The evolution of three-dimensional localized vortices in shear flows. Linear stage

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Abstract

The evolution of a small-amplitude localized vortex disturbance in an unbounded shear flow with the linear velocity profile is investigated. Based on the exact solution of the initial problem, a revision is made of the theoretical approach (suggested by Levinski 1991 and subsequently further developed in a series of other publications) in which the vortex evolution is described in terms of Fluid Impulse of the vortex “core”. Although the theoretical predictions obtained on the basis of that approach were excellently confirmed in subsequent experimental studies, its inconsistency is demonstrated in this study.

According to this solution, the localized vortex increases slowly (as power-law with the time) and attains an almost “horizontal” orientation, unlike the previous theory (Levinski 1991) that predicts the more rapid growth and vortex orientation at the angle of 45° to the flow direction. On the other hand, just the rapid increase and the angle of 45° to the outer flow direction are characteristic for hairpin vortices observed in turbulent boundary layers or artificially synthesized vortices in laminar boundary layers.

Thus the issue of adequate theoretical interpretation of the evolution of localized vortices is again on the agenda. The remaining part of the paper presents the first steps in the solving this problem. In particular, the dynamics of the total enstrophy of vortex as the measure of vortex intensity is followed. The dependence of vortex amplification on its initial orientation is investigated. On this base the validity of the old idea of Theodorsen (1952) on the predominant formation of the 45° vortices is discussed. Also the tensor of enstrophy distribution (TED) is defined and it is shown that it may serve an effective tool in describing of the vortex geometry.

The linear stage of Gaussian vortex evolution presented here provides a very suitable base for testing of further numerical simulation of the nonlinear stage.
I. INTRODUCTION

Two main types of coherent vortex structures, which were first identified using flow visualization technique in experiments reported by Kline et al. (1967), form the basis of the present views of the structure of a turbulent boundary layer. Thus the presence in the wall-bound flow of streaks, along which the streamwise velocity is lower than the average velocity at the same distance from the wall, results from the rise of low-velocity fluid from near-wall layers induced by long-lived vortex structures. These latter represent a pair of counter-rotating vortices extended along the flow direction (Bakewell & Lumley 1967; Smith & Schwartz 1983).

Another phenomenon, widely observed in turbulent boundary layers and referred by Kline et al. (1967) as “bursting”, results from the rapid evolution of the localized vortex having the shape of a hairpin. These vortices were found to be inclined at 45° to the flow direction (Head & Bandyopadhyay 1981), and their typical lifetime are about 5% of the streak lifetime.

The evolution mechanisms of these well-organized vortex structures and their interaction have been subjects for study by an ever increasing number of researchers (see reviews by Robinson 1991 and Smith & Walker 1983).

Despite the fact that both types of coherent vortices have an identical structure of the type of vortex dipole, their properties, and also formation and development mechanisms differ greatly. The slow evolution of near-wall vortices is reasonably well explained by the mechanism of algebraic growth suggested by Benney & Gustavsson (1981) and subsequently further developed in terms of the concept of optimal disturbances by Butler & Farrell (1992), Reddy & Henningson (1993) and Reshotko & Tumin (2001). On the other hand, the derivation of an adequate theoretical model describing the evolution of hairpin vortices is complicated by the high degree of vorticity localization in the core of the hairpin vortex and, hence, by the strong nonlinear character of their development from the outset. This is confirmed by a number of experiments where hairpin vortices were observed only at the transition of the threshold value of a certain parameter corresponding to the mechanism of their generation used in the experiment. Thus in experiments of Asai & Nishioka (1995) where hairpin vortices were generated by using acoustic disturbances, these vortex structures were observed only when the amplitude of the applied disturbance stood out above a certain critical value. In experiments of Malkiel, Levinski & Cohen (1999) the initial disturbance was created by employing suction of fluid through the holes in the wall. Here, as in the previous case, hairpin vortices were observed only at fluid suction rates exceeding a certain critical value.

Based on the aforementioned factors, it is of interest to analyze the theoretical model describing the evolution of a nonlinear localized vortex disturbance in the external plane shear flow first suggested by Levinski (1991) and subsequently generalized to rotating flows (Levinski & Cohen 1995), flows of weakly conducting fluid in a magnetic field (Levinski, Rapoport & Cohen 1997) and to stratified flows (Levinski 2000). In this model the vorticity distribution is characterized by its fluid impulse integral defined as

\[ \mathbf{p} = \frac{1}{2} \int \mathbf{r} \times \mathbf{\omega}(t, \mathbf{r}) \, dV, \]  

\[ (1.1) \]
where $r$ is the position vector, $\omega(t,r)$ is the instantaneous field of vorticity disturbance, and the integration is done over the entire volume of fluid. Accordingly, the fluid impulse dynamics is described by the equation

$$\frac{dp}{dt} = \frac{1}{2} \int r \times \frac{\partial \omega(t,r)}{\partial t} \, dV,$$

(1.2)

where the evolution equation for disturbed vorticity in a steady-state external velocity field ($U$) is obtained by applying the curl operator to the Navier-Stokes equation and by a subsequent substraction of the equation for undisturbed flow:

$$\frac{\partial \omega}{\partial t} + (U \cdot \nabla) \omega - (\omega \cdot \nabla) U = \nu \Delta \omega. \quad (1.3a)$$

Here $\Omega = \text{curl} U$, and $u$ designates the disturbance-induced velocity field,

$$\omega = \text{curl} u. \quad (1.3b)$$

Because the fluid impulse is invariant with respect to the self-induced motion of the vortex disturbance (Batchelor 1967), the nonlinear terms (underlined in equation (1.3a)) make a zero contribution to (1.2). This makes it possible to “linearize” the problem on evolution of a strongly nonlinear localized disturbance (for a more detailed description see the works by Levinski 1991 and Levinski & Cohen 1995, hereinafter L&LC). The fluid impulse integral has also an additional important property. Just by its definition (1.2), the fluid impulse describes both the increase in amplitude and the geometrical growth of the vortex. This is a highly important factor because the experimentally observed growth of hairpin vortices is not necessarily associated with the increase in vorticity amplitude. To cover such a scenario of a localized vortex evolution in terms of classical linear stability theory requires a very extensive analysis of amplitude changes for a great number of modes.

The most important result, obtained on the basis of the fluid impulse approach, is the prediction of an exponential instability of a localized vortex disturbance in plane Couette flow (L&LC). This finding makes it possible to explain the experimentally observed formation and fast development of hairpin vortices in the boundary layer as resulting from a plane shear flow instability to localized vortices originating on the wall inhomogeneities. By applying this approach to circular Couette flow (Malkiel, Levinski & Cohen 1999; Levinski & Cohen 1995), it was possible to predict the growth of hairpin vortices within the range of basic flow parameters where the flow is known to be linearly stable. The criterion obtained for hairpin vortices growth was supported by the results of experiments performed by Malkiel, Levinski & Cohen (1999).

On the other hand, as the procedure of deriving the closed evolution equation for fluid impulse, suggested in L&LC does not contain any formal limitations on the initial disturbance amplitude, this result is in conflict with the results of classical linear stability theory for plane and circular Couette flows (Drazin & Reid 1981, Dikii 1976).

The objective of this paper is to analyze the evolution of a localized vortex disturbance in terms of linear stability theory on the basis of constructing a complete vorticity field. This approach is free from the deficiencies of the description of vortex evolution using the fluid impulse; unfortunately, however, it does not permit us to advance into the region of
strong (nonlinear) vortices by analytical methods. On the other hand, the approach that is developed in this paper, is useful for analyzing the validity of the assumptions made in L&LC in the course of deriving the closed evolution equation for fluid impulse of the disturbance. Moreover, a knowledge of a complete vorticity field in the physical space for an arbitrary instant of time provides the basis for constructing the other integral characteristics of the disturbance vorticity field which can be useful in the analysis of the results of numerical simulations.

This paper is organized as follows.

In §2 we subject to a critical analysis the fluid impulse concept suggested in L&LC, and on the basis of the exact solution of a linearized problem (for the time being, as a Fourier-representation) we demonstrate its invalidity for the present formulation of the problem (i.e. the problem of localized vortex evolution in the external flow with a linear velocity profile).

In §3, on the basis of the exact solution obtained above (in §2), inverse Fourier-transform is used to construct the vorticity field in the physical space (within the linear approximation, of course). By considering an example where a so-called “Gaussian vortex” serves as the initial disturbance, the linear evolution of the vortex is studied for some particular cases of its orientation. It will be shown, in particular, that the symmetry properties of the basic equations forbid the formation of “hairpins” within the framework of a linear problem.

In §4, we introduce the notion of total enstrophy of the vortex as the measure of its intensity. On this basis, we investigate the character of enhancement (attenuation) of the vortex depending on its initial orientation.

In §5, we turn from the description of the vortex development based on its complete vorticity field to the description using a new integral characteristic, namely, the Tensor of Enstrophy Distribution (TED) which we introduce. This integral characteristic makes it possible to describe the vortex using only six independent parameters (and in the case of vortices symmetrical about the plane $z = 0$, even four parameters only).

In §6 we discuss the results obtained and the possible further directions of research.

II. THE EVOLUTION OF THE FLUID IMPULSE. THE EXACT SOLUTION OF THE LINEAR INITIAL PROBLEM FOR THE VORTICITY FIELD IN THE K-SPACE

A. Definition of the modified fluid impulse of vorticity

It follows directly from the definition of fluid impulse integral (1.1) that it exists and is absolutely convergent only if

$$|\omega(t, r)| \leq \frac{A}{|r|^{4+\epsilon}}, \text{ where } \epsilon > 0. \quad (2.1)$$

Initially, a well-localized disturbance induces a velocity field possessing the asymptotic behavior (Batchelor 1967)

$$|u(t = 0; r)| \sim \frac{1}{|r|^3}. \quad (2.2)$$
From substitution of (2.2) into (1.3a) it follows that the vortex field generated at an arbitrary time \( t > 0 \), has the asymptotic representation

\[
|\omega(t, r)| \sim \frac{1}{|r|^4}, \tag{2.3}
\]

which does not satisfy the condition (2.1).

To overcome this problem the approach based on vorticity separation procedure has been suggested by Levinski (1991). Accordingly, the vorticity field of the disturbance is subdivided on a closed vorticity field bounded the region directly adjacent to the initial vortex disturbance \( \omega^I \), and vorticity field \( \omega^{II} \) that includes vortex tails generated in the process of the vortex disturbance development. Furthermore, it is assumed that \( \omega^I \) describes the evolution of the hairpin vortex, whereas \( \omega^{II} \) represents a vortex cloud, which has no substantial impact on the evolution of the concentrated vorticity.

Since the subsequent analysis is based on the exact solution of linearized equations of vorticity dynamics and does not involve any additional assumptions, it is appropriate to introduce the concept of fluid impulse without recourse to the vorticity subdivision procedure. This will permit us, in particular, to analyze the validity of the procedure suggested by L&LC.

For this purpose it is convenient to define the Modified Fluid Impulse (MFI), as follows

\[
\tilde{p}(t) = \lim_{R \to \infty} \frac{1}{2} \int_{r < R} r \times \omega(t, r) \, dV. \tag{2.4}
\]

The definition (2.4) is valid both in the case of the initial disturbance with an infinitely small amplitude as well as in a strongly nonlinear case. The only limitation is the local character of the disturbance at the initial instant of time, which corresponds to the absolute convergence of the fluid impulse integral for the initial distribution of vorticity.

Since in the subsequent discussion the solution to the equations for vorticity dynamics is constructed in the Fourier-space, we shall use a Fourier-transform of the definition of the MFI. Namely, defining the Fourier-transform of the vorticity field as

\[
\omega(t, \mathbf{k}) = (2\pi)^{-3} \int dV \omega(t, r) \exp(-i \mathbf{k} \cdot \mathbf{r}). \tag{2.5}
\]

the MFI (2.4) be represented as

\[
\tilde{p}(t) = \frac{1}{2} i (2\pi)^3 \lim_{k \to 0} \langle \nabla_k \times \omega(t, \mathbf{k}) \rangle, \tag{2.6}
\]

where the angle brackets correspond to averaging over the angles in the \( \mathbf{k} \)-space:

\[
\langle \cdots \rangle = \frac{1}{4\pi} \int_{-\pi/2}^{\pi/2} \cos \beta \, d\beta \int_0^{2\pi} d\phi \langle \cdots \rangle, \tag{2.7}
\]

and \( \beta \) and \( \phi \) are the spherical angles in the \( \mathbf{k} \)-space (with the axis \( y \) as the vertical axis, and the plane \((xz)\) corresponding to \( \beta = 0 \)):

\[
k_1 = k \cos \beta \cos \phi, \quad k_2 = k \sin \beta, \quad k_3 = k \cos \beta \sin \phi. \tag{2.8}
\]
It can be shown that the MFI defined by (2.4) is sufficiently “good” at first glance.

Firstly, it does satisfy the necessary requirement of invariance with respect to the position of the center of a sphere.

It should be noted, however, that in the present case where the vorticity decreases toward the periphery only as \( \sim |r|^{-4} \) (and the fluid impulse in the usual sense does not exist), the velocity field \( \mathbf{u}(r) \) at large distances is no longer a potential one and, in particular, it no longer may be represented in the usual “dipole” form

\[
\mathbf{u} = \frac{1}{4\pi} \text{curl} \left( \frac{\mathbf{p} \times r}{r^3} \right) + O(1/r^4),
\]
even if the MFI \( \tilde{\mathbf{p}} \) serves as \( \mathbf{p} \).

This means in particular that the MFI does not reflect at all the dipole structure of the disturbance vorticity field. As will be shown below, the plane of the vortex core localization, which can be described by the enstrophy distribution, \( L \equiv |\mathbf{\omega}(r)|^2 \), is not necessarily perpendicular to the MFI direction, as should be in the case of the usual vortex dipole. (It can be shown, however, that even in this case the velocity again decreases toward the periphery in inverse proportion to the distance cubed, \( \mathbf{u} \sim |r|^{-3} \), as in the case of the potential velocity field induced by well-localized vortex).

Secondly, it can be shown that the integral (2.4) exists at any instant of time and its value for sufficiently large values of \( R \) is independent on the value of \( R \) provided the MFI is well-defined at the initial instant of time.

Indeed, by taking the time derivative of the expression (2.4) and substituting (1.3) into the right-hand side, we obtain

\[
\frac{d\tilde{\mathbf{p}}(t)}{dt} = -\frac{1}{2} \lim_{R \to \infty} \int_{r<R} \mathbf{r} \times \left[ (\mathbf{U} \nabla) \mathbf{\omega} - (\mathbf{\omega} \nabla) \mathbf{U} - (\mathbf{\Omega} \nabla) \mathbf{u} \right] dV,
\]

Note that in (2.9) there are no contributions from the nonlinear and viscous terms (cf. (1.3)). The volumetric integral of these terms can be transformed to the integral over an infinite surface. The latter is zero by virtue of the asymptotic behavior of the vorticity \( \mathbf{\omega}(r, t) \) and, accordingly, of the velocity field \( \mathbf{u}(r, t) \) induced by it when \( R \to \infty \).

Finally, the equation (2.9) may be transformed into form

\[
\frac{d\tilde{\mathbf{p}}_i(t)}{dt} = -\frac{1}{2} \tilde{p}_j \frac{\partial U_i}{\partial x_j} - \frac{1}{2} \tilde{p}_j \frac{\partial U_j}{\partial x_i} + \lim_{R \to \infty} J_i,
\]

where

\[
J_i = -\frac{1}{2R} \varepsilon_{ijk} \frac{\partial U_l}{\partial x_m} \int_S x_l x_j x_m \omega_k dS
+ \frac{1}{4R} \varepsilon_{ijk} \frac{\partial U_l}{\partial x_k} \int_S x_l x_j x_m \omega_n dS + \frac{1}{4R} \varepsilon_{ijk} \frac{\partial U_l}{\partial x_m} \int_S x_j x_k \omega_l dS.
\]

Here \( \varepsilon_{ijk} \) is the alternating tensor and usual summation convention is applied. By virtue of the asymptotic vorticity behavior, the limit of \( J \) is finite. This, together with (2.10), proves that if the MFI exists at the initial instant of time, then it exists also at an arbitrary instant of time.
Thus the MFI which we have just introduced seems, at first glance, a worthy replacement of the “fluid impulse of the core” introduced in L&LC for describing the evolution of a localized vortex since it does not require the usage of the vortex field subdivision procedure which has not been adequately justified in L&LC.

It will be shown below, however, that any modification of the fluid impulse is unsatisfactory with regards to the capability to describe the structure of localized vortex.

The left side of equation (2.10) together with the first two terms in its r.h.s. represents the evolution equation obtained in L&LC for the fluid impulse components constructed on the basis of a closed vorticity field $\omega^I$, whereas the last term describes the specific contribution from the vortex tails.

Note that the basic equation in L&LC is a linear equation in spite of the fact that it was derived from the exact nonlinear system of equations (1.3) without recourse to the linearization procedure. This means that the theory suggested in L&LC claims, in fact, a possibility of describing not only weak but also strong (nonlinear) vortices. In other words, this theory is insensitive to the vortex disturbance amplitude.

On the one hand, this makes it extremely attractive, which, as a matter of fact, gave impetus to conduct a number of elegant experiments on its basis, and, on the other, if it is true, its predictions must remain valid for weak vortices as well. For weak vortices, however, there is a possibility of drastically simplifying the problem by performing a preliminary linearization of the initial system. This permits us to write the exact solution for the vorticity field and, on its basis, to check the validity of the theory. In particular, it is such a possibility of verifying the theory suggested in L&LC has stimulated this investigation.

Thus, the contribution of vortex tails into (2.10) means that it is impossible to construct the closed equation describing the fluid impulse evolution (and from which the conclusion was drawn in L&LC about exponential instability) without one or another of vorticity subdivision methods.

For that reason, below we make an attempt to give an alternative description to the evolution of a localized vortex without recourse to the evolution equation for fluid impulse. Using the solution obtained for the vorticity components we will also be able to calculate the fluid impulse and check to what extent the solution describing the fluid impulse evolution obtained in L&LC is consistent with what follows from this solution.

**B. The evolution of a localized vortex disturbance in the plane Couette flow**

1. *Solving the initial problem for a localized disturbance*

It will be assumed that the basic flow has a linear velocity profile, $U = (-\Omega y, 0, 0)$ (so that its vorticity is $\Omega = (0, 0, \Omega)$). Note that this choice is not a loss of generality as a consequence of the assumption about the local character of the disturbance. It corresponds to the case where the characteristic size of the disturbance is much less than the characteristic size of variation of the basic velocity field.

The suitable mathematical method for investigation of disturbance evolution in such flow was proposed by Lord Kelvin more than century ago (Kelvin 1887, see also the more
recent publications based on this method, such as Craik & Criminale 1986, Criminale & Drazin 1990, Farrel & Ioannou 1993 and other), and we also apply it here.

In this case the inviscid linearized equation for vorticity dynamics can be represented in the component-wise notation in Cartesian coordinates as

\[
\begin{align*}
\frac{\partial \omega_1}{\partial t} - y \Omega \frac{\partial \omega_1}{\partial x} - \Omega \frac{\partial u_3}{\partial x} &= 0, \\
\frac{\partial \omega_2}{\partial t} - y \Omega \frac{\partial \omega_2}{\partial x} - \Omega \frac{\partial u_2}{\partial z} &= 0, \\
\frac{\partial \omega_3}{\partial t} - y \Omega \frac{\partial \omega_3}{\partial x} - \Omega \frac{\partial u_3}{\partial z} &= 0,
\end{align*}
\]

(2.11)

where the subscripts “1”, “2” and “3” correspond to the x-, y- and z-components of the vectors, respectively. Upon introducing the dimensionless time \( \tau = \Omega t \) and Fourier-transforming the system of equations (2.12), we obtain

\[
\begin{align*}
\frac{\partial \omega_1}{\partial \tau} + k_1 \frac{\partial \omega_1}{\partial k_2} - i k_1 u_3 &= 0, \\
\frac{\partial \omega_2}{\partial \tau} + k_1 \frac{\partial \omega_2}{\partial k_2} - i k_3 u_2 &= 0, \\
\frac{\partial \omega_3}{\partial \tau} + k_1 \frac{\partial \omega_3}{\partial k_2} - i k_3 u_3 &= 0,
\end{align*}
\]

(2.12)

where \( u_i(t, \mathbf{k}) \) represent the Fourier-transform of the disturbed velocity field components \( u_i(\mathbf{r}, t) \).

The solution of the above problem on the evolution of the perturbation in the form of a single plane wave has been presented earlier in the work by Farrel & Ioannou, 1993 (hereinafter F&I). Although the solution presented here below, in fact, is the same, we describe it derivation briefly to emphasize here on the vorticity components (instead of velocity components, as it done in F&I). We use, as in F&I, instead of the set of independent variables \( \tau, k_1, k_2 \) and \( k_3 \), a new set of independent variables \( \tau, k_1, q \) and \( k_3 \), where \( q = k_2 - k_1 \tau \). We will consider \( \omega_i \) as a function of \( \tau, k_1, q, k_3 \), that is, \( \omega_i = \omega_i(\tau, k_1, q, k_3) \). Thus the new variable \( q \) is simply the initial value of the time-varying component of the wave vector \( k_2 \): \( k_2(t) = q + k_1 \tau \).

The new set of variables will be referred to as the Lagrangian coordinates in the \( \mathbf{k} \)-space. The transition to the Lagrangian coordinates makes it possible to transform the system of equations (2.12) to a system of ordinary differential equations

\[
\begin{align*}
\frac{d\omega_1}{d\tau} - \frac{k_1}{k^2(\tau)} [k_2(\tau) \omega_1 - k_1 \omega_2] &= 0, \\
\frac{d\omega_2}{d\tau} - \frac{k_2(\tau)}{k^2(\tau)} [k_2(\tau) \omega_1 - k_1 \omega_2] + \omega_1 &= 0, \\
\frac{d\omega_3}{d\tau} - \frac{k_3}{k^2(\tau)} [k_2(\tau) \omega_1 - k_1 \omega_2] &= 0,
\end{align*}
\]

(2.13)

where \( k^2(\tau) = k_1^2 + [k_2(\tau)]^2 + k_3^2 \), and \( d/d\tau \equiv (\partial/\partial \tau)_q = (\partial/\partial \tau)_{k_2} + k_1(\partial/\partial k_2)_\tau \). In deriving (2.13), we expressed also the Fourier-components of the velocity field in terms of Fourier-
components of the vorticity field $\mathbf{u}(\mathbf{k}) = \frac{i}{k^2}(\mathbf{k} \times \omega(\mathbf{k}))$. As a result we obtain the solution for dynamics of vorticity $\omega(\tau, \mathbf{k})$ as:

$$
\begin{align*}
\omega_1(\tau, \mathbf{k}) &= \omega_1(0, Q) - \frac{k_1^2}{p^2} \tau \omega_2(0, Q) + \frac{k_3 k_2}{p} V(\tau, \mathbf{k}) T(\tau, \mathbf{k}), \\
\omega_2(\tau, \mathbf{k}) &= \omega_2(0, Q) - \frac{k_3 p}{k_1} V(\tau, \mathbf{k}) T(\tau, \mathbf{k}), \\
\omega_3(\tau, \mathbf{k}) &= \omega_3(0, Q) - \frac{k_1 k_3}{p^2} \tau \omega_2(0, Q) + \frac{k_3 k_2^2}{k_1 p} V(\tau, \mathbf{k}) T(\tau, \mathbf{k}).
\end{align*}
$$

(2.14)

Here $p = \sqrt{k_1^2 + k_2^2}$, $Q = (q_1, q_2, q_3) \equiv (k_1, q, k_3)$ is the initial value of the wave vector which in (2.14) must be expressed in terms of $\mathbf{k}$: $Q = (k_1, k_2 - k_1 \tau, k_3)$.

$V(\tau, \mathbf{k}) = (1/p^2)[k_3 \omega_1(0, Q) - k_1 \omega_3(0, Q)]$, $T(\tau, \mathbf{k}) = \arctan(k_2/p) - \arctan[(k_2 - k_1 \tau)/p]$.

Including of the viscosity leads to additional viscous factor in the expressions for $\omega_i$:

$$
\omega_i(\tau; \mathbf{k}) \to \omega_i(\tau; \mathbf{k}) \exp \left( -\nu/\Omega \right) \int_0^\tau k^2(\tau')d\tau'
$$

(see also F&I).

In what follows, we shall use, as the initial vortex disturbance, the Gaussian vortex

$$
\omega(\tau = 0, \mathbf{r}) = \nabla F \times \mathbf{\mu}, \quad F = (\pi^{1/2}\delta)^{-3} \exp \left(-r^2/\delta^2\right).
$$

(2.15)

For numerical simulation of all hydrodynamic quantities (using program packages for 3-D hydrodynamics) it is often also necessary to specify the initial velocity field. It is readily calculated even for the initial isotropic function $F(\mathbf{r})$:

$$
\mathbf{u} = F(\mathbf{r}) \left[ \mathbf{\mu} - \frac{\mathbf{r}(\mathbf{\mu} \cdot \mathbf{r})}{r^2} \right] - \frac{H(\mathbf{r})}{r^3} \left[ \mathbf{\mu} - \frac{3\mathbf{r}(\mathbf{\mu} \cdot \mathbf{r})}{r^2} \right], \quad H(\mathbf{r}) = \int_0^r F(x) x^2 \, dx
$$

(2.16)

Thus we have in the $\mathbf{k}$-representation for the vortex of (2.15)

$$
\omega(0, \mathbf{k}) = \frac{i}{(2\pi)^3} (\mathbf{k} \times \mathbf{\mu}) \exp \left(-\frac{1}{4}k^2\delta^2\right).
$$

(2.17)

Note that for the vortex (2.15) at the initial instant of time the usual fluid impulse $\mathbf{p}$ is also well defined and equal to $\mathbf{\mu}$.

For our further purposes it is also very convenient to use the spherical coordinates in the $Q$-space: $q_1(= k_1) = Q \cos \beta_0 \cos \phi$, $q_2 = Q \sin \beta_0$, $q_3(= k_3) = Q \cos \beta_0 \sin \phi$, and $-\frac{1}{2}\pi \leq \beta_0 \leq \frac{1}{2}\pi$, $0 \leq \phi \leq 2\pi$. In these variables we write finally for the Gaussian vortex

$$
\omega_i(\tau, \mathbf{k}) = \frac{i}{(2\pi)^3} p \exp \left( -\frac{1}{4}Q^2 D^2 \right) \xi_i(\tau; \beta_0, \phi),
$$

(2.18)
\[
\begin{align*}
\hat{\zeta}_1 &= \mu_3 \tan \beta_0 - \mu_2 \sin \phi - \tau \cos^2 \phi (\mu_1 \sin \phi - \mu_3 \cos \phi) - \tan \beta \cos \phi \zeta^*, \\
\hat{\zeta}_2 &= \mu_1 \sin \phi - \mu_3 \cos \phi + \zeta^*, \\
\hat{\zeta}_3 &= \mu_2 \cos \phi - \mu_1 \tan \beta_0 - \tau \sin \phi \cos \phi (\mu_1 \sin \phi - \mu_3 \cos \phi) - \tan \beta \sin \phi \zeta^*, \\
\zeta^*_2 &= (\beta_0 - \beta) \tan \phi [\tan \beta_0 (\mu_1 \cos \phi + \mu_3 \sin \phi) - \mu_2]. 
\end{align*}
\] (2.19)

Here

\[
D = D(\tau; \beta_0, \phi) = \delta \left\{ 1 + \frac{4\tau}{Re} [1 + \tau \cos \beta_0 \sin \beta_0 \cos \phi + \frac{1}{3} \tau^2 \cos^2 \beta_0 \cos^2 \phi] \right\}^{1/2}, \tag{2.20}
\]

and the “Reynolds number of vortex” \( Re \) is defined as \( Re = \Omega \delta^2 / \nu \) (so that in the inviscid case \( D = \delta \)). The spherical angles \( \beta \) (in the \( k \)-space) and \( \beta_0 \) (in the \( Q \)-space) are related by the following expression \( \beta = \beta (\tau; \beta_0, \phi) = \arctan (\tan \beta_0 + \tau \cos \phi) \).

2. Evolution of the Modified Fluid Impulse of a localized disturbance

As has already been discussed in the §1, the objective of this investigation was, in particular, to calculate the dynamics of fluid impulse of localized vortex in the external shear flow without using the vorticity subdivision procedure employed in L&LC. Indeed, the general solution of the vorticity dynamics equations, presented in §2.2.1, makes it possible to calculate the MFI defined by (2.4) at an arbitrary instant of time.

For illustrative purposes we avail ourselves of the Gaussian vortex model (2.15) introduced above. It should be noted that the theory by L&LC is insensitive not only to the vortex disturbance amplitude but also to its form. Therefore, the initial disturbance can be chosen rather arbitrarily. Choosing it in the form a Gaussian vortex (2.15) optimizes calculations substantially, but from the other side it is sufficiently representative model (see also notation in §6). The initial vortex is shown in figure 1, portraying the enstrophy isosurface \( \omega^2(0, r) = \text{const} \) that represents the surface of a torus.
FIGURES

FIG. 1. Surface of constant enstrophy $|\omega(0, r)|^2 = \text{const of the initial Gaussian vortex}$.

Note that for this model the size of the vortex disturbance core is specified by the value of the parameter $\delta$, and the vortex plane and vortex lines that represent concentric circles are normal to the direction of the initial fluid impulse $\mu$.

Substitution of the expressions (2.18)–(2.19) for vorticity components into the expression (2.6) for the MFI gives:

$$\tilde{p}_i(\tau) = \sum_{j=1}^{3} \tilde{\Pi}_{ij}(\tau) \mu_j,$$

$$\tilde{\Pi}_{11} = 1 + \frac{3}{2} \langle \sin^2 \phi \tan \beta_0 (\beta - \beta_0) \rangle = 1 + \frac{1}{2} I_1,$$

$$\tilde{\Pi}_{12} = \frac{3}{2} \langle \sin \phi \tan \phi (\beta - \beta_0) \rangle = \frac{1}{2} I_2,$$

$$\tilde{\Pi}_{21} = \frac{1}{2} \tau, \quad \tilde{\Pi}_{22} = 1, \quad \tilde{\Pi}_{33} = 2 - \tilde{\Pi}_{11} = 1 - \frac{1}{2} I_1,$$

(2.22)

where the angle brackets correspond to averaging over angles in the $k$-space, and $\beta_0 = \beta_0 (\tau; \beta, \phi) = \arctan (\tan \beta - \tau \cos \phi)$. In the expanded form we have

$$\tilde{p}_1(t) = \mu_1 + \frac{1}{2} \mu_1 I_1(t) + \frac{1}{2} \mu_2 I_2(t), \quad \tilde{p}_2(t) = \mu_2 + \frac{1}{2} \mu_1 \Omega t, \quad \tilde{p}_3(t) = \mu_3 - \frac{1}{2} \mu_3 I_1(t).$$

(2.23)

It is easy to obtain the asymptotic expressions for $I_1$ and $I_2$ for small ($t \ll 1/|\Omega|$) and large ($t \gg 1/|\Omega|$) times:

$$I_1(t) = \frac{1}{5} (\Omega t)^2 + O((\Omega t)^3), \quad I_2(t) = \Omega t + O((\Omega t)^3), \quad |\Omega|t \ll 1,$$

(2.24)

and

$$I_1(t) = |\Omega|t - 3 + O(1/|\Omega|t), \quad I_2(\tau) = 3[\ln(|\Omega|t) - 0.6] + O(1/|\Omega|t), \quad |\Omega|t \gg 1.$$

(2.25)

For small $t$ we have from (2.23) and (2.24):

$$\tilde{p}_1 \approx \mu_1 + \frac{1}{2} \mu_2 (\Omega t), \quad \tilde{p}_2 \approx \mu_2 + \frac{1}{2} \mu_1 (\Omega t), \quad \tilde{p}_3 = \mu_3.$$

(2.26)

As would be expected, the MFI dynamics in the case of small times is determined by the first two terms in the evolution equation (2.10), in full agreement with the theory by L&LC. This is because the initial vortex (2.15) is well localized; therefore, the contribution $J$ associated with the presence (at the early stage of evolution still very weak) vorticity “tails” at the periphery of the vortex, $\omega \sim r^{-4}$, is vanishingly small.
In the case of larger times, however, the situation changes radically. The vorticity “tails” now become quite important and begin to affect the fluid impulse dynamics.

This leads, in particular, to the fact that, according to (2.25), at large $t$ the fluid impulse increases as a power law rather than exponentially, as was the case in L&LC.

The fact that both the vorticity itself (including, of course, its “tails” produced in the course of evolution) and the MFI increase not more rapidly than as a power law, makes it possible to prove rigorously that there is no way to subdivide the disturbance vorticity field into two different components as it was suggested in L&LC. Namely, it is impossible to separate a localized vortex core, the fluid impulse of which grows exponentially, from the complete vorticity field (see § 6 for more details).

But it is the assumption about the possibility of such a separation that formed the basis of the approach suggested in L&LC.

This means that the approach by L&LC is incorrect, in spite of a number of predictions obtained on this base which show an excellent agreements with experimental findings. Consequently, the problem of constructing an adequate theory describing the dynamics of localized vortices in shear flows becomes of current importance again.

Nevertheless, it is interesting to point out that the inclination angle $\Psi$ (of the MFI vector to the positive direction of the $x$-axis tends to $45^\circ$ with the time, in exactly the same way as does the fluid impulse $p^I$ constructed from the “core vorticity” in L&LC.

Indeed, when $|\Omega|t \gg 1$ from (2.38) using (2.40) we have

$$\begin{align*}
\tilde{p}_1(t) &= -\frac{1}{2} \mu_1 + \frac{1}{2} \mu_1 |\Omega|t + \frac{3}{2} \mu_2 \left[\ln(|\Omega|t) - 0.6\right], \\
\tilde{p}_2(t) &= \mu_2 + \frac{1}{2} \mu_1 |\Omega|t, \\
\tilde{p}_3(t) &= \frac{5}{2} \mu_3 - \frac{1}{2} \mu_3 |\Omega|t,
\end{align*}$$

(2.27)

so that for vortices symmetric about the plane $z = 0$ ($\mu_3 = 0$) we obtain: $\tan \Psi = \tilde{p}_2(t)/\tilde{p}_1(t) \to 1$. Figure 2 shows the evolution of the quantities $I_1(\tau)$ and $I_2(\tau)$ as well as of the inclination angle $\Psi$ for the case $\mu = (1, 0, 0)$.

![Figure 2](image-url)

**FIG. 2.** The evolution of $I_1(\tau)$ and $I_2(\tau)$ and of the inclination angle $\Psi$ of the modified fluid impulse for the vertically oriented ($\mu_1 = 1$, $\mu_2 = 0$) symmetric ($\mu_3 = 0$) Gaussian vortex.
However, in spite of the relatively good agreement between the orientation of the vortex plane observed in experiments and its orientation following from the description of the vortex using the MFI, it does not mean at all that the MFI is a more acceptable characteristic for describing localized vortices which should replace the “unfortunate” description using the “fluid impulse of the core” $p^I$ suggested previously in L&LC.

It will be shown below that it is also possible to suggest some other methods of modifying the fluid impulse which will also be free from difficulties associated with the convergence of the corresponding integral at large distances, as is the just considered MFI; however, they will lead to a totally different scenario for the vortex geometry evolution.

Actually, this would mean that the fluid impulse is not an adequate characteristic at all for describing the evolution of localized vortices. At least for the statement of the problem of localized vortex disturbance development in the external flow with the linear velocity profile accepted here (as well as in previous publications L&LC, Levinski, Rapoport & Cohen 1995, Malkiel, Levinski & Cohen 1999 and Levinski 2000).

3. The evolution of the “Lagrangian” Modified Fluid Impulse of a localized disturbance

In this section we introduce the concept of the fluid impulse of a selected (“colored”) group of fluid particles and investigate its evolution.

Let the initial position of a fluid particle that resides at the time $\tau$ at a point with the coordinates $\mathbf{r} = (x_1, x_2, x_3)$ be designated as $\mathbf{r}_0$: $\mathbf{r} (\tau = 0) \equiv \mathbf{r}_0$, where the components of the vector $\mathbf{r}_0$ be $s_1, s_2$ and $s_3$: $\mathbf{r}_0 = (s_1, s_2, s_3)$. Then $x_1 = s_1 - s_2 \tau$, $x_2 = s_2$, $x_3 = s_3$, We now select a group of particles which at the initial instant of time are enclosed within a sphere of radius $R$ and mentally paint it: $\mathbf{r}_0 \equiv \sqrt{s_1^2 + s_2^2 + s_3^2} \leq R$. We shall keep track on this painted group and calculate its fluid impulse. At subsequent instants of time, when $\tau \neq 0$, the painted sphere will transform to an ellipsoid $(x_1 + x_2 \tau)^2 + x_2^2 + x_3^2 \leq R^2$.

Thus the fluid impulse of the painted group of particles is

$$\dot{p}_i (\tau, R) = \frac{1}{2} \epsilon_{ijk} \int \mathbf{r} \cdot \mathbf{v}_k (\tau; \mathbf{r}) \, dV,$$

where the integral is taken over the volume of the ellipsoid. By letting further $R \to \infty$ we obtain for the LMFI components: $\dot{p}_i = \dot{\Pi}_{ij} \mu_j$. Omitting the explicit expressions for $\dot{\Pi}_{ik}$, we present here only their asymptotic expressions for $\tau \gg 1$:

$$\begin{align*}
\dot{\Pi}_{11} & \approx -\frac{1}{6} \tau + 2, \quad \dot{\Pi}_{12} \approx -\frac{1}{6} \tau + \frac{3}{2} \ln \tau - 0.414, \quad \dot{\Pi}_{21} \approx -\frac{1}{6} \tau^2 + \frac{4}{3} \tau - \frac{3}{4} \ln \tau, \\
\dot{\Pi}_{22} & \approx -\frac{1}{6} \tau^2 + \frac{1}{2} \tau \ln \tau + 0.026 \tau, \quad \dot{\Pi}_{33} \approx -\frac{1}{2} \tau \ln \tau + 0.64 \tau.
\end{align*}$$

(2.28)

It is evident that for large $\tau$

$$\dot{p}_1 \approx -\frac{1}{6} \tau (\mu_1 + \mu_2), \quad \dot{p}_2 \approx \tau \dot{p}_1 \approx -\frac{1}{6} \tau^2 (\mu_1 + \mu_2), \quad \dot{p}_3 \approx -\frac{1}{2} \tau \ln \tau \mu_3.$$

We can now easily calculate the inclination angle $\Psi$ of the vector $\dot{\mathbf{p}} = (\dot{p}_1, \dot{p}_2, \dot{p}_3)$ in the $(xy)$-plane for different orientation angles of the initial fluid impulse $\dot{\mathbf{p}}(0) (\equiv \mathbf{\mu})$: $\tan \Psi (\tau) = \dot{p}_2 (\tau)/\dot{p}_1 (\tau) = [\Pi_{21} \mu_1 + \Pi_{22} \mu_2]/[\Pi_{11} \mu_1 + \Pi_{12} \mu_2]$. 

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Figure 3 shows the evolution of $\Psi(\tau)$ for 8 orientations of $\mu$: $\alpha_n = (n - 1) \cdot 45^\circ$.

It is evident that at large times the two-dimensional vector $\hat{p} = (\hat{p}_1, \hat{p}_2)$ is directed vertically. The vortices, the directions of the initial fluid impulse of which lie in figure 3 in the shaded and unshaded areas are directed at large $\tau$ downward and upward respectively. It is interesting to note that over a sufficiently long time interval $2 < \tau < 6$ the inclination angle is about $45^\circ$ or $225^\circ$ (according to the initial inclination angle $\alpha$).

Summary. Thus it is evident that two different methods of modifying the fluid impulse, MFI and LMFI, lead to two totally different results as regards their orientation at asymptotically large times: $45^\circ$ for MFI, and $90^\circ$ for LMFI. It will be recalled that we associated intuitively the orientation of the vortex plane with the orientation of the fluid impulse vector $p$ by assuming that, as for the usual dipole structure (such as in magnetostatics if we mean the analogy: $u \rightarrow H$, $\omega \rightarrow j$, $p \rightarrow m$, where $H$ is the magnetic field, $j$ is electric current density, and $m = \frac{1}{2} \int (r \times j) \, dV$ is the magnetic dipole moment), this plane must simply be normal to the fluid impulse direction (L&LC) just as the plane of a ringlet with current is perpendicular to the dipole magnetic moment.

It now becomes clear, however, that the fluid impulse in this problem just cannot describe adequately the vorticity distribution. By choosing in a different manner the form of the domain of integration, we can obtain for the same vorticity distribution not only an arbitrary time dependence of its fluid impulse but also an arbitrary inclination of the vortex plane.

For that reason, there inevitably arises the problem of calculating a complete vorticity field. It is a fairly complicated numerical problem which is being solved to date (preliminary results of these calculations are presented in Suponitsky et al., 2003, 2004), however within the linear approximation, it is actually solved (for single plane wave) by F&I and also in §2.2.1. It will now suffice to perform an inverse Fourier-transform and calculate the vorticity field in the physical space. This is done in §3.
III. CALCULATING THE COMPLETE VORTICITY FIELD IN PHYSICAL SPACE

We have

\[ \omega_i(r) = \int \omega_i(k) \exp(ikr) \, d^3k, \]  

(3.1)

where

\[ \omega_i(k) = \frac{i}{(2\pi)^3} \rho \zeta_i(k), \quad \zeta_i(k) = \hat{\zeta}_i(\beta, \phi; \tau) \exp(-\frac{1}{4}Q^2\delta^2), \]

(3.2)

and the quantities \( \hat{\zeta}_i \) are specified by the expressions (2.19).

The most compact method for evaluating the integral of (3.1) lies in passing from integrating in the \( k \)-space, to integrating in the \( Q \)-space (i.e. in the space of initial wave numbers). Introducing the spherical coordinates \( r_0, \theta_0 \) and \( \phi_0 \) of the point \( r_0 \):

\[
\begin{align*}
    r_0 &= \sqrt{(x_1 + x_2\tau)^2 + x_3^2}, \quad \cos \theta_0 = \frac{x_2}{r_0}, \\
    \cos \varphi_0 &= \frac{x_1 + x_2\tau}{\sqrt{(x_1 + x_2\tau)^2 + x_3^2}}
\end{align*}
\]

(3.3)

and also the angle \( \Theta_0 \) between the vectors \( Q \) and \( r_0 \):

\[ \cos \Theta_0 = \cos \theta_0 \sin \beta_0 + \sin \theta_0 \cos \beta_0 \cos(\phi - \varphi_0). \]

(3.4)

we obtain

\[
\omega_i(\tau; r) = -\frac{2}{\pi^{5/2}} \int_0^{\pi/2} \cos^2 \beta_0 \, d\beta_0 \]

\[
\times \int_0^{2\pi} \frac{d\phi}{D^4} \, \hat{\zeta}_i(\beta_0, \phi; \tau)(r_0 \cos \Theta_0) \left( \frac{3}{2} \frac{r_0^2 \cos^2 \Theta_0}{D^2} \right) \exp \left( -\frac{r_0^2 \cos^2 \Theta_0}{D^2} \right). \]

(3.5)

In spite of the fact that the expression (3.5) is a sufficiently compact one, it still is very difficult for analysis, as it includes double integrals. For that reason, we have to carry out the subsequent analysis numerically.

Note that the applicability of linear theory is limited by the condition \( |\omega|_{\text{max}} \ll |\Omega| \). This means, in particular, that this condition must also be satisfied for the initial vortex, i.e.

\[ |\omega|_{\text{max}} = |\omega(r = \delta/\sqrt{2})| = \sqrt{\frac{2}{\pi^3 c \delta^4}} = 0.154 \frac{\mu}{\delta^4} \ll \Omega, \quad \text{or} \quad \mu \ll 6.49 \Omega \delta^4. \]

In order to investigate the linear evolution of the vortex, we calculated numerically the vorticity field distribution for fixed instants of time \( \tau \) by formula (3.5). Results are presented in Fig. 4 in the form of 3-D isosurfaces of absolute value of vorticity (or, that is the same, of the enstrophy density \( L \)) for fixed instants of time \( \tau \): \( L(\tau; r) \equiv |\omega(\tau; r)|^2 = \text{const} \).
FIG. 4. The linear evolution of the Gaussian vortex: (a), (b) – horizontal and (c) – vertical. Isosurfaces of absolute value of vorticity are shown $\omega(\tau; r) = 0.7 \omega_{\text{max}}(\tau) = \text{const}$, where $\omega_{\text{max}}(\tau)$ is maximum (over volume) absolute value of vorticity at time $\tau$.

From here on we shall confine ourselves to the case of symmetric initial vortices, $\mu_3 = 0$. Furthermore, as is easy to understand, the vortex remains symmetric about the plane $z = 0$ over the course of all subsequent evolution as well. Specifically, for the enstrophy density we have $L(\tau; x, y, z) = L(\tau; x, y, -z)$.

It is apparent from figure 4 that with the passage of time, the initial torus, corresponding to the Gaussian vortex, starts to rotate and deform and eventually turns into two symmetric “sausages” extended along the flow. (Note that only in this figure it is assumed that $-\Omega \equiv dU/dy > 0$ in order to achieve a more usual perception of the vortex plane orientation.)

These “sausages” could, in principle, serve as a source material for the hairpin legs. It can be shown, however, that within the framework of linear theory the sausages cannot turn into a hairpin through the formation of a bridge (a so-called hairpin head) near only one of the ends of the pair of legs. It turns out that this is forbidden by the symmetry properties of the basic equations!

Indeed, it follows from the linearized set of equations (1.3) that if $\omega(0; r) = -\omega(0; -r)$ and $u(0; r) = u(0; -r)$, i.e. if all components of the initial vorticity change their sign with $r \to -r$ and, accordingly, all components of the initial velocity do not alter their values in the case of the substitution $r \to -r$, then this symmetry property remains during the vortex (linear) evolution:

$$\omega(t; r) = -\omega(t; -r) \quad \text{and} \quad u(t; r) = u(t; -r).$$

(3.6)
Note that the Gaussian initial vortex does possess the aforementioned symmetry properties (3.6).

Consequently, for the enstrophy density \( L(t; r) \) of the initial Gaussian vortex at an arbitrary instant of time \( t \) we have \( L(t; x, y, z) = L(t; -x, -y, -z) \). In the case of a symmetric vortex, \( \mu_3 = 0 \), we have an additional \( z \)-symmetry: \( L(t; -x, -y, -z) = L(t; -x, -y, z) \). Consequently, we obtain \( L(t; x, y, z) = L(t; -x, -y, z) \). This means that a current enstrophy distribution must be invariant with respect to the simultaneous replacement \( x \to -x, y \to -y \) even for the same \( z \). Specifically, directly “between the legs”, i.e. in the plane \( z = 0 \), we have

\[
L(t; x, y, 0) = L(t; -x, -y, 0).
\] (3.7)

Hence we cannot obtain the “hairpin” in the course of the Gaussian vortex evolution, since the distribution of enstrophy in hairpin vortex does not have the symmetry property (3.7).

However, the situation is changed drastically if we include in consideration the nonlinear terms in equation (1.3)a). It is easy to see that the nonlinear terms (underlined in (1.3a)) totally destroy the symmetry properties of the linearized version of equation (1.3a) and, hence, the prohibition for the hairpin in linear theory is removed!

Therefore, (numerical) investigation of the nonlinear evolution stage of a localized vortex is strongly needed. Preliminary results of numerical calculations with strong vortices confirm the occurrence of hairpins at a definite evolution stage of the vortex (Suponitsky et al., 2003, 2004).

IV. TOTAL ENSTROPHY OF A LOCALIZED VORTEX AND ITS GROWTH

A. Calculating the total enstrophy of a localized vortex in inviscid flow

In order to be able to describe the enhancement or attenuation of the vortex over the course of the evolution, we introduce, as one of its integral characteristics, the total enstrophy:

\[
\omega^2 = \int \omega^2(r) \, dV
\] (4.1)

The total enstrophy \( \omega \) can serve as the measure of vortex intensity. Note that, at first glance, it seems more natural to take, as the measure of vortex intensity, its total energy \( E \). However, an attempt to introduce its reasonable definition, like \( E = \int [(U + u)^2 - U^2] \, dV \), runs into the same difficulty into which we ran in our attempt to introduce the fluid impulse. Indeed, since \( \int (U \cdot u) \, dV \) is divergent (remember that \( u \sim r^{-3} \) for large \( r \)), such a definition of the vortex energy cannot be recognized as correct. It is also easy to see that a similar introduction of the total enstrophy \( \omega \) is free from such difficulties because \( \int \omega \, dV = 0 \).

The total enstrophy \( \omega \) depends on the time \( \tau \) as well as on the initial fluid impulse \( \mu \). If the viscosity \( \nu \) is included, then \( \omega \) depends also on the Reynolds number \( Re \). It is clear that finite viscosity effects are highly important for the problem of the maximal vortex enhancement (cf. also with F&I), especially in the connection with the problem of hairpin formation in the course of nonlinear evolution of initial weak vortex (which is partly
described by Suponitsky et al., 2003, 2004 and will be considered in details in the following publication).

Thus we have $\mathcal{L} = \mathcal{L}(\tau, \mu)$. We can express the integral in terms of Fourier-variables:

$$\mathcal{L} = (2\pi)^3 \int |\omega(k)|^2 \, d^3 k. \tag{4.2}$$

and use the corresponding expressions for Fourier-components of vorticity. We also introduce the normalized total enstrophy:

$$\hat{\mathcal{L}}(\tau, \mu) = \mathcal{L}(\tau, \mu) / \mathcal{L}_0(|\mu|), \tag{4.3}$$

where $\mathcal{L}_0(|\mu|) = (2\pi)^{-3/2} \delta^{-5} |\mu|^2$ is the total enstrophy when $\tau = 0$. In the linear problem the normalized enstrophy $\hat{\mathcal{L}}$ depends only on the direction of $\mu$. Consequently, it can be put without loss of generality that $|\mu| = 1$.

Finally, for the normalized total enstrophy we obtain

$$\hat{\mathcal{L}}(\tau, \mu) = \ell_{ij}(\tau) \mu_i \mu_j. \tag{4.5}$$

B. Enhancement (attenuation) of the vortex and the relation to the hydrodynamic stability problem

When $\tau \gg 1$ we obtain

$$\ell_{ij} \approx l_{ij} \tau^2 + O(\tau \ln \tau), \tag{4.4}$$

$$\begin{align*}
    l_{11} &= \frac{9\pi^2 + 32}{288} \approx 0.41954, \quad l_{12} = \frac{5}{9} \approx 0.55556, \\
    l_{22} &= \frac{9\pi^2 - 40}{36} \approx 1.35629, \quad l_{33} = \frac{3\pi^2}{32} \approx 0.92529.
\end{align*}$$

For the normalized enstrophy, when $\tau \gg 1$ we have

$$\hat{\mathcal{L}}(\tau) \approx a(\mu) \tau^2, \quad a(\mu) = l_{11} \mu_1^2 + 2 l_{12} \mu_1 \mu_2 + l_{22} \mu_2^2 + l_{33} \mu_3^2. \tag{4.5}$$

By analyzing the coefficient $a(\mu)$, we can readily find the orientation of the vector $\mu$ corresponding to those initial vortices which will become the most enhanced at large times. By fixing $|\mu| = 1$, we find that $a$ is maximum when $\mu = (\cos \alpha, \sin \alpha, 0)$, where

$$\alpha = \frac{\pi}{2} + \frac{1}{2} \arctan \left( \frac{2l_{12}}{l_{11} - l_{22}} \right) \approx 65.07^\circ$$

and is $a = a_{\text{max}} = \frac{1}{2} \left[ (l_{11} + l_{22}) + \sqrt{(l_{11} - l_{22})^2 + 4l_{12}^2} \right] \approx 1.6146$.

Note also that the angle $\sim 155.07^\circ$ with $a_{\text{min}} \approx 0.1613$ corresponds to the orientation of the least enhanced (at large times) vortices, i.e. the normalized enstrophy of the vortices, the initial orientation angle of which is directed along this direction, will be an order of magnitude smaller than the maximum one.

Further, it is assumed again that $\mu_3 = 0$. In this case it will suffice to describe the orientation of the initial vortex by only one inclination angle of its fluid impulse $\alpha$: $\mu_1 = \ldots$
\cos \alpha, \ \mu_2 = \sin \alpha, \ \mu_3 = 0. \text{ In other words, the angle } \alpha \text{ is the angle between the positive direction of the } x\text{-axis and the direction of the initial fluid impulse } \mu.\]

For an arbitrary time \( \tau \) the maximum (over the whole range of initial orientation angles \( \alpha \)) enhancement of the enstrophy corresponds to the initial inclination angle \( \alpha(\tau) \)

\[
\alpha_{opt}(\tau) = \frac{\pi}{2} + \frac{1}{2} \arctan \left[ \frac{2 \ell_{12}(\tau)}{\ell_{11}(\tau) - \ell_{22}(\tau)} \right]. \tag{4.6}
\]

A numerical calculation shows that \( \alpha_{opt}(\tau) \) increases from \( \alpha_{opt}(0) = 45^\circ \) to \( \alpha_{opt}(\infty) = 65.07^\circ \)

![FIG. 5. The inclination angle \( \alpha_{opt}(\tau) \) of the fluid impulse vector for which a maximum enhancement of the enstrophy is reached by a given time \( \tau \).](image)

Figure 6 shows the normalized enstrophy as a function of inclination angle \( \alpha \) for four values of \( \tau \): 0; 0.5; 1.0 and 5.0.

![FIG. 6. The Normalized enstrophy (radial coordinate) as a function of inclination angle of the initial fluid impulse \( \alpha \) (angular coordinate) for four values of \( \tau \): \( \tau = 0 \); 0.5; 1.0 and 5.0.](image)
The calculations show also that at large $\tau$ almost all enstrophy is concentrated in horizontal directions of vortex motion $L_1 \sim L_3 = O(\tau^2 L_2)$. This fact is intuitively consistent with results of numerical calculations of the isosurfaces of enstrophy density (see §3, as well as §5) which show that the localization plane of a weak vortex becomes horizontal asymptotically.

As it follows from (4.5) for all possible orientation angles of the initial vortex we obtain a power-law ($L \sim \tau^2$) growth of the total enstrophy (cf. with law of energy growth of the optimal inviscid excitations in F&I).

We may regard such an asymptotic increase in enstrophy as the manifestation of an instability – the initial vortex starts to be enhanced infinitely (in linear theory at least).

There is nothing surprising in this fact of a power-law enhancement of enstrophy. It is merely a reflection of another well-known fact, namely, that Couette inviscid flow is unstable with respect to 3-D wave disturbances, and this instability is just a power-law one, or, as it is also called, an “algebraic” instability (Ellingsen & Palm 1975). Recall that Couette flow is stable with respect to 2-D wave disturbances ($k_3 = 0$) altogether (that is, there is no not only the exponential instability but also even the algebraic instability).

Note, that the last factor, together with the well-known Squire (1933) theorem stating that if the flow under consideration is stable with respect to 2-D disturbances it is necessarily stable with respect to 3-D disturbances), has long led to a paradox laying in the inconsistency between the experimental fact of the existence of turbulence in Couette flow and the absence of any instability in theory (for more detail see an excellent review by Henningson, Gustavsson & Breuer 1994). This paradox was just resolved by the discovery of the power-law instability ($\omega_1(\tau, k) \sim \omega_3(\tau, k) \sim \tau$, $\omega_2(\tau, k) \sim 1$) of 3-D disturbances caused by a so-called lift-up effect (Landahl 1975) which wasn’t taken into account in the proof of Squire’s theorem.

A localized vortex can be presented as a wave packet composed of 3-D wave disturbances. It is based on this that we have obtained here a power-law increase of the total enstrophy of the vortex.

Precisely the same statement applies for the evolution of a localized vortex on the background of Taylor-Couette (circular) flow considered by Malkiel, Levinski & Cohen 1999. Here also can not be an exponential growth of the localized vortex in the range of parameters where the hydrodynamic stability theory predicts a stability. Now we have obtained the exact solution which describes the development of the weak vortex in the circular flow. It would be presented in separate paper.

It is this fact that reflects the main conflict of the theory L&LC with well-known facts of the classical theory of hydrodynamic stability mentioned in §1.

Note also that including the viscosity is all the more unable to lead to an exponential growth of the vortex. On the contrary, the presence of viscosity leads to the fact that the vortex at some stage of its evolution can cease to increase and subsequently begin to be dissipated (see also F&I). Moreover, a sufficiently large viscosity unavoidably lead to the finiteness of the lifetime of a weak vortex. If, however, the viscosity is not too large and the initial amplitude of the vortex is not too small, the nonlinearity can come into play still before the vortex begins to be decay. This issue requires the further (numerical, of course) investigation.
C. Enhancement of the enstrophy and Theodorsen’s idea of the predominant formation of 45-degree vortices

As early as five decades ago Theodorsen (1952) came up with the hypothesis explaining why the 45-degree direction of orientation of horseshoe vortices, that is, the vortices whose plane is inclined at 45° to the basic flow direction, dominates in experiments.

Since Theodorsen’s (1952) idea is most clearly presented not in his work itself but in a later publication by Head & Bandyopadhyay (1981), we shall follow the presentation of this paper by adapting it to the present case of a linearized problem and an external flow with \( dU/dy = -\Omega = \text{const.} \)

We avail ourselves of the basic equation of the theory (1.3a) omitting the nonlinear (underlined) terms. A scalar multiplication of this equation by \( \omega \) gives the equation describing the enstrophy dynamics of a fluid particle:

\[
\frac{d}{dt} \left( \frac{1}{2} \omega^2 \right) = -\Omega \omega_1 \omega_2 + \frac{\Omega}{\omega} \frac{\partial u}{\partial z} + \nu (\omega \Delta \omega),
\]

(4.7)

where \( d/dt = \partial/\partial t + (U \nabla) \) is the Lagrangian time derivative. The right-hand side of (4.7), with the exception of the viscous term, is the linearized “stretching term” \( \omega^t_1 \omega^t_j \partial u^t_i / \partial x_j \), where the superscript “t” corresponds to the total (non-linearized) value of the physical quantity, \( \omega^t \equiv \Omega + \omega, \ u^t \equiv U + u \).

By analyzing the first term on the right-hand side, it is easy to see that it is maximal when the two-dimensional vector \( (\omega_1, \omega_2) \) (at its fixed absolute value) is directed at the angle of 135° to the positive direction of the axis \( x \), which corresponds to the angle of 45° measured from the direction of the mean velocity \( U \) in the upper half-space \( y > 0 \). (We are reminded that \( dU/dy < 0 \) corresponds to positive values of \( \Omega \).

On the basis of this undeniable fact, Theodorsen arrived to the conclusion that the concentration of the vortex (its enstrophy) would also be maximal in the plane oriented at the angle of 45° to the flow, and this does correspond to experimental findings.

Over the course of the past five decades Theodorsen’s hypothesis has been repeatedly subjected to criticism from different standpoints. Here we want to discuss only two aspects, based on the just obtained (on the basis of the exact solution) results on the evolution of total enstrophy, results of calculations of the 3-D vorticity field presented in § 3, as well as on the results of calculations of the vortex localization plane inclination which will be described later in § 5.

The first aspect implies that the conclusion about the greatest buildup rate of 45-degree vortices was drawn, strictly speaking, from analyzing the structure of only one of the terms on the right-hand side of (4.7). Our analysis shows, however, that the second term, \( \Omega (\omega \partial u / \partial z) \), that represents the other part of the “stretching term” responsible for the distortion of the flow velocity field caused by the vortex is also important. Although the integral contribution of this term to the total enstrophy is exactly zero at the initial instant of time, it becomes substantially larger with the time and can compete with the integral contribution of the first term.

Thus the obviously true statement about the role of the stretching term made by Theodorsen can be applied, strictly speaking, only to the initial instants of time, \( \tau \ll 1 \),

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when the distortion of the flow velocity field still can be neglected. Which one of the initial vortices will turn out to be the most enhanced for a sufficiently large time, in the course of which the direction of velocity is significantly changed, now becomes quite unclear from the reasoning presented.

And the second aspect, which is of course associated with the first one, implies that at this point, taking into consideration the change of the velocity field orientation (actually neglected in Theodorsen’s discussion), it becomes totally unobvious that the initial vortex with the optimal 45°-orientation, which at the initial instant of time was enhanced faster than all the others, would not change the orientation of its plane in the course of evolution.

From the results presented in §4.2 (remember that they refer to the inviscid case) that are most instructively illustrated in figures 5 and 6, it follows that at small $\tau$ the 45-degree vortices are indeed the strongest. However, at larger times, quite different vortices turn out to be most strongly enhanced, i.e. those for which at the initial instant of time the localization plane was more strongly pressed against the flow direction). In the limit $\tau \gg 1$ the vortices, which initially were inclined at $90° - \alpha_{opt}(\tau = \infty) = 25°$ rather than $90° - \alpha_{opt}(\tau = 0) = 45°$ turn out to be the strongest.

It should be noted at this point that, if the relatively small differences between the predicted angles of maximum enhancement are not taken into consideration, Theodorsen’s hypothesis is plausible enough.

As can be shown that including of the viscosity is also in favor of this hypothesis. At a finite Reynolds number, $Re = \Omega \delta^2 / \nu$, the difference between the angle of maximum enhancement and the angle of 45°, as predicted by Theodorsen, becomes still smaller. This can be most easily understood from the expression for total enstrophy at small $\tau$ obtained with the including of viscosity (the details of its derivation are omitted):

$$\hat{L}(\tau; \alpha; Re) = 1 + \frac{1}{2} \tau \left[ \sin(2\alpha) - \frac{20}{Re} \right] + O(\tau^2). \tag{4.8}$$

It follows from (4.8) that initially the 45-degree (or, what is the same, 225-degree) vortices are the strongest. And the vortices whose plane is inclined at 135° are, on the contrary, the weakest (they are even weaker than the initial vortex, i.e. $\hat{L} < 1$). It turns out that in the case of a sufficiently large viscosity the angles that are only very close to 45° are enhanced, and when $Re = Re_{cr} = 20$ the only direction, 45°, is enhanced altogether. Calculation of the total enstrophy with the including of viscosity, shows that vortices of other directions, close to 45°, will start to be enhanced with the passage of time, however, the angles of maximum enhancement remain close around 45° during all time up to the beginning of the dissipation of the vortex.

For illustration we presented in Fig. 7 the contours of $\hat{L}(\tau; \alpha)$ for viscous case with $Re = 20$ and $Re = 40$. The shaded regions of the plane $(\tau, \alpha)$ correspond to $\hat{L}(\tau; \alpha) > 1$, i.e. to the enhancement of the vortex, and the unshaded areas correspond to $\hat{L}(\tau; \alpha) < 1$, that is, to the attenuation of the vortex. We see that if $Re = 20$ the vortex begin to dissipate for all $\alpha$ (except $\alpha = 45°$ and 225°), although with the growth of time some orientations near 45° also (very weakly) enhance.
FIG. 7. Contours of the normalized enstrophy $\hat{L}(\tau; \alpha) = \text{const}$ on the plane $\tau - \alpha$ for $Re = 20$ and $Re = 40$. Unshaded areas correspond to the attenuation of the vortex, $\hat{L} < 1$.

It is interesting to note that this critical Reynolds number is very close to the value $Re_{\text{min}} = 2\pi^2 \approx 19.7$ obtained in F&I for growth of energy of optimal checkerboard excitations with $k_1 = 0$.

The situation with the second point concerning the inclination angle of the localization plane of the vortex is much less favorable. If we keep track on the evolution of the inclination angle of the enstrophy localization plane (vortex plane), we find that at large times this angle tends to zero (i.e. the plane of the vortex tends to become horizontal), rather than to $45^\circ$. This statement is illustrated in figure 4, as well as by calculations of this angle performed on the basis of the tensor of enstrophy distribution (TED) which are presented in §5.

V. THE TENSOR OF ENSTROPHY DISTRIBUTION AND VORTEX GEOMETRY

We can often avoid an unwieldy description of the vortex by specifying its total vector field if we are able to introduce some integral characteristic of the vortex which (although
not reflecting, of course, in full measure the entire vector structure of the vortex) will permit its main geometrical parameters to be described at least roughly.

For this purpose we avail ourselves of the analogy with those approaches which are used in electrostatics in describing the distribution of electric charge. This is customarily done by using so-called multipole moments.

We now introduce the notion of the Tensor of Enstrophy Distribution, TED, which is essentially a usual quadrupole moment of the enstrophy distribution.

We assume to use the following definition of the tensor (see, for example, the book by Levich 1969):

\[ T_{ij} = \int dV \omega^2(r) x_i x_j. \]  \hspace{1cm} (5.1)

Note, that if the displacement of center of vorticity distribution takes place (i.e. if \( X_j \equiv (\int dV \omega^2 x_j) / (\int dV \omega^2) \neq 0 \), as it may be in the case of strong vortices) the definition of TED must be generalized: \( T_{ij} = \int dV \omega^2(r) (x_i - X_i)(x_j - X_j) \).

As any symmetric tensor, it can be transformed to the principal axes, \( x'_i \), where it has a diagonal form:

\[ \hat{T}' = \begin{pmatrix} \lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3 \end{pmatrix}, \]  \hspace{1cm} (5.2)

Here \( \lambda_i \) stands for the eigenvalues of the matrix \( T_{ij} \), i.e. the solution of a characteristic equation \( \text{Det} |T_{ij} - \lambda \delta_{ij}| = 0 \). Then the direction of one of the principal axes that corresponds to the smallest of these three values of \( \lambda \), is the direction which should be identified with a normal to the vortex plane. And the vortex itself is extended along the direction which corresponds to the largest value of \( \lambda \) (see Fig. 8).

![FIG. 8. Illustration to the explanation of TED.](image)

It is also possible to introduce the notion of the size of the vortex \( a_i \) along the corresponding principal axes \( x'_i \):

\[ a_i = \sqrt{ \frac{T'_{ii}}{\mathcal{L}} } = \sqrt{ \frac{\lambda_i}{\int dV \omega^2(r)} } . \]  \hspace{1cm} (5.3)

Let the largest axes be referred to as \( a \), the smallest as \( b \), and let the third axis be \( c \).
For illustration of the meaning of TED let us consider the initial Gaussian vortex. We have in this case: \( T_{ij} = K (2\mu^2 \delta_{ij} - \mu_i \mu_j) \equiv K t_{ij} \), where \( K = 1/(4\sqrt{2\pi}^3/2\delta^3) \). It is easy to see (transforming TED to the principal axes) that the direction along \( \mu \) does correspond to the smallest of the three values of \( \lambda \), i.e. \( \mu^2 \), and in the plane which is normal to this direction, the eigenvalues are identical and are \( 2\mu^2 \). The size ratio along \( \mu \) and in the plane perpendicular to \( \mu \) is \( 1 : \sqrt{2} \).

Thus the TED characterizes rather substantively the distribution of enstrophy for which we have an instructive visual idea from figure 1.

Further, we again restrict our discussion to the case of a symmetric (about the plane \( z = 0 \)) vortex, \( \mu_3 = 0 \). In this case it is easy to see that the axis \( x_3 \) (axis \( z \)) remains one of the principal axes over the course of the entire evolution, and the TED has a more straightforward form

\[
\hat{T} = \begin{bmatrix}
A & C & 0 \\
C & B & 0 \\
0 & 0 & D \\
\end{bmatrix}
\]

(5.4)

In this case the tensor is transformed to the principal axes by a simple rotation of the plane \( (x_1, x_2) \) around the axis \( x_3(=x'_3) \) by an angle \( \Psi \)

\[
\tan 2\Psi = \frac{2C}{A-B'},
\]

(5.5)

and has in these axes a diagonal form. Let the axis \( x'_1 \) coincide with the shortest principal axis. Then in the new axes we obtain

\[
\hat{T}' = \begin{bmatrix}
\frac{1}{2}(A + B - S) & 0 & 0 \\
0 & \frac{1}{2}(A + B + S) & 0 \\
0 & 0 & D \\
\end{bmatrix}
\]

(5.6)

where \( S = \sqrt{(A-B)^2 + 4C^2} \) and the angle between the positive direction of the axis \( x \) and the direction of a normal to the plane of the vortex (i.e. the axis \( x'(\equiv x'_1) \))

\[
\Psi = \frac{1}{2} \arctan \left( \frac{2C}{A-B} \right) + \frac{1}{4}\pi (1 + s), \quad s = \text{sign} (A - B).
\]

(5.7)

Next, using the notion of the TED introduced above, we can employ it to calculate the geometrical characteristics of the vortex and compare them with results that follow from calculations of the 3-D vorticity field by exact formula (3.5).

Results of calculations of the vortex parameters, obtained on the basis of the TED, for four initial directions of the vector \( \mu, \alpha = 0^\circ, 45^\circ, 90^\circ, 135^\circ \) (where \( \mu = (\cos \alpha, \sin \alpha, 0) \)) are shown in Fig. 9.
FIG. 9. The temporal dependence of the vortex parameters. The “axes” $a$, $b$ and $c$ for the initial values of the inclination angle $\alpha$: (a) $\alpha = 0^\circ$, (b) $\alpha = 45^\circ$, (c) $\alpha = 90^\circ$, (d) $\alpha = 135^\circ$ and (e) the inclination angle $\Psi$ as a function of $\tau$ for these four initial values.

It is evident that at sufficiently large $\tau$ the normal to the plane of the vortex is nearly vertical, $\Psi \approx \pi/2 - 1/\tau$.

In order to estimate the effectiveness of TED with respect to description of the vortex geometry, the parameters of the vortex geometry obtained from TED calculations were compared with results following from calculations of the complete vorticity field by exact formula (3.5).

An excellent agreement of the inclination angles $\Psi$ following from the TED with the actual inclination angles of the planes of enstrophy localisation was shown by Suponitsky, Cohen & Bar-Yoseph (2003, 2004).

Thus it can be stated that the TED is a rather convenient and reliable integral characteristic for the description of the vortex dynamics. Of course, it is unable to describe the vector structure of the vortex, yet it can be used to obtain a sufficiently great deal of information about the vortex.

The results on the orientation of vortex plane, obtained in this paragraph, unlike the findings concerning the evolution of total enstrophy, described in §4 are virtually insensitive to the presence of viscosity. (It follows from our calculations of TED in viscous case, not
VI. DISCUSSION

As has been pointed out in the §1, the motivation for this investigation was the analysis of a theoretical model suggested by Levinski (1991) for explaining the evolution mechanism of localized vortices observed in turbulent boundary layers. A key point in this model implies separating the complete vorticity field into the concentric vorticity with vortex lines enclosed within the region immediately surrounding the initial vortex disturbance, and the vorticity field associated with vortex “tails” which are produced in the process of evolution of the initial vortex disturbance. It should be noted that the possibility of such a separation is not strictly substantiated mathematically but is accepted in Levinski (1991) and in subsequent publications (Levinski & Cohen 1995; Levinski, Rapoport & Cohen 1997; Malkiel, Levinski & Cohen 1999 and Levinski 2000) as a physically justified hypothesis. The criterion of correctness of this approach comes from the agreement between results of theoretical analysis and experimental results. In particular, the predictions obtained on the basis of the model suggested in L&LC for rotating Couette flow were confirmed experimentally by Malkiel, Levinski & Cohen (1999).

In this paper the hypothesis about the possibility of separating the vorticity is verified by constructing the complete vorticity field at an arbitrary instant of time for a small amplitude localized disturbance. The problem of the evolution of a weak localized disturbance is analyzed on the basis of exact solution for the external constant shear flow To ease the subsequent analysis of the vorticity field, the initial vortex disturbance was represented by “Gaussian vortex” (2.15) that specifies a very simple localized vortex, having the structure of a vortex dipole.

It has been shown that in accordance with the classical stability theory results (see, for example, Dikii 1976), the vorticity amplitude increases not faster than it does as a power-law. This result contradicts the exponential growth of the fluid impulse obtained in L&LC for the “core” of the vortex disturbance identified in a special way. This could be accounted for by the fact that the generation of a new vorticity in the process of evolution of the vortex disturbance can lead to a fast increase of the “mass” of the vortex “core”. It is this phenomenon that is observed in visualizing vortex structures in turbulent boundary layers. Specifically there is a rapid growth (in the sense of the geometrical growth) of hairpin vortices which represent localized vortex dipoles.

In order to analyze this possibility, we introduce the notion of the modified fluid impulse (MFI) defined as an integral of the dipole moment of vorticity over the infinite spherical volume. This definition coincides formally with the definition of the fluid impulse used in L&LC, but, unlike the latter, it is defined for the complete vorticity field. In doing this, we, using only the property of the localized character of the disturbance and without imposing constraints on its amplitude, show that if the MFI exists at the initial instant of time, then it exists also at any subsequent instant of time and does not depend on the particular coordinate system chosen.

An analysis of the MFI behavior over large times $t \gg 1/|\Omega|$ shows that for any initial vortex orientation the MFI increases not faster than linearly with the time.
This result enables us to verify in a direct manner the hypothesis proposed in L&LC concerning the possibility of separation of the enclosed concentrated vorticity (localized vortex “core”) from the complete disturbed vorticity field. Indeed, let us assume that such a separation is possible, that is, \( \omega = \omega^I + \omega^{II} \), where \( \omega^I \) describes the vortex “core”, and \( \omega^{II} \) describes the vortex “cloud” that includes all “tails” of the complete vorticity field. For each vorticity field, one can determine, in accordance with the expression (2.4), its modified fluid impulse, so that \( \tilde{p} = \tilde{p}^I + \tilde{p}^{II} \) Furthermore, as in L&LC, \( \tilde{p}^I \) is a true fluid impulse. Accordingly, equation (2.10) that describes the dynamics of MFI defined for the complete vorticity field, breaks down into two equations

\[
\frac{dp^I}{dt} = -\frac{1}{2} \tilde{p}^I \frac{dU_i}{dx_j} - \frac{1}{2} \tilde{p}^I \frac{dU_j}{dx_i},
\]

\[
\frac{dp^{II}}{dt} = -\frac{1}{2} \tilde{p}^{II} \frac{dU_i}{dx_j} - \frac{1}{2} \tilde{p}^{II} \frac{dU_j}{dx_i} + \lim_{R \to \infty} J_i(R).
\]

Note that, according to the separation condition, the “tails” of the complete vorticity field make a contribution to the dynamics of \( p^{II} \) only.

An exponential growth of \( \tilde{p}^I \) follows from equation (6.1). The fact that the sum \( \tilde{p}^I + \tilde{p}^{II} \) increases not faster than as a power-law, implies that \( \tilde{p}^{II} \) also increases exponentially fast. It should be noted here that the term \( J \) in equation (6.2), describing the contribution from the vorticity “tails” to the dynamics of \( \tilde{p}^{II} \) grow also not faster than as a power-law. Thus the main contribution to the MFI dynamics for the field \( \omega^{II} \) is made by the region that immediately surrounds the vortex “core” and, hence, the assumption about the possibility of separating the vortex core is invalid.

On the other hand, the fluid impulse, defined for the complete vorticity field, cannot be an adequate characteristic of the evolution of a localized vortex. The formal reason is the fact that the volumetric integral involved in the definition of the fluid impulse is not absolutely convergent, and its value depends on the form of the integration domain when its size is made tend to infinity. In this paper this is illustrated by a comparison of the asymptotic values of the fluid impulse for two cases where the region of integration represents a spherical volume, first in Euler coordinates, and then in Lagrangian coordinates.

In summarizing all attempts to describe the evolution of a localized vortex in the external shear flow, it can be stated that using the moments of the vorticity field in this problem is unjustified.

In order to be able to describe the enhancement or attenuation of the vortex and the variation of its orientation the course of the evolution, we have analyzed the evolution of the total enstrophy of the vortex (4.1) and of the tensor of enstrophy distribution (5.1). They permit the evolution of the vorticity amplitude and the main geometrical characteristic of the vortex to be described by means of only a few independent parameters.

In particular, the effectiveness of the description of the vortex on the basis of the tensor of enstrophy distribution (TED) can be demonstrated by comparing visual pictures of enstrophy density isosurfaces \( |\omega(r)|^2 = \text{const} \), constructed on the basis of the exact solution for the complete vorticity field, with what follows from the description based on TED for the inclination angles of the plane of the vortex. Thus the TED is a reliable alternative
(to the fluid impulse) integral characteristic that enables an instructive representation of its evolution, instead of an unwieldy description using the complete vorticity field.

Thus, the calculations done in this paper show that the linear evolution results in a pair of rollers lying in the horizontal plane and aligned along the flow. It is interesting to note that this result is in excellent agreement with the finding reported in F&I where the evolution of the form of energetic isosurfaces was calculated for optimal checkerboard perturbation. In spite of the large difference of the initial perturbations considered in our paper and in F&I, the outcome of their evolution turned out to be strikingly alike (see figures 9 and 10 in F&I). This indicates the universal character of the mechanisms giving rise to ordered structures in the course of the evolution of 3-D perturbations in shear flows, as declared in F&I, and also lends support to the idea that the particular form of the initial vortex selected in our paper is not very important.

However, these results that follow from the exact solution of the evolution problem for the small amplitude localized vortex, contradict the known experimental facts obtained by visualizing hairpin vortices developing in turbulent boundary layers (Head & Bandyopadhyay 1981) or artificially synthesized in laminar boundary layers (Acalar & Smith 1987a,b).

Moreover, as shown in § 3, the symmetry properties of the basic equations in the linear case, in principle, do not allow the formation of hairpin vortices. For that reason, (numerical) investigation of the nonlinear stage of evolution of a localized vortex is of utmost current importance. Preliminary results of numerical simulations with strong vortices confirm the occurrence of hairpins at a certain stage of vortex evolution (Suponitsky et al, 2003, 2004).

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