Complexity of Extended Dynamical Systems

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Abstract. The purpose of this paper is to present some notions in the theory of complexity in lattice models of extended dynamical systems.

1. Introduction

The time evolution of a dynamical system (DS) with few degrees of freedoms, which we shall call simply ordinary dynamical system, is nicely represented by its trajectory in the phase space which lies generally in a region of $\mathbb{R}^n$. On the opposite, it is very difficult to have such a purely geometrical representation of a spatially extended dynamical system, for, the phase point is to be substituted by a function (or vector field) belonging to a topological space of functions. The level of abstraction of the phase space of an extended dynamical system is very high and the description is less intuitive than in an ordinary one. However, the introduction of a metric in this space yields a good tool to describe the sensitivity to small perturbations of initial conditions, although several non equivalent metrics are possible. It is thus very useful to start with some simple models of Lattice Dynamical Systems (LDS) where the space variable $x \in \mathbb{R}^n$ is discretized. In more specific cases, the field function $u(x, t)$, describing the state of the system, takes only a finite number of values at each point of the lattice. Any function $u(x, 0)$ will be called a spatial configuration the evolution of which will be called a space-time configuration or a pattern. In the simplest case, where the lattice is $\mathbb{Z}$, and the field function two-valued, e.g., $u(i, 0) = 0$ or $1$ for all $i \in \mathbb{Z}$, it is possible to visualize the space-time configuration by drawing a black dot in the site $i$ at time $n$ only when $u(i, t) = 1$ (see fig1). If the function has continuous range of values called the local state space $I$, the visualization of the space-time configurations is much more difficult, although a coarse-graining can be introduced through suitable partitions of the local state space.

In ordinary systems, the complexity of the dynamics is described by using either several concepts: the transitivity, the abundance of periodic orbits of various periods densely filling the phase space, the topological entropy for chaotic systems [15], or complexity functions for hamiltonian systems [5]. Here, in the simplest models of extended systems, like cellular automata and coupled maps lattices, we consider some aspects of the complexity of the dynamics as the abundance of periodic travelling waves and the directional space-time entropy function.
2. Lattice dynamical systems

Cellular automata were introduced by von Neumann, following a suggestion by Ulam in 1952, as self-reproducing model of robots constructing other robots with the same complexity. They have been recently used also as microscopic models of self-organisation and non-equilibrium processes (see [6] for more details).

Let the set $\mathbb{K} = \{0, 1, \ldots, q - 1\}$ be the local state space. Then, the phase space is the set $\Omega = \mathbb{K}^\mathbb{Z}$ of all bi-sided infinite sequences $u = (u_i), i \in \mathbb{Z}$, $u_i \in \mathbb{K}$, called configurations. A cellular automaton (CA) is a transformation $F : \Omega \to \Omega$, defined by a local function, $f$, called the rule of the CA, as follows: the $i^{th}$ coordinate of $Fu$ is given in terms of the coordinates $(u_{i+l}, \ldots, u_{i+r}), l, r \in \mathbb{Z}, l < r$, by:

$$ (Fu)_i = f(u_{i+l}, \ldots, u_{i+r}) $$

Then $F$ defines an iterative dynamical system on $\Omega$, $u(t) = F^t u$, $t \geq 0$, $u_i(t)$ denotes its $i^{th}$ coordinate. In what follows we denote by $\sigma$ the one site shift map to the left: $(\sigma u)_i = u_{i+1}$. Then, the translation invariance of the dynamics (2.1) means that $F$ and $\sigma$ commute. These two commuting transformations on $\Omega$ represent the main new structure of this DS, called a two-parameter topological action.

We shall introduce a family of expansive cellular automata which has good "chaotic" properties. First introduce a distance between two configurations $x, y$ defined by:

$$ d_\alpha(x, y) = \alpha^{L(x,y)} $$

where $L(x, y) = \min\{|i| : x_i \neq y_i\}$ and $\alpha \in ]0, 1[$. For simplicity we can choose $\alpha = 1/2$. To give heuristic picture of the distance between two configurations, $x = (x_i)$ and $y = (y_i)$, we consider the nearest site the origin where $x$ and $y$ start to distinguish, then $L(x, y)$ is the distance between this site and the origin. In other words, $L(x, y) - 1$, is the radius of the greatest window around the origin where $x$ and $y$ coincide. A rule $f$ is called right permutative if the mapping $u_{r-l} \to f(\pi_0, \ldots, \pi_{r-l-1}, u_{r-l})$ is one-to-one for any fixed $(\pi_0, \ldots, \pi_{r-l-1})$. Left permutative rule
is defined similarly. The mathematical literature on these CA is huge (for some references, see e.g. in [8]). Let us remind some of their properties. Right or left permutative CA are surjective (they are generally noninvertible). From the measure theoretical point of view, these CA has as invariant measure the Bernoulli measure on $\Omega$. A rule is called bipermutative if it is both right and left permutative. For example, with $K = \{0, 1\}$, $r = 1$ the rule defined by:

$$f(u_{i-1}, u_i, u_{i+1}) = u_{i-1} \oplus u_{i+1}$$

where $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$ and $1 \oplus 1 = 0$ (the addition mod 2), is bipermutative. Bipermutative rules are expansive maps. That is a "chaotic" property meaning, in ordinary systems, that there is a positive constant $c$ such that any couple of distinct initial points, $(x, y)$, will diverge at some instant by an amount greater than $c$. Expansiveness is something like hyperbolicity in ordinary smooth dynamics. An expansive map is clearly sensitive to initial conditions. Recall that a map is sensitive to initial conditions if there is a constant $c$ such that in every vicinity of a point $x$, there is another point, $y$, whose orbits will diverge at some instant by an amount greater than $c$. Reminding the meaning of the distance, the expansivity property for a CA means that there is a fixed window around the origin such that any right or left far perturbation of a configuration will propagate to show up in this window around the origin. Thus, expansive cellular automata propagate perturbations at long ranges.

Another slightly weaker characteristic properties of chaos in ordinary systems are transitivity and the existence of dense set of periodic orbits which implies sensitivity to initial conditions. Note that expansiveness implies those chaotic properties. (see e.g.[15]). In cellular automata, we considered the complexity of the dynamics through the problem of the density of the set of space and time periodic travelling waves in a large class of CA: the one-sided permutative. However, in the class of one-sided permutative there are both transitive and non-transitive CA (see examples given by Coven [14]), ergodic and nonergodic ones with respect to the Bernoulli measure on $\Omega$ [17].

**Coupled maps lattice.**

Let $I$ be a metric space and let $d(\cdot, \cdot)$ be the corresponding distance. Consider the direct product $I^Z = \{(x = (\cdots, x_{i-1}, x_i, x_{i+1}, \cdots))\}$ which is the set of all bi-sided infinite sequences of real numbers $x_i \in \mathbb{Z}$. Let $F$ be a map from $I^Z$ into itself. This is called a Lattice dynamical system (LDS).

Some examples are Coupled Maps Lattice (CML) of the following type:

(i) Linearly coupled maps lattice, where $I$ is a compact interval and the map is given by

$$(Fu)_s = \sum_{n \in \mathbb{Z}} l_n f(u_{s-n}), \ s \in \mathbb{Z}, \quad (2.3)$$

There is also a huge literature on the physical, topological and ergodic properties of these systems.

(ii) Discrete versions of PDE’s of the evolutionary type,

$$(Fu)_s = f(u_s) + \epsilon(u_{s-1} - 2u_s + u_{s+1}), \ s \in \mathbb{Z}, \quad (2.4)$$

where $\epsilon \geq 0$ and the local map $f$ has a compact absorbing region.

### 3. Travelling waves

Let us start with a special class of travelling waves (TW) that are solutions of the form:

$u_i(t) = H(i + vt)$ for all $i, t$, where $v$ is a given integer called the velocity of the wave. That is: $u_i(t) = u_{i+vt}(0)$ for some integer $v$ and any $i \in \mathbb{Z}$, $t \geq 0$. Their initial conditions $u$ are characterized by the relation: $Fu = \sigma^v u$. 

Let us write, without loss of generality, the rule of a cellular automata in a symmetric way, with $l = -r < 0$. We have the following results [9]: All TW of velocity $|v| > r$ are space and time periodic (this is a result of the finiteness of the state space $\mathbb{K}$). If $f$ is left (resp. right) permutative, then the set of TW of velocity $v > r$ (resp. $v < -r$) is dense in the set of all configurations [13]. In particular any linear cellular automaton defined by $f(u_0, \ldots, u_{2r}) = \sum_{i=0}^{2r} \alpha_i u_i \mod (k)$, and such that $\alpha_0 \neq 0$ or $\alpha_{2r} \neq 0$, satisfies the above assumptions. The set of travelling waves with velocity of propagation $v > r$ is a countable set and the density property means that for any initial configuration and any window of radius $n$, there exists a TW $w$ which coincides with $u$ on this window. Thus, in the large class of one-sided permutative CA, any configuration has a travelling wave behavior at least for some time and in some window. It follows that, in this case, the set of all space periods and the set of all velocities of propagation are infinite (fig1).

We can also consider travelling waves with non integer velocity $v = m/k$, $m, k \in \mathbb{Z}$; they are solutions of the form:

$$u_i(t) = H(ki + mt)$$

(3.5)

It is easily seen that solutions of the form (3.5) satisfy to the equation

$$F^k u(t) = \sigma^m u(t)$$

for each $t \geq 0$. In other words, the initial condition $u$ is a fixed point of the operator $\sigma^{-m} F^k$. More generally, it is possible to represent all fixed points of such operator in the form (3.5).

Let $S$ be the unit circle, $S = \{x, \mod 1\}$. Set $S^\mathbb{Z} = \otimes_{i \in \mathbb{Z}} S_i$ where $S_i$ are copies of $S$ and $\mathbb{R}^\mathbb{Z} = \otimes_{i \in \mathbb{Z}} \mathbb{R}_i$ where $\mathbb{R}_i$ are copies of $\mathbb{R}$. Fix a number $Q > 1$ and introduce a Hilbert space $B = \{x \in \mathbb{R}^\mathbb{Z} | \|x\| < 1\}$, where $\|x\| = \sqrt{<x, y>}$, and the scalar product

$$<x, y> = \sum_{i=-\infty}^{\infty} \frac{x_i y_i}{Q^{|i|}}$$

(3.6)

Let $f : S \to S$ be an expanding $C^r$ differentiable map, $r > 1$, and denote also by $f : \mathbb{R} \to \mathbb{R}$ a lift of it. We assume that:

$$1 < \lambda_0 \leq |f'(x)| \leq \lambda_1 < \infty$$

(3.7)

Without loss of generality we may assume that $f(0) = 0$. Then consider the coupled circle expanding maps lattice $F : S^\mathbb{Z} \to S^\mathbb{Z}$ of the form

$$(Fx)_i = f(x_i) + \beta \sum_{j-i=-r}^{r} a_{j-i}(x_j), \mod 1$$

(3.8)

It is shown the following result [1]: For any coupled circle expanding maps lattice, there exists $\beta_0$ such that for any $\beta \leq \beta_0$ and any fixed positive integer $m$, the set of all travelling waves of the $(m,k)$-type, $k$ taking all positive integer values, is dense in the space of all configurations. A more general setting of this result is given in [2].

Here it is important to notice that this density result is obtained for CML through a subset of TW at the opposite of the one considered in the CA. For CML the density is obtained when the TW run along a range of rational velocities tending to zero (for, $m$ is kepted fixed and $k$ becoming large) while in CA the density is obtained when the velocities of the waves go to infinity).
4. Directional entropy and space time complexity

The topological entropy of an ordinary dynamical system is, roughly speaking, the measure of the exponential increase with time of the maximal number of distinct trajectories up to some precision. How to generalize this notion in the extended systems?

Let us remind it more precisely in ordinary systems: for a given mapping $f$ consider a "window" in the time axis, $W_n = \{1, 2, \ldots, n\}$, and consider the maximal number $N(\epsilon, W_n)$ of distinguished orbits up to $\epsilon$-precision, at some instant in this window (i.e. we say that $x, y$ have $(W_n, \epsilon)$-distinguished segment of trajectories if there is some $t \in W_n$, for which the distance $d(x(t), y(t)) > \epsilon$). If $N(\epsilon, W_n)$ increases exponentially with $n$, then the ordered limit:

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \ln(N(\epsilon, W_n)) = h$$

defines the **topological entropy of** $f$. To have some heuristic picture, let us say that for a given precision $\epsilon$, dividing the phase space into disks with radius $\epsilon$, we count the number of orbits that will separate in distinct disks at some instant of the window $W_n$. This number can increase very rapidly with $n$ if the map is instable, like in the expansive case. The limiting precision $\epsilon$, does not matter in this case, since after taking the limits $n \to \infty$, the number $\lim_{n \to \infty} \frac{1}{n} \ln(N(\epsilon, W_n))$ becomes, for $\epsilon$ small enough, independent of $\epsilon$.

For extended systems, to introduce a measure of the space-time complexity consider a "window" in the $\theta$-direction of the space-time plane (see fig.2), $W(m, T, \theta) = \{(x + y\cos\theta, y\sin\theta), -m < x < m, 0 < y < T\}$

For a Given LDS $F$, let $N(\epsilon, W(m, T, \theta))$ be the maximal number of $\epsilon$-distinguished configurations at some point $(s, t)$ in this window (i.e. we say that two configurations $x, y$
are \((W(m, T, \theta), \epsilon)-\)distinguished if there is a couple of numbers \((s, t) \in W(m, T, \theta)\) for which the distance \(d(x(s, t), y(s, t)) > \epsilon\), where \(x(s, t) = F^t\sigma^sx\). If, for a fixed \(\theta\), \(N(\epsilon, W(m, T, \theta))\) increases exponentially with \(T\), then the ordered limit:

\[
\lim_{\epsilon \to 0} \lim_{m \to \infty} \lim_{T \to \infty} \frac{1}{T} \ln(N(\epsilon, W(m, T, \theta))) = h(\theta)
\]

is called the directional entropy function in the direction \(\theta\). Here also, in the chaotic case the main limit is the first one with respect to \(T\).

However, on account of the finitness of the state space \(\mathbb{K}\), there is huge difference between cellular automata and extended systems. In fact, the more chaotic cellular automata (the bipermutative ones), are topologically equivalent to ordinary dynamical systems with two commuting transformations \((F, \sigma)\). Nevertheless, the calculation of the directional entropy is not always possible. The formulae of the directional entropy for bipermutative cellular automata has been given by Milnor [16], for the CA of the form (2.1), in the special case \(l < 0 < r\) and \(f\) bipermutative. Another expanded proof has been given in [7]. Denoting \(\cotg \theta_l = -l, \cotg \theta_r = -r\), the directional entropy is given by:

\[
h(\theta) = \begin{cases} 
(cos(\theta) + rsin(\theta)) \ln(q) & , \quad \theta \in [0, \theta_l] \\
(r - l)sin(\theta) \ln(q) & , \quad \theta \in [\theta_l, \theta_r] \\
(cos(\theta) + lsin(\theta)) \ln(q) & , \quad \theta \in [\theta_r, \pi]
\end{cases}
\]

(4.10)

It has been shown [7] that, running among all CA on the state space \(\Omega\) with fixed \(k, r, l\), these values are the maximal over all possible values of the topological directional entropy. Moreover, it is the right permutativity property which allows to reach the maximal value in the first sector and the left permutativity which allows to reach the maximal value in the third one. The above results have been also obtained for the measure theoretical directional entropy of CA as explained below.

If instead of cellular automata, we consider coupled maps lattice, then after the limit with respect to \(T\), the following quantity may behave as:

\[
\lim_{T \to \infty} \frac{1}{T} \ln(N(\epsilon, W(m, T, \theta))) \sim h(\theta)(2m + 1)
\]

This can be easily seen in the simple model of the uncoupled map lattice given by (2.4) with \(\epsilon = 0\). Therefore, the density of the directional entropy is defined as the space average:

\[
\lim_{\epsilon \to 0} \lim_{m \to \infty} \frac{1}{2m + 1} \lim_{T \to \infty} \frac{1}{T} \ln(N(\epsilon, W(m, T, \theta))) = \overline{h(\theta)}
\]

(4.11)

As expected, for uncoupled systems, this is the entropy of \(\overline{f}: \overline{h(\theta)} = h(f)\). However, this result holds also if one considers of a class of linearly weakly coupled maps lattice (2.3) where the map \(f\) is a Markov expanding map of the interval [3]. But, the independence of the density of directional entropy of \(\theta\) follows more generally from the homogeneity of the LDS (i.e. its invariance with respect to time translation) and it has been shown [4]) that the directional entropy coincides with the topological entropy of the \(\mathbb{Z}^2\)-action generated by \((F, \sigma)\). In the opposite case, the density may depend on the direction \(\theta\) and an example has been given in [4]).

Now suppose that we are given some probability measure \(\mu\) on the set of configurations \(\Omega\), which is \(F\)-invariant in the sense: \(\mu(F^{-1}(A)) = \mu(A)\) for any measurable set \(A\) of configurations. Then instead of the topological entropy we study the so-called Kolmogorov-Sinai entropy of \(F\).
Figure 3. The directional entropy curves for two bipermutative CA with $r=1$, $l=-1$ (solid line) and $l=-1$, $r=5$ (dashed line).

That is to say, we substitute the $\epsilon$-precision by some measurable partition $\mathcal{P} = \{A_1, \ldots, A_k\}$ of $\Omega$, the $(\epsilon, W_n)$-distinguished trajectories by the $(\mathcal{P}, n)$ itineraries

$$(i_0, i_1, \ldots, i_{n-1}) = \{x \in A_{i_0}, Fx \in A_{i_1}, \ldots, F^n x \in A_{i_{n-1}}\}$$

and the ln of the maximal number of $\epsilon$-distinguished trajectories $\ln(N(\epsilon, W_n))$ by:

$$\sum_{(i_0, i_1, \ldots, i_{n-1})} \mu(i_0, i_1, \ldots, i_{n-1}) \ln \mu(i_0, i_1, \ldots, i_{n-1})$$

We should nevertheless note that, for the cellular automata, in order to define the directional entropy we have to use a measure that is also invariant with respect to the space translation $\sigma$. 
and this structure of \((F, \sigma)\) with such invariant measure \(\mu\) on \(\Omega\) is called a CA-action on \(\Omega\) (e.g. the Bernoulli measure in many cases). In [10], it is shown that the entropy function given by (4.10) is also an upper bound of the directional entropy function for any measurable CA-action on \(\Omega\), reached in the first sector by right permutative CA, in the third sector by left permutative CA and in the second sector by bipermutative CA with \(l < 0 < r\).

A measure theoretical definition of the density of directional entropy in Lattice dynamical systems has been considered in [11] for LDS transformation \(F\) with given invariant measure. It is shown that if \(F\) and \(\sigma\) commute, then the directional entropy coincides with Conze entropy of the \(\mathbb{Z}^2\)-action generated by the commuting transformations \((F, \sigma)\). In the opposite case, the density may depend on the direction \(\theta\) and examples of \(\theta\)-dependent density are given.

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