AREA MINIMIZING SURFACES IN HOMOTOPY CLASSES IN METRIC SPACES

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Abstract. We introduce and study a notion of relative 1–homotopy type for Sobolev maps from a surface to a metric space spanning a given collection of Jordan curves. We use this to establish the existence and local Hölder regularity of area minimizing surfaces in a given relative 1–homotopy class in proper geodesic metric spaces admitting a local quadratic isoperimetric inequality. If the underlying space has trivial second homotopy group then relatively 1–homotopic maps are relatively homotopic. We also obtain an analog for closed surfaces in a given 1–homotopy class. Our theorems generalize and strengthen results of Lemaire, Jost, Schoen-Yau, and Sacks-Uhlenbeck.

1. Introduction

1.1. Background. Let $M$ be a 2–dimensional surface with boundary. A map from $M$ to a Riemannian manifold $N$ is said to span a given collection $\Gamma \subset N$ of Jordan curves if its restriction to $\partial M$ is a weakly monotone parametrization of $\Gamma$. Consider the problem of finding a weakly conformal map of minimal area among maps spanning $\Gamma$. When $M$ is a disc this amounts to the classical Problem of Plateau, with first general solutions going back to [6, 31, 2] for $N = \mathbb{R}^n$ and to [28] for homogeneously regular Riemannian manifolds $N$. When $M$ is a surface of higher topological type, possibly with several boundary components, the problem is known as the Plateau-Douglas problem. It was first considered in [7, 36, 11] with different non-degeneracy conditions; complete modern solutions appeared in [16, 37].

One may further ask whether it is possible to find a weakly conformal map of minimal area spanning $\Gamma$ in a fixed relative homotopy class. In general, such maps need not exist, see [15, 21]. However, Lemaire [22] showed the existence of an area minimizer in a fixed relative homotopy class under the assumption that $N$ has trivial second homotopy group, while Jost [16] proved the existence of an area minimizer inducing the same action on fundamental groups as a given map. Schoen-Yau [35] and Sacks-Uhlenbeck [34] considered the related problem of finding a mapping of minimal area from a closed (i.e. compact and without boundary) surface $M$ to $N$ inducing the same action on fundamental groups as a given map. Finally, White [39] introduced the notion of $d$–homotopy type for Sobolev maps from a closed manifold of any dimension to a Riemannian manifold and proved the existence of mappings of minimal energy in a given $d$–homotopy class for suitable integers $d$.

Recently, the classical Plateau and the Plateau-Douglas problems have been solved in metric spaces of various generality in [29, 17, 27, 30, 24, 8] and [8, 4], respectively. In the present article we strengthen the results of Lemaire [22] and Jost...
mentioned above and generalize them to the setting of proper geodesic metric spaces admitting a local quadratic isoperimetric inequality. For this purpose, we introduce and study a notion of 1–homotopy classes of Sobolev maps relative to a given collection of Jordan curves. Our notion is akin to $d$–homotopy of Sobolev maps defined on a closed manifold introduced by White [39] and studied in [35, 34]. It provides better control than the induced action on the fundamental group. We then solve the Plateau-Douglas problem in relative 1–homotopy classes and show that solutions are locally Hölder continuous and conformal in a weak metric sense. If the underlying space has trivial second homotopy group then relatively 1–homotopic maps are relatively homotopic. To our knowledge, our results are already partially new for Riemannian manifolds. We further obtain an analog for closed surfaces, generalizing the results in [39, 10]. It provides better control than the induced action on the fundamental group. Moreover, we introduce and study a notion of 1–homotopy classes of Sobolev maps relative to a given collection of Jordan curves. Our notion is akin to 1–homotopy of Sobolev maps relative to a given collection of Jordan curves.

A complete metric space $X$ is said to admit a local quadratic isoperimetric inequality if there exist $C, l_0 > 0$ such that every Lipschitz curve $c: S^1 \to X$ of length $\ell(c) \leq l_0$ is the trace of a Sobolev map $u \in W^{1,2}(\mathbb{D}, X)$ with

$$\text{Area}(u) \leq C \cdot \ell(c)^2.$$ 

For the notions related to Sobolev maps we refer to Section 2. The class of spaces admitting a local quadratic isoperimetric inequality contains all homogeneously regular Riemannian manifolds [28], compact Lipschitz manifolds, complete CAT($\kappa$)–spaces, compact Alexandrov spaces, some sub-Riemannian manifolds, and many more spaces, cf. [24, Section 8].

### 1.2. Relative 1–Homotopy classes of Sobolev maps.

Let $\Gamma \subset X$ be the disjoint union of $k \geq 1$ rectifiable Jordan curves in a proper geodesic metric space $X$ admitting a local quadratic isoperimetric inequality. Let $M$ be a smooth compact oriented surface with $k$ boundary components, and let $g$ be an auxiliary Riemannian metric on $M$. We denote by $[\Gamma]$ the family of weakly monotone parametrizations of $\Gamma$, i.e. uniform limits of homeomorphisms $\partial M \to \Gamma$, and by $\Lambda(M, \Gamma, X)$ the family of Sobolev maps $u \in W^{1,2}(M, X)$ such that the trace $\text{tr}(u)$ has a continuous representative in $[\Gamma]$. Let $h: K \to M$ be a $C^1$–smooth triangulation of $M$, and $g: K^1 \to X$ a continuous map such that $g|_{\partial K} \in [\Gamma]$, where $K^1$ denotes the 1–skeleton of $K$ and $\partial K \subset K^1$ is the subset of $K$ homeomorphic to $\partial M$. The homotopy class of $g$ relative to $\Gamma$ is the family

$$[g]|_{\Gamma} := \{g': K^1 \to X \mid g' \text{ continuous }, g'|_{\partial K} \in [\Gamma], g \sim g' \text{ rel } \Gamma\},$$

where $g$ and $g'$ are said to be homotopic relative to $\Gamma$, denoted $g \sim g' \text{ rel } \Gamma$, if there exists a homotopy $F: K^1 \times [0, 1] \to X$ from $g$ to $g'$ with $F(\cdot, t)|_{\partial K} \in [\Gamma]$ for every $t$.

The 1–homotopy class $u_{g,1}[h]$ relative to $\Gamma$ of an element $u \in \Lambda(M, \Gamma, X)$ will be defined in Section 4. In the following theorem we summarize its most important properties. These could in fact be used to give an equivalent definition of $u_{g,1}[h]$, see the remark after the theorem.

**Theorem 1.2.** Every $u \in \Lambda(M, \Gamma, X)$ has a well-defined relative homotopy class $u_{g,1}[h]$ of continuous maps from $K^1$ to $X$ whose restriction to $\partial K$ is in $[\Gamma]$. It satisfies:

(i) If $u$ has a representative $\bar{u}$ which is continuous on the whole of $M$ then

$$u_{g,1}[h] = [\bar{u} \circ h|_{K^1}]_{\Gamma}.$$
(ii) If \( u, v \in \Lambda(M, \Gamma, X) \) satisfy \( u_{\#}, 1[h] = v_{\#}, 1[h] \) then, for every triangulation \( \tilde{K} : \tilde{K} \rightarrow M \) of \( M \), we have
\[
 u_{\#}, 1[\tilde{h}] = v_{\#}, 1[\tilde{h}].
\]
(iii) For every \( L > 0 \) there exists \( \varepsilon > 0 \) such that if \( u, v \in \Lambda(M, \Gamma, X) \) induce the same orientation on \( \Gamma \), and
\[
d_L(u, v) \leq \varepsilon, \quad \max\left\{ E_+^2(u, g), E_+^2(v, g) \right\} \leq L,
\]
then \( u_{\#}, 1[h] = v_{\#}, 1[h] \).

Here, \( E_+^2(u, g) \) denotes the Reshetnyak energy of \( u \) with respect to \( g \), see Section 2.2. Maps in \( \Lambda(M, \Gamma, X) \) can be approximated in the \( L^2 \)-distance by continuous maps in \( \Lambda(M, \Gamma, X) \) with the same trace and control on the energy, see Lemma 4.2. Thus properties (i) and (iii) in Theorem 1.2 imply that the \( 1 \)-homotopy class \( u_{\#}, 1[h] \) is well-defined. The argument used to prove (ii) also shows that, if \( u \in \Lambda(M, \Gamma, X) \) and \( \varphi : M \rightarrow X \) is continuous with \( \varphi|_{\partial M} \in [\Gamma] \), then \( u_{\#}, 1[h] = [\varphi \circ h|_{\partial \tilde{M}}] \) holds for one triangulation \( h \) if and only if it holds for every triangulation. In this case we say that \( u \in \Lambda(M, \Gamma, X) \) is \( 1 \)-homotopic to \( \varphi \) relative to \( \Gamma \), denoted by \( u \sim_1 \varphi \) rel \( \Gamma \).

1.3. Homotopic Plateau-Douglas problem. Let \( \Gamma, X \) be as above, and let \( M \) be a smooth compact oriented and connected surface with \( k \geq 1 \) boundary components. Given a continuous map \( \varphi : M \rightarrow X \) with \( \varphi|_{\partial M} \in [\Gamma] \), set
\[
a(M, \varphi, X) := \inf \{ \text{Area}(u) : u \in \Lambda(M, \Gamma, X), u \sim_1 \varphi \text{ rel } \Gamma \},
\]
where \( \inf \emptyset = \infty \) by convention. Moreover, set \( a'(M, \varphi, X) := \inf a(M^*, \varphi^*, X) \), where the infimum is taken over all primary reductions of \((M, \varphi)\), that is, pairs \((M^*, \varphi^*)\) consisting of
(i) a smooth surface \( M^* \) obtained from \( M \) by cutting \( M \) along a smooth closed simple non-contractible curve \( \alpha \) in the interior of \( M \) and gluing smooth discs to the two new boundary components;
(ii) a continuous map \( \varphi^* : M^* \rightarrow X \) which agrees with \( \varphi \) on \( M \setminus \alpha \).

We say that \( \varphi \) satisfies the homotopic Douglas condition if
\[
a(M, \varphi, X) < a'(M, \varphi, X).
\]
As an illustration, if the induced homomorphism \( \varphi_* : \pi_1(M) \rightarrow \pi_1(X) \) of fundamental groups is injective then \( \varphi \) satisfies the homotopic Douglas condition (1.1) and, in particular, \( a(M, \varphi, X) < \infty \), see Proposition 5.1. In the statement below, we fix \( \Gamma, X, \) and \( M \) as above, and let \( \varphi : M \rightarrow X \) be a continuous map with \( \varphi|_{\partial M} \in [\Gamma] \).

Theorem 1.3. If \( \varphi \) satisfies the homotopic Douglas condition (1.1) then:
(i) There exist \( u \in \Lambda(M, \Gamma, X) \) and a Riemannian metric \( g \) on \( M \) such that \( u \) is \( 1 \)-homotopic to \( \varphi \) relative to \( \Gamma \), \( u \) is infinitesimally isotropic with respect to \( g \), and \( \text{Area}(u) = a(M, \varphi, X) \).
(ii) Any such \( u \) has a representative \( \tilde{u} \) which is locally Hölder continuous in the interior of \( M \) and extends continuously to the boundary \( \partial M \).
(iii) If \( X \) has trivial second homotopy group then \( \tilde{u} \) is homotopic to \( \varphi \) relative to \( \Gamma \).

Moreover, the metric \( g \) can be chosen such that it has constant curvature \(-1, 0, \) or \( 1 \) and \( \partial M \) is geodesic. See Section 2 for the definition of infinitesimal isotropy, which is a metric variant of weak conformality. Here, \( \tilde{u} \) and \( \varphi \) are called homotopic
relative to \( \Gamma \) if they are homotopic through a family of maps whose restriction to \( \partial M \) is in \( [\Gamma] \). We remark that homotopy classes (relative to \( \Gamma \)) need not contain continuous, infinitesimally isotropic area minimizers if \( \pi_2(X) \neq \emptyset \), compare [15], Chapter 5.

Theorem 1.3 generalizes and strengthens [16, Theorem 2.2] and [22, Theorem 1.7], see also [17, Theorem 5.1] for a homotopic variant of the Dirichlet problem in metric spaces. We remark that control on the relative 1–homotopy class is, in general, strictly stronger than the control on the action on fundamental groups in [16], see Example 6.3. An analog of Theorem 1.3 for closed surfaces, generalizing results in [35, 34], will be discussed in Section 6.

We remark that the local quadratic isoperimetric inequality is crucial to the stability statement (iii) in Theorem 1.2. Example 4.7 exhibits a space where the stability of 1–homotopy classes from closed surfaces fails. Compare with [4], where the Plateau-Douglas problem was recently solved in spaces without a local quadratic isoperimetric inequality.

1.4. Outline. The idea for defining the relative 1–homotopy type of a map \( u \in \Lambda(M, \Gamma, X) \) is, like in [39], to consider small perturbations of \( C^1 \)–smooth triangulations of \( M \) in such a way that the restriction of \( u \) to the 1–skeleton of a “generic” perturbed triangulation is essentially continuous. In Section 3 we introduce admissible deformations on \( M \) which accomplish this and prove that the relative homotopy class of such restrictions is essentially independent of the perturbation, see Theorem 3.6. This crucially uses the local quadratic isoperimetric inequality.

In Section 4 we show that the way we perturb a given triangulation does not affect the relative homotopy type of the restrictions to generic 1–skeleta. Together with a continuous approximation of Sobolev maps (see Lemma 4.2) and the results of Section 3, this leads to a well-defined notion of relative 1–homotopy class for Sobolev maps, which is moreover independent of the chosen triangulation. The main results in Section 4 are Theorems 4.1 and 4.6 from which Theorem 1.2 will follow. As already mentioned, our notion of relative 1–homotopy class is related to the \( d \)–homotopy type, studied primarily for Sobolev maps defined on closed manifolds in [39, 10, 11]. While these articles also discuss the case of manifolds with boundary, Sobolev maps in their setting are required to have a fixed Lipschitz trace. This is suitable for solving the Dirichlet problem in \( d \)–homotopy classes but cannot be applied to the Plateau–Douglas problem since it is not possible to control the boundary behaviour of elements of \( \Lambda(M, \Gamma, X) \).

In Sections 5 and 6 we use an approach analogous to that in [8] in order to solve the homotopic Plateau-Douglas problem. Unlike in [8], we need to control the relative 1–homotopy type of the primary reductions appearing in the proofs of Propositions 5.2 and 5.4. Lemma 5.3 provides the necessary technical tool for this. We furthermore provide a simple sufficient condition (see Proposition 5.1) that ensures the homotopic Douglas condition (1.1) is satisfied. Section 6 is devoted to the proof of Theorem 1.3. We present and prove Theorem 6.4 which is an analog of Theorem 1.3 for closed surfaces.

2. Preliminaries

2.1. Terminology. A surface, in this work, refers to a smooth compact oriented surface with (possibly) non-empty boundary, and a closed surface is a surface with empty boundary. We denote by \( \partial M \) and \( \text{int}(M) = M \setminus \partial M \) the boundary and interior
of a surface $M$, respectively. The Euler characteristic of a connected surface $M$ satisfies $\chi(M) = 2 - 2p - k$, where $k \geq 0$ is the number of components of $\partial M$, and $p$ is the genus of the closed surface obtained by gluing a disc along every boundary component of $M$.

For a metric space $X$ and $m \geq 0$, we denote by $\mathcal{H}^m_\delta$ the Hausdorff $m$–measure on $X$. If $X$ is a manifold equipped with a Riemannian metric $g$, we denote $\mathcal{H}^m_g = \mathcal{H}^m_\delta$. The Lebesgue measure of a subset $A \subset \mathbb{R}^m$ is denoted by $|A|$.

2.2. **Triangulations.** A triangulation of a surface $M$ is a homeomorphism $h: K \to M$ from a cell-complex $K$, equipped with the length metric which restricts to the Euclidean metric on every cell $\Delta$ of $K$. We additionally assume throughout the paper that triangulations are $C^1$–diffeomorphisms, i.e. $h|_\Delta$ is a $C^1$–diffeomorphism onto its image for any cell $\Delta$ of $K$ (cells are closed by definition). The $j$–skeleton $K^j$ of $K$ is the union of the cells of $K$ with dimension $\leq j$, and $\partial K \subset K^1$ is the subset of $K$ homeomorphic to $\partial M$.

2.3. **Semi-norms.** The energy of a semi-norm $s$ on (Euclidean) $\mathbb{R}^2$ is defined by
\[
\mathcal{H}^2(s) := \max\{s(v)^2 : v \in \mathbb{R}^2, |v| = 1\}.
\]
The jacobian of a norm $s$ on $\mathbb{R}^2$ is the unique number $J(s)$ such that
\[
\mathcal{H}^2_s(A) = J(s) |A|
\]
for some and thus every subset $A \subset \mathbb{R}^2$ with $|A| > 0$. For a degenerate semi-norm $s$ we set $J(s) := 0$. Notice that we always have $J(s) \leq \mathcal{H}^2(s)$. A semi-norm $s$ on $\mathbb{R}^2$ is called isotropic if $s = 0$ or if $s$ is a norm and the ellipse of maximal area contained in $\{v \in \mathbb{R}^2 : s(v) \leq 1\}$ is a round Euclidean ball.

2.4. **Sobolev maps with metric targets.** Let $(X,d)$ be a complete metric space and let $M$ be a smooth compact $m$–dimensional manifold, possibly with non-empty boundary. Fix a Riemannian metric $g$ on $M$ and let $\Omega \subset M$ be open and bounded.

Denote by $L^2(\Omega ; X)$ the collection of measurable and essentially separably valued maps $u: \Omega \to X$ such that for some and thus every $x \in X$ the function $u_z(x) := d(x, u(z))$ belongs to the classical space $L^2(\Omega)$. For $u, v \in L^2(\Omega ; X)$ we define
\[
d^2_{L^2}(u, v) := \left( \int_{\Omega} d^2(u(z), v(z)) \, d\mathcal{H}^m_g(z) \right)^{\frac{1}{2}},
\]
and we say that a sequence $(u_n) \subset L^2(\Omega ; X)$ converges in $L^2(\Omega ; X)$ to $u \in L^2(\Omega ; X)$ if $d_{L^2}(u_n, u) \to 0$ as $n \to \infty$. The following definition is due to Reshetnyak [32, 33].

**Definition 2.1.** A map $u \in L^2(\Omega ; X)$ belongs to the Sobolev space $W^{1,2}(\Omega ; X)$ if there exists $h \in L^2(\Omega)$ such that $u_z \in W^{1,2}(\text{int}(\Omega))$ and $|\nabla u_z|_g \leq h$ almost everywhere on $\Omega$, for every $x \in X$.

Several other notions of Sobolev spaces exist in the literature and we refer the reader to [14 Chapter 10] for an overview of some of them. We will use in particular Newton-Sobolev spaces which are equivalent to $W^{1,2}(\Omega ; X)$ if $\Omega$ is a bounded Lipschitz domain, see Proposition 2.5 for a precise statement.

If $u \in W^{1,2}(\Omega ; X)$ then for almost every $z \in \Omega$ there exists a unique semi-norm $a \text{p md } u_z$ on $T_z M$ such that
\[
ap \lim \frac{d(u(exp_z(v), u(z))) - a \text{p md } u_z(v)}{|v|_g} = 0,
\]
where \( \text{ap lim} \) is the approximate limit, see e.g. \[13\]. Next, we specialize to the case that \( M \) has dimension \( m = 2 \). We define the notions of energy, jacobian and isotropy of a semi-norm on \((T\pi M, g(z))\) by identifying it with \((\mathbb{R}^2, |\cdot|)\) via a linear isometry.

**Definition 2.2.** Let \( u \in W^{1,2}(\Omega, X) \). The Reshetnyak energy of \( u \) with respect to \( g \) and the parametrized (Hausdorff) area of \( u \) are given, respectively, by

\[
E^2_\pi(u, g) := \int_\Omega \mathbf{1}_\pi^2(\text{ap md } u) \, dH^2_\pi(z), \quad \text{Area}(u) := \int_\Omega \mathbf{J}(\text{ap md } u) \, dH^2_\pi(z).
\]

We have that the parametrized area of a Sobolev map is invariant under precompositions with biLipschitz homeomorphisms, and thus independent of the Riemannian metric \( g \). The energy \( E^2_\pi \) is invariant only under precompositions with conformal diffeomorphisms, and thus depends on \( g \). Our notation reflects these facts. Finally, if \( u \) satisfies Lusin’s property (N) then the area formula \[19\], \[18\] for metric space valued Sobolev maps yields

\[
\text{Area}(u) = \int_X \#u^{-1}(x) \, dH^2_\pi(x).
\]

**Definition 2.3.** A map \( u \in W^{1,2}(\Omega, X) \) is called infinitesimally isotropic with respect to the Riemannian metric \( g \) if for almost every \( z \in \Omega \) the semi-norm \( \text{ap md } u \) on \((Tz M, g(z))\) is isotropic.

If \( X \) is a Riemannian manifold, or more generally a space with property (ET) (cf. \[24\] Definition 11.1), then infinitesimal isotropy is equivalent to weak conformality, see \[24\] Theorem 11.3.

Next, we recall the definition of the trace of a Sobolev map. Let \( \Omega \subset \text{int}(M) \) be a Lipschitz domain. Then for every \( z \in \partial \Omega \) there exist an open neighborhood \( U \subset M \) and a biLipschitz map \( \psi : (0, 1) \times [0, 1) \rightarrow M \) such that \( \psi((0, 1) \times (0, 1)) = U \cap \Omega \) and \( \psi((0, 1) \times \{0\}) = U \cap \partial \Omega \). Let \( u \in W^{1,2}(\Omega, X) \). For almost every \( s \in (0, 1) \) the map \( t \mapsto u \circ \psi(s, t) \) has an absolutely continuous representative which we denote by the same expression. The trace of \( u \) is defined by

\[
\text{tr}(u)(\psi(s, 0)) := \lim_{t \searrow 0} (u \circ \psi)(s, t)
\]

for almost every \( s \in (0, 1) \). It can be shown (see \[20\]) that the trace is independent of the choice of the map \( \psi \) and defines an element of \( L^2(\partial \Omega, X) \).

**Proposition 2.4.** Let \( X \) be a proper metric space admitting a local quadratic isoperimetric inequality. Let \( \Omega \) be a Lipschitz Jordan domain in the interior of \( M \) and let \( u \in W^{1,2}(\Omega, X) \) have a continuous trace. Then for every \( \varepsilon > 0 \) there exists a continuous map \( v : \overline{\Omega} \rightarrow X \) with \( v|_{\partial \Omega} = \text{tr}(u) \), \( v \in W^{1,2}(\Omega, X) \), and

\[
\text{Area}(v) \leq \text{Area}(u) + \varepsilon \cdot E^2_\pi(u, g), \quad E^2_\pi(v, g) \leq (1 + \varepsilon^{-1}) \cdot E^2_\pi(u, g).
\]

It follows, in particular, that if a closed curve \( \gamma \) in \( X \) is the trace of a Sobolev disc then \( \gamma \) is contractible.

**Proof.** By possibly doubling \( M \) we may assume that \( M \) has no boundary. Now, there exists a conformal diffeomorphism from a bounded open subset of \( \mathbb{R}^2 \) onto an open subset of \( M \) which contains \( \overline{\Omega} \). Since area and energy are invariant under conformal diffeomorphisms we may assume that \( \Omega \) is a bounded Lipschitz Jordan domain in \( \mathbb{R}^2 \). We write \( E^2_\pi(u) \) for the energy of \( u \).
Fix $\epsilon > 0$. We first show the existence of a minimizer $v \in W^{1,2}(\Omega, X)$ of

$$A_\epsilon(v) := \text{Area}(v) + \epsilon \cdot E^2_\omega(v),$$

subject to the condition $\text{tr}(v) = \text{tr}(u)$. For this let $(v_n) \subset W^{1,2}(\Omega, X)$ be a minimizing sequence for $A_\epsilon$ with $\text{tr}(v_n) = \text{tr}(u)$ for all $n$. Then $(v_n)$ has bounded energy and thus, by [24] Lemma 4.11 and [20] Theorems 1.13 and 1.12.2], a subsequence converges in $L^2(\Omega, X)$ to a map $v \in W^{1,2}(\Omega, X)$ with $\text{tr}(v) = \text{tr}(u)$. By the lower semi-continuity of area and energy it follows that $v$ is a minimizer of $A_\epsilon$.

Next, we claim that for every Lipschitz domain $\Omega' \subset \Omega$ and every $w \in W^{1,2}(\Omega', X)$ with $\text{tr}(w) = \text{tr}(v|_{\Omega'})$ we have $E^2_\omega(v|_{\Omega'}) \leq \left(1 + \epsilon^{-1}\right) \cdot E^2_\omega(w)$ and thus $v$ is $\left(1 + \epsilon^{-1}\right)$-quasiharmonic in the sense of [23]. Indeed, if $w \in W^{1,2}(\Omega', X)$ satisfies $\text{tr}(w) = \text{tr}(v|_{\Omega'})$ then the map $w'$ which agrees with $w$ on $\Omega'$ and with $v$ on $\Omega \setminus \Omega'$ belongs to $W^{1,2}(\Omega, X)$ and satisfies $\text{tr}(w') = \text{tr}(u)$ by [20] Theorem 1.12.3. Since $v$ minimizes $A_\epsilon$ we obtain

$$\text{Area}(v|_{\Omega'}) + \epsilon \cdot E^2_\omega(v|_{\Omega'}) \leq \text{Area}(w) + \epsilon \cdot E^2_\omega(w) \leq (1 + \epsilon) \cdot E^2_\omega(w)$$

and this implies the claim.

Finally, since $v$ is quasiharmonic and has a continuous trace, it follows from [23] Theorem 1.3 that $v$ has a continuous representative which continuously extends to the boundary. This representative, which we denote again by $v$, satisfies the properties in the statement of the proposition. □

**Proposition 2.5.** Let $\Omega$ be a Lipschitz domain in the interior of $M$. A measurable and essentially separably valued map $u: \Omega \rightarrow X$ satisfies (2.1) for all Lipschitz curves $\gamma: [a,b] \rightarrow \overline{\Omega}$. In this case, we have

$$E^2_\omega(u, g) = \inf \{ \|\rho\|_{L^2(\Omega, \mathbb{R}^m)}^2 : \rho \text{ satisfies (2.1)} \}$$

and $\nu(\cdot) = \text{tr}(u)(\cdot)$ for $\mathcal{H}^1$-almost every $z \in \partial \Omega$. The map $\nu$ in the claim is called a *Newton–Sobolev representative* of $u$, and $\rho$ an *upper gradient* of $\nu$. Inequality (2.1) is known as the upper gradient inequality.

**Proof.** The existence of $\nu$ and $\rho$ as in the claim imply that $u \in W^{1,2}(\Omega, X)$, see [14] Chapter 7. For the opposite implication, by possibly doubling $M$ we may assume $M$ has no boundary. Since $\partial \Omega$ is Lipschitz, there exists a Lipschitz domain $\widehat{\Omega} \subset M$ containing $\overline{\Omega}$ and a map $\hat{u} \in W^{1,2}(\widehat{\Omega}, X)$ with $\hat{u}|_{\partial \widehat{\Omega}} = u$, see the proof of [24] Lemma 3.4. There exists $v: \widehat{\Omega} \rightarrow X$ and $\rho: \widehat{\Omega} \rightarrow [0, \infty]$ satisfying (2.1) for all Lipschitz curves $\gamma: [a,b] \rightarrow \widehat{\Omega}$, cf. [14] Theorems 7.1.20 and 7.4.5. The maps $v|_{\Omega'}$ and $\rho|_{\Omega'}$ satisfy the claim.

The equality (2.2) follows e.g. from [14] Theorem 7.1.20 and Lemma 6.2.2. Let $\psi: (0, 1) \times [0, 1) \rightarrow M$ be as in the definition of the trace, so that $\text{tr}(u)(\psi(s, 0)) = \lim_{t \downarrow 0} u \circ \psi(s, t)$ for a.e. $s \in (0, 1)$. A Fubini-type argument shows that

$$v(\psi(s, 0)) = \lim_{t \downarrow 0} u \circ \psi(s, t)$$

for a.e. $s \in (0, 1)$. This completes the proof. □
We illustrate the use of Newton-Sobolev representatives in the next lemma. Recall that a metric space is said to be \( C \)-quasiconvex if any two points can be joined by a Lipschitz curve of length at most \( C \) times their distance.

**Lemma 2.6.** Let \( h : K \to M \) be a Lipschitz map from a cell-complex \( K \), and \( A \subset K^1 \) a \( C \)-quasiconvex subset of the 1-skeleton. If \( v : M \to X \) is a Newton-Sobolev representative, and \( p \in L^2(\gamma) \) an upper gradient of \( u \) with \( L := \int_A \rho^2 \circ h \, dH^1 < \infty \), then \( v \circ h|_A \) is \( \frac{1}{2} \)-Hölder continuous with constant \( (CL)^{\frac{1}{2}} \text{Lip}(h) \).

**Proof.** For \( x, y \in A \), let \( \gamma : [0, \ell(y)] \to A \) be a simple unit speed curve joining \( x \) and \( y \) with \( \ell(\gamma) \leq C \text{d}(x, y) \). By (2.1) we have
\[
d(v \circ h(x), v \circ h(y)) \leq \int_0^{\ell(\gamma)} \rho \circ h(\gamma(t)) |(h \circ \gamma)'(t)| dt \leq \text{Lip}(h) \int_0^{\ell(\gamma)} \rho \circ h(\gamma(t)) dt
\]
\[\leq \text{Lip}(h) \ell(\gamma)^\frac{1}{2} L \frac{1}{2} \leq (CL)^{\frac{1}{2}} \text{Lip}(h) \text{d}(x, y)^\frac{1}{2}.
\]

\( \square \)

3. Admissible deformations on a surface

The notion of admissible deformation on a surface given below, in the spirit of [11], will be used to define 1–homotopy classes relative to given Jordan curves. We remark that the deformations in [11, 39] keep the boundary fixed and are thus not suitable for studying the Plateau-Douglas problem. The deformations in [38, 39, 10] for closed surfaces also do not adapt to our purposes.

**Definition 3.1.** An admissible deformation on a surface \( M \) is a smooth map \( \Phi : M \times \mathbb{R}^m \to M \), for some \( m \in \mathbb{N} \), such that \( \Phi_\xi := \Phi(\cdot, \xi) \) is a diffeomorphism for every \( \xi \in \mathbb{R}^m \) and \( \Phi_0 = \text{id}_M \), and such that the derivative of \( \Phi^p := \Phi(p, \cdot) \) at the origin satisfies
\[D\Phi^p(0)(\mathbb{R}^m) = \left\{ \begin{array}{ll}
T_pM & \text{if } p \in \text{int}(M) \\
T_p(\partial M) & \text{if } p \in \partial M.
\end{array} \right.\]

If \( \Phi : M \times \mathbb{R}^m \to M \) is an admissible deformation on \( M \) and \( \varphi : M \to M \) is a diffeomorphism then \( \Phi'(p, \xi) := \varphi(\Phi(\varphi^{-1}(p), \xi)) \) also defines an admissible deformation.

**Proposition 3.2.** There exist admissible deformations on every surface.

**Proof.** Let \( \eta_1, \eta_2 : [0, \infty) \to [0, \infty) \) be smooth functions such that \( \eta_1(0) = 0 \), \( \eta'_1(0) > 0 \), \( \eta_2(0) > 0 \) and \( \eta_1(t) = \eta_2(t) = 0 \) for all \( t \geq 1 \). We use \( \eta_1, \eta_2 \) to define smooth vector fields \( X_1, X_2 \) on \( M \) as follows. Each boundary component of \( M \) has a neighborhood which is diffeomorphic to \( S^1 \times [0, 2] \). On such a boundary component we define \( X_1 \) and \( X_2 \) by \( X_1(z, t) = \eta_1(t) \frac{\partial}{\partial z} \) and \( X_2(z, t) = \eta_2(t) \frac{\partial}{\partial z} \), written in coordinates \( (z, t) \in S^1 \times [0, 2] \). Now, extend \( X_1, X_2 \) to all of \( M \) by setting them to be zero outside these neighborhoods. It is easy to see that there exist smooth vector fields \( X_1, \ldots, X_m \) on \( M \), for some \( m \), with support in the interior of \( M \) such that the vectors \( X_1(p), \ldots, X_m(p) \) span \( T_pM \) for every \( p \) in the interior of \( M \). For every \( k = 1, \ldots, m \), the flow \( \varphi_{X_k, t} \) along \( X_k \) is defined for all times \( t \in \mathbb{R} \). Now the map \( \Phi : M \times \mathbb{R}^m \to M \) given by
\[\Phi(p, \xi) := \varphi_{X_1, \xi_1} \circ \varphi_{X_2, \xi_2} \circ \cdots \circ \varphi_{X_m, \xi_m}(p)\]
defines an admissible deformation on \( M \). \( \square \)
Let $M$ be a surface, which we equip with a Riemannian metric $g$. Let $\Phi: M \times \mathbb{R}^m \to M$ be an admissible deformation on $M$ and let $h: K \to M$ be a triangulation of $M$. For $\xi \in \mathbb{R}^m$ let $h_\xi: K \to M$ be the triangulation given by $h_\xi := \Phi_\xi \circ h$. The following variant of [11, Lemma 5] plays a key role throughout the article (see also [10, Lemma 3.3] for closed manifolds).

**Lemma 3.3.** There exist an open ball $B_{\Phi,h} \subset \mathbb{R}^m$ centered at the origin and $C > 0$ such that for every Borel function $\rho: M \to [0, \infty]$ we have

\[(3.1) \quad \int_{B_{\Phi,h}} \left( \int_{K' \cap \partial K} \rho \circ h_\xi \, dH^0(z) \right) \, d\xi \leq C \int_M \rho \, dH^l_g\]

and, for every $l \in \{0, 1, 2\}$,

\[(3.2) \quad \int_{B_{\Phi,h}} \left( \int_{K' \cap \partial K} \rho \circ h_\xi \, dH^l(z) \right) \, d\xi \leq C \int_M \rho \, dH^l_g.

**Proof.** We only prove (3.2) and leave the similar proof of (3.1) to the reader. Let $\Delta$ be a closed cell of some dimension $l$ in $K$ and suppose $\Delta$ is not contained in $\partial K$. Define a map $H: \Delta \times \mathbb{R}^m \to M$ by $H(z, \xi) := \Phi(h(z), \xi)$. The properties of $\Phi$ and $h$ imply that

$$DH(z, 0)(\mathbb{R}^l \times \mathbb{R}^m) = T_{h(z)} M$$

for every $z \in \Delta$ and therefore there exist $\varepsilon, c > 0$ such that the jacobian of the differential of $H$ satisfies

\[(3.3) \quad j(DH(z, \xi)) \geq c\]

for every $(z, \xi) \in \Delta \times \overline{B}(0, 2\varepsilon)$. Since $H|_{\Delta \times B(0, 2\varepsilon)}$ is $C^1$ up to the boundary, we may extend $H$ to a map $\tilde{H}: \tilde{\Delta} \times B(0, 2\varepsilon) \to \tilde{M}$ satisfying (3.3) for some open manifolds $\tilde{\Delta} \subset \mathbb{R}^l$ and $\tilde{M}$ containing $\Delta$ and $M$, respectively, by possibly making $c$ smaller.

We now claim that there exists $L \geq 0$ such that

\[(3.4) \quad \mathcal{H}^{l+m-2}(H^{-1}(x) \cap \Delta \times B(0, \varepsilon)) \leq L\]

for every $x \in M$. In order to prove this, fix $(z, \xi) \in \Delta \times \overline{B}(0, \varepsilon)$. Let $F: \tilde{\Delta} \times B(0, 2\varepsilon) \to \mathbb{R}^{l+m-2}$ be a $C^1$ map such that $F(z, \xi) = 0$ and such that the map

$$\tilde{H}: \tilde{\Delta} \times B(0, 2\varepsilon) \to \tilde{M} \times \mathbb{R}^{l+m-2}$$

given by $\tilde{H} = (H, F)$ satisfies

$$D\tilde{H}(z, \xi)(\mathbb{R}^l \times \mathbb{R}^m) = T_{H(z, \xi)} \tilde{M} \times \mathbb{R}^{l+m-2}.$$ 

There exist $\delta > 0$ and open neighborhoods $U \subset \tilde{\Delta} \times B(0, 2\varepsilon)$ of $(z, \xi)$ and $V \subset \tilde{M}$ of $H(z, \xi)$ such that the restriction of $\tilde{H}$ to $U$ is a biLipschitz homeomorphism with image $V \times B(0, \delta)$. Let $G$ be the inverse of $\tilde{H}|_U$, so that

$$H^{-1}(x) \cap U = G([x] \times B(0, \delta))$$

given by $\tilde{H} = (H, F)$ satisfies

$$\mathcal{H}^{l+m-2}(H^{-1}(x) \cap U, \delta) \leq L'$$

for every $x \in V$. It follows that there exists $L'$ such that

$$\mathcal{H}^{l+m-2}(H^{-1}(x) \cap U) \leq L'$$

for every $x \in V$. Since $\Delta \times \overline{B}(0, \varepsilon)$ is compact we can cover it by finitely many such open sets $U$ and the claim follows for a suitable number $L$. 
Finally, let \( \rho : M \to [0, \infty) \) be a Borel function. From the co-area formula and the inequalities (3.3) and (3.4) we conclude

\[
\int_{B(0,x)} \int_{\Delta} \rho \circ H(z, \xi) \, d\mathcal{H}^1(z) \, d\xi \leq c^{-1} \int_{B(0,x)} \int_{\Delta} \rho \circ H(z, \xi) \, J(DH(z, \xi)) \, d\mathcal{H}^1(z) \, d\xi
\]

\[
= c^{-1} \int_M \rho(x) \cdot \mathcal{H}^{1+m-2}(H^{-1}(x)) \, d\mathcal{H}^2_\xi(x)
\]

\[
\leq \frac{L}{c} \int_M \rho(x) \, d\mathcal{H}^2_\xi(x).
\]

This proves (3.2) with \( B_{\Phi, h} := B(0, \varepsilon) \) and \( C = \frac{L}{c} \).

Lemma 3.3 has the following immediate corollary.

**Corollary 3.4.** If \( N \subset M \) and \( E \subset \partial M \) satisfy \( \mathcal{H}^2_\xi(N) = \mathcal{H}^1_\xi(E) = 0 \) then, for almost every \( \xi \in B_{\Phi, h} \), we have that \( h_\xi(x) \notin N \) for \( \mathcal{H}^1 \)-a.e. \( x \in K^1 \setminus \partial K \) and \( h_\xi(x) \notin E \) for every \( x \in K^0 \setminus \partial K \).

In the next statement, we denote by \( u \circ h_\xi|_{K^1} \) the map which agrees with \( u \circ h_\xi \) on \( K^1 \setminus \partial K \) and with \( \text{tr}(u) \circ h_\xi \) on \( \partial K \).

**Proposition 3.5.** Let \( X \) be a complete metric space and \( u \in W^{1,2}(M, X) \). Then \( u \circ h_\xi|_{K^1 \setminus \partial K} \) is essentially continuous for a.e. \( \xi \in B_{\Phi, h} \). If \( u \) has continuous trace then \( u \circ h_\xi|_{K^1} \) is essentially continuous for a.e. \( \xi \in B_{\Phi, h} \), and extends continuously to \( K \) in case \( X \) is proper and admits a local quadratic isoperimetric inequality.

**Proof.** Let \( v : M \to X \) be a Newton-Sobolev representative of \( u \) with upper gradient \( \rho \in L^2(M) \) (cf. Proposition 2.5), and \( \Lambda := K^1 \setminus \partial K \). Since

\[
\int_{\Lambda} \rho^2 \circ h_\xi \, d\mathcal{H}^1 < \infty
\]

for a.e. \( \xi \in B_{\Phi, h} \) by (3.2), Lemma 2.6 implies that \( v \circ h_\xi|_{\Lambda} \) is continuous for a.e. \( \xi \in B_{\Phi, h} \). The first claim now follows from Corollary 3.4 applied to the null-set \( \{u \neq v\} \).

Suppose \( \text{tr}(u) \) has a continuous representative \( \eta \). The set \( \{v|_{\partial M} \neq \eta\} \subset \partial M \) has null \( \mathcal{H}^1_\xi \)–measure by Proposition 2.5 in particular \( v \circ h_\xi|_{\partial K} = \eta \circ h_\xi|_{\partial K} \) \( \mathcal{H}^1 \)-a.e., for a.e. \( \xi \). The argument above together with Corollary 3.4 applied to \( \{u \neq v\} \) and \( \{v|_{\partial M} \neq \eta\} \) implies that, for a.e. \( \xi \in B_{\Phi, h} \), the map

\[
w_\xi(x) := \begin{cases}
  v \circ h_\xi(x), & x \in K^1 \setminus \partial K \\
  \eta \circ h_\xi(x), & x \in \partial K
\end{cases}
\]

is a continuous representative of \( u \circ h_\xi|_{K^1} \). If \( X \) is proper and admits a local quadratic isoperimetric inequality and \( \Lambda \) is a 2–cell of \( K \) then \( \text{tr}(u \circ h_\xi|_{\Lambda}) = w_\xi|_{\partial \Lambda} \) for a.e. \( \xi \in B_{\Phi, h} \) by Proposition 2.5. Applying Proposition 2.4 on each 2–cell and gluing these together yields the desired continuous extension.

Now, suppose that \( M \) has \( k \geq 1 \) boundary components and let \( \Gamma \) be the disjoint union of \( k \) rectifiable Jordan curves in a proper metric space \( X \) admitting a local quadratic isoperimetric inequality. Recall the definition of homotopy relative to \( \Gamma \) from the introduction.
Theorem 3.6. Let $u \in \Lambda(M, \Gamma, X)$. Then there exists a negligible set $N \subset B_{\Phi,h}$ such that the continuous representatives of $u \circ h_\zeta|_{K'}$ and $u \circ h_\zeta|_{K'}$ are homotopic relative to $\Gamma$ for all $\zeta, \eta \in B_{\Phi,h} \setminus N$.

Proof. Denote by $\eta$ the continuous representative of $\operatorname{tr}(u)$. Let $v : M \to X$ be a Newton-Sobolev representative of $u$ with upper gradient $\rho \in L^2(M)$ as in Proposition 2.5 and set $\tilde{v} := v$ on $\operatorname{int}(M)$ and $\tilde{v} := \eta$ on $\partial M$. By the proof of Proposition 3.5 there exists a null-set $N_0 \subset B_{\Phi,h}$ such that $\tilde{v} \circ h_\zeta|_{K'}$ is the continuous representative of $u \circ h_\zeta|_{K'}$ whenever $\zeta \in B_{\Phi,h} \setminus N_0$.

We claim that there exists $\xi_0 \in B_{\Phi,h} \setminus N_0$ such that the map $H_\xi : K' \times [0, 1] \to M$ given by $H_\xi(x, t) := \Phi(h(x), \xi_0 + t(\xi - \xi_0))$ satisfies

$$
\int_0^1 \int_{K' \setminus \partial K} \rho^2 \circ H_\xi \, dH^l \, dt < \infty, \quad l = 0, 1,
$$

for a.e. $\xi \in B_{\Phi,h}$. Let us first finish the proof assuming (3.5). It is enough to show that there exists a null-set $N \subset B_{\Phi,h}$ containing $N_0$ such that $\tilde{v} \circ h_\zeta|_{K'} \sim \tilde{v} \circ h_\xi|_{K'}$ rel $\Gamma$, whenever $\zeta, \xi \in B_{\Phi,h} \setminus N$. Indeed, from this it follows that $\tilde{v} \circ h_\zeta|_{K'} \sim \tilde{v} \circ h_\xi|_{K'}$ rel $\Gamma$ for every $\zeta, \xi \in B_{\Phi,h} \setminus N$.

Note that $H_\xi(., 0) = \tilde{v} \circ h_\zeta|_{K'}$, $H_\xi(., 1) = \tilde{v} \circ h_\xi|_{K'}$, and that $\tilde{v} \circ H_\zeta|_{\partial K \times [0, 1]}$ is continuous with respect to $\tilde{v} \circ H_\zeta|_{\partial K \times [0, 1]}$ for every $\zeta \in B_{\Phi,h}$ and $t \in [0, 1]$. Fix a $1$-cell $e$ of $K$ not contained in $\partial K$ and let $A := e \times [0, 1]$. We show that $\tilde{v} \circ H_\zeta|_{\partial K}$ is continuous and the trace of a Sobolev map, for a.e. $\zeta \in B_{\Phi,h} \setminus N_0$. By Proposition 2.4 this implies that $\tilde{v} \circ H_\zeta|_{\partial K}$ has a continuous extension to $A$, and choosing a continuous extension for each $A$ we obtain the desired homotopy relative to $\Gamma$ between $v \circ h_\zeta|_{K'}$ and $\tilde{v} \circ h_\xi|_{K'}$, for a.e. $\xi \in B_{\Phi,h} \setminus N_0$.

Since $\operatorname{Lip}(H_\zeta) \cdot \rho \circ H_\zeta|_{\partial A}$ is an upper gradient of $v \circ H_\zeta|_{\partial A}$, it follows from (3.5) and Lemma 2.6 that $v \circ H_\zeta|_{\partial A} \in W^{1,2}(A, X)$ and $\operatorname{tr}(v \circ H_\zeta|_{\partial A})$ is a.e. $\zeta \in B_{\Phi,h} \setminus N_0$. For a.e. $\zeta \in B_{\Phi,h} \setminus N_0$, we have that $\tilde{v} \circ H_\zeta(\zeta_0, \cdot) = v \circ H_\zeta(\zeta_0, \cdot)$ is Hölder continuous for $\zeta_0 \in K^0 \cap \partial K$ by (3.5) and Lemma 2.6 and

$$
\tilde{v} \circ H_\zeta(\zeta_0, t) = v \circ H_\zeta(\zeta_0, t) \quad \text{a.e. } t \in [0, 1]
$$

for $\zeta_0 \in K^0 \cap \partial K$, by Corollary 3.4 and a Fubini-type argument. Thus $\tilde{v} \circ H_\zeta|_{\partial A}$ is the continuous representative of $v \circ H_\zeta|_{\partial A}$ for a.e. $\zeta \in B_{\Phi,h} \setminus N_0$. This completes the proof that $\tilde{v} \circ H_\zeta|_{\partial A}$ is continuous and the trace of a Sobolev function, for a.e. $\zeta \in B_{\Phi,h} \setminus N_0$.

It remains to show (3.5). Define

$$
f(\xi) := \chi_{B_{\Phi,h}}(\xi) \left( \int_{K^0 \setminus \partial K} \rho^2 \circ h_\xi \, dH^0 + \int_{K' \setminus \partial K} \rho^2 \circ h_\xi \, dH^l \right), \quad \xi \in \mathbb{R}^m.
$$

Then $f \in L^1(\mathbb{R}^m)$ by (3.2) and thus there exists $\xi_0 \in B_{\Phi,h} \setminus N_0$ such that the Riesz potential $R_1(f(\xi_0)) := \int_{\mathbb{R}^m} \frac{f(\xi_0 + \xi)}{r^s} \, d\nu = \int_{\mathbb{R}^m} \frac{f(\xi_0 + \xi)}{r^s} \, d\nu$ is finite (cf. [13, Theorem 3.22]). Integrating in spherical coordinates we have

$$
\int_{\mathbb{R}^m} f(\xi_0 + tw) \, dw = R_1(f(\xi_0)) < \infty.
$$

Since

$$
\int_0^1 \int_{K' \setminus \partial K} \rho^2 \circ H_\xi \, dH^l \, dt \leq \int_0^\infty f(\xi_0 + tw) \, dw \, dt = \int_0^\infty f(\xi_0 + tw) \, dw \, dt = \int_0^\infty f(\xi_0 + tw) \, dt \, dw \leq \infty,
$$

for $l = 0, 1$ and $\xi, \eta \in B_{\Phi,h} \setminus \{\xi_0\}$, (3.5) follows. \[\Box\]

We end this section with the following lemma which will be used in the proofs of the theorems in the next section.
Lemma 3.7. Let $u \in W^{1,2}(M, X)$ and let $(u_n) \subset W^{1,2}(M, X)$ be an energy bounded sequence converging to $u$ in $L^2(M, X)$. Then for almost every $\xi \in B_{\Phi,h}$ there exists a subsequence $(u_{n_j})$ such that the continuous representative of $u_{n_j} \circ h^1_{\xi, K^1, \partial K}$ converges uniformly to the continuous representative of $u \circ h^1_{\xi, K^1, \partial K}$ as $j \to \infty$.

Proof. By passing to a subsequence we may assume that $u_n \to u$ almost everywhere in $M$. For each $n \in \mathbb{N}$, let $v_n : M \to X$ be a Newton-Sobolev representative of $u_n$ with upper gradient $\rho_n \in L^2(M)$ satisfying

$$\|\rho_n\|^2_{L^2(M, \mathbb{R})} \leq 2E^2(u_n, g),$$

cf. Proposition 2.5. By the proof of Proposition 3.5 and Corollary 3.4, there exists a negligible set $N_0 \subset B_{\Phi,h}$ such that for every $\varepsilon \in B_{\Phi,h} \setminus N_0$ the map $v_n \circ h^1_{\xi, K^1, \partial K}$ is the continuous representative of $u_n \circ h^1_{\xi, K^1, \partial K}$ for every $n \in \mathbb{N}$ and

(3.6) \hspace{1cm} v_n \circ h^1_{\xi, K^1, \partial K} \to u \circ h^1_{\xi, K^1, \partial K}

$\mathcal{H}^1$–a.e. with $n \to \infty$. Set $A := K^1 \setminus \partial K$. Fatou’s lemma and (3.2) imply that

$$\int_{B_{\Phi,h}} \left( \liminf_{n \to \infty} \int_A \rho_n^2 \circ h \, \mathcal{d} \mathcal{H}^1 \right) \, \mathcal{d} \xi \leq \liminf_{n \to \infty} \int_{B_{\Phi,h}} \int_A \rho_n^2 \circ h \, \mathcal{d} \mathcal{H}^1 \, \mathcal{d} \xi \leq C \liminf_{n \to \infty} \int_M \rho_n^2 \circ h \, \mathcal{d} \mathcal{H}^2 < \infty.$$

Therefore, for almost every $\xi \in B_{\Phi,h} \setminus N_0$, we have

$$\liminf_{n \to \infty} \int_A \rho_n^2 \circ h \, \mathcal{d} \mathcal{H}^1 < \infty.$$ 

By Lemma 2.6, Arzela-Ascoli’s Theorem and (3.6), for such $\xi$ there exists a subsequence $(v_{n_j} \circ h^1_{\xi, K^1, \partial K})_j \subset \mathbb{N}$ which is uniformly $\frac{1}{2}$–Hölder continuous and converges uniformly to the continuous representative of $u \circ h^1_{\xi, K^1, \partial K}$ as $j \to \infty$. \hfill \Box

4. THE RELATIVE 1–HOMOTOPY CLASS OF SOBOLEV MAPS

Throughout this section, let $X$ be a proper geodesic metric space admitting a local quadratic isoperimetric inequality. Let $\Gamma \subset X$ be the disjoint union of $k \geq 1$ rectifiable Jordan curves, and let $M$ be a surface with $k$ boundary components. We fix a Riemannian metric $g$ on $M$.

Let $\Phi : M \times \mathbb{R}^m \to M$ be an admissible deformation on $M$. Theorem 3.6 shows that for every $u \in \Lambda(M, \Gamma, X)$ and every triangulation $h : K \to M$ of $M$ we have

$$[u \circ h^1_{\xi, K^1}]_{\Gamma} = [u \circ h^1_{\xi, K^1}]_{\Gamma}$$

for almost all $\xi, \zeta \in B_{\Phi,h}$. We denote the common relative homotopy class by $u_{\Phi,1}[h]$. The following theorem shows that $u_{\Phi,1}[h]$ is independent of the choice of deformation $\Phi$ and that inducing the same relative homotopy class is independent of the triangulation $h$.

Theorem 4.1. Let $X, \Gamma, M, \Phi$ be as above. Let $u \in \Lambda(M, \Gamma, X)$ and let $h : K \to M$ be a triangulation of $M$. The relative homotopy class $u_{\Phi,1}[h]$ does not depend on the choice of admissible deformation $\Phi$. Moreover, if $v \in \Lambda(M, \Gamma, X)$ is such that $v_{\Phi,1}[h] = u_{\Phi,1}[h]$ then we have $v_{\Phi,1}[	ilde{h}] = u_{\Phi,1}[	ilde{h}]$ for any triangulation $\tilde{h} : K \to M$.

We will need the following two lemmas in the proof.
Lemma 4.2. Let \( u \in W^{1,2}(M, X) \) have continuous trace. Then for all \( \varepsilon, \delta > 0 \) there exists a continuous map \( \hat{u} : M \to X \) in \( W^{1,2}(M, X) \) with \( \hat{u}|_{\partial M} = \text{tr}(u) \), \( d_{E^2}(u, \hat{u}) < \varepsilon \), and

\[
\text{Area}(\hat{u}) \leq \text{Area}(u) + \delta \cdot E^2_+(u, g), \quad E^2_+(\hat{u}, g) \leq (1 + \delta^{-1}) \cdot E^2_+(u, g).
\]

Proof. Let \( u \) be as in the statement of the lemma and let \( \varepsilon, \delta > 0 \). Fix an admissible deformation \( \Phi \) on \( M \) and let \( \varepsilon' > 0 \) be sufficiently small, to be determined later. Choose a triangulation \( h : K \to M \) of \( M \) in such a way that for every \( \xi \in B_{\Phi,h} \) we have \( H^2_+(h_\xi(\Delta)) < \varepsilon' \) for every 2–cell \( \Delta \subset K \).

It follows from (the proof of) Proposition 3.5 that, for almost every \( \xi \in B_{\Phi,h} \), the map \( u \circ h_\xi |_{K^1} \) is essentially continuous and its restriction to the boundary of each open 2–cell \( \Delta \subset K \) coincides with the trace of the Sobolev map \( u \circ h_\xi |_{\Delta} \). Fix such \( \xi \) and abbreviate \( H := h_\xi \). It thus follows that if \( \Delta \) is an open 2–cell then the map \( u|_{H(\partial \Delta)} \) is essentially continuous and the trace of the Sobolev map \( u|_{H(\partial \Delta)} \).

By Proposition 2.4 there thus exists a continuous map \( \hat{u}_\Delta : H(\Delta) \to X \) which extends the continuous representative of \( u|_{H(\partial \Delta)} \), belongs to \( W^{1,2}(H(\Delta), X) \) and satisfies

\[
\text{Area}(\hat{u}_\Delta) \leq \text{Area}(u|_{H(\Delta)}) + \delta \cdot E^2_+(u|_{H(\Delta)}, g)
\]
as well as

\[
E^2_+(u_\Delta, g) \leq (1 + \delta^{-1}) \cdot E^2_+(u|_{H(\Delta)}, g).
\]

It follows from the Sobolev-Poincaré inequality (see [12] Section 2) for closed manifolds, from [20] Corollary 1.6.3 and Hölder’s inequality that

\[
\int_{H(\Delta)} d^2(u_\Delta(z), u(z)) \, d\mathcal{H}^2_g(z) \leq C \cdot \mathcal{H}^2_g(H(\Delta)) \cdot \left[ E^2_+(u_\Delta, g) + E^2_+(u|_{H(\Delta)}, g) \right] \leq C \varepsilon' \left( 2 + \delta^{-1} \right) \cdot E^2_+(u|_{H(\Delta)}, g)
\]

for some constant \( C \) depending on \((M, g)\).

Finally, let \( \hat{u} : M \to X \) be the continuous map obtained by gluing the maps \( u_\Delta \) along their boundaries. Then \( \hat{u} \in W^{1,2}(M, X) \) by [20] Theorem 1.12.3 and, taking the sum over all \( \Delta \) in the three inequalities above, we obtain the inequalities in (4.1) as well as

\[
\int_{M} d^2(\hat{u}(z), u(z)) \, d\mathcal{H}^2_g(z) \leq C \varepsilon' \left( 2 + \delta^{-1} \right) \cdot E^2_+(u, g).
\]

Upon choosing \( \varepsilon' > 0 \) sufficiently small, this yields \( d_{E^2}(\hat{u}, u) < \varepsilon \). \( \square \)

Lemma 4.3. Let \( X, \Gamma, M \) be as above. Then there exists \( \delta > 0 \) with the following property. Let \( h : K \to M \) be a triangulation and let \( \varrho, \varrho' : K^1 \to X \) be continuous such that \( \varrho|_{\partial K}, \varrho'|_{\partial K} \in [\Gamma] \) are homotopic via a family of maps in \([\Gamma]\). If

\[
\sup_{z \in K^1 \setminus \partial K} d(\varrho(z), \varrho'(z)) < \delta
\]

and if for every component \( C \) of \( \partial M \) for which the Jordan curve \( \varrho(C) \) is not contractible in \( X \) we have

\[
\sup_{z \in C} d(\varrho(z), \varrho'(z)) < \delta
\]

then \( \varrho \) and \( \varrho' \) are homotopic relative to \( \Gamma \).
The condition (4.2) cannot be omitted, as easy examples show. The lemma will also be used in the proof of Theorem 4.6 where it will be essential that we do not impose any condition akin to (4.2) for the components $C$ of $\partial K$ which are mapped to contractible Jordan curves.

Proof. Since $X$ is proper, geodesic and admits a local quadratic isoperimetric inequality it follows from [26] Theorem 5.2, [26] Proposition 2.2, and from the proof of [26] Proposition 6.2 that there exists $r_0 > 0$ such that every closed curve in $X$ of diameter at most $4r_0$ is contractible. Recall that $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_2$ is the disjoint union of rectifiable Jordan curves. We may assume that $3r_0 \leq \text{diam}(\Gamma_i)$ for every $i$. There exists $0 < \delta < r_0/3$ such that whenever $x, y \in \Gamma$ satisfy $d(x, y) \leq 9\delta$ then they belong to the same Jordan curve $\Gamma_i$ and one of the two segments of $\Gamma_i$ joining $x$ and $y$ has diameter at most $r_0$.

Let $\varrho, \varrho': K^1 \to X$ be as in the statement of the lemma with this specific choice of $\delta$. After possibly adding vertices to $K^1 \setminus \partial K$ we may further assume that the image under $\varrho$ and $\varrho'$ of any edge in the closure of $K^1 \setminus \partial K$ has diameter at most $\delta$. We now construct a homotopy $H: K^1 \times [0, 1] \to X$ relative to $\Gamma$ between $\varrho$ and $\varrho'$. Let $H(\cdot, 0) = \varrho$ and $H(\cdot, 1) = \varrho'$. For each $z_0 \in K^0 \setminus \partial K$ let $H(z_0, \cdot)$ be a (constant speed) geodesic from $\varrho(z_0)$ and $\varrho'(z_0)$. For each component $C$ of $\partial K$ and each $z \in K^0 \cap C$, let $H(z, \cdot)$ be a weakly monotone parametrization of one of the segments in $\varrho(C)$ joining $\varrho(z)$ to $\varrho'(z)$ in such a way that, for every edge $e \subseteq C$, the map $H|_{\varrho(e) \times [0, 1]}$ is contractible in $\varrho(C)$. By (4.2), in the case that $\varrho(C)$ is not contractible, we may choose $H(z, \cdot)$ to have diameter at most $r_0$ for every $z \in C \cap K^0$.

For every edge $e \subseteq \partial K$ the map $H|_{\varrho(e) \times [0, 1]}$ admits a continuous extension with image in $\Gamma$ such that for each $t \in [0, 1]$ the map $H(\cdot, t)$ is weakly monotone. Moreover, for every edge $e \subseteq K^1$ not intersecting $\partial K$ the curve $H|_{\varrho(e) \times [0, 1]}$ has diameter at most $4\delta$ and thus admits a continuous extension to $e \times [0, 1]$. Finally, let $e \subseteq K^1$ be an edge which intersects (but is not contained in) some component $C$ of $\partial K$. Notice that the image of $H|_{\varrho(e) \times [0, 1]}$ is contained in the $3\delta$-neighbourhood of $\varrho(C)$. Thus, if $\varrho(C)$ is contractible then $H|_{\varrho(e) \times [0, 1]}$ admits a continuous extension to $e \times [0, 1]$. If $\varrho(C)$ is not contractible then, by construction, the image of $H|_{\varrho(e) \times [0, 1]}$ has diameter at most $4r_0$ and hence admits again a continuous extension to $e \times [0, 1]$. □

Proof of Theorem 4.7. Let $u \in \Lambda(M, \Gamma, X)$ and let $h: K \to M$ be a triangulation. We wish to show that the relative homotopy class, which we denote by $[u, h|_{K^1}]$ for the moment, is independent of the choice of admissible deformation $\Phi$. Let $(u_n)$ be a sequence of continuous maps $u_n: M \to X$ converging in $L^2(M, X)$ to $u$, with $u_n|_{\partial M} = \text{tr}(u)$ and $u_n \in W^{1, 2}(M, X)$ for every $n$, and such that the energy of $u_n$ is bounded independently of $n$. Such a sequence exists by Lemma 4.2 and we call it a good approximating sequence for $u$.

We first claim that there exists a subsequence $(n_j)$ such that $u_{n_j}[h, \Phi] = [u_{n_j} \circ h|_{K^1}]$ for all $j \geq 1$. Indeed, by Proposition 3.5 and Theorem 3.6 there exists a negligible subset $N \subseteq B_{\Phi, h}$ such that for $\xi, \xi' \in B_{\Phi, h} \setminus N$ the maps $u \circ h|_{K^1}$ and $u \circ h|_{K^1}$ are essentially continuous and their continuous representatives are homotopic relative to $\Gamma$. Since $u_n|_{\partial M} = \text{tr}(u)$ it follows with Lemma 3.7 that for almost every $\xi_0 \in B_{\Phi, h} \setminus N$ there is a subsequence $(n_j)$ such that the maps $u_{n_j} \circ h_{\xi_0}|_{K^1}$ converge uniformly to the continuous representative of $u \circ h_{\xi_0}|_{K^1}$ as $j \to \infty$. Fix such $\xi_0$ and such a subsequence $(n_j)$. Lemma 4.3 thus implies that there exists $j_0$
such that \( u_n \circ h_{jt}^{j_0} \) is homotopic relative to \( \Gamma \) to the continuous representative of \( u \circ h_{jt}^{j_0} \) for every \( j \geq j_0 \). Since \( u_n \) is continuous the maps \( u_n \circ h_{jt}^{j_0} \) and \( u_n \circ h_t^{j_0} \) are homotopic relative to \( \Gamma \). It thus follows that for all \( j \geq j_0 \) the continuous representative of \( u \circ h_{jt}^{j_0} \) is homotopic relative to \( \Gamma \) to \( u_n \circ h_t^{j_0} \) for every \( \xi \in B_{\Phi, \Gamma} \setminus \mathcal{N} \). Upon reindexing the subsequence we may assume that \( j_0 = 1 \). This proves the claim.

It easily follows from the claim that \( u_{#1}[h, \Phi] \) is independent of \( \Phi \). Indeed, let \( \Phi \) be another admissible deformation on \( M \). On the one hand, the claim shows that there exists a subsequence \( (n_j) \) such that

\[
u_{#1}[h, \Phi] = [u_{n_j} \circ h_t^{j_0}] \Gamma
\]

for all \( j \geq 1 \). Applying the claim again with \( \Phi \) replaced by \( \Phi \) and with \( (u_n) \) replaced by \( (u_{n_j}) \) we see that there is a further subsequence \( (n_{jl}) \) such that

\[
u_{#1}[h, \Phi] = [u_{n_{jl}} \circ h_t^{j_0}] \Gamma
\]

for all \( l \geq 1 \). From this it follows that \( u_{#1}[h, \Phi] = u_{#1}[h, \Phi] \), which proves the first statement of the theorem.

The second statement of the theorem also follows from the claim. Indeed, let \( v \in \Lambda(M, \Gamma, X) \) be such \( v_{#1}[h] = u_{#1}[h] \) and let \( (v_{n_j}) \) be a good approximating sequence for \( v \). The claim shows that we can find a subsequence \( (n_{jl}) \) such that

\[
u_{#1}[h, \Phi] = [u_{n_{jl}} \circ h_t^{j_0}] \Gamma
\]

for all \( j \geq 1 \). Let \( h : \tilde{K} \to M \) be another triangulation. Since \( u_{n_j} \) and \( v_{n_j} \) are continuous it is easy to see that

\[
u_{#1}[h, \Phi] = [v_{n_{jl}} \circ h_t^{j_0}] \Gamma
\]

for all \( j \geq 1 \), compare with [10] Lemma 2.1. The claim now implies that \( u_{#1}[h] = v_{#1}[h] \), which proves the second statement of the theorem.

---

**Proposition 4.4.** Let \( \varphi : M \to X \) be a continuous map such that \( \varphi|_{\partial M} \in [\Gamma] \) and let \( u \in \Lambda(M, \Gamma, X) \). Then

\[
u_{#1}[h] = [\varphi \circ h_t^{j_0}] \Gamma
\]

holds for one triangulation \( h : K \to M \) if and only if it holds for every triangulation.

**Proof.** Let \( h : K \to M \) be a triangulation of \( M \) such that

\[
u_{#1}[h] = [\varphi \circ h_t^{j_0}] \Gamma
\]

and let \( (u_n) \) be a good approximating sequence for \( u \) as in the first paragraph of the proof of Theorem 4.1. By the claim in the second paragraph of that proof, there exists a subsequence \( (n_{jl}) \) such that \( u_{#1}[h] = [u_{n_{jl}} \circ h_t^{j_0}] \Gamma \) for all \( j \geq 1 \) and hence

\[
u_{#1}[h] = [u_{n_{jl}} \circ h_t^{j_0}] \Gamma = [\varphi \circ h_t^{j_0}] \Gamma
\]

for all \( j \geq 1 \). Let \( \tilde{h} : \tilde{K} \to M \) be another triangulation of \( M \). Since \( u_{n_j} \) and \( \varphi \) are continuous

\[
u_{#1}[\tilde{h}] = [u_{n_{jl}} \circ \tilde{h}_t^{j_0}] \Gamma
\]

for all \( j \geq 1 \), compare with [10] Lemma 2.1. After possibly passing to a further subsequence we have \( u_{#1}[\tilde{h}] = [u_{n_{jl}} \circ \tilde{h}_t^{j_0}] \Gamma \) for all \( j \geq 1 \) and hence

\[
u_{#1}[\tilde{h}] = [\varphi \circ \tilde{h}_t^{j_0}] \Gamma
\]

This concludes the proof. □
Definition 4.5. Two maps $u, v \in \Lambda(M, \Gamma, X)$ are said to be 1–homotopic relative to $\Gamma$, denoted $u \sim_1 v \text{ rel } \Gamma$, if for some and thus every triangulation $h$ of $M$ we have $u_{h,1}[h] = v_{h,1}[h]$. If $u \in \Lambda(M, \Gamma, X)$ and $\varphi : M \to X$ is continuous with $\varphi|_{\partial M} \in [\Gamma]$ then $u$ and $\varphi$ are said to be 1–homotopic relative to $\Gamma$, denoted $u \sim_1 \varphi \text{ rel } \Gamma$, if for some and thus every triangulation $h : K \to M$ we have $u_{h,1}[h] = [\varphi \circ h|_{\partial K}]$. If $u, v \in \Lambda(M, \Gamma, X)$, $u \sim_1 v \text{ rel } \Gamma$ and $\psi : M \to M$ is a diffeomorphism then $u \circ \psi \sim_1 v \circ \psi \text{ rel } \Gamma$, see the remark after Definition 3.1.

Theorem 4.6. Let $X, \Gamma, M$ be as above. Then for every $L > 0$ there exists $\varepsilon > 0$ such that if $u, v \in \Lambda(M, \Gamma, X)$ induce the same orientation on $\Gamma$ and satisfy
\[
\max \left\{ E^2(u, g), E^2(v, g) \right\} \leq L \quad \text{and} \quad d_{L^2}(u, v) \leq \varepsilon,
\]
then $u$ and $v$ are 1–homotopic relative to $\Gamma$.

Notice that the theorem does not imply the stability of 1–homotopy classes relative to $\Gamma$, since the $L^2$–limit of a sequence in $\Lambda(M, \Gamma, X)$ with uniformly bounded energy need not belong to $\Lambda(M, \Gamma, X)$. An analog of Theorem 4.6 holds for closed surfaces (where $\Gamma = \emptyset$ and $\Lambda(M, \Gamma, X) = W^{1,2}(M, X)$) and in this case implies the stability of 1–homotopy classes in the presence of a local quadratic isoperimetric inequality. Example 4.7 below shows that the local quadratic isoperimetric inequality is crucial for this.

Proof. We argue by contradiction and assume the statement is not true. Then there exist bounded energy sequences $(u_n), (v_n) \subset \Lambda(M, \Gamma, X)$ such that for every $n \in \mathbb{N}$ we have $d_{L^2}(u_n, v_n) \leq \frac{1}{n}$, that $u_n$ and $v_n$ induce the same orientation on $\Gamma$ but $u_n$ is not 1–homotopic to $v_n$ relative to $\Gamma$. After possibly passing to a subsequence, we may assume by the Rellich-Kondrachov compactness theorem [20, Theorem 1.13] and by [8, Lemma 2.4] that there exists $u \in W^{1,2}(M, X)$ such that the sequences $(u_n)$ and $(v_n)$ both converge to $u$ in $L^2(M, X)$.

Fix an admissible deformation $\Phi$ on $M$ and a triangulation $h : K \to M$. By Proposition 3.5 and Theorem 3.6 there exists a negligible set $N \subset B_{\Phi, h}$ such that for all $\xi, \zeta \in B_{\Phi, h} \setminus N$ and all $n \in \mathbb{N}$ we have $u_n \circ h_{\xi}|_{\partial K}$ and $u_n \circ h_{\zeta}|_{\partial K}$ are essentially continuous and their continuous representatives are homotopic relative to $\Gamma$ and that the same is true when $u_n$ is replaced by $v_n$ and $u$. It moreover follows from Lemma 3.7 that for almost every $\xi_0 \in B_{\Phi, h} \setminus N$ there exists a subsequence $(n_j)$ such that the continuous representatives of $u_{n_j} \circ h_{\xi_0}|_{\partial K}$ and of $v_{n_j} \circ h_{\xi_0}|_{\partial K}$ both converge uniformly to the continuous representative of $u \circ h_{\xi_0}|_{\partial K}$. Fix such $\xi_0$ and denote by $\varrho_j$ and $\varrho'_j$ the continuous representatives of $u_{n_j} \circ h_{\xi_0}|_{\partial K}$ and of $v_{n_j} \circ h_{\xi_0}|_{\partial K}$, respectively. Denote by $C_m$ and $\Gamma_m$, $m = 1, \ldots, k$, the components of $\partial K$ and $\Gamma$, respectively. Notice that the sequences $(\varrho_j)|_{\partial K}$ and $(\varrho'_j)|_{\partial K}$ both converge in $L^2(\partial K, X)$ to $u \circ h_{\xi_0}|_{\partial K}$ by [20, Theorem 1.12.2]. Thus, after possibly relabelling the components, we may assume that
\[
\varrho_j(C_m) = \Gamma_m = \varrho'_j(C_m)
\]
for all sufficiently large $j$ and every $m = 1, \ldots, k$. Since $\varrho_j|_{\partial K}$ and $\varrho'_j|_{\partial K}$ induce the same orientation on $\Gamma$ it follows, in particular, that $\varrho_j|_{\partial K}$ and $\varrho'_j|_{\partial K}$ are homotopic via a family of maps in $[\Gamma]$ for every sufficiently large $j$. Let $m$ be such that $\Gamma_m$ is not contractible. Then it follows from the remark after Proposition 2.4 together with the proof of [8, Proposition 5.1] that the families $(\varrho_j)|_{C_m}$ and $(\varrho'_j)|_{C_m}$ are both equicontinuous. Hence, after possibly passing to a subsequence, we may assume that
both sequences converge uniformly to the continuous representative of $u \circ h_n|_{C_n}$. It thus follows that for every sufficiently large $j$ the maps $\varphi_j$ and $\varphi_j'$ satisfy the hypotheses of Lemma 4.3. In particular, it follows that there exists $j_0$ such that $\varphi_j$ and $\varphi_j'$ are homotopic relative to $\Gamma$ for every $j \geq j_0$. Hence, for every $\xi \in B_{\phi_k} \setminus N$ and every $j \geq j_0$ we have that the continuous representatives of $u_n \circ h|_{K^1}$ and $v_n \circ h|_{K^1}$ are homotopic relative to $\Gamma$. This shows that $u_n$ and $v_n$ are 1–homotopic relative to $\Gamma$, which is a contradiction, concluding the proof.

**Proof of Theorem 1.2**. Statements (ii) and (iii) follow from Theorems 4.1 and 4.6. As for statement (i), suppose $u$ has a continuous representative $\tilde{u} : M \to X$. We have $u \circ h|_{K^1} = \tilde{u} \circ h|_{K^1}$ a.e., for almost every $\xi$ by Corollary 3.4 and hence

$$[u \circ h|_{K^1}]_H = [\tilde{u} \circ h|_{K^1}]_H = [\tilde{u} \circ h|_{K^1}]_H$$

for almost every $\xi$. This proves statement (i).

**Example 4.7.** Consider the surface of revolution $C \subset \mathbb{R}^3$ of the graph of $f : (0, 1] \to [1/3, 1]$, $f(x) = (2 + \sin(1/x))/3$.

The compact set $C \cup \{0\times D \subset \mathbb{R}^3$ equipped with the subspace metric is not geodesic, but by adding a countable number of suitable line segments parallel to the $x$–axis, connecting points on $C$ to $\{0\} \times \mathbb{R}$, we obtain a compact subset of $\mathbb{R}^3$ bi-Lipschitz equivalent to a geodesic space $Y$. It is not difficult to see that $Y$, and thus $X := S^1 \times Y$, fails to admit a local quadratic isoperimetric inequality. Let $x_n \to 0$ be the sequence of local minima of $f$, and $u_n : S^1 \to Y$ the constant speed parametrizations corresponding to the circles $\{x_n\} \times \mathbb{R}^2 \cap C$. The maps $u_n : S^1 \times S^1 \to X$, $(z, z') \mapsto (z, u_n(z'))$

are bi-Lipschitz for each $n$, and converge uniformly to the map $u(z, z') = (z, h(z'))$, where $h : S^1 \to Y$ is the constant speed parametrization of the circle corresponding to $\{(0, z'/3) : z' \in S^1\} \subset Y$. However, one can check that the maps $u_n$ are all non-contractible and pairwise 1–homotopic, while $h$ is contractible. It follows that $u$ cannot lie in the common homotopy class of the maps $u_n$.

The example above can be modified so that the maps $u_n$ form an area minimizing sequence in their common 1–homotopy class. Considering the set $C \cup \{0\} \times \mathbb{R}^3$ with the metric inherited from $\mathbb{R}^3$ in the example above, we obtain a non-geodesic space with a local quadratic isoperimetric inequality where the stability of 1–homotopy classes of maps from closed surfaces fails.

5. The Homotopic Douglas Condition and its Consequences

Let $X$ be a proper geodesic metric space admitting a local quadratic isoperimetric inequality, and let $\Gamma \subset X$ be the disjoint union of $k \geq 1$ rectifiable Jordan curves. Let $M$ be a connected surface with $k$ boundary components, and let $\varphi : M \to X$ be a continuous map such that $\varphi|_{\partial M} \in [\Gamma]$.

**Proposition 5.1.** If the induced homomorphism $\varphi_* : \pi_1(M) \to \pi_1(X)$ on fundamental groups is injective then $\varphi$ satisfies the homotopic Douglas condition (1.4).

**Proof.** We first claim that $a(M, \varphi, X) < \infty$. Let $l_0 > 0$ be as in the definition of the local quadratic isoperimetric inequality. Since $\Gamma$ is a finite union of rectifiable Jordan curves there exists $0 < r_0 < l_0/3$ such that every subcurve of $\Gamma$ of diameter at most $r_0$ has length at most $l_0/3$. Moreover we may choose $r_0$ small enough so
that all closed loops of diameter $\leq 2r_0$ are contractible, cf. the proof of Lemma 4.3.

Now, fix a triangulation of $M$ all of whose 2–cells are triangles. We identify the 1–skeleton of the triangulation with a subset of $M$ and denote it by $M^1$. Choosing the triangulation sufficiently fine we may assume that for each 1–cell $e \subset M^1$ we have $\text{diam}(\varphi(e)) < r_0$. Let $u: M^1 \to X$ be the continuous map which agrees with $\varphi$ on the 0–skeleton $M^0$ and such that for each 1–cell $e \subset M^1$ the following holds: if $e$ is contained in $\partial M$ then $u|_e$ is the constant speed parametrization of $\varphi(e)$; if $e$ is not contained in $\partial M$ then $u|_e$ is a geodesic. It follows that for every 2–cell $\Delta \subset M$ the curve $u|_{\partial \Delta}$ is Lipschitz and has length at most $l_0$ and thus has a continuous Sobolev extension to $\Delta$ (which we denote $u|_{\Delta}$) by the local quadratic isoperimetric inequality and Lemma 4.2. Also note that $u|_e$ is end-point homotopic to $\varphi|_e$ by the choice of $r_0$. The continuous map $\bar{u}: M \to X$ obtained by gluing all the $u|_{\Delta}$ together is a Sobolev map and satisfies $\bar{u}|_{M^1} \sim \varphi|_{M^1}$ rel $\Gamma$. It thus follows that $\bar{u} \sim_1 \varphi$ relative to $\Gamma$. The map $\bar{u}$ has finite area and thus we obtain $\alpha(M, \varphi, X) < \infty$, as claimed.

Since the induced homomorphism $\varphi_*: \pi_1(M) \to \pi_1(X)$ on fundamental groups is injective it follows that if $\alpha$ is a simple closed non-contractible curve in the interior of $M$ then $\varphi \circ \alpha$ is not contractible. Consequently, there are no primary reductions $(M^*, \varphi^*)$ of $(M, \varphi)$ and hence $\alpha^*(M, \varphi, X) = \infty$ by definition. Since $\alpha(M, \varphi, X) < \infty$ this shows that $\varphi$ satisfies the homotopic Douglas condition. \hfill \Box

**Proposition 5.2.** Let $g$ be a Riemannian metric on $M$. Then for every $\eta > 0$ and $L > 0$ the family

$\{\text{tr}(u): u \in \Lambda(M, \Gamma, X), u \sim_1 \varphi \text{ rel } \Gamma, E^2_\eta(u, g) \leq L, \text{Area}(u) \leq \alpha^*(M, \varphi, X) - \eta\}$

is equi-continuous.

A corresponding result without fixing relative 1–homotopy classes is contained in [8 Proposition 5.1]. In order to control the relative 1–homotopy class of the maps that we construct in the proof, we will use the following technical lemma.

In the next statement, $\alpha$ is a smooth closed simple non-contractible curve in the interior of $M$ and let $M^*$ be the smooth surface obtained from $M$ by cutting $M$ along $\alpha$ and gluing smooth discs to the two newly created boundary components.

**Lemma 5.3.** Let $A \subset M$ be a biLipschitz cylinder such that $A \cap \partial M$ is connected and one boundary component of $A$ coincides with $\alpha$. Suppose there is $v \in \Lambda(M^*, \Gamma, X)$ inducing the same orientation on $\Gamma$ as $u$ and satisfying $v|_{M^*} = u$. Then $\varphi \circ \alpha$ is contractible and $v$ is 1–homotopic to $\varphi^*$ relative to $\Gamma$, whenever $\varphi^*: M^* \to X$ is continuous and coincides with $\varphi$ on $M \setminus A$.

**Proof.** Let $A' \subset M$ be a biLipschitz cylinder with piecewise smooth boundary components and such that $A'$ contains a small neighborhood of $A$ in $M$. The boundary component $\alpha'$ of $A'$ which is homotopic to $\alpha$ outside $A$ is contained in the interior of $M$. Let $\beta'$ be the other boundary component of $A'$. If $\gamma := A \cap \partial M$ is not empty then $\beta'$ contains $\gamma$.

Let $h: K \to M$ be a triangulation of $M$ such that $h(K^1)$ contains $\alpha'$ and $\beta'$. Let $K'$ be the sub-complex of $K$ obtained by removing the interior of cells that get mapped to the interior of $A'$. Let $K^*$ be the complex obtained from $K'$ by adding two cells, each glued along the preimage of $\alpha'$ and $\beta'$, respectively, and extend $h|_{K'}$ to a triangulation $h^*: K^* \to M^*$ of $M^*$. Let $C \subset K'^1$ be the preimage of $\beta' \cap \partial M$ under $h$. 

18
Let $U \subset M$ be a small neighborhood of $\alpha$ whose closure is contained in the interior of $A'$. Using vector fields as in the proof of Proposition 2.4, it is not difficult to construct admissible deformations $\Phi: M \times \mathbb{R}^m \to M$ on $M$ and $\Phi^*: M^* \times \mathbb{R}^m \to M^*$ on $M^*$ which agree on $(M \setminus U) \times B(0, \epsilon)$ for some sufficiently small $\epsilon > 0$. On $K^1 \setminus \Gamma$ the maps $h_\epsilon = \Phi_\varepsilon \circ h$ and $h^*_\xi = \Phi^*_{\xi} \circ h^*$ agree for sufficiently small $\xi$ and stay outside $A$, so we have

$$v \circ h^*_\xi|_{K^1 \setminus C} = u \circ h_\xi|_{K^1 \setminus C}$$

for a.e. small $\xi$. Since $u$ and $v$ induce the same orientation on $\Gamma$ it follows that $v \circ h^*_\xi|_{K^1}$ is homotopic to $u \circ h_\xi|_{K^1}$ relative to $\Gamma$ for almost every sufficiently small $\xi$.

Now, $u \circ h_\xi|_{K^1}$ is homotopic to $\varphi \circ h|_{K^1}$ relative to $\Gamma$ for almost every $\xi$ sufficiently small. Let $\Omega \subset M^*$ be the Lipschitz Jordan domain bounded by $\alpha'$. Since $v \circ \Phi^*_{\varepsilon}|_{\Omega}$ is the trace of the Sobolev disc $v \circ \Phi^*_{\xi}|_{\Omega}$ for almost every small $\xi$ it follows from Proposition 2.4 that the continuous representative of $v \circ \Phi^*_{\varepsilon} \circ \alpha'$ is contractible and hence $\varphi \circ \alpha'$ and therefore $\varphi \circ \alpha$ are contractible. Let $\varphi^*: M^* \to X$ be a continuous extension of $\varphi|_{M^1 \alpha}$ to $M^*$. Since

$$\varphi^* \circ h^*_\xi|_{K^1} = \varphi \circ h|_{K^1}$$

and the 1–skeletons of $K^*$ and $K'$ agree it follows that $v \circ h^*_\xi|_{K^1}$ is homotopic to $\varphi^* \circ h^*|_{K^1}$ relative to $\Gamma$ for almost every $\xi$ sufficiently small. This shows that $v$ is 1–homotopic to $\varphi^*$ relative to $\Gamma$.

The proof of Proposition 5.2 is almost the same as that of [8, Proposition 5.1], so we only give a rough sketch.

Proof of Proposition 5.2. Denote by $\mathcal{A}$ the family of maps $u \in \Lambda(M, \Gamma, X)$ such that $u \sim_1 \varphi$ rel $\Gamma$, $E^2(u, g) \leq L$ and $\text{Area}(u) \leq \alpha^*(M, \varphi, X) - \eta$. Suppose the claim is not true. Then there exists $\epsilon_0 > 0$ and, for each $\delta > 0$, a map $u \in \mathcal{A}$ such that the image of some boundary arc with length $\leq \delta$ has length $\geq \epsilon_0$. By considering a conformal chart containing the short boundary arc and using the Courant-Lebesgue lemma [24] Lemma 7.3 we see that there exists an arc $\beta: I \to M$ connecting two boundary points on either side (and outside) of the short boundary arc, for which $u \circ \beta \in W^{1,2}(I, X)$ agrees with the continuous representative of $\text{tr}(u)$ at the endpoints, and $\ell(u \circ \beta) \leq \pi E^2(u, g)/\log(1/\delta)^{1/2}$.

Since $\Gamma$ consists of rectifiable Jordan curves, there exists $\delta' > 0$ so that any points on $\Gamma$ with distance at most $\delta'$ belong to the same component and the shorter of the arcs joining them has length $< \min\{\epsilon_0, \eta\}$, where $0 < \eta' \leq \ell_0/2$ is such that $C(2\eta')^2 < \eta'/2$. Here $C$ and $\ell_0$ are the constants in the local quadratic isoperimetric inequality of $X$. Thus, by choosing $\delta > 0$ small enough, it follows that $\ell(u \circ \beta) < \eta'$ and moreover the image $\Gamma^+$ of the longer boundary arc $\gamma^+$ joining the endpoints of $\beta$ has length $< \eta'$.

Let $\alpha \subset \text{int } M$ be a smooth Jordan curve bounding an annulus $A \subset M$ together with the curve $\alpha' := \gamma^+ \cup \beta$ such that $u \circ \alpha \in W^{1,2}(S^1, X)$. In the surface $M^*$ obtained by cutting $M$ along $\alpha$ and gluing discs to the newly created boundary curves, $\alpha'$ bounds a Lipschitz Jordan domain $\Omega$. If $\Gamma_0$ is the concatenation of $u \circ \beta$ and $\Gamma^+ = \text{tr}(u) \circ \gamma^+$, then $\ell(\Gamma_0) < 2\eta'$ and, by [25] Lemma 4.8, $\Gamma_0$ is the trace of a Sobolev map $w_\Omega \in W^{1,2}(\Omega, X)$ with $\text{Area}(w_\Omega) < C(2\eta')^2 < \eta'/2$.

We define $v$ as $w_\Omega$ and $u|_{M \setminus A}$ on the respective sets. To define $v$ on the remaining smooth disc $\Omega' \subset M^*$, map $A$ diffeomorphically to an annulus $A' \subset \Omega'$ identifying
\( \alpha \) with \( \partial \Omega' \), and \( \alpha' \) with a Jordan curve (compactly contained in \( \Omega' \)) that bounds a copy \( \Omega'' \) of \( \Omega \), and set \( v_{|\Omega''} = w_{|\Omega} \) and \( v_{|\Lambda} = u_{|\Lambda} \) (after the diffeomorphic identifications). The gluing theorem \([20, \text{Theorem 1.12.3}] \) implies that \( v \in W^{1,2}(M^*, X) \) and by construction \( v \in \Lambda(M^*, \Gamma, X) \) with \( v \) and \( u \) inducing the same orientation on \( \Gamma \). Lemma \([5, \text{Lemma 5.3}] \) implies that \( v \) is 1–homotopic to \( \varphi^* \rel \Gamma \) for any primary reduction \( (M^*, \varphi^*) \) of \((M, \varphi)\). Now the estimate

\[
\text{Area}(v) = \text{Area}(u_{|\Lambda}) + 2 \text{Area}(w_{|\Omega}) + \text{Area}(u_{|\Lambda}) < \text{Area}(u) + \eta
\]
yields a contradiction with the fact that \( u \in \mathcal{A} \), completing the proof. \( \square \)

In the next proposition, we assume that the Euler characteristic \( \chi(M) \) of \( M \) is strictly negative so that \( M \) admits a hyperbolic metric, that is, a Riemannian metric on \( M \) of constant curvature \(-1\) and such that \( \partial M \) is geodesic.

**Proposition 5.4.** For every \( \eta > 0 \) and \( L > 0 \) there exists \( \varepsilon > 0 \) with the following property. If \( u \in \Lambda(M, \Gamma, X) \) is 1–homotopic to \( \varphi \) relative to \( \Gamma \) and such that

\[
\text{Area}(u) \leq a^*(M, \varphi, X) - \eta,
\]
and if \( g \) is a hyperbolic metric on \( M \) satisfying \( E^2(u, g) < L \) then the relative systole of \((M, g)\) satisfies \( \text{sys}_{rel}(M, g) \geq \varepsilon \).

The relative systole \( \text{sys}_{rel}(M, g) \) of \((M, g)\) is the minimal length of curves \( \beta \) in \( M \) of the following form. Either \( \beta \) is closed and not contractible in \( M \) via a family of closed curves, or the endpoints of \( \beta \) lie on the boundary of \( M \) and \( \beta \) is not contractible via a family of curves with endpoints on \( \partial M \). The proof of the proposition is almost the same as that of \([8, \text{Proposition 6.1}] \) and we only sketch it. Lemma \([5, \text{Proposition 5.3}] \) will be used again to control the relative 1–homotopy type of the primary reductions appearing in the proof.

**Proof.** Let \( \beta_0 \) be the geodesic realizing the systole \( \lambda := \text{sys}_{rel}(M, g) \). We may use a collar neighbourhood to find a ‘parallel’ Jordan curve \( \beta : I \to M \) for which \( u \circ \beta \in W^{1,2}(I, X) \) and \( \ell(u \circ \beta) \leq 2|\lambda E^2(u, g)|^{1/2} \), see \([8, \text{Lemma 6.2}] \). If \( \beta \) connects two boundary points, then \( I \) is a closed interval and the proof is analogous to that of Proposition \([5, \text{Proposition 5.2}] \). Namely, using the notation from the proof of Proposition \([5, \text{Proposition 5.2}] \) and supposing the relative systole \( \lambda \) is small enough, we may assume the boundary points are on the same boundary component and the image \( \Gamma^+ \) of one boundary arc \( \gamma^+ \) connecting them has small length, so that the concatenation \( \Gamma_0 \) of \( u \circ \beta \) and \( \Gamma^+ \) satisfies \( \ell(\Gamma_0) < 2\eta' \).

We let \( \alpha \subset \text{int} M \) be a closed Jordan curve bounding a (closed) annulus \( A \) with \( \alpha' := \gamma^+ \cup \beta \) such that \( u \circ \alpha \in W^{1,2}(S^1, X) \). In the surface \( M^* \) obtained from \( M \) by cutting along \( \alpha \), \( \alpha' \) bounds a Jordan domain \( \Omega \) containing \( A \) and we let \( w_{|\Omega} \in W^{1,2}(\Omega, X) \) satisfy \( \text{tr}(w_{|\Omega}) = \Gamma_0 \) and \( \text{Area}(w_{|\Omega}) < C(2\eta')^2 < \eta/2 \). Defining \( v \in \Lambda(M^*, \Gamma, X) \) as in the proof of Proposition \([5, \text{Proposition 5.2}] \) we reach the same contradiction with the fact that \( \text{Area}(u) \leq a^*(M, \varphi, X) - \eta \).

If \( \beta_0 \) is a closed geodesic, we construct \( M^* \) and \( v \) essentially as in the proof of \([8, \text{Proposition 6.1}] \) (keeping any components without boundary, and defining \( v \) on them analogously). We omit the details. \( \square \)

6. **Solution of the homotopic Plateau-Douglas problem**

Let \( X \) be a proper geodesic metric space admitting a local quadratic isoperimetric inequality and let \( \Gamma \subset X \) be the union of \( k \geq 1 \) rectifiable Jordan curves. Let
\( M \) be a connected surface with \( k \) boundary components and let \( \varphi: M \to X \) be a continuous map such that \( \varphi|_\partial M \in [\Gamma] \).

**Proposition 6.1.** Suppose \( \chi(M) < 0 \). Let \( (u_n) \subset \Lambda(M, \Gamma, X) \) be a sequence such that each \( u_n \) is 1–homotopic to \( \varphi \) relative to \( \Gamma \) and

\[
\sup_n \text{Area}(u_n) < a^*(M, \varphi, X).
\]

Let \( (g_n) \) be a sequence of hyperbolic metrics on \( M \). Then there exist \( u \in \Lambda(M, \Gamma, X) \) which is 1–homotopic to \( \varphi \) relative to \( \Gamma \) and a hyperbolic metric \( g \) on \( M \) such that

\[
\text{Area}(u) \leq \limsup_{n \to \infty} \text{Area}(u_n) \quad \text{and} \quad E^2_\varphi(u, g) \leq \limsup_{n \to \infty} E^2_\varphi(u_n, g_n).
\]

**Proof.** Let \( (u_n) \) and \( (g_n) \) be as in the statement of the proposition. By [8, Theorem 1.2 and (5.2)] there exist hyperbolic metrics \( \tilde{g}_n \) such that

\[
E^2_\varphi(u_n, \tilde{g}_n) \leq 4 \pi \cdot \text{Area}(u_n) + 1.
\]

After possibly replacing \( g_n \) by \( \tilde{g}_n \) and passing to a subsequence, we may therefore assume that the energies \( E^2_\varphi(u_n, g_n) \) are uniformly bounded and converge to a limit denoted by \( m \).

By Proposition [5,4] the relative systoles of \( (M, g_n) \) are uniformly bounded away from zero. Therefore, by the Mumford compactness theorem (see [8, Theorem 3.3] and [5, Theorem 4.4.1]) for the fact that the diffeomorphisms may be chosen to be orientation preserving, there exist orientation preserving diffeomorphisms \( \psi_n: M \to M \) and a hyperbolic metric \( h \) on \( M \) such that, after possibly passing to a subsequence, the Riemannian metrics \( \psi_n^* g_n \) smoothly converge to \( h \). For \( n \in \mathbb{N} \) define a map by \( v_n := u_n \circ \psi_n \) and notice that \( v_n \in \Lambda(M, \Gamma, X) \). Since \( \psi_n \), when viewed as a map from \( (M, h) \) to \( (M, g_n) \), is \( \lambda_n \)-biliipschitz with \( \lambda_n \to 1 \) it follows that

\[
\lim_{n \to \infty} E^2_\varphi(v_n, h) = m.
\]

By [8, Lemma 2.4] and the metric space valued Rellich-Kondrachov theorem (see [20, Theorem 1.13]) there exists \( v \in W^{1,2}(\overline{M}, X) \) such that a subsequence \( (v_{n_j}) \) converges in \( L^2(M, X) \) to \( v \). The lower semi-continuity of energy implies that \( E^2_\varphi(v, h) \leq m \). Since each \( u_n \) is 1–homotopic to \( \varphi \) relative to \( \Gamma \) and each \( \psi_n \) is orientation preserving it follows that all the maps \( v_n \) induce the same orientation on \( \Gamma \). By Theorem [5,6] there thus exists \( j_0 \in \mathbb{N} \) such that \( v_{n_j} \) is 1–homotopic to \( v_{n_{j_0}} \) for every \( j \geq j_0 \). It follows that for \( j \geq j_0 \) the maps \( w_j := v_{n_j} \circ \psi_{n_{j_0}}^{-1} \in \Lambda(M, \Gamma, X) \) satisfy

\[
w_j \sim_{1} v_{n_{j_0}} \circ \psi_{n_{j_0}}^{-1} = u_{n_{j_0}} \sim_{1} \varphi \text{ rel } \Gamma.
\]

The sequence \( (w_j) \) converges in \( L^2(M, X) \) to the map \( u := v \circ \psi_{n_{j_0}}^{-1} \) and \( g := (\psi_{n_{j_0}})^* h \), we furthermore have

\[
E^2_\varphi(u, g) \leq \lim_{j \to \infty} E^2_\varphi(w_j, h_0) = \lim_{j \to \infty} E^2_\varphi(v_j, h) = m.
\]

Finally, Proposition [5,2] implies that the family \( \{ \text{tr}(w_j): j \in \mathbb{N} \} \) is equi-continuous and hence, after passing to a further subsequence, we may assume that the sequence \( (\text{tr}(w_j)) \) converges uniformly to some continuous map \( \gamma: \partial M \to X \). As the uniform limit of weakly monotone parametrizations of \( \Gamma \), the map \( \gamma \) is also a weakly monotone parametrization of \( \Gamma \). Since \( (\text{tr}(w_j)) \) converges in \( L^2(\partial M, X) \) to \( \text{tr}(u) \) it follows that \( \text{tr}(u) = \gamma \) and hence \( u \in \Lambda(M, \Gamma, X) \). Since \( u \) and \( w_j \) induce
the same orientation on $\Gamma$ and since $w_j$ is $1$–homotopic to $\varphi$ relative to $\Gamma$ for every $j$ sufficiently large, it follows from Theorem 4.6 that $u$ is $1$–homotopic to $\varphi$ relative to $\Gamma$ as well. The lower semi-continuity of area and invariance of area under diffeomorphisms imply that

$$\text{Area}(u) \leq \liminf_{j \to \infty} \text{Area}(w_j) \leq \limsup_{n \to \infty} \text{Area}(u_n).$$

This concludes the proof. □

Proof of Theorem 1.3. Let $X$, $M$, $\Gamma$ be as in the statement of the theorem and let $\varphi : M \to X$ be a continuous map with $\varphi|_{\partial M} \in [\Gamma]$ satisfying the Douglas condition (1.1).

We start by proving (i) in the case $\chi(M) < 0$. The family

$$\Lambda_{\min} := \{u \in \Lambda(M, \Gamma, X) : u \sim_1 \varphi \text{ relative to } \Gamma \text{ and } \text{Area}(u) = a(M, \varphi, X)\}$$

is not empty. Indeed, this follows from Proposition 6.1 applied to a sequence $(u_n) \subset \Lambda(M, \Gamma, X)$ and an arbitrary sequence of hyperbolic metrics such that $u_n$ is $1$–homotopic to $\varphi$ relative to $\Gamma$ for every $n$ and

$$\text{Area}(u_n) \to a(M, \varphi, X)$$

as $n$ tends to infinity. Next, set

$$m := \inf \{E^2(u, g) : u \in \Lambda_{\min}, g \text{ hyperbolic metric} \}$$

and choose sequences $(u_n)$ and $(g_n)$, where $u_n \in \Lambda_{\min}$ and where the $g_n$ are hyperbolic metrics on $M$, such that

$$\lim_{n \to \infty} E^2(u_n, g_n) = m.$$

Applying Proposition 6.1 to these sequences we obtain a map $u \in \Lambda_{\min}$ and a hyperbolic metric $g$ on $M$ such that $E^2(u, g) = m$. It now follows from [9, Corollary 1.3] that $u$ is infinitesimally isotropic with respect to $g$.

We are left with the case $\chi(M) \geq 0$. If $k = 1$ and genus($M$) = 0, the result follows from [8, Theorem 1.2 and 1.4] since in this case any two maps inducing the same orientation on $\Gamma$ are $1$–homotopic.

In the remaining case $k = 2$ and genus($M$) = 0, one uses the Mumford compactness theorem for flat metrics normalized to have volume 1 (see [5, Theorem 4.4.1] for the case of closed surfaces) and a flat collar lemma to prove an analog of Proposition 5.4. Replacing Proposition 5.4 by this analog, the proof of Proposition 6.1 remains valid, and the argument above then works verbatim. See the proof of [8, Theorem 1.2] for more discussion. This concludes the proof of statement (i).

To show (ii) let $u$ and $g$ be as in statement (i). Then $u$ is a local area minimizer and it follows from the proof of [8, Theorem 1.4] that $u$ has a representative $\bar{u}$ which is locally Hölder continuous in the interior of $M$ and continuously extends to the boundary $\partial M$, thus proving statement (ii).

Statement (iii) is a direct consequence of the following lemma.

□

Lemma 6.2. Let $X$ be a metric space, let $\Gamma \subset X$ be the the disjoint union of $k \geq 1$ Jordan curve, and let $M$ be a smooth compact surface with $k$ boundary components. If $X$ has trivial second homotopy group then two continuous maps $\varphi, \psi : M \to X$ with $\varphi|_{\partial M}, \psi|_{\partial M} \in [\Gamma]$ are $1$–homotopic relative to $\Gamma$ if and only if they are homotopic relative to $\Gamma$.

We provide the easy proof for completeness, compare with [22, Lemma 2.1].
Proof. Let $X, M, \Gamma$ be as in the statement of the lemma and let $\varphi, \psi : M \to X$ be continuous maps such that $\varphi|_{\partial M}, \psi|_{\partial M} \in [\Gamma]$. It is clear that if $\varphi$ and $\psi$ are homotopic relative to $\Gamma$ then they are, in particular, 1–homotopic relative to $\Gamma$. In order to prove the opposite direction, suppose $\varphi$ and $\psi$ are 1–homotopic relative to $\Gamma$ and let $F : K^1 \times [0, 1] \to X$ be a homotopy from $\varphi$ to $\psi$ such that $F(\cdot, t) \in [\Gamma]$ for all $t$. Let $G$ be the continuous map which coincides with $F$ on $K^1 \times [0, 1]$ and with $\varphi$ and $\psi$ on $K \times [0]$ and $K \times \{1\}$, respectively. For every 2–cell $\Delta \subset K$ the restriction of $G$ to $\partial(\Delta \times [0, 1])$ extends to a continuous map on $\Delta \times [0, 1]$ since $X$ has trivial second homotopy group. The map $\tilde{G} : K \times [0, 1] \to X$ obtained in this way is a homotopy relative to $\Gamma$, between $\varphi$ and $\psi$. 

Observe that being 1–homotopic is a more restrictive condition than inducing the same action on fundamental groups.

Example 6.3. Let $X = S^1 \times S^1$ be the standard torus, $\Gamma = \{1\} \times S^1 \cup \{e^{i\pi}\} \times S^1 \subset X$, and $M = [0, 1] \times S^1$. The maps $\varphi_k \in \Lambda(M, \Gamma, X)$ given by $\varphi_k(t, z) = (e^{ikt}, z)$ induce the same action $\pi_1(M) \to \pi_1(X)$ and agree on $\partial M$, but are not 1–homotopic relative to $\Gamma$. Note that $\varphi_k$ are both conformal area minimizers in $\Lambda(M, \Gamma, X)$.

We finish the paper by discussing an analog of Theorem 1.3 for closed surfaces, that is, $k = 0$. In this case $\Gamma = \emptyset$ and consequently $\text{tr}(u) \in [\Gamma]$ is a vacuous condition; in particular $\Lambda(M, \Gamma, X) = W^{1,2}(M, X)$. We say that two maps are 1–homotopic if they are 1–homotopic relative to $\Gamma = \emptyset$.

We assume throughout this discussion that $X$ is compact, so that the Rellich-Kondrachov compactness theorem is applicable for any energy bounded sequence in $W^{1,2}(M, X)$. (The assumption $\text{tr}(u) \in [\Gamma]$ prevents a sequence from escaping to infinity when $\Gamma \neq \emptyset$, and we prevent the same here by assuming compactness.) Thus the results in Section 4 about 1–homotopy remain valid with these interpretations. Note that, with the convention $\text{sys}_{\text{rel}}(M) = \text{sys}(M)$, Proposition 5.4 (and thus Proposition 6.1) also remain valid with the same proofs.

The following theorem extends [33, Theorem 4.4] and [34, Theorem 3.1] to non-smooth target spaces.

Theorem 6.4. Suppose $M$ is a closed surface, and $X$ a compact geodesic metric space admitting a local quadratic isoperimetric inequality. If a continuous map $\varphi : M \to X$ satisfies the homotopic Douglas condition, then there exist $u \in W^{1,2}(M, X)$ and a Riemannian metric $g$ on $M$ such that $u$ is 1–homotopic to $\varphi$, $u$ is infinitesimally isotropic with respect to $g$, and

$$\text{Area}(u) = a(M, \varphi, X).$$

Furthermore, any such $u$ has a representative $\bar{u}$ which is Hölder continuous in $M$. If $X$ has trivial second homotopy group then $\bar{u}$ is homotopic to $\varphi$ relative to $\Gamma$.

Proof. The proof Theorem 1.3 (as well as that of Lemma 6.2) remains valid under the hypotheses of the claim (see the discussion above), except for the existence of $u$ and $g$ in the case $\chi(M) \geq 0$, i.e. $M = S^2$ or $M = S^1 \times S^1$.

In the first case we may choose $u \equiv$ constant and $g$ the standard metric on $S^2$, since $\varphi$ is 1–homotopic to a constant map. In the second case $M = S^1 \times S^1$ we use Mumford’s compactness theorem for flat metrics with volume normalized to 1 to obtain analogs of Propositions 5.4 and 6.1 and proceed as in the proof of Theorem 1.3.

\qed
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