$, we obtain $\beta_0 \leq \lambda_{\min}(X)$ and $\lambda_{\max}(X) \leq \alpha_0$. Applying the inequality in (3.7) yields $\beta_0 I \leq X \leq \alpha_0 I$. By induction, it yields that $\beta_i I \leq X \leq \alpha_i I$. Taking limits on both sides of the above inequality, we have $\beta I \leq X \leq \alpha I$. \hfill \Box

**Corollary 3.1.** Every positive definite solution of Eq. (1.1) is in

$$\left[ Q + \frac{1}{\alpha} \sum_{i=1}^{m} A_i^*A_i, \quad Q + \frac{1}{\beta} \sum_{i=1}^{m} A_i^*A_i \right],$$

where $\alpha$ and $\beta$ are defined as in Theorem 3.3.

**Proof.** We suppose that $X$ is a positive definite solution of Eq. (1.1). By Theorem 3.3 it follows that

$$\lambda_{\min}(Q) \leq \beta \leq \lambda_{\min}(X), \quad \lambda_{\max}(Q) \leq \lambda_{\max}(X) \leq \alpha. \quad (3.8)$$

Using $X = Q + \sum_{i=1}^{m} A_i^*X^{-1}A_i$, we obtain $Q + \frac{1}{\lambda_{\max}(X)} \sum_{i=1}^{m} A_i^*A_i \leq X \leq Q + \frac{1}{\lambda_{\min}(X)} \sum_{i=1}^{m} A_i^*A_i$. Applying inequality (3.8) yields $Q + \frac{1}{\alpha} \sum_{i=1}^{m} A_i^*A_i \leq X \leq Q + \frac{1}{\beta} \sum_{i=1}^{m} A_i^*A_i$. \hfill \Box

**Remark 3.1.** Applying (3.1), we obtain

$$Q + \frac{1}{\beta} \sum_{i=1}^{m} A_i^*A_i \leq \left( \lambda_{\max}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\max}(A_i^*A_i)}{\beta} \right) I = \alpha I,$$

$$Q + \frac{1}{\alpha} \sum_{i=1}^{m} A_i^*A_i \geq \left( \lambda_{\min}(Q) + \frac{\sum_{i=1}^{m} \lambda_{\min}(A_i^*A_i)}{\alpha} \right) I = \beta I.$$

That is to say, the estimate of positive definite solution in Corollary 3.1 is more precise than that in Theorem 3.3.
4. Perturbation bounds

Here we consider the perturbed equation

$$\tilde{X} - \sum_{i=1}^{m} \tilde{A}_i^* \tilde{X}^{-1} \tilde{A}_i = \tilde{Q}, \quad (4.1)$$

where $\tilde{A}_i$, $\tilde{Q}$ are small perturbations of $A_i$ and $Q$ in Eq. (1.1), respectively. We assume that $X$ and $\tilde{X}$ are the solutions of Eq. (1.1) and Eq. (4.1), respectively. Let $\Delta X = \tilde{X} - X$, $\Delta Q = \tilde{Q} - Q$ and $\Delta A_i = \tilde{A}_i - A_i$.

In this section we develop three perturbation bounds for the solution of Eq. (4.1). To begin with, a relative perturbation bound for the unique solution $X$ of Eq. (1.1) is derived. The perturbation bound in Theorem 4.1 does not need any knowledge of the actual solution $X$ of Eq. (1.1).

Secondly, based on the matrix differentiation, we use the techniques developed in [8] to derive another perturbation bound in Theorem 4.2. Finally, based on the operator theory, we obtain a sharper perturbation bound in Theorem 4.3.

The next theorem generalizes Theorem 3.2 in Li and Zhang [20] with $m = 1$ to arbitrary integer $m \geq 1$.

Theorem 4.1. Let $b = \beta^2 + \beta \|\Delta Q\| - \sum_{i=1}^{m} \|A_i\|^2$, $s = \sum_{i=1}^{m} \|\Delta A_i\| (2\|A_i\| + \|\Delta A_i\|)$. If

$$0 < b < 2\beta^2 \quad \text{and} \quad b^2 - 4\beta^2 (\beta \|\Delta Q\| + s) \geq 0, \quad (4.2)$$

then

$$\frac{\|\tilde{X} - X\|}{\|X\|} \leq \varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\| \equiv \xi_1, \quad (4.3)$$

where

$$\varrho = \frac{2s}{\sum_{i=1}^{m} \|\Delta A_i\| (b + \sqrt{b^2 - 4\beta^2 (\beta \|\Delta Q\| + s)})}, \quad \omega = \frac{2\beta}{b + \sqrt{b^2 - 4\beta^2 (\beta \|\Delta Q\| + s)}}.$$

Proof. Let

$$\Omega = \{ \Delta X \in \mathcal{H}^{\text{pos}} : \|X^{-1/2} \Delta XX^{-1/2}\| \leq \varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\| \}.$$ 

Obviously, $\Omega$ is a nonempty bounded convex closed set. Let

$$f(\Delta X) = \sum_{i=1}^{m} (\tilde{A}_i^* (X + \Delta X)^{-1} \tilde{A}_i - A_i^* X^{-1} A_i) + \Delta Q, \quad \Delta X \in \Omega.$$ 

Evidently, $f : \Omega \mapsto \mathcal{H}^{\text{pos}}$ is continuous. We will prove that $f(\Omega) \subseteq \Omega$.

For every $\Delta X \in \Omega$, that is $\|X^{-1/2} \Delta XX^{-1/2}\| \leq \varrho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\|$. Thus

$$X^{-1/2} \Delta XX^{-1/2} \geq (-\varrho \sum_{i=1}^{m} \|\Delta A_i\| - \omega \|\Delta Q\|) I,$$
\[ X + \Delta X \geq (1 - \varrho \sum_{i=1}^{m} ||\Delta A_i|| - \omega ||\Delta Q||)X. \]

According to (4.2) and (4.3), we have
\[
\varrho \sum_{i=1}^{m} ||\Delta A_i|| + \omega ||\Delta Q|| = \frac{2(\beta ||\Delta Q|| + s)}{b + \sqrt{b^2 - 4\beta^2(\beta ||\Delta Q||) + s}} \leq \frac{2(\beta ||\Delta Q|| + s)}{b} \leq \frac{b}{2\beta^2} < 1.
\]

Therefore
\[
(1 - \varrho \sum_{i=1}^{m} ||\Delta A_i|| - \omega ||\Delta Q||)X > 0.
\]

From Lemma 2.2, it follows that
\[
\|X^{-\frac{1}{2}} \left[ \sum_{i=1}^{m} A_i^* ((X + \Delta X)^{-1} - X^{-1})A_i \right] X^{-\frac{1}{2}} \|
\leq \left( \|X^{-\frac{1}{2}}XXX^{-\frac{1}{2}}\| + \frac{\|X^{-\frac{1}{2}}\Delta X\|}{1 - \varrho \sum_{i=1}^{m} ||\Delta A_i|| - \omega ||\Delta Q||} \right) \left( \sum_{i=1}^{m} \|X^{-\frac{1}{2}}A_iX^{-\frac{1}{2}}\|^2 \right)
\leq \left( \|X^{-\frac{1}{2}}XXX^{-\frac{1}{2}}\| + \frac{\|X^{-\frac{1}{2}}\Delta X\|}{1 - \varrho \sum_{i=1}^{m} ||\Delta A_i|| - \omega ||\Delta Q||} \right) \left( \frac{1}{b^2} \sum_{i=1}^{m} ||A_i||^2 \right).
\]

Therefore
\[
\|X^{-\frac{1}{2}}f(\Delta X)X^{-\frac{1}{2}}\|
= \|X^{-\frac{1}{2}} \left[ \sum_{i=1}^{m} A_i^* ((X + \Delta X)^{-1} - X^{-1})A_i \right] X^{-\frac{1}{2}} + X^{-\frac{1}{2}}\Delta QX^{-\frac{1}{2}} \|
\leq \left( \sum_{i=1}^{m} \|X^{-\frac{1}{2}}A_i((X + \Delta X)^{-1} - X^{-1})A_iX^{-\frac{1}{2}}\| + \|X^{-\frac{1}{2}}\Delta QX^{-\frac{1}{2}}\| \right)
+ \left( \sum_{i=1}^{m} \|X^{-\frac{1}{2}}[\Delta A_i^*(X + \Delta X)^{-1}(A_i + \Delta A_i) + A_i^*(X + \Delta X)^{-1}\Delta A_i]X^{-\frac{1}{2}}\| \right)
\leq \left( \|X^{-\frac{1}{2}}XXX^{-\frac{1}{2}}\| + \frac{\|X^{-\frac{1}{2}}\Delta X\|}{1 - \varrho \sum_{i=1}^{m} ||\Delta A_i|| - \omega ||\Delta Q||} \right) \left( \frac{1}{b^2} \sum_{i=1}^{m} ||A_i||^2 \right)
+ \frac{\sum_{i=1}^{m} ||\Delta A_i|(|2||A_i|| + ||\Delta A_i||)}{\beta^2(1 - \varrho \sum_{i=1}^{m} ||\Delta A_i|| - \omega ||\Delta Q||)} \frac{||\Delta Q||}{\beta}
\leq \left( \xi_1 + \frac{\xi_1^2}{1 - \xi_1} \right) \left( \frac{1}{b^2} \sum_{i=1}^{m} ||A_i||^2 \right) + \frac{s}{\beta^2(1 - \xi_1)} + \frac{||\Delta Q||}{\beta}
\leq \xi_1.
\]
That is \( f(\Omega) \subseteq \Omega \). By Brouwer fixed point theorem, there exists a \( \Delta X \in \Omega \) such that \( f(\Delta X) = \Delta X \).

Moreover, by Theorem 3.2, we know that \( X \) and \( \bar{X} \) are the unique solutions to Eq.(1.1) and Eq.(4.1), respectively. Then

\[
\frac{\|\bar{X} - X\|}{\|X\|} = \frac{\|\Delta X\|}{\|X\|} = \frac{\|X^{1/2}(X^{-1/2}\Delta XX^{-1/2})X^{1/2}\|}{\|X\|} \\
\leq \|X^{-1/2}\Delta XX^{-1/2}\| \leq \rho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\|.
\]

\[\square\]

**Remark 4.1.** With

\[\rho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\| = \frac{2(\sum_{i=1}^{m} \|\Delta A_i\|)(\sum_{i=1}^{m} \|\Delta A_i\|) + \beta \|\Delta Q\|)}{b + \sqrt{b^2 - 4\beta^2 (\beta \|\Delta Q\| + s)}},\]

we get \( \rho \sum_{i=1}^{m} \|\Delta A_i\| + \omega \|\Delta Q\| \rightarrow 0 \) as \( \Delta Q \rightarrow 0 \) and \( \|\Delta A_i\| \rightarrow 0 \) (\( i = 1, 2, \ldots, m \)). Therefore Eq.(4.1) is well-posed.

Next, with the help of the following lemma, we shall derive a new perturbation bound as shown in Theorem 4.2.

**Lemma 4.1.** Suppose that \( X \) is a unique positive definite solution of Eq.(1.1). If

\[\sum_{i=1}^{m} \|A_i\|^2 < \beta^2, \tag{4.4}\]

then

\[\|dX\| \leq \frac{2\beta \sum_{i=1}^{m} \|A_i\| |dA_i|}{\beta^2 - \sum_{i=1}^{m} |A_i|^2}.\]

**Proof.** According to Lemma 2.3, differentiating on both sides of Eq.(1.1), we have

\[dX = \sum_{i=1}^{m} [dA_i^*(X^{-1}A_i) - (A_i^*X^{-1})dX(X^{-1}A_i) + (A_i^*X^{-1})dA_i] = 0.\]

Therefore,

\[dX = \sum_{i=1}^{m} (A_i^*X^{-1})dX(X^{-1}A_i) = \sum_{i=1}^{m} A_i^*(X^{-1}A_i) + \sum_{i=1}^{m} (A_i^*X^{-1})dA_i\]

and

\[\|dX + \sum_{i=1}^{m} (A_i^*X^{-1})dX(X^{-1}A_i)\| = \| \sum_{i=1}^{m} dA_i^*(X^{-1}A_i) + \sum_{i=1}^{m} (A_i^*X^{-1})dA_i \| \\
\leq \sum_{i=1}^{m} |dA_i^*|\|X^{-1}\||A_i| + \sum_{i=1}^{m} |A_i^*|\|X^{-1}\||dA_i| \\
= 2 \sum_{i=1}^{m} |A_i|\|X^{-1}\||dA_i|\]
are true. By Theorem 3.4, it follows that $\|X^{-1}\| \leq \frac{1}{\beta}$. Then

$$
\|dX + \sum_{i=1}^{m} (A_i^*X^{-1})dX(X^{-1}A_i)\| \leq \frac{2}{\beta^2} \sum_{i=1}^{m} \|A_i\|\|dA_i\|. \tag{4.5}
$$

In addition,

$$
\|dX + \sum_{i=1}^{m} (A_i^*X^{-1})dX(X^{-1}A_i)\| \geq \|dX\| - \| \sum_{i=1}^{m} (A_i^*X^{-1})dX(X^{-1}A_i)\| \geq \|dX\| - \frac{1}{\beta^2} \left( \sum_{i=1}^{m} \|A_i\|^2 \right)\|dX\|
$$

$$
= \left(1 - \frac{1}{\beta^2} \sum_{i=1}^{m} \|A_i\|^2 \right)\|dX\|. \tag{4.6}
$$

By (4.4), it follows that $\left(1 - \frac{1}{\beta^2} \sum_{i=1}^{m} \|A_i\|^2 \right)\|dX\| > 0$.

Combining (4.5) and (4.6), we obtain

$$
\left(1 - \frac{1}{\beta^2} \sum_{i=1}^{m} \|A_i\|^2 \right)\|dX\| \leq \frac{2}{\beta^2} \sum_{i=1}^{m} (\|A_i\|\|dA_i\|),
$$

which means that

$$
\|dX\| \leq \frac{2\sum_{i=1}^{m} (\|A_i\|\|dA_i\|)}{\beta^2 - \sum_{i=1}^{m} \|A_i\|^2}.
$$

\[\square\]

**Theorem 4.2.** Suppose that $X, \tilde{X}$ are the unique positive definite solutions of Eq.(1.1) and Eq. (4.1), respectively. If

$$
\sum_{i=1}^{m} \|A_i\|^2 < \beta^2 \quad \text{and} \quad \sum_{i=1}^{m} (\|A_i\| + \|\Delta A_i\|)^2 < \beta^2, \tag{4.7}
$$

then

$$
\|\tilde{X} - X\| \leq \frac{2\beta \sum_{i=1}^{m} (\|A_i\| + \|\Delta A_i\|)\|\Delta A_i\|}{\beta^2 - \sum_{i=1}^{m} (\|A_i\| + \|\Delta A_i\|)^2}
$$

and

$$
\frac{\|\tilde{X} - X\|}{\|X\|} \leq \frac{2\beta \sum_{i=1}^{m} (\|A_i\| + \|\Delta A_i\|)\|\Delta A_i\|}{(\beta^2 - \sum_{i=1}^{m} (\|A_i\| + \|\Delta A_i\|)^2)\|X\|} \equiv \xi_2
$$

hold true.
Proof. Set $A_i(t) = A_i + t\Delta A_i, t \in [0, 1]$. By Theorem 3.2 we have that for arbitrary $t \in [0, 1]$, the matrix equation

$$X - \sum_{i=1}^{m} A_i'(t)X^{-1}A_i(t) = Q$$

has a unique positive definite solution $X(t)$ satisfying

$$X(0) = X, \quad X(1) = \bar{X}.$$ 
By Lemma 4.1, we have

$$\|\bar{X} - X\| = \|X(1) - X(0)\| = \|\int_0^1 dX(t)\| \leq \int_0^1 \|dX(t)\| dt.$$ 

By mean value theorem of integration, there exists $\varepsilon \in (0, 1]$ satisfying

$$\|\bar{X} - X\| \leq \frac{\int_0^1 2\beta \sum_{i=1}^{m} (\|A_i\| + t\|\Delta A_i\|)\|\Delta A_i\| dt}{\beta^2 - \sum_{i=1}^{m} (\|A_i\| + t\|\Delta A_i\|)^2}.$$ 

Next, based on the operator theory, we derive a sharper perturbation estimate. Subtracting (1.1) from (4.1) we have

$$\Delta X + \sum_{i=1}^{m} B_i'\Delta X B_i = E + h(\Delta X), \quad \text{(4.8)}$$

where

$$B_i = X^{-1}A_i,$$

$$E = \sum_{i=1}^{m} (B_i'\Delta A_i + \Delta A_i'B_i) + \sum_{i=1}^{m} \Delta A_i'X^{-1}\Delta A_i + \Delta Q,$$

$$h(\Delta X) = \sum_{i=1}^{m} B_i'\Delta XX^{-1}\Delta X(I + X^{-1}\Delta X)^{-1}B_i - \sum_{i=1}^{m} \Delta A_i'X^{-1}\Delta X(I + X^{-1}\Delta X)^{-1}X^{-1}\Delta A_i,$$

$$- \sum_{i=1}^{m} \Delta A_i'X^{-1}\Delta X(I + X^{-1}\Delta X)^{-1}B_i.$$
We define the linear operator $L: \mathcal{H}^{\text{pos}} \to \mathcal{H}^{\text{pos}}$ by

$$LW = W + \sum_{i=1}^{m} B_i^* W B_i, \quad W \in \mathcal{H}^{\text{pos}}.$$ 

Since

$$X - \sum_{i=1}^{m} B_i^* X B_i = X - \sum_{i=1}^{m} A_i^* X A_i = X - \sum_{i=1}^{m} A_i^* X^{-1} A_i = Q > 0,$$

by Lemma 3.4.1 and Proposition 3.3.1 in [25], the operator $L$ is invertible. We also define operators $P_i: C^\text{pos} \to \mathcal{H}^{\text{pos}}$ by

$$P_i Z_i = L^{-1}(B_i^* Z_i + Z_i B_i), \quad Z_i \in C^\text{pos}, \quad i = 1, 2, \cdots, m.$$ 

Thus, we can rewrite (4.8) as

$$\Delta X = L^{-1} \Delta Q + \sum_{i=1}^{m} P_i \Delta A_i + L^{-1}(\sum_{i=1}^{m} \Delta A_i^* X^{-1} \Delta A_i) + L^{-1}(h(\Delta X)). \quad (4.9)$$

Define

$$||L^{-1}|| = \max_{W \in \mathcal{H}^{\text{pos}} ||L^{-1} W||, \quad ||L|| = \max_{Z \in C^\text{pos}} ||P_i Z||. \quad \frac{||L^{-1}||}{||W||} = 1, \quad \frac{||P_i||}{||Z||} = 1$$

Now we denote

$$l = ||L^{-1}||^{-1}, \quad \zeta = ||X^{-1}||, \quad m_i = ||A_i||, \quad n_i = ||P_i||, \quad \theta_i = ||B_i||, \quad \theta = \sum_{i=1}^{m} \theta_i, \quad i = 1, 2, \cdots, m,$$

$$\epsilon = \frac{1}{l} ||\Delta Q|| + \frac{m}{l} (n_i ||\Delta A_i|| + \frac{\zeta}{l} ||\Delta A_i||^2), \quad \sigma = \frac{\xi}{l} \sum_{i=1}^{m} ((m_i + ||\Delta A_i||) \zeta + \theta_i) ||\Delta A_i||.$$ 

Then we can state the third perturbation estimate as follows.

**Theorem 4.3.** If

$$\sigma < 1 \text{ and } \epsilon < \frac{l(1 - \sigma)^2}{\xi l + l(\xi + \sigma) + 2 \sqrt{(l\xi + \sigma)(\xi + l)}} \quad (4.10),$$

then

$$||X - \bar{X}|| \leq \frac{2l\epsilon}{l(1 + \xi - \sigma) + \sqrt{l^2(1 + \xi - \sigma)^2 - 4l\xi l(\xi + l)}} \equiv \xi_3.$$ 

**Proof.** Let

$$f(\Delta X) = L^{-1} \Delta Q + \sum_{i=1}^{m} P_i \Delta A_i + L^{-1}(\sum_{i=1}^{m} \Delta A_i^* X^{-1} \Delta A_i) + L^{-1}(h(\Delta X)).$$

Obviously, $f : \mathcal{H}^{\text{pos}} \to \mathcal{H}^{\text{pos}}$ is continuous. The condition (4.10) ensures that the quadratic equation $\xi(l + \theta)\xi^2 - (l + \xi - \sigma)\xi + l\epsilon = 0$ with respect to the variable $\xi$ has two positive real roots. The smaller one is

$$\bar{\xi} = \frac{2l\epsilon}{l(1 + \xi - \sigma) + \sqrt{l^2(1 + \xi - \sigma)^2 - 4l\xi l(\xi + l)}}.$$
Define $\Omega = \{ \Delta X \in \mathcal{H}^{m \times n} : \| \Delta X \| \leq \xi_3 \}$. Then for any $\Delta X \in \Omega$, by (4.10), we have

$$
\| X^{-1} \Delta X \| \leq \| X^{-1} \| \| \Delta X \| \leq \xi \xi_3 \leq \xi \cdot \frac{2l\epsilon}{l(1+\xi \epsilon - \sigma)}
$$

$$
= 1 + \frac{\xi \epsilon + \sigma - 1}{1 + \xi \epsilon - \sigma} \leq 1 + \frac{-2(1-\sigma)(l\epsilon+\theta)}{(l\epsilon+l+2\theta)(1+\xi \epsilon - \sigma)} < 1.
$$

It follows that $I - X^{-1} \Delta X$ is nonsingular and

$$
\| I - X^{-1} \Delta X \| \leq \frac{1}{1 - \| X^{-1} \Delta X \|} \leq \frac{1}{1 - \xi \| \Delta X \|}.
$$

Therefore, we have

$$
\| f(\Delta X) \| \leq \frac{1}{l} \| \Delta Q \| + \sum_{i=1}^{m} \left( n_i \| \Delta A_i \| + \frac{\xi}{l} \| \Delta A_i \| \right) + \frac{1}{l} \sum_{i=1}^{m} \theta_i \frac{\xi \| \Delta X \|}{1 - \xi \| \Delta X \|} + \frac{\theta_i \xi_i}{l(1 - \xi \xi_3)} \cdot \| \Delta X \|
$$

$$
\leq \epsilon + \frac{\sigma \xi \| \Delta X \|}{1 - \xi \| \Delta X \|} + \frac{\theta_i \xi_i^2}{l(1 - \xi \xi_3)} = \xi_1,
$$

for $\Delta X \in \Omega$. That is $f(\Omega) \subseteq \Omega$. According to Schauder fixed point theorem, there exists $\Delta X_0 \in \Omega$ such that $f(\Delta X_0) = \Delta X_0$. It follows that $X + \Delta X_0$ is a Hermitian solution of Eq. (4.1). By Theorem 3.2, we know that the solution of Eq. (4.1) is unique. Then $\Delta X_0 = \tilde{X} - X$ and $\| X - \tilde{X} \| \leq \xi_3$. \hfill \Box

**Remark 4.2.** From Theorem 4.3 we get the first order perturbation bound for the solution as follows:

$$
\| \tilde{X} - X \| \leq \frac{1}{l} \| \Delta Q \| + \sum_{i=1}^{m} n_i \| \Delta A_i \| + O \left( \| (\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q) \|_F \right),
$$

as $(\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q) \to 0$.

Combining this with (4.9) gives

$$
\Delta X = L^{-1} \Delta Q + L^{-1} \sum_{i=1}^{m} \left( B_i \Delta A_i + \Delta A_i^* B_i \right) + O \left( \| (\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q) \|_F \right),
$$

as $(\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q) \to 0$.

### 5. Backward error

In this section, we derive a backward error of an approximate solution for the unique solution to Eq. (1.1) beginning with the lemma.
Lemma 5.1. For every positive definite matrix $X \in \mathcal{H}^{\text{pos}}$, if $X + \Delta X \geq (1/\nu)I > 0$, then

$$
\| \sum_{i=1}^{m} A_i^*(X + \Delta X)^{-1} - X^{-1}A_i \| \leq (\|\Delta X\| + \nu\|\Delta X\|^2) \sum_{i=1}^{m} \|X^{-1}A_i\|^2.
$$

Proof. According to

$$(X + \Delta X)^{-1} - X^{-1} = -X^{-1} \Delta X (X + \Delta X)^{-1} = -X^{-1} \Delta XX^{-1} \Delta X(X + \Delta X)^{-1},$$

it follows that

$$
\| \sum_{i=1}^{m} A_i^*((X + \Delta X)^{-1} - X^{-1})A_i \|
\leq \sum_{i=1}^{m} (\|A_i^* X^{-1} \Delta XX^{-1} A_i\| + \|A_i^* X^{-1} \Delta XX^{-1} \Delta X(X + \Delta X)^{-1} A_i\|)
\leq (\|\Delta X\| + \nu\|\Delta X\|^2) \sum_{i=1}^{m} \|X^{-1}A_i\|^2.
$$

\[\square\]

Theorem 5.1. Let $\tilde{X} > 0$ be an approximation to the solution $X$ of Eq.(1.1). If the residual $R(\tilde{X}) = Q + \sum_{i=1}^{m} A_i^* \tilde{X} - \tilde{X}$ satisfies

$$
\|R(\tilde{X})\| < \frac{(1 - \Sigma)^2}{1 + \Sigma + 2 \sqrt{\Sigma}} \lambda_{\min}(\tilde{X}), \text{ where } \Sigma \equiv \sum_{i=1}^{m} \|\tilde{X}^{-1}A_i\|^2 < 1, \quad (5.1)
$$

then

$$
\|\tilde{X} - X\| \leq \theta \|R(\tilde{X})\|, \quad (5.2)
$$

where

$$
\theta = \frac{2\lambda_{\min}(\tilde{X})}{(1 - \Sigma)\lambda_{\min}(\tilde{X}) + \|R(\tilde{X})\| + \sqrt{(1 - \Sigma)\lambda_{\min}(\tilde{X}) + \|R(\tilde{X})\|^2} - 4\lambda_{\min}(\tilde{X})\|R(\tilde{X})\|)
$$

Proof. Let

$$
\Psi = \{\Delta X \in \mathcal{H}^{\text{pos}} : \|\Delta X\| \leq \theta\|R(\tilde{X})\|\}.
$$

Obviously, $\Psi$ is a nonempty bounded convex closed set. Let

$$
g(\Delta X) = \sum_{i=1}^{m} A_i^* \left[ (\tilde{X} + \Delta X)^{-1} - \tilde{X}^{-1} \right] A_i + R(\tilde{X}).
$$

Evidently $g : \Psi \mapsto \mathcal{H}^{\text{pos}}$ is continuous.

Note that the condition (5.1) ensures that the quadratical equation

$$
x^2 - \left( \lambda_{\min}(\tilde{X})(1 - \Sigma) + \|R(\tilde{X})\| \right) x + \lambda_{\min}(\tilde{X})\|R(\tilde{X})\| = 0
$$

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has two positive real roots, and the smaller one is given by
\[
\mu_* = \frac{2 \lambda_{\min}(\tilde{X}) ||R(\tilde{X})||}{(1 - \Sigma) \lambda_{\min}(\tilde{X}) + ||R(\tilde{X})|| + \sqrt{(1 - \Sigma) \lambda_{\min}(\tilde{X}) + ||R(\tilde{X})||^2 - 4 \lambda_{\min}(\tilde{X}) ||R(\tilde{X})||}}.
\]

Next, we will prove that \(g(\Psi) \subseteq \Psi\).

For every \(\Delta X \in \Psi\), we have
\[
\Delta X \geq -\theta ||R(\tilde{X})|| I.
\]
Hence
\[
\tilde{X} + \Delta X \geq \tilde{X} - \theta ||R(\tilde{X})|| I \geq (\lambda_{\min}(\tilde{X}) - \theta ||R(\tilde{X})||) I.
\]
By (5.2), one sees that
\[
\theta ||R(\tilde{X})|| \leq \frac{2 \lambda_{\min}(\tilde{X}) ||R(\tilde{X})||}{(1 - \Sigma) \lambda_{\min}(\tilde{X}) + ||R(\tilde{X})||} = \lambda_{\min}(\tilde{X}) \left(1 + \frac{||R(\tilde{X})|| - (1 - \Sigma) \lambda_{\min}(\tilde{X})}{(1 - \Sigma) \lambda_{\min}(\tilde{X}) + ||R(\tilde{X})||}\right).
\]
According to (5.1), we obtain
\[
||R(\tilde{X})|| - (1 - \Sigma) \lambda_{\min}(\tilde{X}) \leq \left(\frac{(1 - \Sigma)^2}{1 + \Sigma} - (1 - \Sigma)\right) \lambda_{\min}(\tilde{X}) \leq \frac{-2(1 - \Sigma) \lambda_{\min}(\tilde{X})}{1 + \Sigma} < 0,
\]
which implies that
\[
\theta ||R(\tilde{X})|| \leq \lambda_{\min}(\tilde{X}) \quad \text{and} \quad (\lambda_{\min}(\tilde{X}) - \theta ||R(\tilde{X})||) I > 0.
\]
According to Lemma (5.1), we obtain
\[
||g(\Delta X)|| \leq \left(||\Delta X|| + \frac{||\Delta X||^2}{\lambda_{\min}(X) - \theta ||R(\tilde{X})||} \right) \sum_{i=1}^{m} ||X^{-1} A_i||^2 + ||R(\tilde{X})||
\]
\[
\leq \left(\theta ||R(\tilde{X})|| + \frac{(\theta ||R(\tilde{X})||)^2}{\lambda_{\min}(X) - \theta ||R(\tilde{X})||} \right) \Sigma + ||R(\tilde{X})||
\]
\[
= \theta ||R(\tilde{X})||.
\]
By Brouwer’s fixed point theorem, there exists a \(\Delta X \in \Psi\) such that \(g(\Delta X) = \Delta X\). Hence \(\tilde{X} + \Delta X\) is a solution of Eq. (1.1). Moreover, by Theorem (3.2) we know that the solution \(X\) of Eq. (1.1) is unique. Then
\[
||\tilde{X} - X|| = ||\Delta X|| \leq \theta ||R(\tilde{X})||.
\]

6. Condition number

In this section, we apply the theory of condition number developed by Rice [26] to study condition number of the unique solution to Eq. (1.1).
6.1. The complex case

Suppose that $X$ and $\tilde{X}$ are the solutions of Eq. (1.1) and Eq. (4.1), respectively. Let $\Delta A = \tilde{A} - A$, $\Delta Q = \tilde{Q} - Q$ and $\Delta X = \tilde{X} - X$. Using Theorem 4.3 and Remark 4.2, we have

$$\Delta X = \tilde{X} - X = L^{-1} \Delta Q + L^{-1} \sum_{i=1}^{m} (B^*_i \Delta A_i + \Delta A^*_i B_i) + O \left( \left\| (\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q) \right\|^2 \right), \quad (6.1)$$

as $(\Delta A_1, \Delta A_2, \cdots, \Delta A_m, \Delta Q) \to 0$.

By the theory of condition number developed by Rice [22], we define the condition number of the Hermitian positive definite solution $X$ to Eq. (1.1) by

$$c(X) = \lim_{\delta \to 0} \sup_{\|\Delta X\|_F \leq \delta} \frac{\|\Delta X\|_F}{\|\Delta X\|}.$$

(6.2)

where $\xi$, $\rho$ and $\eta$, $i = 1, 2, \cdots, m$, are positive parameters. Taking $\xi = \eta_1 = \rho = 1$ in (6.2) gives the absolute condition number $c_{abs}(X)$, and taking $\xi = \|X\|_F$, $\eta = \|A\|_F$ and $\rho = \|Q\|_F$ in (6.2) gives the relative condition number $c_{rel}(X)$.

Substituting (6.1) into (6.2), we get

$$c(X) = \frac{1}{\xi} \max_{\Delta A_i \in C^{m \times m}, \Delta \tilde{Q} \in H^{m \times m}} \frac{\|L^{-1}(\Delta Q + \sum_{i=1}^{m} (B^*_i \Delta A_i + \Delta A^*_i B_i))\|_F}{\|L^{-1}(\rho H + \sum_{i=1}^{m} \eta_i (B^*_i E_i + E^*_i B_i))\|_F}.$$

$$= \frac{1}{\xi} \max_{(E_1, E_2, \cdots, E_m, H) \neq 0, E_i \in C^{m \times m}, H \in H^{m \times m}} \frac{\|L^{-1}(\rho H + \sum_{i=1}^{m} \eta_i (B^*_i E_i + E^*_i B_i))\|_F}{\|(E_1, E_2, \cdots, E_m, H)\|_F}.$$

Let $L$ be the matrix representation of the linear operator $L$. Then it is easy to see that

$$L = I \otimes I + \sum_{i=1}^{m} B^T_i \otimes B^*_i = I \otimes I + \sum_{i=1}^{m} (X^{-1} A_i)^T \otimes (X^{-1} A_i)^*.$$

Let

$$L^{-1} = S + i\Sigma,$$

$$L^{-1}(I \otimes B^*_i) = L^{-1}(I \otimes (X^{-1} A_i)^*) = U_{i1} + i\Omega_{i1},$$

$$L^{-1}(B^T_i \otimes I) = L^{-1}((X^{-1} A_i)^T \otimes I) = U_{i2} + i\Omega_{i2},$$

$$S_c = \begin{bmatrix} S & -\Sigma \\ \Sigma & S \end{bmatrix}, \quad U_i = \begin{bmatrix} U_{i1} + U_{i2} & \Omega_{i2} - \Omega_{i1} \\ \Omega_{i1} + U_{i2} & U_{i1} - U_{i2} \end{bmatrix}, \quad i = 1, 2, \cdots, m,$$

(6.3)

$$\text{vec}H = x + iy, \quad \text{vec}E_i = a_i + ib_i, \quad g = (x^T, y^T, a^T_1, b^T_1, \cdots, a^T_m, b^T_m)^T, \quad M = (E_1, E_2, \cdots, E_m, H),$$

where $x, y, a_i, b_i \in \mathbb{R}^d$, $S, \Sigma, U_{i1}, U_{i2}, \Omega_{i1}, \Omega_{i2} \in \mathbb{R}^{d \times d}$, $i = 1, 2, \cdots, m$. $\Pi$ is the vec-permutation matrix such that

$$\text{vec} E^T_i = \Pi \text{vec} E_i.$$
Furthermore, we obtain that

$$c(X) = \frac{1}{\xi} \max_{M \neq 0} \frac{\|L^{-1}(\rho H + \sum_{i=1}^{m} \eta_i (B_i^T E_i + E_i^T B_i))\|_F}{\|(E_1, E_2, \ldots, E_m, H)\|_F}$$

$$= \frac{1}{\xi} \max_{M \neq 0} \frac{\|\rho L^{-1} \text{vec} H + \sum_{i=1}^{m} \eta_i L^{-1} ((I \otimes B_i^T) \text{vec} E_i + (B_i^T \otimes I) \text{vec} E_i^T)\|}{\|(\text{vec}E_1, \text{vec}E_2, \ldots, \text{vec}E_m, \text{vec}H)\|}$$

$$= \frac{1}{\xi} \max_{g \neq 0} \frac{\|\rho (S_c, \eta_1 U_1, \eta_2 U_2, \ldots, \eta_m U_m) g\|}{\|g\|}$$

$$= \frac{1}{\xi} \| (\rho S_c, \eta_1 U_1, \eta_2 U_2, \ldots, \eta_m U_m) \|.$$

Then we have the following theorem.

**Theorem 6.1.** The condition number $c(X)$ defined by (6.2) has the explicit expression

$$c(X) = \frac{1}{\xi} \| (\rho S_c, \eta_1 U_1, \eta_2 U_2, \ldots, \eta_m U_m) \|, \tag{6.4}$$

where the matrices $S_c$ and $U_i$ are defined as in (6.3).

**Remark 6.1.** From (6.4) we have the relative condition number

$$c_{rel}(X) = \frac{\|(\rho S_c, \eta_1 U_1, \eta_2 U_2, \ldots, \eta_m U_m)\|}{\|X\|_F}.$$  

6.2. The real case

In this subsection we consider the real case, i.e., all the coefficient matrices $A_i$, $Q$ of Eq. (6.1) are real. In such a case the corresponding solution $X$ is also real. Completely similar arguments as Theorem 6.1 give the following theorem.

**Theorem 6.2.** Let $A_i$, $Q$ be real and $c(X)$ be the condition number defined by (6.2). Then $c(X)$ has the explicit expression

$$c(X) = \frac{1}{\xi} \| (\rho S_r, \eta_1 U_1, \eta_2 U_2, \ldots, \eta_m U_m) \|,$$

where

$$S_r = \left(I + \sum_{i=1}^{m} (A_i^T X^{-1}) \otimes (A_i^T X^{-1}) \right)^{-1},$$

$$U_i = S_r [I \otimes (A_i^T X^{-1}) + ((A_i^T X^{-1}) \otimes I) \Pi], \quad i = 1, 2, \ldots, m.$$  

**Remark 6.2.** In the real case the relative condition number is given by

$$c_{rel}(X) = \frac{\|(\rho S_r, \eta_1 U_1, \eta_2 U_2, \ldots, \eta_m U_m)\|}{\|X\|_F}.$$
7. Numerical Examples

To illustrate the theoretical results of the previous sections, in this section four simple examples are given, which were carried out using MATLAB 7.1. For the stopping criterion we take
\[ \varepsilon_{k+1}(X) = \| X - \sum_{i=1}^{m} A_i^* X^{-1} A_i - I \| < 1.0e - 10. \]

Example 7.1. We study the matrix equation
\[ X - A_1^* X^{-1} A_1 - A_2^* X^{-1} A_2 = I, \]
with
\[ A_k = \frac{1}{k^2 + 2 \times 10^{-2}} ||A|| A, \quad k = 1, 2, \]
\[ A = \begin{pmatrix}
2 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & 2
\end{pmatrix}. \]

By computation, \( \beta = 1.0009, \alpha = 1.1976. \) Let \( X_0 = 1.1I. \) Algorithm (5.1) needs 11 iterations to obtain the unique positive definite solution \( X = \begin{pmatrix}
1.0643 & 0.0494 & 0.0104 & -0.0009 & -0.0000 \\
0.0494 & 1.0747 & 0.0485 & 0.0104 & -0.0009 \\
0.0104 & 0.0485 & 1.0747 & 0.0485 & 0.0104 \\
-0.0009 & 0.0104 & 0.0485 & 1.0747 & 0.0494 \\
-0.0000 & -0.0009 & 0.0104 & 0.0494 & 1.0643
\end{pmatrix} \in [\beta I, \alpha I] \]
with the residual \( || X - A_1^* X^{-1} A_1 - A_2^* X^{-1} A_2 - I || = 4.8477e - 011, \) which satisfies Theorem 4.2 and Theorem 5.2.

Example 7.2. We consider the matrix equation
\[ X - A_1^* X^{-1} A_1 - A_2^* X^{-1} A_2 = I, \]
with
\[ A_1 = \frac{1}{3} + 2 \times 10^{-2} \frac{||A||}{A}, \quad A_2 = \frac{1}{5} + 3 \times 10^{-2} \frac{||A||}{A}, \]
\[ A = \begin{pmatrix}
2 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & 2
\end{pmatrix}. \]

Suppose that the coefficient matrices \( A_1 \) and \( A_2 \) are perturbed to \( \tilde{A}_i = A_i + \Delta A_i, i = 1, 2, \) where
\[ \Delta A_1 = \frac{10^{-j}}{||C^T + C||} (C^T + C), \quad \Delta A_2 = \frac{3 \times 10^{-j-1}}{||C^T + C||} (C^T + C) \]
and \( C \) is a random matrix generated by MATLAB function \texttt{randn}.

We now consider the corresponding perturbation bounds for the solution \( X \) in Theorem 4.7, Theorem 4.2, and Theorem 4.3.
The conditions in Theorem 4.1 are

\[ \text{con} 1 = 2\beta^2 - b > 0, \quad \text{con} 2 = \beta^2 - \sum_{i=1}^{2} \|A_i\|^2 > 0, \]

\[ \text{con} 3 = (\beta^2 - \sum_{i=1}^{2} \|A_i\|^2)^2 - 4\beta^2 \sum_{i=1}^{2} \|\Delta A_i\| (2\|A_i\| + \|\Delta A_i\|) \geq 0. \]

The condition in Theorem 4.2 is

\[ \text{con} 4 = \beta^2 - \sum_{i=1}^{2} (\|A_i\| + \|\Delta A_i\|)^2 > 0. \]

The conditions in Theorem 4.3 are

\[ \text{con} 5 = 1 - \sigma > 0, \quad \text{con} 6 = \frac{l(1 - \sigma)^2}{\zeta(l + l\sigma + 2\theta + 2\sqrt{(\sigma + \theta)^2(l + l\sigma))} - \epsilon > 0.} \]

By computation, we list them in Table 1.

| j | 4  | 5  | 6  | 7  |
|---|----|----|----|----|
| con1 | 1.1650 | 1.1650 | 1.1650 | 1.1650 |
| con2 | 0.8379 | 0.8379 | 0.8379 | 0.8379 |
| con3 | 0.7018 | 0.7021 | 0.7021 | 0.7021 |
| con4 | 0.8378 | 0.8379 | 0.8379 | 0.8379 |
| con5 | 0.9999 | 1.0000 | 1.0000 | 1.0000 |
| con6 | 0.4802 | 0.4804 | 0.4804 | 0.4804 |

The results listed in Table 1 show that the conditions of Theorem 4.1-4.3 are satisfied.

By Theorem 4.1-4.3, we can compute the relative perturbation bounds \( \xi_1, \xi_2, \nu_\ast = \frac{\xi_3}{\|X\|} \), respectively. These results averaged as the geometric mean of 20 randomly perturbed runs. Some results are listed in Table 2.

| j | 4  | 5  | 6  | 7  |
|---|----|----|----|----|
| \( \|X-X_0\| \) | 2.7093 \times 10^{-5} | 2.5933 \times 10^{-6} | 2.5409 \times 10^{-7} | 2.5031 \times 10^{-8} |
| \( \xi_1 \) | 9.9282 \times 10^{-5} | 9.9853 \times 10^{-6} | 9.7137 \times 10^{-7} | 9.8301 \times 10^{-8} |
| \( \xi_2 \) | 8.6930 \times 10^{-5} | 8.7421 \times 10^{-6} | 8.5042 \times 10^{-7} | 8.6061 \times 10^{-8} |
| \( \nu_\ast \) | 6.4687 \times 10^{-5} | 6.5057 \times 10^{-6} | 6.3287 \times 10^{-7} | 6.4045 \times 10^{-8} |

The results listed in Table 2 show that the perturbation bound \( \nu_\ast \) given by Theorem 4.3 is fairly sharp, the bound \( \xi_2 \) given by Theorem 4.2 is relatively sharp, while the bound \( \xi_1 \) given by Theorem 4.1 which does not depend on the exact solution is conservative.
Example 7.3. We consider
\[ X - A_1^*X^{-1}A_1 - A_2^*X^{-1}A_2 = Q, \]
with
\[
A_1 = \frac{1}{\theta} + 2 \times 10^{-2} \frac{\|A\|}{\|A\|}, \quad A_2 = \frac{1}{\theta} + 3 \times 10^{-2} \frac{\|A\|}{\|A\|}, \quad Q = A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.
\]

Choose \(\bar{X}_0 = A\). Let the approximate solution \(\bar{X}_k\) of \(X\) be given with the iterative method \((3,7)\), where \(k\) is the iterative number.

The residual \(R(\bar{X}_k) \equiv Q + A_1^*\bar{X}_k^{-1}A_1 + A_2^*\bar{X}_k^{-1}A_2 - \bar{X}_k\) satisfies the conditions in Theorem \(5.7\).

By Theorem \(5.7\) we can compute the backward error bound for \(\bar{X}_k\)
\[
\| \bar{X}_k - X \| \leq \theta |R(\bar{X}_k)|,
\]
where
\[
\theta = \frac{2 \lambda_{\min}(\bar{X}_k)}{(1 - \Sigma) \lambda_{\min}(\bar{X}_k) + \|R(\bar{X}_k)\| + \sqrt{((1 - \Sigma) \lambda_{\min}(\bar{X}_k) + \|R(\bar{X}_k)\|)^2 - 4 \lambda_{\min}(\bar{X}_k)|R(\bar{X}_k)|}}
\]

Some results are listed in Table \(3\).

| \(k\) | \(1\) | \(2\) | \(3\) | \(4\) |
|------|------|------|------|------|
| \(\|\bar{X}_k - X\|\) | \(5.0268 \times 10^{-4}\) | \(5.7662 \times 10^{-6}\) | \(6.6162 \times 10^{-8}\) | \(7.5024 \times 10^{-10}\) |
| \(\theta |R(\bar{X}_k)|\) | \(5.1435 \times 10^{-4}\) | \(5.9000 \times 10^{-6}\) | \(6.7689 \times 10^{-8}\) | \(7.7656 \times 10^{-10}\) |

The results listed in Table \(3\) show that the error bound given by Theorem \(5.7\) is fairly sharp.

Example 7.4. We study the matrix equation
\[ X - A_1^*X^{-1}A_1 - A_2^*X^{-1}A_2 = Q, \]
with
\[
A_j = \frac{1}{\theta} + 2 \times 10^{-k} \frac{\|A\|}{\|A\|}, \quad j = 1, 2, \quad A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & 1 & 0 & 9 & 0 \\ 1 & 2 & 1 & 0 & 8 \\ 5 & 1 & 2 & 1 & 6 \\ 9 & 0 & 1 & 2 & 1 \\ 0 & 2 & 3 & 1 & 2 \end{pmatrix}.
\]

By Remark \(6.2\) we can compute the relative condition number \(c_{\text{rel}}(X)\). Some results are listed in Table \(4\).

The numerical results listed in the second line show that the unique positive definite solution \(X\) is well-conditioned.
Table 4: Results for Example 7.4 with different values of k

| k  | 1    | 3    | 5    | 7    | 9    |
|----|------|------|------|------|------|
| $c_{rel}(X)$ | 1.2704 | 1.0951 | 1.0939 | 1.0938 | 1.0938 |

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