Online and Distribution-Free Robustness: Regression and Contextual Bandits with Huber Contamination

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Abstract

In this work we revisit two classic high-dimensional online learning problems, namely regression and linear contextual bandits, from the perspective of adversarial robustness. Existing works in algorithmic robust statistics make strong distributional assumptions that ensure that the input data is evenly spread out or comes from a nice generative model. Is it possible to achieve strong robustness guarantees even without distributional assumptions altogether, where the sequence of tasks we are asked to solve is adaptively and adversarially chosen?

We answer this question in the affirmative for both regression and linear contextual bandits. In fact our algorithms succeed where convex surrogates fail in the sense that we show strong lower bounds categorically for the existing approaches. Our approach is based on a novel way to use the sum-of-squares hierarchy in online learning and in the absence of distributional assumptions. Moreover we give extensions of our main results to infinite dimensional settings where the feature vectors are represented implicitly via a kernel map.

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1 Introduction

1.1 Background

The field of robust statistics was founded over five decades ago by John Tukey [Tuk60, Tuk75], Peter Huber [Hub64] and others and seeks to design estimators that are provably robust to some fraction of their data being adversarially corrupted. However these estimators are generally not efficiently computable in high-dimensional settings [Ber06, HM13]. After a decades long lull we have recently seen considerable progress in algorithmic robust statistics [DKK+19a, LRV16, DKK+17, CSV17, KKM18, DKK+19b, HL18, KSS18, BK20, Kan20, DHKK20]. The first works [DKK+19a, LRV16] focused on robust parameter estimation. The key insight is that uncorrupted data often enjoys various spectral regularity properties that make it possible to efficiently search for low-dimensional projections that can be used to identify corrupted data.

Since then many of these ideas have found a number of exciting further applications, such as performing robust regression [KKM18, BP20, ZJS20, CAT+20] or minimizing a strongly convex function when your gradients can be adversarially corrupted [DKK+19b]. However what these works all share in common is that they are based on assumptions that the uncorrupted data is somehow evenly spread out. These assumptions can either come about by explicitly assuming a generative model, like a Gaussian [DKK+19a] or a mixture of Gaussians [BK20, Kan20, DHKK20], or through a deterministic condition like hypercontractivity [KKM18] or certifiable sub-Gaussianity [HL18, KSS18].

Still, there is a widespread need for provably robust learning algorithms and in many setting these types of evenly spread out assumptions are just not appropriate. This is particularly the case in the context of online prediction [CBL06] which operates in a setting where the input data is ever-changing and potentially even adversarially chosen. This flexibility allows it to capture challenging dynamic settings, as arise in reinforcement learning, where our learning algorithm interacts with the world around it and its decisions may in turn influence the sequence of prediction tasks it is expected to solve. In this work we take an important first step towards answering a much broader question:

\textit{Are there provably robust learning algorithms that can tolerate adversarial corruptions even for challenging high-dimensional and distribution-free online prediction tasks?}

We will study classic online problems like regression/forecasting and linear contextual bandits. In fact, while existing works in algorithmic robust statistics focus on the relationship between the dimension and the error guarantees, many familiar technical problems will arise when we are interested in understanding how the regret guarantees degrade when the dynamic range in predictions is much larger than the intrinsic noise in the model.

In particular, consider the classic online regression problem where we get an adaptive and adversarially chosen sequence of $x_t$’s. Each of these are high-dimensional vectors and our goal is to predict the response according to the model: $y_t = w^T x_t + \xi_t$ where $w$ is unknown and $\xi_t$ is the noise. Now consider what happens when we allow a random $\eta$ fraction of the responses to be adversarially corrupted with full knowledge of the transcript so far (see Section 1.2 for the formal definition). Take the standard scaling where $\|x_t\| \leq 1$ and $\|w\| \leq R$ so that $|w^T x_t| \leq R$, and let $\sigma^2$ be the variance of $\xi_t$. When $\sigma^2$ is comparable to $R^2$, then the robust prediction problem is actually simple to solve. Even seemingly non-robust approaches like ordinary least squares achieve near-optimal regret because, in the contaminated setting, there is not much that can be learned about $w$ in the first place. See Figure 1 for an illustration and Corollary 1.4 for a formal lower bound.
When $\sigma^2$ is comparable to $R^2$, many lines, including the one found by ordinary least squares, fit the data equally well (although the fit is not that good to begin with). When $\sigma^2$ is much smaller than $R^2$, then ordinary least squares fails, but it is clear in principle possible to do much better even in high-dimensions.

Figure 1: Datasets with equal contamination rates but different levels of noise $\sigma$. Clean points denoted by plusses, corrupted points denoted by crosses.

In contrast, when $\sigma^2$ is much smaller than $R^2$, as is natural in many regression settings, existing approaches pay an extra factor of $R$ or $R^2$ in the linear term of the regret. Not only is this not information-theoretically necessary, but it turns out to be a serious obstacle to the prevailing techniques in the area: we show that regression using any convex surrogate (including Huber loss and $L_1$ loss) must pay this price (see Theorem 8.1). So the main high-level question that motivates our work is:

Is it algorithmically possible, in the presence of adversarial corruptions, to achieve limiting average regret that is independent of $R$?

We answer this question in the affirmative for two classic models: online regression with squared loss and linear contextual bandits. Our algorithms succeed where convex surrogates fail, and are based on novel ways to exploit the sum-of-squares hierarchy in online learning and without distributional assumptions.

Finally we remark that the sorts of issues we are broaching are quite relevant in modern reinforcement learning. In particular, there are many sequential tasks (like manipulating a robot arm where we care about the total energy cost over the trajectory) where at each step the variance in the losses/rewards is much smaller than the dynamic range when we consider all possible states that the system could be in. Thus we ask: Is it possible to achieve strong robustness guarantees in model-based reinforcement learning, particularly in settings with large action/state spaces and function approximation? Since contextual bandits are often viewed as a natural “halfway point” between supervised learning and RL (see, for example, the discussion in [ABL03, DHK+11]), this work serves as a first step in this direction.

1.2 Our Models

In this work we study the following robust version of contextual bandits, first introduced in [KPK19].
Definition 1 (Huber-Contaminated Linear Contextual Bandits). Let \( \mathcal{X} \) be an arbitrary Hilbert space, and let \( \mathcal{A} \) be an action space of size \( K \). Fix Huber contamination rate \( \eta \in (0,1/2) \), misspecification rate \( \epsilon \), maximum loss \( R \), and unknown linear function \( f : \mathcal{X} \times \mathcal{A} \to \mathbb{R} \). Ahead of time, an oblivious adversary chooses distributions \( \mathbb{P}_{\epsilon} \) over loss functions \( \ell_t^\epsilon : \mathcal{A} \to [0,R] \) for all possible contexts \( z_t \) and all time steps \( t \in [T] \). We assume these distributions are realized up to misspecification \( \epsilon \) by some linear function \( f \), i.e. for all \( t, z, a \),

\[
\mathbb{E}[\ell_t^\epsilon(a)|z_t = z] = f(z,a) + \epsilon_t(z,a), \quad |\epsilon_t(z,a)| \leq \epsilon. \tag{1}
\]

Let \( \xi_t \) be the random variable which, conditioned on \( z_t = z \), takes on the value

\[
\xi_t \triangleq \ell_t^\epsilon(a) - f(z,a) - \epsilon_t(z,a),
\]

and define noise parameter \( \sigma \) by \( \sigma^2 \triangleq \sup_{z,t} \mathbb{E}[\xi_t^2|z_t = z] \). In each round \( t \in [T] \):

1. Nature chooses \( z_t \), possibly adversarially based on the transcript from previous rounds.
2. Learner chooses action \( a_t \in \mathcal{A} \).
3. A \( \text{Ber}(\eta) \) coin \( \gamma_t \) is flipped to decide whether this round is corrupted.
4. If \( \gamma_t = 0 \), i.e. the round is not corrupted, the learner sees loss \( \ell_t^\epsilon(a_t) \), where \( \ell_t^\epsilon \) is drawn independently from the distribution \( \mathbb{P}_{\epsilon} \).
5. If \( \gamma_t = 1 \), i.e. the round is corrupted, the learner sees an arbitrary loss \( \ell_t(a_t) \) chosen by an adversary based on \( z_t, a_t \), and the transcript from the previous rounds.

The goal of the learner in the adversarial setting is to compete with the best policy in hindsight as measured by the clean losses \( \ell_t^* \) incurred in every round, that is to select a sequence of actions \( a_1, \ldots, a_T \) for which

\[
\hat{\text{Reg}}_{\text{HCB}}(T) = \sup_\pi \mathbb{E} \left[ \sum_{t=1}^T (\ell_t^*(a_t) - \ell_t^*(\pi(z_t))) \right], \tag{2}
\]

is small, where the supremum ranges over all (non-adaptive) policies \( \pi : \mathcal{X} \to \mathcal{A} \) and the expectation is over the randomness of the \( \text{Ber}(\eta) \) coins, the randomness of the rewards, any stochasticity in the choice of contexts, and the randomness of the learner. We say that such a learner achieves clean pseudo-regret \( \hat{\text{Reg}}_{\text{HCB}}(T) \).

In the special case where \( \epsilon = 0 \), we will consider the quantity

\[
\text{Reg}_{\text{HCB}}(T) = \sum_{t=1}^T (\ell_t^*(a_t) - \ell_t^*(\pi^*(z_t)))
\]

where \( \pi^*(z) \triangleq \arg\max_a f(z,a) \). Note that this is a random variable in the same things defining the expectation in (1). We say that a learner achieves clean regret \( \text{Reg}_{\text{HCB}}(T) \). We will establish high-probability bounds on \( \text{Reg}_{\text{HCB}} \).

Without adversarial corruptions this is the familiar linear contextual bandits problem, which has a wide range of applications precisely because in many settings the context is an important component of the prediction task. For example, in online advertising the choice of which ad to display ought to depend on information about the webpage that the ad will be displayed on as well as any information we have about the user we are displaying it to, which can be encoded.
as a high-dimensional vector. In healthcare, when we want to choose between various treatment options again we want to adapt to the relevant context such as the patient history. For additional applications, see the survey [BR19].

However in many of these settings it is natural to imagine that some of the feedback we receive departs in arbitrary ways from the model. This could happen in online advertising due to clickfraud, particularly when malware takes over a user’s account. It could happen in healthcare in the context of drug trials, particularly ones that measure some real valued variable, when there are testing errors or confounding variables that are difficult to model. For all these and many more reasons it is natural to wonder if there could be algorithms for contextual bandits with stronger robustness guarantees.

Remark 1.1. Note that typically in papers on contextual bandits, the range of the loss functions is normalized to $[0, 1]$ for convenience, but in the Huber-contaminated setting we focus on, we crucially want to avoid any $R$ dependence in the dominant term of our pseudo-regret/regret bounds. Equivalently, the scale-invariant quantity which we want to avoid dependence on is the ratio $R/\sigma$.

This problem is closely related to the following robust version of online regression, which we informally introduced earlier. We briefly note that online regression is one of the fundamental problems in online learning that has been extensively studied in the uncontaminated setting, see e.g. [Vov01, AW01, CBL06].

Definition 2 (Huber-Contaminated Online Regression). Let $\eta, f, \epsilon$ be the same as in Definition 1. In each round $t \in [T]$:  
1. Nature chooses $(z_t, a_t)$, possibly adversarially based on the transcript from previous rounds.
2. Learner chooses prediction $\hat{y}_t$.
3. A Ber($\eta$) coin is flipped to decide whether this round is corrupted.
4. If the round is not corrupted, sample $\xi_t$ independently from $\mathcal{D}$. The learner sees $y_t \triangleq y^*_t + \xi_t$, where $y^*_t \triangleq f(z_t, a_t) + \epsilon_t(z_t, a_t)$ for some quantity $\epsilon_t(z_t, a_t)$ satisfying $|\epsilon_t(z_t, a_t)| \leq \epsilon$.
5. If the round is corrupted, the learner sees an arbitrary $y_t$ chosen by an adversary based on $(z_t, a_t)$ and the transcript from the previous rounds.

The goal of the learner, given any $(z_t, a_t)$ in round $t$ (and the transcript from the previous rounds), is to choose a prediction $\hat{y}_t(z_t, a_t)$ such that with high probability over the choice of Ber($\eta$) coins, and for any (possibly adaptively chosen) sequence of feature vectors $\{(z_1, a_1), \ldots, (z_T, a_T)\}$ in the above model, the quantity

$$\text{Reg}_{\text{HSq}}(T) = \sum_{t=1}^{T} (\hat{y}_t(z_t, a_t) - y^*_t)^2.$$  \hspace{1cm} (3)

is small. We say that $A$ achieves clean square loss regret $\text{Reg}_{\text{HSq}}(T)$. Note that $\text{Reg}_{\text{HSq}}$ is a random variable depending on the randomness of the Ber($\eta$) coins, the randomness of the noise $\xi_t$, any stochasticity in the choice of the inputs $(z_t, a_t)$, and the randomness of the learner and adversary. We will establish high-probability bounds on this random variable.

Remark 1.2. As we will rely on a formal connection between contextual bandits and online regression illuminated in [FR20], it will be helpful to situate our definitions in their context. In particular, when $\eta = 0$, Definition 1 specializes to Assumption 4 of [FR20], and an algorithm for Definition 2 achieving clean square loss regret at most $\text{Reg}_{\text{HSq}}(T)$ would satisfy Assumption 2b of [FR20] in the realizable case with $\epsilon$-misspecification.
As the bulk of the technical contributions of this work is focused on obtaining guarantees for Huber-contaminated online regression, we will simplify notation somewhat by referring to \((z_t, a_t)\) in Definition 2 simply as \((x_t, \epsilon_t)\). As \(x_t \in \mathcal{X}\), the linear function \(f : \mathcal{X} \to \mathbb{R}\) is given by \(f(x) = \langle w^*, x \rangle\) for some unknown regressor \(w^* \in \mathcal{X}\), and for any round \(t\) which is not corrupted, we have that

\[
y_t \triangleq y_t^* + \xi_t, \quad y_t^* \triangleq \langle w^*, x_t \rangle + \epsilon_t
\]

for \(\xi_t\) sampled independently from some distribution \(\mathcal{D}\) over \(\mathbb{R}\) and \(|\epsilon_t| \leq \epsilon\). We will make the following assumption on \(\mathcal{D}\):

**Assumption 1.** Let \(\sigma \triangleq \mathbb{E}_{\xi \sim \mathcal{D}}[\xi^2]^{1/2}\). For some absolute constant \(c > 0\) and any real number \(k > 2\), the noise distribution \(\mathcal{D}\) is mean zero and \((c, k)\)-hypercontractive, that is,

\[
\mathbb{E}_{\xi \sim \mathcal{D}}[|\xi|^{\ell}]^{1/\ell} \leq c\sqrt{\ell} \cdot \sigma \quad \forall \ 2 \leq \ell \leq k.
\]

We emphasize in this assumption and our results that \(k\) does not need to be integer, i.e. it can be as taken as small as \(2 + \varepsilon\) for \(\varepsilon > 0\).

We will make the analogous assumption on the noise \(\xi_t\) in Definition 1:

**Assumption 2.** Let \(\xi_t\) be the random variable defined in Definition 1. For some absolute constant \(c > 0\), any real number \(k > 0\), and any round \(t\) and context \(z \in \mathcal{X}\), conditioned on \(z_t = z\), the distribution over \(\xi_t\) is \((c, k)\)-hypercontractive.

Lastly, we adopt the following standard normalization convention.

**Assumption 3.** For any round \(t\), \(\|x_t\| \leq 1\) almost surely and \(\|w^*\| \leq R\).

**Connection to robust mean estimation** Note that regression with Huber contaminations is at least as hard as the problem of mean estimation under Huber contaminations, implying that achieving sublinear regret for Huber-contaminated online regression is impossible:

**Example 1.3.** Let \(d = 1\) and \(\epsilon = 0\), and suppose \(w^* = R\) and \(\mathcal{D} = \mathcal{N}(0, \sigma^2)\). Suppose we only ever see \(x_t = 1\), so that we always have \(y_t^* = R\). Then each uncorrupted \(y_t\) is simply an independent draw from \(\mathcal{N}(R, \sigma^2)\), so the question of producing a good predictor \(\hat{y}\) in this special case is equivalent to that of estimating the mean of a univariate Gaussian with variance \(\sigma^2\) under the Huber contamination model. It is known that one cannot do this to error better than \(\Omega(\eta \sigma)\) (see [DKK18]).

More generally, if we only assume \(\mathcal{D}\) has hypercontractive moments up to degree \(k\), one can devise distributions \(\mathcal{D}\) for which one cannot do better than error \(\Omega(\eta^{1-1/k} \sigma)\) (see e.g. Fact 2 from [HL19]).

**Corollary 1.4.** For any \(0 \leq \eta < 1/2\), any algorithm for Huber-contaminated online regression must incur clean square loss regret at least \(\text{Reg}_{\text{HSq}}(T) = \Omega((\eta^2-2/k) \sigma^2 T)) = \Omega((\eta^2-2/k) \sigma^2 T))\).

**1.3 Our Results**

Our first main result is a pseudo-regret bound for the setting of Huber-contaminated contextual bandits given in Definition 1 whose leading-order term is independent of \(R\).
Theorem 1.5 (Robust contextual bandits, informal version of Theorem 7.1). In the setting of Huber-Contaminated Contextual Bandits (see Definition 1) under Assumption 2, for any fixed \( \eta < \frac{1}{2} \), there is an algorithm which runs in time \( \text{poly}(n, d) \) and selects actions \( a_t \) which satisfy the following clean pseudo-regret bound:

\[
\mathcal{R}_{\text{HCB}}(T) = \sup_{\pi} \mathbb{E} \left[ \sum_{t=1}^{T} (\ell_t'(a_t) - \ell_t'(\pi(z_t))) \right] \lesssim (k^{1/2}\sigma\eta^{k-2} + \epsilon)\sqrt{KT} + \text{poly}(R, \sigma, d) \cdot T^{1+\beta_k}
\]

where \( \beta_k = k^2/(k^2 + k - 2) \in (0, 1) \). In the special case where \( \epsilon = 0 \), the clean regret \( \mathcal{R}_{\text{HCB}}(T) \) is upper bounded by the quantity on the right-hand side of (1.5) with high probability.

Note that if Assumption 2 holds for all even \( k > 2 \), i.e. when the noise \( \xi_t \) is sub-Gaussian for all \( t \), then with \( k = 2\log(1/\eta) \), the leading order term in (1.5) becomes \( \sigma\sqrt{\eta}\log(1/\eta) \cdot \sqrt{KT} \). Our algorithm also has an optimal breakdown point of \( 1/2 \), because its guarantee holds for any \( \eta < 1/2 \). No procedure can attain a similar guarantee to (1.5) when \( \eta \geq 1/2 \), because when the observed data comes from a uniform mixture of two different linear models, it is impossible to tell which is the ground truth and which has been adversarially planted.

The regret bound of Theorem 1.5 explicitly depends on the dimensionality of the contexts, and so it does not explicitly capture even more challenging settings where the features are high-dimensional and/or represented indirectly via a kernel map (as in a kernel ridge regression). Fortunately, we also have a variant where the bound is dimension-free and applies to these settings, at the cost of a small change in the lower order terms in the regret guarantee.

Theorem 1.6 (High-Dimensional Variant of Theorem 1.5, informal version of Theorem 7.2). In the same setting as Theorem 1.5, there exists an algorithm which runs in time \( \text{poly}(n, d) \) and attains the dimension-free regret bound

\[
\mathcal{R}_{\text{HCB}}(T) = \sup_{\pi} \mathbb{E} \left[ \sum_{t=1}^{T} (\ell_t'(a_t) - \ell_t'(\pi(z_t))) \right] \lesssim (k^{1/2}\sigma\eta^{k-2} + \epsilon)\sqrt{KT} + \text{poly}(R, \sigma) \cdot T^{1+\beta_k'}
\]

where \( \beta_k' = (k^2 + k - 2)/(k^2 + 2k - 4) \in (0, 1) \).

We remark that for linear contextual bandits without adversarial corruptions, Foster and Rakhlin [FR20] gave a powerful reduction, which showed that all you need to solve online contextual bandits is a primitive for online regression. We observe that a modified analysis of their reduction carries over in the adversarial setting too (Appendix A). Thus the main question is really: are there efficient algorithms for Huber-contaminated online regression? Indeed, in our second main result, we obtain a regret bound whose leading-order term is independent of \( R \). Ultimately this is our main new ingredient in the proof of Theorem 1.5 - i.e. a new, more powerful primitive for robust regression to build around.

Theorem 1.7 (Robust online regression, informal version of Theorem 6.2). In the setting of Huber-Contaminated Online Regression (see Definition 2) under Assumptions 1 and 3, for any fixed \( \eta < 1/2 \), there exists an algorithm which runs in time \( \text{poly}(n, d) \) and outputs online predictions \( \hat{y}_t \) which satisfy the following clean square loss regret bound:

\[
\mathcal{R}_{\text{HSq}}(T) = \sum_{t=1}^{T} (y_t - \hat{y}_t)^2 \lesssim (k\sigma^2\eta^{k-2} + \epsilon^2)T + \text{poly}(R, \sigma, d, \log(1/\delta)) \cdot T^{\beta_k},
\]

where \( \beta_k \) is defined in Theorem 1.5.
Again, note that if $D$ is sub-Gaussian so that Assumption 1 holds for all even $k > 2$, then with $k = 2 \log(1/\eta)$, the leading order term in (1.7) becomes $\sigma^2 \eta \log(1/\eta) \cdot T$. As with Theorem 1.5, we also have a high-dimensional variant of this result, Theorem 6.4, which eliminates the $d$ dependence, is compatible with the kernel trick, and is used to derive Theorem 7.2.

This guarantee also carries over to the well-studied case where the covariates are stochastic, i.e. sampled i.i.d. from a distribution, where the goal is to obtain good generalization error (see Corollary 4.6). In this setting and the closely related fixed-design setting, the result also naturally applies to other variants of the corrupted response model which have been studied in the literature (see Remark 4.7).

Lastly, we show a no-go theorem against a natural family of algorithms for the problem of Huber-contaminated regression that capture many of the existing techniques for this problem like M-estimation with the $L_1$ or Huber loss (see Section 1.4).

**Theorem 1.8** (Lower bound against convex surrogates, informal version of Theorem 8.1). There is an instance of Huber-contaminated linear regression where the covariates $x_t$ are drawn i.i.d. from a distribution, for which no vector $w$ obtained by minimizing a convex loss with respect to the Huber-contaminated distribution over $(x,y)$’s can achieve square loss better than $\Omega(\eta^3 R)$ on the true distribution.

### 1.4 Related Work

**Robust regression, when both the covariates and responses are corrupted** As discussed in Section 1, our work is closely tied to the long line of recent work on designing efficient algorithms for robust statistics in high dimensions. We refer to [Li18, Ste18, DK19] for comprehensive surveys of this literature and focus here on the results related to regression [KKM18, BP20, ZJS20, PJL20, DKK+19b, PSB+20, DKS19, CAT+20]. These works are for the stochastic setting where the covariates are drawn i.i.d. from some distribution $D_x$ but work in a challenging corruption model where the adversary can arbitrarily alter any $\eta$ fraction of the responses and the corresponding covariates. All of these results operate under the assumption that the underlying distribution $D_x$ is either Gaussian or at least (certifiably) 4-hypercontractive. This is not merely an issue of convenience: in the absence of such assumptions, it is impossible to do anything even in one dimension under this corruption model:

**Example 1.9.** Let $d = 1$ and $\epsilon = 0$, and suppose $w^* = R$. Suppose the distribution over covariates has $1 - \eta$ mass at 0 and $\eta$ mass at 1; when $\eta = o(1)$, this is clearly not $O(1)$-hypercontractive as its fourth moment is $\eta R^4$ while the square of its second moment is $\eta^2 R^4$. Suppose the adversary corrupts an $\eta$ fraction of the pairs $(0,0)$ to be $(1,-R)$. Then it is impossible for the learner to distinguish whether $w^* = R$ or $w^* = -R$.

We note that variants of this example have already appeared previously in the literature, see e.g. Lemma 6.1 in [KKM18] or Theorem D.1 in [CAT+20]. In summary, when there exist rare features in the data then it is not information-theoretically possible to handle corruption in the covariates.

**Robust regression, when just the responses are corrupted** A milder corruption model which has received significant attention in the statistics literature is the setting where a fraction, either randomly or adversarially chosen, of the responses are corrupted, while the covariates are left intact. One popular approach for regression in this setting is M-estimation [L+17, ZBFL18], originally introduced by Huber [Hub64], in which one minimizes a loss function with suitable
robustness properties. Common choices of loss function include the $L_1$ loss and the Huber loss. In addition to the earlier asymptotic results for this approach [BJK78, Hub73, Pol91], by now numerous works have obtained non-asymptotic guarantees for M-estimation under a variety of models for how the responses are corrupted, but predominantly under the assumption that the design is sub-Gaussian or similarly structured [KP18, DT19, SF20, dNS20]. Notably, in [DT19, SF20] it was shown that in the setting of sparse linear regression with Huber-contaminated responses, M-estimation with ($L_1$-regularized) Huber loss is nearly minimax-optimal when the noise distribution $D$ and the covariates are i.i.d. Gaussian.

One exception, and perhaps the result closest in spirit to our results for regression, is that of [Chi20]. One consequence of the results in this work is that in the random-design setting of Definition 2, that is when the covariates are drawn i.i.d. from some distribution $D_x$, then if the function class (equivalently, covariate distribution) is hypercontractive in the sense that for any $w \in W$, $E_{D_x}[(w - w^*, x)^p]^{2/p} \leq E_{D_x}[(w - w^*, x)^2]$ for some $p > 2$, and if the noise distribution $D$ satisfies suitable conditions, then M-estimation with Huber loss achieves the information-theoretically optimal error of $\Theta(\sigma^2 \eta^2)$ in squared loss. It is also possible to modify their proof to show that the same algorithm would yield the information-theoretically optimal error of $\Theta(\sigma \eta)$ in a different metric, the $L_1$ loss, without the hypercontractivity condition. An $L_1$ guarantee is much weaker than the usual $L_2$ (i.e. squared loss) guarantee: for example, it is too weak to give anything interesting for the contextual bandits application.

In fact, as we show in Theorem 8.1, M-estimation with Huber loss, and more generally minimization of any convex surrogate loss, will not achieve squared loss $\Theta(\sigma^2 \eta^2)$ in general when the function class/covariate distribution fails to satisfy this hypercontractivity condition. Instead, we show such estimators must pay squared loss at least $\Omega(\sigma R \eta^3)$. We also mention that to our knowledge, the only work that has explicitly considered online regression with corruptions is [PF20], where they considered Gaussian covariates and a random fraction of responses are corrupted by an oblivious shift. Additionally, another notable line of work to mention in the literature on regression with contaminated responses stems from using hard thresholding [BJK15, BJKK17, SBRJ19], though these works work also make strong regularity assumptions on the covariates.

Lastly, we mention that in the context of classification, there have been a number of recent works giving new algorithmic results for corruption models where the binary labels are corrupted by some process that is halfway between purely stochastic and purely adversarial. For instance, [DGT19, CKMY20, DKTZ20] focus on the Massart noise model which can essentially be viewed as a setting where an adversary can only control a random fraction of the labels, but can change them in an arbitrary way. This can be thought of as the classification version of the Huber-contaminated regression problem that we consider in the present work, and the former two results work in the setting without distributional assumptions. We also note that the very recent work of [DKK++20] considers the stronger model of Tsybakov noise and obtains polynomial-time algorithms under some natural distributional assumptions.

Robustness for bandits There have been a number of notions of robustness proposed in the bandits literature. A classic notion is that of adversarial bandits, a setting where one would like to prove regret bounds even when the rewards are chosen adversarially [ACBFS02]. Many papers have worked to identify ways of interpolating between fully adversarial rewards and stochastically generated ones, including the line of work on “best of both worlds” results [BS12, SS14, AC16, SL17] as well as an interesting model of bandits with adversarial corruptions introduced by [LMPL18] and subsequently studied by [GKT19]. The latter is a setting of multi-armed bandits where rewards are generated stochastically but then perturbed by an adaptive adversary with a fixed budget of how
much he can move the rewards in any given sample path. We stress that the setting of adversarial bandits is somewhat orthogonal to the thrust of the present work: while the adversarial nature of the rewards makes the former quite challenging, it is still possible to achieve sublinear regret for adversarial bandits, whereas in our setting, one cannot do better than $\Omega(\eta^2\sigma^2 T)$. Other notions of robustness that have been considered include the notion of misspecification [FR20, NO20] as in Definition 1 as well as the notion of heavy-tailed reward distributions [BCBL13]. The setting of Huber-contaminated rewards that we study was previously studied in the multi-armed case by [KPK19, ABM19]. [KPK19] also studied Huber-contaminated linear contextual bandits when the contexts are Gaussian or collectively satisfy some RSC-like condition. Even in this distribution-specific setting, their analysis loses a factor of $R$. A recent work [Ano20] also studied the Gaussian context case of Huber-contaminated linear contextual bandits and improved over [KPK19] by getting rid of the $R$ dependence. Lastly, we mention the work of [SS14, ZS19] who considered a different corruption model for the multi-armed case where the contaminations cannot reduce the “gap,” i.e. the difference between the reward of the best arm and that of any other arm, by more than a constant factor in any time step.

1.5 Roadmap

In Section 2, we get an overview of the main techniques in our approach. In Section 3 we record some useful technical facts. In Section 4, we give an SoS relaxation for solving the fixed-design case of Huber-contaminated linear regression. In Section 5, we extend this to the high-dimensional setting. In Section 6, we give a generic recipe for converting our fixed-design guarantees into online ones, thereby proving Theorem 1.7. In Section 7 we apply the reduction of [FR20] to our regression results to obtain our main results for contextual bandits, Theorems 1.5 and 1.6. Lastly, in Section 8, we prove our lower bound, Theorem 1.8. In Appendix A we verify that the reduction in [FR20] applies to our Huber-contaminated setting.

2 Overview of Proof

By a slight modification of the proof of Theorem 5 in [FR20], we can reduce the problem of achieving low clean regret in the contextual bandits setting of Definition 1 to that of producing an oracle for Huber-contaminated online regression which gets low clean square loss regret. In this section, we overview the main ingredients for producing such an oracle. For simplicity, we will focus on the special case where there is zero misspecification, i.e. $\epsilon_t = 0$ at all time steps $t$.

There are two main steps: 1) designing an algorithm for fixed-design Huber-contaminated regression that achieves low square loss, and 2) a generic online-to-offline reduction based on cutting plane methods/online gradient descent.

2.1 Huber-Contaminated Fixed-Design Regression

In the fixed-design setting, we are simply given a collection of pairs $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^d \times \mathbb{R}$, where a random subset of roughly $\eta \cdot n$ of the responses $y_t$ are corrupted adversarially. We are promised that for the indices $i$ for which $y_i$ was not corrupted, $y_i = \langle w^*, x_i \rangle + \xi_i$ for some independent noise $\xi_i \sim D$ satisfying Assumption 1, and the goal is to output $\tilde{w} \in \mathbb{R}^d$ for which $\mathbb{E}_{x \sim D}[\langle w^* - \tilde{w}, x \rangle^2]^{1/2}$ is small.

As mentioned in Section 1.4, a consequence of existing techniques [Chi20] is that M-estimation based on the Huber loss suffices to achieve $L_1$ error $\eta \sigma$ in this setting, but as we show in Theorem 8.1, this approach would fail to achieve low square loss without additional assumptions that
we do not make. Indeed, as we will now see, achieving low square loss in this same setting, which is essential for the reduction in [FR20], is far more challenging.

**Searching for a structured subset**  Similar to existing approaches in the robust statistics literature, our general strategy is to fit a regressor \( w \) to a \((1 - O(\eta))n\)-sized subset of the data which satisfies certain structural properties that the set of uncorrupted points \( S^* \subseteq [n] \) would collectively satisfy and that can be used to certify that the regressor we use is close to \( w^* \).

Before we describe how the structural property that we use fundamentally differs from the ones exploited in prior works on robust regression, it is instructive to walk through how one might certify that a vector \( w \) that we fit to some large “structured” subset \( S \) of the data is close to \( w^* \).

A natural starting point is to try minimizing over all such \( S \) and \( w \) the square error incurred by \( w \) on the points selected by \( S \), that is, to solve the optimization problem

\[
\begin{align*}
  \min_{w, S} & \quad \frac{1}{n} \sum_{t \in S} (y_t - \langle w, x_t \rangle)^2 \quad \text{(6)}
\end{align*}
\]

Finding the optimal \( w, S \) is obviously computationally intractable, but we will ignore this for the time being (as we explain later, there are well-known recipes for handling this issue). Now the main step to showing that \( w \) performs well on the true set of uncorrupted points \( S^* \) is to argue that on the points in \( S^* \) that \( S \) failed to pick out, the square loss incurred by \( w \) is still small.

Formally, we would like to bound \( \frac{1}{n} \sum_{t \in S^* \setminus S} (y_t - \langle w, x_t \rangle)^2 \). Naively, because \( S, S^* \) are both big, \( S^* \setminus S \) is only an \( O(\eta) \)-sized set of points, but naively each point could contribute as much as \( \Omega(R^2) \) to the sum. Instead, we proceed by further decomposing this quantity into:

1. The noise \( \frac{1}{n} \sum_{t \in S^* \setminus S} \xi_t^2 \) over points in \( S^* \setminus S \)
2. The error \( \frac{1}{n} \sum_{t \in S^* \setminus S} (w^* - w, x_t)^2 \) over points in \( S^* \setminus S \) from predicting with \( w \) instead of \( w^* \).

Term (1) can be controlled by using the fact that the noise is hypercontractive, and this step is standard. With term (2) however, we arrive at the key distinction between our approach and that of previous works on robust regression. In prior works, this is the place where one could insist that \( S \) is structured in the sense that along every univariate projection, the empirical moments of the points in \( S \) are \( k \)-hypercontractive for some \( k \geq 4 \), in which case we could use Holder’s to upper bound term (2) in terms of just \( \eta \) and \( \sigma \). This is not applicable to our fixed-design setting where we make no assumptions on the process by which \( x_1, \ldots, x_n \) were generated, so a subset with hypercontractive empirical moments may not even exist.

Instead, our approach is to insist that \( S \) must sub-sample the empirical covariance, i.e. that

\[
\frac{1}{n} \sum_{t \in S} x_t x_t^\top \succeq (1 - O(\eta)) \frac{1}{n} \sum_{t=1}^n x_t x_t^\top - o(1) \cdot \text{Id} \quad \text{(7)}
\]

The intuition for this constraint is that because the points that get corrupted in the Huber contamination setting form a random subset of the data, \( S^* \) will satisfy this constraint with high probability. So for any \( S \) which sub-samples the empirical covariance, ignoring the low-order term in (2.1), we can thus upper bound term (2) by \( O(\eta/n) \cdot \sum_{t \in S^*} (w^* - w, x_t)^2 \). Recall that the whole point of the preceding argument was to upper bound \( \frac{1}{n} \sum_{t \in S^*} (w^* - w, x_t)^2 \), so term (2) is negligible as desired.
Proofs to algorithms Solving the optimization problem in (2.1) under the above definition of structure, i.e. over large $S \subseteq [n]$ which sub-sample the empirical covariance, is computationally infeasible. But a nice feature of the argument sketched above is that it only uses steps that are captured by the sum-of-squares proof system. In particular, instead of searching for $S$ infeasible. But a nice feature of the argument sketched above is that it only uses steps that are
good. We now explain how to use the guarantee of the previous section to get an algorithm for online regression. At a high level, the idea is to use the fixed-design guarantee above to design a
ellipsoid algorithm \[YJY09, TK08\].

2.2 Online-to-Offline Reduction

We now explain how to use the guarantee of the previous section to get an algorithm for online learning \[SS+11\], and the efficient variant of halving for halfspace learning based on the ellipsoid algorithm \[YJY09, TK08\].

Concretely, suppose inductively we have seen samples $(x_1, y_1), \ldots, (x_n, y_n)$ thus far and have used some vector $w$ to predict in the last $m$ steps where we were given $(x_{n-m+1}, y_{n-m+1}), \ldots, (x_n, y_n)$. Let $\Sigma$ be the average of $x_i x_i^T$ over the last $m$ steps. One of two things could be true.

It could be that in these last $m$ steps, $w$ actually performed well, that is, $\|w - w^*\|_\Sigma$ is small, either because $w \in B$ or because $x_{n-m+1}, \ldots, x_n$ mostly lie in the slab of space where $w$ and $w^*$ yield similar predictions. Either way, because the prediction error under $w$ has been small so far, there is no need to update to a new predictor just yet.

Alternatively, if $\|w - w^*\|_\Sigma$ is large, then the gradient of the function $w \mapsto \|w - w^*\|_\Sigma$ would give a separating hyperplane between $w$ and $B$. Of course, the issue with this is that we don’t know $w^*$. But recall from the fixed-design guarantee that if we ran the SoS-based algorithm above on the data $(x_{n-m+1}, y_{n-m+1}), \ldots, (x_n, y_n)$ (assuming $m$ is large enough that things concentrate sufficiently well), then the resulting vector $\tilde{w}$ is close to $w^*$ under $\|\cdot\|_\Sigma$. So to check whether $\|w - w^*\|_\Sigma$ is large, by triangle inequality we can simply check whether $\|w - \tilde{w}\|_\Sigma$ is large! If so, the gradient of $w \mapsto \|w - \tilde{w}\|_\Sigma$ gives us a separating hyperplane that we can actually compute.

To summarize, the contrapositive of this tells us that if we don’t form a separating hyperplane in a given step, then we know $\|w - w^*\|_\Sigma$ is small and we are content to continue using $w$. Conversely, if we do form a separating hyperplane, we know we won’t cut $B$. This is because every point in $B$ is, by design, close to $w^*$ under any norm $\|\cdot\|_\Sigma$ defined by the empirical covariance $\Sigma$ of a sequence of samples.

With these two facts in hand, we can safely run, e.g., Vaidya’s cutting plane algorithm to update our predictor every time we find a separating hyperplane and ensure that after a bounded number of updates, we find a predictor that will achieve low regret on subsequent steps.
Handling the high-dimensional case  The above approach does not work when the dimension is unbounded, e.g. in kernelized settings, because the guarantees of cutting plane methods are inherently dimension-dependent. We now describe an alternative approach based on wrapping online gradient descent around our guarantee for Huber-contaminated fixed-design regression.

Instead of using Vaidya’s algorithm to update the vector $w$ that we predict with whenever the separation oracle returns $\nabla \varphi_t(w)$, we can imagine updating $w$ by simply stepping in the direction of $-\nabla \varphi_t(w)$. The key challenge is to bound the number of times $V$ we get a hyperplane from the separation oracle and have to make such a step, because as long as we don’t receive any new hyperplanes, the predictions we make will incur low square loss.

For this, we can appeal to the the fundamental regret bound for online gradient descent [Zin03]. Specifically, if we receive a sequence of convex losses $\varphi_1, \ldots, \varphi_V$ and play a sequence of inputs $w_1, \ldots, w_V$ where $w_{t+1}$ is given by taking a gradient step with respect to $\varphi_t$ from $w_t$, then the cumulative loss $\sum \varphi_t(w_t)$ incurred only exceeds $\sum \varphi_t(w^*)$ for any single move $w^*$ by an $O(\sqrt{V})$ term (see Theorem 6.3). But because the separation oracle is called only when $\varphi_t(w_t) \approx \varphi_t(w^*)$ is large, this immediately implies that $V$ is bounded.

3 Technical Preliminaries

Here we collect miscellaneous technical facts that will be useful in later sections.

Fact 3.1 (Bernstein’s inequality). For $X_1, \ldots, X_n$ independent and mean-zero, if $|X_i| \leq M$ for all $i$, then for all $t > 0$,

$$\mathbb{P}\left[\frac{1}{n} \sum_{i=1}^{n} X_i \geq t\right] \leq \exp\left(-\Omega\left(\frac{nt^2}{\frac{1}{n} \sum \mathbb{E}[X_i^2] + Mt}\right)\right)$$

Given a matrix $M$, we let $\|M\|$ denote the operator norm of $M$.

Lemma 3.2 (Matrix Azuma, e.g. Corollary 7.2 in [Tro12]). For $Y_1, \ldots, Y_n \in \mathbb{R}^{d \times d}$ a symmetric matrix martingale whose associated difference sequence $\{X_t\}$ satisfies $\|X_t\| \leq 1$ almost surely for all $t$. Then for any $\delta > 0$ we have that

$$\mathbb{P}\left[\|Y_t - \mathbb{E}[Y_t]\| \geq \sqrt{8 \log(d/\delta)/n}\right] \leq \delta.$$

Given a positive semidefinite matrix $\Sigma$, we define the Mahalanobis norm by

$$\|x\|^2_{\Sigma} := \|\Sigma^{1/2} x\|^2 = \langle x, \Sigma x \rangle.$$

Lemma 3.3. For any psd matrix $\Sigma$ which induces a norm $\|\cdot\|_{\Sigma}$, any vector $w^*$, and any degree-2 pseudoexpectation $\mathbb{E}[\cdot]$ over $\mathbb{R}^d$-valued variable $w$, we have that

$$\|\mathbb{E}[w] - w^*\|^2_{\Sigma} \leq \mathbb{E}[\|w - w^*\|^2_{\Sigma}].$$

Proof. By the dual definition of $L_2$ norm, the left-hand side of (3.3) can be written as $\sup_{v \in S^{d-1}} \langle \Sigma v, \mathbb{E}[w] - w^* \rangle^2$. For any $v \in S^{d-1}$,

$$\langle \Sigma v, \mathbb{E}[w] - w^* \rangle^2 = \langle \mathbb{E}[\Sigma v, w - w^*], \mathbb{E}[\Sigma v, w - w^*]^\top \rangle \leq \mathbb{E}[\|\Sigma v, w - w^*\|^2_{\Sigma}],$$

where the first inequality follows by the pseudo-expectation version of SoS Cauchy-Schwarz (see e.g. Lemma A.5 of [BKS14]). Therefore, taking the maximum over all $v \in S^{d-1}$ proves the inequality. 

\[ \square \]
4 An SoS Relaxation for Huber-Contaminated Regression

In this section, we consider the offline version of the Huber-Contaminated Regression Problem (Definition 2). The setup is exactly the same, except that the step where the algorithm creates a prediction $\hat{y}_t$ is deferred to the very end. In other words, the algorithm gets to see all of the data points $x_1, \ldots, x_n$ and (possibly corrupted) labels $y_t$ before it has to pick predictions $\hat{y}_1, \ldots, \hat{y}_n$. We use the notation $n$ for the number of data points instead of $T$ to emphasize that we are in an offline setting. Since we are following the generative process from Definition 2, the adversary still gets to choose the covariate $x_t$ adaptively based off of the noise and coin flips at previous rounds. In Remark 4.7 we describe a slightly more conventional, non-adaptive setup where our result also holds and with an even more powerful adversary.

We briefly recall some of the relevant notation. Let $a_t^*$ be the indicator for whether round $t$ was uncorrupted, i.e. $a_t^* = 1$ when the round is not corrupted and this occurs with probability $1 - \eta$. Recall from (2) that for every $t \in [n]$ corresponding to a round which is not corrupted, we observe $y_t$ given by

$$y_t = y_t^* + \xi_t, \quad y_t^* = \langle w^*, x_t \rangle + \epsilon_t$$

where $w^*$ is the true regressor and $\|w^*\| \leq R$, and $\xi_t$ is independently sampled from the noise distribution $D$ which satisfies Assumption 1, and $|\epsilon_t| \leq \epsilon$ is the misspecification. On the other hand, on corrupted rounds $y_t$ is chosen freely by the adversary. For convenience, define

$$\Sigma_n \triangleq \frac{1}{n} \sum_{t=1}^{n} x_t x_t^\top.$$

Our goal will be to output $\tilde{w}$ such that $\|\tilde{w} - w^*\|_{\Sigma_n}^2$ is small. When there is no misspecification, this is the same as the definition of the usual MSE (Mean Squared Error) objective $\frac{1}{n} \sum_{t=1}^{n} (y_t^* - \langle \tilde{w}, x_t \rangle)^2$. When there is misspecification, it differs by a $O(\epsilon^2)$ error, so we will ultimately get the same guarantees for both upper bounding $\|\tilde{w} - w^*\|_{\Sigma_n}^2$ and the MSE. The algorithm achieving our goal is SoSRegression (defined in Algorithm 1 and analyzed in Theorem 4.1), and it succeeds up to the optimal breakdown point $\eta = 1/2$.

We introduce some notation used throughout the analysis. For $\bar{\eta} \geq \eta$ a parameter to be tuned, it will be convenient to define $a_t' \triangleq a_t^* 1[\xi_t^2 \leq s]$ for $s = \Theta(\sigma^2/\bar{\eta})$ to handle the rounds $t$ for which $\xi_t$ lives in the tails of $D$. It is helpful for the reader to think of $\eta = \Theta(1)$, in which case we will take $\bar{\eta} \approx \eta$ and $s = \Theta(\sigma^2)$. In the following Theorem, the constants in the guarantee must deteriorate slightly as we approach the (optimal) breakdown point $\eta = 1/2$, so we introduce a parameter $\rho$ which tracks the distance to $1/2$; as long as we are strictly bounded away from this point, $\rho$ is upper bounded by a constant and can be ignored.

**Theorem 4.1.** Suppose that $\eta \leq \bar{\eta} < 1/2$, define $\rho > 0$ by $\bar{\eta} = \frac{1}{2 + 2\rho^2}$, and suppose

$$n \geq \tilde{\Omega}\left(d \log(1/\delta)/\bar{\eta}^k\right) \lor \Omega\left(\log(d/\delta)/\alpha^2\right).$$

There is a poly$(n, d)$ algorithm which takes as input $(x_1, y_1), \ldots, (x_n, y_n)$ and outputs a vector $\tilde{w}$ which satisfies

$$\|\tilde{w} - w^*\|_{\Sigma_n}^2 \leq \frac{1}{\rho^4} \cdot O\left(k\sigma^2\bar{\eta}^{-2/2k} + \epsilon^2 + \sigma(R + \sigma)\sqrt{\frac{d}{n} \log(2/\delta) + \rho^2 \alpha R^2}\right) \quad (10)$$
In particular, by taking \( \bar{\eta} \triangleq \eta + \Theta([d \log(1/\delta)/n]^{1/k}) \) and \( \alpha \triangleq \Theta(\sqrt{\log(d/\delta)/n}) \), we get that

\[
\|\bar{w} - w^*\|_{\Sigma_n}^2 \leq \frac{1}{\rho^3} \cdot O \left( k\sigma^2\eta^{1-2/k} + \epsilon^2 + k\sigma^2[d \log(1/\delta)/n]^{1/k-2/k^2} + R\sigma \sqrt{(d/kn) \cdot \log(2/\delta)} + \rho^2 R^2 \sqrt{\log(d/\delta)/n} \right)
\]

(11)

**Algorithm 1: SoSRegression(D)**

**Input:** Dataset \( D = \{(x_1, y_1), \ldots, (x_n, y_n)\} \)

**Output:** Vector \( \bar{w} \) for which \( \|\bar{w} - w^*\|_{\Sigma_n} \) is small (see Theorem 4.1)

1. Let \( \bar{E}[\cdot] \) be the pseudoexpectation optimizing Program 2.1.
2. return \( \bar{E}[w] \).

We first show that certain regularity conditions hold with high probability over the \( \text{Ber}(\eta) \) coins generating \( a_1^*, \ldots, a_n^* \).

**Lemma 4.2.** For any \( \alpha > 0 \), suppose (4.1) holds. For any sequence of \( x_1, \ldots, x_n \) chosen during the process in Definition 2, we have that with probability at least \( 1 - \delta \) over the randomness of the \( \text{Ber}(\eta) \) coins generating \( a_1^*, \ldots, a_n^* \), the following hold:

1. \( \sum a_t^* \geq 1 - \bar{\eta} - \alpha \).
2. \( \left\| \frac{1}{n} \sum_{t=1}^{n} a_t^* \xi_t x_t \right\| \leq \Theta \left( \sigma \sqrt{\frac{d}{n} \log(2/\delta)} \right) \).
3. \( \Sigma_n - \frac{1}{n} \sum_{t=1}^{n} a_t^* x_t x_t^\top \leq \bar{\eta} \Sigma_n + \alpha \cdot \text{Id} \).
4. \( \frac{1}{n} \sum_{t=1}^{n} a_t^* |\xi_t|^k \leq (2c\sigma k^{1/2})^k \)

**Proof.** For part 1, by Chernoff bounds (e.g. Fact 3.1), the fraction of rounds which are corrupted is at most \( \eta + \alpha / 2 \) with probability \( 1 - \delta / 2 \) provided \( n \geq \Omega(\log(1/\delta) / \alpha^2) \). By Chebyshev’s inequality and our choice of \( s \),

\[
\mathbb{P}(\langle y_t - \langle w^*, x_t \rangle \rangle^2 > s) \leq (\bar{\eta} - \eta) / 2,
\]

(12)

so the fraction of rounds for which \( \langle y_t - \langle w^*, x_t \rangle \rangle^2 > s \) is at most \( (\bar{\eta} - \eta) / 2 + \alpha / 2 \) with probability \( 1 - \delta / 2 \) provided \( n \geq \Omega(\log(1/\delta) / \alpha^2) \). So part 1 follows by a union bound.

For part 2, let \( T \subseteq [n] \) denote the set of indices \( t \) for which \( a_t^* = 1 \). For any unit vector \( v \),

\[
\left\langle \frac{1}{n} \sum_{t=1}^{n} a_t^* \xi_t x_t, v \right\rangle = \frac{1}{n} \sum_{t \in T} \xi_t \mathbb{1}[|\xi_t|^2 \leq s] \langle x_t, v \rangle \triangleq Z_v.
\]

Fix the randomness of \( T \) and regard \( Z_v \) as a (mean-zero) random variable in \( \{\xi_t\} \). Noting that \( \frac{1}{n} \sum \langle v, x_t \rangle^2 \leq 1 \), we get from Bernstein’s inequality (Fact 3.1) that

\[
\mathbb{P}[|Z_v| \geq t] \leq \exp \left( -\Omega \left( \frac{nt^2}{\sigma^2 + \sqrt{s}t} \right) \right) = \exp \left( -\Omega \left( \frac{nt^2}{\sigma^2 + \sigma t \sqrt{1/\bar{\eta}}} \right) \right).
\]

By well-known arguments (Exercise 4.4.2 of [Ver18]), we can upper bound

\[
\left\| \frac{1}{n} \sum_{t=1}^{n} a_t^* \xi_t x_t \right\| \leq 2 \max_{v \in \mathcal{V}} Z_v
\]

15
where \( \mathcal{N} \) is a 1/2-net of \( S^{d-1} \); standard covering number bounds (e.g. Corollary 4.2.13 of [Ver18]) let us take \( |\mathcal{N}| \leq 6^d \). Therefore

\[
P \left[ \left\| \frac{1}{n} \sum_{t=1}^{n} a_t' \xi_t x_t \right\| \geq 2u \right] \leq 6^d \exp \left( -\Omega \left( \frac{n u^2}{\sigma^2 + \sigma u \sqrt{1/\eta}} \right) \right).
\]

Taking \( u = \Theta(\sqrt{(d/n) \log(2/\delta)}) \) and requiring \( n = \Omega(d \log(2/\delta)/\eta) \) ensures the event occurs with probability \( 1 - \delta/3 \).

We now show part 3. We can apply matrix Azuma (Lemma 3.2) to the matrix martingale difference sequence

\[
(a_1' - \mathbb{E}[a_1']) \cdot x_1 x_1^\top, (a_2' - \mathbb{E}[a_2']) \cdot x_2 x_2^\top, \ldots, (a_t' - \mathbb{E}[a_t']) \cdot x_t x_t^\top,
\]
to get that if \( n \geq \Omega(\log(d/\delta)/\alpha^2) \), then

\[
P \left[ \left\| \frac{1}{n} \sum_{t=1}^{n} a_t' \cdot x_t x_t^\top - \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[a_t'] \cdot x_t x_t^\top \right\| \geq \alpha \right] \leq \delta,
\]

where the probability is over the randomness of the martingale. For any \( t \in [n] \), we have by (2) that \( \mathbb{E}[a_t'] = (1 - \eta) \mathbb{P}_{\xi \sim D}[\xi^2 \leq s] \geq (1 - \eta)(1 - \frac{\eta n}{2}) \geq 1 - \eta \). We conclude that

\[
\frac{1}{n} \sum_{t=1}^{n} a_t' x_t x_t^\top \geq (1 - \eta) \Sigma_n - \alpha \cdot \text{Id},
\]

from which part 3 follows.

For part 4, note that \( \frac{1}{n} \sum_{t=1}^{n} a_t' \xi_t k = \frac{1}{n} \sum_{t=1}^{n} 1[\xi^2 \leq s] \xi_t^k \) where \( \xi_1, \ldots, \xi_n \) are sampled independently from \( D \). Let \( D' \) be the distribution over \( \xi \cdot 1[\xi^2 \leq s] \) where \( \xi \sim D \). Clearly \( \mathbb{E}_{\xi' \sim D'}[|\xi'|^k] \leq \mathbb{E}_{\xi \sim D}[|\xi|^k] \leq (c \sigma k^{1/2})^k \). So by Chernoff, we conclude that

\[
P_{\xi_1' \ldots \xi_n' \sim D'} \left[ \frac{1}{n} \sum_{t=1}^{n} \xi_t^k \geq 2(c \sigma k^{1/2})^k \right] \leq \exp \left( -\Omega \left( \frac{n(\sigma^2 k^{1/2})^k}{s^k} \right) \right) \leq \exp(-\Omega(n c^2 k^{k/2} \eta^k)),
\]

so as long as \( n \geq \Omega(\log(1/\delta)/(c^2 k \eta)^k) \), part 4 holds with probability at least \( 1 - \delta/3 \).

Henceforth, we will condition on the events of Lemma 4.2 holding for two values of \( s \), to be specified later.

Now consider the following set of polynomial constraints.

**Program 1.** Let \( c, k \) be the parameters from Assumption 1, let \( \eta \geq \eta \) and \( \alpha > 0 \) be parameters to be tuned later. The program variables are \( \{a_t\}_{t \in [n]} \) and \( w \), and the constraints are

1. (Norm bound) \( \sum_{i=1}^{d} w_i^2 \leq R^2 \).
2. (Booleanity) \( a_t^2 = a_t \) for all \( t \in [n] \).
3. (Large fraction of inliers) \( \frac{1}{n} \sum_{t=1}^{n} a_t \geq 1 - \eta - \alpha \).
4. (Outliers sub-sample the empirical covariance)

\[
\frac{1}{n} \sum_{t=1}^{n} (1 - a_t) x_t x_t^\top \leq \eta \Sigma_n + \alpha \cdot \text{Id}.
\]
The program objective is to minimize
\[
\min \mathbb{E} \left[ \sum_{t=1}^{n} a_t(y_t - \langle w, x_t \rangle)^2 \right]
\]
over degree-4 SoS-pseudoexpectations satisfying the above constraints.

We first show that conditioned on the events of Lemma 4.2 holding, there always exists a feasible solution to the above polynomial system.

**Lemma 4.3 (Satisfiability).** For any \( \delta > 0 \), if \( n \) satisfies the bound in (4.1), then for any sequence of \( x_1, \ldots, x_n \) chosen during the process in Definition 2, we have that with probability at least \( 1 - \delta \) over the randomness of the \( \text{Ber}(\eta) \) coins generating \( a^*_1, \ldots, a^*_n \) and over the randomness of \( \xi_1, \ldots, \xi_n \), the choice of \( a_t = a^*_t \) and any \( v \) with \( \|v\| \leq R \) is a feasible solution to Program 1, and in particular, the objective value of Program 1 is at most \( \frac{1}{n} \sum_{t=1}^{n} a^*_t(y_t - \langle v, x_t \rangle)^2 \).

**Proof.** Clearly Constraints 1 and 2 are satisfied. Part 1 of Lemma 4.2 implies that Constraint 3 is satisfied with probability \( 1 - \delta/3 \). Finally, part 4 of Lemma 4.2 implies that Constraint 4 is satisfied with probability at least \( 1 - \delta/3 \). \( \square \)

Let \( v^* \) be defined as
\[
v^* \triangleq \arg \min_{v : \|v\| \leq R} \frac{1}{n} \sum_{t=1}^{n} a^*_t(y_t^* - \langle v, x_t \rangle)^2.
\]

The following Lemma is needed only for the misspecified setting: if \( \epsilon = 0 \) we will trivially have \( v^* = w^* \). In the misspecified setting \( v^* \) will naturally appear in the analysis, instead of \( w^* \), because it gives the optimal bounded norm linear function approximating the true regression function \( x_t \mapsto \langle w^*, x_t \rangle + \epsilon_t \). We define \( \Sigma'_n \triangleq \frac{1}{n} \sum a'_t \cdot x_t x_t^\top \).

**Lemma 4.4.** For \( v^* \) as defined above, we have \( \|v^* - w^*\|_{\Sigma'_n}^2 = O(\epsilon^2) \) and also, if we define
\[
\epsilon'_t \triangleq y_t^* - \langle v, x_t \rangle,
\]
then for all \( w \) with \( \|w\| \leq R \) we have:
\[
\sum_{t=1}^{n} a'_t \epsilon'_t \langle w - v^*, x_t \rangle \leq 0
\]

**Proof.** Since \( \nabla_v (y_t^* - \langle v, x_t \rangle)^2 = -2(y_t^* - \langle v, x_t \rangle)x_t \), we see that the first order optimality condition for (2) implies for any \( w \) with \( \|w\| \leq R \) we have
\[
-\frac{2}{n} \sum_{t=1}^{n} a'_t \epsilon'_t \langle w - v^*, x_t \rangle \geq 0
\]
which gives (4.4).

It remains to upper bound \( \|v^* - w^*\|_{\Sigma'_n}^2 \). By writing it out, we see
\[
\|v^* - w^*\|_{\Sigma'_n}^2 = \frac{1}{n} \sum_{t=1}^{n} a'_t(v^* - w^*, x_t)^2 = \frac{1}{n} \sum_{t=1}^{n} a'_t(y_t^* - \epsilon_t - \langle v^*, x_t \rangle)^2 \leq \frac{2}{n} \sum_{t=1}^{n} a'_t(\epsilon_t^2 + (\epsilon'_t)^2) \leq 2\epsilon^2
\]
where in the second-to-last step we used \( (a + b)^2 \leq 2a^2 + 2b^2 \) and in the last step we used that \( v^* \) minimizes (2). \( \square \)
We can now prove Theorem 4.1.

Proof of Theorem 4.1. Let \( \bar{E}[:]:n \) be the pseudo-expectation optimizing the objective in Program 1, and define \( \tilde{w} \triangleq \bar{E}[w] \). By part 3 of Lemma 4.2 and Constraint 1, we have that

\[
(1 - \eta)\|\tilde{w} - w^*\|^2_{\Sigma_n} \leq \|\tilde{w} - w^*\|^2 + \alpha \|\tilde{w} - w^*\|^2 \leq \bar{E}[\|w - v^*\|^2_{\Sigma_n}] + \alpha R^2 + 2\epsilon^2,
\]

where \( \Sigma_n \triangleq \frac{1}{n} \sum a_i^t \epsilon x_i x_i^\top \) and \( \|r\|_{\Sigma_n} \) is the induced norm, and in the last step we used Lemma 4.4, Lemma 3.3, and Constraint 1.

We can further bound

\[
\bar{E}[\|w - v^*\|^2_{\Sigma_n}]
= \frac{1}{n} \sum_{t=1}^n a_t^i \bar{E}[\langle w - v^*, x_t \rangle^2]
= \frac{1}{n} \sum_{t=1}^n a_t^i \bar{E}[\langle y_t - \langle w, x_t \rangle \rangle - \langle v^*, x_t \rangle)^2]
= \frac{1}{n} \sum_{t=1}^n a_t^i \left[ \bar{E}[\langle y_t - \langle w, x_t \rangle \rangle^2 - \langle y_t - \langle v^*, x_t \rangle \rangle^2 + \frac{2}{n} \sum_{t=1}^n a_t^i \langle y_t - \langle v^*, x_t \rangle \rangle \cdot \langle \bar{E}[w] - v^*, x_t \rangle \right] + \frac{2}{n} \sum_{t=1}^n a_t^i \langle \xi_t + \epsilon_t x_t \rangle
\]

where in the fourth step we used the identity \((a - b)^2 = a^2 - 2b(a - b) + \epsilon^2 : y_t - \xi_t - \langle v^*, x_t \rangle \) as defined in Lemma 4.4.

Because of Lemma 4.4 and \( \|\bar{E}[w]\|^2 \leq R^2 \) from Constraint 1 we know that

\[
\langle \bar{E}[w] - v^* \rangle \sum_{t=1}^n a_t^i \epsilon_t x_t \leq 0
\]
so we can drop this term from 2. Then by Lemma 3.3, Constraint 1, and Cauchy-Schwarz,

\[
\bar{E}[w] - v^* \sum_{t=1}^n a_t^i \epsilon_t x_t \leq O \left( \sigma R \sqrt{\frac{d}{n} \log(\delta)} \right),
\]

where the second step follows by part 2 of Lemma 4.2.

It remains to upper bound 1, and this is the bulk of the analysis. Concretely, we need to show that the constraints of the program SoS-imply an upper bound on the quantity \( \frac{1}{n} \sum_{t=1}^n a_t^i (y_t - \langle w, x_t \rangle)^2 - \frac{1}{n} \sum_{t=1}^n a_t^i (y_t - \langle w^*, x_t \rangle)^2 \) of \( c\|w - v^*\|^2_{\Sigma_n} + O(\cdot) \) with \( c \in [0, 1) \), so that we can solve for an upper bound on \( \|w - v^*\|^2_{\Sigma_n} \). We do so in Lemma 4.5 below and get \( c = \frac{(1 + \rho^2)\eta}{1 - \eta} \). Choosing \( \rho \) to be the solution to \( \eta = \frac{1}{1 + 2\rho^2} \) and observing that

\[
\frac{1}{1 - c} = \frac{1 - \eta}{1 - (2 + \rho^2)\eta} = \frac{1 + 2\rho^2}{\rho^2} = 2 + 1/\rho^2
\]

yields (4.1). Plugging in \( \eta \) and \( \alpha \) gives (4.1). \( \square \)
Lemma 4.5. Conditioned on the four parts of Lemma 4.2 holding, we have for any $\rho \in (0,1]$ that

$$\mathbb{E}\left[\frac{1}{n}\sum_{t=1}^{n} a_t'(y_t - \langle w, x_t \rangle)^2\right] \leq \frac{1}{n}\sum_{t=1}^{n} a_t'(y_t - \langle v^*, x_t \rangle)^2 + \frac{(1 + 2\rho^2)\eta^2}{1 - \eta} \|v^* - w\|^2 + O\left(\frac{k\sigma^2}{\rho^2}e^{1-2/k} + \frac{1}{\rho^2}e^2 + \alpha R^2\right)$$

(15)

as long as $\mathbb{E}[\cdot]$ is a SoS degree-4 pseudoexpectation satisfying the constraints of the program.

Proof. Let $\bigcirc$ denote the quantity inside the pseudoexpectation on the left-hand side of (4.5). Then in the SoS degree-4 proof system we can show the following bound

$$\bigcirc = \frac{1}{n}\sum_{t=1}^{n} a_t'(y_t - \langle w, x_t \rangle)^2 + \frac{1}{n}\sum_{t=1}^{n} a_t'(1 - a_t)(y_t - \langle w, x_t \rangle)^2$$

$$\leq \frac{1}{n}\sum_{t=1}^{n} a_t(y_t - \langle w, x_t \rangle)^2 + \frac{1}{n}\sum_{t=1}^{n} a_t'(1 - a_t)(y_t - \langle w, x_t \rangle)^2$$

$$= \frac{1}{n}\sum_{t=1}^{n} a_t(y_t - \langle w, x_t \rangle)^2 + \frac{1}{n}\sum_{t=1}^{n} a_t'(1 - a_t)(y_t - \epsilon_t - \langle v^*, x_t \rangle + \langle v^*-w, x_t \rangle + \epsilon_t)^2$$

$$\leq \frac{1}{n}\sum_{t=1}^{n} a_t(y_t - \langle w, x_t \rangle)^2 + \frac{2 + 1/\rho^2}{n}\sum_{t=1}^{n} a_t'(1 - a_t)(y_t - \epsilon_t - \langle v^*, x_t \rangle)^2$$

$$+ \frac{1 + 2\rho^2}{n}\sum_{t=1}^{n} a_t'(1 - a_t)(\epsilon_t)^2 + \frac{2 + 1/\rho^2}{n}\sum_{t=1}^{n} a_t'(1 - a_t)(\epsilon_t)^2$$

$$\leq \frac{1}{n}\sum_{t=1}^{n} a_t(y_t - \langle w, x_t \rangle)^2 + \frac{2 + 1/\rho^2}{n}\sum_{t=1}^{n} a_t'(1 - a_t)(\epsilon_t)^2 +$$

$$\frac{1 + 2\rho^2}{n}\sum_{t=1}^{n} a_t'(1 - a_t)(\epsilon_t)^2 + (2 + 1/\rho^2)e^2$$

where in the second step we use Constraint 2 to get $a_t'a_t \leq a_t$, in the fourth step we use the SOS Cauchy-Schwartz inequality to show $(a+b+c)^2 = (\rho a/\rho+b+pc/\rho)^2 \leq (1+2\rho^2)(a^2/\rho^2+b^2+c^2/\rho^2)$, and in the fifth step we used that $\sum_{t=1}^{n} a_t'(\epsilon_t)^2 \leq \sum_{t=1}^{n} a_t'(\epsilon_t)^2 \leq \epsilon^2$ by construction (see (2).).

Therefore, we can upper bound $\mathbb{E}[\bigcirc]$ by

$$\mathbb{E}\left[\frac{1}{n}\sum_{t=1}^{n} a_t(y_t - \langle w, x_t \rangle)^2\right] + \frac{2 + 1/\rho^2}{n}\sum_{t=1}^{n} a_t'(1 - \mathbb{E}[a_t])(\epsilon_t)^2 +$$

$$\left(\frac{1 + 2\rho^2}{n}\sum_{t=1}^{n} a_t'(1 - a_t)(\epsilon_t)^2 + (2 + 1/\rho^2)e^2\right).$$

From the last part of Lemma 4.3, we know $\bigcirc \leq \frac{1}{n}\sum_{t=1}^{n} a_t'(y_t - \langle v^*, x_t \rangle)^2$. 

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By (scalar) Hölder’s,\(^2\)

\[
\begin{align*}
\mathbb{E}\leq (2 + 1/\rho^2) \left( \frac{1}{n} \sum_{t=1}^{n} (1 - \mathbb{E}[a_t])^{k/(k-2)} \right)^{1-2/k} \\ \leq (2 + 1/\rho^2) \left( \frac{1}{n} \sum_{t=1}^{n} a_t^k \mathbb{E}[\xi_t^k] \right)^{2/k} \\
= O((k/\rho^2) \cdot \eta^{1-2/k} \sigma^2)
\end{align*}
\]

where we used in the second inequality that the vector \(v := (1 - \mathbb{E}[a_t])_{t=1}^{n}\) has entries in \([0, 1]\) (from Constraint 2), in the third inequality that \(\|v\|_1 \leq 2\eta n\) (from Constraint 3), and also in the third inequality we used part 4 of Lemma 4.2.

Finally, to bound (iii), we can finally apply Constraint 4. We get that

\[
\frac{1 + 2\rho^2}{n} \sum_{t=1}^{n} \langle v^* - w, x_t \rangle^2 \leq \frac{1 + 2\rho^2}{n} \sum_{t=1}^{n} (1 - a_t) \langle v^* - w, x_t \rangle^2 \\
\leq (1 + 2\rho^2) \eta \sum_{t=1}^{n} \langle v^* - w, x_t \rangle^2 + 3\alpha \|v^* - w\|_2^2 \\
\leq (1 + 2\rho^2) \eta \sum_{t=1}^{n} a_t \langle v^* - w, x_t \rangle^2 + \frac{3\eta\alpha R^2}{1 - \eta} + 3\alpha R^2 \\
= (1 + 2\rho^2) \eta \|v^* - w\|_\Sigma_n^2 + \frac{3\eta\alpha R^2}{1 - \eta} + 3\alpha R^2
\]

the second step follows by Constraint 4 and \(\rho \leq 1\), the third step follows by part 3 of Lemma 4.2 which we are conditioning on in this section, and the fourth step uses the definition of \(\Sigma_n\).

Finally, we note that while the guarantees in this section have been in a “fixed-design” setting, if the \(x_1, ..., x_n\) were sampled independently from some distribution \(D_x\), we can also obtain an analogous bound on \(\mathbb{E}_{x \sim D_x} \langle \langle \tilde{w} - w^*, x \rangle^2 \rangle\) without much extra effort:

**Corollary 4.6** (Stochastic version). Fix any \(\alpha > 0\). In the special case where \(x_1, ..., x_n\) are i.i.d. samples from a distribution \(D_x\) with covariance \(\Sigma^*\), if \(n\) satisfies (4.1) and \(\mathbb{E}[\cdot]\) is the pseudoexpectation minimizing the objective of Program 1, then

\[
\|\tilde{w} - w^*\|_{\Sigma^*}^2 \leq \frac{1}{\rho^4} \cdot O \left( k\sigma^2 \eta^{1-2/k} + \epsilon^2 + k\sigma^2 [d \log(1/\delta)/n]^{1/k-2/k^2} + R\sigma \sqrt{(d/kn) \cdot \log(2/\delta)} + \rho^2 R^2 \sqrt{\log(d/\delta)/n} \right)
\]

with probability at least \(1 - \delta\).

**Proof.** By matrix Chernoff (i.e. Lemma 3.2 specialized to the case where the martingale difference sequence just consists of i.i.d. summands), if \(n \geq \Omega(\log(d/\delta)/\alpha^2)\), then \(\mathbb{P}[\|\Sigma_n - \Sigma^*\| > \alpha] \leq \delta\). In particular, for any vector \(v\), \(\|v\|_{\Sigma^*_n}^2 \leq \alpha \|v\|^2 + \|v\|_{\Sigma^*}^2\). The claim follows by (4.1) in Lemma 4.5, together with Constraint 1 which implies that \(\|\tilde{w} - w^*\|_2^2 \leq O(R^2)\). \(\Box\)

---

\(^2\)We are using degree-\(k\) Hölder’s, but outside of the pseudoexpectation, so \(\mathbb{E}[\cdot]\) does not need to be degree-\(k\) (indeed, this allows us to obtain guarantees even when the noise is only \(k = (2 + \epsilon)\)-hypercontractive).
We note that in the case where the contexts are chosen stochastically, [SLX20] recently showed that a modified version of the reduction from [FR20] can reduce from stochastic contextual bandits to offline regression with stochastic contexts. It should be possible to combine this reduction with Corollary 4.6; however, we omit the details since we will give an algorithm for the more general online setting anyway.

Remark 4.7 (An alternative setup where the analysis works.) The exact same proof can handle a slightly different setup of the problem where the vectors $x_1, \ldots, x_n$ are chosen obliviously, but the misspecification and corruption adversary are slightly more powerful (they can look “into the future”) — this more closely matches the setup in [Chi20]. The guarantee at the end for Theorem 4.1 and Corollary 4.6 is exactly the same. This is the setup:

1. Covariates $x_1, \ldots, x_n$ are arbitrary fixed vectors in the unit ball of $\mathbb{R}^d$, i.e. they are chosen obliviously.

2. For every $t$ from 1 to $n$, a $\text{Ber}(\eta)$ coin is flipped to determine if round $t$ is corrupted or not. Let $a^*_t$ be the indicator for whether round $t$ was uncorrupted, i.e. $a^*_t = 1$ when the round is not corrupted and this occurs with probability $1 - \eta$.

3. For every uncorrupted round, we observe $y_t$ given by

$$y_t = y^*_t + \xi_t, \quad y^*_t = \langle w^*, x_t \rangle + \epsilon_t$$

where $w^*$ is the true regressor and $\|w^*\| \leq R$, and $\xi_t$ is independently sampled from the noise distribution $D$ which satisfies Assumption 1, and $|\epsilon_t| \leq \epsilon$ is the misspecification. The misspecification $\epsilon_t$ can chosen in a completely adversarial fashion: formally, it is a random variable depending arbitrarily on all other randomness in the setup (e.g. it can depend arbitrarily on the noise and the coin flips from all rounds).

4. For every corrupted round, $y_t$ is chosen freely by the adversary. Again, we assume nothing about $y_t$ — it can depend arbitrarily on all other randomness in the problem.

5 Huber-Contaminated Regression in High Dimension

We now consider the case where the inputs $x_t$ are still bounded norm (i.e. $\|x_t\| \leq 1$), but there is no constraint on the dimension of the ambient space they live in. This is the setting relevant to kernel ridge regression, where the norm is the RKHS (Reproducing Kernel Hilbert Space) norm: see e.g. [SSBD14] for a reference.

In this setup, there is a well-known way to reduce from high dimensions to the low-dimensional misspecified setting using the Johnson-Lindenstrauss Lemma (see [AV99, BBV06]). We first briefly review how this reduction works in the context of regression problems, and then state the results that follow by combining this reduction with our misspecified regression results.

Theorem 5.1 (Johnson-Lindenstrauss Lemma, Exercise 5.3.3 of [Ver18]). For any collection of $n$ points $x_1, \ldots, x_n$ in $\mathbb{R}^d$, if $m = \Omega(\log(n/\delta)/\epsilon^2)$ and $P \in \mathbb{R}^{m \times d}$ has i.i.d. entries from $\mathcal{N}(0, 1/m)$ then

$$(1 - \epsilon)\|x_i - x_j\|^2 \leq \|P(x_i - x_j)\|^2 \leq (1 + \epsilon)\|x_i - x_j\|^2$$

for all $i, j \in [n]$ with probability at least $1 - \delta$. 

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Since we have by the Parallelogram law that
\[ \langle x_i, x_j \rangle = \frac{\|x_i + x_j\|^2 - \|x_i - x_j\|^2}{4}, \]
we know by Theorem 5.1 that for a collection of points \( x_1, \ldots, x_n \) in the origin-centered ball of radius 1,
\[ |\langle x_i, x_j \rangle - \langle Px_i, Px_j \rangle| \leq 2\epsilon, \]
provided \( m = \Omega(\log(n/\delta)/\epsilon^2) \). In our setting, this means we can guarantee
\[ |\langle w^*, x_i \rangle - \langle Pw^*, Px_i \rangle| \leq 2R\epsilon \] (16)
for all \( t \in [T] \) provided that \( m = \Omega(\log(T/\delta)/\epsilon^2) \), or equivalently \( \epsilon = \Omega(\sqrt{\log(T/\delta)/m}) \).

**Remark 5.2 (Combining Dimension Reduction with the Kernel Trick).** Dimension reduction can be combined with the kernel trick [BBV06], so we can rely on it to handle infinite-dimensional spaces as well, when we are given access to a kernel map computing the inner products between vectors. Essentially, this is because we can reduce to the finite dimensional setting by factorizing the kernel matrix — see [BBV06] for details.

Based on the Johnson-Lindenstrauss guarantee (5), we immediately derive dimension-free versions of Theorem 4.1 and Corollary 4.6.

**Theorem 5.3 (High-dimensional variant of Theorem 4.1).** Suppose that \( \eta \leq \eta \leq 1/7 \) and
\[ n \geq \Omega \left( \log^2(1/\delta)/\eta^{2k} \right). \] (17)
There is a polynomial time algorithm which takes as input \((x_1, y_1), \ldots, (x_n, y_n)\) and outputs a vector \( \tilde{w} \) for which
\[ \|\tilde{w} - w^*\|_{\Sigma_n}^2 \leq O(k\sigma^2\eta^{1-2/k} + \epsilon^2 + (R^2 \log(n/\delta) + k\sigma^2)n^{-\gamma_k} + R\sigma \sqrt{(1/kn) \cdot \log(2/\delta)n^{(\gamma_k-1)/2}} + R^2 \sqrt{\log(1/\delta)n^{(\gamma_k-1)/2}} \] (18)
with probability at least \( 1 - \delta \) where \( \gamma_k \triangleq (k-2)/(k^2 + k - 2) \in (0,1) \).

**Proof.** In general if we dimension reduce to \( d \) dimensions using JL, then (5) and Theorem 4.1 gives us a regret guarantee
\[ \|\tilde{w} - w^*\|_{\Sigma_n}^2 \leq \frac{1}{\rho^2} \cdot O(k\sigma^2\eta^{1-2/k} + \epsilon^2 + R^2 \log(n/\delta)/d + k\sigma^2[d \log(1/\delta)/n]^{1/k-2/k} + R\sigma \sqrt{(d/\delta) \cdot \log(d/\delta)} + R^2 \sqrt{\log(1/\delta)n^{(\gamma_k-1)/2}} + \rho^2 R^2 \sqrt{\log(d/\delta)/n} \]
Taking \( d = n^{(k-2)/(k^2 + k - 2)} = n^{\gamma_k} \) gives the stated guarantee. \( \square \)

**Corollary 5.4 (High-dimensional variant of Corollary 4.6).** Fix any \( \alpha > 0 \). In the special case where \( x_1, \ldots, x_n \) are i.i.d. samples from a distribution \( D_x \) with covariance \( \Sigma^* \), if \( n \) satisfies (5.3) then there is a polynomial time algorithm which takes as input \((x_1, y_1), \ldots, (x_n, y_n)\) and outputs a vector \( \tilde{w} \) for which
\[ \|\tilde{w} - w^*\|_{\Sigma^*}^2 \leq O(k\sigma^2\eta^{1-2/k} + \epsilon^2 + (R^2 \log(n/\delta) + k\sigma^2)n^{-\gamma_k} + R\sigma \sqrt{(1/kn) \cdot \log(2/\delta)n^{(\gamma_k-1)/2}} + R^2 \sqrt{\log(n/\delta)n^{(\gamma_k-1)/2}} \]
with probability at least \( 1 - \delta \), where \( \gamma_k \) is defined in Theorem 5.3.

**Proof.** This follows from Theorem 5.3 and the basic uniform generalization bounds for linear predictors of bounded Euclidean norm, see e.g. [BM02, KST09, SSBD14]. More specifically, it follows from Corollary 4 of [KST09]. \( \square \)
Algorithm 2: SeparationOracle\((w, x_t, C_0, D)\)

**Input:** Vector \(w \in \mathcal{W}\)

**Output:** Separating hyperplane between \(w\) and the target region \(\{w' : \|w' - w^*\| \leq r\}\), if \(w\) lies outside

1. \(D \leftarrow \emptyset\).
2. for each new point \(x_t\) input by Nature do
3.     Predict \(\hat{y}_t = \langle w, x_t \rangle\) and observe \(y_t\).
4.     Append \((x_t, y_t)\) to \(D\).
5.     \(v_t \leftarrow \text{SOSRegression}(D)\).
6.     \(\Sigma_t \leftarrow \frac{1}{|D|} \sum_{(x_t, y_t) \in D} x_t x_t^\top\). Define \(\varphi_t(u) \triangleq \|u - v_t\|^2_{\Sigma_t}\).
7.     if \(|D| \geq N_0\) and \(\varphi_t(w) \geq C_0\) then
8.         // intersect current feasible region with \(\{u : \langle u - w, \nabla \varphi_t(w) \rangle < 0\}\)
9.         return separating hyperplane given by \(\nabla \varphi_t(w)\).

6 Online Algorithm

6.1 Cutting Plane Algorithm

In this section we leverage the guarantees of Section 4 to design an efficient algorithm for Huber-contaminated online regression. The basic trick we use is to combine the offline regression oracle with a cutting plane method, so that we can keep efficiently cutting down the space of linear predictors until we find one near \(w^*\). Essentially, the algorithm collects a large batch of samples, compares it’s current performance on this batch to the optimal robust regression result in hindsight (estimated by \(\text{SOSRegression}\)), and if it finds its performance is poor it cuts out a large set of possible predictors and updates to use a new predictor.

The algorithm, which we will refer to as \(\text{SOSAndCut}\), can be based upon any central cutting-plane optimization method like ellipsoid or Vaidya’s algorithm; here we use Vaidya’s algorithm since it is oracle-efficient. More specifically, we recall the following guarantee for Vaidya’s algorithm:

**Theorem 6.1** ([Vai89], see e.g. Section 2.3 of [Bub14]). Suppose that \(K\) is an (unknown) convex body in \(\mathbb{R}^d\) which contains a Euclidean ball of radius \(r > 0\) and is contained in a Euclidean ball centered at the origin of radius \(R > 0\). There exists an algorithm which, given access to a separation oracle for \(K\), finds a point \(x \in K\), runs in time \(\text{poly} (\log (R/r), d)\), and makes \(O(d \log (Rd/r))\) calls to the separation oracle.

Now we describe the algorithm. \(C_0\) and \(N_0\) are constants to be determined later. \(\text{SeparationOracle}\) (see Algorithm 2) implements the separation oracle (which is also where most of the interaction with Nature occurs). Here the input \(w\) lies in \(\mathcal{W} = \{w : \|w\| \leq R\}\) and Nature’s inputs are \(x_t\) with \(\|x_t\| \leq 1\). Finally, we note that if \(\text{SeparationOracle}\) gets to the final round \(T\) of the online regression problem, then it may not return to Vaidya’s algorithm (so step 2 of \(\text{SOSAndCut}\) is never reached), but as we will see, even if this happens the algorithm still achieves the correct regret bound.

As far as the choice of constants, based on (4.1) and Theorem 4.1 we will leave \(N_0\) to be
Algorithm 3: SoSandCut($r, R, N_0, C_0, T$)

Input: Radius $r$ of target ball around $w^*$, parameter $R$ from Assumption 3, parameters $N_0, C_0$ to be tuned, number of rounds $T$

Output: Sequence of predictions $\hat{y}_1, \ldots, \hat{y}_T$

1. Let $w$ be the output of running Vaidya’s algorithm [Vai89] with SeparationOracle defined above and parameters $r, R$, and let $\hat{y}_1, \ldots, \hat{y}_t$ be the predictions made in the course of running SeparationOracle.

2. for $t_1 + 1 \leq t \leq T$ do
   3. Given new point $x_t$ input by Nature, predict $\hat{y}_t = \langle w, x_t \rangle$.
   4. return $\hat{y}_1, \ldots, \hat{y}_T$.

optimized later and take

$$C_0 \triangleq 4Rr + \frac{1}{\rho^k} \cdot \Theta \left( k\sigma^2 \eta^{1-2/k} + \epsilon^2 + k\sigma^2 [d \log(1/\delta)/N_0]^{1/k-2/k^2} + R\sigma \sqrt{(d/kN_0) \cdot \log(T/\delta)} + \rho^2 R^2 \sqrt{\log(dT/\delta)/N_0} \right)$$

(19)

based on (4.1) from Theorem 4.1, where $\delta > 0$ is the desired overall probability of success. With this choice of parameters we can guarantee with probability at least $1 - \delta$:

1. At every step where $|D| \geq N_0$ in SeparationOracle, the guarantee (4.1) is satisfied by the vector $v_t$ output by SOSRegression, by applying Theorem 4.1 and the union bound over all rounds.

2. If $w$ lies outside the ball of radius $r$ around $w^*$, the result of SeparationOracle is a valid separating hyperplane between $w$ and the ball. By convexity of $\varphi$, to see that the ball of radius $r$ around $w^*$ is never cut, we just need to show that all $w'$ with $\|w' - w^*\| \leq r$ satisfy $\varphi_t(w') \leq C_0$. For $w^*$ we have the stronger guarantee $\varphi_t(w^*) \leq C_0 - 4Rr$, just from the guarantee of step 1. For other $w'$ in the ball of radius $r$, we deduce the claim by triangle inequality from the guarantee for $w^*$, using that

$$\varphi_t(w') - \varphi_t(w^*) \leq \langle \nabla \varphi_t(w'), w' - w^* \rangle = 2\langle \Sigma_t(w' - v_t), w' - w^* \rangle \leq 4R\|w' - w^*\| \leq 4Rr$$

where the first inequality is by convexity, and the second inequality uses that $\|\hat{\Sigma}_t\| \leq 1$ and that the diameter of $W$ is at most $2R$.

Recall that the separation oracle can only be called $I = O(d \log(R/r))$ many times, since this is the oracle complexity guarantee from Theorem 6.1: after this many rounds the algorithm is guaranteed to return or query a point in the ball of radius $r$ around $w^*$. Let $D_i$ be the collected dataset $D$ built during the $i$-th invocation of the oracle. Since we know by the triangle inequality and AM-GM that

$$\|w - w^*\|^2_{\hat{\Sigma}_t} \leq 2\|w - v_t\|^2_{\hat{\Sigma}_t} + 2\|v_t - w^*\|^2_{\hat{\Sigma}_t}$$

it follows that after $|D_i|$ gets to size $N_0$ and up to the step before returning a hyperplane, we are guaranteed that $\|w - w^*\|^2_{\hat{\Sigma}_t} \leq 4C_0$. For all of the steps before $|D_i|$ gets to size $N_0$, the error incurred per step is trivially upper bounded by $4R^2$. It follows that the regret incurred per call of the separation is upper bounded by $\max\{4N_0R^2, 4|D_i|C_0 + 4R^2\}$. Hence, the total regret incurred
in step 1 of SoSAndCut is upper bounded by

\[
\sum_{i=1}^{t}(4N_0R^2 + 4|D_i|C_0) \leq 4N_0IR^2 + 4C_0T = O\left(N_0dR^2 \log(R/r) + C_0T\right)
\] (20)

using that the total number of oracle calls is \(I = O(d \log(R/r))\), and \(\sum_t|D_i| \leq T\). If \(t_1\) is the time step at which the algorithm enters step 2, then the total regret in step 2 of SoSAndCut is upper bounded by

\[
\sum_{t=t_1}^{T}(\langle w^*, x_t \rangle + \epsilon_t - \langle w, x_t \rangle)^2 \leq \sum_{t=t_1}^{T}(r + |\epsilon_t|)^2 \leq 2T(r^2 + \epsilon^2)
\] (21)

where in the last step we used the basic inequality \((a + b)^2 \leq 2a^2 + 2b^2\). In particular, the leading term in the regret is \(O(k\sigma^2\eta^{1-2/k}T)\) as expected. We formalize this in the following Theorem.

**Theorem 6.2.** For the Huber-Contaminated Online Regression problem with \(\eta \leq \overline{\eta} < 1/2\) and \(\overline{\eta} = \frac{1}{2+2\rho^2}\), Algorithm SoSAndCut with parameters \(R\) and \(r \triangleq 1/T\) satisfies the following regret guarantee:

\[
\sum_{t=1}^{T}(y_t^* - \hat{y}_t)^2 \leq \frac{1}{\rho^4} \cdot \left[(k\sigma^2\eta^{1-2/k} + \epsilon^2)T + (d^{1+\alpha_k}R^2 \log(R)T^k + k\sigma^2d^{1+\alpha_k} \log(1/\delta)^{1/k^2}T^{\beta_k}) \right.
\]

\[
+ R\sigma \sqrt{(d^{1+\alpha_k}/k) \cdot \log(2/\delta)T^{1-\beta_k/2} + \rho^2R^2 \sqrt{\log(d/\delta)/d\alpha_kT^{1-\beta_k/2}}}
\] (22)

with probability at least \(1 - \delta\) over the randomness of the coin flips, where \(\alpha_k \triangleq \frac{-k^2+k-2}{k^2+k-2}\) and \(\beta_k \triangleq \frac{k^2}{k^2+k-2}\).

**Proof.** From the above (4) and (4), we see that the total regret is upper bounded by

\[O\left(N_0dR^2 \log(R/r) + C_0T\right) + 2T(r^2 + \epsilon^2).
\]

and recalling how \(N_0\) appears in \(C_0\) in the terms

\[O(k\sigma^2[d \log(1/\delta)/N_0]^{1/k-2/k^2} + R\sigma \sqrt{(d/kN_0) \cdot \log(2/\delta) + \rho^2R^2 \sqrt{\log(d/\delta)/N_0}}\]

from (4), we see that the bound is approximately minimized by taking

\[N_0 \triangleq d^{(-k^2+k-2)/(k^2+k-2)}T^{k^2/(k^2+k-2)} = d^{\alpha_k}T^{\beta_k}.
\]

Taking \(r \triangleq 1/T\) and simplifying the result, we get the claimed regret bound. \(\square\)

### 6.2 Gradient Descent Algorithm

For the high-dimensional setting, cutting planes don’t work because their guarantees are dimension-dependent. Fortunately, we can fix this by using gradient descent instead. We recall the following guarantee for online gradient descent from [Zin03].

**Theorem 6.3 ([Zin03, Haz19]).** Suppose that \(f_1, \ldots, f_T\) is a sequence of convex functions such that \(\|\nabla f_t(w)\| \leq G\) for any \(w\) with \(\|w\| \leq R\). Let \(w_1 = 0\) and suppose that

\[w_{t+1} \triangleq \Pi_R \left(w_t - \frac{2R}{G\sqrt{T}} \nabla f_t(w_t)\right)
\]
Algorithm 4: SoS-GD($R, N_0, C_1, \gamma, T$)

\textbf{Input}: Parameter $R$ from Assumption 3, number of rounds $T$, parameters $r, N_0, C_1, \gamma$ to be tuned

\textbf{Output}: Sequence of predictions $\hat{y}_1, \ldots, \hat{y}_T$ (via interaction with Nature)

1. Let $w_1 = 0$.
2. While there are more inputs do
   3. Let $g_s$ be the output of SeparationOracle run with parameters $r = 0, R, C_1$ and input $w_s$
   4. Let $w_{s+1} = w_s - \frac{1}{\sqrt{\tau}}g_s$.
   5. Set $s \leftarrow s + 1$.

where $\Pi_R(x) \triangleq \frac{x}{\max(R, \|x\|)}$ is the projection onto the Euclidean ball of norm $R$. Then for any $w^*$ with $\|w^*\| \leq R$,

$$\sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(w^*) \leq \sum_{t=1}^{T} \langle \nabla f_t(w_t), w_t - w^* \rangle \leq 3RG\sqrt{T}.$$ 

We now discuss parameter selection: we define

$C_0 \triangleq \Theta \left(k\sigma^2\eta^{1-2/k} + \epsilon^2 + (R^2 \log(N_0T/\delta) + k\sigma^2)N_0^{-\gamma_k}
+ R\sigma \sqrt{(1/kn) \cdot \log(2T/\delta)}N_0^{(\gamma_k - 1)/2} + \rho^2 R^2 \sqrt{\log(N_0T/\delta)}N_0^{(\gamma_k - 1)/2}\right)$

where $\delta > 0$ is the overall acceptable probability of failure, based upon the right-hand side of (5.3) and take $C_1 \triangleq 2C_0$.

**Theorem 6.4.** For the Huber-Contaminated Online Regression problem with $\eta \leq \eta^* < 1/2$ and $\eta^* = \frac{1}{2 + 2\rho R}$, Algorithm SoS-GD with parameters $R$ and $\gamma = \Theta(1)$ satisfies the following regret guarantee:

$$\sum_{t=1}^{T} (y_t^* - \hat{y}_t)^2 \lesssim k\sigma^2\eta^{1-2/k} + \epsilon^2 + (R^2 \log(T/\delta) + k\sigma^2)T^\beta_k
+ R\sigma \sqrt{(1/kn) \cdot \log(2T/\delta)}T^{1-\beta_k'(\gamma_k - 1)/2} + \rho^2 R^2 \sqrt{\log(T/\delta)}T^{1-\beta_k'(\gamma_k - 1)/2}$$

with probability at least $1 - \delta$ over the randomness of the coin flips, where $\beta_k' \triangleq 1/(1 + \gamma_k) = (k^2 + k - 2)/(k^2 + 2k - 4) \in (0, 1)$ and $\gamma_k$ was defined in Theorem 5.3.

**Proof.** As in the proof of Theorem 6.2, we first bound the regret incurred in a single call of SeparationOracle by $4N_0R^2 + 8|D_i|C_0$ where $D_i$ is the dataset $D$ collected in call $i$. It follows then that if $V$ is the total number of calls made to SeparationOracle then the total clean regret is upper bounded by $O(N_0R^2V + TC_0)$ where we used that $\sum_i |D_i| \leq T$. On the other hand, we know from Theorem 6.3 that if we define $\varphi_i$ to be the function whose gradient is returned at the end of Algorithm SeparationOracle, then

$$C_0V = (C_1 - C_0)V \leq \sum_{s=1}^{V} (\varphi_i(w_s) - \varphi_i(w^*)) \leq 6R^2\sqrt{V}$$
Proof. For the Huber-Contaminated Contextual Bandits problem with contamination rate \( \eta < 1/2 \), misspecification rate \( \epsilon \), range parameter \( R \), noise parameter \( \sigma \), action space of size \( K \), and \( d \)-dimensional contexts, if Assumption 2 holds for some choice of \( c,k \), then there is a \( \text{poly}(n,d) \)-time algorithm which achieves clean pseudo-regret \( \text{Reg}_{\text{HCB}}(T) \) at most

\[
O(C_\eta \sqrt{K}) \left( (\epsilon + k^{1/2} \sigma \eta^{k-2} T + \left( d^{-1/2} R \sqrt{\log(RT)} + \sqrt{k} \sigma d^{-1/2} ((1 - \beta_k) \log T) \frac{k-2}{k^2} \right) T^{1+\beta_k} + \left( k^{-1/4} R^{1/2} \sigma^{1/2} \frac{1}{4}(1 - \beta_k) \log T)^{1/4} + d^{-\alpha_k/4} R ((1 - \beta_k) \log(dT))^{1/4} \right) T^{2-\beta_k} \right),
\]

where \( \alpha_k, \beta_k \) are defined in Theorem 6.2 and \( C_\eta \) is some increasing function of \( \eta \). In particular, for sufficiently large \( T \), this quantity is dominated by \( (\epsilon + k^{1/2} \sigma \eta^{k-2}) \sqrt{KT} \).

In the special case where \( \epsilon = 0 \), for any \( \delta > 0 \), there is a \( \text{poly}(n,d) \)-time algorithm which achieves clean regret \( \text{Reg}_{\text{HCB}}(T) \) at most

\[
O(C_\eta \sqrt{K}) \left( k^{1/2} \sigma \eta^{k-2} T + \left( d^{-1/2} R \sqrt{\log(RT)} + \sqrt{k} \sigma d^{-1/2} \log(1/\delta) \frac{k-2}{k^2} \right) T^{1+\beta_k} + \left( k^{-1/4} R^{1/2} \sigma^{1/2} \frac{1}{4} \log(2/\delta)^{1/4} + d^{-\alpha_k/4} R \log(d/\delta)^{1/4} \right) T^{2-\beta_k} + \sqrt{T} \log(2/\delta) \right)
\]

with probability \( 1 - \delta \). For sufficiently large \( T \), this is dominated by \( k^{1/2} \sigma \eta^{k-2} \sqrt{KT} \).

Proof. For the first part of the theorem, we can apply Theorem 6.2 with \( \delta = T^{\beta_k - 1} \land 1/3 \) to get that the clean square loss regret incurred by SOSAndCut is given by (6.2) with probability at least 2/3 and is otherwise upper bounded by \( R^2 T \). So the expectation of this quantity is at most the quantity in (6.2) plus \( R^2 T^{\beta_k} \), which is dominated by the \( T^{\beta_k} \) term in (6.2). The result then follows from applying the clean pseudo-regret bound of Theorem A.1 and using the elementary fact that for positive numbers \( \{a_i\}_{i \in [s]} \), \( \sum_{i=1}^{s} a_i^{1/2} \leq \sum_{i=1}^{s} \sqrt{a_i} \).

For the second part of the theorem, we can directly apply the high-probability guarantee Theorem 6.2 together with the high-probability guarantee of Theorem A.3 and a union bound. The result follows upon absorbing the constant factor in front of \( \delta \).

\begin{theorem} \textbf{(High-dimensional variant of Theorem 7.1).} \let\eta\eta\let\epsilon\epsilon\let\sigma\sigma\let\K\K \thetaeta, \thetaepsilon, \thetaR, \theta\sigma, \thetaK \text{ be the same as in Theorem 7.1, but now we make no assumptions on the dimension of the context space } \thetaX. \thetaThere
exists an algorithm which runs in time \( \text{poly}(n,d) \) and achieves clean pseudo-regret \( \overline{\text{Reg}}_{\text{SHCB}}(T) \) at most
\[
O(C_\eta \sqrt{K}) \left( (\epsilon + k^{1/2} \sigma \eta^{k/2}) T + (R \sqrt{\log(T/\delta)} + \sigma \sqrt{K}) T^{(1+\beta_k')/2} + (R^2 \sigma^2 (1/kn) \log(2T/\delta))^{1/4} T^{1-\beta_k'(\gamma_k-1)/4} + \rho R \log(T/\delta)^{1/4} T^{1-\beta_k'(\gamma_k-1)/4} \right)
\]
where \( \beta_k' \) is defined in Theorem 6.4, \( \gamma_k \) is defined in Theorem 5.3, and \( C_\eta \) is some increasing function of \( \eta \). In particular, for sufficiently large \( T \), this quantity is dominated by \( C_\eta \left( \epsilon + k^{1/2} \sigma \eta^{k/2} \right) \sqrt{KT} \).

**Proof.** The proof is identical to Theorem 7.1, except that we replaced the use of Theorem 6.2 by Theorem 6.4. \( \square \)

### 8 Lower Bound Against Convex Surrogates

We exhibit an \( \Omega(\eta^3 \sigma R) \) lower bound against regression using convex losses. This lower bound captures natural approaches like Huber regression, L1 (i.e. LAD) regression, and OLS. By rescaling, we can assume \( \sigma = 1 \) without loss of generality, which we do in the statement of the Lemma below.

**Theorem 8.1.** For any convex loss \( h(\cdot) \), there exists a distribution over covariates \( x \sim \mathcal{D}_x \) with support in \([-R,R]\) and true regressor \( \ell \in [-1,1] \). Let \( y \sim \ell \cdot x + \zeta \) with noise \( \zeta \sim \mathcal{N}(0,1) \), and let \( \mathcal{C} \) denote the joint distribution over \((x,y)\). Furthermore, let \( \tilde{y} \) denote the Huber contaminated labels drawn \( y \sim (1-\eta)(\ell \cdot x + \zeta) + \eta Q \) where \( Q \) is an arbitrary distribution with support in \([-R,R]\) for \( R \geq \frac{1}{\eta} \) and \( \eta \in [0,\frac{1}{2}) \). Let \( \mathcal{H} \) be the joint distribution of the contaminated data \((x,\tilde{y})\). Let \( w^* := \arg \min_{w} \mathbb{E}_{(x,\tilde{y}) \sim \mathcal{H}}[h(y - \ell \cdot x)] \) be the minimizer of the loss on contaminated data. Then the square loss of \( w^* \) on clean data is lower bounded by \( \mathbb{E}_{(x,y) \sim c}[(y - w^* \cdot x)^2] \geq \frac{\eta^2 R}{\delta^2} \).

**Proof.** Our hard instance is constructed as follows. Let \( \mathcal{D}_x \triangleq m_1 \delta(1) + (1-m_1)\delta(-R) \) where \( \delta(\cdot) \) is the dirac delta and \( m_1 = 1 - \frac{\eta}{10R} \). Let the true regressor \( \ell = 0 \) so that the uncorrupted \( y \sim \mathcal{N}(0,1) \) for all \( x \in [-R,R] \). Let the corrupted labels be \( \tilde{y} \) defined as follows

\[
\tilde{y} = \begin{cases} 
(1-\eta)\mathcal{N}(0,1) + \eta \delta(R+1) & x = 1 \\
\mathcal{N}(0,1) & x = -R
\end{cases}
\]

Let \( h'(\cdot) \) be the right derivative of \( h(\cdot) \), which is well defined because every convex function on an open convex domain is semi-differentiable. Let \( g(v) \triangleq -\mathbb{E}_{y \sim \mathcal{N}(0,1)}[h'(y - v)] \). By convexity of \( h(\cdot) \) we have the right derivative evaluated at \( w^* \) is greater than or equal to zero.

\[
\lim_{\epsilon \to 0} \mathbb{E}_{(x,y) \sim \mathcal{H}} \left[ h(y - (v + \epsilon) \cdot x) \right] - \mathbb{E}_{(x,y) \sim \mathcal{H}} \left[ h(y - v \cdot x) \right] \bigg|_{v = w^*} = (1-\eta)m_1 \cdot g(w^*) - h'(R + 1 - w^*) \eta \cdot m_1 + (1-m_1)Rg(-Rw^*) \geq 0
\]

Rearranging we obtain
\[
g(w^*) \geq \frac{h'(R + 1 - w^*) \eta \cdot m_1 - (1-m_1)Rg(-Rw^*)}{(1-\eta)m_1} \tag{23}
\]

Let \( g^{-1}(\cdot) \) denote the left inverse of \( g(\cdot) \). Note that \( h(\cdot) \) is convex implies \(-h'(\cdot)\) is monotonically decreasing implies \( g(\cdot) \) is monotonically increasing implies \( g^{-1}(\cdot) \) is monotonically increasing. Thus, applying \( g^{-1}(\cdot) \) to both sides of (8) we obtain
\[
w^* \geq g^{-1} \left( \frac{h'(R + 1 - w^*) \eta \cdot m_1 - (1-m_1)Rg(-Rw^*)}{(1-\eta)m_1} \right) \tag{24}
\]
To lower bound \( w^* \) it suffices to lower bound the argument of \( g^{-1}(\cdot) \). We obtain,

\[
\frac{h'(R + 1 - w^*)\eta \cdot m_1 - (1 - m_1)R \cdot g(-R w^*)}{(1 - \eta)m_1} \geq \frac{h'(R)\eta \cdot m_1 + h'(R)R(1 - m_1)}{(1 - \eta)m_1}
\]

Where we lower bounded the first term in the numerator using the fact that \( h'(\cdot) \) is monotonically increasing and \( w^* \in [-1, 1] \) to conclude \( h'(R + 1 - w^*) \geq h'(R) \). We lower bounded the second term in the numerator using the fact that \( g(\cdot) \) is monotonically increasing and that \( h'(R) \geq \max_{[-R, R]} |h'(x)| \) (monotonicity of \( h'(\cdot) \)) to conclude \( g(-R w^*) \geq g(-R) \geq -h'(R) \). Further lower bounding, we obtain

\[
\frac{h'(R)(\eta m_1 - (1 - m_1)R)}{(1 - \eta)m_1} = \frac{h'(R)(\eta(1 - \frac{\eta}{10R}) - \frac{\eta}{10})}{(1 - \eta)m_1} \geq \frac{\eta}{2(1 - \eta)m_1} \geq \frac{h'(R)\eta}{2}
\]

Where in the first inequality we use that \( R \geq \frac{1}{\eta} \). Substituting this lower bound into (8) we obtain \( w^* \geq g^{-1}(\frac{h'(R)\eta}{2}) \). Once again using the fact that \( h'(R) \geq \max_{[-R, R]} |h'(x)| \) we observe that

\[
g(\rho) - g(g^{-1}(0)) \leq \frac{(\rho - g^{-1}(0))h'(R)}{\sqrt{2\pi}}
\]

for any \( \rho \geq g^{-1}(0) \). This follows by the definition of \( g(\cdot) \) and the fact that the mode of the standard gaussian is \( \frac{1}{\sqrt{2\pi}} \). Setting \( \rho = g^{-1}(\frac{h'(R)\eta}{2}) \) we obtain

\[
\frac{h'(R)\eta}{2} = g(g^{-1}(\frac{h'(R)\eta}{2})) - g(g^{-1}(0)) \leq \frac{(g^{-1}(\frac{h'(R)\eta}{2}) - g^{-1}(0))h'(R)}{\sqrt{2\pi}}
\]

which implies

\[
w^* \geq g^{-1}(\frac{h'(R)\eta}{2}) \geq \eta + g^{-1}(0) \quad (25)
\]

We then have two possibilities.

**Case 1:** Either \( g^{-1}(0) \geq \frac{-\eta}{2} \) in which case the loss is lower bounded by

\[
\mathbb{E}_{(x, y) \sim \mathcal{C}}[(y - w^* \cdot x)^2] \geq \mathbb{E}_{(x, y) \sim \mathcal{C}}[(y - w^* \cdot x)^2 | x = -R] \mathbb{P}_D(x = -R) = (1 - m_1)R^2(w^*)^2 \geq (1 - m_1)R^2(\eta + g^{-1}(0))^2 \geq \frac{\eta^3 R}{40}
\]

Where in the first inequality we use the law of total expectation, and in the second inequality we used (8) and \( g^{-1}(0) \geq \frac{-\eta}{2} \). This is the desired lower bound.

**Case 2:** In the other case we have \( g^{-1}(0) \leq \frac{-\eta}{2} \). Then we flip the sign of the corruptions placed by the adversary. Let the corrupted distribution be

\[
\hat{y} = \begin{cases} 
(1 - \eta)\mathcal{N}(0, 1) + \eta\delta(-R - 1) & x = 1 \\
\mathcal{N}(0, 1) & x = -R 
\end{cases}
\]

Then working through the same calculations flipping signs at the right places we obtain \( w^* \leq g^{-1}(\frac{-h'(R)\eta}{2}) \). Once again, using that

\[
g(\rho) - g(g^{-1}(0)) \geq \frac{(\rho - g^{-1}(0))h'(R)}{\sqrt{2\pi}}
\]

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for any $\rho \leq g^{-1}(0)$, and setting $\rho = g^{-1}(\frac{-h'(R)\eta}{2})$ we obtain

$$-\frac{h'(R)\eta}{2} = g(g^{-1}(\frac{-h'(R)\eta}{2})) - g(g^{-1}(0)) \geq \frac{(g^{-1}(\frac{-h'(R)\eta}{2}) - g^{-1}(0))h'(R)}{\sqrt{2\pi}}$$

Rearranging we obtain

$$w^* \leq g^{-1}(\frac{-h'(R)\eta}{2}) \leq g^{-1}(0) - \eta \leq -\frac{3\eta}{2}$$

Where the last inequality follows by $g^{-1}(0) \leq \frac{-\eta}{2}$. The loss is then lower bounded by

$$\mathbb{E}_{(x,y)\sim C}(y - w^* \cdot x)^2 \geq \mathbb{E}_{(x,y)\sim C}(y - w^* \cdot x)^2 | x = -R| \mathbb{P}_D(x = -R) \geq (1 - m_1)R^2 (w^*)^2 \geq \frac{9\eta^2 R}{40}$$

where in the last inequality we use $w^* \leq \frac{-3\eta}{2}$. This is our desired lower bound.

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A Reduction from Contextual Bandits to Online Regression

In this section we verify that the reduction given in [FR20], specifically the proof of Theorem 5 in their paper, also applies to our Huber-contaminated setting as well. Formally, we show the following:

**Theorem A.1 (Bandits to Regression Reduction).** Given any oracle $O$ for Huber-contaminated online regression achieving clean square loss regret $\text{Reg}^\text{HSq}(T)$ in the sense of Definition 2, we can produce a learner for Huber-contaminated contextual bandits in the sense of Definition 1 that achieves clean pseudo-regret $O\left(\sqrt{KT \cdot \text{Reg}^\text{HSq}(T)} + \epsilon \sqrt{K}T\right)$. 

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We will use the SQUARECB algorithm from [FR20], which draws upon ideas from [AL99], and which we repeat here for completeness:

**Algorithm 5: SQUARECB(A, γ, µ)**

**Input:** Online regression oracle \( \mathcal{O} \), learning rate \( \gamma > 0 \), exploration parameter \( \mu > 0 \)

**Output:** Sequence of actions, in the setting of Definition 1

1. for \( t \in [T] \) do
   2. Get context \( z_t \) from Nature.
   3. For every \( a \in A \), use regression oracle \( \mathcal{O} \) to compute prediction \( \hat{y}_{t,a} \triangleq \hat{y}_t(z_t, a) \).
   4. Define \( b_t \triangleq \arg \min_{a \in A} \hat{y}_{t,a} \).
   5. For any \( a \neq b_t \), define \( p_{t,a} = \frac{1}{t+\gamma(1/\mu - 1)} \) and let \( p_{t,b_t} = 1 - \sum_{a \neq b_t} p_{t,a} \). The numbers \( \{p_{t,a}\}_a \) define a distribution \( p_t \) over actions.
   6. Sample \( a_t \) from \( p_t \) and observe loss \( \ell \), and update \( \mathcal{O} \) with example \( ((x_t, a_t), \ell) \).

**Proof of Theorem A.1.** Fix any policy \( \pi : X \to A \) and consider the learner given by SQUARECB (Algorithm 5) above for a regression oracle \( \mathcal{O} \) achieving square loss \( \text{Reg}_{\text{HSq}}(T) \), which is some random variable depending on the interactions with Nature. Recall that for this choice of learner, \( \text{Reg}_{\text{HCB}}(T) \) is the supremum of

\[
E \left[ \sum_{t=1}^{T} (\ell^*_t(a_t) - \ell^*_t(\pi(z_t))) \right]
\]

over all such \( \pi \). Define the filtration

\[
\mathcal{F}_{t-1} \triangleq \sigma((z_1, a_1, \ell^*_1(a_1), \ell_1(a_1), \gamma_1), \ldots, (z_{t-1}, a_{t-1}, \ell^*_t(a_{t-1}), \ell_t(a_{t-1}), \gamma_{t-1}), (z_t, \gamma_t)).
\]

We can write the sum of conditional expectations of immediate regrets incurred by \( \pi \) as

\[
\sum_{t=1}^{T} E[(\ell^*_t(a_t) - \ell^*_t(\pi(z_t))) | \mathcal{F}_{t-1}] \leq \sum_{t=1}^{T} E[(f(z_t, a_t) - f(\pi(z_t))) | \mathcal{F}_{t-1}] + 2\epsilon T
\]

\[
\leq \sum_{t=1}^{T} E[(f(z_t, a_t) - f(\pi f(z_t))) | \mathcal{F}_{t-1}] + 2\epsilon T
\]

\[
= \sum_{t=1}^{T} \sum_{a \in A} p_{t,a}(f(z_t, a) - f(z_t, \pi f(z_t))) + 2\epsilon T. \tag{26}
\]

where recall from Definition 1 that \( \pi f(z) \triangleq \arg \max_{a} f(z, a) \), and \( p_{t,a} \) is defined in Step 5 of SQUARECB.

The following lemma is a key ingredient in the reduction of [FR20]:

**Lemma A.2** (Lemma 3, [FR20]). For any collection of numbers \( \{\hat{y}_a\}_{a \in A} \subseteq [-R, R]^K \), let \( p \) be the corresponding probability distribution computed in Step 5. For any collection of numbers \( \{f_a\}_{a \in A} \subseteq \{-R, R\}^K \), we have that

\[
\sum_{a \in A} p_a \left[ (f_a - f_a^*) - \frac{\gamma}{4} (\hat{y}_a - f_a)^2 \right] \leq \frac{2K}{\gamma}
\]
Applying Lemma A.2, we can upper bound (6) by
\[
\frac{\gamma}{4} \sum_{t=1}^{T} \mathbb{E}[(\hat{y}_t(z_t, a_t) - f(z_t, a_t))^2 \mid \mathcal{F}_{t-1}] + \frac{2KT}{\gamma} + 2\epsilon T.
\]
By this and law of total expectation, the pseudo-regret incurred by policy π can be upper bounded by
\[
\frac{\gamma}{4} \mathbb{E}[(\hat{y}_t(z_t, a_t) - f(z_t, a_t))^2] + \frac{2KT}{\gamma} + 2\epsilon T.
\] (27)
To bound the prediction error in (6), using the identity \(b^2 \leq (a + b)^2 - 2ab\), we can upper bound \((\hat{y}_t(z_t, a_t) - f(z_t, a_t))^2\) by
\[
(\hat{y}_t(z_t, a_t) - \ell^*_t(a_t))^2 - 2(f(z_t, a_t) - \ell^*_t(a_t))(\hat{y}_t(z_t, a_t) - f(z_t, a_t)).
\] (28)
Recall from (1) that the misspecification adversary is oblivious, that is, conditioned on \(\mathcal{F}_{t-1}\), \(f(z_t, a_t) - \ell^*_t(a_t)\) is equal to \(-\epsilon_t(z_t, a_t)\). Putting this and (6) together and applying law of total expectation, we can bound the expectation of the prediction error in (6) by
\[
\mathbb{E}[(\hat{y}_t(z_t, a_t) - f(z_t, a_t))^2]
\leq \mathbb{E}[	ext{Reg}_{\text{HSq}}(T)] + 2\mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{E}[\epsilon_t(z_t, a_t)(\hat{y}_t(z_t, a_t) - f(z_t, a_t)) \mid \mathcal{F}_{t-1}] \right]
\leq \mathbb{E}[	ext{Reg}_{\text{HSq}}(T)] + 2\mathbb{E} \left[ \sum_{t=1}^{T} \epsilon_t^2(z_t, a_t) + \frac{1}{4} \sum_{t=1}^{T} \mathbb{E}[(\hat{y}_t(z_t, a_t) - f(z_t, a_t))^2 \mid \mathcal{F}_{t-1}] \right]
\leq \mathbb{E}[	ext{Reg}_{\text{HSq}}(T)] + 2\epsilon^2 T + \frac{1}{2} \sum_{t=1}^{T} \mathbb{E}[(\hat{y}_t(z_t, a_t) - f(z_t, a_t))^2],
\]
which upon rearranging gives
\[
\mathbb{E}[(\hat{y}_t(z_t, a_t) - f(z_t, a_t))^2] \leq 2\mathbb{E}[	ext{Reg}_{\text{HSq}}(T)] + 4\epsilon^2 T.
\]
Substituting this into (6), and taking \(\gamma = 2\sqrt{KT/(\mathbb{E}[	ext{Reg}_{\text{HSq}}(T)] + 2\epsilon^2 T)}\) and \(\mu = K\), we conclude that the pseudo-regret incurred by π is upper bounded by
\[
\frac{\gamma}{2}(\mathbb{E}[	ext{Reg}_{\text{HSq}}(T)] + 2\epsilon^2 T) + \frac{2KT}{\gamma} + 2\epsilon T \leq 2\sqrt{KT \cdot \mathbb{E}[	ext{Reg}_{\text{HSq}}(T)] + 5\epsilon \sqrt{KT}}.
\]
as desired.

In the special case where \(\epsilon = 0\), [FR20] also gives a high-probability bound on the regret (see their Theorem 1). By adapting their argument, we can show an analogous statement in this setting:

**Theorem A.3** (Bandits to Regression Reduction). Fix any \(\delta > 0\). Given any oracle \(\mathcal{O}\) for Huber-contaminated online regression achieving clean square loss regret \(\text{Reg}_{\text{HSq}}(T)\) in the sense of Definition 2 with \(\epsilon = 0\), we can produce a learner for Huber-contaminated contextual bandits in the sense of Definition 1 that with probability at least \(1 - \delta\) achieves achieves clean regret at most \(4\sqrt{KT \cdot \text{Reg}_{\text{HSq}}(T)} + 8\sqrt{KT \log(2/\delta)}\).