Integrable geodesic flows on 2-torus: formal solutions and variational principle

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Abstract

In this paper we study quasi-linear system of partial differential equations which describes the existence of the polynomial in momenta first integral of the integrable geodesic flow on 2-torus. We proved in [3] that this is a semi-Hamiltonian system and we show here that the metric associated with the system is a metric of Egorov type. We use this fact in order to prove that in the case of integrals of degree three and four the system is in fact equivalent to a single remarkable equation of order 3 and 4 respectively. Remarkably the equation for the case of degree four has variational meaning: it is Euler-Lagrange equation of a variational principle. Next we prove that this equation for \( n = 4 \) has formal double periodic solutions as a series in a small parameter.

1 Introduction

In this paper we study integrable geodesic flows on two-dimensional torus \( T^2 = \mathbb{R}^2/\Gamma \), where \( \Gamma \subset \mathbb{Z}^2 \) is a lattice. Let \( ds^2 = \sum_{i,j=1}^{2} g_{ij}(q) dq_i dq_j \) be a Riemannian metric on 2-torus. The geodesic flow of the metric is called integrable if the Hamiltonian system

\[
\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}^i = -\frac{\partial H}{\partial q^i}, \quad H = \frac{1}{2} \sum_{i,j=1}^{2} g^{ij}(q)p_ip_j,
\]

has a first integral \( F(q,p) : T^*T^2 \to \mathbb{R} \), i.e.

\[
\dot{F} = \{ F,H \} = \left( \frac{\partial H}{\partial q^1} \frac{\partial F}{\partial p_1} - \frac{\partial H}{\partial p_1} \frac{\partial F}{\partial q^1} \right) + \left( \frac{\partial H}{\partial q^2} \frac{\partial F}{\partial p_2} - \frac{\partial H}{\partial p_2} \frac{\partial F}{\partial q^2} \right) = 0,
\]

such that almost everywhere \( F \) is independent with \( H \). There are two kind of Riemannian metrics having polynomial integrals for the geodesic flow. If the metric has the form

\[
ds^2 = \Lambda(\alpha x + \beta y)(dx^2 + dy^2) \quad \text{or} \quad ds^2 = (\Lambda_1(\alpha_1 x + \beta_1 y) + \Lambda_2(\alpha_2 x + \beta_2 y))(dx^2 + dy^2),
\]

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then there exist polynomial integrals of degree one or degree two. The existence of Riemann metrics with non-reducible polynomial integrals of higher degree is a difficult open problem.

Let us recall some results on integrable geodesic flows on $T^2$. If the geodesic flow is integrable, then on the torus there are global semi-geodesic coordinates $(t, x)$ (see [3]), i.e.

$$ds^2 = g^2(t, x)dt^2 + dx^2, \quad H = \frac{1}{2} \left( \frac{p_1^2}{g^2} + p_2^2 \right).$$

The polynomial first integral has the form

$$F = \frac{a_0}{g^n} p_1^n + \frac{a_1}{g^{n-1}} p_1^{n-1} + \cdots + \frac{a_{n-2}}{g} p_1^{n-2} p_2^{n-2} + p_1 p_2^{n-1} + p_2^n, \quad a_s = a_s(t, x).$$

The condition $\{F, H\} = 0$ is equivalent to the quasi-linear system of partial differential equations

$$U_t + A(U)U_x = 0, \quad (1)$$

where $U^T = (a_0, \ldots, a_{n-1})$, $a_{n-1} = g$.

$$A = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & a_1 \\
0 & a_{n-1} & \cdots & 0 & 0 & 2a_2 - na_0 \\
0 & 0 & a_{n-2} & \cdots & 0 & 3a_3 - (n-1)a_1 \\
0 & 0 & \cdots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\
0 & 0 & \cdots & 0 & a_{n-1} & na_n - 2a_{n-2}
\end{pmatrix}.$$

System (1) has remarkable properties. It can be written in the form of conservation laws, i.e. there is a regular change of variables

$$(a_0, \ldots, a_{n-1}) \mapsto (G_1(a), \ldots, G_n(a))$$

such that for some $F_1(a), \ldots, F_n(a)$ the conservation laws hold true

$$(G_i(a))_t + (F_i(a))_x = 0, \quad i = 1, \ldots, n.$$  

Moreover, in the hyperbolic domain, where eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A$ are real and pairwise distinct there exist change of variable

$$(a_0, \ldots, a_{n-1}) \mapsto (r_1(a), \ldots, r_n(a))$$

such that the system can be written in Riemann invariants.

$$(r_i)_t + \lambda_i(r)(r_i)_x = 0, \quad i = 1, \ldots, n.$$  

Such systems are called semi-Hamiltonian and generalized hodograph method applies [13]. Semi-Hamiltonian systems and in particular systems of hydrodynamic type are very important in mathematical physics (see for example [6], [7], [12]).
It is not clear to us at present if and how the generalized hodograph method can be used to prove non-existence of smooth solutions for a semi-Hamiltonian system. Using the original idea by P. Lax based on analysis along characteristics we proved for the cases $n = 3, 4$ that in the elliptic domain (where matrix $A$ has two complex-conjugated eigenvalues) the behavior of solutions can be analyzed: the integrals of degree three and four are reduced to integrals of degree one or two [4]. Thus non-trivial integrals of degree 3, 4 may exist only in the hyperbolic region of the quasi-linear system (1).

In this paper we study a quasi-linear system (2) which corresponds to the choice of conformal coordinates $(x, y)$ for Riemannian metric $ds^2 = \Lambda(dx^2 + dy^2)$. We assume that the geodesic flow has a polynomial in momenta integral

$$F = a_0(x, y)p_1^n + a_1(x, y)p_1^{n-1}p_2 + \cdots + a_n(x, y)p_2^n.$$ 

Kozlov and Denisova [8] proved that if $\Lambda$ is trigonometric polynomial then the geodesic flow has no irreducible polynomial integrals of degree higher than two (see also [9]). By Kolokoltsov’s [10] theorem

\[
a_{n-1} = c_1 + a_{n-3} - a_{n-5} + \cdots, \quad a_n = c_2 + a_{n-2} - a_{n-4} + \cdots,
\]

where $c_1, c_2$ are some constants. Then the condition $\{H, F\} = 0$, where $H = \frac{p_1^2 + p_2^2}{2\Lambda}$ is equivalent to the system of quasi-linear equations

$$A(U)U_x + B(U)U_y = 0, \quad U = (a_0, a_1, \ldots, a_{n-2}, \Lambda) \quad (2)$$

(see (11) below to specify $A(U), B(U)$ explicitly). This system also can be written in the form of conservation laws and moreover, in the hyperbolic region it admits $n$ Riemann invariants, so the system is semi-Hamiltonian (see [2]).

Let us remind that for semi-Hamiltonian systems the following relations on eigenvalues hold

$$\partial_r \lambda_i \lambda_i - \lambda_k = \partial_r \lambda_j \lambda_j - \lambda_k, \quad i \neq j \neq k \neq i.$$ 

These relations mean that there exists a diagonal metric on the space of field variables

$$ds^2 = H_1^2(r)dr_1^2 + \cdots + H_n^2(r)dr_n^2 \quad (3)$$

with Christoffel symbols satisfying the identities

$$\Gamma_{ki}^k = \frac{\partial_r \lambda_k}{\lambda_i - \lambda_k}, \quad i \neq k.$$ 

Let us formulate now our main results. In Theorems 1–7 we assume that $c_1 = 0$. This can be achieved by a rotation in the plane $x, y$.

**Theorem 1** The metric (3) associated with the semi-Hamiltonian system (2) is a metric of Egorov type, i.e. the rotation coefficients

$$\beta_{ij} = \frac{\partial_r H_j}{H_i}, \quad i \neq j$$

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are symmetric $\beta_{ij} = \beta_{ji}$, or equivalently there is a function $A(r)$ such that
\[
\partial_r A(r) = H^2_i(r).
\]

In fact it follows from theorem of Pavlov and Tsarev [11] that in order to prove Theorem 1 one needs to find two conservation laws of a special form. In the next theorem we state the existence of these conservation laws for (2).

**Theorem 2** The system (2) has two conservation laws of the form
\[
P(U)_x + Q(U)_y = 0, \quad Q(U)_x + R(U)_y = 0. \tag{4}
\]

Functions $P(U), Q(U), R(U)$ are found explicitly in Lemma 1 (see below). Remarkably, for $n = 3, 4$ Theorem 2 allows us to reduce system (2) to a single equation.

**Theorem 3** Let $n = 3$, and $\lambda(x, y)$ be a solution periodic with respect to the lattice $\Gamma$ of the equation
\[
\Delta \lambda = \frac{3c^2}{2} \Lambda - 2a_{11} - 2a_{22}. \tag{5}
\]
Then the function $\lambda$ satisfies the equation
\[
2\lambda_{xx}\lambda_{xy} + \lambda_{yy}(\lambda_{xy} - \lambda_{yy}) + \lambda_{xy}(\lambda_{xx} + \lambda_{yy}) + 4a_{11}\lambda_{xy} + 2a_{12}(\lambda_{xx} + \lambda_{yy}) + 2a_{22}(\lambda_{xy} - \lambda_{yy}) = 0, \tag{6}
\]
where $a_{11}, a_{12}, a_{22}$ are some constants defined by the metric and the integral.

**Theorem 4** Let $n = 4$, and $\lambda(x, y)$ be a solution periodic with respect to the lattice $\Gamma$ of the equation
\[
\Delta \lambda = 2c^2 \Lambda - 2a_{11} - 2a_{22}. \tag{5'}
\]
Then the function $\lambda$ satisfies the equation
\[
\lambda_{xy}(\lambda_{yyy} - \lambda_{xxx}) + 3(\lambda_{yyy}\lambda_{xy} - \lambda_{xxxy}) + 2(\lambda_{yy}\lambda_{xyy} - \lambda_{xxxxy}) + 4a_{22}\lambda_{xyy} - 4a_{11}\lambda_{xxx} + 2a_{12}(\lambda_{yyy} - \lambda_{xxx}) = 0, \tag{7}
\]
where $a_{11}, a_{12}, a_{22}$ are some constants defined by the metric and the integral.

Our next theorem states that the equations (6) and (7) are in fact equivalent to the system (2).

**Theorem 5** Let $\lambda$ be a solution of the equation (6) or (7) respectively periodic with respect to the lattice $\Gamma$ which satisfies the condition $\Delta \lambda + 2(a_{11} + a_{22}) > 0$. Then the corresponding solution of the system (2) is periodic also.
Proof of this theorem is very simple for the case $n = 3$ but requires certain topological argument for the case $n = 4$. This argument is given below in Section 4.

Remarkably equation (7) admits the following variational interpretation.

**Theorem 6** Equation (7) coincides with the Euler-Lagrange equation of the functional

$$
\mathcal{L}(\lambda) = \int \frac{1}{2} \left( 4\lambda_{xy}(a_{22}\lambda_{yy} - a_{11}\lambda_{xx}) + 2a_{12}(\lambda_{yy}^2 - \lambda_{xx}^2) + \lambda_{xy}(\lambda_{yy}^2 - \lambda_{xx}^2) \right) dx 
$$

Let us remark that the functional becomes especially simple

$$
\mathcal{L}(\lambda) = \int \frac{1}{2} \lambda_{xy}(\lambda_{yy} - \lambda_{xx})(\frac{1}{\varepsilon} + \Delta \lambda) dx
dy
$$

for the choice of constants $a_{ij}$ where $a_{11} = a_{22} = \frac{1}{4\varepsilon}, a_{12} = 0$. However, at the present moment we have no significant results on the critical points of this functional.

Let us consider now in more details the equation (7) for $a_{12} = 0, a_{11} = a_{22} = \frac{1}{4\varepsilon}$

$$
\lambda_{xxy} - \lambda_{xyy} = \varepsilon(\lambda_{xy}(\lambda_{yyyy} - \lambda_{xxxx}) + 3(\lambda_{yyyy}\lambda_{xyy} - \lambda_{xyy}\lambda_{xxxx}) +
+ 2(\lambda_{yyyy}\lambda_{xyy} - \lambda_{xyy}\lambda_{xxxx})).
$$

The conformal factor of the metric has the form

$$
\Lambda = \frac{1}{c^2} \left( \frac{\Delta \lambda}{2} + \frac{1}{2\varepsilon} \right).
$$

Let us look for a solution of (8) as a formal power series in $\varepsilon$:

$$
\lambda(x, y) = \lambda_0(x, y) + \lambda_1(x, y)\varepsilon + \lambda_2(x, y)\varepsilon^2 + \ldots,
$$

where $\varepsilon$ is a small parameter. Then from (8) we have a recursion formula.

$$
(\lambda_k)_{xyy} - (\lambda_k)_{xxy} = \sum_{s=0}^{k-1} < \lambda_s, \lambda_{k-s-1} >
$$

where

$$
< \lambda_p, \lambda_q > = \lambda_{p_{yy}}(\lambda_{q_{yyyy}} - \lambda_{q_{xxxx}}) + 3(\lambda_{p_{yyyy}}\lambda_{q_{xyy}} - \lambda_{p_{xyy}}\lambda_{q_{xxxx}}) + 2(\lambda_{p_{yyyy}}\lambda_{q_{xyy}} - \lambda_{p_{xyy}}\lambda_{q_{xxxx}}).
$$

In the rest of this section we shall assume that $\Gamma \subset \mathbb{R}^2$ is the integer lattice $\mathbb{Z}^2$. Given initial doubly periodic function $\lambda_0$ we wish to find all $\lambda_k$ recursively by means of equation (10). It is easy to see that (10) has a periodic solution $\lambda_k$ if the Fourier series of the right hand side does not have monomials of the form

$$
e^{inx}, e^{iny}, e^{in(x+y)}, e^{in(x-y)}.$$

Let us remark also that the periodic solution \( \lambda_k \) of (10) is defined up to addition to \( \lambda_k \) some function of the form

\[
\tilde{\lambda}_k = f_1(x) + f_2(y) + f_3(x-y) + f_4(x+y).
\]

Next theorem states that recursive process is well defined and thus gives formal periodic solution of (8) if the initial function \( \lambda_0 \) and the additions \( \tilde{\lambda}_k \) on every stage \( k \) are chosen symmetric with respect to coordinate axes and diagonals, \( x = 0, y = 0, x = y, x = -y \). It is an important open question if the convergence of the series can be achieved by a good choice of initial function \( \lambda_0 \) and the functions \( \tilde{\lambda}_k \).

**Theorem 7** Let

\[
\lambda_0 = \sum_{n \in \mathbb{N}} \alpha_n (\cos(nx) + \cos(ny)) + \sum_{n \in \mathbb{N}} \beta_n (\cos(n(x-y)) + \cos(n(x+y))),
\]

\[
\tilde{\lambda}_k = \sum_{n \in \mathbb{N}} \alpha^n_k (\cos(nx) + \cos(ny)) + \sum_{n \in \mathbb{N}} \beta^n_k (\cos(n(x-y)) + \cos(n(x+y))),
\]

then the recursion formula gives a well defined formal periodic solution (9) of (10).

We prove Theorem 7 in the last section.

**Remark** It is an interesting fact that substituting into the equation (8)

\[
\lambda = f_1(x) + f_2(y) + f_3(x-y) + f_4(x+y)
\]

one gets an equation on the functions \( f_1, \ldots, f_4 \) which was studied recently in [1]. It was proved later in [5] that there are no new periodic solutions of this equation in this particular form.

### 2 Proof of Theorems 1 and 2

Let us assume that \( ds^2 = \Lambda(x,y)(dx^2 + dy^2) \) is a metric on \( \mathbb{T}^2 \) and \( F \) is a first integral polynomial in momenta for the geodesic flow

\[
F = a_0(x,y)p_1^n + a_1(x,y)p_1^{n-1}p_2 + \cdots + a_n(x,y)p_2^n.
\]

We have

\[
2\Lambda^2\{H, F\} = 2(a_0p_1^n + a_1p_1^{n-1}p_2 + a_2p_1^{n-2}p_2^2 + \cdots + a_n p_2^n)p_1\Lambda +
\]

\[
+ (na_0p_1^{n-1} + (n-1)a_1p_1^{n-2}p_2 + (n-2)a_2p_1^{n-3}p_2^2 + \cdots + a_{n-1}p_2^{n-1})(p_1^2 + p_2^2)\Lambda_x +
\]

\[
2(a_0p_1^n + a_1p_1^{n-1}p_2 + a_2p_1^{n-2}p_2^2 + \cdots + a_n p_2^n)p_2\Lambda +
\]

\[
+ (a_1p_1^{n-1} + 2a_2p_1^{n-2}p_2 + 3a_3p_1^{n-3}p_2^2 + \cdots + na_n p_2^{n-1})(p_1^2 + p_2^2)\Lambda_y,
\]

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Let us recall that by Kolokoltsov’s theorem

\[ 2\Lambda^2\{H, F\} = l_{n+1}p_1^{n+1} + l_n p_1^n p_2 + \cdots + l_0 p_2^{n+1}, \]

\[ l_{n+1} = 2a_{0x}\Lambda + na_0\Lambda_x + a_1\Lambda_y, \]

\[ l_n = 2a_{1x}\Lambda + (n-1)a_1\Lambda_x + 2a_{0y}\Lambda + 2\Lambda_y, \]

\[ l_{n-1} = 2a_{2x}\Lambda + na_0\Lambda_x + (n-2)a_2\Lambda_x + 2a_{1y}\Lambda + a_1\Lambda_y + 3a_3\Lambda_y, \]

\[ l_{n-2} = 2a_{3x}\Lambda + (n-1)a_1\Lambda_x + (n-3)a_3\Lambda_x + 2a_{2y}\Lambda + 2a_2\Lambda_y + 4a_4\Lambda_y, \]

\[ l_{n-3} = 2a_{4x}\Lambda + (n-2)a_2\Lambda_x + (n-4)a_4\Lambda_x + 2a_{3y}\Lambda + 3a_3\Lambda_y + 5a_5\Lambda_y, \]

\[ \dot{\cdots} \dot{\cdots} \dot{\cdots} \dot{\cdots} \dot{\cdots} \]

\[ l_3 = 2a_{n-2x}\Lambda + 4a_{n-4}x + 2a_{n-2}\Lambda_x + 2a_{n-3y}\Lambda + (n-3)a_{n-3}\Lambda_y + (n-1)a_{n-1}\Lambda_y, \]

\[ l_2 = 2a_{n-1x}\Lambda + 3a_{n-3}x + a_{n-1}\Lambda_x + 2a_{n-2}\Lambda + (n-2)a_{n-2}\Lambda_y + na_n\Lambda_y, \]

\[ l_1 = 2a_n\Lambda + 2a_{n-2}\Lambda_x + 2a_{n-1y}\Lambda + (n-1)a_{n-1}\Lambda_y, \]

\[ l_0 = a_{n-1}\Lambda_x + 2a_{n}\Lambda + na_{n}\Lambda_y. \]

Let us recall that by Kolokoltsov’s theorem

\[ a_{n-1} = c_1 + a_{n-3} - a_{n-5} + \cdots, \quad a_n = c_2 + a_{n-2} - a_{n-4} + \cdots \]

We have the following system of differential equations

\[ l_{n+1} = \cdots = l_0 = 0. \]  \hspace{1cm} (11)

**Lemma 1** If \( n \) is even \( n = 2k \), then the system (11) has the following two conservation laws

\[ [(na_0 - (n-2)a_2 + (n-4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})\Lambda]_x + \]

\[ [(-n-2)a_1 + (n-4)a_3 - \cdots + (-1)^{k+1}2a_{n-3} + (-1)^{k+1}(n-1)c_1)\Lambda]_y = 0, \]

\[ [((n-2)a_1 - (n-4)a_3 + \cdots + (-1)^{k}2a_{n-3} + (-1)^{k+1}c_1)\Lambda]_x + \]

\[ [(na_0 - (n-2)a_2 + (n-4)a_4 - \cdots + (-1)^{k+1}2a_{n-2} + (-1)^{k+1}nc_2)\Lambda]_y = 0. \]

If \( n \) is odd \( n = 2k + 1 \) then the system (11) has the following two conservation laws

\[ [((n-1)a_0 - (n-3)a_2 + (n-5)a_4 - \cdots + (-1)^{k+1}2a_{n-3} + (-1)^{k}c_1)\Lambda]_x + \]

\[ [(-n-1)a_1 + (n-3)a_3 - (n-5)a_5 + \cdots + (-1)^{k}2a_{n-2} + (-1)^{k}nc_2)\Lambda]_y = 0, \]

\[ [((n-1)a_1 - (n-3)a_3 + \cdots + (-1)^{k+1}2a_{n-2})\Lambda]_x + \]

\[ [((n-1)a_0 - (n-3)a_2 + \cdots + (-1)^{k+1}2a_{n-3} + (-1)^{k+1}(n-1)c_1)\Lambda]_y = 0. \]
Let us consider the following linear combination of equations (11) in Lemma 1 we get two conservation laws of the form (4), where for
\[ a_n = 1 \]

Theorems 1 and 2 immediately follow from the Lemma 1. Indeed, if we put
\[ c_1 = 0 \]
in Lemma 1 we get two conservation laws of the form (11), where for \( n \) even
\[ P = (na_0 - (n - 2)a_2 + (n - 4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})\Lambda, \]
\[ Q = -(n - 2)a_0 + (n - 4)a_3 - \cdots + (-1)^{k+1}2a_{n-3})\Lambda, \]
\[ R = (-na_0 + (n - 2)a_2 - (n - 4)a_4 + \cdots + (-1)^{k+1}2a_{n-2} + (-1)^knc_2)\Lambda. \]

Analogously for \( n \) odd one has:
\[ P = ((n - 1)a_1 - (n - 3)a_3 + \cdots + (-1)^{k+1}2a_{n-2})\Lambda, \]
\[ Q = ((n - 1)a_0 - (n - 3)a_2 + \cdots + (-1)^{k+1}2a_{n-3})\Lambda, \]
\[ R = -(n - 1)a_1 + (n - 3)a_3 - (n - 5)a_5 + \cdots + (-1)^{k+1}2a_{n-2} + (-1)^knc_2)\Lambda. \]

**Proof of Lemma 1**

Let us consider the case of even degree \( n = 2k \). Then
\[ a_{n-1} = c_1 + a_{n-3} - a_{n-5} + \cdots + (-1)^k a_1, \quad a_n = c_2 + a_{n-2} - a_{n-4} + \cdots + (-1)^{k+1}a_0. \]

Let us consider the following linear combination of equations (11)
\[ nl_{n+1} - (n - 2)l_{n-1} + (n - 4)l_{n-3} - \cdots + (-1)^{k+1}2l_3 = \]
\[ 2(na_0 - (n - 2)a_2 + (n - 4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})\Lambda + \]
\[ (n^2a_0 - (n - 2)na_0 - (n - 2)^2a_2 + (n - 4)(n - 2)a_2 + (n - 4)^2a_4 - \cdots \]
\[ + (-1)^{k+1}2 \cdot 4a_{n-4} + (-1)^{k+1}2a_{n-2})\Lambda_x + \]
\[ 2(-n - 2)a_1y + (n - 4)a_3y - \cdots + (-1)^{k+1}2a_{n-3y})\Lambda + \]
\[ (na_1 - (n - 2)a_1 - (n - 2)3a_3 + (n - 4)3a_3 + (n - 4)5a_5 - \cdots + (-1)^{k+1}2(n - 3)a_{n-3} + \]
\[ (-1)^{k+1}2(n - 1)a_{n-1})\Lambda_y = \]
\[ 2(na_0 - (n - 2)a_2 + (n - 4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})\Lambda + \]
\[ 2(na_0 - (n - 2)a_2 + (n - 4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})\Lambda_x + \]
\[ 2(-n - 2)a_1y + (n - 4)a_3y - \cdots + (-1)^{k+1}2a_{n-3y})\Lambda + \]
\[ 2(a_1 - 3a_3 + 5a_5 - \cdots + (-1)^k(n - 3)a_{n-3} + (-1)^{k+1}a_{n-1}(n - 1))\Lambda_y = \]
\[ 2[(na_0 - (n - 2)a_2 + (n - 4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})\Lambda]_x + \]
\[ 2(-n - 2)a_1y + (n - 4)a_3y - \cdots + (-1)^{k+1}2a_{n-3y})\Lambda + \]
\[ 2(a_1 - 3a_3 + 5a_5 - \cdots + (-1)^k(n - 3)a_{n-3} + (-1)^{k+1}(n - 1)(c_1 + a_{n-3} - a_{n-5} + \cdots + (-1)^ka_1))\Lambda_y = \]
\[ 2[(na_0 - (n - 2)a_2 + (n - 4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})\Lambda]_x + \]
\[ 2(-n - 2)a_1y + (n - 4)a_3y - \cdots + (-1)^{k+1}2a_{n-3y})\Lambda + \]
We have the first required conservation law. By similar calculations, from (4) it follows that
\[
2[(na_0 - (n - 2)a_2 + (n - 4)a_4 - \cdots + (-1)^{k+1}2a_{n-2})]\Lambda_x + \\
2[(-n - 2)a_1 + (n - 4)a_3 - \cdots + (-1)^{k+1}2a_{n-3}]\Lambda_y + 2(-1)^{k+1}(n - 1)c_1\Lambda_y = 0.
\]

We have the first required conservation law. By similar calculations,

\[
nl_n - (n - 2)l_{n-2} + (n - 4)l_{n-4} + \cdots + (-1)^{k+1}2l_2 = \\
2[[(n - 1)a_0 - (n - 3)a_2 + (n - 5)a_4 - \cdots + (-1)^{k+1}2a_{n-3} + (-1)^{k+1}c_1]\Lambda_x + \\
2[(-n - 1)a_1 + (n - 3)a_3 - (n - 5)a_5 + \cdots + (-1)^{k+1}2a_{n-2} + (-1)^{k+1}nc_2]\Lambda_y = 0.
\]

Let us consider the case of odd degree \( n = 2k + 1 \). Then
\[
a_{n-1} = c_1 + a_{n-3} - a_{n-5} + \cdots + (-1)^{k+1}a_0, \quad a_n = c_2 + a_{n-2} - a_{n-4} + \cdots + (-1)^{k+1}a_1.
\]

By direct calculation we get
\[
(n + 1)l_{n+1} - (n - 1)l_{n-1} + (n - 3)l_{n-3} - \cdots + (-1)^{k+1}2l_2 = \\
2[[(n - 1)a_0 - (n - 3)a_2 + (n - 5)a_4 - \cdots + (-1)^{k+1}2a_{n-3} + (-1)^{k+1}c_1]\Lambda_x + \\
2[(-n - 1)a_1 + (n - 3)a_3 - (n - 5)a_5 + \cdots + (-1)^{k+1}2a_{n-2} + (-1)^{k+1}nc_2]\Lambda_y = 0,
\]

and
\[
(n - 1)l_n - (n - 3)l_{n-2} + (n - 5)l_{n-4} - \cdots + (-1)^{k+1}2l_3 = \\
2[[(n - 1)a_1 - (n - 3)a_3 + \cdots + (-1)^{k+1}2a_{n-2})]\Lambda_x + \\
2[[(n - 1)a_0 - (n - 3)a_2 + \cdots + (-1)^{k+1}2a_{n-3} + (-1)^{k+1}(n - 1)c_1]\Lambda_y = 0.
\]

Lemma 1 and Theorems 1, 2 are proved.

## 3 Proof of Theorems 3 and 4

Let us consider the case \( c_1 = 0, n = 3 \) in Lemma 1. Then we have conservation laws (4) where
\[
P = -a_1\Lambda, \quad Q = -a_0\Lambda, \quad R = \left(\frac{3}{2}c_2 + a_1\right)\Lambda.
\]

From (4) it follows that
\[
P = h_{yy}, \quad Q = -h_{xy}, \quad R = h_{xx},
\]

where \( h(x, y) \) is some function. From (12) and (13) we obtain
\[
a_0 = \frac{3c_2h_{xy}}{2\Delta h}, \quad a_1 = -\frac{3c_2h_{yy}}{2\Delta h}, \quad \Lambda = \frac{2\Delta h}{3c_2}.
\]

9
For $n = 3$ the equation $l_4 = 0$ has the form

$$a_1 \Lambda_y + 2a_0, \Lambda + 3a_0 \Lambda_x = 0.$$  

Let us substitute (14) in the last equation. We get

$$2h_{xx}h_{xxy} + h_{yy}(h_{xxy} - h_{yyy}) + h_{xy}(h_{xxx} + h_{xyx}) = 0.$$  

Since $a_0, a_1, \Lambda$ are periodic function we have

$$h = \lambda + a_{11}x^2 + 2a_{12}xy + a_{22}y^2,$$

where $\lambda$ is a function periodic with respect to $\Gamma$. Then $\lambda$ satisfies the equation (6). Theorem 3 is proved.

Let us consider the case $c_1 = 0, n = 4$ in Lemma 1. Then we have conservation laws (4) where

$$P = (2a_0 - a_2)\Lambda, \quad Q = -a_1\Lambda, \quad R = (2c_2 - 2a_0 + a_2)\Lambda.$$  

(15)

We have

$$P = f_{yy}, \quad Q = -f_{xy}, \quad R = f_{xx},$$  

(16)

where $f(x, y)$ is some function. From (15), (16) we get

$$a_1 = 2c_2 f_{xy} \frac{f_{xy}}{\Delta f}, \quad a_2 = -2c_2 f_{yy} \frac{f_{yy}}{\Delta f} + 2a_0, \quad \Lambda = \frac{\Delta f}{2c_2}.$$  

(17)

Using (17) from $l_5 = 0$ and $l_4 = 0$ we get

$$a_{0x} = -\frac{1}{(\Delta f)^2}(c_2 f_{yyyy} f_{xy} + c_2 f_{xy} f_{xxy} + 2a_0 \Delta f (f_{xxx} + f_{xyy})), \quad (18)$$

$$a_{0y} = \frac{1}{(\Delta f)^2}(2f_{yy}((c_2 - a_0)f_{yyy} - a_0 f_{xxy})$$

$$- 2f_{xx}(c_2 f_{xy} + a_0 (f_{yyyy} + f_{xyy})) - c_2 f_{xy} (f_{xyy} + f_{xxx})).$$  

(19)

We differentiate (18) with respect to $y$, (19) — with respect to $x$ and take a difference between the results, and after that we substitute into the result instead of $a_{0x}$ and $a_{0y}$ expressions (18) and (19). It gives an equation on $f$

$$f_{xy}(f_{yyyy} - f_{xxxx}) + 3(f_{yyyy} f_{xy} - f_{xxxx} f_{xy}) + 2(f_{yy} f_{xyy} - f_{xx} f_{xxy}) = 0.$$  

Since $P, Q$ and $R$ are periodic functions, $f$ can be written in the form

$$f = \lambda + a_{11}x^2 + 2a_{12}xy + a_{22}y^2,$$

where $\lambda$ is a function periodic with respect to $\Gamma$. This yields (7). Theorem 4 is proved.
4 Proof of Theorems 5 and 6

Let us prove first Theorem 5. We start with the simple case of $n = 3$. Given a periodic solution $\lambda$ of equation (6) satisfying $\Delta \lambda + 2(a_{11} + a_{22}) > 0$, then it follows from the explicit formulas (14) that the coefficients of the integral $a_0, a_1, a_2, a_3$ as well as the factor $\Lambda$ are periodic functions.

In order to treat the case $n = 4$ we proceed with a topological argument as follows. Let $\lambda$ be a periodic solution of the equation (7) satisfying $\Delta \lambda + 2(a_{11} + a_{22}) > 0$. Coefficients of the integral and the conformal factor $\Lambda$ are determined by the function

$$f = \lambda + a_{11}x^2 + 2a_{12}xy + a_{22}y^2$$

using the formulas (17), (18), (19). Notice that by Kolokoltsov identities and (17) the functions $\Lambda, a_1, a_3$ are periodic with respect to the lattice. However, coefficients $a_0$ and also $a_2, a_4$ are not necessarily periodic. Coefficient $a_0$ is determined by the equations (18), (19) and it is convenient to rewrite them in the form

$$(a_0(\Delta f)^2)_x = -c_2f_{xy}(\Delta f)_y := V \quad (20)$$

$$(a_0(\Delta f)^2)_y = c_2(f_{yy}^2 - f_{xx}^2)_y - c_2f_{xy}(\Delta f)_x := W. \quad (21)$$

The functions $\Delta f, V, W$ are periodic and we need to show that the periods of the 1-form $Vdx + Wdy$ on the torus are zeroes. We have from (20), (21)

$$a_0 = \frac{L(x, y)}{(\Delta f)^2} + \frac{P_0(x, y)}{(\Delta f)^2},$$

where $L$ is a linear function and $P_0$ and $\Delta f$ are periodic functions with respect to the lattice. Due to (17) we have for other coefficients an analogous form:

$$a_2 = \frac{2L(x, y)}{(\Delta f)^2} + \frac{P_2(x, y)}{(\Delta f)^2},$$

and since $a_4 = c_2 + a_2 - a_0$ we get also

$$a_4 = \frac{L(x, y)}{(\Delta f)^2} + \frac{P_4(x, y)}{(\Delta f)^2},$$

where $P_2, P_4$ are periodic functions. In addition, odd coefficients $a_1, a_3$ are periodic functions. Take now two periodic geodesics $\gamma_1, \gamma_2$ on the covering plane of the configuration torus representing two independent homotopy classes $e_1, e_2$ of the lattice. Denote by $z$ the intersection point and by

$$z_1 = z + e_1, \quad z_2 = z + e_2$$

the translations of $z$. Since

$$F = a_0(x, y)p_1^4 + a_1(x, y)p_1^3p_2 + a_2(x, y)p_1^2p_2^2 + a_3(x, y)p_1p_2^3 + a_4(x, y)p_2^4,$$
is the first integral of the geodesic flow we have that the increment of $F$ along the two geodesics $\gamma_1, \gamma_2$ must vanish. But on the other hand we compute:

$$
\Delta F|_{\gamma_1} = \frac{L(e_1)}{(\Delta f)^2} (p'_1)^4 + 2 \frac{L(e_1)}{(\Delta f)^2} (p'_1)^2 (p'_2)^2 + \frac{L(e_1)}{(\Delta f)^2} (p'_2)^4,
$$

$$
\Delta F|_{\gamma_2} = \frac{L(e_2)}{(\Delta f)^2} (p''_1)^4 + 2 \frac{L(e_2)}{(\Delta f)^2} (p''_1)^2 (p''_2)^2 + \frac{L(e_2)}{(\Delta f)^2} (p''_2)^4,
$$

where we used the form of the coefficients $a_i$. Here we used $(p'_1, p'_2)$ (respectively $(p''_1, p''_2)$) for the momenta variables corresponding to the tangent vector $\dot{\gamma}_1(z)$ (respectively $\dot{\gamma}_2(z)$) at the intersection point $z$. But the last two identities reduce to

$$
\Delta F|_{\gamma_1} = \frac{L(e_1)}{(\Delta f)^2} ((p'_1)^2 + (p'_2)^2)^2 = 0,
$$

$$
\Delta F|_{\gamma_2} = \frac{L(e_2)}{(\Delta f)^2} ((p''_1)^2 + (p''_2)^2)^2 = 0.
$$

Thus

$$
L(e_1) = L(e_2) = 0,
$$

which means that the linear function $L$ vanishes. This completes the proof of Theorem 5 for $n = 4$.

Let us finish this section establishing variational form of the equation (7). This becomes clear if one rewrites (7) in the following way:

$$
4a_{22}\lambda_{xyyy} - 4a_{11}\lambda_{xxxx} + 2a_{12}(\lambda_{yyyy} - \lambda_{xxxx}) +
$$

$$
+ (\lambda_{xy}\lambda_{yy})_{yy} - (\lambda_{xy}\lambda_{xx})_{xx} + \frac{1}{2}(\lambda_{yy}^2 - \lambda_{xx}^2)_{xy} = 0.
$$

(22)

It is easy to verify that last equation is indeed Euler-Lagrange equation of the functional of Theorem 6. This completes the proof.

## 5 Proof of Theorem 7

As in the proof of Theorem 6 let us rewrite equation (8) in the form

$$
\lambda_{xyy} - \lambda_{yyy} =
$$

$$
= \varepsilon (\lambda_{xy}\lambda_{yy})_{yy} - (\lambda_{xy}\lambda_{xx})_{xx} + \frac{1}{2}(\lambda_{yy}^2 - \lambda_{xx}^2)_{xy}.
$$

(23)

Then the recursion step (10) looks as follows:

$$
(\lambda_k)_{xyy} - (\lambda_k)_{yyy} = \sum_{p,q \geq 0, p+q=k-1}^{k-1} A_{pq} + B_{pq} + C_{pq},
$$

(24)
where

\[ A_{pq} = ((\lambda_p)_{xy}(\lambda_q)_{yy})_{yy}, \]

\[ B_{pq} = -((\lambda_p)_{xy}(\lambda_q)_{xx})_{xx}, \]

\[ C_{pq} = \frac{1}{2}((\lambda_p)_{yy}(\lambda_q)_{yy} - (\lambda_p)_{xx}(\lambda_q)_{xx})_{xy}. \]

We prove by induction the following claim. All \( \lambda_k \) have no Fourier monomials of the form

\[ e^{inx}, \quad e^{iny}, \quad e^{in(x+y)}, \quad e^{in(x-y)}, \]

and is a function which is symmetric with respect to axes and diagonals.

Assume inductively that for all \( k = 1, ..., K-1 \) the claim holds. In order to construct \( \lambda_K \) one needs that the monomials \( (25) \) do not show up in the right hand side of \( (24) \). Start with \( e^{inx} \). Such a monomial can appear on the right hand side only from \( B_{pq} \). But the functions \( \lambda_p, \lambda_q \) are even with respect to \( y \), therefore \( B_{pq} \) is odd with respect to \( y \) and so the Fourier coefficient of \( e^{inx} \) must vanish. Analogously, \( e^{iny} \) can appear only from \( A_{pq} \). But this is again an odd function on \( x \) and thus the Fourier coefficient of \( e^{iny} \) must vanish. In order to conclude about the monomials \( e^{in(x+y)}, e^{in(x-y)} \) we notice that the equation \( (23) \) and therefore also \( (24) \) is invariant on the rotation of the plane by \(-45^\circ\) and so the previous argument can be applied. Thus monomials \( (25) \) do not appear and \( \lambda_k \) can be found. One can easily see it is also symmetric with respect to the axes and diagonals. This proves the claim.

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