Algebraic degrees of pseudo-Anosov stretch factors

Balázs Strenner

November 22, 2016

Abstract

The motivation for this paper is to justify a remark of Thurston that the algebraic degree of stretch factors of pseudo-Anosov maps on a surface \( S \) can be as high as the dimension of the Teichmüller space of \( S \). In addition to proving this, we completely determine the set of possible algebraic degrees of pseudo-Anosov stretch factors on almost all finite type surfaces. As a corollary, we find the possible degrees of the number fields that arise as trace fields of Veech groups of flat surfaces homeomorphic to \( S \). Our construction gives an algorithm for finding a pseudo-Anosov map on a given surface whose stretch factor has a prescribed degree. One ingredient of the proofs is a novel asymptotic irreducibility criterion for polynomials.

1 Introduction

Let \( S \) be a finite type surface. An element \( f \) of the mapping class group \( \text{Mod}(S) \) is pseudo-Anosov if there is a representative homeomorphism \( \psi \), a number \( \lambda > 1 \) called the stretch factor (or dilatation), and a pair of transverse invariant singular measured foliations \( F_u \) and \( F_s \) such that \( \psi(F_u) = \lambda F_u \) and \( \psi(F_s) = \lambda^{-1} F_s \). The stretch factor \( \lambda \) is an algebraic integer. The main purpose of this paper is to determine the possible algebraic degrees that can be attained.

Thurston’s remark. Thurston announced the classification of elements of \( \text{Mod}(S) \) to finite order, reducible and pseudo-Anosov elements in his seminal bulletin paper in 1988 [Thu88]. On pages 427–428, he provides a bound for the algebraic degree of pseudo-Anosov stretch factors \( \lambda \). He denotes the dimension of the Teichmüller space of \( S \) by \( d \), and writes:

\[
\text{Therefore } \lambda \text{ is an algebraic integer of degree } \leq d. \text{ The examples of Theorem 7 show that this bound is sharp.}
\]

Theorem 7, which describes a construction of pseudo-Anosov mapping classes using Dehn twists, definitely should produce examples realizing the degree \( d \), but Thurston did not explain this in the paper, nor did anyone else since then. The intuition that supports Thurston’s claim is that a random degree \( d \) polynomial should be irreducible. However, this intuition seems difficult to justify, partly because of the lack of irreducibility criteria that apply for the class of polynomials arising from the construction.
Even the fact that the degree can grow linearly with the genus is nontrivial. This is due to Arnoux and Yoccoz [AY81] who constructed a degree g stretch factor on each closed orientable surface $S_g$ of genus $g \geq 3$. (For $S_g$, we have $d = 6g - 6$.) This was improved recently by Shin [Shi16] who realized the degree 2 on $S_g$.

The main result. In this paper, not only do we realize the theoretical maximum $6g - 6$, but we completely answer the question of which degrees appear on $S_g$. Moreover, we also answer this question for most finite type surfaces, including all nonorientable surfaces, for which Thurston’s construction does not apply.

Let $D(S)$ be the set of possible algebraic degrees of stretch factors of pseudo-Anosov elements of $\text{Mod}(S)$. Let $D^+(S) \subset D(S)$ be the set of degrees arising from pseudo-Anosov maps with a transversely orientable invariant foliation. Finally, denote by $[a, b]_{\text{even}}$ and $[a, b]_{\text{odd}}$ the set of even and odd integers, respectively, in the interval $[a, b]$.

Theorem 1.1. Let $g \geq 2$. We have

$$D(S_g) = [2, 6g - 6]_{\text{even}} \cup [3, 3g - 3]_{\text{odd}}$$

and

$$D^+(S_g) = [2, 2g]_{\text{even}} \cup [3, g]_{\text{even}}.$$  

In fact, we prove a more general result in Theorem 2.10, where we also determine $D^+(S)$ for any finite type surface, and $D(S)$ for

- nonorientable surfaces with any number of punctures and
- orientable surfaces with an even number of punctures.

We also almost completely determine $D(S)$ for orientable surfaces with an odd number of punctures. The issue in this last case is that those surfaces are not double covers of nonorientable surfaces, but our construction relies on this covering to realize the highest possible odd degree.

The fact that $D(S_g)$ and $D^+(S_g)$ cannot be larger than stated in Theorem 1.1 is well-known. We have $\min D(S_g) \geq 2$, because only 1 and $-1$ are algebraic units of degree 1. Thurston [Thu88] showed that $\max D(S_g) \leq 6g - 6$ and $\max D^+(S_g) \leq 2g$, and Long [Lon85] showed that if $d \in D(S)$ is odd, then $d \leq 3g - 3$. A similar argument shows that if $d \in D^+(S)$ is odd, then $d \leq g$. (Here and in what follows, we use $d$ to denote any degree, not necessarily the maximal one.)

Irreducibility of polynomials via converging roots. In the work of Arnoux–Yoccoz [AY81] and Shin [Shi16], a construction of pseudo-Anosov maps is given such that $\lambda$ is a root of a polynomial which can be shown to be irreducible. The irreducibility criteria in both cases are specific to one particular sequence of polynomials and cannot easily be generalized to realize other degrees.

In this paper we use a novel irreducibility criterion. To illustrate the idea, we give the criterion below in a special case.
**Lemma 1.2.** Let \( p_n(x) \in \mathbb{Z}[x] \) be a sequence of monic degree \( d \) polynomials whose constant coefficients are \( \pm 1 \). Suppose there is a sequence of real numbers \( \lambda_n \to \infty \) such that \( p_n(\lambda_n) = 0 \) for all \( n \), and suppose that \( p_n(1) \neq 0 \) for all \( n \). If
\[
\lim_{n \to \infty} \frac{p_n(x)}{x - \lambda_n} = x(x - 1)^{d-2},
\]
then \( p_n(x) \) is irreducible for all but finitely many \( n \).

A more general version is stated in Proposition 2.4. The proof is completely elementary. The criteria are cleanest to state and prove for sequences of polynomials, but quantitative versions would also be possible to obtain using Newton’s formulas for coefficients of polynomials in terms of the roots.

**Constructing sequences of polynomials with converging roots.** We also need to construct sequences of pseudo-Anosov maps such that the defining polynomials of the stretch factors satisfy the hypotheses of the irreducibility criteria. For this, we use Penner’s construction of pseudo-Anosov maps. In this construction one has two choices: the choice of a collection of curves and a choice of a product of Dehn twists about these curves. It turns out that it is possible to fix a collection of curves, and vary the Dehn twist product in such a way that the roots of the defining polynomials have the asymptotic behavior as in Lemma 1.2. Proving this involves showing that certain sequences of matrices have converging codimension 1 invariant subspaces, and showing that the actions on these subspaces converge to a linear transformation of the limit subspace. Finally, we show that this limit transformation is a composition of projections, where cancellations occur under certain conditions on the Dehn twist product. In these cases, the limit transformation is a projection, which ensures the desired asymptotic behavior of the eigenvalues.

**Finding stretch factors with prescribed degrees.** So how should one go about finding a stretch factor of prescribed algebraic degree using Penner’s construction? The starting point is the observation that the algebraic degree of a stretch factor depends primarily only on the choice of a collection of curves, and not the Dehn twist product. More specifically, computer experiments have led to the following observation.

*The algebraic degree of a stretch factor arising from Penner’s construction typically equals the rank of the intersection matrix of the collection of curves.*

Unfortunate choices of Dehn twist products may result in a lower algebraic degree than the typical one. The technical tools outlined above allow us to provide criteria that guarantee that the degree of the stretch factor equals the rank of the intersection matrix. In fact, we provide a recipe for writing down a sequence of Dehn twist products so that the degree equals the rank of the intersection matrix if \( n \) is large enough (Theorem 2.8).

**Degrees of trace fields of Veech groups.** Our second result concerns trace fields of Veech groups. A half-translation surface is a surface with a singular Euclidean
structure with trivial or $\mathbb{Z}_2$-holonomy. Its Veech group is the group of its $\text{PSL}(2,\mathbb{R})$-symmetries. Every Veech group is a Fuchsian group, and its trace field is a natural invariant of the half-translation surface. There is a half-translation surface associated with every pseudo-Anosov map $f$ which is defined by the stable and unstable foliations of $f$. The trace field of the Veech group of this surface is $\mathbb{Q}(\lambda + \lambda^{-1})$, where $\lambda$ is the stretch factor of $f$. The degree of the field extension $\mathbb{Q}(\lambda + \lambda^{-1}) : \mathbb{Q}$ is either the algebraic degree of $\lambda$ or half of it. For more details, see [FM12, §11.3], [Zor06], [KS00, §7], [GJ00, §7] or [McM03, §9].

Which Fuchsian groups arise as Veech groups is an open question [HMSZ06, Problem 5]. Whether there is a cyclic Veech group generated by a hyperbolic element is also unknown [HMSZ06, Problem 6]. Also little is known about the number fields that arise as trace fields of Veech groups. As a corollary of our results on the algebraic degrees of stretch factors, we obtain the following.

**Theorem 1.3.** The set of degrees of number fields that arise as trace fields of Veech groups of half-translation surfaces homeomorphic to $S_g$ is $\{1, \ldots, 3g-3\}$.

**Open questions.** The algebraic degree of $\lambda$ is an interesting measure of complexity of $f$. Franks and Rykken [FR99] (see also [GJ00, Theorem 5.5]) showed that if $S$ is orientable and $\mathcal{F}^u$ and $\mathcal{F}^s$ are transversely orientable, then $f$ is a lift of an Anosov mapping class of the torus by a branched covering if and only if $\lambda$ is quadratic. The hope is that this phenomenon generalizes to higher degrees.

**Conjecture 1.4 (Farb).** Given any $d$ there exists $h(d)$ so that any pseudo-Anosov map with degree $d$ stretch factor on a closed orientable surface arises by lifting a pseudo-Anosov map on some surface of genus at most $h(d)$ by a (branched or unbranched) cover.

The Franks–Rykken result is the statement $h(2) = 1$ (in the special case that the invariant foliations are orientable).

There are many other open questions about the degrees of stretch factors. Margalit asked what the possible algebraic degrees of stretch factors in the Torelli group are. (Computer experiments suggest that the same degrees occur in the Torelli group as in the whole mapping class group. We wonder if the methods of this paper can be used to prove this.) One can ask the same question for any other subgroup of $\text{Mod}(S)$. For the point-pushing subgroup, it would be interesting to know if the degree of the stretch factor is related to some property of the corresponding element of the fundamental group.

Instead of only having an algorithm for finding a pseudo-Anosov map with a degree $d$ stretch factor on a surface $S$, one could hope to find an explicit formula for such a pseudo-Anosov map. This could be done by estimating how large $k$ has to be in Theorem 2.8. (This would involve effectivizing Proposition 2.4 and estimating the rate of convergence in Proposition 4.12.)

One can also wonder what degrees are generic in various subgroups of $\text{Mod}(S)$ or strata of the holomorphic quadratic differentials over the moduli space of $S$. Finally, many of these questions have versions for outer automorphisms of free groups.
The structure of the paper. Section 2 gives an overview about the content of the paper. The proof of Theorem 1.1 (and its generalizations to punctured and nonorientable surfaces) and Theorem 1.3 are given in Section 2.6 assuming statements whose proofs are postponed to Sections 3 to 8.

Acknowledgements. This work is an improvement over a part of the Ph.D. thesis of the author. He thanks his advisor, Autumn Kent, for her guidance and Dan Margalit for suggesting the problem and for many valuable comments. He also appreciates helpful conversations with Joan Birman, Jordan Ellenberg, Benson Farb, Eriko Hironaka and Jean-Luc Thiffeault.

The author was partially supported by the grant NSF DMS-1128155.

2 Overview

2.1 Penner’s construction

The standard reference for this section is [Pen88]. See also [Fat92]. Let \( c \) be a two-sided simple closed curve on a surface \( S \). Let \( \theta : \nu_c \to S^1 \times [-1,1] \) be a trivialization of the normal bundle \( \nu_c \) of \( c \). We refer to the pair \((c,\theta)\) as a marked curve.

The Dehn twist \( T(c,\theta) \) about the marked curve \((c,\theta)\) is the identity off \( \nu_c \) and on \( \nu_c \) it is defined by the formula

\[
T(c,\theta)(x) = \theta^{-1} \left( p_1(\theta(x)) e^{p_2(\theta(x)) + 1} \right),
\]

where \( p_j \) denotes the projection to the \( j \)th factor. If \( S \) is oriented, then \( T(c,\theta) \) is the left Dehn twist \( T_c \) if \( \theta \) is orientation-preserving, and it is the right Dehn twist \( T_{-1}c \) otherwise.

Let \((c,\theta_c)\) and \((d,\theta_d)\) be marked curves that intersect at a point \( p \). We say that they are marked inconsistently at \( p \) if the pullbacks by \( \theta_c \) and \( \theta_d \) of the orientation on \( S \times [-1,1] \) disagree near \( p \).

Penner gave the following construction for pseudo-Anosov mapping classes [Pen88].

**Penner’s Construction** (General case). Let \( C = \{(c_1,\theta_1),\ldots,(c_n,\theta_n)\} \) be a collection of marked curves on \( S \) which are in pairwise minimal position and fill. Suppose that they are marked inconsistently at every intersection.

Then any product of the \( T(c_i,\theta_i) \) is pseudo-Anosov provided each twist is used at least once.

When \( S \) is orientable, the hypotheses imply that \( C \) is a union of two multicurves \( A \) and \( B \), and the statement takes the following more well-known form.

**Penner’s Construction** (Orientable case). Let \( A = \{a_1,\ldots,a_n\} \) and \( B = \{b_1,\ldots,b_m\} \) be a pair of filling multicurves on an orientable surface \( S \). Then any product of \( T_{a_j} \) and \( T_{b_k}^{-1} \) is pseudo-Anosov provided that each twist is used at least once.

We remark that if \( C \) is a union of a pair of multicurves and the curves are marked inconsistently at every intersection, then the surface filled by \( C \) is necessarily orientable. Hence if \( S \) is nonorientable, then \( C \) in Penner’s construction cannot be a union of two multicurves.
2.2 Oriented collections of marked curves

Let \((c, \theta)\) be a marked curve. We define its left side as \(\theta^{-1}(S^1 \times [-1, 0])\) and its right side as \(\theta^{-1}(S^1 \times (0, 1])\). Note also that the marking \(\theta\) induces an orientation of \(c\) by pulling back the standard orientation of \(S^1\).

Suppose \((c, \theta_c)\) and \((d, \theta_d)\) are marked inconsistently at some \(p \in c \cap d\). When we follow \(c\) in the direction of its orientation near \(p\), we either cross from the left side of \(d\) to the right side of \(d\) or the other way around. In the first case, we call \(p\) a left-to-right crossing. In the second case, we call it right-to-left crossing. Note that the definition is symmetric in \(c\) and \(d\): if \(c\) crosses from the left side of \(d\) to the right side of \(d\), then \(d\) also must cross from the left side of \(c\) to the right side of \(c\) (Figure 2.1).

![Figure 2.1: A left-to-right and a right-to-left crossing.](image)

Let \(C\) be a collection of marked curves which are inconsistently marked at each intersection. Then \(C\) is called completely left-to-right if all crossings are left-to-right and completely right-to-left if all crossings are right-to-left.

**Proposition 2.1.** If \(C\) is completely left-to-right or completely right-to-left, then the pseudo-Anosov maps constructed from it by Penner’s construction have an orientable invariant foliation.

**Proof.** Penner observed that by smoothing out the intersections of \(C\), one obtains a bigon track \(\tau^+\) invariant under the Dehn twists of \(C\) and hence under any pseudo-Anosov map \(\psi\) constructed from his construction. By choosing the smoothings differently at every intersection, we get a track \(\tau^-\) invariant under \(\psi^{-1}\). The unstable foliation is carried by \(\tau^+\), and the stable foliation is carried by \(\tau^-\).

When \(C\) is completely left-to-right, \(\tau^-\) is transversely orientable. There is a continuously varying set of vectors transverse to \(\tau^-\) such that the vectors point toward the right side of the curves of \(C\) (Figure 2.2). Hence the stable foliation is transversely orientable.

Similarly, when \(C\) is completely right-to-left, \(\tau^+\) is transversely orientable, and the unstable foliation is transversely orientable.

2.3 Description of Penner’s construction by linear algebra

Denote by \(i(a, b)\) the geometric intersection number of the simple closed curves \(a\) and \(b\). For two collections of curves \(A = \{a_j\}\) and \(B = \{b_k\}\), the intersection matrix \(i(A, B)\)
is a matrix whose \((j,k)\)-entry is \(i(a_j, b_k)\).

Suppose we use Penner’s construction on the collection \(C = \{(c_1, \theta_1), \ldots, (c_n, \theta_n)\}\). Let \(\Omega = i(C, C)\). The actions of the twists \(T(c_i, \theta_i)\) on the invariant bigon track \(\tau^+\) (see Proposition 2.1) are described by the matrices \(Q_i\) \((1 \leq i \leq n)\) defined as

\[
Q_i = Q_i(\Omega) = I + D_i \Omega,
\]

where \(I\) denotes the \(n \times n\) identity matrix, and \(D_i\) denotes the \(n \times n\) matrix whose \(i\)th entry on the diagonal is 1 and whose other entries are zero [Pen88, p. 194]. When a product of Dehn twists is pseudo-Anosov, the corresponding product of the \(Q_i\) is Perron–Frobenius, that is, it has nonnegative entries and some power of it has strictly positive entries. Moreover, the stretch factor of the pseudo-Anosov mapping class equals the Perron–Frobenius eigenvalue (the unique largest real eigenvalue) of the corresponding Perron–Frobenius matrix.

Let \(\Gamma(\Omega)\) be the monoid generated by the matrices \(Q_i(\Omega)\) for \(1 \leq i \leq n\), where the operation is the matrix multiplication. Understanding Penner stretch factors boils down to understanding the eigenvalues of Perron–Frobenius matrices in the monoids \(\Gamma(\Omega)\)—a completely algebraic problem. This makes Penner’s construction particularly useful for studying algebraic properties of pseudo-Anosov stretch factors. We explore the basic properties of the monoids \(\Gamma(\Omega)\) in Section 3.

**Single objects versus functions.** Whenever confusion may arise, we use bold font for functions and roman font for single objects. For example, \(Q_i\) are matrices, but \(Q_i\) are a matrix-valued functions.

**The space \(\mathcal{O}\).** Fix \(n\), and denote by \(\mathcal{O}\) the space of \(n \times n\) symmetric matrices with nonnegative real entries whose diagonal entries vanish. The matrix \(i(C, C)\) is an integral point in \(\mathcal{O}\) for some \(n\). Thus \(\mathcal{O}\) can be thought of as the **space of generalized intersection matrices** of size \(n \times n\). Although non-integral points of \(\Omega\) do not arise topologically, it will be useful to have a nice connected parameter space to work with.

For all \(\Omega \in \mathcal{O}\), the formula (2.1) defines the matrices \(Q_1(\Omega), \ldots, Q_n(\Omega)\). These matrices are of size \(n \times n\), have nonnegative real entries and depend continuously on
For any $\Omega \in \mathcal{O}$, let $G(\Omega)$ be the graph on the vertex set $\{1, \ldots, n\}$ where $i$ and $j$ are connected if the $(i, j)$-entry of $\Omega$ is positive. Let $\mathcal{O}_{\text{conn}}$ denote the set of matrices $\Omega \in \mathcal{O}$ such that $G(\Omega)$ is connected.

We show in Corollary 3.11 that $M(\Omega)$ is Perron–Frobenius if and only if $M \in \Gamma_{\text{gen}}$ and $\Omega \in \mathcal{O}_{\text{conn}}$. Note that $M \in \Gamma_{\text{gen}}$ corresponds to the requirement that all twists are used at least once, and $\Omega \in \mathcal{O}_{\text{conn}}$ follows from the fillingness of $C$.

2.4 Asymptotics in the space of Penner matrices

The boundary of $\mathcal{O}$ at infinity. Denote by $\partial_{\infty} \mathcal{O}$ the space of rays from the origin inside $\mathcal{O}$. We think of $\partial_{\infty} \mathcal{O}$ as the boundary at infinity. The space $\mathcal{O} \cup \partial_{\infty} \mathcal{O}$ naturally compactifies $\mathcal{O}$. Our goal is to continuously extend functions on $\mathcal{O}$ related to the pseudo-Anosov stretch factors to $\partial_{\infty} \mathcal{O}$. Some functions will extend without any difficulties, some will not extend at all, and some will extend only to some subset of $\partial_{\infty} \mathcal{O}$.

One function that extends without any problems is $G$, because the graph $G(\Omega)$ is constant along rays in $\mathcal{O}$ from the origin. Let $(\partial_{\infty} \mathcal{O})_{\text{conn}} = \{\Omega^* \in \partial_{\infty} \mathcal{O} : G(\Omega^*) \text{ is connected}\}$.

Perron–Frobenius eigenvalues. Let $\lambda$ be the function that associates to every Perron–Frobenius matrix its Perron–Frobenius eigenvalue. Then $\lambda \circ M$ is a continuous real-valued function on $\mathcal{O}_{\text{conn}}$ for all $M \in \Gamma_{\text{gen}}$.

If $\Omega_k \in \mathcal{O}$ is a sequence converging to a point $\Omega^* \in (\partial_{\infty} \mathcal{O})_{\text{conn}}$, then eventually $\Omega_k \in \mathcal{O}_{\text{conn}}$. So if $M \in \Gamma_{\text{gen}}$, then $M(\Omega_k)$ is eventually Perron–Frobenius. Moreover, $\lambda(M(\Omega_k)) \to \infty$ (Proposition 3.13), so the function $\lambda \circ M$ cannot be extended continuously to $\partial_{\infty} \mathcal{O}$ at all.

Invariant subspaces. For any Perron–Frobenius matrix $M$, let $W(M)$ be the codimension 1 subspace spanned by the right eigenspaces of $M$ corresponding to the eigenvalues different from $\lambda(M)$. The matrix $M$ acts on $W(M)$ by left multiplication and induces a linear map $f(M) : W(M) \to W(M)$.

Encoding products using paths in graphs. Let $G_n$ be the complete graph on the vertex set $\{1, \ldots, n\}$. Let $\gamma = (i_1 \ldots i_K i_1)$ be a closed path in $G_n$, starting and ending at the vertex $i_1$. Let $p = (p_1, \ldots, p_K)$ be positive integers associated to the vertices of $\gamma$.

Associated to $\gamma$ and $p$ is the following element of $\Gamma$:

$$M^p_\gamma = Q^{p_K}_{i_K} \cdots Q^{p_1}_{i_1}.$$ \hfill (2.2)
Every element of $\Gamma$ can be written as $M^p_\gamma$ for some $\gamma$ and $p$.

**Extensions to** $\partial_\infty \mathcal{O}$. For any $M^p_\gamma \in \Gamma_{\text{gen}}$, introduce the notations

$$W^p_\gamma = W \circ M^p_\gamma : \mathcal{O}_{\text{conn}} \to \text{Gr}(n-1,n)$$

and

$$f^p_\gamma = f \circ M^p_\gamma : \mathcal{O}_{\text{conn}} \to \text{End}(n-1,n),$$

where $\text{Gr}(n-1,n)$ is the Grassmannian parametrizing the $(n-1)$-dimensional linear subspaces of $\mathbb{R}^n$, and $\text{End}(n-1,n)$ is the space of linear endomorphisms of the elements of $\text{Gr}(n-1,n)$.

The topology of the space $\text{End}(n-1,n)$ is defined by the following notion of convergence. Let $f_k : W_k \to W_k$ and $f : W \to W$ be linear endomorphisms where $W_k, W \in \text{Gr}(n-1,n)$. Then $f_k \to f$ if

(i) $W_k \to W$ in $\text{Gr}(n-1,n)$ and

(ii) $f_k(v_k) \to f(v)$ holds for all sequences $v_k \to v$, where $v_k \in W_k$ and $v \in W$.

If $f_k \to f$, then the characteristic polynomials $\chi(f_k)$ converge to $\chi(f)$. It is also clear that $W^p_\gamma$ and $f^p_\gamma$ are continuous.

For a closed path $\gamma$ in $G_n$, introduce the notation

$$\partial_\infty \mathcal{O} = \{ \Omega^* \in \partial_\infty \mathcal{O} : \gamma \subset G(\Omega^*) \}. \quad (2.3)$$

**Theorem 2.2.** If $M^p_\gamma \in \Gamma_{\text{gen}}$, then $W^p_\gamma$ and $f^p_\gamma$ extend continuously to $\partial_\infty \mathcal{O}$. Moreover, $W^p_\gamma|_{\partial_\infty \mathcal{O}}$ and $f^p_\gamma|_{\partial_\infty \mathcal{O}}$ depend only on $\gamma$ and not on $p$.

Thus we can write $W_\gamma = W^p_\gamma|_{\partial_\infty \mathcal{O}}$ and $f_\gamma = f^p_\gamma|_{\partial_\infty \mathcal{O}}$.

A corollary of this theorem is that if $\Omega_k \in \mathcal{O}$ is a sequence converging to a point $\Omega^* \in \partial_\infty \mathcal{O}$, then all eigenvalues of $M^p_\gamma(\Omega_k)$ except the Perron–Frobenius eigenvalue converge. It turns out that precisely one eigenvalue of $f_\gamma(\Omega^*)$ converges to zero (Corollary 5.2). This asymptotic behavior proves to be very useful for studying the Galois conjugates of Penner stretch factors, since it helps reducing the problem of understanding the eigenvalues of $M^p_\gamma(\Omega)$ where $\Omega \in \mathcal{O}_{\text{conn}}$ to the problem of understanding the eigenvalues of $f_\gamma(\Omega^*)$ where $\Omega^* \in \partial_\infty \mathcal{O}$.

There are a couple of reasons why the latter problem is considerably simpler:

- the functions $f_\gamma$ have no dependence on $p$,
- they are invariant even under homotopy of $\gamma$ rel the last edge (Proposition 5.5); moreover, the characteristic polynomial of $f_\gamma$ depends only on the free homotopy class of $\gamma$ (Proposition 5.7),
- and most importantly, the functions $f_\gamma$ have very nice descriptions as compositions of projections (Section 5).
2.5 Factorization and irreducibility of polynomials

So far we have introduced some tools to study the eigenvalues of the matrices $M_{p(\Omega)}$. Recall that pseudo-Anosov stretch factors arising from Penner’s construction occur as Perron–Frobenius eigenvalues of such matrices when $\Omega \in \Omega_{\text{conn}}$ is an integral point. For the study of algebraic degrees of stretch factors, we also need to understand how the characteristic polynomials $\chi(M_{p(\Omega)}) \in \mathbb{Z}[x]$ factor to irreducible factors when $\Omega$ is an integral point.

**Proposition 2.3.** Let $M_{p(\Omega)} \in \Gamma_{\text{gen}}$, and let $\Omega_k \in \Omega$ be a sequence of integral points converging to a point $\Omega^* \in \partial_{\mathbb{R}} \Omega$. Let $\lambda_k = \lambda(M_{p(\Omega_k)})$. Furthermore, introduce the notations $u_k(x) = \chi(M_{p(\Omega_k)}) \in \mathbb{Z}[x]$ and $v(x) = \chi(f_{p}(\Omega^*)) \in \mathbb{R}[x]$. Then

(i) $u_k(x)$ is monic and has constant coefficient $\pm 1$,

(ii) $0$ is a single root of $v(x)$,

(iii) $\lambda_k \to \infty$, and

(iv) $\lim_{k \to \infty} \frac{u_k(x)}{x - \lambda_k} = v(x)$.

**Proof.** Since the $Q_i$ have determinant $1$, their products also have determinant $1$, and we obtain (i). We have (ii) by Corollary 5.2 and (iii) by Proposition 3.13. For all $k$, we have

$$\frac{u_k(x)}{x - \lambda_k} = \chi(f_{p, \gamma}(\Omega_k)),$$

so (iv) follows from Theorem 2.2. \hfill \Box

As the following statement shows, the conditions (i)–(iv) imply strong restrictions on the factorization of the polynomials $u_k(x)$ to irreducible factors in $\mathbb{Z}[x]$.

**Proposition 2.4.** Suppose $u_k(x) \in \mathbb{Z}[x]$, $v(x) \in \mathbb{R}[x]$ and $\lambda_k$ satisfy the conditions (i)–(iv) in Proposition 2.3. Let $\theta$ be a root of $v(x)$, and let $\theta_k \to \theta$ be a sequence with $u_k(\theta_k) = 0$ for all $k$ where $\theta_k \neq \theta$ for all but finitely many $k$. (This last condition is satisfied for instance when $\theta$ is not an algebraic unit, since all roots of the $u_k(x)$ are algebraic units.) Then $\theta_k$ and $\lambda_k$ are eventually roots of the same irreducible factor of $u_k(x)$.

**Proof.** For each $k$, choose $\beta_k$ such that $u_k(\beta_k) = 0$ and $\beta_k \to 0$. Note that that $\lambda_k$ and $\beta_k$ are in the same irreducible factor of $u_k(x)$ when $k$ is large enough. Indeed, irreducible factors of $u_k(x)$ have main and constant coefficients $\pm 1$, therefore the the product of the roots of any such factor is $\pm 1$. Since $\lambda_k \to \infty$, and all roots of $u_k(x)$ except $\lambda_k$ and $\beta_k$ converge to a nonzero limit, $\lambda_k$ and $\beta_k$ eventually have to be in the same factor.

Now assume for a contradiction that $\theta_k$ and $\lambda_k$ are in different irreducible factors for infinitely many $k$. By restricting to a subsequence, we may assume that $\theta_k$ and $\lambda_k$ are in different irreducible factors for all $k$. For all $k$, let $w_k(x) \in \mathbb{Z}[x]$ be an irreducible factor with root $\theta_k$. By the argument above, we can further restrict to a subsequence so that $\lambda_k$ and $\beta_k$ are not roots of $w_k(x)$. Since there is a uniform bound on the roots of $w_k(x)$, we can once more restrict to a subsequence where all $w_k(x)$ have the same
degree and \( w_k(x) \to w(x) \) for some \( w(x) \). Since \( w_k(x) \in \mathbb{Z}[x] \), we have \( w(x) \in \mathbb{Z}[x] \) and \( w_k(x) = w(x) \) if \( k \) is large enough. In particular, \( \theta_k = \theta \) if \( k \) is large enough, a contradiction. \( \Box \)

The following will be useful for calculating the degree of the trace field in the proof of Theorem 1.3 in Section 2.6.

**Corollary 2.5.** Let \( \mathbf{M}_p^r \in \Gamma_{\text{gen}} \) and let \( \Omega_k \in \mathbb{O} \) be a sequence converging to a point \( \Omega^* \in \partial^*_\infty \mathbb{O} \). Suppose \( G(\Omega_k) \) is bipartite for all \( k \). Then \( \lambda(\mathbf{M}_p^r(\Omega_k)) \) is a Galois conjugate of its reciprocal when \( k \) is large enough.

**Proof.** By (3.4) and Corollary 3.17, \( \lambda(\mathbf{M}_p^r(\Omega_k))^{-1} \) is a root of \( \chi(\mathbf{M}_p^r(\Omega_k)) \) for all \( k \). The statement now follows from Proposition 2.3 and the argument in the first paragraph of the proof of Proposition 2.4. \( \Box \)

**Complexity.** Define the complexity of a matrix or linear endomorphism to be the number of its eigenvalues, counted with multiplicity, that are different from 1. Denote the complexity by \( \delta \). As another corollary of Propositions 2.3 and 2.4 we have the following.

**Corollary 2.6.** Let \( \mathbf{M}_p^r \in \Gamma_{\text{gen}} \) and let \( \Omega_k \in \mathbb{O} \) be a sequence converging to a point \( \Omega^* \in \partial^*_\infty \mathbb{O} \). If \( \delta(\mathbf{M}_p^r(\Omega_k)) = r \) when \( k \) is large enough and \( \delta(f_r(\Omega^*)) = 1 \), then \( \lambda(\mathbf{M}_p^r(\Omega_k)) \) has algebraic degree \( r \) when \( k \) is large enough.

**Proof.** Zero is always an eigenvalue of \( f_r(\Omega^*) \), so \( \delta(f_r(\Omega^*)) \geq 1 \) always holds. Requiring \( \delta(f_r(\Omega^*)) = 1 \) implies that all other eigenvalues are 1. From \( \delta(\mathbf{M}_p^r(\Omega_k)) = r \), we know that all eigenvalues of \( \mathbf{M}_p^r(\Omega_k) \) are different from 1 (and also from zero, since \( \mathbf{M}_p^r(\Omega_k) \) is invertible). We can therefore apply Proposition 2.4 for all eigenvalues of \( \mathbf{M}_p^r(\Omega_k) \). \( \Box \)

Next, we discuss how to pick a sequence \( \Omega_k \) that satisfies the hypotheses of Corollary 2.6.

**Rank.** Complexity turns out to be strongly related to rank. Since the rank function is constant along rays through the origin, it has a well-defined extension to \( \partial^*_\infty \mathbb{O} \). Denote the level sets \( \{ \Omega \in \mathbb{O} : \text{rank}(\Omega) = r \} \) by \( \mathbb{O}_{\text{rank}=r} \).

It follows from Proposition 3.19 that if \( \mathbf{M} \in \Gamma_{\text{gen}} \), then \( \delta \circ \mathbf{M} = \text{rank} \) and hence \( \delta \circ f \circ \mathbf{M} = \text{rank} - 1 \) on \( \mathbb{O} \). On the ideal boundary, however, \( \delta \circ f_\gamma \) depends on \( \gamma \) and only the inequality \( \delta \circ f_\gamma \leq \text{rank} - 1 \) holds (Proposition 5.3). This inequality is far from sharp: for instance, \( \delta \circ f_\gamma = 1 \) if \( \gamma \) is contractible (Proposition 5.6). In particular, the conditions of Corollary 2.6 are satisfied if \( \Omega_k \) travels in the level set \( \mathbb{O}_{\text{rank}=r} \) and either \( \text{rank}(\Omega^*) = 2 \) or \( \Omega^* \) is arbitrary and \( \gamma \) is contractible. Hence as a corollary of Corollary 2.6, we have the following.

**Corollary 2.7.** Let \( \mathbf{M}_p^r \in \Gamma_{\text{gen}} \) and let \( \Omega_k \in \mathbb{O}_{\text{rank}=r} \) be a sequence converging to a point \( \Omega^* \in \partial^*_\infty \mathbb{O} \). Suppose furthermore that

(i) \( \gamma \) is contractible or
(ii) \( \text{rank}(\Omega^*) = 2 \).
Then \( \lambda(\mathbf{M}_\gamma(\Omega_k)) \) has algebraic degree \( r \) when \( k \) is large enough.

In addition to computer experiments, this result gives support to the intuition that Penner stretch factors arising from a collection of curves with rank \( r \) intersection matrix tend to have algebraic degree \( r \).

This paper only uses part (i) of Corollary 2.7, which is trivial to satisfy. Sequences \( \Omega_k \) satisfying (ii) also exist, and they can be constructed geometrically (see [Str15, Proposition 5.2]).

### 2.6 Proofs of the main theorems

**Theorem 2.8** (Algorithm for constructing degree \( r \) stretch factors). Let \( C = \{ (c_1, \theta_1), \ldots, (c_n, \theta_n) \} \) be a filling collection of curves with inconsistent markings such that \( \text{rank}(i(C,C)) = r \).

Let \( \gamma = (i_1 \ldots i_K i_1) \) be a contractible closed path in \( \mathbf{G}(i(C,C)) \) visiting each vertex at least once and let \( p = (p_1, \ldots, p_K) \) be positive integers. Then the pseudo-Anosov mapping class

\[
f_k = T^{kp_K}_{(c_{i_K}, \theta_{i_K})} \cdots T^{kp_1}_{(c_{i_1}, \theta_{i_1})}
\]

has degree \( r \) stretch factor for all but finitely many positive integer \( k \).

**Proof.** Let \( \Omega = i(C,C) \). The matrix describing \( f_k \) is \( \mathbf{Q}_{(c_{i_K}, \theta_{i_K})} \cdots \mathbf{Q}_{(c_{i_1}, \theta_{i_1})} \). By Corollary 3.4, this can be rewritten as \( \mathbf{Q}_{(c_{i_K}, \theta_{i_K})}(k\Omega) \cdots \mathbf{Q}_{(c_{i_1}, \theta_{i_1})}(k\Omega) = \mathbf{M}_\gamma(\Omega) \). Applying Corollary 2.7 for \( \mathbf{M}_\gamma \) and the sequence \( \Omega_k = k\Omega \) proves the claim. \( \square \)

The following corollary gives a simple way to prove that a given degree can be realized on a given surface.

**Corollary 2.9.** Suppose that the surface \( S \) admits a filling collection \( C \) of curves with inconsistent markings such that \( \text{rank}(i(C,C)) = r \). Then \( r \in D(S) \).

In addition, if \( C \) is completely left-to-right or completely right-to-left, then \( r \in D^+(S) \).

**Proof.** The first conclusion follows immediately from Theorem 2.8. The second follows from Theorem 2.8 and Proposition 2.1. \( \square \)

Denote by \( \overline{S} \) the closed surface obtained from \( S \) by filling in the punctures. Let \( \dim(\text{Teich}(S)) \) denote the real dimension of the Teichmüller space of \( S \). We use the notations \( \left[a, b\right]_{\text{even}} \) and \( \left[a, b\right]_{\text{odd}} \) from Theorem 1.1, and abuse the interval notation \( [a, b] \) to mean the set of integers in the corresponding real interval.

**Theorem 2.10.**

(i) If \( S \) be an orientable surface, then

\[
D^+(S) = \left[ 2, \dim(H_1(\overline{S}, \mathbb{R})) \right]_{\text{even}} \cup \left[ 3, 2 \dim(H_1(\overline{S}, \mathbb{R})) \right]_{\text{odd}}.
\]  

(ii) If \( S \) has an even number of punctures or \( \frac{1}{2} \dim(\text{Teich}(S)) \) is even, then

\[
D(S) = \left[ 2, \dim(\text{Teich}(S)) \right]_{\text{even}} \cup \left[ 3, \frac{1}{2} \dim(\text{Teich}(S)) \right]_{\text{odd}}.
\]
(iii) If $S$ has an odd number of punctures and $\frac{1}{2} \dim(\text{Teich}(S))$ is odd, then either (2.5) holds or

$$D(S) = [2, \dim(\text{Teich}(S))]_{\text{even}} \cup \left[3, \frac{1}{2} \dim(\text{Teich}(S)) - 1\right]_{\text{odd}}.$$ (2.6)

(iv) If $N$ is a nonorientable surface, then

$$D^+(N) = [3, \dim(H_1(\overline{N}, \mathbb{R}))].$$

(v) If, in addition, $N$ is not the closed nonorientable surface $N_3$ of genus 3, then

$$D(N) = [3, \dim(\text{Teich}(N))].$$

The reason why the cases (ii) and (iii) are separated is that we cannot realize the odd degree $\frac{1}{2} \dim(\text{Teich}(S))$ when the surface is not a double cover of a nonorientable surface. It seems possible that realizing this degree is not possible using Penner’s construction. We do not know if it is possible using other constructions. Randomized computer experiments yield almost exclusively even degree stretch factors, so searching for these degrees with computers also seems a nontrivial task.

The exclusion of the surface $N_3$ is necessary, since $\dim(\text{Teich}(N_3)) = 3$, so the interval $[3, \dim(\text{Teich}(N_3))]$ is nonempty. However, this surface does not admit pseudo-Anosov maps (Proposition 7.7).

**Proof.** The fact that the sets cannot be larger than stated follows from the results in Section 7. It remains to prove that all the degrees claimed in the theorem can be realized.

The statements (iv) and (v) follow from Corollary 2.9 and Proposition 6.2 with the exception of two sporadic cases: $6 \in D(N_4)$ and $5 \in D(N_{3,1})$. (Compare Table 2.1 and Table 6.1.) Examples for these two cases are given in Section 8.

| $n \backslash g$ | 1 | 2 | 3 | 4 | 5 |
|-----------------|---|---|---|---|---|
| 0               | $\emptyset$ | $\emptyset$ | $\emptyset$ | E | E |
| 1               | $\emptyset$ | $\emptyset$ | E | E | E |
| 2               | $\emptyset$ | E | E | E | E |
| 3               | E | E | E | E | E |

Table 2.1: Nonorientable surfaces admitting (E) and not admitting ($\emptyset$) pseudo-Anosov maps.

The construction of even degrees for (i), (ii) and (iii) follow from Corollary 2.9 and Proposition 6.1.

It remains to construct odd degrees on orientable surfaces. These are obtained from lifting pseudo-Anosov maps from nonorientable surfaces. Lifting preserves the stretch factor hence also the degree.

Suppose $S$ is an orientable surface with an even number of punctures, and let $S \to N$ be a covering where $N$ is a (unique) nonorientable surface. We have $\dim(\text{Teich}(S)) = \ldots$
2 \dim(\text{Teich}(N)) and \dim(H_1(S)) = 2 \dim(H_1(N)). If \( S \neq S_2 \), then \( N \neq N_3 \), and it follows that \([3, \frac{1}{2} \dim(\text{Teich}(S))] \subset D(S)\) and \([3, \frac{1}{2} \dim(H_1(S))] \subset D^+(S)\). This proves (2.4) and (2.5) when \( S \) has an even number of punctures and \( S \neq S_2 \). In the case \( S = S_2 \), we only need to show that \( 3 \in D(S) \). This is shown by an example in Section 7 of [Shi16]. Lemma 7.2 implies that (2.4) holds also when \( S \) has an odd number of punctures. This completes the proof of (i).

From now on, suppose \( S \) is an orientable surface having an odd number of punctures. The argument in the proof of Lemma 7.2 also shows that if \( S' \) is obtained from \( S \) by filling in one puncture, then \( D(S') \subset D(S) \).

Let \( T_{ij} \) be the \( n \times n \) matrix whose \((i, j)\)-entry is 1 and whose other entries are zero, and denote by \( \omega_{ij} \) the \((i, j)\)-entry of \( \Omega \).

**Lemma 3.1** (Multiplication by \( D_i \)). Multiplying a matrix on the left by \( D_i \) has the effect of keeping the \( i \)-th row unchanged and zeroing out all other rows. Multiplying by \( D_i \) on the right has the effect of keeping the \( i \)-th column unchanged and zeroing out all other columns.

Note that there is no difference between the right hand sides of (2.5) for \( S \) and \( S' \) when \( \frac{1}{2} \dim(\text{Teich}(S)) \) is even, and the difference is \( \frac{1}{2} \dim(\text{Teich}(S)) \) when this number is odd. In the first case, transferring all odd degrees realized on \( S' \) to \( S \) completes the proof of (ii). In the second case, transferring all odd degrees yields all desired odd degrees on \( S \) except \( \frac{1}{2} \dim(\text{Teich}(S)) \). This completes the proof of (iii) and hence the proof of the theorem.

**Proof of Theorem 1.3.** All even degrees between 2 and \( 6g - 6 \) can be realized on \( S_g \) as the algebraic degree of stretch factors by Theorem 2.10. The construction of these examples occurs on the surfaces \( S_g \) themselves (as opposed to odd degrees that are constructed by lifting from nonorientable surfaces). Hence by Corollary 2.5, in these examples \( \lambda \) and \( \lambda^{-1} \) are Galois conjugates. It follows that \( \mathbb{Q}(\lambda) : \mathbb{Q}(\lambda + \lambda^{-1}) = 2 \), and we obtain that the trace field \( \mathbb{Q}(\lambda + \lambda^{-1}) \) can have any degree from 1 to \( 3g - 3 \).

On the other hand the degree of the trace field cannot be larger than \( 3g - 3 \). If that was the case, then \( \lambda \) and \( \lambda^{-1} \) could not be Galois conjugates, otherwise \( \deg(\lambda) \) would be greater than \( 6g - 6 \). But if they are not Galois conjugates, then McMullen’s argument that proves Proposition 7.4 would prove that \( \deg(\lambda) \leq 3g - 3 \).

### 3 Properties of Penner matrices

**3.1 Calculation with the matrices \( D_i \), \( \Omega \) and \( Q_i \)**

To make the calculations more transparent in later sections, here we collect some useful facts about the matrices \( D_i \), \( \Omega \) and \( Q_i \)—mostly without proof—that follow immediately from the definitions.

**Lemma 3.1** (Multiplication by \( D_i \)). Multiplying a matrix on the left by \( D_i \) has the effect of keeping the \( i \)-th row unchanged and zeroing out all other rows. Multiplying by \( D_i \) on the right has the effect of keeping the \( i \)-th column unchanged and zeroing out all other columns.

Let \( T_{ij} \) be the \( n \times n \) matrix whose \((i, j)\)-entry is 1 and whose other entries are zero, and denote by \( \omega_{ij} \) the \((i, j)\)-entry of \( \Omega \).

**Lemma 3.2.** \( D_i \Omega D_j = \omega_{ij} T_{ij} \).

**Corollary 3.3.** \( D_i \Omega D_v = 0 \) if and only if \( ii' \) is a non-edge in \( G(\Omega) \).

**Corollary 3.4.** \( Q^+_k(\Omega) = Q_i(k\Omega) \).
Proof. Since $ii$ is a non-edge, Corollary 3.3 implies that $Q^k_i(\Omega) = (I + D_i \Omega)^k = I + kD_i \Omega = Q_i(k\Omega)$. \hfill \Box

Lemma 3.5 (Multiplication by $T_{ij}$). Multiplying a matrix by $T_{ij}$ on the left has the effect of zeroing out all rows except the $j$th row and then moving the $j$th row to the $i$th row.

Multiplying a matrix by $T_{ij}$ on the right has the effect of zeroing out all columns except the $i$th column and then moving the $i$th column to the $j$th column.

Let $e_1, \ldots, e_n$ be the standard basis vectors. We follow the convention that vectors denoted by bold lowercase letters are column vectors. Row vectors are written as transposes of column vectors. The $i$th row of a matrix $M$ is $e_i^T M$.

Lemma 3.6. Let $Y = \Omega D_{i_1}\Omega \cdots \Omega D_{i_K}\Omega$. Then

$$e_{i_0}^T Y = \omega_{i_0 i_1} \omega_{i_1 i_2} \cdots \omega_{i_{K-1} i_K} e_{i_K}^T \Omega.$$

Proof. By Lemma 3.2, we have

$$D_{i_0} Y = (D_{i_0} \Omega D_{i_1}) (D_{i_1} \Omega D_{i_2}) \cdots (D_{i_{K-1}} \Omega D_{i_K}) \Omega = \omega_{i_0 i_1} \omega_{i_1 i_2} \cdots \omega_{i_{K-1} i_K} T_{i_0 i_1} T_{i_1 i_2} \cdots T_{i_{K-1} i_K} \Omega = \omega_{i_0 i_1} \omega_{i_1 i_2} \cdots \omega_{i_{K-1} i_K} T_{i_0 i_K} \Omega.$$

In addition, $e_{i_0}^T Y = e_{i_0}^T (D_{i_0} Y)$ by Lemma 3.1 and $e_{i_0}^T T_{i_0 i_K} \Omega = e_{i_K}^T \Omega$ by Lemma 3.5. \hfill \Box

The following statement is the analogue of the fact that Dehn twists about disjoint curves commute.

Lemma 3.7. If $ii'$ is a non-edge in $G(\Omega)$, then $Q_i$ and $Q_{i'}$ commute.

Proof. We have $Q_i Q_{i'} = (I + D_i \Omega)(I + D_{i'} \Omega) = I + D_i \Omega + D_{i'} \Omega + D_i \Omega D_{i'} \Omega$, where the last term vanishes by Corollary 3.3. \hfill \Box

3.2 Perron–Frobenius Penner matrices

For $\Omega \in \mathcal{O}$, let $\Gamma_{gen}(\Omega) = \{M(\Omega) : M \in \Gamma_{gen}\}$. A necessary condition for $M \in \Gamma(\Omega)$ to be Perron–Frobenius is that $M \in \Gamma_{gen}(\Omega)$. Indeed, for instance, if $M$ can be written as the product of $Q_2, \ldots, Q_{n+m}$, then the first row of $M$ (and all powers of $M$) equals the first row of the identity matrix. However, this is not a sufficient condition since if $\Omega = 0$, then $Q_1, \ldots, Q_n$ all equal the identity matrix, and no element of $\Gamma_{gen}(\Omega)$ is Perron–Frobenius.

Lemma 3.8. If $M \in \Gamma_{gen}(\Omega)$, then $M \geq I + \Omega$.

Proof. $M = \prod_{k=1}^K Q_{i_k} = \prod_{k=1}^K (1 + D_{i_k} \Omega) \geq 1 + \sum_{k=1}^K D_{i_k} \Omega \geq 1 + \sum_{i=1}^n D_i \Omega = 1 + \Omega$. \hfill \Box

Proposition 3.9. $M \in \Gamma_{gen}(\Omega)$ is Perron–Frobenius if and only if $I + \Omega$ is Perron–Frobenius.
Proof. For any positive integer \( N \) we have \((I + \Omega)^N \leq M^N\) by Lemma 3.8, and
\[
M^N = \left( \prod_{k=1}^K Q_{ik} \right)^N \leq \left( \prod_{k=1}^K (I + \Omega) \right)^N = (I + \Omega)^{KN}
\]
using that \( Q_i \leq I + \Omega \) for \( 1 \leq i \leq n \). So if \( M \) has a power with positive entries, then so does \( I + \Omega \), and vice versa. \(\square\)

**Proposition 3.10.** \( I + \Omega \) is Perron–Frobenius if and only if \( G(\Omega) \) is connected.

Proof. By the binomial formula, we have
\[
(I + \Omega)^N = \sum_{k=0}^N \binom{k}{N} \Omega^k.
\]
Note that the \((i,j)\)-entry of \( \Omega^k \) is positive if and only if there exists a path of length \( k \) between \( i \) and \( j \) in \( G(\Omega) \). Therefore \((I + \Omega)^N > 0\) if and only if there is a path of length at most \( N \) between each pair of vertices. \(\square\)

**Corollary 3.11.** A matrix \( M \in \Gamma(\Omega) \) is Perron–Frobenius if and only if \( \Omega \in O_{\text{conn}} \) and \( M \in \Gamma_{\text{gen}}(\Omega) \).

Proof. This follows immediately from Propositions 3.9 and 3.10 and the necessity of \( M \in \Gamma_{\text{gen}}(\Omega) \) discussed at the beginning of this section. \(\square\)

Recall that if \( M \) is a Perron–Frobenius matrix, then \( \lambda(M) \) denotes its Perron–Frobenius eigenvalue.

**Lemma 3.12.** If \( M_1 \) and \( M_2 \) are Perron–Frobenius matrices such that \( M_1 \leq M_2 \), then \( \lambda(M_1) \leq \lambda(M_2) \).

Proof. The spectral radius \( \sigma(M) \) of a Perron–Frobenius matrix equals \( \lambda(M) \). By Gelfand’s formula for the spectral radius, we have
\[
\lambda(M_1) = \sigma(M_1) = \lim_{k \to \infty} ||M_1^k||^{1/k} \leq \lim_{k \to \infty} ||M_2^k||^{1/k} = \sigma(M_2) = \lambda(M_2),
\]
where \( ||\cdot|| \) is any matrix norm. \(\square\)

**Proposition 3.13.** Let \( \Omega_k \in \partial \) be a sequence converging to a point \( \Omega^* \in (\partial_{\infty} \partial)_{\text{conn}} \), and let \( M \in \Gamma_{\text{gen}} \). Then
(i) \( M(\Omega_k) \) is Perron–Frobenius if \( k \) is large enough, and
(ii) \( \lambda(M(\Omega_k)) \to \infty \).
Proof. Since $\Omega_k \to \Omega^*$, eventually $G(\Omega^*) \subseteq G(\Omega_k)$, therefore $\Omega_k \in \mathcal{O}_{\text{conn}}$. By Proposition 3.10 and Corollary 3.11, $I + \Omega_k$ and $M(\Omega_k)$ are Perron–Frobenius if $k$ is large enough.

Denote by $s_k$ the smallest entry of $\Omega_k$ that is at a position $(i,j)$, where $ij$ is an edge of $G(\Omega^*)$. Since $\Omega_k \to \Omega^*$, we have $s_k \to \infty$. The Perron–Frobenius eigenvalue is bounded below by the smallest row sum [Spi94, Proposition 9.41], and all rows of $I + \Omega_k$ contain an entry which is at least $s_k$, thus $\lambda(I + \Omega_k) \geq s_k$. By Lemma 3.8, we have $M(\Omega_k) \geq I + \Omega_k$, hence $\lambda(M(\Omega_k)) \geq \lambda(I + \Omega_k) \geq s_k$ by Lemma 3.12. It follows that $\lambda(M(\Omega_k)) \to \infty$. \hfill \qed

### 3.3 Induced maps

The matrix $\Omega$ is symmetric, so its left and right nullspaces are the same; we denote them by $\ker(\Omega)$. The row and column spaces are also equal, and we denote them by $\im(\Omega)$. We have an orthogonal decomposition

$$V = \ker(\Omega) \oplus \im(\Omega),$$

where $V = \mathbb{R}^n$.

**Proposition 3.14.** We have

$$\ker(\Omega) = \{ v \in V : Q_i v = v \text{ for all } 1 \leq i \leq n \} = \{ v \in V : M v = v \text{ for all } M \in \Gamma(\Omega) \}.$$

**Proof.** $\Omega v = 0 \iff \forall i D_i \Omega v = 0 \iff \forall i Q_i v = v$, which implies the first equality. The second equality follows from the fact that the $Q_i$ generate $\Gamma(\Omega)$. \hfill \qed

For any $M \in \Gamma(\Omega)$, multiplication on the left by $M$ induces a linear map

$$L_M : V \to V$$

that acts on $\ker(\Omega)$ as the identity by Proposition 3.14. Therefore $L_M$ descends to a linear map

$$\hat{L}_M : \hat{V} \to \hat{V}$$

where

$$\hat{V} = V / \ker(\Omega).$$

If $\text{rank}(\Omega) = r$, then $\dim(\ker(\Omega)) = n - r$ and $\dim(\hat{V}) = r$.

### 3.4 The case when $G(\Omega)$ is bipartite

In this section, suppose that $G(\Omega)$ is bipartite. (This is the case if the surface $S$ in Penner’s construction is orientable.) We also assume that $\Omega$ has the block form

$$\begin{pmatrix}
0 & 0 \\
X & 0
\end{pmatrix},$$

where $X$ is a $a \times b$ matrix. The remaining cases can be reduced to this case by permutation of the basis vectors.
Define the alternating bilinear form \( \langle \cdot, \cdot \rangle_\Delta \) on \( V \) by the formula \( \langle v_1, v_2 \rangle_\Delta = v_1^T \Delta v_2 \), where
\[
\Delta = \begin{pmatrix} 0 & X \\ -X^T & 0 \end{pmatrix}.
\]
The matrices \( \Omega \) and \( \Delta \) are related by the equations \( \Delta = U \Omega = -\Omega U \), where
\[
U = \begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}
\] (3.1)
and \( I_a \) and \( I_b \) are the \( a \times a \) and \( b \times b \) identity matrices.

**Proposition 3.15.** The alternating form \( \langle \cdot, \cdot \rangle_\Delta \) is invariant under \( L_M \) for all \( M \in \Gamma(\Omega) \).

**Proof.** We need to show that \( M^T \Delta M = \Delta \) for all \( M \in \Gamma(\Omega) \). It suffices to prove that \( Q_i^T \Delta Q_i = \Delta \) for \( 1 \leq i \leq n \). Using the fact that the diagonal matrices \( U \) and \( D_i \) commute, we have
\[
Q_i^T \Delta Q_i = (I + \Omega D_i)U\Omega(I + D_i\Omega) = U\Omega + \Omega D_i U\Omega + U\Omega D_i \Omega + \Omega D_i U\Omega D_i \Omega = \Delta - U\Omega D_i \Omega + U\Omega D_i \Omega + \Omega UD_i \Omega D_i \Omega = \Delta,
\]
where the last term vanishes by Corollary 3.3. \( \square \)

**Proposition 3.16.** The matrix \( \Delta \) gives rise to a symplectic form on \( \hat{V} \) that is invariant under \( \hat{L}_M \) for all \( M \in \Gamma(\Omega) \).

**Proof.** If \( v_0, w_0 \in \ker(\Omega) \), then
\[
(w + w_0)^T \Delta (v + v_0) = w^T \Delta v,
\]
because \( \Delta v_0 = U\Omega v_0 = 0 \) and \( w_0^T \Delta = (\Delta w_0)^T = 0 \). As a consequence, \( \Delta \) gives rise to a well-defined alternating form on \( \hat{V} \) that we also denote by \( \langle \cdot, \cdot \rangle_\Delta \). This form is nondegenerate, because for any \( v \in V \setminus \ker(\Omega) \), we have \( \Delta v \neq 0 \), therefore there exists \( w \in V \) with \( w^T \Delta v \neq 0 \). Hence \( \langle \cdot, \cdot \rangle_\Delta \) is a symplectic form on \( \hat{V} \). The invariance under \( \hat{L}_M \) follows from Proposition 3.15. \( \square \)

Recall that a polynomial \( p(x) \in \mathbb{R}[x] \) of degree \( d \) is reciprocal if \( p(x) = x^d p(x^{-1}) \).

**Corollary 3.17.** If \( G(\Omega) \) is bipartite, then the characteristic polynomial \( \chi_{\hat{L}_M}(x) \) is reciprocal for all \( M \in \Gamma(\Omega) \).

**Proof.** Symplectic transformations have reciprocal characteristic polynomials [MHO09, Chapter 2]. \( \square \)

We remark that the converse is also true: if \( G(\Omega) \) is not bipartite and \( M \in \Gamma_{\text{gen}}(\Omega) \), then \( \chi_{\hat{L}_M}(x) \) is not reciprocal. (We will not need this.) One can prove this by “lifting” the matrices \( Q_i \) to the bipartite double cover of \( G(\Omega) \), analogously to lifting a homeomorphism of a nonorientable surface to an orientable surface.
3.5 Multiplicity of the eigenvalue 1

Define a quadratic form \( h \) on \( V \) by the formula

\[
h(v) = \frac{1}{2} v^T \Omega v.
\] (3.2)

It was shown in Proposition 2.1 in [SS15] that

\[
h(Q_i v) - h(v) = ||Q_i v - v||^2
\] (3.3)

for \( 1 \leq i \leq n \) and \( v \in V \). The function \( h \) can be thought of as a height function, and (3.3) says that the matrices \( Q_i \) act on \( V \) by increasing the height. The main purpose of the function \( h \) in [SS15] was to show that no matrix \( M \in \Gamma(\Omega) \) can have an eigenvalue on the unit circle other than 1. Now we show that \( h \) can also be used to determine the multiplicity of the eigenvalue 1.

**Proposition 3.18.** If \( M \in \Gamma_{\text{gen}}(\Omega) \), then 1 is not an eigenvalue of \( \hat{L}_M \).

**Proof.** Suppose on the contrary that \( \hat{L}_M(\tilde{v}) = \tilde{v} \) for some nonzero \( \tilde{v} \in \tilde{V} \). Then there is some \( v \in V \) such that \( v \notin \ker(\Omega) \) and \( M v - v \in \ker(\Omega) \). Using (3.2), we have \( h(Mv) = h(v) \). By (3.3), this is only possible if \( Q_i v = v \) for \( 1 \leq i \leq n \). Hence \( v \in \ker(\Omega) \) by Proposition 3.14, which is a contradiction. \( \square \)

**Proposition 3.19** (Structure of the characteristic polynomials). If \( M \in \Gamma_{\text{gen}}(\Omega) \), then \( \chi_M(x) = (x - 1)^{n-r} p(x) \), where \( r = \text{rank}(\Omega) \), and \( p(x) \) is a degree \( r \) monic polynomial such that \( p(1) \neq 0 \).

**Proof.** Since \( L_M \) acts as the identity on the subspace \( \ker(\Omega) \) of dimension \( n - r \), we have

\[
\chi_M(x) = (x - 1)^{n-r} \chi_{\hat{L}_M}(x).
\] (3.4)

The statement now follows from Proposition 3.18. \( \square \)

4 Continuous extensions

This section is devoted to the proof of Theorem 2.2. Before delving into the details, we outline the main steps of the proof.

Let \( M^p \in \Gamma_{\text{gen}} \) and suppose \( \Omega_k \in \mathcal{O} \) is a sequence converging to \( \Omega^* \in \partial_\infty \mathcal{O} \). Our main goal is to show that \( W^p_\gamma(\Omega_k) \) and \( f^p_\gamma(\Omega_k) \) converge as \( k \to \infty \).

1. In Section 4.1, we provide an asymptotic estimate for the left Perron–Frobenius eigenvectors \( v^T_\lambda(\Omega_k) \) of \( M^p_\gamma(\Omega_k) \).

2. Using this, we show in Section 4.2 that the left invariant subspaces \( \langle v^T_\lambda(\Omega_k) \rangle \) of \( M^p_\gamma(\Omega_k) \) converge in \( \text{Gr}(1,n) \); therefore, the right invariant subspaces \( W^p_\gamma(\Omega_k) \) converge in \( \text{Gr}(n-1,n) \).

3. For the convergence of the linear endomorphisms \( f^p_\gamma(\Omega_k) \), we first show that the left action of \( Q^p_\gamma(\Omega_k) \) on \( W^p_\gamma(\Omega_k) \) is well approximated by the left action of a matrix independent of \( k \) (Proposition 4.11).
4. Using this, we then show that the left action of all other factors $Q^{p_j}_j(\Omega_k)$, $\ldots$, $Q^{p_K}_K(\Omega_k)$ in the product form of $M^p_j(\Omega_k)$ are well approximated by left actions of matrices independent of $k$ (Proposition 4.12).

5. It is then a simple consequence that the maps $f^p(\Omega_k)$—the left actions of the matrices $M^p_j(M_k)$ on $W^p_j(\Omega_k)$—converge to a limit (Proposition 4.14).

4.1 Estimating the left Perron–Frobenius eigenvectors

Let $M \in \Gamma(\Omega)$ be Perron–Frobenius with Perron–Frobenius eigenvalue $\lambda$. It is well-known that the left Perron–Frobenius eigenvector $v_\lambda^T$ can be chosen so that $v_\lambda^T \geq 0$, and all left eigenvectors of $M$ that lie in the positive cone $\mathbb{R}_{>0}^n$ are scalar multiples of $v_\lambda^T$. The standard proof of the existence of an eigenvector in the positive cone builds on the fact that the right action of $M$ maps $\mathbb{R}_{>0}^n$ into itself and uses the Brouwer fixed point theorem to conclude that there is a fixed ray in $\mathbb{R}_{>0}^n$.

Our first step towards locating $v_\lambda^T$ more accurately is finding a smaller invariant cone inside $\mathbb{R}_{>0}^n$. By the same fixed point argument, we are guaranteed to find an eigenvector inside this smaller cone, and this eigenvector has to be a Perron–Frobenius eigenvector.

**Proposition 4.1.** Let

$$C = \{v^T \Omega : v^T \geq 0\}$$

be the cone generated by the rows of $\Omega$. Then we have $CM \subset C$ for all $M \in \Gamma(\Omega)$.

**Proof.** It is sufficient to show that $C$ is invariant under the right action of the generators $Q_i$. For this, observe that

$$\Omega Q_i = \Omega (I + D_i) = (I + \Omega D_i) \Omega = Q_i^T \Omega$$

for $1 \leq i \leq n$. For an arbitrary element $v^T \Omega \in C$, the vector $(v^T \Omega) Q_i = (v^T Q_i^T) \Omega$ is contained in the $C$, since $v \geq 0$ implies $v^T Q_i^T \geq 0$.

As explained above, the fact that $C$ is invariant under the right action of $M$ implies that $v_\lambda^T \in C$. This is an improvement over $v_\lambda^T \in \mathbb{R}_{>0}^n$, but $C$ is still a large cone. For example, it is not contained in the interior of $\mathbb{R}_{>0}^n$.

By multiplying both sides of the containment $CM \subset C$ by $M$, we get $(CM)M \subset CM$, so the smaller cone $CM$ inside $C$ is also invariant under $M$. Hence we have the following.

**Proposition 4.2.** The left Perron–Frobenius eigenvector $v_\lambda^T$ of a Perron–Frobenius matrix $M \in \Gamma(\Omega)$ can be chosen so that $v_\lambda^T \in CM$.

Introduce the following notations:

- $p_{\text{max}} = \max\{p_1, \ldots, p_K\}$,
- $p_{\text{min}} = \min\{p_1, \ldots, p_K\}$,
- $||\Omega||_\infty$ — the maximum of the entries of $\Omega$,
- $||\Omega||_{\text{min}, \gamma}$ — the minimum of those entries of $\Omega$ that are at a position $(i,j)$ such that $ij$ is an edge in $\gamma$. 

20
Recall that we write $\omega_{ij}$ for the $(i, j)$-entry of $\Omega$.

**Proposition 4.3.** Suppose $\gamma \subset G(\Omega)$ and $\gamma$ visits every vertex of $G(\Omega)$ at least once. If $\|\Omega\|_{\min, \gamma} \geq 1$, then left Perron–Frobenius eigenvector $v^T_\lambda$ of $M^p(\Omega)$ can be chosen so that

$$
\|v^T_\lambda - p_1 e^T_{i_1} \Omega - \omega_{i_2 i_1}^{-1} e^T_{i_2} \Omega\|_\infty \leq 2^{K-2} \|\Omega\|^{K-1}_{\infty}.
$$

**Proof.** Note that we can assume that $\gamma$ visits every vertex of $G(\Omega)$ at least twice. This is because the square of $M^p(\Omega)$ is associated to a path visiting every vertex at least twice, has the same Perron–Frobenius eigenvectors as $M^p(\Omega)$, and the quantities in (4.6) are the same for $M^p(\Omega)$ and its square.

The cone $C(\Omega)M^p(\Omega)$ is generated by the rows of $\Omega M^p(\Omega)$.

$$
\Omega M^p(\Omega) = \Omega Q_i^p(\Omega) \cdots Q_i^p(\Omega) = \Omega(I + p_K D_{i_1} \Omega) \cdots (I + p_1 D_{i_1} \Omega) = \\
= \Omega + \sum_{1 \leq t_1 < \cdots < t_k \leq K} p_1 \cdots p_k D_{i_1} \Omega \cdots D_{i_k} \Omega = \\
= \Omega + \sum_{i=1}^n \sum_{1 \leq t_1 < \cdots < t_k \leq K} p_1 \cdots p_k D_{i_1} \Omega \cdots D_{i_k} \Omega.
$$

By Lemma 3.6, we have the expression

$$
e^T_i \Omega M^p(\Omega) = e^T_i \Omega + \sum_{i=1}^n H(i, i'), \quad (4.1)
$$

for the $i'$th row, where

$$
H(i, i') = \sum_{1 \leq t_1 < \cdots < t_k \leq K} p_1 \cdots p_k \omega_{i_1 i_2} \cdots \omega_{i_k i_1} e^T_1 \Omega. \quad (4.2)
$$

A summand on the right hand side is nonzero if and only if $(i_{t_1} \ldots i_{t_k}) \subset G(\Omega)$. In particular, if $i' i'' \notin G(\Omega)$, then all summands vanish and $H(i, i') = 0$.

Now suppose that $i' i'' \in G(\Omega)$. Let $t$ be the largest integer such that $i_t = i$. Since $\gamma$ visits $i$ at least twice, such $t$ exists and $t \geq 3$. Since $(i_1 \ldots i_t)$ and $(i_2 \ldots i_t)$ are paths in $G(\Omega)$, the right hand side of (4.2) contains the nonzero summands

$$
p_1 \cdots p_t \omega_{i_1 i_2} \cdots e^T_1 \Omega,
$$

$$
p_2 \cdots p_t \omega_{i_1 i_2} \cdots e^T_2 \Omega.
$$

Moreover, these two paths are the unique longest and second longest paths of the form $(i_{t_1} \ldots i_{t_1})$ in $G(\Omega)$. So we have

$$
H(i, i') = c(i, i') \left( p_1 e^T_{i_1} \Omega + \omega_{i_2 i_1}^{-1} e^T_{i_2} \Omega + R(i, i') \right), \quad (4.3)
$$

21
where \( c(i, i') = p_2 \cdots p_{i(i'1)} \cdots i_{2i} \) and \( R(i, i') \) is a sum of expressions of the form

\[
\prod p_i \prod \omega_{ij} c_{i'}^T \Omega
\]

such that the number of multiplicands \( \omega_{i'j'} \) in the denominator is at least two more than the number of multiplicands \( \omega_{ij} \) in the numerator. Moreover, since \( \omega_{i'1} \) appears in all terms of the right hand side of (4.2), we can simplify the fractions so that and the multiplicands \( \omega_{i'j'} \) include only \( \omega_{i'1_i}, \ldots, \omega_{i'2i} \). So the supremum norm of each expression of the form (4.4) is bounded from above by

\[
\frac{p_{\text{max}}^{K-2} ||\Omega||_{\infty}^{K-2}}{p_{\text{min}} ||\Omega||_{\min, \gamma}^{K}}.
\]

(This is the first time where the hypothesis \( ||\Omega||_{\min, \gamma} \geq 1 \) is used.)

A trivial upper bound on the number of terms in the sum \( R(i, i') \) is \( 2^K - 1 \), the number of nonempty subsets of \( \{i_1, \ldots, i_K\} \), hence

\[
||R(i, i')|| \leq (2^K - 1) \frac{p_{\text{max}}^{K-2} ||\Omega||_{\infty}^{K-1}}{p_{\text{min}} ||\Omega||_{\min, \gamma}^{K}}.
\]

We remark that (4.3) and (4.5) hold also when \( ii' \) is a non-edge in \( G(\Omega) \) provided \( c(i, i') \) and \( R(i, i') \) are defined to be zero.

By substituting (4.3) into (4.1), we conclude that the generators of \( C(\Omega) \mathbf{M}_\gamma^p(\Omega) \) can be chosen as

\[
\mathbf{w}_{i'}^{T} = \mathbf{e}_{i}^{T} \Omega \mathbf{M}^p(\Omega) = p_1 e_{i1}^T \Omega + \omega_{i2i}^{-1} e_{i2}^T \Omega + \frac{\sum_{i=1}^{n} c(i, i') R(i, i')}{\sum_{i=1}^{n} c(i, i')} + \frac{e_{i}^{T} \Omega}{\sum_{i=1}^{n} c(i, i')}.
\]

The third term is a convex combination of the \( R(i, i') \), so the same bound holds for it as in (4.5). To obtain a bound for the last term, note that \( c(i, i') \geq p_{\text{min}} ||\Omega||_{\min, \gamma}^2 \) whenever \( ii' \in G(\Omega) \), because the number of multiplicands \( \omega_{ij} \) in the definition of \( c(i, i') \) is always at least two. (The hypothesis \( ||\Omega||_{\min, \gamma} \geq 1 \) is used here as well.) Hence

\[
\left\| \sum_{i=1}^{n} c(i, i') \right\|_{\infty} \leq \frac{||\Omega||_{\infty}^{2}}{p_{\text{min}} ||\Omega||_{\min, \gamma}^2} \leq \frac{p_{\text{min}}^{K-2} ||\Omega||_{\min, \gamma}^{K-1}}{p_{\text{min}} ||\Omega||_{\min, \gamma}^K}.
\]

By Proposition 4.2, \( v_{i'}^T \) can be chosen to be a convex combination of the \( w_{i'}^T \), and the statement follows.

Recall the definition of \( \partial^\infty \Omega \) from (2.3).

**Corollary 4.4.** Let \( \mathbf{M}^p(\Omega) \in \Gamma_{\text{gen}} \), where \( \gamma = (i_1 i_2 \ldots i_{K1}) \) and \( p = (p_1, \ldots, p_K) \).

Suppose \( \Omega_k \in \Omega \) is a sequence such that \( \Omega_k \to \Omega^* \in \partial^\infty \Omega \). Then the left Perron–Frobenius eigenvector \( v_{i'}^T(\Omega_k) \) of \( \mathbf{M}^p(\Omega_k) \) can be chosen for all \( k \) so that

\[
\lim_{k \to \infty} \left[ v_{i'}^T(\Omega_k) - \left( p_1 e_{i1}^T \Omega_k + \omega_{i2i1}^{-1} e_{i2}^T \Omega_k \right) \right] = 0.
\]

(Here \( \omega_{i2i1} \) denotes the \((i_2, i_1)\)-entry of \( \Omega_k \). The fact that \( \mathbf{M}^p(\Omega_k) \) is Perron–Frobenius if \( k \) is large enough was shown in Proposition 3.13.)
Proof. If \( k \) is large enough, then \( G(\Omega^*) \subset G(\Omega_k) \), hence \( \gamma \subset G(\Omega_k) \). Since \( ||\Omega_k||_{\min,\gamma} \to \infty \), we may apply Proposition 4.3. Using that \( ||\Omega_k||_{\infty}/||\Omega_k||_{\min,\gamma} \) stays bounded completes the proof. \( \square \)

The intuition behind this result is the following. The cone \( C(\Omega_k)M_p(\Omega_k) \) is the image of the cone \( C(\Omega_k) \) under the right action of \( Q_{i_1}(\Omega_k), \ldots, Q_{i_1}(\Omega_k) \) in this order. Where \( C(\Omega_k) \) ends up after these actions is determined by the most part by the last action. The right action of \( Q_{i_1}(\Omega_k) \) is trivial on all standard basis vectors except one which is translated by \( p_1e_{i_1}^T \Omega_k \). Thus we can think of the right action of \( Q_{i_1}(\Omega_k) \) as a map sending the cone \( C(\Omega_k) \) towards the direction of \( e_{i_1}^T \Omega_k \). This is why \( e_{i_1}^T \Omega_k \) appears in (4.6) and why it appears with the largest weight.

The second to last action is the action of \( Q_{i_2}(\Omega_k) \), sending the \( C(\Omega_k) \) towards the direction of \( e_{i_2}^T \Omega_k \). This action has a smaller effect than the last action, but a more significant one than all the actions before. So \( e_{i_2}^T \Omega_k \) still appears in (4.6), but with a smaller weight than \( e_{i_1}^T \Omega_k \). The rest of the actions turn out to be negligible as \( k \to \infty \).

This line of thought also provides some explanation for why the assumption \( \gamma \in \partial^*_\infty \) is necessary: since \( Q_{i_1}(\Omega_k) \) and \( Q_{i_2}(\Omega_k) \) commute when \( i_1i_2 \) is a non-edge in \( G(\Omega_k) \), the last and second to last actions would not be well-defined if \( i_1i_2i_3 \) was not a path in \( G(\Omega_k) \).

### 4.2 Convergence of subspaces

We have seen that whenever \( \Omega \in \Omega_{\text{conn}} \) and \( M \in \Gamma_{\text{gen}} \), the matrix \( M(\Omega) \) is Perron–Frobenius. Its Perron–Frobenius eigenvalue \( \lambda(M(\Omega)) \) is simple, hence the left eigenspace \( E_{\lambda}(M(\Omega)) \) associated to \( \lambda(M(\Omega)) \) is 1-dimensional. Therefore we have a continuous function

\[
E_{\lambda} \circ M : \Omega_{\text{conn}} \to \text{Gr}(1,n)
\]

for all \( M \in \Gamma_{\text{gen}} \).

The subspaces \( \langle e_{i_1}^T \Omega \rangle \) are constant along rays, which makes \( \langle e_{i_1}^T \Omega^* \rangle \) well-defined for all \( \Omega^* \in \partial^*_\infty \). On the rays where \( e_{i_1}^T \Omega \neq 0 \), we have \( \langle e_{i_1}^T \Omega^* \rangle \in \text{Gr}(1,n) \).

**Proposition 4.5.** If \( M_p \in \Gamma_{\text{gen}} \), then \( E_{\lambda} \circ M_p \) extends continuously from \( \Omega_{\text{conn}} \) to \( \text{Gr}(1,n) \). Moreover, if \( \gamma = (i_1 \ldots i_k \ldots) \), then for all \( \Omega^* \in \partial^*_\infty \)

\[
E_{\lambda}(M_p^p(\Omega^*)) = \langle e_{i_1}^T \Omega^* \rangle
\]

**Proof.** We need to show that

\[
\lim_{k \to \infty} E_{\lambda}(M_p^p(\Omega_k)) = \langle e_{i_1}^T \Omega^* \rangle
\]

whenever \( \Omega_k \in \Omega \) is a sequence such that \( \Omega_k \to \Omega^* \in \partial^*_\infty \). This follows from Corollary 4.4, since the term \( \omega_{j_1i_1(k)}^{-1}e_{i_1}^T \Omega_k \) stays bounded, the term \( p_1e_{i_1}^T \Omega_k \) goes off to infinity, and the lines \( p_1e_{i_1}^T \Omega_k \) converge to \( \langle e_{i_1}^T \Omega^* \rangle \). \( \square \)

Recall from Section 2.4 that for all \( M \in \Gamma_{\text{gen}} \) and \( \Omega \in \Omega_{\text{conn}} \), the subspace \( W(M(\Omega)) \) is defined as the codimension 1 subspace of \( \mathbb{R}^n \) spanned by the right eigenspaces of \( M(\Omega) \) corresponding to eigenvalues other than \( \lambda(M(\Omega)) \). Since a left
and a right eigenvector of a matrix are orthogonal if they correspond to different eigenvalues, we have

\[ \mathbf{W}(\mathbf{M}(\Omega)) = (\mathbf{E}_\lambda(\mathbf{M}(\Omega)))^\perp. \]

Recall also that for an alternating element \( \mathbf{M}_p^\gamma \in \Gamma_{\text{gen}} \), we use the shorthand \( \mathbf{W}_p^\gamma(\Omega) \) for \( \mathbf{W}(\mathbf{M}_p^\gamma(\Omega)). \)

**Proposition 4.6.** Let \( \mathbf{M}_p^\gamma \in \Gamma_{\text{gen}} \), where \( \gamma = (i_1 \ldots i_K) \). Then \( \mathbf{W}_p^\gamma \) extends continuously to \( \partial_\infty^\gamma \mathbb{O} \), and \( \mathbf{W}_p^\gamma|_{\partial_\infty^\gamma \mathbb{O}} = \mathbf{W}_\gamma \), where \( \mathbf{W}_\gamma: \partial_\infty^\gamma \mathbb{O} \rightarrow \text{Gr}(n-1,n) \) is defined as

\[ \mathbf{W}_\gamma(\Omega^*) = (\mathbf{e}_i^T\Omega^*)^\perp. \quad (4.7) \]

**Proof.** Clearly \( \mathbf{W}_\gamma \) is continuous on \( \partial_\infty^\gamma \mathbb{O} \), so we only need to show

\[ \lim_{k \to \infty} \mathbf{W}_p^\gamma(\Omega_k) = (\mathbf{e}_i^T\Omega^*)^\perp \]

for a sequence \( \Omega_k \in \mathbb{O} \) such that \( \Omega_k \rightarrow \Omega^* \in \partial_\infty^\gamma \mathbb{O} \). This follows from Proposition 4.5 by taking the limit of both sides of the equation

\[ \mathbf{W}_p^\gamma(\Omega_k) = (\mathbf{E}_\lambda(\mathbf{M}_p^\gamma(\Omega_k)))^\perp. \]

We remark that the definition (4.7) for \( \mathbf{W}_\gamma \) makes sense even if \( \gamma \) does not visit every vertex of \( G_n \).

### 4.3 The matrices \( Q_{i \leftarrow j} \)

For \( \Omega \in \mathbb{O} \) and an edge \( ij \) in \( \mathbf{G}(\Omega) \), we have \( \omega_{ij} > 0 \), so we may introduce the definition

\[ Q_{i \leftarrow j} = Q_{i \leftarrow j}(\Omega) = I - \omega_{ij}^{-1}T_{ji}^\Omega, \quad (4.8) \]

where \( T_{ji} \) was defined in Section 3.1. In words, \( \Omega_{i \leftarrow j} \) is calculated in the following way. Zero out all rows of \( \Omega \) except the \( i \)th row, move the \( i \)th row to the \( j \)th row (Lemma 3.5), and then normalize it so that the entry on the diagonal is 1. Subtracting this matrix from the identity matrix gives \( Q_{i \leftarrow j} \). As a consequence, we have the following facts.

**Lemma 4.7.** The matrix \( Q_{i \leftarrow j} \) differs from the identity matrix only in its \( j \)th row.

**Lemma 4.8.** The \( j \)th column of \( Q_{i \leftarrow j} \) is zero.

**Proposition 4.9.** The left nullspace of \( Q_{i \leftarrow j} \) is \( \{ \mathbf{e}_i^T \Omega \} \).

**Proof.** In order for \( Q_{i \leftarrow j} \) to be defined, we must have \( \omega_{ij} > 0 \). So \( \mathbf{e}_i^T \Omega \neq 0 \). By Lemmas 4.7 and 4.8, the corank of \( Q_{i \leftarrow j} \) is 1, therefore we only need to show that \( \mathbf{e}_i^T \Omega Q_{i \leftarrow j} = 0 \). One can easily check that \( \mathbf{e}_i^T \Omega T_{ji} = \omega_{ij} \mathbf{e}_i^T \), therefore \( \mathbf{e}_i^T \Omega (\omega_{ij}^{-1}T_{ji}^\Omega) = \mathbf{e}_i^T \Omega \). Rearranging this, we have \( 0 = \mathbf{e}_i^T \Omega (I - \omega_{ij}^{-1}T_{ji}^\Omega) = \mathbf{e}_i^T \Omega Q_{i \leftarrow j} \).

**Proposition 4.10.** Left multiplication by \( Q_{i \leftarrow j} \) is a projection on the hyperplane \( (\mathbf{e}_i^T \Omega)^\perp \) in the direction of \( \mathbf{e}_j \).
Proof. For all \( v \in V \), we have \( 0 = (e_i^T \Omega Q_{i \rightarrow j}) v = e_i^T \Omega (Q_{i \rightarrow j} v) \). Therefore the image of the left multiplication by \( Q_{i \rightarrow j} \) is contained in \( (e_i^T \Omega)^\perp \). Since \( Q_{i \rightarrow j} \) has corank 1, the image actually equals \( (e_i^T \Omega)^\perp \). By Lemma 4.7, left multiplication by \( Q_{i \rightarrow j} \) changes only the \( j \)th coordinate, and the statement follows. \( \square \)

This explains the notation, at least partially. The reason for using \( i \leftarrow j \) instead of \( j \rightarrow i \) is that a product of such matrices, corresponding to a composition of projections, should be read from right to left.

For an edge \( ij \) in \( G_n \), the domain of the continuous function \( Q_{i \rightarrow j} \) is

\[
\{ \Omega \in \mathbb{O} : ij \text{ is an edge in } G(\Omega) \} \subset \mathbb{O}.
\]

Note that \( Q_{i \rightarrow j}(\Omega) \) is constant along rays in \( \mathbb{O} \) from the origin, therefore \( Q_{i \rightarrow j}(\Omega^*) \) is a well-defined matrix when \( ij \) is an edge in \( G(\Omega^*) \). So \( Q_{i \rightarrow j} \) is a continuous function on the extended domain

\[
\{ \Omega \in \mathbb{O} \cup \partial_\infty \mathbb{O} : ij \text{ is an edge in } G(\Omega) \} \subset \mathbb{O} \cup \partial_\infty \mathbb{O}.
\] (4.9)

4.4 Convergence of linear maps

Proposition 4.11. Let \( M_\gamma \in \Gamma_{gen} \) and let \( \Omega_k \in \mathbb{O} \) be a sequence such that \( \Omega_k \rightarrow \Omega^* \in \partial_\infty \mathbb{O} \). If \( v_k \in \mathbb{E}(M_\gamma(\Omega_k))^\perp \) for all \( k \), and \( v_k \rightarrow v^* \), then

\[
\lim_{k \to \infty} Q_{i \rightarrow i}^j(\Omega_k)v_k = Q_{i \rightarrow i}^j(\Omega^*)v^*.
\]

Proof. Clearly \( Q_{i \rightarrow i}^j(\Omega_k) \rightarrow Q_{i \rightarrow i}^j(\Omega^*) \), so \( Q_{i \rightarrow i}^j(\Omega_k)v_k \rightarrow Q_{i \rightarrow i}^j(\Omega^*)v^* \). Therefore it suffices to show that

\[
\lim_{k \to \infty} \left[ Q_{i \rightarrow i}^j(\Omega_k)v_k - Q_{i \rightarrow i}^j(\Omega_k)v_k \right] = 0.
\]

For each \( k \), let \( v_r^j(\Omega_k) \) be a left Perron–Frobenius eigenvector of \( M_\gamma(\Omega_k) \) guaranteed by Corollary 4.4. Then we have

\[
\left| \left( p_1 e_1^T \Omega_k + \omega_{i \rightarrow i}(k) e_2^T \Omega_k \right) v_k \right| = \left| \left( v_r^j(\Omega_k) - \left( p_1 e_1^T \Omega_k + \omega_{i \rightarrow i}(k) e_2^T \Omega_k \right) \right) v_k \right| \leq n \left( \left\| v_r^j(\Omega_k) - \left( p_1 e_1^T \Omega_k + \omega_{i \rightarrow i}(k) e_2^T \Omega_k \right) \right\| \cdot \| v_k \| \right) \rightarrow 0.
\]

The equation holds, because \( v_k \in \mathbb{E}(M_\gamma(\Omega_k))^T \) implies \( v_r^j(\Omega_k)v_k = 0 \). The inequality is a trivial estimate of a scalar product of vectors of length \( n \). Finally, we have convergence to zero by Corollary 4.4. Therefore

\[
\left\| (Q_{i \rightarrow i}^j(\Omega_k) - Q_{i \rightarrow i}^j(\Omega_k)) v_k \right\| \rightarrow 0,
\]

since \( p_1 D_{i \rightarrow i} \Omega_k + \omega_{i \rightarrow i}(k) T_{i \rightarrow i} \Omega_k \) is a matrix whose \( i \)th row equals \( p_1 e_1^T \Omega_k + \omega_{i \rightarrow i}(k) e_2^T \Omega_k \) and whose other rows are zero (see Lemmas 3.1 and 3.5). \( \square \)
For any closed path $\gamma = (i_1i_2 \ldots i_ki_1)$ in $G_n$, let

$$P_\gamma = Q_{i_1 \leftarrow i_k} \cdots Q_{i_k \leftarrow i_1}Q_{i_2 \leftarrow i_1}. \quad (4.10)$$

The domain of $P_\gamma$ restricted to $\partial_\infty \mathcal{O}$ is $\partial_\infty \mathcal{O}$ (cf. (4.9)). Since the $Q_{i \leftarrow j}$ are continuous on their domains, $P_\gamma$ is continuous on $\partial_\infty \mathcal{O}$.

**Proposition 4.12.** Let $MP_\gamma \in \Gamma_{\text{gen}}$ and let $\Omega_k \in \mathcal{O}$ be a sequence such that $\Omega_k \rightarrow \Omega^* \in \partial_\infty \mathcal{O}$. If $v_k \in E_{\lambda}(MP_\gamma(\Omega_k))\perp$ for all $k$, and $v_k \rightarrow v^*$, then

$$\lim_{k \rightarrow \infty} MP_\gamma(\Omega_k)v_k = P_\gamma(\Omega^*)v^*. \quad (4.11)$$

**Proof.** We show the following convergences by induction:

$$\lim_{k \rightarrow \infty} Q_{i_1i_2}^{P_{i_1i_2}}(\Omega_k)v_k = Q_{i_2 \leftarrow i_1}(\Omega^*)v^* \quad (4.11)$$

$$\lim_{k \rightarrow \infty} Q_{i_2i_3}^{P_{i_2i_3}}(\Omega_k)Q_{i_1i_2}^{P_{i_1i_2}}(\Omega_k)v_k = Q_{i_3 \leftarrow i_2}(\Omega^*)Q_{i_2 \leftarrow i_1}(\Omega^*)v^* \quad (4.12)$$

$$\vdots$$

$$\lim_{k \rightarrow \infty} Q_{i_ki_1}^{P_{i_ki_1}}(\Omega_k) \cdots Q_{i_2i_1}^{P_{i_2i_1}}(\Omega_k)v_k = Q_{i_1 \leftarrow i_k}(\Omega^*) \cdots Q_{i_2 \leftarrow i_1}(\Omega^*)v^* \quad (4.13)$$

We have (4.11) by Proposition 4.11, and we want to prove (4.13). We describe in detail how to go from (4.11) to (4.12). The other steps are analogous.

Let $MP_\gamma'$ be the cyclic permutation $Q_{i_1}^{P_{i_1}}Q_{i_2}^{P_{i_2}} \cdots Q_{i_k}^{P_{i_k}}$ of $MP_\gamma$. If $v_\lambda^T(\Omega_k)$ is a left Perron–Frobenius eigenvector of $MP_\gamma(\Omega_k)$, then

$$v_\lambda^T(\Omega_k)Q_{i_ki_1}^{P_{i_ki_1}}(\Omega_k) \cdots Q_{i_2i_1}^{P_{i_2i_1}}(\Omega_k)$$

is a left Perron–Frobenius eigenvector of $MP_\gamma'(\Omega_k)$. So $Q_{i_1i_2}^{P_{i_1i_2}}(\Omega_k)v_k$ is contained in $E_{\lambda}(MP_\gamma'(\Omega_k))\perp$ for all $k$, because

$$0 = \lambda(MP_\gamma(\Omega_k))v_\lambda^T(\Omega_k)v_k = v_\lambda^T(\Omega_k)MP_\gamma(\Omega_k)v_k = (v_\lambda^T(\Omega_k)Q_{i_ki_1}^{P_{i_ki_1}}(\Omega_k) \cdots Q_{i_2i_1}^{P_{i_2i_1}}(\Omega_k))(Q_{i_1i_2}^{P_{i_1i_2}}(\Omega_k)v_k).$$

Therefore we can apply Proposition 4.11 to $MP_\gamma'$ and the sequence of vectors $Q_{i_1i_2}^{P_{i_1i_2}}(\Omega_k)v_k \in E_{\lambda}(MP_\gamma'(\Omega_k))\perp$ that converge to $Q_{i_2 \leftarrow i_1}(\Omega^*)v^*$ by (4.11) to get (4.12).

**Proposition 4.13.** If $\gamma = (i_1 \ldots i_ki_1)$ is a closed path in $G_n$, and $\Omega^* \in \partial_\infty \mathcal{O}$, then the subspace $W_{\gamma}(\Omega^*)$ is invariant under the left action of $P_\gamma(\Omega^*)$.

**Proof.** The image of $P_\gamma(\Omega^*)$ is contained in the image of $Q_{i_1 \leftarrow i_k}(\Omega^*)$ which is $W_{\gamma}(\Omega^*)$ by Proposition 4.10.

As a corollary, for any closed path $\gamma$ in $G_n$ and for all $\Omega^* \in \partial_\infty \mathcal{O}$, left multiplication by $P_\gamma(\Omega^*)$ defines a linear endomorphism $f_\gamma(\Omega^*) \in \text{End}(W_{\gamma}(\Omega^*))$, so we have a function

$$f_\gamma : \partial_\infty \mathcal{O} \rightarrow \text{End}(n-1, n). \quad (4.14)$$
Recall from Section 2.4 that $f(M(\Omega))$ is defined as the linear endomorphism of $W(M(\Omega))$ induced by the left action of $M(\Omega)$ for all $M \in \Gamma_{\text{gen}}$ and $\Omega \in \mathcal{O}_{\text{conn}}$. Thus for all $M \in \Gamma_{\text{gen}}$ we have a function

$$f \circ M : \mathcal{O}_{\text{conn}} \to \text{End}(n - 1, n).$$

It is easy to see that $f_\gamma$ and $f \circ M$ are continuous. Recall also that the shorthand $f_p(\Omega)$ is used for $f(M_p(\Omega))$.

**Proposition 4.14.** If $M_p^\gamma \in \Gamma_{\text{gen}}$, then $f_p^\gamma$ extends continuously to $\partial \mathcal{O}_{\gamma}$. Moreover, we have $f_p^\gamma|_{\partial \mathcal{O}_{\gamma}} = f_\gamma$.

**Proof.** Let $\Omega_k \in \mathcal{O}$ be a sequence such that $\Omega_k \to \Omega^* \in \partial \mathcal{O}_{\gamma}$. Then $f_p^\gamma(\Omega_k) \in \text{End}(W_p^\gamma(\Omega_k))$ for all $k$ and $f_\gamma(\Omega^*) \in \text{End}(W_\gamma(\Omega^*))$. We have $W_p^\gamma(\Omega_k) \to W_\gamma(\Omega^*)$ by Proposition 4.6. It follows from Proposition 4.12 that we have $f_p^\gamma(\Omega_k)(v_k) \to f_\gamma(\Omega^*)(v^*)$ for all sequences $v_k \to v^*$ where $v_k \in W_p^\gamma(\Omega_k)$ and $v^* \in W_\gamma(\Omega^*)$. Hence $f_p^\gamma(\Omega_k) \to f_\gamma(\Omega^*)$ in $\text{End}(n - 1, n)$.

Propositions 4.6 and 4.14 immediately imply

**Theorem 2.2.** If $M_p^\gamma \in \Gamma_{\text{gen}}$, then $W_p^\gamma$ and $f_p^\gamma$ extend continuously to $\partial \mathcal{O}_{\gamma}$. Moreover, $W_p^\gamma|_{\partial \mathcal{O}_{\gamma}}$ and $f_p^\gamma|_{\partial \mathcal{O}_{\gamma}}$ depend only on $\gamma$ and not on $p$.

**Remarks.** We remark that in general $f_p$ definitely does not extend to all of $\partial \mathcal{O}_{\gamma}$, not even to $(\partial \mathcal{O}_{\gamma})_{\text{conn}}$. An example, for instance, is given by $M_p^\gamma = Q_1Q_2Q_3Q_4$ when $n = 4$, where $f_p^\gamma$ doesn’t extend to $\Omega^*$, the ray of

$$\Omega = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix},$$

because there is an eigenvalue that goes to infinity. It is easy to construct lots of such examples by making sure that $\gamma \not\subset \Gamma(\Omega^*)$ and that rank$(\Omega^*)$ is not too small (otherwise most eigenvalues of $M_p^\gamma(\Omega)$ are 1).

It is worth noting that as $k \to \infty$ the eigenvalues of $M_p^\gamma(k\Omega)$ seem to behave in a remarkable way, even when they diverge. For instance, when $M_p^\gamma = Q_1Q_2Q_3Q_4$ and $\Omega$ is as above, then the eigenvalues of $M_p^\gamma(k\Omega)$ are approximately $3k^3$, $k/3$, $-3/k$, $-3/k^3$ asymptotically. This shows that there is more interesting structure hiding inside the monoids $\Gamma(\Omega)$ than we explore in this paper.

### 5 Properties of the extensions

#### 5.1 Compositions of projections

Fix some $\Omega \in \mathcal{O}$ or $\Omega \in \partial \mathcal{O}$, and a closed path $\gamma = (i_1i_2\ldots i_Ki_1)$ in $\Gamma(\Omega)$. Note that we are not assuming here that $\gamma$ visits every vertex. As defined previously in (4.10),
associated to $\gamma$ is the matrix $P_\gamma$, which is well-defined, because the matrices $Q_{i\rightarrow j}$ are well-defined when $ij$ is an edge in $G(\Omega)$.

Introduce the notation $Z_i = (e_i^T \Omega)^{-1}$. If $i$ is not an isolated vertex of $G(\Omega)$, then $e_i^T \Omega \neq 0$ hence $Z_i$ is a hyperplane. Note that $W_\gamma = Z_{i_1}$

Denote the left action of $Q_{i\rightarrow j}$ by $p_{i\rightarrow j}$. By Proposition 4.10, $p_{i\rightarrow j}$ is a projection onto $Z_i$ in the direction of $e_j$.

**Proposition 5.1.** Suppose that $\gamma = (i_1 \ldots i_m)$ is a path in $G(\Omega)$ where $m \geq 3$. Then the image of $p_{i_m \leftarrow i_{m-1}} \circ \cdots \circ p_{i_2 \leftarrow i_1}$ is the codimension 2 subspace $Z_{i_m} \cap Z_{i_{m-1}}$.

**Proof.** The proof goes by induction on $m$. Suppose first that $m = 3$. Note that $e_{i_2} \in \mathbb{Z}_{i_2}$, since the diagonal entries of $\Omega$ vanish. So the projection onto $Z_{i_2}$ in the direction of $e_{i_2}$ followed by the projection onto $Z_{i_3}$ in the direction of $e_{i_3}$ has codimension 2 image, and it equals $Z_{i_2} \cap Z_{i_3}$.

If $m > 3$, then by the induction hypothesis $p_{i_{m-1} \leftarrow i_{m-2}} \circ \cdots \circ p_{i_2 \leftarrow i_1}$ has image $\Sigma = Z_{i_{m-1}} \cap Z_{i_{m-2}}$. Note that $e_{i_{m-1}} \notin \Sigma$, since the $(i_{m-2}, i_{m-1})$-entry of $\Omega$ is nonzero. So $p_{i_m \leftarrow i_{m-1}}(\Sigma)$ is still of codimension 2. Moreover, it clearly contains $Z_{i_m}$, and also $Z_{i_{m-1}}$, because $e_{i_{m-1}} \in Z_{i_{m-1}}$, so $p_{i_m \leftarrow i_{m-1}}$ maps elements of $Z_{i_{m-1}}$ to elements of $Z_{i_{m-1}}$.

Recall that

$$f_\gamma = (p_{i_1 \leftarrow i_K} \circ \cdots \circ p_{i_2 \leftarrow i_1})|_{Z_{i_1}}.$$  

(5.1)

We have the following corollary.

**Corollary 5.2.** Zero is a single eigenvalue of $f_\gamma$.

Finally, we give an estimate on the complexity—defined in Section 2.5—of the maps $f_\gamma(\Omega^*)$.

**Proposition 5.3.** For all closed paths $\gamma$ in $G_n$ and $\Omega^* \in \partial_{\infty}^\gamma \mathcal{O}$, we have

$$\delta(f_\gamma(\Omega^*)) \leq \text{rank}(\Omega^*) - 1.$$  

**Proof.** It follows directly from the definition (4.8) of the matrices $Q_{i\rightarrow j}(\Omega^*)$ that their left action is the identity on the subspace $\ker(\Omega^*)$ of dimension $n - \text{rank}(\Omega^*)$. Since $\ker(\Omega^*) \subset W_\gamma(\Omega^*)$ and $\dim(W_\gamma(\Omega^*)) = n - 1$, it follows that $\delta(f_\gamma(\Omega^*)) \leq \dim(W_\gamma(\Omega^*)) - \dim(\ker(\Omega^*)) = \text{rank}(\Omega^*) - 1$.

### 5.2 Homotopy invariance

The primary goal of this section is investigating how $f_\gamma(\Omega^*)$ changes for a fixed $\Omega^* \in \partial_{\infty}^\gamma \mathcal{O}$ when $\gamma$ is changed by a homotopy. Since the condition for $f_\gamma(\Omega^*)$ to be well-defined is $\Omega^* \in \partial_{\infty}^\gamma \mathcal{O}$, we assume that $\gamma$ stays in $G(\Omega^*)$ throughout the homotopy. To simplify the notation—as in Section 4.3 and Section 5.1—we usually omit $\Omega^*$ from the notation.

**Lemma 5.4.** The following identity hold:

$$p_{i_3 \leftarrow i_2} \circ p_{i_2 \leftarrow i} \circ p_{i_1 \leftarrow i_2} \circ p_{i_2 \leftarrow i_1} = p_{i_3 \leftarrow i_2} \circ p_{i_2 \leftarrow i_1}$$

28
Proof. By Proposition 5.1, the image of $p_{i\leftarrow i_2} \circ p_{i_2\leftarrow i_1}$ is $Z_{i_1} \cap Z_{i_2}$, so the projection onto $Z_{i_2}$ after that does not have any effect. Hence

$$p_{i_3\leftarrow i_2} \circ p_{i_2\leftarrow i_1} \circ p_{i\leftarrow i_2} \circ p_{i_2\leftarrow i_1} = p_{i_3\leftarrow i_2} \circ p_{i_2\leftarrow i_1} \circ p_{i_2\leftarrow i_1}.$$  

But now both $p_{i_3\leftarrow i_2}$ and $p_{i\leftarrow i_2}$ are projections in the direction of $e_{i_2}$, hence $p_{i_3\leftarrow i_2}$ is overwritten by $p_{i_2\leftarrow i_1}$.  

We call a closed path of the form $(iji)$ a backtracking and $i$ its base. We call a sequence of the insertions and removals of backtrackings a discrete homotopy.

**Proposition 5.5.** The map $f_\gamma$ is invariant under discrete homotopy rel the last edge of $\gamma$.

**Proof.** Let $\gamma = (i_1i_2\ldots i_Ki_1)$. We need to show that the removal of the backtracking $(ik'i_k)$ leaves $f_\gamma$ unchanged if $k \neq K$. If $k \neq 1$, then this follows from Lemma 5.4.

If $k = 1$, then

$$f_\gamma = (p_{i_1\leftarrow i_K} \circ \cdots \circ p_{i_2\leftarrow i_1} \circ p_{i_1\leftarrow i'} \circ p_{i'\leftarrow i_1})|_{Z_{i_1}}.$$  

Since $p_{i'\leftarrow i_1}$ maps $Z_{i_1}$ to $Z_{i_1} \cap Z_{i'}$ (Proposition 5.1), $p_{i'\leftarrow i_1}$ does not have any effect. Moreover, $p_{i_2\leftarrow i_1} \circ p_{i'\leftarrow i_1} = p_{i_2\leftarrow i_1}$, so the last term can be omitted from the composition, too.

Note that in general $f_\gamma$ is not invariant under a homotopy that changes the last edge, because domain and codomain of $f_\gamma$ might change.

**Proposition 5.6.** If $\gamma$ is contractible, then $\chi(f_\gamma) = x(x-1)^s$ for some $s$.

**Proof.** It is straightforward to prove [Str15, Proposition B.6.] that if $\gamma$ is contractible, then it is discretely homotopic rel the last edge to the backtracking $(i_1i_Ki_1)$. By Proposition 5.5,

$$f_\gamma = f_{(i_1i_Ki_1)} = (p_{i_1\leftarrow i_K} \circ p_{i_K\leftarrow i_1})|_{Z_{i_1}}.$$  

By Proposition 5.1, the image is $Z_{i_1} \cap Z_{i_K}$, a codimension 1 subspace inside $Z_{i_1}$. The map on this subspace is the identity, hence $\chi(f_\gamma)$ has the stated form.

It is worth mentioning the following useful fact, even though it is not used in this paper.

**Proposition 5.7.** The characteristic polynomial of $f_\gamma$ depends only on the free homotopy class of $\gamma$.

**Proof.** One can see from the formula (5.1) that cyclic permutation of the vertices of $\gamma$ changes $f_\gamma$ by conjugation, hence does not change its characteristic polynomial. This, together with Propositions 5.5 and 5.6, implies the statement.

29
6 Collections of curves

6.1 Orientable surfaces

Proposition 6.1. Let $S$ be an orientable surface. For all $1 \leq r \leq \frac{1}{2} \dim(\text{Teich}(S))$ there is a filling collection $C$ of curves with inconsistent markings on $S$ such that $\text{rank}(i(C,C)) = 2r$.

Moreover, for $1 \leq r \leq \frac{1}{2} \dim(H_1(S))$, the collection $C$ can be chosen to be completely left-to-right or completely right-to-left. (See Section 2.2 for the definitions.)

Proof. Since $S$ is orientable, $C$ is necessarily a union of two multicurves $A$ and $B$, where the curves of $A$ are marked consistently with the orientation of $S$, while the curves of $B$ are marked in the opposite way. Note also that $\text{rank}(i(C,C)) = 2 \text{rank}(i(A,B))$.

Let $S_{g,n}$ be the orientable surface of genus $g$ with $n$ punctures. In the special case $(g,n) = (4,3)$, Figure 6.1 shows a pair of filling multicurves $A$ and $B$ on $S_{g,n}$ with $\frac{1}{2} \dim(\text{Teich}(S)) = 3g - 3 + n$ simple closed curves in each multicurves. This construction generalizes for all $S_{g,n}$ where $g \geq 2$.

![Figure 6.1: A maximal pair of filling multicurves on $S_{4,3}$.

By numbering the curves in each mult curves left-to-right and top-to-bottom, $i(A,B)$ takes the form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\
\end{pmatrix}.
$$

(6.1)

From the pattern, it is not hard to see that $i(A,B)$ has nonzero determinant for all $S_{g,n}$ where $g \geq 2$.

Note that $A$ and the multicurve consisting of the $g$ curves of $B$ around the holes still fill the surface. Therefore $A$ and any submulticurve of $B$ that contains those $g$
curves also fill. This gives examples for pairs of filling multicurves with intersection matrices of rank $r$ for $g \leq r \leq 3g - 3 + n$.

![Figure 6.2: Pairs of multicurves realizing ranks $1 \leq r \leq g$.](image)

To obtain examples for all ranks $1 \leq r \leq g - 1$, link together the $g$ curves around the holes one by one as on Figure 6.2, resulting in multicurves $B'$ consisting of fewer and fewer curves. These multicurves still fill with $A$, and the columns of $i(A, B')$ are linearly independent, because for every curve in $B'$ there is a curve in $A$ that intersects only that curve. Note that when $1 \leq r \leq g$, pairs of multicurves in the examples shown on Figure 6.2 are completely left-to-right if oriented as on the figure, because the red curves always cross the blue curves from left to right (cf. Figure 2.1). This completes the case $g \geq 2$.

![Figure 6.3: The case $g = 1$.](image)

Figures 6.3 and 6.4 show examples with rank($i(A, B)$) = $3g - 3 + n$ in the case $g = 1, n \geq 1$ and $g = 0, n \geq 4$, respectively. In all these cases, there is a curve in $B$ that intersects all curves of $A$ and which alone fills the surface with $A$. Hence once again we can drop curves from $B$ preserving the filling property and decreasing the rank. The rank 1 example for $g = 1$ is completely left-to-right (with the appropriate markings), so this proves the second part of the proposition when $g = 1$, while for $g = 0$ there is nothing to prove since dim($H_1(S_g)$) = 0.

In the remaining cases the formula dim(Teich($S_{g,n}$)) = $6g - 6 + 2n$ does not hold. The case $(g, n) = (1, 0)$ is straightforward to check as we have dim(Teich($S_1$)) = 2. In the cases $g = 0, n < 4$ we have dim(Teich($S_{g,n}$)) = 0, so there is nothing to check. □
6.2 Nonorientable surfaces

Let $N_{g,n}$ be the nonorientable surface of genus $g$—the connected sum of $g$ crosscaps—with $n$ punctures. The main result of this section is the following.

**Proposition 6.2.** Suppose either $g \geq 3$ and $g + n \geq 5$ or $1 \leq g \leq 2$ and $g + n \geq 4$.
Then for all $3 \leq r \leq \dim(\text{Teich}(N_{g,n})) = 3g + 2n - 6$ there is a filling inconsistently marked collection $C$ on $N_{g,n}$ such that $\text{rank}(i(C,C)) = r$.

Suppose either $(g,n) = (4,0)$ or $(3,1)$. Then for all $3 \leq r \leq \dim(\text{Teich}(N_{g,n})) - 1$ there is a filling inconsistently marked collection $C$ on $N_{g,n}$ such that $\text{rank}(i(C,C)) = r$.

Moreover, when $3 \leq r \leq \dim(H_1(N_g,\mathbb{R})) = g - 1$, the collection $C$ on $N_{g,n}$ can be chosen to be completely left-to-right or completely right-to-left.

See Table 6.1 for an illustration of the different cases. The proof is based on Proposition 6.1 and the following two lemmas.

| $n$ | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|
| 0   | $\emptyset$ | $\emptyset$ | $\emptyset$ | -1 | E |
| 1   | $\emptyset$ | $\emptyset$ | -1 | E | E |
| 2   | $\emptyset$ | E | E | E | E |
| 3   | E | E | E | E | E |

Table 6.1: The cases in the first and second paragraphs of Proposition 6.2 are denoted by E and -1, respectively. The surfaces that do not admit pseudo-Anosov maps are marked by $\emptyset$.

Proposition 6.1 and the following two lemmas.

Suppose we have a filling collection $C$ on a surface. Provided $C$ satisfies certain conditions, Lemma 6.3 says that adding a crosscap to the surface allows extending $C$ to a filling collection on the new surface in a way that $\text{rank}(i(C,C))$ increases by 0, 2 or 3. Lemma 6.4 says that adding a puncture allows increasing $\text{rank}(i(C,C))$ by 0 or 2.

Let $N$ (resp. $\bar{N}$) denote a small regular open (resp. closed) neighborhood.

**Lemma 6.3.** Let $C = \{c_i\}$ be a collection of inconsistently marked simple closed curves on a surface $S$ which are in pairwise minimal position. Let $R$ be a component of the
complement of $N(C)$. Note that $\partial R$ is a union of arcs $a_j$, each of which lies on the boundary of some $\bar{N}(c_i)$.

Let $a_1$ and $a_2$ be two arcs such that

- $c_{i_1}$ and $c_{i_2}$ are distinct, non-isotopic and disjoint,
- there exists an arc $b$ inside $R$ connecting $a_1$ and $a_2$ such that the markings of $c_{i_1}$ and $c_{i_2}$ induce different orientations on $N(c_{i_1}) \cup N(b) \cup N(c_{i_2}) \cong S_{0,3}$.

Let $S'$ be the surface obtained by attaching a crosscap to $S$ inside $R$. Let $d_1$ and $d_2$ be curves obtained from $c_{i_1}$ and $c_{i_2}$ by pulling them over the crosscap. Let $e$ be the curve obtained from $c_{i_1}$ and $c_{i_2}$ by isotoping parts of them into the crosscap and applying surgery. (Note that $e$ is simple, since $c_1$ and $c_2$ are disjoint.)

Let

\[
C' = C \cup \{e\}
\]
\[
C'' = C \cup \{e, d_1\}
\]
\[
C''' = C \cup \{e, d_1, d_2\}
\]

Then, with the appropriate marking of $d_1, d_2$ and $e$,

(i) $C'$, $C''$ and $C'''$ are inconsistently marked;
(ii) $C'$, $C''$ and $C'''$ are in pairwise minimal position;
(iii) $\text{rank}(i(C, C)) = \text{rank}(i(C', C')) = \text{rank}(i(C'', C'')) - 2 = \text{rank}(i(C''', C''')) - 3$;
(iv) if $C$ fills $S$, then $C'$, $C''$ and $C'''$ fill $S'$.

Proof. Items (i) and (iv) are clear, while (ii) is straightforward to check using the bigon criterion.

Let $C_0 = C - \{c_{i_1}, c_{i_2}\}$. We can write $i(C''', C''')$ in the following block form.
where $X = i(C_0, c_{i_1})$ and $Y = i(C_0, c_{i_2})$ and where the last relation is the equivalence under column and row operations. The upper left $3 \times 3$ block is $i(C, C)$, and the lower right $3 \times 3$ block is invertible. Hence $\text{rank}(i(C'', C''')) = \text{rank}(i(C, C)) + 3$.

The calculation is analogous for $i(C'', C''')$ and $i(C', C'')$. In the first case, we have the invertible matrix $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ in the lower right corner. In the second case, the lower right corner is a single zero entry, hence $\text{rank}(i(C', C'')) = \text{rank}(i(C, C))$.

\section*{Lemma 6.4.}

Let $C$ be a filling collection of inconsistently marked simple closed curves on a surface $S$ with at least one puncture. Suppose that the curves of $C$ are in pairwise minimal position.

Then there is a point $p \in S - C$ and marked simple closed curves $d$ and $e$ on $S - \{p\}$ such that $C' = C \cup \{d\}$ and $C'' = C \cup \{d, e\}$ are filling inconsistently marked collections, and the curves in each collection are in pairwise minimal position. Moreover,

$$\text{rank}(i(C, C)) = \text{rank}(i(C', C')) = \text{rank}(i(C'', C'')) - 2.$$ 

![Figure 6.6: Creating two new curves by duplicating a puncture.](image)

\begin{proof}

Let $R$ be a component of $S - C$ which is a once-punctured disk. Let $p \in R$. Let $e$ be a curve surrounding the puncture and $p$ inside $R$. Let $d$ be a curve $e$ obtained from a curve on the boundary of $R$ by pulling it over $p$. (See Figure 6.6.) The properties of

\end{proof}
inconsistent marking, filling and minimal position are easy to verify. We have

\[
i(C'', C'') = \begin{pmatrix}
i(C_0, C_0) & \mathbf{x} & \mathbf{x} & 0 \\
\mathbf{x}^T & 0 & 0 & 0 \\
\mathbf{x}^T & 0 & 0 & 2 \\
0 & 0 & 2 & 0
\end{pmatrix},
\]

where \( C_0 = C - c \) and \( \mathbf{x} = i(C_0, c) \). Note that the \( i(C', C') \) is the submatrix obtained by deleting the last row and last column, and \( i(C, C) \) is the submatrix obtained by deleting the last two rows and the last two columns. This proves the equation about the ranks.

Proof of Proposition 6.2. The proof is divided into three parts:

1. giving examples for completely left-to-right collections;
2. when not restricted to completely left-to-right collections and \( g \geq 5 \), the construction mainly uses previously constructed collections on orientable surfaces and Lemma 6.3 to create collections on nonorientable surfaces;
3. when \( g \leq 4 \), we explicitly give finitely many collections on certain nonorientable surfaces, and use Lemmas 6.3 and 6.4 to create examples for all other required cases.

Part 1 (Completely left-to-right collections). We need to construct a rank \( r \) completely left-to-right collection on \( N_{g,n} \) for \( g \geq 4 \), arbitrary \( n \) and \( 3 \leq r \leq g - 1 \).

Figure 6.7: A rank 4 left-to-right collections on \( N_5 \) and \( N_7 \). Red arcs always cross blue arcs from the left to right.

When \( r = g - 1 \) and \( n = 0 \), the \( r \) curves are organized around a central crosscap as shown on the left on Figure 6.7. The intersection matrix is the \( r \times r \) square matrix \( M_r \) whose off-diagonal entries are 1 and whose diagonal entries are 0. Note that \( M_r \) is invertible for all \( r \): the inverse is the matrix whose off-diagonal entries are \( 1/(r-1) \) and whose diagonal entries are \( -(r-2)/(r-1) \).
For the cases \( r < g - 1 \) and \( n = 0 \), we modify the previous arrangement as seen on the right. We claim that this modification does not change the rank of the intersection matrix. Note that we might have increased the number of curves, because the arcs connecting to the central crosscap to the crosscaps on the right may not close up to a single curve but multiple disjoint ones. However, the intersection number of all these curves with the other curves are proportional, so the corresponding columns and rows in the intersection matrix are scalar multiples of each other. Hence the rank of the intersection matrix is the same for the two examples on Figure 6.7.

When \( n > 0 \), we consider the collection \( C \) already constructed in the case \( n = 0 \), add \( n \) disjoint isotopic copies of one of the curves in \( C \), and arrange the \( n \) punctures between the \( n + 1 \) isotopic curves so that no two of them are isotopic in the punctured surface. These \( n + 1 \) curves have the same intersection numbers with the other curves, hence the rank of the intersection matrix remains the same as in the case \( n = 0 \).

**Part 2** (Unrestricted case, \( g \geq 5 \)). Note that gluing a crosscap to \( S_{h,n} \) yields \( N_{2h+1,n} \).

Hence we can obtain nonorientable surfaces of odd (resp. even) genus by gluing one (resp. two) crosscap(s) to an orientable surface.

During the proof of Proposition 6.1, we constructed for many \( h, n, s \) a collection \( C_{h,n,2s} \) on \( S_{h,n} \) such that \( \text{rank}(i(C_{h,n,2s}, C_{h,n,2s})) = 2s \). If \( h \geq 2 \) and \( s \geq 2 \), then \( C_{h,n,2s} \) has at least two complementary regions that allow applying Lemma 6.3: both of the two leftmost regions work on Figures 6.1 and 6.2. What is more, Lemma 6.3 can be applied subsequently for the two regions. That is, after applying it for the first region, the hypotheses of the lemma still hold for the other region. This gives examples for all triples \((g, n, r)\) where \( g \geq 5 \), \( r \neq 3, 5 \) and \( n \) is arbitrary.

![Figure 6.8: A rank 5 collection on \( N_5 \).](image)

Examples for the cases

- \( g \geq 5 \), \( r = 3 \), \( n \) is arbitrary,
- \( g \geq 6 \), \( r = 5 \), \( n \) is arbitrary,

have been given in 1. The only case that remains is \( g = 5 \), \( r = 5 \). Figure 6.8 gives an example when \( n = 0 \). The intersection matrix is

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 2 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 2 & 0 & 1 & 0
\end{pmatrix}
\]
and it is invertible. When \( n > 0 \), add parallel curves separating the punctures as in the end of 1.

**Part 3** (Unrestricted case, \( g \leq 4 \)). The following table summarizes the ranks we need to realize on the surfaces \( N_{g,n} \) when \( g \leq 4 \).

| \( n \backslash g \) | 1   | 2   | 3   | 4   |
|------------------|-----|-----|-----|-----|
| 0                | ∅   | ∅   | ∅   | 3–5 (3,4,5) |
| 1                | ∅   | ∅   | 3–4 (3,4) | 3–8 (8) |
| 2                | ∅   | 3–4 (3,4) | 3–7 (7) | 3–10 |
| 3                | 3 (3) | 3–6 | 3–9 | 3–12 |
| 4                | 3–5 (4) | 3–8 | 3–11 | 3–14 |
| 5                | 3–7 | 3–10 | 3–13 | 3–16 |

Table 6.2: Ranks to realize on nonorientable surfaces of genus at most 4.

We construct only finitely many examples (shown in parentheses in Table 6.2). When \( g \leq 3 \), these examples and Lemma 6.4 take care of all cases. The case \( g = 4 \) is different, since \( N_4 \) is a closed surface and Lemma 6.4 does not apply. However, the rank 3, 4 and 5 collections on \( N_4 \) still fill when a puncture is added, so the same collections realize the ranks 3, 4 and 5 on \( N_{4,1} \). To realize 6 and 7, we apply Lemma 6.3 for our rank 4 collections on \( N_{3,1} \). Finally, we describe a rank 8 example explicitly. Lemma 6.4 can then be used to complete the construction for all \( N_{g,n} \) with \( g = 4 \) and \( n > 1 \).

**Case I** \( (g = 1) \). Figure 6.9 shows collections with

\[
i(C,C) = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \quad \text{and} \quad i(C,C) = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}.
\]

Both matrices are invertible.

**Case II** \( (g = 2) \). Figure 6.10 shows collections with

\[
i(C,C) = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 4 \\ 2 & 4 & 0 \end{pmatrix} \quad \text{and} \quad i(C,C) = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix}.
\]

Both matrices are invertible.

**Case III** \( (g = 3) \). Figure 6.11 shows a filling collection with

\[
i(C,C) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix},
\]

\[\text{(6.2)}\]

37
which has rank 4. Note that there is a complementary region with two disjoint curves on its boundary, hence Lemma 6.3 indeed applies to yield rank 4, 6 and 7 collections on $N_{4,1}$.

By dropping the curve surrounding the hole which intersects only one other curve, the remaining three curves still fill and the intersection matrix is the lower right $3 \times 3$ submatrix of (6.2), which has rank 3.

Figure 6.12 shows a filling collection with

$$i(C,C) = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 & 2 & 4 \\ 2 & 0 & 0 & 2 & 4 & 4 & 4 \\ 2 & 0 & 0 & 2 & 4 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 & 2 & 4 \\ 2 & 4 & 4 & 2 & 0 & 0 & 4 \\ 2 & 4 & 2 & 2 & 0 & 0 & 2 \\ 4 & 4 & 2 & 4 & 4 & 2 & 0 \end{pmatrix}$$

which has rank 7.

**Case IV** ($g = 4$). Figure 6.13 shows a filling collection with

$$i(C,C) = \begin{pmatrix} 0 & 2 & 2 & 2 & 2 \\ 2 & 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 0 & 4 \\ 2 & 2 & 2 & 4 & 0 \end{pmatrix},$$

(6.3)

which has rank 5. By dropping both or one of the dashed curves, the remaining three or four curves still fill. The intersection matrices are the upper left $3 \times 3$ and $4 \times 4$ submatrices of (6.3), which have rank 3 and rank 4, respectively.
Figure 6.10: A rank 3 and rank 4 collection on $N_{2,2}$.

Figure 6.11: A rank 4 collection on $N_{3,1}$.

Figure 6.14 shows a filling collection with

$$i(C, C) = \begin{pmatrix}
0 & 2 & 2 & 2 & 2 & 2 & 4 & 0 \\
2 & 0 & 2 & 2 & 2 & 2 & 4 & 0 \\
2 & 2 & 0 & 4 & 4 & 4 & 8 & 0 \\
2 & 2 & 4 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 4 & 0 & 0 & 2 & 2 & 2 \\
2 & 2 & 4 & 0 & 2 & 2 & 0 & 2 \\
4 & 4 & 8 & 0 & 2 & 2 & 0 & 4 \\
0 & 0 & 0 & 0 & 2 & 2 & 4 & 0
\end{pmatrix}.$$  

It is easy to zero out the upper right and lower left $4 \times 4$ block using row and column operations. The remaining $4 \times 4$ matrices are invertible, hence $i(C, C)$ has rank 8.
7 Bounds on the algebraic degree

Let \( S \) be a finite type surface, and let \( \text{Teich}(S) \) be its Teichmüller space.

**Proposition 7.1** (Thurston, [Thu88]). \( \max D(S) \leq \dim(\text{Teich}(S)) \).

Recall that \( \overline{S} \) denotes the closed surface obtained from \( S \) by filling in the punctures.

**Lemma 7.2.** \( D^+(S) = D^+(\overline{S}) \)

*Proof.* If \( \psi \) is a pseudo-Anosov map on \( S \) with an orientable invariant foliation, then it has no 1-pronged singularities. So the pseudo-Anosov map on \( \overline{S} \) obtained by extending \( \psi \) to the punctures has the same stretch factor as \( \psi \). This shows that \( D^+(S) \subseteq D^+(\overline{S}) \).

If \( \psi \) is a pseudo-Anosov map on \( \overline{S} \) with stretch factor \( \lambda \), then some power of it has as many fixed points as the number of punctures of \( S \). Hence there is a pseudo-Anosov map on \( S \) whose stretch factor is some power of \( \lambda \). The algebraic degree is preserved under powers, hence \( D^+(\overline{S}) \subseteq D^+(S) \).

**Proposition 7.3.** \( \max D^+(S) \leq \dim(H^1(\overline{S}, \mathbb{R})) \).

*Proof.* By Lemma 7.2 it suffices to show that \( \max D^+(S) \leq \dim(H^1(S, \mathbb{R})) \) if \( S \) is closed. This follows from the fact that the orientable invariant foliation is an eigenvector of the action of the \( H^1(S, \mathbb{R}) \).

**Proposition 7.4.** If \( S \) is orientable, \( d \in D(S) \) and \( d \) is odd, then \( d \leq \frac{1}{2} \dim(\text{Teich}(S)) \).
For closed surfaces, this result is due to Long [Lon85, Theorem 3.3]. McMullen later gave an different proof [Shi16, Theorem 10] which works for punctured surfaces as well. Essentially the same argument also yields the following.

**Proposition 7.5.** If $S$ is orientable, $d \in D^+(S)$ and $d$ is odd, then $d \leq \frac{1}{2} \dim(H^1(S, \mathbb{R}))$.

Propositions 7.4 and 7.5 have no analogs for nonorientable surfaces.

**Proposition 7.6.** If $S$ is nonorientable, then $2 \notin D(S)$.

*Proof.* If $\deg(\lambda) = 2$, then its minimal polynomial has the form $x^2 \pm kx \pm 1$ for some $k \in \mathbb{Z}$, since $\lambda$ in algebraic unit. But this is impossible, since $\pm 1/\lambda$ is never a Galois conjugate of $\lambda$ when the pseudo-Anosov map is supported on a nonorientable surface. (This is stated for $1/\lambda$ in [Str16, Proposition 2.3], but the proof—which goes along the same lines as McMullen’s argument above—works for $-1/\lambda$ as well.)

The following fact is well-known. We include here for completeness.

**Proposition 7.7.** The closed nonorientable surface of genus 3 does not admit pseudo-Anosov maps.

*Proof.* There is a unique one-sided curve on the surface whose complement is a one-holed torus [Sch82, Lemma 2.1]. This curve is fixed by every mapping class.

8 Two pseudo-Anosov examples for the sporadic cases

**Proposition 8.1.** $6 \in D(N_4)$ and $5 \in D(N_{3,1})$. 

41
Proof. Consider the triangulated surfaces $\tilde{X}_0$ and $\tilde{X}_1$ shown on Figures 8.1 and 8.2, homeomorphic to $S_{3,2}$ and $S_{2,2}$, respectively. (For both $\tilde{X}_i$, the identifications yield two vertex classes which are considered punctures.)

Each $\tilde{X}_i$ admits an orientation-reversing involution $h_i$ which respects the triangulation and which induces the map

$$\alpha(i) = \begin{cases} 
  i + 1 & \text{if } i \text{ is even} \\
  i - 1 & \text{if } i \text{ is odd}
\end{cases}$$

on the edge labels. The involution $h_0$ can be pictured as a reflection about a horizontal line followed by a (cyclic) horizontal translation. To see $h_1$, reflect about a line with slope 1, then shift the four squares accordingly. The quotients of $\tilde{X}_0$ and $\tilde{X}_1$ by these involutions yield triangulated surfaces $X_0$ and $X_1$ homeomorphic to $N_{4,1}$ and $N_{3,1}$, since the two vertex classes are interchanged by $h_i$ in each case.

Consider the curves $\tilde{a}_i$, $\tilde{b}_i$ and $\tilde{c}_i$ on each $\tilde{X}_i$. These curves are lifts of simple two-sided curves $a_i$, $b_i$ and $c_i$ on $X_i$, because their images under $h_i$ are disjoint from them and they belong to a different isotopy class.

The mapping classes

$$\tilde{f}_0 = T_{\tilde{a}_0}^{-1}T_{h_0(\tilde{a}_0)}T_{\tilde{b}_0}^{-1}T_{h_0(\tilde{b}_0)}T_{\tilde{c}_0}T_{h_0(\tilde{c}_0)} \in \text{Mod}(\tilde{X}_0)$$

$$\tilde{f}_1 = T_{\tilde{a}_1}T_{\tilde{h}_1(\tilde{a}_1)}T_{\tilde{b}_1}T_{\tilde{h}_1(\tilde{b}_1)}T_{\tilde{c}_1}T_{\tilde{h}_1(\tilde{c}_1)} \in \text{Mod}(\tilde{X}_1)$$

are lifts of

$$f_0 = T_{a_0}^{-1}T_{b_0}T_{c_0} \in \text{Mod}(X_0)$$

42
Figure 8.1: $\tilde{X}_0$

Figure 8.2: $\tilde{X}_1$

$f_1 = T_a T_b T_c \in \text{Mod}(X_1)$.

Using Flipper (v. 0.10.3) [Bel16] under Sage or Python, the code below certifies that $\text{deg}(\lambda(f_0)) = 6$ and $\text{deg}(\lambda(f_1)) = 5$. The latter implies $5 \in D(N^{3,1})$. The computation also shows that $f_0$ does not have one-pronged singularities, so the puncture can be filled in to obtain a pseudo-Anosov element of $\text{Mod}(N^4)$ with the same stretch factor as $f_0$. This shows $6 \in D(N^4)$.

Code snippet 1: Input

```python
import flipper

X = [flipper.create_triangulation([(0,4,2), (~4,6,8), (~8,12,16),
                                  (~12,10,~0), (14,~16,7), (3,~14,~10), (~7,5,9), (~5,1,~3),
                                  (13,~9,17), (~1,11,~13), (~6,~17,15), (~2,~11,~15))],
     flipper.create_triangulation([(0,4,2), (~4,~2,6), (~6,10,9),
                                  (~10,8,~0), (~5,1,~3), (7,3,5), (11,~7,~8), (~1,~9,11)])]

lam_list = [[[0,0,0,1,0,1,0,1,0,0,0,0,1,0,0,0], # 2nd 45 degree cylinder
             [0,0,0,0,0,0,0,0,1,0,1,0,1,0,0,0], # 3rd 45 degree cylinder
             [1,0,0,1,1,1,0,0,1,0,1,1,1,1,1,1,0]], # 1st vertical cylinder
            [[1,0,0,0,1,0,1,0,0,0,1,0,0,1,0,0], # 1st NW cylinder
             [1,0,0,0,1,0,1,0,0,0,1,0,0,1,0,0], # 1st NW cylinder
             [1,0,0,0,1,0,1,0,0,0,1,0,0,1,0,0], # 1st NW cylinder
             [1,0,0,0,1,0,1,0,0,0,1,0,0,1,0,0], # 1st NW cylinder
             [1,0,0,0,1,0,1,0,0,0,1,0,0,1,0,0], # 1st NW cylinder
             [1,0,0,0,1,0,1,0,0,0,1,0,0,1,0,0]]]```

43
def swap(l):
    res = [] for i in range(len(l)/2):
        res += [l[2*i+1], l[2*i]]
    return res

lifts_of_twists = [[X[i].lamination(x).encode_twist() *
    X[i].lamination(swap(x)).encode_twist().inverse()
    for x in lam_list[i]] for i in range(2)]

pas = [lifts_of_twists[0][0].inverse() *
    lifts_of_twists[0][1] *
    lifts_of_twists[0][2],
    lifts_of_twists[1][0] *
    lifts_of_twists[1][1] *
    lifts_of_twists[1][2]]

for i in range(2):
    print "Surface of genus {0} with {1} punctures."
    format([i].genus, [i].nut_vertices)
    print "Nielsen-Thurston type of mapping class: Pseudo-Anosov",
    pas[i].nielsen_thurston_type()
    d = pas[i].dilatation()
    print "Stretch factor: {0}", d
    print "Its minimal polynomial: {1} ", d.minimal_polynomial()
    print "Singularity structure: {2} ", pas[i].stratum().values()
    print

Surface of genus 3 with 2 punctures.
Nielsen-Thurston type of mapping class: Pseudo-Anosov
Stretcher factor: 3.3180227
Its minimal polynomial: 1 + x - x^2 - x^4 - 3*x^5 + x^6
Singularity structure: [2, 2, 3, 3, 3, 3, 3, 3, 3, 3]

Surface of genus 2 with 2 punctures.
Nielsen-Thurston type of mapping class: Pseudo-Anosov
Stretcher factor: 3.2510347
Its minimal polynomial: -1 - x + x^2 - x^3 - 3*x^4 + x^5
Singularity structure: [1, 1, 3, 3, 3, 3, 3, 3]

References

[ALM16] Ian Agol, Christopher J. Leininger, and Dan Margalit. Pseudo-Anosov stretch factors and homology of mapping tori. J. Lond. Math. Soc. (2), 93(3):664–682, 2016.

[AY81] Pierre Arnoux and Jean-Christophe Yoccoz. Construction de difféomorphismes pseudo-Anosov. C. R. Acad. Sci. Paris Sér. I Math., 292(1):75–78, 1981.

[Bel16] Mark Bell. flipper (computer software). Version 0.10.3, 2013–2016.
Albert Fathi. Démonstration d’un théorème de Penner sur la composition des twists de Dehn. *Bull. Soc. Math. France*, 120(4):467–484, 1992.

Benson Farb and Dan Margalit. A primer on mapping class groups, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

J. Franks and E. Rykken. Pseudo-Anosov homeomorphisms with quadratic expansion. *Proc. Amer. Math. Soc.*, 127(7):2183–2192, 1999.

Eugene Gutkin and Chris Judge. Affine mappings of translation surfaces: geometry and arithmetic. *Duke Math. J.*, 103(2):191–213, 2000.

Pascal Hubert, Howard Masur, Thomas Schmidt, and Anton Zorich. Problems on billiards, flat surfaces and translation surfaces. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 233–243. Amer. Math. Soc., Providence, RI, 2006.

Richard Kenyon and John Smillie. Billiards on rational-angled triangles. *Comment. Math. Helv.*, 75(1):65–108, 2000.

Darren D. Long. Constructing pseudo-Anosov maps. In *Knot theory and manifolds (Vancouver, B.C., 1983)*, volume 1144 of *Lecture Notes in Math.*, pages 108–114. Springer, Berlin, 1985.

Curtis T. McMullen. Teichmüller geodesics of infinite complexity. *Acta Math.*, 191(2):191–223, 2003.

Kenneth R. Meyer, Glen R. Hall, and Dan Offin. *Introduction to Hamiltonian dynamical systems and the N-body problem*, volume 90 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2009.

Robert C. Penner. A construction of pseudo-Anosov homeomorphisms. *Trans. Amer. Math. Soc.*, 310(1):179–197, 1988.

Martin Scharlemann. The complex of curves on nonorientable surfaces. *J. London Math. Soc. (2)*, 25(1):171–184, 1982.

Hyunshik Shin. Algebraic degrees of stretch factors in mapping class groups. *Algebr. Geom. Topol.*, 16(3):1567–1584, 2016.

Karlheinz Spindler. *Abstract algebra with applications. Vol. I*. Marcel Dekker, Inc., New York, 1994. Vector spaces and groups.

Hyunshik Shin and Balázs Strenner. Pseudo-Anosov mapping classes not arising from Penner’s construction. *Geom. Topol.*, 19(6):3645–3656, 2015.

Balázs Strenner. *Algebraic degrees and Galois conjugates of pseudo-Anosov stretch factors*. ProQuest LLC, Ann Arbor, MI, 2015. Thesis (Ph.D.)–The University of Wisconsin - Madison.

Balázs Strenner. Lifts of pseudo-Anosov homeomorphisms of nonorientable surfaces have vanishing SAF invariant. *Preprint, arXiv:1604.05614*, 2016.

William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.

Anton Zorich. Flat surfaces. In *Frontiers in number theory, physics, and geometry. I*, pages 437–583. Springer, Berlin, 2006.