Quantum Galois theory for compact Lie groups

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Abstract
We establish a quantum Galois correspondence for compact Lie groups of automorphisms acting on a simple vertex operator algebra.

1 Introduction

Suppose that $V$ is a simple vertex operator algebra (VOA) and that $G$ is a compact Lie group, possibly finite, which acts as a continuous group of automorphisms of $V$. Such pairs $(V, G)$ occurs in a number of contexts within conformal field theory, for example the theory of orbifolds [DM1], and the theory of $W$-algebras [BS], and it is of interest to investigate the general situation.

In [DM2] we introduced some techniques for studying the representation of $V^G$, the sub VOA of fixed points of $G$ on $V$, afforded by $V$, and used this to establish a quantum Galois correspondence for certain classes of finite groups $G$. Later, in [DLM], the representation of $V^G$ on $V$ was considered for a general compact Lie group $G$ and the quantum Galois correspondence was established for compact abelian groups. Very recently, Hanaki, Miyamoto and Tambara have considered the correspondence for an arbitrary finite group [HMT].

The purpose of present paper is to establish a quantum Galois correspondence for a general compact Lie group. To describe the main result we need to recall the following result (Theorem 2.4 of [DLM]) which is fundamental to our approach.

Theorem 1 There is a decomposition of $V$, considered as $G \times V^G$-module,

$$V = \bigoplus_{\lambda \in \Lambda} W_\lambda \otimes V_\lambda.$$  (1.1)
Here, $\Lambda$ is a set which indexes the inequivalent, finite-dimensional, continuous, complex, simple, unitary representations of $G$, and $W_\lambda$ is the corresponding left $G$-module. In the decomposition (1.1), the following hold:

(a) For each $\lambda \in \Lambda$, $V_\lambda = \text{Hom}_G(W_\lambda, V)$ is a nonzero, irreducible $V^G$-module.
(b) If $\lambda, \mu \in \Lambda$ and $\lambda \neq \mu$ then $V_\lambda$ and $V_\mu$ are not isomorphic as $V^G$-modules.

Now let $U$ be sub VOA of $V$ which contains $V^G$. From (1.1) we see that

$$U = \bigoplus_{\lambda \in \Lambda} R_\lambda \otimes V_\lambda$$

(1.2)

for subspaces $R_\lambda \subset W_\lambda$. Let $\langle \cdot, \cdot \rangle_\lambda$ be a positive definite, $G$-invariant, hermitian form on $W_\lambda$, and let $R^c_\lambda$ be the orthogonal complement to $R_\lambda$ in $W_\lambda$.

**Definition 2** We say that $U$ is orthogonally complemented in $V$ if the subspace $U^c$ defined by

$$U^c = \bigoplus_{\lambda \in \Lambda} R^c_\lambda \otimes V_\lambda$$

(1.3)

is a $U$-submodule of $V$.

Our main result is the following:

**Theorem 3** Let $V$ be a simple vertex operator algebra and let $G$ be a compact Lie group of automorphisms of $V$ which acts continuously on $V$. Let

$$\gamma : H \mapsto V^H$$

(1.4)

be the map which associates to a subgroup $H$ of $G$ the sub VOA $V^H$ consisting of $H$-fixed points. Then the following hold:

(A) $\gamma$ induces a bijection between the closed Lie subgroups of $G$ and the simple, orthogonally complemented sub VOAs of $V$.

(B) If $G$ is finite then $\gamma$ induces a bijection between the subgroups of $G$ and the sub VOAs of $V$ which contain $V^G$.

Part (A) is the main new result; (B) is a restatement of the result of Hanaki, Miyamoto and Tambara. Of course, the classical Galois correspondence may be stated in the following form: if $K$ is a commutative field and $G$ a (finite) group of automorphisms of $K$ then $K$ is a Galois extension of $K^G$ and $H \mapsto K^H$ set-up a bijection between subgroups of $G$ and subfields of $K$ which contain $K^G$. In this sense the theorem represents a direct generalization of the classical case inasmuch as a commutative field may be considered to be a simple VOA.
One of the new ingredients appearing in (A) is the fact that one has to restrict oneself to simple sub VOAs of \( V \) which contain \( V^G \), a condition that is obviously redundant in the classical case. Notice that the adjectives “simple” and “orthogonally complemented” do not appear in (B). Indeed, given the truth of (A), (B) is equivalent to the assertion that, if \( G \) is finite, every sub VOA of \( V \) which contains \( V^G \) is simple and orthogonally complemented. It should be noted that the simplicity condition appearing in (A) is necessary: there are many examples of sub VOAs of \( V \) which contain \( V^G \) which are not simple. We discuss this and related matters in Section 4. Such sub VOAs cannot appear in a Galois correspondence such as ours. It is likely that the complementary condition in (A) is unnecessary, but we do not know how to remove it at present.

Our proof follows in broad outline that of [DLM] with the new ingredient supplied by [HMT] added in a suitably modified way. We use a basic result from conformal field theory (established in Section 2) together with some standard techniques from Lie group theory (representative functions and the Stone-Weierstrass Theorem).

We thank Hanaki, Miyamoto and Tambara for communicating their pretty paper [HMT] and thereby inspiring the results contained herein.

2 Some results from VOA theory

The following notation will be in force throughout of the paper: \( V = (V, Y, 1, \omega) \) is a simple vertex operator algebra and \( G \leq Aut(V) \) a compact Lie group of automorphisms of \( V \) which acts continuously on \( V \). For a subgroup \( H \leq G \), \( V^H \) is the sub VOA of \( H \)-invariants and \( \gamma \) the map (1.4). We retain the other notation introduced in Section 1.

Recall that \( V \) is a direct sum of finite-dimensional homogeneous subspaces

\[
V = \bigoplus_{n \in \mathbb{Z}} V_n
\]  

(2.1)

where \( V_n = 0 \) for all small enough \( n \). If \( v \in V \) the corresponding vertex operator is denoted by

\[
Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}
\]

(2.2)

so that \( v_n \in \text{End}(V) \). (For basic facts about VOAs, we refer the reader to [FHL].)

We may, and shall, choose each \( W_\lambda \) to be homogeneous i.e., \( W_\lambda \subset V_k \) for some \( k \in \mathbb{Z} \). This is possible because each \( V_k \) is a \( G \)-module.

Now let \( U \) be a sub VOA of \( V \) which satisfies

\[
V^G \subset U \subset V,
\]

(2.3)
with decomposition of $U$ into $V^G$-modules as in (1.2) where $R_\lambda \subset W_\lambda$. We set

$$W = \bigoplus_{\lambda \in \Lambda} W_\lambda \quad (2.4)$$

$$R = \bigoplus_{\lambda \in \Lambda} R_\lambda \quad (2.5)$$

and consider $W$ as a $G$-submodule of $V$. Of course $R \subset W$.

The main idea of [HMT] is to establish the following result:

**Proposition 2.1** Suppose that $\pi$ is a $G$-homomorphism from $W \otimes_c W$ to $W$. Then $\pi(R \otimes R) \subset R$.

Fix $\lambda, \mu \in \Lambda$ and consider the following diagram of $G$-modules and $G$-homomorphisms:

$$\cdots \rightarrow Z_n \xrightarrow{\phi_n} Z_{n+1} \xrightarrow{\psi_n} Z_{n+1} \rightarrow \cdots$$

$$W_\lambda \otimes W_\mu \quad = \quad W_\lambda \otimes W_\mu \quad (2.6)$$

The essential ingredients in (2.6) are defined as follows: for each integer $n$,

$$Z_n = \langle \sum_{m=n}^{\infty} u_m v \mid u \in W_\lambda, v \in W_\mu \rangle \quad (2.7)$$

$$\psi_n : u \otimes v \mapsto \sum_{m=n}^{\infty} u_m v, u \in W_\lambda, v \in W_\mu \quad (2.8)$$

$$\phi_n : \sum_{m=n}^{\infty} u_m v \mapsto \sum_{m=n+1}^{\infty} u_m v, u \in W_\lambda, v \in W_\mu \quad (2.9)$$

We make several observations. First, given $u$ and $v$, we have $u_p v = 0$ for all large enough $p$. So all sums in (2.7)-(2.9) are finite and $Z_n$ is a finite-dimensional $G$-submodule of $V$. Second, the maps $\psi_n$ and $\phi_n$ are well-defined. For $\psi_n$ this is clear; as for $\phi_n$, suppose that $\sum_{m=n}^{\infty} u_m v = 0$. Since $u$ and $v$ are homogeneous (since $W_\lambda, W_\mu$ are so chosen), then $u_m v$ and $u_p v$ have different weights if $m \neq p$. So $u_m v = 0$ for all $m \geq n$ and $\sum_{m=n+1}^{\infty} u_m v = 0$. Now the assertion concerning $\phi_n$ follows. Finally, note that indeed $\phi_n$ and $\psi_n$ are $G$-module homomorphisms. This follows because if $g \in G$ then $gu_m g^{-1} = (gu)_m$.

**Lemma 2.2** For all small enough $n$, $\psi_n$ is an isomorphism.

**Proof:** Assume false. Since each $\psi_n$ and $\phi_n$ is a surjection, there is an integer $q$ such that $\phi_n$ is an isomorphism for $n < q$ and moreover there are $u^i \in W_\lambda, v^i \in W_\mu$, $i = 0, \ldots, t$ with $v^0, \ldots, v^t$ linearly independent and such that

$$0 \neq \sum_{i=0}^{t} u^i \otimes v^i \in \ker \psi_q.$$
We thus have \( \sum_{i=0}^{t} u^i \otimes v^i \in \cap_{n \in \mathbb{Z}} \ker \psi_n \), so for all \( n \in \mathbb{Z} \) we get \( \sum_{i} \sum_{m} w^i_{m} w^i = 0 \). But then for each \( n \in \mathbb{Z} \) we have \( \sum_{i} u_{n}^i v^i = 0 \), so that

\[
\sum_{i} Y(u^i, z)v^i = 0. \tag{2.10}
\]

Since the \( v^i \) are linearly independent, (2.10) forces each \( u^i = 0 \) by Lemma 3.1 of [DM2], a contradiction because \( \sum_{i} u^i \otimes v^i \neq 0 \). This completes the proof of the lemma. \( \square \)

Turning to the proof of Proposition 2.1, let \( \pi \) be as in the statement of the proposition. It is sufficient to assume that \( \pi \) induces a surjection \( \pi : W_\lambda \otimes W_\mu \rightarrow W_\nu \) for some \( \lambda, \mu, \nu \) in \( \Lambda \) and establish that \( \pi(R_\lambda \otimes R_\mu) \subset R_\nu \). There are \( G \)-module homomorphisms

\[
\begin{array}{ccc}
Z_n & \psi_n^{-1} & W_\lambda \otimes W_\mu \\
\downarrow i & \pi & \downarrow \rho \\
V & = & V
\end{array}
\]

where \( i \) is inclusion, \( n \) is chosen so that \( \psi_n \) is an isomorphism (Lemma 2.2), and \( \rho \) is some extension of \( \pi \circ \psi_n^{-1} \) to \( V \).

We claim that \( \rho(U) \subset R \). Indeed it is clear that \( \rho \) annihilates \( W_{\alpha} \otimes V_{\alpha} \) if \( \alpha \neq \nu \) and induces a surjection

\[
\rho : W_\nu \otimes V_\nu \rightarrow W_\nu.
\]

As \( V_\nu \) is a trivial \( G \)-module, Schur’s lemma tells us that if \( v \in V_\nu \) then the restriction of \( \rho \) to \( W_\nu \otimes v \) has the form \( \rho(w \otimes v) = kw \) for \( w \in W_\nu \) and \( k \) a scalar. In particular \( \rho(R_\nu \otimes V_\nu) \subset R_\nu \), so that \( \rho(U) = \rho(R_\nu \otimes V_\nu) \subset R \) as claimed.

Finally, since \( U \) is a sub VOA of \( V \) we see that \( \psi_n(R_\lambda \otimes R_\mu) \subset U \), so that

\[
\pi(R_\lambda \otimes R_\mu) = \rho \circ \psi_n(R_\lambda \otimes R_\mu) \subset \rho(U) \subset R.
\]

This completes the proof of Proposition 2.1.

### 3 Proof of Theorem 1

We will use several standard results from the representation theory of compact Lie groups. For these results and general background, the reader may consult, for example, [BT].

It is a consequence of the Peter-Weyl theorem that if \( H \) is a proper closed subgroup of \( G \) then there is \( \lambda \in \Lambda \) such that the space \( W_\lambda^H \) of \( H \)-invariants is non-zero and \( W_\lambda \) is not the trivial module. Since \( W_\lambda \) is a constituent of \( V \) by Theorem 1 it follows that \( V^H \neq V^G \). Therefore, the map \( \gamma \) (1.4) is certainly an injection on the family of closed subgroups of \( G \).
Lemma 3.1  Let $H \subseteq G$ be a closed subgroup. Then $V^H$ is a simple VOA which is orthogonally complemented in $V$.

Proof:  The simplicity of $V^H$ follows from assertion (a) of Theorem 1 applied with $H$ in place of $G$; indeed, $V^H = V_{1_H}$ where $1_H$ is the trivial $H$-module.

Now take $U = V^H$ as in (1.2), with $U^c$ as in (1.3). It is easy to see that $R_\alpha^c$ is, in this case, the unique $H$-invariant complement to $R_\lambda$ in $W_\lambda$, since $R_\lambda = W_\lambda^H$. Thus, applying (1.1) with $H$ in place of $G$, we see that

$$U^c = \bigoplus_{\alpha} X_\alpha \otimes V_\alpha$$

(3.1)

where $X_\alpha$ ranges over the non-trivial, simple, unitary $H$-modules, and $V_\alpha$ is a simple $U$-module. In particular, $U^c$ is a $U$-module, so that $U$ is indeed orthogonally complemented. The lemma is thus proved. □

It remains to prove that if $U$ is an orthogonally complemented simple sub VOA of $V$, then $U = V^H$ for some closed subgroup $H$ of $G$. In the course of proving this we will also establish (B) of Theorem 1.

We begin with an arbitrary sub VOA $U$ and assume earlier notation. Let $F$ be the space of complex-valued representative functions on $G$; $F$ is a $G$-bimodule via $(g\alpha h)(k) = \alpha(g^{-1}kh^{-1})$ for $\alpha \in F$, $g, h, k \in G$, and there is an isomorphism of $G$-bimodules $F \cong \bigoplus_{\lambda \in \Lambda} \text{End}_C(W_\lambda)$ (3.2)

where $\text{End}_C(W_\lambda) = W_\lambda^* \otimes W_\lambda$ carries the usual $G$-bimodule structure.

Fix a basis for each $W_\lambda$. Then $\text{End}_C(W_\lambda)$ may be identified with a matrix algebra. Embed $W_\lambda$ into the first column of $\text{End}_C(W_\lambda^*)$. This gives an embedding of $W$ into $F$ as a left $G$-submodule. Then $R$ is also a subspace of $F$, and we let $S$ be the right $G$-submodule of $F$ generated by $R$.

Now $F$ is an algebra with respect to pointwise multiplication of functions. We consider this multiplication to be a map of $G$-bimodules $\pi : F \otimes F \to F$. We will establish

Lemma 3.2  $S$ is a subalgebra of $F$.

Proof:  We must show that $\pi(S \otimes S) \subseteq S$, and for this it suffices to establish that
\( \pi(R_\lambda \otimes R_\mu) \subset S \) for all \( \lambda, \mu \in \Lambda \). Consider the diagram of left \( G \)-modules and maps

\[
\begin{array}{ccc}
F \otimes F & \xrightarrow{\pi} & F \\
\uparrow & & \uparrow \\
\text{End } W_\lambda \otimes \text{End } W_\mu & \xrightarrow{\pi} & \bigoplus_\nu \text{End } W_\nu \\
\uparrow & & \uparrow \\
W_\lambda \otimes W_\mu & \xrightarrow{\pi} & \pi(W_\lambda \otimes W_\mu) \\
\downarrow \pi & & \downarrow \rho' \\
\pi(W_\lambda \otimes W_\mu) & \xrightarrow{\rho} & W_\nu
\end{array}
\]

Here, \( \nu \) ranges over those elements of \( \Lambda \) such that \( W_\nu \) is a constituent of \( W_\lambda \otimes W_\mu \), \( \rho \) and \( \rho' \) are projections onto \( W_\nu \) and a summand \( W'_\nu \) isomorphic to \( W_\nu \), and the upper maps are containments. By Proposition 2.1 we obtain the following by restricting the bottom part of the preceding diagram:

\[
\begin{array}{ccc}
R_\lambda \otimes R_\mu & \xrightarrow{\pi} & \pi(R_\lambda \otimes R_\mu) \\
\downarrow \pi & & \downarrow \rho' \\
\pi(R_\lambda \otimes R_\mu) & \xrightarrow{\rho} & R_\nu
\end{array}
\]

Now the space of all such \( W'_\nu \), that is to say \( \text{End}(W_\nu) \), is spanned by \( W_\nu \cdot g \) for \( g \in G \), so the same is true if we replace \( W'_\nu \) by \( R'_\nu \) and \( W_\nu \) by \( R_\nu \). As a result, we get

\[ \pi(R_\lambda \otimes R_\mu) \subset \bigoplus_\nu \sum_{g \in G} R_\nu \cdot g \subset S. \]

The lemma is proved. \( \square \)

We define

\[ H = \{ h \in G | \sigma(h) = \sigma(1), \forall \sigma \in S \} \]  \hspace{1cm} (3.3)

where 1 refers to the identity of \( G \).

Now obviously 1 \( \in \) H. Remembering that \( S \) is a right \( G \)-module, if \( h, k \in H \) and \( \sigma \in S \) then

\[ \sigma(hk) = (\sigma \cdot k^{-1})(h) = (\sigma \cdot k^{-1})(1) = \sigma(k) = \sigma(1) \]

so also \( hk \in H \). Similarly

\[ \sigma(1) = \sigma(hh^{-1}) = (\sigma \cdot h)(h) = (\sigma \cdot h)(1) = \sigma(h^{-1}) \]
so \( h^{-1} \in H \). This shows that \( H \) is a closed subgroup of \( G \), in particular \( H \setminus G \) is a Hausdorff space.

Next, if \( \sigma \in S \), \( h \in H \) and \( t \in G \) then
\[
\sigma(ht) = (\sigma \cdot t^{-1})(h) = (\sigma \cdot t^{-1})(1) = \sigma(t).
\]

Thus \( \sigma \) is constant on the coset \( Ht \) and we may regard \( S \) as a subspace of the space \( C^0(H \setminus G) \) of continuous functions on \( H \setminus G \).

We claim that \( S \) separates points of \( H \setminus G \). Indeed suppose that \( t, t_1 \in G \) are such that \( \sigma(Ht) = \sigma(Ht_1) \) for all \( \sigma \in S \). Then \( \sigma(t) = \sigma(t_1) \), so that
\[
\sigma(tt_1^{-1}) = (\sigma \cdot t_1)(t) = (\sigma \cdot t_1)(t_1) = \sigma(t_1t_1^{-1}) = \sigma(1).
\]
Thus \( tt_1^{-1} \in H \) and \( Ht = Ht_1 \), establishing the claim. We are going to prove

**Lemma 3.3** Assume that either \( G \) is finite, or that \( U \) is simple and orthogonally complemented. Then \( S =HF \) is the space of \( H \)-invariants\( ^3 \) under the left action of \( G \) on \( F \).

If this is true then it follows that \( R = HW \). Then clearly \( U \) is the space of \( H \)-invariants on \( V \), and the main theorem is proved.

There are natural identifications which give containments \( S \subset HF \subset C^0(H \setminus G) \), and we also know that \( S \) separates points of \( H \setminus G \). If \( G \) is finite then this already forces \( S = C^0(G/H) \) and there is nothing more to prove. So part (B) of Theorem is established.

To complete the proof of the lemma we show that \( S \) is dense in \( C^0(H \setminus G) \) (with the supremum norm). Since \( HF \) is a direct sum of finite-dimensional subspaces \( H\text{End}_\mathbb{C}(W_\lambda) \), each of which is closed, we then conclude immediately that \( S = HF \), as required. According to the Stone-Weierstrass theorem, the density of \( S \) in \( C^0(H \setminus G) \) will follow as long as we can show that \( S \) is closed under complex conjugation, since we have already seen that \( S \) separates points of \( H \setminus G \). It is this condition which requires us to assume – as we now do – that \( U \) is also simple and orthogonally complemented.

For \( \lambda \in \Lambda \), let \( \lambda^* \) be the dual of \( \lambda \) in the sense that \( W_\lambda^* = W_\lambda^* = \text{Hom}_\mathbb{C}(W_\lambda, \mathbb{C}) \). We restrict the canonical projection \( W_\lambda^* \otimes W_\lambda \to W_1 = \mathbb{C} \) to a pairing \( R_\lambda^* \otimes R_\lambda \to \mathbb{C} \), and prove next that this latter pairing is non-degenerate.

For convenience we set \( U_\lambda = R_\lambda \otimes V_\lambda \). Since \( U \) is a simple VOA it follows from Proposition 4.1 of \( \text{[DM2]} \) that for any nonzero \( v \in R_\lambda \) we have
\[
U = (u_m v | u \in U, m \in \mathbb{Z}).
\]

\(^3\)Previously we wrote \( V^G \) etc for \( G \)-invariants, even though the action was a left action. Here, however, we need to distinguish between left and right actions of \( G \) on \( F \).
Notice that \( u_m v \in \oplus_{\nu \in \mu \otimes \lambda} U_{\nu} \) if \( u \in U_{\mu} \). Thus \( \langle u_m v | u \in U_{\mu}, \mu \in \Lambda \setminus \{ \lambda^* \}, m \in \mathbb{Z} \rangle \) is contained in \( \oplus_{\nu \neq 1} U_{\nu} \) and \( V^G \subset \langle u_m v | u \in U_{\lambda^*}, m \in \mathbb{Z} \rangle \). Since \( U_{\lambda^*} \) is a \( V^G \)-module generated by \( R_{\lambda^*} \) we see from the associativity of vertex operators that \( \langle u_m v | u \in U_{\lambda^*}, m \in \mathbb{Z} \rangle \) is a \( V^G \)-module generated by \( \langle u_m v | u \in R_{\lambda^*}, m \in \mathbb{Z} \rangle \). The actions of \( u_m \) and \( G \) on \( V \) commute for \( u \in V^G \) and \( m \in \mathbb{Z} \). It follows that there exists \( u \in R_{\lambda^*} \) and \( p \in \mathbb{Z} \) such that the projection of \( u_p v \) into \( V^G \) is nonzero.

Consider the composition of \( G \)-maps

\[ W_{\lambda^*} \otimes W_{\lambda} \overset{\psi_p}{\longrightarrow} Z_p \overset{\tau}{\longrightarrow} Z_p^G \]

where we use the notation of (2.7)-(2.8) and \( \tau \) is projection onto \( G \)-invariants. From the last paragraph it follows that the image of \( R_{\lambda^*} \otimes v \) under \( \tau \circ \psi_p \) is nonzero. Thus \( \tau \circ \psi_p \) generates \( \text{Hom}_G(W_{\lambda^*} \otimes W_{\lambda}, \mathbb{C}) \) and its restriction to \( R_{\lambda^*} \otimes v \) is nonzero. Since \( v \) is an arbitrary non-zero element of \( R_{\lambda} \), and since the same argument applies with \( \lambda \) and \( \lambda^* \) interchanged, this proves that indeed restriction of the canonical pairing to \( R_{\lambda^*} \otimes R_{\lambda} \) is nondegenerate.

Next we claim that the restriction of the pairing \( W_{\lambda^*} \otimes W_{\lambda} \rightarrow \mathbb{C} \) to both \( R_{\lambda^*} \otimes R_{\lambda} \) and \( R_{\lambda^*} \otimes R_{\lambda}^* \) is zero. The proofs of these two assertions being similar we only prove the second. Namely, if \( R_{\lambda^*} \otimes R_{\lambda}^* \rightarrow \mathbb{C} \) is nonzero, application of Lemma 2.2 shows that that there are \( u \in R_{\lambda^*}, v \in R_{\lambda}^* \) and \( m \in \mathbb{Z} \) such that \( u_m v \) has a nonzero projection into \( V^G \). But \( u_m v \) lies in \( U^e \) and \( U^e \cap V^G = 0 \). This contradiction proves the assertion.

Now let us identify \( W_{\lambda^*} \) with \( W_{\lambda} \) via a positive definite, \( G \)-invariant, hermitian form on \( W_{\lambda} \). It follows from what we have established so far, together with the fact that \( R_{\lambda} \) and \( R_{\lambda}^* \) are mutually orthogonal, as are \( R_{\lambda^*} \) and \( R_{\lambda}^* \), that \( R_{\lambda^*} \) is then identified with \( R_{\lambda} \). In terms of coordinate functions, this means that if we embed \( R_{\lambda} \) into \( \text{End} W_{\lambda} \) as the first \( k \)-rows of the first column, i.e., as the coordinate functions \( a_{i1}, 1 \leq i \leq k \), then \( R_{\lambda^*} \) corresponds to the coordinate functions \( \bar{a}_{i1}, 1 \leq i \leq k \).

It follows that for all \( \lambda, R_{\lambda} \subset S \), and hence \( S \) is indeed closed under complex conjugation. This completes the proof of Lemma 3.3 and with it that of Theorem 3.1.

### 4 Further comments

First we wish to point out that the statement of Theorem 3.1 of [DLM] inadvertently omitted the adjective “simple” so that as stated, Theorem 3.1 of (loc. cit.) is incorrect. Of course, the correct statement is contained as a special case of the main theorem of the present paper. Note that in the situation of (loc. cit.), where \( G \) is abelian, the condition of orthogonal complementation is redundant. As we pointed out in the introduction, Galois
correspondences of the type discussed in this paper must necessarily only involve simple sub VOAs, although there are generally many sub VOAs $U$ with $V^G \subset U \subset V$ and $U$ not simple, at least if $G$ is not finite.

Indeed, suppose that $\Lambda$ is as before, and that $\Theta \subset \Lambda$ is a closed subset in the sense that if $\alpha, \beta \in \Theta$, then $\delta \in \Theta$ whenever $W_\delta$ is a constituent of $W_\lambda \otimes W_\beta$. Suppose that $1 \in \Theta$. Then

$$U = \bigoplus_{\lambda \in \Theta} W_\lambda \otimes V_\lambda$$  \hspace{1cm} (4.1)$$

is a sub VOA of $V$ which contains $V^G$. What we have proved implies that $U$ is a simple sub VOA if, and only if, $\Theta$ is closed with respect to conjugation, i.e., if $\lambda \in \Theta$ then also $\lambda^* \in \Theta$. Finally, note that (4.1) represents a typical $G$-invariant sub VOA of $V$ which contains $V^G$, so that there is a bijection between $G$-invariant sub VOAs containing $V^G$ and closed subsets of $\Lambda$ containing 1.

We consider a different kind of example. Let $V_L$ be the simple vertex operator algebra associated to the lattice $L = \mathbb{Z}\alpha$ where $\langle \alpha, \alpha \rangle = 2n$ for positive integer $n$ (cf. [B] and [FLM]). Let $L^\circ = \frac{1}{2n}\mathbb{Z}\alpha$ be the dual lattice of $L$. Then the compact Lie group $G = R\alpha/L^\circ$, which is a circle, acts on $V_L$ in the following way (cf. [DM1]):

$$\beta \cdot (u \otimes e^\gamma) = e^{2\pi i \langle \beta, \gamma \rangle} u \otimes e^\gamma$$

where $\beta \in G$, $u \in \mathbb{C}[\alpha(-1), \alpha(-2), \cdots]$ and $\gamma \in L$. It is easy to see that

$$V_L^G = \mathbb{C}[\alpha(-1), \alpha(-2), \cdots].$$

Let $U = \bigoplus_{m \geq 0} \mathbb{C}[\alpha(-1), \alpha(-2), \cdots] \otimes e^{m\alpha}$. Then $U$ is a sub VOA of $V_L$ which contains $V_L^G$. Note that $U$ is not simple as $\bigoplus_{m \geq 1} \mathbb{C}[\alpha(-1), \alpha(-2), \cdots] \otimes e^{m\alpha}$ is a proper ideal of $U$. Clearly if $g \in G$ is the identity on $U$ then $g$ must be the identity on the whole space $V_L$. Thus $U \not= V_L^H$ for any subgroup $H$ of $G$. This kind of sub VOA was also considered in [D] in order to construct examples of vertex operator algebras with certain properties.

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