Tail processes for stable-regenerative multiple-stable model

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We investigate a family of discrete-time stationary processes defined by multiple stable integrals and renewal processes with infinite means. The model may exhibit behaviors of short-range or long-range dependence, respectively, depending on the parameters. The main contribution is to establish a phase transition in terms of the tail processes that characterize local clustering of extremes. Moreover, in the short-range dependence regime, the model provides an example where the extremal index is different from the candidate extremal index.

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1. Introduction and main results

1.1. The model and background

The objective of this paper is to study the local behavior of extremes of a family of stationary stochastic processes known as the stable-regenerative multiple-stable model that has attracted attention in the studies of stochastic processes with long-range dependence [6, 20, 25]. In an accompanying paper [4] the macroscopic/global limit of extremes are established in terms of convergence of the random sup-measures in the framework of O’Brien, Torfs and Vervaat [18], and a phase transition is revealed. Here, we characterize the microscopic/local limit of extremes, in terms of the tail processes as introduced by Basrak and Segers [5].

The family of processes of our interest has a tail parameter \( \alpha \in (0, 2) \), a memory parameter \( \beta \in (0, 1) \), and a multiplicity parameter \( p \in \mathbb{N} := \{1, 2, \ldots\} \). The representation is intrinsically related to renewal processes, for which we introduce some notation. Consider a discrete-time renewal process with the consecutive renewal times denoted by \( \tau := \{\tau_0, \tau_1, \ldots\} \subset \mathbb{N}_0 := \{0, 1, 2, \ldots\} \). Here \( \tau_0 \) is the initial renewal time, and the inter-renewal times \( (\tau_i - \tau_{i-1})_{i \geq 0} \) are i.i.d. \( \mathbb{N} \)-valued with cumulative distribution function \( F \), that is, \( F(x) = \mathbb{P}(\tau_i - \tau_{i-1} \leq x), i \in \mathbb{N}, x \geq 0 \). Denote the probability mass function by \( f(n) = \mathbb{P}(\tau_i - \tau_{i-1} = n), n \in \mathbb{N} \). Throughout, we assume

\[
\mathcal{F}(x) = 1 - F(x) \sim C_F x^{-\beta} \text{ as } x \to \infty \text{ with } \beta \in (0, 1), \tag{1.1}
\]

which implies an infinite mean, and the following technical assumption

\[
\sup_{n \in \mathbb{N}} \frac{nf(n)}{F(n)} < \infty. \tag{1.2}
\]
By default, a renewal process starts at renewal at time 0, and hence \( \tau_0 = 0 \). Note that our renewal processes may be *delayed*, that is, \( \tau_0 \) is not necessarily zero, and may be a random variable in \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), and we shall be specific when this is the case. An important notion is the *stationary delay measure* of the renewal process, denoted by \( \pi \). More precisely, \( \pi \) is supported on \( \mathbb{N}_0 \) with
\[
\pi(k) \equiv \pi(\{k\}) = \mathcal{F}(k) = 1 - F(k), \quad k \in \mathbb{N}_0.
\]
(For the sake of simplicity, we do not distinguish \( \pi(k) \), the mass function at \( k \in \mathbb{N}_0 \), from \( \pi(\{k\}) \), the measure evaluated at the set \( \{k\} \).) Note that the stationary delay measure \( \pi \) is a \( \sigma \)-finite and infinite measure on \( \mathbb{N}_0 \), since the renewal distribution has infinite mean. More details are in Section 2.

We then provide a series representation of the model. Consider
\[
\sum_{i=1}^{\infty} \delta_{(y_i, d_i)} \overset{d}{=} \text{PPP} \left( (0, \infty) \times \mathbb{N}_0, \alpha x^{-\alpha-1} dx d\pi \right), \tag{1.3}
\]
where the right-hand side is understood as the Poisson point process on \((0, \infty) \times \mathbb{N}_0\) with intensity measure \( \alpha x^{-\alpha-1} dx d\pi \). In addition, let \( \{\pi(\{i\})\}_{i \in \mathbb{N}} \) denote, given the above Poisson point process, conditionally independent delayed renewal processes, each \( \pi(\{i\}) \) with initial renewal time delayed at \( \tau_0 = d_i \) and inter-renewal times following \( F \).

In this paper, we consider the *stable-regenerative multiple-stable model* given by
\[
\{X_k\}_{k \in \mathbb{N}_0} \overset{d}{=} \left\{ \sum_{0 < i_1 < \cdots < i_p} [\varepsilon_{i_1} \varepsilon_{i_2} \cdots \varepsilon_{i_p}] \mathbf{1}_{\{k \in \bigcap_{r=1}^{p} \tau(\{r\}) \}} \right\}_{k \in \mathbb{N}_0}, \tag{1.4}
\]
where \( \{\varepsilon_i\}_{i \in \mathbb{N}} \) are i.i.d. Rademacher random variables independent of the point process in (1.3), \( [\varepsilon_i] = \varepsilon_{i_1} \times \cdots \times \varepsilon_{i_p} \), and similar notation applies for \( [y_i] \). The name ‘stable-regenerative’ comes from the fact that each renewal process \( \tau \), say non-delayed, has a scaling limit as a \( \beta \)-stable-regenerative random closed set [7, 13]. The name ‘multiple-stable’ comes from the fact that the series representation above corresponds to a multiple stochastic integral with respect to a stable random measure. See [2] for our model described explicitly in a multiple-stable-integral representation. We shall work with the series representation only, so we omit the stochastic-integral representation here. The investigation of multiple stochastic integrals has a long history, dated from the celebrated work of Itô [14] for the Gaussian case (a.k.a. Wiener–Itô integrals). See [15, 17, 24, 29] for their extensions to non-Gaussian case, and [26] for series representations. It is worth noting that the exclusion of the diagonals in the multiple stable integral concurs with the exclusion of the diagonals in the multiple sum in (1.4). Our model is actually a simplified version of the one investigated in [2]; yet it preserves the key features and the renewal-process point of view facilitates our analysis.

When \( p = 1 \), the model \( \{X_k\}_{k \in \mathbb{N}_0} \) in (1.4) is a stationary (non-Gaussian) stable process and was first introduced in [23]. It exhibits non-standard asymptotic behaviors in terms of limit theorems for sums and extremes as revealed in [19, 27], and is hence viewed as a model with long-range dependence. For \( p \geq 2 \), a functional central limit theorem has been established recently in [2], under the assumption
\[
\beta_p := p\beta - p - 1 \in (0, 1), \tag{1.5}
\]
with a new family of self-similar multiple-stable processes with stationary increments arising in the limit (see also [1, 3] for variations of the model (1.4) that scale to multiple-Gaussian processes known as the Hermite processes). The functional central limit theorem indicates that under the assumption (1.5), the process exhibits behaviors of long-range dependence.
Tail processes for stable-regenerative model

Our motivation is to understand the limit behavior of extremes for all possible values of $\beta$ and $\rho$. Note that with $\rho = 1$, necessarily $\beta_p = \beta \in (0, 1)$, and hence one only encounters the regime of long-range dependence: an extremal limit theorem in terms of random sup-measures [18] has been established in [27], where the limit random sup-measures exhibit non-trivial dependence structure and in particular their marginal laws may no longer belong to the classical extreme-value distributions when $\beta > 1/2$. It was also shown that when $\beta_p < 0$ the process no longer exhibits long-range dependence, with the limit random sup-measure being independently scattered [4].

Here, we examine the microscopic limit of extremes by investigating the tail processes for a full range of parameters. For a general stationary process with regularly varying tails $\{X_k\}_{k \in \mathbb{N}_0}$, its tail process characterizes the possible local clustering of extremes, and arises in a conditional limit theorem as

$$
\mathcal{L}\left(\frac{X_0}{x}, \ldots, \frac{X_m}{x} \middle| |X_0| > x\right) \to \mathcal{L}(Y_0, \ldots, Y_m), \text{ for all } m \in \mathbb{N},
$$

(1.6)
as $x \to \infty$. The left-hand side is understood as the conditional joint law of $\{X_i/x\}_{i=0,\ldots,m}$ given $|X_0| > x$, and the right-hand side the joint law of $\{Y_0, \ldots, Y_m\}$. If the above holds, then $\{Y_i\}_{i \in \mathbb{N}_0}$ is referred to as the tail process of $\{X_i\}_{i \in \mathbb{N}_0}$. The tail process was originally introduced by Basrak and Segers [5]. Some closely related ideas date back to at least Davis and Hsing [9], Davis and Mikosch [10] (and see references therein for some closely related earlier developments) in the characterization of local clustering of extremes via point-process convergence. See also Kulik and Soulier [16] for the state of the art on the topic.

There are several equivalent characterizations of the tail process. In particular, equivalently to (1.6), one has

$$
\mathcal{L}\left(\frac{X_0}{|X_0|}, \ldots, \frac{X_m}{|X_0|} \middle| |X_0| > x\right) \to (\Theta_0, \ldots, \Theta_m), \text{ for all } m \in \mathbb{N},
$$

and the process $\Theta = \{\Theta_i\}_{i \in \mathbb{N}_0}$ is referred as the spectral tail process. In such a case,

$$
\{Y_i\}_{i \in \mathbb{N}_0} \stackrel{d}{=} \{V_0 \Theta_i\}_{i \in \mathbb{N}_0},
$$

where $V_0$ is an $\alpha$-Pareto random variable ($\mathbb{P}(V_0 > x) = x^{-\alpha}, x \geq 1$) independent from $\{\Theta_i\}_{i \in \mathbb{N}_0}$. Furthermore, $\Theta$, when it exists, can be uniquely extended to a $\mathbb{Z}$-indexed stochastic process.

### 1.2. Main result: a phase transition

We first describe the spectral tail processes that will appear in the limit. Let $\Theta^* = \{\Theta^*_i\}_{i \in \mathbb{N}_0}$ be a $\{0, 1\}$-valued sequence defined as follows: let $\{\tau^{(r)}\}_{r=1,\ldots,p}$ denote i.i.d. copies of a standard (non-delayed) renewal process, with the inter-renewal distribution function $F$ as in (1.1), and consider

$$
\Theta^*_k := \begin{cases} 
1, & \text{if } k \in \eta, \\
0, & \text{otherwise,}
\end{cases} \quad k = 0, 1, \ldots \quad \text{with} \quad \eta := \bigcap_{r=1}^p \tau^{(r)}.
$$

(1.7)

In particular, $\Theta^*_0 = 1$ since $0 \in \tau^{(r)}, r = 1, \ldots, p$ by definition. Moreover, let $\varepsilon$ be a Rademacher random variable independent from $\Theta^*$. We set

$$
\Theta := \varepsilon \Theta^* = \{\varepsilon \Theta^*_i\}_{i \in \mathbb{N}_0}.
$$

(1.8)
We emphasize that the law of the spectral tail process is completely determined by $F$ and $p$. In fact, the intersection $\eta$ is again a non-delayed (i.e., $0 \in \eta$) renewal process. Note that the larger $p$ is, the smaller/sparser the intersection set $\eta$ becomes. In particular, $\eta$ is possibly terminating, namely, $\eta_1 = \infty$ with strictly positive probability; this is the case when the renewal distribution of $\eta$ has a mass at infinity, and we write $\eta = \{0, \eta_1, \ldots, \eta_k\}$ if the $(k + 1)$-th renewal time is the first time with value infinity. The renewal process $\eta$ is terminating if and only if $\beta_p < 0$, where $\beta_p$ is as in (1.5). A quick derivation can be found in Section 1.3 below (see the discussions after (1.10)). The main result of the paper is the following.

**Theorem 1.1.** For all $m \in \mathbb{N}$, 
\[
\mathcal{L} \left( \frac{X_0}{|X_0|}, \ldots, \frac{X_m}{|X_0|} \right| |X_0| > x) \to \mathcal{L}(\varepsilon \Theta_0^*, \ldots, \varepsilon \Theta_m^*),
\]
as $x \to \infty$, with the right-hand side as introduced in (1.8).

**Remark 1.2.** Theorem 1.1 complements our results in [4] so that now we have a complete picture regarding limit extremes at both macroscopic and microscopic levels, as summarized in Table 1. In the table, the extremal index (EI) and the candidate extremal index are two well-known notions in extreme-value theory for stationary sequences. See more background and discussions in Section 1.3 below.

| regime                  | tail process (microscopic) | limit random sup-measure (macroscopic) | candidate EI | EI $\theta$ |
|-------------------------|----------------------------|----------------------------------------|--------------|------------|
| super-critical, $\beta_p > 0$ | non-terminating            | with long-range dependence              | 0            | 0          |
| critical, $\beta_p = 0$  | non-terminating            | independently scattered                 | 0            | 0          |
| sub-critical, $\beta_p < 0$ | terminating                | independently scattered                 | $q_{F,p}$    | $D_{\beta_p \theta_{F,p}}$ |

Table 1. Summary of phase transition. See $q_{F,p}$ and $D_{\beta_p \theta}$ in (1.10) and (1.14) below.

At microscopic level here, Theorem 1.1 reveals that the super-critical ($\beta_p > 0$) and critical ($\beta_p = 0$) regimes have the same type of asymptotic behavior, in the sense that the tail processes are not terminating; while at macroscopic level as described in [4], the critical ($\beta_p = 0$) and sub-critical ($\beta_p < 0$) regimes have the same type of asymptotic behavior, with independently scattered $\alpha$-Fréchet random sup-measures arising in the limit. On the other hand, the limit random sup-measures in the regime $\beta_p > 0$ are of a new type, extending the one characterized in [27]. They again exhibit long-range dependence and, unless $p = 1, \beta < 1/2$, their marginal law goes beyond the family of classical extreme-value distributions due to an aggregation effect (see [30] for an explanation when $p = 1$).

So, the convergence of tail processes reveals a more delicate picture of local behaviors when $\beta_p \leq 0$. At the same time, the tail process at the super-critical regime $\beta_p > 0$ is still of a local nature, as it is impossible to recover from the tail processes the random sup-measures that arise in the macroscopic limit obtained in [4]. It is also remarkable that in the critical regime, while the macroscopic limit is independently scattered, the same as in the sub-critical regime, the normalization is not the same; this is related to the fact that the tail process in the critical regime is again non-terminating, reflecting the infinite size of the local cluster of extremes.
1.3. A notable example: when the candidate extremal index differs from the extremal index

A widely investigated notion for regularly-varying stationary stochastic process is the extremal index. A closely related notion is the candidate extremal index. For both, our reference is again Kulik and Soulier [16]. We first recall the definitions when examining the stable-regenerative model in the sub-critical regime. Throughout this subsection, we assume $\beta_0 < 0$. In this case, the spectral tail process $\Theta$ is terminating. Then the candidate extremal index $\vartheta$ is (writing $a_+ = \max(a, 0)$ for $a \in \mathbb{R}$)

$$\vartheta := \mathbb{E} \left( \sup_{i \geq 0} (\Theta_i)_+ - \sup_{i \geq 1} (\Theta_i)_+ \left| \Theta_0 > 0 \right. \right) = q_{F,p},$$

with

$$q_{F,p} := \mathbb{P}(\eta_1 = \infty) = \left( \sum_{n=0}^{\infty} u(n)p \right)^{-1} = \lim_{n \to \infty} \mathbb{P}_p(n) \in (0, 1),$$

where $u(n) = \mathbb{P}(n \in \tau)$ (see (2.2) below). (See Remark 1.3 on our convention of candidate extremal indices.) To compute $\vartheta = q_{F,p}$, it is key to observe that $(\Theta_0)_+ = \varepsilon_+$ and $(\Theta_i)_+ = \varepsilon_+ \Theta_i^*$ is $\{0, 1\}$-valued for all $i \geq 1$, and that $\varepsilon$ and $\Theta$ are independent, whence $\vartheta = \mathbb{P}(\Theta_i^* = 0, \text{ for all } i \in \mathbb{N} \mid \varepsilon = 1) = \mathbb{P}(\Theta_i^* = 0, \text{ for all } i \in \mathbb{N}) = \mathbb{P}(\eta_1 = \infty)$. At the same time, the sequence $\{\Theta_i^*\}_{i \in \mathbb{N}}$ is all zeros except a finite number, say $\Theta$, of ones, and because of a strong Markov property, $\Theta$ is then geometric with

$$\mathbb{P}(\Theta = k) = \mathbb{P}(\eta_1 = \infty)\mathbb{P}(\eta_1 < \infty)^{k-1}, \quad k \in \mathbb{N},$$

and $\mathbb{E} \Theta = q_{F,p}^{-1}$. At the same time, $\mathbb{E} \Theta = \sum_{n=0}^{\infty} \mathbb{P}(n \in \eta) = \sum_{n=0}^{\infty} u(n) \mathbb{P}(n \in \tau)^p \in (0, \infty)$, whence (1.10) follows.

Next, the extremal index $\theta$, if it exists, can be defined as the unique value from $[0, 1]$ so that the following convergence holds

$$\mathbb{P} \left( \frac{1}{b_n} \max_{k=1, \ldots, n} X_k \leq x \right) = \exp \left( -\theta x^{-\alpha} \right),$$

where $b_n$ is such that (see (3.6) below)

$$\lim_{n \to \infty} n \mathbb{P}(X_1 > b_n) = 1, \quad \text{and in fact} \quad b_n \sim \left( \frac{1}{2} \frac{\log n}{n} \right)^{1/\alpha}$$

as $n \to \infty$. For our model, it has been proved in [4] that

$$\theta = D_{\beta,p} q_{F,p},$$

with

$$D_{\beta,p} := \sum_{s=q_{\beta,p}}^{p} \binom{p}{s} (-1)^{p-s} (-\beta_p)^{p-1} \quad \text{with} \quad q_{\beta,p} := \min \{ q \in \mathbb{N} : \beta q < 0 \}. \quad (1.14)$$

This follows from the convergence of random sup-measure established in [4], formally,

$$\frac{1}{b_n} \left\{ \max_{k \in \mathbb{N}} X_k \right\}_{G \in \mathbb{G}} \stackrel{f.d.d.}{\to} (D_{\beta,p} q_{\beta,p})^{1/\alpha} \left\{ \lambda^{is} \right\}_{G \in \mathbb{G}}, \quad (1.15)$$
where \( G \) is the collection of all open intervals of \([0, 1]\) and \( \mathcal{M}_G^\alpha \) is the independently scattered \( \alpha \)-Fréchet random sup-measure. Indeed, (1.12) is the special case of marginal convergence with \( G = \{0, 1\} \).

**Remark 1.3.** We have used a slightly different convention compared to [16]. In fact, our candidate extremal index \( \vartheta \) is the same as the right-tail candidate extremal index in [16, Eq. (7.5.4b)]. The so-called candidate extremal index in [16] is instead given by (see [16, Eq. (5.6.5)])

\[
\vartheta^0 = \mathbb{E} \left( \sup_{i \geq 0} |\Theta_i|^\alpha - \sup_{i \geq 1} |\Theta_i|^\alpha \right).
\]

The right-tail index (our \( \vartheta \)) concerns only positive extreme values, while the index \( \vartheta^0 \) above concerns extreme values in absolute values. For our model, one readily checks that \( \vartheta \) and \( \vartheta^0 \) happen to coincide.

It is clear from the definitions above that the extremal index \( \theta \) and the candidate extremal index \( \vartheta \) are characteristics of macroscopic and microscopic, respectively, behaviors of extremes. A priori these two values are not necessarily equal, although they can be shown to be equal under an anti-clustering condition and a mixing condition (see Remark 1.4 below). However, we are not aware of any other examples of a regularly varying stochastic process so that \( \theta \) and \( \vartheta \) are both between 0 and 1 and yet not the same. Therefore, our model is of special interest, and we elaborate more the underlying mechanism for this rare phenomenon from the following two aspects.

(i) First, we provide a simplified computation of the extremal index for the case \( p = 2 \) in Section 4. The limit theorem (1.15) above is established in [4] by a different and much more involved proof.

The proof in in Section 4, however, does not apply for \( p \geq 3 \). We hope the presentation here sheds light on the very unusual dependence structure of the model.

(ii) Second, we prove that the so-called anti-clustering condition holds, that is,

\[
\lim_{\ell \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \max_{0 \leq k \leq \ell r_n} |X_k| > b_n \eta \bigg| |X_0| > b_n \eta \right) = 0, \quad \text{for all } \eta > 0, \quad (1.16)
\]

for \( r_n \to \infty, r_n = o(\log b_n) \), in Section 5. (We actually prove a stronger version of it, known as the \( S \) condition in the literature.) This, and combined with the fact that \( \vartheta \neq \theta \), implies immediately that the commonly applied mixing-type condition in the classical approach fails for our model at least for the block size \( r_n = o(\log b_n) \). See Remark 1.4 below for more discussions.

**Remark 1.4.** In the classical approach, in addition to the convergence of the tail processes and the verification of the anti-clustering condition (1.16), if one could also verify for our model the condition

\[
\lim_{n \to \infty} \mathbb{E} \exp \left( - \frac{n}{r_n} f(X_i/b_n) \right) - \left( \mathbb{E} \exp \left( - \frac{r_n}{n} f(X_i/b_n) \right) \right)^{n/r_n} = 0, \quad (1.17)
\]

with \( b_n, r_n \) as in (1.13), then it follows that

\[
\sum_{i=1}^{n} \delta(X_i/b_n, i/n) \mathbbm{1}_{\{X_i > 0\}} \Rightarrow \sum_{i=1}^{\infty} \Theta_i \delta \left( \vartheta^{1/\alpha} r_i^{-1/\alpha}, U_i \right), \quad (1.18)
\]

in the space of \( \mathcal{M}_p((0, \infty] \times [0, 1]) \), where \( \{\Gamma_i\}_{i \in \mathbb{N}} \) are consecutive arrival times of a standard Poisson process, \( \{U_i\}_{i \in \mathbb{N}} \) are i.i.d. uniform random variables over \((0, 1)\), \( \{\Theta_i\}_{i \in \mathbb{N}} \) are i.i.d. copies of \( \Theta \) in
(1.11), and all families are independent (see [16, Corollary 7.3.4]). The convergence (1.18) would imply the extremal index \( \theta = \vartheta \), whence a contradiction. Thus, in the context of our model, the relation (1.17) fails to hold for \( r_n \to \infty \) and \( r_n = o(\log b_n) \).

The idea of the classical approach is as follows. Within each block of size \( r_n \), the local asymptotics is fully characterized by the tail processes, and different blocks behave asymptotically independently due to the condition (1.17). Usually, the condition (1.17) follows from certain strong mixing properties (e.g., \( \beta \)-mixing; see [16, Section 7.4.1]). Our results indicate that our model does not enjoy very strong mixing properties. Nevertheless, we expect to be able to prove (1.18) with \( \theta^{1/\alpha} \) on the right-hand side replaced by \( \theta^{1/\alpha} \). This is left for an upcoming work.

We conclude the introduction with a few more remarks.

**Remark 1.5.** Note that \( \theta = D_{\beta,p} \vartheta \), and here we collect a couple of facts on \( D_{\beta,p} \):

(i) for all \( \beta \in (0,1) \) so that \( \beta_p < 0 \), \( D_{\beta,p} \in (0,1) \),

(ii) for all \( \beta \in (0,1/2) \), \( D_{\beta,p} = 1 - p \beta^{-1} \). In particular, \( \lim_{\beta \downarrow 0} D_{\beta,p} = 1 \).

To see the first, introduce

\[
    f_p(x) = \frac{1}{(p-1)!} \sum_{s=0}^{p} \binom{p}{s} (-1)^{p-s}(s-x)^{p-1} = \frac{1}{(p-1)!} \sum_{s=0}^{p} (-1)^{p-s} \binom{p}{s} (x-s)^{p-1},
\]

which is the probability density function of the so-called Irwin–Hall distribution, the one for the sum of \( n \) i.i.d. uniform random variables over \((0,1)\). Then we can write

\[
    D_{\beta,p} = (p-1)!/(1-\beta)^{p-1} f_p((1-\beta)^{-1}) > 0, \quad \text{for all } \beta \in (0,1-1/p),
\]

or exactly when \( \beta_p < 0 \). To show that \( D_{\beta,p} < 1 \), it is equivalent to show that \( f_p(x) < ((p-1)!)^{-1}x^{p-1} \) for all \( x > 1 \). But, recall that \( f_p(x) \) is the \( p \)-times convolution function of uniform density function over \((0,1)\) evaluated at \( x \). While \( \frac{1}{(p-1)!} x^{p-1} \) is the \( p \)-times convolution of the indicator function over \([0,\infty)\) evaluated at \( x \). The desired relation now follows.

To see the second, first recall that for any polynomial function \( Q(s) \) with degree at most \( p-1 \), one has \( \sum_{s=0}^{p} (-1)^{p-s} \binom{p}{s} Q(s) = 0 \) since this corresponds to a \( p \)-times differencing operation. Then, take \( Q(s) = (-s)^{p-1} = (s(1-\beta) - 1)^{p-1} \) here. For \( \beta \in (0,1/2) \), \( q_{\beta,p} = 2 \), and it then follows that

\[
    D_{\beta,p} = \sum_{s=2}^{p} (-1)^{p-s} \binom{p}{s} (-\beta s)^{p-1} = -(1)^{p} Q(0) - (1)^{p-1} p Q(1) = 1 - p \beta^{p-1}.
\]

The fact \( D_{\beta,p} \in (0,1) \), which implies \( \theta < \vartheta \), is consistent with Kulik and Soulier [16, Lemma 7.5.4]. We note that although the lemma there is stated only for \( \mathbb{R}_{+} \)-valued processes, the proof readily extends to real-valued processes with the right-tail (cf. Remark 1.3) candidate extremal index \( \vartheta \) in (1.9).

**Remark 1.6.** In the case \( \beta_p = 0 \), we have in the critical regime, in place of (1.15),

\[
    P \left( \frac{1}{b_n} \max_{k=1,...,n} X_k \leq x \right) = \exp \left( -\tilde{\theta} x^{-\alpha} \right) \quad \text{with} \quad \tilde{b}_n = \left( \frac{n(\log \log n)^{p-1}}{\log n} \right)^{1/\alpha},
\]

and the value \( \tilde{\theta} > 0 \) explicitly computed in [4]. Since \( b_n/\tilde{b}_n \to \infty \) as \( n \to \infty \) with \( b_n \) as in (1.13), the convergence above implies that the extremal index \( \theta \) defined via (1.12) is zero. One may argue that in the super-critical regime, the extremal index is again \( \theta = 0 \) in the same way.
Remark 1.7. Our example shows that the candidate extremal index should be viewed as a local statistic only. In this regard, we recall that despite the fact that it is commonly interpreted as the reciprocal of the mean cluster size, it is not always the case. In particular, Smith [28] provided an example where the extremal index is less than 1, while there is no extremal clustering from a point-process-convergence perspective. As further elaborated in [16, Example 14.4.5], in this example the candidate extremal index and extremal index are the same.

The paper is organized as follows. Section 2 provides related background, notably on multiple-stable processes and renewal processes. Section 3 proves Theorem 1.1. Section 5 proves the anti-clustering condition and the convergence of cluster point process when \( \beta_p < 0 \). Section 4 provides a computation of the extremal index with \( p = 2, \beta \in (0, 1/2) \).

2. Preliminary results

2.1. Renewal processes with infinite mean

Throughout, our references on renewal processes are Giacomin [13, Appendix A.5] and Bingham, Goldie and Teugels [8]. Besides the notation and properties of renewal processes in Section 1, we shall also use the renewal mass function of a non-delayed renewal process \( \tau \) as follows

\[
  u(k) := \mathbb{P}(k \in \tau), \quad k \in \mathbb{N}_0.
\]

It is known that the assumption

\[
  u(k) \sim \frac{k^{\beta_p - 1}}{\mathcal{C}_F \Gamma(\beta) \Gamma(1 - \beta)}
\]

as \( k \to \infty \) implies the assumption (1.1), and that under the assumption (1.2) the two are equivalent (see [12]).

Note that \( \tau \) is null-recurrent, and the stationary delay measure associated with the inter-renewal distribution \( F \) can be taken as \( \pi(k) = C\mathbb{F}(k) \), \( k \in \mathbb{N}_0 \) (more generally one may set \( \pi(k) = C\mathbb{F}(k) \), and we choose the constant \( C = 1 \) for simplicity). The meaning of stationary delay measure is explained in the Section 2.3, where for the sake of completeness, we present a proof for a system of two-sided renewal processes to be shift-invariant.

Next, we recall some properties of the intersected renewal processes \( \eta = \bigcap_{r=1}^p \tau^{(r)} = \{0, \eta_1, \eta_2, \ldots \} \) as introduced in (1.7). We let \( F_p \) denote the cumulative distribution function of \( \eta_1 \) and we shall need the asymptotics of \( F_p(x) = 1 - F_p(x) \). In summary, we have the following:

\[
  F_p(n) \sim \begin{cases}
    n^{-\beta_p} \frac{(\mathcal{C}_F \Gamma(\beta) \Gamma(1 - \beta))^p}{\Gamma(\beta_p) \Gamma(1 - \beta_p)}, & \text{if } \beta_p > 0; \\
    (\mathcal{C}_F \Gamma(\beta) \Gamma(1 - \beta))^p \frac{\log n}{n}, & \text{if } \beta_p = 0.
  \end{cases}
\]

When \( \beta_p < 0 \), the renewal process is terminating, with the probability \( q_{F,p} = \mathbb{P}(\eta_1 = \infty) \) as given in (1.10). For more details, see [8, 13] and [4].
2.2. An equivalent series representation

Here, we explain a different representation that shall be needed for our proofs, for finite-dimensional distributions of the process. We are interested in the joint law of \((X_0, \ldots, X_m)\), for some \(m\) fixed. Then, for all those \(i \in \mathbb{N}_0\) such that \(\tau^{(i,d_i)} \cap \{0, \ldots, m\} = \emptyset\), they do not contribute in the series representation. Therefore, by a standard thinning argument of Poisson point processes, it follows that

\[
\sum_{i=1}^{\infty} \delta_{\left(\varepsilon_i \Gamma_i^{-1/\alpha}, \tau^{(i,d_i)} \cap \{0, \ldots, m\}\right)} \mathbf{1}_{\{\tau^{(i,d_i)} \cap \{0, \ldots, m\} \neq \emptyset\}} \overset{d}{=} \sum_{i=1}^{\infty} \delta_{\left(w_i^{1/\alpha} \varepsilon_i \Gamma_i^{-1/\alpha}, R_{m,i}\right)},
\]

(2.3)

where on the right-hand side, \(w_m = \sum_{k=0}^{m} \pi(k) = \sum_{k=0}^{m} F(k) \sim \frac{C_F}{1-\beta} \cdot m^{1-\beta}\), the random variables \(\{\Gamma_i\}_{i \in \mathbb{N}}\) are consecutive arrival times of a standard Poisson process, \(\{\varepsilon_i\}_{i \in \mathbb{N}}\) are i.i.d. Rademacher random variables, \(\{R_{m,i}\}_{i \in \mathbb{N}}\) are i.i.d. random closed subsets of \([0, \ldots, m]\) with the law \(R_{m,i} \overset{d}{=} R_m\) described below, and all families are independent. Suppose \(\tau^*\) is a delayed renewal process with the stationary delay measure \(\pi\) and renewal distribution \(F\) defined on a measurable space with respect to an infinite measure \(\mu^*\) (since \(\pi\) is an infinite measure). Then, one can introduce a probability measure \(\mu_m\) on the same measurable space via

\[
\frac{d\mu_m}{d\mu^*} = \frac{\mathbf{1}_{\{\tau^* \cap \{0, \ldots, m\} \neq \emptyset\}}}{\mu^*(\{\tau^* : \tau^* \cap \{0, \ldots, m\} \neq \emptyset\})} = \frac{\mathbf{1}_{\{\tau^* \cap \{0, \ldots, m\} \neq \emptyset\}}}{w_m}.
\]

Then, the law of \(R_m\) is the one induced by \(\tau^*\) with respect to the probability measure \(\mu_m\). Moreover, it is immediately verified that

(i) \(\mathbb{P}(k \in R_m) = 1/w_m, k = 0, \ldots, m\) (shift invariance).
(ii) \(\mathbb{P}(\min(R_m \cap \{k + 1, k + 2, \ldots\}) \leq k + j \mid k \in R_m) = F(j)\) (Markov/renewal property).

By (2.3), we work with the following equivalent representation of (1.4):

\[
\{X_k\}_{k=0, \ldots, m} \overset{d}{=} \left\{X_k^{p/\alpha} w_m^{p/\alpha} \sum_{0<i_1<\cdots<i_p} \frac{[\varepsilon_i]}{\Gamma_i^{1/\alpha}} \mathbf{1}_{\{k \in \bigcap_{r=1}^{p} R_{m,i_r}\}} \right\}_{k=0, \ldots, m},
\]

(2.4)

where on the right-hand side the notation is as in (2.3).

Remark 2.1. Note that the representation in (2.4) is slightly different from the one used in [4]: we include the time zero here, which is more convenient when studying the tail processes. Such a change does not effect the normalization or the limit object.

2.3. A two-sided representation

We provide another series representation of \(\{X_k\}_{k \in \mathbb{Z}}\) based on two-sided renewal processes. We do not need this in the rest of the paper. Such a representation is of its own interest, and helps illuminate the notion of delayed stationary distribution which is now defined in a two-sided manner. The results in this section may have been known in the literature but we were unable to find it.
Recall (1.3) and (1.4). In place of \( \tau^{(i,d_i)} \) now we introduce \( \tau^{(i,d_i,g_i)} \); this is a two-sided renewal processes to be introduced with the first renewal time to the right of the origin (included) is \( d_i \), the first renewal time to the left of the origin (not included) is \(-g_i \) (so \( g_i \in \mathbb{N} \)).

We first introduce

\[
\sum_{i=1}^{\infty} \delta_{(y_i,d_i,g_i)} \overset{d}{=} \text{PPP}((0,\infty) \times \mathbb{N}_0 \times \mathbb{N}, \alpha x^{-\alpha-1} \, dx d\pi),
\]

with the measure \( \pi \) on \( \mathbb{N}_0 \times \mathbb{N} \) determined by the following mass function

\[
\pi(d,g) := \pi(d) \cdot \frac{f(d+g)}{F(d)} = f(d+g), \quad d \in \mathbb{N}_0, \, g \in \mathbb{N}.
\]

Note that the factor \( f(d+g)/F(d) \) is the probability mass function at \( d+g, \, g \in \mathbb{N} \), of the conditional law of a renewal time with respect to \( F \), given that the renewal time is strictly larger than \( d \).

Now we attach independent renewal processes to each pair of \( (d_i,g_i) \). Let each \( \tau^{(i,\cdot)} \) be a copy of \( \tau = \{ \tau_0, \tau_1, \ldots \} \), a renewal process starting from the origin (so \( \tau_0 = 0 \)) with renewal distribution \( F \). \( \tau^{(i,\cdot)} \overset{d}{=} \{0, \tau_1, \tau_2, \ldots \} \). Similarly, let each \( \tau^{(i,\cdot)} \) be a time-reversed renewal process, starting from zero: so with \( \tau \) as before, \( \tau^{(i,\cdot)} \overset{d}{=} \{0, -\tau_1, -\tau_2, \ldots \} \). All the renewal processes are assumed independent from everything else. Then, we set

\[
\tau^{(i,d_i,g_i)} := (d_i + \tau^{(i,\cdot)}) \cup (-g_i + \tau^{(i,\cdot)}),
\]

where \( d_i + \tau^{(i,\cdot)}, -g_i + \tau^{(i,\cdot)} \) are understood as subsets of \( \mathbb{Z} \). Each two-sided renewal \( \tau^{(i,d_i,g_i)} \) takes value in the path space

\[
S = \{ t = \{ t_i \}_{i \in \mathbb{N}} : t_i \in \mathbb{Z}, \text{ all distinct} \}
\]

equipped with the cylindrical \( \sigma \)-field, i.e., the \( \sigma \)-field generated by \( \{ t \in S : k \in t \} \), \( k \in \mathbb{Z} \).

The two-sided series representation is then given by

\[
\{X_k\}_{k \in \mathbb{Z}} \overset{d}{=} \left\{ \sum_{i \in D_p} \varepsilon_i [y_i] 1_{\{ k \in \bigcap_{r=1}^{p} \tau^{(r,d_r,g_r)} \}} \right\}_{k \in \mathbb{Z}}
\]

(2.6)

with \( \{ \varepsilon_i \} \) as in (1.4) which is independent of everything else, and

\[
D_p = \{(i_1, \ldots , i_p) \in \mathbb{N}^p : i_1 < \ldots < i_p \}.
\]

Lemma 2.2. The two-sided representation (2.6) represents a stationary process \( \{X_k\}_{k \in \mathbb{Z}} \) which restricted to \( k \in \mathbb{N}_0 \) has the representation (1.4).

Proof. It is known that the multiple series in (2.6) converges almost surely and unconditionally (e.g., [26, Theorem 1.3 and Remark 1.5]). By construction, (2.6) restricted to \( k \in \mathbb{N}_0 \) is the same as (1.4). It remains to prove stationarity of the process \( \{X_k\}_{k \in \mathbb{Z}} \). This shall follow from a shift-invariance property of the intensity measure of the Poisson point process

\[
\sum_{i=1}^{\infty} \delta_{(y_i, \tau^{(i,d_i,g_i)})}.
\]
(We omit the discussions regarding \( \{ \varepsilon_i \}_{i \in \mathbb{N}} \) for the sake of simplicity.) Introduce the following shift operation \( B \) on \( t \in S: B(t) = t + 1 \). For \( t = \{ t_i \}_{i \in \mathbb{N}} \in S, B(t) = t + 1 = \{ t_i + 1 \}_{i \in \mathbb{Z}} \) (so that \( d + \tau^{(t, \tau)} \in B^k(\tau^{(t, \tau)}) \)). Let \( P_\tau \) denote the probability measure on \( S \) induced by the distribution of the renewal process \( \tau \) as above. Then, the Poisson point process above is on the space \( (0, \infty) \times S \), with its intensity measure denoted by \( \alpha x^{-\alpha-1} dx dQ \), and \( Q \) is defined as the push-forward measure with respect to the construction \( (2.5) \) from the product measure \( \tilde{P} \times P_\tau \times P_\tau \) on \( \mathbb{N}_0 \times \mathbb{N} \times S \).

Now, the stationarity of the process \( \{ X_k \}_{k \in \mathbb{Z}} \) follows from the following shift-invariance property for the measure \( Q \) on \( S \):

\[
Q \circ B^{-1} = Q. \tag{2.7}
\]

To prove the above, we start by writing for every \( t \in S \) and \( k \in \mathbb{Z} \) that

\[
d_k(t) := \min \{ t_i : t_i \geq k \} - k, \quad \text{and} \quad g_k(t) := k - \max \{ t_i : t_i < k \}.
\]

In words, \( d_k(t) \) represents the distance between \( k \) and the first element to the right of \( k \) (included) of \( t \), and \( g_k(t) \) the distance between \( k \) and the first element to the left of \( k \) (not included). Our construction tells that

\[
Q(d_0(t) = d, g_0(t) = g) = \bar{\pi}(d, g) = f(d + g) \text{ for all } d \in \mathbb{N}_0, g \in \mathbb{N}. \tag{2.8}
\]

To show \( (2.7) \), based on a decomposition with respect to \( d_0(t) \) and \( g_0(t) \) and Dynkin’s theorem, it suffices to show

\[
Q \left( s \subset B^{-1}t, \ d_0(B^{-1}t) = d, \ g_0(B^{-1}t) = g \right) = Q \left( s \subset t, \ d_0(t) = d, \ g_0(t) = g \right) \tag{2.9}
\]

for all \( d \in \mathbb{N}_0, g \in \mathbb{N} \) and \( s \in S \) of finite size such that \( d_0(s) = d \) and \( g_0(s) = g \). Write \( s = \{ s_{-m}, \ldots, s_{-1}, s_0, \ldots, s_n \} \), such that \( s_i < s_{i+1} \) and \( s_{-1} = -g \) and \( s_0 = d, m \in \mathbb{N}, n \in \mathbb{N}_0 \). Then the left-hand side of \( (2.9) \) is equal to

\[
Q(s_{i+1} \in t, \ i = -m, \ldots, n, \ d_1(t) = d, \ g_1(t) = g). \tag{2.10}
\]

Now, if \( d \geq 0 \) and \( g \geq 2 \), then \( \{ t \in S : d_1(t) = d, \ g_1(t) = g \} = \{ t \in S : d_0(t) = d + 1, \ g_0(t) = g - 1 \} = \{ t \in S : d_0(t) = s_0 + 1, \ g_0(t) = s_{-1} + 1 \} \). Hence the expression in \( (2.10) \) in this case becomes

\[
f(d + g) \prod_{-m \leq i \leq n-1, i \neq -1} u(s_{i+1} - s_i), \tag{2.11}
\]

where we have used the relation \( (2.8) \) as well as the renewal property on the two sides with the renewal mass function \( u \) as in \( (2.1) \). Next suppose \( d \geq 0 \) and \( g = 1 \). Set \( d'_0(t) = \min(t \cap \{ d_0(t) + 1, d_0(t) + 2, \ldots \}) \), namely, the second element of \( t \) to the right of the origin (included). We have \( \{ t \in S : d_1(t) = d, \ g_1(t) = 1 \} = \{ t \in S : d_0(t) = 0, \ d'_0(t) = d + 1 \} = \{ t \in S : d_0(t) = s_{-1} + 1, \ d'_0(t) = s_0 + 1 \} \). Then by construction and the renewal property, the expression in \( (2.10) \) in this case becomes

\[
Q(d_0(t) = 0) \mathbb{P} \left( \tau_1 = d + 1 \right) \prod_{-m \leq i \leq n-1, i \neq -1} u(s_{i+1} - s_i).
\]

Note that \( Q(d_0(t) = 0) = \pi(0) = \mathbb{F}(0) = 1 \) and \( \mathbb{P}(\tau_1 = d + 1) = f(d + 1) \), and hence the formula above coincides with \( (2.11) \) when \( d \geq 0 \) and \( g = 1 \). Therefore, we have shown that the left-hand side of \( (2.9) \) is equal to the expression in \( (2.11) \) for all \( d \in \mathbb{N}_0 \) and \( g \in \mathbb{N} \). The proof is concluded once noticing that the right-hand side of \( (2.9) \) is readily \( (2.11) \) for all \( d \in \mathbb{N}_0 \) and \( g \in \mathbb{N} \). \( \square \)
3. Convergence for tail processes

We prove Theorem 1.1. Below $C$ will denote a generic positive constant whose value may change from one expression to another.

In order to establish $(m+1)$-dimensional multivariate regular variation, we shall work with the representation (2.4) with $m$ throughout fixed. We introduce some notation. Set

$$
\bar{\ell}(1) := \min \{ i \in \mathbb{N} : k \in R_{m,i} \},
$$

$$
\bar{\ell}(s) := \min \{ i > \bar{\ell}(s-1) : k \in R_{m,i} \},
$$

namely, $\bar{\ell}(1), \bar{\ell}(2), \ldots$ are the successive $i$-indices such that $k \in R_{m,i}$. For $i = (i_1, \ldots, i_p) \in D_p$, we write $\bar{\ell}(i) = (\bar{\ell}(i_1), \ldots, \bar{\ell}(i_p))$. In this way, we write

$$
X_k = u_m^{p/\alpha} \sum_{i \in D_p} \frac{[\varepsilon_{\bar{\ell}(i)}]}{[\Gamma_{\bar{\ell}(i)}]^{1/\alpha}} \quad \text{and} \quad T_k := u_m^{p/\alpha} \frac{[\varepsilon_{\bar{\ell}(1,\ldots,p)}]}{[\Gamma_{\bar{\ell}(1,\ldots,p)}]^{1/\alpha}},
$$

$k = 0, \ldots, m$. (3.2)

Here and below the notational convention is to write the product $\varepsilon_{\bar{\ell}(i_1)} \cdots \varepsilon_{\bar{\ell}(i_p)}$ as $[\varepsilon_{\bar{\ell}(i)}]$, and similarly for $[\Gamma_{\bar{\ell}(i)}]$. Note that the indicator functions are dropped in the representation (3.2). Moreover, in order to study the marginal distribution, taking into account of the thinning probability $\mathbb{P}(k \in R_{m,i}) = u_m^{-1}$, we shall work with the following representation for each $k \in \mathbb{N}_0$ (but not jointly in $k$):

$$
(X_k, T_k) \overset{d}{=} \left( \sum_{i \in D_p} \frac{[\varepsilon_i]}{[\Gamma_i]^{1/\alpha}}, \frac{[\varepsilon_{1,\ldots,p}]}{[\Gamma_{1,\ldots,p}]^{1/\alpha}} \right).
$$

(3.3)

Recall that from [26], we have

$$
q_p(x) := \mathbb{P} \left( \left( \Gamma_{1:p} \right)^{-1} > x \right) \sim \frac{x^{-1} \log^{p-1} x}{p!(p-1)!}.
$$

(3.4)

as $x \to \infty$. Here and below we write $\Gamma_{1:p} = \Gamma_{1,\ldots,p} = \Gamma_1 \times \cdots \times \Gamma_p$. Moreover we have the following.

**Lemma 3.1.** We have $\mathbb{P}(\max_{1,\ldots,p} \notin D_p \Gamma_i^{-1/\alpha} > x) = \mathbb{P}(\Gamma_1 \cdots \Gamma_{p-1} \Gamma_{p+1}^{-1/\alpha} > x)$ and

$$
\limsup_{x \to \infty} \frac{\mathbb{P}(\Gamma_1 \cdots \Gamma_{p-1} \Gamma_{p+1}^{-1/\alpha} > x)}{q_{p-1}(x^{\alpha})} \leq \nu^{-1} \Gamma_2 < \infty.
$$

(3.5)

Moreover, as $x \to \infty$,

$$
\mathbb{P}(|X_k| > x) \sim 2\mathbb{P}(X_k > x) \sim \mathbb{P}(|T_k| > x) = q_p(x^{\alpha}) \sim \frac{\alpha^{p-1} x^{-\alpha} \log^{p-1}(x)}{p!(p-1)!},
$$

(3.6)

and

$$
\mathbb{P}(|X_k - T_k| > x) \leq Cq_{p-1}(x^{\alpha}) = O(x^{-\alpha} \log^{p-2}(x)).
$$

(3.7)
In words, $T_k$ is the product of $\Gamma_i$ for the smallest $p$ $i$-indices such that $k \in R_{m,i}$, and the tails of $X_k$ are determined by this single term alone. Then, by Lemma 3.1, the joint representation (3.2) allows us to replace the $X_k$'s by $T_k$'s and deal with the joint law of $T_0, \ldots, T_m$. The above can be read from [26], which deals with a more general setup. For the sake of convenience we include a proof here.

**Proof of Lemma 3.1.** The first claim (3.5) follows from monotonicity. Let $\Gamma_2 = \Gamma_2$ be a random variable independent from everything else. We then have

$$(\Gamma_1 \cdots \Gamma_{p-1} \Gamma_{p+1})^{-1/\alpha} \overset{d}{=} \left[ \Gamma_{1:p-1}^{-1/\alpha} \Gamma_{p+1} \right]^{-1/\alpha} \leq \left[ \Gamma_{1:p-1}^{-1/\alpha} \Gamma_2 \right]^{-1/\alpha}.$$  

The right-hand side is now a product of two independent random variables. The second part now follows from Breiman’s lemma (e.g. [22]).

Notice that in view of (3.4), the relation (3.7) implies (3.6), and hence it is left to prove (3.7). Let $M$ be a fixed integer such that $M \geq 2p/\alpha + p$. Note that the conclusion concerns only the marginal distribution, and hence we can work with the simplified representation of $X_k$ in (3.3). By (3.5), any finite sum of terms of the form $[\epsilon_i] [\Gamma_i]^{-1/\alpha}$ with $i \neq (1, \ldots, p)$ will have lighter tails than $[\Gamma_{1, \ldots, p}]^{-1/\alpha}$. Since the sets $\{ i \in D_q : i_q \leq M \}, q = 0, \ldots, p - 1$ are finite (when $q = 0$ the set is understood as an empty set), it suffices to show the following: for any fixed $i_1 < \cdots < i_q \leq M$ and $q = 0, 1, \ldots, p - 1$,

$$\mathbb{P} \left( \left[ \Gamma_{i_1, \ldots, q} \right]^{-1/\alpha} \sum_{M < i_{q+1} < \ldots < i_p} \frac{[\epsilon_{i_{q+1}}]}{[\Gamma_i]^{-1/\alpha}} > x \right) = o(x^{-\alpha \log^{p-1} x}), \quad x \to \infty,$$  

(3.8)

here and below we write $i_{i_1} = (i_1, \ldots, i_q), [\Gamma_{i_1, \ldots, q}] = \Gamma_{i_1} \times \cdots \times \Gamma_{i_q}$, and similarly for other terms. If $q = 0$, in view of the inequality $\mathbb{E} \left( [\Gamma_{1:p}]^{-2/\alpha} \right) \leq c(i_1 \ldots i_p)^{-2/\alpha}$ which holds by the choice of $M$ (cf. [26, Eq.(3.2)]), as well as the orthogonality induced by $[\epsilon_{i_1}]$, one can verify that the second moment of the multiple series in (3.8) is finite, and hence the tail decay is $O(x^{-2}) = o(x^{-\alpha \log^{p-1} x})$ as $x \to \infty$. Suppose $q > 0$ below. Then the probability above is bounded by

$$\mathbb{P} \left( \left[ \Gamma_{1, \ldots, q} \right]^{-1/\alpha} \sum_{M < i_{q+1} < \ldots < i_p} \frac{[\epsilon_{i_{q+1}}]}{[\Gamma_i]^{-1/\alpha}} > x \right).$$  

(3.9)

Using $[\Gamma_{i_{q+1}}]^{-1/\alpha} \leq \prod_{r=q+1}^p (\Gamma_i - \Gamma_q)^{-1/\alpha}$, conditioning on $\{ \Gamma_i \} i \in \mathbb{N}$ and applying the contraction principle for multilinear form of Rademacher random variables [11, Theorem 3.6], the probability in (3.9) is bounded, up to a multiplicative constant, by

$$\mathbb{P} \left( \left[ \Gamma_{1, \ldots, q} \right]^{-1/\alpha} \sum_{M < i_{q+1} < \ldots < i_p} \frac{[\epsilon_{i_{q+1}}]}{[\Gamma_i]^{-1/\alpha}} \right) = \mathbb{P} \left( \left[ \Gamma_{1, \ldots, q} \right]^{-1/\alpha} \sum_{M < i_{q+1} < \ldots < i_p} \frac{[\epsilon_{i_{q+1}}]}{[\Gamma_i]^{-1/\alpha}} \right) > x \right).$$

(3.10)

Note that $[\Gamma_{1, \ldots, q}]^{-1/\alpha}$ is independent of the absolute value part above, which can be shown to have a finite second moment similarly as before using the fact $(\Gamma_i - \Gamma_q)_{i \geq M} \overset{d}{=} (\Gamma_i)_{i > M - q}$ with the choice of $M$. Therefore, by Breiman’s lemma again, the probability above is of the same order as $\mathbb{P}(\left[ \Gamma_{1, \ldots, q} \right]^{-1/\alpha} > x) \sim q(x^\alpha) = o(x^{-\alpha \log^{p-1} x})$ as $x \to \infty$ in view of (3.4).
Proof of Theorem 1.1. Write

\[ T_m^* := w_m^{p/\alpha} \frac{[\varepsilon(1,...,p)]}{[T(1,...,\rho)]^{1/\alpha}} \quad \text{and} \quad H_k := 1_{\{\bar{\ell}_k(1,...,p) = (1,...,p)\}}, \quad k = 0, \ldots, m. \]  

(3.10)

Note that \( T_k H_k = T_m^* H_k \). The proof consists of establishing the following asymptotic equivalence in law:

\[ \mathcal{L}\left( \frac{X_0}{|X_0|}, \ldots, \frac{X_m}{|X_0|} \middle| |X_0| > x \right) \sim \mathcal{L}\left( \frac{X_0}{|X_0|}, \ldots, \frac{X_m}{|X_0|} \middle| |T_m^*| > x, H_0 = 1 \right) \]  

(3.11)

\[ \sim \mathcal{L}\left( \frac{X_0}{|T_m^*|}, \ldots, \frac{X_m}{|T_m^*|} \middle| |T_m^*| > x, H_0 = 1 \right) \]  

(3.12)

\[ \sim \mathcal{L}\left( \frac{T_m^* H_0}{|T_m^*|}, \ldots, \frac{T_m^* H_m}{|T_m^*|} \middle| |T_m^*| > x, H_0 = 1 \right) \]  

(3.13)

\[ \sim \mathcal{L}(\varepsilon H_0, \ldots, \varepsilon H_m \mid H_0 = 1). \]  

(3.14)

Here and below, \( \mathcal{L}(\cdots \mid \cdots) \) denote the corresponding conditional law, and \( \mathcal{L}(\cdots) \sim \mathcal{L}(\cdots) \) means that as \( x \to \infty \), the two sides have same limit law, given that the limit law of one of them exists. We will leave the verification of first step (3.11) to the end. We first verify steps (3.12) and (3.13). Write, for each \( k \),

\[ X_k = T_k + X_k - T_k = T_m^* H_k + T_k(1 - H_k) + X_k - T_k. \]

Observe that \( |T_k|(1 - H_k) \leq w_m^{p/\alpha} \max_{1 \leq i \leq \rho} |T_i|^{-1/\alpha} \). In addition, in view of Item (i) before (2.4) and independence, we have \( \mathbb{P}(H_k = 1) = w_m^{-p} \). It then follows from Lemma 3.1 and the independence between \( T_m^* \) and \( H_k \) that

\[ \mathbb{P}(|X_k| > x) \sim \mathbb{P}(|T_m^* H_k| > x) \sim \frac{w_m^{p} \alpha^{-\alpha} \log^{p-1} x}{p!(p-1)!} \quad \text{as} \quad x \to \infty, \]  

(3.15)

and

\[ \mathbb{P}(|X_k - T_m^* H_k| > x) = O(x^{-\alpha} \log^{p-2} x) \]  

(3.16)

as \( x \to \infty \). Next, we need the following facts: for random variables \( Y \) and \( Z \), not necessarily independent, such that \( \mathbb{P}(|Z| > \epsilon x) = o(\mathbb{P}(|Y| > x)) \) as \( x \to \infty \) for any \( \epsilon > 0 \), we have as \( x \to \infty \) the convergences in distribution:

\[ \mathcal{L}\left( \frac{|Y|}{|Y + Z|} \middle| |Y| > x \right) \to \mathcal{L}(1) \quad \text{and} \quad \mathcal{L}\left( \frac{Z}{|Y|} \middle| |Y| > x \right) \to \mathcal{L}(0), \]  

(3.17)

where \( \mathcal{L}(1) \) and \( \mathcal{L}(0) \) denote the laws of constants 1 and 0, respectively. Indeed, the first relation follows from the second one, whereas the second holds since

\[ \mathbb{P}(|Z/Y| > \epsilon \mid |Y| > x) \leq \mathbb{P}(|Z| > \epsilon x)/\mathbb{P}(|Y| > x) \to 0 \]

for any \( \epsilon > 0 \). Now (3.12) follows by writing for each \( k \),

\[ \frac{X_k}{|X_0|} = \frac{X_k}{|T_m^*|} \left( \frac{|T_m^*|}{|T_m^* + X_0 - T_m^*|} \right) \]
and then applying the first relation in (3.17) with \( Y = T^*_m, \ Z = X_0 - T^*_m \), as well as the relations (3.15) and (3.16). The step (3.13) follows by writing for each \( k \),

\[
\frac{X_k}{|T^*_m|} = \frac{T^*_m H_k}{|T^*_m|} + \frac{X_k - T^*_m H_k}{|T^*_m|},
\]

and applying the second relation in (3.17).

The last step (3.14) follows from \([\varepsilon(1,\ldots,p)] \overset{d}{=} \varepsilon \) and independence between \((H_k)_{k=0,\ldots,m} \) and \( T^*_m \).

Comparing the definitions (3.10) (note that \( \ell_k((1,\ldots,p)) = (1,\ldots,p) \) means \( k \in R_{m,i}, i = 1,\ldots,p \) and (1.8), one readily checks that

\[
\mathcal{L}(H_0,\ldots,H_m | H_0 = 1) = \mathcal{L}(\Theta^*_0,\ldots,\Theta^*_m).
\]

Now we return to verify the first step (3.11). Introduce \( B_0(x) := \{|X_0| > x\}, B^*_m(x) := \{|T^*_m| > x\}, \) and

\[
E = \left\{ \left( \frac{X_0}{|X_0|}, \ldots, \frac{X_m}{|X_0|} \right) \in A \right\},
\]

where \( A \) is a Borel set in \( \mathbb{R}^{m+1} \) whose boundary is not charged by the limit law (3.14). It suffices to prove

\[
\lim_{x \to \infty} \mathbb{P}(E \mid |X_0| > x) = \lim_{x \to \infty} \mathbb{P}(E \mid |T^*_m| > x, H_0 = 1). \tag{3.18}
\]

It is clear that, from the established steps (3.12) to (3.14), the (limit of) right-hand side of (3.18) exists. For the left-hand side, write

\[
\mathbb{P}(E \mid |X_0| > x) = \frac{\mathbb{P}(E \cap \{ |X_0| > x, H_0 = 1 \}) + \mathbb{P}(E \cap \{ |X_0| > x, H_0 = 0 \})}{\mathbb{P}(|X_0| > x)}.
\]

In view of (3.6) and (3.7), we have

\[
\mathbb{P}(E \cap \{ |X_0| > x, H_0 = 0 \}) \leq \mathbb{P}(|X_0| > x, H_0 = 0) = o(\mathbb{P}(|X_0| > x))
\]

as \( x \to \infty \). Therefore, with in addition (3.15), we see that \( \mathbb{P}(E \mid |X_0| > x) \) has the same limit as

\[
\frac{\mathbb{P}(E \cap \{ |X_0| > x, H_0 = 1 \})}{\mathbb{P}(|T^*_m| > x, H_0 = 1)} \tag{3.19}
\]

as \( x \to \infty \). For the numerator, we have the following upper and lower bounds:

\[
\mathbb{P}(E \cap \{ |X_0| > x, H_0 = 1 \}) \geq \mathbb{P}(E \cap \{ |T^*_m| > (1 + \varepsilon)x, H_0 = 1 \}) - \mathbb{P}(|X_0 | - T_0 | > \varepsilon x), \\
\mathbb{P}(E \cap \{ |X_0| > x, H_0 = 1 \}) \leq \mathbb{P}(E \cap \{ |T^*_m| > (1 - \varepsilon)x, H_0 = 1 \}) + \mathbb{P}(|X_0 | - T_0 | > \varepsilon x).
\]

The limit of (3.19) with the numerator replaced by the upper and lower bounds above, as \( x \to \infty \), can be determined by applying (3.7), (3.15) and the relations (3.12) to (3.14) (with \( x \) replaced by \( (1 \pm \varepsilon)x \)). Letting \( \varepsilon \downarrow 0 \), we conclude that (3.19) has the same limit as

\[
\frac{\mathbb{P}(E \cap \{ |T^*_m| > x, H_0 = 1 \})}{\mathbb{P}(|T^*_m| > x) \mathbb{P}(H_0 = 1)} = \mathbb{P}(E \mid |T^*_m| > x, H_0 = 1).
\]

This completes the proof of (3.18). \( \Box \)
4. Extremal index in the sub-critical regime, \( p = 2 \)

In this section, we restrict to the sub-critical regime \( p = 2, \beta \in (0, 1/2) \), and compute the extremal index directly. Recall the series representation of \( \{X_k\}_{k=0,\ldots,n} \) in (2.4), and for convenience we repeat here

\[
\{X_k\}_{k=0,\ldots,n} \overset{d}{=} \{X_{n,k}\}_{k=0,\ldots,n} \equiv \left\{ \frac{2/\alpha}{\left\lfloor \frac{1}{\alpha} \sum_{1 \leq i_1 < i_2} \varepsilon_{i_1} \varepsilon_{i_2} \Gamma_{i_1}^{1/\alpha} \Gamma_{i_2}^{1/\alpha} \mathbb{1}\{k \in R_{n,i_1} \cap R_{n,i_2}\} \right\}} \right\}_{k=0,\ldots,n},
\]

and we follow the same notation in earlier sections except that we now have \( n \) instead of \( m \) and will let \( n \to \infty \). We also emphasize on the dependence on \( n \) of the series representation when writing \( X_{n,k} \) instead of \( X_k \). Introduce

\[
b_n = \left( \frac{1}{4} n \log n \right)^{1/\alpha}.
\]

It can be verified based on (3.6) that \( P(X_1 > b_n) \sim 1/n \) as \( n \to \infty \). Then by a classical extreme limit theorem (e.g., [21, Proposition 1.11], for i.i.d. copies \( \{X_k^{(0)}\}_{k \in \mathbb{N}} \) of \( X_1 \), we have

\[
\frac{1}{b_n} \max_{k=1,\ldots,n} X_k^{(0)} \Rightarrow Z_{\alpha},
\]

where \( Z_{\alpha} \) follows a standard \( \alpha \)-Fréchet distribution: \( P(Z_{\alpha} \leq x) = e^{-x^{-\alpha}}, x \geq 0 \).

The goal is to establish a extreme limit theorem for the model (4.1) in the sub-critical case.

**Theorem 4.1.** Assume \( \beta \in (0, 1/2) \). As \( n \to \infty \), we have

\[
\frac{1}{b_n} \max_{k=1,\ldots,n} X_k \Rightarrow \theta^{1/\alpha} Z_{\alpha} \ \text{with} \ \theta = (1-2\beta)q_{F;2},
\]

where \( Z_{\alpha} \) follows a standard \( \alpha \)-Fréchet distribution, and \( q_{F;2} \) is as in (1.10).

Then, in comparison with (4.2) it follows that \( \theta \) is the extremal index of the original process.

The proof below is different from the one presented in [4] and a non-trivial adaption is needed to extend the proof here to \( p \geq 3 \). We shall proceed with following approximation procedure. Fix throughout a sequence of increasing integers \( \{m_n\}_{n \in \mathbb{N}} \) such that

\[
\frac{w_n^2}{n \log^{2/(2-\alpha)} n} \ll m_n \ll \frac{w_n^2}{n}
\]

as \( n \to \infty \), where \( a_n \ll b_n \) means \( a_n/b_n \to 0 \) as \( n \to \infty \). Introduce

\[
\bar{X}_{n,k} := w_n^{2/\alpha} \sum_{1 \leq i_1 < i_2 \leq m_n} \frac{\varepsilon_{i_1} \varepsilon_{i_2}}{\Gamma_{i_1}^{1/\alpha} \Gamma_{i_2}^{1/\alpha}} \mathbb{1}\{k \in R_{n,i_1} \cap R_{n,i_2}\}.
\]

Then Theorem 4.1 follows from the following two results.

**Proposition 4.2.** As \( n \to \infty \), we have

\[
\frac{1}{b_n} \max_{k=1,\ldots,n} \bar{X}_{n,k} \Rightarrow \theta^{1/\alpha} Z_{\alpha}.
\]
Lemma 4.3. As \( n \to \infty \), we have
\[
\frac{1}{b_n} \max_{k=1, \ldots, n} |X_{n,k} - \overline{X}_{n,k}| \overset{p}{\to} 0,
\]
where \( \overset{p}{\to} \) stands for convergence in probability.

4.1. Proof of Proposition 4.2

Let \( \{U_i\}_{i \in \mathbb{N}} \) be i.i.d. uniform random variables independent of everything else. Introduce
\[
V_{i_1,i_2} := \epsilon_{i_1} \epsilon_{i_2} (U_{i_1} U_{i_2})^{-1/\alpha}, \quad R_{n,i_1,i_2} := R_{n,i_1} \cap R_{n,i_2}, \quad i_1, i_2 \in \mathbb{N}, \ i_1 < i_2.
\]

Then by a well-known relation \((\Gamma_1, \ldots, \Gamma_{m_n}) \overset{d}{=} (U_{1,m_n}, \ldots, U_{m_n,m_n}) \Gamma_{m_n+1}\), where \( U_{1,m_n} < \ldots < U_{m_n,m_n} \) are order statistics of \( \{U_1, \ldots, U_{m_n}\} \), we have
\[
\left\{ \overline{X}_{n,k} \right\}_{k=1, \ldots, n} \overset{d}{=}
\left\{ \left( \frac{w_n}{m_n} \right)^{2/\alpha} \sum_{1 \leq i_1 < i_2 \leq m_n} V_{i_1,i_2} \mathbb{1}_{\{k \in R_{n,i_1,i_2}\}} \right\}_{k=1, \ldots, n}.
\]

Noting that \( \Gamma_{m_n+1} \sim m_n \) as \( n \to \infty \) almost surely, we can instead work with
\[
\left\{ X^*_{n,k} \right\}_{k=1, \ldots, n} = \left\{ \left( \frac{w_n}{m_n} \right)^{2/\alpha} \sum_{1 \leq i_1 < i_2 \leq m_n} V_{i_1,i_2} \mathbb{1}_{\{k \in R_{n,i_1,i_2}\}} \right\}_{k=1, \ldots, n}.
\]

Lemma 4.4. We have as \( n \to \infty \),
\[
\rho_n := \mathbb{P} \left( R_{n,1,2} \neq \emptyset \right) \sim \frac{n}{w_n^2}.
\]

**Proof.** Introduce
\[
q_{n,1} := \mathbb{P} \left( R_{n,1,2} \cap \{1, \ldots, n\} \neq \emptyset, \ \max R_{n,1,2} \leq n \right), \\
q_{n,2} := \mathbb{P} \left( R_{n,1,2} \cap \{1, \ldots, n\} \neq \emptyset, \ \max R_{n,1,2} > n \right).
\]

Then, \( q_{n,1} \leq \rho_n \leq q_{n,1} + q_{n,2} \). For \( q_{n,1} \) we apply the last-renewal decomposition and Markov property:
\[
q_{n,1} = \sum_{i=1}^{n} \mathbb{P} \left( \max R_{n,1,2} = i \ \middle| \ i \in R_{n,1,2} \right) \mathbb{P} \left( i \in R_{n,1,2} \right) = \sum_{i=1}^{n} q_{F:2} \frac{1}{w_n^2} = \frac{q_{F:2} n}{w_n^2},
\]
where we have used the fact that conditioning on \( i \in R_{n,1,2} \), the count \( \sum_{k=1}^{\infty} \mathbb{1}_{\{k \in R_{n,1,2}\}} \) follows a geometric distribution with mean \( q_{F:2}^{-1} \). For \( q_{n,2} \), write first by a similar decomposition based on the last renewal before time \( n \),
\[
q_{n,2} = \sum_{i=1}^{n} \mathbb{P} \left( \max (R_{n,1,2} \cap \{1, \ldots, n\}) = i, \ \max R_{n,1,2} > n \right)
\]
\[ \sum_{i=1}^{n} \mathbb{P} (i \in R_{n,1,2}) \mathbb{P} (\max R_{n,1,2} > n \mid i \in R_{n,1,2}) \leq \sum_{i=1}^{n} \frac{1}{w_{n}^{2}} \sum_{j=n-i+1}^{\infty} u(j)^{2}. \]

The last step above follows from the renewal property and the union bound. With \( v(i) := \sum_{j=1}^{\infty} u(j)^{2} \downarrow 0 \) as \( i \to \infty \) due to the fact \( u(j) \leq C j^{\beta-1} \) with \( \beta \in (0, 1/2) \), the last displayed expression becomes

\[ w_{n}^{-2} \sum_{i=1}^{n} v(i) = n w_{n}^{-2} \sum_{i=1}^{n} (v(i)/n) = n w_{n}^{-2} o(1) = o(q_{n,1}), \]

completing the proof. \( \square \)

Introduce the following two counting numbers:

\[ N_{n} := \sum_{1 \leq i_{1} < i_{2} \leq m_{n}} 1 \{ R_{n,i_{1},i_{2}} \cap \{1, \ldots, n\} \neq \emptyset \}, \tag{4.5} \]

and

\[ M_{n} := \sum_{1 \leq i_{1} < i_{2} \leq m_{n}, 1 \leq j_{1} < j_{2} \leq m_{n}, \{i_{1}, i_{2}\} \neq \{j_{1}, j_{2}\}, \{i_{1}, i_{2}\} \cap \{j_{1}, j_{2}\} \neq \emptyset} 1 \{ R_{n,i_{1},i_{2}} \cap \{1, \ldots, n\} \neq \emptyset, R_{n,j_{1},j_{2}} \cap \{1, \ldots, n\} \neq \emptyset \}. \tag{4.6} \]

**Lemma 4.5.** We have as \( n \to \infty \) that

\[ EN_{n} \sim \lambda_{n} := \frac{q_{F,2} nm_{n}^{2}}{2 w_{n}^{2}} \to \infty, \quad \frac{N_{n}}{\lambda_{n}} \to 1 \tag{4.7} \]

and \( EM_{n} \leq C \frac{n^{2} m_{n}^{3}}{w_{n}^{4}} = o(\lambda_{n}) \).

**Proof.** For the first part, we have

\[ EN_{n} \sim \frac{m_{n}^{2} \rho_{n}}{2} = \frac{q_{F,2} nm_{n}^{2}}{2 w_{n}^{2}}, \]

which tends to \( \infty \) due to \( m_{n} \gg w_{n}^{2}/(n \log^{2/(2-\alpha)} n) \), the first part of assumption (4.3). Next, note that

\[ \rho'_{n} := \mathbb{P} (R_{n,1,2} \cap \{1, \ldots, n\} \neq \emptyset, R_{n,1,3} \cap \{1, \ldots, n\} \neq \emptyset) \leq \sum_{k=1}^{n} \sum_{k'=1}^{n} \mathbb{P} (k \in R_{n,1,2}, k' \in R_{n,1,3}) \leq \frac{2}{w_{n}^{2}} \sum_{k=1}^{n} \sum_{k'=1}^{n} u(k' - k) \leq C \frac{n^{2}}{w_{n}^{4}}. \]

Hence

\[ EM_{n} \leq C \rho'_{n} m_{n}^{3} \leq C \frac{n^{2} m_{n}^{3}}{w_{n}^{4}} = o(\lambda_{n}), \]

where the last relation follows from \( m_{n} \ll w_{n}^{2}/n \), the second part of (4.3).

Next, by a decomposition of the double sum over \( 1 \leq i_{1} < i_{2} \leq m_{n} \) and \( 1 \leq j_{1} < j_{2} \leq m_{n} \) according to \( |\{i_{1}, i_{2}\} \cap \{j_{1}, j_{2}\}| = 0, 1, 2 \), we have as \( n \to \infty \),

\[ EN_{n}^{2} = \binom{m_{n}}{2} \binom{m_{n} - 2}{2} \rho_{n}^{2} + EM_{n} + EN_{n} = (EN_{n})^{2} (1 - o(1)) + O(\lambda_{n}), \]

and hence \( \text{Var}(N_{n}) = o((EN_{n})^{2}) \), which concludes the convergence in probability in (4.7). \( \square \)
Proof of Proposition 4.2. We shall work with $X^*_{n,k}$ in (4.4). The key underlying structure is that its partial maximum can be approximated by a collection of $\lambda_n$ (see (4.7)) i.i.d. random variables, as summarized in (4.10) below, and this approximation alone explains why the extra factor $D_{2,\beta} = 1 - 2\beta$ shows up in the extremal index compared to the candidate extremal index.

We start by setting the random index set

$$J_n(k) := \{(i_1, i_2) \in \{1, \ldots, m_n\}^2 : i_1 < i_2, k \in R_{n,i_1,i_2}\}, \quad k = 1, \ldots, n.$$ 

So

$$\max_{k=1,\ldots,n} X^*_{n,k} = \left(\frac{w_n^2}{m_n^2}\right)^{1/\alpha} \max_{k=1,\ldots,n} \left(\sum_{(i_1, i_2) \in J_n(k)} V_{i_1, i_2}\right), \quad (4.8)$$

where the sum over $(i_1, i_2) \in J_n(k)$ is understood as 0 if $J_n(k) = \emptyset$. Define also

$$I_n(k) := \bigcup_{(i_1, i_2) \in J_n(k)} \{i_1, i_2\}.$$ 

In fact $|J_n(k)| = \binom{I_n(k)}{2}$, and hence $|J_n(k)|$ cannot take arbitrary integer values. But for simplicity of notation we shall still write a consecutive integer range for $|J_n(k)|$ below. Let

$$\mathcal{K}_n := \{k \in \{1, \ldots, n\} : |J_n(k)| = 1, \text{ and either } J_n(k') = J_n(k) \text{ or } I_n(k') \cap I_n(k) = \emptyset \forall k' \in \{1, \ldots, n\} \setminus \{k\}\},$$

and set $\mathcal{K}_n^c := \{1, \ldots, n\} \setminus \mathcal{K}_n$. Then by independence,

$$\max_{k \in \mathcal{K}_n} \left(\sum_{(i_1, i_2) \in J_n(k)} V_{i_1, i_2}\right) \overset{d}{=} \max_{\ell = 1, \ldots, N_n} V_{\tilde{\ell}} \quad \text{with} \quad \tilde{N}_n := |\cup_{k \in \mathcal{K}_n} J_n(k)|, \quad (4.9)$$

where $\{V_{\ell}\}_{\ell \in \mathbb{N}}$ are i.i.d. of the same distribution as $V_{1,2}$ and independent of everything else. Here and below, when a maximum is performed over an empty index set, it is understood as 0.

Next, observe that $N_n \leq N_n$ with $N_n$ in (4.5). In addition, any $(i_1, i_2)$ satisfying $R_{n,i_1,i_2} \neq \emptyset$ and $1 \leq i_1 < i_2 \leq m_n$, but not included in $\cup_{k \in \mathcal{K}_n} J_n(k)$, must have been counted at least once by $M_n$ in (4.6). So $0 \leq N_n - \tilde{N}_n \leq M_n$. Recall $\lambda_n$ in (4.7).

In summary, one can prove

$$\lim_{n \to \infty} \mathbb{P}\left( \frac{1}{b_n} \max_{k \in \mathcal{K}_n} X^*_{n,k} \leq x \right) = \lim_{n \to \infty} \mathbb{P}\left( \left(\frac{w_n^2}{b_n^2 m_n^2}\right)^{1/\alpha} \max_{\ell = 1, \ldots, N_n} V_{\ell} \leq x \right)$$

$$= \lim_{n \to \infty} \mathbb{P}\left( \left(\frac{w_n^2}{b_n^2 m_n^2}\right)^{1/\alpha} \max_{\ell = 1, \ldots, \lceil \lambda_n \rceil} V_{\ell} \leq x \right) \quad (4.10)$$

$$= \mathbb{P}\left( b^{1/\alpha} Z_\alpha \leq x \right), \quad \text{for all } x > 0. \quad (4.11)$$

Indeed, the first relation above follows from (4.8) and (4.9). We postpone the proof for (4.10) for a moment. Then (4.11) follows from classical extreme-value limit theorem for $[\lambda_n]$ i.i.d. random
variables of regularly-varying tail (e.g., [21, Proposition 1.11]). The normalization, say $d_n$, for

$$
\frac{1}{d_n} \max_{\ell=1,\ldots,|\lambda_n|} V_\ell \Rightarrow Z_\alpha,
$$

is well-known to be determined by $\lim_{n \to \infty} \lambda_n \mathbb{P}(V_1 > d_n) = 1$. It is elementary to verify that

$$
\mathbb{P}(V_1 < -x) = \mathbb{P}(V_1 > x) = \frac{1}{2} \mathbb{P}(U_1 U_2 < x^{-\alpha}) \sim \frac{\alpha}{2} x^{-\alpha} \log x \quad (4.12)
$$
as $x \to \infty$. Recall also $\lambda_n \sim (q_{F,2}/2)n m_n^2/w_n^2$ and the requirement (4.3) which implies $\log(\lambda_n) \sim \log(w_n^2/n) \sim (1 - 2\beta) \log n$. So we have

$$
d_n \sim \left(\frac{1}{2} \lambda_n \log \lambda_n \right)^{1/\alpha} \sim \left(\frac{1 - 2\beta}{2} \lambda_n \log n \right)^{1/\alpha}
$$

$$
\sim \left(\frac{b_n^2 m_n^2}{w_n^2} \right)^{1/\alpha} (q_{F,2}(1 - 2\beta))^{1/\alpha} = \left(\frac{b_n^2 m_n^2}{w_n^2} \right)^{1/\alpha} \theta^{1/\alpha},
$$
yielding (4.11). Now we check (4.10). Introduce $f(x) := ((1/2)x \log x)^{1/\alpha}$. So $d_n \sim f(\lambda_n)$. Introduce accordingly $\tilde{d}_n := f(\tilde{\lambda}_n) = ((1/2)\tilde{\lambda}_n \log \tilde{\lambda}_n)^{1/\alpha}$. By Lemma 4.5, we have $\tilde{\lambda}_n/\lambda_n \xrightarrow{p} 1$ and $\lambda_n \to \infty$ as $n \to \infty$, and it follows that $\tilde{d}_n/d_n \xrightarrow{p} 1$ since $f$ is a regularly varying function (e.g. [16, Theorem 1.12]). Write $Z_n = f(n^{-1} \max_{\ell=1,\ldots,n} V_\ell)$ and $d_n^{-1} \max_{\ell=1,\ldots,n} V_\ell = (\tilde{d}_n/d_n) \cdot Z_{\tilde{\lambda}_n}$. So to obtain (4.10) it is remains to argue $Z_{\tilde{\lambda}_n}$ and $Z_n$ have the same limit distribution as $n \to \infty$. The last step is an exercise.

To complete the proof, it remains to show that as $n \to \infty$, $b_n^{-1} \max_{k \in \mathcal{K}_n^c} X_{n,k}^* \xrightarrow{p} 0$. We fix $p^*$ large enough so that $1 - (p^* + 1)\beta < 0$ (recall $\beta \in (0, 1/2)$). Then

$$
\mathbb{P}(|J_n(k)| > p^* \text{ for some } k = 1, \ldots, n) \leq \sum_{k=1}^{n} \left(\frac{m_n}{p^* + 1}\right)^{w_n^{-p^*-1} \leq C \frac{w_n^{p^*+1}}{n^{p^*}} \leq C n^{1-(p^*+1)\beta} \to 0},
$$

where we have used (4.3) in the second inequality above. Hence with probability tending to 1 as $n \to \infty$, we have

$$
\frac{1}{b_n} \max_{k \in \mathcal{K}_n^c} X_{n,k}^* = \left(\frac{u_n^2}{b_n^2 m_n^2} \right)^{1/\alpha} \max_{j=1,\ldots,p^*} \max_{k \in \mathcal{K}_n^c (i_1,i_2) \in J_n(k), |J_n(k)|=j} \sum V_{i_1,i_2}.
$$

So it suffices to show for fixed $j = 1, \ldots, p^*$ as $n \to \infty$ that

$$
\left(\frac{u_n^2}{b_n^2 m_n^2} \right)^{1/\alpha} \max_{k \in \mathcal{K}_n^c} W_j(k) \xrightarrow{p} 0 \quad \text{with} \quad W_j(k) = \sum_{(i_1,i_2) \in J_n(k), |J_n(k)|=j} |V_{i_1,i_2}|. \quad (4.13)
$$

Note that $(W_j(k))_{k \in \mathcal{K}_n^c}$ are, given $\mathcal{K}_n^c$, identically distributed but possibly dependent random variables, and the total number of distinct $W_j(k)$’s, say $K_j(n)$, does not exceed $N_n - \tilde{N}_n \leq M_n$. On the other hand in view of (4.12),

$$
\mathbb{P}(W_j(1) > x) \leq j \mathbb{P}(|V_{1,2}| > x/j) \leq C x^{-\alpha} \log x \quad (4.14)
$$
for all $x > 0$ and some constant $C > 0$. Suppose $\{W_j(\ell)\}_{\ell \in \mathbb{N}}$ are i.i.d. copies of $W_j(1)$ and independent of everything else. Then by the tail bound (4.14), the tail estimate (4.12), the extreme-value limit theorem for $|V_\ell|$ and the fact that $M_n/\lambda_n \to 0$ (Lemma 4.5), we have

$$
\mathbb{P} \left( \left( \frac{w_n^2}{b_n^2 m_n^2} \right)^{1/\alpha} \max_{\ell = 1, \ldots, K_j(n)} \tilde{W}_j(\ell) > \epsilon \right) \leq C \mathbb{P} \left( \left( \frac{w_n^2}{b_n^2 m_n^2} \right)^{1/\alpha} \max_{\ell = 1, \ldots, M_n} |V_\ell| > \epsilon \right) \to 0
$$

for any $\epsilon > 0$ as $n \to \infty$. Then (4.13) follows from [25, Proposition 9.7.3].

### 4.2. Proof of Lemma 4.3

Lemma 4.3 follows if one establishes that

$$A_n := \frac{w_n^{2/\alpha}}{b_n} \max_{k=1, \ldots, n} \sum_{1 \leq i_1, i_1 < n} \frac{\epsilon_{i_1} \epsilon_{i_2}}{i_1^{1/\alpha} i_2^{1/\alpha}} 1_{\{k \in R_{n, i_1, i_2}\}} \overset{p}{\to} 0,$$

$$B_n := \frac{w_n^{2/\alpha}}{b_n} \max_{k=1, \ldots, n} \sum_{m_n < i_1 < i_2} \frac{\epsilon_{i_1} \epsilon_{i_2}}{i_1^{1/\alpha} i_2^{1/\alpha}} 1_{\{k \in R_{n, i_1, i_2}\}} \overset{p}{\to} 0,$$

as $n \to \infty$. First, fix a number $m > 4/\alpha$. Assume without loss of generality that $m_n > 4/\alpha$. Then write

$$A_n = A_{n,1} + A_{n,2} := \frac{w_n^{2/\alpha}}{b_n} \max_{k=1, \ldots, n} \sum_{m \leq i_1 < i_2 < m_n} \cdots + \frac{w_n^{2/\alpha}}{b_n} \max_{k=1, \ldots, n} \sum_{1 \leq i_1 < i_2 > m_n} \cdots.$$

Then by bounding the maximum of non-negative numbers with their sum, the orthogonality induced by $\epsilon_{i_1} \epsilon_{i_2}$ and [26, the inequality (3.2)], we have

$$\mathbb{E} A_{n,1}^2 \leq \frac{w_n^{4/\alpha}}{b_n^2} \sum_{k=1}^{n} \left( \sum_{m \leq i_1 < i_2 < m_n} \frac{\epsilon_{i_1} \epsilon_{i_2}}{i_1^{2/\alpha} i_2^{2/\alpha}} 1_{\{k \in R_{n, i_1, i_2}\}} \right)^2 = \frac{w_n^{4/\alpha}}{b_n^2} \sum_{m \leq i_1 < i_2 < m_n} \mathbb{E} \left( \Gamma_{i_1}^{-2/\alpha} \Gamma_{i_2}^{-2/\alpha} \right) \left( \sum_{k=1}^{n} 1_{\{k \in R_{n, i_1, i_2}\}} \right) \leq C \frac{n w_n^{4/\alpha - 2}}{b_n^2} m_n^{-2/\alpha} \to 0,$$

where we have used the relation $\frac{w_n^2}{n \log^2 (2 - \alpha)(n)} \ll m_n$ in the first part of (4.3) in the last step.

For $A_{n,2}$, since $i_1$ takes finitely many values, it suffices to show for fixed $i_1$ that

$$b_n^{-2} w_n^{4/\alpha} \mathbb{E} \left| \max_{k=1, \ldots, n} \sum_{i_2 > m_n} \Gamma_{i_2}^{-1/\alpha} \epsilon_{i_2} 1_{\{k \in R_{n, i_1, i_2}\}} \right|^2 \leq C n b_n^{-2} w_n^{4/\alpha - 2} m_n^{-2/\alpha} \to 0.$$
which follows as above. For \( B_n \), similarly we have
\[
\mathbb{E}B_n^2 \leq Cn b_n^{-2} w_n^{-4/\alpha - 2} \sum_{m_n < i_1 < i_2} i_1^{-2/\alpha} i_2^{-2/\alpha} \leq Cn b_n^{-2} w_n^{-4/\alpha - 2} m_n^{-4/\alpha} \to 0.
\]

5. Anti-clustering condition when \( \beta_p < 0 \)

Assume \( \beta_p < 0 \) in this section. Based on the convergence of the tail processes we can actually prove the convergence of the single-block cluster point process, which is based on the verification of the anti-clustering condition (1.16). Strictly speaking we prove a stronger condition known as the \( S(r_n, a_n) \) condition as in [15, P.243], in Lemma 5.2 below. This condition could have other consequences, notably when proving limit theorems for tail empirical processes [15, Chapter 9]. It remains an interesting question whether one could prove limit theorems for tail empirical processes for our model here: the classical approach relies also on certain mixing-type condition and does not seem applicable here.

Proposition 5.1. Assume \( \beta_p < 0 \). For \( a_n \to \infty \), \( r_n \to \infty \) and \( r_n = o(\log a_n) \), with \( M_{r_n} := \max_{k=1, \ldots, r_n} |X_k| \), the limit of
\[
\mathcal{L} \left( \sum_{i=1}^{r_n} \delta_{X_i/a_n} \mid M_{r_n} > a_n \right)
\]
in the space of \( \mathcal{M}_{\mu}((\mathbb{R} \setminus \{0\}) \) is \( \mathcal{G}_\varepsilon \), a unit mass at a Rademacher random variable \( \varepsilon \) multiplied by a geometric random variable \( \mathcal{G} \) with mean \( 1/q_{F,p} \) and independent from \( \varepsilon \).

Proof. By Theorem 1.1, [5, Theorem 4.3] and a continuous mapping argument (cf. [5, Remark 4.6]), the limit is a point measure
\[
\sum_{i=0}^{\infty} \delta_{\Theta^*_i}
\]
restricted to \( \mathcal{M}_{\mu}((\mathbb{R} \setminus \{0\}) \) given that the anti-clustering condition (1.16) holds [5, e.g. Condition 4.1], which is verified in Lemma 5.2 below. Note that because of the restriction, the point process above is actually \( \langle \sum_{i=0}^{\infty} 1_{\{\Theta^*_i = 1\}} \rangle \delta_{\varepsilon} \) (note that \( \Theta^*_0 = 1 \) always), and the summation in the parenthesis is readily checked to be a geometric random variable with the desired law, by examining the definition of \( \Theta^* \) in (1.7) and the renewal property.

The following lemma implies the anti-clustering condition (1.16) by a simple argument based on union bound and the stationarity of the sequence.

Lemma 5.2 (Condition \( S(r_n, a_n) \)). Assume \( \beta_p < 0 \). For \( a_n \to \infty \), \( r_n \to \infty \) and \( r_n = o(\log a_n) \).
\[
P \left( \max_{\ell \leq k \leq r_n} |X_k| > a_n \eta \right) \mid |X_0| > a_n \eta \right) \leq \frac{2}{P(|X_0| > a_n \eta)} \sum_{k=\ell}^{r_n} P(|X_0| > a_n \eta, |X_k| > a_n \eta) \to 0.
\]

(5.1)

Proof. Unlike in the proof of Theorem 1.1, we only need a series representation for two-dimensional \((X_0, X_K)\), for each \( K \in \mathbb{N} \) fixed. This can be done in the same way as (2.4) is derived, with the
only modification that the restriction \( \tau^* \cap \{0, \ldots, m\} \neq \emptyset \) is now replaced by \( \tau^* \cap \{0, K\} \neq \emptyset \). More precisely, let \( \{\tilde{R}_{K,i}\}_{i \in \mathbb{N}} \) be i.i.d. copies of \( \tilde{R}_K \), of which the law, denoted by \( \tilde{\mu}_K \), is determined by

\[
\frac{d\tilde{\mu}_K}{d\mu^*} = \frac{1_{\{\tau^* \cap \{0, K\} \neq \emptyset\}}}{\mu^* \{\{\tau^* : \tau^* \cap \{0, K\} \neq \emptyset\}\}} = \frac{1_{\{\tau^* \cap \{0, K\} \neq \emptyset\}}}{\tilde{w}_K},
\]

and one can compute by inclusion-exclusion formula that \( \tilde{w}_K = 2 - u(K) \). Then we arrive at

\[
\{X_k\}_{k=0,K} = \left\{ \frac{d}{\tilde{w}_K^{p/\alpha}} \sum_{i \in D_p} \frac{[\varepsilon_i]}{[\Gamma_{\tilde{k}(i)}]}^{1/\alpha} 1_{\{k \in \tilde{R}_{K,i}\}} \right\}_{k=0,K},
\]

where as usual \( \{\tilde{R}_{K,i}\}_{i \in \mathbb{N}} \) are independent from \( \{\varepsilon_i\}_{i \in \mathbb{N}} \) and \( \{\Gamma_i\}_{i \in \mathbb{N}} \).

This time, introduce (similar to (3.1)) for \( k = 0, K \),

\[
\hat{\ell}_k(1) := \min \left\{ j \in \mathbb{N} : k \in \tilde{R}_{K,j} \right\},
\]

\[
\hat{\ell}_k(s) := \min \left\{ j > \hat{\ell}_K,k(s - 1) : k \in \tilde{R}_{K,j} \right\}, s \geq 2.
\]

We can write

\[
\tilde{X}_k := \tilde{w}_K^{p/\alpha} \sum_{i \in D_p} \frac{[\varepsilon_i]}{[\Gamma_{\hat{k}(i)}]}^{1/\alpha} \quad \text{with} \quad \tilde{T}_k := \frac{[\varepsilon_i]}{[\Gamma_{\hat{k}(1,\ldots,p)}]}^{1/\alpha}, k = 0, K. \tag{5.2}
\]

Then, \( (\tilde{X}_0, \tilde{X}_K) \overset{d}{=} (X_0, X_K) \) and \( (\tilde{X}_k, \tilde{T}_k) \overset{d}{=} (X_k, T_k), k = 0, K \). Then

\[
P(|X_0| > a_n\eta, |X_K| > a_n\eta) \leq P\left( |\tilde{T}_0| > a_n\eta/2, |\tilde{T}_K| > a_n\eta/2 \right) + 2P\left( |X_0 - \tilde{T}_0| > a_n\eta/2 \right)
\]

\[
\leq P\left( |\tilde{T}_0| > a_n\eta/2, |\tilde{T}_K| > a_n\eta/2 \right) + Cq_{p-1}((a_n\eta/2)\alpha), \tag{5.3}
\]

where the last step follows from (3.7). Introduce

\[
\hat{H}_k := 1_{\{\tilde{\ell}_K((1,\ldots,p)) = (1,\ldots,p)\}}, k = 0, K.
\]

Therefore we have, using the representation (5.2) again,

\[
P\left( \frac{a_n\eta}{2} \mid \tilde{T}_K > \frac{a_n\eta}{2} \right)
\]

\[
\leq P\left( \frac{(2 - u(K))^{p/\alpha}}{[\Gamma_{1,\ldots,p}]}^{1/\alpha} > \frac{a_n\eta}{2} \right) P\left( \hat{H}_0 = \hat{H}_K = 1 \right) + P\left( \frac{(2 - u(K))^{p/\alpha}}{[\Gamma_{1,\ldots,p}]}^{1/\alpha} > \frac{a_n\eta}{2} \right)
\]

\[
\leq Cq_p((a_n\eta)^{\alpha/2^{p+\alpha}})P\left( \hat{H}_0 = \hat{H}_K = 1 \right) + Cq_{p-1}(\alpha\eta^{2/2^{p+\alpha}}). \tag{5.4}
\]

It is straightforward to compute

\[
P\left( \hat{H}_0 = \hat{H}_K = 1 \right) = P\left( \hat{H}_0 = 1 \right) P\left( \hat{H}_K = 1 \mid \hat{H}_0 = 1 \right) \leq P\left( \hat{H}_K = 1 \mid \hat{H}_0 = 1 \right) = u(K)^p. \tag{5.5}
\]
To sum up, by (5.3), (5.4) and (5.5), the right-hand side of (5.1) is further bounded from above by

$$\frac{Cq_p((a_n \eta)^\alpha / 2^{p+\alpha})}{q_p((a_n \eta)^\alpha)} \sum_{k=\ell}^\infty u(k)^p + \frac{Cr_n q_{p-1}((a_n \eta)^\alpha / 2^{p+\alpha})}{q_p((a_n \eta)^\alpha)} \leq C \sum_{k=\ell}^\infty u(k)^p + C \frac{r_n}{\log a_n}.$$ 

Recall $u(k)$ in (2.2). Since $\beta_p < 0$, $\sum_{k=1}^\infty u(k)^p < \infty$. We have thus proved the desired result.

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