A Flat Wall Theorem for Matching Minors in Bipartite Graphs

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ABSTRACT
In 1913, Pólya asked for which (0,1)-matrices $A$ it is possible to create a new matrix $A'$ by changing some of the signs such that the permanent of $A$ equals the determinant of $A'$. A combinatorial solution to this problem was found by Little in 1975; he found these matrices to be exactly the biadjacency matrices of bipartite graphs excluding $K_{3,3}$ as a matching minor. Utilising ideas from graph minors theory, this characterisation was later shown to yield a polynomial time algorithm to compute the permanent of matrices which satisfy Little’s condition. By a seminal result of Valiant, computing the permanent of (0,1)-matrices in general is #P-hard; however, it can be observed that the tractability of the permanent is closely related to the exclusion of matchings minors in bipartite graphs.

Building on the results of Robertson’s and Seymour’s graph minors theory it was shown that the permanent remains tractable under the exclusion of a planar or a single-crossing matching minor. In this paper, we provide an essential next step in the form of a matching theoretic analogue of the Flat Wall Theorem for bipartite graphs, describing the local structure of bipartite graphs excluding $K_{3,3}$ as a matching minor. Our result builds on a tight relationship between structural digraph theory and matching theory and allows us to deduce a Flat Wall Theorem for digraphs which substantially differs from a previously established directed variant of this theorem.

CCS CONCEPTS
- Mathematics of computing → Graph algorithms; Matchings and factors; Combinatorial problems; Paths and connectivity problems; Graphs and surfaces.

KEYWORDS
Permanent, Perfect Matching, Matching Minor, Flat Wall, Pfaffian Graphs

ACM Reference Format:
Archontia C. Giannopoulou and Sebastian Wiederrecht. 2024. A Flat Wall Theorem for Matching Minors in Bipartite Graphs. In Proceedings of the 56th Annual ACM Symposium on Theory of Computing (STOC ’24), June 24–28, 2024, Vancouver, BC, Canada. ACM, New York, NY, USA, 12 pages. https://doi.org/10.1145/3618260.3649774

1 INTRODUCTION
We consider the structure of bipartite graphs with perfect matchings that exclude a fixed bipartite graph $H$ as a so called “matching minor”. The notion of matching minors originates from the work on Pólya’s Permanent Problem (PPP) by Little [17]. Pólya had the idea to change the signs of some entries of a given (0,1)-matrix $A$ such that the determinant of the resulting matrix would equal the permanent of $A$. Little’s result revealed that the applicability of Pólya’s sign flips to a (0,1)-matrix is closed under certain reduction operations. In particular, every (0,1)-matrix can be interpreted as the biadjacency matrix of a bipartite graph and Little’s operations gave rise to graph theoretic operations that behave analogously to the well known minor operations when applied to graphs with perfect matchings. The biadjacency matrix of a bipartite graph $B = (V_1 \cup V_2, E)$ where $V_1 = \{u_1, \ldots, u_n\}$, $V_2 = \{v_1, \ldots, v_m\}$, and $E \subseteq \{(u, v) \mid u \in V_1, v \in V_2\}$, is the binary $n \times m$ matrix $A = (a_{i,j})$ where $a_{i,j} = 1$ if and only if $(u_i, v_j) \in E(B)$. Little’s result reads as follows: Pólya’s sign flips are applicable to the biadjacency matrix of a bipartite graph $B$ if and only if $B$ does not contain $K_{3,3}$ as a matching minor. By using concepts and techniques reminiscent of the ideas deployed in graph minors theory, McCuaig [23] and, independently, Robertson, Seymour, and Thomas [32], gave a structural characterisation that turned Little’s result into an algorithm that recognizes those matrices for which Pólya’s approach to compute the permanent is applicable and computes their permanent in polynomial time. This result is particularly remarkable as computing the permanent of (0,1)-matrices in general was the first problem shown to be #P-hard [40].

Recent results in matching theory have shown that the exclusion of a planar matching minor or a single-crossing matching minor also yield classes of bipartite graphs for which the permanent of their biadjacency matrices is computable in polynomial time (shown in [8] and [9] respectively). Furthermore, it was shown that the computation of the permanent of a matrix $A$ remains a #P-hard problem on classes of bipartite graphs that are closed under matching minors and contain some shallow vortex [9]. It is worth taking a detour here to note the following. Recently, Curticapean and Xia [3] showed that computing the permanent of symmetric adjacency matrices of $K_4$-minor-free graphs remains #P-hard. In particular, they showed an even stronger result by tying this hardness result to the occurrence of vortices in the celebrated Graph Minors Structure Theorem (GMST) by Robertson and Seymour [31]. This computational lower bound was matched in [36] where a precise description of those $H$ whose minor exclusion allows to avoid the occurrence of vortices in the GMST was given. With this result, a full dichotomy
for computing the permanent of symmetric adjacency matrices was established.

Returning to (general) biadjacency matrices of bipartite graphs, the hardness result in [9] can be considered to be a direct analogue of [3]. Since all currently known algorithmic and intractability results on the permanent of general matrices are based on structural descriptions similar to those appearing in the graph minors series, a fully developed theory for matching minors in bipartite graphs seems like a promising direction towards tackling questions regarding the tractability of the permanent and, in particular, towards hopefully obtaining a similar dichotomy.

In this paper, we provide a first major structural result for bipartite graphs excluding an arbitrary non-planar matching minor. Our result is a matching theoretic analogue of Robertson’s and Seymour’s celebrated Flat Wall Theorem [30] and as such we expect it to mark an important milestone towards a full structure theorem for bipartite graphs excluding a fixed matching minor. Before we discuss in detail the connection of the theory of matching minors in bipartite graphs to the problem of computing the permanent of (0, 1)-matrices let us first place our result within the context of the graph minors theory and present its statement.

1.1 Towards a Theory for Matching Minors in Bipartite Graphs

The general structure theorem for graphs excluding a fixed graph \( H \) as a minor [31] which sits at the heart of the graph minors series by Robertson and Seymour says, in a simplified way, that every graph that excludes \( H \) as a minor can be obtained from a surprisingly small list of ingredients:

(i) Graphs of bounded size, (ii) clique sums, (iii) graphs on surfaces, (iv) apex vertices, and (v) vortices.

The proof of this general structure theorem can be summarised into the following three steps:

(a) Robertson and Seymour provide a structure theorem for \( H \)-minor free graphs where \( H \) is planar, the Grid Theorem [28].

(b) Next, they provide a local structure theorem for \( H \)-minor free graphs where \( H \) is non-planar, the Flat Wall Theorem [30].

(c) Finally, they provide a Global Structure Theorem for \( H \)-minor free graphs where \( H \) is non-planar using all five ingredients from the list above [31].

Each of these three steps gave rise to its own consequences and applications, both algorithmic and structural. Replicating parts of these structural descriptions of proper minor closed families of graphs for other partial orderings of graphs, for example for the notion of butterfly minors in digraphs, has since become a fruitful quest which has given rise to many new discoveries and tools for the structural study of graphs.

In terms of matching minors, Norin proposed a variant of the parameter treewidth, called perfect matching width, designed for the study of graphs with perfect matchings and conjectured the existence of a grid theorem based on matching minors [25, 37]. For non-bipartite graphs, perfect matching width with a grid as a potential dual object could give rise to a polynomial time recognition algorithm for non-bipartite Pfaffian graphs. While the non-bipartite case of Norin’s conjecture remains open, the bipartite case was positively settled in [11] and has since led to additional insight into structural aspects of the study of butterfly minors in digraphs [8].

Our Contribution. This paper aims to mark a milestone in the search for a fully developed structure theory for matching minors in bipartite graphs. The first step towards the global structure theorem for \( H \)-minor free graphs has been replicated in the setting of matching minors in bipartite graphs in [8], where it was shown that a class of bipartite graphs has bounded perfect matching width if and only if it excludes some bipartite and planar graph with a perfect matching as a matching minor, essentially proving the analogue of (a). The next step according to the roadmap above would be to describe the local structure of areas of large perfect matching width in \( H \)-matching minor free bipartite graphs. In this paper we establish a matching theoretic version of the Flat Wall Theorem for bipartite graphs, settling (b). A cornerstone of our result is the recent discovery of a matching theoretic version of the Two Paths Theorem [10].

**Theorem 1.1.** Let \( r, t \in \mathbb{N} \) be positive integers. There exist functions \( \alpha: \mathbb{N} \to \mathbb{N} \) and \( \rho: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that for every bipartite graph \( B \) with a perfect matching \( M \) the following is true: If \( W \) is an \( M \)-conformal matching \( \rho(t, r) \)-wall in \( B \) such that \( M \cap E(W) \) is the canonical matching of \( W \), then either

1. \( B \) has a \( K_{t+1} \)-matching minor grasped by \( W \), or
2. there exist an \( M \)-conformal set \( A \subseteq V(B) \) with \( |A| \leq \alpha(t) \) and an \( M \)-conformal matching \( r \)-wall \( W' \subseteq W - A \) such that \( W' \) is \( \text{Per}(W') \)-flat with respect to \( A \).

An illustration of a cylindrical wall with its canonical matching can be found in Figure 4. Our matching theoretic version of flatness is a close analogue of the original definition from the graph minors project. That is, a wall \( W' \) is \( \text{Per}(W') \)-flat if the subgraph attached to it can be reduced to a planar graph by repeated applications of small local reductions. This attribute is of particular importance as it provides a very well behaved structure. We postpone the formal definitions to later points in this paper. In particular, matching minors will be introduced in the next subsection, while the definition of \( \text{Per}(W') \)-flatness can be found in Section 5. All results in this paper are constructive in the sense that they lead to efficient algorithms for the computation of the respective flat walls and their apex sets. Moreover, our main result has immediate applications to structural (di)graph theory which we highlight in Section 1.3.

1.2 The Permanent and Matching Minors

Permanents. We begin with a brief discussion on the permanents of (0, 1)-matrices and operations on those matrices fit for the study of structural properties of the permanent.

Given an \( n \times n \) matrix \( A = (a_{i,j}) \) the permanent and the determinant of \( A \) are defined as

\[
\text{perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)} \quad \text{and} \\
\text{det}(A) = \sum_{\sigma \in S_n} (\text{sgn}(\sigma) \cdot \prod_{i=1}^{n} a_{i,\sigma(i)})
\]
respectively, where $S_n$ is the set of all possible permutations of the set $[n] = \{1, \ldots, n\}$ and $\text{sgn}(\sigma)$ is the sign of the permutation $\sigma \in S_n$. We write $\text{PERMANENT}$ for the problem of computing the permanent of a matrix.

**Proposition 1.2 (Valiant, 1979 [40]).** The problem $\text{PERMANENT}$ is #P-hard, even when restricted to $(0, 1)$-matrices.

Let $A = (a_{i,j}) \in \{0, 1\}^{n \times n}$ be a matrix for some positive $n$, let $\sigma \in S_n$ and let $A_{\sigma} := \prod_{i \in [n]} a_{i,\sigma(i)}$. Now consider the following two operations on a matrix $A = (a_{i,j}) \in \{0, 1\}^{n \times n}$ with $\text{perm}(A) \neq 0$. A cross deletion is the simultaneous removal of some row and some column of $A$, whose shared entry is equal to one, while an entry deletion is the operation that replaces some non-zero entry of $A$ with a zero. $A (0, 1)$-matrix $A'$ is said to be a conformal submatrix of $A$ if there exists a sequence $A_1, \ldots, A_k$ of $(0, 1)$-matrices with $A_1 = A$, $A_k = A'$, $\text{perm}(A_i) \neq 0$ for all $i \in [n]$, and for every $i \in [n-1]$, the matrix $A_{i+1}$ is obtained from $A_i$ by a cross deletion or an entry deletion. A class $\mathcal{A}$ of $(0, 1)$-matrices is said to be hereditary if for every $A \in \mathcal{A}$, $A$ contains all conformal submatrices of $A$.

A cross deletion of the row $i$ and column $j$ can be seen as the restriction of the possible choices for $\sigma \in S_n$ where $\sigma(i) = j$, while the entry deletion of the entry $a_{i,j}$ is the exclusion of all $\sigma \in S_n$ with $\sigma(i) = j$ in the computation of the permanent. By the requirement that performing either operation does not drop the permanent to zero, both of these operation restrict the formula for $\text{perm}(A)$ to a proper, but non-trivial, subsum of the same expression.

Now suppose our matrix $A$ has a row, say row $i$, (or column) with exactly two non-zero entries, say $a_{i,j}$ and $a_{i,k}$, $i < k$. Since the permanent is invariant under the permutation of rows and columns, we may assume $i = i_2 = n$ and $i_1 = n - 1$. Moreover, the permanent is invariant under transposition and thus w.l.o.g. we may assume we are working with the row $i$. Then we can partition the set $S \subseteq S_n$ of permutations $\sigma$ with $A_{\sigma} \neq 0$ into the sets $S_1, S_2,$ and $S_3$ such that: For $j \in [2]$, $S_j$ contains exactly those $\sigma$ with $\sigma(n) = i_j$ and the entry at $i_j$ of the row $\sigma^{-1}(i_{j+1})$ is $0$, and $S_3$ contains all $\sigma$ for which the entry at $i_j = \sigma(n)$ of the row $\sigma^{-1}(i_{j+1})$ is equal to $1$. Moreover, let $S' \subseteq S_{n-1}$ be the collection of all $\sigma' \in S_{n-1}$ such that there exist $\sigma \in S$ and $j \in [n-1]$ with $\sigma'(k) = \sigma(k)$ for all $k \in [n-1] \setminus \{j\}$, $\sigma(j) \in [n-1]$, and $\sigma'(j) = n - 1$. It follows that $|S'| = |S_1| + |S_2| + \frac{1}{2}|S_3|$. Let us define a new column by setting $a'_{j,i} := \max\{a_{j,i}, a_{i,j}\}$ for every $j \in [n]$. Now let $A'$ be the matrix obtained from $A$ by replacing the column $i$ with $(a'_{j,i})_{j \in [n]}$ and then cross deleting row $i$ and column $i_2$. We call the above operation a bicontraction of the matrix $A$ at the row $i$. In case one of the two columns $i_1$ and $i_2$ also has exactly two non-zero entries, we call the corresponding bicontraction elementary. Notice that in this case we have $S_3 = \emptyset$ and thus an elementary bicontraction does not change the permanent. Hence allowing for elementary bicontractions, or even bicontractions in general, appear to be a reasonable addition to taking conformal submatrices.

Perfect Matchings. Any $(0, 1)$-matrix $A = (a_{i,j})$ can be seen as the biadjacency matrix of a bipartite graph $B(A)$. Valiant proved Proposition 1.2 by using the simple observation that the permanent of a $(0, 1)$-matrix corresponds exactly to the number of perfect matchings in $B(A)$. Building on the previous discussion we now introduce the concept of matchings minors as the graph theoretic analogue of the matrix operations introduced above.

Let $G$ be a graph, a set $F \subseteq E(G)$ of pairwise disjoint edges is called a matching, and a vertex $v \in V(G)$ is said to be covered by $F$ if $F$ contains an edge that has $v$ as an endpoint. The set of all vertices covered by $F$ is denoted by $V(F)$. A matching $M$ is perfect if $V(M) = V(G)$. A set $X \subseteq V(G)$ is conformal if $G - X$ has a perfect matching and a subgraph $H$ of $G$ is conformal if $V(H)$ is conformal.

Consider the term $A_{\sigma}$ and observe that $A_{\sigma} = 1$ if and only if for all $i \in [n]$, we have $a_{i,\sigma(i)} = 1$. Since each $a_{i,j}$ corresponds to an edge of $B(A)$, we may associate with $\sigma$ the set

$$F_{\sigma} := \{\{u, v_{\sigma(i)}\} \mid i \in [n] \text{ and } a_{i,\sigma(i)} = 1\}.$$ 

It follows that $A_{\sigma} = 1$ if and only if $F_{\sigma}$ is a perfect matching of $B(A)$. Hence, $\text{perm}(A)$ equals the total number of perfect matchings in $B(A)$. Since determining whether a bipartite graph has a perfect matching is polynomial time solvable, deciding whether $\text{perm}(A) = 0$ is as well. Moreover, it follows that a bipartite graph $H$ with perfect matching and biadjacency matrix $A_H$ is a conformal subgraph of a bipartite graph $B$ with a perfect matching and biadjacency matrix $A_B$ if and only if $A_H$ is a conformal submatrix of $A_B$.

Notice, that a row (or column) of $A_H$ with exactly two non-zero entries corresponds to a vertex $v$ of $H$ with exactly two neighbours. Hence $v$ is incident with exactly two edges $e_1$ and $e_2$ in $H$. Bicontracting this row (column) in $A_H$ corresponds to the operation of contracting the edges $e_1$ and $e_2$ simultaneously and then removing all resulting loops and parallel edges. This naturally leads to the definition of matching minors.

**Definition 1.3 (Matching minor).** Let $G$ be a graph with a perfect matching. If $v \in V(G)$ is a vertex of degree exactly two, we call the process of contracting both edges incident with $v$ and removing all loops and parallel edges afterwards the bicontraction of $v$. A graph $H$ that can be obtained from a conformal subgraph of $G$ by repeatedly applying bicontractions is called a matching minor.

In light of our discussion on conformal submatrices and (elementary) bicontractions in $(0, 1)$-matrices it is to be expected that the theory of (topological) matching minors in bipartite graphs plays a key role in gaining further insight into the tractability horizon of the permanent of $(0, 1)$-matrices.

**1.3 Consequences for Structural Digraph Theory and Applications**

As a consequence of the flat wall theorem, we also provide a new directed version of the Flat Wall Theorem which is uncomparable with the directed flat wall theorem of Giannopoulou et al. [6] and we use these insights to obtain a new duality theorem for (undirected) treewidth based on matching minors. Such a result is made possible by a well known transformation of bipartite graphs with fixed perfect matchings to digraphs. Let $B = (V_1, V_2, E)$ be a bipartite graph with a perfect matching $M$. The digraph $D(B, M)$ obtained from $B$ by orienting all edges from $V_1$ to $V_2$ and then contracting all edges of $M$ is called the $M$-direction of $B$. See Figure 1 for an illustration.
Figure 1: Left: A bipartite graph $B$ with a perfect matching $M$. Right: The arising $M$-direction $D(B, M)$.

The $M$-direction of $B$ translates many matching theoretic properties of $B$ into digraphic analogues. Of particular importance for our results is the following relation between butterfly minors and matching minors. A butterfly minor of a digraph $D$ is a digraph $H$ which can be obtained from a subgraph of $D$ by “butterfly contractions”. We say that a directed edge is butterfly contractible if it is the only incoming edge of its head or the only outgoing edge of its tail.

**Lemma 1.4 ([22]).** Let $B$ and $H$ be bipartite matching covered graphs. Then $H$ is a matching minor of $B$ if and only if there exist perfect matchings $M \in M(B)$ and $M' \in M(H)$ such that $D(H, M')$ is a butterfly minor of $D(B, M)$.

Lemma 1.4 allows for a precise description of certain infinite anti-chains of butterfly minors. The groundwork for this theory has been established in [8]. Recall the definition of $M$-directions of bipartite graphs and notice that the operation can be reversed to yield a unique bipartite graph with a perfect matching when given a digraph $D$ as input. Let us formalise this inverse operation.

**Definition 1.5 (Split).** Let $D$ be a directed graph. We define $S(D)$ to be the bipartite graph $B$ for which a perfect matching $M$ exists such that $D(B, M) = D$.

The digraph $J$ is a proper butterfly minor of the digraph $D$ if $J$ is a butterfly minor of $D$ and $J \neq D$. We say that $D$ is $J$-minimal if $S(D)$ contains $S(J)$ as a matching minor, but for every proper butterfly minor $D'$ of $D$, $S(D')$ is $S(J)$-free.

**Definition 1.6 (Fundamental Anti-Chain).** Let $D$ be a digraph. The family $\mathcal{A}(D) := \{D' \mid D' \text{ is a $D$-minimal digraph}\}$ is called the fundamental anti-chain based on $D$.

**Lemma 1.7 ([8]).** Let $H$ and $D$ be digraphs. Then $D$ contains a butterfly minor from $\mathcal{A}(H)$ if and only if $S(D)$ contains $S(H)$ as a matching minor.

It follows that a theory of matching minors in bipartite graphs also provides a theory for the handling of (possible infinite) anti-chains of butterfly minors in digraphs. The following digraphic version of Theorem 1.1 is a manifestation of this observation.

**Theorem 1.8.** Let $r, t \in \mathbb{N}$ be positive integers. There exist functions $i : \mathbb{N} \to \mathbb{N}$ and $b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every digraph $D$ the following is true: If $W$ is a cylindrical $\bar{\alpha}(t, r)$-wall in $D$, then either

1. There exists $H \in \mathcal{A}(K_t)$ such that $D$ has an $H$-butterfly minor grasped by $W$, or
2. there exists a set $A \subseteq V(D)$ with $|A| \leq \bar{\alpha}(t)$ and a cylindrical $r$-wall $W' \subseteq W - A$ such that $W'$ is $Per(W')$-flat in $D$ with respect to $A$.

Since the notion of flatness in Theorem 1.8 is inherited from our matching theoretic version of flatness, the above theorem yields a much stronger local structure for digraphs than the directed flat wall theorem from [6]. The trade-off is, that the result from [6] excludes a single butterfly minor, while we have to exclude an entire (possibly infinite) anti-chain.

Any undirected graph can be transformed into a digraph by replacing every undirected edge with two anti-symmetric directed edges. If $G$ is a graph, we denote by $\overrightarrow{G}$ the digraph obtained from $G$ by this construction. The digraphs obtained this way are called symmetric digraphs.

Using Theorem 1.8 we establish a broad generalisation of a classic result by Thomassen [39] by providing a qualitative characterisation of all symmetric digraphs excluding $\mathcal{A}(K_t)$ as butterfly minors.

**Theorem 1.9.** There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $t \in \mathbb{N}$ and every graph $G$ the following two statements hold:

1. If $S(G)$ contains $K_{t, t}$ as a matching minor, then $tw(G) \geq \frac{1}{t}t - 1$, and
2. if $tw(G) \geq f(t)$, then $S(G)$ contains $K_{t, t}$ as a matching minor.

The proofs of the results highlighted here, together with a more in depth explanation and the necessary definitions, can be found in the full version of the paper.

### 1.4 A Sketch of the Proof of Theorem 1.1

We conclude this introduction with a brief summary of the proof of Theorem 1.1. We start by quickly describing the main tools and ideas used in the proof, followed by a rough outline of how everything comes together.

**The Main Tools.** As the overall framework for the proof we make use of the definitions introduced in [6] through the relation between bipartite graphs with perfect matchings and digraphs provided by the $M$-direction. This is particularly helpful as it allows us to handle most cases of the first step in the road map below by directly utilising a lemma (Lemma 3.7) from [6]. However, since the directed setting is essentially a restriction of the world of bipartite graphs with perfect matchings to a single perfect matching $M$, we are able to circumvent some of the inconveniences which drastically increase the complexity of this first step in [6].

The point where the directed setting is not useful for our cause is, when we have to handle local crosses over our wall. The **Directed 2-Disjoint Paths Problem** is famously NP-complete [5] and thus one would not expect a topological description of areas without a cross. Instead, we utilise a result by Giannopoulou and Wiederrecht [10] which links the existence of two crossing alternating paths in a bipartite graph with a perfect matching to the presence of $K_{3, 3}$ as a matching minor. Since this **matching theoretic Two Paths Theorem** requires a pretty tight setup and potentially changes the perfect
matching, we need to introduce some gadgets that make sure our changes are only of a local nature and can be embedded into the majority of our wall-infrastructure. We give more insight into this main tool of our proof in Section 4.

A Brief Summary. The full proof of Theorem 1.1 has been omitted due to space constraints. Here we follow the known proofs of the (Directed) Flat Wall Theorem with some slight alterations. Specifically, given a bipartite graph $G$ with a perfect matching $M$ and a large $M$-conformal matching wall $W$ we show that

1. If there are many pairwise disjoint internally $M$-conformal paths that are internally disjoint from $W$ and have both of their endpoints in $W$ but they are far apart from each other in $W$, we find $K_{t,t}$ as an $M$-minor grasped by $W$. We essentially adapt the tools introduced in [6] to achieve this. In particular, this part of the proof is done mostly in the setting of digraphs.

2. (The second part consists of two steps at once, both of which can be solved by the same technique, but since they are slightly different, we explain both:

(a) In case there are many pairwise disjoint internally $M$-conformal paths that are internally disjoint from $W$ whose endpoints both lie in $W$ and are close together but not in the same cell, we can find many pairwise disjoint matching minor models of $K_{3,3}$. These $K_{3,3}$-matching models yield many local crosses which can be used to construct a matching minor model of $K_{t,t}$ grasped by $W$.

(b) Finally, we know that every internally $M$-conformal path that is internally disjoint from $W$ must have both endpoints on the same cell of $W$. Hence we may associate with every cell of $W$ a bipartite matching covered graph that is otherwise disconnected from $W$. If many of these cells are, essentially, non-Pfaffian we can again find many pairwise disjoint models of $K_{3,3}$ which then can be used to construct a matching minor model of $K_{t,t}$ grasped by $W$.

While the second and third steps are relatively similar to the proof of the undirected Flat Wall Theorem, they differ vastly from their directed analogues. Both steps, (a) and (b), as well as the actual proof of Theorem 1.1 and thus the combination of all three steps have been omitted due to space constraints.

2 ORGANISATION

In the next section we give a more extensive explanation and overview on the Flat Wall Theorem and highlight some of its main features. We also introduce the directed flat wall theorem of Giannopoulou et al. [6] and explain its merits and drawbacks to get a better point of reference for the comparison with our own results.

In Section 4 we give an in depth introduction to the relevant parts of matching theory to make this article as self contained as possible and provide further insight into the solution of the bipartite case of Pfaffian Recognition as these findings make up the corner stones of our notion of flatness.

2.1 Preliminaries

All graphs and digraphs in this article are considered simple, that is we do now allow multiple edges or loops and wherever such objects would arise from contraction, we identify multiple edges and remove loops. For general notation not introduced in this paper we stick to the book on graph theory by Diestel [4] while for digraph theory we recommend [2].

This paper is relatively long and contains several lesser known concepts and highly technical definitions. To increase readability we present key definitions in their own block environment and provide hyperlinks to the definitions wherever those are central to the understanding of statements and proofs.

Since the majority of our findings is focused on bipartite graphs we fix the following convention. Wherever possible we use $B$ as the standard name for a bipartite graph and $G$ for arbitrary graphs if not stated explicitly otherwise. Moreover, we assume every bipartite graph to come with a bipartition into the colour classes $V_1$ and $V_2$, where in our figures $V_1$ is represented by black vertices and the vertices in $V_2$ are depicted white. In case ambiguity arises we either treat $V_1$ as the placeholder for all possible vertices of colour $i \in [2]$, or we write $V_i(B)$ to specify which graph we are talking about. Since we also work a lot on digraphs, we will use $D$ as our standard name for digraphs to make these three different cases more distinguishable.

For integers $i, j \in \mathbb{Z}$ we use the notation $[i, j]$ for the set $\{i, i + 1, \ldots, j\}$, where $[i, j] = \emptyset$ if $i > j$. In case $i = 1$ and $j > 1$ is some other positive integer, we shorten this notation to $[j]$.

If $X$ and $Y$ are two finite sets, we denote the symmetric difference by $X \Delta Y := (X \setminus Y) \cup (Y \setminus X)$.

Given a graph $G$ we denote by $\bar{G}$ its bidirection, that is the digraph with vertex set $V(G)$ and edge set $\{(u, v), (v, u) \mid uv \in E(G)\}$.

If $F$ is a family of (di)graphs we denote by $V(F)$ and $E(F)$ the sets $\bigcup_{H \in F} V(H)$ and $\bigcup_{H \in F} E(H)$, respectively.

Definition 2.1 ((Directed) Separation). Let $D$ be a digraph. A directed separation is a tuple $(X, Y)$ such that $X \cup Y = V(D)$ and there is no directed edge with tail in $Y$ \ $X$ and head in $X$ \ $Y$. The set $X \cap Y$ is called the separator and the order of $(X, Y)$ is $|X \cap Y|$. A tuple $(X, Y)$ is called a separation, or undirected separation if we want to emphasise this fact, if $(X, Y)$ and $(Y, X)$ both are directed separations. If $G$ is a graph, then $(X, Y)$ is a separation if it is a separation in $\bar{G}$.

3 A SHORT INTRODUCTION TO STRUCTURAL (DI)GRAPH THEORY

This section is dedicated to the introduction of the two flat wall theorems predating this paper: the original Flat Wall Theorem of Robertson and Seymour and the Directed Flat Wall theorem of Giannopoulou et al. [6]. We also use this section to introduce the technical details necessary for the proof of the directed flat wall theorem since our proof of Theorem 1.1 heavily relies in the directed setting as well.

3.1 Graph Minor Theory and the Flat Wall

The combination of clique sums and graphs of bounded size captures exactly the classes of $H$-minor free graphs which are of
bounded treewidth. From the Grid Theorem [28] it follows that this class already contains all H-minor free graph classes for which H is planar. The structure of H-minor free graphs where H is non-planar is substantially more complicated. To better illustrate some of the problems one encounters when excluding a non-planar graph H let us digress a bit.

All graphs and digraphs in this paper are considered simple, so they will not contain loops or parallel edges if not explicitly stated to do so.

The Two Paths Theorem. Let C be a cycle in a graph G. A C-cross in G is a path of disjoint paths $P_1$ and $P_2$ with endpoints $v_1, t_1, v_2, t_2$ respectively such that $t_1, t_2, v_2, v_1$ appear on C in the order listed and $P_1, P_2$ are otherwise disjoint from C.

Let $s_1, s_2, t_1, t_2 \in V(G)$ be any four distinct vertices in a graph G. The Two Disjoint Paths Problem asks whether there exist two disjoint paths $P_1$ and $P_2$ in G such that for each i \in \{2\} $P_i$ has endpoints $s_i$ and $t_i$. We can reduce the Two Disjoint Paths Problem to the question whether there exists a C-cross over the 4-cycle $C := (s_1, s_2, t_1, t_2)$. Please note that we may add potentially missing edges of G to C without any impact on or the existence of the two disjoint paths since no solution to the Two Disjoint Paths Problem may use any of the edges of C. It turns out that the existence of a C-cross in general is linked to two special cases of the list of ingredients from above: clique sums (of order at most three) and graphs on surfaces (in particular planar graphs).

Let $X \subseteq V(G)$ be a set of vertices of a graph G and let (A, B) be a separation of order at most three in G with $X \subseteq A$ such that there exists some $v \in B$ for which $A \cap B$ is a minimum separator between $v$ and $X$. Let $H^\ell$ be the graph obtained from $G[A]$ by joining all pairs of vertices of $A \cap B$ with an edge. Then $H^\ell$ is called an elementary $X$-reduction of G. A graph H is an X-reduction of G if it can be obtained by a sequence $(H_1, \ldots, H_\ell)$, $\ell \geq 1$, such that $H_1 = G$, $H_\ell = H$ and for all $i \in \{2, \ldots, \ell\}$ we have that $H_i$ is an elementary X-reduction of $H_{i-1}$.

It follows that taking C-reductions for some cycle C in G preserves the existence of a C-cross. The key observation now is, in case no further C-reduction is possible in H, there does not exist a C-cross in H if $H$ can be drawn into the plane without crossing edges such that C bounds a face. Indeed, this is the only obstruction for the existence of a C-cross in a C-reduction minimal graph as captured by the so called Two Paths Theorem. The theorem has been obtained in many different forms with various techniques by a plethora of authors over the years.

Theorem 3.1 (Two Paths Theorem, [13, 29, 34, 35, 38]). A cycle C in a graph G has no C-cross in G if and only if there exists a C-reduction H of G which is planar and in which C bounds a face.

The Flat Wall Theorem. It is possible to slightly generalise the property of cycles C without a C-cross to connected planar subgraphs J as follows. Let G be a graph and $A \subseteq V(G)$ be a set of vertices, called the apex set. Moreover, let $J \subseteq G-A$ be a subdivision of a 3-connected planar graph and let $C_J$ be the outer face of J. We say that J is A-flat in G if there exists a separation $(X, Y)$ in $G-A$ with $V(C_J) = X \cap Y$ such that $V(J) \subseteq X$ and $G[X]$ has a J-reduction H which is planar and in which $C_J$ bounds a face.

Let $k, t \in \mathbb{N}$ be positive integers. The $k \times t$-grid is the graph with vertex set $\{v_{i,j} \mid i \in \{k\}, j \in \{t\}\}$, and edge set $\{v_{i,j}, v_{i,j'} \mid i_1 = i_2$ and $|j_1 - j_2| = 1$, or $j_1 = j_2$ and $|i_1 - i_2| = 1\}$.

An elementary $k$-wall is obtained from the $k \times 2k$-grid by deleting all edges with endpoints $v_{i_1, 1, j_2}$ and $v_{i_2, 1, 1}$ for all $i \in [[\frac{k}{2}]]$ and $j \in [k]$, and all edges with endpoints $v_{i_1, j_2}$ and $v_{i_2, j_2+1}$ for all $i \in [[\frac{k}{2}]]$ and $j \in [k]$, and then deleting the two resulting vertices of degree one. A subdivision of an elementary $k$-wall is called a $k$-wall.

The following is another form of the Grid Theorem which uses the fact that for any subcubic graph H a graph G contains H as a minor if and only if it contains a subdivision of H as a subgraph.

Theorem 3.2 (Wall Theorem, [28]). There exists a function $g_{\text{autdir}}$ where $g_{\text{autdir}} : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every graph G we have $tw(G) \leq g_{\text{autdir}}(2k)$, or G contains a $k$-wall as a subgraph.

Notice that, if G has a $K_t$-minor, there exist pairwise disjoint sets $X_1, \ldots, X_t \subseteq V(G)$ such that $G[X_i]$ is connected and for every pair $X_i, X_j, i \neq j$, there is an edge $u_i u_j \in E(G)$ with $u_i \in X_i$ and $u_j \in X_j$. We say that the $X_1, \ldots, X_t$ form a model of $K_t$ in G. Let $W$ be some $k$-wall for $k \geq t$ with horizontal paths $P_1, P_2, \ldots, P_k$, and vertical paths $Q_1, Q_2, \ldots, Q_k$. We say that a model of a $K_t$-minor is grasped by $W$ if for every $X_i$ there are distinct indices $i_1, i_2 \in [k]$, and $j_1, j_2 \in [t]$ such that $V(P_{i_1} \cap Q_{j_2}) \subseteq X_{i_2}$ for all $\ell \in [t]$.

The following theorem was first obtained in [30], but there it was stated in a slightly weaker form and the bounds were not given explicitly. So we credit Kawarabayashi et al. [16] for simplifying the proof and providing explicit bounds.

Theorem 3.3 (Flat Wall Theorem, [16, 30]). Let $r, t \in \mathbb{N}$ be positive integers, $R = 49152t^{24}(40t^2 + r)$, and G be a graph. Then the following is true: If W is an R-wall in G, then either

1. G has a $K_t$-minor grasped by W, or
2. there exists a set $A \subseteq V(G)$ with $|A| \leq 12288t^{24}$ and an r-wall $W' \subseteq W-A$ such that $W'$ is A-flat in G.

Indeed, one can take the Flat Wall Theorem and obtain an almost-characterisation of all graphs G that exclude $K_t$ as a minor.

Theorem 3.4 (Weak Structure Theorem, [16]). Let $r, t \in \mathbb{N}$ be positive integers, $R = 49152t^{24}(40t^2 + r)$, and G be a graph. If G has no $K_t$-minor, then for every R-wall W in G there exists a set $A \subseteq V(G)$ with $|A| \leq 12288t^{24}$ and an r-wall $W' \subseteq W-A$ such that $W'$ is A-flat in G. Conversely, if $t \geq 120t^{12}$ and for every R-wall W in G there is a set $A \subseteq V(G)$ of size at most 12288t^{24} and an r-wall $W' \subseteq W-A$ which is A-flat in G, then G has no $K_t$-minor, where $t^r = 2R^2$.

Recall the list of ingredients from the start of the introduction. In the Flat Wall Theorem clique sums and graphs of bounded size are present implicitly since it only treats the case of H-minor free graphs with large enough treewidth. Moreover, clique sums also appear in the form of reductions in the definition of A-flatness where we use small ordered separation to cut away non-planar parts which still exist after deleting A, but cannot contribute to any substantial connectivity over the wall. Additionally, we are given a

2Traditionally the Flat Wall Theorem and the Weak Structure Theorem are used synonymously, but since there are two slightly different statements it might make sense to differentiate between the two.
first glimpse of the ‘graphs on surfaces’ part by the flatness of the wall and apex vertices make a first appearance.

3.2 The Strange Case of Digraphs

A key feature of the Flat Wall Theorem is the definition of flatness itself. The fact that the existence of two disjoint but crossing paths is directly linked to a topological property plays a central role in this theorem. In the case of digraphs one encounters several problems when trying to follow along the three steps from above in order to create a structural theory of minors in digraphs. In this subsection we introduce some of the main concepts currently used in structural digraph theory and discuss why they appear to be unavoidable while still being somewhat flawed when compared to their undirected counterparts.

To this end let us start by introducing some basic concepts. The Graph Minors Series was started to investigate the structure of $H$-minor free graphs. If we want to achieve something similar for digraphs we first need a notion of minors that works well with directed graphs. However, simply allowing to contract any directed edge might lead to the introduction of new directed cycles where no directed cycles might have existed before. In particular, it is possible to give rise to non-trivial strong components in digraphs without any directed cycles. While this is not necessarily a bad thing depending on the application, it is possible that the introduction of new directed cycles might change the reachability of vertices by also introducing directed paths that did not exist before the contraction. Such a behaviour makes it hard to control the structure using the means of digraphs and does not lead to a structural theory that is different from the structure of the underlying undirected graph.

Thus more restricted notions for directed minors were pursued. The first notion that enabled structural characterisations of classes of digraphs, although not under the name they would eventually become known by, was the idea of butterfly minors [33].

**Definition 3.5 (Butterfly minor).** Let $D$ be a digraph and $(u, v) \in E(D)$. The edge $(u, v)$ is **butterfly contractable** if $N^0_D(u) = \{v\}$, or $N^2_D(v) = \{u\}$. As in undirected graphs we identify parallel edges and remove loops after contracting a butterfly contractable edge. A digraph $H$ is a **butterfly minor** of $D$ if it can be obtained from $D$ by a sequence of edge-deletions, vertex-deletions, as well as butterfly contractions.

With this restricted setting of legal contractions it was ensured that no new directed cycles would be created. However, there exist many infinite anti-chains for the butterfly minor relation such as the family of odd bicycles from [33].

In the paper by Johnson et al. [12] it was conjectured that, similar to the undirected case, there would exist a grid like structure that acted as a universal obstruction to small directed treewidth. Moreover, they conjectured this grid to exist as a butterfly minor, so no further additions to the minor relation would be necessary. The directed grid theorem was proven almost 15 years later by Kawarabayashi and Kreutzer.

Let $k \in \mathbb{N}$ be a positive integer. The **cylindrical grid of order** $k$ is the digraph obtained from the cycles $C_1, \ldots, C_k$, with $C_i = (v_{i0}, e_{i0}, v_{i1}, e_{i1}, \ldots, e_{2k-3}, v_{i(2k-3)}, e_{i(2k-3)}, v_{i(2k-2)}, e_{i(2k-2)}, \ldots, e_{2k}, v_{i0})$ for each $i \in [k]$, by adding the directed paths $P_i = v_{2i}^1 v_{2i}^2 \ldots v_{2i}^{k-1} v_{2i}^k$ and $Q_i = v_{2i+1}^1 v_{2i+1}^2 \ldots v_{2i+1}^{k-1} v_{2i+1}^k$ for every $i \in [0, k - 1]$.

**Theorem 3.6 (Directed Grid Theorem, [15]).** There exists a function $g_{dtw} : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every digraph $D$ we have $dtw(D) \leq g_{dtw}(k)$, or $D$ contains the cylindrical grid of order $k$ as a butterfly minor.

An interesting observation about the cylindrical grid is, that of course itself has large directed treewidth. That means any statement for all digraphs of large directed treewidth must also be true for the cylindrical grid itself. In Figure 2 we present two strongly connected and planar digraphs both of which are not butterfly minors of any cylindrical grid.

![Figure 2: Two strongly connected and planar digraphs that are not a butterfly minor of the cylindrical grid. The digraph on the left is strongly planar while the digraph on the right is not.](image)

The fact that such digraphs exist can be seen as an indication that obtaining any structural results beyond the directed grid theorem for digraphs that exclude a fixed butterfly minor would be a difficult task. To date no structural description of $H$-butterfly minor-free digraphs where $H$ is strongly connected and planar is known. One could interpret this as a problem inherited from butterfly minors, but indeed one can observe that the only strong minor contained in the cylindrical grid which has large directed treewidth is the cylindrical grid itself. So by changing the notion of minors we use, we cannot expect much more. Another way to resolve this problem would be to search for an alternative for directed treewidth. But while many such attempts have been made (see [2] for an overview), no structural parameter has been found that could fill this role. Moreover, there are several strong indications, in form of other dualities like brambles and tangles [7], that directed treewidth is indeed the best we can do.

The **Directed Flat Wall Theorem.** Let $H$ be a strongly connected digraph. Suppose we are content with the fact that the class of $H$-butterfly minor free digraphs has bounded directed treewidth if and only if $H$ is a butterfly minor of a cylindrical grid. Then we would still be left with the question, what if $H$ is not a butterfly minor of some cylindrical grid? We know that $H$ might still be planar, but it might as well not be. Note that Theorem 3.3 only considers the complete graph rather than general graphs $H'$. Still, since $H'$ is a subgraph, and hence a minor, of $K_{\lceil |V(H')| \rceil}$, this is enough to roughly capture the structure for all $H'$-minor free graphs. Let us now take a similar route.

As in the undirected case, we can translate the Directed Grid Theorem into a Directed Wall Theorem by simply doubling the
necessary quantities. Since this is just a linear factor we may use the same function as the directed grid theorem and simply double the argument.

**Theorem 3.7 (Directed Wall Theorem).** There exists a function \( g_{dtr} : \mathbb{N} \to \mathbb{N} \) such that for every \( k \in \mathbb{N} \) and every digraph \( D \) we have \( dw(D) \leq g_{dtr}(2k) \), or \( D \) contains a cylindrical \( k \)-wall.

Another problems that must be addressed when attempting to generalise the Flat Wall Theorem into the setting of digraphs is the lack of a directed version of the Two Paths Theorem. It is a well known fact that the Directed \( t \)-Disjoint Paths Problem is NP-complete already for \( t = 2 \) \([5]\). And thus we cannot hope for any nice characterisation for the existence for two crossing directed paths over a cycle that can be used to build a large clique as a butterfly minor. Even worse, since we are dealing with digraphs there might even disconnect directed crossing paths, but they could in fact be useless for our purposes. Moreover, we are somewhat bound to the direction of the concentric cycles within a cylindrical wall, so there might exist many crosses which might even be useful, but if they all appear in a single ‘row’ of the wall it is not possible for us to route through them as often as it would be necessary to create a large clique as a butterfly minor. Because of all these problems Giannopoulou et al. \([6]\) had to settle for a rather relaxed version of ‘flatness’ for their directed flat wall theorem. Due to space constraints the statement of the theorem is omitted.

**4 MATCHING THEORETIC BACKGROUND**

In this section we aim to introduce the matching theoretic concepts necessary for this paper. For a deeper introduction to Matching Theory the reader may consult \([7]\).

Let \( G \) be a graph. If \( M \) is a perfect matching of \( G \) and \( E(G \setminus X) \cap M \) is a perfect matching of \( G \setminus X \), we call \( X \) \( M \)-conformal. A subgraph \( H \) of \( G \) for which \( V(H) \) is \( (M-) \)conformal is called \( (M-) \)conformal.

We say that \( H \) is an \( M \)-minor of \( G \) if \( H \) can be obtained from an \( M \)-conformal subgraph \( H' \) of \( G \) by repeated bicontractions where \( M \) contains a perfect matching of \( H' \). An edge \( e \in E(G) \) is admissible if there exists a perfect matching \( M' \) of \( G \) with \( e \in M' \). A graph \( G \) is matching covered if it is connected and all of its edges are admissible.

The idea of matching minors, despite its close relation to the idea of matching minors, for a bipartite graph to have a Pfaffian Orientation Problem, it is a well known result of Lovász \([8]\) that the braces of a bipartite matching covered graph \( B \) are uniquely determined. Similar to how every 2-connected minor of a graph \( G \) must be a minor of one of its blocks, every brace that is a matching minor of some bipartite matching covered graph \( B \) is a matching minor of some brace of \( B \) \([9]\). Moreover, the braces of a bipartite matching covered graph play a similar role for matching theory as blocks, i.e. maximal 2-connected subgraphs, do in general graph theory. This is even more emphasised by the following classical result of Plummer.

**Definition 4.3 (Extendability).** Let \( k \) be a positive integer. A graph \( G \) is called \( k \)-extendable if it has at least \( 2k + 2 \) vertices and for every matching \( F \subseteq E(G) \) of size \( k \) there exists a perfect matching \( M \) of \( G \) with \( F \subseteq M \).

**Theorem 4.4 \((19)\).** A bipartite graph \( B \) is a brace if and only if it is either isomorphic to \( C_4 \), or it is 2-extendable.

The following theorem is a collection of several different characterisations of \( k \)-extendability in bipartite graphs.

**Theorem 4.5 \((1, 27)\).** Let \( B \) be a bipartite graph and \( k \in \mathbb{N} \) a positive integer. The following statements are equivalent.

1. \( B \) is \( k \)-extendable.
2. \( |V_1| = |V_2| \), and for all non-empty \( S \subseteq V_1 \), \( |N_G(S)| \geq |S| + k \). for all sets \( S_1 \subseteq V_1 \) and \( S_2 \subseteq V_2 \) with \( |S_1| = |S_2| \leq k \) the graph \( B - S_1 - S_2 \) has a perfect matching.
3. \( B \) is matching connected.
4. \( B \) is matching covered.
5. There is a perfect matching \( M \in \mathcal{M}(B) \) such that for every \( v_1 \in V_1 \), every \( v_2 \in V_2 \) there are \( k \) pairwise internally disjoint \( M \)-conformal paths with endpoints \( v_1 \) and \( v_2 \).
6. For every perfect matching \( M \in \mathcal{M}(B) \), every \( v_1 \in V_1 \), every \( v_2 \in V_2 \) there are \( k \) pairwise internally disjoint \( M \)-conformal paths with endpoints \( v_1 \) and \( v_2 \).

While ii) can be seen as a generalisation of Hall’s Theorem, iii) is, in a way, an even stronger version of matching \( k \)-connectivity. The statements iv) and v) can be seen as matching theoretic versions of Menger’s Theorem.

**Theorem 4.6 (Menger’s Theorem \([24]\) ).** Let \( D \) be a digraph and \( X, Y \subseteq V(D) \) be two sets of vertices, then the maximum number of pairwise disjoint directed \( X \)-\( Y \)-paths in \( D \) equals the minimum size of a set \( S \subseteq V(G) \) such that every directed \( X \)-\( Y \)-path in \( D \) contains a vertex of \( S \).

**The Pfaffian Recognition Problem.** To obtain a matching theoretic version of the Flat Wall Theorem, that is a version that interacts with matching minors in bipartite graphs, we need a matching theoretic notion of flatness. To understand where we derive our flatness from we need to know a bit about the structure of matching covered bipartite graphs that exclude \( K_{3,3} \) as a matching minor.

The graph \( K_{3,3} \) plays a key role in the theory of matching minors. It was found to be the singular obstruction, in the sense of matching minors, for a bipartite graph to have a Pfaffian Orientation relatively early \([14]\). At the time however this did not yield a solution for the Pfaffian Recognition problem as no algorithm was known to check for the presence of a specific matching minor. In the following 30 years many equivalent problems would be
discovered by various authors (see [23] for a good overview) but it took a complete structural characterisation of bipartite graphs without a $K_{3,3}$-matching minor that resembles similar results from (regular) minor theory such as, for example, Wagner’s description of $K_5$-minor free graphs [41].

Given a bipartite graph $B$ with a perfect matching we say that $B$ contains $K_{3,3}$ if it has a matching minor isomorphic to $K_{3,3}$, in this case we often say that $B$ is non-Pfaffian. If $B$ does not contain $K_{3,3}$ we say that $B$ is $K_{3,3}$-free or Pfaffian.

**Definition 4.10 (Conformal Cross).** Let $G$ be a graph with a perfect matching and let $C$ be a conformal cycle in $G$. Two paths $P_1$ and $P_2$ where $P_1$ has endpoints $s_1$ and $t_1$ for each $i \in [2]$ are said to form a conformal cross over $C$ if they are disjoint, internally disjoint from $C$, their endpoints $s_1$, $s_2$, $t_1$, and $t_2$ occur on $C$ in the order listed, and the graph $G + P_1 + P_2$ is a conformal subgraph of $G$.

**Definition 4.11 (Alternating Path).** Let $G$ be a graph with a perfect matching $M$. A path $P$ in $G$ is $M$-alternating if there exists a set $S \subseteq V(P)$ of endpoints of $P$ such that $P - S$ is a conformal subgraph of $G$ and we say that $P$ is alternating if there exists a perfect matching $M$ of $G$ such that $P$ is $M$-alternating. $P$ is $M$-conformal if $S = \emptyset$ and $P$ is internally $M$-conformal if $S$ contains both endpoints of $P$.

Please note that, if $P_1$ and $P_2$ form a conformal cross over some conformal cycle $C$ in a graph $G$, then there exists a perfect matching $M$ of $G$ such that both $P_1$ and $P_2$ are internally $M$-conformal and $C$ is an $M$-conformal cycle. The following two results are the key to the main theorem of [10] and will play the role of the Two Paths Theorem in this paper.

**Lemma 4.12 ([10]).** Let $B$ be a brace and $C$ a 4-cycle in $B$, then there is a conformal cross over $C$ in $B$ if and only if $C$ is contained in a conformal bisubdivision of $K_{3,3}$.

**Theorem 4.13 ([10]).** Let $B$ be a brace containing $K_{3,3}$ and $C$ a 4-cycle in $B$, then there exists a conformal bisubdivision of $K_{3,3}$ with $C$ as a subgraph.

### 4.1 A Relation Between Digraphs and Bipartite Graphs with Perfect Matchings

In many cases it suffices to fix a single perfect matching $M$ of a bipartite graph $B$ to analyse its matching theoretic properties. This approach can lead to a simplification of the matter at hand or at least simplify some notation. When working with bipartite graphs, fixing a single perfect matching yields an interesting and deep connection to the setting of digraphs. Let $B$ be a bipartite graph with a perfect matching $M$ and consider the edges of $E(B) \setminus M$. Each such edge connects a black endpoint of some edge $e \in M$ to the white endpoint of another edge $e' \in M \setminus \{e\}$. Instead of encoding the bipartition of $B$ through colours we could also assign a direction to every edge of $E(B) \setminus M$ fixing the convention that the tail of an oriented edge should always be black, while its head should always be white. If we now contract each edge of $M$ individually into a vertex, the resulting graph is simply a digraph.

**Definition 4.14 (M-Direction).** Let $B \equiv (V_1 \cup V_2, E)$ be a bipartite graph and let $M \in \mathcal{M}(G)$ be a perfect matching of $B$. The $M$-direction $\mathcal{D}(B, M)$ of $B$ is defined as follows.

1. $V(\mathcal{D}(G, M)) \equiv M$
2. $E(\mathcal{D}(G, M)) \equiv \{(e, f) \in \binom{M}{2} \mid$ there is $g \in E(B)$ such that $\emptyset \neq e \cap g \subseteq V_1$ and $\emptyset \neq f \cap g \subseteq V_2\}$.

Several properties of matching covered bipartite graphs naturally correspond to properties of digraphs. In particular this is the case for strong connectivity, as one can easily observe that the $M$-alternating cycles of a bipartite graph $B$ with a perfect matching $M$ are in bijection with the directed cycles of its $M$-direction. The following statement is folklore (a proof can be found in [42], but the result was already known by [32]).
Theorem 4.15. Let $B$ be a bipartite graph with a perfect matching $M$ and $k \in \mathbb{N}$ be a positive integer. Then $B$ is $k$-extendable if and only if $\mathcal{D}(B, M)$ is strongly $k$-connected.

The $M$-direction and the relation between matching minors and butterfly minors allows us to state an interesting set of equivalences regarding bipartite Pfaffian graphs.

Theorem 4.16 ([17, 23]). Let $B$ be a bipartite graph with a perfect matching $M$. The following statements are equivalent.

(1) $B$ is Pfaffian.

(2) $B$ does not contain $K_{3,3}$ as a matching minor.

(3) $\mathcal{D}(B, M)$ does not contain a $\tilde{C}_{2k+1}$ for any positive $k \in \mathbb{N}$ as a butterfly minor.

It is worth to note here that in the setting of digraphs an infinitely large family of obstructions must be excluded to achieve the same result as simply excluding $K_{3,3}$ as a matching minor in the bipartite case. In particular, note that the set \( \{ \tilde{C}_{2k+1} \mid k \geq 1, k \in \mathbb{N} \} \) is exactly the anti-chain $\mathcal{A}(K_{3,3})$.

5 THE MATCHING THEOREMIC FLAT WALL

In [11] it was proven that any bipartite graph with large enough perfect matching width contains a huge cylindrical wall as a matching minor. Since the notion of perfect matching width is not needed for the results in this paper we will not introduce it, just note that the main theorem of this article represents the case of bipartite graphs without $K_{3,3}$ as a matching minor whose perfect matching width exceeds a certain threshold which only depends on $t$. Still we need a formal introduction of the matching grid itself.

Definition 5.1 (Cylindrical Matching Grid). The cylindrical matching grid $CG_k$ of order $k$ is defined as follows. Let $C_1, \ldots, C_k$ be $k$ vertex disjoint cycles of length $4k$. For every $i \in [k]$ let $C_i = (v_i, v_{i+1}, \ldots, v_{i+2k-1})$, $V_i := \{v_i \mid j \in \{1, 3, 5, \ldots, 4k-1 \}\}$, $V_j^i := V(C_i) \setminus V_j$, and $M_i := \{v_i, v_{i+1}^j \mid v_i^j \in V_j\}$. Then $CG_k$ is the graph obtained from the union of the $C_i$ by adding
\[
\begin{align*}
&\{v_i^j, v_{i+1}^j \mid i \in [k-1] \text{ and } j \in \{1, 5, 9, \ldots, 4k-3 \}\}, \\
&\{v_i^j, v_{i+1}^j \mid i \in [2, k] \text{ and } j \in \{3, 7, 11, \ldots, 4k-1 \}\}
\end{align*}
\]
to the edge set. We call $M := \bigcup_{i=1}^k M_i$ the canonical matching of $CG_k$. See Figure 4 for an illustration.

Please note that the cylindrical matching grid is indeed a subcubic graph and thus, by Lemma 5.3, one can always find a conformal subdivision of $CG_k$ within a graph $G$ if it contains $CG_k$ as a matching minor. Towards an easier transition between the setting of bipartite graphs with perfect matchings and digraphs we do not use the cylindrical matching grid itself as our wall, but a slight modification.

Definition 5.2 (Matching Wall). Let $k \in \mathbb{N}$ be a positive integer. The elementary matching $k$-wall with its canonical matching $M_i$ is the graph obtained from the cylindrical matching grid of order $2k$ with canonical matching $M$ by deleting the non-$M$-edges whose index in congruent to 1 modulo 4 on each of the concentric cycles with even index and the non-$M$-edges whose index in congruent to 3 modulo 4 on each of the concentric cycles of order $2k$.

Figure 4: The cylindrical matching grid of order 4 with the canonical matching.
disjoint. Observe that reversing a bisubdivision corresponds to elementary bicontractions in the corresponding biadjacency matrix.

**Lemma 5.3 ([20]).** Let $G$ and $H$ be matching covered\(^3\) graphs such that $\Delta(H) = 3$. Then $G$ contains a conformal subdivision of $H$ if and only if it contains $H$ as a matching minor.

Let $M$ be a perfect matching such that $\mu(H)$ is $M$ conformal. We call a matching minor model $\mu : H \rightarrow G$ an $M$-model of $H$ in $G$ if $\mu(H)$ is $M$-conformal.

**Lemma 5.4 ([8]).** Let $G$ and $H$ be graphs with perfect matchings and $M$ a perfect matching of $G$. Then $H$ is isomorphic to an $M$-minor of $G$ if and only if there exists an $M$-model $\mu_M : H \rightarrow G$ in $G$.

Let $k, t \in \mathbb{N}$ be positive integers and $B$ be a bipartite graph with a conformal matching $k$-wall $W$, and let $H$ be some bipartite matching covered graph. Let $M$ be a perfect matching of $B$. We say that $W$ grasps an $H$-matching minor if there exists a matching minor model $\mu : H \rightarrow B$ such that $\mu(H)$ is $M$-conformal, and for every $e \in E(H) \cap M$ the $M$-conformal path $\mu(e)$ is completely contained in $W$. We say that $W$ grasps an $H$-matching minor if there exists a perfect matching $M$ of $B$ such that $W$ grasps $H$.

Let $B$ be a Pfaffian brace and $H$ be a planar brace. We say that $H$ is a summand of $B$ if there exist planar braces $H_1, \ldots, H_r$ such that $B$ can be constructed from the $H_i$ by repeated applications of the trinoma operation, and $H = H_1$.

What is left to do before we can formally state our main theorem is a definition of flatness in the matching theoretic sense. In contrast to the undirected setting, where it makes sense to speak about subgraphs as a means of reductions, in the setting of graphs with perfect matchings, we sometimes have to perform tight cut contractions.

Let $B$ and $H$ be bipartite graphs with a perfect matching such that $H$ has a single brace $J$ that is not isomorphic to $C_4$. We say that $H$ is a $J$-expansion. A brace $G$ of $B$ is said to be a host of $H$ if $G$ contains a conformal subgraph $H''$ that is a $J$-expansion and can be obtained from $H$ by repeated applications of tight cut contractions.

The graph $H''$ is called the remnant of $H$.

It makes sense for us to work with a cylindrical grid/wall rather than a square one. However, the only problem this brings is that any cylindrical wall has two faces which might be considered the natural ‘outer face’. Indeed, we would like the inner-most and the outer-most cycle of the cylindrical wall to both bound faces in an appropriate reduction.

Let $B$, $H$, and $J$ be bipartite graphs with perfect matchings such that $H$ and $J$ are conformal subgraphs of $B$. We say that $H$ is $J$-bound if there exists a subgraph $K \subseteq B - J$ that is the union of elementary components of $B - J$ such that $K \cup J$ is matching covered, and $H$ is a conformal subgraph of $K \cup J$. The graph $K \cup J$ is called a $J$-base of $H$ in $B$.

**Definition 5.5 (P-Flatness).** Let $B$ be a bipartite graph with a perfect matching, and let $H$ be a planar matching covered graph that is a $J$-expansion of some planar brace $J$. Moreover, let $P$ be a collection of pairwise vertex disjoint faces of $H$ such that $P$ is a conformal subgraph of $H$. At last, let $A \subseteq V(B)$ be a conformal set. Then $H$ is $P$-flat in $B$ with respect to $A$ if

\(^3\)A graph $G$ is said to be matching covered if it is connected and each of its edges belongs to at least one perfect matching of $G$.

1. $H$ is a conformal subgraph of $B' := B - A$.
2. some $P$-base of $H$ in $B'$ has a Pfaffian brace $B''$ that is a host of $H$, and
3. $B''$ has a summand $G$ that contains a remnant $H'$ of $H$ such that every remnant of a face from $P$ within $H'$ bounds a face of $G$.

The set $A$ in the definition above is the apex set, similar to the one which occurs in the version of flatness used for the original Flat Wall Theorem. The set $P$ takes on two roles at once, it mimics the separator of the separation $(X, Y)$ in the original definition and also allows us to essentially, prescribe which faces of $H$ should take the role of the outer face. Since we do not require $B - A$ to be a brace or even matching covered, we need to remove all non-admissible edges and just take the matching covered subgraph that contains $H$. This is modelled by selecting a certain $P$-base of $H$ in $B'$. Now that we have reduced $B - A$ to a matching covered graph $B''$, we must go one step further and get rid of the non-trivial tight cuts of $B''$. This is done by selecting $B''$ to be a host of $H$ in $B''$. By requiring $B''$ to be Pfaffian, we ready ourselves for the final reduction. Indeed, since we insist $G$ to be a summand of $B''$ this means $B''$ cannot be isomorphic to the Heawood graph by Theorem 4.8. Since $G$ is a summand of $B''$, it must be a planar brace. To get to this point, some tight cut contractions could have been necessary, and thus we are only able to talk about a remnant $H''$ of $H$, but since $H$ was chosen to be a $J$-expansion of some planar brace $J$, this remnant is well defined. Similarly, the tight cut contractions could have shrunk some of the faces that were selected to form $P$, but we can still make out their remnants and thus (iii) resembles the third requirement of the original definition. With this all necessary definitions for Theorem 1.1 are in place. Please note that the matching minor from Theorem 1.1 is not necessarily conformal for the canonical perfect matching of the respective wall.

In the following, we sometimes say that a conformal matching $k$-wall $W$ is flat in $B$, if $k$ is large enough and the second part of the theorem above holds true for $W$.

With Theorem 1.1 at hand, we can give an approximate characterisation of all bipartite graphs with perfect matchings that exclude $K_{t+1}$ as a matching minor for some $t \in \mathbb{N}$. This weak structure theorem is similar to Theorem 3.3 and in some sense can be seen as a generalisation of Theorem 4.8.

**Theorem 5.6.** Let $r, t \in \mathbb{N}$ be positive integers, $\alpha$ and $\rho$ be the two functions from Theorem 1.1, and $B$ be a bipartite graph with a perfect matching.

- If $B$ has no $K_{t+1}$-matching minor, then for every conformal matching $\rho(t, r)$-wall $W$ in $B$ and every perfect matching $M$ of $B$ such that $M \cap E(W)$ is the canonical matching of $W$, there exist an $M$-conformal set $A \subseteq V(B)$ with $|A| \leq \alpha(t)$ and an $M$-conformal matching $r$-wall $W' \subseteq W - A$ such that $W'$ is $\text{Per}(W')$-flat in $B$ with respect to $A$.

- Conversely, if $t \geq 2$ and $r \geq \sqrt{2\alpha(t)}$, and for every conformal matching $\rho(t, r)$-wall $W$ in $B$ and every perfect matching $M$ of $B$ such that $M \cap E(W)$ is the canonical perfect matching of $W$, there exist an $M$-conformal set $A \subseteq V(B)$ with $|A| \leq \alpha(t)$ and an $M$-conformal matching $r$-wall $W' \subseteq W - A$ such that $W'$ is $\text{Per}(W')$-flat in $B$ with respect to $A$, then $B$ has no matching minor isomorphic to $K_{t', t'}$, where $t' = 16\rho(t, r)^2$. 

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Proof. The first part of the theorem follows immediately from Theorem 1.1, since in case B does not have $K_1,t$ as a matching minor, the first part of Theorem 1.1 can never be true and thus every matching $\rho(t,r)$-wall must be flat in $B$.

For the reverse, note that an elementary matching $\rho(t,r)$-wall has exactly $16\rho(t,r)^2$ vertices. Now suppose $B$ has a matching minor model $μ: K_{t,r'} \to B$. Then there exists a perfect matching $M$ such that $μ$ is $M$-conformal. Indeed, $K_{t,r'}$ contains an $M[K_{t,r'}]'$-conformal elementary matching $ρ(t,r')$-wall, and thus $μ(K_{t,r'})$ contains an $M$-conformal matching $ρ(t,r')$-wall. For every vertex $w$ of degree three in $W$ there exists a unique vertex $w_{neq} \in V(K_{t,r'})$ such that $w \in V(μ(w_{neq}))$, and in case $w ≠ w_{neq}$ are both vertices of degree three in $W$, then $w_{neq} ≠ w_{neq}$. Moreover, if $P$ is a path in $W$ whose endpoints $w$ and $w$ have degree three in $W$ and all internal vertices are vertices of degree two in $W$, then $P(V(μ(w_{neq}))∪V(μ(w_{neq})))$.

By assumption there exist an $M$-conformal set $A \subseteq V(B)$ and an $M$-conformal matching $r$-wall $W' \subseteq W$ such that $W'$ is $Per(W')\text{-flat}$ in $B$ with respect to $A$ and $W'$ has $16r^2$ many vertices of degree three in $W'$, $16r$ of which lie on $Per(W')$. Since $r ≥ \sqrt{2|A(r)|/2}$, we have at least $32r|A(r)|$ many such degree three vertices. Thus, with $|A| ≤ α(t)$ and $t ≥ 2$, there exist $w_1, \ldots, w_6 \in V(W' - Per(W'))$ such that $V(μ(w_{neq}))∪V(μ(w_{neq})) = \emptyset$ for all $i \in [6]$. This, however, means that for every $Per(W')$-base $H$ of $W'$, every brace $J$ of $H$ that is a host of $W'$ must contain $K_{3,3}$ as a matching minor and therefore no such $J$ can be Pfaffian by Theorem 4.16. Hence $W'$ cannot be $Per(W')\text{-flat}$ in $B$ with respect to $A$ and we have reached a contradiction. □

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