A Complete Parameterized Complexity Analysis of Bounded Planning

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Abstract

The propositional planning problem is a notoriously difficult computational problem, which remains hard even under strong syntactical and structural restrictions. Given its difficulty it becomes natural to study planning in the context of parameterized complexity. In this paper we continue the work initiated by Downey, Fellows and Stege on the parameterized complexity of planning with respect to the parameter “length of the solution plan.” We provide a complete classification of the parameterized complexity of the planning problem under two of the most prominent syntactical restrictions, i.e., the so called PUBS restrictions introduced by Bäckström and Nebel and restrictions on the number of preconditions and effects as introduced by Bylander. We also determine which of the considered fixed-parameter tractable problems admit a polynomial kernel and which don’t.

1 Introduction

The (propositional) planning problem has been the subject of intensive study in knowledge representation, artificial intelligence and control theory and is relevant for a large number of industrial applications [28]. The problem involves deciding whether an initial state—an n-vector over some domain D—can be transformed into a goal state via the application of actions (or operators) each consisting of preconditions and post-conditions (or effects) stating the conditions that need to hold before the action can be applied and which conditions will hold after the application of the action, respectively. It is known that the problem of deciding whether a solution exists or not is \(\text{PSPACE}\)-complete [8, 13]. Although various \(\text{NP}\)-complete and even tractable restrictions are known in the literature [8, 12, 13, 38] these are often considered not to coincide well with cases that are interesting in practice. In the authors experience, classical complexity analysis is often too coarse to give relevant results for planning, since most interesting restrictions seem to remain \(\text{PSPACE}\)-complete. Despite this, there has been very few attempts to use alternative analysis methods. The few exceptions include probabilistic analysis [14], approximation [9, 37] and padding [5].

Another obvious alternative is to use the framework of Parameterized Complexity which offers the more relaxed notion of fixed-parameter tractability (FPT). A problem is fixed-parameter tractable if it can be solved in time \(f(k)n^{O(1)}\) where \(f\) is an arbitrary function of the parameter \(k\) (which measures some aspect of the input) and \(n\) is the input size. Indeed, already in a 1999 paper, Downey, Fellows and Stege [18] initiated the parameterized analysis of planning, taking the minimum number of steps from the initial state to the goal state (i.e., the length of the solution plan) as the parameter. However, the parameterized viewpoint did not immediately gain momentum for analysis of planning and it is only during the last few years that we have witnessed a strongly increased interest in this method, cf. [5, 6, 17, 40].
Table 1: Complexity of Bounded SAS+ Planning when restricting the number of preconditions (p) and effects (e). All parameterized results are shown in this paper and all classical results are from Bylander [13]. The classical results apply to STRIPS, while the parameterized results hold for SAS+ (the hardness results hold already for binary domains and the membership results hold for arbitrary domains). We also show that none of the problems that are in FPT admit polynomial kernels.

|         | e = 1     | e = 2     | fixed e > 2 | arbitrary e |
|---------|-----------|-----------|-------------|-------------|
| p = 0   | in P      | in FPT    | W[1]-C      | W[2]-C      |
|         | in P      | NP-C      | NP-C        | NP-C        |
| p = 1   | W[1]-C    | W[1]-C    | W[1]-C      | W[2]-C      |
|         | NP-H      | NP-H      | NP-H        | PSPACE-C    |
| fixed p > 1 | W[1]-C | W[1]-C | W[1]-C | W[2]-C |
|         | NP-H      | PSPACE-C  | PSPACE-C    | PSPACE-C    |
| arbitrary p | W[1]-C | W[1]-C | W[1]-C | W[2]-C |
|         | PSPACE-C  | PSPACE-C  | PSPACE-C    | PSPACE-C    |

In this article, we use the same parameter as Downey et al. and provide a complete analysis of planning under various syntactical restrictions, in particular the restrictions considered by Bylander [13] and by Bäckström and Nebel [8]. These were among the first attempts to understand why and when planning is hard or easy and they have had a heavy influence on later theoretical research in planning. We complement these results by also considering bounds on problem kernels for those planning problems that we prove to be fixed-parameter tractable. It is known that a decidable problem is fixed-parameter tractable if and only if it admits a polynomial-time self-reduction where the size of the resulting instance is bounded by a function f of the parameter [24, 26, 30]. The function f is called the kernel size. By providing upper and lower bounds on the kernel size, one can rigorously establish the potential of polynomial-time preprocessing for the problem at hand.

Our results

We provide a full parameterized complexity analysis of planning with respect to the length of the solution plan, under all combinations of the syntactical P, U, B, and S restrictions previously considered by Bäckström and Nebel [8] as well as under the restrictions on the number of preconditions and effects previously considered by Bylander [13]. Our new parameterized results are summarized in Table 1 and Figure 1 alongside the previously reported classical complexity results. We discuss our results more thoroughly in Section 8. In addition, we examine whether the fixed-parameter tractable subcases, which we obtain, admit polynomial kernels or not. Our results on this are negative throughout—if any of these problems admit polynomial kernels, or even polynomial bi-kernels, then coNP ⊆ NP/poly and the Polynomial-time Hierarchy collapses.

Outline

The rest of the paper is laid out as follows. Section 2 defines some concepts of parameterized complexity theory and Section 3 defines the SAS+ and STRIPS planning languages. The hardness results are collected in Section 4 and the membership results in Section 5. Section 6 is devoted to our tractability results and in Section 7 we show that none of the tractable subcases admits a polynomial kernel. We summarize the results of the paper in Section 8 and discuss some observations and consequences. The paper ends with an outlook in Section 9.
Figure 1: Complexity of BOUNDED SAS$^+$ PLANNING for the restrictions P, U, B and S illustrated as a lattice defined by all possible combinations of these restrictions. Again all parameterized results are shown in this paper and all classical results are from Bäckström and Nebel [8]. Furthermore, as shown in this paper, PUS and PUBS are the only restrictions that admit a polynomial kernel, unless the Polynomial Hierarchy collapses.

2 Parameterized Complexity

We define the basic notions of Parameterized Complexity and refer to other sources [19, 25] for an in-depth treatment. Let $\Sigma$ be a finite alphabet and $\mathbb{N}$ the set of natural numbers. A parameterized decision problem or parameterized problem, for short, is a language $L \subseteq \Sigma^* \times \mathbb{N}$. The instances of the problem are pairs on the form $\langle I, k \rangle$, where $I$ is a string over $\Sigma^*$, which constitutes the main part, and $k$ is the parameter. A parameterized problem is fixed-parameter tractable (FPT) if there exists an algorithm that solves any instance $\langle I, k \rangle$ in time $f(k)n^c$ where $f$ is an arbitrary computable function, $n = |\langle I, k \rangle|$, and $c$ is a constant independent of both $n$ and $k$. FPT is the class of all fixed-parameter tractable parameterized problems. Since the emphasis in parameterized algorithms lies on the dependence of the running time on $k$ we will sometimes use the notation $O^*(f(k))$ as a synonym for $O(f(k)n^c)$.

Parameterized complexity offers a completeness theory, similar to the theory of NP-completeness, that supports the accumulation of strong theoretical evidence that certain parameterized problems are not fixed-parameter tractable. This theory is based on a hierarchy of complexity classes

$$\text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \text{W}[3] \subseteq \ldots$$

where all inclusions are believed to be strict. Each class $\text{W}[i]$ contains all parameterized problems that can be reduced by a parameterized reduction to a certain canonical parameterized problem (known as Weighted $i$-Normalized Satisfiability) under parameterized reductions.

A parameterized reduction or fpt-reduction from a parameterized problem $P$ to a parameterized problem $Q$ is an algorithm that maps instances $\langle I, k \rangle$ of $P$ to instances $\langle I', k' \rangle$ of $Q$ such that:

1. $\langle I, k \rangle \in P$ if and only if $\langle I', k' \rangle \in Q$;  
2. there is a computable function $g$ such that $k' \leq g(k)$; and  
3. there is a computable function $f$ and a constant $c$ such that $\langle I, k \rangle$ is computed in time $O(f(k) \cdot n^c)$, where $n = |\langle I, k \rangle|$.
A bi-kernelization [1] (or generalized kernelization [11]) for a parameterized problem $P$ is a parameterized reduction from $P$ to a parameterized problem $Q$ that maps instances $(I, k)$ of $P$ to instances $(I', k')$ of $Q$ with the additional property that

1. $(I', k')$ can be computed in time that is polynomial in $|I| + k$, and
2. $|I'|$ and $k'$ are both bounded by some function $f$ of $k$.

The output $(I', k')$ is called a bi-kernel (or a generalized kernel). We say that $P$ has a polynomial bi-kernel if $f$ is a polynomial. If $P = Q$, we call the bi-kernel a kernel. Every fixed-parameter tractable problem admits a bi-kernel, but not necessarily a polynomial bi-kernel [15].

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1. $(I', k')$ can be computed in time that is polynomial in $|I| + k$, and
2. $k'$ is bounded by some polynomial $p$ of $k$.

The following result is an adaptation of a result by Bodlaender [10, Theorem 8].

**Proposition 1.** Let $P$ and $Q$ be two parameterized problems such that there is a polynomial parameter reduction from $P$ to $Q$. Then, if $Q$ has a polynomial bi-kernel also $P$ has a polynomial bi-kernel.

**Proof.** Let $(I, k)$ be an instance of $P$. We first apply the polynomial parameter reduction from $P$ to $Q$ to the instance $(I, k)$ and obtain the instance $(I', k')$ of $Q$. Then the instance $(I'', k'')$ of some parameterized problem, say $Q'$, that we obtain by applying the polynomial bi-kernelization algorithm for $Q$ is also a polynomial bi-kernel for $P$. This concludes the proof of the proposition.

For a parameterized problem $P \subseteq \Sigma^* \times \mathbb{N}$, we define its unparameterized version $[P]$ as the corresponding classical problem with the parameter given in unary. That is, every instance $(I, k)$ of $P$ has a corresponding instance $[(I, k)]$ of $[P]$ such that $[(I, k)]$ is the string $I\#^1k$, where $\#$ is a separator symbol and 1 is an arbitrary symbol in $\Sigma$.

We note here that most researchers in kernelization talk about kernels instead of bi-kernels. It is however well known that the approaches to obtain lower bounds for kernels and bi-kernels, respectively, work in the same manner [11]. It is immediate from the definitions that if a problem does not admit a polynomial bi-kernel, then it cannot admit a polynomial kernel either, so super-polynomial lower bounds for the size of bi-kernels imply super-polynomial lower bounds for the size of kernels. Since we will be mostly concerned with lower bounds we will give all our results in terms of bi-kernels.

An OR-composition algorithm for a parameterized problem $P$ maps $t$ instances $(I_1, k_1), \ldots, (I_t, k_t)$ of $P$ to one instance $(I', k')$ of $P$ such that the algorithm runs in time polynomial in $\sum_{1 \leq i \leq t} |I_i| + k$, the parameter $k'$ is bounded by a polynomial in the parameter $k$, and $(I', k') \in P$ if and only if there is an $i$, where $1 \leq i \leq t$, such that $(I_i, k_i) \in P$.

**Proposition 2** (Bodlaender, et al. [11, Lemmas 1 and 2]). If a parameterized problem $P$ has an OR-composition algorithm and its unparameterized version $[P]$ is NP-complete, then it has no polynomial bi-kernel\textsuperscript{1} unless $\text{coNP} \subseteq \text{NP}/\text{poly}$.

Unfortunately, this proposition can only be applied to problems that are contained in $\text{NP}$, while membership in $\text{NP}$ is not known for some of the problems that we consider in this article. Furthermore, to the best of our knowledge, there is no consistent presentation of bi-kernel lower bounds for problems that may not be in $\text{NP}$. Hence, we fill this gap by giving such a presentation from first principles, i.e., in the following we will give an analogue of Proposition 2 for problems that are not (or not known to be) in $\text{NP}$.

A distillation algorithm [11, 27] for a classical problem $P$ is an algorithm that takes $t$ instances $I_1, \ldots, I_t$ of $P$ as the input, runs in time polynomial in $\sum_{1 \leq i \leq t} |I_i|$, and outputs an instance $I$ of some problem $Q$ such that: (1) $I \in Q$ if and only if there is an $i$, where $1 \leq i \leq t$, such that $I_i \in P$ and (2) $|I|$ is polynomial in $\max_{1 \leq i \leq t} |I_i|$.

\textsuperscript{1}The original results refer to kernels but the authors remark that all their results extend also to bi-kernels.
Proposition 3 (Fortnow and Santhanam [27, Theorem 1.2]). Unless coNP ⊆ NP/poly, the satisfiability problem for propositional formulas (SAT) has no distillation algorithm.

Using the fact that SAT is NP-complete and there are polynomial reductions from SAT to any other NP-hard problem, we immediately obtain the following corollary.

Corollary 1. Unless coNP ⊆ NP/poly, no NP-hard problem has a distillation algorithm.

Below, we first introduce a stronger OR-composition concept and then prove generalizations to NP-hard problems of two results known from the literature.

A strong OR-composition algorithm for a parameterized problem \( P \) maps \( t \) instances \( \langle I_1, k_1 \rangle, \ldots, \langle I_t, k_t \rangle \) of \( P \) to one instance \( \langle I, k \rangle \) of \( P \) such that the algorithm runs in time polynomial in \( \sum_{1 \leq i \leq t} |I_i| + \max_{1 \leq i \leq t} k_i \), the parameter \( k \) is bounded by a polynomial in \( \max_{1 \leq i \leq t} k_i \), and \( \langle I, k \rangle \in P \) if and only if there is an \( i \), where \( 1 \leq i \leq t \), such that \( \langle I_i, k_i \rangle \in P \).

The following result is an adaptation of Proposition 2, based on the original proof of this result [11].

Proposition 4. If a parameterized problem \( P \) has a strong OR-composition algorithm and its unparameterized version \( [P] \) is NP-hard, then it has no polynomial bi-kernel unless coNP ⊆ NP/poly.

Proof. We show that if \( P \) satisfies the conditions of the proposition and \( P \) has a polynomial bi-kernel, then \( [P] \) has a distillation algorithm and it thus follows from Corollary 1 that coNP ⊆ NP/poly.

Let \( \langle I_1, k_1 \rangle, \ldots, \langle I_t, k_t \rangle \) be instances of \( P \) and \( \langle [I_1, k_1], \ldots, [I_t, k_t] \rangle \) be the corresponding unparameterized instances of \( [P] \). We give a distillation algorithm for \( [P] \) that consists of two steps.

In the first step the algorithm runs the strong OR-composition algorithm for \( P \) and obtains the instance \( \langle I, k \rangle \) from the instances \( \langle I_1, k_1 \rangle, \ldots, \langle I_t, k_t \rangle \). In the second step the algorithm runs the polynomial bi-kernelization algorithm on \( \langle I, k \rangle \) and obtains the instance \( \langle I', k' \rangle \) of a parameterized problem \( Q \). The algorithm then outputs \( \langle [I', k'] \rangle \).

In the following we will show that the algorithm outlined above is indeed a distillation algorithm for \( [P] \). It is straightforward to verify that \( \langle [I', k'] \rangle \in [Q] \) if and only if there is an \( i \), where \( 1 \leq i \leq t \), such that \( \langle [I_i, k_i] \rangle \in [P] \). The running time of the first step of our algorithm is polynomial in \( \sum_{1 \leq i \leq t} |I_i| + \max_{1 \leq i \leq t} k_i \) (because of the properties of the strong OR-composition algorithm), which in turn is polynomial in \( \sum_{1 \leq i \leq t} |\langle [I_i, k_i] \rangle| \). The running time of the second step is polynomial in \( |\langle [I, k] \rangle| \), which is again polynomial in \( \sum_{1 \leq i \leq t} |\langle [I_i, k_i] \rangle| \). Hence, the running time of the complete algorithm is polynomial in \( \sum_{1 \leq i \leq t} |\langle [I_i, k_i] \rangle| \), as required for a distillation algorithm.

Furthermore, since \( \langle [I', k'] \rangle \) is a polynomial bi-kernel of \( \langle I, k \rangle \), it follows that \( |\langle [I', k'] \rangle| \) is bounded by a polynomial function in \( k' \), and hence in \( k \). Because of the properties of the strong OR-composition algorithm we also obtain that \( k \) is bounded by a polynomial in \( \max_{1 \leq i \leq t} k_i \), which in turn is bounded by \( \max_{1 \leq i \leq t} |\langle [I_i, k_i] \rangle| \). Putting all this together, we obtain that \( |\langle [I', k'] \rangle| \) is bounded by a polynomial function of \( \max_{1 \leq i \leq t} |\langle [I_i, k_i] \rangle| \). This shows that our algorithm is a distillation algorithm for \( [P] \), which, because of Corollary 1, implies that coNP ⊆ NP/poly.

3 Planning Framework

Let \( V = \{v_1, \ldots, v_n\} \) be a finite set of variables over a finite domain \( D \). Implicitly define \( D^+ = D \cup \{u\} \), where \( u \) is a special “undefined” value not present in \( D \). Then \( D^n \) is the set of total states and \( (D^+)^n \) is the set of partial states over \( V \) and \( D \), where \( D^n \subseteq (D^+)^n \). The value of a variable \( v \) in a state \( s \in (D^+)^n \) is denoted \( s[v] \). A SAS+ instance is a tuple \( P = \langle V, D, A, I, G \rangle \) where \( V \) is a set of variables, \( D \) is a domain, \( A \) is a set of actions, \( I \in D^n \) is the initial state and \( G \in (D^+)^n \) is the goal state. Each action \( a \in A \) has a precondition \( \text{pre}(a) \in (D^+)^n \) and an effect \( \text{eff}(a) \in (D^+)^n \). We will frequently use the convention that a variable has value \( u \) in a precondition/effect unless a value is explicitly specified. Let \( a \in A \) and let \( s \in D^n \). Then \( a \) is valid in \( s \) if for all \( v \in V \), either \( \text{pre}(a)[v] = s[v] \) or \( \text{pre}(a)[v] = u \). Furthermore, the result of \( a \) in \( s \) is a state \( t \in D^n \) defined such that for all \( v \in V \), \( t[v] = \text{eff}(a)[v] \) if \( \text{eff}(a)[v] \neq u \) and \( t[v] = s[v] \) otherwise.

Let \( s_0, s_t \in D^n \) and let \( \omega = \langle a_1, \ldots, a_k \rangle \) be a sequence of actions. Then \( \omega \) is a plan from \( s_0 \) to \( s_t \) if either
(1) \( \omega = \emptyset \) and \( \ell = 0 \) or
(2) there are states \( s_1, \ldots, s_{\ell-1} \in D^n \) such that for all \( 1 \leq i \leq \ell \), \( a_i \) is valid in \( s_{i-1} \) and \( s_i \) is the result of \( a_i \) in \( s_{i-1} \).

A state \( s \in D^n \) is a goal state if for all \( v \in V \), either \( G[v] = s[v] \) or \( G[v] = u \). An action sequence \( \omega \) is a plan for \( P \) if it is a plan from \( I \) to some goal state. We will study the following problem:

**Bounded SAS\(^+\) Planning**

*Instance:* A tuple \((P, k)\) where \( P \) is a SAS\(^+\) instance and \( k \) is a positive integer.

*Parameter:* The integer \( k \).

*Question:* Does \( P \) have a plan of length at most \( k \)?

The propositional version of the STRIPS planning language can be treated as the special case of SAS\(^+\) satisfying restriction B. More precisely, propositional STRIPS is commonly used in two variants, differing in whether negative preconditions are allowed or not. Both these variants as well as SAS\(^+\) have been shown to be equivalent under a strong form of polynomial reduction that preserves solution length \([3]\). Hence, we will not treat STRIPS explicitly. It should be noted, though, that while the equivalence holds in the general case, it often breaks down when further restrictions are imposed.

We will mainly consider the following four restrictions, originally defined by Bäckström and Klein \([7]\).

- **Post-unique (P):** For each \( v \in V \) and each \( x \in D \) there is at most one \( a \in A \) such that \( \text{eff}(a)[v] = x \).
- **Unary (U):** For each \( a \in A \), \( \text{eff}(a)[v] \neq u \) for exactly one \( v \in V \).
- **Binary (B):** \(|D| = 2\).
- **Single-valued (S):** For all \( a, b \in A \) and all \( v \in V \), if \( \text{pre}(a)[v] \neq u \), \( \text{pre}(b)[v] \neq u \) and \( \text{eff}(a)[v] = \text{eff}(b)[v] = u \), then \( \text{pre}(a)[v] = \text{pre}(b)[v] \).

For any set \( R \) of such restrictions we write \( R\)-Bounded SAS\(^+\) Planning to denote the restriction of Bounded SAS\(^+\) Planning to only instances satisfying the restrictions in \( R \).

Additionally we will consider restrictions on the number of preconditions and effects as previously considered by Bylander \([13]\). For two non-negative integers \( p \) and \( e \) we write \( (p, e)\)-Bounded SAS\(^+\) Planning to denote the restriction of Bounded SAS\(^+\) Planning to only instances where every action has at most \( p \) preconditions and at most \( e \) effects. Apart from doing a parameterized analysis, we also generalize Bylander’s results to SAS\(^+\): all our membership results hold for arbitrary domain size while all our hardness results apply already for binary domains. All non-parameterized hardness results in Table 1 follow directly from Bylander’s classical complexity results for STRIPS \([13, \text{Fig. 1 and 2}]\). Note that we use results both for bounded and unbounded plan existence, which is justified since the unbounded case is (trivially) polynomial-time reducible to the bounded case. The membership results for PSPACE are immediate since Bounded SAS\(^+\) Planning is in PSPACE. The membership results for NP (when \( m_p = 0 \)) follow from Bylander’s \([13]\) Theorem 3.9, which says that every solvable STRIPS instance with \( m_p = 0 \) has a plan of length \( \leq m \) where \( m \) is the number of actions. It is easy to verify that the same bound holds for SAS\(^+\) instances.

4 Hardness Results

In this section we prove the three main hardness results of this paper. For the first proof we need the following problem, which is \( W[2] \)-complete \([19, \text{p. 464}]\).

**Hitting Set**

*Instance:* A finite set \( S \), a collection \( C \) of subsets of \( S \), and an integer \( k \).

*Parameter:* The integer \( k \).

*Question:* Does \( C \) have a hitting set of cardinality at most \( k \), i.e., is there a set \( H \subseteq S \) with \(|H| \leq k \) and \( H \cap c \neq \emptyset \) for every \( c \in C \)?
**Theorem 1.** \(\{B, S\}\)-Bounded SAS\(^+\) Planning is \(\text{W}[2]\)-hard, even when the actions have no preconditions.

**Proof.** We proceed by a parameterized reduction from Hitting Set. Let \(I = (S, C, k)\) be an instance of this problem. We construct an instance \(\mathbb{P}' = (P, k)\) with \(P = (V, D, A, I, G)\) of the \(\{B, S\}\)-Bounded SAS\(^+\) Planning problem such that \(I\) has a hitting set of size at most \(k\) if and only if there is a plan of length at most \(k\) for \(\mathbb{P}'\) as follows. Let \(V = \{v_c \mid c \in C\}\), let \(D = \{0, 1\}\) and let \(A = \{a_s \mid s \in S\}\) such that \(\text{eff}(a_s)[v_c] = 1\) if \(s \in c\). We set \(I = (0, \ldots, 0)\) and \(G = (1, \ldots, 1)\). Clearly, \(P\) is binary (B) and no action has any preconditions. It follows trivially from the latter observation that \(P\) is also single-valued (S).

It remains to show that \(P\) has a plan of length at most \(k\) if and only if \(I\) has a hitting set of size at most \(k\).

Suppose that \(I\) has a hitting set \(H = \{h_1, \ldots, h_t\}\) of size at most \(k\). Then \(\omega = (a_{h_1}, \ldots, a_{h_t})\) is a plan of length at most \(k\) for \(P\).

For the reverse direction suppose that there is a plan \(\omega = (a_1, \ldots, a_l)\) of length at most \(k\) for \(P\). We will show that the set \(H_p = \{ s \mid a_s \in \omega \}\) is a hitting set of size at most \(k\) for \(I\). Since \(I[v_c] = 0\) and \(G[v_c] = 1\) for every \(c \in C\), it follows that for every \(c \in C\) there has to be an action \(a_s\) with \(s \in c\) in \(\omega\). Hence, \(H_p\) is a hitting set for \(I\) and because \(l \leq k\) it follows that \(|H_p| \leq k\). \(\square\)

We continue with the second result, using the following problem, which is \(\text{W}[1]\)-complete [42].

**Multicolored Clique**

**Instance:** A \(k\)-partite graph \(G = (V, E)\) with a partition \(V_1, \ldots, V_k\) of \(V\) such that \(|V_1| = \cdots = |V_n| = n\).

**Parameter:** The integer \(k\).

**Question:** Are there nodes \(v_1, \ldots, v_k\) such that \(v_i \in V_i\) for all \(1 \leq i \leq k\) and \(\{v_i, v_j\} \in E\) for all \(1 \leq i < j \leq k\) (i.e. the subgraph of \(G\) induced by \(\{v_1, \ldots, v_k\}\) is a clique of size \(k\)?)

**Theorem 2.** \(\{U, B, S\}\)-Bounded SAS\(^+\) Planning is \(\text{W}[1]\)-hard, even for binary instances where every action has at most 1 precondition and 1 effect.

**Proof.** We proceed by a parameterized reduction from Multicolored Clique. Let \(G = (V, E)\) be a \(k\)-partite graph with partition \(V_1, \ldots, V_k\) of \(V\). Let \(k_2 = \frac{k(k-1)}{2}\) and \(k' = 7k_2 + k\), and define \(J_i = \{ j \mid 1 \leq j \leq k\) and \(j \neq i\}\) for every \(1 \leq i \leq k\).

For the \(\{U, B, S\}\)-Bounded SAS\(^+\) Planning instance \(P\) we introduce four kinds of variables:

1. For every \(e \in E\) we introduce an edge variable \(x(e)\).
2. For every \(1 \leq i \leq k\) and \(v \in V_i\) we introduce \(k - 1\) vertex variables \(x(v, j)\) where \(j \in J_i\).
3. For every \(1 \leq i \leq k\) and every \(j \in J_i\) we introduce a checking variable \(x(i, j)\).
4. For every \(v \in V\), we introduce a clean-up variable \(x(v)\).

We also introduce five kinds of actions:

1. For every \(e \in E\) we introduce an action \(a^e\) such that \(\text{eff}(a^e)[x(e)] = 1\).
2. For every \(e = \{v_i, v_j\} \in E\) where \(v_i \in V_i\) and \(v_j \in V_j\), we introduce two actions \(a^e_i\) and \(a^e_j\) such that \(\text{pre}(a^e_i)[x(e)] = 1, \text{eff}(a^e_i)[x(v_i, j)] = 1, \text{pre}(a^e_j)[x(e)] = 1\) and \(\text{eff}(a^e_j)[x(v_j, i)] = 1\).
3. For every \(v \in V_i\) and \(j \in J_i\), we introduce an action \(a^v_j\) such that \(\text{pre}(a^v_j)[x(v, j)] = 1\) and \(\text{eff}(a^v_j)[x(i, j)] = 1\).
4. For every \(v \in V\), we introduce an action \(a_v\) such that \(\text{eff}(a_v)[x(v)] = 1\).
5. For every \(v \in V_i\) for some \(1 \leq i \leq k\), and \(j \in J_i\), we introduce an action \(a^v_i\) such that \(\text{pre}(a^v_i)[x(v)] = 1\) and \(\text{eff}(a^v_i)[x(v, j)] = 0\).
Let $A_1, \ldots, A_5$ be sets of actions corresponding to these five groups, and let $A = A_1 \cup \ldots \cup A_5$ be the set of all actions. Let $I = (0, \ldots, 0)$ and define $G$ such that all checking variables $x(i,j)$ are 1, all vertex variables $x(v,j)$ are 0 and the rest are $u$. Clearly $P$ can be constructed from $G$ in polynomial time. Furthermore, $P$ is binary and no action has more than 1 precondition and 1 effect. The theorem will follow after we have shown the following claim.

Claim 1. $G$ has a $k$-clique if and only if $P$ has a plan of length at most $k'$.

$(\Rightarrow)$ Assume $G$ has a $k$-clique $K = (V_K, E_K)$ where $V_K = \{v_1, \ldots, v_k\}$ with $v_i \in V_i$ for every $1 \leq i \leq k$. We construct a plan $\omega$ for $P$ as follows. For all $1 \leq i < j \leq k$, we apply the actions $a^{(v_i,v_j)} \in A_1$, to select the edges of the clique, and $a^{(v_i,v_j)}_i, a^{(v_i,v_j)}_j \in A_2$, to set the corresponding connection information for the vertices of the clique. This gives $3k_2$ actions. Then for each checking variable $x(i,j)$, for every $1 \leq i \leq k$ and $j \in J_i$, we apply $a^{i,j}_i \in A_3$ to verify that the selected vertices do form a clique. This gives $2k_2$ actions. Now we have all checking variables set to the required value 1, but the vertex variables $x(v_i,j)$, for $1 \leq i \leq k$ and $j \in J_i$, still bear the value 1 which will have to be set back to 0 in the goal state. So we need some actions to “clean up” the values of these vertex variables. First we set up a cleaner for each vertex $v_i$ by applying $a_{v_i} \in A_4$. This gives $k$ actions. Then we use $a^{v_i,j}_j \in A_5$ for all $j \in J_i$ to set the vertex variables $x(v_i,j)$ to 0. This requires $2k_2$ actions. We observe that all the checking variables are now set to 1, and all the vertex variables are set to 0. The goal state is therefore reached from the initial state by the execution of exactly $k' = k + 7k_2$ actions, as required. Hence the forward direction of the claim is shown.

$(\Leftarrow)$ Assume $\omega$ is a plan for $P$ of length at most $k'$. In the following, we use $A^\omega_1$ to denote the set of actions from $A_1$ that occur in the plan $\omega$ and we use $A^\omega$ to denote the set of all actions from $A$ that occur in $\omega$. In the initial state all variables are set to 0 and in the goal state all the $2k_2$ checking variables must be set to 1, so it follows that $|A^\omega_2| \geq 2k_2$ since each action in $A_3$ sets exactly one checking variable to 1. Each action in $A^\omega_2$ requires that a distinct vertex variable is set to 1 first. This can only be accomplished by the execution of an action from $A_2$, hence $|A^\omega_2| \geq 2k_2$. In turn, to make sure that some action in $A^\omega_2$ can be executed, some edge variable must be set to 1 first by an action in $A^\omega_1$. However, one edge variable provides the precondition for at most two actions in $A^\omega_2$. Hence we require $|A^\omega_1| \geq k_2$. The actions in $A^\omega_2$ set at least $2k_2$ vertex variables to 1. In the goal state all vertex variables must have the value 0 again, hence we need to apply at least $2k_2$ actions from $A_3$, and consequently $|A^\omega_3| \geq 2k_2$.

In order to apply an action in $A^\omega_3$, we first need to set a clean-up variable to 1 with an action from $A^\omega_4$. One clean-up variable provides the precondition for at most $k - 1$ actions in $A^\omega_4$, hence $|A^\omega_4| \geq k$. In total we get $|A^\omega| = \sum_{i=1}^3 |A^\omega_i| \geq 7k_2 + k = k'$. Conversely, $k'$ is an upper bound on the length of $\omega$, and the length of $\omega$ is clearly an upper bound on the number of actions on $A^\omega$, hence $|A^\omega| \leq k'$. Thus $|A^\omega| = k'$ and $\omega$ has exactly length $k'$. It follows that in all the above inequalities, equality holds, i.e., we have $|A^\omega_1| = k_2$, $|A^\omega_2| = |A^\omega_3| = |A^\omega_4| = 2k_2$, and $|A^\omega_5| = k$.

We call a variable active if its value gets changed during the execution of $\omega$. All $2k_2$ checking variables are active, and by the above considerations, there are exactly $k_2 = |A^\omega_1|$ active edge variables, $2k_2 = |A^\omega_2|$ active vertex variables, and $2k_2 = |A^\omega_3|$ active clean-up variables. We conclude that each active clean-up variable $x(v)$, $v \in V_i$, must provide the precondition for actions in $A^\omega_4$ to set $k - 1$ vertex variables to 0 (these vertex variables are active). This is only possible if these vertex variables are exactly the $k - 1$ variables $x(v,j)$ for $j \in J_i$. For each $1 \leq i \leq k$ and $j \in J_i$ the checking variable $x(i,j)$ is active, hence there must be some vertex $v \in V_i$ such that the vertex variable $x(v,j)$ is active, in order to provide the precondition for the action $a^{v,j}_v \in A^\omega_3$. We conclude that for each $1 \leq i \leq k$, the set $V_i$ contains exactly one vertex $v_i$ such that $x(v_i)$ and $x(v_i,j)$, $j \in J_i$ are all active. We show that these vertices $v_1, \ldots, v_k$ induce a clique in $G$.

Since we have $k_2$ active edge variables and $2k_2$ active vertex variables, each edge variable $x(e)$ must provide the precondition for two actions in $A^\omega_2$ that make two vertex variables active. This is only possible if $e = \{u,v\}$ for $u \in V_i, v \in V_j$, and the two vertex variables are $x(u,j)$ and $x(v,i)$. We conclude that the active edge variables are exactly the variables $x(\{v_i,v_j\}), 1 \leq i < j \leq k$. Hence, indeed, the vertices $v_1, \ldots, v_k$ induce a clique in $G$. This concludes the proof of the claim. The theorem follows.

$\square$
Theorem 3. (0, 3)-Bounded SAS$^+$ Planning is W[1]-hard, even for binary instances.

Proof. By parameterized reduction from Multicolored Clique. Let $I = (G, k)$ be an instance of this problem where $G = (V, E)$, $V_1, \ldots, V_k$ is the partition of $V$, $|V_1| = \cdots = |V_k| = n$ and parameter $k$. We construct a $(0, 3)$-Bounded SAS$^+$ Planning instance $I' = (P, k')$ with $P = (V, D, A, I, G)$ such that $I$ has a multicolored clique of size at most $k$ if and only if $P$ has a plan of length at most $k'$.

We set $V = V(G) \cup \{ p_{i,j} \mid 1 \leq i < j \leq k \}$, $D = \{ 0, 1 \}$, $I = \{ 0, \ldots, 0 \}$, $G[p_{i,j}] = 1$ for every $1 \leq i < j \leq k$ and $G[v] = 0$ for every $v \in V(G)$. Furthermore, the set $A$ contains the following actions:

- For every $v \in V(G)$ one action $a_v$ with $\text{eff}(a_v)[v] = 0$;
- For every $e = \{ v_i, v_j \} \in E(G)$ with $v_i \in V_i$ and $v_j \in V_j$ one action $a_e$ with $\text{eff}(a_e)[v_i] = 1$, $\text{eff}(a_e)[v_j] = 1$, and $\text{eff}(a_e)[p_{i,j}] = 1$.

Clearly, $P$ is binary and no action in $A$ has any precondition or more than 3 effects. The theorem will follow after we have shown the following claim.

Claim 2. $G$ contains a $k$-clique if and only if $P$ has a plan of length at most $k' = \binom{k}{2} + k$.

Suppose that $G$ contains a $k$-clique with vertices $v_1, \ldots, v_k$ and edges $e_1, \ldots, e_{\binom{k}{2}}$. Then $\omega = \langle a_{e_1}, \ldots, a_{e_{\binom{k}{2}}}, a_{v_1}, \ldots, a_{v_k} \rangle$ is a plan of length $k'$ for $P$.

For the reverse direction suppose that $\omega$ is a plan of length at most $k'$ for $P$. Because $I[p_{i,j}] = 0 \neq G[p_{i,j}] = 1$ the plan $\omega$ has to contain at least one action $a_e$ where $e$ is an edge between a vertex in $V_i$ and a vertex in $V_j$ for every $1 \leq i < j \leq k$. Because $\text{eff}(a_{e=\{v_i,v_j\}})[v_i] = 1 \neq G[v_i] = 0$ and $\text{eff}(a_{e=\{v_i,v_j\}})[v_j] = 1 \neq G[v_j] = 0$ for every such edge $e$ it follows that $\omega$ has to contain at least one action $a_v$ with $v \in V_i$ for every $1 \leq i \leq k$. Because $k' = \binom{k}{2} + k$ it follows that $\omega$ contains exactly $\binom{k}{2}$ actions of the form $a_v$ for some edge $e \in E(G)$ and exactly $k$ actions of the form $a_e$ for some vertex $v \in V(G)$. It follows that the graph $K = (\{ v \mid a_v \in \omega \}, \{ e \mid a_e \in \omega \})$ is a $k$-clique of $G$. \hfill $\square$

5 Membership Results

Our membership results are based on First-Order Logic (FO) Model Checking. For a class $\Phi$ of FO formulas we define the following parameterized problem.

**ϕ-FO Model Checking**

**Instance**: A finite structure $\mathcal{A}$, an FO formula $\varphi \in \Phi$.

**Parameter**: The length of $\varphi$.

**Question**: Does $\mathcal{A} \models \varphi$, i.e., is $\mathcal{A}$ a model for $\varphi$?

Let $\Sigma_1$ be the class of all FO formulas of the form $\exists x_1 \ldots \exists x_t \varphi$ where $t$ is arbitrary and $\varphi$ is a quantifier-free FO formula. For every positive integer $u$, let $\Sigma_{2,u}$ denote the class of all FO formulas of the form $\exists x_1 \ldots \exists x_t \forall y_1 \ldots \forall y_u \varphi$ where $t$ is arbitrary and $\varphi$ is a quantifier-free FO formula. The following connections between model checking and parameterized complexity classes are known.

**Proposition 5** (Flum and Grohe [25, Theorem 7.22]). The problem $\Sigma_1$-FO Model Checking is $\text{W[1]}$-complete. For every positive integer $u$ the problem $\Sigma_{2,u}$-FO Model Checking is $\text{W[2]}$-complete.

We will reduce Bounded SAS$^+$ Planning to ϕ-FO Model Checking. We start by defining a relational structure $\mathcal{A}(I)$ for an arbitrary Bounded SAS$^+$ Planning instance $I = (P, k)$ with $P = (V, D, A, I, G)$ as follows:

- The universe of $\mathcal{A}(I)$ is $V \cup A \cup D^+ \cup \{ \text{dum}_A \}$, where $\text{dum}_A$ is a novel element that represents a “dummy” action (which we need for technical reasons).
- $\mathcal{A}(I)$ contains the unary relations $\text{VAR} = V$, $\text{ACT} = A \cup \{ \text{dum}_A \}$, $\text{DOM} = D^+$, and $\text{DUM}_A = \{ \text{dum}_A \}$ together with the following relations of higher arity:
has a solution if and only if

\[ I \subseteq \text{SAS} \text{ variable } v \text{ inductively defined as follows.} \]

function that only depends on the parameter \( k \).

Finally, we define the formula

\[ \varphi \]

We proceed by parameterized reduction to the problem \( \Sigma_{2,2} \text{-FO Model Checking} \), which is \( \mathsf{W}[2] \)-complete by Proposition 5. Let \( \mathcal{I} = \langle \mathcal{P}, k \rangle \) with \( \mathcal{P} = \langle V, D, A, I, G \rangle \) be an instance of \( \text{Bounded SAS}^+ \) \text{Planning}. We construct an instance \( \mathcal{I}' = \langle A(\mathcal{I}), \varphi \rangle \) of \( \Sigma_{2,2} \text{-FO Model Checking} \) such that \( \mathcal{I} \) has a solution if and only if \( \mathcal{I}' \) has a solution and the length of the formula \( \varphi \) is bounded by some function that only depends on the parameter \( k \). For the definition of \( \varphi \) we need the following auxiliary formulas. In the following let \( 0 \leq i \leq k \). We define the formula \( \text{value}(\langle a_1, \ldots, a_i \rangle, v, x) \) which holds if the variable \( v \) has value \( x \) after applying the actions \( a_1, \ldots, a_i \) to the initial state. This formula is inductively defined as follows.

\[
\text{value}(\langle \rangle, v, x) = \text{INIT}_V v, x \\
\text{value}(\langle a_1, \ldots, a_i \rangle, v, x) = (\text{value}(\langle a_1, \ldots, a_{i-1} \rangle, v, x) \land \neg \text{EFF}_{a_i}, v) \\
\lor \text{EFF}_V a_i, v, x
\]

We also define the formula \( \text{check-pre}(\langle a_1, \ldots, a_i \rangle, v, x) \) which holds if the variable \( v \) has the value \( x \) after the actions \( a_1, \ldots, a_{i-1} \) have been applied to the initial state, whenever \( x \) is the precondition of the action \( a_i \) on the variable \( v \). The formula is defined as follows.

\[
\text{check-pre}(\langle a_1, \ldots, a_i \rangle, v, x) = \text{PRE}_V a_i, v, x \rightarrow \text{value}(\langle a_1, \ldots, a_{i-1} \rangle, v, x)
\]

We further define the formula \( \text{check-pre-all}(\langle a_1, \ldots, a_k \rangle, v, x) \) which holds if the formula \( \text{check-pre}(\langle a_1, \ldots, a_i \rangle, v, x) \) holds for every \( 0 \leq i \leq k \).

\[
\text{check-pre-all}(\langle a_1, \ldots, a_k \rangle, v, x) = \land_{i=1}^k \text{check-pre}(\langle a_1, \ldots, a_i \rangle, v, x)
\]

Finally, we define the formula \( \text{check-goal}(\langle a_1, \ldots, a_k \rangle, v, x) \) which holds if whenever \( x \) is the goal on the variable \( v \), then the variable \( v \) has the value \( x \) after the actions \( a_1, \ldots, a_k \) have been applied to the initial state. The formula is defined as follows.

\[
\text{check-goal}(\langle a_1, \ldots, a_k \rangle, v, x) = \text{GOAL}_V v, x \rightarrow \text{value}(\langle a_1, \ldots, a_k \rangle, v, x)
\]

We can now define the formula \( \varphi \) itself as:

\[
\varphi = \exists a_1 \ldots \exists a_k \forall v \forall x. (\land_{i=1}^k \text{ACT}_{a_i} \land (\text{VAR} v \land \text{DOM} x \rightarrow \\
\text{check-pre-all}(\langle a_1, \ldots, a_k \rangle, v, x) \land \text{check-goal}(\langle a_1, \ldots, a_k \rangle, v, x)))
\]

Evidently \( \varphi \in \Sigma_{2,2} \), the length of \( \varphi \) is bounded by some function that only depends on \( k \) and \( A(\mathcal{I}) \models \varphi \) if and only if \( \mathcal{P} \) has a plan of length at most \( k \). The “dummy” action \( \text{dum}_a \) guarantees that there is a plan of length exactly \( k \) whenever there is a plan of length at most \( k \).

Our next results shows that if we restrict ourselves to unary planning instances then \( \text{Bounded SAS}^+ \) \text{Planning} becomes easier (at least from the parameterized point of view). We show \( \mathsf{W}[1] \)-membership of \( \{U\} \)-\( \text{Bounded SAS}^+ \) \text{Planning} by reducing it to the \( \Sigma_1 \text{-FO Model Checking} \) problem. The basic idea behind the proof is fairly similar to the proof of Theorem 4. However, we cannot directly express within \( \Sigma_1 \) that all the preconditions of an action are satisfied, since we are not allowed to use universal quantifications within \( \Sigma_1 \). Hence, we avoid the universal quantification with a trick: we observe that the preconditions only need to be checked with respect to at most \( k \) “important” variables,
that is, the variables in which the preconditions of an action differ from the initial state. Since we only consider unary planning instances there can be at most \(k\) such variables. Hence, it becomes possible to guess the important variables using only existential quantifiers.

It remains to check that all the important variables are among these guessed variables. We do this without universal quantification by adding dummy elements \(d_1, \ldots, d_k\) and a relation \(\text{DIFF}\_\text{ACT}\) to the relational structure \(A(\mathcal{P})\). The relation associates with each action exactly \(k\) different elements. These elements consist of all the important variables of the action, say the number of these variables is \(k'\), plus \(k-k'\) dummy elements. Hence, by guessing these \(k\) elements and eliminating the dummy elements, the formula knows all the important variables of the action and can check their preconditions without using universal quantification.

To accommodate the “dummy” elements we start by defining a new extended structure \(A^*(\mathcal{P})\) for an arbitrary \(\text{Bounded SAS}^+\text{ Planning}\) instance \(I = \langle \mathcal{P}, k \rangle\) with \(\mathcal{P} = \langle V, D, A, I, G \rangle\). For a partial state \(s \in (D^+)\upharpoonright V\) and an action \(a \in A\) we define the following sets.

\[
\text{diff}(s) = \{ v \in V \mid s[v] \neq u \text{ and } s[v] \neq I[v] \}
\]
\[
\text{diff}(a) = \text{diff}(\text{pre}(a)).
\]

We define the structure \(A^*(\mathcal{I})\) as follows.

- The universe \(A^*\) of \(A^*(\mathcal{I})\) consists of the elements of the universe of \(A(\mathcal{I})\) plus \(k\) novel “dummy” elements \(d_1, \ldots, d_k\).

- \(A^*(\mathcal{I})\) contains all relations of \(A(\mathcal{I})\) and additionally the following relations.
  - A unary relation \(\text{DUM} = \{d_1, \ldots, d_k\}\).
  - A binary relation \(\text{DIFF}\_\text{ACT} = \{(a, v) \in A \times V \mid a \in A \text{ and } v \in \text{diff}(a)\} \cup \{(a, d_i) \in A \times \{d_1, \ldots, d_k\} \mid a \in A \text{ and } 1 \leq i \leq k - |\text{diff}(a)|\}\).
  - A unary relation \(\text{DIFF}\_\text{GOAL} = \{(v) \in V \mid v \in \text{diff}(G)\} \cup \{(d_i) \mid 1 \leq i \leq k - |\text{diff}(G)|\}\).

Before we show that \(\{U\}\text{-Bounded SAS}^+\text{ Planning}\) is in \(\text{W}[1]\) we need some simple observations about planning. Let \(\mathcal{P} = \langle V, D, A, I, G \rangle\) be a \(\text{SAS}^+\text{ Planning}\) instance, \(V' \subseteq V\), and \(s \in (D^+)\upharpoonright V\). We denote by \(s|V'\) the state \(s\) restricted to the variables in \(V'\) and by \(A|V'\) the set of actions obtained from the actions in \(A\) after restricting the preconditions and effects of every such action to the variables in \(V'\). Furthermore, we denote by \(\mathcal{P}|V'\) the \(\text{SAS}^+\text{ Planning}\) instance \(\langle V', D, A|V', I|V', G|V' \rangle\).

**Proposition 6.** Let \(\omega = \langle a_1, \ldots, a_l \rangle\) be a sequence of actions from \(A\). Then \(\omega\) is a plan for \(\mathcal{P}\) if and only if \(\omega\) is a plan for \(\mathcal{P}|V_0\) where \(V_0 = \bigcup_{i=1}^l \text{diff}(a_i) \cup \text{diff}(G) \cup \{v \in V \mid \text{eff}(a_i)[v] \neq u \text{ and } 1 \leq i \leq l\}\).

**Proof.** If \(\omega\) is a plan for \(\mathcal{P}\), then \(\omega\) is a plan for \(\mathcal{P}|V'\) whenever \(V' \subseteq V\). In particular, \(\omega\) is a plan for \(\mathcal{P}|V_0\).

For the reverse direction assume for a contradiction that \(\omega|V_0\) is a plan for \(\mathcal{P}|V_0\) but \(\omega\) is not a plan for \(\mathcal{P}\). There are two possible reasons for this: (1) a precondition of some action \(a_i\) for \(1 \leq i \leq l\), is not met or (2) the goal state is not reached after having completed the plan.

In the first case, consider the state \(s\) that \(a_i\) is applied in. Then there is a variable \(v \in V\) such that \(s[v] \neq \text{pre}(a_i)[v] \neq u\). There are two cases to consider:

1. \(I[v] = x \neq u\) and \(\text{pre}(a_i)[v] = x\). In this case an action \(a_j\) for some \(j < i\) has changed the variable \(v\) and \(v \in \{v \in V \mid \text{eff}(a_i)[v] \neq u \text{ and } 1 \leq i \leq l\}\).
2. \(I[v] = x, \text{pre}(a_i)[v] = x' \neq u,\) and \(x \neq x'\). This implies that \(v \in \text{diff}(a_i)\).

Hence, in both cases we obtain that \(v \in V_0\) and that \(\omega\) is a plan for \(\mathcal{P}\).

In the second case, consider the state \(q\) after applying \(\omega\) to the initial state. Then there is a variable \(v \in V\) such that \(s[v] \neq G[v] \neq u\). There are two cases to consider:

1. \(I[v] = x \neq u\) and \(G[v] = x\). In this case an action \(a_j\) for some \(1 \leq j \leq l\) has changed the variable \(v\) and \(v \in \{v \in V \mid \text{eff}(a_i)[v] \neq u \text{ and } 1 \leq i \leq l\}\).
2. \( I[v] = x, G[v] = x' \neq u, \) and \( x \neq x' \). This implies that \( v \in \text{diff}(G) \).

Hence, in both cases we obtain that \( v \in V_0 \) and that \( \omega \) is a plan for \( \mathbb{P} \). \( \square \)

**Proposition 7.** Let \( \omega = \langle a_1, \ldots, a_l \rangle \) be a plan for \( \mathbb{P} \). Then \( \bigcup_{i=1}^{l} \text{diff}(a_i) \cup \text{diff}(G) \subseteq \{ v \in V \mid \text{eff}(a_i)[v] \neq u \} \). Hence, in both cases we obtain that \( v \in V_0 \) and that \( \omega \) is a plan for \( \mathbb{P} \).

**Proof.** First assume that \( v \in \text{diff}(G) \), i.e. \( v \in \{ v \in V \mid G[v] \neq u \} \). Since \( \omega \) is a plan for \( \mathbb{P} \) it follows that there is an action \( a_j \in \omega \), which changes the value of \( v \), as required.

Then assume that \( v \in \text{diff}(a_j) \) for some \( 1 \leq j \leq l \), i.e. \( v \in \{ v \in V \mid \text{pre}(a_j)[v] \neq u \} \). \( \omega \) is a plan it follows that there is an action \( a_k, k < j \) in \( \omega \), which changes the value of \( v \), as required. \( \square \)

**Corollary 2.** Let \( \omega = \langle a_1, \ldots, a_l \rangle \) be a sequence of actions from \( A \) and \( V_0 = \{ v \in V \mid \text{eff}(a_i)[v] \neq u \} \). Then \( \omega \) is a plan for \( \mathbb{P} \) if and only if \( \bigcup_{i=1}^{l} \text{diff}(a_i) \cup \text{diff}(G) \subseteq V_0 \).

**Theorem 5.** \( \{U\}\)-Bounded SAS\(^+\) Planning is in \( \text{W}[1] \).

**Proof.** We proceed by a parameterized reduction to the \( \text{W}[1]\)-complete problem \( \Sigma_1\)-FO Model Checking. Let \( \mathbb{I} = (\mathbb{P}, k) \) with \( \mathbb{P} = (V, D, A, I, G) \) be an instance of \( \{U\}\)-Bounded SAS\(^+\) Planning. We construct an instance \( \mathbb{I}' = (A^\ast(\mathbb{I}), \varphi) \) of \( \Sigma_1\)-FO Model Checking such that I has a solution if and only if \( \mathbb{I}' \) has a solution and the length of the formula \( \varphi \) is bounded by some function that only depends on the parameter \( k \).

The formula \( \varphi \) uses the following existentially quantified variables:

- The variables \( a_1, \ldots, a_k \). The values of these variables correspond to the at most \( k \) actions of a plan for \( \mathbb{P} \).
- The variables \( v_1, \ldots, v_k \). The values of these variables correspond to the variables that are involved in the effects of the actions assigned to \( a_1, \ldots, a_k \), i.e., it holds that \( \text{eff}(a_i)[v_i] \neq u \) for every \( 1 \leq i \leq k \). Because \( \mathbb{P} \) is unary there is at most one such variable for each of the at most \( k \) actions in a potential plan for \( \mathbb{P} \).
- The variables \( d_1, \ldots, d_k \). These variables are so-called “dummy” variables that we use to check the maximality of certain sets.
- The variables \( x_1,1, \ldots, x_{k,1}, \ldots, x_{k,k} \). These variables are used to check the preconditions of the actions \( a_1, \ldots, a_k \). Here \( x_{i,j} \) represents \( \text{pre}(a_i)[v_j] \) for every \( 1 \leq i, j \leq k \).
- The variables \( x_g,1, \ldots, x_g,k \). These variables are used to check whether all conditions of the goal state are met after the actions \( a_1, \ldots, a_k \) have been executed on the initial state. Here \( x_g,i \) represents \( G[v_i] \).

We define \( \varphi \) in such a way that \( A^\ast(\mathbb{I}) \models \varphi \) if and only if there is a sequence of actions \( a_1, \ldots, a_k \) and a set \( V_0 = \{ v \in V \mid \text{eff}(a_i)[v] \neq u \} \) and \( 1 \leq i \leq k \} \) of variables with \( \bigcup_{i=1}^{k} \text{diff}(a_i) \cup \text{diff}(G) \subseteq V_0 \) such that \( a_1, \ldots, a_k \) is a plan for \( \mathbb{P} \). Because of Corollary 2 it then follows that \( A^\ast(\mathbb{I}) \models \varphi \) if and only if there is a plan of length at most \( k \) for \( \mathbb{P} \). Consequently, the formula \( \varphi \) has to ensure the following properties:

**P1** For every \( 1 \leq i \leq k \) if the variable \( a_i \) is assigned to an action other than the “dummy” action \( (\text{dum}, a_0) \), then the variable \( v_i \) is assigned to the unique variable \( v \in V \) with \( \text{eff}(a_i)[v] \neq u \). In the following we denote by \( V_0 \) the variables in \( V \) that are assigned to the variables \( v_1, \ldots, v_k \).

**P2** For every \( 1 \leq i \leq k \) it holds that \( \text{diff}(a_i) \subseteq V_0 \).

**P3** \( \text{diff}(G) \subseteq V_0 \).

**P4** For every \( 1 \leq i \leq k \) all preconditions of the action \( a_i \) on the variables \( v_1, \ldots, v_k \) are met after the execution of the actions \( a_1, \ldots, a_{i-1} \) on the initial state.
P5 The goal state on the variables \( v_1, \ldots, v_k \) is reached after the execution of the actions \( a_1, \ldots, a_k \) on the initial state.

Observe that the properties P1–P3 ensure that the variables \( v_1, \ldots, v_k \) are assigned to a set of variables \( V_0 \) with \( V_0 = \{ v \in V \mid \text{eff}(a_i)[v] \neq u \text{ and } 1 \leq i \leq k \} \) (or to the “dummy” action) and \( \bigcup_{i=1}^{k} \text{diff}(a_i) \cup \text{diff}(G) \subseteq V_0 \). The properties P4–P5 make sure that the sequence of actions \( (a_1, \ldots, a_k) \) is a plan for \( P|V_0 \).

The formula \( \varphi \) is composed of several auxiliary formulas that we define next. We define a formula \( \text{check-eff}(a_1, \ldots, a_k, v_1, \ldots v_k, x_1, \ldots, x_k) \) that ensures property P1, i.e., \( \text{eff}(a_i)[v_i] = x_i \) or \( a_i = \text{dum} \alpha \) for every \( 1 \leq i \leq k \).

\[
\text{check-eff}(a_1, \ldots, a_k, v_1, \ldots v_k, x_1, \ldots, x_k) = \bigwedge_{i=1}^{k} (\text{EFF}_{\alpha} a_i v_i x_i \lor \text{DUM}_{\alpha} a_i)
\]

For every \( 1 \leq i \leq k \) we define a formula \( \text{diff-op}(a_i, v_1, \ldots, v_k, d_1, \ldots, d_k) \) that holds if \( \text{diff}(a_i) \subseteq V_0 \). To check this, the formula checks that all of the exactly \( k \) tuples in \( \text{DIFF}_{\alpha} \) ACT that contain \( a_i \) (recall the definition of the relation \( \text{DIFF}_{\alpha} \text{ACT} \) in the structure \( \mathcal{A}^\alpha(P) \)) are tuples of the form \( (a_i, v_j) \) or \( (a_i, d_j) \) for some \( 1 \leq j \leq k \).

\[
\text{diff-op}(a_i, v_1, \ldots, v_k, d_1, \ldots, d_k) = \text{DUM}_{\alpha} a_i \lor \\
\bigvee_{j \subseteq \{1, \ldots, k\}} \left[ \bigwedge_{j \neq j' \in J} v_j \neq v_j' \land \bigwedge_{j \in J} \text{DIFF}_{\alpha} \text{ACT} a_i v_j \land \bigwedge_{1 \leq j \leq k-|J|} \text{DIFF}_{\alpha} \text{ACT} a_i d_j \right]
\]

We define a formula that ensures property P2.

\[
\text{diff-op-all}(a_1, \ldots, a_k, v_1, \ldots, v_k, d_1, \ldots, d_k) = \bigwedge_{i=1}^{k} \text{diff-op}(a_i, v_1, \ldots, v_k, d_1, \ldots, d_k)
\]

Similarly to \( \text{diff-op} \) above we define a formula \( \text{diff-goal}(v_1, \ldots, v_k, d_1, \ldots, d_k) \) that ensures property P3, i.e., \( \text{diff}(G) \subseteq V_0 \).

\[
\text{diff-goal}(v_1, \ldots, v_k, d_1, \ldots, d_k) = \\
\bigvee_{j \subseteq \{1, \ldots, k\}} \left[ \bigwedge_{j \neq j' \in J} v_j \neq v_j' \land \bigwedge_{j \in J} \text{DIFF}_{\alpha} \text{GOAL} v_j \land \bigwedge_{1 \leq j \leq k-|J|} \text{DIFF}_{\alpha} \text{GOAL} d_j \right]
\]

For every \( 1 \leq i \leq k \), we define a formula \( \text{value}(a_1, \ldots, a_i, v, x) \) that holds if the variable \( v \) has value \( x \) after the actions \( a_1, \ldots, a_i \) have been executed on the initial state. We define the formulas inductively as follows.

\[
\text{value}(v, x) = \text{INIT}v x
\]

\[
\text{value}(a_1, \ldots, a_i, v, x) = (\text{value}(a_1, \ldots, a_{i-1}, v, x) \land \text{EFF}a_i v) \lor \text{EFF}_{\alpha} a_i v x
\]

For every \( 1 \leq i \leq k \) we define a formula \( \text{check-pre}(a_1, \ldots, a_i, v_1, \ldots, v_k, x_1, \ldots, x_k) \) that holds if all preconditions of the action \( a_i \) defined on the variables \( v_1, \ldots, v_k \) are met after the actions \( a_1, \ldots, a_{i-1} \) have been executed on the initial state.

\[
\text{check-pre}(a_1, \ldots, a_i, v_1, \ldots, v_k, x_1, \ldots, x_k) = \\
\left( \bigwedge_{j=1}^{k} (\text{PRE}_{\alpha} a_i v_j x_j \land \text{value}(a_1, \ldots, a_{i-1}, v_j, x_j)) \lor \text{PRE}a_i v_j \right)
\]

We define a formula \( \text{check-pre-all}(a_1, \ldots, a_k, v_1, \ldots, v_k, x_1, \ldots, x_{k,k}) \) that ensures property P4, i.e., \( \text{check-pre}(a_1, \ldots, a_i, v_1, \ldots, v_k, x_1, \ldots, x_{i,k}) \) for every \( 1 \leq i \leq k \).

\[
\text{check-pre-all}(a_1, \ldots, a_k, v_1, \ldots, v_k, x_1, \ldots, x_{k,k}) = \\
\bigwedge_{i=1}^{k} \text{check-pre}(a_1, \ldots, a_i, v_1, \ldots, v_k, x_{i,1}, \ldots, x_{i,k})
\]

Finally, we define a formula \( \text{check-goal}(a_1, \ldots, a_k, v_1, \ldots, v_k, x_1, \ldots, x_{g,k}) \) that ensure property P5, i.e., all conditions of the goal state on the variables \( v_1, \ldots, v_k \) are met after the actions \( a_1, \ldots, a_k \) have been executed on the initial state.
we introduce some terminology on sequences of actions. For

\[ L = (\mathbb{P}, k) \] with \( \mathbb{P} = (V, A, I, G) \), be an instance of \{P\}-Bounded \text{SAS}\textsuperscript{+} \text{ PLANNING}. First, we introduce some terminology on sequences of actions. For \( l \geq 0 \), let \( \omega = \langle a_1, \ldots, a_l \rangle \) be a sequence of actions from \( A \). We define insert\( (i, a, \omega) = (a_1, \ldots, a_{i-1}, a, a_i, \ldots, a_l) \). Let \( \langle v, x \rangle \in V \times D \). For \( 0 \leq i < j \leq l + 1 \) we say that \( \langle v, x \rangle \) is required in \( \omega \) between positions \( i \) and \( j \) if the following two conditions hold:

1. Either \( j \leq l \) and the \( j \)-th element of \( \omega \) is an action \( a \) with \( \text{pre}(a)[v] = x \), or \( j = l + 1 \) and \( G[v] = x \).

2. \( i \) is the smallest integer such that \( s[v] \neq x \) where \( s \) is the state obtained after applying the actions \( a_1, \ldots, a_i \) to the initial state (with \( i = 0 \) if \( I[v] \neq x \)).

If \( \langle v, x \rangle \) is required in \( \omega \) between positions \( i \) and \( j \), and \( a \) is an action which sets \( v \) to \( x \), then we also say that \( a \) is required in \( \omega \) (between positions \( i \) and \( j \)). Note that there can be at most one such action since \( \mathbb{P} \) is post-unique.

The following claim is immediate from the above definitions.

**Claim 3.** \( \omega \) is a plan for \( \mathbb{P} \) if and only if there is no pair \( \langle v, x \rangle \in V \times D \) which is required in \( \omega \) between some positions \( 0 \leq i < j \leq l + 1 \).

**Claim 4.** If \( \omega \) is a subsequence of some plan \( \omega^* \) for \( \mathbb{P} \), and \( \langle v, x \rangle \) is required in \( \omega \) between some positions \( i \) and \( j \) with \( 0 \leq i < j \leq l + 1 \). Then the following holds:

1. \( \omega^* \) must contain the unique action \( a \) with \( \text{eff}(a)[v] = x \).
2. for some \( i \leq m \leq j \), the sequence \( \text{insert}(m, a, \omega) \) is a subsequence of \( \omega^* \).

Proof. If \( j \leq l \), then clearly without an action \( a \) that sets \( v \) to \( x \) we cannot meet the precondition of the \( j \)-th action of \( \omega \). Similarly, if \( j = l + 1 \), then without an action \( a \) that sets \( v \) to \( x \) we cannot reach the goal state. Since \( \mathcal{P} \) is post-unique, there is at most one such action \( a \), hence the first statement of the claim follows. We further observe that \( \omega^* \) can be obtained from \( \omega \) by inserting actions, and one of these inserted actions must be \( a \), hence the second statement of the claim follows.

The above considerations suggest that we can find a plan by starting with the empty sequence, and as long as there is a required action, guessing its position and insert it into the sequence. Next we describe an algorithm that follows this general idea. It constructs a search tree, where every node of the tree is labeled with a sequence of actions. Each leaf of the tree is marked either as a “success node” if its label is a plan of length at most \( k \), or as a “failure node” if its label is not a subsequence of a plan of length at most \( k \). The algorithm not only decides whether there exists a plan of length at most \( k \), but it even lists all minimal plans of length at most \( k \) (a plan is called minimal if none of its proper subsequences is a plan).

We start with a trivial tree consisting of just a root, labeled with the empty sequence, and recursively extend this tree. Assume that \( T \) is the search tree constructed so far. Consider a leaf \( n \) of \( T \) labeled with a sequence \( \omega \) of length \( l \leq k \). If \( \omega \) is a plan we can mark \( n \) as a success node, and we do not need to extend the search tree below \( n \). Otherwise, by Claim 3, there is some \( \langle v, x \rangle \in V \times D \) and some \( 1 \leq i < j \leq l + 1 \) such that \( \langle v, x \rangle \) is required in \( \omega \) between positions \( i \) and \( j \) (clearly we can find \( v, x, i, j \) in polynomial time). Hence, some action needs to be added to make \( \omega \) a plan. If \( l = k \) or if \( A \) does not contain an action which sets \( v \) to \( x \), we know that \( \omega \) is not a subsequence of any plan of length at most \( k \), and we can mark \( n \) as a failure node. We do not need to extend the search tree below \( n \). It remains to consider the case where \( l < k \) and \( A \) contains an action \( a \) which sets \( v \) to \( x \). Since \( \mathcal{P} \) is post-unique, there is exactly one such \( a \). We need to insert \( a \) into \( \omega \) between the positions \( i \) and \( j \), but we don’t know where. However, there are only \( j - i + 1 \leq k \) possibilities. Therefore we add below \( n \) a child \( n_m \) for each possibility \( m \in \{i, \ldots, j\} \), and we label \( n_m \) with the sequence \( \text{insert}(m, a, \omega) \). Eventually we arrive at a search tree \( T \) where all its leaves are marked either as success or failure nodes. The depth of \( T \) is at most \( k \), since each node of \( T \) of depth \( d \) is labeled with a sequence of length \( d \), and we do not add nodes with sequences of length greater than \( k \). Each node has at most \( k \) children. Hence \( T \) has \( O(k^k) \) many nodes. As the time required for each node is polynomial, building the search tree is fixed-parameter tractable for parameter \( k \).

Claim 5. Let \( \omega \) be a minimal plan of length \( l \leq k \). Then for each \( 0 \leq d \leq l \) the tree \( T \) has a node \( n_d \) of depth \( d \) such that the label of \( n_d \) is a subsequence of \( \omega \).

Proof. We show the claim by induction on \( d \). The claim is evidently true for \( d = 0 \). Let \( d > 0 \) and assume the claim holds for \( d - 1 \). Consequently, there is a tree node \( n_{d-1} \) at depth \( d - 1 \) which is labeled with a subsequence \( \omega' \) of \( \omega \). Since \( \omega' \) is a proper subsequence of \( \omega \), and since \( \omega \) is assumed to be a minimal plan, \( \omega' \) is not a plan; thus \( n_{d-1} \) is not a success node. Since \( \omega' \) is a subsequence of \( \omega \), it is not a failure node either. Hence \( n_{d-1} \) must have children. Consequently there is some pair \( \langle v, x \rangle \in V \times D \) which is required in \( \omega' \) between some positions \( 0 \leq i < j \leq d \), and \( n_{d-1} \) has \( j - i + 1 \) children, each labeled with a sequence \( \text{insert}(m, a, \omega') \), where \( a \) is the unique action from \( A \) that sets \( v \) to \( x \). By Claim 4, at least one of the children of \( n_{d-1} \) is labeled with a subsequence of \( \omega \), hence the induction step holds true, and Claim 5 follows.

Claim 5 entails as the special case \( d = l \) that \( \omega \) appears as the label of a success node. We conclude that each minimal plan of \( \mathcal{P} \) of length at most \( k \) appears as the label of some success node of \( T \). Hence, once we have constructed the search tree \( T \), we can list all minimal plans of length at most \( k \). In particular, we can decide whether there exists a plan of length at most \( k \) and Theorem 6 follows.

In is interesting to note that the same result can be obtained by a slight adaption of the standard partial-ordering planning algorithm by McAllester and Rosenblitt [41]. This suggests that many successful applications of planning might be cases where the problem is “almost tractable” and the algorithm used happens to implicitly exploit this. The details of how to modify this algorithm to obtain an FPT algorithm for \{P\}-BOUNDED SAS\(^*\) PLANNING can be found in one of our previous papers [4].
6.2 (0, 2)-Bounded SAS$^+$ Planning

Before we show that (0, 2)-Bounded SAS$^+$ Planning is fixed-parameter tractable we need to introduce some notions and prove some simple properties of (0, 2)-Bounded SAS$^+$ Planning. Let $P = (V, D, A, I, G)$ be an instance of Bounded SAS$^+$ Planning. We say an action $a \in A$ has an effect on some variable $v \in V$ if $\text{eff}(a)[v] \neq u$. We call this effect good if furthermore $\text{eff}(a)[v] = G[v]$ or $G[v] = u$ and we call the effect bad otherwise. We say an action $a \in A$ is good if it has only good effects, bad if it has only bad effects, and mixed if it has at least one good and at least one bad effect. Note that if a valid plan contains a bad action then this action can always be removed without changing the validity of the plan. Consequently, we only need to consider good and mixed actions. Furthermore, we write $\Delta(V)$ to denote the set of variables $v \in V$ such that $G[v] \neq u$ and $I[v] \neq G[v]$.

The next lemma shows that we do not need to consider good actions with more than 1 effect for (0, 2)-Bounded SAS$^+$ Planning.

**Lemma 1.** There is a parameterized reduction from (0, 2)-Bounded SAS$^+$ Planning to (0, 2)-Bounded SAS$^+$ Planning that maps an instance $I = (P, k)$ to an instance $I' = (P', k')$ where $k' = k(k + 3) + 1$ and no good action of $P$ affects more than one variable. Furthermore, if $P$ is binary, then also $P'$ is binary.

**Proof.** The required instance $I'$ is constructed from $I$ as follows. $V'$ contains the following variables:

- All variables in $V$;
- One binary variable $g$;
- For every action $a \in A$ and every $1 \leq i \leq k + 2$ one binary variable $v_i(a)$;

$A'$ contains the following actions:

- For every mixed action $a \in A$ that has a good effect on the variable $v$ and a bad effect on the variable $v'$, there is
  - one action $a_1(a)$ such that $\text{eff}(a_1(a)) [v'] = \text{eff}(a)[v']$ and $\text{eff}(a_1(a)) [v_1(a)] = 0$,
  - one action $a_i(a)$ for all $1 < i < k + 3$ such that $\text{eff}(a_i(a)) [v_{i-1}(a)] = 1$ and $\text{eff}(a_i(a)) [v_1(a)] = 0$, as well as
  - one action $a_{k+3}(a)$ such that $\text{eff}(a_{k+3}(a)) [v_{k+2}(a)] = 1$ and $\text{eff}(a_{k+3}(a))[v] = \text{eff}(a)[v]$;

- For every good action $a \in A$ that has only one effect on the variable $v$, there is
  - one action $a_1(a)$ such that $\text{eff}(a_1(a))[g] = 1$ and $\text{eff}(a_1(a))[v_1(a)] = 0$,
  - one action $a_i(a)$ for all $1 < i < k + 3$ such that $\text{eff}(a_i(a)) [v_{i-1}(a)] = 1$ and $\text{eff}(a_i(a)) [v_1(a)] = 0$, as well as
  - one action $a_{k+3}(a)$ such that $\text{eff}(a_{k+3}(a)) [v_{k+2}(a)] = 1$ and $\text{eff}(a_{k+3}(a))[v] = \text{eff}(a)[v]$;

- For every good action $a \in A$ that has two effects on the variables $v$ and $v'$, there is
  - one action $a_1(a)$ such that $\text{eff}(a_1(a))[g] = 1$ and $\text{eff}(a_1(a))[v_1(a)] = 0$,
  - one action $a_i(a)$ for all $1 < i < k + 2$ such that $\text{eff}(a_i(a)) [v_{i-1}(a)] = 1$ and $\text{eff}(a_i(a)) [v_1(a)] = 0$,
  - one action $a_{k+2}(a)$ such that $\text{eff}(a_{k+2}(a)) [v_{k+1}(a)] = 1$ and $\text{eff}(a_{k+2}(a))[v] = \text{eff}(a)[v]$, as well as
Furthermore, all new variable domains introduced are binary so \( P \leq \) an instance of \((0,2)\)-Bounded SAS\(^+\) Planning where no good action affects more than 1 variable. Furthermore, all new variable domains introduced are binary so \( P \) has the same maximum domain size as \( P \). It remains to show that \( P \) is equivalent to \( \mathcal{I} \).

Suppose that \( \omega = \langle a_1, \ldots, a_l \rangle \) is a plan of length at most \( k \) for \( \mathcal{I} \). Then
\[
\langle a_{k+3}(a_1), \ldots, a_1(a_1), \ldots, a_{k+3}(a_1), \ldots, a_1(a_1), a_g \rangle
\]
is a plan of length \( l(k+3)+1 \leq k(k+3)+1 \) for \( \mathcal{I}' \).

Let \( \omega = \langle a_1, \ldots, a_l \rangle \) be the (unique) sequence of actions in \( A \) that are used by \( \omega' \) whose order corresponds to the order in which they are used by \( \omega' \). Clearly, \( \omega \) is a plan for \( \mathcal{I} \). It remains to show that \( l \leq k \) for which we need the following claim.

**Claim 6.** If \( \omega' \) uses some action \( a \in A \) then \( \omega' \) contains at least \( k+2 \) actions from \( a_1(a), \ldots, a_{k+3}(a) \).

Let \( i \) be the largest integer with \( 1 \leq i \leq k+3 \) such that \( a_i(a) \) occurs in \( \omega' \). We first show by induction on \( i \) that \( \omega' \) contains all actions in \( \{ a_j(a) \mid 1 \leq j \leq i \} \). Clearly, if \( i = 1 \) there is nothing to show, so assume that \( i > 1 \). The induction step follows from the fact that the action \( a_1(a) \) has a bad effect on the variable \( v_{i-1}(a) \) and the action \( a_{i-1}(a) \) is the only action of \( \mathcal{I}' \) that has a good effect on \( v_{i-1}(a) \) and hence \( \omega' \) has to contain the action \( a_{i-1}(a) \). It remains to show that \( i \geq k+2 \). Suppose for a contradiction that \( i < k+2 \) and consequently the action \( a_{i+1}(a) \) is not contained in \( \omega' \). Because the action \( a_{i+1}(a) \) is the only action of \( \mathcal{I}' \) that has a bad effect on the variable \( v_i(a) \) it follows that the variable \( v_i(a) \) remains in the goal state over the whole execution of the plan \( \omega' \). But then \( \omega' \) without the action \( a_i(a) \) would still be a plan for \( \mathcal{I}' \) contradicting our assumption that \( \mathcal{I}' \) is minimal with respect to sub sequences.

It follows from Claim 6 that \( \omega' \) uses at most \( \frac{l(k+2)}{k+3} \leq \frac{k(k+3)+1}{k+2} < k+1 \) actions from \( A \). Hence, \( l \leq k \) proving the lemma.

We are now ready to show that \((0,2)\)-Bounded SAS\(^+\) Planning is fixed-parameter tractable.

**Theorem 7.** \((0,2)\)-Bounded SAS\(^+\) Planning is fixed-parameter tractable.

**Proof.** We show fixed-parameter tractability of \((0,2)\)-Bounded SAS\(^+\) Planning by reducing it to the following fixed-parameter tractable problem. The problem has originally been shown to be fixed-parameter tractable for undirected graphs [20]. Later Guo, Niedermeier, and Suchy mentioned that this result can be directly transferred to the directed case [31].

**Directed Steiner Tree**

**Instance:** A set of nodes \( N \), a weight function \( w : N \times N \rightarrow (\mathbb{N} \cup \{ \infty \}) \), a root node \( s \in N \), a set \( T \subseteq N \) of terminals, and a weight bound \( p \).

**Parameter:** \( pm = \min \{ w(u,v) \mid u,v \in N \} \).

**Question:** Is there a set of arcs \( E \subseteq N \times N \) of weight \( w(E) \leq p \) (where \( w(E) = \sum_{e \in E} w(e) \))
such that in the digraph \( D = (N, E) \) for every \( t \in T \) there is a directed path from \( s \) to \( t \)?

We will call the digraph \( D \) a directed Steiner Tree (DST) of weight \( w(E) \).

Let \( \mathbb{I} = (\mathbb{P}, k) \) where \( \mathbb{P} = (V, D, A, I, G) \) be an instance of \((0,2)\)-Bounded SAS\(^+\) Planning. Because of Lemma 1 we can assume that \( A \) contains no good actions with two effects. We construct an instance \( \mathbb{I}' = (N, w, s, T, p) \) of Directed Steiner Tree where \( p_M = k \) such that \( \mathbb{P} \) has a plan of length at most \( k \) if and only if \( \mathbb{I}' \) has a directed Steiner tree of weight at most \( p \). Because \( p_M = k \) this shows that \((0,2)\)-Bounded SAS\(^+\) Planning is fixed-parameter tractable.

We are now ready to define the instance \( \mathbb{I}' \). The node set \( N \) consists of the root vertex \( s \) and one node for every variable in \( V \). The weight function \( w \) is \( \infty \) for all but the following arcs:

(i) For every good action \( a \in A \) the arc from \( s \) to the unique variable \( v \in V \) that is affected by \( a \) gets weight 1.

(ii) For every mixed action \( a \in A \) with a good effect on some variable \( v_g \in V \) and a bad effect on some variable \( v_b \in V \), the arc from \( v_b \) to \( v_g \) gets weight 1.

We identify the root \( s \) from the instance \( I \) with the node \( s \), we let \( T \) be the set \( \Delta(V) \), and \( p_M = p = k \).

**Claim 7.** \( \mathbb{P} \) has a plan of length at most \( k \) if and only if \( \mathbb{I}' \) has a DST of weight at most \( p_M = p = k \).

Suppose \( \mathbb{P} \) has a plan \( \omega = (a_1, \ldots, a_l) \) with \( l \leq k \). Without losing generality we can assume that \( \omega \) contains no bad actions. The arc set \( E \) that corresponds to \( \omega \) consists of the following arcs:

(i) For every good action \( a \in \omega \) that has its unique good effect on a variable \( v \in V \), the set \( E \) contains the arc from \( s \) to \( v \).

(ii) For every mixed action \( a \in \omega \) with a good effect on some variable \( v_g \) and a bad effect on some variable \( v_b \), the set \( E \) contains an arc from \( v_b \) to \( v_g \).

Intuitively, for case (ii), note that \( a \) has a bad effect on \( v_b \), i.e. it sets \( v_b \) to a different value than its goal value, so \( a \) must be followed by some sequence \( a_1, \ldots, a_n \) of actions where only the last one is good and the others are mixed. This will provide a path in the DST from the root to \( v_b \).

More formally, it follows that the weight of \( E \) equals the number of actions in \( \omega \) and hence is at most \( p = k \) as required. It remains to show that the digraph \( D = (V, E) \) is a DST, i.e., \( D \) contains a directed path from the vertex \( s \) to every vertex in \( T \). Suppose to the contrary that there is a terminal \( t \in T \) that is not reachable from \( s \) in \( D \). Furthermore, let \( R \subseteq E \) be the set of all arcs in \( E \) such that \( D \) contains a directed path from the tail of every arc in \( R \) to \( t \). It follows that no arc in \( R \) is incident to \( s \). Hence, \( R \) only consists of arcs that correspond to mixed actions in \( \omega \). If \( R = \emptyset \) then the plan \( \omega \) does not contain an action that affects the variable \( t \). But this contradicts our assumption that \( \omega \) is a plan (because \( t \in \Delta(V) \)). Hence, \( R \neq \emptyset \). Let \( a \) be the mixed action corresponding to the arc in \( R \) that occurs last in \( \omega \) (among all mixed actions that correspond to an arc in \( R \)). Furthermore, let \( v \in V \) be the variable that is badly affected by \( a \). Then \( \omega \) cannot be a plan because after the occurrence of \( a \) in \( \omega \) there is no action in \( \omega \) that affects \( v \) and hence \( v \) cannot be in the goal state after \( \omega \) is executed.

To see the reverse direction, let \( E \subseteq N \times N \) be a solution of \( \mathbb{I} \) and let \( D = (N, E) \) be the DST. Without losing generality we can assume that \( D \) is a directed acyclic tree rooted in \( s \) (this follows from the minimality of \( D \)). We obtain a plan \( \omega \) of length at most \( p \) for \( \mathbb{P} \) by traversing the DST \( D \) in a bottom-up manner. More formally, let \( d \) be the maximum distance from \( s \) to any node in \( T \), and for every \( 1 \leq i < d \) let \( A(i) \) be the set of actions in \( A \) that correspond to arcs in \( E \) whose tail is at distance \( i \) from the node \( s \). Then \( \omega = (A(d - 1), \ldots, A(1)) \) (for every \( 1 \leq i \leq d - 1 \) the actions contained in \( A(d - 1) \) can be executed in an arbitrary order) is a plan of length at most \( k = p \) for \( \mathbb{P} \).

\( \square \)

7 Kernel Lower Bounds

In the previous sections we have classified the parameterized complexity of Bounded SAS\(^+\) Planning. It turned out that the problems fall into four categories (see Figure 1):
(i) polynomial-time solvable,
(ii) NP-hard but fixed-parameter tractable,
(iii) \(W[1]\)-complete, and
(iv) \(W[2]\)-complete.

The aim of this section is to further refine this classification with respect to kernelization. The problems in category (i) trivially admit a kernel of constant size, whereas the problems in categories (iii) and (iv) do not admit a bi-kernel at all (polynomial or not), unless \(W[1] = \text{FPT} \) or \(W[2] = \text{FPT}\), respectively. Hence it remains to consider the problems in category (ii), each of them could either admit a polynomial bi-kernel or not. We show that none of them does.

### 7.1 Kernel Lower Bounds for PUBS Restrictions

According to our classification so far, the only problems in category (ii) with respect to the PUBS-restrictions are the problems \(R\)-Bounded SAS\(^+\) Planning, for \(R \subseteq \{P, U, B, S\}\) such that \(P \in R\) and \(\{P, U, S\} \not\subseteq R\).

**Theorem 8.** None of the problems \(R\)-Bounded SAS\(^+\) Planning for \(R \subseteq \{P, U, B, S\}\) such that \(P \in R\) and \(\{P, U, S\} \not\subseteq R\) (i.e., the problems in category (ii)) admits a polynomial bi-kernel unless \(\text{coNP} \subseteq \text{NP}/\text{poly}\).

The remainder of this section is devoted to establish the above theorem. The relationships between the problems as indicated in Figure 1 greatly simplify the proof. Instead of considering all six problems separately, we can focus on the two most restricted problems \(\{P, U, B\}\)-Bounded SAS\(^+\) Planning and \(\{P, B, S\}\)-Bounded SAS\(^+\) Planning. If any other problem in category (ii) would have a polynomial bi-kernel, then at least one of these two problems would have one. This follows by Proposition 1 and the following facts:

1. The unparameterized versions of all the problems in category (ii) are NP-hard. This holds since the corresponding classical problems are strongly NP-hard, i.e., the problems remain NP-hard when \(k\) is encoded in unary (as shown by Bäckström and Nebel [8]);

2. If \(R_1 \subseteq R_2\) then the identity function gives a polynomial parameter reduction from \(R_2\)-Bounded SAS\(^+\) Planning to \(R_1\)-Bounded SAS\(^+\) Planning.

Furthermore, the following result of Bäckström and Nebel even provides a polynomial parameter reduction from \(\{P, U, B\}\)-Bounded SAS\(^+\) Planning to \(\{P, B, S\}\)-Bounded SAS\(^+\) Planning. Consequently, \(\{P, U, B\}\)-Bounded SAS\(^+\) Planning remains the only problem for which we need to establish a super-polynomial bi-kernel lower bound.

**Proposition 8** (Bäckström and Nebel [8, Theorem 4.16]). Let \(I = (P, k)\) be an instance of \(\{P, U, B\}\)-Bounded SAS\(^+\) Planning. Then \(I\) can be transformed in polynomial time into an equivalent instance \(I' = (P', k')\) of \(\{P, B, S\}\)-Bounded SAS\(^+\) Planning such that \(k = k'\).

Hence, in order to complete the proof of Theorem 8 it only remains to establish the next lemma.

**Lemma 2.** \(\{P, U, B\}\)-Bounded SAS\(^+\) Planning has no polynomial bi-kernel unless \(\text{coNP} \subseteq \text{NP}/\text{poly}\).

**Proof.** Because of Proposition 4, it suffices to devise a strong OR-composition algorithm for \(\{P, U, B\}\)-Bounded SAS\(^+\) Planning. Suppose we are given \(t\) instances \(I_1 = (P_1, k_1), \ldots, I_t = (P_t, k_t)\) of \(\{P, U, B\}\)-Bounded SAS\(^+\) Planning where \(P_i = (V_i, D_i, A_i, I_i, G_i)\) for every \(1 \leq i \leq t\). Let \(k = \max_{1 \leq i \leq t} k_i\). According to Theorem 6, \(\{P, U, B\}\)-Bounded SAS\(^+\) Planning can be solved in time \(O^*(S(k))\) (where \(S(k) = k^k\) and the \(O^*\) notation suppresses polynomial factors). It follows that \(\{P, U, B\}\)-Bounded SAS\(^+\) Planning can be solved in polynomial time with respect to \(\sum_{1 \leq i \leq t} |I_i| + k\)
if \( t > S(k) \). Hence, if \( t > S(k) \) this gives us a strong OR-composition algorithm as follows. We first run the algorithm for \{P,U,B\}-Bounded SAS\(^+\) PLANNING on each of the \( t \) instances. If there is some \( i \), where \( 1 \leq i \leq t \), such that \( P_i \) has a plan of length at most \( k_i \), then arbitrarily choose such an \( i \) and output \( \Pi_i \). Otherwise, arbitrarily output any of the instances \( \Pi_1, \ldots, \Pi_t \). This shows that \{P,U,B\}-Bounded SAS\(^+\) PLANNING has a strong OR-composition algorithm for the case where \( t > S(k) \). Hence, in the following we can assume that \( t \leq S(k) \).

Given \( \Pi_1, \ldots, \Pi_t \) we will construct an instance \( I = \langle P', k' \rangle \) of \{P,U,B\}-Bounded SAS\(^+\) PLANNING as follows. For the construction of \( I \) we need the following auxiliary gadget, which will be used to calculate the logical “OR” of two binary variables. The construction of the gadget uses ideas from Bäckström and Nebel [8, Theorem 4.15]. Assume that \( v_1 \) and \( v_2 \) are two binary variables. The gadget \( \text{OR}_2(v_1, v_2, o) \) consists of the five binary variables \( o_1, o_2, a, i_1, \) and \( i_2 \). Furthermore, \( \text{OR}_2(v_1, v_2, o) \) contains the following actions:

- the action \( a_o \) with \( \text{pre}(a_o)[o_1] = \text{pre}(a_o)[o_2] = 1 \) and \( \text{eff}(a_o)[o] = 1 \);
- the action \( a_{o_1} \) with \( \text{pre}(a_{o_1})[i_1] = 1 \), \( \text{pre}(a_{o_1})[i_2] = 0 \) and \( \text{eff}(a_{o_1})[o_1] = 1 \);
- the action \( a_{o_2} \) with \( \text{pre}(a_{o_2})[i_1] = 0 \), \( \text{pre}(a_{o_2})[i_2] = 1 \) and \( \text{eff}(a_{o_2})[o_2] = 1 \);
- the action \( a_{i_1} \) with \( \text{eff}(a_{i_1})[i_1] = 1 \);
- the action \( a_{i_2} \) with \( \text{eff}(a_{i_2})[i_2] = 1 \);
- the action \( a_{v_1} \) with \( \text{pre}(a_{v_1})[v_1] = 1 \) and \( \text{eff}(a_{v_1})[i_1] = 0 \);
- the action \( a_{v_2} \) with \( \text{pre}(a_{v_2})[v_2] = 1 \) and \( \text{eff}(a_{v_2})[i_2] = 0 \);

We now show that \( \text{OR}_2(v_1, v_2, o) \) can indeed be used to compute the logical “OR” of the variables \( v_1 \) and \( v_2 \). We need to show the following claim.

**Claim 8.** Let \( \mathbb{P}(\text{OR}_2(v_1, v_2, o)) \) be a \{P,U,B\}-Bounded SAS\(^+\) PLANNING instance that consists of the two binary variables \( v_1 \) and \( v_2 \), and the variables and actions of the gadget \( \text{OR}_2(v_1, v_2, o) \). Furthermore, let the initial state of \( \mathbb{P}(\text{OR}_2(v_1, v_2, o)) \) be any initial state that sets all variables of the gadget \( \text{OR}_2(v_1, v_2, o) \) to 0 but assigns the variables \( v_1 \) and \( v_2 \) arbitrarily, and let the goal state of \( \mathbb{P}(\text{OR}_2(v_1, v_2, o)) \) be defined by \( G[o] = 1 \). Then \( \mathbb{P}(\text{OR}_2(v_1, v_2, o)) \) has a plan if and only if its initial state sets at least one of the variables \( v_1 \) or \( v_2 \) to 1. Furthermore, if there is such a plan then its length is 6.

Suppose that there is a plan \( \omega \) for \( \mathbb{P}(\text{OR}_2(v_1, v_2, o)) \) and assume for a contradiction that both variables \( v_1 \) and \( v_2 \) are initially set to 0. It is easy to see that the value of \( v_1 \) and \( v_2 \) can not change during the whole duration of the plan and that \( \omega \) has to contain the actions \( a_{o_1} \) and \( a_{o_2} \). Without losing generality we can assume that \( \omega \) contains \( a_{o_1} \) before it contains \( a_{o_2} \). Because of the preconditions of the actions \( a_{o_1} \) and \( a_{o_2} \), the variable \( i_1 \) must have value 1 before \( a_{o_1} \) occurs in \( \omega \) and it must have value 0 before the action \( a_{o_2} \) occurs in \( \omega \). Hence, \( \omega \) must contain an action that sets the variable \( i_1 \) to 0. However, this can not be the case, since the only action setting \( i_1 \) to 0 is the action \( a_{v_1} \), which can not occur in \( \omega \) because the variable \( v_1 \) is 0 for the whole duration of \( \omega \).

To see the reverse direction suppose that one of the variables \( v_1 \) or \( v_2 \) is initially set to 1. If \( v_1 \) is initially set to one then \( \langle a_{o_1}, a_{v_1}, o_1, a_{i_2}, a_{o_2}, a_o \rangle \) is a plan of length 6 for \( \mathbb{P}(\text{OR}_2(v_1, v_2, o)) \). On the other hand, if \( v_2 \) is initially set to one then \( \langle a_{i_2}, a_{o_2}, a_{v_2}, o_1, a_{o_1}, a_o \rangle \) is a plan of length 6 for \( \mathbb{P}(\text{OR}_2(v_1, v_2, o)) \). Hence the claim is true. It should be noted that this is a use-once gadget; when it has computed the disjunction of \( v_1 \) and \( v_2 \) it may not be possible to reset it to do this computation again. This is sufficient for our purpose, however.

We continue by showing how we can use the gadget \( \text{OR}_2(v_1, v_2, o) \) to construct a gadget \( \text{OR}(v_1, \ldots, v_r, o) \) such that there is a sequence of actions of \( \text{OR}(v_1, \ldots, v_r, o) \) that sets the variable \( o \) to 1 if and only if at least one of the external variables \( v_1, \ldots, v_r \) are initially set to 1. Furthermore, if there is such a sequence of actions then its length is at most \( 6\lceil \log r \rceil \). Let \( T \) be a rooted binary tree with root \( s \) that has \( r \) leaves \( l_1, \ldots, l_r \) and is of smallest possible height. For every node \( t \in V(T) \) we make
a copy of our binary OR-gadget such that the copy of a leave node \(l_i\) is the gadget \(\text{OR}_2(v_{2i-1}, v_{2i}, a_i)\) and the copy of an inner node \(t \in V(T)\) with children \(t_1\) and \(t_2\) is the gadget \(\text{OR}_2(a_{t_1}, a_{t_2}, o_t)\) (clearly this needs to be adapted if \(r\) is odd or an inner node has only one child). For the root node with children \(t_1\) and \(t_2\) the gadget becomes \(\text{OR}_2(a_{t_1}, a_{t_2}, o)\). This completes the construction of the gadget \(\text{OR}(\text{v}_1, \ldots, \text{v}_r, o)\). Using Claim 8 it is easy to verify that the gadget \(\text{OR}(\text{v}_1, \ldots, \text{v}_r, o)\) can indeed be used to compute the logical “OR” or the variables \(\text{v}_1, \ldots, \text{v}_r\).

We are now ready to construct the instance \(\mathcal{I}\). \(\mathcal{I}\) contains all the variables and actions from every instance \(\mathcal{I}_1, \ldots, \mathcal{I}_t\) and of the gadget \(\text{OR}(\text{v}_1, \ldots, \text{v}_r, o)\). Furthermore, for every \(1 \leq i \leq t\) and \(k_i \leq j \leq k_h\), the instance \(\mathcal{I}\) contains the binary variables \(p^i_j\) and the actions \(a^i_j\) such that:

- \(\text{pre}(a^i_j) = G_i\) and \(\text{eff}(a^i_j[p^i_k]) = 1\),
- \(\text{pre}(a^i_{k_i+j})[p^i_{k_i+t-1}] = 1\) and \(\text{eff}(a^i_{k_i+j}[p^i_{k_i+t}]) = 1\), for every \(1 \leq l \leq k - k_i\).

Note that the actions \(a^i_j\) and the variables \(p^i_j\) are used to “pad” the different parameter values of the instances \(\mathcal{I}_1, \ldots, \mathcal{I}_t\) to the value \(k_h\).

Additionally, \(\mathcal{I}\) contains the binary variables \(\text{v}_1, \ldots, \text{v}_r\) and the actions \(a_1, \ldots, a_h\) with \(\text{pre}(a_i)[\text{v}_j] = 1\) and \(\text{eff}(a_i)[\text{v}_j] = 1\). Furthermore, the initial state \(I\) of \(\mathcal{I}\) is defined as \(I[v] = I[i][v]\) if \(v\) is a variable of \(\mathcal{I}_i\) and \(I[v] = 0\), otherwise. The goal state of \(\mathcal{I}\) is defined by \(G[o] = 1\) and we set \(k' = k + 1 + 6 \log t\). Clearly, \(\mathcal{I}\) can be constructed from \(\mathcal{I}_1, \ldots, \mathcal{I}_t\) in polynomial time and \(\mathcal{P}\) has a plan of length at most \(k\) if and only if there is some \(i\), where \(1 \leq i \leq t\), such that \(\mathcal{P}_i\) has a plan of length at most \(k_i\). Furthermore, because \(k' = k + 1 + 6 \log t \leq k + 1 + 6 \log 5(k) = k + 1 + 6 \log k^4 = k + 1 + 6/k \log k\), the parameter \(k^4\) is polynomially bounded by the parameter \(k\). This concludes the proof of the lemma. □

### 7.2 Kernel Lower Bounds for \((0, 2)\)-Bounded SAS$^+$ Planning

According to our classification so far, the only problem in category (ii) with respect to restrictions on the number of preconditions and effects is \((0, 2)\)-Bounded SAS$^+$ Planning. The next theorem suggests that \((0, 2)\)-Bounded SAS$^+$ Planning has no polynomial bi-kernel.

**Theorem 9.** \((0, 2)\)-Bounded SAS$^+$ Planning has no polynomial bi-kernel unless \(\text{coNP} \subseteq \text{NP}/\text{poly}\).

**Proof.** It is apparent from the\(\text{NP}\)-completeness proof for \((0, 2)\)-Bounded SAS$^+$ Planning \([13, \text{Theorem 4.6}]\) that the problem is even strongly \(\text{NP}\)-complete, i.e., the unparameterized version of it is \(\text{NP}\)-complete. According to Proposition 2 it is thus sufficient to devise an \(\text{OR}\)-composition algorithm for \((0, 2)\)-Bounded SAS$^+$ Planning to prove the theorem. Suppose we are given \(t\) instances \(\mathcal{I}_1 = (\mathcal{P}_1, k), \ldots, \mathcal{I}_t = (\mathcal{P}_t, k)\) of \((0, 2)\)-Bounded SAS$^+$ Planning where \(\mathcal{P}_1 = (\mathcal{V}_1, D_1, A_1, I_1, G_1)\) for every \(1 \leq i \leq t\). We will now show how we can construct the required instance \(\mathcal{I} = (\mathcal{P}_1, k')\) of \((0, 2)\)-Bounded SAS$^+$ Planning via an \(\text{OR}\)-composition algorithm. Without losing generality, we assume that \(\mathcal{P}_1, \ldots, \mathcal{P}_t\) have disjoint sets of variables and disjoint sets of actions. As a first step we compute the new instances \(\mathcal{I}'_1 = (\mathcal{P}'_1, k'), \ldots, \mathcal{I}'_t = (\mathcal{P}'_t, k')\) from \(\mathcal{I}_1 = (\mathcal{P}_1, k), \ldots, \mathcal{I}_t = (\mathcal{P}_t, k)\) according to Lemma 1. Then \(\mathcal{V}\) consists of the following variables:

- (i) the variables \(\bigcup_{1 \leq i \leq t} V_i\);
- (ii) binary variables \(b_1, \ldots, b_{2k'}\);
- (iii) for every \(1 \leq i \leq t\) and \(1 \leq j < 2k'\) a binary variable \(p_{i,j}\);
- (iv) A binary variable \(r\).

\(\mathcal{A}\) contains the action \(a_r\) with \(\text{eff}(a_r)[r] = 0\) and the following additional actions for every \(1 \leq i \leq t\):

- (i) The actions \(A'_i \setminus a^i_g\), where \(a^i_g\) is the copy of the action \(a_g\) for the instance \(\mathcal{I}'_i\) (recall the construction of \(\mathcal{I}'_i\) given in Lemma 1);
- (ii) An action \(a_i(r)\) with \(\text{eff}(a_i(r))[r] = 1\) and \(\text{eff}(a_i(r))[p_{i,1}] = 0\);
For actions with at most one effect, we have two cases: With no preconditions in Table 1 are derived as follows. For actions with an arbitrary number of effects, the results follow
arbitrary domain sizes $FPT$ completeness results follow from Theorems 2 and 5, and the
Summary of Results Finally, Theorem 8 shows that none of the variants of Bounded SAS$^+$
and Nebel [8], using standard complexity analysis, while dashed lines denote separation results from
previously considered by Bäckström and Nebel [8]. The complexity results for the various combinations
of restrictions P, U, B, and S restrictions previously considered by Bäckström and Nebel [8], using standard complexity analysis, while dashed lines denote separation results from our parameterized analysis. The W[2]-completeness results follow from Theorems 1 and 4, the W[1]-completeness results follow from Theorems 2 and 5, and the FPT results follow from Theorems 6 and 7. Finally, Theorem 8 shows that none of the variants of Bounded SAS$^+$ PLANNING, which are NP-hard and in FPT, admit a polynomial bi-kernel.

Bylander [13] studied the complexity of STRIPS under varying numbers of preconditions and effects, which is natural to view as a relaxation of restriction U in SAS$^+$.

We provide a full classification of the parameterized complexity of planning under Bylander’s restrictions. Table 1 shows such results (for arbitrary domain sizes $\geq 2$) under both parameterized and classical analysis. The parameterized results in Table 1 are derived as follows. For actions with an arbitrary number of effects, the results follow from Theorems 1 and 4. For actions with at most one effect, we have two cases: With no preconditions

(iii) For every $1 \leq j < 2k' - 1$ an action $a_{i,j}$ with $\text{eff}(a_{i,j})[p_{i,j}] = 1$ and $\text{eff}(a_{i,j})[p_{i,j+1}] = 0$;

(iv) An action $a_i(g)$ with $\text{eff}(a_i(g))[p_{i,2k'-1}] = 1$ and $\text{eff}(a_i(g))[g'] = 0$ where $g'$ is the copy of the variable $g$ for the instance $I_i'$ (recall the construction of $I_i'$ given in Lemma 1);

(v) Let $v_1, \ldots, v_r$ for $r \leq k'$ be an arbitrary ordering of the variables in $\Delta(V_i)$ (recall the definition of $\Delta(V_i)$ from Section 6.2). Then for every $1 \leq j \leq r$ we introduce an action $a_i(b_j)$ with $\text{eff}(a_i(b_j))[v_j] = I'_i[v_j]$ and $\text{eff}(a_i(b_j))[b_j] = 0$. Furthermore, for every $r < j \leq k'$ we introduce an action $a_i(b_j)$ with $\text{eff}(a_i(b_j))[v_j] = I'_i[v_j]$ and $\text{eff}(a_i(b_j))[b_j] = 0$.

We set $D = \bigcup_{1 \leq i \leq t} D'_i \cup \{0, 1\}$, $I_i[v] = G'_i[v]$ for every $v \in V'_i$ and $1 \leq i \leq t$, $I_i[v] = 0$ for every $v \in V \setminus (\bigcup_{1 \leq i \leq t} V'_i) \cup \{b_1, \ldots, b_k\}$, $I_i[v] = 1$ for every $v \in \{b_1, \ldots, b_k\}$, $G_i[v] = G'_i[v]$ for every $v \in V'_i$ and $1 \leq i \leq t$, $G_i[v] = 0$ for every $v \in V \setminus (\bigcup_{1 \leq i \leq t} V'_i)$, and $k'' = 4k' + 1$.

We note that all the subinstances corresponding to $P'_1, \ldots, P'_t$ already have their goals satisfied in the initial state $I$. However, since the variables $b_1, \ldots, b_k$ have the wrong value in $I$ it is necessary to include actions of type $a_i(b_j)$ in the plan. This “destroys the goal” for at least one subinstance so the plan must include a subplan to solve also this subinstance.

Clearly, $I_i$ can be constructed from $I_1, \ldots, I_t$ in polynomial time with respect to $\sum_{1 \leq i \leq t} |I_i| + k$ and the parameter $k'' = 4k' + 1 = 4(k(k + 3) + 1) + 1$ is polynomially bounded by the parameter $k$. By showing the following claim we conclude the proof of the theorem.

Claim 9. $P$ has a plan of length at most $k$ if and only if at least one of $P_1, \ldots, P_t$ has a plan of length at most $k$.

Suppose that there is an $1 \leq i \leq t$ such that $P_i$ has a plan of length at most $k$. It follows from Lemma 1 that $P'_i$ has a plan $\omega'$ of length at most $k'$. Then it is straightforward to check that $\omega = \langle a_i(b_1), \ldots, a_i(b_k) \rangle, \omega', \langle a_i(g), a_i, a_i(2k'-2), \ldots, a_i, a_i(r), a_i \rangle$ is a plan of length at most $4k' + 1$ for $P$.

For the reverse direction let $\omega$ be a plan of length at most $k''$. Without losing generality we can assume that for every $1 \leq i \leq t$ the set $\Delta(V_i)$ is not empty and hence every plan for $P'_i$ has to contain at least one action $a \in A'_i$ that corresponds to a good action in $P_i$. Because $\text{eff}(a)[g'] \neq I'_i[g'] = G'_i[g']$ for every such good action $a$ (recall the construction of $I'_i$ according to Lemma 1) it follows that there is an $1 \leq i \leq t$ such that $\omega$ contains all the $2k' + 1$ actions $a_i(g), a_i, a_i(2k'-2), \ldots, a_i, a_i(r), a_i$. Furthermore, because $k'' < 2(2k' + 1)$ there can be at most one such $i$ and hence $\omega \cap \bigcup_{1 \leq j \leq t} A'_j \subseteq A'_i$. Because $\Delta(V) = \{b_1, \ldots, b_k\}$ the plan $\omega$ also has to contain the actions $a_i(b_1), \ldots, a_i(b_k)$. Because of the effects (on the variables in $\Delta(V_i)$) of these actions it follows that $\omega$ has to contain a plan $\omega'_i$ of length at most $4k' + 1 - (2k' + 1) - k'' = k'$ for $P'_i$. It now follows from Lemma 1 that $P_i$ has a plan of length at most $k$. \hfill $\Box$

8 Summary of Results

We have obtained a full classification of the parameterized complexity of planning with respect to the length of the solution plan, under all combinations of the syntactical P, U, B, and S restrictions previously considered by Bäckström and Nebel [8]. The complexity results for the various combinations of restrictions P, U, B and S are displayed in Figure 1. Solid lines denote separation results by Bäckström and Nebel [8], using standard complexity analysis, while dashed lines denote separation results from our parameterized analysis. The W[2]-completeness results follow from Theorems 1 and 4, the W[1]-completeness results follow from Theorems 2 and 5, and the FPT results follow from Theorems 6 and 7. Finally, Theorem 8 shows that none of the variants of Bounded SAS$^+$ PLANNING, which are NP-hard and in FPT, admit a polynomial bi-kernel.

Bylander [13] studied the complexity of STRIPS under varying numbers of preconditions and effects, which is natural to view as a relaxation of restriction U in SAS$^+$.

We provide a full classification of the parameterized complexity of planning under Bylander’s restrictions. Table 1 shows such results (for arbitrary domain sizes $\geq 2$) under both parameterized and classical analysis. The parameterized results in Table 1 are derived as follows. For actions with an arbitrary number of effects, the results follow from Theorems 1 and 4. For actions with at most one effect, we have two cases: With no preconditions
the problem is trivially in \( P \). Otherwise, the results follow from Theorems 2 and 5. The case where the number of effects is bounded by some constant \( m_e > 1 \) can be reduced in polynomial time to the case with only one effect using a reduction by Bäckström [2, proof of Theorem 6.7]. Since this reduction is a parameterized reduction we have membership in \( W[1] \) by Theorem 5. When \( m_p \geq 1 \), then we also have \( W[1]\)-hardness by Theorem 2. For the final case \((m_p = 0)\), we obtain \( W[1]\)-hardness from Theorem 3 and containment in \( W[1] \) from Theorem 5 if the number of effects \( m_e \) is at least 3. The case where also \( m_e = 2 \) is fixed-parameter tractable according to Theorem 7, but Theorem 9 excludes that it admits a polynomial bi-kernel.

Since \( W[1] \) and \( W[2] \) are not directly comparable to the standard complexity classes we get interesting separations from combining the two methods. For instance, we can single out restriction \( U \) as making planning easier than in the general case, which is not possible under standard analysis. Since planning remains as hard as in the general case under restrictions \( B \) and \( S \) also for parameterized analysis, it seems that \( U \) is a more interesting and important restriction than the other two. Furthermore, the results in Table 1 suggest that also the restriction to a fixed number of effects larger than one is an interesting case. Even more interesting is that planning is in \( \text{FPT} \) under restriction \( P \), making it easier than the combination restriction \( US \), while it seems to be rather the other way around for standard analysis where restriction \( P \) is only known to be hard for \( \text{NP} \).

We have also provided a full classification of bi-kernel sizes for all the fixed-parameter tractable fragments. It turns out that none of the nontrivial problems (where the unparameterized version is \( \text{NP}\)-hard) admits a polynomial bi-kernel unless the Polynomial-time Hierarchy collapses. This implies an interesting dichotomy concerning the bi-kernel size: we only have constant-size and super-polynomial bi-kernels, and polynomially bounded bi-kernels that are not of constant size are absent. In order to establish these results, we had to adapt standard tools for kernel lower bounds to parameterized problems whose unparameterized versions are not (or not known to be) in \( \text{NP} \). We think that our notion of a strong \( \text{OR}\)-composition and the corresponding Proposition 4 could be useful for showing kernel lower bounds for other parameterized problems whose unparameterized versions are outside \( \text{NP} \).

### 9 Discussion

This work opens up several new research directions. We briefly discuss some of them below.

The use of parameterized analysis in planning is by no means restricted to using plan length as parameter. For instance, very recently Kronegger et al. [40] obtained parameterized results for several different parameters and combinations of them: one should note that the parameter need not be a single value, it can be a combination of two or more ‘basic’ parameters. A second example is considered by Downey et al. [18]. They show that STRIPS planning can be recast as the \textit{Signed Digraph Pebbling} problem which is modeled as a special type of graph. They analyze the parameterized complexity of this problem considering also the treewidth of the graph as a parameter. A final example is the recent paper by de Haan et al. [17] who study the parameterized complexity of plan reuse, where the task is to modify an existing plan to obtain the solution for a new planning instance by making a small modification.

Our observation that restriction \( U \) makes planning easier under parameterized analysis is interesting in the context of the literature on planning. Although this case remains \( \text{PSPACE}\)-complete under classical complexity analysis, it has been repeatedly stressed in the literature that unary actions are interesting for reasons of efficiency. For instance, Williams and Nayak [44] considered planning for spacecrafts and found that unary actions were often sufficient to model real-world problems in this domain. They noted that one consequence of having only unary actions is that the causal graph for a planning instance must be acyclic, a property which has often been exploited in the literature, both for theoretical results on planning complexity for various structures of the causal graph [12, 29, 39] as well as for practical planning [34, 35]. Of particular interest is a result on causal graphs in general by Chen and Giménez [16]. It is a classical complexity result that is proven under an assumption from parameterized complexity.

There are also close ties between model checking and planning and this connection deserves further study. For instance, model-checking traces can be viewed as plans and vice versa [23], and methods and
results have been transferred between the two areas in both directions [21, 22, 43]. Our reductions from planning to model-checking suggest that the problems are related also on a more fundamental level than just straightforward syntactical translations.

The major motivation for our research is the need for alternative and complementary methods in complexity analysis of planning. However, planning is also an interesting problem per se. It is a very powerful modelling language since it is PSPACE-complete in the general case, while it is also often simple to model other problems as planning problems. For instance, it would be interesting to identify various restrictions that make planning NP-complete but still allow for straightforward modelling of many common NP-complete problems, and analogously for other classes than NP.

Like most other results on complexity analysis of planning in the literature, our results apply to various restrictions of the actual planning language. However, the commonly studied language restrictions usually do not match the restrictions implied by applications. A complementary approach is thus to study the complexity of common benchmark problems for planning, e.g. the blocks world [32] and the problems used in the international planning competitions [33, 36]. Hence, it would be interesting to apply parameterized complexity analysis to these problems to see if it could help to explain the empirical results on which problems are hard and easy in practice. Finding the right parameter(s) would, of course, be crucial for achieving relevant results on this.

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