On 3–decomposable geometric drawings of $K_n$

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Abstract

The point sets of all known optimal rectilinear drawings of $K_n$ share an unmistakeable clustering property, the so–called 3–decomposability. It is widely believed that the underlying point sets of all optimal rectilinear drawings of $K_n$ are 3–decomposable. We give a lower bound for the minimum number of ($\leq k$)–sets in a 3–decomposable $n$–point set. As an immediate corollary, we obtain a lower bound for the crossing number $\overline{\text{cr}}(D)$ of any rectilinear drawing $D$ of $K_n$ with underlying 3–decomposable point set, namely $\overline{\text{cr}}(D) > \frac{15 - \pi^2}{15} \binom{n}{3} + \Theta(n^3) \approx 0.380029 \binom{n}{3} + \Theta(n^3)$. This closes this gap between the best known lower and upper bounds for the rectilinear crossing number $\overline{\text{cr}}(K_n)$ of $K_n$ by over 40%, under the assumption of 3–decomposability.

1 Introduction

Figure 1 shows the point set of an optimal (crossing minimal) rectilinear drawing of $K_9$, with an evident partition of the 9 vertices into 3 highly structured clusters of 3 vertices each:

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  ●
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Figure 1: The points in this optimal drawing of $K_9$ are clustered into 3 sets.

A similar, natural, highly structured partition into 3 clusters of equal size is observed in every known optimal drawing of $K_n$, for every $n$ multiple of 3 (see [4]). Even for those values of $n$ (namely, $n > 27$) for which the exact rectilinear crossing number $\overline{\text{cr}}(K_n)$ of $K_n$ is not known, the best available examples also share this property [4].

In all these examples, a set $S$ of $n$ points in general position is partitioned into sets $A$, $B$, and $C$, with $|A| = |B| = |C| = n/3$ with the following properties:

(i) There is a directed line $\ell_1$ such that, as we traverse $\ell_1$, we find the $\ell_1$–orthogonal projections of the points in $A$, then the $\ell_1$–orthogonal projections of the points in $B$, and then the $\ell_1$–orthogonal projections of the points in $C$;

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(ii) there is a directed line $\ell_2$ such that, as we traverse $\ell_2$, we find the $\ell_2$–orthogonal projections of the points in $B$, then the $\ell_2$–orthogonal projections of the points in $A$, and then the $\ell_2$–orthogonal projections of the points in $C$; and

(iii) there is a directed line $\ell_3$ such that, as we traverse $\ell_3$, we find the $\ell_3$–orthogonal projections of the points in $B$, then the $\ell_3$–orthogonal projections of the points in $C$, and then the $\ell_3$–orthogonal projections of the points in $A$.

**Definition** A point set that satisfies conditions (i)–(iii) above is 3–decomposable. We also say that the underlying rectilinear drawing of $K_n$ is 3–decomposable.

A possible choice of $\ell_1$, $\ell_2$, and $\ell_3$ for the example in Figure 1 is illustrated in Figure 2.

![Figure 2](image)

Figure 2: The 9–point set $S$ gets naturally partitioned into three clusters $A$, $B$, and $C$. The $\ell_1$–, $\ell_2$–, and $\ell_3$–orthogonal projections of $A$, $B$, and $C$ satisfy conditions (i)–(iii), and so $S = A \cup B \cup C$ is 3–decomposable.

### 1.1 The main result

It is widely believed that all optimal rectilinear drawings of $K_n$ are 3–decomposable. One of our main results in this paper is the following lower bound for the number of crossings in all such drawings.

**Theorem 1** Let $D$ be a 3–decomposable rectilinear drawing of $K_n$. Then the number $cr(D)$ of crossings in $D$ satisfies

$$cr(D) \geq \frac{2}{27} \left(15 - \pi^2\right) \left(\frac{n}{4}\right) + \Theta(n^3) \approx 0.380029 \left(\frac{n}{4}\right) + \Theta(n^3).$$

The best known general lower and upper bounds for the rectilinear crossing number $cr(K_n)$ are $0.37968 \binom{n}{4} + \Theta(n^3) \leq cr(K_n) \leq 0.38054 \binom{n}{4} + \Theta(n^3)$ (see [3] and [2]). Thus the
bound given by Theorem 1 closes this gap by over 40%, under the (quite feasible) assumption of 3–decomposability.

To prove Theorem 1 (in Section 2), we exploit the close relationship between rectilinear crossing numbers and \((\leq k)\)-sets, unveiled independently by Ábrego and Fernández–Merchant [1] and by Lovász et al. [8].

Recall that a \((\leq k)\)-set of a point set \(S\) is a subset \(T\) of \(S\) with \(|T| \leq k\) such that some straight line separates \(T\) and \(S \setminus T\). The number \(\chi_{\leq k}(S)\) of \((\leq k)\)-sets of \(S\) is a parameter of independent interest in discrete geometry (see [6]), and, as we recall in Section 2, is closely related to the rectilinear crossing number of the geometric graph induced by \(S\).

The main ingredient in the proof of Theorem 1 is the following bound (Theorem 2) for the number of \((\leq k)\)-sets in 3–decomposable point sets. The bound is in terms of the following quantity (by convention, \(\binom{r}{s} = 0\) if \(r < s\)),

\[
Y(k, n) := 3\binom{k + 1}{2} + 3\binom{k - n/3 + 1}{2} + 3\sum_{j=2}^{b} j(j+1)\left(k + 1 - \frac{j}{2} - \frac{1}{3j(j+1)}\right)^3n - \frac{1}{3} , \tag{1}
\]

where \(b := b(k, n)\) is the unique integer such that \(\binom{b(k,n)+1}{2} < n/(n-2k-1) \leq \binom{b(k,n)+2}{2}\).

**Theorem 2** Let \(S\) be a 3–decomposable set of \(n\) points in general position, where \(n\) is a multiple of 3, and let \(k < n/2\). Then

\[\chi_{\leq k}(S) \geq Y(k, n).\]

The best general lower bound for \(\chi_{\leq k}(S)\) is the sum of the first two terms in (1) (see [3] and [2]). Thus the third summand in (1) is the improvement we report, under the assumption of 3–decomposability.

The proofs of Theorems 1 and 2 are in Sections 2 and 3 respectively. In Section 4 we present some concluding remarks and open questions.

## 2 Proof of Theorem 1

Let \(D\) be a 3–decomposable rectilinear drawing of \(K_n\), and let \(S\) denote the underlying \(n\)-point set, that is, the vertex set of \(D\). Besides Theorem 2 our main tool is the following relationship between \((\leq k)\)-sets and the rectilinear crossing number (see [1] or [8]):

\[
\text{cr}(D) = \sum_{1 \leq k \leq (n-2)/2} (n-2k-1)\chi_{\leq k}(S) + \Theta(n^3). \tag{2}
\]

Combining Theorem 2 and Eq. (2), and noting that both the \(-1\) in the factor \(n-2k-1\) and the summand \(-1/3\) in (1) only contribute to smaller order terms, we obtain:
\[
\mathfrak{cr}(D_n) \geq \sum_{k=1}^{(n-2)/2} [n - 2k] \left( \frac{3}{2}(k+1) + 3\left(\frac{1}{2}\right)^{k+1} \right) + \Theta(n^3)
\]

\[
\geq 24 \left( \frac{(n-2)/2}{n} \right) \left( \frac{3}{2}(k+1) + 3\left(\frac{1}{2}\right)^{k+1} \right) + \Theta(n^3)
\]

\[
= 24 \left( \frac{(n-2)/2}{n} \right) \left( \frac{3}{2} \left( \frac{1-2k/n}{n} \right) \left( k/n - \left( \frac{1}{3} \right) \right)^2 \right) + \Theta(n^3)
\]

\[
= 24 \left( \frac{(n-2)/2}{n} \right) \left( \int_0^{1/2} (3/2)(1 - 2x) dx + \int_{1/3}^{1/2} (3/2)(1 - 2x)(x - 1/3)^2 dx \right) + \Theta(n^3)
\]

Elementary calculations show that
\[
\frac{3}{2} \int_0^{1/2} (1 - 2x)x^2 dx = \frac{3}{8} \int_{1/3}^{1/2} (1 - 2x)(x - 1/3)^2 dx = \frac{1}{216}, \text{ and } \int_{1/3}^{1/2} (1 - 2x)(x - 1/3)^2 dx = \frac{1}{486}j^4(1 + j)^4.
\]

Thus,
\[
\mathfrak{cr}(D_n) \geq 24 \left( \frac{(n-2)/2}{n} \right) \left( \frac{3}{8} + 1/216 + (3/2)\sum_{j=2}^{\infty} \frac{1}{486j^4(j + 1)^2} \right) + \Theta(n^3).
\]

Since
\[
\sum_{j=2}^{\infty} \frac{1}{j(j + 1)^2} = \sum_{j=2}^{\infty} \left( \frac{1}{j^2} - \frac{3}{j} + \frac{3}{j + 1} - \frac{1}{j + 1} \right) = \frac{79}{8} - \pi^2,
\]

then
\[
\mathfrak{cr}(D_n) \geq \frac{2}{27} \left( 15 - \pi^2 \right) \left( \frac{n}{4} \right) + \Theta(n^3).
\]

3 Proof of Theorem 2

The first step to prove Theorem 2 is to obtain an equivalent (actually, more general) formulation in terms of circular sequences (namely Proposition 3 below).

3.1 Circular sequences: reducing Theorem 2 to Proposition 3

All the geometrical information of a point set \( S \) gets encoded in (any halfperiod of) the circular sequence associated to \( S \). We recall that a circular sequence on \( n \) elements is a doubly infinite sequence \( \ldots \pi_{-1}, \pi_0, \pi_1, \ldots \) of permutations of the points in \( S \), where consecutive permutations differ in a transposition of neighboring elements, and, for every \( i \), \( \pi_i \) is the reverse permutation of \( \pi_{n-i} \). Thus a circular sequence on \( n \) elements has period \( 2\left(\begin{array}{c} n \\ 2 \end{array}\right) \), and all the information is encoded in an \( n \)-halfperiod, that is, a sequence of \( \left(\begin{array}{c} n \\ 2 \end{array}\right) + 1 \) consecutive permutations.

Each \( n \)-point set \( S \) has an associated circular sequence \( \Pi_S \), which contains all the geometrical information of \( S \). As we observed above, any \( n \)-halfperiod \( \Pi \) of \( \Pi_S \) contains all the information of \( \Pi_S \), and so \( n \)-halfperiods are usually the object of choice to work
with. In an $n$–halfperiod $\Pi = \pi_0, \pi_1, \ldots, \pi_{\left(\frac{n}{2}\right)}$, the initial permutation is $\pi_0$ and the final permutation is $\pi_{\left(\frac{n}{2}\right)}$.

Not every $n$–halfperiod $\Pi$ arises from a point set $S$. We refer the reader to the seminal work by Goodman and Pollack [7] for further details.

Observe that if $S$ is $3$–decomposable, then there is an $n$–halfperiod $\Pi$ of the circular sequence associated to $S$, whose points can be labeled $a_1, \ldots, a_{n/3}, b_1, \ldots, b_{n/3}, c_1, \ldots, c_{n/3}$, so that:

(i) The initial permutation $\pi_0$ reads $a_{n/3}, a_{n/3} - 1, \ldots, a_1, b_1, \ldots, b_{n/3}, c_1, c_2, \ldots, c_{n/3}$;

(ii) there is an $s$ such that in the $(s + 1)$–st permutation first the $b$’s appear consecutively, then the $a$’s appear consecutively, and then the $c$’s appear consecutively; and

(iii) there is a $t$, with $t > s$, such that in the $(t + 1)$–st permutation first the $b$’s appear consecutively, then the $c$’s appear consecutively, and then the $a$’s appear consecutively.

**Definition** An $n$–halfperiod $\Pi$ that satisfies properties (i)–(iii) above is 3–decomposable.

A transposition that occurs between elements in sites $i$ and $i + 1$ is an $(i, i + 1)$–transposition. An $i$–critical transposition is either an $(i, i + 1)$–transposition or an $(n - i, n - i + 1)$–transposition, and a $(\leq k)$–critical transposition is a transposition that is $i$–critical for some $i \leq k$. If $\Pi$ is an $n$–halfperiod, then $\eta(\Pi)$ denotes the number of $(\leq k)$–critical transpositions in $\Pi$.

The key result is the following.

**Proposition 3** Let $\Pi$ be a 3–decomposable $n$–halfperiod, and let $k < n/2$. Then

$$\eta(\Pi) \geq Y(k, n).$$

**Proof of Theorem** Let $S$ be 3–decomposable, and let $\Pi$ be an $n$–halfperiod of the circular sequence associated to $S$, that satisfies properties (i)–(iii) above. Then $\Pi$ is 3–decomposable. Now, for any point set $T$ and any halfperiod $\Pi_T$ associated to $T$, the $(\leq k)$–critical transpositions of $\Pi_T$ are in one–to–one correspondence with $(\leq k)$–sets of $T$. Applying this to $\Pi$ and $S$, it follows that $\chi(\Pi) = \eta(\Pi)$. Applying Proposition 3, Theorem follows.

We devote the rest of this section to the proof of Proposition 3.

**3.2 Proof of Proposition** Throughout this section, $\Pi = (\pi_0, \pi_1, \ldots, \pi_{\left(\frac{n}{2}\right)})$ is a 3–decomposable $n$–halfperiod, with initial permutation $\pi_0 = (a_{n/3}, a_{n/3} - 1, \ldots, a_1, b_1, \ldots, b_{n/3}, c_1, \ldots, c_{n/3})$.

In order to (lower) bound the number of $(\leq k)$–critical transpositions in 3–decomposable circular sequences, we distinguish between two types of transpositions. A transposition is homogeneous if it occurs between two $a$’s, between two $b$’s, or between two $c$’s; otherwise it is heterogeneous. We let $\eta(\Pi)$ (respectively $\eta(\Pi)$) denote the number of homogeneous (respectively heterogeneous) $(\leq k)$–critical transpositions in $\Pi$, so that

$$\eta(\Pi) = \eta_{\text{hom}}(\Pi) + \eta_{\text{het}}(\Pi).$$
3.2.1 Bounding (actually, calculating) $\eta^{\text{het}}_{\leq k}(\Pi)$

Let us call a transposition an ab-transposition if it involves one a and one b. We similarly define ac- and bc-transpositions. Thus, each heterogeneous transposition is either an ab- or an ac- or a bc-transposition.

Since in $\Pi$ each ab-transposition moves the involved $a$ to the right and the involved $b$ to the left, then (a) for each $i \leq n/3$, there are exactly $i$ i-critical ab-transpositions; and (b) for each $i$, $n/3 < i < 2n/3$, there are exactly $2n/3 - i$ i-critical ab-transpositions. Since the same holds for ac- and bc-transpositions, it follows that for each $i \leq n/3$, there are exactly $3i$ i-critical heterogeneous transpositions, and for each $i$, $n/3 < i < 2n/3$, exactly $2n - 3i$ i-critical heterogeneous transpositions.

Therefore, for each $k \leq n/3$, there are exactly $\sum_{i\leq k} 3i = 3\left(\frac{k+1}{2}\right)$ (at least) critical transpositions, and if $n/3 < k < n/2$, then there are exactly $\sum_{i\leq n/3} 3i + \sum_{n/3 < i \leq k} 2n - 3i + \sum_{n-k-1 < i \leq 2n/3-1} 2n - 3i = 3\left(\frac{n/3+1}{2}\right) + (k - n/3)n$ ($\leq k$) critical transpositions.

We now summarize these results.

**Proposition 4** Let $\Pi$ be a 3-decomposable n-halfperiod, and let $k < n/2$. Then

$$\eta^{\text{het}}_{\leq k}(\Pi) = \begin{cases} 3\left(\frac{k+1}{2}\right) & \text{if } k \leq n/3, \\ 3\left(\frac{n/3+1}{2}\right) + (k - n/3)n & \text{if } n/3 < k < n/2, \end{cases}$$

3.2.2 Bounding $\eta^{\text{hom}}_{\leq k}(\Pi)$

Our goal here is to give a lower bound (see Proposition 10) for the number $\eta^{\text{hom}}_{\leq k}(\Pi)$ of homogeneous ($\leq k$)-critical transpositions in a 3-decomposable $n$-halfperiod $\Pi$.

Our approach is to find an upper bound for $\eta^{\text{aa}}_{\leq k}(\Pi)$, which will denote the number of aa-transpositions that are not ($\leq k$)-critical ($\eta^{\text{bb}}_{\leq k}(\Pi)$ and $\eta^{\text{cc}}_{\leq k}(\Pi)$ are defined analogously). Since the total number of aa-transpositions is $\frac{n(n-1)}{2}$, this will yield a lower bound for the contribution of aa-transpositions (and, by symmetry, for the contribution of bb-transpositions and of cc-transpositions) to $\eta_{\leq k}(\Pi)$.

**Remark 5** For every $k \leq n/3$, it is a trivial task to construct $n$-halfperiods $\Pi$ for which $\eta^{\text{hom}}_{\leq k}(\Pi) = 0$. In view of this, we concentrate our efforts on the case $k > n/3$.

A transposition between elements in positions $i$ and $i + 1$, with $k + 1 \leq i \leq n - k - 1$, is valid. Thus our goal is to (upper) bound the number of valid aa-transpositions.

Let $D^a_{\Pi}$ be the digraph with vertex set $a_1, \ldots, a_{n/3}$, and such that there is a directed edge from $a_i$ to $a_j$ if and only if $i < j$ and the transposition that swaps $a_i$ and $a_j$ is valid. For $j = 1, \ldots, n/3$, we let $[a_j]_1^\Pi$ (respectively $[a_j]_2^\Pi$) denote the outdegree (respectively indegree) of $a_j$ in $D^a_{\Pi}$. We define $D^{bb}_{\Pi}, D^{bc}_{\Pi}, D^{cc}_{\Pi}, [a_1]^{-}_{1}^\Pi, [a_1]^{+}_{1}^\Pi, [a_1]^{-}_{2}^\Pi$, and $[a_1]^{+}_{2}^\Pi$ analogously.

The inclusion of the symbol $\Pi$ in $D^{aa}_{\Pi}, [a_i]_{\Pi}$, etc., is meant to emphasize the dependence on the specific $n$-halfperiod $\Pi$. For brevity we will omit the reference to $\Pi$ and simply write $D^{aa}, D^{bb}, D^{cc}, [a_i]^{-}, [a_i]^{+},$ and so on. No confusion will arise from this practice.

The importance of $D^{aa}, D^{bb}$, and $D^{cc}$ is clear from the following observation.

**Remark 6** For each $n$-halfperiod $\Pi$, the number of edges of $D^{aa}$ equals $\eta^{aa}_{\leq k}(\Pi)$. Indeed, to each valid aa-transposition, that is, each transposition that contributes to $\eta^{aa}_{\leq k}(\Pi)$, there corresponds a unique edge in $D^{aa}$. Analogous observations hold for $D^{bb}$ and $D^{cc}$.
In view of Remark 6, we direct our efforts to bounding the number of edges in \( D^{aa} \). The essential observation to get this bound is the following:

\[
[a_j]^- \leq \min \{n - 2k - 1 + [a_j]^+, (n/3) - j \}.
\]

(4)

To see this, simply note that \([a_j]^- \leq n - 2k - 1 + [a_j]^+\), since \( n - 2k - 1 + [a_j]^+ \) is clearly the maximum possible number of valid moves in which \( a_j \) moves right, and trivially \([a_j]^- \leq (n/3) - j\), since there are only \((n/3) - j \) \(a_i\)'s with \( \ell < j\).

Proposition 7 If \( \Pi \) is a 3-decomposable \( n\)-halfperiod, and \( n/3 < k < n/2 \), then \( D^{aa} \) has at most \( (n/3)^2 - (1/3) \left( Y(k, n) - 3(n/3 + 1) - (k - n/3)n \right) \) edges.

Proof. Let \( D_{k,n} \) denote the class of all digraphs with vertex set \( a_1, \ldots, a_{n/3} \), with every directed edge \( a_i \rightarrow a_j \) satisfying \( \ell < j \) and \([a_j]^- \leq \min \{n - 2k - 1 + [a_j]^+, n/3 - j \} \).

We argue that any graph in \( D_{k,n} \) has at most \( (n/3)^2 - (1/3)Y(k, n) - 3(n/3 + 1) - (k - n/3)n \) edges. This clearly finishes the proof, since \( D^{aa} \in D_{k,n} \).

To achieve this, we note that it follows from the work in Section 2 in [5] that the maximum number of edges of such a digraph is attained in the digraph \( D_{k,n} \) recursively constructed as follows. First define that all the directed edges arriving at \( a_{n/3} \) are the edges \( a_j \rightarrow a_{n/3} \) for \( j = (n/3) - 1, \ldots, (n/3) - n - 2k - 2 \). Now, for \( j + 1 \leq n/3 \), once all the directed edges arriving at \( a_{j+1} \) have been determined, fix that (all) the directed edges arriving at \( a_j \) are \( a_{j} \rightarrow a_{j'} \), for all those \( \ell \) that satisfy \( j - \ell \leq n - 2k - 1 + [a_j]^+ \).

Since no digraph in \( D_{k,n} \) has more edges than \( D_{k,n} \), to finish the proof it suffices to bound the number of edges of \( D_{k,n} \). This is the content of Claim 8 below.

Claim 8 \( D_{k,n} \) has at most \( (n/3)^2 - (1/3)Y(k, n) - 3(n/3 + 1) - (k - n/3)n \) edges.

Sketch of proof. Since we know the exact indegree of each vertex in \( D_{k,n} \), we know the exact number of edges of \( D_{k,n} \), and so the proof of Claim 8 is no more than a straightforward, but quite long and tedious, calculation. 

Corollary 9 If \( \Pi \) is a 3-decomposable \( n\)-halfperiod, and \( n/3 < k < n/2 \), then each of \( D^{bb} \) and \( D^{cc} \) has at most \( (n/3)^2 - (1/3)Y(k, n) - 3(n/3 + 1) - (k - n/3)n \) edges.

Proof. In the proof of Proposition 7, the only relevant property about \( D^{aa} \) is that the \( a \)'s form a set of \( n/3 \) points that in some permutation of \( \Pi \) (namely \( \pi_0 \)) appear all consecutively and at the beginning of the permutation. Since \( \Pi \) is 3-decomposable, this condition is also satisfied by the set of \( b \)'s and by the set of \( c \)'s.

We now summarize the results in the current subsection.

Proposition 10 If \( \Pi \) is a 3-decomposable \( n\)-halfperiod, and \( n/3 < k < n/2 \), then

\[
\eta_{\leq k}^{\text{hom}}(\Pi) \geq Y(k, n) - 3(n/3 + 1) - (k - n/3)n.
\]

Proof. By Remark 6, the number \( \eta_{\leq k}^{aa}(\Pi) \) of \( aa \)-transpositions that are \textit{not} (\( \leq k \))-critical equals the number of edges in \( D^{aa} \), which by Proposition 7 is at most \( (1/3)Y(k, n) - 3(n/3 + 1) - (k - n/3)n \). Since the total number of \( aa \)-transpositions is \( (n/3)^2 \), then the number of \( aa \)-transpositions that contribute to \( \eta_{\leq k}^{aa}(\Pi) \) is at least \( (n/3)^2 - (1/3)Y(k, n) - 3(n/3 + 1) - (k - n/3)n \). By Corollary 9, \( bb \)- and \( cc \)-transpositions contribute in at least the same amount to \( \eta_{\leq k}^{\text{hom}}(\Pi) \), and so the claimed inequality follows.
3.2.3 Proof of Proposition 3

Proposition 3 follows immediately from Eq. (3) and Propositions 4 and 10.

4 Concluding remarks

All the lower bounds proved above remain true for point sets that satisfy conditions (i) and (ii) (and not necessarily condition (iii)) for 3-decomposability.

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Appendix: Proof of Claim

Since $D_{k,n}$ is a well-defined digraph, and we know the exact indegree of each of its vertices, Claim 8 is no more than long and tedious, yet elementary, calculation.

The purpose of this Appendix is to give the full details of this calculation.

We prove Claim 8 in two steps. First we obtain an expression for the exact value of the number of edges of $D_{k,n}$, and then we show that this exact value is upper bounded by the expression in Claim 8.

1 The exact number of edges in $D_{k,n}$

The exact number of edges in $D_{k,n}$ is a function of the following parameters. Let $i, j$ be positive integers with $i \leq j$. Then:

- $b(i, j)$ is the (unique) nonnegative integer such that $\left\lfloor \frac{b(i, j)+1}{2} \right\rfloor < j/i \leq \left\lfloor \frac{b(i, j)+2}{2} \right\rfloor$; and
- $q(i, j)$ and $r(i, j)$ are the (unique) integers satisfying $0 \leq q(i, j) < i, 1 \leq r(i, j) \leq b(i, j) + 1$ and such that

$$j = i\left(\frac{b(i, j)+1}{2}\right) + q(i, j)(b(i, j) + 1) + r(i, j) \quad \text{(A-1)}$$

For brevity, in the rest of the section we let $s := n/3$ and $m := n - 2k - 1$.

The key observation is that we know the indegree of each vertex in $D_{k,n}$.

Proposition 1 (Proposition 19 in [5]) For each integer $1 \leq i \leq s$, and each vertex $a_i$ of $D_{k,n}$, $[a_i]^- = b(i, s)m + q(i, s)$.

The number of edges of $D_{k,n}$ equals the sum of the indegrees over all vertices in $D_{k,n}$. Thus our main task is to find a closed expression for the sum $\sum_{1 \leq i \leq s} [a_i]^-$. This is the content of our next statement.

Proposition 2 (Exact number of edges of $D_{k,n}$) The number $\sum_{1 \leq i \leq s} [a_i]^- $ of edges of $D_{k,n}$ is

$$E(k, n) := 2m^2 \left(\frac{b(m, s)}{3} + 1\right) + \binom{b(m, s)}{2} \binom{m}{2} + 2m \cdot q(m, s) \left(\frac{b(m, s)+1}{2}\right) +$$

$$\left(\binom{q(m, s)}{2} \left(\frac{b(m, s)+1}{2}\right) + r(m, s) \left(m \cdot b(m, s) + q(m, s)\right)\right).$$

Proof. We break the index set of the summation $\sum_{1 \leq i \leq s} [a_i]^{-}$ into three parts, in terms of $\alpha := m^{b(m, s)+1}$ and $\beta := q(m, s)(b(m, s) + 1)$. We let $A = \sum_{1 \leq i \leq \alpha} [a_i]^{-}$, $B = \sum_{\alpha+1 \leq i \leq \alpha+\beta} [a_i]^{-}$, and $C = \sum_{\alpha+\beta+1 \leq i \leq s} [a_i]^{-}$ so that

$$\sum_{1 \leq i \leq s} [a_i]^{-} = A + B + C. \quad \text{(A-2)}$$

We calculate each of $A, B, \text{ and } C$ separately.

Calculating $A$

If $\ell, j$ are integers such that $0 \leq \ell \leq m - 1$ and $0 \leq j \leq b(m, s)$, we define $S_j := \{i : b(i, m) = j\}$ and $T_{j, \ell} := \{i : b(i, m) = j, q(i, m) = \ell\}$. Note that $S_1, S_2, \ldots, S_{b(m, s)}$ is a
partition of \{1, 2, ..., n\} and that for each \(j \leq b(m, s) - 1, T_{j,0}, T_{j,1}, ..., T_{j,m-1}\) is a partition of \(S_j\).

Note that \(A\) can be rewritten as \(\sum_{0 \leq j \leq b(m, s) - 1} \sum_{i \in S_j} [a_i]^{-}\). By Proposition\[\text{this equals}\] \(\sum_{0 \leq j \leq b(m, s) - 1} \sum_{i \in S_j} (m \cdot b(i, m) + q(i, m))\). That is,

\[
A = \sum_{0 \leq j \leq b(m, s) - 1} \left( m \sum_{i \in S_j} b(i, m) + \sum_{i \in S_j} q(i, m) \right).
\]

Since \(0 \leq q(i, m) \leq m - 1\) for all \(i\), and \(T_{j,0}, T_{j,1}, ..., T_{j,m-1}\) is a partition of \(S_j\), then \(\sum_{i \in S_j} q(i, m) = \sum_{0 \leq i \leq m-1} \sum_{i \in T_{j,\ell}} q(i, m)\). Thus,

\[
A = \sum_{0 \leq j \leq b(m, s) - 1} \left( m \sum_{i \in S_j} b(i, m) + \sum_{0 \leq i \leq m-1} \sum_{i \in T_{j,\ell}} q(i, m) \right). \tag{A-3}
\]

On other hand, for \(0 \leq j \leq b(m, s) - 1\) and \(0 \leq \ell \leq m - 1\), it is not difficult to verify that \(|T_{j,\ell}| = j + 1\). This implies that \(|S_j| = m(j + 1)\).

By definition of \(S_j\) we have

\[
\sum_{i \in S_j} b(i, m) = \sum_{i \in S_j} j = j |S_j| = jm(j + 1). \tag{A-4}
\]

By definition of \(T_{j,\ell}\) we have

\[
\sum_{i \in T_{j,\ell}} q(i, m) = \sum_{i \in T_{j,\ell}} \ell = \ell |T_{j,\ell}| = \ell(j + 1). \tag{A-5}
\]

Substituting (A-4) and (A-5) into (A-3) we obtain

\[
A = \sum_{0 \leq j \leq b(m, s) - 1} \left( m(jm(j + 1)) + \sum_{0 \leq \ell \leq m-1} \ell(j + 1) \right)
\]

\[
= \sum_{0 \leq j \leq b(m, s) - 1} \left( 2m^2(j+1) + (j + 1) \sum_{0 \leq \ell \leq m-1} \ell \right)
\]

\[
= \sum_{0 \leq j \leq b(m, s) - 1} \left( 2m^2(j+1) + (j + 1)_{\binom{m}{2}} \right)
\]

\[
= 2m^2 \binom{b(m, s)+1}{3} + \binom{b(m, s)+1}{2} \binom{m}{2}. \tag{A-6}
\]

Calculating \(B\):

Since \(b(i, m) = b(m, s)\) for each \(i \geq \alpha + 1\), and \([a_i]^{-} = m \cdot b(i, m) + q(i, m)\), then

\[
B = \sum_{\alpha + 1 \leq i \leq \alpha + \beta} [a_i]^{-} = \sum_{\alpha + 1 \leq i \leq \alpha + \beta} (m \cdot b(i, m) + q(i, m)).
\]
Therefore
\[
B = \sum_{\alpha+1 \leq i \leq \alpha+\beta} m \cdot b(m, s) + \sum_{\alpha+1 \leq i \leq \alpha+\beta} q(i, m)
= m \cdot b(m, s) \sum_{\alpha+1 \leq i \leq \alpha+\beta} 1 + \sum_{\alpha+1 \leq i \leq \alpha+\beta} q(i, m)
= m \cdot b(m, s) + \sum_{\alpha+1 \leq i \leq \alpha+\beta} q(i, m)
= m \cdot b(m, s)q(m, s)(b(m, s) + 1) + \sum_{\alpha+1 \leq i \leq \alpha+\beta} q(i, m).
\]

On other hand it is easy to check that \(|T_{b(m, s), k}| = b(m, s) + 1\) for every \(k\) such that \(0 \leq k \leq q(m, s) - 1\). Since \(0 \leq q(i, m) \leq q(m, s) - 1\) for every \(i\) such that \(\alpha+1 \leq i \leq \alpha+\beta\), then \(T_{b(m, s), 0}, T_{b(m, s), 1}, \ldots, T_{b(m, s), q(m, s) - 1}\) is a partition of \(\{\alpha+1, \alpha+2, \ldots, \alpha+\beta\}\). Thus,
\[
\sum_{\alpha+1 \leq i \leq \alpha+\beta} q(i, m) = \sum_{0 \leq \ell \leq q(m, s) - 1} \sum_{i \in T_{b(m, s), \ell}} q(i, m).
\]

We note that \(\sum_{i \in T_{b(m, s), \ell}} q(i, m) = \ell |T_{b(m, s), k}| = \ell (b(m, s) + 1)\). Using this fact in (A-7) we obtain
\[
\sum_{\alpha+1 \leq i \leq \alpha+\beta} q(i, m) = \sum_{0 \leq \ell \leq q(m, s) - 1} \ell (b(m, s) + 1) = \binom{q(m, s)}{2} (b(m, s) + 1).
\]

Thus,
\[
B = m \cdot b(m, s)q(m, s)(b(m, s) + 1) + \binom{q(m, s)}{2} (b(m, s) + 1)
= 2m \cdot q(m, s)\binom{b(m, s)+1}{2} + \binom{q(m, s)}{2} (b(m, s) + 1).
\]

**Calculating \( C \)**

Since \(b(i, m) = b(m, s); q(i, m) = q(m, s)\) for each \(i\) such that \(i \geq \alpha+\beta+1\); and \([a_i]^- = m \cdot b(i, m) + q(i, m)\), it follows that
\[
C = \sum_{\alpha+\beta+1 \leq i \leq s} [a_i]^- = \sum_{\alpha+\beta+1 \leq i \leq s} m \cdot b(i, m) + q(i, m)
= \sum_{\alpha+\beta+1 \leq i \leq s} m \cdot b(m, s) + q(m, s)
= (s - \alpha - \beta)\left(m \cdot b(m, s) + q(m, s)\right)
\]

From (A-1) it follows that \(r(m, s) = s - \alpha - \beta\), and so
\[
C = r(m, s)(m \cdot b(m, s) + q(m, s)).
\]

Now from (A-2), (A-8), and (A-9), it follows that \(E(k, n) = A + B + C\), and so Proposition 2 follows from (A-2).
Upper bound for number of edges in $D_{k,n}$:

Proof of Claim 8

First we bound the number of $(\leq k)$-edges in 3-decomposable $n$-halfperiods in terms of the expression $E(k,n)$ in Proposition 2.

Proposition 3 Let $\Pi$ be a 3-decomposable $n$-halfperiod, and let $k < n/2$. Then

$$\eta_{\leq k}(\Pi) \geq \begin{cases} \binom{k+1}{2} & \text{if } k \leq n/3, \\ 3^{\left(\frac{n}{3}+1\right)} + (k-n/3)n + 3\left(\frac{n}{2}\right) - E(k,n) & \text{if } n/3 < k < n/2. \end{cases}$$

Proof. Obviously, $\eta_{\leq k}(\Pi) \geq \eta_{\leq k}^{\text{het}}(\Pi)$ and so the case $k \leq n/3$ follows from Proposition 4. Now suppose that $n/3 < k < n/2$. Recall that $\eta_{\leq k}^{\text{hom}}(\Pi) = \eta_{\geq k}^{bb}(\Pi) + \eta_{\geq k}^{cc}(\Pi) + C_{\leq k}(\Pi)$.

Now the total number of $aa$- and $bb$- and $cc$- transpositions is exactly $\binom{n}{2}$, and so $\eta_{\leq k}^{\text{hom}}(\Pi) = 3\left(\frac{n}{2}\right) - \eta_{\geq k}^{aa}(\Pi) - \eta_{\geq k}^{bb}(\Pi) - \eta_{\leq k}^{cc}(\Pi)$. Thus it follows from Remark 6 and Proposition 2 that $\eta_{\leq k}^{\text{hom}}(\Pi) \geq 3\left(\frac{n}{2}\right) - E(k,n)$. This fact, together with Proposition 4, implies that $\eta_{\leq k}(\Pi) = \eta_{\leq k}^{\text{het}}(\Pi) + \eta_{\leq k}^{\text{hom}}(\Pi) \geq 3\left(\frac{n}{2}\right) + (k-n/3)n + 3\left(\frac{n}{2}\right) - E(k,n)$, as claimed. \[\square\]

Proof of Claim 8. Recall that $s := n/3$ and $m := n-2k-1$. By Remark 5 we know that $k > n/3$, and so $s \geq m$. From (A-11), it follows that

$$q(m,s) = \frac{s - m(b(m,s)+1) - r(m,s)}{b(m,s) + 1}. \quad (A-10)$$

Now by Proposition 3 $\eta_{\leq k}(\Pi) \geq L(k,n)$, where

$$L(k,n) := 3\left(\frac{s+1}{2}\right) + (k-s)n + 3\left(\frac{n}{2}\right) - E(k,n). \quad (A-11)$$

Substituting in $E(k,n)$ the value of $q(m,s)$ given in (A-10), a (long and tedious yet) totally elementary simplification yields

$$L(n,k) - Y(k,n) - 1/3 = \frac{1}{8b(m,s)+1}\left(5b(m,s)^2 + 4b(m,s)^3 + b(m,s)^4 + b(m,s)(-12r(m,s) + 2) + 12r(m,s) - 1) r(m,s) \right).$$

Define $f(b(m,s), r(m,s)) := L(n,k) - Y(k,n) - 1/3$. An elementary calculation shows that $r_0(m,s) := (b(m,s) + 1)/2$ minimizes $f(b(m,s), r(m,s))$. Thus $f(b(m,s), r(m,s)) \geq f(b(m,s), r_0(m,s)) = (b(m,s)+3)(b(m,s)+1)(b(m,s)-1)/8$. Since $b(m,s)$ is a nonnegative integer, it follows that $f(b(m,s), r(m,s)) \geq -1/3$ and therefore $L(n,k) - Y(k,n) \geq 0$. By (A-11), $E(k,n) = (1/3)(b(m,s)+1)(k-s)n + 3\left(\frac{n}{2}\right) - L(k,n))$. Since $-L(n,k) \leq -Y(k,n)$, then $E(k,n) \leq (1/3)(b(m,s)+1)(k-s)n + 3\left(\frac{n}{2}\right) - Y(k,n)) = \frac{1}{2}(1/3)Y(k,n) - (k-s)n$. This proves Claim 8 since $E(k,n)$ is the total number of edges in $D_{k,n}$. \[\square\]