Collective traffic-like movement of ants on a trail: dynamical phases and phase transitions

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The traffic-like collective movement of ants on a trail can be described by a stochastic cellular automaton model. We have earlier investigated its unusual flow-density relation by using various mean field approximations and computer simulations. In this paper, we study the model following an alternative approach based on the analogy with the zero range process, which is one of the few known exactly solvable stochastic dynamical models. We show that our theory can quantitatively account for the unusual non-monotonic dependence of the average speed of the ants on their density for finite lattices with periodic boundary conditions. Moreover, we argue that the model exhibits a continuous phase transition at the critical density only in a limiting case. Furthermore, we investigate the phase diagram of the model by replacing the periodic boundary conditions by open boundary conditions.

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I. INTRODUCTION

It is well known that no phase transition takes place in one-dimensional systems in thermodynamic equilibrium if the interactions are short-ranged. But, systems of interacting particles driven far from equilibrium can exhibit transitions from one dynamical phase to another even in one-dimensional space with only short-ranged interactions. A class of models, which have been receiving lot of attention in the recent years from this perspective, consist of interacting particles each of which is self-propelled rather than driven by external field; the dynamics of these systems are most often formulated in terms of updating rules and, therefore, these models can also be regarded as stochastic cellular automata (CA).

Several different mechanisms are known to give rise to the phase transitions in systems of driven interacting particles. The boundary-induced phase transitions have been studied extensively. Phase transition has also been observed in a special limiting situation of the bus route model that involves two coupled dynamical variables one of which, because of the periodic boundary conditions, is conserved while the other is not. The aim of this paper is to investigate the dynamical phases and phase transitions in a model of interacting self-driven particles, which has been motivated by the collective traffic-like movement of ants on a trail.

In this paper we first utilize the close relation between the ant-trail model (ATM) and zero-range processes (ZRP) to show the existence of a continuous phase transition from an inhomogeneous jammed phase to homogeneous congested phase at a particular density in a special limit of this ATM with periodic boundary conditions; this phenomenon is argued to be closely related to a phase transition in the bus route model. Then, in order to investigate the boundary-induced phase transitions in the ATM we also compute its phase diagram with open boundary conditions for a wide range of values of the relevant parameters.

The paper is organized as follows: the ATM is defined in Section where the corresponding numerical results are summarized briefly. Then we discuss the relation between ATM and ZRP in Section where we also give analytical calculations based on this analogy, including the results in the thermodynamical limit. The phase diagram of the ATM for open boundary conditions is given...
in Section . Conclusions are drawn in Section .

II. ANT TRAIL MODEL

Since our ATM model reduces to the asymmetric simple exclusion process (ASEP) in some special limits, we begin by defining the ASEP \(^{13}\). A fraction of the sites of a one-dimensional lattice are occupied initially randomly by particles each of which can move forward by one lattice spacing, with probability \(q_{\text{eff}}\), if and only if the target site is not already occupied by another particle. The updating is done either in parallel or in a random-sequential manner. The ASEP \(^{13}\) with parallel updating has been used often as an extremely simple model of vehicular traffic on single-lane highways \(^{3}\).

A. Definition of the ATM model

Let us now define ATM which is a simple model for an unidirectional motion of ants on an existing trail. The ants communicate with each other by dropping a chemical (generically called pheromone) on the substrate as they crawl forward \(^{14, 15}\). The pheromone sticks to the substrate long enough for the other following sniffing ants to pick up its smell and follow the trail. The presence of pheromone leads to a higher effective speed of the isolated ants. In ref.8 and 9 we have proposed an extension of the ASEP that takes into account this enhancement of the effective speed.

Each site of our one-dimensional ant-trail model represents a cell that can accommodate at most one ant at a time. The lattice sites are labelled by the index \(i\) \((i = 1, 2, ..., L)\); \(L\) being the length of the lattice. We associate two binary variables \(S_i\) and \(\sigma_i\) with each site \(i\) where \(S_i\) takes the value 0 or 1 depending on whether the cell is empty or occupied by an ant. Similarly, \(\sigma_i = 1\) if the cell \(i\) contains pheromone; otherwise, \(\sigma_i = 0\). The instantaneous state (i.e., the configuration) of the system at any time is specified completely by the set \((\{S\}, \{\sigma\})\).

Since a unidirectional motion is assumed, ants do not move backward. Their forward-hopping probability is higher if it smells pheromone ahead of it. The state of the system is updated at each time step in two stages. In stage I ants are allowed to move. This motion follows rules similar to those of the particles in the ASEP except that the hopping probability now depends on the presence or absence of pheromone at the target site. Here the subset \((\{S(t+1)\})\) at the time step \(t+1\) is obtained using the full information \((\{S(t)\}, \{\sigma(t)\})\) at time \(t\). Stage II corresponds to the evaporation of pheromone. Here only the subset \((\{\sigma(t)\})\) is updated so that at the end of stage II the new configuration \((\{S(t+1)\}, \{\sigma(t+1)\})\) at time \(t+1\) is obtained. In each stage the dynamical rules are applied in parallel to all ants and pheromones, respectively.

Stage I: Motion of ants

An ant in cell \(i\) that has an empty cell in front of it, i.e., \(S_i(t) = 1\) and \(S_{i+1}(t) = 0\), hops forward with

\[
\text{probability} = \begin{cases} 
Q & \text{if } \sigma_{i+1}(t) = 1, \\
q & \text{if } \sigma_{i+1}(t) = 0,
\end{cases}
\]

where, to be consistent with real ant-trails, we assume \(q < Q\).

Stage II: Evaporation of pheromones

At each cell \(i\) occupied by an ant after stage I a pheromone will be created, i.e.,

\[
\sigma_i(t+1) = 1 \quad \text{if} \quad S_i(t+1) = 1. \tag{2}
\]

On the other hand, any ‘free’ pheromone at site \(i\) not occupied by an ant will evaporate with the probability \(f\) per unit time, i.e., if \(S_i(t + 1) = 0\) and \(\sigma_i(t) = 1\), then

\[
\sigma_i(t + 1) = \begin{cases} 
0 & \text{with probability } f, \\
1 & \text{with probability } 1 - f.
\end{cases} \tag{3}
\]

Note that, if periodic boundary conditions are imposed, the dynamics conserves the number \(N\) of ants, but not the number of pheromones; in that case ATM model is a stochastic CA model with two coupled dynamical variables one of which is conserved and the other nonconserved. The stationary states of this ATM model reduces to those of the ASEP \(^{13}\) with \(q_{\text{eff}} = Q\) and \(q_{\text{eff}} = q\), respectively, in limiting cases \(f = 0\) and \(f = 1\).

B. Relation between ATM and the bus route model

The bus route model \(^{16}\) describes a system of buses that move unidirectionally from one bus stop to the next on a circular route and, at each bus stop, a bus picks up the waiting passengers that arrive stochastically since the departure of the last bus from that stop.

In the bus route model \(^{16}\) the bus stops are represented by the sites on a one-dimensional lattice each of which may be labeled by an index \(i\) \((i = 1, 2, ..., L\). Two binary variables \(\sigma_i\) and \(\tau_i\) are assigned to each cell \(i\): (i) If the cell \(i\) is occupied by a bus then \(\sigma_i = 1\); otherwise \(\sigma_i = 0\). (ii) If cell \(i\) has passengers waiting for a bus then \(\tau_i = 1\); otherwise \(\tau_i = 0\). Since a cell cannot have simultaneously a bus and waiting passengers, a cell cannot have both \(\sigma_i = 1\) and \(\tau_i = 1\) simultaneously. Each bus is assumed to hop from one stop to the next.

The random-sequential update rules of the model are as follows: a cell \(i\) is picked up at random. Then, (i) if \(\sigma_i = 0\) and \(\tau_i = 0\) (i.e., cell \(i\) contains neither a bus nor waiting passengers), then \(\tau_i \rightarrow 1\) with probability \(\lambda\), where \(\lambda\) is the probability of arrival of passenger(s) at the bus stop. (ii) If \(\sigma_i = 1\) (i.e., there is a bus at the cell \(i\)) and \(\sigma_i + 1 = 0\), then the hopping rate \(\mu\) of the bus is
defined as follows: (a) if \( \tau_{i+1} = 0 \), then \( \mu = \alpha_b \) but (b) if \( \tau_{i+1} = 1 \), then \( \mu = \beta_b \), where \( \alpha_b \) is the hopping rate of a bus onto a stop which has no waiting passengers and \( \beta_b \) is the hopping rate onto a stop with waiting passenger(s). Generally, \( \beta_b < \alpha_b \), which reflects the fact that a bus has to slow down when it has to pick up passengers. One can set \( \alpha_b = 1 \) without loss of generality. When a bus hops onto a stop \( i \) with waiting passengers \( \tau_i \) is reset to zero as the bus takes all the passengers. Note that, because of the periodic boundary conditions, the density of buses is a conserved quantity whereas that of the passengers is not.

Note that in the ATM the pheromones “appear” at a site when an ant visits the site just as the waiting passengers “disappear” from a site when a bus visits the site. Also note that the “disappearance” of pheromone from a site takes place with a probability \( f \) per unit time, independent of the ants. Similarly, the passengers “appear” at a bus stop with the probability \( \lambda \) per unit time independent of buses. Therefore, the “appearance” of the pheromones is analogous to the “disappearance” of the passengers and vice versa. Naturally, we expect the role of the parameter \( f \) in the ATM to be similar to that of \( \lambda \) in the bus route model. The presence of passenger is analogous to the absence of pheromone. Thus, the ATM is equivalent to the BRM with parallel updating \[^{14}\].

Interestingly, the queuing of the buses in the bus route model, in turn, is very similar to the bunching of macrosteps during the growth of faceted crystals because of impurity absorption \[^{17}\]. If \( y_n(t) \) denotes the position of the particle (representing a bus or an ant or a macrostep) at time \( t \), and \( \tau_n \) is the time elapsed since the particle ahead (which is labelled by \( n + 1 \)) passed the same position \( y_n(t) \), one can write \[^{17}\] \( y_n(t) = y_{n+1}(t - \tau_n) \) and corresponding speed \( v_n(t) = V(\tau_n(t)) \) is a function of \( \tau_n(t) \).

C. Numerical results

We shall use the symbols \( L \) and \( M \) to denote the system size and the number of ants, respectively; hence \( \rho = M/L \) is the density of ants on the trail. The most important quantity of interest in the context of flow properties of the traffic models is the fundamental diagram, i.e., the flux-versus-density relation, where flux \( F \) is the product of the density \( \rho \) and the average speed \( v \). Because of the similarity of the ATM with traffic models, it would be interesting to draw the fundamental diagram of ATM and compare the traffic-like collective movements of ants with vehicular traffic.

The fundamental diagrams of ATM is given in our earlier papers (see, for example, Fig.2(b) of ref.8 or Fig.2(b) of ref.9). First of all, the diagram does not possess particle-hole symmetry for any \( f \) in the range \( 0 < f < 1 \); the particle-hole symmetry observed in the ATM in the special cases of \( f = 0 \) and \( f = 1 \) is a consequence of the fact that in the two special cases \( f = 0 \) and \( f = 1 \), as pointed out earlier, the ATM becomes identical to the ASEP, with parallel updating, corresponding to the effective hopping probabilities \( q_{\text{eff}} = Q \) and \( q_{\text{eff}} = q \), respectively.

Most important feature of the fundamental diagram is that, over a range of small values of \( f \) (\( f \ll 0.01 \)), it exhibits an anomalous behaviour; this is a consequence of the fact that, unlike common vehicular traffic, the average speed of the ants in the ATM is a non-monotonic function of the density of ants on the trail (see, for example, Fig.3(b) of ref.8 or Fig.2(a) of ref.9).

The ATM model also exhibits interesting coarsening of clusters of ants starting from random initial conditions (Fig.1).

![FIG. 1: Spatial-temporal behaviours of loose clusters in the low density case (\( \rho = 0.16 \)). Parameters are \( Q = 0.75, q = 0.25, f = 0.005 \). We see the loose clusters emerge form the random initial configuration, and will eventually merge into one big loose cluster if we take sufficiently long time.](image)

III. PHASE TRANSITION IN ATM WITH PERIODIC BOUNDARY CONDITIONS

In ref.9 we have developed a formalism by introducing the loose cluster approximation and successfully captured the non-monotonicity heuristically with the help of analytical results of ASEP, which is an exactly solvable stochastic model. In this paper, we study the phenomenon in detail by utilizing the analogy of ATM with the zero range process (ZRP), which is exactly solvable \[^{11, 12}\]. The ZRP is a particle-hopping model where the hopping probabilities do not depend on the state of occupation of the target site.

A. Analytical results for finite systems: ZRP and ATM

We now first show the relation between the ATM and the ZRP \[^{11, 12}\]. Note that, in the ATM the effective hopping probability \( u \) of the can be exactly expressed as \[^{8, 9}\]

\[
u = q(1-g) + Qg, \tag{4}\]

\[^{8, 9}\]
where $g$ is the probability that there is a surviving pheromone on the first empty site of a gap in front of an ant. Obviously, $g(t)$ depends on the time $t$ elapsed since the ant immediately in front left the target site. It is straightforward to see that

$$g(t + 1) = (1 - f)g(t)$$

(5)

Alternatively, we can express the probability $g(t)$ as a function $\tilde{g}(x)$ of the gap $x$ between the ant under consideration and that immediately in front of it. Recall that the average speed of the ants is $v$. Since the time interval between the passage of successive ants through any arbitrary site is $x/v$, and since equation (5) holds for all $t$, we obtain

$$\tilde{g}(x) = (1 - f)^{x/v}$$

(6)

after iterating the mapping $5x/v$ times.

Thus, the ATM can be regarded as an ASEP involving fictitious particles where, $u(x)$, the effective hopping probability of a particle having a gap $x$ in front, depends on $x$ through the relation $8$

$$u(x) = q + (Q - q)(1 - f)^{x/v}. \quad (7)$$

Note that in the case of the original ASEP, $u(x) = \text{constant}$ for $x > 0$ and $u(0) = 0$. The form (7) is approximate description of the hopping probabilities of the ants in the ATM, but all the results we derive for this form of $u(x)$ are exact.

As long as $f$ remains non-zero, $u(x)$ decreases with increasing $x$ at a rate much faster than that required for a phase transition to occur $18$. However, since $v$ depends on $f$ and $x$, the trend of variation of $u(x)$ with $x$ is difficult to infer without going through a detailed analysis which we present below.

The configurations of this ASEP can be uniquely described by the gap configurations $\{x_1, x_2, \ldots, x_M\}$. Note that in this ASEP the pheromones do not appear explicitly, but their effect enters through the $f$-dependence of $u(x)$. It is well known $12$ that any ASEP, where the hopping probability of the particles $u(x)$ depends on the gap $x$ in front of the particle, can be mapped onto the ZRP. The advantage of mapping the ATM onto the corresponding ZRP (with parallel updating) is that the the stationary state of the ZRP is given by product measure $\mu$.

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The ATM is formulated with parallel update. The corresponding form of $h(x)$ is known to be given by $21$

$$h(x) = \begin{cases} 
1 - u(1) & \text{for } x = 0, \\
1 - u(x) & \prod_{y=1}^{x-1} 1 - u(y) \text{ for } x > 0 
\end{cases} \quad (12)$$

Our aim is to calculate the average velocity $v$ of ants given by

$$v = \sum_{x=1}^{L-M} u(x)p(x). \quad (13)$$

In order to compute $v$ we need to compute $p(x)$ and $u(x)$. However, the computation of $p(x)$ requires $h(x)$ which, in
turn, depends on $u(x)$. On the other hand, $u(x)$ is given by equation (7) which involves average speed $v$. Therefore, in order to compute $v$ self-consistently, we begin with the initial approximate estimate $v = q$ and, hence, compute $u(x)$ and $p(x)$ to get better estimate of average speed $v$ from equation (7). Through this iterative process, we estimate $v$ self-consistently and, hence, the fundamental diagram.

Fundamental diagrams are given in Fig. 2 with $L = 100$ and $L = 200$. The data points, shown with various symbols in Fig. 2 and connected by dashed curves, have been obtained from computer simulations of the ATM, whereas the continuous curves are the corresponding theoretical predictions. We find that the theoretical curves are almost identical to the numerical ones, thus confirming that the ZRP successfully describes the steady state of the ATM.

![Graphs showing fundamental diagrams of the ATM](image)

**FIG. 2:** Comparison of the theoretically obtained fundamental diagrams of the ATM with for the system sizes (a) $L = 100$ (continuous curve in (a)) and (b) $L = 200$ (continuous curve in (b)) with the corresponding numerical data obtained from computer simulations (represented by the symbol +). The numerical data obtained from simulations of systems of size $L = 1000$ are also plotted (represented by the symbol ×) in both (a) and (b) to show the trend of variation with increasing $L$. The common parameters are $Q = 0.75$, $q = 0.25$, $f = 0.005$.

## B. Thermodynamic limit of ATM

Next we discuss the thermodynamic limit of the ATM, that is, the case $L \to \infty$. From Fig. 2, we see that the curve shows sharp increase near the density region $0.4 < \rho < 0.5$, and the tendency is expected to increase with the increase of $L$. Thus it is important to study whether there is a (second order) phase transition from an inhomogeneous jammed phase, in which the ants are bunched together, to homogeneous congested phase with the increase of the density of the ants in the thermodynamic limit.

Using the integral representation

$$\delta(\mu, \nu) = \int \frac{ds}{2\pi i} \frac{s^{\mu}}{s^{\nu+1}}$$

of the Kronecker delta, we rewrite the partition function $Z(L, M)$ as

$$Z(L, M) = \oint \frac{ds}{2\pi i} \left( \frac{G(s)}{s^{1/\nu+1}} \right)^{M},$$

where $G(s)$ is the generating function of $h$ defined by

$$G(s) = \sum_{x=0}^{\infty} h(x)s^x.$$ // (16)

We evaluate the integral in (15) in the $L \to \infty$ limit, keeping $M/L = \rho$ as constant, by the saddle point method. The partition function reduces to the form

$$Z(L, M) \sim \exp(M \ln G(z) - (L - M) \ln z)$$ // (17)

where the saddle point $s = z$ is given by

$$\frac{1}{\rho} - 1 = z \frac{\partial \ln G(z)}{\partial z}.$$ // (18)

The equation (18) may also be regarded as the relation that defines $z$.

Using the approximate form (17) of the partition function, we have

$$p(x) \simeq h(x) \frac{Z(L-x-1, M-1)}{Z(L, M)} \simeq \frac{h(x)}{G(z)} z^x.$$ // (19)

Then, substituting (19) in expression (18), we get

$$v = \sum_{x=1}^{\infty} \frac{u(x)h(x)}{G(z)} z^x.$$ // (20)

for the average speed $v$ in the thermodynamic limit.

Next we study the properties of the generating function $G(z)$ in detail to examine the possibility of phase transition of the ATM. Since $\lim_{x \to \infty} u(x) = q$ for all $f > 0$, the condition that $G$ converges is given by

$$\lim_{z \to 1} \sup_{x \to \infty} \frac{h(x+1)z}{h(x)} < 1.$$ // (21)

Using (12) for $h(x)$ the convergence condition reduces to

$$\lim_{z \to 1} \sup_{x \to \infty} \frac{h(x+1)z}{h(x)} = \frac{1 - q}{q} z < 1.$$ // (22)

Thus, $G$ converges in the range

$$0 < z < z_c = \frac{q}{1 - q}.$$ // (23)

Using (12) and (23) in (10) we find that the critical value of $G$ at $z = z_c$ is given by

$$G(z_c) = 1 - u(1) + \sum_{x=1}^{\infty} \frac{1 - u(1)}{1 - u(x)} \prod_{y=1}^{x} \left( \frac{1 - u(y)}{u(y)} - \frac{q}{1 - q} \right).$$ // (24)
For all \( f > 0 \), this sum diverges for the class of \( u(x) \) which decays to \( q \) more rapidly than the function \((1 + c/x)q\), where \( c > 1 - q \). In the ATM, \( u(x) = q + (Q - q)(1 - f)^{x/v} \), which decays exponentially to \( q \) as \( x \to \infty \). Hence we conclude that there is no phase transition in the ATM for \( f > 0 \). This is because, from (15), we have \( \rho = 1 \) when \( z = 0 \), and \( \rho = 0 \) at \( z = z_c \) if \( G(z_c) \) diverges. Thus, as long as \( f > 0 \), in the entire density region \( 0 \leq \rho \leq 1 \) there is no singularity in \( G \) and, hence, no phase transition in the ATM.

The situation is drastically different in the limit \( f \to 0 \). In this limit, \( u(x) = Q \), and then \( G(z_c) \) approaches the finite limit

\[
\lim_{z \to z_c} G(z) = \frac{Q(1 - q)}{Q(1 - q) - q(1 - Q)} - Q. \tag{25}
\]

Thus, in the limit \( f \to 0 \) a phase transition takes place at the critical density

\[
\rho_c = \frac{Q - q}{Q - q^2} \tag{26}
\]

which is obtained from (15).

In order to get insight into the nature of the phases involved, we now calculate the average speed \( v = v_c \) at \( z = z_c \). Since \( u(x) \to Q \) as \( f \to 0 \), \( h(x) \to (1 - Q)/Q \) and, hence, from (20) we get

\[
v_c = \sum_{x=1}^{\infty} \frac{Q}{G(z_c)} \left( \frac{1 - Q}{Q} \right)^x z_c^x. \tag{27}
\]

Substituting (25) into (27), we obtain \( v_c = q \).

At first sight the two facts, namely, \( u(x) = Q \) and \( v_c = q \) may appear mutually contradictory. But, the consistency of these is a consequence of the fact that the mutual hindrance leads to the smaller average speed \( v_c = q \) in spite of the higher hopping probability \( u(x) = Q \).

It should be noted that (20) is also obtained by the intersection point of the line \( F = v_c \rho \) and the curve \( F \) that describes the fundamental diagram of the ASEP with parallel updating. Also note that the limits \( L \to \infty \) and \( f \to 0 \) do not commute. If we take \( f \to 0 \) before \( L \to \infty \), then we apparently have (28), which corresponds to the situations in our numerical simulations. On the other hand, if we take \( f \to 0 \) after \( L \to \infty \), then we have the thick curve in Fig.3. Thus, in the latter case, we have proved that the anomalous variation of the average velocity with the density disappears.

IV. PHASES OF ATM WITH OPEN BOUNDARY CONDITIONS

So far we have considered the ATM with only periodic boundary conditions. However, for ant-trails the open boundary conditions are more realistic. Therefore, in this section we study the phases and phase transitions in the ATM with open boundary conditions.

Suppose \( \alpha \) and \( \beta \) denote the probabilities of incoming and outgoing particles at the open boundaries per unit time. The phase diagram of the ASEP in the \( \alpha - \beta \)-plane has been investigated exhaustively [2]. In this section we report the effects of varying the pheromone evaporation probability \( f \) on the phase diagram of the ASEP with parallel updating and open boundary conditions.

![FIG. 3: The thick curve (with no decorating symbols) represents the fundamental diagram of the ATM corresponding to \( f \to 0 \) after taking the thermodynamic limit \( L \to \infty \). Note that these two limits do not commute. The simulation data corresponding to \( f = 0.005(\diamond), 0.001(\circ), 0.005(\bullet), 0.01(\triangle), 0.05(\square), 0.10(\times), 0.25(+), 0.50(*) \).](image)

![FIG. 4: The phase diagram of the ATM with open boundary conditions in the \( \alpha - \beta \)-plane for several different values of the pheromone evaporation probability \( f \) (0 \leq f \leq 1). The values of the hopping parameters are \( Q = 0.75, q = 0.25 \).](image)
the high-density phase and the maximal current phase is
given by \( \beta_c(f) \).

![Image of graph](image.png)

**FIG. 5:** The critical rates, \( \alpha_c \), (denoted by +), which have been obtained from computer simulations of the ATM with open boundary conditions, are plotted against the pheromone evaporation probability \( f \). The discrete data points denoted by the symbols × and * denote the average speeds \( v_{1/2}(f) \) obtained from computer simulations of the same model but with periodic boundary conditions for system sizes \( L = 100 \) and \( L = 500 \), respectively. The corresponding theoretical predictions for \( v_{1/2}(f) \) of our ZRP-based theory of ATM are plotted for the system sizes \( L = 100 \) (the continuous curve) and \( L = 500 \) (the dashed curve), respectively. The values of the hopping parameters are \( Q = 0.75, q = 0.25 \).

The variation of the critical point \( \alpha_c(f) = \beta_c(f) \) with \( f \) is shown in Fig.4. In order to understand this observation let us first examine the limiting values at \( f = 0 \) and \( f = 1 \). It is well known \cite{21} that for ASEP with hopping probability \( q_{\text{eff}} \),

\[
\alpha_c = \beta_c = 1 - \sqrt{1 - q_{\text{eff}}}.
\]

Therefore, we must have

\[
\alpha_c(f = 0) = \beta_c(f = 0) = 1 - \sqrt{1 - Q} \quad \alpha_c(f = 1) = \beta_c(f = 1) = 1 - \sqrt{1 - q} ;
\]

these are consistent with the numerical data \( \alpha_c(f = 0) = 0.5 \) and \( \alpha_c(f = 1) = 0.133 \) for \( Q = 0.75, q = 0.25 \) shown in Fig.4.

Next we try to understand the detailed variation of \( \alpha_c = \beta_c \) with \( f \) by utilizing the relation between the ATM model with periodic and open boundary conditions. Note that, in the maximal current phase in the ASEP with open boundary conditions and parallel updating \cite{21} the current is given by

\[
J = \frac{1 - \sqrt{1 - q_{\text{eff}}}}{2}.
\]

(31)

(in ASEP with parallel updating) while the corresponding bulk density is given by \cite{21}

\[
\rho(x = L/2) = 1/2
\]

(32)

and, consequently, the corresponding average speed should be

\[
v = 1 - \sqrt{1 - q_{\text{eff}}} = \alpha_c = \beta_c.
\]

(33)

In order to check the validity of this argument, we have computed the average speed \( v_{1/2} \) corresponding to \( \rho = 1/2 \) by (a) computer simulation of the ATM with periodic boundary conditions and (b) using the ZRP-based theory, mentioned in the preceding section. All these data are plotted in Fig.5. The simulation data for \( v_{1/2}(f) \) are in good agreement with the simulation data for \( \alpha_c \). But, there are significant differences between these data and \( v_{1/2}(f) \) obtained from our ZRP-based theory in the small \( f \) regime. We believe that this discrepancy arises from the boundary effect. If periodic boundary conditions are imposed, at low densities, the leading ant in the loose cluster can smell the pheromone which is dropped by the last ant in the same cluster. However, this effect would disappear when the periodic boundary conditions are replaced by open boundary conditions.

Since we have numerically estimated \( \alpha_c \) in the ATM as a function of \( f \) by carrying out computer simulations, we utilized the relation (29) to get the \( f \)-dependence of the effective hopping probability \( q_{\text{eff}}(f) \) in the ATM (see Fig.6). The two limiting values \( \lim_{f \to 0} q_{\text{eff}}(f) = Q \) and \( \lim_{f \to 1} q_{\text{eff}}(f) = q \) as well as the nature of the variation of \( q_{\text{eff}} \) with \( f \) are fully consistent with one’s expectation based on physical arguments mentioned earlier in this paper.

![Image of graph](image.png)

**FIG. 6:** \( f \)-dependence of the effective hopping probability, extracted by using the equation (29) from the numerical data obtained by computer simulation of the ATM with open boundary conditions.

V. CONCLUDING DISCUSSIONS

The close similarities between phase transitions in non-living systems and that of foraging behaviour on the ant-trail was pointed out by Beckman et al.\cite{22}. However, their study was concerned with the nature of foraging,
namely, the possibility of a transition from a disordered foraging behaviour (i.e., foraging without a pheromone trail) to ordered foraging (i.e., trail-based foraging). In contrast to this problem of the formation of the trail pattern \[23\], we have studied the phases and phase transitions exhibited by the system when the collective traffic-like movement of ants take place on an existing trail.

The dynamical phases and non-equilibrium phase transitions in systems of interacting self-propelled particles have been among the most challenging problems of investigation in non-equilibrium statistical mechanics. In contrast to equilibrium systems, these intrinsically non-equilibrium systems exhibit phase transitions even in one-dimensions with only short range interactions. In this paper we have studied the phases of a one-dimensional model motivated by the collective traffic-like movements of ants on an existing trail \[8, 9\]. In this model the ants are represented by self-propelled particles which, in addition to the hard-core repulsion, interact indirectly via pheromone. The model is, thus, characterized by two coupled dynamical variables, representing the ants and the pheromone.

In our earlier works \[8, 9\] we had shown that the homogeneous mean-field approximations cannot capture the non-monotonic variation of the average speed with the density of the ants. Even the loose-cluster approximation that we developed \[9\] could account for the simulation data with limited accuracy. In this paper we have reported our new quantitative results on this model with periodic boundary conditions; these results have been derived by utilizing the analogy with ZRP. Moreover we have shown that there is a phase transition in the thermodynamic limit in this model, albeit in a special limit \(f \to 0\).

In our earlier published works \[8, 9\] we imposed periodic boundary conditions. However, in order to capture the real ant-trails in nature imposition of open boundary conditions seems more appropriate. In equilibrium statistical mechanics, the boundary conditions do not play any role in the phase transitions which, strictly speaking, take place only in the thermodynamic limit. In contrast, boundary-induced phase transition \[9\] is a well-known phenomenon in non-equilibrium statistical mechanics. The phase diagrams of the ASEP with open boundary conditions is now well established. Since the ATM can be regarded as an extension of the ASEP, our investigation of the phase diagram of the ATM reported here illuminates the effects of varying the pheromone evaporation rate \(f\) on the phase diagram of the ASEP.

In this paper we have focussed attention on phase transitions in an idealized single-lane ATM. The phase transitions in a more elaborate two-lane ATM will be reported later \[24\]. It would be interesting to test the predictions of the model, particularly the non-monotonic variation of average speed with density, by repeating experiments of the type reported in ref.\[10\] to get sufficiently accurate data for real ants.

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[1] B. Schmittmann and R.K.P. Zia, in: *Phase Transitions and Critical Phenomena*, Vol.17, eds. C. Domb and J.L. Lebowitz (Academic Press, 1995).
[2] G. Schütz, in: *Phase Transitions and Critical Phenomena*, Vol.19, eds. C. Domb and J.L. Lebowitz (Academic Press, 2000).
[3] D. Chowdhury, L. Santen and A. Schadschneider: Phys. Rep. 329 (2000) 199.
[4] S. Wolfram: *Theory and Applications of Cellular Automata* (World Scientific, Singapore, 1986).
[5] B. Chopard, M. Droz: *Cellular Automata Modelling of Physical Systems* (Cambridge University Press, 1998).
[6] J. Krug: Phys. Rev. Lett. 67, (1991) 1882.
[7] O.J. O’Loan, M.R. Evans and M.E. Cates: Phys.Rev.E 58 (1998) 1404; see also Europhys. Lett. 42 (1998) 137.
[8] D. Chowdhury, V. Guttal, K. Nishinari and A. Schadschneider: J. Phys. A:Math. Gen. 35 (2002) L573.
[9] K. Nishinari, D. Chowdhury and A. Schadschneider: Phys. Rev. E, 67 (2003) 036120.
[10] M. Burd, D. Archer, N. Aranwela and D.J. Stradling: American Natur. 159 (2002) 283.
[11] F. Spitzer, F: Advances in Math. 5 (1970) 246.
[12] Evans, M.R., Braz. J. Phys. 30 (2000) 42.
[13] M.R. Evans and R.A. Blythe: Physica A 313 (2002) 110 and references therein.
[14] S. Camazine, J.L. Deneubourg, N. R. Franks, J. Sneyd, G. Theraulaz and E. Bonabeau: *Self-organization in Biological Systems* (Princeton University Press, Princeton, 2001).
[15] A.S. Mikhailov and V. Calenbuhr: *From Cells to Societies* (Springer, Berlin, 2002).
[16] D. Chowdhury and R.C. Desai, Eur. Phys. J B 15, (2000) 375.
[17] J. P. v. d. Eerden and H. Müller-Krumbhaar: Phys. Rev. Lett. 57 (1986) 2431.
[18] M. R. Evans, E. Levine, P. K. Mohanty and D. Mukamel, preprint cond-mat/0405049 (2004).
[19] M.R. Evans, preprint cond-mat/0401341 (2004).
[20] M. R. Evans: J. Phys. A:Math. Gen. 30 (1997) 5669.
[21] N. Rajewsky, L. Santen, A. Schadschneider and M. Schreckenberg: J. Stat. Phys. 92, (1998) 151.
[22] M. Beckman, D.J.T. Sumpter and F.L.W. Ratnieks: Proc. Natl. Acad. Sci. USA 98, (2001) 9703; see also D. Waxman, Trends in Ecology and Evolution, 17 (2002) 103.
[23] D. Chowdhury, K. Nishinari and A. Schadschneider: Phase Transitions 77 (2004) 601.
[24] A. John, A. Schadschneider, K. Nishinari and D. Chowdhury, in preparation.