On lower bounds for integration of multivariate permutation-invariant functions

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  Worst case error
  Known results
  Permutation-invariant subspaces

New lower bounds

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Integration problem for Korobov-type spaces

We study multivariate integration

\[ \text{Int}_d(f) = \int_{[0,1]^d} f(x) \, dx \]

for periodic, complex-valued functions in the Korobov class

\[ E_{d,\alpha} = \left\{ f \in L_1([0,1]^d) \mid \|f\| := \|f \, |\, E_{d,\alpha}\| < \infty \right\} \]

where \( d \in \mathbb{N} \), \( \alpha > 1 \), and

\[ \|f \, |\, E_{d,\alpha}\| = \sup_{k \in \mathbb{Z}^d} \left| \hat{f}(k) \right| \left( k_1 \cdot \ldots \cdot k_d \right)^\alpha. \]

For \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \) we set

\[ \overline{k}_m := \max \{1, |k_m|\} \quad \text{and} \quad \hat{f}(k) := \left\langle f, e^{2\pi i k \cdot} \right\rangle_{L_2}. \]
Worst case error

Without loss of generality, we consider linear cubature rules

\[ \mathcal{A}_{N,d}(f) := \sum_{n=1}^{N} w_n f(t^{(n)}) , \quad N \in \mathbb{N}_0, \]

with nodes \( t^{(n)} \in [0, 1]^d \) and weights \( w_n \in \mathbb{C}, \ n = 1 \ldots, N \).

As usual the \( N \)th minimal worst case error of \( \text{Int} = (\text{Int}_d)_{d \in \mathbb{N}} \) is defined by

\[ e(N, d; \text{Int}_d, E_{d,\alpha}) := \inf_{\mathcal{A}_{N,d}} \sup_{\|f\|_{E_{d,\alpha}} \leq 1} |\text{Int}_d(f) - \mathcal{A}_{N,d}(f)| . \]
Known results

The problem is well-scaled:

\[ e(0, d; \text{Int}_d, E_d, \alpha) = 1 \quad \text{for all} \quad d \in \mathbb{N}. \]

In 1997 Sloan and Woźniakowski [SW97] showed that for every \( d \in \mathbb{N} \)

\[ e(N, d; \text{Int}_d, E_d, \alpha) = e(0, d; \text{Int}_d, E_d, \alpha) \]

provided that \( N < 2^d \).

Hence, we have the *curse of dimensionality* since

\[
\begin{align*}
n(\varepsilon, d) & := n(\varepsilon, d; \text{Int}_d, E_d, \alpha) \\
& := \min \{ N \in \mathbb{N}_0 \mid e(N, d; \text{Int}_d, E_d, \alpha) \leq \varepsilon \} \\
& \geq 2^d
\end{align*}
\]

for all \( \varepsilon \in (0, 1), \ d \in \mathbb{N} \).
Notions of tractability

- *Polynomial tractability*: $\exists C, p > 0$ and $q \geq 0$ s.t.

$$n(\varepsilon, d) \leq C d^q \varepsilon^{-p}, \quad \forall d \in \mathbb{N}, \varepsilon \in (0, 1)$$

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Notions of tractability

- **Polynomial tractability:** \( \exists C, \ p > 0 \) and \( q \geq 0 \) s.t.
  \[
n(\varepsilon, d) \leq C \ d^q \ \varepsilon^{-p}, \quad \forall d \in \mathbb{N}, \ \varepsilon \in (0, 1)
\]

- **Strong polynomial tractability:** polynomial tractability with \( q = 0 \)

- **(s, t)-weak tractability:** \( \exists s, t \in (0, 1] \) s.t.
  \[
  \lim_{\varepsilon^{-1} + d \to \infty} \frac{\log n(\varepsilon, d)}{\varepsilon^{-s} + d^t} = 0
  \]

- **(Classical) weak tractability:** \( (1, 1) \)-weak tractability

- **Uniform weak tractability:** \( (s, t) \)-weak tractability for all \( s, t \in (0, 1] \) (sufficient: \( s = t \in (0, 1] \))
Permutation-invariant subspaces

For each $d \in \mathbb{N}$ take a subset of coordinates

$$\mathcal{I}_d \subseteq \{1, \ldots, d\}.$$ 

A function $f \in E_{d, \alpha}$ is called $\mathcal{I}_d$-permutation-invariant, if for all $x \in [0, 1]^d$ and every permutation $\sigma \in S_d$ of $\mathcal{I}_d$

$$f(x) = f(\sigma(x)).$$
Let $G_{I_d}(E_d, \alpha)$ denote the subspace of all $I_d$-permutation-invariant functions in $E_d, \alpha$, $d \in \mathbb{N}$, and consider $\text{Int} = (\text{Int}_d)_{d \in \mathbb{N}}$ restricted to these sets.

- New kind of additional structure (in contrast to weights)
- Motivated by applications (wave functions and Pauli principle in quantum mechanics)
- Handsome:

$$\hat{f}(k) = \hat{f}(\sigma(k))$$

for all $f \in G_{I_d}(E_d, \alpha)$, every $k \in \mathbb{Z}^d$, and each $\sigma \in S_d$.
- Successfully used for (tensor product) approximation problems
Does (full) permutation-invariance make integration trivial?
Does (full) permutation-invariance make integration **trivial**?

![Diagram](image.png)

**NO!**
Theorem (W., 2013)

Let

\[ N^* := N^*(d, I_d) := (\# I_d + 1) \cdot 2^{d - \# I_d}, \quad d \in \mathbb{N}. \]

Then, for every \( N < N^* \),

\[ e(N, d; \text{Int}_d, \mathcal{S}_{I_d}(E_d, \alpha)) = 1 \]

and

\[ e(N^*, d; \text{Int}_d, \mathcal{S}_{I_d}(E_d, \alpha)) \leq \left(1 + \frac{\zeta(\alpha)}{2^{\alpha-1}}\right)^d - 1 \]

for all \( d \in \mathbb{N} \) and \( \alpha > 1 \). Consequently,

\[ \lim_{\alpha \to \infty} e(N^*, d; \text{Int}_d, \mathcal{S}_{I_d}(E_d, \alpha)) = 0 \]

for all \( d \in \mathbb{N} \).
- $N^*$ is sharp (at least for large smoothness $\alpha$)
- Reformulation:

$$n(\varepsilon, d; \text{Int}_d, \mathcal{G}_{\mathcal{I}_d}(E_{d,\alpha})) \geq N^* = (\#\mathcal{I}_d + 1) \cdot 2^{d-\#\mathcal{I}_d}$$

for all $d \in \mathbb{N}$ and every $\varepsilon \in (0, 1)$.
- If $\mathcal{I}_d = \emptyset$ then $N^* = 2^d$ (generalization of [SW97])
- If $\mathcal{I}_d = \{1, \ldots, d\}$ then still $N^* \geq d + 1$
Corollary (W., 2013)

Let $\alpha > 1$ and set $b_d := d - \#I_d$ for all $d \in \mathbb{N}$.

- If $\text{Int}$ is polynomially tractable with the constants $C, p, q$ then $q \geq 1$ and $(b_d)_{d \in \mathbb{N}} \in \mathcal{O}(\ln d)$.  
  $\implies$ **NO strong polynomial tractability!**

- If $\text{Int}$ is uniformly weakly tractable then $(b_d)_{d \in \mathbb{N}} \in o(d^t)$ for all $t \in (0, 1]$.

- If $\text{Int}$ is $(s, t)$-weakly tractable for some $s, t \in (0, 1]$ then $(b_d)_{d \in \mathbb{N}} \in o(d^t)$.

  In particular, weak tractability implies $(b_d)_{d \in \mathbb{N}} \in o(d)$.

- If $(b_d)_{d \in \mathbb{N}} \notin o(d)$ then we have the curse of dimensionality.

  In turn, already the absence of the curse implies $(b_d)_{d \in \mathbb{N}} \in o(d)$. 
Basic idea of the proof

In [SW97], i.e. without permutation-invariance ($\mathcal{I}_d = \emptyset$):

- For $\mathbf{k} \in \mathbb{Z}^d$ set $e_\mathbf{k} = \exp(2\pi i \mathbf{k} \cdot \cdot)$.
- Choose a bijection $\Psi : \{0, \ldots, 2^d - 1\} \rightarrow \{0, 1\}^d$.
- For $N < N^* = 2^d$ solve the system

$$\sum_{n=0}^{N} a_n \cdot e_{\Psi(n)}(\mathbf{t}(j)) = 0, \quad j = 1, \ldots, N,$$

such that $a_{n^*} = 1 = \max_{n=0,1,\ldots,N} |a_n|$ for some $n^*$.

- Fooling function

$$f_N(\mathbf{x}) := e_{-\Psi(n^*)}(\mathbf{x}) \cdot \sum_{n=0}^{N} a_n \cdot e_{\Psi(n)}(\mathbf{x}).$$

satisfies $\|f_N\| \leq 1$, $\mathcal{A}_{N,d}(f_N) = 0$, and $\text{Int}_d(f_N) = 1$.

- Upper bound via $2^d$-point product-rectangle rule.
Proof in the permutation-invariant case

- “Symmetrize” the old proof using

\[(S_{I_d}e_k)(x) := \frac{1}{\#S_{I_d}} \sum_{\sigma \in S_{I_d}} e_k(\sigma(x)) \quad \text{for} \quad x \in [0, 1]^d,\]

instead of \(e_k\) and replace

\[\{0, 1\}^d \quad \text{by} \quad \{k \in \{0, 1\}^d \mid k_1 \leq \ldots \leq k_{\#I_d}\}.\]

- A lot of nasty calculations caused by the permutations.
Final remarks

Further results for

- more than only one “symmetry set”,
- weighted spaces (product weights).

Work in progress:

- Upper bounds for related spaces endowed with $\ell_2$-norm.

Conclusion:

- Integration of permutation-invariant functions is not trivial (no strong polynomial tractability)!
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  - more than only one “symmetry set”,
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Open problem

*Do we have polynomial tractability for integration of (fully) permutation-invariant functions?*
References

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Thank you!