1. Introduction

We assume that the reader is familiar with elementary differential geometry on manifolds and with differential forms. These lectures explain how to apply the Cartan–Kähler theorem to problems in differential geometry. We want to decide if there are submanifolds of a given dimension inside a given manifold on which given differential forms vanish. The Cartan–Kähler theorem gives a linear algebra test: if the test passes, such submanifolds exist. I will not give a proof or give the most general statement of the theorem, as it is difficult to state precisely.

For a proof of the Cartan–Kähler theorem, see [2], which we will follow very closely, and also the canonical reference work [1] and the canonical textbook [6]. The last two also give proof of the Cartan–Kuranishi theorem, which we will only briefly mention.

2. Expressing differential equations using differential forms

Take a differential equation of second order \(0 = f(x, u, u_x, u_{xx})\). To write it as a first order system, add a new variable \(p\) to represent \(u_x\), and a new equation:

\[
\begin{align*}
u_x &= p, \\
0 &= f(x, u, p, p_x).
\end{align*}
\]

It is easy to generalize this to any number of variables and equations of any order: reduce any system of differential equation to a first order system.
To express a first order differential equation \(0 = f(x, u, u_x)\), add a variable \(p\) to represent the derivative \(u_x\), let \(\vartheta = du - pdx\) on the manifold
\[
M = \{(x, u, p) \mid 0 = f(x, u, p)\}
\]
(assuming it is a manifold). A submanifold of \(M\) of suitable dimension on which \(0 = \vartheta\) and \(0 \neq dx\) is locally the graph of a solution. It is easy to generalize this to any number of variables and any number of equations of any order.

3. The Cartan–Kähler theorem

An integral manifold of a collection of differential forms is a submanifold on which the forms vanish. An exterior differential system is an ideal \(I \subset \Omega^*\) of smooth differential forms on a manifold \(M\), closed under exterior derivative, which splits into a direct sum
\[
I = I^0 \oplus I^1 \oplus \cdots \oplus I^n
\]
of forms of various degrees: \(I^p := I \cap \Omega^p\). Any collection of differential forms has the same integral manifolds as the exterior differential system it generates. An exterior differential system is analytic if it is locally generated by real analytic differential forms.

Some trivial examples: the exterior differential system generated by
1. \(0\),
2. \(\Omega^*\),
3. the pullbacks of all forms via a submersion,
4. \(dx^1 \wedge dy^1 + dx^2 \wedge dy^2\) in \(\mathbb{R}^4\),
5. \(dy - zdx\) on \(\mathbb{R}^3\).

Problem 1. What are the integral manifolds of our trivial examples?

The elements of \(I^0\) are 0-forms, i.e. functions. All \(I\)-integral manifolds lie in the zero locus of these functions. Replace our manifold \(M\) by that zero locus (which might not be a manifold, a technical detail we will ignore); henceforth we add to the definition of exterior differential system the requirement that \(I^0 = 0\).

An integral element at a point \(m \in M\) of an exterior differential system \(I\) is a linear subspace \(E \subset T_m M\) on which all forms in \(I\) vanish. Every tangent space of an integral manifold is an integral element, but some integral elements of some exterior differential systems don’t arise as tangent spaces of integral manifolds.

Problem 2. What are the integral elements of our trivial examples?

The polar equations of an integral element \(E\) are the linear functions
\[
w \in T_m M \mapsto \vartheta(w, e_1, e_2, \ldots, e_k)
\]
where \(\vartheta \in I^{k+1}\) and \(e_1, e_2, \ldots, e_k \in E\). They vanish on a vector \(w\) just when the span of \(\{w\} \cup E\) is an integral element. If an integral element \(E\) is contained in another one, \(E \subset F\), then all polar equations of \(E\) occur among those of \(F\): larger integral elements have more (or at least the same) polar equations.

Problem 3. What are the polar equations of the integral elements of our trivial examples?

A partial flag \(E_\bullet\) is a sequence of nested linear subspaces
\[
E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_p
\]
in a vector space. The increments of a partial flag are the integers measuring how the dimensions increase:

\[
\begin{align*}
\dim E_0, \\
\dim E_1 - \dim E_0, \\
\dim E_2 - \dim E_1, \\
\vdots \\
\dim E_p - \dim E_{p-1}.
\end{align*}
\]

A flag is a partial flag 

\[
E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_p
\]

for which \( \dim E_i = i \). Danger: most authors require that a flag have subspaces of all dimensions; we don’t: we only require that the subspaces have all dimensions \( 0, 1, 2, \ldots, p \) up to some dimension \( p \). In particular, the increments of any flag are \( 0, 1, 1, \ldots, 1 \).

The polar equations of a flag \( E_\bullet \) of integral elements form a partial flag in the cotangent space. The characters \( s_0, s_1, \ldots, s_p \) of \( E_\bullet \) are the increments of its polar equations, i.e. the numbers of linearly independent polar equations added at each increment in the flag.

**Problem 4.** What are the characters of the integral flags of our trivial examples?

The rank \( p \) Grassmann bundle of a manifold \( M \) is the set of all \( p \)-dimensional linear subspaces of tangent spaces of \( M \).

**Problem 5.** Recall how charts are defined on the Grassmann bundle. Prove that the Grassmann bundle is a fiber bundle over the underlying manifold.

The integral elements of an exterior differential system form a subset of the Grassmann bundle. Let us inquire whether this subset is a submanifold of the Grassmann bundle; if so, let us predict its dimension. We say that a flag of integral elements predicts the dimension \( \dim M + s_1 + 2s_2 + \cdots + ps_p \); an integral element predicts the dimension predicted by the generic flag inside it.

**Theorem 1** (Cartan’s bound). Every integral element predicts the dimension of a submanifold of the Grassmann bundle containing all nearby integral elements.

An integral element \( E \) correctly predicts dimension if the integral elements near \( E \) form a manifold of dimension predicted by \( E \). An integral element which correctly predicts dimension is involutive.

**Theorem 2** (Cartan–Kähler). There is an integral manifold tangent to every involutive integral element of any analytic exterior differential system.

If an integral element is involutive, then all nearby integral elements are too, as the nonzero polar equations will remain nonzero. An exterior differential system is involutive if its generic maximal dimensional integral element is involutive.

**Problem 6.** The Frobenius theorem in this language: on a manifold \( M \) of dimension \( p + q \), take an exterior differential system \( \mathcal{I} \) locally generated by \( q \) linearly independent 1-forms together with all differential forms of degree more than \( p \): \( \mathcal{T}^k = \Omega^k \) for \( k > p \). Prove that \( \mathcal{I} \) is involutive if and only if every 2-form in \( \mathcal{I} \) is a sum of terms of the form \( \xi \wedge \vartheta \) where \( \vartheta \) is a 1-form in \( \mathcal{I} \). Prove that this occurs just when the \( (n-k) \)-dimensional \( \mathcal{I} \)-integral manifolds form the leaves of a foliation \( \mathcal{F} \) of \( M \). Prove that then \( \mathcal{T}^1 \) consists precisely of the 1-forms vanishing on the leaves of \( \mathcal{F} \).
4. **Example: Lagrangian submanifolds**

Let

\[ \vartheta := dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n. \]

Let \( \mathcal{I} \) be the exterior differential system generated by \( \vartheta \) on \( M = \mathbb{R}^{2n} \). The integral manifolds of \( \mathcal{I} \) are called **Lagrangian manifolds**. Let us employ the Cartan–Kähler theorem to prove the existence of Lagrangian submanifolds of complex Euclidean space. Writing spans of vectors in angle brackets,

| Flag       | Polar equations | Characters |
|------------|-----------------|------------|
| \( E_0 = \{ 0 \} \) | \{ 0 \} | \( s_0 = 0 \) |
| \( E_1 = \langle \partial_{x_1} \rangle \) | \( \langle dy_1 \rangle \) | \( s_1 = 1 \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( E_n = \langle \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n} \rangle \) | \( \langle dy_1, dy_2, \ldots, dy_n \rangle \) | \( s_n = 1 \) |

The flag predicts

\[ \dim M + s_1 + 2s_2 + \cdots + ns_n = 2n + 1 + 2 + \cdots + n. \]

The nearby integral elements at a given point of \( M \) are parameterized by \( dy = a \, dx \), which we plug in to \( \vartheta = 0 \) to see that \( a \) can be any symmetric matrix. So the space of integral elements has dimension

\[ \dim M + \frac{n(n+1)}{2} = 2n + \frac{n(n+1)}{2}, \]

correctly predicted. Therefore the Cartan–Kähler theorem proves the existence of Lagrangian submanifolds of complex Euclidean space, one (at least) through each point, tangent to each subspace \( dy = a \, dx \), at least for any symmetric matrix \( a \) close to 0.

**Problem 7.** On a complex manifold \( M \), take a Kähler form \( \vartheta \) and a holomorphic volume form \( \Psi \), i.e. closed forms expressed in local complex coordinates as

\[ \vartheta = \frac{\sqrt{-1}}{2} g_{\mu\overline{\nu}} dz^\mu \wedge d\overline{z}^\nu, \]

\[ \Psi = f(z) dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n, \]

with \( f(z) \) a holomorphic function and \( g_{\mu\overline{\nu}} \) a positive definite self-adjoint complex matrix of functions. Prove the existence of **special Lagrangian manifolds**, i.e. integral manifolds of the exterior differential system generated by the pair of \( \vartheta \) and the imaginary part of \( \Psi \).

5. **The last character**

In applying the Cartan–Kähler theorem, we are always looking for submanifolds of a particular dimension \( p \). For simplicity, we can add the hypothesis that our exterior differential system contains all differential forms of degree \( p + 1 \) and higher. In particular, the \( p \)-dimensional integral elements are maximal dimensional integral elements. The polar equations of any maximal dimensional integral element \( E_p \) cut out precisely \( E_p \), i.e. there are \( \dim M - p \) independent polar equations on \( E_p \). We encounter \( s_0, s_1, \ldots, s_p \) polar equations at each increment, so the number of independent polar equations is \( s_0 + s_1 + \cdots + s_p \). Our hypothesis helps us to calculate \( s_p \) from the other characters:

\[ s_0 + s_1 + \cdots + s_{p-1} + s_p = \dim M - p. \]
For even greater simplicity, we take this as a definition for the final character $s_p$, throwing out the previous definition. Now we can ignore any differential forms of degree more than $p$ when we test Cartan’s bound.

6. Example: harmonic functions

We will prove the existence of harmonic functions on the plane with given value and first derivatives at a given point. On $M = \mathbb{R}^5_{x,y,u,u_x,u_y}$, let $\mathcal{I}$ be the exterior differential system generated by

$$\vartheta := du - u_xdx - u_ydy,$$
$$\Theta := du_x \wedge dy - du_y \wedge dx.$$ 

Note that

$$d\vartheta = -du_x \wedge dx - du_y \wedge dy$$

also belongs to $\mathcal{I}$ because any exterior differential system is closed under exterior derivative. An integral surface $X \subset M$ on which $0 \neq dx \wedge dy$ is locally the graph of a harmonic function $u = u(x,y)$ and its derivatives $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$.

Each integral plane $E_2$ (i.e. integral element of dimension 2) on which $dx \wedge dy \neq 0$ is given by equations

$$du_x = u_{xx}dx + u_{xy}dy,$$
$$du_y = u_{yx}dx + u_{yy}dy,$$

for a unique choice of 4 constants $u_{xx}, u_{xy}, u_{yx}, u_{yy}$ subject to the 2 equations $u_{xy} = u_{yx}$ and $0 = u_{xx} + u_{yy}$. Hence integral planes at each point have dimension 2. The space of integral planes has dimension $\dim M + 2 = 5 + 2 = 7$.

Each vector inside that integral plane has the form

$$v = (\dot{x}, \dot{y}, u_x \dot{x} + u_y \dot{y}, u_{xx} \dot{x} + u_{xy} \dot{y}, u_{yx} \dot{x} + u_{yy} \dot{y})$$

Each integral line $E_1$ is the span $E_1 = \langle v \rangle$ of a nonzero such vector. Compute

$$v \cdot (d\vartheta \wedge \Theta) = \left( \dot{x}du_x + \dot{y}du_y - (u_{xx} \dot{x} + u_{xy} \dot{y}) \right) dx - (u_{yx} \dot{x} + u_{yy} \dot{y}) dy$$
$$\quad \text{and} \quad \left( \dot{y}du_x - \dot{x}du_y + (u_{xx} \dot{x} + u_{xy} \dot{y}) \right) dy - (u_{yx} \dot{x} + u_{yy} \dot{y}) dx,$$

and

| Flag | Polar equations | Characters |
|------|----------------|-----------|
| $E_0$ = $\{ 0 \}$ | $\langle \vartheta \rangle$ | $s_0 = 1$ |
| $E_1$ = $\langle v \rangle$ | $\langle \vartheta, v \wedge d\vartheta, v \wedge \Theta \rangle$ | $s_1 = 2$ |

Since we are only interested in finding integral surfaces, we compute the final character from

$$s_0 + s_1 + s_2 = \dim M - 2.$$ 

The Cartan characters are $(s_0, s_1, s_2) = (1, 2, 0)$ with predicted dimension $\dim M + s_1 + 2s_2 = 5 + 2 + 2 \cdot 0 = 7$: involution. We see that harmonic functions exist near any point of the plane, with prescribed value and first derivatives at that point.

7. Generality of integral manifolds

The proof of the Cartan–Kähler theorem (which we will not give) constructs integral manifolds inductively, starting with a point, then building an integral curve, and so on. The choice of the initial data at each inductive stage consists of $s_0$ constants, $s_1$ functions of 1 variable, and so on. Different choices of this initial data give rise to different integral manifolds in the final stage. In this sense, the integral manifolds depend on $s_0$ constants, and so on.
If one describes some family of submanifolds in terms of the integral manifolds of an exterior differential system, someone else might find a different description of the same submanifolds in terms of integral manifolds of a different exterior differential system, with different Cartan characters.

For example, any smooth function \( y = f(x) \) of 1 variable is equivalent information to having a constant \( f(0) \) and a function \( y' = f'(x) \) of 1 variable.

For example, immersed plane curves are the integral curves of \( I = 0 \) on \( M = \mathbb{R}^2 \). Check that any integral flag \( E_0 = \{ 0 \}, E_1 = \langle v \rangle \) has \( (s_0, s_1) = (0, 1) \). But immersed plane curves are also the integral curves of the ideal \( I \) generated by \( \vartheta \).

\[
\vartheta := \sin \phi dx - \cos \phi dy
\]
on \( M := \mathbb{R}^2_{x,y} \times S^1_\phi \). Here \( (s_0, s_1) = (1, 1) \). The last nonzero character does not change. In general, we cannot expect all of the Cartan characters to stay the same for different descriptions of various submanifolds, but we can expect the last nonzero Cartan character to stay the same.

Lagrangian submanifolds of \( \mathbb{C}^n \) depend on 1 function of \( n \) variables. This count is correct: those which are graphs \( y = y(x) \) are precisely of the form

\[
y = \frac{\partial S}{\partial x}
\]
for some potential function \( S(x) \), unique up to adding a real constant. On the other hand, the proof of the Cartan–Kähler theorem builds up each Lagrangian manifold from a choice of one function of one variable, one function of two variables, and so on. Similarly, harmonic functions depend on 2 functions of 1 variable. Summing up: we “trust” the last nonzero Cartan–Kähler \( s_p \) to tell us the generality of the integral manifolds: they depend on \( s_p \) functions of \( p \) variables, but we don’t “trust” \( s_0, s_1, \ldots, s_{p-1} \).
8. Example: triply orthogonal webs

On a 3-dimensional Riemannian manifold $X$, a triply orthogonal web is a triple of foliations whose leaves are pairwise perpendicular. We will see that these exist, locally, depending on 3 functions of 2 variables. Each leaf is perpendicular to a unique smooth unit length 1-form $\eta_i$, up to $\pm$, which satisfies $0 = \eta_i \wedge d\eta_i$, by the Frobenius theorem. Let $M$ be the set of all orthonormal bases of the tangent spaces of $X$, with obvious bundle map $x: M \to X$, so that each point of $M$ has the form $m = (x, e_1, e_2, e_3)$ for some $x \in X$ and orthonormal basis $e_1, e_2, e_3$ of $T_x X$. The soldering 1-forms $\omega_1, \omega_2, \omega_3$ on $M$ are defined by

$$v \cdot \omega_i = \langle e_i, x_*v \rangle.$$ 

Note: they are 1-forms on $M$, not on $X$. Let

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$$

Define cross product $\alpha \times \beta$ on $\mathbb{R}^3$-valued 1-forms by

$$\alpha \times \beta(u, v) = \alpha(u) \times \beta(v) - \alpha(v) \times \beta(u).$$

By the fundamental lemma of Riemannian geometry, there is a unique $\mathbb{R}^3$-valued 1-form $\gamma$ for which $d\omega = \frac{1}{2} \gamma \times \omega$. Our triply orthogonal web is precisely a section
$X \to M$ of the bundle map $M \to X$ on which $0 = \omega_i \wedge d\omega_i$ for all $i$, hence an integral 3-manifold of the exterior differential system $I$ on $M$ generated by the 3-forms

$$\omega_1 \wedge d\omega_1, \ \omega_2 \wedge d\omega_2, \ \omega_3 \wedge d\omega_3.$$ Using the equations above, $I$ is also generated by

$$\gamma_3 \wedge \omega_1 \wedge \omega_2, \ \gamma_1 \wedge \omega_2 \wedge \omega_3, \ \gamma_2 \wedge \omega_3 \wedge \omega_1.$$ The 3-dimensional $I$-integral manifolds on which

$$0 \neq \omega_1 \wedge \omega_2 \wedge \omega_3$$

are locally precisely the triply orthogonal webs. The 3-dimensional integral elements on which

$$0 \neq \omega_1 \wedge \omega_2 \wedge \omega_3$$

are precisely given by

$$\gamma = \begin{pmatrix}
0 & p_{12} & p_{13} \\
p_{21} & 0 & p_{23} \\
p_{31} & p_{32} & 0
\end{pmatrix} \omega$$

for any $p_{ij}$, hence $6 + 6 = 12$ dimensions of integral elements. Since the system is generated by 3-forms, on any integral flag in this integral element, $0 = s_0 = s_1$. Count out $(s_0, s_1, s_2, s_3) = (0, 0, 3, 0)$, predicting 12 dimensions of integral elements: involution.

We conclude: for any orthonormal basis at a point of any real analytic Riemannian 3-manifold, there are infinitely many real analytic triply orthogonal webs, depending on 3 functions of 2 variables, defined near that point, with the tangent spaces of the leaves perpendicular to those basis vectors.

9. EXAMPLE: ISOMETRIC IMMERSION

Take a surface $P$ with a Riemannian metric. Naturally we are curious if there is an isometric immersion $f: P \to \mathbb{R}^3$, i.e. a smooth map preserving the lengths of all curves on $P$. For example, these surfaces

![Surfaces](image)

are isometric immersions of a piece of this paraboloid

More generally, take a Riemannian manifold $P$ of dimension 3. We ask if there is an isometric immersion $f: P \to P$.

On the orthonormal frame bundle $FP$, denote the soldering forms as $\omega = \omega_1 + i\omega_2$. By the fundamental lemma of Riemannian geometry there is a unique 1-form (the connection 1-form) $\gamma$ so that $d\omega = i\gamma \wedge \omega$ and $d\gamma = (i/2)K\omega \wedge \bar{\omega}$. As above, on $FP$ there is a soldering 1-form $\omega$ and a connection 1-form $\gamma$ so that $d\omega = \frac{1}{2}\gamma \wedge \omega$ and $d\gamma = \frac{1}{2}\gamma \wedge \gamma + \frac{1}{2}(\overline{\frac{s}{2} - R})\omega \wedge \omega$.

with Ricci curvature $R_{ij} = R_{ji}$ and scalar curvature $s = R_{ii}$.
If there is an isometric immersion $f: P \to P$, then let $X := X_f \subset M := FP \times FP$ be its adapted frame bundle, i.e. the set of all tuples
\[(p, e_1, e_2, p, e_1, e_2, e_3)\]
where $p \in P$ with orthonormal frame $e_1, e_2 \in T_pP$ and $p \in P$ with orthonormal frames $e_1, e_2, e_3 \in T_pP$, so that $f_* e_1 = e_1$ and $f_* e_2 = e_2$. Let $\mathcal{I}$ be the exterior differential system on $M$ generated by the 1-forms
\[
\begin{pmatrix}
\vartheta_1 \\
\vartheta_2 \\
\vartheta_3
\end{pmatrix}
:=
\begin{pmatrix}
\omega_1 - \omega_1 \\
\omega_2 - \omega_2 \\
\omega_3
\end{pmatrix}.
\]
Along $X$, all of these 1-forms vanish, while the 1-forms $\omega_1, \omega_2, \gamma$ remain linearly independent. Conversely, we will eventually prove that all $\mathcal{I}$-integral manifolds on which $\omega_1, \omega_2, \gamma$ are linearly independent are locally frame bundles of isometric immersions. For the moment, we concentrate on asking whether we can apply the Cartan–Kähler theorem.

Compute:
\[
d\begin{pmatrix}
\vartheta_1 \\
\vartheta_2 \\
\vartheta_3
\end{pmatrix}
= -\begin{pmatrix}
0 & \gamma_3 - \gamma & 0 \\
- \gamma_3 & 0 & 0 \\
-\gamma & 0 & \gamma
\end{pmatrix}
\wedge
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix}
\mod \vartheta_1, \vartheta_2, \vartheta_3.
\]

We count $s_1 = 2, s_2 = 1, s_3 = 0$. Each 3-dimensional integral element has $\omega = \omega$, so is determined by the linear equations giving $\gamma_1, \gamma_2, \gamma_3$ in terms of $\omega_1, \omega_2, \omega_3$ on which $d\vartheta = 0$:
\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 - \gamma
\end{pmatrix}
= \begin{pmatrix}
a & b \\
c & -a \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2
\end{pmatrix}.
\]

Therefore there is a 3-dimensional space of integral elements at each point. But $s_1 + 2s_2 = 4 > 3$: no integral element correctly predicts dimension, so we can’t apply the Cartan–Kähler theorem.

What to do? On every integral element, we said that
\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 - \gamma
\end{pmatrix}
= \begin{pmatrix}
a & b \\
c & -a \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2
\end{pmatrix}.
\]

Make a new manifold $M' := M \times \mathbb{R}^3_{a,b,c}$, and on $M'$ let $\mathcal{I}'$ be the exterior differential system generated by
\[
\begin{pmatrix}
\vartheta_4 \\
\vartheta_5 \\
\vartheta_6
\end{pmatrix}
:=
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 - \gamma
\end{pmatrix}
- \begin{pmatrix}
a & b \\
c & -a \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2
\end{pmatrix}.
\]

10. Prolongation

Take an exterior differential system $\mathcal{I}$ on a manifold $M$. What should we do if there are no involutive integral elements? Let $M'$ be the set of all pairs $(m, E)$ consisting of a point $m$ of $M$ and an $\mathcal{I}$-integral element $E \subset T_mM$. So $M'$ is a subset of the Grassmann bundle over $M$. Locally on $M$, take a local basis $\omega, \vartheta, \pi$ of the 1-forms, with $\vartheta$ a basis for the 1-forms in $\mathcal{I}$. We can write each integral element on which $\omega$ has linearly independent components as the solutions of the linear equations $0 = \vartheta, \pi = a\omega$ for some constants $a$. On an open subset of $M'$, $a$ is a function valued in some vector space. Pull back the 1-forms $\vartheta, \omega, \pi$ to $M'$ via the map $(m, E) \in M' \mapsto m \in M$. On $M'$, let $\vartheta' := \pi - a\omega$. The exterior differential system $\mathcal{I}'$ on $M'$ generated by $\vartheta'$ is the prolongation of $\mathcal{I}$. Inductively, let $M^{(1)} := M', \mathcal{I}^{(1)} := \mathcal{I}', M^{(k+1)} := (M^{(k)})', \mathcal{I}^{(k+1)} := (\mathcal{I}^{(k)})'$, if defined.
Theorem 3 (Cartan–Kuranishi). If each $M^{(k)}$ is a submanifold of the Grassmann bundle over $M^{(k-1)}$, with finitely many connected components, and if each $M^{(k)} \to M^{(k-1)}$ is a submersion, then all but finitely many $\mathcal{I}^{(k)}$ are involutive.

II. BACK TO ISOMETRIC IMMERSION

Returning to our example of isometric immersion of surfaces, we have prolongation given by

$$
\begin{pmatrix}
\vartheta_4 \\
\vartheta_5 \\
\vartheta_6
\end{pmatrix} = \begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 - \gamma
\end{pmatrix} - \begin{pmatrix}
a & b \\
c & -a \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix}.
$$

Note that $0 = d\vartheta_1, d\vartheta_2, d\vartheta_3$ modulo $\vartheta_4, \vartheta_5, \vartheta_6$, so we can forget about them.

Calculate the exterior derivatives:

$$
d\begin{pmatrix}
\vartheta_4 \\
\vartheta_5 \\
\vartheta_6
\end{pmatrix} = -\begin{pmatrix} Da & Db & Dc \\ Da & -Da & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ t\omega_1 \wedge \omega_2 \end{pmatrix} \mod \vartheta_1, \ldots, \vartheta_6.
$$

where

$$
\begin{pmatrix}
Da \\
Db \\
Dc
\end{pmatrix} := \begin{pmatrix}
da + (b + c)\gamma - R_{33}\omega_1 \\
db - 2a\gamma + R_{13}\omega_1 \\
dc - 2a\gamma
\end{pmatrix}
$$

and the torsion is

$$t := \frac{s}{4} - R_{33} - K - a^2 - bc.
$$

This torsion clearly has to vanish on any 3-dimensional $\mathcal{I}'$-integral element, i.e. every 3-dimensional $\mathcal{I}'$-integral element lives over the subset of $M'$ on which

$$0 = \frac{s}{4} - R_{33} - K - a^2 - bc.
$$

To ensure that this subset is a submanifold, we let $M'_0 \subset M'$ be the set of points where this equation is satisfied and at least one of $a, b, c$ is not zero. Clearly $M'_0 \subset M'$ is a submanifold, on which we find $Da, Db, Dc$ linearly dependent. On $M'_0$, every 3-dimensional $\mathcal{I}'$-integral element on which $\omega_1, \omega_2, \gamma$ are linearly independent has $s_1 = 2, s_2 = 0$ and 2 dimensions of integral elements at each point. Therefore the exterior differential system is in involution: there is an integral manifold through each point of $M'_0$, and in particular above every point of the surface. The prolongation exposes the hidden necessary condition for existence of a solution: the relation $t = 0$ between the curvature of the ambient space, that of the surface, and the shape operator.

We won’t prove the elementary:

Lemma 1. Take any smooth 3-dimensional integral manifold $X$ of the linear Pfaffian system constructed above. Suppose that on $X$, $0 \neq \omega_1 \wedge \omega_2 \wedge \gamma$. Every point of $X$ lies in some open subset $X_0 \subset X$ so that $X_0$ is an open subset of the adapted frame bundle of an isometric immersion $P_0 \to P$ of an open subset $P_0 \subset P$.

To sum up:

Theorem 4. Take any surface $P$ with real analytic Riemannian metric, with chosen point $p_0 \in P$ and Gauss curvature $K$. Take any 3-manifold $P$ with real analytic Riemannian metric, with chosen point $p_0$, and a linear isometric injection $F: T_{p_0}P \to T_{p_0}P$. Let $\nu$ be a unit normal vector to the image of $P$. Let $R$ be the Ricci tensor on that 3-manifold and $s$ the scalar curvature. Pick a nonzero quadratic form $q$ on the tangent plane $T_{p_0}P$ so that

$$
\det q = K + R(\nu, \nu) - \frac{s}{4}.
$$
Then there is a real analytic isometric immersion $f$ of some neighborhood of $p_0$ to $P$, so that $f'(p_0) = F$ and so that $f$ induces shape operator $q$ at $p_0$.

12. For further information

For proof of the Cartan–Kähler theorem see [2], which we followed very closely, and also the canonical reference work [1] and the canonical textbook [6]. The last two also give proof of the Cartan–Kuranishi theorem. For more on triply orthogonal webs in Euclidean space, and orthogonal webs in Euclidean spaces of all dimensions, see [3, 4, 7, 8]. For more on isometric immersions and embeddings see [5].

13. Linearization

Take an exterior differential system $\mathcal{I}$ on a manifold $M$, and an integral manifold $X \subset M$. Suppose that the flow of a vector field $v$ on $M$ moves $X$ through a family of integral manifolds. So the tangent spaces of $X$ are carried by the flow of $v$ through integral elements of $\mathcal{I}$. Equivalently, the pullback by that flow of each form in $\mathcal{I}$ also vanishes on $X$. So $L_v \vartheta$ vanishes on $X$ for each $\vartheta \in \mathcal{I}$.

Problem 8. Prove that all vector fields $v$ tangent to $X$ satisfy this equation.

More generally, suppose that $E \subset T_m M$ is an integral element of $\mathcal{I}$. If a vector field $v$ on $M$ carries $E$ through a family of integral elements, then $0 = L_v \vartheta|_E$ for each $\vartheta \in \mathcal{I}$. Take local coordinates $x, y$, say $x = (x^1, x^2, \ldots, x^p)$ and $y = (y^1, y^2, \ldots, y^q)$, where $E$ is the graph of $dy = 0$. We use a multiindex notation where if

$$I = (i_1, i_2, \ldots, i_m)$$

then

$$dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_m}.$$ 

Allow the possibility of $m = 0$, no indices, for which $dx^I = 1$. Let $(-1)^I$ mean $(-1)^m$.

Lemma 2. Take an integral element $E \subset T_m M$ for an exterior differential system $\mathcal{I}$ on a manifold $M$. Take local coordinates $x, y$, say $x = (x^1, x^2, \ldots, x^p)$ and $y = (y^1, y^2, \ldots, y^q)$, where $E$ is the graph of $dy = 0$. For any $\vartheta \in \mathcal{I}$, if we write $\vartheta = c_{I,A} dx^I$ then

$$L_v \vartheta|_E = \frac{\partial c_{I,1}}{\partial x^j} c_{I,3} dx^I + v^a \frac{\partial c_{I,1}}{\partial y^a} dx^I.$$ 

Proof. Expand out

$$\vartheta = c_{I,A}(x, y) dx^I \wedge dy^A.$$ 

and

$$v = v^j \frac{\partial}{\partial x^j} + v^b \frac{\partial}{\partial y^b}.$$ 

Note that

$$v \cdot c_{I,1} dx^I = (-1)^l v^l c_{I,1} dx^{I + K}.$$ 

Commuting with exterior derivative,

$$L_v dx^I = \frac{\partial v^j}{\partial x^j} dx^I + \frac{\partial v^j}{\partial y^a} dy^a,$$

$$L_v dy^a = \frac{\partial v^a}{\partial x^j} dx^j + \frac{\partial v^a}{\partial y^a} dy^b.$$
By the Leibnitz rule,
\[
L_v \vartheta = v^i \frac{\partial c_I}{\partial x^i} dx^I \wedge dy^A + v^a \frac{\partial c_I}{\partial y^a} dx^I \wedge dy^A \\
+ c_{JK} dx^J \wedge \left( \frac{\partial v^i}{\partial x^j} dx^j + \frac{\partial v^i}{\partial y^a} dy^a \right) \wedge dx^K \wedge dy^A \\
+ c_{Ia} dx^I \wedge dy^a \wedge \left( \frac{\partial v^a}{\partial x^j} dx^j + \frac{\partial v^a}{\partial y^b} dy^b \right) \wedge dy^C.
\]

On $E$, $dy = 0$ so
\[
L_v \vartheta|_E = v^i \frac{\partial c_I}{\partial x^i} dx^I + v^a \frac{\partial c_I}{\partial y^a} dx^I \\
+ c_{JK} dx^J \wedge \frac{\partial v^i}{\partial x^j} dx^j \wedge dx^K \\
+ c_{Ia} dx^I \wedge \frac{\partial v^a}{\partial x^j} dx^j.
\]

Write the tangent part of $v$ as
\[
v' = v^i \frac{\partial}{\partial x^i}.
\]

Let $A$ be the linear map $A : E \to E$ given by
\[
A_i^j = \frac{\partial v^i}{\partial x^j},
\]
and apply this by derivation to forms on $E$,
\[
(A\xi) (v_1, \ldots, v_k) = \xi(Av_1, v_2, \ldots, v_k) - \xi(v_1, Av_2, v_3, \ldots, nv_k) + \ldots.
\]

Then
\[
L_v \vartheta|_E = v' \wedge d\vartheta|_E + v^a \frac{\partial c_I}{\partial y^a} dx^I + A\vartheta|_E + c_{Ia} dx^I \wedge \frac{\partial v^a}{\partial x^j} dx^j.
\]

Since $0 = \vartheta|_E = d\vartheta|_E$, we find
\[
L_v \vartheta|_E = v^a \frac{\partial c_I}{\partial y^a} dx^I + c_{Ia} dx^I \wedge \frac{\partial v^a}{\partial x^j} dx^j.
\]

In our coordinates, take any submanifold $X$ which is the graph of some functions $y = y(x)$. Suppose that the linear subspace $E = \{dy = 0\}$ at $(x, y) = (0, 0)$ is an integral element. Pullback the forms from the ideal:
\[
\vartheta|_X = \sum c_{IA}(x, y) dx^I \wedge \frac{\partial y^{a_1}}{\partial x^{j_1}} dx^{j_1} \wedge \cdots \wedge \frac{\partial y^{a_k}}{\partial x^{j_k}} dx^{j_k}.
\]

The right hand side, as a nonlinear first order differential operator on functions $y = y(x)$, has linearization $\vartheta \mapsto L_v \vartheta|_E$. That linearization is applied to sections of the normal bundle $TM|_X / TX$, which in coordinates are just the functions $v^a$. The linearized operator at the origin of our coordinates depends only on the integral element $E = T_m X$, not on the choice of submanifold $X \subset M$ tangent to $E$.

**Problem 9.** Compute the linearization of $u_{xx} = u_{yy} + u_{zz} + u_t^2$ around $u = 0$ by setting up this equation as an exterior differential system.

Take sections $v$ of the normal bundle of $X$ which vanish at $m$. Then the linearization applied to these sections is
\[
L_v \vartheta|_E = c_{Ia} dx^I \wedge \frac{\partial v^a}{\partial x^j} dx^j = A\vartheta|_E,
\]
where $A$ is the linear map
\[
A = \frac{\partial v^a}{\partial x^j}.
\]
the linearization of $v$ around the origin, and $A\vartheta$ as usual means the derivation action of linear maps $A$ on differential forms $\vartheta$: $A\vartheta(u_1, u_2, \ldots, u_k) = \vartheta(Au_1, u_2, \ldots, u_k) - \vartheta(u_1, Au_2, \ldots, u_k) + \cdots + \vartheta(u_1, u_2, \ldots, Au_k)$ if $\vartheta$ is a $k$-form.

The leading order terms of the linearization of an exterior differential system form the tableau:

$$\vartheta_m \in I_m, A \in E^* \otimes (T^* M/E) \mapsto A\vartheta_m|_E,$$

where $I_m$ is the set of all values $\vartheta_m$ of differential forms in $I$.

14. The characteristic variety

Lemma 3. Take an integral manifold $X$ of an exterior differential system $I$ on a manifold $M$. The characteristic variety

$$\Xi_x \subset PE$$

of the linearization of the exterior differential system at any integral element $E$ is precisely the set of hyperplanes in $E$ which lie not only in the integral element $E$ but also in some other integral element different from $E$.

Proof. Take a submanifold $X$ tangent to $E$. Apply the linearization operator $L_{v\vartheta}|_E$ as above to sections $v$ of the normal bundle of $X$. We can also apply this operator formally to sections of the complexified normal bundle. In particular, we compute that for any smooth function $f$ on $X$, and real constant $\lambda$

$$e^{-i\lambda f}L_{e^{i\lambda f}\vartheta} = L_v\vartheta + i\lambda df \wedge (v \vartheta).$$

In particular, the symbol of the linearization is

$$\sigma(\xi)v = \xi \wedge (v \vartheta).$$

The linearization of $I$ is just the sum of the linearizations for any spanning set of forms $\vartheta \in I$. The characteristic variety $\Xi$ at a point $x \in X$ is therefore precisely the set of $\xi \in E^*$ for which there is some section $v$ of the normal bundle of $X$ with $v(x) \neq 0$ and

$$\xi \wedge (v \vartheta)|_E = 0$$

for every $\vartheta \in I$. If we look at the hyperplane $(0 = \xi) \subset E$, the vector $v$ can be added to that hyperplane to make an integral element enlarging the hyperplane. So the characteristic variety $\Xi_x$ of $I$ is precisely the set of hyperplanes in $E$ for which there is more than one way to extend that hyperplane to an integral element: you can extend it to become $E$ or to become this other extension containing $v$. □

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