INTEGRABLE MULTIDIMENSIONAL COSMOLOGY FOR INTERSECTING P-BRANES

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A multidimensional field model describing the behaviour of (at most) one Einstein space of non-zero curvature and \(n\) Ricci-flat internal spaces is considered. The action contains several dilatonic scalar fields \(\varphi^a\) and antisymmetric forms \(A^I\). The problem setting covers various problems with field dependence on a single space-time coordinate, in particular, isotropic and anisotropic homogeneous cosmologies. When the forms are chosen to be proportional to volume forms of “\(p\)-brane” submanifolds manifold, a Toda-like Lagrange representation arises. Exact solutions are obtained when the \(p\)-brane dimensions and the dilatonic couplings obey some orthogonality conditions. General features and some special cases of cosmological solutions are discussed. It is shown, in particular, that all hyperbolic models with a 3-dimensional external space possess an asymptotic with the external scale factor \(a(t) \sim |t|\) (\(t\) is the cosmic time), while all internal scale factors and all scalar fields tend to finite limits. For \(D = 11\) a family of models with one 5-brane and three 2-branes is described.

1. Introduction

This paper continues the trend of studying various exact solutions describing the properties of \(p\)-branes interacting with gravity. Such problems naturally emerge in bosonic sectors of supergravitational models\(^4\) and may be of interest in the context of superstring and M-theories\(^5\). We actually consider gravitational models containing several coupled dilatonic scalar fields and antisymmetric forms\(^6\), namely, the one-variable (in particular, cosmological) sector of the model from\(^7\), i.e. a self-consistent set of \(D\)-dimensional Einstein-Hilbert equations and the equations for interacting dilatonic scalar fields and fields of forms. The model describes generalized intersecting \(p\)-branes (for different aspects of \(p\)-branes see\(^8\) and references therein.) Using a \(\sigma\)-model representation\(^9\), we reduce the equations of motion to a pseudo-Euclidean Toda-like Lagrange system\(^10\) with a zero-energy constraint. In the simplest case of orthogonal vectors in the exponents of the Toda potential we obtain exact solutions. Some general features of the cosmological solutions and their special cases are then discussed.

Special cases of the present models were recently studied by a number of authors\(^11\) and others; thus, spherically symmetric and cosmological models with a conformally invariant generalization of the Maxwell field to higher dimensions were discussed in Refs.\(^12\); in particular, in\(^13\) some integrable cases with perfect fluid sources were indicated. The present treatment deals with the general class of coupled \(p\)-brane and dilatonic fields in the so-called electric case, but with no other material sources; the corresponding magnetic and mixed electro-magnetic models will be discussed elsewhere.

For convenience we give a summary of indices used in the paper and objects they correspond to. Namely:

- \(M, N, \ldots \mapsto\) coordinates of the \(D\)-dimensional Riemannian space \(\mathcal{M}\);
- \(I, J, \ldots \mapsto\) subsets of the finite set \(\Omega\) and antisymmetric forms \(A^I, F^I\);
- \(a, b, \ldots \mapsto\) scalar fields;
- \(i, j, \ldots \mapsto\) subspaces of \(\mathcal{M}\);
- \(m_i, n_i \mapsto\) coordinates in \(\mathcal{M}_i\);
- \(A, B, \ldots\) \mapsto minisuperspace coordinates.

As usual, summing over repeated indices is assumed when one of them is at a lower position and another at an upper one.
2. Initial field model

We start (like [3]) from the action

$$S = \frac{1}{2\kappa^2} \int d^Dz \sqrt{|g|} \left\{ R[g] - \delta_{ab}g^{MN}\partial_M \varphi^a\partial_N \varphi^b - \sum_{I \in \Omega} \frac{\eta_I}{|n_I|} \exp(2\lambda_I \varphi^a)(F^I)^2 \right\} + S_{\text{GH}}, \quad (2.1)$$

in a $D$-dimensional (pseudo-)Riemannian manifold $\mathcal{M}$ with the metric $g = g_{MN}dz^M \otimes dz^N$; $|g| = |\det(g_{MN})|$; $\varphi^a$ are dilatonic scalar fields;

$$F^I = dA^I = \frac{1}{n_I!} F_{M_1...M_n}^I dz^{M_1} \wedge ... \wedge dz^{M_n} \quad (2.2)$$

are $n_I$-forms ($n_I \geq 2$); $\lambda_I$ are coupling constants; $\eta_I = \pm 1$ (to be specified later); furthermore,

$$(F^I)^2 = F_{M_1...M_n}^IF_{N_1...N_m}^I g^{M_1N_1}...g^{M_nN_n} \quad (2.3)$$

$I \in \Omega$, $a \in A$, where $\Omega$ and $A$ are non-empty finite sets. Finally, $S_{\text{GH}}$ is the standard Gibbons-Hawking boundary term [19], essential for a quantum treatment of the model.

The equations of motion corresponding to (2.1) have the form

$$R_{MN} - \frac{1}{2}g_{MN}R = T_{MN}, \quad (2.4)$$

$$\Box \varphi^a = \sum_{I \in \Omega} \frac{\eta_I}{n_I!} \lambda_I e^{2\lambda_I \varphi^a} (F^I)^2, \quad (2.5)$$

$$\nabla_M \left( e^{2\lambda_I \varphi^a} F^{I,M}_{M_2...M_{n_I}} \right) = 0, \quad (2.6)$$

where

$$T_{MN} = \sum_{a \in A} T_{MN}[^{\varphi^a}] + \sum_{I \in \Omega} \frac{\eta_I}{n_I} e^{2\lambda_I \varphi^a} T_{MN}[F^I], \quad (2.7)$$

$$T_{MN}[\varphi^a] = \partial_M \varphi^a \partial_N \varphi^a - \frac{1}{2}g_{MN} \partial_P \varphi^a \partial_P \varphi^a, \quad (2.8)$$

$$T_{MN}[F^I] = \frac{1}{n_I!} \left[ -\frac{1}{2}g_{MN}(F^I)^2 + n_I F^I_{M_2...M_{n_I}} F^I_{N_2...M_{n_I}} \right]. \quad (2.9)$$

Here $\Box$ and $\nabla$ are the Laplace-Beltrami and covariant derivative operators corresponding to $g$, respectively.

We consider the manifold

$$\mathcal{M} = \mathbb{R} \times M_0 \times ... \times M_n \quad (2.10)$$

with the metric

$$g = we^{2\tau(u)} du \otimes du + \sum_{i=0}^n e^{2\phi_i(u)} g^i, \quad (2.11)$$

where $w = \pm 1$, $u$ is a distinguished coordinate which, by convention, will be called “time”;

$$g^i = g^i_{m,n_i}(y_i) dy^m_i \otimes dy^n_i$$

is a metric on $M_i$ satisfying the equation

$$R_{m,n_i}[g^i] = \xi_i g^i_{m,n_i}, \quad (2.12)$$

$m_i, n_i = 1, ... , d_i \ (d_i = \text{dim } M_i); \ \xi_i = \text{const.}$

We assume each manifold $M_i$ to be oriented and connected, $i = 0, ..., n$. Then the volume $d_i$-form

$$\tau_i = \sqrt{|g^i(y_i)|} \ dy^1_i \wedge ... \wedge dy^d_i, \quad (2.13)$$

and the signature parameter

$$\varepsilon(i) = \text{sign det}(g^i_{m,n_i}) = \pm 1 \quad (2.14)$$

are correctly defined for all $i = 0, ..., n$.

Let $\Omega$ in (2.1) be the set of all non-empty subsets of the set of indices $\{0, ..., n\}$, i.e.

$$\Omega = \Omega(0, n) = \{\{0\}, \{1\}, ..., \{n\}, \{0, 1\}, ..., \{0, 1, ..., n\}\} \quad (2.15)$$

The number of elements in $\Omega$ is $|\Omega| = 2^n + 1 - 1$.

For any $I = \{i_1, ..., i_k\} \in \Omega$, $i_1 < ... < i_k$, we put in $2^{23}$

$$A^I = \Phi^I \tau_I \quad (2.16)$$

where

$$\tau_I = \tau_{i_1} \wedge ... \wedge \tau_{i_k}, \quad (2.17)$$

and $\tau_i$ are defined in (2.13). In components Eq. (2.16) reads:

$$A^I_{P_1...P_{d(I)}(u,y)} = \Phi^I(u) \sqrt{|g^{i_1}(y_{i_1})|} \cdots \sqrt{|g^{i_k}(y_{i_k})|} \varepsilon_{P_1...P_{d(I)}}, \quad (2.18)$$

where $\varepsilon_{P_1...P_k}$ is the Levi-Civita symbol and

$$d(I) \equiv d_{i_1} + ... + d_{i_k} = \sum_{I \in I} d_i \quad (2.19)$$

is the dimension of the oriented manifold

$$M_I = M_{i_1} \times ... \times M_{i_k}, \quad (2.20)$$

$(d(\emptyset) = 0)$ and the indices $P_1, ..., P_{d(I)}$ correspond to $M_I$.

It follows from (2.16) that

$$F^I = dA^I = d\Phi^I \wedge \tau_I. \quad (2.21)$$

By construction and according to (2.21),

$$n_I = d(I) + 1, \quad I \in \Omega, \quad (2.22)$$

so that the ranks of the forms $F^I$ are fixed by the manifold decomposition.

For the dilatonic scalar fields we put

$$\varphi^a = \varphi^a(u), \quad a \in A. \quad (2.23)$$

The problem setting as described embraces various classes of models where field variables depend on a single coordinate, such as
(A) cosmological models, both isotropic and anisotropic, where the \( u \) coordinate is timelike, \( w = -1 \), and some of the factor spaces (one in the isotropic case) are identified with the physical space;

(B) static models with various spatial symmetries (spherical, planar, pseudospherical, cylindrical, toroidal) where \( u \) is a spatial coordinate, \( w = +1 \), and time is selected among \( M_i \);

(C) Euclidean models with similar symmetries, or models with a Euclidean “external” space-time; \( w = +1 \).

A simple analysis shows that in all Lorentzian models, in order to have a positive energy density \( T_0^0 \) of the fields \( F^i \), one should choose in (2.4) the solution

\[
\theta = \eta \varepsilon (I) = -w, \quad \varepsilon (I) = \prod_{i \in I} \varepsilon (i),
\]

(2.24)

where, in the case B, it is assumed that \( I \ni i_0 \), where \( M_{i_0} \) contains the time submanifold. Otherwise we simply obtain

\[
\theta = +1.
\]

(2.25)

We also use the following notations for the logarithms of volume factors of the subspaces of \( M \):

\[
\sum_{i=0}^n d_i \phi^i \equiv \sigma_0, \quad \sum_{i \in I} d_i \phi^i \equiv \sigma_I.
\]

(2.26)

3. Sigma model representation

As in [3], we pass to the sigma-model representation. It is easy to verify that the field equations (2.4)-(2.6) for the field configuration (2.11), (2.21), (2.23) may be obtained as the equations of motion corresponding to the action

\[
S_\sigma = \frac{1}{2 \kappa_0^2} \int du e^{\sigma_0 - \gamma} \left\{ -w G_{ij} \dot{\phi}^i \dot{\phi}^j - w \delta_{iab} \dot{\varphi}^a \dot{\varphi}^b - \sum_{i \in \Omega} e^{2\tilde{\lambda}_i} e^{-2\sigma_\gamma} \theta_I (\dot{\phi}^I)^2 - 2 V e^{-2(\sigma_0 - \gamma)} \right\},
\]

(3.1)

where \( \tilde{\varphi} = (\varphi^a) \), \( \tilde{\lambda}_I = (\lambda_{ia}) \), dots denote \( d/du \),

\[
G_{ij} = d_i \delta_{ij} - d_\sigma d_j,
\]

(3.2)

are component of the “pure cosmological” minisuperspace metric and

\[
V = V (\phi) = -\frac{1}{2} \sum_{i=1}^n \xi_i d_i e^{-2\phi^i + 2\sigma_0 (\phi)}
\]

(3.3)

is the potential.

For finite internal space volumes (e.g. compact \( M_i \))

\[
V_i = \int_{M_i} d^d y_i \sqrt{|g_i|} < +\infty,
\]

(3.4)

the action (3.1) coincides with (2.1) if

\[
\kappa^2 = \kappa_0^2 \prod_{i=0}^n V_i.
\]

(3.5)

The representation (3.1) follows as well from a more general \( \sigma \)-model action of [3].

The action (3.1) may be written in the following form corresponding to an arbitrary time gauge:

\[
S_\sigma = \frac{1}{2} \int du \left\{ (-w) NG_{AB} (\zeta) \zeta^A \zeta^B - 2 N^{-1} V (\zeta) \right\}
\]

(3.6)

where \( (\zeta^A) = (\phi^i, \varphi^a) \), \( (3.1) \) is a spatial coordinate,

\[
(\mathcal{G}_{AB}) = \begin{pmatrix}
G_{ij} & 0 \\
0 & \delta_{ab}
\end{pmatrix}
\]

(3.8)

gives the minisuperspace (target space) metric.

Let us, like [20], choose the harmonic \( u \) coordinate (\( u = 0 \)), such that

\[
\gamma = \sigma_0 (\phi) \Rightarrow N = 1.
\]

(3.9)

The minisupernetric \( \mathcal{G}_{AB} d\zeta^A \otimes d\zeta^B \) still remains to be curved. However, the integrability problem is considerably simplified since it is possible to integrate the generalized Maxwell equations:

\[
\frac{d}{du} \left[ e^{2\tilde{\lambda}_i} e^{-2\sigma_\gamma} (\dot{\phi}^I) \right] = 0,
\]

(3.10)

where \( Q_I \) are constants.

Let

\[
Q_I \neq 0, \quad I \in \Omega_*,
\]

(3.11)

where \( \Omega_* \subset \Omega \) is a non-empty subset of \( \Omega \).

For fixed \( Q_I \) the Lagrange equations for \( \dot{\phi}^i \) and \( \varphi^a \), after substitution of (3.10), are equivalent to those for the Lagrangian of a pseudo-Euclidean Toda-like system (see [17, 21, 23]) with a zero-energy constraint:

\[
L_Q = \frac{1}{2} \mathcal{G}_{AB} \dot{x}^A \dot{x}^B - Q_V,
\]

(3.12)

\[
E_Q = \frac{1}{2} \mathcal{G}_{AB} \dot{x}^A + V_Q = 0,
\]

(3.13)

where \( x = (x^A) = (\phi^i, \varphi^a), \ i = 0, \ldots, n; \ a \in A \),

\[
(\mathcal{G}_{AB}) = \begin{pmatrix}
G_{ij} & 0 \\
0 & \delta_{ab}
\end{pmatrix}
\]

(3.14)
and the potential $V_Q$ has the form

$$V_Q = \sum_{j=0}^{n} \left( \frac{w_j}{2} \xi_j d_i \right) e^{2U^i(x)} + \sum_{I \in \Omega_*} \frac{\theta_I}{2} (Q_I)^2 e^{2U^I(x)}, \quad (3.15)$$

where

$$U^i(x) = U_A^i x^A = -\phi^i + \sigma_0, \quad (3.16)$$

$$U^I(x) = U_A^I x^A = \sigma_I - \bar{\lambda}_I \bar{\varphi}, \quad (3.17)$$

or in components

$$(U_A^i) = (-\delta_i^j + d_j, \ 0), \quad (3.18)$$

$$(U_A^I) = (\delta_I d_i, \ -\lambda_I a), \quad (3.19)$$

where

$$\delta_{ij} \overset{\text{def}}{=} \sum_{I \in \Omega} \delta_{ij} \quad (3.20)$$

is an indicator of $i$ belonging to $I$ (1 if $i \in I$ and 0 otherwise).

The constraint (3.13) follows from the $(u)$ component of the Einstein equations (2.4), which contains only first-order derivatives with respect to $u$.

The nondegenerate matrix $\overline{G}_{AB}$ (3.14) defines an $(n + 1 + m^r)$-dimensional minisuperspace metric whose contravariant components form the inverse matrix

$$G^{ij} = \begin{pmatrix} C_{ij} & 0 \\ 0 & \sigma_{ij} \end{pmatrix} \quad (3.21)$$

where, as in (2.4),

$$G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2 - D}, \quad i, j = 0, \ldots, n. \quad (3.22)$$

The integrability of the Lagrange system crucially depends on the scalar products of the vectors $U^i, U^A, U^I$ from (1.16)- (1.17). These products are (by definition, $(U, U') = G^{ij} U_A U'_B$):

$$(U^i, U^j) = \frac{\delta^{ij}}{d_i} - 1, \quad (3.23)$$

$$(U^I, U^I) = -\delta_{I}, \quad (3.24)$$

$$(U^I, U^J) = q(I, J) + \bar{\lambda}_I \bar{\lambda}_J, \quad (3.25)$$

where

$$q(I, J) \equiv d(I \cap J) - \frac{d(I) d(J)}{D - 2} \quad (3.26)$$

The relation (3.23) was found in (22) and (3.25) in (8) ($U_A^i = -L_{AI}$ in the notations of (8)).

4. Solutions for at most one curved factor space

4.1. Solutions with one curved factor space

To solve the field equations, let us adopt the following assumptions:

(i) Among all $M_i$, only $M_0$ has a nonzero curvature, while others are Ricci-flat, i.e., $\xi_i = 0$ for $i > 0$ and (properly normalizing the scale factor $\phi_0$)

$$\xi_0 = (d_0 - 1)K_0, \quad K_0 = \pm 1. \quad (4.1)$$

(ii) Neither of $I \in \Omega_*$ contains the index 0, that is, neither of the $p$-branes under consideration involves the curved subspace $M_0$.

(iii) The vectors $U^I$, $I \in \Omega_*$, are mutually orthogonal with respect to the metric $\overline{G}_{AB}$, that is,

$$d(I \cap J) - \frac{d(I) d(J)}{D - 2} + \bar{\lambda}_I \bar{\lambda}_J = 0 \quad (4.2)$$

for any $I \neq J$.

Thus the potential $V_Q$ (3.13) contains one “timelike” vector $U^0$ with the norm

$$(U^0, U^0) = -(d_0 - 1)/d_0 < 0 \quad (4.3)$$

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and $d_0 > 1$ and $m_+ = [\Omega_*]$ “spacelike” vectors $U^I$ with the norms $1/N_I > 0$:

$$1/N^2_I \overset{\text{def}}{=} (U^I, U^I) = d(I) - 2 - d(I) + (\bar{\lambda}_I)^2 > 0. \quad (4.4)$$

One easily verifies that under the assumptions (i)-(iii) the conditions adopted in the Appendix are fulfilled, with the space $V$ of $(x^A)$ and the scalar product $\langle x, y \rangle = G_{AB} x^A y^B = (x, y)$ defined in the previous section. Therefore we are able to write down the following solution to the field equations:

$$x^A(u) = -\frac{d_0}{d_0 - 1} U^A_0 y_0(u) + \sum_{I \in \Omega_*} N^2_I U^A_0 y_I(u) + c^A u + \varphi^A, \quad (4.5)$$

where $A = i, a$ ($i = 1, \ldots, n; \ a \in A$), the functions $y_0, y_I$ are found from the relations

$$e^{-y_0(u)} = (d_0 - 1) S(\varepsilon_0, h_0, u - u_0), \quad (4.6)$$

the function $S(\varepsilon_0, u)$ is defined in (A.4); the constants $N_I$ in (4.4); recall that $w = \text{sign}\ g_{uu}$. The integration constants $c^A, \varphi^A, h_0, u_0, h_I$ and $u_I$ are connected
by the following relations due to (4.13) and the energy integral (3.13):

\[ c^0(d_0 - 1) + \sum_{i=1}^{n} d_i c^i = 0; \]
\[ \tau^0(d_0 - 1) + \sum_{i=1}^{n} d_i \tau^i = 0; \quad (4.7) \]
\[ \sum_{i \in I} d_i c^i - \lambda_{Ia} c^a = 0; \]
\[ \sum_{i \in I} d_i \tau^i - \lambda_{Ia} \tau^a = 0; \quad I \in \Omega_{a}; \quad (4.8) \]
\[ \frac{d_0}{d_0 - 1} (c^0)^2 + \sum_{j=1}^{n} d_j (c^j)^2 + \delta_{ab} c^a c^b. \quad (4.9) \]

Finally, the contravariant components \( U^{A}_0, U^A_I \) of the vectors \( U^0, U^I \) are found using the metric tensor \( G^{AB} \) \((U^A = G^{AB} U_B)\) presented in (3.21), (3.22). Namely:

\[ U^0_I = -\delta^0_i/d_i, \quad U^0_I = 0, \]
\[ U^I_I = \delta_{Ii} - \frac{d(I)}{D - 2}, \quad U^7_i = -\lambda_{Ia}, \quad (4.10) \]
\[ i = 0, \ldots, n; \quad I \in \Omega_{a}; \quad a \in \mathcal{A}. \]

Thus the logarithms of scale factors in the metric (2.11) and the scalar fields are

\[ \phi^i(u) = \frac{d_i}{d_0 - 1} y_0(u), \]
\[ + \sum_{I \in \Omega_{a, I}} \delta_{Ii} + \frac{d(I)}{D - 2} N^2_I y_I(u) + c^0 u + \tau^0, \quad (4.11) \]
\[ \varphi^a(u) = -\sum_{I \in \Omega_{a, I}} \lambda_{Ia} N^2_I y_I(u) + c^a u + \tau^a \quad (4.12) \]

and the function \( \gamma = \frac{1}{2} \ln |g_{uu}| \equiv \sigma_0 \) is

\[ \gamma(u) = \sum_{i=0}^{n} d_i \phi^i = \frac{d_0}{d_0 - 1} y_0(u), \]
\[ - \sum_{I \in \Omega_{a}} \frac{d(I)}{D - 2} N^2_I y_I(u) + c^0 u + \tau^0, \quad (4.13) \]

Substitution of the relations (4.11)–(4.12) into (3.10) implies

\[ \Phi^I = Q_I e^{2y_I}, \quad (4.14) \]

and hence we get for the forms

\[ F^I = d\Phi^I \wedge \tau_I = Q_I e^{2y_I} d\mu \wedge \tau_I. \quad (4.15) \]

Eqs. (4.11)–(4.15), along with the definitions (4.6), (4.12) and (4.13) and the relations (4.17)–(4.18) among the constants, completely determine the solution. Under the above assumptions (i)–(iii) the solution is general and involves all cases mentioned in items A, B, C at the end of Sec. 2.

The solution describes the behaviour of \((n + 1)\) spaces \((M_0, y_0), \ldots, (M_n, y_n)\), where \((M_0, y_0)\) is an Einstein space of nonzero curvature, and \((M_i, y_i)\) are “internal” Ricci-flat spaces, \(i = 1, \ldots, n\), in the presence of several scalar fields and forms.

The solution also describes a set of charged (by the \(F\) forms) intersecting \(p\)-branes “living” on the manifolds \(M_I\) \((2.20)\), \(I \in \Omega_{a}\), where the set \(\Omega_{a} \subset \{1, 2, \ldots, 1, \ldots, n\}\) does not contain \(0\), i.e. all \(p\)-branes live in internal spaces.

### 4.2. Solutions with Ricci-flat factor spaces

The same solutions can also describe configurations where all \(M_i\) are Ricci-flat (such as spatially flat cosmology) if one just deletes every mentioning of \(M_0\), putting \(y_0\) and the respective constants equal to zero and postulating the decomposition (2.10) in the form \(\mathbb{R} \times M_1 \ldots M_n\).

In this case Eq. (4.11) for \(\phi_i\) \((i = 1, \ldots, n)\) with \(y_0 \equiv 0\) is valid; Eq. (4.14) for \(\varphi^a\) is unchanged; the expression (4.13) for \(\gamma(u)\) with \(y_0 \equiv 0\) is valid if we denote

\[ c^0 \equiv \sum_{i=1}^{n} d_i c^i, \quad \tau^0 \equiv \sum_{i=1}^{n} d_i \tau^i. \quad (4.16) \]

The relations among the constants (4.7) disappear, while (4.8) remain unchanged. The energy constraint may be rewritten using (4.16) in a simplified form:

\[ (c^0)^2 = \sum_{I \in \Omega_{a}} N^2_I h^2_I \gamma(u) + \sum_{i=1}^{n} d_i (c^i)^2 + \delta_{ab} c^a c^b. \quad (4.17) \]

However, in the case of all Ricci-flat factor spaces there can appear forms with \(d(I) = D - 2\) and even \(D - 1\), so that zero or negative norms (4.4) are possible. Then some slight and evident modifications of the scheme are required; we will, however, always assume \(d(I) < D - 2\).

**Remark.** The above solutions evidently do not exhaust all integrable cases. Thus, some of the functions \(y_I\) may coincide, reducing the number of the required orthogonality conditions (4.12) and giving more freedom in choosing the set of input constants \(d(I)\) and \(\lambda_{Ia}\). Such a coincidence of, say, \(y_I\) and \(y_{I'}\) is a constraint on the unknowns and the emerging consistency relation should lead to a reduction of the number of integration constants (e.g. coincidence of charges \(Q_I\) and \(Q_{I'}\)). In other words, it becomes possible to obtain less general solutions but for a more general set of input parameters. An example of such a situation is given in Ref. [24] discussing some spherically symmetric solutions with intersecting electric and magnetic \(p\)-branes.
5. Cosmological solutions

5.1. General features

Among the above solutions, cosmological models correspond to \( w = -1 \). We also assume that the curved subspace \( \mathcal{M}_0 \) (\( K_0 = \pm 1 \)) is the “external” physical space, spherical (\( K = +1 \)) or hyperbolic (\( K = -1 \)). To describe spatially flat models (to be labelled \( K_0 = 0 \)) we assume that in the solution of Subsec. 4.2 (where \( \mathcal{M}_0 \) is absent), one of the subspaces, say, \( M_1 \), is external and is not involved in the \( p \)-branes, i.e., \( 1 \notin I, I \in \Omega_* \).

Due to \( w = -1 \) and positive energy condition (2.24) we immediately obtain

\[
e^{-w} = \frac{|Q|}{h_I N_I^2} \cosh[h_I(u - u_I)], \quad h_I > 0. \tag{5.1}
\]

Another sign factor in (4.4), \( \varepsilon_0 \), is \( \varepsilon_0 = -K_0 \).

The behaviour of the solutions is extremely diverse according to the interplay of the numerous integration constants and input parameters. Thus, different scale factors may be monotone or not, grow to infinity or fall to zero, or tend to constant values. Some general observations can nevertheless be made.

The first observation is that the range of the time coordinate \( u \) is \( \mathbb{R} \) for \( K_0 = 0, \pm 1 \). At both ends of the evolution, \( u \to \pm \infty \), the scale factors \( e^{\phi_i} \) and the function \( g_{uu} = e^{2\gamma} \) behave like

\[
f(u) = (\cosh u)^k e^{qu} \tag{5.2}
\]

where the constants \( k \) and \( c \) are fixed for all \( e^{\phi_i} \) and \( e^{\gamma} \) for each specific set of the input and integration constants.

For spatially flat models, \( K_0 = 0 \), the constant \( k \) is positive for both the “physical” scale factor \( a(u) = e^{\phi_1(u)} \) and \( e^{\gamma} \). This means that \( a(u) \to \infty \) at least one of the asymptotics and so does the cosmic synchronous time \( t = \int e^{\gamma} du \) (but not necessarily at the same asymptotic).

For closed models (\( K_0 = +1 \)) even such small general information cannot be extracted: both \( a(u) \) and \( e^{\gamma} \) behave according to (5.2), but now both constants \( k \) and \( c \) can have arbitrary signs.

One can note that \( f \) can have a finite limit at one end of the evolution (if \( c = \pm k \)), but in the generic case all \( e^{\phi_i(u)} \) and \( t(u) \) have exponential asymptotics with different increments, hence the dependence \( e^{\phi_i(t)} \) has a power-law asymptotic, \( e^{\phi(u)} \sim t^s \). In particular, one can expect \( s \gg 1 \) for some choices of the parameters, i.e., power-law inflation in at least some spatial directions.

For hyperbolic models, \( K_0 = -1 \), we have in (4.9) \( \varepsilon_0 = +1 \) and due to (4.9) \( h_0 > 0 \), so that, with no generality loss, we can write

\[
e^{-\varepsilon_0} = \frac{d_0 - 1}{h_0} \sinh h_0 u \tag{5.3}
\]

and \( u > 0 \). As \( u \to \infty \), all \( e^{\phi_i} \) and \( e^{\gamma} \) behave again like \( f \) in (5.2), with all the above inferences, but, as \( u \to 0 \), all factors like \( e^{qu} \) and \( \cosh k(u - u_0) \) are finite and the model behavior is governed by \( y_0 \). In particular, for the physical dimension \( d_0 = 3 \) we obtain:

\[
e^\gamma \sim u^{-3/2}, \quad a(u) = e^{q_0} \sim 1/\sqrt{u}, \quad t \sim 1/\sqrt{u}. \tag{5.4}
\]

This asymptotic corresponds to infinite linear expansion or contraction in the external space \( (a(t) \sim |t|) \), while all internal scale factors and scalar fields tend to finite limits.

5.2. Some special solutions

Consider the case \( D = 11, \bar{X}_I = 0, \) and \( d_0 = 3 \) (a curved isotropic physical space). From (1.2) we have the following possibilities:

\[
\{d(I), d(J)\} = \{3, 3\}, \{3, 6\}, \{6, 6\} \tag{5.5}
\]

if \( d(I \cap J) = 1, 2, 4 \), respectively, and \( 2N^2_I = 1, I, J \in \Omega_* \). Using the rules (5.5), we can list all possible sets \( \Omega_* \), or collections of intersecting 2-brane and 5-branes. Two variants of maximum possible sets \( \Omega_* \) in the remaining 7 internal dimensions are schematically shown in Fig. 1, where straight lines and circles correspond to possible sets of indices \( I \) and black spots refer to coordinates in the subspaces \( M_i, i > 0 \).

Fig. 1a shows a system of \( n = 7 \) one-dimensional spaces \( M_1 \), so that each black spot depicts both a coordinate and a factor space. Each of the 7 sets \( I \in \Omega_* \) contains 3 elements, i.e. corresponds to a 2-brane.

Fig. 1b shows a system of 2-branes and a 5-brane with \( n = 4: d_0 = 3, d_1 = d_2 = d_3 = 2, d_4 = 1 \). Let us consider this case in more detail. The set \( \Omega_* \), reads:

\[
\Omega_* = \{\{1, 2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\} \tag{5.6}
\]

where the letters \( A, B, C, D \) label the corresponding sets \( I \in \Omega_* \).

Due to the orthogonality relations between \( \varepsilon_i \) and \( \bar{\varepsilon}_i \), all these constants vanish and

\[
3h^2_i = h^2_A + h^2_B + h^2_C + h^2_D. \tag{5.7}
\]

One easily finds that

\[
\begin{align*}
2\phi^0(u) &= y_0 - \Sigma, \\
2\phi^1(u) &= y_A + y_B - \Sigma, \\
2\phi^2(u) &= y_A + y_C - \Sigma, \\
2\phi^3(u) &= y_A + y_D - \Sigma, \\
2\phi^4(u) &= y_B + y_C + y_D - \Sigma, \\
2\gamma(u) &= 3y_0 - \Sigma.
\end{align*} \tag{5.8}
\]

where all \( y(u) \) are defined in (4.9) and

\[
\Sigma \overset{\text{def}}{=} \sum_{I \in \Omega_*} \frac{d(I)}{D - 2} y_I(u) = \frac{2}{3} y_A + \frac{1}{3} (y_B + y_C + y_D). \tag{5.9}
\]
Figure 1: Diagram of maximal sets of intersecting 2-branes. Each I is shown by a straight line or a circle. Enumerated black spots correspond to internal coordinates.

a: Seven 3-dimensional sets $I$. The subspaces $M_i$ ($i=1,\ldots,7$) are 1-dimensional.

b: Three 2-branes and one 5-brane. The subspaces $M_i$ ($i=1,2,3,4$) are described by the coordinates labelled as $\{1,4\}$, $\{2,5\}$, $\{3,6\}$ and $\{7\}$, respectively. The 3-dimensional sets $I=B,C,D$ are shown by straight lines and the 6-dimensional one, $A$, by a circle.

If we assume that $M_0=S^3$, all $y(u)$ are expressed in terms of hyperbolic cosines. Their asymptotics as $u \to \pm \infty$ are

$$y_0 \sim -h_0|u|, \quad y_I \sim -h_Iu.$$  

(5.10)

On the other hand, the logarithms of volume factors of extra dimensions ($\sigma_{\text{extra}}$) and the whole space ($\sigma_0$) are

$$\sigma_{\text{extra}} = \sum_{i=1}^4 d_i \phi^i, \quad \sigma_0 = \gamma = \sum_{i=0}^4 d_i \phi^i.$$  

(5.11)

From $(5.9)$ to $(5.11)$ it follows that, as $u \to \pm \infty$, the volume factor of extra dimensions

$$V_{\text{extra}} = e^{\sigma_{\text{extra}}} \to 0$$  

(5.12)

and, in addition, in view of $(5.7)$,

$$V_{\text{total}} = e^{\sigma_0} = e^\gamma \to 0.$$  

(5.13)

On the other hand, as $\gamma \sim -h_0|u|$ as $u \to \pm \infty$, the integral $t = \int e^{\gamma} du$ converges in the same limits. One has to conclude that in these models both the total volume and that of extra dimensions collapse to zero at some finite times $t$ both in the past and in the future. The physical space, taken separately, may, however, avoid the collapse if one properly chooses the values of $h_I$ (see the first line of $(5.8)$).

All this is true as well for hyperbolic models ($K_0=-1$), but only at one end of the evolution ($u \to \infty$), while at the other ($u \to 0$) the asymptotic is always as described at the end of Subsec. 5.1.

The outlined picture is true for the model with the maximal $\Omega_*$ given in $(5.4)$; non-maximal models, with $\Omega_*$ taken as its subsets, require a separate consideration.

6. Appendix

Let there be an $n$-dimensional real vector space $V$ with a nondegenerate real-valued quadratic form $\langle \ldots \rangle$. Let, further, a real-valued vector function $x(t) \in V$ obey the equations of motion corresponding to the Lagrangian

$$L = \langle \dot{x}, \dot{x} \rangle - \sum_{s=1}^m A_s e^{2\langle b_s, x \rangle},$$  

(A.1)

where $m \leq n$, $A_s = \text{const} \neq 0$ and $b_s \in V$ are constant vectors such that

$$\langle b_s, b_s \rangle \neq 0, \quad \langle b_s, b_p \rangle = 0, \quad s \neq p.$$  

(A.2)

Then, the equations of motion for the Lagrangian $(A.1)$ have the following solutions $(A.3)$:

$$x(t) = \sum_{s=1}^m \frac{b_s}{\langle b_s, b_s \rangle} y_s(t) + tc + c_0,$$  

(A.3)

where $c, c_0 \in V$ (integration constants) satisfy the orthogonality relations

$$\langle c, b_s \rangle = \langle c_0, b_s \rangle = 0,$$  

(A.4)

and the functions $y_s(t) \equiv \langle b_s, x \rangle$ are found from the decoupled equations

$$\ddot{y}_s = -A_s \langle b_s, b_s \rangle e^{2y_s}.$$  

(A.5)

Their solutions may be in turn presented in the form

$$e^{-2y_s(t)} = |A_s(b_s, b_s)| S^2(\varepsilon_s, h_s, t - t_s),$$  

(A.6)

where $h_s, t_s$ are integration constants,

$$\varepsilon_s = -\text{sign} [A_s(b_s, b_s)]$$  

(A.7)

and the function $S(\ldots, \ldots)$ is, by definition,

$$S(1, h, t) = \begin{cases} h^{-1} \sinh ht, & h > 0, \\ t, & h = 0, \\ h^{-1} \sin ht, & h < 0; \end{cases}$$  

$$S(-1, h, t) = h^{-1} \cosh ht; \quad h > 0.$$  

(A.8)

The conserved energy corresponding to the solution $(A.3)$ is

$$E = \langle \dot{x}, \dot{x} \rangle + \sum_{s=1}^m A_s e^{2y_s}$$  

$$= \sum_{s=1}^m \frac{h_s^2 \text{sign} h_s}{\langle b_s, b_s \rangle} + \langle c, c \rangle.$$  

(A.9)
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