Excess number of percolation clusters on the surface of a sphere

Christian D. Lorenz * and Robert M. Ziff†

Michigan Center of Theoretical Physics and Department of Chemical Engineering
University of Michigan, Ann Arbor, MI 48109-2136

(December 30, 2021)

Abstract

Monte Carlo simulations were performed in order to determine the excess number of clusters $b$ and the average density of clusters $n_c$ for the two-dimensional “Swiss cheese” continuum percolation model on a planar $L \times L$ system and on the surface of a sphere. The excess number of clusters for the $L \times L$ system was confirmed to be a universal quantity with a value $b = 0.8841$ as previously predicted and verified only for lattice percolation. The excess number of clusters on the surface of a sphere was found to have the value $b = 1.215(1)$ for discs with the same coverage as the flat critical system. Finally, the average critical density of clusters was calculated for continuum systems $n_c = 0.0408(1)$.

I. INTRODUCTION

One of the interesting characteristics of the percolation model is the existence of universal quantities, which are independent of the microscopic qualities of the system. Critical exponents and amplitude ratios are examples of one class of universal quantities that are only dependent on the dimensionality of the system. Shape-dependent universal quantities, which depend on the shape of the boundary as well as the dimensionality of the system, have been identified in the Ising model by Privman and Fisher, Müller, and Kamieniarz and Blöte. In percolation, recent research has been focused on shape-dependent universal quantities such as the crossing probabilities and the excess number of clusters.

The excess number of clusters $b$ on a two dimensional system is defined by

$$N \sim n_c A + b \quad (1)$$

*cdl@umich.edu

†rziff@umich.edu
as $A \to \infty$, where $N$ is the total number of the clusters in a system of area $A$, and $n_c$ is the number of clusters per unit area in an infinite system. In [1], it is assumed that the system is at criticality, and that there are no boundaries to the system.

The average density of clusters $n_c$ is a non-universal quantity, meaning that it depends on the microscopic qualities of the system and will have different values for different systems. In two-dimensional percolation, exact values of $n_c$ have been determined theoretically for bond percolation on two lattices [18, 19]. Also, simulations have provided precise values of $n_c$ for 2d [16] and 3d [15] lattice percolation.

While $n_c$ is a system-dependent quantity, $b$ is a universal quantity that is independent of the type of percolation but does depend on shape of the system boundary and the dimensionality of the system. Previous work has demonstrated the shape dependence of $b$ for toroidal systems using $L \times L'$ systems with periodic boundary conditions, for which $b(r)$ has been found exactly [17]. For an $L \times L$ system, the theoretical prediction is $b = 0.883576\ldots$ [2], which agrees with the numerical value $b = 0.8835$ found for site and bond percolation on square and triangular lattices. For 3d lattice percolation, theoretical predictions for $\tilde{b}$ (the excess number per unit length) do not exist; however, the universality of $\tilde{b}$ has been confirmed using numerical values for different $L \times L \times L'$ cubic systems [3].

In the present paper, we are interested in finding $b$ for percolation on the surface of a sphere, another two dimensional surface with no boundaries, for which however no theoretical prediction exists. In order to do this, we use the 2d continuum “Swiss cheese” model of percolation, for which the critical density has recently been found to high accuracy [20]. In addition to the spherical systems, we are interested in using this continuum model to determine $b$ for $L \times L$ toroidal systems and comparing it to the theoretical prediction. In order to find $b$, we must also find $n_c$, the number of clusters per unit area for the continuum percolation model. We note that continuum percolation on the surface of a sphere (and hypersphere) has also been studied in a recent publication [21], in the context of diffusion on fractal clusters.

In the following section, we describe the simulations that were used to model the square and spherical system. Then we present and summarize our results for the excess number of clusters and the average density of clusters in these systems.

**II. METHOD**

The basic “cluster counting” algorithm that was used for determining the number of clusters in the $L \times L$ toroidal and spherical continuum systems is identical to the algorithm we used to study similar problems in lattice percolation. However, the implementation of this algorithm was quite different for the two continuum systems. The $L \times L$ system with periodic boundary conditions was first divided into squares of unit area. Discs, whose radius $R$ is equal to 0.5, were distributed into each of the unit squares in the plane using a Poisson function, where the probability $P_n$ that there are $n$ particles in a given volume $V$ is given by

$$P_n = \frac{1}{n!} (\rho_c V)^n e^{-\rho_c V}. \quad (2)$$
Here $\rho_c$ is the critical density of discs ($\rho_c = 1.43628$ for 2d continuum percolation of discs) and $V = 1$ is the volume of each of the unit squares. [Note: The critical density of discs is often referred to as $n_c$ in the continuum percolation literature, but in order to differentiate from the average density $n_c$, we use $\rho_c$ here.] The algorithms in Ref. 22 were used to generate numbers with this distribution. For each unit square, a random number $n$ was generated and if $n > 0$, then $n$ discs were placed within that square. The $x$- and $y$-coordinates of each disc were stored in two one-dimensional arrays, which were indexed by the order that the discs were distributed (i.e., the first sphere placed is numbered 0, the second is numbered 1, ...). The index of the first and last disc distributed in the square were also stored in two one-dimensional pointer arrays. After discs were placed in each unit square, a search was made for clusters, starting with the first disc placed in the first unit square. The search checked only the neighboring eight unit squares, as opposed to the entire system, for discs. If the distance between any two discs was less than or equal to 1, then the two were considered to be in the same cluster. The coordinates of each disc in the cluster were stored in two one-dimensional list ("growth") arrays, which were indexed by the order that the discs were determined to be part of the cluster. After the first disc was checked for overlapping neighbors, then subsequent discs on the list arrays were checked, in the order that they were placed on the list, for overlapping neighbors. This process was continued until a cluster stopped growing; at which time, the same search for each of the unchecked discs in the current square was performed. After each of the discs in a square were checked, the search moved to the next square that had unchecked discs remaining. This cluster search was continued until all discs within the system were determined to be part of a cluster.

When simulating the spherical systems, we encountered two areas of difficulty that were not present when using the $L \times L$ system. First, we weren’t able to divide the surface into smaller sub-sections because of the difficulty in producing equal area sub-sections. Instead, the discs were distributed on the surface of the entire sphere, using (2) to determine the number, and the $\phi$-coordinate and the cosine of the $\theta$-coordinate for each disc were stored in two one-dimensional arrays. The entire list was searched for each neighbor check, resulting in a much slower simulation for larger spherical systems compared to the toroidal system. We began our cluster search algorithm with the first disc that was placed on the sphere. The “great circle” distance between two discs $d$, whose coordinates are $(\theta_1, \phi_1)$ and $(\theta_2, \phi_2)$, is given by

$$d = \sqrt{2r^2(1 - \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2) - \cos \theta_1 \cos \theta_2)},$$

which is the arc length between two points on the surface of a sphere of radius $r$. Two discs were considered to be overlapping if $d \leq 1$. The cluster search algorithm was applied to the spherical system until every disc on the sphere was checked.

The second area of difficulty we encountered was in the selection of a criterion for the critical threshold that is consistent with the criterion used for a $L \times L$ system. Different choices (density of discs, surface area coverage, etc.) lead to different numbers of discs. As a result of this ambiguity, we considered two different methods to determine the average number of discs that were required. In general, the total number of the discs $M$ is defined as

3
\[ M = \frac{MA}{A} \]  

(4)

where \( \overline{A} \) is the area covered by each disc, \( M \) is the coverage of the discs, and \( A \) is the total area of the system. For a finite system, \( M = \rho_c A \), where \( \rho_c \) is the critical density of discs.

First, we used “type 1” discs, where we chose the number of discs per unit area on the sphere to have the same value as the \( L \times L \) critical system. Therefore, the number of discs is \( M = \rho_c (4\pi r^2) \), where \( r \) is the radius of the sphere and \( \rho_c = 1.43628 \) is the critical density of discs for a flat system.

For the second type of discs (“type 2” discs), we determined the number of discs required to keep the critical coverage the same as it was for a flat system. The area covered by each disc is calculated by

\[ A = \int_0^{\theta_{\text{max}}} 2\pi r^2 \sin \theta d\theta = 2\pi r^2 (1 - \cos \theta_{\text{max}}), \]  

(5)

where \( r \) is the radius of the sphere, \( \theta \) is the angle between the horizontal axis of the sphere and a point on the surface of the sphere, and \( \theta_{\text{max}} \) is the angle between the edge of the disc and the horizontal axis as shown in Fig. 2. In this case \( \theta_{\text{max}} = 1/2r \), therefore, the area covered by each type 2 disc is

\[ \overline{A} = 2\pi r^2 (1 - \cos 1/2r). \]  

(6)

The number \( M \) of these discs that would be required to achieve the same coverage as the \( L \times L \) system (\( M = \rho_c \pi / 4 \)) was determined using (4), with the result that

\[ M = \frac{M A}{A} = \frac{\rho_c \pi}{2(1 - \cos(1/2r))} \]  

(7)

where \( r \) is the radius of the sphere, where \( A = 4\pi r^2 \).

Using these two simulations, we were able to study planar \( L \times L \) systems, where \( L = 8, 16, 32, 64, \) and 128, and spherical systems with radius \( r = 5, 6, 10, \) and 15. The simulations counted the number of clusters that existed within the system. Numerous realizations (\( \sim 10^6 \) for spherical systems and \( \sim 10^7 \) for planar systems) were averaged over in order to calculate the density of clusters within each system.

The random numbers used in these simulations were generated by the four-tap shift-register rule \( x_n = x_{n-471} \oplus x_{n-1586} \oplus x_{n-6988} \oplus x_{n-9689} \), where \( \oplus \) is the exclusive-or operation.

III. RESULTS

The number of clusters \( N \) present in a system which has area \( A \) is expected to follow (1). Using our simulations, we were able to calculate the total number of clusters present and then determine the overall density of clusters \( n = N/A \) for each system size, which by rearranging (1) is \( n = n_c + b/A + \ldots \), or
\[ n = n_c + b/(L^2) + \ldots \quad [L \times L \text{ system}] \]  
\[ n = n_c + b/(4\pi r^2) + \ldots \quad [\text{spherical system}] . \] 

Figure 1 is a plot of \( n \) vs. \( 1/A \) for each of the three systems. By fitting these plots with equations (8) and (9), we were able to determine the values of \( b \) and \( n_c \) for these systems from the slope and the intercept, respectively, which are summarized in Table I.

Several observations can be made from these results. First, the average density of clusters for the spherical systems and for the \( L \times L \) system is, as expected, the same (within numerical error). Also, the excess number of clusters \( b \) for the planar \( L \times L \) continuum system is consistent with the theoretical value \( (b = 0.883576 \ldots) \) and the simulation value \( (b = 0.8835) \) found for the \( L \times L \) lattice percolation system, which further confirms the universality of this quantity. Finally, the values of the excess number of clusters \( b \) for the spherical systems are different than that found for the planar system, which is expected because the shape and topology of the boundary is different. However, there is a difference in the value of \( b \) for the two spherical systems, which is a result of the different definitions for the critical number of discs placed on the surface of the sphere.

When the total number of discs \( M \) that are placed on the sphere in each case are compared, the difference in \( b \) becomes more understandable:

\[ M = \rho_c (4\pi r^2) \quad \text{[type 1 discs]} \]  
\[ M = \rho_c (4\pi r^2) + \rho_c (\pi/12) + O(1/r^2 \ldots) \quad \text{[type 2 discs]} . \] 

where the second expression follows from a Taylor series expansion of (7). The first term in the expression for type 2 discs is exactly the number of type 1 discs present on the surface of the sphere. However, the second term shows that of the order of one more disc is present on the surface of the sphere than type 1 disc at the same (critical) density. These relatively small differences in the number of discs present on the sphere can cause a significant difference in the density of clusters. In general, about the critical point, one expects the density of clusters to behave as

\[ n = n_c + a_0 (\rho - \rho_c) + a_1 (\rho - \rho_c)^2 + a_2 (\rho - \rho_c)^{2-\alpha} + \ldots \]  

where \( 2 - \alpha = 8/3 \) in two dimensions, \( \rho = M/A \), and \( A = 4\pi r^2 \). Then the number of clusters \( N \) is

\[ N = N_c + a_0 (M - A\rho_c) + a_1 \frac{(M - A\rho_c)^2}{A} + \ldots . \]

Therefore, if \( M \) changes by an amount of order 1, then \( N \) will also change by an amount of order 1, which is the same order as the \( b \) term in (11). Thus, the slight differences in the two definitions of the total number of discs \( M \) — even though they are asymptotically identical for large \( r \) — lead to non-zero differences in the value of \( b \).
IV. DISCUSSION OF RESULTS

The universality of the excess number of clusters $b$ has been demonstrated using 2-d and 3-d lattice percolation, but it had never been tested with a continuum model. Our result $b = 0.8841$ for the $L \times L$ planar system is the same as found for the $L \times L$ lattice percolation system\cite{17}, thus it further confirms the universality of this quantity.

The excess number of clusters had never been calculated for spherical systems. By using the Swiss cheese continuum model, we were able to study percolation on the surface of sphere. We found that the excess number of clusters ($b$(type 1 discs) = 1.263, $b$(type 2 discs) = 1.215) is slightly dependent upon the definition of the critical density of discs for the spherical system. We believe that the type 2 definition is the most reasonable as it is based upon the idea that the surface coverage is the same as the planar system, which is consistent with the fact that the surface coverage is invariant when the $L \times L$ periodic system is deformed to a torus. Therefore, we propose the value $b = 1.215(1)$ for the spherical surface. While theoretical results exist for $b$ for the $L \times L'$ periodic system, there is no prediction for the sphere.
REFERENCES

1 V. Privman and M. E. Fisher, Phys. Rev. B 30, 322 (1984).
2 B. Müller, Int. J. Mod. Phys. C 9, 1 (1998).
3 G. Kamieniarz and H. W. J. Blöte, J. Phys. A 26, 201 (1993).
4 R. P. Langlands, C. Pichet, P. Pouliot, and Y. Saint-Aubin, J. Stat. Phys. 67, 553 (1992).
5 R. P. Langlands, P. Pouliot, and Y. Saint-Aubin, Bull. AMS 30, 1 (1994).
6 M. Aizenman, The IMA Volumes in Mathematics and its Applications, Springer, Berlin, 1997.
7 J. L. Cardy, J. Phys. A 25, L201 (1992).
8 R. M. Ziff, Phys. Rev. Lett. 69, 2670 (1992).
9 J.-P. Hovi and A. Aharony, Phys. Rev. E 53, 235 (1996).
10 C.-K. Hu, and C.-Y. Lin, Phys. Rev. Lett. 75, 193 (1995).
11 C.-K. Hu, C.-Y. Lin and J.-A. Chen, Phys. Rev. Lett. 77, 8 (1996).
12 G. M. T. Watts, J. Phys. A 29, L363, (1996).
13 H. T. Pinson, J. Stat. Phys. 75, 1167 (1994).
14 R. M. Ziff, C. D. Lorenz, and P. Kleban, Physica A 266, 17 (1999).
15 C. D. Lorenz and R. M. Ziff, J. Phys. A 31, 8147 (1998).
16 R. M. Ziff, S. R. Finch, and V. S. Adamchik, Phys. Rev. Lett. 79, 3447 (1997).
17 P. Kleban and R. M. Ziff, Phys. Rev. B 57, R8075 (1998).
18 H. N. V. Temperley and E. H. Lieb, Proc. R. Soc. A 322, 251 (1971).
19 R. J. Baxter, H. N. V. Temperley, and S. E. Ashley, Proc. R. Soc. A 358, 535 (1978).
20 J. Quintanilla, S. Torquato, and R. M. Ziff, J. Phys. A 33, L399 (2000).
21 P. Jund, R. Jullien, and I. Campbell, [cond-mat/0011494].
22 W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, Numerical Recipes in C, revised 2nd ed. (Cambridge University Press, 1992).
23 R. M. Ziff, Computers in Physics 12, 385 (1998).
24 D. Stauffer and A. Aharony, An Introduction to Percolation Theory, Revised 2nd. Ed. (Taylor and Francis, London, 1994).
TABLE I. Values of the critical average density of clusters $n_c$ and the excess number of clusters $b$ for two-dimensional “Swiss cheese” model. Numbers in parenthesis represent the error in the last digit.

| System                           | $n_c$           | $b$            |
|----------------------------------|-----------------|----------------|
| Planar ($L \times L$)           | 0.04075(5)      | 0.8841(2)      |
| Spherical (“type 1” discs)       | 0.04093(10)     | 1.263(1)       |
| Spherical (“type 2” discs)       | 0.04092(10)     | 1.215(1)       |
FIG. 1. Plot of $N/A$ vs. $1/A$ for all three systems. The three sets of data represent the spherical system with “type 1” discs, the spherical system with “type 2” discs, and the planar $L \times L$ system, from top to bottom. In these plots, the $y$-intercept corresponds to the average density of clusters $n_c$ and the slope is equal to the excess number of clusters $b$. 
FIG. 2. Drawing of spherical with flat disc on the surface. The central angle $\theta_{\text{max}}$ is related to the radius $r$ of the sphere by $\sin \theta_{\text{max}} = \frac{1/2}{r}$. 