Burns’ equivariant Tamagawa invariant $T\Omega^{loc}(N/Q, 1)$ for some quaternion fields

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1 Introduction

Inspired by the work of Bloch and Kato in [2], David Burns constructed several “equivariant Tamagawa invariants” associated to motives of number fields. These invariants lie in relative $K$-groups of group-rings of Galois groups and in [3] Burns made several conjectures (see Conjecture 2.2) about their values. In this paper I shall verify Burns’ conjecture concerning the invariant $T\Omega^{loc}(N/Q, 1)$ for some families of quaternion extensions $N/Q$. The family of quaternion fields which are not covered here will be covered in [8].

The paper is arranged in the following manner. In §2 the relative K-groups of $\mathbb{Z}[G]$ are introduced together with the principle method for constructing elements in them and the Hom-description for representing elements in the case when $G = Q_8$, the quaternion group of order eight. In §3 the formula for $T\Omega^{loc}(N/Q, 1)$ is given together with several simplifying observations which are special to quaternion extensions. This formula involves a sum of “local terms” whose Hom-description representatives are calculated in §4. In §5, when $N/Q$ is a quaternion extension whose decomposition group at 2 is a proper subgroup of $Q_8$, the Hom-descriptions of §3 and §4 are combined to prove Conjecture 2.2, which asserts that $T\Omega^{loc}(N/Q, 1)$ vanishes.

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2 Relative K-groups of group-rings

2.1 The exact sequence

Let $G$ be a finite group. Let $R$ be an integral domain and let $f: \mathbb{Z}[G] \rightarrow R[G]$ be the corresponding inclusion of group-rings. There is an associated exact sequence of algebraic K-groups, constructed in ([17] p.216), of the form
\[
\cdots \longrightarrow K_1(\mathbb{Z}[G]) \longrightarrow K_1(R[G]) \xrightarrow{\delta_1} K_0(\mathbb{Z}[G], R) \xrightarrow{\pi_*} K_0(\mathbb{Z}[G]) \xrightarrow{f_*} K_0(R[G]).
\]

Here we have adopted the notation of [3] for the relative K-group in the middle, which was originally denoted by \(K_0(\mathbb{Z}[G], f)\) in [17].

An arbitrary element of \(K_0(\mathbb{Z}[G], R)\) is represented by a triple \([A, \phi, B]\) in which \(A\) and \(B\) are finitely generated projective (left) \(\mathbb{Z}[G]\)-modules and \(\phi\) is an isomorphism of \(R[G]\)-modules of the form

\[
\phi : A \otimes R \xrightarrow{\cong} B \otimes R.
\]

A presentation for \(K_0(\mathbb{Z}[G], R)\) in terms of these triples is given in ([17] p.215). The homomorphisms in the sequence are defined by

\[
\pi_*[A, \phi, B] = [A] - [B] \in K_0(\mathbb{Z}[G]), \quad f_*[A] = [A \otimes R] \in K_0(R[G]) \quad \text{and if} \quad X \in GL_n(R[G]) \quad \text{is an invertible matrix representing} \quad x \in GL(R[G])^{ab} = K_1(R[G]) \quad \text{then} \quad \delta_1(x) = [R[G]^n, X, R[G]^n] \in K_0(\mathbb{Z}[G], R).
\]

Now consider the situation when \(R = \mathbb{Q}\), the rational field, which is the case of particular interest to us. Incidentally, in this case the relative K-theory exact sequence is just the low-dimensional end of a localisation sequence (cf. [11] §5 Theorem 5). The categorical construction of the localisation sequence in [11] shows that there is an isomorphism of the relative K-groups of the form

\[
K_0(\mathbb{Z}[G]; \mathbb{Q}) \cong \bigoplus_{p \text{ prime}} K_0(\mathbb{Z}_p[G]; \mathbb{Q}_p)
\]

sending \([A, \phi, B]\) to \([A \otimes \mathbb{Z}_p, \phi, B \otimes \mathbb{Z}_p] | p \text{ prime}\}. Here \(K_0(\mathbb{Z}_p[G]; \mathbb{Q}_p)\) is defined in an analogous manner to \(K_0(\mathbb{Z}[G]; \mathbb{Q})\), replacing \(\mathbb{Z}\) by \(\mathbb{Z}_p\) and \(\mathbb{Q}\) by \(\mathbb{Q}_p\).

2.2 Constructing elements of \(K_0(\mathbb{Z}[G], \mathbb{Q})\)

We shall require methods by which to construct and compare elements in \(K_0(\mathbb{Z}[G], \mathbb{Q})\).

Suppose that we have a bounded chain complex of perfect (i.e. finitely generated, projective) \(\mathbb{Z}[G]\)-modules of the form

\[
P^* : 0 \longrightarrow P^n \xrightarrow{d_n} P^{n+1} \xrightarrow{d_{n+1}} \ldots \xrightarrow{d_{m-1}} P^m \longrightarrow 0
\]

together with a given isomorphism

\[
\psi : \bigoplus_j H^{2j+1}(P^*) \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_j H^{2j}(P^*) \otimes \mathbb{Q}.
\]

Associated to this data is a well-defined element of \(K_0(\mathbb{Z}[G], \mathbb{Q}), [P^{\text{odd}}, \phi, P^{\text{even}}]\) in the following manner. Set \(P^{\text{odd}} = \bigoplus_j P^{2j+1}\) and \(P^{\text{even}} = \bigoplus_j P^{2j}\).
Choose $Q[G]$-modules splittings for the right-hand epimorphisms in each of the short exact sequences:

$$0 ightarrow \text{Ker}(d_j) \otimes Q \rightarrow P^j \otimes Q \rightarrow d_j(P^j) \otimes Q \rightarrow 0,$$

$$0 \rightarrow d_{j-1}(P^{j-1}) \otimes Q \rightarrow \text{Ker}(d_j) \otimes Q \rightarrow H^j(P^*) \otimes Q \rightarrow 0.$$

Using these chosen splittings the rational isomorphism $\phi$ is given by the following composition

$$\oplus_j P^{2j+1} \otimes Q \cong (\oplus_j \text{Ker}(d_{2j+1}) \otimes Q) \oplus (\oplus_j d_{2j+1}(P^{2j+1}) \otimes Q)

\cong (\oplus_j d_{2j}(P^{2j}) \otimes Q) \oplus (\oplus_j H^{2j+1}(P^*) \otimes Q) \oplus (\oplus_j d_{2j+1}(P^{2j+1}) \otimes Q)

\cong (\oplus_j d_{2j}(P^{2j}) \otimes Q) \oplus (\oplus_j \text{Ker}(d_{2j}) \otimes Q)

\cong \oplus_j P^{2j} \otimes Q.$$

The Euler characteristic of the perfect complex is defined to be $\chi(P^*) = \sum_j (-1)^{j+1} |P_j| \in K_0(Z[G])$ so that $\pi_*[P^{od}, \phi, P^{ev}] = \chi(P^*)$.

### 2.3 The Hom-description

The Hom-description enables us to represent elements of $K_0(Z[G], Q)$ in terms of functions on the representation ring of $G$, $R(G)$, given by the Grothendieck group of finite-dimensional, complex representations of $G$. Let $\Omega_{Q_p}$ denote the absolute Galois group of the $p$-adic rationals $Q_p$ and let $\overline{Q}_p$ be an algebraic closure. Choosing an isomorphism of $R(G)$ with the Grothendieck ring of $\overline{Q}_p$-representations we can endow $R(G)$ with an action by $\Omega_{Q_p}$. Also the multiplicative group, $(\overline{Q}_p)^*$, has a natural action by $\Omega_{Q_p}$. We have a Hom-description isomorphism of the form

$$K_0(Z_p[G]; Q_p) \cong \frac{Hom_{\Omega_{Q_p}}(R(G), (\overline{Q}_p)^*)}{Det(Z_p[G])^*},$$

which is defined in the following manner. Every element of $K_0(Z_p[G]; Q_p)$ may be written as $[A, \phi, B]$ in which $A, B$ are finitely generated, free $Z_p[G]$-modules. We may choose $Z_p[G]$-bases $x_p = \{x_1, x_2, \ldots, x_n\}$ and $y_p = \{y_1, y_2, \ldots, y_n\}$ for $A$ and $B$ respectively. Now let $\Phi_p \in GL_n(Q_p[G])$ denote the matrix which represents the isomorphism, $\phi : A \otimes Q_p \cong B \otimes Q_p$. Hence $\phi(x_i) = \Phi_p y_j$. Now suppose
that \( \rho : G \to GL_m(\mathbb{Q}_p) \) is a representation of \( G \). Define \( \text{Det}(\Phi_p)(\rho) \in (\mathbb{Q}_p)^* \) by the formula

\[
\text{Det}(\Phi_p)(\rho) = \det(\rho(\Phi_p))
\]

where \( \rho(\Phi_p) \in GL_{nm}(\mathbb{Q}_p) \) is obtained by applying \( \rho \) to the group elements in \( \Phi_p \). This formula uniquely characterises a Galois equivariant homomorphism

\[
\text{Det}(\Phi_p) \in \text{Hom}_{\mathbb{Q}_p}(R(G), (\mathbb{Q}_p)^*)
\]

which gives a Hom-description representative of \([A, \phi, B] \in K_0(\mathbb{Z}_p[G]; \mathbb{Q}_p)\).

Performing this construction prime by prime gives rise to a Hom-description of \( K_0(\mathbb{Z}[G]; \mathbb{Q}) \) in term of Galois equivariant homomorphisms from \( R(G) \) to \( J^*(\mathbb{Q}) \), the idèles of \( \mathbb{Q} \), an algebraic closure of the rationals. The class-group, \( \mathcal{CL}(\mathbb{Z}[G]) \), is defined to be the reduced \( K_0 \)-group

\[
\mathcal{CL}(\mathbb{Z}[G]) = \ker(\text{rank} : K_0(\mathbb{Z}[G]) \to \mathbb{Z}).
\]

A similar construction with local bases yields a Hom-description isomorphism of the form ([6], [14] p.40, [15] p.115)

\[
\text{Det} : \mathcal{CL}(\mathbb{Z}[G]) \to \frac{\text{Hom}_{\mathbb{Q}_p}(R(G), J^*(\mathbb{Q}))}{\text{Hom}_{\mathbb{Q}_p}(R(G), (\mathbb{Q})^*) \cdot \text{Det}(\prod_p (\mathbb{Z}_p[G])^*)}.
\]

We shall hardly need this Hom-description but, in passing, we remark that a Hom-description for \( \pi_A[A, \phi, B] \) is given by the idelic-valued function which is given at the component above \( p \) by any Hom-description representative of \([A \otimes \mathbb{Z}_p, \phi, B \otimes \mathbb{Z}_p] \in K_0(\mathbb{Z}_p[G]; \mathbb{Q}_p)\) and is trivial at Archimedean primes.

2.4 The case when \( G = Q_8 \)

When \( G = Q_8 \), the quaternion group of order eight, all the complex representations have rational characters so that the isomorphism of §2.3 simplifies to the form

\[
K_0(\mathbb{Z}_p[Q_8]; \mathbb{Q}_p) \cong \frac{\text{Hom}(R(Q_8), \mathbb{Q}_p^*)}{\text{Det}(\mathbb{Z}_p[Q_8]^*)}.
\]

Since determinantal functions on \( R(Q_8) \) take values in \( \mathbb{Z}_p^* \) there is a short exact sequence of the form

\[
0 \to \frac{\text{Hom}(R(Q_8), (\mathbb{Z}_p)^*)}{\text{Det}(\mathbb{Z}_p[Q_8]^*)} \to \frac{\text{Hom}(R(Q_8), (\mathbb{Q}_p)^*)}{\text{Det}(\mathbb{Z}_p[Q_8]^*)} \to \text{Hom}(R(Q_8), \mathbb{Z}) \to 0.
\]

From this sequence we see that there is an isomorphism of the form

\[
\text{Tors}K_0(\mathbb{Z}_p[Q_8]; \mathbb{Q}_p) \cong \frac{\text{Hom}(R(Q_8), \mathbb{Z}_p^*)}{\text{Det}(\mathbb{Z}_p[Q_8]^*)}.
\]
When $p$ is an odd prime every function in $\text{Hom}(R(Q_8), \mathbb{Z}_p^*)$ is a determinantal function because in this case $\mathbb{Z}_p[Q_8]$ is a maximal order in $\mathbb{Q}_p[Q_8]$ [6]. Hence, if $p$ is odd then $\text{Tors}_0(\mathbb{Z}_p[Q_8]; \mathbb{Q}_p) = 0$.

Now let us examine the case when $p = 2$. Let $1, \chi_1, \chi_2, \chi_1\chi_2$ denote the four one-dimensional representations of $Q_8$ and $Q_8^{ab}$. To be precise, if $Q_8 = \{x, y \mid x^2 = y^2, y^4 = 1, xyx = y\}$ set $\chi_1(x) = -1 = \chi_2(y)$ and $\chi_1(y) = 1 = \chi_2(x)$. The following result is taken from ([16] Proposition 2.5.37 and Corollaries 2.5.38, 2.5.39)

**Proposition 2.5**

(i) There is an isomorphism of the form

$$\lambda : \text{Tors}_0(\mathbb{Z}_2[Q_8^{ab}]; \mathbb{Q}_2) \cong \frac{\text{Hom}(R(Q_8^{ab}), \mathbb{Z}_2^*)}{\text{Det}(\mathbb{Z}_2[Q_8^{ab}]^*)} \xrightarrow{\cong} (\mathbb{Z}/4)^* \cong \{\pm 1\}$$

given by

$$\lambda[f] = f(1 + \chi_1 + \chi_2 + \chi_1\chi_2) \pmod{4}. \quad \text{(modulo 4)}.$$

(ii) The natural maps yield an isomorphism of the form

$$\text{Tors}_0(\mathbb{Z}[Q_8]; \mathbb{Q}) \xrightarrow{\cong} \mathcal{CL}(\mathbb{Z}[Q_8]) \oplus \text{Tors}_0(\mathbb{Z}[Q_8^{ab}]; \mathbb{Q}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2.$$

(iii) $\text{Tors}_0(\mathbb{Z}[\mathbb{Z}/2]; \mathbb{Q}) = 0$.

**Proposition 2.6**

(i) There is an isomorphism of the form

$$K_0(\mathbb{Z}_2[Q_8^{ab}]; \mathbb{Q}_2) \cong \text{Hom}(R(Q_8^{ab}), \mathbb{Z}) \times (\mathbb{Z}/4)^*.$$

(ii) Suppose that $H \subset Q_8^{ab}$ has order one or two. Then, under the isomorphism of (i), the canonical homomorphism

$$\text{Ind}^{Q_8^{ab}}_H : K_0(\mathbb{Z}_2[Q_8^{ab}]; \mathbb{Q}_2) \cong \frac{\text{Hom}(R(H), \mathbb{Q}_2^*)}{\text{Hom}(R(H), \mathbb{Z}_2^*)} \rightarrow K_0(\mathbb{Z}_2[Q_8^{ab}]; \mathbb{Q}_2)$$

has the form $[f] \mapsto (\chi \mapsto (v_2(f(\text{Res}^{Q_8^{ab}}_H(\chi)))), 1 \pmod{4})$ where $v_2$ denotes the 2-adic valuation.

**Proof**

For part (i) send the class of a function $f : R(Q_8^{ab}) \rightarrow \mathbb{Q}_2^*$ to

$$(\chi \mapsto v_2(f(\chi)), f(1 + \chi_1 + \chi_2 + \chi_1\chi_2)2^{-v_2(f(1 + \chi_1 + \chi_2 + \chi_1\chi_2))} \pmod{4}).$$

This an isomorphism by the discussion of §2.4 and Proposition 2.5(i). Since $\text{Ind}^{Q_8^{ab}}_H$ is induced in terms of the Hom-description by $\text{Res}^{Q_8^{ab}}_H$, part (ii) follows from part (i). □
3 Burns invariant $T\Omega^{loc}(N/Q, 1)$ for quaternion fields

3.1 Let $N/Q$ be a quaternion extension of number fields with Galois group $G(N/Q)$ and with biquadratic subfield, $E$. In this case the equivariant Tamagawa number, defined in ([3] §3), is a canonical Galois structure invariant of the form

$$T\Omega^{loc}(N/Q, 1) \in TorsK_0(\mathbb{Z}[G(N/Q)]; \mathbb{Q}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2.$$ 

For a general Galois extension of number fields this invariant may only be defineable in the relative K-group, $K_0(\mathbb{Z}[G(N/Q)]; \mathbb{R})$, but for quaternion fields the Stark conjecture holds (cf. [18]) in which case it is shown in [3] that $T\Omega^{loc}(N/Q, 1)$ is an element of finite order in $K_0(\mathbb{Z}[G(N/Q)]; \mathbb{Q})$.

The following is a particular case of a conjecture of Burn’s [3], which is rather remarkable since $T\Omega^{loc}(N/Q, 1)$ has a definition which looks far from trivial:

Conjecture 3.2

$$0 = T\Omega^{loc}(N/Q, 1) \in TorsK_0(\mathbb{Z}[G(N/Q)]; \mathbb{Q}).$$

The following result will considerably simplify the calculation of $T\Omega^{loc}(N/Q, 1)$.

Lemma 3.3

The image of $T\Omega^{loc}(N/Q, 1)$ under the isomorphism of Proposition 2.5(ii) is equal to

$$(0, T\Omega^{loc}(E/Q, 1)) \in CL(\mathbb{Z}[Q_8]) \oplus TorsK_0(\mathbb{Z}[Q_8^{ab}]; \mathbb{Q}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2.$$ 

Proof

It is shown in in [3] that the image of $T\Omega^{loc}(N/Q, 1)$ in the class group is equal to $\Omega(N/Q, 2) - W_{N/Q}$, the difference of the second Chinburg invariant and the Cassou-Noguès-Fröhlich class. However this difference is shown to be trivial for all quaternion fields in [7]. This shows that the first component of the image vanishes. By naturality of $T\Omega^{loc}(N/Q, 1)$ with respect to passage to Galois subextensions (cf. [1]) the second component is equal to $T\Omega^{loc}(E/Q, 1)$.

Definition 3.4 The invariant $T\Omega^{loc}(E/Q, 1)$

Let $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ be the biquadratic subfield of the quaternion field $N/Q$, as in §3.1. By virtue of Lemma 3.3, we wish to calculate the invariant $T\Omega^{loc}(E/Q, 1)$ of ([3] Proposition 3.3 p.15). In general this invariant is defined as a difference
\[ T\Omega^{\text{loc}}(E/\mathbb{Q}, 1) = T\Omega^{\text{loc}}(E/\mathbb{Q}, 1, \lambda) - T(\lambda) \in K_0(\mathbb{Z}[G(E/\mathbb{Q})], \mathbb{Q}) \]

depending on an element, \( \lambda \in \mathbb{R}[G(E/\mathbb{Q})]^\ast \). However, when all units in the real group-ring are reduced norms we may choose \( \lambda = 1 \) and then \( T(1) \) is trivial.

The definition of \( T\Omega^{\text{loc}}(E/\mathbb{Q}, 1, \lambda) \) requires the choice of a finite, Galois invariant set of places of \( E \), denoted by \( S \) and containing all the infinite places \( S_\infty(E) \) together with all finite places which ramify in \( E/\mathbb{Q} \). Then, in the notation of ([3] p.13),

\[ T\Omega^{\text{loc}}(E/\mathbb{Q}, 1, \lambda) = T_S(\mathcal{L}, \lambda) - \sum_{p \in S_f} \text{Ind}_{G(E_w/\mathbb{Q}_p)}^{G(E/\mathbb{Q})}(\chi_w^\ast(\mathcal{L}, V_w)) + \sum_{p \in S_f} \hat{\delta}_1(\epsilon'_p(0)). \]

Here \( w \) is a chosen prime of \( E \) above \( p \) and \( \mathcal{L} \) is a lattice of which more presently. Each of the sums is taken over \( S_f \), which is the set of rational primes \( p \) lying below the finite primes of \( E \) in \( S \). The terms \( \hat{\delta}_1(\epsilon'_p(0)) \) of [3] are elements lying in the image of the map \( \delta_1 \) of \( \S 2.1 \).

3.5 If \( \chi : G(E/\mathbb{Q}) \rightarrow \{ \pm 1 \} \) is a one-dimensional representation a Hom-representative of \( \hat{\delta}_1(\epsilon'_p(0)) \in K_0(\mathbb{Z}[G(E/\mathbb{Q})], \mathbb{Q}) \) has the same component at each prime \( l \) given by the function

\[ \chi \mapsto \left( \frac{|G(E_w/\mathbb{Q}_p)|}{|I(E_w/\mathbb{Q}_p)|} \right)^{-\dim(\chi^{G(E_w/\mathbb{Q}_p)})} \det(1 - \text{Frob}_w^{-1}|(\chi^{I(E_w/\mathbb{Q}_p)}/\chi^{G(E_w/\mathbb{Q}_p)})) \]

where \( I(E_w/\mathbb{Q}_p) \subseteq G(E_w/\mathbb{Q}_p) \) is the inertia subgroup at \( p \). Here \( \chi^H \) denotes the subrepresentation given by the \( H \)-fixed points for \( H \subseteq Q_8^{ab} \). Hence the Hom-representative of \( \hat{\delta}_1(\epsilon'_p(0)) \) has the form \( \chi \mapsto 2^{\alpha_p(\chi)} \). Since \( 2 \in \mathbb{Z}_l^\ast \) for all odd primes, a Hom-representative of \( \sum_{p \in S_f} \hat{\delta}_1(\epsilon'_p(0)) \) is given by the function which is given by

\[ \chi \mapsto 2 \sum_{p \in S_f} \alpha_p(\chi) \in \frac{\text{Hom}(R(Q_8^{ab}), Q_2)}{\text{Det}(\mathbb{Z}_2[Q_8^{ab}]^\ast)} \]

in the 2-adic coordinate and is trivial at all odd-primary components.

Combining this discussion with Proposition 2.6(i) yields the following result.

Corollary 3.6

The torsion component of the image of \( \sum_{p \in S_f} \hat{\delta}_1(\epsilon'_p(0)) \) in

\[ K_0(\mathbb{Z}_2[Q_8^{ab}]; \mathbb{Q}_2) \cong \text{Hom}(R(Q_8^{ab}), \mathbb{Z}) \times (\mathbb{Z}/4)^\ast \]

is trivial.
3.7 The element $T_S(L, 1)$

The term, $T_S(L, \lambda)$ (with $\lambda = 1$ in our case) is constructed ([3] p.10) in terms of a free $\mathbb{Z}[G(E/\mathbb{Q})]$-sublattice $L$ of the integers, $O_E$, of $E$ so that $L = \mathbb{Z}[G(E/\mathbb{Q})] < \gamma$ for some $\gamma \in O_E$. In general, $L$ would merely be a projective module but finitely generated, projective $\mathbb{Z}[G(E/\mathbb{Q})]$-modules are all free.

Let $\mathbb{Z} < 2\pi i >$ denote a copy of the integers on which complex conjugation acts by minus the identity. Form the chain complex of perfect $\mathbb{Z}[G(E/\mathbb{Q})]$-modules

$$H^*(L) : 0 \longrightarrow \mathbb{Z} < 2\pi i > \otimes \mathbb{Z}[G(E/\mathbb{Q})] \xrightarrow{(2,0)} \mathbb{Z} < 2\pi i > \otimes \mathbb{Z}[G(E/\mathbb{Q})] \oplus \mathcal{L}$$

in which the non-zero modules are, from left to right, in degrees $-1, 0, 1$, respectively. Hence $H^0 = L$ and $H^1 = \mathbb{Z} < 2\pi i > \otimes \mathbb{Z}[G(E/\mathbb{Q})]$. The class, $T_S(L, 1) \in K_0(\mathbb{Z}[G(E/\mathbb{Q}), \mathbb{Q}])$, is the element associated to this perfect complex together with a rational homology isomorphism defined in the manner of §2.2.

There is a rational isomorphism

$$\Theta : L \otimes \mathbb{Q} \xrightarrow{\simeq} \mathbb{Z} < 2\pi i > \otimes \mathbb{Q}[G(E/\mathbb{Q})]$$

which, for $\gamma \in L$ such that $\gamma \otimes 1$ is a generator for the module $L \otimes \mathbb{Q}$, is given by the formula

$$\Theta(\gamma \otimes 1) = (2\pi i) \otimes \sum_{g \in G(E/\mathbb{Q}), \mathbb{Q}} g(\gamma)g^{-1}.$$ 

The rational isomorphism used in ([3] p.6), obtained by using $\Theta^{-1} : H^{ev} \otimes \mathbb{Q} \xrightarrow{\simeq} H^{od} \otimes \mathbb{Q}$ in §2.2, is equal to

$$\bigoplus_{i=1,2} \mathbb{Z} < 2\pi i > \otimes \mathbb{Q}[G(E/\mathbb{Q})] \xrightarrow{\simeq} \mathbb{Z} < 2\pi i > \otimes \mathbb{Q}[G(E/\mathbb{Q})] \oplus L \otimes \mathbb{Q}$$

given by

$$\mathcal{E}^{-1}_S \begin{pmatrix} 2 & 0 \\ 0 & \Theta^{-1} \end{pmatrix}$$

Here $\mathcal{E}_S \in \mathbf{R}[G(E/\mathbb{Q})]^\ast$ is defined in ([3] p.6) and will be recalled in detail below. Therefore a Hom-representative for $T_S(L, 1)$ is given by the function whose coordinate at each prime $p$ is given on a one-dimensional representation, $\chi$, by

$$\chi \mapsto 2\chi(\mathcal{E}^{-1}_S) \text{det}(\sum_{g \in G(E/\mathbb{Q})} g(\gamma)\chi(g^{-1}))^{-1} = 2\chi(\mathcal{E}^{-1}_S)(\gamma | \chi)^{-1},$$

where $(\gamma | \chi)$ is the resolvent (see [5]) associated to $\gamma \in E$. Finally, for $\chi : G(E/\mathbb{Q}) \longrightarrow \{\pm 1\}$, $\chi(\mathcal{E}^{-1}_S)$ is given by the formula (see [3] §1.2.1(proof))

$$\chi(\mathcal{E}_S) = \frac{L^*(1, \chi)}{L^*(0, \chi)} \prod_{p \in S_f} \text{det}(1 - p^{-1} \text{Frob}_w^{-1}| \chi^{(E_w/\mathbb{Q})})$$
where \( L^*(s_0, \chi) \) denotes the leading coefficient in the Laurent series for the Artin L-function, \( L(s, \chi) \), about \( s = s_0 \).

**Lemma 3.8**

In §3.7

\[
\frac{L^*(1, \chi)}{L^*(0, \chi)} = 2f(\chi)^{-1/2} = 2f(\chi)^{-1/2}
\]

for each one-dimensional representation, \( \chi : G(E/\mathbb{Q}) \rightarrow \{\pm 1\} \).

**Proof**

In order to compute the quotient of the leading coefficients we recall that there is a functional equation ([10], [13] p.253) of the form

\[
f(\chi)^{s/2} \pi^{s/2} \Gamma(s/2)L(s, \chi)
= f(\chi)^{(1-s)/2} \pi^{-(1-s)/2} \Gamma((1-s)/2)L(1-s, \chi)
\]

in which \( f(\chi) \) denotes the Artin conductor of \( \chi \). In the neighbourhood of \( s = 1 \) recall that when \( \chi = 1 \) the L-function has a simple pole (as is seen from the Analytic Class Number Formula (see e.g. [13] p.248)) while \( \Gamma(s/2) \) is continuous and non-zero in the vicinity of \( s = 1 \). At \( s = 0 \) the behaviour is the other way round. When \( \chi \) is non-trivial the L-function tends to a non-zero limit at \( s = 1 \). Therefore re-writing the function equation as

\[
f(\chi)^{s/2} \pi^{s/2} \frac{L(s, \chi)}{\Gamma((1-s)/2)} = f(\chi)^{(1-s)/2} \pi^{-(1-s)/2} \frac{L(1-s, \chi)}{\Gamma(s/2)}.
\]

Consider the case when \( \chi = 1 \). Hence

\[
\lim_{t \to 1} (t - 1)L(t, 1) = \lim_{s \to 0} (1 - s - 1)L(1 - s, 1) = - \lim_{s \to 0} sL(1 - s, 1)
\]

exists so that near \( s = 0 \)

\[
L(1 - s, 1) = L^*(1, 1)/s + \sum_{j=0}^{\infty} z_j s^j.
\]

Similarly it is well known ([9] p.8 and p.491) that near \( s = 0 \)

\[
\Gamma(s/2) = 2/s + \sum_{j=0}^{\infty} y_j s^j.
\]

Letting \( s \) tend to zero in the functional equation yields

\[
\pi^{-1/2} L^*(0, 1) = \frac{L^*(0, 1)}{\Gamma(1/2)} = \pi^{-1/2} \frac{L^*(1, 1)}{2}.
\]
Hence
\[ \frac{L^*(1, 1)}{L^*(0, 1)} = 2. \]

When \( \chi \) is non-trivial the limit of \( L(s, \chi) \) as \( s \) tends to one exists and is non-zero, since \( L(s, \chi) \) is the quotient of two zeta functions, and so
\[ L^*(1, \chi) = \lim_{s \to 1} L(s, \chi) = \lim_{s \to 0} L(1 - s, \chi). \]
Therefore near \( s = 0 \)
\[ L(s, \chi) = L^*(0, \chi) s + \sum_{j=2}^{\infty} u_j s^j \]
and dividing the functional equation by \( s \) and letting \( s \) tend to zero yields
\[ \pi^{-1/2} L^*(0, \chi) = f(\chi)^{1/2} \pi^{-1/2} \frac{L^*(1, \chi)}{2} \]
so that
\[ \frac{L^*(1, \chi)}{L^*(0, \chi)} = 2f(\chi)^{-1/2}, \]
as required. \( \square \)

**Corollary 3.9**

Hence, for each one-dimensional representation \( \chi : G(E/\mathbb{Q}) \to \{\pm 1\} \), a Hom-representative for \( T_S(L, 1) \) of §3.7 is given by the function whose coordinate at each prime is
\[ \chi \mapsto f(\chi) \left( \frac{\gamma}{|\chi|} \prod_{p \in S} det(1 - p^{-1} Frbw | \chi^I(E_w/\mathbb{Q}_p)) \right). \]

**Proposition 3.10**

By Proposition 2.6(ii) the image of \( Ind_{G(E/\mathbb{Q})}^G(E_w/\mathbb{Q}_p)(\chi_w^\bullet(\mathcal{L}), V_w) \) in \( K_0(\mathbb{Z}[Q_{w, 8}^ab]; \mathbb{Q}_2) \cong Hom((R(Q_{w, 8}^ab), \mathbb{Z}) \times (\mathbb{Z}/4)^* \)
has trivial torsion component unless \( G(E_w/\mathbb{Q}_p) = G(E/\mathbb{Q}) \).

### 4 The local terms \( Ind_{G(E_w/\mathbb{Q}_p)}^G(E/\mathbb{Q})(\chi_w^\bullet(\mathcal{L}, V_w)) \)

**Definition 4.1** The terms in the middle expression for \( T\Omega^{loc}(E/\mathbb{Q}, 1, \lambda) \) in Definition 3.4 are the images of elements, \( \chi_w^\bullet(\mathcal{L}, V_w) \in TorsK_0(\mathbb{Z}[G(E_w/\mathbb{Q}_p)], \mathbb{Q}) \) under the canonical homomorphism
\[ Ind_{G(E_w/\mathbb{Q}_p)}^G(E/\mathbb{Q}) : K_0(\mathbb{Z}[G(E_w/\mathbb{Q}_p)], \mathbb{Q}) \longrightarrow TorsK_0(\mathbb{Z}[G(E/\mathbb{Q})], \mathbb{Q}) \]
where \(G(E_w/Q_p)\) denotes the decomposition group at \(p\). Hence, by Proposition 3.10, the torsion part of the local contribution, \(\text{Ind}^{G(E/Q)}_{G(E_w/Q_p)}(\chi_w(L, V_w))\), vanishes unless \(G(E_w/Q_p) = G(E/Q)\). In this section we shall evaluate these local torsion contributions when \(p\) is odd and \(G(E_w/Q_p) = G(E/Q)\). In this case \(p\) is ramified, so that \(p \in S_f\), and the inertia group \(I(E_w/Q_p)\) has order two. In this case, let us describe the class \(\chi_L^{\bullet}(L, V_w)\). Choose \(w \in S\) over \(p\). Let \(L_w \subset E_w\) denote the \(w\)-adic completion of \(L\). According to [3] we may choose any free \(L\) such that the \(p\)-adic exponential is defined and gives an isomorphism, \[\exp : L_w \xrightarrow{=} 1 + L_w.\]

Since an odd ramified prime is tamely ramified, \(O_{E_w}\) is a free \(\mathbb{Z}_p\)-module and we may choose an integer \(m = m(p) \geq 1\) and arrange that \(p^m O_{E_w} = L_w\) (cf. [6], [14]).

Now suppose that the 2-extension \(E_w^* \longrightarrow A \longrightarrow B \longrightarrow \mathbb{Z}\) represents the canonical local fundamental class of class field theory in \[\text{Ext}^2_{\mathbb{Z}[G(E_w/Q_p)]}(\mathbb{Z}, E_w^*) = H^2(G(E_w/Q_p); E_w^*)\] ([12] p.168). Form the 2-extension \(E_w^*/(1 + p^m O_{E_w}) \longrightarrow A/(1 + p^m O_{E_w}) \longrightarrow B \longrightarrow \mathbb{Z}\).

Choose a complex \[P^* : 0 \longrightarrow P^{-2} \xrightarrow{d_{-2}} P^{-1} \xrightarrow{d_{-1}} P^0 \longrightarrow 0\] of finitely generated, projective \(\mathbb{Z}[G(E_w/Q_p)]\)-modules which is quasi-isomorphic to \[0 \longrightarrow A/(1 + p^m O_{E_w}) \longrightarrow B \longrightarrow 0\] in which \(B\) is in degree zero. Hence there are canonical isomorphisms of the form \[H^{-1}(P^*) \cong E_w^*/(1 + p^m O_{E_w}), \quad H^0(P^*) \cong \mathbb{Z}.\]

Therefore the valuation gives a rational isomorphism of the form \[\text{val} : H^{-1}(P^*) \otimes \mathbb{Q} \xrightarrow{\cong} H^0(P^*) \otimes \mathbb{Q}\]

and by the construction of §2.2 we obtain \[\left[P^{-1}, \text{val}, P^{-2} \oplus P^0\right] = \chi_w^{\bullet}(L, V_w) \in K_0(\mathbb{Z}[G(E_w/Q_p)], \mathbb{Q}).\]
4.2 Computing the local terms at odd primes

We continue with the situation of Definition 4.1. In $G(E/Q) = G(E_w/Q_p)$ let $a_p$ denote the non-trivial element of the inertia group at $p$ and let $b_p$ be an element acting like the Frobenius of the residue extension, $F_{p^2}/F_p$.

In order to compute the local term we are first going study the related chain complex

$$
0 \rightarrow A/U_{E_w}^1 \rightarrow B \rightarrow 0
$$
in which we quotient by the principal units, $U_{E_w}^1$, rather than by $1 + p^m \mathcal{L}_w$. Since $p$ is tamely ramified, $U_{E_w}^1$ is a cohomologically trivial $\mathbb{Z}[G(E_w/Q_p)]$-module.

If, as in Definition 4.1, $E^* \rightarrow A \rightarrow B \rightarrow \mathbb{Z}$ respresents the canonical local fundamental class of class field theory then minus the canical class is given by pulling this back along $(-1): \mathbb{Z} \rightarrow \mathbb{Z}$. Hence minus the fundamental class is represented by

$$
E_w^* \rightarrow A \rightarrow B \rightarrow \mathbb{Z}.
$$

Since $p$ is tamely ramified we have a complex of free $\mathbb{Z}[G(E_w/Q_p)]$-modules which is quasi-isomorphic to minus the local fundamental class divided by $U_{E_w}^1$

$$
E_w^*/(U_{E_w}^1) \rightarrow A/(U_{E_w}^1) \rightarrow B \rightarrow \mathbb{Z}
$$

which was first constructed in [4] and is constructed in a slightly different manner in [15] pp.319-329). This quasi-isomorphic complex of free modules has the form

$$
0 \rightarrow \mathbb{Z}[G(E/Q)] < w > \rightarrow \mathbb{Z}[G(E/Q)] < z_1 > \oplus \mathbb{Z}[G(E/Q)] < z_2 > \rightarrow \mathbb{Z}
$$

where

$$
\lambda(w) = (b_p((p+1)/2 + ((p-1)/2)a_p) - 1)z_1 - (a_p - 1)z_2
$$

$$
\phi(z_1) = (a_p - 1)t, \phi(z_2) = (b_p - 1)t.
$$

If the left-hand module is in dimension minus two then the only non-trivial cohomology of this complex is given by

$$
H^{-1} \cong E_w^*/U_{E_w}^1 \cong F_{p^2} \times \mathbb{Z}, \quad H^0 \cong \mathbb{Z}
$$
where the first isomorphism sends \( T = (1 + b_p)z_2 - (1 + a_p)z_1 \) to \((\xi, \pi)\), by ([15] Lemma 7.1.55) – \( \pi \) being a prime of \( E_w \) – and the second isomorphism is induced by the augmentation homomorphism. The valuation on \( E_w \), normalised to ensure that \( \text{val}(\pi) = 1 \), induces a canonical isomorphism, \( H^{-1} \otimes \mathbb{Q} \rightarrow H^0 \otimes \mathbb{Q} \).

Therefore the complex

\[
0 \rightarrow \mathbb{Z}[G(E/\mathbb{Q})] < w > \xrightarrow{\lambda} \mathbb{Z}[G(E/\mathbb{Q})] < z_1 > \oplus \mathbb{Z}[G(E/\mathbb{Q})] < z_2 > \xrightarrow{-\phi} \mathbb{Z}[G(E/\mathbb{Q})] < t > \rightarrow 0
\]

is quasi-isomorphic to the quotient of the local fundamental class by \( U_1^{E_w} \). Let us apply the construction of §2.2 to this complex and rational cohomology isomorphism. We must choose splittings

\[
0 \rightarrow \text{im}(-\phi) \otimes \mathbb{Q} \xrightarrow{i_0} Q[G(E/\mathbb{Q})] < t > \xrightarrow{\rho_0} H^0 \otimes \mathbb{Q} \rightarrow 0,
\]

\[
0 \rightarrow \ker(-\phi) \otimes \mathbb{Q} \xrightarrow{i_1} Q[G(E/\mathbb{Q})] < z_1, z_2 > \xrightarrow{\rho_1} \text{im}(-\phi) \otimes \mathbb{Q} \rightarrow 0,
\]

\[
0 \rightarrow Q[G(E/\mathbb{Q})] < w > \xrightarrow{\lambda} \ker(-\phi) \otimes \mathbb{Q} \xrightarrow{\rho_2} H^{-1} \otimes \mathbb{Q} \rightarrow 0.
\]

Here \( i_j \) is induced by an inclusion map.

The recipe of §2.2 then makes use of the following string of isomorphisms

\[
Q[G(E/\mathbb{Q})] < z_1, z_2 > \xrightarrow{i_{12} + \rho_1} \ker(-\phi) \otimes \mathbb{Q} \oplus \text{im}(-\phi) \otimes \mathbb{Q}
\]

\[
\xrightarrow{(i_{21} + \rho_2, 1)} \text{im}(\lambda) \otimes \mathbb{Q} \oplus \text{im}(-\phi) \otimes \mathbb{Q} \oplus H^{-1} \otimes \mathbb{Q}
\]

\[
\xrightarrow{(\lambda, 1, \text{val})^{-1}} Q[G(E/\mathbb{Q})] < w > \oplus \text{im}(-\phi) \otimes \mathbb{Q} \oplus H^0 \otimes \mathbb{Q}
\]

\[
\xrightarrow{(1, i_0 + \rho_0)} Q[G(E/\mathbb{Q})] < w, t >
\]

The decomposition of the group-ring \( Q[G(E/\mathbb{Q})] \) into the product of one copy of \( \mathbb{Q} \) for each one-dimensional representation is induced by evaluating at the corresponding representation. Hence we can produce the function, \( f \in \text{Hom}(R(G(E/\mathbb{Q}), \mathbb{Q}^\ast)) \), which gives the \( p \)-component (for every \( p \)) of the Hom-representative for the element of \( K_0(\mathbb{Z}[G(E/\mathbb{Q})], \mathbb{Q}) \) associated with the above isomorphism data idempotent by idempotent. In other words, if \( \sigma_\chi \in Q[G(E/\mathbb{Q})] \) is the idempotent corresponding to \( \chi : G(E/\mathbb{Q}) \rightarrow \{\pm 1\} \) then \( f(\chi) \) is given by the determinant of the matrix representing the restriction of the above isomorphism

\[
Q < \sigma_\chi z_1, \sigma_\chi z_2 > \rightarrow Q < \sigma_\chi w, \sigma_\chi t >
\]
Furthermore, the splittings, $\rho_i$, are determined by the splittings, $\sigma_1 \rho_i$.

We begin with $\chi = 1$, the one-dimensional trivial representation, so that $\sigma_1 = (a_p + 1)(b_p + 1)/4$. In this case $\sigma_1 \text{im}(\phi) \otimes Q = 0$ and $\rho_0(\sigma_1 t) = 1 \otimes 1 = \text{val}(\pi \otimes 1) \in H_0 \otimes Q$. Since $\sigma_1 \lambda(w) = (p - 1)\sigma_1 z_1$ and we may choose $\rho_2(\sigma_1 (\pi \otimes 1)) = \sigma_1 T = 2\sigma_1(z_2 - z_1)$ we obtain

$$f(1) = \det \begin{pmatrix} 1/(p - 1) & 1/(p - 1) \\ 0 & 1/2 \end{pmatrix} = 1/(2p - 2) \in Q^*.$$ 

When $\chi$ is a non-trivial one-dimensional representation then $\sigma_\chi = (1 + \chi(a_p) a_p)(1 + \chi(b_p) b_p)/4$ and $\sigma_\chi H^{-1} \otimes Q = 0 = \sigma_\chi H^0 \otimes Q$.

Suppose that $\chi(a_p) = -1 = \chi(b_p)$. Then $(-\phi)(\sigma_\chi z_1) = 2\sigma_1 t = (-\phi)(\sigma_\chi z_2)$ so that we may choose $\rho_1(\sigma_1 t) = \sigma_\chi z_1/2$ since $(a_p - 1)(-\sigma_\chi z_1/2) = \sigma_\chi t = 0$. Also $\lambda(\sigma_\chi w) = 2\sigma_\chi(z_2 - z_1)$. Therefore, in this case, we obtain

$$f(\chi) = \det \begin{pmatrix} 0 & 1/2 \\ 2 & 2 \end{pmatrix} = -1 \in Q^*.$$ 

If $\chi(a_p) = 1$, $\chi(b_p) = -1$ then $\lambda(\sigma_\chi w) = -(p + 1)\sigma_\chi z_1$ and we may choose $\rho_1(\sigma_\chi t) = \sigma_\chi z_2/2$. Therefore we obtain

$$f(\chi) = \det \begin{pmatrix} -1/(p + 1) & 0 \\ 0 & 2 \end{pmatrix} = -2/(p + 1) \in Q^*.$$ 

Finally, if $\chi(a_p) = -1$, $\chi(b_p) = 1$ then $\lambda(\sigma_\chi w) = 2\sigma_\chi z_2$ and we may choose $\rho_1(\sigma_\chi t) = \sigma_\chi z_1/2$. Therefore

$$f(\chi) = \det \begin{pmatrix} 0 & 1/2 \\ 2 & 0 \end{pmatrix} = -1 \in Q^*.$$ 

One may easily verify that this function may be written as the $Q^*$-valued function sending a one-dimensional $\chi$ to

$$(-1)^{\text{codim}(\chi G_{E_w}^G/Q_p)} \left( \frac{[G_{E_w}^G/Q_p]}{[T G_{E_w}^G/Q_p]} \right)^{-\text{dim}(\chi G_{E_w}^G/Q_p)} \text{det}(1 - Frob_w^{-1} | \chi^I G_{E_w}^G/Q_p) / \chi^G G_{E_w}^G/Q_p) \right) \times$$

$$p^{\text{dim}(\chi^G_{E_w}/Q_p)} \text{det}(1 - p^{-1} Frob_w^{-1} | \chi^I G_{E_w}^G/Q_p) \right).$$

We still have to evaluate these local terms using the correct lattice.

The categorical derivation of the localisation sequence in [11] shows that $K_0(Z[G_{E_w}^G/Q_p], Q) \cong K_0(T[Z[G_{E_w}^G/Q_p]])$, the Grothendieck group of the category of finite $Z[G_{E_w}^G/Q_p]$-modules of finite projective dimension. The module $U_{E_w}^1 / (1 + p^m O_{E_w})$ lies in this category. To see this observe that each quotient in
the level filtration \((1 + p^{n-1}O_{E_w})/(1 + p^nO_{E_w})\) is isomorphic to the residue field \(F_{p^2}\), which has a free resolution of the form

\[
0 \rightarrow \mathbb{Z}[G(O_{E_w}/Q_p)] \rightarrow \mathbb{Z}[G(O_{E_w}/Q_p)] \rightarrow F_{p^2} \rightarrow 0.
\]

Here the second map sends 1 to a normal basis element of the residue field. To verify exactness we note that \((p + 1)/2 + (1/2)(p - 1)a_p\) acts on the normal basis element as multiplication by \(p\) so that the composition of the two maps is zero. Since \((p + 1)/2 + (1/2)(p - 1)a_p\) maps to \(1\) or \(p\) under the one-dimensional representations of \(G(O_{E_w}/Q_p)\) one sees that the left-hand map is injective. To complete the verification of exactness we must show that the ideal, \(<p, (a_p-1)\>\) equals the principal ideal generated by \((p+1)/2 + (1/2)(p-1)a_p\). It is clear that \((p+1)/2 + (1/2)(p-1)a_p ∈ <p, (a_p-1)\>\) and to see the reverse inclusion it suffices to observe that

\[
((p+1)/2 + (1/2)(p-1)a_p) - a_p((p+1)/2 + (1/2)(p-1)a_p) = (p+1)/2 + (1/2)(p-1)a_p - (1/2)(p+1)a_p - (1/2)(p-1) = 1 - a_p.
\]

This shows that \([F_{p^2}] \in K_0 T(\mathbb{Z}[G(E_w/Q_p)]) \cong K_0(\mathbb{Z}[G(E_w/Q_p)], Q)\) has a Hom-description representative given by the function whose coordinate at each prime sends a one-dimensional representation \(\chi\) of \(G(E_w/Q_p)\) to

\[
\chi((p+1)/2 + (1/2)(p-1)a_p) = \begin{cases} p & \text{if } \chi(a_p) = 1, \\ 1 & \text{otherwise.} \end{cases}
\]

if we consider \(F_{p^2}\) to be an odd-dimensional homology group. Since \(p = p^{-1} ∈ (\mathbb{Z}/4)^*\), where our final computation takes place, the homological grading dimension of \(F_{p^2}\) will be immaterial.

This discussion shows that the Hom-description representative of \(\chi_w^*(L, V_w)\) differs from the one obtained above, using \(U_{E_w}^1\), by \((χ′ \mapsto p^{\dim(\chi'(E_w/Q_p))})\). As remarked earlier, for our purposes the sign of the exponent will be immaterial.

**Proposition 4.3**

Let \(p\) be an odd prime such that \(G(E/Q) = G(E_w/Q_p)\), as in Definition 4.1 and §4.2. Then a Hom-description representative for the local element

\[
\chi_w^*(L, V_w) ∈ TorsK_0(\mathbb{Z}[G(E_w/Q_p)], Q)
\]

is given by the function whose coordinate at each prime on one-dimension representations \(\chi\) equals

\[
e(\chi) \frac{(\dim G(E_w/Q_p))^{-\dim(\chi(G(E_w/Q_p)))} \det(1 - Frob_w^{-1}(\chi I(E_w/Q_p)/\chi G(E_w/Q_p)))}{p^{\pm \dim(\chi'(E_w/Q_p))} \det(1 - p^{-1}Frob_w^{-1} | \chi(I(E_w/Q_p)))}
\]
where
\[ \epsilon(\chi) = (-1)^{\text{dim}(\chi^{f(E_w/Q_p)})/\chi^{G(E_w/\mathbb{Q})}}. \]

5 Conjecture 2.2 for some quaternion fields

5.1 In this section we shall verify Conjecture 2.2 in the case when the decomposition group at \( p = 2 \) \( |G(N_v/\mathbb{Q}_2)| \) strictly smaller than \( \mathbb{Q}_8 \). The remaining cases correspond to quaternion extensions \( N/\mathbb{Q} \) in Cases A,B or C in [7]. These remaining cases seem to be much more difficult and will hopefully be treated in [8].

Theorem 5.2

Let \( N/\mathbb{Q} \) be a quaternion extension of number fields with Galois group \( G(N/\mathbb{Q}) \). If the decomposition group at \( p = 2 \) is strictly smaller than \( \mathbb{Q}_8 \) then
\[ T\Omega^{\text{loc}}(N/\mathbb{Q}, 1) \in \text{Tors}_{K_0}(\mathbb{Z}[G(N/\mathbb{Q})]; \mathbb{Q}) \cong (\mathbb{Z}/4)^* \]
is trivial, as predicted in Conjecture 2.2.

Proof

Let \( E \) denote the biquadratic subfield of \( N \). By Lemma 3.3 we must show that
\[ T\Omega^{\text{loc}}(E/\mathbb{Q}, 1) \in \text{Tors}_{K_0}(\mathbb{Z}[G(E/\mathbb{Q})]; \mathbb{Q}) \cong (\mathbb{Z}/4)^* \]
is trivial. By Definition 3.4
\[ T\Omega^{\text{loc}}(E/\mathbb{Q}, 1) = T_S(\mathcal{L}, \lambda) - \sum_{p \in S_f} \text{Ind}_{G(E_w/\mathbb{Q}_p)}^{G(E/\mathbb{Q})}(\chi_v^*(\mathcal{L}, V_w)) + \sum_{p \in S_f} \delta_1(\epsilon'_p(0)). \]

Let \( \chi_1, \chi_2 : G(E/\mathbb{Q}) \rightarrow \{\pm 1\} \) be two distinct, non-trivial characters. By Proposition 2.6 and the discussion of §2.4, if \( f : R(G(E/\mathbb{Q})) \rightarrow \mathbb{Q}_2^* \) is the 2-component of a Hom-description representative for \( T\Omega^{\text{loc}}(E/\mathbb{Q}, 1) \in (\mathbb{Z}/4)^* \), then
\[ T\Omega^{\text{loc}}(E/\mathbb{Q}, 1) = f(1 + \chi_1 + \chi_2 + \chi_1\chi_2)2^{-v_2(f(1+\chi_1+\chi_2+\chi_1\chi_2))} \pmod{4}. \]

By Corollary 3.6 and Proposition 3.10 this remains true if \( f \) is replaced by \( h \), a Hom-description representative for
\[ T_S(\mathcal{L}, 1) - \sum_{p \in S_f, G(E_w/\mathbb{Q}_p) = G(E/\mathbb{Q})} \chi_v^*(\mathcal{L}, V_w). \]

Note that, by hypothesis, the primes in the some are all odd.

Let \( h_p \) denote the Hom-description representative given in Proposition 4.3 for the local term corresponding to an odd prime \( p \) in the sum. The factor
\(e(1 + \chi_1 + \chi_2 + \chi_1\chi_2) = e(1)e(\chi_1)e(\chi_2)e(\chi_1\chi_2) = 1\) because there are precisely two distinct, non-trivial characters which are trivial on the inertia group. For the same reason
\[
\prod_{\chi = 1, \chi_1, \chi_2, \chi_1\chi_2} p^{1 + \text{mdim}(x^{I(E_w/Q_p)})}
\]
is a square in \((\mathbb{Z}/4)^*\) and hence trivial. The factors in the numerator are all positive powers of two, which cancel out in \(h_p(1 + \chi_1 + \chi_2 + \chi_1\chi_2)\). Therefore the torsion component of the two of local terms is equal to
\[
\prod_{\chi = 1, \chi_1, \chi_2, \chi_1\chi_2} p^{1 + \text{mdim}(x^{I(E_w/Q_p)})}
\]
modulo 4.

By Corollary 3.9 a Hom-description representative for \(T_S(\mathcal{L}, 1)\) of §3.7 is given by the function whose 2-adic coordinate is
\[
\chi \mapsto f(\chi)^{1/2} \prod_{\chi = 1, \chi_1, \chi_2, \chi_1\chi_2} (\gamma | \chi) \text{det}(1 - p^{-1}Frob_w^{-1} | \chi^{I(E_w/Q_p)})^{-1}.
\]
By [19], the function \((\chi \mapsto f(\chi)^{1/2} (\gamma | \chi)) \in \mathbb{Q}_2^*\) is a determinantal function if 2 is unramified in \(E/\mathbb{Q}\). By the calculations of [7], \((\gamma|\gamma) = \text{Ind}_{G(E_w/Q_p)}^G(\gamma_2|\gamma)\) and \(\gamma_2 = 4(1 + \pi)\) where \(E_w = \mathbb{Q}_2(\pi)\) so that \(\pi = \sqrt{2}\) or \(\pi = \sqrt{10}\) Hence
\[
(\gamma|\gamma) = \begin{cases} 8 & \text{if } \chi(a_2) = 1, \\ 8\pi & \text{otherwise.} \end{cases}
\]
and in all cases when 2 ramifies
\[
\prod_{\chi = 1, \chi_1, \chi_2, \chi_1\chi_2} f(\chi)^{1/2} \prod_{\gamma = 1, \chi_1, \chi_2, \chi_1\chi_2} (\gamma | \chi) = 2^a 5^b
\]
for suitable integers \(a\) and \(b\). Since \(2^a 5^b 2^{1 - \nu_2(2^a 5^b)} = 5^b \equiv 1\) (modulo 4) the torsion component of \(T_S(\mathcal{L}, 1)\) may also be computed by replacing its Hom-description representative by the simplified function
\[
\chi \mapsto \text{det}(1 - p^{-1}Frob_w^{-1} | \chi^{I(E_w/Q_p)})^{-1}.
\]
Therefore the torsion component of
\[
\sum_{\chi \in \mathcal{S}, G(E_w/Q_p)=G(E/Q)} \chi_w^\bullet(\mathcal{L}, V_w) - T_S(\mathcal{L}, 1)
\]
is equal to

$$
\prod_{p \in S_f, G(E_w/Q_p) \neq G(E/Q)} \det(1 - p^{-1} \text{Frob}_w^{-1} \mid \chi^I(E_w/Q_p))2^{-v_2(\det(1 - p^{-1} \text{Frob}_w^{-1} \mid \chi^I(E_w/Q_p)))}
$$

modulo 4. However this expression is a square in $(\mathbb{Z}/4)^*$ and is therefore trivial.

\[\square\]

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