Classical BRST charge and observables in reducible gauge theories

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Abstract

The general solution to the master equation for the classical Becchi-Rouet-Stora-Tyutin (BRST) charge is presented in the case of reducible gauge theories. The BRST observables are constructed.

1 Introduction

The modern quantization method for gauge theories is based on the BRST symmetry [1, 2]. In the framework of the canonical formalism this symmetry is generated by the BRST charge. If the quantum BRST charge exists it is essentially determined by the corresponding classical one. The BRST charge is defined as a solution to the master equation with certain boundary conditions. The BRST construction in the case of reducible gauge theories was given in [3]. The global existence of the classical BRST charge in the reducible case was proved in [4].

In this paper we give the general solution to the classical master equation for the BRST charge in the case of reducible gauge theories. To this aim, we construct a new coordinate system in the extended phase space and transform the equation by changing variables. Then it can be solved by using a simple iterative procedure. We also give a solution to the equation determining the classical BRST observables, and describe a new realization of the observable algebra. In contrast to [4], our approach does not require neither
redefining of the constraints no modification of the reducibility functions. In the framework of the Lagrangian formalism the master equation was solved in \[5\].

The paper is organized as follows. In section 2, we review the BRST construction and derive two auxiliary equations. In section 3, we introduce new variables and construct a generalized inverse of the Koszul-Tate differential operator $\delta$. With respect to the new variables, both $\delta$ and its generalized inverse take a standard form. The solution to the master equation is given in section 4. The BRST observables are described in section 5.

In what follows Grassman parity and ghost number of a function $X$ are denoted by $\epsilon(X)$ and $\text{gh}(X)$, respectively. The Poisson superbracket in phase space $\Gamma = (P_i, Q^i)$, $\epsilon(P_i) = \epsilon(Q^i)$, is given by

$$\{X, Y\} = \frac{\partial X}{\partial Q^i} \frac{\partial Y}{\partial P_i} - (-1)^{\epsilon(X)\epsilon(Y)} \frac{\partial Y}{\partial Q^i} \frac{\partial X}{\partial P_i}.$$ 

Derivatives with respect to generalized momenta $P_i$ are always understood as left-hand, and those with respect to generalized coordinates $Q^i$ as right-hand ones.

## 2 Master equation for the BRST charge

Let $\xi_a$, $a = 1, \ldots, 2m$, be the phase space coordinates, and let $G_{a_0}$, $a_0 = 1, \ldots, m_0$, be the first class constraints which satisfy the following Poisson brackets

$$\{G_{a_0}, G_{b_0}\} = U^{c_0}_{a_0 b_0} G_{c_0},$$

where $U^{c_0}_{a_0 b_0}$ are phase space functions. The constraints are assumed to be of definite Grassmann parity $\epsilon_{a_0}$, $\epsilon(G_{a_0}) = \epsilon_{a_0}$.

We shall consider a reducible theory of $L$-th order. That is, there exist phase space functions

$$Z^{a_k}_{a_{k+1}}, \quad k = 0, \ldots, L - 1, \quad a_k = 1, \ldots, m_k,$$

such that at each stage the $Z$’s form a complete set,

$$Z^{a_k}_{a_{k+1}} \lambda^{a_{k+1}} \approx 0 \Rightarrow \lambda^{a_{k+1}} \approx Z^{a_{k+1}}_{a_{k+2}} \lambda^{a_{k+2}}, \quad k = 0, \ldots, L - 2,$$

$$Z^{a_{L-1}}_{a_L} \lambda^{a_L} \approx 0 \Rightarrow \lambda^{a_L} \approx 0.$$
\[ G_{a_0} Z_{a_1}^{a_0} = 0, \quad Z_{a_k+1}^{a_k} Z_{a_{k+2}}^{a_k} \approx 0, \quad k = 0, \ldots, L - 2. \quad (1) \]

The weak equality \( \approx \) means equality on the constraint surface
\[ \Sigma : \quad G_{a_0} = 0. \]

Following the BRST method the ghost pairs \((P_{a_k}, c_{a_k}), k = 0, \ldots, L,\) are introduced
\[ \epsilon(P_{a_k}) = \epsilon(c_{a_k}) = \epsilon_{a_k} + k + 1, \quad -\text{gh}(P_{a_k}) = \text{gh}(c_{a_k}) = k + 1. \]

The BRST charge \( \Omega \) is defined as a solution to the equations
\[ \{ \Omega, \Omega \} = 0, \quad (2) \]
\[ \epsilon(\Omega) = 1, \quad \text{gh}(\Omega) = 1, \quad (3) \]
and the boundary conditions
\[ \frac{\partial \Omega}{\partial c_{a_0}} \bigg|_{c=0} = G_{a_0}, \quad \frac{\partial^2 \Omega}{\partial P_{a_{k-1}} \partial c_{a_k}} \bigg|_{P=c=0} = Z_{a_{k+1}}^{a_k}. \]

One can write
\[ \Omega = \Omega^{(1)} + M, \quad M = \sum_{n \geq 2} \Omega^{(n)}, \quad \Omega^{(n)} \sim c^n, \quad (4) \]
\[ \Omega^{(1)} = G_{a_0} c_{a_0} + \sum_{k=1}^{L} (P_{a_{k-1}} Z_{a_k}^{a_{k-1}} + N_{a_k}) c_{a_k}, \quad (5) \]
where \( N_{a_1} = 0 \) and \( N_{a_k}, k > 1, \) only involves \( P_{a_s}, s \leq k - 2. \) Eq. \((3)\) implies
\[ N_{a_k} |_{P=0} = 0, \quad M |_{P=0} = 0. \]

Let \( V \) be the space of the formal power series in \((\xi, P, c)\) which vanish on \( \Sigma \) at \( P = 0. \) For \( X, Y \in V \) we have
\[ XY \in V, \quad \{X, Y\} \in V, \]
and therefore \( V \) is a Poisson algebra. It is easily verified that \( \Omega \in V. \)
The bracket \( \{ \ldots \} \) splits as
\[
\{X, Y\} = \{X, Y\}_\xi + \{X, Y\}_\circ - (-1)^{\varepsilon(X)\varepsilon(Y)}\{Y, X\}_\circ,
\]
where \( \{ \ldots \}_\xi \) refers to the Poisson bracket in the original phase space and
\[
\{X, Y\}_\circ = \sum_{k=0}^{L} \frac{\partial X}{\partial c^a_k} \frac{\partial Y}{\partial \mathcal{P}_a_k}.
\]

Let \( \delta : V \rightarrow V \) be defined by
\[
\delta = \{\Omega^{(1)}, \cdot\}_\circ = G_{a_0} \frac{\partial}{\partial \mathcal{P}_{a_0}} + \sum_{k=1}^{L} (\mathcal{P}_{a_{k-1}} Z_{a_k}^{a_{k-1}} + N_{a_k}) \frac{\partial}{\partial \mathcal{P}_{a_k}}.
\]

From (5) we obtain
\[
\delta = \{\Omega^{(1)}, \cdot\}_\circ = G_{a_0} \frac{\partial}{\partial \mathcal{P}_{a_0}} + \sum_{k=1}^{L} (\mathcal{P}_{a_{k-1}} Z_{a_k}^{a_{k-1}} + N_{a_k}) \frac{\partial}{\partial \mathcal{P}_{a_k}}.
\]

Substituting (4) in (2) one obtains
\[
\delta \Omega^{(1)} = 0, \quad \delta M = D,
\]
where
\[
-D = \frac{1}{2} F + AM + \frac{1}{2} \{M, M\}, \quad F = \{\Omega^{(1)}, \Omega^{(1)}\}_\xi,
\]
and the operator \( A \) is given by
\[
A = \{\Omega^{(1)}, \cdot\}_\xi + \{\cdot, \Omega^{(1)}\}_\circ.
\]

Eq. (7) is equivalent to
\[
\delta^2 = 0.
\]

We shall need equations for
\[
S = \delta \Omega^{(1)}, \quad R = \delta M - D.
\]
By using the definition of \( \delta \), we get
\[
\delta S = \{\Omega^{(1)}, S\}_\circ.
\]
If (7) holds, then
\[
R = \{\Omega, \Omega\}.\text{ From the Jacobi identity } \{\Omega, \{\Omega, \Omega\}\} = 0 \text{ it follows that } \{\Omega, R\} = 0, \text{ or equivalently }
\]
\[
\delta R + AR + \{M, R\} = 0.
\]
3 Generalized inversion of $\delta$

In this section we reduce $\delta$ to a standard form and construct its generalized inverse. For $k = L - 2$, eq. (1) reads

$$Z_{a^{L-2}_{a_{L-1}} }^{a_{L-1}} Z_{a_{L-1}}^{a_{L-1}} + Z_{a^{L-2}_{a_{L-1}} }^{a_{L-1}} Z_{a_{L-1}}^{a_{L-1}} \approx 0,$$

(11)

where $a'_{L-1}, a''_{L-1}$ are increasing index sets, such that $a'_{L-1} \cup a''_{L-1} = a_{L-1}$, $|a'_{L-1}| = |a_L|$ and rank $Z_{a_{L-1}}^{a_{L-1}} = |a_L|$. For an index set $i = \{i_1, i_2, \ldots, i_n\}$, we denote $|i| = n$. From (11) it follows that rank $Z_{a_{L-1}}^{a_{L-1}} = |a_{L-1}| - |a_L| = |a''_{L-1}|$, and rank $Z_{a_{L-1}}^{a_{L-1}} = |a''_{L-1}|$.

One can split the index set $a_{L-2}$ as $a_{L-2} = a'_{L-2} \cup a''_{L-2}$, such that $|a'_{L-2}| = |a''_{L-1}|$, and rank $Z_{a_{L-2}}^{a_{L-2}} = |a''_{L-1}|$. For $k = L - 3$, eq. (1) implies

$$Z_{a_{L-2}}^{a_{L-3}} Z_{a_{L-2}}^{a_{L-2}} + Z_{a_{L-2}}^{a_{L-3}} Z_{a_{L-1}}^{a_{L-2}} \approx 0.$$

From this it follows that

$$\text{rank } Z_{a_{L-2}}^{a_{L-3}} = \text{rank } Z_{a_{L-2}}^{a_{L-3}} = |a_{L-2}| - |a''_{L-1}| = |a''_{L-2}|.$$

Using induction on $k$, we obtain a set of nonsingular matrices $Z_{a_k^{a_{L-1}}}^{a_{a_{L-1}}}$, $k = 1, \ldots, L$, and a set of matrices $Z_{a_k^{a_{L-1}}}^{a_{a_{L-1}}}$, $k = 1, \ldots, L$, such that

$$\text{rank } Z_{a_k^{a_{L-1}}}^{a_{a_{L-1}}} = \text{rank } Z_{a_k^{a_{L-1}}}^{a_{a_{L-1}}} = |a''_k|.$$  

Here

$$a'_k \cup a''_k = a_k, \quad k = 1, \ldots, L - 1, \quad a''_L = a_L.$$

Eq. (1) implies

$$G_{a_0} Z_{a_0}^{a_0} + G_{a_0} Z_{a_0}^{a_0} = 0.$$  

(12)

From this it follows that $G_{a_0}$ are independent. We assume that $G_{a_0}$ satisfy the regularity conditions. It means that there are some functions $F_{\alpha}(\xi)$, $\alpha \cup a''_0 = a$, such that $(F_{\alpha}, G_{a''_0})$ can be locally taken as new coordinates in the original phase space.
Let \( f : a''_{k+1} \to a_k, k = 0, \ldots, L-1 \), be an embedding, \( f(a''_{k+1}) = a''_{k+1} \in a_k \), and let \( \alpha_k \) be defined by \( a_k = f(a''_{k+1}) \cup \alpha_k \). Since \( |a''_k| = |\alpha_k| \), one can write \( \alpha_k = g(a''_k) \) for some function \( g \), and hence
\[
a_k = f(a''_{k+1}) \cup g(a''_k), \quad k = 0, \ldots, L - 1.
\]

**Lemma.** The nilpotent operator \( \delta \) is reducible to the form
\[
\delta = \xi'_a \frac{\partial}{\partial \xi'_a} + \sum_{k=1}^{L} P'_f(a''_k) \frac{\partial}{\partial g(a''_k)},
\]
by the change of variables: \((\xi_a, P_{a_0}, \ldots, P_{a_L}) \to (\xi'_a, P'_{a_0}, \ldots, P'_{a_L})\),
\[
\xi'_a = F_a, \quad \xi'_a = G_a,
\]

\[
P'_f(a''_{k+1}) = \delta P''_a_{k+1}, \quad P'_g(a''_k) = P''_a_k,
\]

\[
P'_{a_L} = P_{a_L},
\]
where \( k = 0, \ldots, L - 1, g(a''_k) = a_L. \)

To prove this statement we first observe that eqs. \((14)\) are solvable with respect to \((\xi_a, P_{a_0}, \ldots, P_{a_L})\). The original variables can be represented as
\[
\xi_a = \xi_a(\xi'), \quad P_{a_k} = P_{a_k}(\xi', P'_{a_0}, \ldots, P'_{a_k}), \quad k = 0, \ldots, L.
\]
Here we have used the fact that the \( P_{a_k} \) depends only on the functions \( P'_{a_s} \) with \( s \leq k \). Assume that the functions \( \xi_a(\xi') \) have been constructed. Then from \((14)\) it follows that
\[
P'_{a_k} = (P'_f(a''_{k+1}) - P'_g(a''_{k+1})Z_{a''_{k+1}}(Z(-1))^{a''_{k+1}}), \quad P''_{a_k} = P'_{g(a''_k)},
\]

\[
P'_{a_k} = P''_{a_k},
\]
where \( k = 0, \ldots, L - 1, \)
\[
N'_{a_{k+1}}(\xi', P'_{a_0}, \ldots, P'_{a_k}) = N_{a_{k+1}}(\xi, P_{a_0}, \ldots, P_{a_{k-1}}).
\]
Therefore, the variables
\[(\xi'_a, P'_{a_0}, \ldots, P'_{a_L}) = (\xi'_a, \xi'_a, P'_{g(a')_0}, P'_{g(a')_1}, \ldots, P'_{g(a')_L}, P'_{g(a')_L})\]
are independent. It follows from (14) that
\[\delta \xi'_a = \delta P'_{f(a')_1} = \ldots = \delta P'_{f(a')_L} = 0,\]
\[\delta P'_{g(a''_0)} = \xi''_{a''_0}, \delta P'_{g(a''_k)} = P'_{f(a''_k)}, \quad k = 1, \ldots, L.\] (17)

Eqs. (17) are equivalent to (13).

Let \(n\) be the counting operator
\[n = \xi''_{a''_0} \frac{\partial}{\partial \xi''_{a''_0}} + P'_{g(a''_0)} \frac{\partial}{\partial P'_{g(a''_0)}} + \sum_{s=1}^{L} \left( P'_{f(a''_s)} \frac{\partial}{\partial P'_{f(a''_s)}} + P'_{g(a''_s)} \frac{\partial}{\partial P'_{g(a''_s)}} \right),\]
and let
\[\sigma = P'_{g(a''_0)} \frac{\partial}{\partial \xi''_{a''_0}} + \sum_{k=1}^{L} P'_{g(a''_k)} \frac{\partial}{\partial P'_{f(a''_k)}}.\]

One can directly verify that
\[\sigma^2 = 0, \quad \delta \sigma + \sigma \delta = n, \quad n \delta = \delta n, \quad n \sigma = \sigma n.\] (18)

With respect to the new coordinate system the condition \(X \in V\) becomes
\[X|_{\xi''_{a''_0}=P'_{a''_0}=0} = 0.\]

The space \(V\) splits as
\[V = \bigoplus_{m \geq 1} V_m\] (19)
with \(nX = mX\) for \(X \in V_m\). Hence the operator \(n : V \to V\) is invertible. It is easily verified that \(\delta^+ = \sigma n^{(-1)}\) is a generalized inverse of \(\delta\):
\[\delta \delta^+ \delta = \delta, \quad \delta^+ \delta \delta^+ = \delta^+.\] (20)

From (18) and (19) it follows that for any \(X \in V\),
\[X = \delta^+ \delta X + \delta \delta^+ X.\] (21)
4 Solution of the master equation

In this section we start from eq. (7). One can write
\[ \delta \Omega^{(1)} = \delta N + BN + Q, \]
where
\[ N = \sum_{k=2}^{L} N_{a_k} e^{a_k}, \quad Q = \sum_{k=2}^{L} P_{a_k} Z_{a_{k-1}}^{a_{k-1}} e^{a_k}, \] (22)

\[ B : V \to V \] is defined by \( B = 0 \), if \( L \leq 2 \), and otherwise
\[ BX = \sum_{k=3}^{L} \frac{\partial X}{\partial c_{a_{k-1}}} Z_{a_{k-1}}^{a_{k-1}} e^{a_k}. \]

Then eq. (7) takes the form
\[ \delta N + BN + Q = 0. \]

Changing variables \((\xi, P) \to (\xi', P')\), we get
\[ \delta N' + B'N' + Q' = 0. \] (23)

Here and in what follows, for any \( X(\xi, P, c) \) we denote by \( X' \) the function
\[ X'(\xi', P', c) = X(\xi, P, c). \]

We shall seek the solution to eq. (23) in the form of expansion in power series of variables
\[ P'_{g(a'_{0})}, P'_{g(a'_{1})}, \ldots, P'_{g(a'_{L})}. \]

Applying \( \delta \delta^+ \) to (23) and using (20) we have
\[ \delta N' + \delta \delta^+(B'N' + Q') = 0, \]
and therefore
\[ N' + \delta^+(B'N' + Q') = Y', \] (24)

where
\[ Y' = \sum_{k=2}^{L} Y'_{a_k}(\xi', P') e^{a_k}, \quad \delta Y' = 0, \quad \epsilon(Y') = 1, \quad gh(Y') = 1. \]
Solving (24), we get
\[ N' = (I + \delta^+ B')^{-1}(Y' - \delta^+ Q'), \tag{25} \]
where \( I \) is the identity map, and
\[ (I + \delta^+ B)^{-1} = \sum_{m \geq 0} (-1)^m (\delta^+ B)^m. \]

It remains to show that (25) satisfies (23). We shall use the approach of [6]. With respect to the new coordinate system eq. (2) takes the form
\[ \delta S' = \{\Omega''(1), S''\}'_c, \tag{26} \]
where
\[ S' = \delta N' + B' N' + Q'. \]

If \( N' \) is a solution to (24), then
\[ \delta^+ N' = \delta^+ Y', \]
since \((\delta^+)^2 = 0\), and
\[ \delta^+ S' = \delta^+ \delta N' + \delta^+ (B' N' + Q') = 0. \tag{27} \]

Here we have used (21) and (24). Consider eq. (26) and condition (27), where \( N' \) is the solution to (24). Applying \( \delta^+ \) to (26), and using (27), we get
\[ S' = \delta^+ \{\Omega''(1), S''\}'_c, \]
from which by iterations it follows that \( S' = 0 \).

The functions \( N_{ak}, k = 1, \ldots, L, \) are found by substituting (14) in (16), where
\[ N'_{ak} = \frac{\partial N'}{\partial c^{ak}}. \]
Recall that \( N_{a1} = 0 \). Assume that \( N_{as}, s \leq k, \) have been constructed. It follows from (6) and (14) that the variables \((\xi', \mathcal{P}'_{a0}, \ldots, \mathcal{P}'_{ak-1})\) depends only on the functions \( N_{as}, s \leq k, \) and therefore \( N'_{ak+1} \) is easily computed.
Our next task is to find a solution to eq. (8). Changing variables \((\xi, P) \rightarrow (\xi', P')\), we get
\[
\delta M' = D', \tag{28}
\]
where
\[
-D' = \frac{1}{2} F' + A M' + \frac{1}{2} \{M', M'\}'.
\]
Applying \(\delta \delta^+\) to (28), we have
\[
\delta M' = \delta \delta^+ D',
\]
from which it follows that
\[
M' = W' + \delta^+ D', \tag{29}
\]
where
\[
W' \in V, \quad \delta W' = 0, \quad \epsilon(W') = 1, \quad gh(W') = 1. \tag{30}
\]
Let \(\langle \ldots \rangle : V^2 \rightarrow V\) be defined by
\[
\langle X_1, X_2 \rangle = -\frac{1}{2} (I + \delta^+ A) (-1)^{\delta^+} (\{X_1, X_2\}' + \{X_2, X_1\})',
\]
where \(I\) is the identity map, and
\[
(I + \delta^+ A) (-1) = \sum_{m \geq 0} (-1)^m (\delta^+ A)^m.
\]
One can rewrite (29) as
\[
M' = M'_0 + \frac{1}{2} \langle M', M' \rangle, \tag{31}
\]
where
\[
M'_0 = (I + \delta^+ A) (-1) (W' - \frac{1}{2} \delta^+ F').
\]
Iterating eq. (31) we can construct the solution \(M'\) in the form of a power series expansion in \((P'_{g(a^k)}, e^{a_k}, k = 0, \ldots, L)\):
\[
M' = M'_0 + \frac{1}{2} \langle M'_0, M'_0 \rangle + \ldots. \tag{32}
\]
Changing variables in (10) \((\xi, \mathcal{P}) \to (\xi', \mathcal{P}')\), we get
\[
\delta R' + AR' + \{M', R'\}' = 0,
\]
where
\[
R' = \delta M' + \frac{1}{2}F' + AM' + \frac{1}{2}\{M', M'\}'.
\]
To prove that (32) satisfies (28) consider eq. (33) and the condition
\[
\delta^+ R' = 0,
\]
where \(M'\) is the solution to (29). Applying \(\delta^+\) to eq. (33) and using (34), we get
\[
R' = -\delta^+(AR' + \{M', R'\}').
\]
From (35) by iterations it follows that \(R' = 0\).

It remains to check (34). The solution to (29) satisfies the condition
\[
\delta^+ M' = \delta^+ W'.
\]
By using (21), we have
\[
M' = \delta^+ \delta M' + \delta \delta^+ W'.
\]
From this, (21) and (30) it follows that
\[
M' = \delta^+ \delta M' + W'.
\]
We have
\[
\delta^+ R' = \delta^+ \delta M' + \delta^+ \left(\frac{1}{2}F' + AM' + \frac{1}{2}\{M', M'\}'\right),
\]
and therefore by (36) and (29), \(\delta^+ R' = 0\).

5 BRST observables

Let \(P\) denote the Poisson algebra of the first class functions,
\[
P = \{\varphi(\xi) \mid \{\varphi, G_\alpha\}\big|_{G=0} = 0\}.\]
and let
\[ J = \{ u(\xi) \mid u|_{G=0} = 0 \}. \]

Elements of \( P/J \) are called classical observables.

A function \( \Phi = \Phi(\xi, \mathcal{P}, c) \) is called a BRST-invariant extension of \( \Phi_0 \in P \) if
\[ \Phi = \Phi_0 + \Pi, \quad \Pi = \sum_{n \geq 1} \Phi^{(n)}, \quad \Phi^{(n)} \sim c^n, \quad gh(\Phi) = 0, \]
\[ \{ \Omega, \Phi \} = 0. \] (37)

Let \( \mathcal{U} \) denote the space of all such extensions. The functions \( \Phi_1, \Phi_2 \in \mathcal{U} \) are set to be equivalent if
\[ \Phi_1 - \Phi_2 = \{ \Omega, \Psi \} \] (38)
for some \( \Psi \). Elements of the corresponding factorspace \( \mathcal{U}/\sim \) are called the BRST observables. The Poisson algebras \( P/J \) and \( \mathcal{U}/\sim \) are isomorphic [4].

Let us consider the equation
\[ \{ \Omega, \Psi \} - \Lambda = 0 \] (39)
where \( \Lambda \) is a given function, \( gh(\Lambda) = 0, \{ \Omega, \Lambda \} = 0 \), and \( \Psi \) is an unknown one. The equation implies that \( \Psi, \Lambda \in V \), since \( gh(\Psi) = -1 \). Let us show that for any \( \Lambda \in V \) there exist a solution to (39). One can write (39) in the form
\[ \delta \Psi + A \Psi + \{ M, \Psi \} - \Lambda = 0. \] (40)

Changing variables from \((\xi, \mathcal{P})\) to \((\xi', \mathcal{P}')\), we get
\[ \delta \Psi' + A \Psi' + \{ M', \Psi' \}' - \Lambda' = 0. \] (41)
By using (20), one can write
\[ \Psi' + \delta^+(A \Psi' + \{ M', \Psi' \}' - \Lambda') = \Upsilon', \] (42)
where
\[ \Upsilon' \in V, \quad \delta \Upsilon' = 0, \quad gh(\Upsilon') = 1. \]
From (42) it follows
\[ \Psi' = (I + \delta^+(A + \text{ad} M'))^{-1}(\Upsilon' + \delta^+ \Lambda'), \tag{43} \]
where \( \text{ad} M' = \{M', \cdot\}' \).

Now, let us show that (43) satisfies (41). Denote by \( \Gamma \) the left-hand side of
\[ \Gamma = \{\Omega, \Psi\} - \Lambda. \tag{44} \]
From the Jacobi identity \( \{\Omega, \{\Omega, \Psi\}\} = 0 \) and the BRST invariance of \( \Lambda \) it follows that
\[ \delta \Gamma + A \Gamma + \{M, \Gamma\} = 0. \]
Changing variables \( (\xi, \mathcal{P}) \rightarrow (\xi', \mathcal{P}') \), we get
\[ \delta \Gamma' + A \Gamma' + \{M', \Gamma'\}' = 0, \tag{45} \]
where
\[ \Gamma' = \{\Omega', \Psi'\}' - \Lambda'. \]
It is straightforward to check that if \( \Psi' \) satisfies (42) then \( \delta^+ \Psi' = \delta^+ \Upsilon' \), and
\[ \delta^+ \Gamma' = 0. \tag{46} \]
Consider (44) and (45), where \( \Psi' \) satisfies (42). By using (21), we get
\[ \Gamma' = -\delta^+(A \Gamma' + \{M', \Gamma'\}'), \tag{47} \]
from which it follows that \( \Gamma' = 0 \). From definition (38) we conclude that \( \Phi_1 \sim \Phi_2 \) if and only if \( \Phi_1 - \Phi_2 \in \mathcal{U} \cap \mathcal{V} \).

Let us now turn our attention to eq. (37). It can be written in the form
\[ \delta \Pi + \{\Omega, \Phi_0\} + A \Pi + \{M, \Pi\} = 0. \tag{48} \]
We note that left-hand side of (47) belong to \( \mathcal{V} \). Changing variables from \( (\xi, \mathcal{P}) \) to \( (\xi', \mathcal{P}') \), we get
\[ \delta \Pi' + \{\Omega', \Phi'_0\}' + A \Pi' + \{M', \Pi'\}' = 0. \tag{49} \]
By repeating the same steps as in the case of eq. (41), we obtain the general solution to (48)

$$\Pi' = (I + \delta^+(A + \text{ad} M'))(-1)(X' - \delta^+\{\Omega', \Phi_0\}),$$

(49)

$$X' \in V, \quad \delta X' = 0, \quad \text{gh}(X') = 0.$$  

The condition

$$\delta^+ \Pi' = 0$$  

(50)

implies $X' = 0$. Therefore the solution to (37) with boundary condition (50) is given by

$$\Phi' = L\Phi'_0,$$

(51)

where

$$L = I - (I + \delta^+(A + \text{ad} M'))(-1)\delta^+\text{ad} \Omega'.$$

The operator $L$ is invertible. The inverse $L^{-1}$ is given by

$$L^{-1}\Phi' = \Phi'|_{\mathcal{P} = 0}.$$  

Eq. (51) establishes a one-to-one correspondence between first class functions and solutions to eqs. (37), (50).

Let us denote by $L(P)$ and $L(J)$ the images of $P$ and $J$, respectively, under the mapping $L$. For $\Phi'_1, \Phi'_2 \in L(P)$

$$\{\Phi'_1, \Phi'_2\}|_{\mathcal{P} = 0} = \{\Phi'_1|_{\mathcal{P}' = 0}, \Phi'_2|_{\mathcal{P}' = 0}\}, \quad (\Phi'_1\Phi'_2)|_{\mathcal{P}' = 0} = (\Phi'_1|_{\mathcal{P}' = 0})(\Phi'_2|_{\mathcal{P}' = 0}),$$

from which it follows that $L(P)$ and $P$ are isomorphic as Poisson algebras, and therefore $L(P)/L(J)$ gives a realization of classical observables.

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