DYONS AND S-DUALITY IN N=4 SUPERSYMMETRIC GAUGE THEORY

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Abstract

We analyze the spectrum of dyons in $N = 4$ supersymmetric Yang-Mills theory with gauge group $SU(3)$ spontaneously broken down to $U(1) \times U(1)$. The Higgs fields select a natural basis of simple roots. Acting with S-duality on the $W$-boson states corresponding to simple roots leads to an orbit of BPS dyon states that are magnetically charged with respect to one of the $U(1)$'s. The corresponding monopole solutions can be obtained by embedding $SU(2)$ monopoles into $SU(3)$ and the S-duality predictions reduce to the $SU(2)$ case. Acting with S-duality on the $W$-boson corresponding to a non-simple root leads to an infinite set of new S-duality predictions. The simplest of these corresponds to the existence of a harmonic form on the moduli space of $SU(3)$ monopoles that have magnetic charge $(1, 1)$ with respect to the two $U(1)$'s. We argue that the moduli space is given by $R^3 \times (R^1 \times M)/\mathbb{Z}$, where $M$ is Euclidean Taub-NUT space, and that the latter admits the appropriate normalizable harmonic two form. We briefly discuss the generalizations to other gauge groups.

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1. Introduction

It is conjectured that $N = 4$ supersymmetric gauge theory is invariant under an $SL(2, \mathbb{Z})$ $S$-duality group which includes the interchange of strong and weak coupling \cite{1-4}. As emphasised by Sen \cite[5], the conjecture makes non-trivial predictions about the spectrum of BPS saturated dyons which can be tested at weak coupling using semiclassical techniques. For gauge group $SU(2)$ spontaneously broken down to $U(1)$ he showed that the required dyon states are equivalent to the existence of certain harmonic forms on the moduli space of classical BPS monopole solutions. For monopole charge two he further demonstrated that the moduli space admits the appropriate form. Evidence for the harmonic forms for higher monopole charge has also been found \cite[8,9]. The purpose of this work is to extend these investigations to higher rank gauge groups.

For the most part we will focus on gauge group $SU(3)$ spontaneously broken down to $U(1) \times U(1)$ but we will also discuss how our results apply to other gauge groups with maximal symmetry breaking. A basis of simple roots of the Lie algebra of $SU(3)$ is determined by the Higgs fields \cite[10]. Using this basis, the electric charge vectors, $n^e$, of the $W$-bosons of positive charge corresponding to the simple positive roots are given by $(1,0)$ and $(0,1)$ whilst for the non-simple positive root it is $(1,1)$. The mass and charge of the $W$-boson states saturate a Bogomol’nyi bound and hence they form part of a short BPS multiplet. The magnetic duals of these states then have magnetic charge vectors, $n^m$, given by the same vectors, respectively. Weinberg has argued in \cite[10] that the classical monopole solutions with $n^m = (1,0)$ and $(0,1)$ should be considered to be “fundamental” in the sense that they have a moduli space given by $R^3 \times S^1$ (corresponding to translations and a dyon degree of freedom) and hence have no “internal” degrees of freedom. Furthermore, both the dimension of the moduli space of a general monopole with $n^m = (n_1,n_2)$, and the BPS mass formula are consistent with interpreting it as a multimonompole configuration consisting of $n_1 (1,0)$ and $n_2 (0,1)$ fundamental monopoles.

Using the techniques explained in \cite[11,12] the semiclassical quantization of the fundamental monopoles gives rise to a BPS multiplet of states dual to the corresponding $W$-bosons. On the other hand, the magnetic dual of the $W$-boson BPS multiplet with charge $n^e = (1,1)$ must emerge as a bound state of two fundamental monopoles. Since the monopoles with charge $n^m = (1,1)$ are only neutrally stable into the decay of two fundamental monopoles, the bound state is at threshold\footnote{Bound states at threshold have been recently found in the dyon spectrum of exactly $S$-dual models with $N = 2$ supersymmetry \cite[13,14].}. To determine the existence of these bound states we first need to identify the moduli space of monopoles with charge $n^m = (1,1)$. We will argue that is given by $R^3 \times (R^1 \times M)/\mathbb{Z}$ where $M$ is Euclidean Taub-NUT space. Using the results of \cite[11,12], S-duality then predicts that Taub-NUT space should admit a unique harmonic form. We show it indeed possesses a self-dual harmonic two-form as required.

Thus, we will show that the magnetic duals of the $W$-boson states exist in the quantum spectrum. Our analysis will actually go further than just checking the electric/magnetic $\mathbb{Z}_2$ subgroup of the duality group, as we also discuss the spectrum of dyons as well as how the $SL(2, \mathbb{Z})$-duality predictions of the $SU(2)$ case are embedded in the $SU(3)$ case.
The plan of the rest of the paper is as follows. In section 2 we review some features of S-duality of $N = 4$ Super Yang-Mills theory, magnetic monopoles, and the relevant parts of Weinberg’s analysis concerning the number of zero modes around a monopole solution. Section 3 discusses in detail the S-duality predictions for gauge group $SU(3)$ and contains our main results. Section 4 is a discussion section which includes some comments concerning other gauge groups.

2. N=4 Supersymmetric Gauge Theory, Monopoles and Duality

We consider $N=4$ supersymmetric Yang-Mills with arbitrary simple gauge group. The supermultiplet includes 6 Higgs fields $\phi^I$ and a gauge field, all taking values in the adjoint representation of the gauge group. The bosonic part of the action is

$$S = -\frac{1}{16\pi} \text{Im} \int \tau \text{Tr}(F \wedge F + i * F \wedge F) - \frac{1}{2e^2} \int \left[ \text{Tr} D_\mu \phi^I D^\mu \phi^I + V(\phi^I) \right]$$

(2.1)

where the potential is given by

$$V(\phi^I) = \sum_{1 \leq I < J \leq 6} \text{Tr}[\phi^I, \phi^J]^2,$$  

(2.2)

and $\tau = \theta/2\pi + i4\pi/e^2$. We have normalized the generators of the gauge group so that $\text{Tr} t_a t_b = \delta_{ab}$.

The classical vacua of the theory correspond to solutions of the equations

$$F_{\mu\nu} = 0, \quad D_\mu \phi^I = 0, \quad \text{and } V(\phi^I) = 0.$$  

(2.3)

This last equation implies that $[\phi^I, \phi^J] = 0$ for all $I, J$. Spontaneous symmetry breaking is achieved by demanding

$$\text{Tr} \phi^I \phi^I = v^2,$$  

(2.4)

as a boundary condition at infinity. In the following we will work at a generic point in the moduli space of vacua where the gauge symmetry is broken down to $U(1)^l$, where $l$ is the rank of the gauge group.

A set of conserved electric and magnetic charges may be defined which arise as central charges of the $N = 4$ supersymmetry algebra:

$$Q^I_e = \frac{1}{ev} \int dS \cdot \text{Tr}(E \phi^I),$$

$$Q^I_m = \frac{1}{ev} \int dS \cdot \text{Tr}(B \phi^I),$$

(2.5)

where the electric field $E_i = F_{0i}$, and the magnetic field $B_i = 1/2\epsilon_{ijk}F_{jk}$. For BPS saturated states, i.e. states in the short 16 dimensional representation of the supersymmetry algebra, the mass is exactly given by the formula

$$M^2 = \frac{v^2}{e^2} ((Q^I_e)^2 + (Q^I_m)^2),$$

(2.6)
for $\theta = 0$. A magnetic monopole solution with zero electric charge is BPS saturated if and only if it satisfies the Bogomol’nyi equations

$$B_i = D_i \phi \ , \quad (2.7)$$

where the scalar field $\phi$ is defined by the equations

$$\phi^I = \phi a^I + \hat{\phi}^I, \quad (a^I)^2 = 1 \ , \quad (2.8)$$

with $V(\phi^I) = 0$, $D_i \hat{\phi}^I = 0$, $a^I$ are constant and we must apply the boundary condition (2.4) [4]. For our purposes it will be sufficient to set $\hat{\phi}^I = 0$ in the following which allows us to focus on a single direction in the 6 dimensional Higgs field space. BPS anti-monopoles similarly satisfy $B_i = -D_i \phi$. Dyon states are obtained from the monopole solutions after semiclassical quantization.

Before proceeding further, let us review some relevant properties of Lie algebras. The maximal abelian subalgebra will be denoted $H$, and the $l$ generators $H_i$. The raising and lowering operators satisfy

$$[H_i, E_\alpha] = \alpha_i E_\alpha \ , \quad [E_\alpha, E_{-\alpha}] = \sum \alpha_i H_i \ . \quad (2.9)$$

The $H_i$ and $E_\alpha$ are linear combinations of the generators $t_a$ previously defined. A basis of simple roots, $\beta^{(a)}$ ($a = 1, \cdots, l$), may be chosen such that any root is a linear combination of $\beta^{(a)}$ with integral coefficients all of the same sign. The term positive roots refers to those with positive coefficients.

We may choose the Cartan subalgebra such that $\phi_0 = v \mathbf{h} \cdot \mathbf{H}$ is the asymptotic value of the Higgs field along the positive $z$-axis, and $v$ is the asymptotic value of $\sqrt{\text{Tr} \phi^2}$. If $\alpha \cdot \mathbf{h} = 0$ for some root $\alpha$ then the unbroken gauge group is nonabelian. Otherwise, maximal symmetry breaking occurs, and $\phi_0$ picks out a unique set of simple roots which satisfy the condition $\mathbf{h} \cdot \beta^{(a)} > 0$ [10].

The electric quantum numbers live on the $l$-dimensional root lattice spanned by the simple roots $\beta^{(a)}$,

$$\mathbf{q} = \sum n_a^e \beta^{(a)} \ , \quad (2.10)$$

where the $n_a^e$ are integer. The electric charge $Q_e^I$ is then given by

$$Q_e^I = a^I Q_e \ , \quad Q_e \equiv e \mathbf{h} \cdot \mathbf{q} \ . \quad (2.11)$$

For each root $\alpha$ there is a BPS $W$-boson with $\mathbf{q} = \alpha$. From (2.7) we see that the $W$-bosons corresponding to simple roots are stable, whilst those corresponding to the non-simple roots are only neutrally stable.

Now we consider magnetic quantum numbers which arise from topologically nontrivial field configurations. For any finite energy solution, asymptotically we have

$$B_i = \frac{r_i}{4 \pi r^3} G(\Omega) \ , \quad (2.12)$$
where $G$ is covariantly constant, and takes the value $G_0$ along the positive $z$-axis. The Cartan subalgebra may be chosen so that $G_0 = g \cdot H$. This quantity must satisfy a topological quantization condition \cite{2,15}

$$e^{iG_0} = I .$$

(2.13)

The solution to this equation is

$$g = 4\pi \sum n^m_a \beta^{(a)*} ,$$

(2.14)

where the $n^m_a$ are integers and the $\beta^{(a)*}$ are the duals of the simple roots, defined as

$$\beta^{(a)*} = \frac{\beta^{(a)}}{\beta^{(a)2}} .$$

(2.15)

For maximal breaking, all the $n^m_a$ are conserved topological charges, labeling the homotopy class of the Higgs field configuration \cite{16–18}. The magnetic quantum numbers thus live on the lattice spanned by the $\beta^{(a)*}$.

The topological charge $g$ is related to the charge $Q^l_m$ defined above by the formula

$$Q^l_m = a^l Q^M , \quad Q^M = \frac{1}{e} g \cdot h .$$

(2.16)

Substituting this into the BPS mass formula (2.6) we deduce that the monopoles corresponding to the duals of the simple roots will be stable. Those corresponding to the duals of the nonsimple roots are neutrally stable with respect to decay to simple root monopoles.

A general state may be labeled by the integer valued $l$-vectors $n^e$ and $n^m$. For a BPS state the mass is given by the BPS mass formula (2.6) which, using (2.11) and (2.16), can be recast in the form

$$M = v| (h \cdot \beta^{(a)}) n^e_a + \tau (h \cdot \beta^{(a)*}) n^m_a | ,$$

(2.17)

where we have reinstated $\theta$. The action of $SL(2, \mathbb{Z})$ duality on such a state is given by

$$(n^m, n^e) \rightarrow (n^m, n^e)M^{-1} ,$$

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} ,$$

(2.18)

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - bc = 1$, with $a, b, c, d$ integers. $S$-duality is generated by $S: \tau \rightarrow -1/\tau$ and $T: \tau \rightarrow \tau + 1$. When we act with $S$ we must replace the group $G$ with its dual group $G^*$ \cite{2}. For simply laced groups the $N = 4$ supersymmetric Lagrangian with gauge group $G$ is the same as that of $G^*$ since all fields are in the adjoint representation $\dagger$.

$\dagger$ For non-simply-laced groups this is not true since for example $SO(2N + 1)^* = SP(N)$. In this case one does not expect the theory to be invariant under the full $SL(2, \mathbb{Z})$ duality group, but rather a $\Gamma_0(2)$ subgroup \cite{7}. We restrict our considerations to simply-laced gauge groups in the following.
Starting with the $W$-boson states $S$-duality predicts an infinite number of dyon BPS states. Since the quantum moduli space is assumed to be the same as the classical moduli space, these states should exist for all values of the coupling constant $\tau$ and in particular for weak coupling where semiclassical techniques are reliable.

To determine the semiclassical dyon spectrum one needs to know the moduli space of classical BPS monopole solutions. Using an index theorem Weinberg has argued that the moduli space of monopoles of charge $n^m$ has dimension

$$d = 4 \sum_a n_a^m .$$  \hspace{1cm} (2.19)$$

A number of explicit monopole solutions can be constructed by embedding $SU(2)$ monopoles as follows \cite{3}. Let $\phi^s$, $A^s_i$ be an $SU(2)$ monopole solution with charge $k$ and Higgs expectation value $\lambda$. If we let $\alpha$ be any root satisfying $\alpha \cdot h > 0$ then we can define an $SU(2)$ subgroup with generators

\begin{align*}
t^1 &= (2\alpha^2)^{-1/2}(E_\alpha + E_{-\alpha}) \\
t^2 &= -i(2\alpha^2)^{-1/2}(E_\alpha - E_{-\alpha}) \\
t^3 &= (\alpha^2)^{-1} \alpha \cdot H.
\end{align*}

A monopole with magnetic charge

$$g = 4\pi k\alpha^*$$ \hspace{1cm} (2.21)

is then given by

\begin{align*}
\phi &= \sum_s \phi^s t^s + v(h - \frac{h \cdot \alpha}{\alpha^2} \cdot H) \\
A_i &= \sum_s A_i^s t^s \hspace{1cm} (2.22) \\
\lambda &= v h \cdot \alpha.
\end{align*}

Since the moduli space of $SU(2)$ monopoles with charge $k$ has dimension $4k$ these solutions provide a $4k$ dimensional submanifold of monopoles with charge (2.21). Note that by embedding an $SU(2)$ monopole with charge one we obtain spherically symmetric monopole solutions.

Weinberg has shown that there is a distinguished set of $l$ “fundamental monopoles” with $g = 4\pi \beta^{(a)*}$ i.e., they have magnetic charge vectors $n^m$ consisting of a one in the $a$th position and zeroes elsewhere. The reason for calling them fundamental is twofold. Firstly, they have no “internal” degrees of freedom: all of these solutions can be constructed by embedding an $SU(2)$ monopole of unit charge using the corresponding simple root and consequently they have only four zero modes: three translation zero modes and a $U(1)$ phase zero mode corresponding to dyonic excitations of the same $U(1)$ as where the magnetic charge lies*. Secondly, a general monopole with charge $n^m$ can be considered to be a

\* One can check that the embedded $SU(2)$ solutions are invariant under gauge transformations of the other $U(1)$’s.
multimonopole configuration consisting of $n^a_m$ monopoles of type $a$. The Bogomol’nyi mass formula (2.17) and the dimension of the moduli space (2.19) support this interpretation.

Note that for magnetic monopoles with charge vector $g = 4\pi k\beta^{(a)*}$ i.e. consisting of $k$ fundamental monopoles of the same type, the dimension of moduli space is $4k$. Thus we deduce that these solutions can all be obtained by embedding $SU(2)$ monopoles of charge $k$, using the embedding based on the same simple root.

3. Duality and $SU(3)$ Dyons

In order to simplify the notation we now restrict ourselves to gauge group $SU(3)$. The generalization of the discussion to other gauge groups should be reasonably clear and we will return to this in the discussion section.

The $W$-boson states have electric charge vectors $n^e = \pm(1,0), \pm(0,1)$ and $\pm(1,1)$ corresponding to the two simple roots and the non-simple root, respectively. Note that the $(1,1)$ $W$-boson is only neutrally stable into the decay of two simple root $W$-bosons. Starting with these BPS saturated states, $SL(2,\mathbb{Z})$-duality generates an orbit of electric and magnetic charge vectors $(n^m, n^e)$ given by

$$(p(1,0), q(1,0)), \quad (p(0,1), q(0,1)), \quad (p(1,1), q(1,1)),$$

for relatively prime integers $p$ and $q$. At weak coupling these states should be visible in the semiclassical quantization of the monopole solutions. A monopole solution with charge vector $n^e = (n_1, n_2)$ can be interpreted as being a multimonopole configuration consisting of $n_1$ fundamental monopoles of type $(1,0)$ and $n_2$ fundamental monopoles of type $(0,1)$.

First let us consider the monopoles with $n^m = (p,0)$ or $n^m = (0,p)$ i.e. $p$ fundamental monopoles of the same type. As we mentioned at the end of the last section, all of these $SU(3)$ monopoles can be obtained by embedding $SU(2)$ monopoles with charge $p$ using the appropriate simple root. The dyonic states with charges $(p(1,0), q(1,0))$ and $(p(0,1), q(0,1))$ predicted by duality should thus emerge from the semiclassical quantization of $SU(2)$ monopoles [8]. In this way, the predictions of $S$-duality for gauge group $SU(2)$ are embedded in the $SU(3)$ case.

The new predictions for $SU(3)$ monopoles thus arise in the sectors with both magnetic quantum numbers non-zero. In particular, the $(p(1,1), q(1,1))$ dyon states should arise as bound states of $p (1,0)$ and $p (0,1)$ monopoles. Note from the BPS mass formula that these states are only neutrally stable and consequently they should emerge as bound states at threshold. In the following, we will prove the existence of these states for the case $p = 1$.

To proceed we must quantize the collective coordinates of the 2-monopole solution corresponding to a $(1,0)$ and a $(0,1)$ monopole. This moduli space does not appear to have been studied in the literature, so our first task will be to determine its form. By factoring out the center of mass, we expect that the moduli space is of the form $R^3 \times S^1 \times M$. The $S^1$ should correspond to the overall $U(1)$ charge aligned along the $(1,1)$ direction. We will see later that this is not quite correct and we must replace the $S^1$ with $R^1$ and also make a discrete identification by $\mathbb{Z}$. From (2.19) we deduce that the “relative moduli space”, $M$, is four-dimensional. Since the low-energy dynamics of the monopoles is given
by an $N = 4$ supersymmetric quantum mechanics on the moduli space, $M$ must be hyperkähler. The spherically symmetric $(1,1)$ monopole solution obtained by the $SU(2)$ embedding based on the non-simple positive root corresponds to a single point in $M$.

$M$ also should admit various isometries: as there are two independent $U(1)$ charges which are good quantum numbers $M$ should admit a $U(1)$ group of isometries. In addition there are isometries arising from the the action of spatial rotations, combined in general with a gauge transformation. One might naively think that this gives rise to an $SO(3)$ group of isometries. However, by studying the behavior of the zero modes about the spherically symmetric $(1,1)$ monopole solution we will now argue that the group of isometries is in fact $U(1) \times SU(2)$.

Let us work in $A_0^a = 0$ gauge and consider the solution to the Bogomolʼnyi equation $B_i = D_i \phi$ corresponding to the embedding of the 't Hooft-Polyakov $SU(2)$ monopole in a nonsimple $(1,1)$ root of $SU(3)$. The zero modes of this solution satisfy the linearized equation

$$D_i \delta \phi + [\delta A_i, \phi] - \epsilon_{ijk} D_j \delta A_k = 0 . \hspace{1cm} (3.2)$$

The gauge condition

$$D_i \delta A_i + [\phi, \delta \phi] = 0 , \hspace{1cm} (3.3)$$

is imposed to remove modes corresponding to residual gauge transformations. As noted in [19,20], we may combine $\delta \phi$ and $\delta A_i$ into a 4-vector. Using the decomposition $O(4) = SU(2) \times SU(2)$ this 4-vector may be represented as a $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2) \times SU(2)$, i.e. as the $2 \times 2$ matrix

$$\psi = I \delta \phi + i \sigma_j \delta A_j , \hspace{1cm} (3.4)$$

where the $\sigma_j$ are the Pauli matrices. For real zero modes we deduce that $\psi$ satisfies the reality constraint $\psi^* = \sigma_2 \psi \sigma_2$. In this notation, equations (3.2) and (3.3) take the simple form

$$-i \sigma_j D_j \psi + [\phi, \psi] = 0 . \hspace{1cm} (3.5)$$

Noting that this equation does not mix the two columns of the matrix $\psi$, we see $\psi$ may be constructed from solutions to the 2-component spinor equation

$$-i \sigma_j D_j \chi + [\phi, \chi] = 0 . \hspace{1cm} (3.6)$$

The spinor $\chi$ transforms as the $(\frac{1}{2}, 0)$ rep of $SU(2) \times SU(2)$, as does $\sigma_2 \chi^*$. Spatial rotations correspond to the diagonal $SU(2)$ subgroup of $SU(2) \times SU(2)$. Setting

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} , \hspace{1cm} (3.7)$$

a solution $\psi$ is

$$\psi = \chi \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \sigma_2 \chi^* \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \hspace{1cm} (3.8)$$

Another linearly independent solution is obtained by replacing $\chi$ by $i \chi$.

The solutions to the Dirac equation (3.6) are discussed in [10]. The modes are categorized by their quantum numbers with respect to an $SU(2)$ isospin $t$ and a $U(1)$ hypercharge.
y. Generators of $SU(3)$ which lie in the Cartan subalgebra are isospin singlets with $y = 0$. The roots have

$$t_3 E_\alpha = \frac{\beta \cdot \alpha}{\beta^2} E_\alpha$$

$$y E_\alpha = \left( \frac{h \cdot \alpha}{h \cdot \beta} - t_3 \right) E_\alpha,$$

where $\beta$ is the root used to embed the $SU(2)$ solution. The adjoint of $SU(3)$ decomposes as $8 \rightarrow 3 + 2 + 2 + 1$ with respect to $t$ and the hypercharge depends on the Higgs field.

The number of normalizable modes is as follows [10]:

$$t = \frac{1}{2} : 0 \leq |y| < \frac{1}{2}, \text{ one},$$

$$\frac{1}{2} \leq |y|, \text{ zero},$$

$$t = 1 : 0 \leq |y| < 1, \text{ two},$$

$$1 \leq |y|, \text{ zero}.$$

We will not consider the cases $t = |y| = \frac{1}{2}$ and $t = |y| = 1$ in the following since they do not occur for maximal symmetry breaking. Let us define the generator of spatial rotations $j = L + s$. The zero modes are eigenvectors under the combined rotation and gauge transformation generated by $J = j + t$. Note that the $SU(2)$ embedded solution itself is spherically symmetric with respect to this $SU(2)$. Since the bosonic zero modes are constructed as a tensor product of $\chi$ with a constant $(0, \frac{1}{2})$ spinor (plus a piece with $-i\sigma_2 \chi^* \text{ tensored with another } (0, \frac{1}{2})$ spinor) and similarly for $i\chi$, we see that $\psi$ transforms as a $J \otimes \frac{1}{2}$ representation.

The fermion zero modes arising from the triplet of $SU(2)$ have quantum numbers $t = 1, y = 0$ and $J = 1/2$. The bosonic zero modes thus transform as a $1 \oplus 0$ rep with respect to $J$. This corresponds to the $R^3 \times S^1$ factor of the moduli space, which arises from the three translation zero modes of the center of mass, together with an overall $U(1)$ phase degree of freedom.

When $\beta$ is a simple root, the two doublets with $t = 1/2$ have $|y| > 1/2$ and lead to no further zero modes. On the other hand when $\beta$ is a nonsimple root the two doublets have $0 \leq |y| < 1/2$ (with opposite signs of $y$) and there are an additional pair of normalizable fermion zero modes. The bosonic zero modes then transform as two doublets with respect to $J$. The moduli space we are interested in therefore has a $SU(2)$ subgroup of isometries rather than the usual $SO(3)$ group, since the zero modes are sensitive to the center of $SU(2)$.

So we are led to look for a four dimensional hyperkähler manifold with an $SU(2) \times U(1)$ group of isometries. Moreover, there should be a fixed point, a “NUT”, of the $SU(2)$ action corresponding to the spherically symmetric embedded $SU(2)$ solution. In addition, since the three complex structures are inherited from those on $R^4$ (see e.g., [11]), they should transform as a triplet under the action of $SU(2)$. A classification of such spaces has been carried out by Atiyah and Hitchin [21]. Assuming that the manifold is complete we are led to one of two possibilities: Euclidean positive mass Taub-NUT space and $R^4$. Note that the Atiyah-Hitchin manifold is excluded both because it has just $SO(3)$ isometry...
and the only fixed point set is a two-dimensional “bolt”. Likewise, $R^3 \times S^1$ is ruled out because its isometry group includes $SO(3)$ rather than $SU(2)$. Asymptotically, we expect the manifold to approach $R^3 \times S^1$, with the $S^1$ arising from the relative $U(1)$ orientation of the two monopoles, and the $R^3$ from the separation of the monopole centers. $R^4$ is thus ruled out and Taub-NUT is the unique solution.

The metric for Taub-NUT space is given by

$$ds^2 = V^{-1}dr^2 + V^{-1}r^2((\sigma_1^R)^2 + (\sigma_2^R)^2) + V(\sigma_3^R)^2,$$

with

$$V^{-1} = 1 + \frac{1}{r},$$

where we have scaled out the positive mass parameter and set it equal to $1/2$, and where $\sigma_i^R$ are a basis of left-invariant one-forms on $S^3$ whose explicit form is

$$\sigma_1^R = -\sin \psi d\theta + \cos \psi \sin \theta d\phi$$

$$\sigma_2^R = \cos \psi d\theta + \sin \psi \sin \theta d\phi$$

$$\sigma_3^R = d\psi + \cos \theta d\phi.$$  

(3.13)

By introducing the coordinate $R = 2\sqrt{r}$ it is straightforward to show that the metric is non-singular at $r = 0$ if the period of $\psi$ has period $4\pi$. Choosing this period, Taub-NUT has $SU(2)$ isometry. The generators of the $SU(2)$ isometry are given by the “left vector fields”, $\xi_i^L$, $i = 1, 2, 3$, whose explicit form can be found, for example, in [13]. The generator of the extra $U(1)$ isometry is the “right vector field” $\xi_3^R = \partial/\partial \psi$.

To ensure that semiclassical quantization yields a spectrum of dyons with electric charges lying on the lattice (2.10), we must replace the overall $S^1$ factor by $R^1$ and perform a discrete identification by $\mathbb{Z}$, as we now explain [22]. The electric charge operators $Q_1$ and $Q_2$ defined by

$$Q_1 = \frac{2}{3}(2\beta^{(1)} + \beta^{(2)}) \cdot H,$$

$$Q_2 = \frac{2}{3}(\beta^{(1)} + 2\beta^{(2)}) \cdot H,$$

(3.14)

take integer values $n_i^a$. The total electric charge $Q_\chi$, conjugate to the collective coordinate $\chi$, can be determined in terms of the $Q_i$ by expanding the Bogomol’nyi mass formula (2.17) for $n^m = (1, 1)$ and (small) general $n^i$. The kinetic energy terms in the Hamiltonian for the collective coordinates conjugate to the charges $Q_i$ can also be determined from (2.17) by considering two well separated fundamental monopoles. The relative electric charge $Q_\psi$ conjugate to $\psi$ is then determined by reexpressing these kinetic energy terms in terms of $Q_\chi$ and $Q_\psi$ using the fact that the moduli space is in a factored form. We find

$$Q_\chi = (m_1 Q_1 + m_2 Q_2)/(m_1 + m_2), \quad Q_\psi = \frac{1}{2}(Q_1 - Q_2),$$

(3.15)

where $m_a = v(4\pi/g^2)\beta^{(a)} \cdot H$ is the mass of the $a$th (pure) fundamental monopole. It now follows that we must impose the discrete identification $(\chi, \psi) \sim (\chi + 2\pi, \psi + 4\pi m_2/(m_1 +$
to reproduce the allowed lattice of charges \((\mathbb{Z}/m_1, \mathbb{Z}/m_2)\). In general, the coordinate \(\chi\) is not periodic and hence the moduli space is given by \(R^3 \times (R^1 \times M)/\mathbb{Z}\). Note that if \(m_2/m_1 = p/q\) is rational then \(\chi\) has period \(2\pi(p + q)\) and the moduli space is then \(R^3 \times (S^1 \times M)/\mathbb{Z}_{p+q}\).

The low-energy dynamics of the monopoles is given by an \(N = 4\) supersymmetric quantum mechanics on the moduli space. As discussed in [11,12] the states are given by differential forms on moduli space. The Hamiltonian is the Laplacian acting on these forms. The basis of 16 forms on \(R^3 \times R^1\) leads to a BPS supermultiplet of 16 states with total charge \(Q_\chi = x, x \in R^1\). These should be tensored with forms on Taub-NUT space with relative charge \(Q_\psi = n/2, n \in \mathbb{Z}\), where \(x + nm_2/(m_1 + m_2) \in \mathbb{Z}\) to ensure the state is well defined on the moduli space. The value of \(n\) is simply the eigenvalue of the operator \(-i\xi^R_{3}\).

Recall that \(S\)-duality predicts that there should be a tower of dyon BPS states with magnetic charge \((1, 1)\) and electric charge \(q(1, 1)\), for arbitrary integer \(q\). In order for these states to exist there must be a normalizable harmonic form on \(M\). It should be harmonic to ensure that when it is combined with the forms on \(R^3 \times R^1\), the state saturates the Bogomol’nyi bound as explained in more detail in [1]. Since the electric charge of the predicted states is only in the \((1, 1)\) direction, the harmonic form should carry no relative electric charge which means that it must be invariant under the generator of the \(U(1)\) isometry \(\xi^R_{3}\). The above discussion then implies that \(x\) is integer valued and that \(n^e = x(1, 1)\) as required. Since we only require a single harmonic form, it should either be self-dual or anti-self dual. Taub-NUT space does indeed admit a harmonic self-dual form given by

\[
\omega = \frac{r}{r + 1} \sigma^1 \wedge \sigma^2 + \frac{1}{(r + 1)^2} dr \wedge \sigma_3 = d(V\sigma_3).
\]  

(3.16)

It is straightforward to check that it is well defined at \(r = 0\), is normalizable and is invariant under the action of \(\xi^R_{3}\), exactly as required by duality.

4. Discussion

Starting with the charged \(W\)-boson BPS states we have argued that \(S\)-duality for gauge group \(SU(3)\) predicts an infinite tower of BPS states with electric and magnetic charge vectors given by \((3.1)\). Making the weak assumption that the \(W\)-boson states are the only purely electric charged states in the theory, \(S\)-duality implies that these are the only BPS states in the theory whose electric and magnetic charge vectors are parallel. If other states of this type existed, then acting with \(S\) duality transformations would give a purely electric charged state which was not a \(W\)-boson. Note that if the charge vectors are not parallel then there may exist states which break one quarter of the supersymmetry and form medium size supermultiplets. Such solutions have been found in supergravity (see for example [25]) and it would be interesting to know if they also existed in field theory.

We have shown that the existence of the states in \((3.1)\) with magnetic charge vector \((p, 0)\) or \((0, p)\) is equivalent to their existence in the \(SU(2)\) theory. The main result of the

\[\footnote{The existence of this form was noted in a different context in [23,24].} \]
paper was the verification that the states with magnetic charge vector \((1, 1)\) also exist. The general states with charges \((p(1, 1), q(1, 1))\) should emerge via the existence of certain harmonic forms on the moduli space of \(p (1, 1)\) monopoles. An explicit verification of this seems a difficult undertaking at present since the moduli spaces are not known.

The preceding discussion for \(SU(3)\) has a simple extension to a more general simply-laced gauge group. The W-bosons, with electric quantum numbers \(n^e\), generate an orbit of electric and magnetic quantum numbers given by

\[
(n^m, n^e) = (pn^e, qn^e),
\]

where the integers \(p\) and \(q\) are relatively prime. Exact duality for gauge group \(SU(2)\) implies the existence of bound states when \(n^e\) is that of a simple root W-boson. As before, the new predictions arise when one considers a nonsimple root \(\alpha\). The analysis of section 3 for \(n^m = (1, 1)\) carries over directly when \(\alpha^* = \beta_1^* + \beta_2^*\), with \(\beta_1\) and \(\beta_2\) different simple roots, and when the regular embedding of the \(SU(2)\) monopole solution using the root \(\alpha\) gives rise to fields transforming as a complex doublet of the \(SU(2)\), in addition to the usual triplet. A list of such embeddings may be extracted from table 58 of [26]. The moduli space of these monopole solutions will be the same and the required bound state will exist, as discussed above.

More generally, the \(SU(3)\) predictions for \(n^m = p(1, 1)\) will be embedded in higher rank gauge groups. However, since some of the nonsimple roots must be expressed as a sum over \(n\) \((n > 2)\) simple roots, there will also be new predictions. \(SL(2, \mathbb{Z})\) invariance predicts that the corresponding moduli spaces also admit unique harmonic forms. As far as we know, these moduli spaces are also not yet known.

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