SELF-ENERGY $O(\alpha^2)$ CORRECTION TO THE POSITRONIUM DECAY RATE

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Abstract

Self-energy corrections of order $O(\alpha^2)$ to the parapositronium and orthopositronium decay rates are calculated. Numerical values of the corresponding coefficients are $B_p = -3.74$, $B_o = 2.02$. 

Investigation of the parapositronium and orthopositronium decay rates is one of the important tasks in the QED bound state problem. Especially interesting experimental situation came into existence about orthopositronium decay width \[1, 2, 3, 4\]. While the experiments performed by the University of Michigan group \[5, 6\] lead to strong disagreement between theoretical and experimental values of orthopositronium decay rate (6-9 standard deviations), in the experiments of Tokyo University group this discrepancy keep within the experimental accuracy \[7\]. The theoretical expression for the orthopositronium decay rate, known at present, can be written in the form:

\[
\Gamma^{\text{th}}(o - Ps) = \Gamma_0 \left[ 1 - A \frac{\alpha}{\pi} - \frac{\alpha^2}{3} \ln \frac{1}{\alpha} + B_o \left( \frac{\alpha}{\pi} \right)^2 - \frac{3\alpha^3}{2\pi} \ln^2 \frac{1}{\alpha} + \ldots \right],
\]

(1)

\[\Gamma_0 = \frac{2(\pi^2 - 9)m\alpha^6}{9\pi}, \quad A = -10.286606(10).\]

Numerical value of (1) without \(\alpha^2\) correction is equal to 7.0382 \(\mu s^{-1}\). Experimental results for the orthopositronium lifetimes are:

\[
\Gamma^{\text{exp}}(o - Ps, \text{ gas measurement}) = 7.0514(14) \mu s^{-1} \quad [5],
\]

(2)

\[
\Gamma^{\text{exp}}(o - Ps, \text{ vacuum measurement}) = 7.0482(16) \mu s^{-1} \quad [3],
\]

(3)

\[
\Gamma^{\text{exp}}(o - Ps, \text{ SiO}_2 \text{ measurement}) = 7.0398(29) \mu s^{-1} \quad [7],
\]

The coefficient \(B_o\) should be equal to 41 \[8\] so that the expressions (1) and (3) coincide. Such value of \(B_o\) can be derived on the basis of standard QED calculations \[2, 9\]. During some years the different QED effects of order \(O(\alpha^2)\) in the \(\Gamma^{\text{th}}\) were studied \[10, 11, 12, 13, 14, 15\], but exact value of coefficient \(B_o\) is not known up to now. In the case of parapositronium the corrections \(\alpha^2\) were calculated recently in \[16, 17\]. The theoretical expression for the decay rate of singlet positronium is \[17\]:

\[
\Gamma^{\text{th}}(p - Ps) = \frac{m\alpha^5}{2} \left[ 1 - \left( 5 - \frac{\pi^2}{4} \right) \frac{\alpha}{\pi} + 2\alpha^2 \ln \frac{1}{\alpha} + B_p \left( \frac{\alpha}{\pi} \right)^2 - \frac{3\alpha^3}{2\pi} \ln^2 \frac{1}{\alpha} + \ldots \right],
\]

(4)

where \(B_p = 1.75(30)\). The corresponding experimental quantity \[18\]

\[
\Gamma^{\text{exp}}(p - Ps) = 7990.9 \pm 1.7 \mu s^{-1}
\]

(5)

agrees well with the value from (4):

\[
\Gamma^{\text{th}}(p - Ps) = 7989.50 \mu s^{-1}
\]

(6)

Two-loop contributions to the o-Ps lifetime may be divided into several classes: vertex, vacuum polarization, self-energy and annihilation corrections. Two-loop corrections due to vacuum polarization were calculated in \[11, 12\]. Some annihilation type contributions to \(\Gamma^{\text{th}}(o - Ps)\) were studied in \[14, 15\]. The second order electron self-energy corrections \([13, 20, 21, 22, 23]\) represent the special set of \(\alpha^2\) corrections.
in the decay width of positronium. Such corrections can be studied independently by means of renormalized expression for the mass operator. In this work we calculate $\alpha^2$ corrections to the positronium decay rate, connected with electron self-energy insertions to inner electron lines. We don’t consider here self-energy vacuum polarization corrections, which were calculated in [11, 12]. Corresponding Feynman diagrams are shown on Fig. 1-2.

The decay width of o-Ps into three photons can be written in the form [3, 24]:

$$\Gamma(o-Ps \to 3\gamma) = \frac{1}{3!} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_3}{(2\pi)^3} (2\pi)^4 \delta(P - k_1 - k_2 - k_3) \times$$

$$\times \frac{1}{3} \sum_{\text{spin}} \sum_{\epsilon_1, \epsilon_2, \epsilon_3} |T(o-Ps \to 3\gamma)|^2,$$

$$T(o-Ps \to 3\gamma) = \frac{1}{\sqrt{2\omega_12\omega_22\omega_34m}} M(o-Ps \to 3\gamma),$$

and amplitude $M(o-Ps \to 3\gamma)$ has the following integral representation:

$$M(o-Ps \to 3\gamma) = \int \frac{d^3p}{(2\pi)^3} Tr \left[ \tilde{M}(\epsilon, \vec{p}) \hat{\Pi} \right] \psi(\vec{p}),$$

where $P=(M=2m,0)$ is the four-momentum of the orthopositronium, $k_1, k_2, k_3$ are photon momenta, $\psi(\vec{p})$ is the positronium Coulomb wave function, $\tilde{M}(\epsilon, \vec{p})$ is the amplitude of the annihilation $e^+ + e^- \to 3\gamma$, $\hat{\Pi}$ is the projection operator on the $^3S_1$ electron-positron state:

$$\hat{\Pi} = \frac{\not{P} + M}{2\sqrt{2}M} \hat{\epsilon},$$

$\epsilon^\mu$ is the polarization vector of orthopositronium. In the case of parapositronium we must change $\hat{\epsilon} \to \gamma_5$.

As it is well known the mass operator has no infrared divergences in the Fried-Yennie gauge (FY) [25]. So, we can perform the usual mass-shell subtraction without introduction of the photon mass. The bare mass operator has the following structure:

$$\Sigma(p) = \delta m + \left(1 - Z_2^{-1}\right)(\hat{p} - m) + \Sigma^{(R)}(p).$$
Then the self-energy correction to the electron line is determined by the substitution:

\[
\frac{1}{\hat{p} - m} \rightarrow \frac{1}{\hat{p} - m} \Sigma^{(R)}(p) \frac{1}{\hat{p} - m}.
\] (12)

The renormalized self-energy operator of the second order in the FY gauge can be presented as follows [26]:

\[
\Sigma_{1\gamma}(p) = \alpha \frac{4}{\pi} \left(\frac{\hat{p} - m}{m^2}\right)^2 \int_0^1 dx \frac{x}{x + \rho(1 - x)} = (13)
\]

\[
(\hat{p} - m)^2 \left(-\frac{3\alpha \hat{p}}{4\pi m^2}\right) \frac{1}{1 - \rho} \left[1 + \frac{\rho}{1 - \rho} \ln \rho\right], \quad \rho = \frac{m^2 - p^2}{m^2}.
\]

We have used this expression to obtain loop-after-loop contributions, presented by \(P_1\) and \(O_1\) graphs. Feynman diagrams \(P_2\), \(O_3\) contain one internal and one external loops. To find contributions of these diagrams to the mass operator we take expression (13) for inner loop and again use FY gauge for external photon. Introducing Feynman parameterization, momentum cutoff and subtracting ultraviolet divergences in accordance with (11), we have obtain the following renormalized expression for mass operator with two non-crossed photons:

\[
\Sigma^{(R)}_{2\gamma, a}(p) = \left(\frac{\alpha}{4\pi}\right)^2 \left\{\hat{p} \left[-9\ln(1 - \rho) + \frac{3\pi^2}{2} + \frac{27}{2} - \frac{9}{2(1 - \rho)} - \ln \rho \left(\frac{27}{2} + \frac{9}{2(1 - \rho)^2}\right)\right] + m (18 \ln \rho - 9)\right\}. \] (14)

Explicit expression for the overlapping two-loop self-energy diagram in the FY gauge was derived in [27]. Corresponding renormalized part of the mass operator has the form of integral representation over five Feynman parameters (functions \(\sigma_p\), \(\sigma_m\)):

\[
\Sigma^{(R)}_{2\gamma, b}(p) = \left(\frac{\alpha}{4\pi}\right)^2 \frac{(\hat{p} - m)^2}{m^2} [\hat{p} \sigma_p(p) + m \sigma_m(p)]. \] (15)

We used this relation to calculate contributions of diagrams \(P_3\), \(O_4\). The results of our calculation of \(\alpha^2\) self-energy corrections to the p-Ps and o-Ps decay rates are shown in Tables 1,2.

| Feynman diagram | \(B_p\) |
|-----------------|---------|
| \(P_1\)        | \(\frac{9}{8} (1 - 2 \ln 2)^2 \approx 0.17\) |
| \(P_2\)        | \(-\frac{9}{16} \left(\frac{\pi^2}{8} + 1 - \ln 2\right) \approx -0.87\) |
| \(P_3\)        | -3.04   |

Total contributions of the diagrams \(P_1 - P_3\), \(O_1 - O_4\) to the coefficients \(B_p\), \(B_o\) are equal:

\[
B_p^{SE} = -3.74, \quad B_o^{SE} = 2.02. \] (16)
Figure 2: Feynman diagrams representing two-loop self-energy corrections to orthopositronium decay rate.
The contribution of the term $\sim B^{SE}_o$ to the o-Ps lifetime increases the quantity 7.0382 $\mu s^{-1}$ by the value $8 \cdot 10^{-5}$ $\mu s^{-1}$. Our results for finite part of self-energy contribution to the p-Ps decay rate do not coincide with the results of \[10\]. We think that the reason of such difference lies in the choice of gauge for exchanged photons and in the used renormalization scheme. The calculation of the modulus squared of the amplitude $|M|^2$ was done by means of the system of analytical calculations "Form" [28]. Many self-energy corrections for Ps decay width, presented in the Tables were obtained by the numerical multidimensional integration by means of system "Mathematica" [29]. Integral representations for the corrections to the orthopositronium coefficient $B_o$ are given in the Supplement.

**Table 2. Self-energy $\alpha^2$ corrections to o-Ps decay rate.**

| Feynman diagram | $B_o$   |
|-----------------|---------|
| $O_1$           | 0.50    |
| $O_2$           | 0.20    |
| $O_3$           | 5.05    |
| $O_4$           | -3.73   |

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**A. Contributions of Feynman diagrams $O_1 - O_4$ to the coefficient B.**

\[
B_{O_1} = \frac{27}{8(\pi^2 - 9)} \int_0^1 d\rho_1 \int_{2-\rho_1}^2 d\rho_2 \frac{f(\rho_1)f(\rho_2)}{\rho_1\rho_2(4 - \rho_1 - \rho_2)} \times (19)
\]

\[
\times [8 - 20\rho_1 + \frac{1}{2}\rho_1(14\rho_1 + 15\rho_2) - \frac{1}{4}\rho_1^2(2\rho_1 + 5\rho_2) - \frac{1}{4}\rho_1^2\rho_2(\rho_1 + 2\rho_2)],
\]

\[
f(\rho_i) = \frac{1}{1 - \rho_i} \left(1 + \frac{\rho_i}{1 - \rho_i} \ln \rho_i \right).
\]

\[
B_{O_2} = -\frac{27}{2(\pi^2 - 9)} \int_0^1 d\rho_1 \int_{2-\rho_1}^2 d\rho_2 \frac{(1 - \rho_1 + \rho_1 \ln \rho_1)^2}{(4 - \rho_1 - \rho_2)\rho_1\rho_2^2(1 - \rho_1)^3} \times (18)
\]

\[
\times [16 - 24\rho_1 - 10\rho_2 + 10\rho_1(\rho_1 + \rho_2) - \rho_1^3 - \frac{7}{4}\rho_1^2\rho_2 + \frac{1}{4}\rho_1\rho_2^2 + \frac{1}{4}\rho_2^3 - \frac{1}{2}\rho_1\rho_2(\frac{1}{4}\rho_1^2 + \rho_1\rho_2 + \frac{1}{4}\rho_2^2)],
\]

\[
B_{O_3} = \frac{18}{(\pi^2 - 9)} \int_0^1 d\rho_1 \int_{2-\rho_1}^2 d\rho_2 \frac{1}{(4 - \rho_1 - \rho_2)\rho_1^3\rho_2^2} \times (19)
\]

\[
\times [s_\rho(\rho_1)(8 - 14\rho_1 - 10\rho_2 + \frac{17}{2}\rho_1^2) + \frac{53}{4}\rho_1\rho_2 + 4\rho_2^2 - \frac{9}{4}\rho_1^3 - \frac{39}{8}\rho_1^2\rho_2 - \frac{31}{8}\rho_1\rho_2^2 - \frac{1}{2}\rho_2^3 + \frac{1}{4}\rho_4^4 +
\]

\]
\[ + \frac{3}{8}\rho_1^3\rho_2 + \frac{7}{16}\rho_1^2\rho_2^2 + \frac{3}{8}\rho_1\rho_2^3 + \frac{1}{32}\rho_1^4\rho_2 + \frac{1}{8}\rho_1^3\rho_2^2 + \frac{1}{32}\rho_1^2\rho_2^2 \]
\[ + s_m(\rho_1)(8 - 10\rho_1 - 10\rho_2 + \frac{7}{2}\rho_1^2 + \frac{33}{4}\rho_1\rho_2 + 4\rho_2^2 - \frac{11}{4}\rho_1^3 - \frac{11}{8}\rho_1^2\rho_2 - \frac{15}{8}\rho_1\rho_2^2 - \frac{1}{2}\rho_2^3 - \frac{1}{16}\rho_1^3\rho_2 + \frac{1}{8}\rho_1\rho_2^3) \]
\[ + \frac{3}{4}\text{Li}_2(1 - \rho) - \frac{\pi^2}{8} - \frac{9}{8} + \frac{3}{8(1 - \rho)} + \ln\rho \left( \frac{9}{8} + \frac{3}{8(1 - \rho)^2} \right) \]
\[ s_m = \frac{3}{4} - \frac{3}{2}\ln\rho. \]

\[ B_{O_4} = -\frac{3}{4(\pi^2 - 9)} \int_0^2 d\rho_1 \int_{\rho_1}^2 d\rho_2 \frac{1}{(4 - \rho_1 - \rho_2)\rho_1\rho_2^3} \times (20) \]
\[ [\sigma_p(\rho_1)(8 - 11\rho_1 - \frac{15}{2}\rho_2 + \frac{9}{2}\rho_1^2 + 6\rho_1\rho_2 + 2\rho_2^2 - \frac{7}{8}\rho_1^3 - \frac{7}{8}\rho_1^2\rho_2 - \frac{7}{8}\rho_1\rho_2^2 - \frac{7}{8}\rho_2^3 - \frac{1}{16}\rho_1^3\rho_2^2 - \frac{1}{16}\rho_1\rho_2^3) + \sigma_m(\rho_1)(\rho_1 - \frac{5}{2}\rho_2 - \frac{1}{2}\rho_1^2 + \rho_1\rho_2 + 2\rho_2^2 - \frac{1}{2}\rho_1\rho_2^2 - \frac{3}{8}\rho_2^3)]. \]

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