Dark Energy and Hubble Constant
From the Latest SNe Ia, BAO and SGL

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ABSTRACT

Based on the latest MLCS17 SNe Ia data provided by Hicken et al. (2009),
together with the Baryon Acoustic Oscillation (BAO) and strong gravitational
lenses (SGL), we investigate the dark energy equation-of-state parameter for both
constant $w$ and time-varying $w(z) = w_0 + w_a z/(1 + z)$ in the flat universe, and
its correlation with the matter density $\Omega_M$ and Hubble constant $h$. The con-
straints from SNe data alone arrive at: (a) the best-fit results are $(\Omega_M, w, h) =$
$(0.358, -1.09, 0.647)$, while both $\Omega_M$ and $w$ are very sensitive to the difference
$\Delta h = h - \tilde{h}$ of the Hubble constant deviating to the prior input $\tilde{h} = 0.65$; (b) the
likelihoods of parameters are found to be: $w = -0.88^{+0.31}_{-0.39}$ and $\Omega_M = 0.36^{+0.09}_{-0.15}$,
which is consistent with the $\Lambda$CDM at 95% C.L.; (c) the two parameters in
the time-varying case are found to be $(w_0, w_a) = (-0.73^{+0.23}_{-0.97}, 0.84^{+1.66}_{-10.34})$ after
marginalizing other parameters; (d) there is a clear degeneracy between constant
$w$ and $\Omega_M$, which depresses the power of SNe Ia to constrain both of them;
(e) the likelihood of parameter $w_a$ has a high non-Gaussian distribution; (f)
an extra restriction on $\Omega_M$ is necessary to improve the constraint of the SNe
Ia data on $(w_0, w_a)$. A joint analysis of SNe Ia data and BAO is made to
break the degeneracy between $w$ and $\Omega_M$, and it provides a stringent constrain
with the likelihoods: $w = -0.88^{+0.07}_{-0.09}$ and $\Omega_M = 0.29^{+0.02}_{-0.03}$. For the time-varying
$w(z)$, it leads to the interesting maximum likelihoods $w_0 = -0.94$ and $w_a = 0$.
When marginalizing the parameters $\Omega_M$ and $h$, the fitting results are found to
be $(w_0, w_a) = (-0.95^{+0.45}_{-0.18}, 0.41^{+0.79}_{-0.96})$. After adding the splitting angle statistic
of SGL data, a consistent constraint is obtained $(\Omega_M, w) = (0.298, -0.907)$ and
$(w_0, w_a)$ is further improved to be $(w_0, w_a) = (-0.92^{+0.14}_{-0.10}, 0.35^{+0.47}_{-0.54})$, which indicates that the phantom type models are disfavored.

Keywords: dark energy, type Ia supernova, cosmological parameters
1. INTRODUCTION

Analysis of the distance modulus versus redshift relation of type Ia supernova (SNe Ia) provides a direct evidence that the universe expansion is accelerating in the last few billion years (e.g., Riess et al. 1998, 2004, 2007; Wood-Vasey et al. 2007; Kowalski et al. 2008; Hicken et al. 2009; Biswas and Wandelt 2009). This cosmic image is also supported by many other cosmological observations, like the Cosmic Microwave Background (CMB) (Hinshaw et al. 2009; Komatsu et al. 2009), the Baryon Acoustic Oscillation (BAO) measurement (York et al. 2000; Eisenstein et al. 2005) and the weak gravitational lenses (Weinberg and Kamionkowski 2002; Zhan and Knox 2006). Based on the Friedmann equation, the acceleration can be explained through introducing a negative pressure component in the universe, named dark energy, which is nearly spatially uniform distribution and contributes about 2/3 critical density of universe today. To reveal the property of dark energy, the most of studies, either theoretical models or experiment data analysis, are focused on its equation-of-state parameter $w = p/\rho$ (Albrecht et al. 2006). Here we shall utilize the latest SNe Ia data provided by Hicken et al. (2009) with using MLCS17 light curve fitter, together with the splitting angle statistic of strong gravitational lenses (SGL; Zhang et al. 2009) and the baryonic acoustic oscillations (BAO; Eisenstein et al. 2005) to investigate the constraint for the parameter $w$ of dark energy in the flat cosmology. The influences of Hubble constant $h$ and matter density $\Omega_M$ on the fitting results are carefully demonstrated.

By far, all observed data are consistent with the $\Lambda$CDM cosmology, with dark energy in the form of a cosmological constant $\Lambda$. However, this model raises theoretical problems related to the fine tuned value (see e.g. Padmanabhan 2003). Many other theoretical models, like quintessence models and phantom model (Ratra and Peebles 1988; Caldwell, Davé, and Steinhardt 1998; Caldwell 2002), reveal that dark energy might be a dynamical component and evolves with time. It is usual to parametrize dark energy as an ideal liquid with its equation-of-state (EOS) parameter $w(z) = w_0 + w_a z/(1 + z)$ (Chevallier et al. 2001; Linder 2003). Conveniently, it includes the case of a constant EOS with $(w_0 = w, w_a = 0)$, and the $\Lambda$CDM model $(w_0 = -1, w_a = 0)$. Then theoretical models can be classified in a phase diagram on the $(w_0, w_a)$ plane (see e.g. Barger et al. 2006; Biswas and Wandelt 2009). Thus the accurate measurement of the parameters $(w_0, w_a)$ is helpful for testing a certain theoretical models. The current allowed regions of $(w_0, w_a)$ given by the most observations or their combinations remain surrounding the crucial point $(w_0 = -1, w_a = 0)$, which is the common point in the phase diagram of different classified models. Therefore, the final judgment of models can not be made and more careful works are still needed.

In practice, the measurements of other cosmological parameters, like Hubble constant $h$ and matter density parameter $\Omega_M$, shall affect the determination of the EOS parameter $w$
of dark energy. Hubble constant \( h \) gives the rate of recession of distant galaxies and scales the expansion of the present universe. Parameter \( \Omega_M \) describes the present matter density relative to the critical density, which is essential to understand the structure formation. In the flat cosmology, \( \Omega_M \) is directly related to the present density of dark energy by \( \Omega_{DE} = 1 - \Omega_M \). The precise measurements on the two cosmological parameters are still processing. The results from WMAP five-year observations are \( h = 0.719 \pm 0.03 \) and \( \Omega = 0.258 \pm 0.03 \) in the \( \Lambda \)CDM cosmology, respectively(Hinshaw et al. 2009). The SNe Ia also have been used to determine the Hubble constant, calibrated with their peak luminosities or Cepheid variables in nearby galaxies(Schaefer 1996; Sandage, et al. 1996; Branch 1998; Gibson 2001; Sandage et al. 2006; Riess et al. 2009). Because the Hubble constant degenerates with the distance modulus of SNe Ia, the correct value of the absolute magnitude of the fiducial SN Ia or Hubble constant is not relevant for the dark energy analysis which only make use of differences between SNe Ia magnitudes(Riess et al. 2004). In this case, the value \( h = 0.65 \) is often used as a prior input(Hicken et al. 2009). However, we can still investigate the sensitivity of the analysis results on the change of the Hubble constant, which is not presented in the literatures.

The SNe Ia has homogeneity and extremely high intrinsic luminosity of peak magnitude and thus is widely used to measure the cosmological parameters(e.g. Riess et al. 2004; Wood-Vasey et al. 2007; Kowalski et al. 2008; Hicken et al. 2009). With given density of dark energy in the universe today \( \rho(z = 0) \), the change of equation-of-state parameter \( w \) will bring the change of its density \( \rho(z) \) at the redshift \( z \) and then the distance \( d(z) \). Inversely, measurement of the redshift \( z \) of SNe Ia and corresponding distance \( d(z) \) can constrain the dark energy. In spite of the high accuracy of SNe Ia measurement, its potential of constraining dark energy is not very strong due to the degeneracy of \( w \) and \( \Omega_M \). Therefore, a combining analysis with other observations is often useful.

We will also use the summary parameters of the baryonic acoustic oscillations as reported in previous studies(Eisenstein et al. 2005). The large-scale correlation function of a large sample of luminous red galaxies has been measured in the Sloan Digital Sky Survey and a well-detected acoustic peak was found to provide a standard ruler by which the absolute distance of \( z = 0.35 \) can be determined with 5% accuracy, independent on the Hubble constant \( h \). This ruler is a \( \Omega_M \) prior and can be used to constrain dark energy(e.g. Porciani and Madau 2000; Huterer and Ma 2004; Chae, 2007). We shall also use strong gravitational lensing statistic observation, which provides us a useful probe of dark energy of the universe. Mainly through the comoving number density of dark halos described by Press-Schechter theory and the background cosmological line element, dark energy affects the efficiency with which dark-matter concentrations produce strong lensing signals. Then through comparing the observed number of lenses with the theoretical expected result as a function of image
separation and cosmological parameters, it enables us to determine the allowed range of the parameter $w$. The constraint process also depends on the density profile of dark halos. Here we shall use the two model combined mechanism to reproduce the observed curve of lensing probability to the image splitting angle (Sarbu, Rusin and Ma 2001; Li and Ostriker 2002; Zhang et al. 2009). The redshift of CMB is above 1000 and far larger than 1, and there is no other observation to fill up this redshift gap, thus we would not adopt the CMB data in the present analysis and limit our study on dark energy to the redshift region of $z \sim 1$, which is the characteristic redshift scale of SNe Ia, BAO and SGL statistic.

In our recent work (Zhang et al. 2009), we have present the constraint on the dark energy from the SGL splitting angle statistic. In this paper, by taking the latest analyzed SNe Ia data (Hicken et al. 2009), the baryonic acoustic oscillations (Eisenstein et al. 2005) and the CLASS statistical sample (Browne et al. 2003), we shall make an joint analysis to constrain the dark energy equation of state parameter $w$ and the matter density $\Omega_M$, and to study the possible influence of the Hubble constant $h$. Two model independent assumptions of dark energy are considered: constant $w$ and the time-varying parameterization $w(z) = w_0 + w_a z/(1+z)$. We mainly highlight two issues which have not previously been illuminated. First, we carefully study, based on the latest SNe Ia data, the constraints for the dark energy EOS $w(z)$ and the influences of the matter density $\Omega_M$ and Hubble constant $h$. Second, we investigate the joint analysis of SNe Ia data, BAO and SGL statistic in detail. Our paper is organized as follows: Sect. 2 shows the constraint by the latest SNe Ia data on dark energy. Sect. 3 describes the joint analysis of the SNe Ia, the BAO and the SGL statistic, more stringent constraints on dark energy and matter density are resulted, and the influence of Hubble constant is explicitly demonstrated. The conclusions are presented in the last section.

2. DARK ENERGY CONSTRAINTS BY THE LATEST SNe Ia DATA

As the standard candles of the cosmology, the SNe Ia is used to study the geometry and dynamics of the universe with redshift $z \leq 1.7$. In determinations of cosmological parameters about the accelerating expand and dark energy, the SNe Ia remains a key ingredient. In 1998, the SNe Ia measurement provided the first direct evidence for the presence of dark energy with the negative pressure. Then many SN Ia observations have been done and the total number of SNe Ia sample increases quickly. The SN Ia compilations are often consist of high-redshift ($z \simeq 0.5$) data set and low-redshift ($z \simeq 0.05$) sample at the same time (e.g. Riess et al. 1998; Perlmutter et al. 1999; Wood-Vasey et al. 2007). When combining several independent group’s SNe Ia data sets into one compilation, the consistent analysis method
of light curves and the selection of supernova are crucial. For a certain sample, the different light curve fitter and corresponding different selection of supernova can lead to different constraints on the cosmological parameters (e.g. Hicken et al. 2009).

The fitting results of cosmological parameters from different SN Ia compilations show an obvious difference which could be huge in some cases. For example, the sample provided by Wood-Vasey et al. (2007) led a cosmology with near zero matter component $\Omega_M \approx 0$, while the best-fit result of $\Omega_M$ by Riess et al. (2004) was found to be $\Omega_M \sim 0.5$. The fitting constant $w$ of dark energy for the former sample was found to be $w \sim -0.75$ and for the latter sample, it was found to be $w \sim -2.0$. On the other hand, it is obvious that larger sample of SNe Ia is more powerful for constraining cosmological parameters. Therefore it is useful and attractive to adopt the SNe Ia data as many as possible. There is a conventional method to combine several group’s SNe Ia compilations, namely, by introducing an extra nuisance parameter in the $\chi^2$ statistic of every used SNe Ia sample and marginalizing them over in the fit, all $\chi^2$ statistics of samples can be summed into one total statistics (see e.g. Barger et al. 2006). The nuisance parameters are considered as analysis-dependent global unknown constants in the distances and are degenerate with the Hubble constant $h$. Although this combined mechanism is wildly adopted, the so-called analysis-dependent unknown constant is just an averaged effect of analysis-dependent uncertainties.

For a certain SNe Ia, its ‘measuring results’ from independent groups often have visible differences from disparate light curve fitting functions and analysis procedures. Riess et al. (2004) and Wood-Vasey et al. (2007) provided 157 golden sample and 162 SNe Ia sample, respectively. The two samples both adopt MLCS2k2 light-curve fitter and have 39 common SNe Ia, every SNe Ia of which has two sets of different redshift $z$ and distance moduli $\mu$. From the 39 common data, we can compare the two data sample directly and then obtain the distance uncertainty $\sigma_{o,\mu} = 0.21$ and redshift uncertainty $\sigma_{o,z} = 0.00029$. Therefore it is clear that the analysis-dependent uncertainties from different group are quite large. Partially to solve this problem, Kowalski et al. (2008) provided the Union data set, a compilation of 307 SNe Ia discovered in different surveys. The heterogeneous nature of the data set have been reflected and all SNe Ia sample are analyzed with the same analysis procedure.

In the Union data set, all SNe Ia light curve are fitted by using the spectral-template-based fit method of Guy et al. (2005) (also known as SALT). There are other light curve fitters used in literatures, such as SALT2 (Guy et al. 2007), MLCS2k2 (Jha, Riess, and Kirshner 2007) with $R_V = 3.1$ (MLCS31) and MLCS2k2 with $R_V = 1.7$ (MLCS17). Hicken et al. (2009) compared these light curve fitters and found that SALT produces high-redshift Hubble residuals with systematic trends versus color and larger scatter than MLCS2k2, and MLCS31 overestimates host-galaxy extinction while MLCS17 does not. For a certain SNe
Ia, the analysis outcomes of different light curve fitters are not equal. Here we choose the SNe Ia compilation provided by using MLCS17 light curve fitter with the best cuts $A_V \leq 0.5$ and $\Delta < 0.7$ to constrain the dark energy.

In the flat universe, the Friedmann equation are given by

$$H^2(z) = H_0^2 \left[ \Omega_M (1+z)^3 + \Omega_{DE}(z) \right]$$

$$\Omega_{DE}(z) = \begin{cases} (1 - \Omega_M)(1+z)^{3(1+w)} & \text{for constant } w, \\ (1 - \Omega_M)(1+z)^{3(1+w_0+w_a)} e^{-3w_0 z/(1+z)} & \text{for } w(z) = w_0 + w_a \frac{z}{1+z}, \end{cases}$$

with Hubble constant $H_0 = 100 \, h \, \text{km s}^{-1} \, \text{Mpc}^{-1}$. The influence of cosmological parameter $w$ is focused on the dark energy density $\Omega_{DE}(z)$ and then the Luminosity distance $d_L$, which is defined as

$$d_L(z) = (1+z) \int_0^z \frac{dz'}{H(z')}$$

Analysis of the distance modulus versus redshift relation of SNe Ia can give us the information about the cosmological parameters. Distance estimates of SNe Ia are derived from the luminosity distance, $d_L = (L/4\pi F)^{1/2}$ where $L$ and $F$ are the intrinsic luminosity and observed flux of the SNe Ia, respectively. It is usual to introduce the apparent magnitude $m$ and absolute magnitude $M$. The apparent magnitude $m$ of a celestial body is a measure of its brightness as seen by an observer on Earth and it is defined as $m = \log_{10} \left( \frac{F}{F_0} \right) = -2.5 \log_{10} \left( \frac{F}{F_0} \right)$, where $F_0$ is a reference flux, which is the zero point by definition (used to be the observed flux of Vega star). The absolute magnitude $M$ of a celestial body outside of the solar system is defined to be the apparent magnitude it would have if it were 10 parsecs away, namely

$$M \equiv m(10\text{pc})$$

$$= -2.5 \log_{10} \left( \frac{F(10\text{pc})}{F_0} \right) = -2.5 \log_{10} \left( \frac{F}{F_0} \right) \frac{d_L^2}{(\text{Mpc})^2} 10^{10}$$

$$= m - 2.5 \log \left[ \frac{d_L^2}{(\text{Mpc})^2} \right] - 25$$

(3)

From the definition of the distance moduli $\mu = m - M$, we have

$$\mu = 5 \log d_L / \text{Mpc} + 25.$$  \hspace{1cm} (4)

Using Equations (1), (2) and (4), we can relate the parameter $w$ with the measured redshift $z$ and distance moduli $\mu(z)$ of SNe Ia data. Define $\tilde{d}_L = H_0 d_L$, then we get

$$\mu + 5 \log H_0 = 5 \log \tilde{d}_L + 25.$$  \hspace{1cm} (5)
The right side of the above formula doesn’t depend on the Hubble constant and $h$ directly degenerates with the distance modula $\mu$. For the dark energy analysis, the SNe Ia data often use the certain value of $\tilde{h} = 0.65$ as the prior input and only make use of the differences of the logarithmic SNe Ia distances. However, we can still investigate the influence on the statistic analytic results when changing the Hubble constant $h$. Here $\Delta h = h - \tilde{h} = h - 0.65$ is equivalent to the difference between the input Hubble constant and the real Hubble constant in the universe. Therefore although the certain value of the parameter $h$ loses its physics meaning, $\Delta h$ is still a physics quantity. The likelihood for the parameters $(\Omega_M, w, h)$ can be determined from a $\chi^2$ statistic

$$\chi^2(\Omega_M, w, h) = \sum_i \frac{(\mu^T_i(z_i; \Omega_M, w, h) - \mu^O_i)^2}{\sigma_i^2}$$

where subscript $i$ denotes the $i$th SNe Ia data and $\sigma_i$ is the observed uncertainty. $\mu^O$ and $\mu^T$ are the observed and theoretical distance moduli, respectively.

Let us first discuss the constraints for the constant $w$ case. Using the Powell minimization method (Press, et al 1992), we minimize the likelihood function of the three parameters $(\Omega_M, w, h)$ and find that the coordinate of the best-fit point is $(\Omega_M, w, h) = (0.358, -1.09, 0.647)$. The fitted result of the Hubble constant $h$ is very close to the prior input value $\tilde{h} = 0.65$ of Hicken et al. (2009). It is noted that the value of $\Omega_M$ is large, in comparison with $\Omega_M = 0.258$ of the concordance cosmology provided by WMAP five year data (Komatsu et al. 2009).

To see the correlations among the three parameters $(\Omega_M, w, h)$ and their influences on the fitting results, we consider three different combination fittings which are shown in figure 1. From top to bottom in figure 1 we show the fitting results of $(\Omega_M, w)$, $(w, h)$ and $(\Omega_M, h)$ with other parameter being varying in the resonable range. In every panel, three pairs of contours are presented with a given cosmological parameter: for the upper panel, the deviation of the Hubble constant to the prior input $\Delta h = h - 0.65$ is taken to be: $\Delta h = -0.03$, $\Delta h = -0.005$ and $\Delta h = 0.02$, which are corresponding to $h = 0.620$, 0.645, and 0.670, from top to bottom respectively; for the middle panel, the matter density $\Omega_M$ is taken to be: $\Omega_M = 0.35$, 0.30, and 0.25 from left to right, respectively; for the lower panel, the constant $w$ of dark energy is taken to be: $w = -0.7$, $-0.9$, and $-1.1$ from left to right, respectively. The crosshairs in every panel mark the best fit points: for the upper panel, $(\Omega_M, w) = (0.01, -0.41)$, (0.28, $-0.87$) and (0.48, $-2.34$) for $h = 0.620$, 0.645, and 0.670; for the middle panel, $(w, h) = (-1.05, 0.647)$, $(-0.91, 0.646)$ and $(-0.81, 0.645)$ for $\Omega_M = 0.35$, 0.30, and 0.25; for the lower panel, $(\Omega_M, h) = (0.19, 0.644)$, (0.29, 0.646) and (0.36, 0.647) for $w = -0.7$, $-0.9$, and $-1.1$. It can be seen that the fitting results for $(\Omega_M, w)$ from the SNe Ia data is very sensitive to the value of Hubble constant $h$: when $h$ increases, the fitting
\( \Omega_M(w) \) increases (decreases) very rapidly.

To investigate the constraint on the interested parameters, it is usual to marginalize other parameters. Figure 2 shows the \((\Omega_M, w)\), \((w, h)\) and \((\Omega_M, h)\) contours from top to bottom, respectively. The unshown parameter in every panel has been marginalized through integrating it by applying the likelihood function \( L = \exp(-\chi^2/2) \). The best-fit points are \((\Omega_M, w) = (0.30, -0.92)\), \((w, h) = (-0.86, 0.645)\) and \((\Omega_M, h) = (0.35, 0.647)\). The \((\Omega_M, w)\) contours have been shown in figure 4 of paper by Hicken et al. 2009, and our corresponding results are almost identical with theirs. Here we shall carefully discuss the information from the figures: (a) there is an obvious degeneracy of constant \( w \) and \( \Omega_M \), which depresses the power of SNe Ia to constrain both of them; (b) the fitting results are consistent with the \( \Lambda \)CDM cosmology at 95% C.L.. Comparing with the results obtained from the Union data set (Kowalski et al. 2008), we notice that the confidence regions in our present consideration are similar to those of the Union data set. Figure 3 shows the likelihoods of parameters \( \Omega_M \), \( w \) and \( h \), in which the maximum likelihood points are located at \( \Omega_M = 0.36 \), \( w = -0.88 \) and \( h = 0.646 \), respectively. It is interesting to notice that the parameter \( w \) is restricted to be from \(-2.0 \) to \(-0.5 \) and \( \Omega_M \) is less than \( 0.5 \). Again, our fitted Hubble constant \( \hat{h} \) is consistent with the prior input value \( \tilde{h} = 0.65 \) and its allowed region is very narrow.

We now focus on the time-varying model \( w(z) = w_0 + w_a z/(1 + z) \). Figure 4 shows the contours of \((w_0, w_a)\), the best-fit point is \((w_0, w_a) = (-0.73, 0.84)\). It is seen that the SNe Ia data alone have a poor constraint power on the parameter \( w_a \). In figure 5 we plot the contours of two parameters \((\Omega_M, w_0)\), the best-fit point is found to be \((\Omega_M, w_0) = (0.45, -0.68)\). It is shown that when \( \Omega_M \) increases from \( 0.3 \) to \( 0.45 \), the allowed region of parameter \( w_0 \) is enlarged quickly. For a smaller \( \Omega_M < 0.3 \), it leads to a much better constraint for the parameter \( w_0 \): \( w_0 \sim -1.4 \sim -0.6 \). Figure 6 gives the contours of \((\Omega_M, w_a)\), the best-fit point is \((\Omega_M, w_a) = (0.44, -4.63)\). It is noticed that when \( \Omega_M \) increases from \( 0.34 \) to \( 0.5 \), the allowed region for the parameter \( w_a \) is enlarged rapidly. For a smaller \( \Omega_M < 0.34 \), it also leads to a much better constraint for the parameter \( w_a \): \( w_a \sim -3.0 \sim 2.5 \). From the figure 5 and figure 6, it indicates that an extra restriction on \( \Omega_M \) is necessary to improve the constraint of the SNe Ia data on the parameters \( w_0 \) and \( w_a \). Figure 7 shows the likelihoods of parameters \( w_0 \) and \( w_a \), in which the maximum likelihood points are located at \( w_0 = -0.8 \) and \( w_a = 0.4 \), respectively. It can be seen that the parameter \( w_0 \) is limited in the region \(-2.5 < w_0 < 0.5 \) and the likelihood of parameter \( w_a \) has a high non-Gaussian distribution.
3. JOINT ANALYSIS OF SNe Ia DATA, BAO AND SGL STATISTIC

As we have shown in the previous chapter, over 300 SNe Ia observed so far are not sufficient for determining the cosmological parameters, especially for $w_0$ and $w_a$. Many surveys (e.g. the Dark Energy Survey and Pan-STARRS) are proposed to obtain the SNe Ia sample with enlarged number and improved precision. Here we are going to constrain the dark energy through the combination of the SNe Ia data, the baryon acoustic oscillations as well as the SGL splitting angle statistic.

3.1. Baryon Acoustic Oscillations

In the relativistic plasma of the early universe, ionized hydrogens (protons and electrons) are coupled with energetic photons by Thomson scattering. The plasma density is uniform except for the primordial cosmological perturbations. Driven by high pressure, the plasma fluctuations spread outward at over half the speed of light. After about $10^5$ years, the universe has cooled enough and the protons capture the electrons to form neutral Hydrogen. This decouples the photons from the baryons, which dramatically decreases the sound speed and effectively ends the sound wave propagation. Because the universe has a significant fraction of baryons, these baryon acoustic oscillations leave their imprint on very large scale structures (about 100Mpc) of the Universe.

The measurement of baryon acoustic oscillations was first processed by the Sloan Digital Sky Survey (SDSS; York et al. 2000) and Eisenstein et al. (2005) studied the large-scale correlation function of its sample, which is composed of 46,748 luminous red galaxies over 3816 square degrees and in the redshift range 0.16 to 0.47. The typical redshift of the sample is at $z = 0.35$. The large-scale correlation function is a combination of the correlations measured in the radial (redshift space) and the transverse (angular space) direction (Davis et al. 2007). Thus, the relevant distance measure is modeled by the so-called dilation scale, $D_V(z) = [D_A^2(z)/H(z)]^{1/3}$, with comoving angular diameter distance $D_A(z) = \int_{z}^{\infty} dz'/H(z')$. The dimensionless combination $A(z) = D_V(z) \sqrt{\Omega_M H_0^2}/z$ has no dependence on the Hubble constant $h$ and is found to be well constrained by the SDSS data at $z = 0.35$. A standard ruler is provided as (Eisenstein et al. 2005)

$$A = \frac{\sqrt{\Omega_M}}{H(z_1)/H_0} \left[ \frac{1}{z_1} \int_{z_1}^{\infty} \frac{dz}{H(z)/H_0} \right]^{2/3} = 0.469 \pm 0.017,$$

where $z_1 = 0.35$. This ruler is a $\Omega_M$ prior and can be used to constrain dark energy. Then
the statistic is given by

\[ \chi^2(\Omega_M, w) = \frac{(A(\Omega_M, w) - 0.469)^2}{0.017^2} \]  

(8)

### 3.2. SGL Splitting Angle Statistic

The CLASS statistical sample has provided a well-defined statistical sample with \( N \) = 8958 sources. Totally \( N_l = 13 \) multiple image gravitational lenses have been discovered and all have image separations \( \Delta \theta < 3'' \) (Browne et al. 2003). The SGL statistics are sensitive to the equation-of-state parameter \( w \) of dark energy, which influences the number density of lens galaxies and the distances between the sources and lens. The probability with image separations larger than \( \Delta \theta \) for a source at redshift \( z_s \) on account of the galaxies distribution from the source to the observer can be obtained by (Schneider et al. 1992)

\[ P(> \Delta \theta) = \int_{z_s}^{\infty} \frac{dD_L}{dz}(1 + z)^3 n(M, z) \sigma(> \Delta \theta) dM dz, \]  

(9)

where \( M \) is the mass of a dark halo, \( D_L \) is the proper distance from the observer to the lens, \( n(M, z) \) is the comoving number density of dark halos virialized by redshift \( z \) with mass \( M \) and \( \sigma \) is the cross section for two images with a splitting angle > \( \Delta \theta \).

According to the Press-Schechter theory, the comoving number density with mass in the range \((M, M + dM)\) is given by

\[ n(M, z) dM = \frac{\rho_0}{M} f(M, z) dM, \]  

(10)

with the matter density of universe today \( \rho_0 = \Omega_M \rho_{\text{crit,0}} \) and the critical matter density at present \( \rho_{\text{crit,0}} = 3H_0^2/(8\pi G) \). \( f(M, z) \) is the Press-Schechter function, and we shall utilize the modified form by Sheth and Tormen (1999)

\[ f(M, z) = -\frac{0.383}{\sqrt{\pi}} \frac{\delta_c d\Delta}{\Delta^2 dM} \left[ 1 + \left( \frac{\Delta^2}{0.707\delta_c^2} \right)^{0.3} \right] \times \exp \left[ -\frac{0.707}{2} \left( \frac{\delta_c}{\Delta} \right)^2 \right], \]  

(11)

\[ \Delta^2(M, z) = \int_0^{\infty} \frac{dk}{k} \Delta_k(k, z) W^2(kr) \]  

(12)

where \( \Delta \) is the variance of the mass fluctuations(Eisenstein and Hu 1999) and parameter \( \delta_c(z) \) is the linear overdensity threshold for a spherical collapse(Wang and Steinhardt 1998; Weinberg and Kamionkowski 2002).

For different density profile of dark halo, the lensing cross section \( \sigma \) can be calculated out based on the lensing equation. We shall use the combined mechanism of SIS and NFW
model to explain the whole experimental curve of strong gravitational lensing statistic. For that a new model parameter $M_c$ was introduced by Li and Ostriker (2002): lenses with mass $M < M_c$ have the SIS profile, while lenses with mass $M > M_c$ have the NFW profile. Then the differential probability is given by

$$dP/dM = dP_{\text{SIS}}/dM \vartheta(M - M_c) + dP_{\text{NFW}}/dM \vartheta(M - M_c),$$

where $\vartheta$ is the step function, $\vartheta(x - y) = 1$, if $x > y$ and 0 otherwise. As the splitting angle $\Delta \theta$ is directly proportional to the mass $M$ of lens halos, the contribution to large $\Delta \theta$ of SIS profile is depressed by $M_c$. The lens data require a mass threshold $M_c \sim 10^{13} h^{-1} M_\odot$, which is consistent with the halo mass whose cooling time equals the age of the universe today.

The likelihood function of the SGL splitting angle statistic is defined as

$$L(w) = (1 - p(w))^{N - N_l} \prod_{i=1}^{N_l} q_i(w).$$

(13)

$p(w)$ and $q_i(w)$ represent the model-predicted lensing probabilities and the differential lensing probabilities, respectively. They are related to $P$ in Equation (9) by an integration

$$p(w) \equiv P_{\text{obs}}(> \Delta \theta) = \int \int B \frac{dP(> \Delta \theta)}{dz} \varphi(z_s) dz dz_s,$$

(14)

and

$$q(w) \equiv \frac{dP_{\text{obs}}(> \Delta \theta)}{d\Delta \theta} = \int \int B \frac{d^2P(> \Delta \theta)}{d\Delta \theta dz} \varphi(z_s) dz dz_s.$$

(15)

$B$ is the magnification bias and can be found in our previous work (Zhang et al. 2009). $\varphi(z_s)$ is the redshift distribution of sources. Here we take the Gaussian model by directly fitting the redshift distribution of the subsample of CLASS statistical sample provided by Marlow et al. (2000), which is given by (Zhang et al. 2009)

$$g(z_s) = \frac{N_s}{\sqrt{2\pi}\lambda} \exp \left[ -\frac{(z_s - a)^2}{2\lambda^2} \right],$$

(16)

with $N_s = 1.6125; a = 0.4224; \lambda = 1.3761$.

### 3.3. Joint Analysis and Numerical Results

In this section, we will investigate the constraint on the cosmological parameters from the joint analysis of (SNe + BAO) and (SNe + BAO + SGL), respectively. For the two(or
three) independent observations, the likelihood function of a joint analysis is just given by

\[ L = L_{\text{SNe}} \times L_{\text{BAO}} \times L_{\text{SGL}} \]

\[ = \exp(-\chi^2_{\text{SNe}}/2) \times \exp(-\chi^2_{\text{BAO}}/2) \times L_{\text{SGL}}. \]  \hfill (17)

The statistic significance \( \chi^2_{\text{BAO}} \) and \( \chi^2_{\text{SNe}} \) can be obtained by using Equations (8) and (6), respectively. \( L_{\text{SGL}} \) is the likelihood function of SGL statistic and can be obtained by using Equation (13). The parameter \( h \) in the SNe Ia data and SGL data has different meaning: for the former, we integrate \( h \) in the range around 0.65 and for the latter, the integral range is 0.4 to 0.9. The BAO analysis has no dependence on the Hubble constant \( h \).

Let us first discuss the constraints for the constant \( w \) case from the joint analysis of (SNe + BAO). After the Powell minimization (Press, et al 1992), we get the best fit results of the three parameters \( (\Omega_M, w, h) = (0.287, -0.885, 0.646) \). The fitted \( \Omega_M \) decreases in comparison with the result of SNe data alone, and at the same time the fitted \( w \) increases due to their degeneracy relation. Then in figure 8, we show the constraints on \( \Omega_M \) and the constant \( w \) from the joint analysis of (SNe + BAO). The parameter \( h \) has been marginalized. The best-fit results are \( (\Omega_M, w) = (0.28, -0.88) \). The 95% C.L. allowed regions of constant \( w \) and \( \Omega_M \) are found to be: \(-1.03 \leq w \leq -0.75 \) and \( 0.24 \leq \Omega_M \leq 0.33 \), respectively. It is seen that: (a) the most allowed region of parameter \( w \) is above the line \( w = -1 \); (b) the fitting results are consistent with the ΛCDM cosmology at 95% C.L.. Comparing with the results of SNe Ia data alone, we see that the allowed regions for \( \Omega_M \) and \( w \) are much reduced and the degeneracy between them disappears.

In figure 9, we show the constraints on \( \Omega_M \) and the constant \( w \) from the joint analysis of (SNe + BAO + SGL). The parameter \( h \) has been marginalized again. The best fit result is \( (\Omega_M, w) = (0.29, -0.91) \). The 95% C.L. allowed regions of constant \( w \) and \( \Omega_M \) are found to be: \(-1.06 \leq w \leq -0.77 \) and \( 0.25 \leq \Omega_M \leq 0.34 \). Comparing with the results of (SNe + BAO) case, it is seen that the results have only slight differences and the fitted \( w \) is found to be slightly smaller after adding the SGL data.

Figure 10 plots the likelihoods for the parameters \( \Omega_M, w \) and \( h \) from the joint analysis of (SNe + BAO) and for the parameters \( \Omega_M \) and \( w \) from (SNe + BAO + SGL), respectively. The maximum likelihood points are located at \( \Omega_M = 0.29, w = -0.88 \) and \( h = 0.65 \) for (SNe + BAO) and \( \Omega_M = 0.296 \) and \( w = -0.91 \) for (SNe + BAO + SGL). It is interesting to find that the parameters \( w \) and \( \Omega_M \) are restricted to the range: \(-1.17 \leq w \leq -0.67 \) and \( 0.23 \leq \Omega_M \leq 0.37 \).

After marginalizing the cosmological parameters \( (\Omega_M, h) \), we obtain the constraint on \( (w_0, w_a) \) in figure 11 from the joint analysis of (SNe + BAO) and (SNe + BAO + SGL),
respectively. The crosshairs mark the best-fit point \((w_0, w_a) = (-0.95, 0.41)\) for the (SNe + BAO) case and \((w_0, w_a) = (-0.92, 0.35)\) for the (SNe + BAO + SGL) case. For the (SNe + BAO) case, the 95% C.L. allowed regions for the parameters \(w_0\) and \(w_a\) are found to be: \(-1.22 \leq w_0 \leq -0.66\) and \(-0.92 \leq w_a \leq 1.59\). For the (SNe + BAO + SGL) case, the 95% C.L. allowed regions for the parameters \(w_0\) and \(w_a\) are found to be: \(-1.10 \leq w_0 \leq -0.72\) and \(-0.55 \leq w_a \leq 1.32\). After adding the SGL data, the constraint on the parameter \(w_0\) is improved moderately, but for the parameter \(w_a\), the allowed region decreases by near half. The extra constraint power on the time-varying \(w(z)\) obtained through adding SGL data is due to the larger redshift \(0 < z < 3.0\) of the galaxies in CLASS observational sample, in comparison with the reshift range of SNe data \(0 < z < 1.5\) and the reshift of BAO \(z = 0.35\).

It can be seen for the both cases that: (a) the most allowed region of \(w_a\) is above \(w_a = 0\); (b) in comparison with the cosmological constant \((w_0, w_a) = (-1.0, 0.0)\), the joint analysis for both cases favors more positive \((w_0, w_a)\); (c) in comparison with the results of SNe Ia data alone, the constraint on \(w_a\) is much improved and \(w_0\) also gets better constrained.

Figure 12 plots the likelihoods of parameters \(w_0\) and \(w_a\) from the joint analysis of (SNe + BAO) and (SNe + BAO + SGL), respectively. For (SNe + BAO) case, the maximum likelihood points are located at \(w_0 = -0.94\) and \(w_a = 0\). Note that \(w_a = 0\) implies a constant equation-of-state of dark energy. For (SNe + BAO + SGL) case, the maximum likelihood points are found to be \(w_0 = -0.91\) and \(w_a = 0.34\). We see that the parameters \(w_0\) and \(w_a\) are restricted to be: \(-1.20 \leq w_0 \leq -0.67\) and \(-1.0 \leq w_a \leq 2.0\).

4. CONCLUSIONS

We have carefully investigated, based on the latest SNe Ia data, BAO and SGL statistic, the constraint on the equation-of-state parameter \(w\) of dark energy for both constant and time varying cases in the flat cosmology. The influences of Hubble constant \(h\) and matter density \(\Omega_M\) on the fitting results are carefully demonstrated. The typical redshift measured by the three kinds of observations is \(z \sim 1\) and far smaller than the redshift of CMB involved, their constraints on the parameter \(w\) are effective and significant only for the redshift region \(z < 1.5\).

The influence of the equation-of-state parameter \(w\) on the density \(\rho(z)\) of dark energy in the universe and the distance \(d(z)\) makes SNe Ia data a powerful probe of dark energy. In this paper, we have utilized the latest 324 SNe Ia data provided by Hicken et al. (2009) using MLCS17 light curve fitter with the best cuts \(A_V \leq 0.5\) and \(\Delta < 0.7\) to carefully investigate the constraint on the constant \(w\) and especially the time-varying \(w(z) = w_0 + w_a z/(1 + z)\). The influences of Hubble constant \(h\) and matter density \(\Omega_M\) have also been studied carefully.
For the constant $w$ case, we have shown that: (a) the best-fit results for the three correlated parameters are found to be $(\Omega_M, w, h) = (0.358, -1.09, 0.647)$; it is seen that $\Omega_M$ is somewhat large in comparison with $\Omega_M = 0.26$ of the concordance cosmology provided by WMAP five year data (Komatsu et al. 2009); note that using a different parameterization of dark energy, Huang et al. (2009) presented a best-fitted $\Omega_M = 0.446$ from SNe Ia data, which is even larger but still consistent with our result at 95% C.L.; (b) the fitted results of both $(\Omega_M, w)$ from the SNe Ia data is very sensitive to the value of Hubble constant $h$: when $h$ increases, the fitted $\Omega_M$ increases and $w$ decreases very rapidly; (c) for parameter $\Omega_M$, the SNe Ia data alone can only give a large allowed region $0.0 \leq \Omega_M \leq 0.5$ at 95% C.L.; (d) for the constant $w$ case, the likelihoods are found to be: $w = -0.88^{+0.31}_{-0.39}$ and $\Omega_M = 0.36^{+0.09}_{-0.15}$ after marginalizing other parameters in obtaining each of them, which is consistent with the $\Lambda$CDM at 95% C.L.; (e) there is clear degeneracy of constant $w$ and $\Omega_M$, which depresses the power of SNe Ia to constrain two parameters. It has been shown that the parameter $w$ is limited from $-2.0$ to $-0.5$ and $\Omega_M$ should be less than 0.5.

In particular, we have paid special attention to the constraints on the time-varying case parameterized by two parameters $(w_0, w_a)$. After marginalizing the parameters $\Omega_M$ and $h$, we have obtained the fitting results $(w_0, w_a) = (-0.73^{+0.23}_{-0.97}, 0.84^{+0.06}_{-0.34})$, which indicates that (a) the SNe Ia data alone have only a poor constraint power on the parameter $w_a$, an extra restriction of $\Omega_M$ is necessary, so that the constraint of SNe Ia on the parameters $w_0$ and $w_a$ can be much improved; (b) the likelihood of parameter $w_a$ has a high non-Gaussian distribution.

The summary parameter of BAO can provide a standard ruler by which the absolute distance of $z = 0.35$ can be determined with 5% accuracy. This ruler can be a $\Omega_M$ prior and has been used to constrain dark energy. The strong gravitational lensing (SGL) statistic is a useful probe of dark energy. Through comparing the observed number of lenses with the theoretical expected result, it enables us to constrain the parameter $w$. We have used the latest SNe Ia data together with the BAO (and the CLASS statistical sample) to constraint dark energy. For the constant $w$ case, the results obtained from (SNe + BAO) and (SNe + BAO + SGL) only have a slight difference: (a) for the (SNe + BAO) case, the best fit results of the three parameters $(\Omega_M, w, h)$ are $(0.287, -0.885, 0.65)$ and for the (SNe + BAO + SGL) case, the best fit point is $(\Omega_M, w) = (0.298, -0.907)$ where $h$ has been marginalized; (b) the fitting results of the parameter $\Omega_M$ are found to be $\Omega_M = 0.29^{+0.02}_{-0.03}$ for the (SNe + BAO) case and $\Omega_M = 0.29^{+0.03}_{-0.03}$ for the (SNe + BAO + SGL) case; (c) the fitting results for the constant $w$ case are found to be $w = -0.88^{+0.07}_{-0.09}$ for the (SNe + BAO) case and $w = -0.91^{+0.10}_{-0.10}$ for the (SNe + BAO + SGL) case, which are consistent with the $\Lambda$CDM at 95% C.L.; (d) the most allowed region of parameter $w$ is above the line $w = -1$. Comparing with the fitting results from the SNe Ia data alone, we have found: (a) the allowed region
at 95% C.L. for $\Omega_M$ is reduced to one-fifth; (b) the best fit value of $w$ is almost not changed but its variance is reduced very much.

For the time-varying case $w(z)$ after marginalizing $(\Omega_M, h)$, we have obtained the fitting results $(w_0, w_a) = (-0.95_{-0.18}^{+0.45}, 0.41_{-0.96}^{+0.79})$ for the (SNe + BAO) case and $(w_0, w_a) = (-0.92_{-0.10}^{+0.14}, 0.35_{-0.54}^{+0.47})$ for the (SNe + BAO + SGL) case. It has been seen that the adding of the SGL data makes the constraints on parameter $(w_0, w_a)$ to be much improved. For both cases, the most allowed region of $w_a$ is above $w_a = 0$, which indicates that the phantom type models are disfavored. Comparing with the fitting results from the latest SNe Ia data alone, we have observed that: (a) the best fit values for $w_0$ are decreased by over 0.2 and the variances are approximately reduced to one-fourth; (b) the best fit values of $w_a$ are decreased by 0.49 and the variances are reduced to one-twelfth.

In conclusion, the latest MLCS17 data set given by Hicken et al. (2009) has provided an interesting constraint on the cosmological parameters. The joint analysis of SNe Ia and BAO breaks the degeneracy between $w$ and $\Omega_M$ and leads to a more stringent to constrain on the dark energy and matter density than the SNe Ia data alone, especially for the time-varying case with parameters $(w_0, w_a)$. The adding of the SGL data can further improve the constraint for $(w_0, w_a)$. A large number of SNe Ia samples with reduced systematical uncertainties in the near future, together with possible new observations on BAO and SGL statistic, would be very useful to understand the properties of dark energy and Hubble constant.

**Acknowledgments**

The author (YLW) would like to thank Q.G. Huang and M. Li for useful discussions. This work was supported in part by the National Science Foundation of China (NSFC) under the grant # 10821504, 10491306 and the Project of Knowledge Innovation Program (PKIP) of Chinese Academy of Science.
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Fig. 1.— The constraint results on the cosmological parameters $(\Omega_M, w)$, $(h, w)$ and $(h, \Omega_M)$ from top to bottom, respectively. In the upper panel, three pairs of contours are for the given Hubble constants $h = 0.62, 0.645,$ and $0.67$ from top to bottom, respectively. The crosshairs mark the best fit points $(\Omega_M, w) = (0.01, -0.41), (0.28, -0.87)$ and $(0.48, -2.34)$ from top to bottom. In the middle panel, three pairs of contours are for the given $\Omega_M = 0.35, 0.30,$ and $0.25$ from left to right, respectively. The crosshairs mark the best fit points $(w, h) = (-1.05, 0.647), (-0.91, 0.646)$ and $(-0.81, 0.645)$ from left to right. In the lower panel, three pairs of contours are for the given $w = -0.7, -0.9,$ and $-1.1$ from left to right, respectively. The crosshairs mark the best fit points $(\Omega_M, h) = (0.19, 0.644), (0.29, 0.646)$ and $(0.36, 0.647)$ from left to right.
Fig. 2.—68% C.L. and 95% C.L. allowed regions of $(\Omega_M, w)$, $(h, w)$ and $(h, \Omega_M)$ respectively. The crosshairs in three panels mark the best-fit points are $(\Omega_M, w) = (0.30, -0.92)$, $(w, h) = (-0.86, 0.645)$ and $(\Omega_M, h) = (0.35, 0.647)$, from top to bottom.
Fig. 3.— The likelihoods of the parameters $\Omega_M$, $w$ and $h$. The maximum likelihood points are located at $\Omega_M = 0.36$, $w = -0.88$ and $h = 0.65$, respectively.
Fig. 4.— The \((w_0, w_a)\) contours of SNe Ia data alone. The crosshairs mark the best-fit point \((w_0, w_a) = (-0.73, 0.84)\).
Fig. 5.— The \((\Omega_M, w_0)\) contours of SNe Ia data alone. The crosshairs mark the best-fit point \((\Omega_M, w_0) = (0.45, -0.68)\).
Fig. 6.— The \((\Omega_M, w_a)\) contours of SNe Ia data alone. The crosshairs mark the best-fit point \((\Omega_M, w_a) = (0.44, -4.63)\).
Fig. 7.— The likelihoods of the parameters $w_0$ and $w_a$. The maximum likelihood points are located at $w_0 = -0.8$ and $w_a = 0.4$, respectively. It can be seen that the likelihood of parameter $w_a$ has a high non-Gaussian distribution.
Fig. 8.— 68% C.L. and 95% C.L. allowed regions of $(\Omega_M, w)$ from the joint analysis of (SNe + BAO). The best-fit result is $(\Omega_M, w) = (0.28, -0.88)$. 
Fig. 9.— 68% C.L. and 95% C.L. allowed regions of $(\Omega_M, w)$ from the joint analysis of (SNe + BAO + SGL) (solid lines) in comparison with the joint analysis of (SNe + BAO) (dotted lines). The best-fit result from (SNe+ BAO + SGL) is $(\Omega_M, w) = (0.29, -0.91)$. 
Fig. 10.— The likelihoods of the parameters $\Omega_M$, $w$, and $h$ from the joint analysis of (SNe + BAO) and (SNe + BAO + SGL), respectively. The maximum likelihood points are located at $\Omega_M = 0.29$, $w = -0.88$ and $h = 0.65$ for (SNe + BAO) and $\Omega_M = 0.296$ and $w = -0.91$ for (SNe + BAO + SGL).
Fig. 11.— the 68% C.L. and 95% C.L. allowed regions of \((w_0, w_a)\) from the joint analysis of (SNe + BAO) and (SNe + BAO + SGL), respectively. The crosshairs mark the best-fit point \((w_0, w_a) = (-0.95, 0.41)\) for the (SNe + BAO) case and \((w_0, w_a) = (-0.92, 0.35)\) for the (SNe + BAO + SGL) case.
Fig. 12.— the likelihoods of parameters $w_0$ and $w_a$ from the joint analysis of (SNe + BAO) and (SNe + BAO + SGL), respectively. For (SNe + BAO) case, the maximum likelihood points are located at $w_0 = -0.94$ and $w_a = 0.0$, respectively. For (SNe + BAO + SGL) case, the maximum likelihood points are $w_0 = -0.91$ and $w_a = 0.34$, respectively.