The basis property of generalized Jacobian elliptic functions *

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Abstract
The Jacobian elliptic functions are generalized to functions including the generalized trigonometric functions. The paper deals with the basis property of the sequence of generalized Jacobian elliptic functions in any Lebesgue space. In particular, it is shown that the sequence of the classical Jacobian elliptic functions is a basis in any Lebesgue space if the modulus $k$ satisfies $0 \leq k \leq 0.99$.

1 Introduction
The Jacobian elliptic function $sn(x, k)$ and the complete elliptic integral of the first kind $K(k)$ play important roles in expressing exact solutions of, for example, the pendulum equation $u'' + \lambda \sin u = 0$, a typical bistable equation $u'' + \lambda u(1 - u^2) = 0$, and so on.

Now we will propose new generalization of $sn(x, k)$ and $K(k)$. For constants $p, q \in (1, \infty)$ and $k \in [0, 1)$, we define a generalized Jacobian elliptic function $sn_{pq}(x, k) : [0, K_{pq}(k)] \to [0, 1]$ with a modulus $k$ as

$$x = \int_0^{sn_{pq}(x, k)} \frac{dt}{(1 - t^4)^{\frac{1}{p}}(1 - k^4 t^4)^{\frac{1}{q}}},$$

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where \( p' = p/(p-1) \) and

\[
K_{pq}(k) = \int_0^1 \frac{dt}{(1-t^q)^\frac{1}{p}(1-k^qt^q)^\frac{1}{p'}}.
\]

We extend the domain of \( sn_{pq}(x,k) \) to \( \mathbb{R} \) so that we obtain a 4\( K_{pq}(k) \)-period function like the sine function, and call the extended function \( sn_{pq}(x,k) \) again. Then, \( sn_{22}(x,k) = sn(x,k) \) and \( K_{22}(k) = K(k) \) when \( p = q = 2 \); and \( sn_{pq}(x,0) = sn_{pq} x \) and \( K_{pq}(0) = \pi_{pq}/2 \) when \( k = 0 \), where \( sn_{pq} x \) is the generalized trigonometric function and \( \pi_{pq} \) is the half period of \( sn_{pq} x \), which will be introduced in Section 3 below. Therefore, \( sn_{pq}(x,k) \) is also generalization of both \( sn(x,k) \) and \( sin_{pq} x \).

In the previous paper [22], the author proposed another generalization of \( sn(x,k) \) and \( K(k) \), and applied them to bifurcation problems for \( p \)-Laplacian. As we will see in Section 3, \( sn_{pq}(x,k) \) and \( K_{pq}(k) \) above are defined in a slightly different way from those in [22], but \( sn_{pq}(x,k) \) also satisfies the following equation involving \( p \)-Laplacian nevertheless.

\[
(|u'|^{p-2}u')' + \frac{(p-1)q}{p}|u|^q u(1 + (p-1)k^q - pk^q |u|^q)(1 - k^q |u|^q)^{p-2} = 0.
\]

While the generalization of \( K(k) \) of [22] converges to a finite value as \( k \to 1 \) when \( p > 2 \), the \( K_{pq}(k) \) diverges to \( \infty \) as \( k \to 1 \) for any \( p > 1 \). In this sense, \( sn_{pq}(x,k) \) has closer properties to \( sn(x,k) \) than the function defined in [22].

In the present paper, we will show the basis property of functions

\[
f_n(x,k) = sn_{pq}(2nK_{pq}(k)x,k), \quad n = 1, 2, \ldots,
\]

which means that the family of these functions is a basis in Banach spaces. Here, a sequence \( \{\varphi_n\} \) in a Banach space \( X \) is called a basis for \( X \) if for every \( u \in X \) there exists a unique sequence of scalars \( \{\alpha_n\} \) such that \( u = \sum_{n=1}^\infty \alpha_n \varphi_n \) in the strong sense. In general, when we try to find an approximation of a given function by a family of functions \( \{\varphi_n\} \), it is desirable that \( \{\varphi_n\} \) is a basis which approximates to the function with convergence of higher order as possible. Concerning this, for example, we have known an interesting study [3] of Boulton and Lord. They study the best index \( q \) for which \( \{sin_q (n\pi_q x)\} \) approximates well to the solution of \( p \)-Poisson problem, where \( sin_q x = sin_{qq} x \) and \( \pi_q = \pi_{qq} \). The basis property is quite fundamental to such a stimulating problem.
When \( p = q = 2 \), the sequence \([1.1]\) is the family of Jacobian elliptic functions \( \{ \text{sn}(2nK(k)x, k) \} \). In this case, Craven [4] proves that if the modulus \( k \) satisfies \( 0 \leq k \leq 0.99 \), then the sequence is complete in \( L^2(0, 1) \). Since the sequence is not orthogonal, we have no guarantee of its basis property.

On the other hand, when \( k = 0 \), the sequence \([1.1]\) is the family of generalized trigonometric functions \( \{ \sin_{pq}(n\pi pq x) \} \). The sequence for \( p = q \), Binding et al. [1] first studied the basis property. Recently, Edmonds et al. [10] show that if, for example, \( p'/q < 4/(\pi^2 - 8) \), then the sequence is a basis in \( L^\alpha(0, 1) \) for any \( \alpha \in (1, \infty) \).

This paper deals with the basis property of \([1.1]\) for general \( p, q \in (1, \infty) \) and \( k \in [0, 1) \). Our results involve results of [4] when \( p = q = 2 \) and [10] when \( k = 0 \).

**Theorem 1.1.** Let \( p, q \in (1, \infty) \) and \( r = \max\{p', q\} \). If

\[
\frac{1}{q} B \left( \frac{1}{r}, \frac{1}{r} \right) < \frac{8}{\pi^2 - 8}, \tag{1.2}
\]

then \( \{ f_n(x, k) \} \) forms a Riesz basis of \( L^2(0, 1) \) and a Schauder basis of \( L^\alpha(0, 1) \) for any \( \alpha \in (1, \infty) \) when \( k = 0 \) or

\[
\frac{\tanh_r^{-1} k^q}{k^q} \leq \frac{8q}{\pi^2 - 8} B \left( \frac{1}{r}, \frac{1}{r} \right)^{-1}, \tag{1.3}
\]

where \( B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt \) is the Beta function and \( \tanh_r^{-1} x \) is a generalized inverse hyperbolic function

\[
\tanh_r^{-1} x = \int_0^x \frac{dt}{1 - r^t}.
\]

**Remark 1.2.** The function \( \tanh_r^{-1} x/x \) is a monotone increasing function from \((0, 1)\) onto \((1, \infty)\).

From Theorem [1.1] we obtain an improvement of Craven’s result stated in Remark of [4, Theorem 2].

**Corollary 1.3.** If \( 1 < p' \leq q < \infty \), then \( \{ f_n(x, k) \} \) forms a Riesz basis of \( L^2(0, 1) \) and a Schauder basis of \( L^\alpha(0, 1) \) for any \( \alpha \in (1, \infty) \) when \( k = 0 \) or

\[
\frac{\tanh_q^{-1} k}{k} \leq \frac{8q}{\pi^2 - 8} B \left( \frac{1}{q}, \frac{1}{q} \right)^{-1}.
\]

In particular, the sequence of Jacobian elliptic functions \( \{ \text{sn}(2nK(k)x, k) \} \) does so when \( 0 \leq k \leq 0.99 \).
We will give another corollary of Theorem 1.1, whose conditions are verified easier than (1.2) and (1.3).

**Corollary 1.4.** Let \( p, q \in (1, \infty) \) and \( r = \max\{p', q\} \). If

\[
\frac{r}{q} < \frac{4}{\pi^2 - 8},
\]

then \( \{f_n(x, k)\} \) forms a Riesz basis of \( L^2(0, 1) \) and a Schauder basis of \( L^\alpha(0, 1) \) for any \( \alpha \in (1, \infty) \) when

\[
0 \leq k < \left[ 1 - \left\{ \frac{(\pi^2 - 8)r}{4q} \right\}^{\frac{1}{q}} \right].
\]

**Remark 1.5.** (i) If \( p' \leq q \), i.e., \( r = q \), then (1.2) and (1.4) hold. (ii) Case \( k = 0 \) in (1.5) corresponds to the result of [10, Theorem 4.4]. (iii) When \( r = q = 2 \), the value of the right-hand side of (1.5) is about 0.88, which is not so satisfactory as 0.99 of Corollary 1.3. However, we can check (1.5) much easier than (1.3).

The paper is organized as follows. In Section 2 we give a summary of general properties of bases in Banach spaces. In Section 3 we recall the generalized trigonometric functions and introduce new generalization of Jacobian elliptic functions. In Section 4 we observe properties of the generalized Jacobian elliptic function \( \text{sn}_{pq}(x, k) \) and its quarter period \( K_{pq}(k) \). To show that the sequence (1.1) is a basis in \( L^\alpha(0, 1) \) for any \( \alpha \in (0, 1) \), we depend on the strategy of Binding et al. [1] and Edmunds et al. [10]. Our main device is a linear mapping \( T \) of \( L^\alpha(0, 1) \), satisfying \( Te_n = f_n \), where \( e_n = \sin(n \pi x) \), and decomposing into a linear combination of certain isometries. In Section 5 we show that \( T \) is a bounded operator for \( p \in (1, \infty) \). Section 6 is devoted to the proof of boundedness of the inverse for the ranges (1.2) and (1.3).

## 2 Properties of Bases

In this section we will give a summary of properties of bases in Banach spaces. For details, we can refer to Gohberg and Kreın [12], Higgins [13], and Singer [21].

A sequence \( \{x_n\} \) in an infinite dimensional Banach space \( X \) is called a basis of \( X \) if for every \( x \in X \) there exists a unique sequence of scalars \( \{\alpha_n\} \)
such that $x = \sum_{i=1}^{\infty} \alpha_i x_i$ (i.e., such that $\lim_{n \to \infty} \|x - \sum_{i=1}^{n} \alpha_i x_i\| = 0$). A basis $\{x_n\}$ of a topological linear space $U$ is said to be a Schauder basis of $U$, if all coefficient functionals $f_n, n = 1, 2, \ldots$, are continuous on $U$.

**Proposition 2.1.** Every basis of a Banach space is a Schauder basis of this space.

**Proof.** See [21, Theorem 3.1, p.20]. □

**Definition 2.2.** A basis $\{x_n\}$ of a Banach space $X$ is said to be
(a) a Bessel basis, if
\[ \sum_{i=1}^{\infty} \alpha_i x_i \text{ is convergent} \Rightarrow \sum_{i=1}^{\infty} |\alpha_i|^2 < \infty, \]
i.e., there exists a constant $c > 0$ such that we have
\[ c \sqrt{\sum_{i=1}^{n} |\alpha_i|^2} \leq \sqrt{\sum_{i=1}^{n} \alpha_i x_i} \]
for all finite sequences of scalars $\alpha_1, \ldots, \alpha_n$.

(b) a Hilbert basis, if
\[ \sum_{i=1}^{\infty} |\alpha_i|^2 < \infty \Rightarrow \sum_{i=1}^{\infty} \alpha_i x_i \text{ is convergent}, \]
i.e., there exists a constant $C > 0$ such that we have
\[ \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq C \sqrt{\sum_{i=1}^{n} |\alpha_i|^2} \]
for all finite sequences of scalars $\alpha_1, \ldots, \alpha_n$.

(c) a Riesz basis, if it is both a Bessel basis and a Hilbert basis, i.e., there exist two constants $c > 0$ and $C > 0$ such that we have
\[ c \sqrt{\sum_{i=1}^{n} |\alpha_i|^2} \leq \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq C \sqrt{\sum_{i=1}^{n} |\alpha_i|^2} \]
for all finite sequences of scalars $\alpha_1, \ldots, \alpha_n$. 5
Example 2.3. In the space $X = L^p(-\pi, \pi), \ p \in (1, \infty)$, the sequence $\{x_n\}$, where

$$x_0(t) = \frac{1}{2}, \quad x_{2n-1}(t) = \sin nt, \quad x_{2n}(t) = \cos nt \quad (t \in [-\pi, \pi], \ n = 1, 2, \ldots)$$

is a bounded Bessel basis if $p \geq 2$ and a bounded Hilbert basis if $1 < p \leq 2$. In particular, it is a Riesz basis if $p = 2$.

Proof. See [21, Example 11.1, pp.342–345].

We call two sequences $\{\phi_n\}$ and $\{\psi_n\}$ in Banach space $X$ equivalent if there exists a linear homeomorphism (i.e., bounded, linear and invertible operator) $T$ on $X$ such that $\psi_n = T(\phi_n)$ for every $n$. Note that by ‘invertible’ we mean that $T^{-1}$ exists and is bounded on all of $X$.

Proposition 2.4. If a sequence $\{\psi_n\}$ in a Banach space is equivalent to a basis $\{\phi_n\}$, it too is a basis.

Proof. See [13, Lemma, p.75].

Furthermore, we have

Proposition 2.5 (Bari). Let $H$ be a Hilbert space. The following assertions are equivalent.

(a) The sequence $\{\phi_n\}$ forms a basis of $H$, equivalent to an orthonormal basis.

(b) The sequence $\{\phi_n\}$ is complete and a Riesz basis in $H$.

Proof. See the first and third assertions of [12, Theorem 2.1, pp.310–311].

3 Generalized Functions

This section is devoted to the definitions of two kinds of generalized functions.

3.1 Generalized Trigonometric Functions

Generalized Trigonometric functions were introduced in 1879 by E. Lundberg (see Lindqvist and Peetre [17, pp.113-141]). After that, these functions have been developed mainly by Elbert [11], Lindqvist [15], Drábek and Manásevich [9], and Lang and Edmunds [14].
For any constants $p, q \in (1, \infty)$, we define $\pi_{pq}$ by

$$\pi_{pq} = 2 \int_0^1 \frac{dt}{(1 - t^q)^{\frac{1}{p}}} = 2 \frac{B \left(1 - \frac{1}{p}, \frac{1}{q}\right)}{q \Gamma(1 - \frac{1}{p} + \frac{1}{q})},$$

where $B$ and $\Gamma$ are the Beta and Gamma functions, respectively. Then, for any $x \in [0, \pi_{pq}/2]$ we define $\sin_{pq} x$ by

$$x = \int_0^{\sin_{pq} x} \frac{dt}{(1 - t^q)^{\frac{1}{p}}}.$$

Clearly, $\sin_{pq} x$ is an increasing function in $x$ from $[0, \pi_{pq}/2]$ onto $[0, 1]$. We extend the domain of $\sin_{pq} x$ to $[0, \pi_{pq}]$ by $\sin_{pq} x = \sin_{pq}(\pi_{pq} - x)$, and furthermore, to the whole of $\mathbb{R}$ by $\sin_{pq}(x + \pi_{pq}) = -\sin_{pq} x$, so that $\sin_{pq} x$ has $2\pi_{pq}$-periodicity. We can see that $\pi_{22} = \pi$ and $\sin_{22} x = \sin x$. Moreover, the function $y = \sin_{pq} x$ satisfies that $y, |y'|^{p-2}y' \in C^1(\mathbb{R})$, and $y \in C^2(\mathbb{R})$ if $1 < p \leq 2$.

We agree that $\pi_p$ and $\sin_p x$ denote $\pi_{pp}$ and $\sin_{pp} x$ when $p = q$, respectively. In that case, we can also refer to [5, 6, 7, 8, 11, 15].

Using $\sin_{pq} x$, for $x \in [0, \pi_{pq}/2]$ we also define

$$\cos_{pq} x = (1 - \sin_{pq}^q x)^{\frac{1}{q}}.$$  \hfill (3.1)

Clearly, $\cos_{pq} x$ is a decreasing function in $x$ from $[0, \pi_{pq}/2]$ onto $[0, 1]$. We extend the domain of $\cos_{pq} x$ to $[-\pi_{pq}/2, \pi_{pq}/2]$ by $\cos_{pq} x = \cos_{pq}(-x)$, and furthermore, to the whole of $\mathbb{R}$ in the same way as $\sin_{pq} x$. Then, $\cos_{pq} x$ has $2\pi_{pq}$-periodicity. We can see that $\cos_{22} x = \cos x$. An analogue of $\tan x$ is obtained by defining

$$\tan_{pq} x = \frac{\sin_{pq} x}{\cos_{pq} x}$$

for those values of $x$ at which $\cos_{pq} x \neq 0$. This means that $\tan_{pq} x$ is defined for all $x \in \mathbb{R}$ except for the points $(k + 1/2)\pi_{pq}$ ($k \in \mathbb{Z}$). We denote by $\cos_p x$ and $\tan_p x$ as for the case $\sin_p x$. The functions $\sin_p x$ and $\cos_p x$ are useful for Prüfer transformation of half-linear differential equations. For this, see [6, 7, 11, 19].
These functions satisfy, for \( x \in (0, \pi pq/2) \),
\[
\cos_{pq}^q x + \sin_{pq}^q x = 1, \quad (3.2)
\]
\[
\left(\sin_{pq}^q x\right)' = \cos_{pq}^q x,
\]
\[
\left(\cos_{pq}^q x\right)' = -\sin_{pq}^{q-1} x \cos_{pq}^{1-q} x,
\]
\[
\left(\cos_{pq}^q x\right)' = -\frac{q}{p'} \sin_{pq}^{q-1} x,
\]
\[
\left(\tan_{pq}^q x\right)' = \cos_{pq}^{-1-q} x.
\]

The case \( p = q = r \) for some \( r \in (1, \infty) \) is as follows.
\[
\cos_{r}^r x + \sin_{r}^r x = 1,
\]
\[
\left(\sin_{r}^r x\right)' = \cos_{r} x,
\]
\[
\left(\cos_{r}^r x\right)' = -\sin_{r}^{r-1} x \cos_{r}^{2-r} x,
\]
\[
\left(\cos_{r}^{r-1} x\right)' = -(r-1) \sin_{r}^{r-1} x,
\]
\[
\left(\tan_{r} x\right)' = \cos_{r}^{-r} x.
\]

It is useful to collect formulae for case \( p = r' \) and \( q = r \) for some \( r \in (1, \infty) \).
\[
\cos_{r'}^r x + \sin_{r'}^r x = 1,
\]
\[
\left(\sin_{r'}^r x\right)' = \cos_{r'}^{r-1} x, \quad (3.3)
\]
\[
\left(\cos_{r'}^r x\right)' = -\sin_{r'}^{r-1} x, \quad (3.4)
\]
\[
\left(\tan_{r'} x\right)' = \cos_{r'}^{-2} x.
\]

In particular, it is important that for any \( p, q \in (1, \infty) \)
\[
\left(\left(\sin_{pq}^q x\right)'\right)^p + \sin_{pq}^q x = 1. \quad (3.5)
\]

We can find many other properties of these functions in [10, 14].

Remark 3.1. There are some different definitions of \( \cos_{pq} x \) from (3.1). For example, Drábek and Manásevich [9] define \( \cos_{pq} x \) by
\[
\cos_{pq} x = \left(\sin_{pq}^q x\right)',
\]
and so (3.5) gives
\[
\cos_{pq}^p x + \sin_{pq}^q x = 1,
\]
which is slightly different from (3.2). The fact that \( \sin_{pq} x \) satisfies (3.5) is essential, independently of the definition of \( \cos_{pq} x \).
Proposition 3.2. The number $\pi_{r,r}$ is an increasing function in $r \in (1, \infty)$ and $\lim_{r \to 1+0} \pi_{r,r} = 2$ and $\lim_{r \to +\infty} \pi_{r,r} = 4$.

Proof. Putting $1 - t^r = s$ in the definition of $\pi_{r,r}$, we have

$$\pi_{r,r} = \frac{2}{r} B \left( \frac{1}{r}, \frac{1}{r} \right).$$

(3.6)

It suffices to show that $tB(t,t)$ is decreasing on $(0,1)$. Using a formula of Euler (see Example 36 in [23, p.262]):

$$\log B(s,t) = \log \frac{s + t}{st} + \int_0^1 \frac{(1 - v^s)(1 - v^t)}{(1 - v) \log v} dv, \quad s, t \in (0, \infty),$$

we have

$$\log tB(t,t) = \log t + \log B(t,t)$$

$$= \log 2 + \int_0^1 \frac{(1 - v^t)^2}{(1 - v) \log v} dv, \quad t \in (0, \infty).$$

Clearly, the right-hand side is decreasing in $t$ (note that $\log v < 0$), so that $tB(t,t)$ is also decreasing on $(0,1)$. Furthermore, since

$$\lim_{t \to 1} tB(t,t) = B(1,1) = 1 \quad \text{and} \quad \lim_{t \to 0} \log tB(t,t) = \log 2,$$

we obtain the values of limits.

Remark 3.3. We can find another proof of Proposition 3.2 in [10, Lemma 2.4], in which they use the fact that the area of $r$-circle $|x|^r + |y|^r = 1$ is $\pi_{r,r}$ (see also [16]).

3.2 Generalized Jacobian Elliptic Functions

In the fashion of the classical Jacobian elliptic functions, we define new transcendental functions.

Let $p, q \in (1, \infty)$. For any $k \in [0, 1)$ we define $K_{pq}(k)$ by

$$K_{pq}(k) = \int_0^1 \frac{dt}{(1 - t^q)^{\frac{p}{r}}(1 - k^q t^q)^{\frac{r}{s}}}. \quad (3.7)$$
Then, for any $k \in [0, 1)$ and $x \in [0, K_{pq}(k)]$ we define $sn_{pq}(x,k)$ by

$$x = \int_0^{sn_{pq}(x,k)} \frac{dt}{(1-t^q)^{\frac{1}{2}}(1-k^q t^q)^{\frac{1}{2}}}.$$  \hfill (3.8)

Clearly, $sn_{pq}(x,k)$ is an increasing function in $x$ from $[0, K_{pq}(k)]$ onto $[0, 1]$. We extend the domain of $sn_{pq}(x,k)$ to $[0, 2K_{pq}(k)]$ by $sn_{pq}(x,k) = sn_{pq}(2K_{pq}(k) - x, k)$, and furthermore, to $\mathbb{R}$ by $sn_{pq}(x + 2K_{pq}(k), k) = -sn_{pq}(x, k)$, so that $sn_{pq}(x,k)$ has $4K_{pq}(k)$-periodicity. We can see that $K_{22}(k) = K(k)$, $sn_{22}(x,k) = sn(x,k)$, $K_{pq}(0) = \pi_{pq}/2$, and $sn_{pq}(x,0) = sn_{pq}x$. Moreover, the function $y = sn_{pq}(x,k)$ satisfies that $y, |y'|^{p-2}y' \in C^1(\mathbb{R})$, and $y \in C^2(\mathbb{R})$ if $1 < p \leq 2$.

Using $sn_{pq}(x,k)$, for $x \in [0, K_{pq}(k)]$ we also define

$$cn_{pq}(x,k) = (1 - sn_{pq}^q(x,k))^{\frac{1}{2}},$$

$$dn_{pq}(x,k) = (1 - k^q sn_{pq}^q(x,k))^{\frac{1}{2}}.$$  \hfill (3.9)

Clearly, $cn_{pq}(x,k)$ and $dn_{pq}(x,k)$ are decreasing functions in $x$ from $[0, K_{pq}(k)]$ onto $[0, 1]$. We extend the domains of $cn_{pq}(x,k)$ and $dn_{pq}(x,k)$ to $[-K_{pq}(k), K_{pq}(k)]$ in the same way of $cos_{pq} x$, and furthermore, to $\mathbb{R}$ by $cn_{pq}(x + 2K_{pq}(k), k) = -cn_{pq}(x,k)$ and $dn_{pq}(x + 2K_{pq}(k), k) = dn_{pq}(x,k)$, respectively. This implies that $cn_{pq}(x,k)$ and $dn_{pq}(x,k)$ have $4K_{pq}(k)$- and $2K_{pq}(k)$-periodicity. We can see that $cn_{pq}(x,0) = cos_{pq} x$ and $dn_{pq}(x,0) = 1$.

These functions satisfy, for $x \in (0, K_{pq}(k))$

$$cn_{pq}^q(x,k) + sn_{pq}^q(x,k) = 1,$$

$$dn_{pq}^q(x,k) + k^q sn_{pq}^q(x,k) = 1,$$

$$(sn_{pq}(x,k))' = cn_{pq}(x,k) dn_{pq}(x,k),$$

$$(cn_{pq}(x,k))' = -sn_{pq}^{q-1}(x,k) cn_{pq}^{1-\frac{q}{2}}(x,k) dn_{pq}^{\frac{q}{2}}(x,k),$$

$$(dn_{pq}(x,k))' = -k^q sn_{pq}^{q-1}(x,k) cn_{pq}(x,k) dn_{pq}^{1-\frac{q}{2}}(x,k).$$

In case $p = q = r$ for some $r \in (1, \infty)$, we write $sn_{pq}(x,k)$, $cn_{pq}(x,k)$ and $dn_{pq}(x,k)$ by $sn_r(x,k)$, $cn_r(x,k)$ and $dn_r(x,k)$, respectively. For the case,
the formulae above become as follows.

\[
\begin{align*}
\text{cn}_r (x, k) + \text{sn}_r (x, k) &= 1, \\
\text{dn}_r (x, k) + k^r \text{sn}_r (x, k) &= 1, \\
(s_n_r (x, k))' &= \text{cn}_r (x, k) \text{dn}^{-1}_r (x, k), \\
(cn_r (x, k))' &= -s_n^{-1}_r (x, k) \text{cn}^{2-r}_r (x, k) \text{dn}^{-1}_r (x, k), \\
(dn_r (x, k))' &= -k^r s_n^{-1}_r (x, k) \text{cn}^{-1}_r (x, k).
\end{align*}
\]

We also state the case \( p = r' \) and \( q = r \) for some \( r \in (1, \infty) \).

\[
\begin{align*}
\text{cn}_{r'} (x, k) + \text{sn}_{r'} (x, k) &= 1, \\
\text{dn}_{r'} (x, k) + k^r \text{sn}_{r'} (x, k) &= 1, \\
(s_{n_{r'}} (x, k))' &= \text{cn}_{r'}^{-1} (x, k) \text{dn}_{r'} (x, k), \\
(cn_{r'} (x, k))' &= -s_{n_{r'}}^{-1} (x, k) \text{dn}_{r'} (x, k), \\
(dn_{r'} (x, k))' &= -k^r s_{n_{r'}}^{-1} (x, k) \text{cn}^{-1}_{r'} (x, k). \\
\end{align*}
\]

Moreover, \( y = \text{sn}_{pq} (x, k) \) satisfies

\[
(|u'|^{p-2}u')' + \frac{(p - 1)q}{p} |u|^{q-2}u(1 + (p - 1)k^q - p k^q |u|^q)(1 - k^q |u|^q)^{p-2} = 0.
\]

As mentioned in Introduction, the author [22] has introduced another
generalized Jacobian elliptic functions, which also include both the Jacobian
elliptic functions and the generalized trigonometric functions. However, we
should note that the definitions above of \( K_{pq}(k) \) and \( \text{sn}_{pq}(x, k) \) are slightly
different from those of [22], in which the common index of \( 1 - k^q t^q \) to (3.7)
and (3.8) is not \( 1/p' \) but \( 1/p \). On account of the index, \( K_{pq}(k) \) has similar
asymptotic behavior near \( k = 1 \) as \( K(k) \), indeed, \( \lim_{k\to 1} K_{pq}(k) = \infty \) for any
\( p, q \in (1, \infty) \).

To observe the convergence properties of generalized Jacobian elliptic
functions as \( k \to 1 \), we will prepare \emph{generalized hyperbolic functions},
for which similar definitions are seen in [15].

For \( x \in [0, \infty) \), we define \( \text{sinh}_{pq} x \) by

\[
x = \int_{0}^{\text{sinh}_{pq} x} \frac{dt}{(1 + t^q)^{\frac{p}{2}}},
\]

(3.9)
and extend its domain to \( \mathbb{R} \) by \( \sinh_{pq} x = -\sinh_{pq} (-x) \). Using \( \sinh_{pq} x \), for \( x \in [0, \infty) \), we define

\[
\cosh_{pq} x = (1 + \sinh_{pq}^q x)^{\frac{1}{q}},
\]

and extend its domain to \( \mathbb{R} \) by \( \cosh_{pq} x = \cosh_{pq}(-x) \). The function \( \tanh_{pq} x \) is defined by

\[
\tanh_{pq} x = \frac{\sinh_{pq} x}{\cosh_{pq} x}.
\]

We agree that \( \sinh_p x, \cosh_p x \) and \( \tanh_p x \) denote \( \sinh_{pp} x, \cosh_{pp} x \) and \( \tanh_{pp} x \) when \( p = q \), respectively. Putting \( p = q \) and \( t = \sqrt{1 - s^p} \) in (3.9), we have

\[
x = \int_0^{\tanh_p x} \frac{dt}{1 - t^p}.
\]

Then, it is easy to prove the following properties: for any \( p, q \in (1, \infty) \) and all \( x \in \mathbb{R} \),

\[
\lim_{k \to 1} \sin_{pq}(x, k) = \tanh_q x,
\]

\[
\lim_{k \to 1} \cos_{pq}(x, k) = \lim_{k \to 1} \sec_{pq}(x, k) = \frac{1}{\cosh_q x}.
\]

4 Properties of \( \sin_{pq}(x, k) \) and \( K_{pq}(k) \)

In this section we observe some properties of generalized Jacobian elliptic function \( \sin_{pq}(x, k) \) and its quarter period \( K_{pq}(k) \).

The function \( y = \sin_{pq}(x, k) \) satisfies that \( \sin_{pq}(0, k) = 0 \), \( \sin_{pq}(K_{pq}(k), k) = 1 \), \( 0 < \sin_{pq}(x, k) < 1 \) for \( x \in (0, K_{pq}(k)) \), \( y \in C^1[0, K_{pq}(k)] \), and

\[
y' = (1 - y^q)^{\frac{1}{p}} (1 - k^q y^q)^{\frac{1}{p}} \geq 0.
\]

If \( 1 < p \leq 2 \), then \( y \in C^2[0, K_{pq}(k)] \) and

\[
y'' = -\frac{q}{p} y^{q-1} (1 - y^q)^{\frac{2}{p} - 1} (1 - k^q y^q)^{\frac{2}{p} - 1} (1 - k^q) + pk^q (1 - y^q) \leq 0.
\]

When \( p > 2 \), we see that \( y'' \in L^1(0, K_{pq}(k)) \) and

\[
\int_0^{K_{pq}(k)} |y''| \; dx = -\int_0^{K_{pq}(k)} y'' \; dx = -[y']_0^{K_{pq}(k)} = 1.
\]
To obtain the estimate of $K_{r^r}(k)$ in Lemma 4.7 below, we state Tchebycheff’s integral inequality in \cite{18,20}.

Lemma 4.1. Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a, b] \to \mathbb{R}$ be a positive, integrable function. Then

$$\int_a^b p(x)f(x)\,dx \int_a^b p(x)g(x)\,dx \leq \int_a^b p(x)\,dx \int_a^b p(x)f(x)g(x)\,dx. \quad (4.2)$$

If one of the functions $f$ or $g$ is nonincreasing and the other nondecreasing, then the inequality in (4.2) is reversed.

Concerning the following Lemmas 4.2–4.6, we can refer to \cite{2,10,14} for the corresponding results of $\sin_{\frac{pq}{2\pi}}x$ and $\pi_{\frac{pq}{2\pi}}$, which are in case $k = 0$ of $\sin_{pq}(x,k)$ and $K_{pq}(k)$. Lemma 4.7 extends an estimate of $K_{r^r}(k)$ by Qi and Huang \cite{20, Eq.(10)} to that of $K_{r^r}(k)$ for any $r \in (1, \infty)$.

4.1 Properties of the Function $\sin_{pq}(x,k)$

Since $f_n(x,k) = f_1(nx,k)$, it suffices to observe properties of $f_1(x,k) = \sin_{pq}(2K_{pq}(k)x,k)$ in order to study those of $f_n(x,k)$.

Lemma 4.2. For each $x \in [0, 1]$,

\begin{itemize}
  \item $p \mapsto \sin_{pq}(2K_{pq}(k)x,k)$ is decreasing on $(1, \infty)$ for any fixed $k \in [0, 1)$, $q \in (1, \infty)$.
  \item $q \mapsto \sin_{pq}(2K_{pq}(k)x,k)$ is decreasing on $(1, \infty)$ for any fixed $k \in [0, 1)$, $p \in (1, \infty)$.
  \item $k \mapsto \sin_{pq}(2K_{pq}(k)x,k)$ is increasing on $[0, 1)$ for any fixed $p$, $q \in (1, \infty)$.
\end{itemize}

Proof. First we will show that $\sin_{pq}(2K_{pq}(k)x,k)$ is decreasing in $p \in (1, \infty)$.

Let $1 < p < r < \infty$. Putting

$$f(t) = \frac{\sin_{rq}^{-1}(t,k)}{\sin_{pq}^{-1}(t,k)} \quad \text{for } t \in (0, 1],$$

we have

$$f'(t) = \frac{G(t)}{(\sin_{pq}^{-1}(t,k))^2(1-tq)^{\frac{1}{r}}(1-kqtq)^{\frac{1}{r}}},$$

where

$$G(t) = \sin_{pq}^{-1}(t,k) - g(t)\sin_{rq}^{-1}(t,k)$$
and  
\[ g(t) = (1 - t^q)\frac{1}{p} - \frac{1}{r} (1 - k^q t^q)\frac{1}{r'} - \frac{1}{p'} = \left( \frac{1 - k^q t^q}{1 - t^q} \right)^{\frac{1}{p} - \frac{1}{p'}}. \]

Since \( g(t) \) is increasing in \( t \in (0, 1) \) when \( p < r \), it is easy to see that \( G'(t) = -g'(t) \frac{1}{s_{rq}^{-1}(t, k)} < 0 \),

which means that \( G(t) < 0 \), i.e., \( f'(t) < 0 \) for each \( t \in (0, 1] \). Thus,

\[ \frac{K_{rq}(k)}{K_{pq}(k)} \leq \frac{\frac{1}{s_{rq}^{-1}(t, k)}}{\frac{1}{s_{pq}^{-1}(t, k)}} < 1 \text{ for } t \in (0, 1], \]

namely,

\[ \frac{1}{K_{pq}(k)} s_{pq}^{-1}(t, k) \leq \frac{1}{K_{rq}(k)} s_{rq}^{-1}(t, k) \text{ for } t \in (0, 1]. \]

Therefore we conclude that

\[ s_{pq}(2K_{pq}(k)x, k) \geq s_{rq}(2K_{rq}(k)x, k) \text{ for } x \in [0, 1], \]

so that \( s_{pq}(2K_{pq}(k)x, k) \) is decreasing in \( p > 1 \).

The assertions for \( q \) and \( k \) are proved in a similar way. It is enough to replace \( f(t) \) by

\[ \frac{\frac{1}{s_{pr}^{-1}(t, k)}}{\frac{1}{s_{pq}^{-1}(t, k)}} (1 < q < r) \quad \text{and} \quad \frac{\frac{1}{s_{pq}^{-1}(t, l)}}{\frac{1}{s_{pq}^{-1}(t, k)}} (0 \leq k < l < 1), \]

respectively, and to replace \( g(t) \) by

\[ \left( \frac{1 - t^q}{1 - t^q} \right)^{\frac{1}{p'}} \left( \frac{1 - k^q t^q}{1 - k^q t^q} \right)^{\frac{1}{p'}} \quad \text{and} \quad \left( \frac{1 - l^q t^q}{1 - l^q t^q} \right)^{\frac{1}{p'}}, \]

respectively.

\[ \square \]

**Lemma 4.3.** Let \( p, q \in (1, \infty) \) and \( k \in [0, 1) \). Then, for any \( x \in (0, K_{pq}(k)] \),

\[ \frac{1}{K_{pq}(k)} \leq \frac{s_{pq}(x, k)}{x} < 1. \]
Proof. Let \( y = \text{sn}_{pq}(x, k) \). Putting \( t = ys \) in (3.8), we have
\[
\text{sn}^{-1}_{pq}(y, k) = \int_0^1 \frac{ys \, ds}{(1 - y^q s^q)^{\frac{1}{p}} (1 - k^q y^q s^q)^{\frac{1}{p}'}}.
\]
so that
\[
x = \text{sn}_{pq}(x, k) \int_0^1 \frac{ds}{(1 - s^q \text{sn}^q_{pq}(x, k))^\frac{1}{p} (1 - k^q s^q \text{sn}^q_{pq}(x, k))^\frac{1}{p'}}.
\]
Since \( 0 < \text{sn}_{pq}(x, k) \leq 1 \), we obtain
\[
\text{sn}_{pq}(x, k) < x \leq K_{pq}(k) \text{sn}_{pq}(x, k),
\]
which implies the assertion. \( \square \)

4.2 Properties of the Number \( K_{pq}(k) \)

Lemma 4.4. \( p \mapsto K_{pq}(k) \) is decreasing on \((1, \infty)\) for any fixed \( k \in [0, 1) \), \( q \in (1, \infty) \).

\( q \mapsto K_{pq}(k) \) is decreasing on \((1, \infty)\) for any fixed \( k \in [0, 1) \), \( p \in (1, \infty) \).

\( k \mapsto K_{pq}(k) \) is increasing on \([0, 1)\) and
\[
K_{pq}(0) = \frac{\pi_{pq}}{2}, \quad \lim_{k \to 1} K_{pq}(k) = \infty
\]
for any fixed \( p, q \in (1, \infty) \).

Proof. The assertion on \( p \) immediately follows from

\[
K_{pq}(k) = \int_0^1 \left( \frac{1 - k^q t^q}{1 - t^q} \right)^{\frac{1}{p}} \frac{1}{1 - k^q t^q} \, dt.
\]
The remaining parts also follow from the form of \( K_{pq}(k) \). \( \square \)

Lemma 4.5. Let \( p, q \in (1, \infty) \) and \( k \in [0, 1) \). Then
\[
K_{pq}(k) = \frac{p'}{q'} K_{q'p'}(k^{\frac{1}{q'}}).
\]
Proof. Putting $1 - t^q = x^{p'}$ in (3.7), we have
\[ K_{pq}(k) = \frac{p'}{q(1 - k^q)^{\frac{1}{p'}}} \int_0^1 \frac{dx}{(1 - x^{p'})^{\frac{1}{p'}}(1 + \kappa x^{p'})^{\frac{1}{p'}}}, \]
where $\kappa = k^q/(1 - k^q)$. Furthermore, setting $x = t(1 + \kappa(1 - t^{p'}))^{-\frac{1}{p'}}$, we obtain
\[ K_{pq}(k) = \frac{p'}{q(1 - k^q)^{\frac{1}{p'}}} \int_0^1 \frac{dt}{(1 - t^{p'})^{\frac{1}{p'}}(1 + \kappa(1 - t^{p'}))^{\frac{1}{p'}}}. \]

The integration of the right-hand side is equal to $K_{q'q'}(k^{\frac{q}{p'}})$.

\[ \square \]

**Lemma 4.6.** Let $p, q \in (1, \infty)$ and $k \in [0, 1)$. Then
\[ K_{pq}(k) \leq \frac{r}{q} K_{r'r'}(k^{\frac{q}{p}}), \]
where $r = \max\{p', q\}$.

**Proof.** Case $p' \leq q$ follows from only Lemma 4.2. Case $p' > q$ is also proved similarly after using Lemma 4.5. \[ \square \]

**Lemma 4.7.** Let $r \in (1, \infty)$ and $k \in (0, 1)$. Then
\[ \frac{\pi r \sin^{-1} k}{2} \leq K_{r'r'}(k) \leq \frac{\pi r \tanh^{-1} k}{2} \leq \frac{1}{2(1 - k^r)^{\frac{1}{r}}} \tag{4.3} \]

**Proof.** Putting $t = \sin_{r'} \theta$ in the definition of $K_{r'r'}(k)$ and using (3.5), we have
\[ K_{r'r'}(k) = \int_0^{\pi r'/2} \frac{d\theta}{(1 - k^r \sin_{r'}^2 \theta)^{\frac{1}{r}}}. \]

We follow the proof of Qi and Huang [20], which deals with the case $r = 2$. Let $p(x) = 1, f(x) = (1 - k^r \sin_{r'} \theta)^{-\frac{1}{r}}, g(x) = \cos_{r'}^{-1} \theta$ or $\sin_{r'}^{-1} \theta$, $[a, b] = [0, \pi r'/2]$ in Tchebycheff’s integral inequality (4.2) of Lemma 4.1, then we obtain
\[ K_{r'r'}(k) \int_0^{\pi r'/2} \cos_{r'}^{-1} \theta d\theta \geq \int_0^{\pi r'/2} 1 d\theta \int_0^{\pi r'/2} \frac{\cos_{r'}^{-1} \theta}{(1 - k^r \sin_{r'}^2 \theta)^{\frac{1}{r}}} d\theta \]

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or

\[ K_{r'}(k) \int_0^{\frac{\pi}{2}} \sin_{r'}^{-1} \theta \, d\theta \leq \int_0^{\frac{\pi}{2}} \frac{1}{1 - k^r \sin_{r'} \theta} \sin_{r'}^{-1} \theta \, d\theta. \]

By (3.3) and (3.4), easy calculation gives

\[ \int_0^{\frac{\pi}{2}} \cos_{r'}^{-1} \theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin_{r'}^{-1} \theta \, d\theta = 1. \]

Moreover, putting \( \sin_{r'} \theta = t \), we have

\[ \int_0^{\frac{\pi}{2}} \frac{\cos_{r'}^{-1} \theta}{(1 - k^r \sin_{r'} \theta)^{\frac{1}{2}}} \, d\theta = \int_0^1 \frac{dt}{(1 - k^r t^r)^{\frac{1}{2}}} = \frac{1}{k} \int_0^k \frac{ds}{1 - s^r} = \frac{1}{k} \sin_{r'}^{-1} k. \]

Similarly, putting \( \cos_{r'} \theta = t \) and \( t^r = \frac{1 - k^r}{k^r - 1 - s^r} \), we obtain

\[ \int_0^{\frac{\pi}{2}} \frac{\sin_{r'}^{-1} \theta}{(1 - k^r \sin_{r'} \theta)^{\frac{1}{2}}} \, d\theta = \int_0^1 \frac{dt}{1 - k^r + k^r t^r} = \frac{1}{k} \int_0^k \frac{ds}{1 - s^r} = \frac{1}{k} \tanh_{r'}^{-1} k. \]

Thus, we accomplished the first and second inequalities of (4.3).

Finally, from the equality above, we obtain the third inequality of (4.3) as

\[ \frac{1}{k} \tanh_{r'}^{-1} k = \int_0^1 \frac{dt}{(1 - k^r + k^r t^r)^{\frac{1}{2}}} \leq \frac{1}{(1 - k^r)^{\frac{1}{2}}}. \]

The graphs of terms of (4.3) for \( r = 2 \) can be shown in Figure 1.

5 The Operator \( T \)

Let \( \alpha \in (1, \infty) \) be an arbitrary number. In this section, we will make the functions

\[ f_n(x) = f_n(x, k) = \text{sn}_{pq}(2nK_{pq}(k)x, k) \in L^\alpha(0, 1), \quad n = 1, 2, \ldots, \]

correspond to the sine series

\[ e_n(x) = \sin(n\pi x) \in L^\alpha(0, 1), \quad n = 1, 2, \ldots, \]

which form a basis of \( L^\alpha(0, 1) \).
Figure 1: The graphs of terms of (4.3) for $r = 2$. The black line and the gray lines indicate $K_{r'}(k)$ and the others, respectively.

**Theorem 5.1.** If there is a linear homeomorphism $T$ of $L^\alpha(0,1)$ satisfying $Te_n = f_n$, $n = 1, 2, \ldots$, then $\{f_n(x,k)\}$ forms a Riesz basis of $L^2(0,1)$ and a Schauder basis of $L^\alpha(0,1)$ for any $\alpha \in (1, \infty)$.

**Proof.** By Example 2.3, any function in $L^\alpha(-1,1)$ has a unique sine-cosine series representation. For any $f \in L^\alpha(0,1)$, we can thus represent its odd extension to $L^\alpha(-1,1)$ uniquely in a sine series, so the $e_n$ form a basis of $L^\alpha(0,1)$. Since $\{e_n\}$ and $\{f_n\}$ are equivalent, according to Proposition 2.4 the same is true for the $f_n$. It follows from Proposition 2.1 that they form a Schauder basis of $L^\alpha(0,1)$. The argument for a Riesz basis when $\alpha = 2$ is similar and follows from Proposition 2.5.

In the remainder of this section we define $T$ as a linear combination of certain isometries of $L^\alpha(0,1)$. Then we show that $T$ is a bounded operator satisfying $Te_n = f_n$, $n = 1, 2, \ldots$, for all $p, q \in (1, \infty)$.

The functions $f_n$ have Fourier sine series expansions

$$f_n(x) = \sum_{l=1}^{\infty} \hat{f}_n(l) e_l(x),$$
where
\[ \hat{f}_n(l) = 2 \int_0^1 f_n(x)e_l(x) \, dx. \]

An argument involving symmetry with respect to the middlepoint \( x = 1/2 \) easily shows that \( \hat{f}_1(l) = 0 \) whenever \( l \) is even. On account of this property, we can show \( \hat{f}_n(l) \) by using \( \hat{f}_1(l) \) as follows.

\[ \hat{f}_n(l) = 2 \int_0^1 f_1(nx)e_l(x) \, dx = 2 \sum_{m: \text{odd}} \hat{f}_1(m) \int_0^1 e_{mn}(x)e_l(x) \, dx = \begin{cases} \hat{f}_1(m) & \text{if } mn = l \text{ for some odd } m, \\ 0 & \text{otherwise.} \end{cases} \quad (5.1) \]

In what follows we will often denote \( \hat{f}_1(m) \) by \( \tau_m \). We first find a bound on \( |\tau_m| \) which will be crucial in the definition of \( T \) below. Since \( \tau_m = 0 \) if \( m \) is even, we may assume that \( m \) is odd. Integration by parts ensures that

\[ \tau_m = 4 \int_0^{1/2} f_1(x)e_m(x) \, dx = -\frac{4}{m^2\pi^2} \int_0^{1/2} f''_1(x)e_m(x) \, dx, \]

where the integrals exist because \( f''_1 \in L^1(0,1) \). In fact (4.1) shows that

\[ |\tau_m| \leq \frac{4}{m^2\pi^2} \int_0^{1/2} |f''_1(x)e_m(x)| \, dx < \frac{4}{m^2\pi^2} \int_0^{1/2} |f''_1(x)| \, dx = \frac{8K_{pq}(k)}{m^2\pi^2}. \quad (5.2) \]

In order to construct the linear operator \( T \), we next define isometries \( M_m \) of the Banach space \( L^\alpha(0,1) \) by \( M_m g(x) := g^*(mx) \), \( m = 1, 2, \ldots \), where \( g^* \) is its successive antiperiodic extension of \( g \) over \( \mathbb{R}_+ \) by \( g^* = g \) on \([0,1]\), and

\[ g^*(x) = -g^*(2n - x) \quad \text{if } n < x \leq n + 1, \quad n = 1, 2, \ldots. \]

Notice that \( M_m e_n = e_{mn} \).

**Lemma 5.2.** The maps \( M_m \) are isometric linear transformations of \( L^\alpha(0,1) \) for all \( m = 1, 2, \ldots \) and \( \alpha \in (1, \infty) \).
Proof.

\[ \|M_m g\|_\alpha = \int_0^1 |M_m g(x)|^\alpha \, dx = \int_0^1 |g^*(m x)|^\alpha \, dx \]
\[ = \frac{1}{m} \int_0^m |g^*(u)|^\alpha \, du = \frac{1}{m} \sum_{l=1}^{m} \int_{l-1}^{l} |g^*(u)|^\alpha \, du \]
\[ = \frac{1}{m} \sum_{l=1}^{m} \int_0^1 |g(u)|^\alpha \, du = \int_0^1 |g(u)|^\alpha \, du = \|g\|_\alpha. \]

\[ \square \]

We now define \( T : L^\alpha(0,1) \to L^\alpha(0,1) \) by

\[ T g(x) = \sum_{m=1}^{\infty} \tau_m M_m g(x). \] \( (5.3) \)

Lemma 5.2, the triangle inequality and \( (5.2) \) ensure that \( T \) is a bounded everywhere-defined operator with \( \|T\|_{L^\alpha \to L^\alpha} \leq K_{pq}(k) \).

We conclude this section by showing that \( T e_n = f_n \). Indeed, by virtue of \( (5.1) \),

\[ T e_n = \sum_{m=1}^{\infty} \tau_m e_{mn} = \sum_{m : \text{odd}} \hat{f}_1(m) e_{mn} = \sum_{l=1}^{\infty} \hat{f}_n(l) e_l = f_n. \]

6 Bounded Invertibility of \( T \)

In this section we complete the proof of Theorem 1.1 by showing that \( T \) has a bounded inverse for all \( p, q \) and \( k \) satisfying \( (1.2) \) and \( (1.3) \).

Observe that \( (5.3) \) and the triangle inequality give

\[ \|T g - \tau_1 M_1 g\|_\alpha \leq \sum_{m=3}^{\infty} |\tau_m| \|M_m g\|_\alpha, \]

so that Lemma 5.2 and the fact \( M_1 = I \) (the identity operator on \( L^\alpha(0,1) \)) give

\[ \|T - \tau_1 I\|_{L^\alpha \to L^\alpha} \leq \sum_{m=3}^{\infty} |\tau_m|. \]
Thus, by C. Neumann’s theorem [24, Theorem 2 in p. 69], to show that $T$ has
a bounded inverse, it is sufficient to prove

$$\sum_{m=3}^{\infty} |\tau_m| < |\tau_1|. \quad (6.1)$$

We estimate the left-hand side from above and the right-hand side from below. Inequality (5.2) shows that

$$\sum_{m=3}^{\infty} |\tau_m| \leq \frac{8K_{pq}(k)}{\pi^2} \left( \frac{\pi^2}{8} - 1 \right). \quad (6.2)$$

On the other hand, it follows from Lemma 4.3 that

$$|\tau_1| = 4 \int_0^{\frac{\pi}{2}} \text{sn}_{pq}(2K_{pq}(k)x, k) \sin \pi x \, dx \geq 8 \int_0^{\frac{\pi}{2}} x \sin \pi x \, dx = \frac{8}{\pi^2}. \quad (6.3)$$

Then, we can prove

**Theorem 6.1.** Let $p, q \in (1, \infty)$ and $k \in [0, 1)$. The sequence $\{f_n(x, k)\}$ forms a Riesz basis of $L^2(0, 1)$ and a Schauder basis of $L^\alpha(0, 1)$ for any $\alpha \in (1, \infty)$ if

$$K_{pq}(k) < \frac{8}{\pi^2 - 8}. \quad (6.4)$$

**Remark 6.2.** In case $k = 0$, inequality (6.4) corresponds with $\pi_{pq} < 16/(\pi^2 - 8)$, which is given in [10, Corollary 4.3].

**Proof.** Combining (6.2), (6.4) and (6.3), we obtain (6.1). Thus, $T$ has a bounded inverse and is a linear homeomorphism in $L^\alpha(0, 1)$. Therefore, from Theorem 5.1 we have completed the proof. \qed

Now we are in a position to show Theorem 1.1.

**Proof of Theorem 1.1.** Assume (1.2) and let $k$ be a number satisfying (1.3). From Lemmas 4.6 and 4.7 we have

$$K_{pq}(k) \leq \frac{r \pi_{r' r} \tanh^{-1} k^{\frac{q}{r'}}}{2q}.$$
Using (3.6) to the right-hand side, we obtain

\[ K_{pq}(k) \leq \frac{1}{q} B \left( \frac{1}{r'}, \frac{1}{r} \right) \frac{\tanh^{-1} \frac{k^q}{k' q}}{k^q}. \]

Thus, (1.2) and (1.3) give

\[ K_{pq}(k) < \frac{8}{\pi^2 - 8}, \]

which implies (6.4) of Theorem 6.1. We have thus proved the theorem. \(\square\)

Proof of Corollary 1.3. Let \(1 < p' \leq q < \infty\). Then \(r = q\), and it suffices to show that (1.2) is satisfied. Since we have the inequality \(tB(t, t) \leq 2\) in the proof of Proposition 3.2, we obtain

\[ \frac{1}{q} B \left( \frac{1}{r'}, \frac{1}{r} \right) \leq 2 < \frac{8}{\pi^2 - 8}, \]

so that (1.2) holds.

In particular, we consider the case \(p = q = 2\). Then, \(r = q = 2\), and by Theorem 1.1 the sequence \(\{\text{sn}(2nK(k)x, k)\}\) is a basis in \(L^\alpha(0, 1)\) for any \(\alpha \in (1, \infty)\) when

\[ \frac{\tanh^{-1} k}{k} \leq \frac{16}{(\pi^2 - 8)\pi}, \]

which holds true if \(0 < k \leq 0.9909\ldots\). \(\square\)

Proof of Corollary 1.4. Suppose that \(q\) and \(r\) satisfy (1.4). Since \(tB(t, t) \leq 2\) as the proof of Corollary 1.3, we obtain

\[ \frac{1}{q} B \left( \frac{1}{r'}, \frac{1}{r} \right) \leq \frac{2r}{q} < \frac{8}{\pi^2 - 8}, \]

which implies (1.2). Furthermore, if we assume (1.5), then (4.3) in Lemma 4.7 shows

\[ \frac{\tanh^{-1} \frac{k^q}{k' q}}{k^q} \leq \frac{1}{(1 - k')^2} \leq \frac{4q}{(\pi^2 - 8)r} \leq \frac{8q}{\pi^2 - 8} B \left( \frac{1}{r'}, \frac{1}{r} \right)^{-1}, \]

so that (1.3) holds, and Theorem 1.1 gives Corollary 1.4. \(\square\)
References

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