Cosmological number density in depth from V/V_m distribution

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Abstract. Using distribution p(V/V_m) of V/V_m rather than just mean <V/V_m> in V/V_m- test leads directly to cosmological number density n(z). Calculation of n(z) from p(V/V_m) is illustrated using best sample (of 76 quasars) available in 1981, when method was developed. This is only illustrative, sample being too small for any meaningful results.

Keywords: V/V_m . luminosity volume . cosmological number density . V/V_m distribution

Luminosity-distance and volume
For cosmological populations of objects, distance is measured by (monochromatic) luminosity-distance ℓ_ν(z) (at frequency ν), function of redshift z of object. Similarly, volume of sphere passing through object and centered around observer is (4.π/3).v(z).

Calculation of limiting redshift z_m
For source of (monochromatic radio) luminosity L_ν, flux density S_ν, (radio) spectral index α (≡ dlog S_ν / dlog ν), and redshift z, L_ν = 4.π.ℓ_ν^2(α, z).S_ν. For survey limit S_0, value of limiting redshift z_m is given by ℓ_ν^2(α, z) / ℓ_ν^2(α, z_m) = S_0 / S_ν ≡ s, 0 ≤ s ≤ 1, for source of redshift z and spectral index α. For simplest case, [ℓ_ν(α, z) / ℓ_ν(α, z_m)]^2 = s has single finite solution z_m for given α, z and S_ν, S_0. Different values z_m correspond to different L_ν(α).

Relating n(z) to p(V/V_m)
Let N(z_m).dz_m represent number of sources of limiting redshifts between z_m and z_m + dz_m in sample covering solid angle ω of sky. Then 4.π.N(z_m) / ω is total number of sources of limit z_m per unit z_m-interval. Since volume available to source of limit z_m is V(z_m) = (4.π/3).(c / H_0)^3.v(z_m), (where speed of light c and Hubble constant H_0 together determine linear scale of universe,) number of sources (per unit z_m-interval) per unit volume is

{3.N(z_m) / ω}.(H_0 / c)^3.(1 / v_m), where v_m ≡ v(z_m). Let n_m(z_m, z) be number of sources / unit volume / unit z_m-interval at redshift z. Then, n(z) = ∫_z^∞ dz_m. n_m(z_m, z), and n_m(z_m, z) = \{3.N(z_m) / ω\}.(H_0 / c)^3.(1 / v_m).p_m(v(z) / v(z_m)) for 0 ≤ z ≤ z_m, where p_m(x) is distribution of x ≡ V/V_m for given z_m. For z > z_m, n_m(z_m, z) = 0, since sources with limiting redshift z_m cannot have z > z_m. To get n(z) for all z_m-values, integrate over z_m:

n(z) = \{3 / ω\}.(H_0 / c)^3.∫_z^∞ dz_m.( N(z_m) / v(z_m)).p_m(v(z) / v(z_m)).

Scheme of Calculation
Any real sample has maximum z_max for z_m. So, n(z_max) = 0. In fact, lifetimes of individual sources will come into consideration, as well as structure-formation epoch at some high redshift (say, > 10). Thus, n(z) calculation will give useful results only upto redshift much less than z_max. Formally writing z_max instead of ∞ for upper limit,

n(z) = \{3 / ω\}.(H_0 / c)^3.∫_z^{z_max} dz_m.( N(z_m) / v(z_m)).p_m(v(z) / v(z_m)) for 0 ≤ z ≤ z_max.
To apply to real samples, this must be converted to sum. Divide $z_m$-range 0 to $z_{\text{max}}$ into $k$ equal intervals, each $= z_{\text{max}} / k = \Delta z$. Mid-points are $z_j = (j - \frac{1}{2}) \Delta z = \frac{(j - \frac{1}{2})}{k}.z_{\text{max}}$. Calculate $n(z)$ at these points: $n(z_j)$. Converting integral to sum,

$$\frac{(\omega / 3).(c / H_0)^3.n(z)}{3} = \int z_{\text{max}} dz_m \frac{N_i}{v(z_i)}.v(z_j) / v(z_i) \cdot p_i(x_{ij})$$

where $x_{ij} = v(z_j) / v(z_i)$.

It is useful to use $Z_m = \ln z_m$ as redshift variable. Integral and converted sum are then:

$$n(z) = \frac{3}{\omega}.(H_0 / c)^3.\int z_{\text{max}} dZ_m \frac{N_i}{v(z_i)}.v(z_j) / v(z_i).p_i(x_{ij})$$

In these two forms (with $z_m$ and $Z_m$ as variables), $N_i$ is population of $i$th $z_m$-bin and $L_i$ that of $i$th $Z_m$-bin. There are $K$ bins for $Z_m$, and $K$ and $k$ will, in general, be different.

**Illustrative Calculation in 1981**

Wills & Lynds (1978) have defined carefully sample of 76 optically identified quasars. We use this sample only to illustrate derivation of $n(z)$ from $p(x) \equiv p(V/V_m)$. We use Einstein-de Sitter cosmology or $q_0 = \sigma_0 = \frac{1}{2}$, $k = \lambda_0 = 0$ or $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ world model in von Hoerner’s (1974) notation, for which

$$(H_0 / c)^2.\ell_v^2 = 4.(1 + z)^{\alpha} / \{\sqrt{1 + z} - 1\}^2$$

and

$$(H_0 / c)^3.v(z) = 8.(1 - 1 / \sqrt{1 + z})^3$$

For each quasar, $z_m$ is calculated by iteration with initial guess $z$ for $z_m$. Values of $z, z_m$ are then used to calculate $v(z), v(z_m)$ and hence $x = V/V_m$. All 76 $V/V_m$-values are used to plot histogram. Good approximation for $p(x)$ is $p(x) = 2.x$, which is normalized over $[0,1]$. The limiting redshifts $z_m$ range from 0 to 3.2. Dividing into four equal intervals, bins centered at 0.4, 1.2, 2.0 and 2.8 contain 19, 31, 16 and 10 quasars. Although each of these 4 subsets is quite small, we calculate and plot histograms $p_i(x), i = 1, 2, 3, 4$ for each subset for $x$-intervals of width 0.2 from 0 to 1, thus with 5 intervals centered at $x = 0.1, 0.3, 0.5, 0.7$ and 0.9. Each normalized $p_i(x)$ is also well approximated by $p_i(x) = 2.x$ except $p_4(0.2994)$. So we do calculations using this approximation in addition to using actual values. Finally we calculate $\frac{(\omega / 3).(c / H_0)^3.n(z)}{3}$. (See tables.)

| Table for $p_i(x)$ and $p(x)$ |
|-----------------------------|
| $x$ | No. | $p_1(x)$ | No. | $p_2(x)$ | No. | $p_3(x)$ | No. | $p_4(x)$ | No. | $p(x)$ |
|-----|-----|---------|-----|---------|-----|---------|-----|---------|-----|--------|
| 0.1 | 0   | 0       |     | 0.161   | 0   | 0       | 0   | 0       | 1   | 0.066  |
| 0.3 | 2   | 0.526   | 2   | 0.323   | 3   | 0.9375  | 1   | 0.5     | 8   | 0.526  |
| 0.5 | 3   | 0.789   | 6   | 0.968   | 2   | 0.625   | 1   | 0.5     | 12  | 0.789  |
| 0.7 | 8   | 2.105   | 8   | 1.290   | 7   | 2.1875  | 5   | 2.5     | 28  | 1.842  |
| 0.9 | 6   | 1.580   | 14  | 2.258   | 4   | 1.25    | 3   | 1.5     | 27  | 1.776  |
| Totals | 19 | 31       | 16  | 10       | 76  |          |     |         |     |        |

| Table of $n(z)$ calculation using linear scale for limiting redshifts |
|-----------------------------|
| $j$ | $z_j$ | $N_j$ | $v(z_j)$ | $i = 1$ | $i = 2$ | $i = 3$ | $i = 4$ | $n(z_j)$ |
|-----|-------|------|----------|---------|---------|---------|---------|----------|
| 1   | 0.4   | 19   | 2.97E-2  | 0       | 0.1074  | 0.0492  | 0.0321  | 1307.   |
| 2   | 1.2   | 31   | 0.27666  | 0       | 0.0492  | 0.0321  | 0.2994  | 255.    |
| 3   | 2.0   | 16   | 0.60399  | 0       | 0.2994  | 0.0321  | 0.6536  | 67.     |
| 4   | 2.8   | 10   | 0.92407  | 0       | 0.6536  | 0.0321  | 1       | 22.     |

Notes for second table: (a) $5^\text{th}$ to $8^\text{th}$ columns list $x_{ij}$-values,
(b) $v(z_j) = (H_0 / c)^3.v(z_j) = 8.(1 - 1 / \sqrt{1 + z_j})^3$, and
(c) $n(z_j) = (\omega / 3).(c / H_0)^3.n(z_j)$. 

Use of approximations \( p(x) = 2x \) in evaluating sums (1) for each row \( j = 1, 2, 3, 4 \) gives virtually same results. Table below shows steps in evaluating \( n(z) \) using ln-scale for limiting redshifts, and \( p(x) = 2x \), so that no \( x_{ij} \)-values need be calculated.

| \( j \) | \( Z_m \)-range | mid-\( Z_m \) | \( z_m \) (i.e. \( z_j \)) | \( L_j \) | \( \rightarrow v(z_j) \) | \( \rightarrow n(z_j) \) |
|---|---|---|---|---|---|---|
| 1 | -1.5to-0.9 | -1.2 | 0.3012 | 7 | 0.015012 | 355. |
| 2 | -0.9to-0.3 | -0.6 | 0.5488 | 11 | 0.060673 | 301. |
| 3 | -0.3to+0.3 | 0.0 | 1.0000 | 27 | 0.201010 | 337. |
| 4 | +0.3to+0.9 | +0.6 | 1.8221 | 23 | 0.530388 | 181. |
| 5 | +0.9to+1.5 | +1.2 | 3.3201 | 8 | 1.117620 | 48. |

Number of sources in bin \( j \) is denoted \( L_j \) for ln-scale (instead of \( N_j \) for linear scale).

**Conclusion**

Due to too small sample, results are only indicative. Main aim is illustrating method fully.

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(See longer versions astro-ph/0903.1903 and 0902.2898 for fuller exposition and references.)