Physical states in $N = 1$ supergravity

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Abstract
By solving the supersymmetry constraints for physical wave-functions, it is shown that the only two allowed bosonic states in $N = 1$ supergravity are of the form $\text{const.} \exp (\pm I/\hbar)$, where $I$ is an action functional of the three-metric. States containing a finite number of fermions are forbidden. In the case that the spatial topology is $S^3$, the state $\text{const.} \exp (-I/\hbar)$ is the wormhole ground state, and the state $\text{const.} \exp (I/\hbar)$ is the Hartle–Hawking state. $N = 1$ supergravity has no quantum ultraviolet divergences, and no quantum corrections.

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Usually, quantum supergravity is treated in terms of scattering theory using a path-integral approach. However, the full quantum theory can instead be treated non-perturbatively by studying the quantum constraints acting on the wave function [1], i.e. by taking an approach based on functional differential equations. This approach is used here to show that there are only two purely bosonic wave-functions in $N = 1$ supergravity, which have the simple form $\exp(\pm I/h)$, where $I$ is an action functional of the 3-metric. Further, states containing a finite number of fermions are forbidden.

A wave-function can be taken to be of the form $\Psi\left(e^{AA'}_i(x), \psi^A_i(x)\right)$. Here, using 2-component notation [1], $e^{AA'}_i(x)$ is the spatial tetrad, which gives the 3-metric as $h_{ij} = -e^{AA'}_i e_{AA'}^j$, and $\left(\psi^A_i, \tilde{\psi}^{A'}_i\right)$ is the spatial gravitino field, taken to be an odd Grassmann quantity. The wave-function can equivalently be described by $\tilde{\Psi}\left(e^{AA'}_i(x), \tilde{\psi}^{A'}_i(x)\right)$, which is related to $\Psi$ by a fermionic Fourier transform [1]. A physical wave-function must obey the Lorentz and supersymmetry constraints

$$J^{AB}\Psi = 0, \quad \bar{J}^{A'B'}\Psi = 0, \quad (1), (2)$$
$$S^A\Psi = 0, \quad \bar{S}^{A'}\Psi = 0. \quad (3), (4)$$

The remaining Hamiltonian constraints $H_{AA'}\Psi = 0$, suitably ordered, will be implied by the above constraints [2]. The Lorentz constraints imply that $\Psi$ is invariant under local Lorentz rotations applied to the arguments $\left(e^{AA'}_i(x), \psi^A_i(x)\right)$. Thus all Lorentz indices must be contracted together in $\Psi$. Hence $\Psi$ can only contain an even number $\psi^0, \psi^2, \psi^4, \ldots$ of fermions. Further the supersymmetry constraints do not mix fermion number; thus one can study the states proportional to $\psi^0, \psi^2, \psi^4, \ldots$ separately. The constraint $\bar{S}^{A'}\Psi = 0$ reads

$$\epsilon^{ijk} e_{AA'i} \left(3 s D_j \psi^A_k\right) \Psi - \frac{1}{2} \hbar k^2 \psi^A_i \frac{\delta \Psi}{\delta e^{AA'}_i} = 0. \quad (5)$$
Here \(3^s D_j\) is the 3-dimensional covariant derivative on spinors without torsion \([1]\), and \(\kappa^2 = 8\pi\). The constraint \(S_A \Psi = 0\) reads

\[
3^s D_i \left( \frac{\delta \Psi}{\delta \psi^A_i} \right) + \frac{1}{2} h \kappa^2 \frac{\delta}{\delta e^{AA}_i} \left( D^{BA'}_{ji} \frac{\delta \Psi}{\delta \psi^{B'}_j} \right) = 0 ,
\]

where

\[
D^{BA'}_{ji} = -2i h^{-\frac{1}{2}} e^{BB'}_i e^{CB'}_j n^{CA'} .
\]

Here \(h = \det (h_{ij})\), and \(n^{AA'}\) is the spinor version of the unit future-pointing normal \(n^\mu\) to a surface \(x^0 = \text{constant}\). This is defined as a function of the \(e^{AA'}_i\) by

\[
n^{AA'} e_{AA'}^i = 0 , \quad n^{AA'} n_{AA'} = 1 .
\]

The \(S_A\) constraint is more easily understood in the representation \(\tilde{\Psi} \left( e^{AA'}_i, \tilde{\psi}^{A'}_i \right)\), where it reads

\[
\epsilon^{ijk} e_{AA'}^i \left( 3^s D_j \tilde{\psi}^{A'}_k \right) \tilde{\Psi} + \frac{1}{2} h \kappa^2 \tilde{\psi}^{A'}_i \frac{\delta \tilde{\Psi}}{\delta e^{AA'}_i} = 0 .
\]

We now restrict attention to a purely bosonic wave function \(\Psi \left( e^{AA'}_i(x) \right)\). This automatically obeys the constraint \(S_A \Psi = 0\) \([\text{Eq. (6)}]\). Consider then the constraint \(\overline{S}_A \Psi = 0\) \([\text{Eq. (5)}]\). Since \(\Psi\) is purely bosonic, one can rewrite \(\text{Eq. (5)}\) as

\[
\epsilon^{ijk} e_{AA'}^i \left( 3^s D_j \psi^A_k \right) - \frac{1}{2} h \kappa^2 \psi^A_i \frac{\delta (\ln \Psi)}{\delta e^{AA'}_i} = 0 .
\]

One now remarks that this gives a unique expression for \(\frac{\delta (\ln \Psi)}{\delta e^{AA'}_i(x)}\) as a functional of \(e^{AA'}_i(x)\). To see this, note that the field \(\psi^A_i(x)\), equivalently given by

\[
\psi^A_{BB'} = \psi^A_i e_{BB'}^i ,
\]

(11)
can be decomposed into a spin - $\frac{3}{2}$ part $\gamma_{ABC}$ and a spin - $\frac{1}{2}$ part $\beta_A$ as [1]

$$
\psi_{ABB'} = -2n^C_B\gamma_{ABC} + \frac{2}{3}(\beta_A n_{BB'} + \beta_B n_{AB'}) - 2\epsilon_{ABn}^{C} B' \beta_C ,
$$

(12)

where $\gamma_{ABC} = \gamma_{(ABC)}$ is totally symmetric. One can then decompose $\psi^A_i(x)$ into spinor harmonics, using the bases $\gamma^{(n)}_{ABC}$ and $\beta^{(m)}_A$ which obey

$$
e^{AA'j} 3s D_j \gamma^{(n)}_{ABC} = i\mu_n n^{AA'} \gamma^{(n)}_{ABC} ,
$$

(13)

$$
e^{AA'j} 3s D_j \beta^{(m)}_A = i\lambda_m n^{AA'} \beta^{(m)}_A .
$$

(14)

[The indices $m, n$ have been written as discrete, for a compact 3-manifold, and should be replaced by continuous indices for a non-compact manifold.] One writes

$$
\gamma_{ABC} = \sum_n c_n \gamma^{(n)}_{ABC} ,
$$

(15)

$$
\beta_A = \sum_m b_m \beta^{(m)}_A ,
$$

(16)

where the coefficients $c_n$ and $b_m$ are odd Grassmann quantities, and $\gamma^{(n)}_{ABC}$, $\beta^{(m)}_A$ are even.

One substitutes Eqs. (15, 16) into Eq. (12), to obtain an expansion for

$$
\psi_A^i = -\psi_A^{BB'} e_{BB'}^i .
$$

(17)

The first term in the constraint (10) can then be expanded, using Eqs. (13, 14), as

$$
\epsilon^{ijk} e_{AA'i} (3s D_j \psi^A_k) = -\frac{8}{3} h^{\frac{1}{2}} n^A A' \sum_m \lambda_m b_m \beta_A^{(m)} .
$$

(18)

Since the constraint (10) must hold for all choices of $\psi^A_i(x)$, it must equivalently hold for all choices of the coefficients $b_m$ and $c_n$ in Eqs. (15, 16). Hence

$$
e_{BB'}^{i} n^{CB'} \gamma^{(n)AB} C \frac{\delta (\ln \Psi)}{\delta e^{AA'i}} = 0 ,
$$

(19)
\[-\frac{8}{3} \hbar \frac{2}{3} \lambda_m n^A A' \beta_A^{(m)} \]

\[+ \frac{1}{2} \hbar \kappa^2 e_{BB'}^i \left( \frac{2}{3} \beta^{(m)B} n^A A' - 2 e^{AB} n^{CB'} \beta_C^{(m)} \right) \frac{\delta \ln \Psi}{\delta e_i^{AA'}} = 0 , \tag{20} \]

for all \( m \) and \( n \). These are the same equations as would appear if \( \psi^i_A(x) \) were taken to be bosonic in Eq. (10). One can then contract Eqs. (19, 20) with an arbitrary bosonic field

\[\delta A' = \sum_p d_p \delta^{(p)} A' , \tag{21}\]

where

\[e^{AA'} \gamma_{ij} D_j \delta^{(p)} A' = -i \lambda_p n^{AA'} \delta^{(p)} A' . \tag{22}\]

Eqs. (19, 20) contracted with terms of the form \( d_p n^{AA'} \delta^{(p)} A' \) show that the variation \( \int d^3x \left[ \delta \ln \Psi / \delta e_i^{AA'}(x) \right] \delta e_i^{AA'}(x) \) can be found by writing the general variation \( \delta e_i^{AA'}(x) \) as

\[\delta e_i^{AA'}(x) = 2 e_{BB'}^i n^{CB'} \sum_p \sum_n e_{pn} \delta^{(p)} A' \gamma^{(n)AB}_C \]

\[-\frac{8}{3} e_{BB'}^i n^{AB'} \sum_p \sum_m f_{pm} \delta^{(p)} A' \beta^{(m)}_B , \tag{23}\]

where \( e_{pn} \) and \( f_{pm} \) are even quantities. This variation in \( e_i^{AA'}(x) \) produces the variation

\[\delta (\ln \Psi) = -\frac{16}{3 \hbar \kappa^2} \int d^3x h \frac{2}{3} n^A A' \sum_p \sum_m \lambda_m f_{pm} \delta^{(p)} A' \beta^{(m)}_A . \tag{24}\]

Thus \( \delta (\ln \Psi) \) is determined uniquely for a typical variation \( \delta e_i^{AA'}(x) \), through the \( \overline{S}_A' \Psi = 0 \) constraint (10). An analogous differential equation for a finite-dimensional system is \( \partial f / \partial x = F(x) \), which has a unique solution given a starting value \( f(x_0) \). Similarly, the \( \overline{S}_A' \Psi = 0 \) constraint in the bosonic sector must have a unique solution \( \Psi(e_i^{AA'}(x)) \), up
to a constant factor, related to a starting value $\Psi(e_{(0)i}^{AA'}(x))$, where $e_{(0)i}^{AA'}(x)$ is a reference spatial tetrad.

The wave function $\Psi$ will have a semi-classical expansion

$$\Psi \sim (A_0 + \hbar A_1 + \hbar^2 A_2 + \ldots) \exp (-I/\hbar) ,$$

where $I$ is a certain action functional of the spatial tetrad, and $A_0, A_1, A_2, \ldots$ are loop prefactors. Inserting this into the $\mathcal{S}_A \Psi = 0$ constraint (5), one obtains

$$\epsilon^{ijk} e_{AA'i} 3 \hbar D_j \psi^A_k + \frac{1}{2} \hbar^2 \psi^A_{i} \frac{\delta I}{\delta e^{AA'i}} = 0 .$$

(26)

The uniqueness of the solution of the constraint (10) then implies that

$$\Psi = c_0 \exp (-I/\hbar) .$$

(27)

[Alternatively, one can examine the constraint (10) for $\hbar \ln \Psi$. This differential equation has no $\hbar$ in it, and so one expects that the solution has the form $\hbar \ln \Psi = -I + \text{const.}$] Here $\Psi$ obeys the quantum $S^A$ and $\mathcal{S}^{A'}$ constraints. It obeys the Lorentz rotation constraints, since $I$ must be formed in a Lorentz-invariant way by contraction of all spinor indices. Hence $\Psi = c_0 \exp (-I/\hbar)$ gives the general purely bosonic solution of the quantum constraints. The classical action functional $I$ depends on the 3-geometry only though the 3-metric $h_{ij}$, and its Hamilton–Jacobi trajectories gives solutions of the classical positive-definite Einstein equations.

The above uniqueness refers to a purely bosonic state $\Psi(e^{AA'}_{i}(x))$ containing no fermions. One can also examine the “filled” state in which all fermionic states are filled. In the representation $\bar{\Psi} \left( e^{AA'}_{i}(x), \bar{\Psi}^{A'}_{i}(x) \right)$, this corresponds to the $\bar{\Psi}^0$ part of the wave function. This obeys the $S_A \bar{\Psi} = 0$ constraint (9), which similarly gives a solution

$$\bar{\Psi} = c_1 \exp (I/\hbar) ,$$

(28)
where $I$ is the same action as in Eq. (27). This state is unique among the filled states.

Exponential solutions of the type const. exp(±$I/\hbar$) have previously been found in mini-superspace examples, where supergravity is quantized subject to the Bianchi I or IX Ansatz [3, 4, 5]. It was also found in [4, 5] that no fermionic states were allowed, because of the restrictive nature of the $S_{A}\Psi = 0$ and $\overline{S}_{A'}\Psi = 0$ constraints imposed together. Similarly one can show that there are no fermionic states at levels $\psi^{2n}(n = 1, 2, 3, \ldots)$ in the full theory of supergravity studied here. To see this, note that the supersymmetry constraints (5), (6) each give an equation for $\delta\Psi/\delta e^{AA'_{i}}(x)$ at level $\psi^{2n}$. These equations follow on the lines of Eqs. (23), (24). For consistency, these first-order equations must be identical, otherwise one obtains $\Psi = 0$ at order $\psi^{2n}$. But if the two supersymmetry constraints $\overline{S}_{A'}\Psi = 0$, $S_{A}\Psi = 0$ give identical equations for $\delta\Psi/\delta e^{AA'_{i}}(x)$ at order $\psi^{2n}$, then the form of their anti-commutator contradicts the form of the Hamiltonian constraint $H_{AA'} = 0$ at the corresponding order. Thus there are no fermionic states at levels $\psi^{2n}(n = 1, 2, 3, \ldots)$. The case with an infinite number of fermions might of course be different.

Consider now the case in which the 3-surface has the topology of $S^{3}$. On general grounds, there are two preferred quantum states in this case - the Hartle–Hawking state [6] and the wormhole ground state [7]. The Hartle–Hawking state is defined by a path integral in which one fills in inside the 3-surface subject to the given data on it; the wormhole ground state is defined by a path integral in which one sums over fields outside the 3-surface which are asymptotically Euclidean, i.e. subject to asymptotic flatness at infinity. Since for $N = 1$ supergravity we have only two allowed quantum states, there should be a simple linear relation between these and the Hartle–Hawking and wormhole ground states. This relation can be checked by studying the exact Friedmann case of spherical symmetry with radius $a$ of the 3-sphere [8]. One finds the wormhole state exp $(-3a^{2}/\hbar)$ for the bosonic state exp $(-I/\hbar)$, and the Hartle–Hawking state exp $(3a^{2}/\hbar)$ for the filled state exp $(I/\hbar)$. Hence quite generally (on $S^{3}$) $c_{0}$ exp $(-I/\hbar)$ gives the wormhole ground state,
and $c_1 \exp(I/\hbar)$ gives the Hartle–Hawking state. It would be of interest to investigate this relation for other compact topologies.

Making a loop expansion of the wormhole state, one has

$$c_0 \exp(-I/\hbar) \sim (B_0 + \hbar B_1 + \hbar^2 B_2 + \ldots) \exp(-I_1/\hbar), \quad (29)$$

where $I_1$ is the classical action outside the $S^3$ subject to asymptotic flatness and to the prescribed 3-metric $h_{ij}$ on the $S^3$. Hence $I = I_1$, and $c_0 \exp(-I/\hbar)$ is the wave function for the wormhole state. Similarly $c_1 \exp(I/\hbar)$ is the Hartle–Hawking wave function, where $-I$ is the classical action found by solving the positive-definite Einstein equations inside the $S^3$ with prescribed 3-metric $h_{ij}$ on the $S^3$. This shows that there is a connection between the dynamical field equations of supergravity and the initial or boundary conditions for the wave function.

This highly restrictive form of the solution to the quantum constraints may be a feature only of $N = 1$ supergravity, or perhaps also of higher - $N$ supergravity without couplings. When one turns to supergravity coupled to supermatter, it is immediate that many more solutions to the constraints are possible, essentially because the wave function $\Psi$ depends on more fields, but does not have to obey any more constraints than in pure supergravity. The special form of the solution found here should be a feature of pure supergravity only.

It will, of course, also be of interest to compare the approach of this paper with the standard approach based on scattering theory, to understand why scattering states should be forbidden. It is now clear, however, that $N = 1$ supergravity is a theory without quantum ultraviolet divergences, and without any quantum corrections at all.

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