REAL FORMS ON RATIONAL SURFACES

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Abstract. For any positive integer \( r \), we construct a smooth complex projective rational surface which has at least \( r \) real forms not isomorphic over \( \mathbb{R} \).

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1. Introduction

A real form of a complex algebraic variety \( X \) is a real algebraic variety whose complexification is isomorphic to \( X \). For many families of complex projective algebraic varieties it is known that the number of isomorphism classes over \( \mathbb{R} \) of real forms is finite, as for example in the cases of abelian varieties [Sil82], [DIK00, Appendix D] (based on [BS64] for abelian varieties as algebraic groups), algebraic surfaces of Kodaira dimension greater than or equal to one [DIK00, Appendix D], minimal algebraic surfaces [DIK00, Appendix D], Del Pezzo surfaces [Rus02] and compact hyperkähler manifolds [CF19]. Different authors have raised this finiteness question for nonminimal rational surfaces: [DIK00, pages 232-233], [Ben16a, Problem, page 1128], [DO19, page 943] and [DOY20, Question 1.5]. More precisely, at the time this paper appeared, the following question was still open:

Question 1.1 ([Ben16a, Problem, page 1128], [DOY20, Question 1.5]). Is there a smooth complex projective rational surface with infinitely many mutually nonisomorphic real forms?

The first example of a complex projective variety with infinitely many nonisomorphic real forms was obtained by Lesieutre [Les18, Theorem 2], for varieties of dimension 6. It was then generalised in [DO19, Theorem 1.1] by Dinh and Oguiso to any dimension \( d \geq 2 \) for varieties.
of Kodaira dimension $d - 2$, and in [DOY20] by Dinh, Oguiso and Yu to any dimension greater or equal to three for rational varieties. See also Dubouloz, Freudenburg and Moser-Jauslin [DFMJ20] for examples of rational affine varieties of dimension greater or equal to four. But as expressed by Kharlamov in [Kha02], “[…] what concerns this finiteness problem, the case of non minimal rational surfaces looks to be the most complicated and enigmatic,” or as Cattaneo and Fu [CF19] describe, “[the] remaining biggest challenge for surfaces seems to be the case of rational surfaces.”

In fact, in [Les18, DO19, DOY20], all constructions of projective varieties with infinitely many nonisomorphic real forms come from projective varieties $X$ such that $\text{Aut}(X)/\text{Aut}^0(X)$ is not finitely generated. Whether $\text{Aut}(X)/\text{Aut}^0(X)$ is not finitely generated for projective rational surfaces is still open. As an indication of where to look for potentially infinitely many isomorphism classes of real forms in this situation, Benzega [Ben16a] proved that if such a complex projective rational surface exists, then it will be the blow-up of $\mathbb{P}^2_\mathbb{C}$ in at least ten points and have at least one automorphism of positive entropy. Also, due to [Kim20], the automorphism group of this surface will need to contain a subgroup isomorphic to $\mathbb{Z} \ast \mathbb{Z}$.

A related question is whether for smooth complex projective rational surfaces, there exists an upper bound on the number of isomorphism classes of real forms. We can answer this in the negative:

**Theorem 1.2.** For every positive integer $r$, there exists a smooth complex projective rational surface with at least $r$ real forms not pairwise isomorphic over $\mathbb{R}$.

Our proof of Theorem 1.2 hinges on the following construction: consider a real smooth cubic curve $C \subset \mathbb{P}^2_\mathbb{C}$ whose real locus $C(\mathbb{R})$ has two connected components in the Euclidean topology and for which the group of group automorphisms satisfies $\text{Aut}_{\mathbb{R}}(C) \cong \mathbb{Z}/2\mathbb{Z}$. If we fix an integer $r \geq 3$ and choose points $p_{00}, \ldots, p_{0r} \in C(\mathbb{R})$ satisfying some suitable condition, then we can find exactly four points $p_{1i}, \ldots, p_{4i} \in C(\mathbb{R})$ for each $1 \leq i \leq r$ such that the tangent to $C$ at a point $p_{ij}$ with $j \neq 0$ cuts $C$ in $p_{i0}$. This situation allows us to define of cubic involutions $\sigma_1, \ldots, \sigma_r$ on the blow-up $X$ of $\mathbb{P}^2_\mathbb{C}$ in the $5r$ points $p_{ij}$. For each $i$, the automorphism $\sigma_i$ is the unique involution that fixes pointwise the strict transform of $C$ and preserves the strict transform of a general line through $p_{i0}$. These cubic involutions will be shown to induce $r$ nonisomorphic real forms on $X$, compare with Theorem 10.2. The assumption that $r \geq 3$ is due to some computations in the proof of Theorem 8.3 (see Equations (8.8) and (8.9)); the additional, tedious calculations for $r = 2$ were left out but with some extra work, one could even show that for $r = 2$, there are precisely 2 real forms. In fact, we believe that all of the surfaces constructed in Theorem 10.2 have precisely $r$ forms, though we do not know how much more effort would be required to prove it.

After this paper appeared, Dinh, Oguiso and Yu [DOY21] answered the Question 1.1 in the affirmative, and the author herself [Bot21] succeeded in constructing a smooth affine rational surface defined over $\mathbb{C}$ with uncountably many nonisomorphic real forms.

The structure of this article is the following: In Section 2, we discuss the connection between real forms, real structures and automorphisms on $X$. We then introduce the cubic involutions in Section 3, which will be the candidates for the inequivalent cocycles. We obtain desirable relations in the Picard group of $C$ in Section 4, for which we need to assume certain points fulfilling an extra condition; with this condition, we can then start the translation of geometric
properties of an automorphism of $X$ into arithmetic ones with the help of Section 5. As mentioned, along the way, we will have encountered a few conditions on the points chosen, and therefore, we secure their existence in Section 6. In Sections 7, 8 and 9, we will use the tools collected so far to complete the transition to arithmetic conditions on automorphisms of $X$. This allows us, finally, in Section 10 to show Theorem 10.2, which describes the construction for the smooth complex projective rational surface of Theorem 1.2.

We fix $\mathbb{C}$ as the base field, and write $\mathbb{P}^2$ instead of $\mathbb{P}^2_{\mathbb{C}}$.

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2. Real structures

The complexification of a real algebraic variety $X_0$ is given by $(X_0)_\mathbb{C} := X_0 \times_{\text{Spec} \mathbb{R}} \text{Spec} \mathbb{C}$.

Fix a complex algebraic variety $X$. A real form of $X$ is a real algebraic variety $X_0$ with a $\mathbb{C}$-isomorphism $(X_0)_\mathbb{C} \xrightarrow{\sim} X$. To understand the isomorphism classes of real forms of $X$, we instead study the real structures on $X$.

A real structure on $X$ is an anti-regular involution $\rho : X \to X$, where the anti-regularity means that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\rho} & X \\
\downarrow & & \downarrow \\
\text{Spec} \mathbb{C} & \xrightarrow{z \mapsto \overline{z}} & \text{Spec} \mathbb{C}.
\end{array}
$$

Two real structures $\rho$ and $\rho'$ on the same complex variety $X$ are equivalent if there exists $\varphi \in \text{Aut}(X)$ such that $\rho = \varphi \rho' \varphi^{-1}$.

It is an exercise to show (see [Rus02, Prop. 1.1] or [Ben16b, Thm. 4.1]) that, if $n$ is even, there is, up to equivalence, only one real structure on $\mathbb{P}^n$, namely

$$
[z_0 : \ldots : z_n] \mapsto [\overline{z_0} : \ldots : \overline{z_n}].
$$

If, however, $n = 2k + 1$ is odd, there are two, up to equivalence:

$$
[z_0 : \ldots : z_n] \mapsto [\overline{z_0} : \ldots : \overline{z_n}],
$$

$$
[z_0 : z_1 : \ldots : z_n] \mapsto [-\overline{z_1} : \overline{z_0} : \ldots : -\overline{z_{2k+1}} : \overline{z_{2k}}].
$$

Therefore, any real structure on $\mathbb{P}^2$ is equivalent to

$$
\hat{\rho} : [z_0 : z_1 : z_2] \mapsto [\overline{z_0} : \overline{z_1} : \overline{z_2}]. \quad (2.1)
$$

We can associate a real structure to a real form, and vice versa: considering a real form $X_0$ with complex isomorphism $\varphi : (X_0)_\mathbb{C} \xrightarrow{\sim} X$, we can set $\rho := \varphi^{-1}\rho_0\varphi$ for the real structure $\rho_0 := \text{id} \times \text{Spec}(z \mapsto \overline{z})$ on $(X_0)_\mathbb{C}$. Conversely, given a real structure $\rho$ on a complex variety $X$, the variety $X_0 := X/\langle \rho \rangle$ is a real form of $X$.

As a matter of fact, there is an equivalence between the category of complex quasi-projective algebraic varieties with a real structure and the category of real quasi-projective algebraic
varieties (see for example [Ben16b], Chapter 3.1, for more details). Thus, knowing all real forms of a given complex variety is the same as knowing all real structures on it:

**Theorem 2.1** ([Ben16b, Thm. 3.17]). Any two real forms of a complex quasi-projective variety $X$ are $\mathbb{R}$-isomorphic if and only if their associated real structures are equivalent.

With this theorem under our belt, we turn our attention to real structures. Fix a real structure $\rho$ on a complex variety $X$ and set $G := \langle \rho \rangle$. Let $\rho'$ be another real structure on $X$. Then:

$$\rho' \rho =: a_\rho \in \text{Aut}(X).$$

Conversely, we can analyse the conditions on an automorphism of $X$ such that its composition with $\rho$ is again a real structure.

**Definition 2.2.** An automorphism $a_\rho$ of $X$ is called a cocycle if it verifies $(a_\rho \rho)^2 = \text{id}_X$. Two cocycles $a_\rho, b_\rho$ are called equivalent if there exists $\alpha \in \text{Aut}(X)$ such that $b_\rho \rho = \alpha^{-1} a_\rho \rho \alpha$. In that case, we write $a_\rho \sim b_\rho$. We denote by $Z^1(G, \text{Aut}(X))$ the set of cocycles and by

$$H^1(G, \text{Aut}(X)) := Z^1(G, \text{Aut}(X))/\sim$$

the first Galois cohomology set.

Using $H^1(G, \text{Aut}(X))$, we may describe the equivalence classes of real structures, and therefore of real forms:

**Theorem 2.3** ([BS64, Section 2.6]). Let $X$ be a quasi-projective complex variety and $\rho$ a real structure on $X$. The equivalence classes of real structures are in bijection with the elements of $H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), \text{Aut}(X))$, where the nontrivial element of $\text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\text{Aut}(X)$ by conjugation with $\rho$.

In the context of a blow-up, we would like to preserve the real structure on a complex projective surface:

**Proposition 2.4** ([Sil89, II.6.1]). Let $Y$ be a complex projective surface and $\hat{\rho}$ a real structure on $Y$. The blow-up $\pi : X \rightarrow Y$ in a real point of $Y$ or in a pair of complex conjugated points of $Y$ allows one to give $X$ a real structure $\rho$ in a natural way such that $\pi$ is real, meaning $\pi \rho = \hat{\rho} \pi$.

So if we blow-up $\mathbb{P}^2$ in real points or in pairs of conjugates, the real structure $\hat{\rho}$ given in (2.1) lifts to a real structure $\rho$ on the blow-up $X$. Therefore, by Theorem 2.3, the equivalence classes of real structures on $X$ — and hence the isomorphism classes of real forms of $X$ — are in a one-to-one correspondence with the equivalence classes of cocycles of $X$ with respect to the real structure $\rho$.

**3. Cubic involutions**

Theorem 2.3 replaces our search for a rational surface with $r$ real structures by finding one having $r$ inequivalent cocycles. In fact, we aim to construct a blow-up $X$ of $\mathbb{P}^2$ at real points having $r \geq 3$ automorphisms $\sigma_1, \ldots, \sigma_r$ such that

(i) for all $1 \leq i \leq r$, the equation $(\sigma_i \rho)^2 = \text{id}_X$ holds, where $\rho$ is the standard real structure inherited from $\mathbb{P}^2$ (see Proposition 2.4),
and, provided (i) holds, for \( i \neq j \), the cocycles \( \sigma_i \) and \( \sigma_j \) are not equivalent, meaning that there does not exist any automorphism \( \alpha \in \text{Aut}(X) \) such that \( \alpha \sigma_i \rho = \sigma_j \rho \alpha \).

The construction we propose to achieve this relies on the following classical birational map (see [Giz94, Ex. 3]):

**Definition 3.1.** Let \( C \subset \mathbb{P}^2 \) be a smooth cubic curve. For \( p \in C \), denote by \( \hat{\sigma}_p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \) the cubic involution centred at \( p \), which is the unique birational map defined in the following way: if \( L \) is a general line containing \( p \), then

(i) the map \( \hat{\sigma}_p \) satisfies \( \hat{\sigma}_p(L) = L \),

(ii) the restriction \( \hat{\sigma}_p|_L \) is the unique involution that fixes the two points of \( (L \cap C) \setminus \{p\} \) pointwise.

Note that \( \hat{\sigma}_p \) restricts to the identity on the curve \( C \). In the sequel, we will always assume that \( p \) in the above definition is not an inflection point. Then, by [Bla08, Prop. 12], the base points of \( \hat{\sigma}_p \) are \( p \) and those points \( q \) where the tangent to \( C \) at \( q \) cuts \( C \) in \( p \); and there are precisely four such points. Denote them by \( p_1, \ldots, p_4 \) and call them *associated with \( p \).*

Recall that for a fixed inflection point \( p_0 \in C \) as neutral element we get a group structure on \( C \). Furthermore, we have a group isomorphism

\[
C \cong \text{Pic}^0(C) \leq \text{Pic}(C),
\]

\[
p \mapsto p - p_0.
\]

In \( \text{Pic}(C) \), we write \( p \) for the class of \( p \). The condition that the tangent to the cubic curve at \( p_i \) cuts it in \( p \) can be expressed in \( \text{Pic}(C) \) by \( p + 2p_i = 3p_0 \). Since \( p + 2p_i = p + 2p_j \) for \( 1 \leq i, j \leq 4 \), we can deduce

\[
2(p_i - p_j) = 0,
\]

and hence \( p_i - p_j \in C[2] \), where \( C[2] \) denotes the group of 2-torsion elements of \( \text{Pic}^0(C) \).

In for example [Sil09, Cor. III.6.4], it is shown that \( C[2] \cong (\mathbb{Z}/2\mathbb{Z})^2 \), which we can use in the proof of the following lemma.

**Lemma 3.2.** Fix \( p \) on a smooth cubic curve \( C \subset \mathbb{P}^2 \) which is not an inflection point, and call its associated points \( p_1, \ldots, p_4 \). Then, for every \( 1 \leq i, j \leq 4 \),

\[
p_i - p_j = p_j - p_i,
\]

and if \( \{i, j, k, \ell\} = \{1, 2, 3, 4\} \), then

\[
p_i - p_j = p_k - p_\ell.
\]

Moreover, \( C[2] = \{0, p_1 - p_4, p_2 - p_4, p_3 - p_4\} \).

**Proof.** The first equality follows from \( 2(p_i - p_j) = 0 \). To see that the second equality holds, note that we have — thanks to the first equality — six nonzero elements in total in \( C[2] \):

\[
p_1 - p_2, \ p_1 - p_3, \ p_1 - p_4, \ p_2 - p_3, \ p_2 - p_4, \ p_3 - p_4.
\]

But since \( C[2] \cong (\mathbb{Z}/2\mathbb{Z})^2 \) by [Sil09, Cor. III.6.4], the 2-torsion group \( C[2] \) only contains three nonzero elements. If we equate two elements where an index agrees, then, possibly using (3.1), we find that two of the associated \( p_i \) would have to agree. This is a contradiction to the points \( p_i \) being distinct.
As we have seen, the nontrivial elements of \( C[2] \) can be given as \( p_1 - p_4, p_2 - p_4 \) and \( p_3 - p_4 \), which concludes the proof. \( \square \)

We can now blow up the points \( p, p_1, \ldots, p_4 \) and examine what happens to \( \hat{\sigma}_p \).

**Proposition 3.3.** Let \( C \subset \mathbb{P}^2 \) be a real smooth cubic curve and suppose there exists a real point \( p \in C \) which is not an inflection point. Consider the cubic involution \( \hat{\sigma}_p \) centred at \( p \) and the rational surface \( X \) obtained by blowing up \( p, p_1, \ldots, p_4 \). Then the birational map \( \hat{\sigma}_p \) lifts to an involution \( \sigma_p \in \text{Aut}(X) \) which commutes with the lift \( \rho \) of the anti-regular involution \( \hat{\rho} : [x : y : z] \mapsto [\overline{x} : \overline{y} : \overline{z}] \) of the projective space. Furthermore, this implies \((\sigma_p \rho)^2 = \text{id}_X\), meaning \( \sigma_p \) is a cocycle.

*Proof.* Because \((\hat{\sigma}_p)^2 = \text{id}_X\), and since \( p, p_1, \ldots, p_4 \) are the base points of \( \hat{\sigma}_p \), the birational map \( \hat{\sigma}_p \) lifts to an automorphism on the blow-up of the base points \( p, p_1, \ldots, p_4 \). Denote this lift by \( \sigma_p \in \text{Aut}(X) \).

By [Giz94, Ex. 3], the birational map \( \hat{\sigma}_p \) is defined over \( \mathbb{R} \) so long as \( C \) is defined over \( \mathbb{R} \) and \( p \) is real. Therefore, on the blow-up in \( p, p_1, \ldots, p_4 \), the involutions \( \sigma_p \) and \( \rho \) commute. \( \square \)

Suppose we can choose real points \( p_{10}, \ldots, p_{r0} \) with \( r \geq 3 \) on a real smooth cubic curve \( C \) such that none of them is an inflection point or associated points of one another. There will be further assumption on the points, which will be introduced in Section 4 and Section 5 — the existence of such points will be shown in Section 6. Denote the associated points of \( p_{10} \) by \( p_{11}, \ldots, p_{14} \).

Under these assumptions, we can blow up all the \( p_{ij} \)'s to obtain a rational surface \( X \), and for every \( 1 \leq i \leq r \), the cubic involution \( \hat{\sigma}_{p_{10}} \) lifts to an automorphism \( \sigma_{p_{10}} =: \sigma_i \). These automorphisms \( \sigma_1, \ldots, \sigma_r \in \text{Aut}(X) \) are the candidates for the inequivalent cocycles. Proposition 3.3 shows that they are indeed cocycles.

The rest of the paper is devoted to proving that they are also inequivalent, which shows that there are at least \( r \) real structures. In Section 5, we will express geometric properties of automorphisms of \( X \) arithmetically, a translation needed to prove that the \( \sigma_i \) are not equivalent.

## 4. Independent points and relations in \( \text{Pic}(C) \)

As mentioned in the previous chapter, we will discuss an important condition on the points \( p_{10}, \ldots, p_{r0} \). For one, this condition will be indispensable for the translation of geometric properties into arithmetic ones, but it will also turn out to be stronger than the assumption of noncollinearity on the \( p_{ij} \)'s (see Corollary 4.3):

**Definition 4.1.** Let \( C \subset \mathbb{P}^2 \) a smooth cubic curve and \( p_0 \in C \) an inflection point. Points \( p_1, \ldots, p_s \in C \) are called independent if in \( \text{Pic}^0(C) \), the elements \( (p_1 - p_0), \ldots, (p_s - p_0) \) are \( \mathbb{Z} \)-linearly independent.

Note that independence implies that these points cannot be inflection points; in fact, \( (p_i - p_0) \) is without torsion in \( \text{Pic}^0(C) \), or equivalently, \( p_i \) is not torsion in \( C \). We can show that the condition of independence on the \( p_{10}, \ldots, p_{r0} \) is stronger than the condition of the existence four points out of all the \( p_{ij} \)'s, with no three collinear, for which we first show the following proposition:
Proposition 4.2. Let $r \geq 1$, fix independent points $p_1, \ldots, p_r$ lying on a smooth cubic curve $C \subset \mathbb{P}^2$, choose an inflection point $p_0 \in C$ as neutral element and fix a labelling $\{0, \delta_1, \delta_2, \delta_3\} = C[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$. Then we can order the associated points $p_{i1}, \ldots, p_{i4}$ such that $\delta_k = p_{ik} - p_{i4}$ for $k = 1, 2, 3$. Furthermore:

(i) We have the following relations in Pic$(C)$:

\[
\begin{align*}
3p_0 &= p_{10} + 2p_{14}, \\
p_0 &= p_{10} + 2p_{14} - 2p_{i4}, \\
p_{i3} &= \delta_1 + p_{i4}, \\
p_{i2} &= \delta_2 + p_{i4}, \\
p_{i3} &= \delta_1 + \delta_2 + p_{i4},
\end{align*}
\]

for $1 \leq i \leq r$.

(ii) If there exist $m, n_1, \ldots, n_r, s_1, s_2, d \in \mathbb{Z}$ such that

\[
mp_{10} + \sum_{i=1}^{r} n_i p_{i4} + s_1 \delta_1 + s_2 \delta_2 = 3dp_0,
\]

then $m = d$, $n_1 = 2d$, $n_2 = \ldots = n_r = 0$ and $s_1 \equiv s_2 \equiv 0 \mod 2$.

Proof. By Lemma 3.2, we have $C[2] = \{p_{ik} - p_{i4} \mid k \in \{1, 2, 3, 4\}\}$, so we may choose the order of the associated points such that $\delta_k = p_{ik} - p_{i4}$ for $k = 1, 2, 3$.

As for (i), the first equation is due to the nature of the points chosen, and the second equation follows from

\[
p_{i0} + 2p_{i4} = 3p_0 = p_{10} + 2p_{14}.
\]

The last three equations hold by choice of the order of the associated points, where we use $\delta_3 = \delta_1 + \delta_2$.

To prove (ii), we note that due to degree reason, we have $m + \sum_{i=1}^{r} n_i = 3d$. Then, we multiply the equation by two and use $2\delta_1 = 2\delta_2 = 0$:

\[
2m(p_{10} - p_0) + \sum_{i=1}^{r} 2n_i(p_{i4} - p_0) = 0.
\]

Using $2(p_{i4} - p_0) = -(p_{i0} - p_0)$, we obtain

\[
(2m - n_1)(p_{10} - p_0) - \sum_{i=2}^{r} n_i(p_{i0} - p_0) = 0.
\]

Since the $p_{i0}$’s are independent,

\[
2m = n_1, \quad n_2 = \ldots = n_r = 0,
\]

and thereby,

\[
mp_{10} + 2mp_{14} + s_1 \delta_1 + s_2 \delta_2 = 3dp_0.
\]  \hspace{1cm} (4.1)

As $\delta_1$ and $\delta_2$ are of degree 0, we may deduce $3m = 3d$, implying $m = d$ and $n_1 = 2d$. Due to $p_{i0} + 2p_{i4} = 3p_0$, we find from (4.1) that

\[
s_1 \delta_1 + s_2 \delta_2 = 0.
\]

Applying Lemma 3.2, we conclude that $s_1 \equiv s_2 \equiv 0 \mod 2$. This completes the proof. \hfill \square
With the above proposition, we can show that independence of the $p_{10}, \ldots, p_{r_0}$ implies that no three points out of the $p_{ij}$’s are collinear.

**Corollary 4.3.** If the points $p_{10}, \ldots, p_{r_0} \in C$ with $r \geq 1$ are independent, then no three points out of the $p_{ij}$’s, $1 \leq i \leq r, 0 \leq j \leq 4$, are collinear.

**Proof.** Suppose by contradiction that $p_{ij}$, $p_{kt}$ and $p_{st}$ are collinear. By Proposition 4.2, (i), we know that we can write $p_{ij} + p_{kt} + p_{st} = 3p_0$ as

$$mp_{10} + \sum_{i=1}^{r} n_i p_{i4} + s_1 \delta_1 + s_2 \delta_2 = p_{ij} + p_{kt} + p_{st} = 3p_0,$$

(4.2)

where $m, s_1, s_2 \in \{0, 1, 2, 3\}$ and $n_1, \ldots, n_r \in \mathbb{Z}$. Using (ii), we obtain $m = 1$, $n_1 = 2$, $n_2 = \ldots = n_r = 0$ and $s_1 \equiv s_2 \equiv 0 \mod 2$. This implies, with (i), that exactly one of the indices $j, \ell, t$ is equal to zero. Up to exchanging the points, we may assume $j = 0$ and $\ell, t \neq 0$. Then, again with (i),

$$p_{10} + 2p_{14} = p_{00} + p_{k}\ell + p_{st} = p_{10} + 2p_{14} - 2p_{14} + p_{kt} + p_{st},$$

which implies $2p_{14} = p_{kt} + p_{st}$. Apply once more (i) to obtain

$$2p_{14} = p_{kt} + p_{st} = p_{k4} + p_{s4} + s_1' \delta_1 + s_2' \delta_2,$$

where $s_1', s_2' \in \{0, 1, 2\}$. With (ii), this can only be if $k = s = i$ and $s_1' \equiv s_2' \equiv 0 \mod 2$. This implies $p_{kt} = p_{st}$, a contradiction to the points being distinct.

As a consequence of Proposition 4.2, we can find a specific base of the subgroup of $\text{Pic}(C)$ generated by the $p_{ij}$’s.

**Proposition 4.4.** Consider a smooth cubic curve $C \subset \mathbb{P}^2$ and an inflection point $p_0 \in C$ as neutral element. Fix a labelling $\{0, \delta_1, \delta_2, \delta_3\} = C[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$. Choose, for $r \geq 1$, independent points $p_{10}, \ldots, p_{r_0} \in C$ and label their associated points $p_{ij}$ such that $\delta_k = p_{ik} - p_{i4}$ for $k = 1, 2, 3$. Then we have an isomorphism

$$\mathbb{Z}p_{10} \oplus \bigoplus_{i=1}^{r} \mathbb{Z}p_{i4} \oplus \bigoplus_{j=1}^{2} (\mathbb{Z}/2\mathbb{Z})\delta_j \sim \langle p_{ij} \mid 1 \leq i \leq r, 0 \leq j \leq 4 \rangle,$$

$$(m, n_1, \ldots, n_r, s_1, s_2) \mapsto mp_{10} + \sum_{i=1}^{r} n_i p_{i4} + s_1 \delta_1 + s_2 \delta_2.$$

**Proof.** Surjectivity can be shown by hitting all the generators: this is guaranteed by the relations in (i) of Proposition 4.2. For injectivity, we take $(m, n_1, \ldots, n_r, s_1, s_2)$ such that

$$mp_{10} + \sum_{i=1}^{r} n_i p_{i4} + s_1 \delta_1 + s_2 \delta_2 = 0.$$

By Proposition 4.2, (ii), we find $m = n_1 = \ldots = n_r = 0$ and $s_1 \equiv s_2 \equiv 0 \mod 2$, showing injectivity and finishing the proof.

5. **Conditions on automorphisms of X**

Let $X$ be the blow-up of $\mathbb{P}^2$ in finitely many points $p_{ij}$. The action of $\text{Aut}(X)$ on $\text{Pic}(X)$ gives us a representation

$$1 \to \ker \tau \to \text{Aut}(X) \xrightarrow{\tau} \text{Aut}(\text{Pic}(X)).$$

(5.1)
Any \( g \in \ker \tau \) sends each \((-1\)-curve onto itself. Taking such a \( g \), we find that it descends to an automorphism of \( \mathbb{P}^2 \) which fixes the points \( p_{ij} \) pointwise.

By Corollary 4.3, this situation is granted for points \( p_{ij} \), lying on a smooth cubic curve \( C \), where \( p_{i0}, \ldots, p_{r0} \) are independent and \( p_{ij} \) with \( j \neq 0 \) is associated with \( p_{i0} \). Therefore, we can view \( \text{Aut}(X) \) as a subgroup of \( \text{Aut}(\text{Pic}(X)) \), and any \( g \in \text{Aut}(X) \) by its action \( g^* \) on the Picard group

\[
\text{Pic}(X) = \mathbb{Z}[L] \oplus \bigoplus_{i,j} \mathbb{Z}[E_{ij}],
\]

where \( L \) is the strict transform of a line in \( \mathbb{P}^2 \) not passing through any of the \( p_{ij} \)'s, and the \( E_{ij} \) are the exceptional curves of \( X \).

What about \( \sigma_1, \ldots, \sigma_r \) introduced in Section 3?

**Lemma 5.1** ([Bla08, Lemma 17]). The induced action of \( \sigma_i \) on \( \text{Pic}(X) \) is given by

\[
\sigma_i^* : [L] \mapsto 3[L] - 2[E_{i0}] - [E_{i1}] - \cdots - [E_{i4}],
\]

\[
[E_{i0}] \mapsto 2[L] - [E_{i0}] - [E_{i1}] - \cdots - [E_{i4}],
\]

\[
[E_{ij}] \mapsto [L] - [E_{i0}] - [E_{ij}],
\]

\[
[E_{k\ell}] \mapsto [E_{k\ell}],
\]

for \( j \neq 0 \) and \( k \neq i \).

Now that we know both \( \text{Aut}(X) \leq \text{Aut}(\text{Pic}(X)) \) and what the action of the \( \sigma_i \) looks like on \( \text{Aut}(\text{Pic}(X)) \), we can introduce helpful properties on automorphisms \( g \) of \( X \):

(i) The induced automorphism \( g^* \) of \( \text{Pic}(X) \) preserves the intersection form, meaning that for any two divisors \([D],[D']\), we have

\[
g^*([D]) : g^*([D']) = [D] : [D'].
\]

(ii) The canonical divisor is mapped to itself under \( g^* \), meaning

\[
g^*(K_X) = K_X.
\]

With the help of the first condition, we can infer that automorphisms of \( X \) commute with the anti-regular involution \( \rho \), as long as \( X \) is the blow-up of real points.

**Lemma 5.2.** If \( X \) is the blow-up in real points of \( \mathbb{P}^2 \), then all \((-1\)-curves are real. If, in addition, out of four of these points, no three are collinear, then all automorphisms are real.

**Proof.** Recall that we denote by \( \hat{\rho} \) the real structure on \( \mathbb{P}^2 \) and by \( \rho \) the unique real structure on \( X \) such that the blow-up is real.

Let \( E \) be a \((-1\)-curve on \( X \). Write it as

\[
E \sim dL - \sum_i m_i E_i,
\]

where \( E_i \) are the exceptional curves of \( X \) and \( L \) is the pullback of a line of \( \mathbb{P}^2 \) not passing through the points blown-up. As we only blow up in real points, we have \( \rho(E_i) = E_i \), since \( \hat{\rho} \) maps a real point to itself. We can thus calculate

\[
[\rho^{-1}(E)] = \rho^*([E]) = d\rho^*([L]) - \sum_i m_i \rho^*([E_i]) = d[L] - \sum_i m_i [E_i] = [E],
\]

where we used that \( \rho([L]) = [L] \). Since \( \rho(E) \cdot E = -1 \), we get \( \rho(E) = E \).
For the second claim, consider \(\alpha \in \text{Aut}(X)\). For each \((-1)\)-curve \(E\), we have \(\rho \alpha \rho(E) = \rho \alpha(E)\), since \(\rho(E) = E\). Due to \(\alpha\) preserving the intersection form by \((\clubsuit)\), the curve \(\alpha(E)\) is also of self-intersection \(-1\). As a result, \(\rho \alpha(E) = \alpha(E)\). Hence, \(\rho \alpha \rho(E) = \alpha(E)\). Consequently, \(\alpha^{-1} \rho \alpha \rho\) maps every \((-1)\)-curve to itself; as there are four points where no three are collinear, this implies that \(\alpha^{-1} \rho \alpha \rho = \text{id}_X\), as desired. \(\square\)

Thus, in our set-up, we have to assume that all associated points \(p_{ij}\) are also real, so that for any automorphism \(\alpha \in \text{Aut}(X)\), we have \(\alpha \rho = \rho \alpha\). This will then help in the proof of Theorem 10.1.

The second condition \((\diamondsuit)\) leads to the fact that any automorphism of \(X\) must map the strict transform of the smooth cubic curve \(C\) to itself:

**Lemma 5.3.** Fix a smooth cubic curve \(C \subset \mathbb{P}^2\), and choose at least ten points lying on \(C\). Then any automorphism \(g\) of the blow-up \(X\) in those points restricts to an automorphism of the strict transform \(\tilde{C}\).

**Proof.** Every automorphism \(g \in \text{Aut}(X)\) induces an automorphism \(g^*\) of \(\text{Pic}(X)\) which satisfies \((\diamondsuit)\). Therefore, we find \([g^{-1}(\tilde{C})] = g^*([\tilde{C}]) = g^*(-K_X) = -g^*(K_X) = -K_X = [\tilde{C}]\), which implies that the two irreducible curves \(\tilde{C}\) and \(g^{-1}(\tilde{C})\) lie in the same divisor class. Furthermore, we can calculate \([\tilde{C}]\) as \([\tilde{C}] = 3[L] - \sum[E_i]\), where \(L\) is the pullback of a line in \(\mathbb{P}^2\) not passing through the points blown up, and the \(E_i\) are the exceptional curves. This implies that \([\tilde{C}]^2 = 9 - s\), where \(s \geq 10\) is the number of points blown up. So from that we obtain \(\tilde{C} \cdot g^{-1}(\tilde{C}) < 0\), so \(g^{-1}(\tilde{C}) = \tilde{C}\). Since \(\tilde{C} \cong C\) is smooth, we find that \(g|_{\tilde{C}}\) is an automorphism. \(\square\)

Note that since \(C \cong \tilde{C}\), we can view \(g|_{\tilde{C}}\) also as an automorphism of \(C\). In addition, blowing up at least ten points lying on a smooth cubic curve implies that \(\tau\) in (5.1) is injective: any nontrivial automorphism of \(X\) being mapped to the identity would fix at least ten exceptional curves and thus descend to an automorphism of \(\mathbb{P}^2\) fixing at least ten points. Since any nontrivial automorphism of \(\mathbb{P}^2\) having at least ten fixed points fixes four points lying on a line, this is a contradiction to Bézout.

### 6. Existence of Suitable Points

Consider the following set-up resulting from the discussion in the previous sections: the \(p_{ii}\)’s are assumed to be independent as defined in Definition 4.1, any \(p_{ij}\) with \(j \neq 0\) is an associated point of \(p_{i0}\) and they are all assumed to be real. As observed in Section 4, independence excludes 3-torsion elements, so none of the points can be an inflection point, and hence all have precisely four associated points. Therefore, we would like to prove the existence of points \(p_{10}, \ldots, p_{r0}\) which are real, have only real associated points, and are independent.

To achieve this, we will need to assume that the smooth cubic curve \(C\) is defined over \(\mathbb{R}\) and that the set of real points of \(C\) has two components in the Euclidean topology. This assumption is equivalent to other useful statements:

**Lemma 6.1.** Let \(C \subset \mathbb{P}^2\) be a smooth cubic curve defined over \(\mathbb{R}\) and denote by \(C(\mathbb{R})\) the real points of \(C\), then the following three statements are equivalent:

(i) The set \(C(\mathbb{R})\) has two connected components in the Euclidean topology.
(ii) For every group structure given to \(C(\mathbb{C})\) by choosing a neutral point \(p_0 \in C(\mathbb{R})\), the 2-torsion elements are all real.

(iii) There exists a real point \(p \in C(\mathbb{R})\) with four real associated points. Moreover, if the above conditions are satisfied, then a real point has at least one real associated point if and only if it has only real associated points.

Proof. We prove the implications in the following order: first, we prove the equivalence of (i) and (ii), and then that (ii) and (iii) are equivalent.

The equivalence of (i) and (ii) is well-known; we nevertheless include its proof, as we could not find a reference. First, we make the following observation: choose \(p_0 \in C\) a real inflection point and let the defining equation of \(C\) in Weierstrass form be

\[
C : Y^2Z = F(X, Z),
\]

where \(F\) is homogeneous of degree three and has real coefficients. Dehomogenise this equation at \(Z = 1\), as the only point with \(Z = 0\) is \(p_0 = [0 : 1 : 0]\) at infinity. Using the group law on \(C\), the 2-torsion points will then be precisely those points \([x : y : 1]\) with \(y = 0\), meaning points \([x : 0 : 1]\) \(\in C\) where \(F(x, 1) = 0\). Since \(C\) is smooth, this polynomial of degree three in \(x\) has three distinct solutions \(a_1, a_2, a_3\) in \(\mathbb{C}\), and we can write

\[
y^2 = F(x, 1) = \lambda(x - a_1)(x - a_2)(x - a_3), \quad (6.1)
\]

with \(\lambda \in \mathbb{R}\). Replacing \(x\) with \(\sqrt[3]{\lambda}\), we may assume \(\lambda = 1\).

We can rephrase the equivalence accordingly and instead prove that \(C(\mathbb{R})\) has two connected components in the Euclidean topology if and only if \(a_1, a_2, a_3\) are all real. As \(F(x, 1) = 0\) has real coefficients, there are either three real solutions, or one real solution and two complex conjugate solutions. So we can show the equivalence by making a full classification.

First, if all solutions are real, up to permutation of the indices, we may assume \(a_1 < a_2 < a_3\). Then \(C(\mathbb{R}) \setminus \{[0 : 1 : 0]\}\) is precisely the set of points \([x : y : 1]\) where \(F(x, 1) \geq 0\), as then \(y^2 = F(x, 1)\) has a real solution for \(y\). Hence, we can describe the set \(C(\mathbb{R})\) as

\[
C(\mathbb{R}) = \{[x : \sqrt[3]{F(x, 1)} : 1] | a_1 \leq x \leq a_2\} \cup \{[x : \sqrt[3]{F(x, 1)} : 1] | a_3 \leq x < \infty\} \cup \{[0 : 1 : 0]\}.
\]

The two components are therefore the first set and the union of the latter two sets.

Assume now that, up to permutation of the indices, \(a_1\) is real and \(a_3 = \overline{a_2}\). Since \((x - a_2)(x - \overline{a_2})\) is a polynomial with leading coefficient equal to one, we have \((x - a_2)(x - \overline{a_2}) > 0\) for all \(x \in \mathbb{R}\). So, we can write \(C(\mathbb{R})\) as the set

\[
C(\mathbb{R}) = \{[x : \sqrt[3]{F(x, 1)} : 1] | a_1 \leq x < \infty\} \cup \{[0 : 1 : 0]\},
\]

which is comprised of one component.

Now for the equivalence of (ii) and (iii), call \(p_1, \ldots, p_4\) the associated points of \(p\). We recall that due to Lemma 3.2, the group \(C[2] \subset C(\mathbb{C})\) of (complex) 2-torsion points is equal to

\[
C[2] = \{0, p_1 - p_4, p_2 - p_4, p_3 - p_4\}.
\]

So, if the four associated points \(p_1, \ldots, p_4\) of some real \(p \in C\) are real, then the 2-torsion group \(C[2]\) contains only real points. This shows that (iii) implies (ii).

Suppose (ii) holds and consider the morphism \(\varphi : C \to C\) sending a point \(q\) to the second point on \(C\) lying on the tangent to \(C\) at \(q\). Choose a point \(p \in \varphi(C(\mathbb{R}))\) which is not an
inflection point; this is possible by choosing $p$ as the image of a point other than an inflection point or a 2-torsion point. So $p$ has at least one real associated point. Suppose without loss of generality that this point is $p_4$. Since $p_i - p_4 \in C[2]$ and $C[2] \subset C(\mathbb{R})$ by (ii), the other associated points must also all be real.

As for the last assertion, if $C[2] \subset C(\mathbb{R})$ and $p$ has a real associated point, then by what just preceded, all its associated points must be real. This wraps up the proof. \hfill $\square$

The next lemma introduces an uncountable set which will be crucial in the proof of the existence of suitable points.

**Lemma 6.2.** If $C[2] \subset C(\mathbb{R})$, then the set

$$M := \{ p \in C(\mathbb{R}) \mid \forall q \in C : (p + 2q = 3p_0 \Rightarrow q \in C(\mathbb{R})) \}$$

is uncountable.

**Proof.** Consider the morphism $\varphi : C \to C$ from the previous proof sending a point $q \in C$ to the second point on $C$ lying on the tangent to $C$ at $q$. Restricting it to $C(\mathbb{R})$, we obtain a continuous map $\bar{\varphi} : C(\mathbb{R}) \to C(\mathbb{R})$.

The set $M$ is equal to $\bar{\varphi}(C(\mathbb{R}))$. Indeed, since all 2-torsion elements are real, $p$ having any real associated point is equivalent to $p$ having only real associated points by Lemma 6.1. The morphism $\varphi$ corresponds to multiplication by minus two when considering the group structure on $C$. Therefore, $\varphi$ has finite fibres, which implies the claim. \hfill $\square$

Equipped with these lemmas, we can verify the existence of suitable points.

**Proposition 6.3.** For any real smooth cubic curve $C \subset \mathbb{P}^2$ whose real points $C(\mathbb{R})$ has two components and for any $r \geq 1$ we can find points $p_{10}, \ldots, p_{r0} \in C$ such that for all $1 \leq i \leq r$,

(i) the $p_{10}$’s are independent, and

(ii) the $p_{10}$’s and their associated points are real.

**Proof.** We proceed by induction. Consider the uncountable set $M$ which describes the points of $C(\mathbb{R})$ having real associated points, as stated in Lemma 6.2.

For $r = 1$, independence is equivalent to $p_{10}$ having no torsion. Since by, for example [Sil09, Cor. III.6.4], the group of (complex) $m$-torsion points is $C[m] \cong (\mathbb{Z}/m\mathbb{Z})^2$, the points of $m$-torsion are finite. Thus, the set of torsion points

$$C_{\text{tors}} := \bigcup_{m \geq 1} C[m]$$

is countable. Since $M$ is uncountable, we can always choose a point $p_{10}$ from $M \setminus C_{\text{tors}}$ satisfying conditions (i) and (ii). Note that this also excludes the inflection points, as they are precisely the 3-torsion points.

We assume that the proposition holds for $r - 1 \geq 0$. Suppose we have points $p_{10}, \ldots, p_{r-1,0}$ satisfying (i) and (ii). The set

$$M_r := M \setminus \{ p_{ij} \mid 1 \leq i \leq r - 1, 1 \leq j \leq 4 \}$$

is still uncountable, as we only remove $5(r - 1)$ points.
Now, we would like to remove the points \( q \in M_r \) for which there exist \((m, n_1, \ldots, n_{r-1}) \in \mathbb{Z}^r \setminus \{(0, \ldots, 0)\}\) such that

\[
m(q - p_0) + n_1(p_{10} - p_0) + \cdots + n_{r-1}(p_{r-1,0} - p_0) = 0.
\] (6.2)

Consider the morphism

\[\zeta_m : C \to C, \quad q \mapsto mq,\]

where \(mq\) is the unique point corresponding to \(m(q - p_0) \in \text{Pic}^0(C)\). This morphism has finite fibres, which we can see by working on the complex torus analytically isomorphic to \(C\) and using \(C \cong \mathbb{C}^2 / (\mathbb{Z}/m\mathbb{Z})^2\). Denote by

\[B_m := \zeta_m^{-1}\{\text{point corresponding to } -n_1(p_{10} - p_0) - \cdots - n_{r-1}(p_{r-1,0} - p_0) \mid (n_1, \ldots, n_{r-1}) \in \mathbb{Z}^r\},\]

a countable subset of \(C\). Therefore,

\[B := \bigcup_{m \in \mathbb{Z}} B_m\]

is also countable.

Then \(M_r \setminus B\) is still uncountably infinite, and we may choose \(p_{r0} \in M_r \setminus B\). This proves the induction and therefore the proposition. \(\square\)

7. Map from \(\text{Pic}(X)\) to \(\text{Pic}(C)\)

We would like to use the group structure on \(C\) to find further conditions on automorphisms of \(X\). For this, we take a step back from all the assumptions made so far: let \(X\) be a smooth projective surface and let \(C, D\) be two curves on \(X\) having no common irreducible component.

The intersection multiplicity of \(C\) and \(D\) at a point \(p \in X\) is defined as

\[i_p(C, D) := \dim \left( \mathcal{O}_{X,p} / (f, g) \right),\]

where \(f\) and \(g\) are local equations for \(C\) and \(D\) at \(p\).

Fix the irreducible curve \(C\) and consider the following map for irreducible curves \(D \subset X\) which are not equal to \(C\):

\[[D] \mapsto \sum_{p \in C \cap D} i_p(C, D)p.\] (7.1)

Considering divisors in \(\text{Pic}(X)\) as line bundles, we see that the above map extends to a unique \(\mathbb{Z}\)-linear map \(\Phi : \text{Pic}(X) \to \text{Pic}(\tilde{C})\).

Suppose now that \(\pi : X \to \mathbb{P}^2\) is the blow-up in \(5r \geq 10\) points \(p_{10}, \ldots, p_{r4}\) lying on a smooth cubic curve \(C\) chosen such that the \(p_{i0}\)’s are independent, and \(p_{i1}, \ldots, p_{i4}\) are associated points of \(p_{i0}\). Then we can write the map as

\[\Phi : \text{Pic}(X) \to \text{Pic}(\tilde{C}),\]

\[d[L] - \sum_{i,j} m_{ij}[E_{ij}] \mapsto 3dp_0 - \sum_{i,j} m_{ij}p_{ij},\] (7.2)

where \(\tilde{C}\) is the strict transform of the cubic curve, \(L\) is the strict transform of a general line in \(\mathbb{P}^2\), and the \(E_{ij} := \pi^{-1}(p_{ij})\) are the exceptional curves. As observed in Lemma 5.3, any automorphism \(g \in \text{Aut}(X)\) restricts to an automorphism of \(\tilde{C}\), and we therefore obtain a commutative diagram.
Note that due to \( \tilde{C} \cong C \), we can replace \( \text{Pic}(\tilde{C}) \) by \( \text{Pic}(C) \).

Furthermore, we may connect \( \Phi(\text{Pic}(X)) \) with what we found in Proposition 4.2:

**Lemma 7.1.** Given \( r \geq 2 \) independent points \( p_{10}, \ldots, p_{r} \) lying on a smooth cubic curve \( C \subset \mathbb{P}^2 \), blow up these points and their associated points to obtain a rational surface \( X \). Consider the map \( \Phi \) given in (7.2). Then \( \Phi(\text{Pic}(X)) = \langle p_{10}, \ldots, p_{r} \rangle \).

**Proof.** Since \( \Phi(\text{Pic}(X)) = \langle 3p_0, p_{10}, \ldots, p_{r} \rangle \), we need to prove that \( 3p_0 \in \langle p_{10}, \ldots, p_{r} \rangle \). This is due to \( 3p_0 = p_{10} + 2p_{14} \), as in Proposition 4.2, (i). \( \square \)

### 8. Description of induced automorphisms on \( \text{Pic}(C) \)

Before reaping the benefits of the set-up of our points, it is worth analysing how an automorphism of a smooth cubic curve induces an automorphism of the Picard group of that curve.

**Lemma 8.1.** Fix a smooth cubic curve \( C \subset \mathbb{P}^2 \) with \( \text{Aut}_{\text{sp}}(C) \cong \mathbb{Z}/2\mathbb{Z} \), and choose an inflection point \( p_0 \in C \). Any automorphism \( h \) of \( C \) may be given by

\[
h : C \to C, \quad p \mapsto ap + b,
\]

with \( a = \pm 1 \) and \( b \in C \). Then the induced automorphism \( h^* \) of \( \text{Pic}(C) \) is given by

\[
h^* : \text{Pic}(C) \to \text{Pic}(C), \quad D \mapsto aD + \deg(D)B,
\]

where \( B = b - p_0 \) if \( a = 1 \), or \( B = b + p_0 \) if \( a = -1 \), for a fixed inflection point \( p_0 \).

**Proof.** The assumption on \( \text{Aut}_{\text{sp}}(C) \) implies that every automorphism \( h \) of \( C \) is of the form \( p \mapsto ap + b \) for some \( a \in \{ \pm 1 \} \) and some \( b \in C \). Such an automorphism \( h \) induces an automorphism

\[
h_* : \text{Pic}^0(C) \to \text{Pic}^0(C), \quad (p - p_0) \mapsto a(p - p_0) + (b - p_0).
\]

We first consider \( a = -1 \) and \( (b - p_0) = 0 \). Then \( (p - p_0) \) is mapped to \( h_*(p - p_0) = (h(p) - p_0) \) which satisfies

\[
(p - p_0) + (h(p) - p_0) = 0.
\]

This implies, in \( \text{Pic}(C) \), that \( h(p) = -p + 2p_0 \), and therefore that the automorphism \( h : C \to C, \quad p \mapsto -p \) induces the automorphism

\[
h^* : \text{Pic}(C) \to \text{Pic}(C), \quad D \mapsto -D + 2\deg(D)p_0.
\]
We can analyse a pure translation, meaning \((p - p_0) \mapsto (p - p_0) + (b - p_0)\), in the same manner; we determine the point \(h(p)\) for which \((h(p) - p_0) = h_*(p - p_0) = (p - p_0) + (b - p_0)\), which must be

\[ h(p) = p + b - p_0. \]

We therefore obtain the induced automorphism

\[ h^* : \text{Pic}(C) \to \text{Pic}(C), \]

\[ D \mapsto D + \deg(D)(b - p_0). \]

This covers \(a = 1\). We can obtain the case \(a = -1\) by composing the inversion with a translation. \(\Box\)

Note that the condition \(\text{Aut}_{\text{gp}}(C) \cong \mathbb{Z}/2\mathbb{Z}\) is satisfied for a general smooth cubic curve \(C \subset \mathbb{P}^2\).

The following lemma is a technical result which will be useful in the proof of the theorem thereafter. Its proof is dependent on condition \((\spadesuit)\), meaning on automorphisms preserving the intersection form.

**Lemma 8.2.** Given an automorphism \(g \in \text{Aut}(X)\), denote the corresponding matrix by \(G\) and write \(G_0\) for the first column and \(G_{ij}\) for the column corresponding to the image of \([E_{ij}]\). For any row vector \(v := (a_0, a_{10}, \ldots, a_{r4}) \in \mathbb{Z}^{5r+1}\), we have

\[ a_0[L] - \sum_{i,j} a_{ij}[E_{ij}] = g^*\left( (v \cdot G_0)[L] - \sum_{i,j} (v \cdot G_{ij})[E_{ij}] \right), \tag{8.1} \]

and therefore,

\[ a_0^2 - \sum_{i,j} a_{ij}^2 = (v \cdot G_0)^2 - \sum_{i,j} (v \cdot G_{ij})^2. \tag{8.2} \]

**Proof.** Write

\[ Q := \begin{pmatrix} 1 & & & & \\ -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & 1 & \end{pmatrix} \tag{8.3} \]

for the matrix associated to the intersection form on \(\text{Pic}(X)\), where

\[ [L]^2 = 1, \quad [E_{ij}]^2 = -1, \quad [L] \cdot [E_{ij}] = 0, \quad [E_{ij}] \cdot [E_{k\ell}] = 0 \text{ for } (i, j) \neq (k, \ell). \]

We can express the fact that \(g^*\) preserves the intersection form — as stated in \((\spadesuit)\) — using \(G\) and \(Q\), namely,

\[ Q = GQG^T, \]

where \(G^T\) denotes the transpose of \(G\). Therefore, we have

\[ Qv^T = GQG^Tv^T = GQ(v \cdot G_0, v \cdot G_{10}, \ldots, v \cdot G_{r4})^T = G(v \cdot G_0, -v \cdot G_{10}, \ldots, -v \cdot G_{r4})^T. \]

This implies (8.1). Equation (8.2) follows from (8.1) by taking the self-intersection, and using the fact that any automorphism preserves the intersection form. \(\Box\)
The next theorem is right at the intersection of the geometric and arithmetic world, and will prove crucial in securing inequivalency between the $\sigma_1, \ldots, \sigma_r$.

**Theorem 8.3.** Consider a smooth cubic curve $C \subset \mathbb{P}^2$ with $Aut_{sp}(C) \cong \mathbb{Z}/2\mathbb{Z}$ and let $r \geq 3$. Choose independent points $p_{10}, \ldots, p_{r0} \in C$ and blow them and their associated points up. Call the resulting rational surface $X$. Then, for each $g \in Aut(X)$, the restricted map $g|_{\tilde{C}}$ is a translation by a 2-torsion element or the identity.

**Proof.** Let $g \in Aut(X)$. Then $g$ induces an automorphism of $Pic(X)$, which we may write as

$$g^*([L]) = d[L] - \sum_{i,j} m_{ij}[E_{ij}],$$

$$g^*([E_{kl}]) = n_{kl}[L] - \sum_{i,j} e_{ij}^{k\ell}[E_{ij}],$$

where $L$ is the strict transform of a general line not passing through any of the $p_{ij}$’s, and the $E_{ij}$ are the exceptional curves.

As $r \geq 3$, we have $g(\tilde{C}) = \tilde{C}$ by Lemma 5.3, where $\tilde{C}$ is the strict transform of $C$ in $X$. Using the induced automorphism $(g|_{\tilde{C}})^*$ on $Pic(\tilde{C}) \cong Pic(C)$, we find, thanks to Lemma 8.1, elements $a = \pm 1$ and $B \in Pic(C)$ such that

$$3dp_{10} - \sum_{i,j} m_{ij}p_{ij} = 3ap_{10} + 3B,$$  \hspace{1cm} (8.4)

$$3n_{kl}p_{10} - \sum_{i,j} e_{ij}^{k\ell}p_{ij} = ap_{kl} + B.$$  \hspace{1cm} (8.5)

The aim is to prove $a = 1$ and $B \in C[2]$. Denote the matrix corresponding to $g^*$ by $G$, with columns $G_0$, $G_{ij}$ according to the basis $[L]$, $[E_{ij}]$.

Recall that by Proposition 4.4, we have an isomorphism $\langle p_{10}, \ldots, p_{r4} \rangle \cong \mathbb{Z}^{r+1} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ with basis $p_{10}$, $p_{14}$, $p_{24}$, $\ldots$, $p_{r4}$; and furthermore, thanks to Lemma 7.1, $\Phi(Pic(X)) = \langle p_{10}, \ldots, p_{r4} \rangle$.

Writing (8.4) and (8.5) in this basis gives us more information. We would like to write $B$ in this basis as well. This is possible since $(g|_{\tilde{C}})^*$ sends $\Phi(Pic(X))$ to $\Phi(Pic(X))$, and therefore, for any $p \in \langle p_{10}, \ldots, p_{r4} \rangle$, we have $ap + B \in \langle p_{10}, \ldots, p_{r4} \rangle$, implying $B \in \langle p_{10}, \ldots, p_{r4} \rangle$. Therefore, $B$ can be expressed as $B = (m, n_1, \ldots, n_r, s_1, s_2)$.

We then use the relations in $\langle p_{10}, \ldots, p_{r4} \rangle$ of Proposition 4.2, (i), to write the left hand side of (8.5) as

$$3n_{kl}p_{10} - \sum_{i,j} e_{ij}^{k\ell}p_{ij} = n_{kl}(p_{10} + 2p_{14}) - e_{10}^{k\ell}p_{10} - \sum_{i \geq 2} e_{i0}^{k\ell}(p_{10} + 2p_{14} - 2p_{i4}) - \sum_{i} e_{i1}^{k\ell}(\delta_1 + p_{i4}) - \sum_{i,j} e_{ij}^{k\ell}(\delta_2 + p_{i4}) - \sum_{i} e_{i3}^{k\ell}(\delta_1 + \delta_2 + p_{i4}) - \sum_{i} e_{i4}^{k\ell}p_{i4}.$$  

Next, we regroup and collect the terms to obtain

$$3n_{kl}p_{10} - \sum_{i,j} e_{ij}^{k\ell}p_{ij} = \left( n_{kl} - \sum_{i} e_{i0}^{k\ell} \right)p_{10} + \left( 2n_{kl} - \sum_{i \geq 2} e_{i0}^{k\ell} - \sum_{j \neq 0} e_{ij}^{k\ell} \right)p_{14} + \left( 2e_{k20} - \sum_{j \neq 0} e_{j0}^{k\ell} \right)p_{24} + \cdots + \left( 2e_{r0} - \sum_{j \neq 0} e_{rj}^{k\ell} \right)p_{r4} + \left( - \sum_{i} e_{i1}^{k\ell} - \sum_{i} e_{i2}^{k\ell} \right)\delta_1 + \left( - \sum_{i} e_{i2}^{k\ell} - \sum_{i} e_{i3}^{k\ell} \right)\delta_2.$$
Therefore, (8.5) can be rewritten as:

\[
\begin{pmatrix}
2n_k \ell - \sum_i e_i^k t_i \\
2n_k \ell - \sum_{i \geq 2} 2 e_i^k \ell - \sum_{j \neq 0} e_i^j \ell j \\
2e_{20}^k - \sum_{j \neq 0} e_{2j}^k \\
\vdots \\
2e_{r0}^k - \sum_{j \neq 0} e_{rj}^k \\
-\sum_i (e_i^{k1} + e_i^{k3}) \\
-\sum_i (e_i^{k2} + e_i^{k3})
\end{pmatrix}
= a_{k\ell} + \begin{pmatrix} m \\ n_1 \\ n_2 \end{pmatrix}, \tag{8.6}
\]

where \( a_{k\ell} \) is the vector corresponding to \( p_{k\ell} \) in the basis \( p_{10}, p_{14}, \ldots, p_{r4}, \delta_1, \delta_2 \).

Similarly, for (8.4),

\[
\begin{pmatrix}
d - \sum_i m_i^0 \\
2d - 2\sum_{i \geq 2} m_i^0 - \sum_{j \neq 0} m_{ij} \\
2m_{20} - \sum_{j \neq 0} m_{2j} \\
\vdots \\
2m_{r0} - \sum_{j \neq 0} m_{rj} \\
-\sum_i (m_i^1 + m_i^3) \\
-\sum_i (m_i^2 + m_i^3)
\end{pmatrix}
= a_{l0} + 3 \begin{pmatrix} m \\ n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} a + 3m \\ 2a + 3n_1 \\ 3n_2 \end{pmatrix}, \tag{8.7}
\]

and here, \( a_{l0} \) is the vector corresponding to \( 3p_0 = p_{10} + 2p_{14} \).

We now use Lemma 8.2: to show that \( m = 0 \), consider the row vector \( v = (v_0, v_{10}, \ldots, v_{r4}) \) with entries in the basis \( 3p_0, p_{ij} \) for \( 1 \leq i \leq r, 0 \leq j \leq 4 \) given by

\[
v_0 := 1, \quad v_{ij} := \begin{cases}
1 & j = 0, \\
0 & j \neq 0.
\end{cases}
\]

Then, applying (8.2) of Lemma 8.2, we obtain

\[
1 - r = (v \cdot G_0)^2 - \sum_{k, \ell} (v \cdot G_{k\ell})^2.
\]

Note that

\[
v \cdot G_0 = d - \sum_i m_i^0 \overset{(8.7)}{=} a + 3m,
\]

\[
v \cdot G_{k\ell} = n_{k\ell} - \sum_i e_i^{k0} \overset{(8.6)}{=} \begin{cases}
m & \ell \neq 0, \\
am + m & \ell = 0.
\end{cases}
\]

Hence, we get

\[
1 - r = (a + 3m)^2 - r(a + m)^2 - 4rm^2 = (1 - r)a^2 + 2(3 - r)am + (9 - 5r)m^2.
\]

As \( a^2 = 1 \), we can deduce

\[
m(2(3 - r)a + (9 - 5r)m) = 0, \quad (8.8)
\]
and therefore $m = 0$ or $2(3 - r)a + (9 - 5r)m = 0$.

If $r = 3$, then $2(3 - r)a + (9 - 5r)m = 0$ has the solution $m = 0$, as desired. If $r \geq 4$, we note that $0 < |m| = \frac{2(r-3)}{5r-9} < 1$, so we cannot find another integer solution. Therefore, for any $r \geq 3$, we found $m = 0$.

As for $n_1$, we follow the same structure of argument as for $m$, but with the vector $w$ given by

$$w_0 := 2, \quad w_{ij} := \begin{cases} 1 & i = 1, j \neq 0, \\ 2 & i \geq 2, j = 0, \\ 0 & \text{else.} \end{cases}$$

Observing

$$w \cdot G_0 = 2d - 2\sum_{i \geq 2} m_{i0} - \sum_{j \neq 0} m_{1j} \overset{(8.7)}{=} 2a + 3n_1,$$

$$w \cdot G_{kl} = 2n_{kl} - 2\sum_{i \geq 2} c_{i0}^{kl} - \sum_{j \neq 0} c_{1j}^{kl} \overset{(8.6)}{=} \begin{cases} a + n_1 & k = 1, \ell \neq 0, \\ 2a + n_1 & k \geq 2, \ell = 0, \\ n_1 & \text{else,} \end{cases}$$

we can once more use Lemma 8.2 to find

$$4 - 4 - 4(r - 1) = (2a + 3n_1)^2 - 4(a + n_1)^2 - (r - 1)(2a + n_1)^2 - (4r - 3)n_1^2 = -4a^2(r - 1) - 4(r - 2)an_1 + (9 - 5r)n_1^2.$$

Again using $a^2 = 1$, we see that

$$n_1(-4(r - 2)a + (9 - 5r)n_1) = 0.$$ 

The second factor in the above equation cannot have an integer solution for $n_1$ if $r \geq 3$, as in this case, $0 < |n_1| = \frac{4(r-2)}{5r-9} < 1$. Therefore, $n_1 = 0$.

Next, we analyse $n_s$ for $2 \leq s \leq r$. We choose $u_s$ to be the vector with entries

$$(u_s)_0 := 0, \quad (u_s)_{ij} := \begin{cases} -2 & i = s, j = 0, \\ 1 & i = s, j \neq 0, \\ 0 & \text{else.} \end{cases}$$

Once again, we apply Lemma 8.2 using

$$u_s \cdot G_0 = 2m_{s0} - \sum_{j \neq 0} m_{sj} \overset{(8.7)}{=} 3n_s,$$

$$u_s \cdot G_{kl} = 2c_{s0}^{kl} - \sum_{j \neq 0} c_{sj}^{kl} \overset{(8.6)}{=} \begin{cases} -2a + n_s & k = s, \ell = 0, \\ a + n_s & k = s, \ell \neq 0, \\ n_s & \text{else.} \end{cases}$$

It implies

$$-8 = 9n_s^2 - 5(r - 1)n_s^2 - 4(a + n_s)^2 - (-2a + n_s)^2$$

$$= -8a^2 - 4an_s - (5r - 9)n_s^2,$$
and with \( a^2 = 1 \), we get

\[
ns(4a + (5r - 9)ns) = 0. \tag{8.9}
\]

One solution is \( ns = 0 \), so we need to determine the possible solutions of \( 4a + (5r - 9)ns = 0 \). Any such solution satisfies \( 0 < |ns| = \frac{1}{5r-9} < 1 \) if \( r \geq 3 \). Therefore, the only integer solution is \( ns = 0 \).

Since \( m = n_1 = \ldots = n_r = 0 \), we find \( B = s_1\delta_1 + s_2\delta_2 \in C[2] \). Furthermore, fixing \( k = 1 \), \( \ell = 0 \) and summing up the first \( r + 1 \) entries of \( (8.6) \), we find, as \( \vec{L}_{10} = (1, 0, \ldots, 0)^T \),

\[
3n_{10} - \sum e_{ij}^{10} = a + m + n_1 + \cdots + n_r = a.
\]

Thanks to Lemma 5.3, we get \( g(\tilde{C}) = \tilde{C} \); therefore, using that \( g^* \) preserves the intersection form, we calculate

\[
3n_{10} - \sum e_{ij}^{10} = [\tilde{C}] \cdot g^*([E_{10}]) = g^*([\tilde{C}]) \cdot g^*([E_{10}]) = [\tilde{C}] \cdot [E_{10}] = 1.
\]

We deduce \( a = 1 \), which finishes the proof. \( \square \)

9. Equations on the coefficients

We saw in Theorem 8.3 that we can use the information given by the restriction of an automorphism of \( X \) to an automorphism of the smooth cubic curve \( C \). Using this, we can determine conditions on the coefficients:

**Proposition 9.1.** Consider the blow-up \( X \) of \( \mathbb{P}^2 \) in the points \( p_{ij} \) lying on a smooth cubic curve \( C \), where \( p_{10}, \ldots, p_{r0} \) with \( r \geq 3 \) are independent and if \( j \neq 0 \), then \( p_{ij} \) is associated with \( p_{i0} \). Let \( g \) be an automorphism of \( X \), given by

\[
g^*([L]) = d[L] - \sum_{i,j} m_{ij}[E_{ij}],
\]

\[
g^*([E_{kl}]) = n_{kl}[L] - \sum_{i,j} e_{ij}^{kl}[E_{ij}],
\]

with \( E_{ij} \) the exceptional curves of \( X \) and \( L \) the strict transform of a line in \( \mathbb{P}^2 \) not passing through any of the points \( p_{ij} \). Suppose that the restriction of \( g \) to \( \tilde{C} \) is the identity or a translation by a 2-torsion element. Then, we have the following conditions on the coefficients:

\[
2e_{ij}^0 - 1 = e_{ij}^{i1} + \cdots + e_{ij}^{i4}, \quad 1 \leq i \leq r, \ j \neq 0.
\]

**Proof.** We work with the inverse \( g^{-1} \), for which we know that \( G^{-1} = QG^TQ \), with \( Q \) as given in (8.3) and \( G \) the matrix corresponding to \( g \). Therefore, we deduce that

\[
(g^{-1})^*([L]) = d[L] - \sum_{k,l} n_{kl}[E_{kl}],
\]

\[
(g^{-1})^*([E_{ij}]) = m_{ij}[L] - \sum_{k,l} e_{ij}^{kl}[E_{kl}].
\]
Since \((g^{-1})|_{\mathcal{C}}\) is a translation by a 2-torsion element, we can write, as in the proof of Theorem 8.3,
\[
\begin{pmatrix}
  m_{ij} - \sum_k \epsilon_{ij}^{k0} \\
  2m_{ij} - \sum_{k \geq 2} 2\epsilon_{ij}^{k0} - \sum_{\ell \neq 0} \epsilon_{ij}^{1\ell} \\
  2\epsilon_{ij}^{20} - \sum_{\ell \neq 0} \epsilon_{ij}^{2\ell} \\
  \vdots \\
  2\epsilon_{ij}^{00} - \sum_{\ell \neq 0} \epsilon_{ij}^{\ell} \\
  -\sum_k (\epsilon_{ij}^{k1} + \epsilon_{ij}^{k3}) \\
  -\sum_k (\epsilon_{ij}^{k2} + \epsilon_{ij}^{k3})
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ s_1 \\ s_2 \end{pmatrix},
\]
where \(L_j\) is the vector corresponding to \(p_{ij}\) in the basis \(p_{10}, p_{14}, \ldots, p_{r}\), \(\delta_1, \delta_2\), and \(B = s_1 \delta_1 + s_2 \delta_2\). Consider \(i = 2\) and \(j \neq 0\). Then \(L_j = e_{i+1} + f_{ij}\), where \(f_{ij}\) is a vector with zero entries except maybe in the last two spots, meaning \(f_{ij}\) corresponds to an element in \(C[2]\). From that, we deduce
\[
2\epsilon_{ij}^{00} = \epsilon_{ij}^{11} + \cdots + \epsilon_{ij}^{14} + 1.
\]
For \(i = 1\) and \(j \neq 0\), we analyse the first two entries together. Since \(L_j = e_2 + f_{1j}\), with again \(f_{1j}\) a vector corresponding to an element in \(C[2]\), we see that
\[
\begin{align*}
  m_{1j} - \sum_k \epsilon_{1j}^{k0} &= 0, \\
  2m_{1j} - \sum_{k \geq 2} 2\epsilon_{1j}^{k0} - \sum_{\ell \neq 0} \epsilon_{1j}^{1\ell} &= 1.
\end{align*}
\]
We can subtract the double of (9.1) from (9.2), which implies the desired equation, and hence concludes the proof.

10. AT LEAST \(r\) REAL FORMS

We are finally able to prove that given our setting, the \(\sigma_i\) are inequivalent, which is the last step needed to prove Theorem 1.2:

**Theorem 10.1.** Fix \(r \geq 3\) and a real smooth cubic curve \(C \subset \mathbb{P}^2\) whose set of real points \(C(\mathbb{R})\) has two connected components in the Euclidean topology, and which satisfies \(\text{Aut}_{sp}(C) \cong \mathbb{Z}/2\mathbb{Z}\). Consider points \(p_{ij} \in C\), \(1 \leq i \leq r\), \(0 \leq j \leq 4\), such that the points are all real, the \(p_{ij}\) for a fixed \(i\) and \(1 \leq j \leq 4\) are associated to \(p_{0}\), and the points \(p_{10}, \ldots, p_{r}\) are independent. The blow-up \(X\) of \(\mathbb{P}^2\) in the points \(p_{ij}\) has automorphisms \(\sigma_1, \ldots, \sigma_r\) arising from the cubic involutions centred at \(p_{10}, \ldots, p_{r}\). If \(i \neq j\), then the anti-regular automorphisms \(\sigma_i \rho\) and \(\sigma_j \rho\) are not equivalent, meaning there does not exist an automorphism \(\alpha \in \text{Aut}(X)\) such that \(\sigma_i \rho \alpha = \alpha \sigma_j \rho\).

**Proof.** The existence of points \(p_{ij}\) which are real is granted by Proposition 6.3 and by \(C(\mathbb{R})\) having two connected components in the Euclidean topology. Then, thanks to Lemma 5.2 and all the points being real, we have \(\alpha \rho = \rho \alpha\) for any automorphism \(\alpha \in \text{Aut}(X)\). Therefore, we can show instead that there does not exist \(\alpha \in \text{Aut}(X)\) such that \(\sigma_i \alpha = \alpha \sigma_j\).
We may assume without loss of generality that $i = 1, j = 2$. Suppose by contradiction that there exists some automorphism $\alpha$ such that $\sigma_1 \alpha = \alpha \sigma_2$, where $\alpha$ is given by

$$\alpha^*([L]) = d[L] - \sum_{i,j} m_{ij}[E_{ij}],$$

$$\alpha^*([E_{k\ell}]) = n_{k\ell}[L] - \sum_{i,j} e^{k\ell}_{ij}[E_{k\ell}],$$

where the $E_{ij}$ are the exceptional curves and $L$ is the pullback of a line in $\mathbb{P}^2$ not passing through any of the $p_{ij}$’s. We consider the image of $[L]$ under $\sigma_1^* \alpha^*$ and $\alpha^* \sigma_2^*$ and compare coefficients; in fact, the coefficient of $[E_{21}]$ will be enough. In these calculations, we will use both the action of $\alpha$ on $\text{Pic}(X)$ and Lemma 5.1. Now, on the one hand:

$$\sigma_1^* \alpha^*([L]) = \sigma_1^* (d[L] - \sum_{i,j} m_{ij}[E_{ij}])$$

$$= -m_{21}[E_{21}] + \text{other terms.}$$

On the other hand:

$$\alpha^* \sigma_2^*([L]) = \alpha^* (3[L] - 2[E_{20}] - [E_{21}] - \cdots - [E_{24}])$$

$$= -(3m_{21} - 2e_{21}^{20} - e_{21}^{21} - \cdots - e_{21}^{24})[E_{21}] + \text{other terms.}$$

We can now equate the coefficients

$$-(3m_{21} - 2e_{21}^{20} - e_{21}^{21} - \cdots - e_{21}^{24}) = -m_{21}.$$ 

To this equation we can, thanks to $\text{Aut}_{\text{gp}}(C) \cong \mathbb{Z}/2\mathbb{Z}$ and in turn Theorem 8.3, apply Proposition 9.1 and obtain

$$2m_{21} - 4e_{21}^{20} + 1 = 0.$$ 

But this is a contradiction modulo 2, as the coefficients can only take integer values. This completes the proof. □

The following theorem ascertains the existence of a smooth projective rational surface as stated in Theorem 1.2.

**Theorem 10.2.** Consider a real smooth cubic curve $C \subset \mathbb{P}^2$ with $\text{Aut}_{\text{gp}}(C) \cong \mathbb{Z}/2\mathbb{Z}$ and whose set of real points $C(\mathbb{R})$ has two connected components in the Euclidean topology. Fix $r \geq 3$. Let $p_{ij}, 1 \leq i \leq r, 0 \leq j \leq 4$, be points on $C$ satisfying the conditions:

(i) The $p_{10}, \ldots, p_{r0}$ are independent,

(ii) for a fixed $1 \leq i \leq r$, the points $p_{i1}, \ldots, p_{i4}$ are associated with $p_{i0}$,

(iii) and all the $p_{ij}$ are real.

Call $X$ the blow-up of $\mathbb{P}^2$ in the points $p_{ij}$. Then $X$ has at least $r$ real forms, namely the ones corresponding to the inequivalent cocycles $\sigma_1, \ldots, \sigma_r$, the lifts of cubic involutions centred at $p_{10}, \ldots, p_{r0}$.

**Proof.** The existence of points satisfying (i), (ii) and (iii) is given by Proposition 6.3. We can therefore consider the rational surface $X$ — the blow-up of $\mathbb{P}^2$ in the points $p_{ij}$ — and automorphisms $\sigma_1, \ldots, \sigma_r$ on $X$, which are cocycles by Proposition 3.3.

These automorphisms $\sigma_1, \ldots, \sigma_r$ are pairwise inequivalent by Theorem 10.1. The first cohomology set $H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), \text{Aut}(X))$ therefore contains at least the equivalence classes of these $r$
cocycles. From this, we find by Theorem 2.3 that the rational surface \(X\) we constructed has at least \(r\) real structures, and therefore \(r\) real forms, by Theorem 2.1, concluding the proof. □

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