A fast randomized algorithm for computing a hybrid CUR-type Tucker decomposition

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Abstract

The paper develops a fast randomized algorithm for computing a hybrid CUR-type decomposition of tensors in the Tucker representation. Specifically, to obtain the factor matrices, random sampling techniques are utilized to accelerate the procedure of constructing the classical matrix decompositions, that are, the interpolatory decomposition and singular value decomposition. Compared with the non-random algorithm, the proposed algorithm has advantages in speed with lower computational cost while keeping a high degree of accuracy. We establish a detailed probabilistic error analysis for the algorithm and provide numerical results that show the promise of our approach.

Keywords: randomized algorithm; low-rank approximation; Tucker decomposition; interpolatory decomposition

1 Introduction

Recently, a hybrid CUR-type Tucker decomposition [1] has been proposed for tensors such that for a given tensor \( X \in \mathbb{R}^{n_1 \times \cdots \times n_d} \),

\[
X \approx G \times_1 C_1 \cdots \times_t C_t \times_{t+1} U_{t+1} \cdots \times_d U_d,
\]

(1.1)

where \( G \in \mathbb{R}^{r_1 \times \cdots \times r_d} \) is a core tensor, and the columns of matrices \( \{C_i\}_{i=1}^t \in \mathbb{R}^{n_i \times r_i} \) are extracted from the mode-\( i \) fibers of \( X \), preserving certain important features of the original tensor \( X \) such as sparsity, non-negativity and integer values, while orthonormal matrices \( \{U_j\}_{j=t+1}^d \in \mathbb{R}^{n_j \times r_j} \) are chosen to minimize the approximation error.

This hybrid decomposition can be considered as a generalization of the standard CUR-type tensor decomposition [2,4,9], which provides a decomposition in the Tucker format for a given tensor \( Y \in \mathbb{R}^{n_1 \times \cdots \times n_d} \) such that

\[
Y \approx H \times_1 C_1 \cdots \times_d C_d,
\]

(1.2)

where \( H \in \mathbb{R}^{r_1 \times \cdots \times r_d} \) is a core tensor and factors matrices \( C_i \) are formed from columns of the mode-\( i \) unfolding of \( Y \). The author in [9] introduced a method for choosing \( C_i \) based on the interpolatory decomposition [7], which is named as the higher-order interpolatory decomposition.

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(HOID), and detailed analysis of the computational costs and error bounds are also studied. The idea of the hybrid decomposition is that, in some applications, it is not always important to keep the original entries in all modes, and hence it is suggested to sample fibers only in some modes and not all of them. In this way, the approximation error obtained by the hybrid method is smaller than the HOID, and this difference increases as the tensor dimension $d$ increases. However, in practice, the cost of accurately computing the matrices $C_i$ and $U_j$ can be prohibitively expensive, making it unsuitable for large-scale applications.

It is known that randomized algorithms facilitate the procedure for matrix decompositions [5] and tensor decompositions [3, 10] not only by reducing the computational complexity levels of deterministic algorithms but also reducing the communication among different levels of memories, which is the main bottleneck in modern computing environments for handing large-scale data matrices or tensors. Motivated by this success, we introduce a randomized scheme for efficiently computing a hybrid CUR-type decomposition in the Tucker format, providing a low multilinear rank approximation to a given tensor. Specifically, there are two main computational stages involving the processing of generating the factor matrices $\{C_i\}_{i=1}^{t}$ and $\{U_j\}_{j=t+1}^{d}$. The first stage is to make a reduction in all modes of the unfolding matrices with random sampling methods [7] to construct a low-dimensional subspace that captures the actions of the unfolding matrices. The second stage can be completed with well-established deterministic methods, namely, the interpolatory decomposition (ID) and singular value decomposition (SVD). Compared with the non-random algorithm in [1], our algorithm allows for a comparable accuracy with much lower costs and will be more computationally efficient on large-scale data. Details of the algorithm, theoretical analysis and numerical results are provided to show the effectiveness of our approach.

2 Hybrid CUR-type Tucker decomposition

We first review basic notations and concepts involving tensors and more detailed properties can be found in [6]. The norm of a $d$-dimensional tensor $X$ with entries $x_{j_1,\ldots,j_d}$ is defined as $\|X\|_F = \sqrt{\sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1}^{n_d} |x_{j_1,\ldots,j_d}|^2}$. We denote by matrix $X(k) \in \mathbb{R}^{n_k \times \prod_{j \neq k} n_j}$ the $k$th mode unfolding of the tensor $X$. The multilinear rank of $X$ is a tuple $(r_1,\ldots,r_d)$ where $r_k$ is the rank of $X(k)$. The $k$-mode multiplication of the tensor $X$ with a matrix $U \in \mathbb{R}^{m \times n_k}$ results in a tensor $Y \in \mathbb{R}^{n_1 \times \cdots \times n_{k-1} \times m \times n_{k+1} \times \cdots \times n_d}$ such that

$$Y_{i_1,\ldots,i_{k-1},i,i_{k+1},\ldots,i_d} = (X \times_k U)_{i_1,\ldots,i_{k-1},i,i_{k+1},\ldots,i_d} = \sum_{i_k=1}^{n_k} x_{i_1,\ldots,i_d} u_{i,i_k}. \quad (2.1)$$

We are now set to introduce several algorithms that produce a multilinear rank-$(r_1,r_2,\ldots,r_d)$ approximation to tensors in the Tucker format, i.e. the higher order singular value decomposition (HOSVD) [6], HOID and the hybrid algorithm.

The HOSVD algorithm returns a core tensor $\mathcal{U} \in \mathbb{R}^{r_1 \times \cdots \times r_d}$ and a set of matrices $U_j \in \mathbb{R}^{n_j \times r_j}$, containing the $r_j$ leading left singular vectors of $X(j)$, $j = 1,\ldots,d$ such that

$$X \approx U_1 \times_1 U_2 \times_2 \cdots \times_d U_d.$$
hybrid decomposition is that we can keep the fibers of the original tensor in only one mode, or in more, but not all modes. As in (1.2), fibers from the first $t$ modes are preserved in matrices $\{C_i\}_{i=1}^t$, which are the representative of the mode-$i$ unfolding matrices $X_{(i)}$, while matrices $\{U_j\}_{j=t+1}^d$ contain first $r_j$ left singular vectors of $X_{(j)}$, chosen to minimize the approximation error. We summarize the hybrid approach in Algorithm 1 for the case that only the first mode of the original fibers is preserved.

**Algorithm 1** Hybrid algorithm [1]

Require: $A \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ and desired multilinear rank $(r_1, r_2, \ldots, r_d)$.

1: for $i = 2, \ldots, d$ do
2: Compute matrix $U_i$ containing the leading $r_i$ left singular vectors of $A_{(i)}$.
3: end for
4: Perform the PQR decomposition $A_{(1)}P = QR$.
5: Compute factor matrix $C = X_{(1)}P(:, 1 : r_1) \in \mathbb{R}^{n_1 \times r_1}$.
6: Compute core tensor $G \in \mathbb{R}^{r_1 \times \cdots \times r_d}$ as $G = A \times_1 C^\dagger \times_2 U_2^T \cdots \times_d U_d^T$.

In Algorithm 1, we can also choose to extract fibers from more than one mode of $A$ and the error also increases as the number of modes in which the original fibers are preserved decreases. However, the cost of accurately computing the PQR or the HOSVD can be prohibitively expensive, making it difficult for large-scale applications. Inspired by the randomized methods in [7, 8], we are motivated to design a randomized algorithm for computing the hybrid CUR-type decomposition of tensors.

### 3 Randomization for hybrid decomposition

One can observe that the task of computing the hybrid CUR-type Tucker decomposition can be split naturally into two stages. The first is to construct matrices $C_i$ to capture the features of the mode-$i$ tensor unfolding by the ID. The second is to obtain matrices $U_j$ by performing the SVD for the mode-$j$ tensor unfolding. The two stages can be executed separately with random sampling methods [5, 7].

For given matrix $A \in \mathbb{R}^{m \times n}$, the randomized algorithms in [7] yield approximate ID and SVD with the following error bounds,

\[
\|CB - A\|_2 \leq (\sqrt{2lm\beta^2\gamma^2} + 1(\sqrt{4k(n - k)} + 1 + 1) + \beta\gamma\sqrt{2lm}\sqrt{4k(n - k)} + 1)\sigma_{k+1}, \quad (3.1)
\]

\[
\|Z\Sigma V^T - A\|_2 \leq (2\sqrt{2lm\beta^2\gamma^2} + 1 + 2\sqrt{2lm\beta\gamma})\sigma_{k+1}, \quad (3.2)
\]

with probability not less than $\chi = 1 - \frac{1}{\sqrt{2\pi(l-k+1)}(\frac{e}{l-k+1})^{l-k+1}} - \frac{1}{2(\gamma^2-1)\sqrt{\pi m\gamma^2}(2\gamma^2-1)^m}$, where $C$ contains $k$ columns of $A$, $B \in \mathbb{R}^{k \times n}$, and matrices $Z \in \mathbb{R}^{m \times k}$ and $V \in \mathbb{R}^{n \times k}$ are orthonormal. In addition, $l$ is a user-specified integer with $l = k + p$, $p$ is the oversampling parameter, $\sigma_{k+1}$ is the $(k+1)$th largest singular value of $A$ and $\beta, \gamma$ are positive real numbers such that $\gamma > 1$. For example, choosing $\beta = 3/4, \gamma^2 = 5$ and $p = 20$, we obtain the above error bounds with
probability not less than $1 - 10^{-17}$, and [7] Table 1 contains the similar results obtained by taking other values for $l$, $\beta$ and $\gamma$.

Analysis results including the computational costs and numerical examples illustrate that the two randomized algorithms in [7] accelerate the approximation of matrices via constructing approximations to the SVD and the ID. We present our randomized approach in Algorithm 2, where we exploit the randomization techniques in [7] to accelerate the process of the SVD and the ID to each mode unfolding, providing an approximation $\hat{A}$ in a hybrid CUR-type Tucker format for a given tensor $A$ such that $\hat{A} = G \times_1 C_1^\top \cdots \times_t C_t^\top U_{t+1}^T \cdots \times_d U_d^T$.

**Algorithm 2 Randomized hybrid algorithm**

| Require: $A \in \mathbb{R}^{n_1 \times \cdots \times n_d}$, multilinear rank $(r_1, r_2, \ldots, r_d)$ and oversampling parameter $p$. |
|---|
| 1: for $i = 1, \ldots, t$ do |
| 2: Draw random Gaussian matrix $\Omega \in \mathbb{R}^{(r_i + p) \times n_i}$. |
| 3: Compute $Y = \Omega A_{(i)}$. |
| 4: Apply the pivoted Gram-Schmidt process to the columns of $Y$, leaving the factorization $YP = QR$, |
| where $P$ is a permutation matrix, $Q \in \mathbb{R}^{(r_i + p) \times r_i}$ is orthonormal, and $R \in \mathbb{R}^{r_i \times \prod_{k \neq i} n_k}$ is upper triangular. |
| 5: Form $C_i = A_{(i)}P(:, 1 : r_i)$. |
| 6: end for |
| 7: for $j = t + 1, \ldots, d$ do |
| 8: Draw random Gaussian matrix $\Omega \in \mathbb{R}^{(r_j + p) \times n_i}$. |
| 9: Compute $M = \Omega A_{(j)}$. |
| 10: Compute the SVD of $M$, |
| $M = Z \Sigma V^T$, |
| where $Z \in \mathbb{R}^{\prod_{k \neq j} n_k \times (r_j + p)}$ and $V \in \mathbb{R}^{(r_j + p) \times (r_j + p)}$ are orthonormal, and $\Sigma \in \mathbb{R}^{(r_j + p) \times (r_j + p)}$ is diagonal. |
| 11: Form $Q = Z(:, 1 : r_j)$. |
| 12: Compute $T = A_{(j)}Q$. |
| 13: Compute matrix $U_j$ containing $r_j$ left singular vectors of $A_{(j)}$. |
| 14: end for |
| 15: Compute the core tensor $G \in \mathbb{R}^{r_1 \times \cdots \times r_d}$ as |
| $G = A \times_1 C_1^\top \cdots \times_t C_t^\top U_{t+1}^T \cdots \times_d U_d^T$. |

The following theorem quantifies the error of the approximate hybrid CUR-type Tucker decomposition computed by Algorithm 2.

**Theorem 3.1.** Let $A \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ with $n_i \leq \prod_{k \neq i} n_k$ for $1 \leq i \leq d$. Suppose that $p$ is an oversampling parameter, $\beta$ and $\gamma$ are positive numbers such that $\gamma > 1$, and $\phi = 1 - \frac{1}{\sqrt{2\pi(p+1)}} \left(\frac{e}{(p+1)^{\beta}}\right)^{p+1} - \frac{1}{2(\gamma^2-1)\sqrt{\pi I^2 \gamma^2}} \left(\frac{2^2}{e^{2/1}}\right)^I$, $I' = \min\{n_1, \ldots, n_d\}$. Then we have the
following error bound,
\[ \|A - \hat{A}\|_F^2 \leq \sum_{i=1}^{t} n_i \left[ \sqrt{2(r_i + p)n_i\beta^2\gamma^2 + 1} \left( \sqrt{4r_i(\prod_{k \neq i} n_k - r_i)} + 1 \right) \right]^2 + \beta\gamma \sqrt{2(r_i + p)n_i} \sqrt{4r_i(\prod_{k \neq i} n_k - r_i)} + 1 \left( \sqrt{4r_i(\prod_{k \neq i} n_k - r_i)} + 1 \right) \sigma_{r_{i+1}}^2 \\
+ \sum_{j=t+1}^{d} n_j \left( 2\sqrt{2(r_j + p)n_j\beta^2\gamma^2 + 1} + 2\sqrt{2(r_j + p)n_j\beta\gamma} \right)^2 \sigma_{r_{j+1}}^2 \]

with probability not less than \( \phi \), where \( \sigma_{r_{i+1}} \) is the \( (r_i + 1) \)th largest singular value of \( A(i) \).

**Proof.** Using the property of the mode-\( n \) product, we have
\[ \|A - \hat{A}\|_F^2 = \|A - A \times_1 (C_1C_1^T) \times_2 \cdots \times_t (C_tC_t^T) \times_{t+1} (U_{t+1}U_{t+1}^T) \times \cdots \times_d (U_dU_d^T)\|_F^2. \]
Notice that \( C_tC_t^T \) and \( U_dU_d^T \) are orthogonal projections. Thus, using the following result from [9, Lemma 2.1]:
\[ \|X - X \times_1 \Pi_1 \times_2 \cdots \times_d \Pi_d\|_F^2 \leq \sum_{i=1}^{d} \|X - X \times_i \Pi_i\|_F^2 \]
for projections \( \Pi_1, \ldots, \Pi_d \), it follows that
\[ \|A - \hat{A}\|_F^2 \leq \sum_{i=1}^{t} \|A - A \times_i (C_tC_t^T)\|_F^2 + \sum_{j=t+1}^{d} \|A - A \times_j (U_dU_d^T)\|_F^2 \]
\[ = \sum_{i=1}^{t} \|(I_{n_i} - C_tC_t^T)A(i)\|_F^2 + \sum_{j=t+1}^{d} \|(I_{n_j} - U_dU_d^T)A(j)\|_F^2. \]

As described in Algorithm 2, matrices \( A(i), 1 \leq i \leq t \) and \( A(j), t + 1 \leq j \leq d \) own the interpolatory factorization \( A(i) = C_iB_i + E_i \) and the SVD such that \( A(j) = U_j\Sigma_jV_j^T + H_j. \) Therefore, we obtain that
\[ \|(I_{n_i} - C_tC_t^T)A(i)\|_F^2 = \|(I_{n_i} - C_tC_t^T)(C_iB_i + E_i)\|_F^2 = \|(I_{n_i} - C_tC_t^T)E_i\|_F^2 \leq n_i\|E_i\|_F^2, \]
\[ \|(I_{n_j} - U_dU_d^T)A(j)\|_F^2 = \|(I_{n_j} - U_dU_d^T)(U_j\Sigma_jV_j^T + H_j)\|_F^2 = \|(I_{n_j} - U_dU_d^T)H_j\|_F^2 \leq n_j\|H_j\|_F^2. \]

Plugging these two inequalities into \( (3.3) \), and using the result in \( (3.1) \) and \( (3.2) \), we obtain the desired result. \( \square \)

### 4 Numerical examples

In this section, we implement the randomized hybrid algorithm (Algorithm 2) and the hybrid algorithm (Algorithm 1) on the function related tensors (a more detailed description can be found in [1][8][9])
\[ A(i_1, \ldots, i_d) = \frac{1}{i_1 + i_2 + \cdots + i_d}, \quad B(i_1, \ldots, i_d) = \frac{1}{i_1 + 2\cdot i_2 + \cdots + d\cdot i_d}. \]
and show that the former is numerically advantageous over the latter in terms of the computing time in seconds (denoted as CPU) without a sacrifice in quality (measured by the relative error, denoted as Err). All computations are carried out in MATLAB R2020a on a computer with an AMD Ryzen 5 processor and 16 GB RAM.

We first compare the accuracy of the two methods. For each algorithm, we use the target rank \((r, r, \ldots, r)\), where \(r\) varies from 1 to 10, and we set the oversampling parameter \(p = 5\), \(t = 2\), and \(I_i = 50\) for \(1 \leq i \leq 3\). The relative errors are plotted in Figure 1, where we can see that there is almost no difference between the approximation error of the two algorithms.

![Figure 1: Relative error when the target rank varies. Left: tensor \(B\). Right: tensor \(A\).](image)

We then report the CPU and relative errors of the two algorithms as the size of the dimension increases and the corresponding results are displayed in Table 1. In this experiment, we fix the target rank to be \((5, 5, 5)\) and the oversampling parameter as \(p = 5\), while the original fibers only in the first mode are preserved. We can see that the two algorithms have comparable relative errors. However, there is a significant improvement in the CPU time obtained by the randomized algorithm.

| \(I_1 = I_2 = I_3\) | Method     | Err of \(B\)      | CPU of \(B\)   | Err of \(A\)      | CPU of \(A\)   |
|----------------------|------------|-------------------|----------------|-------------------|----------------|
| 50                   | Algorithm 1| 9.9927e-05        | 1.821          | 2.5769e-04        | 1.351          |
|                      | Algorithm 2| 1.0038e-04        | 0.02379        | 2.6701e-04        | 0.02580        |
| 100                  | Algorithm 1| 3.4099e-04        | 40.98          | 8.6822e-04        | 40.93          |
|                      | Algorithm 2| 3.4162e-04        | 0.05701        | 8.4108e-04        | 0.04297        |
| 150                  | Algorithm 1| 6.0663e-04        | 59.81          | 1.4107e-03        | 42.04          |
|                      | Algorithm 2| 6.0697e-04        | 0.05753        | 1.4459e-03        | 0.05863        |
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