An approach to the problem of generating irreducible polynomials over the finite field $GF(2)$ and its relationship with the problem of periodicity on the space of binary sequences

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Abstract

A method for generating irreducible polynomials of degree $n$ over the finite field $GF(2)$ is proposed. The irreducible polynomials are found by solving a system of equations that brings the information on the internal properties of the splitting field $GF(2^n)$. Also, the choice of a primitive normal basis allows us to build up a natural representation of $GF(2^n)$ in the space of $n$-binary sequences. Illustrative examples are given for the lowest orders.

Keywords: Finite Fields; Generation of Irreducible Polynomials; Canonical Representations; Periodic Orbits; Computation on Binary Sequences; Solution’s Choice.

AMS Subject Classifications: 11R32, 11R09, 11T71, 12E20, 12Y05.
1 Introduction

The problem of determining the irreducible polynomials over a finite field of order \( q \), \( GF(q) \), is a task, from a qualitative mathematical point of view, similar to that of finding the prime numbers in the set of integer numbers \([1, 2]\). Such polynomials are necessary in numerous applications. For instance, the generation of algebraic error-correcting codes and cryptography \([3]\) or the construction of fast transforms are based on finite field properties \([4]\) that require algorithms to obtain the irreducible polynomials \( f_n(x) \) of degree \( n \) over a particular finite field.

Tables of irreducible polynomials are found in the literature \([5]\). These can be built up by different methods \([6, 7]\), including a close analog to the 'sieve' method used in making tables of prime numbers. In this work a method to generate the irreducible polynomials of degree \( n \) over the field \( GF(2) \) is proposed. The strategy comprises the construction of a representation of the splitting field \( GF(2^n) \) over the set of binary \( n \)-digits sequences. Each periodic orbit and its \( m \) permutations, with \( m \) a divisor of \( n \), identify a irreducible polynomial of degree \( m \) and its \( m \) roots on \( GF(2^n) \).

In section II the connection between the field \( GF(2^n) \) and the space of \( n \)-binary sequences, \( B_n \), is identified. The two internal operations over \( B_n \) necessary in order to obtain a particular representation of \( GF(2^n) \) are defined in section III. The method to find the irreducible polynomials \( f_n(x) \) over \( GF(2) \) is disclosed in section IV. Section V summarizes the general scheme to be followed in order to obtain \( f_n(x) \) for all \( n \). Two illustrative examples are given in section VI. Last section is concerned with our conclusions.

2 \( GF(2^n) \) and \( B_n \)

Notation: Let us call \( B_n \) the space of binary sequences of \( n \) digits. \( B_n \) has \( 2^n \) elements. Each element \( O \in B_n \) is said to be an orbit. If \( O \) is given by the binary sequence \((\alpha_{n-1}\alpha_{n-2}\cdots\alpha_1\alpha_0)\), \((\alpha_i = 0 \ or \ 1)\), the left shifted orbit \( O' \) defined by the sequence \((\alpha_{n-2}\alpha_{n-3}\cdots\alpha_1\alpha_0\alpha_{n-1})\) is represented by \( shift(O) \). If \( m \) is the smallest integer such that \( O = shift^m(O) = shift \ ( \cdots \ (shift \ (O) ) \cdots) \) then the orbit \( O \) is said to be of period \( m \) or \( m \)-periodic, and it is represented by \( O_m \). It is obvious that if \( O_m \in B_n \) then \( m \) is a divisor of \( n \), \( m \mid n \). The set \([O_m]\) formed by the \( m \) different shifts of an \( m \)-periodic orbit, \( O_m \), is called an orbital of period \( m \): \([O_m] = \{O_m, shift(O_m), shift^2(O_m), \cdots, shift^{m-1}(O_m)\}\). As an example, \([O_3] = \)
Fundamental Idea: Firstly, let us note that the number \( N_n \) of irreducible polynomials (I.P.) of degree \( n \) over \( GF(2) \), is equal to the number of orbitals \([O_n]\) of period \( n \) (table 1,[8, 9]):

\[
N_n = \frac{2^n - \sum_{i=1}^{k} m_i N_{m_i}}{n},
\]

where \( \{m_1, m_2, \ldots, m_k\} \) are the integer divisors of \( n \) less than \( n \).

The idea is to build up a representation of \( GF(2^n) \) over \( \mathcal{B}_n \), with the constraint that each orbital \([O_m]\) contained in \( \mathcal{B}_n \) has an associated I.P. \( f_m(x) \), of degree \( m \) over \( GF(2) \). Moreover, the \( m \) orbits \( O_m \) that expand the orbital \([O_m]\) are the \( m \) roots of \( f_m(x) \) in \( GF(2^n) \). Thus, two internal operations \((+, \ast)\) are defined over \( \mathcal{B}_n \) in such a way that \((\mathcal{B}_n, +, \ast)\) be the splitting field for all I.P. \( f_m(x) \) over \( GF(2) \), with \( m \mid n \).

3 Field Construction in \( \mathcal{B}_n \)

Choice of a Basis: If we consider \( GF(2^n) \) a vectorial space over \( GF(2) \), a primitive normal basis, \( B_{\sigma_n}=\{\sigma_0, \sigma_1, \ldots, \sigma_{n-1}\} \), can be found in \( GF(2^n) \) [10, 11]. \( B_{\sigma_n} \) to be normal means that \( \sigma^2 = \sigma_{(i+1) \mod n} \). \( B_{\sigma_n} \) to be primitive means that \( \sigma_0 \) generate \( GF(2^n) \) over \( GF(2) \). Therefore, if we denote by \((+, \ast)\) the two internal operations in \( GF(2^n) \), each element \( a \in GF(2^n) \ast \) has a unique representation as addition (+) or as product (\( \ast \)) of the \( B_{\sigma_n} \) elements (tables 2, 3, 4, 5, 6):

(i) \textbf{+\-representation}: each element \( a \in GF(2^n) \) can be expanded in a linear combination of \( B_{\sigma_n} \), \( a = \sum_{i=0}^{n-1} \alpha_i \sigma_i \) with \( \alpha_i \in GF(2) \equiv \{0, 1\} \). The binary sequence \( a_+ = (\alpha_{n-1} \alpha_{n-2} \cdots \alpha_1 \alpha_0) \in \mathcal{B}_n \) is the +\-representation of \( a \).

(ii) \textbf{\( \ast \)-representation}: each element \( a \in GF(2^n) \ast \) is generated by \( \sigma_0 \), consequently \( a = \sigma_0^k \) with \( k \in \{1, 2, \cdots, 2^n - 1\} \). If we take the binary expression of the exponent, \( k = \sum_{i=0}^{n-1} \beta_i 2^i \), the binary sequence \( a_* = (\beta_{n-1} \beta_{n-2} \cdots \beta_1 \beta_0) \in \mathcal{B}_n \) is the \( \ast \)-representation of \( a \). It can also be reformulated as \( a = \prod_{i=0}^{n-1} \sigma_i^{\beta_i} \) where \( \sigma_0^0 = 1 \). It is obvious that \((a^2)_* = \text{shift}(a_*)\).

Example (table 5): If \( B_{\sigma_4}=\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} \) is a primitive normal basis, the following expansions are obtained for some elements of \( GF(2^4) \):

\[
a = \sigma_0^3 = \sigma_0 + \sigma_1 + \sigma_3 \rightarrow a_+ = 1011,
\]
\[
a = \sigma_0^3 = \sigma_0 \ast \sigma_1 \rightarrow a_* = 0011,
\]
\[
b = \sigma_0^4 = \sigma_2 \rightarrow b_+ = 0010,
\]
\[ b = \sigma_0^4 = \sigma_2 \rightarrow b_* = 0010, \]
\[ c = \sigma_0^7 = \sigma_2 + \sigma_3 \rightarrow c_+ = 1100, \]
\[ c = \sigma_0^7 = \sigma_0 \cdot \sigma_1 \cdot \sigma_2 \rightarrow c_* = 0111. \]

The particular elements, \( a = \sigma_0^{2^i} = \sigma_i \) and \( a = \sigma_0^{2^n-1} = \sigma_{n-1} \cdot \cdots \cdot \sigma_1 \cdot \sigma_0 = \sigma_{n-1} + \sigma_{n-2} + \cdots + \sigma_1 + \sigma_0 = 1 \), verify \( a_+ = a_* \). In this case, \((00 \cdots 010 \cdots 00)\) and \((11 \cdots 11)\), respectively.

### Internal operations in \( B_n \):

**+ -operation:** the property \( \sigma_i + \sigma_i = 0 \) (the characteristic of \( GF(2^n) \) is 2) imposes the algorithm for performing this operation. Given \( a, b \in GF(2^n) \) the element \( a + b \) is obtained by summing bit to bit the binary sequences of \( a \) and \( b \) (mod2) in the + -representation. If we have

\[
\begin{align*}
a_+ &= \alpha_{n-1} \alpha_{n-2} \cdots \alpha_1 \alpha_0, \\
b_+ &= \beta_{n-1} \beta_{n-2} \cdots \beta_1 \beta_0, \\
(a + b)_+ &= \gamma_{n-1} \gamma_{n-2} \cdots \gamma_1 \gamma_0,
\end{align*}
\]

the addition + presents the form:

\[
(a + b)_+ = a_+ + b_+ \mod 2,
\]

\[
\gamma_i = (\alpha_i + \beta_i) \mod 2.
\]

\((B_n, +)\) is a representation of the abelian additive group defined over \( GF(2^n) \), where the neutral element is \( 0 \equiv (00 \cdots 00) \) and the inverse element \( \overline{a} \) of \( a \) is the element itself, \( a + a = 0 \), then \( \overline{a} = a \).

**Example** (table 5):

\[
\begin{align*}
(\sigma_0^3)_+ &\rightarrow 1011, \\
(\sigma_0^4)_+ &\rightarrow 0010, \\
(\sigma_0^3 + \sigma_0^4)_+ &\rightarrow 1001 = \sigma_0^{14}.
\end{align*}
\]

**-operation:** the property \( \sigma_i^2 = \sigma_{(i+1) \mod n} \) gives the clue to perform this operation. Given \( a, b \in GF(2^n) \) the element \( a * b \) is obtained after performing the addition with carrier (mod2) of the binary sequences of \( a \) and \( b \) in the *-representation. If we have

\[
\begin{align*}
a_* &= \alpha_{n-1} \alpha_{n-2} \cdots \alpha_1 \alpha_0, \\
b_* &= \beta_{n-1} \beta_{n-2} \cdots \beta_1 \beta_0, \\
(a + b)_* &= \gamma_{n-1} \gamma_{n-2} \cdots \gamma_1 \gamma_0,
\end{align*}
\]
then the multiplication $\ast$ presents the form:

$$\quad (a \ast b)_s \equiv a_s \ast b_s \equiv (a_s + b_s \text{ with carrier}) \mod 2,$$

$$\quad \gamma_i = (\alpha_i + \beta_i + \text{carrier}_i) \mod 2.$$ 

To start we suppose carrier$_0 = 0$. If $\alpha_n - 1 + \beta_n - 1 + \text{carrier}_n < 1$ then the operation is restarted with carrier$_0 = 1$. $(B_n, \ast)$ is a representation of the abelian multiplicative group defined over $GF(2^n)^*$, where the neutral element is $1 \equiv (11 \cdots 11)$ and the inverse element $a^{-1}$ of $a$ is represented by the complementary binary sequence, $(a^{-1})_s = cp(a_s)$ given by digit$_i((a^{-1})_s) = (\text{digit}_i(a_s) + 1 \mod 2)$. This is as consequence of the fact that if $a = \sigma^k_0$ then $a^{-1} = \sigma^{(2^n-1-k)}_0$. The multiplication by zero is $(a \ast 0)_s = a_s \ast (00 \cdots 00) = (00 \cdots 00) \equiv 0$ by definition. It can also be seen that if $(a \ast b)_s = 0$, then only one of the two sequences are null, $a = 0$ or $b = 0$.

**Example:**

carrier$_0 = 0$,

$$\quad (\sigma^{11}_0)_s \rightarrow 1011,$$

$$\quad (\sigma^{14}_0)_s \rightarrow 0010,$$

$$\quad (\sigma^{11}_0 \ast \sigma^{14}_0)_s \rightarrow 1101 = \sigma^{13}_0.$$

carrier$_0 = 1$,

$$\quad (\sigma^{11}_0)_s \rightarrow 1011,$$

$$\quad (1)_s \rightarrow 1111,$$

$$\quad (\sigma^{11}_0 \ast 1)_s \rightarrow 1011 = \sigma^{11}_0.$$

$$\quad a = \sigma^{11}_0 \rightarrow a_s = 1011,$$

$$\quad a^{-1} = \sigma^{14}_0 \rightarrow (a^{-1})_s = 0100.$$ 

**Connection between $+$ and $\ast$:** the central question in order to build up a representation of the field $GF(2^n)$ over $B_n$ is to find the relationship between $a_+$ and $a_*$. This construction can be performed if that relation is known for the elements $\sigma_i \ast \sigma_j$ because all the other elements can be reduced to products of $\sigma_i \ast \sigma_j$ type. Thus, if we write

$$\quad \sigma_i \ast \sigma_j = \sum_{l=0}^{n-1} e_{ijl}\sigma_l,$$

there are $n^3$ constants $e_{ijl} (= 0, 1)$ defining the field $GF(2^n)$ completely. By construction $\sigma_i \ast \sigma_i = \sigma_{i+1}$, then $e_{iii} = \delta_{(i+1)i}$. By the commutative property, $e_{ijl} = e_{jil}$. Also, $(\sigma_i \ast \sigma_j)^2 = \sigma_{i+1} \ast \sigma_{j+1}$, then shift$(\sigma_i \ast \sigma_j)_s = (\sigma_{i+1} \ast \sigma_{j+1})_s$. If we impose
The important point for our proposal is that the coefficients of an I.P. over \( GF(2) \) can be identified if the traces are previously calculated. Therefore, the I.P. can be identified if the traces are calculated. Let us call \( N_{GF}(2) \) the number of traces whose value is 0. Hence, \( N_{GF}(2) \) is equal to the number of those whose value is 1 and \( N_{GF}(2) \), the number of those whose value is 0. Hence, \( N_{GF}(2) = N_{GF}(2) + N_{GF}(2) \).

The important point for our proposal is that the coefficients of an I.P. over \( GF(2) \) can be obtained by adding (mod2) the traces of different orbitals of the field \( GF(2) \). Therefore, the I.P. can be identified if the traces are previously calculated.

4 Construction of the Irreducible Polynomials

Let us recall that \( GF(2) \) is the splitting field of all irreducible polynomials \( f_m(x) \) over \( GF(2) \) of degree \( m \leq n \) and \( m \mid n \). If \( a \in GF(2) \) is a root of \( f_m(x) \), the \( m \) roots of this irreducible polynomial are \( \{a, a^2, a^3, \ldots, a^{2^{(m-1)}}\} \). Therefore, the \( m \) orbits that integrate an orbital \([O_m] \in B_n\) are the \( m \) roots of the irreducible polynomial \( f_m(x) \) in the representation \((B_n, +, \ast)\) of \( GF(2) \), as explained in the last section. For instance, in the case \( n = 4 \) (table 5), the elements \( \{\sigma_0\sigma_1, \sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_0\} \) are the roots of the irreducible polynomial \( x^4 + x^3 + x^2 + x + 1 \). The objective of this section is to establish a method to find the irreducible polynomials over \( GF(2) \) working out this property.

**Trace of an orbital:** The trace of an orbital \([O_m] \in B_n\), \( Tr(O_m) \), is defined as the trace of one of its elements: \( \{a, a^2, a^3, \ldots, a^{2^{(m-1)}}\} \) with \( a \in GF(2) \), that is,

\[
Tr(O_m) = a + a^2 + a^3 + \cdots + a^{2^{(m-1)}}.
\]

The trace is an element of \( GF(2) \), in this case 0 or 1. The number of traces \( N_T^{(n)} \) in \( GF(2) \) is equal to the number of orbitals \([O_m] \in B_n\),

\[
N_T^{(n)} = \sum_m N_m \quad with \quad m \mid n.
\]

Let us call \( N_T^{(n)} \) the number of traces whose value is 1 and \( N_T^{(n)} \) the number of those whose value is 0. Hence, \( N_T^{(n)} = N_T^{(n)} + N_T^{(n)} \).
Example (table 5): (i) the I.P. associated to the orbital \( \{ \sigma_0 \sigma_1, \sigma_1 \sigma_2, \sigma_2 \sigma_3, \sigma_3 \sigma_0 \} \) is a polynomial of degree 4, \( x^4 + dx^3 + ex^2 + fx + 1 \). Its coefficients are functions of the traces:

\[
\begin{align*}
d &= \sigma_0 \sigma_1 + \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_0 = Tr(\sigma_0 \sigma_1), \\
e &= \sigma_0 \sigma_1 + \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_0 + \sigma_0 \sigma_1 \sigma_2 \sigma_3 + \sigma_0 \sigma_1 \sigma_2 \sigma_3 = Tr(\sigma_0 \sigma_1), \\
f &= \sigma_0 \sigma_1 + \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_0 = Tr(\sigma_0 \sigma_1).
\end{align*}
\]

(ii) the I.P. associated to the orbital \( \{ \sigma_0, \sigma_1, \sigma_2, \sigma_3 \} \) is also of degree 4, \( x^4 + ax^3 + bx^2 + cx + 1 \), where:

\[
\begin{align*}
a &= \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 = Tr(\sigma_0), \\
b &= \sigma_0 \sigma_1 + \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_0 + \sigma_0 \sigma_1 \sigma_2 + \sigma_1 \sigma_3 = Tr(\sigma_0 \sigma_1) + Tr(\sigma_0 \sigma_2), \\
f &= \sigma_0 \sigma_1 \sigma_2 + \sigma_1 \sigma_2 \sigma_3 + \sigma_2 \sigma_3 \sigma_0 + \sigma_3 \sigma_0 \sigma_1 = Tr(\sigma_0 \sigma_1 \sigma_2).
\end{align*}
\]

**Newton’s Formulas**: the fundamental theorem on symmetrical polynomials can guide us in finding the different traces in the field \( GF(2^n) \). This theorem can be expressed as follows [12]: Let \( r_1, r_2, \ldots, r_n \) be different variables and let us call \( \theta_1, \ldots, \theta_n \) the following symmetrical polynomials in those variables:

\[
\begin{align*}
\theta_1 &= r_1 + r_2 + \cdots + r_n, \\
\theta_2 &= r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n, \\
\theta_3 &= r_1 r_2 r_3 + r_1 r_2 r_4 + \cdots + r_{n-2} r_{n-1} r_n, \\
&\vdots \\
\theta_n &= r_1 r_2 r_3 \cdots r_n.
\end{align*}
\]

Then every symmetrical polynomial in the variables \( \{ r_1, r_2, \ldots, r_n \} \) and with coefficients over a field \( F \) can be expressed as a polynomial in the new variables \( \{ \theta_1, \theta_2, \ldots, \theta_n \} \) with coefficients in \( F \).

In the particular case of the following symmetrical polynomials \( s_k \):

\[
s_k = r_1^k + r_2^k + \cdots + r_n^k \quad k = 1, 2, 3, \ldots
\]

the expansion in the variables \( \theta_k \) is given by the Newton’s formulas [13]:

\[
\begin{align*}
s_1 - \theta_1 &= 0, \\
s_2 - s_1 \theta_1 + 2\theta_2 &= 0, \\
s_3 - s_2 \theta_1 + s_1 \theta_2 + 3\theta_3 &= 0, \\
&\vdots \\
s_k - s_{k-1} \theta_1 + s_{k-2} \theta_2 - \cdots + (-1)^{k-1} s_1 \theta_{k-1} + (-1)^k \theta_k &= 0,
\end{align*}
\]
where \( \theta_j = 0 \) for \( j > n \) and the symbol \( (k) \) is interpreted as \( (k) = 1 + 1 + \cdots + 1 \).

In our problem, for instance, if we take \( r_i = \sigma_{i-1} \) the expressions of \( s_k \) and \( \theta_k \) are functions of the \( GF(2^n) \) traces. Then the Newton’s formulas give us a system of equations for the different traces of \( GF(2^n) \). By solving this system it is possible build up the I.P. of degree \( n \) over \( GF(2) \).

**Finding the Traces:** If the elements \( \{\sigma_0, \sigma_1, \cdots, \sigma_{n-1}\} \in GF(2^n) \) are taken as the variables \( \{r_1, r_2, \cdots, r_n\} \) in the equations (2), a system of equations for the different traces is obtained. The total number of unknowns (traces) is equal to the total number of orbitals, \( N_T^{(n)} \), in \( GF(2^n) \) (eq. 4). By construction:

\[
Tr(0) = 0, \\
Tr(\sigma_0) = 1, \\
Tr(\sigma_0\sigma_1\cdots\sigma_{n-1}) = 1.
\]

For large \( n \) the number of orbitals is of the same order as

\[
N_T^{(n)} \sim \frac{2^n}{n}.
\]

Noting that \( r_i^{2^n} = \sigma_i^{2^n} = \sigma_i = r_i \) we will have \( 2^n \) equations. Therefore, a system of \( 2^n \) equations with \( 2^n/n \) unknowns (many equations are interdependent) should be solved. Only those solutions verifying that the number of traces with value 1 is equal to \( N_T^{(n)} \) are accepted. If one of these solutions is chosen, then the I.P. are directly calculated.

**Note:** Let us recall a property that allows us to generate the I.P. by pairs. Two polynomials, \( f(x) \) and \( g(x) \), are reciprocal if they can be expressed in the form:

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,
\]

\[
g(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n,
\]

or equivalently, \( g(x) = x^n f(x^{-1}) \). This means that the roots of \( g(x) \) are the inverse roots of \( f(x) \). Consequently, if \( f_m(x) \) is an I.P. of degree \( m \), the reciprocal polynomial \( g_m(x) \) is also an I.P. of degree \( m \). This property can be used as another useful tool to facilitate the finding of all irreducible polynomials. It is necessary to remark at this point that both polynomials can be the same, that is, \( f_m(x) = g_m(x) \).

### 5 General Methodology

Summarizing, the steps to follow for building up a natural representation of the field \( GF(2^n) \) in \( B_n \) and for finding all I.P. of degree \( n \) over \( GF(2) \) are the following:
(1) All the independent \( n \)-binary sequences (orbitals \([O_m]\) with \( m \mid n \)) in \( B_n \) are found (see appendix). Its total number, \( N_T^{(n)} \), is the total number of I.P. over \( GF(2) \). The splitting field is \( GF(2^n) \). Let us observe that each orbital \([O_m]\) has associated an I.P. \( f_m(x) \) of degree \( m \). The roots of \( f_m(x) \) in \((B_n,+,* )\) are the \( m \) roots given by the \( m \) orbits of the orbital \([O_m]\). It also is possible to evaluate \( N_T^{(n)} \) by calculating directly the trace for each orbit \( O_m \),

\[
Tr(O_m) = O_m + shift(O_m) + shift^2(O_m) + \cdots + shift^{m-1}(O_m),
\]

where \( shift(O_m) \) is the left shift of the orbit \( O_m \). If \( \#_1(O_m) \) represents the number of 1 in the orbit \( O_m \) multiplied by \( m/n \) then it can be deduced that

\[
Tr(O_m) = 1 \quad \text{if} \quad \#_1(O_m) \text{ is odd},
\]

\[
Tr(O_m) = 0 \quad \text{if} \quad \#_1(O_m) \text{ is even}.
\]

For instance,

\[
\begin{align*}
\#_1(0011) &= 2 \cdot \frac{4}{4} = 2 \quad \Rightarrow \quad Tr(0011) = 0, \\
\#_1(0111) &= 3 \cdot \frac{4}{4} = 3 \quad \Rightarrow \quad Tr(0111) = 1, \\
\#_1(0101) &= 2 \cdot \frac{2}{4} = 1 \quad \Rightarrow \quad Tr(0101) = 1, \\
\#_1(1111) &= 4 \cdot \frac{1}{4} = 1 \quad \Rightarrow \quad Tr(1111) = 1.
\end{align*}
\]

(2) To calculate the trace associated to each element in \( GF(2^n) \), the \( 2^n \) non-independent equations deriving from the Newton’s formulas are established. There are \( N_T^{(n)} \sim 2^n/n \) unknowns.

(3) A solution of the former equation system is worked out. This solution must be compatible with the constraints: (i) \( Tr(0) = 0, Tr(\sigma_0) = 1 \) and \( Tr(\sigma_0 \sigma_1 \cdots \sigma_{n-1}) = 1 \); (ii) the number of traces with value 1 must be equal to \( N_T^{(n)} \) (calculated in point (1)). The different possible solutions are related with the different possible constructions of \( GF(2^n) \) in \( B_n \).

(4) The irreducible polynomials are explicitly found in a deterministic way from a particular solution obtained at point (3).

(5) The construction of \((B_n,+,* )\) is performed from the I.P. calculated in (4). This is a natural representation of \( GF(2^n) \).

(6) Note: As it can be seen in the tables below, an only representation of \( GF(2^n) \) in \( B_n \) is possible when \( \{n = 1, 2, 3, 4\} \).
6 Examples

In this section the process formerly explained is applied for the cases $n = 4, 5$.

$n=4$: Let us suppose $GF(2^4)$ expanded by the primitive normal basis $B_{\sigma 4} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$. Then $\sigma_i^2 = \sigma_{(i+1) \mod 2}$. The field’s elements $\{0, \sigma_0, \sigma_0^2, \sigma_0^3, \ldots, \sigma_0^{14}, \sigma_0^{15} = 1\}$ are generated by $\sigma_0$. If an element $a \in GF(2^4)$ verifies $a^{2^m} = a$, the $m$ powers of $a$, $\{a, a^2, a^2^2, \ldots, a^{2^m-1}\}$, are the $m$ roots of an I.P. of degree $m$ over $GF(2)$. Obviously, $m$ is a divisor of 4. The different representative elements of this kind in $GF(2^4)$ are:

\[
\begin{array}{|l|l|}
\hline
a \in GF(2^4) & m \\
\hline
0 & 1 \\
\sigma_0 & 4 \\
\sigma_0 \sigma_1 & 4 \\
\sigma_0 \sigma_2 & 2 \\
\sigma_0 \sigma_1 \sigma_2 & 4 \\
\sigma_0 \sigma_1 \sigma_2 \sigma_3 & 1 \\
\hline
\end{array}
\]

The different orbitals $[O_m]$ of period $m$, with $m \mid 4$, in $B_4$ are given by the representative orbits:

\[
\begin{array}{|l|l|l|}
\hline
[O_m] & Tr(O_m) & m \\
\hline
0000 & 0 & 1 \\
0001 & 1 & 4 \\
0011 & 0 & 4 \\
0101 & 1 & 2 \\
0111 & 1 & 4 \\
1111 & 1 & 1 \\
\hline
\end{array}
\]

Thus, $N_T^{(4)} = N_1 + N_2 + N_3 = 6$ and $N_T^{(4)}_{\sigma_1} = 4$ (see Table 1). It can also be observed that the orbit $\{\sigma_0 \sigma_2, \sigma_1 \sigma_3\}$ is of period 2, then its binary $\pm$-representation is $\{0101, 1010\}$ and by direct calculation $Tr(\sigma_0 \sigma_2) = 1$.

If $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ are taken as the roots of an I.P. of degree 4, the equations derived from Newton’s formulas (where $\theta_k = 0$ for $k > 4$) for the different traces can be established. The variables $\theta_k$ and $s_k$ are expressed as follows:

\[
\begin{align*}
\theta_1 &= \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 \\
&= Tr(\sigma_0) = 1, \\
\theta_2 &= \sigma_0 \sigma_1 + \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_0 + \sigma_0 \sigma_2 + \sigma_1 \sigma_3
\end{align*}
\]
\[ = \text{Tr}(\sigma_0 \sigma_1) + \text{Tr}(\sigma_0 \sigma_2), \]
\[ \theta_3 = \sigma_0 \sigma_1 \sigma_2 + \sigma_1 \sigma_2 \sigma_3 + \sigma_2 \sigma_3 \sigma_0 + \sigma_3 \sigma_0 \sigma_1 = \text{Tr}(\sigma_0 \sigma_1 \sigma_2), \]
\[ \theta_4 = \sigma_0 \sigma_1 \sigma_2 \sigma_3 \]
\[ = \text{Tr}(\sigma_0 \sigma_1 \sigma_2 \sigma_3) = 1, \]
\[ s_k = \sigma_0^k + \sigma_1^k + \sigma_2^k + \sigma_3^k \]
\[ = \sum_i \text{Tr}(\sigma^i) \text{ with } k = 1, 2, \ldots, 15. \]

The independent Newton’s formulas from the system (2) are:

\[ \text{Tr}(\sigma_0 \sigma_2) = 1, \]
\[ \text{Tr}(\sigma_0 \sigma_2) + \text{Tr}(\sigma_0 \sigma_1 \sigma_2) = 1, \]
\[ \text{Tr}(\sigma_0 \sigma_1) = 1. \]

The solution is unique:

\[ \text{Tr}(\sigma_0) = 1, \]
\[ \text{Tr}(\sigma_0 \sigma_1) = 1, \]
\[ \text{Tr}(\sigma_0 \sigma_2) = 1, \]
\[ \text{Tr}(\sigma_0 \sigma_1 \sigma_2) = 0, \]
\[ \text{Tr}(\sigma_0 \sigma_1 \sigma_2 \sigma_3) = 1. \]

Finally, the I.P. are:

\[ [O_4] \equiv \{\sigma_0\} \rightarrow x^4 + ax^3 + bx^2 + cx + 1 \rightarrow x^4 + x^3 + 1, \]
\[ [O_4] \equiv \{\sigma_0^{-1} = \sigma_0 \sigma_1 \sigma_2\} \rightarrow x^4 + cx^3 + bx^2 + ax + 1 \rightarrow x^4 + x + 1, \]
\[ a = \text{Tr}(\sigma_0) = 1, \]
\[ b = \text{Tr}(\sigma_0 \sigma_1) + \text{Tr}(\sigma_0 \sigma_2) = 0, \]
\[ c = \text{Tr}(\sigma_0 \sigma_1 \sigma_2) = 0, \]
\[ d = \text{Tr}(\sigma_0 \sigma_1) = 1, \]
\[ e = \text{Tr}(\sigma_0 \sigma_1) = 1, \]
\[ f = \text{Tr}(\sigma_0 \sigma_1) = 1, \]
\[ [O_4] \equiv \{\sigma_0 \sigma_1\} \rightarrow x^4 + dx^3 + ex^2 + fx + 1 \rightarrow x^4 + x^3 + x^2 + x + 1, \]
\[ g = \text{Tr}(\sigma_0 \sigma_2) = 1, \]
\[ [O_2] \equiv \{\sigma_0 \sigma_2\} \rightarrow x^2 + gx + 1 \rightarrow x^2 + x + 1, \]
$[O_1] \equiv \{\sigma_0\sigma_1\sigma_2\sigma_3\} \to \quad x + 1 \quad \rightarrow x + 1,$

$[O_1] \equiv \{0\} \rightarrow \quad x \quad \rightarrow x,$

where $\{\sigma_1 \cdots \sigma_k\}$ is representing all the $GF(2^4)$ elements that correspond to the orbital $[O_m]$.

The step to find the $+$-representation of every $a \in GF(2^4)$ is the following. Let us take $\sigma_0$ with associated I.P.: $x^4 + x^3 + 1$, then:

$$\sigma_0^4 + \sigma_0^3 + 1 = 0,$$

$$\sigma_0^4 = \sigma_2 \quad \rightarrow \quad (\sigma_0^3)_+ = 1 + (\sigma_2)_+ = 1011.$$

Taking the shifts of 1011, we find the $+$-representation for all the elements of the orbital $\{\sigma_3\} \equiv \{\sigma_0\sigma_1\}$. The distributive property determines the $+$-representation of the orbit $\{\sigma_0\sigma_1\sigma_2\}$, with the final result shown in table 5. Let us observe that this construction of $GF(2^4)$ is unique.

**n=5:** We repeat the same process as one of the case $n = 4$. Let us suppose $GF(2^5)$ generated by the primitive normal basis $B_{\sigma_5} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$. Hence $\sigma_2^2 = \sigma_{(i+1) \mod 2}$. The field’s elements $\{0, \sigma_0, \sigma_0^2, \sigma_0^3, \cdots, \sigma_0^{30}, \sigma_0^{31} = 1\}$ are generated by $\sigma_0$. If an element $a \in GF(2^5)$ verifies $a^{2^m} = a$, then $m \mid 5$ and the $m$ powers of $a$, $\{a, a^2, a^3, \cdots, a^{2^{m-1}}\}$, are the $m$ roots of an I.P. of degree $m$ over $GF(2)$. The representative different subsets of this kind in $GF(2^5)$ are:

| $a \in GF(2^5)$ | $m$ |
|-----------------|-----|
| 0               | 1   |
| $\sigma_0$     | 5   |
| $\sigma_0\sigma_1$ | 5   |
| $\sigma_0\sigma_2$ | 5   |
| $\sigma_0\sigma_1\sigma_2$ | 5   |
| $\sigma_0\sigma_1\sigma_3$ | 5   |
| $\sigma_0\sigma_1\sigma_2\sigma_3$ | 5   |
| $\sigma_0\sigma_1\sigma_2\sigma_3\sigma_4$ | 1   |

The different orbitals $[O_m]$ of period $m$, with $m \mid 5$, in $B_5$ are given by the representative orbits:
Thus, $N_T^{(5)} = N_1 + N_5 = 8$ and $N_T^{(5)} = 4$.

If \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\} are taken as the roots of an I.P. of degree 5 the equations derived from Newton's formulas (where $\theta_k = 0$ for $k > 5$) for the different traces can be established. The variables $\theta_k$ and $s_k$ are expressed as follows:

\[
\begin{align*}
\theta_1 &= \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 \\
&= \text{Tr}(\sigma_0) = 1,
\end{align*}
\]

\[
\begin{align*}
\theta_2 &= \sigma_0\sigma_1 + \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_0 \\
&\quad + \sigma_0\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_4 + \sigma_3\sigma_0 + \sigma_4\sigma_1 \\
&= \text{Tr}(\sigma_0\sigma_1) + \text{Tr}(\sigma_0\sigma_2),
\end{align*}
\]

\[
\begin{align*}
\theta_3 &= \sigma_0\sigma_1\sigma_2 + \sigma_1\sigma_2\sigma_3 + \sigma_2\sigma_3\sigma_4 + \sigma_3\sigma_4\sigma_0 + \sigma_4\sigma_0\sigma_1 \\
&\quad + \sigma_0\sigma_1\sigma_3 + \sigma_1\sigma_2\sigma_4 + \sigma_2\sigma_3\sigma_0 + \sigma_3\sigma_4\sigma_1 + \sigma_4\sigma_0\sigma_2 \\
&= \text{Tr}(\sigma_0\sigma_1\sigma_2) + \text{Tr}(\sigma_0\sigma_1\sigma_3),
\end{align*}
\]

\[
\begin{align*}
\theta_4 &= \sigma_0\sigma_1\sigma_2\sigma_3 + \sigma_1\sigma_2\sigma_3\sigma_4 + \sigma_2\sigma_3\sigma_4\sigma_0 \\
&\quad + \sigma_3\sigma_4\sigma_0\sigma_1 + \sigma_4\sigma_0\sigma_1\sigma_2 \\
&= \text{Tr}(\sigma_0\sigma_1\sigma_2\sigma_3),
\end{align*}
\]

\[
\begin{align*}
\theta_5 &= \text{Tr}(\sigma_0\sigma_1\sigma_2\sigma_3\sigma_4) = 1,
\end{align*}
\]

\[
\begin{align*}
s_k &= \sigma_0^k + \sigma_1^k + \sigma_2^k + \sigma_3^k + \sigma_4^k \\
&= \sum_i \text{Tr}(\sigma^i) \text{ with } k = 1, 2, \ldots, 31.
\end{align*}
\]

The independent Newton's formulas from the system (2) are:

\[
\begin{align*}
\text{Tr}(\sigma_0\sigma_2) + \text{Tr}(\sigma_0\sigma_1\sigma_2) + \text{Tr}(\sigma_0\sigma_1\sigma_3) &= 1, \\
\text{Tr}(\sigma_0\sigma_1\sigma_3) + \text{Tr}(\sigma_0\sigma_1\sigma_2\sigma_3) &= 0, \\
\text{Tr}(\sigma_0\sigma_1) + \text{Tr}(\sigma_0\sigma_1\sigma_2) &= 0, \\
\text{Tr}(\sigma_0\sigma_2)(1 + \text{Tr}(\sigma_0\sigma_1\sigma_3)) &= 0, \\
\text{Tr}(\sigma_0\sigma_2)(1 + \text{Tr}(\sigma_0\sigma_1)) &= 0.
\end{align*}
\]
One solution compatible with the constraints is:

\[
\begin{align*}
Tr(\sigma_0) &= 1, \\
Tr(\sigma_0\sigma_1) &= 0, \\
Tr(\sigma_0\sigma_2) &= 0, \\
Tr(\sigma_0\sigma_1\sigma_2) &= 0, \\
Tr(\sigma_0\sigma_1\sigma_3) &= 1, \\
Tr(\sigma_0\sigma_1\sigma_2\sigma_3) &= 1, \\
Tr(\sigma_0\sigma_1\sigma_2\sigma_3\sigma_4) &= 1.
\end{align*}
\]

Finally, the I.P. are calculated from this solution:

\[
\begin{align*}
[O_5] &\equiv \{\sigma_0\} \rightarrow \ x^5 + ax^4 + bx^3 + cx^2 + dx + 1 \rightarrow x^5 + x^4 + x^2 + x + 1, \\
[O_5] &\equiv \{\sigma_0^{-1}\} \equiv \sigma_0\sigma_1\sigma_2\sigma_3 \rightarrow \ x^5 + dx^4 + cx^3 + bx^2 + ax + 1 \rightarrow x^5 + x^4 + x^3 + x + 1, \\
&\quad a = Tr(\sigma_0) = 1, \\
&\quad b = Tr(\sigma_0\sigma_1) + Tr(\sigma_0\sigma_2) = 0, \\
&\quad c = Tr(\sigma_0\sigma_1\sigma_2) + Tr(\sigma_0\sigma_1\sigma_3) = 1, \\
&\quad d = Tr(\sigma_0\sigma_1\sigma_2\sigma_3) = 1, \\
[O_5] &\equiv \{\sigma_0\sigma_1\} \rightarrow \ x^5 + ex^4 + fx^3 + gx^2 + hx + 1 \rightarrow x^5 + x^3 + 1, \\
[O_5] &\equiv \{(\sigma_0\sigma_1)^{-1}\} \equiv \sigma_0\sigma_1\sigma_2 \rightarrow \ x^5 + hx^4 + gx^3 + fx^2 + ex + 1 \rightarrow x^5 + x^2 + 1, \\
&\quad e = Tr(\sigma_0\sigma_1) = 0, \\
&\quad f = Tr(\sigma_0\sigma_2) + Tr(\sigma_0\sigma_1\sigma_2\sigma_3) = 1, \\
&\quad g = Tr(\sigma_0\sigma_1\sigma_3) + Tr(\sigma_0) = 1, \\
&\quad h = Tr(\sigma_0\sigma_1\sigma_2) = 0, \\
[O_5] &\equiv \{\sigma_0\sigma_2\} \rightarrow \ x^5 + jx^4 + kx^3 + lx^2 + mx + 1 \rightarrow x^5 + x^3 + x^2 + x + 1, \\
[O_5] &\equiv \{(\sigma_0\sigma_2)^{-1}\} \equiv \sigma_0\sigma_1\sigma_3 \rightarrow \ x^5 + mx^4 + lx^3 + kx^2 + jx + 1 \rightarrow x^5 + x^4 + x^3 + x^2 + 1, \\
&\quad j = Tr(\sigma_0\sigma_2) = 0, \\
&\quad k = Tr(\sigma_0\sigma_1\sigma_2\sigma_3) + Tr(\sigma_0\sigma_1\sigma_2) = 1, \\
&\quad l = Tr(\sigma_0) + Tr(\sigma_0\sigma_1) = 1,
\end{align*}
\]
\[ m = Tr(\sigma_0 \sigma_1 \sigma_3) = 1, \]

\[ [O_1] \equiv \{\sigma_0 \sigma_1 \sigma_3 \sigma_4\} \rightarrow x + 1 \rightarrow x + 1, \]

\[ [O_1] \equiv \{0\} \rightarrow x \rightarrow x, \]

where \(\{\sigma_i \cdots \sigma_k\}\) is representing all the \(GF(2^5)\) elements that correspond to the orbital \([O_m]\).

The steps to find the +-representation of every \(a \in GF(2^5)\) are the following. Let us take \(\sigma_0\) with associated I.P.: \(x^5 + x^4 + x^2 + x + 1\), then:

\[ \sigma_0^5 + \sigma_0^4 + \sigma_0^2 + \sigma_0 + 1 = 0, \]

\[ \sigma_0^2 = \sigma_1; \quad \sigma_0^4 = \sigma_2 \rightarrow (\sigma_0^5)_+ = 1 + (\sigma_0)_+ + (\sigma_1)_+ + (\sigma_2)_+ = 11000. \]

The shifts of 11000 are the +-representation for all the elements of the orbital \(\{\sigma_0^5\} \equiv \{\sigma_0 \sigma_2\}\).

If we now take \(\sigma_0^3\) with associated I.P.: \(x^5 + x^3 + 1\), then:

\[ (\sigma_0^3)^5 + (\sigma_0^3)^3 + 1 = 0, \]

\[ (\sigma_0^3)_+ = 00110 \rightarrow (\sigma_0^{15})_+ = 1 + (\sigma_0^9)_+ = 11001. \]

The shifts of 11001 are the +-representation for all the elements of the orbital \(\{\sigma_0^{15}\} \equiv \{\sigma_0 \sigma_1 \sigma_2 \sigma_3\}\).

Proceeding now with \(\sigma_0^7\) that has an associated I.P.: \(x^5 + x^2 + 1\), then:

\[ (\sigma_0^7)^5 + (\sigma_0^7)^2 + 1 = 0, \]

\[ \sigma_0^{35} = \sigma_0^4 \rightarrow (\sigma_0^{14})_+ = 1 + (\sigma_0^4)_+ = 11011. \]

The shifts of 11011 are the +-representation for all the elements of the orbital \(\{\sigma_0^7\} \equiv \{\sigma_0 \sigma_1 \sigma_2\}\).

Taking \(\sigma_0^{15}\) with associated I.P.: \(x^5 + x^4 + x^3 + x + 1\), then:

\[ (\sigma_0^{15})^5 + (\sigma_0^{15})^4 + (\sigma_0^{15})^3 + \sigma_0^{15} + 1 = 0, \]

\[ \sigma_0^{60} = 00111; \quad \sigma_0^{45} = 11011 \rightarrow (\sigma_0^{75})_+ = (\sigma_0^{13})_+ = 1 + (\sigma_0^{15})_+ + (\sigma_0^{45})_+ + (\sigma_0^{60})_+ = 11010. \]

The shifts of 11010 are the +-representation for all the elements of the orbital \(\{\sigma_0^{11}\} \equiv \{\sigma_0 \sigma_1 \sigma_3\}\).

Finally, the +-representation for the elements of the orbital \(\{\sigma_0^3\} \equiv \{\sigma_0 \sigma_1\}\) are the shifts of 01001. Other solutions for the Newton’s formulas can allow different natural representations of the field \(GF(2^5)\) in \(B_n\). Table 6 is one of them.
7 Conclusion

A possible method to build up the irreducible polynomials of degree \( n \) over the field \( GF(2) \) has been exposed in this paper. The strategy exploits the properties and the connection between the splitting field \( GF(2^n) \) and the space of \( n \)-binary sequences \( B_n \).

The choice of a primitive normal basis allows us to perform a natural construction of \( GF(2^n) \) in \( B_n \): (i) Two natural operations \((+,\,\,*)\) are defined on \( B_n \) in order to obtain a representation of the splitting field \( GF(2^n) \); (ii) A system of equations for the different traces of the field is derived from the fundamental theorem on symmetrical polynomials; (iii) Different constraints, that have been established, limit the number of possible solutions of these equations; (iv) Any of these solutions gives us the irreducible polynomials of degree \( m \), with \( m \mid n \); (v) The field \((B_n,\,+,\,*)\) is finally built up with the help of the irreducible polynomials.

We want to remark the direct relationship that seems to exist between the irreducible polynomials over \( GF(2) \) and the periodic binary orbits, in such a way that this is a one-to-one application. Following this ideas and under a similar scheme, this method could be applied to find the irreducible polynomials of degree \( n \) over a general finite field \( GF(q) \) by using the connection between the properties of the splitting field \( GF(q^n) \) and the space of \( n \)-q-ary sequences.

Acknowledgements: This work was performed during a postdoctoral stage at the Ecole Normale Supérieure of Paris in 1995. I thank J-M. Couveignes (Paris) and J. Cabeza (Zaragoza) for useful discussions and references on this subject of finite fields, and the Gouvernement Français for a research grant.

Note: This manuscript should be understood as the approach of a 'newbie', plenty of questions, to a subject very far from its common working topics.
Appendix: Calculation of $N_T^{(n)}$ and $N_{T_1}^{(n)}$

An algorithm to calculate $N_T^{(n)}$ and $N_{T_1}^{(n)}$ is proposed in this appendix. The process to find the different orbitals in $B_n$ involves the following steps:

1. The decimal value of the different $n$-digit orbitals is stored in an indexed integer variable $O(i)$, then $0 < i < N_{T_1}^{(n)}$. The initial conditions are: $O(0) = 0$, $O(1) = 1$. Another integer variable $O_p$ is defined and initialized as $O_p = 1$.

2. The operational loop to find $O(k + 1)$ when all orbitals $O(m)$ are known for $m \leq k$ is the following:

   \[
   (* *) \quad O_p = O_p + 1 \\
   \quad do \ i = 0, n - 1 \\
   \quad \quad O_p = O_p \cdot 2^i \mod (2^n - 1) \\
   \quad \quad do \ j = 1, k \\
   \quad \quad \quad if \ (O_p \text{.eq.} O(j)) \ \text{goto} \ (* *) \\
   \quad \quad end \ do \\
   \quad end \ do \\
   \quad O(k + 1) = O_p \\
   \quad k = k + 1 \\
   \quad \text{goto} \ (* *)
   \]

3. This is repeated for all sequences of $B_n$ while $O_p \leq (2^n - 1)$. The maximum index $k$ from the step (2) is $k_{max} = N_T^{(n)} - 1$.

4. The period ($m$) of each orbital $O(i)$ is calculated:

   \[
   \text{per}(O(i)) = m \iff m \text{ is the smallest integer } 1 \leq m \leq n - 1 \\
   \quad \rightarrow O(i) = O(i) \cdot 2^m \mod (2^n - 1)
   \]

5. Each orbital is converted to its binary expression and its trace is calculated:

   \[
   O_b(i) = \text{binary}(O(i)) \\
   Tr(O(i)) \equiv Tr(O_b(i)) = O_b(i) + \text{shift}(O_b(i)) + \text{shift}^2(O_b(i)) + \cdots + \text{shift}^m(O_b(i))
   \]

6. The number of traces equal to 1 when $i$ ranges from 1 to $N_T^{(n)}$ is $N_{T_1}^{(n)}$. 

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Table Captions

**Table 1.** Functions over the natural numbers $N$, that appear in this work, are calculated for the first $n \in N$. $N_n$ is the number of irreducible polynomials of degree $n$ over $GF(2)$. $N_T^{(n)} = \sum_m N_m$ with $m|n$ represents the total number of orbitals in $B_n$. $N_{T1}^{(n)}$ and $N_{T0}^{(n)}$ are the number of orbitals in $B_n$ with trace 1 or 0, respectively.

**Table 2,3,4,5,6.** Natural representation of the finite field $GF(2^n)$ for the cases $n = 1, 2, 3, 4, 5$ in the space of binary sequences $B_n$, where the two internal operations are performed in the following way: (i) addition: $(a_+ + b_+)_+ = a_+ + b_+$; (ii) multiplication: $(a * b)_* = a_* * b_*$. The set $(B_n, +, *)$ is a representation of the finite field $GF(2^n)$. The $m$ orbits that compose each orbital $[O_m]$, with $m | n$, are the $m$ roots of an irreducible polynomial (I.P.) of degree $m$ over $GF(2)$. 
| $n$ | $N_n$ | $N_T(n)$ | $N_T(n)_1$ | $N_T(n)_0$ |
|-----|-------|-----------|-------------|-------------|
| 1   | 2     | 2         | 1           | 1           |
| 2   | 1     | 3         | 2           | 1           |
| 3   | 2     | 4         | 2           | 2           |
| 4   | 3     | 6         | 4           | 2           |
| 5   | 6     | 8         | 4           | 4           |
| 6   | 9     | 14        | 8           | 6           |
| 7   | 18    | 20        | 10          | 10          |

Table 1:

| $GF(2)$ element: $a$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---------------------|-------|-------|-------|-------|-------|-------|
| $x + 1$             | 1     | 1     | 1     | 1     | 1     | 1     |
| $x$                 | 0     | 0     | 0     | 0     | 0     | 0     |

Table 2:

| $GF(2^2)$ element: $a$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|------------------------|-------|-------|-------|-------|-------|-------|
| $x^2 + x + 1$          | 1     | 0     | 1     | 1     | 0     | 1     |
| $x$                    | 0     | 0     | 0     | 0     | 0     | 0     |

Table 3:
### Table 4:

| element: $a$ | $a_+$ | $a_+$ | Trace | I.P. |
|--------------|-------|-------|-------|------|
| $\sigma_0$  | $\sigma_0$ | $\sigma_0$ | 001   | 001  | $x^3 + x^2 + 1$ |
| $\sigma_0^2$ | $\sigma_1$ | $\sigma_2$ | 010   | 010  | 0 |
| $\sigma_0^3$ | $\sigma_2$ | $\sigma_3$ | 100   | 100  | $x^3 + x + 1$ |
| $\sigma_0^4$ | $\sigma_0 + \sigma_2$ | $\sigma_0\sigma_1$ | 011   | 101  | |
| $\sigma_0^5$ | $\sigma_1 + \sigma_0$ | $\sigma_1\sigma_2$ | 110   | 111  | |
| $\sigma_0^6$ | $\sigma_2 + \sigma_1$ | $\sigma_2\sigma_0$ | 101   | 110  | |
| $\sigma_0 + \sigma_1 + \sigma_2$ | $\sigma_0\sigma_1\sigma_2$ | 111   | 111  | 1 |

### Table 5:

| element: $a$ | $a_+$ | $a_+$ | Trace | I.P. |
|--------------|-------|-------|-------|------|
| $\sigma_0$  | $\sigma_0$ | $\sigma_0$ | 0001  | 0001 | $x^4 + x^3 + 1$ |
| $\sigma_0^2$ | $\sigma_1$ | $\sigma_1$ | 0010  | 0010 | 1 |
| $\sigma_0^3$ | $\sigma_2$ | $\sigma_2$ | 0100  | 0100 | $x^4 + x^3 + x^2 + x + 1$ |
| $\sigma_0^4$ | $\sigma_3$ | $\sigma_3$ | 1000  | 1000 | 1 |
| $\sigma_0 + \sigma_1 + \sigma_3$ | $\sigma_0\sigma_1\sigma_3$ | 0111  | 1011 | |
| $\sigma_1 + \sigma_2 + \sigma_0$ | $\sigma_1\sigma_2$ | 0110  | 0111 | |
| $\sigma_2 + \sigma_3 + \sigma_1$ | $\sigma_2\sigma_3$ | 1100  | 1110 | |
| $\sigma_3 + \sigma_0 + \sigma_2$ | $\sigma_3\sigma_0$ | 1001  | 1101 | |
| $\sigma_0^8$ | $\sigma_0 + \sigma_2$ | $\sigma_0\sigma_2$ | 0101  | 0101 | 1 |
| $\sigma_1 + \sigma_3$ | $\sigma_1\sigma_3$ | 1010  | 1010 | |
| $\sigma_0^9$ | $\sigma_0 + \sigma_3$ | $\sigma_0\sigma_3$ | 0111  | 1100 | |
| $\sigma_1 + \sigma_2$ | $\sigma_1\sigma_2$ | 1110  | 1101 | 1 |
| $\sigma_0^{13}$ | $\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3$ | $\sigma_0\sigma_1\sigma_2\sigma_3$ | 1111  | 1111 | 1 |

Table 4:

| element: $a$ | $a_+$ | $a_+$ | Trace | I.P. |
|--------------|-------|-------|-------|------|
| $\sigma_0$  | $\sigma_0$ | $\sigma_0$ | 0001  | 0001 | $x^4 + x^3 + 1$ |
| $\sigma_0^2$ | $\sigma_1$ | $\sigma_1$ | 0010  | 0010 | 1 |
| $\sigma_0^3$ | $\sigma_2$ | $\sigma_2$ | 0100  | 0100 | $x^4 + x^3 + x^2 + x + 1$ |
| $\sigma_0^4$ | $\sigma_3$ | $\sigma_3$ | 1000  | 1000 | 1 |
| $\sigma_0 + \sigma_1 + \sigma_3$ | $\sigma_0\sigma_1\sigma_3$ | 0111  | 1011 | |
| $\sigma_1 + \sigma_2 + \sigma_0$ | $\sigma_1\sigma_2$ | 0110  | 0111 | |
| $\sigma_2 + \sigma_3 + \sigma_1$ | $\sigma_2\sigma_3$ | 1100  | 1110 | |
| $\sigma_3 + \sigma_0 + \sigma_2$ | $\sigma_3\sigma_0$ | 1001  | 1101 | |
| $\sigma_0^8$ | $\sigma_0 + \sigma_2$ | $\sigma_0\sigma_2$ | 0101  | 0101 | 1 |
| $\sigma_1 + \sigma_3$ | $\sigma_1\sigma_3$ | 1010  | 1010 | |
| $\sigma_0^9$ | $\sigma_0 + \sigma_3$ | $\sigma_0\sigma_3$ | 0111  | 1100 | |
| $\sigma_1 + \sigma_2$ | $\sigma_1\sigma_2$ | 1110  | 1101 | 1 |
| $\sigma_0^{13}$ | $\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3$ | $\sigma_0\sigma_1\sigma_2\sigma_3$ | 1111  | 1111 | 1 |

Table 5:
Table 6:

| \( \sigma_0 \) | \( \sigma_0 \) | \( a_* \) | \( a_+ \) | Trace | I.P. |
|---|---|---|---|---|---|
| \( \sigma_0 \) | \( \sigma_0 \) | 00001 | 00001 | \( x^5 + x^4 + x^2 + x + 1 \) | |
| \( \sigma_1 \) | \( \sigma_0 \) | 00010 | 00010 | \( x^5 + x^4 + x^2 + x + 1 \) | |
| \( \sigma_2 \) | \( \sigma_2 \) | 00100 | 00100 | \( x^5 + x^3 + 1 \) | |
| \( \sigma_3 \) | \( \sigma_3 \) | 01000 | 01000 | \( x^5 + x^3 + 1 \) | |
| \( \sigma_4 \) | \( \sigma_4 \) | 10000 | 10000 | \( x^5 + x^2 + 1 \) | |
| \( \sigma_0 + \sigma_3 \) | \( \sigma_0 \sigma_1 \) | 00111 | 01001 | \( x^5 + x^4 + x^3 + x^2 + 1 \) | |
| \( \sigma_1 + \sigma_4 \) | \( \sigma_1 \sigma_2 \) | 01110 | 10010 | \( x^5 + x^4 + x^3 + x^2 + 1 \) | |
| \( \sigma_2 + \sigma_0 \) | \( \sigma_2 \sigma_3 \) | 01100 | 00101 | \( x^5 + x^4 + x^3 + x^2 + 1 \) | |
| \( \sigma_3 + \sigma_1 \) | \( \sigma_3 \sigma_4 \) | 11000 | 01010 | \( x^5 + x^4 + x^3 + x^2 + 1 \) | |
| \( \sigma_4 + \sigma_2 \) | \( \sigma_4 \sigma_0 \) | 10001 | 10100 | \( x^5 + x^4 + x^3 + x^2 + 1 \) | |