ON ISOTOPY AND UNIMODAL INVERSE LIMIT SPACES

H. BRUIN AND S. ŠTIMAC

University of Surrey and University of Zagreb

ABSTRACT. We prove that every self-homeomorphism \( h : K_s \to K_s \) on the inverse limit space \( K_s \) of tent map \( T_s \) with slope \( s \in (\sqrt{2}, 2] \) is isotopic to a power of the shift-homeomorphism \( \sigma^R : K_s \to K_s \).

1. Introduction

The solution of Ingram’s Conjecture constitutes a major advancement in the classification of unimodal inverse limit spaces and the group of self-homeomorphisms on them. This conjecture was posed by Tom Ingram in 1992 for tent maps \( T_s : [0, 1] \to [0, 1] \) with slope \( \pm s, s \in [1, 2] \), defined as \( T_s(x) = \min\{sx, s(1-x)\} \). The turning point is \( c = \frac{1}{2} \) and we denote its iterates by \( c_n = T_s^n(c) \). The inverse limit space \( K_s = \varprojlim ([0, s/2], T_s) \) consists of the core \( \varprojlim ([c_2, c_1], T_s) \) and the 0-composant \( \mathcal{C}_0 \), i.e., the composant of the point \( \bar{0} := (\ldots, 0, 0, 0) \), which compactifies on the core of the inverse limit space. Ingram’s Conjecture reads:

If \( 1 \leq s < s' \leq 2 \), then the corresponding inverse limit spaces \( \varprojlim ([0, s/2], T_s) \) and \( \varprojlim ([0, s'/2], T_{s'}) \) are non-homeomorphic.

The first results towards solving this conjecture were obtained for tent maps with a finite critical orbit \([9, 12, 3]\). Raines and Štimac \([11]\) extended these results to tent maps with a possibly infinite, but non-recurrent critical orbit. Recently Ingram’s Conjecture was solved completely (in the affirmative) in \([2]\), but we still know very little of the structure of inverse limit spaces (and their subcontinua) for the case that \( \text{orb}(c) \) is infinite and recurrent, see \([1, 5, 8]\).

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Given a continuum $K$ and $x \in K$, the \textit{composant} $A$ of $x$ is the union of the proper subcontinua of $K$ containing $x$. For slopes $s \in (\sqrt{2}, 2]$, the core is indecomposable (i.e., it cannot be written as the union of two proper subcontinua), and in this case we also proved \cite{2} that any self-homeomorphism $h : K_s \to K_s$ is pseudo-isotopic to a power $\sigma^R$ of the shift-homeomorphism $\sigma$ on the core. This means that $h$ permutes the composants of the core of $K_s$ in the same way as $\sigma^R$ does, and it is a priori a weaker property than isotopy. This is for instance illustrated by the $\sin \frac{1}{x}$-continuum, defined as the graph $\{(x, \sin \frac{1}{x}) : x \in (0, 1]\}$ compactified with a bar $\{0\} \times [-1, 1]$. There are homeomorphisms that reverse the orientation of the bar, and these are always pseudo-isotopic, but never isotopic, to the identity. Since such $\sin \frac{1}{x}$-continua are precisely the non-trivial subcontinua of Fibonacci-like inverse limit spaces \cite{8}, this example is very relevant to our paper.

In this paper we make the step from pseudo-isotopy to isotopy. To this end, we exploit so-called \textit{folding points}, i.e., points in the core of $K_s$ where the local structure of the core of $K_s$ is not that of a Cantor set cross an arc. In the next section we prove the following results:

**Theorem 1.1.** If $s \in (\sqrt{2}, 2]$, and $h : K_s \to K_s$ is a homeomorphism, then there is $R \in \mathbb{Z}$ such that $h(x) = \sigma^R(x)$ for every folding point $x$ in $K_s$.

Folding points $x = (\ldots, x_{-2}, x_{-1}, x_0)$ are characterized by the fact that each entry $x_{-k}$ belongs to the omega-limit set $\omega(c)$ of the turning point $c = \frac{1}{2}$, see \cite{10}. This gives the immediate corollary for those slopes such that the critical orbit $\text{orb}(c)$ is dense in $[c_2, c_1]$, which according to \cite{7} holds for Lebesgue a.e. $s \in [\sqrt{2}, 2]$.

**Corollary 1.2.** If $\text{orb}(c)$ is dense in $[c_2, c_1]$, then for every homeomorphism $h : K_s \to K_s$ there is $R \in \mathbb{Z}$ such that $h = \sigma^R$ on the core of $K_s$.

The more difficult case, however, is when $\text{orb}(c)$ is not dense in $[c_2, c_1]$. In this case, $h$ can be at best isotopic to a power of the shift, because at non-folding points, where the core of $K_s$ is a Cantor set cross an arc, $h$ can easily act as a local translation. It is shown in \cite{4} that for tent maps with non-recurrent critical point (or in fact, more generally long-branched tent maps), every homeomorphism $h : K_s \to K_s$ is indeed isotopic to a power of the shift. The proof exploits the fact that in this case, so-called $p$-points (indicating folds in the arc-components of $K_s$) are separated from each other, at least in arc-length semi-metric. Here we prove the general result.

**Theorem 1.3.** If $s \in (\sqrt{2}, 2]$, and $h : K_s \to K_s$ is a homeomorphism, then there exists $R \in \mathbb{Z}$ such that $h$ is isotopic to $\sigma^R$. 

The paper is organized as follows. In Section 2 we give basic definitions and prove results on how homeomorphisms act on folding points, i.e., Theorem 1.1 and Corollary 1.2. These proofs depend largely on the results obtained in [2]. In Section 3 we present the additional arguments needed for the isotopy result and finally prove Theorem 1.3.

2. INVERSE LIMIT SPACES OF TENT MAPS AND FOLDING POINTS

Let \( \mathbb{N} = \{1, 2, 3, \ldots \} \) be the set of natural numbers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). The tent map \( T_s : [0, 1] \to [0, 1] \) with slope \( \pm s \) is defined as \( T_s(x) = \min\{sx, s(1-x)\} \). The critical or turning point is \( c = 1/2 \) and we write \( c_k = T_k^k(c) \), so in particular \( c_1 = s/2 \) and \( c_2 = s(1-s/2) \). Also let \( \text{orb}(c) \) and \( \omega(c) \) be the orbit and the omega-limit set of \( c \). We will restrict \( T_s \) to the interval \( I = [0, s/2] \); this is larger than the core \( [c_2, c_1] = [s-s^2/2, s/2] \), but it contains the fixed point 0 on which the 0-composant \( \mathfrak{C}_0 \) is based.

The inverse limit space \( K_s = \lim_{\leftarrow}([0, s/2], T_s) \) is
\[
\{ x = (\ldots, x_{-2}, x_{-1}, x_0) : T_s(x_{i-1}) = x_i \in [0, s/2] \text{ for all } i \leq 0 \},
\]
equipped with metric \( d(x, y) = \sum_{n \leq 0} 2^n |x_n - y_n| \) and induced (or shift) homeomorphism
\[
\sigma(\ldots, x_{-2}, x_{-1}, x_0) = (\ldots, x_{-2}, x_{-1}, x_0, T_s(x_0)).
\]

Let \( \pi_k : \lim_{\leftarrow}([0, s/2], T_s) \to I, \pi_k(x) = x_{-k} \) be the \( k \)-th projection map. Since 0 \( \in I \), the endpoint \( \tilde{0} := (\ldots, 0, 0, 0) \) is contained in \( \lim_{\leftarrow}([0, s/2], T_s) \). The composant of \( \lim_{\leftarrow}([0, s/2], T_s) \) of \( \tilde{0} \) will be denoted as \( \mathfrak{C}_0 \); it is a ray converging to, but disjoint from the core \( \lim_{\leftarrow}([c_2, c_1], T_s) \) of the inverse limit space. We fix \( s \in (\sqrt{2}, 2] \); for these parameters \( T_s \) is not renormalizable and \( \lim_{\leftarrow}([c_2, c_1], T_s) \) is indecomposable. Moreover, the arc-component of \( \tilde{0} \) coincides with the composant of \( 0 \), but for points in the core of \( K_s \), we have to make the distinction between arc-component and composant more carefully.

A point \( x = (\ldots, x_{-2}, x_{-1}, x_0) \in K_s \) is called a \textit{p-point} if \( x_{-p-l} = c \) for some \( l \in \mathbb{N}_0 \). The number \( L_p(x) := l \) is the \textit{p-level} of \( x \). In particular, \( x_0 = T_s^{p+1}(c) \). By convention, the endpoint \( \tilde{0} \) of \( \mathfrak{C}_0 \) is also a \textit{p-point} and \( L_p(\tilde{0}) := \infty \), for every \( p \). The ordered set of all \( p \)-points of the composant \( \mathfrak{C}_0 \) is denoted by \( E_p \), and the ordered set of all \( p \)-points of \textit{p-level} \( l \) by \( E_{p,l} \). Given an arc \( A \subset K_s \) with successive \( p \)-points \( x^0, \ldots, x^n \), the \textit{p-folding pattern} of \( A \) is the sequence
\[
FP_p(A) := L_p(x^0), \ldots, L_p(x^n).
\]

Note that every arc of \( \mathfrak{C}_0 \) has only finitely many \( p \)-points, but an arc \( A \) of the core of \( K_s \) can have infinitely many \( p \)-points. In this case, if \( (u^i)_{i \in \mathbb{Z}} \) is the sequence of successive \( p \)-points of \( A \), then \( FP_p(A) = (L_p(u^i))_{i \in \mathbb{Z}} \). The \textit{folding pattern of the composant} \( \mathfrak{C}_0 \), denoted by \( FP(\mathfrak{C}_0) \),
is the sequence \( L_p(z^1), L_p(z^2), \ldots, L_p(z^n), \ldots \), where \( E_p = \{ z^1, z^2, \ldots, z^n, \ldots \} \) and \( p \) is any nonnegative integer. Let \( q \in \mathbb{N}, q > p \), and \( E_q = \{ y^0, y^1, y^2, \ldots \} \). Since \( \sigma^{q-p} \) is an order-preserving homeomorphism of \( \mathcal{C}_0 \), it is easy to see that \( \sigma^{q-p}(z^i) = y^i \) for every \( i \in \mathbb{N} \), and \( L_p(z^i) = L_q(y^i) \). Therefore, the folding pattern of \( \mathcal{C}_0 \) does not depend on \( p \).

**Definition 2.1.** Let \((s_i)_{i \in \mathbb{N}}\) be a sequence of \( p \)-points of \( \mathcal{C}_0 \) such that \( 0 \leq L_p(x) < L_p(s_i) \) for every \( p \)-point \( x \in (0, s_i) \). We call \( p \)-points satisfying this property *salient*.

Since for every slope \( s > 1 \) and \( p \in \mathbb{N}_0 \), the folding pattern of the 0-compsasant \( \mathcal{C}_0 \) starts as \( \infty 0 1 0 2 0 1 \ldots \), and since by definition \( L_p(s_1) > 0 \), we have \( L_p(s_1) = 1 \). Also, since \( s_i = \sigma^{i-1}(s_1), L_p(s_i) = i \), for every \( i \in \mathbb{N} \). Note that the salient \( p \)-points depend on \( p \): if \( p \geq q \), then the salient \( p \)-point \( s_i \) equals the salient \( q \)-point \( s_{i+p-q} \).

A *folding point* is any point \( x \) in the core of \( K_s \) such that no neighborhood of \( x \) in core of \( K_s \) is homeomorphic to the product of a Cantor set and an arc. In [10] it was shown that \( x = (\ldots, x_{-2}, x_{-1}, x_0) \) is a folding point if and only if \( x_{-k} \in \omega(c) \) for all \( k \geq 0 \). We can characterize folding points in terms of \( p \)-points as follows:

**Lemma 2.2.** Let \( p \) be arbitrary. A point \( x \in K_s \) is a folding point if and only if there is a sequence of \( p \)-points \((x^k)_{k \in \mathbb{N}}\) such that \( x^k \to x \) and \( L_p(x^k) \to \infty \).

**Proof.** \( \Rightarrow \) Take \( m \geq p \) arbitrary. Since \( \pi_m(x) \in \omega(c) \) there is a sequence of post-critical points \( c_{n_i} \to \pi_m(x) \). This means that any point \( y^i = (\ldots, c_{n_i}, c_{n_i+1}, \ldots, c_{n_i+m}) \) is a \( p \)-point with \( p \)-level \( L_p(y^i) = n_i + m - p \). Furthermore, for each \( 0 \leq j \leq m \), \( |\pi_j(y^i) - \pi_j(x^i)| \to 0 \) as \( i \to \infty \).

Since \( m \) is arbitrary, we can construct a diagonal sequence \((x^k)_{k \in \mathbb{N}}\) of \( p \)-points, by taking a single element from \((y^i)_{i \in \mathbb{N}}\) for each \( m \), such that \( \sup_{j \leq k} |\pi_j(x^k) - \pi_j(x^i)| \to 0 \) as \( k \to \infty \). This proves that \( x^k \to x \) and \( L_p(x^k) \to \infty \).

\( \Leftarrow \) Take \( m \) arbitrary. Since \( x^k \to x \), also \( |\pi_m(x^k) - \pi_m(x)| \to 0 \) and \( \pi_m(x^k) = c_n \) for \( n = L_p(x^k) + p - m \). But \( L_p(x^k) \to \infty \), so \( \pi_m(x) \in \omega(c) \).

A continuum is *chainable* if for every \( \varepsilon > 0 \), there is a cover \( \{\ell^1, \ldots, \ell^n\} \) of open sets (called *links*) of diameter \( < \varepsilon \) such that \( \ell^i \cap \ell^j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). Such a cover is called a *chain*. Clearly the interval \([0, s/2]\) is chainable.

**Definition 2.3.** We call \( \mathcal{C}_p \) a *natural* chain of \( \lim([0, s/2], T_s) \) if

1. there is a chain \( \{I_p^1, I_p^2, \ldots, I_p^n\} \) of \([0, s/2]\) such that \( I_p^j := \pi_p^{-1}(I_p^i) \) are the links of \( \mathcal{C}_p \);
2. each point \( x \in \bigcup_{i=0}^p T_s^{-i}(c) \) is the boundary point of some link \( I_p^i \);
3. for each \( i \) there is \( j \) such that \( T_s(I_p^{i+1}) \subset I_p^j \).
If \( \max_j |I_j^p| < \varepsilon s^{-p}/2 \) then \( \text{mesh}(C_p) := \max\{\text{diam}(\ell) : \ell \in C_p\} < \varepsilon \), which shows that \( \lim_{x \to (0, s/2], T_s} \) is indeed chainable. Condition (3) ensures that \( C_{p+1} \) refines \( C_p \) (written \( C_{p+1} \leq C_p \)).

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( h : K_s \to K_s \) be a homeomorphism. Let \( x, y \in K_s \) be folding points with \( h(x) = y \). For \( i \in \mathbb{N}_0 \) let \( q_i, p_i \in \mathbb{N} \) be such that for sequences of chains \( (C_{q_i})_{i \in \mathbb{N}_0} \) and \( (C_{p_i})_{i \in \mathbb{N}_0} \) of \( K_s \) we have

\[
\cdots \prec h(C_{q_{i+1}}) \prec C_{p_{i+1}} \prec h(C_{q_i}) \prec C_{p_i} \prec \cdots \prec h(C_{q_1}) \prec C_{p_1} \prec h(C_q) \prec C_p,
\]

where \( q_0 = q \) and \( p_0 = p \). Let \( (\ell_{q_i})_{i \in \mathbb{N}_0} \) be sequence of links such that \( x \in \ell_{q_i} \in C_{q_i} \), and similarly for \( (\ell_{p_i})_{i \in \mathbb{N}_0} \). Then \( \ell_{q_{i+1}} \subset \ell_{q_i} \), \( \ell_{p_{i+1}} \subset \ell_{p_i} \), and \( h(\ell_{q_i}) \subset \ell_{p_i} \). Let \( (s_{d_i}') \in \mathbb{N} \) be a sequence of salient \( q \)-points with \( s_{d_i}' \to x \) as \( i \to \infty \). Then for every \( i \) there exist \( j_i \) such that \( s_{d_{j_i}}' \in \ell_{q_i} \), \( h(s_{d_{j_i}}') \in \ell_{p_i} \), and \( h(s_{d_{j_i}}') \to y \) as \( i \to \infty \). By [2] Theorem 4.1] the midpoint of the arc component \( A_i \) of \( \ell_{p_i} \), which contains \( h(s_{d_{j_i}}') \) is a salient \( p \)-point \( s_{m_i}'' \). Since \( s_{m_i}, y \in \ell_{p_i} \), for every \( i \) and \( i \to \infty \), we have \( s_{m_i}'' \to y \). Since \( s_{d_{j_i}}' \) is a salient \( q \)-point and \( s_{d_{j_i}}' \in \ell_{q_i}, s_{m_i}'' \) can be also considered as a salient \( p \)-point and is also the midpoint of the arc component \( B_i \supset A_i \) of \( \ell_{p_i} \) which contains \( h(s_{d_{j_i}}') \). Therefore, \( s_{m_i}'' = s_{d_{j_i}+M} \), where \( M \) is as in [2] Theorem 4.1].

Let \( R = M - q + p \). By [2] Corollary 5.3], \( R \) does not depend on \( q, p \) and \( M \). Since \( \sigma^R : K_s \to K_s \) is a homeomorphism, and since \( s_{d_{j_i}}' \to x \) as \( i \to \infty \), we have \( \sigma^R(s_{d_{j_i}}') \to \sigma^R(x) \) as \( i \to \infty \). Note that \( \sigma^R(s_{d_{j_i}}') = s_{d_{j_i}+M} \) and \( s_{d_{j_i}+M} \to y \). Therefore \( \sigma^R(x) = y \), i.e., \( \sigma^R(x) = h(x) \).

**Proof of Corollary 1.2.** If \( \text{orb}(c) \) is dense in \([c_2, c_1] \), every point \( x \) in the core of \( K_s \) satisfies \( \pi_k(x) = \omega(c) \) for all \( k \in \mathbb{N} \). By [11], this means that every point is a folding point, and hence the previous theorem implies that \( h \equiv \sigma^R \) on the core of \( K_s \).

**Remark 2.4.** A point \( x \in K_s \) is an endpoint of an atriodic continuum, if for every pair of subcontinua \( A \) and \( B \) containing \( x \), either \( A \subset B \) or \( B \subset A \). The notion of folding point is more general than that of end-point. An example of a folding point that is not an end-point is the midpoint \( x \) of a double spiral \( S \), i.e., a continuous image of \( \mathbb{R} \) containing a single folding point \( x \) and two sequences of \( p \)-points

\[
\ldots y^k \prec y^{k+1} \prec \cdots \prec x \prec \cdots \prec z^{k+1} \prec z^k \ldots
\]

converging to \( x \) such that the arc-length \( \bar{d}(y^k, y^{k+1}), \bar{d}(z^k, z^{k+1}) \to 0 \) as \( k \to \infty \). Here \( \prec \) denotes the induced order on \( S \).
It is natural to classify arc-components $\mathfrak{A}$ according to the folding points they may contain. For arc-components $\mathfrak{A}$, we have the following possibilities:

- $\mathfrak{A}$ contains no folding point.
- $\mathfrak{A}$ contains one folding point $x$, e.g. if $x$ is an end-point of $\mathfrak{A}$ or $\mathfrak{A}$ is a double spiral.
- $\mathfrak{A}$ contains two folding points, e.g. if $\mathfrak{A}$ is the bar of a $\frac{1}{x}$-continuum.
- $\mathfrak{A}$ contains countably many folding points. One can construct tent maps such that the folding points of its inverse limit space belong to finitely many arc-components that are periodic under $\sigma$, but where there are still countably many folding points.
- $\mathfrak{A}$ contains uncountably many folding points, e.g. if $\omega(c) = [0, 1]$, because then every point in the core is a folding point.

This is clearly only a first step towards a complete classification.

**Definition 2.5.** Let $\ell^0, \ell^1, \ldots, \ell^k$ be those links in $C_p$ that are successively visited by an arc $A \subset C_0$ (hence $\ell^i \neq \ell^{i+1}$, $\ell^i \cap \ell^{i+1} \neq \emptyset$ and $\ell^i = \ell^{i+2}$ is possible if $A$ turns in $\ell^{i+1}$). Let $\mathfrak{A}_i \subset \ell^i$ be the corresponding arc components such that $\text{Cl} \mathfrak{A}_i$ are subarcs of $A$. We call the arc $A$

- $p$-link symmetric if $\ell^i = \ell^{k-i}$ for $i = 0, \ldots, k$;
- maximal $p$-link symmetric if it is $p$-link symmetric and there is no $p$-link symmetric arc $B \supset A$ and passing through more links than $A$.

The $p$-point of $A^{k/2}$ with the highest $p$-level is called the center of $A$, and the link $\ell^{k/2}$ is called the central link of $A$.

### 3. ISOTOPIC HOMEOMORPHISMS OF UNIMODAL INVERSE LIMITS

It is shown in [2] that every salient $p$-point $s_t \in C_0$ is the center of the maximal $p$-link symmetric arc $A_t$. We denote the central link that $s_t$ belongs to by $\ell^{s_t}$. For a better understanding of this section, let us mention that a key idea in [2] is that under a homeomorphism $h$ such that $h(C_q) \prec C_p$, (maximal) $q$-link symmetric arcs have to map to (maximal) $p$-link symmetric arcs, and for this reason $h(s_m) \in \ell^{s_{p}}$ for some appropriate $m \in \mathbb{N}$ (see [2, Theorem 4.1]).

**Lemma 3.1.** Let $h : K_s \to K_s$ be a homeomorphism pseudo-isotopic to $\sigma^R$, and let $q, p \in \mathbb{N}_0$ be such that $h(C_q) \preceq C_p$. Let $x$ be a $q$-point in the core of $K_s$ and let $\ell^{s_{p}} \in C_p$ be the link

\[ \text{An example is the tent-map where } c_1 \text{ has symbolic itinerary (kneading sequence) } \nu = 10010^201^301^401^5 \ldots. \text{ Then the two-sided itineraries of folding points are limits of } \{\sigma^j(\nu)\}_{j \geq 0}. \text{ The only such two-sided limit sequences are } 1^\infty, 1^\infty, \text{ and } \{\sigma^j(1^\infty, 01^\infty) : j \in \mathbb{Z}\}. \text{ Since they all have left tail } \ldots 1111, \text{ these folding points belong to the arc-component of the point } (\ldots, p, p, p) \text{ for the fixed point } p = \frac{1}{1+s}. \text{ This use of two-sided symbolic itineraries was introduced for inverse limit spaces in [6].} \]
containing both \(\sigma^p(x)\) and salient point \(s_l\), where \(l = L_p(\sigma^R(x))\). Suppose that the arc-component \(W_x\) of \(\ell^p\) containing \(\sigma^R(x)\) does not contain any folding point. Then \(h(x) \in W_x\).

**Proof.** Since \(W_x\) does not contain any folding point, it contains finitely many \(p\)-points. Note that \(W_x\) contains at least one \(p\)-point since \(\sigma^R(x) \in W_x\) is a \(p\)-point. Since \(\mathcal{C}_0\) is dense in \(K_s\), there exists a sequence \((W_i)_{i \in \mathbb{N}}\) of arc-components of \(\ell^p\) such that \(W_i \subset \mathcal{C}_0\), \(FP_p(W_i) = FP_p(W_x)\) for every \(i \in \mathbb{N}\), and \(W_i \rightarrow W_x\) in the Hausdorff metric. Let \((x_i)_{i \in \mathbb{N}}\) be a sequence of \(q\)-points such that for every \(i \in \mathbb{N}\), \(L_q(x_i) = L_q(x)\), \(x_i \rightarrow x\) and \(\sigma^R(x_i) \in W_i\). Obviously \((x_i)_{i \in \mathbb{N}} \subset \mathcal{C}_0\), \(L_p(\sigma^R(x_i)) = L_p(\sigma^R(x))\) and \(\sigma^R(x_i) \rightarrow \sigma^R(x)\). Since \(h\) is a homeomorphism, \(h(x_i) \rightarrow h(x)\). It follows by the construction in the proof of [2, Proposition 4.2] that \(h(x_i) \in W_i\) for every \(i \in \mathbb{N}\). Therefore \(h(x) \in W_x\). □

**Corollary 3.2.** Let \(h : K_s \rightarrow K_s\) be a homeomorphism pseudo-isotopic to \(\sigma^R\). Then \(h\) permutes arc-components of \(K_s\) in the same way as \(\sigma^R\).

**Proof.** Since \(h\) is a homeomorphism, \(h\) maps arc-components to arc-components. Let \(A\) be an arc-component of \(K_s\). Let us suppose that \(A\) contains a folding point, say \(x\). Then \(h(x) = \sigma^R(x)\) implies \(h(A) = \sigma^R(A)\).

Let us assume now that \(A\) does not contain any folding point. There exist \(q, p \in \mathbb{N}_0\) such that \(h(C_q) \subset C_p\) and that \(h(A)\) is not contained in a single link of \(C_p\). Then \(A\) is not contained in a single link of \(C_q\). Let \(\ell_q \in C_q\) and \(V \in \ell_q \cap A\) be an arc-component of \(\ell_q\) such that \(V\) contains at least one \(q\)-point, say \(x\). Let \(\ell_p^q \in C_p\) be such that \(l = L_p(\sigma^R(x))\). Let \(W \subset \ell_p^q\) be arc-component containing \(\sigma^R(x)\). Since \(A\) does not contain any folding point, \(h(A)\) does not contain any folding point implying \(W\) does not contain any folding point. Then, by Lemma 3.1 \(h(x) \in W\) implying \(h(A) = \sigma^R(A)\). □

**Lemma 3.3.** Let \(h : K_s \rightarrow K_s\) be a homeomorphism that is pseudo-isotopic to the identity. Then \(h\) preserves orientation of every arc-component \(A\), i.e., given a parametrization \(\varphi : \mathbb{R} \rightarrow A\) (or \(\varphi : [0, 1] \rightarrow A\) or \(\varphi : [0, \infty) \rightarrow A\)) that induces an order \(\prec\) on \(A\), then \(x \prec y\) implies \(h(x) \prec h(y)\).

**Proof.** Let us first suppose that \(h : K_s \rightarrow K_s\) is any homeomorphism. Then, by [2, Theorem 1.2] there is an \(R \in \mathbb{Z}\) such that \(h\), restricted to the core, is pseudo-isotopic to \(\sigma^R\), i.e., \(h\) permutes the composants of the core of the inverse limit in the same way as \(\sigma^R\). Therefore, by Corollary 3.2 it permutes the arc-components of the inverse limit in the same way as \(\sigma^R\).

Let \(A, A'\) be arc-components of the core such that \(h, \sigma^R : A \rightarrow A'\), and let \(x, y \in A, x \prec y\). We want to prove that \(h(x) \prec h(y)\) if and only if \(\sigma^R(x) \prec \sigma^R(y)\). Since \(h\) and \(\sigma^R\) are...
homeomorphisms on arc-components, each of them could be either order preserving or order reversing. Therefore, to prove the claim we only need to pick two convenient points \( u, v \in \mathcal{A} \), \( u \prec v \), and check if we have either \( h(u) \prec h(v) \) and \( \sigma^R(u) \prec \sigma^R(v) \), or \( h(v) \prec h(u) \) and \( \sigma^R(v) \prec \sigma^R(u) \). If \( \mathcal{A} \) contains at least two folding points, we can choose \( u, v \) to be folding points. Then \( h(u) = \sigma^R(u) \) and \( h(v) = \sigma^R(v) \) and the claim follows.

Let us suppose now that \( \mathcal{A} \) contains at most one folding point. Then there exist \( q, p \in \mathbb{N}_0 \) such that \( h(C_q) \leq C_p \) and \( q \)-points \( u, v \in \mathcal{A} \), \( u \prec v \) (on the same side of the folding point if there exists one) such that \( \sigma^R(u) \) and \( \sigma^R(v) \) are contained in disjoint links of \( C_p \) each of which does not contain the folding point of \( \mathcal{A} \), if there exists one.

Let \( \ell^j_p, \ell^k_p \in C_p \) with \( j = L_p(\sigma^R(u)) \) and \( k = L_p(\sigma^R(v)) \) be links containing \( \sigma^R(u) \) and \( \sigma^R(v) \) respectively. Let \( W_u \subset \ell^j_p \) and \( W_v \subset \ell^k_p \) be arc-components containing \( \sigma^R(u) \) and \( \sigma^R(v) \) respectively. Then \( W_u \) and \( W_v \) do not contain any folding point and by Lemma 3.1 \( h(u) \in W_u \) and \( h(v) \in W_v \). Therefore obviously \( h(u) \prec h(v) \) if and only if \( \sigma^R(u) \prec \sigma^R(v) \).

If \( h \) is a homeomorphism that is pseudo-isotopic to the identity, then \( R = 0 \) and the claim of lemma follows.

**Corollary 3.4.** If \( h \) is pseudo-isotopic to the identity, then the arc \( A \) connecting \( x \) and \( h(x) \) is a single point, or \( A \) contains no folding point.

**Proof.** Since \( h \) is pseudo-isotopic to the identity, \( x \) and \( h(x) \) belong to the same composant, and in fact the same arc-component. So let \( A \) be the arc connecting \( x \) and \( h(x) \). If \( x = h(x) \), then there is nothing to prove. If \( h(x) \neq x \), say \( x \prec h(x) \), and \( A \) contains a folding point \( y \), then \( x \prec y = h(y) \prec h(x) \), contradicting Lemma 3.3.

In particular, any homeomorphism \( h \) that is pseudo-isotopic to the identity cannot reverse the bar of a \( \frac{1}{x} \)-continuum, or reverse a double spiral \( S \subset K_s \), see Remark 2.4. The next lemma strengthens Lemma 3.1 to the case that \( W_x \) is allowed to contain folding points.

**Lemma 3.5.** Let \( h : K_s \to K_s \) be a homeomorphism that is pseudo-isotopic to the identity. Let \( q, p \in \mathbb{N}_0 \) be such that \( h(C_q) \leq C_p \). Let \( x \) be a \( q \)-point in the core of \( K_s \) and let \( \ell^x_q \in C_p \) be such that \( l = L_p(x) \). Let \( W_x \subset \ell^x_q \) be an arc-component of \( \ell^x_q \) containing \( x \). Then \( h(x) \in W_x \).

**Proof.** If \( W_x \) does not contain any folding point the proof follows by Lemma 3.1 for \( R = 0 \).

Let \( W_x \) contain at least one folding point. If \( x \) is a folding point, then \( h(x) = x \in W_x \) by Theorem 1.1. If \( W_x \) contains at least two folding points, say \( y \) and \( z \), such that \( x \in [y, z] \subset W_x \), then \( h(x) \in [y, z] \subset W_x \) by Corollary 3.4.
The last possibility is that \( x \in (y, z) \subset W_z \), where \( z \in W_z \) is a folding point, \( y \notin W_x \), i.e., \( y \) is a boundary point of \( W_z \), and \((y, z)\) does not contain any folding point. Since \( \mathcal{C}_0 \) is dense in \( K_s \), there exists a sequence \((W_i)_{i \in \mathbb{N}}\) of arc-components of \( \ell^s_p \) such that \( W_i \subset \mathcal{C}_0 \) and \( W_i \to (y, z) \) in the Hausdorff metric. Note that for the sequence of \( p \)-points \((m_i)_{i \in \mathbb{N}}, \) where \( m_i \) is the midpoint of \( W_i \), we have \( m_i \to z \) and \( L_p(m_i) \to \infty \). Also, for every \( i \) large enough, every \( W_i \) contains a \( q \)-point \( x_i \) with \( L_q(x_i) = L_q(x) \), and for the sequence of \( q \)-points \((x_i)_{i \in \mathbb{N}} \) we have \( x_i \to x \). Obviously \((x_i)_{i \in \mathbb{N}} \subset \mathcal{C}_0 \) and \( L_p(x_i) = L_p(x) \). By the proof of [2, Proposition 4.2] applied for \( R = 0 \) we have \( h(x_i) \in W_i \) for every \( i \). Since \( h \) is a homeomorphism, \( h(x_i) \to h(x) \).

Therefore, \( h(x) \in (y, z) \subset W_x \).

\[ \square \]

**Proposition 3.6.** Let \( h : K_s \to K_s \) be a homeomorphism. If \( z^n \to z \) and \( A^n = [z^n, h(z^n)] \), then \( A^n \to A := [z, h(z)] \) in Hausdorff metric.

**Proof.** We know that \( h \) is pseudo-isotopic to \( \sigma^R \) for some \( R \in \mathbb{Z} \); by composing \( h \) with \( \sigma^{-R} \) we can assume that \( R = 0 \). By Corollary 3.2, \( h \) preserves the arc-components, and by Lemma 3.3 preserves the orientation of each arc-component as well.

Take a subsequence such that \( A^{n_k} \) converges in Hausdorff metric, say to \( B \). Since \( x, h(x) \in B \), we have \( B \supset A \). Assume by contradiction that \( B \neq A \). Fix \( q, p \) arbitrary such that \( h(C_q) \) refines \( C_p \), and such that \( \pi_p(B) \neq \pi_p(A) \) and a fortiori, that there is a link \( \ell \in C_p \) such that \( \ell \cap A = \emptyset \) and \( \pi_p(\ell) \) contains a boundary point of \( \pi_p(B) \).

Let \( d_n = \max\{L_p(y) : y \text{ is } p\text{-point in } A^n\} \). If \( D := \sup d_n < \infty \), then we can pass to the chain \( C_{p+D} \) and find that all \( A^{n_k} \)'s go straight through \( C_{p+D} \), hence the limit is a straight arc as well, stretching from \( x \) to \( h(x) \), so \( B = A \). Therefore \( D = \infty \), and we can assume without loss of generality that \( d_{n_k} \to \infty \).

Since the link in \( \ell \) is disjoint from \( A \) but \( \pi_p(\ell) \) contains a boundary point of \( \pi_p(B) \), the arcs \( A^{n_k} \) intersect \( \ell \) for all \( k \) sufficiently large. Therefore \( A^{n_k} \cap \ell \) separates \( x^{n_k} \) from \( h(x^{n_k}) \); let \( W^{n_k} \) be a component of \( A^{n_k} \cap \ell \) between \( x^{n_k} \) and \( h(x^{n_k}) \). Since \( \pi_p(\ell) \) contains a boundary point of \( \pi_p(B) \), \( W^{n_k} \) contains at least one \( p \)-point for each \( k \). Lemma 3.5 states that there is \( y^{n_k} \in W^{n_k} \) such that \( h(y^{n_k}) \in W^{n_k} \) as well, and therefore \( x^{n_k} \prec y^{n_k} \prec h(x^{n_k}) \) (or \( y^{n_k} \prec x^{n_k} \prec h(x^{n_k}) \prec h(y^{n_k}) \)), contradicting that \( h \) preserves orientation.

\[ \square \]

Let us finally prove Theorem 1.3:
Proof of Theorem 1.3. Fix $R$ such that $h$ is pseudo-isotopic to $\sigma^R$. Then $\sigma^{-R} \circ h$ is pseudo-isotopic to the identity. So renaming $\sigma^{-R} \circ h$ to $h$ again, we need to show that $h$ is isotopic to the identity.

If $x$ is a folding point of $K_s$, then $h(x) = x$ by Theorem 1.1. In this case, and in fact for any point such that $h(x) = x$, we let $H(x, t) = x$ for all $t \in [0, 1]$. If $h(x) \neq x$, then $x$ and $h(x)$ belong to the same arc-component, and the arc $A = [x, h(x)]$ contains no folding point by Corollary 3.4. By Lemma 2.2 $A$ contains only finitely many $p$-points, so there is $m$ such that $\pi_m : A \to \pi_m(A)$ is one-to-one. In this case,

$$H(x, t) = \pi_m^{-1}[(1 - t)\pi_m(x) + t\pi_m(h(x))].$$

Clearly $t \mapsto H(\cdot, t)$ is a family of maps connecting $h$ to the identity in a single path as $t \in [0, 1]$. We need to show that $H$ is continuous both in $x$ and $t$, and that $H(\cdot, t)$ is a bijection for all $t \in [0, 1]$.

Let $z \in K_s$ and $(z^n, t^n) \to (z, t)$. If $h(z) = z$, then $H(z, t) \equiv z$, and Proposition 3.6 implies that $H(z^n, t^n) \to z = H(z, t)$. So let us assume that $h(z) \neq z$. The arc $A = [z, h(z)]$ contains no folding point, so by Lemma 2.2 for all $x \in A$, there is $\varepsilon(x) > 0$ and $W(x) \in \mathbb{N}$ such that $B_{\varepsilon(x)}(x)$ contains no $p$-point of $p$-level $\geq W(x)$. By compactness of $A$, $\varepsilon := \inf_{x \in A} \varepsilon(x) > 0$ and $\sup_{x \in A} W(x) < \infty$, whence there is $m > p + W$ such that $V := \pi_m^{-1} \circ \pi_m(A)$ is contained in an $\varepsilon$-neighborhood of $A$ that contains no $p$-point.

By Proposition 3.6 there is $N$ such that $A^n \subset V$ for all $n \geq N$, and in fact $\pi_m(A^n) \to \pi_m(A)$. It follows that $H(z^n, t^n) \to H(z, t)$.

To see that $x \mapsto H(\cdot, t)$ is injective for all $t \in [0, 1]$, assume by contradiction that there is $t_0 \in [0, 1]$ and $x \neq y$ such that $H(x, t_0) = H(y, t_0)$. Then $x$ and $y$ belong to the same arc-component $\mathfrak{A}$, which is the same as the arc-component containing $h(x)$ and $h(y)$. The smallest arc $J$ containing all four point contains no folding point by Corollary 3.4. Therefore there is $m$ such that $\pi_m : J \to \pi_m(J)$ is injective, and we can choose an orientation on $\mathfrak{A}$ such that $x < y$ on $J$, and $\pi_m(x) < \pi_m(y)$. Since $t \mapsto \pi_m \circ H(x, t)$ is monotone with constant speed depending only on $x$, we find

$$\pi_m(x) < \pi_m(y) < \pi_m \circ H(x, t_0) = \pi_m \circ H(y, t_0) < \pi_m \circ h(y) < \pi_m \circ h(x)$$

This contradicts that $h$ preserves orientation on arc-components, see Lemma 3.3.

To prove surjectivity, choose $x \in K_s$ arbitrary. If $h(x) = x$, then $H(x, t) = x$ for all $t \in [0, 1]$. Otherwise, say if $h(x) > x$, there is $y < x$ in the same arc-component as $x$ such that $h(y) = x$. The map $t \mapsto H(\cdot, t)$ moves the arc $[y, x]$ continuously and monotonically to $[h(y), h(x)] = x$. 


\( [x, h(x)] \). Therefore, for every \( t \in [0, 1] \), there is \( y_t \in [y, x] \) such that \( H(y_t, t) = x \). This proves surjectivity.

We conclude that \( H(x, t) \) is the required isotopy between \( h \) and the identity. \( \square \)

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