WHEN IS A SPACE Menger AT INFINITY?

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Abstract. We try to characterize those Tychonoff spaces $X$ such that $\beta X \setminus X$ has the Menger property.

1. Introduction

A space $X$ is Menger (or has the Menger property) if for any sequence of open coverings $\{U_n : n < \omega\}$ one may pick finite sets $V_n \subseteq U_n$ in such a way that $\bigcup \{V_n : n < \omega\}$ is a covering. This equivals to say that $X$ satisfies the selection principle $S_{\text{fin}}(\mathcal{O}, \mathcal{O})$. It is easy to see the following chain of implications:

$$\sigma\text{-compact} \rightarrow \text{Menger} \rightarrow \text{Lindelöf}$$

An important result of Hurewicz [4] states that a space $X$ is Menger if and only if player 1 does not have a winning strategy in the associated game $G_{\text{fin}}(\mathcal{O}, \mathcal{O})$ played on $X$. This highlights the game-theoretic nature of the Menger property, see [6] for more.

Henriksen and Isbell [3] proposed the following:

Definition 1.1. A Tychonoff space $X$ is Lindelöf at infinity if $\beta X \setminus X$ is Lindelöf.

They discovered a very elegant duality in the following:

Proposition 1.2. [3] A Tychonoff space is Lindelöf at infinity if and only if it is of countable type.

A space $X$ is of countable type provided that every compact set can be included in a compact set of countable character in $X$.

A much easier and well-known fact is:

Proposition 1.3. A Tychonoff space is Čech-complete if and only if it is $\sigma$-compact at infinity.

These two propositions suggest the following:

Question 1.4. When is a Tychonoff space Menger at infinity?

Before beginning our discussion here, it is useful to note these well known facts:

Proposition 1.5. The Menger property is invariant by perfect maps.

Corollary 1.6. $X$ is Menger at infinity if, and only if, for any $Y$ compactification of $X$, $Y \setminus X$ is Menger.

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Fremlin and Miller \cite{Fremlin1984} proved the existence of a Menger subspace $X$ of the unit interval $[0,1]$ which is not $\sigma$-compact. The space $X$ can be taken nowhere locally compact and so $Y = [0,1] \setminus X$ is dense in $[0,1]$. Since the Menger property is invariant under perfect mappings, we see that $\beta Y \setminus Y$ is still Menger. Therefore, a space can be Menger at infinity and not $\sigma$-compact at infinity. Another example of this kind, stronger but not second countable, is Example 3.1 in the last section.

On the other hand, the irrational line shows that a space can be Lindelöf at infinity and not Menger at infinity.

Consequently, the property $\mathcal{M}$ characterizing a space to be Menger at infinity strictly lies between countable type and Čech-complete.

Of course, taking into account the formal definition of the Menger property, we cannot expect to have an answer to Question 1.4 as elegant as Henriksen-Isbell’s result.

2. A characterization

\textbf{Definition 2.1.} Let $K \subset X$. We say that a family $\mathcal{F}$ is a \textbf{closed net at $K$} if each $F \in \mathcal{F}$ is a closed set such that $K \subset F$ and for every open $A$ such that $K \subset A$, there is an $F \in \mathcal{F}$ such that $F \subset A$.

\textbf{Lemma 2.2.} Let $X$ be a $T_1$ space. If $(F_n)_{n \in \omega}$ is a closed net at $K$, for $K \subset X$ compact, then $K = \bigcap_{n \in \omega} F_n$.

\textbf{Proof.} Simply note that for each $x \notin K$, there is an open set $V$ such that $K \subset V$ and $x \notin V$. \hfill \Box

\textbf{Lemma 2.3.} Let $Y$ be a regular space and let $X$ be a dense subspace of $Y$. Let $K \subset X$ be a compact subset. If $(F_n)_{n \in \omega}$ is a closed net at $K$ in $X$, then $(\overline{F_n}^Y)_{n \in \omega}$ is a closed net at $K$ in $Y$.

\textbf{Proof.} In the following, all the closures are taken in $Y$. Let $A$ be an open set in $Y$ such that $K \subset A$. By the compactness of $Y$ and the regularity of $Y$, there is an open set $B$ such that $K \subset B \subset \overline{B} \subset A$. Thus, there is an $n \in \omega$ such that $K \subset F_n \subset B \cap X$. Note that $K \subset F_n \subset \overline{B} \subset A$. \hfill \Box

\textbf{Lemma 2.4.} Let $X$ be a compact Hausdorff space. If $K = \bigcap_{n \in \omega} F_n$, where $(F_n)_{n \in \omega}$ is a decreasing sequence of closed sets, then $(F_n)_{n \in \omega}$ is a closed net at $K$.

\textbf{Proof.} If not, then there is an open set $V$ such that $K \subset V$ and, for every $n \in \omega$, $F_n \setminus V \neq \emptyset$. By compactness, there is an $x \in \bigcap_{n \in \omega} F_n \setminus V = K \setminus V$. Contradiction with the fact that $K \subset V$. \hfill \Box

\textbf{Theorem 2.5.} Let $X$ be a Tychonoff space. $X$ is Menger at infinity if, and only if, $X$ is of countable type and for every sequence $(K_n)_{n \in \omega}$ of compact subsets of $X$, if $(F_p^n)_{p \in \omega}$ is a decreasing closed net at $K_n$ for each $n$, then there is an $f : \omega \longrightarrow \omega$ such that $K = \bigcap_{n \in \omega} F_{f(n)}^n$ is compact and $(\bigcap_{k \leq n} F_k^f)_{n \in \omega}$ is a closed net for $K$.

\textbf{Proof.} In the following, every closure is taken in $\beta X$.

Suppose that $X$ is Menger at infinity. By \cite{Aurichi2020} $X$ is of countable type. Let $(F_p^n)_{p, n \in \omega}$ be as in the statement. Note that, by Lemma 2.3 and Lemma 2.2, $\bigcap_{p \in \omega} F_p^n = \bigcap_{p \in \omega} \overline{F_p^n}$ for each $n \in \omega$. Thus, for each $n \in \omega$, $(V_p^n)_{p \in \omega}$, where $V_p^n = \beta X \setminus \overline{F_p^n}$, is an increasing covering for $\beta X \setminus X$. Since $\beta X \setminus X$ is Menger, there is an $f : \omega \longrightarrow \omega$ such that $\beta X \setminus X \subset \bigcup_{n \in \omega} V_{f(n)}^n$. Note that $K = \bigcap_{n \in \omega} \overline{F_{f(n)}^n}$.
is compact and it is a subset of $X$. By Lemma 2.3, $(\bigcap_{k \leq n} F^k_{f(k)})_{n \in \omega}$ is a closed net at $K$ in $\beta X$, therefore, $(\bigcap_{k \leq n} F^k_{f(k)})_{n \in \omega}$ is a closed net at $K$ in $X$. Conversely, for each $n \in \omega$, let $W_n$ be an open covering for $\beta X \setminus X$. We may suppose that each $W \in W_n$ is open in $\beta X$. By regularity, we can take a refinement $V_n$ of $W_n$ such that, for every $x \in \beta X \setminus X$, there is a $V \in V_n$ such that $x \in V \subset \overline{V} \subset W_V$ for some $W_V \in W_n$. Since $X$ is of countable type, By 2.2 we may suppose that each $V_n$ is countable. Fix an enumeration for each $V_n = (V^n_n)_{k \in \omega}$. Define $A_k^F = \beta X \setminus (\bigcup_{j \leq k} \overline{V}_j^n)$). Note that each $K_n = \bigcap_{k \in \omega} A_k^F$ is compact and a subset of $X$. By Lemma 2.3, $(A_k^F)_{k \in \omega}$ is a closed net at $K_n$. Thus, $(\overline{A_k^F} \cap X)_{k \in \omega}$ is a closed net at $K_n$ in $X$. Therefore, there is $f : \omega \to \omega$ such that $K = \bigcap_{n \in \omega} \overline{A_{f(n)}^F} \cap X$ is compact and $(\bigcap_{k \leq f(n)} \overline{A_k^F} \cap X)_{n \in \omega}$ is a closed net at $K$. So, by Lemma 2.3, $K = \bigcap_{n \in \omega} \overline{A_{f(n)}^F} \cap X$. Since $\bigcap_{n \in \omega} \overline{A_{f(n)}^F} \cap X = \bigcap_{n \in \omega} \overline{A_{f(n)}^F}$ and by the fact that $K \subset X$, it follows that $\beta X \setminus X \subset \bigcup_{n \in \omega} \beta X \setminus \overline{A_{f(n)}^F} \cap X \subset \bigcup_{n \in \omega} \text{Int}(\bigcup_{j \leq f(n)} \overline{V}_j^n) \cap \bigcup_{n \in \omega} \bigcup_{j \leq f(n)} W_{V_j}$. Therefore, letting $U_n = \{W_{V_j} : j \leq f(n)\} \subset W_n$, we see that the collection $\bigcup_{n \in \omega} U_n$ covers $\beta X \setminus X$, and we are done.

Property $M$ given in the above theorem does not look very nice and we wonder whether there is a simpler way to describe it, at least in some special cases.

Recall that a metrizable space is always of countable type. Moreover, a metrizable space is complete if and only if it is $\sigma$-compact at infinity. Therefore, we could hope for a “nicer” $M$ in this case.

**Question 2.6.** What kind of weak completeness characterizes those metrizable spaces which are Menger at infinity?

**Proposition 2.7.** Let $X$ be a Tychonoff space. If $X$ is Menger at infinity then for every sequence $(K_n)_{n \in \omega}$ of compact sets, there is a sequence $(Q_n)_{n \in \omega}$ of compact sets such that:

1. each $K_n \subset Q_n$;
2. each $Q_n$ has a countable base at $X$;
3. for every sequence $(B^n_k)_{k \in \omega}$ such that, for every $n \in \omega$, $(B^n_k)_{k \in \omega}$ is a decreasing base at $K_n$ then there is a function $f : \omega \to \omega$ such that $K = \bigcap_{n \in \omega} \overline{B_{f(n)}^n}$ is compact and $(\bigcap_{k \leq n} \overline{B_k^F})_{n \in \omega}$ is a closed net at $K$.

**Proof.** Suppose $X$ is Menger at infinity. Let $(K_n)_{n \in \omega}$ be a sequence of compact sets. Since $X$ is Menger at infinity, $X$ is Lindelöf at infinity. Thus, by Proposition 1.2, for each $K_n$, there is a compact $Q_n \supset K_n$ such that $Q_n$ has a countable base. Now, let $(B^n_k)_{k \in \omega}$ be as in 3. Since each $Q_n$ is compact and $X$ is regular, each $(B^n_k)_{k \in \omega}$ is a decreasing closed net at $Q_n$. Thus, by Proposition 2.5 there is an $f : \omega \to \omega$ as we need.

We end this section presenting a selection principle that at first glance could be related with the Menger at infinity property.

**Definition 2.8.** We say that a family $\mathcal{U}$ of open sets of $X$ is an almost covering for $X$ if $X \setminus \bigcup \mathcal{U}$ is compact. We call $\mathcal{A}$ the family of all almost coverings for $X$.

Note that the property “being Menger at infinity” looks like something as $S_{\text{fin}}(\mathcal{A}, \mathcal{A})$, but for a narrow class of $\mathcal{A}$. We will see that the “narrow” part is important.
Proposition 2.9. If $X$ satisfies $S_{\aleph_0}(\mathcal{A}, \mathcal{A})$, then $X$ is Menger.

Proof. Let $(U_n)_{n \in \omega}$ be a sequence of coverings of $X$. By definition, for each $n \in \omega$, there is a finite $U_n \subset U_n$ such that $K = X \setminus \bigcup_{n \in \omega} U_n$ is compact. Therefore, there is a finite $W \subset U_n$ such that $K \subset W$. Thus, $X = W \cup \bigcup_{n \in \omega} U_n$. \hfill \Box

Example 2.10. The space of the irrationals is an example of a space that is Menger at infinity but does not satisfy $S_{\aleph_0}(\mathcal{A}, \mathcal{A})$ (by the Proposition 2.7).

Example 2.11. The one-point Lindelöfication of a discrete space of cardinality $\aleph_1$ is an example of a Menger space which does not satisfy $S_{\aleph_0}(\mathcal{A}, \mathcal{A})$.

Example 2.12. $\omega$ is an example of a space that satisfies $S_{\aleph_0}(\mathcal{A}, \mathcal{A})$, but it is not compact.

Proof. Let $(V_n)_{n \in \omega}$ be a sequence of almost coverings for $\omega$. Therefore, for each $n$, $F_n = \omega \setminus \bigcup V_n$ is finite. For each $n$, let $V_n \subset V_n$ be a finite subset such that $F_{n+1} \setminus F_n \subset \bigcup V_n$ and $\min(\omega \setminus \bigcup_{k<n} V_k) \in V_n$. Note that $\omega \setminus \bigcup_{n \in \omega} V_n = F_0$. \hfill \Box

3. More than Menger at infinity

One may wonder whether the hypothesis “player 2 has a winning strategy in the Menger game $G_{\aleph_0}(\mathcal{O}, \mathcal{O})$ played on $\beta X \setminus X$” is strong enough to guarantee that $X$ is Čech-complete. It turns out this is not the case, as the following example shows.

Example 3.1. Take the usual space of rational numbers $\mathbb{Q}$ and an uncountable discrete space $D$. Let $Y = \mathbb{Q} \times D \cup \{p\}$ be the one-point Lindelöfication of the space $\mathbb{Q} \times D$ and then let $X = \beta Y \setminus Y$. Since $Y$ is nowhere locally compact, we have $Y = \beta X \setminus X$. $X$ is not Čech-complete, since $Y$ is not $\sigma$-compact, but player 2 has a winning strategy in $G_{\aleph_0}(\mathcal{O}, \mathcal{O})$ played on $\beta X \setminus X$. The latter assertion easily follows by observing that any open set containing $p$ leaves out countably many points.

Therefore, to ensure the Čech-completeness of $X$, we need to assume something more on the space (see for instance Corollary 3.3 below). Moreover, the first example presented in the introduction shows that a metrizable space (actually a subspace of the real line) can be Menger at infinity, but not favorable for player 2 in the Menger game at infinity (see again Corollary 3.3).

Recall that a space $X$ is sieve complete $\mathcal{S}$ if there is an indexed collection of open coverings $(\{U_i : i \in I_n\} : n < \omega)$ together with maps $\pi_n : I_{n+1} \to I_n$ such that $U_i = X$ for each $i \in I_0$ and $U_i = \bigcup \{U_j : j \in \pi_n^{-1}(i)\}$ for all $i \in I_n$. Moreover, we require that for any sequence of indexes $\langle i_n : n < \omega \rangle$ satisfying $\pi_n(i_{n+1}) = i_n$ if $\mathcal{F}$ is a filterbase in $X$ and $U_{i_n}$ contains an element of $\mathcal{F}$ for each $n < \omega$, then $\mathcal{F}$ has a cluster point.

Every Čech-complete space is sieve complete and every sieve complete space contains a dense Čech-complete subspace. In addition, a paracompact sieve complete space is Čech-complete and a sieve complete space is of countable type $\mathcal{S}$.

Telgársky presented a characterization of sieve completeness in terms of the Menger game played on $\beta X \setminus X$ (note that in $\mathcal{H}(X)$ the Menger game is called the Hurewicz game and is denoted by $H(X)$):

Theorem 3.2 (Telgársky $\mathcal{H}$). Let $X$ be a Tychonoff space. $\beta X \setminus X$ is favorable for player 2 in the Menger game if and only if $X$ is sieve complete.

Since a paracompact sieve-complete space is Čech-complete, we immediately get:
Corollary 3.3. Let $X$ be a paracompact Tychonoff space. $X$ is Čech-complete if and only if player 2 has a winning strategy in the game $G_{\text{fin}}(\mathcal{O}, \mathcal{O})$ played on $\beta X \setminus X$.

In particular:

Corollary 3.4. A metrizable space $X$ is complete if and only if player 2 has a winning strategy in $G_{\text{fin}}(\mathcal{O}, \mathcal{O})$ played on $\beta X \setminus X$.

Corollary 3.5. A topological group $G$ is Čech-complete if and only if player 2 has a winning strategy in $G_{\text{fin}}(\mathcal{O}, \mathcal{O})$ played on $\beta G \setminus G$.

Proof. Every topological group of countable type is paracompact. \hfill \Box

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