Thermofield dynamics and twisted Poincaré symmetry on Moyal space-time

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On Moyal space-time, one can implement twisted Poincaré symmetry with the resultant modification of symmetrization and anti-symmetrization postulates for bosons and fermions. We develop the thermofield approach of Umezawa and Takahashi on such a spacetime preserving the twisted Poincaré symmetry of the underlying quantum field theory (QFT). Implications of this twisted Poincaré symmetry for QFT’s at finite temperature are pointed out.

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I. INTRODUCTION

Quantum fields on noncommutative spacetimes like the Moyal spacetime have been studied in recent times extensively [1, 2]. So also are there studies on fuzzy geometries like coadjoint orbits of compact Lie groups [3]. These studies indicate novel modifications of symmetrisation and antisymmetrisation postulates conventionally used for bosons and fermions. They also establish new phases which are novel [4–6]. The existence of such features makes it interesting to study quantum field theories (QFTs) on the Moyal spacetime at finite temperature. A study along these line was initiated in [7] using linear response theory. Here we will instead examine quantum fields at finite temperature on the noncommutative Moyal space-time using Umezawa and Takahashi’s thermofield dynamics approach [8]. Thus this work is complemented by [7] and also by [9].

The thermofield approach uses “mirror” fields and Bogoliubov transformations to construct the thermal “vacuum” state. Once such a state is constructed the various expectation values of operators in this state give the thermal distributions effectively.

In Sec. II we will briefly review thermofield dynamics for commutative space-time. In Sec. III we will extend the above to the Moyal space-time. In Sec IV, we present an analysis of interacting field theories in the above approach.

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II. REVIEW OF THERMOFIELD DYNAMICS

In an important contribution, Umezawa and Takahashi [8] constructed a quantum field theory known as thermofield dynamics (TFD), in which the thermal average of an observable is given by the vacuum expectation value in an extended Hilbert space. This is achieved by constructing a thermal state \( |0(\beta)\rangle \) such that

\[
\langle A \rangle = \frac{\text{Tr} e^{-\beta H} A}{\text{Tr} e^{-\beta H}} = \langle 0(\beta) | A | 0(\beta) \rangle.
\]

(1)

where \( \beta = \frac{1}{kT} \), \( k \) is the Boltzman constant and \( T \) is the temperature.

For this one starts with a Hamiltonian \( H \) acting on a Hilbert space \( \mathcal{H} \) and following Umezawa and Takahashi extend the latter to \( \mathcal{H} \otimes \tilde{\mathcal{H}} \). Here \( \tilde{\mathcal{H}} \) is also a Hilbert space isomorphic to \( \mathcal{H} \). The total Hamiltonian \( \tilde{H} \) is given by

\[
\tilde{H} = H - \hat{H}
\]

(2)

where we will give the details of \( \hat{H} \) below.

If \( P_\mu \) is the four-momentum for \( \mathcal{H} \) with \( P_0 = H \), then \( \tilde{P}_\mu \) on \( \mathcal{H} \otimes \tilde{\mathcal{H}} \) has a similar form (see below) and \( \tilde{P}_\mu = P_\mu - \hat{P}_\mu \).

In the above \( H \) and \( \tilde{H} \) (\( P_\mu \) and \( \tilde{P}_\mu \)) we mean \( H \otimes I \) and \( I \otimes \tilde{H} \) (\( P_\mu \otimes I \) and \( I \otimes \hat{P}_\mu \)) while \( \hat{P}_0 = \hat{H} \).

In TFD, the degrees of freedom are doubled. To every observable operator \( O \), we assign a tilde conjugate operator \( \tilde{O} \). These conjugations obey the following rules:

\[
(O_1 O_2) = \tilde{O}_1 \tilde{O}_2, \quad (cO) = c' \tilde{O}, \quad (\tilde{O}) = (\tilde{O})^\dagger, \quad (O) = \pm O \quad (+ \text{ for boson}, \quad - \text{ for fermion})
\]

(3)

The untwisted free field \( \phi(x) \left( \tilde{\phi}(x) \right) \) of mass \( m \) has the conventional mode expansion:

\[
\phi(x) \left( \tilde{\phi}(x) \right) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega(k)} \left( c_k (\tilde{\phi}_k) e^{-ik \cdot x} + \tilde{c}_k^\dagger (\tilde{\phi}_k^\dagger) e^{ik \cdot x} \right), \quad \omega_k = + \sqrt{k^2 + m^2}.
\]

(4)

In the above \( c_k, c_k^\dagger \) and \( \tilde{c}_k, \tilde{c}_k^\dagger \) are the standard Fock space annihilation and creation operators of \( \mathcal{H}, \tilde{\mathcal{H}} \).

\[
\left[ c_k, c_k^\dagger \right] = \left[ \tilde{c}_k, \tilde{c}_k^\dagger \right] = (2\pi)^3 (2\omega(k)) \delta^3(k - k').
\]

(5)

Then \( |0(\beta)\rangle \) is the “thermal vacuum state” of annihilation and creation operators \( \alpha_k, \alpha_k^\dagger \) and \( \tilde{\alpha}_k, \tilde{\alpha}_k^\dagger \). They are obtained by a Bogoliubov transformation of the creation (annihilation) operators \( c_k (\tilde{c}_k^\dagger) \) as follows:

\[
\alpha_k = e^{-iG} c_k e^{iG}, \quad \tilde{\alpha}_k = e^{-iG} \tilde{c}_k e^{iG}
\]

(6)

where

\[
G = -i \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega(k)} \Theta(k) (c_k \tilde{c}_k^\dagger - \tilde{c}_k^\dagger c_k^\dagger)
\]

(7)

Here

\[
\tanh^2 \Theta(k) = e^{-\beta \omega(k)},
\]

for free fields.

The exact expression for \( \alpha_k, \tilde{\alpha}_k \) are:

\[
\alpha_k = \cosh \Theta(k)c_k - \sinh \Theta(k)\tilde{c}_k^\dagger
\]

\[
\tilde{\alpha}_k = \cosh \Theta(k)\tilde{c}_k - \sinh \Theta(k)c_k^\dagger
\]
One can define Bogoliubov transformed field $\phi_D(x)\left(\tilde{\phi}_D(x)\right)$:

$$\phi_D(x)\left(\tilde{\phi}_D(x)\right) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega(k)} \left( \alpha_k (\tilde{\alpha}_k) e^{-ik.x} + \alpha_k^\dagger (\tilde{\alpha}_k^\dagger) e^{ik.x} \right).$$ \hspace{1cm} (8)

The thermal “vacuum” state condition is that $|0(\beta)\rangle$ is annihilated by $\alpha_k, \tilde{\alpha}_k$:

$$\alpha_k |0(\beta)\rangle = \tilde{\alpha}_k |0(\beta)\rangle = 0.$$ \hspace{1cm} (9)

It is given by

$$|0(\beta)\rangle = U(\Theta)|0, \tilde{0}\rangle = \int \frac{d^3k}{2\omega(k)} \sum_{n_k} \frac{1}{\sqrt{Z_k(\beta)}} e^{-\frac{\beta E_n(\omega_k)}{2}} |n_k, \tilde{n}_k\rangle,$$

$$E_n(\omega_k) = n\omega(k), \quad U(\Theta) = e^{-iG}.$$ \hspace{1cm} (10)

In the above $Z_k(\beta) = Tr_{n_k} e^{-\beta H}$ (with trace over all states with fixed $n_k$) is the normalisation factor and $E_n(\omega_k)$ is the energy of a state with $n_k$ particles all of momentum $k$. $|n_k, \tilde{n}_k\rangle$ is the state with each of $n_k$ particles, all of momenta $k$, in $\mathcal{H}$ and $\tilde{\mathcal{H}}$. (Thus $E_1(\omega(k)) = \omega(k).$) With this definition of the thermal vacuum, it is obvious that Eq(1) follows.

The operator $N_k = c_k^T c_k - c_k^T \tilde{c}_k$ commutes with $G$. Hence:

$$N_k |0(\beta)\rangle = |0(\beta)\rangle |N_k = 0.$$ \hspace{1cm} (11)

Given the above it is easy to work out the Green’s functions for the free theory at finite temperature. For this purpose consider the two fields $\phi, \tilde{\phi}$ as a column $\Phi$:

$$\Phi(x) = \begin{pmatrix} \phi(x) \\ \tilde{\phi}(x) \end{pmatrix}.$$ \hspace{1cm} (12)

Then

$$i \ G_0(x, y) = \langle 00|T(\Phi(x)\Phi^T(y))|00\rangle$$

$$= \begin{pmatrix} \langle 00|T(\phi(x)\phi(y))|00\rangle & 0 \\ 0 & \langle 00|T(\tilde{\phi}(x)\tilde{\phi}(y))|00\rangle \end{pmatrix}$$

$$= \frac{1}{(2\pi)^4} \int d^4k \ (iG_0(k)) \ e^{-ik.(x-y)}$$ \hspace{1cm} (13)

where

$$G_0(k) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k^2 - m^2 - i\epsilon} \end{pmatrix}.$$ \hspace{1cm} (14)

On the other hand the thermal Green’s function is given by:

$$i \ G_\beta(x, y) = \langle 0, \beta|T(\Phi(x)\Phi(y))|0, \beta\rangle$$

$$= \langle 00|T(U(\Theta)^\dagger \Phi(x)U(\Theta)U(\Theta)^\dagger \Phi(y))U(\Theta)|00\rangle$$

$$= \frac{1}{(2\pi)^4} \int d^4k \ (iG_\beta(k)) \ e^{ik.(x-y)}$$ \hspace{1cm} (15)

where $U(\Theta) = e^{-iG}$ is defined as in Eqs.(6) and

$$G_\beta(k) = \begin{pmatrix} \frac{1}{k^2 - m^2 + i\epsilon} - 2\pi i\delta(k^2 - m^2)N(k) & 2\pi i\delta(k^2 - m^2)N(k) \\ 2\pi i\delta(k^2 - m^2)N(k) & \frac{1}{k^2 - m^2 - i\epsilon} - 2\pi i\delta(k^2 - m^2)N(k) \end{pmatrix}.$$ \hspace{1cm} (16)
where \( N(k) = \sinh^2 \Theta(k) \).

Now one can diagonalise the above thermal Green’s function by a suitable linear combination of \( \phi \) and \( \tilde{\phi} \) fields. For this purpose one starts from Eq(13) and inserts \( U(\Theta) \dagger U(\Theta) \) as follows:

\[
i G_0(x, y) = \langle 0 \tilde{\phi}_0 | T(\Phi(x) \Phi^T(y) U(\Theta) \dagger U(\Theta)) | 0 \tilde{\phi}_0 \rangle
= \langle 0(\beta) | T(\Xi_D(x) \Xi_D^T(y)) | 0(\beta) \rangle
\]

(20)

where we have defined

\[
\Xi_D(x) = U(\Theta) \Phi(x) U(\Theta) \dagger = \left( \begin{array}{c} \chi_D(x) \\ \tilde{\chi}_D(x) \end{array} \right)
\]

which agrees with Eq(8). \((U(\Theta)\) conjugates each entry in the column of \( \Phi(x) \).) In the next section, we will see how this formalism extends to the Moyal plane and what are the changes that can be expected.

III. THERMOFIELD DYNAMICS ON THE MOYAL PLANE

In the Moyal plane \( \mathcal{A}_\theta(R^4) \) described by the commutator,

\[
[X_\mu, X_\nu] = i \theta_{\mu\nu},
\]

(21)

the ideas of TFD can be incorporated.

We will first point out some important features about the Moyal plane and the action of the symmetry transformations on it.

The multiplication map \( m_\theta \) in the algebra \( \mathcal{A}_\theta(R^4) \) is defined by the * product,

\[
m_\theta(f \otimes g)(x) = f(x) * g(x) = \left( f e^{i\Theta_{\mu\nu} \overset{\leftrightarrow}{\partial_\mu} \overset{\leftrightarrow}{\partial_\nu}} g \right)(x)
\]

(22)

where \( f, g \in \mathcal{A}_\theta(R^4) \). Further the Poincaré group action on tensor products \( \mathcal{A}_\theta(R^4) \otimes \mathcal{A}_\theta(R^4) \otimes \cdots \otimes \mathcal{A}_\theta(R^4) \) cannot be implemented in the conventional way. What is required is a twisted action of its group algebra on these tensor products. This leads to important changes in the algebra of creation/annihilation operators and statistics. The outcome are the relations:

\[
a_p a_q = a_q a_p e^{i p \wedge q}, \quad a_p a_q^\dagger = e^{-i p \wedge q} a_q^\dagger a_p + 2 p_0 \delta^3(p - q)
\]

(23)

where \( A \wedge B = A_\mu \theta^{\mu\nu} B_\nu \). For details see Balachandran et al. [10].

The operators \( a_p, a_p^\dagger \) can be obtained from the standard annihilation and creation operators \( c_p, c_p^\dagger \) through the following “dressing transformation”:

\[
a_p = c_p e^{-i p \wedge P} \quad a_p^\dagger = c_p^\dagger e^{i p \wedge P}
\]

(24)

The twisted quantum field \( \phi_\theta \) can also be obtained by a dressing transformation from the untwisted quantum field:

\[
\phi_\theta(x) = \phi_0(x) e^{i P \wedge P}, \quad \phi_0(x) := \phi(x)
\]

(25)

Here \( P_\mu \) is the total momentum operator.
We now discuss how these ‘dressing transformations’ generalise to thermal field theories. We must first construct the twisted analogues $\alpha_k, \alpha_k^\dagger, \tilde{\alpha}_k, \tilde{\alpha}_k^\dagger$ of the creation and the annihilation operators. For this purpose, we need to generalise Eqs.(6) and (7). The important point to note in this context is that, the total momentum operator in thermofield theory is the difference of momentum operators for $\phi$ and $\tilde{\phi}$ fields:

$$P_\mu(\phi_0, \tilde{\phi}_0) = P_\mu(\phi_0) - P_\mu(\tilde{\phi}_0) \equiv P_\mu - \tilde{P}_\mu$$

This is required for the dressing transformation since $\alpha_k$ for example involves the annihilation operators of $\phi$ and creation operators of $\tilde{\phi}$ fields.

The required Bogoliubov transformations for the twisted case are:

$$\alpha_k^\Theta = \left( \cosh \Theta(k) \alpha_k - \sinh \Theta(k) \tilde{\alpha}_k^\dagger \right) e^{-i k \wedge P(\phi, \tilde{\phi})}$$
$$\tilde{\alpha}_k^\Theta = \left( \cosh \Theta(k) \tilde{\alpha}_k - \sinh \Theta(k) \alpha_k^\dagger \right) e^{-i k \wedge P(\phi, \tilde{\phi})}$$

and their adjoints.

The twisted field at zero temperature is:

$$\phi_\theta(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p_0} \left( a_p e^{-ip.x} + a_p^\dagger e^{ip.x} \right)$$
$$\tilde{\phi}_\theta(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p_0} \left( \tilde{a}_p e^{-ip.x} + \tilde{a}_p^\dagger e^{ip.x} \right).$$

In the same way we can define the twisted ‘diagonal’ fields at finite temperature:

$$\chi_{\theta D}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p_0} \left( \alpha_p^\Theta e^{-ip.x} + \alpha_p^\dagger e^{ip.x} \right)$$
$$\tilde{\chi}_{\theta D}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p_0} \left( \tilde{\alpha}_p^\Theta e^{-ip.x} + \tilde{\alpha}_p^\dagger e^{ip.x} \right).$$

As the number operators commute with the twist, the thermal vacuum is independent of $\theta_{\mu\nu}$. The twisted thermal vacuum $|0(\beta)\rangle$ is defined by

$$\alpha_k^\Theta |0(\beta)\rangle = \tilde{\alpha}_k^\Theta |0(\beta)\rangle = 0$$

For later use, we also define the twisted finite temperature fields $\phi_{\theta,\beta}$:

$$\phi_{\theta,\beta} = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p_0} \left( a_p e^{-ip.x} + a_p^\dagger e^{ip.x} \right)$$

In the above in $\phi_{\theta,\beta}(x)$ the subscript $\beta$ is added to emphasize that the twist involves $P_\mu(\phi_0, \tilde{\phi}_0)$ (Eq. 26) which annihilates the thermal vacuum.

We also would like to remark that in our approach to gauge fields, the latter are not twisted and are associated with commutative spacetime. So, its fields operators are not twisted.

**A. Wightman functions and propagators**

One can obtain the Wightman functions for the “free” theory:

$$\langle 0(\beta)|(\phi_\theta(x)\phi_\theta(y))|0(\beta)\rangle = \langle 0, 0 | (U^\dagger(\Theta)\phi_\theta(x)\phi_\theta(y)U(\Theta)) | 0, 0 \rangle$$

$$= i \ W_0^\beta(x-y) \ = i e^{-i\frac{1}{2} \Theta_{x^2+y^2}} \ W_0^\beta(x-y) = i \ W_0^\beta(x-y).$$
Hence we get $\Delta^\beta_{F,\theta}(x-y)$, the Feynman propagator as:

$$\Delta^\beta_{F,\theta}(x-y) = \langle (0(\beta) | T(\phi_\theta(x)\phi_\theta(y)) | 0(\beta)) \rangle = \int \frac{d^4k}{(2\pi)^4} (i\Delta^\beta_{F,\theta}(k)) e^{-ik.(x-y)}$$

(33)

where

$$\Delta^\beta_{F,\theta}(k) = \frac{1}{k^2 - m^2 + i\epsilon} - 2\pi i\delta(k^2 - m^2)N(k) \equiv \Delta^\beta_{F,\theta=0}(k)$$

(34)

The above behaviour of 2-point function is independent of $\theta$ because of translational invariance of the theory on the Moyal spacetime. (We assume that $H$ and hence $\hat{H}$ are spacetime translational invariant.) Even though the two-point function is $\theta$-independent, the higher order functions do depend on $\theta$ and do not factorize into sums of products of two-point functions as it happens in zero temperature field theory.

The 4-point function already reflects the important difference between Moyal and commutative space time. Consider

$$\mathcal{F}(x_1, x_2, x_3, x_4) = \langle (0(\beta) | (\phi_\theta(x_1)\phi_\theta(x_2)\phi_\theta(x_3)\phi_\theta(x_4)) | 0(\beta)) \rangle_{\theta}$$

$$= \langle (0, 0) | (U(\Theta)\phi_\theta(x_1)\phi_\theta(x_2)\phi_\theta(x_3)\phi_\theta(x_4)U(\Theta)) | 0, 0 \rangle$$

$$= \int \prod_i \frac{d^4p_i}{(2\pi)^4} W^\beta_\theta(p_1, p_2, p_3, p_4) e^{i\sum p_i \cdot x_i}.$$ 

Then

$$W^\beta_\theta(p_1, p_2, p_3, p_4) = W^\beta_\theta(p_1)W^\beta_\theta(p_2) [\delta(p_1 - p_2)\delta(p_3 - p_4) + \delta(p_1 - p_4)\delta(p_2 - p_3)]$$

$$+ e^{ip_1 \cdot x} W^\beta_\theta(p_1)W^\beta_\theta(p_2)\delta(p_1 - p_3)\delta(p_2 - p_4)$$

(35)

In the above $W^\beta_\theta(p) = \frac{1}{(2\pi)^4} \int e^{-ik \cdot x} W^\beta_\theta(x) = \frac{1}{(2\pi)^4} \int e^{-ik \cdot x} W^\beta_0(x)$ The higher order functions can also be worked out systematically.

Interestingly Eq.(35) is the Bogoliubov transformed 4-point function of the twisted Poincaré invariant Wightman function given in an earlier zero temperature field theory[10].

Remark: The discussion till now has dealt with free or possibly interaction representation fields. Remarks on Heisenberg fields will be made later.

**IV. SCATTERING THEORY AT FINITE TEMPERATURE**

There are several equivalent approaches to scattering theory on commutative spacetimes, a few of the well-known being the following:

1. The interaction representation scattering theory
2. The LSZ formalism.
3. The Yang-Feldman approach.

These approaches are not necessarily equivalent on noncommutative spacetimes even at zero temperature [11, 12]. Elsewhere [13, 14] we have discussed the first two approaches on the Moyal spacetime at zero temperature in detail. We shall now generalise them to the finite temperature thermofield theory.

**A. Scattering amplitudes in the interaction representation**

As a preliminary, we note that the free Hamiltonian is not affected by $\theta_{\mu\nu}$ or $\beta$, it is the same as for $\theta_{\mu\nu} = T = 0$ for both matter and gauge fields.
Without gauge fields:
We can consider the case of the real scalar field \( \phi_0 \) for \( \theta_{\mu\nu} = 0, T = 0 \) an illustration. In the interaction representation, it is a free field. Consider an interaction Hamiltonian \( H_I \) such as:

\[
H_I = \lambda \int d^3x : \phi_0^\dagger \phi_0 : 
\]

(36)

If \( \theta_{\mu\nu} \neq 0 \), but \( T = 0 \), it becomes \( H_I^\theta \) which for eq.(36) is

\[
H_I^\theta = \lambda \int d^3x : \phi_0 \phi_0 \phi_0 \phi_0 : 
\]

(37)

where \( \phi_0 \) is defined by eq.(29) and normal ordering is with regard to \( a_p, a_p^\dagger \) of eq.(24).

If both \( \theta_{\mu\nu} \) and \( T \) are nonzero, then \( H_I^\theta \) becomes \( H_I^{\theta,\beta} (H^{\theta,\infty} = H_I^\theta) \) which for eq.(37) is:

\[
H_I^{\theta,\beta} = \lambda \int d^3x : \phi_{\theta,\beta} \phi_{\theta,\beta} \phi_{\theta,\beta} : 
\]

(38)

where the normal ordering is with respect to the \( a_k^\beta, a_k^{\beta \dagger} \) operators of eq.(37).

The interaction representation S-matrix \( S_I^\theta \) for \( T = 0 \) is:

\[
S_I^\theta = T \exp \left( -i \int dt H_I^\theta \right). 
\]

(39)

We showed in earliar papers that \( S_I^\theta \) is independent of \( \theta \):

\[
S_I^\theta = S_I^0 = S_I = T \exp \left( -i \int dt H_I \right). 
\]

(40)

The proof uses only properties of the field \( \phi_0 \), the *-product and translational invariance. Hence by the same arguments, the S-matrix \( S_I^{\theta,\beta} \) is also independent of \( \theta_{\mu\nu} \) and \( T \):

\[
S_I^{\theta,\beta} = T \exp \left( -i \int dt H_I^{\theta,\beta} \right) = S_I. 
\]

(41)

The incoming and outgoing state-vectors are affected by \( \theta_{\mu\nu} \) and \( \beta \) since the Fock vacuum | 0, 0 \rangle should be replaced by \( | 0(\beta) \rangle \) and the particle states are created by repeated applications \( a_k^{\beta \dagger} \)’s.

Note that this is the correct procedure since (1) the free Hamiltonian is \( \theta_{\mu\nu} \) and \( T \) independent so that the thermal vacuum has no dependence on \( \theta_{\mu\nu} \), and (2) we do not want to include the tilde excitations in the incoming and outgoing particles, so that particle states which participate in scattering are to be created by \( a_k^{\beta \dagger} \)’s from | 0(\beta) \rangle.

Consider a scattering process at temperature \( T \) on the Moyal plane. If the incoming momenta are \( q_j, j = N + 1, N + 2, \cdots, N + M \) and the outgoing momenta are \( -q_k, k = 1, 2, \cdots, N \), then incoming and outgoing state vectors at temperature \( T \) are:

\[
| q_{N+M}, q_{N+M-1}, \cdots, q_{N+1} \rangle_{\theta,\beta} = a^\dagger_{q_{N+1},\beta} a^\dagger_{q_{N+2},\beta} \cdots a^\dagger_{q_{N+M},\beta} | 0(\beta) \rangle, 
\]

(42)

\[
| -q_N, -q_{N-1}, \cdots, q_1 \rangle_{\theta,\beta} = a^\dagger_{-q_N,\beta} a^\dagger_{-q_{N-1},\beta} \cdots a^\dagger_{-q_1,\beta} | 0(\beta) \rangle. 
\]

(43)

The scattering amplitude is the matrix element of \( S_I \) between these vectors.

With gauge fields:

In our approach, the gauge fields are not twisted. As a consequence, the twisted interaction Hamiltonian \( H_I^{\theta,\beta} \) splits into three parts:

\[
H_I^{\theta,\beta} = H_I^{M,\theta,\beta} + H_I^G + H_I^{M-G,\theta,\beta}. 
\]

(44)
Here $H_i^{M,\theta,\beta}$ contains only matter fields and they are twisted, while $H_i^G$ involves only gauge fields and they are not twisted. The term $H_i^{M-G,\theta,\beta}$ involves both matter and gauge fields with the former alone twisted. In each term, products of matter fields involve $*$, but those of gauge fields with themselves or with matter fields require special treatment. For details see [14].

In the interaction representation, the S-Matrix $S^\theta_I$ now depends on $\theta_{\mu\nu}$:

$$S^\theta_{I} = T \exp \left(-i \int dt H^{\theta,\beta}_I(x) \right)$$ (45)

If there are incoming or outgoing gauge particles, we should include them in eqs.(42) and (43) by acting with their creation operators on $|\ 0(\beta) \rangle$. The scattering amplitude is the expectation value of $S^\theta_{I}$ between these extended vectors (42) and (43).

B. The LSZ formalism

Without gauge fields

We first consider the LSZ scattering amplitude for $\theta_{\mu\nu} = 0$ for both $T = 0$ and $T \neq 0$ and for spinless particles. We use the thermofield formalism.

For $T = 0$, for ingoing and outgoing momenta as in eqs.(42) and (43), the scattering amplitude is

$$S^0(-q_N, -q_{N-1}, \cdots - q_1; q_{N+M}, q_{N+M-1}, \cdots , q_{N+1}) = \int \prod_{i=1}^{N+M} d^4x_i e^{-iq_i \cdot x_i} i(\partial_i^2 + m^2) G^0_{N+M}(x_1, x_2, \cdots , x_{M+N})$$ (46)

where $G^0_{N+M}$ is:

$$G^0_{N+M}(x_1, x_2, \cdots , x_{M+N})) = T e^{* \sum_{I<J} \partial_{x_I} \wedge \partial_{x_J} W^0_{N+M}(x_1, \cdots x_{N+M})} \quad : = T W^0_{N+M}(x_1, \cdots x_{N+M})$$ (47)

and $W^0_{N+M}$ are Wightman functions at $T = 0$:

$$W^0_{N+M} = (0 | \Phi_0(x_1) \cdots \Phi_0(x_{N+M}) | 0)$$ (48)

Here $\Phi_0$ are Heisenberg fields and $| 0)$ is the vacuum state annihilated by $P_\mu$, the four momentum of the fully interacting theory :

$$P_\mu | 0) = 0$$ (49)

In the twisted case, still at $T = 0$ we have argued (see [14]) that $S^0$ is modified to $S^\theta$ where

$$S^\theta(-q_N, -q_{N-1}, \cdots - q_1; q_{N+M}, q_{N+M-1}, \cdots , q_{N+1}) = \int \prod_{i=1}^{N+M} d^4x_i e^{-iq_i \cdot x_i} i(\partial_i^2 + m^2) G^\theta_{N+M}(x_1, x_2, \cdots , x_{M+N})$$ (50)

where $G^\theta_{N+M}$ is:

$$G^\theta_{N+M}(x_1, \cdots x_{N+M}) = T e^{* \sum_{I<J} \partial_{x_I} \wedge \partial_{x_J} W^0_{N+M}(x_1, \cdots x_{N+M})} \quad : = T W^0_{N+M}(x_1, \cdots x_{N+M})$$ (51)
and \(W_{N+M}^\theta\) are Wightman functions with \(\theta_{\mu\nu} \neq 0\).

\[
W_{N+M}^\theta = \langle 0 \mid \Phi_\theta(x_1) \cdots \Phi_\theta(x_{N+M}) \mid 0 \rangle, \quad \Phi_\theta = \Phi_0 e^{\frac{i}{\theta} \mathcal{D} \wedge P}.
\] (52)

The twisted Heisenberg field \(\Phi_\theta\) generalises (25).

In arriving at eq.(52), it is important to note that: (a) \(|0\rangle\) is stable under evolution. It is in fact \(\theta_{\mu\nu}\) independent (b) as the energy-momentum operator \(\mathcal{P}_\mu\) of in- and out- fields \(\mathcal{P}_{\mu}^{\text{in},\text{out}}\) are all equal to \(\mathcal{P}_\mu\), and \(\Phi_0\) approaches in- and out- fields \(\Phi_0^{\text{in},\text{out}}\) as \(x_0 \to \mp \infty\), \(\Phi_\theta\) approaches their twisted versions:

\[
\Phi_\theta \xrightarrow{x_0 \to \mp \infty} \Phi_\theta^{\text{in},\text{out}}, \quad \mathcal{P}_\mu = \mathcal{P}_{\mu}^{\text{in},\text{out}}
\] (53)

We now generalise \(S^\theta\) to the finite temperature scattering amplitude \(S^{\theta,\beta}\). For this purpose we introduce the tilde Heisenberg fields \(\tilde{\Phi}_0\) and their energy momentum operator \(\tilde{\mathcal{P}}_\mu\). Then if we can consistently replace \(\mathcal{P}_\mu\) by \(\mathcal{P}_\mu - \tilde{\mathcal{P}}_\mu\) and define an "exact" Heisenberg vacuum \(|0(\beta)\rangle\) and appropriate twisted Heisenberg field \(\Phi_{\theta,\beta}\) we can write down \(S^{\theta,\beta}\).

But we see that this is straightforward. Thus (a) \(|0(\beta)\rangle\) can be constructed from the in-states of \(\mathcal{P}_\mu\) and \(\tilde{\mathcal{P}}_\mu\):

\[
|0(\beta)\rangle = \sum N \int \frac{1}{\sqrt{T_{RN}}} e^{-\beta P_0} \prod_{i=1}^N d\mu(k_i) e^{\frac{-\beta P_0}{2}} \mid k_1, \cdots k_N \rangle_{\text{in}} \mid \tilde{k}_1, \cdots \tilde{k}_N \rangle_{\text{in}}
\] (55)

\[
d\mu(k_i) = \frac{d^3k_i}{2|k_0|}.
\] (56)

where \(T_{RN}\) denotes trace in the \(N\)-particle sector. Since \(\mathcal{P}_\mu = \mathcal{P}_{\mu}^{\text{in}} = \mathcal{P}_{\mu}^{\text{out}}\) and \(\tilde{\mathcal{P}}_\mu = \tilde{\mathcal{P}}_{\mu}^{\text{in}} = \tilde{\mathcal{P}}_{\mu}^{\text{out}}\) it is clear that:

\[
(\mathcal{P}_\mu - \tilde{\mathcal{P}}_\mu) \mid 0(\beta)\rangle = 0
\] (57)

It is also plausible that

\[
|0(\beta)\rangle = \sum N \int \frac{1}{\sqrt{T_{RN}}} e^{-\beta P_0} \prod_{i=1}^N d\mu(k_i) e^{\frac{-\beta P_0}{2}} \mid k_1, \cdots k_N \rangle_{\text{out}} \mid \tilde{k}_1, \cdots \tilde{k}_N \rangle_{\text{out}}
\] (58)

\[
d\mu(k_i) = \frac{d^3k_i}{2|k_0|}.
\] (59)

(b) The twisted Heisenberg field at temperature \(T\) is:

\[
\Phi_{\theta,\beta}(x) \xrightarrow{x_0 \to \mp \infty} \Phi_0^{\text{in},\text{out}} e^{\frac{i}{\theta} \mathcal{D} \wedge (P - \tilde{P})}
\] (60)

Since \(\Phi_0\) fulfills asymptotic condition and \(\mathcal{P}_\mu - \tilde{\mathcal{P}}_\mu = \mathcal{P}_{\mu}^{\text{in},\text{out}} - \tilde{\mathcal{P}}_{\mu}^{\text{in},\text{out}}\), \(\Phi_{\theta,\beta}\) at least formally obeys the asymptotic conditions

\[
\Phi_{\theta,\beta}(x) \xrightarrow{x_0 \to \mp \infty} \Phi_0^{\text{in},\text{out}} e^{\frac{i}{\theta} \mathcal{D} \wedge (\mathcal{P}_{\mu}^{\text{in},\text{out}} - \tilde{\mathcal{P}}_{\mu}^{\text{in},\text{out}})}
\] (61)

as required.

The in- and out-states now are:

\[
\mid q_{N+M}, q_{N+M-1}, \cdots, q_{N+1}\rangle_{\theta,\beta}^{\text{in}} = a_{q_{N+1,\beta}}^{\text{in}} a_{q_{N+2,\beta}}^{\text{in}} \cdots a_{q_{N+M,\beta}}^{\text{in}} \mid 0(\beta)\rangle,
\] (62)

\[
\mid -q_{N}, -q_{N-1}, \cdots, q_{1}\rangle_{\theta,\beta}^{\text{out}} = a_{-q_{N,\beta}}^{\text{out}} a_{-q_{N-1,\beta}}^{\text{out}} \cdots a_{-q_{1,\beta}}^{\text{out}} \mid 0(\beta)\rangle
\] (63)
The S-matrix element is the scalar product of in- and out-states. Its connected part reads:

\[
S^\theta,\beta(-q_N, -q_{N-1}, \ldots, -q_1; q_{N+M}, q_{N+M-1}, \ldots, q_{N+1}) = \int \prod_{i=1}^{N+M} d^4x_i e^{-i q_i \cdot x_i} i(\partial_i^2 + m^2)G^\theta,\beta_{N+M}(x_1, x_2, \ldots, x_M+N) 
\]

where \( G^\theta,\beta_{N+M} \) is:

\[
G^\theta,\beta_{N+M}(x_1, \ldots, x_{N+M}) = T e^{\frac{i}{\beta} \sum_{J<J} \partial_J^x \partial_J^\beta W^0,\beta_{N+M}(x_1, \ldots, x_{N+M})} \]

\[
= T W^0,\beta_{N+M}(x_1, \ldots, x_{N+M}) 
\]

and \( W^0,\beta_{N+M} \) are the standard Wightman functions for untwisted fields at finite temperature:

\[
W^0,\beta_{N+M}(x_1, \ldots, x_{N+M}) = (0(\beta) | \Phi_0(x_1) \cdots \Phi_0(x_{N+M}) | 0(\beta)). 
\]

This can be derived as in the standard LSZ formalism.

Note that already for \( T = 0, \beta = \infty \), \( S^{\theta \infty} \equiv S^\theta \) and \( S^\theta \) do not agree unless \( \theta = 0 \). This was pointed out in our earlier paper [14] and remarked on at the beginning of this section.

In [14] we have indicated how calculations can be performed using eq.50 for \( T = 0 \). They can be extended to \( T \neq 0 \). An actual calculation will be presented in a future work.

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[1] A. Connes, Noncommutative Geometry (Academic Press, San Diego, CA, 1994) J. Madore, An Introduction To Noncommutative Differential Geometry And Its Physical Applications, Lond. Math. Soc. Lect. Note Ser. 257 (2000) 1. M.R. Douglas and N.A. Nekrasov, Rev. Mod. Phys. 73, 977 (2001) [arXiv:hep-th/0106048]. R.J. Szabo, Phys. Rept. 378, 207 (2003) [arXiv:hep-th/0109162]. N. Seiberg and E. Witten, J. High Energy Phys. 09, 032 (1999) 032. [arXiv:hep-th/9908142]. N. Nekrasov and A. Schwarz, Commun. Math. Phys. 198, 689 (1998) [arXiv:hep-th/9802068].
[2] M. Chaichian, P.P. Kulish, K. Nishijima and A. Tureanu, Phys. Lett. B 604, 98 (2004); J. Wess, [arXiv:hep-th/0408080]; H. Steinacker, J. High Energy Phys. 0712, 049 (2007) [arXiv:hep-th/0708.2426] A P Balachandran and Pramod Padmanabhan AIP conf. Proc. 1196, 18 (2009) [arXiv: hep-th/0908.3888]; T R Govindarajan ibid., 134 (2009) [arXiv:hep-th/0908.3940]; J Lee and H S Yang, [arXiv: hep-th/1004.0745]; A. H. Chamseddine and A Connes, [arXiv:hep-th/1004.0464].
[3] A P Balachandran, S Kurkcuoglu, S Vaidya, Lectures on fuzzy and fuzzy SUSY physics, World Scientific, Singapore (2007)
[4] S S Gubser, S L Sondhi, Nucl.Phys. B605 395 (2001).
[5] Fernando García Flores, Xavier Martin, Denjoe O’Connor, Int.J.Mod.Phys. A24 3917 (2009).
[6] C R Das, S Digal, T R Govindarajan, Mod.Phys.Lett.A24 2693 (2009); Mod.Phys.Lett.A23 1781 (2008).
[7] E Akofer, A P Balachandran, Phys. Rev. D80, 036008 (2009)
[8] H Umezawa, H Matsumoto and M Tachiki, Thermofield dynamics and condensed states (North-Holland, Amsterdam, 1982). H Umezawa, Advanced Field theory, American Institute of Physics, NY 1993. Y Takahashi and H Umezawa Colect.Pheom,2 (1975) 55. F C Khanna, A P C Malbouisson, J M C Malbouisson, A E Santana, Thermal Quantum Field Theory: Algebraic Aspects and Applications (World Scientific, 2009)
[9] P Basu, R Srivastava, S Vaidya, Phys. Rev D82, 025005 (2010) [arXiv:1003.4069 [hep-th]].

[10] A P Balachandran, T R Govindarajan, G Mangano, A Pinzul, B A Qureshi and S Vaidya, Phys. Rev. D 75, 045009 (2007); [arXiv:hep-th/0608179].

[11] S. Doplicher, K. Fredenhagen and J Roberts, Comm. Math. Phys. 172 187 (1995).

[12] D. Bahns, S. Doplicher, K. Fredenhagen and G. Piacitelli, Comm. Math. Phys. 237 221 (2003).

[13] A P Balachandran, T R Govindarajan, G Mangano, A Pinzul, B A Qureshi and S Vaidya, Phys. Rev. D 75 045009 (2007).

[14] A. P. Balachandran, A. Pinzul, B. A. Qureshi, S. Vaidya Phys. Rev. D 76 105025 (2007) ; A. P. Balachandran, T. R. Govindarajan, S. Vaidya, Phys. Rev. D 79, 105020 (2009).