Coinductive Intersection Types are Completely Unsound

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Abstract. Type assignment systems are known to grant or characterize normalization. The grammar of the types they feature is usually inductive. It is easy to see that, when types are coinductively generated, we obtain unsound type systems (meaning here that they are able to type some non head-normalizing terms). Even more, for most of those systems, it is not difficult to find an argument proving that every term is typable (complete unsoundness). However, this argument does not hold for relevant intersection type systems (ITS), that are more restrictive because they forbid weakening. Thus, the question remains: are relevant ITS featuring coinductive types – despite being unsound – still able to characterize some bigger class of terms? We show that it is actually not the case, because they can also type every term. The main theorem of this paper also yields a new non sensible relational model for pure $\lambda$-calculus.

Types Systems assign formulas called types to $\lambda$-terms $t$ under some constraints usually translating the rules of Natural Deduction. In simple type systems (STS), each term variable $x$ is assigned at most one type. In intersection type systems (ITS) [8, 18], a term variable $x$ may be assigned a new type in each new axiom rule typing it. Let us remind that a term $t$ is usually regarded as normalizing, when it can be reduced to a normal form (NF) (i.e. a term that does not hold some kind of redexes), meaning that the execution of $t$ terminates. Usually, STS ensure some kind of normalizability whereas ITS characterize normalizability [14].

The simplest set of normal forms is usually regarded as the set of head normal forms (HNF), i.e. terms $t$ that are of the form $t = \lambda x_1 \ldots x_p . x t_1 \ldots t_q$, where $p, q \geq 0$ and $t_1, \ldots, t_q$ are standard $\lambda$-terms. An example of non HN term is $\Omega = \Delta \Delta$ where $\Delta = \lambda x . x x$. Indeed, $\Omega \rightarrow \Omega$ is the only way to reduce $\Omega$, which never terminates. Most used sets of NF contain the set of HNF. Therefore, if a type system is able to type a non HN term, we say here that it is unsound.

In a typing system, an application $tu$ is typable when we can type $u$ (the argument of the application) with a type $A$ and the left hand side
t with type $A \rightarrow B$, where $B$ is another type. In that case, $tu$ is typed $B$ (modus ponens). The types are usually generated by an inductive grammar, meaning that they are finite.

Now, what happens if we use a coinductive grammar to generate types (roughly meaning that types may be infinite)? It is not difficult to see why this yields an unsound type system: we can build a type $A_\Omega$ satisfying $A_\Omega = A_\Omega \rightarrow o$ (where $o$ is a type variable). Thus, $tu$ can be typed $o$ when $t$ and $u$ have been typed $A_\Omega$. We can then type $\Omega$ with $o$: if $x$ is assigned $A_\Omega$, then $xx$ is typed $o$, so that $\Delta$ is typed $A_\Omega \rightarrow o$ i.e. $A_\Omega$. Thus, we can type $\Omega = \Delta\Delta$ with $o$. To avoid that, coinductive/recursive type or proof systems are usually endowed with a validity criterion [16, 1] or a guard condition [15].

Actually, coinduction allows us to build a reflexive type $A_\rightarrow$ i.e. $A_\rightarrow$ satisfies $A_\rightarrow = A_\rightarrow \rightarrow A_\rightarrow$. With that type, in the coinductive versions of standard STS and irrelevant ITS, we can easily type every term (§1). An ITS is relevant when it forbids weakening: in a relevant ITS, if $\Gamma \vdash t : A$ is derivable, then $\Gamma$ only assigns types to variables that occur freely in $t$ e.g. $\lambda x.y$ will usually have a type of the form $\{\} \rightarrow A$, where $\{\}$ is an empty type, because $x$ does not occur free in $y$ and is thus untyped. The question of characterizing the set of typable terms in a relevant coinductive intersection type systems (RCITS) turns out to be far more difficult: typing rules constrain the empty type to occur in unforeseeable places if we do not consider a NF. But here, we already know that typability does not entail normalizability. Moreover, the possibility to type easily $\Omega$ by using coinductive types is related to the fact that $\Omega$ satisfies a fix-point equation. But many terms do not have such nice regular behaviour. Thus, there may be a chance that some very erratic $\lambda$-terms could not be typable in a RCITS. In that case, RCITS would be able to characterize a class of regular $\lambda$-terms, bigger than that of the HN terms.

Contributions

The goal of this paper is to prove that relevant intersection type systems featuring coinductive types, although seeming more restrictive than other type systems, are also able to type any $\lambda$-term $t$. We present a proof for the set of finite $\lambda$-terms, but this can be adapted for the infinitary $\lambda$-calculus [9,13]. Naively, when $xu$ occurs in $t$, we would like to assign to $x$ a type of the form $A \rightarrow B$, where $A$ is the type of $u$, and proceed by induction. However, $x$ may be substituted in the course of a reduction sequence. In this case, typing constraints on $x$ are not easily readable. In the finite case, we escape this problem by typing normal forms and
then proceeding by expansion. But, as it has been noticed above, in the
coinductive case, no form of normalizability is granted by typability. We
must then proceed differently. We present an original method, that we
introduced in [19]:

– We define a standard RCITS that we call $\mathcal{D}$. The goal of this paper is
to prove that any term is typable in $\mathcal{D}$. For that, we introduce another
RCITS, called $\mathcal{S}$. In $\mathcal{S}$, derivations and types are (parsable) labelled
trees, which allows us to have a more fine-grained control on typing
constraints and type creation e.g. we can track any part of a type, from
its creation in an axiom rule to an application rule which erases it.
System $\mathcal{S}$ features pointers called bipositions and $\mathcal{S}$-derivations have
a bisupport (an extension of the notion of support). Derivations of
$\mathcal{S}$ naturally collapse on derivations of $\mathcal{D}$.
– Let $t$ be a term. Some sets of bipositions may be the bisupport of
a derivation typing $t$, some others are not (e.g. a set that does not
even define a tree). We define several "stability relations" on biposi-
tions ensuring that a set of bipositions is the bisupport of an actuel
derivation. Those stability conditions capture the way "emptiness" is
constrained to occur by relevance. We show that if the root element of
a derivation cannot be reached by emptiness via those relations, then
the term $t$ is actually typable.
– We prove that the root element of a derivation typing $t$ cannot be
reached by emptiness. The points above allow us then to conclude.
There lies the main technical difficulties of this result. We resort to
a finite normalizing strategy on chains of relations, that we call the
collapsing strategy.
– We explain briefly why this result provides us with a new non sensible
relational model for pure $\lambda$-calculus.

1 Type Systems

We present the type systems to be studied here. If $t$ is a $\lambda$-term, $\text{fv}(t)$ is
the set of its free variables.

1.1 A Coinductive Simple Type System

Let $\mathcal{X}$ be a countable set of type variables (metavariable $o$). We consider
the set of simple types generated by the following coinductive grammar:

$$A, B ::= o \mid A \rightarrow B$$
A context (metavariables $\Gamma, \Delta$) is a partial function from the set of variables $\mathcal{V}$ to the set of simple types. If for all $x \in \text{dom}(\Gamma) \cap \text{dom}(\Delta)$, $\Gamma(x) = \Delta(x)$, we write $\Gamma :: \Delta$ for the context of domain $\text{dom}(\Gamma) \cup \text{dom}(\Delta)$ inducing $\Gamma$ and $\Delta$. If $\text{dom}(\Gamma) \cap \text{dom}(\Delta) = \emptyset$, we may write $\Gamma; \Delta$ instead of $\Gamma :: \Delta$.

The set of typing derivations is defined inductively by the following rules:

\[
\begin{align*}
  x : A & \vdash x : A \quad \text{(ax)} \\
  \Gamma \vdash t : A \rightarrow B & \quad \Delta \vdash u : A \\
  \frac{}{\Gamma :: \Delta \vdash tu : B} \quad \text{(app)}
\end{align*}
\]

Notice that app-rule can be applied when $\Gamma :: \Delta$ is defined. The regular, inductive version of this system is known to ensure Strong Normalization.

Now, let $A_{\rightarrow}$ be the type coinductively defined by $A_{\rightarrow} = A_{\rightarrow} \rightarrow A_{\rightarrow}$. Then a straightforward induction on the structure of $t$ shows that $\Gamma \vdash t : A_{\rightarrow}$ is derivable for any context $\Gamma$ s.t. $\text{fv}(t) \subseteq \text{dom}(\Gamma)$ and $\Gamma(x) = A_{\rightarrow}$ for all $x \in \text{dom}(\Gamma)$: this type system is completely unsound.

### 1.2 Systems $\mathcal{D}$ and $\mathcal{D}_w$ (Idempotent Intersection)

The set $\text{Types}_\mathcal{D}$ of types of system $\mathcal{D}$ is defined coinductively:

\[
A, B, A_i ::= o \parallel \{ A_i \}_{i \in I} \rightarrow B
\]

We call $\{ A_i \}_{i \in I}$ a set type and it is always assumed that $I$ is countable. The set types represent intersection and the countable intersection operator $\land$ is the set-theoretic union: $\land_{j \in J} \{ A_i \}_{i \in I(j)} := \bigcup_{j \in J} \{ A_i \}_{i \in I(j)}$.

A context (metavariables $\Gamma, \Delta$) is a function from $\mathcal{V}$ to the set of set types. The domain of $\Gamma$ is given by $\{ x | \Gamma(x) \neq \{ \} \}$. The intersection of contexts $\land_{i \in I} \Gamma_i$ is defined point-wise. We may write $\Gamma; \Delta$ instead of $\Gamma \land \Delta$ when $\text{dom}(\Gamma) \cap \text{dom}(\Delta) = \emptyset$.

The set of typing derivations of system $\mathcal{D}$, named $\text{Deriv}_\mathcal{D}$, is defined inductively by the following rules:

\[
\begin{align*}
  x : A & \vdash x : A \quad \text{(ax)} \\
  \Gamma \vdash t : \{ A_i \}_{i \in I} & \quad \Delta_i \vdash u : A_i \\
  \frac{}{\Gamma \land_{i \in I} \Delta_i \vdash tu : B} \quad \text{(app)}
\end{align*}
\]

Typing derivations of $\mathcal{D}$ may be infinite because an app-rule may have an infinite number of arguments, although having a finite height.
System \( \mathcal{D} \) is relevant because no weakening is allowed: roughly speaking, if \( \Gamma \vdash t : B \) is derivable and \( x \in \text{fv}(t) \), then \( \Gamma(x) \) is the intersection of types of the free occurrences of \( x \) seen as a subterm of \( t \). Thus, relevance ensures some resource-awareness. There are many ways to obtain an irrelevant ITS. A standard one is to replace ax-rule by:

\[
\frac{i_0 \in I}{\Gamma; x : \{A_i\}_{i \in I} \vdash x : A_{i_0}} \text{ (ax)}
\]

yielding an ITS \( \mathcal{D}_\nu \). The proof for the simple type system above can be straightforwardly adapted for \( \mathcal{D}_\nu \) every term is typable in \( \mathcal{D}_\nu \).

From now on, we prove that every term is typable in the RCITS \( \mathcal{D} \).

### 1.3 Type System \( S \) (Sequential Intersection)

We want to have a good control on where types are created and how argument branches occur in an \texttt{app}-rule. For that, we define a type system \( S \), in which intersection is represented by means of sequences. It is inspired from Gardner and de Carvalho’s quantitative system \( R \) \cite{10,7}, in which intersection is represented by means of finite multisets.

We consider the set of types \( \text{Types} \) generated by the following coinductive grammar:

\[
T, S_k ::= o \parallel (S_k)_{k \in K} \to T
\]

Notation \( (S_k)_{k \in K} \) denotes a sequence type, which are the intersection types for \( S \). We always assume that \( K \subseteq \mathbb{N} - \{0,1\} \) and a \( k \in K \) is called a track. If \( k \geq 2 \), then \( k : T \) is the singleton sequence \( (S_l)_{l \in L} \) s.t. \( L = \{k\} \) and \( S_k = T \). We write ( ) for the empty sequence. A family \( ((S_k^i)_{k \in K(i)})_{i \in I} \) of sequence types is compatible if the \( K(i) \) are pairwise disjoint. In that case, we write \( \bigcup_{i \in I} (S_k^i)_{k \in K(i)} \) for the sequence type \( (S_k)_{k \in K(I)} \) s.t. \( K(I) = \bigcup_{i \in I} K(i) \) and \( S_k = S_k^i \) for the unique \( i \in I \) s.t. \( k \in K(i) \).

A context (metavariables \( C, D \)) assigns sequence types to variables. We extend pointwise the notions of compatibility and union. The set of typing derivations of \( S \) (metavariable \( P \)) is defined inductively by the following rules:

\[
\frac{x : k : T \vdash x : T}{(\text{ax})} \quad \frac{C; x : (S_k)_{k \in K} \vdash t : T}{C \vdash \lambda x. t : (S_k)_{k \in K} \to T} \quad \frac{C \vdash t : (S_k)_{k \in K} \to T}{C \cup_{k \in K} D_k \vdash tu : T} \quad \frac{D_k \vdash u : S_k_{k \in K}}{(\text{app})}
\]
Notice that we can apply \texttt{app} only if \( C \) and the \( D_k \) are compatible. If \( x \notin \texttt{fv}(t) \), then \( \lambda x.t \) is typed with \( \lambda \rightarrow T \). We write \( P \vdash C : T \) (or only \( P \vdash t \)) to mean that \( P \) concludes with judgment \( C : T \).

### 1.4 Parsing

We write \( \mathbb{N}^* \) for the set of finite sequences of integers (metavariables \( a, c \)). The length of \( a \in \mathbb{N}^* \) is written \(|a|\), meaning that \( a \) holds \(|a|\) integers, \( \varepsilon \) is the empty sequence and \( a_1 \cdot a_2 \) is the concatenation of \( a_1 \) and \( a_2 \). We write \( a_1 \leq a_2 \) when \( \exists a_3 \in \mathbb{N}^* \text{ s.t. } a_2 = a_1 \cdot a_3 \) (prefix order).

A tree is a subset of \( \mathbb{N}^* \) downward closed for the prefix order (\( A \) is a tree if \( a_2 \in A, a_1 \leq a_2 \) implies \( a_1 \in A \)). A labelled tree \( T \) is a function from a tree \( A \) to a set \( \Sigma \). We set then \( \text{supp}(T) = A \). If \( a \in \text{supp}(T) \), \( T|_a \) is the subtree of \( T \) at position \( a \): \( \text{supp}(T|_a) = \{ a_0 \in \mathbb{N}^* | a \cdot a_0 \in \text{supp}(T) \} \) and \( T|_a(a_0) = T(a \cdot a_0) \).

Terms can be seen as labelled trees: \( \text{supp}(x) = \{ \varepsilon \} \) and \( x(\varepsilon) = x \), \( \text{supp}(\lambda x.t) = \{ \varepsilon \} \cup \text{supp}(t) \) with \( (\lambda x.t)(\varepsilon) = \lambda x, \text{supp}(t u) = \{ \varepsilon \} \cup 1 \cdot \text{supp}(t) \cup 2 \cdot \text{supp}(u) \) with \((t u)(\varepsilon) = @\).

\( S \)-types can also be seen as trees: if \( T = o \), then \( \text{supp}(T) = \{ \varepsilon \} \) and \( T(\varepsilon) = o \) and if \( T_0 = (S_k)_{k \in K} \rightarrow T \), then \( \text{supp}(T_0) = \{ \varepsilon \} \cup 1 \cdot \text{supp}(T) \cup k \cdot \text{supp}(S_k) \), \( T_0(\varepsilon) = \rightarrow, T_0(1 \cdot a) = T(a) \) and \( T_0(k \cdot c) = S_k(c) \) for \( k \in K \).

Derivations can also be seen as labelled trees. We use the same notations as for the typing rules: position \( \varepsilon \) points to the judgment concluding derivation \( P \), if \( P \) types \( \lambda x.t \), its unique depth 1 subderivation is \( P|_0 \) and if \( P \) types \( t u \), \( P|_1 \) is the depth 1 subderivation typing \( t \) and, for \( k \in K \), \( P|_k \) is the subderivation concluded with \( D_k \vdash u : S_k \).

If \( k \in \mathbb{N} \), the collapse of \( k \), written \( \overline{k} \), is \( \min(k, 2) \). If \( |a| = \ell \), \( a = k_0 \cdot k_1 \ldots k_{\ell-1} \), the collapse of \( a \), also written \( \overline{a} \), is \( \overline{k_0} \cdot \overline{k_1} \ldots \overline{k_{\ell-1}} \). Thus, \( \overline{a} \in \{0, 1, 2\}^* \).

Let \( P \) be a derivation typing a term \( t \). Notice that \( \text{supp}(P) \subseteq \text{supp}(t) \) and that \( P(a) \) is a judgment typing \( t|_a \). If \( a \in \text{supp}(P) \), we write then \( t(a) \) and \( t|_a \) instead of \( t(\overline{a}) \) and \( t|_{\overline{a}} \). Moreover, \( P(a) \) may be written \( P(a) = C(a) \vdash t|_a : T(a) \), where \( C(a) \) is a context and \( T(a) \) a type. We set then \( \text{bisupp}(P) = \{ (a, c) | a \in \text{supp}(P), c \in \text{supp}(T(a)) \} \) and we call \( \text{bisupp}(P) \) the \text{bisupport} of \( P \). A \text{biposition} of \( P \) is an element of the bisupport of \( P \). Thus, a biposition (metavariable \( b \)) points to a symbol of type (on the right hand-side of \( \vdash \)) nested in a judgment inside \( P \). We write then \( P(a, c) \) for \( T(a)(c) \) and \( b \in P \) instead of \( b \in \text{bisupp}(P) \).

Some other notations are useful to handle \( S \)-derivations: assume that \( P \) types \( t \). We set \( A = \text{supp}(P) \) and \( B = \text{bisupp}(P) \). If \( x \notin \forall, a \in A, \)
we set $\text{Ax}_{P}^{\lambda}(x) = \{ a_0 \in A \mid a \equiv a_0, t(a) = x, \not\exists a_0', a \equiv a_0' \equiv a_0, t(a_0') = \lambda x \}$ (occurrences of $x$ in $P$ above $a$, that are not bound w.r.t. $a$). If $P(a) = x : k \cdot T \vdash x : T$, then we set $\text{tr}^{P}(a) = k$ and call $\text{tr}^{P}(a)$ the \textbf{axiom track} that has been used at $a$. Since $t$ is finite, we have $C(a)(x) = \bigcup_{a_0 \in \text{Ax}_{P}^{\lambda}(x)} \text{tr}^{P}(a_0) = T(a_0)$. This equality indicates that in a $S$-derivation, contexts and types can be computed from the support $\text{supp}(P)$ and the types created in axiom rules.

We define coinductively a \textbf{collapse} $\pi$ from the set of types of $S$ to the set of types of $D$ by $\pi(o) = o$ and $\pi((S_k)_{k \in K} \rightarrow T) = \{ \pi(S_k) \}_{k \in K} \rightarrow \pi(T)$. This collapse can be straightforwardly extended to a collapse from the set of derivations of $S$ to the set of derivations of $D$, noticing that $(S_k)_{k \in K} = (S_k')_{k \in K'}$ implies $\pi((S_k)_{k \in K}) = \pi((S_k')_{k \in K'})$. Thus:

**Proposition 1.** \textit{If a term $t$ is typable in $S$, then it is typable in $D$.}

## 2 Referents and Consumption

We want to prove that every term $t$ is typable by means of a derivation $P$ of System $S$. We recall that every time we use an axiom rule, we must choose an \textbf{axiom track} such that no conflict occurs. For that, it is enough to arbitrarily fix an injective function $\lfloor \cdot \rfloor : \mathbb{N}^* \rightarrow \mathbb{N} \setminus \{0, 1\}$, whose inverse function is written $\text{pos}$. Let $t$ be a term. We want to prove that there exists a $S$-derivation $P$ typing $t$ s.t., if $a \in \text{supp}(P)$ points to an axiom rule, then $\text{tr}^{P}(a) = \lfloor a \rfloor$ (thus, the value of $\lfloor a \rfloor$ matters only for axioms).

We set $A^t = \{ a \in \mathbb{N}^* \mid \bar{a} \in \text{supp}(t) \}$ and $B^t = (A^t \times \mathbb{N}^*) \cup \{ \bot \}$ (where $\bot$ is an "empty biposition" constant), so that, if $P$ types $t$, then $\text{supp}(P) \subseteq A^t$ and $\text{b supp}(P) \subseteq B^t - \{ \bot \}$. We drop $P$ and $t$ from most notations, in which they are implicit now. We set $A_0(x) = \{ a_0 \in A \mid a \equiv a_0, t(a_0) = x, \not\exists a_0', a \equiv a_0' < a_0, t(a_0') = \lambda x \}$. If $t(a) = \lambda x$, we set $\text{Tr}_{\lambda}(a) = \{ \lfloor a \rfloor \mid a_0 \in A_0(x) \}$. Thus, $\text{Tr}_{\lambda}(a)$ is the set of tracks typing $x$ before it is bound.

Let us have now another look at the syntactic dependencies induced by the typing rules between the types in the right hand side of judgments in a $S$-derivation. For instance, if $C \vdash t : S \rightarrow T$ is the premise of $C' \vdash t u : T$ in an $\text{app}$-rule, the occurrences of $T$ in both judgments are intuitively the same. We say the former is the \textbf{ascendant} of the latter, since it occurs above in the typing derivation. Likewise, $C; x : (S_k)_{k \in K} \vdash t : T$ may yield $C' \vdash \lambda x.t : (S_k)_{k \in K} \rightarrow T$: the first occurrence of $T$ is the ascendant of $T$ in the 2nd judgment. Moreover, assume $5 \in K$ in the last example. Then the occurrence of $S_5$ in $(S_k)_{k \in K} \rightarrow T$ stems from an axiom rule concluding
with $x : 5 \cdot S_5 \vdash x : S_5$ : we say that the occurrence $S_5$ (in $(S_k)_{k \in K} \rightarrow T$) is the polar inverse of the occurrence of $S_5$ in the axiom rule.

Formally, we define relations $\rightarrow_{\text{asc}}$ and $\rightarrow_{\text{pos}}$ not on types but on $\mathbb{B}$ by:

- $(a, c) \rightarrow_{\text{asc}} (a \cdot 1, 1 \cdot c)$ if $t(\overline{a}) = \emptyset$.
- $(a, 1 \cdot c) \rightarrow_{\text{asc}} (a \cdot 0, c)$ if $t(\overline{a}) = \lambda x$. According to the typing constraints, if $b \equiv (a \cdot c)$ occurs at biposition $b$ on $\mathbb{B}$, then there is an axiom rule concluding with $a$ at position $a$.
- $(a, k \cdot c) \rightarrow_{\text{pos}} (\text{pos}(k), c)$ if $t(\overline{a}) = \lambda x$ and $k \in Tr(a)$. For instance, assume $t(a) = \lambda x$, $t|_a = \lambda x. u$ : $(S_k)_{k \in K} \rightarrow T$. Then, the premise of $a$ concludes with $u : T$ at position $a$. Position $1 \cdot c$ of $(S_k)_{k \in K} \rightarrow T$ corresponds to position $c$ of $T$. Moreover, if $k \in Tr(a)$, then there is an axiom rule concluding with $x : k \cdot S_k \vdash x : S_k$ at position $k$. Position $c$ in $S_k$ corresponds to position $k \cdot c$ in $(S_k)_{k \in K} \rightarrow T$ (the type of $\lambda x. u$). Notice that, if $k \geq 2$, $k \notin Tr(a)$, there is no axiom rule typing $x$ using track $k$ above $a$. So $b = (a, k \cdot c)$ cannot be in $\text{bisupp}(P)$, so we relate it with the “empty biposition” $b_\bot$.

Let $\equiv$ be the reflexive, symmetric closure of $\rightarrow_{\text{asc}} \cup \rightarrow_{\text{pos}}$. According to the typing constraints, if $b_1 \equiv b_2$ and $P \triangleright C \vdash t : T$ then $b_1$ is in $P$ iff $b_2$ is in $P$. A referent is an equivalence class of the relation $\equiv$. We write $\text{Ref}$ for the quotient set $\mathbb{B}/\equiv$. The class of $(a, c) \in \mathbb{B}$ is written $\text{Ref}(a, c)$ and we set $r_\varepsilon = \text{Ref}(\varepsilon, \varepsilon)$, $r_\bot = \text{Ref}(b_\bot)$. If $\text{Ref}(b) = r$, we say that $r$ occurs at biposition $b$, also written $r : b$ or $b : r$.

### 2.1 Type Consumption

The notion of consumption is associated with rule $\text{app}$. When $u : (S_k)_{k \in K} \rightarrow T$ and $v : S_k$ for all $k \in K$, $uv$ can be typed with $T$. Each type $S_k$ occurs in $(S_k)_{k \in K} \rightarrow T$ and $v : S_k$. However, it is absent in the type of $uv$: we say it has been consumed. Formally, we set:

$$(a \cdot 1, k \cdot c) \overset{a}{\rightarrow} (a \cdot k, c) \quad \text{if} \quad t(a) = @$$

Indeed, assume $t(a) = @$, $t|_a = uv$: the premise concluding with $u : (S_k)_{k \in K} \rightarrow T$ (resp. with $v : S_k$) is at position $a \cdot 1$ (resp. $a \cdot k$). Position $c \in \text{supp}(S_k)$ corresponds to position $k \cdot c$ in $\text{supp}((S_k)_{k \in K} \rightarrow T)$.

Assume $b_1 \overset{a}{\rightarrow} b_2$ and $P \triangleright t$. Then $b_1 \in P$ iff $b_2 \in P$. We set $\rightarrow = \cup \{\overset{a}{\rightarrow} | a \in K, t(a) = @\}$ and write $\leftarrow$ for the symmetric relation.

### 2.2 Subjugation

If $b_1 \rightarrow b_2$, then $b_1 \notin P$ iff $b_2 \notin P$. However, this is not enough to express the conditions that a subset $B$ of $\mathbb{B}$ should satisfy to be the bisupport of...
a derivation $P$ typing $t$. We define an additional relation $\rightarrow_\ast$ s.t., when $b_1 \rightarrow_\ast b_2$, if $b_1 \notin P$, then $b_2 \notin P$. In that case, we say that $b_1$ subjugates $b_2$.

Relation $\rightarrow_\ast$ is to ensure that the types are correctly defined:

- $(a, c) \rightarrow_{t_1} (a, c \cdot k)$
- $(a, c \cdot 1) \rightarrow_{t_2} (a, c \cdot k)$ for any $k \geq 2$.

Indeed, if $c \notin \text{supp}(T)$, then no $c' \geq c$ can be in $\text{supp}(T)$ (since $T$ is a tree) and if $c \cdot 1 \notin \text{supp}(T)$, then $T(c) \neq \rightarrow$, so that we cannot have $c \cdot k \in \text{supp}(T)$ for $k \geq 2$.

The relation $\rightarrow_{rt}$ is here to grant that, if $\lambda x. u$ is a typed subterm of $t$, then its type cannot be a type variable: if $t(a) = \lambda x$, then $(a, 1) \rightarrow_{rt} (a, \varepsilon)$.

Relation $\rightarrow_{up}$ is included in the reflexive transitive closure of $\rightarrow \cup \rightarrow_{t_1}$ $\cup \rightarrow_{t_2}$, but turns out to be useful (Lemma 9): we set $(a, \varepsilon) \rightarrow_{up} b_\bot$ and $(a, \varepsilon) \rightarrow_{up} (a', c)$ if $a \leq a'$, meaning that if $a$ is not in $P$, no extension of $a$ can be in $P$.

We set $\rightarrow_\ast = \rightarrow \cup \leftarrow \cup \rightarrow_{t_1} \cup \rightarrow_{t_2} \cup \rightarrow_{rt} \cup \rightarrow_{up}$.

### 2.3 Coherence Theory

We set now $r_1 \xrightarrow{a} r_2$ if $\exists b_1, b_2$, $r_1 = \text{ref}(b_1)$, $r_2 = \text{ref}(b_2)$, $b_1 \xrightarrow{a} b_2$ i.e. $r_1 \xrightarrow{a} r_2$ iff $r_1 : b_1 \xrightarrow{a} b_2 : r_2$ for some $b_1, b_2$. In that case, we say that $r_1$ (resp. $r_2$) has been left-consumed (resp. right-consumed) at biposition $b_1$ (resp. $b_2$).

We proceed likewise for $\rightarrow_{t_1}$, $\rightarrow_{t_2}$, $\rightarrow_{rt}$, $\rightarrow_{up}$, $\rightarrow_\ast$, thus defining $\rightarrow_{t_1}$, $\rightarrow_{t_2}$, $\rightarrow_{abs}$, $\rightarrow_{up}$, $\rightarrow_\ast$.

**Definition 1.** A nihilating chain (for short, a chain) is a finite sequence of the form $r_0 \rightarrow_\ast r_1 \rightarrow_\ast \ldots \rightarrow_\ast r_m$ with $r_0 = r_\bot$ and $r_m = r_\varepsilon$.

Intuitively, the existence of a nihilating chain would prove that the (root of the) type of $t$ must be empty. Thus, this would imply that $t$ is not typable. On the contrary, we prove that when such a chain does not exist, then we can build a derivation typing $t$ whose bisupport is minimal.

**Proposition 2.** If there is no nihilating chain, then $t$ is typable in $S$.

**Proof Sketch.** Let $B$ be the set of $b \in B$ s.t. $r_\varepsilon$ is in the reflexive transitive closure of $\{\text{ref}(b)\}$ by $\rightarrow_\ast$. By hypothesis, if $\text{ref}(b) = r_\bot$, then $b \notin B$.

We set $\text{Lves}(B) = \{(a, c) \in B \mid (a, c \cdot 1) \notin B\}$ and we define $P$ on $B$ by $P(b) = o$ if $b \in \text{Lves}(B)$ and $P(b) = \rightarrow$ if not. We verify that $P$ is a correct $S$-derivation using the definition of $\rightarrow_\ast$. 
3 Main proof

Let \( t \) be a term. We want to apply Prop. 2 to \( t \). For that, we show \textit{ad absurdum} that there is no nihilating chain \( r_0 \to \cdots \to r_m \) with \( r_0 = r_{\bot} \) and \( r_m = r_\varepsilon \). However, \( \to \) can be \( \to, \leftarrow, \to_{t_1}, \to_{t_2}, \to_{\text{abs}} \) or \( \to_{\text{up}} \). The proof that this cannot happen goes like this:

- We define (Def. 2) the notion of \textit{syntactic polarity} (positive or negative) for bipositions: roughly, \( \text{Pol}(b) = \oplus \) if \( \text{Asc}(b) \) is in an axiom and \( \text{Pol}(b) = \ominus \) if \( \text{Asc}(b) \) is in a \( \lambda x \).
- A collapsing strategy grants that positivity can occur at suitable places in the chain (collapsing hypothesis). In that case, we say that the nihilating chain is normal.
- We can rule out the occurrences of \( \to, \to_{\text{abs}} \) and \( \to_{\text{up}} \) in a normal nihilating chain.
- Relation \( \leftarrow \) “commutes” with \( \to_{t_1} \) and \( \to_{t_2} \) in normal nihilating chains. This will allow us to assume that the chain is of the form:

\[
r_0 \to t_1 \to t \to t \cdots \to t m' \leftarrow r_{m'+1} \leftarrow \cdots \leftarrow r_m
\]

- This implies that \( r_{m'} = r_{\bot} \). We can then assume that \( m' = 0 \).
- Definition of \( \to_{\text{asc}} \) and \( \to_{\text{pl}} \) shows then that \( r_m \) cannot be \( r_\varepsilon \), which concludes the proof.

3.1 Syntactic Polarity and Consumption

Let \( \equiv_{\text{asc}} \) be reflexive, transitive, symmetric closure of \( \to_{\text{asc}} \). Notice that \( \to_{\text{asc}} \) is functional: if \( b_1 \to_{\text{asc}} b_2 \), we write \( b_2 = \text{asc}(b_1) \). Notice also that \( \text{asc} \) is injective. Thus, \( b_1 \equiv_{\text{asc}} b_2 \) iff \( \exists i \geq 0, b_2 = \text{asc}^i(b_1) \) or \( b_1 = \text{asc}^i(b_2) \). Moreover, by induction:

\[\text{Lemma 1. If } (a_1, c_1) \equiv_{\text{asc}} (a_2, c_2) \text{ then } \exists a_3 \in \{0, 1\}^+, (a_2 = a_1 \cdot a_3 \text{ or } a_1 = a_2 \cdot a_3) \text{ and } \exists i \geq 0, (c_2 = 1^i \cdot c_1 \text{ or } c_1 = 1^i \cdot c_2).\]

We set, for all \( b \in \mathbb{B} \), \( \text{Asc}(b) = \text{asc}^i(b) \), where \( i \) is maximal (i.e. \( \text{asc}^i(b) \) is defined, but not \( \text{asc}^{i+1}(b) \)).

\[\text{Definition 2 (Syntactic Polarity).}\]

- Let \( b \in \mathbb{B} \) and \( (a_0, c_0) = \text{Asc}(b) \). We define the \textit{syntactic polarity} of \( b \) as follows: if \( t(a_0) = x \), then we set \( \text{Pol}(b) = \oplus \) and if \( t(a_0) = \lambda x \), then we set \( \text{Pol}(b) = \ominus \).
If \( \text{ref}(b) = r \) and \( \text{Pol}(b) = \oplus/\ominus \), we say that \( r \) occurs positively/negatively at \( b \).

If \( r \) is left/right-consumed at \( b \) and \( \text{Pol}(b) = \oplus \) (resp. \( \text{Pol}(b) = \ominus \)), we say that \( r \) is left/right-consumed positively (resp. negatively) at \( b \).

Since \( \rightarrow_{p1} \) also defines an injective function and \( b_1 \rightarrow_{p1} b_2 \) implies that \( b_1 \) and \( b_2 \) do not have ascendants:

**Lemma 2.** – For all \( b_1, b_2 \in B \), \( b_1 \equiv b_2, \text{Pol}(a_1, c_1) = \oplus \) and \( \text{Pol}(a_2, c_2) = \ominus \) iff \( \text{Asc}(a_2, c_2) \rightarrow_{p1} \text{Asc}(a_2, c_2) \).

For all \( b \), \( \text{ref}(b) = r_\perp \) iff \( \text{Asc}(b) = (a_0, k \cdot c_0) \) with \( t(a_0) = \lambda x \) and \( k \notin \text{Tr}_{\lambda}(a_0) \).

Then, we write for instance \( r_1 \oplus \overset{a}{\rightarrow} \ominus r_2 \) to mean that \( r_1 \) (resp. \( r_2 \)) is left(resp. right)-consumed positively (resp. negatively).

**Definition 3.** A nihilating chain is **normal** if no referent is left-consumed negatively in it (the chain does not hold a \( r_i \ominus \rightarrow r_{i+1} \) or \( r_{i-1} \rightarrow \ominus r_i \)).

Since a consumed biposition does not have a descendant, Lemma 1 and 2 imply:

**Lemma 3 (Uniqueness of Consumption).** Let \( \ominus \in \{\oplus, \ominus\} \) and \( r \in \text{Ref} \). Then, there is a most one \( r' \) s.t. \( r \ominus \rightarrow r' \) or \( r' \ominus \rightarrow r \).

The following lemmas make explicit possible interactions between consumption and subjugation. We prove them mostly using Lemmas 1 and 2:

**Lemma 4.** If \( r_1 \rightarrow r_2 \) and \( r_1 \rightarrow_{t_1} r_3 \), then, \( \exists r_4, r_2 \rightarrow_{t_1} r_4 \) and \( r_3 \rightarrow r_4 \).

**Lemma 5.** If \( r \ominus \rightarrow r' \), there is no \( r_0 \) s.t. \( r \rightarrow_{\text{abs}} r_0 \) or \( r \rightarrow_{\text{up}} r_0 \).

**Lemma 6.** If \( r \ominus \rightarrow r_2 \) and \( r \rightarrow_{t_2} r_3 \), then, \( \exists r_4, r_2 \rightarrow_{t_2} r_4 \) and \( r_3 \rightarrow r_4 \).

**Lemma 7.** – If \( \text{ref}(b) = r_\perp \), then \( \text{Pol}(b) = \ominus \).

– If \( r_\perp \rightarrow_{t_1} r \) or \( r_\perp \rightarrow_{t_2} r \), then \( r = r_\perp \).

– We cannot have \( r_\perp \rightarrow_{\text{abs}} r \) or \( r_\perp \rightarrow_{\text{up}} r \).
3.2 Residuals

As it has been said in the Introduction, the possibility for a variable $x$ to be substituted in a reduction sequence is problematic. It will turn out (§ 3.4) that we only need to avoid the case of negative left-consumption. It will allow us to use freely Lemmas 4, 5 and 6. The key tool to prove that there is no nihilating chain is a normalizing reduction strategy, called the **collapsing strategy**, which destroys referents that are left-consumed negatively and allows to build a normal chain from any nihilating chain. This reduction strategy is finite, despite the fact normalizability is not granted.

Assume $t|_b = (\lambda x.r)s$ and $t \rightarrow^\beta t'$. We are also going to study the typability of $t'$. We may prove the subject reduction property (as well as subject expansion) for $S$ i.e. if $P \triangleright C \vdash t : T$, then $\exists P' \triangleright C \vdash t' : T$. Informally, such a $P'$ is obtained by suitably moving parts of $P$ and $t$, namely, by replacing each axiom rule typing $x$ by a subderivation typing $s$, whereas the rules typing the root of the redex are destroyed. This observation guides every formal definition to follow in this section.

Here, we are not interested in subject reduction per se but in the way by which the positions are moved or destroyed: the (proper) residual of a position that is not destroyed during reduction will be its new position after reduction. Some (bi)positions are destroyed (e.g. in axiom rules typing $x$), but they can be identified (often, via $\equiv$ or $\rightarrow$) with (bi)positions that have a proper residual: we say those (bi)positions have a quasi-residual. The precise definitions that follow are quite technical (they are given for the sake of verification), but the main point is to obtain Lemma 8, which allows us to define quasi-residuals of referents (not only of (bi)positions), and then, Lemma 9, which states that reduction does not have a bad effect on residuals of referents.

For what concerns residuals, metavariable $a$ will now denote only positions in $A$ s.t. $\overline{a} = b$. Metavariables $\alpha$ and $\gamma$ range over $\mathbb{N}^*$. For instance, $\alpha \neq a$ means that $\overline{\alpha} \neq b$. If $k \in \text{Tr}_\lambda(a)$, $a_k$ is the unique position s.t. $\text{pos}(k) = a \cdot 10 \cdot a_k$.

Let us have a look at the positions in $t$: positions $a$ and $a \cdot 1$ point to the root and the abstraction of the redex, $a \cdot 10$ points to the root of $r$, $a \cdot k$ (with $k \in \text{Tr}_\lambda(a)$) points to the root of $s$, $a \cdot 10 \cdot a_k$ (with $k \in \text{Tr}_\lambda(a)$) points to an occurrence of $x$. Thus:

- We set $\text{QRes}_b(\alpha) = \text{Res}_b(\alpha)$ when $\text{Res}_b(\alpha)$ is defined and $\text{QRes}_b(\alpha, \gamma) = (\text{QRes}_b(\alpha), \gamma)$ when $\text{QRes}_b(\alpha)$ is defined and we set $\text{QRes}_b(b_\bot) = b_\bot$.
- If $\alpha \not\not\in a$, then $\alpha$ is not in the redex. We set $\text{Res}_b(\alpha) = \alpha$. 

– a and a · 1 are destroyed by the reduction. So \( \text{Res}_b(a) \) and \( \text{Res}_b(a) \cdot 1 \) are not defined and \( a \cdot 10 \cdot \alpha \) should become \( a \cdot \alpha \) after reduction (except when \( t(a \cdot 10 \cdot \alpha) = x \)); we set \( \text{Res}_b(a \cdot 10 \cdot \alpha) = a \cdot \alpha \) for \( \alpha \neq a_k \).

– Assume \( k \in \text{Tr}_\lambda(a) \). By the typing constraints and the definition of \( a_k \), notice that \( T(a \cdot k) = T(a \cdot 10 \cdot a_k) \). So argument derivation at \( a \cdot k \) will replace \( \text{ax-rule} \) typing at position \( a \cdot 10 \cdot a_k \) (which is destroyed). So its position after reduction will be \( a \cdot a_k \). We set then \( \text{Res}_b(a \cdot k \cdot \alpha) = a \cdot a_k \cdot \alpha \) and \( \text{QRes}_b(a \cdot 10 \cdot a_k) = \text{Res}_b(a \cdot k) = a \cdot a_k \).

– \( (a, \gamma) \) and \( (a \cdot 1, 1 \cdot \gamma) \) are destroyed but \( (a, \gamma) \rightarrow_{\text{asc}} (a \cdot 1, 1 \cdot \gamma) \rightarrow_{\text{asc}} (a \cdot 10, \gamma) \). We set \( \text{QRes}_b(a, \gamma) = \text{QRes}_b(a \cdot 1, 1 \cdot \gamma) = \text{QRes}_b(a \cdot 10, \gamma) = (a, \gamma) \).

– To grant Lemma 9, we set \( \text{QRes}_b(a \cdot 1, \varepsilon) = \text{QRes}_b(a \cdot 1, 1) = (a, \varepsilon) \).

– Assume \( k \notin \text{Tr}_\lambda(a) \), \( k > 2 \). Then \( (a \cdot 1, k \cdot c) \rightarrow_{\text{pi}} b \perp, (a \cdot 1, k \cdot c) \rightarrow (a \cdot k, c) \) and there is no \( \text{ax-rule} \geq \alpha \) typing \( x \) using track \( k \). So we say that \( (a \cdot 1, k \cdot \gamma) \) and \( (a \cdot k \cdot \alpha, \gamma) \) are erased after reduction. We set \( \text{QRes}_b(a \cdot 1, k \cdot \gamma) = \text{QRes}_b(a \cdot k \cdot \alpha, \gamma) = b \perp \).

Thus, \( \text{QRes}_b \) is a total function on bipositions and \( \text{Res}_b \) is a partial injective function. Moreover, \( t'(\text{Res}_b(\alpha)) = t(\alpha) \) (not true for \( \text{QRes}_b \)).

We set \( B' = B' \) and \( A' = A' \) (i.e. \( A' = \{ \alpha' \in \mathbb{N}^* | \overline{\sigma} \in \text{supp}(t') \} \) and \( B' = A' \times \text{supp}(t) \)). We set \( [\alpha'] = k \) if there is \( \alpha \in A \) s.t. \( t(\alpha) = y \neq x \) and \( \text{Res}_b(\alpha) = \alpha' \). We check that \( \alpha \rightarrow [\alpha'] \) is still an injective function. We set \( \text{pos}'(k) = \alpha' \) if there is \( \alpha \) s.t. \( [\alpha'] = k \). Thus, \( \text{pos}'(k) = \text{Res}_b(\text{pos}(k)) \).

In \( B' \), we also define \( \rightarrow_{\text{asc}}, \rightarrow_{\text{pi}} \) and so on. The distinction between \( B \) and \( B' \) is understood from the context. We also define \( \text{asc}' \) and \( \text{Ref}' \).

By case analysis:

– Assume \( b_1 \rightarrow_{\text{asc}} b_2 \) or \( b_1 \rightarrow_{\text{pi}} b_2 \) and \( b_1 \) is erased. Then \( \text{QRes}_b(b_1) = \text{QRes}_b(b_2) = b \perp \). We assume below that \( b_1 \) is not erased.

– Assume \( b_1 = (a, \gamma) \rightarrow_{\text{asc}} b_2 \). If \( \alpha 
eq a, a \cdot 1 \), then \( \text{QRes}_b(b_1) = \text{QRes}_b(b_2) \).

– If \( b_1 = (a, \gamma) \rightarrow_{\text{pi}} b_2 \) with \( b_2 \neq b \perp \). If \( \alpha 
eq a \cdot 1 \), then \( \text{Res}_b(b_1) = \text{Res}_b(b_2) \).

– Assume \( b = (a, \gamma) \rightarrow_{\text{pi}} b \perp \). If \( t(\alpha) = \lambda x \), then \( \text{Res}_b \rightarrow_{\text{pi}} b \perp \).

This entails:

**Lemma 8.** If \( b_1 \equiv b_2 \), then \( \text{QRes}_b(b_1) \equiv \text{QRes}_b(b_2) \).

This Lemma allows us to define (quasi-)residuals for referents. We set \( \text{QRes}_b(r) = \text{ref}'(\text{QRes}_b(b)) \) for any \( b : r \). Notice that \( \text{QRes}_b(r) = r \).

By case analysis on \( b_1 \rightarrow_{\text{pi}} b_2 \) and \( \text{QRes}_b \) similar to the one above, we prove:
Lemma 9. Let \( r_1, r_2 \in \text{Ref} \). We set \( r'_i = \text{QRes}_b(r_i) \) \( (i = 1, 2) \).

- If \( r_1 \rightarrow r_2 \), then \( r'_1 \rightarrow r_2 \) or \( r'_1 = r'_2 \).
- If \( r_1 \rightarrow t_1 r_2 \), then \( r'_1 \rightarrow t_1 r'_2 \) or \( r'_1 = r'_2 \).
- If \( r_1 \rightarrow t_2 r_2 \), then \( r'_1 \rightarrow t_2 r'_2 \), \( r'_1 \rightarrow \text{up} r'_2 \) or \( r'_1 = r'_2 \).
- If \( r_1 \rightarrow \text{abs} r_2 \), then \( r'_1 \rightarrow \text{abs} r'_2 \) or \( r'_1 = r'_2 \).
- If \( r_1 \rightarrow \text{up} r_2 \), then \( r'_1 \rightarrow \text{up} r'_2 \) or \( r'_1 = r'_2 \).

The above Lemma implies that the length of a chain does not increase by reduction.

3.3 The Collapsing Strategy

We show now that we can assume that, in the chain, no referent is left-consumed negatively. This will allow us to use the Lemmas of §3.1. The idea is that if \( r_1 : b_1 \xrightarrow{a} r_2 \), then either \( t|a \) is a redex and the reduction at \( b = \pi \) ensures that \( \text{Res}_b(r_1) = \text{Res}_b(r_2) \) or there is a redex between \( b_1 \) and \( a \). When we reduce it, the relative height of \( b_1 \) will decrease. Thus, in a finite number of steps, we collapse \( r_1 \) on \( r_2 \).

Assume then \( r_1 : b_1 = (\alpha \cdot 1, k \cdot \gamma) \xrightarrow{\alpha} r_2 \). Then \( \text{Asc}(r_1) = (\alpha, k \cdot \gamma) \) and \( t(a) = \lambda x \) for some \( \alpha, x \). We set \( h = |\alpha| - |a| \) and we call \( h \) the height of the consumption. By Lemma 1, for \( 1 \leq i \leq h \), we may write \( b_i = (\alpha_i, 1^{i-1} \cdot k \cdot \gamma) \) for \( \text{asc}^{-1}(r_1) \) where \( \alpha_{i+1} = \alpha_i \cdot k_i \) for a \( k_i \in \{0, 1\} \).

If \( h = 1 \), then \( \alpha = \alpha \cdot 1 \) and we set \( b = \pi \) so that \( t|b \) is a redex and \( \text{QRes}_b(b_1) = \text{QRes}_b(b_2) \). Thus, \( \text{Res}_b(r_1) = \text{Res}_b(r_2) \).

Assume now \( h > 1 \). Then let \( 1 \leq i_0 \leq h-1 \) be maximal s.t. \( t(a_{i_0}) = \emptyset \). Actually, \( i_0 \leq h-2 \) (if \( i_0 = h-1 \), then \( t|a_{h-1} \) is a redex, so \( a_{h-1} = a \) i.e. \( h = 1 \)).

We set \( b = a_{i_0} \) so that \( t|b \) is a redex. We set \( b'_i = (\alpha'_i, \gamma'_i) = \text{QRes}_b(b_i) \) (so that \( b'_{i_0} = b'_{i_0+1} = b'_{i_0+2} \)). By induction on \( i \), if \( 1 \leq i \leq i_0 \), then \( \text{asc}^{-1}(b_1) = b'_i \) and \( |\alpha'_i| - |\alpha| = i \) if \( i_0 < i \leq h-2 \), then \( \text{asc}^{-1}(b_1) = b'_{i+2} \) and \( |\alpha'_{i+2}| - |\alpha| = i \). Thus, \( \text{asc}^{-1}(b_1) = (\alpha'_i, k \cdot \gamma) \) and \( |\alpha'_i| - |\alpha| = h-2 \). Since \( t'(\alpha_{h-2}) = t(\alpha_{h}) \) (proper residual), \( \text{asc}(b'_i) = b'_i \) and \( \text{Pol}(b'_1) = \emptyset \). Thus, \( r'_1 = \text{Res}_b(r_1) \rightarrow r'_2 = \text{Res}_b(r_2) \), but the height has decreased by 2. This yields the collapsing strategy:

Lemma 10. If \( r_1 \rightarrow r_2 \), then there is a reduction strategy that equalizes \( r_1 \) and \( r_2 \).

3.4 Final Arguments

- We assume the length \( m \) of the chain to be minimal among all the chains proving \( \text{Bot}(r_c) \).
– By Lemma 9, the collapsing strategy does not increase the size of the chain. By Lemma 10, it actually decreases it. Thus, we can assume that no referent is left-consumed negatively (collapsing hypothesis).
– Finally, let \( m_t \) be the minimal integer \( m' \) s.t. \( r_{m'} \rightarrow_t r_{m'+1} \) or \( r_{m'} \rightarrow_t r_{m'+1} \) (we set \( m_t = m \) if this never occurs in the chain). Then, we assume \( m_t \) to be minimal among all the shortest chains proving \( \text{Bot}(r_\varepsilon) \).

\[ \text{• First, we are interested in the first edge } r_0 \rightarrow_\bullet r_1 \text{ with } r_0 = r_\perp. \]

– By Lemma 7, \( \text{ref}(b) = r_\perp \) implies \( \text{Pol}(b) = \emptyset \), so, here, \( \rightarrow_\bullet \) cannot be \( \rightarrow \) (collapsing hypothesis).
– Moreover, \( \rightarrow_\bullet \) cannot be \( \rightarrow_{t_1} \) or \( \rightarrow_{t_2} \) : if it were, by Lemma 7, we would have \( r_1 = r_\perp \) and the chain would not be minimal.
– By Lemma 7, it cannot be \( \rightarrow_{\text{abs}} \) or \( \rightarrow_{\text{up}} \).
Thus, we have \( r_0 \leftarrow r_1 \) and \( m_t > 0 \).

\[ \text{• We prove now that for all } 0 \leq i \leq m - 1, r_i \leftarrow r_{i+1}. \text{ Ad absurdum, let } n \text{ be the minimal rank } i \text{ s.t. } r_i \rightarrow_\bullet r_{i+1} \text{ with } \rightarrow_\bullet \neq \leftarrow. \text{ Notice that } n \geq 1. \]

– By Lemma 5, \( \rightarrow_\bullet \) cannot be \( \rightarrow_{\text{abs}} \) or \( \rightarrow_{\text{up}} \).
– It cannot be \( \rightarrow_{t_1} \) or \( \rightarrow_{t_2} \). Assume the converse: \( r_{n-1} \leftarrow r_n \rightarrow_t r_{n+1} \) and \( n = m_t \). By Lemma 4 or 6, there is \( r'_n \) s.t. \( r_{n-1} \rightarrow_t r'_n \leftarrow r_{n+1} \). The minimality of \( m_t \) would not hold.
– It cannot be \( \rightarrow \). Assume the converse. By the collapsing hypothesis, \( r_{n-1} \leftarrow r_n \oplus \rightarrow r_{n+1} \). Then, by Lemma 3, \( r_{n-1} = r_{n+1} \) and the chain would not be minimal in length.

\[ \text{• Thus, } r_{m-1} \leftarrow \oplus r_m = r_\varepsilon. \text{ This means } r_\varepsilon : (a \cdot 1, k \cdot c) \rightarrow (a \cdot k, c) : r_{m-1} \text{ for some } a, k, c. \text{ By Lemma 1, } \text{asc}^i(\varepsilon, \varepsilon) = (a_0, i_0) \text{ for some } a_0, i_0. \]

– Assume \( \text{Pol}(\varepsilon, \varepsilon) = \oplus \): since \( \text{Pol}(a \cdot 1, k \cdot c) = \emptyset \), by Lemma 1, \( (\varepsilon, \varepsilon) \equiv (a \cdot 1, k \cdot c) \) is impossible.
– Assume \( \text{Pol}(\varepsilon, \varepsilon) = \ominus \): \( \exists b, (a_0, i_0) \rightarrow_{p_1} b \) is impossible. Thus, \( (\varepsilon, \varepsilon) \equiv b \) and \( \text{Pol}(b) = \oplus \) is impossible.
Thus, \( r_{m-1} \leftarrow \oplus r_\varepsilon \) is impossible. No chain can prove that \( \text{Typ}_t \vdash \text{Bot}(r_\varepsilon) \).
It concludes the proof of :

**Proposition 3.** Let \( t \) be a \( \lambda \)-term. Then \( \text{Typ}_t \nvdash \text{Bot}(r_\varepsilon) \).
4 Conclusion

Theorem 1. The Relevant Intersection Type System \( \mathcal{D} \), featuring coinductive types, is completely unsound: every \( \lambda \)-term is typable in \( \mathcal{D} \).

Proof. By Propositions 1, 2 and 3.

If we take the typing rules of \( \mathcal{S} \) coinductively, we can also type every infinitary \( \lambda \)-term [13,9]. The non-inductive definition of residuals (§3.2) can be reused in the coinductive framework. There may be infinite sequences of ascendants and the definition of polarity must be slightly extended, although the collapsing strategy (§3.3) is still finite.

Moreover, derivations of system \( \mathcal{S} \) collapse on system \( \mathcal{D} \) (§1.4), but they also collapse on system \( \mathcal{M} \), the coinductive version of Gardner and de Carvalho’s System \( \mathcal{R} \) [10,7]. Thanks to subject reduction and expansion, this yields a new relational model of pure \( \lambda \)-calculus [6] (finite or infinite) in which, by Theorem 1, no term has a trivial denotation, including the mute terms [12,3]. This means that this model is non sensible [2] and its equational theory should be investigated and compared to that of other models [4]. We may conjecture that this model is able to discriminate terms according to their order (the order of \( t \) is the maximal \( n \) s.t. \( t \rightarrow^* \lambda x_1 \ldots \lambda x_n.t' \)).

We introduced the method presented in this paper (in short, the use of the finite collapsing strategy to prove that some ad hoc first order theory is consistent) to solve another problem in which normalizability was not granted: namely, proving that every derivation of \( \mathcal{M} \) was the collapse of a derivation of \( \mathcal{S} \) [19]. It suggests that the tools introduced here could be of some use to compare coinductive or recursive type systems before they are endowed with some validity or guard condition, or maybe to build other models of pure \( \lambda \)-calculus.

System \( \mathcal{S} \) has other uses: in a restriction of the infinitary calculus called \( \Lambda \) [13], the normal forms are the Böhm trees that do not hold \( \bot \). Thus, the weakly normalizing terms of \( \Lambda \) are the terms whose Böhm trees do not hold \( \bot \). This property is also called hereditary head-normalization (HHN). If we resort to a validity criterion to discard some unsound derivations in \( \mathcal{S} \), we can characterize the set of HHN terms [20]. This gives a positive answer in the coinductive case to a problem referred as Klop’s Question [17]. Moreover, sequential intersection may be connected to Grellois and Melliès infinitary exponential modality [11], as well as Bucciarelli and Ehrhard indexed linear logic [5].
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