UNIQUENESS FOR ELLIPTIC PROBLEMS WITH HÖLDER–TYPE DEPENDENCE ON THE SOLUTION

LUCIO BOCCARDO
Dipartimento di Matematica, Università di Roma La Sapienza
Piazzale Aldo Moro 2, 00185 Roma, Italy.

ALESSIO PORRETTA
Dipartimento di Matematica, Università di Roma Tor Vergata
Via della Ricerca Scientifica 1, 00133 Roma, Italy.

Abstract. We prove uniqueness of weak (or entropy) solutions for nonmonotone elliptic equations of the type

$$-\text{div}(a(x,u)\nabla u) = f$$

in a bounded set $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary conditions. The novelty of our results consists in the possibility to deal with cases when $a(x,u)$ is only Hölder continuous with respect to $u$.

1. Introduction. Let $\Omega$ be a bounded subset of $\mathbb{R}^N$, $N \geq 1$, and consider the nonlinear elliptic Dirichlet problem

$$\begin{cases}
-\text{div}(a(x,u)\nabla u) = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

(1)

In many situations, one is led to consider the case that $a(x,u)$ has either unbounded or singular growth with respect to $u$. For example, in the paper [17], A. G. Kartsatos and I. V. Skrypnik used the differential operator

$$A(v) = -\text{div}([a(x) + e^{2v}]\nabla v)$$

(2)

as a simple example of elliptic operator with strong coefficient growth, where $a(x)$ is a measurable function such that $0 < \alpha \leq a(x) \leq \beta$ for some $\alpha, \beta \in \mathbb{R}^+$. Classical questions as existence or uniqueness of solutions have to be handled with care in order to deal with such examples. Even the formulation itself of the problem, and the notion of solution considered, deserves some attention, since weak solutions may not have sense. To this purpose, the notions of renormalized solution or entropy solution, introduced in [8], [2] respectively, have proved to be suitable, in particular to deal with the case of coefficients with unbounded growth with respect to $u$. For example, a general existence result of entropy solutions of (1) when $f \in L^1(\Omega)$ is proved in [18], which in particular applies to the operator (2). In particular cases, using the growth of the operator, it is also possible to show that such solutions are more regular, for example if $f \in H^{-1}(\Omega)$ and the operator is coercive then $u \in H^1_0(\Omega)$ and even more, see e.g. [5], [22].

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The problem of uniqueness of solutions of (1) has also been the object of several works. First of all, let us recall that the operator $A(u) = -\text{div}(a(x,u)\nabla u)$ is never monotone, unless $a(x,u)$ does not depend on $u$, see [7]. Therefore the question of uniqueness is not trivial. In the case that $a(x,s)$ is coercive and bounded, uniqueness results for $H^1_0(\Omega)$ solutions have been proved in [1], [24] under the assumption that $a(x,u)$ is (globally) Lipschitz continuous with respect to $u$. The method used in these latter papers was developed furtherly in [10]; some generalization to the case that $a(x,s)$ is locally Lipschitz continuous with respect to $s$ can be found in [5], [20], still in the context of finite energy solutions (i.e. $a(x,u)|\nabla u|^2 \in L^1(\Omega)$). Note that in order to consider possibly general growth conditions of the function $a(x,s)$, a generalized concept of solution is needed, to this purpose the use of approximated solutions (so-called SOLA), entropy solutions or renormalized solutions is suitable.

In particular, in the Appendix, we present the method of [5] applied to the Dirichlet problem associated to the operator (2).

Finally, in the two papers [3], [21], uniqueness results for entropy solutions (or renormalized solutions) were proved in case that $a(x,u)$ is locally Lipschitz continuous with respect to $u$ under fairly general growth conditions on the modulus of Lipschitz continuity, and assuming only that $f \in L^1(\Omega)$. For example, Theorem 1.4 in [21] proves the uniqueness of entropy solutions of (1) when the left hand side is given by the operator (2), and $f \in L^1(\Omega)$. Let us mention that some generalization of the above results to $p$–Laplace type operators are recently given in [16].

In this paper, we consider the case where the dependence of $a(x,u)$ with respect to $u$ is not locally Lipschitz, being possibly singular at some point like in the case of Hölder type continuity. As we mentioned above, most, if not all, of the previous uniqueness results for the model problem (1) are confined to the case that $a$ is locally Lipschitz with respect to $u$. The case that $a$ is Hölder continuous has been considered previously only in two cases, either for evolution problems or stationary problems with additional zero order terms (see e.g. [1], [13]) or assuming extra regularity of $a(x,s)$ with respect to $x$ (see e.g. [11], and the recent results in [15]). Here, although we do not give a general result for Hölder type nonlinearities, we introduce a new idea which makes it possible at least to deal with several examples of Hölder continuous functions with exponent $\alpha \in \left(\frac{1}{2}, 1\right)$. The model example of our results can be seen in the following form. We set, for $t \in \mathbb{R}$, $T_1(t) = \min(1, \max(t, -1))$.

**Theorem 1.1.** Assume that $a(x,s)$ is a Carathéodory function such that for every $s, \sigma \in \mathbb{R}$ and a.e. $x \in \Omega$,

$$\alpha \leq a(x,s) \leq \beta, \quad (3)$$

$$|a(x,s) - a(x,\sigma)| \leq H \frac{|s - \sigma|}{|T_1(s)|^{1-\theta} + |T_1(\sigma)|^{1-\theta}}, \quad (4)$$

where $\alpha, \beta > 0$, $H > 0$ and

$$\frac{1}{2} < \theta \leq 1. \quad (5)$$

Let $f \in L^1(\Omega) \cap H^{-1}(\Omega)$. Then problem (1) has a unique weak solution $u \in H^1_0(\Omega)$.

**Theorem 1.1** is clearly modeled on the simplest example of Hölder nonlinearity, given by

$$a(x,u) = \alpha(x) + |u|^\theta, \quad \frac{1}{2} < \theta \leq 1. \quad (6)$$
Our simple idea is to treat such type of Hölder continuous functions as examples where the modulus of Lipschitz continuity becomes singular at some point. Such idea is vaguely inspired by the assumption used in case of $p$–Laplace type operators, see e.g. [12].

Of course, Theorem 1.1 admits a generalized version to many extents. First of all, the bound from above on $a(x, s)$ was assumed in Theorem 1.1 just to allow us to use weak solutions. Such bound can be removed up to using a generalized concept of solution, e.g. that of entropy solution. Secondly, the continuity assumption on $a(x, s)$ will also take a more general form allowing for combinations of Lipschitz and/or Hölder functions with possibly different Hölder exponents at different points. The general version of our result will be the following.

**Theorem 1.2.** Assume that $a(x, s)$ is a Carathéodory function which satisfies the following conditions:

$$
\exists \alpha > 0 : \quad a(x, s) \geq \alpha \quad \forall s \in \mathbb{R}, \ a.e. \ x \in \Omega,
$$

$$
\forall K > 0, \ \exists C_K > 0 : \quad \sup_{|s| \leq K} a(x, s) \leq C_K, \quad a.e. \ x \in \Omega.
$$

Moreover, defining

$$
\omega_\varepsilon(s) := \sup \left\{ \frac{|a(x, s) - a(x, \sigma)|}{|s - \sigma|}, \ (x, \sigma): \ \varepsilon < |\sigma - s| < 2\varepsilon, \ x \in \Omega \right\},
$$

assume that

$$
\exists \bar{\omega} \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) : \quad \omega_\varepsilon(s)^2 \leq \bar{\omega}(s) \quad \forall s \in \mathbb{R}, \forall \varepsilon \leq \varepsilon_0,
$$

for some $\varepsilon_0 > 0$. Let $f \in L^1(\Omega) \cap H^{-1}(\Omega)$. Then problem (1) has a unique entropy solution $u$.

We refer the reader to Section 3 for the definition of entropy solution and some discussion concerning assumption (7). Let us note that, following the ideas in [3], [21], further generalizations of these results are possible in at least two directions. On one hand, by relaxing the assumptions on the modulus of Lipschitz continuity at infinity (i.e. relaxing the conditions on $\bar{\omega}$ at infinity), on the other hand by considering simply $f \in L^1(\Omega)$ (see also the discussion at the end of Section 3). Further possible extensions may also concern the presence of convection terms, at least for nonlinearities which are locally Lipschitz continuous. However, the above extensions require more technicalities which are far beyond the scope of this note, details will appear elsewhere.

On the other hand, we mention that the range $\theta \in (\frac{1}{2}, 1]$ in Theorem 1.1, as well as the assumption $f \in L^1(\Omega)$, are somehow optimal, at least as far as our methods are concerned. We refer to Remark 1 for a short discussion about this point; in particular, Lemma 2.1 would not hold, for the same range of Hölder nonlinearities, only assuming $f \in H^{-1}(\Omega)$.

2. **The model example of Hölder-like dependence.** In this section we study the model example which is the motivation of our results. Let us consider the Dirichlet problem (1), with $f \in L^1(\Omega) \cap H^{-1}(\Omega)$. In particular, if $N > 2$, any $f \in L^\frac{2N}{N+2}(\Omega)$ is admitted, due to Sobolev embedding. We assume here (3). In

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$^1$the sup defined here is an essential supremum as far as $x \in \Omega$ is concerned.
Letting $\epsilon$ we have
\[ \int \nabla f \nabla \varphi \, dx = \langle f, \varphi \rangle, \quad \forall \varphi \in H^1_0(\Omega), \]
where $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

A key point in the proof will be played by the following lemma, which explains the condition (5).

**Lemma 2.1.** Let $u \in H^1_0(\Omega)$ be a weak solution of (1). Then, for every $\theta \in (\frac{1}{2}, 1]$ we have
\[ \int_{\{x: |u(x)| < 1\}} \frac{|\nabla u|^2}{|u|^{2 - 2\theta}} \, dx \leq \frac{\|f\|_{L^1(\Omega)}}{\alpha(2\theta - 1)}. \]

**Proof.** Define $v_\epsilon = \left[ (\epsilon + |T_1(u)|)^{2\theta - 1} - \epsilon^{2\theta - 1} \right] \text{sgn}(u), \epsilon > 0$, and use $v_\epsilon$ as test function in (8). We have
\[ (2\theta - 1) \int \frac{a(x, u) |\nabla T_1(u)|^2}{(\epsilon + |T_1(u)|)^{2 - 2\theta}} \, dx \leq \int f v_\epsilon \, dx, \]
which implies, due to (3) and since $|v_\epsilon| \leq |T_1(u)|^{2\theta - 1} \leq 1$,
\[ \alpha(2\theta - 1) \int_{\{x: |u(x)| < 1\}} \frac{|\nabla u|^2}{(\epsilon + |u|)^{2 - 2\theta}} \, dx \leq \|f\|_{L^1(\Omega)}. \]

Letting $\epsilon$ go to zero, thanks to Fatou's Lemma we deduce (9). \(\square\)

**Remark 1.** The condition $\theta \in (\frac{1}{2}, 1]$ is optimal for Lemma 2.1 to hold. It is enough to consider the case of the Laplace operator and $f \geq 0$ to observe that, in the best situation, we have $u = O(d(x))$ as $x \to \partial \Omega$, where $d(x)$ is the distance function to $\partial \Omega$. This is just consequence of the Hopf boundary lemma, stating in addition that $|\nabla u| \geq \gamma$ at $\partial \Omega$ for some $\gamma > 0$. Therefore, we have, for some $\delta > 0$:
\[ \int_{\{x: |u(x)| < 1\}} \frac{|\nabla u|^2}{|u|^{2 - 2\theta}} \, dx \geq \int_{\{d(x) < \delta\}} \frac{|\nabla u|^2}{|u|^{2 - 2\theta}} \, dx \geq c \int_{\{d(x) < \delta\}} \frac{1}{d(x)^{2 - 2\theta}} \, dx \]
and last integral is not finite for every $\theta \leq \frac{1}{2}$.

We also stress that the $L^1(\Omega)$ character of $f$ is also essential in the above estimate. As easily shown by one dimensional examples, assuming only $f \in H^{-1}(\Omega)$ would not be enough to obtain a similar estimate (at least with the same range of $\theta$).

We can now prove Theorem 1.1, whose proof follows the ideas of [10] in connection with Lemma 2.1.

**Proof of Theorem 1.1.**

Let $u, z$ be two weak solutions of (1). We have then, for any test function $\varphi \in H^1_0(\Omega)$:
\[ \int \{a(x, z) \nabla z - a(x, u) \nabla u\} \nabla \varphi \, dx = 0. \]

We take $\varphi = T_\epsilon [z - u], \epsilon > 0$, and we have
\[ \int \{a(x, z) \nabla z - a(x, u) \nabla u\} \nabla T_\epsilon (z - u) \, dx = 0, \]
which implies
\[\int_{\Omega} a(x, z)|\nabla T_\varepsilon(z - u)|^2 \, dx = \int_{\{0 < |z - u| < \varepsilon\}} [a(x, u) - a(x, z)] \nabla u \nabla (z - u) \, dx.\]

Using the coercivity of \(a(x, s)\) and Young’s inequality we get
\[\frac{\alpha}{2} \int_{\Omega} |\nabla T_\varepsilon(z - u)|^2 \, dx \leq \frac{1}{2\alpha} \int_{\{0 < |z - u| < \varepsilon\}} [a(x, u) - a(x, z)]^2 |\nabla u|^2 \, dx,
\]
which implies, using (4),
\[\frac{\alpha}{2} \int_{\Omega} |\nabla T_\varepsilon(z - u)|^2 \, dx \leq \frac{1}{2\alpha} H^2 \varepsilon^2 \int_{\{0 < |z - u| < \varepsilon\}} \frac{|\nabla u|^2}{(T_1(|u|)^{1-	heta} + T_1(|z|)^{1-	heta})^2} \, dx \leq \frac{H^2}{2\alpha} \varepsilon^2 \int_{\{0 < |z - u| < \varepsilon\}} \frac{|\nabla u|^2}{T_1(|u|)^{2(1-	heta)}} \, dx.
\]

Using Poincaré inequality we get
\[\lambda \frac{\alpha}{2} \int_{\Omega} |T_\varepsilon[z - u]|^2 \, dx \leq \frac{H^2}{2\alpha} \varepsilon^2 \int_{\{0 < |z - u| < \varepsilon\}} \frac{|\nabla u|^2}{T_1(|u|)^{2(1-	heta)}} \, dx.
\]

For every fixed \(\delta > \varepsilon\) we have
\[\int_{\Omega} |T_\varepsilon[z - u]|^2 \, dx \geq \varepsilon^2 \text{meas} \{x : |z - u| > \delta\}
\]
hence we deduce
\[\text{meas} \{x : |z - u| > \delta\} \leq \frac{H^2}{\lambda \frac{\alpha}{2}} \int_{\{0 < |z - u| < \varepsilon\}} \frac{|\nabla u|^2}{T_1(|u|)^{2(1-	heta)}} \, dx. \quad (11)
\]

Remark that
\[\bigcap_{\varepsilon > 0} \{x \in \Omega : 0 < |u(x) - z(x)| < \varepsilon\} = \{x \in \Omega : 0 < |u(x) - z(x)| \leq 0\} = \emptyset.
\]
The decreasing continuity of the measure implies that
\[\text{meas} \{x \in \Omega : 0 < |u(x) - z(x)| < \varepsilon\} \to 0,
\]
as \(\varepsilon \to 0\). Moreover, since
\[\frac{|\nabla u|^2}{T_1(|u|)^{2(1-	heta)}} \leq \frac{|\nabla u|^2}{|u|^2 - 2\theta} \chi(|u| \leq 1) + |\nabla u|^2,
\]
from Lemma 2.1 and the fact that \(u\) belongs to \(H^1_0(\Omega)\) it follows that \(\frac{|\nabla u|^2}{T_1(|u|)^{2(1-	heta)}} \in L^1(\Omega)\). Therefore, we have
\[\lim_{\varepsilon \to 0} \int_{\{0 < |z - u| < \varepsilon\}} \frac{|\nabla u|^2}{T_1(|u|)^{2(1-	heta)}} \, dx = 0.
\]

Passing to the limit as \(\varepsilon \to 0\) in (11), we deduce
\[\text{meas} \{x \in \Omega : |u(x) - z(x)| > \delta\} = 0
\]
for every \(\delta > 0\), hence \(u = z\). \(\square\)
3. Generalizations. In this section we generalize the example of the previous section, which is of course very special to many regards, in particular \( a(x,s) \) was supposed to be singular at only one point. The boundedness assumption on \( a(x,s) \) was also not essential but for considering standard weak solutions. Here we assume that \( a(x,s) \) only satisfies
\[
\exists \alpha > 0 : \ a(x,s) \geq \alpha \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega,
\]
and that
\[
\forall K > 0, \exists C_K > 0 : \sup_{|s| \leq K} a(x,s) \leq C_K, \quad \text{a.e. } x \in \Omega. \tag{13}
\]
In order to deal with possibly unbounded functions \( a(x,s) \), a generalized concept of solution is needed. We recall (see [2]) that a function \( u \) (almost everywhere finite in \( \Omega \)) is said to be an entropy solution of (1) if \( T_k(u) \in H^1_0(\Omega) \) for every \( k > 0 \) and
\[
\int \Omega a(x,u)\nabla u \nabla T_k[u - \varphi] \, dx = \langle f, T_k[u - \varphi] \rangle \tag{14}
\]
\[\forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega), \forall k > 0,\]
where \( \langle \cdot, \cdot \rangle \) denotes the duality between \( H^{-1}(\Omega) \) and \( H^1_0(\Omega) \) and, here and later, by \( \nabla u \) we denote the generalized gradient as defined in [2] for entropy solutions.

Let us now generalize assumption (4). First of all, we set
\[\omega_\varepsilon(s) := \sup \left\{ \frac{|a(x,s) - a(x,\sigma)|}{|s - \sigma|}, (x, \sigma) : \varepsilon < |\sigma - s| < 2\varepsilon, \ x \in \Omega \right\}.\]
Note that, thanks to assumption (13), \( \omega_\varepsilon(s) \) is a locally bounded function. We assume that \( \omega_\varepsilon(s) \) satisfies
\[
\exists \bar{\omega} \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) : \omega_\varepsilon(s)^2 \leq \bar{\omega}(s) \quad \forall s \in \mathbb{R}, \forall \varepsilon \leq \varepsilon_0, \tag{15}
\]
for some \( \varepsilon_0 > 0 \).

Observe that the Lipschitz case corresponds to \( \bar{\omega} \in L^\infty(\mathbb{R}) \). In the model example of the previous section we have \( \omega_\varepsilon(s) \leq \frac{c}{|T_1(s)|^\theta + \varepsilon} \), hence
\[
\omega_\varepsilon^2(s) \leq \frac{c^2}{|T_1(s)|^{2(1-\theta)}} \leq \frac{c^2}{|s|^{2(1-\theta)} \chi_{|s| \leq 1}} + c^2.
\]
Since \( 2(1-\theta) < 1 \) if \( \theta > \frac{1}{2} \), we see that \( \bar{\omega}(s) \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}) \) and (15) is satisfied. Of course, a similar case is satisfied by the function \( a(x,s) = a(x) + |s - s_0|^\theta \) with \( s_0 \in \mathbb{R} \). Combinations of sublinear power nonlinearities and Lipschitz functions are also included in this assumption.

We start with a standard lemma. The proof is essentially contained in [21], however we give the argument here for completeness.

**Lemma 3.1.** Let \( u \) be an entropy solution of (1). Let \( \Psi(s) \) be a locally Lipschitz function, nondecreasing, and let \( \Psi'(s) = \psi(s) \) a.e. in \( \mathbb{R} \). If \( \Psi \) is bounded, we have
\[
\int_{\{|u| < h\}} a(x,u)\psi(u)|\nabla u|^2 \, dx \leq \|\Psi\|_{L^\infty(\mathbb{R})} \|f\|_{L^1(\Omega)} \quad \forall h > 0. \tag{16}
\]

\[\omega_\varepsilon(s) := \varepsilon \sup_{x \in \Omega} \sup_{\sigma : \varepsilon < |\sigma - s| < 2\varepsilon} \left\{ \frac{|a(x,s) - a(x,\sigma)|}{|s - \sigma|} \right\} \]

\footnote{note that the sup here is an essential sup as far as \( x \in \Omega \) is concerned; more precisely,}
Proof. First of all note that, since $\Psi$ is locally Lipschitz and $T_h(u) \in H^1_0(\Omega)$, by Stampacchia’s theorem (see e.g. [19] for a proof) we have $\Psi(T_h(u)) \in H^1_0(\Omega)$ and $\nabla \Psi(T_h(u)) = \psi(T_h(u)) \nabla T_h(u)$ a.e. in $\Omega$. We choose $k = \|\Psi\|_{L^\infty(-\infty,\infty)}$ and $\varphi = T_h(u) - \Psi(T_h(u))$ in the definition of entropy solution for $u$. We have

$$
\int_{\Omega} a(x,u) \nabla u \nabla T_h(u - T_h(u) + \Psi(T_h(u))) \, dx = \int_{\Omega} f T_h(u - T_h(u) + \Psi(T_h(u))) \, dx.
$$

Since $k = \|\Psi\|_{L^\infty(-\infty,\infty)}$, we have

$$
\int_{\Omega} a(x,u) \nabla u \nabla T_h(u - T_h(u) + \Psi(T_h(u))) \, dx \leq k \|f\|_{L^1(\Omega)} = \|\Psi\|_{L^\infty(-\infty,\infty)} \|f\|_{L^1(\Omega)},
$$

and (16) is proved.

Remark 2. Let us recall that assuming $f \in H^{-1}(\Omega)$ implies that entropy solutions have finite energy, in particular

$$
\int_{\Omega} a(x,u) |\nabla u|^2 \, dx \leq C \|f\|^2_{H^{-1}(\Omega)}. \tag{17}
$$

This is a well known fact. Indeed, from the definition of entropy solution we have

$$
\int_{\Omega} a(x,u) |\nabla T_h(u)|^2 \, dx \leq \langle f, T_h(u) \rangle \leq \|f\|_{H^{-1}(\Omega)} \|T_h(u)\|_{H^1_0(\Omega)}.
$$

Using (12) we have

$$
\|T_h(u)\|_{H^1_0(\Omega)}^2 \leq \frac{1}{\alpha} \int_{\Omega} a(x,u) |\nabla T_h(u)|^2 \, dx,
$$

hence

$$
\int_{\Omega} a(x,u) |\nabla T_h(u)|^2 \, dx \leq \|f\|^2_{H^{-1}(\Omega)}.
$$

Letting $k \to \infty$ yields (17).

Theorem 3.2. Assume (12), (13), (15). Let $f \in L^1(\Omega) \cap H^{-1}(\Omega)$. Then the entropy solution of (1) is unique.

Proof.

Step 1. Let $u, z$ be two entropy solutions of (1). Let us set $\psi_\varepsilon(t) = T_\varepsilon(t - T_\varepsilon(t))$, $\varepsilon \leq \varepsilon_0$. We take $k = \varepsilon$ and $\varphi = T_h(u) - \psi_\varepsilon(u - T_h(z))$ in the entropy formulation
Observe that, since \( f \) for \( h \) and \( T_h \), we obtain
\[
\int a(x, z) \nabla z \nabla T_h(z - T_h(z)) dx + \int a(x, u) \nabla u \nabla T_h(u - T_h(z)) dx = \int f(T_h(z) + \psi(z - T_h(u))) T_h(u - T_h(z)) dx.
\]

On the other hand, we have
\[
\int T_h(z) + \psi(z - T_h(u)) + T_h(u - T_h(z)) dx = 0.
\]

Therefore, we have
\[
\int a(x, z) \nabla z \nabla T_h(z - T_h(z) + \psi(z - T_h(u))) dx + \int a(x, u) \nabla u \nabla T_h(u - T_h(z)) dx = o(1),
\]

where \( o(1) \) denotes some quantity going to zero as \( h \to \infty \). Note that, since \( \phi \) is nondecreasing and smaller than \( \varepsilon \), we have
\[
\int a(x, z) \nabla z \nabla T_h(z - T_h(z) + \psi(z - T_h(u))) dx \geq \int a(x, z) \nabla z \nabla \psi(z - u) dx - \int a(x, z) \nabla z \nabla \psi(z - u) dx.
\]

Last integral can be estimated as follows: since \( \nabla \psi(z - u) = 0 \) except if \( \varepsilon < |z - u| < 2\varepsilon \), we have
\[
\int a(x, z) |\nabla z| |\nabla \psi(z - u)| dx \leq c \int a(x, z) |\nabla z|^2 dx + c \int a(x, z) |\nabla u|^2 dx.
\]

On the other hand, we have
\[
a(x, z) \leq a(x, u) + 2\varepsilon \omega(u) \quad \text{in} \quad \{x : \varepsilon < |z - u| < 2\varepsilon\},
\]

hence
\[
\int a(x, z) |\nabla u|^2 dx \leq \int a(x, u) |\nabla u|^2 dx + 2\varepsilon \int \omega(u) |\nabla u|^2 dx
\]
and we deduce from (20)
\[
\int_{\{|z|>h, |u|<h\}} a(x, z) \|
\nabla z\| \nabla \psi_z(z-u) \, dx 
\leq c \int_{\{|h|<|h+2\varepsilon\}} a(x, z) \|
\nabla z\|^2 \, dx 
+ c \int_{\{|h-2\varepsilon|<|u|<h\}} \omega_\varepsilon(u) \|
\nabla u\|^2 \, dx + c \varepsilon \int_{\{|h-2\varepsilon|<|u|<h\}} \omega_\varepsilon(u) \|
\nabla u\|^2 \, dx .
\]
(21)

The first two terms go to zero as \(h \to \infty\) (e.g. by Remark 2 using that \(f \in H^{-1}(\Omega)\), or simply by properties of entropy solutions using that \(f \in L^1(\Omega)\)). Last term is estimated as follows: by assumption (15), we have \(\bar{\omega} = \bar{\omega}_1 + \bar{\omega}_2\) with \(\bar{\omega}_1 \in L^1(\mathbb{R})\) and \(\bar{\omega}_2 \in L^\infty(\mathbb{R})\), with these notations we have
\[
\int_{\{|h-2\varepsilon|<|u|<h\}} \omega_\varepsilon(u) \|
\nabla u\|^2 \, dx \leq \int_{\{|h-2\varepsilon|<|u|<h\}} (1 + \omega_\varepsilon^2(u)) \|
\nabla u\|^2 \, dx 
\leq \int_{\{|h-2\varepsilon|<|u|<h\}} (1 + \bar{\omega}_2(u)) \|
\nabla u\|^2 \, dx + \int_{\{|h-2\varepsilon|<|u|<h\}} (\omega_\varepsilon^2(u) - \bar{\omega}_2(u)) \|
\nabla u\|^2 \, dx .
\]
(22)

Of course we may assume that \(\bar{\omega}_1 \geq 0\). Since \(\bar{\omega}_2 \in L^\infty(\mathbb{R})\), we have
\[
\lim_{h \to \infty} \int_{\{|h-2\varepsilon|<|u|<h\}} (1 + \bar{\omega}_2(u)) \|
\nabla u\|^2 \, dx = 0 .
\]
(23)

Moreover, we split
\[
\int_{\{|h-2\varepsilon|<|u|<h\}} (\omega_\varepsilon^2(u) - \bar{\omega}_2(u)) \|
\nabla u\|^2 \, dx 
\leq \int_{\{|h-2\varepsilon|<|u|<h\}} T_k(\omega_\varepsilon^2(u) - \bar{\omega}_2(u)) \|
\nabla u\|^2 \, dx 
+ \int_{\{|h-2\varepsilon|<|u|<h\}} G_k(\omega_\varepsilon^2(u) - \bar{\omega}_2(u)) \|
\nabla u\|^2 \, dx ,
\]
(24)

where \(G_k(s) = s - T_k(s)\). We use now Lemma 3.1 with \(\Psi(s) = \int_0^s G_k(\omega_\varepsilon^2(t) - \bar{\omega}_2(t)) \, dt\); observe that \(\Psi\) is a locally Lipschitz function and moreover
\[
\|\Psi\|_{L^\infty(\mathbb{R})} \leq \int_\mathbb{R} G_k(\omega_\varepsilon^2(t) - \bar{\omega}_2(t)) \, dt .
\]

Since, by (15), we have \(\omega_\varepsilon^2(t) - \bar{\omega}_2(t) \leq \bar{\omega}_1\), we deduce that
\[
\|\Psi\|_{L^\infty(\mathbb{R})} \leq \int_\mathbb{R} G_k(\omega_\varepsilon^2(t) - \bar{\omega}_2(t)) \, dt \leq \int_\mathbb{R} G_k(\bar{\omega}_1(t)) \, dt .
\]

Therefore, from Lemma 3.1 we obtain
\[
\int_{\{|u|<h\}} G_k(\omega_\varepsilon^2(u) - \bar{\omega}_2(u)) \|
\nabla u\|^2 \, dx \leq \frac{1}{\alpha} \|f\|_{L^1(\Omega)} \|\Psi\|_{L^\infty(\mathbb{R})} ,
\]
which implies

$$\sup_h \int_{\{|u|<h\}} G_k(\omega_1^2(u) - \bar{\omega}_{2}(u))^+ |\nabla u|^2 \, dx \leq \frac{1}{\alpha} \|f\|_{L^1(\Omega)} \|G_k(\bar{\omega}_1)\|_{L^1(\mathbb{R})}. \quad (25)$$

We conclude from (24) that

$$\int_{\{|h-2\epsilon<|u|<h\}} (\omega_1^2(u) - \bar{\omega}_{2}(u))^+ |\nabla u|^2 \, dx \leq k \int_{\{|h-2\epsilon<|u|<h\}} |\nabla u|^2 \, dx + \frac{1}{\alpha} \|f\|_{L^1(\Omega)} \|G_k(\bar{\omega}_1)\|_{L^1(\mathbb{R})}. \quad (25)$$

Letting first $h \to \infty$ and then $k \to \infty$, using that $\bar{\omega}_1 \in L^1(\mathbb{R})$, we get

$$\lim_{h \to \infty} \int_{\{|h-2\epsilon<|u|<h\}} (\omega_1^2(u) - \bar{\omega}_{2}(u))^+ |\nabla u|^2 \, dx = 0. \quad (26)$$

From (22), (23) and (26) we deduce

$$\lim_{h \to \infty} \int_{\{|h-2\epsilon<|u|<h\}} \omega_1(u) |\nabla u|^2 \, dx = 0$$

and therefore we have from (21)

$$\lim_{h \to \infty} \int_{\{|z|h, |u|<h\}} a(x,z) |\nabla z| |\nabla \psi_e(z-u)| \, dx = 0.$$

We deduce from (19) that

$$\int_{\Omega} a(x,z) \nabla z \nabla T_e(z - T_h(z) + \psi_e(z - T_h(u))) \, dx$$
$$\geq \int_{\{|z|h, |u|<h\}} a(x,z) \nabla z \nabla \psi_e(z-u) \, dx + o(1)_h.$$

Similarly we prove that

$$\int_{\Omega} a(x,u) \nabla u \nabla T_e(u - T_h(u) + \psi_e(u - T_h(z))) \, dx$$
$$\geq \int_{\{|z|h, |u|<h\}} a(x,u) \nabla u \nabla \psi_e(u-z) \, dx + o(1)_h$$

so that we get from (18)

$$\int_{\{|z|h, |u|<h\}} a(x,z) \nabla z \nabla \psi_e(z-u) \, dx$$
$$+ \int_{\{|z|h, |u|<h\}} a(x,u) \nabla u \nabla \psi_e(u-z) \, dx \leq o(1)_h.$$
Step 2. We proceed now as in the case of weak solutions. Indeed, we proved that
\[
\int_{\{|z|<h, |u|<h\}} a(x, z) \nabla (z - u) \nabla \psi(z - u) \, dx \\
\leq \int_{\{|z|<h, |u|<h\}} \{a(x, u) - a(x, z)\} \nabla u \nabla \psi(z - u) \, dx + o(1)_h ,
\]
which yields, using (12) and Young’s inequality
\[
\frac{\alpha}{2} \int_{\{|z|<h, |u|<h\}} |\nabla (z - u)|^2 \psi'(z - u) \, dx \\
\leq \frac{1}{2\alpha} \int_{\{|z|<h, |u|<h\}} \{a(x, u) - a(x, z)\}^2 |\nabla u|^2 \psi'(z - u) \, dx + o(1)_h .
\]
Since \(\psi'(t) = \chi_{\{|z|<h, |u|<h\}}\), by definition of \(\omega_\varepsilon\) we have
\[
\frac{\alpha}{2} \int_{\{|z|<h, |u|<h\}} |\nabla \psi(z - u)|^2 \, dx \\
\leq \frac{\alpha}{2} \int_{\{|z|<h, |u|<h\}} \omega_\varepsilon(u)^2 |\nabla u|^2 \, dx + o(1)_h .
\]
We estimate last term as we did before; namely, we have from (15)
\[
\omega_\varepsilon^2 \leq \bar{\omega} = \bar{\omega}_1 + \bar{\omega}_2 , \quad \bar{\omega}_1 \in L^1(\mathbb{R}), \, \bar{\omega}_2 \in L^\infty(\mathbb{R}) .
\]
Therefore we get
\[
\int_{\{|u|<h, |z|<h, |u|<h\}} \omega_\varepsilon(u)^2 |\nabla u|^2 \, dx \leq \int_{\{|u|<h, |z|<h, |u|<h\}} \bar{\omega}_2(u) |\nabla u|^2 \, dx \\
+ \int_{\{|u|<h, |z|<h, |u|<h\}} T_k(\omega_\varepsilon(u)^2 - \bar{\omega}_2(u)) + |\nabla u|^2 \, dx + \int_{\{|u|<h\}} G_k(\omega_\varepsilon(u)^2 - \bar{\omega}_2(u) + |\nabla u|^2 \, dx .
\]
Thanks to (25) we deduce
\[
\int_{\{|u|<h, |z|<h, |u|<h\}} \omega_\varepsilon(u)^2 |\nabla u|^2 \, dx \\
\leq \int_{\{|u|<h, |z|<h, |u|<h\}} (\bar{\omega}_2(u) + k) |\nabla u|^2 \, dx + \frac{1}{\alpha} \|f\|_{L^1(\Omega)} \|G_k(\bar{\omega}_1)\|_{L^1(\mathbb{R})} .
\]
Therefore
\[
\frac{\alpha}{2} \int_{\{|z|<h, |u|<h\}} |\nabla \psi(z - u)|^2 \, dx \\
\leq \frac{\alpha}{2} \int_{\{|z|<h, |u|<h\}} \omega_\varepsilon(u)^2 |\nabla u|^2 \, dx \\
+ \int_{\{|u|<h, |z|<h, |u|<h\}} (\bar{\omega}_2(u) + k) |\nabla u|^2 \, dx + \frac{1}{\alpha} \|f\|_{L^1(\Omega)} \|G_k(\bar{\omega}_1)\|_{L^1(\mathbb{R})} + o(1)_h ,
\]
and letting $h \to \infty$ we obtain
\[
\frac{\alpha}{2} \int_\Omega |\nabla \psi_\varepsilon(z-u)|^2 \, dx
\leq \frac{2}{\alpha} \varepsilon^2 \left\{ \int_{\{\varepsilon<|z-u|<2\varepsilon\}} (\bar{\omega}(u) + k) |\nabla u|^2 \, dx + \frac{1}{\alpha} \|f\|_{L^1(\Omega)} \|G_k(\bar{\omega}_1)\|_{L^1(\mathbb{R})} \right\}.
\]
By Poincaré inequality we have
\[
\int_\Omega |\nabla \psi_\varepsilon(z-u)|^2 \, dx \geq \lambda_1 \int_\Omega |\psi_\varepsilon(z-u)|^2 \, dx
\]
and since, for any fixed $\delta > 2\varepsilon$, we have
\[
\int_\Omega |\psi_\varepsilon(z-u)|^2 \, dx \geq \int_{\{|z-u|>\delta\}} |\psi_\varepsilon(z-u)|^2 \, dx \geq \varepsilon^2 \, \text{meas} \{x \in \Omega : |z-u| > \delta\}
\]
we conclude
\[
\text{meas} \{x \in \Omega : |z-u| > \delta\}
\leq \frac{4}{\lambda_1 \alpha^2} \left\{ \int_{\{\varepsilon<|z-u|<2\varepsilon\}} (\bar{\omega}(u) + k) |\nabla u|^2 \, dx + \frac{1}{\alpha} \|f\|_{L^1(\Omega)} \|G_k(\bar{\omega}_1)\|_{L^1(\mathbb{R})} \right\}
\]
(27)
for every $\delta > 2\varepsilon$. Observe that
\[
\{\varepsilon < |z-u| < 2\varepsilon\} \subset \{0 < |z-u| < 2\varepsilon\}
\]
and since
\[
\text{meas} \{x \in \Omega : 0 < |z-u| < 2\varepsilon\} \to 0
\]
we get
\[
\text{meas} \{\varepsilon < |z-u| < 2\varepsilon\} \to 0.
\]
This implies
\[
\lim_{\varepsilon \to 0} \int_{\{\varepsilon<|z-u|<2\varepsilon\}} (\bar{\omega}(u) + k) |\nabla u|^2 \, dx = 0.
\]
Passing to the limit as $\varepsilon \to 0$ in (27), we conclude that
\[
\text{meas} \{x \in \Omega : |z-u| > \delta\} \leq \frac{4}{\lambda_1 \alpha^3} \|f\|_{L^1(\Omega)} \|G_k(\bar{\omega}_1)\|_{L^1(\mathbb{R})}.
\]
Letting $k \to \infty$, since $\bar{\omega}_1 \in L^1(\mathbb{R})$ we obtain
\[
\text{meas} \{x \in \Omega : |z-u| > \delta\} = 0 \quad \forall \delta > 0
\]
hence $z = u$.

\[\square\]

**Remark 3.** With the same proof one obtains a comparison principle. Namely, if $u_1$, $u_2$ are entropy solutions with data $f_1$, $f_2$ respectively, if $f_1 \leq f_2$ then $u_1 \leq u_2$.

Let us conclude by observing that the above method can also be used to deal with data $f \in L^1(\Omega)$ which may not belong to $H^{-1}(\Omega)$. Indeed, the main ingredient of the arguments used is that $\bar{\omega}(u) |\nabla u|^2 \in L^1(\Omega)$ for an entropy solution $u$, where $\bar{\omega}$ is the function given in (15). It is well known that such regularity holds whenever $\bar{\omega} \in L^1(\mathbb{R})$, only assuming $f \in L^1(\Omega)$; in this case a direct extension of the above proof is possible. Another easy case is given if, for instance, $\bar{\omega}(s) = \frac{1}{|s|^{\alpha+1}}$ as for
a globally Hölder nonlinearity; in that situation, if \( \theta \in \left( \frac{1}{2}, 1 \right] \) and if \( f \in L^m(\Omega) \) with \( m \geq \frac{2N\theta}{(N-2)\theta+4} \), then \( \bar{\omega}(u)|\nabla u|^2 \in L^1(\Omega) \) as a consequence of Lemma 2.1 and of the regularity results in [9] which ensure \( u \in L^{\frac{2N}{N-2m}}(\Omega) \) (if \( N > 2 \)). However, such examples are far from being optimal, they simply allow for a straightforward extension of the above proof. A general result is possible only coupling the above idea for Hölder–type nonlinearities with the methods used in [3], [21] for locally Lipschitz nonlinearities with large growth at infinity. In particular, an adaptation of such methods can provide directly a general result for \( L^1 \)-data.

4. Appendix. In the unpublished paper [5] it is proved the uniqueness of the solutions of the Dirichlet problem

\[
u \in H^1_0(\Omega) : -\text{div}(a(x,u)(1+|u|^\gamma)\nabla u) = f,
\]

under the assumptions (3), (4) with \( \theta = 1, \gamma \geq 0 \) and

\[
f \in L^{\frac{2N}{N+2}}(\Omega), \tag{28}
\]

assuming, for simplicity, that \( N > 2 \). Here we show how the approach of [5] can be adapted to the proof of existence and uniqueness of the solutions of the Dirichlet problem with the differential operator (2) which was treated by I. V. Skrypnik in [17]. Let us recall that the existence can be found also in [18]; for the uniqueness in the framework of entropy solutions, see [21].

**Theorem 4.1.** Assume that \( a(x) \) is a measurable function which satisfies

\[
\alpha \leq a(x) \leq \beta, \tag{29}
\]

for some \( 0 < \alpha, \beta \). Under the assumptions (28), (29) there exists \( u \in H^1_0(\Omega) \) satisfying of

\[
\int_\Omega [a(x) + e^{2u}]|\nabla u| \nabla T_k[u - \varphi] \, dx = \int_\Omega f T_k[u - \varphi] \, dx
\]

for every \( \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega) \), and such that

\[
\int_\Omega [a(x) + e^{2u}]|\nabla u|^2 \, dx = \int_\Omega f u \, dx.
\]

Moreover the solution achieved by approximation is unique.

**Proof.** We consider the approximate problems

\[
u_n \in H^1_0(\Omega) : -\text{div}(a(x) + e^{2u_n})\nabla u_n) = f_n
\]

where \( f_n \) converges to \( f \) in \( L^{\frac{2N}{N+2}}(\Omega) \). It is quite standard to prove, for every \( n \in \mathbb{N} \), thanks to Schauder’s theorem, the existence of a bounded solution \( u_n \). Moreover the use of \( e^{2u_n} - 1 \) as test function yields, dropping positive terms,

\[
\frac{1}{2} \int_\Omega |\nabla (e^{2u_n} - 1)|^2 \, dx = 2 \int_\Omega e^{4u_n} |\nabla u_n|^2 \, dx \leq \int_\Omega |f_n| |e^{2u_n} - 1| \, dx
\]

which implies

\[
C_1 \left( \int_\Omega |e^{2u_n} - 1|^2 \, dx \right)^{\frac{1}{2}} \leq 2 \int_\Omega e^{4u_n} |\nabla u_n|^2 \, dx \leq \|f_n\|_{L^{\frac{2N}{N+2}}(\Omega)} \left( \int_\Omega |e^{2u_n} - 1|^2 \, dx \right)^{\frac{1}{2}}.
\]
Thus the sequence \( \{ e^{2u_n} - 1 \} \) is bounded in \( L^1(\Omega) \), which in turn implies that the sequence \( \{ e^{2u_n} |\nabla u_n| \} \) is bounded in \( L^2(\Omega) \). Moreover, using \( u_n \) as test function and dropping the exponential term, we have
\[
\int_\Omega a(x)|\nabla u_n|^2 \, dx \leq \int_\Omega |f_n||u_n| \, dx \leq C_2 \|f\|_{L^{2^*}(\Omega)} \|u_n\|_{H^1_0(\Omega)},
\]
and by (29) we deduce that \( u_n \) is bounded in \( H^1_0(\Omega) \). Thus, by the above estimates, there exist \( u \in H^1_0(\Omega) \) and a subsequence, still denoted by \( \{u_n\} \), such that
\[
\begin{cases}
\text{\( u_n \) converges weakly to } u \text{ in } H^1_0(\Omega); \\
e^{2u_n} \nabla u_n \text{ converges weakly to } Y \text{ in } (L^2(\Omega))^N.
\end{cases}
\]
Moreover the use of \( T_k[u_n - T_h(u)] \), introduced in [4], as test function yields
\[
\int_\Omega [a(x) + e^{2u_n}] \nabla u_n \nabla T_k[u_n - T_h(u)] \, dx = \int_\Omega f_n T_k[u_n - T_h(u)] \, dx
\]
which implies
\[
\int_\Omega [a(x) + e^{2u_n}] |\nabla T_k[u_n - T_h(u)]|^2 \, dx \\
+ \int_\Omega [a(x) + e^{2u_n}] \nabla T_k(u) \nabla T_k[u_n - T_h(u)] \, dx = \int_\Omega f_n T_k[u_n - T_h(u)] \, dx.
\]
The limit as \( n \to \infty \) yields
\[
\limsup_n \int_\Omega |\nabla T_k[u_n - T_h(u)]|^2 \, dx \leq \frac{1}{\alpha} \int_\Omega f T_k[u - T_h(u)] \, dx
\]
which implies that (see also [4])
\[
\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in } \Omega,
\]
and so
\[
\begin{cases}
\text{\( u_n \) converges weakly to } u \text{ in } H^1_0(\Omega), \\
e^{2u_n} \nabla u_n \text{ converges weakly to } e^{2u} \nabla u \text{ in } (L^2(\Omega))^N.
\end{cases}
\]
Then we can pass to the limit in the weak formulation of (31) and we obtain, for every \( v \in H^1_0(\Omega) \),
\[
\int_\Omega [a(x) + e^{2u}] \nabla u \nabla v \, dx = \int_\Omega f v \, dx.
\]
In particular, we have
\[
\int_\Omega [a(x) + e^{2u}] |\nabla u|^2 \, dx = \int_\Omega f u \, dx.
\]
The use of \( u_n \) as test function in (31) yields
\[
\int_\Omega [a(x) + e^{2u_n}] |\nabla u_n|^2 \, dx = \int_\Omega f_n u_n \, dx \to \int_\Omega f u \, dx = \int_\Omega [a(x) + e^{2u}] |\nabla u|^2 \, dx.
\]
On the other hand (32) and the Fatou Lemma imply
\[
\begin{cases}
\int_\Omega a(x)|\nabla u|^2 \, dx \leq \liminf_{n \to \infty} \int_\Omega a(x)|\nabla u_n|^2 \, dx \\
\int_\Omega e^{2u} |\nabla u|^2 \, dx \leq \liminf_{n \to \infty} \int_\Omega e^{2u_n} |\nabla u_n|^2 \, dx.
\end{cases}
\]
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Since \([a(x) + e^{2u_n(x)}]|\nabla u_n(x)|^2\) converges a.e. and is positive, then we deduce
\[
\begin{cases}
a(x)|\nabla u_n|^2 \text{ converges in } L^1(\Omega), \\
e^{2u_n}|\nabla u_n|^2 \text{ converges in } L^1(\Omega),
\end{cases}
\]
that is
\[
\begin{cases}
u_n \text{ converges strongly to } u \text{ in } H^1_0(\Omega), \\
e^{2u_n}|\nabla u_n|^2 \text{ converges to } e^{2u}|\nabla u|^2 \text{ in } L^1(\Omega).
\end{cases}
\]
Thanks to the strong convergence established, it is possible to pass to the limit when we use \(T_\varepsilon[u_n - \varphi]\) as test function in (31), so that we conclude that \(u\) satisfies (30).
Now we prove that the techniques of the above existence proof can be used to show that the solution \(u\) obtained (with the above approximation method) is unique. This kind of uniqueness (solutions obtained as limit of approximation) is strongly related to the entropy solutions also used in this paper (see [14], [2], [23]). This proof follows the techniques of [5]. Let \(\{g_n\}\) be any sequence of smooth functions converging to \(f\) in \(L^{\frac{2N}{N-2}}(\Omega)\) and let \(w_n\) be the weak solutions of the Dirichlet problem
\[
w_n \in H^1_0(\Omega) : -\text{div}(|a(x) + e^{2w_n}|\nabla w_n) = g_n(x).
\]
The above proof says that \(w_n \in H^1_0(\Omega) \cap L^\infty(\Omega)\) converges in \(H^1_0(\Omega)\) to \(w\), solution of
\[
\int_{\Omega} [a(x) + e^{2w}]\nabla w \nabla T_\varepsilon[w - \varphi] dx = \int_{\Omega} f T_\varepsilon[w - \varphi] dx.
\]
Using \(T_\varepsilon[u_n - w_n]\) as test function in the equations of \(u_n\) and \(w_n\), and subtracting, we have
\[
\int_{\Omega} [a(x) + e^{2u_n}]\nabla (u_n - w_n) \nabla T_\varepsilon[u_n - w_n] dx
= \int_{\Omega} [e^{2u_n} - e^{2u_n}]\nabla w_n \nabla T_\varepsilon[u_n - w_n] dx + \int_{\Omega} (f_n - g_n) T_\varepsilon[u_n - w_n] dx
\]
Thus we get
\[
\int_{\Omega} [a(x) + e^{2u_n}]|\nabla T_\varepsilon[u_n - w_n]|^2 dx
\leq 2\varepsilon \int_{\{|u_n - w_n| < \varepsilon\}} [e^{u_n+2\varepsilon} + e^{u_n}]|\nabla w_n||\nabla T_\varepsilon[u_n - w_n]| e^{u_n} dx
+ \int_{\Omega} (f_n - g_n) T_\varepsilon[u_n - w_n] dx,
\]
which implies
\[
\int_{\Omega} [a(x) + e^{2u_n}]|\nabla T_\varepsilon[u_n - w_n]|^2 dx
\leq 2\varepsilon \int_{\{|u_n - w_n| < \varepsilon\}} [e^{u_n+3\varepsilon} + e^{u_n+\varepsilon}]|\nabla w_n||\nabla T_\varepsilon[u_n - w_n]| e^{u_n} dx
+ \int_{\Omega} (f_n - g_n) T_\varepsilon[u_n - w_n] dx.
\]
We pass now to the limit as \( n \) tends to infinity. Thanks to (35), and using that 
\[ |\nabla T_e[u_n - w_n]| < |\nabla u_n|^\alpha + |\nabla w_n|^\alpha \] 
in the subset \( \{|u_n - w_n| < \varepsilon\} \), we have 
\[ |\nabla w_n| \leq |\nabla u_n| + \varepsilon \] 
for almost every \( \varepsilon > 0 \). Since \( \lim_{\varepsilon \to 0} \chi_{\{|u_n - w_n| < \varepsilon\}} \) converges to \( \chi_{\{|u - w| < \varepsilon\}} \) a.e. in \( \Omega \) as \( n \to \infty \), and therefore we deduce 
\[ \int_\Omega [a(x) + e^{2u}] |\nabla T_e[u - w]|^2 \, dx \leq 2 \varepsilon \int_{\{|0 < |u - w| < \varepsilon\}} [e^{u+3\varepsilon} + e^{u+\varepsilon}] |\nabla w|^2 (\Omega). \]

Moreover, for almost every \( \varepsilon > 0 \), \( \chi_{\{|u_n - w_n| < \varepsilon\}} \) converges to \( \chi_{\{|u - w| < \varepsilon\}} \) a.e. in \( \Omega \) as \( n \to \infty \), and therefore we deduce 
\[ \int_\Omega [a(x) + e^{2u}] |\nabla T_e[u - w]|^2 \, dx \leq 4 \alpha \int_{\{|0 < |u - w| < \varepsilon\}} [e^{u+3\varepsilon} + e^{u+\varepsilon}] |\nabla w|^2 \, dx. \]

Thanks to the Poincaré inequality, we obtain 
\[ \lambda_1 \int_\Omega \left| \frac{T_e[u - w]}{\varepsilon} \right|^2 \, dx \leq \frac{4}{\alpha} \int_{\{|0 < |u - w| < \varepsilon\}} [e^{u+3\varepsilon} + e^{u+\varepsilon}] |\nabla w|^2 \, dx. \]

Then, for \( 0 < \varepsilon < \delta \), we have the inequality 
\[ \lambda_1 \, \text{meas} \{|\delta \leq |u - w|\} \leq \lambda_1 \int_{\{|\delta \leq |u - w|\}} \left| \frac{T_e[u - w]}{\varepsilon} \right|^2 \, dx \leq \frac{4}{\alpha} \int_{\{|0 < |u - w| < \varepsilon\}} [e^{u+3\varepsilon} + e^{u+\varepsilon}] |\nabla w|^2 \, dx. \]

Since \( \lim_{\varepsilon \to 0} \text{meas} \{|0 < |u - w| < \varepsilon\} = 0 \) and \( e^{2u} |\nabla w|^2 \in L^1(\Omega) \), we deduce that, for every \( \delta > 0 \), 
\[ \text{meas} \{|\delta \leq |u - w|\} = 0, \]
hence \( u = w \).

\[ \square \]

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E-mail address: boccardo@mat.uniroma1.it
E-mail address: porretta@mat.uniroma2.it