Some dynamics in real quadratic fields with applications to Euclidean minima

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1 Introduction

Let $D > 1$ be a square-free positive integer and let $K = \mathbb{Q}(\sqrt{D})$ be the associated real quadratic field with ring of integers $\mathcal{O}_K$. Let $N : K \to \mathbb{Q}$ denote the absolute norm $N(a) = |Nm_{K/\mathbb{Q}}(a)| = |a\overline{a}|$, where $a \mapsto \overline{a}$ is Galois conjugation, and recall that the ring $\mathcal{O}_K$ is called norm-Euclidean if for all $a \in K$ there exists $q \in \mathcal{O}_K$ such that $N(a - q) < 1$. The ring of integers $\mathcal{O}_K$ embeds as a lattice in the two-dimensional real vector space $V_K = K \otimes \mathbb{R}$, and we denote the quotient torus by $T_K = V_K/\mathcal{O}_K$. Galois conjugation extends linearly to $V_K$, and the absolute norm extends accordingly to an indefinite quadratic form on $V_K$ that we also denote by $N$. The norm is not $\mathcal{O}_K$-invariant, but the function defined by

$$M(P) = \inf_{Q \in \mathcal{O}_K} N(P - Q)$$

is, and descends to a function on the torus $T_K$ which we also denote by $M$. The function $M$ is upper-semicontinuous ([2], Theorem F).

The Euclidean minimum of $K$ is defined by $M_1(K) = \sup_{P \in K} M(P)$. In particular, $M_1(K) < 1$ implies that $\mathcal{O}_K$ is norm-Euclidean, while $M_1(K) > 1$ implies that it is not. The second Euclidean minimum is defined by

$$M_2(K) = \sup_{M(P) < M_1(K)} M(P)$$

and $M_1(K)$ is said to be isolated if $M_2(K) < M_1(K)$. We may proceed in this fashion producing Euclidean minima $M_i(K)$ until we find a non-isolated one. Note that upper-semicontinuity ensures that each of these suprema is actually achieved by some collection of points on the torus. These Euclidean minima demonstrate a variety of behavior, in some cases producing an infinite sequence of isolated minima while in others we find that $M_2(K)$ already fails to be isolated - see [10] for an overview of results. Barnes and Swinnerton-Dyer conjectured in [3] that $M_1(K)$ is always isolated and rational, and that $M_2(K)$ is taken at a point with coordinates in $K$. Numerous computations by other authors (e.g. [4], [6], [9], [7], [8], [13]) suggest further that all Euclidean minima lie in $K$.

**Theorem 1.** All isolated Euclidean minima lie in $K$. If $M_1(K)$ is isolated, then it lies in $\mathbb{Q}$.

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The first part of this statement follows from the next theorem, which implies more broadly that all isolated points of the Euclidean spectrum $ES(K) = M(T_K)$ lie in $K$. The method of proof establishes that any such isolated point is taken at a point $P$ with coordinates in $K$, and we prove that $M(P) \in K$ for such points. The second part is here for completeness, but was known already to Barnes and Swinnerton-Dyer ([3], Theorem M). The following theorem is our main result, and is proven Section 6.

**Theorem 2.** The set $ES(K) \cap K$ is dense in $ES(K)$.

## 2 The dynamical systems $X_t$

By Dirichlet’s unit theorem, we have $O_K^* = \pm \mathbb{Z}$ for some fundamental unit $\varepsilon$ of infinite order. We will later fix an embedding of $K$ into $\mathbb{R}$ and assume that $\varepsilon$ is chosen so that $\varepsilon > 1$. Multiplication by $\varepsilon$ is absolute norm-preserving and extends by linearity to an endomorphism $\phi$ of $V_K$ that is also absolute norm-preserving. Since $\phi$ preserves the lattice $O_K$, it descends to an endomorphism of the torus $T_K$ with the property that $M(\phi(P)) = M(P)$ for all $P \in T_K$. The eigenvalues of $\phi$ are the embeddings of $\varepsilon$ in to $\mathbb{R}$ and hence not roots of unity, so $\phi$ is an ergodic transformation of $T_K$. This dynamical system, and a symbolic coding of it obtained from a Markov partition of the torus, is our main resource. We note that the subset $K/O_K$ coincides with the set of periodic points for $\phi$.

For $t > 0$, the $\phi$-invariant set $X_t = \{ P \in T_K \mid M(P) \geq t \}$ is closed by upper semicontinuity. We can describe $X_t$ alternatively by first noting that the open set

$$\mathcal{U}(t) = \bigcup_{Q \in O_K} \{ P \in V_K \mid N(P - Q) < t \}$$

is translation-invariant and descends to an open subset of $T_K$, and then observing that $X_t$ is its complement. The sets $X_t$ have Lebesgue measure zero for $t > 0$ since they are proper, closed, and $\phi$-invariant.

**Question 1.** How does the Hausdorff dimension $\dim(X_t)$ vary with $t$?

That $\dim(X_t) \to 2$ as $t \to 0$ is a simple consequence of Theorem 2.3 of [4]. We prove in Corollary 2 that $\dim(X_t)$ is left-continuous everywhere. Right-continuity remains an open question.

We illustrate this dimension in the case $K = \mathbb{Q}(\sqrt{5})$. Davenport computed the Euclidean minima for this field in [3] and [4], finding the infinite decreasing sequence of minima $M_1 = 1/4$, and for $i \geq 1$,

$$M_{i+1} = \frac{f_{6i-2} + f_{6i-4}}{4(f_{6i-1} + f_{6i-3} - 2)}$$

where $f_k$ denotes the $k$th Fibonacci number. Each of these minima is obtained at a finite collection of elements of $K/O_K$, and we have $M_i \to t_\infty = (-1 + \sqrt{5})/8 \approx .1545$. A plot of $\dim(X_t)$ in this case is given below. The zero-dimensional region necessarily covers $t > t_\infty$, since the collection of points giving rise to the Euclidean minima is countable. We prove in [11] that $\dim(X_t) > 0$ for all $t < t_\infty$ and that $\dim(X_t)$ is continuous at $t_\infty$.

The evident plateaus on this graph and its detail in Figure 1 have dynamical significance. The dimensions plotted here are actually upper bounds obtained by symbolically coding the torus dynamical system with a Markov partition and finding subshifts of finite type (SFTs) that contain the coding of $X_t$, as in Section 2.

As we will see in Proposition 3, a plateau will occur wherever it is possible to make such a bound tight and $X_t$ can be described directly by an SFT. The longest such plateau occurs around $t = .15$ (see Figure 1 for a detail), and we give an explicit symbolic coding of the $X_t$ on this plateau in [11].
3 Coordinates and \( K \)-points

Let us now take \( K \) to be a subset of \( \mathbb{R} \) by fixing an embedding, and take \( \varepsilon \) to be a fundamental unit with \( \varepsilon > 1 \). Recall that \( \{1, \alpha_K\} \) is a \( \mathbb{Z} \)-basis of \( \mathcal{O}_K \), where

\[
\alpha_K = \begin{cases} \sqrt{D} & D \equiv 2, 3 \pmod{4} \\ \frac{1 + \sqrt{D}}{2} & D \equiv 1 \pmod{4} \end{cases}
\]

Coordinates with respect to this basis will be denoted \((x, y)\). The choice of embedding gives an isomorphism

\[
V_K = K \otimes_{\mathbb{Q}} \mathbb{R} \sim \mathbb{R} \times \mathbb{R}
\]

of \( \mathbb{R} \)-algebras, and thus another coordinate system. Multiplication by \( \varepsilon \) has the effect of multiplying by \( \varepsilon = \pm \varepsilon^{-1} \) in the first coordinate and \( \varepsilon \) in the second coordinate. Accordingly, these are known as the stable and unstable coordinates and denoted \((s, u)\). Note that the absolute norm is simply \( N(s, u) = |su| \) in these coordinates, and that the coordinate transformations between \((x, y)\) and \((s, u)\) coordinates are \( K \)-linear.

A point \( P \in T_K \) is called determinate if it has a representative \( Q \in V_K \) with \( N(Q) = M(P) \). It is shown in \cite{4} (Theorem 2.6) that the set of determinate points is a meagre \( F_\sigma \) set of measure zero and Hausdorff dimension 2. For a general point \( P \in T_K \), the following two lemmas help relate the value \( M(P) \) to the more concrete values \( N(Q) \) for \( Q \in V_K \).

**Lemma 1** (\cite{4}, Lemma 4.2). Suppose that \( P \in T_K \) satisfies \( M(P) < t \). There exists a point \( Q = (s, u) \in V_K \) representing an element of the orbit of \( P \) satisfying

\[
|s|, |u| < \sqrt{\varepsilon t}
\]

such that \( N(Q) = |su| < t \).

**Lemma 2.** Let \( P \in T_K \). There exists \( Q \in V_K \) representing an element of the orbit closure of \( P \) satisfying

\[
N(Q) = M(Q) = M(P)
\]

**Proof.** Let \( R_{\text{big}} \) denote the rectangle in \( V_K \) given by \( |s|, |u| < \sqrt{\varepsilon(M_1(K) + 1)} \). By Lemma 1, there is for each \( n \in \mathbb{N} \) a point \( Q_n \in R_{\text{big}} \) representing an element of orbit \( \phi^\varepsilon(P) \) with

\[
N(Q_n) < M(P) + \frac{1}{n}
\]

(1)

Since \( R_{\text{big}} \) is bounded, there exists a subsequence \( Q_{k_n} \) converging to some point \( Q \). Observe that

\[
M(Q) \leq N(Q) = \lim_{k \to \infty} N(Q_{k_n}) \leq M(P)
\]

where the last inequality follows from \cite{1}. The definition of \( Q \) ensures that it represents an element of the closure of \( \phi^\varepsilon(P) \). But this implies that \( M(Q) \geq M(P) \) by upper-semicontinuity, since the value \( M(P) \) is common to the entire orbit of \( P \). These inequalities combine to give the desired result. \( \square \)
Let us call a point in $V_K$ with rational $(x, y)$ coordinates a $\mathbb{Q}$-point. Similarly a $K$-point is one whose $(x, y)$ coordinates lie in $K$, or equivalently whose $(s, u)$ coordinates lie in $K$. The set of $\mathbb{Q}$-points coincides with $K/O_K$, which is also the set of periodic points for $\phi$. In particular, if $P$ is a $\mathbb{Q}$-point then the previous lemma immediately implies that $P$ is determinate and $M(P) \in \mathbb{Q}$.

**Proposition 1.** Let $P \in \mathbb{T}_K$ be a $K$-point.

1. There exists $N \in \mathbb{N}$ such that $\phi^k(NP) \to 0$ as $|k| \to \infty$

2. $M(P) \in K$

**Proof.**

1. Since $P$ has $(s, u)$ coordinates in $K$, there exists $N \in \mathbb{N}$ such that $NP$ has $(s, u)$ coordinates in $O_K$. In $(s, u)$ coordinates, the lattice $O_K \subseteq V_K$ is given by the set of pairs $(\alpha, \alpha)$ for $\alpha \in O_K$. It follows by subtracting such elements that $NP$ has a representative whose stable coordinate vanishes, as well as a representative whose unstable coordinate vanishes. Now $\phi^k(NP) \to 0$ as $|k| \to \infty$ follows immediately.

2. By the previous part, the orbit closure of the $K$-point $P$ consists of the orbit of $P$ together with a finite collection of $N$-torsion points on the torus. By Lemma 2, there exists $Q \in V_K$ representing an element of this orbit closure with $N(Q) = M(P)$. Should $Q$ represent an $N$-torsion point, then $M(P) \in \mathbb{Q}$ since torsion points are $\mathbb{Q}$-points. On the other hand, if $Q = (s, u)$ represents an element of the orbit of $P$ then $Q$ is also a $K$-point, so we have $M(P) = N(Q) = |su| \in K$.

4 Markov Partitions

For each $K$, the dynamical system $(\mathbb{T}_K, \phi)$ admits a Markov partition consisting of two open rectangles. Such a partition $\{R_0, R_1\}$ for $K = \mathbb{Q}((\sqrt{5}))$ is pictured in Figure 2 in $(x, y)$ coordinates. Figure 3 furnishes a uniform description in $(s, u)$ coordinates of a two-rectangle Markov partition for any $K$. This description is simply the one provided by Adler in [1] translated into $(s, u)$ coordinates. See also [12], where the construction may originate.

![Fig. 2: \{R_0, R_1\} for \mathbb{Q}(\sqrt{5})](image)

![Fig. 3: Original partition in (s, u) coordinates (scale shown for \mathbb{Q}(\sqrt{5}))](image)

These two-rectangle partitions are typically not generators essentially because the intersections $R \cap \phi(S)$ are generally disconnected. In the case of $\mathbb{Q}(\sqrt{5})$ however, the original partition $\mathcal{P}_0 = \{R_0, R_1\}$ is a generator. Moreover, $R_0 \cap \phi(R_0) = \emptyset$, while the remaining intersections consist of a single nonempty rectangle each. Let $\Sigma$ denote the subset of $\{0, 1\}^\mathbb{Z}$ that avoids the string $00$ and let $\sigma : \Sigma \to \Sigma$ be the shift operator $\sigma(s)_i = s_{i+1}$. The Markov generator property furnishes a map

$$\pi : \Sigma \to \mathbb{T}_K$$
interwining $\phi$ and the shift operator on $\Sigma$ that sends each string of coordinates to the unique point in $T_K$ whose orbit has these coordinates:

$$\pi(s) = \bigcap_{n \in \mathbb{N}} \bigcap_{i=-n}^{n} \phi^{-i}(s_i) = \bigcap_{i \in \mathbb{Z}} \phi^{-i}(s_i)$$

**Remark 1.** This construction ensures that $\phi^k \pi(s) \in s(k)$ for all $k$. It follows that if the coordinate word of $A \in \mathcal{P}_n$ occurs in $s \in \Sigma$, then $\phi^k \pi(s) \in \overline{A}$ for a suitable $k \in \mathbb{Z}$.

The map $\pi$ is continuous, surjective, bounded-to-one, and essentially one-to-one. Moreover, if $X \subseteq T_K$ is a closed, invariant subset then $\pi$ restricts to a map

$$\pi^{-1}(X) \rightarrow X$$

with the same properties, from which it follows that the entropy of $\phi|_X$ coincides with the entropy of the shift restricted to $\pi^{-1}(X)$. In the case of $X = X_t$, this entropy can be approximated by approximating the set $\pi^{-1}(X_t)$ by subshifts obtained by refining the partition $\mathcal{P}_0$ and omitting some rectangles. The refinements are defined by taking $\mathcal{P}_n$ to consist of all nonempty intersections of the form

$$\phi^n(A_{-n}) \cap \cdots \cap \phi(A_{-1}) \cap A_0 \cap \phi^{-1}(A_1) \cap \cdots \cap \phi^{-n}(A_n), \ A_i \in \mathcal{P}_0,$$

and we say that this particular rectangle has coordinate word $A_{-n} \cdots A_0 \cdots A_n$. When a representative rectangle in the plane $V_K$ is needed for a member of $\mathcal{P}_n$, we take the one contained in the original footprint $R_0 \cup R_1$.

The refinement $\mathcal{P}_n$ is also a Markov generator, and we have a refined coding $\pi_n : \Sigma_n \rightarrow T_K$ by the set of admissible strings in the alphabet $\mathcal{P}_n$. Note that $\Sigma_n$ is simply a “block form” of $\Sigma$ and there is a canonical bijection $\Sigma \cong \Sigma_n$ compatible with the shift operator and the two codings of $T_K$. While for general $K$, the partition $\{R_0, R_1\}$ is not a generator, in all cases the connected components of $A_0 \cap \phi^{-1}(A_1)$ for $A_i \in \{R_0, R_1\}$ do comprise a Markov generator (see the proof of Theorem 8.4 of [1]). Thus for any $K$ other than $Q(\sqrt{5})$ we may let $\mathcal{P}_0$ denote this generator and then proceed as in the previous paragraph to produce refinements $\mathcal{P}_n$. In all cases, the diameter of $\mathcal{P}_n$ tends to zero as $n \rightarrow \infty$.

The following explicit construction of $\pi$ will be useful below. Here, $\mathcal{P}$ can be any Markov generator on $T_K$ arising from a collection of rectangles in the plane $V_K$ with sides parallel to the stable and unstable axes. In particular we suppose we have a chosen representative in the plane for each member of $\mathcal{P}$, or equivalently a choice of stable and unstable interval of which this member is the product. Let $s \in \Sigma$, the set of all admissible bi-infinite strings in the alphabet $\mathcal{P}$. First we show how to compute the unstable coordinate of $\pi(s)$. The intersections

$$r_0 = s_0$$
$$r_1 = s_0 \cap \phi^{-1}(s_1)$$
$$r_2 = s_0 \cap \phi^{-1}(s_1) \cap \phi^{-2}(s_2)$$
$$\vdots$$

on the torus can be viewed in the plane as a sequence of rectangles within $s_0$ whose stable interval is constant (and equal to that of $s_0$) and whose unstable interval is shrinking. Up to similarity, the footprint of the unstable interval of $r_{i+1}$ inside that of $r_i$ depends only on the rectangles $s_i$ and $s_{i+1}$ and is independent of $i$. This is because $\phi$ simply scales by the positive number $\varepsilon$ in the unstable direction, preserving similarity.

Given a rectangle in the plane with sides parallel to the stable and unstable axes, let us denote its stable and unstable intervals by $[\alpha_s(A), \beta_s(A)]$ and $[\alpha_u(A), \beta_u(A)]$, and let $\ell_s(A) = \beta_s(A) - \alpha_s(A)$ denote the corresponding lengths. For each pair $A, B \in \mathcal{P}$ with $AB$ admissible, we define

$$\rho_u(A, B) = \frac{\alpha_u(A \cap \phi^{-1}(B)) - \alpha_u(A)}{\ell_u(A)}$$
Pictured in the \((s,u)\) plane, this is the height of the bottom of the subrectangle \(A \cap \phi^{-1}(B)\) inside \(A\), expressed as a fraction of the total height of \(A\), and is a measure of the footprint of this subrectangle in \(A\) alluded to above. The left endpoint of the unstable interval of \(r_i\) is then equal to
\[
\alpha_u(s_0) + \rho_u(s_0, s_1) \ell_u(s_0) + \rho_u(s_1, s_2) \frac{\ell_u(s_1)}{\varepsilon} + \cdots + \rho_u(s_{i-1}, s_i) \frac{\ell_u(s_{i-1})}{\varepsilon^{i-1}},
\]
so the unstable coordinate of \(\pi(s)\) is given by the series
\[
\alpha_u(s_0) + \rho_u(s_0, s_1) \ell_u(s_0) + \rho_u(s_1, s_2) \frac{\ell_u(s_1)}{\varepsilon} + \rho_u(s_2, s_3) \frac{\ell_u(s_2)}{\varepsilon^2} + \cdots \tag{3}
\]

The stable coordinate works the same way if \(\varepsilon > 0\). Some additional care must be taken if \(\varepsilon < 0\), since then \(\phi\) is orientation-reversing in the stable direction and the footprints alternate with their mirror images up to similarity instead of being independent of \(i\). In that case we define coefficients
\[
\rho^+_s(A, B) = \frac{\alpha_s(A \cap \phi(B)) - \alpha_s(A)}{\ell_s(A)}
\]
and
\[
\rho^-_s(A, B) = \frac{\beta_s(A) - \beta_s(A \cap \phi(B))}{\ell_s(A)},
\]
and stable coordinate alternates between these:
\[
\alpha_s(s_0) + \rho^+_s(s_0, s-1) \ell_s(s_0) + \rho^-_s(s-1, s-2) \frac{\ell_s(s-1)}{\varepsilon} + \rho^+_s(s-2, s-3) \frac{\ell_s(s-2)}{\varepsilon^2} + \cdots \tag{4}
\]

If \(s \in \Sigma\) is periodic, then the image \(\pi(s) \in \mathbb{T}_K\) has periodic orbit, and hence is a \(\mathbb{Q}\)-point. The following lemma furnishes a similar description of some \(K\)-pts.

**Lemma 3.** Suppose that \(s\) is eventually periodic in both directions. Then \(\pi(s)\) is a \(K\)-point.

**Proof.** First observe that all members of our partitions \(\mathcal{P}_n\) have coordinates in the field \(K\). If \(s\) is eventually periodic in both directions, then the series \((3)\) and \((4)\) (and its analog in case \(\varepsilon > 0\)) decompose into finitely many geometric series with all terms and coefficients expressible in terms of these coordinates, and the result follows.

**Lemma 4.** If \(t' < t\) and \(X_t \subsetneq X_{t'}\), then there exists a finite word occurring in \(\pi^{-1}(X_{t'})\) that does not occur in \(\pi^{-1}(X_t)\).

**Proof.** Suppose to the contrary that every word appearing in \(\pi^{-1}(X_{t'})\) also occurs in \(\pi^{-1}(X_t)\). We claim this forces \(\pi^{-1}(X_{t'})\) to be contained in the closure of \(\pi^{-1}(X_t)\), which is a contradiction since the latter is closed assumed distinct from the former. Let \(s \in \pi^{-1}(X_{t'})\), and for \(k \in \mathbb{N}\) let \(w_k\) be the word \((-k) \cdots s(0) \cdots s(k)\). By hypothesis, this word occurs in \(\pi^{-1}(X_t)\), and by applying \(\phi\) we may assume that it occurs centrally in some element \(x_k \in \pi^{-1}(X_t)\). In particular, \(x_k\) and \(s\) agree on the index interval \([-k, k]\), and it follows that \(x_k \to s\) as \(k \to \infty\), so \(s\) lies in the closure of \(\pi^{-1}(X_t)\).

5 Upper bounds via trapping rectangles

Given a collection of rectangles \(\mathcal{C} \subseteq \bigcup_n \mathcal{P}_n\), we denote by \(\Sigma(\mathcal{C})\) the subshift of \(\Sigma\) that avoids the coordinate words of elements of \(\mathcal{C}\). If \(\mathcal{C}\) is finite, then there is a largest \(n\) for which \(\mathcal{P}_n\) contains an element of \(\mathcal{C}\). Now every element of \(\mathcal{C}\) breaks up into rectangles in \(\mathcal{P}_n\), and we let \(\mathcal{C}' \subseteq \mathcal{P}_n\) denote the collection of rectangles occurring in this fashion. Under the identification \(\Sigma \cong \Sigma_n\), the subshift \(\Sigma(\mathcal{C}')\) can alternately be described as the collection of \(s \in \Sigma_n\) for which \(s(k) \notin \mathcal{C}'\) for all \(k \in \mathbb{Z}\).

Let \(I \subseteq \Omega_K\) be a finite set of lattice points and let
\[
\mathcal{U}(t, I) = \bigcup_{Q \in I} \{P \in V_K \mid N(P - Q) < t\}
\]
and let
\[ \mathcal{T}_n(t, I) = \{ A \in \mathcal{P}_n \mid \bar{A} \subseteq \mathcal{U}(t, I) \} \]
be the collection of rectangles in \( \mathcal{P}_n \) whose closures are trapped within the norm-distance \( t \) “neighborhood” of some lattice point in \( I \). The following lemma says that \( \Sigma(\mathcal{T}_n(t, I)) \) is an upper bound not only for \( X_t \), but for \( X_{t-\eta} \) for some \( \eta > 0 \).

**Lemma 5.** There exists \( \eta > 0 \) such that \( \pi^{-1}(X_{t-\eta}) \subseteq \Sigma(\mathcal{T}_n(t, I)) \).

**Proof.** The elements of \( \mathcal{T}_n(t, I) \) have closures contained in the \( \mathcal{U}(t, I) \) and thus in \( \mathcal{U}(t-\eta, I) \) for some \( \eta > 0 \) since \( I \) is finite. If \( s \in \Sigma \) contains the coordinates of \( A \in \mathcal{T}_n(t, I) \), then \( \phi^k \pi(s) \) lies in \( \bar{A} \) for some \( k \), by Remark \( \square \) but then \( M(\pi(s)) = M(\phi^k \pi(s)) < t - \eta \). Thus \( s \notin \pi^{-1}(X_{t-\eta}) \). \( \square \)

The elements of \( \phi \) on \( X_t \) is thus bounded above by the shift entropy of \( \Sigma(\mathcal{T}_n(t, I)) \), which is computable by Perron-Frobenius theory. These upper bounds depend on the set \( I \subseteq \mathcal{O}_K \) and improve as \( I \) grows. The following proposition and its corollary ensure that it is possible to choose \( I \) so that the bounds are tight in the limit as \( n \to \infty \).

**Proposition 2.** There exists a finite set \( I_K \) such that if \( I_K \subseteq I \) and \( t' < t \), then for \( n \) sufficiently large we have
\[
\pi^{-1}(X_t) \subseteq \Sigma(\mathcal{T}_n(t, I)) \subseteq \pi^{-1}(X_{t'}) \tag{5}
\]
In particular, for such \( I \) we have
\[
\pi^{-1}(X_t) = \bigcap_{n \geq 0} \Sigma(\mathcal{T}_n(t, I))
\]

**Proof.** The second assertion here follows immediately from the first. Let \( R_{\text{big}} \) denote the rectangle in \( V_K \) given by \( |s|, |u| < \sqrt{\varepsilon(M_1(K) + 1)} \) and let \( I_K \) be the set of all \( q \in \mathcal{O}_K \) such that \( R - q \) meets \( R_{\text{big}} \) for some \( R \in \mathcal{P}_0 \). The set \( I_K \) is finite and necessarily contains any \( q \) for which there exists some \( A \in \mathcal{P}_n \) such that \( A - q \) meets \( R_{\text{big}} \). Since the diameter of \( \mathcal{P}_n \) tends to zero, there exists \( N \in \mathbb{N} \) such that \( n \geq N \) implies that every translate of \( A \) that meets the region defined by \( N < t' \) in \( R_{\text{big}} \) must have closure entirely contained within the region \( N < t \).

The first containment in (5) is clear from the preceding lemma, and we prove the second by contrapositive. Suppose that \( s \in \Sigma \) is not in \( \pi^{-1}(X_{t'}) \). Then with \( P = \pi(s) \) we have \( M(P) < t' \), so
\[
M(P) < t'' = \min(t', M_1(K) + 1)
\]
Thus we may take \( Q = (s, u) \) as in Lemma \( \square \) representing an element of the orbit of \( P \) with \( N(Q) < t'' \) and
\[
|s|, |u| < \sqrt{\varepsilon t''}
\]
In particular, \( Q \in R_{\text{big}} \). For each \( n \), the point \( Q \) lies in the \( \mathcal{O}_K \)-translates of the the closures of one or more members of the partition \( \mathcal{P}_n \). Let \( A \in \mathcal{P}_n \) and \( q \in \mathcal{O}_K \) such that \( Q \in \bar{A} - q \). Thus \( \bar{A} - q \) meets the region defined by \( N < t' \) in \( R_{\text{big}} \), which requires that \( A - q \) meet this region since \( A \) is open, and hence \( q \in I_K \subseteq I \). Now if \( n \geq N \), it follows that \( A \in \mathcal{T}_n(t, I) \).

By Remark \( \square \), the 0th symbolic coordinate of any element of \( \pi^{-1}(Q) \) must be a member of \( \mathcal{T}_n(t, I) \), which implies that each element of \( \pi_n^{-1}(P) \) has some symbolic coordinate in \( \mathcal{T}_n(t, I) \). This is to say that each element of \( \pi_n^{-1}(P) \), including \( s \), contains the coordinates of some element of \( \mathcal{T}_n(t, I) \), and thus \( s \notin \Sigma(\mathcal{T}_n(t, I)) \). \( \square \)

**Corollary 1.** If \( I_K \subseteq I \), then
\[
h(\phi|X_t) = \lim_{n \to \infty} h(\sigma|\Sigma(\mathcal{T}_n(t, I)))
\]

**Proof.** Let \( \mu_n \) be a measure of maximal entropy for \( \Sigma(\mathcal{T}_n(t, I)) \). Extended to \( \Sigma \), this sequence of measures has some weak-* limit point \( \mu \) in the convex, compact space of invariant probability measures on \( \Sigma \). The measure \( \mu \) is supported on the intersection \( \pi^{-1}(X_t) \), and by upper semi-continuity of entropy in subshifts we have
\[
h(\sigma|\pi^{-1}(X_t)) \geq h(\sigma) \geq \lim \sup h_{\mu_n}(\sigma) = \sup h(\sigma|\Sigma(\mathcal{T}_n(t, I))) \geq \lim \inf h(\sigma|\Sigma(\mathcal{T}_n(t, I))) \geq h(\sigma|\pi^{-1}(X_t))
\]
This implies that \( \mu \) is a measure of maximal entropy for \( \pi^{-1}(X_t) \), as well as the claim. \( \square \)
Corollary 2. The function $t \mapsto \dim(X_t)$ is left-continuous at each point.

Proof. The dimension of a closed, invariant subset $X \subseteq \mathbb{T}_K$ is related to the entropy of $\phi$ on $X$ via

$$\dim(X) = \frac{2h(\phi|X)}{\log(\varepsilon)},$$

so it suffices to prove that $t \mapsto h(\phi|X_t)$ is left continuous. Since this function is decreasing, left-discontinuity at $t$ would imply there exists $B > 0$ such that

$$h(\phi|X_{t-\eta}) - h(\phi|X_t) \geq B \quad \text{for all } \eta > 0$$

By the previous corollary we know there exists $n \in \mathbb{N}$ with

$$h(\sigma|\Sigma(\mathcal{J}_n(t, I_K))) - h(\phi|X_t) < B$$

Now Lemma 5 ensures that $\Sigma(\mathcal{J}_n(t, I_K))$ contains $\pi^{-1}(X_{t-\eta})$ for some $\eta > 0$, which implies

$$h(\sigma|\Sigma(\mathcal{J}_n(t, I_K))) \geq h(\sigma|\pi^{-1}(X_{t-\eta})) = h(\phi|X_{t-\eta}),$$

contradicting the inequalities above. $\square$

6 Applications to the Euclidean Spectrum

The plot of $\dim(X_t)$ contains a number of plateaus as illustrated in the case $K = \mathbb{Q}(\sqrt{5})$ above. Sometimes these are actually set-theoretic plateaus, and the following proposition demonstrates that $\pi^{-1}(X_t)$ is particularly simple in such cases.

Proposition 3. Suppose that $X_t = X_{t-\eta}$ for some $\eta > 0$. Then $\pi^{-1}(X_t)$ is a subshift of finite type.

Proof. By Proposition 2 we may choose $n \in \mathbb{N}$ so that

$$\pi^{-1}(X_t) \subseteq \Sigma(\mathcal{J}_n(t, I)) \subseteq \pi^{-1}(X_{t-\eta}) = \pi^{-1}(X_t)$$

Thus $\pi^{-1}(X_t) = \Sigma(\mathcal{J}_n(t, I))$, which is expressible directly as an SFT via a 0-1 matrix when viewed in block form in $\Sigma_m$ for some $m$ (namely, any $m \geq n - 1$). $\square$

Theorem 3. $\text{ES}(K) \cap K$ is dense in $\text{ES}(K)$.

Proof. First suppose that $t \in \text{ES}(K)$ is an isolated point. By the previous proposition, $\pi^{-1}(X_t)$ is a subshift of finite type, which is to say that it can be described by a 0-1 transition matrix when viewed in block form $\Sigma_m$ for some $m$. Since $t$ is isolated, we know by Lemma 4 that $\pi^{-1}(X_t)$ contains a finite word $w$ that does not occur in $\pi^{-1}(X_{\geq 1})$. Let $s = uvw \in \Sigma$ with $M(\pi(s)) = t$. Viewed in $\Sigma_m$, there is by the Pigeonhole Principle a repeated block in both $u$ and $v$. We can then truncate $u$ and $v$ and loop the segment between these books indefinitely to produce an element $s' \in \pi^{-1}(X_t)$ that contains $w$ and is eventually periodic in both directions. Then $\pi(s')$ is a $K$-point by Lemma 2 and $M(\pi(s')) = t$ since $s'$ contains $w$, so $t \in K$ by Proposition 1.

Now suppose that $t \in \text{ES}(K)$ is not isolated, so there is a strictly monotone sequence $(t_k)$ in $\text{ES}(K)$ with $t_k \to t$. Fixing $k \in \mathbb{N}$, we will show that there is a $K$-point $P$ with such that $M(P)$ lies between $t$ and $t_k$, which will finish the density claim. First suppose that $(t_k)$ increases to $t$. Since $t_{k+1} \in \text{ES}(K)$, Lemma 4 ensures that there exists $s \in \pi^{-1}(X_{t_{k+1}})$ containing a word $w$ that does not occur in $\pi^{-1}(X_t)$. Now take $n$ large enough so that

$$\pi^{-1}(X_{t_{k+1}}) \subseteq \Sigma(\mathcal{J}_n(t_{k+1}, I)) \subseteq \pi^{-1}(X_{t_k})$$

as in Proposition 2. Since $s$ belongs to the SFT $\Sigma(\mathcal{J}_n(t_{k+1}, I))$, we can modify it by looping its ends as in the previous paragraph to obtain another element $s'$ of this SFT that also contains $w$. But then we have $t_k \leq M(\pi(s')) < t$, so $P = \pi(s')$ is the desired $K$-point.
Now suppose that \((t_k)\) is decreasing. Since \(t_{k+1} \in \text{ES}(K)\), Lemma 4 ensures there is word \(w\) occurring in \(\pi^{-1}(X_{t_{k+1}})\) that does not occur in \(\pi^{-1}(X_{t_k})\). Now take \(n\) large enough so that
\[
\pi^{-1}(X_{t_{k+1}}) \subseteq \Sigma(T_n(t_{k+1}, I)) \subseteq \pi^{-1}(X_t)
\]
and proceed as before to produce \(s' \in \Sigma(T_n(t_{k+1}, I))\) that contains \(w\) and is eventually periodic in both directions. We have \(t \leq M(\pi(s')) < t_k\), and again \(P = \pi(s')\) is the desired \(K\)-point. 

**Corollary 3.** The isolated Euclidean minima all lie in \(K\). If \(M_1(K)\) is isolated, then \(M_1(K) \in \mathbb{Q}\).

**Proof.** The first statement is immediate from preceding theorem. If \(M_1(K)\) is isolated, then \(X_{M_1(K)}\) is a nonempty SFT and hence contains a periodic point \(P\). But then \(M_1(K) = M(P) \in \mathbb{Q}\) is forced. 

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