Large-time behaviour of solutions to the outflow problem of full compressible Navier–Stokes equations

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Abstract
This paper is concerned with the large-time behaviour of solutions to the outflow problem of full compressible Navier–Stokes equations on the half line \( \mathbb{R}_+ = (0, \infty) \). On the basis of fine analysis, we obtain two results. One is that the non-degenerate boundary layer is stable under partially large initial perturbation. The other is that the superpositions of the boundary layer (including the non-degenerate case) and the 3-rarefaction wave are asymptotically stable under some smallness conditions. The proofs are given by the elementary energy method.

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1. Introduction

In this paper, we consider an initial-boundary-value problem (IBVP) for full compressible Navier–Stokes equations on \( \mathbb{R}_+ = (0, \infty) \). The one-dimensional (1D) motion of viscous and heat-conductive fluid is described by the following system in Eulerian coordinates:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x &= (\mu u)_x, \\
\left[\rho(e + \frac{1}{2}u^2)\right]_t + \left[\rho u(e + \frac{1}{2}u^2) + pu\right]_x &= (\kappa \theta)_x + \mu u u_x.
\end{align*}
\]

(1.1)

Here the five unknowns \( \rho (\geq 0), u, \theta (\geq 0), p (\geq 0) \) and \( e (\geq 0) \) stand for the mass density, the velocity, the absolute temperature, the pressure and the specific internal energy of the fluid, respectively. \( \mu (> 0) \) is the coefficient of viscosity and \( \kappa (> 0) \) is the coefficient of heat conduction. In this paper \( \mu \) and \( \kappa \) are constants, although they are functions of \( \theta \) for real fluids in general. In addition, let \( s \) denote the specific entropy of the fluid. As usual, in thermodynamics any given two of the five thermodynamical variables, \( \rho, \theta, p, e \) and \( s \), can...
express the remaining three. Moreover, if the fluid is an ideal polytropic gas, then it owns the following specific constructive relations:

\[ p(\rho, \theta) = R \rho \theta^\gamma \exp \left( \frac{\gamma - 1}{R} s \right) = p(\rho, s), \quad e(\rho, \theta) = c_v \theta, \quad c_v = \frac{R}{\gamma - 1}, \]

(1.2)

where \( A \) and \( R \) are positive constants, and \( c_v (> 0) \) and \( \gamma (> 1) \) denote the specific heat at constant volume and the adiabatic exponent of the gas, respectively.

The initial data attached to (1.1) are

\[ (\rho_0, u_0, \theta_0)(x) \overset{\Delta}{=} (\rho, u, \theta)(x, 0) \rightarrow (\rho_\pm, u_\pm, \theta_\pm) \quad \text{as} \quad x \rightarrow \infty, \quad \inf_{x \in \mathbb{R}_+} (\rho_0, \theta_0)(x) > 0, \]

(1.3)

where \( \rho_\pm, u_\pm \) and \( \theta_\pm \) are given constants.

As pointed out by Matsumura [24], the boundary conditions of the IBVP of (1.1) can be posed as one of the following three cases:

**Case I.** (negative velocity on the boundary):

\[ u(0, t) = u_- < 0, \quad \theta(0, t) = \theta_-, \quad t > 0; \]

(1.4)\(_1\)

**Case II.** (zero velocity on the boundary):

\[ u(0, t) = u_- = 0, \quad \theta(0, t) = \theta_-, \quad t > 0; \]

(1.4)\(_2\)

**Case III.** (positive velocity on the boundary):

\[ u(0, t) = u_- > 0, \quad \rho(0, t) = \rho_-, \quad \theta(0, t) = \theta_-, \quad t > 0. \]

(1.4)\(_3\)

Here \( \rho_-, u_- \) and \( \theta_- \) are constants too.

The initial data (1.3) satisfy one of (1.4)\(_i\), \( i = 1, 2, 3 \), as the compatibility condition. It is noticeable that in cases I and II, \( \rho \) cannot be imposed a value on the boundary \( \{ x = 0 \} \), while in case III it has to be done. This comes from the theory of well-posedness of IBVP. The three kinds of IBVP (1.1), (1.3) and (1.4)\(_i\), \( i = 1, 2, 3 \), are called the outflow problem, impermeable wall problem and inflow problem, respectively. See [24] for more details.

In the past several decades, there have been many works on the large-time behaviour of solutions to Cauchy problem of 1D compressible Navier–Stokes equations (1.1) (including its isentropic and isothermal cases) where at far fields \( x = \pm \infty \), the end states are

\[ \lim_{x \to k\infty} (\rho, u, \theta)(x, 0) = (\rho_\pm, u_\pm, \theta_\pm). \]

(1.5)

We refer to [2–4, 6, 9, 11, 13, 14, 19, 22, 29] etc, and some references therein. All the works show that the large-time behaviour of solutions to Cauchy problem of (1.1) with (1.5) is basically described by the Riemann solutions to its corresponding hyperbolic system, i.e. compressible Euler equations, just as shock waves and contact discontinuity are replaced by the corresponding (shifted) viscous shock waves and viscous contact wave, respectively.

A natural question of importance and interest is that whether the solutions to the IBVP of (1.1) have a similar large-time behaviour? Unfortunately, the answer is no. In fact, some people found a new wave phenomenon while studying the IBVP for scalar viscous conservation law, see [20, 21, 23], etc. This phenomenon appeared due to the presence of the boundary, and they named it boundary layer (since the wave’s form is the stationary solution, other people still call it stationary solution). From then on, it has attracted some people’s attention to investigate the existence and stability of the boundary layer, including the stability of its superpositions with viscous hyperbolic waves.
This new wave phenomenon also appears in the IBVP of (1.1) including its isentropic case. Taking the outflow problem,
\[
\begin{aligned}
    \rho_t + (\rho u)_x &= 0, & x \in \mathbb{R}_+, & t > 0, \\
    (\rho u)_t + (\rho u^2 + p)_x &= \rho u_{xx}, & p &= K\rho^{\gamma}, K > 0, & \gamma \geq 1, \\
    (\rho, u)(x, 0) &= (\rho_+, u_+) & \text{as } x \to \infty, & u(0, t) &= u_- < 0, & \inf_{x \in \mathbb{R}_+} \rho(x, 0) > 0, 
\end{aligned}
\]  
(1.6)
as an example, we roughly explain this phenomenon. Consider the Riemann problem
\[
\begin{aligned}
    \rho_t + (\rho u)_x &= 0, & x \in \mathbb{R}, & t > 0, \\
    (\rho u)_t + (\rho u^2 + p)_x &= 0, & (p, u)(x, 0) &= \begin{cases} (\rho_+, u_+), & x > 0, \\
    (\rho_-, u_-), & x < 0. \end{cases}
\end{aligned}
\]  
(1.7)
Here the right end states \((\rho_+, u_+)\) are the same in (1.6) and (1.7), while the left end states are not necessarily equal. The system in (1.7) has two eigenvalues \(u - \sqrt{p'(\rho)}\) and \(u + \sqrt{p'(\rho)}\).

Define the rarefaction and shock curves to be \(R_i(\rho_+, u_+)\) and \(S_i(\rho_+, u_+), i = 1, 2,\) respectively. For each given right state \((\rho_+, u_+)\), if the left state \((\rho_-, u_-)\) satisfies one of the following four cases:
\[
\begin{cases} (\rho_-, u_-) \in R_i(\rho_+, u_+), & u_- \geq (-1)^{i-1}\sqrt{p'(\rho_-)}, \\
    (\rho_-, u_-) \in S_i(\rho_+, u_+), & u_- \geq (-1)^{i-1}\sqrt{\rho_i(p(\rho_i) - p(\rho_-))}. \end{cases}
\]  
(1.8)
then the hyperbolic waves of (1.7) have no negative wave speeds, which means they are constant in the region \((x, t) \in \mathbb{R}_- \times \mathbb{R}_+\). It is natural to expect that the part of the viscous version of the solutions to (1.7) with (1.8) in the region \((x, t) \in \mathbb{R}_+ \times \mathbb{R}_+\) can be looked on as the asymptotic profiles of the solutions to the IBVP of the isentropic compressible Navier–Stokes equations with the same left/right end states. For the outflow problem (1.6), due to \(u_- < 0\), one easily knows that for any \(\rho_-\) there exists no solution towards 1-family hyperbolic waves. On the one hand, if there exists a \(\rho_-\) such that it and \(u_-\) in (1.6) satisfy one of (1.8) with \(i = 2\), then it is reasonable to expect that the 2-family viscous hyperbolic waves are the profiles of the solutions. On the other hand, if there never exists such a \(\rho_-\), then there must exist a gap between the value of 2-family hyperbolic waves with non-negative wave speed at \(x = 0\) and \(u_-\) in (1.6). In this situation we expect that a boundary layer which compensates the gap comes up. Such a boundary layer could be constructed by the stationary solution of the compressible Navier–Stokes equations. We refer to [24] for more details.

For the IBVP of the isentropic case, Matsumura [24] proposed a criterion on the existence of the boundary layer. Since then, some people have already verified some conjectures in [24]. See [8, 10, 16, 18, 25–28, 30], etc.

For the IBVP of the non-isentropic case, there exist some works, while it is still an open problem to give the complete classification about the large-time behaviour of the solutions similar to those in [24]. For the inflow problem, Huang et al [5] show the asymptotic stability of non-degenerate boundary layer and its superposition with 3-rarefaction wave. In [32] and [31] Wang and I give the relation between the end states and the existence of a boundary layer and prove the asymptotic stability of the boundary layer (including the degenerate case) as well as its superposition with 1-rarefaction wave, the viscous contact wave and 3-rarefaction wave.

For the outflow problem, Kawashima et al [15] study its boundary layer comprehensively and deeply, and obtain many results including the relation between the existence of the boundary layer and the end states, the asymptotic stability of the boundary layer, and the convergence rate of the solutions towards the boundary layer.
Motivated by [10] and [15–18], we are interested in studying further the outflow problem of full compressible Navier–Stokes equations, i.e. (1.1)–(1.3) and (1.4).

Here, we state our results in a little detail and briefly review some key analytical techniques. First, we show that the non-degenerate boundary layer is stable under partially large initial perturbation. It is known that in studying the solutions with wave profiles of (1.1), if the initial perturbation is small enough, then the corresponding linearized system plays a leading role in the stability analysis. Nevertheless, this method does not work when the initial perturbation is large. The real cause is that in the latter the nonlinear parts will give birth to some overwhelming difficulties, and thus they become the research focus. Precisely speaking, during the stability analysis it is critical to obtain the positive uniform upper/lower bounds of density $\rho$ and absolute temperature $\theta$, while this is slightly trivial under small perturbation. To achieve the bounds of $\rho$ for large perturbation, we have to close the a priori estimate for the perturbation of density $\rho(x, t) - \tilde{\rho}(x)$ (see (3.22) for details). As for the bounds of $\theta$, we require a technical condition $0 < \gamma - 1 \ll 1$ which means the underlying situation is very close to the isothermal case (see (2.16) for details). At the same time, the condition $\gamma - 1 \ll 1$ also means that the initial perturbation of the temperature component is not arbitrarily large but has to be small. That is why we use partially to modify large for a more precise definition in the abstract. This idea comes from Nishihara et al [29]. It is also noted that the initial perturbation of $\rho$ and $u$ is large, but not arbitrarily. Now, there appears to be another problem following along with this condition: the coefficients of $\theta_t$ and the convection term are very large! This means that under the condition above the conservation-of-energy equation may be close to a hyperbolic equation rather than a parabolic one in the sense of the management procedure. However, in thermodynamics, for example see [1], there exists the following important relation between $\kappa$ and $\mu$:

$$ Pr \triangleq \frac{\gamma c_v \mu}{\kappa}, $$

(1.9)

where $Pr$ is the Prandtl number of the fluid. This indicates that the coefficient of heat conduction is also large, and we can manage the third equation like a parabolic one.

Second, we prove that the degenerate boundary layer, the 3-rarefaction wave and the superposition of the boundary layer (including the non-degenerate case) are also stable under small perturbations. Compared with the non-degenerate boundary layer, the degenerate case is much more difficult to study. The key reason is that the non-degenerate case decays exponentially, while the degenerate case decays algebraically, thus Poincaré inequality is out of function in the degenerate case. In the outflow problem of the isentropic case, the degenerate boundary layer monotonically increases w.r.t. $x$, and this a good property in stability analysis, while in the non-isentropic case its counterpart does not have this merit. After carefully analysing the degenerate boundary layer, we find that it possesses the same property far from the boundary $\{x = 0\}$. Now, it is natural to divide the half line into two parts, one is far from the boundary, the other is the remaining. And on the different parts, we use different methods to control it. As for the 3-rarefaction wave, its monotonicity w.r.t. $x$ also plays an important role in the a priori estimates. At the same time, the amplitude of the 3-rarefaction wave is not necessarily weak if the rarefaction wave is far from the boundary.

This paper is organized as follows. In section 2, we state our main results. In section 3, we prove the stability of the non-degenerate boundary layer under partially large perturbation. Section 4 is devoted to prove the stability of the 3-rarefaction wave and its superposition with the boundary layer (including the non-degenerate case) under small perturbation. The appendix is devoted to some details of the boundary layer for a transonic case.
Notation. Throughout this paper, several positive generic constants are denoted by $c$ and $C$ without confusion, and $c(\cdot)$ and $C(\cdot)$ stand for some generic constant depending only on the quantity listed in the parentheses. For function spaces, $L^p(\Omega)$, $1 \leq p \leq \infty$, denotes the usual Lebesgue space on $\Omega \subset \mathbb{R} = (-\infty, \infty)$. $W^{k,p}(\Omega)$ denotes the $k$th order Sobolev space, and if $p = 2$, let $H^k(\Omega) \equiv W^{k,2}(\Omega)$, $\| \cdot \|_{L^2(\Omega)}$ and $\| \cdot \|_{H^k(\Omega)}$ for brevity. Let $T$ and $\mathbb{B}$ be a positive constant and a Banach space, respectively. $C([0,T]; \mathbb{B})$ denotes the space of $\mathbb{B}$-valued continuous functions on $[0,T]$, and $L^2(0,T; \mathbb{B})$ is the space of $\mathbb{B}$-valued $L^2$-function on $[0,T]$. The domain $\Omega$ will be often abbreviated without confusion. For $0 < \alpha < 1$, $B^{\alpha,p}(\Omega)$ denotes the Hölder space of continuous functions which have the $k$th order derivative of Hölder continuity with exponent $\alpha$, and let $| \cdot |_{k,\alpha} \mathbb{B}$ represent its norm. For a domain $\Omega_T = \Omega \times [0,T]$ and the integers $k$ and $l$, $B^{k+\alpha,l+p}(\Omega_T)$ denotes the Hölder space of the functions satisfying $\partial_j^\alpha u, \partial_j^l u \in B^{\alpha,p}(\Omega_T)$ for integers $0 \leq i \leq k, 0 \leq j \leq l$, and $| \cdot |_{k,\alpha,l+p} \mathbb{B}$ denotes its norm. Here $B^{\alpha,p}(\Omega_T)$ denotes the Hölder space of continuous functions which have the Hölder exponents $\alpha$ and $\beta$ with respect to $t$ and $x$, respectively. All of the integers $k$ and $l$ are non-negative. For the sake of simplicity, let $B_T^{k+\alpha,l+p}(\Omega_T)$ represent $B^{k+\alpha,l+p}(\Omega_T)$.

2. Preliminaries and main results

Let

$$c(\rho, s) \triangleq \sqrt{\rho c(\rho, s)} = \sqrt{\gamma \rho \theta}, \quad M(\rho, u, \theta) \triangleq \frac{|u|}{c}, \quad (2.1)$$

be the local sound speed and the Mach number, and let

$$c_\pm \triangleq c(\rho_\pm, \theta_\pm) = \sqrt{\gamma \rho_\pm \theta_\pm}, \quad M_\pm \triangleq M(\rho_\pm, u_\pm, \theta_\pm) = \frac{|u_\pm|}{c_\pm}, \quad (2.2)$$

be the sound speed and the Mach number at the far field $\{x = \infty\}$, respectively. Let quarter 3D space $\{(\rho, u, \theta) | \rho > 0, \theta > 0\}$ be the state space, and divide it into three parts:

$$\Omega_{\text{sub}} \triangleq \{(\rho, u, \theta) | |u| < \sqrt{\gamma \rho \theta}\}, \quad \Gamma_{\text{trans}} \triangleq \{(\rho, u, \theta) | |u| = \sqrt{\gamma \rho \theta}\}, \quad \Omega_{\text{super}} \triangleq \{(\rho, u, \theta) | |u| > \sqrt{\gamma \rho \theta}\}. \quad (2.3)$$

$\Omega_{\text{sub}}, \Gamma_{\text{trans}}$ and $\Omega_{\text{super}}$ are called subsonic, transonic and supersonic regions, respectively. Obviously, $\Gamma_{\text{trans}}$ and $\Omega_{\text{super}}$ are not connected, and if we add the alternative condition $u < 0$ or $u \geq 0$, then we have six connected subsets $\Omega_{\text{sub}}^\pm$, $\Gamma_{\text{trans}}^\pm$ and $\Omega_{\text{super}}^\pm$.

2.1. Boundary layer

It is known that the corresponding hyperbolic system of (1.1) with (1.2) has three characteristic values

$$\lambda_1(\rho, u, \theta) = u - \sqrt{\gamma \rho \theta}, \quad \lambda_2(\rho, u, \theta) = u, \quad \lambda_3(\rho, u, \theta) = u + \sqrt{\gamma \rho \theta}. \quad (2.4)$$

For the outflow problem $u_- < 0$, one easily knows $\lambda_1 < \lambda_2 < 0$ at the boundary $\{x = 0\}$. So if $u_+ < 0$ and $u_-$ is sufficiently close to $u_+$ such that $u_- < 0$ also holds, then a stationary
solution \((\bar{\rho}, \bar{u}, \bar{\theta})(x)\) to (1.1)–(1.3) and (1.4)_1 is expected:

\[
\begin{align*}
\langle \bar{\rho} \bar{u} \rangle &= 0, \\
\langle \bar{\rho} \bar{u}^2 + \bar{p} \rangle &= \mu \bar{u}^\prime, \\
\left[ \bar{p} \bar{u} \left( \frac{R}{y-1} \bar{\theta} + \frac{1}{2} \bar{u}^2 \right) + \bar{\rho} \bar{u} \right] &= \kappa \bar{u}^\prime + \mu (\bar{u} \bar{u}^\prime)', \\
(\bar{u}, \bar{\theta})(0) &= (u_-, \theta_-), \\
(\bar{\rho}, \bar{\theta})(\infty) &= (\rho_+, u_+, \theta_+),
\end{align*}
\]

(2.5)

where \(\bar{\rho} \triangleq p(\bar{\rho}, \bar{\theta}) = R \bar{p} \bar{\theta}\). We call the stationary solution \((\bar{\rho}, \bar{u}, \bar{\theta})(x)\) the boundary layer.

From the fact that \(\bar{\rho}(x) > 0\) and \(u_- < 0\), we have

\[
\bar{u}(x) < 0, \quad \bar{\rho}(x) = \frac{\rho_+ u_+}{u(x)}, \quad \rho_- \triangleq \rho(0) = \frac{\rho_+ u_+}{u_-}.
\]

Thus (2.5) is equivalent to the coupling of (2.6) and the following ODE system:

\[
\begin{align*}
\bar{u}^\prime &= \frac{1}{\mu \kappa} \left[ \rho_+ u_+ (\bar{u} - u_+) + R \rho_+ u_+ \left( \bar{\theta} - \theta_+ \right) \right], \\
\bar{\theta}^\prime &= \frac{1}{\kappa} \left[ p_+ (\bar{u} - u_+) + \frac{R}{y-1} \rho_+ u_+ (\bar{\theta} - \theta_+) - \frac{\rho_+ u_+}{2} (\bar{u} - u_+)^2 \right], \\
(\bar{u}, \bar{\theta})(0) &= (u_-, \theta_-), \quad (\bar{\rho}, \bar{\theta})(\infty) = (\rho_+, u_+, \theta_+),
\end{align*}
\]

(2.7)

where \(\rho_+ \triangleq \rho(u_+, \theta_+)\). The amplitude of the boundary layer is measured by

\[
\bar{\delta} \triangleq |u_+ - \theta_+ - \theta_-|.
\]

(2.8)

Lemma 2.1 (Existence of boundary layer [15]). Suppose that the boundary data \((u_-, \theta_-)\) satisfy

\[
(u_-, \theta_-) \in B_+ \triangleq \left\{ (u, \theta) \in \mathbb{R}^2 : |u - u_+ + \theta - \theta_+| < \delta_0 \right\}
\]

(2.9)

for a certain positive constant \(\delta_0\). Note that condition (2.9) is equivalent to \(\bar{\delta} < \delta_0\).

(i) \(M_+ > 1\), there exists a unique smooth solution \((u, \theta)\) to the problem (2.7) satisfying

\[
\left| \left( u - u_+, \theta - \theta_+ \right)^{(n)} \right| \leq C \bar{\delta} e^{-\xi x}, \quad \frac{d^n}{dx^n}, \quad n = 0, 1, 2, \ldots.
\]

(2.10)

(ii) \(M_+ = 1\), there exists a centre-stable manifold \(M \subset B_+\) consisting of two trajectories \(\Gamma_i \triangleq \{M_{i1}, M_{i2}\}(\xi), i = 1, 2, \xi \in \mathbb{R}_+,\) tangent to the line \(\mu u_+(u - u_+) - (y-1)(\theta - \theta_+) = \) 0 on the opposite directions at \((u_+, \theta_+)\). Depending on the location of \((u_+, \theta_+)\), this case is divided into three subcases:

Subcase 1. For each \((u_-, \theta_-) \in M_1, (\bar{u}, \bar{\theta}) \subset M\) and

\[
\left| \left( \bar{u} - u_+, \bar{\theta} - \theta_+ \right)^{(n)} \right| \leq C \bar{\delta} e^{-\bar{\delta} x}.
\]

(2.11)

Subcase 2. For each \((u_-, \theta_-) \in B_+, \mu u_+(u_+ - u_-) - (y-1)(\theta_+ - \theta_-) < M_{i2}(\xi),\) where \(\xi\) is determined uniquely by \(M_{i1}(\xi) = \frac{\gamma}{\mu y \gamma(y-1)} + \frac{(y-1)R y (\theta_+ - \theta_-)}{|\mu y (y-1)\gamma u^2|}, i = 1 \text{ or } 2,\) there exists a unique solution \((\bar{u}, \bar{\theta}) \subset B_+\) satisfying

\[
\left| \left( \bar{u} - u_+, \bar{\theta} - \theta_+ \right)^{(n)} \right| \leq C \frac{\bar{\delta}^{1+n}}{(1 + \bar{\delta} x)^{1+n}}, \quad \bar{u}_x > 0 \text{ and } \bar{\theta}_x > 0 \text{ for } x \gg 1.
\]

(2.12)

Subcase 3. For each \((u_-, \theta_-) \in B_+, if it does not belong to subcases 1 or 2, then there exists no solution.\)
(iii) \( M_s < 1 \), there exists a curve such that the unique smooth solution \((u, \theta)\) to the problem (2.7) satisfies
\[
\left| \tilde{u} - u_s, \tilde{\theta} - \theta_s \right|^{(\alpha)} \leq C \delta e^{-cx}, \quad n = 0, 1, 2, \ldots.
\]  

(2.13)

**Remark 2.1.** For the transonic case, i.e. \( M_s = 1 \), we do not directly cite the corresponding result of [15]. One of the main reasons is that \( \tilde{u}_s > 0, \tilde{\theta}_s > 0 \) when \( x \gg 1 \) will be fundamental to obtain some energy estimates in the proof of theorem 2.3. See section 4 for details. The proof of case II is given in the appendix. In addition, in [15] the authors show the stability of the degenerate boundary layer through originally introducing a dissipative norm. Perhaps this method also works well for the stability of the superposition of the degenerate boundary layer and the 3-rarefaction wave.

Now we list the result that the non-degenerate boundary layer is stable under partially large initial perturbation.

**Theorem 2.1 (Stability of non-degenerate case).** Assume that \( 0 < \gamma - 1 < 1, \rho_\pm, u_\pm, \) and \( \theta_\pm \) satisfy cases (i), (iii) or subcase 1 of case (ii) in lemma 2.1, and \( |\theta_\pm - \theta_{-\pm}| \leq C (\gamma - 1) \).
\[
\begin{align*}
\begin{cases}
(\rho_0, u_0, \theta_0)(x) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x) \in H^1(\mathbb{R}_+), \\
\rho_0(x) - \tilde{\rho}(x) \in B^{1+\sigma}(\mathbb{R}_+), \\
\| \theta_0 - \tilde{\theta} \|_1 \leq C (\gamma - 1), \\
(u_0, \theta_0)(x) - (\tilde{u}, \tilde{\theta})(x) \in B^{2+\sigma}(\mathbb{R}_+),
\end{cases}
\end{align*}
\]
\[
\text{where } \sigma \in (0, 1) \text{ is some constant. For each } M > 0, \text{ there is a small constant } \delta_0 = \delta_0(M) > 0 \text{ such that if } \delta \leq \delta_0 \text{ and } \| (\rho_0 - \tilde{\rho}, u_0 - \tilde{u}, \theta_0 - \tilde{\theta}) \|_1 \leq M, \text{ where } (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x) \text{ satisfies (2.5), then the outflow problem, (1.1)-(1.3), (1.4) with (1.9), has a unique solution } (\rho, u, \theta)(x, t) \text{ satisfying}
\begin{align*}
\begin{cases}
(\rho, u, \theta)(x, t) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x) \in C \left[ 0, \infty; H^1 \right), \\
\rho(x, t) - \tilde{\rho}(x) \in L^2(0, \infty; L^2), \\
(\rho, u)(x, t) - \tilde{\rho}(x, t) \in B^{1+\frac{\gamma}{2}, 1+\sigma} T, \\
(\rho, u)(x, t) - \tilde{\rho}(x, t) \in B^{1+\frac{\gamma}{2}, 2+\sigma} T
\end{cases}
\end{align*}
\]
\[
\text{for any fixed } T > 0. \text{ Furthermore,}
\lim_{t \to \infty} \sup_{x \in \mathbb{R}_+} |(\rho, u, \theta)(x, t) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x)| = 0.
\]

(2.15)

(2.16)

**Remark 2.2.** In fact, \( 0 < \gamma - 1 \ll 1 \) implies two a priori conditions. One is \( |\theta_\pm - \theta_{-\pm}| \ll 1 \), and the other is that the initial perturbation of the temperature component is small. Specifically speaking, from (1.2) and (1.3) one easily has
\[
\| \theta_0 - \tilde{\theta} \|_1 \leq C (\gamma - 1) \| (\rho_0 - \tilde{\rho}, s_0 - \tilde{s}) \|_1,
\]
\[
\text{where } s_0 \text{ and } \tilde{s} \text{ are the initial data of the entropy of systems (1.1) and (2.5), respectively.}
\]

**Remark 2.3.** The initial perturbation of \( \rho \) and \( u \) is large, but not arbitrarily.

2.2. Rarefaction wave

It is well known that the 3-rarefaction wave curve through the right-hand side state \((\rho_s, u_s, \theta_s)\) is
\[
R_3(\rho_s, u_s, \theta_s) \triangleq \left\{ (\rho, u, \theta) \mid \begin{array}{l}
0 < \rho < \rho_s, \\
u = u_s + \int_{\rho_s}^{\rho} \sqrt{R_{\gamma} \rho \gamma^{\frac{1}{\gamma}} \theta \xi^{1-\gamma}} \, d\xi
\end{array} \right\}.
\]

(2.18)
So for each pair of data \((u_-, \theta_-)\) with the restriction condition
\[
u_+ = u_+ + \int_{\rho_+}^{(\rho_-, \rho_+)} \sqrt{R\gamma \rho_+^{1-\gamma} \theta_+^{\gamma-1}} \, d\xi,
\]
there exists a unique \(\rho_-\) such that \((\rho_-, u_-, \theta_-) \in R_3(\rho_+, u_+, \theta_+).\) The 3-rarefaction wave \((\rho', u', \theta')(x/t)\) connecting \((\rho_-, u_-, \theta_-)\) and \((\rho_+, u_+, \theta_+)\) is the unique weak solution globally in time to the following Riemann problem:
\[
\begin{cases}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2 + p)_x = 0, \quad x \in \mathbb{R}, t > 0,
\end{cases}
\]
\[
\begin{cases}
\rho(\frac{R}{\gamma - 1}(\theta + \frac{1}{2} u^2))_x + [\rho u(\frac{R}{\gamma - 1}(\theta + \frac{1}{2} u^2) + pu)]_x = 0, \quad x > 0, \\
(\rho, u, \theta)(x, 0) = \begin{cases} 
(\rho_-, u_-, \theta_-), & x < 0, \\
(\rho_+, u_+, \theta_+), & x > 0.
\end{cases}
\end{cases}
\]

Here \(\theta_- < \theta_+\) (or equivalently, \(\rho_- < \rho_+, u_- < u_+)\).

In addition, if \(\lambda_3(\rho_-, u_-, \theta_-) \geq 0\), then the rarefaction wave is constant on \((x, t) \in \mathbb{R}_- \times [0, \infty).\) For the outflow problem when \(u_- \in \Omega_{\text{sub}} \cap \Gamma_{\text{trans}},\) then \(\lambda_3(\rho_-, u_-, \theta_-) \geq 0.\) Thus in this situation, one can expect that the solutions to the outflow problem converge towards 3-rarefaction wave which is similar to the Cauchy problem of (1.1). To give the details of the large-time behaviour of the solutions to the outflow problem, it is necessary to construct a smooth approximation \((\tilde{\psi}, \tilde{u}, \tilde{\theta})(x, t)\) of \((\rho', u', \theta')(x/t).\) To this end, we combine the ideas from [3] and [7].

Consider the solution to the following Cauchy problem:
\[
\begin{cases}
\frac{w_t + w w_x}{w(0, x)} = w_-, \quad x \in \mathbb{R}, t > 0, \\
\frac{w(0, x)}{w_0} = w_-, \quad x < 0, \\
\frac{w + C_q \delta_t}{w_0} = w_-, \quad x > 0.
\end{cases}
\]

Here \(\delta_t = w_+ - w_- > 0, 0 < \epsilon < 1 \leq q\) are two constants to be determined later, \(C_q\) is a constant such that \(C_q \int_0^\infty y^q e^{-y} dy = 1.\) Let \(w_{\pm} = \lambda_3(\rho_{\pm}, u_{\pm}, \theta_{\pm})\) and \((\tilde{\psi}, \tilde{u}, \tilde{\theta})(x, t)\) be
\[
\begin{cases}
(\tilde{u} + \sqrt{R\gamma \tilde{\theta}})(x, t) = w(x, 1 + t), \\
(\tilde{\psi}^{1-\gamma})(x, t) = \rho_0^{1-\gamma} \psi_0, \quad x \in \mathbb{R}, t > 0, \\
\tilde{u}(x, t) = u_+ + \int_{\rho_+}^{\rho(x, t)} \sqrt{R\gamma \rho_+^{1-\gamma} \theta_+^{\gamma-1}} \, d\xi.
\end{cases}
\]

where \(\gamma > 1\) is a constant to be determined later.

Due to \(w_+ \geq 0\) and (2.21), one has \(w(x, t) \equiv w_-\) on \(\mathbb{R}_- \times [0, \infty).\) Combining (2.22), one easily knows that \((\tilde{\psi}, \tilde{u}, \tilde{\theta})(x, t)\) is constant on \(\mathbb{R}_- \times [0, \infty)\) too. From here on we still use \((\tilde{\psi}, \tilde{u}, \tilde{\theta})(x, t)\) to represent \((\tilde{\psi}, \tilde{u}, \tilde{\theta})(x, t)|_{x \geq 0}.\) Then one easily has
\[
\begin{cases}
\tilde{\psi}_t + (\tilde{\psi} \tilde{u})_x = 0, \\
(\tilde{\psi} \tilde{u})_t + (\tilde{\psi} \tilde{u}^2 + \tilde{p})_x = 0, \quad x \in \mathbb{R}_+, t > 0, \\
\tilde{p}(\frac{R}{\gamma - 1}(\tilde{\theta} + \frac{1}{2} \tilde{u}^2))_x + [\tilde{p} \tilde{u}(\frac{R}{\gamma - 1}(\tilde{\theta} + \frac{1}{2} \tilde{u}^2) + \tilde{p} \tilde{u})]_x = 0, \\
(\tilde{\psi}, \tilde{u}, \tilde{\theta})(0, t) = (\rho_-, u_-, \theta_-), \quad (\tilde{\psi}, \tilde{u}, \tilde{\theta})(x, 0) \rightarrow \begin{cases} 
(\rho_+, u_+, \theta_+), & x \rightarrow 0, \\
(\rho_+, u_+, \theta_+), & x \rightarrow \infty.
\end{cases}
\end{cases}
\]

where \(\tilde{p} \equiv p(\tilde{\psi}, \tilde{\theta}).\)
Lemma 2.2. \((\tilde{\rho}, \tilde{u}, \tilde{\theta})(x, t)\) satisfies

(i) \(0 \leq \tilde{\rho}_x, \tilde{\theta}_x \leq C\tilde{u}_x, \quad \left| \tilde{\rho}_x, \tilde{\theta}_x \right| \leq C(\tilde{u}_x)\); 

(ii) For any \(p (1 \leq p \leq \infty)\), there exists a constant \(C_{pq}\) such that

\[
\| (\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)(t) \|_{L^p} \leq C_{pq} \min \left\{ \varepsilon^{1-1/p}, (1+t)^{-1+1/p} \right\},
\]

\[
\| (\tilde{\rho}_x, \tilde{u}_x, \tilde{\theta}_x)(t) \|_{L^p} \leq C_{pq} \min \left\{ \varepsilon^{2-1/p}, (1+t)^{-1+1/q} \right\}.
\]

(iii) If \(x \leq (u_\ast + \sqrt{Ry\theta_\ast})(1+t)\), then \((\tilde{\rho}, \tilde{u}, \tilde{\theta})(x,t) = (\rho_\ast, u_\ast, \theta_\ast) \equiv 0\); 

(iv) \(\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |(\tilde{\rho}, \tilde{u}, \tilde{\theta})(x,t) - (\rho', u', \theta')(x/t)| = 0\).

Under the preliminaries above, we give the local stability of the 3-rarefaction wave:

Theorem 2.2. Assume \((\rho_\ast, u_\ast, \theta_\ast) \in \Omega_{sub} \cup \{u \geq 0\}, \theta_\ast < \theta_\ast\) and

\[
u_\ast = u_\ast + \int_{\rho_0}^{\rho_\ast} \frac{\rho^2}{\sqrt{\gamma \rho^2 + \gamma \rho \theta}} \, d\xi = -\sqrt{\gamma \rho \theta_\ast}.
\]

where \(\sigma (0,1)\) is some constant. In addition, for suitably small constant \(\delta_0 > 0\) such that if \(\varepsilon + \| (\rho_0, u_0, \theta_0) - (\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0) \| \leq \delta_0\) then the outflow problem, (1.1)–(1.3) and (1.4), has a unique global solution \((\rho, u, \theta)(x,t)\) satisfying

\[
\begin{align*}
(\rho - \tilde{\rho}, u - \tilde{u}, \theta - \tilde{\theta})(x,t) & \in C([0, \infty); H^1), \\
(\rho - \tilde{\rho})_x(x,t) & \in L^2(0, \infty; L^2), \\
(u - \tilde{u}, \theta - \tilde{\theta})_x(x,t) & \in L^2(0, \infty; H^1), \\
(\rho - \tilde{\rho})(x,t) & \in B^{1+\gamma,1}_{1} + \mathbb{B}^{1+\gamma,2\sigma}_1,
\end{align*}
\]

for any fixed \(T > 0\). Furthermore,

\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} \left| (\rho, u, \theta)(x,t) - (\rho', u', \theta')(x/t) \right| = 0.
\]

2.3. Superposition of boundary layer and rarefaction wave

First, let \((\rho_\ast, u_\ast, \theta_\ast) \in \Omega_{sub} \cup \{u \geq 0\}\). For \((\rho_\ast, u_\ast, \theta_\ast) \in R_3(\rho_\ast, u_\ast, \theta_\ast) \cap (\Omega_{sub} \cup \Gamma_{trans})\), let \(S_\ast \triangleq \{(\rho, u, \theta) | \rho u = \rho_\ast u_\ast\}\) be a family of surfaces. From section 2.2, one knows that for each point \((\rho_\ast, u_\ast, \theta_\ast)\) there exists a uniquely determined 3-rarefaction wave connecting it and \((\rho_\ast, u_\ast, \theta_\ast)\). Among the three variables, \(\rho_\ast, u_\ast, \theta_\ast\) just one is independent, the other two can be determined accordingly. Precisely speaking, if let \(\rho_\ast\) be independent, then

\[
\rho_\ast < \rho_\ast, \quad \rho_\ast^{-1/\gamma} \theta_\ast = \rho_\ast^{-1/\gamma} \theta_\ast, \quad u_\ast = u_\ast + \int_{\rho_0}^{\rho_\ast} \sqrt{Ry\rho_\ast^{-1/\gamma}} \theta_\ast \frac{\xi}{\sqrt{\xi}} \, d\xi.
\]

Obviously, both \(u_\ast\) and \(\theta_\ast\) are strictly increasing and continuously differentiable w.r.t. \(\rho_\ast\). From section 2.1, one easily knows that each boundary layer belongs to one surface of the family. Consider the family \(S_\ast\) to be a function of \(\rho_\ast\), then

\[
\frac{dS_\ast}{d\rho_\ast} = u_\ast + \rho_\ast \frac{d\rho_\ast}{d\rho_\ast} = u_\ast + \sqrt{Ry\theta_\ast}.
\]

Since \((\rho_\ast, u_\ast, \theta_\ast) \in \Omega_{sub} \cup \Gamma_{trans}\), one knows that \(R_3(\rho_\ast, u_\ast, \theta_\ast)\) and each one of \(S_\ast\) owns a unique intersection point, i.e. \((\rho_\ast, u_\ast, \theta_\ast)\). Furthermore, all of \(S_\ast\) never intersect each other, especially when \((\rho_\ast, u_\ast, \theta_\ast) \in \Gamma_{trans}^\ast\).

Now if

\[
0 \neq u_\ast - u_\ast - \int_{\rho_0}^{\rho_\ast} \sqrt{Ry\rho_\ast^{-1/\gamma}} \theta_\ast \frac{\xi}{\sqrt{\xi}} \, d\xi \ll 1,
\]

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then it is expected that there exists a unique point \((\rho_*, u_*, \theta_*) \in R_3(\rho_*, u_*, \theta_*) \cap \Omega_{\text{sub}}\) such that 
\(\rho_*, u_*, \theta_*\) and \(\theta_-\) satisfy (2.5) just when \((\rho_*, u_*, \theta_*)\) is replaced by \((\rho_*, u_*, \theta_* )\) there. As for \((\rho_*, u_*, \theta_*)\) \(\in R_3(\rho_*, u_*, \theta_*) \cap \Omega_{\text{trans}}\), one knows \(u_* = -\sqrt{R\rho}\), which means \((\rho_*, u_*, \theta_*)\) is unique too.

Let
\[
(\tilde{\rho}, \tilde{u}, \tilde{\theta}) = (\tilde{\rho}, \tilde{u}, \tilde{\theta}) + (\rho_*, u_*, \theta_*) ,
\]
and the amplitudes of boundary layer \(\tilde{\delta}\) and rarefaction wave \(\tilde{\delta}\) are \(|u_* - u_-|\) and \(|u_* - u_* \|\), respectively. Now we state the third result.

**Theorem 2.3.** Assume \((\rho_*, u_*, \theta_*) \in \Omega_{\text{sub}} \cup \{ u \geq 0 \} \), and \((\rho_*, u_*, \theta_*) \in R_3(\rho_*, u_*, \theta_*) \cap \Omega_{\text{sub}} \cup \Omega_{\text{trans}}\) and \(\rho_*, u_*, \theta_*, u_-\) and \(\theta_-\) satisfy (2.5) just when \((\rho_*, u_*, \theta_*)\) is replaced by \((\rho_*, u_*, \theta_* )\).

\[
\rho_0 - \tilde{\rho}_0 \in H^1(\mathbb{R}_+) \cap B^{1+\sigma}_\infty(\mathbb{R}_+), \quad (u_0, \theta_0) - (\tilde{u}_0, \tilde{\theta}_0) \in H^1(\mathbb{R}_+) \cap B^{2+\sigma}_\infty(\mathbb{R}_+),
\]
where \(\sigma \in (0, 1)\) is some constant. For \(\varepsilon_0\) suitably small such that if \(\tilde{\delta} + \varepsilon + \| (\rho_0 - \tilde{\rho}_0, u_0 - \tilde{u}_0, \theta_0 - \tilde{\theta}_0) \|_1 \leq \varepsilon_0\), then the outflow problem, (1.1)–(1.3) and (1.4), has a unique solution \((\rho, u, \theta)(x, t)\) satisfying
\[
\left\{ \begin{array}{l}
(\rho - \tilde{\rho}, u - \tilde{u}, \theta - \tilde{\theta})(x, t) \in C([0, \infty); H^1), \\
(\rho - \tilde{\rho})_x(x, t) \in L^2(0, \infty; L^2), \\
(\rho - \tilde{\rho})_x(x, t) \in L^2(0, \infty; H^1), \\
(\rho - \tilde{\rho})(x, t) \in B^{1+\sigma}_{\infty}, \\
(\rho, u - \tilde{u}, \theta - \tilde{\theta})(x, t) \in B^{1+\sigma}_{\infty},
\end{array} \right.
\]
for any fixed \(T > 0\). Furthermore,
\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}_+} \| (\rho, u, \theta)(x, t) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x) - (\rho', u', \theta')(x/t) + (\rho_*, u_*, \theta_*) \| = 0.
\]

3. **Proof of theorem 2.1.**

Replace the perturbation \((\phi, \psi, \theta)(x, t)\) by
\[
(\phi, \psi, \theta)(x, t) = (\rho, u, \theta)(x, t) - (\tilde{\rho}, \tilde{u}, \tilde{\theta})(x),
\]
where \((\tilde{\rho}, \tilde{u}, \tilde{\theta})\) is the boundary layer defined in (2.4). Then the reformulated problem is
\[
\begin{align*}
\phi_x + u\phi_x + \rho\psi_x &= f, \\
\rho(\psi_x + u\psi_x) + (\rho - \tilde{\rho})_x &= \mu \psi_{xx} + g, \\
\gamma - 1 \rho(\tilde{\theta}_x + u\tilde{\theta}_x) + p\psi_x &= \kappa \psi_{xx} + \mu \psi_x^\gamma + h,
\end{align*}
\]
(3.2)
where \(\kappa\) and \(\gamma\) have the relation (1.9), and
\[
\begin{align*}
f &= -\tilde{u}\phi - \tilde{\rho}\psi, \\
g &= -\tilde{u}\phi - \rho \psi, \\
h &= -\frac{R}{\gamma - 1} \tilde{\theta}_x (\tilde{u}\phi + \rho \psi) - \tilde{u}_x (\rho - \tilde{\rho}) + 2\mu \tilde{u}_x \psi_x.
\end{align*}
\]
(3.3)
Define the solution space \(\mathbb{X}(0, T)\) by
\[
\mathbb{X}_{B,h}(0, T) \triangleq \left\{ (\phi, \psi, \theta)(x, t) \bigg| \phi \in B^{1+\sigma}_{\infty}, (\psi, \theta) \in B^{2+\sigma}_{\infty}, \\
(\phi, \psi, \theta) \in C([0, T]; H^1), \phi_x \in L^2(0, T; L^2), (\psi_x, \theta_x) \in L^2(0, T; H^1), \\
\end{align*}
\]
\[
B^{-1} \leq \rho \leq B, \quad b^{-1} \leq \theta \leq b,
\]
(3.4)
where $\sigma \in (0, 1)$ is some constant and $B$ is a positive constant to be determined later, and

$$b \triangleq \max \left\{ \frac{1}{2} \inf_{x \in \mathbb{R}} \phi(x), \frac{1}{4} \sup_{x \in \mathbb{R}} \phi(x) \right\}$$

is a constant which just depends on the left/right end states $\rho_\pm, u_\pm$ and $\theta_\pm$.

**Proposition 3.1 (Local existence).** Assume the conditions in theorem 2.1 hold, and $B^{-1} < \rho_0(x) \leq B$, $b^{-1} < \psi_0(x) \leq b$. If $\| (\phi_0, \psi_0, \theta_0) \|_1 \leq M$, where $M$ is a constant, and $\phi_0 \in B^{1+\sigma}$ and $(\psi_0, \theta_0) \in B^{2+\sigma}$ for a certain $\sigma \in (0, 1)$, then there exists a $t_0 > 0$ depending on the initial data such that the IBVP (3.2) has a unique solution $(\phi, \psi, \theta)(x, t) \in X_2_{R, b}(0, t_0)$ with $\| (\phi, \psi, \theta)(t) \|_1 \leq 2M$.

It is not very difficult to prove the local existence, hence the details are omitted here. To prove theorem 2.1, one needs the following *a priori* estimates.

**Proposition 3.2 (A priori estimates).** Let $(\phi, \psi, \theta) \in X_{R, b}(0, T)$ be a solution to the IBVP (3.2) for some positive $T$, and the conditions in theorem 2.1 hold. Then there exists a positive constant $C$ depending only on the initial data and the left/right end states such that $(\phi, \psi, \theta)(x, t)$ satisfies

$$\left\| \left( \phi, \psi, \frac{\theta}{\sqrt{\gamma - 1}} \right)(t) \right\|_1^2 + \int_0^t \phi^2(0, \tau) \, d\tau$$

$$+ \int_0^t \left( \| \phi_x(\tau) \|^2 + \left\| \left( \psi, \frac{\theta}{\sqrt{\gamma - 1}} \right)(\tau) \right\|_1^2 \right) \, d\tau$$

$$\leq C(b) \left\| \left( \phi_0, \psi_0, \frac{\partial_0}{\sqrt{\gamma - 1}} \right) \right\|_1^2,$$

(3.5)

and

$$\frac{2}{B} \leq \rho(x, t) \leq B, \quad \frac{2}{b} \leq \theta(x, t) \leq b, \quad (x, t) \in \mathbb{R} \times [0, T].$$

(3.6)

**Proof.**

*Step 1.* Define

$$E \triangleq R \bar{\theta} \Phi \left( \frac{\bar{\theta}}{\rho} \right) + \frac{1}{2} \psi^2 + \frac{R}{\gamma - 1} \tilde{\theta} \phi \left( \frac{\theta}{\rho} \right), \quad \Phi(\xi) \triangleq \xi - \ln \xi - 1, \quad \xi \in (0, \infty).$$

(3.7)

Obviously, there exists a positive function $C(\xi)$ such that

$$C(\xi)^{-1}(\xi - 1)^2 \leq \Phi(\xi) \leq C(\xi)(\xi - 1)^2.$$  (3.8)

A direct computation gives

$$(\rho E)_t + \left( \rho u E + \psi (p - \bar{p}) - \mu \psi_x - \kappa \bar{\theta}_x \right) + \frac{\bar{\theta}}{\rho} \psi_x^2 + \kappa \frac{\bar{\theta}_x^2}{\rho} = Q,$$  (3.9)

where

$$Q = -R \rho \bar{\theta}_x \left[ \tilde{u} \Phi \left( \frac{\bar{\theta}}{\rho} \right) + \frac{\bar{u}}{\gamma - 1} \Phi \left( \frac{\theta}{\rho} \right) + \psi \ln \left( \frac{\theta}{\rho} \right) + \psi \ln \left( \frac{\bar{\theta}}{\rho} \right) \right] + \rho \bar{u}_x \psi^2$$

$$+ \left( \frac{R}{\gamma - 1} \bar{u} \bar{\theta}_x + \bar{p} \bar{u}_x \right) \left( \frac{\phi \theta}{\rho \bar{\theta}} + \frac{\psi^2}{\rho} \right) - 2 \frac{\mu \bar{u}_x \psi_x \theta}{\bar{\theta}^2} - \kappa \bar{\theta}_x \bar{\theta}_x + \frac{\mu \bar{u}_x \psi_x \theta}{\bar{\theta}^2}.$$  (3.10)
Integrating (3.9) over $[0, t] \times \mathbb{R}_+$, thanks to the good sign of $u_-$, one has

$$
\int_{\mathbb{R}_+} \rho E \, dx + \int_0^t \left\{ c(B)\psi^2(0, \tau) + c(b) \left( \psi_x, \frac{\phi}{\sqrt{\gamma - 1}}(\tau) \right)^2 \right\} \, d\tau
\leq C \left( \phi_0, \psi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right)^2 + \int_0^t \int_{\mathbb{R}_+} |Q| \, dx \, d\tau,
$$

(3.11)

where we use the important relation in (1.9). Employing Cauchy’s inequality and (2.5), one has

$$
|Q| \leq C(B)C(b) \left( |\tilde{u}_x| + \frac{|\tilde{\theta}_x|}{\sqrt{\gamma - 1}} \right) \left( \phi, \psi, \frac{\phi}{\sqrt{\gamma - 1}} \right)^2 + c(b) \left( \frac{\psi^2}{4} + \frac{\phi^2}{\gamma - 1} \right).
$$

(3.12)

Using a Poincaré-type inequality

$$
|\xi(x)| = \left| \xi(0) + \int_0^x \xi, \, dy \right| \leq |\xi(0)| + \sqrt{x} \| \xi_x \|, \quad x \in \mathbb{R}_+
$$

(3.13)

on $\phi, \psi$ and $\theta$ and setting $\tilde{\delta} \leq \delta_0$ suitably small such that

$$
\left( C(B)C(b) + \frac{1}{\sqrt{\gamma - 1}} \right) \sqrt{\delta_0} \leq \frac{1}{2},
$$

(3.14)

one obtains

$$
\int_{\mathbb{R}_+} \rho E \, dx + \int_0^t \left\{ c(B)\psi^2(0, \tau) + c(b) \left( \psi_x, \frac{\phi}{\sqrt{\gamma - 1}}(\tau) \right)^2 \right\} \, d\tau
\leq C \left( \phi_0, \psi_0, \frac{\phi_0}{\sqrt{\gamma - 1}} \right)^2 + C(B)C(b) \sqrt{\delta} \int_0^t \| \psi_x \|^2 \, d\tau.
$$

(3.15)

**Step 2.** Differentiating (3.2) w.r.t. $x$ and then multiplying it by $\frac{\phi}{\rho^3}$ yield

$$
\left( \frac{\phi_x^2}{2\rho^3} \right)_t + \left( \frac{\psi_x^2}{2\rho^3} \right)_x = \tilde{u}_x \frac{\phi_x^2}{\rho^3} + \tilde{\theta}_x \frac{\phi_x \psi_x}{\rho^3} + \frac{\phi_x \psi_x}{\rho^2} = f_x \phi_x^2.
$$

(3.16)

Multiplying (3.2) by $\frac{\phi_x}{\rho}$ gives

$$
\frac{\phi_x \psi_{xx}}{\rho^2} - \frac{\phi_x}{\rho^2} = \left( \frac{\phi_x \psi}{\rho} \right)_t + \left( \frac{\phi_x \psi}{\rho} + \tilde{\theta}_x \psi_x \right)_x + \frac{\phi_x \psi_x^2}{\rho} + \frac{2\tilde{\theta}_x \psi_x \psi}{\rho} - \psi_x^2 - \tilde{u}_x \psi \frac{\phi_x}{\rho} + \tilde{\theta}_x \phi \psi \frac{\phi_x}{\rho} + \tilde{\theta}_x \phi \psi \frac{\phi_x}{\rho^2} + \tilde{\theta}_x \phi \psi \frac{\phi_x}{\rho^2} + (p - \tilde{\rho}) \psi_x.
$$

(3.17)

Combining (3.16) and (3.17) gives

$$
\left( \frac{\phi_x^2}{2\rho^3} + \frac{\phi_x \psi}{\rho} \right)_t + \left( \frac{\psi_x^2}{2\rho^3} - \frac{\phi_x \psi_x}{\rho} - \frac{\phi_x \psi^2}{\rho} \right)_x + \frac{\rho_x \phi_x^2}{\rho^2}
\leq Q_1 + Q_2.
$$

(3.18)
Integrating (3.18) over \([0, t] \times \mathbb{R}_+\), thanks to the good sign of \(\epsilon\) again, and applying Cauchy’s inequality and (3.15), we have
\[
\int_{\mathbb{R}_+} \frac{\partial_x^2}{\rho^2} \, dx + 4 \int_0^t \int_{\mathbb{R}_+} \frac{\rho}{\rho^2} \phi_x^2 \, dx \, d\tau \leq C \left( \left\| (\phi_0, \psi_0, \frac{\partial_0}{\sqrt{\gamma - 1}}, \phi_c) \right\|^2 + 4(I_1 + I_2) + C(B)C(b) \right).
\]
where \(I_1 \triangleq \int_0^t \int_{\mathbb{R}_+} |Q_i| \, dx \, d\tau, i = 1, 2\).

First, using Cauchy’s inequality and choosing \(\delta\) suitably small than before one has
\[
I_1 \leq \int_0^t \left\{ c(b) \left[ 1 + C(B)\delta^2 \right] \left( \psi_t, \frac{\partial_0}{\sqrt{\gamma - 1}} \right)(\tau) \right\}^2 + \int_0^t \int_{\mathbb{R}_+} \frac{\rho}{\delta \rho^2} \phi_x^2 \, dx \, d\tau \leq C(b) \left( \left\| (\phi_0, \psi_0, \frac{\partial_0}{\sqrt{\gamma - 1}}) \right\|^2 + \frac{1}{4} \int_0^t \int_{\mathbb{R}_+} \frac{\rho}{\rho^2} \phi_x^2 \, dx \, d\tau \right). \tag{3.20}
\]
Here it is noted that \(\frac{\delta \phi_0}{\rho} \leq \frac{\rho}{\delta \rho^2} \phi_x^2 + 2 \frac{\delta}{\delta} \phi_x^2\), thus the coefficient of \(\phi_x^2\) does not relate to \(\rho\) which is fundamental. Second, it is obvious that \(I_2\) is a good term, and for suitably small \(\delta\) one easily has
\[
I_2 \leq C(b) \left( \left\| (\phi_0, \psi_0, \frac{\partial_0}{\sqrt{\gamma - 1}}) \right\|^2 + \frac{1}{4} \int_0^t \int_{\mathbb{R}_+} \frac{\rho}{\rho^2} \phi_x^2 \, dx \, d\tau \right). \tag{3.21}
\]
Here we use the facts \(|\tilde{\rho}_x| \leq C|\tilde{u}_t|\) and \(|\tilde{\rho}_x| \leq C(\tilde{u}_t^2 + |\tilde{u}_{t+1}|)\). Combining (3.15), (3.19)–(3.21) gives
\[
\int_{\mathbb{R}_+} \left( \rho E + \frac{\partial_x^2}{\rho^3} \right) \, dx + \int_0^t \left\{ c(B)\phi^2(0, \tau) + \int_{\mathbb{R}_+} \frac{\rho}{\rho^2} \phi_x^2 \, dx \right\} \, d\tau + c(b) \int_0^t \left\| \left( \psi_t, \frac{\partial_0}{\sqrt{\gamma - 1}} \right)(\tau) \right\|^2 \, d\tau \leq C(b) \left( \left\| (\phi_0, \psi_0, \frac{\partial_0}{\sqrt{\gamma - 1}}, \phi_c) \right\|^2 + \frac{1}{4} \int_0^t \int_{\mathbb{R}_+} \frac{\rho}{\rho^2} \phi_x^2 \, dx \, d\tau \right). \tag{3.22}
\]

**Remark 3.1.** The above computations are formal since there is a term \(\left( \frac{\phi_x^2}{\rho^2} \right)\) in (3.16), while the solution space \(X_{B,h}(0, T)\) is not smooth enough. In order to obtain the \(L^2\)-estimate for \(\phi_c\) under the rigorous procedure, one should use a difference quotient w.r.t. \(x\), \(\psi_h \triangleq \frac{\psi(x+h,t) - \psi(x,t)}{h}\), to replace \(\psi_c\). By a similar process above, one can get the \(L^2\)-estimate for \(\phi_0\). Then setting \(h \to 0\), one obtains the \(L^2\)-estimate for \(\psi_0\). As for the \(L^2\)-estimates of \(\psi_0, \phi_0, \psi_c, \phi_x, \) and others, in the following, the processes are similar. We refer to [16] for more details.

**Step 3.** We now use (3.22) to determine the constant \(B\) stated in (3.4), where this idea comes from [12]. Define
\[
\Psi(\eta) \triangleq \int_1^\eta \frac{\sqrt{\phi(\xi)}}{\xi} \, d\xi, \quad \eta \in \mathbb{R}_+, \quad \eta \to 0_+, \quad \eta \to \infty. \tag{3.23}
\]
then we easily have
\[
\Psi(\eta) \rightarrow \begin{cases} -\infty, & \eta \to 0_+, \\ \infty, & \eta \to \infty. \end{cases} \tag{3.24}
\]
At the same time,
\[
\left| \psi \left( \frac{\rho}{\bar{\rho}} \right) \right| = \left| \int_0^x \Phi \left( \frac{\rho}{\bar{\rho}} \right) \, dy \right| = \left| \int_0^x \Phi \left( \frac{\rho}{\bar{\rho}} \right) \, dy \right| \\
\leq \int_{\mathbb{R}_+} \left\{ \rho \Phi \left( \frac{\rho}{\bar{\rho}} \right) + \frac{\Phi^2}{\rho^3} + C(B) \bar{\rho} \rho \Phi \left( \frac{\rho}{\bar{\rho}} \right) \right\} \, dx \\
\leq C(b) \left\{ \phi_0, \psi_0, \frac{\delta_0}{\sqrt{\gamma - 1}}, \phi_0x \right\}^2, \tag{3.25}
\]
choosing \( \bar{\delta} \leq \delta_0 \) smaller than before such that
\[
C(B)\bar{\delta}^2 \leq 1. \tag{3.26}
\]

In view of \((3.24)\) and \((3.25)\), there exists a positive constant \( \bar{B} \) depending on the initial data and the left/right end states such that
\[
\bar{B}^{-1} \leq \rho(x, t) \leq \bar{B}, \quad (x, t) \in \mathbb{R}_+ \times [0, T]. \tag{3.27}
\]
Choosing \( B = 2 \bar{B} \), one then has the first estimate in \((3.6)\).

**Step 4.** Multiplying \((3.2)2\) by \(-\psi_{xx}\), we have
\[
\left( \frac{\psi_x^2}{2} \right)_x - \left( \psi_x \psi_x + \frac{1}{2} u \psi_x^2 \right)_x + \frac{\mu \psi_{xx}}{\rho} = \frac{(p - \bar{p}) \psi_{xx}}{\rho} - \frac{u_x \psi_x^2}{2} - \psi_{xx} \left( 2 \bar{\rho} - \rho \right) + \frac{\rho^2}{\mu} \psi_x^2. \tag{3.28}
\]
Integrating it over \([0, t] \times \mathbb{R}_+\), using Cauchy’s inequality and the *a priori* estimates \((3.27)\) gives
\[
\| \psi_x (t) \|^2 + \int_0^t \int_{\mathbb{R}_+} \frac{\mu \psi_x^2}{\rho} \, dx \, dt \leq \| \psi_0x \|^2 + \int_0^t |u_-| \psi_x^2 (0, \tau) \, d\tau \\
+ 2 \int_0^t \int_{\mathbb{R}_+} \left( \frac{(p - \bar{p}) \psi_x^2}{\rho} + \frac{\rho^2}{\mu} \psi_x^2 \right) + |u_x| \psi_x^2 + |\psi_x|^3 \, dx \, d\tau. \tag{3.29}
\]
First, employing the Nirenberg inequality to estimate the \( L^\infty \)-norm we have
\[
\int_0^t |u_-| \psi_x^2 (0, \tau) \, d\tau \leq C \int_0^t \| \psi_x (\tau) \|^2_{L^\infty} \, d\tau \\
\leq C(b) \left\{ \phi_0, \psi_0, \frac{\delta_0}{\sqrt{\gamma - 1}}, \phi_0x \right\}^2 + 1 \int_0^t \int_{\mathbb{R}_+} \frac{\mu \psi_x^2}{\rho} \, dx \, d\tau. \tag{3.30}
\]
Second
\[
\int_0^t \int_{\mathbb{R}_+} \left( \frac{(p - \bar{p}) \psi_x^2}{\rho} + \frac{\rho^2}{\mu} \psi_x^2 \right) + |u_x| \psi_x^2 \, dx \, d\tau \\
\leq C(b) \int_0^t \int_{\mathbb{R}_+} \left( |\phi_x, \psi_x, \partial_x| \right)^2 + \left( \tilde{u}^2 + \tilde{\theta}^2 \right) \left( \phi, \psi, \theta \right)^2 \, dx \, d\tau \\
\leq C(b) \left\{ \phi_0, \psi_0, \frac{\delta_0}{\sqrt{\gamma - 1}}, \phi_0x \right\}^2. \tag{3.31}
\]
Third, using the Nirenberg inequality yields
\[
\int_0^t \int_{\mathbb{R}_+} |\psi_x|^3 \, dx \, d\tau \leq C \int_0^t \| \psi_x (\tau) \|^2 \| \psi_x (\tau) \|^2 \, d\tau \\
\leq C(b) \left\{ \phi_0, \psi_0, \frac{\delta_0}{\sqrt{\gamma - 1}}, \phi_0x \right\}^2 \sup_{0 \leq \tau \leq t} \| \psi_x (\tau) \|^4 \\
+ 1 \int_0^t \int_{\mathbb{R}_+} \frac{\mu \psi_x^2}{\rho} \, dx \, d\tau. \tag{3.32}
\]
From (3.29)–(3.32), it is easy to get
\[ \sup_{0 \leq t \leq T} \| \psi_x(t) \|^2 \leq C(b) \left\| \left( \phi_0, \psi_0, \frac{\partial_0}{\sqrt{\gamma - 1}}, \phi_{0x}, \psi_{0x} \right) \right\|^2 \left( 1 + \sup_{0 \leq t \leq T} \| \psi_x(t) \|^2 \right). \]  
(3.33)
which means \( \sup_{0 \leq t \leq T} \| \psi_x(t) \| \) has a bound concerned with initial data. Thus, one has
\[ \| \psi_x(t) \|^2 + \int_0^t \| \psi_{xx}(\tau) \|^2 \, d\tau \leq C(b) \left\| \left( \phi_0, \psi_0, \frac{\partial_0}{\sqrt{\gamma - 1}}, \phi_{0x}, \psi_{0x} \right) \right\|^2 \]. \( \tag{3.34} \)
At the same time, one easily has the following estimate which will be employed to get (3.37)
\[ |u(x, t)| \leq C(b), \quad (x, t) \in \mathbb{R}_+ \times [0, T]. \] \( \tag{3.35} \)

**Step 5.** Multiplying (3.22) by \( -\partial_{xx} \), we have
\[ \frac{R}{\gamma - 1} \left[ \frac{1}{2} (\psi_x^2)_t - (\partial_1 \psi_x)_x - \mu \frac{\partial_{xx}^2}{\rho} + \frac{\gamma^2}{\mu P_r} \frac{\partial_{xx}^2}{\rho} \right] = \frac{R \psi_x \partial_{xx}}{\rho} - \frac{\mu \psi_x \partial_{xx}^2}{\rho} - \frac{h \partial_{xx}}{\rho}. \] \( \tag{3.36} \)
Integrating it over \( [0, t] \times \mathbb{R}_+ \), using Cauchy’s inequality, the estimates (3.22) and (3.27) gives
\[ \left\| \frac{\partial_x(t)}{\sqrt{\gamma - 1}} \right\|^2 + \int_0^t \left\| \frac{\partial_{xx}(\tau)}{\sqrt{\gamma - 1}} \right\|^2 \, d\tau \leq C(b) \left\| \left( \phi_0, \psi_0, \frac{\partial_0}{\sqrt{\gamma - 1}}, \phi_{0x}, \psi_{0x} \right) \right\|^2 + \int_0^t \int_{\mathbb{R}_+} \psi_x^4 + h^2 \, dx \, d\tau. \] \( \tag{3.37} \)
Obviously, it is easy to know \( \int_0^t \int_{\mathbb{R}_+} h^2 \, dx \, d\tau \leq C(b) \left\| \left( \phi_0, \psi_0, \frac{\partial_0}{\sqrt{\gamma - 1}}, \phi_{0x}, \psi_{0x} \right) \right\|^2. \) Now,
\[ \int_0^t \int_{\mathbb{R}_+} \psi_x^4 \, dx \, d\tau \leq \int_0^t \| \psi_x(t) \|^2 \| \psi_x(t) \|^2_{L^\infty} \, d\tau \leq \sup_{0 \leq t \leq T} \| \psi_x(t) \|^2 \int_0^t \| \psi_x(\tau) \|^2 \, d\tau \leq C(b) \left\| \left( \phi_0, \psi_0, \frac{\partial_0}{\sqrt{\gamma - 1}}, \phi_{0x}, \psi_{0x} \right) \right\|^2. \] \( \tag{3.38} \)
Thus one obtains
\[ \left\| \frac{\partial_x(t)}{\sqrt{\gamma - 1}} \right\|^2 + \int_0^t \left\| \frac{\partial_{xx}(\tau)}{\sqrt{\gamma - 1}} \right\|^2 \, d\tau \leq C(b) \left\| \left( \phi_0, \psi_0, \frac{\partial_0}{\sqrt{\gamma - 1}}, \phi_{0x}, \psi_{0x} \right) \right\|^2. \] \( \tag{3.39} \)

**Step 6.** Combining (3.22), (3.34) and (3.39) yields
\[ \left\| \left( \phi, \psi, \frac{\partial}{\sqrt{\gamma - 1}} \right)(t) \right\|_{L^1}^2 + \int_0^t \left( \phi^2(0, \tau) \right) \, d\tau \]
\[ + \int_0^t \left\{ \| \phi_x(\tau) \|^2 + \left\| \frac{\psi_x}{\sqrt{\gamma - 1}}(\tau) \right\|_{L^1}^2 \right\} \, d\tau \]
\[ \leq C(b) \left\| \left( \phi_0, \psi_0, \frac{\partial_0}{\sqrt{\gamma - 1}} \right) \right\|_{L^1}^2. \] \( \tag{3.40} \)

Now the last task is to give the *a priori* upper/lower bounds of \( \theta \). Combining (2.17) and (3.40), we obtain
\[ \sup_{t \in [0, T]} \| \theta(t) \|_{L^\infty} \leq \sup_{t \in [0, T]} \left\{ \| \theta(t) \|_{L^1} \| \theta_x(t) \|_{L^1} \right\} \]
\[ \leq C(b) \sqrt{\gamma - 1} \| \phi_0, \psi_0, s_0 - \delta \|_{L^1}. \] \( \tag{3.41} \)
4. Proofs of theorems 2.2 and 2.3

It is easy to know that the proof of theorem 2.2 is similar to and simpler than that of theorem 2.3 below, thus the details are omitted here. And it is noted that theorem 2.3 concerns two cases of boundary layer: one is non-degenerate, the other is degenerate.

Let
\[
(\hat{\rho}, \hat{u}, \hat{\theta}) = (\tilde{\rho}, \tilde{\theta}) + (\hat{\rho}, \tilde{u}, \hat{\theta}) - (\rho_\ast, u_\ast, \theta_\ast),
\]
then one easily obtains
\[
\begin{align*}
\hat{\rho}_t + \hat{u} \hat{\rho}_x + \hat{\rho} \hat{u}_x &= \hat{f}, \\
\hat{\rho} (\hat{u}_t + \hat{u} \hat{u}_x) + \hat{\rho}_x &= \mu \hat{u}_{xx} + \hat{g}, \\
\frac{R}{\gamma - 1} \hat{\rho} (\hat{\theta}_t + \hat{u} \hat{\theta}_x) + \hat{\rho} \hat{u}_x &= \kappa \hat{\theta}_{xx} + \mu \hat{u}_x^2 + \hat{h}, \\
(\hat{\rho}, \hat{u}, \hat{\theta})(0, t) &= (\rho_\ast, u_\ast, \theta_\ast),
\end{align*}
\]
(4.1)
where \(\rho_\ast \triangleq (\rho_\ast, u_\ast)/u_\ast\), and \(\rho_\ast, u_\ast, \theta_\ast, u_\ast, \theta_\ast\) satisfy (2.5). Here \(\hat{\theta} \triangleq p (\hat{\rho}, \hat{u}) = R \hat{\rho} \hat{\theta}\) and
\[
\begin{align*}
\hat{g} &= \hat{\rho} [(\tilde{u} - u_\ast) \tilde{u}_x + (\tilde{u} - u_\ast) \tilde{u}_x] + (\tilde{\rho} - \rho_\ast) \tilde{u}_x + (\tilde{\rho} - \tilde{\rho}) x - \frac{\tilde{\rho} - \rho_\ast}{\tilde{\rho}} \tilde{u}_x, \\
\hat{h} &= \frac{R}{\gamma - 1} \hat{\rho} [(\tilde{u} - u_\ast) \tilde{\theta}_x + (\tilde{u} - u_\ast) \tilde{\theta}_x] + \frac{R}{\gamma - 1} (\tilde{\rho} - \rho_\ast) \tilde{u}_x \tilde{\theta}_x + (\tilde{\rho} - \tilde{\rho}) \tilde{u}_x - R \hat{\theta} (\tilde{\rho} - \rho_\ast) \tilde{u}_x.
\end{align*}
\]
(4.3)
Combining \(\tilde{u}_x \geq 0\) and (2.5), we know that
\[
|\hat{f}| + |\hat{g}| + |\hat{h}| \leq C (|\tilde{u} - u_\ast| |\tilde{u}_x| + |\tilde{u} - u_\ast| |\tilde{u}_x|).
\]
(4.4)
Replace the perturbation \((\phi, \psi, \vartheta)(x, t)\) by
\[
(\phi, \psi, \vartheta)(x, t) = (\rho, u, \theta)(x, t) - (\hat{\rho}, \hat{u}, \hat{\theta})(x, t),
\]
(4.5)
then the reformulated problem is
\[
\begin{align*}
\phi_t + u \phi_x + \rho \psi_x = f, \\
\rho (\psi_t + u \psi_x) + (p - \hat{\rho})_x &= \mu \psi_{xx} + g, \\
\frac{R}{\gamma - 1} \rho (\vartheta_t + u \vartheta_x) + p \psi_x &= \kappa \vartheta_{xx} + \mu \psi_x^2 + h, \\
(\phi_0, \psi_0, \vartheta_0)(x) &\triangleq (\phi, \psi, \vartheta)(x, 0) \to (0, 0, 0), \\
(\psi, \vartheta)(0, t) &= (0, 0),
\end{align*}
\]
as \(x \to \infty\).
(4.6)
Proposition 3.2 is proved completely.

Remark 3.2. In these proofs, we have used the Hölder regularity of the initial data. The time \(t_0\) in proposition 3.1 depends on the Hölder norms of the initial data. Thus, to complete the proof of the global existence in theorem 2.1, it needs to show the \(a\ priori\) estimates in the Hölder spaces. This process is the same as that in [16] which proves the stability of the boundary layer for the isentropic model, and we omit the details of the proof.
where
\[
\begin{aligned}
  f &= \tilde{u}_t \phi - \frac{\partial_x \rho}{\rho} \psi - \tilde{f}, \\
  g &= -\rho \tilde{u}_t \psi + \frac{\partial_x \rho}{\rho} \mu \tilde{u}_t + \mu \tilde{u}_x - \frac{\rho}{\rho} \phi, \\
  h &= -\frac{R}{\gamma - 1} \rho \tilde{u}_t \psi - R \rho \tilde{u}_t \phi - (\kappa \tilde{u}_t + \mu \tilde{u}_x^2) \phi - \frac{\rho}{\rho} \tilde{h}, \\
  \tau &\equiv \mu \tilde{u}_t + 2 \mu \tilde{u}_x + \mu \tilde{u}_x^2 - \frac{\rho}{\rho} \tilde{h}.
\end{aligned}
\] (4.7)

Define the solution space \( \mathbb{X}(0, T) \) by
\[
\mathbb{X}(0, T) \equiv \left\{ (\phi, \psi, \vartheta)(x, t) \big| \phi \in B^{1, \frac{1}{2} - 1}_{r, \infty}(0, T; L^2), (\psi, \vartheta) \in L^2(0, T; H^1), \right\} \\
\left. \sup_{0 \leq t \leq T} \| (\phi, \psi, \vartheta)(t) \|_1 \leq \varepsilon_0 \right\},
\] (4.8)

where \( \sigma \in (0, 1) \) is some constant and \( \varepsilon_0 \leq \frac{1}{2} \min \left\{ \inf_{x \in \mathbb{R}_+} \tilde{\rho}(x, t), \inf_{x \in \mathbb{R}_+} \frac{\partial_x}{\partial x} \right\} \) is a suitably small constant to be determined later, which ensures the existence of the positive unique upper/lower bounds of \( \rho \) and \( \vartheta \) on \( \mathbb{R}_+ \times [0, T] \). It is also noted that the upper bound of \( \varepsilon_0 \) depends only on the left/right end states \( \rho_\pm, u_\pm \) and \( \theta_\pm \).

**Proposition 4.1 (A priori estimates).** Let \( (\phi, \psi, \vartheta) \in \mathbb{X}(0, T) \) be a solution to the IBVP (4.6) for some positive \( T \), and the conditions in theorem 2.3 hold. Then there exists a positive constant \( C \) only depending on the initial data such that \( (\phi, \psi, \vartheta) \) satisfies
\[
\begin{aligned}
  \| (\phi, \psi, \vartheta)(t) \|_1^2 + \int_0^t \left\{ \| \phi^2(0, \tau) \| + \| \sqrt{\tilde{u}}(\phi, \psi, \vartheta)(\tau) \|_1^2 \right\} d\tau \\
  + \int_0^t \left\{ \| \phi_x(t) \|_1^2 + \| (\psi_x, \vartheta_x)(\tau) \|_1^2 \right\} d\tau \leq C \left\{ \| (\phi_0, \psi_0, \vartheta_0) \|_1^2 + \tilde{\delta} + \varepsilon_0^2 \right\}.
\end{aligned}
\] (4.9)

**Proof.**

**Step 1.** Similarly, define
\[
E \equiv \tilde{R} \tilde{\Phi} \left( \tilde{\rho} \right) + \frac{1}{2} \psi^2 + \frac{R}{\gamma - 1} \tilde{\vartheta} \tilde{\Phi} \left( \frac{\tilde{\vartheta}}{\tilde{\rho}} \right),
\] (4.10)

where \( \Phi(\xi) \) is defined in (3.7). A direct computation gives
\[
(\rho E) + \left( \rho u_x + \psi (p - \tilde{\rho}) - \mu \psi \psi_x - \kappa \frac{\partial_x}{\partial x} \right) + \frac{\mu \tilde{\vartheta}}{\tilde{\rho}} \psi_x^2 + \frac{\kappa \tilde{\vartheta}}{\til\rho} \vartheta_x^2 = Q,
\] (4.11)

where
\[
Q = -\frac{R \tilde{\rho}_t \phi \psi}{\tilde{\rho}} - \frac{R \tilde{\theta}_t \phi}{\tilde{\rho}} + g \psi + \left[ 2 \mu \tilde{u}_t \psi_x + \mu \left( 2 \tilde{u}_t \tilde{u}_x + \tilde{u}_x^2 \right) + \kappa \tilde{u}_x \right] \frac{\partial_x}{\partial x} \\
- \left( \frac{p \tilde{\theta}_x \psi + \kappa \tilde{u}_x (p - \tilde{\rho}) + \mu \tilde{u}_x^2 (p - \tilde{\rho}) + \tilde{p} \tilde{\vartheta}}{\tilde{\rho}} \right) \frac{\partial_x}{\partial x} + \kappa \frac{\partial_x}{\partial x} \frac{\partial_x}{\partial x} \\
+ \rho \left( \frac{R}{\gamma - 1} \tilde{\theta}_x \psi + \kappa \tilde{u}_x \right) \left[ (\gamma - 1) \Phi \left( \frac{\tilde{\rho}}{\tilde{\rho}} \right) + \Phi \left( \frac{\tilde{\vartheta}}{\tilde{\rho}} \right) \right] \\
= Q_1 + Q_2 + Q_3,
\] (4.12)
and

\[
Q_1 = \rho \left( \frac{R}{\gamma - 1} \hat{\rho}_l \psi - R \hat{\theta}_l \right) \left[ (\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\hat{\rho}} \right) \right] + R \hat{\theta}_l \phi \psi - \frac{R \rho \hat{\theta}_l \psi}{(\gamma - 1) \hat{\rho}}
- \rho \hat{\rho}_l \psi^2 - \mu \hat{u}_l \psi \left( \frac{\phi}{\rho} \right)
+ \frac{\rho (\hat{\theta}_l \phi + \mu \hat{u}_l \psi)}{(\gamma - 1) \hat{\rho}} \left[ (\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\hat{\rho}} \right) \right]
- \left[ \frac{\hat{\rho}_l \psi}{\rho} - \mu \hat{u}_l \psi \right] \frac{\phi}{\rho} \left[ (\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\hat{\rho}} \right) \right]
- \frac{2 \mu \hat{u}_l \psi \psi}{\rho} \left[ (\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\hat{\rho}} \right) \right] + \frac{\kappa \hat{\theta}_l \phi \theta}{\rho}.
\]

\[
Q_2 = \rho \left( \frac{R}{\gamma - 1} \hat{\rho}_l \psi - R \hat{\theta}_l \right) \left[ (\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\hat{\rho}} \right) \right] + R \hat{\theta}_l \phi \psi - \frac{R \rho \hat{\theta}_l \psi}{(\gamma - 1) \hat{\rho}}
- \rho \hat{\rho}_l \psi^2 + \mu \hat{u}_l \psi \left( \frac{\phi}{\rho} \right)
+ \frac{2 \mu \hat{u}_l \psi \psi}{\rho} \left[ (\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\hat{\rho}} \right) \right]
- \frac{2 \mu \hat{u}_l \psi \psi}{\rho} \left[ (\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\hat{\rho}} \right) \right] + \frac{\kappa \hat{\theta}_l \phi \theta}{\rho}.
\]

\[
Q_3 = \frac{R \hat{\rho}_l \phi}{\rho} - \frac{\rho \hat{\rho}_l \psi \psi}{\rho} + \frac{2 \mu \hat{u}_l \psi \psi}{\rho} \left[ (\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\hat{\rho}} \right) \right] + \frac{\rho \hat{\rho}_l \psi}{\rho} \left[ (\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\hat{\rho}} \right) \right] - \frac{\kappa \hat{\theta}_l \phi \theta}{\rho}.
\]

Integrating (4.11) over $\mathbb{R}_+ \times [0, t]$, thanks to the good sign of $u_-$, one has

\[
\int_{\mathbb{R}_+} \rho E \, dx + \int_0^t \left\{ \phi^2 (0, \tau) + \parallel (\psi_\tau, \theta_\tau) (\tau) \parallel^2 \right\} \, d\tau \leq C \parallel (\phi_0, \psi_0, \theta_0) \parallel^2 + \sum_{i=1}^3 I_i,
\]

where $I_i \triangleq O(1) \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} Q_i \, dx \, dt, \, i = 1, 2, 3$. 

\[\textbf{\mathbf{\Delta}_i}\] For the estimate of $I_1$, if the boundary layer decays exponentially, then employing a Poincaré-type inequality on $\phi$, $\psi$ and $\theta$, one easily knows that

\[
I_1 \leq C \delta \int_0^t \left\{ \phi^2 (0, \tau) + \parallel (\psi_\tau, \theta_\tau) (\tau) \parallel^2 \right\} \, d\tau + \left( \frac{1}{8} + C \delta \right) \int_0^t \parallel (\psi_\tau, \theta_\tau) (\tau) \parallel^2 \, d\tau.
\]

Here and in the following, $\delta$ represents the amplitude of boundary layer. One easily knows that if the boundary layer is not degenerate, then all the terms concerning boundary layer are easier to control than the counterparts of the degenerate case. Hence from now on, we only consider the degenerate boundary layer.

Using the fact $\frac{R}{\gamma - 1} \hat{\rho}_l \hat{\rho} - \rho \hat{\rho}_l \psi = \kappa \hat{\theta}_l \phi + \mu \hat{u}_l \psi - \hat{\rho}_l$, one easily has

\[
Q_1 = -\rho \hat{\rho}_l \psi \left\{ \frac{\hat{\rho}}{\rho} \left[ (\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\hat{\rho}} \right) \right] \right\}
- \frac{R (\gamma - 1) \psi}{\hat{u}_l} \ln \left( \frac{\hat{\rho}}{\rho} \right) - \frac{R \psi}{\hat{u}_l} \ln \left( \frac{\theta}{\hat{\rho}} \right) + \psi^2 \right\}
+ \left\{ \rho (\hat{\theta}_l \phi + \mu \hat{u}_l \psi) \left[ \frac{1}{\rho} \left[ (\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\hat{\rho}} \right) \right] \right] \right\}
- \frac{1}{\rho \hat{u}_l} \left\{ \left[ (\gamma - 1) \psi \ln \left( \frac{\hat{\rho}}{\rho} \right) + \psi \ln \left( \frac{\theta}{\hat{\rho}} \right) \right] \right\} \left\{ \frac{\mu \hat{u}_l \phi}{\rho} - \frac{\mu \hat{u}_l \psi}{\rho} \left[ (\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\hat{\rho}} \right) \right] + \frac{\kappa \hat{\theta}_l \phi \theta}{\rho} \right\}
\]

\[
Q_3 = Q_{11} + Q_{12}.
\]

For $I_1 \triangleq \int_0^t \int_{\mathbb{R}_+} Q_{11} \, dx \, dt$, it is noted that in subcase 2 of case (ii) $(\hat{u}, \hat{\theta})(x)$ is parallel to the vector $(-\mu \hat{u}_l, (\gamma - 1)\psi)$ at $(u_\tau, \theta_\tau)$. Hence, for each $(\hat{u}, \hat{\theta})(x)$ there exists a constant $M_0 \geq 1$ just depending on $u_\tau, \theta_\tau, \mu, u_\tau, \theta_\tau$ such that if $x > M_0$, then $(\hat{u}, \hat{\theta})(x) \not\parallel (u_\tau, \theta_\tau)$
as \( x \to \infty \). This means \( \tilde{a}_x \geq 0 \) and \( \tilde{\theta}_x \geq 0 \) on \( [M_0, \infty) \). At the same time, one easily obtains the following inequality:

\[
\xi - 1 - \ln \xi \geq \frac{1}{3} \ln^2 \xi, \quad \text{as } |\xi - 1| \leq \frac{1}{4}.
\]  

(4.17)

Finally, choose \( \delta \leq \delta_0 \) suitably small such that

\[
\frac{7}{8} \sqrt{R \gamma \tilde{\theta}} \leq |\tilde{u}| \leq \frac{9}{8} \sqrt{R \gamma \tilde{\theta}},
\]

where it is reasonable because of \( |u_x| = \sqrt{R \gamma \tilde{u}_x} \). Under the preliminaries and conditions above, we divide the integral into two parts

\[
I_{11} = \int_0^1 \left\{ \int_0^{M_0} + \int_{M_0}^{\infty} \right\} Q_{11} \, dx \, d\tau \triangleq I_{111} + I_{112}.
\]  

(4.19)

For \( I_{111} \), employing Poincaré-type inequality on \( \phi, \psi \) and \( \theta \), one has

\[
I_{111} \leq C \delta \int_0^1 \left\{ \phi^2(0, \tau) + \| (\phi_x, \psi_x, \theta_x) \|_2^2 \right\} \, d\tau.
\]  

(4.20)

For \( I_{112} \), using (4.18) and two inequalities

\[
R(\gamma - 1) \frac{\psi}{\tilde{u}} \ln \left( \frac{\hat{\rho}}{\rho} \right) \leq \frac{7R(\gamma - 1)}{24} \ln^2 \left( \frac{\hat{\rho}}{\rho} \right) + \frac{6R(\gamma - 1)}{7\tilde{u}^2} \psi^2,
\]  

(4.21)

and

\[
\frac{R \psi}{\tilde{u}} \ln \left( \frac{\theta}{\tilde{\theta}} \right) \leq \frac{7R}{24} \ln^2 \left( \frac{\theta}{\tilde{\theta}} \right) + \frac{6R}{7\tilde{u}^2} \psi^2,
\]  

(4.22)

one has

\[
R \left( \gamma - 1 \right) \frac{\hat{\rho}}{\rho} \Phi \left( \frac{\hat{\rho}}{\rho} \right) + \Phi \left( \frac{\theta}{\tilde{\theta}} \right) - R(\gamma - 1) \frac{\psi}{\tilde{u}} \ln \left( \frac{\hat{\rho}}{\rho} \right) - \frac{R \psi}{\tilde{u}} \ln \left( \frac{\theta}{\tilde{\theta}} \right) + \frac{\psi^2}{\tilde{u}^2} \geq \frac{1}{8} \left[ R(\gamma - 1) \Phi \left( \frac{\hat{\rho}}{\rho} \right) + R \Phi \left( \frac{\theta}{\tilde{\theta}} \right) + \frac{\psi^2}{\tilde{u}^2} \right].
\]  

(4.23)

where we use the fact \( \frac{\tilde{\theta}}{\theta} \geq 1 \) because of \( \tilde{\theta} - \theta_x \geq 0 \). Combining \( \tilde{a}_x \geq 0 \), \( \forall x \in [M_0, \infty) \), one knows

\[
I_{112} \leq 0.
\]  

(4.24)

Thus,

\[
I_{11} \leq C \delta \int_0^1 \left\{ \phi^2(0, \tau) + \| (\phi_x, \psi_x, \theta_x) \|_2^2 \right\} \, d\tau.
\]  

(4.25)

For \( I_{12} \triangleq \int_0^1 \int_{M_0}^\infty Q_{12} \, dx \, d\tau \), we have

\[
Q_{12} \leq C (\tilde{u}_x^2 + \tilde{\theta}_x^2 + |\tilde{a}_{xx}| + |\tilde{\theta}_{xx}|) \|(\phi, \psi, \theta)\|_2^2 + \frac{1}{8} \left( \psi^2_x + \tilde{\theta}_x^2 \right).
\]  

(4.26)

Using Poincaré-type inequality on \( \phi, \psi, \theta \) again, one easily has

\[
I_{12} \leq C \delta \int_0^1 \left\{ \phi^2(0, \tau) + \| \phi_x(\tau) \|_2^2 \right\} \, d\tau + \left( \frac{1}{8} + C \delta \right) \int_0^1 \| (\psi_x, \theta_x)(\tau) \|_2^2 \, d\tau.
\]  

(4.27)

Thus,

\[
I_1 \leq C \delta \int_0^1 \left\{ \phi^2(0, \tau) + \| \phi_x(\tau) \|_2^2 \right\} \, d\tau + \left( \frac{1}{8} + C \delta \right) \int_0^1 \| (\psi_x, \theta_x)(\tau) \|_2^2 \, d\tau.
\]  

(4.28)
Third, employing Cauchy’s inequality and (2.4), one has

\[ I_3 \leq \frac{C}{\epsilon^{1/3}} \int_0^t \| (\rho E)(\tau) \|_{L^1}^{1/2} \left( \int_{\mathbb{R}_+} \{ \tilde{u}_x \mid u_x \mid + \| \tilde{u}_x \| \{ \tilde{u}_x - u_x \} \right) \frac{1}{d\tau} \]

\[ + C \epsilon \int_0^t \int_{\mathbb{R}_+} \left( \frac{\mu \dot{\theta}}{\theta} \dot{\theta}^2 + \frac{\kappa \dot{\theta}}{\theta^2} x \theta_x + \phi_\theta^2 \right) dx \frac{d\tau}{d\tau}. \]
where $\varepsilon$ is a positive constant to be determined later. Now we consider the integral
\[ \int_{M_0+\bar{\delta}^{-1}r} \left\{ \bar{u}_x |\bar{u} - u_*| + |\bar{u}_x| |\bar{u} - u_*| \right\} \, dx \]
Due to $\bar{u}_x \geq 0$ on $[M_0 + \bar{\delta}^{-1}r, \infty)$, $\forall t \geq 0$, one has
\[ \int_{M_0+\bar{\delta}^{-1}t} \left\{ \bar{u}_x |\bar{u} - u_*| + |\bar{u}_x| |\bar{u} - u_*| \right\} \, dx \]
\[ \leq \| \bar{u}_x(t) \|_{L^\infty} \int_{M_0+\bar{\delta}^{-1}t} \left\{ |\bar{u} - u_*| + x|\bar{u}_x| \right\} \, dx \]
\[ \leq C e^{\bar{\delta} t} (1+ t)^{-\frac{\bar{\delta}}{2}} \ln (1+ t + \bar{\delta} M_0) \leq C e^{\bar{\delta} t} (1+ t)^{-\frac{\bar{\delta}}{2}}, \quad (4.37) \]
and
\[ \int_{M_0+\bar{\delta}^{-1}t} \left\{ (\bar{u} - u_*) (u_* - \bar{u}) - (u_* - \bar{u}) (\bar{u} - u_*) \right\} \, dx \]
\[ = (\bar{u} - u_*) (u_* - \bar{u}) \bigg|_{M_0+\bar{\delta}^{-1}t}^{\infty} - 2 \int_{M_0+\bar{\delta}^{-1}t}^{\infty} (u_* - \bar{u}) (\bar{u} - u_*) \, dx \]
\[ \leq C \bar{\delta} (1+ t)^{-1}. \quad (4.38) \]
Choosing $\varepsilon = \bar{\delta} + \bar{\varepsilon}^{\frac{1}{4}}$, (4.39) we have
\[ I_3 \leq C \left( \bar{\delta} + \bar{\varepsilon}^{\frac{1}{4}} \right) \left( 1 + \int_0^t (1+ \tau)^{-\frac{\bar{\delta}}{4}} \| (\rho E) (\tau) \|_{L^1} \, d\tau \right) \]
\[ + C \left( \bar{\delta} + \bar{\varepsilon}^{\frac{1}{4}} \right) \int_0^t \int_{M_0+\bar{\delta}^{-1}t} \left\{ \frac{\mu}{\rho} \psi^2 \frac{\phi \psi}{\rho^2} + \frac{\kappa}{\rho^2} \phi^2 \right\} \, dx \, d\tau. \quad (4.40) \]
Combining (4.14), (4.28), (4.35) and (4.40), and choosing $\bar{\delta}$ and $\varepsilon$ suitably small we obtain
\[ \| (\phi, \psi, \theta)(\cdot)(t) \|_2 + \int_0^t \left\{ \phi^2 (0, \tau) + \| \sqrt{\rho} \phi \|_{L^2} \right\} \, d\tau \]
\[ \leq C \left\{ \| (\phi_0, \psi_0, \theta_0) \|_2 + \bar{\delta} + \bar{\varepsilon}^{\frac{1}{4}} \right\} \]
\[ + C \left( \bar{\delta} + \bar{\varepsilon}^{\frac{1}{4}} \right) \int_0^t \left\{ \| \phi_x (\tau) \|_2 + (1+ \tau)^{-\frac{\bar{\delta}}{4}} \| (\phi, \psi, \theta)(\cdot)(\tau) \|_2 \right\} \, d\tau. \quad (4.41) \]

**Step 2.** Differentiating (4.6) w.r.t. $x$ and then multiplying it by $\frac{\phi \psi}{\rho^3}$ yield
\[ \left( \frac{\phi \psi}{\rho^3} \right)_t + \left( \frac{\mu \phi \psi}{\rho^3} \right)_x = -\bar{u}_x \frac{\phi^2}{\rho^3} + \bar{\rho}_x \frac{\phi \psi}{\rho^3}, \quad (4.42) \]
Multiplying (4.6) by $\frac{\phi \psi}{\rho^3}$ gives
\[ \left( \frac{\phi \psi}{\rho^3} \right)_t = \left( \frac{\phi \psi}{\rho} + \frac{\bar{\rho}_x \psi}{\rho} \right)_x + \frac{\bar{\rho}_x \bar{u} \phi \psi}{\rho^2} + \frac{\phi \psi}{\rho} \]
\[ + \left( \frac{\bar{\rho}_x \bar{u} - \bar{\rho}_x \bar{u} - \bar{\rho}_x \bar{u}}{\rho^2} \right)_x \frac{\phi \psi}{\rho} + \left( \frac{\bar{\rho}_x \phi \bar{\psi}}{\rho} \right)_x \]
\[ \leq \bar{\rho}_x \bar{u}_x \frac{\phi \psi}{\rho} + \left( \frac{\bar{\rho}_x \bar{u} \psi}{\rho} \right)_x \frac{\phi \psi}{\rho} + \bar{\rho}_x \frac{\phi \psi}{\rho} \]
\[ + \left( \frac{\bar{\rho}_x \phi \bar{\psi}}{\rho} \right)_x \frac{\phi \psi}{\rho} \]
\[ -\psi_x \left( \frac{\mu \phi \psi}{\rho^3} \right)_t = -\frac{\bar{u}_x \phi \psi}{\rho} + \frac{\bar{\rho}_x \phi \psi}{\rho} \]
\[ + \left( \frac{\mu \phi \psi}{\rho^3} \right)_x = \frac{\mu \phi \psi}{\rho^3} + \frac{\phi \psi}{\rho^3}. \quad (4.43) \]
\[ \mu \times (4.42) + (4.43), \text{ we have} \]
\[ \left( \frac{\mu \phi \psi}{2 \rho^3} \right)_t + \left( \frac{\mu \phi \psi}{2 \rho^3} \right)_x + \frac{P_x}{\rho^2} \phi \psi = Q_4. \quad (4.44) \]
where

\[
Q_4 = -\frac{2\mu \lambda \phi_x \psi_x}{\rho^3} - \mu (\ddot{u}_{xx} \phi + \dot{\rho} \dot{u}_x \psi + \dot{\phi}) \phi_x - \frac{\dot{\rho} \dot{u}_x \psi}{\rho^2} - \frac{\dot{\phi}}{\rho} - (\dot{\rho} u_x \psi \dot{\phi} - \frac{\dot{\phi}}{\rho} - \frac{\ddot{u}_x \psi}{\rho^2}) \phi_x
\]

\[
+ \frac{\dot{u}_x \phi \psi_x}{\rho} - (\dot{u} \dot{\phi}_x \dot{\phi} + R \rho \theta_3 + R \dot{\theta}_3 \phi) \phi_x + \dot{\phi} \psi_x + \frac{\dot{\phi}}{\rho} - \frac{\ddot{u}_x \psi_2}{\rho^2}
\]

\[
\lesssim \frac{p}{2\rho^2} \phi^2_x + C(\psi^2_x + \theta^2_x) + C(\dot{f}^2 + \dot{f}_x^2 + \dot{\theta}^2 + \dot{u}_{xx}^2)
\]

\[
+C[(\dot{u}_{xx}^2 + |\dot{u}_{xx}|) + (\dot{u}_x^2 + |\dot{u}_x|)](\phi^2 + \psi^2 + \theta^2).
\]

Integrating (4.44) over \( \mathbb{R}_n \times [0, r] \), thanks to the good sign of \( u_- \) again, we have

\[
\|
\phi_x(t)
\|^2 + \int_0^t \|
\phi_x(\tau)
\|^2 \, d\tau
\]

\[
\lesssim C \left\{ \|
\phi_0, \psi_0, \theta_0, \phi_{0x}
\|^2 + \delta + \epsilon \frac{\delta}{\rho} \right\} + \sum_{i=4}^6 I_i
\]

\[
+ C \left( \delta + \epsilon \frac{\delta}{\rho} \right) \int_0^t \left\{ \|
\phi_x(\tau)
\|^2 + (1 + \tau)^{-\frac{1}{2}} \|
\phi, \psi, \theta(\tau)
\|^2 \right\} \, d\tau.
\]

where

\[
I_4 \leq C \int_0^t \int_{\mathbb{R}_n} \left( \ddot{f}^2 + \dot{f}_x^2 + \dot{\theta}^2 + \ddot{u}_{xx}^2 \right) \, dx \, d\tau
\]

\[
I_5 \leq C \int_0^t \int_{\mathbb{R}_n} (\ddot{u}_x^2 + |\dot{u}_{xx}|)(\phi^2 + \psi^2 + \theta^2) \, dx \, d\tau
\]

\[
I_6 \leq C \int_0^t \int_{\mathbb{R}_n} (\ddot{u}_x^2 + |\dot{u}_x|)(\phi^2 + \psi^2 + \theta^2) \, dx \, d\tau.
\]

Now,

\[
I_4 \leq C \int_0^t \int_{\mathbb{R}_n} \left\{ (\dot{u} - u_x)^2 \ddot{u}_x^2 + (\dot{u} - u_x)^2 \ddot{u}_{xx}^2 + \ddot{u}_{xx}^2 \right\} \, dx \, d\tau \leq C \delta,
\]

where we employ the following estimate on \( \ddot{u} - u_x \)

\[
|\xi(x)| = \left| \xi(0) + \int_0^x \xi \, dy \right| \leq |\xi(0)| + x \| \xi \|_{L^\infty(\mathbb{R}_n)}.
\]

For the estimates of \( I_5 \) and \( I_6 \), they are similar and simpler than those of \( I_2 \) and \( I_3 \), respectively. Combining (4.48) and (4.49), and next choosing \( \tilde{\delta} \) and \( \epsilon \) smaller than before and then using Gronwall’s inequality, one has

\[
\|
\phi_x(t)
\|^2 + \int_0^t \phi^2(0, \tau) \, d\tau
\]

\[
+ \int_0^t \left\{ \|
\sqrt{\dot{u}_x}(\phi, \psi, \theta)(\tau)
\|^2 + \|
\phi_x, \psi_x, \theta_x(\tau)
\|^2 \right\} \, d\tau
\]

\[
\leq C \left\{ \|
\phi_0, \psi_0, \theta_0, \phi_{0x}
\|^2 + \tilde{\delta} + \epsilon \frac{\tilde{\delta}}{\rho} \right\}.
\]
Step 3. For the higher order estimates, since the process is similar to steps 4 and 5 in section 4, we omit the details and only give the following inequality:

$$\| (\psi, \vartheta)(t) \|^2 + \int_0^t \| (\psi_{xx}, \vartheta_{xx})(\tau) \|^2 \, d\tau \leq C \left\{ \| (\psi_0, \vartheta_0) \|^2_t + \delta + \epsilon^{\frac{1}{10}} \right\}. \quad (4.51)$$

Combining (4.50) and (4.51) yields (4.9). Proposition 4.1 is proved completely.

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Appendix

Because of the subtlety of the case $M_* = 1$, we show the relation between the right end state and the existence of boundary layer which is another proof.

Let

$$V \triangleq \begin{pmatrix} U \\ \Theta \end{pmatrix} = \begin{pmatrix} \bar{u} - u_+ \\ \bar{\theta} - \theta_+ \end{pmatrix}, \quad H(V) \triangleq \begin{pmatrix} p_+ U^2 - R p_+ U \Theta \\ \mu u_+ U + u_+ - \frac{R \rho_+ U^2}{2\kappa} \end{pmatrix}, \quad (A.1)$$

then (2.7) is transformed into

$$V' = J_* V + H(V), \quad V(\infty) = 0, \quad (A.2)$$

where

$$J_* \triangleq \begin{pmatrix} (M_*^2 \gamma - 1) p_* & R \rho_* \\ \mu u_+ & \frac{R \rho_+ u_+}{\kappa} \end{pmatrix}, \quad \det(J_*) = \frac{(M_*^2 - 1) R \gamma \rho_* p_*}{(\gamma - 1) \kappa \mu}. \quad (A.3)$$

Then $\det(J_*) = 0$ because of $M_* = 1$, and there exists a matrix $P$ such that $P^{-1} J_* P$ is a standard form

$$P^{-1} J_* P = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}. \quad (A.4)$$

Choosing

$$P = \begin{pmatrix} 1 & 0 \\ \frac{1 - R(\gamma - 1)\gamma \kappa}{[R \gamma \mu + (\gamma - 1)^2 \kappa] u_+} & 1 \end{pmatrix} \begin{pmatrix} 1 & - \frac{R(\gamma - 1)\gamma \kappa}{[R \gamma \mu + (\gamma - 1)^2 \kappa] u_+} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{R(\gamma - 1)\gamma \kappa}{[R \gamma \mu + (\gamma - 1)^2 \kappa] u_+} \\ \frac{1 - R(\gamma - 1)\gamma \kappa}{[R \gamma \mu + (\gamma - 1)^2 \kappa] u_+} & 1 \end{pmatrix}, \quad (A.5)$$
and
\[ W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} (\gamma - 1)^2 \kappa \frac{U}{\mu} + \frac{R(\gamma - 1)\gamma \kappa}{\mu \mu_+ [R \gamma \mu + (\gamma - 1)^2 \kappa] u_+} \theta \\ -\frac{R(\gamma - 1)\gamma \kappa}{(\gamma - 1)\kappa} \frac{U}{\mu} + \Theta \end{pmatrix}, \quad (A.6) \]
we have
\[ W' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} + H(W), \quad (A.7) \]
where \( \lambda = \frac{(R \gamma \mu + (\gamma - 1)^2 \kappa) u_+}{\mu (\mu + (\gamma - 1)^2 \kappa) u_+} < 0 \) and \( H(W) = P^{-1}H(V) \), i.e.
\[
\begin{align*}
H_1(W) &= \frac{(\gamma - 1)^2 \kappa}{\mu [R \gamma \mu + (\gamma - 1)^2 \kappa]} \left( \frac{p_+}{u_+} U^2 - R \rho_+ \frac{U\theta}{U + u_+} \right) - \frac{R(\gamma - 1)\gamma \rho_+}{2[R \gamma \mu + (\gamma - 1)^2 \kappa]} U^2, \\
H_2(W) &= -\frac{\rho_+}{(\gamma - 1)\kappa} \left( \frac{p_+}{u_+} U^2 - R \rho_+ \frac{U\theta}{U + u_+} \right) - \frac{\rho_+ u_+}{2\kappa} U^2.
\end{align*}
\quad (A.8)
\]

Obviously, there exists a suitably small neighbourhood \( \Omega_{\gamma \mu} (0, 0) \) such that \( (H_1, H_2) \) is analytic in it, and \( (0, 0) \) is the solitary singular point of the system \( (A.7) \). In this neighbourhood if \( |W_1| \ll |W_2| \), then
\[ H_2(W) = -\frac{R^2(\gamma - 1)^2 \kappa^2 (\gamma + 1) \rho_+}{2[R \gamma \mu + (\gamma - 1)^2 \kappa]^2 u_+} W_2^2 + O\left(W_2^2\right). \quad (A.9) \]

From the qualitative theory of ODE, for example, refer to [33], we know that \( (0, 0) \) is a saddle-node point. Precisely speaking, there exist two trajectories denoted by \( \Gamma_1, \Gamma_2 \) tangent to \( W_1 \)-axis at \( (0, 0) \) on the opposite directions. For \( i = 1, 2 \), let \( \Gamma_i = (M_{i1}, M_{i2})(\xi), \xi \in \mathbb{R}_+ \), then \( (M_{i1}, M_{i2})(\xi) \) also satisfies \( (A.7) \) just when \( x \) is replaced by \( \xi \), and \( |M_{i2}| \ll |M_{i1}| \). Moreover, we easily have
\[ -c_1 M_{i1} \leq M_{i1} \leq -c_2 M_{i1}, \quad \xi \in \mathbb{R}_+, \quad (i = 1, 2), \quad (A.10) \]
where \( c_1, c_2 \) are two positive constants near \( -\lambda \). Thus there exist positive constants \( c(< |\lambda|) \) and \( C \) such that
\[ |M_{i2}(\xi)| \ll |M_{i1}(\xi)| \leq C \delta_W e^{-c\xi}, \quad \xi \in \mathbb{R}_+, \quad (i = 1, 2), \quad (A.11) \]
where \( \delta_W = \left(\frac{|(M_{i1}, M_{i2})(0)|}{|M_{i1}|}\right) \). Let \( \mathcal{M} = \Gamma_1 \cup \Gamma_2 \), then \( \mathcal{M} \) is a centre-stable manifold. See figure 1.

Now the circular neighbourhood \( \Omega_{\gamma \mu} (0, 0) \) are divided into three parts which is given in the following. Depending on the location of \( W(0) \) we divide this case into three subcases.

**Subcase 1.** \( W(0) \in \mathcal{M} \), then \( \delta_W = |W(0)| \leq \delta_W \), and
\[ |W(x)| \leq C \delta_W e^{-c\xi}, \quad x \in \mathbb{R}_+ \quad (A.12) \]

**Subcase 2.** \( W(0) \in \Omega_{\gamma \mu} (0, 0) \) and \( W_2(0) \leq M_{i2}(\xi) \), where \( \xi \) is determined uniquely by \( M_{i1}(\xi) = W_1(0), \; i = 1 \) or 2. Then there exists a unique solution \( W(x) \) such that \( W(x) \rightarrow (0, 0) \) along \( W_2 \)-axis as \( x \to \infty \). Furthermore, there exists a constant \( L(\geq 1) \) such that if \( x \geq L \) then \( |W_1(x)| \ll |W_2(x)| \), thus
\[ c_3 W_2^2 \leq W_2^2 \leq c_4 W_2^2, \quad x \geq L. \quad (A.13) \]
where \( 0 < c_3 < -\frac{R^2(\gamma - 1)^2 \kappa^2 (\gamma + 1) \rho_+}{2[R \gamma \mu + (\gamma - 1)^2 \kappa]^2 u_+} \leq c_4 \) are two constants. From this one easily obtains that there exists a positive constant \( C \) such that
\[ |W(x)| \leq C \frac{\delta_W}{1 + \delta_W x}, \quad x \in \mathbb{R}_+. \quad (A.14) \]
Subcase 3. \( W(0) \in \Omega_{\delta_0}(0, 0) \) and \( W_2(0) > M_2(\xi) \), where \( \xi \) is determined uniquely by \( M_1(\xi) = W_i(0), \ i = 1 \) or \( 2 \), then there does not exist a solution to (A.7) such that \( W(x) \to (0, 0) \) as \( x \to \infty \).

After inversely transforming, one easily gets the conclusion on \((\tilde{u}, \tilde{\theta})\) in case (ii) of lemma 2.1, where \( \delta_0 > 0 \) is constrained by
\[
\delta_0 \leq \left(1 - \frac{\mu u_\pm}{(\gamma - 1)\kappa}\right)^{-1} \left(1 - \frac{R(\gamma - 1)\gamma \kappa}{[R\gamma \mu + (\gamma - 1)^2 \kappa]u_\pm}\right)^{-1} \delta_W. \tag{A.15}
\]

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