Abstract. We study the one-dimensional Laplace operator with point interactions on the real line identified with two copies of the half-line $[0, \infty)$. All possible boundary conditions that define generators of $C_0$-semigroups on $L^2([0, \infty)) \oplus L^2([0, \infty))$ are characterized. Here, the Cayley transform of the boundary conditions plays an important role, and using an explicit representation of the Green’s functions, it allows us to study invariance properties of semigroups.

1. Introduction

Here, point interactions for the Laplacian on the real line are considered. The real line is realized here as two half-lines $[0, \infty) \cup [0, \infty)$ coupled at the two boundary points. More concretely, we consider realizations $-\Delta(A,B)$ of

$$\frac{d^2}{dx} \text{ in } L^2([0, \infty) ; \mathbb{C}) \oplus L^2([0, \infty) ; \mathbb{C})$$

with boundary conditions of the form

$$A \begin{bmatrix} \psi_1(0) \\ \psi_2(0) \end{bmatrix} + B \begin{bmatrix} \psi_1'(0) \\ \psi_2'(0) \end{bmatrix} = 0 \quad \text{for } A, B \in \mathbb{C}^{2 \times 2},$$

for $(\psi_1, \psi_2)^T \in L^2([0, \infty) ; \mathbb{C}) \oplus L^2([0, \infty) ; \mathbb{C})$. Like in [Mug10, §4], we regard this setting as a toy model of more complicated quantum graphs.

There are many studies on self-adjoint boundary conditions, cf. [BK13] and references therein, boundary conditions leading to so-called spectral operators, cf. [DS71], or boundary conditions related to quadratic forms, cf. [Mug14]. However, a study of all possible boundary conditions of this form seems to be lacking so far. In this note, we turn to classical Hille–Yosida theory and address the issue of semigroup generation by realizations of the Laplacian with point interactions of the above type. It turns out that resolvent estimates for $\Delta(A,B)$ are closely related to the behavior of the Cayley transform.

One could naively expect that imposing two linearly independent boundary conditions is both necessary and sufficient to induce a realization that generates a semigroup, because there are two boundary points and this leads to the rank condition

$$\text{Rank}(A \ B) = 2;$$

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and in fact if \( \text{Rank}(A, B) \neq 2 \), then \( \sigma(-\Delta(A, B)) = \mathbb{C} \), see [HKS15, Prop. 4.2]. However, this rank condition is not yet sufficient to establish basic spectral properties and it turns out that the question of determining when \( A, B \) induce a semigroup generator is not trivial. In a previous work Krejčiřík, Siegl and the first author, see [HKS15], pointed out the importance of the Cayley transform

\[
\mathcal{S}(k; A, B) := -(A + ikB)^{-1}(A - ikB), \quad k \in \mathbb{C},
\]

for basic spectral properties. The condition that \( A + ikB \) is invertible for some \( k \in \mathbb{C} \) has been used in [HKS15] as definition for the notion of regular boundary conditions: on general metric graphs irregular boundary conditions can produce very wild spectral features, ranging from empty spectrum – as in the situation considered here – to empty resolvent set. For the case of one boundary point this cannot occur: the easiest non-trivial case features two boundary points and will be investigated in detail in the following.

In the present setting we find out that realizations \( \Delta(A, B) \) with irregular boundary conditions have empty resolvent set and thus fail to be generators of \( C_0 \)-semigroups; surprisingly, it turns out that there are even some regular boundary conditions that do not define generators of \( C_0 \)-semigroups. We will see that not only the mere existence of the Cayley transform is relevant, but also its asymptotic behavior. The crucial point is that the Cayley transform \( \mathcal{S}(k; A, B) \) appears in a natural way in an explicit formula for the resolvent of \( \Delta(A, B) \), which in turn easily allows us to check the conditions of the Hille–Yosida Theorem in its version for analytic semigroups.

Once generation is assessed, we turn to the issue of qualitative properties of the semigroup generated by \( \Delta(A, B) \), again in dependence of \( A, B \). It is well-known that relevant features of a semigroup – in particular, whether it is positive and/or \( L^\infty \)-contractive – is tightly related to analogous invariance properties of its generator’s resolvent. Using again our machinery, we are then able to formulate sufficient conditions for invariance in terms of properties of \( \mathcal{S}(k; A, B) \). In the context of general metric graphs, positivity and Markovian features of semigroups in dependence of the boundary conditions have been studied already in [KS06, § 5] – however only for self-adjoint boundary conditions (1.1) and giving only sufficient conditions – and in [CM09, § 5–6] for the case of only m-sectorial boundary conditions for which a complete characterisation is obtained, see also [Mug07, Mug10, KKVW09] for related results. The notion of m-sectorial boundary conditions is explained in Section 4 below: roughly speaking, these are boundary conditions that induce realizations of \( \Delta(A, B) \) associated with sesquilinear forms. One step beyond the hitherto discussed invariance properties, we are finally also able to characterize asymptotic positivity of the semigroup – a rather weak property recently introduced in [DGK16].

Our note is organized as follows: In Section 2 we are going to present our setting, including relevant function spaces and the parametrization of our boundary conditions. Section 3 contains our main result, Theorem 3.1 as well as a few examples that show its applicability. The proof of Theorem 3.1 is based on a number of technical lemmata, which will be proved in Sections 4 and 5. Finally, we are going to discuss positivity, asymptotic positivity, and further invariance issues in Section 6.
2. Function spaces, operators and boundary conditions

Whenever \( I \subset \mathbb{R} \) is an interval, denote by \( L^2(I) \) the usual space of complex-valued square integrable function with scalar product \( \langle \cdot, \cdot \rangle_{L^2} \). Moreover, let \( H^1(I) \) and \( H^2(I) \) be the Sobolev spaces of order one and two, and set \( H^2_0(I) := \{ \psi \in H^2(I) : \psi, \psi'|_{\partial I} = 0 \} \). Then one defines minimal and maximal operators in \( L^2([0, \infty)) \oplus L^2([0, \infty)) \) by

\[
\Delta_{\max} \psi = \psi'' , \quad D(\Delta_{\max}) = H^2([0, \infty)) \oplus H^2([0, \infty)),
\]

\[
\Delta_{\min} \psi = \psi'' , \quad D(\Delta_{\min}) = H^2_0([0, \infty)) \oplus H^2_0([0, \infty)).
\]

Since \( D(\Delta_{\max})/D(\Delta_{\min}) \cong \mathbb{C}^4 \), any realization

\[
\Delta_{\min} \subset \Delta \subset \Delta_{\max}
\]

is determined by a subspace \( \mathcal{M} \subset \mathbb{C}^4 \) and \( \Delta = \Delta_{\mathcal{M}} \) with

\[
D(\Delta_{\mathcal{M}}) = \{ \psi \in D(\Delta_{\max}) : [\psi, \psi']^T \in \mathcal{M} \},
\]

where

\[
\psi := \begin{bmatrix} \psi_1(0) \\ \psi_2(0) \end{bmatrix} , \quad \psi' := \begin{bmatrix} \psi_1'(0) \\ \psi_2'(0) \end{bmatrix},
\]

and one sets

\[
[\psi] := \begin{bmatrix} \psi \\ \psi' \end{bmatrix}.
\]

For \( \dim \mathcal{M} = 2 \), \( \mathcal{M} \) can be represented as kernel of a surjective linear map from \( \mathbb{C}^4 \to \mathbb{C}^2 \), and hence the condition \( \dim \mathcal{M} = 2 \) is equivalent to existence of matrices \( A, B \in \mathbb{C}^{2 \times 2} \) with \( \mathcal{M} = \mathcal{M}(A, B) = \ker(A\ B) \) and \( \text{Rank}(A\ B) = 2 \). With respect to our goal of studying the generator property of different realizations of Laplacians on \( L^2([0, \infty)) \oplus L^2([0, \infty)) \), this is the only case which provides enough boundary conditions and we will restrict to it throughout this note. For simplicity, we refer to boundary conditions defined by \( [\psi] \in \mathcal{M}(A, B) = \ker(A\ B) \) in short as boundary conditions \( A, B \).

Boundary conditions \( A, B \) and \( A', B' \) are equivalent if \( \mathcal{M}(A, B) = \mathcal{M}(A', B') \), and one sets

\[
\Delta(A, B) := \Delta_{\mathcal{M}(A, B)}.
\]

Note that \( A' = CA \) and \( B' = CB \) define equivalent boundary conditions whenever \( C \in \mathbb{C}^{2 \times 2} \) is invertible, since \( \ker(A'\ B') = \ker(A\ B) \).

The following notion of regularity of boundary conditions has been introduced in [HKS15 § 3.2]. Note that there are also further notions of regularity, in particular the one introduced by Birkhoff, cf. [Bir08a, Bir08b], see also [DS71]. This regularity assumption does not agree with the one used here, see [HKS15 § 3.3].

**Definition 2.1 (Regular and irregular boundary conditions).** Let \( A, B \) be boundary conditions with \( \text{Rank}(A\ B) = 2 \). These are called **regular** if \( A + i k B \) is invertible for some \( k \in \mathbb{C} \), and **irregular** otherwise.
Remark 2.2. It can be shown that $A, B$ are irregular if and only if $\text{Rank}(A \cdot B) = 2$ and $\text{Ker} A \cap \text{Ker} B \neq \{0\}$, cf. [HKS15, Prop. 3.3].

3. Generator properties and examples

The following is the main result of our paper. Here $\sigma_{\text{ess}}, \sigma_r, \sigma_p$ denote as usual the essential, residual, and point spectrum, respectively.

**Theorem 3.1.** Let the boundary conditions $A, B$ be regular. Then the following assertions hold.

(a) $\sigma_{\text{ess}}(-\Delta(A,B)) = [0, \infty)$, $\sigma_r(-\Delta(A,B)) = \emptyset$, and $\lambda = k^2 \in \sigma_p(-\Delta(A,B))$ if and only if $k$ with $\text{Im} k > 0$ solves $\det(A + ikB) = 0$, and its geometric multiplicity is given by $\dim \text{Ker}(A + ikB)$.

(b) $\Delta(A,B)$ is not the infinitesimal generator of a $C_0$-semigroup on $L^2([0, \infty)) \oplus L^2([0, \infty))$ if and only if $\dim \text{Ker} A = 0$, $\dim \text{Ker} B = 1$, and $P^\perp A^{-1}BP^\perp = 0$, where $P^\perp = 1 - P$ and $P$ denotes the orthogonal projection onto $\text{Ker} B$.

(c) If $\Delta(A,B)$ is a generator, then the $C_0$-semigroup extends to an analytic semigroup.

(d) If $\Delta(A,B)$ is a generator and furthermore if any pole $s$ of $k \mapsto \mathcal{S}(k; A, B)$ or $k \mapsto \mathcal{S}(-k; A, B)^*$ with $\text{Im} s > 0$ satisfies $\text{Re} s > 0$, and $s = 0$ is not a pole of any of these functions, then the semigroup generated by $\Delta(A,B)$ is uniformly bounded.

(e) If $A = L + P$ and $B = P^\perp$ for an orthogonal projection $P$ in $\mathbb{C}^2$, $P^\perp = 1 - P$, and $L \in \mathbb{C}^{2 \times 2}$ with $P^\perp LP^\perp = L$, then $\Delta(A,B)$ is even the generator of a cosine operator function and hence of an analytic semigroup of angle $\frac{\pi}{2}$ on $L^2([0, \infty)) \oplus L^2([0, \infty))$. This semigroup is always quasi-contractive, and in fact contractive if the numerical range of $L$ is contained in $\{ z : \text{Re} z \leq 0 \}$.

If $\Delta(A,B)$ is a generator, then the semigroup consists of operators that are bounded on $L^2$ and map $L^2$ into $H^2 \hookrightarrow L^\infty$, hence are integral operators by the Kantorovich–Vulikh Theorem.

**Remark 3.2** (Irregular boundary conditions do not define generators). If $A, B$ are irregular, $\Delta(A,B)$ cannot be a generator of a $C_0$-semigroup since $\sigma(\Delta(A,B)) = \mathbb{C}$. For the case of general finite metric graphs, determining spectra and resolvent estimates for irregular boundary conditions is more involved.

**Remark 3.3** (Multiplicity of eigenvalues). For regular boundary conditions $A, B$, the geometric multiplicity of an eigenvalue $-k^2$ of $\Delta(A,B)$ is at most two, and equal to two if and only if $A + ikB = 0$. This implies that $\text{Ker} B = \text{Ker} A = \{0\}$, and that there are equivalent boundary conditions $A' = l \cdot 1$ and $B' = 1$ with $\text{Re} l = \text{Im} k > 0$.

Unfortunately, we are not able to determine the semigroup’s analyticity angle in the general case; as a matter of fact, we cannot exclude that $\Delta(A,B)$ is always the generator of a cosine operator function. Indeed, the proof of (c) shows that the spectrum of $\Delta(A,B)$ is always contained in a parabola centered around the real axis; this is a necessary condition for generation of a cosine operator function, cf. [ABHN01, Thm. 3.14.18].

**Proof.** The proof of (a) can be deduced from [HKS15, Sec. 4]: The statement on the residual spectrum follows from [HKS15, Prop. 4.6] for the case of only external edges.
Essential spectra are discussed in [HKS15, Prop. 4.11]. Note that for non-self-adjoint operators there are various notions of the essential spectrum. Five types, defined in terms of Fredholm properties and denoted by $\sigma_{e_j}$ for $j = 1, \ldots, 5$, are discussed in detail in [EES7, Chap. IX]. All these essential spectra coincide for self-adjoint $T$, but for closed non-self-adjoint $T$ in general one has only the inclusions $\sigma_{e_j}(T) \subset \sigma_{e_j}(T)$ for $j < i$. However, here one even has $\sigma_{\text{ess}}(-\Delta(A, B)) = \sigma_{e_j}(-\Delta(A, B))$ for $i = 1, \ldots, 5$. The statement on the eigenvalues follows from the Ansatz for the eigenfunctions

$$\psi(x; k) = \begin{bmatrix} \alpha_1(k)e^{ikx_1} \\ \alpha_2(k)e^{ikx_2} \end{bmatrix},$$

which is square integrable only if $\text{Im} k > 0$, and there are non-trivial $\alpha_j(k)$, $j = 1, 2$, such that $\psi(\cdot; k) \in D(\Delta(A, B))$ if and only if $\det(A + ikB) = 0$, and the geometric multiplicity is given by $\dim \ker(A + ikB)$. For part (b), uniform boundedness of the Cayley transform is characterized in Lemma 4.2, Lemma 4.4 and Lemma 4.6 below. The corresponding resolvent estimates are given in Lemma 5.1 and 5.3 below, where Lemma 5.3 discusses the case of non-generators. Lemma 5.1 implies that for $\omega > |S|$, where $S$ is a set of singularities of $\mathcal{G}(k; A, B)$ defined there, $\Delta(A, B)$ is a closed densely defined operator with $[\omega^2, \infty) \in \rho(\Delta(A, B))$. For any sector

$$(3.1) \quad \Sigma_\theta := \{k \in \mathbb{C} : \text{Im} k > 0, |\text{Re} k| \leq \tan(\theta)|\text{Im} k\}, \quad \theta \in (0, \pi/2),$$

one has $|k| \cong |\text{Re} k| + \text{Im} k \leq (1 + \tan(\theta))|\text{Im} k|$, and therefore

$$\|(-\Delta(A, B) - k^2)^{-1}\| \leq \frac{1}{|k|^2} + \frac{C_\omega}{|k|^2} \leq \frac{1 + C_\omega}{|k|^2}, \quad k \in \Sigma_\theta, \quad |k| > \omega.$$

In particular, $\Delta(A, B)$ is sectorial on the sector $\Sigma_{2\theta} - \omega^2 \sin(\pi - 2\theta)$. Shifting the sector allows avoiding the two poles of $\mathcal{G}(k; A, B)$: this finishes the proof of (b) and (c), whereas (e) is proved in Lemma 5.2.

A necessary condition for boundedness of a semigroup is that the spectrum of its generator $A$ is contained in $\{z \in \mathbb{C} : \text{Re} z \leq 0\}$. To prove (d), recall that by a celebrated result due to Gomilko [Gom99], for semigroups acting on a Hilbert space $H$ boundedness is equivalent to said spectral inclusion and the additional condition

$$\sup_{\delta > 0} \int_{\delta-i\infty}^{\delta+i\infty} \left(\| (A - \lambda)^{-1} f \|^2 + \| (A^* - \lambda)^{-1} f \|^2 \right)|d\lambda| < \infty \quad \text{for all } f \in H.$$  

Here, the kernel of $(-\Delta(A, B) - k^2)^{-1}$ is given below by (5.1) and the kernel of $(-\Delta(A, B)^* - k^2)^{-1}$ is given by the adjoint kernel $r_{A,B}(y, x; -k)^*$. Analogously to Lemma 5.1 one can estimate the resolvent norm away from the singularities of $\mathcal{G}(k; A, B)$ and $\mathcal{G}(-k; A, B)^*$. These singularities are finitely many and have by assumption a finite, strictly positive distance to the imaginary axis. In particular, the estimate in Lemma 5.1 implies sectoriality in sectors with vertex zero

$$\|(-\Delta(A, B) - \lambda)^{-1}\| \leq \frac{C}{|\lambda|} \quad \text{for } \lambda \in \Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\text{arg } z| > \theta\}, \quad \theta \in (0, \pi/2),$$
and an analogous estimate holds for $\| (\Delta(A, B)^* - \lambda)^{-1}\|$. Therefore,

$$
\sup_{\delta > 0} \delta \int_{\delta-i\infty}^{\delta+i\infty} \left( \| (\Delta(A, B) - \lambda)^{-1} f \|^2 + \| (\Delta(A, B)^* - \lambda)^{-1} f \|^2 \right) |d\lambda| \\
\leq C \sup_{\delta > 0} \delta \int_{\delta-i\infty}^{\delta+i\infty} \frac{1}{|\delta + i\lambda|^2} |d\lambda| \leq C \sup_{\delta > 0} \delta \int_{|\lambda|>\delta} \frac{1}{|\lambda|^2} |d\lambda| = 2C < \infty.
$$

This completes the proof. □

The generator property is traced back to the uniform boundedness of the Cayley transform $k \mapsto \mathcal{G}(k; A, B)$ outside a compact set containing its poles, where for irregular boundary conditions one might set $\mathcal{G}(k; A, B) = \infty$. Some cases for the possible behavior of the Cayley transform are illustrated in the following examples.

**Example 3.4** (Boundary conditions defining operators associated with sectorial forms). Let

$$
A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$

for any $A_{11}, A_{22} \in \mathbb{C}$: the boundary conditions $A, B$ correspond to $A \overline{\psi} + B \overline{\psi}' = 0$, i.e.,

$$
A_{11} \psi_1(0) + \psi_1'(0) = 0, \quad A_{21} \psi_1(0) + \psi_2'(0) = 0.
$$

Then $A, B$ are regular since $\det(A + ikB) = ik(\overline{A_{11}} + ik) \neq 0$ for $k \notin \{0, iA_{11}\}$. The Cayley transform

$$
\mathcal{G}(k; A, B) = \begin{bmatrix} -(A_{11} + ik)^{-1}(A_{11} - ik) & 0 \\ (ik)^{-1}A_{12}[(A_{11} + ik)^{-1}(A_{11} - ik) - 1] & 1 \end{bmatrix}
$$

is uniformly bounded away from its singularity $\{0, iA_{11}\}$, where $0$ is in fact a removable singularity. Since $\dim \ker B = 0$, by Theorem 3.4 $\Delta(A, B)$ generates an analytic semigroup; if $\text{Im} iA_{11} > 0$, then $A_{11}^2$ is a (simple, by Remark 3.3) eigenvalue of $\Delta(A, B)$, and

$$
\sigma_{\text{ess}}(-\Delta(A, B)) = [0, \infty).
$$

Note that $-\Delta(A, B)$ is associated with the sesquilinear form defined by

$$
\delta_{A, B}[\psi] = \|\psi\|^2_{L^2} - \langle A\overline{\psi}, \psi \rangle_{\mathbb{C}^2}, \quad \psi \in H^1([0, \infty)) \oplus H^1([0, \infty))
$$

and hence sectorial, in particular, the semigroup generated by $\Delta(A, B)$ is contractive if the numerical range of $A$ is contained in the left halfplane: this is the case if and only if $A_{21} = 0$. We will refer to boundary conditions of this type as $m$-sectorial.

In a more general setting the question if $-\Delta(A, B)$ is associated with a form of the type given in (3.3) is discussed in [Hus14]. The following is a prominent example from the theory of $\mathcal{PT}$-symmetric operators, and it is discussed for instance in [HKS15, Example 3.5] and also in the references given there.

**Example 3.5** (Boundary conditions defining operators not associated with sectorial forms). Consider

$$
A_\tau = \begin{bmatrix} 1 & -e^{i\tau} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_\tau = \begin{bmatrix} 0 & 0 \\ 1 & e^{-i\tau} \end{bmatrix}, \quad \tau \in [0, \pi/2),
$$

\[\]
leading to the boundary conditions

\[ \psi_1(0) = e^{i\tau} \psi_2(0), \quad \psi'_1(0) = -e^{-i\tau} \psi'_2(0). \]

Here, \( \det(A_\tau + ikB_\tau) = 2ki \cos \tau \neq 0 \) and hence by Theorem 3.1 \( \Delta(A_\tau, B_\tau) \) has no eigenvalues. Integration by parts gives

\[ \langle -\Delta(A_\tau, B_\tau) \psi, \psi \rangle_{L^2} = \| \psi' \|^2_{L^2} + (1 - e^{2i\tau}) \psi_2(0) \overline{\psi'_2(0)}, \quad \psi \in D(\Delta(A_\tau, B_\tau)). \]

The trace of the derivative cannot be balanced by \( \| \psi' \|^2_{L^2} \), hence in particular \( \psi \mapsto \langle -\Delta(A_\tau, B_\tau) \psi, \psi \rangle_{L^2} \) does not define a closed sesquilinear form: indeed, the numerical range of this form is the entire complex plane. Nevertheless, \( \Delta(A_\tau, B_\tau) \) does generate an analytic semigroup, as in fact \( \Delta(A_\tau, B_\tau) \) is similar to the one-dimensional Laplacian on \( \mathbb{R} \). Observe that because \( \Delta(A_\tau, B_\tau) \) is not dissipative, the semigroup it generates cannot be contractive; it is bounded, though, due to its similarity with the Gaussian semigroup on \( \mathbb{R} \). Observe that \( A_\tau, B_\tau \) are irregular boundary conditions for \( \tau = \frac{\pi}{2} \).

The following two examples are slight modifications of cases discussed in [DS71, Section XIX.6, page 2373].

**Example 3.6** (Intermediate boundary conditions). Consider

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \]

and the boundary conditions \( A\psi + B\psi' = 0 \), i.e.,

\[ \psi_1(0) = 0, \quad \psi'_1(0) = \psi_2(0). \]

Then \( \det(A + ikB) = 1 \) for all \( k \in \mathbb{C} \), i.e., \( A, B \) are regular. Furthermore, \( \dim \ker A = 0 \) and \( \dim \ker B = 1 \), but \( P^\perp B P^\perp = 0 \) and \( PBP^\perp = -1 \neq 0 \) and

\[ \mathcal{G}(k; A, B) = -\begin{bmatrix} 1 & 0 \\ 2ik & 1 \end{bmatrix}, \quad k \in \mathbb{C}. \]

We conclude from Theorem 3.1 that \( \Delta(A, B) \) does not generate an analytic semigroup on \( L^2([0, \infty)) \oplus L^2([0, \infty)) \), although its (purely essential) spectrum is \( [0, \infty) \).

**Example 3.7** (Totally degenerate boundary conditions). Consider

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \]

Then \( \text{Rank}(A B) = 2 \), but \( \det(A + ikB) = 0 \) for any \( k \in \mathbb{C} \), and hence \( A, B \) are irregular.

4. **Cayley transforms**

In this section we are going to derive properties of the Cayley transform that are essential in the proof of Theorem 3.1.
4.1. **M-sectorial boundary conditions.** For regular boundary conditions the Cayley transform (1.2) is well-defined except for at most two \( k \in \mathbb{C} \). One important class of boundary conditions are related to quadratic forms.

**Definition 4.1.** Boundary conditions \( A, B \) are said to be *m-sectorial* if there exist \( L, P \in \mathbb{C}^{2 \times 2} \) such that \( P \) is an orthogonal projection, \( P^\perp = 1 - P \), and \( L = P^\perp LP^\perp \), and such that \( A = L + P \) and \( B = P^\perp \).

The reason for this name is that whenever \( A, B \) are m-sectorial boundary conditions, \(-\Delta(A, B)\) is associated with the sectorial sesquilinear form, cf. e.g. [Ouh05, Def. 1.7] for this notion.

\[(\delta_{P,L}[\psi]) = \|\psi'\|_{L^2}^2 - \langle LP^\perp \psi, P^\perp \psi \rangle_{L^2}, \psi \in \{\psi \in H^1([0, \infty)) \oplus H^1([0, \infty)) : P\psi = 0\};\]

\(-\Delta(A, B)\) is hence an m-sectorial operator and \( \Delta(A, B) \) generates an analytic semigroup. M-sectorial boundary conditions are in particular regular since \( A + ikB = \begin{bmatrix} L + ikP^\perp & 0 \\ 0 & P \end{bmatrix} \) is invertible for \( k > \|L\| \).

The Cayley transform can be estimated as follows.

We first consider the case \( \dim \ker B = 1 \): then \( \dim \ran L \leq 1 \), and with respect to \( \ran P \) and \( \ran P^\perp \) one obtains a block decomposition

\[ A \pm ikB = \begin{bmatrix} L \pm ik1 & 0 \\ 0 & \pm ik1 \end{bmatrix} \quad \text{and} \quad \mathcal{G}(k; A, B) = \begin{bmatrix} -(L + ik1)^{-1}(L - ik1) & 0 \\ 0 & 1 \end{bmatrix}. \]

For \( A \) invertible and \( B = 1 \) one has

\[ \mathcal{G}(k; A, B) = -(A + ik1)^{-1}(A - ik1) = -(A/ik + 1)^{-1}(A/ik - 1), \]

and hence

\[ \|\mathcal{G}(k; A, B)\| \leq \frac{2}{1 - \frac{\|A\|}{|k|}} \quad \text{for } |k| > \|A\|. \]

Therefore, \( \mathcal{G}(k; A, B) \) is uniformly bounded away from its poles, i.e., outside a compact set.

If \( B = 1 \) and \( \dim \ran L = 1 \), one obtains a block decomposition with respect to \( \ker L \) and \( (\ker L)^\perp \)

\[ A \pm ikB = \begin{bmatrix} L_{11} \pm ik1 & 0 \\ L_{12} & \pm ik1 \end{bmatrix}, \]

and hence

\[ \mathcal{G}(k; A, B) = \begin{bmatrix} -(L_{11} + ik1)^{-1} & 0 \\ -(ik)^{-1}L_{12}(L_{11} + ik1)^{-1} + (ik)^{-1}1 & L_{11} - ik1 \\ -(L_{11} + ik1)^{-1}(L_{11} - ik1) & 0 \\ (ik)^{-1}L_{12}[(L_{11} + ik1)^{-1}(L_{11} - ik1) - 1] & 1 \end{bmatrix}. \]
Similarly to the case of $A$ invertible and $B = 1$, using (4.2) one can show that $\mathcal{S}(k; A, B)$ is uniformly bounded away from its poles for general $m$-sectorial boundary conditions. This is summarized in the following.

**Lemma 4.2.** Let $A, B$ define $m$-sectorial boundary conditions. Then $\mathcal{S}(k; A, B)$ is uniformly bounded outside a compact set.

Depending now on the dimension of $\ker A$ and $\ker B$ one can distinguish the following cases listed in Table 1, where the cases $\dim \ker A = 1$, $\dim \ker B = 2$, and $\dim \ker A = 2$, $\dim \ker B = 1$ collide with the rank condition, and hence are excluded. We have already remarked that for $\text{rank}(AB) \neq 2$ one has $\sigma(\Delta(A, B)) = \mathbb{C}$.

| $\dim \ker A$ | $\dim \ker B$ | equiv. b.c. | $-\Delta(A, B)$ |
|---------------|---------------|-------------|----------------|
| 0             | 0             | $A' = B^{-1}A$ and $B' = 1$ | m-sectorial    |
| 1             | 0             | $A' = B^{-1}A$ and $B' = 1$ | m-sectorial    |
| 2             | 0             | $A' = B^{-1}A$ and $B' = 1$ | m-sectorial    |
| 0             | 2             | $A = 1$ and $B = 0$          | m-sectorial    |
| 0             | 1             | some block representation    | regular        |
| 1             | 1             | some block representation    | regular or irregular |

**Table 1.** Different cases of boundary conditions

4.2. The case $\dim \ker A = 0$ and $\dim \ker B = 1$.

**Lemma 4.3.** Let $\dim \ker A = 0$ and $\dim \ker B = 1$. Then $\text{rank}(AB) = 2$, equivalent boundary conditions are given by

$$A' = 1 \quad \text{and} \quad B' = A^{-1}B,$$

and the boundary conditions $A, B$ are regular.

*Proof.* First, since $A$ is invertible, its columns are linearly independent and therefore $\text{rank}(AB) \geq 2$, and $A', B'$ define equivalent boundary conditions. Furthermore, since $\ker A \cap \ker B = \{0\}$ these boundary conditions are regular. □

Let $P$ be the orthogonal projection onto $\ker B$ and $P^\perp = 1 - P$, then without loss of generality, consider

$$A = 1 \quad \text{and} \quad B = \begin{bmatrix} P^\perp BP^\perp & 0 \\ PBP^\perp & 0 \end{bmatrix}.$$

**Lemma 4.4.** Let $\dim \ker A = 0$ and $\dim \ker B = 1$. Then the Cayley transform $\mathcal{S}(\cdot; A, B)$ is uniformly bounded outside a compact set containing its only possible pole if and only if $P^\perp BP^\perp \neq 0$. If $P^\perp BP^\perp = 0$, and hence $PBP^\perp \neq 0$, then $\|\mathcal{S}(k; A, B)\| = O(|k|)$ for $|k| \to \infty$. 
The Cayley transform is then

\[(A \pm ikB) = \begin{bmatrix} 1 & \pm ikB_{11} \\ \pm ikB_{21} & 1 \end{bmatrix}, \quad (A \pm ikB)^{-1} = \frac{1}{1 \pm ikB_{11}} \begin{bmatrix} 1 & 0 \\ \mp ikB_{21} & 1 \pm ik1 \end{bmatrix}.\]

The Cayley transform is then

\[\mathcal{G}(k; A, B) = -(A + ikB)^{-1}(A - ikB) = -\frac{1}{1 + ikB_{11}} \begin{bmatrix} 1 - ikB_{11} & 0 \\ -2ikB_{21} & 1 \end{bmatrix}.\]

For \(B_{11} \neq 0\) this is uniformly bounded away from the pole \(k = i/B_{11}\). For \(B_{11} = 0\) there are no poles, and \(\dim \ker B = 1\) implies that \(B_{21} \neq 0\). In this case \(\|\mathcal{G}(k; A, B)\| = O(|k|)\) for \(|k| \to \infty\).

4.3. The case \(\dim \ker A = 1\) and \(\dim \ker B = 1\). In this subsection we focus on the case of \(\dim \ker A = \dim \ker B = 1\). Denote by \(Q^\perp\) the orthogonal projection on \(\text{Ran} A\), \(Q = I - Q^\perp\), and as before \(P\) the orthogonal projection on \(\ker B\), where each \(Q, Q^\perp\) and \(P, P^\perp\) have one-dimensional range. Then

\[(4.3) \quad A = \begin{bmatrix} Q^\perp AP^\perp & Q^\perp AP \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} Q^\perp BP^\perp & 0 \\ QBP^\perp & 0 \end{bmatrix}.\]

**Lemma 4.5.** Let \(\dim \ker A = \dim \ker B = 1\). Then \(\text{Rank}(A \ B) = 2\) if and only if \(QBP^\perp \neq 0\). The boundary conditions \(A, B\) are irregular if and only if \(\ker A = \ker B\), i.e. if \(Q^\perp AP = 0\).

**Proof.** From (4.3) one deduces \((A \ B)\) is surjective if and only if \(QBP^\perp \neq 0\). Recall that \(A, B\) are irregular if and only if \(\ker A = \ker B\), see [HKS15, Prop. 3.3], and here (4.3) implies that \(\ker A = \ker B\) if and only if \(Q^\perp AP = 0\).

**Lemma 4.6.** Let \(A, B\) define regular boundary conditions with \(\dim \ker A = \dim \ker B = 1\). Then the Cayley transform \(\mathcal{G}(\cdot; A, B)\) has one possible pole, and away from this \(\mathcal{G}(\cdot; A, B)\) is uniformly bounded.

**Proof.** Note that \(\text{Ran} P^\perp = \text{span}\{p_1\}\) and \(\text{Ran} P = \text{span}\{p_2\}\), where \(\{p_1, p_2\}\) is an orthonormal basis of \(\mathbb{C}^2\). For \(A, B\), regular, in this basis, equivalent boundary conditions are

\[A = \begin{bmatrix} A_{11} & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & 0 \\ 1 & 0 \end{bmatrix}\]

since

Let \(\text{Ran} P = \text{span}\{p_1\}\), \(\text{Ran} P^\perp = \text{span}\{p_2\}\), and \(\text{Ran} Q = \text{span}\{q_1\}\), \(\text{Ran} Q^\perp = \text{span}\{q_2\}\) where \(\{p_1, p_2\}\) and \(\{q_1, q_2\}\) are orthonormal basis of \(\mathbb{C}^2\). Now, a coordinate change from \(\{q_1, q_2\}\) to \(\{p_1, p_2\}\) is given by a unitary \(U\), and hence equivalent boundary conditions \(UA\) and \(UB\) can be written in the basis \(\{p_1, p_2\}\) as

\[A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & 0 \end{bmatrix}.\]
By Lemma 4.5, one has $QBP \perp \neq 0$ and $Q^\perp AP \neq 0$ and hence $B_{21} \neq 0$ and $A_{12} \neq 0$. Therefore equivalent boundary conditions are

$$A = \begin{bmatrix} A_{11} & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & 0 \\ 1 & 0 \end{bmatrix}.$$  

Hence

$$(A \pm ikB) = \begin{bmatrix} A \pm ikB_{11} & 1 \\ 0 & \pm ik \end{bmatrix}, (A \pm ikB)^{-1} = \frac{1}{\pm ik(A_{11} \pm ikB_{11})} \begin{bmatrix} \pm ik & -1 \\ 0 & A_{11} \pm ikB_{11} \end{bmatrix},$$

and

$$\mathcal{S}(k; A, B) = \begin{bmatrix} -\frac{A_{11} - ikB_{11}}{A_{11} + ikB_{11}} & \frac{-2}{A_{11} + ikB_{11}} \\ 0 & -1 \end{bmatrix}.$$  

This is uniformly bounded away from the only possible pole at $k = ik/B_{11}$. □

Our findings are summarized in Table 2, where as before $P$ is the orthogonal projection onto Ker $B$ and $P^\perp = 1 - P$, and uniformly bounded refers to the Cayley transform away from its poles.

| dim Ker $A$ | dim Ker $B$ | Condition | Cayley transform | Ref. |
|-------------|-------------|-----------|-----------------|-----|
| 0           | 0           | none      | uniformly bounded | Lemma 4.2 |
| 1           | 0           | none      | uniformly bounded | Lemma 4.2 |
| 0           | 2           | none      | uniformly bounded | Lemma 4.2 |
| 0           | 1           | $P^\perp BP^\perp \neq 0$ | uniformly bounded | Lemma 4.4 |
| 0           | 1           | $P^\perp BP^\perp = 0$ | $\|\mathcal{S}(k; A, B)\| = O(|k|)$ | Lemma 4.4 |
| 1           | 1           | Ker $A \neq$ Ker $B$ | uniformly bounded | Lemma 4.6 |
| 1           | 1           | Ker $A =$ Ker $B$ | $\mathcal{S}(k; A, B) = \infty$ | Lemma 4.6 |

$A, B$ irregular

| Table 2. Cayley transforms |

5. Resolvent estimates

The keystone of our analysis is that for regular boundary conditions the resolvent

$$(-\Delta(A, B) - k^2)^{-1}$$

is an integral operator, i.e.,

$$(-\Delta(A, B) - k^2)^{-1} f(x) = \int_{[0, \infty) \times \{1, 2\}} r_{A, B}(x, y; k) f(y) \, dy,$$

where

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in L^2([0, \infty)) \oplus L^2([0, \infty)), \ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in [0, \infty) \times \{1, 2\},$$
with kernel
\[
(5.1) \quad r_{A,B}(x, y; k) = \frac{i}{2k} \left\{ \begin{bmatrix} e^{ik|x_1-y_1|} & 0 \\ 0 & e^{ik|x_2-y_2|} \end{bmatrix} + \begin{bmatrix} e^{ikx_1} & 0 \\ 0 & e^{ikx_2} \end{bmatrix} \mathcal{G}(k; A, B) \begin{bmatrix} e^{iky_1} & 0 \\ 0 & e^{iky_2} \end{bmatrix} \right\}. 
\]
whenever \( k \in \mathbb{C} \) such that \( \text{Im} \ k > 0 \) and \( A + ikB \) is invertible, cf. [HKSI15, Prop. 4.7]. We stress that the first addend on the right hand side corresponds to the kernel of the Laplacian on \( \mathbb{R} \) without any point interactions; the second addend can be thus interpreted as a correcting term that mirrors the influence of the point interactions. It is also remarkable that the kernel is bounded and jointly uniformly continuous on \( \mathbb{R} \times \mathbb{R} \), regardless of \( A, B \); in particular, it extends to a bounded linear operator from \( L^1 \) to \( L^\infty \).

Lemma 5.1 (Estimate for uniformly bounded Cayley transform). Let the boundary conditions \( A, B \) be regular and such that \( k \mapsto \mathcal{G}(k; A, B) \) is uniformly bounded away from its poles. Then there exists \( C > 0 \) such that
\[
(5.2) \quad \|(-\Delta(A, B) - k^2)^{-1}\| \leq \frac{1}{\text{dist}(k^2, [0, \infty))} + \frac{C}{|k||\text{Im} k|\text{dist}(S, k)},
\]
where \( \text{Im} k > 0 \) with \( \det(A + ikB) \neq 0 \) and
\[
S = \{ s \in \mathbb{C} : \text{Im} s > 0 \text{ or } s \in [0, \infty), \text{ and } s \text{ non-removable singularity of } \mathcal{G}(k; A, B) \}. 
\]
Proof. By using (5.1) we obtain outside the poles of \( \mathcal{G}(k; A, B) \) the estimate
\[
\|(-\Delta(A, B) - k^2)^{-1}f\| \leq \frac{1}{|k|^2}\|f\| + \frac{1}{|k|}\|\mathcal{G}(k; A, B)\| \cdot \|f\| \cdot \|e^{ik}\|^2.
\]
The first term follows from the standard resolvent estimate for the Laplacian on \( \mathbb{R} \) with no point interactions, while for the second one we have used the product form of the kernel, and moreover \( \|e^{ik}\|^2 = 1/(2\text{Im} k) \). Note that non-removable singularities of \( \mathcal{G}(k; A, B) \) are poles of order one and hence, \( \|\mathcal{G}(k; A, B)\| \leq C\text{dist}(S, k) \). □

In the case of \( m \)-sectorial boundary conditions, a stronger estimate holds.

Lemma 5.2 (Resolvent estimate for \( m \)-sectorial boundary conditions). Let the boundary conditions \( A, B \) be \( m \)-sectorial. Then there exist \( C > 0 \) and \( \omega \geq 0 \) such that
\[
(5.3) \quad \|\lambda(\Delta(A, B) - \lambda^2)^{-1}\| \leq \frac{C}{\text{Re} \lambda - \omega}
\]
all \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda > \omega \); in particular, the spectrum of \( \Delta(A, B) \) is contained in a parabola centered on the real axis and contained in a left half-plane.

Proof. The proof is based on a result due to Lions: If a bounded, \( H \)-elliptic sesquilinear form \( a \) with form domain \( V \) satisfies the additional condition
\[
(5.4) \quad |\text{Im} a(u, u)| \leq M\|u\|_V\|u\|_H \quad \text{for all } u \in V,
\]
then the associated operator \( A \) generates a cosine operator function on \( H \) with associated Kisyński space \( V \) and its resolvent satisfies an estimate corresponding to (5.3), see e.g. [ABHN01, § 3.14] and [Mug14, § 6.2]. Hence, it suffices to observe that (5.4) is
satisfied by the form $\delta_{A,B}$ defined in (4.1): the proof of this fact is analogous to that of [Mug14, Lemma 6.63].

The only cases where the Cayley transform is not uniformly bounded have been discussed in Lemma 4.4.

**Lemma 5.3** (Estimate for the other cases). Let $\dim \ker A = 0$ and $\dim \ker B = 1$ and let $P^\perp B P^\perp = 0$. Then $\sigma_p(-\Delta(A, B)) = \emptyset$ and for some $c > 0$

$$\|(-\Delta(A, B) + \kappa^2)^{-1}\| \geq \frac{c}{\kappa^{3/2}} \text{ as } \kappa \to \infty.$$ 

In particular $\Delta(A, B)$ is not a generator of a $C_0$-semigroup.

**Proof.** As in [HKS15, Section 6] one can show that $\Delta(A, B)$ is unitarily equivalent to $\Delta(A', B')$ with

$$\mathcal{G}(k; A', B') = \begin{bmatrix} -1 & 0 \\ 2ikB_{21} & 1 \end{bmatrix}, \quad B_{21} \neq 0,$$

and there are no eigenvalues nor poles of $\mathcal{G}(k; A', B')$. Consider the function $u = (u_1, u_2)^T = (\chi_{[0,1]}, 0)^T$, where $\chi_{[0,1]}$ denotes the characteristic function of the unit interval. Then

$$(-\Delta(A, B) - k^2)^{-1} u = \frac{i}{2k} \left\{ \int_0^1 e^{ik|x_1-y_1|} dy_1 + \int_0^1 e^{ikx_1} e^{ikx_2} dy_1 \right\},$$

and estimating the second component only

$$\|(-\Delta(A, B) - k^2)^{-1} u\| \geq |B_{12}| \cdot \left| \int_0^1 e^{iky_1} dy_1 \right| \cdot \|e^{ikx_2}\| = |B_{12}| \frac{|e^{ik} - 1|}{|k|} \left( 2\text{Im } k \right)^{1/2}.$$ 

In particular for $k = i\kappa$, $\kappa > 0$,

$$\frac{|e^{-\kappa} - 1|}{\sqrt{2}|\kappa|^{3/2}} \to \frac{1}{\sqrt{2}|\kappa|^{3/2}} \text{ as } \kappa \to \infty,$$

and therefore

$$\|(-\Delta(A, B) - \kappa^2)^{-1}\| \geq \mathcal{O}(\kappa^{-3/2}).$$

In particular, assume that $\Delta(A, B)$ is the generator of a $C_0$-semigroup, then

$$\mathcal{O}(\kappa^{-3/2}) \leq \|\Delta(A, B) - \kappa^2\| \leq M/(\kappa^2 - \omega), \quad \text{for } \omega > 0 \text{ and } M > 0,$$

which multiplying by $\kappa^2$ and passing to the limit $\kappa \to \infty$ leads to a contradiction. Recall that $\Delta(A, B)$ is closed and densely defined. □
6. Invariance properties

Several issues in the qualitative analysis of semigroups associated with sesquilinear forms are made particularly easy by variational methods. In particular, the classical Beurling–Deny criteria have been generalized in [Ohn96; based upon this general criterion, invariance properties for heat equations on metric graphs have been obtained in [CM09]. We can paraphrase [CM09, Prop. 4.3] (see also [Mug14, Thms. 6.71 and 6.72] and obtain the following: given a closed convex subset $C$ of $\mathbb{C}^2$, we denote by $\mathcal{C}$ the induced closed convex subset of $L^2([0, \infty)) \oplus L^2([0, \infty))$ defined by

$$\mathcal{C} := \{ f \in L^2([0, \infty)) \oplus L^2([0, \infty)) : f(x) \in C \text{ for a.e. } x \in [0, \infty) \cup [0, \infty) \}$$

**Proposition 6.1.** Let the boundary conditions $A, B$ be $m$-sectorial. Let $C$ be a closed convex subset of $\mathbb{C}^2$ with $(0, 0) \in C$.

Then the semigroup generated by $\Delta(A, B)$ leaves $\mathcal{C}$ invariant if and only if both the projection onto $\text{Ran } B$ and the semigroup generated by $A + B - 1$ leave $C$ invariant.

The power of our approach lies in the possibility of the explicit representation (5.1) of the resolvent kernel. From this, some semigroup properties can be derived even for boundary conditions that are not $m$-sectorial, when Ouhabaz’ variational methods are not available.

**Lemma 6.2.** Let the boundary conditions be regular and $\Delta(A, B)$ generate a contractive $C_0$-semigroup. Let $C$ be a closed convex subset of $\mathbb{C}^2$. Then the semigroup generated by $\Delta(A, B)$ leaves $\mathcal{C}$ invariant provided

$$\frac{\kappa}{2} \begin{bmatrix} e^{-\kappa|x_1-y_1|} + e^{-\kappa(x_1+y_1)} & e^{-\kappa(x_1+y_2)} \sigma_{11}(i\kappa) & e^{-\kappa(x_2+y_1)} \sigma_{12}(i\kappa) \\ e^{-\kappa(x_2+y_1)} \sigma_{21}(i\kappa) & e^{-\kappa|x_2-y_2|} + e^{-\kappa(x_1+y_2)} \sigma_{22}(i\kappa) \end{bmatrix}$$

leaves $C$ invariant for all $\kappa > 0$ and all $x_i, y_j \in [0, \infty)$, where $\sigma_{ij}(i\kappa) = (\mathcal{S}(i\kappa; A, B))_{ij}, 1 \leq i, j \leq 2$. In particular, the semigroup generated by $\Delta(A, B)$ is $L^\infty$-contractive provided $1 + \mathcal{S}(i\kappa; A, B)$ leaves $\{ \xi \in \mathbb{C}^2 : |\xi_1| + |\xi_2| \leq 1 \}$ invariant for all $\kappa > 0$.

**Proof.** It is well-known that the semigroup leaves $\mathcal{C}$ invariant if and only if so does $\lambda(\lambda - \Delta(A, B))^{-1}$ for all $\lambda > 0$, see [Ohn96, Prop. 2.3]. By (5.1), $\kappa^2 r_{A,B}(x, y; i\kappa)$ is the kernel of $\lambda(\lambda - \Delta(A, B))^{-1}$ for $\lambda = -(i\kappa)^2$; a direct computation shows that for all $\mu \in \mathbb{R}$

\begin{equation}
\mu^2 r_{A,B}(x, y; i\kappa) = \frac{\mu^2}{2\kappa} \begin{bmatrix} e^{-\kappa|x_1-y_1|} + e^{-\kappa(x_1+y_1)} & e^{-\kappa(x_1+y_2)} \sigma_{11}(i\kappa) \\ e^{-\kappa(x_2+y_1)} \sigma_{21}(i\kappa) & e^{-\kappa|x_2-y_2|} + e^{-\kappa(x_1+y_2)} \sigma_{22}(i\kappa) \end{bmatrix}
\end{equation}

and the main claim now follows from closedness of $\mathcal{C}$, taking $\mu = \kappa$. \hfill \Box

Finally, in order to prove the assertion about $L^\infty$-contractivity, it suffices to observe that the matrix in (6.1) is in absolute value no larger than $1 + \mathcal{S}(i\kappa; A, B)$.

If $\mathcal{C}$ is the positive cone, then the assertion can be sharpened as follows.

**Corollary 6.3.** Let the boundary conditions be regular and $\Delta(A, B)$ generate a quasi-contractive $C_0$-semigroup. Then the semigroup generated by $\Delta(A, B)$ is real if and only if $\mathcal{S}(i\kappa; A, B)$ has real entries; in this case, the semigroup is additionally positive whenever $1 + \mathcal{S}(i\kappa; A, B)$ leaves $\{ \xi \in \mathbb{R}^2 : \xi_1 \geq 0, \xi_2 \geq 0 \}$ invariant for some $\kappa_0$ and all $\kappa \geq \kappa_0$. \hfill \Box
Proof. Positivity and reality of a semigroup is unaffected by scalar (real) perturbations of its generator. Furthermore, reality (resp., positivity) of a positive operator is equivalent to reality (resp., positivity) of its kernel, cf. [MN11, Thm. 5.2]. Finally, (6.1) shows that the entries of the resolvent’s kernel at $i\kappa$ are real if and only if so are the entries of $\mathcal{G}(i\kappa; A, B)$. The proof of the positivity follows essentially the proof of [KS06, Thm. 4.6] although there only self-adjoint boundary conditions are considered: we only need to observe that if $1 + \mathcal{G}(i\kappa; A, B)$ has real and positive entries, then for $\mu = \kappa$ the matrix in (6.1) is entry-wise no smaller than

$$
\begin{align*}
\frac{\kappa}{2} \begin{bmatrix}
    e^{-\kappa(x_1+y_1)}(1 + \sigma_{11}(i\kappa)) & e^{-\kappa(x_1+y_2)}\sigma_{12}(i\kappa) \\
    e^{-\kappa(x_2+y_1)}\sigma_{21}(i\kappa) & e^{-\kappa(x_2+y_2)}(1 + \sigma_{22}(i\kappa))
\end{bmatrix},
\end{align*}
$$

whence the claim follows. □

Example 6.4. By Proposition 6.1, the boundary conditions in Example 3.4 define a semigroup that leaves invariant $C$ if and only if the semigroup generated by $A$ leaves $C$ invariant: e.g., the semigroup generated by $\Delta(A, B)$ is positive if and only if $A_{11}$ is real and $A_{21} \geq 0$. That this is a sufficient condition can be deduced from Corollary 6.3 too. Furthermore, the semigroup is $L^\infty$-contractive (and in this case automatically $L^p$-contractive for all $p \in [1, \infty]$) if and only if $\text{Re} A_{11} \leq 0$ and $A_{12} = 0$, cf. [Mug07, Lemma 6.1]. (Observe that the latter condition induces a decoupling of our system, as we are left with two Laplacians on $[0, \infty)$ with Neumann and Robin boundary conditions, respectively.)

Remark 6.5. One criterion for contractivity of the semigroup is that $A, B$ are $m$-sectorial with $\text{Re} L \leq 0$, see [KPS08, Thm. 2.4], or equivalently $\text{Re} AB^* \leq 0$, but as we know the case of $m$-sectorial boundary conditions can be treated more directly by Proposition 6.1 without invoking Lemma 6.2. Example 3.3 shows that non-(quasi-)contractive semigroups can actually arise; we note in passing that in the setting of Example 3.3

$$
\begin{align*}
\mathcal{G}(i\kappa; A\tau, B\tau) &= \frac{1}{\cos \tau} \begin{bmatrix}
    i \sin \tau & 1 \\
    1 & -i \sin \tau
\end{bmatrix}, \quad \kappa > 0,
\end{align*}
$$

(see [HKST15, Example 3.5]), i.e., $1 + \mathcal{G}(i\kappa; A, B)$ does not leave either the positive cone of $\mathbb{R}^2$ or the unit ball of $\ell^\infty \times \ell^\infty$ invariant.

An interesting consequence of Theorem 3.1(a) is that if there is a simple pole $k_0$ of $k \mapsto (A + ikB)^{-1}$ with $\text{Im} k_0 > |\text{Re} k_0|$, then the peripheral spectrum of $\Delta(A, B)$ is finite and consists of simple poles of the resolvent. This paves the way to study semigroups that are merely asymptotically positive; i.e., those semigroups whose orbits starting at positive initial data tend to the lattice’s positive cone, see [DGK16, Def. 8.1].

Proposition 6.6. Let $\Delta(A, B)$ generate a $C_0$-semigroup. Assume the zero $k$ of $\{k : \text{Im} k > 0\} \ni k \mapsto \det(A + ikB) \in \mathbb{C}$ of larger magnitude lies on $i(0, \infty)$, i.e., $k = i\kappa_0$ for some $\kappa_0 > 0$, and let $A \neq \kappa_0 B$. Consider the following assertions:

(i) the semigroup generated by $\Delta(A, B)$ is asymptotically positive;
(ii) the spectral projection of $\Delta(A, B)$ associated with $\kappa_0^2$ is positive;
(iii) the distance to the set $[0, \infty)$ of each entry of $(\kappa - \kappa_0)^2 \mathcal{S}(ik; A, B)$ tends to 0 as $\kappa \searrow \kappa_0$;
(iv) $\lim_{\kappa \searrow \kappa_0} \frac{(\kappa - \kappa_0)^2}{\det(A - \kappa B)} = 0$.

Then (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv).

We refer to [EN00, § IV.1] for the definition of spectral projections of possibly non-selfadjoint operators.

**Proof.** By Theorem [3.1] and Remark [3.3] the main assumptions imply that the peripheral spectrum contains precisely one eigenvalue, which is simple: hence the spectral bound is a dominant spectral value. We are thus in the position to apply [DGK16, Thm. 1.2]: in view of (6.1) we conclude that asymptotic positivity (in the sense of [DGK16, Def. 7.2]) of $\lambda \mapsto (\lambda - \Delta(A, B))^{-1}$ at $\lambda_0 := -(i\kappa_0)^2 = \kappa_0^2$ (which is in turn equivalent to (ii), by [DGK16, Thm. 7.6]) is equivalent to the condition that the distance to $[0, \infty)$ of each entry of

$$
(6.4) \quad \frac{(\kappa - \kappa_0)^2}{2\kappa} \begin{bmatrix}
-\kappa|x_1 - y_1| + \kappa(x_1 + y_1) \sigma_{11}(ik) \\
\kappa(x_2 + y_2) \sigma_{21}(ik) \\
-\kappa|x_2 - y_2| + \kappa(x_1 + y_2) \sigma_{22}(ik)
\end{bmatrix}
$$

tends to 0 as $\kappa \searrow \kappa_0$ for each $x_1, x_2, y_1, y_2 \in [0, \infty)$. Now, observe that if $K$ is a cone in a lattice $X$, $\delta \in (0, 1]$, and $a \in K$, then for any $b \in X$ $\text{dist}(b, K) \geq \text{dist}(a + \delta b, K)$; we conclude that the distance to the positive cone $C^2_+$ of the matrix in (6.4) is no larger than the distance to the same cone of the matrix $(\kappa - \kappa_0)^2 \mathcal{S}(ik; A, B)$, which proves that (i) is implied by (iii). To conclude the proof, it suffices to observe that the poles of $k \mapsto \mathcal{S}(ik; A, B)$ are the zeros of $k \mapsto \det(A + ikB)$, with equal multiplicity. $\square$

**Example 6.7.** Let us come back to the setting in Example [3.4]. We have already seen that if $A_{11} > 0$, then $A_{11}^2$ is a dominant eigenvalue of $\Delta(A, B)$ and we can hence apply Proposition [6.6] to (3.2) and conclude that the semigroup generated by $\Delta(A, B)$ is asymptotically positive for any $A_{11}, A_{12}$.

**Example 6.8.** Consider

$$
A = \begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
$$

Because $\det(A + ikB) = -k^2 - 1$, we obtain that $1 = -(i\kappa_0)^2$ with $\kappa_0 = 1$ is the only eigenvalue of $\Delta(A, B)$; and it is simple, since $A \neq B$. Accordingly, by Proposition [6.1] the semigroup generated by $\Delta(A, B)$ is not positive (nor it is $\ell^\infty$-contractive). Let us sharpen this assertion:

$$
\frac{(\kappa - \kappa_0)^2}{\det(A - \kappa B)} = \frac{(\kappa - 1)^2}{\kappa^2 - 1} = \frac{\kappa - 1}{\kappa + 1} \kappa \searrow_1 0,
$$

and we conclude from Proposition [6.6] that the semigroup generated by $\Delta(A, B)$ is asymptotically positive.
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