Supersymmetric codimension-two branes in six-dimensional gauged supergravity

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Abstract

We consider the six-dimensional Salam-Sezgin supergravity in the presence of codimension-2 branes. In the case that the branes carry only tension, we provide a way to supersymmetrise them by adding appropriate localised Fayet-Iliopoulos terms and localised corrections to the Chern-Simons term and modifying accordingly the fermionic supersymmetry transformations. The resulting brane action has $\mathcal{N} = 1$ supersymmetry (SUSY). We find the axisymmetric vacua of the system and show that one has unwarped background solutions with "football"-shaped extra dimensions which always respect $\mathcal{N} = 1$ SUSY for any value of the equal brane tensions, in contrast with the non-supersymmetric brane action background. Finally, we generically find multiple zero modes of the gravitino in this background and discuss how one could obtain a single chiral zero mode present in the low energy spectrum.

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1 Introduction

For the last decade, it has been an intensive effort to incorporate gravity for solving the particle physics problems. Particularly, in higher dimensional models with branes where the Standard Model (SM) particles are confined [1], the mass scale hierarchies in the SM can be understood from geometric factors in extra dimensions. Moreover, for the minimal supersymmetric extension of the SM (MSSM), the SUSY flavor problem can be ameliorated by a geometrical separation of the hidden sector from the visible sector in extra dimensions, the so called sequestering mechanism [2,3]. In this case, the anomaly mediation [2,4] can be a dominant contribution\(^1\) to the soft mass parameters in the MSSM. The supersymmetric embedding of the brane action in the 5D warped supergravity was studied in [8] and the extension of the analysis to the 6D flat supergravity has been done in [6].

Recently there has been a renewed interest into the 6D Salam-Sezgin supergravity [9], due to the findings of the new warped solutions [10–14]. The warped background has the extra dimensions “spontaneously” compactified by \(U(1)\) flux on the warped product of the 4D Minkowski space and a deformed sphere (or general two-dimensional compact Riemann surfaces). Moreover, the branes with nonzero tensions are accommodated at the conical singularities, without the need of cutting and pasting the extra dimension as in the 5D case. Since the 4D Minkowski space is present as a unique regular solution with maximal symmetry [10], the warped solution has a feature of self-tuning of the cosmological constant [15]\(^2\) (for a review, see [18]). There have been a lot of follow-up works on this model (as well as its non-SUSY analogue [19]), such as the perturbation analysis [20–22], the gravitino spectrum [23], cosmological de-Sitter or scaling solutions [24]\(^3\), regularisation of the conical singularities [26,27], cosmology on a regularised brane [28,29], modulus stabilisation [30], the Casimir effect [31], the effective 4D theory using the gradient expansion [32], exact wave solutions [33], etc. In the literature, however, the branes are regarded as breaking SUSY explicitly at the scale of brane tensions.

In this paper, we consider the supersymmetrisation of the brane tension action in a way compatible with the bulk SUSY in 6D Salam-Sezgin supergravity. We find that a brane-localised Fayet-Ilioupolos (FI) term\(^4\) proportional to each brane tension must be introduced to cancel the SUSY variation of the brane tension term. With a nonzero FI term, we should also add in the action the brane-localised bilinear fermion terms that couple to the \(U(1)_R\) field strength. Furthermore, we should modify the SUSY transformation of the \(U(1)_R\) gaugino with a singular term. The \(Z_2\) orbifold boundary conditions on the branes are also required to project out half of the bulk SUSY. In order to get the right Bianchi identities with the modified gauge field strengths, we also need to add a localised correction to the

\(^1\)The Kähler potential is not of a sequestered form in higher than five dimensions [5,6] but some global symmetry that is not broken by the messenger sector can keep the sequestering [7].

\(^2\)See, however, Refs. [16,17].

\(^3\)See Ref. [25] for old cosmological solutions without the presence of branes.

\(^4\)An arbitrary brane-localised FI term was considered to see the effect on the quantization condition in Refs. [11,17]. In 6D global SUSY, the effect of the FI term on the localisation and the Kaluza-Klein(KK) mass spectrum of bulk fields was discussed in Ref. [34].
Chern-Simons term in the field strength for the Kalb-Ramond field appearing in the action and the SUSY transformation.

Consequently, solving the modified equations of motion with singular FI terms, we find that the axisymmetric warped solution of the non-SUSY brane action is maintained, because the localised FI term is cancelled by a singular piece of the $U(1)_R$ field strength. However, the Wilson line phase of the gauge potential is now fixed to be nonzero at the brane position due to the extra singular term in the gauge field equation. From the SUSY variations of the spinors, we show that the only supersymmetric solution with branes is the unwarped "football"-shaped compactification. Moreover, we find that the FI terms change the flux quantization condition such that the brane tensions are not quantized any more for the same monopole number as in the Salam-Sezgin vacuum [9]. Furthermore, the FI terms affect the number of zero modes of gravitino and we expect that the same is true for any $U(1)_R$ charged bulk field.

By analysing the equation for the 4D component gravitino, we show that even after the $Z_2$ projection around the branes, there are generically multiple normalizable zero modes of the gravitino. In particular, for the "football" solutions, there are multiple chiral zero modes only from the left-handed gravitino: the one with zero winding number and pairs of chiral zero modes with nonzero winding numbers ($m, -m$). The mass terms for them would be forbidden unless the two $U(1)$ gauge symmetries in the system, the $U(1)_Q$ isometry of the axisymmetric extra dimensions and the $U(1)_R$ symmetry, are broken. In this "football" case, we propose that it is possible to have only one chiral zero mode of the 4D gravitino left (with zero winding number), if a linear combination of the $U(1)$ symmetries remains unbroken at low energies. The survival of only one chiral gravitino would be what one should expect from 4D unbroken $\mathcal{N} = 1$ supergravity.

The paper is organized as follows. First we present the bulk action of the 6D Salam-Sezgin supergravity to fix the notations. Then we consider the supersymmetrisation of the brane tension action and derive the required supersymmetric brane-bulk couplings. We go on to discuss the modified solutions with the localised FI terms, identify the supersymmetric football-shaped solution and study the effect on the zero modes of gravitino. Finally, the conclusions are drawn.

2 Six-dimensional Salam-Sezgin supergravity

The six-dimensional Salam-Sezgin supergravity [9] consists of gravity coupled to a dilatonic field $\phi$, a $U(1)_R$ gauge field $A_M$ and a Kalb-Ramond field $B_{MN}$, along with the necessary SUSY fermionic fields, the gravitino $\psi_M$, the dilatino $\chi$ and the gaugino $\lambda$ where all spinors are 6D Weyl. The $U(1)_R$ gauge field corresponds to the gauging of the $R$-symmetry of six-dimensional supergravity. The complete bulk Langrangian up to four fermion terms is given by

$$e_6^{-1} \mathcal{L}_{\text{bulk}} = R - \frac{1}{4} (\partial M \phi)^2 - \frac{1}{12} e^{\phi} G_{MNP} G^{MNP} - \frac{1}{4} e^{\frac{1}{2} \phi} F_{MN} F^{MN} - 8g^2 e^{-\frac{1}{2} \phi}$$

$$+ \bar{\psi}_M \Gamma^{MNP} D_N \psi_P + \bar{\chi} \Gamma^M D_M \chi + \bar{\lambda} \Gamma^M D_M \lambda$$
\[
+ \frac{1}{4} (\partial_M \phi)(\bar{\psi}_N \Gamma^M \Gamma^N \chi + \bar{\chi} \Gamma^N \Gamma^M \psi_N) \\
+ \frac{1}{24} e^{1/4} G_{MNP} (\bar{\psi}^R \Gamma^M N \Gamma^P \Gamma^S \psi^S + \bar{\psi}^R \Gamma^M N \Gamma^P \Gamma^S \chi^S) \\
- \bar{\chi} \Gamma^R \Gamma^M N \psi_R - \bar{\chi} \Gamma^M N \chi + \bar{\lambda} \Gamma^M N \lambda) \\
- \frac{1}{4 \sqrt{2}} e^{1/4} F_{MN} (\bar{\psi}^Q \Gamma^M N \Gamma^Q \lambda + \bar{\lambda} \Gamma^M N \Gamma^Q \psi_Q + \bar{\chi} \Gamma^M N \lambda - \bar{\lambda} \Gamma^M N \chi) \\
+ i \sqrt{2} g e^{-1/4} (\bar{\psi}_M \Gamma^M \lambda + \bar{\lambda} \Gamma^M \psi_M - \bar{\chi} \lambda + \bar{\lambda} \chi).
\]

(1)

The field strengths of the gauge and the Kalb-Ramond fields are defined as

\[
F_{MN} = \partial_M A_N - \partial_N A_M, \quad G_{MNP} = 3 \partial_M [B_{NP}] + \frac{3}{2} F_{[MNP]},
\]

and satisfy the Bianchi identities

\[
\partial_Q F_{MN} = 0, \quad \partial_Q G_{MNP} = \frac{3}{4} F_{[MNP]},
\]

For \( \delta A_M = \partial_M \Lambda \) under the \( U(1)_R \), the Kalb-Ramond field \( B_{MN} \) transforms as

\[
\delta B_{MN} = -\frac{1}{2} \Lambda F_{MN}.
\]

All the spinors have the same charge normalized to \(+1\) under \( U(1)_R \), so the covariant derivative of the gravitino, for instance, is given by

\[
D_M \psi_N = (\partial_M + \frac{1}{4} \omega_{MAB} \Gamma^{AB} - igA_M) \psi_N.
\]

The action for this Lagrangian is invariant under the following local \( \mathcal{N} = 2 \) SUSY transformations (up to the trilinear fermion terms):

\[
\delta e^A_M = \frac{1}{4} (e \Gamma^A \psi_M + \bar{\psi}_M \Gamma^A \varepsilon), \quad \delta \phi = \frac{1}{2} (\bar{\varepsilon} \chi + \bar{\chi} \varepsilon),
\]

\[
\delta \Gamma_{MN} = A_{[M} \delta A_{N]} + \frac{1}{4} e^{-1/4} (e \Gamma_M \psi_N - \bar{\psi}_N \Gamma_M \varepsilon - \bar{\varepsilon} \Gamma_N \psi_M + \bar{\psi}_M \Gamma_N \varepsilon) + \bar{\varepsilon} \Gamma_{MN} \chi - \bar{\chi} \Gamma_{MN} \varepsilon,
\]

\[
\delta \chi = -\frac{1}{4} (\partial_M \phi) \Gamma^M \varepsilon + \frac{1}{24} e^{1/4} G_{MNP} \Gamma^{MNP} \varepsilon,
\]

\[
\delta \psi_M = D_M \varepsilon + \frac{1}{48} e^{1/4} G_{PQR} \Gamma^{PQR} \Gamma_M \varepsilon;
\]

\[
\delta A_M = \frac{1}{2 \sqrt{2}} e^{1/4} F_{MNP} \Gamma^M \varepsilon - i \sqrt{2} g e^{-1/4} \varepsilon.
\]

\[
\delta \lambda = \frac{1}{4 \sqrt{2}} e^{1/4} F_{MNP} \Gamma^M \varepsilon - i \sqrt{2} g e^{-1/4} \varepsilon.
\]
The above spinors are chiral with handednesses
\[
\Gamma^7 \psi_M = +\psi_M, \quad \Gamma^7 \chi = -\chi, \quad \Gamma^7 \lambda = +\lambda, \quad \Gamma^7 \varepsilon = +\varepsilon. \tag{15}
\]
Taking into account that \( \Gamma^7 = \sigma^3 \otimes 1 \) (see Appendix A), the 6D (8-component) spinors can be decomposed to 6D Weyl (4-component) spinors as
\[
\psi_M = (\tilde{\psi}_M, 0)^T, \quad \chi = (0, \tilde{\chi})^T, \quad \lambda = (\tilde{\lambda}, 0)^T, \quad \varepsilon = (\tilde{\varepsilon}, 0)^T. \tag{16}
\]
For later use, we decompose the 6D Weyl spinor \( \tilde{\psi} \) to \( \tilde{\psi} = (\tilde{\psi}_L, \tilde{\psi}_R)^T \), satisfying \( \gamma^5(\tilde{\psi}_L, 0)^T = +(\tilde{\psi}_L, 0)^T \) and \( \gamma^5(0, \tilde{\psi}_R)^T = -(0, \tilde{\psi}_R)^T \).

3 Supersymmetrising the brane tension action

In this section, we will add in the previous action codimension-two branes with nonzero tension. With this addition, the total action is no longer invariant under the transformations (8)-(14). We will, thus, modify our action and SUSY transformations, so that the brane-bulk system is rendered supersymmetric. With the modification that we propose, we show that the bulk action remains supersymmetric while the brane action preserves \( \mathcal{N} = 1 \) SUSY.

3.1 Requirements for the supersymmetric brane action

Let us add to the bulk Lagrangian a term for a brane located at the position \( y = y_i \), where \( y \) is the internal space 2D coordinate. This brane Lagrangian will be given by
\[
\mathcal{L}_{\text{brane}} = -e_4 T_i \delta^{(2)}(y - y_i), \tag{17}
\]
where \( T_i \) is the brane tension and the 2D delta function is defined as \( \int d^2 y \delta^{(2)}(y - y_i) = 1 \).

The SUSY transformation of the brane action is non-vanishing as follows,
\[
\delta \mathcal{L}_{\text{brane}} = -e_4 T_i \frac{1}{4} \delta^{(2)}(y - y_i)(\bar{\psi}_\mu \Gamma^\mu \varepsilon + \text{h.c.}). \tag{18}
\]
On the other hand, because the gravitino is charged under \( U(1)_R \), varying the gravitino kinetic term under (12), it contains a piece of the gauge field strength as
\[
\delta \mathcal{L}_{\text{gravitino}} \supset e_6 \bar{\psi}_M \Gamma^{MNP} D_N D_P \varepsilon = -\frac{i}{2} e_6 g \bar{\psi}_M \Gamma^{MNP} \varepsilon F_{NP} + \cdots. \tag{19}
\]

We can utilise the above term of the gravitino variation to cancel the brane tension term as following. The \( U(1)_R \) field can have in principle FI localised terms [11,17] parameterized by constants \( \xi_i \). We can then define a hatted field strength \( \hat{F}_{MN} \)
\[
\hat{F}_{\mu \nu} = F_{\mu \nu}, \quad \hat{F}_{\mu m} = F_{\mu m}, \tag{20}
\]
\[
\hat{F}_{mn} = F_{mn} - \epsilon_{mn} \xi_i \frac{\delta^{(2)}(y - y_i)}{e_2}, \tag{21}
\]
where $\epsilon_{mn}$ is the 2D volume form, and rewrite the variation of the gravitino kinetic term as

$$\delta \mathcal{L}_{\text{gravitino}} \supset - \frac{i}{2} e_6 g \bar{\psi}_M \Gamma^{MNP} \epsilon \hat{F}_{NP}$$

$$+ e_4 g \xi_i \delta^{(2)}(y - y_i) \bar{\psi}_\mu \Gamma^\mu \gamma^5 \epsilon + \cdots,$$

(22)

where use is made of $\Gamma^{mn} \epsilon_{mn} = 2 \Gamma^{56} = 2 i \sigma^3 \otimes \gamma^5$, the 6D chirality condition, $\sigma^3 \otimes 1 \epsilon = \epsilon$, and $\frac{e_6}{e_2} = e_4$. Then, the first term cancels the variation of the bulk fermion bilinear term, if the $F_{MN}$ in the fermion bilinear term is replaced with $\hat{F}_{MN}$. Most importantly, the second term has the right form to cancel the variation of the brane tension term. The condition for this to happen is that,

$$\left( \gamma^5 - \frac{T_i}{4g\xi_i} \right) \epsilon(y_i) = 0.$$  

(23)

In other words, decomposing the SUSY variation spinor as $\epsilon = (\bar{\epsilon}, 0)^T$ with $\bar{\epsilon} = (\bar{\epsilon}_L, \bar{\epsilon}_R)^T$, the following should be satisfied,

$$\left( 1 - \frac{T_i}{4g\xi_i} \right) \bar{\epsilon}_L(y_i) = 0,$$

(24)

$$\left( 1 + \frac{T_i}{4g\xi_i} \right) \bar{\epsilon}_R(y_i) = 0.$$  

(25)

Thus, fixing the FI terms with the brane tensions as $\xi_i = \frac{T_i}{4g}$ or $-\frac{T_i}{4g}$, one needs to impose that either $\bar{\epsilon}_R$ or $\bar{\epsilon}_L$ vanish on the brane. Therefore, only $\mathcal{N} = 1$ SUSY can be preserved on the brane. For other values of $\xi_i$, both $\bar{\epsilon}_L$ and $\bar{\epsilon}_R$ must vanish at the brane, so there would be no SUSY left. In the bulk action and the SUSY transformations, when $F_{MN}$ is replaced by $\hat{F}_{MN}$, we also need to modify the field strength $G_{MNP}$ by $\hat{G}_{MNP}$ as

$$\hat{G}_{\mu\nu\lambda} = G_{\mu\nu\lambda},$$

(26)

$$\hat{G}_{\mu mn} = 3 \partial_{[\mu} B_{mn]} + \frac{3}{2} F_{[mn} A_{\mu]} - \xi_i A_\mu \epsilon_{mn} \frac{\delta^{(2)}(y - y_i)}{e_2}$$

$$= \hat{G}_{mn\mu} = \hat{G}_{n\mu mn}.$$  

(27)

On the other hand, keeping the form of terms $A_M$ to be the same\(^5\) as in the case with no branes, the modified bulk action is supersymmetric up to four fermion terms.

From now on, we choose $\xi_i = \frac{T_i}{4g}$ for all branes\(^6\) present in the internal space, so that there is $\mathcal{N} = 1$ SUSY remaining in the brane action with a SUSY parameter $\bar{\epsilon}_L$ non vanishing on the branes. This choice is made to agree with the no-brane Salam-Sezgin vacuum [9] where a constant $\bar{\epsilon}_L$ is a Killing spinor.

\(^5\)We note, however, that the solutions for the gauge field and the Kalb-Ramond field can be changed due to the singular FI term compared to the case with no branes, as will be shown later.

\(^6\)When there are different FI terms on the branes, there is no SUSY left, which corresponds to an explicit SUSY breaking by orbifolding.
3.2 Orbifold boundary conditions

Once an FI term has been chosen to make the brane tension action invariant under the SUSY transformations, one has in addition to impose that $\tilde{\varepsilon}_R$ vanishes at the brane position to preserve $\mathcal{N} = 1$ SUSY on the brane. This can be easily accomplished if we assume an orbifold $Z_2$ symmetry around the brane.

If the local complex coordinate around the brane is $z$ (in locally polar coordinates $z = re^{i\theta}$), then the $Z_2$ symmetry corresponds to

$$ z \leftrightarrow -z \quad (\text{or} \quad \theta \leftrightarrow \theta + \pi). \quad (28) $$

The same $Z_2$ was also introduced in [21] to avoid the possible instability of a negative tension brane. We should then assign $Z_2$ parities to all bulk fields and, of course, the SUSY variation parameters $\tilde{\varepsilon}_L$ and $\tilde{\varepsilon}_R$. A consistent choice of parities for the fields and the SUSY variation parameter is

even : $\tilde{\psi}_{aL}, \tilde{\psi}_{aR}, \tilde{\lambda}_{L}, \tilde{\lambda}_{R}, \tilde{\varepsilon}_L, A_\alpha, B_{a\beta}, B_{ab}, \phi,$ \hspace{1cm} (29)

odd : $\tilde{\psi}_{aR}, \tilde{\psi}_{aL}, \tilde{\lambda}_{R}, \tilde{\lambda}_{L}, \tilde{\varepsilon}_R, A_a, B_{\alpha a},$ \hspace{1cm} (30)

where the gauge field, the Kalb-Ramond field and the gravitino have been written with locally flat indices, e.g., $A_A = e^A_M A_M$, so that the parity assignments do not depend on the coordinate system. It is obvious that the above choice of parities forces $\tilde{\varepsilon}_R$ to vanish on the brane position.

In the case with two branes system, the warped vacua of [10] have an axially symmetric internal space. The above $Z_2$ symmetry about both branes present, is just a discrete subgroup of the axial symmetry. On the other hand, for the general warped solutions with multiple branes [13], we require the holomorphic function $V(z)$ in the metric to satisfy the condition $|V(-z + z_i)| = |V(z - z_i)|$, where $z_i$ is the $i$-th brane position.

3.3 The supersymmetric brane-bulk coupling

As a consequence of introducing the localised FI terms, we have seen that the brane tension action is made compatible with the bulk SUSY transformations. The supersymmetric action of the brane-bulk system up to four fermion terms is

$$ e_6^{-1} \mathcal{L}_{\text{SUSY}} = R - \frac{1}{4} (\partial_M \phi)^2 - \frac{1}{12} e^\phi \tilde{G}^{MNP} \tilde{G}^{MNP} - \frac{1}{4} e^{\frac{1}{2} \phi} \hat{F}_{MN} \hat{F}^{MN} - 8g^2 e^{-\frac{1}{2} \phi} $$

$$ + \tilde{\psi}_M \Gamma^{MNP} D_N \psi_P + \tilde{\chi} \Gamma^M D_M \chi + \tilde{\lambda} \Gamma^M D_M \lambda $$

$$ + \frac{1}{4} (\partial_M \phi) (\tilde{\psi}_N \Gamma^M \Gamma^N \chi + \tilde{\chi} \Gamma^N \Gamma^M \psi_N) $$

$$ + \frac{1}{24} e^{\frac{1}{2} \phi} \tilde{G}^{MNP} (\tilde{\psi}_R \Gamma_{[R} \Gamma^{MNP} \Gamma_{S]} \psi^S + \tilde{\psi}_R \Gamma^{MNP} \Gamma^R \chi $$

$$ - \tilde{\chi} \Gamma^R \Gamma^{MNP} \psi_R - \tilde{\lambda} \Gamma^{MNP} \chi + \tilde{\lambda} \Gamma^{MNP} \lambda) $$

$$ - \frac{1}{4 \sqrt{2}} e^{\frac{1}{2} \phi} \hat{F}_{MN} (\tilde{\psi}_Q \Gamma^{MNP} \Gamma^Q \lambda + \tilde{\lambda} \Gamma^Q \Gamma^{MN} \psi_Q + \tilde{\chi} \Gamma^{MN} \lambda - \tilde{\lambda} \Gamma^{MN} \chi) $$

$$ - \frac{1}{4 \sqrt{2}} e^{\frac{1}{2} \phi} \hat{F}_{MN} (\tilde{\psi}_Q \Gamma^{MNP} \Gamma^Q \lambda + \tilde{\lambda} \Gamma^Q \Gamma^{MN} \psi_Q + \tilde{\chi} \Gamma^{MN} \lambda - \tilde{\lambda} \Gamma^{MN} \chi) $$
\[+i\sqrt{2} g e^{-\frac{i}{4}\phi}(\bar{\psi}_M \Gamma^M \lambda + \bar{\lambda} \Gamma^M \psi_M) - \frac{e_4}{e_6} T_I \delta^{(2)}(y - y_i),\]  

where the modified gauge field strengths are

\[
\hat{F}_{MN} = F_{MN} - \delta^m_M \delta^n_N \epsilon_{mn} \xi_i \frac{\delta^{(2)}(y - y_i)}{e_2},
\]

\[
\hat{G}_{MNP} = G_{MNP} - 3 \delta^\mu_M \delta^\nu_N \delta^\rho_P A_\mu \epsilon_{mn} \xi_i \frac{\delta^{(2)}(y - y_i)}{e_2},
\]

with

\[\xi_i = \frac{T_i}{4g}.
\]

All the fermionic SUSY transformations are modified as

\[
\delta \chi = -\frac{1}{4}(\partial_M \phi) \Gamma^M \varepsilon + \frac{1}{24} e^{\frac{1}{4} \phi} \hat{G}_{MNP} \Gamma^{MNP} \varepsilon,
\]

\[
\delta \psi_M = D_M \varepsilon + \frac{1}{48} e^{\frac{1}{4} \phi} \hat{G}_{PQR} \Gamma^{PQR} \Gamma^M \varepsilon,
\]

\[
\delta \lambda = \frac{1}{4\sqrt{2}} e^{\frac{1}{4} \phi} \hat{F}_{MN} \Gamma^{MN} \varepsilon - i\sqrt{2} g e^{-\frac{1}{4} \phi} \varepsilon,
\]

but the bosonic SUSY transformations are the same as eqs. (8)-(10) and (13). The important ingredient of the above modifications is that we have a brane term linear in $F_{MN}$, the brane-localised FI term. In other words, there is a brane coupling to the magnetic flux, which is proportional to the brane tension. Moreover, we get a singular correction to the Chern-Simons term in the field strength for the KR field. We note that the modified field strengths satisfy the Bianchi identities, $\partial [Q \hat{F}_{MN}] = 0$ and $\partial [Q \hat{G}_{MNP}] = \frac{3}{4} \hat{F}_{[MN} \hat{F}_{QP]}$, even with the singular term.

One could be worried by the squared terms of the two-dimensional delta functions appearing in the kinetic term $\hat{F}_{MN} \hat{F}^{MN}$. However, SUSY requires these terms to be present and are a usual ingredient of orbifold supersymmetric theories [34,35]. The delta squared terms, i.e., $\delta^2(0)$, appear naturally in orbifolds, when bulk and brane fields are coupled supersymmetrically. One can obtain the same form $\hat{F}_{MN} \hat{F}^{MN}$ in a 6D off-shell supersymmetric $U(1)$ theory on $T^2/Z_2$, after the auxiliary field of the bulk vector multiplet is eliminated [34]. It has been known that the $\delta^2(0)$ term provides counterterms, which are necessary to maintain supersymmetry in explicit calculations on orbifolds, like the scattering amplitude and the self-energy correction for a brane field [35]. In our case, we have not introduced brane multiplets other than the tension. The case with brane multiplets will be studied elsewhere so the usual discussion on the $\delta^2(0)$ term on orbifolds is expected to hold.

As will be shown in the next section, when one looks for the solutions of the equations of motion of the above system, the singular term in the modified gauge field strength is cancelled by the singular part of the background value of $F_{MN}$, without affecting the
solution of the metric and the dilaton obtained for the non-SUSY brane action. Only the linear term in $F_{MN}$ with arbitrary coefficient has been considered for the non-SUSY brane action [11, 17]. However, in this case, even if $F_{MN}$ acquires a singular piece to satisfy the gauge field equation, it would lead to a problematic two-dimensional delta squared term in the Einstein and dilaton equations of motion [17]. Moreover, when one looks at the low energy effective theory, there is a worrisome singular delta squared term corresponding to the mass term of 4D $U(1)_R$ gauge boson $A_\mu$ from $\tilde{G}_{\mu\nu\rho} \tilde{G}^{\mu\nu\rho}$. However, by solving the linearized equation for $B_{MN}$ and inserting the solution for $B_{\mu m}$ into the action, the singular piece of the $B_{\mu m}$ cancels the contribution of the singular term in $\tilde{G}_{\mu\nu\rho}$, ending up with the regular action where the gauge boson gets a finite mass from the FI terms. Similar cancellations happen in 5D [36] and 6D [6] supergravities coupled to branes.

There are some known anomaly-free models including the non-abelian gauge fields in 6D gauged supergravity [37, 38]. In these cases, an abelian flux can be also turned on in the direction of the non-abelian gauge fields. For instance, in the model with $E_7 \times E_6 \times U(1)_R$ with hyperino $(912, 0)_0$, the $U(1)$ contained in $E_6$ can also develop a nonzero flux, still maintaining the warped solution that was obtained for the Salam-Sezgin supergravity [23]. As a result, $E_6$ is broken down to $SO(10)$ in the bulk and the adjoint fermions of $E_6$ can survive as two chiral 16’s of $SO(10)$ [37]. Even in this more general case, the supersymmetric brane action obtained for the Salam-Sezgin supergravity remains the same.

Furthermore, we can always introduce arbitrary localised FI terms for any abelian factor\(^7\) of the bulk gauge group other than $U(1)_R$ in a supersymmetric way because there is no constraint from the variation of the gravitino kinetic term unlike eq. (22). We only have to modify the field strengths appearing in both the bulk action and the fermionic SUSY transformations like in eqs. (32), (33) and (35)-(37). Thus, it is straightforward to see that the localised FI terms generated in 6D global SUSY case [34] are embedded into a supergravity theory.

4 Modification of the background solution due to the SUSY-brane action

In the present section, we will study the effect of the brane-localised FI terms to the warped axisymmetric solution that was obtained for non-SUSY brane action. We will see that the geometry is not modified by the latter addition, but the gauge field solution and the quantization condition change.

4.1 The modified equations of motion

We will study vacua where the Kalb-Ramond field is consistently (\textit{i.e.}, satisfying its equation of motion) set to zero. Then, the Einstein equations derived from the modified action

\(^7\)This does not include U(1) directions of non-abelian groups, as the one in $E_6$ mentioned above.
\begin{align*}
R_{MN} &= 2g^2 e^{-\frac{1}{2}\phi}g_{MN} + \frac{1}{2}e^{\frac{1}{2}\phi}(\hat{F}_{MP} \hat{F}_N^P - \frac{1}{8}g_{MN}\hat{F}_{PQ}^2) \\
&\quad + \frac{1}{4}\partial_M\phi \partial_N\phi + T^i_{MN},
\end{align*}

where $T^i_{MN} = -\frac{1}{2}\frac{\sqrt{-g}}{\sqrt{g}}T_i(g^A_{\mu\nu}\delta^\mu_M\delta^\nu_N - g_{MN})\delta^{(2)}(y - y_i)$ is the brane tension contribution (with $g^A_{\mu\nu}$ the 4D induced metric). Furthermore, the dilaton and the gauge field equations read

\begin{align*}
\Box^{(6)}\phi &= \frac{1}{4}e^{\frac{1}{2}\phi}\hat{F}_{PQ}^2 - 8g^2 e^{-\frac{1}{2}\phi}, \\
\partial_M(\sqrt{-g}e^{\frac{1}{2}\phi}\hat{F}^{MN}) &= 0.
\end{align*}

### 4.2 The modified warped solution

Assuming axial symmetry in the internal space, the form of the general warped solution of [10–12] is maintained, except that the solution for $F_{mn}$ is being replaced with the hatted one. Thus, the metric, the gauge field and the dilaton solutions are respectively

\begin{align*}
ds^2 &= W^2(r)\eta_{\mu\nu}dx^\mu dx^\nu + R^2(r)\left(dr^2 + \lambda^2\Theta^2(r)d\theta^2\right),
\hat{F}_{r\theta} &= \lambda q \frac{\Theta R^2}{W^6}, \\
\phi &= 4\ln W,
\end{align*}

with

\begin{align*}
R &= \frac{W}{f_0}; \quad \Theta = \frac{r}{W^2}, \\
W^4 &= \frac{f_1}{f_0}, \quad f_0 = 1 + \frac{r^2}{r_0^2}, \quad f_1 = 1 + \frac{r^2}{r_1^2},
\end{align*}

where $q$ is a constant denoting the magnetic flux, and the two radii $r_0, r_1$ are given by

\begin{align*}
r_0^2 &= \frac{1}{2g^2}, \quad r_1^2 = \frac{8}{q^2}.
\end{align*}

In the warped solution, the metric has two conical singularities, one at $r = 0$ and the other at $r = \infty$, which is at finite proper distance from the former one. The deficit angles $\delta_i$ of these singularities (supported by brane tensions $T_i = 2\delta_i$) are given by

\begin{align*}
\frac{\delta_0}{2\pi} &= 1 - \lambda, \\
\frac{\delta_\infty}{2\pi} &= 1 - \lambda\frac{r_1^2}{r_0^2}.
\end{align*}
In the unwarped limit, i.e., for \( r_0 = r_1 \), the two brane tensions must be equal.

Writing the delta function in eq. (32) in polar coordinates around \( r = 0 \) as
\[
\delta(2) (y - y_i) / e_2 = \delta(r) / (2 \lambda \pi r) \quad \text{and} \quad \epsilon_{r\theta} = \lambda r,
\]
eq (42) becomes
\[
F_{r\theta} - \frac{\xi_0}{2\pi} \delta(r) = \lambda q \frac{\Theta R^2}{W^6}.
\tag{49}
\]
Then, applying Stokes theorem around the patch including \( r = 0 \), one obtains that \( A_\theta(0) = \xi_0 / (2\pi) \) and thus the solution of the only non-zero component of the gauge field is
\[
A_\theta = -\frac{4\lambda}{q} \left( \frac{1}{f_1} - 1 \right) + \frac{\xi_0}{2\pi}.
\tag{50}
\]
Likewise, the gauge potential in the patch surrounding \( r = \infty \) is
\[
A_\theta = -\frac{4\lambda}{q} \frac{1}{f_1} - \frac{\xi_\infty}{2\pi}.
\tag{51}
\]
Hence, after connecting the gauge field solutions in two patches by a gauge transformation and requiring that it is single valued under \( 2\pi \) rotations, we find the following quantization condition should hold
\[
\frac{4\lambda g}{q} = n - \frac{g}{2\pi} (\xi_\infty + \xi_0), \quad n \in \mathbb{Z}.
\tag{52}
\]
In other words, we find that the FI terms fix the Wilson line phases of the gauge potential to be non-vanishing on the branes and can contribute to the quantization condition for \( \xi_0 \neq -\xi_\infty \), i.e., when \( T_0 \neq -T_\infty \). Since the covariant derivative has the same form as in the case with no branes, the modified background solution for the gauge potential changes the equations of motions of the other bulk fields and can affect the number of their zero modes. Using the flux quantization (52) with eqs. (47) and (48), we obtain the brane tensions are related as
\[
\left( 1 - \frac{T_0}{4\pi} \right) \left( 1 - \frac{T_\infty}{4\pi} \right) = \left[ n - \frac{g}{2\pi} (\xi_\infty + \xi_0) \right]^2.
\tag{53}
\]
In particular, for the football solution, since \( q = 4g \) and \( \xi_0 = \xi_\infty = \frac{\pi}{g} (1 - \lambda) \), the quantization condition (52) is satisfied for \( n = 1 \) and arbitrary \( \lambda \).

5 Supersymmetry of the background solution

Calculating the fermionic SUSY variations (35), (36), (37) for the above background solution, we can find in which cases the background respects or breaks SUSY. In the general warped background, SUSY is completely broken in the bulk. In the general warped background, this can be seen just from the SUSY transformation of the dilatino,
\[
\delta \chi = -\frac{W''}{W} [\cos \theta \sigma^1 \otimes \gamma^5 + \sin \theta \sigma^2 \otimes 1] \varepsilon,
\tag{54}
\]
which is always non-zero. In the special case of zero warping, \( i.e. \), when \( W' = 0 \), we need to study the remaining SUSY transformations.

When there is no brane present, the solution (41) becomes a sphere compactification, known as the Salam-Sezgin vacuum [9]. The nontrivial SUSY transformations of the fermions are

\[
\delta \lambda = i \sqrt{2} g (\gamma^5 - 1) \varepsilon, \tag{55}
\]

\[
\delta \psi_\theta = \left[ \partial_\theta + \frac{i}{2} \left( 1 + \lambda \left( 1 - \frac{2}{f_0} \right) \right) \gamma^5 + i \lambda \left( \frac{1}{f_0} - 1 \right) - \frac{i g \xi_0}{2 \pi} \right] \varepsilon. \tag{56}
\]

In this case, there exists a constant Killing spinor \( \tilde{\varepsilon}_L \), which means that \( \mathcal{N} = 1 \) SUSY is preserved.

For the “football”-shaped extra dimensions [15], there are two branes of equal tension, \( T_0 = T_\infty \), located at the poles of the sphere. The warp factor is constant, so we have that \( q = 4g \) and \( n = 1 \). In this case, the FI terms make the gauge potential nonzero at the branes and contribute to the quantization condition. In the patch surrounding the brane at \( r = 0 \), the nontrivial fermionic SUSY transformations are

\[
\delta \lambda = i \sqrt{2} g (\gamma^5 - 1) \varepsilon, \tag{57}
\]

\[
\delta \psi_\theta = \left[ \partial_\theta + \frac{i}{2} \left( 1 + \lambda \left( 1 - \frac{2}{f_0} \right) \right) \gamma^5 + i \lambda \left( \frac{1}{f_0} - 1 \right) - \frac{i g \xi_0}{2 \pi} \right] \varepsilon, \tag{58}
\]

where use is made of \( g \xi_0 = \frac{1}{4} T_0 = \pi (1 - \lambda) \) from eq. (47) in the last line. Then, for a non-zero left-handed variation parameter \( \tilde{\varepsilon}_L \), for which the gaugino variation is manifestly zero, the remaining nonzero gravitino variation is

\[
\delta \tilde{\psi}_\theta L = \partial_\theta \tilde{\varepsilon}_L. \tag{59}
\]

So, for any \( \lambda \), \( i.e. \) any brane tension, there exists a constant Killing spinor \( \tilde{\varepsilon}_L \), which is \( \mathbb{Z}_2 \)-even with respect to the \( r = 0 \) brane. Thus, we find that the modified spin connection is cancelled by the nonzero Wilson line phases at the brane positions, so that \( \mathcal{N} = 1 \) SUSY is preserved for the football solution. This is to be compared with the case of non-SUSY brane action in [23], where only the case of odd monopole number \( n \) would allow for \( \mathcal{N} = 1 \) SUSY on the brane.

### 6 The gravitino zero modes

As we have seen in section 4 and in particular in eqs. (50) and (51), there are in general two possible inequivalent Wilson line phases at the conical singularities due to the localised FI terms. In this section, we discuss the effect of these Wilson line phases to the existence of massless modes of the gravitino. We will also note the differences from the result obtained in the case for a non-SUSY brane action [23].

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For comparison with our earlier work [23], let us move to a Gaussian normal coordinate system, where the warped solution is written as

\[ ds^2 = W^2 \eta_{\mu\nu} dx^\mu dx^\nu + d\rho^2 + a^2 d\theta^2, \]

with \( d\rho = Rdr, a = \lambda R\Theta \).

After decomposing the 4D vector part\(^8\) of the 6D Weyl gravitino \( \psi_\mu = (\tilde{\psi}_\mu, 0)^T \) as \( \tilde{\psi}_\mu = (\tilde{\psi}_{\mu L}, \tilde{\psi}_{\mu R})^T \) in terms of the 4D Weyl spinors, we make a Fourier expansion of them as

\[
\tilde{\psi}_{\mu L} = \sum_m \tilde{\psi}_{\mu L}^{(m)}(x) \varphi_{\mu L}^{(m)}(\rho) e^{im\theta},
\]

\[
\tilde{\psi}_{\mu R} = \sum_m \tilde{\psi}_{\mu R}^{(m)}(x) \varphi_{\mu R}^{(m)}(\rho) e^{im\theta}.
\]

By the redefinition of the 4D gravitino, there is no mixing of \( \tilde{\psi}_\mu \) with the other fermionic modes [23]. To obtain the massless modes, we set \( \bar{\sigma}^\alpha \partial_\alpha \tilde{\psi}_{\mu L}^{(m)} = \bar{\sigma}^\beta \partial_\beta \tilde{\psi}_{\mu R}^{(m)} = 0 \). Then, the equations of left-handed and right-handed gravitinos are decoupled [23] and read

\[
\left[ \partial_\rho + \frac{W'}{W} + \frac{1}{a}(m - \frac{1}{2}\omega - gA_\theta) \right] \varphi_{\mu L}^{(m)} = 0,
\]

\[
\left[ \partial_\rho + \frac{W'}{W} + \frac{1}{a}(-m - \frac{1}{2}\omega + gA_\theta) \right] \varphi_{\mu R}^{(m)} = 0,
\]

with \( \omega = 1 - a' \). In the patch surrounding \( r = 0 \), we can find the explicit solution to the above equations as

\[
\varphi_{\mu L}^{(m)} = \frac{1}{W} \exp \left[ \int^\rho \frac{d\rho'}{a} \left( m + \frac{1}{2}\omega - gA_\theta \right) \right] = \frac{N_m}{W \sqrt{a}} \left( \frac{r}{r_0} \right)^{\frac{s}{2}} f_0^{\frac{1-t}{2}},
\]

with

\[
\begin{align*}
  s &= \frac{1}{\lambda} (1 + 2m) - \frac{g\xi_0}{\pi\lambda}, \\
  t &= \frac{1}{\lambda} (m + \frac{1}{2} - n + \frac{g\xi_\infty}{2\pi}) \left( 1 - \frac{r_0^2}{r^2} \right) + \frac{1}{\lambda} \left[ n - \frac{g}{2\pi}(\xi_\infty + \xi_0) \right] + 1,
\end{align*}
\]

where \( N_m \) is the normalization constant. We note that the solution for the right-handed gravitino is given by the one for the left-handed gravitino (65) with \((m,n,\xi_0,\xi_\infty)\) being replaced by \((-m,-n,-\xi_0,-\xi_\infty)\).

\(^8\)We will not be interested in the extra dimensional vector components of the gravitino \( \psi_m \) which are spin-\( \frac{1}{2} \) components.
From the normalisation condition
\[ \int d\theta \int d\rho \, W a \, | \varphi_{L,R}^{(m)} |^2 < \infty, \]  
we determine the normalisation constant of the general solution (65) as
\[ N_{m}^2 = \frac{1}{2\pi r_0} \left( \int_0^{\infty} dx \, \frac{x^s}{(1 + x^2)^t} \right)^{-1} \equiv \frac{\Gamma_m}{2\pi r_0}, \]  
with
\[ \Gamma_m \equiv \frac{2\Gamma[t]}{\Gamma[(1 + s)/2] \Gamma[t - (1 + s)/2]}. \]

Then, in order for a left-handed zero mode to exist, the following normalisability conditions should be respected,
\[ s > -1, \quad s - 2t < -1. \]  
In terms of our original parameters, we require that
\[ -\frac{1}{2} (1 + \lambda) + \frac{g\xi_0}{2\pi} < m < n - \frac{1}{2} \left( 1 - \frac{r_1^2}{r_0^2} \right) - \frac{g\xi_\infty}{2\pi}. \]  
For the right-handed zero mode, the corresponding normalisability condition reads
\[ n + \frac{1}{2} \left( 1 - \frac{r_1^2}{r_0^2} \right) - \frac{g\xi_\infty}{2\pi} < m < \frac{1}{2} (1 + \lambda) + \frac{g\xi_0}{2\pi}. \]

Using the relation between the FI term and the brane tension (34), as well as eqns. (47) and (48), the normalisability condition becomes for the left-handed mode
\[ -\lambda < m < n - 1 + \frac{r_1^2}{r_0^2}, \]  
and for the right-handed mode
\[ n < m < 1. \]

If we compare the above calculation to the one of the non-SUSY brane tensions [23], we see that in the SUSY brane case, due to the localised FI terms, there are corrections to the gravitino wavefunction (65) and consequently to the normalisability conditions (71) and (72). Moreover, it is also expected that there are modifications to the KK massive modes of the gravitino [23].

For the "football"-shaped solutions, we have that \( q = 4g \) and \( n = 1 \). For \( \lambda = 1 \), we obtain the well-known Salam-Sezgin vacuum with one 4D chiral gravitino, the left-handed zero mode \( \varphi_L^{(0)} \). For \( \lambda \neq 1 \), we see that we will always have normalisable left-handed zero modes \( \varphi_L^{(m)} \), but no right-handed ones. The action of the \( Z_2 \) parity on the left-handed modes requires that \( m \) is even. Therefore, for \( [\lambda] \) even, where \( [\lambda] \) is the nearest integer smaller than \( \lambda \), \( ([\lambda] - 1) \) left-handed zero modes are allowed, and for \( [\lambda] \) odd, \( [\lambda] \) left-handed
zero modes survive. In all the cases, $\mathcal{N} = 1$ SUSY is preserved by the background. We note that for $0 < \lambda < 1$, i.e. the positive tension branes, there is only one zero mode coming from the left-handed gravitino.

It would be surprising to find that for $\lambda \geq 3$, the $\mathcal{N} = 1$ unwarped solutions support more than one 4D chiral gravitinos, because one would expect only one surviving in $\mathcal{N} = 1$ 4D effective supergravity. The mass terms for these chiral gravitinos would be forbidden due to the $U(1)$ gauge symmetries: one is the $U(1)_{Q}$ isometry of the axisymmetric extra dimensions and the other is the $U(1)_{R}$ gauge symmetry. The charge operator $\hat{Q}$ of the $U(1)_{Q}$ commutes with the 6D Dirac mass operator [40] and it is given in the 6D spinor basis by

$$\hat{Q} = -i \partial_{\theta} + \frac{1}{2} \sigma^{3} \otimes \gamma^{5}. \quad (75)$$

Let us now consider the 4D effective action for the left-handed zero modes of the gravitino coupled to two $U(1)$ gauge bosons. The part of the effective low energy Lagrangian that is relevant in our discussion, is similar with the non-SUSY bulk model [41], and reads

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu \nu}^{\mu} - \frac{1}{4} F_{\mu \nu}^{\prime 2} + \sum_{m} \bar{\psi}_{\mu L}^{(m)} \gamma^{\mu} \sigma^{\nu} \gamma^{\lambda} \left( \partial_{\nu} + \frac{1}{4} \omega_{\nu \alpha \beta} \sigma^{[\alpha} \sigma^{\beta]} - ig_{4} R A_{\nu} - ig_{4}^{\prime} Q A_{\nu}^{\prime} \right) \psi_{\lambda L}^{(m)} \quad (76)$$

where $A_{\mu}, A_{\mu}^{\prime}$ are the $U(1)_{R}$ and $U(1)_{Q}$ gauge bosons with the 4D effective gauge couplings $g_{4}$ and $g_{4}^{\prime}$, respectively. Here, we note that the $R$ and $Q$ charge operators take the values $+1$ and $m + \frac{1}{2}$ for $\bar{\psi}_{\mu L}^{(m)}$, respectively. Then, after changing the basis of the gauge bosons to $A_{1\mu}$ and $A_{2\mu}$ as

$$A_{1\mu} = \frac{1}{\sqrt{4g_{4}^{2} + g_{4}^{\prime 2}}} \left( g_{4} A_{\mu} - 2g_{4} A_{\mu}^{\prime} \right), \quad (77)$$

$$A_{2\mu} = \frac{1}{\sqrt{4g_{4}^{2} + g_{4}^{\prime 2}}} \left( 2g_{4} A_{\mu} + g_{4}^{\prime} A_{\mu}^{\prime} \right), \quad (78)$$

the above action is rewritten as

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{1\mu \nu}^{2} - \frac{1}{4} F_{2\mu \nu}^{2} + \sum_{m} \bar{\psi}_{\mu L}^{(m)} \gamma^{\mu} \sigma^{\nu} \gamma^{\lambda} \left( \partial_{\nu} + \frac{1}{4} \omega_{\nu \alpha \beta} \sigma^{[\alpha} \sigma^{\beta]} - ig_{1} Q_{1} A_{1\nu} - ig_{2} Q_{2} A_{2\nu} \right) \psi_{\lambda L}^{(m)}, \quad (79)$$

Both of them can be anomaly-free due to the generalised Green-Schwarz mechanism where the $U(1)$ gauge bosons get masses but the theory is still invariant due to the axionic coupling to the gauge boson. The gauge boson mass of the $U(1)_{Q}$ could be read from a possible gravitational Chern-Simons term in the three form field strength, which arises due to the supersymmetric completion of the Green-Schwarz term [39], as in the case of the $U(1)_{R}$ gauge boson. The computation of it, is beyond the scope of the present paper.
where the new charge operators are

$$Q_1 = R - 2Q, \quad Q_2 = \frac{2g_4^2}{g_4^2}R + Q,$$

(80)

and the new gauge couplings are

$$g_1 = \frac{g_4 g_4'}{\sqrt{4g_4^2 + g_4'^2}}, \quad g_2 = \frac{g_4'^2}{\sqrt{4g_4^2 + g_4'^2}}.$$  

(81)

In this case, we note that the $Q_1$ charge of the left-handed zero mode with $m$ winding number is $Q_1 = -2m$.

Let us now suppose that at low energies, only $Q_1$ survives while $Q_2$ is broken\textsuperscript{10}. Then, for the "football" solutions, after the $Z_2$ projection, the remaining left-handed zero modes with nonzero even and opposite $m$ or $Q_1$ charges can be paired up to make a 4D Dirac spinor

$$\Psi^{(m)}_{\mu} = (\tilde{\psi}^{(-m)}_{\mu L}, -i\sigma^2 \tilde{\psi}^{(-m)*}_{\mu L})^T,$$

(82)

so that they get coupled by their Dirac masses. Therefore, there can be only one chiral massless mode of the gravitino with $m = 0$, \textit{i.e.}, the zero mode uncharged under the $U(1)_1$. The above mechanism for pairing the left-handed modes, relies on the VEV of a complex scalar field that breaks the $U(1)_2$, with appropriate quantum numbers which makes a Yukawa coupling with the left-handed modes $Q_2$-invariant. If in addition we write down localised Majorana mass terms on regularised branes [23] for the chiral $m = 0$ massless mode, we can end up with a non-zero mass 4D Majorana gravitino. In this case, the remaining $\mathcal{N} = 1$ SUSY should be also broken by nonzero F-terms on the branes.

For the general warped solution, we find that there are multiple zero modes of left-handed gravitino with even $m$ while there could also exist zero modes of right-handed gravitino with odd $m$. In this case, the number of zero modes depends on the warping and the monopole number.

In the presence of the localised FI terms, for a spin-$\frac{1}{2}$ fermion with the same $U(1)_R$ charge as the gravitino, a similar analysis can be done like in Ref. [21]. There is a difference from the gravitino case only by the warp factor dependence of the wavefunction. The wavefunction of the zero mode is given by eq. (65) with $W$ being replaced by $W^2$. However, for the spin-$\frac{1}{2}$ fermion, the weighting function in the norm integration (67) is changed to $W^3a$, so the normalization condition is the same as eqs. (73) and (74) in the gravitino case. Therefore, a spin-$\frac{1}{2}$ fermion has the same spectrum as the one of the gravitino. Thus, a pair of the spin-$\frac{1}{2}$ zero modes with $(m, -m)$ could be regarded as being eaten by a pair of the zero modes of the gravitino with $(m, -m)$ to make up a massive 4D Dirac gravitino. Consequently, each massive 4D Dirac gravitino should be part of an $\mathcal{N} = 1$ massive spin-$\frac{3}{2}$ supermultiplet.

\textsuperscript{10}If a linear combination $Q_2$ is anomalous, it could be broken due to the corresponding FI terms without breaking SUSY.
7 Conclusions

In this work, we examined the way to supersymmetrise the Salam-Sezgin model in the presence of codimension-2 branes carrying only tension. We have modified the brane action by adding brane localised FI terms and localised corrections to the Chern-Simons term and in addition changed the fermionic SUSY transformations. The resulting brane action respects $\mathcal{N} = 1$ SUSY, if the FI terms are chosen appropriately (related to the brane tension) and requires the presence of a $\mathbb{Z}_2$ symmetry to be realised.

The axisymmetric background solution for the above system is the same for the metric and dilaton fields as for the non-SUSY brane action system [10–12]. However, the gauge field solution acquires an additional Wilson line contribution. The last is important when discussing the SUSY of the background solution. Therefore, we find that the unwarped solution with "football"-shaped internal space does not need a quantized brane tension due to the flux quantization condition and it always respects 4D $\mathcal{N} = 1$ SUSY, in contrast with the non-SUSY brane action system.

The gravitino zero mode equation of motion was then analysed for the above-mentioned background. We found the conditions for which left- and right-handed modes are normalisable. We have focused on the unwarped "football" background case and remarked that always a left-handed mode survives with zero winding number $m$. For positive brane tensions, i.e. $0 < \lambda < 1$, there is only one zero mode of gravitino as in the Salam-Sezgin vacuum. For negative brane tensions with $\lambda \geq 3$, there are additional chiral zero modes with non-zero even $m$. It is conceivable that these extra modes, in some cases, can be paired to Dirac four-dimensional spinors, leaving only one chiral zero mode in the massless spectrum.

A natural continuation of the present study is to include $\mathcal{N} = 1$ matter multiplets (chiral and vector) on the branes with couplings to the bulk fields. This would require a regularisation of the brane, e.g., in the lines of [27], since the brane source terms coupled to the bulk fields other than the brane tension would lead to classical divergences. Then, it is expected that SUSY will completely fix the couplings of the brane with the bulk fields. In this way, we can reconsider the issue of moduli stabilisation [30, 39, 42] in the specific gauged supergravity with the supersymmetric branes. Moreover, if the MSSM fields are localised on one of the branes, one is expected to draw important conclusions about the supersymmetry breaking transmission between the bulk and the branes, or between the two distant branes in the different geometry than a torus. A generalization of the above study to multibrane systems without the axial symmetry [13] could also be interesting in that respect.

In addition, a necessary work that is important to be done is the consistency check of our proposal to eliminate the chiral modes of the gravitino with non-zero winding number $m$. One should study whether it is possible in the specific model to have one of the two $U(1)$'s naturally much heavier than the other, thus leaving one gravitino with a small mass in the low energy spectrum. Moreover, the decoupling of the chiral modes with non-zero $m$ relies on the nonzero VEV of a scalar field which has a right quantum number $Q_2$ for the Yukawa coupling. We plan to investigate the above questions in the near future.
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Appendix A: Notations and conventions

We use the metric signature \((-, +, +, +, +, +)\) for the 6D metric. The index conventions are the following: (1) for the Einstein indices we use \(M, N, \cdots = 0, \cdots, 5, 6\) for the 6D indices, \(\mu, \nu, \cdots = 0, \cdots, 3\) for the 4D indices and \(m, n, \cdots = 5, 6\) for the internal 2D indices, (2) for the Lorentz indices we use \(A, B, \cdots = 0, \cdots, 5, 6\) for the 6D indices, \(\alpha, \beta, \cdots = 0, \cdots, 3\) for the 4D indices and \(a, b, \cdots = 5, 6\) for the internal 2D indices.

We take the gamma matrices in the locally flat coordinates \([9]\), satisfying
\[\{\Gamma_A, \Gamma_B\} = 2\gamma_{AB},\]
where \(\gamma\)'s are the 4D gamma matrices with \(\gamma_5^2 = 1\) and \(\sigma\)'s are the Pauli matrices with
\[\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\] (A.2)
The curved gamma matrices on the other hand are given in terms of the ones in the locally flat coordinates as
\[\Gamma_M = e_M^A \Gamma_A,\]
where \(e_M^A\) is the 6D vielbein. In addition, the 6D chirality operator is given by
\[\Gamma_7 = \Gamma_0 \Gamma_1 \cdots \Gamma_6 = \sigma^3 \otimes \mathbf{1}.\] (A.3)
The convention for 4D gamma matrices is that
\[\gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\bar{\sigma}^\alpha & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\] (A.4)
with \(\sigma^\alpha = (1, \sigma^i)\) and \(\bar{\sigma}^\alpha = (-1, \sigma^i)\). The chirality projection operators are defined as
\[P_L = (1 + \gamma^5)/2 \quad \text{and} \quad P_R = (1 - \gamma^5)/2.\]

Finally, some useful quantities which we use in the text are the following
\[\Gamma^{a5} = \mathbf{1} \otimes \gamma^a \gamma^5, \quad \Gamma^{a6} = i\sigma^3 \otimes \gamma^a, \quad \Gamma^{56} = i\sigma^3 \otimes \gamma^5.\] (A.5)

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