The hamburger theorem

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Abstract

We generalize the ham sandwich theorem to $d+1$ measures in $\mathbb{R}^d$ as follows. Let $\mu_1, \mu_2, \ldots, \mu_{d+1}$ be absolutely continuous finite Borel measures on $\mathbb{R}^d$. Let $\omega_i = \mu_i(\mathbb{R}^d)$ for $i \in [d+1]$, $\omega = \min\{\omega_i; i \in [d+1]\}$ and assume that $\sum_{j=1}^{d+1} \omega_j = 1$. Assume that $\omega_i \leq 1/d$ for every $i \in [d+1]$. Then there exists a hyperplane $h$ such that each open halfspace $H$ defined by $h$ satisfies $\mu_i(H) \leq (\sum_{j=1}^{d+1} \mu_j(H))/d$ for every $i \in [d+1]$ and $\sum_{j=1}^{d+1} \mu_j(H) \geq \min(1/2, 1 - d\omega) \geq 1/(d+1)$. As a consequence we obtain that every $(d+1)$-colored set of $n d$ points in $\mathbb{R}^d$ such that no color is used for more than $n$ points can be partitioned into $n$ disjoint rainbow $(d - 1)$-dimensional simplices.

Keywords: Borsuk–Ulam theorem; ham sandwich theorem; hamburger theorem; absolutely continuous Borel measure; colored point set.

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1 Introduction

It is well-known that if $n$ red points and $n$ blue points are given in the plane in general position, then there exists a noncrossing perfect matching on these points where each edge is a straight-line segment and connects a red point with a blue point. Akiyama and Alon [1] generalized this result to higher dimensions as follows.

For a positive integer $m$, we write $\mathbb{R}^m$ for the $m$-dimensional Euclidean space and $[m]$ for the set $\{1, 2, \ldots, m\}$.

Theorem 1 (Akiyama and Alon [1]). Let $d \geq 2$ and $n \geq 2$ be integers, and for each $i \in [d]$, let $X_i$ be a set of $n$ points in $\mathbb{R}^d$ such that all $X_i$ are pairwise disjoint and no $d + 1$ points of $X_1 \cup X_2 \cup \ldots \cup X_d$ are contained in a hyperplane. Then there exist $n$ pairwise disjoint $(d - 1)$-dimensional simplices, each of which contains precisely one vertex from each $X_i$, $i \in [d]$.

The planar version of Theorem 1 follows, for example, from the simple fact that a shortest geometric red-blue perfect matching is noncrossing [1]. This elementary metric argument does not generalize to higher dimensions, however. Akiyama and Alon [1] proved Theorem 1 using the ham sandwich theorem; see Subsection 1.1.

Recently, Kano, Suzuki and Uno [11] extended the planar version of Theorem 1 to three colors as follows.

Theorem 2 (Kano, Suzuki and Uno [11]). Let $R$, $G$ and $B$ be disjoint finite sets of red, green and blue points in the plane, respectively. Assume that no three points of $R \cup G \cup B$ are collinear. If $|R \cup G \cup B| = 2n$, $|R| \leq n$, $|G| \leq n$ and $|B| \leq n$, then there exists a noncrossing geometric perfect matching on $R \cup G \cup B$ where every edge connects two points of distinct colors; see Figure 1.

Theorem 2 was proved by induction using a result on partitions of 3-colored point sets on a line. Moreover, Theorem 2 is easily extended to four or more colors by merging the smallest color classes together.
Theorem 3 (Kano, Suzuki and Uno [11]). Let \( r \geq 3 \) and \( n \geq 1 \) be integers. Let \( X_1, \ldots, X_r \) be \( r \) disjoint point sets in the plane. Assume that no three points of \( X_1 \cup X_2 \cup \ldots \cup X_r \) are collinear, \( \sum_{i=1}^{r} |X_i| = 2n \), and \( |X_i| \leq n \) for every \( i \in [r] \). Then there exists a noncrossing geometric perfect matching on \( X_1 \cup X_2 \cup \ldots \cup X_r \) where every edge connects two points from different sets \( X_i \) and \( X_j \).

Theorems 2 and 3 can be alternatively proved by the same metric argument as the in case of two colors. First, the condition that no color class contains more than half of the points implies that there is at least one matching of the points such that every edge connects points of different colors. The shortest geometric matching with this property is then noncrossing.

Kano and Suzuki [10] made the following conjecture generalizing Theorem 1 and Theorem 3.

Conjecture 4 (Kano and Suzuki [10]). Let \( r \geq d \geq 3 \) and \( n \geq 1 \) be integers. Let \( X_1, \ldots, X_r \) be \( r \) disjoint point sets in \( \mathbb{R}^d \). Assume that no \( d+1 \) points of \( X_1 \cup \ldots \cup X_r \) lie in a hyperplane, \( \sum_{i=1}^{r} |X_i| = dn \), and \( |X_i| \leq n \) for every \( i \in [r] \). Then there exist \( n \) pairwise disjoint \((d-1)\)-dimensional simplices, each of them having \( d \) vertices in \( d \) distinct sets \( X_i \).

Conjecture 4 holds when \( r = d \) by Theorem 1 or \( d = 3 \) by Theorem 3.

In this paper we prove Conjecture 4 for every \( d \geq 2 \) and \( r = d + 1 \).

Theorem 5. Let \( d \geq 2 \) and \( n \geq 1 \) be integers. Let \( X_1, \ldots, X_{d+1} \) be \( d + 1 \) disjoint point sets in \( \mathbb{R}^d \). Assume that no \( d+1 \) points of \( X_1 \cup \ldots \cup X_{d+1} \) lie in a hyperplane, \( \sum_{i=1}^{d+1} |X_i| = dn \), and \( |X_i| \leq n \) for every \( i \in [d+1] \). Then there exist \( n \) pairwise disjoint \((d-1)\)-dimensional simplices, each of them having \( d \) vertices in \( d \) distinct sets \( X_i \).

The proof of Theorem 5 provides yet another different proof of Theorem 3.

Many related results and problems on colored point sets can be found in a survey by Kaneko and Kano [9].

1.1 Simultaneous partitions of measures

By \( S^n \) we denote the \( n \)-dimensional unit sphere embedded in \( \mathbb{R}^{n+1} \), that is, \( S^n = \{ x \in \mathbb{R}^{n+1}; \|x\| = 1 \} \).

The Borsuk–Ulam theorem plays an important role throughout this paper.

Theorem 6 (The Borsuk–Ulam theorem [12, Theorem 2.1.1]). Let \( f : S^n \to \mathbb{R}^n \) be a continuous mapping. If \( f(-u) = -f(u) \) for all \( u \in S^n \), then there exists a point \( v \in S^n \) such that \( f(v) = 0 = (0, \ldots, 0) \).
Informally speaking, the ham sandwich theorem states that a sandwich made of bread, ham and cheese can be cut by a single plane, bisecting the mass of each of the three ingredients exactly in half. According to Beyer and Zardecki [2], the ham sandwich theorem was conjectured by Steinhaus and appeared as Problem 123 in The Scottish Book [13]. Banach gave an elementary proof of the theorem using the Borsuk–Ulam theorem for $S^2$. A more direct proof can be obtained from the Borsuk–Ulam theorem for $S^3$ [12]. Stone and Tukey [14] generalized the ham sandwich theorem to $d$-dimensional sandwiches made of $d$ ingredients.

**Theorem 7** (The ham sandwich theorem [14], [12, Theorem 3.1.1]). Let $\mu_1, \mu_2, \ldots, \mu_d$ be $d$ absolutely continuous finite Borel measures on $\mathbb{R}^d$. Then there exists a hyperplane $h$ such that each open halfspace $H$ defined by $h$ satisfies $\mu_i(H) = \mu_i(\mathbb{R}^d)/2$ for every $i \in [d]$.

Stone and Tukey [14] proved more general versions of the ham sandwich theorem, including a version for Carathéodory outer measures and more general cutting surfaces. Cox and McKelvey [4] and Hill [7] generalized the ham sandwich theorem to general finite Borel measures, which include measures with finite support. For these more general measures, the condition $\mu_i(H) = \mu_i(\mathbb{R}^d)/2$ must be replaced by the inequality $\mu_i(H) \leq \mu_i(\mathbb{R}^d)/2$. Breuer [3] gave sufficient conditions for the existence of more general splitting ratios. In particular, he showed that for absolutely continuous measures whose supports can be separated by hyperplanes, there is a hyperplane splitting the measures in any prescribed ratio. See Matoušek’s book [12] for more generalizations of the ham sandwich theorem and other related partitioning results.

To prove Theorem 5, we follow the approach by Akiyama and Alon [1]. To this end, we need to generalize the ham sandwich theorem to $d + 1$ measures in $\mathbb{R}^d$. Clearly, it is not always possible to bisect all $d + 1$ measures by a single hyperplane, for example, if each measure is concentrated in a small ball around one vertex of a regular simplex.

Let $r \geq d$ and let $\mu_1, \mu_2, \ldots, \mu_r$ be finite Borel measures on $\mathbb{R}^d$. We say that $\mu_1, \mu_2, \ldots, \mu_r$ are balanced in a subset $X \subseteq \mathbb{R}^d$ if for every $i \in [r]$, we have

$$\mu_i(X) \leq \frac{1}{d} \sum_{j=1}^{r} \mu_j(X).$$

**Theorem 8** (The hamburger theorem). Let $d \geq 2$ be an integer. Let $\mu_1, \mu_2, \ldots, \mu_{d+1}$ be absolutely continuous finite Borel measures on $\mathbb{R}^d$. Let $\omega_i = \mu_i(\mathbb{R}^d)$ for $i \in [d+1]$ and $\omega = \min\{\omega_i; i \in [d+1]\}$. Assume that $\sum_{j=1}^{d+1} \omega_j = 1$ and that $\mu_1, \mu_2, \ldots, \mu_{d+1}$ are balanced in $\mathbb{R}^d$. Then there exists a
hyperplane $h$ such that for each open halfspace $H$ defined by $h$, the measures $\mu_1, \mu_2, \ldots, \mu_{d+1}$ are balanced in $H$ and $\sum_{j=1}^{d+1} \mu_j(H) \geq \min(1/2, 1 - d\omega) \geq 1/(d+1)$.

Our choice of the name for Theorem 8 is motivated by the fact that compared to a typical ham sandwich, a typical hamburger consists of more ingredients, such as bread, beef, bacon, and salad.

Note that the lower bound $\min(1/2, 1 - d\omega)$ on the total measure of the two halfspaces is tight: consider, for example, $d+1$ measures such that each of them is concentrated in a small ball centered at a vertex of the unit $d$-dimensional simplex.

We were not able to generalize Theorem 8 for $d \geq 3$ and $d+2$ or more measures in $\mathbb{R}^d$, even if instead of the condition that the hyperplane cuts at least $1/(d+1)$ of the total measure on each side, we require only that the partition is nontrivial.

**Problem 9.** Let $d \geq 3$ and $r \geq d+2$ be integers. Let $\mu_1, \mu_2, \ldots, \mu_r$ be absolutely continuous positive finite Borel measures on $\mathbb{R}^d$ that are balanced in $\mathbb{R}^d$. Does there exists a hyperplane $h$ such that for each open halfspace $H$ defined by $h$, the total measure $\sum_{j=1}^{r} \mu_j(H)$ is positive and the measures $\mu_1, \mu_2, \ldots, \mu_r$ are balanced in $H$?

It is easy to see that for a given $d$, it would be sufficient to prove Problem 9 for $r \leq 2d-1$. Indeed, suppose that $r \geq 2d$ and that $\mu_1, \mu_2, \ldots, \mu_r$ are measures balanced in $\mathbb{R}^d$. If $\mu_1(\mathbb{R}^d) \geq \mu_2(\mathbb{R}^d) \geq \cdots \geq \mu_r(\mathbb{R}^d)$, then after replacing $\mu_{r-1}$ and $\mu_r$ by a single measure $\mu'_{r-1} = \mu_{r-1} + \mu_r$, the resulting set of $r-1$ measures is still balanced in $\mathbb{R}^d$.

## 2 Proof of the hamburger theorem

In this section we prove Theorem 8. We follow the proof of the ham sandwich theorem for measures from [12].

The set of open half-spaces in $\mathbb{R}^d$, together with the empty set and the whole space $\mathbb{R}^d$, has a natural topology of the sphere $S^d$. We use the following parametrization.

Let $u = (u_0, u_1, \ldots, u_d)$ be a point from the sphere $S^d$, that is, $u_0^2 + u_1^2 + \ldots + u_d^2 = 1$. If $|u_0| < 1$, then at least one of the coordinates $u_1, \ldots, u_d$ is nonzero, and we define two halfspaces as follows:

$$H^-(u) = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d; u_1 x_1 + u_2 x_2 + \cdots + u_d x_d < u_0\},$$

$$H^+(u) = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d; u_1 x_1 + u_2 x_2 + \cdots + u_d x_d > u_0\}.$$
We also define a hyperplane $h(u)$ as the common boundary of $H^+(u)$ and $H^-(u)$. For the two remaining points $(1, 0, \ldots, 0)$ and $(-1, 0, \ldots, 0)$, we set

\[
H^-(1, 0, 0, \ldots, 0) = \mathbb{R}^d, \quad H^+(1, 0, 0, \ldots, 0) = \emptyset, \\
H^-(1, 0, 0, \ldots, 0) = \emptyset, \quad H^+(1, 0, 0, \ldots, 0) = \mathbb{R}^d.
\]

Note that antipodal points on $S^d$ correspond to complementary half-spaces; that is, $H^-(u) = H^+(\bar{u})$ for every $u \in S^d$.

We define a function $f = (f_1, \ldots, f_{d+1}) : S^d \to \mathbb{R}^{d+1}$ by

\[
f_i(u) = \mu_i(H^-(u)).
\]

Since the measures $\mu_i$ are absolutely continuous, $\mu_i(h) = 0$ for every hyperplane $h$. This implies that $f$ is continuous [12].

The image of $f$ lies in the box $B = \prod_{i=1}^{d+1} [0, \omega_i]$. Moreover, $f$ maps antipodal points of the sphere to points symmetric about the center of $B$. The target polytope is the subset of points $(y_1, \ldots, y_{d+1})$ of $B$ satisfying the inequalities

\[
y_i \leq \frac{1}{d} \cdot \sum_{j=1}^{d+1} y_j \quad \text{and} \quad \omega_i - y_i \leq \frac{1}{d} \cdot \sum_{j=1}^{d+1} (\omega_j - y_j).
\]

See Figure 2, left. The subset of the target polytope satisfying the inequalities

\[
\min(1/2, 1-d\omega) \leq y_1 + y_2 + \cdots + y_{d+1} \leq 1 - \min(1/2, 1-d\omega)
\]

is called the truncated target polytope; see Figure 2, right. Our goal is to show that the image of $f$ intersects the truncated target polytope.

We first show a proof using the notion of a degree of a map between spheres. Then we modify it so that it uses only the Borsuk–Ulam theorem.

Let $b = (\omega_1/2, \omega_2/2, \ldots, \omega_{d+1}/2)$ be the center of $B$. The map $g = f - b$ is antipodal, that is, $g(-u) = -g(u)$ for every $u \in S^d$. Clearly, $b$ satisfies both (1) and (2) and thus it belongs to the truncated target polytope. Hence, if $0$ is in the image of $g$, then any hyperplane $h(u)$ such that $g(u) = 0$ satisfies the theorem.

For the rest of the proof we may assume that $0$ is not in the image of $g$. Then we can define an antipodal map $\tilde{g} : S^d \to S^d$ as

\[
\tilde{g}(u) = \frac{g(u)}{\|g(u)\|}.
\]

Using the fact that every antipodal map from $S^d$ to itself has odd degree [6, Proposition 2B.6.], we conclude that $\tilde{g}$ is surjective. Hence, the image of $g$
Figure 2: Left: the target polytope inside $B$ for $d = 2$ and $\omega_1 = \omega_2 = \omega_3 = 1/3$. Right: the truncated target polytope corresponding to hyperplanes cutting at least $1/3$ of the total measure on both sides. The segment $ac$, which is the intersection of the line $\ell$ with $B$, is contained in the truncated target polytope.

intersects every line passing through the origin, equivalently, the image of $f$ intersects every line passing through $b$. Therefore, it is sufficient to find a line $\ell$ through $b$ such that $\ell \cap B$ belongs to the truncated target polytope.

Without loss of generality, we may assume that $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_{d+1} = 1$. Let

$$t = \min \left( \frac{1}{2d}, \frac{1}{d} - \omega_{d+1} \right).$$

We define $\ell$ as the line containing the points

$$a = (t, t, \ldots, t, 0) \quad \text{and} \quad c = (\omega_1 - t, \omega_2 - t, \ldots, \omega_d - t, \omega_{d+1}).$$

Since the measures $\mu_1, \mu_2, \ldots, \mu_{d+1}$ are balanced in $\mathbb{R}^d$, we have $\omega_i \leq 1/d$ for all $i \in [d+1]$. Thus $\omega_d + \omega_{d+1} = 1 - (\omega_1 + \cdots + \omega_{d-1}) \geq 1/d$ and consequently $\omega_d \geq 1/(2d) \geq t$. This implies that both points $a$ and $c$ lie in $B$. Moreover, $a$ and $c$ lie on the opposite facets of $B$, they are symmetric around the center $b$, they both satisfy (1) since $(\omega_1 - t) + (\omega_2 - t) + \cdots + (\omega_d - t) + \omega_{d+1} = 1 - dt \geq d\omega_{d+1}$, and they both satisfy (2) since $dt \leq 1 - dt$. Therefore, the segment $ac$ is the intersection of the line $\ell$ with $B$ and it is contained in the truncated target polytope.

Now we show how to replace the degree argument with the application of the Borsuk–Ulam theorem. Let $\pi_\ell$ be a projection of $\mathbb{R}^{d+1}$ in the direction of the line $\ell$ to a $d$-dimensional subspace orthogonal to $\ell$, which we identify with $\mathbb{R}^d$. Define a map $g' : S^d \to \mathbb{R}^d$ by $g'(u) = \pi_\ell(g(u))$. The map $g'$ is antipodal, and so by the Borsuk–Ulam theorem, there exists $u \in S^d$ such that $g'(u) = 0$, which means that $f(u) \in \ell$. This concludes the proof.
3 A discrete version of the hamburger theorem

Theorem 5 follows by induction from the following discrete analogue of the hamburger theorem.

We say that point sets $X_1, \ldots, X_r$ are balanced in a subset $S \subseteq \mathbb{R}^d$ if for every $i \in [r]$, we have

$$|S \cap X_i| \leq \frac{1}{d} \cdot \sum_{j=1}^r |S \cap X_j|.$$ 

**Theorem 10.** Let $d \geq 2$ and $n \geq 2$ be integers. Let $X_1, \ldots, X_{d+1} \subset \mathbb{R}^d$ be $d+1$ disjoint point sets balanced in $\mathbb{R}^d$. Assume that no $d+1$ points of $X_1 \cup \ldots \cup X_{d+1}$ lie in a hyperplane and that $\sum_{i=1}^{d+1} |X_i| = dn$. Then there exists a hyperplane $h$ disjoint with each $X_i$ such that for each open halfspace $H$ determined by $h$, the sets $X_1, \ldots, X_{d+1}$ are balanced in $H$ and $\sum_{i=1}^{d+1} |H \cap X_i|$ is a positive integer multiple of $d$.

### 3.1 Proof of Theorem 10.

Let $X = \bigcup_{i=1}^{d+1} X_i$. Replace each point $p \in X$ by an open ball $B(p)$ of a sufficiently small radius $\delta > 0$ centered in $p$, so that no hyperplane intersects more than $d$ of these balls. We will apply the hamburger theorem for suitably defined measures supported by the balls $B(p)$. Rather than taking the same measure for each of the balls, we use a variation of the trick used by Elton and Hill [5]. For each $p \in X$ and $k \geq 1$, we choose a small number $\varepsilon_k(p) \in (0, 1/k)$ so that for every $i, j \in [d+1], i \neq j$, and for every two proper nonempty subsets $Y \subset X_i$ and $Z \subset X_j$, we have

$$\sum_{p \in Y} (1 - \varepsilon_k(p)) \neq \sum_{p \in Z} (1 - \varepsilon_k(p)).$$

Moreover, for each $i$ such that $|X_i| = n$, we require that

$$\sum_{p \in X_i} (1 - \varepsilon_k(p)) = \frac{1}{d} \cdot \sum_{p \in X} (1 - \varepsilon_k(p)).$$

Now let $k \geq 1$ be a fixed integer. For each $i \in [d+1]$, let $\mu_{i,k}$ be the measure supported by the closure of $\bigcup_{p \in X_i} B(p)$ such that it is uniform (that is, equal to a multiple of the Lebesgue measure) on each of the balls $B(p)$ and $\mu_{i,k}(B(p)) = 1 - \varepsilon_k(p)$. 

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By the condition (4), the measures \( \mu_{1,k}, \ldots, \mu_{d+1,k} \) are balanced in \( \mathbb{R}^d \) if the numbers \( \varepsilon_k(\mathbf{p}) > 0 \) are sufficiently small. We may thus apply the hamburger theorem to the normalized collection of measures \( \mu_{i,k} / (\sum_{j=1}^{d+1} \mu_{j,k}(\mathbb{R}^d)) \). Let \( h_k \) be the resulting hyperplane. We distinguish two cases.

1) We have \( \mu_{i,k}(H) = 0 \) for some \( i \in [d + 1] \) and some halfspace \( H \) determined by \( h_k \). Since the measures \( \mu_{1,k}, \ldots, \mu_{d+1,k} \) are balanced in \( H \), there is an \( \alpha > 0 \) such that \( \mu_{j,k}(H) = \alpha \) for every \( j \in [d + 1] \setminus \{i\} \).

2) The hyperplane \( h_k \) splits each measure \( \mu_{i,k} \) in a nontrivial way. By the condition (3), \( h_k \) intersects the support of exactly \( d \) of the measures \( \mu_{i,k} \).

For each \( i \in [k + 1] \), let \( \mu_i \) be the limit of the measures \( \mu_{i,k} \) when \( k \) grows to infinity; that is, \( \mu_i \) is uniform on every ball \( B(\mathbf{p}) \) such that \( \mathbf{p} \in X_i \) and \( \mu_i(B(\mathbf{p})) = 1 \). Since the supports of all the measures \( \mu_{i,k} \) are uniformly bounded, there is an increasing sequence \( \{k_m, m = 1, 2, \ldots\} \) such that the sequence of hyperplanes \( h_{k_m} \) has a limit \( h \). More precisely, if \( h_{k_m} = \{\mathbf{x} \in \mathbb{R}^d; \mathbf{x} \cdot \mathbf{v}_m = c_m\} \) where \( \mathbf{v}_m \in S^{d-1} \), then \( h = \{\mathbf{x} \in \mathbb{R}^d; \mathbf{x} \cdot \mathbf{v} = c\} \) where \( \mathbf{v} = \lim_{m \to \infty} \mathbf{v}_m \) and \( c = \lim_{m \to \infty} c_m \). By the absolute continuity of the measures, the measures \( \mu_1, \mu_2, \ldots, \mu_{d+1} \) are balanced in each of the two halfspaces determined by \( h \), and the total measure \( \sum_{j=1}^{d+1} \mu_j \) of each of the two halfspaces is at least \( n/(d + 1) \). One of the cases 1) or 2) occurred for infinitely many hyperplanes \( h_{k_m}, m \geq 1 \). We distinguish the two possibilities.

Suppose that case 1) occurred infinitely many times. Then there is an \( i \in [d + 1] \) such that \( \mu_{i,k_m}(H^+_{k_m}) = 0 \) occurred for infinitely many \( m \geq 1 \) or \( \mu_{i,k_m}(H^-_{k_m}) = 0 \) occurred for infinitely many \( m \geq 1 \), where \( H^+_{k_m} = \{\mathbf{x} \in \mathbb{R}^d; \mathbf{x} \cdot \mathbf{v}_m > c_m\} \) and \( H^-_{k_m} = \{\mathbf{x} \in \mathbb{R}^d; \mathbf{x} \cdot \mathbf{v}_m < c_m\} \). By the absolute continuity of the measures, there is an \( \alpha \geq n/(d + 1) > 0 \) such that for one of the halfspaces \( H \) determined by \( h \) and for every \( j \in [d + 1] \setminus \{i\} \), we have \( \mu_j(H) = \alpha \).

If \( h \) is disjoint from all the balls \( B(\mathbf{p}), \mathbf{p} \in X \), then this hyperplane satisfies the conditions of the theorem. Otherwise, \( h \) intersects one ball from the support of each of the measures \( \mu_j, j \in [d + 1] \setminus \{i\} \). Let \( \bar{h} \) be the translation of \( h \) that touches each of these \( d \) balls and such that \( \bar{h} \) is in the complement of \( H \). Let \( \bar{H} \) be the open halfspace determined by \( \bar{h} \) containing \( H \), and let \( \bar{H}' \) be the open halfspace opposite to \( \bar{H} \). Then for every \( j \in [d + 1] \setminus \{i\} \), we have \( \mu_j(\bar{H}) = [\alpha] \). In particular, the sets \( X_1, \ldots, X_{d+1} \) are balanced in \( \bar{H} \) and \( 0 < [\bar{H} \cap X] = d[\alpha] < dn \). It remains to show that \( X_1, \ldots, X_{d+1} \) are also balanced in \( \bar{H}' \). Let \( \bar{H}' \) be the open halfspace opposite
to $H$. Since the measures $\mu_1, \ldots, \mu_{d+1}$ are balanced in $H'$, we have

$$
\mu_i(\tilde{H}') = \mu_i(H') \leq \frac{1}{d} \cdot \sum_{j=1}^{d+1} \mu_j(H') = \frac{1}{d} \cdot \left( \sum_{j=1}^{d+1} \mu_j(\tilde{H}') \right) + ([\alpha] - \alpha)
$$

$$
= (n - [\alpha]) + ([\alpha] - \alpha) = n - \alpha.
$$

Since $\mu_i(\tilde{H}') = |X_i|$, we can replace the upper bound by the nearest integer, and thus we have

$$
\mu_i(\tilde{H}') \leq n - [\alpha] = \frac{1}{d} \cdot \left( \sum_{j=1}^{d+1} \mu_j(\tilde{H}') \right).
$$

Finally, suppose that case 2) occurred infinitely many times. There is an $i \in [d+1]$ such that for infinitely many $m \geq 1$, the hyperplane $h_{k_m}$ intersects the support of each measure $\mu_{j,k_m}$, $j \in [d+1] \setminus \{i\}$. In particular, for each $j \in [d+1] \setminus \{i\}$, there is a point $p_j \in X_j$ such that for infinitely many $m \geq 1$, the hyperplane $h_{k_m}$ intersects each of the balls $B(p_j)$. It follows that $h$ intersects or touches each of the balls $B(p_j)$.

Call a hyperplane in $\mathbb{R}^d$ admissible if it intersects or touches each of the balls $B(p_j)$, $j \in [d+1] \setminus \{i\}$, and the measures $\mu_1, \ldots, \mu_{d+1}$ are balanced in each of the two open halfspaces determined by the hyperplane. We may assume that among all admissible hyperplanes, $h$ intersects the minimum possible number of the balls $B(p_j)$; equivalently, $h$ touches as many of the balls $B(p_j)$ as possible.

Let $H$ and $H'$ be the two open halfspaces determined by $h$. For $j \in [d+1]$, call a measure $\mu_j$ saturated in a set $S \subseteq \mathbb{R}^d$ if $\mu_j(S) = (\sum_{k=1}^{d+1} \mu_k(S))/d$. Call a measure $\mu_j$ saturated if $\mu_j$ is saturated in $H$ or in $H'$. We may assume, under the conditions above, that $h$ is chosen so that the number of saturated measures is the maximum possible.

**Observation 11.** If $\mu_j$ and $\mu_{j'}$ are saturated and $h$ intersects $B(p_j)$, then $\mu_j(H \cap B(p_j)) = \mu_{j'}(H \cap B(p_{j'}))$.

*Proof.* If $\mu_j$ and $\mu_{j'}$ are saturated in the same halfspace, say, $H$, then $\mu_j(H) = \mu_{j'}(H)$. If $\mu_j$ is saturated in $H$ and $\mu_{j'}$ is saturated in $H'$, then $\mu_j(H) + \mu_{j'}(H') = n$. The observation follows since for each of the measures, each of the balls $B(p)$, $p \in X$, has measure 0 or 1, and $h$ intersects at most one ball from the support of each of the measures. \hfill $\square$

**Observation 12.** There is at most one measure $\mu_j$ such that $\mu_j$ is not saturated and $h$ intersects $B(p_j)$. 

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Proof. Suppose that there are \( j, j' \in [d+1] \setminus \{i\}, j \neq j' \), such that neither of \( \mu_j \), \( \mu_{j'} \) is saturated and \( h \) intersects both \( B(p_j) \) and \( B(p_{j'}) \). We can rotate the hyperplane \( h \) while keeping the values \( \mu_k(H \cap B(p_k)) \) constant for each \( k \in [d+1] \setminus \{i, j, j'\} \), keeping the value \( \mu_j(H \cap B(p_j)) + \mu_{j'}(H \cap B(p_{j'})) \) constant, and decreasing the value \( \mu_j(H \cap B(p_j)) \) while increasing the value \( \mu_{j'}(H \cap B(p_{j'})) \), until one of the measures \( \mu_j \), \( \mu_{j'} \) becomes saturated or \( h \) does not intersect \( B(p_j) \) or \( B(p_{j'}) \) anymore. Observe that \( h \) is admissible all the time during the rotation. \( \blacksquare \)

Observation 13. If \( \mu_i \) is saturated, or if, for some \( j \in [d+1] \setminus \{i\} \), \( \mu_j \) is saturated and \( h \) does not intersect \( B(p_j) \), then \( h \) touches each of the balls \( B(p_k), k \in [d+1] \setminus \{i\} \). Moreover, \( |H \cap X| \) is an integer multiple of \( d \).

Proof. The observation follows by Observations 11 and 12 and from the fact that \( \sum_{k=1}^{d+1} \mu_k(S) = d\mu_j(S) \) for every measure \( \mu_j \) saturated in \( S \). \( \blacksquare \)

Observation 14. If \( \mu_j \) is saturated and \( j \neq i \), then \( h \) does not intersect \( B(p_j) \).

Proof. Suppose for contrary that \( \mu_j \) is saturated and \( h \) intersects \( B(p_j) \). If all the measures \( \mu_k, k \in [d+1] \setminus \{i\} \), are saturated, then by Observation 11, \( h \) intersects each of the balls \( B(p_k), k \in [d+1] \setminus \{i\} \), and moreover, there is an \( \alpha \in (0, 1) \) such that \( \mu_k(H \cap B(p_k)) = \alpha \), for each \( k \in [d+1] \setminus \{i\} \). In this case, we can move \( h \) to a hyperplane \( \tilde{h} \) that touches each of the balls \( B(p_k) \). The measures \( \mu_k, k \in [d+1] \setminus \{i\} \) are saturated all the time during the translation, and \( \mu_i \) can become saturated only at the end when \( h = \tilde{h} \) by Observation 13.

Thus, at most \( d-1 \) of the measures \( \mu_k, k \in [d+1] \setminus \{i\} \), are saturated. Let \( \sigma \subset [d+1] \setminus \{i\} \) be the set of all \( k \in [d+1] \setminus \{i\} \) such that \( \mu_k \) is saturated. Let \( \alpha = \mu_j(H \cap B(p_j)) \). Let \( \beta = \sum_{k \in [d+1] \setminus \{i\} \cup \sigma} \mu_k(H \cup B(p_k)) \) and \( \beta' = \sum_{k \in [d+1] \setminus \{i\} \cup \sigma} \mu_k(H' \cup B(p_k)) \). Since \( \beta + \beta' = d - |\sigma| \), we have \( \beta \geq \alpha(d - |\sigma|) \) or \( \beta' \geq (1 - \alpha)(d - |\sigma|) \). In the first case, we rotate \( h \) while keeping the values \( \mu_k((H \cup B(p_k)) \) equal for all \( k \in \sigma \), keeping all the measures \( \mu_k, k \in \sigma \), balanced, keeping the condition of Observation 12, and decreasing \( \mu_j((H \cap B(p_j)) \) towards \( 0 \). In the other case, we keep the same conditions except that we increase \( \mu_j((H \cap B(p_j)) \) towards \( 1 \). The hyperplane \( h \) stays admissible all the time during the rotation, but eventually it touches more of the balls \( B(p_k) \) than in the beginning, or some measure \( \mu_k, k \in [d+1] \setminus \sigma \), becomes saturated. \( \blacksquare \)

Observation 15. The hyperplane \( h \) touches each of the balls \( B(p_j), j \in [d+1] \setminus \{i\} \).
Proof. By Observations 12 and 14, \( h \) intersects at most one of the balls, say, \( B(p_j) \), and in that case \( \mu_j \) is not saturated. By Observation 13, none of the measures \( \mu_k, k \in [d+1] \), is saturated either. We may thus rotate \( h \) while keeping the property that \( h \) touches each of the balls \( B(p_k), k \in [d+1] \setminus \{i, j\} \), until \( h \) touches \( B(p_j) \), too. The hyperplane stays admissible all the time during the rotation.

If \( h \) satisfies the property that \( |H \cap X| \) is divisible by \( d \), then we are finished, since both \( H \cap X_i \) and \( H' \cap X_i \) are nonempty. Otherwise, by Observation 13, none of the measures \( \mu_j, j \in [d+1] \), is saturated. Therefore, we can rotate \( h \) in order to move some of the balls \( B(p_j) \) from one side of \( h \) to the other, while keeping \( h \) admissible all the time, until \( |H \cap X| \) is divisible by \( d \). This final hyperplane \( h \) satisfies the theorem. \( \square \)

4 Concluding remarks

Our approach for proving the hamburger theorem does not easily generalize to five measures in \( \mathbb{R}^3 \). The problem is that if \( \mu_i(\mathbb{R}^3) = 1/5 \) for each \( i \in [5] \), then the target polytope, now in \( \mathbb{R}^5 \), intersected with the boundary of the box \( B \), does not contain a closed curve symmetric with respect to the center of \( B \). If such a curve existed, we could apply a generalization of the Borsuk–Ulam theorem saying that if \( f : S^k \to S^{k+l} \) and \( g : S^l \to S^{k+l} \) are antipodal maps, then their images intersect [12, Exercise 3.*/116].

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