SYMPLECTIC REPRESENTATIONS OF INERTIA GROUPS

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1. INTRODUCTION AND NOTATION

The aim of this paper is to study finite inertia subgroups of symplectic groups over the field \( \mathbb{Q}_\ell \) of \( \ell \)-adic numbers. A finite group is called an inertia group with respect to a given prime \( p \neq \ell \) if it is a semi-direct product of a finite normal \( p \)-subgroup and a cyclic \( p' \)-group. These groups are exactly the inertia groups of finite Galois extensions of discrete valuation fields of residue characteristic \( p \). The study of semistable reduction of abelian varieties over such fields leads naturally to certain finite inertia subgroups of the symplectic group \( \text{Sp}_{2g}(\mathbb{Q}_\ell) \) [6]. In [7] we constructed examples for every odd prime \( \ell \) of inertia subgroups of \( \text{Sp}_{2g}(\mathbb{Q}_\ell) \) which are not conjugate, even in \( \text{GL}_{2g}(\mathbb{Q}_\ell) \). However, it turns out (and this is the main result of this paper) that every finite inertia subgroup of \( \text{Sp}_{2g}(\mathbb{Q}_\ell) \) is isomorphic to a subgroup of \( \text{Sp}_{2g}(\mathbb{Z}_\ell) \), if \( \ell > 3 \).

See [3] for a study of representations of inertia groups in characteristic 0.

Throughout this paper \( \ell \) is an odd prime, \( K \) is a field that is an unramified finite extension of \( \mathbb{Q}_\ell \), and \( G \) is a finite group that is a semi-direct product \( G = HL \) of a normal \( \ell' \)-subgroup \( H \) and a cyclic \( \ell \)-group \( L \). Note that if \( G \) is a finite inertia group for some prime \( p \), then \( G \) is of this form for every prime \( \ell \neq p \) (see Lemmas 3.2 and 3.3 of [7]). We always assume that the group algebra \( K[H] \) is decomposable, i.e., splits into a direct sum of matrix algebras over fields (this will automatically be the case when \( G \) is an inertia group; see [4]). We write \( \text{Sp}_{2d}(R) \) for the group of \( 2d \times 2d \) symplectic matrices over a ring \( R \). If \( E \) is a field that is a finite extension of \( \mathbb{Q}_\ell \), let \( \mathcal{O}_E \) denote the ring of integers.

The following result is the main result of this paper.

**Theorem 1.1** (Embedding Theorem). Suppose \( d \) is a positive integer, and there is an embedding \( G \hookrightarrow \text{Sp}_{2d}(K) \). If \( \ell \geq 5 \), then there is an embedding \( G \hookrightarrow \text{Sp}_{2d}(\mathcal{O}_K) \).

We will make use of the following result, which we prove in \([2] \). Our proof was inspired by §17.6 in [3] and the proof of Lemma 1.1 in §1 of Chap. X in [2].

**Theorem 1.2** (Extension Theorem). Suppose that \( W \) is a finite dimensional \( K \)-vector space, \( f : W \times W \to K \) is a non-degenerate alternating (resp., symmetric) \( K \)-bilinear form, and \( \tau : H \to \text{Aut}(W, f) \) is a group homomorphism that makes \( W \) into a simple \( K[H] \)-module. Assume that for every \( g \in G \), the representation of \( H \)
\[
\tau_g : H \to \text{Aut}(W), \quad h \mapsto \tau(ghg^{-1})
\]
is isomorphic to \( \tau \). Suppose \( T \) is an \( H \)-stable \( \mathcal{O}_K \)-lattice in \( W \). Then
\[
\tau : H \to \text{Aut}(T, f) \subset \text{Aut}(W, f)
\]

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can be extended to a homomorphism
\[ \tau_G : G \to \text{Aut}(T, f) \subset \text{Aut}(W, f). \]

We will say that two bilinear forms have the same parity if they are either both symmetric or both alternating. Let \( \pi \) denote a uniformizer of \( \mathcal{O}_K \).

2. Proof of the Extension Theorem

Replacing \( f \) by \( \pi^i f \) if necessary, we may assume that \( f(T, T) = \mathcal{O}_K \) and
\[ f : T \times T \to \mathcal{O}_K \]
is perfect.

Let \( E \) denote the centralizer of \( H \) in \( \text{End}_K(W) \). Since \( K[H] \) is decomposable and \( W \) is a simple \( K[H] \)-module, \( E \) is a field that is a finite extension of \( K \). It follows from Theorem 74.5 (especially a description of the center \( K \) in (ii)) of [1] that \( E \) is the field of definition of a certain character of \( H \); in particular, \( E \subseteq K(\mu_{#H}) \). Thus \( E/K \) is unramified, since \( #H \) is prime to \( \ell \). Therefore \( E/Q_\ell \) is also unramified, since \( K/Q_\ell \) is unramified.

Since the \( H \)-module \( W \) is simple and \( H \) is an \( \ell' \)-group, the \( H \)-module \( T/\pi T \) is also simple. This implies that the \( H \)-stable \( \mathcal{O}_K \)-sublattice \( \mathcal{O}_E T \) of \( W \) is of the form \( aT \) with \( a \in K^* \) (see Exercise 15.2 of [5]). Thus \( T = a^{-1}(\mathcal{O}_E T) \) is \( \mathcal{O}_E \)-stable, i.e., \( T \) is an \( \mathcal{O}_E \)-lattice in the \( E \)-vector space \( V \).

It follows from the Jacobson density theorem that the image of \( K[H] \) in \( \text{End}_K(W) \) is \( \text{End}_E(W) \). In other words, \( \text{End}_E(W) \) is the \( K \)-vector subspace of \( \text{End}_K(W) \) generated by the \( \tau(h) \) for \( h \in H \).

By the non-degeneracy of \( f \), there is an involution \( u \mapsto u' \) of \( \text{End}_K(W) \) characterized by
\[ f(ux, y) = f(x, u'y) \quad \forall x, y \in W. \]

By the \( H \)-invariance of \( f \), \( \tau(h)' = \tau(h^{-1}) \) \( \forall h \in H \). Thus the involution \( u \mapsto u' \) sends \( \text{End}_E(W) \) into itself, and therefore sends \( E \), the center of \( \text{End}_E(W) \), into itself. Let
\[ E_0 = \{ u \in E \mid u' = u \}. \]
Then \( K \subseteq E_0 \subseteq E \). Either \( E = E_0 \) or \( E/E_0 \) is a quadratic extension.

Let \( c \) be a generator of \( L \). The homomorphisms
\[ \tau : H \to \text{Aut}_{\mathcal{O}_K}(T), \quad \tau_c : H \to \text{Aut}_{\mathcal{O}_K}(T) \]
define \( \mathcal{O}_K[H] \)-module structures on \( T \) in such a way that the corresponding \( \mathcal{O}_K[H] \)-modules are isomorphic. Since \( H \) is an \( \ell' \)-group, the corresponding \( \mathcal{O}_K[H] \)-modules are isomorphic (see §14.4 and §15.5 of [3]), i.e., there exists \( A \in \text{Aut}_{\mathcal{O}_K}(T) \) such that
\[ \tau(chc^{-1}) = At(h)A^{-1} \quad \forall h \in H. \]
Then
\[ \tau(c^ihc^{-i}) = A^i\tau(h)A^{-i} \quad \forall h \in H, i \in \mathbb{Z}. \]
Since \( c^{#L} = 1 \), we have \( A^{#L} \in \mathcal{O}_E^* \). Further, \( A\text{End}_E(W)A^{-1} = \text{End}_E(W) \). Since \( E \) is the center of \( \text{End}_E(W) \), \( AEA^{-1} = E \). In other words, defining \( \iota(c)(u) = AuA^{-1} \) for \( u \in E \) induces a homomorphism
\[ \iota : L \to \text{Gal}(E/K). \]
Since $[E : E_0]$ divides 2 and $\ell$ is odd, the composition $L \to \text{Gal}(E/K) \to \text{Gal}(E_0/K)$ has the same kernel as $\iota$. Thus

\[ \#\iota(L) \text{ divides } [E_0 : K]. \]

Let $f_A(x, y) = f(Ax, Ay)$ for $x, y \in T$. For all $h \in H$,

\[ f_A(\tau(h)x, \tau(h)y) = f(A\tau(h)x, A\tau(h)y) = f(A\tau(chc^{-1})A^{-1}Ax, A\tau(chc^{-1})Ay) = f_A(x, y), \]

since $chc^{-1} \in H$ and $f$ is $H$-invariant. Thus $f_A$ is $H$-invariant and of the same parity as $f$. Therefore there exists $a \in E_0^*$ such that

\[ f(Ax, Ay) = f(ax, y) = f(x, ay) \quad \forall x, y \in W. \]

Since $A \in \text{Aut}(T)$ and $f : T \times T \to O_K$ is perfect, we have $a \in \text{Aut}(T)$. This implies easily that $a \in O_{E_0}^*$.

Let $\sigma := \iota(c) \in \text{Gal}(E/K)$. Then $aA = A\sigma^{-1}(a)$. There exists $a_1 \in O_{E_0}^*$ such that

\[ a\sigma^{-1}(a) = a_1^2. \]

(Indeed, let $\eta$ be a uniformizer for $O_{E_0}$. Then $\sigma(a)^{-1} \equiv a^{\ell^j} \pmod{\eta}$ for some non-negative integer $j$. Since $\ell$ is odd, $\ell^j + 1$ is even, so $a\sigma^{-1}(a)$ is a square modulo $\eta$. Thus $a\sigma^{-1}(a)$ is a square, since all elements of $O_{E_0}$ congruent to 1 modulo $\eta$ are squares.)

For all $x, y \in T$,

\[ f((A^2a_1^{-1}x, (A^2a_1^{-1})y) = f(aAa_1^{-1}x, Aa_1^{-1}y) = f(A\sigma^{-1}(a)a_1^{-1}x, Aa_1^{-1}y) = f(x, y). \]

Let $A_1 = A^2a_1^{-1} \in \text{Aut}_O(T)$. Then $f$ is $A_1$-invariant, $\det(A_1) = 1$, and conjugation by $A_1$ coincides with conjugation by $c^2$. Thus $A_1^{\#L} = bI$ is a scalar operator of determinant 1 on the $E_0$-vector space $W$. Therefore $b \in E_0^*$ is a root of unity, i.e., $b^\mu = 1$ where $\mu$ is the number of roots of unity in $E_0$. Since $E_0 \subseteq E$, the extension $E_0/Q_\ell$ is unramified, so $\ell$ does not divide $\mu$. Letting $B := A_1^\mu$, then conjugation by $B$ coincides with conjugation by $b_1 := c^{2\mu}$, and $B^{\#L} = I$. Since $b_1$ is a generator of $L$, sending $b_1$ to $B$ defines the desired extension $\tau_G$ of $\tau$.

3. Lemmas for the Embedding Theorem

Assume from now on that we are in the setting of the Embedding Theorem. Therefore there exist a 2d-dimensional $K$-vector space $V$, a non-degenerate alternating $K$-bilinear form

\[ e : V \times V \to K, \]

and a faithful symplectic representation

\[ \rho : G \to \text{Aut}(V, e). \]

**Proposition 3.1.** Suppose that $V$ is simple as a $G$-module but not as an $H$-module. Then either

(i) the $H$-module $V$ is isomorphic to $W^r$ for some simple $H$-module $W$ and some $r > 1$, or
(ii) there exist a normal subgroup $G_1$ of $G$, and a simple symplectic $G_1$-module $V_1$ which is a $K$-vector space of dimension $2d/[G : G_1]$, such that $H \subseteq G_1 \neq G$ and such that if $g_1$ is a non-identity element of $G_1$, then there exists $g \in G$ such that $gg_1g^{-1}$ is not in the kernel of $G_1 \to \text{Aut}(V_1)$.

Proof. We follow the proof of Prop. 24 of [3]. Let $V = \bigoplus_{i=1}^{n} V_i$ be the canonical decomposition of the restriction of $\rho$ to $H$ into a direct sum of isotypic representations. Since the $G$-module $V$ is simple, $G$ permutes the $V_i$ transitively. If $V$ is some $V_i$, then the $H$-module $V$ is isotypic and (i) holds. Assume from now on that (i) does not hold. Let

$$G_1 = \{ s \in G \mid s(V_1) = V_1 \} \subset G.$$  

Then $H \subseteq G_1 \neq G$. Since $G_1$ contains $H$, it is normal in $G$. Thus for every $V_i$,

$$G_1 = \{ s \in G \mid s(V_i) = V_i \} \subset G.$$  

Every $V_i$ is a simple $G_1$-module, because $V$ is a simple $G$-module.

The kernels of the natural maps $G_1 \to \text{Aut}(V_i)$ have trivial intersection and are conjugate in $G$. It follows that if $g_1$ is a non-identity element of $G_1$, then it has a conjugate which does not lie in the kernel of $G_1 \to \text{Aut}(V_1)$.

Since $[G : G_1]$ divides $[G : H]$, it is an $\ell$-power, and therefore odd. Thus $n$ is odd, so at least one of the simple $G_1$-modules $V_i$ is self-dual. This implies easily that all the $V_i$ are self-dual. Suppose $V_i$ is not symplectic. Then none of the $V_i$ are symplectic. Since $V_i$ is a simple $G_1$-module, every $G_1$-invariant alternating bilinear form on $V_i$ is zero. Since the $V_i$ are mutually non-isomorphic simple $G_1$-modules, every $G_1$-invariant bilinear pairing between $V_i$ and $V_j$ for $i \neq j$ induces the zero map $V_i \to V_j^\ast(= V_j)$, and therefore is zero. Therefore, every $G_1$-invariant alternating bilinear form on $V = \bigoplus_{i=1}^{n} V_i$ is zero, contradicting that $V$ is symplectic. Thus $V_1$ is symplectic.

We leave the next lemma as an exercise.

**Lemma 3.2.** If $G_0$ is a finite group, $V_0$ is a finite-dimensional $K$-vector space which is also a faithful $K[G_0]$-module, and $T_0$ is a $G_0$-stable $O_K$-lattice in $V_0$, then

$$e_0((x,f),(y,g)) = g(x) - f(y) \text{ for } x, y \in T_0 \text{ and } f, g \in T_0^\ast := \text{Hom}(O_K(T_0, O_K))$$

defines a perfect alternating $G_0$-invariant form on $T_0 \oplus T_0^\ast$, and induces a natural embedding $G_0 \hookrightarrow \text{Aut}(T_0 \oplus T_0^\ast, e_0)$.

**Lemma 3.3.** If the Embedding Theorem is true for all irreducible $\rho$ (and $G$) then it is valid for all $\rho$.

Proof. The $G$-module $V$ splits into a direct sum of $G$-modules $V'$ such that the restriction of $\rho$ to $V'$ is non-degenerate, and either $V'$ is simple or $V' = V_0 \oplus V_0^\ast$ where $V_0$ is simple. In the latter case choose a $G$-stable $O_K$-lattice $T_0$ in $V_0$ and apply Lemma 3.2.

**Lemma 3.4.** Suppose $V$ is simple as both a $G$-module and an $H$-module Suppose $T$ is a $G$-stable $O_K$-lattice in $V$, and choose $i \in \mathbb{Z}$ so that $\pi^i(e(T, T) = O_K$. Then $\pi^i e : T \times T \to O_K$ is a perfect $G$-invariant alternating bilinear form, and induces an embedding

$$G \hookrightarrow \text{Aut}(T, \pi^i e) \cong \text{Sp}_{2d}(O_K).$$
Lemma 3.7. If \( By Lemma 3.7 \) of \([7]\) there is an injective homomorphism \( \Lambda \) is a finite group and \( G_1 \) is a normal subgroup of \( G_0 \). Suppose that whenever \( g_1 \) is a non-identity element of \( G_1 \) then there exists \( g \in G \) such that \( gg_1g^{-1} \notin \ker(f) \). Then there exist a free \( \mathcal{O}_K \)-module \( T \) of rank \( 2d_1 \), an alternating perfect form \( e : T \times T \to \mathcal{O}_K \), and an injective homomorphism \( \psi : G_0 \hookrightarrow \text{Aut}(T, e) \).

Proof. Let

\[
T = \{ u : G_0 \to T_1 \mid u(xs) = s^{-1}u(x) \quad \forall s \in G_1, \forall x \in G_0 \},
\]

choose a section \( p : G_0/G_1 \to G_0 \), and let

\[
e(u, v) = \sum_{\gamma \in G_0/G_1} e_1(u(p(\gamma))), v(p(\gamma))) \quad \text{for } u, v \in T.\]

Note that \( e \) is independent of the choice of section \( p \). Define a homomorphism \( \psi : G_0 \to \text{Aut}(T, e) \) by \( \psi(g)(u)(x) = u(g^{-1}x) \) for \( g \in G_0 \), \( u \in T \), \( x \in G_0 \). Then the desired conditions are all satisfied.

Corollary 3.6. Suppose \( G_0 \) is a finite group and \( G_1 \) is a normal subgroup of \( G_0 \). Suppose there exists an injective homomorphism \( G_1 \hookrightarrow \text{Sp}_{2d_1}(\mathcal{O}_K) \). Then there exists an injective homomorphism \( G_0 \hookrightarrow \text{Sp}_{2d_1|G_0:G_1|}(\mathcal{O}_K) \).

Lemma 3.7. If \( \Lambda \) is a finite cyclic group of order \( \ell^m \), then there exists an injective homomorphism \( \Lambda \times \{ \pm 1 \} \hookrightarrow \text{Sp}_{\ell^m}(\mathbb{Z}_\ell) \).

Proof. By Lemma 3.7 of \([6]\) there is an injective homomorphism \( \mu_\ell \times \{ \pm 1 \} \hookrightarrow \text{Sp}_{\ell^m}(\mathbb{Z}_\ell) \). Now apply Corollary 3.6 to \( G_1 = \mu_\ell \times \{ \pm 1 \} \subseteq \Lambda \times \{ \pm 1 \} = G_0 \).

4. Proof of the Embedding Theorem

By Lemmas 3.3 and 3.4, we may assume from now on that the \( G \)-module \( V \) is simple and the \( H \)-module \( V \) is not simple.

If (ii) of Proposition 3.1 holds, then induct on \( \#G \), applying Lemma 3.3 to \( G_1 \triangleleft G_0 = G \).

By Proposition 3.1, we may now assume that \( V \cong W^r \) for some simple \( H \)-module \( W \), where \( r > 1 \). It follows that \( W \) is self-dual, i.e., there is an \( H \)-invariant non-degenerate alternating or symmetric \( K \)-bilinear form \( f : W \times W \to K \). We may choose an \( H \)-stable lattice \( T \) in \( W \) and (replacing \( f \) by \( \pi^i f \) for suitable \( i \), if necessary) we may assume that \( f : T \times T \to \mathcal{O}_K \) is perfect.

Let

\[
w = \dim_K(W).
\]

Assume first that \( w = 1 \). Then \( H \subset \text{Aut}_K(W) = K^* \). Since \( V = W^r \) and \( H \subseteq G \subseteq \text{Sp}(V) \), we have \( H \subseteq \{ \pm 1 \} \), and Lemma 3.7 gives the desired embedding.

Assume from now on that \( w \geq 2 \). Let

\[
\tau : H \hookrightarrow \text{Aut}(W, f) \subset \text{Aut}(W)
\]
be the injective homomorphism defining the $H$-module structure on $W$. Since $H$ is normal in $G$, the subspace $gW \subset V$ is an $H$-submodule of $V$ for every $g \in G$. The natural representation $H \to \text{Aut}(gW)$ is isomorphic to the representation

$$\tau_g : H \to \text{Aut}(W), \quad h \mapsto \tau(ghg^{-1}).$$

On the other hand, since $V \cong W^r$ as $H$-modules, therefore $gW \cong W$ as $H$-modules. (Indeed, every $H$-submodule in $V$ is isomorphic to a direct sum of copies of $W$, and $w = \dim(gW)$.) Now apply the Extension Theorem and extend $\tau$ to a homomorphism

$$\tau_G : G \to \text{Aut}(T, f) \subseteq \text{Aut}(V, f).$$

Since $f$ is perfect, we have $T^* = T$. If $f$ is symmetric, then Lemma 3.2 gives a perfect alternating $G$-invariant form $e_0$ on $T \oplus T$, and we let

$$\tau_0 : G \to \text{Aut}(T \oplus T, e_0) \cong \text{Sp}_{2w}(O_K)$$

be the direct sum of two copies of $\tau_G$.

Suppose $\tau_G$ is injective. If $f$ is alternating (resp., symmetric), then $\tau_G$ (resp., $\tau_0$) gives the desired embedding.

Now assume $\tau_G$ is not injective. Then $\ker(\tau_G)$ meets $H$ only at the identity, so $\ker(\tau_G)$ is a normal $\ell$-subgroup of $G$ and therefore is central, and thus contained in the Sylow $\ell$-subgroup $L$. Now retain the notation from the proof of the Extension Theorem.

Suppose $\dim_{E_0}(W) = 1$. Then $H \subseteq E_0^*$, and for $h \in H$, $f(x, y) = f(hx, hy) = f(h^2x, y) \forall x, y \in W$. Thus $h^2 = 1$, i.e., $H \subseteq \{1, -1\}$, and we are done by Lemma 3.7.

Now assume that $\dim_{E_0}(W) \geq 2$. Let

$$L_0 = \{x \in L : \tau_G(x)y = y\tau_G(x) \forall y \in E\}.$$ 

Then $\ker(\tau_G) \subseteq L_0 \subseteq L$. By (3),

$$\#i(L) \leq [E_0 : K] \leq w/2.$$

Let $G_0 = H L_0$, a normal subgroup of $G$. Since $V = W^r$, the restriction of $\tau_G$ to $G_0$ can be viewed as a homomorphism

$$\tau_G : G_0 \to \text{Aut}_E(W) \subset \text{Aut}_E(V).$$

Then $\rho = \tau_G$ on $H \subseteq G_0$, and

$$\rho(z)\rho(h)\rho(z)^{-1} = \tau_G(z)\tau_G(h)\tau_G(z)^{-1} = \tau_G(z)\rho(h)\tau_G(z)^{-1}$$

for every $z \in L_0, h \in H$. Thus

$$\kappa : L_0 \to \text{End}_H(V)^* = \text{GL}_r(E), \quad z \mapsto \tau_G(z)\rho(z)^{-1}$$

is a homomorphism. Since $\tau_G$ is not injective on $L_0$ but $\rho$ is, thus $\kappa$ is injective, since $L_0$ is cyclic. Write $\#L_0 = \ell^t$. Then

$$r \geq \varphi(\ell^t) = (\ell - 1)\ell^{t-1} \geq \ell - 1.$$ 

Since $\#L = \#\ker(\iota)\#i(L) = \ell^t \#i(L)$, we have by (2) and (3):

$$\varphi(\#L) = \varphi(\ell^t)\#i(L) \leq rw/2 \leq (r - \ell - 1)w \leq (r - 2)w$$

since $\ell \geq 5$. Thus

$$2w + \varphi(\#L) \leq rw = \dim_K(V) = 2d.$$
Let
\[ \psi : G/H \cong L \hookrightarrow \text{Sp}_{\varphi(\#L)}(\mathcal{O}_K) \]
be the embedding from Lemma 3.7.

If \( f \) is alternating (resp., symmetric), we are done by taking the direct sum of \( \tau_G \) (resp., \( \tau_0 \)) and \( \psi \).

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