On the Treatment of Neutrino Oscillations
Without Resort to Weak Eigenstates

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Abstract

We discuss neutrino oscillations in the framework of the quantum field theory without introducing the concept of neutrino weak eigenstates. The external particles are described by wave packets and the different mass eigenstate neutrinos propagate between the production and detection interactions, which are macroscopically localized in space-time. The time-averaged cross section, which is the measurable quantity in the usual experimental setting, is calculated. It is shown that only in the extremely relativistic limit the usual quantum mechanical oscillation probability can be factored out of the cross section.
1 Introduction

Neutrino oscillations have long been recognized as a powerful tool to probe the intrinsic properties of neutrinos [1]. Furthermore, it already appears that they may provide an elegant solution to the Solar Neutrino Problem, possibly leading to information on the basic properties of neutrinos such as mass and mixing angle.

If neutrinos are massive and mixed, a weak eigenstate neutrino which is produced in a weak process accompanying a lepton is a linear superposition of mass eigenstates. In the standard treatment of neutrino oscillations [1, 2], the mass eigenstates are assumed to be relativistic and to have the same momentum but different energies. Because of the energy differences, the quantum mechanical probability of finding weak eigenstates becomes a function of the distance from the production point, leading to neutrino oscillations.

Although the standard approach of treating neutrino oscillations with use of the weak eigenstates is physically intuitive and simple, it is, strictly speaking, neither rigorous nor sufficient for a complete understanding of the physics involved in neutrino oscillations. Furthermore, as shown in Ref.[3], the usual “weak eigenstates” \( |\nu_\alpha\rangle = \sum_a U_{\alpha a}^* |\nu_a\rangle \) (\( U \) is the mixing matrix of the neutrino fields and \( |\nu_a\rangle \) are the mass eigenstates) describe correctly the neutrinos produced and detected in weak-interaction processes only in the extremely relativistic limit. Also, energy-momentum conservation in the process in which the neutrino is created implies that the different mass-eigenstate components must have different momenta as well as different energies [4]. On the other hand, if the particles involved in the production (as well as the detection) process are assumed to have definite four-momenta, then the neutrino produced (detected) is forced to have a definite four-momentum, implying that the neutrino is one of the mass eigenstates [5]. This
observation suggests an apparent incompatibility between energy-momentum conservation in the production and detection processes and the neutrino oscillations. In other words, energy-momentum conservation which forces neutrinos to be in mass eigenstates is incompatible with neutrino oscillations. This apparent incompatibility, however, does not arise in a physical situation since a necessary condition for neutrino oscillations to occur is that the neutrino source and detector are localized within a region much smaller than the oscillation length, and hence the neutrino momentum must have at least the corresponding spread given by the uncertainty principle \[6\]. This spread is responsible for neutrino oscillations.

The localization of the neutrino source and detector and the spread of the neutrino momentum imply that a propagating flavor neutrino is not described by a superposition of plane waves, but instead by a superposition of localized wave packets \[7\]. The wave packet treatment necessary for a correct quantum mechanical description of neutrino oscillations has been discussed in Ref.\[7\]. However, neutrino oscillations have, so far, been discussed in the framework of quantum mechanics of neutrino propagation, whereas the effects of the production and detection weak-interaction processes have been neglected. As shown in Ref.\[3\], the neutrino oscillation probability is independent from the details of the production and detection processes only in the extremely relativistic limit. Hence a quantum field theoretical treatment of neutrino oscillations is necessary for the discussion of the case in which some of the mass eigenstates happen to be not extremely relativistic.

It is, of course, expected that a quantum field theoretical treatment must reproduce the quantum mechanical oscillation probability in the extremely relativistic limit.

In this paper, we present the quantum field theoretical treatment of neutrino oscillations by using, as an example, a specific flavor changing process (see Eq.\([1]\)) and by calculating its cross section. The external (initial and final) particles are described by wave packets and the mass eigenstate neutrinos propagate from the production region to a detector which are macroscopically separated in space-time. Since energies and
momenta of the external particles are not precisely defined and energy-momentum is conserved in the interaction vertices within the uncertainty principle, the contributions from the propagation of different mass eigenstate neutrinos can interfere to produce oscillations.

In the quantum field theoretical treatment it is impossible to derive a general formula for neutrino oscillations because the cross sections depend on the details of the specific production and detection interactions involved. Hence in this paper we illustrate a general method using a specific example. Furthermore, strictly speaking, an oscillation probability cannot even be defined because the space dependence of the cross section cannot be factorized out. This does not imply that neutrino oscillation phenomena do not take place. The phenomena can only be inferred from actual measurements of cross sections. It will be shown that the quantum field theoretical treatment yields the standard quantum mechanical oscillation probability only in the extremely relativistic limit.

The plan of this paper is as follows: In Section 2 we present the calculation of the amplitude for a specific flavor changing process. In Section 3 we calculate the cross section and its time average, which is an experimentally measurable quantity.

Finally, we discuss, in Section 4, the case of extremely relativistic neutrinos and reproduce the standard oscillation probability.

2 Amplitude

Let us consider the weak flavor-changing processes

\[ P_I \rightarrow P_F + \mu^+ + \nu \rightarrow \nu + D_I \rightarrow D_F + e^- , \]

occurring through the intermediate propagation of a neutrino, where \( P_I \) and \( P_F \) (\( D_I \) and \( D_F \)) are the initial and final production (detection) particles. For simplicity, we denote
the energy-momentum four vectors for the particles involved in Eq. (1) as
\[ p = p' + p_\mu + p_\nu, \quad p_\nu + k = k' + p_e. \] (2)

We will consider a process in which the production and detection interactions are macroscopically localized at the coordinates \((\vec{X}_P, T_P)\) and \((\vec{X}_D, T_D)\), respectively.

The relevant weak interaction Lagrangians are
\[ \mathcal{L}_P(x) = \frac{G_F}{\sqrt{2}} \sum_a U_{\alpha a}^* \bar{\nu}_a(x) \gamma^\alpha (1 + \gamma_5) \mu(x) J_\mu^P(x) \]
\[ \mathcal{L}_D(x) = \frac{G_F}{\sqrt{2}} \sum_a U_{\alpha a} \bar{e}_a(x) \gamma^\alpha (1 + \gamma_5) \nu_a(x) J_\nu^D(x), \]
where \(J_\mu^P(x)\) and \(J_\nu^D(x)\) are the weak currents of the production and detection particles, respectively and the other notations are self-evident.

The amplitude for the process is
\[ \mathcal{A} = \left\langle P_F, \mu^+, D_F, e^- \right| T \left[ \int d^4x_1 \int d^4x_2 \mathcal{L}_P(x_1) \mathcal{L}_D(x_2) \right] \right| P_I, D_I \right\rangle, \]
where the initial and final particles are described by the wave packets
\[ |P_I\rangle = \int \frac{d\vec{p}}{(2\pi)^3/2} \psi_{P_I}(\vec{p}; \vec{X}_P, T_P, \langle \vec{p} \rangle) |P_I(\vec{p})\rangle \]
\[ \vdots \]
\[ |e^-\rangle = \int \frac{d\vec{p}_e}{(2\pi)^3/2} \psi_e(\vec{p}_e; \vec{X}_D, T_D, \langle \vec{p}_e \rangle) |e^-((\vec{p}_e)\rangle. \]

The form of the wavefunctions \(\psi\) in momentum space is determined by how the initial particles are prepared and how the final particles are detected. In the following we will assume, for simplicity, gaussian wavefunctions, whose form is given by Eq.(25) in Appendix A. In Eq.(4), \(\langle \vec{p}\rangle, \cdots, \langle \vec{p}_e\rangle\) are the average momenta of the particles around which their momenta are spread due to the uncertainty principle. The wave packets are constructed in such a way that at the time \(t = T_P\) the wave packets of the muon and the
production particles overlap at \( \vec{x} \simeq \vec{X}_P \) and at the time \( t = T_D \) the wave packets of the electron and the detection particles overlap at \( \vec{x} \simeq \vec{X}_D \).

The propagators of the mass eigenstate neutrinos are

\[ \langle 0 \mid T \left[ \nu_a(x_2)\overline{\nu_a}(x_1) \right] \mid 0 \rangle = i \int \frac{d^4q}{(2\pi)^4} \frac{\gamma + m_a}{q^2 - m_a^2 + i\epsilon} e^{-iq(x_2-x_1)}, \quad (6) \]

Hence the amplitude in Eq.(4) becomes, with Eqs. (5) and (6),

\[
A \propto \sum_a U_{\mu a}^\ast U_{\nu a} \int d^4x_1 \int d^4x_2 \int \frac{d^4q}{(2\pi)^4} e^{-iq(x_2-x_1)}
\times \int \frac{d\vec{p}}{(2\pi)^3/2} \psi_{P_1} e^{-ip \cdot x_1} \int \frac{d\vec{k}}{(2\pi)^3/2} \psi_{D_1} e^{-ik \cdot x_2}
\times \int \frac{d\vec{p}'}{(2\pi)^3/2} \psi_{P_p}^\ast e^{ip' \cdot x_1} \int \frac{d\vec{p}_\mu}{(2\pi)^3/2} \psi_{\mu}^\ast e^{ip_\mu \cdot x_1}
\times \int \frac{d\vec{p}_e}{(2\pi)^3/2} \psi_{D_F}^\ast e^{i\vec{k} \cdot x_2} \int \frac{d\vec{p}_e'}{(2\pi)^3/2} \psi_{e}^\ast e^{i\vec{p}_e \cdot x_2}
\times J^D_\lambda(k,k') \frac{1}{\mu} e(p_e) \gamma^\lambda \frac{(1 + \gamma_5)q}{q^2 - m_a^2 + i\epsilon} \gamma^\mu v(p_\mu) J^P_\mu(p,p'),
\quad (7)
\]

where \( J^D_\lambda(k,k') \) and \( J^P_\mu(p,p') \) are the matrix elements of the weak currents of the detection and production particles, respectively and we have not explicitly written down the arguments of the wave packets.

The momentum integrations of external particles can easily be carried out if the wave packets in momentum space are sharply peaked around their average momenta (we have assumed, for simplicity, that all the wave packets are gaussian with the same width \( \sigma_x \)). After straightforward but tedious integrations, the amplitude can be written, with appropriate changes of the coordinates, as

\[
A \propto \sum_a U_{\mu a}^\ast U_{\nu a} \int \frac{d^4q}{(2\pi)^4} \frac{\gamma}{q^2 - m_a^2 + i\epsilon} V_P \exp \left[ -i q_0 T + i \vec{q} \cdot \vec{L} \right]
\times \int d^4x_1 \exp \left[ -i (E_P - q_0) t_1 + i (\vec{p}_P - \vec{q}) \cdot \vec{x}_1 - 3 \frac{x_1^2}{4\sigma_x^2} + 3 \frac{\vec{v}_P \cdot \vec{x}_1}{2\sigma_x^2} - t_1 - \frac{\vec{q}^2 + \vec{\sigma}_P^2 + \vec{\sigma}_x^2}{4\sigma_x^2} t_1^2 \right]
\times \int d^4x_2 \exp \left[ -i (E_D + q_0) t_2 + i (\vec{p}_D + \vec{q}) \cdot \vec{x}_2 - 3 \frac{x_2^2}{4\sigma_x^2} + 3 \frac{\vec{v}_D \cdot \vec{x}_2}{2\sigma_x^2} - t_2 - \frac{\vec{q}^2 + \vec{\sigma}_D^2 + \vec{\sigma}_x^2}{4\sigma_x^2} t_2^2 \right],
\quad (8)
\]
where we have defined
\[
E_P \equiv \langle p_0 \rangle - \langle p'_0 \rangle - \langle E_\mu \rangle \\
E_D \equiv \langle k_0 \rangle - \langle k'_0 \rangle - \langle E_e \rangle \\
3\vec{v}_P \equiv \vec{v} + \vec{v}' + \vec{v}_\mu \\
3\vec{v}_D \equiv \vec{u} + \vec{u}' + \vec{v}_e \\
T \equiv T_D - T_P \\
U_D \equiv J_\alpha^D (\langle k \rangle, \langle k' \rangle, \langle p_e \rangle) \gamma^\alpha (1 + \gamma_5) \\
V_P \equiv (1 + \gamma_5)\gamma^\alpha v_\mu (\langle p_\mu \rangle) J_\alpha^P (\langle p \rangle, \langle p' \rangle) .
\]

Also in Eqs.(8) and (9), we have introduced the notation
\[
\vec{v} = \langle \vec{p} \rangle / \langle p_0 \rangle, \quad \vec{v}' = \langle \vec{p}' \rangle / \langle p'_0 \rangle, \quad \vec{v}_\mu = \langle \vec{E}_\mu \rangle / \langle E_\mu \rangle \\
\vec{u} = \langle \vec{k} \rangle / \langle k_0 \rangle, \quad \vec{u}' = \langle \vec{k}' \rangle / \langle k'_0 \rangle, \quad \vec{v}_e = \langle \vec{E}_e \rangle / \langle E_e \rangle.
\]

Carrying out the integrals over \( x_1 \) and \( x_2 \) which are gaussian, we obtain, from Eq.(8),
\[
\mathcal{A} \propto \sum_a \mathcal{U}_{\alpha a}^{\mu a} \mathcal{U}_{e a} \int \frac{d^4q}{(2\pi)^4} U_D \frac{\hat{q}}{q^2 - m^2 + i\epsilon} V_P \exp \left[ -i q_0 T + i \vec{q} \cdot \vec{L} \right] \times \exp \left[ -\frac{(\vec{p}_P - \vec{q})^2}{12\sigma^2_p} - \frac{(E_P - q_0) - (\vec{p}_P - \vec{q}) \cdot \vec{v}_P}{12\sigma^2_p \lambda_P} - \frac{(\vec{p}_D + \vec{q})^2}{12\sigma^2_p} - \frac{(E_D + q_0) - (\vec{p}_D + \vec{q}) \cdot \vec{v}_P}{12\sigma^2_p \lambda_D} \right]
\]
with
\[
\lambda_P \equiv \frac{1}{3} \left( \vec{v}^2 + \vec{v}'^2 + \vec{v}^2_\mu \right) - \vec{v}_P^2 \\
\lambda_D \equiv \frac{1}{3} \left( \vec{u}^2 + \vec{u}'^2 + \vec{v}_e^2 \right) - \vec{v}_D^2 .
\]

We now face the problem of performing the integration over \( q \). In usual calculations of the processes occurring through the propagation of a virtual intermediate particle the integration over its four-momentum \( q \) is easily simplified by the Dirac \( \delta \)-functions arising from energy-momentum conservation in the interaction vertices. On the other hand, in the wave packet treatment of the initial and final particles, energy-momentum is not exactly conserved and there are no Dirac \( \delta \)-functions available for the simplification of the integration over \( q \). However, in our case the production and detection interactions are macroscopically separated, so only the propagation of real neutrinos contribute significantly to the process. This physical fact allows us to perform the integration over \( q_0 \).
by closing the integration path in the lower half of the complex plane. In fact, if the contribution of the additional path in the lower half of the complex plane can be neglected, this procedure picks up only the contribution from the neutrino pole which lies inside the path, whereas the antineutrino contribution is neglected. However, the choice of the contour needs some caution because the term \(-q_0^2\) in the exponent diverges as \(q_0 \to -i\infty\) and this prevents us from using the usual half circle contour that encircles the lower half of the complex plane. Instead, the appropriate integration path has the form of a rectangle whose lower side dissects the imaginary axis at \(q_0 = -i \left[6\sigma_p^2 \lambda_P \lambda_D/(\lambda_P + \lambda_D)\right] T\). The contributions from the three sides except the real axis are negligible. The integrals along the sides at \(\pm\infty\) vanish since the sides are finite in length and are damped by the \(-q_0^2\) term in the exponent. The lower side gives a finite result with an exponentially damping factor \(-3\sigma_p^2 \lambda_P \lambda_D/(\lambda_P + \lambda_D)\) \(T^2\) (easily obtained with a saddle point approximation) which suppresses strongly its contribution for macroscopic time separations. Therefore the integration over \(q_0\) is dominated by the neutrino pole which lies inside the integration contour. The resulting amplitude is given by

\[
\mathcal{A} \propto \sum_a U_{\mu a}^* U_{\nu a} \int \frac{d\vec{q}}{(2\pi)^3} U_D \frac{\gamma^0 E_a(\vec{q}) - \vec{\gamma} \cdot \vec{q}}{E_a(\vec{q})} V_P \exp \left[-iE_a(\vec{q})T + i\vec{q} \cdot \vec{L} - S_a(\vec{q})\right],
\]

where \(E_a(\vec{q}) = \sqrt{\vec{q}^2 + m_a^2}\) and

\[
S_a(\vec{q}) = \frac{(p_P - \vec{q})^2}{12\sigma_P^2} + \frac{[(E_P - E_a(\vec{q})) - (\vec{p}_P - \vec{q}) \cdot \vec{v}_P]^2}{12\sigma_P^2 \lambda_P} + \frac{(p_D + \vec{q})^2}{12\sigma_D^2} + \frac{[(E_D + E_a(\vec{q})) - (\vec{p}_D + \vec{q}) \cdot \vec{v}_D]^2}{12\sigma_D^2 \lambda_D}.
\]

Since \(\sigma_P\) is small and \(\lambda_P\) and \(\lambda_D\) are of order of unity, the integral over \(d\vec{q}\) is dominated by the minimum of \(S_a(\vec{q})\), which occurs at \(\vec{q} = \vec{q}_a\), given by

\[
\left(\frac{\vec{v}_P - \vec{v}_a}{\lambda_P}\right) [E_P - E_a - \vec{v}_P \cdot (\vec{p}_P - \vec{q}_a)] - (\vec{p}_P - \vec{q}_a) + \left(\frac{\vec{v}_D + \vec{v}_a}{\lambda_D}\right) [E_D + E_a - \vec{v}_D \cdot (\vec{p}_D + \vec{q}_a)] - (\vec{p}_D + \vec{q}_a) = 0,
\]

(14)
where \( E_a \equiv E_a(\vec{q}_a) = \sqrt{\vec{q}_a^2 + m_a^2} \) and \( \vec{v}_a \equiv \vec{q}_a/E_a \) are the velocities of the mass eigenstate neutrinos propagating between the two interaction vertices. A saddle point approximation of the integral over \( d\vec{q} \) leads to

\[
\mathcal{A} \propto \sum_a \mathcal{U}_{\mu a}^* \mathcal{U}_{e a} \mathcal{A}_a \exp \left[ -iE_a T + i\vec{q}_a \cdot \vec{L} - S_a(\vec{q}_a) - \frac{1}{2} \left( \vec{L} - \vec{v}_a T \right) \Omega_a^{-1} \left( \vec{L} - \vec{v}_a T \right) \right] \tag{15}
\]

where

\[
\mathcal{A}_a \equiv \frac{1}{\sqrt{\text{Det}(\Omega_a)}} \gamma_0 E_a - \vec{\gamma} \cdot \vec{q}_a \sqrt{V_p} \quad \Omega_a_{ij} \equiv \frac{\delta_{ij}}{3\sigma_p^2} + \frac{(\vec{v}_P - \vec{v}_a)_i (\vec{v}_P - \vec{v}_a)_j}{6\lambda_P\sigma_p^2} + \frac{(\vec{v}_D + \vec{v}_a)_i (\vec{v}_D + \vec{v}_a)_j}{6\lambda_D\sigma_p^2}. \tag{16}
\]

In Eq.(15) \( \vec{V}M\vec{V} \) denotes \( \sum_{ij} V_i M_{ij} V_j \) for arbitrary vector \( \vec{V} \) and matrix \( M \).

The amplitude (15) describes the process under consideration with the assumptions that the wave packets of the external particles are sharply peaked around their average momenta and the production and detection processes are macroscopically separated in space-time. The amplitude contains the space-time dependent phase factor \( \exp \left[ -iE_a T + i\vec{q}_a \cdot \vec{L} \right] \) which gives rise to the conventional neutrino oscillations. The exponential damping factor \( \exp \left[ -S_a(\vec{q}_a) \right] \) implements the overall energy-momentum conservation only within an uncertainty \( \sigma_p \) (if \( |E_P + E_D| \lesssim \sigma_p \) and \( |\vec{p}_P + \vec{p}_D| \lesssim \sigma_p \) then \( \exp[-S_a(\vec{q}_a)] \simeq 1 \)). Due to the damping factor \( \exp \left[ -\frac{1}{2} \left( \vec{L} - \vec{v}_a T \right) \Omega_a^{-1} \left( \vec{L} - \vec{v}_a T \right) \right] \), since the matrices \( \Omega_a \) are proportional to \( 1/\sigma_p^2 \simeq \sigma_x^2 \) (see Eq.(16)), the amplitude in Eq.(15) is non-vanishing if the velocities of the mass eigenstate neutrinos satisfy

\[
\left| \vec{L} - \vec{v}_a T \right| \lesssim \sigma_x. \tag{17}
\]

However, if the mass difference between the mass eigenstates \( \nu_a \) and \( \nu_b \) is such that \( |\vec{v}_a - \vec{v}_b| T \gtrsim \sigma_x \), then the condition (17) cannot be satisfied by both mass eigenstates. In this case, at a given time \( T \) the amplitude has two (or more) separate peaks in space corresponding to the two (or more) mass eigenstate neutrinos and the experiment measures only a constant probability (in space) for the flavor changing process under
consideration. This is due to the fact that the wave packets of the two mass eigenstates are separated by a distance larger than their width and, since they do not overlap, the interference term that produces the neutrino oscillations is damped out.

3 Cross Section

The cross section for the process Eq.(1) is given, from Eq.(15), by

\[
\sigma(\vec{L}, T) \propto \int d\vec{P} \sum_{a,b} A_a A_b^* U_{a}^{\mu a} U_{ea} U_{pb} U_{eb}^* \exp \left[ -S_a (\vec{q}_a) - S_b (\vec{q}_b) \right] \\
\times \exp \left[ -i (E_a - E_b) T + i (\vec{q}_a - \vec{q}_b) \cdot \vec{L} \right] \\
\times \exp \left[ -\frac{1}{2} \left( \vec{L} - \vec{v}_a T \right) \Omega_a^{-1} \left( \vec{L} - \vec{v}_a T \right) - \frac{1}{2} \left( \vec{L} - \vec{v}_b T \right) \Omega_b^{-1} \left( \vec{L} - \vec{v}_b T \right) \right],
\]

(18)

where \( \int d\vec{P} \) represents the integration over the 3-momenta and the sum over the spins of the final particles; one must also include appropriate average over the 3-momenta and spins of the initial particles which are not measured.

In a practical experimental setting, the distance \( \vec{L} \) is usually a fixed and known quantity, whereas the time \( T \) is not. Therefore, the cross section at a given distance \( \vec{L} \) is given by the time average of \( \sigma(\vec{L}, T) \). We take \( \vec{L} \) along the \( z \) direction and integrate over time to obtain

\[
\sigma(L) \propto \int d\vec{P} \sum_{a,b} A_a A_b^* U_{a}^{\mu a} U_{ea} U_{pb} U_{eb}^* \left[ \vec{v}_a \Omega_a^{-1} \vec{v}_a + \vec{v}_b \Omega_b^{-1} \vec{v}_b \right]^{-1/2} \\
\times \exp \left[ -S_a (\vec{q}_a) - S_b (\vec{q}_b) \right] \\
\times \exp \left\{ i \left[ (q_{az} - q_{bz}) - (E_a - E_b) \frac{(\Omega_a^{-1} \vec{v}_a)_z + (\Omega_b^{-1} \vec{v}_b)_z}{\vec{v}_a \Omega_a^{-1} \vec{v}_a + \vec{v}_b \Omega_b^{-1} \vec{v}_b} \right] L \right\} \\
\times \exp \left\{ -\frac{L^2}{2} \left[ \Omega_a^{-1} \right]_{zz} + \left[ \Omega_b^{-1} \right]_{zz} - \frac{\left[ (\Omega_a^{-1} \vec{v}_a)_z + (\Omega_b^{-1} \vec{v}_b)_z \right]^2}{\vec{v}_a \Omega_a^{-1} \vec{v}_a + \vec{v}_b \Omega_b^{-1} \vec{v}_b} \right\} \\
\times \exp \left\{ -\frac{1}{2} \frac{(E_a - E_b)^2}{\vec{v}_a \Omega_a^{-1} \vec{v}_a + \vec{v}_b \Omega_b^{-1} \vec{v}_b} \right\}.
\]

(19)
The first exponential term in Eq.(19) guarantees energy-momentum conservation within the accuracy of the uncertainty principle. The second gives rise to the neutrino oscillation in terms of the distance $L$. The third is a damping factor which describes the coherence of the process allowing significant contributions only from the propagation of the mass eigenstate neutrinos in the $z$ direction. In fact, since $\Omega_a \sim \sigma_x^2$ and $L^2/\sigma_x^2 \gg 1$, the integration over the 3-momenta of the final particles receives its dominant contribution when this damping exponential factor becomes maximal. For each pair $a$ and $b$, this maximum occurs when both $\vec{v}_a$ and $\vec{v}_b$ are in the $z$ direction. From the stationary equation (14), this is realized when all $\vec{p}_P$, $\vec{v}_P$, $\vec{p}_D$ and $\vec{v}_D$ are in the $z$ direction. In this case the matrices $\Omega_a$ and $\Omega_b$ are diagonal (see Eq.(16)). Let us denote with underlines all the quantities evaluated at the maximum of the damping exponential and perform a saddle point approximation of the integration over the angular variables that parameterize the deviation of $\vec{v}_a$ and $\vec{v}_b$ from the $z$ direction. As shown in Appendix B, the result can be written as

$$\sigma(L) \propto \frac{1}{L^2} \int d\vec{P} \sum_{a,b} F_{ab}(L) A_a A_b^* U_{\mu a} U_{\mu b}^* \left[ \vec{v}_a \Omega_a^{-1} \vec{v}_a + \vec{v}_b \Omega_b^{-1} \vec{v}_b \right]^{-1/2}$$

$$\times \exp \left[ -S_a (\vec{q}_a) - S_b (\vec{q}_b) \right]$$

$$\times \exp \left\{ i \left( \vec{q}_{az} - \vec{q}_{bz} \right) - (E_a - E_b) \left[ \frac{1}{\Omega_a} \left[ \Omega_a^{-1} \vec{v}_{az} + \Omega_b^{-1} \vec{v}_{bz} \right] \right] L \right\}$$

$$\times \exp \left\{ -\frac{L^2}{2} \left[ \frac{(\vec{v}_{az} - \vec{v}_{bz})^2}{\Omega_a} + \frac{(\vec{v}_{az} - \vec{v}_{bz})^2}{\Omega_b} \right] \right\}$$

$$\times \exp \left\{ -\frac{1}{2} \left[ \frac{(E_a - E_b)^2}{\Omega_a} + \frac{(E_a - E_b)^2}{\Omega_b} \right] \right\},$$

where $F_{ab}(L)$ is a factor which is weakly dependent on $L$. Since $F_{ab}(L)$ becomes constant for large $L$, as shown in Appendix B, we shall neglect $F_{ab}(L)$ in the following. In Eq.(20), $\int d\vec{P}$ represents the remaining integrations over the momenta of the initial and final particles. The factor $1/L^2$ represents the geometric decrease of the neutrino flux due to the distance of propagation, $L$. It is important to point out here that the second exponential in Eq.(20) cannot be factored out to derive the oscillation probability as in
the usual treatment of oscillations.

Equation (20) also contains a damping factor which decreases exponentially with $L^2$ and measures the coherence of the contributions of the different mass eigenstate neutrinos. The coherence length for $a \neq b$ is defined by

$$L_{ab}^{\text{coh}} \sim \sqrt{\frac{[\Omega_b]_{zz} v_{az}^2 + [\Omega_a]_{zz} v_{bz}^2}{(v_{az} - v_{bz})^2} \sim \sigma_x \sqrt{\frac{v_{az}^2 + v_{bz}^2}{(v_{az} - v_{bz})^2}},}$$

(21)
beyond which neutrinos do not practically oscillate. This coherence length can be very large in the case of relativistic neutrinos, for which $|v_{az} - v_{bz}| \ll 1$.

The last factor in Eq. (20) is due to the time integration and suppresses the interference of the contributions coming from the propagation of different mass eigenstate neutrinos unless $|E_a - E_b| \lesssim \sigma_p$, as it should be from energy conservation in both the production and detection interactions.

4 Relativistic Limit

As we have emphasized, although our general result given in Eq. (20) exhibits characteristics of neutrino oscillations, the oscillation probability could not, in general, be factored out. We now demonstrate that this can be done when intermediate neutrinos are extremely relativistic.

Let us consider a process in which all the intermediate mass eigenstate neutrinos are relativistic, i.e. $m_a \ll E_a$. In this case, the momentum $\vec{q}_{az}$, the energy $E_a$ and the velocity $\vec{v}_{az}$ can be expanded as

$$\vec{q}_{az} = \vec{q}_{0z} + \vec{v}_{az},$$
$$E_a = \vec{q}_{0z} + \vec{v}_{az} + \frac{m_a^2}{2q_{0z}},$$
$$v_{az} = 1 - \frac{m_a^2}{2q_{0z}^2}$$

(22)
where \( q_{0z} \) is the solution of the stationary equation (14) in the \( z \) direction for \( m_a = 0 \) and \( \epsilon_{az} \sim m_a^2/q_{0z} \ll q_{0z} \) is given by the solution of the stationary equation (14) in the \( z \) direction to first order in \( m_a^2/q_{0z} \). To lowest order in the relativistic approximation the space-dependent part of \( \sigma(L) \) can be factorized as \( \sigma(L) = P(L)\sigma_0 \), where \( \sigma_0 \) is the cross section for massless neutrinos. The space-dependent probability \( P(L) \) is then given by

\[
P(L) = \sum_{a,b} U_{\mu a}^* U_{\mu a} U_{\mu b}^* U_{\mu b} \exp \left\{ -i\frac{m_a^2 - m_b^2}{2q_{0z}} L \right\} \times \exp \left\{ -\frac{L^2}{2} \left[ \frac{\Omega^{-1}}{zz} \right] \left( \frac{m_a^2 - m_b^2}{2q_{0z}} \right)^2 - \frac{\left( \epsilon_{az} - \epsilon_{bz} + \frac{m_a^2 - m_b^2}{2q_{0z}} \right)^2}{4 \left[ \frac{\Omega^{-1}}{zz} \right]} \right\}
\]

(23)

The first line of Eq.(23) gives the usual oscillation probability for relativistic neutrinos (which can be obtained from a quantum mechanical treatment of the neutrino oscillations \([1, 2, 6, 8, 9]\)). The second line of Eq.(23) contains an exponent which decreases quadratically with the distance \( L \) and measures the coherence of the contributions due to the wave packets of the different mass eigenstate neutrinos. Since \( \left[ \frac{\Omega^{-1}}{zz} \right] \sim \sigma_0^2 \), the coherence length becomes

\[
L_{coh}^{ab} \sim \sigma_x \frac{2q_{0z}^2}{m_a^2 - m_b^2}.
\]

(24)

The length, \( L_{coh}^{ab} \), is the coherence length for the neutrino oscillations, i.e. the two mass eigenstate neutrinos \( \nu_a \) and \( \nu_b \) contribute coherently to the flavor changing process only when \( L \ll L_{coh}^{ab} \), in which case the probability oscillates as a function of the distance \( L \).

The coherence length given in Eq.(24) is the same as that obtained by physical intuitions in Ref.\([1, 6, 10]\) and from a quantum mechanical wave-packet treatment in Ref.\([7]\). If the distance \( L \) is much smaller than the coherence length \( L_{coh}^{ab} \), the damping factor in the probability (23) becomes approximately unity and hence one obtains the usual oscillation probability. On the other hand, for distances \( L \gg L_{coh}^{ab} \) the two mass eigenstate neutrinos \( \nu_a \) and \( \nu_b \) contribute incoherently to the flavor changing process.
From Eq. (23), the well-known oscillation wavelength $L_{ab}^{\text{osc}}$ is

$$L_{ab}^{\text{osc}} = 2\pi \frac{2q_{0z}}{|m_a^2 - m_b^2|}$$

(25)

so that Eq. (24) can be written as

$$L_{ab}^{\text{coh}} \sim \frac{q_{0z}}{\sigma_p} L_{ab}^{\text{osc}}.$$  

(26)

Hence, the maximum number of observable oscillations before the decoherence of the wave packets of the different mass eigenstate neutrinos is given by $[8, 3, 10]$

$$N_{\text{osc}} = \frac{L_{ab}^{\text{coh}}}{L_{ab}^{\text{osc}}} \sim \frac{q_{0z}}{\sigma_p}.$$  

(27)

Notice that $N_{\text{osc}}$ is independent of the neutrino mass eigenvalues $m_a$. Since we have assumed that $\sigma_p$ is much smaller than the energies of the initial and final particles involved, $\sigma_p \ll q_{0z}$ and $N_{\text{osc}} \gg 1$.

Finally, in the last exponential factor of the probability (23) we have retained a space-independent damping factor. Since

$$\epsilon_{az} - \epsilon_{bz} + \frac{m_a^2 - m_b^2}{2q_{0z}} \sim \frac{m_a^2 - m_b^2}{q_{0z}} \sim \frac{1}{L_{ab}^{\text{osc}}}$$

(28)

the probability (23) for $a \neq b$ does not vanish only when $L_{ab}^{\text{osc}} \gg \sigma_x$. This result is due to the time integration: if the neutrino wave packets are larger than the oscillation length, the interference terms are washed out.

5 Conclusions

In order to illustrate the quantum field theoretical treatment of neutrino oscillations without introducing the concept of weak eigenstates, we have discussed a specific flavor
changing process (see Eq. (1)) in which the external particles are described by wave packets and the mass eigenstate neutrinos propagate between the production and detection interactions which are macroscopically localized in space-time. We have calculated the time-averaged cross section which is the measurable quantity in the usual experimental setting where the distance between the production and detection interactions is known but the time separation is not measured. We have pointed out that in general, it is not possible to factor out of the cross section a space-dependent oscillation probability because the dynamics of the production and detection interactions is not the same for the different mass eigenstate neutrinos. However, we have shown that in the extremely relativistic limit the usual quantum mechanical oscillation probability can be factored out of the cross section.

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A Wave Packet

A gaussian wave packet in the momentum space is given by

\[
\psi(\vec{p}; \vec{X}, T, \langle \vec{p} \rangle) = \left[ \sqrt{2\pi} \sigma_p \right]^{-3/2} \exp \left[ -\frac{(\vec{p} - \langle \vec{p} \rangle)^2}{4\sigma_p^2} - i\vec{p} \cdot \vec{X} + iE(\vec{p})T \right],
\]

(29)

where \( E(\vec{p}) \equiv \sqrt{\vec{p}^2 + m^2} \) and \( \sigma_p \) is the width of the wave packet for simplicity assumed to be the same along the three directions.

In the coordinate space, we have

\[
\psi(\vec{x}, t; \vec{X}, T, \langle \vec{p} \rangle) = \int \frac{d\vec{p}}{(2\pi)^{3/2}} \psi(\vec{p}; \vec{X}, T, \langle \vec{p} \rangle) e^{i\vec{p} \cdot \vec{x} - iE(\vec{p})t}.
\]

(30)

Since the gaussian wave packet in the momentum space is peaked around the average momentum \( \langle \vec{p} \rangle \), neglecting the spreading of the wave packet, one can approximate

\[
E(\vec{p}) \simeq \langle E \rangle + \vec{v} (\vec{p} - \langle \vec{p} \rangle);
\]

\[
\langle E \rangle \equiv E(\langle \vec{p} \rangle) = \sqrt{\langle \vec{p} \rangle^2 + m^2}
\]

(31)

\[
\vec{v} \equiv \left. \frac{\partial E}{\partial \vec{p}} \right|_{\vec{p} = \langle \vec{p} \rangle} = \frac{\langle \vec{p} \rangle}{\langle E \rangle}.
\]

Hence the wave packet in the coordinate space is found to be

\[
\psi(\vec{x}, t; \vec{X}, T, \langle \vec{p} \rangle) \simeq \left[ \sqrt{2\pi} \sigma_x \right]^{-3/2} \exp \left[ i \langle \vec{p} \rangle \cdot (\vec{x} - \vec{X}) - i \langle E \rangle (t - T) - \frac{\left( (\vec{x} - \vec{X}) - \vec{v} (t - T) \right)^2}{4\sigma_x^2} \right].
\]

(32)

At time \( t = T \) the wave packet is peaked at \( \vec{x} = \vec{X} \) with a width \( \sigma_x \) given by

\[
\sigma_x \sigma_p = \frac{1}{2}
\]

(33)
in agreement with the uncertainty principle. The wave packet propagates in space with group velocity $\vec{v}$.

B Saddle Point Approximation

The 2-dimensional integration over the angular variables $\vec{\xi} = (\xi_1, \xi_2)$ that parameterize the deviation of $\vec{v}_a$ and $\vec{v}_b$ from the $z$ direction in Eq.(19) is of the type

$$I = \int d\vec{\xi} \exp \left[ iA(\vec{\xi})L - B(\vec{\xi})L^2 - C(\vec{\xi}) \right].$$

(34)

For large $L$ the integral gets its dominant contribution from the minimum of $B(\vec{\xi})$, which occurs for $\vec{\xi} = \vec{\xi}_0$. We expand all the terms around this minimum:

$$A(\vec{\xi}) = A + \sum_{i=1,2} A_i' (\xi^i - \xi^i_0)$$

$$B(\vec{\xi}) = B + \frac{1}{2} \sum_{i,j=1,2} (\xi^i - \xi^i_0) B_{ij}' (\xi^j - \xi^j_0)$$

$$C(\vec{\xi}) = C + \sum_{i=1,2} C_i' (\xi^i - \xi^i_0).$$

(35)

After a change of variable $(\xi^i - \xi^i_0) \rightarrow \xi^i$, the integral in Eq.(34) can be written in gaussian form

$$I = \exp \left[ iAL - BL^2 - C \right]$$

$$\times \exp \left[ \frac{1}{2} \left( iA_i' - \frac{1}{L} C_i' \right) \left[ B_{ij}'^{-1} \right]^{ij} \left( iA_j' - \frac{1}{L} C_j' \right) \right]$$

$$\times \int d\vec{\xi} \exp \left\{ -\frac{L^2}{2} \left[ \xi^i - \frac{1}{L} (iA_i' - \frac{1}{L} C_i') \left[ B_{ij}'^{-1} \right]^{ij} \left[ \xi^j - \frac{1}{L} (iA_j' - \frac{1}{L} C_j') \left[ B_{ij}'^{-1} \right]^{ij} \right] \right\}.$$

(36)

The final result is

$$I = \frac{2\pi}{L^2} \frac{F(L)}{\sqrt{\det(B_{ij}'^{-1})}} \exp \left[ iAL - BL^2 - C \right]$$

(37)

with

$$F(L) = \exp \left[ \frac{1}{2} \left( iA_i' - \frac{1}{L} C_i' \right) \left[ B_{ij}'^{-1} \right]^{ij} \left( iA_j' - \frac{1}{L} C_j' \right) \right].$$

(38)

From Eq.(38) it is clear that the space dependence of $F(L)$ is negligible for large $L$. 

17
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