DERIVATIONS WITH VALUES IN QUASI-NORMED BIMODULES OF LOCALLY MEASURABLE OPERATORS

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Abstract. We prove that every derivation acting on a von Neumann algebra \( \mathcal{M} \) with values in a quasi-normed bimodule of locally measurable operators affiliated with \( \mathcal{M} \) is necessarily inner.

1. Introduction

One of the important results in the theory of derivations in Banach bimodules is the Theorem of J. R. Ringrose on automatic continuity of every derivation acting on a \( C^* \)-algebra \( \mathcal{M} \) with values in a Banach \( \mathcal{M} \)-bimodule \[15\]. This theorem extends the well-known result that every derivation of a \( C^* \)-algebra \( \mathcal{M} \) is automatically norm continuous \[17\]. In the case when \( \mathcal{M} \) is an \( AW^* \)-algebra (in particular, \( W^* \)-algebra), every derivation on \( \mathcal{M} \) is inner \[14\], \[17\].

Significant examples of \( W^* \)-modules are non-commutative symmetric spaces of measurable operators affiliated with a von Neumann algebra. At the present time the theory of symmetric spaces is actively developed (see, e.g. \[7\], \[10\]) and it gives useful applications both in the geometry of Banach spaces and in the theory of unbounded operators. Every non-commutative symmetric space is a solid linear space in the \( \tau \)-algebra \( S(\mathcal{M}, \tau) \) of all \( \tau \)-measurable operators affiliated with a von Neumann algebra \( \mathcal{M} \), where \( \tau \) is a faithful normal semifinite trace on \( \mathcal{M} \) \[13\]. The algebra \( S(\mathcal{M}, \tau) \) equipped with the natural topology \( t_\tau \) of convergence in measure generated by the trace \( \tau \) is a complete metrizable topological algebra. In its turn the algebra \( S(\mathcal{M}, \tau) \) represents a solid \( \tau \)-subalgebra of the \( \tau \)-algebra \( LS(\mathcal{M}) \) of all locally measurable operators, affiliated with a von Neumann algebra \( \mathcal{M} \) \[18\], \[21\], in addition, the latter algebra \( LS(\mathcal{M}) \) with the natural topology \( t(\mathcal{M}) \) of convergence locally in measure is a complete topological \( \tau \)-algebra \[21\].

In \[1\], \[2\], \[3\], \[4\] there are established significant results in description of derivations of these algebras. In particular, it is proved that for every \( t(\mathcal{M}) \)-continuous derivation \( \delta \) acting on the algebra \( LS(\mathcal{M}) \) there exists \( a \in LS(\mathcal{M}) \), such that \( \delta(x) = \delta_a(x) = ax - xa = [a, x] \) for all \( x \in LS(\mathcal{M}) \), that is the derivation \( \delta \) is inner. One of the main corollary of this result provides a full description of derivations \( \delta \) acting from \( \mathcal{M} \) into a Banach \( \mathcal{M} \)-bimodule \( \mathcal{E} \) of locally measurable operators affiliated with \( \mathcal{M} \). More precisely it is shown that any derivation an a von Neumann algebra \( \mathcal{M} \) with values in a Banach \( \mathcal{M} \)-bimodule \( \mathcal{E} \) is inner \[4\].

In the present paper we establish the similar property of derivations in more general case when \( \mathcal{E} \) is a quasi-normed \( \mathcal{M} \)-bimodule of locally measurable operators affiliated with \( \mathcal{M} \). The proof proceed in several stages. Firstly, in Section 2 we

1991 Mathematics Subject Classification. 46L51, 46L52, 46L57.

Key words and phrases. Derivation, von Neumann algebra, quasi-normed bimodule, locally measurable operator.
establish the version of Ringrose Theorem for arbitrary quasi-normed $C^*$-bimodules without the assumption of their completeness. After that in section 3 we supply the proof of the main result of the present paper (Theorem 4.7) showing that every derivation $\delta : \mathcal{M} \to \mathcal{E}$ is necessarily inner, i.e. $\delta = \delta_d$ for some $d \in \mathcal{E}$. In particular, $\delta$ is a continuous derivation from $(\mathcal{M}, \| \cdot \|_{\mathcal{M}})$ into $(\mathcal{E}, \| \cdot \|_{\mathcal{E}})$. In addition, the operator $d \in \mathcal{E}$ may be chosen so that $\|d\|_{\mathcal{E}} \leq 2C_{\mathcal{E}}\|\delta\|_{\mathcal{M}\to\mathcal{E}}$, where $C_{\mathcal{E}}$ is the modulus of concavity of the quasi-norm $\| \cdot \|_{\mathcal{E}}$.

We use terminology and notations from the von Neumann algebra theory [8, 20] and the theory of locally measurable operators from [12, 24].

2. Preliminaries

Let $B(H)$ be the $*$-algebra of all bounded linear operators acting in a Hilbert space $H$, and let $1$ be the identity operator on $H$. Let $\mathcal{M}$ be a von Neumann algebra acting on $H$, and let $\mathcal{Z}(\mathcal{M})$ be the centre of $\mathcal{M}$. Denote by $\mathcal{P}(\mathcal{M}) = \{p \in \mathcal{M} : p = p^2 = p^*\}$ the lattice of all projections in $\mathcal{M}$ and by $\mathcal{P}_{fin}(\mathcal{M})$ the set of all finite projections in $\mathcal{M}$.

A densely-defined closed linear operator $x$ affiliated with $\mathcal{M}$ is said to be measurable with respect to $\mathcal{M}$ if there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{M})$ such that $p_n \uparrow 1$, $p_n(H) \subset \mathcal{D}(x)$ and $p_n^+ = 1 - p_n \in \mathcal{P}_{fin}(\mathcal{M})$ for every $n \in \mathbb{N}$, where $\mathcal{D}(x)$ is the domain of $x$ and $\mathbb{N}$ is the set of all natural numbers. The set $S(\mathcal{M})$ of all measurable operators is a unital $*$-algebra over the field $\mathbb{C}$ of complex numbers with respect to strong sum $x + y$, strong product $xy$ and the adjoint operation $x^*$ [19].

A densely-defined closed linear operator $x$ affiliated with $\mathcal{M}$ is called locally measurable with respect to $\mathcal{M}$ if there is a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ such that $z_n \uparrow 1$, $z_n(H) \subset \mathcal{D}(x)$ and $x z_n \in S(\mathcal{M})$ for all $n \in \mathbb{N}$.

The set $LS(\mathcal{M})$ of all locally measurable operators (with respect to $\mathcal{M}$) is a unital $*$-algebra over the field $\mathbb{C}$ with respect to the same algebraic operations as in $S(\mathcal{M})$, in addition $\mathcal{M}$ and $S(\mathcal{M})$ are $*$-subalgebras of $LS(\mathcal{M})$.

For every subset $E \subset LS(\mathcal{M})$, the set of all self-adjoint (respectively, positive) operators in $E$ is denoted by $E_h$ (respectively, $E_+$). The partial order in $LS_h(\mathcal{M})$ is defined by its cone $LS_+(\mathcal{M})$ and is denoted by $\leq$.

Let $x$ be a closed linear operator with the dense domain $\mathcal{D}(x)$ in $H$, let $x = u|x|$ be the polar decomposition of the operator $x$, where $|x| = (x^* x)^{\frac{1}{2}}$ and $u$ is a partial isometry in $B(H)$ such that $u^* u$ is the right support $r(x)$ of $x$. It is known that $x \in LS(\mathcal{M})$ (respectively, $x \in S(\mathcal{M})$) if and only if $|x| \in LS(\mathcal{M})$ (respectively, $|x| \in S(\mathcal{M})$) and $u \in \mathcal{M}$ [12, §2.2, 2.3]. For every $x \in LS(\mathcal{M})$ the projection $s(x) = l(x) \vee r(x)$, where $l(x)$ and $r(x)$ are left and right supports of $x$ respectively, is called the support of $x$. If $x \in LS_h(\mathcal{M})$, then the spectral family of projections $\{E_h(x)\}_{\lambda \in \mathbb{R}}$ for $x$ belongs to $\mathcal{M}$ [12, §2.1], in addition, $s(x) = l(x) = r(x)$.

Denote by $\| \cdot \|_{\mathcal{M}}$ the $C^*$-norm in the von Neumann algebra $\mathcal{M}$. We need the following property of partial order in the algebra $LS(\mathcal{M})$.

**Proposition 2.1.** [4, Proposition 6.1] Let $\mathcal{M}$ be a von Neumann algebra acting on the Hilbert space $H$, $x, y \in LS_+(\mathcal{M})$ and $y \leq x$. Then $y^{1/2} = ax^{1/2}$ for some $a \in s(x)MS(x)$, $\|a\|_{\mathcal{M}} \leq 1$, in particular, $y = ax^*a$. 

Now, let us recall the definition of the local measure topology. If $\mathcal{M}$ is a commutative von Neumann algebra, then $\mathcal{M}$ is $*$-isomorphic to the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$.
of all essentially bounded measurable complex-valued functions defined on a measure space \((\Omega, \Sigma, \mu)\) with the measure \(\mu\) satisfying the direct sum property (we identify functions that are equal almost everywhere) (see e.g. [20], Ch. III, §1). The direct sum property of a measure \(\mu\) means that the Boolean algebra \(\mathcal{P}(L^\infty(\Omega, \Sigma, \mu))\) of all projections of a commutative von Neumann algebra \(L^\infty(\Omega, \Sigma, \mu)\) is order complete, and for any non-zero \(\chi_E \in \mathcal{P}(L^\infty(\Omega, \Sigma, \mu))\) there exists a non-zero projection \(\chi_F \leq \chi_E\) such that \(\mu(F) < \infty\), where \(E, F \in \Sigma, \chi_E(\omega) = 1, \omega \in E\) and \(\chi_E(\omega) = 0,\) when \(\omega \notin E\).

Consider the \(*\)-algebra \(LS(M) = S(M) = L^0(\Omega, \Sigma, \mu)\) of all measurable almost everywhere finite complex-valued functions defined on \((\Omega, \Sigma, \mu)\) (functions that are equal almost everywhere are identified). Define on \(L^0(\Omega, \Sigma, \mu)\) the local measure topology \(t(L^\infty(\Omega))\), that is, the Hausdorff vector topology, whose base of neighbourhoods of zero is given by

\[
W(B, \varepsilon, \delta) := \{ f \in L^0(\Omega, \Sigma, \mu) : \text{there exists a set } E \subseteq \Sigma \text{ such that } E \subseteq B, \mu(B \setminus E) \leq \delta, f\chi_E \in L^\infty(\Omega, \Sigma, \mu), \|f\chi_E\|_{L^\infty(\Omega, \Sigma, \mu)} \leq \varepsilon\},
\]

where \(\varepsilon, \delta > 0, B \in \Sigma, \mu(B) < \infty\).

The topology \(t(L^\infty(\Omega))\) does not change if the measure \(\mu\) is replaced with an equivalent measure [21].

Now let \(M\) be an arbitrary von Neumann algebra and let \(\varphi\) be a \(*\)-isomorphism from \(\mathcal{Z}(M)\) onto the \(*\)-algebra \(L^\infty(\Omega, \Sigma, \mu)\). Denote by \(L^+(\Omega, \Sigma, m)\) the set of all measurable real-valued functions defined on \((\Omega, \Sigma, \mu)\) and taking values in the extended half-line \([0, \infty)\) (functions that are equal almost everywhere are identified). Let \(D : \mathcal{P}(M) \rightarrow L^+(\Omega, \Sigma, \mu)\) be a dimensional function on \(\mathcal{P}(M)\) [19].

For arbitrary scalars \(\varepsilon, \delta > 0\) and a set \(B \subseteq \Sigma, \mu(B) < \infty\), we set

\[
V(B, \varepsilon, \delta) := \{ x \in LS(M) : \text{there exist } p \in \mathcal{P}(M), z \in \mathcal{P}(\mathcal{Z}(M)), \text{such that }\]
\[
\phi(xp) \in M, \|xp\|_M \leq \varepsilon, \varphi(z^+), D(z^+) \leq \varepsilon \varphi(z)\}.
\]

It was shown in [21] that the system of sets

\[(x + V(B, \varepsilon, \delta) : x \in LS(M), \varepsilon, \delta > 0, B \in \Sigma, \mu(B) < \infty)\]

defines a Hausdorff vector topology \(t(M)\) on \(LS(M)\) such that the sets of (1) form a neighbourhood base of an operator \(x \in LS(M)\). The topology \(t(M)\) on \(LS(M)\) is called the local measure topology. It is known that \((LS(M), t(M))\) is a complete topological \(*\)-algebra, and the topology \(t(M)\) does not depend on a choice of dimension function \(D\) and on a choice of \(*\)-isomorphism \(\varphi\) (see e.g. [12], §3.5, 3.5).

Let us mention the following important property of the topology \(t(M)\).

**Proposition 2.2.** [1, Proposition 2.5] The von Neumann algebra \(M\) is dense in \((LS(M), t(M))\).

Let \(M\) be a semifinite von Neumann algebra acting on the Hilbert space \(H\), let \(\tau\) be a faithful normal semifinite trace on \(M\). An operator \(x \in S(M)\) is called \(\tau\)-measurable if for any \(\varepsilon > 0\) there exists a projection \(p \in \mathcal{P}(M)\) such that \(p(H) \subseteq \mathcal{D}(x)\) and \(\tau(p^+) < \varepsilon\).

The set \(S(M, \tau)\) of all \(\tau\)-measurable operators is a \(*\)-subalgebra of \(S(M)\) and \(M \subseteq S(M, \tau)\). If the trace \(\tau\) is finite, then \(S(M, \tau) = S(M) = LS(M)\).
Let $t_\tau$ be the measure topology \cite{13} on $S(M, \tau)$ whose base of neighbourhoods of zero is given by

$$U(\varepsilon, \delta) = \{x \in S(M, \tau) : \text{ there exists a projection } p \in P(M),$$

such that $\tau(p^+) \leq \delta$, $xp \in M$, $\|xp\|_M \leq \varepsilon\}, \varepsilon > 0, \delta > 0$.

The pair $(S(M, \tau), t_\tau)$ is a complete metrizable topological $*$-algebra. Here, the topology $t_\tau$ majorizes the topology $t(M)$ on $S(M, \tau)$ and, if $\tau$ is a finite trace, the topologies $t_\tau$ and $t(M)$ coincide \cite{6}.

3. Continuity of derivations on a $C^*$-algebra $M$ with values in a quasi-normed $M$-bimodule

Let $M$ be a $C^*$-algebra with identity $1$ and $X$ be an arbitrary bimodule over $M$ with bilinear mappings $(a, x) \rightarrow ax, (a, x) \rightarrow xa : M \times X \rightarrow X$ such that $1x = x1$ for all $x \in X$. By introduction of algebraic operation $\lambda x := (\lambda 1)x, \lambda \in \mathbb{C}, x \in X$, the $M$-bimodule $X$ become a complex linear space.

Recall that a real function $\| \cdot \|$ on a complex linear space $X$ is called a quasi-norm on $X$, if there exists a constant $C \geq 1$ such that for all $x, y \in X, \alpha \in \mathbb{C}$ the following properties hold:

(i) $\|x\| \geq 0, \|x\| = 0 \iff x = 0$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$;

(iii) $\|x + y\| \leq C(\|x\| + \|y\|)$.

The couple $(X, \| \cdot \|)$ is called a quasi-normed space and the least of all constants $C$ satisfying the inequality (iii) above is called the modulus of concavity of the quasi-norm $\| \cdot \|$ and denoted by $C_X$. Every quasi-normed space $(X, \| \cdot \|)$ is a locally bounded Hausdorff topological vector space (see e.g. \cite{9}).

If an $M$-bimodule $X$ is equipped with a quasi-norm $\| \cdot \|_X$, then the couple $(X, \| \cdot \|_X)$ is called a quasi-normed $M$-bimodule if for all $x, a, b \in M$ the equality

$$\|axb\|_X \leq \|a\|_M \|b\|_M \|x\|_X,$$

holds, where $\| \cdot \|_M$ is the $C^*$-norm on $M$.

Let $X$ be an $M$-bimodule over a $C^*$-algebra $M$. A linear mapping $\delta : M \rightarrow X$ is called a derivation, if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in M$. A derivation $\delta : M \rightarrow X$ is called inner, if there exists an element $d \in X$, such that $\delta(x) = [d, x] = dx - xd$ for all $x \in M$.

If $(X, \| \cdot \|_X)$ is a quasi-normed $M$-bimodule, then, by \cite{2} every inner derivation $\delta : M \rightarrow X$ is a continuous linear mapping from $(M, \| \cdot \|_M)$ into $(X, \| \cdot \|_X)$.

In the following theorem we show that every derivation $\delta : M \rightarrow X$ is continuous and strengthen the Ringrose Theorem by omitting the assumption of completeness of the space $(X, \| \cdot \|_X)$. Our proof uses the original proof of J.Ringrose \cite{15} Theorem 2).

Theorem 3.1. Let $(X, \| \cdot \|_X)$ be a quasi-normed $M$-bimodule. Then every derivation $\delta : M \rightarrow X$ is a continuous mapping from $(M, \| \cdot \|_M)$ into $(X, \| \cdot \|_X)$.

Proof. Set

$$J = \{x \in M| \text{ the mapping } u \mapsto \delta(xu) \text{ from } M \text{ into } X \text{ is continuous }\}.$$

Exactly as in \cite{15} Theorem 2 we obtain that $J$ is a two-sided ideal in $M$. 
Since \( \delta(xu) = \delta(x)u + x\delta(u) \) we have
\[
J = \{ x \in \mathcal{M} | S_x(u) := x\delta(u) \text{ is continuous map from } \mathcal{M} \text{ into } X \}.
\]
Let us show that \( J \) is a closed ideal in \( \mathcal{M} \). Let \( \{ x_n \}_{n=1}^{\infty} \subset J, x \in \mathcal{M} \) and \( \| x_n - x \|_\mathcal{M} \to 0 \) as \( n \to \infty \). Then every \( S_{x_n} \) is a continuous mapping from \( \mathcal{M} \) into \( X \) and for every fixed \( u \in \mathcal{M} \) we have
\[
\| S_x(u) - S_{x_n}(u) \|_X = \| x\delta(u) - x_n\delta(u) \|_X \leq \| x - x_n \|_\mathcal{M} \| \delta(u) \|_X \to 0.
\]

The principle of uniform boundedness (see e.g. \[10, \text{Theorem 2.8}] implies that \( S_x \) is continuous, that is \( x \in J \). Thus, \( J \) is a closed two-sided ideal in \( \mathcal{M} \).

Now, we show that the restriction \( \delta|_J \) is continuous. Suppose the contrary. Then there exists a sequence \( \{ x_n \}_{n=1}^{\infty} \subset J \) such that
\[
\sum_{n=1}^{\infty} \| x_n \|_\mathcal{M}^2 \leq 1 \text{ and } \| \delta(x_n) \|_X \to \infty.
\]
Set \( y = (\sum_{n=1}^{\infty} x_n x_n^*)^{1/2} \). Then \( y \in J, \| y \|_\mathcal{M} \leq 1 \) and \( x_n x_n^* \leq y^4 \). By \[13, \text{Lemma 1}] there exists \( z_n \in J \) with \( \| z_n \|_\mathcal{M} \leq 1 \) such that \( x_n = yz_n \). Hence, \( \| \delta(yz_n) \|_X = \| \delta(x_n) \|_X \to \infty \), that is the mapping \( u \mapsto \delta(uy) \) is unbounded. That contradicts the inclusion \( y \in J \). Thus, the restriction \( \delta|_J \) is a continuous mapping.

Next, we claim that the quotient algebra \( \mathcal{M}/J \) is finite-dimensional. Assume the contrary and as in \[15, \text{Theorem 2}] choose a sequence \( y_j \in \mathcal{M}_+ \) such that
\[
\| y_j \|_\mathcal{M} \leq 1, y_j^2 \notin J \text{ and } y_j y_i = 0 \text{ for } j \neq i.
\]
Since \( y_j^2 \notin J \) the mapping \( u \mapsto \delta(y_j^2 u), u \in \mathcal{M} \), is unbounded, therefore there exist \( u_j \in \mathcal{M} \) such that
\[
\| u_j \|_\mathcal{M} \leq 2^{-j} \text{ and } \| \delta(y_j^2 u_j) \|_X \geq C_X(\| \delta(y_j) \|_X + j).
\]
By setting \( z = \sum_{j=1}^{\infty} y_j u_j \in \mathcal{M} \) we have \( \| z \|_\mathcal{M} \leq 1 \) and \( y_j z = y_j^2 u_j \).

The inequality \( \| x \|_X \leq C_X \| x - y \|_X + C_X \| y \|_X, x, y \in X, \) implies that
\[
\| y_j \delta(z) \|_X = \| \delta(y_j z) - \delta(y_j) z \|_X \geq C_X^{-1} \| \delta(y_j^2 u_j) \|_X - \| \delta(y_j) z \|_X
\]
\[
\geq \| \delta(y_j) \|_X | x + j - \| \delta(y_j) \|_X \| z \|_\mathcal{M} \geq j.
\]
Now, using the inequality \( \| y_j \|_\mathcal{M} \leq 1 \) we obtain that
\[
1 \leq j^{-1} \| y_j \delta(z) \|_X \leq j^{-1} \| y_j \|_\mathcal{M} \| \delta(z) \|_X \to 0 \text{ as } j \to \infty,
\]
that is a contradiction, thus the quotient algebra \( \mathcal{M}/J \) is finite-dimensional.

Therefore, \( \mathcal{M} \) is a direct sum \( Y \oplus J \), where \( Y \) is a finite-dimensional subspace in \( \mathcal{M} \) and \( J \) is a closed ideal in \( (\mathcal{M}, \| \cdot \|_\mathcal{M}) \). Since the restrictions \( \delta|_J \) and \( \delta|_Y \) are continuous it follows that the derivation \( \delta : (\mathcal{M}, \| \cdot \|_\mathcal{M}) \to (X, \| \cdot \|_X) \) is continuous too. \( \square \)

4. DESCRIPTION OF DERIVATIONS WITH VALUES IN QUASI-NORMED \( \mathcal{M} \)-BIMODULES OF LOCALLY MEASURABLE OPERATORS

In this section we establish the main result of the present paper, which give description of all derivations acting on a von Neumann algebra \( \mathcal{M} \) with values in a quasi-normed \( \mathcal{M} \)-bimodule \( E \) of locally measurable operators affiliated with \( \mathcal{M} \).

Let \( \mathcal{M} \) be an arbitrary von Neumann algebra. A linear subspace \( E \) of \( LS(\mathcal{M}) \), is called an \( \mathcal{M} \)-bimodule of locally measurable operators if \( axb \in E \) whenever \( x \in E \) and \( a, b \in \mathcal{M} \). It is clear that \( E \) is a bimodule over the \( C^* \)-algebra \( \mathcal{M} \) in the sense of
section 3. If $\mathcal{E}$ is a $\mathcal{M}$-bimodule of locally measurable operators, $x \in \mathcal{E}$ and $x = v|x|$ is the polar decomposition of operator $x$ then $|x| = v^*x \in \mathcal{E}$ and $x^* = |x|v^* \in \mathcal{E}$. In addition, by Proposition 2.1 every $\mathcal{M}$-bimodule satisfies the following condition

\begin{equation}
\text{if } |x| \leq |y|, \quad y \in \mathcal{E}, \quad x \in LS(\mathcal{M}) \text{ then } x \in \mathcal{E}.
\end{equation}

If an $\mathcal{M}$-bimodule of locally measurable operators $\mathcal{E}$ is equipped with a quasi-norm $\| \cdot \|_\mathcal{E}$, satisfying the inequality (2), then $(\mathcal{E}, \| \cdot \|_\mathcal{E})$ is called a quasi-normed $\mathcal{M}$-bimodule of locally measurable operators.

Examples of quasi-normed $\mathcal{M}$-bimodules of locally measurable operators, which are not normed $\mathcal{M}$-bimodules, are the noncommutative $L_p$-space $L_p(\mathcal{M}, \tau)$ associated with a faithful normal semifinite trace $\tau$ for $p \in (0, 1)$.

It is easy to see that for the quasi-norm $\| \cdot \|_\mathcal{E}$ on a quasi-normed $\mathcal{M}$-bimodule of locally measurable operators $\mathcal{E}$ the following properties hold:

\begin{equation}
\|x\|_\mathcal{E} = \|x^*\|_\mathcal{E} = \|x\|_\mathcal{E} \text{ for any } x \in \mathcal{E};
\end{equation}

\begin{equation}
\|y\|_\mathcal{E} \leq \|x\|_\mathcal{E} \text{ for any } x, y \in \mathcal{E}, \quad 0 \leq y \leq x \quad \text{(see Proposition 2.1)}.
\end{equation}

Recall that two projections $e, f \in \mathcal{P}(\mathcal{M})$ are called equivalent (notation: $e \sim f$) if there exists a partial isometry $u \in \mathcal{M}$ such that $u^*u = e$ and $uu^* = f$. For projections $e, f \in \mathcal{P}(\mathcal{M})$ notation $e \leq f$ means that there exists a projection $q \in \mathcal{P}(\mathcal{M})$ such that $e \sim q \leq f$.

**Proposition 4.1.** Let $(\mathcal{E}, \| \cdot \|_\mathcal{E})$ be a quasi-normed $\mathcal{M}$-bimodule of locally measurable operators, $p, q, k \in \mathcal{P}(\mathcal{M}) \cap \mathcal{E}, k = 1, 2, \ldots, q \in \mathcal{P}(\mathcal{M}), q \leq p$. Then

\begin{equation}
\sup_{1 \leq k \leq n} p_k \leq p, \quad \|p_k\|_\mathcal{E} \leq \sum_{k=1}^{n} C^k_p \|p_k\|_\mathcal{E}.
\end{equation}

**Proof.** If $q \sim e \leq p$ and $u$ is a partial isometry from $\mathcal{M}$ such that $u^*u = q, uu^* = e$, then $q = u^*e u \in \mathcal{E}$ and $\|q\|_\mathcal{E} \leq \|u^*\|_\mathcal{M} \|u\|_\mathcal{M}\|\|_\mathcal{E} \leq \|p\|_\mathcal{E}$.

Since $p_1 \lor p_2 - p_1 \sim p_2 - p_1 \land p_2 \leq p_2$ we have that $p_1 \lor p_2 - p_1 \in \mathcal{E}$ and $\|p_1 \lor p_2 - p_1\|_\mathcal{E} \leq \|p_2\|_\mathcal{E}$. Hence, $p_1 \lor p_2 = (p_1 \lor p_2 - p_1) + p_1 \in \mathcal{E}$ and $\|p_1 \lor p_2\|_\mathcal{E} = \|p_1 \lor p_2 - p_1 + p_1\|_\mathcal{E} \leq C_{\mathcal{E}}(\|p_1 \lor p_2 - p_1\|_\mathcal{E} + \|p_1\|_\mathcal{E}) \leq C_{\mathcal{E}}\|p_1\|_\mathcal{E} + C_{\mathcal{E}}\|p_2\|_\mathcal{E}$, in particular, $\|p_1 \lor p_2\|_\mathcal{E} \leq C_{\mathcal{E}}\|p_1\|_\mathcal{E} + C_{\mathcal{E}}^2\|p_2\|_\mathcal{E}$ since $C_{\mathcal{E}} \geq 1$.

Further, proceed by the induction we have that $\sup_{1 \leq k \leq n} p_k \in \mathcal{E}$ and

\begin{equation}
\sup_{1 \leq k \leq n} p_k \leq \|p_1 \lor (\sup_{2 \leq k \leq n} p_k)\|_\mathcal{E} \leq C_{\mathcal{E}}\|p_1\|_\mathcal{E} + C_{\mathcal{E}}\|p_{k+1}\|_\mathcal{E} \leq n \sum_{k=1}^{n-1} C^k_{\mathcal{E}}\|p_{k+1}\|_\mathcal{E} + n C^k_{\mathcal{E}}\|p_k\|_\mathcal{E}.
\end{equation}

Let $\delta : \mathcal{M} \to LS(\mathcal{M})$ be an arbitrary derivation, that is $\delta$ is a linear mapping such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{M}$. In [3, Lemma 3.1] it is proved that $\delta(x) = 0$ and $\delta(xz) = z\delta(x)$ for all $x \in \mathcal{M}, z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$. In particular, $\delta(z\mathcal{M}) \subseteq zLS(\mathcal{M})$ and the restriction $\delta(z)$ of the derivation $\delta$ to $z\mathcal{M}$ is a derivation on $z\mathcal{M}$ with values in $zLS(\mathcal{M}) = LS(z\mathcal{M})$. 

Let $\delta$ be a derivation from $\mathcal{M}$ with values in a $\mathcal{M}$-bimodule $\mathcal{E}$ of locally measurable operators. Let us define a mapping
$$\delta^* : \mathcal{M} \to \mathcal{E},$$
by setting $\delta^*(x) = (\delta(x^*))^*$, $x \in \mathcal{M}$. A direct verification shows that $\delta^*$ is also a derivation from $\mathcal{M}$ with values in $\mathcal{E}$.

A derivation $\delta$ is said to be self-adjoint, if $\delta = \delta^*$. Every derivation $\delta$ from $\mathcal{M}$ with values in a $\mathcal{M}$-bimodule $\mathcal{E}$ of locally measurable operators can be represented in the form $\delta = \text{Re}(\delta) + i\text{Im}(\delta)$, where $\text{Re}(\delta) = (\delta + \delta^*)/2$, $\text{Im}(\delta) = (\delta - \delta^*)/2i$ are self-adjoint derivations from $\mathcal{M}$ with values in $\mathcal{E}$.

For the proof of the assertion that every derivation $\delta : \mathcal{M} \to \mathcal{E}$ is inner we need the following significant theorem establishing that every $t(\mathcal{M})$-continuous derivation acting on the *-algebra $LS(\mathcal{M})$ is inner.

**Theorem 4.2.** [4, Theorem 4.1] Every derivation on the algebra $LS(\mathcal{M})$ continuous with respect to the topology $t(\mathcal{M})$ is inner derivation.

For application of the Theorem 4.2 in our case, when $\delta$ is a derivation acting on $\mathcal{M}$ with values in quasi-normed $\mathcal{M}$-bimodule $(\mathcal{E}, \| \cdot \|_\mathcal{E})$ of locally measurable operators, we need $t(\mathcal{M})$-continuity of $\delta$. In the following Proposition 4.5 using Theorem 3.1 we prove this property of derivation $\delta$.

Let $\tau$ be a faithful normal finite trace $\tau$ on the von Neumann algebra $\mathcal{M}$. In this case, the algebra $\mathcal{M}$ is finite. Moreover, $LS(\mathcal{M}) = S(\mathcal{M}) = S(\mathcal{M}, \tau)$, $t(\mathcal{M}) = t_\tau$ and $(LS(\mathcal{M}), t(\mathcal{M}))$ is an $F$-space [12, §§3.4,3.5].

The following lemma shows that quasi-normed $\mathcal{M}$-bimodule $(\mathcal{E}, \| \cdot \|_\mathcal{E})$ of locally measurable operators is continuously embedded into $(LS(\mathcal{M}), t(\mathcal{M}))$.

**Lemma 4.3.** If $\{a_n\}_{n=1}^\infty \subset \mathcal{E}$ and $\|a_n\|_\mathcal{E} \to 0$, then $a_n \xrightarrow{t(\mathcal{M})} 0$.

**Proof.** Since the trace $\tau$ is finite, we have that $t(\mathcal{M}) = t_\tau$, and therefore it is sufficient to show that $a_n \xrightarrow{t_\tau} 0$. Suppose the contrary. Passing to a subsequence, if necessary, we may choose $\varepsilon, \delta > 0$ such that

(6) $\tau(1 - E_\varepsilon(|a_n|)) > \delta$;

(7) $\|a_n\|_\mathcal{E} < (2C_\mathcal{E})^{-n}\varepsilon$

for all $n \in \mathbb{N}$.

Set
$$p_n = 1 - E_\varepsilon(|a_n|), q_n = \sup_{m \geq n} p_m, q = \inf_{n \geq 1} q_n.$$ Since $\tau$ is a normal finite trace and $\tau(p_n) > \delta$ (see (6)), we have that

(8) $\tau(q) \geq \delta$.

The inequalities (7) and $0 \leq \varepsilon p_n \leq |a_n|$ imply that

(9) $\|p_n\|_\mathcal{E} \leq \varepsilon^{-1}\|a_n\|_\mathcal{E} < (2C_\mathcal{E})^{-n}$.

Set
$$r_{n,s} = \left( \bigvee_{m=n}^{n+s} p_m \right) \wedge q.$$
It is clear that $r_{n,s} \leq r_{n,s+1}$. Since $\tau(1) < \infty$ the set $\mathcal{P}(M)$ is an orthocomplemented complete modular lattice, and therefore $\mathcal{P}(M)$ is a continuous geometry \[1\]. In particular,

$$
\sup_{s \geq 1} r_{n,s} = \left( \bigvee_{m=n}^{\infty} p_m \right) \wedge q = q_n \wedge q = q.
$$

Consequently, for all $n \in \mathbb{N}$ there exists an integer $s_n$ such that $\tau(q - r_{n,s_n}) < 2^{-n}$. Set $e_n = \inf_{m \geq n} r_{m,s_m}$. The sequence of projections $\{e_n\}$ is increasing, moreover, $e_n \leq q$ and

$$
\tau(q - e_n) = \tau(q - \inf_{m \geq n} r_{m,s_m}) = \tau(\sup_{m \geq n} (q - r_{m,s_m})) < 2^{-(n-1)}.
$$

Therefore, $e_n \uparrow q$.

By Proposition 4.1 we have

$$
\|e_n\| \leq \|r_{n,s_n}\| \leq \left\| \bigvee_{k=n}^{n+s} p_k \right\| \leq \sum_{k=n}^{n+s} C_k \|p_k\| \leq \sum_{k=n}^{n+s} 2^{-k} < 2^{-(n-1)}.
$$

Since the sequence $e_n$ is increasing, the last inequality implies that $e_n = 0$ for all $n \in \mathbb{N}$. Using the convergence $e_n \uparrow q$ we obtain $q = 0$, that contradicts \(8\). $\square$

For the proof of the following Proposition 4.5 on $t(M)$-continuity of a derivation $\delta : M \to \mathcal{E}$ we need the following lemma from \[1\].

**Lemma 4.4.** \[4\] Lemma 6.8] If $M$ is a von Neumann algebra with a faithful normal finite trace $\tau$, $\{a_n\}_{n=1}^{\infty} \subset LS(M)$ and $a_n \xrightarrow{t(M)} 0$, then there exists a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $a_{n_k} = b_k + c_k$, where $b_k \in M$, $c_k \in LS(M)$, $k \in \mathbb{N}$, $\|b_k\|_M \to 0$ and $s(|c_k|) \xrightarrow{t(M)} 0$.

The following proposition is crucial step in the proof that every derivation $\delta : M \to \mathcal{E}$ is inner.

**Proposition 4.5.** Let $M$ be a von Neumann algebra with a faithful normal finite trace $\tau$, let $(\mathcal{E}, \| \cdot \|_{\mathcal{E}})$ be a quasi-normed $M$-bimodule of locally measurable operators and let $\delta$ be a derivation on $LS(M)$ such that $\delta(M) \subset \mathcal{E}$. Then the derivation $\delta$ is $t(M)$-continuous.

**Proof.** Since $(LS(M), t(M))$ is an $F$-space for the proof of $t(M)$-continuity of the mapping $\delta$ it is sufficient to show that the graph of the linear operator $\delta$ is closed.

Suppose that the graph of the operator $\delta$ is not closed. Then there exist a sequence $\{a_n\}_{n=1}^{\infty} \subset LS(M)$ and $0 \neq b \in LS(M)$ such that $a_n \xrightarrow{t(M)} 0$ and $\delta(a_n) \xrightarrow{t(M)} b$.

By Lemma 4.4 passing, if necessary, to a subsequence, we may assume that $a_n = b_n + c_n$, where $b_n \in M$, $c_n \in LS(M)$, $n \in \mathbb{N}$, $\|b_n\|_M \to 0$ and $s(|c_n|) \xrightarrow{t(M)} 0$. As $n \to \infty$.

Since the restriction $\delta|_M$ of the derivation $\delta$ to the von Neumann algebra $M$ is a derivation from $M$ into the quasi-normed $M$-bimodule $(\mathcal{E}, \| \cdot \|_{\mathcal{E}})$, by Theorem 3.1 we have $\|\delta(b_n)\|_{\mathcal{E}} \to 0$. Lemma 4.3 implies that $\delta(b_n) \xrightarrow{t(M)} 0$.

By verbatim repetition of the second part of the proof \[4\] Lemma 6.9 we obtain that $\delta(c_n) \xrightarrow{t(M)} 0$. 


Thus, $\delta(a_n) = \delta(b_n) + \delta(c_n) \xrightarrow{t(M)} 0$, that contradicts to the inequality $b \neq 0$.
Consequently, the operator $\delta$ has closed graph, therefore $\delta$ is $t(M)$-continuous. □

For the proof of the main result we also need the following property of the algebra $LS(M)$.

**Theorem 4.6.** [3] Theorem 1] Let $M$ be a von Neumann algebra and $a \in LS_n(M)$. Then there exist a self-adjoint operator $c$ in the centre of the $*$-algebra $LS(M)$ and a family $\{u_\varepsilon\}_{\varepsilon > 0}$ of unitary operators from $M$ such that

$$
\|a, u_\varepsilon\| \geq (1 - \varepsilon)|a - c|.
$$

Now, we give the main result of this paper.

**Theorem 4.7.** Let $M$ be a von Neumann algebra and let $(E, \| \cdot \|_E)$ be a quasi-normed $M$-bimodule of locally measurable operators. Then any derivation $\delta : M \to E$ is inner, that is there exists an element $d \in E$ such that $\delta(x) = [d, x], x \in M$, in addition $\|d\|_E \leq 2C_E \|\delta\|_{M \to E}$. If $\delta^* = \delta$ or $\delta^* = -\delta$ then $d$ may be chosen so that $\|d\|_E \leq \|\delta\|_{M \to E}$.

**Proof.** Let $\overline{\delta}$ be a derivation on $LS(M)$ such that $\overline{\delta}(x) = \delta(x)$ for all $x \in M$ (see [3] Theorem 4.8).

Choose pairwise orthogonal central projections $\{z_\varepsilon, z_j\}_{j \in J}$ such that $z_\varepsilon + \sup_{j \in J} z_j = 1$, $z_\varepsilon M$ is a properly infinite von Neumann algebra and on every von Neumann algebra $z_j M$ exists a faithful normal finite trace. By [3] Theorem 3.3 the derivation $\overline{\delta}^{(z_\varepsilon)} := \overline{\delta}_{LS(z_\varepsilon M)} : LS(z_\varepsilon M) \to LS(z_\varepsilon M)$ is $t(z_\varepsilon M)$-continuous. Since the von Neumann algebra $z_j M$ is finite with a faithful normal finite trace and for the derivation $\overline{\delta}^{(z_j)} := \overline{\delta}_{LS(z_j M)} : LS(z_j M) \to LS(z_j M)$ the inclusion $\overline{\delta}^{(z_\varepsilon)}(z_j M) = \delta(z_j)(z_j M) \subset z_j E$ holds, Proposition 4.5 implies that $\overline{\delta}^{(z_j)}$ is also $t(z_j M)$-continuous for all $j \in J$. Therefore, by [3] Corollary 2.8], the derivation $\overline{\delta}$ is $t(M)$-continuous. By Theorem 4.2 the derivation $\overline{\delta}$ is inner, that is there exists an element $a \in LS(M)$, such that $\overline{\delta}(x) = [a, x]$ for all $x \in LS(M)$. It is clear that $[a, M] = \overline{\delta}(M) = \delta(M) \subset E$. Let us show that we can choose $d \in E$ such that $\delta(x) = [d, x]$ for all $x \in M$.

Let $a_1 = Re(a) = (a + a^*)/2, a_2 = Im(a) = (a - a^*)/2i$. Since $[a^*, x] = -[a, x^*] \in E$ for any $x \in M$, it follows that $[a_1, x] = [a + a^*, x]/2 \in E$ and $[a_2, x] = [a - a^*, x]/2i \in E$ for all $x \in M$.

By Theorem 4.6 and taking $\varepsilon = 1/2$ in (10) we obtain that there exist self-adjoint operators $c_1, c_2$ in the centre of the $*$-algebra $LS(M)$ and unitary operators $u_1, u_2 \in M$ such that

$$2|a_i, u_1| \geq |a_i, c_1|, \; i = 1, 2.$$

Since $[a_i, u_1] \in E$ and $E$ is $M$-bimodule of locally measurable operators we have that $d_i := a_i - c_1 \in E, \; i = 1, 2$ (see [3]). Therefore $d = d_1 + id_2 \in E$. Since $c_1, c_2$ are central elements from $LS(M)$ it follows that $\delta(x) = [a, x] = [d, x]$ for all $x \in M$.

Now, suppose that $\delta^* = \delta$. In this case, $[d + d^*, x] = [d, x] - [d, x^*] = \delta(x) - \delta(x^*)^* = \delta(x) - \delta^*(x) = 0$ for any $x \in M$. Consequently, the operator $Re(d) = (d + d^*)/2$ commutes with every elements from $M$, and by Proposition 2.2 $Re(d)$ is a central element in the algebra $LS(M)$. Therefore we may suggest that $\delta(x) = [d, x], \; x \in M$, where $d = ia, \; a \in E_A$. According to Theorem 4.6 there exist $c = c^*$
from the centre of the algebra $LS(M)$ and a family \( \{u_x\}_{\varepsilon > 0} \) of unitary operators from $M$ such that

\[
||a, u_x|| \geq (1 - \varepsilon)|a - c|.
\]

For $b = ia - ic$ and $\varepsilon = 1/2$ we have

\[
||b|| = ||a - c|| \leq 2||a, u_{1/2}|| = 2||-id, u_{1/2}|| = 2||d, u_{1/2}|| \in E.
\]

Consequently, $b \in E$, moreover,

\[
\delta(x) = [d, x] = [ia, x] = [b, x]
\]

for all $x \in M$. Since

\[
(1 - \varepsilon)||b|| = (1 - \varepsilon)||a - c|| \leq ||a, u_x|| = ||d, u_x|| = ||\delta(u_x)||,
\]

it follows that

\[
(1 - \varepsilon)||b||_E \leq ||\delta(u_x)||_E \leq ||\delta||_{M \to E}
\]

for all $\varepsilon > 0$, that implies the inequality $||b||_E \leq ||\delta||_{M \to E}$.

If $\delta^* = -\delta$, then taking $Im(d)$ instead of $Re(d)$ and repeating previous proof we obtain that $\delta(x) = [b, x]$, where $b \in E$ and $||b||_E \leq ||\delta||_{M \to E}$.

Now, suppose that $\delta \neq \delta^*$ and $\delta \neq -\delta^*$. Equality \( 1 \)
implies that

\[
||\delta^*||_{M \to E} = \sup\{||\delta(x)^*||_E : ||x||_M \leq 1\}
\]

\[= \sup\{||\delta(x)||_E : ||x||_M \leq 1\} = ||\delta||_{M \to E}.
\]

Consequently,

\[
||Re(\delta)||_{M \to E} = 2^{-1}||\delta + \delta^*||_{M \to E} \leq C_E ||\delta||_{M \to E}.
\]

Similarly, $||Im(\delta)||_{M \to E} \leq C_E ||\delta||_{M \to E}$. Since $(Re(\delta))^* = Re(\delta)$, $(Im(\delta))^* = Im(\delta)$, there exist $d_1, d_2 \in E$, such that $Re(\delta)(x) = [d_1, x]$, $Im(\delta)(x) = [d_2, x]$ for all $x \in M$ and $||d_i||_E \leq ||\delta||_{M \to E}$, $i = 1, 2$. Taking $d = d_1 + id_2$, we have that $d \in E$, $\delta(x) = (Re(\delta) + i \cdot Im(\delta))(x) = [d_1, x] + id_2, x = [d, x]$ for all $x \in M$, in addition $||d||_E \leq 2C_E ||\delta||_{M \to E}$. \( \square \)

Now, let us give an application of Theorem 4.7 to derivations on $M$ with values in quasi-normed symmetric spaces.

Let $M$ be a semifinite von Neumann algebra and let $\tau$ be a faithful normal semifinite trace on $M$. Let $S(M, \tau)$ be the $*$-algebra of all $\tau$-measurable operators affiliated with $M$. For each $x \in S(M, \tau)$ and $t > 0$ we define the decreasing rearrangement (or generalised singular value function) by setting

\[
\mu_t(x) = \inf\{\lambda > 0 : \tau(E^+_\lambda(||x||)) \leq t\}
\]

\[= \inf\{||x(1 - e)||_M : e \in \mathcal{P}(M), \tau(e) \leq t\}.
\]

Let $E$ be a linear subspace in $S(M, \tau)$ equipped with a quasi-norm $||\cdot||_E$ satisfying the following condition:

If $x \in S(M, \tau)$, $y \in E$ and $\mu_t(x) \leq \mu_t(y)$ then $x \in E$ and $||x||_E \leq ||y||_E$.

In this case, the pair $(E, ||\cdot||_E)$ is called quasi-normed symmetric spaces of $\tau$-measurable operators. It is easy to see that every quasi-normed symmetric space of measurable operators is a quasi-normed $M$-bimodule of locally measurable operators, and therefore Theorem 4.7 implies the following
Corollary 4.8. Let $(\mathcal{E}, \| \cdot \|_{\mathcal{E}})$ be a quasi-normed symmetric spaces of $\tau$-measurable operators, affiliated with a semifinite von Neumann algebra $\mathcal{M}$ with a faithful semifinite normal trace $\tau$. Then any derivation $\delta : \mathcal{M} \to \mathcal{E}$ is continuous and there exists $d \in \mathcal{E}$ such that $\delta(x) = [d, x]$ for all $x \in \mathcal{M}$ and $\|d\|_{\mathcal{E}} \leq 2C_{\mathcal{E}} \|\delta\|_{\mathcal{M} \to \mathcal{E}}$.

In particular, Corollary 4.8 implies that every derivation on $\mathcal{M}$ with values in noncommutative $L_p$-spaces $L_p(\mathcal{M}, \tau)$, $0 < p \leq \infty$, is inner.

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