Fighting Contextual Bandits with Stochastic Smoothing

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Abstract

We introduce a new stochastic smoothing perspective to study adversarial contextual bandit problems. We propose a general algorithm template that represents random perturbation based algorithms and identify several perturbation distributions that lead to strong regret bounds. Using the idea of smoothness, we provide an $O(\sqrt{T})$ zero-order bound for the vanilla algorithm and an $O(T^{2/3})$ first-order bound for the clipped version. These bounds hold when the algorithm use with a variety of distributions that have a bounded hazard rate. Our algorithm template includes EXP4 as a special case. The contextual bandit setting usually involves a fixed set of mappings from contexts to actions. The role of the context can be unambiguous in the general expert learning setting. However, following [Bubeck et al. 2012], we use the term contextual bandits to denote the expert learning setting.

We are in the loss setting, and the learner tries to minimize its cumulative loss. We use $L_t \in \mathbb{R}^N$ to denote the cumulative losses of the experts up to iteration $t$:

$$L_{t,i} = \sum_{\tau=1}^{t} l_{\tau,E_i(x_\tau)}.$$

We define the learner’s regret by the difference between the learner’s cumulative loss and the cumulative loss of the best expert in hindsight:

$$R(T) = \sum_{i=1}^{T} l_{t,i} - \min_{i \in [N]} L_{T,i}.$$

Finally, we will use $L_T^* = \min_i L_{T,i}$ to give the first-order regret bound. In case $L_T^* = o(T)$, the first-order bound can be tighter than the zero-order bound, which is given as a function of the time horizon $T$.

1.1. Main Results

A famous contextual bandits algorithm, EXP4, proposed by [Auer et al. 2002] is shown to have a worst-case regret bound of $O(\sqrt{T})$, which is optimal. Also, [Agarwal et al. 2017] prove that a clipped version of EXP4 can achieve $O(T^{2/3})$ bound using the argument made by [Allenberg et al. 2006]. The first-order bound can be tighter than the zero-order bound if $L_T^* = o(T^{3/4})$.

In an orthogonal direction, [Abernethy et al. 2013] analyze a general family of gradient-based prediction algorithms for multi-armed bandit problems. This family of algorithms include EXP3, the multi-armed bandit version of EXP4, which can be interpreted as a perturbation-based algorithm using the Gumbel distribution. Using the notion of the hazard rate (see [1] for definition), they identify other distributions that can also be used as a perturbation and achieve the same optimal zero-order regret bound. Their work paved the way for the analysis of perturbation based algorithms for...
partial information settings beyond multi-armed bandits. In this paper, we provide an extension of their work to contextual bandits.

We develop a general gradient-based prediction algorithm template (GBPA, see Algorithm 1) that can be applied in contextual bandit problems. The template can represent many existing algorithms either regularization-based or perturbation-based. Our main contribution is to show a zero-order bound and a first-order bound for this family of algorithms when equipped with certain distributions that have a finite hazard rate. The distributions include the Gamma, Gumbel, Weibull, Frechet, and Pareto distributions. Two regret bounds match the previous results that were shown to hold only for EXP4.

We have two slightly different versions of algorithms, first of which has the following zero-order regret bound:

**Theorem 1.** (see Corollary 2) Suppose GBPA algorithm uses the stochastic smoothing using a distribution in Table 1. Then the expected regret of the algorithm satisfies the following bound:

\[ ER(T) \leq O(\sqrt{KT\log N}). \]

The second version of the algorithm has an additional step of clipping and enjoys the following first-order regret bound:

**Theorem 2.** (see Corollary 11) Suppose GBPA algorithm uses the stochastic smoothing using a distribution in Table 1. If its final sampling distribution is clipped by a calibrated threshold, then its regret satisfies the following bound with high probability:

\[ R(T) \leq \tilde{O}(K^{1/3}L_T^{2/3}(\log N)^{1/3} + K\log N). \]

Even though we restrict the choice of distribution to Table 1, the algorithm can use other distributions as long as they satisfy certain conditions. We discuss the conditions in Section 2.3.

Note that the first-order bound holds with high probability, which is stronger than the expectation argument in the zero-order bound. This is possible because the clipping provides an easier control of the variance.

## 2. NOTATION

We denote a zero vector by \( \mathbf{0} \) and standard basis vectors by \( e_i \). We state the dimension of a vector unless it is clear from the context. For an integer \( n \), define \( [n] = \{1, \ldots, n\} \). Given a set \( A \), we write the family of distributions over \( A \) by \( \Delta_A \) and shorten \( \Delta_{[n]} \) to \( \Delta_n \). Indicator functions are denoted as \( \mathbb{1}(.). \)

For a distribution \( \mathcal{D} \), we write its cumulative distribution function and density by \( F_D \) and \( f_D \), respectively. We will also use the hazard rate of a distribution \( \mathcal{D} \), which is a well-known tool in survival analysis:

\[ h_D(x) = \frac{f_D(x)}{1 - F_D(x)}. \quad (1) \]

We are mostly interested in the supremum:

\[ \sup h_D = \sup_{x \in \text{supp}(\mathcal{D})} h_D(x). \]

For simplicity, we use the hazard rate of a distribution to denote this supremum.

Given a convex function \( F \), the Bregman divergence is

\[ D_F(p, q) = F(p) - F(q) - \nabla F(q) \cdot (p - q). \]

Throughout the paper, we will frequently use a potential function \( \hat{\Phi} \), which is concave. Since \( -\hat{\Phi} \) is convex, we write

\[ D_{-\hat{\Phi}}(p, q) = -\hat{\Phi}(p) + \hat{\Phi}(q) + \nabla \hat{\Phi}(q) \cdot (p - q). \]

Note that the Bregman divergence is always non-negative and convex in the first argument.

## 3. ALGORITHMS

Our algorithm is based on the Gradient-Based Prediction Algorithm of Abernethy et al. (2013). The original algorithm was designed for the multi-armed bandit problems, and we slightly modify it to apply it to contextual bandits. Its generality that can encompass many Follow-the-Regularized-Leader (FTRL) and Follow-the-Perturbed-Leader (FTPL) algorithms is inherited by the modified version. In this paper, however, we will focus on the FTPL perspective and use a stochastically smoothed potential function when discussing the regret bounds.

### 3.1. Algorithm Template

The GBPA needs a potential function \( \hat{\Phi} \) that is differentiable and concave. Additionally, its partial derivatives should be non-negative. Abernethy et al. (2013) required the potential to be convex because they were in the gain setting in contrast to our loss setting. The GBPA also has an optional input \( \rho \), which is a threshold when we clip the sampling distribution for the variance control. The zero-order regret bound proof does not require this clipping, but the first-order regret bound proof does.

Algorithm 1 summarizes the template. The algorithm first computes a sampling distribution \( p_t \in \Delta_N \) over the experts that is proportional to the gradient of the potential. Step 6 is the extra step that we add to apply the algorithm in the contextual bandit setting. A context \( x_t \), determines a mapping from \( i \in [N] \) to \( j \in [K] \) in a
**Algorithm 1 GBP A Template for Contextual Bandits**

1. **Input:** potential \( \Phi \), (optional) threshold \( \rho \)
2. **Initialize:** \( \hat{L}_0 = 0 \in \mathbb{R}^N \)
3. for \( t = 1, \ldots, T \) do
   4. Receive a context \( x_t \)
   5. Compute \( p_t \in \Delta_N \) such that \( p_{t,i} \propto \nabla \hat{\Phi}(\hat{L}_{t-1}) \)
   6. Convert \( p_t \in \Delta_N \) to \( q_t \in \Delta_K \)
   7. Draw \( J_t \) based on \( q_t \)
   8. Observe and suffer the loss \( l_{t,J_t} \)
   9. Estimate \( \hat{l}_t = \frac{1}{q_t,J_t} e_{J_t} \in \mathbb{R}^K \)
   10. Update \( \hat{L}_t = \hat{L}_{t-1} + \phi_{x_t}(\hat{l}_t) \) (see (3))
4. end for

way that \( E_t(x_t) = j \). Using this mapping, we define two transformations \( \psi_{x_t} : \mathbb{R}^N \rightarrow \mathbb{R}^K \) and \( \phi_{x_t} : \mathbb{R}^K \rightarrow \mathbb{R}^N \):

\[
\psi_{x_t}(p_t) = \sum_{j=1}^K \sum_{i:E_t(x_t) = j} p_{t,i} e_j, \tag{2}
\]

\[
\phi_{x_t}(l_t) = \sum_{j=1}^K \sum_{i:E_t(x_t) = j} l_{t,j} e_i. \tag{3}
\]

It is easy to check that

\[ p_t \cdot \phi_{x_t}(l_t) = \psi_{x_t}(p_t) \cdot l_t. \]

If the threshold \( \rho \) is not specified, we will simply use \( q_t = \psi_{x_t}(p_t) \) for the final sampling distribution. If the threshold is specified, then we will use \( q_t = \omega_\rho \circ \psi_{x_t}(p_t) \), where \( \omega_\rho \) is the clipping function as defined below:

\[
\omega_\rho(q_t) = \begin{cases} 1 - \sum_{i:q_{t,i} < \rho} q_{t,i} & \text{if } q_{t,i} \geq \rho \\ 0 & \text{if } q_{t,i} < \rho \end{cases} \tag{4}
\]

In words, it puts zero weights on the minor actions whose original weights are less than the threshold and scales other weights to keep the output a distribution. Once applied, the output distribution has weights that are either zero or greater than the threshold.

When the final sampling distribution \( q_t \) is computed, the algorithm draws an action \( J_t \) based on it and suffers the loss \( l_{t,J_t} \). Then we estimate the loss by \( \hat{l}_t = \frac{l_{t,J_t}}{q_{t,J_t}} e_{J_t} \). This underestimates the loss in that

\[
\text{E}_{J_t \sim q_t} \hat{l}_t = \begin{cases} l_{t,j} & \text{if } q_{t,j} > 0 \\ 0 & \text{if } q_{t,j} = 0 \end{cases}. \tag{5}
\]

However, the algorithm’s expected loss is unbiased:

\[ \text{E}_{J_t \sim q_t} q_t \cdot \hat{l}_t = q_t \cdot l_t. \]

The last step is to update the estimate of the cumulative loss by adding \( \phi_{x_t}(l_t) \).

### 3.2. Stochastic Smoothing and Hazard Rate

As mentioned earlier, we will particularly focus on the FTPL perspective. Specifically, we define the potential function as

\[
\tilde{\Phi}(L; D) = \mathbb{E}_{Z_1, \ldots, Z_N \sim D} \min_{i} \{ L_i - Z_i \}, \tag{6}
\]

where \( D \) is a continuous distribution over an unbounded support and has a mean equal to zero. This is a stochastic smoothing of the concave function \( \min_{i} L_i \). Note that FTPL (Follow The Leader) uses the gradients of the non-smooth minimum function which leads to instability and lack of strong regret bounds. We record few useful properties of this potential function. Most of them are proven by Abernethy et al. (2014, 2015), but we present the proof for completeness.

**Proposition 3.** Suppose we use the potential in (7).

Then it satisfies the following properties for all \( L \in \mathbb{R}^N \):

1. \( \nabla \tilde{\Phi}(L) \in \Delta_N \), and we may write \( p_t = \nabla \tilde{\Phi}(\hat{L}_{t-1}) \).
2. \( \nabla^2_{ii} \tilde{\Phi}(L) \leq 0 \) and \( \nabla^2_{ij} \tilde{\Phi}(L) \geq 0 \) for \( i \neq j \).
3. \( \sum_{j=1}^N \nabla^2_{jj} \tilde{\Phi}(L) = 0 \) for all \( i \).
4. \( -\nabla^2_{ii} \tilde{\Phi}(L) \leq \sup_D \nabla \tilde{\Phi}(L) \).

Proof. According to Bertsekas (1973), we can swap the order of expectation and differentiation to get

\[
\frac{\partial \tilde{\Phi}}{\partial L_i} = \mathbb{E}_{Z_1, \ldots, Z_N} \mathbb{I}(L_i - Z_i < L_j - Z_j, \forall j \neq i) = \mathbb{E}_{L \sim D} \mathbb{P}_{Z_1}(Z_i > L_i - \hat{L}_i) = \mathbb{E}_{L \sim D} \mathbb{P}_{Z_1}(Z_i > L_i - \hat{L}_i) = \mathbb{E}_{L \sim D} \mathbb{P}_{Z_1}(Z_i > L_i - \hat{L}_i), \tag{7}
\]

where \( \hat{L}_i = \min_{j \neq i} L_i - Z_j \). As we assume that the distribution of \( Z_i \) is continuous, we can ignore the case \( L_i - Z_i = L_j - Z_j \) for some \( i \neq j \). The first equality justifies that the entries of \( \nabla \tilde{\Phi}(L) \) are non-negative and sum up to one, which proves the first statement. Also, the function \( 1(L_i - Z_i < L_j - Z_j, \forall j \neq i) \) is decreasing in \( L_i \) and increasing in other directions, which shows the second statement.

Since \( \sum_{j=1}^N \nabla_j \tilde{\Phi}(L) = 1 \), which is a constant, the third statement follows by taking derivative. Taking derivative of the last term in (7), we get

\[
-\frac{\partial^2 \tilde{\Phi}}{\partial L_i^2} = \mathbb{E}_{L \sim D} f_D(L_i - \hat{L}_i) \leq \sup_D \mathbb{E}_{L \sim D} \frac{\partial \tilde{\Phi}}{\partial L_i},
\]

which completes the last part of the proof. \( \square \)

Essentially, we want to use \( \eta Z \) instead of \( Z \) and optimize the scaling parameter \( \eta > 0 \) afterward. Observe that \( \eta Z \) has the cumulative distribution function \( F_D(x/\eta) \) and thus the density function \( \frac{1}{\eta} f_D(x/\eta) \). From this, we can deduce that the hazard rate of \( \eta Z \) becomes \( \frac{1}{\eta} \sup_D h_D \).
Table 1: Distributions with Optimal Parameters

| Distribution | $\sup D \mathbb{E}_{Z_1,\ldots,Z_N \sim D} \max_i Z_i$ |
|--------------|----------------------------------|
| Gamma ($\alpha = 1, \beta = 1$) | $O(1)$ | $O(\log N)$ |
| Gumbel ($\mu = 0, \beta = 1$) | $O(1)$ | $O(\log N)$ |
| Weibull ($\lambda = 1, k = 1$) | $O(1)$ | $O(\log N)$ |
| Frechet ($\alpha = \log N$) | $O(\log N)$ | $O(1)$ |
| Pareto ($x_m = 1, \alpha = \log N$) | $O(\log N)$ | $O(1)$ |

3.3. Distributions with Bounded Hazard Rate and Optimal Parameters

Our zero-order and first-order regret bounds involve the term (see Section 4)

$$\sup D \mathbb{E}_{Z_1,\ldots,Z_N \sim D} \max_i Z_i,$$

where $D$ is the distribution that is used for stochastic smoothing. This is the main reason that we want the hazard rate to be bounded. Abernethy et al. (2015) identified such distributions, which include Gamma, Gumbel, Weibull, Frechet, and Pareto distributions. Table 1 summarizes the optimal choice of parameters for each of the distributions that gives the smallest value of $\mathbb{E}_{Z_1,\ldots,Z_N \sim D} \max_i Z_i$. We adopt the parameterization of Abernethy et al. (2015), and the interested readers can refer to their paper for mathematical derivations. Interestingly, for the listed distributions, we can find parameters such that

$$\sup D \mathbb{E}_{Z_1,\ldots,Z_N \sim D} \max_i Z_i = O(\log N).$$

4. REGRET BOUNDS

In this section, we provide a zero-order regret bound for GBPA($\bar{\Phi}$) and a first-order regret bound for GBPA($\bar{\Phi}, \rho$). The proofs use a concept of smoothness resulting from the bounded hazard rate of the perturbation distribution $D$.

4.1. Zero-Order Regret Bound

We remind the readers that GBPA($\bar{\Phi}$) does not specify the threshold $\rho$ and uses $q_t = \psi_{x_t}(p_t)$ without clipping. Define $\Phi(L) = \min_i L_i$. The following lemma is a slight variation of Lemma 2.1 in Abernethy et al. (2015).

Lemma 4. (General Regret Bound) The expected regret of GBPA($\bar{\Phi}$) satisfies the following bound:

$$\mathbb{E} R(T) \leq \mathbb{E}[\bar{\Phi}(\hat{L}_T) - \Phi(\hat{L}_T) + \sum_{t=1}^T D_{\ldots \bar{\Phi}}(\hat{L}_t, \hat{L}_{t-1})] - \bar{\Phi}(0),$$

where the expectations are over $J_1,\ldots,J_T$.

Proof. This version of GBPA($\bar{\Phi}$) without clipping uses $q_t = \psi_{x_t}(p_t)$ for the final sampling distribution. Using (2), (3), and (5), we can deduce that

$$q_t \cdot l_t = \mathbb{E}_{J_t} q_t \cdot l_t = \mathbb{E}_{J_t} \psi_{x_t}(p_t) \cdot l_t = \mathbb{E}_{J_t} p_t \cdot \phi_{x_t}(\hat{L}_t).$$

Therefore, the expected regret of the algorithm can be written as below:

$$\mathbb{E} R(T) = \sum_{t=1}^T q_t \cdot l_t - \min_i L_{T,i}$$

$$= \mathbb{E}_{J_1,\ldots,J_T} \sum_{t=1}^T p_t \cdot \phi_{x_t}(\hat{L}_t) - \min_i L_{T,i}$$

$$\leq \mathbb{E}_{J_1,\ldots,J_T} \sum_{t=1}^T p_t \cdot \phi_{x_t}(\hat{L}_t) - \min_i \hat{L}_{T,i},$$

where the last inequality holds because the minimum function is concave and $\hat{L}_T$ underestimates $L_T$.

Note that $p_t = \nabla \Phi(\hat{L}_{t-1}), \phi_{x_t}(\hat{L}_t) = \hat{L}_t - \hat{L}_{t-1}$, and $\min_i \hat{L}_{T,i} = \Phi(\hat{L}_T)$. Using these, we can write

$$\sum_{t=1}^T p_t \cdot \phi_{x_t}(\hat{L}_t) - \min_i \hat{L}_{T,i}$$

$$= -\Phi(\hat{L}_T) + \sum_{t=1}^T \nabla \Phi(\hat{L}_{t-1}) \cdot (\hat{L}_t - \hat{L}_{t-1})$$

$$= -\Phi(\hat{L}_T) + \sum_{t=1}^T \nabla \Phi(\hat{L}_{t-1}) + \bar{\Phi}(\hat{L}_t) - \Phi(\hat{L}_{t-1})$$

$$= -\Phi(\hat{L}_T) + \Phi(\hat{L}_T) - \Phi(0) + \sum_{t=1}^T \nabla \Phi(\hat{L}_t, \hat{L}_{t-1}).$$

Plugging the last result back to (4) completes the proof. ■

Recall that we use $\eta Z$ perturbation to compute the potential. Since it has a mean zero, by using the Jensen’s inequality, we get

$$\bar{\Phi}(\hat{L}_T) = \mathbb{E}_{Z_1,\ldots,Z_n} \min_i \{\hat{L}_{T,i} - \eta Z_i\}$$

$$\leq \min_i \mathbb{E}_{Z_1,\ldots,Z_n} \{\hat{L}_{T,i} - \eta Z_i\}$$

$$= \min_i \hat{L}_{T,i} = \Phi(\hat{L}_T).$$

The last term in Lemma 4 can be written

$$-\Phi(0) = -\mathbb{E}_{Z_1,\ldots,Z_N} \min_i -\eta Z_i$$

$$= \eta \mathbb{E}_{Z_1,\ldots,Z_N} \max_i Z_i$$

(11)

To bound the divergence term in Lemma 4 we prove the following lemma.

Lemma 5. Suppose we use the potential in (4). Then we can bound the divergence term as follows:

$$\mathbb{E}_{J_1,\ldots,J_T} D_{\ldots \bar{\Phi}}(\hat{L}_t, \hat{L}_{t-1}) \leq \frac{\sup D}{2} K.$$
Proof. We first bound the conditional expectation
\[
E_J[D_{-\hat{\Phi}}(\hat{L}_t, \hat{L}_{t-1}) | \hat{L}_{t-1}] \leq \sup_{h_D} \frac{h_D}{2} K,
\]
which will imply the final bound.

Let \( f_j = \phi(e_j) \in \mathbb{R}^N \), which has ones at the indices \( i \) where \( E_j(x_i) = j \) and zeros otherwise. Given an action index \( j \in [K] \), define
\[
g_j(r) = D_{-\hat{\Phi}}(\hat{L}_{t-1} + r f_j, \hat{L}_{t-1})
= \hat{\Phi}(\hat{L}_{t-1}) - \hat{\Phi}(\hat{L}_{t-1} + r f_j) + r f_j \cdot \nabla \hat{\Phi}(\hat{L}_{t-1}).
\]
It is convex and \( g_j(0) = g_j'(0) = 0 \). We can write
\[
g_j''(r) = -f_j^T \cdot \nabla^2 \hat{\Phi}(\hat{L}_{t-1} + r f_j) \cdot f_j.
\]
From Proposition 3, we have that the diagonal entries of \( \nabla^2 \hat{\Phi}(L) \) are non-positive and the non-diagonal entries are non-negative. Using this, we can check for \( r > 0 \),
\[
g_j''(r) \leq \text{diag}(-\nabla^2 \hat{\Phi}(\hat{L}_{t-1} + r f_j)) \cdot f_j
\leq \sup h_D \nabla \Phi(\hat{L}_{t-1} + r f_j) \cdot f_j
\leq \sup h_D \nabla \hat{\Phi}(\hat{L}_{t-1}) \cdot f_j
= \sup h_D \, p_i \cdot \phi(e_j)
= \sup h_D \, q_{t,j},
\]
where \( \text{diag}(A) \) is the vector that consists of the diagonal entries of a matrix \( A \). The second inequality results from Proposition 3 and the third inequality holds because the potential function is concave. Now we can rewrite the conditional expectation as below:
\[
E_J[D_{-\hat{\Phi}}(\hat{L}_t, \hat{L}_{t-1}) | \hat{L}_{t-1}]
= \sum_{j: q_{t,j} > 0} q_{t,j} \cdot g_j(1/q_{t,j})
= \sum_{j: q_{t,j} > 0} q_{t,j} \int_0^{1/q_{t,j}} \int_0^v g_j''(u) \, du \, dv
\leq \sum_{j: q_{t,j} > 0} \int_0^{1/q_{t,j}} \int_0^v \sup h_D \, q_{t,j} \, du \, dv
\leq \frac{\sup h_D}{2} K,
\]
which completes the proof. \[ \square \]

As we are using \( \eta Z \) perturbation, the hazard rate term in Lemma 5 becomes \( \sup_{h_D} \frac{h_D}{\eta} \). Plugging the results of (10), (11), and Lemma 9 to Lemma 4, we have our first main result.

Theorem 6. (Zero-Order Regret Bound) Suppose we use \( \eta Z \) perturbation to build the potential \( \Phi \) where \( Z \) follows a distribution \( D \). Then the expected regret of GBPA(\( \Phi \)) satisfies the following bound:
\[
ER(T) \leq \frac{\eta \sum_{1 \leq i \leq N} \max Z_i + \sup h_D}{2\eta} KT.
\]
If we choose \( \eta = \sqrt{KT \sup h_D \max Z_i} \), then we get the bound
\[
O(\sqrt{KT \sup h_D \max Z_i}).
\]

Using the parameters in Table 1 we can make the bound become \( O(\sqrt{KT \log N}) \). We record this as a corollary.

Corollary 7. Suppose we use \( \eta Z \) perturbation to build the potential \( \Phi \) where \( \eta = \sqrt{KT \sup h_D \max Z_i} \) and \( Z \) follows a distribution in Table 1. Then the expected regret of GBPA(\( \Phi \)) satisfies the following bound:
\[
ER(T) \leq O(\sqrt{KT \log N}).
\]

The bound has the optimal \( O(\sqrt{T}) \) scaling in \( T \). \cite{agarwal2012} present a lower bound of \( \Omega(\sqrt{KT \log N / \log K}) \), which matches our bound up to a \( \sqrt{\log K} \) factor. Using the Gumbel distribution results in the well-known EXP4 algorithm by \cite{auer2002}, which is already shown to have the optimal regret bound up to a \( \sqrt{\log K} \) factor. However, our framework is more general in that a similar bound can be obtained by using other distributions that have a finite hazard rate.

4.2. First-Order Regret Bound

In this section, we will bound the regret in terms of the loss of the best hindsight expert, \( L^*_T \), instead of the time horizon \( T \) as in the previous section. The bound can be even tighter if one of the experts has a very small loss such that \( L^*_T = o(T) \).

We remind the readers that we use the threshold \( \rho \) to clip the final sampling distribution \( q_t \):
\[
q_t = \omega \circ \psi_x (p_t).
\]
This ensures that \( q_{t,j} \) is either zero or greater than \( \rho \). One main advantage of using clipping is that we have \( \|\hat{\ell}_t\|_\infty < 1/\rho \) for all \( t \).

The main idea of proving the regret bound comes from the differential privacy. See a survey by \cite{dwork2014} for an overview of the area. Following the notation introduced by \cite{dwork2010}, we define the max divergence of two distributions \( p, q \in \Delta_n \) as:
\[
D_\infty(p \parallel q) = \max_{j: q_{t,j} > 0} \frac{p_j}{q_j}.
\]
One useful property of the max divergence is that if \( D_\infty(p \parallel q) \leq \epsilon \) for \( p, q \in \Delta_K \), then it is easy to check that for any loss vector \( \ell \in \mathbb{R}_+^K \), we have
\[
p \cdot \ell \leq \epsilon q \cdot \ell.
\]
(12)

The preprint by \cite{abernethy2017} brings the differential privacy perspective into the multi-armed bandit problems. The following lemma is a contextual bandit version of Theorem 3.2 in the preprint.
Lemma 8. Suppose \( A^1, A^2 \) are algorithms for the contextual bandits that sample the actions using \( q^1_t, q^2_t \in \Delta_K \) at time \( t \). If \( D_\infty(q^1_t \| q^2_t) \leq \epsilon \leq 1 \) for all \( t \), then their expected regrets satisfy the following inequality:

\[
\mathbb{E}R_{A^1}(T) \leq 2T L^*_T + 3\mathbb{E}R_{A^2}(T).
\]

Proof. Let \( l_t \in \mathbb{R}^K_+ \) be the loss vector at time \( t \). Then using (12), we have

\[
q^1_t \cdot l_t \leq e^\epsilon q^2_t \cdot l_t.
\]

Summing this over \( t \), we can write

\[
\sum_{t=1}^T q^1_t \cdot l_t \leq e^\epsilon \sum_{t=1}^T q^2_t \cdot l_t
= e^\epsilon (L^*_T + \mathbb{E}R_{A^2}(T))
\leq (1 + 2\epsilon)(L^*_T + \mathbb{E}R_{A^2}(T)),
\]

where the last inequality holds because \( \epsilon \leq 1 \).

Since \( \mathbb{E}R_{A^1}(T) = \sum_{t=1}^T q^1_t \cdot l_t - L^*_T \), we get

\[
\mathbb{E}R_{A^1}(T) \leq 2T L^*_T + (1 + 2\epsilon)\mathbb{E}R_{A^2}(T)
\leq 2T L^*_T + 3\mathbb{E}R_{A^2}(T),
\]

which concludes the argument.

Once our algorithm predicts the actions \( J_1, \ldots, J_T \), we construct a new imaginary adversary just for the purpose of analyzing the regret. The imaginary adversary runs for the same time length \( T \) and reveals the context \( x_t \) along with the full information loss \( \hat{l}_t = \frac{1}{\eta} \cdot x_t \cdot l_t \). This loss is deterministic as the original algorithm is already completed with fixed \( J_1, \ldots, J_T \).

Then we run two algorithms against this imaginary adversary to apply Lemma [3] For the first algorithm, we use the GBPA. As this is the full information setting, however, the algorithm does not need an estimate step (step 9 in Algorithm 1) and tracks the exact cumulative loss. To differentiate this version from the original GBPA, we name the algorithm as GBPA-NE (no estimation step). The key observation is that two algorithms GBPA(\( \tilde{\Phi}, \rho \)) on the original adversary and GBPA-NE(\( \tilde{\Phi}, \rho \)) on the imaginary adversary use the same sampling distribution at each step because their cumulative loss \( \tilde{l}_t \) remains identical. Also, to emphasize this imaginary setting, we denote the expected regret in this setting as below:

\[
\mathbb{E}\tilde{R}(T) = \sum_{t=1}^T q_t \cdot \tilde{l}_t - \tilde{L}^*_T,
\]

where \( q_t \) is the sampling distribution of the algorithm at round \( t \). In particular, if we run GBPA-NE(\( \tilde{\Phi}, \rho \)), then by the fact that its sampling distribution is identical to the one in the original GBPA algorithm, we can write

\[
\mathbb{E}\tilde{R}_{NE}(T) = \sum_{t=1}^T l_{t,J_t} - \tilde{L}^*_T.
\]

After putting GBPA-NE(\( \tilde{\Phi}, \rho \)) on \( A^1 \) in Lemma 8, we use a strong algorithm for \( A^2 \) that cheats in a sense that it observes the loss \( l_t \) before deciding its sampling distribution at time \( t \). Specifically, the algorithm, which we call \( A^+ \), chooses \( q_t = \phi_{x_t}(p_t + 1) \) at time \( t \), where \( p_{t+1} = \nabla \tilde{\Phi}(\tilde{l}_t) \). Note that it does not clip the final sampling distribution. This algorithm is actually called as Be-The-Perturbed-Leader (BTPL) by Kalai and Vempala [2004]. The authors show that the BTPL algorithm suffers a small regret that does not depend on the time horizon \( T \):

\[
\mathbb{E}\tilde{R}_{BTPL}(T) \leq \mathbb{E}Z_1, \ldots, Z_N \max_i \eta Z_i,
\]

provided that the potential is built upon \( \eta Z \) perturbation. We will show that the divergence between the sampling distributions of GBPA-NE(\( \tilde{\Phi}, \rho \)) and \( A^+ \) at time \( t \),

\[
\mathbb{E}_t(\omega_{\rho} \circ \psi_{x_t}(p_t) \| \psi_{x_t}(p_{t+1}))
\]

is not too big. To do so, we first show that the divergence between \( p_t \) and \( p_{t+1} \) is bounded.

Lemma 9. Suppose we use the potential in (9). Then for any round \( t \), we have

\[
D_\infty(p_t \| p_{t+1}) \leq 2\sup h_D \| \hat{l}_t \| \infty.
\]

Proof. From Proposition 3, we know

\[
p_t = \nabla \tilde{\Phi} (\tilde{l}_{t-1}) \quad \text{and} \quad p_{t+1} = \nabla \tilde{\Phi} (\tilde{l}_t).
\]

Since \( \sum_{j=1}^N \nabla^2_{ij} \tilde{\Phi}(L) = 0 \) and only \( \nabla^2_{ii} \tilde{\Phi}(L) \) is negative, we have \( \| \nabla^2_{ii} \tilde{\Phi}(L) \| \leq -2 \nabla^2_{ii} \tilde{\Phi}(L) \). Also note that \( \tilde{l}_t = \tilde{l}_{t-1} + \phi_{x_t}(\hat{l}_t) \) and \( \| \phi_{x_t}(\hat{l}_t) \| \leq \| \hat{l}_t \| \infty \) by the definition of \( \phi_{x_t} \) (see [3]). Define \( f_t(r) = \nabla \tilde{\Phi} (\tilde{l}_t - r \phi_{x_t}(\tilde{l}_t)) \). The derivative is

\[
f'_t(r) = -\nabla^2_{ii} \tilde{\Phi}(\tilde{l}_t - r \phi_{x_t}(\hat{l}_t)) \cdot \phi_{x_t}(\hat{l}_t)
\leq \| \nabla^2_{ii} \tilde{\Phi}(\tilde{l}_t - r \phi_{x_t}(\hat{l}_t)) \| \| \phi_{x_t}(\hat{l}_t) \| \| \hat{l}_t \| \infty
\leq 2\sup h_D \nabla \tilde{\Phi} (\tilde{l}_t - r \phi_{x_t}(\hat{l}_t)) \| \hat{l}_t \| \infty
= 2\sup h_D f_t(r) \| \hat{l}_t \| \infty,
\]

where the second inequality holds by the last item in Proposition 3. Since \( f_t(r) \geq 0 \), we have

\[
\frac{f'_t(r)}{f_t(r)} = \frac{d}{dr} \log f_t(r) \leq 2\sup h_D \| \hat{l}_t \| \infty.
\]

Using this, we can deduce

\[
\log \frac{p_{t+1,i}}{p_{t,i}} = \log f_t(1) - \log f_t(0)
= \int_0^1 \frac{d}{dr} \log f_t(r) dr \leq 2\sup h_D \| \hat{l}_t \| \infty.
\]
As in Section 4.1 we use \( \eta Z \) perturbation where \( Z \) follows a distribution \( D \). Recall that the hazard rate of \( \eta Z \) becomes \( \frac{\eta}{\sup h_{\mathcal{D}}} \). Using Lemma 8 and the definition of \( \psi_x \) in (2), we get

\[
D_{\infty}(\psi_x(p_t) \parallel \psi_x(p_{t+1})) \leq 2 \frac{\sup h_{\mathcal{D}}}{\eta} \| \hat{t}_i \|_{\infty}.
\]

Note that \( \psi_x(p_{t+1}) \) is actually the final sampling distribution by \( A^+ \) at round \( t \). Furthermore, as \( \text{GBP}(\hat{\Phi}, \rho) \) has a clipping threshold, we can bound \( \| \hat{t}_i \|_{\infty} \leq 1/\rho \).

Suppose we use \( \rho = \frac{1}{MK} \) for some \( M > 2 \). By definition of \( \omega_{\rho} \) in (4), we have \( \frac{\omega_{\rho}}{\eta} \leq 1 + \frac{1}{M} \), for \( q_j > 0 \) and \( \tilde{\eta} = \omega_{\rho}(q) \). From this, we can derive

\[
D_{\infty}(\omega_{\rho} \circ \psi_x(p_t) \parallel \psi_x(p_{t+1})) 
\leq 2 \frac{\sup h_{\mathcal{D}}}{\eta} \| \hat{t}_i \|_{\infty} + \log(1 + \frac{1}{M - 1})
\leq 2 \frac{MK \sup h_{\mathcal{D}}}{\eta} + \frac{2}{M},
\]

where we use the relation \( \log(1 + x) \leq x \) for all \( x \) and \( M > 2 \). Then the choice of \( M = \sqrt{\frac{\eta}{\sup h_{\mathcal{D}}}} \) provides

\[
D_{\infty}(\omega_{\rho} \circ \psi_x(p_t) \parallel \psi_x(p_{t+1})) \leq \frac{4}{\sqrt{k}} \frac{\sup h_{\mathcal{D}}}{\eta}.
\]

We need \( \eta > 4K \sup h_{\mathcal{D}} \) to ensure \( M > 2 \). In fact, we will use a stronger condition \( \eta > 16K \sup h_{\mathcal{D}} \) to bound the divergence by 1. Applying Lemma 8 along with the result (13) in the imaginary full information setting provides the following first-order regret bound of GBPAN-NE(\( \Phi, \rho \)):

\[
E\hat{R}_{\text{NE}}(T) \leq 8 \sqrt{\frac{K \sup h_{\mathcal{D}}}{\eta}} \hat{L}^*_T + 3\eta E Z_1, \ldots, Z_N \max_i Z_i.
\]

Plugging (13) in this, we get

\[
\sum_{t=1}^T 1_{l_t \neq t} \leq (1 + 8 \sqrt{\frac{K \sup h_{\mathcal{D}}}{\eta}}) \hat{L}^*_T + 3\eta E \max_i Z_i.
\]

Finally, since \( \hat{L}_T \) underestimates \( L_T \) and each \( \hat{t}_i \) is bounded by \( \frac{1}{\rho} \), the Freedman’s inequality (see, e.g., Lemma 8.8 in Cesàro-Bianchi and Lugosi (2008)) provides with probability \( 1 - \delta \),

\[
\hat{L}^*_T \leq L^*_T + 2 \sqrt{\frac{L^*_T \log 1/\delta}{\rho}} + \frac{\log 1/\delta}{\rho}.
\]

We postpone the proof of the inequality to Appendix.

Combining the last two results, we get our first-order regret bound.

**Theorem 10. (First-Order Regret Bound)** Suppose we use \( \eta Z \) perturbation in the potential \( \Phi \) where \( Z \sim \mathcal{D} \) and \( \eta > 16K \sup h_{\mathcal{D}} \). We apply the clipping \( \omega_{\rho} \) with \( \rho = \sqrt{\frac{\sup h_{\mathcal{D}}}{\eta}} \) to get \( q_t \). Then the regret of \( \text{GBP}(\hat{\Phi}, \rho) \) has the following bound with probability \( 1 - \delta \):

\[
R(T) \leq \hat{O}(\sqrt{\frac{K \sup h_{\mathcal{D}}}{\eta}} L^*_T + \eta E \max_i Z_i + K),
\]

where \( \hat{O}(\cdot) \) suppresses dependence on \( \log \frac{1}{\delta} \).

If we choose \( \eta = \max\{ (\frac{KL^*_T \sup h_{\mathcal{D}}}{\eta \max_i Z_i})^2, 17K \sup h_{\mathcal{D}} \} \), then the bound becomes

\[
\hat{O}(KL^*_T ^2 \sup h_{\mathcal{D}} E \max_i Z_i)^{1/3} + K \sup h_{\mathcal{D}} E \max_i Z_i).
\]

Using the distributions in Table 4 we can replace \( \sup h_{\mathcal{D}} E \max_i Z_i \) by \( \log N \) to get the next corollary.

**Corollary 11.** Suppose we use \( \eta Z \) perturbation in the potential \( \Phi \) where \( Z \) follows a distribution in Table 4 and \( \eta = \max\{ (\frac{KL^*_T \sup h_{\mathcal{D}}}{\eta \max_i Z_i})^2, 17K \sup h_{\mathcal{D}} \} \). Additionally, we use \( \rho = \sqrt{\frac{\sup h_{\mathcal{D}}}{\eta \max_i Z_i}} \) for the clipping threshold. Then the regret of \( \text{GBP}(\hat{\Phi}, \rho) \) has the following bound with probability \( 1 - \delta \):

\[
R(T) \leq \hat{O}(K^{1/3} L^*_T ^{2/3} \log N)^{1/3} + K \log N),
\]

where \( \hat{O}(\cdot) \) suppresses dependence on \( \log \frac{1}{\delta} \).

We want to emphasize that the first-order bound can be shown in a probabilistic sense because the variance of the learner’s predictions is controlled due to clipping. Otherwise, it is impossible to use the Freedman’s inequality, which is the case in the zero-order bound proof. To the best of our knowledge, the vanilla EXP4 does not have probabilistic regret bounds.

**5. DISCUSSION**

In this paper, we extended the general GBP template from the multi-armed bandit problem to the contextual bandit problem. In particular, we focused on FTPL based algorithms that utilize stochastic smoothing to construct the potential function. Many distributions with a bounded hazard rate can be used as the source of perturbation, which makes our framework general. The famous EXP4 algorithm is included in our framework.

Two versions of GBP are introduced. The vanilla GBP has the zero-order regret bound \( O(\sqrt{T}) \), which is optimal, and the clipped version has the first-order regret bound \( O(L^*_T^{2/3}) \). These bounds were already shown for EXP4 and its clipped version. However, the previous results specifically relied on the property of EXP4 algorithm, and it remained unclear if they generalize to other algorithms. The ideas of smoothness and the differential privacy in our theory are general in that they
can be applied to any distribution with a bounded hazard rate.

Finally, note that our first-order bound $O(L^*T^{2/3})$ is weaker than the one proved for the recent algorithm MYGA (Allen-Zhu et al. 2018). Proving $O(\sqrt{LT})$ bound for contextual bandits was an open problem until recently (see Agarwal et al. 2017). MYGA achieves the optimal first-order bound, but the algorithm is not simple in that it has to maintain $\Theta(T)$ auxiliary experts in every round. In contrast, our algorithm is simple as it is basically a version of EXP4 algorithm along with the clipping idea. We hope that this work can bring a new perspective to the contextual bandits, which leads to a simple algorithm that enjoys the optimal first-order regret bound.

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8
A. DETAIL PROOFS

We provide a detailed proof that is omitted in the main part of the manuscript.

A.1. Bounding $\hat{L}_T^*$

We prove the inequality (16) that with probability $1 - \delta$,

$$\hat{L}_T^* \leq L_T^* + 2\sqrt{\frac{L_T^* \log \frac{1}{\delta}}{\rho} + \frac{\log \frac{1}{\delta}}{\rho}}.$$  

We first record a concentration inequality introduced in Lemma A.8 by Cesa-Bianchi and Lugosi (2006).

**Lemma 12.** Let $X_1, \ldots, X_T$ be a bounded martingale difference sequence with respect to the filtration $\mathcal{F} = (\mathcal{F}_t)_{t=1,\ldots,T}$ and with $|X_t| \leq b$ for all $t$. Let

$$S_T = \sum_{t=1}^T X_t$$

be the associated martingale. Denote the sum of the conditional variances by

$$\Sigma_T^2 = \sum_{t=1}^T \mathbb{E}[X_t^2 | \mathcal{F}_{t-1}].$$

Then for all constants $\epsilon, v > 0$, we have

$$\mathbb{P}(S_T > \sqrt{2v\epsilon} + \frac{\sqrt{2}}{3} b_\epsilon \text{ and } \Sigma_T^2 \leq v) \leq e^{-\epsilon}.$$  

Now we apply this lemma to prove the inequality.

**Proof.** Without loss of generality, we may assume that the first expert is the best hindsight expert and that $L_{T,1} = L_T^*$. We can write

$$\hat{L}_{T,1} = \sum_{t=1}^T \hat{i}_{t,j_t},$$

where $j_t$ is the prediction of the first expert at round $t$. Note that

$$\hat{i}_{t,j_t} = \begin{cases} \frac{i_{t,j_t}}{q_{t,j_t}} & \text{with probability } q_{t,j_t} \\ 0 & \text{otherwise} \end{cases}$$

where $q_t$ is the sampling distribution of the algorithm at time $t$. In fact, $q_{t,j_t}$ can be zero, in which case $\hat{i}_{t,j_t}$ is constantly zero. Then we get for $p = q_{t,j_t} > 0$,

$$\text{Var}(\hat{i}_{t,j_t} | J_1, \ldots, J_{t-1}) = (1-p)\hat{i}_{t,j_t}^2 + p\left(\frac{1}{p} - 1\right)^2 \hat{i}_{t,j_t}^2$$

$$= \frac{1 - p}{p} \hat{i}_{t,j_t}^2$$

$$\leq \frac{1}{\rho} \hat{i}_{t,j_t}.$$