SMOOTH DEPENDENCE ON PARAMETERS OF SOLUTION OF COHOMOLOGY EQUATIONS OVER ANOSOV SYSTEMS AND APPLICATIONS TO COHOMOLOGY EQUATIONS ON DIFFEOMORPHISM GROUPS

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Abstract. We consider the dependence on parameters of the solutions of cohomology equations over Anosov diffeomorphisms. We show that the solutions depend on parameters as smoothly as the data. As a consequence we prove optimal regularity results for the solutions of equations taking value in diffeomorphism groups. These results are motivated by applications to rigidity theory, dynamical systems, and geometry.

In particular, in the context of diffeomorphism groups we show: Let $f$ be a transitive Anosov diffeomorphism of a compact manifold $M$. Suppose that $\eta \in C^{k+\alpha}(M, \text{Diff}^r(N))$ for a compact manifold $N$, $k, r \in \mathbb{N}$, $r \geq 1$, and $0 < \alpha \leq \text{Lip}$. We show that if there exists a $\varphi \in C^{k+\alpha}(M, \text{Diff}^1(N))$ solving

$$\varphi f(x) = \eta_x \circ \varphi_x$$

then in fact $\varphi \in C^{k+\alpha}(M, \text{Diff}^r(N))$.

1. Introduction

Let $f$ be a diffeomorphism of a compact manifold $M$. A cohomology equation over $f$ is an equation of the form

$$\varphi f(x) = \eta_x \cdot \varphi_x$$

(1.1)

where $\eta$ is given and we are to determine $\varphi$. The interpretation of $\cdot$ varies depending on the context but is typically either a group operation or the composition of linear maps.

Equations of the form (1.1) have been studied for many different maps (including rotations, and horocycle flows). In this paper we will always consider $f$ to be an Anosov diffeomorphism, usually a transitive Anosov diffeomorphism.

It is relevant to point out that if $f^n(p) = p$ then, by applying (1.1) repeatedly, we obtain

$$\varphi f^n(p) = \eta f^{n-1}(p) \cdots \eta_p \cdot \varphi_p.$$
Hence, if there is a solution \( \varphi \) then, since \( \varphi_{f^n(p)} = \varphi_p \), we must have
\[
\eta_{f^{n-1}p} \cdots \eta_p = \text{Id}.
\tag{1.2}
\]

When \( f \) is an Anosov diffeomorphism a great deal of effort, starting with the pioneering work \[\text{Liv71}\], has been devoted to showing that \( (1.2) \), supplemented with regularity and “localization” assumptions, is sufficient for the existence and regularity of \( \varphi \).

The main goal of this paper is to study the parameter dependence of these solutions given that the data depends smoothly on parameters.

In this paper, we will not be concerned with the existence of solutions of \( (\text{1.1}) \), rather, we will assume that a solution exists and establish smoothness with respect to parameters. Of course, there is an extensive literature establishing the existence of solution, see for example the references in \[\text{dW07}\].

There are several contexts in which to study cohomological equations \( (\text{1.1}) \). The most classical one is when \( \varphi, \eta \) take values in a Lie group \( G \). In dynamical systems and geometry, \( (\text{1.1}) \) also appears in some different contexts. Given a bundle \( E \) over \( M \), we take \( \varphi \) to be an object defined on the fiber \( E_x \) and \( \eta_x \) is an action transporting objects in \( E_x \) to objects in \( E_{f(x)} \). For example, in \[\text{dW07}\], one can find an application where \( \varphi_x \) are conformal structures on \( E_x \) (a subspace of \( T_x M \)) and \( \eta_x \) is the natural transportation of the conformal structure by the differential. We note that it is not necessary to assume that the bundle \( E \) is finite dimensional. It suffices to assume that the linear operators lie on a Banach algebra \[\text{BN98}\]. Of course, when the bundle \( E \) is trivial, the linear operators on \( E_x \) can be identified with a matrix group, so that this geometric framework reduces to the Lie group framework with \( G \) a matrix group (or, more generally, a Banach algebra of operators).

One very interesting example, which indeed serves as the main motivation for this paper is when \( \varphi_x \) and \( \eta_x \) are supposed to be \( C^r \) diffeomorphisms of a compact manifold \( N \) and the operation in the right hand side of \( (\text{1.1}) \) is just composition. Though \( \text{Diff}^r(N) \) is certainly a group under composition, it is not a Lie group since the group operation is not differentiable as a map from \( \text{Diff}^r(N) \) to \( \text{Diff}^r(N) \). The problem was considered in \[\text{NT96}\] where it was shown that results on \( (\text{1.1}) \) have implications for rigidity. In \[\text{NT96}\] one can find results on existence of solutions when \( N = \mathbb{T}^d \) and, in \[\text{dW07}\] for a general \( N \).

Both in \[\text{NT96}, \text{dW07}\], from the fact that \( \eta_x \in \text{Diff}^r(N) \) one can only reach the conclusion that \( \varphi_x \in \text{Diff}^{r-R}(N) \) (in \[\text{dW07}\] the regularity loss \( R = 3 \), but in \[\text{NT96}\] \( R \) depends on the dimension of \( N \)).
The main goal of this paper is to overcome this loss of regularity and show that if \( \eta_x \in \text{Diff}^r(N) \) for \( r \geq 1 \), and \( \varphi \in \text{Diff}^1(N) \), then \( \varphi \in \text{Diff}^r(N) \). See Theorem 6 for a more precise formulation.

As it turns out, the main technical tool in the proof of Theorem 6 is to establish results on the smooth dependence on parameters for the solutions of (1.1) which may be of independent interest. We formulate them as Theorem 3, Theorem 5.

When \( G \) is a commutative group smooth dependence on parameters is rather elementary, see Proposition 2, and our main technique is to reduce to the commutative case.

1.1. Sketch of the proofs. In this section we informally present the main ideas behind the proof omitting several of the details which we will treat later.

1.1.1. Some remarks on the relation between bootstrap of regularity and dependence on parameters. Going over the proofs in [NT96] and, more explicitly, in [dW07] it becomes apparent that, the loss of regularity of the solutions is closely related to the fact that the group operation is not differentiable.

Hence, we find it convenient to turn the tables. Rather than considering \( \varphi : M \times N \to N \) as a function from \( M \) to a space of mappings on \( N \), we consider \( \varphi \) as a mapping from \( M \) to \( N \) with parameters from \( N \). In this application, \( N \) plays two roles: as the ambient space for the diffeomorphisms and as the parameter. For the sake of clarity, we will develop our main technical results denoting the target space as a \( N \) and the set of parameters as \( U \). This is, of course, more general, and in many cases, the parameters appear independently of the target space.

More precisely, if we fix \( u \in U \) and suppose that \( \varphi \) is differentiable with respect to \( u \) then the chain rule tells us that

\[
D_u \varphi_{f(x)}(u) = D_u \eta_x \circ \varphi_x(u) \cdot D_u \varphi_x(u)
\]  

(1.3)

where \( D_u \) denotes the derivative with respect to the parameter \( u \in U \). Note that for typographical convenience we will often use \( \eta^u_x \) in place of \( \eta(x,u) \), and \( \varphi^u_x \) in place of \( \varphi(x,u) \).

We see that for each \( u \in U \) (1.3) is an equation of the form (1.1) for the cocycle \( D_u \varphi_x(u) \) with generator

\[
\tilde{\eta}^u_x = D_u \eta_x \cdot \varphi^u_x
\]

(1.4)

In this case \( E \) is the bundle of linear maps from \( T_u N \) to \( T_{\varphi_x(u)} N \).

If \( \varphi^u_x \) depends \( C^1 \) on \( u \) and \( \eta_x \) depends \( C^2 \) on \( u \), then \( \tilde{\eta}_x \) is \( C^1 \) in \( u \). Using the regularity theory for commutative cohomology equations we obtain that \( D_u \varphi^u_x \) is \( C^1 \) with respect to
This means that \( \varphi_x^u \) is \( C^2 \) with respect to \( u \). This argument to improve the regularity can be repeated so long as we can differentiate \( \eta_x \).

1.1.2. The dependence on parameters. The idea of the proof of the smooth dependence on parameters for (1.1) is more involved that the bootstrap of regularity since we need to justify the existence of the first derivative.

For simplicity of notation, we interpret (1.1) in the linear bundle maps framework. This will be the crucial technical result for the bootstrap of regularity in diffeomorphism groups. The case of Lie group valued cocycles will be dealt with in Section 2.3.

We want to show that if \( \varphi^u, \eta^u \) solve (1.1) and \( \eta^u \) depends smoothly on parameters, then \( \varphi^u \) depends smoothly on parameters.

However solutions of (1.1) are not unique. Taking advantage of this it is very easy to construct solutions which depend badly on parameters. It is therefore necessary to impose some additional condition such as that \( \varphi_x^u \) is smooth in \( u \in U \) for some fixed \( p \in M \). For convenience we will take \( p \in M \) to be a periodic point.

In order to prove differentiability we first find a candidate for the derivative using Livšic methods and then argue that this candidate is the true derivative. The key observation is that if we formally differentiate (1.1) with respect to the parameter \( u \) we obtain

\[
D_u \varphi^u_f(x) = D_u \eta_x^u \cdot \varphi_x^u + \eta_x^u \cdot D_u \varphi_x^u
\]

Using (1.1) we get

\[
D_u \varphi^u_f(x) = D_u \eta_x^u \cdot \varphi_x^u + \eta_x^u \cdot (\varphi_x^u)^{-1} \cdot D_u \varphi_x^u
\] (1.5)

Multiplying on the right both sides of (1.5) by \((\varphi_x^u)^{-1}\), we obtain

\[
(\varphi_x^u)^{-1} \cdot D_u \varphi_x^u = (\varphi_x^u)^{-1} \cdot D_u \eta_x^u \cdot \varphi_x^u + (\varphi_x^u)^{-1} \cdot D_u \varphi_x^u
\] (1.6)

where \((\varphi_x^u)^{-1}\) refers to the inverse of the linear map \( \varphi_x^u \). This equation (1.6) is a cohomology equation over a commutative group for the new unknown \( \xi_x = (\varphi_x^u)^{-1} D_u \varphi_x^u \).

We will show that the periodic obstruction (1.2) corresponding to (1.6) is the derivative with respect to parameters of the periodic obstruction corresponding to (1.1).

Hence, applying the results on Livšic equations, we obtain a solution of (1.6). It remains to show that this candidate is the true derivative.

As mentioned before, the solutions of (1.6) are not unique, but, under the hypothesis that \( \varphi_p(u) := \gamma(u) \) is smooth in \( u \) for some \( p \), which we made at the outset, it is natural
to impose that the solution to (1.6) satisfy the normalization

\[ D_u \varphi_p = D_u \gamma. \]

Once we have a candidate for a derivative, we will use a comparison argument, Proposition 4 to show that indeed it is a derivative.

Of course, once we have established the existence of the first derivative, we have also shown that it satisfies (1.6). Hence, to establish the existence of higher derivatives it suffices to use the (much simpler) theory of dependence of parameters in commutative cohomology equations.

The above argument uses very heavily that we are dealing with the framework of linear operators on bundles. The adaptation of the above argument to general Lie groups requires some adaptations of the geometry, see Section 2.3.

2. Smooth dependence on parameters of cohomology equation

In this section, we make precise the previous arguments and establish smooth dependence on parameters for the solutions of (1.1). In Section 2.1 we make precise what we mean by smooth dependence on parameters. In Section 2.2 we present the results when (1.1) is interpreted as an equation between linear bundle maps. This section will be crucial for the case of diffeomorphism groups. In Section 2.3 we present the results for cocycles taking values in a Lie group.

2.1. Notation on regularity. We will understand that a function is \( C^r \) when it admits \( r \) continuous derivatives. When the spaces we consider are not compact, we will assume that all the derivatives are bounded.

For \( 0 < \alpha \leq \text{Lip} \) we will define real valued \( C^\alpha \) functions on a metric space \( N \) in the usual way

\[ H_\alpha(N) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\alpha} \]

As it is well known, for functions taking values on a manifold, there are several equivalent definitions of Hölder functions but not a natural norm [HP69]. For our purposes, any of these definitions will be enough.

We will now introduce the function spaces we will consider. Both are of the “rectangular” type common in dependence on parameters arguments [CFdlL03].

Definition 1. Let \( M \) and \( N \) be compact smooth manifolds, and \( U \subseteq \mathbb{R}^n \) be an open ball. Let \( 0 < \alpha \leq \text{Lip} \) and \( k, r \in \mathbb{N} \). Let \( h : M \times U \to N \).
We will write \( h \in C^{k+\alpha,r} \) if

1. for all \( 0 \leq i \leq r \) and all \( 0 \leq j \leq k \) the derivative \( D_x^i D_u^j \varphi \) exists and is continuous on \( M \times U \).

2. for all \( 0 \leq i \leq r \) and all \( u \in U \) the derivative \( D_u^i \varphi(\cdot, u) \in C^\alpha(M) \) and the Hölder constant does not depend on \( u \).

We also define \( \hat{C}^{k+\alpha,r}(U,M) = C^r(U,C^{k+\alpha}(M)) \). That, is the set of functions \( h : M \times U \rightarrow N \) such that \( u \rightarrow h(\cdot, u) \) is \( C^r \) when we give \( \varphi(\cdot, u) \) the \( C^{k+\alpha} \) topology.

The reason to introduce these spaces is that \( C^{k+\alpha,r} \) is natural when performing geometric constructions involving derivatives. The space \( \hat{C}^{k+\alpha,r} \) is natural when we consider dependence on parameters of commutative Livšic equations. Of course, the two spaces are closely related. The relation is formulated in the following proposition.

**Proposition 1.** For any \( 0 < \alpha' < \alpha \leq \text{Lip}, k, r \in \mathbb{N} \), we have:

\[
\hat{C}^{k+\alpha,r} \subset C^{k+\alpha,r} \subset \hat{C}^{k+\alpha',r} \tag{2.1}
\]

The simple example \( \varphi(u,x) = (x-u)^{k+\alpha} \) satisfies \( \varphi \in C^{k+\alpha,0}, \varphi \notin \hat{C}^{k+\alpha,0} \). Integrating with respect to \( u \) one can obtain similar examples for all \( r \in \mathbb{N} \).

**Proof.** The first inclusion of (2.1) is obvious.

Let \( \varphi \in C^{k+\alpha,r} \). Let \( \alpha' = \alpha \cdot \theta \) for \( 0 < \theta < 1 \). We wish to show that \( \varphi \in C^r(U,C^{k+\alpha'}(M)) \). This is equivalent to \( \partial_u^i \varphi \in C^{0}(U,C^{k+\alpha'}(M)) \) for all multi-indices \( i \) with \( 0 \leq |i| \leq r \). Fix \( u \in U \) and consider a compact \( K \) with \( u \in K \subset U \). Restricted to \( M \times K \) the function \( \varphi \) and all its partial derivatives are uniformly continuous. So we have

\[
\| \partial_x^j \partial_u^i \varphi(\cdot, u) - \partial_x^j \partial_u^i \varphi(\cdot, u') \|_0
\]

is controlled by \( d(u,u') \), for all multi-indices \( j \) with \( 0 \leq |j| \leq k \). It remains to show that for \( j = k \) we have

\[
\| \partial_x^j \partial_u^i \varphi(\cdot, u) - \partial_x^j \partial_u^i \varphi(\cdot, u') \|_{\alpha'}
\]

is controlled. For any multi-index \( j \) with \( |j| = k \) we have from the Kolmogorov-Hadamard inequalities \([dLLO99]\)

\[
\| \partial_x^j \partial_u^i \varphi(\cdot, u) - \partial_x^j \partial_u^i \varphi(\cdot, u') \|_{\alpha'} \leq C \| \partial_x^j \partial_u^i \varphi(\cdot, u) - \partial_x^j \partial_u^i \varphi(\cdot, u') \|_{\alpha} \| \partial_x^j \partial_u^i \varphi(\cdot, u) - \partial_x^j \partial_u^i \varphi(\cdot, u') \|_{0-\theta}
\]
By definition of $\varphi \in C^{k+\alpha,r}$ we have that $\|\partial^i_u \partial^j_x \varphi(\cdot,u)\|_\alpha$ is bounded independent of $u \in U$ and hence
\[ \|\partial^i_u \partial^j_x \varphi(\cdot,u) - \partial^i_u \partial^j_x \varphi(\cdot,u')\|_0 \]
is controlled since $\partial^i_u \partial^j_x \varphi$ is uniformly continuous. Consequently $\varphi \in C^r(U, C^{k+\alpha'}(M))$. Note that we do not claim uniform continuity in $u$. □

Next, we discuss the regularity properties of the commutative cohomology equation. The results for $k = 0$ appear in [Liv71, Liv72] and the results for $k > 0$ appear in [dlLMM86]. The following result will be one of our basic tools later.

**Proposition 2.** Let $M$ be a compact manifold and $f$ a transitive Anosov diffeomorphism on $M$. Let $p \in M$ be a periodic point for $f$.

Consider the commutative cohomology equation
\[ \varphi^u \circ f(x) - \varphi^u(x) = \eta^u(x) \]
\[ \varphi(p, \cdot) \in C^r(U) \] (2.2)
Assume that for all values of $u$, the periodic orbit obstruction with $\eta^u$ vanishes. Then,

1. if $\eta \in \hat{C}^{k+\alpha,r}$, then $\varphi \in \hat{C}^{k+\alpha,r}$, and
2. if $\eta \in C^{k+\alpha,r}$, then $\varphi \in C^{k+\alpha,r}$.

**Proof.** The first statement is obvious since we are considering the solution of a linear equation and [dlLMM86] showed that the solution operator is continuous from $C^{k+\alpha}$ to $C^{k+\alpha}/\mathbb{R}$.

The second statement follows because, by the previous argument, we have that $\varphi \in \hat{C}^{k+\alpha',r}$. In particular for $|i| \leq r$, $\partial^i_u \varphi(\cdot,u) \in C^{k+\alpha'}$ but this function satisfies
\[ \partial^i_u \varphi^u \circ f - \partial^i_u \varphi^u = \partial^i_u \eta^u \]
Since we assumed that $\eta \in C^{k+\alpha,r}$ we have that $\partial^i_u \eta(\cdot,u) \in C^{k+\alpha}(M)$ and, by the results of [dlLMM86], $\partial^i_u \varphi(\cdot,u) \in C^{k+\alpha}(M)$. □

2.2. **Cohomology Equations for Bundle Maps.** As we have mentioned before, cohomology equations for bundle maps appear naturally when we consider the linearization of a dynamical system or cohomology equations for cocycles taking values in diffeomorphism groups.

**Theorem 3.** Let $M$ and $N$ be compact smooth manifolds, $U \subseteq \mathbb{R}^n$ open, and $E$ a bundle over $N$. Let $0 < k + \alpha \leq 1$ and $r \in \mathbb{N}$ with $r \geq 1$. 


Let $f : M \to M$ be a transitive $C^r$ Anosov diffeomorphism, $\sigma : U \to N$ with $\sigma \in C^r(N)$ and $\tau : M \times U \to N$ with $\tau \in C^{k+\alpha,r}$ (resp. $\tau \in \hat{C}^{k+\alpha,r}$).

Suppose that we have linear maps
\[
\varphi^u_x : E_{\sigma(u)} \to E_{\tau(x,u)}, \\
\eta^u_x : E_{\tau(x,u)} \to E_{\tau(f(x),u)}.
\]
such that for all $u \in U$ we have
\[
\varphi^u_{f(x)} = \eta^u_x \cdot \varphi^u_x. \tag{2.3}
\]

If $\eta \in C^{k+\alpha,r}$ (resp. $\eta \in \hat{C}^{k+\alpha,r}$) and there exists a periodic point $p \in M$ such that $\varphi(p, \cdot) \in C^r(U)$ then $\varphi \in C^{k+\alpha,r}$ (resp. $\varphi \in \hat{C}^{k+\alpha,r}$).

The most difficult case of Theorem 3 is the case when $r = 1$. The higher differentiability cases follow from this one rather straightforwardly. The case $r = 1$ of Theorem 3 will be the inductive step for the bootstrap of regularity for cohomology equations on diffeomorphism groups.

The proof we present of the $r = 1$ case uses that $f$ is transitive but the subsequent bootstrap argument does not require that $f$ is transitive.

As indicated before, we will obtain a candidate for a derivative and, then, we argue it is indeed a true derivative.

Smooth dependence is a local question. For that reason we consider a local trivialization of the bundle $E$ about the points $\sigma(u)$ and $\tau(x,u)$. The base space portion of the map $\varphi^u_x$ is as smooth as the minimum smoothness of $\sigma$ and $\tau$. We will concentrate therefore on the smoothness of the linear operator in the fibers.

Proof. We note that if we could take derivatives in (2.3) then $D_u \varphi^u_x$, the derivative of $\varphi^u_x$ with respect to $u$, would satisfy
\[
D_u \varphi^u_{f(x)} = D_u \eta^u_x \cdot \varphi^u_x + \eta^u_x \cdot D_u \varphi^u_x \tag{2.4}
\]
Observing that $\eta^u_x = \varphi^u_{f(x)} \cdot (\varphi^u_x)^{-1}$ we get
\[
(\varphi^u_{f(x)})^{-1} \cdot D_u \varphi^u_{f(x)} = (\varphi^u_{f(x)})^{-1} \cdot D_u \eta^u_x \cdot \varphi^u_x + (\varphi^u_x)^{-1} \cdot D_u \varphi^u_x. \tag{2.5}
\]
Writing
\[
\xi^u_x := (\varphi^u_x)^{-1} \cdot D_u \varphi^u_x \tag{2.6}
\]
we see that $\xi^u_x$ would satisfy
\[
\xi^u_{f(x)} - \xi^u_x = (\varphi^u_{f(x)})^{-1} \cdot D_u \eta^u_x \cdot \varphi^u_x \tag{2.7}
\]
Note that (2.7) is a commutative cohomology equation with the group operation being addition on a vector space.

In this case, as we will show, the methods of [Liv72, dLMM86] to obtain existence and regularity for $\xi$. Using (2.6) we will obtain a candidate for $D_u\varphi^u_x$.

Since we are assuming that there are solutions of (2.3) for all $u$ in an open set $U$, we have that for any $p$, $f^n(p) = p$

$$\eta^{u}_{f^{n-1}(p)} \cdots \eta^{u}_{p} = \text{Id}. \quad (2.8)$$

Differentiating (2.8) with respect to $u$ we obtain

$$0 = D_u \eta^{u}_{f^{n-1}(p)} \cdot \eta^{u}_{f^{n-2}(p)} \cdots \eta^{u}_{p} + \eta^{u}_{f^{n-1}(p)} \cdot D_u \eta^{u}_{f^{n-2}(p)} \cdot \eta^{u}_{f^{n-3}(p)} \cdots \eta^{u}_{p}$$

$$+ \cdots + \eta^{u}_{f^{n-1}(p)} \cdot \eta^{u}_{f^{n-2}(p)} \cdots \eta^{u}_{f^{1}(p)} \cdot D_u \eta^{u}_{p}.$$ Using the (2.3) to express the products of $\eta$, we obtain

$$0 = D_u \varphi^{u}_{f^{n-1}(p)} \cdot \varphi^{u}_{f^{n-1}(p)} \cdot (\varphi^{u}_{p})^{-1}$$

$$+ \varphi^{u}_{p} \cdot (\varphi^{u}_{f^{n-1}(p)})^{-1} \cdot D_u \eta^{u}_{f^{n-2}(p)} \cdot \varphi^{u}_{f^{n-2}(p)} \cdot (\varphi^{u}_{p})^{-1}$$

$$+ \cdots + \varphi^{u}_{p} \cdot (\varphi^{u}_{f^{1}(p)})^{-1} \cdot D_u \eta^{u}_{p}.$$

Multiplying (2.9) on the left by $(\varphi^{u}_{p})^{-1}$ and multiplying by $\varphi^{u}_{p}$ on the right we obtain

$$0 = (\varphi^{u}_{p})^{-1} \cdot D_u \eta^{u}_{f^{n-1}(p)} \cdot \varphi^{u}_{f^{n-1}(p)} + (\varphi^{u}_{p})^{-1} \cdot D_u \eta^{u}_{f^{n-2}(p)} \cdot \varphi^{u}_{f^{n-2}(p)}$$

$$+ \cdots + (\varphi^{u}_{p})^{-1} \cdot D_u \eta^{u}_{p} \cdot \varphi^{u}_{p}, \quad (2.10)$$

which shows that the periodic orbit obstruction for the existence of a solution to the cohomology equation (2.7) vanishes. Hence we have a solution $\xi^u_x$ to (2.7).

Using (2.6) we see that we have a candidate for the derivative,

$$D_u \varphi^u_x = \varphi^u_x \cdot \xi^u_x.$$ It remains to show that this candidate, which we obtained by a formal argument is actually the derivative.

We start by proving that all the solutions of (2.3) are differentiable in the stable manifold of a periodic point. For the sake of notation, and without any loss of generality, we will consider a fixed point $p$.

Applying (2.3) repeatedly we have

$$\varphi^{u}_{f^{n+1}(x)} = \eta^{u}_{f^n(x)} \cdots \eta^{u}_{x} \cdot \varphi^u_x. \quad (2.11)$$

For $x \in W^s_p$ we have $d(f^n(x), p) \leq C^u \lambda^n$ for some $0 < \lambda < 1$, $n \geq 0$. Since $\eta^{u}_{x}$ is Hölder in $x$, and $\eta^{u}_{p} = \text{Id}$ is continuous, in any smooth local trivialization around $p$ we obtain

$$\|\eta^{u}_{f^n(x)} - \text{Id}\| \leq C \lambda^n.$$
Therefore we can pass to the limit in (2.11) and obtain for \( x \in W^s_p \)
\[
\varphi^u_p = \lim_{n \to \infty} \eta^u_{f^n(x)} \cdots \eta^u_x \cdot \varphi^u_x
\]  
(2.12)
where the convergence in (2.12) is uniform in bounded sets of \( W^s_p \) (in the topology of \( W^s_p \)).

We will show that (2.12) can be differentiated with respect to \( u \). By the product rule
\[
D_u \left( \eta^u_{f^n(x)} \cdots \eta^u_x \right)
= D_u \eta^u_{f^n(x)} \cdot \eta^u_{f^{n-1}(x)} \cdots \eta^u_x + D_u \eta^u_{f^{n-1}(x)} \cdot \eta^u_{f^{n-2}(x)} \cdots \eta^u_x
+ \cdots + \eta^u_{f^n(x)} \cdots \eta^u_x D_u \eta^u_x
= \varphi^u_{f^{n+1}(x)} \cdot \left( \varphi^u_{f^n(x)} \right)^{-1} \cdot D_u \eta^u_{f^n(x)} \cdot \varphi^u_{f^n(x)} \cdot \left( \varphi^u_x \right)^{-1} + \varphi^u_{f^{n+1}(x)} \cdot \left( \varphi^u_f(x) \right)^{-1} \cdot D_u \eta^u_x \cdot \varphi^u_x \cdot \left( \varphi^u_x \right)^{-1}
= \varphi^u_{f^{n+1}(x)} \cdot \left( \sum_{i=0}^{n-1} \left( \varphi^u_{f^{i+1}(x)} \right)^{-1} \cdot D_u \eta^u_{f^i(x)} \cdot \varphi^u_{f^i(x)} \right) \cdot \left( \varphi^u_x \right)^{-1}
\]  
(2.13)
where we have used again (2.3). Since \( p \) is a fixed point using (2.9) we obtain that
\[
\left( \varphi^u_p \right)^{-1} \cdot D_u \eta^u_p \cdot \varphi^u_p = 0.
\]
Because of the assumed regularity on \( \varphi_x \), \( D_u \eta^u_x \) and the exponentially fast convergence of \( f^n(x) \) to \( p \), we obtain that the general term in (2.13) converges to zero exponentially.

By the Weierstrass M-test we obtain that \( D_u \left( \eta^u_{f^n(x)} \cdots \eta^u_x \right) \) converges uniformly on compact subsets of \( W^s_p \). Therefore, the limit is the derivative of \( \lim_{n \to \infty} \eta^u_{f^n(x)} \cdots \eta^u_x \) and from that, it follows immediately that \( \varphi^u_p \) is differentiable with respect to \( u \) for \( x \in W^s_p \).

Of course, the argument so far allows only to conclude that \( D_u \varphi^u_x \) is bounded on bounded sets of \( W^s_p \). This does not show that it is bounded on \( M \) since \( W^s_p \) is unbounded.

Nevertheless, we realize that the derivative solves (2.4) on \( W^s_p \). Of course, so does the candidate for the derivative that we produced earlier. From the fact that these two functions satisfy the same functional equation in \( W^s_p \), and that they agree at zero, we will show that they agree. This will allow us to conclude that the candidate is indeed the derivative in \( W^s_p \). Using that it is a continuous function on the whole manifold, there is a standard argument that shows it is the derivative everywhere.

**Proposition 4.** Let \( p \in M \) be a fixed point of \( f \). Let \( \xi_1, \xi_2 \) be continuous functions on \( W^s_p \). Assume that \( \xi_1(p) = \xi_2(p) \) and that both of them satisfy (2.7). Then \( \xi_1(x) = \xi_2(x) \) for all \( x \in W^s_p \).
Proof. Note that \( A(x) \equiv \xi_1(x) - \xi_2(x) \) satisfies \( A(x) = A(f(x)) \). For every \( x \in W^s_p \) we have \( \lim_{n \to \infty} f^n(x) = p \). Since \( A \) is continuous, and \( A(p) = 0 \) we see that
\[
A(x) = A(f^n(x)) = \lim_{n \to \infty} A(f^n(x)) = A(\lim_{n \to \infty} f^n(x)) = A(p)
\]
\[= 0.\]
Thus \( \xi_1(x) = \xi_2(x) \) for all \( x \in W^s_x \). \( \square \)

We have now shown that the function is differentiable in a leaf of the foliation which is dense. Furthermore, \( D_u \varphi^u_x \) extends continuously – indeed Hölder continuously – to the whole manifold. In this case, an elementary real analysis argument, which we will present now, shows that \( \varphi^u_x \) is everywhere differentiable.

Since we just want to show that the continuous function is indeed a derivative in the stable leaf, we can just take a trivialization of the bundles in a small neighborhood. We will use the same notation for the objects in the trivialization and the corresponding one in the bundles. This allows us to subtract objects in different fibers.

If \( \psi(s) \) is any smooth path in \( U \) contained in the trivializing neighborhood with \( \psi(0) = u_0, \psi_1 = u_1 \) we have, for \( x \in W^s_p \)
\[
\varphi^u_{x1} - \varphi^u_{x2} = \int_0^1 D_u \varphi_x^{\psi(s)} \psi'(s) \, ds \tag{2.14}
\]
Both sides of the equation are Hölder in \( x \). Therefore, we can pass to the limit in (2.14) and conclude that the set of points \( x \) for which (2.14) holds is closed. But we had already shown that (2.14) held for \( x \in W^s_p \), which is a dense set since \( f \) is transitive. Thus (2.14) holds for every \( x \in M \). This shows immediately that our candidate is indeed the true derivative and concludes the proof of the case \( r = 1 \) of Theorem 3.

It is now relatively simple to obtain higher derivatives. The auxiliary function \( \xi \) satisfies the commutative cohomology equation (2.7). By assumption \( D_u \eta \in C^{k+\alpha,r-1} \) (resp. \( D_u \eta \in \hat{C}^{k+\alpha,r-1} \)) hence, if \( \varphi \in C^{k+\alpha,m} \) (resp. \( \varphi \in \hat{C}^{k+\alpha,m} \)) for \( m \leq r - 1 \) then the right hand side of (2.7) is in \( C^{k+\alpha,m} \) (resp. \( \hat{C}^{k+\alpha,m} \)). Applying Proposition 2 we then obtain \( D_u \varphi \in C^{k+\alpha,m} \) (resp. \( D_u \varphi \in \hat{C}^{k+\alpha,m} \)) and thus \( \varphi \in C^{k+\alpha,m+1} \) (resp. \( \varphi \in \hat{C}^{k+\alpha,m+1} \)). The induction stops at \( m = r - 1 \) when we have exhausted the regularity of \( D_u \eta \). At this point \( \varphi \in C^{k+\alpha,r} \) (resp. \( \varphi \in \hat{C}^{k+\alpha,r} \)) as required. \( \square \)
2.3. Cohomology Equations for Lie Group Valued Cocycles. In this section we
consider the dependence on parameters of the solution to (1.1) when \( \eta \), and \( \varphi \) are func-
tions taking values in a Lie group \( G \). We denote the Lie algebra of the Lie group \( G \) by \( g \)
and we denote the identity in \( G \) by \( e \).

The proof follows along the same lines as the proof of Theorem 3. We derive a func-
tional equation for a candidate for the first derivative, show that there is a solution for
this equation, and that the candidate is a true derivative. Finally we use a bootstrapping
argument to get full regularity. The main difference with the previous section is that we
have to deal with the fact that the group operation is not just the product of linear
operations, so that the derivatives of the functional equation involve the derivatives of
the group operation.

We introduce the notation
\[
L_g h = g \cdot h \\
R_g h = h \cdot g
\]
(2.15)

where \( \cdot \) denotes the group operation.

It follows directly from the definitions of \( L, R \) that
\[
L_g \circ L_h = L_{g \cdot h} \\
R_g \circ R_h = R_{h \cdot g}
\]
(2.16)

It is immediate to show that if \( g^u, h^u \) are smooth families
\[
D_u(g^u \cdot h^u) = DL_{g^u}(h^u)D_u h^u + DR_{h^u}(g^u)D_u g^u
\]
(2.17)

**Theorem 5.** Let \( M \) be a compact manifold, \( U \subset \mathbb{R}^d \) open, \( G \) be a Lie group, \( f \) a
transitive Anosov diffeomorphism of \( M \), and \( p \in M \) a periodic point for \( f \). Suppose that
\( \eta : M \times U \to G \) with \( \eta \in C^{k+\alpha,r} \) (resp. \( \eta \in \hat{C}^{k+\alpha,r} \)) for \( 0 < \alpha \leq \text{Lip} \) and \( k, r \in \mathbb{N} \) with
\( r \geq 1 \).

If \( \varphi : M \times U \to G \) solves
\[
\varphi^u_f(x) = \eta^u_x \cdot \varphi^u_x
\]
and \( \varphi_p \in C^r(U, G) \) then \( \varphi \in C^{k+\alpha,r} \) (resp. \( \varphi \in \hat{C}^{k+\alpha,r} \)).

The definition of the spaces \( C^{k+\alpha,r} \) and \( \hat{C}^{k+\alpha,r} \) appears in Section 2.1

**Proof.** Taking derivatives of (1.1) and applying the product rule we obtain that if there
is a derivative of \( D_u \varphi^u_x \), it should satisfy:
\[
D_u \varphi^u_f(x) = DL_{\eta^u_x}(\varphi^u_x)D_u \varphi^u_x + DR_{\varphi^u_x}(\eta^u_x)D_u \eta^u_x
\]
(2.18)
We introduce a function \( \xi : M \times U \to \mathfrak{g} \) by
\[
D_u \varphi_p^u = DL_{\varphi_p^u}(e)\xi_p^u
\]  
(2.19)

Introducing the notation \([2.19]\) is geometrically natural because we want to transport all the infinitesimal derivatives to the identity, so that \( \xi_x^u \) takes values in the Lie algebra \( \mathfrak{g} \).

In terms of \( \xi \), the equation \((2.18)\) becomes
\[
DL_{\varphi_f(x)}(e)\xi_f^u(x) = DL_{\varphi_f^u(x)}(e)\xi_x^u + DR_{\varphi_f^u(x)}(\eta_x^u)D_u\eta_x^u.
\]  
(2.20)

The first factor of the first term in \((2.20)\) can be simplified
\[
DL_{\eta_f^u}(\varphi_f^u(x)) \cdot DL_{\varphi_f^u(x)}(e) = D_u(L_{\eta_f^u} \circ L_{\varphi_f^u})(e) = D_u(L_{\eta_f^u} \varphi_f^u)(e) = DL_{\varphi_f^u(x)}(e).
\]  
(2.21)

Substituting \((2.21)\) into \((2.20)\) we obtain
\[
DL_{\varphi_f^u(x)}(e)\xi_f^u(x) = DL_{\varphi_f^u(x)}(e)\xi_x^u + DR_{\varphi_f^u(x)}(\eta_x^u)D_u\eta_x^u.
\]  
(2.22)

Multiplying \((2.22)\) in the left by \((DL_{\varphi_f^u(x)}(e))^{-1}\) we obtain:
\[
\xi_f^u(x) = \xi_x^u + (DL_{\varphi_f^u(x)}(e))^{-1}DR_{\varphi_f^u(x)}(\eta_x^u)D_u\eta_x^u.
\]  
(2.23)

Equation \((2.23)\) can be simplified further; from \((2.16)\), we have
\[
\text{Id} = L_{(\varphi_f^u(x))^{-1}} \circ L_{\varphi_f^u(x)}
\]
where \( \text{Id} \) is the identity map on \( G \). Thus, by the chain rule, we have
\[
\text{Id} = DL_{(\varphi_f^u(x))^{-1}}(\varphi_f^u(x)) DL_{\varphi_f^u(x)}(e)
\]
where \( \text{Id} \) is the identity map on \( \mathfrak{g} \). Hence
\[
(DL_{\varphi_f^u(x)}(e))^{-1} = DL_{(\varphi_f^u(x))^{-1}}(\varphi_f^u(x)).
\]  
(2.24)

Therefore, \((2.23)\) can be rewritten as
\[
\xi_f^u(x) = \xi_x^u + DL_{(\varphi_f^u(x))^{-1}}(\varphi_f^u(x)) DR_{\varphi_f^u(x)}(\eta_x^u) D_u\eta_x^u.
\]  
(2.25)

We are assuming \( \varphi^u \) is a \( C^r \) function of \( u \). Thus, taking derivatives, and using \((2.19)\) we obtain
\[
D_u \varphi_p^u = DL_{\varphi_p^u}(e)\xi_p^u
\]  
(2.26)

so we see that \( \xi_p : U \to \mathfrak{g} \) is \( C^r \).

In summary, if the function \( \varphi_x^u \) is differentiable then the \( \mathfrak{g} \) valued function \( \xi \) introduced in \((2.19)\) would satisfy \((2.25)\). Equation \((2.25)\) is a cohomology equation for functions taking values in \( \mathfrak{g} \). It is therefore a commutative cohomology equation. Thus a necessary and sufficient condition for the existence of a smooth solution is the vanishing of the periodic orbit obstruction.
Finally we observe that

$$e = \eta_{f^{n-1}(p)}^u \cdots \eta_p^u$$

$$= (\varphi_p^u)^{-1} \cdot \eta_{f^{n-1}(p)}^u \cdot \eta_p^u \cdot \varphi_p^u$$

Differentiating with respect to $u$, we obtain:

$$0 = DR_{\eta_{f^{n-1}(p)}^u \cdots \eta_p^u \cdot \varphi_p^u}((\varphi_p^u)^{-1}) D_u(\varphi_p^u)^{-1}$$

$$+ DL(\varphi_p^u)^{-1}(\eta_{f^{n-1}(p)}^u \cdots \eta_p^u \cdot \varphi_p^u) DR_{\eta_{f^{n-2}(p)}^u \cdots \eta_p^u \cdot \varphi_p^u}((\eta_{f^{n-1}(p)}^u) D_u \eta_{f^{n-1}(p)}^u$$

$$+ \cdots + DL(\varphi_p^u)^{-1}(\eta_{f^{n-2}(p)}^u \cdots \eta_p^u \cdot \varphi_p^u) DR_{\varphi_p^u}((\eta_p^u) D_u \eta_p^u$$

$$+ DL(\varphi_p^u)^{-1}(\eta_p^u \cdot \varphi_p^u) D_u \varphi_p^u$$

(2.27)

Using (1.1) to reduce the products, we transform (2.27) into

$$0 = DR_{\varphi_p^u}((\varphi_p^u)^{-1}) D_u(\varphi_p^u)^{-1}$$

$$+ DL(\varphi_p^u)^{-1}(\eta_{f^{n-1}(p)}^u) DR_{\varphi_p^u}((\eta_{f^{n-1}(p)}^u) D_u \eta_{f^{n-1}(p)}^u$$

$$+ DL(\varphi_p^u)^{-1}(\eta_{f^{n-2}(p)}^u) DR_{\varphi_p^u}((\eta_{f^{n-2}(p)}^u) D_u \eta_{f^{n-2}(p)}^u$$

$$+ \cdots + DL(\varphi_p^u)^{-1}(\eta_{f^{n}(p)}^u) DR_{\varphi_p^u}((\eta_{f^{n}(p)}^u) D_u \eta_{f^{n}(p)}^u$$

$$+ DL(\varphi_p^u)^{-1}(\varphi_p^u) D_u \varphi_p^u$$

(2.28)

Finally we observe that

$$DR_{\varphi_p^u}((\varphi_p^u)^{-1}) D_u(\varphi_p^u)^{-1} + DL(\varphi_p^u)^{-1}(\varphi_p^u) D_u \varphi_p^u = D_u((\varphi_p^u)^{-1} \cdot \varphi_p^u) = 0$$

and hence (2.28) becomes

$$0 = DL(\varphi_p^u)^{-1}(\eta_{f^{n-1}(p)}^u) DR_{\varphi_p^u}((\eta_{f^{n-1}(p)}^u) D_u \eta_{f^{n-1}(p)}^u$$

$$+ DL(\varphi_p^u)^{-1}(\eta_{f^{n-2}(p)}^u) DR_{\varphi_p^u}((\eta_{f^{n-2}(p)}^u) D_u \eta_{f^{n-2}(p)}^u$$

$$+ \cdots + DL(\varphi_p^u)^{-1}(\eta_{f^{n}(p)}^u) DR_{\varphi_p^u}((\eta_{f^{n}(p)}^u) D_u \eta_{f^{n}(p)}^u$$

$$+ DL(\varphi_p^u)^{-1}(\varphi_p^u) D_u \varphi_p^u$$

which shows the vanishing of the periodic orbit obstruction for (2.25).

The rest of the argument does not need any modification from the argument in the previous case and we just refer to the previous section. We argue that $\varphi$ is differentiable in $W^s_p$ for some $p \in M$ and that the candidate for the derivative is indeed the true derivative. \qed
3. Cohomology equations on diffeomorphism groups

In this section, we consider (1.1) when $\eta$ and $\varphi$ take values in the diffeomorphism group of a compact manifold. As before $f$ is a transitive Anosov diffeomorphism. In order to emphasize that the parameter space and target space are the same smooth manifold $N$ and to match the notation in [dW07] we use $y$ in place of $u$.

**Theorem 6.** Let $\eta \in C^{k+\alpha}(M, \text{Diff}^r(N))$ and $p$ is a periodic point of $f$.

If $\varphi \in C^{k+\alpha}(M, \text{Diff}^1(N))$ solves

$$\varphi_{f(x)} = \eta_x \circ \varphi_x$$

and $\varphi_p \in \text{Diff}^r(N)$ then $\varphi \in C^{k+\alpha}(M, \text{Diff}^r(N))$.

Notice that questions of differentiability is entirely local and hence the global structure of the diffeomorphism group, which is quite complicated, does not enter into the argument. This should be compared with the existence argument in [dW07] for $k=0$ which devotes considerable effort to defining the metric on $\text{Diff}^r(N)$. The local differential structure on $\text{Diff}^r(N)$ can be found in [Ban97]. We remark that to show that $\varphi : M \to \text{Diff}^r(N)$ is $C^k + \alpha$ it suffices to show that all partial derivatives in $N$ are $C^k + \alpha$ uniformly.

**Proof.** Taking derivatives of (1.1) with respect to the variable $y$ in the manifold $N$ we obtain

$$D_y \varphi_{f(x)}(y) = D_y \eta_x \circ \varphi_x(y) \cdot D_y \varphi_x(y).$$

This is possible since for all $x$ we have $\varphi_x \in \text{Diff}^1(N)$. Now we write $\psi^y = D_y \varphi_x(y)$ and $\hat{\eta}^y_x = D_y \eta_x \circ \varphi_x(y)$. The equation is then

$$\psi^y_{f(x)} = \hat{\eta}^y_x \cdot \psi^y_x$$

We have

$$\psi^y_x : T_y N \to T_{\varphi_x(y)} N$$

$$\hat{\eta}^y_x : T_{\varphi_x(y)} N \to T_{\varphi_{f(x)}(y)} N.$$ 

Thus we have the set up of Theorem 3 with $\sigma(y) = y$ and $\tau(x, y) = \varphi^y_x$. Clearly $\sigma \in C^\infty(N)$.

If we assume that $\varphi \in C^{k+\alpha}(M, \text{Diff}^m(N))$ then $\tau \in C^{k+\alpha,m}$. By hypothesis $\eta \in C^{k+\alpha}(M, \text{Diff}^r(N))$ thus for $m \leq r - 1 \hat{\eta} \in C^{k+\alpha,m}$. Applying Theorem 3 we obtain that $D_y \varphi \in C^{k+\alpha,m}$. Thus $\varphi \in C^{k+\alpha,m+1}$. We proceed by induction until $m = r - 1$ at which point we have $\varphi \in C^{k+\alpha,r}$. 

We have addressed on $\varphi_x$ and not $(\varphi_x)^{-1}$ however since we know that $\varphi_x \in \text{Diff}^1(N)$ the inverse function theorem shows that $(\varphi_x)^{-1}$ is as smooth as $\varphi_x$ and depends on parameters with the same smoothness.

□

**Remark.** Note that the main result of [dW07] is that if the periodic orbit obstruction is met, $f$ is hyperbolic enough, and $\eta \in C^\alpha(M, \text{Diff}^r(N))$, $r \geq 4$ is close enough to the identity, then the cohomology equation (3.1) has a solution $\varphi \in C^\alpha(M, \text{Diff}^1(N))$. Hence we can apply Theorem 6 to show that in fact $\varphi \in C^\alpha(M, \text{Diff}^r(N))$.

**Remark.** Though the existence result requires both a localization assumption and a relation between the Hölder exponent and the hyperbolicity of $f$ the bootstrap result requires none of these assumptions.

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