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Constant vorticity water flows with full Coriolis term

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Abstract
We consider here three-dimensional water flows governed by the geophysical water wave equations exhibiting full Coriolis term. More precisely, under mild assumptions we determine all possible flow solutions to the governing equations that exhibit constant vorticity vector. That is, we show that the vertical component of the velocity vanishes, the horizontal components are constant and the free surface is necessarily flat. Our investigation features three-dimensionality, nonlinearity, Coriolis effects and vorticity, the last aspect being one of relevance in relation to the issue of turbulence.

Keywords: full Coriolis term, vorticity, three-dimensional water flows
Mathematics Subject Classification numbers: 35Q35, 76B15, 76U05

1. Introduction

The study of geophysical fluid dynamics (GFD) has relied for decades on data driven approaches, ad hoc modeling and numerical simulations. Indeed, the systematic analytical approaches are clearly outnumbers by experimental or observational studies. The scarcity of rigorous mathematical investigations pertaining to GFD is probably due to the complications which are inherent in the (general) understanding of fluid flows. Among these peculiarities exhibited by fluid flows, we mention their nonlinear character, the presence of swirling motions (captured by the vorticity vector) and the three-dimensional nature of many water flows. On sufficiently large time scales, Coriolis effects, i.e. those stemming from the Earth’s
rotation, need also to be accounted for in a thorough treatment, be it of analytical, numerical or experimental type.

One of the particularities mentioned before about water flows is their tendency to whirl, the measure by which they fail to be irrotational being embodied in the vorticity vector-defined as the curl of the velocity field. The (non-zero) vorticity serves as a tool for describing interactions of waves with non-uniform currents; we remark here the wave-current interactions at the Columbia river entrance, where the wave height doubles in just a few hours [28]. More precisely, as documented by numerous studies [28, 38, 41], the vorticity is, for two-dimensional flows, instrumental for the description of the vertical structure of the current profile. The importance of the vorticity in the realistic modeling of ocean flows is highlighted in the very recent studies by Constantin and Johnson [14], Martin [33] and Wheeler [44].

The successful incorporation of the previous mentioned features had its inception recently and relatively recently through the analytical studies by Constantin [6–9] who, in the context of equatorial ocean flows, has constructed exact solutions describing equatorially trapped waves propagating to the East. The flow pattern of these solutions resembles at a fixed latitude the one of the Gerstner wave solution [2, 7–9, 13, 17, 19, 34–36]. The possibility of including (nearly) uniform currents was also explored, see [20]. While the previous studies refer to the f- or β-plane approximations, they are directly relevant to the equatorial undercurrent (EUC) and to the Antarctic circumpolar current (ACC) and exhibit the observed vertical structure, very much ignored in the reduced-gravity shallow water equations on the β-plane. Other specialized exact solutions pertain to (EUC) and (ACC) and display a preferred direction for the velocity field, see [10, 11, 22–24, 30]. Despite of building quite a variety, these exact solutions represent only a starting point in the study of geophysical flows. These solutions lie at the foundations of more extensive studies by means of asymptotic and perturbative methods. Indeed, an asymptotic approach by Constantin and Johnson [12] (set up by letting the ratio depth/wavelength converge to zero) retrieves a three-dimensional fully nonlinear dynamical model that captures simultaneously fundamental oceanic phenomena, which are closely interrelated (such as upwelling/downwelling, zonal depth-dependent currents with flow reversal, and poleward divergence along the equator). We note here that a unified approach towards exact solutions regarding geophysical flows was presented by Johnson [27].

Lying at the heart of the previous analyses is the vorticity, a feature that tremendously determines the dimensionality of the flow. While it would seem that constant vorticity is more likely to allow for three-dimensional solutions to the Euler equations and their boundary conditions, the contrary is, in fact, true. Indeed, recent results by Constantin [5], Constantin and Kartashova [3], Martin [29], Basu and Martin [1] and Wahlen [43] show that gravity, capillary and capillary-gravity-wave trains at the surface of water in a flow with constant non-zero vorticity with a flat bed can only occur if the flow is two-dimensional and if the vorticity vector has only one non vanishing component that points in the horizontal direction orthogonal to the direction of wave propagation. For similar results concerning solitary waves we refer the reader to the works by Craig [16] and Stuhlmeier [40]. Removing the assumption of a steady free surface wave utilized in [1, 3, 5, 29, 43] and that of a time independent flow made in [43] it was shown very recently by Martin [31] that all potential solutions to the time dependent water wave problem exhibiting constant non vanishing vorticity vector present no variation in the direction orthogonal to the direction of motion. In agreement with the conclusion of two-dimensionality of water flows with constant non-vanishing vorticity is the study by Xia and Francois [45] showing that in thick fluid layers, large-scale coherent structures can shear off the vertical eddies and reinforce the planarity of the flow. Reinforcing our conclusion on the non-existence of three-dimensional water flows presenting a constant non-vanishing vorticity vector are the recent studies by Constantin [8, 9] and Constantin and Johnson [12]. Indeed,
[8, 9, 12] show that geophysical water flows that display a non-constant vorticity vector are inherently three-dimensional.

Rather different is the setting of irrotational flows since this is a scenario that permits the utilization of a velocity potential. The latter is a tool used (in the absence of geophysical effects) to prove the existence of doubly periodic capillary-gravity waves by Reeder and Shinbrot [39], the existence of traveling three-dimensional capillary-gravity water waves by Craig and Nicholls [15], the existence of three-dimensional oblique traveling gravity-capillary water waves, see Groves and Haragus [18] or of three-dimensional doubly periodic gravity water waves by Iooss and Plotnikov [25, 26].

One needs to say here that [15, 18, 25, 26, 39] do not include geophysical effects. The insertion of the latter into the governing equations leads to the somewhat surprising conclusion that, in some specific scenarios, meaningful solutions to the water wave problem displaying a constant vorticity vector cease to exist. Indeed, it was proved in [32] that, in the \( \beta \)-plane setting, the only flow exhibiting a constant vorticity vector—including thus the irrotational case—is the stationary flow with vanishing velocity field and flat surface.

In this paper we leave aside the \( \beta \)-plane setting and take a broader view by allowing for the full Coriolis terms into the governing equations. We still assume a constant vorticity vector. This perspective somehow delivers a better outcome—when compared with the \( \beta \)-plane setting—from the point of view of relevant solutions to the water wave problem. Indeed, we will show in this paper that allowing the full Coriolis terms into the nonlinear Euler equations leads to solutions that, in spite of displaying a flat free surface, possess non-vanishing horizontal velocity field. The importance of a nonlinear framework in the modeling of oceanic flows is highlighted in the recent paper [21].

The main result of the paper is presented in theorem 2.2 following the introduction of the general setting we work in.

2. The Euler equations with full Coriolis term

We introduce here the governing equations for geophysical ocean waves exhibiting full Coriolis term, see [8, 37, 42].

We choose a rotating framework with the origin at a point on the Earth’s surface, with the \( x \)-axis pointing horizontally due east, the \( y \)-axis horizontally due north and the \( z \)-axis upward. The governing equations for inviscid, homogeneous geophysical ocean flows are the Euler’s equations

\[
\begin{align*}
    u_t + uu_x + vu_y + wu_z + 2\omega w \cos \phi - 2\omega v \sin \phi &= -P_x, \\
    v_t + uv_x + vv_y + wv_z + 2\omega u \sin \phi &= -P_y, \\
    w_t + uw_x + vw_y + ww_z - 2\omega u \cos \phi &= -P_z - g,
\end{align*}
\]

(2.1)

and the equation of mass conservation

\[u_x + v_y + w_z = 0,\]

(2.2)

that are satisfied within the water flow domain bounded below by a rigid bed \( z = -d(d > 0) \) and above by the free surface \( z = \eta(x, y, t) \). Here \( t \) represents the time variable, \( \phi \) is the latitude, \( \omega = 7.29 \cdot 10^{-5} \text{ rad s}^{-1} \) is the (constant) rotational speed of the Earth round the polar axis toward the east and \( g \) is the gravitational constant. The velocity field \((u, v, w)\), the pressure \(P\) and \(\eta\) are assumed to be smooth enough.
The specification of the water wave problem is completed by the boundary conditions pertaining to the free surface \( z = \eta(x, y, t) \) and to the bed \( z = -d \). These are the kinematic boundary conditions

\[
  w = \eta_t + u\eta_x + v\eta_y \quad \text{on} \quad z = \eta(x, y, t)
\]

(2.3)

and

\[
  w = 0 \quad \text{on} \quad z = -d,
\]

(2.4)

together with the dynamic boundary condition

\[
  P = P_{\text{atm}} \quad \text{on} \quad z = \eta(x, y, t),
\]

(2.5)

which decouples the motion of the motion of the water flow from the motion of the air above it, see Constantin [4]. By a solution of the water wave problem (2.1)–(2.5) we understand a tuple \((u, v, w, P, \eta)\) whose components satisfy the mentioned equations. A bounded solution of (2.1)–(2.5) is a solution \((u, v, w, P, \eta)\) whose components are bounded in the \( L^\infty \) norm.

The local spin in the water flow is captured by the vorticity vector field

\[
  \Omega = (w_y - v_z, u_z - w_x, v_x - u_y) =: (\Omega_1, \Omega_2, \Omega_3).
\]

(2.6)

Remark 2.1. According to the discussion in Constantin [6] the magnitude of the equatorial undercurrent’s relative vorticity (about \( 25 \cdot 10^{-3} \text{ m s}^{-1} \)) is much larger than that of the planetary vorticity \( 2\omega \approx 1.46 \cdot 10^{-4} \text{ s}^{-1} \). Therefore, throughout the paper we will make the assumptions

\[
  \Omega_2 + 2\omega \cos \phi \neq 0 \quad \text{and} \quad \Omega_3 + 2\omega \sin \phi \neq 0.
\]

(2.7)

We state now the main result of the paper.

Theorem 2.2. We assume that \( \phi \neq 0 \) and that the vorticity vector is constant throughout the flow and also satisfies (2.7). Then a tuple \((u, v, w, P, \eta)\) (with bounded components) is a solution to the water wave problem (2.1)–(2.5) if and only if \( w = 0 \), there are constants \( c_1, c_2, \eta_0 \in \mathbb{R} \) such that

\[
  u(x, y, z, t) = u(t) = c_1 \cos (2\omega (\sin \phi) t) + c_2 \sin (2\omega (\sin \phi) t),
\]

(2.8)

\[
  v(x, y, z, t) = v(t) = -c_1 \sin (2\omega (\sin \phi) t) + c_2 \cos (2\omega (\sin \phi) t),
\]

(2.9)

\[
  \eta(x, y, t) = \eta_0,
\]

(2.10)

for all \( x, y, t \) and

\[
  P(x, y, z, t) = P_{\text{atm}} + (-g + 2\omega u(t) \cos \phi)(z - \eta_0) \quad \text{for all} \quad x, y, z, t,
\]

(2.11)

for which \(-d \leq z \leq \eta_0\).

Proof. The vorticity components \( \Omega_1, \Omega_2, \Omega_3 \) satisfy the equations

\[
  \Omega_1 u_x + (\Omega_2 + 2\omega \cos \phi) u_y + (\Omega_3 + 2\omega \sin \phi) u_z = 0,
\]

\[
  \Omega_1 v_x + (\Omega_2 + 2\omega \cos \phi) v_y + (\Omega_3 + 2\omega \sin \phi) v_z = 0,
\]

\[
  \Omega_1 w_x + (\Omega_2 + 2\omega \cos \phi) w_y + (\Omega_3 + 2\omega \sin \phi) w_z = 0.
\]

(2.12)
Since we assume (2.7) we have from the last equation in (2.12) that \(w\) is constant in a direction that is not parallel to the horizontal plane \(z = -d\). By (2.4) we obtain that \(w\) vanishes throughout the flow. We then immediately have that

\[
u_z(x, y, z, t) = \Omega_2 \quad \text{for all} \quad x, y, z, t, \tag{2.13}\]
\[
u_x(x, y, z, t) = -\Omega_1 \quad \text{for all} \quad x, y, z, t. \tag{2.14}\]

From the last two equations we obtain that there are functions \((x, y, t) \to \bar{\pi}(x, y, t)\) and \((x, y, t) \to \bar{\nu}(x, y, t)\) such that

\[
u(x, y, z, t) = \bar{\pi}(x, y, t) + \Omega_2 z, \tag{2.15}\]
\[
u(x, y, z, t) = \bar{\nu}(x, y, t) - \Omega_1 z, \tag{2.15}\]

for all \(x, y, z, t\) for which \(-d \leq z \leq \eta(x, y, t)\). Moreover, the functions \(\bar{\pi}\) and \(\bar{\nu}\) satisfy the equation

\[
\bar{\pi}_x(x, y, t) + \bar{\nu}_t(x, y, t) = 0, \quad \text{for all} \quad x, y, t. \tag{2.16}\]

The previous relation entails the existence of a function \((x, y, t) \to \psi(x, y, t)\) such that

\[
\bar{\pi}(x, y, t) = \psi_y(x, y, t) \quad \text{and} \quad \bar{\nu}(x, y, t) = -\psi_x(x, y, t), \tag{2.16}\]

for all \(x, y, t\).

Consequently, the first two equations in (2.12) can be written as

\[
\Omega_1 \psi_{xy} + (\Omega_2 + 2\omega \cos \phi)\psi_{yy} + (\Omega_3 + 2\omega \sin \phi)\Omega_2 = 0, \tag{2.17}\]
\[
\Omega_1 \psi_{xx} + (\Omega_2 + 2\omega \cos \phi)\psi_{xy} + (\Omega_3 + 2\omega \sin \phi)\Omega_1 = 0, \tag{2.17}\]

while from the third component in (2.6) we obtain

\[
\Delta \psi = -\Omega_3. \tag{2.18}\]

From (2.17) and (2.18) we obtain

\[
\psi_{yy} = \frac{2\omega \Omega_1^2 \sin \phi - \Omega_2 \Omega_3 \Omega_2}{\Omega_1^2 + \Omega_2^2} =: A \tag{2.19}\]
\[
\psi_{xy} = -\frac{\Omega_1 (\Omega_2 \Omega_3 + 2\omega \Omega_1 \sin \phi)}{\Omega_1^2 + \Omega_2^2} =: B \tag{2.19}\]
\[
\psi_{xx} = \frac{2\omega \Omega_2 (\Omega_2 \sin \phi - \Omega_3 \cos \phi) - \Omega_1 \Omega_3 - 2\omega \Omega_1^2 \sin \phi}{\Omega_1^2 + \Omega_2^2} =: C \tag{2.19}\]

where

\[
\Omega_2 := \Omega_2 + 2\omega \cos \phi, \quad \Omega_3 := \Omega_3 + 2\omega \sin \phi. \]

Therefore, by means of (2.19), there exist functions \(t \to a(t), t \to b(t), t \to k(t)\) such that

\[
\psi(x, y, t) = A \frac{y^2}{2} + Bxy + C \frac{x^2}{2} + a(t)y + b(t)x + k(t) \quad \text{for all} \quad x, y, t. \tag{2.20}\]
By means of the previous formula and employing also (2.15) we find that
\[ u(x, y, z, t) = Ay + Bx + a(t) \]  
(2.21)
\[ v(x, y, t) = -Cx + By - b(t) \]  
(2.22)
for all \( x, y, z, t \). Invoking now the boundedness of \( u \) and \( v \) and utilizing (2.21) and (2.22) we infer that
\[ A = B = C = 0. \]
We claim now that \( \Omega_1 = 0 \). To prove this claim, we assume for the sake of contradiction that \( \Omega_1 \neq 0 \). Then, since \( B = 0 \), we conclude from the formula for \( B \) in (2.19) that
\[ \Omega_2 \Omega_3 + 2\omega \Omega_2 \sin \phi = 0. \]
Multiplying the previous equation with \( \Omega_2 \) and adding the result to the equation \( A = 0 \) we obtain the relation
\[ 2\omega (\Omega_1^2 + \Omega_2^2) \sin \phi = 0, \]
which is impossible, since we assumed \( \Omega_1 \neq 0 \) and we work under the assumption that \( \phi \neq 0 \). Therefore, the assumption \( \Omega_1 \neq 0 \) can not be true, hence the claim that \( \Omega_1 = 0 \) is proved.

Recalling that \( C = 0 \) we obtain from the third formula in (2.19) that \( \Omega_3 \cos \phi = 0 \), from which we derive that \( \Omega_3 = 0 \). Equations (2.13) and (2.14) now imply that
\[ u_x(x, y, z, t) = v_x(x, y, z, t) = 0 \]  
(2.23)
which obviously implies that
\[ u_x(x, y, z, t) = v_x(x, y, z, t) = 0 \]  
(2.24)
Availing now of \( \Omega_3 = 0 \) and of the incompressibility condition (2.2) (both used in conjunction with (2.24)) we find that
\[ u_x(x, y, z, t) = v_x(x, y, z, t) = 0 \]  
(2.25)
We notice now that (2.23)–(2.25) imply that \( u \) and \( v \) are functions of \( t \) only. Using now all the inferences pertaining to the velocity field we observe that the Euler’s equations become
\[ u'(t) - 2\omega v(t) \sin \phi = -P_x(x, y, z, t) \]
\[ v'(t) + 2\omega u(t) \sin \phi = -P_y(x, y, z, t) \]
\[ 2\omega u(t) \cos \phi = P_z(x, y, z, t) + g \]  
(2.26)
for all \( x, y, z, t \), that is the functions \( P_x, P_y, P_z \) depend only on the time variable \( t \).

By differentiating with respect to \( x \) and \( y \) in the dynamic boundary condition (2.5) we obtain
\begin{align}
P_x(x, y, \eta(x, y, t), t) + P_z(x, y, \eta(x, y, t), t)\eta_t(x, y, t) &= 0, \quad (2.27) \\
P_y(x, y, \eta(x, y, t), t) + P_z(x, y, \eta(x, y, t), t)\eta_y(x, y, t) &= 0, \quad (2.28)
\end{align}

for all \(x, y, t\). The previous conclusion on the independence of \(P_x, P_y, P_z\) of \(x, y, z\) and relations (2.27)–(2.28) show that \(\eta_t\) and \(\eta_y\) do not depend on \(x\) and \(y\) at all times \(t\). This means that the outward pointing unit normal vector \(\frac{1}{\sqrt{1+\eta_t^2+\eta_y^2}}(-\eta_t, -\eta_y, 1)\) at a point \((x, y, \eta(x, y, t))\) on the surface \(z = \eta(x, y, t)\) does not depend on \((x, y)\). Obviously, the latter is possible if and only if the surface is a plane. Appealing now to the boundedness of \(\eta\) we conclude that the free surface must be a horizontal plane. That is, there is a function \(t \to \eta_0(t)\) such that
\[
\eta(x, y, t) = \eta_0(t) \quad \text{for all } x, y, t.
\]

Since \(\eta_t(x, y, t) = \eta_y(x, y, t) = 0\) we see from the kinematic boundary condition (2.3) that \(\eta_{0t}(t) = 0\) for all \(t\). Another conclusion from (2.27)–(2.28) is that
\[
P_x(x, y, \eta(x, y, t), t) = P_y(x, y, \eta(x, y, t), t) = 0 \quad \text{for all } x, y, t, \quad (2.29)
\]
from which (using also the independence of \(P_x, P_y, P_z\) of \(x, y, z\)– established in (2.26)) we derive that
\[
P_x(x, y, z, t) = P_y(x, y, z, t) = 0 \quad \text{for all } x, y, t. \quad (2.30)
\]

Hence the system (2.26) specializes to
\begin{align}
u'(t) - 2\omega v(t) \sin \phi &= 0 \\
v'(t) + 2\omega u(t) \sin \phi &= 0 \\
2\omega u(t) \cos \phi &= P_z(z, t) + g \quad \text{for all } z, t. \quad (2.31)
\end{align}

From the first two equations above we deduce that
\[
u''(t) + (2\omega^2 \sin^2 \phi)u(t) = 0, \quad (2.32)
\]
which delivers
\begin{align}
u(t) &= c_1 \cos(2\omega(\sin \phi)t) + c_2 \sin(2\omega(\sin \phi)t) \\
v(t) &= -c_1 \sin(2\omega(\sin \phi)t) + c_2 \cos(2\omega(\sin \phi)t) \quad \text{for all } t. \quad (2.33)
\end{align}

Clearly, from the third equation in (2.31) and from the dynamic boundary condition (2.5) we have that
\[
P(x, y, z, t) = P_{\text{atm}} + (-g + 2\omega u(t) \cos \phi)(z - \eta_0) \quad \text{for all } x, y, z, t. \quad (2.34)
\]

To prove the sufficiency, we note that the velocity field with \(u, v\) given by (2.33), \(w = 0\) and \(P\) as in (2.34) satisfy the governing equations (2.1)–(2.2) and their boundary conditions (2.3)–(2.5) on the bed \(z = -d\) and on the (flat) surface \(z = \eta_0\).

**Remark 2.3.** One can satisfy to some extent the curiosity related to the case when the longitude \(\phi\) vanishes. The case \(\phi = 0\) is, in fact, the equatorial \(f\)-plane approximation. More precisely, let \(a, b, \eta_0, \Omega_1, \Omega_2\) be constants and let \(P_{\text{atm}}\) denote the constant atmospheric pres-
sure. Then, the tuple \((u, v, w, P, \eta)\) representing the velocity field, the pressure and the free surface, given by

\[
\begin{align*}
  u(x, y, z, t) &= \Omega_2 z + a, \\
  v(x, y, z, t) &= -\Omega_1 z + b, \\
  w(x, y, z, t) &= 0, \\
  P(x, y, z, t) &= (2\omega a - g)(z - \eta_0) + \omega\Omega_2(z^2 - \eta_0^2) + P_{\text{atm}}, \\
  \eta(x, y, t) &= \eta_0,
\end{align*}
\]

satisfies the Euler equations in the equatorial \(f\)-plane approximation

\[
\begin{align*}
  u_t + uu_x + vv_y + ww_z + 2\omega w &= -P_x, \\
  v_t + uv_x + vv_y + wv_z &= -P_y, \\
  w_t + uw_x + vv_y + wv_z - 2\omega u &= -P_z - g,
\end{align*}
\]

as well as the equation of mass conservation

\[
u_x + v_y + w_z = 0
\]

in the domain bounded below by the bed \(z = -d\) and above by the flat surface \(z = \eta_0\). Clearly, the dynamic boundary condition (2.5) and the kinematic boundary conditions (2.3) and (2.4) are also satisfied. Moreover, the vorticity of the flow (2.35) is given by the (constant) vector \((\Omega_1, \Omega_2, 0)\).

However, it is not clear whether solutions (2.35) are the only ones satisfying the governing equations of the equatorial \(f\)-plane approximation.

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**References**

[1] Basu B and Martin C I 2018 Resonant interactions of rotational water waves in the equatorial \(f\)-plane approximation *J. Math. Phys.* 59 103101

[2] Constantin A 2001 On the deep water wave motion *J. Phys. A: Math. Gen.* 34 1405–17

[3] Constantin A and Kartashova E 2009 Effect of non-zero constant vorticity on the nonlinear resonances of capillary water waves *Europhys. Lett.* 86 29001

[4] Constantin A 2011 *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis* (CBMS-NSF Regional Conf. Series in Applied Mathematics vol 81) (Philadelphia, PA: SIAM)

[5] Constantin A 2011 Two-dimensionality of gravity water flows of constant non-zero vorticity beneath a surface wave train *Eur. J. Mech. B/Fluids* 30 12–6

[6] Constantin A 2012 On the modelling of equatorial waves *Geophys. Res. Lett.* 39 L05602

[7] Constantin A 2012 An exact solution for equatorially trapped waves *J. Geophys. Res.: Oceans* 117 C05029

[8] Constantin A 2013 Some three-dimensional nonlinear equatorial flows *J. Phys. Oceanogr.* 43 165–75

[9] Constantin A 2014 Some nonlinear, equatorially trapped, nonhydrostatic internal geophysical waves *J. Phys. Oceanogr.* 44 781–9
[10] Constantin A and Johnson R S 2016 An exact, steady, purely azimuthal equatorial flow with a free surface J. Phys. Oceanogr. 46 1935–45
[11] Constantin A and Johnson R S 2016 An exact, steady, purely azimuthal flow as a model for the Antarctic circumpolar current J. Phys. Oceanogr. 46 3585–94
[12] Constantin A and Johnson R S 2017 A nonlinear, three-dimensional model for ocean flows, motivated by some observations of the Pacific equatorial undercurrent and thermocline Phys. Fluids 29 056604
[13] Constantin A and Monismith S G 2017 Gerstner waves in the presence of mean currents and rotation J. Fluid Mech. 820 511–28
[14] Constantin A and Johnson R S 2018 Steady large-scale ocean flows in spherical coordinates Oceanography 31 42–50
[15] Craig W and Nicholls D P 2000 Travelling two and three dimensional capillary gravity water waves J. Fluid Mech. 405 337–49
[16] Craig W 2002 Non-existence of solitary water waves in three dimensions. Recent developments in the mathematical theory of water waves (Oberwolfach, 2001) R. Soc. Lond. Phil. Trans. Ser. A: Math. Phys. Eng. Sci. 360 2127–35
[17] Gerstner F 1809 Theorie der Wellen samt einer daraus abgeleiteten theorie der Deichprofile Ann. Phys. 2 412–45
[18] Groves M D and Haragus M 2003 A bifurcation theory for three-dimensional oblique travelling gravity-capillary water waves J. Nonlinear Sci. 13 397–447
[19] Henry D 2008 On Gerstner’s water wave J. Nonlinear Math. Phys. 15 87–95
[20] Henry D 2013 An exact solution for equatorial geophysical water waves with an underlying current Eur. J. Mech. B: Fluids 38 18–21
[21] Henry D 2018 Nonlinear features of equatorial ocean flows Oceanography 31 22–7
[22] Henry D and Martin C I 2018 Exact, purely azimuthal stratified equatorial flows in cylindrical coordinates Dyn. PDE 15 337–49
[23] Henry D and Martin C I 2019 Free-surface, purely azimuthal equatorial flows in spherical coordinates with stratification J. Differ. Equ. 266 6788–808
[24] Hsu H-C and Martin C I 2017 On the existence of solutions and the pressure function related to the Antarctic circumpolar current Nonlinear Anal. 155 285–93
[25] Iooss G and Plotnikov P I 2009 Small divisor problem in the theory of three-dimensional water gravity waves Mem. Am. Math. Soc. 200
[26] Iooss G and Plotnikov P I 2011 Asymmetrical three-dimensional travelling gravity waves Arch. Ration. Mech. Anal. 200 789–880
[27] Johnson R S 2018 Applications of the ideas and techniques of classical fluid mechanics to some problems in physical oceanography Phil. Trans. R. Soc. A 376 20170092
[28] Jonsson I G 1990 Wave-current interactions The Sea, in Ocean Engineering Science ed B Le Méhauté, vol 9(A) (New York: Wiley) pp 65–120
[29] Martin C I 2017 Resonant interactions of capillary-gravity water waves J. Math. Fluid Mech. 19 807–17
[30] Martin C I 2017 On the existence of free-surface azimuthal equatorial flows Appl. Anal. 96 1207–14
[31] Martin C I 2018 Non-existence of time-dependent three-dimensional gravity water flows with constant non-zero vorticity Phys. Fluids 30 107102
[32] Martin C I 2019 On constant vorticity flows in the β-plane approximation J. Fluid Mech. 865 762–74
[33] Martin C I 2018 On the vorticity of mesoscale ocean currents Oceanography 31 28–35
[34] Maticic A-V 2012 An exact solution for geophysical equatorial edge waves over a sloping beach J. Phys. A: Math. Theor. 45 365501
[35] Maticic A-V 2013 Exact geophysical waves in stratified fluids Appl. Anal. 92 2254–61
[36] Maticic A-V 2014 On the particle motion in geophysical deep water waves traveling over uniform currents Q. Appl. Math. 72 455–69
[37] Pedlosky J 1979 Geophysical Fluid Dynamics (Berlin: Springer)
[38] Peregrine D H 1976 Interactions of water waves and currents Adv. Appl. Mech. 16 9–117
[39] Reeder J and Shinbrot M 1981 Three-dimensional, nonlinear wave interaction in water of constant depth Nonlinear Anal. 5 303–23
[40] Stuhlmeier R 2012 On constant vorticity flows beneath two-dimensional surface solitary waves J. Nonlinear Math. Phys. 19 1240004
[41] Thomas G P and Klopman G 1997 Wave-current interactions in the nearshore region *Gravity Waves in Water of Finite Depth* ed J N Hunt (Southampton: Computational Mechanics Publications) pp 255–319

[42] Vallis G K 2006 *Atmospheric and Oceanic Fluid Dynamics* (Cambridge: Cambridge University Press)

[43] Wahlén E 2014 Non-existence of three-dimensional travelling water waves with constant non-zero vorticity *J. Fluid Mech.* 746 R2

[44] Wheeler M H 2018 Simplified models for equatorial waves with vertical structure *Oceanography* 31 36–41

[45] Xia H and Francois N 2017 Two-dimensional turbulence in three-dimensional flows *Phys. Fluids* 29 111107