On the Chekhov-Fock coordinates of dessins d’enfants.
G. Shabat and V. Zolotarskaia

Introduction

There are several ways to associate a complex structure to a ribbon graph. The examples are provided by the construction of Kontzevich [2], the construction of Penner [3], the construction of dessins d’enfants [4]. In the first two constructions the ribbon graph is considered with the additional structure - a number is associated to each edge. Varying these parameters we obtain different riemann surfaces. The statement is that this way we cover a cell of the corresponding moduli space. In the construction of dessins d’enfants a single riemann surface is associated to each graph. (Usually we say dessin d’enfants, which in this paper means exactly the same as ribbon graph). We call it the Grothendieck model of a ribbon graph.

The problem is: which parameters for the edges of graph in the first two constructions should be chosen to obtain its Grothendieck model? This problem was solved in [2], [3] and it turns out that these parameters should all be equal to 1.

The goal of the present paper is to discuss one more such construction - that of Chekhov-Fock and to prove that putting all parameters of the edges of the graph in this construction equal to 0, we obtain the Grothendieck model of this graph.

Cartography

Let $\Gamma$ be any trivalent ribbon graph, that is a graph with the valencies of all the vertices equal to 3 and with the given cyclic order on the origins of the edges in each vertex. Let $E$ be the
set of the oriented edges of $\Gamma$. We have the so-called cartography group $C^+_2$ acting on $E$. This group is generated by the elements $\rho_0$ and $\rho_1$. The element $\rho_0$ turns the current edge counterclockwise around the origin of the edge, $\rho_1$ changes the orientation of the edge (see [4]). Formally we can write

$$C^+_2 := \langle \rho_0, \rho_1 | \rho_1^2 = \rho_0^2 = 1 \rangle.$$

Having in mind the trivalency of our graph we define

$$C^+_2[3] := \langle \rho_0, \rho_1 | \rho_1^2 = \rho_0^3 = 1 \rangle$$

also acting on $E$.

Fix $\epsilon \in E$. Let $B(E, \epsilon)$ be the borel subgroup of $C^+_2[3]$ corresponding to the edge $\epsilon$ (see [4]), that is

$$B(E, \epsilon) = \{ w \in C^+_2[3] | w\epsilon = \epsilon \}.$$

**The Chekhov-Fock construction**

We also have the additional structure:

$$z : E \rightarrow \mathbb{R}, \quad z(\rho_1 \gamma) = z(\gamma) \quad \forall \gamma \in \Gamma.$$

Now given $\Gamma, z, \epsilon$ we define the map

$$CHF : C^+_2[3] \rightarrow PSL_2(\mathbb{R})$$

inductively, setting

$$CHF(1) = 1,$$

$$CHF(\rho_0 w) = L \times CHF(w)$$

$$CHF(\rho_1 w) = X_{z(\epsilon \omega)} \times CHF(w)$$

where

$$X_a = \begin{pmatrix} 0 & -e^{a/2} \\ e^{-a/2} & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$
**Notation.** \( \text{chf} := \text{CHF}|_{B(E, \epsilon)} \)

**Statement 1.** The map \( \text{chf} \) is a homomorphism.

This statement, as most of the other statements of present paper, becomes obvious after thinking about it for a while, but here is the formal

**Proof:** We should prove that

\[
\text{chf}(w_2 w_1) = \text{chf}(w_2) \text{chf}(w_1)
\]

for any \( w_1, w_2 \in B(E, \epsilon) \).

It is sufficient to show that

\[
\text{CHF}(w_2 w_1) = \text{CHF}(w_2) \text{CHF}(w_1)
\]

for any \( w_2 \in C_2^+[3], w_1 \in B(E, \epsilon) \).

We will do it using induction over length of \( w_2 \).

If the length of \( w_2 \) is 1 then either \( w_2 = \rho_0 \),
or \( w_2 = \rho_1 \).

In the first case we have

\[
\text{CHF}(w_2 w_1) =
\]

\[
\text{CHF}(\rho_0 w_1) = (\text{by the definition of \text{CHF}}) \\
L \times \text{CHF}(w_1) = (\text{by the definition of \text{CHF}}) \\
\text{CHF}(\rho_0) \text{CHF}(w_1) = \\
\text{CHF}(w_2) \text{CHF}(w_1)
\]

In the second case we have

\[
\text{CHF}(w_2 w_1) =
\]

\[
\text{CHF}(\rho_1 w_1) = (\text{by the definition of \text{CHF}}) \\
X_z(w_1 \epsilon) \times \text{CHF}(w_1) = (\text{by the definition of \text{B}(E, \epsilon)}) \\
X_z(\epsilon) \times \text{CHF}(w_1) = (\text{by the definition of \text{CHF}})
\]
\[ \text{CHF}(\rho_1) \text{CHF}(w_1) = \]
\[ \text{CHF}(w_2) \text{CHF}(w_1) \]

In the general case if \( w_2 = \rho_0 w_3 \) we have
\[ \text{CHF}(w_2 w_1) = \]
\[ \text{CHF}(\rho_0 w_3 w_1) = \quad (\text{by the definition of CHF}) \]
\[ L \times \text{CHF}(w_3 w_1) = \quad (\text{by the induction}) \]
\[ L \times \text{CHF}(w_3) \text{CHF}(w_1) = \quad (\text{by the definition of CHF}) \]
\[ \text{CHF}(\rho_0 w_3) \text{CHF}(w_1) = \]
\[ \text{CHF}(w_2) \text{CHF}(w_1) \]

Finally if \( w_2 = \rho_1 w_3 \) we have
\[ \text{CHF}(w_2 w_1) = \]
\[ \text{CHF}(\rho_1 w_3 w_1) = \quad (\text{by the definition of CHF}) \]
\[ X_{z(w_3 w_1 \epsilon)} \times \text{CHF}(w_3 w_1) = \quad (\text{by the definition of } B(E, \epsilon)) \]
\[ X_{z(w_3 \epsilon)} \times \text{CHF}(w_3) \text{CHF}(w_1) = \quad (\text{by the definition of CHF}) \]
\[ \text{CHF}(\rho_1 w_3) \text{CHF}(w_1) = \]
\[ \text{CHF}(w_2) \text{CHF}(w_1) \]

which finishes the proof. ■

The Chekhov-Fock net

Let us denote the image of \( chf \) by \( \Delta(\Gamma, z, \epsilon) \). To explain the properties of \( \Delta(\Gamma, z, \epsilon) \) we need the notion of Chekhov-Fock net. It is some ideal triangulation of the upper plane \( \mathcal{H} \) with the numbers associated to the edges of its dual trivalent graph.
Here is the construction of Chekhov-Fock net, associated to the graph $\Gamma$. The first triangle of Chekhov-Fock net will be the ideal triangle $T_0$ with the vertices in $-1, 0$ and $\infty$. Denote $[x, y]$ the Lobachevsky line joining $x$ and $y$. Let us call $\epsilon^*$ the edge of the trivalent graph, intersecting the edge $[0, \infty]$ of the triangle (we need the orientation of the edges, so let us say that the origin of $\epsilon^*$ which is situated inside $T_0$ is its ”beginning”. The number, associated to this edge of the graph, will be $z^*(\epsilon^*) = z(\epsilon)$. Now (using the fact that $C_2^+ [3]$ acts on our trivalent graph and that this action is transitive) all the numbers associated to all the edges of the graph can be determined in the following way: $z^*(w\epsilon^*) = z(we)$, where $w \in C_2^+ [3]$. Now we have to explain, how to obtain inductively all the other triangles. For example we have constructed the triangle $T$ with the edges $a, b$ and $c$ and we want to construct the triangle, which will also contain edge $c$. Let $\alpha \in PSL_2(\mathbb{R})$ be a transformation of $\mathcal{H}$, which takes triangle $T$ to the triangle $T_0$, so that $\alpha(c) = [0, \infty]$. Then the $(\alpha^{-1} X_{z^*(c)} \alpha)T$ is the desired triangle.

**Fact.** $\forall w \in C_2^+ [3]$ $z^*((CHF(w))^{-1} \epsilon^*) = z(we)$

**Statement 2.** For any $w \in C_2^+ [3]$ $\exists$ triangle $T$ of Fock-Chehov net such that $(CHF(w))^{-1}T_0 = T$.

**Proof:** Let us use the induction over the length of $w$. if this length is 0 then the statement is trivial. Now if $w = \rho_0 w_1$,

$$(CHF(w))^{-1}T_0 = \quad (by \ the \ definition \ of \ CHF)$$

$$(L \times CHF(w_1))^{-1}T_0 =$$

$$(CHF(w_1))^{-1}L^{-1}T_0 =$$

$$(CHF(w_1))^{-1}T_0 = T.$$
\[(CHF(w_1))^{-1}(X_{z(w_1)})^{-1}T_0 =
((CHF(w_1))^{-1}(X_{z(w_1)})^{-1}CHF(w_1))(CHF(w_1))^{-1}T_0 =
((CHF(w_1))^{-1}(X_{z(w_1)})^{-1}CHF(w_1))T = \text{(see Note)}
((CHF(w_1))^{-1}(X_{z((CHF(w_1))^{-1}w)})^{-1}CHF(w_1))T
\]

And the last expression by the definition gives us the next triangle of Chekhov-Fock net. ■

**Corollary:** \(\Delta(\Gamma, z, \epsilon)\) is a fuchsian group.

Triangle \(T_0\) is its fundamental domain.

**The Chekhov-Fock coordinates of dessins d’enfants**

Now let us put \(z \equiv 0\).

**Statement 3:** The riemann surface \(\mathcal{H}/\Delta(\Gamma, 0, \epsilon)\) is equivalent to the Grothendieck model of \(\Gamma\).

**Proof:** If \(z \equiv 0\) then \(\Delta(\Gamma, z, \epsilon) = \Delta(\Gamma, 0, \epsilon)\) is the subgroup of \(PSL_2[\mathbb{Z}]\) Let us first consider the case of normal \(B(E, \epsilon)\), (we call such dessin regular). Consequently \(\Delta(\Gamma, 0, \epsilon)\) is normal in \(PSL_2[\mathbb{Z}]\). So we can factorize \(\mathcal{H}/\Delta(\Gamma, 0, \epsilon)\) by \(PSL_2[\mathbb{Z}]\). Now if identify \(\mathcal{H}/PSL_2[\mathbb{Z}]\) and Riemann sphere, we get the function \(\beta_\Gamma\) defined on \(\mathcal{H}/\Delta(\Gamma, 0, \epsilon)\) with the only critical values 0, 1 and \(\infty\), that is Belyi function corresponding to this dessin.

Now if we take any graph \(\Gamma\) and if \(D\) is the corresponding dessin d’enfant, there exists regular dessin d’enfant \(D'\) and the covering \(\chi : D' \rightarrow D\). Let us denote the graph corresponding to this dessin by \(\Gamma'\). Now we can define function \(\beta_\Gamma\) defined on \(\Sigma/\Delta(\Gamma, 0, \epsilon)\) such that \(\beta_\Gamma \circ \chi = \beta_\Gamma'\)

**Examples.**

**Notation** By the case \(< a_1, \ldots, a_n | b_1, \ldots, b_m >\) we denote graph with the valences of vertices equal to \(a_1, \ldots, a_n\) and the valences of vertices of dual graph equal to \(b_1, \ldots, b_m\).

**Example 1. Case** \(\langle 3, 3 | 2, 2, 2 \rangle\) Let the numbers, corresponding to the edges \(A, B, C\), be \(a, b, c\), the chosen oriented edge be B
with the orientation, so that edge $A$ is to the left of the end of $B$ and edge $C$ - to the right. Then the generators of the corresponding fuchsian group are

$\gamma_1 = X_bRX_cR$
$\gamma_2 = RX_aRX_b$
$\gamma_3 = LX_cRX_aL$

with the relation $\gamma_3\gamma_2\gamma_1 = 1$.

Using the fact that $\gamma_i$ are parabolic we get

$$\begin{align*}
  a + b &= 0 \\
  a + c &= 0 \\
  b + c &= 0
\end{align*}$$

This system has the only solution

$$\begin{align*}
  a &= 0 \\
  b &= 0 \\
  c &= 0
\end{align*}$$

So we get the subgroup of $PSL_2(\mathbb{R})$ with generators:

$\gamma_1 = \begin{pmatrix} 1 & 0 \\
                         2 & 1 \end{pmatrix}$, the invariant point is 0.

$\gamma_2 = \begin{pmatrix} 1 & -2 \\
                         0 & 1 \end{pmatrix}$, the invariant point is $\infty$.

$\gamma_3 = \begin{pmatrix} -1 & -2 \\
                         2 & 3 \end{pmatrix}$, the invariant point is $-1$.

with the relation $\gamma_3\gamma_2\gamma_1 = 1$.

**Example 2. Case** $\langle 3, 3, 3, 3 | 3, 3, 3 \rangle - \text{tetrahedron}$.

Let the edges $A, B, C$ follow clockwise around some vertice, edge $D$ be the opposite to the edge $A$, $E$ to $B$, $F$ to $C$. Taking $a, b, c, d, e, f$ by numbers, corresponding to the edges $A, B, C, D, E, F$ and using the fact that all elements of fuchsian group are parabolic

$$\begin{align*}
  a + e + c &= 0 \\
  b + d + c &= 0 \\
  a + f + b &= 0 \\
  e + f + d &= 0
\end{align*}$$

we get the system
The solution of this system is
\[
\begin{align*}
  a &= d \\
  b &= e \\
  c &= f \\
  a + b + c &= 0
\end{align*}
\]

So we get the family of Riemann surfaces with two parameters. The Riemann surface corresponding to the dessin can be obtained putting \(a = b = c = 0\), because of its symmetry.

So we get the following matrices
\[
\begin{align*}
  \gamma_1 &= \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}, \text{ the invariant point is } \infty \\
  \gamma_2 &= \begin{pmatrix} 2 & 3 \\ -3 & -4 \end{pmatrix}, \text{ the invariant point is } -1 \\
  \gamma_3 &= \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \text{ the invariant point is } 0 \\
  \gamma_4 &= \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}, \text{ the invariant point is } 1
\end{align*}
\]
with the relation \(\gamma_4 \gamma_3 \gamma_2 \gamma_1 = 1\).

**Example 3. Case** \(\langle 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4 \rangle\) Using the usual technique of Fock-Chehov we get the generators of the fuchsian group
\[
\begin{align*}
  \gamma_1 &= \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \text{ the invariant point is } 0 \\
  \gamma_2 &= \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}, \text{ the invariant point is } 1 \\
  \gamma_3 &= \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}, \text{ the invariant point is } \infty \\
  \gamma_4 &= \begin{pmatrix} 7 & 16 \\ -4 & -9 \end{pmatrix}, \text{ the invariant point is } -2 \\
  \gamma_5 &= \begin{pmatrix} 3 & 4 \\ -4 & -5 \end{pmatrix}, \text{ the invariant point is } -1 \\
  \gamma_6 &= \begin{pmatrix} 7 & 4 \\ -16 & -9 \end{pmatrix}, \text{ the invariant point is } -1/2
\end{align*}
\]
with the relation \(\gamma_6 \gamma_5 \gamma_4 \gamma_3 \gamma_2 \gamma_1 = 1\).
Also we can find the automorphism of this picture: \( \alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \)

Factorizing this child’s picture by this automorphism we get the another picture:

**Example 4. Case** \langle 3, 3 | 4, 1, 1 \rangle with generators

\( \gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), the invariant point is \( \infty \)

\( \gamma_2 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \), the invariant point is 0

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