Non-Universal Critical Behaviour of Two-Dimensional Ising Systems

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Abstract

Two conditions are derived for Ising models to show non-universal critical behaviour, namely conditions concerning 1) logarithmic singularity of the specific heat and 2) degeneracy of the ground state. These conditions are satisfied with the eight-vertex model, the Ashkin-Teller model, some Ising models with short- or long-range interactions and even Ising systems without the translational or the rotational invariance.
§ 1 Introduction

The universality of critical exponents is one of the most important concepts in critical phenomena. According to this universality hypothesis, critical exponents depend only upon the dimensionality, the symmetry and the interaction range of Hamiltonians, namely they are independent on the details of the relevant Hamiltonian such as the strength of the interactions in ordinary situations.

It is quite interesting to study in what condition this hypothesis is violated from the form of the Hamiltonian.

The first example violating the universality hypothesis is the eight-vertex model solved by Baxter\cite{1}. This model can be mapped\cite{2}\cite{3} onto the two-layered square-lattice Ising model in which two layers interact each other via a four-body interaction $J_4$. The critical exponents of this model vary with the parameter $\mu$ which is a function of interaction energies. The exponents are obtained\cite{1} as $\alpha = 2 - \pi/\mu$, $\beta = \pi/16\mu$, $\nu = \pi/2\mu$, and the scaling hypothesis or the weak universality hypothesis\cite{4} insists that $\gamma = 7\nu/4 = 7\pi/8\mu$, $\delta = 15$, $\eta = 1/4$. Kadanoff and Wegner\cite{3} have shown that the existence of marginal operators is a necessary condition for appearance of continuously varying critical exponents. Kadanoff and Brown\cite{5} have shown that the long-range behaviour of the correlations of the eight-vertex and the Ashkin-Teller models are asymptotically the same as those of the Gaussian model.

There exists another model which consists of short-range two-body interactions and which is believed to have continuously varying critical exponents. The $s = 1/2$ square-lattice Ising model with the nearest-neighbour interaction $J$ and the next-nearest neighbour interaction $J'$ have been studied by many authors\cite{6}-\cite{12} to obtain the phase diagram. The numerical calculations of van Leeuwen,\cite{13} Nightingale\cite{14} and Swendsen and Krinsky\cite{15} are the first to show non-universal critical behaviour of this model. The singular part of the free energy is calculated perturbatively by Barber\cite{16}. The results of the high temperature expansion by Oitmaa\cite{17} agrees with these calculations. The coherent-anomaly method (CAM) is applied\cite{18}-\cite{22} to this model and the continuously varying critical exponents are estimated with errors smaller than $\sim 1\%\cite{21}\cite{22}$. The degeneracy of the ground state energy and the existence of a multi-component order parameter have been studied by Jüngling\cite{23} and by Krinsky and Muhamel\cite{24}. A more general Hamiltonian has been investigated by the present authors\cite{22} which includes the
above two models as special cases of it and which has continuously varying 
critical exponents.

In the present paper, we phenomenologically derive a sufficient condition 
to have continuously varying critical exponents. This study is the general- 
ization of the argument reported in ref.[25]. Our condition is satisfied in the 
eight-vertex model, the Ashkin-Teller model and the $s = 1/2$ square-lattice Ising model with the next-nearest-neighbour interaction. Our condition is 
also satisfied with some systems including long-range interactions and with 
some systems without the translational or the rotational invariance.

Brief explanations of the relevant models are given in section 2, together 
with some explicit conditions on continuously varying critical exponents. A 
phenomenological perturbation scheme[26] is explained in section 3 and ap- 
plicated to the eight-vertex model in order to demonstrate that our scheme 
provides the exact first-order derivative of the critical temperature and the 
exponent $\gamma$ of the eight-vertex model in section 4. The temperature depen- 
dence of correlation functions and the characteristic cancellations of interaction energies at the ground state are discussed in section 5. The symmetry 
of the relevant model is studied in section 6. The perturbational scheme 
explained in section 3 is applied, in section 7, to the model defined in section 
2. It is derived that there exist finite first- or second-order derivatives (with 
respect to the interaction energies) of the critical exponent $\gamma$ and hence it 
varies continuously as a function of interaction energies. Finally, the condition 
in section 2 is generalized to include a more general type of interactions in 
section 8.

§ 2 Models

Let us consider the following Hamiltonian

$$\mathcal{H} = \sum_k \mathcal{H}_k + \sum_{kl} \mathcal{H}_{kl}$$  \hspace{1cm} (2.1)

where $\{\mathcal{H}_k\}$ is a finite set of two-dimensional Ising systems. In the present 
paper, we derive that the model with the Hamiltonian (2.1) have continuously 
varying critical exponents provided it satisfies the following two conditions 
1) and 2).
1) The specific heat of \( \mathcal{H}_k \) shows the logarithmic singularity at the critical temperature \( T_c \) which is independent on \( k \).

2) The ground state energy of the total Hamiltonian \( \mathcal{H} \) is invariant for the spin inversion of each \( \mathcal{H}_k \). Here \( \mathcal{H}_{kl} \) is written as

\[
\mathcal{H}_{kl} = -J_{kl} \sum_i O^{(k)}_i O^{(l)}_i,
\]

where \( O^{(k)}_i \) and \( O^{(l)}_i \) are the \( n^{(k)} \)-body and \( n^{(l)} \)-body spin product of spins belonging to \( \mathcal{H}_k \) and \( \mathcal{H}_l \), respectively.

The logarithmic singularity in the condition 1) results in the temperature dependence of the correlation functions of the form

\[
\langle s_i s_j \rangle = c_0 + c_1 \epsilon \ln \epsilon,
\]

where \( \epsilon = (T - T_c)/T_c \) and \( c_0 \) and \( c_1 \) are some constants. This formula is derived in section 5.

The restriction on \( O^{(k)}_i \) and \( O^{(l)}_i \) in the condition 2) can be partially removed and more complicated interactions are permitted, i.e. \( n^{(k)} \) and \( n^{(l)} \) can depend on the region \( \mathcal{R}_m \) in (5.6). This generalization is explained in section 8.

§ 3 Perturbational scheme

To derive non-universal critical behaviour of these models, let us consider the perturbational expansion[26] with respect to interaction energies. The susceptibility \( \chi \) is assumed to behave as

\[
\chi \sim \epsilon(J)^{-\gamma(J)}, \quad \epsilon(J) = (T - T_c(J))/T_c(J),
\]

where \( J \) is an interaction energy. Differentiating it with respect to \( J \), we obtain

\[
\left( \frac{\partial \chi}{\partial J} \right)_{J=0} \simeq \chi_0 [ -\gamma(0) \frac{1}{\epsilon(0)} \left( \frac{\partial \epsilon}{\partial J} \right)_0 - \left( \frac{\partial \gamma}{\partial J} \right)_0 \log \epsilon(0) ],
\]

where the subscript 0 denotes \( J = 0 \). Hence we can obtain \( (\partial \gamma/\partial J)_0 \) from the coefficient of the temperature dependence \( \chi_0 \log \epsilon \). On the other hand, the susceptibility is expressed by the two-spin correlation functions in the form

\[
\chi = \beta \mu_B^2 \sum_{i\alpha j\beta} g_{ij\alpha\beta} \langle s_i s_j \rangle,
\]
where $\beta = 1/k_B T$, $\mu_B$ is the Bohr magneton and $\{g_{i0j0}\}$ denote the signs coming from the emerging order. We differentiate (3.3) and estimate $(\partial \gamma/\partial J)_0$ by comparing with (3.2). The existence of a finite and non-vanishing derivative of the exponent $\gamma$ is an evidence for continuously varying critical exponents to appear.

§ 4 Example

Here we give two important examples. Let us consider the zero-field eight-vertex model. The Hamiltonian $H_{8V}$ of this model is written as $H_{8V} = H_1 + H_2 + H_{12}$, where $H_1$ and $H_2$ denote the following Hamiltonians

$$H_1 = -J' \sum_{i+j=\text{even}} (s_{ij}s_{i+1j+1} + s_{ij}s_{i+1j-1}),$$

$$H_2 = -J' \sum_{i+j=\text{odd}} (s_{ij}s_{i+1j+1} + s_{ij}s_{i+1j-1}),$$

(4.1)

and $H_{12}$ is

$$H_{12} = -J_4 \sum s_{ij}s_{i+1j+1}s_{i+1j}s_{ij+1}.$$  (4.2)

This model decouples into the two square-lattice Ising models $H_1$ and $H_2$ when $J_4 = 0$. The interaction $H_{12}$ has even symmetry for the spin inversion of each subsystem $H_1$ and $H_2$ and hence the energy of $H_{8V}$ is four-fold degenerated in the whole temperature region. This model obviously satisfies the conditions 1) and 2) in section 2. The weight $g_{i0j0}$ equal 1 for $J' > 0$. Differentiating (3.3) with respect to $J_4$ and using the fact that the model decouples into two layers when $J_4 = 0$, we obtain the following expression

$$(\partial \chi/\partial J_4)_{J_4=0} = \beta^2 \mu_B^2 \sum_{i0j0ijkl} \langle s_{ij} s_{kl}s_{k+l+1} \rangle_0 - \langle s_{ij} s_{ij} \rangle_0 \langle s_{kl} s_{k+l+1} \rangle_0 \langle s_{kl+1} s_{k+l} \rangle_0,$$

(4.3)

where $\langle \rangle_0$ denotes the expectation value for $J_4 = 0$. This expression coincides with

$$(\partial \chi/\partial J')_{J_4=0} \times \omega_0 = \frac{\chi_0}{e(0)} \frac{\gamma(0)}{J'} \omega_0.$$  (4.4)
Here $\omega_0$ denotes the nearest-neighbour two-spin correlation function which shows the following behaviour

$$\omega_0 \simeq \frac{1}{\sqrt{2}} + \frac{4J'}{\pi k_B T_c(0)} \epsilon(0) \log \epsilon(0)$$

(4.5)

at the critical point. Comparing (4.4) and (4.5) with (3.2), we obtain

$$\left(\frac{\partial T_c}{\partial J_4}\right)_{J_4=0} = \frac{T_c(0)}{\sqrt{2}J'}$$

and

$$\left(\frac{\partial \gamma}{\partial J_4}\right)_{J_4=0} = -\frac{4\gamma(0)}{\pi k_B T_c(0)},$$

(4.6)

which are identical with the derivatives obtained from the exact result by Baxter.

The square-lattice Ising model with antiferromagnetic next-nearest-neighbour interactions is another example. The Hamiltonian of this model is given in the form $\mathcal{H}_{SAF} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_{12}$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are given by (4.1) for $J' < 0$ and $\mathcal{H}_{12}$ is

$$\mathcal{H}_{12} = -J \sum_{ij} (s_{ij}s_{i+1j} + s_{ij}s_{ij+1}).$$

(4.7)

The ground state of this model is ordered as the Néel state in each sublattice for the interaction region $|J/J'| < 2$, in which the ground state energy is invariant for the spin inversion of each sublattice. It is in this interaction region $|J/J'| < 2$ that this model is considered to have continuously varying critical exponents. This model also satisfies the condition in section 2. It is derived in ref.[16] and [25] that this model has continuously varying critical exponents with the derivatives $(\partial \gamma / \partial J)_0 = 0$ and $(\partial^2 \gamma / \partial J^2)_0 \neq 0$.

§ 5 Preliminary formulas

The temperature dependence of correlation functions is specified from the condition 1). For the purpose to obtain multi-spin correlation functions, let us consider $\mathcal{H}_k = \mathcal{H}_k - \tilde{J}\mathcal{O}$, where $\mathcal{O}$ denotes a certain spin product. The free energy $\tilde{f}_{ks}$ of this Hamiltonian is differentiable with respect to $\tilde{J}$. Assuming that $\tilde{f}_{ks}$ shows logarithmic or power-law behaviour for $\tilde{J} = 0$, it should have the form

$$\tilde{f}_{ks} = C_1 \epsilon^{2-\alpha_1} \log \epsilon + C_2 \epsilon^{2} \frac{1-\epsilon^{-\alpha_2}}{\alpha_2},$$

(5.1)
where $C_1$ and $C_2$ are some constants, $\alpha_1$ and $\alpha_2$ are functions of $\tilde{J}$ with $\alpha_1(\tilde{J} = 0) = 0$ and $\alpha_2(\tilde{J}) \to 0$ for $\tilde{J} \to 0$. The second term converges to $C_2 \epsilon^2 \log \epsilon$ when $\tilde{J} \to 0$. From (5.1),

$$\langle O \rangle = \frac{1}{\beta} \left( \frac{\partial \tilde{f}_{ks}}{\partial \tilde{J}} \right)_{\tilde{J}=0} \propto 2\epsilon \left( \frac{\partial \epsilon}{\partial \tilde{J}} \right)_{0} \log \epsilon + O(\epsilon),$$

(5.2)

where contributions vanishing faster than $\epsilon \log \epsilon$ (for $\epsilon \to 0$) are included in $O(\epsilon)$. The regular part of the free energy yields a certain constant term. Then we generally obtain the temperature dependence of correlation functions in the form

$$\langle O \rangle \cong c_0 + c_1 \epsilon \log \epsilon.$$  

(5.3)

The condition 2) results in the strong cancellations of interaction energies at the ground state. Here we introduce notations for the ground-state configuration of the system as $\{g_i\}$ and the spin product $O_i = s_{i_1} \cdots s_{i_n}$ in $H_{kl}$ at the ground state as $G_i = g_{i_1} \cdots g_{i_n}$. The $J_{kl}$-dependent part of the ground state energy, $\epsilon_G(J_{kl})$, is written as

$$\epsilon_G(J_{kl}) = -J_{kl} \sum_i G_i.$$  

(5.4)

In the case where $\sum_i O_i$ has even symmetry for the spin inversion of $H_k$ and $H_l$ (i.e. both $n^{(k)}$ and $n^{(l)}$ are even), the condition 2) is automatically satisfied. Otherwise, the condition 2) yields

$$-J_{kl} \sum_i G_i = -J_{kl} \sum_i (-G_i) \text{ that is } \sum_i G_i = 0.$$  

(5.5)

We exclude, from our arguments, the case that the condition 2) is asymptotically satisfied only in the thermodynamic limit. Then we can rewrite the condition (5.5) as

$$\sum_i G_i = \sum_m \sum_{i \in R_m} G_i, \text{ and } \sum_{i \in R_m} G_i = 0,$$  

(5.6)

where $\{R_m\}$ denote a set of finite regions containing a finite number of spins and the symbol $i \in R_m$ denotes that all the spins in $G_i$ are included in $R_m$. 

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§ 6 Symmetries and the vanishing derivatives

6.1 Basic symmetries

In this section, we derive some properties which depend only on the symmetry of the relevant model. We consider here the case $N = 2$, i.e., the following Hamiltonian $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_{12}$ for $\epsilon > 0$.

We assume that $\mathcal{H}_{12}$ has odd symmetry for the spin invergion of $\mathcal{H}_2$. Then the interaction $\mathcal{H}_{12}$ has the form

$$\mathcal{H}_{12} = -J \sum_i \mathcal{O}_i^{(1)} \mathcal{O}_i^{(2)},$$

where $\mathcal{O}_i^{(2)}$ changes its sign for the spin inversion of $\mathcal{H}_2$. Hereafter we omit the subscript of $J_{12}$ and write $J_{12}$ as $J$ for simplicity.

Let us consider the transformation of spin configurations in which all the spins on $\mathcal{H}_2$ are reversed and the spins on $\mathcal{H}_1$ are unchanged. The energy of the model $\mathcal{H}$ with the interaction $J$ for the spin configuration $C$ equals to the energy of the model $\mathcal{H}$ with the interaction $-J$ for the transformed spin configuration $C'$. As a result, we can find one-to-one correspondence of the Boltzmann factor coming from the configuration $C$ and $C'$. Summing over all configurations, we find that the partition functions $Z(J)$ and $Z(-J)$ are the same

$$Z(J) = Z(-J).$$

Since we consider the partition function without an external field, we obtain from (6.2) for the critical temperature $T_c(J)$ and the exponent $\alpha(J)$ of the specific heat as

$$T_c(J) = T_c(-J) \quad \text{and} \quad \alpha(J) = \alpha(-J).$$

From (6.3) we can conclude that any odd derivatives of $T_c(J)$ and $\alpha(J)$ are equal to zero when they exist.

Next we study the symmetry of the correlation functions $\omega_{i_0j_0}(J)$ which are the expectation values of the following two-spin product with the weight corresponding the emerging order as

$$\omega_{i_0j_0}(J) \equiv \frac{\text{Tr} g_{i_0j_0} s_i s_j \exp[-\beta \mathcal{H}]}{Z(J)} \sim \exp[-r \epsilon(J) \nu(J)],$$

(6.4)
and also we study the symmetry of the susceptibility

\[ \chi(J) \equiv \sum_{i_0,j_0} \omega_{i_0j_0}(J) \sim \epsilon(J)^{-\gamma(J)}, \tag{6.5} \]

where \( \{g_{i_0j_0}\} \) are the sign corresponding to the emerging order and \( r \) is the distance between the site \( i_0 \) and \( j_0 \). We can assume without loss of generality that the site \( i_0 \) lies on \( \mathcal{H}_1 \). Corresponding to the change of the interaction \( J \) to \(-J\), the quantity \( g_{i_0j_0} \) changes its sign when the site \( j_0 \) belongs to \( \mathcal{H}_2 \) and then \( g_{i_0j_0}s_{i_0}s_{j_0} \) is even for the spin inversion of \( \mathcal{H}_2 \). As a result, we again arrive at

\[ \omega_{i_0j_0}(J) = \omega_{i_0j_0}(-J) \quad \text{and} \quad \chi(J) = \chi(-J), \tag{6.6} \]

and hence

\[ \nu(J) = \nu(-J) \quad \text{and} \quad \gamma(J) = \gamma(-J). \tag{6.7} \]

### 6.2 Vanishing cases

The above argument cannot exclude the case that the critical temperature and exponents do not differentiable at \( J = 0 \). The critical coefficient, indeed, behaves as a cusp and do not differentiable at \( J = 0 \) when the system is not translationally invariant. Here we show using the scaling hypothesis that the first-order derivatives of the critical temperature and exponents exist and they are vanishing. Let us consider the free energy \( f \) without the external field as

\[ f \simeq C \epsilon^{2-\alpha} \log \epsilon, \tag{6.8} \]

where \( C \) is some constant. Differentiating (6.8) with respect to the interaction \( J \) and taking a limit \( J \to 0 \), we obtain

\[ 0 = \left( \frac{\partial C}{\partial J} \right)_0 \epsilon^2 \log \epsilon + C \epsilon^2 \frac{1}{\epsilon} \left( \frac{\partial \epsilon}{\partial J} \right)_0 \]

\[ + \ C \epsilon^2 \log \epsilon \left[ (2-\alpha) \frac{1}{\epsilon} \left( \frac{\partial \epsilon}{\partial J} \right)_0 - \left( \frac{\partial \alpha}{\partial J} \right)_0 \log \epsilon \right] \tag{6.9} \]

and as a result

\[ \left( \frac{\partial C}{\partial J} \right)_0 = 0, \left( \frac{\partial \alpha}{\partial J} \right)_0 = 0 \quad \text{and} \quad \left( \frac{\partial \epsilon}{\partial J} \right)_0 = 0. \tag{6.10} \]
Next let us consider the free energy $f$ with the external field $h$. We assume the scaling form of $f$ as

$$f_J(\lambda^p \epsilon, \lambda^q h) = \lambda f_J(\epsilon, h), \tag{6.11}$$

where $\lambda$ is some parameter and $p$ and $q$ are numbers independent on $\epsilon$ and $h$. They are related to the exponents $\alpha$ and $\gamma$ as

$$\alpha = 2 - \frac{1}{p} \quad \text{and} \quad \gamma = \frac{2q - 1}{p}. \tag{6.12}$$

From (6.10) and (6.12), we obtain $(\partial p/\partial J)_0 = 0$. The partial derivative of $f$ shows the same scaling form as

$$\frac{\partial f_J(\lambda^p \epsilon, \lambda^q h)}{\partial J} = \lambda \frac{\partial f_J(\epsilon, h)}{\partial J}. \tag{6.13}$$

Differentiating (6.11) with respect to $J$, a straightforward calculation yields

$$M(0, 1) \frac{\log h}{q} \frac{\partial q}{\partial J}_{J=0} = 0, \tag{6.14}$$

for $\epsilon \to 0$, $J \to 0$ and $\lambda^q h = 1$, where $M(\epsilon, h) = \partial f_J(\epsilon, h)/\partial h$ is the magnetization. Note that $(\partial q/\partial J)$ is independent on $h$. From (6.14), we obtain $(\partial q/\partial J)_{J=0} = 0$, and as a result

$$\left(\frac{\partial \gamma}{\partial J}\right)_{J=0} = 0. \tag{6.15}$$

§ 7 The nonvanishing derivatives

In this section, we show the non-universal behaviour of the relevant system when it satisfies the conditions 1) and 2). We use the formulas (5.3) and (5.6) derived from 1) and 2), respectively, and (6.10) and (6.15) result from the symmetry of the model. The properties (5.6), (6.10) and (6.15) are valid when $n^{(k)}$ or $n^{(l)}$ (or both $n^{(k)}$ and $n^{(l)}$) are odd. Let us consider the weighted susceptibility

$$\chi = \beta \mu_B^2 \sum_{i,j} g_{ij} \langle s_i s_j \rangle, \tag{7.1}$$
where $g_{i_0j_0}$ is the sign corresponding to the emerging order as $g_{i_0j_0} = \text{sgn}(g_{i_0}g_{j_0})$. All we have to do here is to differentiate (7.1) in terms of $J_{kl}$ and to show that the second dominant term shows the logarithmic singularity $\chi_0 \log \epsilon$ with \( \epsilon = (T - T_c)/T_c \).

Let us introduce the following notations. The interaction $\mathcal{H}_{kl}$ is expressed as

$$\mathcal{H}_{kl} = -J \sum_i \mathcal{O}_i, \quad \mathcal{O}_i = \mathcal{O}_i^{(k)} \mathcal{O}_i^{(l)} \tag{7.2}$$

where $J_{kl}$ is written as $J$ for simplicity. The vector $r_i$ denotes the coordinate of the spin $s_i$ and $R_i = \{r_{i_1}, \ldots, r_{i_n}\}$ denotes the coordinate of the spin product $\mathcal{O}_i = s_{i_1} \cdots s_{i_n}$. The expectation value of $\mathcal{O}_i$ is expressed as a function of $R_i$ as

$$\langle \mathcal{O}_i \rangle = c_0(R_i) + c_1(R_i) \epsilon \log \epsilon \tag{7.3}.$$  

We also write $R_i \subset \mathcal{R}_m$ (or $i \in \mathcal{R}_m$) when $r_s \in \mathcal{R}_m$ for all $r_s \in R_i$.

Here we derive finite derivatives of the exponent $\gamma$. The first-order derivative of (7.1) is

$$\langle \frac{\partial \chi}{\partial J} \rangle_{J=0} = \beta^2 \mu_B^2 \sum_{i_0j_0} g_{i_0j_0} \sum_i [(s_{i_0}s_{j_0} \mathcal{O}_i^{(k)} \mathcal{O}_i^{(l)})_0 - \langle s_{i_0}s_{j_0} \rangle \langle \mathcal{O}_i^{(k)} \mathcal{O}_i^{(l)} \rangle_0], \tag{7.4}$$

where $\langle \rangle_0$ denotes the expectation value taken for $J = 0$. All the terms cancel except the cases:

**I-1)** $s_{i_0}, s_{j_0} \in \mathcal{H}_k$ (or $s_{i_0}, s_{j_0} \in \mathcal{H}_l$), and both $n^{(k)}$ and $n^{(l)}$ are even, and

**I-2)** $s_{i_0} \in \mathcal{H}_k$, $s_{j_0} \in \mathcal{H}_l$, and both $n^{(k)}$ and $n^{(l)}$ are odd.

The derivatives $(\partial \gamma/\partial J)_0$, $(\partial T_c/\partial J)_0$ of the latter case I-2) vanish as shown in section 6. As a result, we have only to treat the case I-1) as the first-order derivative. This case is a simple generalization of the argument for the eight-vertex model in section 4. The derivative (7.4) is written as

$$\langle \frac{\partial \chi}{\partial J} \rangle_{J=0} = \beta^2 \mu_B^2 \sum_{i_0j_0} g_{i_0j_0} \sum_i [(s_{i_0}s_{j_0} \mathcal{O}_i^{(k)} \mathcal{O}_i^{(l)})_0 - \langle s_{i_0}s_{j_0} \rangle \langle \mathcal{O}_i^{(k)} \mathcal{O}_i^{(l)} \rangle_0]. \tag{7.5}$$

This is a generalization of (4.3). From (3.2) and (5.3), we obtain the following temperature dependence

$$\langle \frac{\partial \chi}{\partial J} \rangle_{J=0} \simeq \chi_0 \sum_i G_i \left[ a_0(R_i) \frac{1}{\epsilon} + a_1(R_i) \log \epsilon \right] [c_0(R_i) + c_1(R_i) \epsilon \log \epsilon], \tag{7.6}$$

where 

$$G_i = \sum_{j_0} |g_{i_0j_0}|^2 \frac{1}{1 - n^{(l)}_{i_0j_0}} \sum_{k_0} |g_{i_0k_0}|^2 \frac{1}{1 - n^{(k)}_{i_0k_0}}.$$
where \(a_0(R_i), a_1(R_i), c_0(R_i)\) and \(c_1(R_i)\) are some constants. The term \(G_i = G_i^{(k)}G_i^{(l)}\) is factorized so that the inside of the brackets \([\cdots][\cdots]\) in (7.6) is positive. We generally cannot omit the term \(a_1(R_i)\). The sum \(\sum_i G_i^{(k)}a_0(R_i)\) and \(\sum_i G_i^{(l)}a_1(R_i)\) appear as the coefficient of \((\partial \tilde{\chi}_k / \partial \tilde{J}_k)_{\tilde{J}_k=0}\) where \(\tilde{\chi}_k\) is the susceptibility of the model described by the Hamiltonian \(\tilde{H}_k = H_k - \tilde{J}_k \sum_i O_i^{(k)}\) and they are finite in the thermodynamic limit. The coefficient of \(\log \epsilon\), namely
\[
\sum_i G_i(a_0(R_i)c_1(R_i) + a_1(R_i)c_0(R_i)), \tag{7.7}
\]
is finite because \(\sum_i G_i^{(k)}a_0(R_i), \sum_i G_i^{(l)}a_1(R_i), G_i^{(l)}c_0(R_i)\) and \(G_i^{(l)}c_1(R_i)\) are all finite. This is the first order derivative of \(\gamma\).

For all the cases except when both \(O_i^{(k)}\) and \(O_i^{(l)}\) have even symmetry (i.e. the case I-1: both \(n^{(k)}\) and \(n^{(l)}\) are even), we obtained from (6.10) and (6.15)
\[
(\frac{\partial T_c}{\partial J})_0 = 0 \text{ and } (\frac{\partial \gamma}{\partial J})_0 = 0. \tag{7.8}
\]
In these cases, we have to show that the second-order derivatives are non-vanishing and finite. From (3.1) and (7.8), the second-order derivative of the susceptibility \(\chi\) is
\[
(\frac{\partial^2 \chi}{\partial J^2})_{J=0} \simeq \chi_0 [-\gamma(0) \frac{1}{\epsilon(0)} (\frac{\partial^2 \epsilon}{\partial J^2})_0 - (\frac{\partial^2 \gamma}{\partial J^2})_0 \log \epsilon(0)]. \tag{7.9}
\]
Hence the existence of the logarithmic singularity is the sign of continuously varying critical exponents. All we have to do is to find a term proportional to \(\chi_0 \log \epsilon\) in the second-order derivative of \(\chi\) and to show the coefficient of \(\chi_0 \log \epsilon\) is finite.

Differentiating \(\chi = \beta \mu_B^2 \sum \langle s_i s_j \rangle\) with respect to \(J\) twice, we obtain
\[
(\frac{\partial^2 \chi}{\partial J^2})_{J=0} = \beta^2 \mu_B^2 \sum_{i \neq j} g_{ij} \sum_i \sum_j \left[ \langle s_i s_j O_i^{(k)} O_j^{(k)} \rangle_0 - \langle s_i s_j \rangle_0 \langle O_i^{(k)} O_j^{(k)} \rangle_0 \langle O_i^{(l)} O_j^{(l)} \rangle_0 \right], \tag{7.10}
\]
where we have used that \(n^{(k)}\) or \(n^{(l)}\) (or both \(n^{(k)}\) and \(n^{(l)}\)) are odd and the expectation value \(\langle s_{i_1} \cdots s_{i_n} \rangle_0\) equals zero when \(n\) is odd. The following case remains nonvanishing:

**II-1** \(s_{i_0}, s_{j_0} \in \mathcal{H}_k\) (or \(s_{i_0}, s_{j_0} \in \mathcal{H}_l\)), \(n^{(k)}\) is even and \(n^{(l)}\) is odd, and
II-2) $s_{i_0}, s_{j_0} \in \mathcal{H}_k$ (or $s_{i_0}, s_{j_0} \in \mathcal{H}_l$), and both $n^{(k)}$ and $n^{(l)}$ are odd.

Both cases can be treated simultaneously. From (3.2) and (5.3), the derivative (7.10) shows the following temperature dependence

$$
\chi_0 \sum_i \sum_j G_i G_j \left[ a_0 (R_i, R_j) \frac{1}{\epsilon^2} + a_1 (R_i, R_j) \log \epsilon \right] \left[ c_0 (R_i, R_j) + c_1 (R_i, R_j) \epsilon \log \epsilon \right],
$$

$$
\equiv \chi_0 \sum_i \sum_j G_i G_j \Omega(R_i, R_j), \quad (7.11)
$$

where $\Omega(R_i, R_j)$ is positive and $a_0 (R_i, R_j), a_1 (R_i, R_j), c_0 (R_i, R_j)$ and $c_1 (R_i, R_j)$ are some constants. This summation can be regrouped by $\{R_{m} \}$ and using the condition (5.6) (i.e. the condition 2: $\sum_{i \in R_m} G_i = 0$), we obtain

$$
\chi_0 \sum_{mm'} \sum_{i \in R_m} \sum_{j \in R_{m'}} G_i G_j \Omega(R_{m} + \Delta R_{mi}, R_{m'} + \Delta R_{m'j})
$$

$$
\simeq \chi_0 \sum_{mm'} \sum_{i \in R_m} \sum_{j \in R_{m'}} G_i G_j (\Delta R_{mi} \cdot \nabla) (\Delta R_{m'j} \cdot \nabla') \Omega(R_{m}, R_{m'}), \quad (7.12)
$$

where $R_m \subset \mathcal{R}_m$ and $R_{m'} \subset \mathcal{R}_{m'}$ are fixed for each $\mathcal{R}_m$ and $\mathcal{R}_{m'}$, respectively, and we have used the following notation

$$
\Delta R_{mi} \cdot \nabla \equiv \Delta r_{mi_1} \cdot \nabla_{i_1} + \cdots + \Delta r_{mn_i_n} \cdot \nabla_{i_n}
$$

$$
\Delta R_{m'j} \cdot \nabla' \equiv \Delta r_{m'j_1} \cdot \nabla'_{j_1} + \cdots + \Delta r_{m'n_j_n} \cdot \nabla'_{j_n} \quad (7.13)
$$

where $\nabla_i$ and $\nabla'_i$ are the gradient operating to the coordinate of $s_i$ and $\Delta_{mi} \cdot \nabla$ and $\Delta_{m'j} \cdot \nabla'$ operate on the first and the second arguments of $\Omega(R_{m}, R_{m'})$.

The first summation in (7.12) is classified by the distance between $\mathcal{R}_m$ and $\mathcal{R}_{m'}$ as

$$
\sum_{mm'} = \sum_r \sum_{mm' \mid \lvert m - m' \rvert = r}, \quad (7.14)
$$

where $\lvert m - m' \rvert = r$ denotes that $r \leq \min \{ \lvert r_i - r_{i'} \rvert \mid r_i \in \mathcal{R}_m, r_{i'} \in \mathcal{R}_{m'} \} < r + \Delta r$ and $\Delta r$ is a constant comparable to the mean size of $\{\mathcal{R}_m\}$.

As $\Omega$ is a smooth and decreasing function of $r$ and all $\{\mathcal{R}_m\}$ denote finite regions containing a finite number of spins, each term in (7.12) is bounded by

$$
G_i G_j (\Delta R_{mi} \cdot \nabla) (\Delta R_{m'j} \cdot \nabla') \Omega \leq \text{const.} (\Delta R \cdot \nabla) (\Delta R \cdot \nabla') \Omega, \quad (7.15)
$$

where $\Delta R$ is the difference between $\mathcal{R}_m$ and $\mathcal{R}_{m'}$.
\[ \Delta R \cdot \nabla \equiv \Delta r \cdot \nabla_i + \cdots + \Delta r \cdot \nabla_n, \quad (7.16) \]

and a similar equation is defined for \( \Delta R' \cdot \nabla' \).

Then the coefficient of \( \chi_0 \log \epsilon \) in (7.12) is bounded as

\[ \sum_r (\Delta R_{mi} \cdot \nabla) (\Delta R_{m'j} \cdot \nabla') F(r, \{ R_i \}) < \sum_{r<r_0} (\Delta R_{mi} \cdot \nabla) (\Delta R_{m'j} \cdot \nabla') F(r, \{ R_i \}) + \text{const}, \quad (7.17) \]

where \( F(r, \{ R_i \}) = \sum_{mm'} \sum_{|m-m'|=r} \mathcal{G}_i \mathcal{G}_j [a_0(R_i, R_j)c_1(R_i, R_j) + a_1(R_i, R_j)c_0(R_i, R_j)] \quad (7.18) \)

is a finite function of \( r \) (see (7.7)).

Our purpose is to show (7.17) is finite. The first sum is of course finite and the second term is bounded by

\[ A \int_{r_0}^{\infty} d^2 r \nabla^2 F(r, \{ R_i \}) = A \int_{r=r_0}^{\infty} dS \frac{\partial F}{\partial r} \approx A \cdot 2\pi r_0 \frac{\partial F}{\partial r} \big|_{r=r_0} < \infty, \quad (7.19) \]

where \( A \) is some constant.

**§ 8 Generalization**

The condition 2) can be generalized in the following two points.

At first, it is straightforward to generalize the form of the interaction \( \mathcal{H}_{kl} \) as

\[ \mathcal{H}_{kl} = \sum_p \mathcal{H}_{kl}^{(p)} \quad \text{and} \quad \mathcal{H}_{kl}^{(p)} = -J_{kl}^{(p)} \sum G_{p(i)} G_{p(l)}, \quad (8.1) \]

where each \( \mathcal{H}_{kl}^{(p)} \) satisfies the condition 2).

Next, the values of \( n^{(k)} \) and \( n^{(l)} \) have been fixed. This condition is necessary for showing the cancellations of correlations in the zeroth- and the first-order terms in (7.12) using the condition \( \sum_{i \in \mathcal{R}_m} \mathcal{G}_i = 0 \). However, our argument is valid for more complicated interactions i.e. the case that \( n^{(k)} \)
and $n^{(l)}$ depend on the region $R_m$. Each contribution from each $R_m$ to the derivatives is classified into the cases shown in sections 6 and 7. As for the first-order derivatives, there is no difference. For the second-order derivative, all the contributions to $(\partial^2 \gamma/\partial J^2)_0$ coming from $R_m$ and $R_m'$ vanish except when both $n^{(k)}_m + n^{(k)}_{m'}$ and $n^{(l)}_m + n^{(l)}_{m'}$ are even. (Otherwise the argument in section 6 is valid for each term and the contributions to the derivative is vanishing.) The nonvanishing case can be treated in the same way as in section 7.

§ 9 Conclusion

We have derived that some Ising systems satisfying the conditions 1) and 2) in section 2 show non-universal critical behaviour. The perturbational expansion in terms of the interaction $J$ is performed. This method results in the exact first-order derivatives (4.6) in the case of the eight-vertex model. We have used the conditions (5.3) and (5.6), which is derived from the conditions 1) and 2), respectively, and (6.10) and (6.15) which result from the symmetry of the model. The existence of finite and non-zero derivatives $(\partial \gamma/\partial J)_0$ or $(\partial^2 \gamma/\partial J^2)_0$ is an evidence of continuous variation of critical exponents. These derivatives are derived in (7.4)-(7.7) and (7.10)-(7.19) for the first- and the second-order derivatives, respectively. Finally, the straightforward generalization of the condition 2) is commented in section 8. This condition is valid for generalized spin S Ising models or can easily be generalized for other classical systems.

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