Max–Min Representation of Piecewise Linear Functions

Sergei Ovchinnikov
Mathematics Department
San Francisco State University
San Francisco, CA 94132
sergei@sfsu.edu

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Abstract
It is shown that any piecewise linear function can be represented as a Max–Min polynomial of its linear components.

1 Introduction

The goal of the paper is to establish a representation of a piecewise linear function on a closed convex domain in $\mathbb{R}^d$ as a Max–Min composition of its linear components.

The paper is organized as follows. In Section 2 we introduce a hyperplane arrangement associated with a given piecewise linear function $f$ on a closed convex domain $\Gamma$. This arrangement defines a set $\mathcal{T}$ of regions with closures forming a cover of $\Gamma$. A distance function on $\mathcal{T}$ is introduced and its properties are established in Section 3. This distance function is an essential tool in our proof of the main result which is found in Section 4 (Theorem 4.1). Some final remarks are made in Section 5.

The ‘standard’ text on convex polytopes is [1]. More information on hyperplane arrangements is found in [2] and [3].

2 Preliminaries

We begin with the following definition.

Definition 2.1. Let $\Gamma$ be a closed convex domain in $\mathbb{R}^d$. A function $f : \Gamma \to \mathbb{R}$ is said to be piecewise linear if there is a finite family $\mathcal{Q}$ of closed domains such that $\Gamma = \cup \mathcal{Q}$ and $f$ is linear on every domain in $\mathcal{Q}$. A unique linear function $g$ on $\mathbb{R}^d$ which coincides with $f$ on a given $Q \in \mathcal{Q}$ is said to be a component of $f$. 

In this definition, an (affine) linear function is a function in the form

\[ h(x) = a \cdot x + b = a_1 x_1 + a_2 x_2 + \cdots + a_d x_d + b. \]

The equation \( h(x) = 0 \) defines an (affine) hyperplane provided \( a \neq 0 \).

Clearly, any piecewise linear function on \( \Gamma \) is continuous.

Let \( f \) be a piecewise linear function on \( \Gamma \) and \( \{g_1, \ldots, g_n\} \) be the family of its distinct components. In what follows, we assume that \( f \) has at least two distinct components.

Since components of \( f \) are distinct functions, the solution set of any equation in the form \( g_i(x) = g_j(x) \) for \( i < j \) is either empty or a hyperplane. We denote \( H \) the set of hyperplanes defined by the above equations that have a nonempty intersection with the interior \( \text{int}(\Gamma) \) of \( \Gamma \). A simple topological argument shows that \( H \neq \emptyset \); thus \( H \) is an (affine hyperplane) arrangement. Let \( T \) be the family of nonempty intersections of the regions of \( H \) with \( \text{int}(\Gamma) \). The elements of \( T \) are the connected components of \( \text{int}(\Gamma) \setminus \cup H \). Clearly, they are convex sets.

Note that \( \cup T \) is dense in \( \Gamma \). We shall use the same name ‘region’ for elements of \( T \). The closure \( \bar{Q} \) of \( Q \in T \) is the intersection of a polyhedron with \( \Gamma \). The intersections of facets of this polyhedron with \( \Gamma \) will be also called facets of \( Q \).

We have the following trivial but important property of \( T \).

**Proposition 2.1.** The restrictions of the components of \( f \) to any given region \( Q \in T \) are linearly ordered, i.e., for all \( i \neq j \), either \( g_i(x) > g_j(x) \) for all \( x \in Q \), or \( g_i(x) < g_j(x) \) for all \( x \in Q \).

In the rest of the paper, we shall use this property of components without making explicit reference to it.

### 3 Metric structure on \( T \)

We use a straightforward geometric approach to define a distance function on \( T \). It is the same distance function as in [1, Section 4.2] where it is defined as the graph distance on the tope graph.

For given \( P, Q \in T \), let \( S(P, Q) \) denote the separation set of \( P \) and \( Q \), i.e., the set of all hyperplanes in \( H \) separating \( P \) and \( Q \).

Let \( p \in P \) and \( q \in Q \) be two points in distinct regions \( P \) and \( Q \). Suppose \( S(P, Q) = \emptyset \). Then the interval \([p, q]\) belongs to a connected component of \( \text{int}(\Gamma) \setminus \cup H \) implying \( P = Q \), a contradiction. Thus we may assume \( S(P, Q) \neq \emptyset \).

The interval \([p, q]\) is a subset of \( \text{int}(\Gamma) \) and has a single point intersection with any hyperplane in \( S(P, Q) \). Moreover, we can always choose \( p \) and \( q \) in such a way that different hyperplanes in \( S(P, Q) \) intersect \([p, q]\) in different points. Let us number these points in the direction from \( p \) to \( q \) as follows

\[ r_0 = p, r_1, \ldots, r_{k+1} = q. \]
Each open interval \((r_i, r_{i+1})\) is an intersection of \([p, q]\) with some region which we denote \(R_i\) (in particular, \(R_0 = P\) and \(R_k = Q\)). Moreover, by means of this construction, points \(r_i\) and \(r_{i+1}\) belong to facets of \(R_i\). We conclude that regions \(R_i\) and \(R_{i+1}\) are adjacent for all \(i = 0, 1, \ldots, k - 1\).

Let us define \(d(P, Q) = |S(P, Q)|\) for all \(P, Q \in \mathcal{T}\). It follows from the argument in the foregoing paragraph that the function \(d\) satisfies the following conditions:

(i) \(d(P, Q) = 0\) if and only if \(P = Q\).

(ii) \(d(P, Q) = 1\) if and only if \(P\) and \(Q\) are adjacent regions.

(iii) If \(d(P, Q) = m\), then there exists a sequence \(R_0 = P, R_1, \ldots, R_m = Q\) of regions in \(\mathcal{T}\) such that \(d(R_i, R_{i+1}) = 1\) for \(0 \leq i < m\).

From an obvious relation \(S(P, Q) = S(P, R) \Delta S(R, Q)\) it follows that

(iv) \(d(P, Q) \leq d(P, R) + d(R, Q)\), and

(v) \(d(P, Q) = d(P, R) + d(R, Q)\) if and only if \(S(P, Q) = S(P, R) \cup S(R, Q)\).

We summarize these properties of \(d\) in the following proposition.

**Proposition 3.1.** The function \(d(P, Q) = |S(P, Q)|\) is a metric on \(\mathcal{T}\) satisfying the following properties:

(i) \(d(P, Q) = 1\) if and only if \(P\) and \(Q\) are adjacent regions.

(ii) If \(d(P, Q) = m\) then there exists a sequence \(R_0 = P, R_1, \ldots, R_m = Q\) such that \(d(R_i, R_{i+1}) = 1\) for \(0 \leq i < m\).

(iii) \(d(P, Q) = d(P, R) + d(R, Q)\) if and only if \(S(P, Q) = S(P, R) \cup S(R, Q)\).

## 4 Main theorem

In this section, we prove the following theorem.

**Theorem 4.1.** (a) Let \(f\) be a piecewise linear function on \(\Gamma\) and \(\{g_1, \ldots, g_n\}\) be the set of its distinct components. There exists a family \(\{S_j\}_{j \in J}\) of subsets of \(\{1, \ldots, n\}\) such that

\[
f(x) = \bigvee_{j \in J} \bigwedge_{i \in S_j} g_i(x), \quad \forall x \in \Gamma.
\]

(b) Conversely, for any family of distinct linear functions \(\{g_1, \ldots, g_n\}\) the above formula defines a piecewise linear function.

Here, \(\bigvee\) and \(\bigwedge\) are operations of maximum and minimum, respectively. The expression on the right side in (1) is a Max–Min (lattice) polynomial in the variables \(g_i\)’s.
For a given $P \in \mathcal{T}$ we denote $f^P$ (resp. $g^P_i$) the restriction of $f$ (resp. $g_i$) to $P$. The functions $g^P_i$ are linearly ordered for a fixed $P$. Since the restriction of $f$ to $P$ is one of the functions $g^P_i$, there is a unique number $n(P)$ such that $f^P = g^P_{n(P)}$.

**Lemma 4.1.** For any $P, Q \in \mathcal{T}$ there exists $k$ such that 

$$g^P_k \leq g^P_{n(P)} \quad \text{and} \quad g^Q_k \geq g^Q_{n(Q)},$$

or, equivalently,

$$g_k(x) \leq f(x), \forall x \in P, \quad \text{and} \quad g_k(x) \geq f(x), \forall x \in Q.$$ 

**Proof.** The proof is by induction on $d(P, Q)$.

(i) $d(P, Q) = 1$. By Proposition 3.1(i), $P$ and $Q$ are adjacent regions. Let $F$ be the common facet of the closures of $P$ and $Q$ and let $H$ be the affine span of $F$. Since functions $f, g_{n(P)}, g_{n(Q)}$ are continuous, $g_{n(P)}(x) = g_{n(Q)}(x)$ for all $x \in F$ and therefore for all $x \in H$. We may assume that $g^P_{n(P)} < g^P_{n(Q)}$ (the other case is treated similarly). Then $g^Q_{n(P)} > g^Q_{n(Q)}$, and $k = n(P)$ satisfies conditions of the lemma.

(ii) $d(P, Q) > 1$. By Proposition 3.1(ii) and (i), there is a region $R$ adjacent to $P$ such that $d(R, Q) = d(P, Q) - 1$. By the induction hypothesis, there is $r$ such that

$$g^R_r \leq g^R_{n(R)} \quad \text{and} \quad g^Q_r \geq g^Q_{n(Q)}.$$ 

If $g^P_r \leq g^P_{n(P)}$, then $k = r$ satisfies conditions of the lemma. Otherwise, we have $g^P_r > g^P_{n(P)}$. By Proposition 3.1(iii), the unique hyperplane $H \in \mathcal{H}$ that separates $P$ and $R$ also separates $P$ and $Q$. The same argument as in (i) shows that $g_{n(P)}(x) = g_{n(R)}(x)$ for all $x \in H$. Since $g^P_r > g^P_{n(P)}$ and $g^R_r \leq g^R_{n(R)}$, we have $g_r(x) = g_{n(P)}(x)$ for all $x \in H$. Consider function $g = g_r - g_{n(P)}$. It is zero on the hyperplane $H$ and positive on the full-dimensional region $P$. Thus it is positive on the open halfspace containing $P$. Hence, it must be negative on the open halfspace containing $R$ and $Q$. We conclude that $g^Q_r \geq g^Q_{n(Q)}$. Since $g^Q_r \geq g^Q_{n(Q)}$, $k = n(P)$ satisfies conditions of the lemma.

Now we proceed with the proof of Theorem 4.1.

(a) For a given $P \in \mathcal{T}$ we define $S_P = \{i : g^P_i \geq g^P_{n(P)}\} \subseteq \{1, \ldots, n\}$. Let $F_P(x)$ be a function defined on $\Gamma$ by the equation

$$F_P(x) = \bigwedge_{i \in S_P} g_i(x).$$

Clearly, $F_P(x) = g_{n(P)}(x) = f(x)$ for all $x \in P$. 


Suppose $F_P(y) < F_Q(y)$ for some $y \in P$ and $Q \neq P$, i.e.,
\[
\bigwedge_{i \in S_P} g_i(y) < \bigwedge_{j \in S_Q} g_j(y).
\]
Then $g_{n(P)}^P < g_j^P$ for all $j \in S_Q$. Thus for any $j$ such that $g_j^Q \geq g_{n(Q)}^Q$, we have $g_{n(P)}^P < g_j^P$. This contradicts Lemma 4.1.

Hence, $F_Q(x) \leq F_P(x) = f(x)$ for all $x \in P$ and $Q \in T$.

Consider function $F(x)$ defined by
\[
F(x) = \bigvee_{P \in T} F_P(x) = \bigvee_{P \in T} \bigwedge_{i \in S_P} g_i(x)
\]
for all $x \in \Gamma$. Clearly, $f(x) = F(x)$ for all $x \in \bigcup T$. Since $\bigcup T$ is dense in $\Gamma$ and $f$ and $F$ are continuous functions on $\Gamma$, we conclude that
\[
f(x) = \bigvee_{P \in T} \bigwedge_{i \in S_P} g_i(x)
\]
for all $x \in \Gamma$.

(b) Let $\{g_1, \ldots, g_n\}$ be a family of distinct linear functions on $\Gamma$ and let $f$ be defined by (1). Consider sets $H_{ij} = \{x : g_i(x) = g_j(x), i > j\}$. If the intersections of these sets with $\text{int}(\Gamma)$ are empty, then the functions $g_i$’s are linearly ordered over $\text{int}(\Gamma)$ and, by (1), $f$ is a linear function. Otherwise, let $\mathcal{H}$ be the family of sets $H_{ij}$’s with nonempty intersections with $\text{int}(\Gamma)$. Let $Q$ be a region of the arrangement $\mathcal{H}$. Since $Q$ is connected, the functions $g_i$’s are linearly ordered over $Q$ and, by (1), there is $i$ such that $f(x) = g_i(x)$ for all $x \in Q$. The same is also true for the closure of $Q$.

This completes the proof of Theorem 4.1.

Corollary 4.1. Let $\Gamma$ be a star-like domain in $\mathbb{R}^d$ such that its boundary $\partial \Gamma$ is a polyhedral complex. Let $f$ be a function on $\partial \Gamma$ such that its restriction to each $(d-1)$-dimensional polyhedron in $\partial \Gamma$ is a linear function on it. Then $f$ admits representation (1).

Proof. Let $a$ be a central point in $\Gamma$. For $x \in \mathbb{R}^d$, $x \neq a$, let $\tilde{x}$ be the unique intersection point of the ray from $a$ through $x$ with $\partial \Gamma$. We define
\[
\tilde{f}(x) = \begin{cases} 
\frac{\|x-a\|}{\|\tilde{x}-a\|} f(\tilde{x}), & \text{for } x \neq a, \\
0, & \text{for } x = a.
\end{cases}
\]

Clearly, $\tilde{f}$ is a piecewise linear function on $\mathbb{R}^d$ and $\tilde{f}|_{\partial \Gamma} = f$. Thus $f$ admits representation (1).

Note that the previous corollary holds for any polyhedron in $\mathbb{R}^d$. 

\[5\]
5 Concluding remarks

1. The statements of Theorem 4.1 also hold for piecewise linear functions from \( \Gamma \) to \( \mathbb{R}^m \). Namely, let \( f : \Gamma \to \mathbb{R}^m \) be a piecewise linear function and \( \{g_1, \ldots, g_n\} \) be the set of its distinct components. We denote

\[
f = (f_1, \ldots, f_m) \quad \text{and} \quad g_k = (g_1^{(k)}, \ldots, g_m^{(k)}), \quad \text{for } 1 \leq k \leq n.
\]

There exists a family \( \{S^k_j\}_{j \in J, 1 \leq k \leq n} \) of subsets of \( \{1, \ldots, n\} \) such that

\[
f_k(x) = \bigvee_{j \in J} \bigwedge_{i \in S^k_j} g_i^{(k)}(x), \quad \forall x \in \Gamma, \; 1 \leq k \leq m.
\]

The converse is also true.

2. The convexity of \( \Gamma \) is an essential assumption. Consider, for instance, the domain in \( \mathbb{R}^2 \) which is a union of three triangles defined by the sets of their vertices as follows:

\[
\Delta_1 = \{(−1,0), (−1,−1), (0,0)\}, \quad \Delta_2 = \{(0,0), (1,1), (1,0)\}, \quad \text{and} \quad \Delta_3 = \{(−1,0), (1,0), (0,−1)\}.
\]

Let us define

\[
f(x) = \begin{cases} 
-x_2, & \text{for } x \in \Delta_1, \\
x_2, & \text{for } x \in \Delta_2 \cup \Delta_3,
\end{cases}
\]

where \( x = (x_1, x_2) \). This piecewise linear function has two components, \( g_1(x) = -x_2 \) and \( g_2(x) = x_2 \), but is not representable in the form (1).

3. Likewise, (1) is not true for piecewise polynomial functions as the following example (due to B. Sturmfels) illustrates. Let \( \Gamma = \mathbb{R}^1 \). We define

\[
f(x) = \begin{cases} 
0, & \text{for } x \leq 0, \\
x^2, & \text{for } x > 0.
\end{cases}
\]

4. It follows from Theorem 4.1 that any piecewise linear function on a closed convex domain in \( \mathbb{R}^d \) can be extended to a piecewise linear function on the entire space \( \mathbb{R}^d \).

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