STOCHASTIC ACCRETION AND THE VARIABILITY OF SUPERGIANT FAST X-RAY TRANSIENTS

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ABSTRACT

In this paper, we consider the variability of the luminosity of a compact object (CO) powered by the accretion of an extremely inhomogeneous (clumpy) stream of matter. The accretion of a single clump results in an X-ray flare; we adopt a simple model for the response of the CO to its arrival, and derive a stochastic differential equation (SDE) for the accretion-powered luminosity \( L(t) \). We set the SDE in the equivalent form of an equation for the flare luminosity distribution (FLD) and discuss its solution in the stationary case. We apply our formalism to the analysis of the FLDs of supergiant fast X-ray transients (SFXTs), a peculiar sub-class of high-mass X-ray binary (HMXB) systems. We compare our theoretical FLDs to the distributions observed in the SFXTs IGR J16479–4514, IGR J17544–2619, and XTE J1739–302. Despite its simplicity, our model agrees well with the observed distributions and allows us to predict some properties of the stellar wind. Finally, we discuss how our model may explain the difference between the broad FLDs of SFXTs and the much narrower FLDs of persistent HMXBs.

Key words: methods: analytical – methods: numerical – X-rays: binaries – X-rays: individual (IGR J16479–4514, IGR J17544–2619, XTE J1739–302)

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1. INTRODUCTION

In several astrophysical contexts, mass accretion onto a compact object (CO; a black hole, a neutron star (NS), or a white dwarf) cannot be considered as a continuous process. The accretion may be discrete: the mass does not flow as a continuous stream, but consists of a volley of clumps whose accretion on the CO results in a flaring activity observed in the X-ray/\( y \)-ray spectral window. Inhomogeneous flows occur in the Polar (a.k.a. AM Herculis) cataclysmic variables (e.g., Warner 1995) or in supergiant fast X-ray transients (SFXTs; Sguera et al. 2006; Negueruela et al. 2006; in’t Zand 2005). Understanding the properties of clumpy accretion on the super-massive black holes located at the cores of galaxy clusters is also relevant in the context of the “cold feedback model” (e.g., Pizzolato & Soker 2005).

The luminosity \( L(t) \) powered by the accretion on a CO results from the interplay of two factors: the properties of the accreting stream and the response of the CO to its arrival. The response process may be quite complex, but it is essentially deterministic. On the other hand, if the accretion occurs from a population of clumps with random masses arriving at random times, the accretion-driving term is a stochastic process. The accretion-powered luminosity \( L(t) \) is an irregular function of time, and it requires an adequate mathematical tool to deal with it. This is provided by the theory of stochastic differential equations (SDEs) which replaces the rules of ordinary calculus to be able to treat highly irregular functions (see e.g., Øksendal 2003; Gardiner 2009).

In this paper, we consider the variability of the X-ray luminosity of a CO powered by the accretion of a family of clumps with randomly distributed masses. We derive (Section 2) a simple SDE for the luminosity \( L(t) \) and illustrate some of the subtleties involved in handling these equations. A way to use this SDE is to compute a sample path of \( L(t) \), to be compared to an observed light curve. An equivalent approach (presented in Section 3) consists in associating with the SDE for \( L(t) \) an integro-differential equation (known as a “generalized Fokker–Planck equation” (GFPE)) for the probability function \( p(L, t) \) that the source has luminosity \( L \) at time \( t \). We discuss the properties of the distribution \( p \) and how it mirrors the features of the accreting stream. As a case study we apply our model in Section 4 to SFXTs, an interesting class of high-mass X-ray binaries (HMXBs) composed of an NS (or a black hole) accreting an extremely inhomogeneous wind blown by its massive companion. We discuss our findings and in particular how our model helps us to explain the difference between the flare luminosity distributions (FLDs) of SFXTs and those of the (much more common) persistent X-ray sources. We summarize in Section 5.

2. A SIMPLE STOCHASTIC ACCRETION MODEL

Analysis of the long-term variability of the X-ray light curve generated by inhomogeneous accretion requires some modeling of the response of the CO to the accretion process of a single clump. To this end, it is helpful to imagine the CO as a “black box” that emits X-rays in response to the accretion of mass from the surrounding environment. If \( \dot{M}_c \) is the mass capture rate and \( L \) is the luminosity produced by the CO, we postulate the linear response,

\[
L(t) = \int_{\mathbb{R}} dt' \ W(p, t-t') \dot{M}_c(t'),
\]

where the function \( W \) (dependent on the parameters \( p \)) describes the response of the CO.

The analytical shape of \( W \) may be exceedingly complex, since it must embody a number of physical processes: tidal effects, interaction of the accretion flow with the magnetic field of the CO, geometry and plasma instabilities of the accretion stream, radiation effects, and so on. Since this paper focuses on the basic properties of the accretion process, we steer clear of these details and demand for \( W \) just a few very basic properties: (1) \( W(t) \) is causal: it vanishes for \( t < 0 \), i.e., before the capture of a clump, and (2) it decays to zero for \( t \gg 0 \), i.e., long after the clump has
been captured. The simplest function with these properties is

\[
W(t) = \begin{cases} 
0 & t < 0 \\
\frac{GM}{R_{st}} \frac{e^{-t/\tau}}{\tau} & t \geq 0.
\end{cases}
\] (2)

This response gives the accretion-powered luminosity of a unit mass on a CO of mass \( M \), down to a “stopping radius” \( R_{st} \). If the CO is able to accrete the flow down to its radius \( R \) (or event horizon), then \( R_{st} = R \). In some cases, e.g., a rapidly spinning magnetized NS, the so-called propeller effect (see, e.g., Lipunov 1987; Illarionov & Sunyaev 1975; Davies et al. 1979; Davies & Pringle 1981), prevents the stream from flowing beyond the magnetosphere, and in this case \( R_{st} \) is the Alfvén radius \( R_A \). The timescale \( \tau \) represents the time taken by the CO to “process” the accretion stream. For example, if the stream has an appreciable angular momentum, it forms an accretion disk before falling on the CO, and the mass accretion rate on the CO decays as a power law on the viscous timescale:

\[
\tau \approx 7.5 \times 10^5 \text{ s} \alpha^{-4/5} \left( \frac{\dot{M}}{10^{-11} \text{ M}_\odot \text{ yr}^{-1}} \right)^{-3/10} \times \left( \frac{M_{ns}}{1.4 \text{ M}_\odot} \right)^{1/4} \left( \frac{R_{out}}{10^{10} \text{ cm}} \right)^{5/4}, \] (3)

(Equation (5.63) in Frank et al. 2002), where \( R_{out} \) is the outer radius and \( \alpha \leq 1 \) is the viscosity parameter (Shakura & Sunyaev 1973).

It is instructive to consider Equations (1) and (2) in the simple case of the accretion of a single clump of mass \( m \), with \( M_c(t) = m \delta(t) \) (where \( \delta \) is the Dirac delta function). With our choice (2) of the response \( W \), the luminosity is zero at \( t < 0 \), i.e., before the clumps arrive, and

\[
L = \frac{GM}{R_{st}} \frac{m}{\tau} e^{-t/\tau}
\] (4)

for \( t \geq 0 \). The luminosity has a sharp peak and then decays exponentially with the e-folding time \( \tau \). Plugging the ansatz (2) into Equation (1) we find

\[
\tau \frac{dL}{dt} = \frac{GM}{R_{st}} \dot{M}_c - L.
\] (5)

We introduce the mass accretion rate \( \dot{M} \) down to the stopping radius \( R_{st} \)

\[
L \equiv \frac{GM}{R_{st}} \dot{M}_c
\] (6)

so Equation (5) becomes

\[
\tau \frac{d\dot{M}}{dt} = \dot{M}_c - \dot{M}.
\] (7)

Since the function \( \dot{M}_c(t) \) is random, the function \( \dot{M}(t) \) is highly irregular and certainly not differentiable, so Equation (7) is to be interpreted as an SDE (see, e.g., Øksendal 2003) and not as an ordinary differential equation.

Thus far we have provided a simple model for the “response” of the CO to the accretion of a clump of matter. We now turn our attention to the stochastic driving term \( \dot{M}_c \), which embodies both the arrival rate of the clumps and their mass distribution.

Neglecting the finite size of the clumps, we model their arrival rate as a train of delta pulses

\[
\dot{M}_c(t) = \sum_{k=1}^{n(t)} m_k \delta(t - t_k),
\] (8)

where \( m_k \) is the mass of the clump accreted at time \( t_k \) and \( n(t) \) is a Poisson counting process, described by the probability

\[
P[n(t) = n] = \frac{e^{-\lambda t} (\lambda t)^n}{n!},
\] (9)

where \( \lambda \) is the clumps’ arrival rate. The masses \( \{m_k\} \) are distributed according to the probability distribution function \( \varphi(m) \); we assume that \( \varphi \) is stationary and that the random variables \( \{m_k\} \) and \( \{t_k\} \) are uncorrelated. The statistical properties of \( \dot{M}_c \) are readily derived:

\[
\langle \dot{M}_c \rangle = \lambda \langle m \rangle
\] (10a)

\[
\langle \dot{M}_c(t) \dot{M}_c(t') \rangle = \lambda \langle m^2 \rangle \delta(t-t').
\] (10b)

The properties (10) show that the Poisson process (8) is a white noise with non-zero mean.

From Equations (7), (8), and (10), it is possible to derive some properties of the random function \( \dot{M}(t) \). They not only provide a useful check against our more elaborate approach developed in the following sections, but also help us to point out some subtle key properties of SDEs.

Integrating Equation (7), we obtain the expression

\[
\tau d\dot{M} = \delta m(t) - \dot{M}(t) dt,
\] (11)

where

\[
\delta m(t) = \int_t^{t+dt} dt' \dot{M}_c(t').
\] (12)

With the aid of Equations (10) we derive

\[
\langle \delta m \rangle = \int_t^{t+dt} dt' \langle \dot{M}_c(t') \rangle = \lambda \langle m \rangle dt
\] (13a)

\[
\langle (\delta m)^2 \rangle = \int_t^{t+dt} dt' \int_t^{t+dt} dt'' \langle \dot{M}_c(t') \dot{M}_c(t'') \rangle
\]

\[
= \lambda \langle m^2 \rangle dt.
\] (13b)

From the first of these expressions we find the mean of Equation (11), i.e., the ordinary differential equation

\[
\tau \frac{d\langle M \rangle}{dt} = \lambda \langle m \rangle - \langle \dot{M} \rangle
\] (14)

(the order of the mean and the derivative may be exchanged, see, e.g., Reif 1965).

For \( t \gg \tau \) the mean mass accretion rate \( \langle M \rangle \) relaxes on the stationary value

\[
\langle M \rangle = \lambda \langle m \rangle.
\] (15)

The evaluation of the second moment is trickier, on account of the stochastic nature of the equation. We write

\[
\tau d(M^2) = \tau [(M + dM)^2 - M^2] = \tau [2M dM + (dM)^2],
\] (16)
where in the expansion we must keep the second-order term \((dM)^2\) for reasons that will soon be clear. Plugging Equation (11) into this expression we get

\[
\tau d(M^2) = 2M(\delta m - M(t)\, dt) + (\delta m)^2 + O(dt),
\]

and taking the mean using Equations (13) we find the ordinary differential equation

\[
\tau \frac{d\langle M^2 \rangle}{dt} = 2\lambda^2 \langle m^2 \rangle - 2 \langle M^2 \rangle + \frac{\lambda}{\tau} \langle m^2 \rangle.
\]

For \(t \gg \tau\) the second moment \(\langle M^2 \rangle\) relaxes on the stationary value

\[
\langle M^2 \rangle = \frac{\lambda^2}{2\tau} \langle m^2 \rangle,
\]

corresponding to the variance

\[
\sigma^2_M = \langle M^2 \rangle - \langle M \rangle^2 = \frac{\lambda}{2\tau} \langle m^2 \rangle.
\]

Some remarks are in order. First, the stochastic nature of the function \(M(t)\) makes \(M(t)\) an extremely irregular function of time. On account of this irregular behavior, the rules of ordinary calculus are not applicable to \(M(t)\). A consequence is that the second-order differential \((dM)^2\) is proportional to \(dt\), and therefore it cannot be neglected in the formal manipulations, as we have done in Equation (16). In order to work with such irregular functions a new kind of differential calculus (known as “Itô calculus”) has been developed as a cornerstone of the theory of SDEs (see, e.g., Øksendal 2003). Some knowledge of SDEs is therefore essential to master the modeling of random processes, such as that studied in this paper.

The moments \(\langle M^p \rangle\) exist only if the correspondent moments \(\langle m^p \rangle\) of the clumps’ distribution are defined. This is not always the case, e.g., when the distribution \(\varphi\) has a fat tail. In order to deal with this possibility, it is necessary to extend the simple approach presented in this section. We shall explain how to do this in Section 2.1.

The mean accretion rate \((15)\) depends on the clumps’ arrival rate \(\lambda\), but not on the relaxation time \(\tau\), which is an intrinsic property of the accretor. The variance \((20)\) features \(\tau\) with an inverse power: a long \(\tau\) corresponds to a narrow dispersion of the observed values of \(M\) around the mean. Indeed, if \(\tau\) is long, several clumps may be accreted before the accretor is able to respond, and an observer cannot distinguish between several elementary accretion processes, which are perceived as a single accretion of a large clump. Clearly, this has the effect of reducing the dispersion of the observed distribution.

2.1. Direct SDE versus the Fokker–Planck Approach

In principle, Equation (7) can be solved numerically. Several computing techniques have been developed to tackle the numerical solution of SDEs (see, e.g., Kloeden & Platen 1999). A sample of the population of the clumps’ masses and arrival times is extracted from the distributions \(\varphi\) and \(P[n(t)]\) (defined by Equation (9)). A sample path of the process \(M(t)\) is computed numerically, and \(M(t)\) is converted to the accretion luminosity \(L(t)\) via Equation (6). A sample path \(L_1(t)\) may be very different from another path \(L_2(t)\), since the clumps’ masses and arrival times extracted from the distributions \(\varphi\) and \(P[n(t)]\) used to compute \(L_1(t)\) numerically may be very different from those extracted to compute \(L_2(t)\). For this reason, it is meaningless to compare a single sample path of \(L(t)\) with a real light curve. Any comparison of a stochastic model with the observations will involve some statistics on a sample of the computed paths and light curves.

We shall adopt an equivalent approach to compare a theoretical stochastic light curve with a real light curve. It is possible to associate with Equation (7) a new equation (called a GFPE) for the probability \(p(L, t)\) that the source has luminosity \(L\) at time \(t\). The distribution \(p\) is directly comparable to the histogram of the flares’ luminosities observed in a real source over a long time span.

The approach based on the luminosity distribution \(p(L, t)\) allows us to deal quite straightforwardly with a complication we have overlooked thus far. The luminosity distribution is defined as a function of the X-ray luminosity \(L\), not of the mass accretion rate \(M\). The general relation between the luminosity distribution \(p(L, t)\) and the mass accretion rate distribution \(P(M, t)\) is

\[
p(L, t) = \left| \frac{dM}{dL} \right| P(M, t).
\]

If all the mass can reach the surface of the NS (located at radius \(R\)), then \(R_m = R\). In this case the distributions \(p(L)\) and \(P(M)\) are simply proportional to each other:

\[
p(L, t) = \frac{R}{GM} P(M, t).
\]

In some systems, however, the accreting stream cannot reach the surface: this is typically the case for a magnetized rapidly spinning NS in which the matter is captured by the star’s gravitational field, but is prevented from reaching its surface by a “propeller” barrier. At a very basic level, the “propeller” effect may be described as follows. The matter inside the magnetosphere (located at the Alfvén radius \(R_A\)) is forced to rotate with the same angular velocity \(\omega_{\text{ms}}\) as the NS. On the other hand, the angular velocity of the mass stream outside the magnetosphere (\(r > R_A\)) is approximately Keplerian, \(\omega \simeq \omega_K(r)\). If \(\omega_{\text{ms}} > \omega_K(R_A)\), no accretion is possible, and the matter is stopped at the Alfvén radius. The radius \(R_A\) depends on the geometry of the accretion flow. If the NS accretes a clumpy wind in a detached binary system, the accretion is approximately spherical, and there are two possible regimes. The NS crosses a wind blown by the companion star with a velocity \(v_w\). This wind is focused by the NS’s own gravitational field within the capture radius

\[
R_G = \frac{2GM_{\text{ms}}}{v_w^2} = 3.74 \times 10^{10} \text{ cm} \left( \frac{M_{\text{ms}}}{1.4M_\odot} \right) \left( \frac{v_w}{10^3 \text{ km s}^{-1}} \right)^{-2}.
\]

If the magnetic field of the NS is “weak,” then the wind is first focused by the NS’s gravitational field, and only then is it affected by the NS’s magnetic force. In this case, the correct expression of the Alfvén radius is (e.g., Lipunov 1987)

\[
R_A = \left( \frac{\mu^4}{2GM_{\text{ms}}M^2} \right)^{1/7} \text{ if } R_A < R_G,
\]

where \(\mu\) is the magnetic moment of the NS. On the other hand, if the magnetic field is strong, then the wind feels the NS’s magnetic field before its gravity, and in this regime

\[
R_A = \left( \frac{4\mu^2G^2M_{\text{ms}}^2}{Mv_w^5} \right)^{1/6} \text{ if } R_G < R_A.
\]
The critical mass accretion rate \( \dot{M}_0 \) at which the propeller barrier sets in can be derived from Equation (23) with \( R_A \) given by either Equation (25) or Equation (26). If \( M < \dot{M}_0 \) the ram pressure exerted by the flow is unable to overcome the magnetic pressure at the magnetospheric radius, and the flow is stopped at the Alfvén radius \( R_A \). The stopping radius is then

\[
R_{st} = \begin{cases} 
R & \dot{M} > \dot{M}_0 \\
R_A & \dot{M} < \dot{M}_0 .
\end{cases}
\] (27)

Plugging this stepwise function \( R_{st}(\dot{M}) \) into Equation (21) we find

\[
p(L, t) = \begin{cases} 
\frac{R}{R_A} P(\dot{M}, t) & L < L_1 \\
0 & L_1 < L < L_{\infty} , \\
\frac{R_A}{R} P(\dot{M}, t) & L > L_{\infty} ,
\end{cases} \] (28)

where \( L_1 = GM \dot{M}_1 R_A \) and \( L_{\infty} = GM \dot{M}_{\infty} R_A \).

For typical parameters of an NS (\( \mu \approx 10^{30} \text{ G} \)) and a stellar wind (\( v_{sw} \approx 10^3 \text{ km s}^{-1} \)) the luminosity \( L_1 \) is below \( \approx 10^{32} \text{ erg s}^{-1} \), which is barely observable. Therefore, we neglect the tail below \( L_1 \) and approximate Equation (28) by

\[
p(L, t) \approx \begin{cases} 
0 & \dot{M} < \dot{M}_0 \\
\frac{R_A}{R} P(\dot{M}, t) & \dot{M} > \dot{M}_0 .
\end{cases} \] (29)

In the presence of an accretion barrier, the luminosity distribution \( p(L) \) has a sharp lower cutoff at the luminosity \( L_{\infty} \).

Whether an accretion barrier is present or not, \( p \) must be computed from the mass accretion distribution \( P(\dot{M}, t) \). If no accretion barrier is present, \( p \) is given by \( P(\dot{M}, t) \) and Equation (22), otherwise it is computed from \( P(\dot{M}, t) \) and Equation (29).

In the next section we derive and discuss the GFPE for \( P(\dot{M}, t) \) associated with the stochastic differential Equation (7).

3. THE GENERALIZED FOKKER–PLANCK EQUATION

The method to derive the GFPE from an SDE is described by Denisov et al. (2009). We leave the technical details of the derivation to the Appendix, and reproduce only the results here. The GFPE associated with Equation (7) reads

\[
\frac{\tau}{\partial \dot{P}} = \frac{\partial (\dot{M} \dot{P})}{\partial \dot{M}} - \rho P(\dot{M}, t) 
+ \rho \int_0^{\tau M} dm \varphi(m) P(\dot{M} - m/\tau, t), \] (30a)

where

\[
\rho = \lambda \tau \] (30b)

is the accretion parameter. The solution of this equation is unique once we impose the normalization

\[
\int_0^{\infty} d\dot{M} P(\dot{M}, t) = 1 \] (31)

and a suitable initial condition.

A dimensional analysis (e.g., Barenblatt 1996) helps us to write Equation (30a) in a non-dimensional form. Suppose that \( \varphi \) depends on the characteristic mass \( m_0 \) as the only dimensional parameter. The only dimensional parameters in the distribution \( P \) are then \( M, t, \tau, \) and \( m_0 \). Note that the clumps’ arrival rate \( \lambda \) appears in the equation only in the dimensionless combination \( \rho = \lambda \tau \), and therefore it does not appear as a separate variable. The only way to combine the governing dimensional variables with dimensionless variables (on which \( P \) must depend) is

\[
x = M \tau / m_0 \quad s = t / \tau, \] (32)

so

\[
P = P \left( \frac{\dot{M} \tau}{m_0}, t / \tau \right) \] (33)

(we do not indicate the dependence on other possible dimensionless parameters). Plugging this expression into Equation (30a) we retrieve

\[
\frac{\partial \tilde{P}}{\partial s} = x \frac{\partial \tilde{P}}{\partial x} + (1 - \rho) \tilde{P} + \rho \int_0^x dy \tilde{\varphi}(x - y) \tilde{P}(y, s), \] (34)

where \( \tilde{P} = P (m_0 / \tau) \) and \( \tilde{\varphi} = \varphi m_0 \) are the scaled distributions \( P \) and \( \varphi \). In the following we refer to the dimensionless Equation (34), omitting the tildes on the scaled quantities.

The problem (30) depends on some initial condition. After a long time \( t \gg \tau \), the compact star has gathered a large sample of clumps, and the probability \( P \) is expected to relax on an equilibrium, time-independent configuration. In the remainder of this paper we assume that such an equilibrium has been achieved, and we work with the stationary distribution, which is a solution of

\[
x \frac{dP}{dx} = (\rho - 1) P - \rho \int_0^x dy \varphi(x - y) P(y). \] (35a)

This is a Volterra integro-differential equation of the second kind, whose solution is unique once we impose the normalization

\[
\int_0^{\infty} dx P(x) = 1. \] (35b)

The solution to the problem (35) is particularly convenient via a Fourier (or Laplace) transform. This not only allows a numerical solution, but also yields some exact results on the moments of the distribution.

3.1. Solutions to the Generalized Fokker–Planck Equation

In this section we derive a solution to the GFPE. Although an exact solution does not exist for a general form of \( \varphi \), the Fourier transform of \( P \) is analytical. This allows us to compute all the moments of \( P \) as a function of the moments of \( \varphi \) (when they exist), and makes it possible to compute \( P(x) \) numerically by evaluating a Fourier integral.

The logarithm of the Fourier transform of Equation (35a) reads

\[
\ln P_k = -\rho \int_0^k \frac{dk'}{k'} (1 - \varphi_{k'}). \] (36)

where \( \varphi_k \) is the Fourier transform of \( \varphi \), and where we have introduced the normalization (35b) as \( P_{k=0} = 0 \). Plugging \( \varphi_k \) into the last equation,

\[
\ln P_k = \rho \int_0^{\infty} dy \varphi(y) \text{Ein}(i k y), \] (37)

with

\[
\text{Ein}(\zeta) = \int_0^\zeta ds \frac{1 - e^{-s}}{s} \] (38)
being the entire exponential integral (see, e.g., Abramowitz & Stegun 1972). The inverse Fourier transform,

$$P(x) = \int_{\mathbb{R}} \frac{dk}{2\pi} \exp[-ikx + ln P_k],$$  

(39)

is rather involved and cannot be further simplified for general $\varphi$. Yet it is possible to derive some exact results. The function $\ln P_k$ is the cumulant distribution function (CDF) of $P(x)$. We Taylor-expand the entire exponential integral in Equation (37) and compare the resulting expression with the definition of the CDF:

$$\ln P_k \equiv \sum_{r=1}^{\infty} \frac{(i k)^r \langle x_r \rangle}{r!},$$  

(40)

to obtain the cumulants

$$\langle x_r \rangle = \varphi \langle x^r \rangle/r,$$

(41)
or, in dimensional units,

$$\langle \langle \dot{x} \rangle \rangle = \frac{\varphi}{r} \langle x^r \rangle.$$  

(42)

From the cumulants it is easy to retrieve the first moments of the distribution $P$ in dimensional variables

$$\langle \dot{M} \rangle = \lambda \langle m \rangle,$$  

(43a)

$$\sigma_M^2 = \frac{\lambda}{2\tau} \langle m^2 \rangle,$$  

(43b)

$$\gamma_1 = \frac{2\sqrt{2}}{3\rho^{1/2}} \frac{\langle m^3 \rangle}{\langle m^2 \rangle^{3/2}},$$  

(43c)

$$\gamma_2 = \frac{1}{\rho} \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2},$$  

(43d)

where $\langle m \rangle$, $\sigma_M^2$, $\gamma_1$, and $\gamma_2$ are, respectively, mean, variance, skewness, and excess kurtosis of $P$ (see, e.g., Section 14.1 of Press et al. 2007 for the definitions).

The mean and variance coincide with the values worked out in Section 2 from our simple analysis of Equation (7). The higher-order moments provide further qualitative detail on the shape of the accretion distribution function $P$. The skewness $\gamma_1$ is always positive, meaning that $P$ has a long tail for large $x$. The kurtosis $\gamma_2$ is always positive as well, i.e., $P(x)$ is leptokurtic: it has a sharp peak and a heavy tail for large $x$. If the terms $\langle m^n \rangle$ are finite, the standardized moment of order $n$ of $P$ scales as $\rho^{1-n/2}$.

Computation of the Fourier integral (39) cannot be performed analytically for any $\varphi$, but it can be attempted numerically. This will be done later for some choices of the distribution $\varphi$.

If $\varphi(m)$ is bounded for $m \to 0$, the asymptotic behavior of $P$ for $x \ll 1$ is independent of $\varphi$. For small $x$ the integral in Equation (35a) is negligible, and

$$P(x) \sim x^{\rho-1} \quad \text{for} \quad x \ll 1.$$  

(44)

For small values of $x$, $P$ is suppressed if $\rho \gg 1$, and diverges for $\rho \ll 1$. This behavior has a clear physical explanation. For simplicity, we assume that there is no accretion barrier, so $P$ may be read as a luminosity distribution. Consider the accretion of two small clumps, the second one hitting the system a time $\lambda^{-1}$ after the first one. The inequality $\rho < 1$ means that the relaxation time $\tau$ is shorter than $\lambda^{-1}$; the CO has time to process the first clump before the arrival of the second one, and their accretion triggers two successive flares. The accretion of each clump is “recorded” in the low-luminosity tail, and therefore the value of $P$ may be high. On the other hand, if $\rho \gg 1$, the CO cannot process the first clump before the arrival of the second one. The accretion of two closely separated clumps cannot trigger two different low-luminosity flares. Such accretion episodes are “recorded” as the accretion of a single larger clump, and the low-luminosity tail of $P$ is depressed by this effect, which for convenience will be referred to as “$\rho$ suppression.”

### 3.2. A Dirac Delta Mass Clump Distribution

Before applying the present model to a concrete astrophysical problem, it is useful to sketch the properties of $P$ in the relatively simple case in which all the clumps have the same mass. The appropriate mass distribution is then (in dimensionless units)

$$\varphi(x) = \delta(x - 1),$$  

(45)

where $\delta$ is the Dirac delta function. The generalized Fokker–Planck Equation (35a) reduces to the delay differential equation

$$\frac{dP}{dx} = (\rho - 1) P - \dot{\vartheta}(x - 1) P(x - 1),$$  

(46)

where $\dot{\vartheta}(u)$ is the Heaviside step function, $\dot{\vartheta} = 0$ for $u < 0$ and $\dot{\vartheta} = 1$ for $u > 0$. Equation (46) can be easily solved numerically. Some plots of $P(x)$ are shown in Figure 1 for several values of the accretion parameter $\rho$.

The mean of $P$ is $\langle x \rangle = \rho$ and the standard deviation is $\sigma = \sqrt{\rho/2}$. On account of the lack of massive clumps in the distribution (45), the distribution $P$ does not display a significant tail. If $\rho > 1$, $P$ is also suppressed for $x \ll 1$. These effects make the distribution $P$ quite narrow around its mean. The limit distribution solution of (46) will be useful later as a benchmark for more complex cases.

### 3.3. A Log-normal Clump Mass Distribution

We now explore in some detail the solution of the GFPE (30a) with a more general distribution of the clumps’ masses. For our purposes, it is particularly convenient to adopt the log-normal distribution

$$\varphi(x) = \frac{1}{x \sigma \sqrt{2\pi}} \exp\left[\frac{-\ln^2 x}{2\sigma^2}\right],$$  

(47)

defined for any $x > 0$. This choice has several advantages. First, the log-normal is extremely flexible: by tuning the shape parameter $\sigma$, it may mimic distributions as different as a Dirac delta (for $\sigma \ll 1$), or a power law (for $\sigma \gg 1$). The distribution (47) may be written (up to a constant factor)

$$\varphi \sim x^{-\zeta},$$  

(48)

where $\zeta = 1 + \ln x/2\sigma^2$. The exponent $\zeta$ is a slowly varying function of the range $x$, and for $x \ll \exp\sqrt{x/2}$ the log-normal $\varphi$ closely resembles a power law with index $\zeta \gtrsim 1$. All the moments of a log-normal are defined, being

$$\langle x^n \rangle = \exp(n^2 \sigma^2/2).$$  

(48)
Equations (43) and (48) give the moments of the distribution $P$

$$\langle \dot{M} \rangle = \lambda m_0 e^{\sigma^2/2}$$

(49a)

$$\sigma^2 = \frac{\lambda}{2 \tau} m_0^2 e^{2 \sigma^2}$$

(49b)

$$\gamma_1 = \frac{2 \sqrt{2}}{3 \rho^{1/2}} e^{3 \sigma^2/2}$$

(49c)

$$\gamma_2 = e^{4 \sigma^2}/\rho.$$  

(49d)

It is clear that the moments of $P$ are defined for any order $n$, but their magnitude rapidly increases with $\sigma > 1$, making $P$ heavy tailed.

The Fourier integral (39) cannot be computed exactly if $\varphi$ is a log-normal; it must be evaluated numerically. The numerical calculation of Fourier integrals poses a well-known problem, widely discussed in the mathematical literature. A very convenient numerical method is a variant of the double exponential quadrature (see, e.g., Mori & Sugihara 2001) suggested by Ooura & Mori (1999).

In Figure 2 we show some theoretical curves computed from a log-normal $\varphi$ for several values of $\sigma$ and $\rho$. If $\rho$ is fixed (top panel), as $\sigma$ increases, the distribution becomes increasingly broad, stretching its tail to high values of $x$. The peak of $P$ moves to the right with increasing $\sigma$, and its height lowers to preserve the normalization. For moderately large values of $\sigma$, the interval over which $P$ significantly differs from zero spans several orders of magnitude.

For a fixed $\sigma$, on the other hand (bottom panel), for increasing values of $\rho$ the distribution $P$ is suppressed for low $x$. This general property of $P$ has already been discussed at the end of Section 3.1.

4. APPLICATION TO SUPERGIANT FAST X-RAY TRANSIENTS

We apply the formalism developed in this paper to the analysis of the FLDs observed in SFXTs. We first outline the main properties of this class of objects and then analyze them in the framework of stochastic accretion.

4.1. Presentation of SFXTs

SFXTs are a class of rare transient X-ray sources (about 10 members are known to date), associated with OB supergiant stars (see Sidoli 2011 for a recent review), making them a subclass of the HMXBs.

The short and bright X-ray emission from several members of this class was discovered by the INTEGRAL satellite while monitoring observations of the Galactic plane at hard X-ray energies (Sguera et al. 2006; Negueruela et al. 2006). About half of them are X-ray pulsars and the mass flow from the massive donor to the compact star (usually assumed to be an NS in all members of the class) occurs via a stellar wind.

SFXTs spend most of their time in a relatively low-level brightness state (below $L_X \sim 10^{34} \text{ erg s}^{-1}$; Sidoli et al. 2008), reaching $L_X \sim 10^{32} \text{ erg s}^{-1}$ in quiescence, but show an occasional "outbursting" activity for a few days when the mean X-ray luminosity is much higher than usual, and is punctuated by a sequence of short flares (from a few minutes to a few hours long) peaking at X-ray luminosities of $10^{36}$–$10^{37} \text{ erg s}^{-1}$ (e.g., Romano et al. 2007; Rampy et al. 2009). Their X-ray transient activity over such a wide dynamic range (from two to five orders of magnitude) is puzzling, since the components of these binary systems seem very similar to persistently accreting classical X-ray pulsars (e.g., Vela X-1). Two kinds of explanation have been proposed, although neither is as yet able to completely explain the whole phenomenology of all the members of the class.

In the first type of model, the CO accretes mass from an extremely inhomogeneous "clumpy" stellar wind (in’t Zand 2005), which is believed to be present in massive OB stars (see, e.g., the review of Puls et al. 2008). In a few cases, a preferential plane for the outflowing clumpy wind is suggested (Sidoli et al. 2007; Drave et al. 2010) while a spherically symmetric morphology is usually assumed (Negueruela et al. 2008). It is still unclear whether the flow is able to form a (possibly transient) accretion disk (see, e.g., Ducci et al. 2010).
In this section, we interpret the observed luminosity distributions plotted in Figure 3 in the light of the stochastic setup in the previous sections. Some of the questions we address are the following.

1. The SFXTs’ luminosity distribution drops quite abruptly below $L_X \simeq 10^{33} \text{–} 10^{34} \text{ erg s}^{-1}$. Provided that this is not a selection effect in our observations, what is the origin of this feature?

2. What properties of the accretion stream and accretion process can we infer by comparing our model with the observations? How do the properties of the stellar wind we deduce compare with the existing literature?

3. Can we explain the difference between the wide dynamic range of luminosities observed in SFXTs and the much

---

Figure 2. Two samples of theoretical curves computed from the log-normal distribution (47). In the top panel, $\rho$ is fixed at $\rho = 10$ and the curves refer to different values of $\sigma$. In the bottom panel, $\sigma$ has been fixed at $\sigma = 1$ and the curves refer to different values of $\rho$.
narrower one seen in persistent sources, such as, e.g., Vela X-1?

Our comparison of the model with the observed FLDs is not intended to be a “best fitting” procedure, for several reasons. First of all, our model (2) for the response of the accretor may be somewhat oversimplified. The theoretical luminosity distribution $P$ computed from it, therefore, is not expected to provide an accurate model of the SFXT FLD observed in reality. Second, only the FLDs of three sources are available for our analysis, probably too few to draw definite conclusions. Third, not all the flares may have been spotted during the observation campaign, and the FLDs we work with might not describe a fair sample of all the flares. Finally, the calculation of a model for given values of $\rho$ and $\sigma$ is quite time consuming, and a quantitative determination of those parameters via a best-fitting procedure is not viable at the moment.

When we compare an observed FLD with its theoretical model, we aim to get a hint at the properties of the accretion process (described by the parameter $\rho$) and the properties of the family of clumps (described by the log-normal distribution (47)). A more physically motivated response is clearly necessary in order to make more quantitatively definite statements.

Despite its all too obvious limitations, we believe that our model has intrinsic value. Its qualitative predictions are expected to hold even for a more sophisticated model; in addition, it points out the fundamental properties of stochastic accretion that can be easily obscured by rather opaque mathematical complications that we leave to a forthcoming paper. That said, we are now ready to compare our theoretical predictions with the observed FLDs in three SFXTs: IGR J16479 − 4514, IGR J17544 − 2619, and XTE J1739 − 302.

We aim to determine the best-fitting parameters of our model from the FLDs of a given SFXT. In general, the free parameters of the model are the accretion parameter $\rho$, the log-normal shape factor $\sigma$, the ratio $m_0/\tau$, and the cutoff mass accretion rate $M_\infty$. Due to the action of an accretion barrier (see Equation (29)), in order to keep things as simple as possible, in the following analysis we assume that no accretion barrier is present (i.e., $M_\infty = 0$), and the model is completely defined by the three parameters $\rho$, $\sigma$, and $m_0/\tau$. What is the best way to determine these parameters from the data of an observation campaign of a given SFXT? As we shall see, our choice is rather limited.

One might think, for instance, to determine the parameters by comparing the mean, variance, skewness, and excess kurtosis computed from our model (see Equation (43)) with those observed, provided we are able to convert the mass accretion rate $M$ to an X-ray luminosity. If there is no accretion barrier, the modeled luminosity distribution is simply $p(L) = P(x)/L_s$, where $P(x)$ is the dimensionless distribution computed from the GFPE and

$$L_s = \frac{GM_{\text{ns}} m_0}{R_{\text{ns}} \tau}, \quad (50)$$

where $M_{\text{ns}} \sim 1.4 M_\odot$ and $R_{\text{ns}} \sim 10$ km are the typical mass and radius of an NS. A direct comparison of the moments of the theoretical and the computed distributions, however, is not viable. The use of high-order momenta such as the skewness or the kurtosis of a heavy tailed distribution (like our $P(x)$) is not recommended for any statistical analysis (e.g., Press et al. 2007). The problem is that these moments are not robust, i.e., they are too sensitive to the outliers. Any small change in the number of the observed high-luminosity flares would alter the values of these moments considerably, rendering them of little use for any statistical purpose.

Since we cannot use the moments, we are forced to determine the parameters by comparing the full shapes of the observed and theoretical distributions. The data from Romano et al. (2011) available to our analysis are already binned in luminosity intervals: we know the fraction $p^{(\text{obs})}$ of the flares observed in the luminosity interval $[L_i, L_{i+1}]$, but we do not have access to the luminosity of each single flare. This prevents us from using any variant of the Kolmogorov–Smirnov test to compare the theoretical and observed flares’ distributions.

For these reasons, we must resort to a least-squares method to determine the parameters $\rho$, $\sigma$, and $m_0/\tau$. Suppose for the moment that $\rho$ and $\sigma$ are known, and the only parameter to be determined is $m_0/\tau$. Then, one can use a least-squares method to determine $m_0/\tau$ by comparing the observed distribution $p^{(\text{obs})}$ with the theoretical distribution $P$, i.e.,

$$\sum_i \left( p^{(\text{obs})}_i - P(L_i) \right)^2$$

This method is not in general adequate, since the theoretical distribution $P$ is not necessarily a good fit to the observed distribution $p^{(\text{obs})}$, and the least-squares method is not in general satisfactory in the presence of heavy tails (e.g., Press et al. 2007). A more sophisticated method is needed, and we use a least-squares method to determine $m_0/\tau$, provided that $\rho$ and $\sigma$ are known.

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Figure 3. Distribution of the X-ray luminosity of three SFXTs we calculated from the X-ray count rate distribution observed with Swift, over a two-year campaign. Data have been taken from Romano et al. (2011).

(A color version of this figure is available in the online journal.)

1 Maybe the response (2) is not always bad as a result. The SFXT IGR J16479−4514, for instance, is known to exhibit flares that, after a fast rise, decay exponentially (Sguera et al. 2005; Ducci et al. 2010). Sguera et al. (2005) were able to fit the light curve of a flare with sufficient statistics following an exponential law with an e-folding time $\tau = 15 \pm 9$ minutes.
determined is \( m_0 / \tau \). For a (tentative) value of \( m_0 / \tau \) we compute the scale luminosity (50) and the expected fraction \( P^{(\text{cal})}_i \) of flares within the luminosity interval \([L_i, L_{i+1}]\). We compute the chi-squared value

\[
\chi^2 = \sum_i \left[ \frac{P^{(\text{obs})}_i - P^{(\text{cal})}_i}{\sigma_i^2} \right]^2, \quad (51)
\]

where we assume the errors on the observed fractions \( \sigma_i^2 \sim P^{(\text{cal})}_i \). Finally, we find the value of \( m_0 / \tau \) that minimizes \( \chi^2 \). A minimization procedure to determine all the parameters \( \rho, \sigma, \) and \( m_0 / \tau \) would be very time consuming, and therefore is not viable at the moment. For this reason we have built a grid of values of \( \rho \) and \( \sigma \), and for each couple \((\rho, \sigma)\) we have computed the correspondent \( P(\lambda) \) and the value of \( m_0 / \tau \) that minimizes the \( \chi^2 \) for that \( P \). Finally, we chose as our best estimates for \( \rho, \sigma, \) and \( m_0 / \tau \) values that return the smallest \( \chi^2 \). On account of the (expected) poor quality of the fits, we do not report the values of the \( \chi^2 \) statistics, but only discuss our results below.

Figure 4 shows the FLDs for our three SFXTs superposed to their “best-fitting” models. On account of the asymptotic behavior (44) of the theoretical luminosity distribution, a glance at Figure 3 hints that in all cases it must be \( \rho > 1 \) to match the observed FLDs. In addition, the observed high dynamical ranges of the flares’ luminosities require a heavy tailed mass distribution of clumps, i.e., a relatively high log-normal shape parameter \( \sigma \).

IGR J17544–2619 is the source with the lowest dynamic range, since it displays flares ranging in the luminosity interval between \(3.9 \times 10^{33} \) erg s\(^{-1}\) and \(2.7 \times 10^{36} \) erg s\(^{-1}\). There is no apparent low-luminosity cutoff (which might be interpreted as the effect of an accretion barrier). The FLD of this source can be reproduced quite nicely by a model with \( \rho = 8, \sigma = 4, \) and \( m_0 / \tau = 1.0 \times 10^{12} \) g s\(^{-1}\).

The source IGR J16479–4514 has the highest dynamic range of all the considered sources, with flares’ luminosities ranging from \(1.4 \times 10^{34} \) erg s\(^{-1}\) to \(3.5 \times 10^{37} \) erg s\(^{-1}\). In this case the best value of the log-normal shape is \( \sigma = 5 \) (slightly larger than the value found for IGR J17544–2619). The best estimate of the accretion parameter is \( \rho = 10 \), and even in this case it is probably not necessary to invoke an accretion barrier to explain the left tail of the observed FLD. The last parameter is \( m_0 / \tau = 5.5 \times 10^{11} \) g s\(^{-1}\).

The flares of XTE J1739–302 show luminosities between \(3.3 \times 10^{33} \) erg s\(^{-1}\) and \(5.3 \times 10^{36} \) erg s\(^{-1}\). Its steep “wall” that cuts off \( S(L) \) below \( L = 2.1 \times 10^{33} \) erg s\(^{-1}\) is difficult to reproduce. Following our discussion of the “\( \rho \) suppression” at the end of Section 3.1, one might expect that a very high value of \( \rho \) is necessary to cut off abruptly the low tail of the luminosity distribution. Indeed, our best fitting model for this source gives \( \rho = 40, \sigma = 8, \) and \( m_0 / \tau = 1.0 \times 10^{3} \) g s\(^{-1}\). We note, however, that the space of parameters characterized by high values of \( \rho \) and \( \sigma \) is quite difficult to explore with our method. For this reason, and also on account of the relatively poor quality of the “best fit” shown at the bottom panel of Figure 4, we cannot be certain that the \( \rho \) suppression is winning over the “propeller barrier” process discussed in Section 2.1.

To resume, we may reproduce at least the gross properties of the sources’ FLDs, with moderate values of the accretion parameter \( \rho \) and a relatively high-shape parameter \( \sigma \) of the clumps’ log-normal. At least two of our sources do not require any accretion barrier to explain their low-luminosity tail, which may be justified by the “\( \rho \) suppression.” The source XTE J1739–302 is more problematic, and here a propeller barrier is not excluded.

Our model has three degrees of freedom: \( m_0 / \tau, \sigma, \) and the accretion parameter \( \rho \equiv \lambda \tau \). The mass arrival rate \( \lambda \) is degenerate since it only appears in a product with \( \tau \) in the accretion parameter \( \rho \), and so it cannot be determined directly.
from a comparison of our model with the observations. We may, however, get a hint from the analysis of the flares’ light curves and from the geometry of the system.

In their analysis of the light curve of IGR J16479−4514 Sguera et al. (2005) found that the flares lasted from ∼30 minutes to ∼3 h. In one case, they were able to analyze a single light curve which, after a fast rise decayed exponentially with an e-folding time τ = 15 ± 9 minutes. Exponentially decaying light curves are not uncommon for the flares occurring in this source (Ducci et al. 2010). If the decay time τ ∼ 103 s can be taken as typical, then for ρ ∼ 10 we estimate the time interval between the successive accretion of two clumps to be λ−1 ∼ 102 s. If the NS orbits a massive star of M ∼ 20 M⊙ with a period Porb ∼ 3 d (typical of IGR J16479−4514), the distance covered by the NS between two successive encounters is of the order of a few times 109 cm, a fraction of the stellar radius. From the values of m0/τ and τ ∼ 103 s we find values of m0 between ∼1010 g and ∼1015 g for our three sources. On account of the large values of σ, the mean masses are much larger than these (compare with Equation (48)), and are of the order of 1018–1021 g. These values for the clumpy wind are in broad agreement with estimates obtained by other means (see, e.g., Ducci et al. 2009).

We now present a simple argument showing how the parameters m0/τ, σ, and ρ may not be fully independent of each other. Consider a giant donor star with mass M* and radius R*, that blows a clumpy wind. Assuming that the mass carried by the wind is all in clumps, then the mean stellar mass outflow rate at the distance r from the star is

$$M_w = 4\pi r^2 n_{cl} v_{cl} \langle m \rangle,$$

where ncl is the mean number of clumps per unit volume and vcl is the mean clump velocity. As the NS orbits the giant star, it crosses this clumpy wind and encounters clumps with the mean rate

$$\lambda = \pi R_G^2 v_{orb} n_{cl}.$$

where vorb is the orbital velocity of the NS and R_G is the capture radius defined by Equation (24). Eliminating ncl from Equations (52) and (53) and postulating that the population of the masses is log-normally distributed with shape parameter σ and median mass m0 we find

$$\lambda = \left( \frac{R_G}{2r} \right)^2 \frac{M_w}{m_0} \frac{v_{orb}}{v_{cl}} e^{-\sigma^2/2}.\quad (54)$$

The rate λ is a strongly decreasing function of the clumps’ mass distribution shape σ for the following simple reason. If σ is large, the log-normal distribution is skewed to the right, and most of the mass blown by the wind is carried by large, heavy clumps. A given Mw is carried by relatively few clumps, and their encounters with the orbiting NS are rare, i.e., λ is small. Multiplying the last equation by τ, we find

$$\rho = \left( \frac{R_G}{2r} \right)^2 \frac{M_w}{m_0/\tau} \frac{v_{orb}}{v_{cl}} e^{-\sigma^2/2}.\quad (55)$$

This shows that in our model the parameters σ, ρ, and m0/τ are not independent of each other: ρ is proportional to (m0/τ)−1 and decreases exponentially with σ2/2. It is difficult to use this equation to make quantitative predictions about the magnitude of ρ as a function of the other parameters. Indeed, when made explicit, its right-hand side is found to be very sensitive to the values of rather uncertain quantities (e.g., the velocity of the stellar wind), and even a small change in their value may vary ρ by orders of magnitude. We can plug the values of λ and ρ into Equations (49) to find

$$\frac{\sigma_L}{\langle L \rangle} = \left( 2 \frac{m_0/\tau}{M_w} \frac{v_{cl}}{v_{orb}} \right)^{1/2} \frac{r}{R_G} e^{\sigma^2/4},\quad (56)$$

where (L) and σL are, respectively, the mean accretion luminosity and the standard deviation of L, if L ∝ M. The exponential factor at the right-hand side of Equation (56) suggests an explanation for the difference between the ordinary, persistent X-ray sources occurring in the high-mass binaries and the SFXTs. Indeed, according to Equation (56), even moderate values of σ (say, σ ∼ 4–5) greatly amplify the dispersion around the mean σL/(L). This means that the wide dynamical range of luminosities observed in SFXTs is nothing other than the effect of the accretion of a population of clumps with a wide mass spectrum φ(m). On the other hand, if σ ≪ 1, i.e., all the accreting clumps have very similar masses, there is no such strong amplification, and σL/(L) may be small. The FLD generated by the accretion of clumps with narrowly distributed masses should resemble the distribution P discussed in Section 3.2.

According to this model, then, the difference between persistent HMXBs and SFXTs is entirely due to a substantial difference between the winds blown by the donor stars hosted in these systems. Since SFXTs are much rarer than ordinary HMXBs, it is likely that the occurrence of a clumpy stellar wind with a wide range of masses is also a rare event.

5. SUMMARY AND CONCLUSIONS

In this paper, we have set up a simple stochastic model for the accretion of a clumpy stream on a CO. We have modeled the response of the CO to the accretion with a simple exponential law (2) characterized by the relaxation time τ. The accretion stream is described as a train of pulses hitting the CO with a mean rate λ. The clumps have a mass distribution described by the function φ(m). With these ingredients we have written an SDE for the mass accretion rate ˙M(t). We have also derived the GFPE associated with the SDE. Its solution P(M, t) is the probability that the mass accretion rate is M at time t. We have then restricted ourselves to analysis of the stationary solution P(M). In general P cannot be written in a closed form for any choice of ψ, but it is nevertheless possible to express its moments as functions of the moments of ψ (if they exist). In addition, the asymptotic behavior of P(M) for small M is P(M) ∝ M−1, for any ψ(m) limited for small M. For ρ ∝ 1, then, P(M) is negligible for M ≪ m0/τ. This “ρ suppression” of P occurs when the relaxation time τ is long with respect to the clumps’ arrival rate, and the CO cannot respond promptly enough to the rapid succession of the clumps’ arrivals (Section 3.1).

We have then studied in some detail the function P for two choices of ψ: a delta function and a log-normal distribution. In the case of a log-normal ψ, P depends on three parameters: the accretion parameter ρ ≡ λτ, m0/τ, and σ, where m0 and σ are the location and shape parameters of the log-normal.

As a case study, we have applied our formalism (with a log-normal ψ) to the physics of SFXTs, a peculiar sub-class of HMXBs.

We have compared our model to the flares’ X-ray luminosity distributions of the SFXTs IGR J17544−2619, IGR J16479−4514, and XTE J1739−302 observed over a two-year campaign with Swift.
We have found that, however simplified, our model may reproduce the features of the observed flares’ distributions. The typical parameters giving the best agreement between the model and the observations are \( \rho \sim 10 \) and \( \sigma \sim 5 \), corresponding to a long tail in the mass distributions. The parameters \( \tau \) and \( \lambda \) cannot be determined separately, but taking the value \( \lambda \sim 10^7 \) s suggested by some observations, we find \( \lambda^{-1} \sim 10^2 \) s, or an inter-clump average distance of \( 10^{0.5} - 10^{10} \) cm. The average masses of the clumps are in the order of \( 10^{18} - 10^{21} \) g, in broad agreement with the estimates found in the existing literature (e.g., Ducci et al. 2009). In at least two of the SFXTs we analyzed, we were able to reproduce the FLDs without invoking an accretion barrier.

The reason why SFXTs show a much higher dynamic range than ordinary HMXBs is still unclear: according to our model, the difference is entirely due to the different properties of the streams of matter accreting the COs hosted in those systems. HMXBs accrete from clumpy winds with a narrow mass distribution, while SFXTs from winds with an extremely wide clump mass distribution. In a recent paper, Oskinova et al. (2012) pointed out that a stochastic component in the clumps’ velocities is probably essential in shaping the light curves of SFXTs. In the present paper, we have neglected the possible contributions to \( L(t) \) from a stochastic component of the velocity of the accreting stream, but it must clearly be included in any (more) realistic model.

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APPENDIX

DERIVATION OF THE GENERALIZED FOKKER–PLANCK EQUATION

In this section we closely follow Denisov et al. (2009) to derive the GFPE (30a) from the Langevin Equation (7). According to the Ito interpretation, the solution of Equation (7) in the (short) time interval \( \Delta t \) is

\[
\dot{M}(t + \Delta t) = \dot{M}(t) - \Delta t \frac{\dot{M}(t)}{\tau} + \delta M_e(t)/\tau, \tag{A1}
\]

where

\[
\delta M_e(t) = \int_t^{t+\Delta t} dt' \dot{M}_e(t') = \sum_{k=1}^{n(\Delta t)} m_k, \tag{A2}
\]

with \( n(\Delta t) \) being the number of clumps accreted in the time interval \( \Delta t \).

In order to derive the GFPE we introduce the probability \( p(\Delta M, \Delta t) \) that the mass \( \Delta M \) is accreted by the CO in the interval \( \Delta t \):

\[
p(\Delta M, \Delta t) = \langle \delta(\Delta M - \delta M_e(t)) \rangle. \tag{A3}
\]

From this definition and Equation (A3), after some algebra it can be shown that

\[
p(\Delta M, \Delta t) = P_0(\Delta t) \delta(\Delta M) + W(\Delta M, \Delta t), \tag{A4a}
\]

where

\[
W(\Delta M, \Delta t) = \sum_{k=1}^{\infty} P_k(\Delta t) \int dm_1 \varphi(m_1) \int dm_2 \varphi(m_2) \cdots \\
\times \int dm_k \varphi(m_k) \delta \left( \Delta M - \sum_{j=1}^{k} m_j \right), \tag{A4b}
\]

\( P_n(\Delta t) \) being the Poisson probability that \( n \) clumps are accreted in the time interval \( \Delta t \), and \( \varphi(m) \) is the mass distribution of the clumps. If \( \Delta t \) is small, we may expand \( p \) to first order in \( \Delta t \) and take

\[
p(\Delta M, \Delta t) = (1 - \lambda \Delta t) \delta(\Delta M) + \lambda \Delta t \varphi(\Delta M) + O(\Delta t). \tag{A5}
\]

The accretion probability \( P(\dot{M}, t) \) is defined as

\[
P(\dot{M}, t) = \langle \delta(\dot{M} - \dot{M}_e(t)) \rangle. \tag{A6}
\]

In order to proceed, we need a couple of formulae to express the averages of the functions \( \langle F(\dot{M}(t)) \rangle \) and \( \langle F(\dot{M}(t), \delta M_e(t)) \rangle \) in terms of the accretion probability distributions \( P(\dot{M}, t) \) and \( P(\dot{M}, \Delta t) \). Since the random variables \( \dot{M}(t) \) and \( \delta M_e(t) \) are independent, then

\[
\langle F(\dot{M}(t)) \rangle = \int d\dot{M} P(\dot{M}, t) F(\dot{M}) \tag{A7a}
\]

and

\[
\langle F(\dot{M}(t), \delta M_e(t)) \rangle = \int d\dot{M} \int d(\Delta M) P(\dot{M}, t) \times p(\Delta M, \Delta t) F(\dot{M}, \Delta M). \tag{A7b}
\]

We now introduce the time Fourier transformation to derive the GFPE. From the definition it is clear that

\[
P_k(t) = \int d\dot{M} P(\dot{M}, t) e^{-ik\dot{M}} = \langle e^{-ik\dot{M}} \rangle. \tag{A8}
\]

We calculate the increment \( \delta P_k(t) = P_k(t + \Delta t) - P_k(t) \) from Equation (A1) as

\[
\delta P_k(t) = \langle e^{-ik(\dot{M}(t + \Delta t) - \dot{M}(t))} \rangle. \tag{A9}
\]

Expressing \( \dot{M}(t + \Delta t) \) with Equation (A1) and retaining only the terms linear in \( \Delta t \),

\[
\delta P_k(t) = \langle e^{-ik\dot{M}(t)} [e^{-ik\dot{M}(t)/\tau} - 1] \rangle + ik \frac{\Delta t}{\tau} \langle \dot{M}(t) e^{-ik\dot{M}(t)} \rangle + O(\Delta t). \tag{A10}
\]

Introducing the probability distributions \( P \) and \( p \),

\[
\delta P_k(t) = \int d\dot{M} P(\dot{M}, t) e^{-ik\dot{M}} \int d(\Delta M) p(\Delta M, \Delta t) \times [e^{-ik\dot{M}(t)/\tau} - 1] \tag{A11}
\]

\[
+ ik \frac{\Delta t}{\tau} \int d\dot{M} P(\dot{M}, t) \dot{M} e^{-ik\dot{M}} + O(\Delta t). \tag{A12}
\]

Plugging Equation (A5) into this expression we obtain

\[
\delta P_k(t) = \int d\dot{M} P(\dot{M}, t) e^{-ik\dot{M}} \lambda \Delta t \int d(\Delta M) \varphi(\Delta M) \times [e^{-ik\delta M(t)/\tau} - 1] \tag{A13}
\]
\[ + i k \frac{\Delta t}{\tau} \int dM P(\dot{M}, t) \dot{M} e^{-i k M} + \mathcal{O}(\Delta t). \]  \hspace{1cm} (A14)

Dividing by $\Delta t$ and sending $\Delta t \rightarrow 0$

\[ \frac{\partial P_k}{\partial t} = \lambda P_k \varphi_{k/\tau} - \lambda P_k + i \frac{k}{\tau} \int dM P(\dot{M}, t) \dot{M} e^{-i k M}. \]  \hspace{1cm} (A15)

Finally, the inverse Fourier transform yields

\[ \frac{\partial P}{\partial t} = -\lambda P + \lambda \int_0^M dm \varphi(m) P(\dot{M} - m/\tau, t) + \frac{1}{\tau} \frac{\partial}{\partial \dot{M}} (\dot{M} P), \]  \hspace{1cm} (A16)

which is the desired Fokker–Planck equation.

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