I. INTRODUCTION

The collective quantization of \( SU(3) \) flavor hedgehog soliton models (eg. the Skyrme model) was first implemented more than two decades ago \[1\,2\]. The approach appeared to be a natural extension of the collective quantization of two flavored Skyrmions introduced by Adkins, Nappi and Witten (ANW) \[3\] with important constraints imposed due to the Witten-Wess-Zumino term. The results of this approach had numerous phenomenological successes in the description of the octet and decuplet baryons. However, as was noticed long ago by Praszalowicz \[4\] and emphasized more recently by Diakonov, Petrov and Polyakov (DPP) \[5\] such a procedure also predicts the existence of collective states fall into representations of \( SU(3) \) flavor labeled by \((p,q)\) and are given by \((2J,\frac{S}{2}-J)\) where \(J = \frac{1}{2}, \frac{3}{2}, \cdots\) is the spin of the collective state. States with strangeness \( S > 0 \) do not arise as collective states from this procedure; thus the \( \theta^+ \) (pentaquark) resonance does not arise as a collective excitation in models of this type.

Numerous experimental groups, including many with much higher statistics, did not see the \( \theta^+ \). Particularly significant are the two recent high statistics experiments reported by CLAS, which found no evidence for the \( \theta^+ \) and were done at similar dynamics to previous experiments which claimed to observe the resonance \[6\]. As a result of this there is considerable skepticism in the community as to whether the \( \theta^+ \) is real.

The theoretical situation is also complicated. Although the predictions of the octet and decuplet baryons from the collective quantization of the \( SU(3) \) soliton appears to be sound, it has been argued that the predictions of exotic states such as the \( \theta^+ \) are not consistent with the underlying large \( N_c \) dynamics \[7\,8\,9\,10\,11\,12\,13\].

The crux of the issue is the use of the rigid rotator scheme to quantize the hedgehog solitons. The rigid rotator approach was developed by ANW for the \( SU(2) \) Skyrme model. It is generally agreed that this method correctly describes the large \( N_c \) collective dynamics of the Skyrme model, although it must be modified for models with explicit quarks or vector meson degrees of freedom \[14\]. The quantization of \( SU(3) \) solitons in Refs. \[1\,2\] borrows the ANW method directly and the predictions of exotic states by Praszalowicz \[15\] and DPP \[5\] depend on this assumption.

However, the rigid rotator approach for exotic states of \( SU(3) \) hedgehog models is known to be inconsistent with the large \( N_c \) dynamics used to justify the models \[10\,11\,13\,16\]. This can be seen in a number of indirect ways. Perhaps most compelling is the fact that collective excitations have a level spacing which goes to zero at large \( N_c \), corresponding to slow classical motion which decouples from the intrinsic degrees of freedom. However, the excitation energy of the exotic states is of order \( N_c^0 \). Initially this was controversial; Diakonov and Petrov made several counterarguments justifying the rigid rotor approach \[17\]. However these were shown...
to be invalid\cite{11} and in a recent paper Diakonov and Petrov now agree that the approach is invalid at large $N_c$. \cite{18}.

While the controversy regarding whether exotic states appear as collective states of the soliton model appears to be resolved, an important theoretical question remains open: how does one consistently quantize these models? There is an obvious \textit{ad hoc} approach — simply use the procedures and results of the rigid rotor quantization of Refs. \cite{1,2}, but restrict their application to the non-exotic states of the theory. Such an approach has one cardinal virtue — \textit{a priori} its results are virtually certain to be correct at large $N_c$. The reason for this is simply that in implementing this scheme one produces exactly the same collective spectrum as emerges from a model-independent analysis based on large $N_c$ consistency rules \cite{14} which, not coincidentally, turns out to be identical to that of a large $N_c$ constituent quark model.

However, this \textit{ad hoc} approach is deficient in two ways. First, from a purely theoretical perspective, making such an \textit{ad hoc} prescription is unsatisfactory. One needs to understand what is wrong with the rigid rotor analysis and how to do it correctly. Second, from a more pragmatic perspective, if one wishes to treat these models in a consistent $1/N_c$ expansion beyond leading order, it is essential to do the bookkeeping correctly by separating the collective degrees of freedom from the non-collective ones in a transparent manner.

As will be discussed below, the key issue turns out to be the one identified in Refs. \cite{11,16}; namely, the correct identification of zero modes in the spectrum of small fluctuations around the soliton. As is standard in soliton physics, the degrees of freedom associated with zero modes get promoted to collective degrees of freedom which decouple from the other degrees of freedom in the problem in the semiclassical limit (large $N_c$ in the present context). There is an important subtlety for $SU(3)$ solitons which is not present for $SU(2)$ solitons; namely, that there are fewer dynamic zero modes than static ones. The distinction between dynamic and static zero modes will be discussed in the following section.

Before embarking on this analysis we should note that at a formal level this approach is based on exact $SU(3)$ flavor symmetry. Any attempt to connect this to the real world, where there is a nontrivial level of $SU(3)$ breaking, requires an additional perturbative treatment in the $SU(3)$ breaking term. There is an alternative formalism due to Callan and Klebanov \cite{20} which encodes the $SU(3)$ violations from the outset, and in which fluctuations which change the strangeness are always regarded as non-collective vibrations. We make no claims here regarding the relative efficiency of the two approaches. Indeed we fully expect the approaches to be essentially equivalent for small quark masses. Nevertheless it is important at least at the level of mathematical physics to understand how to collectively quantize $SU(3)$ solitons.

While the general approach discussed here can be used for any $SU(3)$ chiral soliton model at large $N_c$, we focus on a simple variant of the Skyrme model in order to keep the discussion focused.

The plan of this paper is as follows: In the next section we review a few basic issues about zero modes and collective quantization in soliton physics. There we focus on the circumstances in which rigid rotations are valid, and discuss how to collectively quantize the theory when rigid rotation fails. We also make the distinction between static and dynamic zero modes and show how there can be more of the former than of the latter. Following this is a section introducing the Skyrme model. Next, we focus on identifying the flavor zero-modes of the hedgehog Skyrmion solution, and show that the system has only ten zero modes (seven static and three dynamic ones) rather than fourteen (seven static and seven dynamic modes), as is implicitly assumed in rigid-rotator quantization. We derive a collective Hamiltonian, and describe the states that arise in the quantization of the resulting collective coordinates. These turn out to be precisely those obtained in the model-independent large $N_c$ approach of Ref. \cite{19}.

II. ZERO MODES AND THE COLLECTIVE QUANTIZATION OF SOLITONS

A. Collective quantization: Generalities

There is a standard procedure for quantizing solitons \cite{21,22}. Suppose we have a quantum field-theoretic Lagrangian that, when treated classically, admits solitonic solutions. For simplicity, in this discussion we will restrict our attention to theories such as Skyrme’s original model\cite{22} in which all fields are Lorentz scalars. Theories which include vector or spinor degrees of freedom add some minor technical complications which make the argument a bit less transparent, but do not alter the essential details of the problem. The solitonic solutions can be interpreted as the first approximations to the quantum ground state of the system in a semiclassical expansion. In the present context this expansion turns out to be equivalent to the $1/N_c$ expansion. Small oscillations around the classical soliton are then interpreted as corrections to this approximation. Zero modes in the fluctuation spectrum turn out to play a central role in the analysis.

The Lagrangian necessarily has continuous symmetries which are broken by the classical soliton configuration. While the Lagrangian is Poincaré invariant, the soliton breaks translational invariance. Other symmetries of the Lagrangian may also be broken. By applying symmetry operations at the classical level it is therefore possible to continuously change a solitonic configuration to obtain distinct solitonic configurations of the same energy. A small fluctuation
in an energetically flat direction is a zero mode — there is no restoring force. It should be apparent that these classical soliton configurations cannot directly correspond an energy eigenstate of the quantum system; the eigenstates have well defined values of the symmetry operators while the classical solitons do not. Thus, for example, the fact the soliton breaks translational invariance means that the classical soliton cannot correspond to a quantum state with good momentum. To make a connection with the physical states, it thus becomes essential to quantize the degrees of freedom associated with the symmetry breaking, which, as we have seen, are the zero modes. These are denoted the collective degrees of freedom. If one is working in the extreme semiclassical regime (eg. at large $N_c$) the collective degrees of freedom should completely decouple from the remaining degrees of freedom. In this limit it is possible to quantize them separately by imposing the appropriate commutation relations.

### B. Static and dynamic zero modes

Formally one can find the zero modes by solving the eigenvalue equation for small classical harmonic fluctuations of the fields around the static soliton. The zero modes are so-named because they have zero frequency. Generally the equation for small amplitude fluctuations is second order in time. The most general fluctuation equation is of the form

$$\delta \phi = A \delta \phi + B \delta \phi,$$

where the $\delta \phi$ are the first order fluctuations of the fields and $A, B$ are differential operators in space. If there are no terms in the action linear in the time derivatives of the fields, then $B = 0$ in Eq. (2.1). Harmonic fluctuations with finite frequency are given by a generalized eigenvalue equation

$$-\omega^2 \delta \phi = A \delta \phi - i \omega B \delta \phi.$$  \hspace{1cm} (2.2)

In addition to the finite frequency solutions of the preceding form, there are zero modes. First consider the case $B = 0$ (no terms linear in the time derivative). In such a case the zero modes necessarily come in pairs. Suppose there exists a static field configuration $\psi$ which satisfies $A \psi = 0$. Then $\delta \phi(t) = \psi$ and $\delta \phi(t) = \psi t$ are both solutions of the small fluctuation equation, Eq. (2.1), but they do not correspond to finite frequency harmonic oscillations; they are zero modes. Physically the first of these two solutions corresponds to a static displacement of the fields while the second corresponds to a “kick” in which the fields receive an impulse. We will refer to these two type of solutions as “static” and “dynamic” zero modes, respectively.

The distinction between static and dynamic zero modes is usually not made. For the situation described above in which there are no terms in the action linear in the time derivatives and $B = 0$ there is no need to distinguish them; they necessarily come as a pair and it is sufficient to label the entire pair as the zero mode.

Suppose, however, that there are terms in the action linear in the time derivatives of the fields and, accordingly, $B \neq 0$. Again, let us suppose that there exists a static field configuration $\psi$ which satisfies $A \psi = 0$. In this case there will again be a static zero mode of the form $\delta \phi(t) = \psi$ independent of the nature of $B$. However, from the form of Eq. (2.1), it is apparent that $\delta \phi(t) = \psi t$ is not generically a dynamic zero mode. Dynamical zero modes may be generated from the static ones provided $B \psi$ is either zero or orthogonal to all zero modes of $A$. In this case, it is easy to see that the dynamical zero modes are of the form:

$$\delta \phi(t) = \psi t + \Phi \text{ with } A \Phi = -B \psi.$$  \hspace{1cm} (2.3)

The shift $\Phi$ is well known in the context of many-body physics where it is referred to as arising from “cranking” [14].

Note, however, that Eq. (2.3) only has solutions when $B \psi$ is either zero or orthogonal to all the zero modes of $A$; if this is not true the condition $A \Phi = -B \psi$ cannot be satisfied. Thus, the number of static zero modes is the number of linearly independent configurations satisfying $A \psi = 0$, while the number of dynamic zero modes is the number of linearly independent configurations simultaneously satisfying $A \psi = 0$ and $B \psi$ orthogonal to all the zero modes of $A$. Thus, the number of dynamical zero modes is less than or equal to the number of static modes. In considering the collective quantization of three-flavored hedgehog models such as the Skyrmion, we shall see that the number of dynamical zero modes is less than the number of static zero modes; this fact will be central to the analysis.

To illustrate the difference between static and dynamic zero modes more clearly, consider a simple example in classical mechanics. Let a particle with position $r$ move in $R^3$ with a potential $U$ (depending only on $z$) with a minimum at $z = 0$. The Lagrangian is

$$L = \frac{1}{2} m (\partial_t \vec{r})^2 - U(\vec{r}), \quad i = x, y, z \quad \text{and} \quad \partial_z U = \partial_y U = 0$$

and the equation of motion is simply

$$m (\partial_t r_i)^2 + \partial_i U(\vec{r}) = 0.$$  \hspace{1cm} (2.5)

Clearly $\vec{r} = \vec{r}_0 = (0,0,0)$ is a time-independent solution to Eq. (2.5). The Lagrangian is invariant under displacements in the $x - y$ plane. Thus, it is easy to see that $\vec{r} = \vec{r}_0 + \hbar \vec{k}$ is also a solution with the same energy for any displacement $\hbar$, and similarly for displacements in the $y$ direction. Since any solution of Eq. (2.5) can be displaced time-independently in the $x, y$ directions without any change in energy, we know that this system possesses two static zero modes correspond-
ing to displacements in the \((x, y, 0)\) plane.

There are also solutions of the form \(\vec{r}(t) = \vec{r}_0 + vt\hat{x}\), where \(v\) is a constant, as well as analogous solutions with a term in the \(\hat{y}\) direction. Since \(v\) can be arbitrarily small, and the energy is quadratic in \(v\), these solutions are essentially degenerate in energy with the original one. We identify the infinitesimal fluctuation with finite velocity as the dynamic zero modes of the system. With this identification, we see that in this system each static zero mode is associated with a dynamic zero mode in the same direction.

The reason for this association between static and dynamical zero modes is the absence of any velocity-dependent forces in Eq. (2.6). To see how such forces can break the association, consider modifying the system above by giving the particle an electric charge \(e\), and adding a constant magnetic field \(\vec{B} = g\hat{z}\). It is apparent that the new system still has two static zero modes corresponding to static displacements in the \((x, y, 0)\) plane. However, there are no dynamic zero modes: the magnetic field bends the particle into a closed orbit; the frequency of this orbital motion is simply the cyclotron frequency. Thus we see that a velocity-dependent term in the equation of motion has removed the association between static and dynamical zero modes, and that in presence of such a term, not every static zero mode has a corresponding dynamical zero mode. In terms of the framework given in Eqs. (2.4) and (2.3), this situation is explained by the fact that \(\mathbb{F}\) is now non-zero due to the magnetic field, and the condition \(A\Phi = -\mathbb{F}\psi\) cannot be satisfied.

Let us now return to the soliton problem. The motion associated with the zero modes can occur classically at arbitrarily slow speeds. At infinitesimally slow speeds these modes completely decouple from all of the other degrees of freedom in the system. If the expansion governing the semiclassical expansion (such as the \(1/N_c\) expansion) is valid, then this decoupling is valid quantum mechanically up to corrections which are higher order in the expansion. The quantum theory at lowest order is quite simple conceptually: motion in the direction of the zero modes is expressed in terms of a set of collective degrees of freedom; appropriate commutation relations for these are imposed. The remaining degrees of freedom are harmonic to lowest order and are quantized as harmonic degrees of freedom. Coupling between the collective and non-collective (vibrational) degrees of freedom or between distinct non-collective degrees of freedom are higher order and can be treated perturbatively.

C. Collective quantization

Here we focus on the quantization of the collective degrees of freedom. One essential point is that these degrees of freedom are associated with zero modes which in turn arise from the breaking of symmetries at the classical level. When these collective degrees of freedom are quantized properly, the symmetries of the underlying quantum system are thereby restored.

There are many equivalent ways to implement the quantization of the collective degrees of freedom. For example, if explicit forms for a classical collective coordinate and its conjugate momenta can be found, then canonical quantization rules can be implemented in a straightforward manner \[21, 22\]. However, this may be cumbersome in many cases, particularly in cases such as the Skyrmeion where the soliton breaks multiple symmetries with non-commuting generators.

There is a simple and general strategy for dealing with such cases \[14\]. The procedure has three basic steps: i) First, one obtains an explicit expression for the most general classical collective motion \((i.e., a\) time-dependent field configuration which solves the classical equation of motion for arbitrarily slow motion and which is directed along a local zero mode). These classical configurations are uniquely specified by a set of parameters; the number of parameters is equal to the number of zero modes, both static and dynamic. ii) Next, field-theoretic expressions need to be obtained for the symmetry generators broken by the classical soliton. The insertion of the configuration associated with collective motion of the soliton then gives an explicit expression for the generators in terms of the parameters which specify the collective motion. iii) The known commutation rules for the quantum generators can only be satisfied if the parameters specifying the collective motion are promoted to quantum ‘collective’ operators, the commutation relations of which are derived from those of the known quantum operators. By imposing these derived commutation rules one quantizes the collective motion in a manner consistent with the underlying symmetries of the theory.

III. THE SKYRME MODEL

In this section we briefly review the Skyrme model which we will use to illustrate the issues of collective quantization. As noted in the Introduction, the methods are generally applicable, but it is far more simple to discuss matters in the context of a specific model. Here we will use a Lagrangian identical in form to Skyrmie’s original model \[22\] but generalized to three flavors. The Skyrme Lagrangian density is given by

\[
\mathcal{L}_S = -\frac{f^2}{4} \text{Tr}(L_\mu L^\mu) + \frac{c^2}{4} \text{Tr}([L_\mu, L^\nu]^2).
\]  

(3.1)

Here the left chiral current is defined as \(L_\mu = U_\dagger \partial_\mu U\), with \(U \in SU(3)\), and \(U\) can be written as \(U = e^{i\lambda_i \phi_i / f}\), where the \(\lambda_i\) are the Gell-Mann matrices, and the \(\phi^i\) are the Goldstone
boson fields.

As noted by Witten [24], a Lagrangian of the form of Eq. (3.4) does not correctly encode the anomaly structure of QCD. Moreover, no local term in a Lagrangian is capable of doing this. However, a nonlocal term in the action can. Thus, to reproduce QCD’s low energy anomalous structure, a term nonlocal in $3 + 1$ dimensions (but local in $4 + 1$ dimensions) must be added to the action. This is the Witten-Wess-Zumino term:

$$\Gamma = \pm \frac{i}{240\pi^2} \int_{D^5} d^5 x \, \epsilon_{\mu \nu \alpha \beta \gamma} \text{Tr}(L_\mu L_\nu L_\alpha L_\beta L_\gamma).$$  

(3.2)

Here the boundary of $D^5_5 = S^3 \times S^1 \times [0, \pm 1]$ is compactified space-time. (The sign ambiguity is resolved by Stoke’s theorem.) The full action is then given by

$$S = n\Gamma + \int d^4 x \, L_S, \quad n \in \mathbb{Z}. \quad (3.3)$$

The restriction $n \in \mathbb{Z}$ was shown by Witten to emerge directly from a topological consistency condition in an analysis formally analogous to Dirac’s electric charge quantization argument [22]. To reproduce QCD’s anomaly structure one must take $n = N_c$. With this addition the action above is known to encode the correct scaling behaviors of large $N_c$ QCD, provided the parameters in Eq. (3.3) scale according to

$$f_\pi \sim N_c^{1/2}, \quad \epsilon \sim N_c^{1/2}. \quad (3.4)$$

The equations of motion of the model are obtained in the standard way by varying the action. A convenient way to express these is:

$$-\partial^\mu L_\mu - \frac{2}{f_\pi^2} \partial^\mu \partial^\nu \left[ L_\nu \left[ L_\mu, L_\nu \right] \right] + \frac{iN_c}{24\pi^2 f_\pi} \epsilon_{\alpha \beta \gamma \nu} L_\alpha L_\beta L_\gamma L_\nu = 0.$$  

(3.5)

Observe that the WWZ term is first order in the time derivative in the equations of motion.

The equations of motion above admit topologically non-trivial soliton solutions. To see what this means, note that if working in $SU(2)$, the set of solution maps $U : S^3 \to S^3$ split into homotopically distinct classes, and we can associate a winding number with these homotopy classes [20]. The winding number can be obtained from a current, the Witten-Wess-Zumino term:

$$B^\mu = \frac{i}{24\pi^2} \epsilon_{\mu \nu \alpha \beta} \text{Tr}[L_\nu L_\alpha L_\beta],$$  

(3.6)

which is algebraically conserved: $\partial_\mu B^\mu = 0$. The winding number $B$ is simply the spatial integral of $B^\mu$. $B$ is an integer valued function of the fields ($B : \pi_3(S^3) \to \mathbb{Z}$) and distinguishes between the homotopy classes. The generalization to SU(3) still yields a conserved current and integer values of $B$. The current $B^\mu$ has a simple physical interpretation—it represents the baryon current [23]. Thus soliton solutions which have $B = 1$ corresponds to baryons.

The form of the lowest energy solution to Eq. (3.1) with $B = 1$ is the well-known hedgehog configuration:

$$U = e^{i\pi F(r)}$$  

(3.7)

with $\tau_i = \lambda_i$ and $i \in \{1, 2, 3\}$. The function $F(r)$ is a solution of the following radial equation [23]:

$$\left(\frac{1}{4} \tilde{r}^2 + 2\sin^2 F\right) F'' + \frac{1}{2} \tilde{r} F' + \sin 2F F'' - \frac{1}{4} \sin 2F - \frac{\sin^2 F \sin 2F}{\tilde{r}^2} = 0,$$  

(3.8)

where we use the dimensionless variable $\tilde{r} = \frac{L}{\pi f_\pi} r$. The function $F(r)$ has the boundary conditions:

$$F(r = 0) = \pi, \quad F(r \to \infty) \to 0.$$  

These boundary conditions ensure that $B = 1$.

In the Skyrme model the angular momentum and isospin are individually conserved. However, due to the $\tilde{r} \tilde{r}$ structure of Eq. (3.7), they are correlated within the hedgehog ansatz. This correlation leads to the breaking of both rotational and isorotational invariance at the level of the classical solution. As discussed in Sec. II B if one quantizes the collective degrees of freedom, the resulting states will restore the broken symmetries and allow one to identify well-defined physical states.

### IV. ZERO MODES

The goal of this section is to identify all of the static and dynamical zero modes associated with flavor symmetry. One way to do this is to find explicit expressions for $A$ and $B$ from Sec. II B by expanding the full equation of motion around a static hedgehog. One can then determine the zero eigenmodes of $A$ to obtain the static zero modes, and then construct dynamical zero modes using Eq. (3.6). However, this is not necessary. One can use the known symmetry properties of a given classical soliton to establish the static zero modes. They are simply the infinitesimal changes in the energetically flat directions which are allowed by the symmetries.

The dynamical zero modes are a bit more involved. One can consider a configuration which is slowly rotating in the direction of one of the static zero modes. If the equation of motion is satisfied to leading order in the rotation frequency $\omega$, then in the language of Eq. (2.3) we have $B \psi = 0$ and we automatically have a dynamical zero mode. Conversely,
if $B\psi \neq 0$, then the equation of motion will be violated by an amount proportional to $\omega B\psi$. To construct a dynamical zero mode in such a case we need to invert the relation $B\psi = A\Phi$ to find $\Phi$—provided it is invertible. In general this requires knowledge of the detailed form of $A$. However, we know the preceding relation is not invertible and no dynamic zero mode exists unless $B\psi$ is orthogonal to all of the zero modes of $A$. As we will see, in the Skyrmion model all of the rotations either yield $B\psi = 0$ and directly give a dynamic zero mode independent of the form of $A$, or they are not orthogonal to all of the zero modes of $A$, and thus one does not obtain a dynamical zero mode. All dynamical zero modes in the Skyrmion model can therefore be found without explicitly constructing $A$.

We begin our analysis of zero modes in the Skyrmion model by embedding the $SU(2)$ hedgehog ansatz in the up-down (u-d) subspace of $SU(3)$ \[ U_H = \begin{pmatrix} \exp i(\vec{\omega} \cdot \vec{r}) F(r) & 0 \\ 0 & 1 \end{pmatrix}. \] (4.1)

The hedgehog ansatz $U_H$ solves the static Skyrmion equation of motion (Eq. 3.5). Note that the choice of the u-d subspace embedding above was arbitrary since we are free to choose any $SU(2)$ subspace. As a result of this freedom, it can be seen that a statically rotated hedgehog is also a solution to the equation of motion, Eq. 3.5. That is, the rotated hedgehog below also solves the equation of motion:

$$ U = A U_H A^\dagger, A \in SU(3). \quad (4.2) $$

However, since $\lambda_8$ commutes with $U_H$ we take

$$ A \in SU(3)/U(1)_{\lambda_8} $$

so that $A$ only depends on seven parameters. This implies there are seven static zero modes associated with such changes in initial configuration. They are given explicitly by

$$ \delta U_4(\lambda_j) = [-i \lambda_j, U_H] = \sin(F(r))[\lambda_j, \vec{r} \cdot \vec{r}] \quad (4.3) $$

for $j = 1, \ldots, 7$.

We now analyze the dynamical zero modes. We can construct rigidly rotating hedgehogs by replacing $U_H$ with $A(t)U_H A^\dagger(t)$, where $A(t) = \exp i(\vec{\lambda}_5, \omega) t$. Here $(\vec{\lambda}_5, \omega)$ is an arbitrary linear combination of $SU(3)/U(1)_{\lambda_8}$ generators, and $\omega$ is of order $N_c^{-1}$ and thus the rotations are slow. In the language of Sec. 3.B this form automatically gives the dynamical zero modes if $B = 0$ and gives us $B\psi$ otherwise. Without the WWZ term (that is, for just Eq. 3.1 and $B = 0$), it can be seen by direct substitution that $A(t)U_H A^\dagger(t)$ is an $O(\omega^2)$ approximate solution to the equation of motion.

Now consider the effect of the WWZ term on dynamical rotations. For simplicity let $A(t) = \exp i(\lambda_5, \omega) t$, where $\lambda_5$ is one of the generators of $SU(3)/U(1)_{\lambda_8}$. When we substitute $U = A(t)U_H A^\dagger(t)$ into $\Gamma_{wzw} = -i e^{\alpha \beta \gamma \nu} L_\alpha L_\beta L_\gamma L_\nu$, we see that to first order in $\omega$

$$ \Gamma_{wzw}(\lambda_5) = \omega e^{ijk} [\{[\lambda_5, U_H]^\dagger, L_i], L_j L_k]. \quad (4.4) $$

In the preceding $\{ , \}$ denotes an anticommutator. Explicitly, we find:

$$ \Gamma_{wzw}(\lambda_{1,2,3}) = 0 \quad \Gamma_{wzw}(\lambda_5) = \omega(b \lambda_4 + a \lambda_5 + d \lambda_6 + c \lambda_7) \quad (4.5) $$

$$ \Gamma_{wzw}(\lambda_7) = \omega(-d \lambda_4 + c \lambda_5 + b \lambda_6 - a \lambda_7), $$

where

$$ a = \frac{5i}{r^2} \cos \theta \sin^3 (F(r)) F'(r) \quad b = \frac{48i}{r^2} \cos^2 \left( \frac{F(r)}{2} \right) \sin^4 \left( \frac{F(r)}{2} \right) F'(r) \quad (4.6) $$

$$ c = \frac{4i}{r^2} \cos \phi \sin \theta \sin^3 (F(r)) F'(r) \quad d = \frac{4i}{r^2} \sin \phi \sin \theta \sin^3 (F(r)) F'(r). $$

\[ \]

From Eqs. 4.5, 4.6 we explicitly see that dynamically rotating solutions generated by $\lambda_{4,5,6,7}$ do not satisfy the equations of motion, as the coefficients $a, b, c, d$ are generally non-zero. On the other hand, dynamically rotating solutions generated by $\lambda_{1,2,3}$ do satisfy the equations of motion. In the approach of Sec. 3.B the right-hand sides of Eqs. 4.5, 4.6 are identified as $B\psi$. The construction in Eq. 4.5 can be used to produce dynamical zero modes provided $B\psi$ (i.e. $\Gamma_{wzw}(\lambda_j)$)
has no components in the directions of the static zero modes. For \( j = 1, 2, 3 \) this is trivially true since \( \Gamma_{wwz}(\lambda_{1,2,3}) = 0 \) and these three directions have dynamical zero modes associated with them. This is expected; the WWZ term does not contribute to the equation of motion for these modes since we effectively work in \( SU(2) \) (see Ref. [3]).

On the other hand it is also straightforward to see, by explicit computation, that all of the \( \Gamma_{wwz}(\lambda_{4,5,6,7}) \) have non-vanishing overlaps with the static zero modes of Eq. (4.3). We define the overlap of two functions \( f, g \) as

\[
\langle f|g \rangle = \frac{1}{2} \int d^3x \text{Tr}[f(x)g(x)].
\]

(4.7)

With this definition, it can be shown that

\[
\langle \delta U_s(\lambda_5)\Gamma_{wwz}(\lambda_4) \rangle = \langle \delta U_s(\lambda_7)\Gamma_{wwz}(\lambda_6) \rangle = 9\pi^2
\]

(4.8)

\[
\langle \delta U_s(\lambda_4)\Gamma_{wwz}(\lambda_5) \rangle = \langle \delta U_s(\lambda_6)\Gamma_{wwz}(\lambda_7) \rangle = -9\pi^2,
\]

and all other overlaps are zero. This shows that each of the \( \Gamma_{wwz}(\lambda_{4,5,6,7}) \) has components along one of the static zero-mode directions. Therefore we see that there are no collective dynamical zero modes along the \( \lambda_{4,5,6,7} \) directions, i.e., for dynamical rotations out of the u-d subspace.

The preceding demonstration shows that including the WWZ term in the \( SU(3) \) Skyrme Lagrangian eliminates four of the dynamical zero modes. There are only three collective dynamical zero modes. This is significant since the standard treatment of three flavor hedgehog models has been in the context of a rigid rotor treatment \( [1, 2, 5] \), which is based implicitly on the assumption that there are seven collective dynamical zero modes.

V. QUANTIZATION

There are ten total zero-modes (seven static and three dynamical) that we can use as collective variables to quantize the soliton. We will adopt the strategy of Sec. II(B). We construct an ansatz which allows for the most general collective motion consistent with the zero mode structure. This ansatz is labeled by ten parameters (one for each zero mode) which will be promoted to quantum operators. The appropriate ansatz for collective rotations of the soliton is given by

\[
A e^{i(\vec{\tau} \cdot \vec{\phi}) t} U e^{-i(\vec{\tau} \cdot \vec{\phi}) t} A^\dagger
\]

(5.1)

where \( A \in SU(3)/U(1)_{\lambda_3} \), and \( (\vec{\tau} \cdot \vec{\phi}) \) represents a linear combination of the first three Gell-Mann matrices. The seven parameters that specify \( A \), along with the three parameters that specify \( \vec{\phi} \), can now be used to collectively quantize the Skyrmion following the procedure of Sec. II(B).

Our initial goal here is modest: we seek only to characterize the quantum numbers of the states that would result from such a quantization procedure. In what follows we will obtain such a characterization by using general group-theoretic arguments. We would like to identify the representations of \( SU(3) \) that can arise as a result of quantization. That is, we want to determine the \( (p, q) \) that characterize these representations. To find these \( (p, q) \) we will focus on the maximum hypercharge in each representation.

The dimension of an \( SU(3) \) representation is given by \( 2Y_{\text{max}} + 1 \), where \( Y_{\text{max}} \) is the maximum hypercharge of the representation. If we define \( B \) as the baryon number and \( S \) as the strangeness, then the hypercharge for arbitrary \( N_c \) is given by \( Y = \frac{N_c B}{2} + S \). Note that the dimension of each physically allowable representation scales as \( N_c^2 \) and becomes infinite-dimensional in the large \( N_c \) limit. To make the problem tractable we consider \( N_c \) to be finite but arbitrarily large.

We now seek to determine \( Y_{\text{max}} \) for the collectively rotated hedgehog \( (B = 1) \). In Appendix A we present an explicit computation of \( Y_{\text{max}} \) by constructing the Noether current associated with the hypercharge. We find that at large \( N_c \),

\[
Y_{\text{max}} = \frac{N_c}{3}
\]

(5.2)

independently of the parameters specifying the collective motion. By eliminating the possibility of producing collective states with \( Y > N_c/3 \), we see that all physically allowable states must have \( S \leq 0 \).

Having found the dimensions of the physically allowable representations, we can determine the \( (p, q) \) that characterize them. We use Young tableaux to enumerate \( SU(3) \) representations. First, note that for \( SU(3) \) we need only consider Young tableaux with one or two rows. Representations of the form \( (p, q) \) are known to have a maximum hypercharge given by \( Y_{\text{max}} = 2q/3 + p/3 \). Equating this with the value in Eq. (5.2) yields \( N_c = 2q + p \) and implies that the number of boxes for any allowable representation must be equal to \( N_c \). Thus the relevant Young tableaux are those that have the form \( (p, N_c - p/2) \):

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

The hypercharge is maximized when \( S = 0 \). Let us focus on the states with maximal hypercharge. These states correspond to the situation where each box in a tableau can be thought of as carrying either a u or d — that is, isospin \( I = \pm 1/2 \). However, each of the \( q \) double rows is an anti-symmetric combination which contributes no net isospin. Thus for the states of maximal hypercharge we have \( I = p/2 \). In the u-d subspace, the Skyrmion is known to have \( I = J \), implying that \( J = p/2 \). Using this information we can identify (by giving \( (p, q) \), \( Y, I \) and \( J \)) all of the physically allowable states resulting from collective quantization of the
Skyrmion.

| $J=I$ | $p$ | Collective State |
|-------|-----|------------------|
| $\frac{3}{2}$ | 1 | p.n. . . |
| $\frac{5}{2}$ | 3 | $\Delta$'s . . . |
| $\frac{7}{2}$ | 5 | Large $N_c$ artifact |

TABLE I: Allowable Representations

(In this model, states with $J > \frac{3}{2}$ are assumed to be artifacts of the large $N_c$ world.)

Note that these states are identical to those predicted in the model-independent results of Ref. [19]: they are also the same as those obtained from the naive quark model. Also observe that while we find the same non-exotic states as predicted by rigid rotor quantization, we do not find any exotic states with $S > 0$ such as the $\theta^+$ pentaquark. This is one of the principal differences between this systematic treatment of the quantization and the ad hoc rigid rotor approach of Refs. [1] [2].

Finally let us construct the collective Hamiltonian. This can be done using the approach of Sec. II B We insert the ansatz Eq. (5.1) into the full Skyrme Hamiltonian to obtain a collective Hamiltonian. The parameters of the ansatz are again promoted to collective quantum mechanical operators. The collective Hamiltonian is simply given by:

$$H = M_0 + \frac{1}{2} \mathcal{I} \omega^2$$  \hspace{1cm} (5.3)

where the mass $M_0$ and the moment of inertia $\mathcal{I}$ are the standard Skyrme expressions [3]. Implementation of the procedure in Sec. II B for the Noether current for angular momentum relates $\omega$ to the angular momentum $\mathcal{J} = \mathcal{I} \dot{\omega}$. We arrive at the following expression for the collective Hamiltonian:

$$H = M_0 + \frac{j^2}{2} + O(N_c^{-2})$$  \hspace{1cm} (5.4)

where $\hat{j}$ is the quantum angular momentum operator and $j$ is the spin of the state. This Hamiltonian gives the energy splittings for the states given in Table I.

It is instructive to compare the collective Hamiltonian with that obtained using the rigid rotor approach. A convenient way to write the rigid rotor collective Hamiltonian is given in Ref. [3]:

$$H = M_0 + \frac{j(j+1)}{2\mathcal{I}} + O(N_c^{-2}) + O(N_c^{-2}).$$  \hspace{1cm} (5.5)

Equation (5.5) disagrees in a critical way with Eq. (5.3): there are of two moments of inertia in Eq. (5.5) with one representing motion out of the u-d subspace. This second moment of inertia is in fact not present in the collective Hamiltonian (5.3), since there are no dynamical zero modes for rotations out of the u-d subspace, and thus no moment of inertia can be associated with such rotations. Despite this profound difference, the rigid rotor approach does capture much of the correct physics. In particular, if one uses the rigid rotor expression in Eq. (5.5) but restricts one’s attention to the physically allowable representations of Table I with $q = \frac{\mathcal{J}}{2\mathcal{I}} - J$ and $p = 2J$, the spectrum of the rigid rotor is found to be

$$H = M_0 + \frac{j(j+1)}{2\mathcal{I}} + \frac{N_c}{4\mathcal{I}^2} + O(N_c^{-2}).$$  \hspace{1cm} (5.6)

Note that this differs from the correct spectrum only by the $\frac{N_c}{4\mathcal{I}^2}$ term, which contributes an overall constant at a subleading order (relative order $1/N_c$).

VI. CONCLUSION

We have shown how to collectively quantize the three-flavor Skyrmion. The basic approach can be used with minor modifications for any hedgehog soliton model with three degenerate flavors. The key to the approach is the distinction between dynamic and static zero modes. We have demonstrated explicitly that the Witten-Wess-Zumino term eliminates four of the seven possible dynamical zero modes. Using group-theoretic arguments together with the zero mode results, we showed that the resulting states must have $S \leq 0$, and must lie in representations characterized by $(p,q) = (p, \frac{N_c}{2\mathcal{I}} - \frac{1}{2})$. To complete the specification of the physically allowed collective states we demonstrated that $J = p/2$. The $S \leq 0$ constraint implies that exotic baryons, such as the $\theta^+$ (1540) pentaquark resonance, are not predicted as collective states in the context of the $SU(3)$ flavor Skyrme model.
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APPENDIX A: THE MAXIMUM HYPERCHARGE

In this appendix, we will derive a formula for the hypercharge for collectively rotating hedgehogs in the Skyrme model, and will show that in the large \( N_c \) limit it is bounded from above by \( \frac{N_c B}{3} \).

To begin, observe that the action in Eq. (3.3) is invariant under rotations generated by \( \lambda_8 \). We can construct a conserved current associated with this symmetry. Combining the expression for the current associated with the WWZ term found by Witten [24] with the Noether current obtained from the Skyrme Lagrangian (Eq. (3.1)), we find

\[
y^\mu = \frac{1}{\sqrt{3}} \text{Tr} \left[ \frac{f^2}{N_c} \lambda_8 \left( L^\mu + R^\mu \right) - 2 \lambda_8 \epsilon^{\mu \rho \sigma} (L_\nu [L^\rho, L^\sigma]) \right] + \frac{N_c B}{3} \lambda_8 \epsilon^{\mu \rho \sigma} (L_\rho R_\sigma) + f_{\chi \nu \tau} \epsilon^{\mu \nu \tau},
\]

where \( R^\mu = U (\partial_\mu U^\dagger) \) and \( L^\mu = U^\dagger (\partial_\mu U) \). The current \( y^\mu \) is conserved (\( \partial_\mu y^\mu = 0 \)), and the integral of \( y^0 \) is the appropriately normalized hypercharge \( Y \).

We now insert the ansatz for collective rotations (Eq. (5.1)) into Eq. (A1). Expanding to first order in \( \omega \sim 1/N_c \), we can write the hypercharge density as \( y_0 = y_0^{(0)} + \omega y_0^{(1)} \), where

\[
y_0^{(1)} = \frac{i}{2\sqrt{3}} N_c \text{Tr} \left[ A^\dagger \lambda_8 A \left( -\frac{f^2}{N_c} [U_H^\dagger, \Omega] - 2 \frac{f^2}{N_c} \left( [\hat{L}_k, [U_H^\dagger \Omega, \hat{L}_k]] - [\hat{R}_k, [U_H^\dagger \Omega, \hat{R}_k]] \right) + \frac{\epsilon^{ijk}}{48\pi^2} \left( \hat{i}(\vec{n} \cdot \hat{\omega}), \hat{l}_i \hat{l}_j \hat{l}_k - \hat{r}_i \hat{r}_j \hat{r}_k \right) \right],
\]

\[
y_0^{(0)} = \frac{i}{2\sqrt{3}} N_c \text{Tr} \left[ A^\dagger \lambda_8 A \frac{\epsilon^{ijk}}{48\pi^2} (\hat{l}_i \hat{L}_j \hat{L}_k - \hat{r}_i \hat{r}_j \hat{r}_k) \right],
\]

with \( \hat{R}_k = U_H (\partial_\mu U_H^\dagger) \), \( \hat{L}_i = U_H^\dagger (\partial_\mu U_H) \) and \( \Omega = i(\vec{n} \cdot \hat{\omega}), U_H^\dagger \).

The hypercharge of the collectively rotating hedgehog Skyrmion, \( Y_{\text{col}} \), is given by

\[
Y_{\text{col}} = \int d^3x y_0.
\]

If we consider only static rotations within the u-d subspace \( (\omega = 0 \) and \( A \) in the u-d subspace of \( SU(3) \)), we find that the hypercharge is given by

\[
Y_{\text{col}} = \frac{1}{2\sqrt{3}} N_c \left( \frac{\epsilon^{ijk}}{24\pi^2} \int d^3x \text{Tr} [\lambda_8 (\hat{l}_i \hat{l}_j \hat{l}_k)] \right).
\]

Since the expression in parentheses is proportional to the baryon number \( B \) defined in Eq. (3.0) (specifically, it is just \( \frac{2B}{\sqrt{3}} \)), we obtain the relation \( Y_{\text{col}} = \frac{N_c B}{3} \) for the hypercharge of a hedgehog Skyrmion undergoing static rotations in the u-d subspace.

We now claim that in the large \( N_c \) limit \( Y_{\text{col}} = \frac{N_c B}{3} \) is an upper bound. To see this, let us relax the preceding restrictions and consider a hedgehog that is undergoing slow \( (\omega \sim 1/N_c) \) collective dynamical rotations, and allow general \( SU(3) \) static rotations. After the spatial integral in Eq. (A3) is evaluated, only a term proportional to \( \hat{\omega} \cdot \vec{n} \) can survive in the term coming from \( y_0^{(1)} \). To see this, observe that integrating \( y_0^{(0)} \) over space is equivalent to integrating over isospace due to the correlation of space and isospace in a hedgehog embedded in the u-d space; the only direction that can remain is the one set by \( \vec{n} \). Integrating \( y_0^{(0)} \) over space yields

\[
\frac{N_c B}{3} \text{Tr} \left[ A^\dagger \lambda_8 A \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \right] = \frac{N_c B}{3} \text{Tr} [A^\dagger \lambda_8 A (\frac{2}{3} \mathbf{1} + \lambda_8 / \sqrt{3})].
\]

Thus we see that the hypercharge of a collectively rotating hedgehog \( (B = 1) \) is given by

\[
Y_{\text{col}} = a N_c \omega \text{Tr} \left[ A^\dagger \lambda_8 A(\vec{n} \cdot \vec{\omega}) + \frac{N_c B}{3} \text{Tr} \left[ A^\dagger \lambda_8 A (\frac{2}{3} \mathbf{1} + \lambda_8 / \sqrt{3}) \right] \right],
\]

where \( a \) is a real coefficient of order \( N_c^0 \) that is proportional to the moment of inertia \( I \) of an \( SU(2) \) hedgehog: \( a N_c = \frac{3}{\sqrt{3}} L \).

Since \( Y_{\text{col}} \) is an isoscalar, the orientation of \( \vec{\omega} \) is irrelevant, and thus without loss of generality we set \( \vec{\omega} = \omega \hat{z} \).
We can write $A^†λ_8A = \sum_{i=1}^{8} c_iλ_i$ with the condition that $\sum_{i=1}^{8} |c_i|^2 = 1$. Since a unit matrix in the u-d subspace of $SU(3)$ can be written as $\frac{8}{3}1 + \sqrt{3}λ_8$, we can rewrite the hypercharge as

$$Y_{\text{col}} = 2aωNc_3 + Nc_3 \sqrt{1 - \sum_{i=1}^{8} |c_i|^2}. \quad (A6)$$

Maximizing the hypercharge with respect to the condition $\sum_{i=1}^{8} |c_i|^2 = 1$, we see that $c_3 = 6aω \sim 1/N_c$ and $c_i = 0$ for all $i \neq 3, 8$. Thus in the large $N_c$ limit $c_3$ vanishes. This leads to the condition $|c_3|^2 = 1$ and $Y_{\text{col}} = \frac{N}{2}$. This shows that in the large $N_c$ limit, $Y_{\text{col}} = \frac{N}{2}$ is a global maximum for the hypercharge of a collectively rotating hedgehog Skyrmion.

[1] E. Guadagnini, Nucl. Phys. B 236, 35 (1984).
[2] P. O. Mazur, M. A. Nowak and M. Praszalowicz, Phys. Lett. B 147, 137 (1984).
[3] M. Chemtob, Nucl. Phys. B 256, 600 (1985).
[4] S. Jain and S. R. Wadia, "Chiral Model," Nucl. Phys. B 258, 713 (1985).
[5] A. V. Manohar, Nucl. Phys. B 248, 19 (1984).
[6] G. S. Adkins, C. R. Nappi and E. Witten, Nucl. Phys. B 228, 552 (1983).
[7] M. Praszalowicz, arXiv:hep-ph/0506030.
[8] D. Diakonov, V. Petrov and M. V. Polyakov, Z. Phys. A 359, 305 (1997) arXiv:hep-ph/9703373.
[9] T. Nakano et al. (LEPS Collaboration), Phys. Rev. Lett. 91, 012002 (2003);
[10] V. V. Barmin et al. (DIANA Collaboration), Phys. At. Nucl. 66, 1715 (2003) [Yad. Fiz. 66, 527 (2003)];
[11] A. A. Arzatyan, A. G. Dolgolenko and M. A. Kubantsev, Phys. At. Nucl. 67, 682 (2004) [Yad. Fiz. 67, 704 (2004)];
[12] V. Kubarovsky et al. (CLAS Collaboration), Phys. Rev. Lett. 92, 032001; 92, 049902(E) (2004);
[13] A. Airapetian et al. (HERMES Collaboration), Phys. Lett. B 585, 213 (2004);
[14] S. Chekanov et al. (ZEUS Collaboration), ibid. 591, 7 (2004);
[15] M. Abdel-Bary et al. (COSY-TOF Collaboration), ibid. 595, 127 (2004);
[16] A. Aleev et al. (SVD Collaboration), hep-ex/0401024.
[17] J. Barth et al. (SAPHIR Collaboration), Phys. Lett. B 572, 127 (2003),
[18] S. Stepanyan et al. (CLAS Collaboration), Phys. Rev. Lett. 91, 252001 (2003).
[19] T. D. Cohen, Phys. Lett. B 581, 175 (2004) arXiv:hep-ph/0309111.
[20] T. D. Cohen, Phys. Rev. D 70, 014011 (2004) arXiv:hep-ph/0312191.
[21] N. Itzhaki, I. R. Klebanov, P. Ouyang and L. Rastelli, Nucl. Phys. B 684, 264 (2004) arXiv:hep-ph/0309305.
[22] P. V. Pobylitsa, Phys. Rev. D 69, 074030 (2004) arXiv:hep-ph/0310221.
[23] T. D. Cohen and W. Broniowski, Phys. Rev. D 34, 3472 (1986).
[24] M. Praszalowicz, Phys. Lett. B 575, 234 (2003) arXiv:hep-ph/0308114.
[25] A. Chernyak, T. D. Cohen and A. Nellore, Phys. Rev. D 70, 096003 (2004) arXiv:hep-ph/0408209.
[26] D. Diakonov and V. Petrov, Phys. Rev. D 69, 056002 (2004) arXiv:hep-ph/0309203.
[27] D. Diakonov and V. Petrov, arXiv:hep-ph/0505201.
[28] R. Dashen, E. Jenkins and A.V. Manohar, Phys. Rev. D 49, 4713 (1994); 51, 2489(E) (1995).
[29] C. G. Callan and I. R. Klebanov, Nucl. Phys. B 262, 365 (1985).
[30] For a review see for example R. Jackiw, Rev. Mod. Phys. 49 (1977) 681.
[31] R. Rajaraman, Solitons and Instantons: An introduction to Solitons and Instantons in Quantum Field Theory, (Elsevier Science Publishers, Netherlands, 1987)
[32] T. H. R. Skyrme, Proc. Roy. Soc. Lond. A 260, 127 (1961).
[33] E. Witten, Nucl. Phys. B 223, 422 (1983).
[34] E. Witten, Nucl. Phys. B 223, 433 (1983).
[35] I. Zahed and G. E. Brown, Phys. Rept. 142, 1 (1986).