On the rationality and continuity of logarithmic growth filtration of solutions of $p$-adic differential equations

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Abstract

We study the asymptotic behavior of solutions of Frobenius equations defined over the ring of overconvergent series. As an application, we prove Chiarellotto-Tsuzuki’s conjecture on the rationality and the right continuity of Dwork’s logarithmic growth filtrations associated to ordinary linear $p$-adic differential equations with Frobenius structures.

Contents

1 Introduction .......................... 2

2 Summary of notation .................. 4

2.1 Coefficient rings .................... 4

2.2 Various rings of functions ............ 4

2.3 Filtration and Newton polygon ....... 6

3 Chiarellotto-Tsuzuki’s conjectures and main theorem .......... 6

3.1 σ-modules .......................... 6

3.2 Log-growth filtration ................ 7

3.3 Chiarellotto-Tsuzuki’s conjectures .... 8

3.4 Main theorem ....................... 9

4 Log-growth of analytic ring ........... 9

4.1 Overconvergent rings ............... 9

4.2 Log-growth filtration over $\Gamma_{\text{con}}[p^{-1}]$ .......... 10

4.3 Chiarellotto-Tsuzuki’s conjecture over $\Gamma_{\text{con}}[p^{-1}]$ .... 13

4.4 Example: $p$-adic differential equations with nilpotent singularities .... 13

5 Generic cyclic vector ................ 14

5.1 Proof of Theorem 5.2 ............. 15

6 Frobenius equation and log-growth .......... 16

6.1 Estimation of upper bound .......... 17

6.2 Estimation of lower bound .......... 18

6.3 Proof of Theorem 6.1 ............. 21

7 Proof of Theorem 4.19 ............... 21

8 Appendix: diagram of rings .......... 24

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1 Introduction

We consider an ordinary linear $p$-adic differential equation

$$Dy = \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = 0,$$

whose coefficients are bounded on the $p$-adic open unit disc $|x| < 1$. We define its solution space by

$$\text{Sol}(D) := \{ y \in \mathbb{Q}_p[x]; Dy = 0 \}.$$

In her study of $p$-adic elliptic functions, Lutz proves that any solution $y$ of $Dy = 0$ has a non-zero radius of convergence $r$ ([Lut37, Théorème IV]). In the paper [Dwo73], Dwork studies the asymptotic behavior of $y$ near the boundary $|x| = r$ assuming that any solution of $Dy = 0$ converges in a common open disc $|x| < r$. For simplicity, we assume $r = 1$. The most general result in this viewpoint is that $y$ has a logarithmic growth (log-growth) $n - 1$, that is,

$$\sup_{|x|=\rho} |y(x)| = O((\log (1/\rho))^{1-n}) \text{ as } \rho \uparrow 1.$$

Dwork also defines the so-called special log-growth filtration of $\text{Sol}(D)$ by

$$\text{Sol}_\lambda(D) := \{ y \in \text{Sol}(D); \sup_{|x|=\rho} |y(x)| = O((\log (1/\rho))^{-\lambda}) \text{ as } \rho \uparrow 1 \}.$$

We assume that the $a_i$’s are rational functions over $\mathbb{Q}_p$. Over the $p$-adic field, a naïve analogue of analytic continuation fails. In particular, the existence of local solutions of $Dy = 0$ at the disc $|x-a| < 1$ for any $a$ does not imply the existence of global solutions. Even the $p$-adic exponential series $e^x = 1 + x + 2^{-1}x^2 + \ldots$, which is a solution of $dy/dx = y$, has radius of convergence $p^{-1/(p-1)}$. Hence, it is natural to ask how the special log-growth filtration varies from a disc to disc. Assume $p \neq 2$. In [Dwo82], Dwork gives an answer to this question for the hypergeometric differential equation

$$Dy = x(1-x) \frac{d^2 y}{dx^2} + (1-2x) \frac{dy}{dx} - \frac{1}{4} y = 0,$$

which arises from the Legendre family of elliptic curves over $\mathbb{F}_p$

$$E_2 : z^2 = w(w-1)(w-x), \ x \neq 0, 1.$$

Due to its geometric origin, the hypergeometric differential equation admits a Frobenius structure: let $\bar{a} \in \mathbb{F}_p$, $\bar{a} \neq 0, 1$ and $a \in \mathbb{Z}_p$, a lift of $\bar{a}$. The Frobenius slopes of the solution space of $Dy = 0$ at the disc $|x-a| < 1$ are $0, 1$ if $E_{\bar{a}}$ is ordinary, and $1/2, 1/2$ if $E_{\bar{a}}$ is supersingular. Dwork proves that the special log-growth filtration at the disc $|x-a| < 1$ coincides with the Frobenius slope filtration at the disc $|x-a| < 1$.

In the last few decades, $p$-adic differential equations are extensively studied in many perspectives. As for the existence of solutions, André, Kedlaya, and Mebkhout ([And02],[Ked04],[Meb02]) independently prove the $p$-adic local monodromy theorem which asserts the quasi-unipotence of $p$-adic differential equations defined over the Robba ring with Frobenius structures. Additionally, several striking applications of $p$-adic differential equations emerge: for example, Berger relates a certain $p$-adic representation of the absolute Galois group of $\mathbb{Q}_p$ to a $p$-adic differential equation over the Robba ring, then proves Fontaine’s $p$-adic monodromy conjecture by using the $p$-adic local monodromy theorem ([Ber02]).

However, Dwork’s works on the log-growth of solutions of $p$-adic differential equations have been forgotten for a long time until Chiarellotto and Tsuzuki drew an attention to it in [CT09]. We briefly summarize some recent developments on this subject.

- In [CT09], Chiarellotto and Tsuzuki formulate a fundamental conjecture on the log-growth filtrations for $p$-adic differential equations with Frobenius structures (see Conjecture 3.3). Their conjecture is two-fold. The first part can be stated as follows:

**Conjecture A** (Conjecture 3.3 (i)). Let $Dy = 0$ be a $p$-adic differential equation with a Frobenius structure. Then, the breaks of the filtration $\text{Sol}_\lambda(D)$ are rational and $\text{Sol}_\lambda(D) = \cap_{\mu>\lambda} \text{Sol}_\mu(D)$ for all $\lambda \in \mathbb{R}$. 


The second part is about a comparison of the log-growth filtration and the Frobenius slope filtration under a certain technical assumption, which is based on Dwork’s work on the hypergeometric differential equation. They prove the conjecture in the rank 2 case in [CT09]. They also give a complete answer to a generic version of their conjecture in [CT11].

- In [And08], André proves Dwork’s conjecture on a specialization property for the log-growth filtration, which is an analogue of Grothendieck-Katz specialization theorem on Frobenius structure.

- In [Ked10], Kedlaya studies effective convergence bounds on the solutions of $p$-adic differential equations with nilpotent singularities, which allows the $a_i$’s to have a pole at $x = 0$. Then, he proves a partial generalization of Chiarellotto-Tsuzuki’s earlier works to $p$-adic differential equations with nilpotent singularities.

Our main result in this paper is

**Main Theorem** (Theorem 3.7 (i)). *Conjecture A is true.*

Under a certain technical assumption, we also prove the second part of Chiarellotto-Tsuzuki’s conjecture (Theorem 3.7 (ii)).

**Strategy of proof**

We sketch the proof of the rationality of breaks of the filtration $\text{Sol}_i(D)$. Let $\mathbb{Q}_p[x]_0 := \mathbb{Z}_p[x][p^{-1}]$ be the ring of bounded functions on the open unit disc, and $\sigma$ a $\mathbb{Q}_p$-algebra endomorphism of $\mathbb{Q}_p[x]_0$ such that $\sigma(x) = x^p$. Instead of a naïve $p$-adic differential equation $Dy = 0$, we consider a finite free $\mathbb{Q}_p[x]_0$-module $M$ of rank $n$ endowed with an action of $d/dx$. The existence of a Frobenius structure of $Dy = 0$ is equivalent to the existence of a $\sigma$-semi-linear structure $\varphi$ on $M$ compatible with $\nabla$. In [CT09], Chiarellotto and Tsuzuki establish a standard method for studying the log-growth filtration associated to $M$ as follows. We fix a cyclic vector $e$ of $M$ as a $\sigma$-module over the fraction field of $\mathbb{Q}_p[x]_0$. Let $V(M)$ be the set of horizontal sections of $M$ after tensoring with the ring of analytic functions over the open unit disc. Let $v \in V(M)$ be a Frobenius eigenvector, i.e., $\varphi(v) = \lambda v$ for some $\lambda \in \mathbb{Q}_p$. If we write $v$ as a linear combination of $e, \varphi(e), \ldots, \varphi^{n-1}(e)$, then the coefficient $f$ of $\varphi^{n-1}(e)$ satisfies a certain Frobenius equation

$$b_n f^{\sigma^n} + b_{n-1} f^{\sigma^{n-1}} + \cdots + b_0 f = 0, \quad b_i \in \mathbb{Q}_p[x]_0.$$ 

Then, the rationality of breaks of $\text{Sol}_i(D)$ is reduced to the rationality of the log-growth of $f$, i.e., the existence of $\lambda \in \mathbb{Q}$ such that

$$\sup_{|x|=\rho} |f(x)| = O((\log (1/\rho))^{-\lambda}) \text{ as } \rho \uparrow 1$$

and

$$\sup_{|x|=\rho} |f(x)| \neq O((\log (1/\rho))^{-\mu}) \text{ as } \rho \uparrow 1$$

for any $\mu < \lambda$. The rationality of the log-growth of $f$ is proved by Chiarellotto and Tsuzuki in [CT09] when $n = 2$, then by Nakagawa in [Nak13] when $n$ is arbitrary under the assumption that the number of breaks of the Newton polygon of $b_n X^n + b_{n-1} X^{n-1} + \cdots + b_0$ as a polynomial over the Amice ring $\mathcal{E}$ is equal to $n$. Nakagawa’s assumption is too strong since it is equivalent to assuming that the number of breaks of the Frobenius filtration of $M$ tensored with $\mathcal{E}$ is equal to $n$. Unfortunately, a naïve attempt to generalize Nakagawa’s result without the assumption on the Newton polygon seems to fail.

To overcome difficulty, we carefully choose a cyclic vector $e$ in § 5: by definition, the Newton polygon of $b_n X^n + b_{n-1} X^{n-1} + \cdots + b_0$ is the lower convex hull of some set of points associated to the $b_i$’s. Our requirement for $e$ is that each plotted point belongs to the Newton polygon. The construction of $e$ is done after a certain base change which is described in Kedlaya’s framework of analytic rings. By using our cyclic vector $e$, the corresponding Frobenius equation is defined over Kedlaya’s ring. Hence, we need to introduce a notion of log-growth on Kedlaya’s ring (§ 4). Then, we generalize Nakagawa’s calculation in § 6. Finally, we obtain the rationality of the log-growth filtration of $V(M)$ in § 7.
2 Summary of notation

We summarize our notation in this paper. Basically, we adopt the notation in [CT11]. In the appendix, we have a diagram describing relations between various rings defined in the following.

2.1 Coefficient rings

\( p \) : a prime number.

\( K \) : a complete discrete valuation field of characteristic \((0, p)\).

\( \mathcal{O}_K \) : the integer ring of \( K \).

\( k_K \) : the residue field of \( K \).

\( \pi_K \) : a uniformizer of \( \mathcal{O}_K \).

\(| \cdot | \) : the \( p \)-adic absolute value on \( K \)\(^{\text{alg}} \) associated to a valuation of \( K \), normalized by \(| p | = p^{-1} \).

\( q \) : a positive power of \( p \).

\( q^s \in \mathbb{Q} \) : Let \( s \) be a rational number and write \( s = a/b \) with relatively prime \( a, b \in \mathbb{Z} \). The notation \( "q^s \in \mathbb{Q}" \) means that \( b \) divides \( f \), and we put \( q^s := p^{a/b} \).

\( \sigma \) : a \( q \)-Frobenius on \( \mathcal{O}_K \), i.e., local ring endomorphism of \( \mathcal{O}_K \) such that \( \sigma(a) \equiv a^q \mod \pi_K \).

\( K^\sigma \) : the injective limit of \( K \)

\[ K \xrightarrow{\sigma} K \xrightarrow{\sigma} \ldots. \]

We regard \( K^\sigma \) as an extension of \( K \). Then, \( K^\sigma \) is a Henselian discrete valuation field, whose value group coincides with the value group of \( K \), with residue field \( k_K^{p^{-\infty}} \).

\( K^{\sigma, \text{ur}} \) : the completion of the maximal unramified extension of \( K^\sigma \). Then \( K^{\sigma, \text{ur}} \) is a complete discrete valuation field, whose value group coincides with the value group of \( K \), with the residue field \( k_K^{\text{alg}} \).

Moreover, \( \sigma \) induces a \( q \)-Frobenius on \( K^{\sigma, \text{ur}} \).

2.2 Various rings of functions

\( x \) : an indeterminate.

\(| \cdot |_{0, \text{naive}}(\rho) \) : the multiplicative map

\[ K[x] \to \mathbb{R}_{\geq 0} \cup \{ \infty \}; \sum a_n x^n \mapsto \sup_{n \in \mathbb{N}} |a_n| \rho^n \]

defined for \( \rho \in [0, 1] \).

\( K\{x\} \) : the \( K \)-algebra of analytic functions on the open unit disc \(|x| < 1 \), i.e.,

\[ K\{x\} := \left\{ \sum_{n \in \mathbb{N}} a_n x^n \in K[x]; |a_n| \rho^n \to 0 \ (n \to \infty) \forall \rho \in [0, 1) \right\}. \]

Note that \(| \cdot |_{0, \text{naive}}(\rho) \) defines a multiplicative non-archimedean norm on \( K\{x\} \).

\( K\{x\}_\lambda \) : the Banach \( K \)-subspace of power series of logarithmic growth (log-growth) \( \lambda \) in \( K\{x\} \) for \( \lambda \in \mathbb{R}_{\geq 0} \), i.e.,

\[ K\{x\}_\lambda := \left\{ \sum_{n \in \mathbb{N}} a_n x^n \in K[x]; \sup_{n \in \mathbb{N}} |a_n|/(n+1)^\lambda < \infty \right\} \]

\[ = \left\{ f \in K\{x\}; |f|_{0, \text{naive}}(\rho) = O((\log (1/\rho))^{-\lambda}) \text{ as } \rho \uparrow 1 \right\}. \]
where the last equality follows from [And08, Lemma 2.2.1 (iv)]. Note that \( K[[x]]_0 \) coincides with the ring of bounded functions on the open unit disc \( |x| < 1 \), i.e.,

\[
K[[x]]_0 = \mathcal{O}_K[x][\pi_K^{-1}] = \left\{ \sum_{n \in \mathbb{N}} a_n x^n \in K[x] ; \sup_{n \in \mathbb{N}} |a_n| < \infty \right\}.
\]

We define \( K[x]_\lambda := 0 \) for \( \lambda \in \mathbb{R}_{<0} \). Note that \( K[x]_\lambda \) is stable under the derivation \( d/dx \).

\( \mathcal{E} \) : the fraction field of the \( p \)-adic completion of \( \mathcal{O}_K[[x]][x^{-1}] \), i.e.,

\[
\mathcal{E} := \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \in K[[x]][x^{-1}] ; \sup_{n \in \mathbb{Z}} |a_n| < \infty, |a_n| \to 0 \text{ as } n \to -\infty \right\}.
\]

Note that \( \mathcal{E} \) is canonically endowed with a norm which is an extension of \(|·|_0^{\text{naive}}(1)\). Then, \((\mathcal{E}, |·|_0^{\text{naive}}(1))\) is a complete discrete valuation field of mixed characteristic \((0,p)\) with uniformizer \( \pi_K \) and residue field \( k_K((x)) \).

\( \mathcal{E}^\dagger \) : the ring of overconvergent power series in \( \mathcal{E} \), i.e.,

\[
\mathcal{E}^\dagger := \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \in \mathcal{E} ; |a_n| \rho^n \to 0 \text{ (}n \to -\infty\text{) for some } \rho \in (0,1) \right\}.
\]

Note that \((\mathcal{E}^\dagger, |·|_0^{\text{naive}}(1))\) is a Henselian discrete valuation field whose completion is \( \mathcal{E} \).

\( \mathcal{R} \) : the Robba ring with variable \( x \) and coefficient \( K \), i.e.,

\[
\mathcal{R} := \left\{ \sum_{n \in \mathbb{Z}} a_n x^n \in K[[x]][x^{-1}] ; |a_n| \rho^n \to 0 \text{ (}n \to \pm\infty\text{) }\forall \rho \in (\rho_0,1) \text{ for some } \rho_0 \in (0,1) \right\}.
\]

\( \sigma \) : a \( q \)-Frobenius on \( \mathcal{O}_K[[x]] \), which is an extension of \( \sigma \), defined by fixing \( \sigma(x) \in x^q \mod \mathfrak{m}_K \mathcal{O}_K[[x]] \). Note that \( \sigma \) induces ring endomorphisms on \( K[[x]], K\{x\}, \mathcal{E}, \mathcal{E}^\dagger \) and \( \mathcal{R} \), and \( K[[x]]_\lambda \) is stable under \( \sigma \) ([Chr83, 4.6.4]).

\( t \) : another indeterminate. In the literature, \( t \) is called Dwork’s generic point.

\( \mathcal{E}_t \) : a copy of \( \mathcal{E} \) in which \( x \) is replaced by \( t \).

\( \mathcal{E}_t[X - t]_0 \) : the ring of bounded functions on \( |X - t| < 1 \) with variable \( X - t \) and coefficient \( \mathcal{E}_t \). We endow \( \mathcal{E}_t[X - t]_0 \) with \( \mathcal{E} \)-algebra structure by the \( K \)-algebra homomorphism

\[
\tau : \mathcal{E} \to \mathcal{E}_t[X - t]_0; f \mapsto \sum_{n \in \mathbb{N}} \frac{1}{n!} \left( \frac{d^n f}{dx^n} \right) \bigg|_{x=t} (X - t)^n.
\]

Since \( \tau(K) \subset \mathcal{E}_t \) and \( \tau(x) = X \), \( \tau \) is equivariant under the derivations \( d/dx \) and \( d/dX \). We define a \( q \)-Frobenius on \( \mathcal{E}_t[X - t]_0 \) by \( \sigma|_{\mathcal{E}_t} = \sigma \) (by identifying \( t \) as \( x \)) and \( \sigma(X - t) = \tau(\sigma(x) - \sigma(x)|_{x=t}) \). Then, \( \tau \) is also \( \sigma \)-equivariant.

\( \mathcal{E}_t[X - t]_\lambda \) : the Banach \( \mathcal{E}_t \)-subspace of power series of log-growth \( \lambda \) in \( \mathcal{E}_t\{X - t\} \).

Let \( R \) be either \( K[[x]]_0, K\{x\}, \mathcal{E}, \mathcal{E}^\dagger, \mathcal{E}_t, \mathcal{R} \). We define \( \Omega^n_R := Rdx \) with a \( K \)-linear derivation \( d : R \to \Omega^n_R; f \mapsto (df/dx)dx \). We also endow \( \Omega^n_R \) with a semi-linear \( \sigma \)-action defined by \( \sigma(dx) := d\sigma(x) \). For \( \mathcal{E}_t[X - t]_0 \) and \( \mathcal{E}_t\{X - t\} \), we also define a corresponding \( \Omega^n_\bullet \) by replacing \( K \) and \( x \) by \( \mathcal{E}_t \) and \( X - t \) respectively.
2.3 Filtration and Newton polygon

Let $V$ be a finite dimensional vector space over a field $F$. Let $V^\bullet = \{V^\lambda\}_{\lambda \in \mathbb{R}}$ be a decreasing filtration of $V$. Then, we define

$$V^{\lambda^-} := \bigcap_{\mu < \lambda} V^\mu, \quad V^{\lambda^+} := \bigcup_{\mu > \lambda} V^\mu.$$ 

We say that $\lambda \in \mathbb{R}$ is a break of $V^\bullet$ if $V^{\lambda^-} \neq V^{\lambda^+}$. We also define the multiplicity of $\lambda$ as $\dim_F V^{\lambda^-} - \dim_F V^{\lambda^+}$. We say that $V^\bullet$ is rational if all breaks of $V^\bullet$ are rational. We say that $V^\bullet$ is right continuous if $V^\lambda = V^{\lambda^+}$ for all $\lambda \in \mathbb{R}$. We say that $V^\bullet$ exhaustive or separated if $\bigcup_\lambda V^\lambda = V$ or $\bigcap_\lambda V^\lambda = 0$ respectively.

Similarly, for an increasing filtration $V^\bullet = \{V_\lambda\}_{\lambda \in \mathbb{R}}$ of $V$, we define

$$V_{\lambda^-} := \bigcup_{\mu > \lambda} V^\mu, \quad V_{\lambda^+} := \bigcap_{\mu < \lambda} V^\mu.$$ 

We also define a break, a rationality, and a right continuity of $V^\bullet$ by replacing superscripts by subscripts.

We define the Newton polygon of a filtration as follows ([CT09, 3.3]). Let $\{V^\lambda\}_{\lambda \in \mathbb{R}}$ (resp. $\{V_\lambda\}_{\lambda \in \mathbb{R}}$) be a decreasing (resp. increasing) filtration of $V$. Let $\lambda_1 < \cdots < \lambda_n$ be the breaks of the filtration $V^\bullet$ (resp. $V^\bullet$) with multiplicities $m_1, \ldots, m_n$. We define the Newton polygon of $V^\bullet$ (resp. $V^\bullet$) as the piecewise linear function in the $xy$-plane whose left endpoint is $(0, 0)$, with slopes $\lambda_1, \ldots, \lambda_n$ whose projections to the $x$-axis have lengths $m_1, \ldots, m_n$.

3 Chiarellotto-Tsuizuki’s conjectures and main theorem

We first recall the definition of $(\sigma, \nabla)$-modules over $K[x]_0$ and $E$. Then, we recall the definition of the log-growth filtrations for $(\sigma, \nabla)$-modules over $K[x]_0$ and $E$, and recall Chiarellotto-Tsuizuki’s conjectures. After recalling known results on the conjectures, we state our main results. Our basic references are [CT09], [CT11], and [Ked10].

3.1 $\sigma$-modules

Let $R$ be a commutative ring with a ring endomorphism $\delta$. We denote $\delta(r)$ by $r^\delta$ if no confusion arises. A $\delta$-module $M$ is a finite free $R$-module $M$ endowed with an $R$-linear isomorphism $\varphi : \delta^* M := R \otimes_R M \to M$. We can view $M$ as a left module over the twisted polynomial ring $R\{\delta\}$ ([Ked10, 14.2.1]). If we regard $\varphi$ as a $\delta$-linear endomorphism of $M$, then $(M, \varphi^n)$ for $n \in \mathbb{N}$ is a $\delta^n$-module over $R$. For $\alpha \in R^\times$, $(M, \alpha \varphi)$ is also a $\delta$-module over $R$.

Let $M$ be a $\sigma$-module over $K$ ($K$ might be $E$). We recall the Frobenius slope filtration of $M$ ([CT09, § 2]). We say that $M$ is étale if there exists an $\mathcal{O}_K$-lattice $\mathfrak{M}$ of $M$ such that $\varphi(\mathfrak{M}) \subset \mathfrak{M}$ and $\varphi(\mathfrak{M})$ generates $\mathfrak{M}$. We say that $M$ is pure of slope $\lambda \in \mathbb{R}$ if there exists $n \in \mathbb{N}_{>0}$ and $\alpha \in K$ such that $\log_{q^n} |\alpha| = -\lambda$ and $(M, \alpha^{-1} \varphi^n)$ is étale ([CT09, 2.1]). For a $\sigma$-module $M$ over $K$, there exists a unique increasing filtration $\{S_\lambda(M)\}_{\lambda \in \mathbb{R}}$, called the slope filtration, of $M$ such that $S_\lambda(M)/S_{\lambda^-}(M)$ is pure of slope $\lambda$. We call the breaks of $S_\lambda(M)$ the Frobenius slopes of $M$. The following are basic properties of the slope filtration:

- The slope filtration of $M$ is exhaustive, separated, and right continuous.

- The Frobenius slopes of $M$ are rational.

- The slope filtration of $(M, \varphi^n)$ is independent of the choice of $n \in \mathbb{N}_{>0}$.

Assume that $k_K$ is algebraically closed. Then, any short exact sequence of $\sigma$-modules splits ([Ked10, 14.3.4, 14.6.6]). Moreover, let $M$ be a $\sigma$-module over $K$ such that $q^n \in \mathbb{Q}$ for any Frobenius slope $\lambda$ of $M$. Then, $M$ admits a basis consisting of elements of the form $\varphi(v) = q^n v$ ([Ked10, 14.6.4]); we call $v$ a Frobenius eigenvector of slope $\lambda$. In this situation, for any $\sigma$-submodules $M'$ and $M''$ of $M$, we have $M' \subset M''$ if and only if any Frobenius eigenvector $v$ of $M'$ belongs to $M''$. 

6
3.2 Log-growth filtration

Let $R$ be either $K[[x]]_0$ ($K$ might be $E$, $E^\dagger$, $E$, or $\mathbb{R}$). A $\nabla$-module over $R$ is a finite free $R$-module $M$ endowed with a connection, i.e., a $K$-linear map

\[ \nabla : M \rightarrow M \otimes_R \Omega^1_R = Mdx \]

satisfying

\[ \nabla(am) = m \otimes da + a\nabla(m) \]

for $a \in R$ and $m \in M$. A $(\sigma, \nabla)$-module over $R$ is a $\sigma$-module $(\nabla, \varphi)$ over $R$ with a connection $\nabla$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
M & \xrightarrow{\nabla} & M \otimes_R \Omega^1_R \\
\varphi \downarrow & & \varphi \otimes \sigma \\
M & \xrightarrow{\nabla} & M \otimes_R \Omega^1_R.
\end{array}
\]

(1) Special log-growth filtration ([CT09, 4.2])

Let $M$ be a $(\sigma, \nabla)$-module of rank $n$ over $K[[x]]_0$. We define the space of horizontal sections of $M$ by

\[ V(M) := (M \otimes_{K[[x]]_0} K\{x\})^{\nabla=0} \]

and define the space of solutions of $M$ by

\[ \text{Sol}(M) := \text{Hom}_{K[[x]]_0}(M, K\{x\})^{\nabla=0} \]

\[ := \{ f \in \text{Hom}_{K[[x]]_0}(M, K\{x\}); d(f(m)) = (f \otimes \text{id})(\nabla(m)) \forall m \in M \}. \text{ (see [Ked10, p. 82])} \]

Both $V(M)$ and $\text{Sol}(M)$ are known to be $K$-vector spaces of dimension $n$, and there exists a perfect pairing

\[ V(M) \otimes_K \text{Sol}(M) \rightarrow K \]

induced by the canonical pairing $M \otimes_{K[[x]]_0} M^\vee \rightarrow K[[x]]_0$, where $M^\vee$ denotes the dual of $M$. For $\lambda \in \mathbb{R}$, we define

\[ \text{Sol}_\lambda(M) := \text{Hom}_{K[[x]]_0}(M, K[[x]]_\lambda) \cap \text{Sol}(M), \]

which induces an increasing filtration of $\text{Sol}(M)$. We say that $M$ is solvable in $K[[x]]_\lambda$ if $\dim_K \text{Sol}_\lambda(M) = n$. We define

\[ V(M)^\lambda := \text{Sol}_\lambda(M)^\perp, \]

where $(\cdot)^\perp$ denotes the orthogonal space with respect to the the above pairing. We call the decreasing filtration $\{V(M)^\lambda\}_\lambda$ the special log-growth filtration of $M$. Note that $\text{Sol}_\bullet(M)$ and $V(M)^\bullet$ are exhaustive and separated. Moreover, $V(M)^\lambda$ (resp. $\text{Sol}_\lambda(M)$) is a $\sigma$-submodule of $V(M)$ (resp. $\text{Sol}(M)$) ([CT09, 4.8]).

Example. Let

\[ Dy = \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_0 y = 0, \quad a_i \in K[[x]]_0 \]

be an ordinary linear $p$-adic differential equation. As in the introduction, we define

\[ \text{Sol}(D) := \{ y \in K[[x]]; Dy = 0 \} \supset \text{Sol}_\lambda(D) := \{ y \in K[[x]]_\lambda; Dy = 0 \}. \]

We define a $\nabla$-module $M := K[[x]]_{0e_0} \oplus \cdots \oplus K[[x]]_{0e_{n-1}}$ by

\[ \nabla(e_i) = \begin{cases} e_{i+1} dx & \text{if } 0 \leq i \leq n-2 \\ -(a_{n-1} e_{n-1} + \cdots + a_0 e_0) dx & \text{if } i = n-1. \end{cases} \]

Then, we have the canonical isomorphism

\[ \text{Sol}(M) \rightarrow \text{Sol}(D); f \mapsto f(e_0), \]

under which we have

\[ \text{Sol}_\lambda(M) = \text{Sol}_\lambda(D). \]
(II) Generic log-growth filtration ([CT09, § 4.1])

Let $M$ be a $(\sigma, \nabla)$-module over $\mathcal{E}$. We denote by $\tau^* M$ the pull-back of $M$ under $\tau : \mathcal{E} \to \mathcal{E}/[X - t], 0$, which is a $(\sigma, \nabla)$-module over $\mathcal{E}/[X - t], 0$. By a theorem of Robba, there exists a unique $(\sigma, \nabla)$-submodule $M^\lambda$ of $M$ for $\lambda \in \mathbb{R}$ characterized as a minimal $(\sigma, \nabla)$-submodule of $M$ such that $\tau^*(M/M^\lambda)$ is solvable in $\mathcal{E}/[X - t], 0$ ([CT09, 4.1]). We call the decreasing filtration $\{M^\lambda\}_{\lambda \in \mathbb{R}}$ of $M$ the log-growth filtration of $M$. Note that $M^\bullet$ is exhaustive and separated, and if $M \neq 0$, then $M^\lambda \neq M$ for $\lambda \in \mathbb{R}_{\geq 0}$.

There exists a dual version of the log-growth filtration: for $\lambda \in \mathbb{R}$, we set $M_\lambda := ((M^\vee)^\lambda)^\perp$, where $(\cdot)^\perp$ denotes the orthogonal space with respect to the canonical pairing $M \otimes \mathcal{E} \otimes M^\vee \to \mathcal{E}$. Then, $M_\lambda$ is a maximal $(\sigma, \nabla)$-submodule of $M$ such that $\tau^* M_\lambda$ is solvable in $\mathcal{E}/[X - t], 0$. Note that if $M \neq 0$, then $M_\lambda \neq 0$ for $\lambda \in \mathbb{R}_{\geq 0}$ by $(M^\vee)^\lambda \neq M$.

Note that the Frobenius slope filtration of $M$ is stable under the action of $\nabla$ ([CT09, 6.2]).

**Definition 3.1.** Let $M$ be a $(\sigma, \nabla)$-module over $K[\![x]\!]_0$.

(i) The Frobenius slope filtration $S_\bullet(V(M))$ of $V(M)$ is called the special Frobenius filtration of $M$ ([CT09, 6.7]). We call a Frobenius slope of $V(M)$ a special Frobenius slope of $M$.

(ii) We put $M_\varepsilon := \mathcal{E} \otimes_{K[\![x]\!]} M$, which is a $(\sigma, \nabla)$-module over $\mathcal{E}$. The Frobenius slope filtration $S_\bullet(M_\varepsilon)$ of $M_\varepsilon$ is called the generic Frobenius filtration of $M$ ([CT09, 6.1]). We call a Frobenius slope of $M_\varepsilon$ a generic Frobenius slope of $M$.

### 3.3 Chiarellotto-Tsuzuki’s conjectures

Dwork potentially observes that the log-growth and Frobenius slope filtrations can be compared. To formulate conjectures based on his observation, Chiarellotto and Tsuzuki introduce the following technical conditions:

**Definition 3.2.**

(i) ([CT11, 6.1]) Let $M$ be a $(\sigma, \nabla)$-module over $\mathcal{E}$. We say that $M$ is pure of bounded quotient (PBQ for short) if $M/M^0$ is pure as a $\sigma$-module.

(ii) ([CT11, 5.1]) Let $M$ be a $(\sigma, \nabla)$-module over $K[\![x]\!]/0$. We say that $M$ is PBQ if $M_\varepsilon$ is PBQ. We say that $M$ is horizontal of bounded quotient (HBQ for short) if there exists a quotient $\overline{M}$ of $M$ as a $(\sigma, \nabla)$-module such that there exists a canonical isomorphism $\overline{M}_\varepsilon \cong M_\varepsilon/M_\varepsilon^0$. Finally, we say that $M$ is horizontally pure of bounded quotient (HPBQ for short) if $M$ is PBQ and HBQ.

The following conjectures are first formulated by Chiarellotto and Tsuzuki in [CT09, § 6.4]. In this paper, we use the equivalent forms in [CT11].

**Conjecture 3.3** (the conjecture LGF$_{K[\![x]\!]}$ ([CT11, 2.5])). Let $M$ be a $(\sigma, \nabla)$-module over $K[\![x]\!]/0$.

(i) The special log-growth filtration of $M$ is rational and right continuous.

(ii) Let $\lambda_{\text{max}}$ be the highest Frobenius slope of $M_\varepsilon$. If $M$ is PBQ, then we have

$$V(M)^\lambda = (S_{\lambda - \lambda_{\text{max}}}(V(M^\vee)))^\perp$$

for all $\lambda \in \mathbb{R}$. Here, $(\cdot)^\perp$ denotes the orthogonal space with respect to the canonical pairing $V(M) \otimes K V(M^\vee) \to K$.

**Conjecture 3.4** (the conjecture LGF$_{\mathcal{E}}$ ([CT11, 2.4])). Let $M$ be a $(\sigma, \nabla)$-module over $\mathcal{E}$.

(i) The log-growth filtration of $M$ is rational and right continuous.

(ii) Let $\lambda_{\text{max}}$ be the highest Frobenius slope of $M$. If $M$ is PBQ, then we have

$$M^\lambda = (S_{\lambda - \lambda_{\text{max}}}(M^\vee))^\perp$$

for all $\lambda \in \mathbb{R}$. Here, $(\cdot)^\perp$ denotes the orthogonal space with respect to the canonical pairing $M \otimes _{\mathcal{E}} M^\vee \to \mathcal{E}$.

To prove Chiarellotto-Tsuzuki’s conjectures, we may assume that $k_K$ is algebraically closed as remarked in [CT11, p. 42]. In the following, we recall known results on Chiarellotto-Tsuzuki’s conjectures.
Theorem 3.5 ([CT11, Theorem 7.1, 7.2]). The conjecture LGF is true.

Hence, the remaining part of Chiarellotto-Tsuuzuki's conjectures is the conjecture LGF_{K[x]_0}.

Theorem 3.6. Let M be a (σ, ∇)-module of rank n over K[[x]]_0.

(i) ([CT09, Theorem 7.1 (2)]) The conjecture LGF_{K[x]_0} is true if n ≤ 2.

(ii) ([CT11, Theorem 8.7]) The conjecture LGF_{K[x]_0} (i) is true if M is HBQ.

(iii) ([CT09, Theorem 6.17]) For all λ ∈ R, we have

\[ V(M)^λ \subset (S_{λ−λ\max}(V(M^\vee)))\perp. \]

(iv) ([CT11, Theorem 6.5]) The conjecture LGF_{K[x]_0} (ii) is true if M is HPBQ.

(v) ([CT11, Proposition 7.3]) If the conjecture LGF_{M} (ii) is true for an arbitrary M, then the conjecture LGF_{K[x]_0} (ii) is true for an arbitrary M.

3.4 Main theorem

Our main result of this paper is

Theorem 3.7. (i) The conjecture LGF_{K[x]_0} (i) is true for an arbitrary M.

(ii) The conjecture LGF_{K[x]_0} (ii) is true if the number of Frobenius slopes of M_{E} is less than or equal to 2.

As mentioned in the introduction, we will study (σ, ∇)-modules over E⁺ rather than over K[[x]]_0. Theorem 3.7 will follow from Theorem 4.19, which is a counterpart of Theorem 3.7 for (σ, ∇)-modules over E⁺.

4 Log-growth of analytic ring

In [Ked04] and [Ked05], Kedlaya gives functorial constructions of various analytic rings associated to a certain extension of k_{K}(x). We recall some of his construction. After defining a notion of log-growth on Kedlaya's analytic rings, we develop a theory of log-growth filtrations for (σ, ∇)-modules over E⁺.

Notation 4.1. We put \( \Gamma := \mathcal{O}_{E} \subset \Gamma_{\text{alg}} := \mathcal{O}_{E^\times,ur} \) for a compatibility with the notation in the references. We denote the norm \( |·|_{\text{native}(1)} \) on \( \Gamma[p^{-1}] \) by \( |·|_{0}(1) \), and extend \( |·|_{0}(1) \) to \( \Gamma_{\text{alg}}[p^{-1}] \).

Remark 4.2. The definition of \( \Gamma_{\text{alg}} \) in [Ked04], which coincides with our \( \Gamma_{\text{alg}} \), is different from that in [Ked05]: the latter contains our \( \Gamma_{\text{alg}} \), but the residue field is the completion of \( k_{K}(x)_{\text{alg}} \). Since the definition of \( \Gamma_{\text{alg}} \) comes out the same as mentioned in [Ked05, 2.4.13], we may make (careful) use of the results of [Ked05].

4.1 Overconvergent rings

We define subrings \( \Gamma_{\text{con}} \) and \( \Gamma_{\text{con}}^{\text{alg}} \) of \( \Gamma \) and \( \Gamma_{\text{alg}} \) respectively as follows: For \( f \in \Gamma_{\text{alg}}[p^{-1}] \), we have a unique expression

\[ f = \sum_{i \geq -\infty} \pi_{k_{K}[x]}(\pi_{k_{K}[x]}(\cdots(\pi_{k_{K}[x]}[x])) \right). \]

with \( \pi_{k_{K}[x]}(\pi_{k_{K}[x]}(\cdots(\pi_{k_{K}[x]}[x])) \right) \) denotes Teichmüller lift. We define the partial valuation \( v_{n} : \Gamma_{\text{alg}}[p^{-1}] \rightarrow \mathbb{R} \cup \{\infty\} \) by

\[ v_{n}(f) := \min_{i \leq n} \{v(i)\}. \]

For \( r \in \mathbb{R}_{\geq 0} \), we denote by \( \Gamma_{r}^{\text{alg}} \) the subring of \( f \in \Gamma_{\text{alg}} \) such that \( \lim_{n \rightarrow \infty}(rv_{n}(x) + n) = \infty \). On \( \Gamma_{r}^{\text{alg}}[p^{-1}] \setminus \{0\} \), we define the non-archimedean valuation

\[ w_{r}(f) := \min_{n \in \mathbb{N}} (rv_{n}(x) + n). \]
Lemma 4.7. We have the following:

\[ \text{Fil} \Gamma \text{ close to 1 from the left. We define } \Gamma_{\text{alg}}^{\text{con}} \text{ ([Ked05, 2.3.7]). Both } \Gamma_{\text{alg}}^{\text{con}} \text{ and } \Gamma_{\text{alg}} \text{ define the slopes of Definition 4.3 ([Ked04, 3.5]) and hence } \Gamma_{\text{alg}}^{\text{con}} \text{ is stable under } \sigma \text{ and } |\sigma(\cdot)|_0(\rho) = |\cdot|_0(\rho^0) \text{ for } \rho \in (0, 1). \]

Definition 4.3 ([Ked04, 3.5]). Let \( f \in \Gamma_{\text{con}}^{\text{alg}}[p^{-1}] \) a non-zero element. We define the Newton polygon NP\((f)\) of \( f \) as the lower convex hull of the set of points \((v_n(f), n)\), minus any segments of slopes less than \(-r\) from the left end and/or any segments of non-negative slope on the right end of the polygon. We define the slopes of \( f \) as the negatives of the slopes of NP\((f)\). We also define the multiplicity of a slope \( s \in (0, r) \) of \( f \) as the positive difference in y-coordinate between the endpoints of the segment of NP\((f)\) of slope \(-s\).

The following simple fact is one of the key ingredients in this paper.

Lemma 4.4 (cf. [Nak13, Lemma 2.6]). Let \( f \in \Gamma_{\text{con}}^{\text{alg}}[p^{-1}] \). Then, there exists \( \rho_0 \in \mathbb{R}_{>0} \) and \( a \in \mathbb{Q} \) such that

\[ |f|_0(\rho) = \rho^a|f|_0(1) \text{ for all } \rho \in (\rho_0, 1]. \]

Proof. We may assume \( f \neq 0 \) and \( f \in \Gamma_{\text{con}}^{\text{alg}}[p^{-1}] \) for some \( r > 0 \). Since the number of the slopes of NP\((f)\) with non-zero multiplicities is finite by [Ked05, 2.4.6], we may assume that \( f \) has no slopes after choosing \( r \) sufficiently small. By [Ked05, 2.4.6] again, there exists a unique integer \( n \) such that \( v_n(f) = v_{n,s}(f) \) for all \( s \in (0, r) \). Then, we have \( |f|_0(p^{-s}) = (p^{-s})^{v_n(f)/e_K}|\pi_K|^n \) for any \( s \in (0, r) \), where \( e_K \) is the absolute ramification index of \( K \). By taking the limit \( s \downarrow 0 \), we obtain the assertion for \( \rho_0 = p^{-r} \) and \( a = v_n(f)/e_K \).

4.2 Log-growth filtration over \( \Gamma_{\text{con}}[p^{-1}] \)

Throughout this section, let \( \bullet \in \{ , \text{alg} \} \). Let \( \Gamma_{\text{an}, \bullet}^\ast \) be the Fréchet completion of the ring \( \Gamma_{\text{an}, \bullet}^\ast[p^{-1}] \) with respect to the family of valuations \( \{w_n\}_{s \in (0, r]} \) ([Ked04, 3.3]). We define \( \Gamma_{\text{an}, \bullet}^{\text{con}} := \cup_{\rho \in \mathbb{R}_{>0}} \Gamma_{\text{an}, \bullet}^\ast \). Then, we have \( \Gamma_{\text{an}, \bullet}^{\text{con}} = \mathcal{R} \), in particular, \( \Gamma_{\text{an}, \bullet}^{\text{con}} \) contains \( K\{x\} \). By continuity, \( \Gamma_{\text{an}, \bullet}^{\text{con}} \) is endowed with a family of non-archimedean valuations induced by \( \{v_n\}_{n \in \mathbb{Z}} \) and \( \{w_n\}_{s \in (0, r]} \). Also, the norm \( |\cdot|_0(p^{-r}) \) extends to \( \Gamma_{\text{an}, \bullet}^{\text{con}} \). As before, we can define a value \( |f|_{0, \bullet}[p^{-1}] = \mathcal{E}^\dagger \) and \( \Gamma_{\text{an}, \bullet}^{\text{con}} = \mathcal{R} \) as rings. However, the partial norms \( |\cdot|_{0, \text{naive}}^{\text{naive}} \) on \( \Gamma_{\text{an}, \bullet}^{\text{con}} \) and \( |\cdot|_0 \) on \( \mathcal{R} \) coincide with each other only when \( \rho \) is sufficiently large ([Ked05, 2.3.5]). For this reason, we will distinguish \( \Gamma_{\text{con}}[p^{-1}] \) and \( \Gamma_{\text{an}, \bullet}^{\text{con}} \) from \( \mathcal{E}^{\dagger} \) and \( \mathcal{R} \) respectively as normed rings.

Definition 4.6 (Log-growth of analytic ring (cf. [Nak13, 2.8])). For \( \lambda \in \mathbb{R} \), we denote by \( \text{Fil}_\lambda \Gamma_{\text{an}, \bullet}^{\text{con}} \) the subspace of \( f \in \Gamma_{\text{an}, \bullet}^{\text{con}} \) such that

\[ |f|_{0, \bullet}(\rho) = O((\log(1/\rho))^{-\lambda}) \text{ as } \rho \uparrow 1. \]

Note that \( \text{Fil}_\lambda \Gamma_{\text{an}, \bullet}^{\text{con}} = 0 \) for \( \lambda \in \mathbb{R}_{<0} \) by Lemma 4.7 (i) below. Also, Lemma 4.7 implies that \( \text{Fil}_\lambda \Gamma_{\text{an}, \bullet}^{\text{con}} \) forms an increasing filtration of \( \sigma \)-stable \( \Gamma_{\text{con}}[p^{-1}] \)-subspaces of \( \Gamma_{\text{an}, \bullet}^{\text{con}} \).

Lemma 4.7. We have the following:

(i) For a non-zero \( f \in \Gamma_{\text{an}, \bullet}^{\text{con}} \),

\[ \liminf_{\rho \uparrow 1} |f|_{0, \bullet}(\rho) > 0. \]

(ii) \( \text{Fil}_0 \Gamma_{\text{an}, \bullet}^{\text{con}} = \Gamma_{\text{con}}[p^{-1}], \text{ Fil}_0 \Gamma_{\text{an}, \bullet}^{\text{algebraic}} \supset \Gamma_{\text{con}}[p^{-1}]. \)

(iii) \( K\{x\} \cap \text{Fil}_\lambda \Gamma_{\text{an}, \bullet}^{\text{con}} = K[x]_\lambda \) for \( \lambda \in \mathbb{R} \).

(iv) \( \sigma(\text{Fil}_\lambda \Gamma_{\text{an}, \bullet}^{\text{con}}) \subset \text{Fil}_\lambda \Gamma_{\text{an}, \bullet}^{\text{con}} \) for \( \lambda \in \mathbb{R} \).
Proof. (i) We choose \( r > 0 \) sufficiently small such that \( f \in \Gamma_{an,r} \). Then, we have \( w_r(f) \neq 0 \) by \( f \neq 0 \).

In particular, there exists \( n \in \mathbb{Z} \) such that \( v_n(f) \neq 0 \). By definition, \( w_n(f) \leq sv_n(f) + n \) for all \( s \in (0, r) \). Therefore, \( \lim_{s \to 0} w_n(f) \leq n < \infty \), which implies the assertion.

(ii) By Lemma 4.4, we have only to prove that \( f \in \Gamma_{an,\log}\Gamma \) belongs to \( \Gamma_{an,\log}^{[p^{-1}]} \). Since \( |f|_0(\rho) = O(1) \) as \( \rho \uparrow 1 \), there exist a constant \( C \) and \( r > 0 \) such that \( C < w_n(f) \) for all \( s \in (0, r) \). If \( v_n(f) < \infty \), then we have \( C < n \) by taking the limit \( s \to 0 \) in the inequality \( w_n(f) \leq sv_n(f) + n \). Hence, we have \( v_n(f) = \infty \) for all sufficiently small \( n \in \mathbb{Z} \). If we take \( l \in \mathbb{Z} \) such that \( v_n(\pi_K f) = \infty \) for all \( n < 0 \), then we have \( \pi_K f \in \Gamma L \) by [Ked05, 2.3.5], i.e., \( f \in \Gamma_{an,\log}[\pi_K^{-1}] = \Gamma_{an,\log}^{[p^{-1}]} \).

(iii) It follows from the fact that for \( f \in \Gamma \{ x \} \), we have \( |f|_0(\rho) = |f|^{\text{naive}}_0(\rho) \) for \( \rho \) sufficiently close to 1 from the left ([Ked05, 2.3.5]).

(iv) It follows from \( |\sigma(\gamma)|_0(\rho) = |\cdot|_0(\rho^\gamma) \).

(v) The assertion follows from the multiplicativity of the norm \( |\cdot|_0(\rho) \).

\[ \Box \]

Remark 4.8. In (i), the equality in the latter case does not hold. Indeed, there exists \( f \in \Gamma_{an,\log}^{\text{alg}} \) such that \( v_n(f) = \infty \) for \( n \in \mathbb{Z}_{\leq 0} \), but \( f \notin \Gamma_{an,\log}^{\text{alg}} \) ([Ked05, 2.4.13]).

Definition 4.9 (A log extension of \( \Gamma_{an,\log,\log} \)). We put \( \Gamma_{an,\log,\log} := \Gamma_{an,\log}[\log x] \), where \( \log x \) is an indeterminate. We can extend \( \sigma \) to \( \Gamma_{an,\log,\log} \) as follows:

\[ \sigma(\log x) := q \log x + \sum_{i=1}^{+\infty} (-1)^{i-1} \left( \frac{\sigma(x)}{x} - 1 \right)^i \]

Moreover, we extend \( d/dx \) to \( \Gamma_{an,\log,\log} = R[\log x] \) by

\[ \frac{d}{dx}(\log x) = \frac{1}{x} \]

or

We also define the notion of \( (\sigma, \nabla) \)-modules over \( \Gamma_{an,\log,\log} \) by setting \( R = \Gamma_{an,\log,\log} \) and \( \Omega_R^1 = \Gamma_{an,\log,\log,\log} \).

For \( \rho \in (0, 1) \), we put \( r := -\log \rho, \rho \) and extend \( |\cdot|_0(\rho) \) to \( \Gamma_{an,r}^{\text{log,\log}} \) by

\[ |\sum_{i \in \mathbb{N}} a_i(\log x)^i|_0(\rho) := \sup_{i \in \mathbb{N}} |a_i|_0(\rho) \cdot (\log (1/\rho))^{-i}, \]

Lemma 4.10. The function \( |\cdot|_0(\rho) \) is a multiplicative non-archimedean norm on the ring \( \Gamma_{an,r}^{\text{log,\log}}[\log x] \).

Proof. We have only to check the multiplicativity of \( |\cdot|_0(\rho) \). Let \( f = \sum a_i(\log x)^i \), \( g = \sum b_j(\log x)^j \in \Gamma_{an,r}^{\text{log,\log}}[\log x] \). We have \( |fg|_0(\rho) \leq |f|_0(\rho) \cdot |g|_0(\rho) \) by definition. We prove the converse. Let \( i_0 \) (resp. \( j_0 \)) be the minimum \( i \) (resp. \( j \)) such that \( |f|_0(\rho) = |a_{i_0}|_0(\rho) \cdot (\log (1/\rho))^{-i_0} \) (resp. \( |g|_0(\rho) = |b_{j_0}|_0(\rho) \cdot (\log (1/\rho))^{-j_0} \)). For \( i_1 < i_0 \) and \( j_0 \leq j_1 \), we have

\[ |a_{i_1}b_{j_0}(\log x)^{i_1+j_0}|_0(\rho) > |a_{i_1}|_0(\rho) \cdot (\log (1/\rho))^{-i_1} \cdot |b_{j_0}|_0(\rho) \cdot (\log (1/\rho))^{-j_0} \geq |b_{j_1}|_0(\rho) \cdot (\log (1/\rho))^{-j_1}, \]

and hence \( |a_{i_1}b_{j_0}(\log x)^{i_1+j_0}|_0(\rho) > |a_{i_1}|_0(\rho) \cdot (\log (1/\rho))^{-i_1+j_1} \). Similarly, we have

\[ |a_{i_1}b_{j_0}(\log x)^{i_1+j_0}|_0(\rho) > |a_{i_1}b_{j_1}(\log x)^{i_1+j_1}|_0(\rho) \]

for \( i_1 \geq i_0 \) and \( j_0 > j_1 \). Therefore, we have

\[ |fg|_0(\rho) \geq \sum_{i+j=i_0+j_0} a_i b_j (\log x)^{i+j}|_0(\rho) = |f|_0(\rho) \cdot |g|_0(\rho). \]

\[ \Box \]
Definition 4.11. We define a log-growth filtration of $\Gamma_{\log,\an,\con}$ by

$$\Fil_\lambda \Gamma_{\log,\an,\con} := \bigoplus_{i=0}^{|\lambda|} \Fil_{\lambda-i} \Gamma_{\an,\con} \cdot (\log x)^i$$

for $\lambda \in \mathbb{R}_{\geq 0}$ and $\Fil_\lambda \Gamma_{\log,\an,\con} := 0$ for $\lambda \in \mathbb{R}_{<0}$. Here, $|\lambda|$ denotes the greatest integer less than or equal to $\lambda$. For $\lambda \in \mathbb{R}$, we say that $y \in \Gamma_{\log,\an,\con}$ has a log-growth $\lambda$ if $y \in \Fil_\lambda \Gamma_{\log,\an,\con}$. Moreover, we say that $f$ is bounded if $f$ has a log-growth $0$. Let

$$\log\text{-}\text{submodule of } V$$

be equal to $\lambda$.

Remark 4.14. (i) The assertion follows from Lemma 4.7 (i).

(ii) The assertion follows from $\sum_{i=1}^\infty \frac{(-1)^{i-1}}{i} \left( \frac{\sigma(x)}{x} - 1 \right)^i \in \Gamma_{\con}[p^{-1}] = \Fil_0 \Gamma_{\an,\con}$ ([Ked04, 6.5]).

(iii) The assertion follows from Lemma 4.7 (iv).

Lemma 4.12. (i) For $f \in \Gamma_{\log,\an,\con}$ and $\lambda \in \mathbb{R}$, we have $f \in \Fil_\lambda \Gamma_{\log,\an,\con}$ if and only if

$$|f|_0(\rho) = O((\log (1/\rho))^{-\lambda})$$

as $\rho \uparrow 1$.

(ii) $\sigma(\Fil_\lambda \Gamma_{\log,\an,\con}) \subset \Fil_\lambda \Gamma_{\log,\an,\con}$ for $\lambda \in \mathbb{R}$.

(iii) $\Fil_{\lambda_1} \Gamma_{\log,\an,\con} \cdot \Fil_{\lambda_2} \Gamma_{\log,\an,\con} \subset \Fil_{\lambda_1+\lambda_2} \Gamma_{\log,\an,\con}$ for $\lambda_1, \lambda_2 \in \mathbb{R}$.

Proof. (i) The assertion follows from Lemma 4.7 (i).

(ii) The assertion follows from $\sum_{i=1}^\infty \frac{(-1)^{i-1}}{i} \left( \frac{\sigma(x)}{x} - 1 \right)^i \in \Gamma_{\con}[p^{-1}] = \Fil_0 \Gamma_{\an,\con}$ ([Ked04, 6.5]).

(iii) The assertion follows from Lemma 4.7 (iv).

Definition 4.13 (Log-growth filtration). Let $M$ be a $(\sigma, \nabla)$-module of rank $n$ over $\Gamma_{\con}[p^{-1}]$. We put

$$\mathfrak{M}(M) := (\Fil_{\log,\an,\con} \otimes_{\Gamma_{\con}[p^{-1}]} M)^{\nabla=0},$$

$$\mathfrak{Sol}(M) := \Hom_{\Gamma_{\con}[p^{-1}]}(M, \Fil_{\log,\an,\con})^{\nabla=0} \cong \mathfrak{M}(M^\vee).$$

We say that $M$ is solvable in $\log_{\an,\con}$ if $\dim_K \mathfrak{M}(M) = n$. In this case, we define

$$\mathfrak{Sol}_1(M) := \Hom_{\Gamma_{\con}[p^{-1}]}(M, \Fil_1 \Gamma_{\log,\an,\con}) \cap \mathfrak{Sol}(M)$$

and

$$\mathfrak{M}(M)^{\lambda} := \mathfrak{Sol}_1(M)^{\perp}$$

where $(\cdot)^\perp$ denotes the orthogonal space with respect to the canonical pairing $\mathfrak{M}(M) \otimes_K \mathfrak{Sol}(M) \rightarrow K$.

We call the decreasing filtration $\{\mathfrak{M}(M)^{\lambda}\}$ the special log-growth filtration of $M$.

Note that if $M$ is a $(\sigma, \nabla)$-module over $\Gamma_{\con}[p^{-1}]$ solvable in $\log_{\an,\con}$, then $\mathfrak{M}(M)$ is a $\sigma$-module over $K$ by the injectivity of $\varphi : \mathfrak{M}(M) \rightarrow \mathfrak{M}(M)$. By Lemma 4.12 (ii), $\mathfrak{M}(M)^{\lambda}$ (resp. $\mathfrak{Sol}_1(M)$) is a $\sigma$-submodule of $\mathfrak{M}(M)$ (resp. $\mathfrak{Sol}(M)$).

Remark 4.14. We can define a special log-growth filtration for an arbitrary $(\sigma, \nabla)$-module over $\Gamma_{\con}[p^{-1}]$ as follows. Let $M$ be a $(\sigma, \nabla)$-module over $\Gamma_{\con}[p^{-1}]$. Then, there exists a finite étale extension $\Gamma^\prime/\Gamma$, corresponding to a certain finite separable extension $l/k(x)$, such that $M^\prime := \Gamma_{\con}^l[p^{-1}] \otimes_{\Gamma_{\con}[p^{-1}]} M$ is solvable in $\Gamma_{\log,\an,\con}$ by the log version of the $p$-adic local monodromy theorem ([Ked04, 6.13]). Similarly as above, we may define a log-growth filtration of $M^\prime$.

The log-growth filtrations are compatible with the base change $K[x]_0 \rightarrow \mathcal{E}^\prime = \Gamma_{\con}[p^{-1}]$.

Lemma 4.15. Let $M$ be a $(\sigma, \nabla)$-module over $K[x]_0$. Then, the $(\sigma, \nabla)$-module $\Gamma_{\con}[p^{-1}] \otimes_{K[x]_0} M$ over $\Gamma_{\con}[p^{-1}]$ is solvable in $\log_{\an,\con}$. Moreover, the canonical map

$$\iota : V(M) \rightarrow \mathfrak{M}(\Gamma_{\con}[p^{-1}] \otimes_{K[x]_0} M)$$

is an isomorphism, and preserves the Frobenius filtrations and the log-growth filtrations.

Proof. Since the natural inclusion $K\{x\} \rightarrow \Gamma_{\log,\an,\con}$ is compatible with Frobenius and differentials, $\Gamma_{\con}[p^{-1}] \otimes_{K[x]_0} M$ is solvable in $\log_{\an,\con}$, and $\iota$ is an isomorphism of $\sigma$-modules over $K$. The rest of the assertion follows from $\Fil_1 \Gamma_{\log,\an,\con} \cap K\{x\} = \Fil_1 \Gamma_{\an,\con} \cap K\{x\} = K\{x\}_\lambda$ (Lemma 4.7 (iii)).
4.3 Chiarellotto-Tsuzuki’s conjecture over $\Gamma_{\text{con}}[p^{-1}]$

We formulate an analogue of Theorem 3.7 for $(\sigma, \nabla)$-modules over $\Gamma_{\text{con}}[p^{-1}]$.

**Assumption 4.16.** In this section, we assume that $k_K$ is algebraically closed for simplicity.

**Definition 4.17.** Let $M$ be a $(\sigma, \nabla)$-module over $\Gamma_{\text{con}}[p^{-1}]$ solvable in $\Gamma_{\log, \text{an, con}}$.

(i) We call a Frobenius slope of $\mathfrak{M}(M)$ a special Frobenius slope of $M$.

(ii) We put $M_\mathcal{E} := \mathcal{E} \otimes_{\Gamma_{\text{con}}[p^{-1}]} M$, which is a $(\sigma, \nabla)$-module over $\mathcal{E}$. We call a Frobenius slope of $M_\mathcal{E}$ a generic Frobenius slope of $M$.

**Proposition 4.18.** Let $M$ be a $(\sigma, \nabla)$-module over $\Gamma_{\text{con}}[p^{-1}]$ solvable in $\Gamma_{\log, \text{an, con}}$. Let $\lambda_{\text{max}}$ be the highest Frobenius slope of $M_\mathcal{E}$.

(i) (Analogue of [CT09, Theorem 6.17]) We have

$$\mathfrak{M}(M)^\lambda \subset \left(S_{\lambda-\lambda_{\text{max}}} (\mathfrak{M}(M^\vee))\right)^\perp$$

for all $\lambda \in \mathbb{R}$. Here, $(\cdot)^\perp$ denotes the orthogonal space with respect to the canonical pairing $\mathfrak{M}(M) \otimes_K \mathfrak{M}(M^\vee) \to K$.

(ii) If $M_\mathcal{E}$ is PBQ, then

$$\mathfrak{M}(M)^0 = \left(S_{-\lambda_{\text{max}}} (\mathfrak{M}(M^\vee))\right)^\perp.$$ 

**Theorem 4.19** (Generalization of Theorem 3.7). Let $M$ be a $(\sigma, \nabla)$-module over $\Gamma_{\text{con}}[p^{-1}]$ solvable in $\Gamma_{\log, \text{an, con}}$.

(i) The special log-growth filtration of $M$ is rational and right continuous.

(ii) Let $\lambda_{\text{max}}$ be the highest Frobenius slope of $M_\mathcal{E}$. Assume that $M_\mathcal{E}$ is PBQ and the number of the Frobenius slopes of $M_\mathcal{E}$ is less than or equal to 2. Then,

$$\mathfrak{M}(M)^\lambda = \left(S_{\lambda-\lambda_{\text{max}}} (\mathfrak{M}(M^\vee))\right)^\perp$$

for all $\lambda \in \mathbb{R}$.

Theorem 3.7 follows from Theorem 4.19 by Lemma 4.15. The proofs of Proposition 4.18 and Theorem 4.19 will be given in § 7.

**Remark 4.20.** Obviously, one can formulate an analogue of Conjecture 3.3 (ii) for a $(\sigma, \nabla)$-module over $\Gamma_{\text{con}}[p^{-1}]$ such that $M_\mathcal{E}$ is PBQ.

4.4 Example: $p$-adic differential equations with nilpotent singularities

In [Ked10, § 18], Kedlaya studies effective bounds on the solutions of $p$-adic differential equations with nilpotent singularities. As an application, he proves a nilpotent singular analogue of Theorem 3.6 (iii) ([Ked10, Remark 18.4.4, Theorem 18.4.5]). In this subsection, we explain that a nilpotent singular analogue of Theorem 3.7 follows from Theorem 4.19.

In the following, we assume $\sigma(x) = x^q$. We define $\Omega^1_{K[x]}(\log)$ as a $\sigma$-module of rank 1 over $K[x]_0$ with basis $\text{dx}/x$ such that $\sigma^*(1 \otimes \text{dx}/x) := q\text{dx}/x$. Let $d : K[x]_0 \to \Omega^1_{K[x]}(\log) = K[x]_0 dx/x : f \mapsto xdf/\text{dx} \cdot dx/x$ be the canonical derivation on $K[x]_0$. We define a log $(\sigma, \nabla)$-module over $K[x]_0$ similarly as in § 3.2 by setting $R = K[x]_0$ and $\Omega^1_R = \Omega^1_{K[x]}(\log)$.

As in Definition 4.11, we define a log-growth filtration of $K[x][\log x]$ as

$$K[x][\log x]_\lambda := \bigoplus_{i=0}^{\lfloor \lambda \rfloor} K[x]_{\lambda-i}(\log x)^i$$

for $\lambda \in \mathbb{R}_{\geq 0}$ and $K[x][\log x]_\lambda := 0$ for $\lambda \in \mathbb{R}_{<0}$. For a log $(\sigma, \nabla)$-module $M$ over $K[x]_0$, we define $V(M) := (K[x][\log x]_0 \otimes_{K[x]_0} M)^{\nabla=0}$. By Dwork’s trick, $V(M)$ is of dimension $n$ ([Ked10, Corollary 17.2.4]). We define a special log-growth filtration $V(M)^\bullet$ of $M$ as in § 3.2 by replacing $K[x]_\lambda$ by $K[x][\log x]_\lambda$. 

13
Example. (i) A \((\sigma, \nabla)\)-module over \(K[[x]]_0\) can be regarded as a log \((\sigma, \nabla)\)-module over \(K[[x]]_0\) by identifying \(dx\) as \(x \cdot dx/x\). The special log-growth filtration of \(M\) as a non-log or log \((\sigma, \nabla)\)-module coincides with each other.

(ii) Let \(M := K[[x]]_0 e_1 \oplus K[[x]]_0 e_2\) be the log \((\sigma, \nabla)\)-module of rank 2 over \(K[[x]]_0\) define by
\[
\nabla(e_1, e_2) = (0, e_1 dx/x), \quad \varphi(e_1, e_2) = (e_1, q e_2).
\]
Then, \(V(M)\) has a basis \(\{e_1, -\log x \cdot e_1 + e_2\}\). Moreover, the Frobenius slopes of \(V(M)\) are 0, 1, and we have
\[
V(M)^\lambda = \begin{cases} 
V(M) & \text{if } \lambda < 0 \\
ke & \text{if } 0 \leq \lambda < 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Our main result in this subsection is

Theorem 4.21. An analogue of Theorem 3.7 for log \((\sigma, \nabla)\)-modules over \(K[[x]]_0\) holds.

Proof. It follows from Theorem 4.19 thanks to Lemma 4.22 below.

Lemma 4.22. Let \(M\) be a log \((\sigma, \nabla)\)-module over \(K[[x]]_0\). Then, the \((\sigma, \nabla)\)-module \(\Gamma_{\text{con}}[p^{-1}] \otimes_{K[X]} M\) over \(\Gamma_{\text{con}}[p^{-1}]\) is solvable in \(\Gamma_{\text{log, an, con}}\). Moreover, the canonical map
\[
\iota : V(M) \to \mathfrak{M}(\Gamma_{\text{con}}[p^{-1}] \otimes_{K[X]} M)
\]
is an isomorphism, and preserves the Frobenius filtrations and the log-growth filtrations.

Proof. Similar to the proof of Lemma 4.15.

5 Generic cyclic vector

In this section, we prove a key technical result in this paper concerned with a \(\sigma\)-module over \(\Gamma_{\text{con}}^{\text{alg}}[p^{-1}]\).

Definition 5.1. Let \(R\) be either \(\Gamma_{\text{con}}^{\text{alg}}[p^{-1}]\) or \(\Gamma_{\text{con}}^{\text{alg}}[p^{-1}]\).

(i) For \(f(\sigma) = a_0 + a_1 \sigma + \cdots + a_n \sigma^n \in R[\sigma]\) a twisted polynomial, we define the Newton polygon \(\text{NP}(f(\sigma))\) of \(f(\sigma)\) as the lower convex hull of the set of points
\[
\{(i, -\log q |a_i|_0(1)); 0 \leq i \leq n\}.
\]
A slope of \(\text{NP}(f(\sigma))\) is called a slope of \(f(\sigma)\) (cf. [Ked10, 2.1.3]). We consider the following condition \((*)\) on \(f(\sigma)\):
\[
(*) : \text{ each point } (i, -\log q |a_i|_0(1)) \text{ belongs to } \text{NP}(f(\sigma)).
\]

(ii) Let \(M\) be a \(\sigma\)-module of rank \(n\) over \(R\). When \(R = \Gamma_{\text{con}}^{\text{alg}}[p^{-1}]\), we call a Frobenius slope (resp. the Newton polygon) of \(\Gamma_{\text{con}}^{\text{alg}}[p^{-1}] \otimes_R M\) a generic Frobenius slope (resp. the generic Newton polygon) of \(M\). We say that an element \(e \in M\) is a cyclic vector if \(e, \varphi(e), \ldots, \varphi^{n-1}(e)\) is a basis of \(M\) over \(R\). For a cyclic vector \(e\), we have a unique relation
\[
\varphi^n(e) = -(a_{n-1} \varphi^{n-1}(e) + \cdots + a_0 e)
\]
with \(a_i \in R\). We put \(f_e(\sigma) := a_0 + a_1 \sigma + \cdots + \sigma^n \in R[\sigma]\). Note that \(\text{NP}(f_e(\sigma))\) coincides with the (generic) Frobenius Newton polygon of \(M^e\) ([Ked10, 14.5.7]).

We say that a cyclic vector \(e \in M\) is generic if \(f_e(\sigma)\) satisfies the condition \((*)\).

Theorem 5.2. Let \(M\) be a \(\sigma\)-module over \(\Gamma_{\text{con}}^{\text{alg}}[p^{-1}]\) (resp. \(\Gamma_{\text{con}}^{\text{alg}}[p^{-1}]\)). Assume \(q^* \in \mathbb{Q}\) for any (resp. generic) Frobenius slope \(s\) of \(M\). Then, there exists a generic cyclic vector of \(M\).

In the next subsection, we will see that there exists a non-empty open subset \(U\) of \(M\) such that \(v \in U\) is a generic cyclic vector. In this sense, there exist plenty cyclic vectors satisfying the condition \((*)\). So, the condition \((*)\) is referred to being generic.
5.1 Proof of Theorem 5.2

To prove Theorem 5.2, we first construct a generic cyclic vector over $\Gamma^{\mathrm{alg}}[p^{-1}]$. Then, we deform it to obtain a generic cyclic vector over $\Gamma^{\mathrm{can}}[p^{-1}]$. We first recall Kedlaya’s algorithm to compute an annihilator of an element of a $\sigma$-module over $\Gamma^{\mathrm{alg}}[p^{-1}]$ ([Ked05, 5.2.4]).

**Construction 5.3.** Let $R := \Gamma^{\mathrm{alg}}[p^{-1}]$. Let $M$ be a $\sigma$-module of rank $n$ over $R$ with Frobenius slopes $s_1 \leq \cdots \leq s_n$ with multiplicities. Assume $q^{e_i} \in \mathbb{Q}$ for all $i$. Then, we can choose an $R$-basis $e_1, \ldots, e_n$ of $M$ such that $\varphi(e_i) = q^{e_i} e_i$ for all $i$. Fix $x_1, \ldots, x_n \in R$ and put $v := x_1 e_1 + \cdots + x_n e_n$. We will define $v_l \in M$ for $1 \leq l \leq n$ by induction on $l$. Put $v_1 := v$. Given $v_l$, write $v_l = x_{l,1} e_1 + \cdots + x_{l,n} e_n$ with $x_{l,i} \in R$ and define

$$b_l := \begin{cases} q^{s_l} \cdot \sigma(x_{l,i})/x_{l,i} & \text{if } x_{l,i} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and $v_{l+1} := (\sigma - b_l)v_l$. Then, we have $v_l \in R e_1 + \cdots + R e_n$ and $(\sigma - b_n) \cdots (\sigma - b_1)v = 0$. We write

$$(\sigma - b_n) \cdots (\sigma - b_1) = \sigma^n + \sum_{c_0 \in R} \cdots + \sum_{c_{n-1} \in R} c_i R$$

in $R[\sigma]$. By construction, we may regard $x_{l,i} = x_{l,i}(x)$, $b_l = b_l(x)$, and $c_i = c_i(x)$ as functions of $x = (x_1, \ldots, x_n) \in R^n$ with values in $R$. We also regard $v = v(x)$ as a function of $x$ with values in $M$.

**Lemma 5.4.** We keep the notation in Construction 5.3.

(i) For $x \in R^n$, $v(x)$ is a cyclic vector of $M$ if and only if $x_{1,1}(x)x_{2,2}(x) \cdots x_{n,n}(x) \neq 0$.

(ii) For $x \in R^n$, $v(x)$ is a generic cyclic vector of $M$ if and only if $x_{1,1}(x)x_{2,2}(x) \cdots x_{n,n}(x) \neq 0$ and $-\log_q |c_l(x)|_{(1)} = s_1 + \cdots + s_{n-l}$ for all $i$.

(iii) Let $x^{(0)} \in R^n$. Assume that any of $b_1(x^{(0)}), \ldots, b_n(x^{(0)})$ is non-zero. Then, there exists an open neighborhood $U \subset R^n$ of $x^{(0)}$ (with respect to the topology induced by $| \cdot |_{(1)}$) such that all $b_l$ and $x_{l,i}$ are continuous on $U$. In particular, all $c_i$ are also continuous on $U$.

**Proof.** (i) By construction, there exists an upper triangular matrix $T$ whose diagonals are $(1, \ldots, 1)$ such that $(v_1, v_2, \ldots, v_n) = (v, \sigma(v), \ldots, \sigma^{n-1}(v))T$. Since $\{x_{l,1}, \ldots, x_{l,n}\}$ is an upper triangular matrix, we obtain the assertion.

(ii) It follows from (i) and the fact that the slopes of $\sigma^n + \sum_{c_0 \in R} \cdots + \sum_{c_{n-1} \in R} c_i R$ with multiplicities.

(iii) By induction on $l \in \{1, \ldots, n\}$, we construct an open neighborhood $U_l \subset R^n$ of $x^{(0)}$ such that $x_{1,1}, \ldots, x_{l,n}$ and $b_l$ are continuous on $U_l$, and $x_{l+1}$ is non-zero on $U_l$. Once we construct the $U_l$’s, $U := U_1 \cap \cdots \cap U_n$ satisfies the desired condition. First, note that $x_{l,1}(x^{(0)}) \neq 0$ for all $l$ by assumption. The assertion is trivial for $l = 1$ by setting $U_1 := \{x \in R^n; x_1(x) \neq 0\}$. Given $U_{l-1}$, let $U_l' := U_{l-1} \cap \{x \in R^n; x_{l,1}(x) \neq 0\}$, which is an open neighborhood of $x^{(0)}$. By the induction hypothesis, $x_{l,i} = \sigma(x_{l,i})q^{s_i} - b_{l-1}x_{l-1,i}$ is continuous on $U_l'$. We set $U_l := U_l' \cap \{x \in R^n; x_{l,1}(x) \neq 0\}$. Then, $U_l \subset R^n$ is open neighborhood of $x^{(0)}$ on which $b_l$ is continuous on $U_l$ as desired.

**Lemma 5.5.** Let $s_1 \leq s_2 \leq \cdots \leq s_n$ be rational numbers such that $q^{s_i} \in \mathbb{Q}$. Then, the slopes of $f(\sigma) := (\sigma - q^{s_1} x) \cdots (\sigma - q^{s_n} x) \in \Gamma[p^{-1}][\sigma]$ are $-s_n \leq \cdots \leq -s_1$ with multiplicities. Moreover, $f(\sigma)$ satisfies the condition (*)

**Proof.** We write

$$f(\sigma) = \sigma^n + a_{n-1}\sigma^{n-1} + \cdots + a_0, \quad a_i \in \Gamma[p^{-1}].$$

Then, we have

$$a_{n-i} = \sum_{1 \leq j(1) < \cdots < j(i) \leq n} (-1)^i q^{s_{j(1)} + \cdots + s_{j(i)}} \cdot x_{j(i)-1}^{(i)-1} \cdots x_{j(1)-1}^{(1)-1}. $$

Hence, $a_{n-i}/q^{s_{j(1)} + \cdots + s_{j(i)}}$ is a polynomial over $\mathbb{Z}$ whose lowest term is equal to $(-1)^i q^{s_{j(1)} + \cdots + s_{j(i)}}$. In particular, $-\log_q |a_{n-i}|_0(1) = s_1 + \cdots + s_i$, which implies the assertion. \[\square\]
Proof of Theorem 5.2. We first consider the case where $M$ is a $\sigma$-module over $R := \Gamma^{\text{alg}}[p^{-1}]$. Let $s_1 \leq s_2 \leq \cdots \leq s_n$ be the Frobenius slopes of $M$ with multiplicities. By Lemma 5.5, the slopes of the $\sigma$-module $M' := R(\sigma)/R(\sigma)(\sigma - q^s x) \cdots (\sigma - q^n x)$ are $s_1, \ldots, s_n$ with multiplicities. Recall that $\sigma$-modules over $R$ are classified by its slopes with multiplicities by Dieudonné-Manin theorem ([Ked10, 14.6.3]). Hence, there exists an isomorphism of $\sigma$-modules $M \cong M'$. By Lemma 5.5, $1 \in M'$ is a generic cyclic vector.

We consider the case where $M$ is a $\sigma$-module over $\Gamma^{\text{alg}}[p^{-1}]$. We have only to prove that there exists $f \in M$ which is a generic cyclic vector of $M^{\text{alg}} := \Gamma^{\text{alg}}[p^{-1}] \otimes_{\Gamma^{\text{alg}}[p^{-1}]} M$. We apply Construction 5.3 to $M^{\text{alg}}$. We choose a generic cyclic vector $e$ of $M^{\text{alg}}$ and write $e = v(x^{(0)})$ with $x^{(0)} \in R^n$. By Lemma 5.4 (ii) and (iii), there exists an open neighborhood $U \subset R^n$ of $x^{(0)}$ such that $v(x)$ is a generic cyclic vector of $M^{\text{alg}}$ for all $x \in U$. We choose a $\Gamma^{\text{alg}}[p^{-1}]$-basis $f_1, \ldots, f_n$ of $M$. For $y = (y_1, \ldots, y_n) \in R^n$, we define $w(y) := y_1 f_1 + \cdots + y_n f_n$. For $x \in R^n$, there exists a unique $y = y(x) \in R^n$ such that $v(x) = w(y)$, and the map $x \mapsto y(x)$ is a homeomorphism ([Ked10, 1.3.3]). Hence, there exists an open neighborhood $V \subset R^n$ of $y(x^{(0)})$ such that $w(y)$ is a generic cyclic vector of $M^{\text{alg}}$ for all $y \in V$. Since $\Gamma^{\text{alg}}$ is dense in $\Gamma^{\text{alg}}$, $w(y) \in M$ for $y \in V \cap (\Gamma^{\text{alg}}[p^{-1}])^n \neq \emptyset$ is a generic cyclic vector of $M^{\text{alg}}$. \hfill \qed

6 Frobenius equation and log-growth

In [CT09, § 7.2], Chiarellotto and Tsuzuki compute the log-growth of a solution $y$ of a Frobenius equation

$$ay + by^\sigma + cy^2 = 0, \quad a, b, c \in K[x]_0.$$  

In [Nak13], Nakagawa proves a generalization of Chiarellotto-Tsuzuki's result for a Frobenius equation

$$a_0 y + a_1 y^\sigma + \cdots + a_n y^n = 0, \quad a_i \in \mathcal{E}^j$$  

under the assumption that the number of breaks of the Newton polygon of $a_0 + a_1 \sigma + \cdots + a_n \sigma^n$ is equal to $n$. We generalize Nakagawa's result without any assumption on the Newton polygon:

**Theorem 6.1** (A generalization of Nakagawa's theorem ([Nak13, 1.1])). Let $f(\sigma) = a_0 + a_1 \sigma + \cdots + a_n \sigma^n \in \Gamma^{\text{alg}}_{\text{con}}[p^{-1}](\sigma)$, $a_0 \neq 0$, $a_n \neq 0$, $n \geq 1$

be a twisted polynomial satisfying the condition (*) in Definition 5.1 (ii) with slopes $-s_1 < \cdots < -s_k$. If $y \in \Gamma^{\text{alg}}_{\log, \text{an, con}}$ is a solution of the Frobenius equation

$$f(\sigma)y = a_0 y + a_1 y^\sigma + \cdots + a_n y^n = 0,$$  

then $y$ is either bounded or exactly of log-growth $s_j$ for some $j$ such that $s_j > 0$.

**Remark 6.2.** For a $(\sigma, \nabla)$-module $M$ of rank $n$ over $\Gamma_{\text{con}}[p^{-1}]$, we will construct a Frobenius equation $f(\sigma)y = 0$ satisfying the assumption of Theorem 6.1 (see Construction 7.1). The ambiguity of the log-growth of $y$ in Theorem 6.1 is due to the fact that $M_E/M_E^p$ may not be pure as a $\sigma$-module. One can expect that if $M$ is PBQ, then $y$ is exactly of log-growth $s_1$, as is the case for $n = 2$ ([CT09, 7.3]).

We divide the proof into two parts: the first part is an estimation of an upper bound of the log-growth of $y$ (easier), and the second part is an estimation of a lower bound of the log-growth of $y$ (harder). The condition (*) will be used only in the second part. The integer $j$ in Theorem 6.1 will be determined in § 6.3.

**Notation 6.3.** In this section, we keep the notation in Theorem 6.1. Let $0 = i(0) < i(1) < \cdots < i(k) = n$ be the $x$-coordinates of the vertices of NP$(f(\sigma))$. By Lemma 4.4, there exists a real number $\rho_0$ sufficiently close to 1 from the left such that for all $i \in \{0, 1, \ldots, n\}$, we have

$$a_i \in \Gamma^{\text{alg}}_r, \quad y \in \Gamma^{\text{alg}}_{\log, \text{an, r}},$$  

where $r = -\log(p \rho_0^n)$ and

$$|a_i|_{0}(\rho) = \rho^{|a_i|_{0}(1)} \quad \forall \rho \in [\rho_0^n, 1)$$  

for some $\alpha(i) \in \mathbb{Q}$; we fix such a $\rho_0$. 

16
6.1 Estimation of upper bound

Proposition 6.4 (A refinement of [CT09, 6.12]). Let $j \in \{0, \ldots, k-1\}$. We assume

\[
\sup_{i(j-1) \leq i \leq i(j)} |a_iy^{\sigma}|_0(\rho) \leq \sup_{i(j)+1 \leq i \leq n} |a_iy^{\sigma}|_0(\rho) \ \forall \rho \in [\rho_0, 1);
\]

(2)

when $j = 0$, we set $\sup_{i(j-1) \leq i \leq i(j)} |a_iy^{\sigma}|_0(\rho) = |a_0y|_0(\rho)$.

(i) For any $\rho \in [\rho_0, 1)$ and $m \in \mathbb{N}$, there exist an integer $N \in \{0, \ldots, n-1\}$, which depends only on $m$, and a sequence $\varepsilon_{iu}$ of integers, which depends on $\rho$ and $m$, defined for

\[I_m := \{(i, u) \in \mathbb{Z}^2; i(j) + 1 \leq i \leq n, -m - i(j) \leq u \leq 0\}\]

satisfying the following conditions:

(a) \[
\log |y|_0(\rho^{q^m}) - \log |y|_0(\rho^{q^N}) \leq \sum_{(i, u) \in I_m} \varepsilon_{iu} \log |a_i/a_{i(j)}|_0(\rho^{q^m});
\]

(b) $\varepsilon_{iu} \in \{0, 1\}$ and

\[
\sum_{(i, u) \in I_m} (i - i(j))\varepsilon_{iu} = m - N.
\]

(ii) $y$ has a log-growth $s_{j+1}$.

Proof. (i) Fix $\rho$ and we proceed by induction on $m$. When $m \leq n-1$, we set $N = m$ and $\varepsilon_{iu} \equiv 0$.

Then, we have nothing to prove. Assume that the assertion is true for the integers less than or equal to $m-1$ with $m \geq n$. By (2) for $\rho = \rho^{q^m-i(j)}$, we have

\[
|a_{i(j)}y^{\sigma(i(j))}|_0(\rho^{q^m-i(j)}) \leq \sup_{i(j)+1 \leq i \leq n} |a_iy^{\sigma}|_0(\rho^{q^m-i(j)}).
\]

(3)

We choose $i' \in \{i(j) + 1, \ldots, n\}$ such that

\[
\sup_{i(j)+1 \leq i \leq n} |a_iy^{\sigma}|_0(\rho^{q^m-i(j)}) = |a_{i'}y^{\sigma_{i'}}|_0(\rho^{q^m-i(j)}).
\]

(4)

By (3) and (4),

\[
\log |y|_0(\rho^{q^m}) - \log |y|_0(\rho^{q^m-i(j)+i'}) \leq \log |a_{i'}/a_{i(j)}|_0(\rho^{q^m-i(j)}).
\]

(5)

By the induction hypothesis for $m + i(j) - i'$, there exist an integer $N \in \{0, \ldots, n-1\}$ and a sequence $\varepsilon_{iu}'$ of 0 or 1 defined for $I_{m+i(j)-i'}$ such that

\[
\log |y|_0(\rho^{q^m-i(j)+i'}) - \log |y|_0(\rho^{q^N}) \leq \sum_{(i, u) \in I_{m+i(j)-i'}} \varepsilon_{iu}' \log |a_i/a_{i(j)}|_0(\rho^{q^m}),
\]

(6)

\[
\sum_{(i, u) \in I_{m+i(j)-i'}} (i - i(j))\varepsilon_{iu}' = m + i(j) - i' - N.
\]

(7)

For $(i, u) \in I_m$, we define

\[
\varepsilon_{iu} := \begin{cases} 
\varepsilon_{iu}' & \text{if } (i, u) \in I_{m+i(j)-i'} \\
1 & \text{if } (i, u) = (i', -m - i(j)) \\
0 & \text{otherwise.}
\end{cases}
\]

Then, by adding (5) to (6), the inequality in (a) follows. The condition $\varepsilon_{iu} \in \{0, 1\}$ follows by construction, and the equality in (b) follows from (7).
Proof. For $0 < \rho \in [\rho_0, 1)$, by applying (i) to $(\rho, m) = (\rho^m, m)$, there exist an integer $N(m) \in \{0, \ldots, n-1\}$ and a sequence $\varepsilon_{iu}^{(m)}$ of 0 or 1 defined for $(i, u) \in \mathcal{I}_m$ such that
\[
\log |y|_{0}(\rho) - \log |y|_{0}(\rho^{m-N(m)}) \leq \sum_{(i, u) \in \mathcal{I}_m} \varepsilon_{iu}^{(m)} \log |a_i/a_{i(j)}|_{0}(\rho^{m})
\]
(8)
\[
\sum_{(i, u) \in \mathcal{I}_m} (i - i(j)) \varepsilon_{iu}^{(m)} = m - N(m).
\]
(9)
For $i(j) + 1 \leq i \leq n$, there exist $v(i) \in \mathbb{Q}$ such that
\[
|a_i/a_{i(j)}|_{0}(\rho) = \rho^{v(i)}|a_i/a_{i(j)}|_{0}(1) \forall \rho \in [\rho_0, 1)
\]
by Notation 6.3. Moreover,
\[
\frac{1}{i - i(j)} \log |a_i/a_{i(j)}|_{0}(1) \geq -s_{j+1}
\]
(11)
by the convexity of the Newton polygon of $f(\sigma)$. By (10) and (11),
\[
\text{RHS of (8)} \leq \sum_{(i, u) \in \mathcal{I}_m} \varepsilon_{iu}^{(m)} q^u v(i) \log \rho + \sum_{(i, u) \in \mathcal{I}_m} (i - i(j)) \varepsilon_{iu}^{(m)} s_{j+1} \log q.
\]
(12)
Let $v := \max\{v(i) ; i(j) + 1 \leq i \leq n\}$. Then, the first summation in RHS of (12) is bounded above by
\[
\sum_{(i, u) \in \mathcal{I}_m} \varepsilon_{iu}^{(m)} q^u \log (1/\rho) \leq \sum_{(i, u) \in \mathcal{I}_m} q^u \log (1/\rho) = (n - i(j)) \frac{1 - q^{-m-i(j)-1}}{1 - q^{-1}} v \log (1/\rho) \leq n \frac{q}{q-1} v \log (1/\rho_0).
\]
By (9), the second summation in RHS of (12) is equal to
\[
(m - N(m)) s_{j+1} \log q.
\]
Thus, (8) leads to
\[
|y|_{0}(\rho) \leq C |y|_{0}(\rho^{m-N(m)}) \cdot q^{m-N(m)} s_{j+1}
\]
\[
= C |y|_{0}(\rho^{m-N(m)}) \cdot (\log (1/\rho^{m-N(m)})^{s_{j+1}} \cdot (\log (1/\rho))^{-s_{j+1}},
\]
(13)
where $C := \exp\{u(q-1)^{-1} v \log (1/\rho_0)\}$ is a constant independent of $\rho$. Since $\rho^{m-N(m)} \in [\rho_0^{-q^{-1}} ; \rho_0^{-q^{-1} + 1}]$, the functions $|y|_{0}(\rho^{m-N(m)})$ and $(\log (1/\rho^{m-N(m)}))^{s_{j+1}}$ are bounded when $\rho$ runs over $[\rho_0, 1)$; note that the function $|\rho_0, 1) \rightarrow \mathbb{R} ; \rho \mapsto |y|_{0}(\rho)$ is continuous. Thus, (13) implies the desired estimation
\[
|y|_{0}(\rho) = O((\log (1/\rho))^{-s_{j+1}}) \text{ as } \rho \uparrow 1.
\]
\[
\square
\]

6.2 Estimation of lower bound

We start with converting the condition (*) into the lemma:

Lemma 6.5. For any $j \in \{0, \ldots, k-1\}$, $i \in \{i(j+1) + 1, \ldots, n\}$, and $i' \in \{0, \ldots, i(j+1) - 1\}$, we have
\[
\log |a_{i'}|_{i(i+1)+1} |0(1) - \log |a_{i'}|_{0(1)} > \log |a_{i'}|_{0(1)} - \log |a_{i(i+1)}|_{0(1)}.
\]

Proof. For $0 \leq i \leq n$, we denote by $P_i$ the point $(i, -\log_{\rho_0} |a_i|_{0(1)})$. We also denote by $L_1$ and $L_2$ the segments $P_i P_{i + (j+1)}$ and $P_{i(j+1)} P_i$ respectively. Let $a$ and $b$ be the slopes of $L_1$ and $L_2$ respectively. We have only to prove $a < b$. Let us consider separately the cases where $i' - i(j+1) + i \leq i(j+1)$ or $i' - i(j+1) + i > i(j+1)$. In the first case, we have $a < -s_{j+1}$ by the condition (*). By the convexity of the Newton polygon of $f(\sigma)$, we have $-s_{j+1} \leq b$. Hence, $a < b$. In the latter case, the segment $L_1$ intersects with $L_2$. Since $P_{i(j+1)}$ is under $L_1$, we have $a < b$.

\[
\square
\]

18
Proof. When we have Assumption in Proposition 6.4. By Lemmas 4.4 and 6.5, after choosing $\rho_0$ sufficiently large if necessary, we may assume the following condition: for any $j \in \{0, \ldots, k - 1\}$, $i \in \{i(j + 1) + 1, \ldots, n\}$, and $i' \in \{0, \ldots, i(j + 1) - 1\}$, we have

$$
\log |a_{i' - i(j+1)+1}|_{0}(\rho_2) - \log |a_{i'}|_{0}(\rho_2) > \log |a_{i}|_{0}(\rho_3) - \log |a_{i(j+1)}|_{0}(\rho_3) \quad \forall \rho_2, \rho_3 \in [\rho_0, 1);
$$

(14)
due to the both sides of the inequality are continuous with respect to $\rho_2$ and $\rho_3$ respectively, and converge to $\log |a_{i' - i(j+1)+1}|_{0}(1) - \log |a_{i'}|_{0}(1)$ and $\log |a_{i}|_{0}(1) - \log |a_{i(j+1)}|_{0}(1)$ as $\rho_2, \rho_3 \uparrow 1$ respectively.

To estimate the function $|y|_{0}(\rho)$ of $\rho$ from above, we need to consider several inequalities such as the assumption in Proposition 6.4.

Assumption 6.7. Let $j \in \{0, \ldots, k - 1\}$. In the rest of this subsection, we assume the following:

$$
\sup_{i(0) \leq i \leq i(1)} |a_{i} y^{\sigma'}|_{0}(\rho) \leq \sup_{i(1) + 1 \leq i \leq n} |a_{i} y^{\sigma'}|_{0}(\rho) \forall \rho \in [\rho_0^{-n}, 1),
$$

$$
\sup_{i(1) \leq i \leq i(2)} |a_{i} y^{\sigma'}|_{0}(\rho) \leq \sup_{i(2) + 1 \leq i \leq n} |a_{i} y^{\sigma'}|_{0}(\rho) \forall \rho \in [\rho_0^{-n}, 1),
$$

$$
\vdots
$$

$$
\sup_{i(j-1) \leq i \leq i(j)} |a_{i} y^{\sigma'}|_{0}(\rho) \leq \sup_{i(j) + 1 \leq i \leq n} |a_{i} y^{\sigma'}|_{0}(\rho) \forall \rho \in [\rho_0^{-n}, 1);
$$

when $j = 0$, we set $\sup_{i(j-1) \leq i \leq i(j)} |a_{i} y^{\sigma'}|_{0}(\rho) := |a_{0} y|_{0}(\rho)$. We also assume

$$
\sup_{i(j) \leq i \leq i(j+1)} |a_{i} y^{\sigma'}|_{0}(\rho) > \sup_{i(j+1) + 1 \leq i \leq n} |a_{i} y^{\sigma'}|_{0}(\rho) \exists \rho \in [\rho_0^{-n}, 1);
$$

when $j = k - 1$, we set $\sup_{i(j+1) + 1 \leq i \leq n} |a_{i} y^{\sigma'}|_{0}(\rho) := 0$.

Lemma 6.8. Assume that $\rho_1 \in [\rho_0^{-n}, 1)$ satisfies

$$
\sup_{i(j) \leq i \leq i(j+1)} |a_{i} y^{\sigma'}|_{0}(\rho_1) > \sup_{i(j+1) + 1 \leq i \leq n} |a_{i} y^{\sigma'}|_{0}(\rho_1).
$$

(i) We have

$$
\sup_{0 \leq i \leq i(j+1) - 1} |a_{i} y^{\sigma'}|_{0}(\rho_1) \geq |a_{i(j+1)} y^{\sigma'(j+1)}|_{0}(\rho_1).
$$

(ii) Let $i' \in \{0, \ldots, i(j+1) - 1\}$ be an integer such that

$$
|a_{i'} y^{\sigma'}|_{0}(\rho_1) = \sup_{0 \leq i \leq i(j+1) - 1} |a_{i} y^{\sigma'}|_{0}(\rho_1).
$$

Then, we have

$$
\sup_{i(j) \leq i \leq i(j+1)} |a_{i} y^{\sigma'}|_{0}(\rho_1^{j'-i(j+1)}) > \sup_{i(j+1) + 1 \leq i \leq n} |a_{i} y^{\sigma'}|_{0}(\rho_1^{j'-i(j+1)}).
$$

Proof. (i) Suppose the contrary. Then, we have $|a_{i(j+1)} y^{\sigma'(j+1)}|_{0}(\rho_1) > \sup_{i \neq i(j+1)} |a_{i} y^{\sigma'}|_{0}(\rho_1) \geq 0$ by Assumption 6.7. Since $|\cdot|_{0}(\rho)$ is a non-archimedean norm, we have $|a_{i(j+1)} y^{\sigma'(j+1)}|_{0}(\rho_1) = 0$ by (1), which is a contradiction.

(ii) By (i) and Assumption 6.7, we have

$$
|a_{i'} y^{\sigma'}|_{0}(\rho_1) = \sup_{0 \leq i \leq n} |a_{i} y^{\sigma'}|_{0}(\rho_1).
$$

(15)
For $i \in \{i(j+1) + 1, \ldots, n\}$, we have

\[
\log |y^{a_i(j+1)}(\rho_1) - a_i y^{a_i(j+1)}(\rho_1)| - \log |y^{a'_{i(j+1)}}| - \log |y^{a'_{i(j+1)+1}}| = \log |y| o(\rho_1^{a_i}) - \log |y| o(\rho_1^{a'_{i(j+1)+1}})
\]

\[
= \log |a_i y^{a_i(j+1)}(\rho_1) - a'_i y^{a'_{i(j+1)+1}}| - \log |y| o(\rho_1) + \log |a_i y^{a_i}| - \log |y| o(\rho_1^{a'_{i(j+1)+1}}).
\]

Thus, we obtain

\[
|a_i y^{a_i(j+1)}(\rho_1^{a_i(j+1)+1}) > \sup_{i(j+1)+1 \leq i \leq n} |a_i y^{a_i(j+1)}| |a_i y^{a_i}| o(\rho_1^{a_i(j+1)+1}),
\]

which implies the assertion. □

**Construction 6.9.** Fix $\rho_1 \in [\rho_0, 1)$ such that

\[
\sup_{i(j+1) \leq i \leq i(j+1)+1} |a_i y^{a_i(j+1)+1}| o(\rho_1) > \sup_{i(j+1)+1 \leq i \leq n} |a_i y^{a_i}| o(\rho_1).
\]

By induction on $l \in \mathbb{N}$, we will construct a strictly decreasing sequence $\{m(l)\}_l$ of integers less than or equal to $i(j+1)$, and a sequence $\epsilon_{iu}^{(l)}$ of integers defined for

\[
\mathcal{I}_l := \{(i, u) \in \mathbb{Z}^2 : 0 \leq i \leq i(j+1) - 1, m(l) - i(j+1) \leq u \leq 0\}
\]

satisfying the following conditions:

(a) \[
\log |y| o(\rho_1^{m(l)}) - \log |y| o(\rho_1^{a_i(j+1)+1}) \geq \sum_{(i, u) \in \mathcal{I}_l} \epsilon_{iu}^{(l)} \log |a_i y^{a_i(j+1)+1}| o(\rho_1^{a_i}).
\]

(b) $\epsilon_{iu}^{(l)} \in \{0, 1\}$ and \[
\sum_{(i, u) \in \mathcal{I}_l} (i(j+1) - i) \epsilon_{iu}^{(l)} = i(j+1) - m(l).
\]

(c) \[
\sup_{i(j+1) \leq i \leq i(j+1)+1} |a_i y^{a_i(j+1)+1}| o(\rho_1^{a_i(j+1)+1}) > \sup_{i(j+1)+1 \leq i \leq n} |a_i y^{a_i}| o(\rho_1^{a_i(j+1)+1}).
\]

We set $m_0 := i'$ where $i'$ is defined in Lemma 6.8 (ii), and define

\[
\epsilon_{iu}^{(0)} := \begin{cases} 1 & \text{if } (i, u) = (i', 0) \\ 0 & \text{otherwise.} \end{cases}
\]

Since $|a_i y^{a_i}| o(\rho_1) \geq |a_i y^{a_i(j+1)+1}| o(\rho_1)$ by Lemma 6.8 (i), the condition (a) follows. The condition (b) follows by definition. The condition (c) follows from Lemma 6.8 (ii).

Given $m(l)$ and $\epsilon_{iu}^{(l)}$, we can apply Lemma 6.8 to $\rho_1 = \rho_1^{a_i(j+1)+1}$ by the condition (c) for $m(l)$; let $i' \in \{0, \ldots, i(j+1) - 1\}$ be the integer defined in Lemma 6.8 (ii). Since $|a_i y^{a_i(j+1)+1}| o(\rho_1^{a_i(j+1)+1}) \geq |a_i y^{a_i(j+1)+1}| o(\rho_1^{a_i(j+1)+1})$ by Lemma 6.8 (i), we have

\[
\log |y| o(\rho_1^{a_i(j+1)+1}) - \log |y| o(\rho_1^{a_i(j+1)+1}) \geq \log |a_i y^{a_i(j+1)+1}| o(\rho_1^{a_i(j+1)+1}).
\]

We put $m(l+1) := m(l) - i(j+1) + i' < m(l)$ and define $\epsilon_{iu}^{(l+1)}$ for $(i, u) \in \mathcal{I}_{l+1}$ by

\[
\epsilon_{iu}^{(l+1)} := \begin{cases} \epsilon_{iu}^{(l)} & \text{if } (i, u) \in \mathcal{I}_l \\ 1 & \text{if } (i, u) = (i', m(l) - i(j+1)) \\ 0 & \text{otherwise.} \end{cases}
\]

We verify the conditions (a), (b), and (c). By adding (16) to the inequality in (a) for $m(l)$, the condition (a) follows. The condition (b) follows by construction. The condition (c) follows from Lemma 6.8 (ii).
Proposition 6.10. If $y$ is non-zero and has a log-growth $\alpha \in \mathbb{R}_{\geq 0}$, then $\alpha \geq s_{j+1}$.

Proof. Obviously, we may assume $s_{j+1} > 0$. For $i \in \{0, \ldots, i(j+1) - 1\}$, there exists $v(i) \in \mathbb{Q}$ such that

$$|a_{i(j+1)}/a_i|_0(\rho) = \rho^{v(i)}|a_{i(j+1)}/a_i|_0(1) \quad \forall \rho \in [\rho_0^n, 1),$$

where $\rho_0 = 0$. Therefore, the assumption follows from Propositions 6.4 and 6.10.

Moreover,

$$-\frac{1}{i(j+1)-t}\log_q |a_{i(j+1)}/a_i|_0(1) \leq -s_{j+1}.$$

by the convexity of the Newton polygon of $f(\sigma)$.

We keep the notation in Construction 6.9. By (17), (18), and the inequality (a) for $m(l)$, we have

$$\log |y|_0(\rho^{m(l)}) - \log |y|_0(\rho^{q^{i(j+1+1)})}) \geq \sum_{(i,u) \in I_i} \varepsilon_{iu}^{(l)} q^u v(i) \log \rho_1 + \sum_{(i,u) \in I_i} (i(j+1)-i)\varepsilon_{iu}^{(l)} s_{j+1} \log q.$$

Let $v := \inf\{\pm v(i); 0 \leq i \leq i(j+1) - 1\}$. Then, the first summation in RHS of (19) is bounded below by

$$\sum_{(i,u) \in I_i} \varepsilon_{iu}^{(l)} q^u v(1/\rho_1) \geq \sum_{(i,u) \in I_i} q^u v \log (1/\rho_1) = (i(j+1)-1)q^{m(l)-i(j+1)-1} \log (1/\rho_1) \geq nq/q - v \log (1/\rho_1).$$

By the condition (b) for $m(l)$, the second summation in RHS of (19) is equal to

$$(i(j+1) - m(l))s_{j+1} \log q.$$

Therefore, (19) leads to

$$|y|_0(\rho^{m(l)}) \geq C |y|_0(\rho^{q^{i(j+1+1)})} q^{i(j+1)-m(l)s_{j+1+1}} = C |y|_0(\rho^{q^{i(j+1+1)})} q^{i(j+1)s_{j+1+1}} (1/\rho_1)^{s_{j+1+1}},$$

where $C := \exp\{nu(q - 1)^{-1}v \log (1/\rho_1)\}$. Note that

$$C |y|_0(\rho^{q^{i(j+1+1)})} q^{i(j+1)s_{j+1+1}} (1/\rho_1)^{s_{j+1+1}}$$

is a positive constant independent of $l$. Since $m(l) \rightarrow -\infty$ as $l \rightarrow \infty$, (20) implies

$$|y|_0(\rho) \neq O((\log (1/\rho))^{-\beta})$$

as $\rho \uparrow 1$

for any $\beta \in \mathbb{R}_{< s_{j+1}}$. \hfill \Box

6.3 Proof of Theorem 6.19

Let $\rho_0$ be as in Notation 6.6. For $j \in \{0, 1, \ldots, k-2\}$, we consider the following condition on $y$:

$$(C_j) : \quad \sup_{i(j) \leq i \leq i(j+1)} |a_i y^\sigma|_0(\rho) \leq \sup_{i(j+1) + 1 \leq i \leq n} |a_i y^\sigma|_0(\rho) \quad \forall \rho \in [\rho_0, 1).$$

Let $j \in \{0, \ldots, k-2\}$ be the least integer such that the condition $(C_j)$ does not hold; if the condition $(C_j)$ holds for all $j$, then we set $j = k - 1$. Then, $j$ satisfies the assumption in Proposition 6.4: when $j = 0$, the assumption follows from (1). Also, $j$ satisfies Assumption 6.7; when $j = k - 1$, the assumption follows from $y \neq 0$. Therefore, the assertion follows from Propositions 6.4 and 6.10.

7 Proof of Theorem 4.19

In this section, we assume that $k_K$ is algebraically closed as in Assumption 4.16. For a $(\sigma, \nabla)$-module over $K[x]_0$, Chiarello and Tsuuki define a Frobenius equation ([CT09, Proof of Theorem 6.17 (i)]). Then, they interpret their conjecture $LG_{K[\tau]_0}$ as a problem on the Frobenius equation. For a $(\sigma, \nabla)$-module over $\Gamma_{\text{con}}[p^{-1}]$, their method can be applied as follows.
Lemma 7.2. We keep the notation in Construction 7.1. By [CT11, 4.2], the above exact sequence splits as a sequence of (i) Since we have:

\[ v = y_0 e + y_1 \varphi(e) + \cdots + y_{n-1} \varphi^{n-1}(e), \quad y_i \in \Gamma_{\log, an, con}^{\text{alg}}. \]

Then, we obtain the relation

\[
\begin{pmatrix}
1 & -a_0 \\
& \ddots & \ddots & \ddots \\
& & 1 & -a_{n-1}
\end{pmatrix}
\begin{pmatrix}
y_0 \\
y_1 \\
\vdots \\
y_{n-1}
\end{pmatrix}
= \gamma
\begin{pmatrix}
y_0 \\
y_1 \\
\vdots \\
y_{n-1}
\end{pmatrix}.
\]

(21)

By elimination, \( y := y_{n-1} \) satisfies the following Frobenius equation:

\[
y = - \sum_{0 \leq i \leq n-1} \frac{\sigma^i (a_{n-i-1})}{\gamma \sigma(\gamma) \cdots \sigma^i(\gamma)} \sigma^i(y). \tag{22}
\]

Note that the slopes of the twisted polynomial

\[ 1 + \frac{a_{n-1}}{\gamma} + \cdots + \frac{a_0}{\gamma \sigma(\gamma) \cdots \sigma^{n-1}(\gamma)} \sigma^n \]

are \(-s_k - s < \cdots < -s_1 - s\), where \( s = -\log_q |\gamma| \).

Lemma 7.2. We keep the notation in Construction 7.1.

(i) For \( \lambda \in \mathbb{R} \), we have \( v \in \mathfrak{sol}_\lambda(M) \) if and only if \( y \in \text{Fil}_\lambda \Gamma_{\log, an, con}^{\text{alg}} \).

(ii) We have either \( v \in \mathfrak{sol}_0(M) \) or \( v \in \mathfrak{sol}_{s+j}(M) \setminus \mathfrak{sol}_{(s+j)-}(M) \) for some \( j \) such that \( s + j > 0 \).

Proof. (i) Since we have

\[
\mathfrak{sol}_\lambda(M) \subset \text{Fil}_\lambda \Gamma_{\log, an, con} \otimes_R M^\vee \subset \text{Fil}_\lambda \Gamma_{\log, an, con}^{\text{alg}} \otimes_R M^\vee \cong \text{Fil}_\lambda \Gamma_{\log, an, con}^{\text{alg}} \otimes_{\Gamma_{\log, an, con}^{\text{alg}}} \Gamma_{\log, an, con}^{\text{alg}} \otimes_R M^\vee,
\]

\( v \in \mathfrak{sol}_\lambda(M) \) implies \( y \in \text{Fil}_\lambda \Gamma_{\log, an, con}^{\text{alg}} \). Assume \( y \in \text{Fil}_\lambda \Gamma_{\log, an, con}^{\text{alg}} \). By (21) and Lemma 4.12, we have \( y_i \in \text{Fil}_\lambda \Gamma_{\log, an, con}^{\text{alg}} \otimes_R M^\vee \). Since \( \text{Fil}_\lambda \Gamma_{\log, an, con}^{\text{alg}} \cap \Gamma_{\log, an, con} \) by induction on \( j \). Hence, \( v \in \text{Fil}_\lambda \Gamma_{\log, an, con}^{\text{alg}} \otimes_R M^\vee \). Since \( \text{Fil}_\lambda \Gamma_{\log, an, con}^{\text{alg}} \cap \Gamma_{\log, an, con} = \text{Fil}_\lambda \Gamma_{\log, an, con} \), by definition, we have \( v \in \text{Fil}_\lambda \Gamma_{\log, an, con} \otimes_R M^\vee \), i.e., \( v \in \mathfrak{sol}_\lambda(M) \).

(ii) The assertion follows from (i) and Theorem 6.1.

We will deduce Proposition 4.18 and Theorem 4.19 from Lemma 7.2 (ii) and the following lemma.

Lemma 7.3. Let \( M \) be a \( (\sigma, \nabla) \)-module over \( \mathcal{E} \) and \( \lambda_{\max} \) the highest Frobenius slope of \( M \). If \( M \) is PBQ, then \( (M^\vee)_0 \) is pure of slope \(-\lambda_{\max}\) as a \( \sigma \)-module.

Proof. Note that \( (M^\vee)_0 \cong (M/M^0)^\vee\) is pure as a \( \sigma \)-module by assumption. Moreover, the Frobenius slope \( \lambda \) of \((M^\vee)_0\) is greater than or equal to \(-\lambda_{\max}\). Suppose that the assertion is false, i.e., \( \lambda > -\lambda_{\max} \). Let \( M' \) be the inverse image of \( M'' := S_{-\lambda_{\max}} (M^\vee/(M^\vee)_0) \) under the canonical projection \( M^\vee \to M^\vee/(M^\vee)_0 \). By assumption, \( M'' \neq 0 \) and there exists a short exact sequence of \((\sigma, \nabla)\)-modules over \( \mathcal{E} \):

\[
0 \longrightarrow (M^\vee)_0 \longrightarrow M' \longrightarrow M'' \longrightarrow 0.
\]

By [CT11, 4.2], the above exact sequence splits as a sequence of \((\sigma, \nabla)\)-modules. Since \( M''_0 \neq 0 \), we have \((M^\vee)_0 \subseteq M'_0 \subset (M^\vee)_0\), which is a contradiction.
Proof of Proposition 4.18. By replacing \((M, \varphi, \nabla)\) by \((M, \varphi^h, \nabla)\) for sufficiently large \(h \in \mathbb{N}\), we may assume \(q^* \in \mathbb{Q}\) for all special and generic Frobenius slopes \(s\) of \(M\). Then, we may apply Construction 7.1 to \(M\). We keep the notation in Construction 7.1. Note that \(s_k = \lambda_{\max}\) by definition.

(i) Let \(v \in \mathfrak{Sol}(M)\) be a non-zero Frobenius eigenvector of slope \(s\). By Grothendieck-Katz specialization theorem ([Ked10, 15.3.2]), we have \(s \geq -s_k\). By Lemma 7.2 (ii), we have \(v \in \mathfrak{Sol}_{s+s_j}(M)\). Hence, we have \(S_{\lambda-\lambda_{\max}}(\mathfrak{Sol}(M)) \subset \mathfrak{Sol}_\lambda(M)\) for all \(\lambda \in \mathbb{R}\). By taking \((-)^+\) with respect to the canonical pairing \(\mathfrak{U}(M) \otimes_K \mathfrak{Sol}(M) \rightarrow K\), we obtain \((S_{\lambda-\lambda_{\max}}(\mathfrak{U}(M')))^+ \subset \mathfrak{U}(M)^\lambda\).

(ii) Since \(\mathfrak{Sol}_0(M) = (M^\vee)^{\mathbb{Q}=0}\), we have \(\mathfrak{Sol}_0(M) \subset (M^\vee)_0\) by the characterization of \((M^\vee)_0\). By Lemma 7.3, \((M^\vee)_0\), and hence \(\mathfrak{Sol}_0(M)\) is pure of slope \(-\lambda_{\max}\) as a \(\sigma\)-module, i.e., \(\mathfrak{Sol}_0(M) \subset S_{-\lambda_{\max}}(\mathfrak{Sol}(M))\). Similarly as in the proof of (i), we obtain \((S_{-\lambda_{\max}}(\mathfrak{U}(M')))^+ \subset \mathfrak{U}(M)^0\).

\(\blacksquare\)

Proof of Theorem 4.19. Similarly as in the proof of Proposition 4.18, we may apply Construction 7.1 to \(M\) again.

(i) By the definition of \(\mathfrak{U}(M)^*\), we have only to prove that the filtration \(\mathfrak{Sol}_0(M)\) is rational and right continuous.

We first prove the rationality of breaks \(\lambda\) of \(\mathfrak{Sol}_0(M)\). We may assume \(\lambda > 0\). Since \(\mathfrak{Sol}_{\lambda-}(M)\) is a direct summand of \(\mathfrak{Sol}_{\lambda+}(M)\) as a \(\sigma\)-module, we can choose a Frobenius eigenvector \(v \in \mathfrak{Sol}_{\lambda+}(M)\setminus \mathfrak{Sol}_{\lambda-}(M)\) of slope \(s\). By \(v \notin \mathfrak{Sol}_0(M)\) and Lemma 7.2 (ii), we have \(v \in \mathfrak{Sol}_{s+s_j}(M)\setminus \mathfrak{Sol}_{(s+s_j)-}(M)\) for some \(j\) such that \(s + s_j > 0\), i.e., \(\lambda = s + s_j \in \mathbb{Q}\).

We prove the right continuity of \(\mathfrak{Sol}_0(M)\). Suppose the contrary, i.e., there exists \(\lambda \in \mathbb{R}_{\geq 0}\) such that \(\mathfrak{Sol}_\lambda(M) \neq \mathfrak{Sol}_{\lambda+}(M)\). Let \(\Delta(M)\) be the set of rational numbers consisting of \(0\) and \(s + s_j\) where \(s\) is a Frobenius slope of \(\mathfrak{Sol}(M)\). Fix \(\lambda' \in \mathbb{R}_{\geq 0}\) sufficiently close to \(\lambda\) such that \(\mathfrak{Sol}_{\lambda+}(M) = \mathfrak{Sol}_{\lambda'}(M)\) and \(\Delta(M) \cap (\lambda, \lambda'] = \emptyset\). Since \(\mathfrak{Sol}_\lambda(M)\) is a direct summand of \(\mathfrak{Sol}_{\lambda+}(M)\) as a \(\sigma\)-module, we can choose a Frobenius eigenvector \(v \in \mathfrak{Sol}_{\lambda+}(M)\setminus \mathfrak{Sol}_\lambda(M)\) of slope \(s\). By Lemma 7.2 (ii), we have either \(v \in \mathfrak{Sol}_0(M)\) or \(v \in \mathfrak{Sol}_{s+s_j}(M)\setminus \mathfrak{Sol}_{(s+s_j)-}(M)\) for some \(j\) such that \(s + s_j > 0\). In the first case, we have \(v \in \mathfrak{Sol}_\lambda(M)\), which is a contradiction. In the latter case, we have \(s + s_j > \lambda\) by \(v \in \mathfrak{Sol}_{s+s_j}(M)\). Since \(v \notin \mathfrak{Sol}_{(s+s_j)-}(M)\), we have \(\lambda' \leq s + s_j\). Hence, we have \(s + s_j \in \Delta(M) \cap (\lambda, \lambda'] = \emptyset\), which is contradiction.

(ii) By Proposition 4.18 (i), we have only to prove \(\mathfrak{Sol}_\lambda(M) \subset S_{\lambda-\lambda_{\max}}(\mathfrak{Sol}(M))\) for all \(\lambda \geq 0\). Let us consider separately the cases where \(k = 1\) or \(k = 2\). In the first case, since \(\mathfrak{Sol}(M)\) is pure of slope \(-\lambda_{\max}\) by Grothendieck-Katz specialization theorem, the assertion is trivial. In the latter case, let \(v \in \mathfrak{Sol}_\lambda(M)\) be a non-zero Frobenius eigenvector of slope \(s\). By Grothendieck-Katz specialization theorem, we have \(-s_2 \leq s \leq -s_1\). Hence, we have either \(v \in \mathfrak{Sol}_0(M)\) or \(v \in \mathfrak{Sol}_{s+s_2}(M)\setminus \mathfrak{Sol}_{(s+s_2)-}(M)\) by Lemma 7.2 (ii). In the first case, \(v \in \mathfrak{Sol}_0(M) = S_{-s_2}(\mathfrak{Sol}(M)) \subset S_{\lambda-s_2}(\mathfrak{Sol}(M))\) by Proposition 4.18 (ii). In the latter case, we have \(s + s_2 \leq \lambda\). Hence, \(v \in S_s(\mathfrak{Sol}(M)) \subset S_{\lambda-s_2}(\mathfrak{Sol}(M))\).

\(\blacksquare\)

Remark 7.4. Let \(M\) be a \((\sigma, \nabla)\)-module over \(\Gamma_{\text{con}}[p^{-1}]\) solvable in \(\Gamma_{\log, \text{an, con}}\). We can expect that any break \(\lambda\) of the special log-growth filtration of \(M\) is of the form \(-s + \lambda_{\max}\) where \(s\) is a special Frobenius slope of \(M\). At this point, as in the proof of Theorem 4.19 (i), we can prove only that \(\lambda\) is of the form \(-s + s'\) where \(s\) (resp. \(s'\)) is a special (resp. generic) Frobenius slope of \(M\) such that \(-s + s' \geq 0\).
8 Appendix: diagram of rings

For $0 \leq \lambda_1 \leq \lambda_2$, we have the following diagram of rings: all the morphisms are the natural inclusions.

\[
\begin{array}{ccccccc}
K[[x]]_0 & \to & K[[x]]_{\lambda_1} & \to & K[[x]]_{\lambda_2} & \to & K[[x]] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma[p^{-1}] & \to & \Gamma_{\text{an,con}}[p^{-1}] & \to & \text{Fil}_{\lambda_1} \Gamma_{\text{an,con}} & \to & \text{Fil}_{\lambda_2} \Gamma_{\text{an,con}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{E} & \to & \mathcal{E}^\dagger & \to & \Gamma_{\text{an,con}} & \to & \mathcal{R}.
\end{array}
\]

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24