A generalized volume law for entanglement entropy on the fuzzy sphere

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We investigate entanglement entropy in a scalar field theory on the fuzzy sphere. The theory is realized by a matrix model. In our previous study, we confirmed that entanglement entropy in the free case is proportional to the square of the boundary area of a focused region. Here we argue that this behavior of entanglement entropy can be understood by the fact that the theory is regularized by matrices, and further examine the dependence of entanglement entropy on the matrix size. In the interacting case, by performing Monte Carlo simulations, we observe a transition from a generalized volume law, which is obtained by integrating the square of area law, to the square of area law.

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1. Introduction

It is widely recognized that noncommutative field theories are deeply connected to quantum gravity and string theory. On the other hand, since the discovery of the Ryu–Takayanagi formula (Ref. [1]), the connection between geometry and quantum entanglement has been revealed. One can, therefore, expect to gain insight into quantum gravity by studying quantum entanglement in noncommutative field theories.

Indeed, by studying a gravity dual of noncommutative super Yang–Mills theory (NCSYM) proposed in Refs. [2,3], it was conjectured in Refs. [4,5] that entanglement entropy (EE) in NCSYM is proportional to the volume of a focused region when the volume is small and to the area of the boundary of the region when the volume is large. While EE is proportional to the area in ordinary local field theories, the above volume law in NCSYM would originate from the UV/IR mixing (Ref. [6]) due to nonlocal interactions. Indeed, in Ref. [7], the volume law for EE is obtained in nonlocal theories.

In this paper, we use the words “volume” and “area” for real area and length, respectively, on a sphere. In Refs. [8,9],\textsuperscript{1} EE in a scalar field theory on the fuzzy sphere was studied at zero temperature in the free case.\textsuperscript{2} In Ref. [12], Okuno and the present authors reported the results for EE in the above theory. We verified that EE on the fuzzy sphere in the free case is proportional to the square of the area of the boundary, which was suggested in Ref. [9]. Moreover, we showed the first Monte Carlo

\begin{footnote}{\textsuperscript{1} For earlier studies, see Refs. [10,11].}
\end{footnote}

\begin{footnote}{\textsuperscript{2} Throughout this paper, the case where the action consists of only quadratic terms is called the “free case”, while the case where the action includes higher-order terms is called the “interacting case”.}
\end{footnote}
results\(^3\) for the interacting case, where we found that the behavior of EE is quite different from that in the free case. We also found that the finite temperature effect is governed by the volume law in the interacting case as well as in the free case. In calculating EE, we used a method that is different from the one in Refs. [8,9]. This method was developed in Ref. [19] and used in Refs. [19–21].

In this paper, we continue the study of EE in the scalar field theory on the fuzzy sphere. We present further results for the free case as well as for the interacting case. In the free case, we discuss why EE obeys the square of area law, and further examine the dependence of EE on the matrix size. By performing Monte Carlo simulations in the interacting case, we observe a transition from a generalized volume law, which corresponds to the integral of the square of the area, to the square of area law, when the volume of a focused region is increased. This phenomenon should be attributed to the UV/IR anomaly discovered in Refs. [22,23], which is the counterpart of the UV/IR mixing in field theories on compact noncommutative manifolds.

Another aim of our work is to elucidate geometry in matrix models. This is in particular important in the context of the study of matrix models proposed as nonperturbative formulation of string theory (Refs. [24–26]). Indeed, the above scalar field theory on the fuzzy sphere is realized in a matrix model as a regularized theory. Following a prescription given in Refs. [8,9], we divide the matrices into two parts, each one corresponding to one of the two regions on the sphere. We would like to see how well this division in the matrix works.

This paper is organized as follows. In Sect. 2, we review a matrix model that realizes a scalar field theory on \(S^1 \times \text{fuzzy sphere}\). In Sect. 3, we describe how we calculate EE in this theory. We present the results for EE in the free case in Sect. 4 and in the interacting case in Sect. 5. Section 6 is devoted to conclusion and discussion.

2. Scalar field theory on the fuzzy sphere

2.1. Scalar field theory on the fuzzy sphere realized by a matrix model

The commutative counterpart of the noncommutative scalar field theory we consider in this paper is defined on \(S^1 \times S^2\) as

\[
S_C = \frac{R^2}{4\pi} \int_0^{\beta} dt \int d\Omega \left( \left. \frac{1}{2} \dot{\Phi}(t, \Omega)^2 - \frac{1}{2R^2} \left[ \mathcal{L}_i \Phi(t, \Omega) \right]^2 + \frac{\mu^2}{2} \Phi(t, \Omega)^2 + \frac{\lambda}{4} \Phi(t, \Omega)^4 \right), \tag{2.1}
\]

where \(R\) is the radius of \(S^2\), \(d\Omega = \sin \theta \, d\theta \, d\phi\) is the invariant measure for the unit sphere, \(\beta\) is the circumference of \(S^1\) that corresponds to inverse temperature, and the dot stands for the derivative with respect to \(t\) that parametrizes \(S^1\). \(\mathcal{L}_i\) (\(i = 1, 2, 3\)) are the orbital angular momentum operators given by

\[
\mathcal{L}_\pm \equiv \mathcal{L}_1 \pm i \mathcal{L}_2 = e^{\pm i\phi} \left( \pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right),
\]

\[
\mathcal{L}_3 = -i \frac{\partial}{\partial \phi} \tag{2.2}
\]

To obtain the noncommutative field theory, we replace \(S^2\) with the fuzzy sphere in Eq. (2.1). The resultant theory is realized by a matrix model that is defined by

\[
S_{NC} = \frac{R^2}{2j + 1} \int_0^{\beta} dt \, \text{tr} \left( \frac{1}{2} \dot{\Phi}(t)^2 - \frac{1}{2R^2} \left[ \mathcal{L}_i, \Phi(t) \right]^2 + \frac{\mu^2}{2} \Phi(t)^2 + \frac{\lambda}{4} \Phi(t)^4 \right), \tag{2.3}
\]

\(^3\) For Monte Carlo studies concerning the fuzzy sphere, see Refs. [13–18].
By using Eq. (2.4), it is easy to show that

$$|\Omega\rangle = \sum_{m=-j}^{j} \left( \frac{2j}{j+m} \right)^{1/2} \left( \cos \frac{\theta}{2} \right)^{j+m} \left( \sin \frac{\theta}{2} \right)^{j-m} e^{i(j-m)\varphi} |jm\rangle,$$

(2.4)

where \( L_{\pm} = L_1 \pm iL_2 \) and \(|jm\rangle (m = -j, -j + 1, \ldots, j)\) are the standard basis for the spin \( j \) representation of the SU(2) algebra satisfying

\[ L_{\pm}|jm\rangle = (j \mp m)(j \pm m + 1)^{1/2} |jm \pm 1\rangle, \]

\[ L_3|jm\rangle = m|jm\rangle. \]

(2.5)

By using Eq. (2.4), it is easy to show that

\[ \frac{2j+1}{4\pi} \int d\Omega \, |\Omega\rangle\langle\Omega| = 1 \]

(2.6)

and

\[ |\langle\Omega_1|\Omega_2\rangle|^2 = \left( \cos \frac{x}{2} \right)^{2j} \text{ with } x = \arccos(\vec{n}_1 \cdot \vec{n}_2), \]

(2.7)

where \( \vec{n}_1 = (\sin \theta_1 \cos \varphi_1, \sin \theta_1 \sin \varphi_1, \cos \theta_1) \) and \( \vec{n}_2 = (\sin \theta_2 \cos \varphi_2, \sin \theta_2 \sin \varphi_2, \cos \theta_2) \).

Equation (2.7) implies that the width of the Bloch coherent state is given by \( \Delta = \frac{R}{\sqrt{2j}} \).

We also introduce the Berezin symbol (Ref. [32]) defined by \( f_{\phi(t)}(\Omega) = \langle \Omega|\phi(t)\Omega\rangle \), which is identified with \( \phi(t, \Omega) \) in Eq. (2.1) in the \( j \rightarrow \infty \) limit. First, by using Eq. (2.4), it can easily be shown that

\[ f_{[L, \phi]}(\Omega) = L f_{\phi}(\Omega). \]

(2.8)

Second, the star product for the two Berezin symbols is defined by

\[ f_A(\Omega) \star f_B(\Omega) \equiv f_{AB}(\Omega) = \frac{2j+1}{4\pi} \int d\Omega' \langle \Omega|A|\Omega'\rangle\langle\Omega'|B|\Omega\rangle. \]

(2.9)

where Eq. (2.6) is used. The star product coincides with the ordinary one at the tree level in the \( j \rightarrow \infty \) limit, while it gives rise to the UV/IR anomaly at the quantum level. Indeed, the noncommutative parameter is given by \( \Theta = \frac{R^2}{4j} \) (Refs. [8,30]), which vanishes in the \( j \rightarrow \infty \) limit. Third, by using Eq. (2.6), the trace over a matrix is translated into the integral over \( S^2 \). In this manner, the theory

\[^4\text{See also Refs. [28–31].}\]
(2.3) coincides with the theory (2.1) at the tree level in the \( j \to \infty \) limit, while it differs from the theory (2.1) at the quantum level even in the \( j \to \infty \) limit due to the UV/IR anomaly. The length scale of nonlocality of the interaction that gives rise to the UV/IR anomaly is given by \( \Theta \Lambda \sim R \).

2.2. Division of the fuzzy sphere
Following the prescription in Ref. [8], we divide the fuzzy sphere into two regions.

First, let us see the relationship between the Berezin symbol and the matrix elements \( \langle jm | \Phi | jm' \rangle \). We have a relation

\[
    f_{\phi}(\Omega) = \sum_{m,m'} \langle \Omega | jm \rangle \langle jm' | \Omega \rangle \langle jm | \Phi | jm' \rangle. \quad (2.10)
\]

By using Eq. (2.4), we find that

\[
    \langle \Omega | jm \rangle \langle jm' | \Omega \rangle \sim \left( \cos \frac{\theta}{2} \right)^{2j+m+m'} \left( \sin \frac{\theta}{2} \right)^{2j-m-m'} e^{i(m-m')\phi}. \quad (2.11)
\]

It is easy to show that Eq. (2.11) has a sharp peak at (Ref. [8])

\[
    \cos \theta = \frac{m + m'}{2j} \quad (2.12)
\]

with width \( \Delta \theta \sim \frac{1}{2j} \). This observation shows that the matrix elements \( \langle jm | \Phi | jn-m \rangle \) correspond to the field \( \phi \) at \( \cos \theta = \frac{n}{2j} \) (Ref. [8]).

Next, using the relation (2.12), we assign regions \( A \) and \( B \) on the sphere in Fig. 1(a) to parts \( A \) and \( B \), respectively, of the matrix \( \Phi \) in Fig. 1(b). In order to parametrize region \( A \) on the sphere, we introduce a parameter \( x \), which is the “volume” of region \( A \) divided by \( 2\pi R^2 \):

\[
    x = 1 - \cos \theta. \quad (2.13)
\]

Note that the “area” of the boundary between regions \( A \) and \( B \) is given by

\[
    2\pi R \sin \theta = 2\pi R \left( 2x - x^2 \right)^{1/2}. \quad (2.14)
\]

The condition that the element \( \langle jm | \Phi | jm' \rangle \) is located in part \( A \) is given by

\[
    m + m' > 2j - u, \quad (2.15)
\]
where \( u = 0, 1, 2, \ldots, 4j \). From Eqs. (2.12), (2.13), and (2.15), we find that
\[
x = \frac{u}{2j}.
\]
(2.16)

We can put \( R = 1 \) without loss of generality. In the following sections, we further put \( \mu = 1 \) for simplicity, and denote the matrix size \( 2j + 1 \) by \( N \).

3. Calculation of EE

3.1. Entanglement entropy

The division of the matrix \( \Phi \) in Fig. 1(b) corresponds to decomposing the Hilbert space in the theory (2.3) to a tensor product,
\[
\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B.
\]
(3.1)

EE for subsystem \( A \) is defined by
\[
S_A(x) = -\text{Tr}(\rho_A \log \rho_A),
\]
(3.2)

where \( x = \frac{u}{2j} \). Here \( \rho_A \) is defined by
\[
\rho_A = \text{Tr}_B(\rho_{\text{tot}}),
\]
(3.3)

where \( \text{Tr}_B \) stands for the partial trace over \( \mathcal{H}_B \), and \( \rho_{\text{tot}} \) is the total density matrix. We regard \( S_A \) as EE for region \( A \) in Fig. 1(a). Note that as a general property of EE the following relation holds at zero temperature:
\[
S_A = S_B,
\]
(3.4)

which implies that
\[
S_A(x) = S_A(2 - x),
\]
\[
\frac{\partial S_A}{\partial x}(x) = -\frac{\partial S_A}{\partial x}(2 - x),
\]
(3.5)

reflecting the symmetry of the system. The second relation in Eq. (3.5) will be used in checking the calculation and deriving the finite temperature effect.

3.2. Method to calculate EE

To calculate EE, we use the method developed in Ref. [19]. This method is based on the replica method, in which the definition of EE (3.2) is rewritten as
\[
S_A = \lim_{\alpha \to 1} \left[ -\frac{\partial}{\partial \alpha} \text{Tr} \rho_A^\alpha \right] = \lim_{\alpha \to 1} \left[ -\frac{\partial}{\partial \alpha} \log(\text{Tr} \rho_A^\alpha) \right],
\]
(3.6)

where \( \alpha \) is originally the number of replicas and extended to a real number.

We introduce \( \alpha \) replicas \( \Phi_n(t) \) \((n = 1, \ldots, \alpha)\) for \( \Phi(t) \) in Eq. (2.3). The boundary condition on \( \Phi_n(t) \) is depicted in Fig. 2:
\[
\Phi_n(\beta, m, m') = \Phi_{n+1}(0, m, m') \quad \text{for part } A,
\]
\[
\Phi_n(\beta, m, m') = \Phi_n(0, m, m') \quad \text{for part } B,
\]
(3.7)
where \( n = 1, \ldots, \alpha \) and \( \alpha + 1 \) is identified with 1 in the first line. Then, we find a relation
\[
\text{Tr} \rho^\alpha_A = \frac{Z(x, \alpha)}{Z^\alpha}, \tag{3.8}
\]
where \( Z(x, \alpha) \) is the partition function of the theory in which the boundary condition for the replicas is given in Eq. (3.7), and \( Z \) corresponds to the case with \( \alpha = 1 \) and is independent of \( x \). By substituting Eq. (3.8) into Eq. (3.6), we obtain an expression for \( S_A \):
\[
S_A(x) = -\lim_{\alpha \to 1} \frac{\partial}{\partial \alpha} \ln \left( \frac{Z(x, \alpha)}{Z^\alpha} \right). \tag{3.9}
\]
EE for the ground state is given in the \( \beta \to \infty \) limit, while EE for finite \( \beta \) includes the finite temperature effect.

\[\text{It is much easier to calculate the derivative of } S_A \text{ with respect to } x \text{ than } S_A \text{ itself: It is expressed as} \]
\[
\frac{\partial S_A(x)}{\partial x} = \frac{\partial}{\partial x} \left[ -\lim_{\alpha \to 1} \frac{\partial}{\partial \alpha} \ln \left( \frac{Z(x, \alpha)}{Z^\alpha} \right) \right] = \lim_{\alpha \to 1} \frac{\partial}{\partial x} \frac{\partial}{\partial \alpha} F(x, \alpha), \tag{3.10}
\]
where \( F(x, \alpha) = -\ln Z(x, \alpha) \). Here we approximate the derivative with respect to \( \alpha \) as\(^5\)
\[
\lim_{\alpha \to 1} \frac{\partial}{\partial x} \frac{\partial}{\partial \alpha} F(x, \alpha) \rightarrow \frac{\partial}{\partial x} (F(x, \alpha = 2) - F(x, \alpha = 1)) = \lim_{j \to \infty} \frac{F(x + \varepsilon, \alpha = 2) - F(x, \alpha = 2)}{\varepsilon}, \tag{3.11}
\]
where \( \varepsilon = \frac{1}{2j} \).

\[\text{We discretize the time direction with the lattice spacing } a.\]

In the free case where \( \lambda = 0 \), we calculate \( F(x, \alpha = 2) \) by numerically evaluating the determinant. The method is explained in Appendix B in Ref. [12].

In the interacting case where \( \lambda \neq 0 \), we perform a Monte Carlo simulation. We consider an interpolating action \( S_{\text{int}} = (1 - \gamma)S_{x+\varepsilon} + \gamma S_x \), where \( S_{x+\varepsilon} \) and \( S_x \) are the actions that would give \( F(x + \varepsilon, \alpha = 2) \) and \( F(x, \alpha = 2) \), respectively. Then the last expression in Eq. (3.11) reduces to
\[
\frac{F(x + \varepsilon, \alpha = 2) - F(x, \alpha = 2)}{\varepsilon} = 2j \int_0^1 d\gamma \langle S_{x+\varepsilon} - S_x \rangle_\gamma, \tag{3.12}
\]
\(^5\) To be conservative, what we calculate is the derivative of the Rényi entropy with respect to \( x \), where the Rényi parameter is equal to 2.
Fig. 3. The quantity \( \frac{1}{2} \frac{\delta A}{\delta x} \) at \( \lambda = 0 \), \( N = 16 \), and \( \beta = 1.0 \) is plotted against \( x \). The data for \( a = 0.125 \), \( 6.250 \times 10^{-2} \), \( 4.167 \times 10^{-2} \), \( 3.125 \times 10^{-2} \) are represented by diamonds, triangles, circles, and squares, respectively. The data for \( a = 3.125 \times 10^{-2} \) are fitted to \( \frac{1}{2} \frac{\delta A}{\delta x} = cx + d \) for \( 0.333 \leq x \leq 1.800 \), which gives \( c = -0.1672 \) (26) and \( d = 0.2623 \) (32).

where \( \langle \cdots \rangle_\gamma \) represents the expectation value with respect to the canonical weight \( e^{-S_{\text{int}}} \). We take \( \gamma \) from 0 to 1 in steps of 0.1, and calculate \( \langle S_{x+\epsilon} - S_x \rangle_\gamma \) for each \( \gamma \) by Monte Carlo simulation. Then, we finally apply the Simpson formula to the integral over \( \gamma \) in Eq. (3.12).

4. Results for the free case

4.1. Behavior of EE in the free case

In this subsection, we show our results for the free case (\( \lambda = 0 \)).\(^6\)

We first calculate \( F(x, \alpha = 2) \) numerically using the method given in Appendix B in Ref. [12]. Then, following Eq. (3.11), we calculate \( \frac{\delta S_A}{\delta x} \).

We observe that at \( N = 16 \) and \( \beta = 1.0 \) the data for odd \( u \) behave smoothly while the data for even \( u \) behave smoothly in a different way (note that \( x = \frac{u}{2j} \)). This difference almost disappears at \( \beta = 4.0 \). This difference is considered to originate from a finite \( N \) effect that becomes stronger at high temperature. Indeed, we find that the continuum limit in the time direction can be taken at \( N = 16 \) and \( \beta = 1.0 \) using only the data for odd \( u \) or even \( u \) in such a way that the two continuum limits for odd \( u \) and for even \( u \) differ only by the finite temperature effect. We plot only the data for odd \( u \) in what follows.

In Fig. 3, we plot \( \frac{1}{2j} \frac{\delta S_A}{\delta x} \) against \( x \). We plot the data for four values of the lattice spacing \( a \) to study the continuum limit in the time direction at \( N = 16 \) and \( \beta = 1.0 \). We see that the data for \( a = 4.167 \times 10^{-2} \) and \( a = 3.125 \times 10^{-2} \) almost agree. This implies that \( a = 4.167 \times 10^{-2} \) is close enough to the continuum limit. We fit the data for \( a = 3.125 \times 10^{-2} \) to the linear function \( \frac{1}{2j} \frac{\delta S_A}{\delta x} = cx + d \). In this fitting, we exclude some data points around \( x = 0 \) and \( x = 2.0 \), where the volume of region \( A \) or region \( B \) on the sphere is so small that there should be an ambiguity in the

\(^6\) Figs. 3 and 4 were also presented in Ref. [12].
Fig. 4. The quantity $\frac{1}{2j} \frac{\partial S_A}{\partial x}$ is plotted against $x$ at $\lambda = 0$ and $N = 16$. The data for $\beta = 3.0$ and $a = 0.125$, $6.250 \times 10^{-2}$, $4.167 \times 10^{-2}$, $3.125 \times 10^{-2}$ are represented by diamonds, triangles, inverted triangles, and circles, respectively, while the data for $\beta = 4.0$ and $a = 4.167 \times 10^{-2}$ are represented by squares. The data for $\beta = 3.0$ and $a = 3.125 \times 10^{-2}$ are fitted to $\frac{1}{2j} \frac{\partial S_A}{\partial x} = cx + d$ for $0.200 \leq x \leq 1.800$, which gives $c = -0.1612(29)$ and $d = 0.1629(33)$.

boundary between the two regions due to a finite $N$ effect. We use the range $0.333 \leq x \leq 1.8$ and obtain $c = -0.1672(26)$ and $d = 0.2623(32)$.

In Fig. 4, we perform the same analysis at $N = 16$ and $\beta = 3.0$ as $N = 16$ and $\beta = 1.0$. We see that $a = 4.167 \times 10^{-2}$ is close enough to the continuum limit also in this case. Using the range $0.2 \leq x \leq 1.8$, we fit the data for $a = 3.125 \times 10^{-2}$ to the linear function and obtain $c = -0.1612(29)$ and $d = 0.1629(33)$. Namely, the function is proportional to $1 - x$ within the fitting error. This function is consistent with Eq. (3.5). This implies that $\beta = 3.0$ is close enough to the zero temperature limit (the $\beta \to \infty$ limit). Indeed, we also plot the data for $N = 16$, $\beta = 4.0$, and $a = 4.167 \times 10^{-2}$ in Fig. 4. The data almost agree with those for $N = 16$, $\beta = 3.0$, and $a = 4.167 \times 10^{-2}$. This supports the statement that $\beta = 3.0$ is close enough to the zero temperature limit. Thus we find a square of area law at zero temperature:

$$S_A \propto 2x - x^2 = \sin^2 \theta. \quad (4.1)$$

We see that the difference between the two functions $\frac{1}{2j} \frac{\partial S_A}{\partial x} = cx + d$ fitted to the data for $\beta = 1.0$ and $\beta = 3.0$ is almost constant. This means that the finite temperature effect in $S_A$ is proportional to $x$, namely the volume of region $A$. This volume law for the finite temperature effect is in general seen in local field theories. Fitting the data with even $u$ for $N = 16$, $\beta = 1.0$, and $a = 3.125 \times 10^{-2}$ to $\frac{1}{2j} \frac{\partial S_A}{\partial x} = cx + d$ for $0.133 \leq x \leq 1.6$ gives $c = 0.1626(26)$ and $d = 0.2690(22)$. As we stated, the differences between the data for odd $u$ and the data for even $u$ arise in the finite temperature effect.

In Fig. 5, we plot $\frac{\partial S_A}{\partial x}$ (not divided by $2j$) at $\beta = 1.0$, $a = 4.167 \times 10^{-2}$, $16$, $24$, $32$ against $x$ to examine the large-$N$ (large-$j$) limit, which corresponds to the continuum limit of the fuzzy sphere. We see that the data for all $N$’s coincide at $x = 1$ and that the data for each $N$ can be fitted to the linear function $\frac{\partial S_A}{\partial x} = px + q = -p(1 - x) + p + q$, where $p + q$ is independent of $N$. This is consistent with the above observation that the finite temperature effect is proportional to the volume, and further
Fig. 5. The quantity $\frac{\partial S_A}{\partial x}$ (not divided by $2j$) is plotted against $x$ at $\lambda = 0$, $\beta = 1.0$, and $a = 4.167 \times 10^{-2}$. The triangles, circles, and squares represent the data for $N = 16$, 24, and 32, respectively. The solid line is a fit of the data for $N = 16$ to $\frac{\partial S_A}{\partial x} = px + q$ for $0.333 \leq x \leq 1.667$, which gives $p = -2.497(37)$ and $q = 3.912(40)$. The dashed line is a fit of the data for $N = 24$ to $\frac{\partial S_A}{\partial x} = px + q$ for $0.217 \leq x \leq 1.783$, which gives $p = -3.668(29)$ and $q = 5.087(32)$. The dotted line is a fit of the data for $N = 32$ to $\frac{\partial S_A}{\partial x} = px + q$ for $0.161 \leq x \leq 1.839$, which gives $p = -4.671(20)$ and $q = 6.090(22)$.

Fig. 6. The values of $p$ obtained in Fig. 5 are plotted against $2j$. The solid line is a fit of the data to $p = 2jv + w$, which gives $v = -0.1359(61)$ and $w = -0.49(14)$.

implies that the finite temperature effect is independent of $N$. We exclude a shorter range of $x$ in fitting the data as $N$ increases. This supports the statement that the ambiguity of the boundary is a finite $N$ effect so that it vanishes in the $N \to \infty$ limit. As we stated, we observe that the differences between the data for odd $u$ and the data for even $u$ become smaller as $N$ increases.

In Fig. 6, we plot the values of $p$ obtained in Fig. 5 against $2j$. We obtain a good fit of the data to $p = 2jv + w$. 

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To summarize, we find that in the free case EE behaves in the $a \to 0$ limit with large $j$ as

$$S_A = \left( v|j| + \frac{|w|}{2} \right) \sin^2 \theta + g(1 - \cos \theta),$$

(4.2)

where $v$ and $w$ are constants independent of $j$ and $\beta$, and $g$ is a constant\(^7\) independent of $j$ and vanishes in the $\beta \to \infty$ limit. Namely, the first term corresponds to EE at zero temperature, which is proportional to the square of area and depends on the UV cutoff $j$. The second term corresponds to the finite temperature effect. It is governed by the volume law and is independent of the UV cutoff $j$.

The $j$ dependence of the first term in Eq. (4.2) is observed in Refs. [8,9] and the $\theta$ dependence in the first term is suggested in Ref. [9]. Thus we verified the behavior of EE at zero temperature by using a method different from the one in Refs. [8,9]. This indicates the validity of our method to calculate EE.

### 4.2. Origin of the square of area law

In local field theories, the leading contribution to EE of a focused region $A$ at zero temperature obeys the area law. Namely, it is proportional to $|\partial A|/\epsilon^{d-1}$, where $|\partial A|$ is the area of the boundary of region $A$, $\epsilon$ is a UV cutoff, and $d$ is the space dimension. This behavior is understood from the fact that region $A$ interacts with the outside through the boundary in local field theories. On the other hand, we have confirmed that the leading contribution to EE at zero temperature in the free case is proportional to the square of area, namely $N\sin^2 \theta$, although we had naively expected it to obey the area law. Because this depends on the UV cutoff $N$, we need to go back to a regularized theory (2.3) to discuss the origin of this square of area law.

For $\lambda = 0$, the matrix model action (2.3) is local with respect to the matrix elements because $L_i$ are tridiagonal in the standard basis (2.5). Thus the degree of freedom in the boundary between parts $A$ and $B$ of the matrix $\Phi$ in Fig. 1(b) should contribute to EE at zero temperature. The number of states $|jm\rangle$ that effectively contribute to the degree of freedom in the boundary should be proportional to the area of the boundary divided by the width of the Bloch coherent states $\Delta = \frac{R}{\sqrt{j}}$:

$$2\pi R \sin \theta \times \frac{1}{\Delta} = 2\pi \sqrt{j} \sin \theta. \quad (4.3)$$

Moreover, the matrix elements $\langle jm|\Phi|jm'\rangle$ are bilocal in the sense that they have two indices $m$ and $m'$. It is therefore natural that the leading contribution to EE at zero temperature is proportional to

$$(\sqrt{j} \sin \theta)^2 \sim N\sin^2 \theta. \quad (4.4)$$

### 5. Results for the interacting case

In this section, we study the interacting case. We perform Monte Carlo simulations at $N = 16$, $\beta = 1.0$, and $a = 3.125 \times 10^{-2}$.

We again see a difference between odd $u$ and even $u$, similarly to the free case. We plot only the data for odd $u$ in the interacting case also.

In Fig. 7, we plot $\frac{1}{2} \frac{\partial S_A}{\partial x}$ in the interacting case, together with that in the free case, against $x$. We plot the data for $\lambda = 3.0, 12.0$ (the interacting case) and for $\lambda = 0$ (the free case) after we subtract each value at $x = 1$ from the data for each $\lambda$. We see that the shifted data are consistent with Eq. (3.5). This

\(^7\) $g = p + q.$
Fig. 7. After the value at $x = 1$ is subtracted from the data, $\frac{1}{2} \frac{\partial S_A}{\partial x}$ is plotted against $x$ at $N = 16$, $\beta = 1.0$, and $a = 3.125 \times 10^{-2}$. The triangles, circles, and squares represent the data for $\lambda = 0$, 3.0, and 12.0, respectively.

Fig. 8. After the value at $x = 1$ is subtracted from the data, $\frac{1}{2} \frac{\partial S_A}{\partial x}$ is plotted against $x$ at $\lambda = 12.0$, $N = 16$, $\beta = 1.0$, and $a = 3.125 \times 10^{-2}$. The solid line is a fit of the first four data points ($x = 0.067 \sim 0.467$) to $\frac{1}{2} \frac{\partial S_A}{\partial x} = b (2x - x^2)^{1/2}$, which gives $b = 0.0161(2)$. The dotted line is a fit of the next four data points ($x = 0.6 \sim 1.0$) to $\frac{1}{2} \frac{\partial S_A}{\partial x} = c(1 - x)$, which gives $c = 0.0318(25)$.

suggests that the finite temperature effect in $S_A$ is also proportional to the volume in the interacting case. The shape of the data in the interacting case is different from that in the free case, which should be attributed to nonlocality of the interaction. Moreover, the magnitude in the interacting case is quite smaller than that in the free case.

As we stated in Sect. 2.1, the length scale of nonlocality of the interaction is of order $R = 1$. Therefore we can, in general, expect that there is a transition from the “volume law” to the square of area law, where the “volume law” is given by the integral of the square of the area. We assume that
the transition happens around $\theta = \theta_0$. Namely, for $\theta_0 \lesssim \theta \lesssim \frac{\pi}{2}$, EE behaves as the square of area law

$$S_A = jc \sin^2 \theta,$$

which leads to

$$\frac{1}{2j} \frac{\partial S_A}{\partial x} = c(1 - x),$$

while for $0 \leq \theta \lesssim \theta_0$ EE behaves as

$$S_A = jc \int_0^\theta \sin^2 \theta' d\theta',$$

which leads to

$$\frac{1}{2j} \frac{\partial S_A}{\partial x} = \frac{c}{2} (2x - x^2)^{1/2}.$$

In Fig. 8, we again plot the data for $\lambda = 12.0$ and fit the first four data points ($x = 0.067 \sim 0.467$) to $\frac{1}{2j} \frac{\partial S_A}{\partial x} = b \left(2x - x^2\right)^{1/2}$ and the next four data points ($x = 0.6 \sim 1.0$) to $\frac{1}{2j} \frac{\partial S_A}{\partial x} = c(1 - x)$. We obtain $b = 1.61 \times 10^{-2}(2)$ and $c = 3.18 \times 10^{-2}(25)$. These values are consistent with Eq. (5.2) and Eq. (5.4). We apply the same analysis to the data for $\lambda = 10.0$ and again obtain a good fit consistent with Eqs. (5.2) and (5.4).

6. Conclusion and discussion

In this paper, we calculated EE in the scalar field theory on the fuzzy sphere using the method developed in Ref. [19]. In the free case, we confirmed that EE at zero temperature is proportional to the square of the area of the boundary and the leading contribution to it is proportional to the UV cutoff $N$. We discussed the reason for this peculiar law. These behaviors are consistent with the observations in Refs. [8,9]. We also found that the finite temperature effect in EE is proportional to the volume and independent of the UV cutoff $N$. This property of the finite temperature effect is shared with ordinary local field theories.

In the interacting case, we performed Monte Carlo simulations to calculate EE. We found that the magnitude of EE in the interacting case is quite small compared to that in the free case. We observed a transition from the “volume law” to the square of area law for EE as the volume is increased. The former is obtained by integrating the latter over $\theta$. This transition should originate from the nonlocal nature of the interaction. We also found that the finite temperature effect in EE is proportional to the volume in the interacting case.

The fact that we can interpret our results geometrically following the division of the sphere in Fig. 1(a) shows that the division of the matrix in Fig. 1(b) works well.

In the interacting case, we need to study renormalization and the continuum limit. Indeed, at $N = 16$, $\beta = 1.0$, and $a = 3.125 \times 10^{-2}$, we obtained a good fit to the transition from the “volume law” to the square of area law for $\lambda = 10.0, 12.0$ but not for $\lambda = 3.0$. This suggests that the former is closer to the continuum limit than the latter.
In Ref. [9], it was shown that mutual information (MI) in the free case agrees with that in the ordinary scalar field theory on $R \times S^2$. This result is reasonable because MI is independent of the UV cutoff and the matrix model (2.3) with $\lambda = 0$ reduces to the ordinary scalar field theory on $S^1 \times S^2$ in the limit where the UV cutoff goes to infinity. We expect differences in the interacting case between MI in the noncommutative theory and that in the ordinary theory.

We hope to report progress in the above issues in the near future.

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