Sobolev Trace Inequalities

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Abstract

The existence of extremal functions for the Sobolev trace inequalities is studied using the concentration compactness theorem. The conjectured extremal, the function of conformal factor, is considered and is proved to be an actual extremal function with extra symmetry condition on functions. One of the limiting cases of the Sobolev trace inequalities is investigated and the best constant for this case is computed.

1 Introduction

The classical Sobolev inequalities on $\mathbb{R}^n$ and the Sobolev trace inequalities on $\mathbb{R}_+^{n+1}$ are given by

$$\left( \int_{\mathbb{R}^n} |f(x)|^s dx \right)^{r/s} \leq c_{r,s} \left( \int_{\mathbb{R}^n} |\nabla f(x)|^r dx \right), \quad \frac{1}{s} = \frac{1}{r} - \frac{1}{n}, \quad (1)$$

where $c_{r,s}$ is a positive constant independent of the function $f$, and

$$\left( \int_{\mathbb{R}^n} |f(x)|^q dx \right)^{p/q} \leq A_{p,q} \left( \int_{\mathbb{R}_+^{n+1}} |\nabla u(x, y)|^p dxdy \right), \quad \frac{1}{q} = \frac{n+1}{np} - \frac{1}{n},$$

where $u$ is an extension of $f$ to the upper half-space, and $A_{p,q}$ is a positive constant independent of the function $u$. In general, Sobolev inequalities provide estimates of lower order derivatives of a function in terms of its higher order derivatives. Recently, the importance of having the sharp form of the inequalities has been recognized. For example, the solution to the Yamabe problem turns out to depend on knowledge of the best constant of (1). In order to obtain the sharp form of inequalities, we often consider the variational problem associated with it. Then, we ask if an extremal function (a minimizer or maximizer) exists subject to some constraints. In fact, the question of existence of an extremal function of the inequality is directly related to that of existence of a solution to the partial differential equation (Euler-Lagrange equation) corresponding to the variational problem.

The sharp form of the Sobolev trace inequality for the case $p = 2$ and $n > 1$ is

$$\left( \int_{\mathbb{R}^n} |f(x)|^{2n/(n-1)} dx \right)^{(n-1)/n} \leq \frac{1}{\sqrt{\pi}} \frac{1}{n-1} \left[ \frac{\Gamma(n)}{\Gamma(n/2)} \right]^{\frac{1}{2}} \left( \int_{\mathbb{R}_+^{n+1}} |\nabla u(x, y)|^2 dxdy \right),$$

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and extremal functions for this inequality are given by \( f(x) = (1 + |x|^2)^{-(n-1)/2} \). Since this inequality is conformally invariant, the extremal function given above is unique up to a conformal automorphism. W. Beckner proved this by inverting the inequality to a fractional integral on the dual space and using a special case of the sharp Hardy-Littlewood-Sobolev inequality. Independently, J. Escobar proved this by exploiting the conformal invariance of this inequality and using characteristics of an Einstein metric. He defined a new metric conformal to the Euclidean metric on the ball, and proved that the metric is, in fact, an Einstein metric based on the information obtained from the Euler-Lagrange equation of the inequality, which implies that the new metric has zero curvature with constant mean curvature on the boundary.

An extremal function for the Sobolev trace inequality for \( 1 < p < n + 1 \)

\[
\left( \int_{\mathbb{R}^n} |f(x)|^q dx \right)^{\frac{q}{p}} \leq A_{p,q} \left( \int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^p dxdy \right), \quad \frac{1}{q} = \frac{n+1}{np} - \frac{1}{n} \quad (2)
\]

was conjectured as the function of the form \( f(x) = (1 + |x|^2)^{-(n+1-p)/(2(p-1))} \).

First, we will be concerned with the existence of extremal functions for the Sobolev trace inequalities when \( 1 < p < n + 1 \). In the study of the existence of extremal functions, a compactness problem arises when we deal with inequalities defined on the spaces which are invariant under dilations and translations. In the case of the Sobolev trace inequalities, this question can be put in the following context. Let \( T \) be the trace operator mapping \( W^{1,p}(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) where \( \frac{1}{q} = \frac{n+1}{np} - \frac{1}{n} \). Then \( T \) is a bounded linear operator. Now we consider the smallest positive constant \( A_{p,q} \) with which the inequality (2) holds for all \( u \) in \( W^{1,p}(\mathbb{R}^n) \) and we ask if the best constant \( A_{p,q} \) is attained for some function \( u \). The question concerning the constant \( A_{p,q} \) is equivalent to the following minimization problem:

\[
\inf \left\{ \int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^p dxdy : u \in W^{1,p}(\mathbb{R}^n), \int_{\mathbb{R}^n} |f(x)|^q dx = 1 \right\},
\]

where \( u \) is an extension of \( f \) to the upper half-space. It is evident that (2) remains unchanged if we replace \( u \) by \( \sigma^{-n/q} u(\cdot/\sigma) \) for \( \sigma > 0 \). This implies possible defects of compactness on minimizing sequences of the problem in the sense that if \( u \) is a minimizer, then \( u_\sigma = \sigma^{-n/q} u(\cdot/\sigma) \) will be another minimizer for each \( \sigma \), and if we let \( \sigma \to 0 \) or \( \sigma \to \infty \), then \( (u_\sigma) \) converges weakly to 0 (which is certainly not a minimizer) and \( (|u_\sigma|^q) \) either converges weakly to a Dirac delta function as \( \sigma \to 0 \), or spreads out as \( \sigma \to \infty \). Using the concentration compactness principle of P. L. Lions, it is proved that any minimizing sequence of the variational problem of the Sobolev trace inequality is relatively compact in \( L^q(\mathbb{R}^n) \) up to translations and dilations, and there exists an extremal function.

We will look at the conjectured extremal function for the Sobolev trace inequality. We can prove that this function is an actual extremal if we assume extra symmetry for the functions considered. In particular, we will consider a
space of functions of conformal factor \([(1 + y)^2 + |x|^2\)], where \((x, y) \in \mathbb{R}^{n+1}_+\).

Then, by simple argument, we can easily show that it is indeed a minimizer for the Sobolev trace inequality restricted on the functions of conformal factor.

We will treat the Sobolev trace inequality for the case with \(p = 1\) separately. The existence of the extremal function for this case is not guaranteed by the argument used for \(p\) with \(1 < p < n + 1\). This case can be thought of as one of the limit cases of the inequality and is very closely related to the isoperimetric inequality. We will show that the extremal function does not exist for this particular case. The sharp constant will be computed using a rearrangement technique on the functions on \(\mathbb{R}^{n+1}_+\).

2 Concentration compactness lemmas and the existence of an extremal function

The Sobolev trace theorem tells us that there is a bounded linear operator from \(W^{1,p}(\mathbb{R}^{n+1}_+)\) to \(L^q(\mathbb{R}^n)\) (which is called the trace operator) where \(1/q = (n + 1)/(np - 1)/n\). This means that there exists a positive constant \(C_0\) for which the following inequality holds for any \(u \in W^{1,p}(\mathbb{R}^{n+1}_+)\):

\[
\left( \int_{\mathbb{R}^n} |u(x,0)|^q dx \right)^{1/q} \leq C_0 \left( \int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^p dxdy \right)^{1/p}.
\]

The question we want to ask is whether there exists an extremal function for which the best constant is attained. To that end, we look at the following minimization problem:

\[
\inf \left\{ J(u) \equiv \int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^p dxdy : \int_{\mathbb{R}^n} |u(x,0)|^q dx = 1, u \in W^{1,p}(\mathbb{R}^{n+1}_+) \right\}. \tag{3}
\]

If an extremal function for (3) exists, then it must satisfy the following Euler-Lagrange equation: for a positive constant \(C\),

\[
\begin{align*}
\text{div}(|\nabla u|^{p-2} \nabla u) &= 0, & \text{on } \mathbb{R}^{n+1}_+ \\
|\nabla u|^{p-2} \frac{\partial u}{\partial y} + C |u|^{q-2} u &= 0, & \text{on } \partial \mathbb{R}^{n+1}_+.
\end{align*}
\tag{4}
\]

Consider a minimizing sequence \((u_k)\) for (3). From the trace theorem, we know that the infimum is finite and we denote it by \(I\). So we have

\[
I = \lim_{k \to \infty} J(u_k),
\]

with \(u_k \in W^{1,p}(\mathbb{R}^{n+1}_+)\) and \(\int_{\mathbb{R}^n} |u_k(x,0)|^q dx = 1\) for each \(k\).

Now \((u_k)\) is a bounded sequence in \(W^{1,p}(\mathbb{R}^{n+1}_+)\) and in \(L^q(\mathbb{R}^n)\). We can find a subsequence (which we will also denote \((u_k)\)) and \(u \in W^{1,p}(\mathbb{R}^{n+1}_+)\) such that
(\(u_k\)) converges weakly to \(u\) in \(W^{1,p}(\mathbb{R}^{n+1})\) and \((u_k(x,0))\) converges weakly to \(u(x,0)\) in \(L^q(\mathbb{R}^n)\). Since the integrand of \(J(\cdot)\) is convex, \(J(\cdot)\) is lower semicontinuous and we have

\[
J(u) \leq \liminf_{k \to \infty} J(u_k) = I
\]

and

\[
\|u\|_{L^q(\mathbb{R}^n)} \leq \liminf_{k \to \infty} \|u_k\|_{L^q(\mathbb{R}^n)} = 1.
\]

If \(\|u\|_{L^q(\mathbb{R}^n)} = 1\), then \(u\) is a minimizer. So the real question is whether or not \(\|u\|_{L^q(\mathbb{R}^n)} = 1\). Since \(J(\cdot)\) and the \(L^q(\mathbb{R}^n)\) norm are invariant under the translations and under the scaling

\[
\{v(\cdot) \mapsto \sigma^{-\frac{n}{q}} v\left(\frac{\cdot}{\sigma}\right)\}
\]

for any \(\sigma > 0\), we may be so unfortunate as to choose a minimizing sequence which has possibilities of failures of the compactness. But a good news is that we can design translations and dilations to avoid the failure of compactness by the concentration compactness theorem. The proof of the existence of an extremal function for the Sobolev trace inequality was sketched by P. L. Lions in his paper \([12]\). His proof is based on the concentration compactness theorem.

We start by stating the concentration compactness lemmas. Hereafter \(B_r(x)\) represents the ball centered at \(x\) with radius \(r\) in \(\mathbb{R}^n, \mathbb{R}^{n+1}\), or \(\mathbb{R}^N\), which will be clear in the context.

**Lemma 1 (Concentration Compactness)**

Let \((\rho_k)\) be a sequence in \(L^1(\mathbb{R}^N)\) satisfying \(\rho_k \geq 0\) in \(\mathbb{R}^N\) and \(\int_{\mathbb{R}^N} \rho_k \, dx = \lambda\) (\(\lambda\) fixed). Then there exists a subsequence \((\rho_{k_j})\) of \((\rho_k)\) satisfying one of the following possibilities:

(i) **(Compactness)** there exists a sequence \((y_j)\) in \(\mathbb{R}^N\) so that for any \(\varepsilon > 0\) there exists \(R \in (0, \infty)\) such that

\[
\int_{B_R(y_j)} \rho_{k_j}(x) \, dx \geq \lambda - \varepsilon.
\]

In this case, \((\rho_{k_j}(\cdot + y_j))\) is called **tight**.

(ii) **(Vanishing)** for any positive real number \(R\),

\[
\lim_{j \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} \rho_{k_j}(x) \, dx = 0.
\]

(iii) **(Dichotomy)** there exists \(\alpha \in (0, \lambda)\) such that for \(\varepsilon > 0\), there exist \(j_0 \geq 1\) and sequences \((\eta_j), (\xi_j)\) in \(L^1(\mathbb{R}^N)\) satisfying for \(j \geq j_0\),

\[
\|\rho_{k_j} - (\eta_j + \xi_j)\|_{L^1(\mathbb{R}^N)} < \varepsilon,
\]

\[
\left| \int_{\mathbb{R}^N} \eta_j(x) \, dx - \alpha \right| \leq \varepsilon, \quad \left| \int_{\mathbb{R}^N} \xi_j(x) \, dx - (\lambda - \alpha) \right| \leq \varepsilon,
\]

and

\[
\text{dist}(\text{supp } \eta_j, \text{supp } \xi_j) \to \infty \quad \text{as } j \to \infty,
\]

where \(\text{dist}(A, B) \equiv \inf\{d(a, b) \mid a \in A \text{ and } b \in B\}\).
Thus, in particular,
\[ \sum_{\gamma} L(\gamma) \] and for some constant C > 0

\[ \nu = \sum_{l \in L} \nu_l \delta_{x_l}, \quad \mu \geq C^{-p} \sum_{l \in L} \nu_l^{p/q} \delta_{x_l}. \]

Thus, in particular, \( \sum_{l \in L} \nu_l^{p/q} < \infty \). If, in addition, \( \nu(R^N)^{1/q} \geq C \mu(R^N)^{1/p} \), then \( L \) reduces to a single point and \( \nu = \gamma \delta_{x_0} = \gamma^{-p/q} C^p \mu \), for some \( x_0 \in R^N \) and for some constant \( \gamma \geq 0 \).

We want to prove that there exists a function for which the following infimum \( I \) is attained:

\[ \inf \left\{ J(u) \equiv \int_{R_n^{n+1}} |\nabla u(x, y)|^p dx dy : \int_{R_n} |u(x, 0)|^q dx = 1, u \in W^{1, p}(R^{n+1}) \right\}. \] (6)

We will replace \( W^{1, p}(R^{n+1}) \) by \( W^{1, p}(R^{n+1}) \) without loss of generality. In this section, we will assume that \( p > 1 \) and this will ensure that \( p < q \), which we need to apply Lemma 2 in the proof of the following theorem. The case for \( p = 1 \) will be treated later separately.

**Theorem 3** Let \( (u_k) \) be a minimizing sequence of (3). Then there exist \( (\sigma_k) \) in \((0, \infty)\) and \( (w_k) \) in \( R^n \) such that the new minimizing sequence \( (\tilde{u}_k) \) given by

\[ \tilde{u}_k(x, y) \equiv \sigma_k^{-\frac{1}{q}} u_k \left( \frac{x - w_k}{\sigma_k}, \frac{y}{\sigma_k} \right), \quad x \in R^n, \quad y \in R \]

is relatively compact in \( L^q(R^n) \). In particular, (3) has a minimum.

**Proof:** Let \( P_k(x, y) \equiv |\nabla u_k(x, y)|^p + |u_k(x, 0)|^q \otimes \delta_0(y) + |u_k(x, y)|^{\frac{p+1}{q}} \). Then \( P_k \geq 0 \) and \( \int_{R^{n+1}} P_k(x, y) dx dy \to L \geq 1 + 1 \) by the Sobolev embedding theorem. The idea is to show that we can prevent vanishing and dichotomy occurring for this sequence of functions by judicial choice of dilations and translations, so that we conclude the claim of the theorem by Lemma 2. Consider the concentration function \( Q_k \) of \( P_k \) defined as

\[ Q_k(t) = \sup_{(x, y) \in R^n \times R} \int_{B_1((x, y))} P_k(w, s) dw ds \quad \text{for} \quad t > 0. \]

Then \((Q_k)\) is a sequence of non-decreasing continuous functions on \( R^+ \). For \( \sigma > 0 \), consider the concentration function \( \tilde{Q}_k \) of

\[ \tilde{P}_k(x, y) \equiv |\nabla \tilde{u}_k(x, y)|^p + |\tilde{u}_k(x, 0)|^q \otimes \delta_0(y) + |\tilde{u}_k(x, y)|^{\frac{p+1}{q}}, \]
where $\tilde{u}_k(x, y)$ is defined as in the statement of the theorem with $\sigma_k = \sigma$. Then we have $Q_k^\sigma(t) = Q_k(\frac{t}{\sigma})$. So, we see a chance of vanishing occur. In order to avoid that, we take a sequence $(\sigma_k)$ of dilations so that $Q_k^\sigma(1) = \frac{1}{2}$. We can see that

$$\lim_{k \to \infty} \sup_{(x, y) \in \mathbb{R}^n \times \mathbb{R}} \int_{B_R((x, y))} \tilde{P}_k(w, s)dwds \geq \frac{1}{2} \quad \text{for} \quad R \geq 1$$

\hspace{0.5cm} since $Q_k^\sigma(t) \geq \frac{1}{2}$ for $t \geq 1$. We prevented vanishing occurring by the choice of dilations. We will denote the new minimizing sequence

$$(\sigma_k, u_k(x/\sigma_k, y/\sigma_k))$$

by $(u_k)$. Now we show that dichotomy does not occur.

**Lemma 4** The dichotomy does not occur.

**Proof:** Suppose it occurs. Then there exists $\lambda^* \in (0, L)$ such that for any $\varepsilon > 0$ there exist $(w_k, \tilde{w}_k) \in \mathbb{R}^n \times \mathbb{R}$ and $R_k$, $k = 0, 1, 2, \cdots$ with $R_k > R_0$ (for $k = 1, 2, \cdots$) and $R_k \to \infty$ so that

$$\left| \lambda^* - \int_{B_{R_0}((w_k, \tilde{w}_k))} P_k(x, y)dxdy \right| < \varepsilon,$$

$$\left| (L - \lambda^*) - \int_{[B_{R_k}((w_k, \tilde{w}_k))]^C} P_k(x, y)dxdy \right| < \varepsilon,$$

$$\int_{R_0 < |(x, y) - (w_k, \tilde{w}_k)| < R_k} \supp [P_k \chi_{B_{R_0}((w_k, \tilde{w}_k))}] \subset B_{R_0}((w_k, \tilde{w}_k)),$$

$$\supp [P_k(1 - \chi_{B_{R_k}((w_k, \tilde{w}_k))})] \subset [B_{R_k}((w_k, \tilde{w}_k))]^C,$$

$$\text{dist}\left(\supp [P_k \chi_{B_{R_0}((w_k, \tilde{w}_k))}], \supp [P_k(1 - \chi_{B_{R_k}((w_k, \tilde{w}_k))})]\right) \geq \text{dist}(B_{R_0}((w_k, \tilde{w}_k)), [B_{R_k}((w_k, \tilde{w}_k))]^C) \to \infty \quad \text{as} \quad k \to \infty.$$ Consider $\xi, \eta \in C_b^\infty(\mathbb{R}^{n+1})$ satisfying $0 \leq \xi, \eta \leq 1$, and

$$\xi(x, y) = \begin{cases} 1 & \text{if} \quad |(x, y)| \leq 1 \\ 0 & \text{if} \quad |(x, y)| \geq 2, \end{cases}$$

$$\eta(x, y) = \begin{cases} 0 & \text{if} \quad |(x, y)| \leq \frac{1}{2} \\ 1 & \text{if} \quad |(x, y)| \geq 1. \end{cases}$$

We may take $R_1$ so that $4R_1 \leq R_k$ for $k = 2, 3, \cdots$. Define

$$\xi_k(x, y) \equiv \xi(\frac{x - w_k}{R_1}, \frac{y - \tilde{w}_k}{R_1}) \quad \text{and} \quad \eta_k(x, y) \equiv \eta(\frac{x - w_k}{R_k}, \frac{y - \tilde{w}_k}{R_k}).$$

We look at the following quantity: for $k$ large enough,

$$M = \int_{\mathbb{R}^{n+1}} |\nabla u_k|^p dxdy - \int_{\mathbb{R}^{n+1}} |\nabla (u_k \xi_k)|^p dxdy - \int_{\mathbb{R}^{n+1}} |\nabla (u_k \eta_k)|^p dxdy$$
that the assumptions in the beginning of the lemma, we show

Using Hölder’s inequality and the Sobolev embedding theorem together with the assumptions in the beginning of the lemma, we show

\[ M_2^{1/p} = \left( \int_{B_{2R_1} - B_{R_1}} |\nabla u_k|^p \, dx \, dy \right)^{1/p} + \left( \int_{B_{2R_1} - B_{R_1}} |u_k|^p |\nabla \xi_k|^p \, dx \, dy \right)^{1/p} \leq \varepsilon^{1/p} + \left( \int_{R^{n+1}} |\nabla \xi_k|^{p+1} \, dx \, dy \right)^{1/p} \]

\[ \leq \varepsilon^{1/p} + C \varepsilon^{\frac{np}{n+1}}. \]

(All balls in the above are centered at \((w_k, \tilde{w}_k)\).) Similarly, we can show that \( M_3^{1/p} < \varepsilon + C \varepsilon^{\frac{np}{n+1}}. \)

Denote \( u_{1k} \equiv u_k \xi_k, u_{2k} \equiv u_k \eta_k. \) By combining these estimates, we finally have

\[ |M| = \left| \int_{R^{n+1}} |\nabla u_k|^p \, dx \, dy - \int_{R^{n+1}} |\nabla u_{1k}|^p \, dx \, dy - \int_{R^{n+1}} |\nabla u_{2k}|^p \, dx \, dy \right| < \varepsilon + C \varepsilon^{\frac{np}{n+1}}. \]

In other words,

\[ I = \lim_{k \to \infty} \int_{R^{n+1}} |\nabla u_k|^p \, dx \, dy = \lim_{k \to \infty} \int_{R^{n+1}} |\nabla u_{1k}|^p \, dx \, dy + \lim_{k \to \infty} \int_{R^{n+1}} |\nabla u_{2k}|^p \, dx \, dy. \]

It follows from the assumptions at the beginning that

\[ \left| \int_{R^n} |u_{2k}|^q \, dx - \left( \int_{R^n} |u_k|^q \, dx - \int_{R^n} |u_{1k}|^q \, dx \right) \right| \leq \int_{B_{R_k}((w_k, \tilde{w}_k)) - B_{R_1}((w_k, \tilde{w}_k))} |u_k(x, 0)|^q \otimes \delta_0(y) \, dx \, dy \]

\[ \leq \int_{R_0 \leq |(x, y) - (w_k, \tilde{w}_k)| \leq R_k} |u_k(x, y)|^q \otimes \delta_0(y) \, dx \, dy < \varepsilon. \]  

Let \( \alpha_k \equiv \int_{R^n} |u_{1k}(x, 0)|^q \, dx, \) and \( \beta_k \equiv \int_{R^n} |u_{2k}(x, 0)|^q \, dx. \) By taking a subsequence, if necessary, we may assume that \( \alpha_k \to \alpha, \) and \( \beta_k \to \beta. \) We can see that

\[ 0 \leq \alpha, \beta \leq 1 \quad \text{and} \quad |\beta - (1 - \alpha)| < \varepsilon. \]
Use the estimates for $M$ to observe that
\[
\int_{\mathbb{R}^{n+1}} \left| \nabla u_{1k}(x,y) \right|^p + \left| u_{1k}(x,y) \right|^{\frac{n+1}{n}} + \left| u_{1k}(x,0) \right|^q \otimes \delta_0(y) \, dx \, dy - \lambda^* < \varepsilon,
\]
\[
\int_{\mathbb{R}^{n+1}} \left| \nabla u_{2k}(x,y) \right|^p + \left| u_{2k}(x,y) \right|^{\frac{n+1}{n}} + \left| u_{2k}(x,0) \right|^q \otimes \delta_0(y) \, dx \, dy - (L - \lambda^*) < \varepsilon.
\]

We can also see that $\int_{\mathbb{R}^{n+1}} \left| \nabla u_{ik}(x,y) \right|^p \, dx \, dy \geq \gamma > 0$ for $i = 1, 2$, and $\gamma$ a positive constant using the Sobolev embedding theorem and the Sobolev trace inequalities together with the estimates above. Now we look at all the possible values for $\alpha$ and $\beta$. They are:

(a) $\alpha_k \to 0 (\beta_k \to 1)$,  
(b) $\alpha \neq 0 (\beta \neq 1)$,  
(c) $\alpha_k \to 1 (\beta_k \to 0)$,  
(d) $\beta \neq 0 (\alpha \neq 1)$.

By exchanging the roles of $\alpha_k$ and $\alpha$ with $\beta_k$ and $\beta$, the cases (c) and (d) reduce to the cases (a) and (b). In the case (b), it follows from the estimates for $M$ that $I \geq \gamma + I - \varepsilon$ for all small $\varepsilon$, which leads to a contradiction that $I \geq \gamma + I > I$. For the case (b), we define $I_\alpha$ as
\[
I_\alpha \equiv \inf \left\{ J(u) \equiv \int_{\mathbb{R}^{n+1}} \left| \nabla u(x,y) \right|^p \, dx \, dy + \int_{\mathbb{R}^n} \left| u(x,0) \right|^q \, dx : u \in W^{1,p}(\mathbb{R}^{n+1}) \right\}.
\]
It easily follows from the definition that $I = I_1$ and $I_\alpha = \alpha^{p/q}I$. It can be also shown that
\[
I < I_\alpha + I_{1-\alpha} \quad \text{for} \quad 0 < \alpha < 1.
\]
This is called \textit{Strict Subadditivity}. Now, in the case (b), we have $I \geq I_\alpha + I_{1-\alpha} - \varepsilon$ for all small $\varepsilon > 0$ which violates the strict subadditivity. This completes the proof of this lemma.

Since we have shown that vanishing and dichotomy can not occur, we now conclude by Lemma 1 that we have the compactness as follows: there exists a sequence $(w_k, \tilde{w}_k) \in \mathbb{R}^n \times \mathbb{R}$ so that for any $\varepsilon > 0$, there is $R \in (0, \infty)$ such that
\[
\int_{|B_R((w_k,\tilde{w}_k))|^C} |\nabla u_k(x,y)|^p \, dx \, dy + \int_{|B_R((w_k,\tilde{w}_k))|^C} |u_k(x,y)|^{\frac{n+1}{n}} \, dx \, dy + \int_{|B_R((w_k,\tilde{w}_k))|^C \cap \mathbb{R}^n \times \{0\}} |u_k(x,0)|^q \, dx < \varepsilon. \tag{8}
\]

\textbf{Remark 5} We may choose $\tilde{w}_k = 0$. 

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Proof: If \( \varepsilon < 1 \), then \( |\tilde{w}_k| \leq R \). Otherwise, we would have \( B_{R}( (w_k, \tilde{w}_k) ) \subset \mathbb{R}^n \times (\mathbb{R} - \{0\}) \), implying

\[
\int_{\mathbb{R}^n} |u_k(x, 0)|^q \, dx \leq \int_{\mathbb{R}^n \times (\mathbb{R} - \{0\})} P_k(x, y) \, dx \, dy < \varepsilon < 1,
\]

which violates the assumption that \( \int_{\mathbb{R}^n} |u_k(x, 0)|^q \, dx = 1 \). Take \( (w_k, 0) \in \mathbb{R}^n \times \{0\} \) and replace \( R \) by \( 2R \). Then we have the compactness we had before

\[
\int_{|B_{2R(w_k, 0)}|} P_k(x, y) \, dx \, dy \leq \int_{|B_{R(w_k, \tilde{w}_k)}|} P_k(x, y) \, dx \, dy < \varepsilon. \quad \Box
\]

We denote by \( (u_k) \) the new minimizing sequence \( (\tilde{u}_k) \) defined by \( \tilde{u}_k(x, y) \equiv u_k(x + w_k, y) \) for all \( (x, y) \in \mathbb{R}^n \times \mathbb{R} \). We may assume that

\[
\begin{align*}
    u_k &\to u \quad \text{a.e. in} \quad \mathbb{R}^{n+1}, \\
u &\to u \quad \text{a.e. in} \quad \mathbb{R}^n \times \{0\}, \\
    u_k &\to u \quad \text{in} \quad W^{1,p}(\mathbb{R}^{n+1}), \\
u &\to u \quad \text{in} \quad L^{q}(\mathbb{R}^n \times \{0\}).
\end{align*}
\]

Lemma 6 (Concentration Compactness III) Let \( (u_k) \) be a bounded sequence in \( W^{1,p}(\mathbb{R}^{n+1}) \) such that \( (|\nabla u_k(x, y)|^p) \) is tight. We may assume \( u_k \to u \) a.e. in \( \mathbb{R}^{n+1} \) and \( (|\nabla u_k(x, y)|^p) \) converge weakly to some bounded nonnegative measures \( \mu, \nu \) on \( \mathbb{R}^{n+1} \) and \( \text{supp} (\nu) \subset \mathbb{R}^n \times \{0\} \). Then (i) There exist some at most countable set \( \mathcal{L} \) and two families \( (x_l)_{l \in \mathcal{L}} \) of distinct points in \( \mathbb{R}^n \), \( (\nu_l)_{l \in \mathcal{L}} \) in \( (0, \infty) \) such that

\[
\nu = |u(x, 0)|^q \otimes \delta_0(y) + \sum_{l \in \mathcal{L}} \nu_l \delta_{x_l, 0},
\]

\[
\mu \geq |\nabla u(x, y)|^p + \sum_{l \in \mathcal{L}} I \nu_l \delta_{x_l, 0}.
\]

(ii) If \( u = 0 \) and \( \mu(\mathbb{R}^{n+1}) \leq I \nu(\mathbb{R}^{n+1})^{p/q} \), then \( \mathcal{L} \) is a singleton and \( \nu = c_0 \delta_{(x_0, 0)} \), and \( \mu = I c_0^{p/q} \delta_{(x_0, 0)} \) for some \( c_0 > 0 \), and for some \( x_0 \in \mathbb{R}^n \).

Proof: We first look at the case \( u \equiv 0 \). By the Sobolev trace inequality, we have for \( \varphi \in C_0^\infty(\mathbb{R}^{n+1}) \)

\[
\left( \int_{\mathbb{R}^n} |\varphi(x, 0)u_k(x, 0)|^q \, dx \right)^{1/q} \leq I^{-1/p} \left( \int_{\mathbb{R}^{n+1}} |\nabla(\varphi(x, y)u_k(x, y))|^p \, dx \, dy \right)^{1/p}. \tag{9}
\]

The left-hand side of (9)

\[
\left( \int_{\mathbb{R}^n} |\varphi(x, 0)u_k(x, 0)|^q \, dx \right)^{1/q} = \left( \int_{\mathbb{R}^{n+1}} |\varphi(x, y)u_k(x, 0)|^q \otimes \delta_0(y) \, dx \, dy \right)^{1/q}
\]
Lemma 6. Now consider the general case that the weak limit $u_0$. Let $v = (\int_{\mathbb{R}^{n+1}} |\varphi(x,y)|^p d\mu)^{1/p}$. Taking $k \to \infty$ in (9) yields for $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$,

$$\left(\int_{\mathbb{R}^{n+1}} |\varphi(x,y)|^q d\nu\right)^{1/q} \leq \Gamma^{1/p} \left(\int_{\mathbb{R}^{n+1}} |\varphi(x,y)|^p d\mu\right)^{1/p}.$$

By applying Lemma 2 to two measures $\mu$ and $\nu$ on $\mathbb{R}^{n+1}$, we obtain the results of Lemma 3. Now consider the general case that the weak limit $u$ is not necessarily 0. Let $v_k = u_k - u$. By applying to $(v_k)$ what we have proved for $(u_k)$ above and using Brézis-Lieb lemma saying that for $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\varphi|^q |u_k|^q dx - \int_{\mathbb{R}^n} |\varphi|^q |v_k|^q dx \text{ converges to } \int_{\mathbb{R}^n} |\varphi|^q |u|^q dx,$$

we have the representation for $\nu = |u(x,0)|^q \otimes \delta_0(y) + \sum_{l \in \mathcal{L}} \nu_l \delta_{(x_l,0)}$ for some countable set $\mathcal{L}$. We have that for $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$,

$$\Gamma^{1/p} \left(\int_{\mathbb{R}^{n+1}} |\varphi(x,y)|^q |u_k(x,y)|^q \otimes \delta_0(y) dxdy\right)^{1/q} \leq \left(\int_{\mathbb{R}^{n+1}} |\varphi(x,y)|^p |\nabla u_k(x,y)|^p dxdy\right)^{1/p} + \left(\int_{\mathbb{R}^{n+1}} |\nabla \varphi(x,y)|^p |u_k(x,y)|^p dxdy\right)^{1/p} \quad (10)$$

and

$$\int_{\mathbb{R}^{n+1}} |\nabla \varphi|^p |u_k|^p dxdy \text{ converges to } \int_{\mathbb{R}^{n+1}} |\nabla \varphi|^p |u|^p dxdy,$$

since $|\nabla \varphi|$ has compact support. Passing to the limit in (10), we have

$$\Gamma^{1/p} \left(\int_{\mathbb{R}^{n+1}} |\varphi|^q d\nu\right)^{1/q} \leq \left(\int_{\mathbb{R}^{n+1}} |\varphi|^p d\mu\right)^{1/p} + \left(\int_{\mathbb{R}^{n+1}} |\nabla \varphi|^p |u|^p dxdy\right)^{1/p}. \quad (11)$$

Take $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$ satisfying $0 \leq \varphi \leq 1$, $\varphi(0) = 1$, and supp $\varphi = B_1(0)$. Apply (11) to $\varphi(x_l,0)$ for $l \in \mathcal{L}$ and $\varepsilon$ positive and small enough, to have

$$\Gamma^{1/p} \left(\int_{B_\varepsilon(x_l_0)} |\varphi(x,y)|^q d\nu\right)^{1/q} \leq \mu(B_\varepsilon(x_l,0))^{1/p} + \left(\int_{B_\varepsilon(x_l_0)} |\nabla \varphi(x_l,0)|^p |u(x,y)|^p dxdy\right)^{1/p}.$$

By the Sobolev embedding theorem, we have

$$\Gamma^{1/p} \left(\int_{B_\varepsilon(x_l_0)} |\nabla \varphi(x_l,0)|^p |u(x,y)|^p dxdy\right)^{1/p}$$
Then it gives a contradiction saying
and
\(|\nabla \phi(x, y)|^{n+1} dxdy\rangle^{1/(n+1)}
\leq D \left( \int_{B_x(x,0)} |u(x,y)|^{\frac{n+1}{n}} dxdy \right)^{\frac{n}{n+1}}
where D is a positive constant. Taking \(\varepsilon \to \infty\) yields
\[I^{1/p}\mu \left( \{ (x_1,0) \} \right)^{1/q} \leq \mu \left( \{ (x_1,0) \} \right)^{1/p},\]
then
\[I^{1/p}\mu_1^{1/q} \leq \mu \left( \{ (x_1,0) \} \right)^{1/p},\]
and so,
\[\mu \geq I^{p/q}\delta_{(x_1,0)} \text{ for } l \in \mathcal{L}.\]
Thus, \(\mu \geq \sum_{l \in \mathcal{L}} I^{p/q}_{l} \delta_{(x_1,0)}\). Let \(\sum_{l \in \mathcal{L}} I^{p/q}_{l} \delta_{(x_1,0)} = \mu_1\). By the fact that two measures \(\mu_1\) and \(|\nabla u|^p\) are orthogonal, and \(\mu \geq |\nabla u|^p\) by the weak convergence, we conclude that \(\mu \geq |\nabla u|^p + \sum_{l \in \mathcal{L}} I^{p/q}_{l} \delta_{(x_1,0)}\) to complete the proof. \(\square\)

**Lemma 7** \(u \not\equiv 0\).

**Proof** : Suppose \(u \equiv 0\). Then \((u_k)\) converges weakly to \(0\) in \(W^{1,p}(\mathbb{R}^{n+1})\). We know that \(|\nabla u_k(x,y)|^p\) converges weakly to \(\mu\) tightly in the space of measures and \(|u_k(x,0)|^q \otimes \delta_0(y)\) converges weakly to \(\nu\) \((\text{supp}(\nu) \subset \mathbb{R}^n \times \{0\})\) from (8).

We can see that
\[
\int_{\mathbb{R}^{n+1}} d\mu = \lim_{k \to \infty} \int_{\mathbb{R}^{n+1}} |\nabla u_k(x,y)|^p dxdy = I,
\]
\[
\int_{\mathbb{R}^{n+1}} d\nu = \lim_{k \to \infty} \int_{\mathbb{R}^{n+1}} |u_k(x,0)|^q \otimes \delta_0(y) dxdy = 1.
\]
In other words,
\[\mu(\mathbb{R}^{n+1}) = I = I \nu(\mathbb{R}^{n+1}).\]

By Lemma \[\[\], there exists \(x_0 \in \mathbb{R}^n\), and so that \(\nu = \delta_{(x_0,0)}\) and \(\mu = I \delta_{(x_0,0)}\). Then it gives a contradiction saying
\[
\frac{1}{2} = Q_k(1) \geq \int_{B_1(x_0,0)} |u_k(x,0)|^q \otimes \delta_0(y) dxdy \to \int_{B_1(x_0,0)} d\nu = \nu(B_1(x_0,0)) = 1,
\]
so we complete the proof. \(\square\)

Let \(\int_{\mathbb{R}^n} |u(x,0)|^q ddx = \int_{\mathbb{R}^{n+1}} |u(x,0)|^q \otimes \delta_0(y) dxdy = \alpha\). From Lemma \[\[\], we have \(0 < \alpha \leq 1\). Now it is sufficient to show that \(\alpha = 1\) in order to prove Theorem \[\[\]. So suppose \(\alpha \neq 1\). By Lemma \[\[\], there exist a set \(\mathcal{L}\) at most countable, \((x_l)_{l \in \mathcal{L}} \subset \mathbb{R}^n\) and \((\nu_l)_{l \in \mathcal{L}} \in (0, \infty)\) such that
\[\nu = |u(x,0)|^q \otimes \delta_0(y) + \sum_{l \in \mathcal{L}} \nu_l \delta_{(x_l,0)}\] (so, \(1 = \alpha + \sum_{l \in \mathcal{L}} \nu_l\),
\[ \mu \geq |\nabla u(x, y)|^p + \sum_{l \in L} l^{p/q} \delta(x_l, 0). \]

This leads us to a following contradiction:

\[
\begin{align*}
I_\alpha & \leq \int_{\mathbb{R}^{n+1}} |\nabla u(x, y)|^p dxdy \\
& \leq \int_{\mathbb{R}^{n+1}} d\mu - \int_{\mathbb{R}^{n+1}} \sum_{l \in L} l^{p/q} \delta(x_l, 0) dxdy \\
& = I - \sum_{l \in L} l^{p/q} = I(1 - \sum_{l \in L} l^{p/q}) \\
& < I(1 - \sum_{l \in L} l^{p/q}) = I^{p/q} = I_\alpha.
\end{align*}
\]

The last inequality holds since \( \sum_{l \in L} l^{p/q} = 1 - \alpha \neq 0 \). So we conclude that \( \alpha = 1 \), and this proves that there exists an extremal function for the trace inequality. This completes the proof of Theorem 3. \( \square \)

3 Conjectured extremal function

Any extremal function for the Sobolev trace inequality satisfies the following Euler-Lagrange equation: for a positive constant \( C \),

\[
\begin{align*}
\text{div}(|\nabla u|^{p-2} \nabla u) &= 0 & \text{on } \mathbb{R}^{n+1}_+ \\
|\nabla u|^{p-2} \frac{\partial u}{\partial y} + C|u|^{q-2} u &= 0 & \text{on } \partial \mathbb{R}^{n+1}_+
\end{align*}
\]

(12)

It can be easily proved that there is no radial function in all the variables in \((x, y) \in \mathbb{R}^{n+1}_+\) satisfying the equation (12) due to the boundary condition. As a way to identify a function which satisfies (12), we will look at a restricted class of functions. In particular, we will restrict our attention to the functions of conformal factor, \( [(1 + y)^2 + |x|^2]^{-\frac{n+1}{2(p-1)}} \), where \((x, y) \in \mathbb{R}^{n+1}_+\). This means we assume an extra symmetry for possible extremal functions for the Sobolev trace inequality. This choice of symmetry is not surprising if we look at the extremal function for the special case of the Sobolev trace inequality with \( p = 2 \). This choice also specifies the function on the boundary as a function of \([1 + |x|^2]^{-\frac{n+1}{2(p-1)}}\). This condition is not at all strict since it suffices to consider radial decreasing functions on \( \mathbb{R}^n \) for extremal functions by using a rearrangement technique. J. Escobar conjectured the extremal function for the Sobolev trace inequality in \( \mathbb{R}^n \) as \( [(1 + y)^2 + |x|^2]^{-\frac{n+1}{2(p-1)}} \). The following remark will make it clear that it is the only possible choice of function for the extremal function.

Remark 8 (the conjectured extremal for the Sobolev trace inequality)

Suppose that \( f \) is an extremal function for the Sobolev trace inequality and that \( f \) is a function of \([1 + y]^2 + |x|^2]^{-\frac{n+1}{2(p-1)}}\). We may also assume that \( f \) is decreasing in \(|x|\), and in \( y \). Then \( f(x, y) \) is exactly the same function that was conjectured.
Proof : Let \( f(x, y) \equiv \Phi(v(x, y)) \), where \( \Phi \) is a function of one variable and \( v(x, y) \equiv (1 + y)^2 + |x|^2 \). This gives the following equations:

\[
\frac{\partial f}{\partial x_j}(x, y) = 2x_j \Phi'(v) \quad \frac{\partial f}{\partial y}(x, y) = 2(1 + y)\Phi'(v) \\
\frac{\partial^2 f}{\partial x_j^2}(x, y) = 2[2x_j^2 \Phi''(v) + \Phi'(v)] \quad \frac{\partial^2 f}{\partial y^2}(x, y) = 2[2(1 + y)^2 \Phi''(v) + \Phi'(v)],
\]

where \( \cdot \) denotes the derivative with respect to \( v \). These equations and the fact that \( f \) satisfies (12) since \( f \) is an extremal function yield

\[
\text{div}(|\nabla f|^{p-2} \nabla f) = 2^{p-1} \Phi'^{p-2} v^{\frac{p}{2} - 1} [2(p-1)\Phi''(v)v + (n + p - 1)\Phi'(v)] = 0.
\]

Since \( f \) is not a constant function, we have the equation that \( \Phi \) must satisfy:

\[
[2(p-1)\Phi''(v)v + (n + p - 1)\Phi'(v)] = 0.
\]

From this, we have

\[
[\ln |\Phi'(v)|]' = \frac{\Phi''(v)}{\Phi'(v)} = \frac{-n + p - 1}{2(p - 1)} v.
\]

Hence we obtain \( \Phi(v) = c_0 v^{-\frac{n+1-p}{2(p-1)}} = c_0 [(1+y)^2 + |x|^2]^{-\frac{n+1-p}{2(p-1)}} \), for some constant \( c_0 \), which can be determined uniquely by the condition that \( \|f\|_{L^p(\mathbb{R}^n)} = 1 \). This function is the very function that Escobar conjectured. \qed

The following proposition characterizes this function as the minimizer of the Sobolev trace inequality functional when restricted to the class of functions of conformal factor. For this we define

\[
\mathcal{J}(\omega) \equiv \int_{\mathbb{R}^{n+1}} |\nabla \omega(x, y)|^p dxdy,
\]

where \( \omega \) belongs to the admissible set

\[
\mathcal{A} \equiv \{ \omega \in W^{1,p}(\mathbb{R}^{n+1}_+) : \omega \text{ is a function of } [(1+y)^2 + |x|^2], \quad \omega(x,0) = c_0(1+|x|^2)^{-\frac{n+1-p}{2(p-1)}} \}.
\]

Proposition 9 Let \( f \) be the conjectured extremal function for the Sobolev trace inequality. Then

\[
\mathcal{J}(f) = \min_{\omega \in \mathcal{A}} \mathcal{J}(\omega),
\]

in other words, the infimum of \( \mathcal{J}(\cdot) \) on \( \mathcal{A} \) is attained at \( f \).
**Proof :** Take any \( \omega \in A \) and consider \( f - \omega \). Since \( f \) satisfies the equation (12), we have

\[
\text{div} \left( |\nabla f|^{p-2} \nabla f \right) (f - \omega) = 0 \text{ on } \mathbb{R}^{n+1}.
\]

An integration by parts yields

\[
0 = \int_{\mathbb{R}^{n+1}} |\nabla f|^{p-2} (|\nabla f|^2 - \nabla f \cdot \nabla \omega) \, dx dy
\]

and there is no boundary term since \( f - \omega = 0 \) on \( \partial \mathbb{R}^{n+1} \) by the fact that both \( f \) and \( \omega \) belong to \( A \). Now Young's inequality gives

\[
J(f) = \int_{\mathbb{R}^{n+1}} |\nabla f|^p \, dx dy \leq (1 - \frac{1}{p}) \int_{\mathbb{R}^{n+1}} |\nabla f|^{p-2} (\nabla f \cdot \nabla \omega) \, dx dy.
\]

We obtain

\[
J(f) \leq J(\omega) \quad (\omega \in A).
\]

4 **Sobolev trace inequality with \( p = 1 \)**

In this section, we will treat the Sobolev trace inequality for the case when \( p = 1 \) (thus \( q = 1 \)) separately. The existence of the extremal function for the Sobolev trace inequality for the case when \( p = 1 \) (\( q = 1 \)) is not guaranteed by the argument used for \( p \) with \( 1 < p < n + 1 \). This is one of the limit cases of the inequality and is closely related to the isoperimetric inequality.

The Sobolev trace inequality for \( p = 1 \) is given by

\[
\int_{\mathbb{R}^n} |u(x,0)| \, dx \leq C \int_{\mathbb{R}^{n+1}} |\nabla u(x,y)| \, dx dy
\]

for a positive constant \( C \). To find the best constant for this inequality, we look at the following quotient:

\[
J(u) \equiv \frac{\int_{\mathbb{R}^{n+1}} |\nabla u(x,y)| \, dx dy}{\int_{\mathbb{R}^n} |u(x,0)| \, dx},
\]

where \( u \in W^{1,1}(\mathbb{R}^{n+1}) \) and \( u \neq 0 \). The best constant \( I \) is defined by

\[
I \equiv \inf \{J(u) : u \in W^{1,1}(\mathbb{R}^{n+1}), \ u \neq 0 \}.
\]

Define \( B \equiv \{ g \in W^{1,1}(\mathbb{R}^{n+1}) : g \geq 0 \text{ on } \mathbb{R}^{n+1}, \int_{\mathbb{R}^n} g(x,0) \, dx = 1 \} \). It is sufficient to consider functions in \( B \) to compute the best constant, since \( J(\cdot) \) is dilation invariant and \( J(u) = J(|u|) \). Moreover, we will use a rearrangement
technique to reduce further the functions to consider to a class of functions with a special property. Namely, we will take $\Phi_S^*$ to be the Steiner rearrangement of $\Phi$. Here $\Phi_S^*$ is symmetric radial decreasing in $x$, and is decreasing in $y$. Then we know that

$$
\int_{\mathbb{R}^n} |\Phi(x, 0)| dx = \int_{\mathbb{R}^n} |\Phi_S^*(x, 0)| dx = \int_{\mathbb{R}^n} \int_0^\infty -\frac{\partial \Phi_S^*}{\partial y}(x, y) dydx
\leq \int_{\mathbb{R}^{n+1}_+} |\nabla \Phi_S^*(x, y)| dxdy
\leq \int_{\mathbb{R}^{n+1}_+} |\nabla \Phi(x, y)| dxdy. \quad (13)
$$

By the above observation, it suffices to consider functions in $\mathcal{B}$ having the following property $(P)$:

$$(P): g \text{ is symmetric radial decreasing in } x \text{ and decreasing in } y.$$

For any function $g$ having the property $(P)$, the inequality $(13)$ becomes equality. It is now clear that $\inf \{ J(g) \mid g \text{ has the property (P), } g \in \mathcal{B} \} \geq 1.$

**Theorem 10**

$$I \equiv \inf \{ J(u) : u \in W^{1,1}(\mathbb{R}^{n+1}), \ u \neq 0 \} = 1.$$

**Proof:** We will look at the inequalities above. The inequality $(14)$ becomes equality, since we choose $f$ with the property $(P)$. The question is when the inequality $(13)$ becomes equality. For that we require that $f$ satisfy

$$\left| \frac{\partial f}{\partial y}(x, y) \right| = |\nabla f(x, y)| \text{ on } \mathbb{R}^{n+1}_+.$$

This means that

$$\frac{\partial f}{\partial x_j}(x, y) = 0 \text{ on } \mathbb{R}^{n+1}_+ \text{ for } j = 1, 2, 3, \cdots, n.$$

From this, we can see that $f$ should be a function of $y$ variable only. On the other hand, $f(x, 0)$ is a function in $L^p(\mathbb{R}^n)$, so we need some restriction on the function. Any function of $y$ with appropriate decay multiplied by a characteristic function in the $x$ variable will be an extremal function. The problem is that such functions do not belong to $W^{1,1}(\mathbb{R}^{n+1}_+)$, which means that the extremal function does not exist. However, we can use an approximation argument to compute the best constant. Take a function $f(x, y) = \phi(y)\chi_B(x)$, where $\phi$ is a positive non-increasing function of $y$ variable and $B$ is the unit ball centered at the origin in $\mathbb{R}^n$. Then we have

$$\int_{\mathbb{R}^{n+1}_+} |\nabla f(x, y)| dxdy = \int_{\mathbb{R}^n} |f(x, 0)| dx + \sigma_n \int_0^\infty \phi(y)dy$$
where $\sigma_n$ is the surface area of the unit ball in $\mathbb{R}^n$. If we can make the second term in the right hand side go away, then we get the claim we made. Let 

$$\phi_\varepsilon(y) = \exp\left(-\frac{\pi y^2}{\varepsilon}\right).$$

Then 

$$\int_0^\infty \phi_\varepsilon(y) dy = \sqrt{\varepsilon},$$

so that we can make it as small as we want.

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