Application of Lie-group symmetry analysis to an infinite hierarchy of differential equations at the example of first order ODEs

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Abstract

This study will explicitly demonstrate by example that an unrestricted infinite and forward recursive hierarchy of differential equations must be identified as an unclosed system of equations, despite the fact that to each unknown function in the hierarchy there exists a corresponding determined equation to which it can be bijectively mapped to. As a direct consequence, its admitted set of symmetry transformations must be identified as a weaker set of indeterminate equivalence transformations. The reason is that no unique general solution can be constructed, not even in principle. Instead, infinitely many disjoint and thus independent general solution manifolds exist. This is in clear contrast to a closed system of differential equations that only allows for a single and thus unique general solution manifold, which, by definition, covers all possible particular solutions this system can admit. Herein, different first order Riccati-ODEs serve as an example, but this analysis is not restricted to them. All conclusions drawn in this study will translate to any first order or higher order ODEs as well as to any PDEs.

Keywords: Ordinary Differential Equations, Infinite Systems, Lie Symmetries and Equivalences, Unclosed Systems, General Solutions, Linear and Nonlinear Equations, Initial Value Problems;

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1. Introduction, motivation and a brief overview what to expect

Infinite systems of ordinary differential equations (ODEs) appear naturally in many applications (see e.g. Dolph & Lewis (1958); Bellman (1973); Temam (1997); Robinson (2001); Pustyl’nikov (2002); Haragus & Iooss (2010)). Mostly they arise when regarding a certain partial differential equation (PDE) as an ordinary differential equation on a function space. This idea ranges back to the time of Fourier when he first introduced his method of representing PDE solutions as an infinite series of trigonometric basis functions, through which he basically formed the origin of functional analysis which then a century later was systematically developed and investigated by Volterra, Hilbert, Riesz, and Banach, to name only a few.

The key principle behind the idea that a PDE can be viewed as a ordinary differential equation within an infinite dimensional space, is that every PDE by construction describes the change of a system involving infinitely many coupled degrees of freedom, where each degree then changes according to an ODE. In this regard, let us sketch five simple examples to explicitly see how naturally and in which variety such infinite dimensional ODE systems can arise from a PDE:

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Example 1 (Partial Discretization): Using the numerical method of finite differences, e.g. on the nonlinear initial-boundary value problem for the 2-dimensional function \( u = u(t, x) \)

\[
\partial_t u = \partial_x^2 u + u^2, \quad \text{with} \quad u(0, x) = \phi(x), \quad \text{and} \quad u(t, 0) = \phi(0) = \phi(1) = u(t, 1), \tag{1.1}
\]

in respect to the spatial variable \( x \), one formally obtains the following infinite hierarchy of coupled ODEs for the 1-dimensional functions \( u \)

\[
\frac{du_k}{dt} = \left( \frac{u_{k+1} - 2u_k + u_{k-1}}{\Delta^2} \right) + u_k^2, \quad \text{for all} \quad k = 0, 1, 2, \ldots, N \to \infty, \tag{1.2}
\]

when approximating the second partial derivative by the equidistant central difference formula, where \( \Delta = 1/N \) is the discretization size of the considered interval \( 0 \leq x \leq 1 \). Through the given initial and boundary conditions, this system is restricted for all \( k \) and \( t \) by

\[
u_k(0) = \phi_k, \quad \text{and} \quad u_0(t) = \phi(0) = \phi(1) = u_N(t), \quad \text{for} \quad N \to \infty. \tag{1.3}
\]

To obtain numerical results, the infinite system (1.2) needs, of course, to be truncated, which then leads to an approximation of the original PDE initial-boundary value problem (1.1). The higher \( N \), or likewise, the smaller \( \Delta \), the more exact the approximation (depending on the numerical rounding errors of the computational system, which itself will always be limited).

Example 2 (Power Series Expansion): Let’s assume that the linear initial value problem

\[
\partial_t u + \alpha_1 \cdot \partial_x u + \alpha_2 \cdot u = 0, \quad \text{with} \quad u(0, x) = \phi(x), \tag{1.4}
\]

allows for an analytical function as solution with respect to \( x \) in the interval \( a \leq x \leq b \). Then the solution may be expanded into the power series

\[
u(t, x) = \sum_{n=0}^{\infty} p_n(t) \cdot x^n, \tag{1.5}
\]

which, when applied to (1.4), results into the following restricted infinite ODE system for the expansion coefficients

\[
\frac{dp_n}{dt} + \alpha_1 \cdot (n + 1) \cdot p_{n+1} + \alpha_2 \cdot p_n = 0, \quad \text{for all} \quad n \geq 0, \quad \text{with} \quad \sum_{n=0}^{\infty} p_n(0) \cdot x^n = \phi(x). \tag{1.6}
\]

Example 3 (Fourier’s Method): Consider the initial value problem of the Burgers’ equation

\[
\partial_t u + u \cdot \partial_x u = \nu \cdot \partial_x^2 u, \quad \text{with} \quad u(0, x) = \phi(x), \tag{1.7}
\]

where \( \phi(x) \) is periodic of period \( 2\pi \). If we are looking for such a \( 2\pi \)-periodic solution \( u(t, x) \), we can write

\[
u(t, x) = \sum_{n=-\infty}^{\infty} u_n(t) e^{i n x}. \tag{1.8}
\]

Substitution in (1.7) and equating the coefficients leads to the infinitely coupled ODE relations

\[
\frac{du_n}{dt} + \sum_{k+l=n} i \cdot k \cdot u_k \cdot u_l = -\nu \cdot n^2 \cdot u_n, \quad \text{for} \quad (n, k, l) \in \mathbb{Z}^3, \tag{1.9}
\]

being restricted by the initial conditions \( u_n(0) = \phi_n \), which are determined by the Fourier expansion \( \phi(x) = \sum_{n=-\infty}^{\infty} \phi_n e^{inx} \).
Example 4 (Vector Space Method): This example is the generalization of the two previous ones. Let’s consider the linear initial-boundary value problem for the diffusion equation

\[ \partial_t u = \partial_x^2 u, \quad \text{with} \quad u(0, x) = \phi(x), \quad \text{and} \quad u(t, a) = \phi(a), \quad u(t, b) = \phi(b), \tag{1.10} \]

where \( a < b \) are two arbitrary boundary points of some interval \( I \subset \mathbb{R} \), which both can be also placed at infinity. Consider further the vector space

\[ \mathcal{V} = \left\{ v(x), \text{differentiable functions on } I, \text{with} \quad v(a) = \phi(a), \quad v(b) = \phi(b) \right\}, \tag{1.11} \]

with \( \{w_n\}_{n=1}^{\infty} \) as a chosen basis, for which at this point we do not know their explicit functional expressions, and assume that the solution \( u(t, x) \) for each value \( t \) of the initial-boundary value problem (1.10) is an element of this space \( \mathcal{V} \). Then we can write the solution as an element of \( \mathcal{V} \) in terms of the chosen basis vectors

\[ u(t, x) = \sum_{n=1}^{\infty} u_n(t) \cdot w_n(x), \tag{1.12} \]

which, when inserted into (1.10), induces the following two infinite but uncoupled sets of linear eigenvalue equations

\[ \frac{d u_n(t)}{dt} = -\lambda_n^2 \cdot u_n(t), \quad \frac{d^2 w_n(x)}{dx^2} = -\lambda_n^2 \cdot w_n(x), \quad \text{for all } n \geq 1, \tag{1.13} \]

one infinite (uncoupled) set for the expansion coefficients \( u_n \), and one infinite (uncoupled) set for the basis functions \( w_n \), where \(-\lambda_n^2\) is the corresponding eigenvalue to each order \( n \). Both systems are restricted by the given initial and boundary conditions in the form

\[ \sum_{n=1}^{\infty} u_n(0)w_n(x) = \phi(x); \quad w_n(a) = \phi(a), \quad w_n(b) = \phi(b), \quad \text{for all } n \geq 1, \tag{1.14} \]

and, if the boundary values \( \phi(a) \) or \( \phi(b) \) are non-zero, then the system for \( u_n \) is further restricted by

\[ \sum_{n=1}^{\infty} u_n(t) = 1, \quad \text{for all possible } t. \tag{1.15} \]

This situation can now be generalized further when using a different basis \( \{\psi_n\}_{n=1}^{\infty} \) of \( \mathcal{V} \) (1.11) than the PDE’s optimal basis \( \{w_n\}_{n=1}^{\infty} \), which itself is defined by the PDE induced infinite (uncoupled) ODE system (1.13). For that, it is helpful to first introduce an inner product on the outlaid vector space \( \mathcal{V} \) (1.11). Given two functions \( f, g \in \mathcal{V} \), we will define the inner product between these two functions as the following symmetric bilinear form on \( I \subset \mathbb{R} \)

\[ \langle f, g \rangle = \int_a^b f(x) g(x) \, dx. \tag{1.16} \]

Now, to construct the corresponding induced infinite ODE system for the new representation of the PDE’s solution

\[ u(t, x) = \sum_{n=1}^{\infty} \tilde{u}_n(t) \cdot \psi_n(x), \tag{1.17} \]

it is expedient to identify the differential operator \( \partial_x^2 \) in the PDE (1.10) as a linear operator \( A \) acting on the given vector space \( \mathcal{V} \) (1.11)

\[ \partial_t u = \partial_x^2 u \quad \simeq \quad \partial_t u = A(u), \tag{1.18} \]
which then can also be written in explicit matrix-vector form as

$$\frac{d\mathbf{u}}{dt} = \mathbf{A} \cdot \mathbf{u},$$  \hspace{1cm} (1.19)

where the infinite vector $\mathbf{u} = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n, \ldots)$ is composed of the expansion coefficients of the solution vector $u(t, x)$ (1.17), and where the matrix elements of the infinite matrix $\mathbf{A}$ are given as

$$A_{mn} = \langle \psi_m, A(\psi_n) \rangle = \int_a^b \psi_m(x) \frac{\partial^2}{\partial x^2} \psi_n(x) \, dx.$$  \hspace{1cm} (1.20)

The explicit element values of both $\mathbf{u}$ and $\mathbf{A}$ depend on the choice of the basis $\{\psi_n\}_{n=1}^\infty \subseteq \mathcal{V}$, while the solution vector $u(t, x)$ of the underlying PDE (1.18) itself stays invariant under a change of base, i.e., for example, the expansion (1.12) relative to the basis $\{w_n\}_{n=1}^\infty$ represents the same solution as the expansion (1.17) relative to the basis $\{\psi_n\}_{n=1}^\infty$.

Of particular interest is now to investigate whether $\mathbf{A} = (A_{nm})_{n,m\geq 1}$ (1.20) represents a symmetric matrix or not. If not, are there then any conditions such that a symmetric matrix can be obtained? Performing a double partial integration in (1.20) will yield the expression

$$A_{mn} = \int_a^b \psi_m(x) \frac{\partial^2}{\partial x^2} \psi_n(x) \, dx$$

$$= \int_a^b \psi_n(x) \frac{\partial^2}{\partial x^2} \psi_m(x) \, dx + \left[ \psi_m(x) \frac{\partial}{\partial x} \psi_n(x) - \psi_n(x) \frac{\partial}{\partial x} \psi_m(x) \right]_{x=a}^{x=b}$$

$$= A_{nm} + \left[ \psi_m(x) \frac{\partial}{\partial x} \psi_n(x) - \psi_n(x) \frac{\partial}{\partial x} \psi_m(x) \right]_{x=a}^{x=b},$$  \hspace{1cm} (1.21)

which implies that in order to obtain a symmetric matrix, we have to choose the boundary values $\phi(a)$ and $\phi(b)$ such that they satisfy the symmetry relation

$$\left[ \psi_m(x) \frac{\partial}{\partial x} \psi_n(x) - \psi_n(x) \frac{\partial}{\partial x} \psi_m(x) \right]_{x=a}^{x=b} = 0, \text{ for all } n, m \geq 1.$$  \hspace{1cm} (1.22)

For example, the simplest choice is to enforce vanishing boundary values $\phi(a) = \phi(b) = 0$. This will satisfy the condition (1.22) independently on the choice of the basis functions, with the advantageous effect then that the matrix $\mathbf{A}$ is a real symmetric matrix to any chosen basis of the vector space $\mathcal{V}$ (1.11). In particular, since the eigenvectors of a real symmetric matrix are orthogonal, the matrix elements $A_{mn}$ for the optimal basis $\{w_n\}_{n=1}^\infty$, according to (1.13), form the diagonal matrix

$$A_{mn} = -\lambda_n^2 \cdot \|w_n\|^2 \delta_{mn} = \int_a^b w_m(x) \frac{\partial^2}{\partial x^2} w_n(x) \, dx,$$  \hspace{1cm} (1.23)

where no summation over the repeated index $n$ is implied. Hence, if we choose the alternative basis $\{\psi_n\}_{n=1}^\infty$ such that it’s orthonormal\footnote{If $\{\psi_n\}_{n=1}^\infty$ is an orthonormal basis of $\mathcal{V}$, then each $\psi_n$ must be collinear to $w_n$, in particular $\psi_n = w_n/\|w_n\|$.}

$$\langle \psi_m, \psi_n \rangle = \delta_{mn},$$  \hspace{1cm} (1.24)

then, according to (1.14) and (1.15), the PDE’s induced infinite ODE system (1.19) will only be restricted by the initial conditions

$$\tilde{u}_n(0) = \int_a^b \phi(x) \psi_n(x) \, dx, \text{ for all } n \geq 1.$$  \hspace{1cm} (1.25)
Example 5 (Method of moments): Consider the initial value problem (Cauchy problem) of the more generalized linear diffusion equation
\[ \partial_t u = a \cdot \partial^2_x u + b \cdot x \cdot \partial_x u + (b + c \cdot x^2) \cdot u, \quad \text{with } u(0, x) = \phi(x). \] (1.26)

If one is interested in the moments
\[ u_n(t) = \int_{-\infty}^{\infty} x^n \cdot u(t, x) \, dx, \quad n \geq 0, \] (1.27)

by multiplying the PDE (1.26) with \( x^n \) and integrating over \( \mathbb{R} \), and if one assumes that partial integration is justified, i.e. when assuming the natural boundary conditions
\[ \lim_{x \to \pm\infty} u(t, x) = 0, \quad \lim_{x \to \pm\infty} \partial_x u(t, x) = 0, \] (1.28)

then one obtains the following infinite system of coupled ODEs for all \( n \geq 0 \)
\[ \frac{du_n}{dt} = a \cdot n \cdot (n - 1) \cdot u_{n-2} - b \cdot n \cdot u_n + c \cdot u_{n+2}, \quad \text{with } u_n(0) = \int_{-\infty}^{\infty} x^n \cdot \phi(x) \, dx. \] (1.29)

Note that for \( c = 0 \) the solution \( u = u(t, x) \) of (1.26) has the property of a probability measure, for example in that it can be interpreted as a probability density of a particle undergoing Brownian motion. Only for \( c = 0 \) the PDE (1.26) attains the structure of a Fokker-Planck equation with the drift coefficient \( D_1(t, x) = -b \cdot x \) and the diffusion coefficient \( D_2(t, x) = a > 0 \)
\[ \partial_t u = \partial_x (-D_1 \cdot u + D_2 \cdot \partial_x u), \] (1.30)

which describes the time evolution of the probability distribution \( u(t, x) \geq 0 \) such that no probability is lost, i.e. conserved for all \( t \geq 0 \)
\[ \int_{-\infty}^{\infty} u(t, x) \, dx = 1, \] (1.31)

due to the defining structure of equation (1.30) being a conservation law with the probability current \( J = -D_1 \cdot u + D_2 \cdot \partial_x u \). Hence, for \( c = 0 \) the corresponding infinite ODE system (1.29) is thus further (automatically) restricted by \( u_0(t) = 1 \) for all times \( t \geq 0 \).

However, if \( c \neq 0 \) then the zeroth moment (1.31) in general is not constant in time; it will rather evolve according to some prescribed 'normalization' function \( N(t) \)
\[ \int_{-\infty}^{\infty} u(t, x) \, dx = N(t). \] (1.32)

To exogenously enforce a certain function \( N(t) = N_0(t) \) as an additional (non-local) boundary condition onto the Cauchy problem (1.26) would result into an overdetermined system, for which no solutions may exist. The Cauchy problem itself, e.g. of equation (1.26), is well-posed and allows, up to a normalization constant, for a unique solution. The normalization constant can be fixed by posing at \( t = 0 \) a normalized initial condition, e.g. \( \int_{-\infty}^{\infty} \phi(x) \, dx = N(0) = 1 \).

Note that although such integral overdetermination conditions as (1.32) are widely used to study associated inverse problems, e.g. to find corresponding heat sources and diffusion coefficients (see e.g. Dehghan (2007); Kanca & Ismailov (2012); Hazanee et al. (2013)), we will not consider them here.
These five examples discussed above show that there is a multitude of possibilities in how a PDE can induce an ODE in an infinite dimensional space. If the PDE is restricted by initial or boundary conditions they are transcribed to the infinite system accordingly. Mostly, only the initial conditions get directly transferred, while the boundary conditions are only needed as auxiliary conditions to actually perform the reduction process, e.g., as in the case of Example 4. Note that all examples only considered the reduction of \((1 + 1)\)-dimensional parabolic PDEs, but it’s obvious that this concept extends to any type of PDEs of any dimension. The result is then not a single infinite system, but rather a collective hierarchy of several infinite systems of coupled ODEs.

As all examples showed, it should be clear that the associated infinite system of ODEs is \textit{not} identical to the PDE. It only represents a reduction of the PDE, since always a certain Ansatz of the PDE’s solution manifold has to be made in order to obtain its associated infinite ODE system. Stated differently, the PDE operates on a higher level of abstraction than its induced infinite system of \((\text{lower level})\) ODEs, which, although infinite dimensional, nevertheless depends on assumptions and in particular on the choice of the reduction method used. That is, a \textit{single} PDE can always be reduced to a multitude of functionally and structurally \textit{different} infinite systems of ODEs depending on the choice of method. This insight can be transferred to differential equations which operate on an even higher abstraction level than PDEs, e.g., so-called functional equations which involve functional derivatives. Then an infinite hierarchy of PDEs instead of ODEs takes the place of the reduced system.

A natural question which arises is whether the infinite set of reduced equations is easier to analyze than the original PDE? In general, the answer to this question is “no”. However, if the infinite system is truncated and approximated to a low-dimensional form, then often qualitative analysis is possible, and useful insights into the dynamics of the original system can be obtained. Also from a numerical point of view many interesting stability questions arise when the system is truncated, because, in order to obtain numerical results from an infinite system, some method of truncation must be employed. Surely, the quality of the subsequent approximation towards a consistent finite dimensional system strongly depends on this method in how the system was truncated, which is part of the theories of closure and differential approximation.

The formal mathematical environment to study and analyze an infinite (non-truncated) sequence of differential equations is set by the infinite-dimensional theory of Banach spaces. Questions regarding existence and uniqueness of solutions can only be properly dealt with from the perspective of a Banach space in defining and constructing appropriate functional norms. Such systematic investigations, however, are beyond the scope of this article; for that, the rich literature on this topic has to be consulted (see e.g. Tikhonov (1934); Valeev & Zhautykov (1974); Deimling (1977); Samoilenko & Teplinskii (2003); Hájek & Vivi (2010); Fabian et al. (2011)). Instead, we will only make a small excursion into the uniqueness issue of these solutions when restricting the infinite system by a sufficient set of initial conditions, but only to show where still the problems lie and not on how to solve these problems.

The main focus of this article will be based on the \textit{unrestricted} infinite set of ODEs, and to primarily study the \textit{general} solutions they admit. By taking the perspective of a Lie group based symmetry analysis (Stephani, 1989; Fushchich \textit{et al.}, 1993; Olver, 1993; Ibragimov, 1994; Bluman \textit{et al.}, 2010), we can demonstrate by example that eventually any unrestricted infinite set of differential equations, which is based on a \textit{forward} recurrence relation, must be identified as an unclosed and thus indeterminate system, although, in a formal one-to-one manner, one can associate to each equation in the hierarchy a corresponding unknown function. As a consequence, such unclosed differential systems do not allow for the construction of a unique \textit{general} solution. Any desirable \textit{general} solution can be generated. The side-effect of this result is that each symmetry transformation then only acts in the weaker sense of an equivalence transformation (Ovsiannikov, 1982; Meleshko, 1996; Ibragimov, 2004; Vaneeva \textit{et al.}, 2014). Such an
identification is necessary in order to allow for a consistent invariance analysis among an infinite set of differential equations.

At first sight it may seem to be a trivial observation that an \textit{unrestricted} infinite set of ODEs has the property of an underdetermined system. Because if it represents a specific reduction of an \textit{unrestricted} PDE, i.e. of a PDE which is not accompanied by any initial or boundary conditions, its general solution is only unique up to certain integration functions. And since this arbitrariness on the higher abstraction level of the PDE is transferred down to the lower abstraction level of the reduced ODE system, it is not surprising that the latter system is somehow arbitrary as well. But, by closer inspection there is no one-to-one correspondence, because, for example, for any evolutionary PDE with fixed spatial boundary conditions, the degree of arbitrariness in its general solution only depends on the order of the time derivative, which in turn is directly linked to the number of initial condition functions needed to generate a unique solution from the general one. However, for its reduced ODE system the degree of arbitrariness is differently larger in that it not only depends on the temporal differential order as the underlying PDE does, but also, additionally, on the direction and the order of the spatial recurrence relation which this system inherently defines. For example, if the recurrence relation is a \textit{forward} recurrence of order one, then, independent of the temporal differential order, one unknown function anywhere in the ODE hierarchy can be specified freely; if its a \textit{forward} recurrence of order two, then two unknown functions can be specified freely, and so on. This freedom in choice has no correspondence on the higher abstraction level of the PDE. Appendix A.1 and A.2 provide a preview demonstration of these statements by considering again Example 5. In Section 3.1 and Section 3.2 this insight will then be investigated in more detail by also involving different examples.

In particular, it was exactly Example 5 with its properties discussed in Appendix A.1 and A.2 which motivated this study. The inherent principle that the (higher abstraction level) PDE (1.26) represents a closed system while the correspondingly reduced (lower abstraction level) ODE system of its moments (1.29), although being infinite in dimension, constitutes an unclosed system if the recurrence is of forward direction (see Appendix A.2), obviously transfers to an even higher abstraction level of description, as seen, for example, when formulating the statistical description of Navier-Stokes turbulence. There the functional Hopf equation formally serves as the (higher abstraction level) \textit{closed} equation while its correspondingly induced (lower abstraction level) infinite Friedmann-Keller PDE system of multi-point moments is unclosed; for more details on this issue, see Frewer et al. (2014) and Frewer (2015).

Part of the current study is to mathematically clarify this point in statistical turbulence research, namely where any formally \textit{closed} set of equations which operates on a higher statistical level always induces an \textit{unclosed} infinite system on the lower statistical level of the \textit{moment} equations, and, where thus, both levels of description are not equivalent. The reason for this is that due to the nonlinearity of the Navier-Stokes problem a \textit{forward} recurrence relation is always generated on the lower abstraction level of the statistical moments (in the sense similar to the problem demonstrated in Appendix A.2), turning thus the corresponding infinite Friedmann-Keller PDE hierarchy inherently into an unclosed system. It is necessary to clarify this point, because it seems that in the relevant literature on turbulence there still exists a misconception on this issue, in particular in the studies of Oberlack et al. (Oberlack & Rosteck, 2010; Oberlack & Zieleniewicz, 2013; Avsarkisov et al., 2014; Wacławczyk et al., 2014; Oberlack et al., 2014). A detailed discussion on this misconception is given in Frewer (2015).

The paper is organized as follows: Section 2 first considers a single (closed) ODE to define the concept of a \textit{unique} general solution manifold from the perspective of an invariance analysis. In Section 3 this concept will be applied to an (unclosed) infinite ODE system based on a \textit{forward} recurrence relation. Both a linear (Section 3.1) as well as a nonlinear system (Section 3.2) will be investigated, which both stem as special cases from a generalized hierarchy of
first order Riccati-ODEs. To which higher level PDE this infinite ODE system belongs to is an inverse problem, which will not be investigated since it’s clearly beyond the scope of this article. Based on the results presented herein, we can conclude that any unrestricted infinite system which follows a forward recurrence relation must be identified as an unclosed system, which then, as consequence, only leads to non-unique general solution manifolds and which, instead of symmetry transformations, only admits the weaker equivalence transformations. This twofold conclusion is independent of whether an infinite hierarchy of ODEs or whether an infinite hierarchy of PDEs is considered.

2. Lie-point symmetries and general solution of a single Riccati-ODE

In general a Riccati equation is any first-order ODE that exhibits a quadratic nonlinearity of the form

\[ y'(x) = q_0(x) + q_1(x)y(x) + q_2(x)y^2(x), \]  

(2.1)

with \( q_i(x) \) being arbitrary functions (see e.g. Reid (1972)).\(^1\) In this section, however, we only want to consider the following specific Riccati-ODE

\[ y' - \frac{y}{x} - \frac{y^2}{x^3} = 0, \]  

(2.2)

which is also categorized as a specific Bernoulli differential equation of the quadratic rank (see e.g. Parker (2013)). Its unique general solution\(^2\) is given by

\[ y(x) = \frac{x^2}{1 + c \cdot x}, \quad c \in \mathbb{R}, \]  

(2.3)

involving a single free integration parameter \( c \). Since (2.2) is a single first order ODE it has the special property of admitting an infinite set of Lie-point symmetries (see e.g. Stephani (1989); Bluman & Kumei (1996); Ibragimov (2004)). The symmetries are generated by the tangent field \( X = \xi(x,y)\partial_x + \eta(x,y)\partial_y \), which, in the considered case (2.2), satisfies the following underdetermined relation for the infinitesimals \( \xi = \xi(x,y) \) and \( \eta = \eta(x,y) \):

\[
0 = \xi \cdot (3y^2x^2 + yx^5) - \eta \cdot (2yx^3 + x^5) \tag{2.4}
\]

\[- \partial_x \xi \cdot (y^2x^3 + yx^5) - \partial_y \xi \cdot (y^4 + 2y^3x^2 + y^2x^4) + \partial_x \eta \cdot (x^6) + \partial_y \eta \cdot (y^2x^3 + yx^5). \]

Note that in constructing the general solution of equation (2.4) only one function can be chosen arbitrarily, either \( \xi \) or \( \eta \), but not both. Without restricting the general case, we will choose \( \xi \) as the free infinitesimal, which, once chosen in (2.4), then uniquely fixes the second infinitesimal \( \eta \). For the present, it is sufficient to only consider monomials in the normalized form \( x^n \) with \( n \geq 0 \) as a functional choice for \( \xi \). According to (2.4), the corresponding tangent field \( X \) up to order \( n \) is then given by

\[
T_n : \quad X_n = x^n\partial_x + \left[ x^{n+2} \cdot \left( \frac{y}{x^3} + \frac{y^2}{x^9} \right) + F_n \left( \frac{y-x^2}{yx} \right) \cdot \frac{y^2}{x} \right] \partial_y, \quad n \geq 0, \tag{2.5}
\]

where the \( F_n \) are arbitrary integration functions with argument \( (y-x^2)/yx \). For the sake of simplicity it is convenient to choose these functions such that for each order \( n \) the lowest degree

\(^1\)Note that the nonlinear Riccati-ODE (2.1) can always be reduced to a linear ODE of second order by making use of the transformation \( y(x) = -z'(x)/(q_2(x) \cdot z(x)) \).

\(^2\)Transforming the nonlinear ODE (2.2) according to \( y(x) = x/z(x) \) will reduce it to a linear ODE of first order which can be solved then by a simple integration.
of complexity is achieved. For example, for the first four elements in this chain (2.5) we choose the $F_n$ such that

$$
\begin{align*}
T_0 &: X_0 = \partial_x + \left(\frac{3y}{x} - x\right) \partial_y, \\
T_1 &: X_1 = x \partial_x + 2y \partial_y, \\
T_2 &: X_2 = x^2 \partial_x + xy \partial_y, \\
T_3 &: X_3 = x^3 \partial_x + \left(y^2 + yx^2\right) \partial_y,
\end{align*}
$$

which, according to Lie’s central theorem (see e.g. Bluman & Kumei (1996)), are equivalent to the 1-parameter symmetry group transformations

$$
\begin{align*}
T_0 &: \tilde{x} = x + \varepsilon_0, \quad \tilde{y} = \left(\frac{y - x^2}{x^3} + \frac{1}{x + \varepsilon_0}\right) (x + \varepsilon_0)^3, \\
T_1 &: \tilde{x} = \varepsilon_1 x, \quad \tilde{y} = \varepsilon_2 y, \\
T_2 &: \tilde{x} = \frac{x}{1 - \varepsilon_2 x}, \quad \tilde{y} = \frac{y}{1 - \varepsilon_2 x}, \\
T_3 &: \tilde{x} = \frac{x}{\sqrt{1 - 2\varepsilon_3 x^2}}, \quad \tilde{y} = \frac{yx^2}{y \cdot (1 - 2\varepsilon_3 x^2) - (y - x^2) \cdot \sqrt{1 - 2\varepsilon_3 x^2}}.
\end{align*}
$$

By construction each of the above transformations leaves the considered differential equation (2.2) invariant, i.e. when transforming (2.2) according to one of the transformations (2.7) will thus result into the invariant form

$$
y' - \frac{\tilde{y}}{\tilde{x}} - \frac{\tilde{y}^2}{\tilde{x}^3} = 0.
$$

Note that for any point transformations, as in the case (2.7), the transformation for the first order ordinary derivative is induced by the relation

$$
y' = \frac{dy}{d\tilde{x}} = \frac{\partial \tilde{y}}{\partial \tilde{x}} dx + \frac{\partial \tilde{y}}{\partial \tilde{y}} dy + \frac{\partial y}{\partial x} \frac{dy}{dx} + \frac{\partial y}{\partial y} \frac{dy}{dy} = \left(\frac{\partial \tilde{y}}{\partial \tilde{x}}\right)^{-1} \left(\frac{\partial y}{\partial x} + \frac{\partial y}{\partial y} \frac{dy}{dy}\right),
$$

where the last equality only stems from the fact that $\partial \tilde{x} / \partial y = 0$ for all transformations (2.7).

Now, if the general solution (2.3) would not be known beforehand, then the symmetries (2.7) can be used to construct it. For any first order ODE, as in the present case for (2.2), at least one symmetry is necessary to determine its general solution, which, for example, can be achieved by making use of the method of canonical variables (see e.g. Stephani (1989)). But, instead of performing this construction, the opposite procedure will be investigated, namely to validate in how far the function (2.3) represents a unique general solution\footnote{By definition a unique general solution of a differential equation or a system of differential equations should cover all particular (special) solutions this system can admit. In other words, every special solution that can be constructed must be covered through this general solution by specifying a corresponding initial condition, otherwise the given general solution is not complete or unique.} of (2.2) when transforming it according to the symmetries (2.7). The result to expect is that if function (2.3) represents the unique general solution of (2.2), then it either must map to

$$
\tilde{y}(\tilde{x}) = \frac{\tilde{x}^2}{1 + \tilde{c} \tilde{x}}, \quad \tilde{c} \in \mathbb{R},
$$

where the last equality only stems from the fact that $\partial \tilde{x} / \partial y = 0$ for all transformations (2.7).
with a new free transformed parameter \( \tilde{c} = c(c) \) as a function of the old untransformed parameter \( c \), or it must invariantly map to

\[
\tilde{y}(\tilde{x}) = \frac{\tilde{x}^2}{1 + c \cdot \tilde{x}}, \quad c \in \mathbb{R}, \tag{2.11}
\]

with an unchanged free parameter \( c \) before and after the transformation. The reason is that since all transformations (2.7) form true symmetries, which map solutions to new solutions of the underlying differential equation (2.2), any solution which forms a unique general solution of this equation can thus only be mapped into itself, either into the non-invariant form (2.10) or into the invariant form (2.11), because no other functionally independent solution exists to which the symmetries can map to. If this is not the case, we then have to conclude that either the considered transformation is not a symmetry transformation or that the given solution is not the general solution.

Hence, since we definitely know that (2.3) is the unique general solution of equation (2.2), which again admits the symmetries (2.7), these symmetry transformations only need to be classified into two categories, namely into those which reparametrize the general solution (2.10) and into those which leave it invariant (2.11). For example, the symmetries \( T_0, T_1 \) and \( T_2 \) reparametrize the general solution (2.3) with \( \tilde{c} = c/(1 - c \varepsilon_0) \), \( \tilde{c} = e^{-\varepsilon_1}c \), and \( \tilde{c} = c + \varepsilon_3 \) respectively, while symmetry \( T_3 \) keeps it invariant. This game can then be continued for all higher orders of \( n \) in (2.5), or even for any other functionally different symmetry using the general determining relation (2.4).

To conclude this section, it is helpful to formalize the above insights: Let \( f_\lambda \) formally be a parameter dependent solution of a differential equation \( E \), and \( S \) any transformation which leaves this differential equation invariant \( S(E) = E \). The transformation \( S \) on \( f_\lambda \) is called a reparametrization if \( S(f_\lambda) = f_{s(\lambda)} \), which includes the special case of an invariant transformation if \( s(\lambda) = \lambda \), where the parameter mapping \( s \) is induced by the variable mapping \( S \). Then, based on these conditions, the following two statements are equivalent:

\[
\begin{align*}
\text{\( f_\lambda \) is a unique general solution} & \implies S(f_\lambda) = f_{s(\lambda)}, \\
\text{\( S(f_\lambda) \neq f_{s(\lambda)} \) & \implies \text{\( f_\lambda \) is not a unique general solution},}
\end{align*}
\tag{2.12}
\]

where in each case the opposite conclusion is, of course, \textit{not} valid, i.e.

\[
\begin{align*}
\text{\( S(f_\lambda) = f_{s(\lambda)} \) & \implies \text{\( f_\lambda \) is a unique general solution},} \\
\text{\( f_\lambda \) is not a unique general solution} & \implies S(f_\lambda) \neq f_{s(\lambda)}.}
\end{align*}
\tag{2.13}
\]

3. Lie-point symmetries and general solution for an infinite system of ODEs

Let’s consider the following \textit{unrestricted} infinite and \textit{forward} recursive hierarchy of ordinary differential equations based on the Riccati ODE (2.1)

\[
y''_n(x) - q_0(x) - q_1(x)y_n(x) = q_2(x)y_{n+1}^2(x) + q_3(x)y_{n+1}(x), \quad n = 1, 2, 3, \ldots \tag{3.1}
\]

A solution of such a system is defined as an infinite set of functions \( \{y_1(x), y_2(x), \ldots, y_n(x), \ldots\} \) for which all the equations of the system hold identically. Without restricting the general case, we will consider two specifications: a linear and a nonlinear one.
3.1. Infinite linear hierarchy of first order ODEs

In this section we will consider the linear specification \( q_0 = q_1 = q_2 = 0, q_3 = -1 \) of (3.1)

\[
y_n' = -y_{n+1}, \quad n = 1, 2, 3, \ldots,
\]

(3.2)

which also can be equivalently written in vector form as

\[
y' = -A \cdot y,
\]

(3.3)

where \( A \) is the infinite but bounded bi-diagonal matrix

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\end{pmatrix},
\]

along with the infinite dimensional solution vector \( y^T = (y_1, y_2, y_3, \ldots, y_n, \ldots) \) of (3.3). Naively one would expect that the unique general solution to (3.3) is given by

\[
y = e^{-xA} \cdot c,
\]

(3.4)

where \( c \) is the infinite dimensional integration constant \( e^T = (c_1, c_2, c_3, \ldots, c_n, \ldots) \). When evolving the exponential function into its power series with its infinite radius of convergence, the general solution (3.4) can be equivalently written as

\[
\begin{align*}
y_1(x) &= c_1 - c_2 \cdot x + \frac{1}{2!} c_3 \cdot x^2 - \frac{1}{3!} c_4 \cdot x^3 + \cdots \\
y_2(x) &= c_2 - c_3 \cdot x + \frac{1}{2!} c_4 \cdot x^2 - \frac{1}{3!} c_5 \cdot x^3 + \cdots \\
y_3(x) &= c_3 - c_4 \cdot x + \frac{1}{2!} c_5 \cdot x^2 - \frac{1}{3!} c_6 \cdot x^3 + \cdots \\
\vdots & \quad \vdots & \quad \vdots
\end{align*}
\]

(3.5)

or compactly as

\[
y_n(x) = \sum_{k=0}^{\infty} c_{n+k} \frac{(-1)^k}{k!} x^k, \quad n = 1, 2, 3, \ldots,
\]

(3.6)

which, naively considered, might then serve as the unique general solution for (3.2). Of course, the precondition for it is that for any given initial condition the constant component values of the infinite dimensional vector \( c \) must be given such that the matrix product (3.4) is converging. That (3.4), or equivalently (3.6), represents a general solution to (3.3) is obvious, because to every (first order) differential equation of the hierarchy (3.2) one can associate a solution \( y_n \) involving a free integration parameter \( c_n \).

Now, let us see in how far the general solution (3.4) represents a unique general solution of (3.3). For that we first consider one of the equations’ scaling symmetries admitted by (3.3)

\[
L_1 : \quad \ddot{x} = e^{-x} x, \quad \dot{y} = D(\varepsilon) \cdot y,
\]

(3.7)

\footnote{The operator \( e^{-x}A \) is called the flow of the differential equation (3.3), as it takes the initial state \( y = c \) at \( x = 0 \) into the new state \( y = e^{-x}A \cdot c \) at position \( x \neq 0 \). If (3.3) represents an evolution equation with its forward marching time \( t \geq 0 \) as the independent variable, then the set of all operator elements \( e^{-tA} \) only forms a semi-group. The operator \( A \) is then said to be the infinitesimal generator of this semi-group.}
That means, regarding symmetry transformation identified as a reparametrization of the integration constant which in global form reads as (see derivation (B.1)), we obtain a fundamentally different general solution \( \tilde{\phi} \) into itself:

\[
\tilde{\phi} = D(\varepsilon) \cdot (e^{-\varepsilon \tilde{x}} A \cdot c) = e^{-\tilde{x} A} \cdot (D(\varepsilon) \cdot c) = e^{-\tilde{x} \tilde{A}} \cdot \tilde{c}.
\] (3.8)

That means, regarding symmetry transformation \( L_1 \) (3.7) the general solution (3.4) represents itself as unique general solution indeed. But this is no longer the case if we consider for example the following symmetry transformation

\[
L_2 : \quad X = \xi(x, y_1, y_2, \ldots) \partial_x + \sum_{n=1}^{\infty} \eta_n(x, y_1, y_2, \ldots) \partial_{y_n},
\] (3.9)

with \( \xi = x^2 \), and \( \eta_n = y_n + (n-1)(n-2) y_{n-1} - 2(n-1) x y_n, \)

which in global form reads as (see derivation (B.1)-(B.2))

\[
L_2 : \begin{align*}
\tilde{x} &= \frac{x}{1 - \varepsilon x}, & \tilde{y}_1 &= e^\varepsilon y_1, \\
\tilde{y}_n &= \sum_{k=1}^{n-1} B_{n,k} e^{n-k-1}(1 - \varepsilon x)^{n+k-1} e^\varepsilon y_{k+1}, \quad \text{for all } n \geq 2,
\end{align*}
\] (3.10)

where e.g. the first three explicit elements in this hierarchy are given as

\[
L_2 : \begin{align*}
\tilde{x} &= \frac{x}{1 - \varepsilon x}, & \tilde{y}_1 &= e^\varepsilon y_1, \\
\tilde{y}_2 &= (1 - \varepsilon x)^2 e^\varepsilon y_2, & \tilde{y}_3 &= 2(1 - \varepsilon x)^3 e^\varepsilon y_2 + (1 - \varepsilon x)^4 e^\varepsilon y_3, & \cdots
\end{align*}
\] (3.11)

Because when transforming the general solution (3.4) according to the above symmetry transformation (3.10), which in matrix-vector form reads as

\[
L_2 : \quad \tilde{x} = \frac{x}{1 - \varepsilon x}, \quad \tilde{y} = G(x, \varepsilon) \cdot y,
\] (3.12)

where \( G \) is the infinite group matrix

\[
G(x, \varepsilon) = e^\varepsilon
\]

we obtain a fundamentally different general solution \( \tilde{y} \), which, for all \( x \in \mathbb{R} \setminus \{0\} \), can not be identified as a reparametrization of the integration constant \( c \mapsto \tilde{c} \) of the primary solution (3.4) anymore:

\[
\tilde{y} = G \left( \frac{\tilde{x}}{1 + \varepsilon x} \right) \cdot (e^{-\frac{\tilde{x}}{1 + \varepsilon x}} A \cdot c) \neq e^{-\tilde{x} A} \cdot \tilde{c}.
\] (3.13)

\(^1\)To obtain the second relation in (3.8) one has to use the non-commutative property \( D \cdot A^n = e^{-n^2}(A^n \cdot D). \)
The fundamental difference between the solutions $\tilde{y}$ (3.13) and $y$ (3.4) already shows itself in the fact that the former one $\tilde{y} = \tilde{y}(\tilde{x}, \varepsilon)$ has a permanent non-removable singularity at $\tilde{x} = -1/\varepsilon$ independently of how $c$ is chosen, which thus implies that the transformed solution $\tilde{y}$ has one essential parameter more than the primary solution $y$, namely the group parameter $\varepsilon$, which can not be generally absorbed into the integration constant $c$.\footnote{Note that although the transformation for $x \to \tilde{x}$ in $T_2$ (2.7) is identical to the one in $L_2$ (3.11) by also showing a non-removable singularity at the inverse value of the group parameter, the full transformation $T_2$, however, does not induce this singularity into the transformed general solution (2.10), simply because the corresponding transformation $y \to \tilde{y}$ in $T_2$ (2.7) annihilates this singularity. Hence, in contrast to transformation $L_2$ (3.11), the group parameter in $T_2$ (2.7) is non-essential since it can be absorbed into the integration constant to give the reparametrized general solution (2.10).}

Yet, $\tilde{y}$ (3.13) is not to (3.4) the only functionally different general solution which can be constructed by a symmetry transformation. Infinitely many different general solutions can be obtained by just relaxing the specification $\xi = x^2$ in $L_2$ (3.9) and considering, for example, the more general symmetry transformation

$$L_2^f: \quad X = \xi(x, y_1, y_2, \ldots) \partial_x + \sum_{n=1}^{\infty} \eta_n(x, y_1, y_2, \ldots) \partial_{y_n}, \quad (3.14)$$

with $\xi = f(x)$, $\eta_n = y_n + \sum_{k=1}^{n-1} (-1)^{n-k} \frac{d^n f(x)}{dx^{n-k}} y_{k+1}$,

where $f$ is some arbitrary function. Hence, no unique and thus no privileged general solution can be found for the infinite hierarchy of differential equations (3.2). As a consequence, the infinite hierarchy (3.2) must be identified as an unclosed and thus indeterminate set of equations, irrespective of the fact that to every differential equation in the hierarchy (3.2) one can formally associate a solution function to it, which then, in a unique way, is coupled to the next higher order equation.

And, once accepted that the hierarchy (3.3) is unclosed, all invariant transformations which are admitted by this system, as e.g. $L_1$ (3.7), $L_2$ (3.9) and $L_2^f$ (3.14), must then be identified not as symmetry transformations, but only as weaker equivalence transformations which map between unclosed systems (see e.g. Ovsiannikov (1982); Meleshko (1996); Ibragimov (2004); Frewer et al. (2014)); in this case they even constitute indeterminate transformations. This identification is clearly supported when studying the most general invariant transformation which the system (3.3) can admit. It is given by two arbitrary functions $f$ and $g$, one for the independent infinitesimal $\xi = f(x, y_1)$ and one for the lowest order dependent infinitesimal $\eta_1 = g(x, y_1)$, which then both uniquely assign the functional structure for all remaining infinitesimals $\eta_n$ in the form $\eta_n = \eta_n(f(x, y_1), g(x, y_1), y_2, y_3, \ldots, y_n)$, for all $n \geq 2$.\footnote{Note that each dependent infinitesimal $\eta_n$ only shows a dependence up to order $n$ and not beyond, i.e. it only depends on all dependent variables $y_m$ which appear below a considered level $n$, i.e. where $m \leq n$.} This result shows complete arbitrariness in the choice for the transformation of $x$ and $y_1$ to invariantly transform system (3.3), which, after all, is actually a trivial result since the complete infinite hierarchy (3.2) can also be equivalently written in the form of an underdetermined solution as

$$y_{n+1} = (-1)^n \frac{d^n y_1}{dx^n}, \quad n = 1, 2, 3, \ldots, \quad (3.15)$$

where it’s more than obvious now that the considered system (3.3) is not closed, since, through relation (3.15), all higher-order functions $y_{n+1}$ are predetermined by the lowest order function $y_1$, but which itself can be chosen completely arbitrarily. However, note that $y_1$ is not privileged in the sense that only this function can be chosen arbitrarily. Any function $y_{n^*}$ in the hierarchy (3.2) can be chosen freely, where $n = n^*$ is some arbitrary but fixed order in this hierarchy. Its
underdetermined general solution can then be written as

\[
\begin{align*}
y_1 &= (-1)^{n^*-1} \int y_n^* \, d^{n^*-1}x \\
& \vdots \\
y_{n^*-k} &= (-1)^k \int y_n^* \, dx \\
& \vdots \\
y_{n^*-1} &= - \int y_n^* \, dx \\
y_{n^*+1} &= - \frac{dy_n^*}{dx} \\
y_{n^*+2} &= \frac{d^2 y_n^*}{dx^2} \\
& \vdots \\
y_{n^*+l} &= (-1)^l \frac{d^l y_n^*}{dx^l} \\
& \vdots
\end{align*}
\]  

(3.16)

A corresponding invariance analysis certainly sees the same effect, namely that one function, anywhere in the infinite hierarchy (3.2), can be chosen freely. Hence, since through (3.16) any arbitrary general solution \( y \) can be constructed, system (3.3) does not allow for a unique general solution. The primary general solution (3.4) is thus only one among an infinite set of other, different possible general solutions which this system can admit.

It should be noted here that our study only reveals the property of global non-uniqueness when constructing a general solution for an infinite system of differential equations which is unrestricted; for example as for the plain system (3.3) when no restrictions or any further conditions on the solution manifold are imposed. In particular, our statements do not invalidate the local uniqueness principle which may exist for a system of ODEs once its restricted to satisfy an initial condition.\(^\dagger\) Independent of whether this principle (Picard-Lindelöf theorem) uniquely applies to infinite dimensional ODE initial value systems or not, for the simple linear and homogeneous ODE structure (3.3), however, it is straightforward to show that local uniqueness in the solution for this particular infinite system must exist, when specifying an initial condition at \( x = x_0 \in \mathcal{I} \) inside some given local interval \( \mathcal{I} \subset \mathbb{R} \) (for the proof, see Appendix C). But, this local uniqueness interval \( \mathcal{I} \) can be quite narrow, and, depending on the chosen functions, can be even of point-size only. In how narrow this local interval \( \mathcal{I} \) can be successively made is studied at a simple example in Appendix C.

Besides this, when specifying a particular initial condition, say \( y(x_0) = y_0 \), or in component form \( y_n(x_0) = y_{(0)n} \) for all \( n \geq 1 \), then infinitely many and functionally independent invariant (equivalence) transformations can be constructed which all are compatible with this arbitrary but specifically chosen initial condition. Because, since e.g. the infinitesimals \( \xi = f(x, y_1) \) and

\(^\dagger\)It is not exactly clear yet in how far the well-defined local uniqueness principle for a system of ODE initial value problems (Picard-Lindelöf theorem) applies to systems which are infinite in dimension. Because, for example, since for \( A \) (3.3) not all matrix norms are finite, they cannot be regarded as equivalent anymore. That means, in order to guarantee the necessary Lipschitz continuity for the function \( A \cdot y \) on some interval, the infinite matrix \( A \) needs to satisfy the condition \( \|A\| \leq L \) for some finite Lipschitz constant \( L \), thus leading to a conclusion which now depends on the matrix norm used: For the maximum, row and column norm, which give \( \|A\|_{\text{max}} = \|A\|_{\infty} = \|A\|_1 = 1 \) respectively, the function \( A \cdot y \) is Lipschitz continuous, while for a norm which gives an infinite value, e.g. like the Euclidean norm \( \|A\|_2 \to \infty \), the function \( A \cdot y \) is not Lipschitz continuous.
\[ \eta = g(x, y) \] can be chosen arbitrarily, one only has to guarantee that the initial condition \( y_n(x_0) = y(0_0) \), for all \( n \geq 1 \), gets mapped invariantly into itself. This is achieved by demanding all infinitesimals to satisfy the restrictions

\[
\xi(x, y) \bigg|_{\{x=x_0; y=y_0\}} = 0, \quad \eta_1(x, y) \bigg|_{\{x=x_0; y=y_0\}} = 0, \quad (3.17)
\]

\[
\eta_n = \eta_n \left( \xi(x, y_1), \eta_1(x, y_1), y_2, y_3, \ldots, y_n \right) \bigg|_{\{x=x_0; y=y_0\}} = 0, \quad n \geq 2, \quad (3.18)
\]

where only the two infinitesimals \( \xi \) and \( \eta_1 \) can be chosen freely, while the remaining infinitesimals \( \eta_n \), for all \( n \geq 2 \), are predetermined differential functions of their indicated arguments. The conditions (3.17), in accordance with (3.18), can be easily fulfilled e.g. by restricting the arbitrary functions \( \xi \) and \( \eta_1 \) to

\[
\xi(x, y_1) = f_0(x, y_1) \cdot e^{-\frac{\gamma_2^2}{(x-x_0)^2}}, \quad \eta_1(x, y_1) = g_0(x, y_1) \cdot e^{-\frac{\gamma_2^2}{(y_1-y(0))}^2}, \quad (3.19)
\]

where \( f_0 \) and \( g_0 \) are again arbitrary functions, however, now restricted to the class of functions which are increasing slower than \( e^{1/r^2} \) at \( r = 0 \), where \( r = \sqrt{(x-x_0)^2 + (y_1-y(00))}^2/\gamma_2^2 \).

And, since in this case all differential functions \( \eta_n \), for \( n \geq 2 \), have the special non-shifted affine property \( \eta_n \big|_{\{\xi=0; \eta=0\}} = 0 \), the conditions (3.18) all are automatically satisfied by the above restriction (3.19). Hence, an *infinite* set of functionally independent (non-privileged) invariant solutions \( y = y(x) \) can be constructed from (3.19) which all satisfy the given initial condition \( y(x_0) = y_0 \).

**Remark on partial overlapping and analytic continuation:**

Before closing this section let’s briefly revisit the result (3.13). Important to mention here is that if we expand the function \( \frac{\tilde{x}}{1+\tilde{x}^2} \) into a power series around some arbitrary point \( \tilde{x} = a \neq -1/\varepsilon \), then the alternative general solution \( \tilde{y} \) (3.13) will map into a reparametrization of the primary solution (3.4), but only in a very restrictive manner due to the existence of three restrictions in order to ensure overall convergence (see derivation (D.1) and (D.5)): If, in \( \mathbb{R} \), the chosen values for \( \tilde{x}, \varepsilon \) and \( a \) satisfy the following three restrictions simultaneously\(^1\)

\[
\left| \frac{(\tilde{x} - a) \varepsilon}{1 + a \varepsilon} \right| < 1, \quad \left| \frac{a \varepsilon}{1 + a \varepsilon} \right| < 1, \quad \text{and} \quad |\tilde{x}| < 1, \quad (3.20)
\]

then, and only then, the symmetry transformation \( L_2 \) (3.12) allows for a reparametrization of the primary solution (3.4)

\[
\tilde{y} = \mathbf{G} \left( \frac{\tilde{x}}{1+\tilde{x}^2}, \varepsilon \right) \cdot \left( e^{-\frac{\tilde{x}^2\mathbf{A}}{1+\tilde{x}^2}} \cdot \mathbf{c} \right) = e^{-\tilde{x}^2\mathbf{A}} \cdot \tilde{\mathbf{c}}, \quad (3.21)
\]

where the reparametrized integration constant \( \tilde{\mathbf{c}} \) is then given by (D.3), or equivalently by (D.6), which both take the same form

\[
\tilde{\mathbf{c}} = \mathbf{G} (0, \varepsilon) \cdot \mathbf{c}. \quad (3.22)
\]

However, for the remaining wide range of values \( \tilde{x}, \varepsilon \) and \( a \) within \( \mathbb{R} \), namely for all those values in which at least one of the three restrictions (3.20) is violated, we still have, as was discussed before, a second, fundamentally different general solution \( \tilde{y} \) (3.13) than as given by the primary general solution \( y \) (3.4).

\(^1\)Note that since there are three restrictions for three values, \( \tilde{x}, \varepsilon \) and \( a \), they can not be chosen arbitrarily and independently anymore, i.e. all three values depend on each other according to (3.20). In general this combined set of restrictions leads to a very narrow radius of convergence.
Further note that this particular example even allows for an interesting special case when choosing $x_0 = a = 0$ as the position, and $y_0 = \tilde{c}$ (3.22) as the value for an initial condition $y(x_0) = y_0$ of system (3.3). Because, since on the one side the symmetry transformation (3.12) invariantly maps the initial condition’s space point from $x_0 = 0$ to $\tilde{x}_0 = 0$, and since on the other side there exist a common domain (3.20) in which the transformed general solution (3.21) matches the primary general solution (3.4), the former (transformed) solution serves as the functional continuation of the latter (primary) solution; but only if, of course, both solutions originate from the same initial condition, and if the primary solution (3.4) converges on a given (untransformed) integration constant $c$. To be explicit, we state that when considering the initial value problem

$$y'(x) = -A \cdot y(x), \quad \text{with } y(0) = y_0,$$

where $A$ is the infinite matrix given by (3.3) and $y_0$ some arbitrary constant, then two solutions $y^A$ and $y^B$ exist, namely the primary solution $y^A$ (3.4) and the transformed solution $y^B$ (3.21)

$$y^A(x) = e^{-xA} \cdot y_0,$$

$$y^B(x) = G\left(\frac{x}{1+\varepsilon x}, \varepsilon\right) \cdot \left[ e^{-\frac{x}{1+\varepsilon x}A} \cdot \left(G^{-1}(0, \varepsilon) \cdot y_0\right)\right],$$

which both satisfy the same initial condition $y^A(0) = y^B(0) = y_0$, but where, according to the local uniqueness principle for ODE initial value problems, each solution serves as the functional continuation of the other solution on a domain where it is not converging anymore. This domain depends on the explicit initial value $y_0$ — here $G$ is the infinite group matrix (3.12) with its inverse $G^{-1}(0, \varepsilon) = G(0, -\varepsilon)$ (see (E.3)). In particular, if we choose the initial value as given in (3.22), i.e. if $y_0 = \tilde{c}$, with $\tilde{c} = G(0, \varepsilon) \cdot c$, and where the primary (untransformed) integration constant is e.g. fixed as $c \sim 1$, then the transformed solution $y^B$ significantly extends the primary solution range of $y^A$. The reason is that, since according to (3.20) we are considering the special case $a = 0$, the primary solution $y^A$ (3.24) for this initial condition only converges in the restricted domain $|x| < 1$, while the transformed solution $y^B$ (3.25) converges in the complete range $x \in \mathbb{R} \{ -\frac{1}{\varepsilon} \}$. Symbolically we thus have the relation $y^A \subset y^B$. For more details and for a graphical illustration of these statements, see Appendix F. Surely, if we choose the initial value $y_0$ such that on a specific domain the solution $y^B$ is not converging, then $y^A$ serves as its continuation, i.e. we then have the opposite relation $y^B \subset y^A$. See Table 1 in Appendix F for a collection of several choices in the initial value $y_0$ and the corresponding domains for which the solutions $y^A$ (3.24) and $y^B$ (3.25) are converging.

### 3.2. Infinite nonlinear hierarchy of first order ODEs

In this section we will consider the nonlinear specification $q_0 = q_3 = 0, q_1 = 1/x, q_2 = 1/x^3$ of (3.1):

$$y'_n - \frac{y_n}{x} = \frac{y_{n+1}^2}{x^3}, \quad n = 1, 2, 3, \ldots$$

(3.26)

This coupled system represents a genuine system of nonlinear equations, since it cannot be reduced to a linear set of equations as in the case of the corresponding single Riccati-ODE (2.2) via the transformation $y(x) = x/z(x)$. In addition, its underdeterminate solution cannot be written compactly in closed form anymore as it was possible for the previously considered linear system (3.2), either through (3.15), or, more generally, through (3.16). But nevertheless, since (3.26) is an infinite forward recurrence relation of first order, i.e. where each term $y_{n+1}$ in the sequence depends on the previous term $y_n$ in the functional form $y_{n+1} = F[y_n] = \pm \sqrt{x^4 y_n^2 - x^2 y_n}$, it naturally acts again as an unclosed (underdetermined) system, where, on any level in the prescribed hierarchy (3.26), exactly one function can be chosen freely. This statement is again
Symmetry analysis for an infinite hierarchy of differential equations

... supported when performing an invariance analysis upon system (3.26), namely in the same way as it was already discussed in the previous section: To transform system (3.26) invariantly, complete arbitrariness exists in that two arbitrary functions are available in order to perform the transformation, one for the independent variable \( x \) and one for any arbitrary but fixed chosen dependent variable \( y_n^* \), i.e. where in effect the transformation of one function \( y_n^* = y_n^*(x) \) can be chosen absolutely freely. Hence, again as in the previous case, infinitely many functionally independent equivalence transformations can be constructed in the sense of (3.19), all being then compatible with any specifically chosen initial condition \( y(x_0) = y_0 \).

As explained in the previous section in detail, an unclosed set of differential equations, e.g. such as (3.2) or (3.26), does not allow for the construction of a unique general solution that can cover all possible special solutions these systems can admit. To explicitly demonstrate this again for the nonlinear system (3.26), we first construct its most obvious general solution based on the power series

\[
y_n(x) = x^2 \cdot \sum_{k=0}^{\infty} \lambda_{n,k} (x - a)^k, \quad n \geq 1,
\]

where \( a \in \mathbb{R} \) is some arbitrary expansion point. To be a solution of (3.26), the expansion coefficients have to satisfy the following first order recurrence relation (see Appendix G)

\[
k \cdot \lambda_{n,k} + a \cdot (k + 1) \cdot \lambda_{n,k+1} + \lambda_{n,k} = \sum_{l=0}^{k} \lambda_{n+1,k-l} \cdot \lambda_{n+1,l}, \quad \text{for all } n \geq 1, \quad k \geq 0.
\]

For arbitrary but fixed given initial values \( c_n := \lambda_{n,0} \), this recurrence relation can be explicitly solved for all higher orders relative to the expansion index \( k \geq 1 \). If \( a \neq 0 \), the first four expansion coefficients, for all \( n \geq 1 \), are then given as

\[
\begin{align*}
\lambda_{n,0} & = c_n, \\
\lambda_{n,1} & = - \frac{1}{a} \cdot (c_n - c_{n+1}^2), \\
\lambda_{n,2} & = \frac{1}{a^2} \cdot (c_n - 2c_{n+1} + c_{n+1} \cdot c_{n+2}^2), \\
\lambda_{n,3} & = - \frac{1}{3a^3} \cdot (3c_n - 9c_{n+1} + 9c_{n+1} \cdot c_{n+2}^2 - 2c_{n+1} \cdot c_{n+2} \cdot c_{n+3}^2 - c_{n+2}^4), \\
\vdots
\end{align*}
\]

while if \( a = 0 \), they are given as\(^1\)

\[
\begin{align*}
\lambda_{n,0} & = e^{2^{1-n} \cdot \sigma_1} = c_n, \\
\lambda_{n,1} & = e^{(2^{1-n} - 1) \cdot \sigma_1} \cdot \sigma_2, \\
\lambda_{n,2} & = 2^{-n} \cdot e^{(2^{1-n} - 2) \cdot \sigma_1} \cdot \left( \sigma_2^2 \cdot (2^n - 3^n) + 2 \cdot 3^{1+n} \cdot e^{\sigma_1} \cdot \sigma_3 \right), \\
\lambda_{n,3} & = 2^{-1-n} \cdot e^{(2^{1-n} - 3) \cdot \sigma_1} \cdot \left( \sigma_2^3 \cdot (2^{1+n} + 2^2 2^n - 2 \cdot 3^{1+n}) + (4 \cdot 3^n - 4^{1+n}) \cdot e^{\sigma_1} \cdot \sigma_2 \cdot \sigma_3 + 4^n \cdot e^{2\sigma_1} \cdot \sigma_4 \right), \\
\vdots
\end{align*}
\]

\(^1\)For \( a = 0 \) the recurrence relation (3.28) can be solved by making use for example of a generating function or the more general \( \mathcal{Z} \)-transform (see e.g. Zeidler et al. (2004)). Hereby should be noted that relation (3.28) is a 1-dimensional recurrence relation of first order for any arbitrary but fixed order \( n \).
where \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots) \) is a new infinite set of integration constants, which now, instead of \( c = (c_1, c_2, \ldots, c_n, \ldots) \), take the place for the freely selectable parameters in the general solution (3.27) as soon as the expansion point \( a \) turns to zero; simply because in this singular case all constants \( c_n \) within (3.28) cannot be chosen independently anymore as they would all depend on the single choice of the first parameter \( \sigma_1 \) as given above in the first line of (3.30).

That solution (3.27), along with either (3.29) or (3.30), represents a general solution is obvious, since to each solution \( y_n \) of the first order system (3.26) one can associate to it a free parameter for all \( n \geq 1 \), either \( c_n \) if \( a \neq 0 \), or \( \sigma_n \) if \( a = 0 \). But this general solution is not unique as it does not cover all possible special solutions which that system (3.26) can admit.

For example, if we consider the following independent\(^1\) special solutions of (3.26)

\[
y^{(1)}_n(x) = x^2, \quad \text{for all } n \geq 1, \tag{3.31}
\]

\[
y^{(2)}_n(x) = \begin{cases} 
-1, & \text{for } n = 1, \\
x, & \text{for } n = 2, \\
0, & \text{for all } n \geq 3,
\end{cases} \tag{3.32}
\]

\[
y^{(3)}_n(x) = \frac{1}{5^{n-n}} \left[ \prod_{k=0}^{n-1} \left( 1 + \frac{1}{2^{k-2}} \right)^{2^{k-n}} \right] \cdot x^{2^{n-n}}, \quad \text{for all } n \geq 1, \tag{3.33}
\]

only solution \( y^{(1)}_n \) is covered by the general solution (3.27) for all \( n \geq 1 \), in choosing either \( c_n = 1 \) (if \( a \neq 0 \)), or \( \sigma_n = 0 \) (if \( a = 0 \)). However, the special solution \( y^{(2)}_n \), and in general also \( y^{(3)}_n \), are not covered. The reason for \( y^{(2)}_n \) (3.32) is obvious, because it’s a polynomial with a smaller degree than \( y^{(3)}_n \) (3.27), which itself is always at least of second order for all \( n \geq 1 \). And to see the reason for \( y^{(3)}_n \) (3.33), it’s sufficient to explicitly evolve it up to third order in \( n \):

\[
y^{(3)}_1 = x^4, \quad y^{(3)}_2 = \sqrt{3} \cdot x^3, \quad y^{(3)}_3 = \sqrt{2} \cdot \sqrt{3} \cdot x^{5/2}, \quad \ldots \tag{3.34}
\]

Because, if \( a = 0 \), one has to choose \( \sigma_1 \rightarrow -\infty \) in order to obtain the lowest order particular solution \( y^{(3)}_1 \), which then turns into a contradiction when trying to determine \( \sigma_2 \) for the next higher order particular solution \( y^{(3)}_2 \). This indeterminacy will then propagate through all remaining orders \( n \geq 3 \), i.e. for \( a = 0 \) no consistent set of expansion coefficients \( \lambda_{n,k} \) (3.30) can be determined to generate the special solution (3.33) from the general solution (3.27). If, however, the expansion point of the general solution (3.27) is chosen to be \( a \neq 0 \), then one obtains a coverage for the special solution \( y^{(3)}_n \) (3.33), but only partially, namely only in the domain \( |x-a| < |a| \) (for the proof, see Appendix H). Hence, the general solution (3.27) is not unique, since other general solutions of (3.26) must exist in order to completely cover for example the special solutions \( y^{(2)}_n \) (3.32) and \( y^{(3)}_n \) (3.33) for all \( x \in \mathbb{R} \).

To close this investigation it is worthwhile to mention that the infinite nonlinear hierarchy of coupled equations (3.26) allows for two invariant Lie group actions which are uncoupled. For all \( n \geq 1 \), these are:

\[
\begin{align*}
T^\infty_1 &: \quad \tilde{x} = e^{\varepsilon_1}x, \quad \tilde{y}_n = e^{2\varepsilon_1}y_n, \\
T^\infty_2 &: \quad \tilde{x} = \frac{x}{1 - \varepsilon_2 x}, \quad \tilde{y}_n = \frac{y_n}{1 - \varepsilon_2 x},
\end{align*} \tag{3.35}
\]

being the equivalent invariances to the scaling symmetry \( T_1 \) and projective symmetry \( T_2 \) of the corresponding single Riccati-ODE (2.2) in (2.7) respectively. Their infinitesimal form has

\(^1\)The only dependence which exists between these three different solutions is that \( \lim_{n \to \infty} y^{(3)}_n = y^{(1)}_n = x^2 \) for all \( n^* \geq 1 \).
the structure
\[
\begin{align*}
T_1^\infty: & \quad X_1^\infty = x\partial_x + 2y_1\partial_{y_1} + 2y_2\partial_{y_2} + \cdots + 2y_n\partial_{y_n} + \cdots, \\
T_2^\infty: & \quad X_2^\infty = x^2\partial_x + xy_1\partial_{y_1} + xy_2\partial_{y_2} + \cdots + xy_n\partial_{y_n} + \cdots,
\end{align*}
\]
and, as proven in Appendix J, they are the two only possible non-coupled Lie-point invariances which the coupled hierarchy (3.26) of first order Riccati-ODEs can admit. As was demonstrated in the previous section, these invariant (equivalence) transformations (3.35) can now be used to either generate new additional special solutions or to generate functionally different general solutions by just transforming (3.27) respectively. Hereby note that the special solution \( y_n^{(1)} \) (3.31) is an invariant solution with respect to the scaling invariance \( T_1^\infty \) (3.35), which even can be prolonged to the more general invariant solution
\[
y_n^{(1)}(x; \tau) = e^{-2^{1-n}} \cdot x^2,
\]
which then involves a free parameter \( \tau \in \mathbb{R} \) for all \( n \geq 1 \).

4. Conclusion

At the example of first order ODEs this study has shown that an infinite and forward recursive hierarchy of differential equations carries all features of an unclosed system, and that, conclusively, all admitted invariance transformations must be identified as equivalence transformations only. To obtain from such systems an invariant solution which shows a certain particular functional structure is ultimately without value, since infinitely many functionally different and non-privileged invariant solutions can be constructed, even if sufficient initial conditions are additionally imposed. In order to obtain valuable results, the infinite system needs to be closed by posing modelling assumptions which have to reflect the structure of the underlying (higher abstraction level) equations from which the infinite system emerges.

It is clear that this insight is not restricted to ODEs, but that it holds for differential equations of any type as soon as the infinite hierarchy is of a forward recursive nature. For example, as it’s the case for the infinite Friedmann-Keller hierarchy of PDEs for the multi-point moments in statistical turbulence theory. As it is discussed in detail in Frewer et al. (2014) and further in Frewer (2015), this infinite system is undoubtedly unclosed and that it’s simply without any value, therefore, to determine particular invariant solutions if no prior modelling assumptions are invoked on that system.

A. Infinite backward versus infinite forward differential recurrence relations

A.1. Example for an infinite backward differential recurrence relation

(Closed system with unique solution manifold)

Let us consider the Cauchy problem (1.26) of Example 5 in the simplified form \( a = 1 \) and \( b = c = 0 \), along with an initial function \( \phi \) which is normalized to \( \int_{-\infty}^{\infty} \phi(x)dx = 1 \). Then this initial value problem (1.26) has the unique solution
\[
u(t, x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{4t}} \phi(x') dx', \quad \text{for } t \geq 0,
\]
which by construction, due to \( c = 0 \), automatically satisfies the normalization constraint (1.31).

To study this uniqueness issue on the corresponding moment induced ODE system (1.29), let us first consider the unrestricted system
\[
\frac{du_n}{dt} = n \cdot (n-1) \cdot u_{n-2}, \quad n \geq 0,
\]
It is necessary to realize that the following nonlinear point transformation (Polyanin, 2002)

\[
\frac{d^{n+1}u_{2n+2}(t)}{dt^{n+1}} = (2n + 2)! \cdot u_0(t), \quad n \geq 0,
\]

and

\[
\frac{d^m u_{2m+1}(t)}{dt^m} = (2m + 1)! \cdot u_1(t), \quad m \geq 1.
\]

This system can then be uniquely integrated to give the general solution

\[
u_0(t) = c_0,
\]

\[
u_{2n+2}(t) = (2n + 2)! \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_0} u_0(t') dt' dt_0 \cdots dt_{n-1} + \sum_{k=0}^{n} q_{n,k}^{(1)} t^k, \quad n \geq 0,
\]

\[
u_1(t) = c_1,
\]

\[
u_{2m+1}(t) = (2m + 1)! \int_0^{t_m} \int_0^{t_{m-1}} \cdots \int_0^{t_1} u_1(t') dt' dt_1 \cdots dt_{m-1} + \sum_{k=0}^{m-1} q_{m,k}^{(2)} t^k, \quad m \geq 1,
\]

with the expansion coefficients given as

\[
q_{n,k}^{(1)} = \frac{(2n + 2)!}{(2n + 2 - 2k)!} c_{2n+2-2k}, \quad n \geq 0; \quad 0 \leq k \leq n,
\]

\[
q_{m,k}^{(2)} = \frac{(2m + 1)!}{(2m + 1 - 2k)!} c_{2m+1-2k}, \quad m \geq 1; \quad 0 \leq k \leq m - 1,
\]

where all \(c_n\) for \(n \geq 0\) are arbitrary integration constants. Hence we see that the unrestricted system (A.2) provides a general solution (A.4) which only involves arbitrary constants, i.e. the unrestricted system (A.2) provides a unique general solution. Because, when restricting this system to the underlying PDE’s initial condition \(u(0, x) = \phi(x)\), with \(\int_{-\infty}^{\infty} \phi(x) dx = 1\), which for the ODE system (A.2) takes the form

\[
u_n(0) = \int_{-\infty}^{\infty} x^n \cdot \phi(x) dx, \quad n \geq 0, \quad \text{with} \quad u_0(0) = 1,
\]

it will turn the general solution (A.4) into a unique and fully determined solution, where the integration constants are then given by

\[
c_n = u_n(0), \quad n \geq 0, \quad \text{with} \quad c_0 = 1.
\]

**A.2. Example for an infinite forward differential recurrence relation**

(Unclosed system with non-unique solution manifold)

Now, let’s consider the case \(a = 1, b = 0\) and \(c = -1\), where again the initial condition function \(\phi\) is normalized to \(\int_{-\infty}^{\infty} \phi(x) dx = 1\). To solve this initial value problem (Cauchy problem)

\[
\partial_t u = \partial_x^2 u - x^2 u, \quad \text{for} \quad t \geq 0, \quad \text{with} \quad u(0, x) = \phi(x),
\]

it is necessary to realize that the following nonlinear point transformation (Polyanin, 2002)\(^{1}\)

\[
\tilde{t} = \frac{1}{4} \cdot (e^{4t} - 1), \quad \tilde{x} = x \cdot e^{2t}, \quad \tilde{u} = u \cdot e^{-\frac{1}{2}x^2 - t},
\]

\(^{1}\)This continuous point transformation is not a group transformation, as it neither includes a group parameter nor does it include the unique continuously connected identity transformation from which any infinitesimal mapping can emanate.
which has the unique inverse transformation
\[
t = \frac{1}{4} \ln(1 + 4t), \quad x = \frac{x'}{\sqrt{1 + 4t}}, \quad u = \tilde{u} \cdot \sqrt{1 + 4t} \cdot e^{\frac{1}{4} x'x'}, \tag{A.10}
\]
maps the original Cauchy problem (A.8) into the following Cauchy problem for the standard diffusion equation with constant coefficients:
\[
\partial_t \tilde{u} = \partial_x^2 \tilde{u}, \quad \text{for } \tilde{t} \geq 0, \quad \text{with } \tilde{u}(0, \tilde{x}) = \phi(\tilde{x}) \cdot e^{-\frac{1}{4} \tilde{x}'^2}. \tag{A.11}
\]
Important to note here is that the initial time \( t = 0 \) as well as the relevant time range \( t \in [0, \infty) \) both get invariantly mapped to \( \tilde{t} = 0 \) and \( \tilde{t} \in [0, \infty) \) respectively. Hence, the unique solution of the transformed Cauchy problem (A.11) is thus again given by (A.1), but now in the form
\[
\tilde{u}(\tilde{t}, \tilde{x}) = \frac{1}{4\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(\tilde{x} - x')^2}{4t}} \phi(\tilde{x}') e^{-\frac{1}{4} \tilde{x}'^2} d\tilde{x}', \quad \text{for } \tilde{t} \geq 0, \tag{A.12}
\]
which then, according to transformation (A.9), leads to the unique solution for the original Cauchy problem (A.8)
\[
u(t, x) = \frac{e^{\frac{1}{4} x'^2 + t}}{\sqrt{\pi (e^{4t} - 1)}} \int_{-\infty}^{\infty} e^{-\frac{(x - x')^2}{e^{4t} - 1}} \phi(x') e^{-\frac{1}{4} x'^2} dx', \quad \text{for } t \geq 0. \tag{A.13}
\]
Considering, however, the corresponding moment induced infinite ODE system (1.29) for (A.8), we will now show that this system is not uniquely specified and thus has to be identified as an unclosed system even if sufficient initial conditions are imposed. In clear contrast to its associated higher level PDE system (A.8), which, as a Cauchy problem, is well-posed by providing the unique solution (A.13). To see this, let us first again consider the unrestricted ODE system
\[
\frac{d u_n}{d t} = n \cdot (n - 1) \cdot u_{n-2} - u_{n+2}, \quad n \geq 0, \tag{A.14}
\]
which can be rewritten into the equivalent and already solved form\(^\dagger\)
\[
\begin{align*}
u_{2n+2}(t) &= (-1)^{n+1} \sum_{i=0}^{\infty} A^{(1)}_i(n) \frac{d^{n+1-2i}}{dt^{n+1-2i}} u_0(t), \quad n \geq 0, \tag{A.15}\\
u_{2m+1}(t) &= (-1)^m \sum_{j=1}^{\infty} A^{(2)}_j(m) \frac{d^{m+2-2j}}{dt^{m+2-2j}} u_1(t), \quad m \geq 1,
\end{align*}
\]
where the coefficients \( A^{(1)}_i(n) \) and \( A^{(2)}_j(m) \) are recursively defined as:
\[
\begin{align*}
\text{• Initial seed for } &A^{(1)}_i(n): \quad A^{(1)}_0(-1) = 1, \quad \text{and } A^{(1)}_i(n) = 1, \quad \text{for all } n \geq 0, \\
A^{(1)}_i(n) &= \sum_{k=0}^{n-2i-1} (2n - 2k) \cdot (2n - 1 - 2k) \cdot A^{(1)}_{i-1}(n - 2 - k), \quad i \geq 1, \quad n \geq 0, 
\end{align*}
\]
\[
\begin{align*}
\text{• Initial seed for } &A^{(2)}_j(m): \quad A^{(2)}_1(0) = 1, \quad \text{and } A^{(2)}_j(m) = 1, \quad \text{for all } m \geq 1, \\
A^{(2)}_j(m) &= \sum_{k=1}^{m-2j-3} (2m - 2k) \cdot (2m + 1 - 2k) \cdot A^{(2)}_{j-1}(m - 1 - k), \quad j \geq 2, \quad m \geq 1.
\end{align*}
\]
\(^\dagger\)In the following we agree on the definitions that \( \frac{d^n}{dt^n} = 1, \quad \frac{d^{n+1}}{dt^{n+1}} = 0, \) and \( \sum_{i=0}^{n} = 0. \)
In contrast to the general solution (A.4) of the previously considered unrestricted system (A.2), we see that the degree of underdeterminedness in the above determined general solution (A.15) is fundamentally different and higher than in (A.4). Instead of integration constants \( c_n \), we now have two integration functions \( u_0(t) \) and \( u_1(t) \) which can be chosen freely. Their (arbitrary) specification will then determine all other solutions for \( n \geq 0 \) and \( m \geq 1 \) according to (A.15). The reason for having two free functions and not infinitely many free constants is that system (A.14) defines a forward recurrence relation (of order two)\(^\dagger\) that needs not to be integrated in order to determine its general solution, while system (A.2), in contrast, defines a backward recurrence relation (of order two) which needs to be integrated to yield its general solution.

To explicitly demonstrate that (A.15) is not a unique general solution, we have to impose the corresponding initial condition \( u(0, x) = \phi(x) \), with \( \int_{-\infty}^{\infty} \phi(x) \, dx = 1 \), which led to the unique solution (A.13) of the underlying PDE system (A.8). For the current ODE system (A.14) this condition takes again the same form (A.6) as it did for previous ODE system (A.2):

\[
u_n(0) = \int_{-\infty}^{\infty} x^n \cdot \phi(x) \, dx, \quad n \geq 0, \quad \text{with} \quad u_0(0) = 1.
\] (A.17)

The easiest way to perform this implementation is to choose the two arbitrary functions \( u_0(t) \) and \( u_1(t) \) as analytical functions which can be expanded as power series

\[
u_0(t) = \sum_{k=0}^{\infty} \frac{c_k^{(1)}}{k!} t^k, \quad \nu_1(t) = \sum_{k=0}^{\infty} \frac{c_k^{(2)}}{k!} t^k,
\] (A.18)

where \( c_k^{(1)} \) and \( c_k^{(2)} \) are two different infinite sets of constant expansion coefficients. By inserting this Ansatz into the general solution (A.15) and imposing the initial conditions (A.17) will then uniquely specify these coefficients in a recursive manner as

\[
\begin{align*}
c_k^{(1)} &= 0, \quad k < 0; \quad c_k^{(1)} = (-1)^k \cdot u_{2k}(0) - \sum_{i=1}^{\infty} A_i^{(1)}(k-1) \cdot c_{k-2i}^{(1)}, \quad k \geq 0, \\
c_k^{(2)} &= 0, \quad k < 0; \quad c_k^{(2)} = (-1)^k \cdot u_{2k+1}(0) - \sum_{i=1}^{\infty} A_i^{(2)}(k) \cdot c_{k-2i}^{(2)}, \quad k \geq 0.
\end{align*}
\] (A.19)

Indeed, the two solutions (A.18) with the above determined coefficients (A.19) form the analytical part of the corresponding unique PDE moment solutions relative to (A.13). For example, choosing the non-symmetric and to one normalized initial condition function \( \phi(x) = \frac{1}{\sqrt{\pi}} e^{-(x-1)^2} \) will give the first two unique PDE moment solutions as

\[
\begin{align*}
u_0(t) &= \int_{-\infty}^{\infty} x^0 \cdot u(t, x) \, dx = 2 e^{-\frac{1}{4} \left(1 - \frac{4}{1 + 3e^{4t}}\right) + t} \sqrt{1 + 3e^{4t}}, \quad t \geq 0, \\
u_1(t) &= \int_{-\infty}^{\infty} x^1 \cdot u(t, x) \, dx = 8 e^{-\frac{1}{4} \left(1 - \frac{4}{1 + 3e^{4t}}\right) + 3t} \sqrt{(1 + 3e^{4t})^3}, \quad t \geq 0.
\end{align*}
\] (A.20)

and, if these were Taylor expanded around \( t = 0 \), they would exactly yield the first two power series solutions (A.18) of the associated infinite ODE system (A.14). But, the Taylor expansions of both functions (A.20) only converge in the limited range \( 0 \leq t < \frac{1}{4} \sqrt{\pi^2 + (\ln 3)^2} \sim 0.83 \).

\(^\dagger\)The order of the recurrence relation is defined relative to the differential operator.
That means, our initial assumption that the first two ODE solutions \( u_0(t) \) and \( u_1(t) \) are analytical functions on the global and unlimited scale \( t \in [0, \infty) \) is thus not correct. Only for a very limited range this functional choice (A.18) is valid. But, if we don’t know the full scale PDE solutions (A.20) beforehand, how then to choose these two unknown functions \( u_0(t) \) and \( u_1(t) \) for the infinite ODE system (A.15)? The clear answer is that there is no way without invoking a prior modelling assumption on the ODE system itself. Even if we would choose specific functions \( f_0(t) \) and \( f_1(t) \), which for \( u_0(t) \) and \( u_1(t) \) are valid on any larger scale than the limited analytical Ansatz (A.18), we still have the problem that this particular solution choice is not unique, because one can always add to this choice certain independent functions which give no contributions when evaluated at the initial point \( t = 0 \). For example, if \( u_0(t) = f_0(t) \) and \( u_1(t) = f_1(t) \), and if both functions \( f_0 \) and \( f_1 \) satisfy the given initial conditions at \( t = 0 \), then

\[
u_0(t) = f_0(t) + \psi_0(t) \cdot e^{-\frac{t^2}{\gamma}}, \quad \nu_1(t) = f_1(t) + \psi_1(t) \cdot e^{-\frac{t^2}{\gamma^2}}, \tag{A.21}\]

is also a possible solution choice which satisfies the same initial conditions, where \( \psi_0(t) \) and \( \psi_1(t) \) are again arbitrary functions, with the only restriction that, at the initial point \( t = 0 \), they have to increase slower than \( e^{\alpha t^2/\gamma^2} \) and \( e^{\gamma t^2/\gamma^2} \) respectively.

That no unique solution can be constructed a priori provides the reason that the PDE induced ODE system (A.15), although infinite in dimension, has to be treated as an unclosed system. It involves more unknown functions than there are determining equations, although formally, in a bijective manner, to each function within the hierarchy a corresponding equation can be mapped to. But, since the hierarchy (A.14) can be equivalently rewritten into the form (A.15), it explicitly reveals the fact that exactly two functions \( u_0(t) \) and \( u_1(t) \) in this hierarchy remain unknown, and without the precise knowledge of their global functional structure all remaining solutions \( u_n(t) \) for \( n \geq 2 \) then remain unknown too. And, since the equivalently rewritten form (A.15) already represents the general solution of the original infinite ODE system (A.14), the general solution itself is unclosed as well. In other words, the general solution (A.15) is not a unique general solution. The degree of arbitrariness in having two unknown functions cannot be reduced, even when imposing initial conditions, simply due to the existing modus operandi in the sense of (A.21) when constructing possible valid solutions.

Hence, posing any initial conditions are thus not sufficient to yield a unique solution for the (lower abstraction level) ODE system (A.15) as they are for the (higher abstraction level) PDE equation (A.8). Without a prior modelling assumption on the ODE system (A.15), this system remains unclosed. Fortunately, the solutions of this particular case (A.20) possessed an analytical part in their functions for which the assumed Ansatz (A.18) expressed the correct functional behavior, though only in a very narrow and limited range. But, of course, for more general cases such a partial analytical structure is not always necessarily provided, and an Ansatz as (A.18) would then be misleading.

B. Alternative method in constructing a global transformation

Besides Lie’s central theorem, a more efficient way to determine the global 1-parametric symmetry transformation of \( y_n \) from its infinitesimal form (3.9) for \( n \geq 2 \) is, in this particular case, to make use of the underlying recurrence relation (3.2) along with the transformation rule (2.9) in its more general form:

\[
\tilde{y}_n = -\tilde{y}'_{n-1} = -\frac{d\tilde{y}_{n-1}}{dx} = -\frac{d\tilde{y}_{n-1}(x, y_1, y_2, \ldots, y_{n-1})}{dx} = -(\frac{\partial x}{\partial x})^{-1}
\left[ \sum_{q=1}^{n-1} \frac{\partial \tilde{y}_{n-1}}{\partial y_q} y'_q + \frac{\partial \tilde{y}_{n-1}}{\partial x} \right].
\]
\( \bar{y}_n = \left( \frac{\partial x}{\partial x} \right)^{-1} \left( \sum_{q=1}^{n-1} \frac{\partial \bar{y}_{n-1}}{\partial y_q} y_{q+1} - \frac{\partial \bar{y}_{n-1}}{\partial x} \right) = (1 - \varepsilon x)^2 \left( \sum_{q=1}^{n-1} \frac{\partial \bar{y}_{n-1}}{\partial y_q} y_{q+1} - \frac{\partial \bar{y}_{n-1}}{\partial x} \right) \)

\[ = \sum_{k=1}^{n-1} B_{n,k} \varepsilon^{n-k-1}(1 - \varepsilon x)^{n+k-1} e^x y_{k+1}, \quad \text{for all} \quad n \geq 2, \quad (B.1) \]

where the coefficients \( B_{n,k} \) are defined via the following 2-dimensional recurrence relation:\footnote{In order to obtain from a (1+1)-dimensional recurrence relation a unique solution it has to be supplemented by one initial condition and two zero-dimensional boundary conditions; in full analogy to the situation for PDEs.}

\[ B_{n,k} = (n - 2 + k) \cdot B_{n-1,k} + B_{n-1,k-1}, \quad \text{for} \quad n \geq 3 \text{ and } k = 1, 2, \ldots, n - 1, \quad (B.2) \]

with the initial condition \( B_{2,1} = 1 \), and the boundary conditions \( B_{n,0} = 0 \) (left boundary) and \( B_{n,n} = 0 \) (right boundary) for all \( n \geq 2 \).

C. Local uniqueness proof and an example on its range

**Proposition:** Given is the following initial value problem for the infinite ODE system (3.3)

\[ y' = -A \cdot y, \quad \text{with} \quad y(x_0) = y_0, \quad (C.1) \]

in some local interval \( \mathcal{I} \subset \mathbb{R} \), where \( x_0 \in \mathcal{I} \). Then this (restricted) differential system (C.1) only has the one solution

\[ y(x) = e^{-(x-x_0)A} \cdot y_0, \quad \text{for all} \quad x \in \mathcal{I}. \quad (C.2) \]

In particular, if \( y_0 = 0 \) then \( y(x) = 0 \) is the only solution for all \( x \in \mathcal{I} \).

**Proof:** Let \( y = y(x) \) be any solution which satisfies the initial value problem (C.1) in the given interval \( \mathcal{I} \). Then we can formulate the obvious relation

\[ \frac{d}{dx} \left( e^{(x-x_0)A} \cdot y \right) = \left( A \cdot e^{(x-x_0)A} \right) \cdot y + e^{(x-x_0)A} \cdot y' = \left( A \cdot e^{(x-x_0)A} \right) \cdot y - e^{(x-x_0)A} \cdot (A \cdot y) = 0, \]

since the infinite matrix \( A \) commutes with itself, i.e. \( [A, A^n] = 0 \) for all \( n \in \mathbb{N} \). This relation then implies that

\[ e^{(x-x_0)A} \cdot y(x) = c, \quad \text{for all} \quad x \in \mathcal{I}, \quad (C.3) \]

where \( c \) is some integration constant. But since \( y \) satisfies the initial condition \( y(x_0) = y_0 \), we obtain the result that \( c = y_0 \) and that thus the considered solution \( y \) can only have the unique form:

\[ y(x) = e^{-(x-x_0)A} \cdot y_0, \quad \text{for all} \quad x \in \mathcal{I}. \quad \square \quad (C.4) \]
Figure 1: Plots of the first four solutions of the initial value problem (C.5). The solid lines display the solutions $y_1^I$ (C.6) and the dashed lines the solutions $y_1^{II}$ (C.7) for $\gamma = 1$. For each order $n$, the highlighted region on the $x$-axis indicates the local uniqueness interval $I_n$ where $y_1^I = y_1^{II}$. The size of each interval $|I_n|$ decreases as the order $n$ of the solution increases. In the limit $n \to \infty$ the corresponding interval narrows down to point-size, i.e. $\lim_{n \to \infty} |I_n| \to 0$. Hence, the size of the common uniqueness interval $I$ of the initial value problem (C.5), which is the intersection of all intervals $I = \bigcap_{n=1}^\infty I_n$, thus converges to point-size too.

Figure 2: Plots of the first order solutions $y_1^I$ (C.6), solid lines) and $y_1^{II}$ (C.7), dashed lines) for decreasing $\gamma$. Hence, for $\gamma \to 0$ the size of the local uniqueness interval $I_n^*$ for each arbitrary but fixed order $n = n^*$ diminishes to point-size.
Example: In how narrow this local uniqueness interval \( I \) can be made up to point-size, we want to demonstrate at the following specific initial value problem (C.1)

\[
y' = -A \cdot y, \quad \text{with} \quad y(0) = 1,
\]

(C.5)

in some local interval \( I \subset \mathbb{R} \) for \( x \) around the initial point \( x_0 = 0 \). Of course, globally, i.e. for all \( x \in \mathbb{R} \), the solution of the initial value problem (C.5) is not necessarily unique. Indeed, at least two global solutions \( y = (y_n)_{n \in \mathbb{N}} \) can be found, e.g.

\[
y_n = e^{-x}, \quad \text{for all} \quad n \geq 1,
\]

(C.6)

\[
y_n^I = \begin{cases} 
  e^{-x} + e^{-\frac{x}{n}}, & \text{for } n = 1, \gamma > 0, \\
  (-1)^{n-1} \frac{d^{n-1} y^II}{dx^{n-1}}, & \text{for } n \geq 2,
\end{cases}
\]

(C.7)

which both satisfy (C.5) for all \( n \in \mathbb{N} \), and all \( x \in \mathbb{R} \). Figure 1 and 2 shows this for constant and different \( \gamma \) respectively.

D. Remapping of the general solutions’ integration constant

D.1. Reparametrization of solution \( n = 1 \)

\[
\tilde{y}_1 = e^x y_1
\]

\[
= e^x \sum_{k=0}^{\infty} c_{1+k} \frac{(-1)^k}{k!} x^k = e^x \sum_{k=0}^{\infty} c_{1+k} \frac{(-1)^k}{k!} \left( \frac{\tilde{x}}{1 + \varepsilon \tilde{x}} \right)^k
\]

\[
= e^x \sum_{k=0}^{\infty} c_{1+k} \frac{(-1)^k}{k!} \left( \sum_{l=0}^{\infty} (-1)^l \frac{k(k+1)! \varepsilon^l}{l! (1 + \varepsilon)^{k+l+1}} \right), \quad \text{for } \left| \frac{\tilde{x} - a \varepsilon}{1 + a \varepsilon} \right| < 1, a \neq -\frac{1}{\varepsilon},
\]

\[
= e^x \sum_{k=0}^{\infty} c_{1+k} \frac{(-1)^k}{k!} \left( \sum_{l=0}^{\infty} \sum_{q=0}^{l} \frac{(-1)^{l+q}}{1 + \varepsilon} \frac{1}{k! (k+l)!} \frac{\varepsilon^l}{l!} \frac{k(k+1)! \varepsilon^l}{l!} \frac{\tilde{x}^{k+l}}{(1+\varepsilon)^{k+l}} \right), \quad \text{for } \left| \frac{\tilde{x} - a \varepsilon}{1 + a \varepsilon} \right| < 1, a \neq -\frac{1}{\varepsilon},
\]

\[
= e^x \sum_{k=0}^{\infty} c_{1+k} \frac{(-1)^k}{k!} \left( \sum_{m=0}^{\infty} \sum_{r=0}^{m} \frac{(-1)^m a^r}{(1 + a \varepsilon)^{k+m+r}} \frac{m+r}{r!} \frac{k(k+m+r)!}{k! (k+m+r)!} \frac{\varepsilon^{m+r}}{(1+\varepsilon)^{k+m}} \frac{\tilde{x}^{k+m}}{(1+\varepsilon)^{k+m}} \right)
\]

\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{m} e^x c_{1+k} \frac{(-1)^k}{k!} \frac{a^r}{(1 + a \varepsilon)^{k+m+r}} \frac{m+r}{r!} \frac{k(k+m+r)!}{k! (k+m+r)!} \frac{\varepsilon^{m+r}}{(1+\varepsilon)^{k+m}} \frac{\tilde{x}^{k+m}}{(1+\varepsilon)^{k+m}}
\]

\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^i a^r \varepsilon^{i-j+r}}{(1 + a \varepsilon)^{i+j+r}} \frac{j(i+r)!}{j! (i+r)!} \frac{\tilde{x}^{i+j+r}}{(1+\varepsilon)^{i+j+r}}
\]

\[
= \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} \sum_{r=0}^{j} \frac{(-1)^i a^r \varepsilon^{i-j+r}}{(1 + a \varepsilon)^{i+j+r}} \frac{j(i+r)!}{j! (i+r)!} \frac{\tilde{x}^{i+j+r}}{(1+\varepsilon)^{i+j+r}} \right)
\]

\[
= \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} \sum_{r=0}^{j} \frac{(-1)^i a^r \varepsilon^{i-j+r}}{(1 + a \varepsilon)^{i+j+r}} \frac{j(i+r)!}{j! (i+r)!} \frac{\tilde{x}^{i+j+r}}{(1+\varepsilon)^{i+j+r}} \right)
\]

\[
= \sum_{i=0}^{\infty} \frac{(-1)^i \tilde{x}^i}{i!}, \quad \text{for } \left| \tilde{x} \right| < 1,
\]

(D.1)
where we made use of the Cauchy product rule in both directions:

\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} f_k \cdot g_l \cdot h_{k+l} = \sum_{i=0}^{\infty} \sum_{j=0}^{i} f_j \cdot g_{i-j} \cdot h_i. \tag{D.2}
\]

Note that the reparametrization of the integration constant

\[
c_{1+i} \mapsto \tilde{c}_{1+i} = \begin{cases} e^\varepsilon c_1, & \text{for } i = 0, \\ \sum_{j=0}^{i} e^\varepsilon c_{1+j} \frac{i! \varepsilon^{i-j}}{j! (i-j)! j!}, & \text{for } i \geq 1, \end{cases} \tag{D.3}
\]

is independent of the expansion point \(a\) for all \(i \geq 0\). In particular, relation (D.3) represents the reparametrization for \(a = 0\), which explains the third and last constraint \(|\tilde{x}\varepsilon| < 1\) in (D.1).

Note that in vector form relation (D.3) can be condensed to

\[
c \mapsto \tilde{c} = G(0, \varepsilon) \cdot c, \tag{D.4}
\]

where the infinite matrix \(G\) is defined by (3.12).

### D.2. Reparametrization of all remaining solutions \(n \geq 2\)

\[
\tilde{y}_n = \sum_{k=1}^{n-1} B_{n,k} \varepsilon^{n-k-1}(1 - \varepsilon x)^{n+k-1} \varepsilon^x y_{k+1}, \quad \text{for all } n \geq 2,
\]

\[
= \sum_{k=1}^{n-1} B_{n,k} \varepsilon^{n-k-1}(1 - \varepsilon x)^{n+k-1} \varepsilon^x \sum_{l=0}^{\infty} c_{k+1+l} \frac{(-1)^l}{l!} x^l
\]

\[
= \sum_{k=1}^{n-1} B_{n,k} \varepsilon^{n-k-1} \sum_{l=0}^{\infty} \varepsilon^x c_{k+1+l} \frac{(-1)^l}{l!} \left( \frac{\tilde{x}^l}{1 + \varepsilon \tilde{x}} \right)^{(n+k)}
\]

\[
= \sum_{k=1}^{n-1} \sum_{l=0}^{\infty} B_{n,k} \varepsilon^{n-k-1} \frac{(-1)^l}{l!} \left( \frac{\tilde{x}^l}{(1 + \varepsilon \tilde{x})^{n+k+l-1}} \right)
\]

\[
\cdot \frac{\sum_{m=0}^{\infty} (-1)^m (n + k + l + m - 2)! \varepsilon^m \tilde{x}^l (\tilde{x} - a)^m}{(n + k + l - 2)! m! (1 + a \varepsilon)^{n+k+l+m-1}}
\]

for \(|(\tilde{x} - a) \varepsilon| < 1, \ a \neq -\frac{1}{\varepsilon}\),

\[
= \sum_{k=1}^{n-1} \sum_{l=0}^{\infty} B_{n,k} \varepsilon^{n-k-1} \frac{(-1)^l}{l!} \left( \frac{\sum_{m=0}^{m} (-1)^{m+q} q^q}{(1 + a \varepsilon)^{n+k+l+m-1}} \frac{(m) (n + k + l + m - 2)! \varepsilon^m \tilde{x}^l}{(n + k + l - 2)! m! m-q} \right)
\]
\[
\tilde{y}_n = \sum_{k=1}^{n-1} \sum_{l=0}^{\infty} B_{n,k} \varepsilon^{n-k-1} e^x c_{k+1+l} \left( -1 \right)^l / l!
\]
\[
\cdot \left( \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^p a^r}{(1 + a \varepsilon)^{n + k + l + p + r - 1}} \frac{(n + k + l + p + r - 2)!}{(n + k + l - 2)!} \frac{\varepsilon^{p+r}}{(p+r)!} x^{l+p} \right)
\]
\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} \sum_{k=1}^{i} B_{n,k} \varepsilon^{n-k-1} e^x c_{k+1+j} \left( -1 \right)^j / j!
\]
\[
\cdot \left( \sum_{r=0}^{\infty} \frac{(-1)^{i-j} a^r}{(1 + a \varepsilon)^{n + k + i + r - 1}} \frac{(n + k + i + r - 2)!}{(n + k + i - 2)!} \frac{\varepsilon^{i-j+r}}{(i-j)! (i-j)!} x^i \right)
\]
\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} \sum_{k=1}^{i} B_{n,k} \varepsilon^{n+i-j-k-1} e^x c_{k+1+j} \left( -1 \right)^i / i!
\]
\[
\cdot \left( \sum_{r=0}^{\infty} \frac{(a \varepsilon)^r}{(1 + a \varepsilon)^{n + k + i + r - 1}} \frac{(n + k + i + r - 2)!}{(n + k + i - 2)!} \frac{\varepsilon^{i-j}}{(i-j)! (i-j)!} x^i \right), \text{ and if } \frac{a \varepsilon}{1 + a \varepsilon} < 1, \text{ then:}
\]
\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} \sum_{k=1}^{i} B_{n,k} \varepsilon^{n+i-j-k-1} e^x c_{k+1+j} \left( -1 \right)^i / i!
\]
\[
\cdot \left( \sum_{r=0}^{\infty} \frac{(a \varepsilon)^r}{(1 + a \varepsilon)^{n + k + i + r - 1}} \frac{(n + k + i + r - 2)!}{(n + k + i - 2)!} \frac{\varepsilon^{i-j}}{(i-j)! (i-j)!} x^i \right)
\]
\[
= \sum_{i=0}^{\infty} \frac{(-1)^i c_{n+i}}{i!} \tilde{x}^i, \text{ for } | \tilde{x} \varepsilon | < 1, \text{ and } n \geq 2. \quad (D.5)
\]

Note that the reparametrization of the integration constant

\[
c_{n+i} \mapsto \tilde{c}_{n+i} = \sum_{k=1}^{n-i} \sum_{j=0}^{i} B_{n,k} \varepsilon^{n+i-j-k-1} e^x c_{k+1+j} \frac{i!}{j! (i-j)!} \frac{(n + k + i - 2)!}{(n + k + j - 2)!}, \quad n \geq 2, \quad (D.6)
\]

is again independent of the expansion point \( a \) for all \( i \geq 0 \) and \( n \geq 2 \), and that it basically represents the exact reparametrization for \( a = 0 \), which thus again explains the third and last constraint \( | \tilde{x} \varepsilon | < 1 \) in (D.5). Obviously, when evaluated, (D.6) must give the same result for the transformed integration constant \( \tilde{c}^T = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \ldots, \tilde{c}_n, \ldots) \) as (D.3), which is (D.4).

E. Inverse infinite group matrix

This section demonstrates how the inverse of the infinite group matrix \( G (3.12) \) is constructed. Since, by construction, the group transformation \( L_2 (3.12) \) is based on an additive composition law of the group parameter \( \varepsilon \), the inverse transformation of \( L_2 (3.12) \) is thus given as

\[
L_2^{-1} : \quad x = \frac{\tilde{x}}{1 + \varepsilon \tilde{x}}, \quad y = G(\tilde{x}, -\varepsilon) \cdot \tilde{y}. \quad (E.1)
\]

And since the transformation \( y \mapsto \tilde{y} \) in \( L_2 (3.12) \) can be formally written as

\[
y = G^{-1}(x, \varepsilon) \cdot \tilde{y}, \quad (E.2)
\]

the infinite inverse matrix \( G^{-1} \) is thus defined as

\[
G^{-1}(x, \varepsilon) = G\left(\frac{x}{1 - \varepsilon x}, -\varepsilon\right). \quad (E.3)
\]
F. Explicit forms and graphs of the solutions $y^A$ and $y^B$

The explicit componential form of the primary solution $y^A$ (3.24) is given by (3.6)

$$y^A_n(x) = \sum_{k=0}^{\infty} y(0)_{n+k} \frac{(-1)^k}{k!} x^k, \quad n \geq 1,$$  \hspace{1cm} (F.1)

while to bring the transformed solution $y^B$ (3.25) into its corresponding componential form, one first has to recognize that its matrix-vector structure is iteratively composed as

$$y^B(x) = G(x^*, \varepsilon) \cdot y^*(x^*), \quad \text{with} \quad x^* = \frac{x}{1 + \varepsilon x},$$  \hspace{1cm} (F.2)

where

$$y^*(x^*) = e^{-x^* A} \cdot y^*_0,$$  \hspace{1cm} (F.3)

and

$$y^*_0 = G(0, -\varepsilon) \cdot y_0.$$  \hspace{1cm} (F.4)

Then, according to (3.10), the componential form of (F.2) is given as

$$y^B_n(x) = \begin{cases} e^r y^*_1(x^*), & \text{for } n = 1, \\ \sum_{k=1}^{n-1} B_{n,k} \varepsilon^{n-k-1} (1 - \varepsilon x^*)^{n+k-1} e^r y^*_{k+1}(x^*), & \text{for } n \geq 2, \end{cases}$$  \hspace{1cm} (F.5)

where (F.3), according to (3.6), has the form

$$y^*_q(x^*) = \sum_{l=0}^{\infty} y^*_q(l) \frac{(-1)^l}{l!} (x^*)^l, \quad q \geq 1,$$  \hspace{1cm} (F.6)

and (F.4), again according to (3.10), but now for $y = y_0$, goes over into

$$y^*_r(0) = \begin{cases} e^{-\varepsilon y(0)_1}, & \text{for } r = 1, \\ \sum_{m=1}^{r-1} B_{r,m} (-\varepsilon)^{r-m-1} e^{-\varepsilon y(0)_{m+1}}, & \text{for } r \geq 2, \end{cases}$$  \hspace{1cm} (F.7)

which then needs to be inserted back into (F.6), and this result again back into (F.5) to finally give the componential form of $y^B$ (3.25). For a fixed set of initial conditions, Figure 3 displays the solutions $y^A$ and $y^B$. The convergence domain for each solution for different initial conditions is given in Table 1.
Figure 3: Plots of the first four solutions of the initial value problem (3.23). The solid lines display the solutions $y_n^A$ (3.24), while the solutions $y_n^B$ (3.25) are given by the solid lines along with the extensions displayed by the dashed lines. The initial condition was set $y_0 = G(0, \varepsilon) \cdot c$, with $c = e^{-\varepsilon} \cdot 1$, and $\varepsilon = 1$.

Table 1: Convergence domains for the solutions $y_A$ (3.24) and $y_B$ (3.25) for a collection of various different initial values $y_0$, where all $\alpha$’s are arbitrary global constants. The domains were determined by using the Cauchy-Hadamard root test.

| $y_0 = (y_{(0)i})_{i \in \mathbb{N}}$ | $y^A$ | $y^B$ |
|-----------------------------------|-------|-------|
| $y(0)i = \sum_{j=0}^{\infty} G_{ij}(0, \varepsilon) c_j$, with $c_j = \begin{cases} \alpha_1 j^n, & \text{for any finite } n \in \mathbb{R} \\ \alpha_2, & \text{for all } j \in \mathbb{N} \\ \alpha_3 \frac{i!}{j!} \end{cases}$ | $|x\varepsilon| < 1 \quad x \in \mathbb{R}\backslash\{-\frac{1}{\varepsilon}\}$ | $|x| < 1 \quad \left| \frac{x(1-x)}{1+\varepsilon x} \right| < 1$ | $x = 0 \quad x = 0$ |
G. Derivation of a general solution for the nonlinear system

Given is the infinitely of first order coupled system of Riccati-ODEs (3.26)

\[ y'_n - \frac{y_n}{x} = \frac{y_{n+1}^2}{x^3}, \quad n \geq 1, \quad (G.1) \]

which, if a power series solution around some arbitrary expansion point \( x = a \in \mathbb{R} \) is sought, first should be transformed into an adequate form. This is achieved by transforming the function values as \( y_n = x^2 \cdot z_n \) to give the equivalent differential system to (G.1):

\[ (x-a) \cdot z'_n + a \cdot z'_n + z_n = z_{n+1}^2. \quad (G.2) \]

Inserting then the general Ansatz solution

\[ z_n(x) = \sum_{k=0}^{\infty} \lambda_{n,k} (x-a)^k, \quad n \geq 1, \quad (G.3) \]

will turn this system of equations (G.2) into

\[
0 = (x-a) \sum_{k=0}^{\infty} k \cdot \lambda_{n,k} (x-a)^{k-1} + a \sum_{k=0}^{\infty} k \cdot \lambda_{n,k} (x-a)^{k-1} \\
+ \sum_{k=0}^{\infty} \lambda_{n,k} (x-a)^k - \left( \sum_{k=0}^{\infty} \lambda_{n+1,k} (x-a)^k \right) \cdot \left( \sum_{k=0}^{\infty} \lambda_{n+1,k} (x-a)^k \right) \\
= \sum_{k=0}^{\infty} k \cdot \lambda_{n,k} (x-a)^k + a \sum_{k=1}^{\infty} k \cdot \lambda_{n,k} (x-a)^{k-1} \\
+ \sum_{k=0}^{\infty} \lambda_{n,k} (x-a)^k - \sum_{k=0}^{\infty} \sum_{l=0}^{k} \lambda_{n+1,l} \lambda_{n+1,k-l} (x-a)^k \\
= \sum_{k=0}^{\infty} (x-a)^k \left[ k \cdot \lambda_{n,k} + a \cdot (k+1) \cdot \lambda_{n,k+1} + \lambda_{n,k} - \sum_{l=0}^{k} \lambda_{n+1,l} \cdot \lambda_{n+1,k-l} \right], \quad (G.4) \\
\]

which, termwise equated, gives the following recurrence relation for the expansion coefficients

\[ k \cdot \lambda_{n,k} + a \cdot (k+1) \cdot \lambda_{n,k+1} + \lambda_{n,k} = \sum_{l=0}^{k} \lambda_{n+1,k-l} \cdot \lambda_{n+1,l}, \quad \text{for all } n \geq 1, \quad k \geq 0. \quad (G.5) \]

For every arbitrary but fixed order \( n \), the above relation represents a 1-dimensional recurrence relation of first order relative to index \( k \), which can be uniquely solved by imposing for all \( n \geq 1 \) at \( k = 0 \) an initial condition \( \lambda_{n,0} = c_n \), where \( c_n \) is some arbitrary constant. Note that for the singular case \( a = 0 \) the solution for the expansion coefficients \( \lambda_{n,k} \) will be different to those for all \( a \neq 0 \).

H. Proof that the general solution can be partially matched to a special solution

The proposition is that for \( a \neq 0 \) the general solution \( y_n \) (3.27) can only be matched to the special solution \( y_n^{(3)} \) (3.33) in the domain \(|x-a| < |a|\). This can be straightforwardly seen when performing the following two steps: Firstly, equating these two solutions relative to \( x^2 \)

\[ \frac{y_n^{(3)}(x)}{x^2} = \frac{y_n(x)}{x^2}, \quad \text{for all } n \geq 1, \quad (H.1) \]
will give the matching relation
\[
\frac{1}{5^{2-n}} \left[ \prod_{k=0}^{n-1} \left( 1 + \frac{1}{2^k-2} \right)^{2^n-1} \right] \cdot x^{2^n-2} = \sum_{k=0}^{\infty} \lambda_{n,k} \cdot (x-a)^k, \text{ for all } n \geq 1, \tag{H.2}
\]
which, then, secondly, will undergo the transformation \( x \mapsto \hat{x} = x - a \) to finally give the equivalent matching relation
\[
\frac{1}{5^{2-n}} \left[ \prod_{k=0}^{n-1} \left( 1 + \frac{1}{2^k-2} \right)^{2^n-1} \right] \cdot (\hat{x} + a)^{\frac{n-1}{2^n}} = \sum_{k=0}^{\infty} \lambda_{n,k} \cdot \hat{x}^k, \text{ for all } n \geq 1, \tag{H.3}
\]
which, in contrast to (H.2), is easier to match. In order to explicitly determine the coefficients \( \lambda_{n,k} \) such that equality (H.3) is satisfied for all orders \( n \), it is necessary to expand the power term on the left-hand side
\[
(\hat{x} + a)^\beta = \sum_{k=0}^{\infty} \frac{a^{\beta-k}}{k!} \cdot \frac{\Gamma(\beta + 1)}{\Gamma(\beta - k + 1)} \cdot \hat{x}^k, \text{ with } \beta = \frac{1}{2^n-2} \geq 0, \text{ for all } n \geq 1. \tag{H.4}
\]
Now, since this (transformed) expansion (H.4) only converges for \(|\hat{x}| < |a|\), the original (non-transformed) matching relation (H.1) will therefore only be valid for \(|x-a| < |a|\), with the corresponding matched coefficients
\[
\lambda_{n,k} = \frac{1}{5^{2-n}} \left[ \prod_{i=0}^{n-1} \left( 1 + \frac{1}{2^i-2} \right)^{2^{i-n}} \right] \cdot \frac{a^{\frac{1}{2^n-2} - k}}{k!} \cdot \frac{\Gamma(\frac{1}{2^n-2} + 1)}{\Gamma(\frac{1}{2^n-2} - k + 1)}, \text{ for all } n \geq 1, \text{ } k \geq 0. \tag{H.5}
\]
Note that since the general solution \( y_n \) (3.27) was matched to a genuine solution of (3.26), namely to the special solution \( y_n^{(3)} \) (3.33), and not to some arbitrary function, the matched coefficients (H.5) will thus automatically satisfy the corresponding solved relations (3.29) for \( a \neq 0 \), i.e. at least one set of constants \( c_n \) for all \( n \geq 1 \) can be found which then, according to (3.29), uniquely represent the expansion coefficients \( \lambda_{n,k} \) (H.5).

J. The existence of only two uncoupled Lie point group invariances

Performing a systematic Lie point group invariance analysis on the infinite system of first order Riccati-ODEs (3.26), and looking out only for uncoupled solutions in the overdetermined system for the generating infinitesimals, which themselves can then only take the consistent form
\[
\xi(x, y_1, y_2, \ldots) = \phi(x), \text{ and } \eta_n(x, y_1, y_2, \ldots) = \psi_n(x, y_n), \text{ for all } n \geq 1, \tag{J.1}
\]
one obtains the following infinite recursive set of constraint equations
\[
\left[ \frac{\partial \psi_n}{\partial x} \phi^3 + \left( \frac{\partial \psi_n}{\partial y_n} \frac{\psi_n}{y_n} - \frac{d \phi}{dx} \phi x \right) y_n x^2 \right] + \left[ \left( \frac{\partial \psi_n}{\partial y_n} - \frac{2 \psi_{n+1}}{y_n} \phi^2 \phi \right) \frac{d \phi}{dx} \right] = 0. \tag{J.2}
\]
This equation can only be fulfilled if the terms in each of the two square brackets vanish separately, because, due to that the first square bracket only depends on \( y_n \) and the second one

\footnote{Note that the coordinate transformation \( x \mapsto \hat{x} = x - a \) is a permissible transformation within the determination process for the expansion coefficients \( \lambda_{n,k} \) according to (3.29), simply because the process itself is not affected by this transformation.}
on $y_{n+1}$, both square brackets are independent of each other. The above equation thus breaks apart into the following two equations

\[
\frac{\partial \psi_n}{\partial x} + \left( \frac{\partial \psi_n}{\partial y_n} \frac{\psi_n}{y_n} - \frac{d\phi}{dx} \frac{\psi_n}{y_n} \right) y_n = 0, \text{ (J.3)}
\]

\[
\frac{\partial \psi_n}{\partial y_n} - \frac{2\psi_{n+1}}{y_{n+1}} \frac{d\phi}{dx} + \frac{3\phi}{x} = 0. \text{ (J.4)}
\]

The last equation (J.4), however, is only consistent if $\psi_n$ is restricted to be a non-shifted linear function of $y_n$, i.e. if

\[
\psi_n(x, y_n) = \alpha_n(x) \cdot y_n, \text{ (J.5)}
\]

which then reduces the system (J.3)-(J.4) respectively to

\[
\frac{d\alpha_n}{dx} - \frac{d\phi}{dx} + \frac{\phi}{x} = 0, \text{ (J.6)}
\]

\[
\alpha_n - 2\alpha_{n+1} - \frac{d\phi}{dx} + \frac{3\phi}{x} = 0. \text{ (J.7)}
\]

Since (J.6) leads to the result that $\alpha_n = \alpha_{n+1}$, equation (J.7) gives the solution for $\alpha_n$ in terms of $\phi$

\[
\alpha_n = -\frac{d\phi}{dx} + \frac{3\phi}{x}. \text{ (J.8)}
\]

Inserting this result back into (J.6) leads to the following differential equation for $\phi$

\[
x^2 \frac{d^2\phi}{dx^2} - 2\frac{d\phi}{dx} + 2\frac{\phi}{x} = 0, \text{ (J.9)}
\]

which has the general solution

\[
\phi(x) = c_1 \cdot x + c_2 \cdot x^2, \text{ (J.10)}
\]

which finally, according to (J.8), implies that

\[
\alpha_n(x) = 2c_1 + c_2 \cdot x. \text{ (J.11)}
\]

Hence, the only possible combination in the infinitesimals which lead to uncoupled Lie point group invariances in the infinite system (3.26) is given by the 2-dimensional Lie sub-algebra

\[
\xi(x, y_1, y_2, \ldots) = c_1 \cdot x + c_2 \cdot x^2, \quad \eta_n(x, y_1, y_2, \ldots) = (2c_1 + c_2 \cdot x) y_n, \text{ for all } n \geq 1, \text{ (J.12)}
\]

with $[X_1^\infty, X_2^\infty] = X_2^\infty$, where the to (J.12) corresponding scalar operators $X_1^\infty$ and $X_2^\infty$ are given by (3.36).
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