WAVES AND DIFFUSION ON METRIC GRAPHS WITH GENERAL VERTEX CONDITIONS

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Abstract. We prove well-posedness for general linear wave- and diffusion equations on compact or non-compact metric graphs allowing various conditions in the vertices. More precisely, using the theory of strongly continuous operator semigroups we show that a large class of (not necessarily self-adjoint) second order differential operators with general (possibly non-local) boundary conditions generate cosine families, hence also analytic semigroups, on $L^p(\mathbb{R}_+, C^\ell) \times L^p([0,1], C^m)$ for $1 \leq p < +\infty$.

1. Introduction. It is well-known (for details see [17, Sect. II.6] and [6, Sect. 3.14]) that first and second order abstract Cauchy problems of the form

\begin{align}
(ACP_1) \quad \begin{cases}
\dot{x}(t) = Gx(t), & t \geq 0, \\
x(0) = x_0,
\end{cases}
\quad \text{and} \quad
(ACP_2) \quad \begin{cases}
\ddot{x}(t) = Gx(t), & t \geq 0, \\
x(0) = x_0, \\
\dot{x}(0) = x_1,
\end{cases}
\end{align}

for a linear (in general unbounded) operator $G : D(G) \subset X \to X$ on a Banach space $X$ are well-posed if and only if $G$ generates a strongly continuous semigroup and a cosine family on $X$, respectively. It follows by [6, Thm. 3.14.17] that generators of cosine families generate analytic semigroups of angle $\frac{\pi}{2}$. Hence, well-posedness of (ACP_2) always implies well-posedness of (ACP_1).

In this paper we are concerned with such Cauchy problems for second order elliptic differential operators $G$ acting on spaces of $L^p$-functions defined on a finite union of intervals. Operators of this type appear, e.g., in the modeling of diffusion- and wave equations on metric graphs. In this case the intervals represent the edges
of the graph while its structure is encoded in the boundary conditions appearing in the domain $D(G)$ of $G$. In the simplest case we can take $X = (L^2([0, 1]))^m = L^p([0, 1], \mathbb{C}^m)$ and

$$G = \lambda(\cdot) \cdot \frac{d^2}{ds^2},$$

$$D(G) = \left\{ f \in W^{2,p}([0, 1], \mathbb{C}^m) \left| \begin{array}{l} V_0 f(0) + V_1 f(1) = 0 \\ W_0 f'(0) - W_1 f'(1) + U_1 f(1) + U_0 f(0) = 0 \end{array} \right. \right\},$$

where $\lambda(s) = \text{diag}(\lambda_j(s))^{m}_{j=1}$ for positive, Lipschitz continuous “diffusion” coefficients $\lambda_j(\cdot)$ and suitable “boundary” matrices $V_0, V_1 \in M_{k_0 \times m}(\mathbb{C})$ and $U_0, U_1, W_0, W_1 \in M_{k_1 \times m}(\mathbb{C})$, for $k_0, k_1 \in \mathbb{N}$ satisfying $k_0 + k_1 = 2m$.

Our main result, Theorem 2.3, gives for such operators a condition implying the generation of a cosine family, hence the well-posedness of (1.1). For example, by Corollary 2.11, the operator $G$ in (1.2) generates a cosine family if

$$\det \begin{pmatrix} V_1 & V_0 \\ W_1 \cdot \mu(1)^{-1} & W_0 \cdot \mu(0)^{-1} \end{pmatrix} \neq 0,$$

where $\mu(s) := \sqrt{\lambda(s)} = \text{diag}(\sqrt{\lambda_j(\cdot)})^{m}_{j=1}$. In particular, (1.3) implies that for $G$ as in (1.2) both Cauchy problems in (1.1) are well-posed.

Motivated by different problems from physics, chemistry, biology, and engineering, the study of dynamical processes on metric graphs (also called networks or one-dimensional ramified spaces) has received much attention in the last decades. Diffusion equations on networks were first considered in the 1980s, the earliest references include [35, 39, 38, 41]. Since then many authors used functional analytic methods to treat such problems, we only mention [20, 12, 5, 30, 8, 36]. The study of wave equation on networks was initiated about at the same time by [3, 4], see also [33, 31, 13, 34, 14, 30, 26, 25].

Almost simultaneously, another community of theoretical physicists was mainly interested in the Schrödinger equation on a network structure (calling it a quantum graph), see [18, 29, 28, 32, 9, 40]. They also considered so-called non-compact graphs, where some edges are allowed to be infinite.

All these problems were initially treated in a $L^2$-setting using Hilbert-spaces techniques. Then interpolation was used to generalize the results to $L^p$-spaces. Typically, in this context only self-adjoint operators are considered.

On the contrary, we use methods from the theory of operator semigroups and work on $L^p$-spaces directly. The novelty of our approach is manifold. In fact, it allows us to

- study non-self-adjoint generators $G$,
- treat very general (also non-diagonal) “diffusion coefficient matrices” $a(\cdot, \cdot)$, cf. (2.1),
- treat very general boundary conditions of the form

$$\Phi_0 f = 0, \quad \Phi_1 (f' + B f) = 0,$$

for appropriate boundary functionals $\Phi_0, \Phi_1$ and a bounded operator $B$, cf. (2.3),
- consider state spaces $X = L^p(\mathbb{R}_+, \mathbb{C}^d) \times L^p([0, 1], \mathbb{C}^m)$ with application to non-compact graphs,
- treat all cases for $p \in [1, +\infty)$ simultaneously without using interpolation arguments,
• explicitly compute the phase space \( \ker(\Phi_0) \times X \) of \( G \), cf. Theorem 2.3.

Our reasoning is based on a recent result for boundary perturbations of domains of generators developed in [1] (which we recall in Theorem A.3) and the fact that squares of group generators generate cosine families, cf. [6, Expl. 3.14.15]. Roughly speaking, we start from a simple first-order differential operator \( A \) generating a semigroup. Then we perturb its domain to obtain \( \mathcal{G} \) whose square is closely related to \( G \). Moreover, since we arrange \( \mathcal{G} \) to be similar to \( -\mathcal{G} \), it automatically generates a group. Hence, \( \mathcal{G}^2 \) and consequently also \( G \) generate cosine families. To obtain our main theorem in its most general form we use similarity transformations and bounded perturbations. In this way we are able to generalize the boundary conditions for non-self adjoint and non-compact graphs given in [27, 24], see Example 2.12, as well as the general boundary conditions in terms of “boundary subspaces” presented in [36, Sect. 6.5], see Subsection 3.6. We can also treat different non-local boundary conditions (for example those studied in [37], see Example 2.10).

This paper is organized as follows. In Section 2 we introduce our setup, state and prove the main generation result (Theorem 2.3) and apply it to two important classes of boundary conditions (Corollary 2.11 and Corollary 2.16). This facilitates the verification of the generation conditions (2.32) and (2.39) used in Section 3 to show well-posedness of diffusion- and wave equations on (possibly non-compact) graphs for a wide variety of boundary conditions. In the appendix we recall a perturbation result from [1] which is the main tool for our approach.

Our notation closely follows [17].

2. Generation of cosine families.

2.1. The setup. Throughout this section we make the following assumptions. Although the results presented here are abstract, the terminology already suggests that our main motivation arises from the study of dynamical processes on (possibly non-compact) metric graphs. In the sequel we use the notation \( \mathbb{R}_+ := [0, +\infty) \).

**Assumption 2.1.** Consider for some fixed \( p \in [1, \infty) \), \( \ell \in \mathbb{N}_0 \), and \( m \in \mathbb{N}_0 \) satisfying \( \ell + m > 0 \)

(i) the space \( X^e := L^p(\mathbb{R}_+, \mathbb{C}^\ell) \) of functions on \( \ell \) “external edges”,

(ii) the space \( X^i := L^p([0, 1], \mathbb{C}^m) \) of functions on \( m \) “internal edges”,

(iii) the “state space” \( X := X^e \times X^i \) of functions on all \( \ell + m \) edges,

(iv) two “boundary spaces” \( Y_0, Y_1 \subseteq \mathbb{C}^{\ell+2m} \) satisfying \( Y_0 \oplus Y_1 = \mathbb{C}^{\ell+2m} =: \mathcal{Y} \),

(v) two “boundary functionals” \( \Phi_0 = (\Phi_0^1, \Phi_0^2) \in \mathcal{L}(\mathcal{C}(\mathbb{R}_+, \mathbb{C}^\ell) \times C([0, 1], \mathbb{C}^m), Y_0) \) and \( \Phi_1 = (\Phi_1^1, \Phi_1^2) \in \mathcal{L}(\mathcal{C}(\mathbb{R}_+, \mathbb{C}^\ell) \times C([0, 1], \mathbb{C}^m), Y_1) \) (to determine the boundary conditions),

(vi) a “boundary operator” \( B \in \mathcal{L}(X) \) (appearing in the boundary conditions) leaving \( W^{1,p}(\mathbb{R}_+, \mathbb{C}^\ell) \times W^{1,p}([0, 1], \mathbb{C}^m) \) invariant,

(vii) a Lipschitz continuous function\(^1\) \( a(\bullet, \bullet): \mathbb{R}_+ \times [0, 1] \to M_{\ell+m}(\mathbb{C}) \), the so-called “diffusion coefficients matrix” of the form

\[
\begin{pmatrix}
q^e(\bullet) & 0 \\
0 & q(\bullet)
\end{pmatrix}
\cdot
\begin{pmatrix}
\lambda^e(\bullet) & 0 \\
0 & \lambda(\bullet)
\end{pmatrix}
\cdot
\begin{pmatrix}
q^e(\bullet) & 0 \\
0 & q(\bullet)
\end{pmatrix}^{-1}
\]

\[
:= q(\bullet, \bullet) \cdot \lambda(\bullet, \bullet) \cdot q(\bullet, \bullet)^{-1},
\]

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\(^1\)This implies that the multiplication operator induced by \( a'(\bullet, \bullet) \) is bounded which is needed in our approach.
where \( q^f(\cdot) \in L^\infty(\mathbb{R}_+, M_\ell(\mathbb{C})) \) and \( q^f(\cdot) \in L^\infty([0, 1], M_m(\mathbb{C})) \) are Lipschitz continuous (pointwise) invertible with bounded inverses, and
\[
\lambda^f(s) = \text{diag}(\lambda^f_k(s))_{k=1}^{\ell} \in M_\ell(\mathbb{C}), \quad s \in \mathbb{R}_+,
\lambda^l(s) = \text{diag}(\lambda^l_j(s))_{j=1}^{m} \in M_m(\mathbb{C}), \quad s \in [0, 1],
\]
with entries satisfying for some \( \varepsilon > 0 \)
\[
\varepsilon < \lambda^f_k(s) < \varepsilon^{-1} \quad \text{for all } s \in \mathbb{R}_+, \quad k = 1, \ldots, \ell, \tag{2.2}
\]
\[
0 < \lambda^l_j(s) \quad \text{for all } s \in [0, 1], \quad j = 1, \ldots, m.
\]
Note that (vii) implies that \( q(\cdot, \cdot)^{-1} \), and the functions \( \lambda^f_k(\cdot) \in C(\mathbb{R}_+), \lambda^l_j(\cdot) \in C[0, 1] \) for \( k = 1, \ldots, \ell, \) \( j = 1, \ldots, m \), are all Lipschitz continuous.

Using that \( W^{1,p}(\mathbb{R}_+, \mathbb{C}^\ell) \subset C_0(\mathbb{R}_+, \mathbb{C}^\ell) \) and \( W^{1,p}([0, 1], \mathbb{C}^m) \subset C([0, 1], \mathbb{C}^m) \) (see [11, Sect.8.2]) we then define on \( X = L^p(\mathbb{R}_+, \mathbb{C}^\ell) \times L^p([0, 1], \mathbb{C}^m) \) the operator
\[
G := a(\cdot, \cdot) \cdot \frac{d^2}{ds^2},
\]
\[
D(G) := \{ f \in W^{2,p}(\mathbb{R}_+, \mathbb{C}^\ell) \times W^{2,p}([0, 1], \mathbb{C}^m) : \Phi_0 f = 0, \Phi_1 (f' + B f) = 0 \}.
\]
Our aim is to give conditions on the boundary functionals \( \Phi_0, \Phi_1 \) implying that \( G \) generates a cosine family on \( X \), hence by [6, Thm. 3.14.17] also an analytic semigroup of angle \( \frac{\pi}{2} \). As we will see in Theorem 2.3 below this can be achieved (independently on \( B \)) through an invertibility condition on the operator \( R_{t_0} \) defined in (2.11) below.

We note that we do not consider the case \( p = \infty \). In fact, this would yield a non-densely defined operator \( G \) which cannot be a generator. More generally, it is well-known that on \( L^\infty \)-spaces strongly continuous semigroups are uniformly continuous, i.e., have a bounded generator. Hence, an operator \( G \subset a(\cdot, \cdot) \cdot \frac{d^2}{ds^2} \) will never, independently on the domain, generate a \( C_0 \)-semigroup on \( L^\infty(\mathbb{R}_+, \mathbb{C}^\ell) \times L^\infty([0, 1], \mathbb{C}^m) \).

In order to state our main result rigorously we need some more notations. For \( 1 \leq k \leq \ell \) we define
\[
\mu^f_k(\cdot) := \sqrt{\lambda^f_k(\cdot)} \in C(\mathbb{R}_+) \quad \text{and set}
\mu^f(\cdot) := \text{diag}(\mu^f_k(\cdot))_{k=1}^{\ell} \in C(\mathbb{R}_+, M_\ell(\mathbb{C})). \tag{2.4}
\]
Moreover, we put
\[
\varphi^f_k(s) := \int_0^s \frac{dr}{\mu^f_k(r)}, \quad s \in \mathbb{R}_+.
\]
Then it follows from (2.2) that all \( \varphi^f_k : \mathbb{R}_+ \to \mathbb{R}_+ \) are Lipschitz continuous, surjective and strictly increasing, hence invertible with Lipschitz continuous inverses \( (\varphi^f_k)^{-1} : \mathbb{R}_+ \to \mathbb{R}_+ \).

Similarly, for \( 1 \leq j \leq m \) we define
\[
\mu^l_j(\cdot) := \sqrt{\lambda^l_j(\cdot)} \in C[0, 1] \quad \text{and set}
\mu^l(\cdot) := \text{diag}(\mu^l_j(\cdot))_{j=1}^{m} \in C([0, 1], M_m(\mathbb{C})). \tag{2.5}
\]
Furthermore, we consider
\[
\varphi^l_j(s) := \int_0^s \frac{dr}{\mu^l_j(r)}, \quad \bar{c}_j := \frac{1}{\varphi^l_j(1)}, \quad \varphi^l_j(s) := \bar{c}_j \cdot \varphi^l_j(s), \quad s \in [0, 1]. \tag{2.6}
\]
Then all $\bar{\varphi}_j : [0,1] \to [0,1]$ are Lipschitz continuous, surjective and strictly monotone, hence invertible with Lipschitz continuous inverses $(\bar{\varphi}_j)^{-1} : [0,1] \to [0,1]$.

Next we put
\[
\varphi^e := (\varphi_1^e, \ldots, \varphi_\ell^e)^T : \mathbb{R}_+ \to \mathbb{C}_e, \quad (\varphi^e)^{-1} := ((\varphi_1^e)^{-1}, \ldots, (\varphi_\ell^e)^{-1})^T : \mathbb{R}_+ \to \mathbb{C}_e,
\]
\[
\bar{\varphi}^i := (\bar{\varphi}_1^i, \ldots, \bar{\varphi}_m^i)^T : [0,1] \to \mathbb{C}_m, \quad (\bar{\varphi}^i)^{-1} := ((\bar{\varphi}_1^i)^{-1}, \ldots, (\bar{\varphi}_m^i)^{-1})^T : [0,1] \to \mathbb{C}_m,
\]
\[
c := (c_1, \ldots, c_m) \in \mathbb{C}_m,
\]
\[
c^{-1} := (c_1^{-1}, \ldots, c_m^{-1}) \in \mathbb{C}_m.
\]

Using this notation we define the transformations
\[
\begin{align*}
J_{\varphi^e} &\in \mathcal{L}(X^e), \quad J_{\varphi^e} f^e := f^e \circ \varphi^e \quad \text{for } f^e \in X^e, \\
J_{\bar{\varphi}^i} &\in \mathcal{L}(X^i), \quad J_{\bar{\varphi}^i} f^i := f^i \circ \bar{\varphi}^i \quad \text{for } f^i \in X^i.
\end{align*}
\]
(2.7)

Then $J_{\varphi^e}$ and $J_{\bar{\varphi}^i}$ are invertible with bounded inverses $J_{\varphi^e}^{-1} = J_{(\varphi^e)^{-1}} \in \mathcal{L}(X^e)$ and $J_{\bar{\varphi}^i}^{-1} = J_{(\bar{\varphi}^i)^{-1}} \in \mathcal{L}(X^i)$. These maps will be used in Lemma 2.4 to transform space dependent diffusion coefficients into constant ones.

Next for fixed $t_0 > 0$ and $t \in [0, t_0]$ we introduce the bounded linear operators $Q_t \in \mathcal{L}(L^p([0, t_0], \mathbb{C}_e), X^e)$ and $R_t, S_t \in \mathcal{L}(L^p([0, t_0], \mathbb{C}_m), X^i)$ by
\[
Q_t u^e := \bar{u}^e(t - \bullet), \quad R_t u^i := \bar{u}^i(t - \zeta), \quad \text{and} \quad S_t u^i := \bar{u}^i(t + \zeta^{-1}),
\]
(2.9)

where $\bar{u}$ denotes the extension of a function $u$ defined on $I \subset \mathbb{R}$ to $\mathbb{R}$ by the value 0. Observe that $S_t = \psi R_t$ for $\psi \in \mathcal{L}(X^i), (\psi f^i)(\bullet) := f^i(1 - \bullet)$. Moreover, we define
\[
\bar{\Phi}_1 = (\Phi_1, \bar{\Phi}_1) := \Phi_1 \cdot c(\bullet, \bullet)^{-1} \in \mathcal{L}(C_0(\mathbb{R}_+, \mathbb{C}_e) \times C([0,1], \mathbb{C}_m), Y_1),
\]
where
\[
c(\bullet, \bullet) := \sqrt{a(\bullet, \bullet)} = \begin{pmatrix} q^e(\bullet) & 0 \\ 0 & q^i(\bullet) \end{pmatrix} \cdot \begin{pmatrix} \mu^e(\bullet) & 0 \\ 0 & \mu^i(\bullet) \end{pmatrix} \cdot \begin{pmatrix} q^e(\bullet) & 0 \\ 0 & q^i(\bullet) \end{pmatrix}^{-1}
\]
\[
= q^e(\bullet) \cdot \mu^e(\bullet) \cdot q^i(\bullet)^{-1},
\]
(2.10)

for $\mu^e(\bullet)$ and $\mu^i(\bullet)$ given in (2.4) and (2.5), respectively.

Now we are ready to introduce the operator $\mathcal{R}_{t_0}$ as follows. Note that
\[
\forall^p([0, t_0], Y) = \mathbb{L}^p([0, t_0], Y_0) \times \mathbb{L}^p([0, t_0], Y_1)
\]
\[
= \mathbb{L}^p([0, t_0], \mathbb{C}_e) \times \mathbb{L}^p([0, t_0], \mathbb{C}_m) \times \mathbb{L}^p([0, t_0], \mathbb{C}_m).
\]
Lemma 2.2. The operator $\mathcal{R}_{t_0} : W_{0}^{1,p}([0,t_0],Y) \subset L^p([0,t_0],Y) \to L^p([0,t_0],Y)$ given by

$$(\mathcal{R}_{t_0} u)(t) := (\Phi_t \circ \Phi_0) \cdot \left( \begin{array}{c} q(\cdot,*) \\ 0 \\ 0 \\ 0 \end{array} \right) \cdot \left( \begin{array}{cccc} J_{\varphi^*} & 0 & 0 & 0 \\ 0 & J_{\varphi^*} & J_{\varphi^*} & 0 \\ 0 & J_{\varphi^*} & 0 & 0 \\ -J_{\varphi^*} & 0 & 0 & -J_{\varphi^*} \end{array} \right) \cdot \left( \begin{array}{cccc} Q_t & 0 & 0 & 0 \\ 0 & S_t & 0 & 0 \\ 0 & 0 & 0 & R_t \end{array} \right) \cdot u$$

is well-defined and has a unique bounded extension to $L^p([0,t_0],Y)$ denoted again by $\mathcal{R}_{t_0}$.

The operator $\mathcal{R}_{t_0}$ plays a crucial role in our main result, see Theorem 2.3. As we will see later, in many important cases of boundary conditions involving just the boundary values, cf. Subsection 2.3, the operator $\mathcal{R}_{t_0}$ reduces to a matrix. Before starting the proof, we note that in this section we equip all subspaces $Z \subseteq \mathbb{C}^n$ with the maximum norm, i.e., we define

$$\| (v_1, \ldots, v_n)^T \|_Z := \max \{ |v_1|, \ldots, |v_n| \}.$$ 

Proof of Lemma 2.2. Observe that

$$(\Phi_t^e - \Phi_0^e)q^e(\cdot)J_{\varphi^*} \in \mathbb{L}(C_0(\mathbb{R}^+, \mathbb{C}^\ell), Y), \quad \text{and}$$

$$(\Phi_t^i + \Phi_0^i)q^i(\cdot)J_{\varphi^*} \psi, \ (\Phi_t^i - \Phi_0^i)q^i(\cdot)J_{\varphi^*} \in \mathbb{L}(C([0,1], \mathbb{C}^m), Y)$$

are well-defined. Hence, it suffices to show that the operator

$$U_{t_0} : W_0^{1,p}([0,t_0], \mathbb{C}^k) \subset L^p([0,t_0], \mathbb{C}^k) \to L^p([0,t_0], Y),$$

$$(U_{t_0} u)(t) := \Phi \hat{u}(t - \vartheta(\cdot)), \ t \in [0,t_0],$$

is well-defined and has a bounded extension in $\mathbb{L}(L^p([0,t_0], \mathbb{C}^k), L^p([0,t_0], Y))$ where for the

- external part $k = \ell$, $\Phi \in \mathbb{L}(C_0(I, \mathbb{C}^\ell), Y)$ and $\vartheta(s) := s$, $s \in I := \mathbb{R}^+$,
- internal part $k = m$, $\Phi \in \mathbb{L}(C(I, \mathbb{C}^m), Y)$ and $\vartheta(s) := \frac{s}{n}$, $s \in I := [0,1].$

In both cases the assumption $u(0) = 0$ implies that the function $\hat{u}(t - \vartheta(\cdot)) \in W^{1,p}(I, \mathbb{C}^k)$ has compact support and hence $\Phi \hat{u}(t - \vartheta(\cdot))$ is well-defined for all $t \in [0,t_0]$. Moreover, since $\hat{u}|_{(-\infty,t_0]}$ is uniformly continuous, the map $[0,t_0] \ni t \mapsto \hat{u}(t - \vartheta(\cdot)) \in C_0(I, \mathbb{C}^k)$ is continuous and therefore $[0,t_0] \ni t \mapsto \Phi \hat{u}(t - \vartheta(\cdot)) \in Y$ is continuous as well. Summing up, this shows that the operator $U_{t_0}$ is well-defined.

Next we verify that $U_{t_0}$ is bounded. Since $\Phi \in \mathbb{L}(C_0(I, \mathbb{C}^k), Y)$ and $Y$ is finite dimensional, by the Riesz–Markov representation theorem there exists a function $\eta : I \to \mathbb{L}(\mathbb{C}^k, Y)$ of bounded variation such that $\Phi$ is given by the Riemann–Stieltjes integral

$$\Phi h = \int_I d\eta(s) \ h(s) \quad \text{for all} \quad h \in W^{1,p}(I, \mathbb{C}^k).$$  \hfill (2.12)

\footnote{Note that by definition $C_0([0,1], \mathbb{C}^k) = C([0,1], \mathbb{C}^k)$.}
Then by Hölder’s inequality and Fubini’s theorem we conclude for \( u \in W^{1,p}_0([0,t_0], \mathbb{C}^k) \) that
\[
\|U_{t_0} u\|_p^p = \int_0^{t_0} \left\| \hat{u}(t - \vartheta(\bullet)) \right\|_Y^p \, dt \\
= \int_0^{t_0} \left\| \int_I d\eta(s) \, \hat{u}(t - \vartheta(s)) \right\|_Y^p \, dt \\
\leq \int_0^{t_0} \left( \int \left\| \hat{u}(t - \vartheta(s)) \right\|_{C^k} \, d|\eta|(s) \right)^p \, dt \\
\leq (|\eta|(I))^{p-1} \cdot \int_0^{t_0} \int_I \left\| \hat{u}(t - \vartheta(s)) \right\|_{C^k}^p \, d|\eta|(s) \, dt \\
\leq |\eta|^{p-1} \cdot \int_I \int_0^{t_0} \left\| \hat{u}(t - \vartheta(s)) \right\|_{C^k}^p \, dt \, d|\eta|(s) \\
\leq |\eta|^{p-1} \cdot \|u\|_{C^k}^p,
\]
where \( |\eta| : I \to \mathbb{R}_+ \) denotes the positive Borel measure defined by the total variation of \( \eta \) and \( \|\eta\| := |\eta|(I) \). Since \( W^{1,p}_0([0,t_0], \mathbb{C}^k) \) is dense in \( L^p([0,t_0], \mathbb{C}^k) \) this implies that \( U_{t_0} \) has a unique bounded extension as claimed.

### 2.2. The main result

We are now ready to state our main generation result.

**Theorem 2.3.** Let Assumption 2.1 be satisfied. If there exists \( t_0 > 0 \) such that the operator \( R_{t_0} \in \mathcal{L}(L^p([0,t_0], \mathbb{C}^m \times \mathbb{C}^m)) \) given by (2.11) is invertible, then the operator \( G \) defined in (2.3) generates a cosine family on \( X = L^p(\mathbb{R}_+, \mathbb{C}^l) \times L^p([0,1], \mathbb{C}^m) \) with phase space \( V \times X \) for \( V := \ker(\Phi_0) \).

The proof is split into four parts where in the first three we assume \( B = 0 \). We start by showing the result under the hypothesis that the operator matrix \( \mathcal{G} \) in (2.13) below generates a semigroup. Then, using a series of lemmas we give the proof that \( \mathcal{G} \) indeed is a generator, first in case \( q(\bullet, \bullet) \equiv \text{diag}(Id, Id) \), then for general \( q(\bullet, \bullet) \). Finally, we prove the result for \( B \neq 0 \).

**Proof of Theorem 2.3, 1st part.** Assume that \( B = 0 \) and \( q(\bullet, \bullet) \equiv \text{diag}(Id, Id) \). Hence, \( a(\bullet, \bullet) = \lambda(\bullet, \bullet) \) and \( c(\bullet, \bullet) = \mu(\bullet, \bullet) \) are diagonal matrices. By \( \lambda' \) we denote the derivative of the corresponding diagonal entries, i.e.,
\[
\lambda'(\bullet, \bullet) := \text{diag} \left( (\lambda^i)'(\bullet), (\lambda^j)'(\bullet) \right).
\]

On \( \mathcal{X} := X \times X \) we consider the operator matrix
\[
\mathcal{G} := \begin{pmatrix} 0 & D_{\Phi_0} \\ D_{\Phi_1} & 0 \end{pmatrix}, \quad D(\mathcal{G}) := D(D_{\Phi_1}) \times D(D_{\Phi_0}),
\]
where
\[
D_{\Phi_0} := c(\bullet, \bullet) \cdot \frac{d}{d\nu}, \quad D(D_{\Phi_0}) := \{ g \in W^{1,p}(\mathbb{R}_+, \mathbb{C}^l) \times W^{1,p}([0,1], \mathbb{C}^m) : \Phi_0 \, g = 0 \}, \\
D_{\Phi_1} := c(\bullet, \bullet) \cdot \frac{d}{d\nu}, \quad D(D_{\Phi_1}) := \{ f \in W^{1,p}(\mathbb{R}_+, \mathbb{C}^l) \times W^{1,p}([0,1], \mathbb{C}^m) : \Phi_1 \, f = 0 \}.
\]

Then \( \mathcal{G} \) and \( -\mathcal{G} \) are similar via the similarity transformation induced by operator matrix \( \text{diag}(Id, -Id) \). Hence, if for the time being we assume that \( \mathcal{G} \) generates a \( C_0 \)-semigroup, by [17, Sect. II.3.11] it already generates a group. By [6, Expl. 3.14.15]
this implies that $G^2$ generates a cosine family with phase space $V \times X$ for $V := [D(G)]$. However, $G^2$ is given by the diagonal matrix with diagonal domain

$$G^2 := \begin{pmatrix} D\Phi_0 D\Phi_1 & 0 \\ 0 & D\Phi_1 D\Phi_0 \end{pmatrix}, \quad D(G^2) := D(D\Phi_0 D\Phi_1) \times D(D\Phi_1 D\Phi_0).$$

Hence, $\tilde{G} := D\Phi_1 D\Phi_0$ generates a cosine family with phase space $V \times X$ for $V := [D(D\Phi_0)] = \ker(\Phi_0)$. Since $\lambda$ is Lipschitz continuous, $X' \in L^\infty(\mathbb{R}_+ \times [0, 1], M_{\ell+m}(\mathbb{C}))$ induces a bounded multiplication operator on $X$ and therefore $P := \frac{4}{\lambda^2} \cdot \frac{d}{dx} \in \mathcal{L}(V, X)$. However, by Corollary A.8, $D(G) = D(\tilde{G})$ and $G = \tilde{G} - P$, hence by [6, Cor. 3.14.13] it follows that $G$ generates a cosine family with phase space $V \times X$ as claimed.

Next we verify the generator property of the matrix $G$. To do so we proceed in several steps. First we assume again that $c(\bullet, \bullet) = \text{diag}(\mu^e(\bullet), \mu^i(\bullet))$ is diagonal, i.e., $q(\bullet, \bullet) \equiv \text{diag}(Id, Id)$. The case of general $q(\bullet, \bullet)$ as in (2.1) then follows by similarity and bounded perturbation.

We start by simplifying $G$ by rearranging the coordinates of $X'$ and by normalizing the matrices $\mu^e(\bullet), \mu^i(\bullet)$. Recall that $\tilde{C}$ is defined in (2.7) and $J_{\mu^e} \in \mathcal{L}(X', \mu^e \in \mathcal{L}(X^i))$ are given by (2.8).

**Lemma 2.4.** Let $q(\bullet, \bullet) = \text{diag}(Id, Id)$. Then the operator matrix $G$ on $X = X \times X = (X^e \times X^i) \times (X^e \times X^i)$ given in (2.13) is similar to $\tilde{G}$ on $\tilde{X}' := X^e \times X^i = (X^e \times X^i) \times (X^i \times X^i)$ where

$$\tilde{G} := \text{diag}(\frac{d}{dx} - \frac{d}{dx}, \tilde{C} \cdot \frac{d}{dx}, -\tilde{C} \cdot \frac{d}{dx}), \quad D(\tilde{G}) := \ker(\Phi)$$

(2.14)

and

$$\Phi := (\Phi^e_1 + \Phi^i_0) \cdot J_{\mu^e}, (\Phi^i_1 - \Phi^0_0) \cdot J_{\mu^i}, (\Phi^i_1 + \Phi^0_0) \cdot J_{\mu^i}, (\Phi^i_1 - \Phi^0_0) \cdot J_{\mu^i} \quad (2.15)$$

$$\in \mathcal{L}(C_0(\mathbb{R}_+, \mathbb{C}^\ell) \times C_0(\mathbb{R}_+, \mathbb{C}^\ell) \times C([0, 1], \mathbb{C}^m) \times C([-1, 1], \mathbb{C}^m), Y).$$

**Proof.** Consider the invertible transformation

$$S := \frac{1}{2} \begin{pmatrix} J_{\mu^e} & J_{\mu^e} & 0 & 0 \\ 0 & 0 & J_{\mu^i} & J_{\mu^i} \\ J_{\mu^e} & -J_{\mu^i} & 0 & 0 \\ 0 & 0 & -J_{\mu^i} & J_{\mu^i} \end{pmatrix} \in \mathcal{L}(\tilde{X}, X) \quad \text{with inverse}\n
S^{-1} := \begin{pmatrix} J_{\mu^e}^{-1} & 0 & J_{\mu^e}^{-1} & 0 \\ 0 & 0 & -J_{\mu^i}^{-1} & 0 \\ J_{\mu^e}^{-1} & 0 & J_{\mu^i}^{-1} & 0 \\ 0 & 0 & J_{\mu^i}^{-1} & -J_{\mu^i}^{-1} \end{pmatrix} \in \mathcal{L}(X, \tilde{X}).$$

We claim that

$$\tilde{G} = S^{-1} \cdot G \cdot S. \quad (2.16)$$

Since $\text{rg}(\Phi^0_0, \Phi^i_0) \subseteq Y_0, \text{rg}(\Phi^i_1, \Phi^e_1) \subseteq Y_1$ and $Y_0 \cap Y_1 = \{0\}$ we have $D(G) = \ker(\Phi^e_1, \Phi^i_1, \Phi^0_0, \Phi^i_0)$. Using this a simple computation shows that $D(S^{-1}GS) = \ker((\Phi^i_1, \Phi^e_1, \Phi^0_0, \Phi^i_0) \cdot S) = D(\tilde{G})$. Next $(\phi^{-1})' = \mu^e(\bullet) \circ \phi^{-1}$ implies

$$J_{\mu^e} \cdot \frac{d}{dx} \cdot J_{\mu^e}^{-1} = \mu^e(\bullet) \cdot \frac{d}{dx}. \quad \text{Similarly, since} \quad (\tilde{C} \cdot (\tilde{\varphi}^{-1})')' = \mu^i(\bullet) \circ (\tilde{\varphi}^{-1})'$$

we obtain

$$J_{\tilde{\varphi}^{-1}} \cdot \tilde{C} \cdot \frac{d}{dx} \cdot J_{\tilde{\varphi}^{-1}'} = \mu^i(\bullet) \cdot \frac{d}{dx}.$$
We now represent $\hat{G}$ as a domain perturbation of a simpler generator $A$ which can be treated by a (slight modification of a) recent perturbation result from [1] (see Theorem A.3). Thanks to Lemma 2.4 we can consider the external and internal part separately.

**External Part.** We introduce on $X^e = L^p(\mathbb{R}_+, \mathbb{C}^l)$ the operators

$$D_m^e := \frac{d}{ds}, \quad D(D_m^e) := W^{1,p}(\mathbb{R}_+, \mathbb{C}^l),$$

$$D_0^e := \frac{d}{ds}, \quad D(D_0^e) := \{ f \in D(D_0^e) : f(0) = 0 \},$$

and define on $X^e = X^e \times X^e$ the operator matrices

$$A_m^e := \text{diag}(D_m^e, -D_m^e), \quad D(A_m^e) := D(D_m^e) \times D(D_m^e), \quad (2.17)$$

$$A^e := \text{diag}(D_m^e, -D_0^e), \quad D(A^e) := D(D_m^e) \times D(D_0^e). \quad (2.18)$$

Note that $D_m^e$ and $-D_0^e$ generate the strongly continuous left- and right-shift semigroups $(T^e(t))_{t \geq 0}$ and $(T^e(t))_{t \geq 0}$ on $X^e$, respectively, given by

$$(T^e(t)f)\ast(t) := f(t + s), \quad (T^e(t)f)\ast(0) := \hat{f}(t - s), \quad (2.19)$$

where $\hat{f}$ denotes the extension of the function $f : \mathbb{R}_+ \rightarrow \mathbb{C}^l$ to $\mathbb{R}$ by the value 0. This gives immediately the following result.

**Lemma 2.5.** The operator $A^e$ defined in (2.18) generates a $C_0$-semigroup $(T^e(t))_{t \geq 0}$ given by

$$T^e(t) = \text{diag}(T^e(t), T^e(t)), \quad t \geq 0.$$ 

Note that in the context of Subsection A.1 we have $A^e \subset A_m^e$ with domain

$$D(A^e) = \left\{ (l_0^e) \in D(A_m^e) : L^e(l_0^e) = 0 \right\} = \ker(L^e)$$

for

$$L^e := \{(0, \delta_0) \in \mathcal{L}([D(A_m^e)], \partial X^e)\}, \quad (2.20)$$

where $\delta_0$ denotes the point evaluation in $s = 0$ and $\partial X^e := \mathbb{C}^l$. Now the following follows easily by inspection.

**Lemma 2.6.** Let the operators $A_m^e$ and $L^e$ be defined by (2.17) and (2.20), respectively. Then for $t_0 > 0$ and given $u \in W^{2,p}_0([0, t_0], \partial X^e)$ the function $x : [0, t_0] \rightarrow X^e = X^e \times X^e$ defined by

$$x(t, \bullet) := \left(0, \hat{u}(t - \bullet)\right)^\top \quad (2.21)$$

is a classical solution of the boundary control system

$$\begin{cases}
    x(t) = A_m^e x(t), & 0 \leq t \leq t_0, \\
    L^e x(t) = u(t), & 0 \leq t \leq t_0, \\
    x(0) = 0.
\end{cases}$$

**Internal Part.** Recall that $\tilde{C} := \text{diag}(\tilde{c}_1, \ldots, \tilde{c}_n)$ with $\tilde{c}_i$ defined in (2.6). Then we introduce on $X^i = L^p([0, 1], \mathbb{C}^m)$ the operators

$$D_m^i := \tilde{C} \cdot \frac{d}{ds}, \quad D(D_m^i) := W^{1,p}([0, 1], \mathbb{C}^m),$$

$$D_0^i := \tilde{C} \cdot \frac{d}{ds}, \quad D(D_0^i) := \{ f \in D(D_0^i) : f(0) = 0 \},$$

$$D_1^i := \tilde{C} \cdot \frac{d}{ds}, \quad D(D_1^i) := \{ f \in D(D_1^i) : f(1) = 0 \},$$

$$\tilde{A}_m^e := \text{diag}(D_m^i, -D_m^i), \quad D(\tilde{A}_m^e) := D(D_m^i) \times D(D_m^i), \quad (2.17)$$

$$\tilde{A}^i := \text{diag}(D_m^i, -D_0^i), \quad D(\tilde{A}^i) := D(D_m^i) \times D(D_0^i). \quad (2.18)$$

Note that $D_m^i$ and $-D_0^i$ generate the strongly continuous left- and right-shift semigroups $(\tilde{T}^e(t))_{t \geq 0}$ and $(\tilde{T}^e(t))_{t \geq 0}$ on $X^i$, respectively, given by

$$(\tilde{T}^e(t)f)\ast(t) := f(t + s), \quad (\tilde{T}^e(t)f)\ast(0) := \hat{f}(t - s), \quad (2.19)$$

where $\hat{f}$ denotes the extension of the function $f : \mathbb{R}_+ \rightarrow \mathbb{C}^l$ to $\mathbb{R}$ by the value 0. This gives immediately the following result.
and define on $X^i = X^i \times X^i$ the operator matrices
\begin{align}
\mathcal{A}_m^i &:= \begin{pmatrix} D_{m}^i & 0 \\ 0 & -D_{m}^i \end{pmatrix}, \quad D(\mathcal{A}_m^i) := D(D_m^i) \times D(D_m^i), \\
\mathcal{A}^i &:= \begin{pmatrix} D_1^i & 0 \\ 0 & -D_0^i \end{pmatrix}, \quad D(\mathcal{A}^i) := D(D_1^i) \times D(D_0^i).
\end{align}
(2.22)
(2.23)

Then $D_1^i$ and $-D_0^i$ generate the strongly continuous nilpotent left- and right-shift semigroups $(T_1^i(t))_{t \geq 0}$ and $(T_0^i(t))_{t \geq 0}$ on $X^i$, respectively, given by
\begin{align}
(T_1^i(t)f)(*) := \hat{f}(\bullet + \bar{c} \cdot t), \quad (T_0^i(t)f)(*) := \hat{f}(\bullet - \bar{c} \cdot t),
\end{align}
where $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_m)$. This gives immediately the following result.

**Lemma 2.7.** The operator $\mathcal{A}^i$ defined in (2.23) generates a $C_0$-semigroup $(\mathcal{T}^i(t))_{t \geq 0}$ given by
\begin{align}
\mathcal{T}^i(t) = \begin{pmatrix} T_1^i(t) & 0 \\ 0 & T_0^i(t) \end{pmatrix}, \quad t \geq 0.
\end{align}

As before we observe that in the context of Subsection A.1 we have $\mathcal{A}^i \subset \mathcal{A}_m^i$ with domain
\begin{align}
D(\mathcal{A}^i) = \left\{ (l_g^i) \in D(\mathcal{A}_m^i) : \mathcal{L}^i(l_g^i) = 0 \right\} = \ker(\mathcal{L}^i),
\end{align}
for
\begin{align}
\mathcal{L}^i := \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_0 \end{pmatrix} \in \mathcal{L}(D(\mathcal{A}_m^i), \partial \mathcal{X}^i),
\end{align}
(2.25)
where $\delta_s$ denotes the point evaluation in $s \in \{0, 1\}$ and $\partial \mathcal{X}^i := \mathbb{C}^m \times \mathbb{C}^m$.

**Lemma 2.8.** Let the operators $\mathcal{A}_m^i$ and $\mathcal{L}^i$ be defined by (2.22) and (2.25), respectively. Then for $t_0 := \min\{e_1^{-1}, \ldots, e_m^{-1}\} > 0$ and given $\nu = (v, w)^{\top} \in W_0^{2,p}([0, t_0], \partial \mathcal{X}^i)$ the function $x : [0, t_0] \to \mathcal{X}^i = X^i \times X^i$ defined by
\begin{align}
x(t, s) := \left( \hat{\nu}(t + \frac{s-1}{2}), \hat{\nu}(t - \frac{s}{2}) \right)^{\top}, \quad t \in [0, t_0], \quad s \in [0, 1]
\end{align}
(2.26)
is a classical solution of the boundary control system
\begin{align}
\begin{cases}
\dot{x}(t) = \mathcal{A}_m^i x(t), & 0 \leq t \leq t_0, \\
\mathcal{L}^i x(t) = \nu(t), & 0 \leq t \leq t_0, \\
x(0) = 0.
\end{cases}
\end{align}

We are now well-prepared to continue the proof of our main result.

**Proof of Theorem 2.3, 2nd part.** We show that $\tilde{G}$ given by (2.14) generates a semigroup on $\tilde{X} := \mathcal{X}^c \times \mathcal{X}^i$. By Lemma 2.4 and the first part of the proof this proves Theorem 2.3 in case $q(\bullet, \bullet) = \operatorname{diag}(\operatorname{Id}, \operatorname{Id})$ and $B = 0$.

For the operators $\mathcal{A}_m^c, \mathcal{A}^c$ and $\mathcal{A}_m^i, \mathcal{A}^i$ given by (2.17)–(2.18) and (2.22)–(2.23), respectively, we define on $\tilde{X}$ the matrices
\begin{align}
\mathcal{A}_m := \operatorname{diag}(\mathcal{A}_m^c, \mathcal{A}_m^i), \quad D(\mathcal{A}_m) := D(\mathcal{A}_m^c) \times D(\mathcal{A}_m^i), \\
\mathcal{A} := \operatorname{diag}(\mathcal{A}^c, \mathcal{A}^i), \quad D(\mathcal{A}) := D(\mathcal{A}^c) \times D(\mathcal{A}^i).
\end{align}
(2.27)
Then $\tilde{G}, A \subset A_m$ with domains $D(\tilde{G}) = \ker(\Phi)$ and $D(A) = \ker(L)$, where $\Phi$ is given by (2.15) in Lemma 2.4 and

$$L := \text{diag}(L^e, L^i) = \begin{pmatrix} 0 & \delta_0 & 0 & 0 \\ 0 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & \delta_0 \end{pmatrix} : D(A_m) \to \partial\mathcal{X} := \partial\mathcal{X}^e \times \partial\mathcal{X}^i = C^\ell + 2m.$$

Moreover, by Lemma 2.5 and Lemma 2.7, $A$ generates a $C_0$-semigroup $(T(t))_{t \geq 0}$ given by

$$T(t) = \text{diag}(T^e(t), T^i(t)), \quad t \geq 0. \tag{2.28}$$

Hence, the assertion follows if we verify the assumptions (i)–(iv) in Corollary A.5 adapted to the present context. Let $t_0 := \min\{\tilde{c}_1^{-1}, \ldots, \tilde{c}_m^{-1}\}$.

(i) For $t \in [0, t_0]$ and $u \in L^p([0, t_0], \partial\mathcal{X}^e)$ define $B^e_t u \in \mathcal{X}^e = X^e \times X^i$ by the right-hand-side of (2.21). Similarly, for $\nu = (v, w)^T \in L^p([0, t_0], \partial\mathcal{X}^i)$ define $B^i_t \nu \in \mathcal{X}^i = X^i \times X^i$ by the right-hand-side of (2.26) and put

$$B_t := \text{diag}(B^e_t, B^i_t) : \partial\mathcal{X} = \partial\mathcal{X}^e \times \partial\mathcal{X}^i \to \tilde{X} = X^e \times X^i.$$ 

Then $(B_t)_{t \in [0, t_0]} \subset \mathcal{L}(L^p([0, t_0], \partial\mathcal{X}), \tilde{X})$ is strongly continuous. Moreover, by Lemma 2.6 and Lemma 2.8, for given $(u, v) \in W^2_p([0, t_0], \partial\mathcal{X}^e) \times W^2_p([0, t_0], \partial\mathcal{X}^i)$ we see that $x(t) := B_t(u, v)^T$ solves

$$\begin{cases} \dot{x}(t) = A_m x(t), & 0 \leq t \leq t_0, \\
L x(t) = (\nu(t))^T, & 0 \leq t \leq t_0, \\
x(0) = 0, & 
\end{cases} \tag{2.29}$$

hence the assertion follows from Lemma A.6.

(ii) Using the terminology introduced in Remark A.4, we have to show that $\Phi$ in (2.15) is $p$-admissible for the semigroup $(T(t))_{t \geq 0}$ generated by $A$. By the representations of $T(t)$ in (2.28) this follows if we verify the $p$-admissibility of every

- $\Phi \in \mathcal{L}(C_0(I, C^k), Y)$ for $I = \mathbb{R}_+$ and $k = \ell$ with respect to the semigroups $(T^e_i(t))_{t \geq 0}$ and $(T^i_i(t))_{t \geq 0}$ given in (2.19), and
- $\Phi \in \mathcal{L}(C(I, C^k), Y)$ for $I = [0, 1]$ and $k = m$ with respect to the semigroups $(T^e_i(t))_{t \geq 0}$ and $(T^i_i(t))_{t \geq 0}$ given in (2.24).

Let $T(t) \in \{T^e_i(t), T^i_i(t), T^e_i, T^i_i\}$ and define

$$c := \begin{cases} (1, \ldots, 1) \in \mathbb{C}^\ell & \text{in the external case,} \\
\bar{c} = (\bar{c}_1, \ldots, \bar{c}_m) \in \mathbb{C}^m & \text{in the internal case.} 
\end{cases}$$

Then similarly as in the proof of Lemma 2.2 it follows that

$$\int_0^{t_0} \|\Phi T(t)f\|_Y^p \ dt = \int_0^{t_0} \left\| \int_0^s \eta(s) \tilde{f}(s \pm c \cdot t) \right\|_Y^p \ dt \leq \|\eta\|^{p-1} \cdot \int_0^{t_0} \left\| \tilde{f}(s \pm c \cdot t) \right\|_{C^k}^p \ dt \ d\|\eta\|(s) \leq \|\eta\|^p \cdot M^p \cdot \|f\|_{p'},$$

where $\eta$ is given by (2.12) and $M := \|c_1^{-1}\|_{C^k}$. This completes the proof of (ii).
(iii)&(iv) By (i), Lemma 2.6, Lemma 2.8, and Lemma A.6 the controllability maps for the problem (2.29) are given by

\[ B_t = \begin{pmatrix} 0 & 0 & 0 \\ Q_t & 0 & 0 \\ 0 & S_t & 0 \\ 0 & 0 & R_t \end{pmatrix} \in \mathcal{L}(L^p([0,t_0],\partial X'),\hat{X}), \quad t \in [0,t_0], \]

where \(Q_t, S_t\) and \(R_t\) are defined in (2.9). Hence, for all \(u \in W_0^{2,p}([0,t_0], Y)\),

\[ Q_{t_0}u = \Phi B_t u = R_{t_0}u, \]

where the last equality is obtained by direct computation using definitions of \(\Phi\) in (2.15) and \(R_{t_0}\) in (2.11). This, combined with Lemma 2.2, implies (iii) and also (iv) follows immediately from the invertibility assumption on \(R_{t_0}\).

Summing up we conclude that for \(q(\cdot, \cdot) \equiv \text{diag}(Id, Id)\) the matrix \(\hat{G}\) given in (2.14), hence by the similarity in Lemma 2.4 also \(G\) in (2.13), generate \(C_0\)-semigroups if \(R_{t_0}\) given by (2.11) for \(q(\cdot, \cdot) \equiv \text{diag}(Id, Id)\) is invertible. \(\square\)

Next we consider general \(q(\cdot, \cdot)\), i.e., the case of possibly non-diagonal diffusion coefficient matrices \(a(\cdot, \cdot)\).

**Proof of Theorem 2.3, 3rd part.** Assume that \(a(\cdot, \cdot), c(\cdot, \cdot)\) are given by (2.1) and (2.10), respectively, where \(q(\cdot, \cdot), q(\cdot, \cdot)^{-1}\) are Lipschitz continuous and bounded. Then via the similarity transformation induced by \(\text{diag}(q(\cdot, \cdot), q(\cdot, \cdot))\) we obtain that the operator matrix

\[ G = \begin{pmatrix} 0 & c(\cdot, \cdot) \frac{d}{dt} \\ c(\cdot, \cdot) \frac{d}{dt} & 0 \end{pmatrix}, \quad D(G) = \ker(\Phi_1 \cdot c(\cdot, \cdot)^{-1}) \times \ker(\Phi_0), \]

defined in (2.13), is similar to the operator matrix

\[ \hat{G} \simeq \begin{pmatrix} \mu(\cdot, \cdot)q^{-1}(\cdot, \cdot) \frac{d}{dt} & 0 \\ 0 & \mu(\cdot, \cdot)q^{-1}(\cdot, \cdot) \frac{d}{dt} \end{pmatrix} \]

\[ = \begin{pmatrix} 0 & \mu(\cdot, \cdot)q^{-1}(\cdot, \cdot) \frac{d}{dt} \\ \mu(\cdot, \cdot)q^{-1}(\cdot, \cdot) \frac{d}{dt} & 0 \end{pmatrix} =: \hat{G} + \mathcal{P}, \]

where we used that \(q(\cdot, \cdot)^{-1}c(\cdot, \cdot) = \mu(\cdot, \cdot)q(\cdot, \cdot)^{-1}\) and \(\frac{d}{dt}q(\cdot, \cdot) = q'(\cdot, \cdot) + q(\cdot, \cdot) \frac{d}{dt}\). Since \(q\) and \(q^{-1}\) are Lipschitz continuous and bounded, we have \(\mathcal{P} \in \mathcal{L}(X \times X)\).

Moreover, note that

\[ D(\hat{G}) = \ker(\hat{\Phi}_1 \cdot \mu(\cdot)^{-1}) \times \ker(\hat{\Phi}_0), \]

for \(\hat{\Phi}_1 := \Phi_1 \cdot q(\cdot, \cdot), \hat{\Phi}_0 := \Phi_0 \cdot q(\cdot, \cdot)\). Hence, by similarity and bounded perturbation \(G\) is a generator iff \(\hat{G}\) is. However, by what we proved previously for \(q(\cdot, \cdot) = \text{diag}(Id, Id)\), i.e., for diagonal \(c(\cdot, \cdot) = \mu(\cdot, \cdot)\), the operator \(\hat{G}\) is a generator if \(R_{t_0}\) given by (2.11) is invertible. \(\square\)

We conclude the proof by considering non-zero boundary operators \(B \in \mathcal{L}(X)\).

**Proof of Theorem 2.3, 4th part.** It remains to prove the result for \(B \neq 0\) satisfying the regularity condition

\[ B\left(W^{1,p}(\mathbb{R}_+, C') \times W^{1,p}([0,1], C^m)\right) \subseteq W^{1,p}(\mathbb{R}_+, C') \times W^{1,p}([0,1], C^m). \quad (2.30) \]
To this end we put \( \bar{B} := c(\cdot, \cdot) \cdot B \in \mathcal{L}(X) \) and perturb the matrix \( \mathcal{G} \) in (2.13) by

\[
B := \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X).
\]

Then, by Part 3 and the bounded perturbation theorem, \( \mathcal{G}_B := \mathcal{G} + B \) generates a group. Now a simple computation using Corollary A.8 shows that

\[
\mathcal{G}_B^2 := \begin{pmatrix} (D\Phi_0 + \bar{B})D\Phi_1 & 0 \\ 0 & \tilde{\mathcal{G}}_B \end{pmatrix}, \quad D(\mathcal{G}_B^2) := D(D\Phi_0 \cdot D\Phi_1) \times D(\tilde{\mathcal{G}}_B),
\]

where

\[
\tilde{\mathcal{G}}_B := D\Phi_1 (D\Phi_0 + \bar{B}),
\]

\[D(\tilde{\mathcal{G}}_B) := \{ f \in W^{2,p}(\mathbb{R}_+, \mathbb{C}^d) \times W^{2,p}([0, 1], \mathbb{C}^m) : \Phi_0 f = 0, \Phi_1 (f' + Bf) = 0 \}.\]

Hence, \( \tilde{\mathcal{G}}_B \) with domain \( D(\tilde{\mathcal{G}}_B) = D(\mathcal{G}) \) generates a cosine family on \( X \) with phase space \( V \times X \) for \( V := [D(D\Phi_0)] = \ker(\Phi_0) \). Now as in Part 1 we have \( P := \frac{d^2}{dx^2} \), \( \frac{d}{dx} \in \mathcal{L}(V, X) \). Moreover, since \( B \in \mathcal{L}(X) \), the regularity property (2.30) combined with Corollary A.8 and the closed graph theorem imply \( \bar{B} \in \mathcal{L}(W^{1,p}(\mathbb{R}_+, \mathbb{C}^d) \times W^{1,p}([0, 1], \mathbb{C}^m)) \). Hence, \( Q := c \cdot \frac{d}{dx} \cdot B \in \mathcal{L}(V, X) \) and since \( G = \tilde{\mathcal{G}}_B - P - Q \), the claim follows from [6, Cor. 3.14.13] as in Part 1.

**Remark 2.9.** (i) As in the 1st part of the proof of Theorem 2.3 one can see that operators \( A := a(\cdot, \cdot) \cdot \frac{d^2}{dx^2} \) and \( A_d := \frac{d}{dx} (a(\cdot, \cdot) \cdot \frac{d}{dx}) \) differ only by a bounded perturbation from \( V \to X \). Hence, \( A \) generates a cosine family on \( X \) with phase space \( V \times X \) iff \( A_d \) generates a cosine family on \( X \) with phase space \( V \times X \).

(ii) Note that by [6, Cor. 3.14.13] the sum \( G + P \) of the generator \( G \) of a cosine family with phase space \( V \times X \) and a perturbation \( P \in \mathcal{L}(V, X) \) still generates a cosine family with the same phase space. Here in the context of Theorem 2.3 we have \( V \xrightarrow{\mathcal{G}} W^{1,p}(\mathbb{R}_+, \mathbb{C}^d) \times W^{1,p}([0, 1], \mathbb{C}^m) \) which implies \( (\frac{d}{dx}, \frac{d}{dx}) \in \mathcal{L}(V, X) \). Thus, boundedness and invertibility of \( \mathcal{R}_{\Phi_0} \) in (2.11) imply that for arbitrary \( b(\cdot, \cdot), q(\cdot, \cdot) \in L^\infty(\mathbb{R}_+, \mathbb{C}^d) \times L^\infty([0, 1], \mathbb{C}^m) \) also \( G_P := G + P \) for

\[
P := b(\cdot, \cdot) \cdot \frac{d}{dx} + q(\cdot, \cdot)
\]

with domain \( D(G_P) := D(G) \) generates a cosine family with the same phase space.

(iii) By [6, Thm. 3.14.17] every generator of a cosine family generates an analytic semigroup of angle \( \frac{\pi}{2} \). Hence, the previous remark gives also conditions implying that \( G + P \) generates an analytic semigroup of angle \( \frac{\pi}{2} \).

(iv) It is quite remarkable that \( \mathcal{G} \) in (2.13) might generate a \( C_0 \)-semigroup even if none of its entries \( D\Phi_0 \) and \( D\Phi_1 \) are generators. For example for \( \ell = 0, m = 1 \) and \( Y_0 = C^{t+2m} = C^2, Y_1 = \{0\} \) we can choose \( c(\cdot) \equiv 1, \Phi_0 = (\delta_1), \Phi_1 = 0 \) and \( B = 0 \). Then \( \sigma(D\Phi_0) = \sigma(D\Phi_1) = \mathbb{C} \) and hence both operators do not generate semigroups. On the other hand, the assumptions of Theorem 2.3 are satisfied, hence \( G = D\Phi_1 \cdot D\Phi_0 = \Delta_D \), i.e., the Laplacian with Dirichlet boundary conditions, generates a cosine family on \( L^p[0, 1] \) for all \( p \in [1, \infty) \). Similarly, by reversing the roles of \( \Phi_0 \) and \( \Phi_1 \) (or by looking at the upper diagonal entry \( D\Phi_0 \cdot D\Phi_1 \) of \( G^2 \)) it follows that also \( G = \Delta_N \), i.e., the Laplacian with Neumann boundary conditions, generates a cosine family on \( L^p(\mathbb{R}_+) \). Similarly, choosing \( \ell = 1 \) and \( m = 0 \) it follows easily that the Laplacian with Dirichlet or Neumann boundary conditions generates a cosine family on \( L^p(\mathbb{R}_+) \).
(v) Another remarkable fact is that while $\mathcal{A}$ in (2.27) only generates a semigroup, its perturbation $\tilde{G}$ in (2.14) always generates a group.

(vi) We mention that even for smooth positive definite valued $a(\bullet, \bullet)$ a representation as in (2.1) is not always possible. Assume for simplicity that $\ell = 0^3$.

If $q(\bullet) \in C^\infty[0, 1]$ such that $0 < q(s) \leq \frac{1}{2}$ for $s \neq \frac{1}{2}$ and $q^{(k)}(\frac{1}{2}) = 0$ for all $k = 0, 1, 2, \ldots$, then

\[
a(s) := \begin{cases} 
\begin{pmatrix} 1+q(s) & 0 \\
0 & 1-q(s) \end{pmatrix} & \text{if } s \in [0, \frac{1}{2}], \\
\begin{pmatrix} 1 & q(s) \\
q(s) & 1 \end{pmatrix} & \text{if } s \in (\frac{1}{2}, 1],
\end{cases}
\]

cannot be diagonalized by means of a continuous $q(\bullet)$ even though $a(\bullet) \in C^\infty([0, 1], \mathbb{C}^2)$. This follows from the fact that the eigenvectors of $a(s)$ are given by

\[
\begin{cases} 
\begin{pmatrix} 1 \\
0 \end{pmatrix}, \begin{pmatrix} 0 \\
1 \end{pmatrix} & \text{if } s \in [0, \frac{1}{2}], \\
\begin{pmatrix} 1 \\
1 \end{pmatrix} & \text{if } s \in (\frac{1}{2}, 1].
\end{cases}
\]

However, if $a(\bullet)$ is analytic or if $a(s)$ has $m$ distinct eigenvalues for all $s \in [0, 1]$ then it can always represented as in (2.1). Nevertheless, in case $p = 2$ one can drop this assumption for an important class of boundary functional, cf. [15].

Feller [19] has characterized the boundary conditions in the domain of the generator of the transition semigroup corresponding to one-dimensional diffusion processes. Besides Dirichlet and Neumann boundary conditions (which we discuss in Example 2.13), these include also non-local integral conditions which we discuss next. Note that this also generalizes the well-posedness results in [37].

Example 2.10. We consider a diffusion operator $G \subseteq \frac{d^2}{dt^2}$ on $L^p[0, 1]$ with non-local boundary conditions. More precisely, for $h_0, h_1 \in L^q[0, 1]$ where $\frac{1}{p} + \frac{1}{q} = 1$, we define the domain

\[
D(G) := \left\{ f \in W^{2,p}[0, 1] \mid f(j) = \int_0^1 h_j(s)f(s)\,ds, j = 0, 1 \right\}.
\]

In our setting this corresponds to $\ell = 0$, $m = 1$, the diffusion coefficient $a(\bullet) \equiv 1$, the state space $X = L^p[0, 1]$, the boundary spaces $Y_1 = \{0\}, Y_0 = \mathbb{C}^2$, and the boundary functionals $\Phi_1 = 0$, $\Phi_0 = (\bar{V}_1-V_1)$, where $V_j f := \int_0^1 h_j(s)f(s)\,ds$. This implies $J_{\infty} = Id$, $\ell = 1$, $q(\bullet) \equiv 1$, and for the operators $R_t, S_t$ defined in (2.9) we obtain for $u \in L^p[0, t_0], 0 < t_0 < 1$,

\[
\delta_0 R_t u = \delta_1 S_t u = u(t) \quad \text{and} \quad \delta_1 R_t u = \delta_0 S_t u = 0, \quad t \in [0, t_0].
\]

Moreover, a simple computation shows that

\[
\begin{align*}
\mathcal{V}_j R_t u &= \int_0^t h_j(s)u(t-s)\,ds = (h_j * u)(t) =: (\mathcal{K}_j u)(t), \\
\mathcal{V}_j S_t u &= \mathcal{V}_j \psi R_t u = \int_0^t (\psi h_j)(s)u(t-s)\,ds = ((\psi h_j) * u)(t) =: (\tilde{\mathcal{K}}_j u)(t).
\end{align*}
\]

Hence, the operator $\mathcal{R}_{t_0}$ in (2.11) is given by

\[
\mathcal{R}_{t_0} = Id - \begin{pmatrix} \tilde{\mathcal{K}}_1 & -\mathcal{K}_1 \\
\mathcal{K}_0 & \tilde{\mathcal{K}}_0 \end{pmatrix},
\]

\footnote{In the case of empty external part ($\ell = 0$) we write $q(\bullet) = q(t), \lambda(\bullet) = \lambda(t), \mu(\bullet) = \mu(t), f = f^t$ and $a(\bullet)$ and $c(\bullet)$ instead of $a(\bullet, \bullet)$ and $c(\bullet, \bullet)$.}
where by Young’s inequality each convolution operator $K \in \{K_j, \tilde{K}_j : j = 0, 1\} \subset \mathcal{L}(L^p([0, t_0]))$ with kernel $h \in \{h_j, \psi h_j : j = 0, 1\} \subset L^p[0, 1] \subset L^1[0, 1]$ satisfies
\[
\|K\|_{\mathcal{L}(L^p([0, t_0]))} \leq \|h\|_{[0, t_0]} \to 0 \quad \text{as} \ t_0 \to 0.
\]
This implies that $\mathcal{R}_{t_0}$ is invertible for $t_0 \in (0, 1]$ sufficiently small and by Theorem 2.3 we conclude that $G$ generates a cosine family on $X$.

We close this section by considering two very common and important classes of boundary conditions. The first one uses a set of “boundary matrices” to impose the values in the end points, the second one uses two “boundary spaces” $Y_0, Y_1$ instead. As we will see, in both cases our main assumption in Theorem 2.3, the invertibility of the map $\mathcal{R}_{t_0}$ given by (2.11), reduces to a condition which can be easily verified. More precisely, in the first case we obtain the determinant condition (2.32), in the second one the direct sum condition (2.39).

### 2.3. Boundary conditions via “boundary matrices”

For $k_0, k_1 \in \mathbb{N}_0$ satisfying $k_0 + k_1 = \ell + 2m$ we choose matrices
\[
V_{0}^i \in M_{k_0 \times t}, \quad W_{0}^i \in M_{k_0 \times t}, \quad V_{1}^i, W_{1}^i \in M_{k_1 \times m}(\mathbb{C}) \quad \text{and} \quad W_{0}, W_{1} \in M_{k_1 \times m}(\mathbb{C}).
\]
Moreover, we define
\[
\tilde{W}_{0}^i := W_{0}^i \cdot \mu^e(0)^{-1}, \quad \tilde{W}_{1}^i := W_{1}^i \cdot \mu^l(0)^{-1}, \quad W_{1}^i := W_{1}^i \cdot \mu^l(1)^{-1},
\]
for $\mu^e(\cdot)$ and $\mu^l(\cdot)$ given in (2.4) and (2.5), respectively. Next we will use the matrices $V_{0}^i, V_{1}^i, W_{1}^i$ to specify $k_0$ conditions containing only values at the endpoints, while the matrices $W_{0}^i, W_{0}, W_{1}^i$ will determine $k_1$ (linear independent) conditions regarding derivatives at the endpoints.

**Corollary 2.11.** Let $a(\cdot, \cdot)$ be as in (2.1) and assume that $B^e \in \mathcal{L}(L^p(\mathbb{R}_+, \mathcal{C}^e), L^p(\mathbb{R}_+, \mathcal{C}^k))$ and $B^l \in \mathcal{L}(L^p([0, 1], \mathcal{C}^m), L^p([0, 1], \mathcal{C}^k))$ map $W^{1,p}$ into $W^{1,p}$, i.e.,
\[
B^e W^{1,p}(\mathbb{R}_+, \mathcal{C}^e) \subseteq W^{1,p}(\mathbb{R}_+, \mathcal{C}^k) \quad \text{and} \quad B^l W^{1,p}([0, 1], \mathcal{C}^m) \subseteq W^{1,p}([0, 1], \mathcal{C}^k).
\]
For $f = (f_i^e) \in W^{2,p}(\mathbb{R}_+, \mathcal{C}^e) \times W^{2,p}([0, 1], \mathcal{C}^m)$ consider the boundary conditions
\[
\begin{cases}
V_{0}^e f^e(0) + V_{0}^i f^i(0) + V_{1}^i f^i(1) = 0, \\
W_{0}^e (f^e)'(0) + W_{0}^i (f^i)'(0) - W_{1}^i (f^i)'(1) + (B^e f^e)(0) + (B^l f^l)(0) = 0.
\end{cases}
\tag{2.31}
\]
If the determinant
\[
\det \begin{pmatrix}
V_{0}^e & V_{1}^i \\
W_{0}^e & W_{1}^i
\end{pmatrix} \neq 0,
\tag{2.32}
\]
then the operator
\[
G := a(\cdot, \cdot) \cdot \frac{d^2}{dx^2},
\]
\[
D(G) := \left\{ f = (f_i^e) \in W^{2,p}(\mathbb{R}_+, \mathcal{C}^e) \times W^{2,p}([0, 1], \mathcal{C}^m) \mid f \text{ satisfies (2.31)} \right\},
\tag{2.33}
\]
generates a cosine family on $X = L^p(\mathbb{R}_+, \mathcal{C}^e) \times L^p([0, 1], \mathcal{C}^m)$ with phase space $V \times X$ for
\[
V := \left\{ (f_i^e) \in W^{1,p}(\mathbb{R}_+, \mathcal{C}^e) \times W^{1,p}([0, 1], \mathcal{C}^m) : V_{0}^e f^e(0) + V_{0}^i f^i(0) + V_{1}^i f^i(1) = 0 \right\}.
\]
Proof. In order to fit this setting into our general framework let
\[ Y_0 := C^{k_0} \times \{0\}^{k_1} \subseteq C^{l+2m} \quad \text{and} \quad Y_1 := \{0\}^{k_0} \times C^{k_1} \subseteq C^{l+2m} \]
and define \( \Phi_j \in \mathcal{L}(C_0([0,1], C^\ell), C([0,1], C^m), Y_j), j = 0, 1, \) by
\[
\Phi_0 := \begin{pmatrix} V_0^i \cdot \delta_0 & V_0^j \cdot \delta_0 + V_1^j \cdot \delta_1 \\ 0 & 0 \end{pmatrix}, \quad \Phi_1 := \begin{pmatrix} 0 & W_0^i \cdot \delta_0 \\ W_0^j \cdot \delta_0 - W_1^j \cdot \delta_1 \end{pmatrix}.
\]
Our next aim is to rewrite the boundary conditions (2.31) as
\[
\Phi_0(f_j^r) = 0, \quad \Phi_1(f_j^r + B(f_j^r)) = 0
\]
for a suitable operator \( B \in \mathcal{L}(X) \) leaving \( W^{1,p} \) invariant. To this end first note that by (2.32) there exist matrices \( R_0^e \in M_{k \times k_1}(C), R_1^e \in \mathbb{M}_{m \times k_1}(C) \) such that
\[
\begin{align*}
W_0^e & \cdot R_0^e - W_1^e \cdot R_1^e + \hat{W}_0^e \cdot R_0^i = Id_{C^{k_1}}. \\
\end{align*}
\]
Denote by \( \Gamma \in \mathcal{L}(L^p([0,1], C^{k_1}), L^p([0,1], C^{k_1})) \) the restriction operator, i.e., \( \Gamma f := f|_{[0,1]} \) for \( f \in L^p([0,1], C^{k_1}) \). Moreover, let \( E \in \mathcal{L}(L^p([0,1], C^{k_1}), L^p([0,1], C^{k_1})) \) be an extension operator such that \( \Gamma Ef = g \) for all \( g \in L^p([0,1], C^{k_1}) \) and \( E(W^{1,p}([0,1], C^{k_1})) \subset W^{1,p}([0,1], C^{k_1}) \). Now put
\[
B := \begin{pmatrix} \mu^e(0)^{-1} \cdot R_0^e \cdot B^e & \mu^e(0)^{-1} \cdot R_0^i \cdot EB^i \\ C \cdot \Gamma B^e & C \cdot B^i \end{pmatrix} \in \mathcal{L}(X^e \times X^i),
\]
where
\[
C := (\mathbb{1} - s) \cdot \mu^i(0)^{-1} R_0^i + s \cdot \mu^i(1)^{-1} R_1^i \cdot \psi \in \mathcal{L}(L^p([0,1], C^{k_1}), L^p([0,1], C^m))
\]
for \( \mathbb{1}(s) = 1, s(s) = s, s \in [0,1] \) and \( \psi(g) := g(1 - \cdot) \) for \( g \in L^p([0,1], C^{k_1}) \). Then we have\( B(W^{1,p}([0,1], C^m) \times W^{1,p}) \subseteq W^{1,p}([0,1], C^m) \times W^{1,p}([0,1], C^m) \) and a simple computation using (2.34) shows that for \( f = (f_j^r) \in W^{2,p}([0,1], C^\ell) \times W^{2,p}([0,1], C^m) \)
\[ f \text{ satisfies } (2.31) \iff \Phi_0 f = 0, \Phi_1(f^r + Bf) = 0. \]
Moreover, for the operators \( Q_t, R_t, S_t \) defined in (2.9) we have
\[
\delta_0 Q_t u = u(t), \quad \delta_0 R_t v = v(t), \quad \delta_1 R_t v = 0, \quad \delta_0 S_t v = 0, \quad \delta_1 S_t v = v(t),
\]
where \( u \in W^{1,p}([0,t_0], C^\ell), v \in W^{1,p}([0,t_0], C^m) \) and \( 0 < t_0 \leq \min\{\varphi_1(1), \ldots, \varphi_m(1)\} \). Note also that
\[
\delta_0 J_{\varphi^e} = \delta_0 J_{\varphi^i} = \delta_0 \quad \text{and} \quad \delta_1 J_{\varphi^e} = \delta_1.
\]
Using all this we compute the operator \( \mathcal{R}_{t_0} \) given in (2.11) as
\[
\mathcal{R}_{t_0} = \begin{pmatrix} V_0^i & -V_0^i \\ W_0^i & -W_0^i \end{pmatrix} \cdot \text{diag}(q^e(0), q^i(1), q^i(0)) \in \mathcal{L}(L^p([0,t_0], C^{l+2m})).
\]
Since the matrix \( \text{diag}(q^e(0), q^i(1), q^i(0)) \in M_{l+2m}(C) \) is always invertible, the assertion follows from Theorem 2.3.

We give some possible choices for the operators \( B^e, B^i \) appearing in Corollary 2.11.
Example 2.12. (i) For matrices $U_0^c \in M_{k_1 \times \ell} (\mathbb{C})$, $U_0^i, U_1^i \in M_{k_1 \times m} (\mathbb{C})$ define the operators
\[
B^c : = U_0^c \in L^p(\mathbb{R}_+, \mathbb{C}^\ell), L^p(\mathbb{R}_+, \mathbb{C}^{k_1}),
\]
\[
B^i : = U_0^i + U_1^i \cdot \psi \in L^p([0, 1], \mathbb{C}^m), L^p([0, 1], \mathbb{C}^{k_1}),
\]
where $\psi f(*) := f(1 - *)$ for $f \in X^i$. Then $B^c, B^i$ map $W^{1,p}$ into $W^{1,p}$ and for $f = (f_i) \in W^{2,p}(\mathbb{R}_+, \mathbb{C}^\ell) \times W^{2,p}([0, 1], \mathbb{C}^m)$ the second condition in (2.31) gives the mixed boundary condition
\[
W_0^c(f^c)'(0) + W_0^i(f^i)'(0) - W_1^c(f^c)'(1) + U_0^c f^c(0) + U_0^i f^i(0) + U_1^i f^i(1) = 0.
\]
In particular, this covers the boundary conditions considered in [27, 24].

(ii) For arbitrary operators $T^c \in L^p(\mathbb{R}_+, \mathbb{C}^{k_1})$ and $T^i \in L^p([0, 1], \mathbb{C}^m), (\mathbb{C}^{k_1})$ define
\[
B^c \in L^p(X^c, L^p(\mathbb{R}_+, \mathbb{C}^{k_1})), \quad (B^c f^c)(s) = e^{-s} \cdot T^c f^c,
\]
\[
B^i \in L^p(X^i, L^p([0, 1], \mathbb{C}^{k_1})), \quad (B^i f^i)(s) = T^i f^i.
\]
Then again $B^c, B^i$ map $W^{1,p}$ into $W^{1,p}$ and for $f = (f_i) \in W^{2,p}(\mathbb{R}_+, \mathbb{C}^\ell) \times W^{2,p}([0, 1], \mathbb{C}^m)$ the second boundary condition in (2.31) reduces to
\[
W_0^c(f^c)'(0) + W_0^i(f^i)'(0) - W_1^i(f^i)'(1) + T^c f^c + T^i f^i = 0.
\]

Note that by choosing operators $T^c, T^i$ properly (e.g. as an integral) we thus obtain various non-local boundary conditions.

(iii) We can also combine the two examples above and obtain the second condition in (2.31) of the form
\[
W_0^c(f^c)'(0) + W_0^i(f^i)'(0) - W_1^i(f^i)'(1) + U_0^c f^c(0) + U_0^i f^i(0) + U_1^i f^i(1) = 0.
\]

We now show some applications of Corollary 2.11, first to simple scalar examples.

Example 2.13. We consider the second derivative $G_p$ and $G_D$ with periodic- and Dirichlet boundary conditions, respectively, on $[0, 1]$, that is $G_p, G_D \subset \frac{d^2}{dx^2}$ on $X := L^p[0, 1]$ with domains
\[
D(G_p) := \{ f \in W^{2,p}[0, 1] \mid f(0) = f(1) \text{ and } f'(0) = f'(1) \},
\]
\[
D(G_D) := \{ f \in W^{2,p}[0, 1] \mid f(0) = f(1) = 0 \}.
\]
In order to write these boundary conditions as in (2.31) we choose $\ell = 0^4$ and $m = 1$. Moreover, in case of $G_p$ we take $k_0 = k_1 = 1$ and scalars $V_0 = 1, V_1 = -1, W_0 = W_1 = 1$. Then the determinant condition (2.32) is fulfilled, hence $G_p$ generates a cosine family. To handle $G_D$ one might be tempted to choose again $k_0 = k_1 = 1$. Then the first boundary condition $f(0) = 0$ can be implemented by choosing $V_0 = 1, V_1 = 0$ while the second condition $f(1) = 0$ follows from Example 2.12.(i) if we take $W_0 = W_1 = U_0 = 0$ and $U_1 = 1$. However, by doing so (2.32) is not fulfilled nevertheless it is well-known that $G_D$ generates a cosine family. At a first glance, Corollary 2.11 fails in this simple example, thus only gives a sufficient but in general not necessary generation criterion.

\footnote{As before, in the case of empty external part ($\ell = 0$) we shorten the notation and omit the superscript ‘i’ for the boundary matrices.}
However, as pointed out earlier, the matrices $W_0, W_1$ should be used to implement $k_1$ linear independent conditions regarding the derivatives at the endpoints. In case of $G_D$ this means that we have to choose $k_0 = 2$, $k_1 = 0$ and the two boundary matrices $V_0 = \{0\}, V_1 = \{1\}$. For this choice (2.32) is fulfilled, yielding the desired generation result.

We leave it to the reader to check that also problems with Neumann- or mixed boundary conditions on an interval can be handled in the same way.

Next we consider an example showing that the generator property of $G$ not only depends on the matrices $V_0, V_1, W_0$ and $W_1$ which determine the domain $D(G)$ in (2.33) but also on the values of the diffusion coefficients $\mu(s) = \sqrt{\lambda(s)}$ for $s = 0, 1$, appearing in the definitions of $W_0$ and $W_1$.

**Example 2.14.** For some Lipschitz continuous, positive function $a(\bullet) : [0, 1] \to (0, +\infty)$ consider on $X := L^p[0, 1], 1 \leq p < +\infty$, the operator

$$G := a(\bullet) \cdot \frac{d^2}{ds^2},$$

$$D(G) := \{f \in W^2_p[0, 1] \mid f(0) + f(1) = 0, \ f'(0) - f'(1) = 0\}. \quad (2.36)$$

Then $G$ is given by (2.33) for $\ell = 0, m = 1 = k_0 = k_1$, and $V_0 = W_0 = W_1 = V_1 = 1$. This gives for $c(\bullet) := \sqrt{a(\bullet)}$ the condition

$$\det \left( \begin{array}{cc} V_1 & V_0 \\ W_1 & W_0 \end{array} \right) = c(0)^{-1} - c(1)^{-1} \neq 0 \iff a(0) \neq a(1).$$

Hence, by the previous result $G$ in (2.36) generates a cosine family if $a(0) \neq a(1)$.

We note that in case $a(0) = a(1)$ the operator $G$ never generates a cosine family or even an analytic semigroup. To prove this assertion, we denote the operator obtained by (2.36) for $a(\bullet) \equiv 1$ by $G_1$. Then for each $\lambda \in \mathbb{C}$ we have $e_\lambda \in \ker(\lambda - G_1)$ where

$$e_\lambda(s) := e^{\sqrt{s} \cdot s} + e^{\sqrt{s} \cdot (1-s)}, \ s \in [0, 1].$$

Hence, $\sigma(G_1) = \mathbb{C}$ implying that $G_1$ cannot be a generator.

For Lipschitz continuous, positive $a(\bullet) \in C[0, 1]$ one can use the similarity transformation induced by $J_{\varphi'} \in \mathcal{L}(X), J_{\varphi'} f := f \circ \varphi'$ to show that in case $a(0) = a(1)$ the operators $G$ and $G_1$ are similar up to the perturbation $P := \frac{a'}{2} \cdot \frac{d}{ds}$. Since $P$ is relatively bounded with bound 0, this implies the claim.

Summing up, in this example Corollary 2.11 gives an optimal result which demonstrates the sharpness of the underlying perturbation argument from Subsection A.1.

We continue with an example on a simple star-shaped non-compact metric graph, cf. Figure 1. Further applications to general metric graphs are presented in Section 3.

**Example 2.15.** We consider a diffusion process described by $G \subset \frac{d^2}{ds^2}$ along the edges of the non-compact star graph presented in Figure 1. The two compact edges $e^1_1, e^2_1$ are parametrized as $[0, 1]$, with 0 in the common vertex, while $e^1_2 = \mathbb{R}_+$. In the central vertex we assume continuity and an additional boundary condition for the derivatives, i.e.,

$$f^1_2(0) = f^1_1(0) = f^2_1(0),$$

$$\alpha \cdot (f^1_2)'(0) + \beta \cdot (f^1_1)'(0) + \gamma \cdot (f^2_1)'(0) = 0,$$
for some $\alpha, \beta, \gamma \in \mathbb{C}$, while at the remaining endpoints we set the following Neumann- and Robin condition, respectively, that is

$$(f^1_i)'(1) = 0, \quad \delta \cdot f^1_i(1) + \varepsilon \cdot (f^2_i)'(1) = 0$$

for some $\delta, \varepsilon \in \mathbb{C}$. Then $\ell = 1$, $m = 2$ and if $\varepsilon \neq 0$ we choose $k_0 = 2$, $k_1 = 3$ and boundary matrices

$$V^0_e = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad V^1_e = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$W^0_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad W^1_e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$U^0_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad U^1_e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$

Then the determinant in (2.32) equals $\varepsilon \cdot (\alpha + \beta + \gamma)$. In case $\varepsilon = 0$ the boundary conditions are essentially different and in order to apply Corollary 2.11 one has to take $k_0 = 3$, $k_1 = 2$. For accordingly modified boundary matrices this gives determinant $\delta \cdot (\alpha + \beta + \gamma)$. Hence, if $(|\delta| + |\varepsilon|) \cdot (\alpha + \beta + \gamma) \neq 0$, by Corollary 2.11 the problem is well-posed while for $\varepsilon = \delta = 0$ it is clearly under-determined.

### 2.4. Boundary conditions via “boundary spaces”

We consider another way to impose conditions at the end points using two “boundary spaces” $Y_0, Y_1 \subseteq \mathbb{C}^{\ell+2m}$ and two operators $B^e \in \mathcal{L}(X^e, \ell^p(\mathbb{R}_+), \mathbb{C}^{\ell+2m})$, $B^i \in \mathcal{L}(X^i, \ell^p([0, 1], \mathbb{C}^{\ell+2m}))$ satisfying

$$B^e W^1_p(\mathbb{R}_+, \mathbb{C}^{\ell}) \subseteq W^1_p(\mathbb{R}_+, \mathbb{C}^{\ell+2m}),$$
$$B^i W^1_p([0, 1], \mathbb{C}^{m}) \subseteq W^1_p([0, 1], \mathbb{C}^{\ell+2m}).$$

(2.37)

Then for $f = (f^e, f^i) \in W^2_p(\mathbb{R}_+, \mathbb{C}^{\ell}) \times W^2_p([0, 1], \mathbb{C}^{m})$ we consider the boundary conditions

$$\left( \begin{array}{c} f^e(0) \\ f^i(0) \\ f^i(1) \end{array} \right) \in Y_1, \quad \left( \begin{array}{c} \mu^e(0) \cdot f^e(0) \\ \mu^i(0) \cdot f^i(0) \\ -\mu^i(0) \cdot f^i(0) \end{array} \right) + (B^e f^e)(0) + (B^i f^i)(0) \in Y_0.$$

(2.38)

Applying Theorem 2.3 to this setting yields the following.

**Corollary 2.16.** Let $a(\bullet, \bullet)$ and $c(\bullet, \bullet)$ be given by (2.1) and (2.10), respectively. If for subspaces $Y_0, Y_1 \subseteq \mathbb{C}^{\ell+2m}$ we have

$$Y_0 \oplus Y_1 = \mathbb{C}^{\ell+2m},$$

(2.39)
then for every $B^c \in \mathcal{L}(X^c, L^p(\mathbb{R}_+, \mathbb{C}^{\ell+2m}))$, $B^t \in \mathcal{L}(X^t, L^p([0, 1], \mathbb{C}^{\ell+2m}))$ satisfying (2.37) the operator

$$G := a(\cdot, \cdot) \cdot \frac{d^2}{ds^2},$$

$$D(G) := \left\{ f = \left( f_1^t \right) \in W^{2, p}(\mathbb{R}_+, \mathbb{C}^t) \times W^{2, p}([0, 1], \mathbb{C}^m) : f \text{ satisfies } (2.38) \right\},$$

generates a cosine family on $X = L^p(\mathbb{R}_+, \mathbb{C}^t) \times L^p([0, 1], \mathbb{C}^m)$ with phase space $V \times X$ for

$$V := \left\{ f \in W^{1, p}([0, 1], \mathbb{C}^m) : (f^t(0), f^t(0), f^t(1))^\top \in Y_1 \right\}.$$

**Proof.** Using that for $I = [0, 1]$ or $\mathbb{R}_+$ we have $L^p(I, \mathbb{C}^{\ell+2m}) = L^p(I, \mathbb{C}^t) \times L^p(I, \mathbb{C}^m) \times L^p(I, \mathbb{C}^m)$ we decompose $B^c$ and $B^t$ accordingly, i.e., we write

$$B^c = \begin{pmatrix} b_1^c & 0 \\ b_2^c & b_3^c \end{pmatrix} : L^p(\mathbb{R}_+, \mathbb{C}^t) \to L^p(\mathbb{R}_+, \mathbb{C}^t) \times L^p(\mathbb{R}_+, \mathbb{C}^m) \times L^p(\mathbb{R}_+, \mathbb{C}^m),$$

$$B^t = \begin{pmatrix} b_1^t & 0 \\ b_2^t & b_3^t \end{pmatrix} : L^p([0, 1], \mathbb{C}^t) \to L^p([0, 1], \mathbb{C}^t) \times L^p([0, 1], \mathbb{C}^m) \times L^p([0, 1], \mathbb{C}^m).$$

Denote by $\Gamma \in \mathcal{L}(L^p(\mathbb{R}_+, \mathbb{C}^m), L^p([0, 1], \mathbb{C}^m))$ the restriction operator, i.e., $\Gamma f := f|_{[0,1]}$ for $f \in L^p(\mathbb{R}_+, \mathbb{C}^m)$. Further, let $E \in \mathcal{L}(L^p([0, 1], \mathbb{C}^m), L^p(\mathbb{R}_+, \mathbb{C}^m))$ be an extension operator, i.e. $\Gamma Ef = g$ for all $g \in L^p([0, 1], \mathbb{C}^m)$, such that $E W^{1, p}([0, 1], \mathbb{C}^m) \subset W^{1, p}(\mathbb{R}_+, \mathbb{C}^m)$. Finally, we consider the projection $P \in \mathcal{L}(\mathbb{C}^{\ell+2m})$ associated to the representation (2.39), that is $\ker(P) = \text{rg}(Id - P) = Y_0$ and $\text{rg}(P) = \ker(Id - P) = Y_1$. Then $f = \left( \frac{f_1^t}{f_1^t} \right) \in W^{2, p}(\mathbb{R}_+, \mathbb{C}^t) \times W^{2, p}([0, 1], \mathbb{C}^m)$ satisfies (2.38) if and only if $\Phi_0 f = 0$ and $\Phi_1 (f + Bf) = 0$ for

$$\Phi_0 := (Id - P) \cdot \begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_0 \end{pmatrix} \in \mathcal{L}(C_0(\mathbb{R}_+, \mathbb{C}^t) \times C([0, 1], \mathbb{C}^m), Y_0),$$

$$\Phi_1 := P \cdot \begin{pmatrix} \mu^t(0) - \delta_0 & 0 \\ 0 & -\mu^t(1) - \delta_1 \end{pmatrix} \in \mathcal{L}(C_0(\mathbb{R}_+, \mathbb{C}^t) \times C([0, 1], \mathbb{C}^m), Y_1),$$

and $B \in \mathcal{L}(X^c \times X^t)$ given by

$$B := \begin{pmatrix} \mu^c(0) & \mu^c(0) - \delta_0 & \mu^c(0) - \delta_1 \\ (1 - s) \cdot \mu^t(0) - \delta_0 & \mu^t(0) - \delta_1 & \mu^t(0) - \delta_1 \\ (1 - s) \cdot \mu^c(0) - \delta_0 & \mu^c(0) - \delta_1 & \mu^c(0) - \delta_1 \end{pmatrix},$$

where $s(s) = s, \mathbb{1}(s) = 1$ for $s \in [0, 1]$ and $\psi(h) := h(1 - \cdot)$ for $h \in X^c$. Note that $B$ leaves $W^{1, p}(\mathbb{R}_+, \mathbb{C}^t) \times W^{1, p}([0, 1], \mathbb{C}^m)$ invariant, hence Assumption 2.1 is satisfied. Now choose $t_0 := \min\{\varphi_1(1), \ldots, \varphi_m(1)\} > 0$. Then a simple computation using (2.35) and

$$\tilde{\Phi}_1 = \Phi_1 \cdot c(\cdot, \cdot, -1) = (\tilde{\Phi}_1^c, \tilde{\Phi}_1^t) = P \cdot \begin{pmatrix} \delta_0 & 0 \\ 0 & -\delta_0 \end{pmatrix} \in \mathcal{L}(C_0(\mathbb{R}_+, \mathbb{C}^t) \times C([0, 1], \mathbb{C}^m), Y_1)$$

yields that $\mathcal{R}_{t_0}$ in (2.11) is constant and given by

$$\mathcal{R}_{t_0} = \begin{pmatrix} q^e(0) & 0 \\ 0 & q^t(0) \end{pmatrix} \in \mathcal{L}(L^p([0, t_0], \mathbb{C}^{\ell+2m})).$$

Hence, $\mathcal{R}_{t_0}$ is invertible and Theorem 2.3 implies the claim. 

We give two possible choices for the operators $B^c$, $B^t$ appearing in the boundary condition (2.38).
Example 2.17. (i) For matrices $U_0^i \in M_{(\ell+2m)\times (\ell+2m)}(\mathbb{C})$ and $U_0^i, U_1^i \in M_{(\ell+2m)\times m}(\mathbb{C})$ define

\[
B^e := U_0^e \in \mathcal{L}(X^e, L^p(\mathbb{R}_+), \mathbb{C}^{(\ell+2m)})), \quad B^i := U_0^i + U_1^i \cdot \psi \in \mathcal{L}(X^i, L^p(\mathbb{R}_+), \mathbb{C}^{(\ell+2m)})).
\]

Then $B^e, B^i$ satisfy (2.37) and for $f = (f^e) \in W^{2,p}(\mathbb{R}_+, \mathbb{C}^\ell) \times W^{2,p}([0, 1], \mathbb{C}^m)$ the second boundary condition in (2.38) simplifies to

\[
\begin{pmatrix}
\mu^e(0) \cdot (f^e)'(0) \\
\mu^i(0) \cdot (f^i)'(0) \\
-\mu^e(1) \cdot (f^e)'(1)
\end{pmatrix}
+ U_0^e f^e(0) + U_0^i f^i(0) + U_1^i f^i(1) = Y_0.
\]

This generalizes for example the boundary conditions considered in [36, Sect. 6.5], see also Subsection 3.6.

(ii) For operators $T^e \in \mathcal{L}(L^p(\mathbb{R}_+), \mathbb{C}^{(\ell+2m)})$ and $T^i \in \mathcal{L}(L^p([0, 1], \mathbb{C}^m), \mathbb{C}^{(\ell+2m)})$ define

\[
B^e \in \mathcal{L}(X^e, L^p(\mathbb{R}_+), \mathbb{C}^{(\ell+2m)})), \quad (B^e f^e)(s) = e^{-s} \cdot T^e f^e,
\]

\[
B^i \in \mathcal{L}(X^i, L^p([0, 1], \mathbb{C}^{(\ell+2m)})), \quad (B^i f^i)(s) = T^i f^i.
\]

Then again $B^e, B^i$ satisfy (2.37) and for $f = (f^e) \in W^{2,p}(\mathbb{R}_+, \mathbb{C}^\ell) \times W^{2,p}([0, 1], \mathbb{C}^m)$ the second boundary condition in (2.38) simplifies to

\[
\begin{pmatrix}
\mu^e(0) \cdot (f^e)'(0) \\
\mu^i(0) \cdot (f^i)'(0) \\
-\mu^e(1) \cdot (f^e)'(1)
\end{pmatrix}
+ T^e f^e + T^i f^i = Y_0.
\]

3. Applications to waves and diffusion on metric graphs.

3.1. Introduction. In this section we use our abstract results to show the well-posedness of wave- and diffusion equations on networks. That is, we study first and second order abstract initial-boundary value problems of the form (1.1). The structure of the graph is encoded in the boundary conditions contained in the domain $D(G)$.

We consider a finite metric graph (network) with $n$ vertices $v_1, \ldots, v_n$, $m$ internal edges $e_1, \ldots, e_m$, which we parametrize on the unit interval $[0, 1]$, and $\ell$ external edges $e_1^\ell, \ldots, e_\ell^\ell$, parametrized on the half-line $\mathbb{R}_+$. A graph without external edges is called compact. The structure of the graph is given by the $n \times m$ internal incidence matrices $\Phi^{i-} := (\varphi^{i-}_{rs})$, and $\Phi^{i+} := (\varphi^{i+}_{rs})$, where

\[
\varphi^{i-}_{rs} := \begin{cases} 1, & \text{if } e^i_r(0) = v_r, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad \varphi^{i+}_{rs} := \begin{cases} 1, & \text{if } e^i_r(1) = v_r, \\ 0, & \text{otherwise}, \end{cases}
\]

and the $n \times \ell$ external incidence matrix $\Phi^{e-} := (\varphi^{e-}_{rs})$, where

\[
\varphi^{e-}_{rs} := \begin{cases} 1, & \text{if } e^e_{r}(0) = v_r, \\ 0, & \text{otherwise}. \end{cases}
\]

By using the incidence matrices we obtain the diagonal matrices with in- and out-degrees of all vertices on the diagonal as

\[
D^\dagger := \Phi^\dagger (\Phi^\dagger)^\top, \quad \dagger \in \{(i,-), (i,+), (e,-)\},
\]

and the joint vertex degree matrix

\[
D := D^{i-} + D^{i+} + D^{e-} = \text{diag} \left( \text{deg}(v_r) \right)_{r=1}^n \in M_n(\mathbb{C}).
\]
The diffusion- and wave equation on a metric graph is defined by considering on each edge the heat equation
\[
\frac{d}{dt} u_j^i(t, s) = \lambda_j^i(s) \cdot \frac{d^2}{ds^2} u_j^i(t, s), \quad t \geq 0, \ s \in (0, 1), \ j = 1, \ldots, m,
\]
or the wave equation
\[
\frac{d^2}{dt^2} u_j^i(t, s) = \lambda_j^i(s) \cdot \frac{d^2}{ds^2} u_j^i(t, s), \quad t \geq 0, \ s \in (0, 1), \ j = 1, \ldots, m,
\]
respectively, for some Lipschitz continuous functions \( \lambda_j^i(\bullet) \in C(\mathbb{R}_+, C^\prime) \times C([0, 1], C^m) \) for \( k = 1, \ldots, \ell, j = 1, \ldots, m \), satisfying (2.2). Additionally, one needs to impose some transmission conditions in the vertices. These types of problems for compact graphs were studied for example in [30].

Next we present some types of these transmission conditions and show how our results apply in these examples.

3.2. Standard conditions. The most natural assumption for the solutions to either heat or wave equations on a metric graph is continuity in the vertices. We say that a function \( f = (f_j) \in C(\mathbb{R}_+, C^\prime) \times C([0, 1], C^m) \), defined on the edges of the graph, is continuous on the graph if its values at the endpoints of the contiguous edges coincide, i.e., whenever two edges \( e_j \) and \( e_k \) (both internal, both external or mixed) have a common vertex \( v \) then for the appropriate functions it holds \( f_j(v) = f_k(v) \). Here, \( f_j(v) := f_j(s) \) if \( e_j(s) = v \) for \( s = 0 \) or \( s = 1 \). A direct computation shows that the continuity property of \( f \) can be easily expressed using the incidence matrices as
\[
\exists c \in C^n \text{ such that } (\Phi c^-)^\top c = f^c(0), \ (\Phi c^i)^\top c = f^i(0) \text{ and } (\Phi c^+)^\top c = f^+(1)
\]
which is equivalent to
\[
\left( \begin{array}{c}
   f^c(0) \\
   f^i(0) \\
   f^+(1)
\end{array} \right) \in \text{rg} \left( \begin{array}{c}
   (\Phi c^-)^\top \\
   (\Phi c^i)^\top \\
   (\Phi c^+)^\top
\end{array} \right).
\]
Furthermore, in each of the vertices \( v_r, r = 1, \ldots, n \), we infer the standard Kirchhoff (also called Neumann) conditions
\[
\sum_{e_j \in \Gamma(v_r)} \lambda_j(v_r) \cdot \frac{\partial f_j}{\partial s}(v_r) = 0,
\]
where \( \Gamma(v_r) \) denotes the set of all edges incident to the vertex \( v_r \) and \( \frac{\partial f_j}{\partial s}(v_r) \) is the normal derivative of \( f_j \) computed at the appropriate endpoint of the edge \( e_j \). Using incidence matrices we can express this condition more accurately as
\[
\sum_{k=1}^{\ell} \varphi_{rk}^- \cdot \lambda_k(0) \cdot (f^c)_k(0) + \sum_{j=1}^{m} \varphi_{rj}^i \cdot \lambda_j(0) \cdot (f^i)_j(0) = \sum_{j=1}^{m} \varphi_{rj}^+ \cdot \lambda_j(1) \cdot (f^+)_j(1).
\]
Moreover, letting
\[
\lambda^e(s) := \text{diag}(\lambda^e_k(s))_{k=1}^\ell \in M_\ell(\mathbb{C}), \quad s \in \mathbb{R}_+,
\]
\[
\lambda^i(s) := \text{diag}(\lambda^i_j(s))_{j=1}^m \in M_m(\mathbb{C}), \quad s \in [0, 1],
\]
we can rewrite the Kirchhoff condition in matrix form as
\[
\Phi^{e,-} \cdot \lambda^e(0) \cdot (f^e)'(0) + \Phi^{i,-} \cdot \lambda^i(0) \cdot (f^i)'(0) = \Phi^{i,+} \cdot \lambda^i(1) \cdot (f^i)'(1).
\] (3.5)

Let \(a(\cdot,\cdot) := \begin{pmatrix} \lambda(\cdot,\cdot) & 0 \\ 0 & \lambda(\cdot,\cdot) \end{pmatrix} \) and
\[
G := a(\cdot,\cdot) \cdot \frac{d^2}{dx^2},
\]
\(D(G) := \{ f = (f_r^a) \in W^{2,p}(\mathbb{R}_+; \mathbb{C}^\ell) \times W^{2,p}([0,1], \mathbb{C}^m) : f \text{ satisfies (3.4) and (3.5)} \}\).

Then the diffusion- and wave equations on a network transform into the abstract Cauchy problems given in (1.1). We will see that by Corollary 2.16 both problems are well-posed.

First we show how the boundary conditions in the domain \(D(G)\) can be written as (2.38) for spaces \(Y_0, Y_1\) satisfying (2.39). To this end we define
\[
Y_1 := \text{rg} \begin{pmatrix} (\Phi^{e,-})^\top \\ (\Phi^{i,-})^\top \\ (\Phi^{i,+})^\top \end{pmatrix} \quad \text{and} \quad B^e = B^i := 0.
\]

Moreover, we note that for \(\mu^{e}(\cdot) := \sqrt{\lambda^{e}(\cdot)}\) and \(\mu^{i}(\cdot) := \sqrt{\lambda^{i}(\cdot)}\),
\[
(3.5) \quad \iff \begin{pmatrix} \mu^{e}(0)(f^e)'(0) \\ \mu^{i}(0)(f^i)'(0) \\ -\mu^{i}(1)(f^i)'(1) \end{pmatrix} \in \text{ker} \begin{pmatrix} \Phi^{e,-} - \mu^{e}(0), \Phi^{i,-} - \mu^{i}(0), \Phi^{i,+} + \mu^{i}(1) \end{pmatrix},
\]
\[
\iff \begin{pmatrix} \mu^{e}(0)(f^e)'(0) \\ \mu^{i}(0)(f^i)'(0) \\ -\mu^{i}(1)(f^i)'(1) \end{pmatrix} \in \text{rg} \begin{pmatrix} \mu^{e}(0)(\Phi^{e,-})^\top \\ \mu^{i}(0)(\Phi^{i,-})^\top \\ \mu^{i}(1)(\Phi^{i,+})^\top \end{pmatrix} =: Y_0.
\]

Hence, \(Y_0 = CY_1^\perp\) for the invertible diagonal matrix \(C := \text{diag}(\mu^{e}(0)^{-1}, \mu^{i}(0)^{-1}, \mu^{i}(1)^{-1})\). Next we verify that \(Y_0 \cap Y_1 = \{0\}\). Let \(y \in Y_0 \cap Y_1\). Then there exists \(z \in Y_1^\perp\) such that \(y = Cz\) which gives
\[
0 = \langle y, z \rangle = \langle Cz, z \rangle.
\]

Since \(C\) is positive definite, we conclude that indeed \(z = 0 = y\). Moreover, we have
\[
\dim(Y_0) + \dim(Y_1) = \dim(Y_1^\perp) + \dim(Y_1) = 2m + \ell.
\]

This implies (2.39) and hence Corollary 2.16 applies.

3.3. \(\delta\)-type conditions. This condition appears in the literature on quantum graphs, see [9]. It consists of the continuity condition (3.3) and the condition
\[
\sum_{e_j \in \Gamma(v_r)} \lambda_j(v_r) \cdot \frac{\partial f_j}{\partial s}(v_r) = \alpha_r \cdot f(v_r),
\]
in every vertex \(v_r\), \(r = 1, \ldots, n\). Here \(f(v_r)\) denotes the common value of the functions \(f_j\) corresponding to the edges \(e_j \in \Gamma(v_r)\) that meet in vertex \(v_r\), and \(\alpha_r\) are some fixed complex coefficients. Again we can rewrite this using incidence matrices as
\[
\sum_{k=1}^\ell \phi_{rk}^e \cdot \lambda_k^e(0) \cdot (f^e)'(0) + \sum_{j=1}^m \phi_{rj}^i \cdot \lambda_j^i(0) \cdot (f^i)'(0) - \sum_{j=1}^m \phi_{rj}^i \cdot \lambda_j^i(1) \cdot (f^i)'(1) = \alpha_r \cdot c_r,
\]
\(r = 1, \ldots, n\), where \(c = (c_1, \ldots, c_n)^\top\) is the vector appearing in the continuity condition (3.3). In order to obtain the appropriate matrix form first note that by (3.3), (3.1) and (3.2) we have
\[
Dc = \Phi^{e,-} f^e(0) + \Phi^{i,-} f^i(0) + \Phi^{i,+} f^i(1).
\]
Let $L := \text{diag}(\alpha_r)_{r=1}^n \in \mathbb{M}_n(\mathbb{C})$. Since every column of an incidence matrix consists of exactly one nonzero entry corresponding to the appropriate endpoint of an edge, there are $m \times m$ and $\ell \times \ell$ diagonal matrices $\tilde{D}^i$, such that

$$LD^{-1} = \Phi^i \tilde{D}^i,$$  

$\dagger \in \{(i,-),(i,+),(e,-)\}$.

Hence we can rewrite $\delta$-type conditions in the matrix form as

$$\Phi^{e_i} \cdot \lambda^+(0) \cdot (f^e)'(0) + \Phi^{i_e} \cdot \lambda^i(0) \cdot (f^i)'(0) - \Phi^{i_e} \cdot \lambda^+(1) \cdot (f^i)'(1)$$

$$= \Phi^{e_i} \cdot \tilde{D}^{e_i} \cdot f^e(0) + \Phi^{i_e} \cdot \tilde{D}^{i_e} \cdot f^i(0) + \Phi^{i_e} \cdot \tilde{D}^{i_e} \cdot f^i(1).$$

Defining $Y_0$ and $Y_1$ as in Subsection 3.2 and the operators

$$B^e := -\begin{pmatrix} \mu^e(0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B^i := -\begin{pmatrix} \mu^i(0)^{-1} & 0 \\ 0 & \mu^i(1)^{-1} \end{pmatrix},$$

where $\psi(h) := h(1-\bullet)$, our boundary conditions are of the form (2.38) and Corollary 2.16 applies again.

### 3.4. Non-local boundary conditions

We now further generalize the standard boundary conditions, taking the continuity condition (3.4) together with the condition

$$\Phi^{e_i} \cdot \lambda^+(0) \cdot (f^e)'(0) + \Phi^{i_e} \cdot \lambda^i(0) \cdot (f^i)'(0) - \Phi^{i_e} \cdot \lambda^+(1) \cdot (f^i)'(1)$$

$$= \Phi^{e_i} \cdot M^e \cdot f^e(0) + \Phi^{i_e} \cdot M^i \cdot f^i(0) + \Phi^{i_e} \cdot M^i \cdot f^i(1),$$

for some matrices $M^e \in M_\ell(\mathbb{C})$ and $M^i, M^i, M^i_i, M^i \in M_m(\mathbb{C})$. Note that in this way the Kirchhoff conditions in a vertex are supplemented with a linear combination of values in some other – even non-adjacent – vertices. This models, for example, a network, in which some nodes are able to communicate instantly and directly with another network, atop of the one under consideration. To treat this case we may again define $Y_0$ and $Y_1$ as in Subsection 3.2, take the boundary operators

$$B^e := -\begin{pmatrix} \mu^e(0)^{-1}M^e \\ 0 \end{pmatrix} \quad \text{and} \quad B^i := -\begin{pmatrix} \mu^i(0)^{-1}M^i \\ \mu^i(1)^{-1}M^i \end{pmatrix},$$

with $\psi(h) := h(1-\bullet)$ and apply Corollary 2.16.

### 3.5. Matrix mixed conditions

Motivated by applications in population dynamics, in [7, 8] a diffusion problem on a compact network with the general boundary condition

$$(f_{f(1)}^e(0)) = \mathbb{K}(f_{f(1)}^i(0))$$

for a matrix $\mathbb{K} \in M_{2m}(\mathbb{C})$ is considered. In this case Corollary 2.16 applies directly by choosing $\ell = 0$, $a(\bullet, \bullet) \equiv \text{Id}$, $Y_0 := \{0\}$, $Y_1 := \mathbb{C}^{2m}$ and

$$B^i := -\begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \cdot \mathbb{K} \cdot \text{Id}_\psi,$$

where as usual $\psi f(\bullet) := f(1-\bullet)$.

### 3.6. Generalized node conditions

In [36, Sect. 6.5] the boundary condition

$$(f_{f(1)}^e(0)) \in Y, \quad -\lambda(0)^{-1} f_{f(1)}^i(0) + W(f_{f(1)}^i(0)) \in Y^\perp$$

appears for compact graphs, where $Y \subseteq \mathbb{C}^{2m}$ and $W \in \mathbb{L}(Y)$. We show that also this condition fits in the setting of Corollary 2.16 for $\ell = 0$. To this end we define $Y_1 := Y$, $Y_0 := CY^\perp$ for

$$C := \text{diag}(\mu(0)^{-1}, \mu(1)^{-1}) \quad \text{and} \quad B^i := CW\psi^\perp,$$
where $\mu(\cdot) := \sqrt{\lambda(\cdot)}$, $\psi f(\cdot) := f(1 - \cdot)$. Then a simple computation shows that for these choices (3.6) is equivalent to (2.38). Next, the representation $Y_0 := CY_1^+$ for positive definite $C$ implies, by the same reasoning as at the end of Subsection 3.2, condition (2.39). Hence, Corollary 2.16 applies to the operator $G = a(\cdot) \cdot \frac{d^2}{dx^2}$ satisfying the boundary conditions (3.6). Moreover, this condition can be easily generalized to the non-compact metric graphs.

Appendix A. Perturbation and auxiliary results.

A.1. Domain perturbation for generators of $C_0$-semigroups. In this appendix we briefly recall a perturbation result from [1, Sect. 4.3] which is our main tool to prove Theorem 2.3 (similar see also [22, 23]). Moreover, we give an admissibility criterion which significantly simplifies the computation of the so-called controllability- and input-output maps. To explain the general setup we consider

- two Banach spaces $X$ and $\partial X$, called “state” and “boundary” spaces, respectively;
- a closed, densely defined “maximal” operator $5$ $A_m : D(A_m) \subseteq X \to X$;
- the Banach space $[D(A_m)] := (D(A_m), \|\cdot\|_{A_m})$ where $\|f\|_{A_m} := \|f\| + \|A_m f\|$ is the graph norm;
- two “boundary” operators $L, C \in \mathcal{L}([D(A_m)], \partial X)$.

Then define two restrictions $A, G \subset A_m$ by

$$
D(A) := \{f \in D(A_m) : Lf = 0\} = \ker(L),
$$

$$
D(G) := \{f \in D(A_m) : Lf = Cf\} = \ker(L - C). \tag{A.1}
$$

Hence, one can consider $G$ with boundary condition $Lf = Cf$ as a perturbation of the operator $A$ with abstract “Dirichlet type” boundary condition $Lf = 0$. In order to proceed we make the following

**Assumption A.1.** (i) The operator $A$ generates a $C_0$-semigroup $(T(t))_{t \geq 0}$ on the space $X$;

(ii) the boundary operator $L : D(A_m) \to \partial X$ is surjective.

Under these assumptions the following lemma is shown in [21, Lem. 1.2].

**Lemma A.2.** Let Assumption A.1 be satisfied. Then for each $\lambda \in \rho(A)$ the operator $L|_{\ker(\lambda - A_m)}$ is invertible and $L_\lambda := (L|_{\ker(\lambda - A_m)})^{-1} : \partial X \to \ker(\lambda - A_m) \subseteq X$ is bounded.

In what follows, the extrapolated space $X_{-1}$ associated with $A$ is the completion of $X$ with respect to the norm

$$
\|x\|_{-1} := \|R(\lambda_0, A)x\|, \quad x \in X,
$$

for some fixed $\lambda_0 \in \rho(A)$, $T_{-1}(t) \in \mathcal{L}(X_{-1})$ is the unique bounded extension of the operator $T(t)$ to $X_{-1}$, and $A_{-1}$ is the generator of the extrapolated semigroup $(T_{-1}(t))_{t \geq 0}$ with domain $D(A_{-1}) = X$. For more details on extrapolated spaces and semigroups we refer to [17, Sect. II.5.a].

Now one can verify that the operator

$$
L_A := (\lambda - A_{-1})L_\lambda \in \mathcal{L}(\partial X, X_{-1})
$$

5 “Maximal” concerns the size of the domain, e.g., a differential operator without boundary conditions.
Theorem A.3. Assume that there exist cf. [1, Rem. 7]. Now by [1, Thm. 10] the following holds. L
is independent of λ ∈ ρ(A) and that G = (A_{-1} + L_A \cdot C)|_X. Before stating the perturbation result [1, Cor. 22], we note that from the assumptions (i)–(iii) in Theorem A.3 below it follows that there exists a bounded “input-output map” F_{t_0} ∈ L(L^p([0,t_0], \partial X)) such that
\( (F_{t_0}u)(\bullet) = C \int_0^{t_0} T_{-1}(\bullet - s)L_Au(s) \, ds \quad \text{for all } u \in W_0^{2,p}([0,t_0], \partial X), \) (A.2)
cf. [1, Rem. 7]. Now by [1, Thm. 10] the following holds.

**Theorem A.3.** Assume that there exist \( 1 \leq p < +\infty, \ t_0 > 0 \) and \( M \geq 0 \) such that

(i) \( \int_0^{t_0} T_{-1}(t_0 - s)L_Au(s) \, ds \in X \quad \text{for all } u \in L^p([0,t_0], \partial X), \)

(ii) \( \int_0^{t_0} \left\| CT(s)x \right\|_{\partial X}^p \, ds \leq M \cdot \| x \|_X^p \quad \text{for all } x \in D(A), \)

(iii) \( \int_0^{t_0} \left\| C \int_0^r T_{-1}(r - s)L_Au(s) \, ds \right\|_{\partial X}^p \, dr \leq M \cdot \| u \|_p^p \quad \text{for all } u \in W_0^{2,p}([0,t_0], \partial X), \)

(iv) \( 1 \in \rho(F_{t_0}), \) where \( F_{t_0} \in L(L^p([0,t_0], \partial X)) \) is given by (A.2).

Then \( G \subset A_m \) generated by (A.1) generates a \( C_0 \)-semigroup on the Banach space \( X. \)

**Remark A.4.** If assumption (ii) in Theorem A.3 is satisfied, then the operator \( C \) is called a \( p \)-admissible observation operator for \( (T(t))_{t \geq 0}. \) In this case there exist a unique “observability map” \( C_{t_0} \in \mathcal{L}(X, L^p([0,t_0], \partial X)) \) such that
\[ C_{t_0}x = C \cdot T(\bullet)x \quad \text{for all } x \in D(A). \]

If we put \( \Phi := L - C \in \mathcal{L}([D(A_m)], \partial X) \) we obtain the following slight modification of Theorem A.3 which fits better our needs in Section 2.

**Corollary A.5.** Assume that there exist \( 1 \leq p < +\infty, \ t_0 > 0 \) and \( M \geq 0 \) such that

(i) \( \int_0^{t_0} T_{-1}(t_0 - s)L_Av(s) \, ds \in X \quad \text{for all } v \in L^p([0,t_0], \partial X), \)

(ii) \( \int_0^{t_0} \left\| \Phi T(s)x \right\|_{\partial X}^p \, ds \leq M \cdot \| x \|_X^p \quad \text{for all } x \in D(A), \)

(iii) \( \int_0^{t_0} \left\| \Phi \int_0^r T_{-1}(r - s)L_Av(s) \, ds \right\|_{\partial X}^p \, dr \leq M \cdot \| v \|_p^p \quad \text{for all } v \in W_0^{2,p}([0,t_0], \partial X), \)

(iv) \( \mathcal{Q}_{t_0} \) is invertible, where \( \mathcal{Q}_{t_0} \in L(L^p([0,t_0], \partial X)) \) is given by
\[ (\mathcal{Q}_{t_0}v)(\bullet) = \Phi \int_0^{t_0} T_{-1}(\bullet - s)L_Av(s) \, ds \quad \text{for all } v \in W_0^{2,p}([0,t_0], \partial X), \]

Then \( G \subset A_m, \ D(G) := \ker(\Phi), \) generates a \( C_0 \)-semigroup on the Banach space \( X. \)

**Proof.** Since the assumptions (i) are the same, it suffices to show that the hypotheses (ii)–(iv) imply the corresponding assumptions in Theorem A.3.

(ii) This is clear since \( LT(s)x = 0 \) for all \( x \in D(A) = \ker(L) \) and \( s \geq 0. \)

(iii) Using integration by parts twice, one sees that for all \( v \in W_0^{2,p}([0,t_0], \partial X) \)
\[ L \int_0^r T_{-1}(r - s)L_Av(s) \, ds = v(r), \quad r \in [0,t_0] \]
which implies (iii) in the previous result.
(iv) By the previous point it also follows that $Id - F_{t_0} = Q_{t_0}$ which implies the corresponding assumption in Theorem A.3. \hfill \Box

In [16] we showed two versions of variation of parameters formula for the solutions to boundary control problems. By using them we obtain the following equivalence which is quite helpful to verify the first assumption in the previous two results.

**Lemma A.6.** For $p \in [1, \infty)$ the following are equivalent.

1. Assumption (i) in Theorem A.3 and Corollary A.5 is satisfied.
2. There exists $t_0 > 0$ and a strongly continuous family $(B_t)_{t \in [0, t_0]} \subset L^p([0, t_0], \partial X)$ such that for every $u \in W^{2,p}_0([0, t_0], \partial X)$ the function $x : [0, t_0] \to X$, $x(t) := B_t u$ is a classical solution of the boundary control problem

$$\begin{cases}
    \dot{x}(t) = A_{x}(t)x(t), & 0 \leq t \leq t_0,
    
    Lx(t) = u(t), & 0 \leq t \leq t_0,
    
    x(0) = 0.
\end{cases} \tag{A.3}$$

In this case for $t \in (0, t_0]$ the operator $B_t$ coincides with the “controllability map”, i.e.,

$$B_t u = \int_0^t T_{t-s} L_A u(s) \, ds \quad \text{for } u \in L^p[0, t_0], \partial X). \tag{A.4}$$

**Proof.** (a)$\Rightarrow$(b) By assumption, equation (A.4) defines a bounded operator from $L^p([0, t_0], \partial X)$ to $X$ satisfying $\text{rg}(B_t) \subset X$. By the closed graph theorem and [10, Cor. 3.16] this implies that $(B_t)_{t \in [0, t_0]} \subset L(L^p([0, t_0], \partial X), X)$ is strongly continuous. Finally, by [16, Prop. 2.8] for $u \in W^{2,p}_0([0, t_0], \partial X)$ the function $x(t) := B_t u$ gives the unique classical solution of (A.3).

(b)$\Rightarrow$(a) Define $\tilde{B}_{t_0} \in L(L^p([0, t_0], \partial X), X)$ by the right-hand-side of (A.4) for $t = t_0$. Then by [16, Prop. 2.7] we have

$$\tilde{B}_{t_0} |_{W^{2,p}_0([0, t_0], \partial X)} = B_{t_0} |_{W^{2,p}_0([0, t_0], \partial X)}.$$ 

Since $W^{2,p}_0([0, t_0], \partial X) \subset L^p([0, t_0], \partial X)$ is dense this implies $\tilde{B}_{t_0}$ coincides with $B_{t_0}$, i.e., (a) is satisfied. \hfill \Box

For more details and examples regarding the above perturbation results we refer to [1, 2, 10].

A.2. Two auxiliary results. We state and prove two results concerning the inverse and derivative of matrix-valued functions.

**Lemma A.7.** Let $I \subset \mathbb{R}$ be an interval and $d(\bullet) : I \to M_n(\mathbb{C})$ be Lipschitz continuous and bounded, such that $\sigma(d(s)) \subset (0, +\infty)$ for all $s \in I$. If $I$ is not compact assume in addition that there exists $\varepsilon > 0$ such that $\sigma(d(s)) \subset [\varepsilon, \frac{1}{\varepsilon}]$ for all $s \in I$. Then $d^{-1}(\bullet) : I \to M_n(\mathbb{C})$ is Lipschitz continuous and bounded as well.

**Proof.** By assumption or by compactness of $I$ there exists $\varepsilon > 0$ such that $\sigma(d(s)) \subset [\varepsilon, \frac{1}{\varepsilon}]$ for all $s \in I$. This implies $\det(d(s)^{-1}) \in [\varepsilon^n, \frac{1}{\varepsilon^n}]$ for all $s \in I$, hence by the inversion formula for matrices there exists $M > 0$ such that $\|d(s)^{-1}\| \leq M$ for all $s \in I$. Moreover,

$$\|d^{-1}(s) - d^{-1}(r)\| \leq \|d^{-1}(r)\| \cdot \|d(r) - d(s)\| \cdot \|d^{-1}(s)\| \leq M^2 \cdot \|d(r) - d(s)\|$$

for all $s, r \in I$, i.e. $d^{-1}(\bullet)$ is bounded and Lipschitz continuous as claimed. \hfill \Box
Corollary A.8. Let $d(\cdot)$ be as in Lemma A.7. Then for $f : I \to \mathbb{C}^n$ we have
\[ f \in W^{1,p}(I, \mathbb{C}^n) \iff d \cdot f \in W^{1,p}(I, \mathbb{C}^n) \]
and in this case $(d \cdot f)' = d' \cdot f + d \cdot f'$. 

Proof. $\Rightarrow$: Since $d(\cdot) \in W^{1,p}_{\text{loc}}(I, M_n(\mathbb{C}))$, by [11, Cor. 8.10] we conclude that $d \cdot f \in W^{1,p}_{\text{loc}}(I, \mathbb{C}^n)$ and $(d \cdot f)' = d' \cdot f + d \cdot f'$. By assumption or by compactness of $I$ there exists $\varepsilon > 0$ such that $\sigma(d(s)) \subseteq [\varepsilon, \frac{1}{\varepsilon}]$ for all $s \in I$. If $L$ denotes the Lipschitz constant for $d(\cdot)$ this implies
\[ \|d \cdot f\|_{W^{1,p}(I, \mathbb{C}^n)} \leq \frac{1}{\varepsilon^p} \cdot \|f\|_p + L^p \cdot \|f\|_p + \frac{1}{\varepsilon^p} \cdot \|f'\|_p < +\infty, \]
hence $d \cdot f \in W^{1,p}(I, \mathbb{C}^n)$. To show $\Leftarrow$ we write $f = d^{-1} \cdot df$ and use Lemma A.7.

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REFERENCES

[1] M. Adler, M. Bombieri and K.-J. Engel, On perturbations of generators of $C_0$-semigroups, Abstr. Appl. Anal., (2014), Art. ID 213020, 13pp.
[2] M. Adler, M. Bombieri and K.-J. Engel, Perturbation of analytic semigroups and applications to partial differential equations, J. Evol. Equ., 17 (2017), 1183–1208.
[3] F. Ali Mehmeti, Problèmes de transmission pour des équations des ondes linéaires et quasilinéaires, in Hyperbolic and Holomorphic Partial Differential Equations, Travaux en Cours, Hermann, Paris, (1984), 75–96.
[4] F. Ali Mehmeti, Nonlinear Waves in Networks, Akademie-Verlag, Berlin, 1994.
[5] F. Ali Mehmeti, J. von Below and S. Nicaise, eds. Partial Differential Equations on Multi-structures, Marcel Dekker Inc., New York, 2001.
[6] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, Vector-Valued Laplace Transforms and Cauchy Problems, Birkhäuser/Springer Basel AG, 2nd edition, Basel, 2011.
[7] J. Banasiak, A. Falkiewicz and P. Namayanja, Asymptotic state lumping in transport and diffusion problems on networks with applications to population problems, Math. Models Methods Appl. Sci., 26 (2016), 215–247.
[8] J. Banasiak, A. Falkiewicz and P. Namayanja, Semigroup approach to diffusion and transport problems on networks, Semigroup Forum, 93 (2016), 427–443.
[9] G. Berkolaiko and P. Kuchment, Introduction to Quantum Graphs, American Mathematical Society, 2013.
[10] M. Bombieri and K.-J. Engel, A semigroup characterization of well-posed linear control systems, Semigroup Forum, 88 (2014), 366–396.
[11] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext. Springer, New York, 2011.
[12] C. Cattaneo, The spectrum of the continuous Laplacian on a graph, Monatsh. Math., 124 (1997), 215–235.
[13] C. Cattaneo and L. Fontana, D’Alembert formula on finite one-dimensional networks, J. Math. Anal. Appl., 284 (2003), 403–424.
[14] R. Däger and E. Zuazua, Wave Propagation, Observation and Control in 1-d Flexible Multi-structures, Springer-Verlag, Berlin, 2006.
[15] K.-J. Engel, Generator property and stability for generalized difference operators, J. Evol. Equ., 13 (2013), 311–334.
[16] K.-J. Engel, M. Kramar Fijavž, B. Klöss, R. Nagel and E. Sikolya, Maximal controllability for boundary control problems, Appl. Math. Optim., 62 (2010), 205–227.
[17] K.-J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer-Verlag, New York, 2000.
[18] P. Exner, A model of resonance scattering on curved quantum wires, Annalen der Physik, 47 (1990), 123–138.
[19] W. Feller, Diffusion processes in one dimension, Trans. Amer. Math. Soc., 77 (1954), 1–31.
[20] B. Gaveau, M. Okada and T. Okada, Explicit heat kernels on graphs and spectral analysis, in Several Complex Variables (Stockholm, 1987/1988), Princeton Univ. Press, Princeton, NJ, (1993), 364–388.
[21] G. Greiner, Perturbing the boundary conditions of a generator, Houston J. Math., 13 (1987), 213–229.
[22] S. Hadd, Unbounded perturbations of $C_0$-semigroups on Banach spaces and applications, Semigroup Forum, 70 (2005), 451–465.
[23] S. Hadd, R. Manzo and A. Rhandi, Unbounded perturbations of the generator domain, Discrete Contin. Dyn. Syst., 35 (2015), 703–723.
[24] A. Hussein, D. Krejčířík and P. Siegl, Non-self-adjoint graphs, Trans. Amer. Math. Soc., 367 (2015), 2921–2957.
[25] B. Jacob, K. Morris and H. Zwart, $C_0$-semigroups for hyperbolic partial differential equations on a one-dimensional spatial domain, J. Evol. Equ., 15 (2015), 493–502.
[26] B. Klöss, The flow approach for waves in networks, Oper. Matrices, 6 (2012), 107–128.
[27] S. Hadd, R. Manzo and A. Rhandi, Unbounded perturbations of the generator domain, Discrete Contin. Dyn. Syst., 35 (2015), 2921–2957.
[28] V. Kostrykin, J. Potthoff and R. Schrader, Contraction semigroups on metric graphs, in: Analysis on Graphs and its Applications, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 77 (2008), 423–458.
[29] V. Kostrykin and R. Schrader, Kirchhoff's rule for quantum wires, J. Phys. A, 32 (1999), 595–630.
[30] T. Kottos and U. Smilansky, Quantum chaos on graphs, Physical Review Letters, 79 (1997), 4794–4797.
[31] M. Kramar Fijavž, D. Mugnolo and E. Sikolya, Variational and semigroup methods for waves and diffusion in networks, Appl. Math. Optim., 55 (2007), 219–240.
[32] P. Kuchment, Graph models for waves in thin structures, Waves Random Media, 12 (2002), R1–R24.
[33] P. Kuchment, Quantum graphs: An introduction and a brief survey, in Analysis on Graphs and Its Applications, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 77 (2008), 291–312.
[34] J. E. Lagnese, G. Leugering and E. J. P. G. Schmidt, Modeling, Analysis and Control of Dynamic Elastic Multi-link Structures, Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994.
[35] Y. Le Gorrec, H. Zwart and B. Maschke, Dirac structures and boundary control systems associated with skew-symmetric differential operators, SIAM J. Control Optim., 44 (2005), 1864–1892.
[36] G. Lumer, Espaces ramifiés, et diffusions sur les réseaux topologiques, C. R. Acad. Sci. Paris Sér. A-B, 291 (1980), A527–A530.
[37] D. Mugnolo, Semigroup Methods for Evolution Equations on Networks, Understanding Complex Systems. Springer, Cham, 2014.
[38] D. Mugnolo and S. Ricaise, Well-posedness and spectral properties of heat and wave equations with non-local conditions, J. Differential Equations, 256 (2014), 2115–2151.
[39] S. Ricaise, Some results on spectral theory over networks, applied to nerve impulse transmission, in Orthogonal Polynomials and Applications (Bar-le-Duc, 1984), Springer, Berlin, 1171 (1985), 532–541.
[40] J.-P. Roth, Le spectre du laplacien sur un graphe, in Théorie du Potentiel (Oursay, 1983), Springer, Berlin, 1096 (1984), 521–539.
[41] C. Schubert, C. Seifert, J. Voigt and M. Waurick, Boundary systems and (skew-)self-adjoint operators on infinite metric graphs, Mathematische Nachrichten, 288 (2015), 1776–1785.
[42] J. von Below, A characteristic equation associated to an eigenvalue problem on $c^2$-networks, Linear Algebra Appl., 71 (1985), 309–325.

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