Lower Bounds and Algorithm for Partially Replicated Causally
Consistent Shared Memory *

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Abstract

Distributed shared memory systems maintain multiple replicas of the shared memory locations. Maintaining causal consistency in such systems has received significant attention in the past. However, much of the previous literature focuses on full replication wherein each replica stores a copy of all the locations in the shared memory. In this paper, we investigate causal consistency in partially replicated systems, wherein each replica may store only a subset of the shared data. To achieve causal consistency, it is necessary to ensure that, before an update is performed at any given replica, all causally preceding updates must also be performed. Achieving this goal requires some mechanism to track causal dependencies. In the context of full replication, this goal is often achieved using vector timestamps, with the number of vector elements being equal to the number of replicas. Building on the past work, this paper makes three key contributions:

- We develop lower bounds on the size of the timestamps that must be maintained in order to achieve causal consistency in partially replicated systems. The size of the timestamps is a function of the manner in which the replicas share data, and the set of replicas accessed by each client.
- We present an algorithm to achieve causal consistency in partially replicated systems using simple vector timestamps.
- We present some optimizations to improve the overhead of the timestamps required with partial replication.

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1 Introduction

The focus of this paper is on achieving causal consistency in a partially replicated distributed shared memory (DSM) system that provides the abstraction of shared read/write registers. Each replica (which is maintained by a server) may be partial in the sense that it stores copies of a subset of the shared registers. Full replication is obtained as a special case when each replica stores copies of all the shared registers. Figure 1 illustrates the client-server architectures considered here. In the basic architecture in Figure 1a, client i only accesses replica i, and no other client directly communicates with replica i. Thus, client i performs read/write operations only on registers whose copies are stored at replica i. In the general architecture in Figure 1b, each client is associated with an arbitrary subset of the replicas. For instance, as shown, client 1 is associated with replicas 1 and 2, whereas client 2 only communicates with replica 1. Each client may only perform read/write operations on the registers whose copy is stored in at least one of the replicas associated with that client. For instance, in Figure 1b, client 1 may only read/write the registers whose copy is stored at replicas 1 and/or 2.

We primarily consider the basic architecture in Figure 1a and describe how the results are extended to the general architecture.

In recent years, causal consistency model for the shared memory has received significant attention due to its emerging applications, for instance, in the context of social networking. Intuitively, causal consistency ensures that before a client can view an update to a shared register is must be able to view all the causally preceding updates (as noted above, each client may only access registers stored at one or subset of the replicas, depending on which architecture we consider).

In the context of full replication, there has been significant effort in designing and implementing causally consistent shared memory systems, such as Lazy Replication [16], COPS [18], Orbe [8], SwiftCloud [32] and GentleRain [9]. While much of the past work on shared memory has addressed full replication, there is growing interest in partial replication [5, 19, 2, 18, 13, 6], due to the potential storage efficiencies that can be attained with partial replication. Several researchers have observed that partial replication can require a large amount of metadata in order to track causal dependencies accurately under partial replication [2, 18, 13, 6]. Most relevant to this paper is the work of Helary and Milani [13, 23]. In this paper, we will use the notion of share graph.
introduced in their work for the basic architecture (Figure 1a), and also augment it to apply it to the general architecture (Figure 1b). The main contributions of this paper are as follows:

1. **Lower bound on timestamp size:** We consider algorithms to achieve causal consistency that assign a timestamp to the state of each replica. Vector timestamps have been used in prior work on full replication, with the vector length being equal to the number of replicas (e.g., [16]). The timestamps are used to ensure that updates are performed on a replica in a causal order. We obtain lower bounds on the size of such timestamps for partial replication and full replication both. The size of the timestamp is dependent on which replicas maintain a copy of each shared register. Secondly, for the general architecture (Figure 1b), the lower bound is also dependent on the set of replicas associated with each client. In deriving these lower bounds, we also make a correction to a claim presented in prior work in [13, 23].

2. **Upper bound:** We present an algorithm that achieves causal consistency under partial replication. The algorithm requires each replica to maintain timestamps that counts the number of updates performed on an appropriately chosen subset of shared registers.

3. **We present optimizations that help reduce the size of the timestamps under partial replication.**

**Related work:** Aside from the past work cited above, the prior work on timestamps for capturing causality in message-passing is also relevant to this paper, in particular, the work on vector timestamps [20, 11], and mechanisms to reduce timestamp size by exploiting additional information such as communication topology [26, 21]. For message-passing model, the lower bounds for vector timestamps [11] and general timestamps [22] are well-studied. However, the previously obtained lower bounds for the message-passing model do not directly apply to the shared memory model. In principle, the bounds for the basic architecture (Figure 1a) would be analogous to bounds that one may obtain for causal group communication with overlapping groups. However, such bounds for overlapping groups are not obtained previously, to the best of our knowledge. Importantly, for the general architecture (Figure 1b) there is no meaningful analog in the message-passing context. We present results for both architectures in Figure 1.

Lazy Replication [16] is a classic client-server framework for providing causal consistency via maintaining vector clocks, with the vector size being equal to the number of replicas. The algorithm presented in this paper uses a similar structure as Lazy Replication, with the key difference being the definition of the timestamps. Milani has systematically studied mechanisms to implement causal consistency [23], and presented a protocol for achieving causal consistency [3]. [25, 4] studied protocols for implementing partially replicated causal objects, which is similar to the architecture in Figure 1a, but the size of the metadata is large. Hélary and Milani identified the difficulty of efficient implementation under causal consistency for partial replication, and studied several weakened consistency models that lie between causal consistency and PRAM consistency [13]. Reynal and Ahamad proposed an algorithm that piggybacked control message of size \(O(mn)\) in most general case, where \(n\) is the number of replicas and \(O(m)\) is the number of objects. Shen et al. [27] proposed two algorithms, Full-Track and Opt-Track, to achieve causal consistency for partial replication under relation \(\rightarrow_{co}\) proposed by Milani [3]. Kshemkalyani and Hsu’s research on approximate causal consistency sacrifices accuracy of causal consistencies to reduce the meta-data [15, 14]. Our prior work on this problem [31] proposed an algorithm of achieving causal consistency
in partial replication for the general structure as in Figure 1b, but with timestamps that are not optimal. Also, no lower bound of the timestamp size is presented in [31].

In a somewhat different line of research, concurrent timestamp systems for shared memory, which enable processes to order operations using bounded timestamps have been explored [7, 10, 12]; the problem addressed in our work is distinct from this prior work.

2 System Model

The system is assumed to be asynchronous.

Replicas: The system consists of \( R \) replicas numbered 1 through \( R \). Each replica \( i \) stores copies of a subset of shared registers named \( X_i \). With full replication, \( X_i = X_j \) for all replicas \( i, j \). With partial replication, it is possible that \( X_i \neq X_j \) for \( i \neq j \). We also define \( X_{ij} = X_i \cap X_j \), the set of registers stored at replicas \( i \) and \( j \) both. For instance, in partial replication with four replicas, we may have

- \( X_1 = \{x\} \)
- \( X_2 = \{x, y\} \)
- \( X_3 = \{y, z\} \)
- \( X_4 = \{z\} \)

where \( x, y, z \) are shared read/write registers. In this case, \( X_{23} = \{y\} \).

Clients: There are \( C \) clients numbered 1 through \( C \). In the basic architecture (Figure 1a), client \( i \) is associated with replica \( i \). In the general architecture (Figure 1b), client \( i \) is associated with a subset of replicas \( R_i \). Each client \( i \) may only perform read/write operations on registers in \( \cup_{r \in R_i} X_r \), as discussed in Section 1 as well.

Communication model: Each replica can communicate with all the other replicas using point-to-point message-passing channels that are reliable, asynchronous and non-FIFO.

3 Causally Consistent Shared Memory

For convenience, hereafter we assume that each write operation on any given register writes a unique value.

Happened-before relation [17]: Let \( o_1, o_2 \) be two operations. \( o_1 \) happened-before \( o_2 \), denoted as \( o_1 \rightarrow o_2 \), if and only if at least one the following conditions is true: (i) Both \( o_1 \) and \( o_2 \) are performed by the same client, and \( o_1 \) occurs before \( o_2 \). (ii) \( o_1 \) is a write operation, and \( o_2 \) is a read operation that returns the value written by \( o_1 \). (iii) There exists operation \( o_3 \) such that \( o_1 \rightarrow o_3 \) and \( o_3 \rightarrow o_2 \).

We assume the following safety and liveness requirements for causally consistent systems.

Definition 1 (Causal consistency). Safety: Suppose that read operation \( o_3 \) on some register \( x \) returns the value written by write operation \( o_1 \) on register \( x \). Then there must not exist another write operation \( o_2 \) on register \( x \) such that \( o_1 \rightarrow o_2 \rightarrow o_3 \). Liveness: For a write operation \( o_1 \) by some client that writes value \( v \) in register \( x \), all replicas that store copies of register \( x \) should eventually be updated with the value \( v \).
In the rest of this section, and in Sections 4 and 5, we assume that the system conforms to the basic architecture (Figure 1a), even if the assumption may not necessarily be stated elsewhere. In Section 6 we extend our results to the general architecture (Figure 1b). We consider causally consistent memory systems that are implemented using the prototype algorithm presented below for the clients and replicas. Many prior systems (e.g., Lazy Replication [16]) have used algorithms with similar structures to achieve causal consistency under full replication.

**Client operations:** `read(x)` is a read operation on register `x`, and `write(x, v)` is a write operation on register `x` that writes value `v`. When performing `read(x)` or `write(x, v)` operation on register `x ∈ X_i`, client `i` sends a request to a replica `i`, and awaits the replica’s response. The response to a write operation is an acknowledgement, and the response to a read operation is a returned value.

As noted above, presently we assume that the system conforms to the basic architecture in Figure 1a. We will defer discussion of the general architecture until Section 6.

The replica operation below is specified for the basic architecture. Each replica `i` maintains a timestamp `τ_i`, which is suitably initialized. Exact implementation of the timestamp `τ_i`, and functions `advance`, `merge` and predicate `J` used in the prototype below can potentially be instantiated in many different ways.

**Replica operations:**

1. When replica `i` receives a `read(x)` request from client `i`, replica `i` responds with the value of the local copy of register `x`.

2. When replica `i` receives a `write(x, v)` request from client `i`, replica `i` performs the following steps atomically: (i) write `v` into the local copy of register `x`, (ii) update timestamp `τ_i` using function `advance`, as follows: `τ_i := advance(τ_i, i, x)`, and (iii) multicast `update(i, τ_i, x, v)` message to each other replica `k ∈ V` such that `x ∈ X_k`.

   When replica `i` has completed Step 2 as specified above, we will say that replica `i` has applied `update(i, τ_i, x, v)`, where `τ_i` here is replica `i`’s timestamp after performing step (ii) above.

3. When replica `i` receives a message `update(k, T_k, x, v)`, it adds the message to a local data structure `pending_i`.

4. At replica `i`, for any message `update(k, T_k, x, v) ∈ pending_i`, when predicate `J(τ_i, i, T_k, k)` evaluates True, replica `i` performs the following operations atomically: (i) replica `i` writes value `v` to its local copy of register `x`, (ii) updates its timestamp as `τ_i := merge(τ_i, i, T_k, k)`, and (iii) removes `update(k, T_k, x, v)` from `pending_i`.

   When replica `i` has completed Step 4 as specified above, we will say that replica `i` has applied `update(k, T_k, x, v)`.

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1 For the general architecture in Figure 1b, the client `i` may operate only on `x ∈ ∪_{j∈R_i} X_j`. For an operation on register `x`, client `i` may send its request to any replica `j ∈ R_i` such that `x ∈ X_j`. 
The timestamp $\tau_i$ and predicate $J$ above must ensure that safety and liveness properties in Definition 1 can be satisfied. The lower bound on the timestamp size presented later in the paper is obtained using this observation.

Each update is represented by a tuple of the form $(k, T_k, x, v)$, as shown in Step 4 above. We will also use short labels (such as $u_1$) to represent an update when convenient.

We say that an update $(k, T_k, x, v)$ is issued by replica $k$. We say that an update $(k, T_k, x, v)$ is applied by replica $i$ when replica $i$ modifies its local copy of register $x$ to value $v$ (note that this may occur in Step 2 above when $k = i$, and Step 4 above when $k \neq i$).

Now we introduce a happened-before relation on updates, analogous to the happened-before relation above for read/write operations.

**Definition 2** (Happened-before relation $\hookrightarrow$ for updates). Given updates $u_1$ and $u_2$, $u_1 \hookrightarrow u_2$ if and only if at least one of the following conditions is true: (i) $u_1$ and $u_2$ are both issued by the same replica, and $u_1$ is issued first. (ii) $u_1$ is applied at a replica before the same replica issues $u_2$. (iii) There exists an update $u_3$ such that $u_1 \hookrightarrow u_3$ and $u_3 \hookrightarrow u_2$.

In Step 4 above, let $u_1 = (k, T_k, x, v)$. To ensure safety property in Definition 1 the predicate $J(\tau_i, i, T_k, k)$ must not evaluate True if there exists an update $u_2$ for a write operation on a register in $X_i$ such that (i) $u_2 \hookrightarrow u_1$, and (ii) replica $i$ has not yet applied $u_2$. Also, to satisfy the liveness property in Definition 1 the predicate must eventually evaluate as True for each update in pending.

**Definition 3** (Causal past). In a given execution, let $S$ be the state of replica $i$ after applying the updates in set $U$. Causal past of state $S$ of replica $i$, named $C(S)$, is defined as the following set of updates:

$$C(S) = \{u' \mid u' \hookrightarrow u, \; u \in U\}$$

That is, causal past of the state of a replica $i$ consists of all the updates that were performed at replica $i$ in reaching that state, as well as any other updates that happened-before the updates performed at replica $i$ (note that the latter updates may not necessarily be applied at replica $i$, depending on whether the update is for a register in $X_i$ or not). Note that the dependency graph is a function of the execution in which above state $S$ occurs. It would be more precise to define the notation $C(S)$ as $C^E(S)$, where $E$ is the given execution. For brevity, we do not make this dependence on the execution explicit in the notation. The execution being considered will be clear from the context in our discussion.

**Definition 4** (Causal dependency graph). In a given execution, let $S$ denote the state of some replica at a certain time, and $C(S)$ is the causal past of this state. Causal dependency graph of state $S$ is defined as a directed acyclic graph wherein each vertex represents an update in $C(S)$, and for $u_1, u_2 \in C(S)$, an edge exists from vertex $u_1$ to $u_2$ if and only if $u_1 \hookrightarrow u_2$.

From the definition above, causal dependency graph records updates in the causal past as well as the happened-before relation between them. Notice that different causal dependency graphs may correspond to the same causal past.

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2 As such, condition (i) in Definition 2 is obviated by condition (ii), because a replica applies its own update when the update is issued. However, we retain (i) for clarity, and to maintain a correspondence with the conditions in Definition 3.
The share graph defined below was introduced by Hélary and Milani [13]. The share graph will be used when obtaining lower bounds for the basic architecture. To obtain results for the general architecture, we will introduce in Section 6 an augmented version of the share graph. Recall that $X_{ij} = X_i \cap X_j$, i.e., the set of registers stored at replicas $i$ and $j$ both.

**Definition 5 (Share Graph).** Share graph is defined as $G = (V, E)$, where $V = \{1, 2, \ldots, R\}$, with vertex $i \in V$ representing replica $i$, and there exists an edge $(i, j) \in E$ if and only if $X_{ij} \neq \emptyset$.

For convenience, we assume that the share graph is connected. However, the results here trivially extend to the case when $G(V, E)$ is partitioned.

- We represent a directed edge from $i$ to $j$ as $e_{ij}$ and an undirected edge between $i$ and $j$ as $(i, j)$. The edges in $E$ are necessarily undirected, however, it is convenient to view an undirected edge $(i, j)$ as consisting of two directed edges $e_{ij}$ and $e_{ji}$.
- An update is said to occur on edge $e_{jk}$ if the update is issued by replica $j$ for a register in $X_j \cap X_k$. Intuitively, an update that occurs "on edge $e_{jk}$" results in an update message being sent from replica $j$ to replica $k$.
- A causal past $S$ is valid for replica $i$, if replica $i$’s causal past may equal $S$ at some point during some execution.
- Let $S$ be a causal past, defined as a set of updates (as per Definition 3).
  - For $e_{jk} \in E$, $S|_{e_{jk}}$ denotes the set of updates in $S$ that occur on edge $e_{jk}$.
  - For $e_{jk} \notin E$, define $S|_{e_{jk}} = \emptyset$ for convenience.
  - For $i \in V$, $S|_i$ is the set of updates in $S$ that are issued by replica $i$.

**Observation 1:** To achieve causal consistency, it is necessary and sufficient that, before a replica $i$ applies an update $u_1$, it has applied any other update $u_2$ on any of its incoming edges such that $u_2 \hookrightarrow u_1$. Then all read operations will return the latest value of the registers, satisfying safety property of causal consistency. Thus, once replica $i$ has received update messages for all the updates issued by $i$’s neighbors that are happened-before $u_1$, replica $i$ will eventually be able to apply update $u_1$. Then the liveness property of causal consistency is also guaranteed. While the order in which the updates are received by replica $i$ from its neighbors may affect how long the updates are buffered in $pending_i$ (in Steps 3-4 of the prototype algorithm), the order does not affect the ability to apply update $u_1$ after all the causally preceding updates are received. To reiterate, once all of the updates from $i$’s neighbors that happened-before $u_1$ are applied at $i$, update $u_1$ can be applied at $i$. We will make use of this observation in our proofs.

# 4 Algorithm for Causal Consistency with Partial Replication

In this section, we define our algorithm for achieving causal consistency in the basic architecture (Figure 1a). Section 6 discusses how the algorithm can be modified for the general architecture.

We first define graph $G_i$, $i \in V$, which will be used to state the lower bounds as well as the algorithm in this section. We refer to $G_i$ as the timestamp graph of replica $i$. Definition of $G_i$ uses
the following definition of an \((i,j,k)\)-loop. Note that a simple loop begins and ends in the same vertex, but there are no other repetitions of any vertex in the loop.

**Definition 6 \((i,j,k)\)-loop.** Figure 3 illustrates the notation used here. Consider a simple loop \((i = a_0 = b_{s+1}, a_1, a_2, \ldots, a_t, k = a_{t+1}, j = b_0, b_1, \ldots, b_s, i = a_0 = b_{s+1})\) in graph \(G\), \(t \geq 0\), \(s \geq 0\). Note that \(i = a_0 = b_{s+1}, j = b_0, k = a_{t+1}\). This loop is said to be an \((i,j,k)\)-loop provided that (i) \(X_{b_{p+1}} = \bigcup_{1 \leq q \leq t+1} X_{b_{p}a_q} \neq \emptyset\), for \(0 \leq p \leq s\), and (ii) \(X_{j_{k}} - \bigcup_{1 \leq q \leq t} X_{j_{k}a_q} \neq \emptyset\).

Let us define the set of “chords” in the above \((i,j,k)\)-loop as

\[
H = \{ e_{b_{p}a_{q}} \mid 0 \leq p \leq s + 1, 1 \leq q \leq t + 1 \}
\]

Note that we include edge \(e_{b_{0}a_{t+1}} = e_{j_{k}}\) in \(H\). Condition (i) in Definition 6 implies that it is possible to perform an update on edge \(e_{b_{p}b_{p+1}}, 1 \leq p \leq s\), that is not propagated on the edges in \(H\) (and an update on edge \(e_{b_{p}b_{1}}\) that is not propagated on the edges in \(H - \{e_{j_{k}}\}\)). Condition (ii) similarly implies that it is possible to perform an update on edge \(e_{j_{k}}\) that is not propagated on the remaining edges in \(H\).

**Definition 7 (Timestamp graph).** Timestamp graph of replica \(i\) is \(G_i = (V_i, E_i)\), where

\[
E_i = \{ e_{ij} \mid e_{ij} \in E \} \cup \{ e_{ji} \mid e_{ji} \in E \} \cup \{ e_{jk} \mid \exists \text{ (i,j,k)-loop in } G \}
\]

\[
V_i = \{ u, v \mid e_{uv} \in E_i \}
\]

Note that the timestamp graph is a directed graph. In particular, for some \(j, k\), it may be the case that \(e_{j_{k}} \in E_i\) but \(e_{k_{j}} \not\in E_i\).

Before presenting the algorithm, we present a simple result that hints at the efficacy of the timestamps used by the algorithm.

**Theorem 1.** To guarantee that causal consistency is achieved, for each edge \(e_{j_{k}} \in E_i\), there must exist two causal pasts \(S_1\) and \(S_2\) of replica \(i\) that satisfy the following two properties, and are assigned different timestamps: (i) \(S_1 | e_{j_{k}} \neq S_2 | e_{j_{k}}\), and (ii) \(S_1 | v - S_1 | e_{j_{k}} = S_2 | v - S_2 | e_{j_{k}}\) for \(v \in V\).

**Proof.** The proof is presented in Appendix A. \(\square\)

Intuitively, Theorem 1 implies that replica \(i\)’s timestamp must “keep track” of the updates occurring on each edge \(e_{j_{k}} \in E_i\). Section 5 presents specific bounds on timestamp.

Combined with the correctness of the algorithm presented below, the above theorem indicates that the set of edges our algorithm “keeps track” of is necessary and sufficient. This result corrects a claim in [13], as discussed in Appendix B.

**Proposed algorithm (unoptimized version):** The algorithm is based on the prototype in Section 3. Thus, we only need to define replica timestamps, predicate \(J\), and functions \(\text{advance}\) and \(\text{merge}\) that are used in the prototype.

An optimization that \textit{compresses} the timestamps to reduce their storage requirement is discussed later in this section.
We observe that the different elements of the vector \( \tau_i \): The timestamp can be viewed as a vector indexed by edges in \( E_i \). Thus, the length of the timestamp vector \( \tau_i \) is \( |E_i| \). The length may be different for different replicas, and the indices may be different as well, because it is possible that \( E_j \neq E_i \) for some \( j \neq i \).

By convention, if a timestamp \( \tau \) does not contain an element indexed by \( e \), then \( \tau[e] = 0 \).

As seen below, for \( e_{jk} \in E_i \), the timestamp element \( \tau_i[e_{jk}] \) counts the number of updates on edge \( e_{jk} \) that belong to the causal past of the state of replica \( i \). Thus, the timestamp vector keeps track of precisely those set of edges that Theorem 1 also enumerates.

- **Function** \( \text{advance}(\tau, i, x) \) returns vector \( T \) below:
  
  \[
  T[e_{ij}] := \begin{cases} 
  \tau[e_{ij}] + 1, & \text{for each } e_{ij} \in E_i \text{ such that } x \in X_i \cap X_j, \\
  \tau[e], & \text{for all other edges } e \in E_i 
  \end{cases}
  \]

  \( \text{advance}(\tau, i, x) \) increments elements of \( \tau \) corresponding to edges to only those replicas that also store register \( x \).

- **Function** \( \text{merge}(\tau_i, i, T_k, k) \) at replica \( i \) returns vector \( T_i \) as follows:
  
  \[
  T_i[e] := \begin{cases} 
  \max(\tau_i[e], T_k[e]), & \text{for each edge } e \in E_i \cap E_k, \\
  \tau_i[e], & \text{for each edge } e \in E_i - E_k 
  \end{cases}
  \]

- \( J(\tau_i, i, T_k, k) = \text{True} \) if and only if (i) \( \tau_i[e_{ki}] = T_k[e_{ki}] - 1 \) and (ii) \( \tau_i[e_{ji}] \geq T_k[e_{ji}] \), for \( \forall e_{ji} \in E_i \cap E_k, j \neq k \).

The proof for the correctness of the algorithm is provided in Appendix C. The timestamp compression optimization below reduces the timestamp size above. Also, noting that full replication is a special case of partial replication, the use of the compression scheme below ensures that our timestamp reduces to the standard vector timestamp (of length equal to number of replicas) when applied in the context of full replication.

### 4.1 Timestamp Compression

We observe that the different elements of the vector \( \tau_i \) at replica \( i \) are not necessarily independent. For instance, suppose that \( e_{j1}, e_{j2}, e_{j3}, e_{j4} \) are the only outgoing edges at \( j \) that are in \( E_i \), and suppose that \( X_{j1} = \{x\}, X_{j2} = \{y\}, X_{j3} = \{z\} \) and \( X_{j4} = \{x, y, z\} \). (Recall that \( X_{ji} = X_j \cap X_i \).)

The number of updates on these four edges is *not linearly independent*. In particular, the number of updates on edge \( e_{j4} \) can be computed as the sum of the number of updates on edges \( e_{j1}, e_{j2}, e_{j3} \). Thus, we do not need to explicitly store a vector element corresponding to \( e_{j4} \) in the timestamp \( \tau_i \).

More generally, for each replica \( j \in V_i \), the timestamp \( \tau_i \) of replica \( i \) only needs to store \( I(E_i, j) \) elements, where \( I(E_i, j) \) denotes the number of maximum independent outgoing edges of replica \( j \) that are in \( E_i \). Then \( I(E_i) = \sum_{j \in V_i} I(E_i, j) \) is the total number of elements that replica \( i \)'s timestamp needs to store in its timestamp vector \( \tau_i \). Further details of the timestamp compression scheme are provided in Appendix D.

For full replication, all replicas store identical set of registers. Thus, each replica has only one independent edge, and our timestamps will have the same overhead as the traditional vector timestamps, with the number of vector elements being equal to the number of replicas.
5 Timestamp Lower bounds

As seen in Section 3, the algorithms of interest assign a timestamp $\tau_i$ to the replica of state $i$. Different constraints may be imposed on how the timestamp is chosen. The constraint can potentially affect the size of the timestamps. We consider the following strategies:

- **Constraint 0: Timestamp as a function of the causal dependency graph**: Under constraint 0, the timestamp of a replica state may be chosen using its causal dependency graph. Thus, the causal past of the state and the corresponding happened-before relation may both affect the chosen timestamp.

  As an example, the causal dependency graph of the replica state itself may be chosen as the timestamp of the replica state. Alternatively, a suitably chosen subgraph of the causal dependency graph may be used as the timestamp. The COPS scheme for causal consistency under full replication uses a similar strategy [18].

- **Constraint 1: Timestamp as a function of the causal past**: In this case, the timestamp assigned to a replica state is dependent only on its causal past. For instance, if the causal past of the state of replica $i$ equals $\{u_1, u_2, u_3\}$ in any execution, then the timestamp corresponding to that state is identical in all those executions. Also, since the timestamp only depends on the causal past, it is independent of whether $u_1 \rightarrow u_2$ or not. Thus, for a given replica, its two states with identical causal pasts will be assigned the same timestamps regardless of whether those states occur in the same execution or not.

Any timestamp scheme that satisfied Constraint 1 also satisfies Constraint 0. In this section, we present lower bounds on the worst-case size of the timestamps.

**Definition 8. Worst-case timestamp space size $\sigma^i_l(m)$ of replica $i$ under Constraint $l$:** Consider the set of executions of a causally consistent memory in which each replica issues up to $m$ updates. The worst-case timestamp space size of replica $i$ under Constraint $l$, denoted as $\sigma^i_l(m)$, is the maximum number of distinct timestamps that replica $i$ must assign to its state over all the executions in the above set.

Note that replica $i$ may not use all the distinct $\sigma^i_l(m)$ timestamps in the same execution. However, over all possible executions, replica $i$ will need to use at least $\sigma^i_l(m)$ distinct timestamps. From the definition of the three constraints above, it follows that

$$\sigma^i_0(m) \leq \sigma^i_1(m)$$

### 5.1 Lower bound for full replication

While the lower bound for a fully replicated system is relatively easy to obtain, its proof serves as a warm-up for the proof for a partially replicated system. Recall that $R$ is the number of replicas.

**Theorem 2.** For full replication, under Constraint 0, $\sigma^i_0(m) \geq (m + 1)^R$ for any replica $i$.

The proof is presented in Appendix E. Since $\sigma^i_0(m) \leq \sigma^i_1(m)$, the above lower bound also applies under Constraints 1. Additionally, the lower bound is trivially tight, because the traditional vector timestamps satisfy this bound (similar to the timestamps used by Lazy Replication [16] when applied to the basic architecture in Figure 1).
5.2 Lower bounds for partial replication

In this section, we present a lower bound for partial replication under Constraint 1. To derive the lower bound, consider any replica $i$. For the chosen replica $i$, we define a conflict graph $H_i$ below. In the conflict graph, an edge is added between any two causal pasts of replica $i$ that must be assigned distinct timestamp. Then, the chromatic number for the conflict graph is a lower bound on timestamp space size.

Since the conditions in Definition 9 may appear complicated at a first glance, we also include an intuitive interpretation of the conditions. Recall that $G(V, E)$ is the share graph defined previously.

**Definition 9 (Conflict graph $H_i$).** In the conflict graph $H_i$ for replica $i$ there is a vertex corresponding to each possible causal past at replica $i$. For two possible causal pasts of replica $i$, say $S_1$ and $S_2$, we add edge $(S_1, S_2)$ to the conflict graph $H_i$ if the following conditions hold:

- $\forall e \in E$, $S_1|e \neq \emptyset \neq S_2|e$, and
- At least one of the following three conditions is also true:
  1. $\exists e_{ij} \in G$, such that $S_1|e_{ij} \subset S_2|e_{ij}$
  2. $\exists e_{ji} \in G$, such that $S_1|e_{ji} \subset S_2|e_{ji}$
  3. $\exists$ a simple loop
     $(i = a_0 = b_{a+1}, a_1, \cdots, a_t, k = a_{t+1}, j = b_0, b_1, \cdots, b_s, i) \in G$ such that
     
     - $(3.1)$ $S_1|e_{jk} \subset S_2|e_{jk}$, and
     - $(3.2)$ $S_1|e_{bp_{aq}} = S_2|e_{bp_{aq}}$ for $e_{bp_{aq}} \in H - \{e_{bq_{at+1}}\}$, and
     - $(3.3)$ $S_1|e_{bp_{pq}} - \bigcup_{1 \leq q \leq t+1} S_1|e_{bp_{aq}} \neq \emptyset$ for $0 \leq p \leq s$, and
     - $(3.4)$ $S_2|e_{bp_{pq}} - \bigcup_{1 \leq q \leq t+1} S_2|e_{bp_{aq}} \neq \emptyset$ for $0 \leq p \leq s$, and
     - $(3.5)$ $(S_2 - S_1)|e_{jk} \cap \bigcup_{1 \leq q \leq t} S_2|e_{jaq} = \emptyset$.

Recall that $S|e$ is defined as $\emptyset$ when $e \not\in E$.

**Remark:** From Definition 9, we can obtain a subset of edges $E^*_i$ of the share graph, such that $e \in E^*_i$ if and only if there exists an edge $(S_1, S_2)$ in the conflict graph $H_i$ with $S_1|e \subset S_2|e$. The edge set $E_i$ defined previously for the timestamp graph is similar to $E^*_i$ with a small difference. The difference can be reconciled by modifying Conditions (3.3) and (3.4) slightly, which changes the conflict graph somewhat. The discussion of this modification is in Appendix G.
Theorem 3. Let $\chi(H_i)$ denote the chromatic number of conflict graph $H_i$. Then, for partial replication, $\sigma_1^i(m) \geq \chi(H_i)$ for any replica $i$.

Proof. The proof is presented in Appendix F.

6 Extending Results to the General Architecture

The lower bound for full replication in Theorem 2 remains tight for the general architecture in Figure 1b as well, independent of the number of clients. To extend the lower bound for partial replication to the general architecture, we define an augmented share graph using the share graph $G = (V, E)$ defined previously.

Definition 10 (Augmented Share Graph). An augmented share graph $\hat{G} = (\hat{V}, \hat{E})$ is an undirected graph, where $\hat{V} = V$, and $\hat{E} = E \cup \{(s, t) \mid \exists$ client $i$ such that \{s, t\} $\subseteq R_i\}$.

The lower bound in Theorem 3 extends naturally to the general architecture, simply by replacing $G$ by $\hat{G}$ in Theorem 3. The claim and proof are omitted here for brevity. Intuitively, a client alternately communicating with different replica propagates causal past from one replica to another. This is captured in the augmented share graph by adding an edge between those replicas (even though the replicas may not share any registers in common).

The algorithm presented in Section 4 can be modified for the general architecture using $\hat{G}$ instead of $G$. However, unlike the basic architecture, now the clients must also maintain timestamps. The timestamp maintained by client $i$ contains elements corresponding to the timestamps maintained by all the replicas in $R_i$. Thus, if a replica in $R_i$ has an element in its timestamp corresponding to some edge $e$, then the timestamp of client $i$ also has such an element. The algorithm is provided in Appendix H.

7 Optimize Timestamp Size in Practice

From previous sections, we know that achieving causal consistency in partial replication systems is expensive. Therefore in practice, any solutions that can reduce the size of the timestamps would be appealing. In Appendix I, we will discuss several methods to reduce the timestamp cost, by exploiting the trade-off between timestamp size, accuracy of the causal dependencies, and underlying communication topology.

8 Summary

This paper investigates partially replicated causally consistent shared memory systems (of which full replication is a special case). We obtain lower bounds on the timestamps required for such systems, and also present an algorithm that achieves causal consistency in a partially replicated system. We present some optimizations that help reduce the size of the timestamps under partial replication.
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Appendices

A Proof of Theorem 1

Theorem 1 To guarantee that causal consistency is achieved, for each edge $e_{jk} \in E_i$, there must exist two causal pasts $S_1$ and $S_2$ of replica $i$ that satisfy the following two properties, and are assigned different timestamps: (i) $S_1|_{e_{jk}} \neq S_2|_{e_{jk}}$, and (ii) $S_1|_v - S_1|_{e_{jk}} = S_2|_v - S_2|_{e_{jk}}$ for $v \in V$.

Intuitively speaking, $S_1$ and $S_2$ above only differ in updates performed by $j$ on registers in $X_{jk}$.

Proof. Recall that $X_{jk} = X_j \cap X_k$.

We prove the theorem by contradiction. Considering the definition of $E_i$, we address three cases.

Case 1: There exists edge $e_{ij} \in E_i$ such that any pair of causal pasts of replica $i$ that only differ in updates issued by $i$ on registers in $X_{ij}$ have the same timestamp.

Consider the very beginning of an execution. Suppose that replica $i$ initially issues update $u_1$ on edge $e_{ij}$ (i.e., for a register in $X_{ij}$), followed by another update $u_2$ on edge $e_{ij}$. Due to the property assumed above for every pair of causal pasts at $i$, the timestamp attached to both the corresponding update messages sent to $j$ will be identical. Thus, replica $j$ cannot determine the correct order in which these two updates were sent (recall that the channel is not FIFO). Thus, causal consistency cannot be assured.

Case 2: There exists edge $e_{ji} \in E_i$ such that any pair of causal pasts of replica $i$ that only differ in updates issued by $j$ on registers in $X_{ji}$ have the same timestamp.

Consider the very beginning of an execution. Suppose that replica $j$ initially issues update $u_1$ on edge $e_{ji}$ (i.e., for a register in $X_{ij}$), followed by another update $u_2$ on edge $e_{ji}$.

Due to the property assumed above for causal pasts of $i$ has identical timestamps before receiving $u_1, u_2$, and after receiving either or both of these updates. Thus, when replica $i$ receives update $u_2$, it cannot differentiate between the following two cases: (i) $i$ has already received and applied update $u_1$, and thus, it can now apply update $u_2$. (ii) $i$ has not yet received update $u_1$, so it must wait for that update message before applying $u_2$. If replica $i$ applies $u_2$ when $u_2$ arrives (i.e., without waiting for another update message), but the situation is as in (ii), then safety requirement of causal consistency is violated. On the other hand, if replica $i$ decides to wait, but the situation is as in (i), then another update may never be received from $j$, and liveness requirement of causal consistency is violated.

Case 3: There exists edge $e_{jk} \in E_i$ such that there is an $(i, j, k)$-loop in $G$, and any pair of causal pasts of replica $i$ that only differ in updates issued by $j$ on registers in $X_{jk}$ have the same timestamp.

Let the $(i, j, k)$-loop be $(i = a_0, a_1, \cdots, a_t, k = a_{t+1}, j = b_0, b_1, \cdots, b_s, i = b_{s+1})$. Note that $b_{s+1} = i \neq a_0$, $a_{t+1} = k \neq b_0$, and $a_t = j$. By the definition of the $(i, j, k)$-loop, $X_{b_{p}s+1}-\bigcup_{1 \leq q \leq s+1} X_{b_{p}a_q} \neq \emptyset$, for $0 \leq p \leq s$, and $X_{jk} \cap X_{ja_q} \neq \emptyset$.
Case 3.1: \( X_{jb_1} - \bigcup_{1 \leq q \leq t+1} X_{ja_q} \neq \emptyset \). From the definition of an \((i,j,k)\)-loop, we also have \( X_{jb_p b_{p+1}} - \bigcup_{1 \leq q \leq t+1} X_{ja_q} \neq \emptyset \) for \( 1 \leq p \leq s \) and \( X_{jk} - \bigcup_{1 \leq q \leq t} X_{ja_q} \neq \emptyset \).

Consider the following execution.

- Initially, replica \( j \) issues an update, labelled \( u_{jk} \), on edge \( e_{jk} \) on a register in \( X_{jk} - \bigcup_{1 \leq q \leq t} X_{ja_q} \). Note that \( j = b_0 \).
- Replica \( j \) then issues update \( u_{jb_1} \) on edge \( e_{jb_1} \) on a register in \( X_{jb_1} - \bigcup_{1 \leq q \leq t+1} X_{ja_q} \). Let us call this update \( u_{jb_1} \). The corresponding update message is next delivered to \( b_1 \).
- For \( p = 1 \) to \( s \): \( b_p \) receives an update message from \( b_{p-1} \), applies the update, and then issues an update on edge \( e_{b_p b_{p+1}} \) on a register in \( X_{b_p b_{p+1}} - \bigcup_{1 \leq q \leq t+1} X_{ja_q} \). Let us call this update \( u_{b_p b_{p+1}} \).

Thus, we have constructed a sequence of updates so far such that \( u_{jk} \rightarrow u_{jb_1} \rightarrow u_{b_1 b_2} \rightarrow u_{b_2 b_3} \rightarrow \cdots \rightarrow u_{b_{s+1}} \), where \( b_{s+1} = i \).

- Subsequently, \( i \) issues an update \( u_{ia_1} \) on edge \( e_{ia_1} \). \( a_1 \) receives the update message, applies the update, and then issues an update \( u_{a_1 a_2} \) on edge \( e_{a_1 a_2} \). Continuing in this manner, we build a sequence of updates such that \( u_{ia_1} \rightarrow u_{a_1 a_2} \rightarrow u_{a_2 a_3} \rightarrow \cdots \rightarrow u_{a_t a_{t+1}} \), where \( a_{t+1} = k \), and update \( u_{ab} \) in this chain is issued by replica \( a \) on edge \( e_{ab} \).

- Combining the two sequences of updates, we obtain the following sequence. (Recall that \( b_{s+1} = i = a_0 \).)

\[
\begin{align*}
    u_{jk} & \rightarrow u_{jb_1} \rightarrow u_{b_1 b_2} \rightarrow u_{b_2 b_3} \rightarrow \cdots \rightarrow u_{b_{s+1}} \rightarrow u_{ia_1} \rightarrow u_{a_1 a_2} \rightarrow u_{a_2 a_3} \rightarrow \cdots \rightarrow u_{a_t a_{t+1}},
\end{align*}
\]

Now consider a second execution in which replica \( j \) does not initially perform update \( u_{jk} \), but the remaining sequence of updates are performed.

The timestamp of replica \( i \) when issuing update \( u_{ia_1} \) will be identical in both executions, because the causal pasts at \( i \) when issuing update \( u_{ia_1} \) only differ by updates on edge \( u_{jk} \).

By induction, we can easily show that the timestamp attached to the update \( u_{a_t a_{t+1}} \) received by \( k \) from \( a_t \) will be identical in both executions. (Recall that \( a_{t+1} = k \).) In performing the induction, we make use of the fact that no updates are issued in the above executions on edges in \( H - \{e_{jk}\} \). In the first execution \( u_{jk} \rightarrow u_{a_1 a_2 a_3} \), but this is not the case in the second execution. If the update message from \( j \) to \( k \) is not delivered before \( k \) receives the update from \( a_t \), then replica \( k \) cannot determine whether it should wait for an update from \( j \) or not, and either safety or liveness condition may be violated (similar to Case 1).

Case 3.2: \( X_{jb_1} - \bigcup_{1 \leq q \leq t+1} X_{ja_q} = \emptyset \). From the definition of the \((i,j,k)\)-loop, \( X_{jb_1} - \bigcup_{1 \leq q \leq t} X_{ja_q} \neq \emptyset \), and \( X_{b_p b_{p+1}} - \bigcup_{1 \leq q \leq t+1} X_{ja_q} \neq \emptyset \) for \( 1 \leq p \leq s \).

This implies there exists a register \( w \in X_{jb_1} \) such that \( w \in X_{jk} \) but \( w \notin \bigcup_{1 \leq q \leq t} X_{ja_q} \). We build two executions similar to Case 3.1.

For the first execution, replica \( j \) issues an update \( u_{jk} \) on the register \( w \). Since \( w \in X_{jb_1} \) but \( w \notin \bigcup_{1 \leq q \leq t} X_{ja_q} \), \( u_{jk} \) will be delivered to \( b_1 \), but no update message is sent on edges in \( H - \{e_{jk}\} \).
Since update $u_{jk}$ occurs on edge $e_{jb_1}$, also, unlike Case 3.1, here no other update is performed on edge $e_{jb_1}$. The remaining sequence of updates is similar to Case 3.1. This results in the following happened-before relation.

$$u_{jk} \rightarrow u_{b_1b_2} \rightarrow u_{b_2b_3} \rightarrow \cdots \rightarrow u_{b_kb_{k+1}} \rightarrow u_{a_1a_2} \rightarrow u_{a_2a_3} \rightarrow \cdots \rightarrow u_{a_ta_{t+1}}$$

For the second execution, replica $j$ does not issue update $u_{jk}$, but the remaining sequence of updates are performed.

By similar argument as in Case 3.1, the timestamp attached to the update $u_{a_ta_{t+1}}$ will be identical in both executions, and replica $k$ cannot determine whether it should wait for an update from $j$ or not, and either safety or liveness condition for causal consistency may be violated. 

\[\square\]

**B  Correction on a Claim in [13]**

The original definition of minimal $x$-hoop in [13] states the following.

**Definition 11 (Hoop [13]).** Given a register $x$ and two replicas $r_a$ and $r_b$ in $C(x)$ where $C(x)$ is the set of the replicas that stores $x$, we say that there is a $x$-hoop between $r_a$ and $r_b$, if there exists a path $(r_a = r_0, r_1, \cdots, r_k = r_b)$ in share graph $G$ such that:

i) $r_h \notin C(x)$ ($1 \leq h \leq k - 1$) and

ii) each consecutive pair $(r_{h-1}, r_h)$ shares a register $x_h$ such that $x_h \neq x$ ($1 \leq h \leq k$)

**Definition 12 (Minimal Hoop [13]).** An $x$-hoop $(r_a = r_0, r_1, \cdots, r_k = r_b)$ is said to be minimal, if and only if

i) each edge of the hoop can be labelled with a different register and

ii) none of the edge label is shared by replica $r_a$ and $r_b$.

The following claim in [13] states the tight condition for the system to be causally consistent.

**Lemma 1 ([13]).** A replica stores $x$ or belongs to a “minimal $x$-hoop” if and only if the replica has to transmit some information about a register $x$.

However, the lemma does not seem correct with the above definition of minimal $x$-hoop [13].

Consider the share graph in Figure 3a as an example. In the figure, the label on each edge $(c,d)$ shows set $X_{cd}$. The share graph consists replicas $i, a_1, a_2, k, j, b_1, b_2$. Replicas $j$ and $k$ share variable $x$, replicas $b_1, b_2, a_1$ share variable $y$, and replicas $b_2, a_1, a_2$ share variable $z$. Labels on other edges are unique and distinct from $x, y, z$.

The loop $(j, b_1, b_2, i, a_1, a_2, k)$ is considered a “minimal $x$-hoop” by Definition 12 because (i) the label on each edge in the loop is distinct, (ii) none of the edge label is shared by $j$ and $k$. The claim in Milani [13] implies that replica $i$ must transmit (or keep) information about updates to register $x$ by $j,k$. However, it can be shown that presence of the two edges labeled $y$ (and the manner they are situated) makes it unnecessary for replica $i$ to be aware of updates to $x$ by $j$. Similarly, replica $i$ does not need to transmit updates to $x$ by $k$. Theorem 1 does not require $i$ to keep track of these updates. Hence Lemma 1 does not appear to accurately capture the necessary condition.
We also considered a modification of the Definition 12 shown below in an attempt to remedy the above issue. However, we can show that the modified condition is still not sufficient.

**Definition 13** (Minimal Hoop (Modified version)). An $x$-hoop $(r_a = r_0, r_1, \ldots, r_k = r_b)$ is said to be minimal, if and only if

i) each edge of the hoop can be labelled with a different register and

ii) none of the edge label is shared by more than two replicas in the hoop.

Condition (ii) above means that a register used as a label on one of the edges in the $x$-hoop is not shared by more than two replicas in the hoop.

Consider the share graph in Figure 3b as an example. Replica $j$ and $k$ share variable $x$, and replica $b_1, b_2, a_1$ share variable $y$.

The loop $(j, b_1, \ldots, b_2, i, a_1, a_2, k)$ is not considered a “minimal $x$-hoop” by Definition 13 because the edge label $y$ is also stored at replica $a_1$. If we were to apply Lemma 1 with the modified definition of minimal hoop, then it would imply that replica $i$ does not need to transmit (or keep) information about updates to register $x$ by $j,k$. (Note that there is only one simple loop containing $i,j,k$.) However, by Theorem 1, replica $i$ must keep track of edge $e_{kj}$, or equivalently, updates to register $x$ by replica $k$.

**C Correctness of the Algorithm in Section 4**

In order to simplify the proof, we first assume that each replica $i$ stores larger timestamps than required for correctness of the algorithm in Section 4. This simplification makes it easier to prove correctness.

In Section C.2 we show correctness of the algorithm using smaller timestamps.
C.1 Simplified Proof

Definition 14 (Relaxed Timestamp graph). We define \( G_i = (\bar{V}_i, \bar{E}_i) \) as the relaxed timestamp graph of replica \( i \) on share graph \( G \), where

\[
\begin{align*}
\bar{E}_i &= \{e_{ij} \in G \} \cup \{e_{ji} \in G\} \\
&\quad \cup \{e_{b_t} \in G \mid \exists \text{ a simple loop } (i, a_t, \ldots, a_{t+1}, b_0, b_1, \ldots, b_s, i) \in G\}, \\
\bar{V}_i &= \{u, v \mid e_{uv} \in \bar{E}_i\}.
\end{align*}
\]

Thus, \( G_i \) defined in Section 4 is a subgraph of \( \bar{G}_i \).

Lemma 2. Let \( u \) be an update of write \( w \) with timestamp \( T \) from replica \( j \) to \( i \). Let \( u' \) be an update of write \( w' \) with timestamp \( T' \) from replica \( k \) to \( i \) such that \( w' \to w \). Then \( T[e_{ki}] \geq T'[e_{ki}] \) when \( k \neq j \), and \( T[e_{ki}] > T'[e_{ki}] \) when \( k = j \).

Proof. When \( k = j \), \( w' \) and \( w \) are both updating registers shared by replica \( j \) and \( i \). By the algorithm where write requests from client are handled, the counter of replica \( j \)'s timestamp on edge \( e_{ji} \) is incremented by one via function \textit{advance} for each write. Hence \( T[e_{ji}] > T'[e_{ji}] \) when \( w' \to w \).

When \( k \neq j \), in order to have the happen-before relation, there exists a chain of causally related operations that consists of writes and reads, starting with \( w' \) and ending with \( w \). For instance, suppose the chain of operations are \( w' \to w_1 \to r_2 \to w_2 \to \cdots \to w_{s-1} \to r_{s-1} \to w \), where \( w', w_1 \) are issued by client 1 on replica \( k \), \( r_i, w_i \) are issued by client \( p \) on replica \( p \) for \( 2 \leq p \leq s-1 \), \( r_s, w \) are issued by client \( s \) on replica \( j \) (we denote as replica \( s \) as well for convenience). Write \( w' \) is on edge \( e_{ki} \), write \( w_i \) is on edge \( e_{pp+1} \) for \( 1 \leq p \leq s-1 \), and \( w \) is on edge \( e_{ji} \). And every read operation reads the value of the immediate previous write. By our algorithm, the timestamps of all replicas in this loop maintain a counter for edge \( e_{ki} \). Let \( T_{w_p} \) denote the timestamp of the update message for write \( w_p \). By function \textit{merge} we defined when replica handles updates from other replicas, we have \( T_{w_{p+1}}[e_{ki}] \geq T_{w_p}[e_{ki}] \) for any adjacent write pair \( w_p \to w_{p+1} \) in the above chain of operations. Therefore we have \( T[e_{ki}] \geq T'[e_{ki}] \) for \( w' \to w \).

Lemma 3. Let \( u \) be an update of write \( w \) with timestamp \( T \) from replica \( j \) to replica \( i \). When \( \tau_i[e_{ji}] \geq T[e_{ji}] \), \( u \) is already applied at replica \( i \), namely write \( w \) has been performed.

Proof. By the algorithm the only way for replica \( i \) to increment \( \tau_i[e_{ji}] \) is merging \( \tau_i \) with timestamp \( T \) of some update, and \( \tau_i[e_{ji}] \) is incremented by 1 each time. Consider the first moment when \( \tau_i[e_{ji}] = T[e_{ji}] \) after merging with \( T' \) of an update \( u' \). If \( u' \) is issued by replica \( j \), we must have \( u' = u \), since \( T'[e_{ji}] = T[e_{ji}] \) and both \( u' \) and \( u \) are on edge \( e_{ji} \). Then \( u \) is applied at replica \( i \). If \( u' \) is issued by replica \( k \) where \( k \neq j \), the merge will not increase \( \tau_i[e_{ji}] \), since in order to pass the predicate condition, we already have \( \tau_i[e_{ji}] \geq T'[e_{ji}] \). Hence \( u' \) cannot be issued by replica other than \( j \).

Lemma 4. When a write \( w \) is applied in a replica, namely the value has been written to the register, any causal dependency of \( w \) is already applied in the replica.
Proof. If \( w \) is issued by the client associated with the replica, clearly the lemma holds.

If \( w \) is propagated as an update from another replica \( j \) to replica \( i \), let \( w' \) be a write in the causal dependency of \( w \). That is, \( w' \rightarrow w \) and \( w \) writes on some register that stored by replica \( i \). If \( w' \) is issued on replica \( i \), then it is already applied at replica \( i \) by the algorithm. Suppose \( w' \) is issued on some replica \( k \), and propagated as an update from \( k \) to \( i \). Let \( T \) be the timestamp with \( w \), and \( T' \) be the timestamp with \( w' \). By Lemma 2, \( T[e_{ki}] \geq T'[e_{ki}] \) when \( k \neq j \), and \( T[e_{ki}] > T'[e_{ki}] \) when \( k = j \).

First consider the case \( k \neq j \). When the update of \( w \) passes predicate \( J \), we have \( T[e_{ki}] \leq \tau_i[e_{ki}] \) where \( \tau_i \) is the timestamp of replica \( i \). This implies \( T'[e_{ki}] \leq T[e_{ki}] \leq \tau_i[e_{ki}] \). By Lemma 3, \( w' \) is already applied in the replica \( i \).

Then consider the case \( k = j \). When the update of \( w \) passes predicate \( J \), we have \( T[e_{ki}] - 1 \leq \tau_i[e_{ki}] \) where \( \tau_i \) is the timestamp of replica \( i \). This also implies \( T'[e_{ki}] \leq \tau_i[e_{ki}] \). By Lemma 3, \( w' \) is already applied in the replica \( i \).

Theorem 4. The registers in the replicas are causally consistent.

Proof. Implied by Lemma 4.

C.2 Revised Algorithm Proof: Reducing the timestamp sizes

From Section 4, recall the definition of \((i,j,k)\)-loop and timestamp graph \( G_i \) of replica \( i \).

Definition 6 \((i,j,k)\)-loop. Figure 3 illustrates the notation used here. Consider a simple loop 
\( i = a_0 = b_{s+1}, a_1, a_2, \ldots, a_t, k = a_{t+1}, j = b_0, b_1, \ldots, b_s, i = a_0 = b_{s+1} \) in graph \( G \), \( t \geq 0 \), \( s \geq 0 \). Note that \( i = a_0 = b_{s+1}, j = b_0, \) and \( k = a_{t+1} \). This loop is said to be an \((i,j,k)\)-loop provided that 
(i) \( X_{b_pb_{p+1}} - \bigcup_{b_p a_q \neq jk} X_{b_p a_q} \neq \emptyset \), for \( 0 \leq p \leq s \), and 
(ii) \( X_{jk} - \bigcup_{1 \leq q \leq t} X_{ja_q} \neq \emptyset \).

Definition 7 (Timestamp graph). Timestamp graph of replica \( i \) is \( G_i = (V_i, E_i) \), where
\[
E_i = \{e_{ij} \mid e_{ji} \in E\} \cup \{e_{ji} \mid e_{ji} \in E\} \cup \{e_{jk} \mid \exists (i,j,k)\text{-loop in } G\},
\]
\[
V_i = \{u, v \mid e_{uv} \in E_i\}
\]

We are going to prove by induction that it is sufficient for each replica \( i \) to have counters for edges in \( E_i \) instead of \( \bar{E}_i \) in order to maintain causal consistency by our algorithm.

We define some more terminology for convenience.

- For vertex \( b_p, 0 \leq p \leq s \), let \( A_p = \{a_q \mid (b_p, a_q) \in E\} \)

- **Condition (1) for replica \( i \):** \( X_{jk} - \bigcup_{a_q \in A_p} X_{ja_q} = \emptyset \).

- **Condition (2) for replica \( i \):** Condition (1) does not hold, and there exists \( p, 0 \leq p \leq s \), such that \( X_{b_pb_{p+1}} - \bigcup_{a_q \in A_p \cap \bar{A}_q} X_{b_p a_q} = \emptyset \).
Observe that at most one of the above two conditions can hold at the same time (because, by above definition, for Condition (2) to hold, Condition (1) must not hold).

Recall from Section 3 that, in our algorithm, replica $i$’s timestamp includes an element corresponding to each edge in $E_i$. Then it follows that, if a loop $(i = a_0 = b_{s+1}, a_1, a_2, \ldots, a_t, k = a_{t+1}, j = b_0, b_1, \ldots, b_s, i = a_0 = b_{s+1})$ exists in $G$, and $e_{jk} \not\in E_i$, then either Condition (1) or Condition (2) above must be true.

**Lemma 5.** Suppose that $e_{jk} \not\in E_i$.

- Recall that $j = b_0$. If Condition (1) holds, then for each register $w \in X_{jk}$, there exists a replica $c_w^0 \in A_0 - \{k\}$ such that (i) $w \in X_{ja_0}$ and (ii) replica $c_w^0$’s timestamp includes an element corresponding to $e_{jk}$.

- If Condition (2) holds, let $p$ be the smallest value for which $X_{bp bp+1} - \bigcup_{b_p a_q \neq jk} A_{p} \neq \emptyset$. Then (i) the timestamp at $b_p$ contains an element corresponding to $e_{jk}$, and (ii) for each register $w \in X_{bp bp+1}$, there exists a replica $c_w^p \in A_{p} - \{k\}$ for $p = 0$, and $c_w^p \in A_{p}$ for $p \geq 1$, such that (iii) $w \in X_{bp c_p}$ and (iv) replica $c_w^p$’s timestamp includes an element corresponding to $e_{jk}$.

Note that only one of Conditions (1) and (2) can hold true.

**Proof.** If Condition (1) holds, then $X_{jk} - \bigcup_{a_q \in A_q} X_{ja_q} = \emptyset$. For any $w \in X_{jk}$, there exists a non-empty set of replicas $A_{0}^w \subseteq A_0 - \{k\}$ such that $\forall a_q \in A_{0}^w, w \in X_{ja_q}$. We can show that the replica $a_q \in A_0^w$ with the largest subscript $q$ will have a timestamp that includes an element corresponding to $e_{jk}$, namely $e_{jk} \in E_{a_q}$. The proof that $e_{jk} \in E_{a_q}$ is by contradiction. Consider loop $(a_q, a_{q+1}, \ldots, k, j, a_q)$. If $e_{jk} \not\in E_{a_q}$, Condition (1) for replica $a_q$ must be true. However, this implies there exist $a_{q'}$ with $q' > q$ such that $w \in X_{ja_{q'}}$, which contradicts the fact that $a_q$ has the largest subscript. Hence $e_{jk} \in E_{a_q}$ and we can choose $c_w^0 = a_q$ (i.e., largest $q$ corresponding to replica in $A_0^w$).

If Condition (2) holds, then Condition (1) does not hold, and there exists $p, 0 \leq p \leq s$, such that $X_{bp bp+1} - \bigcup_{b_p a_q \neq jk} A_{p} = \emptyset$. Let $p$ the smallest value for which $X_{bp bp+1} - \bigcup_{b_p a_q \neq jk} A_{p} = \emptyset$.

If $p = 0$ (recall $b_0 = j$), for any $w \in X_{jb_1}$, there exists a non-empty set of replicas $A_0^w \subseteq A_0 - \{k\}$ such that $\forall a_q \in A_0^w, w \in X_{ja_q}$. We can show that the replica $a_q$ with the largest subscript $q$ will have a timestamp that includes an element corresponding to $e_{jk}$, namely $e_{jk} \in E_{a_q}$. Otherwise consider loop $(a_q, a_{q+1}, \ldots, k, j, a_q)$. Notice that Condition (1) for replica $a_q$ does not hold, otherwise it will imply that Condition (1) for replica $i$ holds, contradicting the assumption. Since $e_{jk} \not\in E_{a_q}$ and Condition (1) for replica $a_q$ does not hold, Condition (2) for $a_q$ must be true. However, this implies there exist $a_{q'}$ with $q' > q$ such that $w \in X_{ja_{q'}}$, which contradicts the fact that $a_q$ has the largest subscript. Hence $e_{jk} \in E_{a_q}$ and we can choose $c_w^0 = a_q$.

If $p > 0$, for any $w \in X_{bp bp+1}$, there exists a non-empty set of replicas $A_p^w \subseteq A_p$ such that $\forall a_q \in A_p^w, w \in X_{bp a_q}$. We can show that the replica $a_q$ with the largest subscript $q$ will have a timestamp that includes an element corresponding to $e_{jk}$, namely $e_{jk} \in E_{a_q}$. Otherwise consider loop $(a_q, a_{q+1}, \ldots, k, j, b_1, \ldots, b_p, a_q)$. Since $e_{jk} \not\in E_{a_q}$ and Condition (1) for replica $a_q$ does not hold, Condition (2) for $a_q$ must be true. There may be two cases for Condition (2) for $a_q$ to
be true. In the first case, there exist $a_{q'}$ with $q' > q$ such that $w \in X_{b_p a_{q'}}$, which contradicts the fact that $a_q$ has the largest subscript. In the second case, there exists $p' < p$, such that $X_{b_{p'} a_{q'}} - \bigcup_{a_q \in A_{p'}} X_{b_{p'} a_q} = \emptyset$. However, this contradicts the fact that $b_p$ has the smallest subscript. Hence $e_{jk} \in E_{a_q}$ and we can choose $c^w_{p'} = a_q$.

**Proof of the algorithm:**

To prove the causal consistency is achieved, we show that for every edge $e_{jk}$ and an update $u$ on that edge, any update $u'$ applying at $k$ such that $u \rightarrow u'$ will be applied after $u$ is applied, namely the causal ordering $u \rightarrow u'$ can be ensured. We denote $e = e_{jk}$ in the following proof for brevity. If $u'$ is also on $e$, it is trivially true because both $j, k$ keep a counter for $e$. If $u'$ is on another edge $e_{rh}$, there exists a simple path $(j = r_0, r_1, r_2, \ldots , r_h, r_{h+1} = k)$ starting from $j$ and ending at $k$, that each replica $r_i$ in the path issues an update $u_i$ to the next replica $r_{i+1}$ such that $u \rightarrow u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_h = u'$. We prove that the timestamp of $u'$ has the correct value of counter for edge $e$ where the value is $\geq$ the number of updates on $e$ after $u$ is issued, and therefore $u'$ will wait until $u$ is applied by the algorithm.

We prove by induction on the length of the path. For the base case, consider a path $(j, i, k)$. By the definition of the timestamp graph, the timestamps of replica $i, j, k$ all have a counter for edge $e$. By Lemma 4, the causal ordering of $u \rightarrow u'$ can be maintained for the update chain on this path.

Suppose for simple path of length $\leq h$, algorithm guarantees that $u$ is applied before $u'$ is applied at the same replica when $u \rightarrow u'$. Now consider the case where the simple path has length $h + 1$. We will prove that replica $k$ will have the correct counter value for edge $e$. To prove that, we follow the path $(j, r_1, r_2, \ldots , r_h, k)$ to see when a replica does not have counter for $e$ in its timestamp. Recall that by the definition of timestamp graph, replica $j$’s timestamp must have a counter for $e$ because it is a neighbor edge of $j$.

Then we consider the case when $r_1$ does not have a counter for $e$, that means either Condition (1) or Condition (2) for replica $r_1$ must be true.

First consider the case when Condition (1) for $r_1$ is true. Without loss of generality, suppose update $u_0$ is on register $w \in X_{jk}$. By Lemma 5, there exists a replica $r_i$ where $2 \leq i \leq h$ such that $u_0$ is also on edge $e_{jr_i}$ and replica $r_i$’s timestamp includes a counter for edge $e$. By induction assumption, we can ensure that update $u_{r_{i-1}}$ is applied at $r_i$ after $u_0$ is applied at $r_i$, since the path $(j, r_1, \ldots , r_i)$ has length $\leq h$. Hence replica $r_i$’s timestamp will have the correct counter value for edge $e$ because the timestamp attached with $u_0$ has a counter for $e$.

Consider the case when Condition (2) for $r_1$ is true. Without loss of generality, suppose update $u_j$ is on register $w \in X_{jr_1}$. By Lemma 5, there exists a replica $r_i$ where $2 \leq i \leq h$ such that $u_j$ is also on edge $e_{jr_i}$ and replica $r_i$’s timestamp includes a counter for edge $e$. Similarly, by induction assumption we can ensure that update $u_{r_{i-1}}$ is applied at $r_i$ after $u_j$ is applied at $r_i$, then replica $r_i$’s timestamp will have the correct counter value for edge $e$.

If the timestamps of $r_1, \ldots , r_{i-1}$ all have counter for $e$, but the timestamp of $r_i$ does not, it must be the case that Condition (2) for $r_i$ is true. Without loss of generality, suppose update $u_{l-1}$ is on register $w \in X_{jr_{l-1}}$. By Lemma 5 and the fact that $r_{l-1}$’s timestamp has a counter for edge $e$, we know that there exists a replica $r_z$ where $l + 1 \leq z \leq h + 1$ (recall $r_{h+1} = k$) such that $u_{l-1}$ is also on edge $e_{r_{l-1}r_z}$ and replica $r_z$’s timestamp includes a counter for edge $e$. Then similarly
replica $r_z$'s timestamp will have the correct counter value for edge $e$.

By repeating the above argument on path $(j, r_1, \cdots, r_h, k)$ using induction, replica $k$ will have the correct counter value. Then replica $k$ can perform $u \to u'$ correctly, because the timestamp of $u'$ has the correct counter value which indicates that $u$ should be applied first.

## D Compressing timestamps

The timestamp maintained by each replica $i$ can be compressed by observing that the number of updates issued on each outgoing edge at $j$ are not necessarily independent. In particular, for neighbors $k, l$ of $j$ in share graph, if $X_{jk} \cap X_{jl} \neq \emptyset$, then the number of updates issued on edges $e_{jk}$ and $e_{jl}$ both contain updates to the common registers in $X_{jk} \cap X_{jl}$.

We use this observation to compress the timestamp at replica $i$. Let $O_j$ denote the set of outgoing edges of $j$ that are in $E_i$. That is, $O_j = \{e_{jk} \mid e_{jk} \in E_i\}$. We identify the smallest subset of $O_j$, say $I_j$, such that the number of updates on all edges in $O_j - I_j$ can be computed as linear combinations of the updates on the edges in $I_j$.

Then, for each replica $j$, replica $i$ only needs to store vector elements corresponding to the edges in $I_j$. To perform operations such as merge and advance in the algorithm, the vector elements corresponding to edges in $O_j - I_j$ for each $j$ can be computed whenever needed.

In Figure 4, suppose that $E_i$ contains replica 1’s edges to replicas 2, 3, 4, 5. Suppose that the label on edge $(j, k)$ represents $X_{jk} = X_j \cap X_k$. Then it is easy to see that, in this example, knowing the number of updates on any three outgoing edges of replica 1, we can compute the number of updates on the last outgoing edge.

![Figure 4: Illustration for timestamp compression](image)

We can develop the above idea further to possibly reduce the size of each counter. Instead of counting the number of updates on all registers on each edge in set $I_j$, replica $i$ can potentially count the number of updates on only a subset of registers on that edge, thus reducing the counter size. For example, if $I_j$ contains three edges, which have registers $x$, $xy$, $xyz$ respectively, replica $i$ can simply count the number of updates on $x$, $y$ and $z$ separately, instead of counting the number of updates on $x$, $xy$ and $xyz$. 

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Proof of Theorem 2 for Full Replication

Proof.

Sufficiency: Omitted here for brevity. Sufficiency follows trivially from existing algorithms such as Lazy Replication [16].

Necessity: The proof is by contradiction. Suppose that \( \sigma^i_0(m) < (m + 1)^R \) for some replica \( i \). Then replica \( i \) must use fewer than \( (m + 1)^R \) distinct timestamps. Recall that \( R \) is the number of replicas.

Recall the definition of dependency graph in Definition 3.

We say that the dependency graph of a causal past is sparse, (i) if the dependency graph of the updates in the causal past only has \( \rightarrow \) relationship between updates issued by the same replica, and (ii) has no \( \rightarrow \) dependencies between updates issued by different replicas.

Keep in mind the necessary condition under which a replica may apply an update \( u \), namely that any update \( u_1 \xrightarrow{\rightarrow} u \) on an incoming neighbor edge must already be applied.

For each replica \( i \), we consider a particular sequence of \( m \) updates. By “program order” at replica \( i \), we refer to any prefix of this sequence (thus, the prefix contains between 0 and \( m \) updates). In this proof, we consider executions with the following property: each replica issues up to \( m \) updates in the program order, before receiving update messages from any of the other replicas. Given any two such executions, updates issued by a replica \( i \) in one of the executions is contained in the updates issued by replica \( i \) in the other execution. It should be easy to see that \( (m + 1)^R \) different possible causal pasts may occur at replica \( i \) in this set of executions (each possible causal past contains a different combination of number of updates issued by the \( R \) replicas, with number of updates from each replica being between 0 and \( m \)). Because each replica issues its updates before applying updates from other replicas, the dependency graph of each of the above causal pasts is sparse by construction.

Consider replica \( i \) whose timestamp size \( \sigma^i_0(m) < (m + 1)^R \), there exists at least two causal pasts \( S_1, S_2 \) from above set of \( (m + 1)^R \) causal pasts that are assigned the same timestamp \( T \) by replica \( i \). By the construction of the executions, we know that \( S_1|k \subseteq S_2|k \) or \( S_2|k \subseteq S_1|k \) for any replica \( k \). Now we consider two cases.

Case I: \( S_1|i \neq S_2|i \)

Without loss of generality, suppose \( S_1|i \subset S_2|i \). Let \( U \) denote the set of updates that belong to \( S_2|i \) but not \( S_1|i \), namely \( U = S_2|i - S_1|i \).

As illustrated in Figure 5, there exists an execution \( \mathcal{E}_1 \), where replica \( j \) has causal past \( S = S_1 \cup S_2 - U \) (with a sparse dependency graph), when receiving an update \( u \) from replica \( i \), with timestamp \( T \) corresponding to causal past \( S_1 \) at replica \( i \). (By assumption, \( u \) is an update on edge \( e_{ij} \).) Such an execution can be obtained as follows. Replica \( i \) issues its updates in \( S_1|i \) in the program order before receiving any update messages from other replicas. Concurrently, each other replica \( k \) issues its updates in \( (S_1 \cup S_2)|k \) in program order before receiving any update messages. Note that \( S_1|k \subseteq S_2|k \) or \( S_2|k \subseteq S_1|k \) for any replica \( k \), hence \( (S_1 \cup S_2)|k \) equals either \( S_1|k \) or \( S_2|k \).

The set of updates issued above (by all replicas combined) equals \( S_1 \cup S_2 - U \) because the only
updates in $S_1 \cup S_2$ that are not issued are those in $U$.

When all these updates have been applied at $j$, its causal past will equal $S = S_1 \cup S_2 - U$. Also, suppose that all updates in $S_1$ are delivered to $i$, and delivery of any other updates to $i$ is delayed for a sufficiently long time (beyond the interval of interest in this proof). Once $i$ has received and applied all the updates in $S_1$, its causal past will be $S = S_1 \cup S_2 - U$. Also, suppose that all updates in $S_1$ are delivered to $i$, and delivery of any other updates to $i$ is delayed for a sufficiently long time (beyond the interval of interest in this proof). Once $i$ has received and applied all the updates in $S_1$, its causal past will be $S_1$. Then, replica $i$ issues update $u$ to $j$. In the above execution, the dependency graphs corresponding to causal past $S_1$ at replica $i$ is sparse.

By our assumption about $S_1, S_2$, replica $i$’s timestamp attached to update $u$ will be $T$. By a similar argument, there exists another execution $E_2$, where replica $j$ also has causal past $S = S_1 \cup S_2 - U$ (with a sparse dependency graph), when receiving an update $u$ from replica $i$ with timestamp $T$ corresponding to causal past $S_2$. By a similar construction as the previous execution, we can ensure that replicas $i$ and $j$ simultaneously have causal pasts $S_2$ and $S = S_1 \cup S_2 - U$, respectively. Notice that we delay for an arbitrary interval all update messages sent by $i$ to $j$ corresponding to updates that are in $S_2|_i - S_1|_i = U$, because the communication is asynchronous and non-FIFO. Then replica $i$ issues update $u$ with timestamp $T$, corresponding to causal past $S_2$ (as per assumption about $S_1, S_2$ made previously). Because the channel is non-FIFO, we can assume that update $u$ from $i$ is delivered to $j$ before the updates in $U$.

Observe that in both executions, $j$ has identical causal pasts with identical dependency graphs. Also, in both executions, the timestamps of the update $u$ are the same. Thus, on receipt of $u$, replica $j$ cannot distinguish which execution it is in, and specifically, whether update $u$’s timestamp corresponds to causal past $S_1$ or $S_2$ at $i$. If replica $j$ assumes that the causal past at $i$ (when sending $u$) is $S_1$ but the actual causal past is $S_2$, $j$ may apply $u$ before receiving updates in $U$, violating the safety condition for causal consistency. If replica $j$ assumes that the causal past $S_2$, it will wait for updates in $U$. However, the actual causal past of $i$ may be $S_1$, and replica $i$ may never issue updates in $U$. Then replica $j$ will never apply $u$, even if all dependencies of $u$ have been applied, violating the liveness condition for causal consistency.

**Case II:** $S_1|_i = S_2|_i$, and $S_1|_j \neq S_2|_j$ for some $j \neq i$

Without loss of generality, suppose that $S_1|_j \subset S_2|_j$. Let $U$ denote the set of updates that belong to $S_2|_j$ but not $S_1|_j$, namely $U = S_2|_j - S_1|_j$.

Here we consider two cases separately: $R = 2$ and $R > 2$. Recall that $R$ is the number of replica.

As such, the argument below for $R = 2$ also applies for $R > 2$. However, we present a different argument for $R > 2$ to make it applicable to the case when channels may be FIFO, as discussed.
later (we assume non-FIFO channels in much of this report).

(Case II.1) \( R = 2 \)

As illustrated in Figure 6, there exists an execution \( \mathcal{E}_1 \), where replica \( i \) has causal past \( S_1 \) with sparse dependency graph when receiving update \( u \) from \( j \) with causal past \( S = S_2 \mid j \), and the updates in \( S \) follow the program order dependency. The local timestamp of replica \( i \) is \( T \) by assumption. Such an execution can be created similar to Case I. In this case, by assumption, \( u \) is an update on edge \( e_{ji} \).

There exists another execution \( \mathcal{E}_2 \), where replica \( i \) has causal past \( S_2 \) with sparse dependency graph when receiving update \( u \) from \( j \) also with causal past \( S = S_2 \mid j \) whose updates follow the program order. The local timestamp of replica \( i \) is \( T \) by assumption.

The causal dependency graph for replica \( j \)'s state \( S_1 \mid j \) is the same in both the executions. Thus, replica \( j \) will issue update \( u \) with the same timestamp. Also, the local timestamps at replica \( i \) are also identical in two executions. Then replica \( i \) cannot distinguish which execution it is in, or which causal past it has, \( S_1 \) or \( S_2 \), when receiving update \( u \). Note that each update in \( U \mapsto u \). If replica \( i \) assumes its causal past is \( S_1 \), but the actual causal past of replica \( i \) is \( S_2 \), it will wait for updates in \( U \) forever, violating liveness\(^3\). If replica \( i \) assumes its causal past is \( S_2 \), but the actual causal past is \( S_1 \), it may apply the update \( u \) before receiving all updates in \( U \), violating safety property of causal consistency. Hence in both situations, causal consistency conditions are violated.

(Case II.2) \( R > 2 \)

Let \( k \neq i, j \).

As illustrated in Figure 7, there exists an execution \( \mathcal{E}_3 \), where replica \( k \) has causal past \( S = S_1 \cup S_2 - U \) (with a sparse dependency graph), when receiving an update \( u \) with timestamp \( T \), corresponding to causal past \( S_1 \) at replica \( i \). Such an execution can be obtained similarly as in Case I. Replica \( i \) issues its updates in \( S_1 \mid i = S_2 \mid i \) in the program order before receiving any updates. Concurrently, replica \( j \) issues its updates in \( S_1 \mid j \) in the program order before receiving any updates, and each other replica \( p \neq i, j \) issues its updates in \( (S_1 \cup S_2) \mid p \) in the program order

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\(^3\)Even if we allow the algorithm to repeatedly send identical updates from one replica to another, when replica \( i \) receives all updates in \( U \), it still cannot distinguish which causal past it has, since \( S_1 \) and \( S_2 \) are correspond to the same timestamp \( T \). We can apply the above argument repeatedly, and replica \( i \) will never apply \( u \) even if all the dependencies of \( u \) are already applied.
before receiving any updates.

After all the above updates are applied at \( k \), the causal past of \( k \) will be \( S = S_1 \cup S_2 - U \) (recall that \( S_1|_i = S_2|_i \)), and the dependency graph of \( S \) is sparse. Similarly, we can deliver to \( i \) all updates in \( S_1 \) and delay the rest. After all updates in \( S_1 \) are applied at \( i \), the causal past of \( i \) will be \( S_1 \), and the dependency graph of \( S_1 \) is sparse. Then replica \( i \) issues update \( u \) to replica \( k \). The timestamp of \( u \) will be \( T \) as per our assumption.

By a similar argument, there exists another execution \( \mathcal{E}_4 \), where replica \( k \) has causal past \( S = S_1 \cup S_2 - U \) (with a sparse dependency graph), when receiving an update \( u \) with timestamp \( T \), corresponding to causal past \( S_2 \) at replica \( i \). \( \mathcal{E}_4 \) can be obtained by a similar construction as the previous execution, and the fact that we can delay delivery of update messages sent by \( j \) to \( k \) for updates in \( S_2|_j - S_1|_j = U \).

In both executions, the timestamps of the update \( u \) is \( T \). Also, process \( k \)'s causal past in both executions has the same dependency graph. Thus, replica \( k \) cannot distinguish which execution it is in, and whether update \( u \)'s timestamp corresponds to causal past \( S_1 \) or \( S_2 \) at \( i \). If replica \( k \) assumes the causal past at \( i \) (when sending \( u \)) is \( S_1 \) but the actual causal past is \( S_2 \), replica \( k \) may apply \( u \) before receiving updates in \( U \) from replica \( j \), violating safety condition for causal consistency. If replica \( k \) assumes that the causal past \( S_2 \), it will wait for updates in \( U \). However, the actual causal past of \( i \) may be \( S_1 \), and replica \( j \) may never issue updates in \( U \). Then replica \( k \) will never apply \( u \), even if all dependencies of \( u \) have been applied, violating liveness condition for causal consistency.

\[ \square \]

## F Proof of Theorem 3 for Partial Replication

We will use the following terminology often:

- **Propagating causal past**: Replica \( i \) is said to propagate causal past \( S \) to replica \( j \) if replica \( i \) send an update message to replica \( j \) with the causal past of replica \( i \) being \( S \).

- **Set difference** \( A - B \) is defined as \( A - B = \{ a \mid a \in A, \ a \notin B \} \).
Growing the causal past: We say that, after a certain step, the causal past of a replica grows by $S$, provided that the causal past of the replica after that step is the union of $S$ with its casual past before the said step is performed.

Proof of Theorem 3

Proof. The proof is by contradiction. Suppose that there exists replica $i$, such that $\sigma^i_1(m) < \chi(H_i)$. Then there exist two causal pasts of replica $i$, say $S_1$ and $S_2$, that correspond to adjacent vertices in the conflict graph such that both are assigned the same timestamp. By Definition 9, $S_1$ and $S_2$ satisfy $\forall e \in G, |S_1|_e \geq 1$ and $|S_2|_e \geq 1$.

Additionally, $S_1$ and $S_2$ satisfy at least one of Conditions 1, 2 and 3 in Definition 9. We consider each condition separately.

Case 1: Condition 1 holds: Thus, there exists $e_{ij} \in G$ such that $S_1|_{e_{ij}} \subset S_2|_{e_{ij}}$.

Let $U_1 = S_2|_{e_{ij}} - S_1|_{e_{ij}}$. $U_1$ is non-empty because $S_1|_{e_{ij}} \subset S_2|_{e_{ij}}$. $U_1$ contains updates issued by replica $i$ that are on edge $e_{ij}$ in $S_2$ but not in $S_1$ (i.e., the updates correspond to registers in $X_i \cap X_j$).

Now we construct two different executions, $E_1$ and $E_2$, with the following properties: After execution $E_1$, replica $i$ will have causal past $S_1$, and after execution $E_2$, replica $i$ will have causal past $S_2$. After both executions, replica $j$ will have an identical causal past, which we will name $S^*$. We will then extend both executions by issuing an update at replica $i$, and derive a contradiction.

Recall from Definition 3 that a causal past can be represented as a set of updates – in particular, the happened-before relation is not explicitly included in the causal past.

In order to create the desired executions, we will use a propagation procedure that specifies the order of operations performed at various replicas. This propagation procedure is presented below. The procedure takes as input a rooted spanning tree $Tree$, identifier $a$ of a replica in the spanning tree, and a causal past $S$ that is feasible at the specified replica $a$.

In the propagation algorithm, $\pi(b)$ denotes the parent of $b$ in the rooted tree $Tree$.

---

**Procedure Propagation** ($Tree, a, S$)

if $a$ has at least one child in $Tree$ then

foreach child $c$ of $a$ in $Tree$ chosen in a predefined order do

Propagation($Tree, c, S$)

end

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else
Replica $a$ issues all the updates in $S|_a$ in a sequential order such that (i) the updates in $S|_a - S|_{ea(a)}$ are all issued before any update in $S|_{ea(a)}$ is issued, (ii) update messages sent to replicas that are not ancestors (including descendents) of $a$ in $Tree$ are not delivered until a later time (the proof will elsewhere specify when these “held back” update messages are delivered).

For each ancestor $i$ of $a$, wait until all updates in $S|_{ea_i}$ are applied at $i$. (Note that updates in $S|_{ea_i}$ will be eventually applied at $i$. This is true because the dependencies of these updates are either updates issued by $a$ or by replicas in the subtree rooted at $a$ in $Tree$. Such updates have been propagated and performed at $i$ already.)
end

A simple example of the spanning tree constructed in the procedure $Propagation$ is illustrated in Figure 8b, which is based on share graph in Figure 8a. The directed edges in Figure 8b represents child-father relation in the spanning tree. The blue dotted edges connect neighbors in the share graph such that one replica is an ancestor of another in the spanning tree. The brown dotted edges connect the rest of the neighbors in the share graph. The procedure $Propagation$ essentially let the replicas issue their updates in the post-order of their positions in the spanning tree, and the updates will forward along the spanning tree until reaching the root. In this example the order of replica issuing updates is $b, f, c, a, d, e, j, i$. Notice that we let the updates to ancestors to be applied, but delay those are not. For example, updates from $f$ to $a$ are applied, whereas updates from $f$ to $j$ are delayed.

Figure 8: Illustration for Propagation

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$^4$These updates include ones to replicas that are not ancestors of $v$ in the spanning tree $Tree$, and ones to replicas that are not in $Tree$. 

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The following claim is easy to prove, due to the manner in which the updates are performed during the Propagation procedure.

**Claim 1.** Consider replica \(b\) in the subtree of Tree rooted at \(a\). Then after Propagation\((Tree, a, S)\), the causal past at replica \(b\) grows by \(\bigcup_{c \in \text{subtree}_b} S|_c\) where subtree\(_b\) is the subtree rooted at \(b\) in Tree.

The procedure CreateExecution described next uses procedure Propagation above. Note that the procedure takes edge \(e_{ij}\) as input.

**Procedure CreateExecution\((S_f, S_l, e_{ij})\)**

- Recall that we assume that graph \(G\) is connected. Then there exists a spanning tree \(SP\) that is rooted at replica \(i\), such that \(j\) is a child of \(i\) in the spanning tree, and all the descendents of \(j\) in the tree only have a path to \(i\) via \(j\) (namely no descendent of \(j\) is a direct neighbor of \(i\)).

Such a spanning tree necessarily exists because \(G\) is connected, and \(i\) and \(j\) are neighbors in the share graph. The following steps are performed starting from the initial states at all the replicas.

- Replica \(i\) issues updates in \(S_f|_{e_{ij}}\): The corresponding update messages are delivered to replica \(j\), and \(j\) applies these updates. However, the update messages corresponding to these updates are not delivered to any other replica until the end of the (finite) duration of interest in this proof.

- Perform procedure Propagation\((SP, i, S_f - S_f|_{e_{ij}})\)

After the above steps, the causal past at \(i\) is \(S_f\), the causal past at \(j\) is \(S_f|_{e_{ij}} \cup \left(\bigcup_{b \in \text{subtree}_j} S_f|_b\right)\), and that at any other replica \(k\) is \(\bigcup_{b \in \text{subtree}_k} S_f|_b\).

- Let \(S\) be a set of updates containing at least one update on each edge of the spanning tree \(SP\).

For each child \(c\) of \(i\) in \(SP\), perform procedure Propagation\((\text{subtree}_c, c, (S_l - S_f) \cup S)\). Observe that, in this step, \(i\) does not issue any updates, nor apply any updates.

After the above steps, the causal past at \(i\) remains \(S_f\), the causal past at \(j\) is \(S_f|_{e_{ij}} \cup \left(\bigcup_{b \in \text{subtree}_j} (S_f \cup S_l \cup S)|_b\right)\), and that at any other replica \(k\) in the spanning tree is \(\bigcup_{b \in \text{subtree}_k} (S_f \cup S_l \cup S)|_b\).

- During the instantiations of the Propagation procedure above, updates sent by neighbor \(k\) of replica \(j\), such that \(k \neq i\) and \(k\) is not a descendent of \(j\) in \(SP\), are “held back” (i.e., delayed in the communication channels). We now allow all of those updates to be delivered to \(j\). Observation 1 ensures that these updates can be applied once \(j\) has received all the update messages. After these updates have been applied, each neighbor \(k \neq i\) of \(j\) in \(G\) (\(e_{kj}\) is not necessarily an edge in the spanning tree) issues an additional update \(u_k\) on edge \(e_{kj}\) that is subsequently applied at \(j\). \(u_k\) can be applied, since all dependencies on the incoming neighbor edges of \(j\) have been applied. These \(u_k\) updates are held back on all other edges.

\(^4\)Partitioned \(G\) can be handled similarly without affecting the results.
on which they may be propagated. The $u_k$ updates are meant to ensure that $j$ will have in its causal past above all the updates from $j$’s neighbors that are also in $i$’s causal past ($S_f$).

After this step, the causal past at $j$ is

\[
S_f|_{e_{ij}} \bigcup \left( \bigcup_{b \in \text{subtree}_j} (S_f \cup S_l \cup S)|_b \right) \bigcup \left( \bigcup_{e_{kj} \in G, k \neq i} (S_f \cup S_l \cup S)|_b \right) \bigcup \left( \bigcup_{e_{kj} \in G, k \neq i} u_k \right) \tag{1}
\]

Create executions $\mathcal{E}_1$ and $\mathcal{E}_2$

We create executions $\mathcal{E}_1$ and $\mathcal{E}_2$ such that at the end of these executions the causal pasts of $i$ are $S_1$ and $S_2$, respectively, and the causal past at $j$ is identical in both cases.

- Execution $\mathcal{E}_1$ is created by performing $CreateExecution(S_1, S_2, e_{ij})$, i.e., $S_f = S_1$ and $S_l = S_2$. At the end of execution $\mathcal{E}_1$, the causal past at $i$ is $S_1$, and by $[\square]$, the causal past at $j$ is

\[
S^* = S_1|_{e_{ij}} \bigcup \left( \bigcup_{b \in \text{subtree}_j} (S_1 \cup S_2 \cup S)|_b \right) \bigcup \left( \bigcup_{e_{kj} \in G, k \neq i} (S_1 \cup S_2 \cup S)|_b \right) \bigcup \left( \bigcup_{e_{kj} \in G, k \neq i} u_k \right).
\]

- Recall that $U_1 = S_2|_{e_{ij}} - S_1|_{e_{ij}}$. Execution $\mathcal{E}_2$ is created by first performing $CreateExecution(S_2 - U_1, S_1, e_{ij})$, followed by $i$ issuing updates in $U_1$, but with the delivery of the update message corresponding to $U_1$ being delayed at $j$.

After performing $CreateExecution(S_2 - U_1, S_1, e_{ij})$, the causal past at $i$ is $S_2 - U_1$, and by $[\square]$, the causal past at $j$ is

\[
(S_2 - U_1)|_{e_{ij}} \bigcup \left( \bigcup_{b \in \text{subtree}_j} ((S_2 - U_1) \cup S_1 \cup S)|_b \right) \bigcup \left( \bigcup_{e_{kj} \in G, k \neq i} ((S_2 - U_1) \cup S_1 \cup S)|_b \right) \bigcup \left( \bigcup_{e_{kj} \in G, k \neq i} u_k \right).
\]

Observe that $S_2$ and $S_2 - U_1$ only differ over outgoing edges at replica $i$, and by definition of $U_1$, we have $(S_2 - U_1)|_{e_{ij}} = S_1|_{e_{ij}}$. Therefore, the causal past at $j$ after $CreateExecution(S_2 - U_1, S_1, e_{ij})$ is also

\[
S^* = S_1|_{e_{ij}} \bigcup \left( \bigcup_{b \in \text{subtree}_j} (S_1 \cup S_2 \cup S)|_b \right) \bigcup \left( \bigcup_{e_{kj} \in G, k \neq i} (S_1 \cup S_2 \cup S)|_b \right) \bigcup \left( \bigcup_{e_{kj} \in G, k \neq i} u_k \right).
\]

This is identical to the causal past at $j$ after execution $\mathcal{E}_1$. In $\mathcal{E}_2$, after $CreateExecution(S_2 - U_1, S_1, e_{ij})$, replica $i$ performs updates $U_1$ on $e_{ij}$, but the update messages are not delivered to process $j$ until a later time. Then, after execution $\mathcal{E}_2$, the causal past of $i$ will be $S_2$ and the causal past at $j$ remains same as that shown above.

By Constraint 1, the local timestamps of replica $j$ only depends on its causal past, and thus, at the end of both executions above, $j$ has the same timestamp. In other words, replica $j$ cannot determine whether the execution is $\mathcal{E}_1$ or $\mathcal{E}_2$. Also, by assumption, replica $i$ assigns the same timestamps for causal pasts $S_1$ and $S_2$, thus, replica $i$ also has the same timestamp at the end of the two executions.

Now we extend both the executions by replica $i$ issuing an update $u^*$ on edge $e_{ij}$. Update message for update $u^*$ is delivered to replica $j$—note that the update messages for updates in $U_1$
in execution $\mathcal{E}_2$ have not been delivered. This is feasible because the communication channel is not FIFO.

**Deriving contradiction**

When update $u^*$ is received by replica $j$ from replica $i$, replica $j$ must decide whether it is appropriate to apply this update. From replica $j$’s perspective, the two executions are indistinguishable at the time it receives update $u^*$.

- On receipt of $u^*$, if replica $j$ assumes that it is in execution $\mathcal{E}_1$ but the actual execution is $\mathcal{E}_2$, then replica $j$ may apply $u^*$ before receiving update messages in $U_1$, which will violate causal consistency.

- On receipt of $u^*$, if replica $j$ assumes that it is in execution $\mathcal{E}_2$, it will wait to receive the delayed update message (corresponding to $U_1$). However, if the actual execution is $\mathcal{E}_1$, then replica $j$ will wait forever for these messages (which were not sent by $i$). Then replica $j$ will never apply update $u^*$, even if all the dependencies of $u^*$ have been applied, which violates the liveliness requirement for causal consistency.

The above contradictions show that replica $i$ must assign different timestamps for causal pasts $S_1$ and $S_2$.

**Case 2: Condition 2 holds:** Thus, there exists $e_{ji} \in G$ such that $S_1 | e_{ji} \subset S_2 | e_{ji}$.

Let $U_2 = S_2 | e_{ji} - S_1 | e_{ji}$. $U_2$ is non-empty because $S_1 | e_{ji} \subset S_2 | e_{ji}$.

**Create executions $\mathcal{E}_3$ and $\mathcal{E}_4$**

We now define two executions similar to Case 1 above, using the spanning tree $SP$ and $\text{subtree}_j$ defined previously in $\text{CreateExecution}$.

- Execution $\mathcal{E}_3$: To construct execution $\mathcal{E}_3$, first $\text{Propagation}(SP, i, S_1)$ is performed. After this procedure, the causal past at $i$ is $S_1$, and the causal past at $j$ is $\bigcup_{b \in \text{subtree}_j} S_1|b$.

  Next, procedure $\text{Propagation}(\text{subtree}_j, j, (S_2 - S_1) \cup S)$ is performed. Note that updates in $U_2$ are issued by $j$ but not delivered to $i$ in the above procedure. After this procedure, the causal past at $i$ remains $S_1$, and the causal past at $j$ is $\bigcup_{b \in \text{subtree}_j} (S_1 \cup S_2 \cup S)|b$.

- Execution $\mathcal{E}_4$: To construct execution $\mathcal{E}_4$, first $\text{Propagation}(SP, i, S_2)$ is performed. After this procedure, the causal past at $i$ is $S_2$, and the causal past at $j$ is $\bigcup_{b \in \text{subtree}_j} S_2|b$.

  Next, procedure $\text{Propagation}(\text{subtree}_j, j, (S_1 - S_2) \cup S)$ is performed. After this procedure, the causal past at $i$ remains $S_2$, and the causal past at $j$ is $\bigcup_{b \in \text{subtree}_j} (S_1 \cup S_2 \cup S)|b$.

By Constraint 1, the local timestamps of replica $j$ only depends on its causal past, and thus, at the end of both executions above, $j$ has the same timestamp. In other words, replica $j$ cannot determine whether the execution is $\mathcal{E}_3$ or $\mathcal{E}_4$. Also, by assumption, replica $i$ assigns the same timestamps for causal pasts $S_1$ and $S_2$, thus, replica $i$ also has the same timestamp at the end of the two executions.

Now we extend both the executions by replica $j$ issuing an update $u^*$ on edge $e_{ji}$. Update message for update $u^*$ is delivered to replica $i$ – note that the update messages for updates in $U_2$
issued by replica $j$ in execution $E_3$ have not been delivered at replica $i$ yet. This is feasible because the communication channel is not FIFO.

**Deriving contradiction**

When update $u^*$ is received by replica $i$ from replica $j$, replica $i$ must decide whether it is appropriate to apply this update. From replica $i$’s perspective, the two executions are indistinguishable at the time it receives update $u^*$.

- On receipt of $u^*$, if replica $i$ assumes that it is in execution $E_4$ but the actual execution is $E_3$, then replica $i$ may apply $u^*$ before receiving update messages for $U_2$, which will violate causal consistency.

- On receipt of $u^*$, if replica $i$ assumes that it is in execution $E_3$, it will wait to receive the delayed update message (corresponding to $U_2$). However, if the actual execution is $E_4$, then replica $i$ will wait forever for these messages (which have been previously applied by $i$ already). Then replica $i$ will never apply update $u^*$, even if all the dependencies of $u^*$ have been applied, which violates the liveliness requirement for causal consistency.

The above contradictions show that replica $i$ must assign different timestamps for causal pasts $S_1$ and $S_2$.

**Case 3: Condition 3 holds:** Then there exists $e_{jk} \in G$ and a simple loop $(i, a_1, \ldots, a_t, k, j, b_1, \ldots, b_s, i) \in G$ such that

1. $S_1|e_{jk} \subset S_2|e_{jk}$, and
2. $S_1|e_{bpaq} = S_2|e_{bpaq}$ for $e_{bpaq} \in H - \{e_{jk}\}$, and
3. $S_1|e_{bpaq} - \bigcup_{1 \leq q \leq t+1} S_1|e_{bpaq} \neq \emptyset$ for $0 \leq p \leq s$, and
4. $S_2|e_{bpaq} - \bigcup_{1 \leq q \leq t+1} S_2|e_{bpaq} \neq \emptyset$ for $0 \leq p \leq s$, and
5. $(S_2|e_{jk} - S_1|e_{jk}) \cap \bigcup_{1 \leq q \leq t} S_2|e_{jaq}) = \emptyset$.

Recall that $i = a_0 = b_{s+1}$, $k = a_{t+1}$, and $j = b_0$. Note that when $t = 0$, edge $(i, k)$ is in the loop, and when $s = 0$ edge $(i, j)$ is in the loop.

Let $U_3 = S_2|e_{jk} - S_1|e_{jk}$.

The proof in this case is analogous to the proof of Case 1. Condition (3.2) is key to this proof. In particular, Condition (3.2) makes it possible to ensure that the updates in $U_3$ appear in the causal past of replica $i$ without them being in the causal past replicas $a_q, 1 \leq q \leq t + 1$. We will construct two executions $E_5$ and $E_6$ below. These executions will satisfy the following two properties:

- The causal past of replica $i$ at the end of these executions is $S_1$ and $S_2$, respectively.
- The causal past of replica $a_q, 1 \leq q \leq t + 1$, is identical after both executions, and, in particular, the causal past does not include $U_3$. 


The two executions will then be used to arrive at a contradiction, similar to Case 1.

In graph $G$, there exists a directed spanning tree $T$ rooted at $i$ such that the paths $i = a_0, a_1, \cdots, a_{t+1} = k$, and $i = b_{s+1}, b_s, \cdots, b_1, b_0 = j$ belong to this spanning tree.

Recall that $H = \{e_{bp}a_q \mid 0 \leq p \leq s + 1, 1 \leq q \leq t + 1\}$

We will show how to build executions $E_5$ and $E_6$ below. Since the steps in building the two executions are quite similar, we will present the two executions together.

1. **Step 1 for $E_5$ and $E_6$:** In step 1, each replica $b_p$, $0 \leq p \leq s + 1$, issues updates in $S_1|e_{bp}a_q$ for $e_{bp}a_q \in H$. From condition (3.2), we know that for each edge $e_{bp}a_q \in H - \{e_{jk}\}$, $S_1|e_{bp}a_q = S_2|e_{bp}a_q$. It is assumed that in both executions $E_5$ and $E_6$ these updates are applied at replicas in $a_1, a_2, \cdots, a_{t+1} = k$ in an identical order. Thus, each replica has an identical causal past at this point in both $E_5$ and $E_6$. In particular, replica $b_p$, $1 \leq p \leq s + 1$, has causal past equal to $\cup_{1 \leq q \leq t+1} S_1|e_{bp}a_q = \cup_{1 \leq q \leq t+1} S_2|e_{bp}a_q$. The causal past for $j = b_0$ equals $\cup_{1 \leq q \leq t+1} S_1|e_{bp}a_q = \cup_{1 \leq q \leq t+1} S_2|e_{bp}a_q - U_3$.

**Step 1.1 for $E_6$:** Replica $j$ issues updates in $U_3$, however, the corresponding update messages are not yet delivered to the recipient replicas (including $k$). By condition (3.5), the updates in $U_3$ are not on any edge $e_{ja_q}$ for $1 \leq q \leq t$, hence the causal past for $a_1, a_2, \cdots, a_{t+1}$ remains unchanged after this step. Step 1.1 only grows the causal past of $j$ in $E_6$ by $U_3$ to become $\cup_{1 \leq q \leq t+1} S_2|e_{bp}a_q$. The causal pasts of other replicas remain unchanged.

2. **Step 2:** Recall that, by condition (3.3), $S_1|b_{p}b_{p+1} - \cup_{1 \leq q \leq t+1} S_1|e_{bp}a_q \neq \emptyset$ for $0 \leq p \leq s$. Hence, in execution $E_5$, there exists at least one more update in causal history $S_1$ on each edge on the path $j, b_1, b_2, \cdots, b_s, i$ in tree $T$ that has not been issued in the above step. Similar property holds for $S_2$ in execution $E_6$. These properties are necessary for the desired outcome below from performing Propagation procedure. (Recall that tree $T$ is defined in the earlier discussion of Case 3.)
Step 2 for Execution $\mathcal{E}_5$: Procedure $\text{Propagation}(T, i, S_1 - \bigcup_{e_{bpaq} \in H} S_1|e_{bpaq})$ is performed. After the $\text{Propagation}$ step, causal past at replica $i$ will be $S_1$.

Step 2 for Execution $\mathcal{E}_6$: Procedure $\text{Propagation}(T, i, S_2 - \bigcup_{e_{bpaq} \in H} S_2|e_{bpaq})$ is performed. After the $\text{Propagation}$ step, causal past at replica $i$ will be $S_2$.

3. **Step 3:** Let $S$ be a set of updates that includes at least one update on each edge from a child node to a parent node in the spanning tree $T$, with the constraint that any update on edge $e_{bp}b_{p+1}$ occurs on a register in $X_{bp}b_{p+1} - \bigcup_{e_{bpaq} \in H} X_{bpaq}$ for $0 \leq p \leq s$. Conditions (3.3) and (3.4) imply that such registers necessarily exist. The intent here is to prevent the future updates (below) at replicas $b_0, b_1, \cdots, b_s$ from affecting the causal pasts at $a_1, a_2, \cdots, a_{t+1}$.

Step 3 for Execution $\mathcal{E}_5$: For each child $c$ of $i$ in tree $T$, perform procedure $\text{Propagation}(\text{subtree}_c, c, (S_2 - S_1) \cup S)$, where $\text{subtree}_c$ is the sub-tree of $T$ rooted at $c$. In this step, $i$ does not issue any updates, nor perform any updates.

Step 3 for Execution $\mathcal{E}_6$: For each child $c$ of $i$ in tree $T$, perform procedure $\text{Propagation}(\text{subtree}_c, c, (S_1 - S_2) \cup S)$. In this step, $i$ does not issue any updates, nor perform any updates.

Recall from condition (3.2) that $(S_1 - S_2)|e_{bpaq} = \emptyset = (S_2 - S_1)|e_{bpaq}$ for $b_p a_q \in H - \{e_{jk}\}$. That is, in this step, replicas $b_p$ do not issue updates on the edges in $H - \{e_{jk}\}$. This guarantees that information about the number of updates performed on edge $e_{jk}$ in this step will not leak to replicas $a_1, \cdots, a_{t+1}$ in subsequent steps.

After step 3, the causal past at $i$ remains unchanged (i.e., $S_1$ in Execution $\mathcal{E}_5$, and $S_2$ in $\mathcal{E}_6$). In both executions, the causal past at replica $d \in V - \{a_0 = i, a_1, \cdots, a_t, a_{t+1} = k\}$ is identical, specifically,

$$\bigcup_{p \in \text{subtree}_i}(S_1 \cup S_2 \cup S)|_p$$

Similarly, it should be easy to see that the causal past at $a_q$, $1 \leq q \leq t + 1$ is also identical in both the executions after step 3. To ensure this outcome, it is important that in Step 1 of both executions, the updates are issued by each $b_p$, $0 \leq p \leq s + 1$, in identical order.

4. **Step 4 for $\mathcal{E}_5$ and $\mathcal{E}_6$:** The goal in Step 4 is to ensure that the causal past of replica $a_q \in \{a_1, \cdots, a_t, a_{t+1} = k\}$ includes in its causal past all the updates in $S_1 \cup S_2 - U_3$ that modify the registers stored at $a_q$. Except for the update messages corresponding to $U_3$ on edge $e_{jk}$, any pending updates (from Propagation procedures above) from neighbors of $a_q$ ($1 \leq q \leq t + 1$) are delivered to $a_q$.

From prior steps, observe that the updates in $U_3$ are only in the causal pasts of replicas $b_0 = j, b_1, \cdots, b_s$ in both executions, and also in the causal past of $i$ in execution $\mathcal{E}_6$. Note that there are no pending updates on edges in $H - \{e_{jk}\}$, since we applied the updates on edges in $H - \{e_{jk}\}$ in Step 1 of both executions, and no further updates on those edges are issued in step 3. Thus, $U_3$ is not in the causal history of any pending updates delivered in Step 4 above. This, together with Observation 1 implies that these newly delivered updates can be applied at $a_q$ after all of these update messages have been delivered to $a_q$. 

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The set of above updates applied at $a_q$ is identical in both executions, however, their causal pasts may differ in the two executions. It is because in execution $E_5$ updates in $S_1$ are issued first and then those in $(S_2 - S_1) \cup S$, but in execution $E_6$ updates in $S_2$ are issued first and then those in $(S_1 - S_2) \cup S$. The different order of how updates are issued may result in different causal past of the set of above updates delivered to $a_q$. To equalize the causal pasts at replicas $a_q$, $1 \leq q \leq t+1$, we let each neighbor of $a_q$ except $b_j = b_1, \ldots, b_s, b_{s+1} = i$, issue one more update on the edge to $a_q$, and this update is then applied at $a_q$. Each above update carries the same causal past in both executions, since the causal past at the neighbors where the update is issued are identical at the time when issuing the update. Then the causal pasts at $a_q$ are equalized. After these steps, causal past at replica $a_q$, $1 \leq q \leq t+1$ is identical in both executions. Since, in this step, no additional update on $e_{b_p a_q} \in H$ is applied at $a_q$, $1 \leq q \leq t+1$, the causal past at replicas $a_1, a_2, \ldots, a_{t+1} = k$ does not contain $U_3$ after both executions above.

5. **Step 5 for $E_5$ and $E_6$:** Both executions in Step 5 issue a chain of updates $(u_0, u_1, \ldots, u_t)$ along the path $(i, a_1, a_2, \ldots, a_t, k)$. Specifically, replica $i = a_0$ issues update $u_0$ on edge $e_{i a_1}$, $a_1$ issues $u_1$ on edge $e_{a_1 a_2}$, $\ldots$, $a_t$ issues update $u_t$ on edge $e_{a_t k}$. Observe that in $E_6$, $u_t$ depends on updates in $U_3$, which means $u_t$ should be applied only after updates in $U_3$ are applied at replica $k$.

We now argue that in both executions $E_5$ and $E_6$, an identical timestamp is attached to the update message for $u_t$ sent by $a_t$ (i.e., the timestamp for the causal past of $a_t$ when performing update $u_t$ is identical in both executions).

By Assumption 1, the timestamp of a replica only depends on its causal past. Hence the local timestamps at $a_q$ for $1 \leq q \leq t+1$ are identical after Step 4 of $E_5$ and $E_6$ both. In step 5, when replica $i$ issues update $u_0$, the timestamp of $u_0$ is $T$ in both executions, since both causal pasts $S_1, S_2$ correspond to timestamp $T$. Recall that, the dependencies of $u_0$ are already applied at $a_1$ by Step 4 of both executions, hence $u_0$ can be applied at $a_1$. As we mentioned previously, the timestamp at $a_1$ are identical when receiving $u_0$ in both executions, thus, the timestamp of $a_1$ after applying $u_1$ will also be identical in both executions. Therefore, $a_1$ will issue the update $u_1$ with the same timestamp in both executions. By simple induction along $a_1, a_2, \ldots, a_t$, we know that $a_t$ will issue the update $u_t$ with the same timestamp in both executions. Recall that the causal past, and thus the local timestamp at replica $k$, is also identical in both executions.

We can now derive a contradiction. On receiving $u_t$, replica $k$ cannot distinguish which execution it is in, $E_5$ or $E_6$, since the timestamp attached with $u_t$ and its local timestamp are identical in both executions. If replica $k$ assumes it is in execution $E_6$ (i.e., the causal past of $u_t$ contains $U_3$), it will wait for update messages for $U_3$ from replica $j$. However, the actual execution may be $E_5$, and updates in $U_3$ may never be issued. Then replica $k$ will never apply $u_t$ even if the updates in the causal past of $u_t$ have already been applied, violating the progress property of the causal consistency. If replica $k$ assumes it is in execution $E_5$ (i.e., the causal past of $u_t$ does not contain $U_3$). However, the actual execution may be $E_6$, and replica $k$ may apply $u_t$ before receiving updates in $U_3$ from replica $j$, violating safety condition of causal consistency. Hence, in both situations, causal consistency is violated.

Therefore causal consistency can be maintained only if $\sigma_1^i(m) \geq \chi(H_i)$ for $\forall i$. ☐
G Definition of New Conflict Graph

Definition 15 (New conflict graph $H'_i$). In the new conflict graph $H'_i$ for replica $i$ there is a vertex corresponding to each possible causal past at replica $i$. For two possible causal pasts of replica $i$, say $S_1$ and $S_2$, we add edge $(S_1, S_2)$ to the new conflict graph $H'_i$ if the following conditions hold:

- $\forall e \in G$, $S_1|e \neq \emptyset \neq S_2|e$, and
- At least one of the following three conditions is also true:
  1. $\exists e_{ij} \in G$, such that $S_1|e_{ij} \subset S_2|e_{ij}$
  2. $\exists e_{ji} \in G$, such that $S_1|e_{ji} \subset S_2|e_{ji}$
  3. $\exists an (i, j, k)-\text{loop}$
     $i = a_0 = b_{s+1}, a_1, \ldots, a_t, k = a_{t+1}, j = b_0, b_1, \ldots, b_s, i) \in G$ such that
     \begin{align*}
     & (3.1) S_1|e_{jk} \subset S_2|e_{jk}, \text{ and} \\
     & (3.2) S_1|e_{p_{paq}} = S_2|e_{p_{paq}} \\
     & \text{for } e_{p_{paq}} \in \mathcal{H} - \{e_{p_{0(a+1)}}\}, \text{ and} \\
     & (3.3) S_1|e_{p_{paq}} - \bigcup_{1 \leq q \leq t+1} S_1|e_{p_{pq}} \neq \emptyset \forall 0 \leq p \leq s, \text{ and} \\
     & (3.4) S_2|e_{p_{paq}} - \bigcup_{1 \leq q \leq t+1} S_2|e_{p_{paq}} \neq \emptyset \forall 0 \leq p \leq s, \text{ and} \\
     & (3.5) (S_2 - S_1)|e_{jk} \cap (\bigcup_{1 \leq q \leq t} S_2|e_{jq}) = \emptyset. \\
     & (3.6) \text{if } S_1|e_{jq} - \bigcup_{1 \leq q \leq t+1} S_1|e_{jq} = \emptyset \text{ or} S_2|e_{jq} - \bigcup_{1 \leq q \leq t+1} S_2|e_{jq} = \emptyset, \text{ then} \\
     & S_1|e_{ij} = S_2|e_{ij} \text{ for all } e_{ij} \in G.
     \end{align*}

Interpretation of the conditions:

Condition 1: Causal past $S_2$ contains all the updates in $S_1$ on edge $e_{ij}$, and at least one update that is not in $S_1$ on edge $e_{ij}$.

Conditions 2 and 3.1 are similar to Condition 1.

Condition 3.2: This condition does not apply when $e_{bpaq} = e_{bpa_{t+1}}$, i.e., $e_{jk}$. $S_1$ and $S_2$ include identical set of updates on all other $e_{bpaq}$ edges that exist in the share graph (i.e., updates by $b_p$ on registers in $X_{b_p} \cap X_{aq}$).

Condition 3.3: For each $b_p$, $S_1$ includes at least one update by $b_p$ on a register in $X_{b_p} \cap X_{aq}$ that is not in $X_{aq}$, $1 \leq q \leq t+1$ and $b_pa_q \neq jk$. Condition 3.4 is similar but applied to $S_2$.

Condition 3.5: The updates in $S_2$ on edge $e_{jq}$ that are not included in $S_1$, are not included in $S_2$ on edges $e_{jaq}$, $1 \leq q \leq t$.

Condition 3.6: If the condition evaluates true, $S_1$ and $S_2$ need to include identical set of updates on all incoming neighbor edges of $j$, i.e., $e_{ij} \in G$.

Compared with Definition 9, the conflict graph defined in Section 5, Condition 3.3 and 3.4 are slightly different, which close the gap between the subset of edges inferred by the definition, and the edge set $E_i$ defined for upper bound. This is also discussed in the Remark in Section 5. In addition, Condition 3.6 are needed for proving conflict between two causal pasts.

Theorem 5. Let $\chi(H'_i)$ denote the chromatic number of new conflict graph $H'_i$. Then, for partial replication, $\sigma'_1(m) \geq \chi(H'_i)$ for any replica $i$.

Proof. The proof is analogous to the one for Theorem 3 and omitted from the report.

H Results for the General Architecture

In this section, we discuss how the algorithm presented previously for the basic architecture (Figure 1a) may be modified to use under the general architecture (Figure 1b), and discuss lower bounds
for the general architecture. In the general case, each client $i$ may send its read/write request to any replica in the set of replicas $R_i$.

For a replica set $R_i$, define $X_{R_i} = \bigcup_{j \in R_i} X_j$, that is, the set of registers stored in replicas in $R_i$.

For the general architecture, we consider algorithms with the following prototype structure. The structure is similar to that specified in Section 3 for the basic architecture, with the key difference that each client also now maintains a timestamp. Functions structure is similar to that specified in Section 3 for the basic architecture, with the key difference that each client also now maintains a timestamp. Functions $\text{advance}_1$, $\text{advance}_2$, $\text{merge}$, and predicates $\mathcal{J}_1$ and $\mathcal{J}_2$ used below will be specified later.

For convenience, we sometimes denote client $i$ as $c_i$ and replica $j$ as $r_j$.

- **Client operations:** Each client $i$ maintains a timestamp $\mu_i$, which is suitably initialized. A client $i$ may only perform read/write operations on any register $x \in X_{R_i}$.
  - When client $i$ wants to read a shared register $x \in X_{R_i}$, client $i$ sends $\text{read}_i(x, \mu_i)$ request to a replica $j \in R_i$; note that the request include client $i$’s timestamp and awaits replica’s response containing the register value and a timestamp $\tau$. Then client updates its timestamp using $\text{merge}$ function as $\mu_i = \text{merge}(\mu_i, c_i, \tau, r_j)$.
  - When client $i$ wants to write value $v$ to a shared register $x \in X_{R_i}$, client $i$ sends $\text{write}_i(x, v, \mu_i)$ request to a replica $j \in R_i$, and awaits the replica’s response containing a timestamp $\tau$. Then client updates its timestamp as $\mu_i = \text{merge}(\mu_i, c_i, \tau, r_j)$.

- **Replica operations:** Each replica $j$ maintains a timestamp $\tau_j$, which is suitably initialized. Exact implementation of the timestamp will be discussed below. The timestamp is a representation of the causal history of the local state of the replica $j$.
  - When replica $j$ receives a $\text{read}_i(x, \mu)$ request from client $i$, replica $j$: The request is buffered until predicate $\mathcal{J}_1(\tau_j, j, \mu)$ evaluates true; once the predicate evaluates true, replica $j$ responds to client $i$ with the value of the local copy of register $x$ and its timestamp $\tau_j$.
  - When replica $j$ receives a $\text{write}_i(x, v, \mu)$ request from client $i$: The request is buffered until predicate $\mathcal{J}_2(\tau_j, j, \mu)$ evaluates true; once the predicate evaluates true, replica $j$ performs the following steps: (i) write $v$ into the local copy of register $x$, appropriately advance its timestamp $\tau_j$ using function $\text{advance}_1$, as $\tau_j = \text{advance}_1(\tau_j, j, x)$, and merge its timestamp $\tau_j$ with $\mu$, as $\tau_j = \text{merge}(\tau_j, \mu_j, \mu, \mu_i)$, (ii) appropriately advances timestamp $\mu$ using function $\text{advance}_2$, as $\mu = \text{advance}_2(\mu, j, x, \tau_j)$, and multicasts $\text{update}(j, \mu, x, v)$ to all other replicas $k \in V$ such that $x \in X_k$, and (iii) return timestamp $\mu$ in the reply message to client $i$. Step (i) above needs to be an atomic operation.
  - When replica $j$ receives a message $\text{update}(k, \mu, x, v)$: The update is buffered until predicate $\mathcal{J}_3(\tau_j, j, k, \mu, k)$ evaluates true; when the predicate evaluates true, replica $j$ writes value $v$ to its local copy of register $x$, and updates its timestamp $\tau_j$ as $\tau_j = \text{merge}(\tau_j, \tau_j, \mu, c_i)$.

The lower bounds obtained previously for the basic architecture can be extended for the algorithms satisfying the above prototype for the general architecture. Additionally, we can also generalize the algorithm in Section 4. We now discuss the general algorithm, followed by lower bounds for the general architecture.

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5Instead of the actual timestamp, client $i$ may possibly send a function of its timestamp.
H.1 Generalized Algorithm

Recall the augmented share graph \( \hat{G} = (\hat{V}, \hat{E}) \) defined in Section 6.

The algorithm presented in Section 4 can be modified for the general architecture using \( \hat{G} \) instead of \( G \). In particular, the general algorithm uses a modified definition of Timestamp Graph \( G_i \) for replica \( i \). The new definition is presented below makes use of the augmented share graph.

**Definition 16 (Generalized Timestamp graph).** We define \( G_i = (V_i, E_i) \) as the timestamp graph of replica \( i \) as follows.

\[
E_i = \{ (e_{ij} \in \hat{E}) \cup \{ e_{ji} \in \hat{E} \} \cup \{ e_{jk} \in \hat{E} | \exists (i,j,k) \text{-loop in } \hat{G} \} \},
\]

\[
V_i = \{ u, v | e_{uv} \in E_i \}.
\]

Now we specify the timestamps maintained by the replicas and clients. Unlike the basic architecture, with the general architecture, clients also maintain timestamps. Note that the timestamp of replica \( i \) referred here is the modified version defined above.

- Replica \( i \)’s timestamp: \( \tau_i = \{ (e,k) | e \in E_i \} \) is a set of timestamps \( (e,k) \), where \( e \) is a directed edge in timestamp graph \( E_i \) and \( k \) is a sequence number, initialized to 0

- Client \( i \)’s timestamp: \( \mu_i = \{ (e,k) | e \in \bigcup_{j \in R_i} E_j \} \) is a set of timestamps \( (e,k) \), where \( e \) is a directed edge in the union of timestamp graphs of all the replicas in \( R_i \) (i.e., replicas that client \( i \) communicates with), and \( k \) is a sequence number, initialized to 0. For client \( i \) and replica \( j \), let us also define \( \mu_{ij} = \{ (e,k) \in \mu_i | e \in E_j \} \)

For a timestamp \( \tau \), if \( (e,k) \in \tau \), then \( \tau[e] = k \). Also, if \( (e,k) \notin \tau \), then \( \tau[e] = 0 \).

We now specify the predicates \( J_1, J_2 \), and functions \( advance_1 \), \( advance_2 \) and \( merge \).

- \( J_1(\tau, j, \mu) = true \) if and only if \( \tau[e_{kj}] \geq \mu[e_{kj}] \), for \( \forall e_{kj} \in E_j \).

- \( J_2(\tau, j, \mu, k) = true \) if and only if
  \[
  \tau_j[e_{kj}] = T[e_{kj}] - 1 \]
  \[
  \tau_j[e_{pj}] \geq T[e_{pj}], \text{ for } \forall e_{pj} \in E_j \cap E_k, p \neq k
  \]

- Function \( advance_1(\tau, j, x) \) increments \( \tau[e_{jk}] = \tau[e_{jk}] + 1 \), for \( \forall e_{jk} \in E_j \) such that \( x \in e_{jk} \).

- Function \( advance_2(\mu, j, x, \tau) \) sets \( \mu[e_{jk}] = \tau[e_{jk}] \), for \( \forall e_{jk} \in E_j \) such that \( x \in e_{jk} \).

- Function \( merge(\tau_i, i, T_k, k) \) at replica \( i \) returns vector \( T_i \) as follows:
  \[
  T_i[e] := \begin{cases} 
  \max(\tau_i[e], T_k[e]), & \text{for each edge } e \in E_i \cap E_k, \\
  \tau_i[e], & \text{for each edge } e \in E_i - E_k
  \end{cases}
  \]

Using techniques from Appendix 1, the timestamp of replica \( i \) can be compressed. Notice that there is no register on the augmented edges we add in the augmented share graph, hence after compression no replica will maintain counters for the augmented edges. In other words, the set of edges that any replica keeps counters for will be a subset of edges in the original share graph.

We omit the proof of correctness of the above generalized algorithm for brevity.
H.2 Lower Bounds for the General Architecture

The lower bound in Theorem 3 extends naturally to the general architecture, simply by replacing $G$ by $\hat{G}$ in Theorem 3. The claim and proof are omitted here for brevity. Intuitively, a client alternately communicating with different replica propagates causal past from one replica to another. This is captured in the augmented share graph by adding an edge between those replicas (even though the replicas may not share any registers in common).

I Optimization

As shown in previous sections, the timestamp sizes required to maintain causal consistency is expensive. In this section, we will discuss several optimization strategies to reduce the timestamp sizes in practice. These techniques exploit trade-off between operation latency, timestamp size, and false dependencies.

False dependencies: A false dependency occurs when application of an update $u_1$ is delayed at some replica, waiting for some update $u_2$ to be applied, even though $u_2 \not\rightarrow u_1$. By allowing false dependencies to be introduced, it is possible to reduce timestamp size required to maintain causal consistency.

Let us introduce one such approach. In our prototype algorithms, replicas $i, j$ send updates to each other if and only if $X_{ij} \neq \emptyset$. Such updates contain values of updated registers in $X_{ij}$ as well as timestamps used to track causality. Now suppose that $x \in X_i$ and $x \notin X_j$. Suppose that we introduce a “dummy” copy of register $x$ at replica $j$. This copy of $x$ at $j$ is “dummy” in the sense that no client will ever send a request to $j$ for an operation on $x$. Nevertheless, when $i$ issues an update on $x$, replica $j$ will be sent the update message, and eventually apply the update. Since $x$ is dummy at $j$, it is not really necessary to send the value (or data) associated with $x$ to $j$, and it suffices to send the timestamp (metadata) to $j$. This approach has advantages and disadvantages.

- To see the advantage, consider the following instantiation of the above approach. At each replica $j$, we introduce a dummy copy of every register that $j$ does not store. This effectively emulates full replication, with the important caveat that the dummy copies are never operated on. Thus, while the overhead of storing register copies remains identical to the original partial replication scheme, the timestamps can now be smaller. In particular, vector timestamps of length $R$ suffice with this emulation of full replication.

In general, instead of emulating full replication, we can use dummy register more selectively, and yet reduce the size of necessary timestamps significantly. Instead of introducing a dummy copy for every register that replica $j$ does not store, only the registers stored at $j$’s neighbors and those in the loops that pass through $j$ in the share graph are necessary. The timestamp of replica $j$ for this scheme only stores counters corresponding to neighbor replicas of $j$ and those in the loops that pass through $j$ in the shared graph. As a trade-off, this solution has the following disadvantages.

- The first disadvantage is the increase in the number of update messages. In the example above, with our partial replication prototype algorithms, updates for register $x$ are not sent to replica $j$. However, if $j$ maintains a dummy copy of $x$, then such updates will be sent...
to $j$ (even if the updates contain only the metadata, or timestamps, there is still additional overhead).

The second disadvantage is the introduction of false dependencies. In the above example, suppose that replica $i$ issues update $u_1$ on $x \notin X_{ij}$ and replica $j$ issues update $u_2$ on $y \notin X_{ij}$. Also suppose that there are no other updates by any replica. With the original partial replication algorithm, since replicas $i, j$ will not apply each other’s updates, in any execution of above updates, $u_1 \not\rightarrow u_2$ and $u_2 \not\rightarrow u_1$. Now suppose that $j$ maintains a dummy copy of $x$, and the update for $x$ is applied at $j$ before $j$ issues $u_2$. This will introduce the false dependency $u_1 \rightarrow u_2$. The false dependency may potentially result in additional delay in applying $u_2$ at some other replica $k$.

Provided the system has some guarantees on message delay, we can reduce timestamp sizes without introducing false dependencies. Consider the case where the system is loosely synchronous, which guarantees that message propagation through a path of length $\geq l$ will be slower than message propagation through one hop. In this case, replicas do not need to store counters for loops that have length $\geq l + 1$, since the update travels through a long path will always arrive later than its dependent update which travels only one hop. Hence the timestamp of a replica only needs to store counters for its neighbor edges, and edges in the loops that have length $\leq l$, while guarantee that there is no false-dependency in the system.

**Restricting inter-replica communication patterns:** In our discussion so far, we have assumed that any pair of replicas that are adjacent in the share graph may communicate with each other directly. In the message-passing context, it is known that restricted communication graphs can allow dependency tracking with a lower overhead [21, 30]. A similar observation applies in the case of partial replication too.

We illustrate by an example how such benefits may be achieved. For this example, suppose that the share graph consists of a ring of the $R$ replica. Thus, each replica shares a unique register with each of its neighbors in the ring, and does not share registers with any other replica. Such a ring is illustrated in Figure 10 for the case of $R = 6$. Our previous results show that if we could “break” the ring, the timestamp size may be reduced.

To achieve this goal, we introduce virtual registers (these have similarities to the dummy registers). The virtual registers may be shared by the different replicas in an arbitrary manner, resulting an appropriate share graph corresponding to the virtual registers.

In our example in Figure 10, suppose that we want to break the ring by disallowing direct communication between replicas 1 and 6. However, these replicas share register $x$, thus, need to be able to send to each other updates to $x$. This goal can also be achieved by simulating an update message for $x$ from replica 1 to 6, but a sequence of updates to virtual variables, namely $u_i$, $i = 1, 2, 3, 4, 5$ from replica $i$ to replica $i + 1$, to propagate the value of $x$ from replica 1 to replica 6. When replica 6 receives update $u_5$, it would update the register $x$. However, with this scheme, we can redefine the share graph by assuming that $x \notin X_{16}$, while adding shared virtual registers between replicas $i$ and $i + 1$, $1 \leq i \leq 5$. In this case, we are “piggybacking” updates to $x$ on updates to virtual registers. Of course, since virtual registers are themselves never accessed, only metadata needs to be maintained by the virtual registers.
In general, the assignment of the virtual registers to replicas, and which registers are used to piggyback updates for registers shared by a certain pair of replicas, will dictate the communication path taken by the piggybacked update.

In the extreme case, all the updates may be propagated through a single replica, resulting in star graph. However, more general topologies may also be created, while trading off between the overhead of the timestamps, delay in propagating updates, and false dependencies.

Sacrificing causality: While the above solutions introduce false dependencies, an alternate approach that may be desirable for some applications is to sacrifice causality. For instance, in the timestamp graph $E_i$ defined earlier in the paper, we may choose to include a smaller set of edges. In particular, for $j \neq i \neq k$, we may include edge $e_{jk}$ only if there exists an $(i,j,k)$-loop containing at most $h$ edges, for some choice of $h$. Under this restriction, causal consistency will still not be violated so long as single-hop messages (or updates) are delivered faster than messages propagated over $h$ hops. However, when this condition does not hold, causality may be violated.

Analogous ideas have been explored for full replication [29, 24], and partial replication as well [15, 14].

We leave exploration of these ideas for future work. Other optimization are also possible to improve performance of causally consistent systems.