Statistical distributions for prediction of stress intensity factor using the bootstrap S-version finite element model

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Abstract. Stress intensity factor (SIF) is one of the most fundamental and useful parameters in all of fracture mechanics. The SIF describes the stress state at a crack tip, is related to the rate of crack growth, and used to establish failure criteria due to fracture. The SIF is determined to define whether the crack will grow or not. The aims of this paper is to examine the best sampling statistical distributions in SIF analysis along the crack front of a structure. Box-Muller transformation is used to generate the statistical distributions which is in normal and lognormal distributions. This method transformed from the random number of the variables within range zero and one. The SIFs are computed using the virtual crack-closure method (VCCM) in bootstrap S-version finite element model (BootstrapS-FEM). The normal and lognormal distributions are represented in 95% of confidence bounds from the one hundred of random samples. The prediction of SIFs are verified with Newman-Raju solution and deterministic S-FEM in 95% of confidence bounds. The prediction of SIFs by BootstrapS-FEM in different statistical distribution are accepted because of the Newman-Raju solution is located in between the 95% confidence bounds. Thus, the lognormal distribution for SIFs prediction is more acceptable between normal distributions.

Keywords. Stress intensity factor; Statistical distributions; Box-Muller transformation; Random samples; Regression analysis

1. Introduction
Defects on the material are essential to investigate because it caused the catastrophic failure. It became critical points for the fatigue strength and lifetime of the materials component. Defects have potential sites for crack initiation from the surface crack growth. Thus, it is caused failure of the component structure. The stress concentration is occurred at the surface defects that leads to inhomogeneous stress field. Nevertheless, the endurance limit of the materials is explored from fatigue tests with un-notched materials specimen. In this condition, the stress concentration is inhomogeneous stress field with critical cross section [1]. The SIF is important to define because it can affected the crack growth propagation. Fracture mechanics is introduced SIF parameter in crack growth problems.

Fracture mechanics is used to study of the propagation of crack in the materials structures. The investigation of crack propagation is useful based on the fracture mechanics. It can evaluate the long
term performance by using the different materials [2, 3]. The method from fracture mechanics is used with several theoretical and parametric applications. Ability of this method is to predict the influence of different mechanical, geometrical and microstructural parameters in its definition [4]. The failure time is obtained through integration of crack speed. By using different fracture mechanics test types, it is give more affected when compared with environment performance on an initiation time or crack speed basis. The experiment was conducted to investigate the lifetime based on fracture mechanics. The data from it is analysed in terms of the SIF at the crack tip for any given crack size. Linear elastic fracture mechanics (LEFM) is assumed that the material in isotropic and linear elastic. The stress field near the crack tip is computed based on the theory of elasticity.

LEFM is the concept or theory in the all energy dissipation that is related with the fracture process. It is showed that the deformation is occurred in linear elastic region but not in plasticity region. SIF is defined as a function applied stress, specimen geometry and crack length. However, the LEFM concept is becomes invalid when the size of plastic zone at the crack tip becomes large compared to the crack length or the remaining life [5]. LEFM supposed that a linear elastic body is consisted a sharp crack. Thus, the LEFM is described about the energy change that occurs in the linear elastic body can go through a large or increase in crack area. It is means that the fracture increases as the crack size grows to propagate the crack [6]. The virtual crack closure method (VCCM) is used to calculate the SIF based on the LEFM.

The VCCM is suitable for the applications with $p$-version of the finite element method (FEM) to compute mixed mode energy release rates. The method provides one global total energy release rate as global forces on a structural level are multiplied with global deformations. This way is calculate the energy available to advance the crack. The total energy release rate is computed locally at the crack front. An additional computation, the stiffness matrix of the elements is involved in calculation that affected by the virtual crack extension. The method is yields the total energy release rate as a function of the direction in which the crack was extended virtually. The yielding information depends on the growth direction [7]. The FEM software is complicated to develop that depends on the theory, implementation aspect, computational experience and engineering. It is related with computer hardware and manpower cost to solve by engineering community [8, 9]. The FEM is applied to LEFM to generated suitable mesh for crack model in two-dimensional or in multiple cracks. The interaction integrals is formulated which to applied in different types of the materials [10].

The existing integrals infinite element is used for linear hyperbolic problems including simple element such as triangular and quadrilateral element in two-dimensional. FEM is allowed to use the general polygonal or polyhedral meshes which helpful features in adaptive mesh refinements. FEM is come up with a symmetric and positive definite system [11]. The singular stress field is created by refining mesh at the crack tip or using special types of elements such as quarter point elements [12]. A suitable mesh is handled by advanced re-meshing algorithms. The fine mesh is used at the vicinity of the crack tip and crack front so that the singular stress field is determined accurately [11]. The extended version is applied to improve the quality of FEM calculations by using the existing adaptive techniques such as $h$-version and $p$-version. The combination of two methods are produced the $S$-version method which is increased a polynomial order and finer mesh.

By using the $S$-version, the problems are solved significantly and give more impact in program of engineering development. Especially, the program STRIPE (Aeronautical Research Institute of Sweden), Applied Structure (Rasna Corp., California, USA), PHLEX (Computational Mechanics, Texas, USA) and MSC/PROBE (MacNeal Schwendler, California, USA) [8]. The mathematical theorem is important to conduct the software in two and three dimensional model [13, 14]. There are many major commercial based on the $h$-version of FEM such as MSC/ANASTRAN, ADINA, ANSYS and etc. There are only two of commercial programs such as FIESTA and MSC/PROBE based on $p$-version and $hp$-version [15]. Thus, SIF is evaluated along the crack front. The calculation of SIF is required great computational effort due to geometry modeling of the crack growth propagation. Suga, Kikuchi [16] is developed combination of S-FEM and auto-meshing method to simulate the fatigue
crack growth. Probabilistic method is useful to predict the SIF in hundreds random of samples. It is avoided from the scatter prediction of SIFs.

The probabilistic is defined the input parameters as distributions and predict the output of distributions and bounds of performance [17]. The types of distributions are normally used such as normal, lognormal, Weibull distributions and etc. The data of the crack growth can used to estimate in the types of distributions [18, 19]. In the commercial software, the probabilistic simulation is widely used in the ABAQUS software. The distribution is randomly generated by using the distribution function based on the determined parameters [10, 17]. Thus, the probabilistic method is widely applied in the engineering studies.

See that failure by fracture include the growth of cracks it created that monitoring the size of a crack in a specific structure give a method for evaluating quantitatively the strength before failure completely. The stress analysis was carried out from the BootstrapS-FEM and the parameters of fracture mechanics were computed. The prediction of SIFs are simulated in BootstrapS-FEM based on the statistical distribution between normal and lognormal distributions. The SIFs are compared between numerical Newman-Raju solution and deterministic S-FEM.

2. Methodology
This section is explaining the principle of generating normal and lognormal distributions using Box-Muller transform algorithm. The normal and lognormal distributions are developed from the random samples using the Box Muller transformation method. The SIFs value is predicted by BootstrapS-FEM for one hundred of random samples. The SIF and energy release rate are the parameters that will figured out the crack growth in LEFM. The fracture parameters are predicted by using VCCM method.

2.1. Box-Muller transform algorithm
Box and Muller [20] proposed another efficient transform algorithm. This method is generated two independent samples from a standard normal distribution. Their proposed Box-Muller transformation algorithm is competent to sample bivariate standard normal random variables, \( Z_j \sim N(z_j;0,1) \) and \( Z_k \sim N(z_k;0,1) \) where \( Z_j \) and \( Z_k \) are two independent random variables. Suppose that \( X_j \sim U(x_j;0,1) \) and \( X_k \sim U(x_k;0,1) \) are two independent uniform random variables with \( X_j \in [x_j,1] \) and \( X_k \in [x_k,1] \). In addition, suppose that the relationship between \( x_j, x_k, z_j \) and \( x_k \) can be expressed as

\[
    x_j = \exp\left(\frac{z_j^2 + z_k^2}{2}\right)
\]

\[
    x_k = \frac{1}{2\pi} \tan^{-1}\left(\frac{z_k}{z_j}\right) \tag{2}
\]

By applying the change-of-variable techniques, the joint probability density function (PDF) corresponds to two independent standard normal random variables, \( f(z_j, z_k) \) is resulted as follows.

\[
    f(z_j, z_k) = f(x_j(z_j, z_k), x_k(z_j, z_k)) \left| \frac{\partial(x_j, x_k)}{\partial(z_j, z_k)} \right|
\]

The cumulative distribution function is
The idea behind the Box-Muller transform is to imagine two independent samples $Z_j, Z_k \sim N(0,1)$ are plotted in the Cartesian plane as shown in figure 1. This is represented as a polar coordinates which it needed the distance, $R$ between $(Z_j, Z_k)$ and the origin along with the angle, $\theta$ in x-axis.

![Figure 1. Polar form in the Cartesian plane.](image)

The equation is presented from the origin, $R = \sqrt{Z_j^2 + Z_k^2}$ and simplified to $R^2 = Z_j^2 + Z_k^2$. The polar coordinates for two independent standard normal by converting back to Cartesian as

\begin{align}
Z_j &= R \cos \theta \\
Z_k &= R \sin \theta
\end{align}

(5)

(6)

where $R = \sqrt{-2 \log x_j}$ and $\theta = 2\pi x_k$.

Equation (4) consolidated that the equation (2) and equation (3) is competent to transform variables $X_j$ and $X_k$ into $Z_j$ and $Z_k$ respectively. By manipulating equation (2) and equation (3), both equations can be rewritten based on equation (5) and equation (6) as

\begin{align}
z_j &= \sqrt{-2 \log(x_j)} \cos(2\pi x_j) \\
z_k &= \sqrt{-2 \log(x_k)} \cos(2\pi x_k)
\end{align}

(7)

(8)

Where $\partial()$ represents the partial derivative function and $\log()$ represents the natural logarithm function. Since this study merely focused on univariate random variable, therefore the efficiency of equation (7) and equation (8) of the transformation algorithms are evaluated, respectively.
2.2. Transformation a standard normal random variable into normal and lognormal random variable

Based on table 1, the mean and standard deviation for the crack aspect ratio are used in transform normal and lognormal as $(a/c) \sim N(\mu, \sigma^2)$. Therefore, transforming the standard normal random variables into a normal random variables is indeed much needed. The normal random variables is resulted when the rules for transforming normal random variables is applied, including scalar multiplication and adding a constant. In mathematics,

$$ (a / c)_N = \mu + z\sigma \quad (9) $$

Based on the probability theory, the lognormal random variables are resulted from taking exponential of equation (9), such that

$$ (a / c)_L = \exp(z) \quad (10) $$

2.3. Virtual crack closure method (VCCM)

The SIFs are calculated based on the energy release rate, $G$ using the VCCM. The equation (11) can be expressed in terms of the SIFs of the areas of $S_1$ and $S_2$, as shown in figure 2, by

$$ G_i = \frac{K_i}{E} = \frac{1}{2 \left( S_1 - \frac{1}{4} (S_1 + S_2) \right)} \sum_{i=1}^{5} v_i^i P_i^3 \quad (11) $$

The energy release rate for the remaining failure modes are expressed by

$$ G_{ii} = \frac{K_{ii}}{E} = \frac{1}{2 \left( S_1 - \frac{1}{4} (S_1 + S_2) \right)} \sum_{i=1}^{5} v_i^i P_i^j \quad (12) $$
\[ G_{\text{III}} = \frac{K_{\text{III}}^2}{2\mu} = \frac{1}{2\mu} \sum_{i=1}^{5} \nu_i^2 P_i^2 \]  

(13)

Where \( \mu \) is the shear modulus. Each component of the energy release rate is represented by a subscript at \( G \), whereby the sum of \( G_I, G_{II} \) and \( G_{III} \) produces \( G_{Total} \). The energy release rate can be changed to the SIF, as shown in equation (11), (12) and (13). For further details of derivation of element arrangement at crack front can be referred to Okada, Higashi [21].

3. Results and discussion

The BootstrapS-FEM generates the normal and lognormal distribution for one hundred of samples. The value of SIFs are computed based on the equation (11) for mode I crack growth propagation. Figure 3 shows the three-dimensional model that subjected to tension load with semi-elliptical crack shape. The model was used for verification of SIF between BootstrapS-FEM and Newman and Raju [22]. The comparison of SIF was represented in two different distributions which is normal and lognormal distribution.

![Figure 3. Comparison of normalised SIFs along the crack front.](image)

| Crack depth, \( a \) (mm) | Crack length, \( c \) (mm) | Crack shape aspect ratio, \( a/c \) | Crack size aspect ratio, \( \alpha/t \) | Model width aspect ratio, \( c/b \) | Tension load, MPa |
|---------------------------|---------------------------|-------------------------------|---------------------------|-------------------------------|-----------------|
| 1                         | 2.5                       | 0.4                           | 0.01                      | 0.2                           | 0.1             | 10              |

Table 1. Details parameter of the model.
The model was considered to show the ability of the BootstrapS-FEM in producing SIF values. Thus, the prediction of SIFs results are compared with numerical method Newman-Raju solution. The parameter details of the model is shown in table 1.

Figure 4 shows the comparison of the normalised SIFs along the crack front for the tension model. The SIF curve that constructed using the BootstrapS-FEM with the numerical solution by Newman and Raju [22] and deterministic S-FEM. The BootstrapS-FEM was generated in normal and lognormal distribution. Two means of BootstrapS-FEM was represented in figure 4 for distinguish the distributions. The means of the BootstrapS-FEM were constructed build upon a 100 samples.

![Comparison of normalised SIFs along the crack front.](image)

Figure 4. Comparison of normalised SIFs along the crack front.

Figure 5 shows the results of the normalized SIFs by the Newman-Raju solution and mean BootstrapS-FEM in normal distribution. The bounds in figure 3 had a 95% confidence interval based on the maximum and minimum 100 samples of SIFs. The 95% of confidence bounds are the range within which 95% of the result of BootstrapS-FEM in normal distribution can be defined. The SIFs of Newman–Raju solution was located inside the 95% confidence bounds of upper limit and lower limit for mean BootstrapS-FEM (Normal). This was significantly agreed with the Newman-Raju solution because the points falls within the range of 95% bounds by BootstrapS-FEM. Furthermore, the prediction of SIFs by BootstrapS-FEM (Normal) examined valid based on the 95% confidence bounds.

The mean BootstrapS-FEM of normalized SIFs for lognormal statistical distribution are shown in figure 6 including the 95% confidence bounds. The results are compared with Newman-Raju solution for validation process as a prediction. The comparison shows that the mean of BootstrapS-FEM (Lognormal) slightly contrasting between numerical Newman-Raju solution. The mean of BootstrapS-FEM (Lognormal) is more closer than BootstrapS-FEM (Normal) towards Newman-Raju solution. The SIFs of Newman-Raju solution is discovered inside the 95% confidence bounds of the mean BootstrapS-FEM (Lognormal). Thus, the estimation of SIFs by the mean BootstrapS-FEM is agreed with the Newman-Raju solution.
Figure 5. Normalised SIFs along the crack front for normal distribution with 95% upper and lower bound.

Figure 6. Normalised SIFs along the crack front for lognormal distribution with 95% upper and lower bound.
Based on the figure 5 and figure 6, the 95% confidence bounds of the BootstrapS-FEM (Lognormal) is narrower than the BootstrapS-FEM (Normal) which is slightly wider. As a consequence, boths BootstrapS-FEM of SIFs predictions are valid because the numerical Newman-Raju falls in range of 95% confidence bounds.

The prediction of SIFs by BootstrapS-FEM in normal and lognormal distributions are validated based on the figure 4 respectively. The prediction of SIFs by BootstrapS-FEM in different distribution shows good agreement with the Newman-Raju numerical solution and deterministic S-FEM.

4. Conclusions

The SIFs is the main parameter in analysis process since it will affects the remaining life of structures. Thus, the prediction of SIFs by BootstrapS-FEM is useful for statistical analysis. This paper used normal and lognormal distribution to predict the SIFs along the crack front. The result for both distributions are analysed and validated by Newman-Raju solution and deterministic S-FEM. The Bootstrap (Normal) and BootstrapS-FEM (Lognormal) are considered valid based on the 95% confidence bounds. The Newman-Raju solution is indicated in between the 95% confidence bounds. Furthermore, the best distribution by BootstrapS-FEM is lognormal distribution. The lognormal distribution is more accurate than normal distributions around one percent error differences against Newman-Raju solution.

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