The Physics of crypto-nonlocality

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In 2003, Leggett introduced his model of crypto-nonlocality based on considerations on the reality of the photon polarization. In this note, we prove that, contrary to hints in subsequent literature, crypto-nonlocality does not follow naturally from the postulate that polarization is a realistic variable. More explicitly, consider physical theories where: a) faster-than-light communication is impossible; b) all physical photon states have a definite polarization; and c) given two separate photons, if we measure one of them and post-select on the result, the other system is still a photon. We show that the outcomes of any two-photon polarization experiment in these theories must follow the statistics generated by measuring a separable two-qubit quantum state. Consequently, in such experiments any instance of entanglement detection -and not necessarily a Leggett inequality violation- can be regarded as a refutation of this class of theories.

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In 1935, Einstein, Podolski and Rosen (EPR) argued that quantum mechanics could not be considered a complete theory, as it did not assign definite values to the different observable quantities [1]. This property was called reality, and EPR believed that some future physical theory would combine this feature with the astounding predictive powers of quantum mechanics. Bell later reformulated the EPR argument in a device-independent way: by considering experiments conducted by two separate parties, he showed that the predictions of quantum mechanics were incompatible with any realistic theory where distant observables could not influence each other at superluminal speeds. Very recently, Bell’s argument has been extended to show that realistic theories with arbitrarily high (but finite) influence propagation speed can not reproduce the predictions of quantum mechanics [3].

Inspired by Bell’s work, Leggett proposed in 2003 a family of theories where the correlations generated in two-photon polarization measurements admit a very particular decomposition, that he termed crypto-nonlocal model [4]. As it turns out, certain correlations admitting a crypto-nonlocal model allow violations of local realism [5], and thus they were not ruled out by previous experiments on non-locality. Such experiments were conducted in [6–8].

Subsequent works on crypto-nonlocality have motivated/described Leggett’s model by invoking the image of a two photon polarization experiment as a separable two-photon quantum state. Consequently, the general perception is that crypto-nonlocal theories satisfy the intuitive postulate that physical photon states have a definite polarization. We will call this axiom the realistic polarization principle.

In this paper we show that the realistic polarization principle actually enforces constraints stronger than those captured by Leggett’s crypto-nonlocal model. As a result, we find that the polarization measurement statistics of a multi-photon experiment in any no-signalling physical theory compatible with the realistic polarization principle must necessarily coincide with the correlations observed when measuring a fully separable state in quantum mechanics. Hence, all such theories are local realistic, and any quantum experiment verifying entanglement between the polarization degrees of freedom of N > 1 photons can be considered a refutation of the realistic polarization principle.

First, we will introduce the concept of crypto-nonlocality, as formulated by Leggett [4]. We will then prove our result for the case of two photons and ideal measurements and discuss how to extend our arguments to more than two photons and non-ideal polarizers. Finally, we will present our conclusions.

Consider an experimental scenario where a source distributes pairs of photons to two parties, call them Alice and Bob. Alice (Bob) conducts a measurement of the polarization of her (his) photon in the direction \( \hat{x} \) and outputs the value \( a = 0 \) (\( b = 0 \)) if the photon is detected or \( a = 1 \) (\( b = 1 \)), otherwise. Following Leggett [4], we say that Alice and Bob’s statistics admits a crypto-nonlocal model iff the probabilities \( P(a, b|\hat{x}, \hat{y}) \) satisfy:

\[
P(a, b|\hat{x}, \hat{y}) = \sum_{\hat{u}, \hat{v}} P(\hat{u}, \hat{v}) P(a, b|\hat{x}, \hat{y}, \hat{u}, \hat{v}),
\]

where \( P(\hat{u}, \hat{v}) \) is an arbitrary distribution of unitary vectors \( \hat{u}, \hat{v} \in \mathbb{C}^2 \), and

\[
P_A(a|\hat{x}, \hat{u}) = \sum_{\hat{v}} P(a, b|\hat{x}, \hat{y}, \hat{u}, \hat{v}) = \text{tr}\{\Pi_{a}^{x}|\hat{u}\rangle \langle \hat{u}|\},
\]

\[
P_B(b|\hat{y}, \hat{v}) = \sum_{\hat{u}} P(a, b|\hat{x}, \hat{y}, \hat{u}, \hat{v}) = \text{tr}\{\Pi_{b}^{y}|\hat{v}\rangle \langle \hat{v}|\},
\]

with \( \Pi_{a}^{x} = c_{2a} + (-1)^{a} \frac{\hat{x} \cdot \hat{u}}{\langle \hat{x} \rangle} \). Notice that \( P_A(0|\hat{x}) = |\hat{x} \cdot \hat{u}|^2 \), \( P_B(0|\hat{y}) = |\hat{y} \cdot \hat{v}|^2 \), i.e., locally, the subensembles satisfy Malus’ law.

Remarkably, there exist non-local distributions compatible with these equations, see [6–8] for a clarification of the relations between Leggett’s crypto-nonlocality, Bell’s
nonlocality and quantum separability. Optimizing Bell-type functionals over all distributions admitting a crypto-
nonlocal model is seemingly a very complicated mathemat-
ical problem [7].

Even though Leggett’s model is inspired from a num-
ber of physical considerations, like the reality of pho-
ton polarization, it has not been shown to be implied
by them. In [6], it is nevertheless claimed that crypto-
nonlocal theories “are based on the following assump-
tions: (1) all measurement outcomes are determined by
pre-existing properties of particles independent of the
measurement (realism); (2) physical states are statisti-
cal mixtures of subensembles with definite polarization,
where (3) polarization is defined such that expectation
values taken for each subensemble obey Malus’ law”. In
the same line, it has been stated that “the basic assump-
tion of Leggett’s model is that locally everything happens
as if each single quantum system would always be in a
pure state” [8], or “roughly speaking, it [the concept of
crypto-nonlocality] says that all individual subsystems
of a composite system should locally behave as if they were
in a pure quantum state, with well-defined properties” [9].

These statements seem to appeal to the intuition that
physical photon states ‘should’ have a well-defined po-
larization (or pure quantum state). In the following, we
will show that photon polarization experiments in any
reasonable physical theory where this realistic polariza-
tion principle holds cannot exhibit correlations beyond
those obtained by measuring a separable quantum state.

The class of physical theories that we will be consider-
ing satisfies the following axioms:

(a) Faster-than-light communication is impossible (no-
signalling condition).

(b) Physical photon states have a definite polarization
(realistic polarization principle).

(c) Given two separate photons, if we measure one of
them and post-select on the result, the other system
is still a photon.

Axiom (a) is a consequence of relativistic causality [3],
and axiom (b) is a strong interpretation of axiom (2) in
[4], namely: “physical states are statistical mixtures of
subensembles with definite polarization”, see the discus-
sion below. Axiom (c) is a just re-statement of the in-
tuition that post-selection is a type of preparation. To our
best knowledge, this principle is satisfied in any physical
model proposed so far, and it is implicit in the formal-
ism of generalized probabilistic theories [10–12]. Further-
more, in our view, the concept of ‘two photons’ encom-
passes this principle. We include it in the list of axioms
in case that our notion of ‘two photons’ is not shared by
the reader.

Let us now investigate what kind of correlations should
Alice and Bob expect to observe in bipartite photon po-
larization experiments should assumptions (a), (b), (c)
hold. W.l.o.g., suppose that Alice measures first, i.e., at
time \( t \) \((t + \epsilon)\), Alice (Bob) carries out an action, consist-
ing in either measuring her (his) system or not. Then
axiom (a) implies that Alice and Bob’s correlations must
be of the form (1), with the difference that, in prin-
ciple, Alice’s measurement can potentially influence Bob’s
physical state at a distance. Correspondingly,

\[
P(a, b|\vec{x}, \vec{y}, \vec{u}, \vec{v}) = \text{tr}\{\Pi^x_a|\vec{y}\rangle \langle \vec{u}|\}
\]

if Alice conducts measurement \( x \) at time \( t \), and

\[
P(\emptyset, b|\emptyset, \vec{y}, \vec{u}, \vec{v}) = \text{tr}\{\Pi^y_b|\vec{v}\rangle \langle \vec{v}|\}
\]

if Alice chooses not to measure her system at time \( t \). Due
to axiom (a), though, Bob’s marginal statistics cannot
depend on whether Alice measured her system or not
(otherwise, Alice could signal Bob superluminally), and
so we arrive at Leggett’s model.

Note that, if we interpret axiom (2) in [6] in the
sense that “physical states are particular mixtures of
subensembles with definite polarization”, rather than
arbitrary mixtures – and replace axiom (b) accordingly–
then it is not clear at all why the subensembles
\( P(a, b|\vec{x}, \vec{y}, \vec{u}, \vec{v}) \) in eq. (1) –which would in general not
represent physical states– should satisfy the no-signalling
condition. In principle, the no-signalling property of the
physical state with distribution \( P(a, b|\vec{x}, \vec{y}) \) could be recov-
ered after averaging over \( P(\vec{u}, \vec{v}) \), in which case it can be
seen that the resulting model would reproduce the
correlations generated by measurements of any quantum
state, separable or not [12].

However, we are not finished yet. Despite its apparent
innocuity, axiom (b) imposes extra constraints over eqs.
(1), (2). Indeed, imagine that Alice and Bob share the
physical state with statistics \( P(a, b|\vec{x}, \vec{y}, \vec{u}, \vec{v}) \), and Alice
measures the polarization of her photon in the direction
\( \vec{x} \in \mathbb{C}^2 \), obtaining the outcome \( a \). By consecutive appli-
cation of axioms (a) and (b), we have that Bob’s result-
ning photon state is described by a convex combination
of photon states with definite polarization. That is, there
exist probability distributions \( \rho_a^{\vec{x}, \vec{u}, \vec{v}}(\vec{w}) \) over the unitary
vectors \( \vec{w} \in \mathbb{C}^2 \) such that

\[
P(b|\vec{x}, a, \vec{u}, \vec{v}) = \sum_{\vec{w}} \rho_a^{\vec{x}, \vec{u}, \vec{v}}(\vec{w})\text{tr}\{\vec{w}|\vec{w}\rangle \Pi^y_b\}
\]

This implies that

\[
\text{tr}\{\vec{v}\rangle \langle \vec{v}|\Pi^y_b\} = P(b|\vec{x}, \vec{y}, \vec{u}, \vec{v}) =
\]

\[
= \sum_a P(a|\vec{x}, \vec{u}, \vec{v})P(b|\vec{x}, a, \vec{y}, \vec{u}, \vec{v}) =
\]

\[
= \text{tr}\left(\sum_a P(a|\vec{x}, \vec{u}, \vec{v})\rho_a^{\vec{u}, \vec{v}}\right)\Pi^y_b,
\]

with \( \rho_a^{\vec{u}, \vec{v}}(\vec{w}) \equiv \sum_{\vec{w}} \rho_a^{\vec{x}, \vec{u}, \vec{v}}(\vec{w})|\vec{w}\rangle \langle \vec{w}|.\)
Since this relation holds for all unitary vectors $\hat{y}$, we have that
\[
|\hat{v}\rangle = \sum_{a} P(a|x, \hat{u}, \hat{v}) \rho_a^x(\hat{u}, \hat{v}). \tag{7}
\]

Now, $|\hat{v}\rangle |\hat{v}\rangle$ is a pure state, so the above equation can only hold if $\rho_a^x(\hat{u}, \hat{v}) = |\hat{v}\rangle |\hat{v}\rangle$ for all $x, a$. We thus conclude that
\[
P(a, b|x, \hat{u}, \hat{v}) = P(a|x, \hat{u}, \hat{v}) P(b|x, \hat{u}, \hat{v}) = \text{tr}(|\hat{u}\rangle \langle \hat{u}| \Pi_a^x) \text{tr}(|\hat{v}\rangle \langle \hat{v}| \Pi_b^x). \tag{8}
\]
The statistics $\{P(a, b|x, \hat{y})\}$ hence arise from measurements of the separable state
\[
\sum_{\hat{u}, \hat{v}} P(\hat{u}, \hat{v}) |\hat{u}\rangle |\hat{v}\rangle. \tag{9}
\]

Conversely, it is easy to see that physical theories where only separable quantum polarization states can be prepared are compatible with axioms (a), (b), (c). Such axioms are then equivalent to separable quantum mechanics, and so any two-qubit entanglement witness can be used to refute such a family of physical theories.

This has to be contrasted with the results of [5], where it is shown that the correlations generated by certain two-qubit entangled states admit a crypto-nonlocal model. From our derivation it thus follows that such instances, while complying with Leggett’s definition of crypto-nonlocality, cannot be present in any reasonable physical theory where the realistic polarization principle holds.

If we replace axiom (c) by its obvious multiphoton extension (c’), our arguments can be generalized to more than two parties to conclude that only fully separable $N$-qubit states can be prepared in physical theories complying with axioms (a), (b), (c’). Indeed, suppose that $N$ parties share a multiphoton state with well-defined individual polarizations, and imagine that the first $N-1$ parties measure first, thus projecting the state of the $N$th party into a mixture of photons with definite polarizations. Again, (7) holds, and therefore, the last party is uncorrelated from the rest. The general result hence follows by induction.

Finally, accounting for imperfect measurements is easy: following Leggett’s original paper [3], the response of non-ideal polarizers can be modeled by the measurement operator $\Pi_a^x \equiv \epsilon^{(2)} + (1-\epsilon) \Pi_u^x$, with $0 \leq \epsilon' < 1$. Such operators are tomographically complete, and so, as in the ideal case, it is legitimate to infer eq. (7) from eq. (8). The rest of the proof is identical to the ideal case.

Conclusion

We have shown that a reasonable interpretation of the intuition that “polarization is well-defined” leads, not to Leggett’s definition of crypto-nonlocality, but to the strictly stronger notion quantum separability. In view of our results, future research on crypto-nonlocality will have to be identified with entanglement theory or else be concerned with the mathematical study of a correlation model whose physical significance is yet to be established.

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[13] Indeed, consider any two-qubit state (or even entangled entanglement witness) $\rho_{AB}$, with $\rho_A = \sum_{\hat{a}} P_{\hat{A}}(\hat{u}|\hat{v})$, $\rho_B = \sum_{\hat{b}} P_{\hat{B}}(\hat{v}|\hat{v})$, and define $\rho_{B|\hat{f},a} = \frac{\text{tr}(\rho_{AB}|\Pi_f^x)}{\text{tr}(\rho_{AB}|\Pi_f^x)}$. Then, one can decompose $P(a,b|\hat{x},\hat{y}) = \text{tr}(\Pi_f^x \otimes \Pi_f^y) \rho_{AB}$ as $P(a,b|\hat{x},\hat{y}) = \sum_{\hat{a}} P_{\hat{A}}(\hat{u}|\hat{v}) P_B(\hat{v}|\hat{v}) P(a,b|\hat{x},\hat{y},\hat{u},\hat{v})$, with $P(a,b|\hat{x},\hat{y},\hat{u},\hat{v}) = \text{tr}(\hat{a}|\hat{u} \Pi_f^x \rho_{AB} |\Pi_f^y)$, with $\rho_{B|\hat{f},a}$ can be interpreted as a statistical mixture of ensembles with definite polarization. The statistics resulting from measurements of $\rho_{AB}$ are hence compatible with (a), (c) and the weak interpretation of axiom (2) of [3].