Fractional Aharonov-Bohm oscillation of a two-layer ring with two electrons

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When a circular ring suffers a special topological transformation, it becomes a two-layer ring. Due to the special topology of the two-layer ring, orbital angular momenta are allowed to be a half-integer, this would affect the traditional Aharonov-Bohm oscillation (ABO). In this paper the fractional ABO of the ground state energy, persistent current, and dipole transition of a two-layer ring with two electrons has been studied. Collective and internal coordinates \( \theta_C \) and \( \varphi \) have been introduced. Based on them a very simple formula for the current has been obtained, the symmetry constraint imposed on the dipole transition has been clarified, a strict relation between the photon energies of the dipole radiation and the persistent current of the ground state has been found. Comparing with the one-layer rings, the period of the fractional ABO of the two-layer rings becomes much shorter.

(1) Introduction

The quantum ring\(^1\) is an important member of micro-devices and is promising in application. Therefore it has been extensively studied theoretically and experimentally in recent years. It is well known that the quantum ring has special properties due to the special ring geometry. A distinguished feature is the Aharonov-Bohm oscillation (ABO) and the fractional ABO (FABO) of the eigenenergies and persistent currents\(^2\)-\(^5\). On the other hand, due to the progress in experimental techniques, distorted rings (say, a ring undergoes a topological transformation) with specific geometries containing a given number of electrons can be in principle fabricated. It is believed that, after a topological transformation, the physical properties would be accordingly changed. This is a way to control the properties of micro-devices, therefore the effect of topological transformations on the rings is worthy to be studied.

In this paper we shall consider a special case of topological transformation. Let a one-dimensional ring be twisted into a "8"-shape as shown in Fig.1a, where upper and lower circles have the same size. Then, this shape is further bended around the point \( P_2 \) so that \( P_1 \) and \( P_3 \) are close to each other as shown in Fig.1b. This system is called a two-layer ring. In such a device, when an electron in the lower (upper) layer is passing through the terminal \( P_2 \), it must go to the upper (lower) layer, and this is the only one choice. Although the two layers are close to each other (the interdistance is considered as zero in the following calculation), the electrons are not allowed to penetrate from one layer to the other one except at \( P_2 \). Thus, the domain of the azimuthal angle \( \theta \) of each electron is no more \([0, 2\pi]\) but \([0, 4\pi]\). Accordingly, the periodicity of the system has been changed. This change definitely will cause a number of physical consequences, e.g., orbital angular momenta with half-integers are allowed as shown below. The aim of this paper is to clarify the effect of the above topological transformation on the energy spectra, particle correlations, persistent currents, and optical properties of the system. As a first step, the two-layer ring is assumed to contain two electrons (the simplest case having electron-electron correlation). Comparison with the results of the usual one-layer rings is made. The emphasis is to demonstrate how the (F)ABO is affected.

Let a magnetic field \( B \) be applied perpendicular to the plane of the rings. The Hamiltonian of the two-layer ring with two electrons reads

\[
H = T + V_{12} + H_{\text{Zeeman}}
\]

\[
T = \sum_{j=1}^{2} \frac{\hbar^2}{2m} \left( -i \frac{\partial}{\partial \theta_j} + \Phi \right)^2, \quad G = \frac{k^2}{2m^* R^2} \tag{1}
\]

where \( \theta_j \) is the azimuthal angle of the \( j \)-th electron with a range from 0 to \( 4\pi \), \( \Phi = \pi R^2 B/\Phi_0 \), where \( \Phi_0 = hc/e \) is the flux quantum, \( m^* \) is the effective mass. \( H_{\text{Zeeman}} \) is the well known Zeeman energy. \( V_{12} \) is the Coulomb interaction which can be adjusted as

\[
V_{12} = \frac{e^2}{2\epsilon \sqrt{d^2 + R^2 \sin^2 \theta}} \tag{2}
\]

where \( \epsilon \) is the dielectric constant, and the parameter \( d \) is introduced in ref.\(^6\) to embody the effect of finite width of the ring.

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To diagonalize the Hamiltonian, we introduce a set of basis functions
\[ \phi_{k_1 k_2} = \frac{1}{\sqrt{2}} e^{i(k_1 \theta_1 + k_2 \theta_2)/2} \]
where \(k_1\) and \(k_2\) must be integers to assure the periodicity and \((k_1 + k_2)/2\) is just the total orbital angular momentum \(L\), which is a good quantum number. Due to the special topology as mentioned, \(L\) is allowed to be a half-integer, this is greatly different from usual one-layer rings. \(\phi_{k_1 k_2}\) must be further anti-symmetrized (symmetrized) when the total spin \(S\) is \(0\) (for \(S=1\)).

In this paper \(meV\), \(nm\) and \(Tesla\) are used as units, \(m^* = 0.063m_e\), \(e = 12.4\) (for InGaAs), \(R = 30\), \(d = 0.05R\) is adopted. The magnitude of \(d\) is not sensitive to the qualitative results. The amount of basis functions should be large enough to assure the accuracy. Actually, we found that about fifteen hundreds basis functions are enough (to have six effective figures). The numerical results and related analysis are shown as follows.

(2) Separability of the Hamiltonian and the spectra

We define
\[ \theta_c = \frac{\theta_1 + \theta_2}{2}, \quad \varphi = \theta_2 - \theta_1 \]
(4)
to describe the collective and internal motions, respectively. Then the Hamiltonian is rewritten as
\[ H = H_{coll} + H_{int} \]
(5)
where
\[ H_{coll} = \frac{G}{2} (-i \frac{\partial}{\partial \theta} + 2\Phi)^2 + H_{Zeeman} \]
\[ H_{int} = 2G (-i \frac{\partial}{\partial \theta} + \Phi)^2 + V_{12} \]
(6)
The collective Hamiltonian \(H_{coll}\) depends only on \(\theta_c\) and is equivalent to the Hamiltonian of a single particle with a double mass and a double charge. The internal Hamiltonian \(H_{int}\) depends only on \(\varphi\) and is irrelevant to \(B\). It implies that the eigen-states of \(H_{int}\), namely, the internal states, do not depend on \(B\).

The basis functions can be rewritten as
\[ \phi_{k_1 k_2} = \frac{1}{\sqrt{4\pi}} e^{iL\theta_c} e^{i\theta_1/2} \varphi \]
(7)
The separability of the Hamiltonian leads to the separability of the eigen-energies and eigen-states. Thereby the eigen-energy
\[ E = \frac{G}{2} (L + 2\Phi)^2 + E_{int} - S_z \mu \Phi \]
where the first term \(G/2 \delta (L + 2\Phi)^2\) is the kinetic energy of the collective motion, the second term \(E_{int}\) is the internal energy and the third term \(-S_z \mu \Phi\) is the well known Zeeman energy. In our units, \(G = 604.8/R^2\), \(\mu = 33.33/R^2\). The eigen-state with the good quantum number \(L\) reads
\[ \Psi = \frac{1}{\sqrt{4\pi}} e^{iL\theta_c} \psi_{int} \]
(9)
where \(\frac{1}{\sqrt{4\pi}} e^{iL\theta_c}\) describes the collective motion, while \(\psi_{int}\) is the normalized internal state depending only on \(\varphi\). The eigen-states having the same \(L\) and \(S\) may be different in \(\psi_{int}\), they form a series. From now on the label \((L, S, i)\) denotes the \(i\)-th state of the series. In particular, the \(i = 1\) state is called the first-state of the series, they are candidates of the ground state depending on the flux \(\Phi\). After the diagonalization of the Hamiltonian, the numerical results are obtained as follows

FIG. 2: (color online). The spectra of a two-layer ring (left column) and a one-layer ring (right column) against \(L\) with \(B = 0\). The associated internal energies \(E_{int}\) are also plotted. In 2a all the \(i = 1\) levels are joined by a dotted curve to guide the eyes, this curve splits into two in 2c.

When \(B = 0\), the low-lying eigen-energies \(E_{(L, S, i)}\) against \(L\) are plotted in Fig.2a (for \(S = 1\)) and 2c (for \(S = 0\)), where, for each \((L, S)\)-series, the lowest three levels with \(i = 1, 2, 3\) are plotted. It was shown in both 2a and 2c that the levels lie along three parabolic curves (in 2a one of the parabolic curve is explicitly plotted via a dotted curve). All the levels of each curve have the same internal energy \(E_{int}\), while their collective energies are equal to \(G L^2\); this leads to the parabolic behavior. The associated lowest three internal energies are plotted in Fig.2b and 2d. The two sets of internal energies with \(S = 1\) and \(0\), respectively, are one-to-one extremely close to each other. It implies that the internal structures depend on \(S\) very weakly. Fig.2e to 2h are for a one-layer ring (with also two electrons and with the same \(R\)) plotted for a comparison. There is a great difference between the spectra of the two- and one-layer rings. Firstly, the spectrum of the two-layer ring is much denser because
orbital angular momenta are allowed to be not only integers but also half-integers, and because the excitation energies of the internal states are lower (say, comparing 2b and 2f). Secondly, each of the three internal energies in 2b and 2d splits into two internal energies as shown in 2f and 2h, one is for ̂L even states, another for ̂L odd states. Accordingly, each parabolic curve in 2a and 2c splits into two in 2e and 2g (e.g., the dotted curve in 2a splits into two dotted curves plotted in 2e).

The splitting of the internal energies implies that the internal structures of one-layer rings depend seriously on (−1)̂L. This fact has a profound symmetry background. It is noted that the most favorable configuration arises when the two electrons are far away from each other, namely, |θ1 − θ2| = π, where the e-e repulsion is minimized. At this configuration, a spatial interchange of the electrons is equivalent to a rotation of the system by π. The former imposes a factor (−1)S on the wave function due to the fermionic statistics, while the latter imposes the well known factor e−iπ̂L = (−1)̂L. Thus, the favorable configuration is allowed only if (−1)S = (−1)̂L. Otherwise, the wave function would contain a node at the configuration. Obviously, the appearance of the node would cause an increase of energy, this is the origin of the splitting as shown clearly in 2f and 2h. In 2f (S = 1) the Eint with ̂L odd is lower, while in 2h (S = 0) the Eint with ̂L even is lower. The above mechanism, namely, the equivalence of the interchange and rotation, does not hold for two-layer rings, thus the associated splitting does not appear in 2b and 2d.

When B ≠ 0, the low-lying eigen-energies against Φ are plotted in Fig.3a to 3c to be compared with those of one-layer rings plotted in 3d to 3f, where the oscillation of the ground state (GS) energy can be clearly seen. The associated GS angular momentum ̂L0 are marked by the curves. Since the internal energy does not depend on B, when B varies the variation of the GS energy is caused by the collective term |1/2G(L0 + 2Φ)|2, where the increase of Φ leads to a decrease of ̂L0 step by step, this is the mechanism of the (F)ABO. For one-layer rings as shown in 3f, the decrease of ̂L0 is each step by 1, accordingly the period of Φ is 1/2. Furthermore, the transition of ̂L0 is accompanied by the transition of the GS total spin ̂S0 to keep (−1)̂L0 = (−1)̂S0. This is a necessary condition for the GS to be free from the inherent node at the |θ1 − θ2| = π configuration. However, for two-layer rings, the decrease of ̂L0 is each step by 1/2 as shown in 3c, accordingly the period of Φ is 1/4 implying a much denser oscillation. Besides, the constraint (−1)̂L0 = (−1)̂S0 is no more valid, there is no transition of ̂S0, instead, the total spin remains to be 1 due to the Zeeman energy.

(3) Structure of internal states

The density of an internal state is defined as

ρ1(ϕ) = ψ∗ int(ϕ)ψ int(ϕ) (10)

Examples are given in Fig.4 for i = 1 to 4 states. There are the following features:

(i) For each case of i the curves with distinct L and S nearly overlap, it implies that ψ int(ϕ) depends on L and S very weakly. On the contrary, the internal states of one-layer rings depend on (−1)̂L and S very strongly.8

(ii) More nodes would be contained if i is larger as
clearly shown from a to d. However, since \( \psi_{\text{int}} \) must be a superposition of \( e^{i\mathcal{L} \varphi} \) (refer to eq. (7)), therefore the periodicity of \( \psi_{\text{int}} \) depends on \( L = (k_1 + k_2)/2 \). When \( L \) is an integer, the period of \( \varphi \) is \( 4\pi \), when \( L \) is an half-integer, the period of \( \varphi \) is \( 8\pi \). Thus, due to the distinction in periodicity, the structures of the internal states might be different. It was found that the effect of the periodicity is very weak when \( i \) is small. However, when \( i \geq 3 \) this effect can be seen clearer as shown in Fig.4c and 4d. Incidentally, the periodicity would also affect the internal energies and can be seen if \( i \) is larger as shown by the \( i = 3 \) levels in Fig.2b and 2d.

(iii) The favorable configuration, namely, \( |\theta_1 - \theta_2| = \pi \) or \( 3\pi/2 \) is possessed by all the first-states distinct in \( L \) and \( S \). This is shown in Fig.4a.

(4) Persistent current

It is well known that, from the conservation of mass, the current of the first electron is

\[
J_1 = \frac{e}{2} \int [\Psi^*(-i\frac{\partial}{\partial \theta_1} + \Phi) \Psi + c.c.] d\theta_2
\]

where \( g = \hbar/(m^* R^2) \). Since \( J_1 \) does not depend on \( \theta_2 \), it can be integrated over \( \theta_2 \) and then divided by \( 4\pi \). Thus the total persistent current \( J = J_1 + J_2 \) can be written as

\[
J = \frac{g}{8\pi} \int [\Psi^*(-i\frac{\partial}{\partial \theta_1} - i\frac{\partial}{\partial \theta_2} + 2\Phi) \Psi + c.c.] d\theta_1 d\theta_2
\]

where both \( \theta_1 \) and \( \theta_2 \) are integrated from 0 to \( 4\pi \). Since \( \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2} = \frac{\partial}{\partial \theta} \). By using the arguments \( \theta_c \) and \( \varphi \) and making use of the separability, the integration can be easily performed, and we have

\[
J = \frac{1}{12} g (L + 2\Phi)
\]

Since the current and the eigen-energy both contain the factor \((L + 2\Phi)\), they have the same mechanism of oscillation (namely, the transition of \( L \) each step by \( 1/2 \) in accord with \( \Phi \)) with the same period \( \Phi = 1/4 \), their oscillations match with each other. The maximum of \( J \) arises when \( L \) undergoes a transition, we have \( |J_{\text{max}}| = g/16\pi \).

On the other hand, for one-layer two-electron rings, \( J = \frac{g}{8\pi} g(L + 2\Phi) \). When \( B \) is not large, the period of \( \Phi \) is \( 1/2 \) and the maximal current \( J_{\text{max}} = g/4\pi \). When \( B \) is sufficiently large and enters into the ABO region (so that the GS is dominated by \( S = 1 \) states), the period of \( \Phi \) is 1 and the maximal current \( |J_{\text{max}}| = g/2\pi \).

Thus, comparing with the one-layer rings, the strength of \( J \) of the two-layer rings is at least four times weaker and the period is at least two times shorter. This is shown in Fig.5a and 5c. There is one more point noticeable. For one-layer rings, the FABO will be gradually changed to ABO during the increase of \( B \). This is shown in 5c where the amplitude of the current increases with \( B \) until \( B \) arrives at the border of the ABO region. Such an increase does not appear for two-layer rings.

(5) Dipole transitions and related photon energies

Let the labels \((\text{init}) \) and \((f) \) be used to denote the initial and final states of a dipole transition. For the transition from \( \Psi_{L,S,i}^{\text{(init)}} \) to \( \Psi_{L',S',i'}^{(f)} \), the probability is

\[
P_{(\text{init}) \pm}^{(f)} = \frac{2\gamma^2 (\omega)^3 R^2}{\hbar} \left| < \Psi_{L',S',i'}^{(f)} | (e^{\pm i\theta_1} + e^{\pm i\theta_2}) | \Psi_{L,S,i}^{\text{(init)}} > \right|^2
\]

where \( \hbar \omega = E_{(f)} - E_{(\text{init})} \) is the photon energy. Since \( e^{\pm i\theta_1} + e^{\pm i\theta_2} \equiv 2e^{\pm i\theta_0} \cos(\varphi/2) \), during the transition not only the total orbital angular momentum arising from the collective rotation should be changed by \( \pm 1 \), the internal structure would be changed also. Using the internal degree of freedom and making use of the separability, the probability can be rewritten as

\[
P_{(\text{init}) \pm}^{(f)} = \frac{8\gamma^2 (\omega)^3 R^2}{\hbar} \delta_{L',L \pm 1} \left| < \psi_{\text{int}}^{(f)} | \cos \frac{\varphi}{2} | \psi_{\text{int}}^{\text{(init)}} > \right|^2
\]

It is noted that, in the domain of \( \varphi \) from 0 to \( 2\pi \), \( \cos \frac{\varphi}{2} \) is antisymmetric with respect to \( \pi \) (i.e., \( \cos \frac{\varphi - \pi}{2} = -\cos \frac{\varphi + \pi}{2} \)). Therefore \( \psi_{\text{int}}^{(f)} \) must also be antisymmetric in this domain, otherwise the associated integration in the domain would be zero. This statement holds also for the domain of \( \varphi \) from \( 2\pi \) to \( 4\pi \), thus the transition of the internal states might be constrained. It turns out that, when the initial state \( \Psi_{L,S,i}^{\text{(init)}} \) and the final state \( \Psi_{L',S',i'}^{(f)} \) together have \( i + i' \) an even integer, \( < \psi_{\text{int}}^{(f)} | \cos \frac{\varphi}{2} | \psi_{\text{int}}^{\text{(init)}} > \) is zero. When \( i + i' \) is odd and \( |i' - i| \geq 3 \), the integration is not zero but very small.
due to the nodal structures of the internal states. Consequently, the transition is essentially concentrated into the final state with \( L' = L \pm 1, S' = S, \) and \(|i' - i| = 1\), the associated probability depends essentially on the internal structures and the energy difference \( \hbar \omega \).

Let \((o)\) denote the GS. For the GS transition, \( L' = L_{(o)} \pm 1, S' = S, \) \( i' = i + 1 = 2 \). An example is given in Fig.6, where the FABO with the period 1/4 is caused by the oscillation of \( \omega \). The probability of the GS jumping to a higher state (say, \( i' \geq 4 \)) is very small, e.g., when \( i' = 4 \), it is smaller by two order.

\[
\hbar \omega_{+} = \frac{e}{2 c} \left( 1 \pm 2 \left( L_{(o)} + 2 \Phi \right) \right) + E_{int}^{(f)} - E_{int}^{(o)}
\]

which is plotted against \( \Phi \) in Fig.6b. For a comparison, those of a one-layer ring is plotted in 5d. Both figures have the FABO with periods 1/4 (5b) and 1/2 (5d, when \( \Phi \) is small), respectively. Furthermore, the maximal amplitude in 5b is a constant, while that in 5d is increasing until arriving at the border of the ABO region.

We define \( \Delta \hbar \omega = \hbar \omega_{+} - \hbar \omega_{-} \). Due to eq.(13) we have

\[
\Delta \hbar \omega = 2 \hbar J_{(o)}
\]

where \( \hbar \) is the Planck’s constant. This relation is a new finding, it relates \( \Delta \hbar \omega \) directly with \( J_{(o)} \). The fact that these two kinds of oscillations match with each other exactly is a common feature for various narrow rings. For an example, the one-layer two-electron rings have a roughly similar relation \( \Delta \hbar \omega = \hbar J_{(o)} \). Since the photon energy is much easier to be measured accurately, this relation provides an effective way to measure the currents.

In summary, we have studied the FABO of the eigen-energies, persistent current, and dipole transition of the two-layer rings with two electrons. Collective and internal coordinates \( \theta_C \) and \( \varphi \) have been introduced. Based on them the underlying physics can be better understood. The following points are reminded. (i) we have derived a very simple formula for the current eq.(13), which is different from that arising from the famous formula \( c.J = -e\phi \omega \) for the one-layer ring by a factor 1/2 due to the topology. However, if both the currents in the two layers are taken into account, their sum recovers the result of the above famous formula. (ii) the symmetry constraint imposed on dipole transitions can be revealed (eq.(15), which explains the selection rule of \( i' \), (iii) the strict relation between \( \Delta \hbar \omega \) and the current has been established (eq.(17), which is different from the associated formula derived in [8] for one-layer rings by also a factor 2), etc.. The two-layer rings do not have the ABO, but only the FABO. The period of the FABO is found to be two times shorter than that of the one-layer rings when \( B \) is not large, and four times shorter when \( B \) has entered into the ABO region.

The above discussion can be generalized to the case with many layers (namely, a multi-layer ring). E.g., for a seven-layer ring with two electrons, fractional orbital angular momenta \( (L = I/7) \) would emerge, the period of the FABO becomes 1/14. The very short period of the FABO is a noticeable feature of multi-layer rings.

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