On bilinear estimates and critical uniqueness classes for Navier-Stokes equations

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Abstract

We are concerned with bilinear estimates and uniqueness of mild solutions for the Navier-Stokes equations in critical spaces. For that, we construct general settings in which estimates for the bilinear term of the mild formulation hold true without using auxiliary norms such as Kato time-weighted ones. We first obtain necessary conditions in abstract critical spaces and then consider further structures to obtain the estimates in general classes of Besov, Morrey and Besov-Morrey spaces based on Banach spaces. Examples of applications are provided in different spaces as well as for other PDEs. In particular, as far as we know, the bilinear estimate and the uniqueness property obtained in the framework of Besov-weak-Herz spaces are not available in the existing literature. The proofs are mainly based on characterizations and estimates on the corresponding predual spaces.

Keywords: Navier-Stokes equations; Bilinear estimates; Uniqueness; Mild solutions; Critical spaces

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1 Introduction

In this work we are concerned with the uniqueness of mild solutions for the Navier-Stokes equations

\begin{align*}
\partial_t u - \Delta u + P(u \cdot \nabla)u &= 0 & \text{if } x \in \mathbb{R}^n, t > 0, \\
\nabla \cdot u &= 0 & \text{if } x \in \mathbb{R}^n, t \geq 0, \\
u(x, 0) &= u_0(x) & \text{if } x \in \mathbb{R}^n,
\end{align*}

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and the corresponding key estimate in critical spaces for the bilinear term

$$\mathcal{B}(u, v)(t) = -\int_0^t \nabla U(t - s) \cdot \mathbb{P}(u \otimes v)(s) \, ds,$$

(1.4)

where $n \geq 3$, $u = (u_1, \ldots, u_n)$ is a vector field in $\mathbb{R}^n$, the Leray projector $\mathbb{P}$ in matrix notation is given by $(\mathbb{P})_{kj} = \delta_{kj} + \mathcal{R}_k \mathcal{R}_j$, where $\mathcal{R}_j$ ($j = 1, \ldots, n$) stands for the $j$-th Riesz transform, and the family $\{U(t)\}_{t \geq 0}$ is the heat semigroup, namely $U(t)f = \Phi(t, x) * f$ with $\Phi$ denoting the heat kernel.

It is well known that if $u(x, t)$ is a solution of $(1.1)$-$(1.3)$, then $u^{(\lambda)}(x, t) := \lambda u(\lambda x, \lambda^2 t)$ is also a solution with initial data $\lambda u_0(\lambda x)$, for all $\lambda > 0$. So, we have the scaling map $u \mapsto u^{(\lambda)}$ and the one for the initial data

$$u_0(x) \mapsto \lambda u_0(\lambda x).$$

(1.5)

Related to this scaling, a Banach space $Z \subset S'(\mathbb{R}^n)$, where $S'(\mathbb{R}^n)$ denotes the space of tempered distributions, is called critical for the Navier-Stokes equations whether its norm is invariant under $(1.5)$ in the sense that $\|u_0\|_Z \approx \|\lambda u_0(\lambda x)\|_Z$, for all $u_0 \in Z$ and $\lambda > 0$, where the rescaling operations in $\lambda u_0(\lambda x)$ should be meant in the sense of distributions.

Considering certain classes of critical Banach spaces $Z$, our intent is to develop bilinear estimates for $\mathcal{B}(u, v)$ in the natural critical space $L^\infty((0, T); Z)$ for the flow associated to the Cauchy problem $(1.1)$-$(1.3)$ with $u_0 \in Z$ and $\nabla \cdot u_0 = 0$. Roughly speaking, under suitable conditions on $Z$, we obtain the bilinear estimate

$$\|\mathcal{B}(u, v)\|_{L^\infty((0,T);Z)} \leq K \|u\|_{L^\infty((0,T);Z)} \|v\|_{L^\infty((0,T);Z)},$$

(1.6)

for all $u, v \in L^\infty((0, T); Z)$ and $T \in (0, \infty)$. A feature of $(1.6)$ is that it provides a control on the norm of $L^\infty((0, T); Z)$ without employing auxiliary norms (e.g., Kato time-weighted norms). As a consequence of $(1.6)$ and some further basic properties of spaces, we obtain the uniqueness of solutions in the natural critical class $C([0, T]; \tilde{Z})$ for $T \in (0, \infty)$, regardless the size of solutions, where $\tilde{Z}$ stands for the maximal closed subspace of $Z$ in which the heat semigroup $\{U(t)\}_{t \geq 0}$ is strongly continuous (see $[31],[27]$).

In existence results of mild solutions for $(1.1)$-$(1.3)$, it is relatively common the use of auxiliary norms in addition to the natural norm $\|\cdot\|_{L^\infty((0,T);Z)}$. In this direction, we have the so-called Kato approach, which consists in a fixed point argument in a time-dependent critical Banach space with an auxiliary norm of the type

$$\sup_{t \leq \sigma} t^{\rho} \|u\|_Y \quad \text{with } \rho > 0 \text{ and a Banach space } Y.$$

The auxiliary norm is used to control some integral terms arising in the estimates for the bilinear operator $\mathcal{B}(\cdot, \cdot)$. Then, in general, that kind of approach itself prevents getting uniqueness in the class $C([0, T]; \tilde{Z})$, providing this property in a strict subspace of it. In the sequel, referring the reader to the review books $[27, 29]$ and without making a complete list, we mention some works that employ approaches with two (or more) norms to estimate $\mathcal{B}(\cdot, \cdot)$ and obtain global-in-time well-posedness of $(1.1)$-$(1.3)$ with divergence-free small data in critical spaces: there are results in homogeneous Sobolev space $H^{1/2}(\mathbb{R}^3)$ $[13]$, Lebesgue space $L^p(\mathbb{R}^n)$ $[20]$, Marcinkiewicz space $L^{(n, \infty)}(\mathbb{R}^n)$ $[2]$, homogeneous Besov spaces $B_{p, \infty}^{\frac{n}{p} - \frac{1}{q}}(\mathbb{R}^n)$ with $p > n$ $[6]$, homogeneous Fourier-Besov-type spaces $F \dot{B}_{p, q}^{2 - \frac{2}{p}}(\mathbb{R}^3)$ and $F \dot{N}_{p, q}^{2 - \frac{2}{p}}(\mathbb{R}^3)$ $[9, 18, 25]$, Morrey spaces $\dot{M}^{\alpha}_{p,q}(\mathbb{R}^n)$ $[15, 21, 32]$, homogeneous Besov-Morrey spaces $\dot{B}^{\frac{n}{p} - \frac{1}{q}}_{p, \infty}(\mathbb{R}^n)$ with $r > n/2$ $[24, 30]$, homogeneous Besov-weak-Herz spaces $\dot{B}W\dot{K}_{p,q}^{\alpha,s}(\mathbb{R}^n)$ $[11]$, $BMO^{-1}(\mathbb{R}^n)$ $[22]$, among others. We also quote $[19]$ where results were obtained in abstract critical Banach spaces by means of the Kato approach (see also $[34]$).

In general, estimates in the form $(1.6)$ are more difficult to obtain than those with auxiliary norms and involve more subtle arguments. As far as we know, that kind of bilinear estimate has been proved
in \( L^{(n,\infty)}(\mathbb{R}^n) \) [31, 35], pseudomeasure space \( \mathcal{P}\mathcal{M}'^{n-1}(\mathbb{R}^n) \) [8, 26], Fourier-Besov spaces \( F\mathcal{B}_{p,\infty}^{2, -\frac{2}{p}}(\mathbb{R}^3) \) with \( p > 3 \) [23], homogeneous weak-Herz space \( W\dot{K}_0^{0,\infty}(\mathbb{R}^n) \) [33], weak-Morrey spaces \( \mathcal{M}_{(p,\infty)}^{n}(\mathbb{R}^n) \) with \( 2 < p \leq n \) [28, 10], \( B_{p,\infty}^{\frac{n}{p}-1}(\mathbb{R}^n) \) with \( 2 < p \leq r < n \) and \( n/2 < r \) [12].

First we construct an abstract framework of Banach spaces for (1.1)-(1.3) in which the estimate (1.6) holds true with a general feature (see Lemma 3.3 and Theorem 3.4). In subsection 3.1, we present examples of applications which recover some previous results. In the case of the weak-Herz space \( Z = W\dot{K}_0^{0,\infty}(\mathbb{R}^n) \), it is worth noting that our construction provides a different proof for estimate (1.6) (see Remark 3.8). In fact, the work [33] follows the spirit of [27, 31] while here we are inspired by arguments in [35] (see details more below). Technically speaking, the proofs of Lemma 3.3 and Theorem 3.4 are relatively straightforward as long as we are putting the conditions we need to carry out the predual approach. Their aim is to serve as a basis for unifying some proofs when considering examples of spaces with more structure. Also, Lemma 3.3 depends on a scaling condition connecting two predual spaces \( E \) and \( E_0 \), even though in the case of the Navier-Stokes equations we should consider the dual \( Z = E' \) as being critical, that is, with the Navier-Stokes scaling \( \sigma_{E'} = 1 \) (see Remark 3.5). The more general condition on the predual spaces allows to apply the theory to other nonlinear PDEs.

In Section 4 we introduce further structure in the first framework in order to treat certain general classes of Besov spaces in several spaces, such as \( L^{n,\infty}(\mathbb{R}^n) \), \( W\dot{K}_0^{0,\infty}(\mathbb{R}^n) \), and \( \mathcal{M}_{(p,\infty)}^{n}(\mathbb{R}^n) \), as well as the Besov-type spaces \( B_{p,\infty}^{\frac{n}{p}-1}(\mathbb{R}^n) \) and \( B\mathcal{W}_{p,\infty}^{l,\frac{q}{p}-1}(\mathbb{R}^n) \). However, our estimate (1.6) in the context of Besov-weak-Herz spaces \( B\mathcal{W}_{p,\infty}^{l,\frac{q}{p}-1}(\mathbb{R}^n) \) seems to be a new contribution to the existing literature (see subsection 4.1). As a byproduct (see Theorem 7.1 and its applications in Section 7), adapting an argument by [31] yields a uniqueness class for mild solutions of (1.1)-(1.3), namely the class

\[
C([0, T) ; \mathcal{Z}) \text{ with } Z = B\mathcal{W}_{p,\infty}^{l,\frac{q}{p}-1}(\mathbb{R}^n) = B\left[ W\dot{K}_0^{0,\infty}(\mathbb{R}^n) \right]^{\frac{q}{p}-1}. \tag{1.7}
\]

In a certain sense, the present work provides bilinear estimates and uniqueness results in classes of critical spaces presenting functional structures compatible with (1.4). The main constructions encompass characterizations, basic properties and estimates on the predual spaces, as well as careful use and choices of spaces with suitable interpolation properties, Hölder-type inequality and heat semigroup estimates, among other ingredients. For that, we are motivated by Yamazaki approach in [35] as well as the works [10, 12]. In fact, our approach can be seen as an adaptation of that in [35] to other spaces with a more intricate structure, such as the Besov-weak-Herz space above.

Comparing with previous references, we point out that the approach to obtaining bilinear estimates (1.6) in abstract spaces developed in [28] (see Proposition 4.1 therein) follows the spirit of [31] by relying on the boundedness of the Riesz operator \((-\Delta)^{-1/2}\), the Hardy-Littlewood maximal operator and the boundedness of the pointwise product, and employing spaces of pointwise multipliers, among others. Also, in order to obtain a \( L^{p_0}\)-estimate for \( \mathcal{B}(u, v)(t) \), the pointwise estimate of the kernel of \( e^{\Delta t}\mathbb{P} \) by \( C(t^{2/3} + |x|^4)^{-1} \) plays a central role in [28, 31]. It is worth mentioning that some basic conditions on the base space \( X \), such as inclusion in \( S'(\mathbb{R}^n) \), translation invariance, dilatation control (or scaling), and product estimates (Hölder-type estimates), are common to other abstract frameworks employed previously to analyze the Navier-Stokes
equations; for example, adapted spaces [5],[31], shift-invariant spaces of distributions and local measures [27], and adequate Banach spaces [19].

Finally, we intend to show the versatility of the approach presented here by analyzing a reaction-diffusion system with quadratic reaction term and a nonlocal advection-diffusion system. In this part, the space $M[X]^l$ is chosen to illustrate the theory. We obtain estimates in the spirit of (1.6) and, consequently, the uniqueness property in $C([0, T) : Z)$ with $Z = M[X]^l$. The estimates for the corresponding bilinear terms are obtained for general conditions on $X$ (see Theorems 8.1 and 8.6) and afterwards applications are presented in the specific case of $X$ being a Lorentz space. In order to analyze the nonlocal model, we need to obtain some properties in $M[X]^l$-spaces for fractional derivatives via the Riesz potential. For that, we extend and adapt some ideas of [1].

Let us describe the organization of this work. Section 2 is devoted to recall some notations and give some preliminaries about interpolation theory in sequence spaces. In Section 3, we provide our first abstract framework and present the applications. In Section 4, keeping in mind the construction carried out in Section 3, we present the abstract construction in Besov-type spaces and give the applications. In Sections 5 and 6, we obtain the bilinear estimate for $X$-Morrey spaces $Z = M[X]^l$ and Besov-X-Morrey type spaces $Z = \tilde{B} [M[X]^l]^{\tilde{\theta}}$, respectively. The subject of Section 7 is a uniqueness result compatible with our abstract frameworks constructed in the previous sections. Finally, in Section 8 we analyze other nonlinearities and PDEs.

2 Preliminaries

First, let us set out some notations for the rest of the work. Given a real number $1 \leq r \leq \infty$, we denote by $r'$ the Hölder conjugate of $r$, that is, the number such that $1 = 1/r + 1/r'$. Also, if $E$ is a Banach space we denote its dual by $E'$. Since all the spaces used in this work are defined in the whole space $\mathbb{R}^n$, we omit the notation $\mathbb{R}^n$ in their notation. For example, $S' (\mathbb{R}^n)$ will be denoted by $S'$, $L^{p,\infty} (\mathbb{R}^n)$ by $L^{p,\infty}$, and so on. Given a Banach space $X$ and two $\lambda$-families of nonnegative functionals $F_\lambda, G_\lambda : X \rightarrow \mathbb{R}$, where $\lambda$ is a positive parameter, we use the notation $F_\lambda \approx G_\lambda, \forall \lambda > 0$, to mean that there exist constants $a, b > 0$ such that $aF_\lambda (f) \leq G_\lambda (f) \leq bF_\lambda (f)$ for all $\lambda > 0$ and $f \in X$. For two norms $F, G$ in $X$, note that we recover the definition of equivalent norms.

Now, we briefly recall some definitions and properties of sequence spaces. Let $s \in \mathbb{R}$, $1 \leq r \leq \infty$, and let $E$ be a Banach space. The space $\ell^s_r (E)$ is the set of all sequences $a = (a_k)_{k \in \mathbb{Z}}$ such that $a_k \in E$, for all $k$, and

$$\|a\|_{\ell^s_r (E)} = \left( \sum_{k=-\infty}^{\infty} 2^{ksr} \|a_k\|^r_E \right)^{1/r} < \infty.$$

Most of the spaces that we consider throughout this paper present the structure of sequence spaces $\ell^s_r (E)$ with some suitable Banach space $E$. The next two lemmas contain real interpolation properties for that kind of spaces (see, e.g., [3]).

Lemma 2.1. Suppose that $1 \leq r_0, r_1, r \leq \infty$ and $s_0 \neq s_1$. Then, we have that

$$\left( \ell^{s_0}_{r_0} (E), \ell^{s_1}_{r_1} (E) \right)_{\theta,r} = \ell^s_r (E),$$

where $0 < \theta < 1$ and $s = (1 - \theta)s_0 + \theta s_1$. If $s = s_0 = s_1$, it follows that

$$\left( \ell^{s}_{r_0} (E), \ell^{s}_{r_1} (E) \right)_{\theta,r} = \ell^s_r (E)$$

provided that

$$\frac{1}{r} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}.$$
Lemma 2.2. Let $1 \leq r_0, r_1 < \infty$ and $s_0 \neq s_1$. Then,
\[
\left(\delta^0_{r_0}(E_0), \delta^1_{r_1}(E_1)\right)_{\theta,r} = \delta^0_{r}(E_0, E_1)_{\theta,r},
\]
where $0 < \theta < 1$, $s = (1 - \theta)s_0 + \theta s_1$ and $\frac{1}{r} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}$.

In the sequel we recall a basic duality property for sequence spaces (see [17]).

Lemma 2.3. Assume that $s \in \mathbb{R}$ and $1 \leq r < \infty$. Then, we have the duality relation
\[
\left(\delta^s_r(E)\right)' = \left(\delta^{-s}_{-r}(E')\right).
\]

3 Bilinear estimate in a general framework

In the present section, we provide conditions on suitable Banach spaces which yield the bilinear estimate (1.6) for the Navier-Stokes equations. We start with a basic definition that introduces a scaling property for Banach spaces of distributions. Similar definitions can be found in [19], [28].

Definition 3.1. For each $f \in S'$ and $\lambda > 0$, let $f_{\lambda}(-\cdot) = f(\lambda \cdot)$. We denote by $\mathcal{G}$ the class of all Banach spaces $F \subset S'$ for which there exists $\sigma_F \in \mathbb{R}$ such that $f_{\lambda} \in F$ and $\|f_{\lambda}\|_F \leq C \lambda^{\sigma_F} \|f\|_F$, for all $\lambda > 0$ and $f \in F$, where $C > 0$ is a constant.

Replacing $\lambda$ with $\lambda^{-1}$, it is straightforward to check that $\frac{1}{C} \lambda^{\sigma_F} \|f\|_F \leq \|f_{\lambda}\|_F$. We point that in almost all cases considered by us, we have that $\sigma_F < 0$. If fact, we think in $\|\cdot\|_F$ as a way to measure the volume below of the graphic of the function $f$, then it is at least intuitive that, for $\sigma_F < 0$, the volume of $f_{\lambda}$ is being spread out as $\lambda \to \infty$. This fact can be seen by the inequality $\|f_{\lambda}\|_F \leq C \lambda^{\sigma_F} \|f\|_F$.

Let $E$ and $E_0$ be Banach spaces satisfying the following conditions:

(H1) $E, E_0 \in \mathcal{G}$.

(H2) The Riesz transform $\mathcal{R}_j : E_0' \to E_0'$, defined by
\[
\mathcal{R}_j(f) = c_n \text{PV.} \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x - y) dy, \quad \text{for } j = 1, \ldots, n, \tag{3.1}
\]
is bounded, where $c_n = \frac{\Gamma(n+1)}{\pi(n+1/2)}$.

(H3) For every $f, g \in E'$, the product $f \cdot g \in E_0'$; and, for some universal constant $C > 0$, we have the estimate
\[
\|fg\|_{E_0'} \leq C \|f\|_{E'} \|g\|_{E'}.
\]

(H4) Denote by $\nabla U(t) : E \to E_0$ the operator
\[
\nabla U(t)f = (\nabla \Phi(t, x)) * f,
\]
where
\[
\Phi(t, x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}
\]
is the heat kernel. We assume that $\nabla U(t)$ is a bounded operator and that the estimate
\[
\int_0^\infty s^{\frac{n}{2}} (\sigma_{E_0} - \sigma_E)^{-\frac{1}{2}} \|\nabla U(s) f\|_{E_0} ds \leq C \|f\|_E \tag{3.4}
\]
holds true.
With respect to the condition (H3), we have the following remark.

**Remark 3.2.** Let \( F, G, H \in \mathcal{G} \) be such that for every \( f \in F \) and \( g \in G \) we have that \( f \cdot g \in H \) and
\[
\| f \cdot g \|_H \leq C \| f \|_F \| g \|_G.
\]
Then necessarily \( \sigma_H = \sigma_F + \sigma_G \).

In fact, by replacing \( f, g \) with \( f_\lambda, g_\lambda \) in the previous inequality, we arrive to
\[
\lambda^{\sigma_H - (\sigma_F + \sigma_G)} \| f \cdot g \|_H \leq C \| f \|_F \| g \|_G.
\]
So, if \( \sigma_F + \sigma_G - \sigma_H \neq 0 \), we can take either the limit as \( \lambda \to 0 \) or \( \lambda \to \infty \) to get a contradiction.

Now, recall that if \( U : F \to G \) is a bounded operator between two normed spaces, then the dual operator \( U' : G' \to F' \), defined by
\[
\langle U' g', f \rangle = \langle g', U f \rangle,
\]
is also bounded. Let \( F, G \in \mathcal{G} \) be such that \( \nabla U(t) : F \to G \) is bounded. It is well known that the dual operator of \( \nabla^m U(t) : F \to G \) is given by \( (\nabla^m U(t))^t = (-1)^m \nabla^m U(t) \), for each \( m \in \mathbb{N}_0 \). The following result employs this fact and is based on a duality argument in order to show the \( L^\infty \)-boundedness of the bilinear form (1.4).

**Lemma 3.3.** Let \( E \) and \( E_0 \) verify (H1) and (H4) and assume that \( \sigma_{E_0} - \sigma_E - 1 = 0 \). Given \( f \in L^\infty((0, \infty); E'_0) \), define the linear functional \( T(f) \in E' \) by
\[
\langle T(f), h \rangle = -\int_0^\infty \langle \nabla U(s) f, h \rangle ds, \text{ for all } h \in E.
\]
Then, there exists a constant \( C > 0 \) such that
\[
\| T(f) \|_{E'} \leq C \sup_{t > 0} \| f(t) \|_{E_0'}, \quad (3.5)
\]
for all \( f \in L^\infty((0, \infty); E_0') \).

**Proof.** First, using duality, we obtain that
\[
\| T(f) \|_{E'} = \sup_{|h|_{E}=1} | \langle T(f), h \rangle |
\]
\[
\leq \sup_{|h|_{E}=1} \int_0^\infty | \langle \nabla U(s) f, h \rangle | ds
\]
\[
= \sup_{|h|_{E}=1} \int_0^\infty | \langle f(s), \nabla U(s) h \rangle | ds
\]
\[
\leq \sup_{|h|_{E}=1} \int_0^\infty \| f(s) \|_{E_0'} \| \nabla U(s) h \|_{E_0} ds
\]
\[
\leq \sup_{t > 0} \| f(t) \|_{E_0'} \sup_{|h|_{E}=1} \int_0^\infty \| \nabla U(s) h \|_{E_0} ds.
\] (3.6)

Now, the condition (H4) and \( \sigma_{E_0} - \sigma_E - 1 = 0 \) lead us to
\[
\text{R.H.S. of (3.6) = } \sup_{t > 0} \| f(t) \|_{E_0'} \sup_{|h|_{E}=1} \int_0^\infty s^{\frac{1}{2}(\sigma_{E_0} - \sigma_E) - \frac{1}{2}} \| \nabla U(s) h \|_{E_0} ds
\]
\[
\leq C \sup_{t > 0} \| f(t) \|_{E_0'} \sup_{|h|_{E}=1} \| h \|_E
\]
\[
= C \sup_{t > 0} \| f(t) \|_{E_0'},
\]
which, together with (3.6), yield the desired estimate.

With the estimate (3.5) in hand, we are in position to show the bilinear estimate for (1.4).

**Theorem 3.4.** Let $E$ and $E_0$ verify (H1)-(H4) and assume that $\sigma_{E_0} - \sigma_E - 1 = 0$. Then, we have the bilinear estimate
\[
\sup_{0 < t < T} \|B(u, v)(t)\|_{E'} \leq K \sup_{0 < t < T} \|u(t)\|_{E'} \sup_{0 < t < T} \|v(t)\|_{E'},
\]
for all $u, v \in L^\infty((0, T); E')$ and $T \in (0, \infty)$, where $K > 0$ is a universal constant.

**Remark 3.5.** In the case of Navier-Stokes equations, the conditions in Lemma 3.3 lead to the space $E'$ to be critical for those equations, that is, $\sigma_{E'} = 1$. However, we prefer to maintain the conditions that force to take $\sigma_{E'} = 1$ separately and independently of the others, such as the balance scaling condition $\sigma_{E_0} - \sigma_E - 1 = 0$. A reason is that Lemma 3.3 is useful by itself and depends on the latter condition which gives more versatility and applicability of the theory to other evolution PDEs (see Section 8).

**Proof of Theorem 3.4.**

Let $0 < T \leq \infty$ and $t \in (0, T)$. The bilinear term $B(u, v)$ can be written as
\[
B(u, v)(t) = -\int_0^t \nabla_x U(t - s) \mathbb{P} f(\cdot, s) ds = T(f_t),
\]
where $f_t(x, s)$ is defined by
\[
f_t(\cdot, s) = \mathbb{P}(u \otimes v)(\cdot, t - s), \text{ a.e. } s \in (0, t),
\]
\[
f_t(\cdot, s) = 0, \text{ a.e. } s \in (t, \infty).
\]

It follows from Lemma 3.3 that
\[
\|B(u, v)(t)\|_{E'} = \|T(f_t)\|_{E'} \leq C \sup_{s > 0} \|f_t(s)\|_{E_0'}.
\]

Moreover, using (H2) and (H3), we obtain that
\[
\sup_{0 < s < T} \|f_t(s)\|_{E_0'} \leq C \sup_{0 < s < T} \|(u \otimes v)(\cdot, t - s)\|_{E_0'}
\]
\[
\leq C \sup_{0 < s < T} \|u(\cdot, t - s)\|_{E'} \|v(\cdot, t - s)\|_{E'}
\]
\[
\leq C \sup_{0 < s < T} \|u(\cdot, s)\|_{E'} \sup_{0 < s < T} \|v(\cdot, s)\|_{E'}.
\]

Estimate (3.7) follows by inserting (3.9) into (3.8).

The next result gives sufficient conditions to ensure the condition (H4). In fact, inequality (3.4) can be seen as one of the main ingredients to obtain the bilinear estimate for the Navier-Stokes equations. In general lines, it is a consequence of a careful use of real interpolation tools.

**Lemma 3.6.** Let $E, E_0, E_1, E_2 \in G$ be Banach spaces verifying the following conditions:

(H4) $\|\nabla U(s) f\|_{E_0} \leq C S^{\frac{1}{2} (\sigma_{E_1} - \sigma_{E_0}) - \frac{1}{2}} \|f\|_{E_1}$, for $i = 1, 2$. 

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(H4) Assume that \( z_1 \) and \( z_2 \) satisfy \( 0 < z_1 < 1 < z_2 \) and \( \frac{1}{z_i} = \frac{1}{2} (\sigma_{E_i} - \sigma_{E_i}) + 1 \), for \( i = 1, 2 \).

(H4') If \( \theta \in (0, 1) \) is such that \( 1 = \frac{1-\theta}{z_1} + \frac{\theta}{z_2} \) with \( z_1, z_2 \) as in (H4), then \( E \leftrightarrow (E_1, E_2)_{\theta, 1} \).

Then, the integral estimate (3.4) holds true.

Remark 3.7. Note that, in (H4'), the order taken on the spaces \( E_1 \) and \( E_2 \) is not relevant, but we keep it for convenience of exposition.

Proof of Lemma 3.6. The proof extends some ideas in [35] and we sketch it for the reader convenience (see also [10, Lemma 5.1]). Considering \( h_f(s) = s^{\frac{1}{2}((\sigma_{E_i} - \sigma_{E_i}) - \frac{1}{2})} \| \nabla U(s) f \|_{E_i} \) and using (H4'), we can estimate

\[
h_f(s) \leq C s^{\frac{1}{2}((\sigma_{E_i} - \sigma_{E_i}) - \frac{1}{2})} \| f \|_{E_i} = C s^{-\frac{1}{z_i}} \| f \|_{E_i}, \quad i = 1, 2.
\]

It follows that \( h_f \in L^{z_1, \infty} \cap L^{z_2, \infty} \) and

\[
\| h_f \|_{L^{z_i, \infty}} \leq C \| f \|_{E_i}.
\]

Using interpolation and (H4'), we conclude that

\[
\| h_f \|_{L^1} = \| h_f \|_{(L^{z_1, \infty}, L^{z_2, \infty})_{\theta, 1}} \leq C \| f \|_{(E_1, E_2)_{\theta, 1}} \leq C \| f \|_E,
\]

which is (3.4).

\[\diamond\]

3.1 Applications

In this section we present two cases where Theorem 3.4 applies in a more or less direct way.

3.1.1 Bilinear estimate on weak-\( L^p \) spaces

By taking \( E = L^{\frac{n}{n-1}, 1} \) and \( E_0 = L^{\frac{n}{n-2}, 1} \), we obtain the bilinear estimate (3.7) in the space \( E' = L^{n, \infty} \) (see [35]). In fact, (H1)-(H3) are well-known and we check (H4)-(H4'). For this, take \( E_i = L^{(p_i, 1)} \) with \( 1 < p_2 < \frac{n}{n-1} < p_1 < \frac{n}{n-2} \), then it follows (H4') and (H4'). Choosing \( \theta \in (0, 1) \) such that \( \frac{n-1}{n} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2} \), we have that \( L^{p_i, 1} = (L^{(p_1, 1)}, L^{(p_2, 1)})_{\theta, 1} \), by interpolation theory, and then it follows (H4').

3.1.2 Bilinear estimate on Lorentz-Herz spaces

Now we show that the Lorentz-Herz spaces \( E = \dot{K}_{n-1, 1, 1}^{0, \alpha} \) and \( E_0 = \dot{K}_{n-2, 1, 1}^{0, \alpha} \) fulfill the conditions (H1)-(H4). Therefore, we obtain the bilinear estimate (3.7) in the space \( E' = W \dot{K}_{n, \infty}^{0, \alpha} \).

Remark 3.8. It is worthy to comment that, in comparison with [33], this approach provides a new proof of (3.7) in \( E' = W \dot{K}_{n, \infty}^{0, \alpha} \). In fact, the proof in [33] relies on the so-called Meyer approach [31].

Let \( 1 \leq p \leq \infty \) and \( 1 \leq d, q \leq \infty \), with \( d = \infty \) if \( p = \infty \). Recall that the Lorentz-Herz space \( \dot{K}_{(p, d), q}^{\alpha} \) is defined as a class of functions \( f \) such that

\[
\| f \|_{\dot{K}_{(p, d), q}^{\alpha}} = \| f \|_{L^{p, d}(A_k)}_{L^q } < \infty,
\]

where the sets \( A_k \) are defined as

\[
A_k = \left\{ x \in \mathbb{R}^n; 2^{k-1} \leq x < 2^k \right\}.
\]

Next we collect some properties about Lorentz-Herz spaces that will be useful in the sequel. For more details, see [33, 11].
(a) \( \|f(\lambda)\|_{K^0_{p,d,q}} \approx \lambda^{-\frac{n}{p}} \|f\|_{K^0_{p,d,q}}, \forall \lambda > 0. \)

(b) \( \left( \hat{K}^0_{p',d',q'} \right)' = \hat{K}^0_{p,d,q}, \) for \( 1 < p < \infty \) and \( 1 < d, q < \infty. \)

(c) The Riesz transform is bounded in \( \hat{K}^0_{p,d,q}, \) for \( 1 < p < \infty \) and \( 1 \leq d, q \leq \infty. \)

(d) We have the H"older-type inequality
\[
\|fg\|_{K^0_{p,d,q}} \leq C \|f\|_{\hat{K}^0_{p_1,d_1,q_1}} \|g\|_{\hat{K}^0_{p_2,d_2,q_2}},
\]
where \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{d} = \frac{1}{d_1} + \frac{1}{d_2} \) and \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}. \)

(e) \( \|\nabla U(s)f\|_{\hat{K}^0_{p,d,q}} \leq C s^\theta \left( \frac{1}{p} - \frac{1}{q} \right)^{-\frac{1}{2}} \|f\|_{\hat{K}^0_{p_0,d_0,q_0}}, \forall s > 0, \) for \( 1 < p_0 \leq p < \infty \) and \( 1 \leq d, q \leq \infty. \)

(f) \( \|\psi * f\|_{L^\infty} \leq C(\psi) \|f\|_{\hat{K}^0_{p,d,q}}, \) for \( \psi \in \mathcal{S}. \)

(g) \( \|\psi * f\|_{\hat{K}^0_{p,d,q}} \leq C \max\{\|\psi\|_{L^1}, \|\psi\|_{L^\infty}\} \|f\|_{\hat{K}^0_{p_0,d_0,q_0}}, \) for \( \psi \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n, \, |\cdot|^n \, dx). \)

(h) \( \|\Delta_0 f\|_{\hat{K}^0_{p,d,q}} \leq C \|f\|_{\hat{K}^0_{p_0,d_0,q_0}}, \) for \( 1 < p_0 \leq p < \infty \) and \( 1 \leq d, q \leq \infty. \) (see (4.1) for the definition of the operator \( \Delta_0 \)).

In what follows, we are going to prove (3.4) in the context of Lorentz-Herz spaces. We start with the lemma below.

**Lemma 3.9.** Let \( 1 < p, p_1, p_2 \leq \infty \) and \( 1 \leq d_1, d_2, q_1, q_2 \leq \infty \) be such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{\theta}{p_2} \) with \( \theta \in (0, 1) \).
If \( X_1 \) and \( X_2 \) are Banach spaces and \( T \) is a continuous linear operator such that
\[
T : \hat{K}^0_{(p_1,d_1),q_1} \to X_1 \quad \text{and} \quad T : \hat{K}^0_{(p_2,d_2),q_2} \to X_2,
\]
with the operator norms \( C_1 \) and \( C_2 \), respectively. Then
\[
T : \hat{K}^0_{(p,d),1} \to (X_1, X_2)_{\theta,d},
\]
with the operator norm bounded by \( \tilde{C} = (C_1)^{1-\theta} (C_2)^{\theta}. \)

**Proof.** For every \( k \in \mathbb{Z}, \) it follows from (3.10) that
\[
T : L^{p_1,d_1}(A_k) \to X_1 \quad \text{and} \quad T : L^{p_2,d_2}(A_k) \to X_2.
\]
Denoting \( N^k_1 \) and \( N^k_2 \) the respective operator norms in (3.11), we get \( N^k_1 \leq C_1 \) and \( N^k_2 \leq C_2. \) Next, using interpolation in Lorentz spaces (see [3]), we obtain that
\[
T : L^{p,d}(A_k) \to (X_1, X_2)_{\theta,d},
\]
with operator norm \( N^k \leq (N^k_1)^{1-\theta} (N^k_2)^{\theta} \leq (C_1)^{1-\theta} (C_2)^{\theta} = \tilde{C}. \)

Now consider the dense subspace of \( \hat{K}^0_{(p,d),1} \) given by
\[
Y = \left\{ f^m = \sum_{k=-m}^{m} \chi_{A_k} f; \, m \in \mathbb{N} \, \text{and} \, f \in \hat{K}^0_{(p,d),1} \right\}.
\]
For $f^m$ in $Y$, we have that

$$
\|T(f^m)\|_{(X_0,X_1)_{\theta,d}} = \sum_{k=-m}^{m} \|T(\chi_{A_k} f)\|_{(X_0,X_1)_{\theta,d}} \leq \sum_{k=-m}^{m} \tilde{C} \|f\|_{L^p,d(A_k)} = \sum_{k=-m}^{m} \tilde{C} \|f^m\|_{L^p,d(A_k)} \leq \tilde{C} \|f^m\|_{K^0_{(p,d),1}},
$$

and then we conclude the proof by density.

\[\circ\]

**Remark 3.10.** Note that, under the conditions of Lemma 3.9, we have the continuous inclusion

$$
\dot{K}^0_{(p,d),1} \hookrightarrow \left( \dot{K}^0_{(p_1,d_1),q_1}, \dot{K}^0_{(p_2,d_2),q_2} \right)_{\theta,d}. \tag{3.13}
$$

In particular,

$$
\dot{K}^0_{(p_1),1} \hookrightarrow \left( \dot{K}^0_{(p_1,1),1}, \dot{K}^0_{(p_2,1),1} \right)_{p,1}. \tag{3.14}
$$

Now, we turn to complete the proof of the bilinear estimate (3.7) in $W\dot{K}^0_{n,\infty}$. Let $p_1$ and $p_2$ be such that $0 < p_1 < \frac{n}{n-1} < p_2 < \frac{n}{n-2}$ and define $z_i$ as $\frac{1}{z_i} = \frac{1}{2} \left( -\frac{n}{p_i} + \left(-\frac{n}{p_i}\right) \right) + 1$, for $i = 1, 2$. Note that $0 < z_1 < 1 < z_2 < \infty$, and then we can take $\theta \in (0, 1)$ in such a way that $1 = \frac{\theta}{z_1} + \frac{1-\theta}{z_2}$ and $\frac{1}{n-1} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. Consider the spaces $E = \dot{K}^0_{n-1,1,1}$, $E_0 = \dot{K}^0_{n-2,1,1}$, $E_1 = \dot{K}^0_{p_1,1,1}$ and $E_2 = \dot{K}^0_{p_2,1,1}$ with $p_1$ and $p_2$ as before. Using these choices, we get directly (H4'). The property (e) implies (H4') and the inclusion (3.14) gives (H4'''). By Lemma 3.6, we get (3.4) and then (H4). Conditions (H1), (H2) and (H3) follow from properties (a), (c) and (d), respectively.

With the above properties in hand and applying Theorem 3.4, we obtain the bilinear estimate (3.7) with $E' = \dot{K}^0_{n,\infty} := W\dot{K}^0_{n,\infty}$; that is, we reobtain the bilinear estimate proved in [33] by using a different method.

### 4 Bilinear estimates in Besov-type spaces

In what follows, the functions $\varphi$ and $\Theta$ are radially symmetric and satisfy (see, e.g., [3])

$$
\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\}), \quad \text{supp} \varphi \subset \left\{ x \mid \frac{3}{4} \leq |x| \leq \frac{8}{3} \right\},
$$

$$
\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \text{where} \quad \varphi_j(\xi) := \varphi \left( 2^{-j} \xi \right),
$$

and

$$
\Theta(\xi) = \left\{ \begin{array}{ll}
\sum_{j \leq -1} \varphi_j(\xi), & \text{if } \xi \neq 0 \\
1, & \text{if } \xi = 0.
\end{array} \right.
$$

Thus, $\Theta \in C_c^\infty(\mathbb{R}^n)$, supp $\Theta \subset \left\{ x \mid |x| \leq \frac{4}{3} \right\}$ and

$$
\Theta(\xi) + \sum_{j \geq 0} \varphi_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.
$$
Moreover, we also define \( \hat{\varphi}_j = \sum_{|k-j| \leq 1} \varphi_k \), \( \hat{\Theta} = \Theta + \varphi \) and \( \hat{D}_j = D_{j-1} \cup D_j \cup D_{j+1} \), where \( D_j = \{ x; \frac{2}{3} 2^j \leq |x| \leq \frac{8}{3} 2^j \} \) and \( j \in \mathbb{Z} \). So, we have that \( \hat{\varphi}_j \equiv 1 \) on \( D_j \), \( \varphi_j = \hat{\varphi}_j \varphi_j \) for all \( j \in \mathbb{Z} \) and \( \Theta = \hat{\Theta} \).

The operators \( \Delta_j \) and \( S_j \) are defined as

\[
\Delta_j f = \varphi_j (D) f = \left( \varphi_j f \right) \gamma \quad \forall j \in \mathbb{Z},
\]

\[
S_j f = \Theta_j (D) f = \left( \Theta_j f \right) \gamma \quad \forall j \in \mathbb{Z},
\]

where \( \Theta_j (\xi) = \Theta (2^{-j} \xi) \). It is not hard to show the cancellation identities

\[
\Delta_j \Delta_k f = 0 \quad \text{if} \quad |j - k| \geq 2, \quad \text{and} \quad \Delta_j (S_k - 2^j \Delta_k f) = 0 \quad \text{if} \quad |j - k| \geq 3.
\]

Now we recall a general definition of Besov-type spaces based on a Banach space \( F \). This definition has been used by some authors, see e.g. [14, 27, 30].

**Definition 4.1.** Let \( F \subset S' \) be a Banach space, \( 1 \leq r \leq \infty \) and \( s \in \mathbb{R} \). The homogeneous Besov-F space, denoted by \( \dot{B} [F]_r^s \), is defined as

\[
\dot{B} [F]_r^s = \left\{ f \in S'(\mathbb{R}^n)/\mathcal{P}; \| f \|_{\dot{B}[F]_r^s} < \infty \right\},
\]

where \( \mathcal{P} \) stands for the set of all polynomials and

\[
\| f \|_{\dot{B}[F]_r^s} := \left\{ \begin{array}{ll}
\left( \sum_{j \in \mathbb{Z}} 2^{jsr} \| \Delta_j f \|_F^r \right)^{\frac{1}{r}} & \text{if} \quad r < \infty \\
\sup_{j \in \mathbb{Z}} 2^{js} \| \Delta_j f \|_F & \text{if} \quad r = \infty.
\end{array} \right.
\]

The homogeneous Besov-F space is Banach provided that the underlying space \( F \) verifies some basic conditions (see [19],[30] for related ones). This is the subject of the next lemma.

**Lemma 4.2.** Let \( F \) be a Banach space verifying the conditions

(B1) \( F \in \mathcal{G} \);

(B2) \( \| \Delta_0 f \|_{L^\infty} \leq C \| f \|_F \).

Then, \( \dot{B} [F]_r^s \) is a Banach space and \( \dot{B} [F]_r^s \rightarrow S'/\mathcal{P} \).

**Proof.** Using \( \Delta_j f = \Delta_0 \left( f \left( \frac{\cdot}{2^j} \right) \right) \left( 2^j \cdot \right) \), we can see that

\[
\| \Delta_j f \|_{L^\infty} = \| \Delta_0 \left( f \left( \frac{\cdot}{2^j} \right) \right) \left( 2^j \cdot \right) \|_{L^\infty} \leq C \| \Delta_0 \left( f \left( \frac{\cdot}{2^j} \right) \right) \|_{L^\infty}
\]

\[
\leq C \left\| f \left( \frac{\cdot}{2^j} \right) \right\|_F \leq C 2^{-jsF} \| f \|_F.
\]

The same arguments lead us to

\[
\| \Delta_j f \|_{L^\infty} = \| \tilde{\Delta}_j \Delta_j f \|_{L^\infty} \leq C 2^{-j\sigma_F} \| \Delta_j f \|_F,
\]

and then \( \dot{B} [F]_r^s \rightarrow \dot{B}_{r',\gamma}^\omega \) with \( \gamma = s + \sigma_F \). The rest of the proof follows the same ideas in [19, Lemma 4.2] and [30, Lemma 2.21].

\( \diamond \)
Remark 4.3. Assume that $F, G \in \mathcal{G}$ present the property $\| \Delta_0 f \|_G \leq C \| f \|_F$, $\forall f \in F$. Proceeding as in the previous proof, we obtain that

$$\| \Delta_j f \|_G = \| \Delta_0 \left( f \left( \frac{\cdot}{2^j} \right) \right) \|_G \leq C 2^j \| \Delta_0 \left( f \left( \frac{\cdot}{2^j} \right) \right) \|_G \leq C 2^j \| f \|_F.$$  

Additionally, consider the following condition on the space $F$ which is an extension of the convolution property in shift-invariant spaces of distributions. In fact, taking $(B3)$, we have

$$(B3) \quad \| \psi \ast f \|_F \leq C \| \psi \|_M \| f \|_F$$

for all $f \in F$ and $\psi \in S$, where the norm $\| \cdot \|_M$ is such that $\| 2^m \psi (2^m \cdot) \|_M \approx \| \psi \|_M$ and $\| \psi \|_M \leq C \max \{ \| \psi \|_{L^1}, \| \cdot \|_{L^\infty} \}$.

Lemma 4.4. Let $F \subset S'$ be a Banach space verifying $(B3)$, $m \in \mathbb{R}$, and $P$ a $C^\infty$-function on $\tilde{D}_j$ such that $|\partial^\rho P (\xi)| \leq C 2^{(m-|\rho|)j}$, $\forall \xi \in \tilde{D}_j$ and $\forall \rho \in \mathbb{N}^\mathbb{N}$ with $|\rho| \leq n$. Then,

$$\left\| \left( P \hat{f} \right) \ast \right\|_F \leq C 2^{jm} \| f \|_F,$$

for all $f \in F$ such that $\text{supp} \hat{f} \subset D_j$.

Proof. Define $K(x) = (P(\cdot) \varphi_j(\cdot))(x)$, where $\varphi_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$. Since $\text{supp} \hat{f} \subset D_j$, it follows that

$$P(\xi) \hat{f}(\xi) = P(\xi) \varphi_j(\xi) \hat{f}(\xi).$$

So, $(P \hat{f}) \ast = (P \varphi_j \hat{f}) \ast = K \ast f$. Using $(B3)$, we obtain that

$$\left\| \left( P \hat{f} \right) \ast \right\|_F \leq \| K \|_M \| f \|_F.$$

We claim that $\| K \|_M \leq C 2^{jm}$. For this, we choose $N \in \mathbb{N}$ such that $n/2 < N \leq \lceil n/2 \rceil + 1$ in order to get

$$\| K \|_{L^1} = \int_{D(0,2^{-j})} K(y) + \int_{|y| \geq 2^{-j}} K(y) \leq \left( \int_{D(0,2^{-j})} \right)^{1/2} \left( \int_{D(0,2^{-j})} |K(y)|^2 \right)^{1/2} + \left( \int_{|y| \geq 2^{-j}} \right)^{1/2} \left( \int_{|y| \geq 2^{-j}} |y|^{-2N} |K(y)|^2 \right)^{1/2} \leq C 2^{-j} \| P \varphi_j \|_{L^2} + C 2^{-j} \| P \varphi_{j-1} \|_{L^2} \leq C 2^{jm}.$$
Corollary 4.5. Let \( F \subset S' \) be a Banach space verifying (B3), \( 1 \leq r \leq \infty \), and \( m, s \in \mathbb{R} \). Assume that \( P \in C^\infty (\mathbb{R}^n \setminus \{0\}) \) satisfies \( |\partial^\rho P (\xi)| \leq C |\xi|^{(m-|\rho|)} \), for all \( \rho \in \mathbb{N}_0^n \) and \( |\rho| \leq n \). Then, we have the estimate
\[
\|P (D) f\|_{\dot{B}[F]^{-m}_r} \leq C \|f\|_{\dot{B}[F]^{-m}_r},
\]
for all \( f \in \dot{B}[F]^{s}_r \), where \( C > 0 \) is a universal constant.

Proof. Note that, for every \( j \in \mathbb{Z} \) and \( \xi \in \tilde{D}_j \), we have that \( |\xi|^{m-|\rho|} \leq C 2^{j(m-|\rho|)} \). So, \( |\partial^\rho P (\xi)| \leq C 2^{j|\rho|} \). On the other hand, since \( \text{supp} \tilde{\Delta}_j f \subset D_j \), we can use Lemma 4.4 to obtain
\[
\|\Delta_j (P (D) f)\|_F = \|P (D) (\Delta_j f)\|_F \leq C 2^{j m} \|\Delta_j f\|_F.
\]
Multiplying by \( 2^{j(s-m)} \) and taking the \( l^r \)-norm yield the result.

The lemma below contains an auxiliary result that relates the sequence spaces defined in Section 2 to the Besov-type spaces. This result has already been showed in some particular cases (see [3], [30]).

Lemma 4.6. Let \( s \in \mathbb{R} \), \( 1 \leq r \leq \infty \), and let \( F \) be a Banach space verifying (B1) and (B3). Then, the space \( \dot{B}[F]^{s}_r \) is a retract of the sequence space \( \dot{l}^r(F) \).

Proof. We define the operator \( I \) as \( (I (f))_j := \Delta_j f \), for all \( f \in S' \). It is clear that \( I : \dot{B}[F]^{s}_r \rightarrow \dot{l}^r(F) \) is bounded. We also define the operator \( K \) as \( K(\beta) = \sum_{k=\infty}^\infty \Delta_k \beta_j, \forall \beta_j \in \dot{l}^r(F) \). Since \( \varphi_j = \tilde{\varphi}_j \varphi_j \), it follows that \( \Delta_j K(\beta) = \Delta_j \beta_j \), and
\[
\|K(\beta)\|_{\dot{B}[F]^{s}_r} = \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \|\Delta_j \beta_j\|_F^r \right)^{\frac{1}{r}} \leq C \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \|\beta_j\|_F^r \right)^{\frac{1}{r}} \leq C \|\beta\|_{\dot{l}^r(F)}.
\]
Therefore, \( K : \dot{l}^r(F) \rightarrow \dot{B}[F]^{s}_r \) is bounded. Moreover, \( K \circ I \) is the identity in \( \dot{B}[F]^{s}_r \), as desired.

Corollary 4.7. Let \( F \) satisfy the conditions of Lemma 4.6 and assume that \( 1 \leq r_0, r_1 \leq \infty \) and \( s_0, s_1 \in \mathbb{R} \) with \( s_0 \neq s_1 \). Then,
\[
\left( \dot{B}[F]^{s_0}_{r_0}, \dot{B}[F]^{s_1}_{r_1} \right)_{\theta, r} = \dot{B}[F]^{s}_r, \forall r \in [1, \infty],
\]
where \( \theta \in (0, 1) \) and \( s = (1-\theta) s_0 + \theta s_1 \). In the case \( s_0 = s_1 = s \), we have that
\[
\left( \dot{B}[F]^{s}_{r_0}, \dot{B}[F]^{s}_{r_1} \right)_{\theta, r} = \dot{B}[F]^{s}_r
\]
provided that \( \frac{1}{r} = \frac{\theta}{r_0} + \frac{\theta}{r_1} \).

Proof. It follows from Lemma 2.1 that
\[
\left( \dot{l}^{s_0}_{r_0} (F), \dot{l}^{s_1}_{r_1} (F) \right)_{\theta, r} = \dot{l}^{s}_r (F).
\]
Now we can conclude the proof by employing Lemma 4.6.
Lemma 4.8. Let $F_0$ and $F_1$ verify the conditions (B1)-(B3), $\theta \in (0, 1)$ and $1 \leq r \leq \infty$. Then, the interpolation space $F = (F_0, F_1)_{\theta,r}$ also verifies (B1)-(B3) with $\sigma_F = (1 - \theta) \sigma_{F_0} + \theta \sigma_{F_1}$.

Proof. Let $j \in \mathbb{Z}$ fixed and note that $\Delta_j : F_i \to L^\infty$ with operator norm bounded by $C 2^j (1 - \theta) \sigma_{F_0} + \theta \sigma_{F_1}$ ($i = 0, 1$). Then, by interpolation, we obtain that

$$\|\Delta_j f\|_{L^\infty} = \|\Delta_j f\|_{(L^\infty, L^\infty)_{\theta,r}} \leq C 2^j ((1 - \theta) \sigma_{F_0} + \theta \sigma_{F_1}) \|f\|_{(F_0, F_1)_{\theta,r}}.$$ 

This proves (B2) for $F$. The property (B3) can be proved in the same way.

In the sequel we show (B1). The fact $F \subset S'$ follows trivially. On the other hand, for $\lambda > 0$, we have that

$$K (t, f(\lambda)) = \inf_{f(\lambda) = f_0 + f_1} (\|g_0\|_{F_0} + t \|g_1\|_{F_1}) \leq \inf_{f(\lambda) = f_0 + f_1} (\|f_0(\lambda)\|_{F_0} + t \|f_1(\lambda)\|_{F_1})$$

$$\leq \inf_{f = f_0 + f_1} (\|\lambda \sigma_{F_0} f_0\|_{F_0} + t \lambda \sigma_{F_1} \|f_1\|_{F_1}) = \lambda \sigma_{F_0} \inf_{f = f_0 + f_1} (\|f_0\|_{F_0} + t \lambda \sigma_{F_1 - \sigma_{F_0}} \|f_1\|_{F_1})$$

$$= \lambda \sigma_{F_0} K (\lambda \sigma_{F_1 - \sigma_{F_0}} t, f) .$$

Thus,

$$\Phi_{\theta,r} (K (t, f(\lambda))) = \left( \int_0^\infty \left( \int_0^u \frac{u}{t} dt \right)^r du \right)^{1/r} \leq \left( \int_0^\infty \left( \int_0^u \frac{u}{t} \lambda \sigma_{F_0} K (\lambda \sigma_{F_1 - \sigma_{F_0}} t, f) \right)^r de \right)^{1/r}$$

$$= \lambda \sigma_{F_0} \left( \int_0^\infty \left( \int_0^u \frac{u}{t} \lambda \sigma_{F_0 - \sigma_{F_1}} u \right)^r du \right)^{1/r}$$

$$= \lambda (1 - \theta) \sigma_{F_0} + \theta \sigma_{F_1} \Phi_{\theta,r} (K (t, f)) ,$$

which yields $\|f(\lambda)\|_F \leq \lambda \sigma_F \|f(\lambda)\|_F$ with $\sigma_F = (1 - \theta) \sigma_{F_0} + \theta \sigma_{F_1}$, and we are done.

From the previous results, we have another one about interpolation of Besov-type spaces.

Corollary 4.9. Let $s_0, s_1 \in \mathbb{R}$, $1 \leq q_0, q_1 < \infty$, $\theta \in (0, 1)$, and let $F_0, F_1$ be two Banach spaces verifying the conditions (B1)-(B3). Then

$$\left( \tilde{B} [F_0]_{r_0}^{s_0}, \tilde{B} [F_1]_{r_1}^{s_1} \right)_{\theta,r} = \tilde{B} \left[ (F_0, F_1)_{\theta,r} \right]^s ,$$

where $s = (1 - \theta)s_0 + \theta s_1$ and $\frac{1}{r} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}$.

Proof. It follows from Lemma 2.2 that

$$\left( \tilde{i}_{r_0}^{s_0} (F_0), \tilde{i}_{r_1}^{s_1} (F_1) \right)_{\theta,r} = \tilde{i}_r^s \left( (F_0, F_1)_{\theta,r} \right) .$$

Using again Lemma 4.6, we are done.

The following theorem provides a characterization for the dual of Besov-type spaces.
Theorem 4.10. Let \( s \in \mathbb{R}, \ 1 \leq r < \infty \) and let \( F \) be a Banach space verifying the conditions (B1) and (B3) such that \( F' \) also verifies (B1) and (B3). Then
\[
\left( \hat{B} [F]_r^s \right)' = \hat{B} [F']_{r'}^{-s}.
\]

Proof. From Lemma 2.3, we have that
\[
\left( \hat{i}_r^s (F') \right)' = \left( \hat{i}^{-s}_{r'} (F') \right).
\]
Thus, the result follows by employing Lemma 4.6.

Proposition 4.11. Let \( F \) be a Banach space verifying (B1) and (B3), \( 1 \leq r \leq \infty \) and \( s, \sigma \in \mathbb{R} \). Then, there exists \( C > 0 \) independent of \( t > 0 \) such that
\[
\| U (t) f \|_{\hat{B}[F]_r^s} \leq C t^{(s-\sigma)/2} \| f \|_{\hat{B}[F]_r^s}, \text{ for all } f \in \hat{B}[F]_r^s. \tag{4.4}
\]
Moreover, if \( s < \sigma \), then
\[
\| U (t) f \|_{\hat{B}[F]_r^s} \leq C t^{(s-\sigma)/2} \| f \|_{\hat{B}[F]_r^s}, \text{ for all } f \in \hat{B}[F]_r^s. \tag{4.5}
\]

Proof. Note that, for every \( \beta \in \mathbb{N}_0^n \), there exists a polynomial \( p_\rho (\cdot) \) of degree \( |\rho| \) such that
\[
\partial^\rho \exp \left( -t |\xi|^2 \right) = t^{|\rho|/2} p_\rho (\sqrt{t} \xi) \exp \left( -t |\xi|^2 \right).
\]
Thus, for some \( C > 0 \), it follows that
\[
\left| \partial^\rho \exp \left( -t |\xi|^2 \right) \right| \leq C t^{-m/2} |\xi|^{-m-|\rho|}.
\]
Now, using Corollary 4.5, we obtain
\[
\| U (t) f \|_{\hat{B}[F]_r^{s+m}} \leq C t^{-m/2} \| f \|_{\hat{B}[F]_r^s}.
\]
Taking \( m = \sigma - s \), we conclude the estimate (4.4).

Next we turn to prove (4.5). For \( s < \sigma \), from (4.4) with \( r = \infty \), we get
\[
\| U (t) f \|_{\hat{B}[F]_\infty^{2\sigma-s}} \leq C t^{s-\sigma} \| f \|_{\hat{B}[F]_\infty^s}
\]
and
\[
\| U (t) f \|_{\hat{B}[F]_\infty^s} \leq C \| f \|_{\hat{B}[F]_\infty^s}.
\]
Using Corollary 4.7, we arrive at
\[
U(t) : \hat{B}[F]_\infty^s \longrightarrow \left( \hat{B}[F]_\infty^{2\sigma-s}, \hat{B}[F]_\infty^{2\sigma-s} \right)_{\frac{1}{2}, 1} = \hat{B}[F]_1^s,
\]
with \( \| U (t) \|_{\hat{B}[F]_\infty^s \rightarrow \hat{B}[F]_1^s} \leq C t^{(s-\sigma)/2} \), as required.

In order to prove the bilinear estimate in the context of Besov-type spaces we need a product estimate in such spaces. In the next lemma we establish some conditions to ensure an appropriate product estimate.
Lemma 4.12. Let $s \in \mathbb{R}$ and $F \in \mathcal{G}$ verify the following conditions:

(B4) $F' \in \mathcal{G}$ and verifies (B3).

(B5) There exist Banach spaces $G_0, G_1, G_2 \in \mathcal{G}$ such that
\[
\|fg\|_{F'} \leq C \|f\|_{G_0} \|g\|_{G_0} \quad \text{with} \quad \sigma_{F'} = 2\sigma_{G_0},
\]
\[
\|\Delta_0 f\|_{G_0} \leq C \|f\|_{F'},
\]
\[
\|\Delta_0 f\|_{G_1} \leq C \|f\|_{F'},
\]
\[
\|\Delta_0 f\|_{F'} \leq C \|f\|_{G_2},
\]
\[
\|fg\|_{G_2} \leq C \|f\|_{G_0} \|g\|_{G_1} \quad \text{with} \quad \sigma_{G_2} = 2\sigma_{G_1}.
\]

(B6) $\sigma_{G_1} - \sigma_{F'} - s < 0$ and $-\frac{\sigma_{F'}}{2} - s > 0$.

Then, there exists $\rho_0 > 0$ such that for all $\rho \in [0, \rho_0)$ and for $s_0 = \sigma_{F'} + 2s$, we have the product estimate
\[
\|fg\|_{\dot{B}^s_{2\varphi}} \leq C \|f\|_{\dot{B}^s_{\varphi}} \|g\|_{\dot{B}^s_{\varphi}},
\]
where $C > 0$ is a universal constant.

Proof. Let $\rho_0$ be such that $-\frac{\sigma_{F'}}{2} - s - \rho_0 = 0$. Then, for $\rho \in [0, \rho_0)$ we have that $-\frac{\sigma_{F'}}{2} - s - \rho > 0$. Now, for every $j \in \mathbb{Z}$, we can write
\[
\Delta_j (fg) = \sum_{|k-j| \leq 4} \Delta_j (S_{k-2f} \Delta_k g) + \sum_{|k-j| \leq 4} \Delta_j (S_{-2g} \Delta_k f) + \sum_{k \geq j - 2} \Delta_j (\Delta_k f \tilde{\Delta}_k g)
\]
\[
= I_1^j + I_2^j + I_3^j. \tag{4.6}
\]

We estimate $I_1^j$ as follows:
\[
\|I_1^j\|_{F'} \leq C \sum_{|k-j| \leq 4} \|S_{k-2f}\|_{G_0} \|\Delta_k g\|_{G_0}
\]
\[
\leq C \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} \|\Delta_m f\|_{G_0} \right) \|\Delta_k g\|_{G_0}
\]
\[
\leq C \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m (\sigma_{G_0} - \sigma_{F'}) \|\Delta_m f\|_{F'} \right) 2^k (\sigma_{G_0} - \sigma_{F'}) \|\Delta_k g\|_{F'}
\]
\[
\leq C \|f\|_{\dot{B}^s_{\varphi}} \|g\|_{\dot{B}^s_{\varphi}} \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m (-\frac{\sigma_{F'}}{2} - s) \right) 2^k (-\frac{\sigma_{F'}}{2} - s - \rho).
\]

Thus,
\[
\|I_1^j\|_{F'} \leq C \|f\|_{\dot{B}^s_{\varphi}} \|g\|_{\dot{B}^s_{\varphi}} 2^j (-s_0 - \rho). \tag{4.7}
\]

For $I_2^j$, we proceed similarly in order to obtain
\[
\|I_2^j\|_{F'} \leq C \|f\|_{\dot{B}^s_{\varphi}} \|g\|_{\dot{B}^s_{\varphi}} \sum_{|k-j| \leq 4} \left( \sum_{m \leq k-2} 2^m (-\frac{\sigma_{F'}}{2} - s - \rho) \right) 2^k (-\frac{\sigma_{F'}}{2} - s),
\]

which leads us to
\[
\|I_2^j\|_{F'} \leq C \|f\|_{\dot{B}^s_{\varphi}} \|g\|_{\dot{B}^s_{\varphi}} 2^j (-s_0 - \rho). \tag{4.8}
\]
Now we estimate $I_3^{ij}$:
\[
\|I_3^{ij}\|_{F'} \leq \sum_{k \geq j - 2} \| \Delta_j \left( \Delta_k f \Delta_k g \right) \|_{F'} \\
\leq \frac{\lambda}{\gamma} \sum_{k \geq j - 2} 2^{j (\sigma_{F'} - \sigma_{F_2})} \| \Delta_k f \Delta_k g \|_{F_2} \\
\leq \frac{\lambda}{\gamma} \sum_{k \geq j - 2} 2^{j (\sigma_{F'} - \sigma_{F_2})} \| \Delta_k f \|_{F_1} \| \Delta_k g \|_{F_1}.
\]

Moreover,
\[
\| \Delta_k f \|_{G_1} \leq C 2^{k (\sigma_{F_1} - \sigma_{F'})} \| \Delta_k f \|_{F'} \leq C 2^{k (\sigma_{F_1} - \sigma_{F'})} \| f \|_{\tilde{B}[F']_{x}^{s'}}.
\]

Similarly, we have that
\[
\| \Delta_k g \|_{G_1} \leq C 2^{k (\sigma_{F_1} - \sigma_{F'})} \| \Delta_k g \|_{F'} \leq C 2^{k (\sigma_{F_1} - \sigma_{F'})} \| g \|_{\tilde{B}[F']_{x}^{s'}}.
\]

Thus,
\[
\|I_3^{ij}\|_{F'} \leq C \| f \|_{\tilde{B}[F']_{x}^{s'}} \| g \|_{\tilde{B}[F']_{x}^{s'}} \sum_{k \geq j - 2} 2^{k (\sigma_{F_1} - \sigma_{F'})} \| \Delta_k f \|_{F_1} \| \Delta_k g \|_{F_1} \\
\leq C \| f \|_{\tilde{B}[F']_{x}^{s'}} \| g \|_{\tilde{B}[F']_{x}^{s'}} \sum_{k \geq j - 2} 2^{k (\sigma_{F_1} - \sigma_{F'})} \| \Delta_k f \|_{F_1} \| \Delta_k g \|_{F_1} \\
\leq C \| f \|_{\tilde{B}[F']_{x}^{s'}} \| g \|_{\tilde{B}[F']_{x}^{s'}} \sum_{k \geq j - 2} 2^{k (\sigma_{F_1} - \sigma_{F'})} \| \Delta_k f \|_{F_1} \| \Delta_k g \|_{F_1} \\
\leq C \| f \|_{\tilde{B}[F']_{x}^{s'}} \| g \|_{\tilde{B}[F']_{x}^{s'}} \sum_{k \geq j - 2} 2^{k (\sigma_{F_1} - \sigma_{F'})} \| \Delta_k f \|_{F_1} \| \Delta_k g \|_{F_1}.
\]

Putting together (4.7)-(4.9), and using (4.6), we conclude the proof.

Finally, we prove our bilinear estimate in the context of Besov-type spaces.

**Theorem 4.13.** Let $s \in \mathbb{R}$ and $F \in G$ verify (B1)-(B6). Then, we obtain (H1)-(H4) for the spaces $E = \dot{B}[F]_{x}^{s}$ and $E_0 = \dot{B}[F]_{x}^{-s_i}$. As a consequence, we get the bilinear estimate (3.7) with $E' = \dot{B}[F]_{x}^{-s_i}$.

**Proof.** The inclusion $\dot{B}[F]_{x}^{s_i} \subset S'$ follows by definition; moreover, a direct calculation shows that $\| f (\lambda \cdot) \|_{\dot{B}[F]_{x}^{s_i}} \approx \lambda^{-s_i} \| f \|_{\dot{B}[F]_{x}^{s}}$, and then we obtain (H1). The condition (H2) follows directly by using Corollary 4.5, and the condition (H3) follows by Lemma 4.12. Finally, note that Proposition 4.11 gives the estimate
\[
t^{(s_0 + s_F - (s + s_F))/2 - 1/2} \| \nabla U (t) f \|_{\dot{B}[F]_{x}^{s_0}} \leq t^{(s_i + s_F - (s + s_F))/2 - 1} \| f \|_{\dot{B}[F]_{x}^{s_i}},
\]

in other words, we have that
\[
t^{(s_E_0 - s_E)}/2 - 1/2 \| \nabla U (t) f \|_{E_0} \leq t^{(s_E_i - s_E)}/2 - 1 \| f \|_{E_i},
\]

with $E_i = \dot{B}[F]_{x}^{-s_i}, i = 1, 2$. Taking $-s_1 = -s + \epsilon$ and $-s_2 = -s - \epsilon$ for $0 < \epsilon \ll 1$, and defining $-s_i = 1/2 \sigma_{E_i} + 1$, we obtain the relations
\[
\begin{align*}
\frac{1}{z_1} &= \frac{1}{2} (-s + \sigma_F - (-s + \epsilon + \sigma_F)) + 1 = \frac{1}{2} \epsilon + 1 > 1, \\
0 < \frac{1}{z_2} &= \frac{1}{2} (-s + \sigma_F - (-s - \epsilon + \sigma_F)) + 1 = \frac{1}{2} (-\epsilon) + 1 < 1.
\end{align*}
\]
Therefore, $0 < z_1 < 1 < z_2 < \infty$ and, for $\theta = 1/2$, we have $1 = \frac{\theta}{z_1} + \frac{1-\theta}{z_2}$. Finally, since $-s = (1-\theta)(-s_1) + \theta(-s_2)$, by using Corollary 4.7, we get

$$E \leftrightarrow (E_1, E_2)_{\theta, 1} = E.$$  

The previous calculations show the conditions (H4')-(H4''). Condition (H4) follows by Lemma 3.6. Finally, the bilinear estimate (3.7) follows from Theorems 3.4 and 4.10.

\[\square\]

### 4.1 Applications

#### 4.1.1 Bilinear estimate in Besov-weak-Herz spaces

Using the previous results, we can obtain the bilinear estimate (3.7) for the Besov-weak-Herz space $E' = \tilde{B}\left[W\hat{K}_{p,\infty}^0\right]_{\tilde{p}-1}^\frac{\tilde{p}}{p}$, where $\frac{\tilde{p}}{2} < p < n$. For that, consider $F$ as the Lorentz-Herz space $\hat{K}_{p',1,1}^0$. The properties (B1)-(B4) follow by (a), (f), (g) and (b) in page 9. Moreover, taking $G_0 = W\hat{K}_{2p,\infty}^0$, $G_1 = W\hat{K}_{p,\infty}^0$, and $G_2 = W\hat{K}_{p/2,\infty}^0$ with $p < \tilde{p} < n$, and using the properties (d) and (e) in page 9, we obtain (B5) and (B6). Then, the bilinear estimate follows by Theorem 4.13. As far as we know, this estimate in $\tilde{B}\left[W\hat{K}_{p,\infty}^0\right]_{\tilde{p}-1}^\frac{\tilde{p}}{p}$ has not been previously obtained in the literature, at least we were unable to locate it. Furthermore, as pointed out in Introduction, using heat-semigroup estimates in Besov-Lorentz-Herz spaces (see [11]), estimate (3.7) in $\tilde{B}\left[W\hat{K}_{p,\infty}^0\right]_{\tilde{p}-1}^\frac{\tilde{p}}{p}$, and an argument by [31], one can obtain the uniqueness of mild solutions in $C([0,T] \times \tilde{Z})$ with $Z = \tilde{B}\left[W\hat{K}_{p,\infty}^0\right]_{\tilde{p}-1}^\frac{\tilde{p}}{p}$.

#### 4.1.2 Bilinear estimate in Besov and Besov-weak-$L^p$ spaces

In this subsection we reobtain the bilinear estimate in the Besov space $\tilde{B}_{p,\infty}^\frac{\tilde{p}}{p}$ found in [7]. Also, we obtain an estimate in the Besov-weak-$L^p$ space $\tilde{B}[L^{p,\infty}]_{\tilde{p}-1}^\frac{\tilde{p}}{p}$. In fact, for the Besov space $\tilde{B}_{p,\infty}^\frac{\tilde{p}}{p}$, it is sufficient to consider $F = L^{p'}$, $G_0 = L^{2p}$, $G_1 = L^{\tilde{p}}$, $G_2 = L^{\tilde{p}}$ and $E' = \tilde{B}_{p,\infty}^\frac{\tilde{p}}{p}$, where $\frac{n}{2} < p < \tilde{p} < n$. For the Besov-weak-$L^p$ space $\tilde{B}[L^{p,\infty}]_{\tilde{p}-1}^\frac{\tilde{p}}{p}$, it is sufficient to consider $F = L^{p',1}$, $G_0 = L^{2p,\infty}$, $G_1 = L^{\tilde{p},\infty}$, $G_2 = L^{\tilde{p},\infty}$ and $E' = \tilde{B}[L^{p,\infty}]_{\tilde{p}-1}^\frac{\tilde{p}}{p}$, where $\frac{n}{2} < p < \tilde{p} < n$. With these considerations, we can show (B1)-(B6) and then employ Theorem 4.13 in order to obtain the bilinear estimate (3.7) with $E' = \tilde{B}_{p,\infty}^\frac{\tilde{p}}{p}$ and $E' = \tilde{B}[L^{p,\infty}]_{\tilde{p}-1}^\frac{\tilde{p}}{p}$, according to the corresponding case. The details of checking for the properties (B1)-(B6) can be carry out as in the previous case of Besov-weak-Herz spaces and are left to the reader.

### 5 Bilinear estimate in $X$-Morrey spaces

This section is devoted to obtaining bilinear estimates in the framework of abstract Morrey spaces $\mathcal{M}[X]^f$. For our approach, we need to introduce and develop some core properties in block spaces based on Banach spaces, which are the preduals of $\mathcal{M}[X]^f$. For that, we need to assume some natural properties on the base space $X$. A set of equivalent conditions have already been considered in [28, Appendix A]. In fact, mainly in view of the block spaces, we assume a little more than [28], namely the property of passing the $X$-norm into the integral.
From now on, we denote by $D(a, r)$ the ball in $\mathbb{R}^n$ with center $a \in \mathbb{R}^n$ and radius $r > 0$. Also, for a set $A$, the symbol $1_A$ stands for the characteristic function of $A$.

Let us introduce $\mathcal{H} \subset \mathcal{G}$ as being the class of all Banach spaces $X$ of measurable functions such that

1. $X \hookrightarrow L^1_{loc}$;
2. If $f, g \in X$ and $g \in L^\infty$, then $\|fg\|_X \leq C \|g\|_{L^\infty} \|f\|_X$;
3. If $f \in X$ and $x_0 \in \mathbb{R}^n$, then $\tau_{x_0} f(x) = f(x - x_0) \in X$ and $\|\tau_{x_0} f\|_X \leq C \|f\|_X$;
4. For $g \in L^1_{loc}$ and $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $K(\cdot, y) \in X$, for every $y \in \mathbb{R}^n$, it follows that
   \[ \left\| \int_{\mathbb{R}^n} g(y) K(\cdot, y) dy \right\|_X \leq C \left\| g \right\|_X \|K(\cdot, y)\|_X dy; \]
5. Denoting by $D(0, 1)$ the unit ball in $\mathbb{R}^n$, we have that $1_D(0, 1) \in X$;
6. If $f \in L^1_{loc}$ is such that $f1_{D(x_0, R)} \in X$, for all $x_0 \in \mathbb{R}^n$ and $R > 0$, and $\sup_{x_0 \in \mathbb{R}^n, R > 0} \|f1_{D(x_0, R)}\|_X < \infty$, then $f \in X$ and
   \[ \|f\|_X \leq C \sup_{x_0 \in \mathbb{R}^n, R > 0} \|f1_{D(x_0, R)}\|_X. \]

Moreover, we say that $X \in \mathcal{H}$ if $X \in \mathcal{H}$ and the following condition (M7) is verified:

7. The Riesz transforms are bounded on $X$.

We list some consequences of conditions (M1)-(M6) in the following result. Related properties can be found in [28, Appendix A].

**Proposition 5.1.** Let $X \in \mathcal{H}$, then the following assertions hold true:

1. For all $x_0 \in \mathbb{R}^n$ and $R > 0$, we have $1_{D(x_0, R)} \in X$ and
   \[ \|1_{D(x_0, R)}\|_X \leq C R^{-\sigma x} \|1_D(0, 1)\|_X. \]
   In particular, it follows by (M2) that $1_A \in X$, for all bounded measurable set $A \subset \mathbb{R}^n$.
2. If $f \in X$, then
   \[ \sup_{x_0 \in \mathbb{R}^n, R > 0} \|f1_{D(x_0, R)}\|_X \leq C \|f\|_X. \]  \hspace{1cm} (5.1)
3. For all $x_0 \in \mathbb{R}^n$ and $R > 0$, we have
   \[ \int_{D(x_0, R)} |f(x)| dx \leq C \|f\|_X R^{n + \sigma x}. \]  \hspace{1cm} (5.2)
   Moreover, $\sigma_X = -n$ implies that $X = L^1$, provided that $\|\cdot\|_X$ is lower semicontinuous with respect to a.e. convergence, and $\sigma_X = 0$ implies $X = L^\infty$.  

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Proof. Item (i) follows directly from the identity $1_{D(x_0,R)} = \tau_{x_0} \left((1_{D(0,1)}^{-1})^{-1}\right)$, $X \in \mathcal{G}$, and conditions (M3), (M5). Since $1_{D(x_0,R)} \in L^\infty$, condition (M2) implies (ii). For item (iii), note that the continuous inclusion in (M1) implies that

$$\int_{D(0,1)} |f(x)|\,dx \leq C \|f\|_X.$$ 

Thus, changing variables and using (M3), we get (5.2). Now, assume that $\sigma_X = -n$, it is clear from (5.2) that $X \hookrightarrow L^1$. Moreover, we get from (M4) that

$$\left\| \int_{\mathbb{R}^n} f(y) \frac{1}{R^n} 1_{D(0,R)}(x-y) \,dy \right\|_X \leq \|f\|_{L^1},$$

for all $f \in L^1$. By taking the limit when $R \to 0$ and using the lower semicontinuity on the norm, we get that $X = L^1$ with equivalent norms. Now, if $\sigma_X = 0$, by item (iii) and the Lebesgue Differentiation Theorem, it follows that $X \hookrightarrow L^\infty$. On the other hand, given $f \in L^\infty$, (M2) implies that $f1_{D(x_0,R)} \in X$, for every $D(x_0,R)$, and

$$\|f1_{D(x_0,R)}\|_X \leq C\|f\|_{L^\infty} \|1_{D(x_0,R)}\|_X \leq C\|f\|_{L^\infty},$$

where above we have used item (i). Then, (M6) implies that $f \in X$ with $\|f\|_X \leq C\|f\|_{L^\infty}$.

\[ \end{proof} \]

Remark 5.2. Note that, in order to work with nontrivial spaces, estimate (5.2) implies that $-n \leq \sigma_X \leq 0$, for all $X \in \mathcal{H}$. Also, item (ii) in Proposition 5.1 and (M6) imply $\|f\|_X \approx C \sup_{x_0 \in \mathbb{R}^n, R > 0} \|1_{D(x_0,R)}f\|_X$ which was assumed by [28] as a hypothesis on $X$.

Next, we define Morrey spaces and their preduals based on abstract Banach spaces $X$. The aim is to ensure the requirements for applying the results developed in Section 3. The definition of generalized Morrey spaces in item (ii) below can be found in [28].

**Definition 5.3.** Let $X \in \mathcal{H}$ and $1 \leq l \leq \infty$.

(i) Assume that $\frac{n}{l} + \sigma_X \geq 0$. We say that a measurable function $b$ is a $(l,X)$-block if there exist $a \in \mathbb{R}^n$ and $\rho > 0$ such that $\text{supp}(b) \subset D(a,\rho)$ and

$$\rho^{\frac{n}{l}+\sigma_X} \|1_{D(a,\rho)}b\|_X \leq 1.$$

We define the $(l,X)$-block space, denoted by $\mathcal{PD}[X]^l$, as

$$\mathcal{PD}[X]^l = \left\{ h \in \mathcal{S}'; \exists \{\alpha_k\}_{k \in \mathbb{N}} \subset \mathbb{R}, \sum_{k=1}^{\infty} |\alpha_k| < \infty \text{ and } h = \sum_{k=1}^{\infty} \alpha_k b_k, \text{ a.e. in } \mathbb{R}^n, \text{ where } b_k \text{ is a } (l,X) \text{-block} \right\}.$$

(ii) Assume that $\frac{n}{l} + \sigma_X \leq 0$. Define the $X$-Morrey space $\mathcal{M}[X]^l$ as

$$\mathcal{M}[X]^l = \left\{ g \in \mathcal{S}'; \sup_{x \in \mathbb{R}^n, \rho > 0} \rho^{\frac{n}{l}+\sigma_X} \|1_{D(x,\rho)}g\|_X < \infty \right\}.$$
Following the ideas as the classic case $X = L^p(\mathbb{R}^n)$, it is possible to show that the space $\mathcal{P}D[X]^{\ell}$ is Banach with the norm $\|h\|_{\mathcal{P}D[X]^{\ell}} := \inf \left\{ \sum_k |\alpha_k| : h = \sum_{k=1}^{\infty} \alpha_kb_k \right\}$, where the infimum is taken over the sequences $(\alpha_k)_{k \in \mathbb{N}}$ such that $h = \sum_{k=1}^{\infty} \alpha_kb_k$. In the same way, it follows that the space $\mathcal{M}[X]^{\ell}$ is Banach with the norm $\|g\|_{\mathcal{M}[X]^{\ell}} = \sup_{x \in \mathbb{R}^n, \rho > 0} \rho^{\frac{\sigma}{\ell} + \sigma_X} \|1_{D(x, \rho)}g\|_{X}$.

**Lemma 5.4.** Let $X \in \mathcal{H}$, then $\mathcal{P}D[X]^{\ell} \subset S'$. Recall that, by the definition of $\mathcal{H}$, we have that $X \in \mathcal{G}$. Let $\lambda > 0$ fixed, if $b$ is a $(l, X)$-block, then $\frac{b \lambda}{C\lambda^{\frac{\sigma}{\ell}}} \in X$ is a $(l, X)$-block, where $C > 0$ is the constant given at the definition of the family $\mathcal{G}$ (see Definition 3.1). In fact, if $b$ is $(l, X)$-block, then there exist $a \in \mathbb{R}^n$ and $\rho > 0$ such that

$$\rho^{\frac{\sigma}{\ell} + \sigma_X} \|1_{D(a, \rho)}b\|_{X} \leq 1.$$ 

Using the identity $1_{D(x, r)}b_{\lambda} = \left(1_{D(x, r)} \left(\lambda^{-1} \cdot b\right)_{\lambda}\right) = \left(1_{D(\lambda x, \lambda r)}b\right)_{\lambda}$, we arrive at

$$\|1_{D(x, r)}b\|_{X} \leq C\lambda^{\sigma_X} \|1_{D(\lambda x, \lambda r)}b\|_{X}.$$ 

Now, defining $\bar{a} = a/\lambda$ and $\bar{\rho} = \rho/\lambda$, we have that

$$\rho^{\frac{\sigma}{\ell} + \sigma_X} \left\|1_{D(\bar{a}, \bar{\rho})}b_{\lambda}\right\|_{X} \leq C\rho^{\frac{\sigma}{\ell} + \sigma_X} \lambda^{\frac{\sigma}{\ell}} \left\|1_{D(\lambda \bar{a}, \lambda \bar{\rho})}b\right\|_{X} = \rho^{\frac{\sigma}{\ell} + \sigma_X} \|1_{D(a, \rho)}b\|_{X} \leq 1.$$ 

Thus, if $h = \sum_{k=1}^{\infty} \alpha_kb_k$, then $h_{\lambda} = \sum_{k=1}^{\infty} \left(\frac{C\lambda_{\frac{\sigma}{\ell}}\alpha_k}{\lambda^{\frac{\sigma}{\ell}}}b_{\lambda}\right)$, which gives $\|h\|_{\mathcal{P}D[X]^{\ell}} \leq C\lambda^{-\frac{\sigma}{\ell}} \|h\|_{\mathcal{P}D[X]^{\ell}}$, as desired.

In view of the previous proof, we can see that $\sigma_{\mathcal{P}D[X]^{\ell}} = -\frac{\sigma}{\ell}$. As expected, we recover the duality result between $(l, X)$-block spaces and their counterpart $(l', X')$-Morrey spaces.

**Theorem 5.5.** Let $1 \leq l \leq \infty$ and $X \in \mathcal{H}$ be such that $X' \in \mathcal{H}$ and $\frac{\sigma}{\ell} + \sigma_X \geq 0$. Then,

$$\left(\mathcal{P}D[X]^{\ell}\right)' = \mathcal{M}[X']^{\ell},$$

where $l'$ is the conjugate exponent of $l$.

**Proof.** Let $g \in \mathcal{M}[X']^{\ell}$, for $f \in \mathcal{P}D[X]^{\ell}$ we have that

$$\langle g, f \rangle \leq \sum_{k=1}^{\infty} |\alpha_k| \langle g, f_{\lambda} \rangle = \sum_{k=1}^{\infty} |\alpha_k| \langle g, 1_{D(a_k, \rho_k)}b_{\lambda} \rangle$$

$$\leq \sum_{k=1}^{\infty} |\alpha_k| \left\|1_{D(a_k, \rho_k)}g\right\|_{X'} \left\|1_{D(\alpha_k, \rho_k)}b_{\lambda}\right\|_{X}$$

$$= \sum_{k=1}^{\infty} |\alpha_k| \rho_k^{-\left(\frac{\sigma}{\ell} + \sigma_X\right)} \left\|1_{D(a_k, \rho_k)}g\right\|_{X'} \rho_k^{\frac{\sigma}{\ell} + \sigma_X} \left\|1_{D(a_k, \rho_k)}b_{\lambda}\right\|_{X}$$

$$\leq \sum_{k=1}^{\infty} |\alpha_k| \rho_k^{\frac{\sigma}{\ell} + \sigma_X'} \left\|1_{D(\alpha_k, \rho_k)}g\right\|_{X'} \|g\|_{\mathcal{M}[X']^{\ell}} \sum_{k=1}^{\infty} |\alpha_k|.$$
Taking the infimum in the right-hand side of the above inequality, we obtain
\[ |\langle g, f \rangle| \leq \|g\|_{\mathcal{M}[X']^\prime} \|f\|_{\mathcal{P}D[X]}, \]
which implies that
\[ \mathcal{M}[X']^\prime \hookrightarrow (\mathcal{P}D[X])^\prime. \]

Conversely, let \( \Psi \in (\mathcal{P}D[X])^\prime \). Note that, for every \( \rho > 0 \) and \( f \in X \), we have that \( \rho^{-\left(\frac{n}{p}+\sigma_X\right)}1_{D(0,\rho)}f \in \mathcal{P}D[X] \) with \( \|\rho^{-\left(\frac{n}{p}+\sigma_X\right)}1_{D(0,\rho)}f\|_{\mathcal{P}D[X]} \leq \|f\|_{X} \). Defining
\[ T_\rho(f) = \Psi \left( \rho^{-\left(\frac{n}{p}+\sigma_X\right)}1_{D(0,\rho)}f \right), \]
we get that \( T_\rho \in X' \hookrightarrow L^{1}_{loc} \), and then there exists \( g_\rho \in X' \) such that \( T_\rho(f) = \langle g_\rho, f \rangle \). Let \( (\rho_k)_{k \in \mathbb{N}} \) be such that \( \rho_k \to \infty \) and set \( g(x) = g_{\rho_k}(x) \) if \( x \in D(0, \rho_k) \). We intend to show that \( g \in \mathcal{M}[X']^\prime \). For this, let \( a \in \mathbb{R} \) and \( r > 0 \), and take \( k \) such that \( D(a, r) \subset D(0, \rho_k) \). Thus, we have that
\[ \rho^{\frac{n}{p}+\sigma_X'} \left\| 1_{D(a,r)}g \right\|_{X'} = \rho^{\frac{n}{p}+\sigma_X'} \sup_{|f|_X=1} |\langle 1_{D(a,r)}g, f \rangle| = \rho^{\frac{n}{p}+\sigma_X'} \sup_{|f|_X=1} |\langle g_{\rho_k}, 1_{D(a,r)}f \rangle| \]
\[ = \rho^{\frac{n}{p}+\sigma_X'} \sup_{|f|_X=1} |\langle \Psi, 1_{D(a,r)}f \rangle| \leq C \|\Psi\| \sup_{|f|_X=1} \|\rho^{-\left(\frac{n}{p}+\sigma_X\right)}1_{D(a,r)}f\|_{\mathcal{P}D[X]} \]
\[ \leq C \|\Psi\| \sup_{|f|_X=1} \|f\|_{X} \leq C \|\Psi\|. \]
From the above, we get that \( \|g\|_{\mathcal{M}[X']^\prime} \leq C \|\Psi\| \), and then \( (\mathcal{P}D[X])^\prime \hookrightarrow \mathcal{M}[X']^\prime \).

It is possible to show a Hölder-type inequality in Morrey spaces by assuming that the base spaces also satisfy a Hölder-type inequality.

**Lemma 5.6.** Let \( G, G_0, G_1 \in \mathcal{H} \) be such that \( f \in G_0 \) and \( g \in G_1 \) imply \( f \cdot g \in G \) with
\[ \|f \cdot g\|_{G} \leq C \|f\|_{G_0} \|g\|_{G_1}, \quad (5.4) \]
where the constant \( C > 0 \) is independent of \( f, g \). Assume also that \( \frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1} \). Then, for every \( f \in \mathcal{M}[G_0] \) and \( g \in \mathcal{M}[G_1] \), we have that \( f \cdot g \in \mathcal{M}[G] \) with the Hölder-type inequality
\[ \|f \cdot g\|_{\mathcal{M}[G]} \leq C \|f\|_{\mathcal{M}[G_0]} \|g\|_{\mathcal{M}[G_1]}, \]
where \( C \) is the same constant of \( (5.4) \).

**Proof.** First, by Remark 3.2, recall that \( \sigma_G = \sigma_{G_0} + \sigma_{G_1} \). Since \( 1_{D(x_0,\rho_0)}f g = 1_{D(x_0,\rho_0)}f 1_{D(x_0,\rho_0)}g \), we have that
\[ \|1_{D(x_0,\rho_0)}f g\|_{G} \leq C \|1_{D(x_0,\rho_0)}f\|_{G_0} \|1_{D(x_0,\rho_0)}g\|_{G_1}, \]
which leads us to
\[ \rho_0^{\frac{n}{p}+\sigma_G} \|1_{D(x_0,\rho_0)}f g\|_{G} \leq C \rho_0^{\frac{n}{p}+\sigma_{G_0}} \|1_{D(x_0,\rho_0)}f\|_{G_0} \rho_0^{\frac{n}{p}+\sigma_{G_1}} \|1_{D(x_0,\rho_0)}g\|_{G_1} \]
\[ \leq C \|f\|_{\mathcal{M}[G_0]} \|g\|_{\mathcal{M}[G_1]}, \]
The result follows by taking the supremum in the left-hand side of the above inequality.

The result below is due to Lemarie-Rieusset [28, Proposition B.1.].
Lemma 5.7. Let \( G \in \mathcal{H} \) and \( 1 \leq l \leq \infty \) be such that \( \frac{7}{4} + \sigma_G \leq 0 \). Then, the Riesz transforms are bounded operators on \( \mathcal{M} \left[ G \right] \).

The subject of the next lemma is a basic estimate for convolution that allows us to get, in particular, a heat semigroup inequality.

Lemma 5.8. Let \( 1 \leq l \leq \infty \) and \( G \in \mathcal{H} \) be such that \( \frac{7}{4} + \sigma_G \geq 0 \), and let \( b \) be a \((l,G)\)-block with \( \text{supp}(b) \subset D(x_0, \rho_0) \). Then, we have that

\[
\| \psi \ast b \|_{L^\infty} \leq C,
\]

for every radially symmetric \( \psi \in \mathcal{S} \), where \( C = C(\psi) > 0 \) is independent of \( x_0 \) and \( \rho_0 \).

**Proof.** Defining \( \mu (\rho) = \int_{D(0,\rho)} b(x) dx \), we proceed as follows

\[
(\psi \ast b) (0) = \int_{\mathbb{R}^n} \psi (y) b(y) dy = \int_0^\infty \psi (\rho) d\mu (\rho) = - \int_0^\infty \frac{\partial \psi (\rho)}{\partial \rho} \mu (\rho) d\rho.
\]

Moreover, by item \((iii)\) of Proposition 5.1, we obtain that

\[
\mu (\rho) = \int_{D(0,\rho)} 1_{D(x_0,\rho_0)} (x) b(x) dx = \int_{D(0,\rho) \cap D(x_0,\rho_0)} b(x) dx \leq C \min \left\{ \rho^{n+\sigma_G}, \rho_0^{n+\sigma_G} \right\} \| 1_{D(x_0,\rho_0)} b \|_{G^0} \leq C \min \left\{ \rho^{n+\sigma_G} \rho_0^{-\left(\frac{7}{4}+\sigma_G\right)}, \rho_0^{n-\frac{7}{4}} \right\}.
\]

Thus,

\[
|(\psi \ast b) (0)| \leq C \int_0^\infty \left| \frac{\partial \psi (\rho)}{\partial \rho} \right| \min \left\{ \rho^{n+\sigma_G} \rho_0^{-\left(\frac{7}{4}+\sigma_G\right)}, \rho_0^{n-\frac{7}{4}} \right\} d\rho
\]

\[
= C \int_0^{\rho_0} \left| \frac{\partial \psi (\rho)}{\partial \rho} \right| \rho^{n+\sigma_G} \rho_0^{-\left(\frac{7}{4}+\sigma_G\right)} d\rho + C \int_{\rho_0}^\infty \left| \frac{\partial \psi (\rho)}{\partial \rho} \right| \rho_0^{n-\frac{7}{4}} d\rho
\]

\[
\leq C \int_0^{\rho_0} \left| \frac{\partial \psi (\rho)}{\partial \rho} \right| \rho^{n-\frac{7}{4}} d\rho + C \int_{\rho_0}^\infty \left| \frac{\partial \psi (\rho)}{\partial \rho} \right| \rho_0^{n-\frac{7}{4}} d\rho
\]

\[
\leq C (\psi).
\]

For \( x \in \mathbb{R}^n \), note that \( (\psi \ast b) (x) = \left( \psi \ast \tau_x \hat{b} \right) (0) \). Then, using \((M3)\) and proceeding as before, we get

\[
|(\psi \ast b) (x)| \leq C. \tag{5.5}
\]

Now we can complete the proof by taking the supremum over \( x \in \mathbb{R}^n \) in (5.5).

\( \diamond \)

As a consequence of the previous lemma, we have the following estimate in \((l,X)\)-block spaces.

**Corollary 5.9.** Let \( \psi \in \mathcal{S} \) be radially symmetric and \( 1 \leq l \leq \infty \). Then, there exists a constant \( C > 0 \) depending on \( \psi \) such that

\[
\| \psi \ast f \|_{L^\infty} \leq C \| f \|_{\mathcal{PD} \left[ X \right]}^{l},
\]

for all \( f \in \mathcal{PD} \left[ X \right] \).

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Proof. We have that \( \psi * f = \sum_{k=1}^{\infty} \alpha_k \psi * b_k \), where \( f = \sum_{k=1}^{\infty} \alpha_k b_k \) is a decomposition in \((l, X)\)-blocks for \( f \). Using Lemma 5.8, we can estimate

\[
\| \psi * f \|_{L^\infty} \leq \sum_{k=1}^{\infty} |\alpha_k| \| \psi * b_k \|_{L^\infty} \leq C \sum_{k=1}^{\infty} |\alpha_k|,
\]

which gives the desired estimate after taking the infimum over all the decompositions of \( f \).

In the next two lemmas, we show the basic convolution estimates in \( X \)-Morrey spaces useful for our purposes.

**Lemma 5.10.** Let \( \psi \in \mathcal{S} \) be radially symmetric, \( G \in \mathcal{H} \) and \( 1 \leq l \leq \infty \). Then, there exists a constant \( C > 0 \) depending on \( \psi \) such that

\[
\| \psi * f \|_{L^\infty} \leq C \| f \|_{M[G]^l},
\]

(5.6)

for all \( f \in M[G]^l \).

**Proof.** Proceeding as in the proof of Lemma 5.8, we have that

\[
(\psi * f) (0) = - \int_0^\infty \frac{\partial \psi (\rho)}{\partial \rho} \mu (\rho) \, d\rho,
\]

where \( \mu (\rho) = \int_{D(0, \rho)} f(x) \, dx \). Again, by Proposition 5.1 (iii), it follows that

\[
|\mu (\rho)| \leq C \| 1_{D(0, \rho)} f \|_{L^1} \rho^{n+\sigma G} = C \rho^{n+\sigma G} \rho^{- (n+\sigma G)} \rho^{n+\sigma G} \| 1_{D(0, \rho)} f \|_{L^1} \leq C \rho^{\sigma G} \| f \|_{M[G]^l}.
\]

Therefore,

\[
| (\psi * f) (0) | \leq C \int_0^\infty \left| \frac{\partial \psi (\rho)}{\partial \rho} \right| \rho^{\sigma G} \| f \|_{M[G]^l} \, d\rho
\]

\[
\leq C \| f \|_{M[G]^l} \int_0^\infty \left| \frac{\partial \psi (\rho)}{\partial \rho} \right| \rho^{n} \, d\rho
\]

\[
\leq C \| f \|_{M[G]^l}.
\]

For \( x \in \mathbb{R}^n \), noting \( (\psi * f) (x) = (\psi * \tau_x f) (0) \) and using (M3), we arrive at

\[
| (\psi * f) (x) | \leq C \| f \|_{M[G]^l},
\]

which gives (5.6) by taking the supremum over \( x \in \mathbb{R}^n \).

**Lemma 5.11.** Let \( G \in \mathcal{H} \) and \( 1 \leq l \leq \infty \). Then

\[
\| \psi * g \|_{M[G]^l} \leq C \| \psi \|_{L^1} \| g \|_{M[G]^l},
\]

(5.7)

for all \( g \in M[G]^l \) and \( \psi \in L^1 \), where \( C > 0 \) is a universal constant.
Proof. For \( x_0 \in \mathbb{R}^n \) and \( \rho_0 > 0 \), we can express
\[
1_{D(x_0, \rho_0)}(x) (\psi * g)(x) = \int_{\mathbb{R}^n} 1_{D(x_0, \rho_0)}(x) \psi(y) g(x - y) \, dy = \int_{\mathbb{R}^n} \psi(y) 1_{D(x_0 - y, \rho_0)}(x - y) g(x - y) \, dy.
\]
Thus, condition (M4) with \( K(x, y) = 1_{D(x_0 - y, \rho_0)}(x - y) g(x - y) \) implies that
\[
\left\| 1_{D(x_0, \rho_0)}(\psi * g) \right\|_G \leq \int_{\mathbb{R}^n} |\psi(y)| \left\| \tau_y (1_{D(x_0 - y, \rho_0)}g) \right\|_G \, dy \leq C \int_{\mathbb{R}^n} |\psi(y)| \left\| 1_{D(x_0 - y, \rho_0)}g \right\|_G \, dy
\]
\[
\leq C \rho_0^{-(\frac{n}{2} + \sigma_G)} \left\| \psi \right\|_{L^1} \left\| g \right\|_{\mathcal{M}[G]^l},
\]
and then
\[
\rho_0^{\frac{n}{2} + \sigma_G} \left\| 1_{D(x_0, \rho_0)}(\psi * g) \right\|_G \leq C \left\| \psi \right\|_{L^1} \left\| g \right\|_{\mathcal{M}[G]^l}.
\]
Taking the supremum over \( x_0 \in \mathbb{R}^n \) and \( \rho_0 > 0 \), we are done. \( \diamond \)

From duality and the previous lemma, we obtain the corollary below.

**Corollary 5.12.** Let \( X \in \mathcal{H} \) be such that \( X' \in \mathcal{H} \) and \( 1 \leq l \leq \infty \). Then
\[
\left\| \psi * f \right\|_{\mathcal{P}\mathcal{D}[X]^l} \leq C \left\| \psi \right\|_{L^1} \left\| f \right\|_{\mathcal{P}\mathcal{D}[X]^l},
\]
for all \( f \in \mathcal{P}\mathcal{D}[X]^l \) and \( \psi \in L^1 \), where \( C > 0 \) is a universal constant.

**Proof.** By (5.3) and the canonical embedding into the bidual, we have the linear isometry (not surjective)
\( \mathcal{P}\mathcal{D}[X]^l \hookrightarrow (\mathcal{M}[X]^l)' \). Using it and estimate (5.7), we proceed as follows
\[
\left\| \psi * f \right\|_{\mathcal{P}\mathcal{D}[X]^l} = \sup_{\left\| g \right\|_{\mathcal{M}[X]^l} = 1} |\langle \psi * f, g \rangle| = \sup_{\left\| g \right\|_{\mathcal{M}[X]^l} = 1} |\langle f, \hat{\psi} * g \rangle| \leq \left\| f \right\|_{\mathcal{P}\mathcal{D}[X]^l} \sup_{\left\| g \right\|_{\mathcal{M}[X]^l} = 1} \left\| \psi * g \right\|_{\mathcal{M}[X]^l} \leq C \left\| \psi \right\|_{L^1} \left\| f \right\|_{\mathcal{P}\mathcal{D}[X]^l} \sup_{\left\| g \right\|_{\mathcal{M}[X]^l} = 1} \left\| g \right\|_{\mathcal{M}[X]^l} \leq C \left\| \psi \right\|_{L^1} \left\| f \right\|_{\mathcal{P}\mathcal{D}[X]^l},
\]
as required. \( \diamond \)

Now, using interpolation theory, we can extend Lemma 5.11 as follows.

**Lemma 5.13.** Let \( \psi \in \mathcal{S} \) be radially symmetric and \( N, Q \in \mathcal{H} \). Suppose that \( (L^\infty, N)_{\theta,d} \hookrightarrow Q \) with \( \sigma_Q = \theta \sigma_N \) for some \( 0 < \theta < 1 \) and \( 1 \leq l, d \leq \infty \). Then, there exists a constant \( C > 0 \) depending on \( \psi \) such that
\[
\left\| \psi * f \right\|_{\mathcal{M}[Q]^l_{\theta}} \leq C \left\| f \right\|_{\mathcal{M}[N]^l},
\]
for all \( f \in \mathcal{M}[N]^l \).
\textbf{Proof.} First, it follows from Lemma 5.10 that
\[
\|1_{D(x_0,p_0)} \psi * f\|_{L^\infty} \leq C_1(\psi) \|f\|_{\mathcal{M}[N]^t}.
\]  
(5.8)

On the other hand, by Lemma 5.11, we know that
\[
\|1_{D(x_0,p_0)} \psi * f\|_N \leq p_0^{-\frac{m}{N} + \sigma_N} C_2 \|\psi\|_{L^1} \|f\|_{\mathcal{M}[N]^t}.
\]
(5.9)

Using (5.8)-(5.9) and interpolation, we arrive at
\[
\|1_{D(x_0,p_0)} \psi * f\|_Q \leq C \|1_{D(x_0,p_0)} \psi * f\|_{(L^\infty)_\theta,d} \leq p_0^{-\frac{\theta}{N} + \sigma_N} C_3(\psi) \|f\|_{\mathcal{M}[N]^t},
\]
which yields the desired estimate after multiplying both sides by $p_0^{-\theta} \frac{\sigma_N}{N}$ and taking the supremum over $x_0 \in \mathbb{R}^n$ and $p_0 > 0$.

Recall that $U(t)$ denotes the heat semigroup, which is defined for $t > 0$ as $U(t)f = \exp(t\Delta)f = \Phi(t) * f$, for each $f \in \mathcal{S}'$, where $\Phi(x,t) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$ is the heat kernel. For each $m \in \mathbb{N}_0$, it follows that
\[
|\langle \nabla^m \Phi \rangle (x,t)| \leq h(|x|,t),
\]
where $h(x,t) = t^{-\frac{m+n}{2}} \tilde{h} \left( \frac{x}{\sqrt{t}} , 1 \right)$, $\tilde{h}(\rho,t) \in \mathcal{S}(\mathbb{R})$ and $\tilde{h}(\rho,t) = t^{-\frac{m+n+1}{2}} (\tilde{\psi}_\rho h) \left( \frac{\rho}{\sqrt{t}}, 1 \right)$, for all $t > 0$.

We are in position to present our estimates of the heat semigroup in the context of $X$-blocks and $X$-Morrey spaces.

\textbf{Corollary 5.14.} Let $N, Q \in \mathcal{H}$ and $m \in \mathbb{N}_0$. Suppose that $(L^\infty,N)_{\theta,d} \hookrightarrow Q$ and $\sigma_Q = \theta \sigma_N$ for some $0 < \theta < 1$ and $1 \leq l, d \leq \infty$. Then,
\[
\|\nabla^m U(t)f\|_{\mathcal{M}[Q]^t} \leq C t^{-\frac{1}{2} \left( \frac{n}{2} - \theta \frac{n}{2} \right) - \frac{m}{2}} \|f\|_{\mathcal{M}[N]^t},
\]
for all $f \in \mathcal{M}[N]^t$, where the constant $C > 0$ is independent of $t > 0$.

\textbf{Proof.} First, we have the inequalities
\[
|\langle \nabla^m U(t)f \rangle (\cdot)| \leq (\tilde{h} (\cdot, t) * |f|) (\cdot) \leq t^{-\frac{m+n}{2}} \left( \tilde{h} \left( t^{-\frac{1}{2}}, 1 \right) * |f| \right) (\cdot) = t^{-\frac{m+n}{2}} \left( \tilde{h} (\cdot, 1) * |f| \left( t^{\frac{1}{2}} \right) \right) \left( t^{-\frac{1}{2}} \right).
\]

So, Lemma 5.13 yields
\[
\|\nabla^m U(t)f\|_{\mathcal{M}[Q]^t} \leq \left\| t^{-\frac{m+n}{2}} \left( \tilde{h} (\cdot, 1) * |f| \left( t^{\frac{1}{2}} \right) \right) \left( t^{-\frac{1}{2}} \right) \right\|_{\mathcal{M}[Q]^t} \leq C t^{-\frac{m+n}{2}} \left\| \tilde{h} (\cdot, 1) * |f| \left( t^{\frac{1}{2}} \right) \right\|_{\mathcal{M}[Q]^t} \leq C t^{-\frac{m+n}{2}} \left\| f \left( t^{\frac{1}{2}} \right) \right\|_{\mathcal{M}[N]^t} \leq C t^{-\frac{m+n}{2}} \left( 1 - \theta \right) \|f\|_{\mathcal{M}[N]^t}.
\]

\[\diamondsuit\]
Corollary 5.15. Let \( m \in \mathbb{N}_0 \) and let \( F, G \in \mathcal{H} \) be such that \( F', G' \in \mathcal{H} \), and suppose that \( (G', L^\infty)_{\theta,d} \hookrightarrow F' \) and \( \sigma_{F'} = \theta \sigma_{G'} \), for some \( 0 < \theta < 1 \) and \( 1 \leq l, d \leq \infty \). Then, for \( \bar{l} = \theta/[1 - (1 - \theta)l] \) we have that

\[
\| \nabla^m U(t)f \|_{\mathcal{P}D[G]^\bar{l}} \leq Ct^{-\frac{1}{2} \left( \frac{m}{2} - (\frac{n}{2} - \frac{q}{2}) \right) - \frac{m}{2}} \| f \|_{\mathcal{P}D[F]} ,
\]

for all \( f \in \mathcal{P}D[F]^\bar{l} \), where the constant \( C > 0 \) is independent of \( t > 0 \).

**Proof.** Using duality, the previous lemma, and noting that \( l' = \bar{l}/\theta \), we can estimate

\[
\| \nabla^m U(t)f \|_{\mathcal{P}D[G]^\bar{l}} = \sup \| \langle \nabla^m U(t)f, g \rangle \|_{\mathcal{P}D[F]^\bar{l}} \\
= \sup \| \langle f, \nabla^m U(t)g \rangle \|_{\mathcal{P}D[F]^\bar{l}} \\
\leq \sup \| f \|_{\mathcal{P}D[F]^\bar{l}} \| \nabla^m U(t)g \|_{\mathcal{M}[G']^p} \\
\leq C \sup \| f \|_{\mathcal{P}D[F]^\bar{l}} t^{-\frac{1}{2} \left( \frac{m}{2} - (\frac{n}{2} - \frac{q}{2}) \right) - \frac{m}{2}} \| g \|_{\mathcal{M}[G']^p} \leq Ct^{-\frac{1}{2} \left( \frac{m}{2} - (\frac{n}{2} - \frac{q}{2}) \right) - \frac{m}{2}} \| f \|_{\mathcal{P}D[F]^\bar{l}} .
\]

The following interpolation result will be useful for our ends.

Lemma 5.16. Let \( X, X_1, X_2 \in \mathcal{H} \) be such that \( \sigma_X = (1 - \theta)\sigma_{X_1} + \theta \sigma_{X_2} \) and \( X \hookrightarrow (X_1, X_2)_{\theta,d} \) for some \( \theta \in (0, 1) \) and \( d \in [1, \infty] \). If \( F_1 \) and \( F_2 \) are Banach spaces and \( T \) is a continuous linear operator such that

\[
T : \mathcal{P}D[X]^1 \longrightarrow F_1 \quad \text{and} \quad T : \mathcal{P}D[X]^2 \longrightarrow F_2 ,
\]

with the operator norms \( C_1 \) and \( C_2 \), respectively. Then, for \( \frac{1}{2} = \frac{1 - \theta}{l_1} + \frac{\theta}{l_2} \) we have

\[
T : \mathcal{P}D[X]^1 \longrightarrow (F_1, F_2)_{\theta,d} ,
\]

with the operator norm bounded by \( \tilde{C} = (C_1)^{1-\theta} (C_2)^{\theta} \).

**Proof.** Note that, for every ball \( D(x, \rho) \), we can define

\[
T_{(x, \rho)} : X_1 \longrightarrow F_1 , \quad f \longrightarrow T (1_{D(x, \rho)}f) ; \quad T_{(x, \rho)} : X_2 \longrightarrow F_2 , \quad f \longrightarrow T (1_{D(x, \rho)}f) ,
\]

with norms \( C_1 \leq C \rho^{\frac{n}{2} + \sigma_{X_1}} \) and \( C_2 \leq C \rho^{\frac{n}{2} + \sigma_{X_2}} \). Thus,

\[
T_{(x, \rho)} : X \longrightarrow (F_1, F_2)_{\theta,d} ,
\]

with norm \( C \leq C_1^{\theta} C_2^{1-\theta} \leq C \rho^{(1-\theta)\left(\frac{n}{2} + \sigma_{X_1}\right) + \theta\left(\frac{n}{2} + \sigma_{X_2}\right)} \leq C \rho^{\frac{n}{2} + \sigma} \). For every block \( b \) with \( \text{supp} b \subset D(x, \rho) \), we have that

\[
\| T (1_{D(x, \rho)}b) \|_{(F_1, F_2)_{\theta,d}} = \| T_{(x, \rho)}(1_{D(x, \rho)}b) \|_{(F_1, F_2)_{\theta,d}} \leq C \rho^{\frac{n}{2} + \sigma} \| 1_{D(x, \rho)}b \|_X \leq C .
\]

Now, consider the space

\[
\mathcal{L}D[X]^l = \left\{ f \in \mathcal{P}D[X]^l : f = \sum_{k=1}^m \alpha_k b_k \text{ where } m \in \mathbb{N} \text{ and } b_k \text{ is an } (l, X) \text{-block} \right\} .
\]
Then, for \( f \in \mathcal{LD}[X]^l \) we obtain
\[
\|T(f)\|_{(F_1,F_2)_{\theta,d}} \leq \sum_{k=1}^{m} |\alpha_k| \|T(1_D(x_k,\rho_k)b_k)\|_{(F_1,F_2)_{\theta,d}} \leq C \sum_{k=1}^{m} |\alpha_k| \leq C \|f\|_{\mathcal{PD}[X]^l}.
\]
Using that \( \mathcal{LD}[X]^l \) is dense in \( \mathcal{PD}[X]^l \), we can conclude the proof.

\[ \diamond \]

**Remark 5.17.** Note that in the conditions of the previous lemma we have that
\[
\mathcal{PD}[X]^l \hookrightarrow \left( \mathcal{PD}[X]^{l_1}, \mathcal{PD}[X]^{l_2} \right)_{\theta,d}.
\]

With the above properties in hand, we are in a position to show the bilinear estimate for the Navier-Stokes equations in \( X \)-Morrey spaces.

**Theorem 5.18.** Let \( 1 \leq l < l_0 < \infty \) and \( X,0 \subset \mathcal{H} \) be such that \( X_0',X' \subset \mathcal{H} \), \( \sigma_X < \sigma_{X_0} \) and \( l'_0 \sigma_{X_0} = l' \sigma_{X'} \). Assume the following conditions:

**\( \text{(i)} \)** There exist Banach spaces \( X_1, X_2 \subset \mathcal{H} \) such that
\[
\theta_1 < \frac{l'_0}{l_1} < \theta_2 < \min\left\{ \frac{l'_0}{l_1} + 2 \frac{l'_0}{l_1} \frac{1}{n} \right\}, \tag{5.11}
\]
where \( \frac{\sigma_{X_i}}{\sigma_{X_0}} = \theta_i \), \( i = 1,2 \).

**\( \text{(ii)} \)** For \( \theta \in (0,1) \) such that \( \frac{l'_0}{l_1} = \theta \theta_2 + (1 - \theta) \theta_1 \), it follows that \( X \subset (X_1, X_2)_{\theta,1} \) and \( (L^\infty, X_0')_{\theta,d_i} \subset X_1', \) for some \( 1 \leq d_i \leq \infty \), \( i = 1,2 \).

**\( \text{(iii)} \)** \( X_0' \subset \mathcal{H} \) and \( \|fg\|_{X_0'} \leq C \|f\|_{X'} \|g\|_{X'} \).

Then, under \( \text{(i)} \) and \( \text{(ii)} \), for \( E = \mathcal{PD}[X]^l \) and \( E_0 = \mathcal{PD}[X_0]^l \), conditions \( \text{(H1)} \) and \( \text{(H4)} \) hold true. Moreover, if we assume \( \text{(iii)} \), we obtain \( \text{(H2)} \) and \( \text{(H3)} \). In particular, for \( l' = n \), we obtain the bilinear estimate (3.7) in the space \( E' = \mathcal{M}[X]^n \).

**Proof.** First, we assume \( \text{(i)} \) and \( \text{(ii)} \). Condition \( \text{(H1)} \) is a direct consequence of Lemma 5.4. To show \( \text{(H4)} \), we need only to check the conditions in Lemma 3.6. In fact, define \( l'_i = \frac{l'_0}{l_1} \) and \( E_i = \mathcal{PD}[X_i]^l \), for \( i = 1,2 \). Using Corollary 5.15 with \( G = X_0, F = X_i \) and \( \theta = \theta_i \), by item \( \text{(ii)} \) we get

\[
\|\nabla U(t)f\|_{\mathcal{PD}[X_0]^l_{\theta}} \leq C l_{\frac{n}{2}} \left( \frac{\theta}{\theta_2} \right)^{-\frac{n}{2}} \|f\|_{\mathcal{PD}[X_i]^l_{\theta}}, \text{ for } i = 1,2,
\]
which gives \( \text{(H4)} \) by observing that \( \sigma_{E_0} = -\frac{n}{l_0} \) and \( \sigma_{E_i} = -\frac{n}{l_i} \).

Next, by taking \( \frac{1}{z_2} = \frac{1}{2} \left( \frac{n}{l_i} - \frac{n}{l'_i} \right) + 1 \), inequalities in (5.11) imply that \( \frac{1}{z_2} > 1 \) and \( 0 < \frac{1}{z_2} < 1 \). Moreover, by the choice of \( \theta \), we arrive at
\[
\frac{\theta}{z_2} + \frac{1 - \theta}{z_1} = \frac{n}{2} \left( \frac{1}{l'_0} - \frac{\theta}{l_0} - \frac{1 - \theta}{l'_0} \right) + 1 = \frac{n}{2} \left( \frac{1}{l'_0} - \frac{\theta \theta_2 + (1 - \theta) \theta_1}{l'_0} \right) + 1 = 1,
\]

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and then we obtain \((H4^*)\).

In order to show \((H4^*)\), we first note that the choice of \(\theta\) in item \((ii)\) is equivalent to \(\sigma_X = \theta \sigma_{X_2} + (1 - \theta) \sigma_{X_1}\). Then, Lemma 5.16 and Remark 5.17 give

\[
\mathcal{PD}[X] \overset{\text{L}}{\hookrightarrow} \left( \mathcal{PD}[X_1]^{l_1}, \mathcal{PD}[X_2]^{l_2} \right)_{\theta, 1},
\]

as required. Since we have showed \((H4^*)-\text{(H4)}\), condition \((H4)\) follows from Lemma 3.6.

Assuming now \((iii)\), a direct application of Lemma 5.7 yields \((H2)\). On the other hand, item \((iii)\) implies that \(\frac{l'}{l_0} = \frac{\sigma_X}{\sigma_{X'}} = 2\), and thus Lemma 5.6 implies \((H3)\).

Finally, if \(l' = n\), the conditions in Theorem 3.4 are fulfilled by using the spaces \(E = \mathcal{PD}[X]^l\) and \(E_0 = \mathcal{PD}[X_0]^{l_0}\), and then we obtain \((3.7)\) with \(E' = \mathcal{M}[X]^n\), as desired.

\(\bigcirc\)

Remark 5.19. Note that the condition \((iii)\) in Theorem 5.18 and Remark 5.2 imply that \(\sigma_{X'} > -\frac{n}{2}\).

5.1 Application (bilinear estimate in weak-Morrey spaces)

We wish to apply our result for the weak-Morrey spaces \(\mathcal{M}^{n^p}_{(p', \infty)} = \mathcal{M} \left[ L^{(p', \infty)}_i \right] \) and obtain the corresponding bilinear estimate in \([10, 28]\). For that, take \(p, p_0, l, l_0\) such that \(l' = n, l_0' = n/2, 2 < p' \leq n\) and \(p_0' = p' / 2\), and consider \(X = L^{(p, 1)}, X_0 = L^{(p_0, 1)}\). Note that we can take \(l_1, l_2, p_1, p_2\) satisfying

\[
1 < p_1 < p < p_2 < p_0, \quad p \leq l, \quad \frac{l'}{l_0'} = \frac{p'}{p_1'}, \quad \text{for } i = 1, 2, \text{ and } \frac{1}{p} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_2} \text{ for some } \theta \in (0, 1).
\]

Considering \(X_1 = L^{(p_1, 1)}\) and \(X_2 = L^{(p_2, 1)}\), it follows that \(L^{(p_1, 1)} = (L^{(p_1, 1)}, L^{(p_2, 1)})_{\theta, 1}\). Moreover, we have that

\[
\left( L^{(p_1', \infty)}, L^{(p_2', \infty)} \right)_{\frac{p'}{p_1'}, \frac{p'}{p_2'}} = L^{(p', \infty)}_i \quad \text{and} \quad \left( L^{(p_0', \infty)}, L^{(\infty)}_0 \right)_{\frac{p_0'}{p_1'}, \frac{p_0'}{p_2'}} = L^{(p', \infty)}.
\]

Thus, we obtain the properties \((i)-(iii)\) in Theorem 5.18 for the spaces \(X_0, X_1\) and then the bilinear estimate \((3.7)\) with \(E' = \mathcal{M}^{n^p}_{(p', \infty)}\), as desired.

6 Bilinear estimate in Besov-X-Morrey spaces

In this part we combine some ideas from Morrey and Besov-type spaces in order to obtain the bilinear estimate \((3.7)\) in Besov-Morrey-type spaces.

Lemma 6.1. Let \(X \in \mathcal{H}\) be such that \(X' \in \mathcal{H}\), and suppose that:

\(\text{(BM1)}\) There exist \(H_0 \in \mathcal{H}\) such that \(\|fg\|_{X'} \leq C \|f\|_{H_0} \|g\|_{H_0}\). Moreover, \((L^\infty, X')_{\frac{1}{2}, d_0} \hookrightarrow H_0\) for some \(1 \leq d_0 \leq \infty\).

\(\text{(BM2)}\) There exist \(H_1\) and \(H_2\) such that \(\|fg\|_{H_2} \leq C \|f\|_{H_1} \|g\|_{H_1}\). Moreover, \(\sigma_{X'} = \theta \sigma_{H_2}\) and \((L^\infty, H_2)_{\theta, d_2} \hookrightarrow X'\) with \(\frac{1}{2} < \theta < 1\) and \(1 \leq d_2 \leq \infty\). Additionally, suppose that \((L^\infty, X')_{\frac{1}{2^q}, d_1} \hookrightarrow H_1\) with \(1 \leq d_1 \leq \infty\).

Then, we obtain the condition \((\mathcal{B}5)\) for the spaces \(F = \mathcal{PD}[X]^l, G_0 = \mathcal{M}[H_0]^{2l'}, G_1 = \mathcal{M}[H_1]^{2l_1}\) and \(G_2 = \mathcal{M}[H_2]^{l_2}\), where \(l\) satisfies \(\frac{q}{4} + \sigma_X \geq 0\) and \(l' \geq \frac{1}{\theta}\), and \(d_2 = 0l'\).
Proof. Using (BM1) and Lemma 5.6, we get
\[ \|fg\|_{\mathcal{M}[X']^r} \leq C \|f\|_{\mathcal{M}[H_0]^{2r}} \|g\|_{\mathcal{M}[H_0]^{2r}}. \]
Moreover, using again (BM1) and Lemma 5.13, we arrive at
\[ \|\Delta_0 f\|_{\mathcal{M}[H_0]^{2r}} \leq C \|f\|_{\mathcal{M}[X']^r}. \]

Now, condition (BM2) and Lemma 5.6 yield
\[ \|fg\|_{\mathcal{M}[H_2]^{2r}} \leq C \|f\|_{\mathcal{M}[H_1]^{2r}} \|g\|_{\mathcal{M}[H_1]^{2r}}. \]
In turn, using again (BM2) and Lemma 5.13, it follows that
\[ \|\Delta_0 f\|_{\mathcal{M}[X']^r} \leq C \|f\|_{\mathcal{M}[H_2]^{2r}} \quad \text{and} \quad \|\Delta_0 f\|_{\mathcal{M}[H_1]^{2r}} \leq C \|f\|_{\mathcal{M}[X']^r}. \]

In the sequel we prove the bilinear estimate in the case of Besov-Morrey-type spaces.

**Theorem 6.2.** Let \( \frac{n}{2} < l' < n, \ s = 1 - \frac{n}{2p} \) and let \( X \in \mathcal{H} \) be such that \( X' \in \mathcal{H} \). Moreover, assume (BM1) and (BM2). Then, we obtain the conditions (B1)-(B6) for the space \( F = \mathcal{PD}[X]' \). As a consequence, the bilinear estimate (3.7) with \( E' = \hat{B}\mathcal{M}[X']_{\infty}^{l',\frac{p}{p}-1} = \hat{B}[\mathcal{M}[X']_{l'}^{l}]^{\frac{p}{p}-1} \) holds true.

**Proof.** The property (B1) follows by Lemma 5.4. The conditions (B2), (B3) and (B4) follow respectively from Lemma 5.9, Corollary 5.12 and Theorem 5.5. Also, we get (B5) by Lemma 6.1. Finally, (B6) follows by taking \( \theta \) in Lemma 6.1 close enough to \( 1/2 \) such that \( 2l_2 = 2\theta l' < n \). Finally, we conclude by using Theorem 4.13 with \( E' = \hat{B}[\mathcal{M}[X']_{l'}^{l}]_{\infty}^{\frac{p}{p}-1}, \ F' = \mathcal{M}[X']^{l} \) and \( \sigma_{E'} = -\frac{n}{p} \).

6.1 Applications (bilinear estimate in Besov-Lorentz-Morrey spaces)

We can take \( X = L^{(p,d)} \) with \( 2 < p' < l' < n, \ \frac{n}{2} < l' < n \) and \( 1 < d < \infty \) in order to obtain the bilinear estimate (3.7) with the Besov-Lorentz-Morrey \( E' = \hat{B}\mathcal{L}^{l',\frac{p}{p}-1} = \hat{B}\mathcal{L}[L^{(p,d)}]^{l',\frac{p}{p}-1} \), which is a generalization of the Besov-Morrey space \( \hat{B}\mathcal{M}^{l',\frac{p}{p}-1} = \hat{B}\mathcal{M}[L^{(p,d)}]_{\infty}^{l',\frac{p}{p}-1} \) defined in [24]. This result recovers and extends the bilinear estimate obtained in [12]. In the case \( d = p \), we obtain \( X = L^{(p,p)} = L^{p} \) and (3.7) in the Besov-Morrey space \( E' = \hat{B}\mathcal{M}^{l',\frac{p}{p}-1} \).

7 Uniqueness for Navier-Stokes equations

We finish the study of Navier-Stokes equations by proving a uniqueness result, regardless of the size of the initial data and solutions. For that, we employ the bilinear estimate (3.7) and consider some further standard properties for our framework.

**Theorem 7.1.** Let \( E \in \mathcal{G} \) be a Banach space with \( \sigma_{E'} = -1 \), such that the bilinear estimate (3.7) holds true. Moreover, assume that there exist Banach spaces \( E_3, E_4 \in \mathcal{G} \) verifying the following conditions:

(UI) \[-1 < \sigma_{E'_4} < 1;\]
(U2) \[ \| U(s)f \|_{E_3'} \leq C \| f \|_{E_3'}; \]

(U3) \[ \| U(s)f \|_{E_3'} \leq C S^\frac{1}{2} (\sigma_{E'} - \sigma_{E_3'}) \| f \|_{E'}; \]

(U4) \[ \| f \cdot g \|_{E_4'} \leq C \| f \|_{E'} \| f \|_{E_3'}; \]

(U5) \[ \| \nabla U(s) P f \|_{E'} \leq C S^\frac{1}{2} (\sigma_{E_4'} - \sigma_{E'})^{-\frac{1}{2}} \| f \|_{E_4'}. \]

Then, if \( u \) and \( v \) are two mild solutions of (1.1)-(1.3) in \( C([0, T); E') \) with the same initial data \( u_0 \in E' \), it follows that \( u(\cdot, t) = v(\cdot, t) \) in \( E' \) for all \( t \in [0, T) \). Here, \( E' \) stands for the maximal closed subspace of \( E' \) in which the heat semigroup \( \{ U(t) \}_{t \geq 0} \) is strongly continuous.

**Proof.** The proof is based on an argument due to Meyer [31, p. 188]. First, we prove that there exists \( 0 < T_1 < T \) such that \( u(\cdot, t) = v(\cdot, t) \) in \( E' \) for all \( t \in [0, T_1) \). Denoting \( w = u - v \), \( w_1 = U(t)u_0 - u \) and \( w_2 = U(t)v_0 - v \), we have that

\[
\begin{align*}
  u \otimes u - v \otimes v &= w \otimes u + v \otimes w \\
  &= w \otimes U(t)u_0 + U(t)v_0 - w \otimes w_1 - w_2 \otimes w.
\end{align*}
\]

We can estimate the difference \( w(t) \) in \( E' \) as follows

\[
\begin{align*}
  \| w(t) \|_{E'} &= \left\| \int_0^t \nabla U(t-s)P(u \otimes u - v \otimes v)ds \right\|_{E'} \\
  &\leq \left\| \int_0^t \nabla U(t-s)P(w \otimes w_1 + w_2 \otimes w)ds \right\|_{E'} \\
  &\quad + \left\| \int_0^t \nabla U(t-s)P(w \otimes U(s)u_0 + U(s)v_0 \otimes w)ds \right\|_{E'} \\
  &=: I_1(t) + I_2(t).
\end{align*}
\]

From the bilinear estimate (3.7), we conclude that

\[
I_1(t) \leq K \sup_{0 < t < T_1} \| w \|_{E'} \left( \sup_{0 < t < T_1} \| w_1 \|_{E'} + \sup_{0 < t < T_1} \| w_2 \|_{E'} \right).
\]

Now we estimate \( I_2(t) \). First we use (U5) and then (U4) to obtain
\[ I_2(t) \leq C \int_0^t \| \nabla U(t-s) \mathbb{P} (w \otimes U(s) u_0 + U(s) u_0 \otimes w) \|_{E'} \, ds \]
\[ \leq C \int_0^t (t-s)^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}})^{-\frac{1}{2}} \| w \otimes U(s) u_0 + U(s) u_0 \otimes w \|_{E_4} \, ds \]
\[ \leq C \int_0^t (t-s)^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}})^{-\frac{1}{2}} \| w \otimes U(s) u_0 \|_{E_4} \, ds \]
\[ \leq C \int_0^t (t-s)^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}})^{-\frac{1}{2}} \| w \|_{E'} \| U(s) u_0 \|_{E_3'} \, ds \]
\[ \leq C \max_{0< t < T_1} \| w \|_{E'} \sup_{0< t < T_1} t^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}})^{-\frac{1}{2}} \| U(t) u_0 \|_{E_4} \int_0^t (t-s)^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}})^{-\frac{1}{2}} \, ds \]
\[ \leq C \max_{0< t < T_1} \| w \|_{E'} \sup_{0< t < T_1} t^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}})^{-\frac{1}{2}} \| U(t) u_0 \|_{E_3'} \int_0^t (1-\rho)^{-\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}})^{-\frac{1}{2}} \, d\rho \]
\[ \leq C \max_{0< t < T_1} \| w \|_{E'} \sup_{0< t < T_1} t^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}})^{-\frac{1}{2}} \| U(t) u_0 \|_{E_3'} . \]

In the above computations, the finiteness of the last integral is guaranteed because condition (U4) implies that \( \sigma_{E_3' - \sigma_{E'}} > 0 \) (see Remark 3.2) and then condition (U1) implies that \(-1 < \frac{1}{2} (\sigma_{E_3' - \sigma_{E'}}) - \frac{1}{2} < 0 \) and \(-1 < \frac{1}{2} (\sigma_{E_3' - \sigma_{E'}}) < 0 \). With all the above developments, we arrive at

\[ \max_{0< t < T_1} \| w(t) \|_{E'} \leq CZ(T_1) \max_{0< t < T_1} \| w \|_{E'} , \]

where

\[ Z(T_1) = \max_{0< t < T_1} \| w_1 \|_{E'} + \max_{0< t < T_1} \| w_2 \|_{E'} + \max_{0< t < T_1} t^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}})^{-\frac{1}{2}} \| U(t) u_0 \|_{E_4^{'}} . \]

Note that from the conditions for \( u, v \) and \( u_0 \), we have that \( U(t) u_0, u, v \to u_0 \) in \( E' \) as \( t \to 0^+ \), and then

\[ \lim_{t \to 0^+} \| w_1 \|_{E'} = \lim_{t \to 0^+} \| w_2 \|_{E'} = 0. \]

Now, we prove that

\[ \limsup_{t \to 0^+} t^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}}) \| U(t) u_0 \|_{E_3'} = 0. \]

To do so, we define \( u_{0k} = U \left( \frac{1}{k} \right) u_0 \) for all \( k \in \mathbb{N} \). It follows by (U3) that \( u_{0k} \in E_3' \), and from (U2) that \( \| U(t) u_0 \|_{E_3'} \leq C \| u_{0k} \|_{E_3'} \). Using again that \( u_{0k} \to u_0 \) in \( E' \) as \( k \to \infty \), we can proceed as follows:

\[ \limsup_{t \to 0^+} t^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}}) \| U(t) u_0 \|_{E_3'} \]
\[ \leq \limsup_{t \to 0^+} t^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}}) \| U(t) (u_0 - u_{0k}) \|_{E_3'} + \limsup_{t \to 0^+} t^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}}) \| U(t) u_{0k} \|_{E_3'} \]
\[ \leq C \| u_0 - u_{0k} \|_{E'} + C \| u_{0k} \|_{E_3'} \limsup_{t \to 0^+} t^{\frac{1}{2}} (\sigma_{E_3' - \sigma_{E'}}) \]
\[ \leq C \| u_0 - u_{0k} \|_{E'} \to 0, \text{ as } k \to \infty . \]
Consequently, there exists $T_1 > 0$ small enough such that $CZ \ (T_1) < 1$ and then $w(t) = 0$ for all $t \in [0, T_1)$. The remainder of the proof is to show that in fact $T_1 \in (0, T]$ can be arbitrary. Define

$$T_* = \sup \left\{ \tilde{T}; 0 < \tilde{T} < T, u(t) = v(t) \text{ for all } t \in [0, \tilde{T}) \right\}.$$ 

If $T_* = T$ we are done; otherwise, we have that $u(t) = v(t)$ for $t \in [0, T_*)$ which implies that $u (T_*) = v (T_*)$ due to the time-continuity of $u$ and $v$. It follows from the first part of the proof that there exists $T_2 > 0$ small enough such that $u(t) = v(t)$ for $t \in [T_*, T_* + T_2)$. Therefore $u(t) = v(t)$ for $t \in [0, T_* + T_2)$ which contradicts the definition of $T_*$. \hfill \Box

### 7.1 Applications

In this section we present some examples of spaces $E'$ where the bilinear estimate (3.7) holds true and then the uniqueness follows in a direct way by applying Theorem 7.1.

- Consider the space $E' = L^{n, \infty}$. In this case the uniqueness result follows by choosing $E'_3 = L^{q, \infty}$ and $E'_4 = L^{r, \infty}$ where $n < q < \infty$ and $\frac{1}{r} = \frac{1}{n} + \frac{1}{q}$.

- Similarly, if we consider $E' = W^{K^0, n, \infty}$, the uniqueness result follows by choosing $E'_3 = W^{K^0, q, \infty}$ and $E'_4 = W^{K^0, r, \infty}$ where $n < q < \infty$ and $\frac{1}{r} = \frac{1}{n} + \frac{1}{q}$.

- For the space $E' = \hat{B} \left[ W^{K^0, p, \infty} \right]^{\frac{1}{p} - 1}$, where $\frac{n}{2} < p < n$, we can consider $E'_3 = \hat{B} \left[ W^{K^0, p, \infty} \right]^{\frac{1}{p} + \rho}$ and $E'_4 = \hat{B} \left[ W^{K^0, p, \infty} \right]^{\frac{1}{p} - 2 + \rho}$ with $\rho > 0$ small enough. All the conditions in Theorem 7.1 follow from Lemma 4.11 and Lemma 4.12.

### 8 Other nonlinearities and PDEs

The present section is devoted to give some other examples of nonlinearities and PDEs to which the theory developed in the previous sections can be adapted.

**Reaction-diffusion system:** we can consider the bilinear form

$$B_{1,i}(u, v)(t) = \int_0^t U(t - s) \langle A_i \cdot u(s), v(s) \rangle \, ds \text{ for } i = 1, \ldots, n,$$  

which is related to the system of reaction-diffusion equations

$$u_{it} = \Delta u_i + \langle A_i \cdot u, u \rangle, \text{ for } i = 1, \ldots, n,$$  

where $A_i$ denotes constant $n \times n$-matrices and $u = (u_1, \ldots, u_n) : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^n$.

**Nonlocal advection-diffusion system:** consider the bilinear form

$$B_2(u, v)(t) = - \int_0^t \nabla U(t - s) \, (u(s) \otimes S(v)) \, ds,$$  

which is related to the nonlocal parabolic equation

$$u_t = \Delta u + \nabla \cdot (u \otimes S(u)),$$
where \( S(u) \) is a vector field given by the convolution of \( u \) with a vector function whose components are \( b \)-homogeneous singular kernels \( \psi \), that is,

\[
S(u) = \psi \ast u,
\]

where \( \psi(\lambda x) = \lambda^b \psi(x) \) and \( |\psi(x)| \leq |x|^b \) for some \( b < 0 \). In this case we are concerned with values \( b = -n + \delta \) for \( 0 < \delta < n \). For the reader convenience, examples of such nonlocal operators are given below.

- **Models appearing in nonequilibrium statistical mechanics:** \( \psi(x) = \nabla G_n(x) \) (\( \delta = 1 \)), where \( G_n \) is the fundamental solution of the Laplacian in \( \mathbb{R}^n \), see [4].

- **The Debye system from the theory of electrolytes** presents the form

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= \Delta u_1 - \nabla \cdot (u_1 \psi) \\
\frac{\partial u_2}{\partial t} &= \Delta u_2 + \nabla \cdot (u_2 \psi),
\end{align*}
\]

where \( \psi = \nabla G_n \ast (u_1 - u_2) \) (\( \delta = 1 \)).

- **Aggregation diffusion equations for multi-species** is a system with the form

\[
\frac{\partial u_i}{\partial t} = \Delta u_i - \nabla \left[ u_i \nabla \sum_{j=1}^{n} h_{ij} K \ast u_j \right], \quad i = 1, 2, \ldots, n,
\]

where \( u_i(x,t) \) stands for the density of the population \( i \) and \( h_{ij} \) are constants denoting attraction \( (h_{ij} > 0) \) or repulsion \( (h_{ij} < 0) \) of the population \( i \) to population \( j \).

Recall the nonlocal operator

\[
(-\Delta)^{-\delta/2} (f) = c_\delta \int_{\mathbb{R}^n} |x - y|^{-n+\delta} f(y) \, dy,
\]

where \( c_\delta > 0 \) is a constant and \( 0 < \delta < n \). For the bilinear forms \( B_1 \) and \( B_2 \) related to the reaction-diffusion equation and the nonlocal parabolic equation, we have an analogous result to Theorem 5.18. Let us state the result for the former case.

**Theorem 8.1.** Let \( 1 \leq l < l_0 < \infty \) and \( X_0, X \in \mathcal{H} \) be such that \( \sigma_X < \sigma_{X_0} \), \( l'_0 \sigma_{X_0} = l' \sigma_X \), and \( X_0', X' \in \mathcal{H} \). Assume that the conditions (i) and (ii) in Theorem 5.18 and the Hölder-type inequality \( \|fg\|_{X'} \leq C\|f\|_{X'}\|g\|_{X'} \) hold true. If \( n > 4 \), then we have the bilinear estimate

\[
\sup_{t>0} \|B_1(u,v)(t)\|_{\mathcal{M}[X]^{n/2}} \leq K \sup_{t>0} \|u(t)\|_{\mathcal{M}[X_0']^{n/2}} \sup_{t>0} \|v(t)\|_{\mathcal{M}[X_0']^{n/2}}.
\]

**Proof.** We only give some steps. First, from items (i) and (ii) in Theorem 5.18 we can perform the same arguments as in Lemma 3.6 in order to get

\[
\int_0^\infty s^{-\frac{\delta}{2}}((s - \frac{\delta}{2}) - 1) \|U(s)f\|_{\mathcal{P}[X_0]} \, ds \leq C\|f\|_{\mathcal{P}[X_0]}.
\]

Next, the Hölder-type inequality and condition \( \frac{n}{l} - \frac{n}{l_0} - 2 = 0 \) imply that \( l' = n/2 \) and \( l'_0 = n/4 \). Thus, one can obtain (8.8) by proceeding as in Lemma 3.3 and Theorem 3.4.
Next, we perform the bilinear estimate for the form $B_2(\cdot, \cdot)$ defined in (8.3). This part employs some ideas of [1, Chapter 7] in order to control the nonlocal operator (8.5). Similar estimates for fractional operators can be found in [28, 16].

**Definition 8.2.** Given $\kappa > 1$, we define the family $\mathcal{H}_\kappa$ of all spaces $X \in \mathcal{H}$ such that the function

$$f \mapsto \| |f|^\kappa \|_X^{1/\kappa}$$

satisfies the triangle inequality and the norm of $X$ has the following monotonicity property:

$$|f(x)| \leq |g(x)|, \quad \text{for a.e. } x \in \mathbb{R}^n \Rightarrow \| f \|_X \leq \| g \|_X. \quad (8.10)$$

For $X \in \mathcal{H}_\kappa$, it is easy to see that $X_\kappa = \{ f \in L^1_{loc} : |f|^\kappa \in X \}$ is a Banach space and satisfies (M1), (M2), (M3), (M5) and (M6). Now, let us recall the centered maximal operators

$$M_\alpha f(x) = \sup_{R>0} \frac{1}{R^{n-\alpha}} \int_{D(x,R)} |f(y)| \, dy,$$

for $f \in L^1_{loc}$, where $0 \leq \alpha \leq n$. We start with the following preliminary result.

**Lemma 8.3.** Let $X \in \mathcal{H}$ and $l > 1$ be such that $\frac{\alpha}{\alpha} + \sigma_X \leq 0$, the monotonicity property (8.10) is satisfied, and the maximal operator $M_0 : X \to X$ is bounded. Then, $M_0 : \mathcal{M}[X]^l \to \mathcal{M}[X]^l$ is bounded.

**Proof.** Take $x_0 \in \mathbb{R}^n$ and $R_0 > 0$, and consider $f(x) = f(x)1_{D(x_0,2R_0)}(x) + f(x)1_{D(x_0,2R_0)^c}(x) = f_1(x) + f_2(x)$. Since $M_0$ is subadditive, i.e.

$$M_0 f \leq M_0 f_1 + M_0 f_2,$$

we can estimate

$$\| M_0 f_1 \cdot 1_{D(x_0,R_0)} \|_X \leq \| M_0 f_1 \|_X \leq C \| f 1_{D(x_0,2R_0)} \|_X \leq C R_0 \| f \|_{\mathcal{M}[X]^l},$$

where we have used the boundedness of $M_0$ on $X$. On the other hand, using Proposition 5.1, for $x \in D(x_0, R_0)$ and $r > 0$, we have that

$$\frac{1}{r^n} \int_{D(x,r)} |f_2(y)| \, dy = \frac{1}{r^n} \int_{D(x,r) \cap D(x_0,2R_0)} |f(y)| \, dy 1_{(R_0, +\infty)}(r) \leq C r^{\sigma_X} \| f 1_{D(x,r)} \|_X 1_{(R_0, +\infty)}(r) \leq C R_0^{-\frac{\alpha}{\alpha} + \sigma_X} \| f \|_{\mathcal{M}[X]^l} 1_{(R_0, +\infty)}(r).$$

This last inequality leads us to

$$\| M_0 f_2 \cdot 1_{D(x_0,R_0)} \|_X \leq C R_0^{-\frac{\alpha}{\alpha} + \sigma_X} \| f \|_{\mathcal{M}[X]^l} \| 1_{D(x_0,R_0)} \|_X \leq C R_0^{-\frac{\alpha}{\alpha} + \sigma_X} \| f \|_{\mathcal{M}[X]^l}.$$

Combining all the above inequalities, we are done.
The X-Morrey spaces provide a suitable framework to control the maximal operators $M_\alpha$, for $0 < \alpha \leq n$. In fact, using again Proposition 5.1, note that

$$\frac{1}{R^{n-\alpha}} \int_{D(x,R)} |f(y)| \, dy \leq \frac{C}{R^{\sigma_X-\alpha}} \|f1_{D(x,R)}\|_X \leq C\|f\|_{\mathcal{M}[X]^{n/\alpha}},$$

for all $f \in \mathcal{M}[X]^{n/\alpha}$. It follows that

$$M_\alpha f(x) \leq C\|f\|_{\mathcal{M}[X]^{n/\alpha}}, \quad (8.12)$$

provided that $\alpha + \sigma_X \leq 0$.

With the above developments in hand, we are in a position to show the boundedness of the operator $(-\Delta)^{-\delta/2}$ in the framework of X-Morrey spaces.

**Proposition 8.4.** Consider $X \in \mathcal{H}_{\alpha,0}$ with $0 < \delta < \alpha \leq n$ and such that $\alpha + \sigma_X \leq 0$, and suppose that the maximal operator $M_0$ is bounded in $X$. Then, the fractional operator $(-\Delta)^{-\delta/2} : \mathcal{M}[X]^{\frac{n}{\alpha}} \to \mathcal{M}
\left[X_{\alpha,\frac{\sigma_X}{\alpha}}\right]^{\frac{\alpha-\delta}{\alpha}}$ is bounded.

**Proof.** We employ the so-called Hedberg trick (see [1, Thm. 7.1]) to obtain

$$\left|(-\Delta)^{-\delta/2} f(x)\right| \leq C(M\alpha f(x))^{\frac{\delta}{\alpha}}(M_0 f(x))^{1-\frac{\delta}{\alpha}} \leq C\left(\|f\|_{\mathcal{M}[X]^{n/\alpha}}\right)^{\frac{\delta}{\alpha}}(M_0 f(x))^{\frac{\alpha-\delta}{\alpha}}.$$

Multiplying by $1_{D(x,R)}$, using (M2), and taking the norm of $X_{\alpha,\frac{\sigma_X}{\alpha}}$, we arrive at

$$\|(-\Delta)^{-\delta/2} f \cdot 1_{D(x,R)}\|_{X_{\alpha,\frac{\sigma_X}{\alpha}}} \leq C\left(\|f\|_{\mathcal{M}[X]^{n/\alpha}}\right)^{\frac{\delta}{\alpha}}\|M_0 f \cdot 1_{D(x,R)}\|_{X^{\alpha-\delta}}.$$

Finally, multiplying the last inequality by $R^{(\alpha-\delta)(1+\sigma_X/\alpha)}$, taking the supremum, and applying Lemma 8.3, we conclude the proof.

**Remark 8.5.** The previous result also works well if, instead of assuming that $X \in \mathcal{H}_{\alpha,0}$, we assume that there exists $Y \in \mathcal{H}$ satisfying the compatibility condition

$$\|f^{\alpha-\delta}\|_Y \leq C\|f\|_{X^{\alpha-\delta}},$$

for all $f \in X$ such that $f \geq 0$. In this case, we have that $(-\Delta)^{-\delta/2} : \mathcal{M}[X]^{\frac{n}{\alpha}} \to \mathcal{M}[Y]^{\frac{n}{\alpha-\delta}}$ is bounded.

Finally, we give the statement of the bilinear estimate for $B_2$ in the framework of X-Morrey spaces.

**Theorem 8.6.** Let $n > \delta + 2$, $1 \leq l < l_0 < \infty$ and let $X_0, X \in \mathcal{H}$ be such that $\frac{n}{l} - \frac{n}{l_0} = 1$, $X_0', X \in \mathcal{H}$, $\sigma_X < \sigma_X'$, and $\tau_0' \sigma_X' = \tau' \sigma_X$. Assume the conditions (i) and (ii) in Theorem 5.18. Moreover, assume that the Hölder-type inequality $\|f \cdot g\|_{X_0'} \leq C\|f\|_{X^\kappa}\|g\|_{X^{\bar\kappa}}$, holds true for $\kappa = \frac{\alpha}{\alpha-\delta}$ with $0 < \delta < \alpha \leq n$. Then,

$$\sup_{t>0} \|B_2(u,v)(t)\|_{\mathcal{M}[X']^{\kappa}} \leq K \sup_{t>0} \|u(t)\|_{\mathcal{M}[X'^{\kappa}]} \sup_{t>0} \|v(t)\|_{\mathcal{M}[X'^{l_0\alpha}]} \leq 1, \quad (8.13)$$

for $\frac{n}{n+\frac{\alpha'}{\kappa}} \leq 0$. In addition, suppose that $X'$ satisfies the hypotheses of Proposition 8.4 and $l_1 = \frac{n}{\alpha-\delta}$, then we obtain $\alpha = \delta + 1$ and, in particular, by taking $\tau' = \frac{n}{\delta+1}$ we have the bilinear estimate

$$\sup_{t>0} \|B_2(u,v)(t)\|_{\mathcal{M}[X']^{n/(\delta+1)}} \leq K \sup_{t>0} \|u(t)\|_{\mathcal{M}[X'^{n/(\delta+1)}]} \sup_{t>0} \|v(t)\|_{\mathcal{M}[X'^{l_1}]} \cdot (8.14)$$

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Proof. It was already showed that conditions (i) and (ii) in Theorem 5.18 imply (H4). So, Lemma 3.3 and the Hölder-type inequality yield

\[
\|B_2(u, v)\|_{M[X', Y']} \leq C \sup_{t > 0} \|u S(v)\|_{M[X'_0]} \leq C \sup_{t > 0} \|u\|_{M[X', Y']} \|S(v)\|_{M[X'_0]}.
\]

By taking \(l_1 = \frac{n}{\alpha - p}\), using the Hölder-type inequality and the identity \(\frac{p_0}{p} - \frac{p'}{q} = 1\), it follows that \(\alpha = \delta + 1\). Finally, noting that \(|S(v)(x)| \leq (-\Delta)^{-\delta/2}|v|(x)\) and applying Proposition 8.4, we obtain (8.14).

Remark 8.7. Note that in Theorem 8.6 we are forced to take \(n > \delta + 2\) for the \(X_0'\)-Morrey space to be well defined. Also, Theorem 8.6 works well under the condition stated in Remark 8.5 with the obvious modifications.

8.1 Applications

8.1.1 Reaction-diffusion system

As a case that realizes Theorem 8.1, we take \(X' = L^{p', \infty}\) and \(X'_0 = L^{p'_0, \infty}\) with \(2p'_0 = p'\). Also, we can choose \(p_1, p_2 > 1\) such that

\[
\frac{p'_0}{p_1} < \frac{1}{2} < \frac{p'_0}{p_2} < 1.
\]

The rest of parameters can be chosen as in Example 5.1. Then, we obtain (8.8) in the weak-Morrey space \(M^{n/2}_{(p', \infty)} = \mathcal{M}(L^{p', \infty})\) for \(n > 4\) and \(p' \leq n/2\). As a consequence of (8.8), by adapting an argument by [31], we can get uniqueness of mild solutions in the class \(C([0, T]; \tilde{Z})\) with \(Z = M^{n/2}_{(p', \infty)}\) for system (8.2).

8.1.2 Nonlocal advection-diffusion system

We conclude this work with an application for the nonlocal parabolic equation. From Theorem 8.6, we obtain the bilinear estimate for \(B_2\) in weak-Morrey spaces by taking \(X' = L^{p', \infty}\), \(X'_0 = L^{p'_0, \infty}\), \(\alpha = \delta + 1\) and \(p'_0 = \left(\frac{\delta + 1}{\delta + 2}\right)p'\). Recall that the boundedness of the maximal operator \(M_0\) is well established in the context of weak-L\(^r\) spaces, for all \(r > 1\). The necessary Hölder-type inequality reduces to

\[
\|f \cdot g\|_{L^{p'_0, \infty}} \leq C \|f\|_{L^{p', \infty}} \|g\|_{L^{p'/(\delta + 1), \infty}}.
\]

Therefore, the bilinear estimate for \(B_2\) is verified in the weak-Morrey space \(M^{n/(1 + \delta)}_{(p', \infty)} = \mathcal{M}(L^{p', \infty})\) with \(\frac{\delta + 1}{\delta + 2} < p' \leq \frac{n}{\delta + 1}\). As an application of that estimate, we also obtain the uniqueness of mild solutions in the class \(C([0, T]; \tilde{Z})\) with \(Z = M^{n/(1 + \delta)}_{(p', \infty)}\) for equation (8.4).

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