Superstrings on NSNS PP-Waves and Their CFT Duals

Yasuaki Hikida
hikida@hep-th.phys.s.u-tokyo.ac.jp

Department of Physics, Faculty of Science, University of Tokyo
Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan

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Abstract

We investigate the correspondence between superstring theory on pp-wave background with NSNS-flux and superconformal field theory on a symmetric orbifold. This correspondence can be regarded as the “Penrose limit” of \( AdS_3/CFT_2 \) correspondence. Superstring theory on the Penrose limit of \( AdS_3 \times S^3(\times M^4) \) (\( M^4 = T^4 \) or \( K3 \)) with NSNS-flux can be described by a generalization of Nappi-Witten model. We quantize this system in the covariant gauge and obtain the spectrum of superstring theory. In the dual CFT point of view, the Penrose limit means concentrating on the subsector of almost BPS states with large R-charges. We show that stringy states can be embedded in the single-particle Hilbert space of symmetric orbifold theory.
Contents

1 Introduction 2

2 AdS$_5$/CFT$_4$ Correspondence and BMN Conjecture 5
   2.1 AdS$_5$/CFT$_4$ Correspondence 5
   2.2 PP-Waves as Penrose Limits 8
   2.3 BMN Conjecture 10

3 NSNS PP-Wave as a Penrose Limit of AdS$_3 \times S^3$ 15
   3.1 AdS$_3$/CFT$_2$ Correspondence 15
   3.2 Penrose Limit of AdS$_3$/CFT$_2$ Correspondence 18
   3.3 Nappi-Witten Group and Lie Algebra 20
   3.4 Superstring on NSNS PP-Wave in the Light-Cone Gauge 23

4 Superstring Theory on NSNS PP-Wave 30
   4.1 $H_6$ Super WZW Model as a Penrose Limit 30
   4.2 Hilbert Space of $H_6$ Super WZW Model 35
   4.3 Physical Vertices of Superstring on NSNS PP-Wave 37
   4.4 Spectrum of Physical States 41

5 Comparison with SCFT on Sym$^M(T^4)$ 46
   5.1 Review of SCFT on $Sym^M(T^4)$ 46
   5.2 Comparison with Superstring Spectrum 51
   5.3 Comments on Case with RR-Flux 56

6 Superstring Theory on $H_6 \times T^4/Z_2$ and SCFT on $Sym^M(T^4/Z_2)$ 59
   6.1 Spectrum of Superstring on $H_6 \times T^4/Z_2$ 59
   6.2 Comparison with SCFT on $Sym^M(T^4/Z_2)$ 61

7 Conclusion 64

A Gamma Matrices 67

B Super PP-Wave Algebra from Super AdS$_3 \times S^3$ Algebra 68

C Supersymmetries on NSNS PP-Waves Based on Killing Spinors 71
1 Introduction

Superstring theory\(^1\) has been investigated intensively since it is the only known consistent theory describing quantum gravity. In recent years, important aspects of superstring theory, which are closely related to string duality, have been revealed and D-branes play important roles in these developments. While D-branes can be treated as black-brane solutions in supergravity theory, effective field theory on D-branes is given by super Yang-Mills theory. The fact that we can deal with D-branes in two ways implies a duality between supergravity theory (or superstring theory) and supersymmetric gauge theory.

This duality is called as \(AdS/CFT\) correspondence [3] (for a review, see [4]) and the most famous example is \(AdS_5/CFT_4\) correspondence. Let us consider large \(N\) D3-branes. If we take the near horizon limit, corresponding supergravity solution becomes \(AdS_5 \times S^5\). On the other hand, effective field theory on D-branes is given by \(\mathcal{N} = 4\) \(SU(N)\) super Yang-Mills theory. Therefore, supergravity on \(AdS_5 \times S^5\) is supposed to be dual to \(\mathcal{N} = 4\) \(SU(N)\) super Yang-Mills theory. Because the region where supergravity limit is valid corresponds to strong coupling region of super Yang-Mills theory, it is expected to understand some non-perturbative aspects of super Yang-Mills theory by using classical supergravity theory. For this reason, \(AdS_5/CFT_4\) correspondence has been investigated by many authors.

It is natural to expect that there is a correspondence beyond supergravity level. However, it is difficult to quantize superstrings on \(AdS_5 \times S^5\) because of the existence of RR-charges. Recently, an important progress has been made on this aspect. It was found in [5] that there is a maximally supersymmetric background with RR-flux in type IIB supergravity in addition to the flat Minkowski space and the \(AdS_5 \times S^5\) background. This background is called as maximally supersymmetric pp-wave background and it was shown in [6, 7] that this background is obtained by so-called Penrose limit [8, 9] of \(AdS_5 \times S^5\) background. Moreover, it was realized that superstring theory on the pp-wave background with RR-flux can be quantized by using Green-Schwarz formalism in the light-cone gauge [10, 11]. Motivated by these facts, the authors of [12] applied the “Penrose limit” to the \(AdS_5/CFT_4\) correspondence. They have shown that the Penrose limit of superstring theory on \(AdS_5 \times S^5\) corresponds to the subsector of “almost BPS states” with large R-charges in \(\mathcal{N} = 4\) super Yang-Mills theory and identified stringy excitations.

\(^1\)There are good text books on superstring theory [1, 2]. Please refer to them for the background of superstring theory.
with single trace operators.

The other famous example is $AdS_3/CFT_2$ correspondence. This is the correspondence between superstring theory on $AdS_3 \times S^3 \times M^4$ ($M^4 = T^4$ or $K3$) and two dimensional superconformal field theory on a symmetric orbifold $\mathbb{R}^3/\mathbb{Z}_3$. The correspondence is related to D1/D5 system, which is the configuration of $Q_5$ D5-branes wrapped on a small $M^4$ and $Q_1$ D1-branes put parallel to the extra $1 + 1$ dimensions. Since D1/D5 system was used for the microscopic description of black holes [14], the investigation of $AdS_3/CFT_2$ correspondence may give insights to black hole physics. In the IR limit, effective field theory on the two dimensional D-brane configuration is believed to be given by $\mathcal{N} = (4, 4)$ supersymmetric non-linear sigma model on the symmetric orbifold

$$Sym^{Q_1Q_5}(M^4) \equiv (M^4)^{Q_1Q_5}/S_{Q_1Q_5}, \tag{1.1}$$

where $S_N$ represents the symmetric group of $N$ elements. The near horizon geometry of D1/D5 system becomes $AdS_3 \times S^3 \times M^4$, and hence superstring theory on $AdS_3 \times S^3 \times M^4$ has been conjectured to be dual to superconformal field theory on $Sym^{Q_1Q_5}(M^4)$.

There are many configurations related to D1/D5 system by U-duality and in this thesis we concentrate on F1/NS5 system (the system with $Q_1$ fundamental strings and $Q_5$ NS5-branes), which is S-dual to D1/D5 system. The near horizon geometry of F1/NS5 system can be described by superstring theory on $AdS_3 \times S^3 \times M^4$ with NSNS-flux. This superstring theory can be written in terms of $SL(2; \mathbb{R}) \times SU(2)$ WZW models, and hence we can quantize this system. The superstring theory is dual to superconformal field theory on $Sym^{Q_1Q_5}(M^4)$ and the correspondence has attracted much attention since it can be examined beyond supergravity approximation (see, e.g., [15, 16, 17]).

One of the important tests of this duality is the comparison of spectrum. There is a moduli space in the dual CFT and it is difficult to analyze spectrum at general moduli point. Since BPS states do not depend on the deformation of moduli parameter, we can use BPS spectrum at the orbifold point, where we can obtain general spectrum by using the orbifold theory technique [18,19,20]. Therefore, we can obtain BPS spectrum and compare the supergravity modes of superstring theory. BPS states with small R-charge are explicitly identified in [21,22] and higher R-charged BPS states are also reproduced successfully in [23,24] by making use of the spectral flow symmetry on the $AdS_3$ string theory [25,26]. However, it is still difficult to investigate general spectrum other than BPS states.

In order to compare more general spectrum, we apply a Penrose limit to $AdS_3/CFT_2$
correspondence [27]. The Penrose limit of $AdS_3 \times S^3(\times M^4)$ with NSNS-flux becomes 6 dimensional pp-wave background with NSNS-flux and superstring theory on this background can be described by a generalization of Nappi-Witten model [28]. This model was investigated in a different context by several authors [29,30,31,32,33,34,35,36,37,38,39,40] and it can be quantized by the current algebra approach [29,39,41] and by the sigma model approach [42,43,44]. Taking a Penrose limit corresponds to concentrating on almost BPS states with very large R-charges in the dual CFT, where we can use spectrum at the orbifold point in this limit. We show that stringy states can be embedded in the Hilbert space of the symmetric orbifold theory. The other works on the Penrose limit of $AdS_3/CFT_2$ correspondence are given in [45,46,47,48,49,50].

The organization of this thesis is given as follows. In section 2 we first review the $AdS_5/CFT_4$ correspondence. Then, we see what the Penrose limit is and how we apply the limit to the $AdS_5/CFT_4$ correspondence. In section 3 we begin with the $AdS_3/CFT_2$ correspondence and then apply the Penrose limit to it. After explaining that a generalization of Nappi-Witten model appears in the Penrose limit of $AdS_3 \times S^3 \times M^4$, we review the Nappi-Witten group (and Lie algebra) and quantize this model by the sigma model approach in the light-cone gauge. In section 4 we quantize superstring theory on this pp-wave background by using the current algebra approach in the covariant gauge and obtain Hilbert space by explicitly constructing DDF operators. In section 5 we first review superconformal field theory on the symmetric orbifold $Sym^{Q_1,Q_5}(T^4)$ and then compare (almost) BPS spectrum with string spectrum on the pp-wave background. Then we comment on the case with RR-flux in subsection 5.3. We also discuss an extension to the case of $M^4 = T^4/Z_2^2$ in section 6 and we conclude this thesis in section 7. The notation of the Gamma matrices is summarized in appendix A. The space-time super pp-wave algebra is constructed by using the contraction of the super $AdS_3 \times S^3$ algebra in appendix B. In appendix C we also investigate space-time supersymmetry on pp-waves with NSNS-flux by examining Killing spinors. In particular, we show that there are 24 supersymmetries in the $AdS_3 \times S^3 \times T^4$ background.

\footnote{The K3 surface is obtained by resolving the singularity of $T^4/Z_2$ and we use $T^4/Z_2$ as a solvable case of $K3$.}
2 AdS$_5$/CFT$_4$ Correspondence and BMN Conjecture

Before examining AdS$_3$/CFT$_2$ correspondence, we first review AdS$_5$/CFT$_4$ correspondence in next subsection in order to show the general concept of AdS/CFT correspondence. After explaining Penrose limit in subsection 2.2, we apply the Penrose limit to the both sides of AdS$_5$/CFT$_4$ correspondence and we explicitly construct single trace operators in super Yang-Mills theory corresponding to states of superstrings in subsection 2.3.

2.1 AdS$_5$/CFT$_4$ Correspondence

As we mentioned in introduction, AdS/CFT correspondence was deduced from the fact that there are two ways to describe D-branes. One is the supergravity or superstring description and the other is the super Yang-Mills description. Therefore, it is natural to expect that there is a correspondence between a supergravity or superstring theory and a super Yang-Mills theory. The concrete example is given by Maldacena [3] as AdS/CFT correspondence. Although we concentrate on AdS$_3$/CFT$_2$ correspondence in this thesis, we first overview AdS$_5$/CFT$_4$ correspondence in order to see the general concept of AdS/CFT correspondence.

Let us begin with D-branes. A D-brane can be defined as a hypersurface which open strings are attached to as in figure 1 [51]. Open strings have their ends and we have to assign some boundary conditions to the fields at the ends. The conditions consistent with the equations of motion are given by Neumann or Dirichlet boundary conditions. Let us assign the Neumann boundary condition for $p+1$ coordinates and the Dirichlet boundary conditions for $9-p$ coordinates. More precisely, we assign

$$\partial_\sigma X^\mu(\tau, \sigma)|_{\sigma=0,\pi} = 0 \quad (\mu = 0, 1, \cdots, p),$$

$$X^\mu(\tau, \sigma)|_{\sigma=0,\pi} = \text{const.} \quad (\mu = p + 1, \cdots, 9). \quad (2.1)$$

We have assumed the flat background and we use $(\tau, \sigma)$ as the coordinates of world-sheet. The open strings can move only along $p+1$ dimensional hypersurface and the hypersurface is called as Dp-brane. $N$ D-branes can be described by open strings with $U(N) \times U(N)$ Chan-Paton factors at their ends. In the limit that the string length $l_s = \sqrt{\alpha'} \to 0$, only the massless excitations of open strings remain and low energy effective theory on
Figure 1: Dp-branes are p + 1 dimensional hypersurfaces and open strings are stretched between the branes.

N D3-branes becomes U(N) super Yang-Mills theory. In p = 3 case, the effective theory is given by 4 dimensional \( \mathcal{N} = 4 \) super Yang-Mills theory.

We can describe D-branes also in terms of supergravity. The supergravity solution corresponding to multiple Dp-branes can be constructed \[52\] and for N D3-branes it is given by

\[
\begin{align*}
    ds^2 &= f^{-\frac{1}{2}}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2), \\
    f &= 1 + \frac{R^4}{r^4}, \quad R^4 = 4\pi g_s \alpha'^2 N, \quad (2.2)
\end{align*}
\]

supported by nontrivial 5-form RR-field strength. The string coupling is denoted as \( g_s \). In the super Yang-Mills description, there are two decoupling sectors; One is the super Yang-Mills theory on D-branes and the other is the supergravity theory on flat background. In the supergravity description, we have also two decoupling regions; One is the region near D-branes and the other is the region away from D-branes. In order to extract the information at the region near D-branes, we take the near horizon limit of D3-brane solution \( (2.2) \). By taking the limit \( r \ll R \), the geometry becomes \( AdS_5 \times S^5 \)

\[
    ds^2 = \frac{r^2}{R^2}(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2. \quad (2.3)
\]

Since the supergravity theory on the flat background away from the D-branes are the same in the both descriptions, the super Yang-Mills theory on D3-branes are conjectured
to be dual to superstring theory on the $AdS_5 \times S^5$ background [3]. The $AdS_5$ space has a codimension one boundary and the dual CFT is believed to live in the 4 dimensional boundary.

One evidence of the conjecture is the correspondence of symmetry. 4 dimensional $\mathcal{N} = 4$ super Yang-Mills theory is known to be a superconformal field theory. The combination of 3 + 1 dimensional Poincaré symmetry, dilatation symmetry and special conformal symmetry is described by $SO(4, 2)$, and there is $SU(4)$ R-symmetry in $\mathcal{N} = 4$ supersymmetric theory. The isometry of $AdS_5 \times S^5$ space is also given by $SO(4, 2) \times SO(6)(\simeq SU(4))$ and precisely reproduce the symmetry of $\mathcal{N} = 4$ superconformal field theory.

In the large $N U(N)$ Yang-Mills theory, effective coupling is given by ’t Hooft coupling $\lambda = g_{YM}^2 N$. By using the relation $g_{YM}^2 = 4\pi g_s$ and (2.2), the perturbative region is given by

$$g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4} \ll 1 \ .$$

(2.4)

On the other hand, the supergravity approximation is valid when

$$\frac{R^4}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1 \ ,$$

(2.5)

thus we may investigate super Yang-Mills theory at strong coupling region $\lambda \gg 1$ by using the supergravity on $AdS_5 \times S^5$.

In a conformal field theory, basic objects are operators. In the context of $AdS/CFT$ correspondence, it was proposed in [33, 54] that the correlation functions of operators are obtained from the following relations as

$$\langle e^{\int dx^4\phi_0(x)}\mathcal{O}(x) \rangle_{\text{CFT}} = e^{-S_{\text{string}}[\phi|_{\text{boundary}} = \phi_0(x)]} \ .$$

(2.6)

In the CFT side, $\phi_0$ is a source coupling to a operator and in the AdS side $\phi_0$ is the value of a field $\phi$ at the boundary. The left hand side is the generating function of correlation function and the right hand side is the partition function of superstring theory on $AdS_5 \times S^5$. This equation implies that the operator $\mathcal{O}$ in the conformal field theory corresponds to the field $\phi$ in the superstring theory. In fact, it is shown that there is a one-to-one correspondence between Kaluza-Klein modes compactified on $S^5$ in the supergravity and single trace BPS operators in the super Yang-Mills theory. In the CFT side, we obtain a correlation function by differentiating source currents $\phi_0$. In the AdS side, the correlation function corresponds to S-matrix for the fields $\phi$. For example, 4-point
functions of operators $\mathcal{O}$ in the super Yang-Mills theory can be calculated by Feynman diagrams with the 4 external fields $\phi$ as in figure 2.

### 2.2 PP-Waves as Penrose Limits

In [8] Penrose pointed out that pp-wave (plane fronted wave with parallel rays) geometry is realized as a limit of any geometry. Furthermore, this limiting procedure is generalized in [9] to supergravity on ten or eleven dimensional Lorentzian space-time $M$.

Let us denote a null geodesic of $M$ without conjugate points by $\gamma$. In a neighborhood of $\gamma$, the metric of any geometry can be written by introducing local coordinates $U$, $V$, $Y^i$ as

$$ g = dV \left( dU + \alpha(U, V, Y^k) dV + \sum_i \beta_i(U, V, Y^k) dY^i \right) + \sum_{i,j} C_{ij}(U, V, Y^k) dY^i dY^j. \quad (2.7) $$

We use $\alpha$ and $\beta_i$ as real numbers and $C_{ij}$ as a symmetric positive-definite matrix. In this coordinate, the null geodesic is given by the surface with $V$, $Y^i = \text{const.}$ and affine parameter is $U$.

In the Penrose limit, we see the neighborhood of the null geodesic very closely. In other words, it is obtained by rescaling the coordinates as

$$ U = u, \quad V = \Omega^2 v, \quad Y^i = \Omega y^i, \quad (2.8) $$

and taking the limit of $\Omega \to 0$. Under this limiting procedure, the metric become

$$ \bar{g} = \lim_{\Omega \to 0} \Omega^{-2} g = du dv + \sum_{i,j} \bar{C}_{ij}(u) dy^i dy^j. \quad (2.9) $$

Notice that the matrix $\bar{C}_{ij}$ only depends on $u$ after taking the limit. This metric is of the form of pp-waves written by Rosen coordinates.
In supergravity theory, there are a dilaton $\Phi$ and gauge potentials $A_p$. By using the gauge freedom, we choose a gauge such as

$$A_{U i_1 \cdots i_{p-1}} = A_{UV i_1 \cdots i_{p-2}} = 0.$$  

(2.10)

We use the dilaton and the gauge potentials after taking the Penrose limit as

$$\bar{\Phi} = \lim_{\Omega \to 0} \Phi(\Omega), \quad \bar{A}_p = \lim_{\Omega \to 0} \Omega^{-p} A_p(\Omega).$$  

(2.11)

In this limit, the non-trivial components of gauge potentials are only transverse ones and field strengths are of the forms

$$\bar{F}_{p+1} = du \wedge \frac{d}{du} \bar{A}_p(u).$$  

(2.12)

Because supergravity Lagrangians transform homogeneously under the scaling of metric (2.9) and dilaton and gauge potentials (2.11), the Penrose limits of supergravity solutions are still solutions to the supergravity equations of motion.

In the following section, we will use the expression of pp-wave metric different from (2.9), which is written by Brinkman coordinates as

$$\bar{g} = -4 dx^+ dx^- + \sum_{i,j} A_{ij}(x^+) x^i x^j dx^+ dx^- + \sum_i dx^i dx^i.$$  

(2.13)

In eleven dimensional supergravity, there are three types of the maximally supersymmetric solutions, i.e., flat, $AdS_4 \times S^7$, and pp-wave background. The maximally supersymmetric solution of pp-wave type was found in [55] and its metric is written as

$$g = -4 dx^+ dx^- + \left( \sum_{i=1}^3 \left( \frac{\mu}{3} \right)^2 x^i x^i \right) dx^+ dx^- + \sum_{i=1}^9 \left( \frac{\mu}{6} \right)^2 x^i x^i dx^i dx^i.$$  

(2.14)

The normalization factor $\mu$ can be changed by using the redefinition of light-cone coordinates $x^\pm$. Remarkably, this pp-wave background can be obtained as a Penrose limit of $AdS_4 \times S^7$. In type IIB supergravity, flat and $AdS_5 \times S^5$ backgrounds are know to be maximally supersymmetric solutions. Recently, it was found in [5] that there is another maximally supersymmetric solution given by pp-wave geometry

$$g = -4 dx^+ dx^- + \sum_{i=1}^8 \mu^2 x^i x^i dx^+ dx^- + \sum_{i=1}^8 dx^i dx^i.$$  

(2.15)

This metric can be also obtained as a Penrose limit of $AdS_5 \times S^5$. In next subsection, we apply this fact to the $AdS_5/CFT_4$ correspondence.
The above two coordinates (Rosen and Brinkman coordinates) can be exchanged by performing the following transformation

\[ u = -4x^+ , \quad v = x^- + \sum_{i,j} \frac{1}{4} M_{ij}(x^+) x^i x^j , \quad y^i = \sum_j Q^i_j(x^+) x^j , \quad (2.16) \]

where \( Q^i_j \) is a sort of inverse of \( C_{ij} \) and \( M_{ij} \) is chosen as

\[ C_{ij} Q^i_k Q^j_l = \delta_{kl} , \quad M_{ij} = C_{kl} Q^k_l Q^i_j . \quad (2.17) \]

We use \( \hat{} \) as the derivative \( d/dx^+ \). The relation between \( C_{ij} \) and \( A_{ij} \) can be calculated as

\[ A_{ij} = -C'_{kl} Q^l_j Q^i_k - C_{kl} Q^{kl}_j Q^i_k . \quad (2.18) \]

Using the coordinate transformation \( (2.16) \), we obtain the field strength \( (2.12) \) in terms of Brinkman coordinates as

\[ \bar{F}_{p+1} = \sum_{i_k,j_k} \frac{d}{dx^+} A_{i_1\cdots i_p} (-4x^+) Q_{j_1}^{i_1} \cdots Q_{j_p}^{i_p} dx^+ \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_p} . \quad (2.19) \]

### 2.3 BMN Conjecture

As mentioned in the previous subsection, the maximally supersymmetric pp-wave arises as a Penrose limit of \( AdS_5 \times S^5 \). To see this, it is convenient to rewrite the metric of \( AdS_5 \times S^5 \) \( (2.3) \) in terms of global coordinates as

\[ ds^2 = R^2 \left[ -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + \cos^2 \theta d\psi^2 + d\theta^2 + \sin^2 \theta d\Omega_3^2 \right] . \quad (2.20) \]

We choose a null geodesic as the trajectory of a particle moving very fast along the \( S^5 \), more precisely, moving along the \( \psi \) direction and staying at \( \theta = 0 \) and \( \rho = 0 \). (See figure \( \text{3} \)) The Penrose limit corresponds to concentrating on this particle. In terms of the coordinates, we rescale as

\[ x^+ = \frac{1}{2} (t + \psi) , \quad x^- = R^2 \frac{1}{2} (t - \psi) , \quad r = R \rho , \quad y = R \theta , \quad (2.21) \]

and take the limit of \( R = 1/\Omega \to \infty \). The resultant metric is of the form of the maximally supersymmetric pp-wave \( (2.15) \)

\[ ds^2 = -4dx^2 dx^- - \mu^2 \bar{z}^2 (dx^+)^2 + d\bar{z}^2 , \]

\[ F_{+1234} = F_{+5678} = \text{const.} \times \mu , \quad (2.22) \]
Figure 3: We focus on a particle moving along an equator of $S^5$ very fast and staying at the center of $AdS_5$. This corresponds to taking a Penrose limit of $AdS_5 \times S^5$.

with $\mu = 1$. The vector $\vec{z}$ represents the points on $R^8$ and the constant in front of $\mu$ is fixed by choosing a normalization of field strengths.

Superstrings on the pp-wave (2.22) can be quantized by using Green-Schwarz formalism in the light-cone gauge $[10]$. In the gauge $x^+ = \tau$, the action becomes

$$S = \frac{1}{2\pi\alpha'} \int d\tau \int_0^{2\pi\alpha'p^+} d\sigma \left[ \frac{1}{2} (\partial_\tau \vec{z})^2 - \frac{1}{2} (\partial_\sigma \vec{z})^2 - \frac{1}{2} \mu^2 \vec{z}^2 + i\bar{S} (\theta + \mu \Gamma^{1234} S) \right],$$

with the world-sheet coordinates $(\tau, \sigma)$. Green-Schwarz fermion $S$ consists of eight components, and they are Majorana spinors on the world-sheet and space-time spinors with positive chirality of $SO(8)$.

The boundary condition and equation of motion for bosonic field $z^I$ are

$$z^I(\sigma + 2\pi\alpha'p^+, \tau) = z^I(\sigma, \tau),$$

$$(\partial^2_\tau - \partial^2_\sigma) z^I + \mu^2 z^I = 0.$$  

The solution to these equations is given by the following mode expansion as

$$z^I = z^I_0 \cos \mu \tau + p^I_0 \cos \mu \tau + i \sum_{n \neq 0} \frac{1}{\omega_n} \left( \alpha_n e^{-i(\omega_n \tau - k_n \sigma)} + \bar{\alpha}_n e^{-i(\omega_n \tau + k_n \sigma)} \right).$$

11
where
\[ \omega_n = \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} \quad (n > 0) , \quad \omega_n = -\sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} \quad (n < 0) , \]
\[ k_n = \frac{n}{\alpha' p^+} \quad (n = \pm 1, \pm 2, \cdots) . \quad (2.26) \]

By replacing the coefficients \( \alpha^l_n \) and \( \tilde{\alpha}^l_n \) with the corresponding operators, we can quantize the system and obtain the light-cone Hamiltonian as
\[ 2p^- = -p_+ = H_{l.c.} = \sum_n N_n \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} . \quad (2.27) \]

The number operator \( N_n \) counts one for \( \alpha_n \) and for \( \tilde{\alpha}_{-n} \). By virtue of large supersymmetry, the spectrum of fermions are the same as that of bosons, and the number operator \( N_n \) also counts the fermionic oscillators in the way similar to the bosonic ones. The level matching condition is expressed in this notation as
\[ \sum_n nN_n = 0 . \quad (2.28) \]

Since the pp-wave geometry can be obtained as a Penrose limit of \( AdS_5 \times S^5 \), there is a subsector of super Yang-Mills theory dual to superstrings on the pp-wave. As we mentioned before, the symmetry on conformal field theory corresponds to the isometry of \( AdS_5 \) and the R-symmetry corresponds to the isometry of \( S^5 \). In the superconformal field theory, one of the important quantities is the conformal weight \( \Delta \), which represents how an operator behaves under the dilatation transformation, and another important quantity is the \( U(1) \) R-charge \( Q \) of \( SU(4) \) R-symmetry. The conformal weight \( \Delta \) and the \( U(1) \) R-charge \( Q \) can be identified as the energy \( E = i\partial_t \) and the angular momentum along \( S^5 \) \( J = -i\partial_\psi \) in the coordinates of \( (2.20) \).

Using this correspondence and the coordinate transformation \( (2.21) \), the light-cone momenta of superstrings on the pp-wave are written as
\[ 2p^- = -p_+ = i(\partial_t + \partial_\psi) = \Delta - Q , \]
\[ 2p^+ = -p_- = \frac{1}{R^2} i(\partial_t - \partial_\psi) = \frac{\Delta + Q}{R^2} . \quad (2.29) \]
Because the light-cone momenta should be finite in the limit of \( R \to \infty \), we have to consider the states with
\[ \Delta - Q \sim O(1) , \quad \Delta + Q \sim O(R^2) . \quad (2.30) \]
The BPS bound is given as $\Delta \geq |Q|$, thus these states are almost BPS with very large R-charge $Q$.

Using the relation (2.29), we can see which operators in the super Yang-Mills theory should be identified as states in the superstring theory. Since we have found that the spectrum of superstrings on the pp-wave is given by (2.27), the corresponding operators should have the following quantity

$$ (\Delta - Q)_n = \sqrt{1 + \frac{4\pi g_s N n^2}{Q^2}}. $$

We have used the relation (2.2). In the remaining part of this section, we construct these operators explicitly.

$\mathcal{N} = 4$ $U(N)$ super Yang-Mills theory consists of four components of gauge field $A_i$ ($i = 1, 2, 3, 4$), six scalar fields $\phi^i$ ($i = 1, 2, 3, 4$), $Z = \phi^5 + i\phi^6$, $\bar{Z} = \phi^5 - i\phi^6$ and sixteen components of gaugino $\chi^a_J = 1/2$, $\chi^a_J = -1/2$ ($a = 1, \cdots, 8$). We use the adjoint representation of $U(N)$ for these fields and summarize conformal weights $\Delta$ and R-charges $Q$ of these fields in Table 1.

We are considering large $N$ limit and small fixed $g_{YM}$. The operators we want to construct have large R-charge $Q \sim \sqrt{N}$ but finite $\Delta - Q$. The operator with $\Delta - Q = 0$ can be constructed by using many $Z$’s because this is only the field with $\Delta - Q = 0$. This operator is identified with the vacuum state in the superstring theory, thus we have

$$ \text{Tr}[Z^Q] \longleftrightarrow |0, p_+\rangle. \quad (2.32) $$

We neglect the normalization for simplicity.

| Field | $\Delta$ | $Q$ |
|-------|---------|-----|
| $A_i$ ($i = 1, 2, 3, 4$) | 1 | 0 |
| $\phi^i$ ($i = 1, 2, 3, 4$) | 1 | 0 |
| $Z = \phi^5 + i\phi^6$ | 1 | 1 |
| $\bar{Z} = \phi^5 - i\phi^6$ | 1 | -1 |
| $\chi^a_{J=1/2}$ ($a = 1, \cdots, 8$) | 3/2 | 1/2 |
| $\chi^a_{J=-1/2}$ ($a = 1, \cdots, 8$) | 3/2 | -1/2 |

Table 1: The conformal weights $\Delta$ and the R-charges $Q$ of fields in $\mathcal{N} = 4$ super Yang-Mills theory.
Next, we construct the operators with $\Delta - Q = 1$. We can find the fields with $\Delta - Q = 1$ from table [I] as

$$\phi^i (i = 1, 2, 3, 4), \quad D_i Z = \partial_i Z + [A_i, Z], \quad \chi^a_j (a = 1, \cdots, 8). \quad (2.33)$$

They correspond to eight bosonic oscillators and eight fermionic oscillators

$$a^I_0 (I = 1, \cdots, 8), \quad S^a_0 (a = 1, \cdots, 8). \quad (2.34)$$

We obtain the supergravity modes by acting these zero modes to the vacuum. The corresponding operators can be constructed by considering the following single trace operators

$$\frac{1}{Q} \sum_l \text{Tr}[Z^l \phi^i Z^{Q-l}] = \text{Tr}[\phi^i Z^Q] \longleftrightarrow a^{iti+4}_0 |0, p_+\rangle. \quad (2.35)$$

We have used the symmetric trace in order to make the operators gauge invariant. The operators corresponding to other supergravity modes are of the form similar to (2.35).

Let us turn to the stringy modes. The stringy modes can be created by acting oscillators $a^I_0$ and $S^a_0$ to the vacuum state. It was proposed in [12] that corresponding operators are given by single trace operators with appropriate phase factors. For example, they proposed the correspondence

$$\sum_l \text{Tr}[\phi^3 Z^l \phi^4 Z^{Q-l}] e^{2\pi i l/Q} \longleftrightarrow a^{18}_n a^{17}_{-n} |0, p_+\rangle. \quad (2.36)$$

Notice that we need at least two oscillators in order to satisfy the level matching condition (2.28). In general, the action of oscillators is replaced by the insertion of operators by using following rules

$$a^i (i = 1, 2, 3, 4) \longleftrightarrow D_i Z,$$

$$a^i (i = 5, 6, 7, 8) \longleftrightarrow \phi^{i-4},$$

$$S^a (a = 1, \cdots, 8) \longleftrightarrow \chi^a_j = 1/2, \quad (2.37)$$

with appropriate phase factors. It was shown that these operators correctly reproduce the spectrum (2.31) at the first non-trivial order in [12], at the next non-trivial order in [56] and at the all order in [57].
3 NSNS PP-Wave as a Penrose Limit of $AdS_3 \times S^3$

In this thesis, we concentrate on $AdS_3/CFT_2$ correspondence rather than $AdS_5/CFT_4$ correspondence. The $AdS_3/CFT_2$ correspondence is related to $D1/D5$ system. The near horizon geometry of the system is $AdS_3 \times S^3(\times M^4)$ with $M^4 = T^4$ or $K3$ and the effective field theory is given by a two dimensional superconformal field theory. Therefore, it is conjectured that superstrings on $AdS_3 \times S^3$ is dual to the two dimensional superconformal field theory. One reason why we analyze this case is that two dimensional superconformal field theory is easier to deal with than other dimensional one since symmetry on two dimensional CFT is enhanced to be infinite dimensional.

There is U-duality family of $D1/D5$ system and we use S-dual one, namely, $F1/NS5$ system, which is the system with fundamental strings and NS5-branes. Superstring theory on the near horizon geometry includes only NSNS-flux and hence the superstrings can be analyzed. This is another reason why we investigate on this case. For these reasons, many authors have worked on this subject as in, for example, [15, 16, 17].

In subsequent sections, we will investigate the “Penrose limit” of this correspondence. As we will see below, we can improve the study of correspondence of spectrum by applying the Penrose limit. In subsection 3.2 we see what the “Penrose limit” of $AdS_3/CFT_2$ correspondence means. Then, we show that superstrings on the Penrose limit is described by a generalization of Nappi-Witten model [28]. The Nappi-Witten model is the WZW model associated with the Nappi-Witten group, which is introduced in subsection 3.3. In subsection 3.4 we obtain the spectrum of superstrings on the Penrose limit by using the sigma model approach in the light-cone gauge [42, 43, 44].

3.1 $AdS_3/CFT_2$ Correspondence

As mentioned in subsection 2.1 the $AdS_5/CFT_4$ correspondence is related to the D3-branes. For the $AdS_3/CFT_2$ correspondence, the role of D3-branes are played by D1/D5 system. The D1/D5 system is a configuration with D5-branes and D1-branes\(^3\). $Q_5$ D5-branes are wrapped on $T^4$ whose volume is fixed small as $vl_s^4$. $Q_1$ D1-strings are set parallel to the extra 1 + 1 dimensions. This configuration can be seen as a one dimensional object in 5 + 1 dimensional space-time. We can replace $T^4$ with $K3$ and the case of $K3$ will be investigated in section 6.

\(^3\)See figure 4
Figure 4: The configuration of D1/D5 system. $Q_5$ D5-branes are wrapped on a small $T^4$ (6,7,8,9 directions) and $Q_1$ D1-branes are put on D5-branes along 0,1 directions.

The supergravity solution corresponding to the D1/D5 system is given by

$$ds^2 = f_1^{-\frac{1}{2}} f_5^{-\frac{1}{2}} (-dx_0^2 + dx_1^2) + f_1^{\frac{1}{2}} f_5^{\frac{1}{2}} (dr^2 + r^2 d\Omega_3^2) + f_1^{-\frac{1}{2}} f_5^{-\frac{1}{2}} dx^A dx^A,$$

where

$$f_1 = \left(1 + \frac{g\alpha' Q_1}{v r^2}\right), \quad f_5 = \left(1 + \frac{g\alpha' Q_5}{r^2}\right). \quad (3.1)$$

There is also non-zero 3-form RR-field strength. We have set $g$ as the string coupling constant at infinity. The near horizon geometry of the solution becomes $AdS_3 \times S^3 \times T^4$

$$ds^2 = \frac{r^2}{R^2} (-dx_0^2 + dx_1^2) + \frac{R^2}{r^2} dr^2 + r^2 d\Omega_3^2 + \sqrt{\frac{Q_1}{v Q_5}} dx^A dx^A, \quad (3.2)$$

where

$$g_6 = \frac{g}{\sqrt{v}}, \quad R^2 = \alpha' g \sqrt{Q_1 Q_5}. \quad (3.3)$$

We should notice that the volume of $T^4$ is fixed as $Q_1/Q_5$ in the near horizon limit.

According to the conjecture of Maldacena [3], superstring theory on $AdS_3 \times S^3 \times T^4$ is dual to the effective theory on this D-brane configurations at the IR limit. The effective theory is known as $\mathcal{N} = (4, 4)$ superconformal field theory with central charge $c = 6Q_1 Q_5$ and it has two decoupled sectors called as Coulomb and Higgs branches.

In the D1/D5 system, there are open strings stretched between the same type of D-branes and the different type of D-branes. The former ones are called as (1, 1) and (5, 5).
strings. In the open string spectrum, there are massless scaler fields corresponding to the positions of D-branes. When these scalar fields have non-trivial values, the theory is at the Coulomb branch. The latter ones are called as $(1, 5)$ and $(5, 1)$ strings. The massless scalar fields in the open string spectrum represent how much the D1-branes are smeared into the D5-branes. We are interested in the case when these scalar fields have non-trivial values. In other words, only the Higgs branch is considered.

Among the excitations of a $(1, 5)$ string, there are two massless bosons and two massless fermions. Taking account of two possible orientations and the $Q_1$ and $Q_5$ degrees of freedom from $U(Q_1) \times U(Q_5)$ Chan-Paton factors, there are $4Q_1Q_5$ bosons and $4Q_1Q_5$ bosons. In fact, it is conjectured in [14] that the effective theory on the D1/D5 system is given by $N = (4, 4)$ non-linear sigma model on the symmetric orbifold

$$Sym^{Q_1Q_5}(T^4) = (T^4)^{Q_1Q_5}/S_{Q_1Q_5},$$

(3.4)

where $S_N$ represents the symmetric group of $N$ elements. The degrees of freedom of massless fields are reproduced by the analysis of massless excitations of open strings and the symmetry of $S_N$ is a reminiscent of $U(Q_1) \times U(Q_5)$ Chan-Paton factors.

The form of target space (3.4) is explained as follows. By applying T-dualities along $x^0$ and $x^1$ directions, we obtain D($-1$)/D3 system, and $Q_1$ D($-1$)-brane can be interpreted as $Q_1$ instanton solutions in $SU(Q_5)$ Yang-Mills theory. It is supposed that a D($-1$)-brane separates into $Q_5$ fractional D($-1$) branes with $1/Q_5$ D($-1$)-brane charge. The position of a fractional D-instanton corresponds to one $T^4$ and the fact that these fractional D-instantons are indistinguishable leads to the $Q_1Q_5$ symmetric products of $T^4$. This is a heuristic explanation of the reason why the moduli space of this instanton solution is given by (3.4).

Let us perform S-duality to the D1/D5 system. The S-duality transforms D1-strings to fundamental strings and D5-branes to NS5-branes. Thus, we obtain F1/NS5 system with $Q_5$ NS5-branes wrapped on the $T^4$ and $Q_1$ fundamental string set parallel to the extra $1+1$ directions. By using the S-duality, the metric of supergravity solution is obtained as

$$ds^2 = f_1^{-1}(-dx_0^2 + dx_1^2) + f_5(dr^2 + r^2d\Omega_3^2) + dx^A dx^A,$$

$$f_1 = \left(1 + \frac{g'^2\alpha'Q_1}{vr^2}\right), \quad f_5 = \left(1 + \frac{\alpha'Q_5}{r^2}\right).$$

(3.5)

There is also non-zero 3-form NSNS-field strength.
Its near horizon limit is also $AdS_3 \times S^3 \times T^4$

$$ds^2 = Q_5 \frac{r^2}{\alpha'} (-dx_0^2 + dx_1^2) + \frac{Q_5 \alpha'}{r^2} dr^2 + Q_5 \alpha' d\Omega_3^2 + dx^A dx^A, \quad (3.6)$$

where we use the fact that the string coupling in this limit is $g^2 = v Q_5 / Q_1$. Superstring theory on curved space supported by NSNS-flux may be described by a super WZW model, which is a solvable model, and we can use a super WZW model as superstring theory on $AdS_3 \times S^3$ with NSNS-flux. The isometry of $AdS_3 \times S^3$ is given by

$$SO(2, 2) \times SO(4) \cong SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R \times SU(2)_L \times SU(2)_R, \quad (3.7)$$

therefore we use $SL(2, \mathbb{R}) \times SU(2)$ WZW model. The level of this model is related with the radius $R^2$ and given by $Q_5$. We will concentrate on this case because this model is easier to deal with than superstrings with RR-flux.

We have to apply the S-duality also to the effective field theory, however we do not know much about the moduli space of non-linear sigma model on $Sym^{Q_1 Q_5} (T^4)^4$. Because of this fact, we can only use objects independent of the moduli parameter. For example, BPS states are independent of the moduli parameters and they are identified as supergravity modes in superstrings on $AdS_3 \times S^3 \times T^4$ \cite{21, 22, 23, 24}.

As investigated in the previous section, we can compare non BPS states with stringy modes by applying the Penrose limit to the $AdS_5/CFT_4$ correspondence. Thus it is natural to expect that we can obtain the similar results by applying the Penrose limit to the $AdS_3/CFT_2$ correspondence. Since we concentrate on almost BPS states with very large R-charge, these states are not sensitive to the deformation of moduli parameter and can be compared with the stringy modes\cite{5}.

### 3.2 Penrose Limit of $AdS_3/CFT_2$ Correspondence

In this subsection, we will see the “Penrose limit” of $AdS_3/CFT_2$ correspondence. It is convenient to express the metric of $AdS_3 \times S^3 \times T^4$ as ($R^2 = Q_5 l_s^2$)

$$ds^2 = R^2 \left[ - \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\varphi_1^2 + \cos^2 \theta d\psi^2 + d\theta^2 + \sin^2 \theta d\varphi_2^2 \right]$$

$$+ dx^A dx^A, \quad (3.8)$$

\footnote{The discussions on the moduli space of D1/D5 system are given, for example, in \cite{52, 60}.

\footnote{However, there is a subtlety on this discussion and we will comment on the subtlety in subsection \cite{60}.}
where $x^A$ ($A = 6, 7, 8, 9$) are the coordinates of $T^4$. As discussed in the previous section, the Penrose limit corresponds to concentrating on the particles moving at the $\psi$ direction (one of the $S^3$ coordinates) very fast and staying at $\rho = \theta = 0$. More precisely, we rewrite as

$$\rho = \frac{r_1}{R}, \quad \theta = \frac{r_2}{R}, \quad t = -fu + \frac{1}{4fR^2}v, \quad \psi = fu + \frac{1}{4fR^2}v,$$

and take the limit $R \to \infty$. Then we obtain the plane wave metric as

$$ds^2 = dudv - f^2x^i x^i du^2 + dx^i dx^i + dx^A dx^A,$$

which is supported by the NSNS-flux $H_{u12} = H_{u34} = -2f$. Here we have introduced the coordinates $x^i$ ($i = 1, 2, 3, 4$) as

$$dx^i dx^i = dr_1^2 + r_1^2 d\varphi_1^2 + dr_2^2 + r_2^2 d\varphi_2^2.$$

It is known that the superstring theory on this background supported by the NSNS-flux is described by a (generalized) Nappi-Witten model [28]. Contrary to the case with RR-flux, we can quantize this system in the covariant gauge since we can use the technique of WZW models. Some years ago, the Nappi-Witten model was investigated as a solvable model for string theory on a non-trivially curved background [29,30,31,32,33,34,35,36,37,38,39,40]. Recently, this model attracts some interests again in the context of PP-Wave/CFT duality [41,42,43,44]. The sigma model approach was taken in [42,43,44] and the current algebra method was developed in [29,39,41] by using the free field realization.

Then, consider what this limiting procedure means in the context of the dual CFT. Using the redefinition of the light-cone coordinates (3.9), we find

$$p_u = -i\partial_u = f(i\partial_t - i\partial_\psi) = f(-\Delta + Q),$$

$$p_v = -i\partial_v = \frac{-i\partial_t - i\partial_\psi}{4fR^2} = \frac{\Delta + Q}{4fR^2},$$

where we denote $\Delta$ as the conformal weight and $Q$ as the R-charge. The momenta of states in the superstring theory are finite, thus the corresponding states in the dual CFT are the states with $\Delta - Q \sim O(1)$ and $\Delta + Q \sim O(R^2)$. This is the exactly same as the case in subsection 2.3.

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6We can use arbitrary $f$ in front of $x^i x^i du^2$ by rescaling $u$ and $v$. 19
3.3 Nappi-Witten Group and Lie Algebra

Nappi-Witten model \[28\] is a WZW model associated with the central extension of the two dimensional Euclidean group, which is called as Nappi-Witten group\(^7\). Although we will use a generalization of Nappi-Witten model, we first focus on the original Nappi-Witten group for simplicity.

We denote an element of the two dimensional Euclidean group as \(h(\theta, w)\), which acts to two dimensional coordinate \(z\) as
\[
h(\theta, w) \cdot z = e^{-i\theta}z + w.
\]
In other words, the parameters \(\theta\) and \(w\) represent rotation and translation, respectively, thus we can use \(h(\theta, w) = T(w)R(\theta)\) with
\[
T(w) \cdot z = z + w, \quad R(\theta) \cdot z = e^{-i\theta}z.
\]
In order to define a group we have to determine a multiplication law and in this case we can read as
\[
h(\theta_1, w_1)h(\theta_2, w_2) = h(\theta_1 + \theta_2, w_1 + e^{-i\theta_1}w_2),
\]
or equivalently,
\[
R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2), \quad R(\theta)T(w) = T(e^{-i\theta}w)R(\theta).
\]

The Nappi-Witten group is given by the central extension of the two dimensional Euclidean group and representation is characterized by a group cocycle as
\[
T(w_1)T(w_2) = T(w_1 + w_2)\exp\left(\frac{1}{2} \text{Im}(w_1\bar{w}_2)\right).
\]
Introducing \(Z(\kappa)\), we can rewrite as
\[
T(w_1)T(w_2) = T(w_1 + w_2)Z\left(\frac{1}{2} \text{Im}(w_1\bar{w}_2)\right).
\]
Then, an element of Nappi-Witten group is defined as
\[
g(\theta, w, \kappa) = T(w)R(\theta)Z(\kappa),
\]
and the multiplication law is given by
\[
g(\theta_1, w_1, \kappa_1)g(\theta_2, w_2, \kappa_2) = g\left(\theta_1 + \theta_2, w_1 + e^{-i\theta_1}w_2, \kappa_1 + \kappa_2 + \frac{1}{2} \text{Im}(w_1e^{i\theta_1}\bar{w}_2)\right).
\]
\(^7\)A nice review is given in \[61\] and we follow their discussion.
We can easily see that \( g(0,0,0) \) is the identity and \( g(\theta, w, \kappa)^{-1} = g(-\theta, -e^{i\theta}w, -\kappa) \).

The generators of Nappi-Witten Lie algebra can be defined from the Nappi-Witten group as

\[
T(w) = \exp \left( \frac{i}{\sqrt{2}} (wP^* + \bar{w}P) \right), \quad R(\theta) = \exp(i\theta J), \quad Z(\kappa) = \exp(i\kappa F),
\]

and their Lie brackets can be read from the multiplication laws as

\[
[J, P] = P, \quad [J, P^*] = -P^*, \quad [P, P^*] = F,
\]

which is nothing but a Heisenberg algebra. Using the Lorentzian metric \( \langle P, P^* \rangle = 1 \) and \( \langle J, K \rangle = 1 \), we obtain the metric \( ds^2(g) = \langle g^{-1}dg, g^{-1}dg \rangle \) as

\[
ds^2 = 2d\theta d\kappa + \frac{i}{2}(wd\bar{w} - \bar{w}dw)d\theta + dwd\bar{w}.
\]

Considering the WZW model associated with the Nappi-Witten group, NSNS-flux is included from Wess-Zumino term as \( H_{\theta w\bar{w}} = -i/2 \). The generalization to higher dimensional one is obtained by replacing \( w \to w_i, P \to P_i \) and \( P^* \to P^*_i \) \((i = 1, \cdots, I)\), which correspond to the generators of Heisenberg algebra \( H_{2I+2} \). In this thesis, we concentrate on the case of \( I = 2 \) since this case is obtained by the Penrose limit of superstring theory on \( AdS_3 \times S^3 \).

By redefining the coordinates as

\[
w_1 \to e^{-i\theta/2}y_1, \quad w_2 \to e^{-i\theta/2}y_2,
\]

we obtain the plane wave metric of the type \( (3.10) \)

\[
ds^2 = 2d\theta d\kappa - \frac{1}{4}y^i\bar{y}^j d\theta^2 + dy^i d\bar{y}^i.
\]

The relation to the metric \( (3.10) \) is given by \( v \leftrightarrow 2\kappa, u \leftrightarrow \theta \) and \( f = 1/2 \). The Killing vectors on this metric can be read from the group multiplication law as \( (ii) \)

\[
F = -i\partial_\kappa, \quad J_L = -i\partial_\theta + \frac{1}{2}(y^j \partial_{y^j} - \bar{y}^j \partial_{\bar{y}^j}), \quad J_R = -i\partial_\theta - \frac{1}{2}(y^j \partial_{y^j} - \bar{y}^j \partial_{\bar{y}^j}),
\]

\[
P_{iL} = e^{i\theta/2}(\sqrt{2}i\partial_{y^i} + \frac{1}{2\sqrt{2}}y^j \partial_{\bar{y}^j}), \quad P_{iR} = e^{i\theta/2}(\sqrt{2}i\partial_{\bar{y}^i} + \frac{1}{2\sqrt{2}}y^j \partial_{y^j}),
\]

\[
P^*_{iL} = e^{-i\theta/2}(\sqrt{2}i\partial_{y^i} - \frac{1}{2\sqrt{2}}\bar{y}^j \partial_{\bar{y}^j}), \quad P^*_{iR} = e^{-i\theta/2}(\sqrt{2}i\partial_{\bar{y}^i} - \frac{1}{2\sqrt{2}}y^j \partial_{y^j}).
\]

\( ^8 \)Since the generator \( F \) is central, \( F_L \) and \( F_R \) are not independent of each other (\( F = F_L = F_R \)).
We can see that the commutation relation of these generators correctly reproduce (3.22).

Let us see classical scalar wave functions on these background, which satisfy

\[(\Delta - m^2)\phi = 0 \quad \Delta = 2\partial_\theta \partial_\kappa + \frac{1}{4}y^i\bar{y}^i\partial_\kappa \partial_\kappa + 4\partial_\nu \partial_\nu . \quad (3.27)\]

The ground state is given by

\[\phi(\theta, \kappa, y^i) = \exp \left( ip - \kappa + ip_+ + \theta - \frac{1}{4}|p - |y^i\bar{y}^i| \right) , \quad (3.28)\]

and the arbitrary states are obtained by the action of \(n_{iL}\) times of \(P_{iL}\) and \(n_{iR}\) times of \(P_{iR}\) for \(p_+ > 0^9\). For these states, the condition (3.27) becomes

\[2p_-p_+ + |p_-| \sum_{i=1}^2 (n_{iL} + n_{iR} + 1) + m^2 = 0 \quad , \quad (3.29)\]

and the helicity in the transverse space becomes

\[h = J_L - J_R = \text{sgn}(p_-) \sum_{i=1}^2 (n_{iL} - n_{iR}) \quad . \quad (3.30)\]

For \(p_- = 0\) states, the wave function could have the transverse momenta but there is no solution for general \(m^2\).

The above results can be reproduced by the systematic analysis of unitary irreducible representation of \(H_6\) Lie algebra [29, 39, 41]. There are two Casimir operators in the \(H_6\) Lie algebra, which are given by the generator \(F\) and

\[C = P_iP_i^* + P_i^*P_i + 2JF \quad . \quad (3.31)\]

We define the eigenstates of these operators as

\[J|j, \eta\rangle = j|j, \eta\rangle \quad , \quad F|j, \eta\rangle = \eta|j, \eta\rangle \quad , \quad C|j, \eta\rangle = c|j, \eta\rangle \quad . \quad (3.32)\]

The eigenvalues of \(F\) and \(C\) are not changed in the same irreducible representation. By using the Hermitian property \((P_i)^\dagger = P_i^*\), we find

\[\langle j, \eta|P_iP_i^*|j, \eta\rangle = \frac{c}{2} - \eta(j - 1) \geq 0 \quad , \quad (3.33)\]

Thus, the following representations are obtained.

\[^9\text{For } p_- < 0 \text{ we should replace } P_{iL} \text{ and } P_{iR} \text{ with } P_{iL}^* \text{ and } P_{iR}^*, \text{ respectively.}\]
Type I There are no highest weight state and lowest weight state. The labels take the values of $\eta = 0$, $j \in \mathbb{R}$ and $c > 0$. The eigenvalue of $J$ becomes $\ldots, j-1, j, j+1, \ldots$, therefore the fractional part of $j$ labels the irreducible representation.

Type II There are the highest weight states $P_i|j, \eta\rangle = 0$ with $c = 2\eta(j + 1)$. This representation is characterized by $\eta > 0$ and $j \in \mathbb{R}$ and the eigenvalue of $J$ is $j, j - 1, \ldots$.

Type III There are the lowest weight states $P^*_i|j, \eta\rangle = 0$ with $c = 2\eta(j - 1)$. This representation is characterized by $\eta < 0$ and $j \in \mathbb{R}$ and the eigenvalue of $J$ is $j, j + 1, \ldots$.

Type IV There is the state which is both the lowest weight and the highest weight. The labels take the values of $\eta = 0$, $j \in \mathbb{R}$ and $c = 0$. This is the one dimensional representation and we will not consider this representation.

Now, we can compare with the analysis of classical scalar wave function. The eigenvalues can be identified as $\eta_R = \eta_L = p_-$, $j_L = p_+ + h/2$ and $j_R = p_- - h/2$. Thus, the Type I representation corresponds to the case of $p_- = 0$. The Type II representation corresponds to the case of $p_- > 0$ and the ground state can be identified as the lowest weight state. Then, we can show that the Casimir operator $C$ and the operator $h = j_L - j_R$ correctly reproduce the mass square $\langle 3.29 \rangle$ and the helicity $\langle 3.30 \rangle$, respectively. For the Type III representation, we can make the similar analysis with $p_- < 0$.

3.4 Superstring on NSNS PP-Wave in the Light-Cone Gauge

In this subsection, we quantize the superstring on the pp-wave with NSNS-flux by using the sigma model approach \[42, 43, 44\]. The metric of the type (3.23) is considered first and the metric of the type (3.10) is analyzed by using the coordinate transformation. In the light-cone gauge, we can explicitly show that physical spectrum does not include negative norm states. In the next section, we compare it with the spectrum from the BRST quantization.

Let us first consider the following sigma model action. The bosonic part of the action is given by

$$S_B = \frac{1}{2\pi\alpha'} \int d^2\sigma \left[ \partial_+ u \partial_- v + \partial_+ z^i \partial_- \bar{z}^i + i f \partial_+ u (z^i \partial_- \bar{z}^j - \bar{z}^i \partial_- z^j) + \partial_+ x^A \partial_- x^A \right], \quad (3.34)$$
where \( i = 1, 2 \) and \( z^i \) is the complex conjugate of \( z^i \). The coordinates of \( T^4 \) sector are given by \( x^A (A = 6, 7, 8, 9) \) and we neglect these coordinates and the corresponding fermionic parts for a while. We use the notation of world-sheet coordinates as \( \sigma_{\pm} = \tau \pm \sigma \) and \( \partial_\pm = \frac{1}{2} (\partial_0 \pm \partial_1) \). The fermionic part of the action is given by

\[
S_F = \frac{1}{2 \pi \alpha'} \int d^2 \sigma [i \psi^R_L \partial_- \psi^i_L + i \bar{\psi}^i_L \partial_+ \psi^R_L - f (\partial_- \bar{z}^i \psi^R_L - \partial_+ z^i \bar{\psi}^i_L) + i \psi^R_R \partial_+ \psi^i_R + i \bar{\psi}^i_R \partial_+ \psi^R_R - 2f \partial_+ u \bar{\psi}^i_R \psi^R_R],
\]

where the hatted indices are the ones for tangent space. The equations of motion can be calculated as

\[
\partial_- \partial_+ u = 0, \quad \partial_+ [\partial_- v + if(z^i \partial_- z^i - \bar{z}^i \partial_+ z^i) - 2f \bar{\psi}^i_R \psi^R_R],
\]

\[
\partial_+ \partial_- z^i + 2if \partial_+ u \partial_- z^i - f \partial_- (\psi^R_L \psi^i_L) , \quad \partial_+ \partial_- \bar{z}^i - 2if \partial_+ u \partial_- \bar{z}^i + f \partial_- (\bar{\psi}^i_L \psi^R_L),
\]

for the bosonic part and

\[
\partial_- \psi^R_L = 0, \quad \partial_- \psi^i_L - f (\partial_- \bar{z}^i \psi^R_L - \partial_+ z^i \bar{\psi}^i_L) = 0,
\]

\[
\partial_- \psi^i_L - f \partial_- \bar{z}^i \psi^R_L = 0, \quad \partial_- \bar{\psi}^i_L + f \partial_- \bar{z}^i \psi^R_L = 0,
\]

\[
\partial_+ \psi^R_R = 0, \quad \partial_+ \psi^i_R = 0, \quad \partial_+ \bar{\psi}^i_R + 2if \partial_+ u \psi^i_R = 0, \quad \partial_+ \bar{\psi}^i_R - 2if \partial_+ u \bar{\psi}^i_R = 0,
\]

for the fermionic part.

Since the light-cone coordinate \( u \) and the corresponding fermions \( \psi^R_L \) and \( \psi^R_R \) satisfy the equations of motion for free fields, we can adopt the light-cone gauge as \( u = u_0 + k_+ \sigma_+ + k_- \sigma_- \) and \( \psi^R_L = \psi^R_R = 0 \). Using the super Virasoro constraints \( T_{++} = T_{--} = G_+ = G_- = 0 \), the coordinate \( v \) and the corresponding fermions \( \psi^i_L \) and \( \psi^i_R \) can be written by the transverse coordinates and the corresponding fermions as in the flat case.

The general solutions for the transverse coordinates and the corresponding fermions are given by

\[
\bar{z}^i = e^{-2ifk_+ \sigma_+} X^i, \quad z^i = e^{2ifk_+ \sigma_+} \bar{X}^i, \quad \psi^i_L = e^{-2ifk_+ \sigma_+} \eta^i_-, \quad \bar{\psi}^i_L = e^{2ifk_+ \sigma_+} \bar{\eta}^i_-, \quad \psi^i_R = e^{-2ifk_- \sigma_-} \eta^i_+, \quad \bar{\psi}^i_R = e^{2ifk_- \sigma_-} \bar{\eta}^i_+, \quad (3.38)
\]

where \( X^i = X^i_+(\sigma_+) + X^i_-(\sigma_-) \) and \( \eta^i_- \) are free complex bosons and fermions, respectively. Since the equations of motion for the fermions \( \psi^i_L \) and \( \bar{\psi}^i_L \) are the ones for the free fermions.
in the light-cone gauge, they can be expanded as
\[ \psi_L^i = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \tilde{a}_n e^{-i n \sigma^+} \quad \text{(R sector)}, \]
\[ \psi_R^i = \sqrt{2\alpha'} \sum_{r \in \mathbb{Z}+1/2} \tilde{c}_r e^{-i r \sigma^+} \quad \text{(NS sector)}. \] (3.39)

In order to recover periodicity conditions \( z^i(\sigma + 2\pi, \tau) = z^i(\sigma, \tau) \) and \( \psi_R^i(\sigma + 2\pi, \tau) = \pm \psi_R^i \), it is convenient to redefine the fields as
\[ X_+^i = e^{2i f k^+ \sigma^+} \chi_+^i, \quad X_-^i = e^{-2i f k^+ \sigma^-} \chi_-^i, \quad \eta_-^i = e^{-2i f k^+ \sigma^-} \chi_-^i, \] (3.40)
with periodicity conditions \( \chi_+^i(\sigma + 2\pi, \tau) = \chi_+^i(\sigma, \tau) \) and \( \chi_-^i(\sigma + 2\pi, \tau) = \pm \chi_-^i(\sigma, \tau) \). The mode expansions of these fields are given by
\[ \chi_+^i = i \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} a_n^i e^{-i n \sigma^+} \quad \text{(R sector)}, \]
\[ \chi_-^i = i \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} a_n^i e^{-i n \sigma^-} \]
\[ \chi_-^i = \sqrt{2\alpha'} \sum_{r \in \mathbb{Z}+1/2} c_r^i e^{-i r \sigma^-} \quad \text{(NS sector)}. \] (3.41)

We can calculate stress tensors in this system as
\[ T_{--} = \partial_- \partial_- \psi + \partial_- \partial_i \bar{\partial}_- \bar{\partial}_i^i + if \partial_- u(z^i \partial_- \bar{z}^i - \bar{z}^i \partial_- z^i) + i \bar{\psi}_R^i \partial_- \psi_i^i - 2f \partial_- u \bar{\psi}_R^i \psi_R^i, \]
\[ T_{++} = \partial_+ \partial_+ \psi + \partial_+ \partial_i \bar{\partial}_+ \bar{\partial}_i^i + if \partial_+ u(z^i \partial_+ \bar{z}^i - \bar{z}^i \partial_+ z^i) + i \bar{\psi}_L^i \partial_+ \psi^i_L, \] (3.42)
and zero mode parts are obtained as
\[ L_0 = \frac{1}{4\pi \alpha'} \int_0^{2\pi} d\sigma T_{--} = \frac{k_- q_-}{2\alpha'} + \frac{1}{4} \sum_n (n + 2f k_+) n \bar{a}_n^i a_n^i + \sum_n n \bar{d}_n^i d_n^i, \]
\[ \tilde{L}_0 = \frac{1}{4\pi \alpha'} \int_0^{2\pi} d\sigma T_{++} = \frac{k_+ q_+}{2\alpha'} + \frac{1}{4} \sum_n (n - 2f k_+) n \bar{a}_n^i a_n^i + \sum_n n \bar{d}_n^i d_n^i, \] (3.43)
for the R sector\(^{11}\). We have defined as
\[ y = \frac{1}{2}(u + v) = y_0 + k \tau + \cdots, \quad t = \frac{1}{2}(-u + v) = t_0 + l \tau + \cdots, \] (3.44)

\(^{10}\)The (+) sign and the (−) sign correspond to R-sector and NS sector, respectively.

\(^{11}\)Virasoro generators for the NS sector can be obtained in the similar way.
and hence

\[ u = u_0 + k_+ \sigma_+ + k_- \sigma_-, \quad k_\pm = \frac{1}{2} (k - l), \]

\[ v = v_0 + q_+ \sigma_+ + q_- \sigma_- + \cdots, \quad q_\pm = \frac{1}{2} (k + l), \] (3.45)

where we denote \( \cdots \) as higher modes. The momentum canonically conjugate to \( y \) is given by

\[ p_y = \frac{1}{4 \pi \alpha'} \int_0^{2\pi} d\sigma \partial_0 y + if (z^i \partial_+ \bar{z}^i - \bar{z}^i \partial_+ z^i + 2i \bar{\psi}_R^i \psi_R^i) = \frac{1}{2} \alpha' \partial_0 t - 2 f J_R, \] (3.46)

and the momentum canonically conjugate to \( t \) corresponding to conserved energy is obtained as

\[ E = -\frac{1}{4 \pi \alpha'} \int_0^{2\pi} d\sigma \partial_0 t + if (z^i \partial_+ \bar{z}^i - \bar{z}^i \partial_+ z^i + 2i \bar{\psi}_R^i \psi_R^i) = -\frac{1}{2} \alpha' \partial_0 t + 2f J_R. \] (3.47)

We have defined \( J_R \) as the operator corresponding to negative of right-mover part of angular momentum

\[ J_R = \frac{1}{4} \sum_n (n + 2f k_+) \bar{a}_n^i a_n^i + K, \]

\[ K = \sum_n \bar{d}_n^i d_n^i \ (R \ sector), \quad K = \sum_r \bar{c}_r^i c_r^i \ (NS \ sector). \] (3.48)

Let us turn to the quantization of the model. We can quantize by replacing the coefficients \( a_n^i, \bar{a}_n^i, d_n^i, \bar{d}_n^i, c_r^i, \bar{c}_r^i \) with the corresponding operators whose (anti-)commutation relations are

\[ [a_n^i, \bar{a}_m^i] = \frac{4}{n + 2f k_+} \delta_{nm}, \quad [\bar{a}_n^i, \bar{a}_m^i] = \frac{4}{n - 2f k_+} \delta_{nm}, \]

\[ \{d_n^i, \bar{d}_m^i\} = \delta_{nm}, \quad \{c_r^i, \bar{c}_s^i\} = \delta_{rs}, \quad \{d_n^i, \bar{d}_m^i\} = \delta_{nm}, \quad \{c_r^i, \bar{c}_s^i\} = \delta_{rs}, \] (3.49)

which are determined to be consistent with canonical (anti-)commutation relations. For the bosonic modes, it is convenient to rewrite as \( n > 0 \)

\[ b_{n,+}^{i\dagger} = a_{-n}^i \omega_{n,-}, \quad b_{n,+}^i = \bar{a}_{-n}^i \omega_{n,-}, \quad b_{n,-}^{i\dagger} = \bar{a}_{n}^i \omega_{n,+}, \quad b_{n,-}^i = a_{n}^i \omega_{n,+}, \]

\[ \tilde{b}_{n,+}^{i\dagger} = \bar{a}_{-n}^i \omega_{n,+}, \quad \tilde{b}_{n,+}^i = a_{-n}^i \omega_{n,+}, \quad \tilde{b}_{n,-}^{i\dagger} = a_{n}^i \omega_{n,-}, \quad \tilde{b}_{n,-}^i = \bar{a}_{n}^i \omega_{n,-}, \] (3.50)
with $\omega_{n,\pm} = \frac{1}{2}\sqrt{(n \pm 2f k^+)}$. We assume $0 < 2f k^+ < 1$ and we relabel the modes if $f k^+$ is not in this range. This procedure corresponds to the enlargement of Hilbert space by spectral flow symmetry \textcolor{red}{(13.30)} as we will see below. For the zero modes, we use
\begin{align*}
  b_0^+ = \frac{1}{2}\sqrt{2f k^+}a_0^+ , \quad b_i^+ = \frac{1}{2}\sqrt{2f k^+}a_i^+ , \quad \bar{b}_0^+ = \frac{1}{2}\sqrt{2f k^+}\bar{a}_0^+ , \quad \bar{b}_i^+ = \frac{1}{2}\sqrt{2f k^+}\bar{a}_i^+ .
\end{align*}
These new modes satisfy the standard commutation relations
\begin{align*}
  [b^i_{n,\pm}, b^j_{m,\pm}] = \delta_{nm} , \quad [\tilde{b}^i_{n,\pm}, \tilde{b}^j_{m,\pm}] = \delta_{nm} , \quad [b^i_0, b^j_0] = 1 , \quad [\tilde{b}^i_0, \tilde{b}^j_0] = 1 .
\end{align*}
Fock vacuum is defined by $b^i_{n,\pm}|0\rangle = \bar{d}^{i}_{n}|0\rangle = d_{n}^{i}|0\rangle = 0$ for $n > 0$ and $\bar{c}^i_r|0\rangle = c^i_r|0\rangle = 0$ for $r > 0$. The Fock vacuum for the left-movers is defined in the similar way.

Now, we can obtain the spectrum from the quantum expressions of Virasoro constraints. In order to do this, we include the coordinates of $T^4$ sector $x^A \quad (A = 6, 7, 8, 9)$ and the corresponding fermions $\psi^A_L$ and $\psi^A_R$. Mode expansions of these fields are defined in the way similar to the other fields. It is convenient to use the quantum expression of $J_R \quad \textcolor{red}{(3.48)}$
\begin{align*}
  \hat{J}_R &= b^+_i b^+_0 + \frac{1}{2} - \sum_{n=1}^{\infty} (b^+_i b^+_n - b^+_n b^+_i) + \hat{K} , \\
  \hat{K} &= \bar{d}^+_0 d^+_0 - \frac{1}{2} + \sum_{n=1}^{\infty} (\bar{d}^+_n d^+_n + d^+_n \bar{d}^+_n) \quad (\text{R sector}) , \\
  \hat{K} &= \sum_{r=1/2}^{\infty} (c^+_r c^+_r + \bar{c}^+_r \bar{c}^+_r) \quad (\text{NS sector}) ,
\end{align*}
and operators $N_L$ and $N_R$ (for the R-sector)
\begin{align*}
  N_R &= \sum_{n=1}^{\infty} n(b^+_i b^+_n + b^+_n b^+_i) + b^+_n b^+_A + b^+_n d^+_n + d^+_n \bar{d}^+_n + \bar{d}^+_n d^+_n ) , \\
  N_L &= \sum_{n=1}^{\infty} n(\bar{b}^+_i \bar{b}^+_n + \bar{b}^+_n \bar{b}^+_i) + \bar{b}^+_n \bar{b}^+_A + \bar{b}^+_n \bar{d}^+_n + \bar{d}^+_n \bar{d}^+_n + \bar{d}^+_n \bar{d}^+_n ) .
\end{align*}
The operators for the NS-sector can be defined in the similar way. By using the usual $\zeta$-regularization, we can calculate zero energy shift and obtain the quantum expressions of Virasoro operators \textcolor{red}{(3.54)} as
\begin{align*}
  \hat{L}_0 &= \frac{1}{2} \alpha'(-E^2 + p_A^2 + p_y^2) + \hat{N}_R + 2\alpha'(E + p_y) f \hat{J}_R , \\
  \hat{L}_0 &= \frac{1}{2} \alpha'(-E^2 + p_A^2 + p_y^2) + \hat{N}_L + 2\alpha'(E + p_y) f \hat{J}_R .
\end{align*}
We use $\hat{N}_{R,L} = N_{R,L} - a$ with $a = 0$ for the R-sector and $a = 1/2$ for the NS-sector. Virasoro conditions $\hat{L}_0 = \hat{\tilde{L}}_0 = 0$ lead to level matching condition $\hat{N}_L = \hat{N}_R$ and

$$E^2 - p_A^2 = \frac{1}{\alpha'}(\hat{N}_L + \hat{N}_R) + p_y^2 + 4f(p_y + E)\hat{J}_R. \quad (3.56)$$

From this condition, the light-cone energy can be read as

$$H_{l,c} = -p_u = \frac{1}{2}(E - p_y) = \frac{p_A^2}{4p_v} + \frac{1}{4\alpha'p_v}(\hat{N}_L + \hat{N}_R) + 2f\hat{J}_R, \quad (3.57)$$

with $p_v = \frac{1}{2}(p_y + E)$.

Next, let us consider the case with the metric (3.10). Although spectrum in this case can be calculated straightforwardly, it is simpler by making use of the coordinate transformation. In the polar coordinates $z^1 = r_1e^{i\varphi_1}$ and $z^2 = r_2e^{i\varphi_2}$, the Lagrangian (3.34) is written as

$$L = \partial_+ u \partial_- v + 2f(r_1^2 \partial_+ u \partial_- \varphi_1' + r_2^2 \partial_+ u \partial_- \varphi_2')$$

$$\quad + \partial_+ r_1 \partial_- r_1 + r_1^2 \partial_+ \varphi_1 \partial_- \varphi_1' + \partial_+ r_2 \partial_- r_2 + r_2^2 \partial_+ \varphi_2 \partial_- \varphi_2'. \quad (3.58)$$

By rotating the frame as

$$\varphi_1' = \varphi_1 - fu, \quad \varphi_2' = \varphi_2 - fu, \quad (3.59)$$

the Lagrangian becomes that for sigma model on the metric (3.10) as

$$L = \partial_+ u \partial_- v - f^2(r_1^2 + r_2^2)\partial_+ u \partial_- u$$

$$\quad + \partial_+ r_1 \partial_- r_1 + r_1^2 \partial_+ \varphi_1 \partial_- \varphi_1' + \partial_+ r_2 \partial_- r_2 + r_2^2 \partial_+ \varphi_2 \partial_- \varphi_2$$

$$\quad + fr_1^2(\partial_+ u \partial_- \varphi_1 - \partial_- u \partial_+ \varphi_1) + fr_2^2(\partial_+ u \partial_- \varphi_2 - \partial_- u \partial_+ \varphi_2). \quad (3.60)$$

The explicit form of the spectrum can be obtained from (3.57) by using the coordinate transformations (3.59) and

$$\hat{J} = -i \frac{\partial}{\partial \varphi_1} - i \frac{\partial}{\partial \varphi_2} \leftrightarrow -i \frac{\partial}{\partial \varphi_1'} - i \frac{\partial}{\partial \varphi_2'} = \hat{J} \equiv \hat{J}_L - \hat{J}_R. \quad (3.61)$$

Under the coordinate transformations, we find

$$E = -i \frac{\partial}{\partial t} \leftrightarrow -i \frac{\partial}{\partial t} - if \frac{\partial}{\partial \varphi_1'} - if \frac{\partial}{\partial \varphi_2'} = E + f\hat{J},$$

$$p_y = -i \frac{\partial}{\partial y} \leftrightarrow -i \frac{\partial}{\partial y} + if \frac{\partial}{\partial \varphi_1'} + if \frac{\partial}{\partial \varphi_2'} = p_y - f\hat{J}, \quad (3.62)$$
and hence \( p_u \leftrightarrow p_u - f \hat{J} \) and \( p_v \leftrightarrow p_v \). Therefore, the spectrum is given by

\[
H_{t,c} = -p_u = \frac{1}{2}(E - p_y) = \frac{p_A^2}{4p_v} + \frac{1}{4\alpha'p_v}(\hat{N}_L + \hat{N}_R) + f(\hat{J}_L + \hat{J}_R)
\]

(3.63)

We will compare this spectrum with the one obtained from the BRST quantization in the next section.
4 Superstring Theory on NSNS PP-Wave

In the previous section, we have shown that superstring theory on the pp-wave background with NSNS-flux can be obtained as the Penrose limit of superstring theory on $AdS_3 \times S^3(\times M^4)$ ($M^4 = T^4$ or $K3$) with NSNS-flux. Moreover, we have reviewed the (generalized) Nappi-Witten model and quantized this model by using the sigma model approach in the light-cone gauge. The same analysis can be given by using the current algebra approach. Superstring theory on $AdS_3 \times S^3(\times M^4)$ with NSNS-flux can be described by the $SL(2; \mathbb{R}) \times SU(2)$ super WZW model and the Penrose limit is realized by contracting currents on this model $[30, 31, 38]$. The superstring theory in the Penrose limit is given by the super WZW model whose target space is the 6 dimensional Heisenberg group $H_6$ as we saw in subsection 3.2. In this section, we analyze the $H_6$ super WZW model by using the free field realization in the current algebra approach $[29, 39, 41]$. We use this approach because the similarity with the analysis developed in $[15]$ for the case of $AdS_3 \times S^3$ becomes more transparent.

4.1 $H_6$ Super WZW Model as a Penrose Limit

The superstring theory on $AdS_3 \times S^3$ can be described by the $SL(2; \mathbb{R}) \times SU(2)$ super WZW model. We set the level of each current algebra to positive integer $k$, which corresponds to the number of NS5-branes $Q_5$. (The bosonic parts of $SL(2; \mathbb{R})$ and $SU(2)$ current algebras have the levels $k+2$ and $k-2$, respectively.) It is convenient to use the superfield formalism with supercoordinates $(z, \theta)$. The supercurrents of this system are given by

$$J^A(z, \theta) = \sqrt{\frac{k}{2}} \psi^A(z) + \theta J^A(z) \quad (\text{for } SL(2; \mathbb{R})),$$

$$K^a(z, \theta) = \sqrt{\frac{k}{2}} \chi^a(z) + \theta K^a(z) \quad (\text{for } SU(2)). \quad (4.1)$$

The total currents $J^A(z)$ and $K^a(z)$ ($A,a = \pm, 3$) satisfy the following operator product expansions (OPEs)

$$J^3(z)J^3(w) \sim -\frac{k}{2(z-w)^2}, \quad J^3(z)J^\pm(w) \sim \frac{\pm J^\pm(w)}{z-w},$$

$$J^+(z)J^-(w) \sim \frac{k}{(z-w)^2} - \frac{2J^3(w)}{z-w} \quad (4.2)$$
\[ K^3(z)K^3(w) \sim \frac{k}{2(z-w)^2}, \quad K^3(z)K^\pm(w) \sim \frac{\pm K^\pm(w)}{z-w}, \]

\[ K^+(z)K^-(w) \sim \frac{k}{(z-w)^2} + \frac{2K^3(w)}{z-w}. \quad (4.3) \]

The free fermions \( \psi^A \) and \( \chi^a \) are defined by the OPEs

\[ \psi^3(z)\psi^3(w) \sim -\frac{1}{z}, \quad \psi^+(z)\psi^-(w) \sim \frac{2}{z-w}, \quad (4.4) \]

\[ \chi^3(z)\chi^3(w) \sim \frac{1}{z}, \quad \chi^+(z)\chi^-(w) \sim \frac{2}{z-w}, \quad (4.5) \]

and transformed by the action of the total currents as follows

\[ J^3(z)\psi^\pm(w) \sim \psi^3(z)J^\pm(w) \sim \frac{\pm \psi^\pm(w)}{z-w}, \]

\[ J^\pm(z)\psi^\mp(w) \sim \frac{2\psi^3(w)}{z-w}, \quad (4.6) \]

\[ K^3(z)\chi^\pm(w) \sim \chi^3(z)K^\pm(w) \sim \frac{\pm \chi^\pm(w)}{z-w}, \]

\[ K^\pm(z)\chi^\mp(w) \sim \frac{2\chi^3(w)}{z-w}. \quad (4.7) \]

The Penrose limit of this model is given by a noncompact super WZW model associated with the 6 dimensional Heisenberg group \( H_6 \), which is a natural generalization of the Nappi-Witten model \[28\]. According to \[30, 31, 38\], we can obtain the supercurrents of this model by contracting those of the \( SL(2;\mathbb{R}) \times SU(2) \) WZW model. We redefine the supercurrents \[4.1\] as

\[ \mathcal{J}(z,\theta) = K^3(z,\theta) + J^3(z,\theta), \quad \mathcal{F}(z,\theta) = \frac{1}{k}(K^3(z,\theta) - J^3(z,\theta)), \]

\[ \mathcal{P}_1(z,\theta) = \frac{1}{\sqrt{k}} J^+(z,\theta), \quad \mathcal{P}_1^*(z,\theta) = \frac{1}{\sqrt{k}} J^-(z,\theta), \]

\[ \mathcal{P}_2(z,\theta) = \frac{1}{\sqrt{k}} K^+(z,\theta), \quad \mathcal{P}_2^*(z,\theta) = \frac{1}{\sqrt{k}} K^-(z,\theta), \quad (4.8) \]

and take the limit \( k \to \infty \) with keeping the eigenvalues of \( K^3_0 - J^3_0 \) order \( \mathcal{O}(k) \) but the eigenvalues of \( K^3_0 + J^3_0 \) much smaller than \( k \). Then we obtain the supercurrent algebra of
the $H_6$ super WZW model as

$$
\mathcal{J}(\theta, z) = \psi J(z) + \theta J(z) , \quad \mathcal{F}(\theta, z) = \psi F(z) + \theta F(z) ,
$$

$$
\mathcal{P}_i(\theta, z) = \psi P_i(z) + \theta P_i(z) , \quad \mathcal{P}_i^*(\theta, z) = \psi P^*_i(z) + \theta P^*_i(z) ,
$$

(4.9)

where $i = 1, 2$. The total currents $J(z), F(z), P_i(z)$ and $P^*_i(z)$ satisfy the OPEs

$$
J(z)P_i(w) \sim \frac{P_i(w)}{z - w} , \quad J(z)P^*_i(w) \sim -\frac{P^*_i(w)}{z - w} ,
$$

$$
P_i(z)P^*_j(w) \sim \delta_{ij} \left( \frac{1}{(z - w)^2} + \frac{F(w)}{z - w} \right) ,
$$

$$
J(z)F(w) \sim \frac{1}{(z - w)^2} .
$$

(4.10)

Other OPEs have no singular terms. Their superpartners $\psi J$, $\psi F$, $\psi P_i$ and $\psi P^*_i$ are free fermions defined by

$$
\psi P_i(z)\psi P^*_j(w) \sim \frac{\delta_{ij}}{z - w} , \quad \psi J(z)\psi F(w) \sim \frac{1}{z - w} .
$$

(4.11)

The total currents $J, F, P_i$ and $P^*_i$ non-trivially act on these free fermions as

$$
J(z)\psi P_i(w) \sim \psi J(z)P_i(w) \sim \frac{\psi P_i}{z - w} ,
$$

$$
J(z)\psi P^*_i(w) \sim \psi J(z)P^*_i(w) \sim -\frac{\psi P^*_i}{z - w} ,
$$

$$
P_i(z)\psi P^*_j(w) \sim \psi P_i(z)P^*_j(w) \sim \delta_{ij} \frac{\psi F}{z - w} .
$$

(4.12)

In terms of the metric in subsection 3.2, $-J^3_0$ and $K^3_0$ corresponds to the space-time energy $-i\partial_t$ and the angular momentum $-i\partial_\psi$, respectively. Therefore, the contraction of the currents is identified as the transformation of the coordinates (3.9), see also (3.12). In fact, this contraction can be regarded as a stringy extension of Penrose limit since this contraction defines a transformation from superstrings on $AdS_3 \times S^3$ to superstrings on the pp-wave. In the dual CFT side, the eigenvalue of $-J^3_0$ and $K^3_0$ are interpreted as conformal weight $\Delta$ and R-charge $Q$, respectively and the contraction means focusing on almost BPS states with

$$
\Delta + Q \sim k \gg 1 , \quad \Delta - Q \ll k ,
$$

(4.13)
as in subsection 3.2.

\( \mathcal{N} = 1 \) superconformal symmetry is realized as follows. Because the total currents have non-trivial OPEs with fermions, it is convenient to introduce bosonic currents, which can be treated independently of fermions. They are defined as

\[
\hat{J} = J - \psi_P \psi_P^* - \psi_P \psi_P^* , \quad \hat{F} = F ,
\]

\[
\hat{P}_i = P_i - \psi_P \psi_P , \quad \hat{P}_i^* = P_i^* + \psi_P \psi_P^* ,
\]

which again satisfy the same OPEs (4.10) but have no singular OPEs with the free fermions. \( \mathcal{N} = 1 \) supercurrent is now defined in the standard fashion

\[
G = \hat{J} \psi_P + \hat{F} \psi_P + \sum_{i=1}^2 (\hat{P}_i \psi_P^* + \hat{P}_i^* \psi_P + \psi_F \psi_P^* \psi_P^* )
\]

\[
= J \psi_P + F \psi_P + \sum_{i=1}^2 (\hat{P}_i \psi_P^* + \hat{P}_i^* \psi_P ) ,
\]

and the total currents can be given by acting \( G \) on the fermions \( \psi_J , \psi_F , \psi_P , \) and \( \psi_P^* \) as it should be.

We can continue the analysis by using the abstract current algebra techniques in principle. However, it is easier to make use of the following free field realizations as given in \([29, 39]\). We introduce free bosons \( X^+ , X^- , Z_i , Z_i^*( i = 1, 2) \) defined by the OPEs

\[
X^+(z)X^-(w) \sim -\ln(z - w) , \quad Z_i(z)Z_j^*(w) \sim -\delta_{ij} \ln(z - w) ,
\]

and rewrite the fermions \( \psi_J , \psi_F , \psi_P , \) and \( \psi_P^* \) as

\[
\psi_F = \psi^+ , \quad \psi_J = \psi^- , \quad \psi_P = \psi_i e^{iX_i} , \quad \psi_P^* = \psi_i^* e^{-iX_i} ,
\]

where the new fermions are defined by

\[
\psi^+(z)\psi^-(w) \sim \frac{1}{z - w} , \quad \psi_i(z)\psi_j^*(w) \sim \frac{\delta_{ij}}{z - w} .
\]

The total currents can be now expressed as

\[
F = i\partial X^+ , \quad J = i\partial X^- ,
\]

\[
P_i = e^{iX_i} (i\partial Z_i + \psi^+ \psi_i) , \quad P_i^* = e^{-iX_i} (i\partial Z_i^* - \psi^+ \psi_i^*) ,
\]
and the bosonic currents are written as

\[ \hat{F} = i\partial X^+ , \quad \hat{J} = i\partial X^- \psi_1\psi_1^* - \psi_2\psi_2^* - 2i\partial X^+ , \]

\[ \hat{P}_i = e^{iX^+}i\partial Z_i , \quad \hat{P}_i^* = e^{-iX^+}i\partial Z_i^* . \]  

(4.20)

In terms of these free fields the superconformal current is rewritten as the standard form of flat background

\[ G = \psi^+ i\partial X^- + \psi^- i\partial X^+ + \psi_i^* i\partial Z_i + \psi_i i\partial Z_i^* . \]  

(4.21)

We also introduce free fields \( Y^i \) and \( \lambda^i \) \((i = 1, 2, 3, 4)\) to describe the remaining \( T^4 \) sector as

\[ \lambda^i(z)\lambda^j(w) \sim \frac{\delta^{ij}}{z-w} , \quad Y^i(z)Y^j(w) \sim -\delta^{ij} \ln(z-w) . \]  

(4.22)

In order to analyze space-time fermions, we have to introduce spin fields, which are defined by using bosonized fermions. We bosonize the fermions as

\[ \psi_{\pm 0} = \psi_{\pm} = e^{\pm iH_0} , \]

\[ \psi^{+j} = \psi_j = e^{+ iH_j} , \]

\[ \psi^{-j} = \psi_j^* = e^{- iH_j} , \]

\[ \psi^{\pm 3} = \frac{1}{\sqrt{2}}(\lambda^1 \pm i\lambda^2) = e^{\pm iH_3} , \]

\[ \psi^{\pm 4} = \frac{1}{\sqrt{2}}(\lambda^3 \pm i\lambda^4) = e^{\pm iH_4} , \]  

(4.23)

and define the spin fields as

\[ S^{\epsilon_0\epsilon_1\epsilon_2\epsilon_3\epsilon_4} = \exp \left( \frac{i}{2} \sum_{j=0}^{4} \epsilon_j H_j \right) . \]  

(4.24)

The GSO condition imposes \( \prod_{j=0}^{4} \epsilon_j = +1 \) in the convention of this thesis. Precisely speaking, the OPEs including the spin fields are affected by cocycle factors and they depend on the notation of gamma matrices, which is summarized in appendix A.
4.2 Hilbert Space of $H_6$ Super WZW Model

The irreducible representations of the current algebra of generalized Nappi-Witten model (that is the $H_6$ WZW model) are classified in [29, 39]. We shall here focus on the Type II representation (corresponding to the highest weight representation of the zero-mode subalgebra in subsection 3.3) and later we discuss the other types of representations. The vacuum state (in the NS sector) is characterized by

$$J_0|j, \eta⟩ = j|j, \eta⟩,$$
$$F_0|j, \eta⟩ = \eta|j, \eta⟩,$$
$$P_{i,n}|j, \eta⟩ = 0, \quad (\forall n \geq 0),$$
$$P^*_{i,n}|j, \eta⟩ = 0, \quad (\forall n > 0),$$
$$\Psi_r|j, \eta⟩ = 0, \quad (\forall r > 0, \ r \in \frac{1}{2} + \mathbb{Z}), \quad (4.25)$$

where $\Psi$ represents all the fermionic fields $\psi_J, \psi_F, \psi_P$ and $\psi_P^*$. We assume that $j \in \mathbb{R}$ and $0 < \eta < 1$.

It is useful to rewrite this representation (4.25) in terms of the free fields $X^\pm, Z_i, Z_i^*; \psi^\pm, \psi_i$ and $\psi^*_i$. This is nothing but a Fock representation with the Fock vacuum defined by the vertex operator

$$V = \exp \left( ijX^+ + i\eta X^- \right) \sigma_\eta, \quad (4.26)$$

where $\sigma_\eta$ is the (chiral) twist field. This field imposes the boundary conditions

$$i\partial Z_i(e^{2\pi i}z) = e^{-2\pi i \eta} i\partial Z_i(z), \quad \psi_i(e^{2\pi i}z) = e^{-2\pi i \eta} \psi_i(z),$$
$$i\partial Z_i^*(e^{2\pi i}z) = e^{2\pi i \eta} i\partial Z_i^*(z), \quad \psi_i^*(e^{2\pi i}z) = e^{2\pi i \eta} \psi_i^*(z), \quad (4.27)$$

which ensure the locality of $H_6$ supercurrents. More precisely, $\sigma_\eta$ should have the following OPEs

$$i\partial Z_i(z)\sigma_\eta(w) \sim (z-w)^{-\eta} t^i_\eta(w), \quad i\partial Z_i^*(z)\sigma_\eta(w) \sim (z-w)^{\eta-1} t'^i_\eta(w),$$
$$\psi_i(z)\sigma_\eta(w) \sim (z-w)^{-\eta} t^i_\eta(w), \quad \psi_i^*(z)\sigma_\eta(w) \sim (z-w)^\eta t'^i_\eta(w), \quad (4.28)$$

where $t^i_\eta, t'^i_\eta, t^i_\eta$ and $t'^i_\eta$ are descendant twist fields. This twist operator $\sigma_\eta$ has the conformal weight

$$h(\sigma_\eta) = 2 \times \frac{1}{2} \eta(1 - \eta) + 2 \times \frac{1}{2} \eta^2 = \eta. \quad (4.29)$$
As already discussed in [29, 39, 41], there is the spectral flow symmetry

\[ J_n \rightarrow J_n, \quad F_n \rightarrow F_n + p\delta_n,0, \quad P_{i,n} \rightarrow P_{i,n+p}, \quad P^*_{i,n} \rightarrow P^*_{i,n-p}, \]

\[ \psi_{J,r} \rightarrow \psi_{J,r}, \quad \psi_{F,r} \rightarrow \psi_{F,r}, \quad \psi_{P_{i,r}} \rightarrow \psi_{P_{i,r}} + p, \quad \psi_{P^*_{i,r}} \rightarrow \psi_{P^*_{i,r}} - p, \]  

(4.30)

and hence we also consider the flowed representation as the natural extensions of (4.25). The vacuum states are given by (where we use \( p \in \mathbb{Z} \) as the spectral flow number)

\[ J_0|j,\eta,p\rangle = j|j,\eta,p\rangle, \quad F_0|j,\eta,p\rangle = (\eta + p)|j,\eta,p\rangle, \]

\[ P_{i,n}|j,\eta,p\rangle = 0, \quad (\forall n \geq -p), \quad P^*_{i,n}|j,\eta,p\rangle = 0, \quad (\forall n > p), \]

\[ \psi_{J,r}|j,\eta,p\rangle = 0, \quad (\forall r > 0), \quad \psi_{F,r}|j,\eta,p\rangle = 0, \quad (\forall r > 0), \]

\[ \psi_{P_{i,r}}|j,\eta,p\rangle = 0, \quad (\forall r > -p), \quad \psi_{P^*_{i,r}}|j,\eta,p\rangle = 0, \quad (\forall r > p), \]  

(4.31)

where \( n \in \mathbb{Z} \) and \( r \in 1/2 + \mathbb{Z} \). This spectral flow symmetry (4.30) is actually the counterpart of that for \( SL(2; \mathbb{R}) \times SU(2) \) WZW model [25, 26] and it is not difficult to confirm it directly by taking the contraction (4.8). We will later focus on the sectors with non-zero spectral flow number \( p \) to realize the (almost) BPS states and discuss the correspondence with the symmetric orbifold theory. We can again realize the flowed representation (4.31) by means of the Fock representation, in which the Fock vacuum corresponds to the vertex operator

\[ V = \exp \left( ijX^+ + i(\eta + p)X^- \right) \sigma_\eta. \]  

(4.32)

We should comment on the other types of irreducible representations of \( H_6 \) current algebra. (The irreducible representations of the zero-mode subalgebra are given in subsection 3.3.) The Type III representations have the lowest weights and the eigenvalues of \( F \) take \(-1 < \eta < 0\). The Type I representations have neither the highest nor lowest weights and the eigenvalues of \( F \) are \( \eta = 0 \). There are also their spectrally flowed representations. As in the case of \( SL(2; \mathbb{R}) \) WZW model, (see, for example, [26]) the spectral flow symmetry interchanges the Type III representation with the Type II representation. In fact, we can easily see that

\[ \mathcal{H}^{(II)}_{j,\eta,p} \cong \mathcal{H}^{(III)}_{j,\eta-1,p+1}, \]  

(4.33)
where \( H^{(II)}_{j,\eta,p} \) is the flowed Type II representation defined by using the vacuum (4.31) and \( H^{(III)}_{j,\eta,p} \) is the flowed Type III representation defined in the similar way. (We must also make the redefinitions of fermionic oscillators as \( \psi'_{P_i,r} := \psi_{P_i,r+1} \) and \( \psi'^*_{P_i,r} := \psi^*_{P_i,r-1} \) to equate the both sides of (4.33).) Therefore, we only have to consider the Type II representation if assuming the spectral flow symmetry. We also note that \( p \geq 0 \) representations correspond to positive energy states and \( p < 0 \) representations to negative energy ones in the original \( AdS_3 \) string theory.

On the other hand, the Type I representations seem to be the counterparts of the principal continuous series in string theory on \( AdS_3 \). Because there are no twisted fields in this sector, the vacua of Type I representations correspond to the vertex operators

\[
V = \exp \left( ipX^- + i(j + n)X^+ + ip_i^* Z_i + ip_i Z_i^* \right), \quad n \in \mathbb{Z}.
\]

(4.34)

If we do not consider the spectral flow \( (p = 0) \), there are only trivial massless states with zero energy, that do not propagate along the transverse plane \( (Z_i, Z_i^* \text{ directions}) \) just as the classical wave functions with \( p_- = 0 \) in subsection 3.3. For the cases of flowed Type I representations \( (p \neq 0) \), many physical states are possible. The spectra of light-cone energies in these sectors are continuous and the strings freely propagate along the transverse plane. It is quite natural to identify these sectors with "long string sectors" in string theory on \( AdS_3 \) \cite{26}. On the other hand, the Type II (and Type III) representation should correspond to "short string sectors", because they are the counterparts of the discrete series in string theory on \( AdS_3 \). The strings in these sectors cannot freely propagate along the transverse plane because the string coordinates \( Z_i, Z_i^* \) are twisted.

### 4.3 Physical Vertices of Superstring on NSNS PP-Wave

In this section we turn to physical vertex operators in superstring theory on the pp-wave background with NSNS-flux. We use the BRST quantization and physical states correspond to the elements of the BRST cohomology. We shall concentrate on the Type II representation with \( p \geq 0 \) for the time being and we will discuss later the Type I representation. In order to construct the physical vertices, it is convenient to make use of the free field representation previously discussed. We fix the Fock vacuum as

\[
|j, \eta, p \rangle = \sigma_\eta e^{ijX^+ + i(\eta + p)X^-} |0 \rangle, \quad 0 < \eta < 1, \quad p \in \mathbb{Z}_{\geq 0}.
\]

(4.35)

The light-cone formalism is manifestly ghost free, however the covariant formalism needs a no-ghost theorem. In order to prove the no-ghost theorem, it is convenient to construct
DDF operators which are BRST invariant and generate the physical spectrum. In our case, the spectrum in the light-cone gauge is obtained in (3.57) or (3.63), and we construct DDF operators which generate these spectrum in this subsection. Here we introduce the superghosts \((\gamma, \beta)\) or the bosonized ones \((\phi, \xi, \eta)\). The BRST charge has the standard form of the free superstring theory as \([62]\)
\[
Q_{\text{BRST}} = \oint \left[ c \left( T - \frac{1}{2}(\partial \phi)^2 - \partial^2 \phi - \eta \partial \xi + \partial cb \right) + \eta e^\phi G - b\eta \partial \eta e^{2\phi} \right],
\]
(4.36)
where \(T\) and \(G\) are the total stress tensor and the superconformal current constructed from the free fields \(X^\pm, Z_i, Z_i^*, Y^i, \psi^\pm, \psi_i, \psi_i^*\) and \(\lambda^i\).

The most important vertex operators are the generators of space-time supersymmetry algebra \([B.6], [B.9]\) and \([B.10]\), which are defined in appendix \([B]\). The generators of bosonic part are nothing but the zero-modes of world-sheet \(H_6\) (total) currents
\[
\mathcal{J} = \oint \psi^- e^{-\phi} = \oint i\partial X^- = J_0,
\]
\[
\mathcal{F} = \oint \psi^+ e^{-\phi} = \oint i\partial X^+ = F_0,
\]
\[
\mathcal{P}_i = \oint \psi_i e^{iX^+} e^{-\phi} = \oint (i\partial Z_i + \psi^+ \psi_i) e^{iX^+} = P_{i,0},
\]
\[
\mathcal{P}_i^* = \oint \psi_i^* e^{-iX^+} e^{-\phi} = \oint (i\partial Z_i^* - \psi^+ \psi_i^*) e^{-iX^+} = P_{i,0}^*,
\]
(4.37)
where we implicitly identify the operators by using the picture changing operator\(^{12}\). The generators of fermionic part are obtained by using the spin fields \([4.24]\) as (in the \((-1/2)\) picture)
\[
Q^{++a} = \oint S^{+++aa} e^{iX^+} e^{-\phi/2}, \quad Q^{--a} = \oint S^{++-aa} e^{-iX^+} e^{-\phi/2},
\]
\[
Q^{+-a} = \oint S^{+-+aa} e^{-\phi/2}, \quad Q^{-+a} = \oint S^{--+aa} e^{-\phi/2}.
\]
(4.38)
These operators manifestly BRST invariant and they can locally act on the Fock space associated with \(|j, \eta, p\rangle\) (irrespective of the values of \(j, \eta\) and \(p\)). We can directly check that they generate the super pp-wave algebra \([B.6], [B.9]\) and \([B.10]\). In particular, we note that
\[
\{Q^{+-a}, Q^{--b}\} = \epsilon^{ab} \mathcal{J}, \quad [\mathcal{J}, Q^{\pm+a}] = 0,
\]
(4.39)
\(^{12}\)The picture means the number \(n\) for \(e^{-n\phi}\) in vertices. We can change the picture of the vertices by making use of the picture changing operators without affecting the BRST cohomology.
which indicates that $Q^{-a}$ and $Q^{+a}$ play the role of supercharges with the “Hamiltonian” $J$. In terms of appendix C, these are the “dynamical” supercharges.

In order to analyze the spectrum of physical states we further need to introduce DDF operators. We propose the following DDF operators as the “affine extension” of (4.37) and (4.38)

$$P_{i,n} = \frac{1}{\sqrt{p + \eta}} \int \psi_i e^{i\frac{a}{p+\eta}X^+} e^{-\phi},$$

$$P^*_{i,n} = \frac{1}{\sqrt{p + \eta}} \int \psi_i e^{i\frac{a}{p+\eta}X^+} e^{-\phi},$$

$$Q^{++a}_n = \frac{1}{\sqrt{p + \eta}} \int S^{+++aa} e^{i\frac{a}{p+\eta}X^+} e^{-\phi/2},$$

$$Q^{--a}_n = \frac{1}{\sqrt{p + \eta}} \int S^{+--aa} e^{i\frac{a}{p+\eta}X^+} e^{-\phi/2}.$$

These operators are BRST invariant and locally act on the Fock space associated with $|j, \eta, p \rangle$ (of the fixed $\eta$ and $p$). Later we will compare the spectrum generated by these DDF operators with the light-cone spectrum. It is obvious that

$$\sqrt{p + \eta}P_{i,p} = P_i, \quad \sqrt{p + \eta}P^*_{i,-p} = P^*_i,$$

$$\sqrt{p + \eta}Q^{++a}_p = Q^{++a}, \quad \sqrt{p + \eta}Q^{--a}_p = Q^{--a}.$$

Note that the supercharges $Q^{\pm\mp a}$ do not have such affine extensions because the BRST invariance cannot be preserved. The DDF operators (4.40) satisfy the following (anti-)commutation relations (up to the picture changing and BRST exact terms)

$$[P_{i,m}, P^*_{j,n}] = \frac{m + \eta}{p + \eta} \delta_{ij} \delta_{m+n,0}, \quad \{Q^{--a}_m, Q^{++b}_n\} = \epsilon^{ab} \delta_{m+n,0},$$

$$[J, P_{i,n}] = \frac{n + \eta}{p + \eta} P_{i,n}, \quad [J, P^*_{i,n}] = \frac{n - \eta}{p + \eta} P^*_{i,n},$$

$$[J, Q^{++a}_n] = \frac{n + \eta}{p + \eta} Q^{++a}_n, \quad [J, Q^{--a}_n] = \frac{n - \eta}{p + \eta} Q^{--a}_n.$$

It is also useful to remark that $(P_{i,n}, Q^{++a}_n)$ and $(P^*_{i,n}, Q^{--a}_n)$ are the supermultiplets with respect to the supercharges $Q^{+-a}$ and $Q^{-+a}$. More precisely, we find the relations

$$P_{i,n} \xrightarrow{Q^{--a}} Q^{++a} \xrightarrow{Q^{+-(-a)}} P_{2,n} \xrightarrow{Q^{+-+}} 0$$

$$0 \xrightarrow{Q^{+-+}} P_{i,n} \xrightarrow{Q^{+-(-a)}} Q^{++a} \xrightarrow{Q^{+-(-a)}} P_{2,n}.$$

(4.43)
and the explicit forms of (anti-)commutation relations are
\[\{Q^{-a}, P_{1,n}\} = \frac{n+\eta}{p+\eta} Q_{n}^{++a}, \quad [Q^{+a}, P_{1,n}^\ast] = \frac{n-\eta}{p+\eta} Q_{n}^{-a},\]
\[\{Q^{+a}, P_{2,n}\} = \frac{n+\eta}{p+\eta} Q_{n}^{++a}, \quad [Q^{-a}, P_{2,n}^\ast] = -\frac{n-\eta}{p+\eta} Q_{n}^{-a},\]
\[\{Q^{-a}, Q_{n}^{++b}\} = -\epsilon^{ab}P_{1,n}, \quad \{Q^{+a}, Q_{n}^{-b}\} = \epsilon^{ab}P_{1,n}^\ast,\]
\[\{Q^{-a}, Q_{n}^{++b}\} = \epsilon^{ab}P_{2,n}, \quad \{Q^{+a}, Q_{n}^{-b}\} = \epsilon^{ab}P_{2,n}^\ast.\]

In order to construct the remaining DDF operators for the $T^4$ directions, it is convenient to relabel the fermions $\lambda^i$ as
\[\lambda^+ = \frac{1}{\sqrt{2}}(\lambda^1 + i\lambda^2), \quad \lambda^- = \frac{1}{\sqrt{2}}(-\lambda^1 + i\lambda^2),\]
\[\lambda^+ = \frac{1}{\sqrt{2}}(-\lambda^3 - i\lambda^4), \quad \lambda^- = \frac{1}{\sqrt{2}}(-\lambda^3 + i\lambda^4),\]
and the free bosons $Y^{aa}$ as in the similar way. These fields satisfy the following OPEs
\[\lambda^{a\bar{a}}(z)\lambda^{b\bar{b}}(w) \sim \frac{\epsilon^{ab}\epsilon^{\bar{a}\bar{b}}}{z-w}, \quad Y^{a\bar{a}}(z)Y^{b\bar{b}}(w) \sim -\epsilon^{ab}\epsilon^{\bar{a}\bar{b}}\ln(z-w).\]

Then the DDF operators can be given by
\[A_n^{a\bar{a}} = \frac{1}{\sqrt{p+\eta}} \int \lambda^{a\bar{a}} e^{i\frac{a\bar{a}}{p+\eta} X^+} e^{-\phi} = \frac{1}{\sqrt{p+\eta}} \int i\partial Y^{a\bar{a}} e^{i\frac{a\bar{a}}{p+\eta} X^+},\]
\[B_n^{++} = -\frac{1}{\sqrt{p+\eta}} \int S^{++\phi} e^{i\frac{\phi}{p+\eta} X^+} e^{-\frac{\phi}{2}},\]
\[B_n^{-\phi} = \frac{1}{\sqrt{p+\eta}} \int S^{++\phi} e^{i\frac{\phi}{p+\eta} X^+} e^{-\frac{\phi}{2}},\]
and they satisfy
\[[A_m^{a\bar{a}}, A_n^{b\bar{b}}] = \frac{m}{p+\eta} \delta_{m+n} \epsilon^{ab}\epsilon^{\bar{a}\bar{b}}, \quad [B_m^{a\bar{a}}, B_n^{b\bar{b}}] = \delta_{m+n} \epsilon^{ab}\epsilon^{\bar{a}\bar{b}},\]
\[[J, A_n^{a\bar{a}}] = \frac{n}{p+\eta} A_n^{a\bar{a}}, \quad [J, B_n^{a\bar{a}}] = \frac{n}{p+\eta} B_n^{a\bar{a}}.\]
The pair of \((A_n^{\hat{a}}, B_n^{\hat{a}}})\) again become supermultiplets with respect to \(Q^\pm \to 0\)
\[
0 \xleftrightarrow{Q^\pm \to 0} A_n^{\hat{a}} \xleftrightarrow{Q^\pm \to 0} B_n^{\hat{a}} \xleftrightarrow{Q^\pm \to 0} A_n^{\hat{a}},
\]
more precisely,
\[
\begin{align*}
\{Q^\pm, A_n^{\hat{a}}\} &= -A_n^{\hat{a}}, & \{Q^\pm, B_n^{\hat{a}}\} &= -A_n^{\hat{a}}. 
\end{align*}
\]

### 4.4 Spectrum of Physical States

Let us consider the spectrum which is generated by the DDF operators constructed in the previous subsection. More precisely speaking, we are interested in the physical states corresponding to almost BPS states in the dual CFT characterized by

\[
\Delta + Q \sim k \gg 1, \quad \Delta - Q \ll k.
\]

This condition is equivalent to\(^{13}\)

\[
\mathcal{F} \gtrsim 1, \quad |\mathcal{J}| \ll k,
\]
and all string excitations satisfy this condition in the Penrose limit \(k \to +\infty\). BPS states correspond to the ones with \(\mathcal{J} = 0\) and belong to the short multiplets of the superalgebra (4.37) and (4.38). The condition \(\mathcal{F} \gtrsim 1\) only leads to the restriction \(p \geq 1\) with respect to the spectral flow number \(p\). (We only consider the positive energy states.)

We concentrate on the sectors with no momenta along \(T^4\) direction for the time being. It is not difficult to write down the complete list of BPS states for each of the fixed \(p\) and \(\eta\). In the NS sector, we obtain (where we focus on the left-movers only)

\[
|\omega^0; \eta, p\rangle = \psi_{1, -\frac{1}{2} + \eta} |0, \eta, p\rangle \otimes ce^{-\phi} |0\rangle_{gh},
\]

\[
|\omega^2; \eta, p\rangle = \psi_{2, -\frac{1}{2} + \eta} |0, \eta, p\rangle \otimes ce^{-\phi} |0\rangle_{gh},
\]

and we have two more BPS states in the R-sector as

\[
|\omega^{1\pm}; \eta, p\rangle = (S^{++\mp\pm})_{\frac{1}{8} + \eta} |0, \eta, p\rangle \otimes ce^{-\phi} |0\rangle_{gh}.
\]

\(^{13}\)In our convention, the BPS inequality is equivalent to \(\mathcal{J} \leq 0\).
Here we point out the next relation, which is useful for our discussion:

$$\langle \omega^0; \eta, p \rangle \xrightarrow{\mathcal{B}^{\dot{a}}_+} \langle \omega^1; \eta, p \rangle \xrightarrow{\mathcal{B}^{(-\dot{a})}_+} \langle \omega^2; \eta, p \rangle \xrightarrow{\mathcal{B}^{\dot{a}}_+} 0$$

$$0 \xleftarrow{\mathcal{B}^{-\dot{a}}_0} \langle \omega^0; \eta, p \rangle \xrightarrow{\mathcal{B}^{-(-\dot{a})}_0} \langle \omega^1; \eta, p \rangle \xrightarrow{\mathcal{B}^{-\dot{a}}_0} \langle \omega^2; \eta, p \rangle .$$

(4.56)

There is an obvious correspondence between the BPS states and the “chiral part” of the cohomology ring of $T^4$ by identifying $\mathcal{B}^{\dot{a}}_+ \dot{a}$ with holomorphic one-form $dZ^\dot{a}$ on $T^4$. Therefore, emphasizing the correspondence to $H^*(T^4)$, the non-chiral BPS states can be explicitly written as

$$\langle \omega^{(q, \bar{q})}; \eta, p \rangle = \langle \omega^q; \eta, p \rangle \otimes \langle \omega^{\bar{q}}; \eta, p \rangle , \quad (\forall \omega^{(q, \bar{q})} \in H^{q, \bar{q}}(T^4)) .$$

(4.57)

These states have degenerate charge $\mathcal{F} = p + \eta$ for each sector of $p$ and $\eta$.

We can construct the other types of physical states by making the DDF operators (4.40) and (4.48) act on these BPS states. We first note that the BPS states are actually the Fock vacua with respect to (4.40) and (4.48):

$$\mathcal{P}_{i,n} |\omega; \eta, p\rangle = 0 , \quad (\forall n \geq 0) , \quad \mathcal{P}_{i,n}^\ast |\omega; \eta, p\rangle = 0 , \quad (\forall n > 0) ,$$

$$Q_{n}^{a+} |\omega; \eta, p\rangle = 0 , \quad (\forall n \geq 0) , \quad Q_{n}^{-a} |\omega; \eta, p\rangle = 0 , \quad (\forall n > 0) ,$$

$$A_{n}^{a+} |\omega; \eta, p\rangle = 0 , \quad (\forall n \geq 0) , \quad B_{n}^{a-} |\omega; \eta, p\rangle = 0 , \quad (\forall n > 0) .$$

(4.58)

(For $\mathcal{B}_{0}^{\dot{a}}$, see (4.56).) We hence obtain

$$\mathcal{B}_{-n_1}^{\dot{a}_1} \cdots Q_{-m_1}^{a+b_1} \cdots Q_{-k_1}^{-c_1} \cdots \otimes \mathcal{B}_{-\bar{n}_1}^{\bar{a}_1} \cdots \bar{Q}_{-\bar{m}_1}^{a+b_1} \cdots \bar{Q}_{-\bar{k}_1}^{-c_1} \cdots |\omega; \eta, p\rangle ,$$

$$n_i, \bar{n}_i, m_i, \bar{m}_i > 0 , \quad k_i, \bar{k}_i \geq 0 , \quad \forall \omega \in H^*(T^4) ,$$

(4.59)

as typical physical states. These states become almost BPS under the condition

$$|\mathcal{J} + \bar{\mathcal{J}}| \equiv \frac{1}{p + \eta} \left[ \left( \sum_i n_i + \sum_i (m_i - \eta) + \sum_i (k_i + \eta) \right) + \left( \sum_i \bar{n}_i + \sum_i (\bar{m}_i - \eta) + \sum_i (\bar{k}_i + \eta) \right) \right] \ll k ,$$

(4.60)

which is always satisfied for sufficiently large $k$. Other states can be obtained by multiplying the supercharges $Q^{\pm a}$. Recall that we should include the helicity in the transverse
plane $\mathcal{J} - \bar{\mathcal{J}} = h \in \mathbb{Z}$ as in (3.30). Then the level matching condition becomes
\[
\left( \sum_i n_i + \sum_i (m_i - \eta) + \sum_i (k_i + \eta) \right) \\
- \left( \sum_i \bar{n}_i + \sum_i (\bar{m}_i - \eta) + \sum_i (\bar{k}_i + \eta) \right) \in (p + \eta)\mathbb{Z}.
\] (4.61)

We should note that under the limiting procedure $k \to \infty$, a huge number of stringy excitations of the original $AdS_3$ superstring theory are included in our physical Hilbert space of pp-wave superstring theory. These states could correspond to very massive states, (which could possess very large energies $-J_0^3$) although they have the small $\mathcal{J}$-charges. This fact gives us a theoretical ground for making it possible to identify a lot of stringy excitations with the objects in the dual theory as in [12].

In order to complete our discussion, we also consider the sectors with non-trivial momenta along $T^4$ direction. Although there are no BPS states in these sectors, it is possible to construct almost BPS states. We only consider a rectangular torus for simplicity and use $R_a (a = 1, 2, 3, 4)$ as the radii. The momenta of $T^4$ sector can be written as
\[
p_a = \frac{n_a}{R_a} + \frac{w^a R_a}{2}, \quad \bar{p}_a = \frac{n_a}{R_a} - \frac{w^a R_a}{2}, \quad (n_a, w^a \in \mathbb{Z}),
\] (4.62)

where $n_a$ and $w^a$ are KK momenta and winding modes, respectively. We have used the convention $\alpha' \equiv l_s^2 = 2$. The simplest physical states (in the NSNS sector) have the next form
\[
\psi_{i, -\frac{1}{2} + \eta} \bar{\psi}_{\bar{i}, \frac{1}{2} + \eta} |j, \bar{j}, \eta, p; n_a, w^a\rangle \otimes \bar{c}\bar{c}e^{-\phi - \bar{\phi}} |0\rangle_{gh},
\] (4.63)

where $|j, \bar{j}, \eta, p; n_a, w^a\rangle$ corresponds to the vertex operator
\[
e^{i(jX^+(p+\eta)X^- + p_a Y^a)} \otimes e^{i(\bar{j}\bar{X}^+(p+\eta)\bar{X}^- + \bar{p}_a \bar{Y}^a)}.
\] (4.64)

The on-shell condition leads to
\[
j = -\frac{1}{2(p + \eta)} \sum_a p_a^2, \quad \bar{j} = -\frac{1}{2(p + \eta)} \sum_a \bar{p}_a^2,
\] (4.65)

and the level matching condition $j - \bar{j} \in \mathbb{Z}$ amounts to
\[
\sum_a n_a w^a \in (p + \eta)\mathbb{Z}.
\] (4.66)

The general physical states are obtained by making the DDF operators (4.40) and (4.48) and supercharges $Q^{\pm \tau a}$ act on the above states (4.63). The condition for the almost BPS states is again given by
\[
|\mathcal{J} + \bar{\mathcal{J}}| \ll k,
\] (4.67)
and the level matching condition is

\[ J - \bar{J} \in \mathbb{Z} \, . \quad (4.68) \]

We have again a huge number of stringy states for sufficiently large \( k \).

Now we obtain the complete spectrum generated by the DDF operators, thus we can compare the spectrum with the one from the light-cone analysis \((3.63)\). In our case, the spectrum of light-cone energies is given by

\[ H_{\text{l.c.}} = -\frac{1}{2}(J + \bar{J}) \]

\[ = \frac{1}{2\eta}(N + \bar{N}) + \frac{1}{2}(J + \bar{J}) + \frac{1}{4\eta} \left( \sum_a p_a^2 + \sum_a \bar{p}_a^2 \right) \, , \quad (4.69) \]

where \( N \) and \( \bar{N} \) are the mode counting operators and \( J \) and \( \bar{J} \) are the “angular momentum operators”. These operators act on the DDF operators as

\[ [N, O_n] = -n O_n \, , \quad (O_n = P_{i,n}, P_{i,n}^*, Q_n^{++}, Q_n^{--}, A_n^{a\bar{a}}, B_n^{a\bar{a}}) \, , \]

\[ [J, O_n] = -O_n \, , \quad (O_n = P_{i,n}, Q_n^{++}) \, , \]

\[ [J, O_n] = O_n \, , \quad (O_n = P_{i,n}^*, Q_n^{--}) \, , \]

\[ [J, O_n] = 0 \, , \quad (O_n = A_n^{a\bar{a}}, B_n^{a\bar{a}}) \, . \quad (4.70) \]

The level matching condition is expressed as \( J - \bar{J} \equiv h \in \mathbb{Z} \), which leads to the conditions \( N - \bar{N} = 0 \) and \( J - \bar{J} \equiv h \in \mathbb{Z} \) for generic value of \( \eta \).

In order to compare the spectrum \((3.63)\), we identify \( N \) and \( \bar{N} \) as \( \hat{N}_L \) and \( \hat{N}_R \) and \( J \) and \( \bar{J} \) as \( \hat{J}_L \) and \( \hat{J}_R \) without zero mode shifts. The zero mode shifts are included in the definition of the “Fock vacua” \((4.58)\). We also identify \( f = 1/2, \eta = 2\alpha'p_v \) and \( p_A^2 = \alpha'p_a^2 + \alpha'\bar{p}_a^2 \). Precisely speaking, the fermions used in subsection \(3.3\) is the ones in the light-cone RNS formalism, on the other hand, the fermions used in this section corresponding to the ones in the light-cone Green-Schwarz formalism. We can exchange these two notations of fermions by using the triality symmetry on \( SO(8) \) Lie algebra or the bosonization of fermions. Using these relations, we can show that the spectrum \((4.69)\) is consistent with the result given in \((3.63)\).

\[ ^{14}\text{We can extend this correspondence to the case of general } p \text{ by relabeling the modes as we mentioned just below } (3.50). \]
To conclude the analysis of Hilbert space of superstrings on the NSNS pp-wave, we should also comment on physical states in the sectors of spectrally flowed Type I representations. The analysis is quite simple because it can be described by the usual free fields without twist operators. As we have already mentioned, it is plausible to suppose that they correspond to long strings in string theory on $AdS_3$ \cite{63, 64, 26}. More precisely, the strings in these sectors possess a continuous spectrum of the light-cone energies and can freely propagate along the transverse plane. The corresponding excitations do not seem to exist in the symmetric orbifold theory, which will be analyzed in the next section, because it only includes the discrete spectrum. The D1/D5 system (and also the F1/NS5 system) has the singularity related to the moduli point where the D1-branes are emitted from the D5-branes. The existence of the long strings are related to this singularity \cite{64}, however the symmetric orbifold theory does not have the singularity. Therefore we expect that we obtain the corresponding excitations by correctly deforming the symmetric orbifold theory. More detailed discussions will be given in subsection 5.3.
5 Comparison with SCFT on $Sym^M(T^4)$

Superstring theory on $AdS_3 \times S^3 \times M^4$ has been proposed to be dual to $\mathcal{N} = (4,4)$ non-linear sigma model on the symmetric orbifold space $Sym^M(T^4) \equiv (T^4)^M/S_M$ [3,13]. In the case of F1/NS5 system with $Q_1$ fundamental strings and $Q_5$ NS5-branes, $M$ is given by $M = Q_1 Q_5$. By taking the near horizon limit of F1/NS5 system, we can identify $Q_5$ as the level $k$ of $SL(2; \mathbb{R})$ and $SU(2)$ WZW models as we mentioned above. Since the sum of the winding numbers of world-sheet is $Q_1$ (see, e.g., [17]), the number $Q_1$ should be the implicit upper bound of winding number $p$, which corresponds to the spectral flow index. In our analysis, we can use any $p$ since we take $Q_1 \to \infty$ limit. In the next subsection, we review non-linear sigma model on the symmetric orbifold $Sym^M(T^4)$. In subsection 5.2 we analyze the spectrum of BPS and almost BPS states with large R-charges and compare the spectrum of short string sectors with positive energies. We have found many missing states in the superstring side and we discuss this point by comparing the case with RR-flux in subsection 5.3.

5.1 Review of SCFT on $Sym^M(T^4)$

$\mathcal{N} = (4,4)$ superconformal field theory defined by supersymmetric sigma model on the symmetric orbifold $Sym^M(T^4) \equiv (T^4)^M/S_M$ is described as follows. We use $4M$ free bosons $X^{\dot{a} a}_A (A = 0, 1, 2, \ldots, M-1, a, \dot{a} = \pm)$ and free fermions $\Psi^{\dot{a} a}_A$ and $\bar{\Psi}^{\dot{a} a}_A$ as fundamental fields. The superconformal symmetry is realized by the following currents (where we only write the left-mover)\(^{15}\)

\[
T(z) = -\frac{1}{2} \sum_A \epsilon_{ab} \epsilon_{\dot{a} \dot{b}} \partial X^{\dot{a} a}_A \partial X^{\dot{b} b}_A - \frac{1}{2} \sum_A \epsilon_{\alpha \beta} \epsilon_{\dot{a} \dot{b}} \Psi^{\dot{a} a}_A \partial \Psi^{\dot{b} b}_A ,
\]

\[
G^{\alpha a}(z) = i \sum_A \epsilon_{\dot{a} \dot{b}} \Psi^{\dot{a} a}_A \partial X^{\dot{b} b}_A ,
\]

\[
K^{\alpha \beta}(z) = -\frac{1}{2} \sum_A \epsilon_{\dot{a} \dot{b}} \Psi^{\dot{a} a}_A \Psi^{\dot{b} b}_A .
\]

\(^{15}\)We should emphasize that $(z, \bar{z})$ are the coordinates of space-time superconformal field theory and not related to that of the string world-sheet used in the previous sections.
In our convention, we set $\epsilon^{+-} = \epsilon^{-+} = 1$ and $\epsilon^{-+} = \epsilon^{+ -} = -1$ and the OPEs of free fields are written as

$$X^{ab}_{(A)}(z)X^{bc}_{(B)}(0) \sim -\delta_{AB} \epsilon^{ab} \ln z, \quad \Psi^{a\dot{a}}(z)\Psi^{b\dot{b}}(0) \sim \delta_{AB} \epsilon^{a\dot{a}}\epsilon^{b\dot{b}}. \quad (5.2)$$

The usual convention of $SU(2)$ current is given by

$$K^3 = K^{++} = K^{--}, \quad K^+ = K^{++}, \quad K^- = -K^{--}. \quad (5.3)$$

These currents generate $N = 4$ (small) superconformal algebra (SCA) with central charge $c = 6M$ and their OPEs are

$$T(z)T(w) \sim \frac{3M}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w},$$

$$T(z)K^{\alpha\beta}(w) \sim \frac{K^{\alpha\beta}(w)}{(z-w)^2} + \frac{\partial K^{\alpha\beta}(w)}{z-w},$$

$$K^{\alpha\beta}(z)K^{\gamma\delta}(w) \sim -\frac{M(\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} + \epsilon^{\alpha\delta}\epsilon^{\beta\gamma})}{2(z-w)^2} - \frac{\epsilon^{\alpha\gamma}K^{\beta\delta}(w) + \epsilon^{\alpha\delta}K^{\beta\gamma}(w) + \epsilon^{\beta\gamma}K^{\alpha\delta}(w) + \epsilon^{\beta\delta}K^{\alpha\gamma}(w)}{2(z-w)},$$

$$T(z)G^{\alpha a}(w) \sim \frac{3G^{\alpha a}(w)}{2(z-w)^2} + \frac{\partial G^{\alpha a}(w)}{z-w},$$

$$G^{\alpha a}(z)K^{\beta\gamma}(w) \sim -\frac{\epsilon^{\alpha\gamma}G^{\beta a}(w)}{2(z-w)},$$

$$G^{\alpha a}(z)G^{\beta b}(w) \sim \epsilon^{ab}\left(\frac{2M\epsilon^{\alpha\beta}}{(z-w)^3} + \frac{2K^{\alpha\beta}(w)}{(z-w)^2} + \frac{\epsilon^{\alpha\beta}T(w) + \partial K^{\alpha\beta}(w)}{z-w}\right). \quad (5.4)$$

The subalgebra of “zero-modes” $\{L_{\pm 1}, L_0, G^{\alpha a}_{\pm 1/2}, K^{\alpha\beta}_0\}$ and the counterpart of the right-movers compose super Lie algebra $PSU(1,1|2)_L \times PSU(1,1|2)_R$. This subalgebra corresponds to the supersymmetric algebra on $AdS_3 \times S^3$ geometry discussed in appendix [B].

According to the general approach to orbifold conformal field theories [18, 19, 20], we have various twisted sectors. The Hilbert space of each twisted sector is defined with the following boundary condition ($z \equiv e^{\tau + i\sigma}$) as

$$\Phi_{(A)}(\tau, \sigma + 2\pi) = \Phi_{g(A)}(\tau, \sigma), \quad (5.5)$$

47
Figure 5: The configuration corresponding to a twisted sector of the symmetric orbifold theory. There are connected world-sheets with length $2\pi n_i$ in the $\mathbb{Z}_{n_i}$-twisted sector. In the notation in this figure, $\sum_{i=1}^{k} n_i = M$ and $n_i$ can take the same value as $n_j$ for $i \neq j$.

where $\Phi_{(A)}(\tau, \sigma)$ represents $X^{\alpha\dot{\alpha}}_{(A)}$, $\Psi^{\alpha\dot{\alpha}}_{(A)}$ and $\bar{\Psi}^{\dot{\alpha}\dot{\beta}}_{(A)}$. The Hilbert space of the symmetric orbifold theory can be decomposed as (see, for example, [65])

$$\mathcal{H}(Sym^M(T^4)) = \bigoplus_{\gamma} \mathcal{H}^{C_{\gamma}}_{\gamma},$$

(5.6)

where we denote $\gamma$ as a conjugacy class, which labels a twisted sector. The Hilbert space of the twisted sector has to be invariant under centralizer subgroup $C_{\gamma}$ whose element $g \in S_M$ satisfies $ghg^{-1} = h$ for $h \in \gamma$.

The conjugacy class of symmetric permutation can be written as the form

$$\gamma = (1)^{N_1}(2)^{N_2} \cdots (l)^{N_l},$$

(5.7)

with $\sum_{n} nN_n = M$. Each $(n)$ denotes the cyclic permutation of $n$ elements and $N_n$ denotes the multiplicity of the cycle $(n)$. The centralizer group $C_{\gamma}$ can be given by the permutation of the $N_n$ cycles $(n)$ and the rotation of the cycle $(n)$. The Hilbert space of the twisted sector is then given by

$$\mathcal{H}^{C_{\gamma}}_{\gamma} = \bigotimes_{n > 0} S^{N_n} \mathcal{H}_{(n)}^{\mathbb{Z}_n},$$

(5.8)

where we denote $S^{N_n} \mathcal{H}$ as the (graded) symmetrization of tensor products of $N_n$ Hilbert spaces $\mathcal{H}$. Because of the twisted boundary condition (5.3), there are connected world-sheets with length $2\pi n$ in the Hilbert space $\mathcal{H}_{(n)}^{\mathbb{Z}_n}$ (see figure 5). The states in the Hilbert space $\mathcal{H}_{(n)}^{\mathbb{Z}_n}$ are invariant under the $\mathbb{Z}_n$ action and hence the sector of these states is called as $\mathbb{Z}_n$-twisted sector. States in the $\mathbb{Z}_n$-twisted sector correspond to single-particle states and states in the total Hilbert space generically correspond to multi-particle states. In
order to compare with the physical spectrum of first quantized superstring theory, it is
enough to focus on the single-particle Hilbert space. Hence, we shall concentrate on the
$\mathbb{Z}_N$-twisted sector ($N \leq M$) from now on. More precisely speaking, we consider the
conjugacy class
$$\gamma = (1)^{M-N}(N) \ ,$$
and only employ identity states in the Hilbert space $S^{N_{M-N}}\mathcal{H}_{(1)}$.

We label the objects in the $\mathbb{Z}_N$-twisted sector by the index $[A_0, A_1, \cdots, A_{N-1}] = [0, 1, \cdots, N-1]$. The coordinates of this sector are defined on the world-sheet $0 \leq \sigma \leq 2\pi$, which is rescaled from the $N$-times one $0 \leq \sigma \leq 2\pi N$. These coordinates are given by
$$\Phi(\tau, \sigma) = \Phi(n)(\tau, N\sigma - 2\pi n) \quad \text{for} \quad \frac{2\pi n}{N} \leq \sigma \leq \frac{2\pi(n+1)}{N},$$
with $n = 0, 1, \ldots, N-1$. These variables $X^{a\dot{a}}, \Psi^{\dot{a}\dot{a}}$ and $\bar{\Psi}^{\dot{a}\dot{a}}$ can be used to construct $\mathcal{N} = 4$ superconformal currents $\{L_n, G^a_{\alpha\dot{a}}, \hat{K}^{\alpha\beta}_n\}$ with central charge $c = 6$ in the manner similar to (5.1). The $\mathbb{Z}_n$ action to the Hilbert space $\mathcal{H}_{(N)}^{\mathbb{Z}_N}$ is given by the action of $\Phi(n) \rightarrow \Phi(n+1)$. The invariance under this action amounts to imposing
$$L_0 - \bar{L}_0 \in N\mathbb{Z} \ ,$$
on the physical Hilbert space.

The superconformal currents compatible with the condition (5.11), which can act on the physical Hilbert space independently of the right-movers, consist of only the modes of $n \in N\mathbb{Z}$. More precisely, the superconformal currents properly describing the $\mathbb{Z}_N$-twisted sector $\{\hat{L}_n, \hat{G}^a_{\alpha\dot{a}}, \hat{K}^{n\alpha\beta}_n\}$ should be defined as follows [66, 67] (from now on, we shall only present the NS sector)
$$\hat{L}_n = \frac{1}{N}L_{nN} + \frac{N^2 - 1}{4N}\delta_{n0} \ ,$$
$$\hat{G}^a_{\alpha\dot{a}} = \begin{cases} 
\frac{1}{\sqrt{N}}G^{a\alpha}_{N\tau}(NS), & (N = 2q + 1) \\
\frac{1}{\sqrt{N}}G^{a\alpha}_{N\tau}(R), & (N = 2q) 
\end{cases} \ ,$$
$$\hat{K}^{n\alpha\beta}_n = K^{\alpha\beta}_{nN} \ .$$
(5.12)

One can check directly that these operators generate $\mathcal{N} = 4$ superconformal algebra with $c = 6N$. The anomaly term in the expression of $\hat{L}_n$ corresponds essentially to the
Schwarzian derivative of the conformal mapping \( z \rightarrow z^N \). We should note that the modes of hatted current are counted by \( \hat{L}_0 \). From now on, we also count the modes of objects in this sector by \( \hat{L}_0 \) and hence the fractional modes are allowed.

From these definitions \((5.12)\), we can find that the vacuum \( |0; N\rangle \) of \( \mathbb{Z}_N \)-twisted sector possesses the following properties as

- \( N = 2q + 1 \)
  \[
  \hat{L}_0 |0; N\rangle = \frac{N^2 - 1}{4N} |0; N\rangle ,
  \]
  \[
  \hat{K}^3_0 |0; N\rangle = 0 .
  \] (5.13)

- \( N = 2q \)
  \[
  \hat{L}_0 |0; N\rangle = \left( \frac{N^2 - 1}{4N} + \frac{1}{4N} \right) |0; N\rangle \equiv \frac{N}{4} |0; N\rangle ,
  \]
  \[
  \hat{K}^3_0 |0; N\rangle = -\frac{1}{2} |0; N\rangle .
  \] (5.14)

When \( N \) is even, the supercurrent \( \hat{G}_r^{\alpha a} \) in the NS sector is made of the one in the R sector before imposing the \( \mathbb{Z}_N \)-invariance. The extra vacuum energy \( "\frac{1}{4N}" \) and the extra R-charge \( "-\frac{1}{2}" \) in the case of \( N = 2q \) originate from this fact.

Now, we focus on BPS states (chiral primary states) in the \( \mathbb{Z}_N \)-twisted sector, which are defined by the next conditions

\[
\hat{L}_n |\alpha\rangle = \bar{\hat{L}}_n |\alpha\rangle = 0 , \ (\forall n \geq 1) ,
\]

\[
\hat{G}_r^{+a} |\alpha\rangle = \bar{\hat{G}}_r^{+a} |\alpha\rangle = 0 , \ (\forall r \geq -\frac{1}{2}) ,
\]

\[
\hat{G}_r^{-a} |\alpha\rangle = \bar{\hat{G}}_r^{-a} |\alpha\rangle = 0 , \ (\forall r \geq \frac{1}{2}) .
\] (5.15)

These conditions lead inevitably to

\[
(\hat{L}_0 - \hat{K}^3_0) |\alpha\rangle = 0 .
\] (5.16)

At this point it is not difficult to present the explicit forms of all the possible BPS states in the \( \mathbb{Z}_N \)-twisted sector. (See, for example, \[69\].) They are written as

\[
|\omega^{(q, \bar{q})}; N\rangle = |\omega^q; N\rangle \otimes |\omega^{\bar{q}}; N\rangle , \ \forall \omega^{(q, \bar{q})} \in H^{q, \bar{q}}(T^4) , \ (q, \bar{q} = 0, 1, 2) ,
\] (5.17)

where the left(right)-moving parts are defined by
\( N = 2q + 1 \)

\[
|\omega^{0}; N\rangle = \prod_{i=0}^{q-1} \Psi_{-\frac{1}{2}}^{++}(\frac{i}{N}, \frac{N}{2}) \Psi_{-\frac{1}{2}}^{+-}(\frac{i}{N}, \frac{N}{2}) |0; N\rangle ,
\]

\[
|\omega^{1}; N\rangle = \Psi_{-\frac{1}{2}}^{+\dot{a}} |\omega^{0}; N\rangle ,
\]

\[
|\omega^{2}; N\rangle = \Psi_{-\frac{1}{2}}^{+\dot{a}} \Psi_{-\frac{1}{2}}^{-\dot{a}} |\omega^{0}; N\rangle .
\] (5.18)

\( N = 2q \)

\[
|\omega^{0}; N\rangle = \prod_{i=0}^{q-1} \Psi_{-\frac{1}{2}}^{++} \Psi_{-\frac{1}{2}}^{+-} |0; N\rangle ,
\]

\[
|\omega^{1}; N\rangle = \Psi_{-\frac{1}{2}}^{+\dot{a}} |\omega^{0}; N\rangle ,
\]

\[
|\omega^{2}; N\rangle = \Psi_{-\frac{1}{2}}^{+\dot{a}} \Psi_{-\frac{1}{2}}^{-\dot{a}} |\omega^{0}; N\rangle .
\] (5.19)

It is easy to check that

\[
\hat{\mathcal{L}}_{0} |\omega^{(q, \bar{q})}; N\rangle = \hat{K}_{0}^{3} |\omega^{(q, \bar{q})}; N\rangle = \frac{q + N - 1}{2} |\omega^{(q, \bar{q})}; N\rangle ,
\]

\[
\bar{\hat{\mathcal{L}}}_{0} |\omega^{(q, \bar{q})}; N\rangle = \bar{K}_{0}^{3} |\omega^{(q, \bar{q})}; N\rangle = \frac{\bar{q} + N - 1}{2} |\omega^{(q, \bar{q})}; N\rangle .
\] (5.20)

### 5.2 Comparison with Superstring Spectrum

We have obtained the spectrum of superstrings on pp-wave in section 4 and in the previous subsection we have reviewed the symmetric orbifold theory. Therefore, it is time to compare the both Hilbert spaces. In this subsection we compare the (almost) BPS states with large R-charges in the \( \mathbb{Z}_{N} \)-twisted sector with the states in superstring theory on the pp-wave. In the context of \( AdS_{3}/CFT_{2} \) correspondence, \( \hat{\mathcal{L}}_{0} \) and \( \hat{K}_{0}^{3} \) are identified with \(-J_{0}^{3}\) and \( K_{0}^{3}\), respectively, which are the zero-modes of the total currents in \( SL(2; \mathbb{R}) \) and \( SU(2) \) super WZW models. Recalling the relations (4.8), we must take the identification

\[
\mathcal{J} \leftrightarrow \hat{K}_{0}^{3} - \hat{\mathcal{L}}_{0} , \quad \mathcal{F} \leftrightarrow \frac{1}{k}(\hat{K}_{0}^{3} + \hat{\mathcal{L}}_{0}) .
\] (5.21)

The condition of almost BPS states are (4.52)

\[
\hat{\mathcal{L}}_{0} - \hat{K}_{0}^{3} \ll k , \quad \hat{\mathcal{L}}_{0} + \hat{K}_{0}^{3} \sim k .
\] (5.22)

51
Since the subalgebra of “zero-modes” \{\hat{L}_{\pm1}, \hat{L}_0, \hat{G}^{\alpha\alpha}_{\pm1/2}, \hat{K}_{0\beta}^{\alpha}\} corresponds to superalgebra on \textit{AdS}_3 \times S^3, we can define the Penrose limit on this subalgebra just as (B.5) and (B.8). In order to be consistent with the relation (5.21), it is natural to redefine as

\[ J' = \hat{K}^3_0 - \hat{L}_0, \quad F' = \frac{1}{k} (\hat{K}^3_0 + \hat{L}_0), \]

\[ \mathcal{P}_1' = -\frac{1}{\sqrt{k}} \hat{L}_1, \quad \mathcal{P}_1'' = -\frac{1}{\sqrt{k}} \hat{L}_1, \]

\[ \mathcal{P}_2' = \frac{1}{\sqrt{k}} \hat{K}^+_0, \quad \mathcal{P}_2'' = \frac{1}{\sqrt{k}} \hat{K}^-_0, \]

\[ Q^{++\pm'} = \pm \frac{1}{\sqrt{k}} \hat{G}^{++}_{1/2}, \quad Q^{--\pm'} = \pm \frac{1}{\sqrt{k}} \hat{G}^{--}_{-1/2}, \]

\[ Q^{-+\pm'} = \pm \hat{G}^{+-}_{-1/2}, \quad Q^{+-\pm'} = \pm \hat{G}^{-+}_{1/2}. \quad (5.23) \]

By taking \( k \to \infty \) limit, these operators satisfy the commutation relations of super pp-wave algebra (B.6), (B.9) and (B.10).

First, let us consider the BPS states. As we observed in (5.20), the BPS states in the \( \mathbb{Z}_N \)-twisted sector satisfy

\[ \hat{L}_0 - \hat{K}^3_0 = 0, \quad \hat{L}_0 + \hat{K}^3_0 \approx N. \quad (5.24) \]

The BPS states in the string theory side have the eigenvalues \( \mathcal{F} = p + \eta \) in the sector with fixed \( p, \eta \), the above relations and (5.21) imply the following identification

\[ \eta = \frac{l}{k}, \quad N = pk + l, \quad (5.25) \]

with \( l = 1, 2, \cdots, k - 1 \). Under this identification, we can show the correspondence between the BPS states (4.54) and (4.55) in the string theory side and the BPS states (5.18) or (5.19) in the dual CFT side.

At least with respect to the BPS states, we expect that the equal number of physical states exist in both of superstring theory on the \textit{AdS}_3 \times S^3 \times T^4 and the pp-wave background. This statement is valid as long as superstring theory on the pp-wave is defined

\[ ^{16}\text{The threshold values } l = 0 \text{ and } k \text{ cannot be included because there is a restriction } 0 < \eta < 1 \text{ in the type II representations. Moreover, the states with } l = k - 1 \text{ corresponds to the missing states in the original } \textit{AdS}_3 \times S^3 \text{ string theory for each } p. \text{ (See, for example, [69, 23].) One might worry that } \eta \text{ should be continuous in principle. These subtleties are, however, harmless for sufficiently large } k. \]
by the contraction \eqref{eq:contraction}. Let us fix the spectral flow number \( p \geq 1 \). It is known \cite{23,24} that there are about \( k \times \dim H^*(T^4) \) BPS states with the R-charges \( \frac{1}{2}kp \lesssim Q \lesssim \frac{1}{2}k(p+1) \) in superstring theory on \( AdS_3 \times S^3 \times T^4 \). This values of R-charges amounts to \( F = p + l/k \) (\( 0 \leq l \leq k \)) for each of the spectrally flowed sector. This is consistent with the assumption \( \eta = l/k \) \((l = 1, 2, \ldots, k-1)\) \eqref{eq:eta} and the following physical Hilbert space

\[
\mathcal{H}_{pp}\text{-wave}^{(p)} = \bigoplus_l \mathcal{H}_{pp}\text{-wave}(p, \eta = l/k) \tag{5.26}
\]

includes the equal number of BPS states as those of superstring theory on \( AdS_3 \times S^3 \times T^4 \).

Turning to the symmetric orbifold theory, we consider the following direct sum of the single-particle Hilbert spaces of the \( \mathbb{Z}_{N(l)} \)-twisted sectors, (where we set \( N(l) \equiv kp + l \) \((0 < l < k)\))

\[
\mathcal{H}_{symm}^{(p)} = \bigoplus_l \mathcal{H}_{symm}(N(l) = kp + l) .
\tag{5.27}
\]

It is obvious that \( \mathcal{H}_{symm}^{(p)} \) and \( \mathcal{H}_{pp}\text{-wave}^{(p)} \) have the equivalent spectrum of BPS states. We should also point out that the identification \( B_0^{\lambda^a} (\equiv \Psi_{-\frac{1}{2}}) = B_0^{\lambda^a} \) is consistent with the relation between \( (4.56) \) and \( (5.18), (5.19) \).

Next, let us consider the almost BPS states. For a moment, we again neglect the sectors with momenta along \( T^4 \) sector. The almost BPS states in the dual CFT side are obtained by acting the free oscillators \( \Psi_{\pm\frac{1}{2}} \) and \( i\partial X_{a\lambda}^{\alpha\dot{a}} \) on the BPS states under the condition \( (5.22) \). Since we have confirmed the correspondence of the BPS states, the task is to look for the operators corresponding to the DDF operators. For the DDF operators of \( T^4 \) sector, we can find the corresponding operators as

\[
B_n^{\pm\dot{a}^\prime} = \Psi_{\mp\frac{1}{2}}^{\pm\dot{a}^\prime} , \quad A_n^{\dot{a}^\prime} = \frac{1}{\sqrt{N}} i\partial X_{a\lambda}^{\alpha\dot{a}} ,
\tag{5.28}
\]

which satisfy the same (anti-)commutation relations as those of the DDF operators \( (4.49) \).

For the transverse coordinates of pp-wave background, we also construct the following

\footnote{Strictly speaking, the mode expansions of \( \Psi_{a\lambda}^{\alpha\dot{a}} \) and \( i\partial X_{a\lambda}^{\alpha\dot{a}} \) depend on whether \( N \) is even or odd. However, we can safely neglect the difference \( 1/2N \) under the assumption of large \( N \).}
operators

\[ \mathcal{P}_{1,n}' = -\frac{1}{\sqrt{N}} \left\{ \frac{N}{nk + l} \hat{L}_{nk+l} - \left( \frac{N}{nk + l} - 1 \right) \hat{K}_{nk+l}^3 \right\} , \]

\[ \mathcal{P}_{1,n}^* = \frac{1}{\sqrt{N}} \left\{ \frac{N}{nk - l} \hat{L}_{nk-l} - \left( \frac{N}{nk - l} + 1 \right) \hat{K}_{nk-l}^3 \right\} , \]

\[ \mathcal{P}_{2,n}' = \frac{1}{\sqrt{N}} \hat{K}_{nk+1}^3 , \quad \mathcal{P}_{2,n}^* = \frac{1}{\sqrt{N}} \hat{K}_{nk+1}^3 , \]

\[ Q_n^{++} = \pm \sqrt{\frac{N}{nk + l}} \hat{G}_{nk+l}^+ \hat{G}_{nk+l}^- \]

Under the identification of (5.25) and the large \( N \) limit, we can show that these operators satisfy the same (anti-)commutation relations as those of the corresponding DDF operators (4.42) and (4.49).

It is quite important to note that there are the equal number of degrees of freedom after taking such large \( N \) limit. The Hilbert space of \( \mathbb{Z}_N \)-twisted sector is spanned by the free oscillators \( \Psi_{nk+l}^+ \) and \( i\hat{\partial}X_{nk+l}^- \) and for each energy level there are the equal number of bosonic and fermionic oscillators as those defined in (5.28) and (5.29). Thus, the almost BPS states in the symmetric orbifold theory can be written in terms of the operators (5.28) and (5.29) as

\[ B_{n_1 m_1}^{+\pm} \cdots Q_{-m_1}^{++(l_i)} \cdots Q_{-1}^{--(l_j)} \cdots \otimes B_{\bar{n}_1 \bar{m}_1}^{+\pm} \cdots \bar{Q}_{-\bar{m}_1}^{++(\bar{l}_i)} \cdots \bar{Q}_{-\bar{1}}^{--(\bar{l}_j)} \cdots |\omega; N(l)\rangle , \]

\[ n_1, \bar{n}_1, m_i, \bar{m}_i > 0 , \quad k_i, \bar{k}_i \geq 0 \quad \forall \omega \in H^*(T^4) , \quad N(l) = kp + l , \]  

and the ones generated by acting (5.23). By considering the case with \( l_i = \bar{l}_i = l \), we obtain the states corresponding to the string states (4.59). The non-trivial consistency check is only the level matching condition. In the symmetric orbifold theory side, the level matching condition is given by

\[ \hat{L}_0 - \bar{\hat{L}}_0 \in \mathbb{Z} . \]  

This condition is consistent with the result of string theory side (4.61) under the above correspondence of DDF operators.

However, we should note here that the string Hilbert space \( \mathcal{H}_{(p)\text{pp-wave}} \) is strictly smaller than that of the symmetric orbifold theory;

\[ \mathcal{H}_{(p)\text{pp-wave}} \subsetneq \mathcal{H}_{(p)\text{symm}} . \]  

54
In fact, the general states (i.e., \( l_i \neq \bar{l}_i \neq l \)) have no counterparts in the string side since we cannot define the corresponding DDF operators as local operators on the string Hilbert space. These missing states in the string spectrum may be compensated by non-perturbative excitations. Because we are now assuming the small string coupling, the non-perturbative excitations usually become very massive. Under the assumption of large \( k \), the space of almost BPS states can include such very massive excitations in principle. However, our world-sheet analysis as a perturbative string theory cannot include such excitations.

Finally we consider the sectors with non-vanishing momenta along \( T^4 \). As in the spectrum of superstring theory, there are no BPS states in these sectors but there are many almost BPS states. Recall that \( J' = \tilde{K}_0^3 - \bar{L}_0 \) and \( \bar{L}_0 = \frac{k}{N} L_0 + \frac{N^2 - 1}{4N} \). The operator \( L_0 \) includes the contribution of momenta with the standard normalization. Hence, we find that the contributions of momenta to \( J' \) and \( \bar{J}' \) are given as follows (for the sector \( H_{\text{symm}}(N(\ell)) \) with \( N(\ell) \equiv kp + l \))

\[
\Delta J' = -\frac{1}{2N(\ell)} \sum_a \left( \frac{n'_a}{R_a} + \frac{w'^a R_a}{2} \right)^2, \quad \Delta \bar{J}' = -\frac{1}{2N(\ell)} \sum_a \left( \frac{n'_a}{R_a} - \frac{w'^a R_a}{2} \right)^2, \quad (5.33)
\]

where \( n'_a \) and \( w'^a \) are KK momenta and winding modes as before. The level matching condition for the vacuum state becomes

\[
\sum_a n'_a w'^a \in N(\ell) \mathbb{Z}. \quad (5.34)
\]

By comparing the analysis given in the last section (under the identification \( \eta = l/k \)), we obtain the following correspondence between the spectrum of superstring theory and that of the symmetric orbifold theory as

\[
n'_a = \sqrt{k} n_a, \quad w'^a = \sqrt{k} w^a, \quad (5.35)
\]

where \( n_a \) and \( w^a \) are KK momenta and winding modes \(^{18}\) in the string theory side.

We have observed that there are again many missing states in the string theory side. The essentially same aspect was already pointed out in the context of string theory on \( AdS_3 \times S^3 \times T^4 \) \(^{21}\). It may be worthwhile to comment on how such discrepancy is removed if assuming fractional string excitations. These excitations do not exist in the perturbative string spectrum and may be at least explained in the S-dual picture as discussed in section 3.1. The existence of \( k \equiv Q_5 \) NS5 leads to fractional strings with tension \( \tilde{T} = T/k \), where

\(^{18}\)We can treat \( \sqrt{k} \) as an integer number since we assume that \( k \) is very large.
$T$ represents the tension of fundamental string. If we measure the radii of $T^4$ by the unit of string length $l_s = 1/\sqrt{T}$, namely, $R_a = r_a l_s$, the momenta (4.62) becomes

$$p_a = \frac{1}{l_s} \left( \frac{n_a}{r_a} + w^a r_a \right), \quad \bar{p}_a = \frac{1}{l_s} \left( \frac{n_a}{r_a} - w^a r_a \right). \quad (5.36)$$

For the fractional strings, we should replace the string length $l_s$ with $\tilde{l}_s \equiv \sqrt{k} l_s$. Thus, we obtain

$$p_a = \frac{1}{\sqrt{k}} \times \frac{1}{l_s} \left( \frac{n_a}{r_a} + w^a r_a \right), \quad \bar{p}_a = \frac{1}{\sqrt{k}} \times \frac{1}{l_s} \left( \frac{n_a}{r_a} - w^a r_a \right), \quad (5.37)$$

and the extra factor $1/\sqrt{k}$ completely compensates the spectrum of missing states.

### 5.3 Comments on Case with RR-Flux

As we saw in the last subsection, there are many missing states in the string side. This is because the superstring theory is compared with the supersymmetric sigma model at the orbifold point, which is a different moduli point from the one of the dual CFT. The dual CFT can be obtained by deforming from the orbifold point and we expect that almost BPS states with large R-charge are not sensitive to the deformation. However, the dual CFT is at a singular moduli point, so this expectation is too naive. In fact, there are some missing states in the superstring side even for the BPS states and they are related to the divergence due to the singularity as we said before.

One way to remove this difference is to deform the superstring theory in order to include some non-perturbative excitations as mentioned before. The other way is to deform the symmetric orbifold theory by using the corresponding marginal operator. In our case, we do not know how to deform the theory from the orbifold limit, but in the case with RR-flux, the corresponding deformation is conjectured in [48] and the spectra of two theories are compared in [50] by making use of the deformation. In order to clarify the point, we study the correspondence between superstrings on 6 dimensional pp-waves with RR-flux and non-linear sigma model on a resolution of $Sym^{Q_1 Q_5}(T^4)$ [47, 48, 50] in this subsection.

The Penrose limit of $AdS_3 \times S^3$ with RR-flux is given by 6 dimensional pp-wave background with RR-flux. The metric of pp-wave is the same as that with NSNS-flux (3.10), but now NSNS-flux is replaced by RR-flux. Superstrings on the pp-wave with RR-flux can be quantized by Green-Schwarz formalism in the light-cone gauge [12, 44].
This is the same situation as superstrings on the maximally supersymmetric pp-wave with RR-flux and spectrum can be obtained in the way similar to \((2.27)\) as

\[
2p^- = -p_+ = H_{1,c.} = \sum_n N_n \sqrt{1 + \frac{n^2}{(\alpha'p^+)^2} + \frac{L_0^{a} + \bar{L}_0^{a}}{\alpha'p^+}}.
\] (5.38)

We have the relation between the light-cone momenta in the superstring theory and conformal weights \(\Delta\) and R-charges \(Q\) of operators in the space-time SCFT. Using that relation, the spectrum of the dual CFT is given by

\[
\Delta - Q = \sum_n N_n \sqrt{1 + \left(\frac{ng_sQ_5}{Q}\right)^2 + g_sQ_5 \left(\frac{L_0^{T^4} + \bar{L}_0^{T^4}}{Q}\right)},
\] (5.39)

where we use \(R^2 = \alpha'g_sQ_5\) \((5.3)\). Kaluza-Klein modes in \(T^4\) sector can be explained by assuming the fractional D-strings as before. For the other modes, it was conjectured in \([47, 48]\) that the general modes of four bosons and four fermions in the light-cone spectrum correspond to

\[
\hat{L}_{-1-\frac{a}{2}}, \quad \tilde{L}_{-1-\frac{a}{2}}, \quad \hat{K}_{-\frac{a}{2}}, \quad \tilde{K}_{-\frac{a}{2}},
\]

\[
\hat{G}_{-\frac{1}{2}+\frac{a}{2}}, \quad \tilde{G}_{-\frac{1}{2}+\frac{a}{2}}.
\] (5.40)

They reproduce the first order of the spectrum \((5.39)\). We should notice that these operators correspond to the states missing in our analysis since now we take the limit of \(n \ll Q\). For this reason, we cannot compare this result with the previous one.

In order to see the next order, we have to deform the theory from the orbifold limit. The moduli space of \(\mathcal{N} = (4, 4)\) supersymmetric sigma model on \(\text{Sym}^M(T^4)\) \((M = Q_1Q_5)\) is twenty dimensional. The sixteen moduli parameters are the metric \(G_{ij}\) and the antisymmetric tensor \(B_{ij}\) on \(T^4\). The other four moduli parameters are the blowing up modes which resolve singularities. The singularities appear at the fixed points since we divide \(M\) products of \(T^4\) by \(M\)-th symmetric group. Near the orbifold point, operators corresponding to the blowing up modes are given by

\[
m^{ab} = \sum_{0 \leq i \neq j \leq M-1} \sigma_{ij}^{\alpha\beta} G_{-\frac{1}{2}+\frac{\alpha}{2}}^\alpha \bar{G}_{-\frac{1}{2}+\frac{\beta}{2}}^\beta \epsilon_{\alpha\gamma} \epsilon_{\beta\delta},
\] (5.41)

where \(\sigma_{ij}^{\alpha\beta}\) is \(\mathbb{Z}_2\) twist operators acting on the covering space \((T^4)^M\). The \(\mathbb{Z}_2\) twist operators \(\sigma_{ij}^{\alpha\beta}\) exchange the \(i\)-th and \(j\)-th \(T^4\) and their \(SU(2)_R \times SU(2)_L\) R-charges are \((\alpha/2, \beta/2)\). The label \((a, b)\) is related to the global \(SU(2)_I\) symmetry and \(G_{-1/2}^\alpha\) and \(\bar{G}_{-1/2}^\beta\) are doublets under the symmetry. By taking the tensor products, we obtain \(m_{\text{triplet}}\) and \(m_{\text{singlet}}\). Here we choose \(m_{\text{singlet}}\) because \(m_{\text{singlet}}\) is related to the \(D1\) and \(D5\) charges and \(m_{\text{triplet}}\) is
related to the charges of $D3$-branes wrapped on some two-cycles of $T^4$. In summary, we use the following marginal deformation

$$\delta S = \lambda \int d^2 z \, m_{\text{singlet}}(z, \bar{z}) . \quad (5.42)$$

In [48] it was proposed that $\lambda \sim g_6$ and a heuristic explanation was given. In fact, it was shown in [50] that we can reproduce the first non-trivial order of the spectrum (5.39) by using the marginal deformation.
6 Superstring Theory on $H_6 \times T^4/\mathbb{Z}_2$ and SCFT on $Sym^M(T^4/\mathbb{Z}_2)$

In this section, we extend our previous analysis to the correspondence between superstring theory on $H_6 \times T^4/\mathbb{Z}_2$ background and superconformal field theory on the symmetric orbifold $Sym^M(T^4/\mathbb{Z}_2)$. The general K3 surface can be obtained by resolving the singularity of $T^4/\mathbb{Z}_2$ and we investigate the case of the orbifold $T^4/\mathbb{Z}_2$ as a solvable example of K3 surface. We again find the good correspondence for the (almost) BPS spectrum.

6.1 Spectrum of Superstring on $H_6 \times T^4/\mathbb{Z}_2$

In this subsection, we investigate the superstrings on $H_6 \times T^4/\mathbb{Z}_2$ background. The $\mathbb{Z}_2$-orbifold action acts on the coordinates of $T^4$ sector as

\[ \lambda^{a\dot{a}} \rightarrow - \lambda^{a\dot{a}}, \quad Y^{a\dot{a}} \rightarrow - Y^{a\dot{a}}. \]  

Moreover, we assume the action on the free bosons $H_3$ and $H_4$, which are the bosonizations of $\lambda^{a\dot{a}}$

\[ H_3 \rightarrow H_3 + \pi, \quad H_4 \rightarrow H_4 - \pi, \]  

so that the space-time supersymmetry is preserved. In fact, we can directly check that all the generators of super pp-wave algebra \{\mathcal{J}, \mathcal{F}, P_i, P_i^*, Q^{\alpha\beta a}\} defined in (4.37) and (4.38) are invariant under the $\mathbb{Z}_2$-orbifold action (6.1) and (6.2). As for the DDF operators (4.40) and (4.48), the orbifold action is given by

\[ A_{n}^{a\dot{a}} \rightarrow - A_{n}^{a\dot{a}}, \quad B_{n}^{a\dot{a}} \rightarrow - B_{n}^{a\dot{a}}, \]  

and the other DDF operators are invariant under this orbifold action.

Now, we can write down the spectrum of (almost) BPS states. First, let us consider untwisted sector and start with BPS states. All the task we have to do is to single out $\mathbb{Z}_2$-invariant BPS states. Only the NS-NS and R-R BPS states are left and the NS-R and R-NS BPS states are projected out. Thus, we obtain the 8 BPS states for each $p$ and $\eta$ and they are identified with the even cohomology of $T^4$. As for almost BPS states, we first consider the following Fock vacua as

\[ |i, \tilde{i}; j, \tilde{j}; \eta, p; n_a, w^a; (\pm)\rangle = |i, \tilde{i}; j, \tilde{j}; \eta, p; n_a, w^a\rangle \pm |i, \tilde{i}; j, \tilde{j}; \eta, p; -n_a, -w^a\rangle, \]  

59
where $|i, \tilde{i}; j, \tilde{j}; \eta, p; n, w^a\rangle$ represents physical state with non-vanishing momenta along $T^4$, which was defined in (4.63). NS-NS BPS states are realized as the special cases of such physical states as $|i, \tilde{i}; 0, 0; \eta, p; 0, 0; (+)\rangle$ and R-R BPS states can be obtained by multiplying $B^+_0$ and $\bar{B}^+_{0\dot{a}}$. General physical states are constructed by multiplying the DDF operators and supercharges $Q_{\pm\mp\dot{a}}$ as in the case of $T^4$. We only need the following additional constraint as

$$
\sum_{\alpha} \{B^\alpha_{-n}, A^{-\alpha}_{-m}\} + \sum_{\bar{\alpha}} \{\bar{B}^{\bar{\alpha}}_{-n}, \bar{A}^{-\bar{\alpha}}_{-m}\} = \text{even}, \text{ (for the Fock vacua } |i, \tilde{i}; \cdots; (+)\rangle\text{)} ,
$$

from the condition of $\mathbb{Z}_2$ invariance.

Next, we consider twisted sectors. There are 16 twisted sectors including stringy excitations around each fixed point of orbifold action. For each twisted sector, we need to consider the following boundary conditions as

$$
Y^{\dot{a}a}(e^{2\pi i}z) = -Y^{a\dot{a}}(z) ,
$$

$$
\lambda^{a\dot{a}}(e^{2\pi i}z) = -\lambda^{\dot{a}a}(z) , \text{ (for NS sector)} ,
$$

$$
\lambda^{\dot{a}a}(e^{2\pi i}z) = \lambda^{a\dot{a}}(z) , \text{ (for R sector)} ,
$$

and moreover,

$$
H_3(e^{2\pi i}z) = H_3(z) + \pi, \quad H_4(e^{2\pi i}z) = H_4(z) - \pi .
$$

BPS states are given as follows. In the NS vacua, there are both of bosonic and fermionic twist fields; $\sigma^b_{\mathbb{Z}_2}$, $\sigma^f_{\mathbb{Z}_2}$, whose conformal weights are equal to

$$
h(\sigma^b_{\mathbb{Z}_2}) = \bar{h}(\sigma^b_{\mathbb{Z}_2}) = 4 \times \frac{1}{16} = \frac{1}{4}, \quad h(\sigma^f_{\mathbb{Z}_2}) = \bar{h}(\sigma^f_{\mathbb{Z}_2}) = 4 \times \frac{1}{16} = \frac{1}{4} .
$$

Based on this fact we can observe that there are no BPS states in the NS-NS, NS-R and R-NS sectors. On the other hand, the R vacua can include only the bosonic twist field $\sigma^b_{\mathbb{Z}_2}$ and only the spin fields along the $H_6$ direction, which we express as $S^{a|e|\mathbb{Z}_2}$. This fact leads us to a unique R-R BPS state (per each twisted sector), which is explicitly written as

$$
S^{-++}_{-\frac{3}{4}+\eta} \bar{S}^{-++}_{-\frac{3}{4}+\eta} \sigma^b_{\mathbb{Z}_2}|0, \eta, p\rangle \otimes e^c c e^{-\frac{\phi}{2} - \frac{\bar{\phi}}{2}}|0\rangle_{gh} .
$$

(6.9)
In this way, we have found the 16 R-R BPS states in the twisted sectors for each $p$ and $\eta$. They correspond to the blow-up modes of $T^4/\mathbb{Z}_2$ orbifold and reproduce the cohomology ring of $K3$ together with the contributions from the untwisted sector.

The other physical states are also constructed straightforwardly. Contrary to the untwisted sector, the states with non-trivial momenta along $T^4$ are not allowed. Thus, we only have to consider the states created by the actions of DDF operators over the BPS states (6.9). The only non-trivial point is that the modes of DDF operators $B_{\alpha a}$ should be half integers $r \in \frac{1}{2} + \mathbb{Z}$ in this case. This fact originates from the boundary condition (6.7). We again need the constraint

$$\sharp\{B_{-r}, A_{-m}\} + \sharp\{B_{r}, \bar{A}_{-m}\} = \text{even},$$

(6.10)

to preserve the $\mathbb{Z}_2$-invariance.

### 6.2 Comparison with SCFT on $Sym^M(T^4/\mathbb{Z}_2)$

In the last subsection, we have obtained the spectrum of superstring theory. In this subsection, we study superconformal field theory on the symmetric orbifold $Sym^M(T^4/\mathbb{Z}_2)$ and compare with the spectrum of the superstring theory. The $\mathbb{Z}_2$-orbifoldization of non-linear sigma model on $Sym^M(T^4)$ is defined by the action

$$X^{\alpha a}_{(A)} \rightarrow -X^{\alpha a}_{(A)}, \quad \Psi^{\alpha \dot{a}}_{(A)} \rightarrow -\Psi^{\alpha \dot{a}}_{(A)}.$$  

(6.11)

This action preserves the $\mathcal{N} = (4,4)$ superconformal symmetry. We again study the single-particle Hilbert space of $\mathbb{Z}_N$-twisted sector with $N = pk + l$ ($l = 1, 2, \cdots, k - 1$) as in the previous section.

We first consider the spectrum of BPS states. In the untwisted sector of $\mathbb{Z}_2$ orbifoldization, only 8 $\mathbb{Z}_2$-invariant BPS states survive. They correspond to the even cohomology of $T^4$.

The analysis of twisted sectors is more complicated. Focusing on one of the twisted sectors corresponding to 16 fixed points, we can observe the following aspects:

- $N = 2q + 1$

We have the mode expansions $i\partial X^{\alpha \dot{a}}_{\frac{1}{N} + \frac{1}{N}}$ and $\Psi^{\alpha \dot{a}}_{\frac{n}{N}}$, ($n \in \mathbb{Z}$) and we obtain for the
\[\hat{L}_0 |0; N\rangle^{(t)} = \left( \frac{N^2 - 1}{4N} + \frac{1}{2N} \right) |0; N\rangle^{(t)} \equiv \frac{N^2 + 1}{4N} |0; N\rangle^{(t)},\]

\[\hat{K}_0^3 |0; N\rangle^{(t)} = -\frac{1}{2} |0; N\rangle^{(t)}. \quad (6.12)\]

- \(N = 2q\)

We have the mode expansions \(i\partial X^a_{\frac{a\bar{a}}{N} + \frac{1}{N}}\) and \(\Psi^a_{\frac{a\bar{a}}{N} + \frac{1}{N}}, (n \in \mathbb{Z})\) and we obtain for the NS vacuum \(|0; N\rangle^{(t)}\)

\[\hat{L}_0 |0; N\rangle^{(t)} = \left( \frac{N^2 - 1}{4N} + \frac{1}{4N} \right) |0; N\rangle^{(t)} \equiv \frac{N}{4} |0; N\rangle^{(t)},\]

\[\hat{K}_0^3 |0; N\rangle^{(t)} = 0. \quad (6.13)\]

In these expressions the extra zero-point energies and the R-charges assigned to the NS vacua are due to these twisted mode expansions. Based on these aspects, we can find out the following BPS state that is unique for each of the twisted sectors as

\[|\omega^{(1,1)}; N\rangle^{(t)} = |\omega^1; N\rangle^{(t)} \otimes |\omega^1; N\rangle^{(t)}, \quad (6.14)\]

\[|\omega^1; N\rangle^{(t)} = \prod_{i=0}^{q-1} \Psi^+_{\frac{i}{N}} \Psi^-_{\frac{i}{N}} |0; N\rangle^{(t)}, \quad (N = 2q + 1),\]

\[|\omega^1; N\rangle^{(t)} = \prod_{i=0}^{q-1} \Psi^+_{\frac{i}{N} - \frac{1}{2N}} \Psi^-_{\frac{i}{N} - \frac{1}{2N}} |0; N\rangle^{(t)}, \quad (N = 2q), \quad (6.15)\]

and we obtain

\[\hat{L}_0 |\omega^1; N\rangle^{(t)} = \hat{K}_0^3 |\omega^1; N\rangle^{(t)} = \frac{N}{2} |\omega^1; N\rangle^{(t)}. \quad (6.16)\]

In summary, we have obtained the \((8+16 = 24)\) BPS states, which precisely correspond to the cohomology of \(K3\) for each of the \(\mathbb{Z}_N\)-twisted sectors. They have (approximately) degenerate charges \(F = (\hat{K}_0^3 + \hat{L}_0)/k = p + l/k\) and \(J' = \hat{K}_0^3 - \hat{L}_0 = 0\). As in the case of \(T^4\), we have the good correspondence between the superstring theory and symmetric orbifold theory under the identifications \(N = kp + l\) and \(\eta = l/k \quad (5.25)\).

With respect to the almost BPS states, the discussion is almost parallel to the case of \(T^4\). We can explicitly write down the almost BPS states in the symmetric orbifold theory and compare them with the states in the superstring theory by using the identification.
of DDF operators as before. However, in this case we must identify the DDF operators $B^{\alpha \dot{a}}_{n+\frac{1}{2}}$ (where $n \in \mathbb{Z}$) in the twisted sectors with $\Psi^{\pm \dot{a}}_{\pm \frac{1}{2} + \frac{n}{2} + \frac{k}{N}}$ rather than $\Psi^{\pm \dot{a}}_{\pm \frac{1}{2} + \frac{k}{N}}$. (We again neglect the small difference of mode $1/2N$.) The short string spectrum is again completely embedded in the Hilbert space of symmetric orbifold theory and there are many missing states.
7 Conclusion

In this thesis we have investigated the “Penrose limit” of $AdS_3/CFT_2$ correspondence. The $AdS_3/CFT_2$ correspondence is the correspondence between superstring theory on $AdS_3 \times S^3 \times M^4$ ($M^4 = T^4$ or $T^4/Z_2$) and superconformal field theory on the symmetric orbifold $Sym^{Q_1,Q_2}(M^4)$. The Penrose limit of superstring theory on $AdS_3 \times S^3 \times M^4$ with NSNS-flux is given by superstring theory on NSNS pp-wave background. This theory can be described by a generalization of Nappi-Witten model [28] and it can be quantized by the sigma model approach [42, 40, 43] in the light-cone gauge and by the current algebra approach in the covariant gauge [29, 39, 41]. By using the free field realization in the current algebra approach, we have constructed the complete set of DDF operators. The spectrum of physical states is classified by short string sectors and long string sectors, as in superstring theory on $AdS_3$ [26].

In the dual CFT side, the Penrose limit corresponds to focusing on the subsector of almost BPS states with large R-charges. We have compared the general short string excitations with single-particle states in the dual CFT. We have shown that all the short string states are successfully embedded into the Hilbert space of symmetric orbifold theory. We have also found many missing states in the Hilbert space of superstring theory. We may interpret them as non-perturbative excitations or remove the extra spectrum in the dual CFT by correctly deforming from the orbifold point. In the case with RR-flux, spectra of the two theories are the same at only the leading order at the orbifold point [47, 48] as discussed in subsection 5.3. In that case, we can reproduce the spectrum at the first non-trivial order by using a marginal deformation [50]. We want to investigate on this point in detail in near future.

The subjects we have investigated are some specific aspects of $AdS/CFT$ duality. In general, a duality is very powerful tool since we may map from one theory with strong coupling constant to its dual theory with weak coupling constant. Although we can only deal with a perturbation theory, we obtain some information of a theory at non-perturbative region by using the duality. As for $AdS_5/CFT_4$ correspondence, the CFT side is at strong coupling region and the AdS side is at weak coupling region. Therefore, it is expected that we can calculate, for example, correlation functions by using the S-matrix of supergravity on $AdS_5$. In this thesis we have concentrated on the $AdS_3/CFT_2$ correspondence. This duality is related to the D1/D5 system and hence the black hole physics. Superstrings on the background including black holes are very difficult to deal
with since the coupling constant is very strong inside the black holes. In future we may describe superstrings on black hole solutions by using its dual CFT.

The duality is very strong tool but there was the limitation that we can treat only BPS quantities protected by supersymmetry. Recently, the authors of [12] showed that if the quantities are very near BPS, we can analyze beyond BPS quantities. We have followed their strategy and obtained some information about the correspondence of non BPS spectrum. The CFT side of $\text{AdS}_5/C\text{FT}_4$ correspondence is 4 dimensional $\mathcal{N} = 4$ super Yang-Mills theory and that of $\text{AdS}_3/C\text{FT}_2$ correspondence is 2 dimensional $\mathcal{N} = (4, 4)$ supersymmetric sigma model on $\text{Sym}^{Q_1 Q_2}(T^4)$. They are completely different, and hence our results are a non-trivial evidence of the PP-Wave/CFT correspondence.

There are a lot of works left to investigate more on the $\text{AdS}/\text{CFT}$ correspondence. We can interpret the $\text{AdS}/\text{CFT}$ correspondence as an example which realizes holographic principle [70, 71]. According to the holographic principle, degrees of freedom of gravity modes can be projected into a low dimensional screen. As mentioned in [21], we have a holographic map from states in supergravity on $\text{AdS}_5 \times S^5$ to operators in super Yang-Mills theory [53, 54]. It is know that it is difficult to interpret the PP-Wave/CFT correspondence in a holographic way\(^{19}\), so it is important to study on this aspect. The other important direction is to investigate the correspondence of more general spectrum. By studying this further, we may be able to complete the dictionary of $\text{AdS}/\text{CFT}$ correspondence.

The correspondence between superstring theory on pp-wave background and subsector of super Yang-Mills theory is very interesting since it provides an explicit example of the string/gauge theory duality. There have been subsequent works after [12] and they gave deep insights into both string theory side and gauge theory side. In particular, if we understand the duality between superstring theory and supersymmetric gauge theory more, we will obtain the information of superstring theory on non-trivial background which we do not know how to deal with (for example, a background including black hole solutions) by using a simpler supersymmetric gauge theory.

\(^{19}\)However, there are several works on this subject [72, 73, 74].
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A Gamma Matrices

The cocycle factors of spin fields are defined by using gamma matrices. In this thesis we use the following Gamma matrices

\[
\Gamma_{\pm 0} = \sigma_{\pm} \otimes 1 \otimes 1 \otimes 1 \otimes 1 ,
\]

\[
\Gamma_{\pm 1} = \sigma_3 \otimes \sigma_{\pm} \otimes 1 \otimes 1 \otimes 1 ,
\]

\[
\Gamma_{\pm 2} = \sigma_3 \otimes \sigma_3 \otimes \sigma_{\pm} \otimes 1 \otimes 1 ,
\]

\[
\Gamma_{\pm 3} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_{\pm} ,
\]

\[
\Gamma_{\pm 4} = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_{\pm} ,
\]

(A.1)

with Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(A.2)

and \( \sigma_{\pm} = \frac{1}{2} (\sigma_1 \pm i \sigma_2) \). Charge conjugation matrix can be written as

\[
C = \epsilon \otimes \sigma_1 \otimes \epsilon \otimes \sigma_1 \otimes \epsilon , \quad \epsilon = i \sigma_2 ,
\]

(A.3)

which has the property as

\[
C \Gamma_\mu C^{-1} = -(\Gamma_\mu)^T , \quad C^\dagger = C^{-1} .
\]

(A.4)

OPEs including spin fields are given by

\[
\psi^\mu(z) S^A(w) \sim \frac{1}{(z - w)^{\mp 1}} (\Gamma^\mu)^A_B S^B(w) ,
\]

\[
\psi^\mu \psi^\nu(z) S^A(w) \sim -\frac{1}{z - w} (\Gamma^{\mu \nu})^A_B S^B(w) ,
\]

\[
S^A(z) S^B(w) \sim \frac{1}{(z - w)^{\mp 1}} (\Gamma_\mu C)^{AB} \psi^\mu(w) .
\]

(A.5)
B Super PP-Wave Algebra from Super \( AdS_3 \times S^3 \) Algebra

In this appendix we examine space-time supersymmetry algebra on pp-wave background by contracting that of \( AdS_3 \times S^3 \). It is known that supersymmetry on \( AdS_3 \times S^3 \) is represented by super Lie group \( PSU(1,1|2) \times PSU(1,1|2) \). Its even part corresponds to the isometry of this background. The isometry of \( AdS_3 \) space is identified as \( SU(1,1) \times SU(1,1) \cong SO(2,2) \) and the isometry of \( S^3 \) is identified as \( SU(2) \times SU(2) \cong SO(4) \). We denote the generators of \( SU(1,1) \) Lie algebra as \( m^\alpha_\beta (\alpha, \beta = 1, 2) \) and the generators of \( SU(2) \) Lie algebra as \( m^i_j (i,j = \hat{1}, \hat{2}) \) by following the notation of \[75\]. (We concentrate on holomorphic sector and anti-holomorphic sector can be analyzed in the similar way.)

Their commutation relations are given by

\[
[m^\alpha_\beta, m^\gamma_\delta] = \delta^\gamma_\beta m^\alpha_\delta - \delta^\alpha_\delta m^\gamma_\beta , \quad [m^i_j, m^k_n] = \delta^k_j m^i_n - \delta^i_n m^k_j , \quad (B.1)
\]

and Hermitian conjugations are defined as

\[
(m^1_1)^\dagger = m^1_1 \quad , \quad (m^2_1)^\dagger = m^1_2 \quad , \quad (m^1_2)^\dagger = m^2_1 , \quad (m^2_1)^\dagger = -m^1_2 \quad , \quad (m^1_2)^\dagger = -m^2_1 . \quad (B.2)
\]

The generators of odd sector \( q^i_\alpha \) and \( q^i_\alpha \) correspond to \( 8(+8) \) supercharges and their commutation relations are given by

\[
[m^\alpha_\beta, q^\gamma_k] = -\delta^\gamma_\beta q^\alpha_k + \frac{1}{2} \delta^\alpha_\beta q^\gamma_k \quad , \quad [m^i_j, q^\gamma_k] = \delta^\gamma_j q^\alpha_i - \frac{1}{2} \delta^i_j q^\gamma_k ,
\]

\[
[m^i_j, q^\alpha_k] = -\delta^i_k q^\alpha_j + \frac{1}{2} \delta^i_j q^\alpha_k \quad , \quad [m^\alpha_\beta, q^\gamma_k] = \delta^\gamma_k q^\alpha_\beta - \frac{1}{2} \delta^\alpha_\beta q^\gamma_k ,
\]

\[
\{q^i_\alpha, q^j_\beta\} = i(\delta^i_j m^\alpha_\alpha + \delta^\alpha_\alpha m^i_j) . \quad (B.3)
\]

Hermitian conjugations are defined as

\[
(q^1_1)^\dagger = -iq^1_1 \quad , \quad (q^2_2)^\dagger = -iq^2_2 \quad , \quad (q^1_2)^\dagger = iq^1_2 \quad , \quad (q^2_1)^\dagger = iq^2_1 ,
\]

\[
(q^1_1)^\dagger = iq^1_1 \quad , \quad (q^2_2)^\dagger = iq^2_2 \quad , \quad (q^1_2)^\dagger = -iq^1_2 \quad , \quad (q^2_1)^\dagger = -iq^2_1 . \quad (B.4)
\]
Now, we take a light-cone basis and a Penrose limit. First, we consider the even sector. We redefine the generators as

\[
J = -m_1^1 + m_1^1 , \quad F = -\frac{1}{R} \left( m_1^1 + m_1^1 \right) ,
\]

\[
P_1 = \frac{i}{\sqrt{R}} q_1^1 , \quad P_1^* = -\frac{i}{\sqrt{R}} m_2^1 ,
\]

\[
P_2 = -\frac{i}{\sqrt{R}} m_2^1 , \quad P_2^* = -\frac{i}{\sqrt{R}} m_2^1 , \quad (B.5)
\]

and take the limit of \( R \to \infty \). Then we obtain the commutation relations as

\[
[J, P_i] = P_i , \quad [J, P_i^*] = -P_i^* , \quad [P_i, P_j] = \delta_{ij} F , \quad (B.6)
\]

which is the same as the ones of \( H_6 \) Lie algebra. Hermitian conjugations are given by

\[
(J)^\dagger = J , \quad (F)^\dagger = F , \quad (P_i)^\dagger = P_i^* , \quad (P_i^*)^\dagger = P_i . \quad (B.7)
\]

The analysis of odd part can be done just like the even part. We redefine the generators in the odd sector as

\[
Q^{--} = -\frac{i}{\sqrt{R}} q_1^1 , \quad Q^{++} = -\frac{i}{\sqrt{R}} q_2^1 ,
\]

\[
Q^{++} = -\frac{1}{\sqrt{R}} q_1^1 , \quad Q^{--} = -\frac{1}{\sqrt{R}} q_2^1 ,
\]

\[
Q^{+-} = q_2^1 , \quad Q^{+--} = q_1^1 ,
\]

\[
Q^{+-} = iq_2^1 , \quad Q^{+--} = iq_1^1 , \quad (B.8)
\]

and take the limit of \( R \to \infty \). Their commutation relations with the generators in the even sector becomes

\[
[J, Q^{++}] = Q^{++} , \quad [J, Q^{--}] = -Q^{--} ,
\]

\[
[P_1, Q^{++}] = -Q^{++} , \quad [P_1^*, Q^{--}] = Q^{--} ,
\]

\[
[P_2, Q^{++}] = -Q^{++} , \quad [P_2^*, Q^{--}] = -Q^{--} , \quad (B.9)
\]
and the other commutation relations vanish. The anti-commutation relations of the generators in the odd sector are obtained as

\[
\{Q^{++a}, Q^{--b}\} = \epsilon^{ab} F, \quad \{Q^{+-a}, Q^{-+b}\} = \epsilon^{ab} J,
\]

\[
\{Q^{++a}, Q^{+-b}\} = \epsilon^{ab} P_1, \quad \{Q^{+-a}, Q^{-+b}\} = \epsilon^{ab} P_1^*,
\]

\[
\{Q^{--a}, Q^{++b}\} = \epsilon^{ab} P_2, \quad \{Q^{++a}, Q^{--b}\} = \epsilon^{ab} P_2^*,
\]

and Hermitian conjugations are given by \((Q^e, \epsilon_1, \epsilon_2, \epsilon_3) \dagger = Q^{-e_1, -e_2, -e_3}\). The authors of [76, 77] obtained superalgebras on the pp-wave backgrounds by contracting superalgebras on 

\[AdS_5 \times S^5, AdS_4 \times S^7\] and \[AdS_7 \times S^4\]. Our result is the counterpart of the case of \(AdS_3 \times S^3\)

and a natural supersymmetric extension of \(H_6\) Lie algebra is obtained.
C Supersymmetries on NSNS PP-Waves Based on Killing Spinors

We consider Killing spinors in Type IIB supergravity on pp-wave backgrounds with NSNS-flux\(^{20}\). RR-fields are not considered since we only introduce non-trivial NSNS-flux. The relevant part of Type IIB supergravity action is given by

\[
S = \frac{1}{2} \int d^{10}x \sqrt{-g} e^{-2\phi} (-R + 4(\partial \phi)^2 - \frac{1}{3} H^2) ,
\]

where \(\phi\) is dilaton and \(H\) is field strength \(H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}\). We only consider backgrounds with constant dilaton \(\phi\). Unbroken supersymmetry can be seen from the bosonic part of supersymmetry transformation of fermions, which are given by

\[
\delta \psi_\mu = (\partial_\mu - \frac{1}{4} \Omega_{+\mu\hat{\nu}\hat{\rho}} \Gamma_{\hat{\nu}\hat{\rho}}) \epsilon ,
\]

\[
\delta \lambda = (\Gamma^\mu \partial_\mu \phi - \frac{1}{6} H_{\mu\nu\rho} \Gamma^{\mu\nu\rho}) \epsilon ,
\]

where \(\Omega\) are spin connections with torsion

\[
\Omega_{+\mu\hat{\nu}\hat{\rho}} = \omega_{\mu\hat{\nu}\hat{\rho}} + H_{\mu\hat{\nu}\hat{\rho}} .
\]

Here \(\mu, \nu, \rho, \cdots\) are space-time indices and hatted ones represent the indices of tangent space. Killing spinor conditions are given by the equations that the right hand sides of \(\text{(C.2)}\) are zero.

From now on we concentrate on a specific configuration. Here we use the model of [44] and its bosonic part is given by

\[
L = \partial u \partial \bar{v} + \mathcal{F}_{ij} x^i \partial u \partial \bar{x}^j + \partial x^i \partial \bar{x}^i ,
\]

where \(\mathcal{F}_{ij}\) is a constant and \(i = 1, \cdots, 8\). The nontrivial components of \(B\)-field are \(B_{u j} = \frac{1}{2} \mathcal{F}_{ij} x^i\) and the field strength is \(H_{u ij} = -\frac{1}{2} \mathcal{F}_{ij}\). The spin connections can be calculated straightforwardly [79, 44] and the nontrivial components are

\[
\Omega_{+u i j} = -\mathcal{F}_{ij} .
\]

Therefore, the Killing spinor conditions become

\[
(\partial_u + \frac{1}{4} \mathcal{F}_{ij} \Gamma^{ij}) \epsilon = 0 , \quad \partial_v \epsilon = 0 , \quad \partial_i \epsilon = 0 ,
\]

\(^{20}\)We follow the notation of [78].
and
\[ \mathcal{F}_{ij} \Gamma^{ai} j \epsilon = 0 \, . \tag{C.7} \]

The condition (C.6) can be always satisfied by the following form of spinor as
\[ \epsilon(u) = \exp\left(-\frac{1}{4} \int^u \mathcal{F}_{ij} \Gamma^{ij}\right) \epsilon_0 \, . \tag{C.8} \]

with a constant spinor \( \epsilon_0 \). However, the condition (C.7) can not be always satisfied. In general, this condition breaks a half of supersymmetries by
\[ \Gamma^a \epsilon = 0 \, . \tag{C.9} \]

It is known that pp-wave backgrounds always preserve partial supersymmetries called as “kinematical” supersymmetries. However, there are special cases with enhanced supersymmetries called as “dynamical” supersymmetries. The well-known example is the Penrose limit of \( AdS_5 \times S^5 \) \[5, 6, 7\] and this background preserves 16 dynamical supersymmetries in addition to 16 kinematical supersymmetries. The relatively less-known examples are given in M-theory, Type IIA and Type IIB theory \[80, 81, 82, 83, 84, 85, 86\].

Let us see the example which is obtained by the Penrose limit of \( AdS_3 \times S^3 \times T^4 \). This case is given by \( \mathcal{F}_{ij} = 2 f \epsilon_{ij} (i, j = 1, 2) \) and \( \mathcal{F}_{kl} = 2 f \epsilon_{kl} (k, l = 3, 4) \). (See (3.34).) By using the notation of gamma matrices in Appendix A, the condition (C.7) implies
\[ \Gamma^{+0}(\Gamma^{+1} \Gamma^{-1} - \Gamma^{-2} \Gamma^{+2}) \epsilon = 0 \, . \tag{C.10} \]

Thus, besides the Killing spinors satisfying \( \Gamma^{+0} \epsilon = 0 \), there are 8 Killing spinors satisfying \( (\Gamma^{+1} \Gamma^{-1} - \Gamma^{-2} \Gamma^{+2}) \epsilon = 0 \). In the notation of supercharges in this thesis, the former Killing spinors correspond to 16 kinematical supercharges \(4.38, 4.48\)
\[ Q^{++a} \sim \oint S^{++a} e^{iX^+}, \quad Q^{--a} \sim \oint S^{--a} e^{-iX^+}, \]
\[ B_0^{+a} \sim \oint S^{+-(-a)a}, \quad B_0^{-a} \sim \oint S^{+-(-a)a}, \tag{C.11} \]

and the latter Killing spinors correspond to 8 dynamical supercharges \(4.38\)
\[ Q^{-+a} \sim \oint S^{-+a}, \quad Q^{+-a} \sim \oint S^{+-a} \, . \tag{C.12} \]

In appendix B, we have analyzed the supersymmetry on the pp-wave background from the contraction of super \( AdS_3 \times S^3 \) algebra and we have also found 8 dynamical supersymmetries \(C.12\), which is consistent with the analysis in this appendix. The dynamical
supersymmetries are very important for our analysis because they correspond to the linearly realized supersymmetries in the light-cone gauge. More precisely, the dynamical supercharges generate the super transformations which preserve the light-cone Hamiltonian.

In [49] we have proposed the other superstring vacua on NSNS pp-waves with enhanced supersymmetry. The superstring theories on these backgrounds are defined by the combination of 4 dimensional Nappi-Witten model and a general \( \mathcal{N} = 2 \) rational superconformal field theory (RCFT) with \( c = 9 \). The Nappi-Witten model and the RCFT are non-trivially related to each other due to the GSO projection. In these configurations, it seems difficult to analyze in the similar way, because there are no naive geometric interpretations. However, the existence of extra supercharges implies cancellation between NSNS-fluxes like (C.10). On the other hand, in the case of Penrose limit of the near horizon of NS5-branes [87], there is only one type of NSNS-flux and hence there is no cancellation as in (C.10). Therefore, the model admits only 16 kinematical supersymmetries and no dynamical supersymmetries.
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