MIN-MAX SOLUTIONS FOR PARAMETRIC CONTINUOUS STATIC GAME UNDER ROUGHNESS
(PARAMETERS IN THE COST FUNCTION AND FEASIBLE REGION IS A ROUGH SET)

Yousria A. Aboelnaga
Higher Technological Institute,
Tenth of Ramadan City, 44629, Egypt
yousria_naga@yahoo.com

Mai F. Zidan
Faculty of Engineering, Tanta University,
Al-Geish St., Tanta, 31512, Egypt
mai.fathi@f-eng.tanta.edu.eg

Abstract: Any simple perturbation in a part of the game whether in the cost function and/or conditions is a big problem because it will require a game re-solution to obtain the perturbed optimal solution. This is a waste of time because there are methods required several steps to obtain the optimal solution, then at the end we may find that there is no solution. Therefore, it was necessary to find a method to ensure that the game optimal solution exists in the case of a change in the game data. This is the aim of this paper. We first provided a continuous static game rough treatment with Min-Max solutions, then a parametric study for the processing game and called a parametric rough continuous static game (PRCSG). In a Parametric study, a solution approach is provided based on the parameter existence in the cost function that reflects the perturbation that may occur to it to determine the parameter range in which the optimal solution point keeps in the surely region that is called the stability set of the 1st kind. Also the sets of possible upper and lower stability to which the optimal solution belongs are characterized. Finally, numerical examples are given to clarify the solution algorithm.

Keywords: Continuous static game, Rough programming, Non-linear programming, Rough set theory, Parametric linear programming, Parametric non-linear programming.

1. Introduction

Rough programming is introduced in [7, 8, 13, 19]. It is classified into three classes based on the roughness detected [13]; 1st class: the roughness exists in a feasible set with crisp objective function; 2nd class: the roughness exists in the objective function with a crisp feasible set; 3rd class: the roughness exists in both feasible set and objective function. Therefore, the nonlinear programming problem [2, 3, 11] with a rough representation is called a rough nonlinear programming problem (RNPP) and can be defined by roughness, which may be in constraints and/or objective functions. The RNPP has two solution sets: possibly and surely feasible optimal solution sets in the 1st class. Parametric optimization [6] is a helpful universal tool that has recently got many applications in process engineering systems [1, 7, 14, 20]. Given a constrained optimization problem, including a set of variables and bounded parameters, it provides the optimal value of the variables as an explicit function of the parameters without completely enumerating the entire space of the parameters. This means that if the value of the parameter changes within the given bounds, the optimal solution can
be obtained by a simple evaluation of the explicit function without having to resolve an optimization problem. This is a powerful result that has been successfully applied in multiobjective optimization [10, 15], stochastic optimization [4, 16], flexibility analysis, hybrid, and robust model-based control [5, 17]. This paper investigates the analysis of basic concepts in a parametric rough continuous static game when parameters are in the cost functions and roughness is in the constraints which are not discussed before. A survey of rough optimality, rough set theory (RST), and rough functions is necessary for parametric studying of RNPP.

This article is structured as follows: Section 2 introduces Min-Max solutions for the rough continuous static game of 1st class with an illustrative example to explain the solution steps. In Section 3, the Rough Continuous Static game parametric study when roughness exists in the constraints and parameters in the objective functions is discussed with a numerical example to clear up the presented concepts. Section 4 covers the conclusion and future studies.

2. 1st class of Rough Continuous Static game (RCSG)

The continuous static game (CSG) [9, 17, 18] is defined as an optimization problem where each player in the game controls a specified subset of the system parameters and seeks to minimize his own cost criterion subject to specified constraints. In these games control notation is used and each player \( i = 1, \ldots, r \) selects his control vector \( u^i \in E^s \) seeking to minimize a scalar valued criterion

\[
G_i(x, u),
\]

subject to \( n \) equality constrains

\[
g(x, u) = 0,
\]

where \( x \in E^s \) is the state and

\[
u = (u^1, \ldots, u^r) \in E^s, \quad s = s_1 + \ldots + s_r
\]

is the composite control. The composite control is required to be an element of a regular control constraint set \( \Omega \subseteq E^s \) of the form:

\[
\Omega = \{ u \in E^s | h(x, u) \geq 0 \},
\]

where \( x = \zeta(u) \) is the solution of (2.2) given \( u \). The functions

\[
G_i(x, u) : E^n \ast E^s \rightarrow E^1, \quad g(x, u) : E^n \ast E^s \rightarrow E^n, \quad h(x, u) : E^n \rightarrow E^s \rightarrow E^q
\]

are assumed to be \( e^1 \), with

\[
\left| \frac{\partial g(x, u)}{\partial x} \right| \neq 0
\]

in a ball around a solution point \( (x, u) \). The above problem can be written as:

\[
\min G_i(x, u),
\]

S.T.

\[
M = \{ x \in E^n, u \in E^s | g(x, u) = 0, h(x, u) \geq 0 \},
\]

where \( G_i \) is called the cost function for each player \( i = 1, \ldots, r \), and \( M \) is called the feasible set of the problem.
Definition 1. Let $A(U, R)$ be an approximation space, $U$ be the universe and $R$ be an equivalence relation on $U$. Let $M$ be a subset of $U$, i.e. $M \subseteq U$. Then, the 1st class of RCSG can be defined as follows:

$$\min G_i(x, u),$$
$$S.T.$$
$$M_* \subseteq M \subseteq M^*,$$

where $M_*, M^*$ are lower and upper approximation of the feasible set, respectively. The solution sets of the 1st class of RCSG are called a surely optimal solution set and a possibly optimal solution set. To find such solutions, first solve the following problem:

$$\min G_i(x, u),$$
$$S.T.$$
$$m \in M^*.$$

The optimal solution set of the above game is

$$O = \{ \tilde{m} \in M^* | G_i(x, \tilde{m}) = \min_{m \in M^*} G_i(x, u) \}.$$

The optimal solution value of the cost function on upper approximation set is

$$G^* = \min_{m \in M^*} G_i(x, u).$$

There are now two possibilities: one providing surely optimal solution, possibly optimal solution and another one leads us to resolve the game on the lower approximation as follows:

- If $O_1 = O \cap M_* \neq \emptyset$, then a solution set $O_1$ is called a surely optimal solution set and $O \sim O_1$ is possibly optimal solution set.

- If $O_1 = O \cap M_* = \emptyset$, then $O \subseteq M_{BN}$ (boundary approximation of feasible region). In this case, the game does not have surely optimal solution set. Therefore, it will be resolved on the lower approximation to obtain $O_2$, which is an optimal solution set on the lower approximation.

Where, the game on the lower approximation is

$$\min G_i(x, u),$$
$$S.T.$$
$$M_* \subseteq M,$$

and the optimal solution value of the cost function on lower approximation set is

$$G_* = \min_{m \in M_*} G_i(x, u).$$

Definition 2. If $O_1 \neq \emptyset$, then $O_1$ contains all surely optimal solutions, hence it is called the surely optimal set $O_1 \subseteq M$.

Definition 3. $O \sim O_1$ contains possibly optimal solutions, hence it is called the possibly optimal set because $O \sim O_1 \subseteq M_{BN}$. 
2.1. Min-Max solutions for the 1st class of RCSG

For this game player \( i \) chooses his control under the assumption that all players have formed a coalition to maximize his cost. In other words, all players expect one cooperate against the remaining player \( i \).

**Definition 4.** A point \( w_\ast = (u_\ast, v_\ast) \) is a completely regular lower point if and only if \( w_\ast \) is a regular point of \( \Omega \) and \( u_\ast \) is a regular point of \( U_i \) for each \( i = 1, \ldots, r \). Here \( u_\ast \) is a lower control vector for each player \( i = 1, \ldots, r \) and \( v_\ast \) is the composite lower control of the remaining players, \( h_\ast(\cdot) \) defines lower inequality constraints and

\[
U_\ast = \{ u_\ast \in E^n \mid h_\ast(\zeta(u_\ast, v_\ast), u_\ast, v_\ast) \geq 0 \}. \tag{2.5}
\]

**Definition 5.** A point \( \tilde{u}_\ast \in \Omega \) is a rough min-max lower point for player \( i \) if and only if

\[
G_i[\zeta(\tilde{u}_\ast, v_\ast), \tilde{u}_\ast, v_\ast] \leq G_i[\zeta(u_\ast, v_\ast), u_\ast, v_\ast] \leq G_i[\zeta(u_\ast, \tilde{v}_\ast), u_\ast, \tilde{v}_\ast]
\]

for all \( u_\ast \in U_i \) where \( \tilde{u}_\ast = (\tilde{u}_i, \tilde{v}_i) \), \( U_i \) is defined by (2.5) and \( x_\ast = \zeta(u_\ast) \) is the solution to \( M_i(x_\ast, \tilde{u}_\ast) = 0 \). For a local rough min-max lower point replace \( U_\ast \) by \( B_i \cap U_\ast \) for some ball \( B_i \subset E^n \) centered at \( \tilde{u}_\ast \).

**Lemma 1.** If \( \tilde{u}_\ast = (u_\ast, v_\ast) \in \Omega \) is a local min-max lower point for player \( i \) for the game (2.3)–(2.4), and if \( x_\ast = \zeta(u_\ast) \) is the solution to \( M_i(x_\ast, \tilde{u}_\ast) = 0 \), then there exists a vector \( \gamma_\ast \in E^n \) defined by

\[
\frac{\partial J_{\ast_i}[x_\ast, \tilde{u}_\ast, \gamma_\ast_i]}{\partial x_\ast} = 0, \quad \frac{\partial J_{\ast_i}[x_\ast, \tilde{u}_\ast, \gamma_\ast_i]}{\partial u_\ast} e^i = 0,
\]

for all \( e^i \in T_{\ast_i} \) and

\[
\frac{\partial J_{\ast_i}[x_\ast, \tilde{u}_\ast, \gamma_\ast_i]}{\partial v_\ast} e^v = 0,
\]

for all \( e^v \in T_{\ast_v} \), where

\[
J_{\ast_i}[x_\ast, u_\ast, \gamma_\ast_i] = G_{\ast_i}(x_\ast, u_\ast) - \gamma_\ast_i^T g_{\ast_i}(x_\ast, u_\ast), \tag{2.6}
\]

and the tangent cones \( T_{\ast_i} \) and \( T_{\ast_v} \) are given by

\[
T_{\ast_i} = \{ e^i \in E^n \mid \frac{\partial h_\ast}{\partial u_\ast} - \frac{\partial h_\ast}{\partial x_\ast} [\frac{\partial g_\ast}{\partial x_\ast}]^{-1} \frac{\partial g_\ast}{\partial u_\ast} e^i \geq 0, \}
\]

\[
T_{\ast_v} = \{ e^v \in E^{n-s_i} \mid \frac{\partial h_\ast}{\partial v_\ast} - \frac{\partial h_\ast}{\partial x_\ast} [\frac{\partial g_\ast}{\partial x_\ast}]^{-1} \frac{\partial g_\ast}{\partial v_\ast} e^v \geq 0, \}
\]

where \( h_\ast \) denotes the active inequality constraints at \( u_\ast = (u_\ast, v_\ast) \).

**Proposition 1.** If \( \tilde{u}_\ast = (u_\ast, v_\ast) \in \Omega \) is a completely regular rough min-max lower point for the player \( i \) and \( x_\ast = \zeta(\tilde{u}_\ast) \) is the solution to \( M_i(x_\ast, \tilde{u}_\ast) = 0 \), then there exist vectors \( \lambda_\ast \in E^n \),
\( \lambda_s \in E^n, \mu_s \in E^n \) and \( \bar{\mu}_s \in E^n \) such that:

\[
\frac{\partial L_s}{\partial x_s} [x_s, \bar{u}_s, \lambda_s, \mu_s] = 0, \tag{2.7}
\]

\[
\frac{\partial L_s}{\partial x_s} [x_s, \bar{u}_s, \lambda_s, \bar{\mu}_s] = 0, \tag{2.8}
\]

\[
\frac{\partial L_s}{\partial u_s} [x_s, \bar{u}_s, \lambda_s, \mu_s] = 0, \tag{2.9}
\]

\[
\frac{\partial L_s}{\partial v_s} [x_s, \bar{u}_s, \lambda_s, \bar{\mu}_s] = 0, \tag{2.10}
\]

\[
M_s(x_s, \bar{u}_s) = 0,
\]

\[
\mu_s^T h_s(x_s, \bar{u}_s) = 0,
\]

\[
\bar{\mu}_s^T h_s(x_s, \bar{u}_s) = 0,
\]

\[
h_s(x_s, \bar{u}_s) \geq 0,
\]

\[
\mu_s \geq 0,
\]

\[
\bar{\mu}_s \leq 0,
\]

where

\[
L_s[x_s, \bar{u}_s, \lambda_s, \mu_s] = G_i(x_s, u_s) - \lambda_s M_s(x_s, u_s) - \mu_s h_s(x_s, u_s),
\]

\[
L_s[x_s, \bar{u}_s, \lambda_s, \bar{\mu}_s] = G_i(x_s, u_s) - \lambda_s M_s(x_s, u_s) - \bar{\mu}_s h_s(x_s, u_s).
\]

**Proof.** For player \( i \) consider the cones

\[
K_{s_i} = \left\{ y_{s_i} \in E^{s_i} \mid y_{s_i}^T = \mu_{s_i}^T \left[ \frac{\partial h_s}{\partial u_{s_i}} - \frac{\partial h_s}{\partial x_s} \frac{\partial g_s}{\partial x_s}^{-1} \frac{\partial g_s}{\partial u_{s_i}} \right], \mu_{s_i}^T h_s = 0, \mu_{s_i} \geq 0 \right\},
\]

\[
K_{s_2} = \left\{ y_{s_2} \in E^{s-s_i} \mid y_{s_2}^T = \mu_{s_2}^T \left[ \frac{\partial h_s}{\partial v_{s_2}} - \frac{\partial h_s}{\partial x_s} \frac{\partial g_s}{\partial x_s}^{-1} \frac{\partial g_s}{\partial v_{s_2}} \right], \mu_{s_2}^T h_s = 0, \mu_{s_2} \geq 0 \right\},
\]

and their polars, respectively,

\[
K_{s_i}^P = \left\{ Z_{s_i} \in E^{s_i} \mid y_{s_i}^T Z_{s_i} \geq 0 \ \forall y_{s_i} \in K_{s_i} \right\},
\]

\[
K_{s_2}^P = \left\{ Z_{s_2} \in E^{s-s_i} \mid y_{s_2}^T Z_{s_2} \geq 0 \ \forall y_{s_2} \in K_{s_2} \right\}.
\]

Since \( u_{s_i} \) is a regular point of \( U_{s_i} \) the tangent cone \( T_{s_i} \) to \( U_{s_i} \) is given by

\[
T_{s_i} = K_{s_i}^P, \quad T_{s_v} = K_{s_2}^P.
\]

From this result and Lemma 1 we have

\[
\frac{\partial J_s}{\partial u_{s_i}} [x_s, \bar{u}_s, \gamma_{s_i}] \in K_{s_i}, \tag{2.16}
\]

\[
\frac{\partial J_s}{\partial v_{s}} [x_s, \bar{u}_s, \gamma_{s_i}] \in K_{s_2}, \tag{2.17}
\]

where \( J_{s_i} \) is defined by (2.6) and \( \gamma_{s_i} \) is defined by

\[
\frac{\partial J_s}{\partial x_s} [x_s, \bar{u}_s, \gamma_{s_i}] = 0. \tag{2.18}
\]
Define
\[
\lambda_i^T = \gamma_i^T - \mu_i^T \frac{\partial h}{\partial x_i} \left[ \frac{\partial g_i}{\partial x_i} \right]^{-1},
\]
\[
\tilde{\lambda}_i^T = \gamma_i^T - \mu_i^T \frac{\partial h}{\partial x_i} \left[ \frac{\partial g_i}{\partial x_i} \right]^{-1},
\]
with \(L_{x_i}\) defined by (2.15) and choosing \(\mu_i - \bar{\mu}_i = 0\), equation (2.7) follows from (2.18) by substituting for \(\gamma_i\) from (2.19). Similarly (2.9), (2.11) and (2.13) follow from (2.16), (2.19) and the definition of \(k_{x_i}\). Choosing \(\tilde{\mu}_i = -\bar{\mu}_i \leq 0\) and substituting for \(\gamma_i\) from (2.20) we see that the equation (2.8) follows from (2.18). Similarly (2.10), (2.12) and (2.14) follow from (2.17) and from the definition of \(k_{x_2}\).

\[\square\]

**Definition 6.** A point \(w^* = (u_i^*, v^*)\) is a completely regular upper point if and only if \(w^*\) is a regular point of \(\Omega\) and \(u_i^*\) is a regular point of \(U_i^*\) for each \(i = 1, \ldots, r\). Here \(u_i^*\) is a lower control vector for each player \(i = 1, \ldots, r\) and \(v^*\) is the composite lower control of the remaining players, \(h^*(\cdot)\) defines lower inequality constraints and
\[
U_i^* = \{u_i^* \in E^s | h^*[\zeta(u_i^*, v^*), u_i^*, v^*] \geq 0\}.
\]

**Definition 7.** A point \(\tilde{u}^* \in \Omega\) is a rough min-max upper point for player \(i\) if and only if
\[
G_i[\zeta(\tilde{u}^*_i, v^*), \tilde{u}^*_i, v^*] \leq G_i[\zeta(u_i^*, \tilde{v}_i), u_i^*, \tilde{v}_i] \leq G_i[\zeta(u_i^*, v^*_i), u_i^*, \tilde{v}_i]
\]
for all \(u_i^* \in U_i^*\) where \(\tilde{u}^* = (\tilde{u}^*_i, \tilde{v}^*_i), U_i^*\) is defined by (2.21) and \(x^* = \zeta(u^*)\) is the solution to \(M^*(x^*, \tilde{u}^*) = 0\). For a local rough min-max lower point replace \(U_i^*\) by \(B_i \cap U^*\) for some ball \(B_i \subset E^s\) centered at \(\tilde{u}^*\).

**Proposition 2.** If \(\tilde{u}^* = (u^*, v^*) \in \Omega\) is a completely regular rough min-max upper point for the player \(i\) and \(x^* = \zeta(\tilde{u}^*)\) is the solution to \(M^*(x^*, \tilde{u}^*) = 0\), then there exist vectors \(\lambda_i^* \in E^n, \tilde{\lambda}_i^* \in E^n, \mu_i^* \in E^s\) and a vector \(\tilde{\mu}_i^* \in E^s\) such that:
\[
\frac{\partial L_i^*[x^*, \tilde{u}^*, \lambda_i^*, \mu_i^*]}{\partial u^*} = 0, \quad \frac{\partial L_i^*[x^*, \tilde{u}^*, \tilde{\lambda}_i^*, \tilde{\mu}_i^*]}{\partial u^*} = 0, \quad \frac{\partial L_i^*[x^*, \tilde{u}^*, \lambda_i^*, \mu_i^*]}{\partial \mu^*} = 0, \quad \frac{\partial L_i^*[x^*, \tilde{u}^*, \tilde{\lambda}_i^*, \tilde{\mu}_i^*]}{\partial \mu^*} = 0,
\]
where
\[
M^*(x^*, \tilde{u}^*) = 0, \quad \mu_i^T h^*(\tilde{x}, \tilde{u}^*) = 0, \quad \tilde{\mu}_i^T h^*(\tilde{x}, \tilde{u}^*) = 0, \quad h^*(\tilde{x}, \tilde{u}^*) \geq 0, \quad \mu_i^* \geq 0, \quad \tilde{\mu}_i^* \leq 0,
\]

The numerical example below will explain how we can find the min-max optimal solution point for the 1st class RCGS.

**Example 1.** Two firms sell substitutable products and seek to maximize their profits through advertising. The steady-state profits of firms 1 and 2 are taken respectively, as
\[
H_1(\cdot) = 5x - u - v,
\]
\[
H_2(\cdot) = 3x - v - 2u.
\]
The equilibrium (steady-state) fraction of the market $x$ that firms receive is given by

$$-7x + u - 4vx = 0 \subseteq M(\cdot) \subseteq -2x + v - 2xu = 0,$$

where $u$ and $v$ are control vectors of firm 1 and firm 2 respectively.

**Solution 1.** Two firms seek to maximize $H_1(\cdot)$ and $H_2(\cdot)$. Thus, firms seek to minimize

$$G_1(\cdot) = -5x + u + v,$$

$$G_2(\cdot) = -3x + v + 2u.$$

Let us solve the problem (*upper approximation*) using Min-Max concept:

For player 1: the game (2.22), (2.23) will be the following

$$\min_{u^* \in M^*} G_1(\cdot),$$

S.T.

$$M^* = -2x^* + v^* - 2x^*u^* = 0.$$

Here we have

$$L_1^* = G_1(\cdot) - \lambda_1^* M^* = -5x^* + u^* + v^* - \lambda_1^*(-2x^* + v^* - 2x^*u^*),$$

$$\frac{\partial L_1^*}{\partial x^*} = -5 - \lambda_1^*(-2 - 2u^*) = 0 \implies u^* = \frac{5 - 2\lambda_1^*}{2\lambda_1^*},$$

$$\frac{\partial L_1^*}{\partial u^*} = 1 - \lambda_1^*(-2x^*) = 0 \implies x^* = -\frac{1}{2\lambda_1^*},$$

$$\frac{\partial L_1^*}{\partial v^*} = 1 - \lambda_1^* = 0 \implies \lambda_1^* = 1 \implies u^* = \frac{3}{2}, \quad x^* = -\frac{1}{2}.$$

Since $M^* = 0 \Rightarrow V^* = -5/2$. So, Rough Min-Max upper point is $\tilde{u}^* = (3/2, -5/2)$ and the optimal solution value of the cost function on upper approximation is $G_1^* = 1.5$.

By testing this point, we see that the lower approximation is satisfied. So the solution we have got is a surely optimal solution for player 1 and there is no need for a solution at the lower approximation.

For player 2: the game (2.22), (2.24) will be the following

$$\min_{v^* \in M^*} G_2(\cdot),$$

S.T.

$$M^* = -2x^* + v^* - 2x^*u^* = 0.$$

Here we have

$$L_2^* = G_2(\cdot) - \lambda_2^* M^* = -3x^* + v^* + 2u^* - \lambda_2^*(-2x^* + v^* - 2x^*u^*),$$

$$\frac{\partial L_2^*}{\partial x^*} = -3 - \lambda_2^*(-2 - 2u^*) = 0 \implies u^* = \frac{3 - 2\lambda_2^*}{2\lambda_2^*},$$

$$\frac{\partial L_2^*}{\partial u^*} = 2 - \lambda_2^*(-2x^*) = 0 \implies x^* = -\frac{1}{\lambda_2^*},$$

$$\frac{\partial L_2^*}{\partial v^*} = 1 - \lambda_2^* = 0 \implies \lambda_2^* = 1 \implies u^* = \frac{1}{2}, \quad x^* = -1.$$
Since $M^* = 0 \Rightarrow V^* = -3$. So, Rough Min-Max upper point is $\bar{u}^* = (1/2, -3)$ and the optimal solution value of the cost function on upper approximation is $G_2^* = 1$. By testing this point, the lower approximation is not achieved. So the game for player 2 does not have a surely optimal solution and it will be resolved at the lower approximation.

Let us continue consideration (lower approximation).

For player 2: the game is the following

$$\begin{align*}
&\min_{v^* \in M^*} G_2(\cdot), \\
&S.T. \\
&M_2 = -7x_s + u_s - 4x_sv_s = 0.
\end{align*}$$

Here we have

$$\begin{align*}
L_2 = G_2(\cdot) - \lambda_2, M_2 = -3x_s + v_s + 2u_s - \lambda_2, (-7x_s + u_s - 4x_s v_s), \\
\frac{\partial L_2}{\partial x_s} = -3 - \lambda_2, (-7 - 4v_s) = 0 \Rightarrow v_s = \frac{3 - 7\lambda_2}{4\lambda_2}, \\
\frac{\partial L_2}{\partial v_s} = 1 + \lambda_2, (4x_s) = 0 \Rightarrow x^* = -\frac{1}{4\lambda_2}, \\
\frac{\partial L_2}{\partial u_s} = 2 - \lambda_2 = 0 \Rightarrow \lambda_2 = 2 \Rightarrow v_s = -\frac{11}{8}, x_s = -\frac{1}{8}.
\end{align*}$$

Since $M_s = 0 \Rightarrow u_s = -3/16$. So, Rough Min-Max lower point is $\bar{u}_s = (-3/16, -11/8)$ and the optimal solution value of the cost function on lower approximation is $G_2^* = -1.375$.

3. Parametric Rough Continuous Static game

A parametric study of a Rough Continuous Static game often provides a new insight into the Rough Continuous Static game. The parameter existence in Parametric Rough Continuous Static game (PRCSG) results in two cases, the first one: roughness is in the constraints and the parameter is in the cost function and the second case: the parameter is in the constraints and roughness is in the cost function. In this paper, the first case is what our study is about and the basic concepts of the parametric convex programming which are used here are established in Osman papers [5, 12] i.e. the solvability set, the stability set of the first kind.

3.1. The 1st case of Parametric Rough Continuous Static game (PRCSG)

Here we consider the case when the parameters are in the cost function and roughness is in the constraints. It can be defined as:

$$\begin{align*}
&\min_{u_i, P_i} G_i(\cdot), \\
&S.T. \\
&m \in M,
\end{align*}$$

where $G_i$ is a cost function for each player $i = 1, \ldots, r$ and $M$ is the feasible region of the game which is a rough convex set defined by $M_s \subseteq M \subseteq M^*$, and $P_i$ are real parameters, $i = 1, \ldots, r$.

**Definition 8.** The solvability set of the game (3.1), (3.2) denoted by $S$ is defined by two sets $S_s \subseteq S \subseteq S^*$, where

$$S_s = \{ P_i \in R^r \mid \min_{m \in M_s} G_i(u_i, P_i) \text{ exists} \}, \quad S^* = \{ P_i \in R^r \mid \min_{m \in M^*} G_i(u_i, P_i) \text{ exists} \}.$$
for any $P_i \in S$. If $\bar{m}$ is a surely optimal solution then the stability of the $1^{st}$ kind can be defined and obtained.

**Definition 9.** Suppose that $P_i \in S$ with a corresponding surely optimal solution $\bar{m} \in O_1(P_i)$ where
\[
O_1(P_i) = \{ \bar{m} \in M_* | G_i(\bar{m}, P_i) = \min_{m \in M_*} G_i(u_i, P_i) \}.
\]
Then the stability set of the $1^{st}$ kind for the game (3.1), (3.2) corresponding to $\bar{m}$ denoted by $S_1(\bar{m})$ is defined by
\[
S_1(\bar{m}) = \{ P_i \in S | G_i(\bar{m}, P_i) = \min_{m \in M_*} G_i(u_i, P_i) \}
\]
for any $P_i \in S$. If $\bar{m}$ is a possible optimal solution then the stability of the $3^{rd}$ kind can be defined and obtained.

**Definition 10.** Suppose that $P_i \in S$ with a corresponding lower optimal solution $\bar{m}_* \in O_2(P_i)$, where
\[
O_2(P_i) = \{ \bar{m}_* \in M_* | G_i(\bar{m}_*, P_i) = \min_{m \in M_*} G_i(u_i, P_i) \}.
\]
Then the possibly lower stability set for the game (3.1), (3.2) corresponding to $\bar{m}_*$ denoted by $S_2(\bar{m}_*)$ is defined by
\[
S_2(\bar{m}_*) = \{ P_i \in S | G_i(\bar{m}_*, P_i) = \min_{m \in M_*} G_i(u_i, P_i) \}.
\]

**Definition 11.** Suppose that $P_i \in S$ with a corresponding upper optimal solution $\bar{m}_* \in O_3(P_i)$ where
\[
O_3(P_i) = \{ \bar{m}_* \in M_* | G_i(\bar{m}_*, P_i) = \min_{m \in M_*} G_i(u_i, P_i) \}.
\]
Then the possibly upper stability set for the game (3.1), (3.2) corresponding to $\bar{m}_*$ denoted by $S_2(\bar{m}_*)$ is defined by
\[
S_2(\bar{m}_*) = \{ P_i \in S | G_i(\bar{m}_*, P_i) = \min_{m \in M_*} G_i(u_i, P_i) \}.
\]

### 3.2. Further steps

The steps below lead us to obtain the stability set of the $1^{st}$ kind, the possibly upper and lower stability sets when the parameters are found in cost function and roughness is in the constraints:

1. Formulate the game and include all the parameters that need to be examined.

2. For player 1, begin with a certain $P_i \in S$ and use Proposition 2 to get the optimal solution at upper approximation.

3. If the point that is obtained from the previous step is a surely optimal solution point then formulate Proposition 2 at it to find the stability set of $1^{st}$ kind. Otherwise, for a possible optimal solution point use Propositions 1 and 2 to obtain the possibly lower and upper stability sets.

4. Repeat for each one of the remaining players.
Example 2. Consider the game

\[ G_1(\cdot) = u^2 + v^2 + P_1u - v, \]  
\[ G_2(\cdot) = u^2 + v^2 - 5u + P_2v, \]  
S.T.  
\[ (u + v \leq 1) \subseteq M(\cdot) \subseteq (u^2 + v^2 \leq 9, \ u + v \leq 5). \]  

(3.3)  
(3.4)  
(3.5)

Solution 2 (Upper approximation). Using Min-Max concept we find the following results.

For player 1: Suppose \( P_1 = -7 \) and rewrite the game (3.3), (3.5) to be

\[
\begin{align*}
\min \limits_{u^* \in M^*} G_1(\cdot), \\
\text{S.T.} \\
M^* = (u^*^2 + v^*^2 \leq 9, \ u^* + v^* \leq 5).
\end{align*}
\]

We have here

\[
\begin{align*}
L_1^* &= u^*^2 + v^*^2 - 7u^* - v^* + \mu_1^*(u^*^2 + v^*^2 - 9) + \mu_2^*(u^* + v^* - 5) = 0, \\
\frac{\partial L_1^*}{\partial u^*} &= 2u^* - 7 + 2u^*\mu_1^* + \mu_2^* = 0, \\
\frac{\partial L_1^*}{\partial v^*} &= 2v^* - 1 + 2v^*\mu_1^* + \mu_2^* = 0, \\
\mu_1^*(u^*^2 + v^*^2 - 9) &= 0, \\
\mu_2^*(u^* + v^* - 5) &= 0.
\end{align*}
\]

When \( \mu_1^* > 0 \) and \( \mu_2^* = 0 \), we conclude

\[
\begin{align*}
2u^* - 7 + 2u^*\mu_1^* &= 0 \implies u^* = \frac{7}{2(1 + \mu_1^*)}, \\
2v^* - 1 + 2v^*\mu_1^* &= 0 \implies v^* = \frac{1}{2(1 + \mu_1^*)}.
\end{align*}
\]

Since \( \mu_1^* > 0 \) \( \implies u^*^2 + v^*^2 - 9 = 0 \). Then, \( \mu_1^* = 0.178511 \implies (u^*, v^*) = (2.9698, 0.42445) \).

When \( \mu_2^* > 0 \) and \( \mu_1^* = 0 \), we conclude

\[
\begin{align*}
2u^* - 7 + \mu_2^* &= 0 \implies u^* = \frac{7 - \mu_2^*}{2}, \\
2v^* - 1 + \mu_2^* &= 0 \implies v^* = \frac{1 - \mu_2^*}{2}.
\end{align*}
\]

Since \( \mu_2^* > 0 \) \( \implies u^* + v^* - 5 = 0 \). Then, \( \mu_2^* = -1 \). This solution is refused.

So, the upper optimal solution point \((u^*, v^*) = (2.9698, 0.42445)\) is a surely optimal solution point then the stability of the 1\(^{st}\) kind will be obtained as follows:

\[ 2(2.9698) + P_1 + 2(2.9698)\mu_1^* = 0 \implies \mu_1^* = \frac{-5.94225 - P_1}{5.94228}. \]

Since \( \mu_1^* > 0 \) \( \implies -5.94225 - P_1 > 0 \) \( \implies P_1 < -5.94225 \).

So the stability of the 1\(^{st}\) kind is:

\[ (P_1 < -5.94225). \]
For player 2: Suppose $P_2 = -4$ and rewrite the game (3.4), (3.5) to be

$$
\min_{u^* \in M^*} G_2(\cdot), \\
\text{S.T.} \\
M^* = (u^{*2} + v^{*2} \leq 9, u^* + v^* \leq 5).
$$

We have here

$$
L_1^* = u^{*2} + v^{*2} - 5u^* - 4v^* + \mu_1^*(u^{*2} + v^{*2} - 9) + \mu_2^*(u^* + v^* - 5) = 0,
$$

$$
\frac{\partial L_1^*}{\partial u^*} = 2u^* - 5 + 2u^* \mu_1^* + \mu_2^* = 0, \\
\frac{\partial L_1^*}{\partial v^*} = 2v^* - 4 + 2v^* \mu_1^* + \mu_2^* = 0,
$$

$$
\mu_1^*(u^{*2} + v^{*2} - 9) = 0, \\
\mu_2^*(u^* + v^* - 5) = 0.
$$

When $\mu_1^* > 0$ and $\mu_2^* = 0$, we have

$$
2u^* - 5 + 2u^* \mu_1^* = 0 \Rightarrow u^* = \frac{5}{2(1 + \mu_1^*)},
$$

$$
2v^* - 4 + 2v^* \mu_1^* = 0 \Rightarrow v^* = \frac{4}{2(1 + \mu_1^*)}.
$$

Since $\mu_1^* > 0 \Rightarrow u^{*2} + v^{*2} - 9 = 0$. Then, $\mu_1^* = 0.06718 \Rightarrow (u^*, V^*) = (2.3426, 1.874).

When $\mu_2^* > 0$ and $\mu_1^* = 0$, we have

$$
2u^* - 5 + \mu_2^* = 0 \Rightarrow u^* = \frac{5 - \mu_2^*}{2},
$$

$$
2v^* - 4 + \mu_2^* = 0 \Rightarrow v^* = \frac{4 - \mu_2^*}{2}.
$$

Since $\mu_2^* > 0 \Rightarrow u^* + v^* - 5 = 0$. Then, $\mu_2^* = -12$. This solution is refused.

So, the upper optimal solution point $(u^*, V^*) = (2.3426, 1.874)$ is a surely optimal solution point then the stability of the 1st kind will be obtained as follows:

$$
2(1.874) + P_2 + 2(1.874) \mu_1^* = 0 \Rightarrow \mu_1^* = \frac{-3.748 - P_2}{3.748}.
$$

Since $\mu_1^* > 0 \Rightarrow -3.748 - P_2 > 0 \Rightarrow P_2 < -3.748.$

So the stability of the 1st kind corresponds to the case

$$
(P_2 < -3.748).
$$

4. Conclusion

This paper presented a parametric study of the rough continuous static game when parameters exist in the crisp cost function and constraints are rough sets. Furthermore, the paper provided new concepts including the solvability set and the stability set of the 1st kind and the possibly stability set in a rough environment. Also, the surely and possibly optimal solution sets of RCSG in the 1st class are introduced. Future work may focus on the parametric study of the PRCSG when parameters are found in the constraints and the cost function is a rough function.
REFERENCES

1. Bank B., Guddat J., Klatte D., Kummer B., Tammer K. *Non-Linear Parametric Optimization*. Basel: Birkhäuser, 1982. 228 p. DOI: 10.1007/978-3-0348-6328-5

2. Bazaraa M.S., Sherali H.D., Shetty C.M. *Nonlinear Programming: Theory and Algorithms*. 3rd Ed. Verlag: J. Wiley & Sons Inc., 2013. 872 p.

3. Bertsekas P.D. *Nonlinear Programming*. 2nd Ed. Belmont, Massachusetts: Athena Scientific, 1999. 791 p.

4. Budhiraja A., Dupuis P. Representations for functional of Hilbert space valued diffusions. In: *Stochastic Analysis, Control, Optimization and Applications*. McEneaney W.M., Yin G.G., Zhang Q. (eds.) Ser. Systems Control Found. Appl. Boston, MA: Birkhäuser, 1999. P. 1–20. DOI: 10.1007/978-1-4612-1784-8

5. Elsisy M.A., Eid M.H., Osman M.S.A. Qualitative analysis of basic notions in parametric rough convex programming (parameters in the objective function and feasible region is a rough set). *OPSEARCH*, 2017. Vol. 54. P. 724–734. DOI: 10.1007/s12597-017-0300-2

6. Jongen H.Th., Jonker P., Twilt F. *Nonlinear Optimization in Finite Dimensions*. Boston, MA: Springer, 2000. 513 p. DOI: 10.1007/978-1-4615-0017-9

7. Kalaiselvi R., Kousalya K. Statistical modelling and parametric optimization in document fragmentation. *Neural Comput. Applic.*, 2020. Vol. 32. P. 5909–5918. DOI: 10.1007/s00521-019-04068-1

8. Lijun X., Yijia Z., Bo Y. Robust Optimization Model with Shared Uncertain Parameters in Multi-Stage Logistics Production and Inventory Process. *Mathematics*, 2020. Vol. 8, No. 2. Art. no. 211. P. 1–12. DOI: 10.3390/math8020211

9. Matsumoto A., Szidarovszky F. Continuous Static Games. In: *Game Theory and Its Applications*. Tokyo: Springer, 2016. P. 21–47. DOI: 10.1007/978-4-319-35786-6

10. Miettinen K. *Nonlinear Multiobjective Optimization*. Ser. Internat. Ser. Oper. Res. Management Sci., vol. 12. NY: Springer, 1998. 298 p. DOI: 10.1007/978-1-4615-5563-6

11. Nguyen V., Gupta S., Rana S. et al. Filtering Bayesian optimization approach in weakly specified search space. *Knowl. Inf. Syst.*, 2019. Vol. 60. P. 385–413. DOI: 10.1007/s10115-018-1238-2

12. Osman M.S.A. Qualitative analysis of basic notions in parametric convex programming. I. Parameters in the constraints. *Aplikace Matematiky*, 1977. Vol. 22., No. 5. P. 318–332. DOI: 10.21136/AM.1977.103710

13. Osman M., Lashein E. F., Youness E. A., Elsayed T. Mathematical programming in rough environment. *Optimization*, 2011. Vol. 60, No. 5. P. 603–611. DOI: 10.1080/02303687.2010.535969

14. Patil A., Desai A. D. Parametric optimization of engine performance and emission for various n-butanol blends at different operating parameter condition. *Alexandria Eng. J.*, 2020. Vol. 59, No. 2. P. 851–864. DOI: 10.1016/j.aej.2020.02.006

15. Sawaragi Y., Nakayama H., Tanino T. *Theory of Multiobjective Optimization*. Math. Sci. Eng., vol. 176. Academic Press, 1985. 322 p.

16. Schneider J.J., Kirkpatrick S. *Stochastic Optimization*. Berlin Heidelberg: Springer-Verlag, 2006. 568 p. DOI: 10.1007/978-3-540-34560-2

17. Sun W., Yuan Y.-X. *Optimization Theory and Methods: Nonlinear Programming*. Springer Optim. Appl., vol. 1. US: Springer-Verlag, 2006. 688 p. DOI: 10.1007/b106451

18. Tuy H. Minimax: existence and stability. In: *Pareto Optimality, Game Theory and Equilibria*. A. Chinchuluun, P.M. Pardalos, A. Migdalas, L. Pitsoulis (eds.). Springer Optim. Appl., vol. 17. NY: Springer. P. 3–21. DOI: 10.1007/978-0-387-77247-9

19. Youness E. Characterizing solutions of rough programming problems. *European J. Oper. Res.*, 2006. Vol. 168, No. 3. P. 1019–1029. DOI: 10.1016/j.ejor.2004.05.019

20. Zhang J., Liu N., Wang S. A parametric approach for performance optimization of residential building design in Beijing. *Build. Simul.*, 2019. Vol. 13. P. 223–235. DOI: 10.1007/s12273-019-0571-z