ALL 2-TRANSITIVE GROUPS HAVE THE EKR-MODULE PROPERTY

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Abstract. We prove that every 2-transitive group has a property called the EKR-module property. This property gives a characterization of the maximum intersecting sets of permutations in the group. Specifically, the characteristic vector of any maximum intersecting set in a 2-transitive group is a linear combination of the characteristic vectors of the stabilizers of points and their cosets. We also consider when the derangement graph of a 2-transitive group is connected and when a maximum intersecting set is a subgroup or a coset of a subgroup.

1. Introduction

The Erdős-Ko-Rado (EKR) Theorem [11] is a major result in extremal set theory. This famous result gives the size and the structure of the largest collections of pairwise intersecting $k$-subsets of an $n$-set. The Erdős-Ko-Rado Theorem has been generalized in many different ways. One generalization is to show that a version of the theorem holds for different objects. To date, a version of the EKR Theorem has been shown to hold for the following objects: $k$-subsets of an $n$-set [11, 30], integer sequences [26], $k$-dimensional subspaces of an $n$-dimensional vector space over a finite field [13], signed sets [5], partitions [20] and perfect matchings [15], as well as many other objects.

The commonality relating these results is that a largest set of (pairwise) intersecting objects must be a set of objects that intersect in a “canonical” way. For example, a largest set of intersecting $k$-sets is the collection of all $k$-sets that contain a common point. A largest set of intersecting $k$-subspaces is the set of all subspaces that contain a common 1-dimensional subspace. Similarly, a largest set of intersecting perfect matchings is the collection of all perfect matchings that contain a fixed pair. In all of these cases, the objects are sets of elements and two objects are said to intersect if they contain a common element. And for all the cases named above, a largest set of intersecting objects is the collection of all objects that contain a fixed element—these are the canonical intersecting sets.

In general, whenever we have objects formed from elements we can ask “what is the size and structure of a largest set of intersecting objects?”. If a largest intersecting set must be a canonical intersecting set, then we say that a version of the EKR Theorem holds.

In this paper we consider permutations. Two permutations $g, h \in \text{Sym}(n)$ intersect if there exists an $i \in \{1, \ldots, n\}$ with $i^g = i^h$. (Here a permutation $g$ is the object, and the elements that form it are the pairs $(i, j)$ where $i^g = j$.)
Let $G$ be a transitive subgroup of $\text{Sym}(n)$. Clearly the stabilizer in $G$ of a point, or the coset of a stabilizer of a point, is an intersecting set of permutations. These sets are denoted by

$$S_{i,j} = \{ g \in G | i^g = j \},$$

where $i, j \in \{1, \ldots, n\}$ and we call them the canonical intersecting sets.

The intersecting sets of largest size in $G$ are called the maximum intersecting sets. We say that a group $G$ has the EKR property if the canonical intersecting sets are maximum intersecting sets. The group $G$ is further said to have the strict-EKR property if the canonical intersecting sets are the only maximum intersecting sets. (Note that these properties depend on the group action.) Many specific groups have been shown to have either the strict-EKR property, or the EKR property [2, 21, 24, 22, 29]. One of the most general results is the following, which is equivalent to every 2-transitive group having the EKR property.

**Theorem 1.1 (Theorem 1.1 [25]).** Let $G$ be a finite 2-transitive permutation group on the set $\{1, \ldots, n\}$. The cardinality of a largest intersecting set in $G$ is $|G|/n$.

Clearly any group that has the strict-EKR property has the EKR property. There are 2-transitive permutation groups that do not have the strict-EKR property, for example $\text{PGL}_n(q)$ has the strict-EKR property if and only if $n = 2$ [23, 29].

In this paper we consider a property related to the EKR property, and the strict-EKR property; this property is called the EKR-module property. The EKR-module property was first defined in [22], the definition we give here is slightly different, but equivalent.

Before defining the EKR-module property, we need some notation. The regular module of $G$ is the (complex) vector space with basis $G$. We can think of its elements as vectors of length $|G|$. For any $S \leq G$ define the characteristic vector of $S$ to be the vector with entry 1 in position $g$ if $g \in S$ and 0 otherwise; this vector is denoted by $v_S$. We denote the characteristic vector of $S_{i,j}$ by $v_{i,j}$.

**Definition 1.2.** A transitive permutation group $G$ has the EKR-module property if, for any maximum intersecting set of permutations $S$ in $G$, the characteristic vector $v_S$ is a linear combination of the vectors $v_{i,j}$ with $i, j \in \{1, \ldots, n\}$.

Like the EKR property and strict-EKR property, this is a property of the group action. The main result of this paper is the following.

**Theorem 1.3.** Any 2-transitive group has the EKR-module property.

This result was conjectured in [25 Conjecture 1.3]. We feel that this is the most general statement for all 2-transitive groups, in the context of EKR-type results. The theorem also gives information about the structure of the maximum intersecting sets in a 2-transitive group; this is described in detail in Section 7.

Part of the motivation for Definition 1.2 comes from several papers [14, 21, 24, 29] which prove that a group has the strict-EKR property by first showing that the group has the EKR property, and then showing the group has the EKR-module property. The final characterization is achieved by showing the only linear combinations of the vectors $\{v_{i,j} | i, j \in \{1, \ldots, n\}\}$ that give a characteristic vector of an intersecting set have exactly one non-zero coefficient. The EKR-module property is an essential step in the characterization of the maximum intersecting sets.

It is obvious that the strict-EKR property implies the EKR-module property. It is, however, possible for a transitive group to have the EKR-module property, but not the EKR property. Indeed this occurs if the largest intersecting set of permutations is the union of two or more canonical cocliques. An example is the group $\text{Alt}(4)$ acting on unordered pairs from $\{1, 2, 3, 4\}$. For this transitive group,
the set of permutations mapping a pair \(A\) to either \(A\) or its complement \(\overline{A}\) is an intersecting set of maximum size; this set is the union of two canonical cocliques.

We will define a graph with the property that an intersecting set in a group corresponds to a coclique in the graph. This graph has the property that it is connected if and only if the set of derangements generate the entire group. There are many examples of groups where this graph is not connected and, because of this, there can be many different maximum cocliques, and hence non-canonical maximum intersecting sets. In Section 5 we consider different cases when this graph is connected.

For 2-transitive groups with a connected derangement graph, all known examples of non-canonical maximum intersecting sets are, like canonical ones, either subgroups or cosets of subgroups. In Section 6, we describe one way in which such non-canonical subgroups can arise for a 2-transitive group with a regular normal subgroup. These subgroups correspond to elements of the first cohomology group of a point stabilizer with values in the regular normal subgroup. We consider two examples that illustrate what can happen in this situation.

The main result in this paper is that the characteristic vector of a maximum intersecting set in a 2-transitive group is a linear combination of the characteristic vectors for the canonical sets. In Section 7 we prove that this result gives information about the structure of the set. Using an association scheme on the elements of the group, we can prove that any two maximum intersecting sets have the same inner distribution. This is a count of the number of pairs \((g, h)\) of elements in the set that have \(hg^{-1}\) in a given conjugacy class.

2. Background

In this paper we only consider 2-transitive permutation groups, so throughout this paper \(G\) is assumed to be a 2-transitive group acting faithfully on a set \(X\) of size \(n\). For each such group we let \(\chi_G\) denote the permutation character of this 2-transitive action. Since \(G\) is 2-transitive, \(\chi_G\) is the sum of the trivial character (denoted \(1_G\)) and an irreducible character which we will denote by \(\psi_G\).

Let \(\mathbb{C}[G]\) be the complex group algebra. The regular module can be identified with the vector space \(\mathbb{C}[G]\) and given the structure of a left \(\mathbb{C}[G]\)-module by left multiplication. Thus, \(\mathbb{C}[G]\) also becomes identified with a subalgebra of the \(|G| \times |G|\)-matrices.

For any irreducible character \(\phi\) of \(G\), let \(E_{\phi}\) to be the \(|G| \times |G|\)-matrix with the \((g,h)\)-entry equal to \(\frac{\phi(1)}{|G|} \phi(hg^{-1})\). Then \(E_{\phi} \in \mathbb{C}[G]\) is the primitive central idempotent corresponding to \(\phi\). We call the image of \(E_{\phi}\) (considered as a linear operator on \(\mathbb{C}[G]\)) the \(\phi\)-module. It is an ideal of \(\mathbb{C}[G]\) of dimension \(\phi(1)^2\). For the trivial representation the central idempotent is \(E_{1_G} = \frac{1}{|G|} J\), where \(J\) is the all ones matrix. We set \(E_{\chi_G} = E_{1_G} + E_{\psi_G}\) and define the \(\chi_G\)-module to be the image of \(E_{\chi_G}\), an ideal of dimension \(1 + (n-1)^2\) in \(\mathbb{C}[G]\). This leads to an equivalent definition of the EKR-module property, from which its name originates.

**Lemma 2.1.** A 2-transitive group \(G\) has the EKR-module property if and only if the characteristic vector of any maximum intersecting set is in the \(\chi_G\)-module. Equivalently, \(G\) has the EKR-module property if and only if \(E_{\chi_G} v_S = v_S\) for any maximum intersecting set \(S\).

**Proof.** This follows from two results from [2]. First, Lemma 4.1 of [2] states that if \(G\) is 2-transitive, then every \(v_{i,j}\) is in the \(\chi_G\)-module. Lemma 4.2 of the same paper states that the vectors \(v_{i,j}\) are a spanning set for the module. □

We also state a simple corollary of this lemma that gives the result in a form that can be more convenient.
Corollary 2.2. If a 2-transitive group $G$ has the EKR-module property then for any maximum intersecting set $S$,
\begin{equation}
E_{\psi_G}v_S = v_S - \frac{1}{n}1.
\end{equation}

Proof. From Theorem 1.1 if $S$ is a maximum intersecting set, then $E_{\psi_G}v_S = \frac{1}{n}1$ where 1 denotes the all-ones vector. Then Lemma 2.1 implies the equation. \qed

A common approach to EKR theorems is to convert the problem to a graph problem, and then apply techniques from algebraic graph theory (see [16] for details and examples). This is the approach that we will use as well. The derangement graph of $G$ is the graph with vertices the elements of $G$, in which two vertices are adjacent if they are not intersecting. The set of derangements (permutations with no fixed points) in $G$ is denoted by $\text{Der}_G$, and the derangement graph of $G$ is denoted by $\Gamma_G$. The derangement graph is the Cayley graph on $G$ with connection set $\text{Der}_G$. A coclique (or independent set) in $\Gamma_G$ is equivalent to a set of intersecting permutations in $G$. Theorem 1.1 can be expressed as the size of a maximum coclique in $\Gamma_G$ is $|G|$ for any 2-transitive group $G$.

Using this graph structure allows us to use results from graph theory. For example the clique-coclique bound ([16, Corollary 2.1.2]) easily translates to the following.

Lemma 2.3. Let $\omega(\Gamma_G)$ denote the size of the largest clique in $\Gamma_G$, and $\alpha(\Gamma_G)$, the size of the largest coclique. Then
\begin{equation}
\omega(\Gamma_G)\alpha(\Gamma_G) \leq |G|.
\end{equation}
Further, if equality holds, then each maximum clique intersects each maximum coclique in exactly one vertex. \qed

We define a normal Cayley graph to be a Cayley graph with a connection set that is closed under conjugation. The graph $\Gamma_G$ is a normal Cayley graph since its connection set is the set of derangements in $G$. The eigenvalues of a normal Cayley graph can be calculated from the irreducible representations of $G$. The eigenvalue of $\Gamma_G$ belonging to the irreducible character $\phi$ of $G$ is
\begin{equation}
\lambda_\phi = \frac{1}{\phi(1)} \sum_{d \in \text{Der}_G} \phi(d).
\end{equation}
This result is usually attributed to Babai [4], or Diaconis and Shahshahani [9]; a proof may be found in [16, Section 11.12]. The eigenvalue belonging to the trivial character is clearly $d_G := |\text{Der}_G|$, and it is not difficult to see that the eigenvalue belonging to $\psi_G$ is $-\frac{d_G}{n}$. Equation (2.1) implies that if a 2-transitive group $G$ has the EKR-module property, then for any maximum coclique $S$
\begin{equation}
A(\Gamma_G) \left( v_S - \frac{1}{n}1 \right) = -\frac{d_G}{n-1} \left( v_S - \frac{1}{n}1 \right)
\end{equation}
(where $A(\Gamma_G)$ is the adjacency matrix of $\Gamma_G$).

In his classic book Burnside showed [5, §134, Theorem IX] that a 2-transitive group has a unique minimal normal subgroup. If this minimal normal subgroup is regular, then it is elementary abelian, and otherwise it is a non-abelian, primitive simple group (see also [10, Theorem 4.1B]). We use this fact to divide the 2-transitive groups into two cases. In the next section we will prove Theorem 1.3 for 2-transitive groups in which the minimal normal subgroup is abelian. Section 4 we will prove the result for the groups in which the minimal normal subgroup is not abelian; here we will need to use the classification of the almost simple 2-transitive groups. We will consider when $\Gamma_G$ is connected in Section 5. Section 6 considers
when the maximum intersecting sets are groups or cosets of groups. In Section 7 we show that Theorem 1.3 gives information about the structure of the maximum intersecting sets. Finally we discuss some questions for further investigation in Section 8.

3. 2-TRANSITIVE GROUPS WITH A REGULAR NORMAL SUBGROUP

In this section we consider 2-transitive permutation groups \((G, X)\), with \(|X| = n\), that have a regular normal subgroup \(N\). In this case, \(N\) is an elementary abelian \(p\)-group for some prime \(p\). Further, \(G\) is the semidirect product \(NG_x\) where \(G_x\) is the stabilizer of a point \(x \in X\). In particular, \(G_x\) is a transversal of \(N\) in \(G\) and \(G_x\) is a coclique in \(\Gamma_G\).

**Proposition 3.1.** The elements in \(N\) form a clique of size \(n\) in \(\Gamma_G\).

*Proof.* Since \(N\) is regular, it has size \(n\) and every non-identity element is a derangement. For any distinct \(n_1, n_2 \in N\), \(n_1n_2^{-1}\) is a non-identity element of \(N\), and is a derangement. \(\square\)

By the clique-coclique bound (Lemma 2.3), Proposition 3.1 implies that the size of a maximum coclique in \(\Gamma_G\) is bounded by \(\frac{|G|}{n}\). Since \(G_x\) is a coclique of this size we have \(\alpha(\Gamma_G) = \frac{|G|}{n}\). This shows that all of these groups have the EKR property. Further, any maximum coclique \(S\) in \(\Gamma_G\) intersects \(N\) (and any coset of \(N\)) in exactly one element. So any coclique \(S\) of maximum size is a transversal of \(N\) in \(G\). This can also be seen since for any two distinct elements \(s\) and \(t\) of \(S\), the element \(st^{-1}\) has a fixed point so does not belong to \(N\).

The following is a well-known result that we state in this context.

**Lemma 3.2.** Let \(g = uh\) with \(u \in N\) and \(h \in G_x\). If \(g\) is \(G\)-conjugate to an element of \(G_x\), then the following hold:

(a) \(g = uh\) can be conjugated to \(h\) by an element of \(N\); and

(b) \(h\) is the unique \(N\)-conjugate of \(g\) in \(G_x\).

*Proof.* By hypothesis there exists \(a \in G\) such that \(a^{-1}ga \in G_x\). We may write \(a = mk\), where \(m \in N\) and \(k \in G_x\). Then \(k^{-1}m^{-1}(uh)km \in G_x\), so \(m^{-1}(uh)m \in kG_xk^{-1} = G_x\).

As \(G_x\) is a transversal of \(N\) in \(G\), two elements of \(G_x\) with the same image in \(G/N\) must be equal. Therefore the only possible \(N\)-conjugate of \(uh\) in \(G_x\) is \(h\). So \(m^{-1}uhm = h\) and both parts of the lemma are proved. \(\square\)

Let \(S\) be a maximum coclique in \(\Gamma_G\), for any elements \(s, t \in S\) (including \(s = t\)), write \(st^{-1} = uh\) with \(u \in N\) and \(h \in G_x\). As \(uh\) has a fixed point, it is \(G\)-conjugate to an element of \(G_x\), hence \(N\)-conjugate to \(h\) by Lemma 3.2. If we fix \(t\) and let \(s\) run over \(S\), then each element \(h \in G_x\) is obtained in this way exactly once, since \(St^{-1}\) is also a transversal of \(N\) in \(G\). These observations will allow us, in the next lemma, to generalize to arbitrary cocliques a calculation that was made for canonical cocliques in \([\text{2}]\ Lemma 4.1\).

Recall that \(\psi_G\) denotes the irreducible character of \(G\) of degree \(n - 1\) from the 2-transitive action.

**Lemma 3.3.** Let \(S\) be a coclique and \(y \in G\).

\[
\sum_{s \in S} \psi_G(sy^{-1}) = \begin{cases} 
\frac{|G_x|}{|N|} & \text{if } y \in S, \\
\frac{|G_x|}{n-1} & \text{if } y \notin S. 
\end{cases}
\]
Proof. First suppose that \( y \in S \). Write \( sy^{-1} = uh \), where \( u \in N \) and \( h \in G_x \). We know from the previous lemma that \( sy^{-1} \) is \( G \)-conjugate to \( h \), and so \( \psi_G(sy^{-1}) = \psi_G(h) \). Moreover, as \( s \) runs over \( S \) we obtain each \( h \in G_x \) once, so
\[
\sum_{s \in S} \psi_G(sy^{-1}) = \sum_{h \in G_x} \psi_G(h) = |G_x|.
\]

Next suppose \( y \notin S \). Since \( S \) is a transversal of \( N \) in \( G \), we can write \( y = mt \), with \( t \in S \) and \( m \) a nonidentity element of \( N \). Suppose \( st^{-1} = uh \), where \( u \in N \) and \( h \in G_x \). By Lemma 3.2 there exists \( v \in N \) such that \( v(st^{-1})v^{-1} = v(uh)v^{-1} = h \). Then
\[
\psi_G(sy^{-1}) = \psi_G(st^{-1}m) = \psi_G(vst^{-1}m^{-1}v^{-1}) = \psi_G(vst^{-1}v^{-1}m^{-1}) = \psi_G(hm^{-1}).
\]
Here we used the fact that \( N \) is abelian. Moreover, the transversal property of \( St^{-1} \) means that, as \( s \) runs over \( S \), each element of \( G_x \) is conjugate to \( st^{-1} \) for exactly one \( s \). Hence
\[
(3.2) \quad \sum_{s \in S} \psi_G(sy^{-1}) = \sum_{h \in G_x} \psi_G(hm^{-1}).
\]
Note that the right-hand side does not depend on \( S \). This allows us to proceed as in the proof of [2] Lemma 4.1. The right-hand side of (3.2) is the sum of \( \psi_G \) over a coset of \( G_x \) that is not equal \( G_x \). By the 2-transitivity of \( G \) the value of this sum is the same for all cosets of \( G_x \) other than \( G_x \) itself. Then, since \( \sum_{g \in G} \psi_G(g) = 0 \) and \( \sum_{g \in G_x} \psi_G(g) = |G_x| \), it follows that
\[
\sum_{h \in G_x} \psi_G(hm^{-1}) = \frac{|G_x|}{n-1}.
\]

As in [2], the sum computed in the Equation (3.1) is the coefficient of \( y \) when the element \( \frac{|G|}{\psi_G(1)}E_{\psi_G}vS \in \mathbb{C}[G] \) is expressed in the group basis. It follows as in [2], that
\[
E_{\psi_G} \left( vS - \frac{1}{n} \right) = vS - \frac{1}{n} 1,
\]
which shows that \( vS \) lies in the 2-sided ideal of \( \mathbb{C}[G]E_{\psi_G} \) of \( \mathbb{C}[G] \). This shows that \( G \) has the EKR-module property, so Theorem 1.3 holds for any 2-transitive group with a regular normal subgroup.

4. 2-TRANSITIVE GROUPS OF ALMOST SIMPLE TYPE

In this section we consider the 2-transitive groups that do not have a regular abelian normal subgroup \( N \); these are the 2-transitive groups of almost simple type. In this section, we assume that \( G \) is such a group and \( K \leq G \) is the minimal nonabelian normal subgroup of \( G \). These groups are listed in Table I. With the exception of \( G = \text{Ree}(3) \), for each of these groups the subgroup \( K \) is 2-transitive. The eigenvalues of the group \( \text{Ree}(3) \) can all be directly calculated, and \( \psi_{\text{Ree}(3)} \) is the only irreducible character affording the minimal eigenvalue. Thus \( \text{Ree}(3) \) has the EKR-module property. So we will restrict to the case where \( K \) is 2-transitive.

We will show if \( K \) has the EKR-module property, then \( G \) also has the EKR-module property. Then we will prove that each of these groups, the minimal normal subgroup has the EKR-module property.

We assume that \( G \) and \( K \) are both acting on an \( n \)-set. We denote character from this 2-transitive action of \( G \) by \( \chi_G \), and \( \chi_K \) is the representation of \( K \) for its 2-transitive action. Similarly, we use \( \psi_G \) and \( \psi_K \) for the irreducible character of degree \( n - 1 \) that is a component of \( \chi_G \) and \( \chi_K \).
Lemma 4.1. Let $G$ be a 2-transitive group. If $S$ is a maximum coclique in $\Gamma_G$, then $v_s - \frac{1}{n}$ is a $-\frac{d_G}{n^2}$-eigenvector of $A(\Gamma_G)$.

Proof. From Theorem 1.1 the size of $S$ is $\frac{|G|}{n}$. Since $S$ is a maximum coclique and $\Gamma_G$ is $d_G$-regular, the number of edges between vertices in $S$ and vertices in $V(\Gamma_G)\setminus S$ is $d_G|S|$. So the quotient graph of $\Gamma_G$ with the partition $\{S,V(\Gamma_G)\setminus S\}$ is

$$
\begin{bmatrix}
0 & d_G\left(\frac{|S|}{|G|} - \frac{|S|}{|G|}\right) \\
d_G\left(\frac{|S|}{|G|} - \frac{|S|}{|G|}\right) & d_G\left(1 - \frac{|S|}{|G|}\right)
\end{bmatrix}.
$$

The eigenvalues of this quotient graph are $d_G$ and $-\frac{d_G}{n^2}$. These eigenvalues interlace the eigenvalues of $\Gamma_G$. Further, $d_G$ is the eigenvalue of $\Gamma_G$ afforded by the trivial representation and $-\frac{d_G}{n^2}$ is the eigenvalue afforded by $\psi_G$. Since the eigenvalues of the quotient graph are eigenvalues of the graph, the interlacing is tight. This means that $\{S,\Gamma_G\setminus S\}$ is an equitable partition [17, Lemma 9.6.1]. So each vertex in $G\setminus S$ is adjacent to exactly $d_G\frac{|S|}{|G|}$ vertices in $S$ and $d_G\left(1 - \frac{|S|}{|G|}\right)$ vertices not in $S$. By direct calculation of $A(\Gamma_G)(v_S - \frac{1}{n})$, the vector $v_S - \frac{1}{n}$ is a $-\frac{d_G}{n^2}$-eigenvector of $\Gamma_G$. \hfill $\square$

Lemma 4.2. Suppose $H$ and $G$ are 2-transitive groups with $H \leq G$. Then there exist derangements in $G$ that are not in $H$.

Proof. We have

$$
\sum_{g \in G} \chi_G(g) = |G| \quad \text{and} \quad \sum_{h \in H} \chi_H(h) = |H|,
$$

so

$$
(4.1) \quad \sum_{x \in G \setminus H} \chi_G(x) = |G \setminus H|.
$$

Suppose $\text{Der}_G \subseteq H$. Then $\chi_G(x) \geq 1$ for all $x \in G \setminus H$ so, by (4.1), we must have $\chi_G(x) = 1$ and $\psi_G(x) = 0$ for all $x \in G \setminus H$.

Since $G$ and $H$ both act 2-transitively, both $\psi_G$ and its restriction to $H$ are irreducible characters. We have

$$
\sum_{g \in G} \psi_G(g)^2 = |G| \quad \text{and} \quad \sum_{h \in H} \psi_H(h)^2 = |H|.
$$

so

$$
\sum_{x \in G \setminus H} \psi_G(x)^2 = |G \setminus H|.
$$

Therefore, there exists $x \in G \setminus H$, with $\psi_G(x) \neq 0$. This contradiction completes the proof. \hfill $\square$

Theorem 4.3. Let $G$ be a 2-transitive group with minimal nonabelian normal subgroup $K$. Assume $K$ is 2-transitive and that $\psi_K$ is the unique character of $K$ affording the least eigenvalue $-\frac{d_K}{n^2}$ of $\Gamma_K$. Then for any maximum coclique $S$ of $\Gamma_G$, $v_S - \frac{1}{n}$ is in the $\psi_G$-module.

Proof. Assume that $S$ is any maximum coclique of $\Gamma_G$. Since $G$ is 2-transitive, by Theorem 1.1 $G$ has the EKR property, so the size of $S$ is $\frac{|G|}{n}$. By Lemma 4.1 $v_S - \frac{1}{n}$ is a $-\frac{d_G}{n^2}$-eigenvector of $A(G)$.

Since $K$ is a subgroup of $G$, the graph $\Gamma_G$ contains $[G : K]$ copies of $\Gamma_K$ as a subgraph. Let $A$ be the adjacency matrix for the $[G : K]$ copies of $\Gamma_K$. This is a weighted adjacency matrix for $\Gamma_G$ where the edge $\{\sigma, \pi\}$ is weighted by one if $\sigma \pi^{-1}$ is in the intersection of the derangements of $G$ and $K$ (so $\sigma \pi^{-1}$ is a derangement in $G \setminus H$).
$K$), and zero otherwise. The matrix $A$ has the form $A = I_{[G:K]} \otimes A(\Gamma_K)$. Further, if $G = \bigcup_{i=1}^{[G:K]} x_i K$ is the decomposition of $G$ into cosets of $K$, then each $S_i = S \cap x_i K$ is a coclique of size $\frac{|K|}{n}$ and each $\nu_S - \frac{1}{n} I$ is a $-\frac{d_G}{n-1}$-eigenvector for $A$. This means that $\nu_S - \frac{1}{n} I$ is a $-\frac{d_G}{n-1}$-eigenvector for $A$. The eigenvalues of $A$ are the same as the eigenvalues of $\Gamma_K$, but the multiplicities of the eigenvalues for $A$ are equal to the multiplicities of $\Gamma_K$ multiplied by $[G:K]$. In particular, the eigenvalue $-\frac{d_G}{n-1}$ has multiplicity $[G:K](n-1)^2$ in $A$.

The induced character $\text{ind}_G(1_K)$ has decomposition

\[ \text{ind}_G(1_K) = \sum_i \phi_i(1) \phi_i, \]

where the $\phi_i$ are the distinct irreducible characters of $G$ having $K$ in the kernel (which we may view as characters of $G/K$). We choose notation so that $\phi_1 = 1_G$. Then

\[ \text{ind}_G(\nu_S) = \text{ind}_G(1_K) \psi_G = \sum_i \phi_i(1) \phi_i \psi_G. \]

Each $\phi_i \psi_G$ is an irreducible character of $G$ ([19 Corollary 6.17]). Since the restriction of $\text{ind}_G(\nu_S)$ to $K$ equals $[G:K] \nu_K$, the eigenvalue of $A$ afforded by each $\phi_i \psi_G$ is $-\frac{d_G}{n-1}$. The dimension of the sum of the $\phi_i \psi_G$-modules in $\mathbb{C}[G]$ equals $\sum_i (\phi_i(1) \psi_G(1))^2 = (n-1)^2 \sum_i \phi_i(1)^2 = (n-1)^2 [G:K]$, so this sum is the entire $-\frac{d_G}{n-1}$-eigenspace of $A$. Therefore, $\nu_S - \frac{1}{n} I$ lies the sum of the $\phi_i \psi_G$-modules.

Next we will use the fact that $\nu_S - \frac{1}{n} I$ is also a $-\frac{d_G}{n-1}$-eigenvector for the adjacency matrix of $\Gamma_K$ to show that it is entirely contained in the $\phi_1 \psi_G$-module.

Consider

\[ \lambda_\phi \psi_G = \frac{1}{(n-1) \phi_1(1)} \sum_{d \in \text{Der}_G} \phi_i(d) \psi_G(d) = \frac{-1}{(n-1) \phi_1(1)} \sum_{d \in \text{Der}_G} \phi_i(d). \]

By Lemma 4.2 there are derangements in $G$ that are not in $K$, so some $d$ we have $\phi_i(d) \neq \phi_1(1)$. So, if $\phi_1 \neq 1_G$, then

\[ \frac{1}{\phi_1(1)} \sum_{d \in \text{Der}_G} \phi_i(d) < \frac{1}{\phi_1(1)} \sum_{d \in \text{Der}_G} \phi_i(1) = d_G. \]

So no $\phi_i \psi_G$ affords $-\frac{d_G}{n-1}$ as an eigenvector, other than $\phi_1 = 1_G$. Since $\nu_S - \frac{1}{n} I$ is both a $-\frac{d_G}{n-1}$-eigenvector and in the sum of the $\phi_i \psi_G$-modules, it must in fact be in the $\phi_1 \psi_G$-module.

The classification of finite simple groups has allowed for the complete classification the finite 2-transitive groups. Below is Table 1 from [25] which lists the finite 2-transitive groups of almost simple type. (This table was extracted from [7] page 197].)

**Proposition 4.4.** For $n \geq 5$ the least eigenvalue of $\Gamma_{\text{Alt}(n)}$ is given by $\psi_{\text{Alt}(n)}$ and no other representations, and the largest eigenvalue is given by the trivial character and no other.

**Proof.** The number of derangements in $\text{Alt}(n)$ is known [27] Sequence A003221], and for $n \geq 5$ we have

\[ d_{\text{Alt}(n)} = \frac{n!}{2} \sum_{i=0}^{n-2} (-1)^i \frac{1}{i!} + (-1)^{n-1} (n-1) \]

\[ \geq \frac{n!}{2} \left( 1 - 1 + \frac{1}{2} - \frac{1}{6} \right) = \frac{n!}{6}. \]
The inequality clearly holds for $n$ odd. If $n$ is even and at least 6, then the inequality follows since \( \frac{n!}{2^{(n/2)!}} - \frac{n^2}{2^{(n/2)!}} - (n-1) \) is positive.

Using Lemma 2.4 from [25], if the character $\phi \neq \psi_{\text{Alt}}$ of the alternating group affords the minimum eigenvalue of $\Gamma_{\text{Alt}(n)}$, then

$$
\phi(1) \leq (n-1) \left( \frac{|\text{Alt}(n)|}{d_{\text{Alt}(n)}} - 2 \right)^{\frac{1}{2}}.
$$

Since

$$
(n-1) \left( \frac{|\text{Alt}(n)|}{d_{\text{Alt}(n)}} - 2 \right)^{\frac{1}{2}} \leq (n-1) \left( \frac{n!}{2^{(n/2)!}} \left( \frac{n!}{6} \right)^{-1} - 2 \right)^{\frac{1}{2}} = n-1,
$$

any character giving the minimal eigenvalue must have dimension no more than $n-1$. Since the only representations with degree no more than $n-1$ are the trivial representation and $\psi_{\text{Alt}}$, it follows that $\psi_{\text{Alt}}$ is the unique irreducible representation affording the minimum eigenvalue. Note that this also implies that only the trivial representation gives the largest eigenvalue. 

\[ \square \]

**Theorem 4.5.** The group $K$ of each type in Table 1 has $\chi_K$ as the only irreducible character that gives the eigenvalue $-\frac{d_{\text{G}}}{n-1}$. (Here, as usual, we exclude $K = \text{PSL}(2,8)$ in row 12, as it is not 2-transitive.)

**Proof.** The previous result shows this holds for $\text{Alt}(n)$ with $n \geq 5$. For $\text{PSL}_2(q)$, this fact can be read off the tables in Simpson and Frame [28], for $\text{P}S\ell_3(q)$ it is in [24] Table 5, and for $\text{PSL}_m(q)$ with $m \geq 4$ it is stated in [25] Proposition 8.3. For the groups in lines 3 and 4, $\text{Sp}_{2m}(2)$ this result is from [25] Proposition 9.1 for $m \geq 7$. For $\text{PSU}_3(q)$ this is from [22] Table 5 and Table 6. For $\text{Sz}(q)$ the result is given in [25] Proposition 4.1 and for $\text{Ree}(q)$ this is [25] Proposition 5.1. The eigenvalues of the Mathieu groups are given in [2] Lemma 5.1. For all the other finite groups all the eigenvalues can be calculated from the character table, and only $\chi_K$ gives the eigenvalue $-\frac{d_{\text{G}}}{n-1}$. \[ \square \]
We have now shown that all 2-transitive groups of almost simple type have the EKR-module property. In §3 the EKR-module property was established for 2-transitive groups with a regular normal subgroup. Thus, the proof of Theorem 1.3 is complete.

Remark 4.6. We have proven that the characteristic vector of every maximum coclique in the derangement graph for a 2-transitive group is a linear combination of the characteristic vectors $v_{i,j}$ of the canonical cocliques. We can say a little more about the coefficients in this linear combination. If $Q[G] \subset C[G]$ is the rational group algebra and $V$ is the intersection of $Q[G]$ with the $\chi$-module, then $V$ is a $Q[G]$-module whose dimension over $Q$ equals the complex dimension of the $\chi$-module. Since $E_{\chi} \in Q[G]$, it follows that $E_{\chi}(Q[G]) = V$. The characteristic vector $v_G$ of any maximum coclique lies in $Q[G]$ and by the EKR-module property we have $v_G = E_{\chi}(v_G) \in V$. Moreover, the canonical characteristic vectors $v_{i,j}$ span $V$ over $Q$. Therefore, our proof actually shows that $v_G$ is a rational linear combination of the $v_{i,j}$.

5. Connected derangement graphs

Consider the example of a 2-transitive Frobenius group $G$ with Frobenius kernel $N$ and Frobenius complement $H$. (The group $\text{AGL}(1, q)$ where $q$ is a prime power is an example of such a group.) In this case, the cosets of $N$ are cliques in the derangement graph of $G$. In fact, the derangement graph is exactly the disjoint union of these cliques. Since any transversal of $N$ is a coclique, as long as $|H| > 2$, there are non-canonical cocliques of the form $H \setminus \{h\} \cup \{uh\}$, where $h \in H$ and $u \in N$ are nonidentity elements. The 2-transitive Frobenius groups with $|H| > 2$ are a family of groups that do not satisfy the strict-EKR property. Further the non-canonical independent sets just described are neither subgroups, nor cosets of subgroups.

In this section we will consider other groups that have a disconnected derangement graph; this occurs exactly when the derangements do not generate the group.

Lemma 5.1. Suppose $G$ contains a proper 2-transitive subgroup $H$. Then $G$ is generated by $H \cup \text{Der}_G$. In particular, if $H$ is generated by $\text{Der}_H$, then $G$ is generated by $\text{Der}_G$.

Proof. Suppose for a contradiction that the subgroup $M$ of $G$ generated by $H \cup \text{Der}_G$ is proper. Then we may apply Lemma 4.2 to the group $G$ and the subgroup $M$, to obtain a derangement outside $M$. This is a contradiction and hence $M = G$. The last statement of the lemma follows immediately. 

For all the groups $G$ in Table 1, with the exception of $\text{Ree}(3)$, this corollary applies. Proposition 4.4 implies that the derangement graph for the Alternating group is connected. The fact that the minimal groups $K$ in lines 2-7 of Table 1 have a connected derangement graph can be read from [25] (with results from [18] for lines 3 and 4). The groups in lines 8-11 and 13-14 are finite, and the eigenvalues of the derangement graphs for the minimal group can be directly calculated and individually checked. With these facts, we have the following corollary.

Corollary 5.2. With the exception of $\text{Ree}(3)$ (isomorphic to $\text{PSL}_2(8)$ with its action on 28 points), the derangement graph for any 2-transitive group of almost-simple type is connected.

Proof. $\text{PSL}_2(8)$ is a subgroup with index 3 in $\text{PGL}_2(8)$. Every element in $\text{PSL}_2(8)$ that is not in $\text{PSL}_2(8)$ has order 3, 6 or 9 and $\psi_{\text{PSL}_2(8)}$ vanishes on these points. So all derangement of $\text{PSL}_2(8)$ are in $\text{PSL}_2(8)$. 


Next we focus on the 2-transitive groups $G$ with a regular normal subgroup $N$. We begin with an immediate consequence of the fact that $\text{Der}_G$ is a union of conjugacy classes.

**Lemma 5.3.** Let $G$ be a 2-transitive finite permutation group, with a regular normal subgroup $N$. If $G/N \cong G_x$ is a simple group and there are derangements outside $N$, then the derangement graph of $G$ is connected.

Fix an element $x$, from the set on which $G$ acts, and let $H = G_x$ be its stabilizer. Then, by definition of the regular normal subgroup, there is a map $N \rightarrow X$ defined by $u \mapsto u(x)$ that is an isomorphism of $N$ sets where $N$ acts on itself by left multiplication. This is also an isomorphism of $H$-sets where $H$ acts on $N$ by conjugation. That is to say, for all $h \in H$ and $u \in N$ we have $h(u(x)) = (huh^{-1})(x)$.

Under this identification of $N$ with $X$, the action of $G$ on $X$ is equivalent to an action of $G$ on $N$ given as follows. Each element of $G$ has the unique form $mh$ for $m \in N$ and $h \in H$. Then $m(hu) = m(huh^{-1})$ for all $u \in N$. We will make use of this $G$-action on $N$ in the following lemmas.

**Lemma 5.4.** Let $G$ be a 2-transitive finite permutation group with a regular normal subgroup $N$ and point stabilizer $H$. Then for $h \in H$, the coset $Nh$ contains a derangement if and only if $h$ centralizes a nonidentity element of $N$.

**Proof.** Consider the map $f_h : N \rightarrow N$ defined by

$$f_h(u) = huh^{-1}u^{-1}.$$ 

Then $h$ centralizes a nonidentity element of $N$ if and only if $f_h$ is not injective, which in turn is equivalent to $f_h$ not being surjective.

Suppose $f_h$ is not surjective, and let $m \in N$ be an element not in the image of $f_h$. We claim that $m^{-1}h$ is a derangement. Here we use the identification of $X$ with $N$ described above. Supposed $m^{-1}h$ is not a derangement, then is has a fixed point. So

$$(5.1) \quad u = (m^{-1}h)(u) = m^{-1}huh^{-1}$$

and it follows that $f_h(u) = m$, a contradiction. Thus if $f_h$ is not surjective then $Nh$ contains a derangement.

Conversely, if $f_h$ is surjective, then for every $m \in N$, there exists $u \in N$ such that $f_h(u) = m^{-1}$. This equation can be written as $m(huh^{-1}) = u$, that is $(mh)(u) = u$. Thus every element of $Nh$ has a fixed point. \hfill \Box

**Theorem 5.5.** Let $G$ be a 2-transitive finite permutation group with a regular normal subgroup $N$ and point stabilizer $H = G_x$. Then the subgroup of $G$ generated by $\text{Der}_G$ is equal to the subgroup generated by $N$ and the two-point stabilizer $H_y$, for $y \neq x$.

**Proof.** Let $M$ be the subgroup of $G$ generated by $\text{Der}_G$. Then $N \subseteq M$. By Lemma [5.3] a coset $Nh$, with $h \in H$ contains a derangement if and only if $h$ centralizes a nonidentity element of $N$. In this case, the whole coset $Nh$ will be contained in $M$ since $N$ is contained in $M$. Thus, $M$ is equal to the subgroup generated by those cosets $Nh$ for which $h$ centralizes a nonidentity element of $N$.

As the conjugation action of $H$ on $N$ is isomorphic to the permutation action of $H$ on $X$, an element $h$ centralizes a nonidentity element of $N$ if and only if $h$ lies in $H_y$ for some $y \in X$, $y \neq x$. This completes the proof. \hfill \Box

**Proposition 5.6.** Let $G$ be a 2-transitive finite permutation group, with a regular normal subgroup $N$. Then $G$ is a Frobenius group if and only if $\text{Der}_G = N \setminus \{1\}$. 
Proof. If $G$ is a Frobenius group then it is immediate that $\text{Der}_G = N \setminus \{1\}$.

Suppose that $G$ is not a Frobenius group. Then there is a nonidentity element $h \in H$ that centralizes a nonidentity element of $N$. Then by Lemma 5.4, the coset $Nh$ contains a derangement.

$\square$

Corollary 5.7. Let $G$ be a 2-transitive finite permutation group, with a regular normal subgroup $N$. Then $G$ is a Frobenius group if and only if $\Gamma_G$ is the union of disjoint complete graphs.

Proof. It is not hard to see that if $G$ is a Frobenius group, then $\Gamma_G$ is the union of complete graphs on $n$ vertices, see [3, Theorem 3.6] for details. If $\Gamma_G$ is the union of disjoint complete graphs then, since a point stabilizer is a coclique of size $|G|/n$, no complete subgraph has more than $n$ vertices. In particular, the identity element can have no more than $n-1$ neighbors. However the set of neighbors of the identity element is $\text{Der}_G$, which contains $N \setminus \{1\}$, a set of size $n-1$. Thus, $\text{Der}_G = N \setminus \{1\}$, and by Proposition 5.6 $G$ is a Frobenius group.

$\square$

There are many 2-transitive groups with a regular normal subgroup that are not Frobenius groups and have disconnected derangement graphs. For example, as we shall see, the groups $\AGL_1(p^e)$, for $p > 2$ and $e \geq 2$, are 2-transitive groups with a disconnected derangement graphs, and further examples may be found among their subgroups. Each of these groups have the EKR-property, the EKR-module property, but not the strict-EKR property. Further, for each of these groups there are maximum cocliques that are neither subgroups, nor cosets of subgroups.

Proposition 5.8. If $p > 2$ is prime and $e \geq 2$ then $\AGL_1(p^e)$ is a 2-transitive group with a disconnected derangement graph.

Proof. Let $N$ be the regular normal subgroup of $\AGL_1(p^e)$, consisting of the translations of the form $x \mapsto x + b$ with $b \in \mathbb{F}_{p^e}$. The two-point stabilizers of $\AGL_1(p^e)$ all have order $e$ and are generated by transformations of the form $x \mapsto a^{(p-1)x}p + b$ where $a, b \in \mathbb{F}_q$ and $a \neq 0$. These permutations do not generate all of $\AGL_1(p^e)$.

$\square$

6. Non-canonical Cocliques that are Cosets of Subgroups

In this section we describe examples of groups that have a connected derangement graph, but also have noncanonical cocliques that are of maximum size. These come from considering non-canonical cocliques that are subgroups in 2-transitive finite permutation groups $G$ with a regular normal subgroup $N$.

Since any coclique in $\Gamma_G$ must be a transversal of $N$, any subgroup that is a non-canonical coclique must be complementary to $N$, but not conjugate to $G_x$. The $G$-conjugacy classes of subgroups that are complementary to $N$, are classified by the first cohomology group $H^1(G_x, N)$, where $N$ is viewed as a $G_x$-module by conjugation. The trivial element of $H^1(G_x, N)$ corresponds to the $G$-conjugacy class of $G_x$ and, if $H^1(G_x, N)$ is not trivial, each nontrivial element corresponds to a $G$-conjugacy class of nonstandard complements, by which we mean subgroups complementary to $N$, but not $G$-conjugate to $G_x$.

The following is a necessary and sufficient condition for a nonstandard complement to be a maximum coclique in $\Gamma_G$.

Lemma 6.1. A complement $K$ to $N$ in $G = NG_x$ is a coclique in $\Gamma_G$ if and only if every element of $K$ is $G$-conjugate to an element of $G_x$.

Proof. Assume that $K$ is a complement to $N$ that is a coclique in $\Gamma_G$. Since $1 \in K$, each element of $K$ must have a fixed point (as it intersects with the identity element). Thus any element of $K$ lies in a point stabilizer and is $G$-conjugate to an element of $G_x$.
Conversely, if each element of a subgroup $K$ is $G$-conjugate to an element of $G_2$, then every element has a fixed point. So for any $h, k \in K$, the element $hk^{-1} \in K$ has a fixed point which implies that $K$ is a coclique. \hfill \Box

Using the notation of the previous proof, let $g \in K$ and let $g_p$ be its $p$-part. It follows from the injectivity of the restriction of $H^1((g), N) \to H^1((g_p), N)$ (see [5, Ch.XII, Theorem 10.1]) that we may replace the condition in Lemma 6.1 that every element of $K$ be $G$-conjugate to an element of $H$, by the same condition on $p$-elements only.

**Theorem 6.2.** For $e \geq 2$, the group $\text{ASL}_2(2^e)$ of affine transformations of $X = \mathbb{F}_2^e$, does not have the strict-EKR property.

**Proof.** Let $G = \text{ASL}_2(2^e)$ be the group of affine transformations of $X = \mathbb{F}_2^e$ generated by the linear group $H = \text{SL}_2(2^e)$ (the stabilizer of the zero vector) and the group $N = \mathbb{F}_2^e$ of translations, where $uh : x \mapsto hx + u$, for $x \in X$, $h \in H$ and $u \in N$. It is well known that $H^1(H, N) \cong \mathbb{F}_2$ when $e \geq 2$ [12, Lemma 14.7].

Since $H^1(H, N)$ is not trivial, this group has a non-standard complement, say $K$. The group $K$ is not conjugate to $H$, but it is isomorphic to it. This implies that every element of $K$ is either an involution or an element of odd order. Moreover, there is a single $K$-conjugacy class of involutions and each involution in $K$ has the form $ut$, where $t \in H$ and $u \in C_N(t)$.

If we regard $N$ as a $\mathbb{F}_2^e$-vector space and $t \in H$ as a linear map, then $C_N(t) = \text{Ker}(t - 1)$, and a simple calculation shows that $\text{Ker}(t - 1) = \text{Im}(t - 1)$. It follows that for any $u \in N$ that exists $m \in N$ such that $u = t^{-1}mtm^{-1}$, so $ut = tu = mtm^{-1}$ is conjugate to $t \in H$. As every odd order element of $K$ is conjugate to an element of $H$, the Schur-Zassenhaus Theorem, Lemma 6.1 shows that $K$ (and its cosets) are non-canonical cocliques in $\Gamma_G$. \hfill \Box

**Example 6.3.** For an explicit example, let $e = 2$ and $\alpha$ be a primitive element of $\mathbb{F}_4$. We can think of $\text{ASL}_2(4)$ as the subgroup of $\text{SL}_3(4)$ consisting of matrices of the block form

$$
\begin{bmatrix}
A & v \\
0 & 1
\end{bmatrix}
$$

where $A \in \text{SL}_2(4)$ and $v \in \mathbb{F}_2^2$.

Consider the elements

$$
t = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad u = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}, \quad s = \begin{bmatrix}
0 & 1 & 0 \\
1 & \alpha & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

of orders 2, 2 and 5 respectively.

The standard complement $H$ is generated by the elements $t$ and $s$, while $tu$ and $s$ generate a nonstandard complement. It is interesting to note that this nonstandard complement has an orbit of size $6$ in $\mathbb{F}_4^2$ which is a maximal arc of degree 2. (An arc of degree 2 is a subset in which no three points are collinear, and in $\mathbb{F}_4^2$ such a subset can have at most 6 points.)

Many other examples of non-canonical cocliques arising from nonstandard complements can be found. However, it is not always the case that a nonstandard complement will yield a non-canonical coclique in the derangement graph, as it may fail to satisfy the hypotheses of Lemma 6.1, as in the following example.

**Example 6.4.** Let $G = \text{AGL}_3(2) = NH$, with $H = \text{GL}_3(2)$ and $N = \mathbb{F}_2^3$, acting on $X = \mathbb{F}_2^3$ by affine transformations $uh : x \mapsto hx + u$, for $x \in X$, $h \in H$ and $u \in N$. We can view $G$ as the subgroup of $\text{GL}_4(2)$ consisting of matrices of the following
block form

\[
\begin{bmatrix}
A & v \\
0 & 1
\end{bmatrix}
\]

where \( A \in \text{GL}_3(2) \) and \( v \in \mathbb{F}_2^3 \).

Consider the elements

\[
a = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\quad
u = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix},
\quad
s = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

of orders 2, 2 and 7 respectively.

It is well known and easy to show by direct calculation that \( H^1(H, N) \cong \mathbb{F}_2 \).

The standard complement \( H \) is generated by the elements \( a \) and \( s \), while \( au \) and \( s \) generate a nonstandard complement. If \( au \) were \( G \)-conjugate to any element of \( H \), it would be conjugate under \( N \) to some element of \( H \), as \( G = NH \), and that element would have to have the same image as \( au \) in \( G/N \). Thus \( au \) would be conjugate to \( a \). However they are not conjugate in \( G \), as \( a^{-1} \) and \( au^{-1} \) have different ranks. So there are no subgroups that are also nonstandard cocliques. This particular group can be shown directly to have the strict-EKR property using the method described in \([2]\).

7. Inner Distributions

We have proven that in any 2-transitive group the characteristic vector of any maximum intersecting set is a linear combination of the characteristic vectors of the canonical cocliques. For some groups, this fact has been used to show that the group has the strict-EKR property \([14, 21, 23]\). For other 2-transitive groups, that do not have the strict-EKR property, this fact has been used to characterize all the of the maximum intersecting sets \([24, 29]\). It does not seem feasible to characterize the maximum intersecting sets for a general 2-transitive group, but in this section, we will prove that the EKR-module property does give us some extra information about the structure of the maximum intersecting sets. The number of pairs of elements \((g, h)\) in a set that have \( hg^{-1} \) in a given conjugacy class is called the inner distribution of the set. We will show for a 2-transitive group, every maximum intersecting set has the same inner distribution. This gives information about the pair-wise intersection of elements within an intersecting set. In fact, this can been seen as a refinement of Lemma \([6, 7]\). In Lemma \([6, 7]\) it was shown that if a complement is a coclique, then every element is conjugate to an element in \( G_x \). The result on the inner distribution that we will prove in this section implies that if a subgroup is a maximum coclique, then it has the same number of elements in each conjugacy class of \( G \) as \( G_x \) has.

To do this, we will consider the conjugacy class scheme on the group \( G \). This is the association scheme that has the elements of \( G \) as its vertices and one class for each conjugacy class of \( G \). Two elements \( g, h \in G \) are adjacent in a class if \( hg^{-1} \) is in the corresponding conjugacy class. The matrices in this association scheme are indexed by the conjugacy classes, and denoted by \( A_c \). The idempotents are indexed by the irreducible representations of \( G \), and denoted by \( E_\phi \).

Let \( S \) be any maximum intersecting set in \( G \). Let \( v_S \) denote the characteristic vector of \( S \). Then the inner distribution of \( S \) is the sequence

\[
\left( \frac{v_S^T A_c v_S}{|S|} \right)_c
\]

taken over the conjugacy classes \( c \) of \( G \). This gives a count of how many pairs of elements in \( S \) are \( i \)-related in the association scheme. The dual distribution is
defined to be the sequence
\[
\left( \frac{v^T E_{\phi} v_S}{|S|} \right)_\phi
\]
taken over the irreducible representations $\phi$ of $G$.

Lemma 2.1 implies for any maximum intersecting set $S$ in $G$ that $v^T E_{\phi} v_S = 0$, unless $\phi = 1_G$ or $\phi = \psi_G$. From the comments following Lemma 2.1 we have
\[
\frac{v^T E_{1_G} v_S}{|S|} = \frac{|S|^2}{|G||S|} = \frac{1}{n}
\]
and
\[
\frac{v^T E_{\psi_G} v_S}{|S|} = 1 - \frac{1}{n}.
\]
Thus all maximal intersecting sets have the same dual distribution. It is known (see [16, Theorem 3.5.1]) that in any association scheme the following equation holds
\[
\sum_c \frac{v^T A_c v_S}{|S|} A_c = \sum_\phi \frac{v^T E_{\phi} v_S}{|S|} E_{\phi}.
\]
In particular, for any maximum intersecting set in $G$
\[
\sum_c \frac{v^T A_c v_S}{|S|} A_c = \frac{1}{n} E_{1_G} + \left( 1 - \frac{1}{n} \right) E_{\psi_G}.
\]
In the conjugacy class association scheme the sets $\{A_c\}$ and $\{E_\phi\}$ are both bases and the matrix of eigenvalues for the association scheme is a change-of-basis matrix. The above equation implies that the inner distribution for $S$ can be found by multiplying the dual distribution by the inverse of the matrix of eigenvalues. In particular, we obtain the following result.

**Lemma 7.1.** Let $G$ be a 2-transitive group and let $S$ be any maximum intersecting set in $G$. Then $S$ has the same inner distribution as the stabilizer of a point. □

8. Further Work

There have been many papers looking at specific groups to determine the structure of the maximum cocliques in the derangement graph. Theorem 1.3 gives a strong characterization of the maximum cocliques in any 2-transitive groups. We end with an open problem and a direction for further work.

Our only examples of groups that have non-canonical maximum cocliques in their derangement graphs, that are neither subgroups nor cosets, have the property that the derangement graphs are not connected. This leads to our remaining question.

**Question 8.1.** Are there 2-transitive groups $G$, with connected derangement graphs, that have a maximum coclique that is neither a subgroup nor a coset of a subgroup?

Finally, in this paper we only consider 2-transitive groups. The definition of the EKR-module property can be considered for any group, with the key difference being that, in general, the permutation module is not the sum of the trivial module and a single irreducible module. This situation will be more complicated, as there are transitive groups which satisfy neither the EKR property, nor the EKR-module property, nor the strict-EKR property. The first groups to consider are the rank 3 groups.

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