Initial conditions for models of dynamical systems

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Abstract

The long-time behaviour of many dynamical systems may be effectively predicted by a low-dimensional model that describes the evolution of a reduced set of variables. We consider the question of how to equip such a low-dimensional model with appropriate initial conditions, so that it faithfully reproduces the long-term behaviour of the original high-dimensional dynamical system. Our method involves putting the dynamical system into normal form, which not only generates the low-dimensional model, but also provides the correct initial conditions for the model. We illustrate the method with several examples.

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1 Introduction

The evolution of many physical systems may be described by ordinary differential equations (ODEs) for the “normal modes” of the system. It is often the case that most of the modes are strongly damped (these are “stable modes”) while the others (“critical modes”) are undamped, or nearly so. In this case, the solution of an initial-value problem rapidly approaches a low-dimensional centre manifold (denoted by $M$), which may be parameterised by the amplitudes of the critical modes. The subsequent evolution of the system on $M$ is slower than during the initial rapid approach to $M$.

The centre manifold is an important theoretical and applicable tool for several reasons. Firstly, we are often concerned with the stability of a fixed point to small disturbances. The stability and bifurcations of the fixed point may be completely determined by analysing just the low-dimensional evolution restricted to $M$ [3, 4]. Rational approximations to $M$ may be computed, that is, there exist constructive schemes for approximately eliminating the stable variables [5, 7, 13, 19, 22, 23, 24]. Such schemes involve approximating $M$ by (the first few terms in) a power series in the amplitudes of the critical modes. Secondly, if the fixed point is stable then to any solution $P(t)$ of the original problem there corresponds a solution $Q(t)$ on $M$ which $P(t)$ approaches exponentially quickly. The long-time behaviour of the full problem is therefore determined by the low-dimensional (and hence tractable) centre-manifold model.

The subject of this paper is the relationship between the full solution $P(t)$ and its approximation $Q(t)$ on $M$. Specifically, we consider how to derive the
initial value $Q(0)$ from $P(0)$. For although the theory of centre manifolds guarantees the existence of $Q(t)$, it gives no general constructive means of determining $Q(0)$ (a method for small $|P(0) - Q(0)|$ was given by Roberts [24]). Once $Q(0)$ is known, the solution $Q(t)$ at later times is computed by integration of a low-dimensional system of ODEs.

Computing $Q(0)$ by a perturbation expansion in a small parameter related to the time-scale of the approach to the centre manifold has been discussed by van Kampen [14] see also the references therein]. However, his method requires the solution of a succession of ordinary differential initial-value problems, which is often impractical. In Sections 2 and 3 we develop a simpler procedure for calculating $Q(0)$, by algebraic manipulations alone: no ODEs need be solved, as we illustrate in a simple example in Section 4, where we compare our method with van Kampen's. The heart of our method is putting the evolution equations into normal form [4, 7]. A well-known by-product of the reduction to normal form is the centre manifold [3]. The novelty of our work lies in the observation that appropriate initial conditions for the centre-manifold model follow naturally from the normal-form reduction.

We further illustrate our ideas in Section 5 with Taylor's model of shear dispersion in a channel, where both the original system and the model are partial differential equations. Nevertheless, the model, which involves just one spatial dimension, is considerably simpler than the original problem, which involves two space dimensions. However, at present there is little theory to support an infinite-dimensional centre manifold, so the treatment is necessarily formal. As a further extension of our approach, we discuss in an appendix the use of normal form transformations in constructing initial conditions for dynamical models that involve both unstable and stable modes.

2 A system with centre and stable modes

We consider in this section a nonlinear system of the form

\begin{align*}
\dot{x} &= Ax + M(x, y) \quad x \in \mathbb{R}^m \quad (1) \\
\dot{y} &= By + N(x, y) \quad y \in \mathbb{R}^n, \quad (2)
\end{align*}

where the eigenvalues $\alpha_k$ of the $m \times m$ matrix $A$ satisfy $\Re(\alpha_k) = 0$, and the eigenvalues $\beta_k$ of the $n \times n$ matrix $B$ satisfy $\Re(\beta_k) < -\beta < 0$. The functions $M$ and $N$ are strictly nonlinear smooth functions, of $O\left(\|x, y\|^2\right)$.

Under these conditions centre manifold theory may be applied [8] to deduce the following results.

CM1 There exists an $m$-dimensional centre manifold $\mathcal{M}$ of the form $y = h(x)$, with $h(x) = O\left(\|x\|^2\right)$. On $\mathcal{M}$ the $m$-dimensional dynamics of $x$ are described by (1) restricted to $\mathcal{M}$, namely

\[ \dot{x} = Ax + M(x, h(x)). \quad (3) \]
CM2 Corresponding to each solution \( P(t) \equiv (x(t), y(t)) \) of (1–2) there is a solution \( x = u(t) \) of (3), such that
\[
|P(t) - Q(t)| = \mathcal{O}(e^{-\beta t}) \quad \text{as } t \to \infty,
\]
where \( Q(t) \equiv (u(t), h(u(t))) \).

A sufficient condition for this property to hold is that the origin should be stable. This condition is not, however, necessary, and we assume that CM2 holds regardless of the stability of the origin.

CM3 If an approximation \( \phi(x) \) to \( h(x) \) satisfies
\[
B\phi + N(x, \phi) - \frac{\partial \phi}{\partial x} [A x + M(x, \phi)] = \mathcal{O}(|x|^p) \quad \text{as } |x| \to 0
\]
then \( h = \phi(x) + \mathcal{O}(|x|^p) \).

Solutions of (1–2) therefore are exponentially attracted to \( M \) (at least for sufficiently small initial values of \( x \) and \( y \)), and at large times the essential dynamical behaviour of (1–2) is captured by (3). This is an important result for practical purposes because (3) is of low dimension, \( m \), compared with (1–2), which is of dimension \( m + n \). Indeed, centre manifold theory also applies to problems with infinitely many stable modes (3), for example, the Kuramoto-Sivashinsky equation (3). Therefore, when interested only in the long-term behaviour of (1–2) we may as well compute solutions \( Q(t) \) of (3) rather than solutions \( P(t) \) of (1–2). Furthermore, an important property of (1–2) is that if \( |\beta_j| \gg 1 \) for some of the eigenvalues \( \beta_j \) then the system is stiff, which makes reliable numerical computations expensive. The model (3) is free of such undesirable stiffness (although it may have the different problem of rapid oscillations), which facilitates numerical computations, particularly over long times.

In general \( h(x) \) cannot be computed exactly. But by a straightforward algorithm involving algebraic manipulations based on (3), a power series in \( x \) may be developed for \( h(x) \) (3, 21). In the following subsections we describe a similarly straightforward algorithm, based on a normal form transformation, for determining the correct initial value \( Q(0) \) from the initial value \( P(0) \), so that the exponential approach (3) is achieved. Only with a correct initial condition can long-term quantitative predictions be made by the low-dimensional model (3).

2.1 Normal form and centre manifold

Our method for constructing appropriate initial conditions relies on the structure of the normal form transformation. As shown by Elphick et al. (3), for example, it is always possible to find a nonlinear change of coordinates, to variables \( \chi \in \mathbb{R}^m \) and \( \eta \in \mathbb{R}^n \), of the form
\[
\begin{align*}
x &= \chi + F(\chi, \eta)\eta \\
y &= \eta + g(\chi, \eta)
\end{align*}
\]
such that the dynamical evolution of (1–2) may be written in normal form

\[
\dot{\chi} = A\chi + a(\chi) \quad (8)
\]
\[
\dot{\eta} = B\eta + \tilde{B}(\chi, \eta)\eta. \quad (9)
\]

In these equations: \(F\) is an \(m \times m\) matrix, \(B\) is an \(n \times n\) matrix, and both are at least linear functions of their arguments. The functions \(g \in \mathbb{R}^n\) and \(a \in \mathbb{R}^m\) are strictly nonlinear. The nonlinear terms that remain in the normal form equations (8–9) arise from resonances between eigenvalues of the linear operators \(\tilde{B}\).

Note from (9) that the centre manifold \(\eta = 0\) is invariant under the evolution of the system. For small \(\chi\) and \(\eta\) the linear term \(B\eta\) dominates the nonlinear terms in (9), and so \(\eta \to 0\) exponentially quickly. In terms of the original variables, \(M\) is given by

\[
x = \chi, \quad y = g(\chi, 0),
\]

or \(y = h(x)\), where \(h(x) = g(x, 0)\). Equations (3) and (8) describe the same long-term dynamics.

### 2.2 Initial conditions

We now show that the normal form transformation gives as a by-product an appropriate initial condition \(Q(0)\), so that we can make long-term predictions with (3).

First we observe that if the initial value \(\chi_0\) is fixed, all solutions of the transformed equations (8–9) from initial conditions \((\chi_0, \eta_0)\) have the same long-term dynamics, regardless of the value of \(\eta_0\) (provided it is small enough to guarantee approach to \(M\)). This is because the evolution equation (8) for \(\chi\) is independent of \(\eta\). Therefore \((\chi_0, 0)\) is the initial condition on \(M\) for a solution that soon evolves identically to the solution from the initial condition \((\chi_0, \eta_0)\) off \(M\).

Now it is clear how to derive an initial condition \(Q(0)\) for the model (3) from the initial condition \((x(0), y(0)) = (x_0, y_0)\).

1. Use the coordinate transformation (6–7) to determine the corresponding initial condition \((\chi_0, \eta_0)\) for the normal form equations.

2. As noted above, the corresponding initial condition on \(M\) is \((\chi(0), \eta(0)) = (\chi_0, 0)\).

3. By the coordinate transformation (3), the initial condition for the model (3) is then \(x(0) = \chi_0\); that is, \(Q(0) = (\chi_0, g(\chi_0, 0))\).

Since the change of variables is determined in powers of the new variables \(\chi\) and \(\eta\), the evolution equations of the normal form, (8) and (9), are equivalent.
to \((\chi, \eta)\) only up to terms smaller than any power of \(|(\chi, \eta)|\) as \(|(\chi, \eta)| \to 0\). The differences will become significant if our initial point is close to the boundary between the basins of attraction of different attractors on \(M\) (particularly this will be a problem if the attractors have fractal basin boundaries) or if the solution on \(M\) has a positive Lyapunov exponent. In these cases our method may, like almost any other approximation, fail reliably to predict the long-term behaviour of the system. An analysis of this limitation is beyond the scope of this article.

Provided the initial value of \(\eta\) is not too large, all initial conditions for \((8–9)\) in a given plane of constant \(\chi\) evolve in essentially the same way at large times. In the original \((x, y)\)-space, the planes \(\Pi(\chi_0) = \{(\chi, \eta) \mid \chi = \chi_0\}\) appear as curved manifolds \(\Sigma\):

\[
\Sigma(\chi_0) = \{(x, y) \mid x = \chi_0 + F(\chi_0, \eta)\eta, \ y = \eta + g(\chi_0, \eta)\},
\]

where \(\chi_0\) is fixed on each manifold, and \(\eta\) parameterises it.

The manifolds \(\Sigma(\chi_0)\), which Roberts [24] has termed “isochronic manifolds”, are a generalisation of the concept of “isochrons” introduced by Winfree [31] in the context of resetting biological clocks. Winfree considered a nonlinear oscillator with a stable limit cycle \(\Gamma\) as a model for a biological clock. Each point on \(\Gamma\) can be assigned a phase \(\phi\) which increases uniformly in time, \(\dot{\phi} = \omega\). If a solution on the limit cycle is perturbed away from \(\Gamma\) then it will relax back to \(\Gamma\)—the biological clock will reset itself—but with a slightly altered phase. Winfree proposed the concept of isochrons to describe how different disturbances induce different changes to the phase of the solution. To each point \(Q\) on \(\Gamma\), with phase \(\phi\) say, he associated a surface \(\Sigma(\phi)\), transverse to \(\Gamma\), passing through \(Q\). This isochron \(\Sigma(\phi)\) consisted of all points \(P\) which, after their transient approach to \(\Gamma\), have the same phase as \(Q\) has at the same time. Later, Guckenheimer [11] proved the existence of isochrons for a nonlinear oscillator, using the invariant manifold theorem.

Our method for computing appropriate initial conditions for a centre-manifold model, is founded on the fact that projection along the planes \(\Pi\) is simpler than nonlinear projection along the curved surfaces \(\Sigma\).

### 2.3 An example

Consider the system

\[
\begin{align*}
\dot{X} &= X(1 - X - \beta Y) \\
\dot{Y} &= Y(1 - Y - \alpha X),
\end{align*}
\]

(10)

(11)

which models a range of phenomena from population dynamics to competing modes near a multiple bifurcation point. To demonstrate our method we examine the degenerate case of \(\beta = 1\), and make the linear change of variables

\[
X = x, \quad Y = 1 + y - \alpha x
\]
in order to bring (10–11) into the form (1–2), with
\[
\dot{x} = -(1 - \alpha)x^2 - xy
\]
\[
\dot{y} = -y - \alpha(1 - \alpha)x^2 - y^2.
\]

We now apply the first nonlinear stage of a normal form transformation to remove unnecessary quadratic terms from (12–13), by setting
\[
x = \chi + a\chi^2 + b\chi \eta + c\eta^2
\]
\[
y = \eta + d\chi^2 + e\chi \eta + f\eta^2,
\]
with constants \(a-f\) to be chosen. Under this change of variable (12–13) become
\[
\dot{\chi} - b\chi \eta - 2c\eta^2 = -(1 - \alpha)\chi^2 - \chi \eta + \mathcal{O} \left( |(\chi, \eta)|^3 \right)
\]
\[
\dot{\eta} - e\chi \eta - 2f\eta^2 = -\eta - [d + \alpha(1 - \alpha)]\chi^2 - e\chi \eta - [f + 1]\eta^2 + \mathcal{O} \left( |(\chi, \eta)|^3 \right).
\]
To simplify the equations that govern \(\dot{\chi}\) and \(\dot{\eta}\) we choose \(b = 1, c = 0, d = -\alpha(1 - \alpha), f = 1\). The coefficients \(a\) and \(e\) remain at our disposal and we choose to set them to zero. The evolution equations for \(\chi\) and \(\eta\) are then, to quadratic order,
\[
\dot{\chi} = -(1 - \alpha)\chi^2
\]
\[
\dot{\eta} = -\eta.
\]

Suppose now that we are given the initial condition \(P(0) = (x_0, y_0)\) for (12–13) near \((x_0, y_0) = (0, 0)\). Then by (14–15) the corresponding values for \(\chi\) and \(\eta\) are
\[
\chi_0 = x_0 - x_0y_0 + \mathcal{O} \left( |(x_0, y_0)|^3 \right)
\]
\[
\eta_0 = y_0 + \alpha(1 - \alpha)x_0^2 - y_0^2 + \mathcal{O} \left( |(x_0, y_0)|^3 \right).
\]
According to (17) \(\eta \to 0\) exponentially as \(t \to \infty\), and \(\chi\) evolves independently of \(\eta\), so the long-term evolution in (14–17) is the same as from the initial condition \((\chi_0, 0)\). This corresponds to the initial condition \(Q(0) = (x_0^*, y_0^*)\) for (12–13), where from (14–15)
\[
x_0^* = \chi_0 = x_0 - x_0y_0
\]
\[
y_0^* = -\alpha(1 - \alpha)x_0^2.
\]

### 3 Slow-manifold models

Suppose now that the rapidly decaying modes of (1–2) have been eliminated, and that the dynamics have been reduced to (3) on the centre manifold. Some eigenvalues of \(A\) represent slowly evolving modes, others fast oscillations. In
this section we discuss models in which fast oscillations are eliminated, as the rapid transients were eliminated in the previous section. This is the essence of many physical approximations, and is a useful tool for numerical integrations. This procedure finds application in fluid mechanics, where high-frequency sound waves are ignored under the incompressible approximation [14, §9]; in beam theory, where fast ringing modes are ignored [26]; in meteorology, where weather data must be “initialised” before numerical forecasts can be computed, in order to remove spurious, relatively high-frequency gravity wave oscillations [6, 15, 16].

Suppose the linear spectrum of $A$ consists of a zero eigenvalue repeated $m$ times, together with $n/2$ complex-conjugate pairs of purely imaginary eigenvalues $\pm i\omega_k$. Then with some renaming of variables we may split (3) into slow and fast components,

\[
\dot{x} = Ax + M(x,y), \quad \dot{y} = By + N(x,y),
\]  

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $A$ has $m$ eigenvalues that are precisely zero, and $B$ has purely imaginary eigenvalues, $(\beta_1, \ldots, \beta_n)$ with $\beta_{2k} = -\beta_{2k-1}$. Sijbrand [27] shows that, since there is no resonant forcing by $x$ of the fast oscillations in $y$, this system has a sub-centre manifold $M_S$ of the form $y = h(x)$: a “slow manifold” that may be derived for this system by the same formal means as the centre manifold described earlier. However, the slow manifold does not in general attract neighbouring solutions, but instead acts as a slowly evolving “centre” for their rapid oscillations.

For a slow-manifold model the normal form for $\dot{\chi}$ cannot in general be made independent of $\eta$—it takes the form

\[
\dot{\chi} = A\chi + a(\chi, \eta),
\]

with $a(\chi, \eta) = O(|\eta|^2)$ as $|\eta| \to 0$. Quadratic terms in $\eta$ arise from resonant forcing of the slow modes by nonlinear combinations of two conjugate modes $\pm i\omega_k$. The evolution equation for the fast variables $\eta$ may still be written in the form (18). The subcentre manifold devoid of the fast oscillations characterised by the imaginary eigenvalues of $B$ is given by $\eta = 0$; the evolution of the slow variables $\chi$ is given by (18) restricted to $\eta = 0$.

### 3.1 Initial conditions

Integrating (18) numerically can be costly if the rapid oscillations (potentially of large amplitude) are present. Unfortunately, since (18) cannot in general be made independent of $\eta$, solutions on and off the slow manifold (where $\eta = 0$ and $\eta \neq 0$, respectively) evolve in essentially different ways. It is therefore impossible in general to project an initial condition $(x_0, y_0)$ to $M_S$ so as to maintain the same essential dynamical behaviour for all time. However, it is possible to match the behaviours for a time of $o(|\eta|^{-2})$ in the presence of fast
oscillations of small amplitude $|\eta|$. This approximation, where only those terms up to linear order in $\eta$ are kept, is described by Roberts \[24\].

A more favourable situation occurs when the system (18) is symmetric. If $M(-x,-y) = -M(x,y)$, and similarly for $N$, then a fast mode cannot resonate with its conjugate alone to force the slow modes. Instead a combination of at least three modes of at least two different frequencies is needed to force the slow modes. Hence in this case $a(\chi, \eta) = O(|\eta|^q)$ for some $q \geq 3$ and the effects of any fast oscillations on the long-term evolution are much weaker.

### 3.2 A simple example

Consider the system

\[
\dot{x} = y_2^2, \quad \dot{y}_1 = -y_2, \quad \dot{y}_2 = y_1,
\]

which is of the form (18). The $y$-variables execute fast oscillations with period $2\pi$, while the variable $x$ is constant according to the linearised equations, but is forced quadratically by the rapid oscillations. Here the normal form transformation can be accomplished exactly. We set $\chi = x + y_1y_2/2$, $\eta_1 = y_1$, and $\eta_2 = y_2$, so that the normal form equations become

\[
\dot{\chi} = \frac{(\eta_1^2 + \eta_2^2)}{2}, \quad \dot{\eta}_1 = -\eta_2, \quad \dot{\eta}_2 = \eta_1,
\]

or

\[
\dot{\chi} = r^2/2, \quad \dot{r} = 0, \quad \dot{\theta} = 1,
\]

(20)

where we have written $\eta_1 = r \cos \theta$ and $\eta_2 = r \sin \theta$. Thus $r$ is the amplitude of the oscillations, and $\theta$ is their phase. From (20), the invariant slow manifold, $M_S$, is given simply by $\eta = 0$ (i.e. $r = 0$); the evolution of a solution $Q(t)$ on $M_S$ is trivial: $\dot{\chi} = 0$. So solutions $Q(t)$ have $\chi$ constant and $\eta_1 = \eta_2 = 0$. However, for solutions $P$ off $M_S$, $r$ is a non-zero constant, and so $\chi$ drifts at a (non-zero) constant rate: $\dot{\chi} = r^2/2$. It is not possible to match solutions $P(t)$ and $Q(t)$ for large times by any choice of initial condition $Q(0)$.

### 4 A comparison of our method with others

In order to show how the method we have described for determining $P(0) - Q(0)$ compares with other methods, we describe now three alternative approaches for a simple problem. (The difference $P(0) - Q(0)$ is sometimes termed “initial slip” by analogy with the boundary slip of an inviscid fluid.) These are: in the first instance to solve it exactly; secondly to apply our normal-form method described above, and finally to use explicit perturbation expansions for the solution (rather than for the governing equations, as is the case for the normal form method) to derive the same results. In general, for nonlinear problems, the first approach is not available to us, and the algebraic details of the third quickly
become overwhelming. A further method for computing the “initial slip”, which we shall not discuss here, has been derived by Geigenmüller et al. [9], who apply a systematic perturbation procedure to linear systems of a certain form.

The example involves a simple chemical reaction [14, §2] which in scaled variables that represent two concentrations is written

\[
\begin{align*}
\dot{x} &= -\epsilon(x - y) \\
\dot{y} &= kx - y.
\end{align*}
\]  

(21)

(22)

In the first equation \(x\) evolves slowly, because we consider \(\epsilon\) to be small. These equations are linear and so the system may be solved exactly, in particular the behaviour of the system at large times is determined exactly.

### 4.1 Exact solution

Equations (21–22) may be rewritten as

\[
\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix}, \text{ where } M = \begin{bmatrix} -\epsilon & \epsilon \\ k & -1 \end{bmatrix}.
\]  

(23)

The eigenvalues \(\lambda_0\) and \(\lambda_s\) of the matrix \(M\) are

\[
\lambda_0, \lambda_s = \frac{1}{2} \left( -(1 + \epsilon) \pm \sqrt{(1 - \epsilon)^2 + 4\epsilon k} \right),
\]

As \(\epsilon \to 0\), the eigenvalues take the form \(\lambda_0 \sim \epsilon(k - 1), \lambda_s \sim -1\), corresponding to slow evolution and rapid decay, respectively.

The solution of (23) is

\[
\begin{align*}
x(t) &= ae^{\lambda_0 t} + be^{\lambda_s t}, \\
y(t) &= \frac{ka}{1 + \lambda_0}e^{\lambda_0 t} + \frac{kb}{1 + \lambda_s}e^{\lambda_s t},
\end{align*}
\]

where the constants \(a\) and \(b\) are found from the initial values:

\[
\begin{bmatrix} a \\ b \end{bmatrix} = \frac{(1 + \lambda_0)(1 + \lambda_s)}{k(\lambda_0 - \lambda_s)} \begin{bmatrix} -y_0 + kx_0/(1 + \lambda_s) \\ y_0 - kx_0/(1 + \lambda_s) \end{bmatrix}.
\]

Since \(1 \gg \epsilon(k - 1)\), the \(\lambda_s\)-mode decays to zero much more rapidly than the \(\lambda_0\)-mode evolves, so that after some time the solution is given, to within exponentially small terms, by

\[
\begin{align*}
x(t) &\sim ae^{\lambda_0 t}, \\
y(t) &\sim \frac{ka}{1 + \lambda_0}e^{\lambda_0 t} = \frac{k}{1 + \lambda_0}x
\end{align*}
\]  

(24)

This solution is equivalent to the solution of the one-dimensional model equation

\[\dot{X} = \lambda_0 X\]

(25)

with initial condition

\[
X(0) = \frac{(1 + \lambda_0)(1 + \lambda_s)}{k(\lambda_0 - \lambda_s)} \begin{bmatrix} -y_0 + \frac{kx_0}{1 + \lambda_s} \end{bmatrix},
\]

(26)

where \(x = X\) and \(y = kX/(1 + \lambda_0)\).
4.2 Normal form calculation

The normal form for (21–22) is obtained by making the substitution

\[ x = \chi + f\eta, \quad y = \eta + g\chi, \]  

(27)

where \( f(\epsilon) \) and \( g(\epsilon) \) are constants. Wycoff & Balasz [32] have considered linear systems of a form that includes (21–22), and have derived a substitution of the form (27) by considering a multiple-time-scale perturbation expansion. Our results for this problem agree with theirs for the more general case. Equations (21–22) fail to satisfy the conditions of (1–2) in two ways: firstly, there is no mode with precisely zero growth rate; secondly, the system has not been decomposed into critical and stable subspaces. The first point is dealt with by using a standard trick of bifurcation theory [3], namely considering the parameter \( \epsilon \) as an extra dynamical variable with the trivial evolution equation

\[ \dot{\epsilon} = 0. \]  

(28)

Then the term \(-\epsilon(x - y)\) is nonlinear in the extended system (21, 22, 28), and so the linearised evolution of \( x \) is just \( \dot{x} = 0 \). The second point is unimportant in practice, and we diagonalise the system as we go.

Substituting (27) into the governing equations (21–22) we find

\[ \dot{\chi} + f\dot{\eta} = -\epsilon\chi - \epsilon f\eta + \epsilon\eta + \epsilon g\chi \]
\[ \dot{\eta} + g\dot{\chi} = -\eta - g\chi + k\chi + kf\eta. \]

Separating out terms proportional to \( \chi \) and \( \eta \) we find that the slow and fast variables evolve according to

\[ \dot{\chi} = -\epsilon(1 - g)\chi \]
\[ \dot{\eta} = -(1 - kf)\eta, \]  

(29)

(30)

where \( f \) and \( g \) must satisfy quadratic equations whose roots are

\[ f = \frac{1}{2k} \left( 1 - \epsilon - \sqrt{(1 - \epsilon)^2 + 4\epsilon k} \right) \sim -\epsilon \quad \text{as} \ \epsilon \to 0 \]
\[ g = \frac{1}{2\epsilon} \left( \epsilon - 1 + \sqrt{(1 - \epsilon)^2 + 4\epsilon k} \right) \sim k \quad \text{as} \ \epsilon \to 0. \]

The fact that \( g \) is not small as \( \epsilon \to 0 \) reflects the fact that the original system is not diagonalised.

The centre manifold may now be read off from the normal form (27) by setting \( \eta = 0 \). That is,

\[ x = \chi, \quad y = g\chi. \]  

(31)

Since \( g(1 + \lambda_0) = k \), this expression for the centre manifold agrees with (24) derived from the exact solution (as it should). Since \(-\epsilon(1 - g) = \lambda_0 \), the
evolution of the slow variable \( \chi \) in (29) is precisely that derived from the exact solution, where we denoted the slow variable by \( X \) in (25).

Now we follow the procedure described in Section 2.2 to determine the appropriate initial condition \( Q(0) \) on the centre manifold which gives rise to the same long-term dynamics as \( P(0) = (x_0, y_0) \). Using (27) we find that \( P(0) \) is written in terms of \( \chi \) and \( \eta \) as

\[
(\chi_0, \eta_0) = \frac{1}{1 - fg} (x_0 - fy_0, -gx_0 + y_0).
\]

The expression for the corresponding \( Q(0) \) on \( \mathcal{M} \) is \((\chi_0, 0)\). Transforming this point back to the original variables yields the expression

\[
(x^*_0, y^*_0) = \frac{1}{1 - fg} (x_0 - fy_0, g[x_0 - fy_0]).
\]

A little algebra shows that (32) agrees with the exact solution (20).

### 4.3 Perturbation expansion of the solution

Finally, we treat the problem with a technique based upon a perturbation expansion for the solution during its rapid approach to the centre manifold. Such a method is described by van Kampen [14] for problems that may be nonlinear, and we follow his argument here. The essential idea is that during the rapid approach to the centre manifold, the fast variables change a great deal, while the slow variables change little.

We begin by expanding the slow variable \( x(t) \) as a power series in \( \epsilon \),

\[
x(t) = x^0(t) + \epsilon x^1(t) + \epsilon^2 x^2(t) + \cdots,
\]

and then we substitute this expansion into (21–22). Note the important difference between this perturbation expansion of the solution and the expansion of the governing equations for the normal form calculation. Approximations to the fast variable \( y(t) \) are constructed from (22), with \( x \) replaced by a finite truncation of the series (33).

At \( O(\epsilon^0) \) in (21), \( x^0 = 0 \), so

\[
x^0 = \text{constant} = x_0.
\]

To leading order then, during the approach of the solution to \( \mathcal{M} \), \( x \) does not change. Now we compute an approximation to \( y(t) \) during the approach to \( \mathcal{M} \) using (34) and (22),

\[
\dot{y} = -y + kx^0,
\]

which may be solved to give \( y(t) = y_0 e^{-t} + kx_0(1 - e^{-t}) \). Now we use this solution for \( y(t) \) to compute \( x_1 \), by considering (21) at \( O(\epsilon^2) \):

\[
\dot{x}^1 = -x^0 + y(t),
\]
whose solution is
\[ x^1(t) = -x_0 t + y_0 (1 - e^{-t}) + ky_0 (t + e^{-t} - 1). \]

Now we re-assemble the solution \( x(t) \) to \( O(\epsilon) \) and find
\[ x(t) \sim \{ x_0 + \epsilon(y_0 - kx_0) \} + \epsilon t \{ -(1 - k)x_0 \} + \epsilon e^{-t} \{-y_0 + kx_0 \}. \]

The first two terms are the start of a Taylor expansion of the solution in powers of \( t \), while the last term is an exponentially decaying transient. The initial value for the Taylor-expansion component, \( x_0 + \epsilon(y_0 - kx_0) \), is just the initial value for the solution \( Q(t) \) on \( \mathcal{M} \). This result agrees with the previous two calculations of the initial condition \( Q(0) \), as we see by expanding (32) to give
\[ x_0^* = \frac{x_0 - \int y_0}{1 - f g} \sim \frac{x_0 + \epsilon y_0}{1 + \epsilon k} = x_0 + \epsilon(y_0 - kx_0) + O(\epsilon^2). \]

To continue the expansion requires the solution of a succession of ODEs: in general, even if possible, the computations rapidly become exhausting.

5 An infinite-dimensional example: shear dispersion

We now apply the technique we have described for determining initial conditions to a generalisation of Taylor’s model [28] for shear dispersion. This model idealises the spreading of some contaminant released into a river. The river is modelled as a channel, and the water is supposed to flow parallel to the banks, with the flow being fastest closest to the centre of the channel. The variation in water speed with distance from the bank increases contaminant gradients, while molecular diffusion tends to smooth gradients out. A balance between these competing physical mechanisms occurs at large times after the release of the contaminant; eventually the contaminant is approximately evenly spread in any cross-section of the river. The cross-sectionally averaged concentration varies slowly in the downstream direction. The slow evolution of the averaged concentration obeys a simplified model equation which was derived through the techniques of centre manifold theory by Mercer & Roberts [17, 18].

The following analysis can be made rigorous in Fourier space [17], but this rigour is generally unavailable in other problems where the centre manifold itself is of infinite dimension. In the absence of rigorous theory, the best we can do is to apply our techniques formally.

5.1 The model and its linear behaviour

Consider the dispersion of a contaminant, with concentration \( c(x, y, t) \), in a channel \( 0 < y < 1, -\infty < x < \infty \). If longitudinal spatial variations (that is,
variations in $x$) occur on a large scale than $c$ satisfies the partial differential equation (PDE)

$$\frac{\partial c}{\partial t} = -u(y)\frac{\partial c}{\partial x} + \frac{\partial^2 c}{\partial y^2},$$

(35)

subject to the boundary conditions that no contaminant escapes through the walls of the channel,

$$\frac{\partial c}{\partial y} = 0 \text{ at } y = 0, 1.$$  

(36)

The variables have been made dimensionless, and $u(y)$ is the velocity in the $x$-direction.

The PDE (35) has two space dimensions; the “centre manifold” we derive by performing a normal form transformation involves just one space dimension. We concentrate on the structure of the normal form transformation; the details have been discussed elsewhere [17, 18].

The problem (35–36) is linear in the concentration. The concentration evolves so that after some time the spatial variations in $x$ are small, and so we treat all $x$-derivatives as “nonlinear” terms in the same way that a small parameter was treated earlier. With this interpretation, the “linearised” dynamics are given by

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial y^2},$$

which physically describes cross-channel diffusion. The eigenmodes $e^{\sigma_n t}c_n(y)$ are of the form

$$c_n = \cos n\pi y \quad (n = 0, 1, \ldots),$$

with corresponding eigenvalues $\sigma_n = -n^2\pi^2$. According to this “linear” picture, all modes decay except $c_0$. The system is therefore analogous to (1–2), with one critical mode but infinitely many damped modes. Note that the critical space is spanned by the single mode, $c_0$, and that all other eigenmodes have zero $y$-average. For a given concentration field $c$ we can determine the component in the critical space by taking a $y$-average to give $\overline{c}$; then the component in the stable space is $c' \equiv c - \overline{c}$.

5.2 Normal form

Here we derive a “normal form” for (35). The normal form transformation consists of choosing new variables $\chi$ and $\eta$ in terms of which the governing PDE (35) is simplified. The general form of this transformation is

$$\overline{c} = \chi + g_1(\chi, \eta), \quad c' = \eta + g_2(\chi, \eta).$$

Since the governing equation is linear in $c$ then so are $g_1$ and $g_2$. They are “nonlinear” in the sense that they vanish when $x$-derivatives are ignored. Using experience from the previous sections, we choose $\overline{c}$ to reduce to $\chi$ when
\( \eta = 0, \) and \( c' \) to reduce to \( \eta \) when \( \chi = 0. \) Furthermore, we suppose that \( \chi \) has no component in the stable space (\( \chi = \chi' \)) and that \( \eta \) has no component in the centre space (\( \eta = \eta' \)). Such considerations imply that the “nonlinear” terms in the normal form transformation must be of the form \( g_1(\chi, \eta) = \theta \eta, \)

\( g_2(\chi, \eta) = h \chi, \) where \( h \) and \( \theta \) are linear operators such that \( \theta \eta = \theta \eta \) and \( h \chi = 0. \) For definiteness we define \( \theta \) so that \( \theta f = 0 \) for all \( f. \) We have therefore decomposed \( c(x, y, t) \) as follows:

\[
\begin{align*}
\varepsilon = \chi + h \chi + \eta + \theta \eta. \quad (37)
\end{align*}
\]

Because the original problem is linear in \( c \) we may deal with the \( \chi \)- and the \( \eta \)-components separately. After substituting (37) into (35) we consider first those terms involving \( \chi. \) We find that

\[
(1 + h) \frac{\partial \chi}{\partial t} = -u \frac{\partial}{\partial x}[(1 + h)\chi] + \frac{\partial^2}{\partial y^2} h \chi. \quad (38)
\]

This equation serves to define both the operator \( \chi \) and the derivative \( \partial \eta/\partial t. \) The average of this equation over \( y \) gives an expression for \( \partial \chi/\partial t: \)

\[
\frac{\partial \chi}{\partial t} = -u(1 + h) \frac{\partial \chi}{\partial x}. \quad (39)
\]

This is the evolution equation for \( \chi \) on the centre manifold, and represents a simplification of the original problem (35). Not only is (39) of lower dimension than (35), because there is no \( y \)-dependence in \( \chi(x, t), \) but there are only “nonlinear” terms in (39)—that is, every term on the right-hand side of (39) is differentiated at least once with respect to \( x. \)

Substituting (39) back into (38), we obtain an equation that governs the operator \( h: \)

\[
- (1 + h)u(1 + h) \frac{\partial}{\partial x} = -u(1 + h) \frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} h. \quad (40)
\]

In order to solve this equation, we expand \( h \) in powers of \( \partial_x, \)

\[ h \sim \sum_{j=1}^{\infty} \alpha_j(y) \partial_x^j, \]

substitute this expression into (40), and equate powers of \( \partial_x. \) Retaining only the first term in the expansion of \( h, \) we find (39) to be approximately

\[
\frac{\partial \chi}{\partial t} = -u \frac{\partial \chi}{\partial x} - \frac{\partial^2 \chi}{\partial x^2}. \]

The concentration \( \chi \) is advected with the mean velocity, \( \overline{v}, \) and diffuses with diffusion coefficient \(-\overline{v}.\)
A similar argument for the terms involving \( \eta \) yields
\[
(1 + \theta) \frac{\partial \eta}{\partial t} = -u \frac{\partial}{\partial x} [(1 + \theta) \eta] + \frac{\partial^2 \eta}{\partial y^2}.
\]
Averaging with respect to \( y \) gives
\[
\theta \frac{\partial \eta}{\partial t} = - \frac{\partial}{\partial x} u(1 + \theta) \eta.
\]
Subtracting one equation from the other gives an evolution equation for \( \eta \):
\[
\frac{\partial \eta}{\partial t} = - \frac{\partial}{\partial x} \left[ u(1 + \theta) \eta - u \theta \eta \right] + \frac{\partial^2 \eta}{\partial y^2},
\] (41)
and so \( \theta \) satisfies
\[
\theta \left\{ - \frac{\partial}{\partial x} \left[ u(1 + \theta) \eta + \frac{\partial^2 \eta}{\partial y^2} \right] \right\} = - \frac{\partial}{\partial x} u(1 + \theta) \eta.
\]
This equation may be solved by expanding \( \theta \) in powers of \( \partial_x \). Note that the manifold \( \eta = 0 \) is invariant (from (41)) and the evolution equation (41) for \( \eta \) contains both “linear” and “nonlinear” terms. Since \( \eta = 0 \) then the eigenvalues of the “linear” operator are \( \lambda_1, \lambda_2, \ldots \), which are all negative. Consequently, \( \eta \to 0 \) exponentially quickly.

If we compute the terms in the expansions for \( \theta \) we find that \( \theta \eta = -h \eta \), so the normal form transformation (37) puts \( c \) into the form
\[
c = \chi + h \chi + \eta - h \eta.
\] (42)
Note that when we apply the operator \( 1 + h \) to (42) and average with respect to \( y \) we obtain
\[
(1 + h)c = (1 + h)^2 \chi.
\] (43)
This enables us to determine \( \chi \) from a given \( c \). To then determine the corresponding value for \( \eta \), we use
\[
\eta = c - \chi - h \chi.
\] (44)

### 5.3 An appropriate initial condition

To derive initial conditions for \( \chi \) so that its long-term evolution in (35) agrees asymptotically with that of \( c \) in (33) from a given initial condition \( c = c_0(x, y) \) we first apply (33–44) to obtain the values \( (\chi_0, \eta_0) \) that correspond to \( c_0 \). Then we note that the solution of (33–41) from \( P(0) = (\chi_0, \eta_0) \) exponentially approaches the solution from \( Q(0) = (\chi_0, 0) \). Transferring \( Q(0) \) back to the physical variable \( c \), using (33), we obtain a modified initial condition, \( c'_0 \), for (35) that lies on the centre manifold and has the same long-time behaviour as from \( c_0 \). Of course, if we want to integrate the model equation (35) instead of (33) itself, then all we need is the value \( \chi_0 \).
5.4 Comparison with systematic adiabatic elimination

In order to emphasise the relative simplicity of our method for determining “initial slip” by purely algebraic manipulations, we now describe an alternative method (Haake and Lewenstein [12]): that of systematic adiabatic elimination. (See also a treatment of (33) in a different context by Titulaer [29].) The governing equation (33) for the concentration \( c(x, y, t) \) is written in the form

\[
\frac{\partial c}{\partial t} = (L_0 + L_1) c = Lc, \tag{45}
\]

where for the shear dispersion problem

\[
L_0 = \frac{\partial^2}{\partial y^2}, \quad L_1 = -u(y) \frac{\partial}{\partial x}.
\]

Formally, the solution may be written in terms of the initial concentration distribution as \( c(x, y, t) = e^{Lt} c(x, y, 0) \).

An evolution equation is sought for a “reduced” variable \( C(x, t) = \bar{c} \). This equation will be of the form

\[
\frac{\partial C}{\partial t} = \ell C,
\]

for which the formal solution, subject to the initial condition \( C(x, 0) = C_{\text{eff}}(x) \) is \( C(x, t) = e^{\ell t} C_{\text{eff}}(x) \). Our goal is to calculate the operator \( \ell \), and to determine \( C_{\text{eff}}(x) \) so that the reduced description of the dynamics matches the full solution, that is,

\[
e^{Lt} c(x, y, 0) \sim e^{\ell t} C_{\text{eff}}(x)
\]

for large times. In the notation of the previous subsections \( \ell = -u(1 + h) \partial_x \) and \( C_{\text{eff}}(x) = \chi(x, 0) \).

The computation of these quantities by systematic adiabatic elimination proceeds as follows. First we use (45) and the definition of \( C \) to write

\[
C(t) = \bar{c}(t) = \bar{c}(0) + \int_0^t Lc(\tau)d\tau = \bar{c}(0) + \int_0^t L_{\ell} c(0)d\tau, \tag{46}
\]

where we have suppressed the dependence of \( c \) and \( C \) on \( x \) and \( y \) for notational brevity. Now we assume that \( \partial_x \), and so \( L_1 \), is small, and we carry out a small-\( t \), small-\( L_1 \) expansion for \( C \). The first step is to approximate \( \exp(Lt) \) by \( \exp(L_0 t) \). Then we note the decomposition

\[
e^{Lt} c(0) = c(0) + [e^{Lt} c(0) - c(0)].
\]

The term in brackets is just

\[
L_0^{-1} \frac{\partial}{\partial t} e^{Lt} c(0),
\]
where the “inverse” of $L_0$ has been taken to have zero $y$-mean. Therefore, collecting these results and substituting them into (46), we find

$$C(t) \approx \frac{c(0) + \int_0^t L_1 c(0) d\tau + \int_0^\infty L_1 L_0^{-1} \frac{\partial}{\partial \tau} e^{L_0 \tau} c(0) d\tau}{c(0) + tL_1 c(0) - L_1 L_0^{-1} c(0) + \cdots}.$$

Here we have extended the range of one of the integrals to infinity, and incur exponentially small errors as a result. This expression represents the Taylor series expansion of $C(t)$ for small $t$, with initial condition

$$C_{\text{eff}}(0) = \frac{c(0) - L_1 L_0^{-1} c(0)}{c(0) + \frac{\partial}{\partial x} u(y) L_0^{-1} c(0)} = c(0) + \frac{\partial}{\partial x} c(0) L_0^{-1} u(y).$$

This result agrees with the previous calculation (where we use the result that $\alpha_1 = L_0^{-1} u(y)$).

To solve for even this leading-order contribution to the initial slip (there are also contributions from all higher powers of $\partial_x$) we had to evaluate an integral of the exponential of some operator. For shear dispersion, the integral can be evaluated explicitly, although at higher orders and in truly nonlinear problems the exact computation of such an integral will not be possible. By contrast, the method we have proposed for computing initial slip by purely algebraic manipulations is much more straightforward.

### 6 Conclusion

We have described a method that uses a normal form transformation to determine appropriate initial conditions for low-dimensional models of evolving systems. The nonlinear coordinate transformation allows one to compute the initial slip by solving a succession of algebraic problems, rather than the differential problems associated with other methods. The process has been illustrated with several examples. Using a normal form transformation is simpler than other methods, and also shows, for example, how nonlinear resonances make it impossible, in general, to find good initial conditions for slow sub-centre manifold models such as the quasi-geostrophic approximation or beam theory.

Generally, the calculation of the initial slip will be carried out only approximately (usually by resorting to the truncation of a power series), so the solutions of the model and the full system will differ to some order, after a period of rapid exponential approach. Since the change in variables is made as a power series in the new variables, the approximate normal form equations differ from their exact counterparts by, say, terms of order $|\{\chi, \eta\}|^N$ as $|\{\chi, \eta\}| \to 0$. The differences
will become significant if the initial point is close to the boundary between the basins of attraction of different attractors on \(\mathcal{M}\), or if the attractor on \(\mathcal{M}\) has a positive Lyapunov exponent. However, these are inhospitable circumstances to which to expose our technique, and we have shown elsewhere [6] that for a simple atmospheric model the procedure we have described for determining the initial slip produces a very good agreement between \(P(t)\) and \(Q(t)\) over a reasonable time-scale for forecasting, even when only the first few terms in the power series are kept.

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A Other invariant manifolds

In this appendix we generalise the statements we have so far made about systems with a centre manifold to other types of invariant manifolds. In the context of forming low-dimensional dynamical models this is useful because, for example: some systems, when linearised, also have eigenvalues with positive real parts; some decaying modes may not decay fast enough to be reasonably eliminated in a given application.

A.1 Including linearly unstable modes

First consider the case of a system where some eigenvalues of the linear operator have positive real parts:

\[
\begin{align*}
\dot{x} &= Ax + M(x, y, z) \quad x \in \mathbb{R}^m \\
\dot{y} &= By + N(x, y, z) \quad y \in \mathbb{R}^n \\
\dot{z} &= Cz + L(x, y, z) \quad z \in \mathbb{R}^\ell,
\end{align*}
\]

where the eigenvalues \( \gamma_k \) of \( C \) satisfy \( \Gamma \geq \Re(\gamma_k) > \gamma > 0 \), those of \( A \) satisfy \( \Re(\alpha_k) = 0 \), and those of \( B \) satisfy \( \Re(\beta_k) < -\beta < 0 \). The normal form evolution equations are of the form

\[
\begin{align*}
\dot{\chi} &= A\chi + a(\chi) + \tilde{a}(\chi, \eta, \zeta) \\
\dot{\eta} &= B\eta + \tilde{B}(\chi, \eta, \zeta)\eta \\
\dot{\zeta} &= C\zeta + \tilde{C}(\chi, \eta, \zeta)\zeta,
\end{align*}
\]
where $a(0) = a(\chi, 0, \zeta) = \tilde{a}(\chi, \eta, 0) = 0$.

Immediately we identify five non-trivial invariant subspaces of this system of equations:

- the centre space with $\eta = \zeta = 0$;
- the stable space with $\chi = \zeta = 0$;
- the unstable space with $\chi = \eta = 0$;
- the centre-unstable space with $\eta = 0$;
- the centre-stable subspace with $\zeta = 0$.

In terms of the original variables these five spaces respectively form five invariant manifolds: a centre manifold $M$; a stable manifold $M_S$; an unstable manifold $M_U$; a centre-unstable manifold $M_{CU}$; and a centre-stable manifold, $M_{CS}$.

In the context of constructing low-dimensional models which capture the long-term dynamics of the system, only one of these five invariant manifolds is of real interest. Solutions on the centre manifold, the centre-stable manifold and the stable manifold are all unstable to perturbations in the linearly unstable variables near the origin. Solutions on the unstable manifold, while stable to perturbations in $\eta$, may be unstable to perturbations in the centre variables $\chi$. This leaves the centre-unstable manifold $M_{CU}$ which is invariant and, at least near the origin, attracts all solutions in its neighbourhood at a rate like $e^{-\beta t}$.

On the centre-unstable manifold, the dynamical behaviour is described by

$$\dot{\zeta} = C \zeta + \tilde{C}(\chi, 0, \zeta) \zeta, \quad \dot{\chi} = A \chi + a(\chi).$$

(49)

Unlike the case without linearly unstable modes, the normal form does not in general allow the derivation of initial conditions for a solution $Q(t)$ on $M_{CU}$ that matches a solution $P(t)$ off $M_{CU}$. The reason is possible resonances between the stable and the unstable modes, which only affect the short-term transients (as $\eta \to 0$) but which cannot be removed from the normal form equations. This is most easily seen in a simple example.

Consider the normal form equations

$$\dot{\chi} = \eta \zeta^2, \quad \dot{\eta} = -2\eta, \quad \dot{\zeta} = \zeta - \zeta^3.$$  

(50)

These equations have the centre-unstable manifold $\eta = 0$ which attracts all neighbouring solutions as $\eta = \eta_0 \exp(-2t) \to 0$. Therefore any neighbouring solution $P(t)$ will asymptote exponentially quickly to a solution $Q(t)$ on $M_{CU}$; hence for all neighbouring initial conditions $P(0)$ there is a corresponding initial condition $Q(0)$ on $M_{CU}$. By solving (50) explicitly we find that the solution starting from

$$\chi^* = \chi_0 - \frac{\eta_0 \zeta_0^2 \log \zeta_0}{1 - \zeta_0^2}, \quad \eta^* = 0, \quad \zeta^* = \zeta_0$$

(51)
has the same long-term dynamics as that starting from \((\chi_0, \eta_0, \zeta_0)\).

The presence of the logarithm in this expression is novel. It indicates that the assumed power-series representation for the normal form transformation is not sufficiently general for our purposes. Indeed if we make the nonlinear change of variable

\[ \theta = \chi - \frac{\eta \zeta^2 \log \zeta}{1 - \zeta^2} \]

and leave \(\eta\) and \(\zeta\) unchanged then (50) becomes

\[ \dot{\theta} = 0, \quad \dot{\eta} = -2\eta, \quad \dot{\zeta} = \zeta - \zeta^3. \]

The new centre variable \(\theta\) is not forced by any resonant term, and the projection of initial conditions onto the centre-unstable manifold is trivial.

Let \(\Gamma\) be a realistic upper bound on \(\Re(\gamma_k)\) of \(C\) and let \(-\beta\) be a realistic upper bound on \(\Re(\beta_k)\) of \(B\) (see text just after (47)). Then if \(\Gamma > \beta\), that is, if the most rapidly growing mode grows faster than the slowest decaying mode decays, just one of the decaying modes may directly interact with just one of the growing modes—the resonance could be as direct as a second-order \(\eta_i \zeta_j\) term. However, if \(\Gamma < \beta\), that is, if the slowest decay occurs more rapidly than the most rapid growth, then any resonance must be of order \(\eta_i \zeta_j^{\beta/\Gamma}\) or higher. Thus the normal form projection of initial conditions can be carried out to this order of accuracy. Observe that as \(\Gamma \to 0\), when the centre-unstable manifold becomes a centre manifold, \(\beta/\Gamma \to \infty\), and the order of the resonance moves out to infinity. This limit recovers the case of a centre manifold where there is no problem in computing an appropriate initial condition \(Q(0)\).

### A.2 Keeping some decaying modes

In a particular application it may be that some of the linearly stable modes of \((\ref{eq:47})\) are dynamically significant, for example, if their decay is relatively slow. An example is shear dispersion in a channel or pipe, where the slowest modes decay on a cross-stream diffusion time-scale that may be comparable with other physical processes of interest. We then try to construct an invariant manifold \(M_I\) that includes not only the critical \(x\)-modes together with the unstable modes \(z\), but also the slower-decaying of the \(y\)-modes. For shear dispersion this was done by Watt & Roberts \([30]\).

After renaming the \(z\) variables and some of the \(y\) variables of \((\ref{eq:47})\) as \(x\) variables, we consider a dynamical system given by (1–2) but where the eigenvalues \(\alpha_k\) of \(A\) and those of \(B\) satisfy

\[ \Re(\beta_k) < -\beta < -\gamma < \Re(\alpha_k) < \Gamma \quad \beta, \gamma > 0. \]

Again without loss of generality we take \(A\) and \(B\) to be diagonal matrices. The variables \(x\) are termed master variables in this context \([3]\), while the variables \(y\) are termed slave variables.
As before, we make a nonlinear change of variables. Resonance may now generate a term that includes purely master variables as a forcing in the slave equations. Thus, even in the normal form, the slaved variables do not necessarily evolve quickly to zero. This means that they may evolve non-trivially at large times; that is, on the long-term time-scale of the master variables (“short-term” then refers to the time-scale of the slave variables’ decay). A resonance occurs at lowest order when $j$ master modes combine to force one slave mode; using the bounds on the eigenvalues, the lowest order at which the resonance can occur is

$$j_s = \frac{\beta}{\gamma}.$$ 

This resonance is symptomatic of the non-uniqueness of invariant manifolds. The non-uniqueness first appears at this order; as discussed by Roberts [23] it appears as a zero divisor in the construction of the invariant manifold — there is little point in computing higher orders in the expansions because the higher order differences are smaller than those ignored by the model.

Resonant interactions between purely master modes are the important interactions in the long term. However, any interaction with a slave mode results in the undesirable feature that during the evolution onto $M_I$ the master dynamics depend upon the slave dynamics—an effect which we want to ignore. The lowest order at which this resonance may appear is if one slave mode interacts with $j - 1$ of the most rapidly growing master modes to force the most rapidly decaying master mode. Thus the lowest order for this resonance is

$$j_m = 1 + \left(\frac{\beta - \gamma}{\Gamma}\right).$$

Such resonance makes it hard to prescribe correct initial conditions on the low-dimensional manifold $M_I$ for any given initial condition of the full system. However, in many examples, such as (50), this reflects the limitations of the standard power-series representation.

Observe that these effects show up at high-order only if the spectral gap between the master and the slave modes, $[-\beta, -\gamma]$, is large, both relatively (large $\beta/\gamma$) and absolutely (large $\beta - \gamma$), when compared to the natural growth rates in the dynamics. Thus such a large spectral gap is desirable in constructing low-dimensional dynamical models.