Groups of components of Néron models of Jacobians and Brauer groups

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Abstract. Let $X$ be a proper, smooth, and geometrically connected curve over a non-archimedean local field $K$. In this paper, we relate the component group of the Néron model of the Jacobian of $X$ to the Brauer group of $X$.

1. Introduction

Let $K$ be a non-archimedean local field. Thus $K$ is a complete discrete valuation field with finite residue field $k$. Let $K_{nr}$ be the maximal unramified extension of $K$. Let $X$ be a proper, smooth, and geometrically connected curve over $K$, and $X_{nr} = X \otimes_K K_{nr}$ be the corresponding curve over $K_{nr}$. Let $\delta$ and $\delta'$ denote, respectively, the index and the period of $X$, and let $\delta_{nr}$ and $\delta'_{nr}$ denote the corresponding quantities associated to $X_{nr}$. Let $A$ be the Jacobian variety of $X$ over $K$, $\Phi_A$ the $k$-group scheme of connected components of the Néron model of $A$, and $c_A = \#\Phi_A(k)$ the corresponding Tamagawa number of $A$ at $K$. Consider the Brauer-Grothendieck group $Br(X) = H^2(X, \mathbb{G}_m)$. Let $Br_0(X)$ denote the image of $Br(K) \to Br(X)$, and $Br_{nr}(X)$ denote the kernel of $Br(X) \to Br(X_{nr})$. In this paper, we prove

**Theorem 1.1** (Main Theorem). There exists an exact sequence

$$0 \to \text{Hom} \left( Br_{nr}(X)/Br_0(X), \mathbb{Q}/\mathbb{Z} \right) \to \Phi_A(k) \to \mathbb{Z}/d\mathbb{Z} \to 0$$

where $d = \delta'/\delta_{nr}'$. It follows that

**Corollary 1.2.** $Br_{nr}(X)/Br_0(X)$ is a finite group of order $c_A/d$.

Corollary 1.2 has an interesting application. To explain this, consider a global field $K$. Thus $K$ is either a finite extension of $\mathbb{Q}$ i.e. a number field, or is finitely generated and of transcendence degree 1 over a finite field $k$ i.e. a function field. In the number field case, let $U$ denote a nonempty open subscheme of $\text{Spec} \mathcal{O}_K$, and in the function field case, let $U$ denote a nonempty open subscheme of the unique, smooth, complete, and irreducible curve $V$ over $k$ whose function field
is $K$. Consider a regular, connected scheme $\mathcal{X}$ of dimension 2 with a proper morphism $\pi: \mathcal{X} \to U$ such that its generic fiber $X = \mathcal{X} \otimes_U K$ is a smooth and geometrically connected curve over $K$. Let $S$ be the set of primes of $K$ not corresponding to a point of $U$, and $\overline{K}$ be the separable algebraic closure of $K$. Note that $S$ contains all archimedean primes of $K$ in the number field case, and that it may be empty in the function field case. For each prime $v \notin S$, let $K_v$ denote the completion of $K$ at $v$, and let $X_v = X \otimes_K K_v$. Let $\delta$ and $\delta'$ be, respectively the index and period of $X$ while $\delta_v$ and $\delta'_v$ be the corresponding quantities associated to $X_v$. It is known that $\delta_v \neq 1$ for only finitely many primes $v$, and that either $\delta_v = \delta'_v$ or $\delta_v = 2\delta'_v$ [Lic69, Theorem 8]. Let $Br(\mathcal{X})$ denote the Brauer group of $\mathcal{X}$ and define $Br(\mathcal{X})'$ by the exactness of the sequence

$$0 \to Br(\mathcal{X})' \to Br(\mathcal{X}) \to \bigoplus_{v \in S} Br(X_v)$$

Now let $A/K$ be the Jacobian variety of $X$ over $K$, and denote by $\Sha(A/K)$ the Shafarevich-Tate group of $A/K$. Generalizing the work of Artin [Tat68] and Milne [Mil82], Gonzalez-Aviles has shown [Gon03] that

**Theorem 1.3** (Gonzalez-Aviles). Suppose that the integers $\delta'_v$ are pair-wise co-prime and that $\Sha(A/K)$ contains no nonzero infinitely divisible elements. Then there is an exact sequence

$$0 \to T_0 \to T_1 \to Br(\mathcal{X})' \to \Sha(A/K)/T_2 \to T_3 \to 0$$

in which $T_0, T_1, T_2$ and $T_3$ are finite groups of orders

$$\#T_0 = \delta/\delta'$$
$$\#T_1 = 2^e$$
$$\#T_2 = \delta'/\prod \delta'_v$$
$$\#T_3 = \frac{\delta'/\prod \delta'_v}{2^f}$$

where

$$e = \max(0, d' - 1)$$

and

$$f = \begin{cases} 1 & \text{if } \delta'/\prod \delta'_v \text{ is even and } d' \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Here $d'$ is the number of primes $v$ for which $\delta_v = 2\delta'_v$. In particular, if one of $\Sha(A/K)$ or $Br(\mathcal{X})'$ is finite, then so is the other, and their
orders are related by
\[ \delta \delta' \# \text{Br}(X)' = 2^{e+f} \prod_v (\delta_v')^2 \# \text{III}(A/K) \]

Now let \( K_v^{nr} \) be the maximal unramified extension of \( K_v \), and \( X_v^{nr} = X_v \otimes_{K_v} K_v^{nr} \) be the fiber over \( K_v^{nr} \), with index \( \delta_v^{nr} \) and period \( \delta_v^{nr'} \). Let \( \text{Br}_0(X_v) \) be the image of the map \( \text{Br}(K_v) \to \text{Br}(X_v) \), and \( \text{Br}_m(X_v) \) be the kernel of the map \( \text{Br}(X_v) \to \text{Br}(X_v^{nr}) \). Let \( c_{A,v} \) the Tamagawa number of \( A \) at \( v \). Note that Theorem 1.1 applies to \( X_v \). Combining Corollary 1.2 with Theorem 1.3, we obtain

**Corollary 1.4.** Suppose that the integers \( \delta_v' \) are pairwise co-prime, and \( \text{III}(A/K) \) is finite. Then
\[ \# \text{III}(A/K) \prod_v c_{A,v} = M \# \text{Br}(X)' \prod_v \# \left( \text{Br}_m(X_v)/\text{Br}_0(X_v) \right) \]

where \( M \) is a rational number given by
\[ M = \frac{\delta \delta'}{2^{e+f} \prod_v \delta_v \delta_v^{nr'}} \]

Of course, the left-hand term in the above formula appears in the statement of the well-known Birch and Swinnerton-Dyer Conjecture.

**Remark 1.5.** The hypothesis that the integers \( \delta_v' \) are pairwise coprime in Theorem 1.3 can be dropped when \( K \) is a function field, and \( S = \emptyset \). More precisely, assume that the curve \( V/k \) introduced above is also geometrically connected, and consider \( X \) to be a smooth, proper, and geometrically connected surface endowed with a proper and flat morphism \( f : X \to V \) whose generic fiber is \( X \to \text{Spec} \ K \). In this case, it is shown in [LLR05, Cor. 3] that if, for some prime \( l \), the \( l \)-part of the group \( \text{Br}(X) \) or of the group \( \text{III}(A/K) \) is finite, then
\[ \delta^2 \# \text{Br}(X) = \prod_v \delta_v \delta_v' \# \text{III}(A/K) \]

and \( \# \text{Br}(X) \) is a square. It follows that, in this case, we get
\[ \# \text{III}(A/K) \prod_v c_{A,v} = N \# \text{Br}(X) \prod_v \# \left( \text{Br}_m(X_v)/\text{Br}_0(X_v) \right) \]

where the rational number \( N \) is given by
\[ N = \frac{\delta^2}{\prod_v \delta_v \delta_v^{nr'}} \]
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2. Preliminaries

2.1. Component Groups, Tamagawa numbers. Let $K$ be a complete, discretely valued field with finite residue field $k$. Let $A/K$ be an abelian variety over $K$, and let $\mathcal{A}$ be the Néron model \cite{BLR90} of $A/K$ over Spec $\mathcal{O}_K$. The closed fiber $\mathcal{A}_k$ of $\mathcal{A}$ is a $k$-group scheme, not necessarily connected. Let $\mathcal{A}^0_k$ be the connected component of $\mathcal{A}_k$ containing the identity. Over Spec $k$, there is an exact sequence of group schemes

$$0 \to \mathcal{A}^0_k \to \mathcal{A}_k \to \Phi_A \to 0$$

where the quotient $\Phi_A$ is a finite, étale group scheme over $k$. Equivalently, $\Phi_A$ is a finite abelian group with a continuous action of $\text{Gal}(\overline{K}/k)$ on it. The group scheme $\Phi_A = \mathcal{A}_k/\mathcal{A}^0_k$ is called the component group of $A$. The group of rational points $\Phi_A(k)$, called the arithmetic component group of $A$, counts the number of connected components of $\mathcal{A}_k$ which are geometrically connected and $c_A = \#\Phi_A(k)$ is called the Tamagawa number of $A/K$. Now let $K^{nr}$ be the maximal unramified extension of $K$. The inclusion $\text{Gal}(\overline{K}/K^{nr}) \subset \text{Gal}(\overline{K}/K)$ induces a map $H^1(K, A) \to H^1(K^{nr}, A)$, whose kernel corresponds to the unramified subgroup of $H^1(K, A)$. The map may also be given as $WC(A/K) \to WC(A/K^{nr})$ where, $WC(A/K) \cong H^1(K, A)$ denotes the Weil-Châtelet group of $A$ over $K$. We denote this kernel by $\text{TT}(A/K)$ and call it the group of Tamagawa torsors of $A$ over $K$ \cite{Bis13}.

**Theorem 2.1.** There exists a canonical isomorphism of finite abelian groups

$$\text{TT}(A/K) \cong H^1(k, \Phi_A)$$

**Proof.** The inflation-restriction sequence

$$0 \to H^1(K^{nr}/K, A(K^{nr})) \to H^1(K, A) \to H^1(K^{nr}, A)$$

identifies the set of Tamagawa torsors with the injective image of the group $H^1(K^{nr}/K, A(K^{nr}))$ in $H^1(K, A)$. There is an isomorphism \cite{Mil86}*§Prop I.3.8* $H^1(K^{nr}/K, A(K^{nr})) \cong H^1(k, \Phi_A)$ and the latter group is finite, since $\Phi_A$ is finite. \qed
Corollary 2.2. Suppose that $A/K$ is a Jacobian variety. Then there exists a canonical perfect pairing of finite abelian groups

$$\text{TT}(A/K) \times \Phi_A(k) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

In particular, $\text{TT}(A/K)$ has order $c_A$.

Proof. This follows from Theorem 2.1, the fact that $A$ is a self-dual abelian variety, and the perfectness of the pairing $[\text{McC}86 \ (4.5)]$ induced by Grothendieck’s pairing $[\text{McC}86 \ (2.1)]$. 

Thus, for any Jacobian variety $A/K$, there is an isomorphism

$$\Phi_A(k) \cong \text{Hom}(\text{TT}(A/K), \mathbb{Q}/\mathbb{Z})$$

of finite, abelian groups.

Remark 2.3. Since $\Phi_A$ is finite, its Herbrand quotient is 1 which implies that $\#H^1(k, \Phi_A) = \#H^0(k, \Phi_A) = c_A$. Thus, for any abelian variety $A/K$ (and not just Jacobians), it follows from Theorem 2.1 that $\#\text{TT}(A/K) = c_A$.

2.2. Picard Groups, Jacobian Varieties, and Brauer Groups.

Let $X$ be a smooth, projective, geometrically connected curve defined over any field $K$. Let $\overline{K}$ be a separable closure of $K$, and let $\overline{X} = X \otimes_K \overline{K}$. Let $\text{Div}(\overline{X})$ be the group of divisors of $\overline{X}$ i.e. the free abelian group generated by the points of $X(\overline{K})$. Note that $\text{Div}(\overline{X})^{G_K} = \text{Div}(X)$, where $G_K = \text{Gal}(\overline{K}/K)$. There is a natural summation map $\text{Div}(X_K) \rightarrow \mathbb{Z}$ whose image is $\delta \mathbb{Z}$, where $\delta$ is the index of $X$. Equivalently, $\delta$ is the least positive degree of a divisor in $\text{Div}(X_K)$.

Let $P = \text{Pic}_X$ be the Picard scheme of $X$ so that $P(\overline{K}) = \text{Pic}(\overline{X})$. It follows that $P(K) = P(\overline{X})^{G_K} = \text{Pic}(\overline{X})^{G_K}$. The Picard scheme is a smooth group scheme over $K$ whose identity component $A = \text{Pic}^0_X$ is called the Jacobian variety of $X$. There is an exact sequence of $G_K$-modules

$$0 \rightarrow A(\overline{K}) \rightarrow P(\overline{K}) \xrightarrow{\text{deg}} \mathbb{Z} \rightarrow 0$$

where $\text{deg}$ is the degree map on $\text{Pic}(\overline{X})$. Taking $G_K$-invariants of $\overline{2.2}$ we obtain the exact sequence

$$0 \rightarrow A(K) \rightarrow P(K) \xrightarrow{\delta'} \mathbb{Z} \rightarrow 0$$

where $\delta'$ is the period of $X$. Equivalently, $\delta'$ is the least positive degree of a divisor class in $P(K) = \text{Pic}(\overline{X})^{G_K}$. The image of the map $\text{Div}(X) \rightarrow P(K)$ is denoted by $\text{Pic}(X)$. Furthermore, it is known
that \( \text{Pic}(X) = H^1(X, \mathbb{G}_m) \). Let \( \text{Br}(X) = H^2(X, \mathbb{G}_m) \) be the Brauer-Grothendieck group of \( X \). By Lemma 2.2 in [Mil82], there is an exact sequence
\[
0 \to \text{Pic}(X) \to P(K) \to \text{Br}(K) \to \text{Br}(X) \to H^1(K, P) \to 0
\]
The zero on the right-hand end follows from [Mil86, Cor.I.4.21].

3. The Main Theorem

In this section, we prove

**Theorem 3.1.** Let \( X \) be a proper, smooth, geometrically connected curve over a non-archimedean local field \( K \) having finite residue field \( k \), with index \( \delta \) and period \( \delta' \). Let \( X^{nr} = X \otimes_K K^{nr} \) be the corresponding curve over \( K^{nr} \), with index \( \delta^{nr} \) and period \( \delta^{nr'} \). Let \( \text{Br}_0(X) \) denote the image of \( \text{Br}(K) \to \text{Br}(X) \), and \( \text{Br}_{nr}(X) \) denote the kernel of the map \( \text{Br}(X) \to \text{Br}(X^{nr}) \). Let \( A \) be the Jacobian variety of \( X \) over \( K \). Then there exists an exact sequence
\[
0 \to \text{Hom} \left( \text{Br}_{nr}(X)/\text{Br}_0(X), \mathbb{Q}/\mathbb{Z} \right) \to \Phi_A(k) \to \mathbb{Z}/d\mathbb{Z} \to 0
\]

where \( \Phi_A \) is the component group of \( A \), and \( d = \delta'/\delta^{nr'} \).

**Proof.** The short exact sequence
\[
0 \to A \to P \to \mathbb{Z} \to 0
\]
over \( K \) and \( K^{nr} \) gives rise, respectively, to the exact rows of the commutative diagram
\[
\begin{array}{cccccc}
P(K) & \to & \mathbb{Z} & \to & H^1(K, A) & \to & H^1(K, P) & \to & 0 \\
P(K^{nr}) & \to & \mathbb{Z} & \to & H^1(K^{nr}, A) & \to & H^1(K^{nr}, P) & \to & 0 \
\end{array}
\]
The columns in the diagram are induced by the inclusion \( K \subset K^{nr} \). The image of the map \( P(K) \to \mathbb{Z} \) is, by definition, \( \delta'\mathbb{Z} \) while that of \( P(K^{nr}) \to \mathbb{Z} \) is \( \delta^{nr}\mathbb{Z} \). Thus we have the diagram
\[
\begin{array}{cccccc}
0 & \to & \mathbb{Z}/\delta'\mathbb{Z} & \to & H^1(K, A) & \to & H^1(K, P) & \to & 0 \\
0 & \to & \mathbb{Z}/\delta^{nr}\mathbb{Z} & \to & H^1(K^{nr}, A) & \to & H^1(K^{nr}, P) & \to & 0 \
\end{array}
\]
Since \( \delta^{nr} \) divides \( \delta' \), the leftmost vertical map is surjective. The kernel of this map is \( \mathbb{Z}/d\mathbb{Z} \) where \( d = \delta'/\delta^{nr'} \). The middle vertical map has kernel \( H^1(K^{nr}/K, A(K^{nr})) \cong \text{TT}(A/K) \) by Theorem 2.1. The kernel
of the rightmost vertical map is \( H^1(K_{nr}/K, P(K_{nr})) \). Snake Lemma then gives an exact sequence

\[
0 \to \mathbb{Z}/d\mathbb{Z} \to \text{TT}(A/K) \to H^1(K_{nr}/K, P(K_{nr})) \to 0
\]

We now describe the right-most term in the exact sequence 3.1. Consider the commutative diagram

\[
\begin{array}{c}
P(K) \to \text{Br}(K) \to \text{Br}(X) \to H^1(K, P) \to 0 \\
P(K_{nr}) \to \text{Br}(K_{nr}) \to \text{Br}(X_{nr}) \to H^1(K_{nr}, P) \to 0
\end{array}
\]

where the top and bottom row are obtained by applying 2.4 to \( X \) and \( X_{nr} \) respectively. Since \( \text{Br}(K_{nr}) = 0 \) [Ser79], the above diagram reduces to

\[
\begin{array}{c}
0 \to \text{Br}_{0}(X) \to \text{Br}(X) \to H^1(K, P) \to 0 \\
0 \to \text{Br}(X_{nr}) \to H^1(K_{nr}, P) \to 0
\end{array}
\]

Snake Lemma then yields the exact sequence

\[
0 \to \text{Br}_{0}(X) \to \text{Br}_{nr}(X) \to H^1(K_{nr}/K, P(K_{nr})) \to 0
\]

which yields the isomorphism

\[\text{Br}_{nr}(X)/\text{Br}_{0}(X) \cong H^1(K_{nr}/K, P(K_{nr}))\]

The exact sequence 3.1 can now be given as

\[
(3.2) \quad 0 \to \mathbb{Z}/d\mathbb{Z} \to \text{TT}(A/K) \to \text{Br}_{nr}(X)/\text{Br}_{0}(X) \to 0
\]

Dualizing this sequence, we obtain

\[
(3.3) \quad 0 \to \left(\text{Br}_{nr}(X)/\text{Br}_{0}(X)\right)^\vee \to \text{TT}(A/K)^\vee \to \mathbb{Z}/d\mathbb{Z}^\vee \to 0
\]

Here \( M^\vee = \text{Hom}(M, \mathbb{Q}/\mathbb{Z}) \) denotes the Pontryagin dual of \( M \). By \( \text{TT}(A/K)^\vee \cong \Phi_A(k) \). On the other hand, the homomorphism \( \alpha : 1 \mapsto \left(\frac{1}{d} + \mathbb{Z}\right) \) has order \( d \) and generates \( \text{Hom}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \), so that \( \text{Hom}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z} \). Thus 3.3 can be given as

\[
0 \to \text{Hom}\left(\text{Br}_{nr}(X)/\text{Br}_{0}(X), \mathbb{Q}/\mathbb{Z}\right) \to \Phi_A(k) \to \mathbb{Z}/d\mathbb{Z} \to 0
\]

The following corollary is immediate

**Corollary 3.2.** The quotient group \( \text{Br}_{nr}(X)/\text{Br}_{0}(X) \) is finite of order \( c_A/d \).
Remark 3.3. By [CT13, Prop 2.1](which applies to any proper, smooth, geometrically integral variety and not just a curve), the quotient group \(\text{Br}_{nr}(X)/\text{Br}_0(X)\) is finite. The order, however, does not seem to be recorded in the literature.

**Corollary 3.4.** Suppose that \(\delta' = \delta_{\text{ur}}\) (this happens, for example, if \(X\) has a \(K\)-rational point). Then there exists a canonical perfect pairing of finite abelian groups

\[
\text{Br}_{nr}(X)/\text{Br}_0(X) \times \Phi_A(k) \rightarrow \mathbb{Q}/\mathbb{Z}
\]

**Proof.** \(\delta' = \delta_{\text{ur}}\) implies that \(d = 1\). The exact sequence 3.2 shows that, in this case, there exists a canonical isomorphism

\[
\text{TT}(A/K) \cong \text{Br}_{nr}(X)/\text{Br}_0(X)
\]

The pairing of the statement is then induced by the pairing of Corollary 2.2.

**Remark 3.5.** Theorem 3.1 shows that \(c_A = \#\Phi_A(k)\) can be expressed as the product of \(#(\text{Br}_{nr}(X)/\text{Br}_0(X))\) and \(\delta'/\delta_{\text{ur}}\). On the other hand, Theorem 1.17 in [BL99] expresses \(#(\text{Ker}(\beta)/\text{Im}(\alpha))\) as the product of a term \(#(\ker(\beta)/\text{Im}(\alpha))\) and \(\delta/(q\delta_{\text{ur}})\) (Note that the notations for \(\delta\) and \(\delta_{\text{ur}}\) are different in [BL99]). Both \(\text{Ker}(\beta)\) and \(\text{Im}(\alpha)\) are certain subgroups of the group of Weil divisors on \(X\) with support in the special fiber \(X_k\). Letting \(g\) be the genus of \(X\), we have that \(q = 1\) if \(\delta\) divides \(g - 1\), and \(q = 2\) otherwise. As D. Lorenzini explained to us, by [Lic69, Thm 7.1], \(\delta/q = \delta'\) so that \(\delta/(q\delta_{\text{ur}}) = \delta'/\delta_{\text{ur}}\). The commutative diagram in the proof of [Lic69, Theorem 3] then implies that since \(\text{Br}(K_{nr}) = 0\), we have \(\delta_{\text{ur}} = \delta_{\text{ur}}'\). Thus we have \(\delta'/\delta_{\text{ur}} = \delta'/\delta_{\text{ur}} = \delta/(q\delta_{\text{ur}})\). It follows that \(#(\text{Br}_{nr}(X)/\text{Br}_0(X)) = #(\text{Ker}(\beta)/\text{Im}(\alpha))\). Furthermore, comparing Cor 3.4 with Cor 1.12 in [BL99] yields the isomorphism

\[
(\text{Ker}(\beta)/\text{Im}(\alpha)) \cong \text{Hom}(\text{Br}_{nr}(X)/\text{Br}_0(X), \mathbb{Q}/\mathbb{Z})
\]

when \(d = 1\).

**Remark 3.6.** The surjective map \(\Phi_A(k) \xrightarrow{\beta} \mathbb{Z}/d\mathbb{Z}\) in Theorem 3.1 can be made explicit. Consider the commutative diagram

\[
\begin{array}{ccc}
\Phi_A(k) & \xrightarrow{\beta} & \mathbb{Z}/d\mathbb{Z} \\
\downarrow{\alpha_1} & & \downarrow{\alpha_3} \\
\text{Hom}(\text{TT}(A/K), \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\alpha_2} & \text{Hom}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Q}/\mathbb{Z})
\end{array}
\]

Here \(\alpha_1\) is an isomorphism via the canonical perfect pairing of finite abelian groups \(\langle \, , \, \rangle : \Phi_A \times \text{TT}(A/K) \rightarrow \mathbb{Q}/\mathbb{Z} [\text{McC86, (4.5)}]\) which in
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The turn is induced by Grothendieck’s pairing $\Phi_A \times \Phi_A \to \mathbb{Q}/\mathbb{Z}$ [McC86, (2.1)]. The map $\alpha_3$ is an isomorphism as explained in the proof of Theorem 3.1 above. Finally, the horizontal map $\alpha_2$ is induced by the injective map $\mathbb{Z}/d\mathbb{Z} \to \text{TT}(A/K)$ in the exact sequence 3.2. Note that $\Delta$ is induced by the connecting homomorphism of the long exact sequence induced by the exact sequence $0 \to A \to P \to \mathbb{Z} \to 0$ over $K^{nr}/K$. If $x \in \Phi_A(k)$, then the composition $(\alpha_2 \circ \alpha_1)(x)$ is the homomorphism $\sigma : 1 \mapsto \langle x, \Delta(1) \rangle$. Since $\text{Hom}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is generated by $\alpha : 1 \mapsto \frac{1}{d} + \mathbb{Z}$, $\sigma = ma$ for some $0 < m \leq d - 1$. Then $\alpha_3(\sigma) = m$, and we let $\beta(x) = m$. On the other hand, the injective map $\text{Hom} (\text{Br}_{nr}(X)/\text{Br}_0(X), \mathbb{Q}/\mathbb{Z}) \to \Phi_A(k)$ in Theorem 3.1 is induced by $\text{TT}(A/K) \to H^1(K^{nr}/K, P(K^{nr})) \cong \text{Br}_{nr}(X)/\text{Br}_0(X)$. As shown in the proof of Theorem 3.1 above, the map $\text{TT}(A/K) \to H^1(K^{nr}/K, P(K^{nr}))$ is the restriction of the map $H^1(K, A) \to H^1(K, P)$ to $\text{TT}(A/K)$, and the latter map is induced by the surjective map $A \to P$. Finally, the isomorphism $H^1(K^{nr}/K, P(K^{nr})) \cong \text{Br}_{nr}(X)/\text{Br}_0(X)$ is induced by the map $\text{Br}(X) \xrightarrow{\psi} H^1(K, P)$ as in 2.4. The map $\psi$ is described explicitly in [Mil82, Rem. 2.3].
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