Graph model overview, events scales structure and chains of events.

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Abstract. We present a graph model for a background independent, relational approach to spacetime emergence. The general idea and the graph main features, detailed in [1], are discussed. This is a combinatorial (dynamical) metric graph, colored on vertexes, endowed with a classical distribution of colors probability on the graph vertexes. The graph coloring determines the graph structure in clusters of graph vertices (events) that can be monochromatic (homogeneous loops) or polychromatic (inhomogeneous loops). The probability is conserved after the graph conformal expansion from an initial seed graph state to higher (conformally expanded) graph states. The emerging structure has self-similar characteristics on different scales (states). From the coloring, different levels of vertices and thus graph levels arise as new aggregates of colored vertexes. In this second (derived) graphs level, the derived graph vertices correspond to the polychromatic edges (with differently colored vertexes) of the initial graph. Vertex aggregates are related, as some levels (graph states) to plexors and twistors (involving Clifford statistics). Two metric levels are defined on the colored graph, the first level is a natural metric defined on the graph, the second level emerges from the first and related, due to symmetries. Metric structure reflects the graph colored structure under conformal transformations evolving with its states under conformal expansion. In some special cases vertices/events chains could be related to strings generalizations.

1 Introduction

We introduce a (decomposition) graph for a background independent model of relational emergent spacetime in the frame of (dynamical) quantum graph, here we summarize some ideas underlying the model and we sketch the graph (lattice) main features, a more extensive discussion is in [1]. Eventually, with an adaptation of the metric structures (metric graph) to a dynamical quantum graph one goal is to address in a combination of top-down and down-top strategy the frame of spacetime emergence. The graph closes the decompositions of the spacetime texture in graph vertexes (events) and edges, the spacetime topological, causal and metric structure naturally change in the graph frame. The replacement of the spacetime (differential) manifold with a graph grounds the search for a common language for different scales, for (assumed) different events scales. Space-time can be assumed to emerge at certain scales of the model, connected to graph conformal transformations in higher graph states (conformal expansion). This brief note, discussing the decomposition graph main properties and it can be considered as a model overview and a graph proposal. The spacetime emergence problem is translated into events scale emergence and structures of a "crystallized" graph. The (combinatorial) graph (close to a Cayley graph) is colored and characterized by a classic definition of color probabilities (frequencies) distribution. A relevant aspect of the analysis consists in the construction of vertexes and clusters of vertexes with inner colored structure (monochromatic-homogenous loops ϵ, or polychromatic-inhomogeneous in colors- loops) that can be studied as emerging from the conformal transformation of the colored graph, described through the concept of event algebras $\mathcal{A}$ and created through evolutive operators (introducing a vertex order relation) and shift operators (for color shift), related to graph edges, and acting on vertexes. Lower graph states (scales) admit fluctuations in their structure, understood, in a first approximation, as variation of algebra distribution in the adjacent chain vertexes (in the sense of the order relation in the chain) for color and for graph state and paths (chain). Therefore algebra inhomogeneity (related to a definition of graph radius and inertia) leads to inhomogeneous in algebra chain with valence two vertexes (algebraic valence ($\pm 1, 0$)). We could compare this approach with theories with graphs as an underlying language: the (causal) loop quantum gravity (LQG), spin network and causal set theories. This model also shares some aspects with the analysis in combinatorial spacetime, stochastic metric and quantum computation spacetime and more generally with it-from (qu)bit ideas–[12,13,14,15,16,17]. Chains of events have an event strings-formulation as collateral: chains might be translated as strings generalizations. Our analysis may be also read into quantum graphity framework (QGF) as a
graph dynamical model [2][3][4][5]. Quantum graphity may be defined as a model of emergent geometry (locality) in the context of quantum gravity theory, based on a background independent formulation of condensed matter systems on graphs that can be coupled to a heat bath with temperature. In this frame a quantum model is attached as an Hilbert space to some graph elements, for example the dynamical graph vertices and therefore defining system states which are ultimately given by bosonic (or fermionic) vertex degrees of freedom. An hamiltonian is constructed an Hilbert space to some graph elements, for example the vertex approach, constitutes a pre-geometric (quantum) system

determined the graph "dynamics" and structure. There are however different versions of the QGFs. Generally the approach followed in QGF is the graph dynamic one. In this work however we aim to combine a dynamical and metric graph. The dynamical setup is pursued on the elements associated with parts of the graph which, according to the realization of the theory, are the graph edges (lines) or loops. We start from a vertex approach, the coloring is on the vertices, in this way it is immediate to define loops as clusters of vertices characterized by their coloring. Other levels of events/vertices are associated with connections between different colored vertices of a new different associated graph of higher level, \(\delta\)events ("partial"-events), thus a derived graph (composed by these elements) and a derived metric structure follow. The metric reflects this internal graph clustering providing, in first approximation, a loop measure. In QGF, in general, matter would emerge only after the phase transition from the (net) condensation of the graph as emergent symmetry in the phase transition to the ordered phase graph. At this point we can say that the graph lives in two different regimes: a (final) low energy and the (initial, cosmological) high energy regime. At the high temperature, there is a disordered phase where notions of geometry, dimension and topology drastically changes. In this phase physics seems compelled to be described in quantum mechanical terms while a (phase) transition occurs and the low temperature phase, where the system is ordered and a background and fields emerge, living on a low dimensional spacetime (low temperature and large volume) manifold with a metric, implying the geometry of general relativity (gravity geometriztions). Therefore in QGF the high energy regime features a not local theory where the graph is completely connected and invariant under the full symmetric group acting on the vertices, graph symmetries are associated to the symmetries of permutation. In the second phase of QGF, a graph ground state is defined, the system turns by this cooling process ordered, local and, low-dimensional. This dynamics breaks the permutation symmetry into translations and rotations. In this scenario a U(1) gauge theory can emerge in the ground state by a mechanism of string-net condensation as Levin-Wen mechanism [6]. Maintaining the discretization, the transition should clearly leave a latex with a translation group. Emergence of geometry by such a phase transition is indicated as geometrgenesis. We retain in our investigation this aspect developing a combinatorial colored graph approach although we do not foresee a development in graph structure evolution eras with a cosmological geometro-genesis phase.

More specifically on regards on the graph construction, which is one main focus on this article, we note that in LQG, (space) geometry is on graphs, with lines "colored" with a half-integral number associated to SU(2) representations. Thus, a spin network represents the quantum state of space at a certain point in time, more precisely the states of the spin network basis are the eigenvectors of operators area (quanta on the links) and volume (quanta on the intersections) and are a basis for the (kinematical) LQG states. Introduced by Penrose as a combinatorial foundation for Euclidean 3-space spin network was conceived mainly as a trivalent graphs with spins assigned edges, with vertices labelled by intertwining operators [9][10][11]. LQG has been formulated canonically in the frozen time formalism where a state space is the direct sum Hilbert spaces associated to the graphs. The configuration variable turned to be then an SU(2)-connection (where LQG path integral can be realized as a sequence of spin-network states (3-geometries) as transition terms of spin foam models). On the other hand, these analysis, featuring a combinatorial model, consider cluster of colored vertices, and it is clear that a Clifford algebra will derive, this should guarantee a spinorial representation also after conformal transformations and algebraic re-parametrization (as algebra homogenization i.e. a graph and metric chains parametrization in different events bases adapted to the coloring, it is then important to consider transformations between graphs states with different bases.). We here question more generally the graph model underlying the spin structure selection (emergence), addressing this problem from general considerations on vertex structure and the graph main properties; there is no (a-priori) imposition of a specific emerging spinorial structure (a specific Clifford or Grassmann algebra), arguing a "best-fit" description from general considerations described through the graph language. A "spinorial" structure would emerge from the dynamics of colored graph elements and graph scales related to the graph conformal transformations and associated to a first level (boson) structure, as a "super-symmetric" model where however the associated elements are not simply conserved for state and graph realization (chain) transitions.

Vertexes and clusters could be considered as (geometry) fundamental elements, like in several other approaches, where the fundamental spacetime building blocks are considered as qubit: in these modes space would be a collection of qubits, where the empty space (therefore the vacuum) would correspond to a ground state of qubits, and elementary particles as collective excitations above the ground states. One crucial aspect of these approaches consists in the determination of qubit (and multibit) organizations and, eventually, interaction (and dimensionality of associated Hilbert space). That is a gate might be a superposed quantum state of vertices or clusters in
the graph, multi-(qu)bit states can be constructed considering aspects of the combinatorial graphs and these should be built by graph generations associated with an initial seed graph and emerging after conformal graph expansion where concept of loops, algebra and inertia on the graph and events scale structure are introduced. The role of the graph conformal expansion is relevant as related to the large scales and eventually has a role in the bit- to-qubit transition in this model. We implemented in [1] a simple logic signal representation, using mainly a 2-level logic signal representation (bichromatic graph) with a set of $\delta$ events of a “reference” (i.e. an “inner” parameter) chain. This analysis however has been applied including more basis (coloring), introducing the concept of complementary-chains, immediate in the signals representation relating the signal frequency to the vertex algebra in a first approximation of a bundle chain (a path graph state formulations featuring all possible histories at fixed graph state). The special case of an equiprobable graph i.e. with equiprobable colors distribution, and where there is a maximum in the colors probability distribution are investigated, these aspects regulate the graph vertex loops and their algebra variation after conformal transformation, the presence of a maximum decomposition in events of a vertex sequence, and the $\delta$ events emergence. Using such representations, events-aggregates, according to the permutations, obey Clifford statistics (vectors as plexors-for permutation-and spinors-for rotation-in some way as plexors of aggregates [1]. Emergent symmetries are explored, using correspondent “lightcone coordinates”, and metric transformations are investigated within certain approximations. The (1+1) dimensions (coloring) case is particularly considered. A doubled metric structure is adapted to this model. The first metric level, $\sigma(\cdot, \cdot)$, is adapted to the edges of the graph acting on its vertexes and functions of the loops number on vertex (cluster of homogeneous in color vertices attached to one event). This is an on-vertex approach, edges are “flat” and not-colored. The second metric structure, $g(\cdot, \cdot)$, is a composition of the firsts and can be derived from this. We detail the transformations and symmetries of the colored graph assumed as reference, (related to “world sheet” inner coordinates) binding the $g(\cdot, \cdot)$ to $\sigma(\cdot, \cdot)$ metric. There are several graphs representations of graph states and states evolution. Mainly we will use a graph with maximum valence 2 per vertex, which are called chains $C$ (if the events component are related by an “evolutive” operator $D$ establishing an order relation between the events) or sequences $S$ (if components are related by shift operator $D^x$ exchanging colors on the vertices but without order relation among vertices- graph with not connected homogeneous in colors loops). At fixed graph state $G_m$, each sequence $S$ has a chain $C$ degeneration, each sequence corresponds to a set of chains, we detail in [1]. The graph state is constituted by its sequences (and chains), and the set of these is called the graph sequence (chains) decomposition (or structure). In [1] we study this inner structure, and the decompositions under conformal transformation; the doubled graph metric structure is adapted to the graph degeneracy in chains and sequences. We give the chain (and sequence) degeneration of the graph, the definition of associated entropy of the graph state $G_m$ with this structure and the transformation laws under conformal graph expansion. In this set, it is clear the self-similar structure of the graph. Metrics reflect this structure. The transformations (in fact at fixed graph state $G_m$ this is reduced to action of permutation matrices and for different states to generalized permutation which can be matched with typical graph matrices) act on events/vertexes for the first metric level $\sigma(\cdot, \cdot)$, and on edges for the second metric $g(\cdot, \cdot)$. To highlight symmetries and clarify the graph seeds replicas embedded in superior states (conformally expanded graph), we have decomposed the graph in chains with maximum valence 2 for vertex. However, we can superimpose these decomensions on the seed graph (through an edges approach) and consider a complete graph (where we can define a quantum graph). Then the relation of our graph model with the standard simplex appear to be not immediate, because of the vertex valence 2 and the conformal transformation which in fact reshuffles any ordered relations among different (ordered) clusters of vertices in one graph (chains of events $C$). The metric graph should describe events (vertices) “bubbling up” after conformal transformation, where geometro-genesis will be as an asymptotic re-stitching of the previously dismantled spacetime structure, producing a metric dynamical structure with the reductions of the symmetries between first and second metric levels. (As pointed out in QGF the invariance under permutations of all the vertices would be

1 Clifford algebra can be realized by gamma matrices satisfying a set of canonical anti-commutation relations, the spinors are the column vectors on which these matrices act. It might be convenient to decompose into a pair of so-called “half-spin” or Weyl representations if the dimension is even. (Note the Weyl group is the symmetric group $S_n$, which is represented by signed permutation matrices). We note that the Schur-Weyl duality (the theorem relating irreducible finite-dimensional representations of the general linear group and the symmetric group) connects the irreducible representations of the symmetric group to irreducible algebraic representations of the general linear group of a complex vector space.

2 In a vertex approach, where the graph is isotropic (edges are equal in measure and colors/flat) but not necessarily homogeneous in algebra, it would be natural to consider the n-vertices graph as a standard (or unit) (n-1) dimensional -simplex (or n - 1-simplex), defined as a $\mathbb{R}^n$ subset. Therefore a seed graph minimal sequence can be seen as a n-standard simplex having loops of order (algebra) $\mathbb{C}_2$ on a vertex. In general we remind that volume under the n-simplex (i.e.n+1 vertices) graph (i.e. between the origin and the simplex in $\mathbb{R}^{n+1}$) is $V_n = 1/(n+1)!$, with (n + 1) vertices (and n+1 cells), the simplex has $m(n+1)/2$ 1-faces (polytope edges), faces number $m(n^2 - 1)/6$, with simplicial hypervolume $\mathcal{V}_G = n(n^2 - 1)/6$, and edge length $\mathcal{L}_G$. (Cayley-Menger determinant). The dihedral angle is arccos(1/n), and the angle that the simplex center forms with its two vertices is arccos(-1/n), graph spectrum is $n^{\ell}(-1)^\ell$, and number of cells is $n+1$, with a hypersurfaces area $\sqrt{n2^{\ell}-(n+1)}\mathcal{V}_G^{\ell-1}/(n-1)!$.
lost to the translation group in scale prospective.) There is an emerging events relational structure, and associated events and graph definition, arising from the polychromatic loops and from the conformal graph expansion. The conformal expansion preserves the graph seed structure (“graph-information”), creating isomorphic parts of the expanded graph, and new events depending on probability distribution. Thus the idea of an overlap of metric and dynamical graph model is combined with the existence of two metric levels and an emerging metric structure highlighted by events emergence from the graph inhomogeneity. The selection of the metric level depends on scales and conserved quantities under some specific transformations. Further relevant aspects of this model is the graph coloring and probability role, the graph conformal transformation (expansion), the introduction of vertex algebra in the metric and graph transformations and the emerging graph self-similarity, clear in the graph sequences and chains and graph conformal expansion. More specifically, the events are macroevents (homogeneous in colors “loops” which serve as graph re-parametrization with a color adapted inner base) as derived events (from the inhomogeneity). These notions lead also to investigate the graph entropy role, concept of graph immersion and interaction, briefly discussed here in the metric approach. Concerning the general idea framing the model and in relation to graph expansion, self-similarity after conformal transformation and transmitted information (graph structure), we could use the figurative image of a “spacetime-DNA” for the original seed graph, which propagates (through conformal expansion with constant classical colors probability distribution) to higher graph states, providing the overlap of new states of the graph and its multiple isomorphic parts. In this “events-DNA” analogy, we could more generally thinking possible to write a geometric code encoding a procedure to extract, duplicate and, eventually, modify such genetic code of our spacetime, providing a characterization of the spacetime cells and organisms (dealing somehow with spacetime knowledge as a spacetime engineering). Graph seed would have the role to contain such “genetic information” of the spacetime which replicates, propagates and it is preserved and, eventually, may be manipulated. This would be a spacetime code whose information propagates interacting eventually with other geometric organisms where here the idea of duplication is rendered in the conformal expansion.

In this work we introduce for the first time a graph for a new spacetime emergent model based on the construction of states for the systems as modeled by a graph

Comparing the model with the similar structures present in literature we proceed with the construction on main characteristics of the model discussing possible applications introducing the concept of “spacetime engineering”. We show that the doubled metric structure reflects the graph colored structure under conformal expansions and in some special cases chains could be related to strings generalizations. As first attempt to present and describe the graph model, the article focuses on the introduction of a number of new definitions, therefore the formalism in this article is developed in great details and it is extensively explained in a consistent first part of this manuscript.

In this article we introduce the general idea underlining the model and we sketch the graph, whereas further details and developments will be addressed in [1]. We use a combinatorial graph considered through its polychromatic clusterings, the realizations, the conformal transformation and the different events levels, reflected then in a doubled graph metric structure. The graph is metric as we define a doubled metric structure adapted to the polychromaticity of the graph elements, i.e. its vertex and edges. In a dynamic model the graph would be described by the graph constituents dynamics, vertices and edges, intended as graph “particles”, as əvents. This article is structured in three parts: In the first part we introduce some operators and notation, we discuss the conceptual bases of the graph introduction. Second part builds the model introducing the graph realizations, the entropy notion and the analysis of the effects of the conformal transformation. Third part introduces the graph metric structures. In details: in Sec. 1.1 we start discussing the graph model providing the list of the main graph features, and introducing events as graph vertices. We give a definition of sums and products of events, evolution, ԏ, and shift, Ӟ, operators defining graph events chains and sequences respectively. These definitions lead to develop the concept of macroevents and algebra  of events and operators algebra : events compositions and decompositions follow as example of graph structure and clusterization. We discuss the assumption of minimum algebras and homogenization in algebra of sequences and chains, and particularly graph conformal transformations and graph loops. Sequences and chains of colored graphs

3 Spacetime (or geometric) organism (also in the graph pre-geometric phase) as an organized system, characterized by specific relations, made up by a set of interdependent connected parts in functional relations, preserving and, eventually re-integrating and reproducing its own form (substantialism). Although this work does not enter into the specifics of the mechanisms called into action by the Quantum Darwinism, the general view of this model overlaps in some language proximity; in this respect a similar analysis would be seen as a “Mendelian” investigations, precursors of modern genetics.

4 In the relationalism of the graph frame the actual knowledge of a physical object is based of (two) objects (contact) interaction. We exploit this frame by considering events and əvents definitions grounded upon the graph poly-chromaticity. Entropy has been associated in Sec. 1.1.2 to əvents emergence as expression of graph chains poly-chromaticity. Homogeneous (in colors and algebras) graph chains may be seen as precision (conformally expanded rate) “clocks”, where the existence of a minimum local algebra regulates also the conformal transformations role. These concepts, framed together in the graph, may be simply expressed in a relativistic scenario, in a causal structure terms where mass would be intrinsically related to a time concept for with no mass (time-like causal relations) it would imply having no clocks (events) and, therefore, no distances, then we would have no metric structure, but in fact a conformal structure (light-like causal relations, here related to a topological notion of graph scale definition)
are thus thoroughly considered in Sec. 1.1.1. Presence of loops and events emergence for a state graph and after conformation graph expansion are constrained depending on the colors probability distributions. Entropy notions are also explored in Sec. 1.1.2. We then consider an events matrix $Q_j$ preceding definition of the colored metric graph. Two metrics levels $\sigma_1$ and $\sigma_2$ are introduced in Sec. 2. The overview of the graph metric structures closes in Sec. 2.1 with brief notes on metric graph symmetries and transformations in relation to vertexes algebra in a first approximation. Concluding remarks follow in Sec. 3. Appendix (A) follows on some general texes algebra in a first approximation. Concluding remarks are explored in detail in [1]. Here, however, we note these definitions establish a vertex/operator correspondence. We can consider an operators chain, writing an events chain as an associated chain of operators and vice versa. From the events algebra $\mathcal{A}_0$ the "operatorial" algebra $\mathcal{A}_0^* \mathcal{A}_0^*$ follows associated to $\mathcal{A}_0^*$ where $\forall \{a_i\} \in \mathcal{A}_0^* \exists \mathcal{D}\{a_i\} \equiv \{a_i^{\dagger}\} \in \mathcal{A}_0^*$, with $\mathcal{D} \in \mathcal{A}_0$ and $\{a_i\} \neq \{a_{i+1}\}$. The operatorial algebra $\mathcal{A}_0^*$ is associated to the primary events algebra $\mathcal{A}_0$. Similarly to the events, operators can be composed in elements of superior algebra or also decomposed in inferior algebras. The shift operator $(\mathcal{D}^\dagger)$ acts as "projector" relating two events in a sequence without an order relation (changing in fact the vertex color in a colored graph). In general, composition rules of $\mathcal{D}^\dagger$ follow similarly to $\mathcal{D}$. A sequence of events or operators is homogeneous, if all the elements of the sequence belong to the same algebra. 

Transformations relating homogeneous chains

1.1 Main graph features

Here we list the main characteristics graph referring to \[ for further discussion. We consider a graph (body) $\mathcal{G}$ as the set of $n$ vertices (events) $|a_i\rangle$ (events notation $|a_i\rangle$ or $\langle a_i|$ is here used equivalently for graph vertexes) and connections (edges) $c_{ij}$ among vertices ($|a_i\rangle, |a_j\rangle$). The sum $\sum_{i=1}^n |a_i\rangle$ is a sequence of $n$ (not ordered and with zero valence) events $|a_i\rangle$. This is generally associated to an event $|A\rangle$ (macroevent as vertex cluster) of the order $n$ with respect to the $|a_i\rangle$ events, equal to the sequence dimension (cardinality of the set of events). A vertex/event can be in general decomposed in a sequence of $n \geq 1$ lower order events. The $|A\rangle$ decomposition constitutes an $|A\rangle$ (sub)structure. A graph $\mathcal{G}$ can be composed by vertexes which are clusters of other events, providing therefore the first notion of events clusterization in the graph $\mathcal{G}$, this leads to the concept of event algebra $\mathcal{A}$ and algebra $\mathcal{A}_0$ of an event $|A\rangle$. The algebra $\mathcal{A}_0^* \mathcal{A}_0^*$: The dimension (degree) $Q$ of an algebra $\mathcal{A}_0^* \mathcal{A}_0^*$ is thus a relative quantity relating events and it corresponds to the cardinality of the vertexes decompositions: the macroevent $|A\rangle$ belongs to the algebra of the order (or degree) $n$ ($\mathcal{A}_0^* = n$) with respect to the algebra of its constituents $|a_i\rangle$. In the limiting case of $\mathcal{A}_0^* = 1$, the event $|A\rangle$ is in correspondence with, "decomposed" in one event. The vertexes of a sequence can belong to equal (homogeneous in) algebras or different (inhomogeneous in) algebras, in this case the algebra degree will be not constant for event decompostion index. We assume that there is an irreducible, ordinary or primary algebra $\mathcal{A}_0^* \mathcal{A}_0^*$ of the order 0 considered therefore homogeneous and providing a procedure to define reference algebras (the minimum algebra of one vertex). Evolutive operator $\mathcal{D}_i$: It is useful to introduce the evolutive operator $\mathcal{D}_i: \mathcal{D}_i |a_i\rangle = |a_{i+1}\rangle$ relating two vertexes with the introduction of a vertex order relation specified by the index. These definitions have predominantly a conceptual meaning in this article, explicating the creation of clusters by the color shift and vertexes order. A chain, or graph path, is the ordered set of elements related by $\mathcal{D}$ (directed graph edges, order corresponds to the $\mathcal{D}$ action). Two vertexes of a chain are consecutive (or adjacent), according to $\mathcal{D}$, if they can be ordered by the action of $\mathcal{D}$ as $|a_{i-1}\rangle < |a_i\rangle < |a_{i+1}\rangle$. The chain/sequences concepts can be expressed by defining the product opera-
tor $\mathcal{D}^\dagger$ as follows $\mathcal{D}^\dagger |a_0\rangle \equiv \prod |e_i\rangle \mathcal{D} |a_0\rangle$ or sums

$$\mathcal{D} \sum_{i=1}^{n-1} |e_i\rangle = \sum_{i=1}^{n-1} \mathcal{D} |e_i\rangle = \mathcal{D}(|e_0\rangle + |e_1\rangle + |e_2\rangle + ... \rangle \equiv (1)$$

($k$ in $\mathcal{D}^k$ is the application degree ordering and relating vertexes graph). Sums and products properties, in the context of different aspects of the graph construction, are explored in detail [1] in \[. Here, however, we note these definitions establish a vertex/operator correspondence. We can consider an operators chain, writing an events chain as an associated chain of operators and vice versa. From the events algebra $\mathcal{A}_0^* \mathcal{A}_0^*$ the "operatorial" algebra $\mathcal{A}_0^* \mathcal{A}_0^*$ follows associated to $\mathcal{A}_0^*$ where $\forall \{a_i\} \in \mathcal{A}_0^* \exists \mathcal{D}\{a_i\} \equiv \{a_i^{\dagger}\} \in \mathcal{A}_0^*$, with $\mathcal{D} \in \mathcal{A}_0\{a_i\} = \{a_{i+1}\}$. The operatorial algebra $\mathcal{A}_0^*$ is associated to the primary events algebra $\mathcal{A}_0$. Similarly to the events, operators can be composed in elements of superior algebra or also decomposed in inferior algebras. The shift operator $(\mathcal{D}_i^\dagger)$ acts as "projector" relating two events in a sequence without an order relation (changing in fact the vertex color in a colored graph). In general, composition rules of $\mathcal{D}_i^\dagger$ follow similarly to $\mathcal{D}$. A sequence of events or operators is homogeneous, if all the elements of the sequence belong to the same algebra. 

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5 The combinatorial, polychromatic graph can be related to causal-set approaches is a clear way. The relation order, through the introduction of $\mathcal{D}$ operator, provides the polychromatic structure with a reflexive, antisymmetric (a poset) and transitive order between graph elements, combined with the algebra homogenization and metric structures considering differently the chromatic symmetries. The graph is ordered and characterized by introduction of the state conformal expansion and the state realizations contrasting the arbitrariness of the colors order. Here an operator is intended as a quantity performing specific (logical) actions (evidently connecting elements of the same space). It is clear that the action of $\mathcal{D}$ is doublefold: 1. to provide a strict (irreflexive) partial order and thus ensuring the chain-of-events/chain-of-operators definition and relations. 2. Quantity $\mathcal{D}$ is related to the coloring (values), i.e. it provides a further degree of freedom and the clustering after chromaticity and identity in loop of monochromatic vertices. Action of $\mathcal{D}$ is therefore related to the definition of events and can be associated to a color transition or not. For this reason in \[, we used a colors shift $\mathcal{D}^\dagger$ operator, having all composition properties of $\mathcal{D}$, to distinguish conceptually the two operations. The colors shift produces only the equality $(Q = P)$ relation for two events $(P, Q)$ (at the base of monochromatic loops definitions and indistinguishable events) and negation $Q \neq P$. This structure is complicated by the introduction of $\mathcal{D}$ events, conceptually related to (acceleration notion in \[) no-locality (intended as vertices adjacency) affected by the clustering in monochromatic or polychromatic loops, seen as the vertices in a $\mathcal{D}G$ graph characterized by algebras, an order relation and two colors.

6 Homogeneity implies a "linearity" in the events indices $(\mathcal{A}_0 \mathcal{A}_0^* \mathcal{A}_0 \mathcal{A}_0^*)$, reflected in the algebra transformations. We gen-

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will be particularly important as they can leave conserved many properties of the graph structure. Chains homogenization is a transformation of a inhomogeneous chain into an homogeneous one by a proper algebra adaptation to macroevents (colored) structure. Graph vertexes are generally clustered in macroevents through equal coloring (loops), a colored algebra adapted to this colors structure arises. An events base is a set of vertices (inhomogeneous or homogeneous algebra coinciding with $\mathcal{A}^0$ or a conformally expanded $\mathcal{A}^n$). We denote as $\text{ber}$ an events set (base) with adapted algebra, the "reference" ber ("inner") base has homogeneous algebra, generally a conformal $\mathcal{A}^0$ "expansion". For convenience we consider ber as bichromatic, we will use and 2-level logic signal (or 4-level, introducing the "complementary-chains", leading more generally to a $(r + r)$-levels logic signals).

1.1.1 Graph realizations, sequences and chains, $\text{ber}$ events and loops

We here investigate the properties of the colored graph $G$, laying the groundwork for a dynamic graph [1]. To begin with we consider one graph, assuming an ordinary $\mathcal{A}^n$ ber we discuss the emergent inhomogeneous basis from the graph coloring and the particular case of conformally related bases, where indexes are set accordingly to the graph coloring and the particular case of conformally related sequences. We address the connections emergence introducing the "complementary-chains", leading more generally to a $(r + r)$-levels logic signals).

$\text{ber}$-levels logic signals). For instance the graph $G$ is defined by fixing the values distribution $[a_i]$ and the probability distribution $[p_i]$ The state $G_\alpha$ of a graph $G$ is defined by fixing the cardinality $n$ of the vertex set. The graph state realization is determined by its sequences $S$ and chains $C$. However a graph $G$ may be equipped with some constraints governing the clusterization in sequences or chains, and in this way reducing the degeneration. The conformal expansion. Then, if $m \geq n$ is the events number in the ber, the graph $G$ underwent (transition) from the state $G_n$ with vertices to the state $G_m$ with vertices. Conformal transformation (expanded or contracted graph) between graph states is related to the conformal metric transformations. To simplify our discussion we assume the following $N$-constraint: $m/n \equiv [m/n]$, where there is $\sum_i m_i = m$, we shall assume the conditions $m_i = p_i n_i = [m/n]$ or $m_i/n_i = ([m/n])/(\sum_i n_i) = (1/r)\sum_i (m_i/n_i) = [m/n] \forall i$, thus $m_i$ is the macroevents number in the reference basis associated to the color $a_i$.

For convention here we consider a seed graph state as $G_0$, with a minimum $n = \min m$. In [1] we include more colored graphs, realizing the same sequence. In this representation we clearly decompose a complete graph into its paths (also in paths bundles of disjoint or crossing chain[7] in [1] see also Figs [1]-left) with chain and sequence algebra $\mathcal{Q}_G = \mathcal{Q}_\alpha \equiv m$ respectively in ber (and $\mathcal{Q}(m) \in [r, m]$ in adapted bem) for a graph $G_m$ imposing the maximum valence of graph $2$. The graph state: A graph $G$ is defined by fixing the values distribution $[a_i]$ and the probability distribution $[p_i]$ The state $G_\alpha$ of a graph $G$ is defined by fixing the cardinality $n$ of the vertex set. The graph state realization is determined by its sequences $S$ and chains $C$. However a graph $G$ may be equipped with some constraints governing the clusterization in sequences or chains, and in this way reducing the degeneration.

[1] A graph bundle $B$ is the collection of bounded-i.e. with common fixed boundary vertices/events for any bundle path-disjoint paths (chains) of the graph state $G$. Figs [1]-left. The disjoint paths of the bundle do not intersect apart at the two extreme vertexes constituting the boundaries. We can consider a state $G_\alpha$ totally decomposed in bundles. At fixed state $G_n$, we can ask if there is at least a couple of colors (which can be also equal) such that $G_n$ can be totally decomposed in disjoint bundles or one bundle of paths and if this property is transmitted for conformal graph expansion. There are some trivial cases: for example $S_{\text{un}}$ where for each couple of colors there are $S_{\text{unmax}} = (r - 2)!$ possible ways to realize the path. The total number of possible boundaries for such bundle, and therefore the total number of bundles, is $2S_{\text{unmax}} = r(r - 1)/2$ (not considering the symmetric $c_i = c_j$). For a path $\gamma$ to be a complete (total chain, made of $n$ vertices for a $G_\alpha$ state, differently from which is an homogenized chain with colored adapted algebra) $G_\alpha$ replica in $G_m$ (embedded in a chain) it has to be $m = n\Omega$ where $\Omega \in N$. Necessary condition for the graph to be always decomposed in $\kappa$ disjoint replicas is $\Omega = n\kappa$.

[7] More basis (colorings) can be defined on a graph. A polychromatic graph can be clearly decomposed in $r$ monochromatic chains, or in an $(r + r)$-model, with $r$ complementary chains related to each color, which can be seen as a base coloring, with event value index $a_i \in [1, \infty]$. We considered in [1] a particles/holes for particles/events $p$ and holes $h = p$ related to the complements for the graph $G_\alpha$. In this case, the particles total number in ber (for state and complete realization) is $Z(p) = m$ (independent by the colors number or the probabilities) and $Z(p) = m(r - 1)$. The ratio $Z(p)/Z[p] = (r - 1)$ does not depend on the graph state and the probability distribution, not distinguishing different graphs having equal color numbers and relating, particularly, not-isomorphic graphs.
discussion on graph relational structure intended as set of realizations after conformal expansion. Consider different states complete realizations, i.e. with $m$ vertices for a $G_m$ state, if $m > n$ then any realization of $G_n$ is embedded in the realizations of $G_m$, and $G_n$ has different replicas in $G_m$. Then $G_m$ realizations can be set in one-one correspondence, in different ways, by a conformal transformation, with the entire realizations of the $G_m$-state, in this sense we can say that the set of $m$ realizations contains the $G_n$ realizations (graph self-similarity and scale structures). On the other hand, new relational structure is generated after expansion [1]. We characterize a graph by its states and the degenerations and we show particularly how the conformal transformations do not preserve the graph structure in general, but the sequences of an expanded graph includes all the conformally expanded sequences (the respective algebras are conformally expanded) of the original graph. Some realization properties are not preserved by the application of the group of permutations on $n$ vertices. Some graph properties are invariant for state transition, whereas other graph features are transformed according to the probability distribution. On the other hand, graphs with same $p_i$ and number $(r,n)$ but different bases have clearly equal structures, therefore we defined these graphs as equivalent (isomorphic) graphs. We discuss here in details the chains and sequences degeneracy, much of this discussion mirrors some aspects of the combinatorial graphs. We consider the sequences degeneration and we discuss a criterion for the eligibility of sequences and chains as part of a state degeneration, that is a criterion to establish those chains and sequences emerging as graph state realizations, in adapted base or $ber$, based on the analysis of the necessary and sufficient conditions for the occurrence of equal colors vertices adjacencies. These quantities are then studied under state transitions, particularly relevant is the minimal $S_{\text{min}}$ sequence ($r$ vertices for $r$-colors graph with maximum algebra $\Phi_i$ for each color $i$) and the supremum $S_{\text{sup}}$ sequences (with $m$ vertices for a graph state $G_m$ with minimum algebra per vertex $\varnothing = 1$) and associated chains $C_{\text{min}}$ and $C_{\text{sup}}$ respectively, constituting $\min_{\text{sup}}$ and $\min_{\text{sup}}$ degeneracies. The existence of these "boundary" realizations, and especially $S_{\text{sup}}$ and $C_{\text{sup}}$, turns to be a relevant graph feature. We considered in particular two special cases: the equiprobable graph where there is $p_i = 1/r$ for any color $i \in \{1,\ldots,r\}$ and secondly we investigate the effects of presence of a maximum in the distribution of $p_i$ ratios, and more generally the probability distributions in the graph realizations and conformal expansion. The exploration of the two limiting cases will be crucial in determining the graph loop on vertex and their variation by state, the presence of a maximum $S_{\text{sup}}$ and $C_{\text{sup}}$, and the $\delta$events emergence. We focus on the loop (macroevent) sets for different graph states and the existence of maximum (algebra $\max \varnothing_i$) loop, discussing the existence of the minimum number, $\min 2\ell$, of loops $\ell$ and their algebra $\varnothing_i$ and loops emergence after state transitions. Using these results we investigate the emergence of $\delta$events and other aspects of the graphs conformal expansion–Figs [1].

1.1.2 Inhomogeneities of the graph state realizations: degeneracy in chains, webs and sequences

We evaluate the chains and sequences degeneracy for a colored graph state $G_n$ and after conformal transformation. The metric structures and the spinorial emerging structure would be eventually adapted to the colored graph realizations [1]. There is a generation of different events and graphs $(\partial G)$ whose vertices are $\delta$events) associated with the graph inhomogeneity (differently colored vertices). The metric structure rests on the graph states which is made by evolving (conformally related) self-similar blocks. The change in algebra and colors, in this framework, is closely related to the events definition. For a fixed state, the sequences realizations have different macroevents (multiplicity $N_i$ for $i$ color) and, the different chains, for a fixed state, have different $\delta$events, whose number (for fixed vertex number $n$), and the cardinality $\exists C$ of chains for graph state $G_m$, are constrained. The graph conformal expansion (a state transition) changes also the maximally decomposed realizations ($C_{\text{sup}}, S_{\text{sup}}$) having maximum number of differently colored macroevents in adapted algebra, and the the loops and $\delta$events emerge for the graph state transition and graph transition from one realization to another, constrained by the probability distribution. In $ber$, the total number of chains, $\exists C_n$, of a $G_n$ state is the multinomial distribution

$$\exists C_n = \frac{n!}{\prod_{i=1}^r (p_i m)!} \cdot \quad \text{(2)}$$

where

$$\forall n \quad \exists C_{\text{min}} = r! \geq \exists C, \quad \exists C_{\text{sup}} \neq \exists C \quad \text{and} \quad \exists C_{\text{min}} = \exists C(\exists C_n - \exists C_{\text{min}}) = \exists C_n - \exists C_{\text{min}} = \exists C_n - r! > 0,$$

$\exists C_n(a)$ determines in how many ways it is possible to distribute $n$ distinct events (as $ber$ events, they are all distinct as ordered) in $r$ different boxes in each of which

9 Clearly a graph state $G_n$ of $r$ colors is associated to a polynomial: $\sum_{i_1, i_2, \ldots, i_n} (x_1 + x_2 + \cdots + x_r)^n = (x_1 + x_2 + \cdots + x_r)^{n k}$, $\delta$events for a state $C_{\text{sup}}$, (graphs) determined by the connection set but not the colors frequency (for example in the anyons model developed in [1]), as $\sum_{y^{m-1}} \left( x^{v^{m+1}} + y^{v^{m+1}} \right)$, (sum is understood as a composition and $x^0 = 1$)

10 Algebra inhomogeneities can be considered in the decomposed $r$ monochromatic chains, coupled with constraints related to the state and conservation of (classical) probability, or can be considered in an $r + r$ model of monochromatic chains $C_r$ and their complements $\bar{C}_r$ (having different state number with $m(1 - p_i)$ vertices). The polychromatic chains (and the graph) decoupling into $r$ monochromatic chains (graphs) is determined clearly by the loops partitions, regulated by the multinomial distribution [2]. In the $(r+r)$ models, the monochromatic chains and the complementary numbers are not independent and the latter are subjected to constraints different from the first. In first approximation loops are associated to the symmetric part (in color) of a polychromatic chain, whereas $\delta$events (loops free), eventually determining the spinorial structure, to the antisym-
there can be at most \( \mathcal{C}_2 \equiv p,r,n \) different elements (events) for the \( i \)-th color (therefore the permutations \( P^{p,r,n}_{p+r+n} \) act on the chain vertices). In this way, however, we will make no distinction between elements belonging to the same “box-value”, i.e. in the homogeneous in color loops. (Note that we are neglecting the distinctness of vertices of a monochromatic loops, which are induced by the order relation for application of \( \mathcal{D} \), considering only the distinctness, we discuss this \( [1] \) in terms of associated statistical distribution). In this counting the value-boxes are distinct, since they have \( r \) different (not ordered) labels. The reference, chain \( \mathcal{C} \) events order the events in \( r \) classes of \( \mathcal{C}_2 \) elements all equal and not ordered. We count macroevents afferent to the same color but different algebras as different elements all equal and not ordered. We count macroevents afferent to the same color but different algebras as different vertices because clustered with different loops.

On the webs: Cardinality \( \mathcal{C}_n \) of the graph \( G \) chains can be parameterized in a colors adapted homogenized algebra, where equal-color vertices having different algebras are considered equal, leading to the web definition. In the webs only the number \( N \) of macroevents, associated with a color \( i \), is considered ignoring their respective loops algebras in the reference base. For any sequence of a \( G_n \) state, the total macroevents number is \( N \equiv [r,n] \), correspondingly there is the minimum \( S_{\text{min}}(r) \) sequence \( (N = r) \) and relative chains \( C_{\text{min}}(r) \) (with \( \mathcal{C}(C_{\text{min}}(r)) = r! \)) and a superior sequence \( S_{\text{sup}} \) and chain \( C_{\text{sup}} \). Condition \( n = \text{sup}(N) = \text{max}(N) \) is realized only in special circumstances, where \( S_{\text{max}} \) corresponds to \( N = n \) vertices, this property rarely occurs and it is usually lost for conformal graph expansion depending on the colors probability distribution \( [1] \). We say that all these \( S_n \) sequences are “included” in the set \( S_n \subseteq \{S_{\text{min}}, S_{\text{sup}}\} \), with multiplicity, \( S^{\circ} \), for fixed macroevent multiplicity but not (algebra) loop vertex, defining the web where vertices with different loops (algebras) are indistinguishable in \( \mathcal{C} \). For a graph state is, the cardinality of the webs:

\[
\mathcal{W}_{N,N}(a) = \frac{N!}{\prod_{i=1}^{N} N_i !} \left( \sum_{s=0}^{q} \left( \frac{N + s - \sum_{j=1}^{r} N_j !}{\prod_{j=1}^{r} N_j !} \right) \right) \text{ where } N_i = 1 \forall i \in \{q + 1, ..., r\} \text{ and } q \leq r.
\]

The last term subtracted in \( \mathcal{W}_{N,N}(a) \) of Eq. (3) is \( r! \). This cardinality questions how many are the chains of \( N+1 \) connections (devents) for \( N \) macroevents with different multiplicity \( N_i \), then the sequences \( S^\circ \) (in \( \mathcal{C} \)), for a state \( G_n \), so that a chain segment does not connect two macroevents afferent to the same value, or the equal value macroevents are never adjacent, particularly in the expanded states there could be web chains or \( \mathcal{C} \) events chains, that is, a chain has a multiple multiplicity \( (N_i > 1) \) where a (monochromatic) cluster is considered as a vertex, depending on the colors probability distributions and evolving with the conformation transformations \( [1] \). This is the cardinality of the set of chains with adjacent macroevents, subtracted for the algebra reduction (homogenization), to sequences with lower macroevents number, given by the second term of Eq. (3). In this counting, the (not distinct) macroevents of multiplicity \( N_i \equiv [1, r] \) (instead of \( n = n_p \) ) are considered. Note that \( (N_i) \) in the web, are algebra independent.

Finally, we note that the ordinary \( \mathcal{A} \) basis definition provides an "absolute" for the \( \mathcal{C} \) definition, using \( \mathcal{C} \) indexs, the cardinality in the ordinary base \( \mathcal{A} \), is obviously greater (or equal) than the cardinality where chains are in different indexes. It is therefore straightforward to introduce (Shannon) entropy \( S_m \) for the graph state \( G_m \) associated to the degeneracy Eq. \([2] \)–\([1]\), and similarly we can define an entropy for a monochromatic sub-chain as follows:

\[
\forall i \in \{1, ..., r\} \quad S_i = k_i \ln \Omega_i, \quad \Omega_i = \frac{(mp_i)!}{\prod_{j=1}^{r} \Omega_j ^{s}}, \quad S_i(\text{min}) = 1, \quad S_i(\text{max}) = (mp_i)! \text{ (sum on any subchain vertex)}.
\]

Thus:

\[
\text{Thus: } \frac{1}{m} \log \Omega = \frac{1}{m} \left[ \log m! - \sum_{i=1}^{r} \log((mp_i)!)) \right], \quad \text{where } \lim_{m \to \infty} \left( \frac{1}{m} \log \Omega \right) = - \sum_{i=1}^{r} p_i \log p_i,
\]

or we have

\[
S(m) = \ln(1(m)) - \sum_{j=1}^{r} \ln[(1)_{\circ j}] = \ln[\Gamma(m+1)] - \sum_{j=1}^{r} \ln[\Gamma(\circ j + 1)]
\]

(\( \circ j \) is the Pochhammer symbol and \( \Gamma(\circ j) \) is the Euler gamma function. (The minimal sequence entropy, not considering loops algebra, could be expressed as \( S_{\text{min}} = \ln(1.r) \).) For a graph \( G \), there is minimum entropy \( S \) associated to the graph seed \( G_n \) (\( n < m \)). We study this quantity after graph conformal expansion. The entropy (state independent in the Stirling approximation) as the degeneracy is an algebra and state dependent concept Figs \([2] \)–\([1]\).

Adjacencies, loops and \( \theta \)-criterion This analysis fixes the presence of a totally decomposed sequence, the devents emergence and transformation after conformal expansion, the minimum and maximum loop algebra for a \( G_m \) state. The \( G_s \) realization could have, according to the probability distribution, a necessary adjacency of equal color events in any chain of \( S_{\text{sup}} \). Therefore, for a graph state \( G_s \), there may not be a macroevents sequence realizing the supremum \( (N_{\text{max}} = N_{\text{sup}} = n) \), i.e. a totally decomposed sequence (in other words we address the issue questioning the prevalence of loops or devents after conformal transformation). The adjacency would lead to a higher algebra macroevent and consequentially there is a reduction of \( N = n \) vertices according to the number of the adjacencies and this sequence must be rejected as coincident with one other sequence of the graph realizations. In other words there is in general \( \text{max} N \leq \text{sup} N \) (while \( \text{min} (N) = r \) and \( \exists S_{\text{min}} \). It is clear that if \( \text{max}(N) = \text{sup}(N) = r \), then...
exists, and as for \( S_{\text{min}} \), there is no degeneracy in color or in algebra, but \( S_{\text{min}} \) and its \( r' \) chains is an intrinsic (state independent relational structure) graph characteristic. Then \( N = \mathbb{Z}(d) + 1 \), where \( \mathbb{Z}(d) \) is the \( d \)-event number for the \( i \)-sequence excluding the boundary of the first and last event of a \( n \)-events chain of a \( G \) state the vertex boundaries in graph vertex definition is also an issue of the metric graph \( \mathbb{Z}(d) \).

We can write the set of \( S_N \) sequences, \((S_N \text{ degeneration class in bem})\) as:

\[
S_N = \{S_N^{S_{\text{max}}} = \prod_{a=1}^{r} \sum_{a=1}^{N_{\text{max}}} \left[H[1 - \Theta] \right], (5)
\]

\[
\mathbb{Z}S_N = \prod_{a=1}^{r} \Theta_a - \mathbb{Z}[H[\Theta - 1]]^{r}_{a=1}, \quad \text{where}
\]

\[
\Theta = \left\lbrack \max \{N_\Theta^{r'}_{a=1} - \sum_{a=1}^{r - \max} N_a \} \right\rbrack, \quad \text{then}
\]

\[
\exists S_N \equiv 1 \} 0, \ldots, 1 \} 0, \}
\]

\[
S_{\text{sup}} \equiv (\Theta_1, \ldots, \Theta_3) \} \{1^0, \ldots, 1^0 \} 0.
\]

\( a \in [1, r] \) is the color index, \( N_a \in [1, \Theta_0] \) is the number of macroevents for a color \( a \), the sum \( \sum_{a=1}^{r} N_a \) is the decomposition, at the fixed color, of its algebra (integer \( \Theta_0 \) partition). In the sum, \( r_{\text{max}} \) refers to the color associated to the maximum of the macroevent number \( N_{\text{max}} \) in the selected sequence. \( \prod_{a=1}^{r} \) is the symbol of composition in sequence, acting on the sums by composing elements of the sum of the partition with the decomposition (distributive property). Round brackets, \((N_1, \ldots, N_r) \) \( (r\text{-entries}) \), denote the macroevents number for each color \( i \in [0, r] \), and square brackets \([0, \ldots, 0] \) \([r\text{-entries}) \), denote the algebra distribution \((\Theta_1, \ldots, \Theta_r) \), between the \( N_i \) macroevents for each equal-value \( i \) macroevent. Where \( H[x] \) is the unit step function (Heaviside step function), such that \( H[x] = 0 \) for \( x < 0 \) and \( H[x] = 1 \) for \( x \geq 0 \). This equation provides the actually eligible (effective) vertices, eliminating the necessary adjacency, however it does not discern the difference in algebra between equal color macroevents in the sequence, which therefore constitutes a further degeneration in addition to chain degeneracy. The quantity in \( \Theta \) can also be negative and it serves to eliminate those particular sequences in the product, that would not give any macroevent implying a necessary equal color vertices adjacency. Hence, the threshold term \( \Theta \leq 1 \) \( (\Theta\text{-criterion}) \) excludes from the counting the cardinality of excluded sequences, and it obviously depends on the graph state. (This criterium is investigated in details in \[1\] where we study this under conformal transformations. Particularly we explored equiprobable distributions closely associated to the conformal homogeneous algebras). The equiprobable systems always have a maximum degeneracy in sequence. The degeneracy in algebra of supremum sequence is obviously 1. We note that this question is tantamount to asking whether, given the particular choice of macroevents \((N_1, \ldots, N_r) \), there is at least one acceptable chain realization (according to the no-adjacency condition) and for conformal transformation. (If \( \exists \max \{N_\Theta^{r'}_{a=1} \} \) for a sequence, there is \( N_\Theta(1, 1, \ldots, 1) = N_\Theta S_{\text{max}} \) which, in the case of maximal sequence satisfies the \( \Theta\text{-criterion}) \). Complete analysis of these cases is in \[1\] \[Pi\]. A further related aspect, signifi-

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11 A polychromatic connection (event) corresponds to a \( 0 \text{G} \) vertex. In contract with a \( 1 \text{G} \)-vertex, which is always monochromatic and with an algebra \( \Theta \), a \( 0 \text{G} \) vertex is dichromatic, with an order \( (a \text{ values } z) \) and algebras \((1,1), (1,2)\), with \( (1,2) \equiv a + b \Theta - c \), (where \( i \in [1,2] \) with values \((a, b, c) = (\pm 1, 0)\), depending on the loop counting on first or second (respect to the connection order) vertex.

---

12 In \[Pi\] we particularly focus on partitons and combinatorial problems in the construction of graph states and their
icant in this analysis, is the loop symmetries (equal color vertices). Discussion on the loops distribution involved partitions of equal-value graph vertices: it is in fact the problem of partition \( P(\mathbb{Z}_n) \) of an integer \( n \) into \( k \) summands in one of the \( \mathbb{Z}_n \) partition. Algebra analysis is related to the identification of sequences loops distribution, determining metric symmetries (evolution) for graph under conformal expansion and also the related to the graph entropy definition. 

Discussion on loops involves also analysis in terms of specific polynomials (associated to \( D_\alpha \) dihedral group of order \( \alpha \)). For polynomials of equiprobable distributions, we can describe this problem in terms of an \( r \)-th degree polynomials in \([n/r]\) variables. There are however several expressions with polynomials of this partition and combinatorial problem. In the reference algebra \( \mathcal{A}_\ell \), for any color \( i \) there is \( \Theta_i \in [1, \Theta_j] \), where \( \ell \) stays for a \( \mathcal{A}_\ell \) vertex of the algebra (order) of loop \( \ell \), with respect to \( \ell \). Thus there is \( \theta_i = \Theta_j \) algebra of the macroevent \( j \) associated to the value \( a_i \). The cardinality \( \mathcal{Z}(\ell) \) of the set of loops with \( i \) color, is the \( i \)-color loops (macroevents) number. For a chain, if \( \exists ! i : \exists ! \text{ min } \mathcal{Z}(\ell) = 1 \), then there is \( \varTheta \text{ min } \mathcal{Z}(\ell) = \Theta_{a_i} \), where \( \varTheta \) is introduced in Eq. (5), and \( \mathcal{Z}(\ell) \) is macroevent associated to the value \( a_i \) to which the minimum loop belongs, and that is also the necessary loop for that chain. Therefore, if the minimum order (algebra) of the necessary loop (according to the \( \Theta \)-criterion) is \( \text{min } \mathcal{Z}(\ell) \), then there must be \( \text{min } \mathcal{Z}(\ell) \) vertices that admit only \( 2 \)-valence connections between them, according to \( \Theta \)-criteria. The investigation of the minimum order of loop means, in the first place, to assess the necessary existence of a vertex with loop, and this case is realized only by a one simple vertex (macroevent multiplicity and loop cardinality equal 1) and, as we discuss [1], the minimum necessary loop has to be unique. Graph loops number and its state-dependence, the existence of \( \text{max } \mathcal{Z}(\ell) \) and the order \( \text{max } \mathcal{Z}(\ell) \) can be studied considering a decomposition in polynomials according to vertex algebra. We faced the problem of determining the maximum number of loops, \( \text{max } \mathcal{Z}(\ell) \), and the maximum loop order, \( \text{max } \mathcal{Z}(\ell) \), for a graph and the variation of these quantities following a conformal graph expansion in [1]. In particular, for the equiprobable (\( \varTheta \)) case, there is \( \mathcal{Z}(\ell)(\ell) = 2^{\ell} - 1 - \frac{2}{\ell} \), \( \mathcal{Z}(\ell)(\ell) = r(2^{\ell} - 1) - m \). Figs. [2]. These relations explicitly consider the change for conformal transformation through direct dependence on \( m \) and the growing of the range \( \varTheta_i \in [2, \varTheta_j] \), (note when \( m = n \), for the seed state, then \( \mathcal{Z}(\ell)(\ell) = 0 \), \( \mathcal{Z}(\ell)(\ell) = 0 \)). Therefore we assess in this way the existence of the minimum number of loops \( \text{min } \mathcal{Z}(\ell) \) and their order \( \text{min } \mathcal{Z}(\ell) \) and \( \text{min } \ell \) and loop emergence. We particular focus in [1] on the minimum order (algebra) of the necessary loop for a graph state, and the minimum necessary number of loops, \( \text{min } \mathcal{Z}(\ell) \), and after a graph conformal transformation.

The loop algebra is connected with the metric invariants and the graph inertia \( \mu \), moreover these quantities are related to the graph radius definition. These considerations deal also with the graph (self-similarity in sense of chains embedding and graph replicas in higher states) properties after conformal expansion. Note that vertices in an homogeneous loop considered here are symmetric graph (sub)states. Considering the symmetry group related to loop polygons (actually a polygonal with open boundary conditions), quantity should transform accordingly to the \( D_\alpha \) associated to loop \( \mathcal{Z}(\ell) \), having proper boundary conditions (closed) for “reflection” of indexes \( a_i \)-color (equal color vertex) coincides with the \( a_i \) of the loop with rotation for angle \( 2\pi/\varTheta_i \) (loop radius [2]). Adapting a metric structure to the colored graph, we encounter in the problem to provide a suitable topological structure and appropri-

13 The rigid movements of a regular polygon can be thought as vertices permutations. Then, each group is isomorphic to one subgroup of a group of permutations \( (S_\alpha) \). According to Cayley’ s theorem if \( G \) is a finite group of order \( n \), then \( G \) is isomorphic to a subgroup of \( S_n \), (i.e. we could say Cayley theorem implies that each group can be considered as a particular group of permutations). In general, however, not all the permutations of the \( n \) vertices of a regular polygon are induced by symmetries of the polygon. In general, the symmetries of a regular polygon having \( n \) sides are: the identity \( id \), the rotations \( r \), (amplitude \( 2\pi/n \) here related to the loop radius) around the origin; \( n \) axial symmetries \( s_1, \ldots, s_n \) with respect to a beam of \( n \) straight lines passing through the center the axial symmetries depending if \( n \) is odd or even, \( D_n = (id, r, \ldots, r^{n-1}, s_1, s_2) \) is a not abelian–(axial rotations) group of order \( 2n \) isomorphic to a subgroup of \( S_n \) (Cayley theorem). All the \( D_n \) elements are determined once the rotation \( r \) has been assigned and any axial symmetry \( s \). Finally, the question if a (monochromatic) loop should be considered open or closed, in the sense of graph cluster, touches some aspects of simplex-graph relations and loop inner symmetries. Here we associate to any loop a (outer, boundary) valence \( 2 \).

14 Concerning the loops and their representations, multisets (or bags) formalism can be a valid homogeneous loops representation, for sequences but not for chains (i.e. \( c_{ij} = c_{j} = c_{ij} \)) where the value of multiset coefficients (number of multisets of cardinality \( \varTheta_a \), with elements taken from a finite set of cardinality \( \varTheta_m \)) can be given explicitly as \( \mathcal{N} = \sum_{c_{ij}} \mathcal{N} \).
ate analysis of graph isomorphism under conformal transformation, we discuss these aspects in [1]. For a metric graph it would be important to provide a proper graph “topology” notion, in particular concerning a separability definition. Since the graph admits isomorphic parts and propagates (in the sense of conformal expansion) transmitting seeds-copies and isomorphic parts then a question to be addressed regards the events graph isomorphism.

15 A further point to be addressed is whether an events chain can be seen as a Cauchy sequence (in the sense also of clarifying how and when we can eventually provide a notion of Hilbert space or Banach space properties). More generally, given a metric space \((X, d)\), a sequence \(x_1, x_2, x_3, \ldots\) is Cauchy if, for every positive real number \(e > 0\) there is a positive integer \(N\) such that for all positive integers \(m > N\), the distance \(d(x_m, x_n) < e\). (In this model a vertex norm is never null and then the distance between vertices is never null, it can be null instead \(\sigma(-, \cdot)\) and \((\cdot, \cdot))\). Generally, for a discrete metric, any Cauchy sequence of elements must be constant beyond some fixed point, and converges to the eventually repeating terms. The notion of convergence of sum of vertexes has to be considered through an appropriate topological notion adapted to this graph framework. A further related problem consists in the research of graph “fixed-points”: given graph property \(f_G(m)\), we can ask if it is satisfied \(f_G(m) = f_G(m_0) = f_G(m_0m)\). We face this (per sistence) problem also in [1].

16 More precisely here intend a bijective application \(f\) from the vertices of \(G\) to the vertices of \(G\) that preserves the “relational structure”, in the sense that there is an edge from the vertex \(a\) to the vertex \(a_i\) if and only if there is a similar connection from the vertex \(f(a)\) at the vertex \(f(a_i)\) in \(G\). This notion has to be considered into a dynamical graph framework for transformations preserving relational structures. The relational structure is transmitted in the graph conformal expansions to larger states while a new relational structure appears, expansion of the graph structure as discussed in [1]. We could use a conformal expansion model with constant graph vertex number after expansions, and equal to the seed initial state graph \(G_o\), the expansion would be on charge of the connections number. Conformal expansion connects vertexes of the seed graph \(G_0\) to the twin graph \(G_2\) for the first step of expansion. At the \((m-1)\)-step there is the \(G_m\) (\(m-1\)-copy) completely connected to the \(m-G_0\) copy (all the \(G_0\) vertices, which have always zero valence in \(G_0\), are connected with all the vertices of the next-and according to the model to the former- copy \(G_0\)). The connection between the two copies in the \(1\)-step process inherits the connections of the former states \(G_0\) with the \((m-2)\) copies, according to the specific transmission model fixed by some “boundary data”. (Since connections are all “flat” (equal) this model closes the simplex). Particularly useful in the equiprobable graphs data”. (Since connections are all “flat” (equal) this model closes the other simplex). Particularly useful in the equiprobable graphs.

Two graphs are isomorphic if one can be transformed into the other simply by renaming its vertices, this has been considered in the metric graph context. The loops study faces a different aspect of the conformal transformations and on the \(N\)-constraint and on the choice of color base \(r\). (A further related problem is in what extent the events chains in conformal expansion the \(G_0\) from the conformal expansion in the variable vertex number mode). Then \(N^r(x)\) provides the total number of outgoing connections per vertex from the state of (colored direct) graph \(G_0\) (with “memory” in the sense of transmitting structure/connections as explained below) from former state \(G_m\) to expanded \(G_k (x > m_0)\) inheriting or not previous connections (of former state of the expansion) depending on the values attributed to the boundary data \((r'(x), r''(x))\), \(x\) is a graph state variable and stands for the number of steps in the conformal expansion, while \(r\) number of graph vertices assumed to be colored. There can be then \(r'(x) = [0,1] \) and \(r''(x) = [0,1] \) for each state (particularly \(x > m_0\) and depending on the chosen model; each connection is differently considered according to be a loop or antisymmetric conditions, on \(r'(x), r''(x)\) deals also with this aspect, \(r'(0) = [0,1]\) considers the possible previous history to a zero state \(m_0\), more specifically we intend a graph symmetric -connection, the absence of this condition \(1\) would give rise to a not completely connected graph copies and can be seen as a model for signal transmission from a source-vertex to others of a simplex, it is also clear that the outgoing-sign of the “directed” conformal expansion can be also read as a back-transmission, a “reply”-direction, to the simplex vertexes which are always “trapped” in a (constant) \(R^{+1}\) dimension (or superior) spaces thus the condition of not-reflexive connection. The explicit choice of \(r'(x), r''(x)\) makes possible to change the structure transmission model for expansion in accordance with each process state. To obtain the total connections number (during a transition) it is necessary to multiply \(N^r_i\) or \(N^r_i\) for the number of vertices per state which is always \(r\). Clearly any graph chain of \(m\) vertices can be a well chosen path in the graph sequences \(\Sigma^m G_i,\) the sum is clearly intended here in the sense of clusterization introduced for graph vertexes.

17 However, the problem of isomorphism of subgraphs (appearing here in the scale invariance and conformal transformation) is \(NP\)-complete, although it is evidently in \(NP\), it is suspected that the graph isomorphism problem is neither \(P\) nor \(NP\)-complete. Note that a graph is said to be strongly connected or disconnected if every vertex is reachable from every other vertex. The strongly connected components or disconnected components of an arbitrary directed graph form a partition into subgraphs that are themselves strongly connected. It is possible to test the strong connectivity of a graph, or to find its strongly connected components, in linear time. Note that this problem of connectivity is indeed strongly related to locality problem through valence notion. We address this issue in relation to equidistant (connected with equal edges) vertices in [1].

18 Then, a proper characterization of the graph state transition has to be included for the case \(m = \infty\). Loops distribution, \(P(\Theta,\omega)\), has always cardinality greater then \(\omega\); thus being \(\omega = \infty\) the (countable) cardinality is cardinal. On the other hand, the number of ways in which \(n\) events of \(r\) colors can be combined, considering also a loop on one value with \(\omega = n\) (not satisfying the conservation of \(\rho\) at \(G_o\), having role in the
could be considered as Markovian chains). Large conformal expansions would represent the asymptotic states, while the inhomogeneity changes with realization at equal state. We could say that the larger is the inhomogeneity (in color or algebra) and more articulated and complex are graph (and metric) levels, more articulated is the emerging spinorial structure depending on the colors probability distribution. Chains transitions are related to change in the loop-distribution, where loops presence is balanced by the \( \ell \) events in the chains.

**Connections, events and inhomogeneous vertex aggregates**

Inhomogeneous in color connections, \( \ell \) events, and more generally not-identical aggregates of any order, arising from permutation (and rotation) groups representations, would play a relevant role in the metric and dynamical frame considered in this approach, with the analysis of different representations of graph chain realizations. In this frame it can be simpler to explicit the related Clifford algebra: using such representations for antisymmetric connections, events-aggregates, according to the permutations, obey Clifford statistics, thus vectors as plexxors (for permutation) and spinors (for rotation) would appear, where spinors can be seen in some way as plexxors of aggregates. We conclude with some considerations on the emergence of polychromatic connections, whereas we deepen the number of connections per vertex of the graph (valence in a fully connected graph). We note that for \( m \ell \) loops, number of effective vertices for a general graph state \( G \), the following relations always hold: \( \bar{\mathbb{C}}(m) = m \ell \bar{\mathbb{C}}(m) \) and \( \mathbb{C}(m) = m \ell - 1 \) (See Figs 2). We need however to establish the exact form of \( \bar{\mathbb{C}}(m) \) and the transformation for a conformal expansion, from an initial state \( m \ell \) and reduction in the effective number of vertices by considering loops \( \ell \). The **effective number** \( \bar{m} \) of vertices for sequence \( n \bar{C} = m+1-\min \varnothing(m) \) where \( \min \varnothing(m) = (\Theta(m)H[\Theta(m)-2], \) (total order of two connections)–Figs 2. The \( \ell \) events number (and density \( \mathbb{C}(m)/m \) for \( G_m \)) increases with a graph conformal expansion. The extremes in cardinality is an graph property depending on \( r \) and \( \{p_l\} \) (the maximum probability \( p_{\max} \)). Regular permutations build the degeneracy class of the minimal sequence \( S_{\text{min}} \), this is not the case for other sequences. The supremum occurs in maximally decomposed sequence, \( S_m \), and the \( \ell \) events number is bounded in \( \mathbb{C}(m) < [r, N-1] \subseteq [r, m-1] \), being \( N \) in the effective number of vertices in a sequence (homogenized algebra). In general the degeneracy in chain is not invariant under permutation, accordingly to the \( \Theta \)-criterion. We could use \( \Theta \) as an index of the peak of the colors probability distribution and the colors homogeneity. The situation depends on the gap in the probability distribution with a maximum (or the presence of more maxima), for a gap increasing, the number of \( \ell \) events decreases. Loops number on vertex can vary at \( \max \mathbb{C}(m) = \text{constant} \)–Figs 2. (These relations, including the +1 term hold properly considering the algebra of a vertex without loop equal to \( \Theta = 0 \) or vice versa \( \Theta = 1 \), in fact we consider these possibilities differently in this work, this choice depends on how the loop boundary is considered). These quantities are regulated by the \( \Theta \)-criteria and the \( G \) probability distribution. It is thus important to assess \( \ell \) events transformation with conformal graph expansion. As a general result, the number of \( \ell \) events is minimum in the minimal sequence \( S_{\text{min}} \) (a state invariant-conformally expanded in algebra), then there is

\[
\min \mathbb{C}(m) = r-1 \forall m \text{ for } S_{\text{min}}, \text{ and }
\forall m \exists S_m : \partial_m \max \mathbb{C}(m) > 0,
\]

and the following two cases are possible

\[
\begin{align*}
(1) & \quad \max \mathbb{C}(m) = 2m(1-p_{\max}) < m-1, \\
& \partial_n \max \mathbb{C}(m) = 2(1-p_{\max}) < 1 \text{ if } \exists ! p_{\max} \geq \frac{1}{2} + \frac{1}{m}, \text{ otherwise (7)}
\end{align*}
\]

\[\text{(total order of two connections–Figs 2). The } \ell \text{ events number (and density } \mathbb{C}(m)/m \text{ for } G_m \text{) increases with a graph conformal expansion. The extremes in cardinality is an graph property depending on } r \text{ and } \{p_l\} \text{ (the maximum probability } p_{\max} \text{). Regular permutations build the degeneracy class of the minimal sequence } S_{\text{min}}, \text{ this is not the case for other sequences. The supremum occurs in maximally decomposed sequence, } S_m, \text{ and the } \ell \text{ events number is bounded in } \mathbb{C}(m) < [r, N-1] \subseteq [r, m-1], \text{ being } N \text{ in the effective number of vertices in a sequence (homogenized algebra). In general the degeneracy in chain is not invariant under permutation, accordingly to the } \Theta \text{-criteria. We could use } \Theta \text{ as an index of the peak of the colors probability distribution and the colors homogeneity. The situation depends on the gap in the probability distribution with a maximum (or the presence of more maxima), for a gap increasing, the number of } \ell \text{ events decreases. Loops number on vertex can vary at } \max \mathbb{C}(m) = \text{constant} \text{–Figs 2. (These relations, including the +1 term hold properly considering the algebra of a vertex without loop equal to } \Theta = 0 \text{ or vice versa } \Theta = 1, \text{ in fact we consider these possibilities differently in this work, this choice depends on how the loop boundary is considered)). These quantities are regulated by the } \Theta \text{-criteria and the } G \text{ probability distribution. It is thus important to assess } \ell \text{ events transformation with conformal graph expansion. As a general result, the number of } \ell \text{ events is minimum in the minimal sequence } S_{\text{min}} \text{ (a state invariant-conformally expanded in algebra), then there is} \]

\[
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\forall m \exists S_m : \partial_m \max \mathbb{C}(m) > 0,
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\[
\begin{align*}
(1) & \quad \max \mathbb{C}(m) = 2m(1-p_{\max}) < m-1, \\
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\end{align*}
\]
number of events, including the symmetric chain is
\[
\mathcal{Z}_{\mathbf{c}}(\mathcal{N}, N) = \mathcal{N}(N - 1) - \sum_{i=1}^{r} N_i(N_i - 1) = \mathcal{N}^2 - \sum_{i=1}^{r} N_i^2
\]
and
\[
\mathcal{Z}(n) = \sum_{i=1}^{r} p_i^2 - 1.
\]
(8)

\[
\mathcal{Z}_{\mathbf{c}}(n) = \bar{n}^2 - \sum_{i=1}^{r} p_i^2 = \bar{n}^2 - \sum_{i=1}^{r} p_i(1 - p_i)
\]
and
\[
\bar{n} = n + 1 - \min \mathcal{O}(\ell(n)),
\]
and particularly we find
\[
\mathcal{S}_{\min} \mathcal{Z}_{\mathbf{c}}(N, N) = r(r - 1) \quad \text{and for}
\]
\[
\mathcal{S}_2 \mathcal{Z}_{\mathbf{c}}^{(N)}(n) = n^2 \left(1 - 1/r\right), \quad \mathcal{Z}_{\mathbf{c}}(n) = \frac{n^2}{r} - 1.(9)
\]
for \( r = 2 \) we have
\[
\mathcal{Z}_{\mathbf{c}}(N, N) = 2(N_1N_2)
\]
and
\[
\mathcal{Z}_{\mathbf{c}}(n) = 2n^2p_1p_2.
\]
(10)

These relations were related to graph inertia \( \mu \) and radius \( R \). We considered the effective macroevents ("normalized" by the algebra homogeneization considering the equal color events adjacency). Then \( \mathbf{Z} \) is the total number of possible loops (of the order 2); we take into account all the possible connections, for the sequence, \( (N(N - 1)) \) and all loops are subtracted. Thus, quantity \( \mathcal{Z}(N) = \sum_{i=1}^{r} N_i(N_i - 1) \) and, respectively, \( \mathcal{Z}(n) = n(n - 1) \) for that sequence. In this counting, however, we have considered all multiple events related to the same color as necessarily separated, while we have seen not all combinations allow the separations but some adjacencies emerge, then a second level of rationalization or reduction in effective vertices is needed. Connections \( \sum_{i=1}^{r} N_i(N_i - 1) \) are the totality of those connecting equal values (loop), while \( N(N - 1) \) is the totality of connections, and there is \( \theta - N_i(N_i - 1) = -(N_i - 1)(N_i - 2)/[1 - \theta] \). That is, we excluded loops derived from not-necessary connections between same vertices, however we did not rule out the symmetric (z) connections or connections between identical vertices providing a distribution of macroevents with multiplicity greater than 1. (Importantly this implies that these procedures can be applied also for inhomogeneous algebra respect to \( R^0 \).) The introduction of a color algebra, changes also the metric structure associated to the original not colored ber graph, and consequently we can characterize a graph according to the different (dynamic) metrics \( r(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) defined on the different graph paths (chains). Below we introduce an events matrix, preceding definition of the graph first level metric based on the vertices composition and decomposition and the ber-bem relation, through this formalism metrics transformations can be also discussed.

**Matrix Q of the events** Before addressing the discussion on the metric graph we introduce a matrix \( Q \), this will serve to the presentation of some conceptual features of the graph that have findings in the graph adapted metric structures. For the colored graph, the \( n \times m \) matrix \( Q \) has \( m \) columns of (ber) events in chains, \( \bar{n} \) being the degree of application of \( D, m \) refers to an adapted index (rows are made of sequences elements, for a multi-bases graph there is a multiple matrix \( Q_{ij} \)). Events matrix \( Q_{ij} \) corresponds to an operator matrix \( Q_{ij} \) of elements \( D^r \times D^j \), according to the vertex-operator correspondence:
\[
D^rQ_{ij} = (D^i \times 1)Q_{ij} = Q_{ik}j = (D^j \times D^i)Q_{00} = (D^j \times 1)(1 \times D^i)Q_{00}.
\]
There is \( \bar{i}/ \bar{j} \in [0, 1] \) for the ratio between column (shift) and row (evolution) operator algebras, related to \( D^r \times D^j \) on the initial vertex \( Q_{ij} \) whose indexes are not constrained (a colored graph vertex has an algebra greater or equal then the ber-bem) With an initial event \( Q_{ij} \) on the "diagonal", the upper-triangle part of the matrix is "inaccessible

---

20 Chains are (colored) oriented realizations. \( \mathbf{C}^r \) with an a-priori ordering (orientation) always admits an inverse \( \mathbf{C}^r \), equal to \( \mathbf{C}^r \) but with an opposite orientation of vertices ordering. \( \mathbf{C}^r(\mathbf{a}) = \mathbf{C}^r(\mathbf{a}) \cup \mathbf{C}^r(\mathbf{a}) \) is a (not-homogeneous) loop on the first (last) vertex of \( \mathbf{C}^r(\mathbf{a}) \); this is a "not-abelian" composition, i.e. \( \mathbf{C}^r(\mathbf{a}) \neq \mathbf{C}^r(\mathbf{a}) \) is a loop on the first (last) vertex of \( \mathbf{C}^r(\mathbf{C}^r) \). In a graph, cycles or totally symmetric chains, \( \mathbf{C}^r \), are possible (an homogeneous loop of algebra \( \mathcal{O} \), is a cycle of length \( \mathcal{O} \), endowed with several symmetry properties. Necessary condition for a chain to be a cycle is the periodicity (according to color, algebra) with 1 step. We say that the (open) chain is totally symmetric if (necessary condition) there is at most one element, a part the boundaries with multiplicity (macroevent number in adapted frame) 1, if this is unique it is called the chain center \( a_0 \), or symmetry pole. With the exclusion of \( a_0 \), which has multiplicity 1, there is \( \min N_i = 2 \) for any vertices of the chain, particularly for the chain boundaries. Then \( \forall a_0: a_j < a < a_i, \text{microreversibility} \) or local symmetry with center \( a_0 \). According to this scheme if there is a chain then there is also a reverse, except if it is totally symmetric where \( \mathbf{C}^r = \mathbf{C}^s \). Note homogeneous loops are always reversible. We discussed extensively the influence of chain ordering in the graph metric symmetries, an interesting question is how, considering a probability distribution, the number of symmetric and reversible chains, or the micro-reversibility increases for superior states under graph conformal expansion.

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21 According to the constraints on the operators indexes, two matrixes definitions could be given: (1) A matrix excluding not-homogenous loops: \( 0 < \bar{i} \geq \bar{j} \geq 0 \). (For \( \bar{j} = 0 \) there is an homogeneous loop). There is then: (0) \( 0 < r_2 - r_1 = \bar{j} \geq c_2 - c_1 = \bar{j} \geq 0 \), where \( r_1, c_1 \) are indices following the first or second application (according to chain order) and (8): \( c_2 \geq \bar{j} \) and \( r_2 \geq r_1 - c_1 \) for any starting element. (2) A matrix including not-homogenous loops (in some ways with "memory property"):
\[
0 < r_2 - r_1 = \bar{j} \geq \bar{i} \geq c_2 - c_1 \geq 0.
\]
There is (6) \( c_2 - c_1 = 0 \), an homogeneous loop; (7) \( c_2 \leq c_1 \) then if \( c_2 \in \{c_1 \} \), \( c_2 \) belongs to the "past" of indices \( c \) it is a loop \( (\bar{i} = 1) \), or if \( c_2 \in c \), \( c_2 \) belongs to the "future" of indices \( c \) then \( c_2 = \bar{i} + \text{max }c_1 \). Algebra relation \( \sigma(\bar{i}) \) could be generic according to constraints.
paths confinement), and evolution of $\gamma_{\text{causal}}$, of elements on the main diagonal, corresponds to a maximally decomposed chain $C_{\text{max}}$, and any chain $\gamma_{\text{causal}}$ could be recovered as deformation of $\gamma_{\text{causal}}$, locating the $\gamma_{\text{upper}}$ boundary in the sense of matrix confinement

$$
\gamma_{\text{causal}} = T(\gamma) = \delta_{ij}(D^i \times D^j)Q_{00} = \delta_{ij}g_{ij},
$$

(a further term can be added to the $\delta$-term to consider events, here $D^i \times D^j : Q_{00} \rightarrow Q_{ij}$). $\delta_{ij}$ is the Kronecker delta (sum on repeated index is intended composition in chain), while $g_{ij}$ is a $\delta_{ij}$ deformation and $g_{ij}$ are deformation matrices constrained by the assumption on the algebra degrees $i/j$ mirrored in the metric graph approach.

## 2 The metric graph $G_n$

In this section we introduce the metric graph, discussing aspects of the polychromatic graph symmetries, while we will deepen this feature of the graph model in [1]. After introducing the double metric structure, we then conclude in Sec. [2] by exploring the color symmetries in the polychromatic metric graphs.

A doubled metric structure is overlapped to the colored graph $G$ realizations with two chain-adapted metric definitions, $\sigma(\cdot, \cdot)$ and $g(\cdot, \cdot)$, to be defined on $G_n$ chains, and therefore differently reflecting chains symmetries and chains transformation [2]. A metric transformation connects chains of the same graph state or graph different

$$
\sigma(\cdot, \cdot) - \text{metric } \sigma = h_{AB}w_{xy}^{AB}; \quad \sigma = h_{AB}w_{xy}^{CD},
$$

$$
g(\cdot, \cdot) - \text{metric } g = h_{ABCD}w_{xy}^{ABCD}; \quad g = h_{ABCD}w_{xy}^{ABCD},
$$

coincident in special cases depending on the adaptation to graph realizations. Here we do not include the $\sigma$ and $g$ forms. An "induced metric" level has been also defined

$$
\sigma(\cdot, \cdot) - \text{metric } \sigma = h_{AB}w_{xy}^{AB}; \quad \sigma = h_{AB}w_{xy}^{CD},
$$

$$
g(\cdot, \cdot) - \text{metric } g = h_{ABCD}w_{xy}^{ABCD}; \quad g = h_{ABCD}w_{xy}^{ABCD},
$$

$$
\sigma(\cdot, \cdot) - \text{metric } \sigma = h^{AB}w_{xy}^{AB}; \quad \sigma = h^{AB}w_{xy}^{CD},
$$

$$
g(\cdot, \cdot) - \text{metric } g = h^{ABCD}w_{xy}^{ABCD}; \quad g = h^{ABCD}w_{xy}^{ABCD},
$$

(we do not focus on the induced metrics but it is clear that an induced metric structure exists to which a graph can be associated. In some way, also $g(\cdot, \cdot)$ and $\sigma(\cdot, \cdot)$ metrics in Eqs (13) and (14) are composed by properly chosen induced (h-metrics). Quantities $h_t$, with $h_t \in \{h, (h)\}$ are general matrices we detail below, $F$ are for indices $\{A, B, C, D, ...\}$ of $\text{basis}$ or colors in polychromatic or doubled monochromatic approaches to graph chain realizations. Indices $\{x, y, z, t, ...\}$ are "inner indices" (ber, defined in an "inner space-graph"). These quantities will be discussed in more detail in Sec. [2] with an example which, adopting a fictitious exemplification, will focus especially on symmetries.

22 Multilevel structures associated with the higher levels graphs should be also considered. More generally the concept of structure has become increasingly important over the last few decades, becoming also one of the fundamental notions of modern mathematics, and particularly structure in a relational theory is clearly a predominant concept. In the spacetime (Lorentzian) manifold the (one-level) metric structure overlaps in some extent to the causal structure, describing the causal relation between points in the manifold, rendered and reflected also by the metric definition. As described here, metric structure is the (polychromatic) graph structure reflected and readjusted, in the sense of the role of symmetries, in $g$-metric. In $g(\cdot, \cdot)$ the invariance for colors inversion results in a gluing of realizations in chains which are related by certain predetermined transformations. This approach can be considered therefore a form of structuralism where objects "intrinsic" properties are defined by their (external) relations. Structure, intended as information organization, in the case of the combinatorial graph we are considering here involves the way in which a graph is partitioned into interrelated components. An event structure represents in a broader sense, the set of objects included in $G$ and their relationships, understood as reciprocal relations of $G$ constituent elements, where different levels and different clusters emerge. This structure includes a hierarchy of relations (conformal levels- monochromatic clusters, or the polychromatic $\sigma$-events) where the higher-level structure contains multiple copies of the lower-level structures (the replicas, reflecting the isomorphism and replicas of all the $G_n$ states of a reflect and conformal expanded graph [1]. Algebra homogenization is also translated into metric transformation. Metric $\sigma(\cdot, \cdot)$ depends on and discerns the colors (and eventually subgraphs) symmetries of the polychromatic chains, as defined in Sec. [1.1] mirrored in the metric structure defining metric symmetries. Metric $g(\cdot, \cdot)$ is invariant for a set of chain transformations, we define this metric-level as emergent from $\sigma(\cdot, \cdot)$ level, describing classes of graphs realizations related by specific assumed transformations, in Sec. (2.1) we discuss this aspect considering a simple frame. The $\sigma(\cdot, \cdot)$-metric and $g(\cdot, \cdot)$-metric can be written in different forms as:

$$
\sigma(\cdot, \cdot) - \text{metric } \sigma = h_{AB}w_{xy}^{AB}; \quad \sigma = h_{AB}w_{xy}^{CD},
$$

$$
g(\cdot, \cdot) - \text{metric } g = h_{ABCD}w_{xy}^{ABCD}; \quad g = h_{ABCD}w_{xy}^{ABCD},
$$

$\{h_t, \} \in \{h, (h)\}$ being the basis of Pauli matrices.
Fig. 2. Cardinality $\mathbb{Q}$ of the set of $Q$-quantities and algebra (order) $\otimes Q$ of the element $Q$, where $\ell$ is for the graph loop, $C$ to chains, $\circ$ indicates equiprobable distributions, $r$ is the colors number (where not otherwise specified), $(m,n)$ refer to graph states $(G_m,G_n)$, $\Omega(Q_1,Q_2)$ refers to the distributions considered in Sec. (1.1.1) and particulary Sec. (1.1.2) re-parameterized in terms of quantities $(Q_1,Q_2)$. We highlighted particularly the presence of maxima and minima of entropy and cardinality, the peculiarity of equiprobable distributions and, in the case of not-equiprobable graph, the cases with a single or multiple maximum of probability, the dependence on colors number $r$ and the change for conformal transformation. The curves relate different graphs at equal states creating classes of different graph with similar properties. $U/\epsilon$ refers to a system $\ell$ of $\otimes \ell$ indistinguishable, not-interacting (no events), particles, let $\epsilon(n)$ denotes the energy of a particle (a quantity related to the algebra $(\mathbb{1} \otimes \mathbb{1})$ in ber). Since particles do not interact, the total energy ($U$) is the sum of single-particle energy. This explores the system of $m$ oscillators describing, in the limit of infinite $m$ a Klein-Gordon scalar field. The fundamental difference in our case here is that we do not consider the possibility that a macroevent could be associated to $0$ event. The Stirling approximation can be used within the hypothesis $m \gg 1$ and $k = n - m \gg 1$, implying $r \gg 1$ (large conformal expansion). We note that $S \approx \ln \Omega$ in the Stirling approximation is symmetric for change of $k$ and $m$ ($U/\epsilon \approx k/m$). Last panel: shows $\mathbb{Q}_\ell(m)$ function of $r$.

Quantities $\omega^A$ and $\omega^{AB}$

Quantity $\omega^{AB}$ refers to distance notion for a couple of colored graph vertexes (we drop the inner indexes for convenience while sum on repeated indices and contractions are clear from the context). Element $\omega^A$ in the first level graph $\sigma(\cdot,\cdot)$ considering mainly loops, refers to a vertex in colored bases indexes $A$. We have implicitly considered an homogenized algebra, whereas in a monochromatic loop internal symmetries act (group of polynomial $D$) which we will ignore (undistinguishable vertexes). Equivalently, because of the vertex/events definitions we intend $\omega^A$ as measure related to colored indexed vertex, or event index and base index, related to polychromatic or couples of monochromatic chains with complements, and in first approximation providing the vertex algebra in a given base. Decomposition $\omega^{AB} = \omega^A \omega^B$ follows from assumption on the $\omega^{AB}$ evaluation and the action of matrix $h^{AB}$. Similarly for $\omega^{ABCD}$, this general quantity refers to a couple of vertexes (a chain connection in some colored algebra). Actually metric $g(\cdot,\cdot)$ couples polychromatic chains related by certain transformations. Decomposition $\omega^{ABCD} = \omega^{AB} \omega^{CD}$ binds together specific couples, eventually we bind through this assumption $g(\cdot,\cdot)$ to $\sigma$-metric.

On the matrices $h$ and $(h)$

Here we consider matrices $h$ and $(h)$ in Eqs (13, 14) and (15, 16) in $\sigma$- and $g$-metric. Matrix $h_{AB}$ can be been
considered be symmetric or antisymmetric, we can include an antisymmetric part clearly related to the events or connections. Using a first approximation, in Sec. (2.1) we consider "Lorentzian" metric $h_{AB} = \eta_{AB} = \text{diag}(1, -1)$. A second main assumption concerns the form of $h_{ABCD}$. We already partially addressed this issue discussing quantities $\omega^A$. Considering the symmetries in $g(\cdot, \cdot)$ metric structure, we consider the following two possibilities:

$$\eta\text{-form} : \quad g = h_{ABCD} \omega^{ABCD} ;$$

$$\sigma Q\gamma\text{-form} : \quad g = Q_{ab} \sigma^a_0 \sigma^b_0 \ldots ;$$

In $\eta$-form, $h_{ABCD}$ is a general matrix, in the $\sigma Q\gamma$-form, metric $g(\cdot, \cdot)$ is related to $\sigma(\cdot, \cdot)$ metric, emerging from a set $\{\sigma^a_0\}$ of $\sigma(\cdot, \cdot)$ metrics through the combination of metrics on which generic transformations $Q\gamma$ act, adapted to chains of the same graph state $G\gamma$, related by permutations. These chains are not distinguished in $g(\cdot, \cdot)$ metric structure. (Symbolic expressions of $\sigma^a_0$ in the second term of (17) indicate in a generic way these metricized chains.) In general $Q\gamma$ defined on a color base, corresponds to an inversion of the graph dichromatic connection orientation, acting here directly on the matrix $h_{ABCD}$ indexes (in this case $Q\gamma$ acts on the right terms) and we mainly consider chain couples and their colors inversion using the $\Phi$ form.

The inner indices

General inner indices $(x, y)$ can be as

$$(x, y) = [(a, b), (a_A, b_B), (a_A, a_B)].$$

Within these different choices, from Eq. (13) and (14) we obtain

$$\sigma\text{-metric} : \quad \sigma = h_{AB} \omega^A_{ab} \equiv \sigma_{ab} \omega_{ab},$$

$$\sigma\text{-metric} - \text{chain-form} : \quad \sigma = h_{AB} \omega^A_{ab} \equiv \sigma_{aA} = \sigma_{aA} \omega^{A0},$$

$$\sigma\text{-metric} - \text{string-form} : \quad \sigma_{ab} \equiv h_{AB} \omega^A_{ab};$$

and if $\eta_{AB} = h_{AB}$ then $\sigma_{ab} = \sigma_{aA} = \sigma_{aA} \omega^{A0}$

(\text{where} \ (A, B) \ \text{have values in} \ (0, 1)). It is worth noting that the particularizations presented in Eq. (15) refer to assumptions on inner indices, and on the particular inner space. These different cases turn equivalent only under special assumptions. The indexes $(a_A, b_B)$ and $(a_A, a_B)$ carry color base indexes, the inner and color spaces are consequently attached to each other.

$g\text{-metric}$

Options $[(x, y)] = [(a, b), (a_A, b_B), (a_A, a_B)]$, present in $g$-metric are reflected and inherited by second level $g$-metric, according to Eqs (13), (14) and Eq. (17). We note that there is a clear formal analogy between $\sigma(\cdot, \cdot)$ metric of $g(\cdot, \cdot)$ metric in $\eta$-form of Eqs (13), (14). Because of this, we can understand, in this discussion, properties described for $\sigma(\cdot, \cdot)$ metric holding, in form, also for $g(\cdot, \cdot)$ in $\eta$-form. In general however, we can specify the indices introducing notation $g_{uv}$, where $u \equiv (ab)(ah)\eta$, notation (') for the second pair of indices would fit the $\sigma Q\gamma$-form with the $\Phi$ form of Eq. (17), where $u$ and $v'$ are related to a $Q\gamma$ transformation acting on the capital, (color) indexes ($Q\gamma$ acts on $h_{AB}$ metric). Therefore with $g_{uv}$ we mean this special $\sigma Q\eta\gamma$-form with the $\Phi$ case as the generic case where the indexes are $(a_A, b_B, cc, dd)$ - i.e. $u' = (cc, dd)$. We consider different cases: a "chain-form" $\sigma_{aA} = \sigma_{aA} \omega_{aA}$, $\sigma_{aA} \omega_{aA} \epsilon_{iA} \alpha_{ij} \omega^{A0}$ corresponding to indexes $(a_A, a_B)$ from Eq. (18). Particular assumptions on $h_{AB}$, we obtain $g_{aA} = g_{aA} \omega^{A0} \epsilon_{iA} \alpha_{ij} \omega^{A0}$, reducing to the string form $g_{aA} = h_{AB} \omega^A_{ab} \omega^A_{cd}$ and to $G_{aA} = h_{AB} Q_{aA} \omega^A_{ab} \omega^A_{cd}$ depending on transformations acting only on the color capital indexes. We note that all the variants can coincide, depending on assumptions on the inner space of small indexes.

Inner indexes and the inner space definition

We can introduce a general quantity $\chi^A = (a_A, b_A)$ or $\chi^E = (a, b)$, where $a = a_0\chi^A$ and similarly for $b$ ($\chi^A$ refers to the inner space-color space attribution). Then, the index choice (18) can be reduced through an appropriate choice of inner indexes, relating $a$ to $b$ indexes, and transformations on the colors ones, through assumptions on $h^T$ matrices and $Q\gamma$ transformations. Mainly, we consider adapted bases in $(1 + 1)$-models, where $(a, b) \in (\sigma, \tau)$ and the capital color indexes vary in $(0, 1)$. Schematically, we set the indexes in accordance with $(x, y) = y = (a, b)$, string form, where in general $\chi \in S_A \times S_B$, $S_A$ being index $A$ definition values space), and $(x, y) = \chi = (a_A, b_B)$ for the general case, where $\chi_a \in S_A$, and $\eta_{AB}$, and finally in the chain-form with $(x, y) = \eta_{iA}$, where $u = (A, B)$. In Eqs (18) the inner indexes could be independent from the colors indexes, as in the string-choice, or also related to the color indexes. (These different cases overlap and coincide in special cases. In fact the indexes $a_0, b_0, a_1, b_1$ are not fixed, in other words the indexes $a_0$ can be as $a_0 \in S_0$, which may coincide with one of the others, depending on the specific model and inner space).
this definition reflects the chromatic order relations of the graph chain connection and thus considers the chromatic symmetries of the chains. This construction clearly refers to the situation where we take into account the inner indexes reflecting the events structures. (In this section we use an approximation where we neglect the events and the metrics reduce to algebra relations. For example, concluding this section, we discuss a simple doubled metric model using the Q-transformations as color order inversion to overcome these issues, evaluating loops, the monochromatic, symmetrical connections.

However, to explicitly consider the higher terms derived from the polychromatic vertices, as raised from first level events and the (colors) symmetries describing events, we can define a quantity $\nabla h \rightarrow (\partial x + \Lambda)$, considering connections as new vertices, with the introduction of a term we specify below). Therefore, considering $\sigma(\cdot)$ and $g(\cdot)$, for the $\sigma(\cdot)$ metric we could write $\sigma(\cdot) = h_{ABCD}$ or $\sigma(\cdot) = h_{AB}X^C X^D \equiv h_{ABCD}^\chi$. For the $g(\cdot)$ metric structure, considering the different forms, we can write

$$g_{\alpha\beta} = h_{ABCD}^\chi \omega_{\alpha\beta} \omega_{\gamma\delta} = h_{ABCD}^\chi \omega_{\alpha\beta} \omega_{\gamma\delta} \omega_{\zeta\xi}$$

$$g = h_{AB} \omega_{\alpha\beta} \omega_{\gamma\delta} = g_{\alpha\beta} \delta^{\gamma\delta}$$

(19)

$$g = h_{AB} \omega_{\alpha\beta} \omega_{\gamma\delta} = g_{\alpha\beta} \delta^{\gamma\delta}$$

(20)

(we used the metric operator $h_{xy} \equiv \eta_{AB} \partial x \cdot \partial y$, and grouped the colors indexes $\omega_{\alpha\beta} \rightarrow \omega_{\alpha\beta}$. Several assumptions on color indexes in fact are equivalent to assumptions in the symmetrical part of the matrix and quantities $\omega$. Analogously, for the induced, $g$-metric, considering Eqs (16), for example we find $g_{ABCD} = (h)^{xy} \partial x^A \partial x^B \partial x^C \partial x^D = (h)^{xy} \partial x^A \partial x^B$ where, eventually, $(h)^{xy} = (h)_\tau \Omega_{\tau} \tau$, using the Q-form for an induced metric.

Notes on $\sigma(\cdot)$ and $g(\cdot)$ metrics

Graphs chains metric notion, as in Eqs (20) and Eqs (19) shows similarities in form as strings generalizations. Nevertheless $\sigma(\cdot)$ and $g(\cdot)$ metrics result to be conceptually very different. Metric $\sigma(\cdot)$ can be considered a natural graph coloring adapted to the coloring for vertices defined on a graph ordered realization. Metric $\sigma(\cdot)$, acting directly on vertices, compares different events, mixing the event and basis indexes. On the other hand, $g$-metric can be constructed using $\sigma$-metrics–see Eqs (17). Thus $\sigma(\cdot)$ metric as in Eq. (13) constructs vertex distances with elements with different color indexes. More specifically, consider $X^A \rightarrow (x,y)$ with $\omega_{\alpha\beta} \equiv dx \equiv x \partial x$ for a base $x$, and notation $\otimes$ denoting composition in metric, then metric $g(\cdot)$, in form Eq. (17)-2, could include term $dx dy$ in contrast with $\sigma(\cdot)$ where elements $[x^A y^B, dx \equiv x \partial x, dy \equiv y \partial y]$ can be considered. The term $x \partial y$ is not defined in $g(\cdot)$ of last Eq. (17)-2, defined by connections and not vertices. Note that if $h$ is a diagonal matrix, then $\sigma(\cdot)$ has no $x \otimes y$ term. Metric $g(\cdot)$, compares distances in events of different algebras and event basis relating $g$-metric transformations to events transformations. Metrics $\sigma(\cdot)$ and $g(\cdot)$ differ also in the role of symmetries for directed colored connections.

Transformations and symmetries

We can consider transformations on inner indexes and transformations on colored indexes. These different transformations can be related in some cases. We can also introduce quantities $L_r$ and $L_x$, for $\sigma$ and $g$ metrics, respectively, from contraction of Eqs (13)-(14) with a matrix $h_\tau$ or $(h)_\tau$, where $\tau \in \{A,B,C,...\}$ are colored indices, and $b = \{x,y,z,...\}$ are inner indices, assuming different forms depending on the model chosen for the inner indices and on the matrices symmetries. (Here we can take advantage of the formal analogy between $\sigma(\cdot)$ metric of Eqs (13), (14), and $g(\cdot)$ metric in $\eta$-form. We can adopt a general matrix for the contraction as in $L \equiv g_{ABCD} h_{ABCD} \equiv g_{ABCD}$, and similarly for other cases as $L = h_{ABCD} \omega_{\alpha\beta} \delta^{\gamma\delta}$, and transformations on these indexes can be represented as graph vertex transformations as in $L = h_{ABCD} \omega_{\alpha\beta} \delta^{\gamma\delta} \delta^{\eta\zeta}$, mixing inner and colored indexes.

More generally we could in fact construct a "Nambu–Goto"-like action of the form $S_{NG} \equiv \int \sqrt{-\det(h)} \delta \Sigma$, or a "Polyakov-like" action form $S_P \equiv \int \left(\sqrt{-\det(h)} \delta \Sigma - (p-1) \delta \bar{\Sigma}\right)$. As $L \equiv (h)_\tau \eta_{ABD} \omega_{\alpha\beta} \delta^{\gamma\delta}$ (where $p$ is a term due to the dimensionality $(1+p)$ of "worldsheet area"–the inner space–in the "covariant Polyakov" action which we will here not explore and we can consider $p = 1$. However here we do not specify the "worldsheet invariant area" $\sqrt{-\det(h)} \delta \Sigma$ and the proportional factor $23$.

Concluding we note that graph $G_{\eta}$ metric structures is adapted to the each directed path $G_{\eta}$, which can be considered closed or open, and grouped chains bundles of Figs [1], and then considered after conformal transformations. The $G_{\eta}$ cluster substructure plays an important role in $\sigma$-metric, in the "inner" structure of $X^A = X^\chi$, being loop or connection where color inhomogeneity has to be considered through events definitions, therefore as transformation on vertex index. (Vertices of the first level graph are monochromatic, in different levels there is an articulated structure due to the multiple chromaticity.) Aspects of this graph model, as the loops- and connections sets after paths and states transitions, and the generations of different events structures (graphs), ap-

23 Note that string loops in space are due to closed bosonic string, considering closed string theory compactified on a circle of radius $R$: $X = X + 2\pi R \theta$ implying momentum is quantized (string can wrap the circle: $X = X + 2\pi R \theta$, which is $\theta \in \pi$). This (topological) notion of open or closed string must be translated into the definition of loop (closed chain) and connection in graph representations. A further issue consists in the conditions imposed for example for open strings (Lorentz invariance-leading to Neumann or Dirichlet conditions). For closed string we can parameterize the equation of motion ($X^\chi, h_{\chi}$): using $\xi^\chi(x) = x \otimes x \otimes x$ obtaining $\partial \partial X^\chi = \partial \partial X^\chi = 0$, with a solution (with "chirality") $X^\chi = X^\chi(\sigma + \tau) + X^\chi(\sigma - \tau)$.
proach some conceptual and formal aspects of supersymmetric models [11]. We can consider additional (anticommuting) terms in the \( \alpha \) quantities related to homogeneous loops (with an internal symmetry linked to \( D_2 \) group) and the antisymmetric connections. Conveniently, specifying the \( \Delta \) object introduced above, we could use \( \partial_a \Phi^k \rightarrow \left( \delta^k_c \partial_c + \delta^k_b \partial_b \right) X^C \equiv \hat{\omega}_a^{cb} X^C \) enclosing coupling terms, and \( \hat{\omega} \) indicates the \( \Delta \)events level (any higher levels \( \hat{\omega}^{(w+1)} \) are related to \( \hat{\omega} \) one), related to the vertices of a sequence of derived graphs.\(^{24}\) We stress here that in supersymmetric models there are as many (physical) fermions as (physical) bosons (in versions for each scalar \( X \) there is a Majorana spinor—e.g. each of these fermions a two-component Majorana, real spinors). Considering the antisymmetric terms we could consider then \( S = \mathcal{L} + (h_a^{bc} \hat{\omega}_a^{cb} \Phi_{MN}) \) (string-like formulation \( \Phi_{MN} \) would be formally an antisymmetric spin 2 tensor similarly to a Neveu–Schwarz "B"-field).

2.1 Exploring symmetries in graph metric structures

Metric structures \( \sigma(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) reflect differently the chromatic (and in some cases algebraic) symmetries of the graph chain realizations discussed in Sec. (1.1.1). Color symmetries interest colored graph vertexes and polychromatic clusters, and they are mirrored in the graph metrics playing, eventually, a role in the dynamical graph. Metric structure \( g(\cdot, \cdot) \) does not distinguish chains related by a set of \( Q \) transformations as in Eqs (17). In the example explored in this section, we consider \( Q \) as chromatic metric in the ordered graph. On the other hand, it is clear that generalized permutations reflect on the metric as the conformal graph expansion (following conformal metric transformations).\(^{25}\) To discuss some aspects on algebra and colors chains symmetries in metric structures we adopt here a simplified model for the graph metrics considering only the loops. In this way we also point out some relations between the \( \sigma(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) metric levels. To simplify our discussion we have chosen the chain-form introduced in Eq. (15) and Eq. (21), considering \( h_{AB} \equiv \eta_{AB} = \text{diag}(1,-1) \).

In first approximation the metric could be expressed into colored vertex algebras relations, by introducing a graph vertices metric and vertex norm, with a (natural) metric element adapted to the graph coloring and related with the vertex algebra \( \otimes \). A minimal (not-null) norm is assumed associated to the ordinary events \( |e_0 \rangle \in \mathcal{A}^0 \) (and "minimal" distance \( \delta^{(e)} \) between two adjacent vertices of the minimum \( \mathcal{A}^0 \) ber). A ber can be of local (related to a embedded sub-graph concept) \( \mathcal{A}^0 \) minimum algebra.\(^{26}\) Conveniently, in this simplified framework the sequences and chains homogeneity conditions can be reduced to a metric condition, through an analysis of algebra relations where metrics provide a definition of and a criterion to establish the algebra homogeneity. (A chain can be inhomogeneous with respect to the ber assumed homogenous, in this case the metric could be "evolved" considering action of evolution \( D \) on its elements).\(^{11}\) Exploiting, with \( Q \) applications, the colors symmetries for the directed connections, we consider on similar levels in the metrics, connections \( (c_{ij}, c_{ji}) = (−c_{ij},c_{ji}) \) (color index \( i, j \)). We discuss these aspects in details for the combinatorial graph in [1] with a graph \( G \) characterized by a base of \( r \) colors, having therefore a total number of \( r^2 \) distinct kinds of connections (\( r \) loops plus \( rt \) \( t \) connections) and one ber. In these approximations we discuss the \( \sigma(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) transformations. Using explicitly the notion of algebra \( \mathcal{O}_\ell \) of a loop \( \ell \) on a graph \( G \) vertex, we compare graphs algebras and structures. We consider here a \( 1 + 1 \) dimensional framework a ber-bem couple, including an emerging ordinary algebra associated to an emerging base: by considering metrics transformations a new ber is related to a derived colored base. Two main approaches can be explored, depending to the decomposition base, the algebra relations and the polychromatic order. We start by considering \( \sigma(\cdot, \cdot) \) in chain form with \( h_{AB} = \eta_{AB} \) and using the simplified notation \( \omega^{(x)} \equiv \Delta x \) for the base \( x \). We explored, for the directed graphs, the quantities \( d_{\eta_{mn}}(\alpha) = \pm \delta^{(x)}_{\alpha} \Delta m n \) with \((\alpha, \delta) = (\pm, x, t)\), where \( \omega_0 = \pm 1 \) and \( (\theta, \delta, \alpha) \) respectively, being \( \theta \equiv \pm (\pm, \pm) \), notation \( \mathcal{O}_\ell \) for the ordinary algebra \( \mathcal{A}^0 \). We mainly adopt inner bases with \( \theta = t \) and \( \omega_0 = \pm 1 \). A relevant quantity within this choice of signs is \( \Gamma \equiv e_1 (\mathcal{O}_\ell^{(a)} - \mathcal{O}_\ell^{(a)}) \), where \( (e_1, e_2) = \pm 1 \) and \( a = 1(2) \) for \( \sigma(\cdot, \cdot) \) and \( g(\cdot, \cdot) \), expressing the graph inhomogeneity and symmetries of \( \sigma(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) metrics. In [11] we also

\(^{24}\) Thus, there is \( f_{\alpha}^{\beta} \hat{\omega}_a^{\alpha \beta} X^C = \hat{\omega}_a^{cb} X^C + \hat{\omega}_a^{bc} X^C + \hat{\omega}_a^{cb} X^C = 0 \) in the sum (w’ ≠ w). In strings theory models we could use lightlike gauge for the inner indices, then \( \mathcal{L} = \eta_{\alpha \beta} \hat{\omega}_a^{\alpha \beta} X^C = \eta_{\alpha \beta} \hat{\omega}_a^{cb} X^C + \eta_{\alpha \beta} \hat{\omega}_a^{bc} X^C + \{\text{coupling}\} \) containing additional terms. Eventually using \( \mathcal{P}_M = \hat{\omega}_a^{cb} X^C + \eta_{\alpha \beta} \hat{\omega}_a^{cb} X^C + \{\text{additional terms}\} \) we obtain \( \lambda_{\alpha \beta} = \eta_{\alpha \beta} \hat{\omega}_a^{cb} X^C = 0 \) for the elements of \( \xi \equiv (\pm \Delta m n \Delta m n) \) with \((\delta, \theta, \alpha) = (\pm, x, t)\), where \( \omega_0 = \pm 1 \) and \( (\theta, \delta, \alpha) \) respectively.\(^{26}\) A colored G graph has a structure composed by loops and \( \Delta \)events, from these a new \( \mathcal{O}_\ell \) level is generated with a related derived metric structure. As a first approximation and to fix the ideas, the actual distance notion for connection is not specified. In complexity models distance definition is differently provided and widely debated. This depends from the emerging spinorial structure. In this sense \( \sigma(\cdot, \cdot), g(\cdot, \cdot) \) are to be considered approximations of more complex metrics including greater levels \( \mathcal{O}_\ell \).
consider $\epsilon g = +1$ adapted for the description of the graph realizations properties related to the colors probability distribution.

From $\sigma$-metric to g-metric

A g-metric transformation can be reduced to $\sigma$-metrics transformation and these to a part of the graph (algebra) conformal transformations (and generalized permutations on vertexes). By using Eq. (17)-[2] as decompositions in $\sigma$-metric, a graph with a general colored $\mathcal{A}^g$ algebra is compared with a $\mathcal{A}^\sigma$-graph which can be considered an embedding (cell) graph, since the graph with $\mathcal{A}_x$ algebra can be seen an immersed graph or an overlapped coloring, of the original, embedding, graph. To enlighten this situation and the colored graphs symmetries we can write the $\sigma(\cdot)$ metrics as follows: $\sigma^\sigma(x) : \sigma^\sigma(L) = \sigma^\sigma \circ \gamma(L) \equiv \Gamma^\sigma$, where arrows indicate the connections $\iota_{\omega_{\xi}}$ sign (fixing quantity $\xi$), subscript ($z$) refers to the connections sign relation for the directed graph $X^z \equiv (x, y)$ colors bases. Then metric $\sigma^\sigma(x)$, for a $x$-base graph with algebra $\mathcal{A}_x \succ \mathcal{A}_L$, can be written as the combination of a $x$-base metric $\sigma^\sigma(x)$, with ordinary algebra, and $\sigma^\sigma \circ \gamma(L)$ in a $\mathcal{A}^g$ graph in a fictions derived $y$-base. Transformations $\mathcal{Q}$ as colors inversion $P$ and $T$ for the color base ($x, y$) respectively relate $g(\cdot)$ and $\sigma(\cdot)$ graph metrics and, consequently, graphs endowed with ordinary $\mathcal{A}^\sigma$-algebra base to a general $\mathcal{A}^g$-algebra one. Metrics $\sigma(\cdot)$ and $g(\cdot)$ transform differently for $\mathcal{Q}$-applications, involving differently colored connections symmetries for the directed-antisymmetric graph connections. General $\mathcal{A}^g$ algebra metric can be related to minimal $\mathcal{A}^\sigma$ metrics composition describing $\sigma$-to-$g$ transformations for a $\mathcal{A}^\sigma$ graph and for a $\mathcal{A}^g$ graph as follows

$$\begin{align*}
T &- 1|\sigma^\sigma = \sigma^\sigma(L) = 0, \\
T &- 1|\sigma^\sigma(x) = \left( (T - 1) - \sigma^\sigma \circ \gamma(L) \right) = 0, \\
T &- 1|\sigma^\sigma(L) \equiv \sigma^\sigma \circ \gamma(L) = 0, \\
T &- 1|\sigma^\sigma(L) = \sigma^\sigma \circ \gamma(L), \\
T &- 1|\sigma^\sigma(L) = \sigma^\sigma \circ \gamma(L), \\
T &- 1|\sigma^\sigma(L) = \sigma^\sigma \circ \gamma(L),
\end{align*}$$

(k, n \in \mathbb{N}, and action of $\mathcal{Q}$ follows associative and distributive property on the elements on which it applies). Metrics $\sigma(\cdot)$ and $g(\cdot)$ transformations by the chromatic inversions $\mathcal{Q} \in \{T, P\}$, differ for a graph with $\mathcal{A}^\sigma$ algebra or the minimal $\mathcal{A}^g$ algebra. On this ground, we introduce concepts of local ordinary algebras in metric frame, immersion graph emergence. The g-metric emerges from $\sigma$-metric as in Eq. (17) with $\mathcal{Q}$ applications, differently for $g \circ \sigma(L)$-metrics or general algebra. The chain inhomogeneity (i.e. $\Gamma$'s introduction, also defined for graph conformal transformations) affects both symmetries and algebras $\mathcal{A}^g$ and $\mathcal{A}^\sigma$ relations. No inhomogeneity can be introduced for a metric related to $\mathcal{A}^g$ algebra (graph). Therefore the $\mathcal{A}^g$ symmetric, relation, by $\mathcal{Q}$-applications, are equivalent i.e. they describe the same graph structure (connection $c_{ij} = -c_{ji}$, $\delta$events distribution...). This property characterizes the metric structures adapted to the colored graphs with colored $\mathcal{A}^g$ algebra (maximally decomposed $\mathcal{C}_{\sigma}$ chains). Metric $g(\cdot)$ and $\mathcal{A}^g$-graphs do not capture many of the graph structure properties, even for $\mathcal{A}^g$ maximally decomposed sequence level, which are instead considered in $\sigma$-decompositions [1]. The properties appear in the $\sigma(\cdot)$-decompositions as "degeneracies" (a set of $\sigma$-metrics correspond to a colored adapted g-metric). The inhomogeneity $\Gamma$-terms introduction destroys part of these symmetries. Similarly, however, it should be noted that the graph inhomogeneity, generated by the colors, is reflected in mixing the $x$ and (colored adapted) $y$ bases, with an $\mathcal{A}^g$ graph. This fact, together with the $\Gamma$ definitions leads, by enucleating the “ordinary algebra part” of the $\sigma(\cdot)$ or $g(\cdot)$-metrics, to define a "local" $\mathcal{A}^g$-I. Eventually $\sigma g \circ \gamma(L)$ becomes the metric with an ordinary algebra ($\mathcal{A}_L$) in the colored adapted $y$-base, implementing the concept of local algebra of a colored graph ($\sigma \circ \gamma(L) = 0$ and respectively $g \circ \gamma(L) = 0$). The inhomogeneity term becomes characteristic of the graph coloration or the graph path. We consider the ordinary $\mathcal{A}^g(L)$ algebra, decomposed in $\mathcal{A}^g(x)$ as the local algebra of the embedded graph in the embedding $x$ basis graph. Metric $\sigma \circ \gamma(L) = 0$ in the adapted base, indicates that if the graph is assumed as embedding, any other "immersed" graph has to have algebra to be compared with the local graph algebra which is actually the local minimum algebra. In [1] we explore also a particle/graph relation where to each embedding graph we can associate a particle/event with an algebra $\mathcal{A}_L = \mathcal{A}_x$ (model scales emergence). (The g-metric graph with ordinary local base with deformed $\mathcal{A}_x$ could be represented as a particle with an inertia immersed in a $g \circ \gamma(L)$ graph with ordinary algebra within therefore the particle-graph equivalence, we can introduce an $S$ (action) quantity and inertia). The $g(\cdot)$-metric decomposition reflects in the $\mathcal{Q}$ events structure, when emerging element dy represents a measure of graph homogeneity together with $\Gamma$s.

Metric structures, transformations and symmetries

We end this section with further notes on metric graph transformations, already partially addressed concluding the Sec. (2), with the introduction of the quantities $L_\sigma$ and $L_g$. Then, considering metric $\sigma(\cdot)$ and $g(\cdot)$ transformations, we can look for matrices, $\Lambda$ and $\Lambda$, in colors and inner bases respectively, considering three possible cases: (1) $\delta \sigma = \eta^{g\circ \gamma} \Lambda_\sigma^g \Lambda_\gamma^g \Lambda_\gamma^\sigma \Lambda_\sigma^g \delta \sigma$, (2) $\delta \sigma = \eta^{g\circ \gamma} \Lambda_\sigma^g \Lambda_\gamma^g \Lambda_\gamma^\sigma \delta \sigma$, or (3) $\delta \sigma = \eta^{g\circ \gamma} \Lambda_\sigma^g \Lambda_\gamma^g \Lambda_\gamma^\sigma \delta \sigma$. For $\gamma(\cdot)$-metric (where it can also be $\Lambda = \Lambda (L)$, it is clear that an analogue problem can be addressed for the induced metric in Eqs (17) and on the other hand, we explore the conditions for the three cases coincide).
ric transformations are connected with the spherical wave transformations (the conformal group includes the subgroups of Lorentz and the Poincare group and the La
guerre group, while Cayley’s group theorem ensures a con-
nection with the symmetric group). The associated sym-
metrics can be explored by considering the permutation group action (and Cayley group) [1]. We can write metric transformations from the graphs colored structure, using the notation \( \sigma^{ijk} = d_{x} \) for the color base and \( de_{ij} \) for an ordinary bases with inner indexes (\( i, j, ... \)), \( (\eta = \text{diag}(1, -1)) \), we used also notation \( \sigma \) of the base composition, thus \( \sigma \equiv \sigma^{ij}dx_{ij} \equiv \eta^{AB}G^{A}_{ij}de_{ij} \). Matrix \( \sigma^{ijk} \) arises as \( \eta_{ab} \) deformation through the \( x \)-basis events decom-
position in one only \( \text{ber} e_{i} \), with coefficients \( G_{ij} \equiv \left[ (6A_{i})'(6bj) \right] \) as “soldiering objects”, thus reducing the bases indexes into event indexes (the relation between base-
vertex indexes does not automatically translate into an “up-to-down” index correspondence, as it is not meant as a change between the adapted bases, a \text{ber} into a \text{ber}).

Thus, for the \( g \)-metric, we find \( g^{\text{m}}(x) = g_{\text{m}}^{i(m)}(i)de_{ij}de_{mn} = \eta_{AB}\eta_{CD}e_{i}^{m}(x)\eta_{ij}(x)e_{mn}dx_{mn} \equiv \sigma^{ij}dx_{ij} \equiv \eta^{AB}G^{A}_{ij}de_{ij} = -(\sigma^{2} - \sigma^{2})d_{x} \equiv -(i^{r}de_{i})^{2} \equiv dsd_{x} \equiv -(i^{r}de_{i})^{2} \), or \( ds^{2} = \eta_{ab}G_{ab}G^{b}_{ij}de_{ij} \equiv g_{\text{m}} de_{ij} \). Doubled notation in round brackets \( (i)(j) \) clarifies the origin of the double terms with inner indexes. We used the arrow-sign conven-
tion in Eq. (22), introducing the emergent base \( s \) (cor-
responding “light-cone frame”) decomposed in \text{ber}. Metric deformation reduced to graph vertexes deformations. (However, metric deformations are also written metric oper-
ator deformations, since the graph vertexes-evolutive oper-
ator (graph edges) correspondence considered in Sec. [11].

3 Concluding remarks

We considered a polychromatic multi-scales graph model with conformal graph expansion defining the graph states.

28 The correspondence between \( Q \), \text{matrix and metric} \( c(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) implies particularly that \( Q \) is a \text{ber} constraint (derived from a minimal algebra assumption \( A \)) are inherited at metric levels as metric constraints. Recalling that \( Q \) elements are events with indexes referring to chain \text{ber} events and \text{ber}, we adopt the following compact representation: \( Q^{\text{m}} = (Q_{1}, Q_{2}) \), \( d\sigma = [1 \otimes (1 \times D)]Q_{1} - [1 \otimes (1 \times D)]Q_{2} = Q^{\text{m}}(\eta)^{\text{m}}G^{\text{m}}, \eta_{\text{m}} = \text{diag}(1 \times D), -(1 \otimes (1 \times D)](C^{\text{m}} \otimes 1) \otimes (1 \otimes D)Q_{1} = (C^{\text{m}} \otimes 1) \otimes (1 \otimes D)Q_{2} = (C^{\text{m}} \otimes 1) \otimes (1 \otimes D)Q_{1} = (C^{\text{m}} \otimes 1) \otimes (1 \otimes D)Q_{2} \equiv dQ_{1}dQ_{2}, (C^{\text{m}} \otimes 1) \otimes (1 \otimes D)Q_{1} = [dQ_{1}dQ_{2}] \equiv dQ_{1}dQ_{2} \equiv dQ_{1}dQ_{2} \equiv \eta_{ab}G_{ab}G^{b}_{ij}de_{ij} \equiv g_{\text{m}} de_{ij} \). Doubled notation in round brackets \( (i)(j) \) clarifies the origin of the double terms with inner indexes. We used the arrow-sign conven-
tion in Eq. (22), introducing the emergent base \( s \) (cor-
responding “light-cone frame”) decomposed in \text{ber}. Metric deformation reduced to graph vertexes deformations. (However, metric deformations are also written metric oper-
ator deformations, since the graph vertexes-evolutive oper-
ator (graph edges) correspondence considered in Sec. [11].

Growing of events (vertices) for conformal expansion leads to the self-similar graph states notion, defining graph scales and events clusterings (structure). Different events are de-

29 The framework we consider here is mainly a relational frame, where a topology definition is a particular relevant is-

30 Metric space, as a topological space, requires a notion of separability, grounding also a differential structure. Thus, a toponological notion, in our context, is grounded on considera-
tions on symmetric and reflexivity relation given through the chromatic clustering of vertices/events in the graph, with con-
formal transformations and definition of different realizations, grouping together all the ordered connections, preserving the graph self-similarity and generation of new structure. It this scenario in fact we remain with a conformal geometry and not properly a metric geometry. A further aspect of this frame is the existence of an ordinary algebra constraining the graph vertex relations (with a somehow degenerate metric notion).
constituents to a smooth (in some way rigid) background spacetime.

This polychromatic graph model is characterized by the idea of an overlapping of metric and (dynamical) colored graph, the use of two metric levels and symmetries role, including multiple events definitions, macroevents related to loops, and the definition of multiple levels of events at every scale consequent to linking the events emergence to color exchange which implies also the elaboration of more structured levels of events where metric structures can be defined according to these. Color change in the sense of inhomogeneity is the essential process defining the events. (A color and algebra change, with respect to be, is associated to an event emergence, “if there is no change there is no clock”, conformal transformations are adapted to this situation.) Sec. (2) and particularly Sec. (2.1) explores some features of the graph in an illustrative approach, considering several approximations.

We investigated the persistence of isomorphic parts of the graph under conformal expansions, detailed in [1], where we also introduce new (existential quantifications) operator. We include here a brief outline of these aspects, which are deepened in [1], emphasizing some conceptual features of the graph and anticipating further developments of this model. “Negation” operator, related to the complementary chains, increases the graphs state possibilities and therefore the associated metric structures. The inclusion of new existential operators increases in graphs states possibility in the logic signal representations of chains.

\[ \exists, \neg \exists = Q(\cdot), \neg Q(\cdot) \]

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30 The model is based on a strong “logic” frame typically of it-from-bit approaches. (The introduction of \( I \) and \( J \) operators, in the logic signal formulations, refers to the view of gravity as a manifestation of the “existence” in the sense of [7,8].) The formalization of this general assertion (existence role) is connected to events definition (related, more precisely, to antisymmetric relation), then gravity must emerge as a phenomenon related to graph events.) Graph composite states, considering combinations of vertices clusters, change the concept of probabilities used here (eventually to explore from pure to mixed states in the quantum approach). Generally, we can consider chain is a collection of (homogenized) connections \( c_i \), and we usually main a 2-level logic signal representation with a set of (color independent) events of reference chain, as doublets \((0,1),(1,0)\).

We introduce therefore \( I \) and \( J \), and negation \( Q \)-operators on a quantity \( Q \) (in a graph diagram constructed in [1] there are doubled regions, \( I \) and \( R \), where vertices can be defined introducing a further possible definition). Introduction of \((Q, I, J)\) allows to expand and enhance the events interaction possibilities by combining different possible realizations. In general, for two events \( Q \cdot P \), there are states \((Q, I, J, P, Q)\). These are used from a transition from \( P \)-to- \( P \) loop, and from \( P \)-to-Q. (Events quadruplets and quadruplet transitions can be formed from doublets as follows: (i) \( (Q) = (Q, I, Q) = (P, I, P) \), similarly for \((P);\) (ii)Secondly there is \((Q, Q, Q) \), or non-existence \( P = (P, P, P) \). (iii) Finally, we can consider “values-couples” as \((Q, Q)\) and \( (P, P) \). Within the correspondences \[ ([I]) \equiv \exists (\cdot), \quad ([J]) \equiv \neg \exists (\cdot), \quad ([P]) \equiv \bot, \quad ([Q]) \equiv \top \], we have the operator \[ \exists \top (\cdot) / \equiv \exists (\cdot) \equiv \top (\cdot) \] and \[ \top (\exists (\cdot) ) \equiv \exists (\top (\cdot)) \]. We obtain \[ 2 \sqrt{2} (\exists \top (\cdot)) / = 1 + (\exists \top (\cdot)) \]. (We remind that, according to Eq. (1) the application on vertex of \( \exists \) correspond to \( \kappa + 1 \) vertices). The introduction of these operators

\[ \exists, \neg \exists = Q(\cdot), \neg Q(\cdot) \]

In this transcription of the graph realizations where, for example, a bichromatic graph chain is a 4-levels (from 2-levels) logic signal the introduction of a pair of (existential) operators takes into account the fact that the graph, considering its chain structure, is not complete and, therefore reduces the graph structure to a “chains overlapping” (there is an algebraic valence \((\pm 1,0)\) considering a persistence of each realization in the graph state, increasing therefore the logical levels. The graph frame opens up a serie of possibilities to be discerned concerning the role of the two metric levels, the “lower-level” \( G \)-metric or the “higher” level \( \sigma \)-metric depending also on the emergence of different symmetries, the spinnorial structure (eventually defined for the polychromatic structure) and the choices on the index composition in the explicit chains metrics approach addressed in Sec. (2). For a general metric graph there is currently no indication on a metric level prevalence choice. A further aspect to be explored is the \( D\sigma \) role in the metric approach and the coupling terms \( \Theta_{MN} \), which we visualized here in a possible supersymmetric model and more generally at the core of the spinorial structure analysis through vertexes clusterization (which is also related to the from multi-bit to multi-qubit frame transition). It should be pointed out then a qubit-related spacetime metric might be associated to the topological structure following the notion of spacetime and quantum complexity and particularly Black Hole (BH) complexity. One crucial aspect of

31 Graph entropy has been related to graph energy, in the graph–particle equivalence introduced briefly in Sec. (2.1) for the embedded graph in an embedding graph with different local bases and maximum (and minimum) entropy. Entropy is upper bounded by the area in the sense that entropy is related to geometry and, on the other hand, energy is related to geometry, there is a connection between entropy and gravity. In fact, concerning, eventually, the emergence of an Hilbert space we should consider that the maximum entropy reachable in a region of a space appears to be in fact the BH entropy which we may correspond to the Hilbert space dimensionality , i.e. the entropy \( e^S \) where \( S \) is the BH entropy, which is therefore importantly finite in every manifold region.
approaches considering collection of qubits as fundamental languages consists in the determination of the qubit (and multibit) organizations and representation\(^\text{[32]}\). The second crucial aspect would be the dimensionality of the associated Hilbert space (for a space of states of \(n\) qubits, the relative Hilbert space has dimensionality \(2^n\), number \(n\) would be associated to order the BH entropy analogously to the classical entropy for a system defined by \(2^n\) degrees of freedom). Concerning the relation with spin network, in the approach considered here we would have, for example, a \(SU(2^n)\) for a bichromatic (\(r=2\)) graph of state \(G_n\) (not related to valence). More generally addressing the colored graph and particularly the conformal transformations, it is clear then that one could think to express states as representations of \(SU(r)^m\) (or a reduced \(SU(r)^m\) with \(m < n\)). The basis states are the \(2^n\) chains of \(n\)-events and an overlapping of these. In here we could eventually interpret the graph realizations in terms of quantum bits (qubits) or more precisely \(r\)-levels \(n\)-multibits (to be multitwist) for a complete sequence of graph state \(G_n\) with \(r\) color\(^\text{[33]}\).

In this article the graph and the general ideas underlying the model are discussed, while in the future investigation using the logic signal formalism we study the dynamics of a classical system with complexity for a system of generalized Pauli matrices (thus for 1 qubit the base is the 3 Pauli for \(SU\) Bell states. In general the considered unitary group is a maximally entangled state can be written as a product of time-evolution operator evolves, and this is relevant in the shortest path between state gives the computational complexity of unitary evolution operator linearly growing (a \(m\) asymptotic graph addressing the role of higher level events by exploring the spinorial texture emerging from the graph poly-chromaticity, considering again the graph-particles formalism of Sec.\(^\text{[2.1]}\)).

In conclusion we can distinguish in this model the following three main aspects: 1. the multiple events definitions, defined as macro-events or loops (monochromatic clusterings of graph vertices) or polychromatic clustering and \(\text{events}\) (events are intrinsically related to concept of color change) conceptually identical to event-loops at any scale. 2. A second characterizing factor is the admission that gravity interests the existence, (substantialism and universalism) in the sense\(^\text{[7,8]}\), hence a geometric (relational) description of gravity. 3. The conformal expansion and the isomorphism of graph parts, defining the events scales structures, constitute a third relevant element\(^\text{[44]}\). As in other approaches, the prevalence of event/existence concept leads the distance between matter and spacetime to disappear in a graph model in the

\(^{32}\) Gravity has been variously related to a ("statistical") complexity grown where, in these approaches, complexity definition stands as one key issue. Complexity geometry, for example\(^\text{[15]}\), defines the computational complexity within differential geometrical formalism with a notion of complexity distance, which can be considered in some ways more refined then metric on quantum states provided by the inner-product distance (a large inner product meaning close states, with a supremum distance in the orthogonal state case). So called "gate complexity" corresponds to the number of primitive gates in the smallest path (quantum circuit) that leads to the two states transformation with non null tolerance \(\varepsilon\) to reach the target (while the length of the shortest path between state gives the computational complexity). One key issue however of this frame is the rapidity or, more generally, how in the computational complexity the time-evolution operator evolves, and this is relevant in BH complexity too. A BH could be modeled by \(m\) qubits with a BH complexity of unitary evolution operator linearly growing (a conjecture with time exponential in \(m\)). Therefore \(2m\) qubits maximally entangled state can be written as a product of \(2m\) Bell states. In general the considered unitary group is \(SU(2^n)\) for \(m\) qubits with a complete basis provided by \(2^{2m} - 1\) generalized Pauli matrices (thus for 1 qubit the base is the 3 Pauli matrices plus the identity), and they possibly correspond to elements of the graph. Notably, as mentioned above the quantum complexity for a system of \(m\) qubits is similar to the entropy of a classical system with \(2^n\) degrees of freedom. The systems is a set of \(m\) classical bits, \(m\) binary digits string (element of space state here a chain graph).

\(^{33}\) A physical qubit is a quantum system with dimension two, then a classical bit (i.e. two distinguishable states or a bichromatic graph with \(r = 2\) can be embedded into a qubit with corresponding classical probabilities appearing along the diagonal of the density matrix, of course off-diagonal elements in the density matrix shall remain, here rendered through combination/organization of different states through conformal transformation). The dimensionality of the Hilbert space for a system of \(n\) qubits grows as \(2^n\) or a reduced \(2^n\).

\(^{34}\) Spacetime manifold is dissolved into more fundamental elements and structures, the graph vertices (events) and their relations (polychromatic connections), which are set conceptually in our frame on the same level of vertices/events by introducing concept of \(\text{events}\). This setup addresses therefore also the issue of event definition. Different events definitions, emerging geometry, gravity and notion of "existence"\(^\text{[7,8]}\) are in the same graph frame. Minimal distance is translated in logic terms, \(G\) inertia is related to existence of local algebra. The gravity "universality", in the general relativistic transcription as a geometric spacetime, is formally understood as logic entity made of events and a property relative to existence--also the issue of event definition. Different events definitions, emerging geometry, gravity and notion of "existence"\(^\text{[7,8]}\) could be a point in a spacetime transcription. In the graph model, a loop could be seen a conformal expanded point (reported to special relativistic frame we could think to light-like distances as closest picture of graph conformal event/vertex expansion). Conformal transformation in this sense turns to be a gravity intrinsic property. This aspect is at base of the scale introduction and re-scaling after a conformal expansion, graph self-similarity and generation of new structure. Event is defined as objects interaction, gravity is then doubly related to events and their relations. Observations are interpreted as a set of (related) events, determined by the observer (base), transcribed as logic signal in a graph, where the order relation is defined by its poly-chromaticity. At certain scales we break the (classic) logic signal order, considering the combinatorial graph to construct the realizations. The clusterization is then determined by the dynamics (in this context, considering also notion of graph embedding as a sort of spacetime cell, as in \(\mu\) matter there is not distinction between matter and geometry concepts). On the other hand, entropy is naturally related to geometry and here, through graph poly-chromaticity, entropy connects more events levels. The gap in colors probability distribution regulates the inhomogeneity and the persistence loops (as in Sec.\(^\text{[1.1.1]}\) and Sec.\(^\text{[1.1.2]}\) in the chains \(C\), the maximum probability color regulates the monochromatic clusters, while at the other extreme case the equiprobable cases admit a local, in the sense of graph scale, inhomogeneity in colors. In the equiprobable graph at certain scales there are fluctuations of clusters with respect to the homogeneous minimum or maximum chains.
sense of \( \mu \) matter of QFT conceiving an embedded graphs represented with the figurative idea of spacetime cell and DNA for the graph and graph structures, (a collateral consequence of relationalism and the logic approaches of these frames, yet decoration, graph coloring, is not dissociable from such “spacetime-DNA” containing the instruction and the information–manifold substantialism) \[7.\] Graph state or graph realizations have, at any state, embedded replicas as graph \( G_n \) “seed-genetic-code” of the events structure reproducing at any state transitions. Following a similar idea then, a graph \( G_n \) contains, at inferior scale, as its substructure a disjoint collection of different \( G_n \) graph seeds, as \( G_n \) clusters. We might use the figurative image of a "geometric code" as a kind of "spacetime-DNA", to express the graph and its properties (within this picture, we might say that this investigation is more a proposal description of the “spacetime-phenotype” rather than the “spacetime genotype” of such geometric code). A “spacetime genetics” would come here from the observations on the hereditary characteristics. In this model, considering this analogy, we may deal similarly with inheritance (replicas of the original seed graph) and graph (geometric code) modifications. The possibility of describing, and eventually manipulating, such spacetime geometric code would appear to be an intriguing applicative prospect, as a “spacetime engineering” starting as theory of transmission and coding of information.

A Some notes on logic nature of spacetime notions in graphs and relativity

The polychromatic graph of events is framed as background independent, spacetime emergent model close to it-from bit ideas, where the geometrization of gravity is passed in a logicization of it. We can then retrace a logic nature for general concepts of time and space notion in the graph, which are closed here by the colors order relation, the events decomposition and composition in sum \( \Sigma \) and product \( \Pi \) in Sec. (1.1), related to shift \( D^r \) evolution \( D \) operators respectively, and correspondingly clustered in sequences \( S \) and chains \( C \) of Sec. (1.1.1). Minimal (non-zero) algebra \( \mathcal{F} \) (formalized by the introduction of logical quantifiers) grounds the graph spacetime texture (related to relativistic light-like distances, rephrasing graph conformal transformations), determines the vertex (cluster) algebra and loop radius and constrains the particle/graph inertia and the logic signal wavelength. As discussed in Sec. (1.1) a body (event-graph), associated to \( \Sigma \) and product \( \Pi \) related to a logic signal, is decomposed (decoded) in the polychromatic graph, and vice versa a set of events (bodies) are composed in one object (graph realizations as in sums, sequences \( S \)) at any scale, and decomposed in the polychromatic graphs-clusters. Realizations and states constructions are regulated by colors probabilities. In the equiprobable case, local (colors-algebras) fluctuations, in the sense of Sec. (1.1.2), could be observed. The presence of a maximum of the color probability is a further relevant case (the color associated to the maximum of probability is considered as a color loop “attractor”, governing the color permanence and heritage in states and their realizations). Thus generalized permutations and algebra homogenization, investigated in Sec. (1.1.2), are rendered as metric transformations in the approximation of Sec. (2) and are frequency and phase transformations of the associated logic signals. In \( \Pi \) we recovered the (homogeneous) Lorentz group as algebras transformations, in terms of \( \Theta^r \) with \( e = [\pm 1, \pm 2] \) for a \( \sigma \) metric (\( e = [\pm 1] \)) or \( g \)-metric (\( e = [\pm 2] \)), function of the logic signal wavelength. Thus, within the particle/graph relation the corresponding equations of motion relate metric graph bases. In this sense the correspondent relativistic frame is a theory of (minimum) information, where as in Eq. (2) and (3) (graph-state) entropy is indeed related to \( \partial L \). The equivalence in Eq. (1), \( \Pi \Sigma \) equivalence, provides insight on the notion of body and evolution. To enlighten conceptually this fact in \( \Pi \) we used an events composition/clustering operator, acting on a set \( \{a_i\} \) of events \( f : \{a_i\} \rightarrow \Sigma a_i = a \) composing in sum and in product the event \( a \) and decomposition \( \hat{g} : b \rightarrow \{b_i\} = \Sigma b_i \) with compositions \( \hat{H} = \hat{f} \hat{g} \) and inverse \( \hat{H} = \hat{g} \hat{f} \). Action of \( \hat{f} \) and \( \hat{g} \), in the logic signal formalism can be interpreted as wavelength shift (read, in the metric approach, as a "graph Doppler-effect").

In this context the meaning of the \( \Pi \Sigma \) equivalence, interesting the graph the structure in monochromatic or polychromatic loop, may reveal of a far reach significance. Evolution of body (the chain \( C \)) corresponds to the object set (the sequence \( S \)) defined by the body evolution. Properties of sequences or body chain evolution (elements of dynamic graph) are similarly related. The \( \Pi \Sigma \) equivalence refers to the chain/sequence relations, thus \( D \equiv D^r \) action that can be followed along the column and rows of \( Q_{ij} \) of Eq. (1.1.2)\[12\] \( D \) We can think, two bodies/graphs \( G_i = \{a^i_j\}_n \) and \( G_j = \{b^j_k\}_m \) are composed into in \( G_{ij} = G_i \cup G_j = \{a^i_j\}_n \cup \{b^j_k\}_m \), and decomposing an associated logic signal. We consider the evolution of a body in the sense of \( D \) application \( G \equiv \sum_{k=0}^\infty D^k G_0 = \sum_{k=0}^\infty D^k \hat{G}_0 \), applying Eq. (1) (note the existence of operators decomposition mirroring the events decompositions as discussed in Sec. (1.1)). The evolution leads to graph states with graphs (sphere) radius regulated by \( \partial L \) which regulates and establishes the irreversibility of the process (in the sense of chains symmetries), the conformal transformations, the persistence of structure (inheritance through conformal transformation-self-similarity), the generation of new structure (through conformal expansion, in this

\[35\] We could define \( g_r \equiv d^2 f - d^2 f \equiv df = \sigma \) referring to Sec. (2.3), where however we considered a set of functions associated to the graph, modified for the inhomogeneity terms (\( f, df \)) and decoupling also the embedding and embedded graphs. (Colors symmetries, and particularly color reversibility considered in the metric definition for zero level graph, do not appear to be a symmetry for higher order events). In this example we introduced a function \( f(\mu, r) \), where \( r \) colors space index, \( \mu \) a bar index, according to the convention used in Sec. (2), we can write \( \partial_\mu f = \pm \partial f \), quantities \( \sigma_r = \partial_\mu f = \partial_\mu f \) refers to the first level graph \( \sigma_r(\cdot, \cdot) \), in general a no-zero quantity, \( (\partial_\mu, \partial_\mu \) associated to evolution-shift.) sign \( \pm \) refers to color symmetries.
sense primordial idea of space-time generation). There is $\mathcal{D}^n S_0 \subset \mathcal{D}^{n-1} S_0$ at any scale $(n)$ discussed in Sec. 1.1.1[1], where we also discussed the (chain-structure) irreversibility regulated by $\varphi$. These properties are represented in the matrix in Eq. (11) and Figs (3).

The monochromatic loop of order two is totally reversible in the context of the persistence of states in the ordered chain. Graph structure is regulated by entropy in Eq. (2), energy and inertia as $[1]$, the larger the inhomogeneity colors-(constrained by existence of $C_{\text{om}}$) the greater the inertia and the graph characteristic irreversibility. The decomposition constituents $s_j$ graph of $G_{ij0}$ depend on the graph state and mostly on the probability distribution coming as a "color-attractor" in the union-graph realization. In the graphs union concept, metric transformations read in the particle/graph equivalence reveals useful to discern, as (equivalently relativistic longitudinal and transverse) Doppler effect, the change in frequency (and wavelength) of the associate logic signal (3) Considering Eq. (1) an object $S$ is decomposed in the evolved copies of $S$, The copies are not connected except in the sense of persistence (in the sense we can define $S_2 = D S_1 \in S_1$) and immersion, in the analogy therefore the motion of the observer corresponds to a change of structure of the body, seen as conformal transformation through the equivalent graph Doppler effect (relevant also for the higher order symmetries is that acceleration ordinary in the resting system is an invariant even in this model).

Fig. 3. Schemes for the representation of the states persistence in conformal expanded graph states, each cylinder is a different part (vertex) of a graph. The event matrix $Q_{ij}$ is represented schematically in the left panel with the role of evolution operators representing the cylinders. The different configurations represent different chains configurations and therefore different symmetries, the irreversibility and persistence of parts of the graph is evidenced.

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