Derivation of a homogenized nonlinear plate theory from 3d elasticity

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Abstract

We derive, via simultaneous homogenization and dimension reduction, the Γ-limit for thin elastic plates whose energy density oscillates on a scale that is either comparable to, or much smaller than, the film thickness. We consider the energy scaling that corresponds to Kirchhoff’s nonlinear bending theory of plates.

Keywords: elasticity, dimension reduction, homogenization, nonlinear plate theory, two-scale convergence.

1 Introduction

Kirchhoff’s nonlinear plate theory associates with a deformation \( u : S \rightarrow \mathbb{R}^3 \) of a two-dimensional stress-free reference configuration \( S \subset \mathbb{R}^2 \) the bending energy

\[
\int_S Q_2(\Pi(x')) \, dx',
\]

where \( Q_2 \) is the quadratic form from linear elasticity, and \( \Pi \) denotes the second fundamental form associated with \( u \). The key condition on the admissible deformations \( u \) is that they must satisfy the isometry constraint

\[
\partial_\alpha u \cdot \partial_\beta u = \delta_{\alpha\beta}, \quad \alpha, \beta \in \{1, 2\}
\]

where \( \delta_{\alpha\beta} \) denotes the Kronecker delta. Physically, (1) describes the elastic energy stored in a deformed plate that can undergo large deformations but not shearing or stretching.

In [FJM02] Kirchhoff’s nonlinear plate theory was rigorously derived as a zero-thickness Γ-limit from 3d nonlinear elasticity. In this article we combine their result with homogenization. We consider a plate with thickness \( h \ll 1 \) made of a composite material that periodically oscillates with period \( \varepsilon \ll 1 \) in in-plane directions. We shall derive a homogenized plate model via simultaneous homogenization and dimension reduction in the limit \( (h, \varepsilon) \rightarrow 0 \) when the material period \( \varepsilon \) and the thickness \( h \) are either comparable or behave as \( \varepsilon \ll h \), see Theorem 2.4 below. The derived model is sensitive to the relative scaling of \( h \) and \( \varepsilon \). Our result generalizes recent results from [Neu12] where the one-dimensional case is studied. Regarding plates, related results have been
obtained for different energy scalings: In [BFF00, BB06] the membrane regime has been considered. Recently, the energy scaling corresponding to the von-Kármán plate model was studied in [NeuVel].

This article is organized as follows. Section 2 introduces the general framework and discusses the main results. In Section 3 we recall the notion of two-scale convergence and characterize the two-scale limit of nonlinear strains. In Section 4 we prove our main result.

2 General framework and main result

From now on, $S \subset \mathbb{R}^2$ denotes a bounded Lipschitz domain whose boundary is piecewise $C^1$. The piecewise $C^1$-condition is necessary only for the proof of the upper bound and can be slightly relaxed, cf. [Ho11b].

For $h > 0$ and $I := (-\frac{1}{2}, \frac{1}{2})$, we denote by $\Omega_h := S \times hI$ the reference configuration of the thin plate of thickness $h$. The elastic energy per unit volume associated with a deformation $v^h : \Omega_h \rightarrow \mathbb{R}^3$ is given by

$$\frac{1}{h} \int_{\Omega_h} W(z', \varepsilon \nabla v^h(z)) \, dz.$$  \hspace{1cm} (3)

Here and below $z' = (z_1, z_2)$ stands for the in-plane coordinates of a generic element $z = (z_1, z_2, z_3) \in \Omega_h$ and $W$ is a energy density that models the elastic properties of a periodic composite.

Assumption 2.1. We assume that $W : \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \rightarrow [0, \infty], \quad (y, F) \mapsto W(y, F)$ is measurable and $[0,1)^2$-periodic in $y$ for all $F$. Furthermore, we assume that for almost every $y \in \mathbb{R}^2$, the map $\mathbb{R}^{3 \times 3} \ni F \mapsto W(y, F) \in [0, \infty]$ is continuous and satisfies the following properties:

(frame indifference) \hspace{1cm} (FI) \quad W(y, RF) = W(y, F) \quad \text{for all } F \in \mathbb{R}^{3 \times 3}, R \in \text{SO}(3);

(non degeneracy) \hspace{1cm} (ND) \quad W(y, F) \geq c_1 \text{dist}^2(F, \text{SO}(3)) \quad \text{for all } F \in \mathbb{R}^{3 \times 3};

W(y, F) \leq c_2 \text{dist}^2(F, \text{SO}(3)) \quad \text{for all } F \in \mathbb{R}^{3 \times 3} \text{ with } \text{dist}^2(F, \text{SO}(3)) \leq \rho;

(quadratic expansion at identity) \hspace{1cm} (QE) \quad \lim_{G \rightarrow 0} \frac{W(y, I + G) - Q(y, G)}{|G|^2} = 0

for some quadratic form $Q(y, \cdot)$ on $\mathbb{R}^{3 \times 3}$.

Here $c_1, c_2$ and $\rho$ are positive constants which are fixed from now on.
We define $\Omega := S \times I$. As in [FJM02] we rescale the out-of-plane coordinate: for $x = (x', x_3) \in \Omega$ consider the scaled deformation $u^h(x', x_3) := v^h(x', h x_3)$. Then (3) equals

$$E^{h, \varepsilon}(u^h) := \int_\Omega W\left(\frac{x'}{\varepsilon}, \nabla_h u^h(x)\right) dx,$$

where $\nabla_h u^h := (\nabla' u^h, \frac{1}{h} \partial_3 u^h)$ denotes the scaled gradient, and $\nabla' u^h := (\partial_1 u^h, \partial_2 u^h)$ denotes the gradient in the plane.

We recall some known results on dimension reduction in the homogeneous case when $W(y, F) = W(F)$. As explained in [FJM06] a hierarchy of plate models can be derived from $E^h := E^{h, 1}$ in the zero-thickness limit $h \to 0$. The different limiting models are distinguished by the scaling of the elastic energy relative to the thickness. In [LDR95] it is shown that the scaling $E^h \sim 1$ leads to a membrane model, which is a fully nonlinear plate model for plates without resistance to compression. In the regime $E^h \sim h^4$ finite energy deformations converge to rigid deformations and, as shown in [FJM06], $h^{-4} E^h$ converges to a plate model of “von-Kármán”-type.

In this article we study the bending regime $E^h \sim h^2$, which, as shown in [FJM02], leads to Kirchhoff’s nonlinear plate model: as $h \to 0$ the energy $h^{-2} E^h$ $\Gamma$-converges to the functional (1), with $Q_2 : \mathbb{R}^{2 \times 2} \to \mathbb{R}$ given by the relaxation formula

$$Q_2(A) = \min_{d \in \mathbb{R}^3} Q\left(\sum_{\alpha, \beta = 1}^2 A_{\alpha\beta} (e_\alpha \otimes e_\beta) + d \otimes e_3\right);$$

here, $Q$ denotes the quadratic form from (QE).

We will see that in the non-homogeneous case the effective quadratic form $Q_2$ is determined by a relaxation formula that is more complicated and requires the solution of a corrector problem. In particular, our analysis shows that in-plane oscillations of the deformation couple with the behavior in the out-of-plane direction. As a consequence the effective behavior will depend on the relative scaling between the thickness $h$ and the material period $\varepsilon$. To make this precise we assume that $\varepsilon$ and $h$ are coupled as follows:

**Assumption 2.2.** Let $\gamma \in (0, \infty]$ denote a constant which is fixed throughout this article. We assume that $\varepsilon = \varepsilon(h)$ is a monotone function from $(0, \infty)$ to $(0, \infty)$ such that $\varepsilon(h) \to 0$ and $\frac{\varepsilon(h)}{h} \to \gamma$ as $h \to 0$.

The effective behavior of the homogenized plate with reduced dimension can be computed by means of a relaxation formula that we introduce next. We need to introduce some function spaces of periodic functions. From now on, $Y = [0, 1)^2$, and we denote by $Y$ the set $Y$ endowed with the torus topology, so that functions on $Y$ will be $Y$-periodic.

We write $C(Y)$, $C^k(Y)$ and $C^\infty(Y)$ for the Banach spaces of $Y$-periodic functions on $\mathbb{R}^2$ that are continuous, $k$-times continuously differentiable and smooth, respectively. Moreover, $H^1(I \times Y)$ denotes the closure of $C^\infty(I, C^\infty(Y))$ with respect to the norm in $H^1(I \times Y)$ and we write $H^1(I \times Y)$ for the subspace of functions $f \in H^1(S \times Y)$ with $\int_{I \times Y} f = 0$. The definitions extend in the obvious way to vector-valued functions.
Definition 2.3 (Relaxation formula). Let $Q$ be as in Assumption 2.1. For $x_3 \in I$ and $A,B \in \mathbb{R}^{2 \times 2}$, define
\[
\Lambda(x_3,A,B) := \left( \sum_{\alpha,\beta=1}^{2} (B_{\alpha\beta} + x_3 A_{\alpha\beta})(e_{\alpha} \otimes e_{\beta}) \right).
\]

(a) For $\gamma \in (0,\infty)$ we define $Q_{2,\gamma} : \mathbb{R}^{2 \times 2}_{\text{sym}} \to [0,\infty)$ by
\[
Q_{2,\gamma}(A) := \inf_{B,\phi} \int_{I \times Y} Q \left( y, \Lambda(x_3, A, B) + \langle \nabla_y \phi, \frac{1}{\gamma} \partial_3 \phi \rangle \right) \, dy \, dx_3
\]
where the infimum is taken over all $B \in \mathbb{R}^{2 \times 2}_{\text{sym}}$ and $\phi \in H^1(I \times \mathcal{Y}, \mathbb{R}^3)$.

(b) For $\gamma = \infty$ we define $Q_{2,\infty} : \mathbb{R}^{2 \times 2}_{\text{sym}} \to [0,\infty)$ by
\[
Q_{2,\infty}(A) := \inf_{B,\phi} \int_{I \times Y} Q \left( y, \Lambda(x_3, A, B) + \langle \nabla_y \phi, d \rangle \right) \, dy \, dx_3
\]
where the infimum is taken over all $B \in \mathbb{R}^{2 \times 2}_{\text{sym}}$, $\phi \in L^2(I, H^1(\mathcal{Y}, \mathbb{R}^3))$ and $d \in L^2(I, \mathbb{R}^3)$.

Kirchhoff’s plate model is defined for pure bending deformations of $S$ into $\mathbb{R}^3$; precisely:
\[
W^{2,2}_{\delta}(S, \mathbb{R}^3) := \left\{ u \in W^{2,2}(S, \mathbb{R}^3) : u \text{ satisfies (2) a.e. in } S \right\}.
\]

With each $u \in W^{2,2}_{\delta}(S)$ we associate its normal $n := \partial_1 u \wedge \partial_2 u$, and we define its second fundamental form $\Pi : S \to \mathbb{R}_{\text{sym}}^{2 \times 2}$ by defining its entries as
\[
\Pi_{\alpha\beta} = \partial_\alpha u \cdot \partial_\beta n = -\partial_\alpha \partial_\beta u \cdot n.
\]

We write $\Pi^{h}$ and $n^{h}$ for the second fundamental form and normal associated with some $u^{h} \in W^{2,2}_{\delta}(S, \mathbb{R}^3)$. The $\Gamma$-limit is a functional of the form (1) trivially extended to $L^2(\Omega, \mathbb{R}^3)$ by infinity: for $\gamma \in (0,\infty]$ define $E_{\gamma} : L^2(\Omega, \mathbb{R}^3) \to [0,\infty],$
\[
E_{\gamma}(u) := \begin{cases} \int_{S} Q_{2,\gamma}(\Pi(x')) \, dx' & \text{if } u \in W^{2,2}_{\delta}(S, \mathbb{R}^3) \\ + \infty & \text{otherwise.} \end{cases}
\]

We tacitly identify functions on $S$ with their trivial extension to $\Omega = S \times I$: above $u \in W^{2,2}_{\delta}(S, \mathbb{R}^3)$ means that $u(x', x_3) = \Pi(x') := \int_{I} u(x', z) \, dz$ for almost every $x_3 \in I$, and $\Pi \in W^{2,2}_{\delta}(S, \mathbb{R}^3)$. Our main result is the following:

Theorem 2.4. Suppose that Assumptions 2.1 and 2.2 are satisfied. Then:

(i) (Lower bound). If $\{u^{h}\}_{h>0}$ is a sequence with $u^{h} - \int_{\Omega} u^{h} \, dx \to u$ in $L^2(\Omega, \mathbb{R}^3)$, then
\[
\liminf_{h \to 0} h^{-2} E_{\gamma}(u^{h}) \geq E_{\gamma}(u).
\]
(ii) (Upper bound). For every \( u \in W^{2,2}_0(S, \mathbb{R}^3) \) there exists a sequence \( \{u^h\}_{h>0} \) with \( u^h \to u \) strongly in \( L^2(\Omega, \mathbb{R}^3) \) such that

\[
\lim_{h \to 0} h^{-2} \mathcal{E}^{h, \varepsilon}(u^h) = \mathcal{E}_\gamma(u).
\]

This theorem is complemented by the following compactness result from [FJM02], which in particular shows that \( \{\mathcal{E}^{h, \varepsilon(h)}\}_{h>0} \) is equi-coercive on \( L^2(\Omega, \mathbb{R}^3) \).

**Theorem 2.5** ([FJM02, Theorem 4.1]). Suppose a sequence \( u^h \in H^1(\Omega, \mathbb{R}^3) \) has finite bending energy, that is

\[
\limsup_{h \to 0} \frac{1}{h^2} \int_\Omega \text{dist}^2(\nabla h u^h(x), \text{SO}(3)) \, dx < \infty.
\]

Then there exists \( u \in W^{2,2}_0(S, \mathbb{R}^3) \) such that

\[
\begin{align*}
u^h - \frac{1}{\Omega} \int_{\Omega} u^h \, dx & \to u, \quad \text{strongly in } L^2(\Omega, \mathbb{R}^3), \\
\nabla h u^h & \to (\nabla u, n) \quad \text{strongly in } L^2(\Omega, \mathbb{R}^{3 \times 3}),
\end{align*}
\]

as \( h \to 0 \) after passing to subsequences and extending \( u \) and \( n \) trivially to \( \Omega \).

Theorem 2.4 and Theorem 2.5 imply by standard arguments from the theory of \( \Gamma \)-convergence that minimizers of functionals of the form

\[
h^{-2} \mathcal{E}^{h, \varepsilon(h)}(\cdot) + "(rescaled) dead loads",
\]

subject to certain boundary conditions, converge to minimizers of

\[
\mathcal{E}_\gamma(\cdot) + "\text{dead loads},"
\]

subject to certain boundary conditions. For details see [FJM06].

In the special case when \( W(y, F) = W(F) \) is homogeneous Theorem 2.4 reduces to the result in [FJM02]. The proof of our main result emulates their argument as far as possible.

We now explain our approach. The bending regime is a borderline case in the hierarchy of plate models. On one hand it allows for large deformations, on the other hand it corresponds to small strains: By Theorem 2.5 a sequence \( \{u^h\}_{h>0} \) with finite bending energy in general converges to a non-trivial deformation. However, the associated non-linear strain \( \sqrt{(\nabla h u^h)^t(\nabla h u^h)} - I \) converges to zero. Indeed, let

\[
E^h := \sqrt{(\nabla h u^h)^t(\nabla h u^h)} - I
\]

(7)

denote the scaled non-linear strain associated with \( u^h \). Then due to the elementary inequality \( \sqrt{F^t F} - I \leq \text{dist}(F, \text{SO}(3)) \) we find that \( \{E^h\}_{h>0} \) is bounded in \( L^2 \) when \( \{u^h\}_{h>0} \) has finite bending energy.

The smallness of the nonlinear strain is crucial for our extension to simultaneous homogenization and dimension reduction: By \((\text{QE})\) the elastic energy is related to the nonlinear strain in a quadratic way – indeed, we formally have

\[
\frac{1}{h^2} \mathcal{E}^{h, \varepsilon}(u^h) \approx \int_\Omega Q(\frac{\varepsilon'}{\varepsilon}, E^h(x)) \, dx.
\]

(8)
Heuristically, the right-hand side is obtained by linearizing the stress-strain relation, while preserving the geometric non-linearity. Due to the convexity of the right-hand side in (8) only oscillations of \( \{E^h\}_{h>0} \) that emerge precisely on scale \( \varepsilon \) are relevant for homogenization. A tool to describe such oscillations is two-scale convergence. In Section 3 we characterize (partially) the possible two-scale limits of \( \{E^h\}_{h>0} \). This is the main ingredient for the lower bound in Theorem 2.4.

Assume \( u^h \) converges to some bending deformation with second fundamental form \( II \). Then any two-scale accumulation point of \( \{E^h\}_{h>0} \) can be written in the form

\[
x_3 \begin{pmatrix} II(x') & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \tilde{E}(x, y)
\]

where \( \tilde{E} : \Omega \times Y \to \mathbb{R}^{3 \times 3} \) is a relaxation field that captures oscillations and is a priori “unknown”. In Proposition 3.2 we prove that \( \tilde{E} \) has to be of specific form. The \( \Gamma \)-limit of \( h^{-2}E^h \) is then obtained by relaxation:

\[
\inf_{\tilde{E}} \int_{\Omega} \int_{Y} Q(y, x_3 \begin{pmatrix} II(x') & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \tilde{E}(x, y)) 
\]

where the infimum is taken over all \( \tilde{E} \) of the specific form given in Proposition 3.2.

We conclude this section by discussing the dependency of our limiting model on the parameter \( \gamma \), which describes the relative scaling between \( h \) and \( \varepsilon \). The relaxed quadratic form \( Q_{2, \gamma} \) continuously depends on \( \gamma \). In fact, with [NeuVel, Lemma 5.2] at hand one can easily identify the limits

\[
\lim_{\gamma \to 0} Q_{2, \gamma}(A) \quad \text{and} \quad \lim_{\gamma \to \infty} Q_{2, \gamma}(A) \quad (A \in \mathbb{R}^{2 \times 2}),
\]

which yield proper quadratic forms on \( \mathbb{R}^{2 \times 2} \) that vanish on skew-symmetric matrices and are positive definite on symmetric matrices. In particular, the limit for \( \gamma \to \infty \) coincides with \( Q_{2, \infty} \). The limit for \( \gamma \to 0 \) can be identified as well: We introduce the dimension reduced quadratic form \( Q_2(y, A) \) for all \( A \in \mathbb{R}^{2 \times 2} \) via

\[
Q_2(y, A) = \min_{d \in \mathbb{R}^3} Q(y, \sum_{\alpha, \beta=1}^2 A_{\alpha\beta} (e_\alpha \otimes e_\beta) + d \otimes e_3)
\]

Then \( Q_{2, \gamma}(A) \) converges for \( \gamma \to 0 \) to

\[
Q_{2,0}(A) := \inf_{B, \zeta, \varphi} \int_{I \times Y} Q_2(y, A + x_3 B + \text{sym}(\nabla_y \zeta + x_3 \nabla_y^2 \varphi)) \, dy \, dx_3
\]

where the infimum is taken over all \( B \in \mathbb{R}^{2 \times 2}, \zeta \in H^1(Y, \mathbb{R}^2) \) and \( \varphi \in H^2(Y) \).

A similar behavior has been observed in [Neu12, NeuVel] where also the case \( \gamma = 0 \) is considered (for rods and von-Kármán plates, respectively). In the von-Kármán case (see [NeuVel]) it turns out that in the regime \( h \ll \varepsilon(h) \) the limit \( \gamma \to 0 \) of the quadratic energy density indeed recovers the energy density obtained via \( \Gamma \)-convergence. It is not clear whether or not this picture extends to the bending regime.
3 Two-scale limits of the nonlinear strain

Two-scale convergence was introduced in [Ngu89, All92] and has been extensively applied to various problems in homogenization. In this article we work with the following variant of two-scale convergence which is adapted to dimension reduction.

**Definition 3.1 (two-scale convergence).** We say a bounded sequence \( \{f^h\}_{h>0} \) in \( L^2(\Omega) \) two-scale converges to \( f \in L^2(\Omega \times Y) \) and we write \( f^h \rightharpoonup f \), if

\[
\lim_{h \to 0} \int_{\Omega} f^h(x)\psi(x, \frac{x'}{\epsilon(h)}) \, dx = \int_{\Omega \times Y} f(x,y)\psi(x,y) \, dy \, dx
\]

for all \( \psi \in C_0^\infty(\Omega, C(Y)) \). When \( ||f^h||_{L^2(\Omega)} \to ||f||_{L^2(\Omega \times Y)} \) in addition, we say that \( f^h \) strongly two-scale converges to \( f \) and write \( f^h \rightharpoonup f \). For vector-valued functions, two-scale convergence is defined componentwise.

Since we identify functions on \( S \) with their trivial extension to \( \Omega \), the definition above contains the standard notion of two-scale convergence on \( \Omega \times Y \) as a special case. Indeed, when \( \{f^h\}_{h>0} \) is a sequence in \( L^2(S) \), then \( f^h \rightharpoonup f \) is equivalent to

\[
\lim_{h \to 0} \int_{S} f^h(x')\psi(x', \frac{x'}{\epsilon(h)}) \, dx' = \int_{S \times Y} f(x',y)\psi(x',y) \, dy \, dx'
\]

for all \( \psi \in C_0^\infty(S, C(Y)) \).

The main ingredient in the proof of the lower bound part of Theorem 2.4 is the following characterization of the possible two-scale limits of nonlinear strains.

**Proposition 3.2.** Let \( \{u^h\}_{h>0} \) be a sequence of deformations with finite bending energy, let \( u \in W_2^{2,2}(S, \mathbb{R}^3) \) with second fundamental form \( H \), and assume that

\[
u^h - \frac{1}{|\Omega|} \int_{\Omega} u^h \, dx \to u \quad \text{strongly in } L^2(\Omega, \mathbb{R}^3),
\]

\[
E^h := \sqrt{(\nabla u^h)^T \nabla u^h - I} \xrightarrow{\mathcal{h}} E \quad \text{weakly two-scale}
\]

for some \( E \in L^2(\Omega \times Y; \mathbb{R}^{3 \times 3}) \).

(a) If \( \gamma \in (0, \infty) \) then there exist \( B \in L^2(S, \mathbb{R}^{2 \times 2}) \), and \( \phi \in L^2(S, H^1(I \times Y, \mathbb{R}^3)) \) such that

\[
E(x,y) = \begin{pmatrix} x_3 \Pi(x') + B(x') & 0 \\ 0 & 0 \end{pmatrix} + \text{sym} \left( \nabla_y \phi(x,y), \frac{1}{\gamma} \partial_3 \phi(x,y) \right).
\]

(b) If \( \gamma = \infty \) then there exist \( B \in L^2(S, \mathbb{R}^{2 \times 2}) \), \( \phi \in L^2(S, H^1(Y, \mathbb{R}^3)) \), and \( d \in L^2(\Omega, \mathbb{R}^3) \) with

\[
E(x,y) = \begin{pmatrix} x_3 \Pi(x') + B(x') & 0 \\ 0 & 0 \end{pmatrix} + \text{sym} \left( \nabla_y \phi(x,y), d(x) \right).
\]
Remarks. (i) In [FJM02] a coarser characterization of possible weak limits of \( \{ E^h \}_{h>0} \) was obtained: In the situation of the previous proposition let \( (E^h)' \) denote the 2 \( \times \) 2 matrix obtained from \( E^h \) by deleting the third row and column. Then it was shown in [FJM02] that

\[
(E^h)' \rightharpoonup x_3 \Pi(x') + B'(x') \quad \text{weakly in } L^2(\Omega, \mathbb{R}^{2 \times 2}_{\text{sym}}).
\]

Proposition 3.2 refines this by capturing, in addition, oscillations on the scale \( \varepsilon \).

(ii) Proposition 3.2 still only yields an incomplete characterization of the possible structure of the two-scale limiting strain \( E \): it is not true that every \( E \) in the form of (10) (resp. (11)) can be recovered as a two-scale limit of a sequence of nonlinear strains. For instance, when \( u \) is affine, i.e. \( \Pi = 0 \), then not every two-scale limiting strain of the form (10) with \( B \) arbitrary and \( \phi = 0 \) can emerge.

In our construction of recovery sequences a special role is played by the matrix \( B \), which is “recovered” by corrections of the isometry of order \( h \). More precisely, these corrections are obtained by solving the equation

\[
B = \text{sym} \nabla' g + \alpha \Pi \quad \text{for } g : S \to \mathbb{R}^d \text{ and } \alpha : S \to \mathbb{R}.
\]

(12)

As shown in [Sch07, Lemma 3.3], equation (12) can be solved locally on regions where \( \Pi \neq 0 \), provided that \( u \) is smooth. On the level of these “order \( h \) corrections” the deformed plate behaves like a shell and the condition \( \Pi \neq 0 \) corresponds to the property that the shell is developable without affine region.

An important observation is that, in spite of not giving an exhaustive characterization of limiting strains, the result of Proposition 3.2 is just sharp enough to obtain the optimal lower bound for \( h^{-2} E^h,\varepsilon(h) \). This is because on regions where \( \Pi = 0 \), corrections associated to \( B \) can be ignored, since they do not reduce the energy (as \( Q(y,F) \) is minimal for \( F = 0 \)).

In contrast, for rods and von-Kármán plates, exhaustive characterizations were obtained in [Neu12, Theorem 3.5] and [NeuVel, Proposition 3.3].

(iii) A key technical ingredient in the proof of Proposition 3.2 is Lemma 3.8 below. It allows us to work with piecewise constant \( SO(3) \)-valued approximations of the deformation gradient, as opposed to smooth \( SO(3) \)-valued approximations. The latter were used in the proof of the 1d case given in [Neu12]. In the 2d case studied here, the use of such a smooth \( SO(3) \)-valued approximation would require small limiting energy, cf. [FJM06, Remark 5]. Thanks to Lemma 3.8, our result is not restricted to small limiting energy. Incidentally, the use of this lemma also simplifies the proof of the convergence statement in the 1d case.

The starting point of the proof of the previous Proposition is [FJM06, Theorem 6], which we combine with the last remark in [FJM02, Section 3] in order to allow for \( \gamma_0 < 1 \).
Lemma 3.3. Let $\gamma_0 \in (0,1]$ and let $h, \delta > 0$ with $\gamma_0 \leq \frac{h}{\delta \leq \frac{1}{h}}$. There exists a constant $C$, depending only on $S$ and $\gamma_0$, such that the following is true: if $u \in H^1(\Omega,\mathbb{R}^3)$ then there exists a map $R : S \to SO(3)$ which is piecewise constant on each cube $x + \delta Y$ with $x \in \delta \mathbb{Z}^2$ and there exists $\tilde{R} \in H^1(S,\mathbb{R}^{3 \times 3})$ such that

$$\|\nabla_h u - R\|_{L^2(\Omega)}^2 + \|R - \tilde{R}\|_{L^2(S)}^2 + h^2\|\nabla \tilde{R}\|_{L^2(\Omega)}^2 \leq C\|\text{dist} (\nabla_h u, SO(3))\|_{L^2(\Omega)}^2.$$ 

Let us recall some well-known properties of two-scale convergence. We refer to [All92, Vis06, MT07] for proofs in the standard two-scale setting and to [Neu10] for the easy adaption to the notion of two-scale convergence considered here.

Lemma 3.4. (i) Any sequence that is bounded in $L^2(\Omega)$ admits a two-scale convergent subsequence.

(ii) Let $\bar{f} \in L^2(\Omega \times Y)$ and let $f^h \in L^2(\Omega)$ be such that $f^h \overset{2,\gamma}{\rightharpoonup} \bar{f}$. Then $f^h \to \int_Y \bar{f}(\cdot, y)dy$ weakly in $L^2(\Omega)$.

(iii) Let $f^0$ and $f^h \in L^2(\Omega)$ be such that $f^h \to f^0$ weakly in $L^2(\Omega)$. Then (after passing to subsequences) we have $f^h \overset{2,\gamma}{\rightharpoonup} f^0(x) + \tilde{f}$ for some $\tilde{f} \in L^2(\Omega \times Y)$ with $\int_Y \tilde{f}(\cdot, y)dy = 0$ almost everywhere in $S$.

(iv) Let $f^0$ and $f^h \in H^1(\Omega)$ be such that $f^h \to f^0$ strongly in $L^2(\Omega)$. Then $f^h \overset{2,\gamma}{\rightharpoonup} f^0$, where we extend $f^0$ trivially to $\Omega \times Y$.

(v) Let $f^0$ and $f^h \in H^1(S)$ be such that $f^h \to f^0$ weakly in $H^1(S)$. Then (after passing to subsequences)

$$\nabla^i f^h \overset{2,\gamma}{\rightharpoonup} \nabla^i f^0 + \nabla_y \phi$$

for some $\phi \in L^2(S,H^1(Y))$.

The following lemma is an immediate consequence of Lemma 3.4, cf. [Neu10, Theorem 6.3.3] for a proof.

Lemma 3.5. Let $u^0$ and $u^h \in H^1(\Omega,\mathbb{R}^3)$ be such that $u^h \rightharpoonup u^0$ weakly in $H^1(\Omega,\mathbb{R}^3)$.

(a) If $\gamma \in (0,\infty)$ then there exists $\phi \in L^2(S,\dot{H}^1(I \times Y,\mathbb{R}^3))$ such that (after passing to subsequences)

$$\nabla_h u^h \overset{2,\gamma}{\rightharpoonup} (\nabla^i u^0, 0) + (\nabla_y \phi, \frac{1}{\gamma} \partial_3 \phi).$$

(b) If $\gamma = \infty$ then there exist $\phi \in L^2(S,\dot{H}^1(I \times Y,\mathbb{R}^3))$ and $d \in L^2(\Omega,\mathbb{R}^3)$ such that (after passing to subsequences)

$$\nabla_h u^h \overset{2,\gamma}{\rightharpoonup} (\nabla^i u^0, 0) + (\nabla_y \phi, d).$$
At several places in our proof we will need to make sense of a two-scale limit for sequences which might be unbounded in $L^2$, but which nevertheless have controlled oscillations on the scale $\varepsilon$. In order to capture these oscillations, we ‘renormalize’ the sequence by throwing away the (divergent) part which does not oscillate on the scale $\varepsilon$. (For bounded sequences, this latter part gives rise to the weak limit, but the point here is that our sequences may be unbounded.)

Equivalently, we weaken the notion of two-scale convergence by restricting the admissible test functions to functions with vanishing cell average.

More precisely, for a sequence $\{f_h\}_{h>0} \subset L^2(\Omega)$ and $\tilde{f} \in L^2(\Omega \times Y)$ with $\int_Y \tilde{f}(\cdot, y) \, dy = 0$ almost everywhere in $\Omega$, we write

$$f_h \overset{osc,\gamma}{\rightarrow} \tilde{f}$$

if

$$\lim_{h \to 0} \int_\Omega f_h(x) \varphi(x) g(\frac{\varepsilon(x)}{\varepsilon(h)}) \, dx = \int_{\Omega \times Y} \tilde{f}(x, y) \varphi(x) g(y) \, dy \, dx$$

for all $\varphi \in C_0^\infty(\Omega)$ and $g \in C^\infty(Y)$ with $\int_Y g \, dy = 0$. (13)

The proof of the following lemma is straightforward.

**Lemma 3.6.** Let $f^0$ and $f_h \in L^2(\Omega)$ be such that $f_h \to f^0$ weakly in $L^2(\Omega)$ and $f_h \overset{osc,\gamma}{\rightarrow} \tilde{f}$. Then $f_h \overset{2,\gamma}{\rightarrow} f^0 + \tilde{f}$ weakly two-scale.

For the proof of Proposition 3.2 we have to identify the oscillatory part of two-scale limits for renormalized functions of the form $1/\varepsilon(h)^2 f_h$ where $f_h$ is either a sequence bounded in $H^1(S)$ or piecewise affine with respect to the lattice $\varepsilon(h)Z^2$. The following two lemmas treat these situations.

**Lemma 3.7.** Let $f^0$ and $f_h \in H^1(S)$ be such that $f_h \to f^0$ weakly in $H^1(S)$ and assume that

$$\nabla' f_h \overset{2,\gamma}{\rightarrow} \nabla' f^0 + \nabla_y \phi$$

for some $\phi \in L^2(S, H^1(Y))$ with $\int_S \phi(\cdot, y) \, dy = 0$ almost everywhere in $S$. Then

$$f_h \overset{\varepsilon(h),\gamma}{\rightarrow} \phi.$$ 

**Proof.** Since $f_h$ is independent of $x_3$, we must show that

$$\frac{1}{\varepsilon(h)} \int_S f_h(x') g(\frac{x'}{\varepsilon(h)}) \psi(x') \, dx' \to \int_{S \times Y} \phi(x', y) g(y) \psi(x') \, dx' \, dx'$$

(14)

for all $g \in C^\infty(Y)$ with $\int_Y g \, dy = 0$ and $\psi \in C_0^\infty(S)$. For simplicity we write $\varepsilon$ instead of $\varepsilon(h)$. Let $G$ denote the unique solution in $C^2(Y)$ to

$$-\triangle_y G = g, \quad \int_Y G \, dy = 0.$$
Set $G^h(x') := \varepsilon G(\frac{x'}{\varepsilon})$ so that
\[
\triangle G^h(x') = \frac{1}{\varepsilon} g(x')/\varepsilon).
\]
Hence, the right-hand side of (14) equals
\[
\int_S f^h \triangle G^h \psi \, dx'
= \int_S f^h \left( \triangle (G^h \psi) - 2 \nabla G^h \cdot \nabla \psi - G^h \triangle \psi \right) \, dx'
= - \int_S \nabla f^h \cdot \nabla (G^h \psi) \, dx' - 2 \int_S f^h \nabla G^h \cdot \nabla \psi \, dx' - \int_S f^h G^h \triangle \psi \, dx'.
\]
By the chain rule and the definition of $G^h$ we have
\[
\nabla (G^h \psi)(x') = \nabla G^h(x') \psi(x') + G^h(x') \psi'(x')
= \nabla G(\frac{x'}{\varepsilon}) \psi(x') + \varepsilon G(\frac{x'}{\varepsilon}) \psi'(x').
\]
Since the right-hand side strongly two-scale converges to $\nabla y G(y) \psi(x)$, and because $\nabla f^h \nabla \psi \to f^0 \nabla \psi$ strongly in $L^2(S)$, we deduce that
\[
- \int_S \nabla f^h \cdot \nabla (G^h \psi) \, dx' \\
\rightarrow - \int_{S \times Y} \left( \nabla f^0(x') + \nabla_y \phi(x', y) \right) \cdot \left( \nabla_y G(y) \psi(x') \right) \, dy \, dx'
= \int_{S \times Y} \phi(x', y) \Delta_y G(y) \psi(x') \, dy \, dx'
= \int_{S \times Y} \phi(x', y) g(y) \psi(x') \, dy \, dx'.
\]
Hence it suffices to show that the second and third integral on the right-hand side of (15) vanish for $h \to 0$. We treat the second integral. Since $\nabla G^h(x') = \nabla_y G(\frac{x'}{\varepsilon})$ strongly two-scale converges to $\nabla_y G(y)$, and because $f^h \nabla \psi \to f^0 \nabla \psi$ strongly in $L^2(S)$, we deduce that
\[
-2 \int_S f^h (\nabla G^h \cdot \nabla \psi) \, dx' \rightarrow - \int_{S \times Y} f^0(x') \nabla \psi(x') \cdot \nabla_y G(y) \, dy \, dx' = 0.
\]
The third integral on the right-hand side of (15) vanishes simply because $f^h \nabla \psi$ is bounded in $L^2(S)$ and $G^h \to 0$ in $L^2(S)$.

**Lemma 3.8.** Let $f^0$ and $f^h \in L^\infty(S)$ be such that $f^h \xrightarrow{\ast} f^0$ weakly* in $L^\infty(S)$. Assume that $f^h \in L^\infty(S)$ is constant on each cube $x + \varepsilon(h) Y$, $x \in \varepsilon(h) Z^2$. Then we have
\[
\frac{1}{\varepsilon(h)} \int_S f^h(x') \psi(x') g \left( \frac{x'}{\varepsilon(h)} \right) \, dx' \rightarrow \int_S f^0(x') \nabla \psi(x') \, dx' \cdot \int_Y g(y) y \, dy \quad (16)
\]
for all $g \in C(Y)$ with $\int_Y g = 0$ and $\psi \in C^\infty_0(S)$. In particular, if $f^0 \in W^{1,2}(S)$ we have
\[
\frac{1}{\varepsilon(h)} f^h \xrightarrow{\ast} - (y \cdot \nabla') f^0.
\]
Here we write

\[(y \cdot \nabla') f^0(x') = \sum_{\alpha=1,2} y_\alpha \partial_\alpha f^0(x').\]

**Proof.** We first argue that (16) combined with \(f^0 \in W^{1,\infty}(S)\) implies the convergence of \(\frac{1}{\varepsilon(h)} f^h\). Indeed, since \(f^h\) is independent of \(x_3\) it suffices to consider test functions \(g\) and \(\psi\) as in identity (16). Now the statement simply follows from the observation that the right-hand side of (16) becomes

\[- \iint_{S \times Y} (y \cdot \nabla') f^0(x') \psi(x') g(y) \, dy \, dx'\]

by an integration by parts. We prove (16). For simplicity we write \(\varepsilon\) instead of \(\varepsilon(h)\). We denote by \(\tilde{\psi}^h\) an approximation of \(\psi\) that is constant on each of the cubes \(\xi + \varepsilon Y\), \(\xi \in \varepsilon\mathbb{Z}^2\), say \(\tilde{\psi}^h(x) := \psi(\xi_x)\) where \(\xi_x \in \varepsilon\mathbb{Z}^2\) denotes the cube \(\xi_x + \varepsilon Y\) in which \(x\) lies. Then we have

\[
\int_S \frac{f^h(x)}{\varepsilon} \psi(x) g \left( \frac{x}{\varepsilon} \right) \, dx = \int_S \frac{f^h(x)}{\varepsilon} \psi(x) - \tilde{\psi}^h(x) \frac{g \left( \frac{x}{\varepsilon} \right)}{\varepsilon} \, dx
\]

(17)

because

\[
\int_S f^h(x) \tilde{\psi}(x) g \left( \frac{x}{\varepsilon} \right) \, dx = 0,
\]

since \(g\) has zero average over \(Y\), and \(f^h\) and \(\tilde{\psi}^h\) are both piecewise constant. Let us compute the right-hand side of (17). As \(\xi_x \in \varepsilon\mathbb{Z}^2\) and \(g \in C(Y)\), we have

\[
g \left( \frac{x - \xi_x}{\varepsilon} \right) = g \left( \frac{x}{\varepsilon} \right),
\]

and see (after extending \(f^h\) to \(\mathbb{R}^2\) by zero)

\[
\int_S \frac{f^h(x)}{\varepsilon} \psi(x) - \tilde{\psi}^h(x) \frac{g \left( \frac{x}{\varepsilon} \right)}{\varepsilon} \, dx
\]

\[
= \sum_{\xi \in \varepsilon\mathbb{Z}^2} f^h(\xi) \int_{\xi + \varepsilon Y} \frac{\psi(x) - \tilde{\psi}^h(\xi)}{\varepsilon} g \left( \frac{x}{\varepsilon} \right) \, dx
\]

\[
= \sum_{\xi \in \varepsilon\mathbb{Z}^2} f^h(\xi) \int_{\xi + \varepsilon Y} \left( \int_0^1 (\nabla' \psi(\xi + t x) \cdot \frac{x - \xi}{\varepsilon}) g \left( \frac{x}{\varepsilon} \right) \, dx \right. \]

\[
= \sum_{\xi \in \varepsilon\mathbb{Z}^2} f^h(\xi) \varepsilon^2 \int_Y \left( \int_0^1 (\nabla' \psi(\xi + t \varepsilon y) \cdot y) g(y) \, dy \right.
\]

\[
+ \sum_{\xi \in \varepsilon\mathbb{Z}^2} f^h(\xi) \varepsilon^2 \int_Y (\nabla' \psi(\xi + t \varepsilon y) - \nabla' \psi(\xi)) \cdot y g(y) \, dy.
\]
The first term on the right-hand side converges to zero as \( h \to 0 \) because
\[
|\nabla'\psi(\xi + t\varepsilon y) - \nabla'\psi(\xi)| \leq C\varepsilon
\]
for all \( t \in [0, 1] \), simply because \( \nabla'\psi \) is Lipschitz.

Hence it remains to compute:
\[
\sum_{\xi \in \mathbb{Z}^d} \varepsilon^2 f^h(\xi) \nabla'\psi(\xi) \cdot \int_Y yg(y) \, dy
\]
\[
= \sum_{\xi \in \mathbb{Z}^d} \varepsilon^2 \left( f^h(\xi) \nabla'\psi(\xi) - \int_{\xi + \varepsilon Y} f^h(z) \nabla'\psi(z) \, dz \right) \cdot \int_Y yg(y) \, dy
\]
\[
+ \sum_{\xi \in \mathbb{Z}^d} \varepsilon^2 f^h(z) \nabla'\psi(z) \, dz \cdot \int_Y yg(y) \, dy
\]
\[
\sum_{\xi \in \mathbb{Z}^d} \varepsilon^2 \left( f^h(\xi) \nabla'\psi(\xi) - \int_{\xi + \varepsilon Y} f^h(z) \nabla'\psi(z) \, dz \right) \cdot \int_Y yg(y) \, dy
\]
\[
+ \int_{\mathbb{R}^2} f^h(x) \nabla'\psi(x) \, dx \cdot \int_Y yg(y) \, dy.
\]
Since \( \text{spt } \psi \subset S \), the last term equals
\[
\int_S f^h(x) \nabla'\psi(x) \, dx \cdot \int_Y yg(y) \, dy.
\]
The claim follows because \( f^h \rightharpoonup f^0 \) and because
\[
\sum_{\xi \in \mathbb{Z}^d} \varepsilon^2 \left( f^h(\xi) \nabla'\psi(\xi) - \int_{\xi + \varepsilon Y} f^h(z) \nabla'\psi(z) \, dz \right) \cdot \int_Y yg(y) \, dy \to 0
\]
as \( h \to 0 \). To see this, we compute recalling that \( f^h(x) = f^h(\xi) \) for all \( x \in \xi + \varepsilon Y \):
\[
f^h(\xi) \nabla'\psi(\xi) - \int_{\xi + \varepsilon Y} f^h(z) \nabla'\psi(z) \, dz = f^h(\xi) \left( \nabla'\psi(\xi) - \int_{\xi + \varepsilon Y} \nabla'\psi(z) \, dz \right) \leq Ch,
\]
again because \( \nabla'\psi \) is Lipschitz.

\( \square \)

**Proof of Proposition 3.2, case \( \gamma \in (0, \infty) \). Step 1.** Without loss of generality we assume that all \( u^h \) have average zero. Theorem 2.5 then implies that
\[
\nabla_h u^h \to R := (\nabla' u, n) \quad \text{strongly in } L^2(\Omega, \mathbb{R}^{3 \times 3}) \quad (18)
\]
where \( n \) denotes the normal to \( u \). Let \( R^h, \tilde{R}^h \) be the maps obtained by applying Lemma 3.3 to \( u^h \) with \( \delta(h) = \varepsilon(h) \). Due to the uniform bound on \( \nabla'\tilde{R}^h \) given by Lemma 3.3, \( R^h \) and \( \tilde{R}^h \) are precompact in \( L^2(S, \mathbb{R}^{3 \times 3}) \). Hence, (18) combined with \( \|R - \nabla_h u^h\|_{L^2} \to 0 \) (which also follows from Lemma 3.3) shows that \( R^h \) and \( \tilde{R}^h \) strongly converge in \( L^2(S, \mathbb{R}^{3 \times 3}) \) to \( R \). Following [FJM02], we introduce the approximate strain
\[
G^h(x) = \frac{(R^h)^t \nabla_h u^h(x) - I}{h}. \quad (19)
\]
We set $\pi^h(x') = \int_I u^h(x', x_3) \, dx_3$ and define $z^h \in H^1(\Omega, \mathbb{R}^3)$ via

$$u^h(x', x_3) = \pi^h(x') + h x_3 \tilde{R}^h(x') e_3 + h z^h(x', x_3).$$  \tag{20}

Then clearly $\int_I z^h(x', x_3) \, dx_3 = 0$ and we compute

$$\frac{\nabla_h u^h}{\mu} - R_h = \left( \nabla' \pi^h - (R^h)' \right) \frac{2}{\gamma} + x_3 \nabla' \tilde{R}^h e_3, \quad \frac{1}{\mu} \left( \tilde{R}^h e_3 - R^h e_3 \right) = \frac{1}{\gamma} (y \cdot \nabla')^R(x') + \nabla z^h. \tag{21}
$$

For a given matrix $M \in \mathbb{R}^{3 \times 3}$, we denote by $M'$ the $3 \times 2$-matrix obtained by deleting the third column. We use the notation $(y \cdot \nabla')^R(x') := y_1 \partial_1 R(x') + y_2 \partial_2 R(x')$.

**Step 2.** Let us for the moment take for granted that there exist $B' \in L^2(S, \mathbb{R}^{3 \times 2})$, $\tilde{z} \in L^2(S, H^1(\Omega \times \mathbb{R}^3))$, $\tilde{v}, \tilde{w} \in L^2(S, H^1(\mathcal{Y}, \mathbb{R}^3))$ and $u^0 \in L^2(S, \mathbb{R}^3)$, such that, after passing to a subsequence,

$$\nabla_h z^h \xrightarrow{2} \left( \nabla_y \tilde{z}, \frac{1}{\gamma} \partial_3 \tilde{z} \right), \tag{22}
$$

$$\frac{\nabla' \pi^h - (R^h)'}{\mu} \xrightarrow{2} B'(x') + \frac{1}{\gamma} (y \cdot \nabla') R^h(x') + \nabla y \tilde{v}(x', y), \tag{23}
$$

$$x_3 \nabla' \tilde{R}^h e_3 \xrightarrow{2} x_3 \nabla' R(x') e_3 + x_3 \nabla_y \tilde{w}(x', y), \tag{24}
$$

$$\frac{1}{\mu} \left( \tilde{R}^h e_3 - R^h e_3 \right) \xrightarrow{2} \frac{1}{\gamma} (y \cdot \nabla') R(x') e_3 + u^0(x') + \tilde{w}(x', y). \tag{25}
$$

We now proceed to prove that the proposition follows from these convergences.

First notice that it suffices to identify the symmetric part of the two-scale limit $G$ of the sequence $G^h$. Indeed, since $\sqrt{(I + hF)/(I + hF)} = I + h \text{sym} F$ up to terms of higher order, the convergence $G^h \xrightarrow{2} G$ implies $E = \text{sym} G$ (see e.g. [Neu12, Lemma 4.4] for a proof).

We now identify sym $G$. By combining (22) – (25) with identity (21), we find that $R^h G^h$ weakly two-scale converges to

$$(B', 0) + \left( x_3 \nabla' R(x') e_3, 0 \right) + \left( \nabla_y \bar{\phi}, \frac{1}{\gamma} \partial_3 \bar{\phi} \right) + (y \cdot \nabla') R(x') \tag{26}$$

where

$$\bar{\phi}(x, y) := \tilde{z}(x, y) + \tilde{v}(x', y) + x_3 \tilde{w}(x', y) + x_3 u^0(x').$$

Due to the strong $L^2$-convergence $R^h \to R$, we deduce that $G^h$ weakly two-scale converges to (26) multiplied with $R'$ from the left. The first and second term yield

$$\begin{pmatrix} \tilde{B}(x') + x_3 \Pi(x') & 0 \\ x_3 b_1(x') & x_3 b_2(x') \end{pmatrix},$$

where $\tilde{B}$ denotes the $2 \times 2$-matrix obtained by deleting the third column of $R' B'$ and $(b_1, b_2)$ are defined as the entries of the third row of $R' B'$. Upon left multiplication by $R'$, the last term in (26) yields a skew-symmetric term. Thus we have shown:

$$\text{sym} G(x, y) = \begin{pmatrix} \text{sym} \tilde{B} + x_3 \Pi & 0 \\ 0 & 0 \end{pmatrix} + \text{sym} \left( \nabla_y \phi, \frac{1}{\gamma} \partial_3 \phi \right).$$

14
\[ \phi(x, y) := R^\gamma(x') \tilde{\phi}(x, y) + \gamma x_3 \begin{pmatrix} b_1(x') \\ b_2(x') \\ 0 \end{pmatrix}. \]

**Step 3.** It remains to prove (22) – (25). Since \( \nabla_h z^h \) is uniformly bounded in \( L^2 \) and since \( \int_I z^h \, dx = 0 \) by construction, (22) directly follows from Lemma 3.5.

Next we prove (23). By Lemma 3.3 and Lemma 3.4 (i) there exists \( V \in L^2(S \times Y, \mathbb{R}^{3 \times 2}) \) such that (after taking subsequences)

\[
\frac{\nabla^\gamma h - (R^h)' \, 2 \gamma}{h} \overset{2 \gamma}{\to} V.
\] (27)

Let us verify that

\[ V(x', y) = B'(x') + (y \cdot \nabla') R'(x') + \nabla_y \tilde{v}(x', y), \]

where \( B' := \int_Y V(., y) \, dy \) and \( \tilde{v} \in L^2(S, H^1(\mathcal{Y})) \). This is equivalent to showing that

\[
\int_{S \times \mathcal{Y}} V(x', y) : (\nabla^\gamma G(\gamma) \psi(x') \, dy \, dx' = \int_{S \times \mathcal{Y}} (y \cdot \nabla') R'(x') : (\nabla^\gamma G(\gamma) \psi(x') \, dy \, dx', (28)
\]

for all \( G \in C^1(\mathcal{Y}, \mathbb{R}^3) \), \( \psi \in C_0^\infty(S) \). Here and below \( \nabla^\gamma_y := (-\partial_{y_2}, \partial_{y_1}) \). Set \( G^h(x') := \varepsilon(h) G(\frac{x'}{\varepsilon(h)}) \), so that \( (\nabla')^\gamma G^h(x') = (\nabla^\gamma_y G)(\frac{x'}{\varepsilon(h)}) \).

To prove (28), note that since \( \int_S \nabla^\gamma h : \nabla^\gamma (G^h \psi) = 0 \), we have

\[
\int_S \frac{\nabla^\gamma h(x')}{h} : (\nabla^\gamma_y G) \left( \frac{x'}{\varepsilon(h)} \right) \psi(x') \, dx' \]

\[
= \int_S \frac{\nabla^\gamma h(x')}{h} : \nabla^\gamma (G^h(x') \psi(x')) \, dx' \]

\[
= - \int_S \frac{\nabla^\gamma h}{h} : G^h(x') \otimes \nabla^\gamma \psi(x') \, dx' \]

\[
= - \frac{\varepsilon(h)}{h} \int_S \nabla^\gamma h : G(\frac{x'}{\varepsilon(h)}) \otimes \nabla^\gamma \psi(x') \, dx'.
\]

The right-hand side converges to 0, since \( \frac{\varepsilon(h)}{h} \nabla^\gamma h \) strongly converges in \( L^2 \) and \( G(\frac{x}{\varepsilon(h)}) \to 0 \) weakly in \( L^2 \). In addition, Lemma 3.8 yields

\[
\frac{R^h}{h} = \frac{\varepsilon(h)}{h} \frac{1}{\varepsilon(h)} R^h \overset{\text{osc,} \gamma}{\Rightarrow} \frac{1}{\gamma} (y \cdot \nabla') R(x'),
\] (29)

and thus

\[
\int_S \frac{\nabla^\gamma h(x') - (R^h)'(x')}{h} : (\nabla^\gamma_y G) \left( \frac{x'}{\varepsilon(h)} \right) \psi(x') \, dx' \]

\[
\to - \int_{S \times \mathcal{Y}} \frac{1}{\gamma} (y \cdot \nabla') R'(x') : \nabla^\gamma_y G(\gamma) \psi(x') \, dy \, dx'.
\]

15
On the other hand, the left-hand side converges to
\[
\int_{S \times Y} V(x', y) : \nabla_y^\perp G(y) \psi(x') \, dy \, dx'.
\]
Hence, (28) and thus (23) follows.

We prove (24) and (25). By Lemma 3.3 the right-hand side in (25) is uniformly bounded in \(L^2(S, \mathbb{R}^3)\) and thus we have (after passing to subsequences)
\[
\left( \frac{\tilde{R}^h - R^h}{h} \right)_{e_3} \xrightarrow{\gamma} w(x', y)
\]
for some \(w \in L^2(S \times Y, \mathbb{R}^3)\). Set \(w^0(x') := \int_Y w(x', y) \, dy\). Since \(\tilde{R}^h_{e_3} \rightharpoonup R_{e_3}\) weakly in \(H^1(S, \mathbb{R}^3)\), we know from Lemma 3.4 (v) that there exists \(\tilde{w} \in L^2(S, H^1(I \times Y, \mathbb{R}^3))\) such that
\[
\nabla' \tilde{R}^h_{e_3} \xrightarrow{\gamma} \nabla' R_{e_3} + \nabla_y \tilde{w}.
\]
This implies (24). The combination of (30) with Lemma 3.7 yields \(\frac{\tilde{R}^h - R^h}{h} \xrightarrow{\text{osc} \gamma} \gamma^{-1} \tilde{w}(x', y) + \gamma^{-1} (y \cdot \nabla') R(x') e_3\), and (25) follows from Lemma 3.6.

Proof of Proposition 3.2, case \(\gamma = \infty\). The argument is similar to the case \(\gamma \in (0, \infty)\). Therefore we only indicate the required modifications. Step 1 (of the proof for \(\gamma \in (0, \infty)\)) holds verbatim modulo the following change: As a difference to \(\gamma \in (0, \infty)\), in the case \(\gamma = \infty\) we set \(\delta(h) := \lceil \frac{h}{\epsilon(h)} \rceil \epsilon(h)\) where \(\lceil \cdot \rceil\) denotes the smallest, positive integer larger or equal to \(s\). By construction \(\delta(h)\) is an integer multiple of \(\epsilon(h)\) and we have \(\frac{\delta(h)}{\epsilon(h)} \sim 1\). Hence, Lemma 3.3 yields maps \(R^h\) and \(\tilde{R}^h\) with bounds uniform in \(h\), and moreover \(R^h\) is constant on each cube \(x + \epsilon(h) Y, x \in \epsilon(h) Z^2\).

Similar to Step 2 (of the proof for \(\gamma \in (0, \infty)\)) the statement of the proposition can be reduced to show that (up to subsequences)
\[
\nabla_{h} z - h \frac{2 \gamma}{\epsilon(h)} (\nabla_y \tilde{w}, d'), \quad (31)
\]
\[
\nabla^\perp \Gamma - (R^h)' \left( \frac{2 \gamma}{\epsilon(h)} \right) B'(x') + \nabla_y \tilde{w}(x', y), \quad (32)
\]
\[
\nabla' \tilde{R}^h_{e_3} \left( \frac{2 \gamma}{\epsilon(h)} \right) x_3 \nabla' R(x') e_3 + x_3 \nabla_y \tilde{w}(x', y), \quad (33)
\]
\[
\frac{1}{h} (\tilde{R}^h_{e_3} - R^h_{e_3}) \left( \frac{2 \gamma}{\epsilon(h)} \right) w^0(x'), \quad (34)
\]
where \(R\) is defined as in the case \(\gamma \in (0, \infty)\), and \(B' \in L^2(S, \mathbb{R}^{3 \times 3})\), \(\tilde{z} \in L^2(\Omega, H^1(\mathcal{Y}, \mathbb{R}^3))\), \(d' \in L^2(\Omega, \mathbb{R}^3)\), \(\tilde{v}, \tilde{w} \in L^2(S, H^1(\mathcal{Y}, \mathbb{R}^3))\) and \(w^0 \in L^2(S, \mathbb{R}^3)\). Indeed, by the same arguments as for \(\gamma \in (0, \infty)\), (31) – (34) imply that
\[
sym G(x, y) = \begin{pmatrix}
sym \tilde{B} + x_3 I & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \text{sym} (\nabla_y \phi(x, y), d(x)),
\]
\[
16
\]
where
\[ \phi(x,y) := R^d(z + v + x_3w), \quad d = R^d' + R^d_w + \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}, \]
and \( \tilde{B}, R \) and \((b_1, b_2)\) are defined as in the case \( \gamma \in (0, \infty) \).

The proof of (31) – (34) is similar to Step 3 of the proof for the case \( \gamma \in (0, \infty) \).

4 Proof of Theorem 2.4

As a preliminary step we need to establish some continuity properties of the quadratic form appearing in (QE) and its relaxed version introduced in Definition 2.3.

Lemma 4.1. Let \( W \) be as in Assumption 2.1 and let \( Q \) be the quadratic form associated to \( W \) through the expansion (QE). Then

(Q1) \( Q(\cdot, G) \) is \( Y \)-periodic and measurable for all \( G \in \mathbb{R}^{3 \times 3} \),

(Q2) for almost every \( y \in \mathbb{R}^2 \) the map \( Q(y, \cdot) \) is quadratic and satisfies
\[ c_1 |\text{sym } G|^2 \leq Q(y, G) = Q(y, \text{sym } G) \leq c_2 |\text{sym } G|^2 \]
for all \( G \in \mathbb{R}^{3 \times 3} \).

Furthermore, there exists a monotone function \( r : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\} \) that can be chosen only depending on the parameters \( c_1, c_2 \) and \( \rho \), such that \( r(\delta) \to 0 \) as \( \delta \to 0 \) and
\[ \forall G \in \mathbb{R}^{3 \times 3} : |W(y, I + G) - Q(y, G)| \leq |G|^2 r(|G|) \]
for almost every \( y \in \mathbb{R}^2 \).

(For a proof see [Neu12, Lemma 2.7].)

Lemma 4.2. (a) Let \( \gamma \in (0, \infty) \). For all \( A \in \mathbb{R}^{2 \times 2}_{\text{sym}} \) there exist a unique pair \((B, \phi)\) with \( B \in \mathbb{R}^{2 \times 2}_{\text{sym}} \) and \( \phi \in \dot{H}^1(I \times \mathcal{Y}, \mathbb{R}^3) \) such that:
\[ Q_{2, \gamma}(A) = \int_{I \times \mathcal{Y}} Q(y, \Lambda(x_3, A, B) + (\nabla y \phi, \frac{1}{\gamma} \partial_3 \phi)) \, dy dx_3 \]
The induced mapping \( \mathbb{R}^{2 \times 2}_{\text{sym}} \ni A \mapsto (B, \phi) \in \mathbb{R}^{2 \times 2}_{\text{sym}} \times \dot{H}^1(I \times \mathcal{Y}, \mathbb{R}^3) \) is bounded and linear.

(b) Let \( \gamma = \infty \). For all \( A \in \mathbb{R}^{2 \times 2}_{\text{sym}} \) there exist a unique triple \((B, \phi, d)\) with \( B \in \mathbb{R}^{2 \times 2}_{\text{sym}}, \phi \in L^2(I, \dot{H}^1(\mathcal{Y}, \mathbb{R}^3)) \) and \( d \in L^2(I, \mathbb{R}^3) \) such that:
\[ Q_{2, \infty}(A) = \int_{I \times \mathcal{Y}} Q(y, \Lambda(x_3, A, B) + (\nabla y \phi, d)) \, dy dx_3 \]
The induced mapping \( \mathbb{R}^{2 \times 2}_{\text{sym}} \ni A \mapsto (B, \phi, d) \in \mathbb{R}^{2 \times 2}_{\text{sym}} \times L^2(I, \dot{H}^1(\mathcal{Y}, \mathbb{R}^3)) \times L^2(I, \mathbb{R}^3) \) is bounded and linear.
Proof of Lemma 4.2. The proof is standard.
We only comment on the proof of part (a), i.e. for $\gamma \in (0, \infty)$. We start with
the following Korn-type inequality
\[
\forall \psi \in H^1(I \times \gamma, \mathbb{R}^3) : \quad \|\psi\|^2_{H^1(I \times Y, \mathbb{R}^3)} \leq C \iint_{I \times Y} |\text{sym}(\nabla_y \psi, \frac{1}{\gamma} \partial_3 \psi)|^2 \quad (36)
\]
for some constant $C > 0$. Since $Q$ is elliptic in the sense of Lemma 4.1 (Q2), and since $Q$ vanishes for skew-symmetric matrices, for each pair $(A, B)$ of symmetric $2 \times 2$ matrices we can find a unique function $\phi = \phi_{A, B} \in H^1(I \times \gamma, \mathbb{R}^3)$ minimizing the integral
\[
\tilde{Q}_\gamma(A, B) := \iint_{I \times Y} Q \left( y, A(x, 3, A, B) + (\nabla_y \phi, \frac{1}{\gamma} \partial_3 \phi) \right) dy dx_3. \quad (37)
\]
Evidently $\phi$ depends linearly on $(A, B)$. In particular, for $i = 1, \ldots, 3$ there exist $E_i \in H^1(I \times Y, \mathbb{R}^{2 \times 2}_\text{sym})$ such that
\[
\phi_i(x, 3, y) = E_i(x, 3, y) : A + F_i(x, 3, y) : B. \quad (38)
\]
As a consequence $\tilde{Q}_\gamma$ is a quadratic form and it is easy to check that there exist positive constants $c_{\gamma, 1} < c_{\gamma, 2}$ such that
\[
c_{\gamma, 1}(|A|^2 + |B|^2) \leq \tilde{Q}_\gamma(A, B) \leq c_{\gamma, 2}(|A|^2 + |B|^2).
\]
Hence, we conclude that there exists a bounded, positive definite operator $A : \mathbb{R}^{2 \times 2}_\text{sym} \times \mathbb{R}^{2 \times 2}_\text{sym} \to \mathbb{R}^{2 \times 2}_\text{sym} \times \mathbb{R}^{2 \times 2}_\text{sym}$ such that
\[
\tilde{Q}_\gamma(A, B) = \langle A(A, B), (\text{sym} A, B) \rangle,
\]
where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{2 \times 2}_\text{sym} \times \mathbb{R}^{2 \times 2}_\text{sym}$. By a block-wise decomposition of the right-hand side we get
\[
\tilde{Q}(A, B) = A_1 A : A + A_2(A) : B + A_3 B : B
\]
where $A_1, A_2, A_3 \in L(\mathbb{R}^{2 \times 2}_\text{sym})$ are bounded operators on $\mathbb{R}^{2 \times 2}_\text{sym}$ and $A_1$ and $A_3$ are positive definite. A straightforward calculation shows that this expression is minimized (with respect to $B$) by $B_A = -\frac{1}{c_{\gamma, 2}} A_3^{-1} A_2 \text{sym} A$. Since this expression is obviously linear in $A$, the desired pair of functions associated with $A$ is given by $B_A$ and $\phi = \phi_{A, B_A}$. \qed

Proof of Theorem 2.4 (lower bound). Without loss of generality we may assume that $\int_{\Omega} u_h dx = 0$ and $\limsup_{h \to 0} h^{-2} \mathcal{E}^{h, \varepsilon}(u_h) < \infty$. We only consider the case $\gamma \in (0, \infty)$. The proof in the case $\gamma = \infty$ holds verbatim. In view of (ND), the sequence $u_h$ has finite bending energy and the sequence $E_h$, see (7), is bounded in $L^2(\Omega, \mathbb{R}^{3 \times 3})$. Hence, from Theorem 2.5 we deduce that $u \in W_0^{2, 2} (S, \mathbb{R}^3)$. By Lemma 3.4 (i) and Proposition 3.2 (i) we can pass to a subsequence such that, for some $E \in L^2(\Omega \times Y; \mathbb{R}^{3 \times 3})$,
\[
E_h \xrightarrow{a.s.} E,
\]

18
where $E$ can be written in the form of (10). As in [FJM02] a careful Taylor expansion of $W(\frac{1}{E(x)}, I + hE(x))$ combined with the lower semi-continuity of convex integral functionals with respect to weak two-scale convergence (see e.g. [Vis07, Proposition 1.3]) yields the lower bound

$$\liminf_{h \to 0} \frac{1}{h^2} E^{h,\varepsilon}(h) \geq \int_{\Omega \times Y} Q(x',y,E(x,y)) \, dy \, dx.$$ 

Combined with (10) the right-hand side becomes

$$\int_{\Omega \times Y} Q(x',y,\Lambda(II(x'),B(x') + \nabla \phi(x,y),1) \partial_3 \phi(x,y)) \, dy \, dx,$$

where we used that $Q(x',y,F)$ only depends on $F$. Minimization over $B \in L^2(S,\mathbb{R}^2 \times \mathbb{R}^2)$ and $\phi \in L^2(S,H^1(I \times Y,\mathbb{R}^3))$ yields

$$\liminf_{h \to 0} \frac{1}{h^2} E^{h,\varepsilon}(h) \geq \int_{S} Q_{2,\gamma}(II(x')) \, dx' = \mathcal{E}(u).$$

It remains to prove the upper bound. As in [Sch07] and other related results, the key ingredient here is the density result for $W^{2,2}$ isometric immersions established in [Ho11a, Ho11b] (cf. also [Pa04] for an earlier result in this direction). It is the need for the results in [Ho11b] that forces us to consider domains $S$ which are not only Lipschitz but also piecewise $C^1$ – more precisely, we only need that the outer unit normal be continuous away from a subset of $\partial S$ with length zero.

For brevity, we denote by $\mathcal{A}(S)$ the set of all $u \in W^{2,2}_\delta(S,\mathbb{R}^3) \cap C^\infty(\overline{S},\mathbb{R}^3)$ with the following property:

For each $B \in C^\infty(\overline{S},\mathbb{R}^{2 \times 2})$ with satisfying $B = 0$ in a neighborhood of $\{x \in S: \Pi(x) = 0\}$, there exist $\alpha \in C^\infty(\overline{S})$ and $g \in C^\infty(\overline{S},\mathbb{R}^2)$ such that

$$B = \text{sym} \nabla'g + \alpha \Pi. \tag{39}$$

The key ingredient in the proof of the upper bound is the following lemma.

**Lemma 4.3.** The set $\mathcal{A}(S)$ is dense in $W^{2,2}_\delta(S)$ with respect to the strong $W^{2,2}$-topology.

**Proof.** This follows by combining the construction from [Ho11a, Ho11b] with the arguments in [Sch07] leading to his Lemma 3.3. His result was recently re-derived in [HLP] in a slightly different context.

Thanks to Lemma 4.3 it will be enough to construct recovery sequences for limiting deformations belonging to $\mathcal{A}(S)$. First we present a construction assuming the existence of $\alpha$ and $g$ satisfying (39).

**Lemma 4.4.** For $u \in W^{2,2}_\delta(S,\mathbb{R}^3) \cap W^{2,\infty}(S,\mathbb{R}^3)$, $\alpha \in W^{1,\infty}(S)$ and $g \in W^{1,\infty}(S,\mathbb{R}^2)$ define

$$B(x') = \text{sym} \nabla'g(x') + \alpha(x') \Pi(x').$$
(a) Let $\gamma \in (0, \infty)$ and $\phi \in C^\infty_c(S, C^\infty(I \times Y, \mathbb{R}^3))$. Then there exists a sequence $\{u^h\}_{h>0} \subset H^1(\Omega, \mathbb{R}^3)$ such that

$$u^h \to u, \quad \nabla_h u^h \to (\nabla' u, n) \quad \text{uniformly in } \Omega$$

and

$$\lim_{h \to 0} \frac{1}{h^2} E^{\gamma, \varepsilon(h)}(u^h) = \iint_{\Omega_\varepsilon} Q\left(y, \Lambda(\Pi(x'), B(x')) + (\nabla_y \phi, \frac{1}{\gamma} \partial_3 \phi)\right) \, dy \, dx \, dx'. \quad (40)$$

(b) Let $\gamma = \infty$, $\phi \in C^\infty_c(\Omega, C^\infty(Y, \mathbb{R}^3))$ and $d \in C^\infty_c(\Omega, \mathbb{R}^3)$. Then there exists a sequence $\{u^h\}_{h>0} \subset H^1(\Omega, \mathbb{R}^3)$ such that

$$u^h \to u, \quad \nabla_h u^h \to (\nabla' u, n) \quad \text{uniformly in } \Omega$$

and

$$\lim_{h \to 0} \frac{1}{h^2} E^{h, \varepsilon(h)}(u^h) = \iint_{\Omega_\varepsilon} Q\left(y, \Lambda(\Pi(x'), B(x')) + (\nabla_y \phi, d)\right) \, dy \, dx \, dx'. \quad (41)$$

Proof. We start with the case $\gamma \in (0, \infty)$ and follow [Sch07, Theorem 3.2 (ii)]. Consider

$$v^h(x) := u(x') + h \left[(x_3 + \alpha(x'))n(x') + (g(x') \cdot \nabla')u(x')\right],$$

$$R(x') := (\nabla' u(x'), n(x')).$$

Here $(g \cdot \nabla')u$ stands for $\sum_{\alpha=1,2} g_\alpha \partial_\alpha u$. A direct calculation (see [Sch07]) shows that

$$R^t \nabla v^h = I + hG$$

with

$$G := \begin{pmatrix} x_3 \Pi(x') + B & 0 \\ 0 & -b' \end{pmatrix}, \quad B := \nabla' g + \alpha \Pi, \quad b := -\left(\frac{\partial_1 \alpha}{\partial_2 \alpha}\right) + II g.$$ 

To $v^h$ we add an oscillating correction and a term compensating for $b$:

$$u^h(x) := v^h(x) + h\varepsilon(h) \tilde{\phi}(x, \frac{x'}{\varepsilon(h)}) \quad (42)$$

where

$$\tilde{\phi}(x, y) := R(x') \left[\phi(x, y) + \gamma x_3 \begin{pmatrix} b \\ 0 \end{pmatrix}\right].$$

Since $\alpha, g$ and $\phi$ are sufficiently smooth, the uniform convergence of $u^h$ and $\nabla_h u^h$ directly follows from the construction. Moreover, we have

$$R^t \nabla u^h = I + h(G + G^b)$$

20
where
\[ G^h := \left( \nabla_y \phi(x, \frac{x'}{\varepsilon(h)}), \; \frac{1}{\gamma} \partial_{33} \phi(x, \frac{x'}{\varepsilon(h)}) + \frac{\varepsilon(h)}{h} \left( \begin{array}{c} b \\ 0 \end{array} \right) \right) + C^h. \]

Here the remainder term \( C^h \) satisfies \( \limsup_{h \to 0} \frac{|C^h|}{h} < \infty \). Again, since \( u, \phi, \alpha \) and \( g \) are sufficiently smooth, we have
\[
\limsup_{h \to 0} \sup_{x \in \Omega} \left( |G(x)| + |G^h(x)| \right) = 0. \tag{43}
\]

Hence, (FI), (QE) and (35) yield
\[
\limsup_{h \to 0} \frac{1}{h^2} e^{g_\gamma(x)}(u^h) - \int_{\Omega} Q(x', G(x) + G^h(x)) \, dx = 0,
\]
and it suffices to show that
\[
\lim_{h \to 0} \int_{\Omega} Q(x', G(x) + G^h(x)) \, dx = [\text{R. H. S. of (40)}]. \tag{44}
\]

By construction the sequence \( G + G^h \) strongly two-scale converges to some limit \( \tilde{G} \in L^2(\Omega \times Y, \mathbb{R}^{3 \times 3}) \) with
\[
\text{sym} \tilde{G} = \Lambda(\Pi, \text{sym} B) + \text{sym} \left( \nabla_y \phi(x, y), \; \frac{1}{\gamma} \partial_{33} \phi(x, y) \right).
\]

Hence, by the continuity of convex integral functionals with respect to strong two-scale convergence we can pass to the limit in (44) (e. g. see [Neu12, Lemma 4.8]). Since \( Q(y, F) \) only depends on the symmetric part of \( F \), this completes the proof for the case \( \gamma \in (0, \infty) \).

The proof for \( \gamma = 0 \) is similar to the above reasoning. Essentially, we only need to replace (42) by
\[
u^h(x) = v^h(x) + h \varepsilon(h) \tilde{\phi}(x, \frac{x'}{\varepsilon(h)}) + h^2 \tilde{d}(x),
\]
where
\[
\tilde{\phi}(x, y) = R(x') \phi(x, y) \quad \text{and} \quad \tilde{d}(x) = R(x') \left[ \int_{-1/2}^{x_3} d(x', t) \, dt + x_3 \left( \begin{array}{c} b(x') \\ 0 \end{array} \right) \right].
\]

\[\square\]

Proof of Theorem 2.4 (Upper bound). We only consider the case \( \gamma \in (0, \infty) \) since the argument for \( \gamma = \infty \) is the same. We may assume that \( E_\gamma(u) < \infty \), so \( u \in W^{2,2}_\delta(S, \mathbb{R}^3) \). Moreover, since \( Q_{2,\gamma} \) (see Lemma 4.2) is continuous, it suffices to prove the statement for a dense subclass of \( W^{2,2}_\delta(S, \mathbb{R}^3) \). Hence, by virtue of Lemma 4.3, we may assume without loss of generality that \( u \in \mathcal{A}(S) \).

By Lemma 4.2 there exist \( B \in L^2(S, \mathbb{R}^{2 \times 2}) \) and \( \phi \in L^2(S, H^1(I \times Y, \mathbb{R}^3)) \) such that
\[
E^\gamma(u) = \int_{\Omega \times Y} Q(y, \Lambda(\Pi, B) + \text{sym}(\nabla_y \phi, \frac{1}{\gamma} \partial_{33} \phi)) \, dy \, dx. \tag{45}
\]
Since $B(x')$ linearly depends on $\Pi(x')$ we know in addition that $B(x') = 0$ for $x' \in \{ \Pi = 0 \}$.

By a density argument it suffices to show the following: There exists a doubly indexed sequence $u^{h,\delta} \in H^1(\Omega, \mathbb{R}^3)$ such that

$$
\lim_{\delta \to 0} \limsup_{h \to 0} \| u^{h,\delta} - u \|_{L^2(\Omega, \mathbb{R}^3)} = 0, \quad (46)
$$

$$
\lim_{h \to 0} \left| \frac{1}{h^2} E^{h,\varepsilon}(h)(u^{h,\delta}) - \mathcal{E}_\gamma(u) \right| < \delta. \quad (47)
$$

Indeed, if this is the case then we obtain the desired sequence by extracting a diagonal sequence (e. g. by appealing to [Att84, Corollary 1.16]).

We construct $u^{h,\delta}$ as follows: By a density argument we may choose for each $\delta > 0$ functions $B^\delta \in C^\infty(S, \mathbb{R}^{2 \times 2}_{\text{sym}})$ and $\phi^\delta \in C^\infty_c(S, C^\infty(I \times \mathcal{Y}, \mathbb{R}^3))$ such that

$$
\| B^\delta - B^\delta \|_{L^2(S)} + \| \nabla_y \phi^\delta - \nabla_y \phi^\delta \|_{L^2(\Omega \times \mathcal{Y})} + \| \partial_3 \phi^\delta - \partial_3 \phi \|_{L^2(\Omega \times \mathcal{Y})} \leq \delta^2, \quad (48)
$$

$$
B^\delta = 0 \text{ in a neighborhood of } \{ \Pi = 0 \}. \quad (49)
$$

Because $u \in \mathcal{A}(S, \mathbb{R}^3)$ and due to (49) we can find for each $\delta > 0$ smooth functions $\alpha^\delta$ and $g^\delta$ such that $B^\delta = \text{sym} \nabla_y g^\delta + \alpha^\delta \Pi$. We apply Lemma 4.4 to $u, g^\delta, \alpha^\delta$ and $\phi^\delta$. We obtain a sequence $\{ u^{h,\delta} \}_{h>0}$ that uniformly converges to $u$ as $h \to 0$. Hence, (46) is satisfied. Moreover, Lemma 4.4 yields

$$
\lim_{h \to 0} \frac{1}{h^2} E^{h}(u^{h,\delta}) = \iint_{\Omega \times \mathcal{Y}} Q(y, \Lambda(\Pi, B^\delta) + \text{sym}(\nabla_y \phi^\delta, \frac{1}{\gamma} \partial_3 \phi^\delta)) dydx.
$$

By continuity of the functional on the right-hand side, combined with (48) and (45), the bound (47) follows.

Acknowledgements. Hornung and Velčić were supported by Deutsche Forschungsgemeinschaft grant no. HO4697/1-1.

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22
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