New stochastic equation for a harmonic oscillator:
Brownian motion with adhesion

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Abstract
In addition to the usually considered stochastic harmonic oscillator
with an external random force (Brownian motion) or with random
frequency and random damping, we consider an oscillator with a
random mass for which the particles of the surrounding medium
adhere to the oscillator for some (random) time after the collision,
thereby changing the oscillator mass. We have calculated the first
two moments and the Lyapunov exponent, which describes the
stability of the fixed point. This model can be useful for the analysis
of chemical and biological solutions as well as for
nano-technological devices.

A simple, general and widely used model is that of a harmonic oscillator of
unit mass [1], which is described by the following equation

\[
\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 x = 0
\] (1)

Thousands of articles have been devoted to applications of this model to
different phenomena in physics, chemistry, biology, economics, etc. However,
all phenomena in Nature are subject to random forces, and for the adequate
description of these phenomena, one has to supplement the dynamic equation
(1) with a random force [2]. This can be done in different ways.

1. Brownian motion

\[
\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 x = \xi(t)
\] (2)

This is universally known equation (with \(\omega^2 = 0\)) [3] describes the motion of
nano-particle subject to a systematic viscous force, \(2\gamma dx/dt\), and a random force
\(\xi(t)\), having its origin from non-equal (random) numbers of molecular collisions
from opposite sides and is responsible for the zigzag motion of a Brownian
particle. The \(\omega^2 x\) term in Eq. (2) describes a Brownian motion in an external
quadratic potential.
2. Random frequency.

\[ \frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 [1 + \xi (t)] x = 0 \]  

(3a)

The many applications of this model include different fields in physics, such as wave propagation in a random medium [4], spin precession in a random external field [5], turbulent flow on the ocean surface [6], as well as biology (population dynamics [7]), economics (stock market prices [8]) and so on.

3. Random damping

\[ \frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 x = 0 \]  

(4)

The many applications of this model include water waves influenced by a turbulent wind field [9], the Ginzburg-Landau equation with a convective term [10], mean flow passing through a region under study, such as the phase transitions under shear [11], open flows of liquids [12], dendritic growth [13], chemical waves [14], motion of vortices [15], etc.

In response to an external periodic field, the second moment of a harmonic oscillator with random frequency or random damping shows non-monotonic dependence on the noise strength and correlation length (stochastic resonance in linear systems [16]).

4. Here we study still another possibility for introducing randomness in the oscillator equation (1) by introducing a random mass, which is described by the following equation,

\[ [1 + \xi (t)] \frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 x = 0 \]  

(5)

or, for a concurrent action with the "Brownian" random force,

\[ [1 + \xi (t)] \frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 x = \eta (t) \]  

(6)

with

\[ \langle \eta (t_1) \eta (t_2) \rangle = D_1 \delta (t_2 - t_1) \]  

(7)

The latter equation describes Brownian motion with adhesion, which means that the molecules surrounding a Brownian particle not only collide with it, but also adhere to the Brownian particle for some random time interval. As a result of this adhesion, the mass of a Brownian particle becomes a random quantity. There are many biological or chemical solutions of particles of different sizes which perform random collisions and adhesion. A comprehensive list of references can be found in [17]. Recent applications of this model include a nano-mechanical resonator which randomly absorbs and desorbs molecules [18].

We restrict our consideration to colored noise \( \xi (t) \) with an exponential correlator,

\[ \langle \xi (t_1) \xi (t_2) \rangle = \sigma^2 \exp [-\lambda |t_2 - t_1|] \equiv D \lambda \exp [-\lambda |t_2 - t_1|] \]  

(8)
where the random variable $\xi(t)\) may take the values $\pm\sigma$, so that $\langle \xi^2(t) \rangle = \sigma^2$, with mean waiting time $(\lambda/2)^{-1}$ in each of these two states. This noise is called dichotomous noise (random telegraph signal). The transition to white noise occurs when $\lambda \to \infty$.

Since an arbitrary function of dichotomous noise is linear, one can write

$$1 + \xi(t) = \frac{1 - \sigma^2}{1 - \xi(t)},$$

which transforms Eq. (6) into the following form

$$(1 - \sigma^2) \frac{d^2x}{dt^2} + (1 - \xi(t)) \left(2\gamma \frac{dx}{dt} + \omega^2\right) x = 0 \quad (10)$$

Contrary to all other types of random forces in Eq. (1), the strength of the mass fluctuations is restricted. Due to the positivity of the oscillator mass, the condition $\xi^2 < 1$, and hence $\sigma^2 < 1$, is satisfied in Eq. (10). For the asymptotic case of small oscillations of the mass, $\sigma^2 \ll 1$, this equation can be rewritten as follows

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 x = \xi(t) \left(2\gamma \frac{dx}{dt} + \omega^2\right) x \quad (11)$$

Therefore, small dichotomous fluctuations of mass are equivalent to simultaneous fluctuations of frequency and damping.

In order to find the first moment $\langle x \rangle$, described by the second-order differential equation (6), it is convenient to rewrite this equation in the form of two first-order differential equations,

$$\frac{dx}{dt} = y; \quad \frac{dy}{dt} = -\xi \frac{dy}{dt} - 2\gamma y - \omega^2 x \quad (12)$$

Multiplying the last equation in (12) by $\xi$, averaging the original and obtained equations and using the Shapiro-Loginov splitting procedure [19], one obtains [20] a fourth-order differential equation for $\langle x \rangle$. The use of dichotomous noise with $\langle \xi(t)^2 \rangle = \sigma^2$ allows us to split the hierarchy of higher-order correlation functions and get the expression of $\langle x \rangle$. In the case of white noise, one can compare the results for different types of noise. For additive noise, there is no change to the deterministic case. For the case of random frequency, one obtains a slight change (replacing $\omega$ by a renormalized frequency $\sqrt{\omega^2 - \gamma^2}$). For the case of random damping, the damping coefficient $\gamma$ is replaced by $\gamma(1 - 2\gamma D)$, and, finally, for the case of random mass, $\omega$ and $\gamma$ are replaced by $\omega\sqrt{1 - 2\gamma D}$ and $\gamma(1 - 2\gamma D)$, respectively.

In a similar way, one can find [20] the second moments $\langle x^2 \rangle$ associated with Eq. (6), which is transformed to six equations in six variables, $\langle x^2 \rangle, \langle y^2 \rangle, \langle xy \rangle, \langle \xi x \rangle, \langle \xi y \rangle$ and $\langle \xi xy \rangle$. We bring here only the results for $\langle x^2 \rangle$, obtained for some limit cases of equation (6). 1. Stationary solutions ($d/dt = 0$): $\langle x^2 \rangle = \frac{D_1}{2\eta\sigma^2}$ as for pure Brownian motion. 2. White noise $\xi(t)$: One obtains
two equations for $\langle x^2 \rangle$ and $\langle y^2 \rangle$. 3. In the absence of multiplicative noise ($\xi = 0$), one obtains the same results as those of Brownian motion in a harmonic potential.

The problem of stability of the solutions obtained for the stochastic equation (5) is much more complicated than the calculation of the first moments [21]. Indeed, for a deterministic equation, the stability is defined by the sign of $\alpha$, found from the solution of the form $\exp(\alpha t)$ of a linearized equation. The situation is quite different for a stochastic equation. The first moment $\langle x(t) \rangle$ and higher moments become unstable for some values of the parameters. However, the usual linear stability analysis, which leads to instability thresholds, turns out to be different for different moments, making them unsuitable for the stability analysis. The rigorous mathematical analysis of random dynamic systems shows [22] that, similar to the order–deterministic chaos transition in nonlinear deterministic equations, the stability of a stochastic differential equation is defined by the sign of Lyapunov exponents. This means that for stability analysis, one has to go from the Langevin-type equations (2)-(6) to the associated Fokker-Planck equation, which describe properties of statistical ensembles. Here, we restrict our analysis to the case of the small fluctuations of the mass.

Introducing the dimensionless time $\tau = \omega t$, Eq. (11) becomes

$$\frac{d^2 x}{d\tau^2} + 2\gamma \frac{d}{d\tau} x + \omega x = \xi(t) \left( 2\gamma \frac{d}{d\tau} \omega + 1 \right) x$$

(13)

For stability analysis, it is convenient to introduce into Eq. (13) the variable $z = (dx/dt)/x$, yielding

$$\frac{dz}{d\tau} = \frac{d^2 x/d\tau^2}{x} - \frac{\langle d^2 x/d\tau^2 \rangle x}{x^2} = \frac{d^2 x/d\tau^2}{x} - z^2$$

(14)

In Eq. (13), replacing the variable $x$ by the variable $z$ leads to

$$\frac{dz}{d\tau} + z^2 + 2\gamma \frac{\omega z}{\omega} + 1 = \xi \left( 2\gamma \frac{\omega z}{\omega} + 1 \right)$$

(15)

The general identity which connects the Lyapunov exponent $\lambda = \lim_{t \to \infty} \langle (1/t) \langle \log E \rangle \rangle$ with the stationary average of the solution of Eq. (15), is [23]

$$\lambda = \int_{-\infty}^{\infty} z P_{st}(z) dz$$

(16)

where $P_{st}(z)$ is the stationary value of the distribution function $P(z, t)$, and the integral is to be understood to mean the principal value. We continue with the Fokker-Planck equation associated with Eq. (15). The Langevin equation

$$\frac{dz}{d\tau} = A(z) + B(z) \xi$$

(17)

is associated with the Fokker-Planck equation of the following form (Stratonovich interpretation) [24],

$$\frac{\partial P(z, \tau)}{\partial \tau} = -\frac{\partial}{\partial z} [A(z) P] + \Delta \frac{\partial}{\partial z} B(z) \left[ \frac{\partial}{\partial z} B(z) P \right]$$

(18)
where $J$ is the constant probability current,

$$B(z) = 1 + \frac{2\gamma}{\omega}; \quad A(z) = -z^2 - B(z)$$

(19)

and

$$\langle \xi(\tau_1) \xi(\tau_2) \rangle = 2\Delta \delta(\tau_1 - \tau_2)$$

(20)

The stationary solution of Eq. (18) has the following form [21]

$$P_{st} = P_{st}(w) = J\omega N\Delta\gamma(w)^{-s-1} \exp \left[-g \left(w - \frac{1}{w}\right)\right]$$

(21)

$$\ast \int_{-\infty}^{\infty} dx \, x^{s-1} \exp \left[g \left(x - \frac{1}{x}\right)\right]$$

where

$$c = \frac{\omega}{\Delta\gamma} \left(1 - \frac{\omega^2}{2\gamma^2}\right); \quad g = \frac{\omega^3}{4\Delta\gamma^3}$$

(22)

There is no need to perform an analysis of Eq. (21) since the analogous calculation has been performed for the case of random damping (Eq. (4 ) yielding [25]

$$P_{st}(z) = \frac{2J\omega}{ND} (w)^{-s-1} \exp \left[-b \left(w - \frac{1}{w}\right)\right]$$

(23)

$$\ast \int_{-\infty}^{\infty} dx \, x^{s-1} \exp \left[b \left(x - \frac{1}{x}\right)\right]$$

with

$$a = 4\gamma D; \quad b = \frac{\omega}{D}$$

(24)

and

$$\lambda = \frac{8}{\pi^2} \int_0^\infty du \, K_1(8b \sinh u) \sinh \left[(1 - a) u\right]$$

(25)

$$\pi^2 \left[ J_{2s/D} \left(\frac{1}{D}\right) + Y_{2s/D} \left(\frac{1}{D}\right) \right]$$

where $K_1$ is a modified Bessel function of the second kind, and $J$ and $Y$ are Bessel functions of the first and second kind, respectively. The Bessel functions are always positive, and the sign of the Lyapunov exponent $\lambda$ is the same as the sign of the hyperbolic sinh, i.e., the sign of $1 - a$.

Returning to Eq. (23), we conclude that an oscillator with fluctuating mass becomes unstable when $|c| = (\omega/\Delta\gamma) \left[1 - \frac{\omega^2}{2\gamma^2}\right] < 1$, i.e., the instability of the fixed point $z = 0$ occurs for the noise strength $\Delta > \omega/\gamma \left|1 - \frac{\omega^2}{2\gamma^2}\right|$, which reduces to $\Delta > \frac{\Delta}{2}$ for $\omega < \sqrt{2}\gamma$ and $\Delta > \omega^2/2\gamma^2$ for $\omega > \sqrt{2}\gamma$.

We expect that the model of an oscillator with a random mass will find many applications in modern science.
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