Sequential stability of weak martingale solutions to stochastic compressible Navier-Stokes equations with viscosity vanishing on vacuum

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Abstract

In this paper, we investigate the compressible Navier-Stokes equations with degenerate, density-dependent, viscosity coefficient driven by multiplicative stochastic noise. We consider three-dimensional periodic domain and prove that the family of weak martingale solutions is sequentially compact.

1 Introduction

In this article we investigate the system of equations describing the flow of a compressible fluid in the 3-dimensional domain with the periodic boundary condition. The dynamics of such fluid is characterised by the total mass density \( \bar{\rho} = \rho(t, x) \) and the velocity vector field \( \mathbf{u} = u(t, x) \). The following equations express the physical laws of conservation of mass and momentum, respectively

\[
\begin{align*}
\partial_t \bar{\rho} + \text{div}(\bar{\rho} \mathbf{u}) &= 0, \\
\text{d}(\bar{\rho} \mathbf{u}) + [\text{div}(\bar{\rho} \otimes \mathbf{u}) - \text{div}(\bar{\rho} \nabla \mathbf{u}) + \nabla p] \ dt &= \bar{\rho} f \text{d}W \quad \text{in} \quad (0, T) \times \mathcal{O},
\end{align*}
\]

(1.1)

Here by \( \mathcal{O} = \mathbb{T}^3 \) we denote the 3-dimensional torus, by \( \bar{\rho} f \text{d}W \) special multiplicative noise – a stochastic external force, and by \( p \) the pressure depending on the density \( \bar{\rho} \) via the following formula:

\[ p = p(\bar{\rho}) = \bar{\rho}^\gamma, \quad \gamma \in (1, 3). \]

(1.2)

The aim of this paper is to prove the stability of weak martingale solutions to system (1.1). We refer to this system as the stochastic compressible \textit{degenerate} Navier-Stokes equations, in
contrast to the classical stochastic compressible Navier-Stokes equations in which the stress tensor is independent of the density. The first result concerning the existence of global in time solutions to the stochastically perturbed compressible, viscous multidimensional system, was actually given for the degenerate case, see Tornatore [45]. The author of that paper considered a specific, rather un-physical, choice of the stress tensor, studied earlier in the deterministic setting by Vaigant and Kazhikhov [46]. More than decade later, Feireisl, Maslowski and Novotný in [29] investigated the stochastic classical 3-d system. Since then the study of stochastically driven compressible Navier-Stokes equations took off for even more general multiplicative noise $\Phi(\rho, \rho u) \, dW$. This research was initiated by Breit and Hofmanova in [8], and continued in collaboration with Feireisl in a series of papers [4, 5, 6]. We refer to their book [3], for a complete account of the mathematical literature on that system, including some singular limits results, and to paper [7] devoted to the existence of stationary solutions. The underlying theory for all of these contributions is the one developed for the classical deterministic system. Here the literature is much broader and so we refer only to the first major contributions, i.e. the pioneering work of by Lions [34] for compressible barotropic Navier-Stokes with $\gamma > \frac{9}{5}$ and the proof of the existence of solutions covering more physical cases $\gamma \geq \frac{5}{3}$ by Feireisl [27]. The overview of these methods can be found in the monograph [39] by Novotný and Stráskraba.

Let us also mention that there is an abundant amount of literature corresponding to the stochastically driven incompressible fluids. An interested reader is referred to the original paper [30] by Flandoli and Gątarek, as well as three more recent publications [2, 17, 20] and the references therein). The main idea of the above cited papers is to find some suitable a priori estimates for solutions to appropriate approximating problems, then to use appropriate compactness results, more classical in [30] and less classical, based on the weak topologies, in [17, 20], to pass to the limit and finally to identify the limit as a solution to the investigated problem. The paper [20] relies on a novel approach to proving the passage to the limit based on the Skorokhod-Jakubowski Theorem [33] which is a generalization to a large class of non-metric spaces of the classical Skorokhod Theorem from [44].

For the compressible, degenerate Navier-Stokes system the literature is much more sparse, and for the stochastic version, apart from the mentioned result of Tornatore [45], none. For the deterministic version of system (1.1), i.e. when $dW = 1$, the sequential stability of weak solutions was proven by Mellet and Vasseur [36]. The complete existence result was achieved nearly ten years later by Vasseur and Yu [47, 48], after couple of other attempts including approximation by cold pressure, drag terms or quantum force [38, 12]. The main difficulty concerning systems with the viscosity coefficients vanishing when density equals 0, is that the velocity vector field is no longer defined. This degeneracy leads to some further problems in the analysis, when compared to the constant-viscosity case of Lions [34] and Feireisl [27]. However, for certain forms of density-dependent viscosities, this degeneracy proves to be beneficial. Namely, it provides a particular mathematical structure, called the BD-entropy inequality, that yields a global in time integrability of $\nabla \varphi(q)$ for an appropriate increasing function $\varphi$. This mathematical structure was introduced for the first time by Bresch, Desjardins & Lin [9] for the Korteweg equations. In the following work [10] by Bresch and Desjardins, the same concept was applied to 2-dimensional viscous shallow water model, without capillarity but with additional drag force. The contribution of Mellet and Vasseur [36] goes one step further and solves a problem of passing to the limit in the convective term without any further regularisations. These authors combined the BD-entropy method with an additional energy-type estimate $\bar{q}|u|^2$ in the space $L^\infty(0, T; L \log L(O))$. 

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The present paper is concerned with the sequential stability theory for the stochastic compressible Navier-Stokes Equations with density-dependent stress tensor $S(\rho, \nabla u) = \rho \nabla u$. To the best of our knowledge, this work is the first attempt to extend the deterministic results to the stochastic setting. Our proof of the sequential stability of the weak martingale solutions provides an additional essential insight into the compactness argument. The proof of the existence of those solutions, presumably lengthy and technical, will be considered in the future.

In simple terms, our main result says that given a sequence of martingale solutions satisfying given uniform bounds, there exists a weak limit, possibly defined on a new probability space, which is also a martingale solution. The set of uniform bounds is a stochastic generalization of the deterministic ones from [36], with the exception of the estimate of $\rho |u|^2 \log(1 + |u|^2)$. To make use of Mellet and Vasseur’s idea we had to replace the symmetric part of the velocity gradient in the stress tensor by the full gradient. Let us now shortly discuss the main difficulties and the content of the paper.

The principle difficulty in proving convergence of solutions to problem (1.1) stems from only non-negativity of the density process $\rho$. Had this process been bounded from below by some constant $\varepsilon > 0$ one would be able to use the second equation in (1.1) as the equation for the unknown $u$. Because $\rho$ can be equal to 0 on a set of positive space-time Lebesgue measure, and because the velocity field $u$ is always preceded by the density function $\rho$, passage to the limit in nonlinear terms involving $u$ becomes particularly difficult. The corresponding deterministic problem has been investigated in a number of papers, starting from [35] by Mellet and Vasseur.

The present article will also use the approach from the deterministic variant of the problem to some extent. To be precise we first use the following classical functions associated to a solution $(\rho, u)$.

$$\int_{\mathcal{O}} \left( \frac{1}{2} \rho(t, x) |u(t, x)|^2 + \frac{1}{\gamma - 1} \rho^\gamma(t, x) \right) \, dx \quad \text{the energy}$$

$$\int_{\mathcal{O}} \left( \frac{1}{2} \rho(t, x) |u(t, x)| + \nabla \log \rho(t, x) |^2 + \frac{1}{\gamma - 1} \rho^\gamma(t, x) \right) \, dx \quad \text{the B-D enstrophy}$$

$$\int_{\mathcal{O}} \frac{1}{2 + \delta} \rho(t, x) |u(t, x)|^{2 + \delta} \, dx \quad \text{the M-V energy}$$

for some $\delta \in (0, 1)$, as a source of the a’priori estimates on the solutions. For the convenience of the reader, we present all details of these a-priori estimates in the stochastic setting, assuming that the equations have regular enough solutions to justify all the passages, see Lemmas 3.2, 3.4, 3.5 and and 3.6. However, in the next steps we cannot use these functions directly and simply apply the stochastic counterparts of the classical deterministic compactness theorems. Instead we introduce the following auxiliary functions

$$\vartheta_n := \sqrt{\rho_n}, \quad m_n := \rho_n u_n, \quad q_n := \sqrt{\rho_n} u_n, \quad r_n := \rho_n u_n^{-1}$$

and

$$\mu_n = \left( \vartheta_n, m_n, q_n, r_n, W \right),$$

and using the just estimates we prove that the laws of the processes $\mu_n$ are tight on some appropriately chosen functional space $\mathcal{X}_T$ defined in (4.33), see Lemmata 4.1, 4.2, 4.3, 4.6 and Corollary 4.8. Let us point out here that, contrary to the deterministic case, the tightness
deduced from these a priori estimates is insufficient to conclude the proof of our result. Fortunately we are able to employ the Jakukowski-Skorokhod Theorem and deduce that there exists a subsequence \((\mu_{k_n})_{n=1}^\infty\), a new probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\), a sequence \((\tilde{\mu}_n)_{n=1}^\infty\) of \(\mathcal{X}_T\)-valued random variables, as well as a \(\mathcal{X}_T\)-valued random variable \(\tilde{\mu} = (\tilde{\varrho}, \tilde{\partial}, \tilde{\mathbf{m}}, \tilde{\varphi}, \tilde{\mathbf{r}}, \tilde{\mathbf{W}})\), all defined on the new probability space, and such that (i) \(\mathcal{L}(\tilde{\mu}_n) = \mathcal{L}(\mu_{k_n})\), \(n \in \mathbb{N}\) and (ii) \(\tilde{\mu}_n\) converges \(\tilde{P}\)-almost surely in \(\mathcal{X}_T\) to \(\tilde{\mu}\). However, in the present setting, the limiting random variable \(\tilde{\mathbf{m}}\) does not define a candidate for the velocity field \(\tilde{\mathbf{u}}\). In fact, even the approximate random variables \(\tilde{\mu}_n\) do not keep track of their original structure. We solve this problem by proving that \(\tilde{P}\)-a.s.

\[
(\tilde{\varrho}_n, \tilde{\mathbf{m}}_n, \tilde{\varphi}_n) = (\sqrt{\tilde{\varrho}_n, \tilde{\mathbf{m}}_n}, \tilde{\varphi}_n, \tilde{\mathbf{r}}_n),
\]

\[
(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{\varphi}) = (\sqrt{\tilde{\varrho}, \tilde{\mathbf{m}}}, \tilde{\varphi}, \tilde{\mathbf{r}}).
\]

The proof of these two identities requires measurability of the corresponding nonlinear maps. These two identities allow us to define vector-valued random variables \(\tilde{\varrho}_n\) and \(\tilde{\mathbf{u}}_n\) on our new probability space such that \(\tilde{P}\)-a.s.

Let us conclude the short introduction with an overview of the paper. The rigorous formulation of the assumptions and the main result are presented below, in Section 2. Then, in Section 3 we recall deterministic a-priori estimates and prove their stochastic equivalents for smooth enough solutions. The actual proof of our the main result is the content of Section 4. In Section 5 we show that the limiting process is a martingale solution to Problem (1.1). Finally in the Appendices we recall some known facts about the measurability of functions, simple consequences of the Hölder inequality, and a handy collection of facts from stochastic analysis.

2 The main result

We consider system (1.1) with the initial conditions

\[ \varrho(0, x) = \varrho^0(x), \quad \varrho \mathbf{u}(0, x) = \mathbf{m}^0(x), \]  

satisfying the following assumption.

Assumption 2.1. Assume that \(\varrho^0, \mathbf{m}^0\) are measurable functions from \(\mathcal{O}\) taking values in \(\mathbb{R}\) or \(\mathbb{R}^3\) such that \(\varrho_0 \geq 0\) and \(\mathbf{m}^0 = \mathbf{0}\) on the set \(\{\varrho^0 = 0\} = \{x \in \mathcal{O} : \varrho^0(x) = 0\}\).

In addition, denoting \(\mathbf{u}^0 = \frac{\mathbf{m}^0}{\varrho^0} \mathbf{1}_{\{\varrho^0 > 0\}}\) we assume that

\[
\mathbb{E} \left( \int_{\mathcal{O}} \varrho^0 \mathbf{u}^0 \, d\mathbf{r} \right)^p + \mathbb{E} \left( \int_{\mathcal{O}} \varrho^0 |\nabla \log \varrho^0|^2 \, d\mathbf{r} \right)^p + \mathbb{E} \left( \int_{\mathcal{O}} (\varrho^0)^\gamma \, d\mathbf{r} \right)^p \leq C \tag{2.2}
\]

\[
\mathbb{E} \left( \frac{1}{2+\delta} \int_{\mathcal{O}} \varrho^0 |\mathbf{u}^0|^{2+\delta} \, d\mathbf{r} \right) \leq C \tag{2.3}
\]

for \(\delta \in (0, 1), p \geq 1\) and some positive constant \(C\).

Concerning the external force, we assume what follows.
Assumption 2.2. For every $p \geq 1$, $f \in L^p(\Omega; L^3(\mathcal{O}))$ is $\mathcal{F}_0$-measurable.

Throughout the paper we will use the following definition of the solution to (1.1).

Definition 2.3 (Martingale solution). Let $T > 0$ be arbitrary, let $g^0$, $m^0$ satisfy Assumption (2.1) and $f$ satisfy Assumption (2.2). We say that the system $(U, W, \varrho, u)$ is a martingale solution to the problem (1.1), with the initial data $g^0$ and $m^0$ as described in (2.1), if and only if $U := (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is a stochastic basis with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$, $W$ is a $\mathbb{R}$-valued Wiener process on $U$, and $(\varrho, u)$ are processes such that

(i) $\varrho$ is a continuous $\mathbb{F}$-progressively measurable $L^\gamma(\mathcal{O})$-valued process,

(ii) there exists a weakly continuous $\mathbb{F}$-progressively measurable $L^{3/2}(\mathcal{O})$-valued process $m$, such that $m = \varrho u$ a.e. in $\mathcal{O} \times (0,T)$ $\mathbb{P}$-a.s.,

(iii) for all $\phi \in W^{1,\infty}(\mathcal{O})$, $\psi \in W^{2,\infty}(\mathcal{O})$, $t \in [0,T]$, $\mathbb{P}$-a.s.

\[
\begin{align*}
\int_{\mathcal{O}} g(t) \phi \, dx &= \int_{\mathcal{O}} g^0 \phi \, dx + \int_0^t \int_{\mathcal{O}} g \phi \cdot \nabla \phi \, dx \, ds, \\
\int_{\mathcal{O}} m(t) \cdot \psi \, dx &= \int_{\mathcal{O}} m^0 \cdot \psi \, dx + \int_0^t \int_{\mathcal{O}} g \otimes u \cdot \nabla \psi \, dx \, ds \\
&\quad + \int_0^t \int_{\mathcal{O}} g \varrho \cdot \Delta \psi \, dx \, ds + \int_0^t \int_{\mathcal{O}} \nabla \otimes u : \nabla \psi \, dx \, ds \\
&\quad + \int_0^t \int_{\mathcal{O}} \varrho \nabla \varrho \cdot \nabla \psi \, dx \, ds + \int_0^t \int_{\mathcal{O}} \varrho f \cdot \psi \, dW. \tag{2.5}
\end{align*}
\]

Then, the aim of this work is to prove the sequential stability of the martingale solutions of (1.1). More precisely, we want to show that a sequence $(U_n, W_n, \varrho_n, u_n)_{n \in \mathbb{N}}$ of martingale solutions to (1.1), satisfying Definition 2.3 and some uniform bounds specified below, with initial data $\varrho_n(0) = g^0_n$ and $m_n(0) = m^0_n$, converges to a martingale solution $(\tilde{U}, \tilde{W}, \tilde{\varrho}, \tilde{u})$ to (1.1) in an appropriate sense.

We will also be using the following assumption on the boundedness of solution in the energy class to establish our main theorem.

Assumption 2.4. There exists a positive constant $C$ such that for any $\delta \in (0,1)$ and a corresponding $c_\delta > 0$, and for any $p \geq 1$ the following bounds are satisfied uniformly in $n \in \mathbb{N}$:

\[
\begin{align*}
\mathbb{E} \left( \sup_{t \in [0,T]} \int_{\mathcal{O}} \varrho_n |u_n|^2 \, dx \right)^p &\leq C, \\
\mathbb{E} \left( \int_0^T \int_{\mathcal{O}} |\nabla u_n|^2 \, dx \, dt \right)^p &\leq C, \\
\mathbb{E} \left( \int_0^T \int_{\mathcal{O}} \nabla \cdot u_n^0 |u_n|^2 \, dx \, dt \right)^p &\leq C, \\
\mathbb{E} \left( \frac{1}{2 + \delta} \sup_{t \in [0,T]} \int_{\mathcal{O}} \varrho_n |u_n|^{2+\delta} \, dx + c_\delta \int_0^T \int_{\mathcal{O}} \varrho_n |\nabla u_n|^2 \, dx \, dt \right) &\leq C. \tag{2.8}
\end{align*}
\]

Theorem 2.5. Let $(U_n, W_n, \varrho_n, u_n)_{n \in \mathbb{N}}$ be a sequence of martingale solutions to problem (1.1) with the initial data $g^0_n$ and $m^0_n$ satisfying Assumption 2.1 uniformly w.r.t. $n$. Let $f$
satisfy Assumption 2.2 and let the uniform estimates from Assumption 2.4 be satisfied. Furthermore, we assume that there exist random variables \( \varrho^0 \in L^7(O) \) and \( \mathbf{m}^0 \in L^{3/2}(O) \) such that
\[
\varrho^0 \geq 0, \quad \varrho^0_n \to \varrho^0 \text{ in } L^7(O) \quad \text{and} \quad \mathbf{m}^0_n \to \mathbf{m}^0 \text{ weakly in } L^{3/2}(O).
\]

Then, there exists a stochastic basis \( \tilde{U} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathcal{F}}) \), \( \mathbb{R} \)-valued Wiener process \( \tilde{W} \) on \( \tilde{U} \) and \( \tilde{\mathcal{F}} \)-measurable processes \( \tilde{\varrho} \) and \( \tilde{\mathbf{u}} \) such that the sequence \( \{ (\tilde{\varrho}_n, \tilde{\mathbf{u}}_n) \}_{n \in \mathbb{N}} \) converges to \( (\tilde{\varrho}, \tilde{\mathbf{u}}) \). Moreover, \( (\tilde{U}, \tilde{W}, \tilde{\varrho}, \tilde{\mathbf{u}}) \) is a martingale solution to (1.1) such that \( L(\tilde{\varrho}(0)) = L(\varrho^0) \) and \( L(\tilde{\varrho}(0)) = L(m^0) \).

**Remark 2.6.** The sequence \( \{ (\tilde{\varrho}_n, \tilde{\mathbf{u}}_n) \}_{n \in \mathbb{N}} \) converges to \( (\tilde{\varrho}, \tilde{\mathbf{u}}) \) weakly. The precise meaning of it is explained in the proof.

**Remark 2.7.** Each martingale solution is a tuple consisting of the stochastic basis, Wiener process and the measurable processes. Since, we consider a sequence of martingale solutions we are dealing with a sequence of stochastic bases as well as a sequence of Wiener processes. However, Jakubowski in [33], showed that one can consider a common stochastic basis, see also [8], \( U = (\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}) \) and can assume without loss of generality one Wiener process \( W \) and we adopt this philosophy. This is the reason that in (2.6)–(2.8) we can consider a common probability measure \( \mathbb{P} \) incorporated in the definition of the expectation \( E \).

**Remark 2.8.** The result of Theorem 2.5 can be generalised in the following two ways:

(i) by considering time dependent stochastic diffusion, i.e. \( f : \Omega \times [0, T] \times O \to \mathbb{R}^3 \) and

(ii) by taking infinite dimensional noise, for e.g. a Hilbert space valued Wiener process \( W \).

### 3 A priori estimates

In this section we state the a priori estimates, being derived under the hypothesis that all quantities in question are smooth enough to justify our manipulations. We derive the estimates for both, deterministic (in the absence of Wiener process, but in the presence of deterministic external force “\( f dt \)”) and stochastic system (in the presence of Wiener process “\( f dW \)”). The indexes \( n \) are dropped, but the reader should notice that all these estimates are uniform in \( n \).

#### 3.1 Conservation of mass

We start with the conservation of mass. Integrating the continuity equation (1.1) over \( O \) we deduce that
\[
\frac{d}{dt} \int_O \varrho \, dx = 0,
\]
i.e. knowing that \( \int_O \varrho \, dx = M \) we have \( \int_O \varrho(t, x) \, dx = M \) for any \( t \in [0, T] \). The conservation of mass identity can be interpreted as a following uniform bound
\[
\| \varrho \|_{L^\infty(0, T; L^1(O))} \leq C,
\]
for some \( C > 0 \). The above bound holds for every \( \omega \in \Omega \) and the constant \( C \) is independent of \( \omega \) and only depends on \( M \).
3.2 The energy equality

In Theorem 2.5 we assume that the sequence \((q_n, u_n)_{n \in \mathbb{N}}\) satisfies certain energy inequality and a priori estimates. However, in this section we showcase the steps and techniques used to establish these estimates. One crucial step is the application of the Itô Lemma which is not completely justified as we are working on infinite dimensional Banach spaces. In principal, one could obtain these estimates in a finite dimensional setting (by studying a finite dimensional approximation of (1.1) with the help of Faedo-Galerkin type approximation, where application of the Itô Lemma is justified) and extend those to infinite dimensions through limiting procedure.

In what follows we derive the usual energy estimate, for deterministic as well as stochastic case.

For the whole section we choose and fix \(T > 0\), random variables \(q_0, m_0\) satisfying Assumption (2.1) and \(f\) satisfying Assumption (2.2) and a system \((U, W, q, u)\) which is a martingale solution to the problem (1.1) in the sense of Definition 2.3 with the initial data \(q_0, m_0\) such that Assumption 2.4 is satisfied. Here \(U := (\Omega, \mathcal{F}, P, \mathcal{F})\) is a stochastic basis with filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}\) satisfying the so called usual assumptions and \(W\) is a \(\mathbb{R}\)-valued Wiener process on \(U\). We will prove a series of additional a’priori energy estimates under a vague assumption that the process \((q, u)\) is sufficiently smooth.

Lemma 3.1. In the deterministic case, i.e. with \(dW = dt\), we have

\[
\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} q(t, x)|u(t, x)|^2 + \frac{1}{\gamma - 1} q^\gamma(t, x) \right) \, dx + \int_{\Omega} q|\nabla u(t, x)|^2 \, dx = \int_{\Omega} q(t, x)f(t, x) \cdot u(t, x) \, dx.
\]

(3.2)

Proof of Lemma 3.1. Multiplying the momentum equation of system (1.1) by \(u\) then integrating by parts and using the following relation

\[
\int_{\Omega} \nabla q^\gamma \cdot u \, dx = \int_{\Omega} \frac{\gamma}{\gamma - 1} q^\gamma \nabla q^{-1} \cdot u \, dx = -\int_{\Omega} \frac{\gamma}{\gamma - 1} q^{-1} \nabla q \, dx
\]

(3.3)

along with the following identity (due to the boundary conditions)

\[
\int_{\Omega} \nabla (q \nabla u) \cdot u \, dx = -\int_{\Omega} q|\nabla u|^2 \, dx,
\]

(3.4)

we infer the energy identity (3.2).

\[\square\]

Lemma 3.2. The following equality is satisfied for every \(t \in [0, T], P\)-a.s.,

\[
\int_{\Omega} \left( \frac{1}{2} q(t, x)|u(t, x)|^2 + \frac{1}{\gamma - 1} q^\gamma(t, x) \right) \, dx + \int_{0}^{t} \int_{\Omega} \frac{1}{2} m_0^2(x) + \frac{1}{\gamma - 1} q_0^\gamma(x) \, dx + \frac{1}{2} \int_{0}^{t} \int_{\Omega} \frac{1}{2} q(s, x)|f(s, x)|^2 \, dx \, ds
\]

\[+ \int_{0}^{t} \int_{\Omega} u(s, x) \cdot f(s, x) q(s, x) \, dx \, dW(s).
\]

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Proof of Lemma 3.2. We use the Itô Lemma to obtain these estimates. We will repeatedly use the following notation $m = \varphi u$. Let us introduce a function $\varphi$ defined by

$$\varphi(g, m) = \int_{\Omega} \frac{1}{2} g(x) |u(x)|^2 \, dx = \int_{\Omega} \frac{1}{2} \frac{|m(x)|^2}{\varphi(x)} \, dx. \quad (3.6)$$

For the $\varphi$ defined above we have following

$$\frac{\partial \varphi(g, m)}{\partial g}(y) = - \int_{\Omega} \frac{1}{2} g(x) \frac{|m(x)|^2}{\varphi^2(x)} \, dx, \quad \frac{\partial \varphi}{\partial m}(z) = \int_{\Omega} \frac{1}{2} \frac{m(x)z(x)}{\varphi(x)} \, dx,$n

$$\frac{\partial^2 \varphi(z_1, z_2)}{\partial m^2}(z_1, z_2) = \int_{\Omega} \frac{1}{2} z_1(x)z_2(x) \, dx. \quad (3.7)$$

Therefore, by the application of Itô Lemma to the function $\varphi$ and the processes $\varphi$ and $m$ whose differentials are given by (1.1), we get

$$\int_{\Omega} \frac{1}{2} \varphi |u|^2 \, dx = \int_{\Omega} \frac{1}{2} \frac{|m_0|^2}{\varphi_0} \, dx - \frac{1}{2} \int_0^t \int_{\Omega} \varphi |u|^2 \, dx \, ds + \int_0^t \int_{\Omega} u \cdot d(\varphi u) \, dx \, ds + \frac{1}{2} \int_0^t \int_{\Omega} |f|^2 \, dx \, ds. \quad (3.8)$$

Using (1.1), we obtain

$$\int_{\Omega} \frac{1}{2} \varphi |u|^2 \, dx = \int_{\Omega} \frac{1}{2} \frac{|m_0|^2}{\varphi_0} \, dx + \frac{1}{2} \int_0^t \int_{\Omega} \text{div}(\varphi u) |u|^2 \, dx \, ds$$

$$- \int_0^t \int_{\Omega} u \cdot \text{div}(\varphi u \otimes u) \, dx \, ds + \int_0^t \int_{\Omega} u \cdot \text{div}(\varphi \nabla u) \, dx \, ds$$

$$- \int_0^t \int_{\Omega} u \cdot \nabla \varphi \, dx \, ds + \int_0^t \int_{\Omega} \varphi u \cdot f \, dx \, ds \, dW + \frac{1}{2} \int_0^t \int_{\Omega} |f|^2 \, dx \, ds$$

$$:= \sum_{i=1}^7 J_i. \quad (3.9)$$

Now we investigate each of these terms, individually. For $J_3$, the integration by parts along with the boundary conditions, gives us

$$J_3 = - \int_0^t \int_{\Omega} u \cdot \text{div}(\varphi u \otimes u) \, dx \, ds = - \frac{1}{2} \int_0^t \int_{\Omega} \text{div}(\varphi u) |u|^2 \, dx \, ds, \quad (3.10)$$

and this term thus cancels out $J_2$.

For $J_4$ again, the integration by parts results in

$$J_4 = \int_0^t \int_{\Omega} u \cdot \text{div}(\varphi \nabla u) \, dx \, ds = - \int_0^t \int_{\Omega} \varphi |\nabla u|^2 \, dx \, ds. \quad (3.9)$$

Finally, for $J_5$ we write

$$J_5 = - \int_0^t \int_{\Omega} \nabla \varphi \cdot u \, dx \, ds = - \int_0^t \int_{\Omega} \frac{\gamma}{\gamma - 1} \varphi \nabla \varphi^{-1} \cdot u \, dx \, ds$$

$$= \int_0^t \int_{\Omega} \frac{\gamma}{\gamma - 1} \varphi^{-1} \text{div}(\varphi u) \, dx \, ds$$

$$= - \frac{1}{\gamma - 1} \int_{\Omega} \varphi \, dx + \frac{1}{\gamma - 1} \int_{\Omega} \varphi_0 \, dx. \quad (3.11)$$

Using (3.9)–(3.11) in (3.8) and on rearranging, we obtain (3.5).
3.3 The Bresch-Desjardins entropy

To proceed we need to find some better estimate of the norm of density than $L^\infty(0, T; L^\gamma(O))$. It will be a consequence of integrability of gradient of $\rho$ obtained by a modification of entropy inequality proved for the first time by Bresch & Desjardins [10]. We will roughly recall most important steps from the original proof.

**Lemma 3.3.** In the deterministic case, i.e. with $dW = dt$, the following equality is satisfied

$$\frac{d}{dt} \int_{O} \left( \frac{1}{2} \rho(t,x)|u + \nabla \log \rho(t,x)|^2 + \frac{1}{\gamma - 1} \rho^\gamma(t,x) \right) \, dx \quad (3.12)$$

$$+ \int_{O} \nabla \log \rho(t,x) : \nabla p(t,x) \, dx + 2 \int_{O} \rho |\mathcal{A}u(t,x)|^2 \, dx$$

$$= \int_{O} \rho(t,x)f(t,x)(u(t,x) + \nabla \log \rho(t,x)) \, dx,$$

where the anti-symmetric gradient $\mathcal{A}u$ is given by

$$\mathcal{A}u = \frac{1}{2} \left( \nabla u - \nabla^\perp u \right). \quad (3.13)$$

**Proof of Lemma 3.3.** We will expand the LHS of equality (3.12), i.e. we will compute

$$\frac{d}{dt} \int_{O} \left( \frac{1}{2} \rho|u + \nabla \log \rho|^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) \, dx$$

$$= \frac{d}{dt} \int_{O} \left( \frac{1}{2} \rho|u|^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) \, dx + \frac{d}{dt} \int_{O} \frac{1}{2}\rho |\nabla \log \rho|^2 \, dx + \frac{d}{dt} \int_{O} \rho \nabla \log \rho \cdot \nabla \rho \, dx \quad (3.14)$$

$$= I_1 + I_2 + I_3$$

The first integral $I_1$ is clearly the same as in the classical energy estimate so we drop it, for the second one we write

$$I_2 = \frac{d}{dt} \int_{O} \frac{1}{2} \rho |\nabla \log \rho|^2 \, dx$$

$$= \int_{O} \frac{1}{2} \rho \partial_t |\nabla \log \rho|^2 \, dx - \int_{O} \frac{1}{2} \text{div}(\rho u) |\nabla \log \rho|^2 \, dx$$

$$= \int_{O} \rho \nabla \log \rho \cdot \nabla \left( \frac{\partial_t \rho}{\rho} \right) \, dx - \int_{O} \frac{1}{2} \text{div}(\rho u) |\nabla \log \rho|^2 \, dx$$

$$= - \int_{O} \rho \nabla \log \rho \cdot \nabla \left( \frac{u \cdot \nabla \rho}{\rho} + \text{div} u \right) \, dx - \int_{O} \frac{1}{2} \text{div}(\rho u) |\nabla \log \rho|^2 \, dx$$

$$= - \int_{O} \rho \nabla \log \rho \cdot \nabla \text{div} u \, dx - \int_{O} \rho \nabla \log \rho \otimes u : \nabla \log \rho \, dx$$

$$- \int_{O} \rho \nabla u : \nabla \log \rho \otimes \nabla \log \rho \, dx - \int_{O} \frac{1}{2} \text{div}(\rho u) |\nabla \log \rho|^2 \, dx$$

$$= \int_{O} \rho \Delta \log \rho \text{div} u \, dx + \int_{O} \rho |\nabla \log \rho|^2 \text{div} u \, dx$$

$$- \int_{O} \text{div}(\rho \nabla \log \rho \nabla \log \rho \otimes u) \, dx + \int_{O} \rho u \Delta \log \rho \cdot \nabla \log \rho \, dx + \int_{O} |\nabla \log \rho|^2 \text{div}(\rho u) \, dx$$

$$- \int_{O} \rho \nabla u : \nabla \log \rho \otimes \nabla \log \rho \, dx - \int_{O} \frac{1}{2} \text{div}(\rho u) |\nabla \log \rho|^2 \, dx$$

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Therefore, coming back to (3.16) we note that a lot of terms cancel and we are left

\[ I_3 = \frac{d}{dt} \int_{\Omega} \rho \nabla \log \rho \cdot \nabla \rho \, dx + \int_{\Omega} \partial_t (\rho u) \cdot \nabla \rho \, dx + \int_{\Omega} (\nabla \log \rho)^2 \, dx. \]  

Next, for \( I_3 \) we first write

\[ I_3 = \frac{d}{dt} \int_{\Omega} \rho \nabla \log \rho \cdot \nabla \rho \, dx = \int_{\Omega} \partial_t (\rho u) \cdot \nabla \rho \, dx + \int_{\Omega} (\nabla \log \rho)^2 \frac{1}{\rho} \, dx \]  

and we focus on the first part of this expression. Thanks to the momentum equation of system (1.1) we have

\[ \int_{\Omega} \partial_t (\rho u) \cdot \nabla \log \rho \, dx \]

\[ = - \int_{\Omega} \nabla (\rho \nabla u) \cdot \nabla \log \rho \, dx - \int_{\Omega} \nabla \rho \cdot \nabla \rho \, dx + \int_{\Omega} \nabla f \cdot \nabla \log \rho \, dx + \int_{\Omega} \rho f \cdot \nabla \log \rho \, dx. \]

In particular, we can rearrange the third term as

\[ \int_{\Omega} \nabla (\rho \nabla u) \cdot \nabla \rho \, dx = - \int_{\Omega} \rho \nabla u \cdot \nabla^2 \log \rho \, dx \]

\[ = - \int_{\Omega} \nabla u \cdot \nabla^2 \rho \, dx + \int_{\Omega} \nabla u \cdot \nabla \log \rho \otimes \nabla \rho \, dx \]

\[ = - \int_{\Omega} \partial_t u \cdot \nabla \rho \, dx + \int_{\Omega} \nabla u \cdot \nabla \log \rho \otimes \nabla \rho \, dx \]

\[ = \int_{\Omega} \nabla \partial_t u \cdot \nabla \rho \, dx + \int_{\Omega} \nabla u \cdot \nabla \log \rho \otimes \nabla \rho \, dx \]

\[ = - \int_{\Omega} \nabla u (\rho \nabla \log \rho + \rho |\nabla \log \rho|^2) \, dx + \int_{\Omega} \nabla u \cdot \nabla \log \rho \otimes \nabla \rho \, dx. \]

Therefore, coming back to (3.16) we obtain

\[ \frac{d}{dt} \int_{\Omega} \rho \nabla \log \rho \cdot \nabla \rho \, dx + \int_{\Omega} \nabla \rho \cdot \nabla \log \rho \, dx \]

\[ = - \int_{\Omega} \nabla (\rho \nabla u) \cdot \nabla \log \rho \, dx + \int_{\Omega} (\nabla \log \rho)^2 \frac{1}{\rho} \, dx + \int_{\Omega} \rho f \cdot \nabla \log \rho \, dx \]  

\[ - \int_{\Omega} \nabla u (\rho \nabla \log \rho + \rho |\nabla \log \rho|^2) \, dx + \int_{\Omega} \nabla u \cdot \nabla \log \rho \otimes \nabla \rho \, dx. \]
\[
\frac{d}{dt} \int_O \frac{1}{2} \rho |\nabla \log \rho|^2 \, dx + \frac{d}{dt} \int_O \rho \mathbf{u} \cdot \nabla \log \rho \, dx + \int_O \nabla \rho \cdot \nabla \log \rho \, dx
= - \int_O \div(\rho \mathbf{u} \otimes \mathbf{u}) \cdot \nabla \rho \, dx + \int_O (\div(\rho \mathbf{u}))^2 \frac{1}{\rho} \, dx + \int_O \rho \mathbf{f} \cdot \nabla \log \rho \, dx. \tag{3.18}
\]

Moreover, using the continuity equation we have

\[
- \int_O \div(\rho \mathbf{u} \otimes \mathbf{u}) \cdot \nabla \rho \, dx = - \int_O \div(\rho \mathbf{u}) \mathbf{u} \cdot \nabla \rho \, dx - \int_O \rho (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \nabla \rho \, dx
= - \int_O (\div(\rho \mathbf{u}))^2 \frac{1}{\rho} \, dx + \int_O \div(\rho \mathbf{u}) \mathbf{u} \cdot \nabla \rho \, dx - \int_O (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \nabla \rho \, dx.
\]

Therefore we infer that

\[
\frac{d}{dt} \int_O \frac{1}{2} \rho |\nabla \log \rho|^2 \, dx + \frac{d}{dt} \int_O \rho \mathbf{u} \cdot \nabla \log \rho \, dx + \int_O \nabla \rho \cdot \nabla \log \rho \, dx
= \int_O \div(\rho \mathbf{u}) \mathbf{u} \cdot \nabla \rho \, dx - \int_O (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \nabla \rho \, dx + \int_O \rho \mathbf{f} \cdot \nabla \log \rho \, dx \tag{3.19}
\]

Finally noticing that

\[
|\nabla \mathbf{u}|^2 - \nabla \mathbf{u} : \nabla \mathbf{u}^T = 2|\mathbf{A}\mathbf{u}|^2
\]

we can sum up (3.19) with (3.2) to get precisely (3.12).

Now we are ready to present the main contribution of the present subsection, i.e. a generalization of Lemma 3.3 to the stochastic case.

**Lemma 3.4.** The following equality is satisfied for every \( t \in [0, T] \), \( \mathbb{P} \)-a.s.,

\[
\int_O \left( \frac{1}{2} \rho |\nabla \log \rho|^2 + \frac{1}{\gamma - 1} \rho \cdot \rho \right) \, dx
+ \int_0^t \int_O \nabla \log(\rho) \cdot \nabla p(\rho) \, dx \, ds + 2 \int_0^t \int_O \rho |\mathbf{A}\mathbf{u}|^2 \, dx \, ds
= \int_O \left( \frac{1}{2} \rho \left| \frac{\mathbf{m}}{\rho_0} + \nabla \log \rho_0 \right|^2 + \frac{1}{\gamma - 1} \rho_0 \right) \, dx + \frac{1}{2} \int_0^t \int_O \rho |\mathbf{f}|^2 \, dx \, ds
+ \int_0^t \int_O \rho \mathbf{f} \cdot (\mathbf{u} + \nabla \log \rho) \, dx \, dW, \tag{3.20}
\]

where the anti-symmetric gradient \( \mathbf{A}\mathbf{u} \) is defined in (3.13).
Proof of Lemma 3.4. We will expand the LHS of equality (3.20) in the differential form, i.e. we will compute
\[
\begin{align*}
\text{d} \left( \int_{\mathcal{O}} \left( \frac{1}{2} \varrho |u| + \nabla \log \varrho \right)^2 + \frac{1}{\gamma - 1} \varrho^\gamma \right) \, dx \\
= \text{d} \left( \int_{\mathcal{O}} \left( \frac{1}{2} \varrho |u|^2 + \frac{1}{\gamma - 1} \varrho^\gamma \right) \right) + \text{d} \left( \int_{\mathcal{O}} \frac{1}{2} \varrho |\nabla \log \varrho|^2 \, dx \right) + \text{d} \left( \int_{\mathcal{O}} \varrho u \cdot \nabla \log \varrho \, dx \right)
= I_1 + I_2 + I_3
\end{align*}
\]
(3.21)

$I_1$ is already computed in (3.5). Since, $I_2$ contains only $\varrho$, a deterministic function it can be dealt as in Lemma 3.3. Now for $I_3$, we apply the Itô product rule for the function
\[
\varphi(\varrho, m) := \int_{\mathcal{O}} m \cdot \nabla \log \varrho \, dx = \int_{\mathcal{O}} \varrho u \cdot \nabla \log \varrho \, dx
\]
(3.22)
and the processes $m$ and $\varrho$. Note that the function $\varphi$ defined above in (3.22) differed from the function defined earlier in (3.6). Moreover, the correction term will not appear since $\varrho$ is a deterministic function.

Thus, we have
\[
\text{d} \left( \int_{\mathcal{O}} m \cdot \nabla \log \varrho \, dx \right) = \int_{\mathcal{O}} \text{d}m \cdot \nabla \log \varrho \, dx + \int_{\mathcal{O}} v \cdot \nabla (\text{d} \varrho) \frac{1}{\varrho} \, dx.
\]

Integrating the above with respect to time, the integration by parts and the continuity equation (1.1) results in
\[
\int_{\mathcal{O}} \varrho u \cdot \nabla \log \varrho \, dx = \int_{\mathcal{O}} m_0 \cdot \nabla \log \varrho_0 \, dx + \int_0^t \int_{\mathcal{O}} d(\varrho u) \cdot \nabla \log \varrho \, dx \, ds
+ \int_0^t \int_{\mathcal{O}} \frac{1}{\varrho} \text{div}(\varrho u)^2 \, dx \, ds.
\]
(3.23)

Now, using the calculations from Lemma 3.3 (for $I_3$ and the following computations), on integrating (3.21) w.r.t. $t \in [0, T]$ and summing up the expressions for $I_1$, $I_2$ and $I_3$, we obtain (3.20).

3.4 Uniform estimates

The following uniform estimates are consequences of Lemmata 3.2 and 3.4 and the Burkholder-Davis-Gundy inequality, see [42].

Lemma 3.5. Assume that $p \geq 1$. Then there exists a constant $C$ depending only on the initial data and the external force such that the following estimates are satisfied.

\[
\sup_{t \in [0, T]} \mathbb{E} \left( \int_{\mathcal{O}} \left( \frac{1}{2} \varrho |u|^2 + \frac{1}{2} \varrho^2 + \frac{1}{\gamma - 1} \varrho^\gamma \right) \, dx \right) \leq C,
\]
(3.24)

\[
\mathbb{E} \left( \sup_{t \in [0, T]} \int_{\mathcal{O}} \varrho |u|^2 \, dx \right)^p + \mathbb{E} \left( \sup_{t \in [0, T]} \int_{\mathcal{O}} (\varrho |\nabla \log \varrho|^2) \, dx \right)^p + \mathbb{E} \left( \sup_{t \in [0, T]} \int_{\mathcal{O}} \varrho^\gamma \, dx \right)^p \leq C
\]
(3.25)

\[
\mathbb{E} \left( \int_0^T \int_{\mathcal{O}} \varrho |\nabla u|^2 \, dx \, dt \right)^p + \mathbb{E} \left( \int_0^T \int_{\mathcal{O}} |\nabla \varrho^\gamma|^2 \, dx \, dt \right)^p \leq C,
\]
(3.26)

where the constant $C$ depends on initial data, time and $f$.
Proof of Lemma 3.5. We first sum up inequalities (3.5) with (3.20). Noticing that \( \int_0^t \int\rho |A u|^2 \, dx \, ds \geq 0 \) and can be disregarded we obtain the inequality

\[
\int_\mathcal{O} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{2} \rho |u + \nabla \log \rho|^2 + \frac{2}{\gamma - 1} \rho^\gamma \right) \, dx + \int_0^t \int_\mathcal{O} \nabla \log \rho \cdot \nabla p(\rho) \, dx \, ds + \int_0^t \int_\mathcal{O} \rho |\nabla u|^2 \, dx \, ds
\]
\[
\leq \int_\mathcal{O} \left( \frac{1}{2} \rho_0 \left| \frac{m_0}{\rho_0} \right|^2 + \frac{1}{2} \rho_0 |\nabla \log \rho_0|^2 + \frac{2}{\gamma - 1} \rho_0^\gamma \right) \, dx
\]
\[
+ \int_0^t \int_\mathcal{O} \rho |f|^2 \, dx \, ds + \int_0^t \int_\mathcal{O} \rho u \cdot f \, dx \, dW + \int_0^t \int_\mathcal{O} \rho (u + \nabla \log \rho) \cdot f \, dx \, dW \tag{3.27}
\]

Note that the integrand in second term on the LHS of (3.27) equals

\[
\nabla \log \rho \cdot \nabla p(\rho) = \gamma \rho^\gamma - 2 |\nabla \rho|^2 = C_\gamma |\nabla \rho|^2.
\]

Using the Sobolev embedding, we can therefore estimate

\[
C \int_0^t \|\rho^{1/2}\|_{L^6(\mathcal{O})}^2 \, ds \leq C_\gamma \int_0^t \|\rho^{1/2}\|_{L^2(\mathcal{O})}^2 \, ds = \int_0^t \int_\mathcal{O} \nabla \log \rho \cdot \nabla p(\rho) \, dx \, ds,
\]

where the constant on the LHS comes from the Sobolev embedding. Inequality (3.27) thus gives

\[
\int_\mathcal{O} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{2} \rho |u + \nabla \log \rho|^2 + \frac{2}{\gamma - 1} \rho^\gamma \right) \, dx + C \int_0^t \|\rho^{1/2}\|_{L^6(\mathcal{O})}^2 \, ds + \int_0^t \int_\mathcal{O} \rho |\nabla u|^2 \, dx \, ds
\]
\[
\leq \int_\mathcal{O} \left( \frac{1}{2} \rho_0 \left| \frac{m_0}{\rho_0} \right|^2 + \frac{1}{2} \rho_0 |\nabla \log \rho_0|^2 + \frac{2}{\gamma - 1} \rho_0^\gamma \right) \, dx
\]
\[
+ \int_0^t \int_\mathcal{O} \rho |f|^2 \, dx \, ds + \int_0^t \int_\mathcal{O} \rho u \cdot f \, dx \, dW + \int_0^t \int_\mathcal{O} \rho (u + \nabla \log \rho) \cdot f \, dx \, dW \tag{3.30}
\]
\[
:= \sum_{i=1}^4 J_i.
\]

In order to obtain the first estimate (3.24) we first take mathematical expectation and then supremum over time in (3.30). Note that \( \mathbb{E}(J_3 + J_4) = 0 \), and thus (3.30) reduces to

\[
\sup_{t \in [0,T]} \mathbb{E} \left( \int_\mathcal{O} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{2} \rho |u + \nabla \log \rho|^2 + \frac{2}{\gamma - 1} \rho^\gamma \right) \, dx \right)
\]
\[
+ \sup_{t \in [0,T]} \mathbb{E} \left( C \int_0^t \|\rho^{1/2}\|_{L^6(\mathcal{O})}^2 \, ds + \int_0^t \int_\mathcal{O} \rho |\nabla u|^2 \, dx \, ds \right)
\]
\[
\leq \mathbb{E} \left( \int_\mathcal{O} \left( \frac{1}{2} \rho_0 \left| \frac{m_0}{\rho_0} \right|^2 + \frac{1}{2} \rho_0 |\nabla \log \rho_0|^2 + \frac{2}{\gamma - 1} \rho_0^\gamma \right) \, dx \right)
\]
\[
+ \sup_{t \in [0,T]} \mathbb{E} \left( \int_0^t \int_\mathcal{O} \rho |f|^2 \, dx \, ds \right).
\]

(3.31)
The first term on the r.h.s. of (3.31) is controlled by the Assumption 2.1. To estimate the second term we use H"older and Young inequalities respectively

\[
J_2 = \int_0^T \phi |f|^2 \, dx \, ds \leq C \int_0^T \phi \|L^\gamma(\mathcal{O})\| \|f\|^2 \, ds \leq C \int_0^T \phi |\tilde{\omega}^2| \|L^\delta(\mathcal{O})\| \|f\|^2 \, ds \\
\leq \varepsilon \int_0^T \|\tilde{\omega}^2\|_{L^6(\mathcal{O})} \, ds + TC_\varepsilon \|f\| \gamma^{\frac{2\gamma}{\gamma-1}} \|L^{\gamma}(\mathcal{O})\|,
\]

and so,

\[
\sup_{t \in [0,T]} \mathbb{E} \left( \int_0^T \phi |f|^2 \, dx \, ds \right) \leq \varepsilon \sup_{t \in [0,T]} \mathbb{E} \left( \int_0^T \|\tilde{\omega}^2\|_{L^6(\mathcal{O})} \, ds \right) + TC_\varepsilon \|f\| \gamma^{\frac{2\gamma}{\gamma-1}} \|L^{\gamma}(\mathcal{O})\|. \tag{3.33}
\]

Choosing \( \varepsilon \) sufficiently small the first term can be absorbed by the LHS of (3.31). For the second term, note that \( \frac{\gamma}{\gamma-1} \leq 3 \) for \( \gamma \in (1,3) \), and so it is bounded provided \( f \) is bounded in \( L^3(\mathcal{O}) \).

To prove the rest of the estimates in Lemma 3.5 we need to first apply the supremum over time to both sides of (3.30), and then take the mathematical expectation of the \( \gamma \)-th power, for \( \gamma \geq 1 \). This results in

\[
\begin{align*}
\mathbb{E} \left( \sup_{t \in [0,T]} \int_0^T \frac{1}{2} \phi |\mathbf{u}|^2 \, dx \right)^{\gamma} &+ \mathbb{E} \left( \sup_{t \in [0,T]} \int_0^T \frac{1}{2} \phi |\mathbf{u}| \, dx \right)^{\gamma} + \mathbb{E} \left( \sup_{t \in [0,T]} \int_0^T \phi |\mathbf{u}|^2 \, dx \right)^{\gamma} \\
+ C \mathbb{E} \left( \int_0^T \|\tilde{\omega}^2\|_{L^6(\mathcal{O})} \|^2 \right)^{\gamma} &+ \mathbb{E} \left( \int_0^T \|\nabla \mathbf{u}\|^2 \, dt \right)^{\gamma} \\
&\leq \mathbb{E} \left( \int_0^T \left( \frac{1}{2} \phi |\mathbf{u}|^2 \right) + \frac{1}{2} \phi \left| \frac{m_0}{\phi} \right|^2 + \frac{1}{2} \phi \left| \nabla \log \phi \right|^2 + \frac{2}{\gamma-1} \phi \right) \, dx \right)^{\gamma} \\
&+ \mathbb{E} \left( \sup_{t \in [0,T]} \int_0^T \phi |f|^2 \, dx \, ds \right)^{\gamma} + \mathbb{E} \left( \sup_{t \in [0,T]} \int_0^T \phi |\mathbf{f}| \, dx \, dW \right)^{\gamma} \\
&+ \mathbb{E} \left( \sup_{t \in [0,T]} \int_0^T \phi (\mathbf{u} + \nabla \log \phi) \cdot \mathbf{f} \, dx \, dW \right)^{\gamma} := \sum_{i=1}^4 I_i,
\end{align*}
\]

where we have used the algebraic inequality

\[
a^{\gamma} + b^{\gamma} \leq (a + b)^{\gamma} \leq C(\gamma)(a^{\gamma} + b^{\gamma}) \quad a > 0, b > 0 \text{ and } \gamma \geq 1.
\]

Estimate of \( I_2 \) uses (3.32), from which we deduce

\[
I_2 = \mathbb{E} \left( \sup_{t \in [0,T]} \int_0^T \phi |f|^2 \, dx \, ds \right)^{\gamma} \leq \varepsilon \mathbb{E} \left( \int_0^T \|\tilde{\omega}^2\|_{L^6(\mathcal{O})} \, ds \right) + TC_\varepsilon \|f\| \gamma^{\frac{2\gamma}{\gamma-1}} \|L^{\gamma}(\mathcal{O})\|, \tag{3.35}
\]

which again can be absorbed by the LHS of (3.34) and assumptions on \( f \). To estimate \( I_3 \) and \( I_4 \) we use the Burkholder-Davis-Gundy inequality, the H"older and the Young inequalities

\[
I_3 = \mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_0^T \phi \mathbf{u} \cdot \mathbf{f} \, dx \, dW \right| \right)^{\gamma}
\]

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\begin{align*}
\leq & \mathbb{E}\left(\int_0^T \left| \frac{1}{T} \int_{\Omega} \varrho \mathbf{u} \cdot \mathbf{f} \, dx \right|^2 \, dt \right)^{\frac{\gamma}{2}} \\
\leq & \mathbb{E}\left( \int_0^T \left| \sqrt{\varrho} \mathbf{u} \right|_{L^2}^2 \left| \sqrt{\varrho} \mathbf{f} \right|_{L^\frac{\gamma}{\gamma-1}}^2 \, dt \right)^{\frac{\gamma}{2}} \\
\leq & \mathbb{E}\left( \int_0^T \left| \frac{\partial \varrho}{\partial t} \right|_{L^\infty}^2 \sup_{t \in [0,T]} \left| \sqrt{\varrho} \mathbf{u} \right|_{L^2}^2 \left| \sqrt{\varrho} \mathbf{f} \right|_{L^\frac{\gamma}{\gamma-1}}^2 \, dt \right)^{\frac{\gamma}{2}} \\
\leq & \varepsilon \mathbb{E}\left( \int_0^T \left| \frac{\partial \varrho}{\partial t} \right|_{L^\infty}^2 \, ds \right)^{\frac{\gamma}{2}} \varepsilon \mathbb{E}\left( \sup_{t \in [0,T]} \left| \sqrt{\varrho} \mathbf{u} \right|_{L^2}^2 \right)^{\frac{\gamma}{2}} + C(\varepsilon) T^p \mathbb{E}\left| \mathbf{f} \right|_{L^\frac{2p\gamma'}{\gamma-1}}^{2p\gamma'},
\end{align*}

where \( \varepsilon \) is sufficiently small for the first two terms of (3.36) to be absorbed by the LHS of (3.34). Note that \( \frac{6\gamma}{\gamma-1} \leq 3 \) for \( \gamma \in (1, 3) \) and so, the last term is bounded provided \( \mathbf{f} \) is bounded in \( L^3(\Omega) \). Exactly the same argument can be repeated for \( I_4 \) replacing \( \mathbf{u} \) by \( \mathbf{u} + \nabla \log \varrho \). This finishes the proof. \( \square \)

To summarise, the estimates from Lemma 3.5 give us the following uniform bounds

\begin{align*}
\left| \sqrt{\varrho} \mathbf{u} \right|_{L^p(\Omega; L^{\infty}(0, T; L^2(\Omega)))} & \leq C, \\
\left| \frac{\partial \varrho}{\partial t} \right|_{L^p(\Omega; L^{\infty}(0, T; L^1(\Omega)))} & \leq C, \\
\left| \sqrt{\varrho} \mathbf{u} \right|_{L^p(\Omega; L^2(0, T; L^2(\Omega)))} & \leq C, \tag{3.37} \end{align*}

as well as

\begin{align*}
\left| \nabla \sqrt{\varrho} \right|_{L^p(\Omega; L^{\infty}(0, T; L^2(\Omega)))} & \leq C, \\
\left| \nabla \varrho^{\gamma/2} \right|_{L^p(\Omega; L^2(0, T; L^2(\Omega)))} & \leq C. \tag{3.38} \end{align*}

From (3.38), we have \( \varrho^{\gamma} \) is uniformly bounded in \( L^p(\Omega; L^1(0, T; L^3(\Omega))) \) and using the interpolation, one can infer that for \( p \geq 1 \) there exists a constant \( C > 0 \) such that

\begin{align*}
\left| \varrho^{\gamma} \right|_{L^p(\Omega; L^{5/3}(0, T) \times \Omega)} & \leq C \left| \varrho^{\gamma/2} \right|_{L^p(\Omega; L^{\infty}(0, T; L^1(\Omega)))}^{3/5} \left| \varrho^{\gamma/2} \right|_{L^p(\Omega; L^1(0, T; L^3(\Omega)))}, \tag{3.39} \end{align*}

### 3.5 The Mellet-Vasseur estimate

Our ultimate goal before the limit passage is to improve the integrability of the convective term.

**Lemma 3.6.** Let \( \delta \in (0, 1) \). Then there exist constants \( c_\delta > 0 \) and \( C_\delta > 0 \) depending only on the initial data such that the following inequality is satisfied

\begin{align*}
\frac{1}{2 + \delta} \int_0^t \varrho(t, x) |\mathbf{u}(t, x)|^{2+\delta} \, dx + c_\delta \int_0^t \int_\Omega \varrho(s, x) |\mathbf{u}(s, x)|^4 |\nabla \mathbf{u}(s, x)|^2 \, dx \, ds \\
\leq & \left( \int_0^t \left( \int_\Omega \left( \frac{p(\varrho(s, x))^{\frac{2}{\gamma}} \varrho(s, x)}{\varrho(s, x)} \right)^2 \, dx \right)^{\frac{\gamma}{2}} \, ds \right)^{\frac{2+\delta}{2}} \\
+ & \frac{1+\delta}{2} \int_0^t \varrho(s, x) |\mathbf{f}(s, x)|^2 |\mathbf{u}|^4 \, dx \, ds + \int_0^t \varrho(s, x) |\mathbf{f}(s, x) \cdot \mathbf{u}(s, x)|^\delta \mathbf{u}(s, x)^\delta \, dx \, dW(s). \tag{3.40} \end{align*}
Remark 3.7. Note that because $\gamma < 3$, the above result implies the equi-integrability of the acceleration term $\varphi u \otimes u$.

Proof of Lemma 3.6. When $dW = dt$ this follows the same strategy as in the work of Mellet & Vasseur [36, Lemma 3.2]. We basically test the momentum equation of (1.1) by $u|u|^{\delta}$, and repeat the energy estimate. We can show that the r.h.s. can be controlled using the left hand side and estimates from the previous lemmas and the Gronwall inequality.

In the stochastic case $dW \neq dt$ we again denote

$$\varphi(\varrho, v) := \frac{1}{2 + \delta} \int_{\mathcal{O}} \varrho \frac{|v|^{2+\delta}}{\varrho^{2+\delta}} \, dx = \frac{1}{2 + \delta} \int_{\mathcal{O}} \varrho |u|^{2+\delta} \, dx.$$

For the $\varphi$ defined above we have following

$$\frac{\partial \varphi(\varrho, v)}{\partial \varrho}(y) = -\frac{1 + \delta}{2 + \delta} \int_{\mathcal{O}} \frac{|v|^{2+\delta}}{\varrho^{2+\delta}} \, y \, dx = -\frac{1 + \delta}{2 + \delta} \int_{\mathcal{O}} |u|^{2+\delta} \, y \, dx,$$

$$\frac{\partial \varphi}{\partial v}(z) = \int_{\mathcal{O}} \frac{|v|^{2+\delta}}{\varrho^{2+\delta}} \, z \, dx = \int_{\mathcal{O}} |u|^{\delta} \, z \, dx,$$

$$\frac{\partial^2 \varphi}{\partial v^2}(z_1, z_2) = (1 + \delta) \int_{\mathcal{O}} \frac{1}{\varrho^{2+\delta}} |v|^{\delta} z_1 z_2 \, dx = (1 + \delta) \int_{\mathcal{O}} \frac{|u|^{\delta}}{\varrho} z_1 z_2 \, dx. \quad (3.41)$$

Therefore, by the application of the Itô Lemma for the function $\varphi$ and the processes $\varrho$ and $v$ whose differentials are given by (1.1), we obtain

$$\frac{1}{2 + \delta} \int_{\mathcal{O}} \varrho(t)|u(t)|^{2+\delta} \, dx = \frac{1}{2 + \delta} \int_{\mathcal{O}} \frac{|m_u|^{2+\delta}}{\varrho_0^{1+\delta}} \, dx - \frac{1 + \delta}{2 + \delta} \int_{\mathcal{O}} \int_{0}^{t} d\varrho |u|^{2+\delta} \, dx$$

$$+ \int_{0}^{t} \int_{\mathcal{O}} d(\varrho u) \cdot u |u|^{\delta} \, dx + \frac{1}{2} \int_{0}^{t} \int_{\mathcal{O}} \varrho |f|^2 |u|^{\delta} \, dx \, ds$$

$$= \frac{1}{2 + \delta} \int_{\mathcal{O}} \frac{|m_u|^{2+\delta}}{\varrho_0^{1+\delta}} \, dx + \frac{1 + \delta}{2 + \delta} \int_{0}^{t} \int_{\mathcal{O}} \text{div}(\varrho u) |u|^{2+\delta} \, dx \, ds \quad (3.42)$$

$$- \int_{0}^{t} \int_{\mathcal{O}} \text{div}(\varrho u \otimes u) \cdot u |u|^{\delta} \, dx \, ds + \int_{0}^{t} \int_{\mathcal{O}} \text{div}(\varrho \nabla u) \cdot u |u|^{\delta} \, dx \, ds$$

$$- \int_{0}^{t} \int_{\mathcal{O}} \nabla p(\varrho) \cdot u |u|^{\delta} \, dx \, ds + \int_{0}^{t} \int_{\mathcal{O}} \varrho f \cdot u |u|^{\delta} \, dx \, dW$$

$$+ \frac{1 + \delta}{2} \int_{0}^{t} \int_{\mathcal{O}} \varrho |f|^2 |u|^{\delta} \, dx \, ds = \sum_{i=1}^{7} I_i.$$

We observe that the extra term on the r.h.s. of (3.40) in comparison with the deterministic case, see [36, Lemma 3.2] for details, is

$$\frac{1 + \delta}{2} \int_{0}^{t} \int_{\mathcal{O}} \varrho |f|^2 |u|^{\delta} \, dx \, ds.$$

Integrating by parts in terms $I_2$ and $I_3$ and using the boundary conditions we get cancellation. Integrating by parts in $I_4$ we obtain two terms

$$I_4 = \int_{0}^{t} \int_{\mathcal{O}} \text{div}(\varrho \nabla u) \cdot u |u|^{\delta} \, dx \, ds$$

$$= - \int_{0}^{t} \int_{\mathcal{O}} \left( |u|^{\delta} \varrho |\nabla u|^2 - \delta |u|^{\delta-2} u_i u_k \partial_j u_l \partial_j u_k \right) \, dx \, ds \quad (3.43)$$

$$\leq - (1 - \delta) \int_{0}^{t} \int_{\mathcal{O}} |u|^{\delta} \varrho |\nabla u|^2 \, dx \, ds,$$
that can be moved to the l.h.s of (3.41) as it has the right sign provided \( \delta < 1 \). Finally, for \( I_5 \) we have

\[
I_5 = - \int_0^t \int_\Omega \nabla p(\varrho) \cdot \mathbf{u}|\mathbf{u}|^\delta \, dx \, ds
\]

\[
= \int_0^t \int_\Omega p(\varrho) \text{div} \mathbf{u}|\mathbf{u}|^\delta \, dx \, ds + \delta \int_0^t \int_\Omega p(\varrho)|\mathbf{u}|^{\delta-2} \mathbf{u} \cdot \nabla \mathbf{u} \, dx \, ds
\]

\[
\leq \sqrt{3 + \delta} \int_0^t \int_\Omega p(\varrho)|\mathbf{u}|^{\delta} \nabla \mathbf{u} \, dx \, ds
\]

\[
\leq \frac{1 - \delta}{2} \int_0^t \int_\Omega \varrho|\mathbf{u}|^{\delta} \nabla \mathbf{u}^2 \, dx \, ds + C_\delta \int_0^t \int_\Omega \frac{p(\varrho)^2}{\varrho}|\mathbf{u}|^\delta \, dx \, ds
\]

\[
\leq \frac{1 - \delta}{2} \int_0^t \int_\Omega \varrho|\mathbf{u}|^{\delta} \nabla \mathbf{u}^2 \, dx \, ds
\]

\[
+ C_\delta \int_0^t \left( \int_\Omega \left( \frac{p(\varrho)^2}{\varrho} \varrho^{\frac{\delta}{2}} \right)^{\frac{2}{\delta-3}} \, dx \right) \left( \int_\Omega \varrho|\mathbf{u}|^2 \, dx \right)^{\frac{\delta}{2}} \, ds,
\]

and so the first term can be controlled by the contribution coming from \( I_4 \), while for the second term we use that \( \varrho|\mathbf{u}|^2 \) is uniformly bounded in \( L^p(\Omega; L^\infty(0, T; L^1(\Omega))) \) to conclude the proof of (3.40).

Now, as from (3.39) we know that for \( p \geq 1 \), \( \varrho^\gamma \in L^p(\Omega; L^{5/3}((0, T) \times \Omega)) \), the pressure term on the r.h.s. of (3.40) is bounded provided \( \frac{p-2}{p} = \varrho^{2\gamma-1} \) can be controlled by \( \varrho^{5\gamma/3} \), which holds for \( \gamma < 3 \).

In the following we show how to estimate the rest of the terms on the r.h.s. of (3.40). We start with the third term:

\[
E \left( \int_0^t \int_\Omega \varrho|\mathbf{f}|^2 |\mathbf{u}|^\delta \, dx \, ds \right)
\]

\[
\leq E \left( \int_0^t \left[ \left( \int_\Omega \left( \varrho^{1-\frac{\delta}{2}} |\mathbf{f}|^2 \right)^{\frac{2}{2-\delta}} \, dx \right)^{\frac{2-\delta}{2}} \left( \int_\Omega \varrho|\mathbf{u}|^2 \, dx \right)^{\frac{\delta}{2}} \right] \, ds \right)
\]

\[
\leq \frac{2 - \delta}{2} E \int_0^t \int_\Omega \varrho|\mathbf{f}|^{\frac{4}{2-\delta}} \, dx \, ds + \frac{\delta}{2} E \int_0^t \int_\Omega \varrho|\mathbf{u}|^2 \, dx \, ds
\]

\[
\leq \frac{2 - \delta}{6} \left( E \|\mathbf{f}\|^{\frac{8(2-\delta)}{L_{2-\delta}(\Omega)^{2-\delta}}} \int_0^t \|\varrho\|_{L^6} \, ds \right) + \frac{\delta t}{3} E \left( \sup_{t \in [0, T]} \int_\Omega \varrho|\mathbf{u}|^2 \, dx \right)
\]

\[
\leq \frac{2 - \delta}{6} \left( E \|\mathbf{f}\|^{\frac{8(2-\delta)}{L_{2-\delta}(\Omega)^{2-\delta}}} \right)^{1/2} \left( E \|\varrho\|^2_{L^1(0, T; L^3(\Omega))} \right)^{1/2} + \frac{\delta t}{3} E \left( \sup_{t \in [0, T]} \int_\Omega \varrho|\mathbf{u}|^2 \, dx \right). \tag{3.44}
\]

Since for every \( \gamma \in (1, 3) \) there exists a \( \delta \in (0, 1) \) such that \( \frac{4\gamma}{(3\gamma-1)(2-\delta)} < 1 \), the assumptions on \( \mathbf{f}, \mathbf{f} \in L^p(\Omega; L^3(\Omega)) \), and a priori estimates from Lemma 3.2 can be used to bound the r.h.s. of (3.44).

Next we are required to bound the following term:

\[
E \left( \sup_{t \in [0, T]} \int_0^t \int_\Omega \varrho \cdot \mathbf{u}|\mathbf{u}|^\delta \, dx \, dW \right)
\]
By the Burkholder-Davis-Gundy inequality, Hölder and Young inequalities we obtain

\[
\mathbb{E} \sup_{t \in [0,T]} \left( \int_0^t \int_\Omega \varrho(s) f \cdot u |u|^\delta \, dx \, dW \right) \\
\leq \mathbb{E} \left[ \int_0^T \left( \int_\Omega \varrho(t) f \cdot u |u|^\delta \, dx \right)^2 \, dt \right]^{1/2} \\
\leq \mathbb{E} \left[ \int_0^T \left( \int_\Omega \varrho |u|^2 \, dx \right) \left( \int_\Omega \varrho |u|^{2+\delta} \, dx \right) \left( \int_\Omega \varrho |f|^{2(2+\delta)} \, dx \right)^{\frac{2-\delta}{2+\delta}} \, dt \right]^{1/2} \\
\leq \mathbb{E} \left[ \left( \sup_{t \in [0,T]} \int_\Omega \varrho |u|^2 \, dx \right)^{1/2} \left( \int_0^T \int_\Omega \varrho |u|^{2+\delta} \, dx \, dt \right)^{\frac{\delta}{2+\delta}} \left( \int_0^T \int_\Omega \varrho |f|^{2(2+\delta)} \, dx \, dt \right)^{\frac{2-\delta}{2+\delta}} \right] \\
\leq \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0,T]} \int_\Omega \varrho |u|^2 \, dx \right] + \frac{2\delta}{2+\delta} \mathbb{E} \left[ \sup_{t \in [0,T]} \int_\Omega \varrho |u|^{2+\delta} \, dx \right] \\
+ \frac{2-\delta}{2(2+\delta)} T^{\frac{2\delta}{2+\delta}} \mathbb{E} \left[ \int_0^T \int_\Omega \varrho |f|^{2(2+\delta)} \, dx \, dt \right] \\
\tag{3.45}
\]
\[ \| \sqrt{\varrho_n} \nabla u_n \|_{L^p(\Omega; L^2(0,T;L^2(\mathcal{O})))} \leq C, \]  
(4.3) 
\[ \| \sqrt{\varrho_n} \|_{L^p(\Omega; L^{\infty}(0,T;H^1(\mathcal{O})))} \leq C, \]  
(4.4) 
\[ \| \nabla \varrho_n^{1/2} \|_{L^p(\Omega; L^2(0,T;L^2(\mathcal{O})))} \leq C, \]  
(4.5) 
\[ \| \varrho_n \|_{L^{2+\delta}(\Omega; L^{\infty}(0,T;L^{2+\delta}(\mathcal{O})))} \leq C. \]  
(4.6) 

As observed after the proof of Lemma 3.5 inequalities (4.2), (4.1), (4.3), (4.4) and (4.5) are a consequence of that Lemma. Let us emphasize that Lemma 3.5 is a consequence of our first new a’priori estimates on the Bresch-Desjardins entropy from our Lemma 3.4. Moreover, inequalities (4.6) is a consequence of the first of our new a’priori estimates on of the Mellet-Vasseur from Lemma 3.6. Let us point out that we do not use the second of the a’priori estimates following from Lemma 3.6. Interestingly, this estimate is neither used by the authors of the deterministic paper [36].

We recall that by Remark 2.7, it is possible to choose a common stochastic basis on which this sequence of martingale solutions is defined.

### 4.1 Tightness of density and momentum

With these a priori estimates at hand we are able to pass to the weak limit for the subsequence of \( \varrho_n \). Our next purpose is to show tightness of the laws \( L(\varrho_n), L(\varrho_n u_n) \) according to the Definition D.9. To that purpose we will use some intuitions coming from the deterministic case and the compact embedding result as stated in Lemma D.10, see also [43, Corollary 4].

Our first result is as follows.

**Lemma 4.1.** Assume that \( 1 \leq q < 3 \). Then the family of measures \( \{L(\varrho_n) : n \in \mathbb{N}\} \) is tight on \( C([0,T]; L^q(\mathcal{O})) \).

**Proof of Lemma 4.1.** Let us fix \( q \in [1,3) \). Firstly, we observe that from the uniform bound (4.4), we have \( \varrho_n \in L^1(\Omega; L^\infty(0,T;W^{1,\frac{d}{2}}(\mathcal{O}))) \). Indeed, 
\[ \nabla \varrho_n = 2\sqrt{\varrho_n} \nabla (\sqrt{\varrho_n}) \] 
in a weak sense

and since \( \sqrt{\varrho_n} \) is uniformly bounded in \( L^2(\Omega; L^\infty(0,T;L^d(\mathcal{O}))) \), by the Sobolev embedding theorem (as \( d = 3 \)) and by the a priori estimates (4.4), and \( \nabla \sqrt{\varrho_n} \) is uniformly bounded in \( L^2(\Omega; L^\infty(0,T;L^2(\mathcal{O}))) \), again by the a priori estimates (4.4), we infer that 
\[ \mathbb{E}\|\nabla \varrho_n\|_{L^{\infty}(0,T;L^{\frac{d}{2}}(\mathcal{O}))} \leq C. \]  
(4.7)

By a very similar argument we can prove, using a priori estimates (4.2) and (4.4), together with the Sobolev embedding theorem, that 
\[ \mathbb{E}\|\varrho_n u_n\|_{L^{\infty}(0,T;L^{\frac{d}{2}}(\mathcal{O}))} \leq C. \]  
(4.8)

Therefore, by continuity equation we infer 
\[ \mathbb{E}\|\partial_t \varrho_n\|^p_{L^{\infty}(0,T;W^{-1,\frac{d}{2}}(\mathcal{O}))} \leq C. \]  
(4.9)

Summing up, we proved that 
\[ \mathbb{E}\|\varrho_n\|_{L^{\infty}(0,T;W^{1,\frac{d}{2}}(\mathcal{O}))) \cap L^{1,\infty}(0,T;W^{-1,\frac{d}{2}}) \leq C. \]  
(4.10)
Since \( q < 3 \), the embedding \( W^{1,\frac{4}{3}}(\mathcal{O}) \hookrightarrow L^q(\mathcal{O}) \) is compact. Thus, by Lemma D.10 with \( X = W^{1,\frac{4}{3}}(\mathcal{O}) \), \( B = L^q(\mathcal{O}) \), and \( Y = W^{-1,\frac{4}{3}}(\mathcal{O}) \), we infer that the embedding

\[
L^\infty((0,T; W^{1,\frac{4}{3}}(\mathcal{O})) \cap W^{1,\infty}(0,T; W^{-1,\frac{4}{3}}(\mathcal{O})) \hookrightarrow C([0,T]; L^q(\mathcal{O})) \text{ is compact.} \tag{4.11}
\]

Let us choose and fix \( \varepsilon > 0 \). Put \( \eta = \frac{C}{\varepsilon} \), where \( C \) is from estimate (4.10). Set

\[
B_\eta := \{ f \in L^\infty(0,T; W^{1,\frac{4}{3}}(\mathcal{O})) \cap W^{1,\infty}(0,T; W^{-1,\frac{4}{3}}(\mathcal{O})) : \| f \|_{L^\infty((0,T; W^{1,\frac{4}{3}}(\mathcal{O})) \cap W^{1,\infty}(0,T; W^{-1,\frac{4}{3}}(\mathcal{O})))} \leq \eta \}.
\]

Then by the Chebyshev inequality and uniform estimates (4.10) obtained above, we deduce that

\[
\mathbb{P}\{ \varrho_n \in B_\frac{1}{\eta} \} \leq \frac{1}{\eta} \mathbb{E}\| \varrho_n \|_{L^\infty((0,T; W^{1,\frac{4}{3}}(\mathcal{O})) \cap W^{1,\infty}(0,T; W^{-1,\frac{4}{3}}(\mathcal{O})))} \leq C \frac{1}{\eta} = \varepsilon. \tag{4.12}
\]

Since by (4.11), set \( B_\frac{1}{\eta} \) is a precompact set in \( C([0,T]; L^q(\mathcal{O})) \), tightness of laws of \( \varrho_n \) on \( C([0,T]; L^q(\mathcal{O})) \) follows directly from the Definition D.9.

In a similar manner we prove the tightness of laws for \( \sqrt{\varrho_n} \). In what follows, if \( X \) be a topological vector space, then by \( (X, w) \) we mean that the set \( X \) is equipped with weak topology. If additionally, \( X \) is isomorphic to a dual of another topological vector space, then by \( (X, w^*) \), we denote the set \( X \) equipped with the weak-star topology.

**Lemma 4.2.** Assume that \( 1 \leq r < 6 \). Then the family of measures \( \{ \mathcal{L}(\sqrt{\varrho_n}) : n \in \mathbb{N} \} \) is tight on \( C([0,T]; L^r(\mathcal{O})) \cap (L^\infty(0,T; H^1(\mathcal{O})), w^*) \).

**Proof of Lemma 4.2.** Let us fix \( r \in [1,6) \). To verify tightness in the space \( C([0,T]; L^r(\mathcal{O})) \) we proceed as previously, using the compactness of the embedding \( H^1(\mathcal{O}) \hookrightarrow L^r(\mathcal{O}) \) and noticing that \( \sqrt{\varrho_n} \) satisfies the equation

\[
\partial_t\sqrt{\varrho_n} + \text{div}(\sqrt{\varrho_n} \mathbf{u}_n) - \frac{1}{2} \sqrt{\varrho_n} \text{div} \mathbf{u}_n = 0. \tag{4.13}
\]

Since, by the Banach-Alaoglu theorem a closed ball in \( L^\infty(0,T; H^1) \cong (L^1(0,T; H^{-1}))^* \) is compact in the weak-star topology, the tightness of \( \{ \mathcal{L}(\sqrt{\varrho_n}) : n \in \mathbb{N} \} \) on the space \( (L^\infty(0,T; H^1(\mathcal{O})), w^*) \) follows straight from the Chebyshev inequality and a priori estimate (4.4).

**Lemma 4.3.** Let \( r \in [1,\infty) \) and \( \alpha \in \left[\frac{3}{2},\frac{3}{2}\right] \). Then the family of measures \( \{ \mathcal{L}(\varrho_n \mathbf{u}_n) : n \in \mathbb{N} \} \) is tight on \( L^r(0,T; L^\alpha(\mathcal{O})) \).

**Proof of Lemma 4.3.** Let us fix \( r \in [1,\infty) \) and \( \alpha \in \left[\frac{3}{2},\frac{3}{2}\right] \). Let us also choose and fix auxiliary exponents \( p \in [1,\infty) \) and \( p^* \in [1,2) \) such that \( p^* = \frac{2p}{p+2} \), i.e.

\[
\frac{1}{p} + \frac{1}{2} = \frac{1}{p^*}.
\]

From the uniform estimates (4.2) and (4.4) and the Sobolev embedding \( H^1(\mathcal{O}) \hookrightarrow L^6(\mathcal{O}) \), we infer that \( \sqrt{\varrho_n} \mathbf{u}_n \in L^p(\Omega; L^\infty(0,T; L^2(\mathcal{O}))) \) and \( \sqrt{\varrho_n} \in L^2(\Omega; L^\infty(0,T; L^6(\mathcal{O}))) \) and are uniformly bounded in the respective spaces. Therefore, by applying the Hölder inequality, see inequality (C.4), we infer that

\[
\varrho_n \mathbf{u}_n \in L^{p^*}(\Omega; L^\infty(0,T; L^{3/2}(\mathcal{O}))) \text{ and is uniformly bounded.} \tag{4.14}
\]
We can also prove that $\nabla (\varrho_n u_n) \in L^p(\Omega; L^2(0, T; L^1(\mathcal{O})))$ uniformly. Indeed, since
\[
\nabla (\varrho_n u_n) = 2(\nabla \sqrt{\varrho_n})(\sqrt{\varrho_n} u_n) + \sqrt{\varrho_n}(\sqrt{\varrho_n} \nabla u_n)
\]
it is sufficient to prove that the first term on the r.h.s. above belongs to $L^p(\Omega; L^\infty(0, T; L^1(\mathcal{O})))$ and is uniformly bounded and the second term belongs to $L^p(\Omega; L^2(0, T; L^{3/2}(\mathcal{O})))$ and is uniformly bounded.

For the first one, in view of the Hölder inequality, see (C.6), it is sufficient to observe that by (4.2) and (4.4), $\sqrt{\varrho_n} u_n \in L^p(\Omega; L^\infty(0, T; L^2(\mathcal{O})))$ and $\nabla \sqrt{\varrho_n} \in L^2(\Omega; L^{\infty}(0, T; L^2(\mathcal{O})))$ (uniformly). For the second one, we observe that as proved above $\sqrt{\varrho_n} \in L^2(\Omega; L^\infty(0, T; L^6(\mathcal{O})))$ (uniformly) and, by (4.3), $\sqrt{\varrho_n} \nabla u_n \in L^p(\Omega; L^2(0, T; L^2(\mathcal{O})))$ (also uniformly), the claim follows by applying (C.5).

Hence we can conclude that
\[
\varrho_n u_n \in L^p(\Omega; L^2(0, T; W^{1,1}(\mathcal{O}))) \text{ and is uniformly bounded.}
\]

Next we will verify that the sequence of random variables $\varrho_n u_n$ satisfies the Aldous condition [A], see Definition D.12, in a Banach space $Y$, which will be chosen later in the proof. For this aim we choose and fix $\psi \in C^\infty(\mathcal{O})$. Then the momentum equation of system of (1.1) gives us
\[
\int_{\mathcal{O}} \varrho_n(t, x) u_n(t, x) \psi(x) \, dx - \int_{\mathcal{O}} m^0_0 \cdot \psi(x) \, dx
\]
\[
= \int_0^t \int_{\mathcal{O}} \varrho_n(s, x) u_n(s, x) \cdot \nabla \psi(x) \, dx \, ds - \int_0^t \int_{\mathcal{O}} \varrho_n(s, x) \nabla u_n(s, x) \cdot \nabla \psi(x) \, dx \, ds
\]
\[
+ \int_0^t \int_{\mathcal{O}} \varrho_n(s, x) \text{div} \psi(x) \, dx \, ds + \int_0^t \int_{\mathcal{O}} \varrho_n(s, x) \mathbf{f}(x) \psi(x) \, dx \, dW(s),
\]
\[
:= J_1^\psi(t) + J_2^\psi(t) + J_3^\psi(t) + J_4^\psi(t), \text{ for all } t \in [0, T]
\]
and therefore
\[
\left| \int_{\mathcal{O}} \varrho_n((\tau_n + h) \wedge T, x) u_n((\tau_n + h) \wedge T, x) \psi(x) \, dx - \int_{\mathcal{O}} \varrho_n(\tau_n, x) u_n(\tau_n, x) \psi(x) \, dx \right|
\]
\[
\leq \sum_{k=1}^4 \left| J_k^\psi((\tau_n + h) \wedge T) - J_k^\psi(\tau_n) \right|
\]
for any sequence $(\tau_n)_{n \in \mathbb{N}}$ of $[0, T]$-valued stopping times and $h > 0$. Hence, for every $\eta > 0$,
\[
\mathbb{P} \left\{ \| \varrho_n((\tau_n + h) \wedge T) u_n((\tau_n + h) \wedge T) - \varrho_n(\tau_n) u_n(\tau_n) \|_Y \geq \eta \right\}
\]
\[
\leq \sum_{k=1}^4 \mathbb{P} \left\{ \sup_{\psi \in Y'} \left| J_k^\psi((\tau_n + h) \wedge T) - J_k^\psi(\tau_n) \right| \geq \frac{\eta}{4} \right\},
\]
where $Y'$ is the functional dual of $Y$. We aim to apply the Chebyshev inequality and estimate the expected value of each term in the sum.

In order to continue with our proof we choose and fix: $p \in [1, \infty)$ and $p^* \in [1, 2)$. We claim that due to the uniform estimates (4.2)–(4.4), the sequences below are uniformly bounded in the corresponding spaces:
(i) $\varrho_n u_n \otimes u_n$ in $L^p(\Omega; L^\infty(0, T; L^1(\mathcal{O})))$;
(ii) $\varrho_n^2$ in $L^p(\Omega; L^\infty(0, T; L^1(\mathcal{O})))$.
(iii) $\varrho_n \nabla u_n$ in $L^p(\Omega; L^2(0, T; L^2(\mathcal{O})))$.

With all of these uniform bounds we will analyse each term in (4.17) individually.

\[
E \left| J_1^\psi((\tau_n + h) \land T) - J_1^\psi(\tau_n) \right| = E \int_{\tau_n}^{(\tau_n+h)\land T} \int_\mathcal{O} (\varrho_n u_n \otimes u_n) : \nabla \psi \, dx \, ds \\
\leq h E \left( \sup_{s \in [0,T]} \| \varrho_n u_n \otimes u_n \|_{L^1(\mathcal{O})} \right) \| \nabla \psi \|_{L^\infty(\mathcal{O})} \leq C_T \| \psi \|_{W^{1,\infty}(\mathcal{O})} h. \tag{4.18}
\]

For the second term $J_2^\psi$ we have

\[
E \left| J_2^\psi((\tau_n + h) \land T) - J_2^\psi(\tau_n) \right| = E \int_{\tau_n}^{(\tau_n+h)\land T} \int_\mathcal{O} \varrho_n \nabla u_n(s, x) \cdot \nabla \psi \, dx \, ds \\
\leq h^{1/2} E \left( \int_{\tau_n}^{(\tau_n+h)\land T} \| \varrho_n \nabla u_n \|_{L^2} \| \nabla \psi \|_{L^3} \, ds \right)^{1/2} \| \nabla \psi \|_{L^3} \leq C_T \| \psi \|_{W^{1,3}(\mathcal{O})} h^{1/2}. \tag{4.19}
\]

For the pressure term, we get

\[
E \left| J_3^\psi((\tau_n + h) \land T) - J_3^\psi(\tau_n) \right| = E \int_{\tau_n}^{(\tau_n+h)\land T} \int_\mathcal{O} \varrho_n^2 \text{div} \psi \, dx \, ds \\
\leq h E \left( \sup_{s \in [0,T]} \| \varrho_n^2 \|_{L^1} \| \text{div} \psi \|_{L^\infty} \, ds \right) \tag{4.20}
\]

\[
\leq h E \left( \sup_{s \in [0,T]} \| \varrho_n^2 \|_{L^1} \right) \| \text{div} \psi \|_{L^\infty} \leq C_T \| \psi \|_{W^{1,\infty}(\mathcal{O})} h.
\]

Note in particular that so far the most restrictive assumption on $\psi$ has been made during the estimates on $J_1^\psi$, $J_3^\psi$ and for those we have that

\[W^{3,2}(\mathcal{O}) \hookrightarrow W^{1,\infty}(\mathcal{O}).\]

Finally, for the stochastic term, using the BDG inequality, assumptions on $f$ and uniform estimates (4.4), we obtain

\[
E \left| J_4^\psi((\tau_n + h) \land T) - J_4^\psi(\tau_n) \right| = E \int_{\tau_n}^{(\tau_n+h)\land T} \int_\mathcal{O} \varrho_n f \cdot \psi \, dx \, dW \\
\leq E \left( \int_{\tau_n}^{(\tau_n+h)\land T} \left[ \int_\mathcal{O} \varrho_n f \cdot \psi \, dx \right]^2 \, ds \right)^{1/2} \\
\leq E \left( \int_{\tau_n}^{(\tau_n+h)\land T} \| \varrho_n \|_{L^2}^2 \| f \|_{L^2} \| \psi \|_{L^\infty} \, ds \right)^{1/2}.
\]

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\[
\leq h^{1/2}E \left( \sup_{s \in [0,T]} \| \varrho_n \|_{L^3} \right)^{1/2} \left( \| f \|_{L^{3/2}} \right)^{1/2} \| \psi \|_{L^\infty}
\leq h^{1/2} \left( \mathbb{E} \left( \sup_{s \in [0,T]} \| \varrho_n \|_{L^3} \right) \right)^{1/2} \left( \mathbb{E} \| f \|_{L^{3/2}}^2 \right)^{1/2} \| \psi \|_{L^\infty} \leq C_{T, f} \| \psi \|_{L^\infty} h^{1/2}.
\]

(4.21)

By the Chebyshev inequality, we obtain for given \( \eta > 0 \)
\[
\mathbb{P} \left\{ \sup_{\psi \in Y'} \left| J^\psi_k ((\tau_n + h) \wedge T) - J^\psi_k (\tau_n) \right| \geq \frac{\eta}{4} \right\}
\leq \frac{4}{\eta} \sup_{\psi \in Y'} \mathbb{E} \left| J^\psi_k ((\tau_n + h) \wedge T) - J^\psi_k (\tau_n) \right|, \quad k = 1, \ldots, 4.
\]

Therefore, for \( Y = W^{-3,2}(O) \), using estimates (4.18)–(4.21) in (4.17), we have shown that
\[
\mathbb{P} \{ \| \varrho_n((\tau_n + h) \wedge T) u_n ((\tau_n + h) \wedge T) - \varrho_n(\tau_n) u_n(\tau_n) \|_Y \geq \eta \} \leq \frac{8}{\eta} C h^{1/2},
\]
where \( C \) is a constant independent of \( h \). Let us fix \( \varepsilon > 0 \). With the above estimate we can choose \( \theta > 0 \) such that
\[
\mathbb{P} \{ \| \varrho_n((\tau_n + h) \wedge T) u_n ((\tau_n + h) \wedge T) - \varrho_n(\tau_n) u_n(\tau_n) \|_Y \geq \eta \} \leq \varepsilon
\]
for all \( n \in \mathbb{N} \) and \( 0 < h \leq \theta \) and therefore, the Aldous condition \([A]\) holds in \( Y = W^{-3,2}(O) \).

Therefore by Lemma A.2 there exists a measurable subset \( A_\varepsilon \subset L^1(0,T;Y) \) such that
\[
\mathbb{P}^{X_n}(A_\varepsilon) \geq 1 - \frac{\varepsilon}{2}, \quad \lim_{h \to 0} \sup_{u \in A_\varepsilon} \| \tau_n u - u \|_{L^1(0,T;Y)} = 0.
\]

Thus, we can apply Lemma D.11 with \( X = W^{1,1}(O) \), \( B = L^{3/2}(O) \) and \( Y = W^{-3,2}(O) \), and set \( F = \{ \varrho_n(\omega) u_n(\omega) \}_{n \in \mathbb{N}} \) to show that the set \( F \) is relatively compact in \( L'((0,T);L^3(O)) \).

Let \( \eta > 0 \) and define the following closed ball \( B_\eta \)
\[
B_\eta := \{ v \in F : \| v \|_{L^\infty((0,T);L^3(O))} \leq \eta \}.
\]

Now, by the Chebyshev inequality and uniform bounds from above, we have
\[
\mathbb{P}(\{ \Omega : \varrho_n(\omega) u_n(\omega) \in B_\eta \}) \leq \frac{1}{\eta} \mathbb{E} \| \varrho_n u_n \|_{L^\infty((0,T);L^3(O))},\ |L^1(0,T;W^{1,1})} \leq \frac{C}{\eta}.
\]

(4.22)

Taking \( \varepsilon = \frac{C}{\eta} \), and choosing \( K_\varepsilon = B_{1/\varepsilon} \) we obtain from the first part that \( K_\varepsilon \) is a relatively compact set in \( L'((0,T);L^3(O)) \) and hence we establish tightness of laws of \( \varrho_n u_n \) on \( L'((0,T);L^3(O)) \) directly from the Definition D.9.

\[ \square \]

In order to prove the tightness of laws induced by \( \varrho_n u_n \) we will take the approach of Breit et al., see [3, Proposition 4.3.8], namely we will prove the time regularity of \( \varrho_n u_n \), which holds uniformly in \( n \) and we will do so by proving the time regularity for the deterministic part and the stochastic part of (4.16) separately.
Proposition 4.4. Let \( T > 0 \). Then there exists a \( C > 0 \) and \( \kappa \in (0, \frac{1}{2}) \) such that for every \( n \in \mathbb{N} \), the process \( v \) defined by the deterministic part of \( (4.16) \), i.e. by

\[
\langle Y_n(t), \psi \rangle := \int_{\mathcal{O}} m_n^0 \cdot \psi \, dx - \int_0^t \int_{\mathcal{O}} (\varrho_n u_n \otimes u_n) \cdot \nabla \psi \, dx \, ds \\
+ \int_0^t \int_{\mathcal{O}} \varrho_n \nabla u_n \cdot \nabla \psi \, dx \, ds - \int_0^t \int_{\mathcal{O}} \varrho_n \text{div} \psi \, dx \, ds, \quad \psi \in W^{3,2}(\mathcal{O}),
\]

satisfies

\[
\mathbb{E}\|Y_n\|_{C^\kappa([0,T];W^{-2,2}(\mathcal{O}))} \leq C.
\]

Proof of Proposition 4.4. Proposition 4.4 can be proved by following the steps from the first half of the proof of Lemma 4.3.

Let us recall that if \( U \) is a Banach space, then by \( C^\infty([0,T];U) \) we denote the Banach space of all continuous functions \( u : [0,T] \to U \) such that

\[
\|u\|_{C^\infty([0,T];U)} := |u(0)|_U + \sup\left\{ \frac{|u(t) - u(s)|_U}{|t - s|^\kappa} : s < t \in [0,T] \right\} < \infty.
\]

The above norm is equivalent to another one, more commonly used, in which the term \( |u(0)|_U \) is replaced by \( \sup\{ |u(t)|_U : t \in [0,T] \} \).

Proposition 4.5. Let \( T > 0 \). Then there exists \( C' > 0 \) and \( \kappa^* > 0 \) such that for every \( n \in \mathbb{N} \), the process \( M_n \) defined by

\[
M_n(t) := \int_0^t \int_{\mathcal{O}} \varrho_n f \, dx \, dW, \quad t \in [0,T],
\]

satisfies, \( \mathbb{P} \)-almost surely, \( M_n \in C^{\kappa^*}([0,T];W^{-2,2}(\mathcal{O})) \) and

\[
\mathbb{E}\|M_n\|_{C^{\kappa^*}([0,T];W^{-2,2}(\mathcal{O}))} \leq C'.
\]

Proof of Proposition 4.5. Let us begin the proof with the following observation which is a consequence of the Hölder inequality and of the continuity the embedding \( W^{-2,2}(\mathcal{O}) \to L^\infty(\mathcal{O}). \)

\[
\|\varrho f\|_{W^{-2,2}(\mathcal{O})} = \sup\{ \int_{\mathcal{O}} \varrho(x) f(x) \psi(x) \, dx : |\psi|_{W^{2,2}(\mathcal{O})} \leq 1 \} \leq \sup\{ \|\varrho\|_{L^\infty} \|f\|_{L^{3/2}} \|\psi\|_{L^\infty} : |\psi|_{W^{2,2}(\mathcal{O})} \leq 1 \} \leq C' \|\varrho\|_{L^3} \|f\|_{L^{3/2}}.
\]

Let us fix \( T > 0 \) and \( q > 2 \). Then for \( 0 \leq s < t \leq T \). Since \( W^{-2,2}(\mathcal{O}) \) is a Hilbert space, by applying the Doob inequality we get

\[
\mathbb{E}\left( |M_n(t) - M_n(s)|_{W^{-2,2}(\mathcal{O})}^q \right) = \mathbb{E}\left( \int_s^t \varrho_n(r) f \, dW(r) \right)^q_{W^{-2,2}(\mathcal{O})} \leq C_q \mathbb{E}\left( \int_s^t \varrho_n(r) f \, dW(r) \right)^q_{W^{-2,2}(\mathcal{O})} \leq C_q \mathbb{E}\left( \int_s^t \varrho_n(r) f \, dW(r) \right)^q_{W^{-2,2}(\mathcal{O})} \leq C_q |t - s|^{q/2} \mathbb{E}\left( \sup_{r \in [0,T]} \|\varrho_n(r)\|_{L^3} \|f\|_{L^{3/2}}^q \right).
\]
\[ CC_q |t - s|^{q/2} \left( \sqrt{\mathbb{E} \sup_{r \in [0,T]} \|g_n(r)\|_L^q} \right)^{1/2} \left( \mathbb{E} \|f\|_{L^{3/2}}^q \right)^{1/2} \leq C |t - s|^{q/2}. \]

Since the above estimate holds true for every \( q > 2 \) and hence we can conclude by the Kolmogorov continuity theorem, see Theorem D.7, that there exists \( \kappa^* > 0 \) such that \( M_n \in C^{\kappa^*}([0,T]; W^{-2,2}(O)) \) almost surely and (4.25) holds. The proof is now complete. \[ \square \]

**Lemma 4.6.** The family of measures \( \{ \mathcal{L}(\varrho_n u_n) : n \in \mathbb{N} \} \) is tight on \( C_w([0,T]; L^{3/2}(O)) \).

**Proof of Lemma 4.6.** By Proposition 4.4 and Proposition 4.5 we infer that for some \( \kappa > 0 \),
\[
\varrho_n u_n \in L^1(\Omega; C^{\kappa}([0,T]; W^{-3,2}(O))).
\]
Moreover, in the proof of Lemma 4.3 we showed that \( \varrho_n u_n \in L^1(\Omega; L^\infty(0,T; L^{3/2}(O))) \). Thus, proof of the lemma can be concluded by observing that the embedding
\[
L^\infty(0,T; L^{3/2}(O)) \cap C^{\kappa}([0,T]; W^{-3,2}(O)) \hookrightarrow C_w([0,T]; L^{3/2}(O))
\]
is compact; which follows directly from Lemma A.4, and by application of the Chebyshev inequality. \[ \square \]

**Lemma 4.7.** Assume that \( \delta \in (0,1) \) is as in Lemma 3.6. The families of measures \( \{ \mathcal{L}(\varrho_n u_n) : n \in \mathbb{N} \} \), and resp. \( \{ \mathcal{L}(\tilde{\varrho}_n u_n) : n \in \mathbb{N} \} \), are tight on \( L^\infty(0,T; L^2_w(O)) \), resp. \( L^\infty(0,T; L^{2,4}w^\kappa(O)) \).

**Proof of Lemma 4.6.** Tightness of \( \{ \mathcal{L}(\sqrt{\varrho}_n u_n) : n \in \mathbb{N} \} \) in \( (L^\infty(0,T; L^2(O)), w^*) \) follows directly from the Banach-Alaoglu theorem, the Chebyshev inequality and a priori estimates (4.2), as in Lemma 4.2. Similarly we deduce tightness of \( \{ \mathcal{L}(\tilde{\varrho}_n u_n) : n \in \mathbb{N} \} \) using a priori estimates (4.6).

For \( 1 \leq r < \infty, \ \delta \in (0,1), \ q \in [1,3], \ \alpha \in [1, \frac{3}{2}) \), we introduce the following notation
\[
X_r := C([0,T]; L^q(O)),
\]
\[
X_{\sqrt{\varrho}} := C([0,T]; L^2(O)) \cap (L^\infty(0,T; H^1(O)), w^*),
\]
\[
X_{\varrho u} := L'(0,T; L^2(O)) \cap C_w([0,T]; L^{3/2}(O)),
\]
\[
X_{\sqrt{\varrho} u} := (L^\infty(0,T; L^2(O)), w^*),
\]
\[
X_{\tilde{\varrho} u} := \left( L^\infty(0,T; L^{2+\delta}(O)), w^* \right),
\]
\[
X_W := C([0,T]; \mathbb{R})
\]
and
\[
X_T := X_r \times X_{\sqrt{\varrho}} \times X_{\varrho u} \times X_{\sqrt{\varrho} u} \times X_{\tilde{\varrho} u} \times X_W.
\]

Let us denote by \( \mu_{\varrho_n}, \mu_{\sqrt{\varrho}_n}, \mu_{\varrho_n u_n}, \mu_{\sqrt{\varrho}_n u_n}, \mu_{\tilde{\varrho}_n u_n} \) and \( \mu_W \) the laws of \( \varrho_n, \sqrt{\varrho}_n, \varrho_n u_n, \sqrt{\varrho}_n u_n, \tilde{\varrho}_n u_n \) and \( W_n \) respectively, on the corresponding path space. The joint law of all variables on \( X_T \) is denoted by \( \mu_n \).

Using the above notation one could rewrite previous results on tightness in a bit more compact way. For this let us recall definition (1.7) of a sequence \( (\mu_n)_{n=1}^{\infty} \).

From Lemmata 4.1, 4.2, 4.3, 4.6 and 4.6 we can deduce the following corollary.
Corollary 4.8. The family of measures \( \{ \mathcal{L}(\mu_n) : n \in \mathbb{N} \} \) is tight on the space \( X_T \) defined above in (4.33).

Having this we can apply the Skorokhod representation theorem in order to deduce the existence and convergence of the random variables on new probability spaces. Below we state the generalised version of the Skorokhod-Jakubowski Theorem in the framework of non-metric spaces, see [33].

Theorem 4.9 (Skorokhod-Jakubowski). Let \( (Z, \mathcal{Z}) \) be a topological space such that there exists a sequence \( (f_m)_{m \in \mathbb{N}} \) of continuous functions \( f_m : Z \to \mathbb{R} \) that separate points of \( Z \). Let \( \mathcal{S} = \sigma((f_m)_{m \in \mathbb{N}}) \) be the \( \sigma \)-algebra generated by \( (f_m)_{m \in \mathbb{N}} \). Then

1. Every compact subset of \( Z \) is metrizable.

2. If \( (\mu_m)_{m \in \mathbb{N}} \) is a tight sequence of probability measures on \( (Z, \mathcal{S}) \), then there exists a subsequence \( (\mu_{m_k})_{k \in \mathbb{N}} \), a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), and \( Z \)-valued Borel measurable random variables \( w_k \) and \( w \) such that (i) \( \mu_{m_k} \) is the law of \( w_k \) and (ii) \( w_k \to w \) almost surely on \( \Omega \).

Remark 4.10. Let us observe that in the framework of Theorem 4.9,

\[ \mathcal{S} \subset \mathcal{B}(Z), \]

where \( \mathcal{B}(Z) := \sigma(Z) \) is the Borel \( \sigma \)-field on \( Z \). It may happen, see e.g. Remark 4.11, that the equality holds as well.

In the remaining parts of this subsection we will show that Theorem 4.9 is applicable in the context of our Corollary 4.8.

Remark 4.11. Assume that \( X \) is a separable Banach space with it’s dual denoted by \( X^* \). For example, \( X = L^1(0,T;L^r(\mathcal{O})) \) so that \( X^* = L^\infty(0,T;L^{r'}(\mathcal{O})) \), where we suppose that \( r \in (1,\infty) \) and \( 1/r' + 1/r = 1 \).

Let us point out that on the space \( X^* \) equipped with the \( w^* \) topology, there exists a sequence \( (f_n)_{n=1}^\infty \) of continuous \( \mathbb{R} \)-valued functionals which separate points. Indeed, it is sufficient to consider a sequence \( (x_n)_{n=1}^\infty \) which is dense in the unit ball of \( X \), and define

\[ f_n : X^* \ni x^* \mapsto \langle x^*, x_n \rangle \in \mathbb{R}, \]

where \( \langle \cdot, \cdot \rangle \) denotes the duality between \( X^* \) and \( X \). Note that the sequence \( (f_n)_{n=1}^\infty \) separates the points on \( X^* \). It follows, that the space \( (X^*, w^*) \) satisfies the assumptions of the Skorokhod-Jakubowski Theorem from [33], see Theorem 4.9.

By [13, Theorem 3.28] every ball in \( X^* \) equipped with the weak* topology is metrizable. Moreover, by inspection of the proof of [13, Theorem 3.28] the weak* topology on ball in \( X^* \) is generated by the functionals \( (f_n) \). Therefore, the Borel \( \sigma \)-field on \( X^* \) generated by the \( w^* \) topology, is equal to the \( \sigma \)-field generated by the functionals \( (f_n) \).

Since a ball in \( X^* \) equipped with the weak* topology is also compact by the Banach-Alaoglu Theorem, we infer that it is a Polish compact space.

Moreover, if \( Y \) is a topological space (with \( \sigma(Y) \) denoting the smallest \( \sigma \)-field on \( Y \) containing all open subsets) and function \( F : X^* \to Y \) is sequentially continuous, then \( F \) is \( \mathcal{B}(X^*, w^*)/\sigma(Y) \) measurable. Indeed, for every \( r > 0 \), the restriction \( F|_{B_{X^*}(r)} \) of \( F \) to the
Lemma 4.13. To simplify the notation we are going to denote a subsequence \( \tilde{\mu} \)

Proof of Proposition 4.12. There exists a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) with \(\mathcal{X}_T\)-valued Borel-measurable random variables \(\tilde{\mu}_n = (\tilde{q}_n, \tilde{\vartheta}_n, \tilde{m}_n, \tilde{q}_n, \tilde{r}_n, \tilde{W}_n)\), \(n \in \mathbb{N}\) and \(\tilde{\mu} = (\tilde{q}, \tilde{\vartheta}, \tilde{m}, \tilde{q}, \tilde{r}, \tilde{W})\) and there exists a subsequence \((\mu_{k_n})_{n=1}^{\infty}\), such that the following hold:

1. The law of \(\tilde{\mu}_n\) is equal to the law of \(\mu_{k_n}\), \(n \in \mathbb{N}\).

2. \(\tilde{\mu}_n\) converges \(\tilde{\mathbb{P}}\)-almost surely in \(\mathcal{X}_T\) to \(\tilde{\mu}\).

Proof of Proposition 4.12. This result is a direct consequence of Corollary 4.8 and Theorem 4.9\(^1\)

In view of the definition \((4.33, 4.27, 4.28, 4.29)\) of the space \(\mathcal{X}_T\), the property (2) from Proposition 4.12 implies that the following hold \(\tilde{\mathbb{P}}\)-almost surely,

\[
\begin{align*}
\tilde{q}_n(0) &\to \tilde{q}(0) \text{ in } L^q(\mathcal{O}), \\
\tilde{\vartheta}_n(0) &\to \tilde{\vartheta}(0) \text{ in } L^{2q}(\mathcal{O}), \\
\tilde{m}_n(0) &\to \tilde{m}(0) \text{ weakly in } L^{3/2}(\mathcal{O}).
\end{align*}
\]

4.2 Identification of the limiting random variables

We are now going to identify \(\tilde{\vartheta}_n, \tilde{m}_n, \tilde{q}_n\) and \(\tilde{\vartheta}, \tilde{m}, \tilde{q}\).

In the remaining parts of this subsection we choose and fix \(\delta \in (0, 1)\) and \(q \in [1, 3]\) such that

\[
q(1 + 2\delta) \geq 3(1 + \delta). \tag{4.38}
\]

To simplify the notation we are going to denote a subsequence \(\mu_{k_n}\) by \(\mu_n\), for \(n \in \mathbb{N}\).

Lemma 4.13. \(\tilde{\mathbb{P}}\)-a.s. the following equalities hold for every \(n \in \mathbb{N}\), a.e. in \(\mathcal{O} \times (0, T)\)

\[
(\tilde{\vartheta}_n, \tilde{m}_n, \tilde{q}_n) = (\sqrt{\tilde{q}_n}, \sqrt{\tilde{q}_n^{1+\delta}}, \sqrt{\tilde{q}_n^{2(2+\delta)}}, \tilde{r}_n).
\]

Proposition 4.14. Under assumptions of Lemma 4.13, \(\tilde{\mathbb{P}}\)-a.s. the following identity holds in \(\mathcal{O} \times (0, T)\)

\[
(\tilde{\vartheta}, \tilde{m}, \tilde{q}) = (\tilde{\vartheta}, \sqrt{\tilde{q}^{1+\delta}}, \sqrt{\tilde{q}^{2(2+\delta)}}, \tilde{r}).
\]

Proof of Lemma 4.13. Let us choose and fix \(n \in \mathbb{N}\). Let us denote, on \(\mathcal{B}(\mathcal{X}_q \times \mathcal{X}_{\sqrt{\tilde{q}}})\),

\[
\Upsilon_n := \mathcal{L}(\tilde{q}_n, \sqrt{\tilde{q}_n}), \quad \tilde{\Upsilon}_n := \mathcal{L}(\tilde{q}_n, \tilde{\vartheta}_n).
\]

\(^1\)The assumptions of the theorem can be verified similarly as in [21, Corollary 3.12].
Since by Proposition 4.12
\[
\mathcal{L} \left( \tilde{\varrho}_n, \tilde{\omega}_n, \tilde{m}_n, \tilde{q}_n, \tilde{r}_n \right) = \mathcal{L} \left( \varrho_n, \sqrt{\varrho_n}, \varrho_n u_n, \sqrt{\varrho_n} u_n, \varrho_n^{\frac{1}{n}} u_n \right) \text{ on } \mathcal{B}(\mathcal{X}_T),
\]
we infer that \( \tilde{\Upsilon}_n = \Upsilon_n \) on \( \mathcal{B}(\mathcal{X}_\varrho \times \mathcal{X}_\varrho) \). Since \( \varrho_n = \left( \sqrt{\varrho_n} \right)^2 \), \( \mathbb{P}\text{-a.s.} \) we infer that
\[
\mathbb{E} \| \varrho_n - \left( \sqrt{\varrho_n} \right)^2 \|_{L^1(0,T;L^1(\Omega))} = 0.
\]
Let us observe that the map
\[
\beta : \mathcal{X}_\varrho \ni x \mapsto x^2 \in \mathcal{X}_\varrho \quad \text{is continuous (4.41)}
\]
and hence Borel measurable. Moreover, the map \( \| \cdot \| : \mathcal{X}_\varrho \to \mathbb{R} \) is continuous, and linear combination of measurable maps is measurable. Combining all this with equality \( \Upsilon_n = \tilde{\Upsilon}_n \) on \( \mathcal{B}(\mathcal{X}_\varrho \times \mathcal{X}_\varrho) \) we get following chain of identities
\[
0 = \mathbb{E} \| \varrho_n - \beta(\sqrt{\varrho_n}) \|_{\mathcal{X}_\varrho} = \int_\Omega \| \varrho_n(\omega) - \beta(\sqrt{\varrho_n}(\omega)) \|_{\mathcal{X}_\varrho} d\mathbb{P}(\omega)
= \int_{\mathcal{X}_\varrho \times \mathcal{X}_\varrho} \| x - \beta(y) \|_{\mathcal{X}_\varrho} d\Upsilon_n(x,y) = \int_{\mathcal{X}_\varrho \times \mathcal{X}_\varrho} \| x - \beta(y) \|_{\mathcal{X}_\varrho} d\tilde{\Upsilon}_n(x,y)
= \int_\Omega \| \tilde{\varrho}_n(\omega) - \beta(\tilde{\varrho}_n(\omega)) \|_{\mathcal{X}_\varrho} d\mathbb{P}(\omega) = \mathbb{E} \| \tilde{\varrho}_n - \beta(\tilde{\varrho}_n) \|_{\mathcal{X}_\varrho}.
\]
Thus we infer that
\[
\tilde{\varrho}_n \text{-a.s. } \tilde{\varrho}_n = \sqrt{\varrho_n} \text{ in } \mathcal{X}_\varrho. \quad \text{(4.42)}
\]

We will use a similar argument to identify \( \tilde{m}_n \). We start by defining a map
\[
\Gamma : \mathcal{X}_\varrho \times \mathcal{X}_\varrho \ni (\nu, \psi) \mapsto \nu \frac{1+4}{1+3} \psi \in \left( L^\infty(0,T;L^{3/2}(\Omega)), u^* \right). \quad \text{(4.43)}
\]

Firstly, we observe that \( \Gamma \) is a well-defined function, see also Lemma B.1. Secondly by Lemma B.1, we infer that for every \( \psi \in L^1(0,T;L^3(\Omega)) \), the function
\[
\mathcal{X}_\varrho \times \mathcal{X}_\varrho \ni (\nu, \psi) \mapsto \langle \Gamma(\nu, \psi), \psi \rangle \in \mathbb{R},
\]
where \( \langle \cdot, \cdot \rangle \) denotes the duality between \( L^\infty(0,T;L^{3/2}(\Omega)) \) and \( L^1(0,T;L^3(\Omega)) \), is sequentially continuous.

Due to the sequential continuity of the map \( \Gamma \), the map \( g \)
\[
g : \mathcal{X}_\varrho \times \mathcal{X}_\varrho \ni (\nu, \psi) \mapsto \int_0^T \int_\Omega (\nu(t,x) - \Gamma(\nu(t,x), \psi(t,x))) \psi(t,x) dx dt
\]
is sequentially continuous too and hence, by Remark 4.11, measurable. Since, for every \( \omega \in \Omega \)
\[
\varrho_n u_n = \varrho_n^{\frac{1}{n}} \frac{1}{n} u_n,
\]
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by the measurability of the map $g$ and of random variables $\varrho_n, \varrho_n^{-1} u_n$ and $\varrho_n u_n$ from $\Omega$ to $X_\varrho, X_{\varrho^{-1} u}$ and $X_{\varrho u}$ respectively, we infer that for every $\psi \in L^1(0, T; L^3(\Omega))$

$$
\mathbb{E}g(\varrho_n, \varrho_n^{-1} u_n, \varrho_n u_n, \psi) = \mathbb{E} \int_0^T \int_{\Omega} \left( \varrho_n u_n(t, x) - \varrho_n^{-1} u_n(t) \varrho_n^{-1}(t, x) u_n(t, x) \right) \cdot \psi(t, x) \, dx \, dt = 0. \tag{4.45}
$$

Let us denote $\mathcal{L} \left( \varrho_n, \varrho_n^{-1} u_n, \varrho_n u_n \right)$ and $\mathcal{L} (\tilde{\varrho}_n, \tilde{\varrho}_n, \tilde{m}_n)$ on $\mathcal{B}(X_\varrho \times X_{\varrho^{-1} u} \times X_{\varrho u})$ by $\nu_n$ and $\tilde{\nu}_n$ respectively. From the equality of joint laws of

$$
\left( \varrho_n, \sqrt{\varrho_n} \varrho_n u_n, \sqrt{\varrho_n} \varrho_n u_n, \varrho_n^{-1} u_n \right) \quad \text{and} \quad \left( \tilde{\varrho}_n, \tilde{\theta}_n, \tilde{m}_n, \tilde{q}_n, \tilde{r}_n \right) \tag{4.46}
$$

on $X_T$, which is a consequence of Proposition 4.12, we have

$$
\nu_n = \tilde{\nu}_n \quad \text{on} \quad \mathcal{B} \left( X_\varrho \times X_{\varrho u} \times X_{\varrho^{-1} u} \right). \tag{4.47}
$$

Thus, from identity (4.45) and equality of laws (4.47) we have for $\psi \in L^1(0, T; L^3(\Omega))$

$$
0 = \mathbb{E} \int_0^T \int_{\Omega} \left( \varrho_n u_n(t, x) - \varrho_n^{-1} u_n(t) \varrho_n^{-1}(t, x) u_n(t, x) \right) \cdot \psi(t, x) \, dx \, dt
\begin{align*}
&= \int_\Omega \left( \int_0^T \int_{\Omega} \left( \varrho_n u_n(\omega) - \Gamma(\varrho_n(\omega), \varrho_n^{-1} u_n(\omega)) \right) \cdot \psi \, dx \, dt \right) \, d\mathbb{P}(\omega) \\
&= \int_{X_\varrho \times X_{\varrho^{-1} u} \times X_{\varrho u}} \left( \int_0^T \int_{\Omega} (z - \Gamma(v, w)) \cdot \psi \, dx \, dt \right) \, d\nu_n(v, w, z)
\end{align*}
$$

$$
= \int_{X_\varrho \times X_{\varrho^{-1} u} \times X_{\varrho u}} g(v, w, z, \psi) \, d\nu_n(v, w, z)
$$

$$
= \int_{X_\varrho \times X_{\varrho^{-1} u} \times X_{\varrho u}} g(v, w, z, \psi) \, d\tilde{\nu}_n(v, w, z)
$$

$$
= \int_{X_\varrho \times X_{\varrho^{-1} u} \times X_{\varrho u}} \left( \int_0^T \int_{\Omega} (z - \Gamma(v, w)) \cdot \psi \, dx \, dt \right) \, d\tilde{\nu}_n(v, w, z)
$$

$$
= \tilde{\mathbb{E}} \int_\Omega \int_{\Omega} \left( \tilde{m}_n(\omega) - \Gamma(\tilde{\varrho}_n(\omega), \tilde{r}_n(\omega)) \right) \cdot \psi \, dx \, dt \, d\tilde{\mathbb{P}}(\omega)
$$

$$
= \tilde{\mathbb{E}} \int_\Omega \int_{\Omega} \left( \tilde{m}_n(t, x) - \tilde{\varrho}_n^{-1}(t, x) \tilde{r}_n(t, x) \right) \cdot \psi(t, x) \, dx \, dt,
$$

where we have used the measurability of $\tilde{m}_n, \tilde{\varrho}_n$ and $\tilde{r}_n$ (which is due to Proposition 4.12) and the measurability of the map $g$ defined in (4.44). Therefore, in view of separability of the space $L^1(0, T; L^3(\Omega))$ we deduce that

$$
\tilde{m}_n = \tilde{\varrho}_n^{-1} \tilde{r}_n \quad \text{a.e. in} \quad \Omega \times (0, T), \quad \tilde{\mathbb{P}}\text{-a.s.} \tag{4.48}
$$
We can similarly identify \( \tilde{q}_n \). For this purpose we consider another pair of maps \( \hat{\Gamma} \) and \( \hat{g} \) defined by
\[
\hat{\Gamma}: \mathcal{X}_\theta \times \mathcal{X}_{\frac{1}{\theta} + u} \ni (v, w) \mapsto v_{\frac{1}{\theta} + x}w \in \left(L^\infty(0, T; L^2(\mathcal{O})), w^*\right),
\]
\[
\hat{g}: \mathcal{X}_\theta \times \mathcal{X}_{\frac{1}{\theta} + u} \times \mathcal{X}_{\theta^2 u} \times L^1(0, T; L^2(\mathcal{O})) \to \mathbb{R}
\]
\[
(v, w, z, \psi) \mapsto \int_0^T \int_\mathcal{O} \left(z(t, x) - \hat{\Gamma}(v(t, x), w(t, x))\right) \cdot \psi(t, x) \, dx \, dt
\]
\[
(4.49)
\]
\[
(4.50)
\]
\( \hat{\Gamma} \) is well-defined as well as sequentially continuous, see Lemma B.4 for details, and so is \( \hat{g} \).

Arguing as above we can check that
\[
0 = E \int_0^T \int_\mathcal{O} \left( \sqrt{\theta_n} u_n(t, x) - \theta_n^{\frac{1}{\theta^2 + x}}(t, x) \theta_n^{\frac{1}{\theta^2 + x}}(t, x) u_n(t, x) \right) \cdot \psi(t, x) \, dx \, dt
\]
\[
= \int_\mathcal{O} \left( \int_0^T \int_\mathcal{O} \left( \sqrt{\theta_n} u_n(\omega) - \hat{\Gamma}(\theta_n(\omega), \theta_n^{\frac{1}{\theta^2 + x}} u_n(\omega)) \right) \cdot \psi \, dx \, dt \right) \, d\hat{P}(\omega)
\]
\[
= \int_{\mathcal{X}_\theta \times \mathcal{X}_{\frac{1}{\theta^2 + x} u}} \hat{g}(v, w, z) \, d\nu_n(v, w, z)
\]
\[
= \int_{\mathcal{X}_\theta \times \mathcal{X}_{\frac{1}{\theta^2 + x} u}} \hat{g}(v, w, z) \, d\nu_n(v, w, z)
\]
\[
= \int_{\mathcal{X}_\theta \times \mathcal{X}_{\frac{1}{\theta^2 + x} u}} \hat{g}(v, w, z) \, d\nu_n(v, w, z)
\]
\[
= \int_\mathcal{O} \left( \int_0^T \int_\mathcal{O} \left( \sqrt{\theta_n} u_n(\omega) - \hat{\Gamma}(\theta_n(\omega), \theta_n^{\frac{1}{\theta^2 + x}} u_n(\omega)) \right) \cdot \psi \, dx \, dt \right) \, d\hat{P}(\omega)
\]
\[
= E \int_0^T \int_\mathcal{O} \left( \tilde{q}_n(t, x) - \theta_n^{\frac{1}{\theta^2 + x}}(t, x) \tilde{r}_n(t, x) \right) \cdot \psi(t, x) \, dx \, dt,
\]
holds for all test functions \( \psi \in L^1(0, T; L^2(\mathcal{O})) \). Here \( \nu_n \) and \( \tilde{\nu}_n \) denote \( L(\theta_n, \theta_n^{\frac{1}{\theta^2 + x}}, \sqrt{\theta_n} u) \) and \( L(\tilde{\theta}_n, \tilde{r}_n, \tilde{\nu}_n) \) respectively and \( \nu_n = \tilde{\nu}_n \) on \( B(\mathcal{X}_\theta \times \mathcal{X}_{\frac{1}{\theta^2 + x} u} \times \mathcal{X}_{\theta^2 u}) \). Therefore, we deduce that
\[
\tilde{q}_n = \theta_n^{\frac{1}{\theta^2 + x}} \tilde{r}_n \text{ a.e. in } \mathcal{O} \times [0, T], \quad \hat{P} \text{-a.s..}
\]
\[
(4.51)
\]
This concludes the proof of Lemma 4.13.

\[\square\]

**Proof of Proposition 4.14.** In order to identify the limit \( \tilde{\theta} \) we note that
\[
\tilde{\theta}_n \tilde{\theta}_n \to \left( \tilde{\theta} \right)^2 \text{ strongly in } L^1(0, T; L^1(\mathcal{O})), \quad \hat{P} \text{-a.s.}
\]
\[
(4.52)
\]
as a consequence of the strong convergence of \( \tilde{\vartheta}_n \) in \( \mathcal{X}_{\sqrt{\tilde{\varphi}}} \). But by (4.42) and second assertion of Proposition 4.12 we have that
\[
\tilde{\vartheta}_n \tilde{\vartheta}_n = \tilde{\varphi}_n \to \tilde{\varphi} \quad \text{strongly in } L^1(0, T; L^1(\mathcal{O})), \quad \tilde{\mathbb{P}}\text{-a.s.}
\] (4.53)

Therefore \( \tilde{\varphi} = (\tilde{\vartheta})^2 \), a.e. in \( \mathcal{O} \times (0, T) \) \( \tilde{\mathbb{P}}\)-a.s.. To identify \( \tilde{m} \) we first notice that thanks to strong convergence of \( \tilde{\varphi}_n \) and weak convergence of \( \tilde{\vartheta}_n \)
\[
\tilde{\vartheta}_n^{\frac{1+\delta}{\delta}} \tilde{\varphi}_n \to \tilde{\vartheta}_{\delta}^{\frac{1+\delta}{\delta}} \tilde{\varphi}, \quad \text{weakly* in } L^\infty(0, T; L^2(\mathcal{O})), \quad \tilde{\mathbb{P}}\text{-a.s.},
\] (4.54)

and then because of previous identification and Proposition 4.12
\[
\tilde{\vartheta}_n^{\frac{1+\delta}{\delta}} \tilde{\varphi}_n = \tilde{\vartheta}_n \to \tilde{\vartheta} \quad \text{a.e. in } \mathcal{O} \times (0, T) \tilde{\mathbb{P}}\text{-a.s.}
\] (4.55)

Therefore \( \tilde{\varphi} = (\tilde{\vartheta})^2 \tilde{\varphi} \tilde{\vartheta} \), a.e. in \( \mathcal{O} \times (0, T) \tilde{\mathbb{P}}\)-a.s.. Identification of \( \tilde{q} \) follows in a similar manner. This concludes the proof of Proposition 4.14. \( \square \)

**Lemma 4.15.** Let \( n \in \mathbb{N} \). Then, there exists a random variable \( \tilde{u}_n \) defined on the new probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \), obtained in Proposition 4.12, for which
\[
(\tilde{q}_n, \tilde{m}_n) = (\sqrt{\tilde{\varphi}_n} \tilde{u}_n, \tilde{\varphi}_n \tilde{u}_n) \quad \text{a.e. in } \mathcal{O} \times (0, T) \tilde{\mathbb{P}}\text{-a.s.}
\] (4.56)

**Proof of Lemma 4.15.** From the equality of joint laws and estimate (4.2), we deduce that
\[
||\tilde{r}_n||_{L^{2+\delta}(\tilde{\Omega}; L^\infty(0, T; L^{2+\delta}(\mathcal{O})))} \leq C,
\] (4.57)

from this it follows that \( \tilde{r}_n \) is bounded a.e. in \( \tilde{\Omega} \times (0, T) \times \mathcal{O} \). Therefore, by identity (4.39) it follows that
\[
\tilde{\varphi}_n = 0 \implies \tilde{q}_n = \tilde{\varphi}_n^{\frac{\delta}{2+\delta}} \tilde{r}_n = 0
\] (4.58)

a.e. in \( \tilde{\Omega} \times (0, T) \times \mathcal{O} \). Now we define
\[
\tilde{u}_n = \begin{cases} 
\tilde{r}_n \tilde{\varphi}_n^{\frac{-\delta}{2+\delta}} & \text{if } \tilde{\varphi}_n \neq 0, \\
0 & \text{if } \tilde{\varphi}_n = 0.
\end{cases}
\] (4.59)

Therefore, from (4.58) it follows that
\[
\tilde{q}_n = \begin{cases} 
\sqrt{\tilde{\varphi}_n} \tilde{u}_n & \text{for } \tilde{\varphi}_n \neq 0, \\
0 & \text{for } \tilde{\varphi}_n = 0.
\end{cases}
\] (4.60)

but due to definition of \( \tilde{u}_n \), (4.59), we have that
\[
\tilde{q}_n = \sqrt{\tilde{\varphi}_n} \tilde{u}_n
\] (4.61)

a.e. in \( \tilde{\Omega} \times (0, T) \times \mathcal{O} \). Using the same argument for \( \tilde{m}_n \) we get the second part of the equality. \( \square \)
4.3 Convergence of the convective term and of the stress tensor

**Lemma 4.16.** Let \( \tilde{\mathbf{u}}_n \) be as defined in Lemma 4.15, then there exists a random variable \( \tilde{\mathbf{u}} \) such that

\[
\tilde{\beta}_n \tilde{\mathbf{u}}_n \otimes \tilde{\mathbf{u}}_n \rightarrow \tilde{\beta} \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} \quad \text{in} \quad L^1(0, T; L^1(\Omega)) \quad \tilde{\mathbb{P}}\text{-a.s.}
\]  

(4.62)

**Proof of Lemma 4.16.** So far we know that \( \tilde{\mathbb{P}}\text{-a.s.} \)

\[
\tilde{\mathbf{m}}_n \rightarrow \tilde{\mathbf{m}} \quad \text{in} \quad L^r(0, T; L^\alpha(\Omega)) \cap C_w([0, T]; L^{3/2}(\Omega)), \quad r \in [1, \infty), \ \alpha \in [1, \frac{3}{2})
\]

(4.63)

Let us point out that for fixed \( (\omega, t) \in \tilde{\Omega} \times (0, T) \) and that \( \tilde{\mathbf{m}}_n \rightarrow \tilde{\mathbf{m}} \) in \( L^r(0, T; L^\alpha(\Omega)) \cap C_w([0, T]; L^{3/2}(\Omega)) \) we deduce only the weaker notion of convergence for \( \tilde{\beta}_n \tilde{\mathbf{u}}_n \). From (4.63) and due to the lower semi-continuity of norms we have from (4.57)

\[
\|\tilde{\mathbf{r}}\|_{L^{2+\delta}(\tilde{\Omega}; L^{\infty}(0, T; L^{3/2}(\Omega)))} \leq C.
\]  

(4.65)

Indeed, knowing that \( \tilde{\mathbf{m}}_n = \tilde{\beta}_n \tilde{\mathbf{u}}_n \), a.e. in \( \Omega \times (0, T) \) and that \( \tilde{\mathbf{m}}_n \rightarrow \tilde{\mathbf{m}} \) in \( L^r(0, T; L^\alpha(\Omega)) \cap C_w([0, T]; L^{3/2}(\Omega)) \) we deduce only the weaker notion of convergence for \( \tilde{\beta}_n \tilde{\mathbf{u}}_n \). From (4.63) and due to the lower semi-continuity of norms we have from (4.57)

\[
\|\tilde{\mathbf{r}}\|_{L^{2+\delta}(\tilde{\Omega}; L^{\infty}(0, T; L^{3/2}(\Omega)))} \leq C.
\]  

(4.65)

Proceeding as in the proof of Lemma 4.15 we can define a new random field \( \tilde{\mathbf{u}} \)

\[
\tilde{\mathbf{u}} = \begin{cases} \tilde{\beta} \tilde{\mathbf{m}} + \tilde{\mathbf{r}} & \text{for} \quad \tilde{\beta} \neq 0, \\ 0 & \text{for} \quad \tilde{\beta} = 0. \end{cases}
\]  

(4.66)

Let us point out that for fixed \( (\omega, t) \in \tilde{\Omega} \times [0, T] \), This definition in conjunction with the 2nd and 3rd parts of identification (4.40) in Proposition 4.14 implies that \( \tilde{\mathbb{P}}\text{-a.s.} \)

\[
(\tilde{\mathbf{m}}, \tilde{\mathbf{q}}) = (\tilde{\beta} \tilde{\mathbf{u}}, \sqrt{\tilde{\beta}} \tilde{\mathbf{u}}), \quad \text{a.e. in} \quad (0, T) \times \Omega.
\]  

(4.67)

In particular, by Proposition 4.12 and definitions (4.29-4.30), \( \tilde{\mathbb{P}}\text{-a.s.}, (\tilde{\beta} \tilde{\mathbf{u}}, \sqrt{\tilde{\beta}} \tilde{\mathbf{u}}) \) belongs to \( \mathcal{X}_{\tilde{\mathbf{u}}} \times \mathcal{X}_{\sqrt{\tilde{\beta}} \tilde{\mathbf{u}}} \), i.e. to \( \left( L^r(0, T; L^\alpha(\Omega)) \cap C_w([0, T]; L^{3/2}(\Omega)) \right) \times \left( L^\infty(0, T; L^2(\Omega)) \right), w^* \).

Therefore, to conclude we only need to improve the convergence result from (4.64) to show that

\[
\sqrt{\tilde{\beta}_n} \tilde{\mathbf{u}}_n \rightarrow \sqrt{\tilde{\beta}} \tilde{\mathbf{u}} \quad \text{in} \quad L^2(0, T; L^2(\Omega)) \quad \tilde{\mathbb{P}}\text{-a.s.}
\]  

(4.68)

To prove it, we proceed similarly to Mellet and Vasseur. As in (4.65), by applying the Fatou Lemma, we deduce that

\[
\int_{\Omega} \tilde{\beta} |\tilde{\mathbf{u}}|^{2+\delta} \, dx \leq \int_{\Omega} \tilde{\beta}^{2+\delta} \, dx = \int_{\Omega} \liminf_{n \to \infty} \tilde{\beta}^{2+\delta} \, dx \leq \liminf_{n \to \infty} \int_{\Omega} \tilde{\beta}^{2+\delta} \, dx < \infty.
\]  

(4.69)

Note that the left inequality follows from the definition of (4.66) for \( \tilde{\beta} \neq 0 \), when \( \tilde{\beta} = 0 \) this integral is equal to 0. We will now show that

\[
\sqrt{\tilde{\beta}_n} \tilde{\mathbf{u}}_n = \frac{\tilde{\mathbf{m}}_n}{\sqrt{\tilde{\beta}_n}} \rightarrow \frac{\tilde{\mathbf{m}}}{\sqrt{\tilde{\beta}}} = \sqrt{\tilde{\beta}} \tilde{\mathbf{u}} \quad \text{almost everywhere.}
\]  

(4.70)

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Firstly, due to (4.66) and strong convergence of \( \tilde{m}_n \) and \( \tilde{\varrho}_n \), the convergence (4.70) is true on the set \( \{ \tilde{\varrho}(t, x) \neq 0 \} \). Secondly, this is also true on the set where \( \tilde{u}_n \) is bounded and \( \{ \tilde{\varrho}(t, x) = 0 \} \). Indeed, we have \( \sqrt{\tilde{\varrho}_n \tilde{u}_n} \chi_{|\tilde{u}_n| \leq M} \leq M \sqrt{\tilde{\varrho}_n} \to 0 \).

To conclude that

\[
\sqrt{\tilde{\varrho}_n \tilde{u}_n} \chi_{|\tilde{u}_n| \leq M} \to \sqrt{\tilde{\varrho} \tilde{u}} \chi_{|\tilde{u}| \leq M} \quad \text{almost everywhere.} \tag{4.71}
\]

we use that

\[
\{ |\tilde{u}_n| \leq M \} = \left\{ \left| \frac{\tilde{m}_n}{\tilde{\varrho}_n} \right| \leq M \right\},
\]

and so for \( \tilde{\varrho} \neq 0 \) using the strong convergence of \( \tilde{\varrho}_n, \tilde{m}_n \) we have

\[
\chi_{|\tilde{m}_n|/\tilde{\varrho}_n \leq M} \to \chi_{|\tilde{m}|/\tilde{\varrho} \leq M} = \chi_{|\tilde{u}| \leq M} \quad \text{almost everywhere.}
\]

We are now ready to conclude, we first split

\[
\begin{align*}
\int_0^T \int_{\mathcal{O}} \left| \sqrt{\tilde{\varrho}_n \tilde{u}_n} - \sqrt{\tilde{\varrho} \tilde{u}} \right|^2 \, dx \, dt &\leq \int_0^T \int_{\mathcal{O}} \left| \sqrt{\tilde{\varrho}_n \tilde{u}_n} \chi_{|\tilde{u}_n| \leq M} - \sqrt{\tilde{\varrho} \tilde{u}} \chi_{|\tilde{u}| \leq M} \right|^2 \, dx \, dt \\
&\quad + 4 \int_0^T \int_{\mathcal{O}} \left| \sqrt{\tilde{\varrho}_n \tilde{u}_n} \chi_{|\tilde{u}_n| \geq M} \right|^2 \, dx \, dt \\
&\quad + 4 \int_0^T \int_{\mathcal{O}} \left| \sqrt{\tilde{\varrho} \tilde{u}} \chi_{|\tilde{u}| \geq M} \right|^2 \, dx \, dt. \tag{4.72}
\end{align*}
\]

Due to (4.71) and uniform boundedness of the integrand the first term converges to 0 when \( n \to \infty \). For the other two terms we make use of estimate (4.57) and inequality (4.69), we therefore have

\[
4 \int_0^T \int_{\mathcal{O}} \left| \sqrt{\tilde{\varrho}_n \tilde{u}_n} \chi_{|\tilde{u}_n| \geq M} \right|^2 \, dx \, dt \leq \frac{4}{M^\delta} \int_0^T \int_{\mathcal{O}} |\tilde{u}_n|^{2+\delta} \, dx \, dt \\
\leq \frac{4}{M^\delta} \int_0^T \int_{\mathcal{O}} |\tilde{R}_n|^{2+\delta} \, dx \, dt \leq \frac{C}{M^\delta}, \tag{4.73}
\]

and

\[
4 \int_0^T \int_{\mathcal{O}} \left| \sqrt{\tilde{\varrho} \tilde{u}} \chi_{|\tilde{u}| \geq M} \right|^2 \, dx \, dt \leq \frac{4}{M^\delta} \int_0^T \int_{\mathcal{O}} |\tilde{u}|^{2+\delta} \, dx \, dt \\
\leq \frac{4}{M^\delta} \int_0^T \int_{\mathcal{O}} |\tilde{R}|^{2+\delta} \, dx \, dt \leq \frac{C}{M^\delta}, \tag{4.74}
\]

and so both terms converge to 0 as \( M \to \infty \) (recall \( \delta > 0 \)).

In the next Lemma we will show that the random field \( u \) is weakly differentiable in some specific sense. To explain it’s meaning let us observe that if a locally integrable \( u \) is weakly differentiable with \( \nabla u \) being also locally integrable and \( \varrho \) is a locally bounded function, then for every test function \( \psi \in C^\infty(\mathcal{O}) \) the following identity holds

\[
\int_{\mathcal{O}} \text{div}(\varrho \nabla u) \psi \, dx = \int_{\mathcal{O}} \varrho u \cdot \Delta \psi \, dx + \int_{\mathcal{O}} \nabla \varrho \otimes u : \nabla \psi \, dx. \tag{4.75}
\]
Lemma 4.17 (Convergence of the diffusion terms). Let $\tilde{u}_n$, $\tilde{u}$ be as defined in Lemma 4.15 and Lemma 4.16 respectively. Then $\tilde{\mathbb{P}}$-a.s.
\[
\text{div}(\tilde{\varrho}_n \nabla \tilde{u}_n) \to \text{div}(\tilde{\varrho} \nabla \tilde{u}) \quad \text{in } \mathcal{D'},
\]
in the following sense. For every $\psi \in W^{2,\infty}(\mathcal{O})$ and every $t \in [0,T]$, $\tilde{\mathbb{P}}$-a.s., the following equality holds
\[
\lim_{n \to \infty} \left( \int_0^t \int_{\mathcal{O}} \sqrt{\tilde{\varrho}_n} \Delta \psi \, dx \, ds + 2 \int_0^t \int_{\mathcal{O}} \nabla \sqrt{\tilde{\varrho}_n} \otimes \sqrt{\tilde{\varrho}_n} \tilde{u}_n : \nabla \psi \, dx \, ds \right) = \int_0^t \int_{\mathcal{O}} \tilde{\varrho} \tilde{u} \cdot \Delta \psi \, dx \, ds + \int_0^t \int_{\mathcal{O}} \nabla \tilde{\varrho} \otimes \tilde{u} : \nabla \psi \, dx \, ds.
\]
(4.76)

Proof of Lemma 4.17. It is enough to prove the Lemma for $t = T$. Let us choose and fix a test function $\psi \in C^{\infty}(\mathcal{O})$ and define
\[
I_1^n := \int_0^T \int_{\mathcal{O}} \sqrt{\tilde{\varrho}_n} \nabla \tilde{u}_n : \nabla \psi \, dx \, dt,
\]
\[
I_2^n := 2 \int_0^T \int_{\mathcal{O}} \nabla \sqrt{\tilde{\varrho}_n} \otimes \sqrt{\tilde{\varrho}_n} \tilde{u}_n : \nabla \psi \, dx \, dt.
\]
From Proposition 4.12 and Lemma 4.13 we know that $\sqrt{\tilde{\varrho}_n} \to \sqrt{\tilde{\varrho}}$ strongly in $C([0,T]; L^{2q}(\mathcal{O}))$, $q \in [1,3)$ and weakly* in $L^{\infty}(0,T; H^1(\mathcal{O}))$ $\tilde{\mathbb{P}}$-a.s.. Moreover, from Lemma 4.16 we have $\sqrt{\tilde{\varrho}_n} \tilde{u}_n \to \sqrt{\tilde{\varrho}} \tilde{u}$ strongly in $L^2(0,T; L^2(\mathcal{O}))$ $\tilde{\mathbb{P}}$-a.s.. Using these convergence results we infer that for $\psi \in C^{\infty}(\mathcal{O})$ or in particular for $\psi \in W^{2,\infty}(\mathcal{O})$
\[
I_1^n \to \int_0^T \int_{\mathcal{O}} \tilde{\varrho} \tilde{u} \cdot \Delta \psi \, dx \, dt \quad \tilde{\mathbb{P}}\text{-a.s.},
\]
\[
I_2^n \to \int_0^T \int_{\mathcal{O}} \nabla \tilde{\varrho} \otimes \tilde{u} : \nabla \psi \, dx \, dt \quad \tilde{\mathbb{P}}\text{-a.s.},
\]
and hence we infer that
\[
I_1^n + I_2^n \to \int_0^T \int_{\mathcal{O}} \tilde{\varrho} \tilde{u} \cdot \Delta \psi \, dx \, dt + \int_0^T \int_{\mathcal{O}} \nabla \tilde{\varrho} \otimes \tilde{u} : \nabla \psi \, dx \, dt \quad \tilde{\mathbb{P}}\text{-a.s.}
\]
Once we have proved Lemma for $\psi \in C^{\infty}(\mathcal{O})$ we can can easily deduce it for an arbitrary $\psi \in W^{2,\infty}(\mathcal{O})$. \hfill \square

5 Existence of martingale solutions

Our aim is to show that the limiting system $\left(\tilde{U}, \tilde{W}, \tilde{\varrho}, \tilde{\mu}\right)$, where $\tilde{U} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, is a martingale solution to (1.1), where $\tilde{\mu}$ is as defined in the proof of Lemma 4.16 and we use notation introduced in Proposition 4.12. The very first step in this direction is to show that $\tilde{W}_n$, $n \in \mathbb{N}$, and $\tilde{W}$ are indeed $\mathbb{R}$-valued Brownian motion. This follows directly from the following lemmas taken from [19, Lemma 5.2 and proof]. The approach presented in this section is similar to the one given in paper [15].

Lemma 5.1. Suppose that a process $\left(\tilde{W}_n(t)\right)_{t \in [0,T]}$, defined on $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$, has the same law on $C([0,T]; \mathbb{R})$ as the $\mathbb{R}$-valued Brownian motion $W$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then $\tilde{W}_n$ is also an $\mathbb{R}$-valued Brownian motion on $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$.
Lemma 5.2. The process \( \left( \tilde{W}(t) \right)_{t \in [0,T]} \) is an \( \mathbb{R} \)-valued Brownian motion on \( (\tilde{\Omega}, \tilde{F}, \tilde{P}) \). If \( s \in [0,T) \), then the increments \( \tilde{W}(t) - \tilde{W}(s) \), \( t \in [s,T] \), are independent of the \( \sigma \)-algebra generated by \( \tilde{u}(r) \) and \( \tilde{W}(r) \) for \( r \in [0,s] \).

Below we collect all the convergence results obtained in Section 4, which will be used later to prove the existence of a martingale solution. Let \( \tilde{u}_n \) and \( \tilde{u} \) be the processes as specified in Lemmas 4.15–4.17 and \( q \in [1,3) \). Then, the following assertions hold \( \tilde{P} \)-a.s.

\[
\tilde{g}_n \to \tilde{g} \quad \text{in} \quad C([0,T]; L^q(\mathcal{O})),
\]

\( \sqrt{\tilde{g}_n} \to \sqrt{g} \quad \text{in} \quad (L^\infty(0,T; H^1(\mathcal{O})), \mathbb{P}^\ast) \),

\[
\sqrt{\tilde{g}_n} \tilde{u}_n \to \sqrt{g} \tilde{u} \quad \text{in} \quad L^2(0,T; L^2(\mathcal{O})),
\]

\[
\tilde{g}_n \tilde{u}_n \otimes \tilde{u}_n \to \tilde{g} \tilde{u} \otimes \tilde{u} \quad \text{in} \quad L^1(0,T; L^1(\mathcal{O})),
\]

\[
\text{div}(\tilde{g}_n \nabla \tilde{u}_n) \to \text{div}(\tilde{g} \nabla \tilde{u}) \quad \text{in} \quad D'(((0,T) \times \mathcal{O})).
\]

Lemma 5.3. Let \( (\tilde{u}_n) \) be the sequence from Lemma 4.15. Then for every \( p \in [1, \infty) \) the random variables \( \tilde{g}_n \) and \( \tilde{u}_n \) satisfy the following uniform estimates

\[
\left\| \sqrt{\tilde{g}_n} \tilde{u}_n \right\|_{L^p(\Omega; L^\infty(0,T; L^2(\mathcal{O})))} \leq C,
\]

\[
\left\| \tilde{g}_n \right\|_{L^p(\Omega; L^\infty(0,T; L^1(\mathcal{O})))} \leq C,
\]

\[
\left\| \sqrt{\tilde{g}_n} \right\|_{L^p(\Omega; L^\infty(0,T; H^1(\mathcal{O})))} \leq C.
\]

Proof of Lemma 5.3. The estimates for the random variables \( \tilde{g}_n \) and \( \tilde{u}_n \) are the consequence of the fact that the law of \( \left( \tilde{g}_n, \tilde{u}_n, \tilde{m}_n, \tilde{q}_n, \tilde{r}_n \right) \) on \( \mathcal{X}_T \) (see (4.33)) is given by \( \mu_n, n \in \mathbb{N} \) (see Proposition 4.12 (1)) and identification of the random variables established in Lemma 4.13 and Lemma 4.15.

Lemma 5.4. Let \( \tilde{u}_n \) and \( \tilde{u} \) be as defined in Lemma 4.15 and Lemma 4.16 respectively. Let \( \phi : \mathcal{O} \to \mathbb{R} \) and \( \psi : \mathcal{O} \to \mathbb{R}^3 \) be test functions such that \( \phi \in W^{1,\infty}(\mathcal{O}) \) and \( \psi \in W^{2,\infty}(\mathcal{O}) \). Then

\[
\lim_{n \to \infty} \tilde{E} \left| \int_0^T \int_\mathcal{O} (\tilde{g}_n - \tilde{g}) \phi \ dx \ dt \right| = 0,
\]

\[
\lim_{n \to \infty} \tilde{E} \left| \int_0^T \int_\mathcal{O} (\tilde{g}_n \tilde{u}_n - \tilde{g} \tilde{u}) \cdot \nabla \phi \ dx \ ds \right| \ dx \ dt = 0,
\]

\[
\lim_{n \to \infty} \tilde{E} \left| \int_0^T \int_\mathcal{O} (\tilde{g}_n \nabla \tilde{u}_n - \tilde{g} \nabla \tilde{u}) \cdot \psi \ dx \ dt \right| = 0,
\]

\[
\lim_{n \to \infty} \tilde{E} \left| \int_0^T \int_\mathcal{O} (\tilde{g}_n - \tilde{g}) \div \psi \ dx \ ds \right| = 0,
\]

\[
\lim_{n \to \infty} \tilde{E} \left| \int_\mathcal{O} (\tilde{g}_n(0,\cdot) - \tilde{g}(0,\cdot)) \phi \ dx \right| = 0,
\]

\[
\lim_{n \to \infty} \tilde{E} \left| \int_\mathcal{O} (\tilde{m}_n(0,\cdot) - \tilde{m}(0,\cdot)) \cdot \psi \ dx \right| = 0.
\]
Proof of Lemma 5.4. Let us fix $\phi \in W^{1,\infty}(O;\mathbb{R})$ and $\psi \in W^{2,\infty}(O;\mathbb{R}^3)$.

**Step 1. Proof of (5.9).** From (5.1), we know that $\tilde{g}_n \to \tilde{g}$ in $C([0,T];L^2(O))$ $\tilde{P}$-a.s., which implies that

$$\lim_{n \to \infty} \int_0^T \int_O \tilde{g}_n \phi \, dx \, dt = \int_0^T \int_O \tilde{g} \phi \, dx \, dt \quad \tilde{P}$-a.s. \quad (5.17)$$

We have for some $r > 1$, by (5.7)

$$\mathbb{E} \left( \left| \int_0^T \int_O \tilde{g}_n \phi \, dx \, dt \right|^r \right) \leq C \| \tilde{\varphi} \|_{L^2(O)}^r \mathbb{E} \left( \int_0^T \| \tilde{g}_n(t) \|_{L^2(O)}^r \, dt \right) \leq C.$$ 

This bound provides the equi-integrability of $\int_0^T \int_O \tilde{g}_n(t,x) \phi(x) \, dx \, dt$. Taking into account the convergence (5.17), the Vitali convergence theorem D.8 then shows that (5.9) holds.

**Step 2. Proof of (5.10).** From (5.1) and (5.3), $\sqrt{\tilde{g}_n} \to \sqrt{\tilde{g}}$ strongly in $C([0,T];L^2(O))$ $\tilde{P}$-a.s. and $\sqrt{\tilde{g}_n} \hat{u}_n \to \sqrt{\tilde{g}} \hat{u}$ strongly in $L^2(0,T;L^2(O))$ $\tilde{P}$-a.s. respectively. Therefore, for a.a. $t \in (0,T]$ $\tilde{P}$-a.s.

$$\lim_{n \to \infty} \int_0^t \int_O \tilde{g}_n \hat{u}_n \cdot \nabla \phi \, dx \, ds = \int_0^t \int_O \tilde{g} \hat{u} \cdot \nabla \phi \, dx \, ds. \quad (5.18)$$

Furthermore, by estimates (5.6) and (5.7), we find that

$$\mathbb{E} \left( \left| \int_0^t \int_O \tilde{g}_n \hat{u}_n \cdot \nabla \phi \, dx \, ds \right|^2 \right) \leq \| \nabla \phi \|_{L^\infty(O)}^2 \mathbb{E} \left( \left| \int_0^t \| \sqrt{\tilde{g}_n} \sqrt{\tilde{g}_n} \hat{u}_n \|_{L^1(O)} \, ds \right|^2 \right) \leq \| \nabla \phi \|_{L^\infty(O)}^2 \mathbb{E} \left( \left| \int_0^t \| \tilde{g}_n(t) \|_{L^2(O)} \| \sqrt{\tilde{g}_n} \hat{u}_n \|_{L^2(O)} \, ds \right|^2 \right) \leq \| \nabla \phi \|_{L^\infty(O)}^2 T^2 \left( \mathbb{E} \| \tilde{g}_n \|_{L^\infty(0,T;L^2(O))}^4 \right)^{1/2} \left( \mathbb{E} \| \sqrt{\tilde{g}_n} \hat{u}_n \|_{L^\infty(0,T;L^2(O))}^4 \right)^{1/2} \leq C.$$ 

This bound and the $\tilde{P}$-a.s. convergence (5.18) allows us to apply again the Vitali convergence theorem D.8 to infer that (5.10) holds.

**Step 3.** We can similarly establish the convergence (5.11).

**Step 4. Proof of (5.12).** From the convergence (5.4), we have for $t \in [0,T]$ $\tilde{P}$-a.s.

$$\lim_{n \to \infty} \int_0^t \int_O \tilde{g}_n \hat{u}_n \otimes \hat{u}_n : \nabla \psi \, dx \, ds = \int_0^t \int_O \tilde{g} \hat{u} \otimes \hat{u} : \nabla \psi \, dx \, ds. \quad (5.19)$$

Using the estimate (5.6), we get

$$\mathbb{E} \left( \left| \int_0^t \int_O \tilde{g}_n \hat{u}_n \otimes \hat{u}_n : \nabla \psi \, dx \, ds \right|^2 \right) \leq \| \nabla \psi \|_{L^\infty(O)}^2 \mathbb{E} \| \sqrt{\tilde{g}_n} \hat{u}_n \|_{L^2(0,T;L^2(O))}^4 \leq C.$$ 

Using the bound obtained above, the $\tilde{P}$-a.s. convergence (5.19) and the Vitali convergence theorem D.8 we conclude that (5.12) holds.
Step 5. Proof of (5.13). Let us choose and fix \( t \in [0, T] \). The convergence \( \text{div}(\tilde{\vartheta}_n \nabla \tilde{u}_n) \to \text{div}(\vartheta \nabla \tilde{u}) \) in \( \mathcal{D}' \) (as shown in Lemma 4.17), shows that \( \mathbb{P}\)-a.s.

\[
\lim_{n \to \infty} \int_0^t \int_\Omega \tilde{\vartheta}_n \nabla \tilde{u}_n : \nabla \psi \; dx \; ds = \int_0^t \int_\Omega \vartheta \nabla \tilde{u} : \nabla \psi \; dx \; ds. \tag{5.20}
\]

Moreover, using the estimates (5.6) and (5.8), we have

\[
\tilde{\mathbb{E}} \left| \int_0^t \int_\Omega \tilde{\vartheta}_n \nabla \tilde{u}_n : \nabla \psi \; dx \; ds \right|^2
\]

\[
= \tilde{\mathbb{E}} \left| \int_0^t \int_\Omega \tilde{\vartheta}_n \tilde{u}_n : \Delta \psi \; dx \; ds + \int_0^t \int_\Omega \nabla \tilde{\vartheta}_n \otimes \tilde{u}_n : \nabla \psi \; dx \; ds \right|^2
\]

\[
\leq C \left( \tilde{\mathbb{E}} \left| \int_0^t \int_\Omega \sqrt{\tilde{\vartheta}_n} \nabla \tilde{u}_n : \Delta \psi \; dx \; ds \right|^2 + 4 \tilde{\mathbb{E}} \left| \int_0^t \int_\Omega \nabla \sqrt{\tilde{\vartheta}_n} \otimes \sqrt{\tilde{\vartheta}_n} \tilde{u}_n : \nabla \psi \; dx \; ds \right|^2 \right)
\]

\[
\leq C \| \Delta \psi \|^2_{L^\infty(\mathcal{O})} \tilde{\mathbb{E}} \left| \int_0^t \| \sqrt{\tilde{\vartheta}_n} \|_{L^2(\mathcal{O})} \| \sqrt{\tilde{\vartheta}_n} \tilde{u}_n \|_{L^2(\mathcal{O})} ds \right|^2
\]

\[
+ C \| \nabla \psi \|^2_{L^\infty(\mathcal{O})} \tilde{\mathbb{E}} \left| \int_0^t \| \nabla \sqrt{\tilde{\vartheta}_n} \|_{L^2(\mathcal{O})} \| \sqrt{\tilde{\vartheta}_n} \tilde{u}_n \|_{L^2(\mathcal{O})} ds \right|^2
\]

\[
\leq C \| \Delta \psi \|^2_{L^\infty(\mathcal{O})} T^2 \left( \tilde{\mathbb{E}} \left| \int_0^t \| \sqrt{\tilde{\vartheta}_n} \|^4_{L^\infty(0,T;L^2(\mathcal{O}))} ds \right|^{1/2} \right)^2
\]

\[
+ C \| \nabla \psi \|^2_{L^\infty(\mathcal{O})} T^2 \left( \tilde{\mathbb{E}} \left| \int_0^t \| \nabla \sqrt{\tilde{\vartheta}_n} \|^4_{L^\infty(0,T;L^2(\mathcal{O}))} ds \right|^{1/2} \right)^2
\]

The above bound, the \( \tilde{\mathbb{P}}\)-a.s. convergence (5.20) and the Vitali convergence theorem D.8 are enough to deduce (5.13).

Step 6. Proof of (5.14). Let us begin with recalling that \( \gamma < 3 \). We can prove the convergence result similarly as above by using the strong convergence (5.1) of \( \tilde{\vartheta}_n \), estimate (5.7) and the Vitali convergence theorem D.8.

Step 7. Proof of (5.15) and (5.16). These assertions follow from (4.35) and (4.37) respectively.

The proof of Lemma is complete. \( \square \)

Lemma 5.5. Let Assumption 2.2 hold. Moreover, let \( \tilde{W}_n \) and \( \tilde{W} \) be \( \mathbb{R} \)-valued Brownian motions as in Lemma 5.1 and Lemma 5.2 respectively, where \( \tilde{W}_n \to \tilde{W} \) in \( \mathcal{C}([0,T];\mathbb{R}) \) as \( n \to \infty \) \( \mathbb{P} \)-almost surely. Then, for every \( \psi \in L^\infty(\mathcal{O}) \), the following holds

\[
\lim_{n \to \infty} \tilde{\mathbb{E}} \int_0^T \left| \int_0^t \int_\Omega \left( \tilde{\vartheta}_n f \, d\tilde{W}_n - \vartheta f \, d\tilde{W} \right) \cdot \psi \; dx \right|^2 = 0. \tag{5.22}
\]

Proof of Lemma 5.5. Since \( \tilde{W}_n \to \tilde{W} \) in \( \mathcal{C}([0,T];\mathbb{R}) \), due to results in [1, Section 4.3.5], [31, Lemma 5.1], [24, Lemma 2.1] and [3, Lemma 2.6.5 and Lemma 2.6.6], it is sufficient to show that

\[
\lim_{n \to \infty} \tilde{\mathbb{E}} \int_0^T \left| \int_0^t \int_\Omega \left( \tilde{\vartheta}_n - \vartheta \right) f \, d\tilde{W} \cdot \psi \; dx \right|^2 = 0.
\]
We estimate for $\psi \in L^\infty(O)$

$$
\int_0^t \left\| \int_O (\tilde{\vartheta}_n f - \tilde{\vartheta} f) \cdot \psi \, dx \right\|^2 \, ds \leq \int_0^t \left\| \tilde{\vartheta}_n - \tilde{\vartheta} \right\|_{L^2(O)}^2 \left\| f \right\|_{L^{q'}(O)}^2 \left\| \psi \right\|_{L^\infty(O)}^2 \\
\leq \left\| \tilde{\vartheta}_n - \tilde{\vartheta} \right\|_{L^2(0,T;L^q(O))}^2 \left\| f \right\|_{L^{q'}(O)}^2 \left\| \psi \right\|_{L^\infty(O)},
$$

where $q' = \frac{q}{q-1}$. Recall that $\tilde{\vartheta}_n \to \tilde{\vartheta}$ in $C([0,T];L^q(O))$ $\tilde{\mathbb{P}}$-a.s. for $q < 3$, by (5.1). In particular, $q' < 3$ for $q$ large enough. Therefore under the Assumption 2.2 on $f$, we infer that for a.a. $t \in [0,T]$, $\omega \in \Omega$

$$
\lim_{n \to \infty} \int_0^t \left| \int_O (\tilde{\vartheta}_n f - \tilde{\vartheta} f) \cdot \psi \, dx \right|^2 \, ds = 0. \tag{5.23}
$$

We conclude from (5.8) and assumptions on $f$ that

$$
\tilde{\mathbb{E}} \left| \int_0^t \left| \int_O (\tilde{\vartheta}_n f - \tilde{\vartheta} f) \cdot \psi(x) \, dx \right|^2 \, ds \right|^2 \\
\leq C \tilde{\mathbb{E}} \left( \left\| \psi \right\|_{L^\infty(O)}^2 \left\| f \right\|_{L^{3/2}(O)}^4 \int_0^t \left( \left\| \tilde{\vartheta}_n(s) \right\|_{L^4(O)}^4 + \left\| \tilde{\vartheta}(s) \right\|_{L^4(O)}^4 \right) \, ds \right) \\
\leq C \left\| \psi \right\|_{L^\infty(O)}^4 T \left( \tilde{\mathbb{E}} \left| f \right|_{L^{3/2}(O)}^8 \right)^{1/2} \left( \tilde{\mathbb{E}} \left( \left\| \tilde{\vartheta}_n \right\|_{L^\infty(0,T;L^4(O))}^8 + \left\| \tilde{\vartheta} \right\|_{L^\infty(0,T;L^4(O))}^8 \right) \right)^{1/2} \leq C.
$$

With this bound, convergence (5.23) and the Vitali convergence theorem D.8 we obtain for all $\psi \in L^\infty(O)$

$$
\lim_{n \to \infty} \tilde{\mathbb{E}} \left| \int_0^t \left| \int_O (\tilde{\vartheta}_n f - \tilde{\vartheta} f) \cdot \psi \, dx \right|^2 \, ds \right| = 0.
$$

Hence, by the Itô isometry for $t \in [0,T]$ and $\psi \in L^\infty(O)$,

$$
\lim_{n \to \infty} \tilde{\mathbb{E}} \left| \int_0^T \int_O (\tilde{\vartheta}_n f - \tilde{\vartheta} f) \, d\tilde{W} \cdot \psi \, dx \right|^2 = 0. \tag{5.24}
$$

We use the Itô isometry again and estimate (5.8) for $n \in \mathbb{N}$, strong convergence (5.1) along with the lower semicontinuity of norms to infer

$$
\tilde{\mathbb{E}} \left| \int_0^T \int_O (\tilde{\vartheta}_n f - \tilde{\vartheta} f) \, d\tilde{W} \cdot \psi(x) \, dx \right|^2 \\
= \tilde{\mathbb{E}} \left( \int_0^T \left| \int_O (\tilde{\vartheta}_n f - \tilde{\vartheta} f) \cdot \psi(x) \, dx \right|^2 \, ds \right) \\
\leq C \tilde{\mathbb{E}} \left( \left\| \psi \right\|_{L^\infty(O)}^2 \left\| f \right\|_{L^{3/2}(O)}^4 \int_0^t \left( \left\| \tilde{\vartheta}_n(s) \right\|_{L^4(O)}^4 + \left\| \tilde{\vartheta}(s) \right\|_{L^4(O)}^4 \right) \, ds \right) \\
\leq C \left\| \psi \right\|_{L^\infty(O)}^2 T \left( \tilde{\mathbb{E}} \left| f \right|_{L^{3/2}(O)}^4 \right)^{1/2} \left( \tilde{\mathbb{E}} \left( \left\| \tilde{\vartheta}_n \right\|_{L^\infty(0,T;L^4(O))}^4 + \left\| \tilde{\vartheta} \right\|_{L^\infty(0,T;L^4(O))}^4 \right) \right)^{1/2} \leq C.
$$

This bound and the convergence (5.24) allows us to apply the dominated convergence theorem to conclude that for all $\psi \in L^\infty(O)$,

$$
\lim_{n \to \infty} \tilde{\mathbb{E}} \int_0^T \left| \int_0^T (\tilde{\vartheta}_n f - \tilde{\vartheta} f) \, d\tilde{W} \cdot \psi(x) \, dx \right|^2 \, dt = 0.
$$

This finishes the proof of the lemma. \qed
Let \( \phi \in W^{1,\infty}(O; \mathbb{R}) \) and \( \psi \in W^{2,\infty}(O; \mathbb{R}^3) \). Moreover, let \( \tilde{u}_n, \tilde{u} \) be the processes as defined in Lemma 4.15 and Lemma 4.16 respectively and define

\[
\begin{align*}
\Lambda^1_n(\tilde{\varrho}_n, \tilde{u}_n, \tilde{u}, \phi)(t) &= \int_Q \tilde{\varrho}_n(0, x) \cdot \phi(x) \, dx + \int_0^t \int_Q \tilde{\varrho}_n(s) \tilde{u}_n(s) \cdot \nabla \phi \, dx \, ds, \\
\Lambda^1(\tilde{\varrho}, \tilde{u}, \phi)(t) &= \int_Q \tilde{\varrho}(0, x) \cdot \phi(x) \, dx + \int_0^t \int_Q \tilde{\varrho}(s) \tilde{u}(s) \cdot \nabla \phi \, dx \, ds, \\
\Lambda^2_n(\tilde{\varrho}_n, \tilde{u}_n, \tilde{W}_n, \psi)(t) &= \int_Q \tilde{m}_n(0) \cdot \psi(x) \, dx + \int_0^t \int_Q \tilde{\varrho}_n(s) \tilde{u}_n(s) \otimes \tilde{u}_n(s) : \nabla \psi \, dx \, ds \\
&\quad - \int_0^t \int_Q \tilde{\varrho}_n(s) \nabla \tilde{u}_n(s) : \nabla \psi \, dx \, ds + \int_0^t \int_Q \tilde{\varrho}_n(s) \operatorname{div} \psi \, dx \, ds \\
&\quad + \int_0^t \int_Q \tilde{\varrho}_n(s) f \cdot \psi \, dx \, dz \tilde{W}_n(s), \\
\Lambda^2(\tilde{\varrho}, \tilde{u}, \tilde{W}, \psi)(t) &= \int_Q \tilde{m}(0) \cdot \psi(x) \, dx + \int_0^t \int_Q \tilde{\varrho}(s) \tilde{u}(s) \otimes \tilde{u}(s) : \nabla \psi \, dx \, ds \\
&\quad - \int_0^t \int_Q \tilde{\varrho}(s) \nabla \tilde{u}(s) : \nabla \psi \, dx \, ds + \int_0^t \int_Q \tilde{\varrho}(s) \operatorname{div} \psi \, dx \, ds \\
&\quad + \int_0^t \int_Q \tilde{\varrho}(s) f \cdot \psi \, dx \, dz \tilde{W}(s),
\end{align*}
\]

for \( t \in [0, t] \), where the diffusion terms should be understood in the following sense

\[
\begin{align*}
- \int_0^t \int_Q \tilde{\varrho}_n(s) \nabla \tilde{u}_n(s) : \nabla \psi \, dx \, ds &= \int_0^t \int_Q \tilde{\varrho}_n(s) \tilde{u}_n(s) : \Delta \psi \, dx \, ds \\
&\quad + \int_0^t \int_Q \nabla \tilde{\varrho}_n(s) \otimes \tilde{u}_n(s) : \nabla \psi \, dx \, ds, \\
- \int_0^t \int_Q \tilde{\varrho}(s) \nabla \tilde{u}(s) : \nabla \psi \, dx \, ds &= \int_0^t \int_Q \tilde{\varrho}(s) \tilde{u}(s) : \Delta \psi \, dx \, ds \\
&\quad + \int_0^t \int_Q \nabla \tilde{\varrho}(s) \otimes \tilde{u}(s) : \nabla \psi \, dx \, ds.
\end{align*}
\]

Then, the following corollary is essentially a consequence of Lemmata 5.4 and 5.5.

**Corollary 5.6.** Let us assume that \( \phi \in W^{1,\infty}(O; \mathbb{R}) \) and \( \psi \in W^{2,\infty}(O; \mathbb{R}^3) \). Then the following convergences hold:

\[
\begin{align*}
\lim_{n \to \infty} \left\| \int_Q \tilde{\varrho}_n(x) \cdot \phi(x) \, dx - \int_Q \tilde{\varrho}(x) \cdot \phi(x) \, dx \right\|_{L^1(\bar{O} \times [0,T])} &= 0, \quad (5.25) \\
\lim_{n \to \infty} \left\| \Lambda^1_n(\tilde{\varrho}_n, \tilde{u}_n, \tilde{u}, \phi) - \Lambda^1(\tilde{\varrho}, \tilde{u}, \phi) \right\|_{L^1(\bar{O} \times [0,T])} &= 0, \quad (5.26) \\
\lim_{n \to \infty} \left\| \int_Q \tilde{\varrho}_n(x) \tilde{u}_n(x) \cdot \psi(x) \, dx - \int_Q \tilde{\varrho}(x) \tilde{u}(x) \cdot \psi(x) \, dx \right\|_{L^1(\bar{O} \times [0,T])} &= 0, \quad (5.27) \\
\lim_{n \to \infty} \left\| \Lambda^2_n(\tilde{\varrho}_n, \tilde{u}_n, \tilde{W}_n, \psi) - \Lambda^2(\tilde{\varrho}, \tilde{u}, \tilde{W}, \psi) \right\|_{L^1(\bar{O} \times [0,T])} &= 0. \quad (5.28)
\end{align*}
\]

**Proof of Corollary 5.6.** The first and the third convergences follow directly from the identities

\[
\left\| \int_Q \tilde{\varrho}_n(x) \cdot \phi(x) \, dx - \int_Q \tilde{\varrho}(x) \cdot \phi(x) \, dx \right\|_{L^1(\bar{O} \times [0,T])}
\]
For Lebesgue-a.e. $t \in [0,T]$ and $\bar{\omega}$-a.e. $\omega \in \bar{\Omega}$, we deduce that
\[
\int_{\mathcal{O}} \tilde{\varrho}(t,x) \cdot \phi(x) \, dx - \Lambda^1(\tilde{\varrho}, \tilde{\mathbf{u}}, \phi)(t) = 0,
\]
Hence, for Lebesgue-a.e. $t \in [0,T]$ and $\bar{\omega}$-a.e. $\omega \in \bar{\Omega}$, we deduce that
\[
\int_{\mathcal{O}} \tilde{\varrho}(t,x) \cdot \phi(x) \, dx - \Lambda^1(\tilde{\varrho}, \tilde{\mathbf{u}}, \phi)(t) = 0,
\]
and convergences (5.9) and (5.11) respectively. For (5.26), by the Fubini’s theorem we have
\[
\|\Lambda^1_n(\tilde{\varrho}_n, \tilde{\mathbf{u}}_n, \phi) - \Lambda^1(\tilde{\varrho}, \tilde{\mathbf{u}}, \phi)\|_{L^1(\bar{\Omega} \times [0,T])} = \int_0^T \mathbb{E} \left| \Lambda^1_n(\tilde{\varrho}_n, \tilde{\mathbf{u}}_n, \phi) - \Lambda^1(\tilde{\varrho}, \tilde{\mathbf{u}}, \phi) \right| dt.
\]
Convergence (5.10) shows that the term in the definition of $\Lambda^1_n(\tilde{\varrho}_n, \tilde{\mathbf{u}}_n, \phi)$ tends to the corresponding term in $\Lambda^1(\tilde{\varrho}, \tilde{\mathbf{u}}, \phi)$ at least in the space $L^1(\bar{\Omega} \times [0,T])$. Similarly, with the help of the Fubini’s theorem, the definition of the maps $\Lambda^2_n, \Lambda^2$ and convergences (5.12)–(5.14) we can deduce the convergence (5.28).

Since $(\varrho_n, \mathbf{u}_n)$ is a solution to (1.1), it satisfies
\[
\int_{\mathcal{O}} \varrho_n(t,x) \cdot \phi(x) \, dx = \Lambda^1_n(\varrho_n, \mathbf{u}_n, \phi)(t) \quad \mathbb{P}\text{-a.s.}
\]
\[
\int_{\mathcal{O}} \varrho_n(t,x) \mathbf{u}_n(t,x) \cdot \psi(x) \, dx = \Lambda^2_n(\varrho_n, \mathbf{u}_n, W_n, \psi)(t) \quad \mathbb{P}\text{-a.s.}
\]
for all $t \in [0,T]$ and $\phi \in W^{1,\infty}(\mathcal{O}; \mathbb{R})$, $\psi \in W^{2,\infty}(\mathcal{O}; \mathbb{R}^3)$ and in particular, we have
\[
\int_0^T \mathbb{E} \left| \left( \int_{\mathcal{O}} \varrho_n(t,x) \cdot \phi(x) \, dx \right) - \Lambda^1_n(\varrho_n, \mathbf{u}_n, \phi)(t) \right| dt = 0,
\]
\[
\int_0^T \mathbb{E} \left| \int_{\mathcal{O}} \varrho_n(t,x) \mathbf{u}_n(t,x) \cdot \psi(x) \, dx - \Lambda^2_n(\varrho_n, \mathbf{u}_n, W_n, \psi)(t) \right| dt = 0.
\]
Since the laws $\mathcal{L} \left( \tilde{\varrho}_n, \tilde{\varrho}_n, \tilde{\mathbf{m}}_n, \tilde{\mathbf{c}}_n, \tilde{\mathbf{r}}_n, \tilde{W}_n \right)$ and $\mathcal{L} \left( \varrho_n, \sqrt{\varrho_n}, \varrho_n \mathbf{u}_n, \sqrt{\varrho_n} \mathbf{u}_n, \varrho_n^{1/2} \mathbf{u}_n, W_n \right)$ coincide and by the identifications from Lemma 4.13 and Lemma 4.15, we find that
\[
\int_0^T \mathbb{E} \left| \left( \int_{\mathcal{O}} \tilde{\varrho}_n(t,x) \cdot \phi(x) \, dx \right) - \Lambda^1_n(\tilde{\varrho}_n, \tilde{\mathbf{u}}_n, \phi)(t) \right| dt = 0,
\]
\[
\int_0^T \mathbb{E} \left| \int_{\mathcal{O}} \tilde{\varrho}_n(t,x) \tilde{\mathbf{u}}_n(t,x) \cdot \psi(x) \, dx - \Lambda^2_n(\tilde{\varrho}_n, \tilde{\mathbf{u}}_n, \tilde{W}_n, \psi)(t) \right| dt = 0.
\]
By Corollary 5.6, the limit $n \to \infty$ in these equations yield
\[
\int_0^T \mathbb{E} \left| \int_{\mathcal{O}} \tilde{\varrho}(t,x) \cdot \phi(x) \, dx - \Lambda^1(\tilde{\varrho}, \tilde{\mathbf{u}}, \phi)(t) \right| dt = 0,
\]
\[
\int_0^T \mathbb{E} \left| \int_{\mathcal{O}} \tilde{\varrho}(t,x) \tilde{\mathbf{u}}(t,x) \cdot \psi(x) \, dx - \Lambda^2(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{W}, \psi)(t) \right| dt = 0.
\]
\[ \int_{\mathcal{O}} \tilde{\varrho}(t, x) \tilde{u}(t, x) \cdot \psi(x) \, dx - \Lambda^2 \left( \tilde{\varrho}, \tilde{u}, \tilde{W}, \psi \right)(t) = 0. \]

By definition of \( \Lambda^1 \) and \( \Lambda^2 \), this means that for Lebesgue-a.e. \( t \in [0, T] \) and \( \tilde{\mathbb{P}} \)-a.e. \( \omega \in \tilde{\Omega} \),

\[
\int_{\mathcal{O}} \tilde{\varrho}(t, x) \cdot \phi(x) \, dx = \int_{\mathcal{O}} \tilde{\varrho}(0, x) \cdot \phi(x) \, dx + \int_{0}^{t} \int_{\mathcal{O}} \tilde{\varrho}(s) \tilde{u}(s) \cdot \nabla \phi \, dx \, ds,
\]

\[
\int_{\mathcal{O}} \tilde{\varrho}(t, x) \tilde{u}(t, x) \cdot \psi \, dx = \int_{\mathcal{O}} \tilde{u}(0) \cdot \psi \, dx + \int_{0}^{t} \int_{\mathcal{O}} \tilde{\varrho}(s) \tilde{u}(s) \otimes \tilde{u}(s) : \nabla \psi \, dx \, ds
\]

\[- \int_{0}^{t} \int_{\mathcal{O}} \tilde{\varrho}(s) \tilde{u}(s) : \nabla \psi \, dx \, ds + \int_{0}^{t} \int_{\mathcal{O}} \tilde{\varrho}(s) \text{div} \psi \, dx \, ds
\]

\[+ \int_{0}^{t} \int_{\mathcal{O}} \tilde{\varrho}(s) f \cdot \psi \, dx \, d\tilde{W}(s). \]

Setting \( \tilde{U} := \left( \tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{F}} \right) \), we infer the system \( \left( \tilde{U}, \tilde{W}, \tilde{\varrho}, \tilde{u} \right) \) is a martingale solution to (1.1).

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A  Some compactness and tightness results

We now move to the tightness of law for the momentum. For the compact embeddings in this case we will be using Lemma D.11 [43, Theorem 6], Lemma A.4 and the Aldous condition [A], which we recall in Appendix D.

The following lemma (see [37, Lemma A.7 and Lemma A.8]) gives us a useful conclusion of the Aldous condition [A].

**Lemma A.1.** Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of continuous stochastic processes indexed by \(t \in [0,T]\) and taking values in a Banach space \(E\). Assume that this sequence satisfies the Aldous condition [A]. Then, for every \(\epsilon > 0\) there exists a measurable subset \(A_\epsilon \subset C([0,T];E)\) such that

\[
\mathbb{P}^{X_n}(A_\epsilon) \geq 1 - \epsilon, \quad \lim_{h \to 0} \sup_{u \in A_\epsilon} \sup_{|t-s| \leq h} \|u(t) - u(s)\|_E = 0.
\]

Here, by \(\mathbb{P}^{X_n}\) we denote the law of \(X_n\), which is a Borel probability measure on the Banach space \(C([0,T];E)\).

We will use the following modification of the above lemma.

**Lemma A.2.** Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of stochastic processes with integrable trajectories indexed by \(t \in [0,T]\) and taking values in a Banach space \(E\). Assume that this sequence satisfies the Aldous condition [A]. Then, for every \(\epsilon > 0\) there exists a Borel measurable subset \(A_\epsilon \subset L^1([0,T];E)\) such that

\[
\mathbb{P}^{X_n}(A_\epsilon) \geq 1 - \epsilon, \quad \lim_{h \to 0} \sup_{u \in A_\epsilon} \|\tau_h u - u\|_{L^1([0,T];E)} = 0.
\]

**Remark A.3.** Lemma A.2 is a modification of [37, Lemma A.7] where we have replaced the space of càdlàg functions taking values in a separable and complete metric space, \(D(0,T;\mathbb{S})\), with the space of integrable functions taking values in a Banach space, \(L^1(0,T;E)\).

By closely observing the proof of [37, Lemma A.7] and [37, Lemma A.8] (see also [32, Theorem 2.2.2]) it is clear that the space \(D(0,T;\mathbb{S})\) can be replaced by \(L^1(0,T;E)\).

We have used the following compactness result to establish the tightness of \(\{L(\rho_n u_n) : n \in \mathbb{N}\}\) on \(C_w([0,T];L^{3/2}(\mathcal{O}))\). The exact statement and the proof can be found in [3, Theorem 1.8.5].

**Lemma A.4.** Let \(\kappa \geq 0\), \(1 < p < \infty\), and \(\ell \in \mathbb{R}\). Then

\[
L^\infty(0,T;L^p(\mathcal{O})) \cap C^\kappa([0,T];W^{\ell,2}(\mathcal{O})) \hookrightarrow C_w([0,T];L^p(\mathcal{O}))
\]

(A.1)

continuously. Moreover, if \(\kappa > 0\), then the embedding (A.1) is sequentially compact.

B  Measurability of some functions

**Lemma B.1.** Let us assume that \(\delta > 0\) and

\[
q \geq \frac{3 + \delta}{1 + 2\delta}.
\]

Then the map

\[
\Gamma : C([0,T];L^q(\mathcal{O})) \times (L^\infty(0,T;L^{2+\delta}(\mathcal{O})), w^*) \ni (\rho, w) \mapsto \rho^{\frac{3+\delta}{2+\delta}} w \in (L^\infty(0,T;L^{3/2}(\mathcal{O})), w^*)
\]

is (well-defined and) sequentially continuous.
Remark B.2. Since \( \frac{1+\delta}{1+2\delta} < 1 \) for \( \delta > 0 \), we can always find \( q \) such that the above and \( q < 3 \), i.e.
\[
\frac{1+\delta}{1+2\delta} < q < 3.
\]

So the assumption (B.1) is not void.

Lemma B.3. Assume that \( \alpha \in (0,1) \). Consider maps \( g_2 : \mathbb{R} \ni x \mapsto \text{sgn}(x)|x|^{\alpha} \in \mathbb{R} \) and \( g_1 : \mathbb{R} \ni x \mapsto |x|^{\alpha} \in [0,\infty) \). Then \( g_1 \) and \( g_2 \) are globally \( \alpha \)-Hölder continuous with constants \( 1 \) and resp. \( 2 \), i.e.
\[
|g_i(x_2) - g_i(x_1)| \leq |x_2 - x_1|^\alpha, \quad x_1, x_2 \in \mathbb{R}, \quad i = 1, 2, \quad (B.3)
\]
Then, for \( i = 1, 2 \), the Nemytskii map \( G_i \) associated with \( g_i \), is globally \( \alpha \)-Hölder from the Lebesgue space \( L^p(O) \) to \( L^{q/\alpha}(O) \) for any \( p \in [1,\infty] \), with the same constant as \( g_i \).

Proof of Lemma B.3. It is a standard observation that \( g_i, i = 1, 2 \) restricted to the interval \([0,\infty)\) is globally \( \alpha \)-Hölder with constant 1. Then the first part of the result for \( i = 2 \) follows by applying another standard argument, see e.g. the proof of Proposition 3.2 [14]. Then the first part of the result for \( i = 2 \) follows by observing that function \( g_1 \) is even. The proof of the second part is also standard.

Proof of Lemma B.1. Let us fix \( \delta > 0 \). Put \( \alpha = \frac{1+\delta}{1+2\delta} \). Since by Lemma B.3 the map
\[
C([0,T];L^q(O)) \ni \rho \mapsto \rho^\alpha \in C([0,T];L^{q/\alpha}(O))
\]
is globally \( \alpha \)-Hölder continuous, it is sufficient to consider the following map (denoted, in order to avoid introducing additional symbols) by \( \Gamma \) as well,
\[
\Gamma : C([0,T];L^{q/\alpha}(O)) \times (L^\infty(0,T;L^{2+\delta}(O)),w^*) \ni (v,w) \mapsto vw \in (L^\infty(0,T;L^{3/2}(O)),w^*) \quad (B.4)
\]
First of all, let us observe that since by (B.1)
\[
\frac{1}{q/\alpha} + \frac{1}{2+\delta} \leq \frac{1}{3/2}, \quad (B.5)
\]
in view of Proposition C.1 the auxiliary map \( \Gamma \) defined in (B.6) is well defined, bilinear and bounded with respect to the normed spaces involved. However, we need to prove the continuity in the weak* topologies as shown.

Since \( L^\infty(0,T;L^{3/2}(O)) \) is the dual of \( L^1(0,T;L^3(O)) \), it is sufficient to prove that for every \( \psi \in L^1(0,T;L^3(O)) \) the map
\[
C([0,T];L^{q/\alpha}(O)) \times (L^\infty(0,T;L^{2+\delta}(O)),w^*) \ni (v,w) \mapsto \langle \Gamma(v,w),\psi \rangle \in \mathbb{R}, \quad (B.6)
\]
is sequentially continuous. Here \( \langle \cdot,\cdot \rangle \) denotes the duality between \( L^\infty(0,T;L^{3/2}(O)) \) and \( L^1(0,T;L^3(O)) \).

For this aim let us choose and fix an element \( \psi \in L^1(0,T;L^3(O)) \), a sequence \( \{v_n\} \) convergent to \( v \) in \( C([0,T];L^q(O)) \) and a sequence \( \{w_n\} \) convergent to \( w \) in \( (L^\infty(0,T;L^{2+\delta}(O)),w^*) \).

We will show that the sequence \( \langle \Gamma(v_n,w_n),\psi \rangle \) converges to \( \langle \Gamma(v,w),\psi \rangle \).
By Proposition C.1 in view of (B.5) we have the following train of equalities or inequalities

\[
\langle \Gamma(v_n, w_n), \psi \rangle - \langle \Gamma(v, w), \psi \rangle = \langle \Gamma(v_n, w_n) - \Gamma(v, w), \psi \rangle
\]
\[
= \langle \Gamma(v_n, w_n) - \Gamma(v, w_n), \psi \rangle + \langle \Gamma(v, w_n) - \Gamma(v, w), \psi \rangle
\]
\[
= \int_0^T \int_\mathcal{O} (v_n(t, x) - v)(w_n(t, x)) \cdot \psi(t, x) \, dx \, dt
\]
\[
+ \int_0^T \int_\mathcal{O} (v(t, x)(w_n(t, x) - w(t, x))) \cdot \psi(t, x) \, dx \, dt
\]
\[
\leq \|v_n - v\|_{L^\infty(0, T; L^{2+\delta}(\mathcal{O}))} \|w_n(t)\|_{L^\infty(0, T; L^{2+\delta}(\mathcal{O}))} \|\psi(t)\|_{L^1(0, T; L^2(\mathcal{O}))}
\]
\[
+ \int_0^T \int_\mathcal{O} (w_n(t, x) - w(t, x)) v(t, x) \cdot \psi(t, x) \, dx \, dt
\]
\[
:= I_1^n + I_2^n.
\]

Since \(v_n \to v\) in \(C([0, T]; L^{q/\alpha}(\mathcal{O}))\) we infer that \(I_1^n \to 0\) as \(n \to \infty\). For the term \(I_2^n\), let us observe that \(v\psi \in L^1(0, T; L^{2+\delta}(\mathcal{O}))\) by Proposition C.1. Since \(L^\infty(0, T; L^{2+\delta}(\mathcal{O}))\) is the dual of \(L^1(0, T; L^{2+\delta}(\mathcal{O}))\) and \(w_n \to w\) weakly* in \(L^\infty(0, T; L^{2+\delta}(\mathcal{O}))\). Therefore, \(I_2^n \to 0\) as \(n \to \infty\). Hence, the result follows.

**Lemma B.4.** Let \(\delta > 0\) and \(q \geq 1\). The map

\[
\hat{\Gamma} : C([0, T]; L^q(\mathcal{O})) \times (L^\infty(0, T; L^{2+\delta}(\mathcal{O})), w^*) \ni (v, w) \mapsto v \frac{q}{2+\delta} \in \left( L^\infty(0, T; L^2(\mathcal{O})), w^* \right)
\]

is sequentially continuous.

**Proof of Lemma B.4.** Let us choose and fix \(\delta > 0\). It is sufficient to consider the case \(q = 1\). Denote \(\beta = \frac{\delta}{2(2+\delta)} \in (0, \frac{1}{2})\), i.e. \(\frac{1}{2} = \frac{1}{2+\delta} + \beta\). As in the proof of Lemma B.1 it is sufficient to consider a modified function \(\hat{\Gamma}\) defined by

\[
\hat{\Gamma} : C([0, T]; L^{1/\beta}(\mathcal{O})) \times (L^\infty(0, T; L^{2+\delta}(\mathcal{O})), w^*) \ni (v, w) \mapsto v w \in \left( L^\infty(0, T; L^2(\mathcal{O})), w^* \right)
\]

and prove that this new function \(\hat{\Gamma}\) is sequentially continuous. First of all let us observe that since \(\frac{1}{2} = \frac{1}{2+\delta} + \beta\), by Proposition C.1 the maps \(\hat{\Gamma}\) is defined. In order to prove that it is sequentially continuous, let consider a sequence \(\{v_n\}\) convergent to \(v\) in \(C([0, T]; L^{1/\beta}(\mathcal{O}))\) and a sequence \(\{w_n\}\) convergent to \(w\) in \(C([0, T]; L^{2+\delta}(\mathcal{O})), w^*\).

We will show that the sequence \(\hat{\Gamma}(v_n, w_n), \psi\) converges to \(\hat{\Gamma}(v, w), \psi\). We have

\[
\langle \hat{\Gamma}(v_n, w_n), \psi \rangle - \langle \hat{\Gamma}(v, w), \psi \rangle = \langle \hat{\Gamma}(v_n, w_n) - \hat{\Gamma}(v, w_n), \psi \rangle + \langle \hat{\Gamma}(v, w_n) - \hat{\Gamma}(v, w), \psi \rangle
\]
\[
= \int_0^T \int_\mathcal{O} (v_n(t, x)w_n(t, x) - v w_n(t, x)) \cdot \psi(t, x) \, dx \, dt
\]
\[
+ \int_0^T \int_\mathcal{O} (v(t, x)w_n(t, x) - v w(t, x)) \cdot \psi(t, x) \, dx \, dt
\]
\[
\leq \|v_n - v\|_{L^\infty(0, T; L^{1/\beta}(\mathcal{O}))} \|w_n(t)\|_{L^\infty(0, T; L^{2+\delta}(\mathcal{O}))} \|\psi(t)\|_{L^1(0, T; L^2(\mathcal{O}))}
\]
\[
+ \int_0^T \int_\mathcal{O} (w_n(t, x) - w(t, x)) v(t, x) \cdot \psi(t, x) \, dx \, dt
\]
\[
:= I_1^n + I_2^n.
\]

Since \(v_n \to v\) in \(C([0, T]; L^{1/\beta}(\mathcal{O}))\), \(I_1^n \to 0\) as \(n \to \infty\). For \(I_2^n\), we observe that since \(\beta + \frac{1}{2} = \frac{1+\delta}{2+\delta}\) by Proposition C.1 we infer that \(v\psi \in L^1(0, T; L^{2+\delta}(\mathcal{O}))\). Since \(L^\infty(0, T; L^{2+\delta}(\mathcal{O}))\) is the

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dual of $L^1(0,T;L^{2+\delta}(\mathcal{O}))$ and $w_n \to w$ weakly* in $L^\infty(0,T;L^{2+\delta}(\mathcal{O}))$, we infer that $I^n_y \to 0$ as $n \to \infty$. Hence, the result follows. \hfill \Box

C About some simple consequences of the Hölder inequality

In many places we have used the following simple consequence of the Hölder inequality.

Proposition C.1. Assume that real numbers $r \in [1, \infty]$ and $p,q \in [r, \infty]$ are such that

$$\frac{1}{p} + \frac{1}{r} = \frac{1}{q}. \quad (C.1)$$

Assume that $X,Y$ and $Z$ are Banach spaces and

$$\beta : X \times Y \to Z$$

is bilinear continuous map. \quad (C.2)

If $\mu$ is a non-negative measure on $(M,M)$, then

$$\|\beta(f,g)\|_{L^q(M,Z)} \leq \|f\|_{L^p(M,X)} \|g\|_{L^r(M,Y)} \quad (C.3)$$

for all elements $f \in L^p(M,X)$ and $g \in L^r(M,Y)$.

We have the following two simple corollaries.

Corollary C.2. Assume that $q \in [1,2]$ and $p \in [q, \infty)$.

$$\frac{1}{p} + \frac{1}{2} = \frac{1}{q}$$

Then, for appropriate $f$ and $g$ we have

$$\|fg\|_{L^q(\Omega, L^\infty(0,T;L^{2+\delta}(\mathcal{O})))} \leq \|f\|_{L^2(\Omega, L^\infty(0,T;L^6(\mathcal{O})))} \|g\|_{L^p(\Omega, L^\infty(0,T;L^2(\mathcal{O})))}, \quad (C.4)$$

$$\|fg\|_{L^q(\Omega, L^\infty(0,T;L^{2+\delta}(\mathcal{O})))} \leq \|f\|_{L^2(\Omega, L^\infty(0,T;L^6(\mathcal{O})))} \|g\|_{L^p(\Omega, L^\infty(0,T;L^2(\mathcal{O})))}, \quad (C.5)$$

$$\|fg\|_{L^q(\Omega, L^\infty(0,T;L^1(\mathcal{O})))} \leq \|f\|_{L^2(\Omega, L^\infty(0,T;L^2(\mathcal{O})))} \|g\|_{L^p(\Omega, L^\infty(0,T;L^2(\mathcal{O})))}. \quad (C.6)$$

D Stochastic preliminaries

Definition D.1. A sequence of measures $\{\mu_n\}_{n \in \mathbb{N}}$ on $(E, \mathcal{B}(E))$ is said to be weakly convergent to a measure $\mu$ if for every $\phi \in C_b(E)$, we have

$$\lim_{n \to \infty} \int_E \phi(x) \mu_n(dx) = \int_E \phi(x) \mu(dx).$$

Definition D.2. The family $\Lambda$ is said to be compact (respectively relatively compact), if an arbitrary sequence $\{\mu_n\}_{n \in \mathbb{N}}$ of elements from $\Lambda$ contains a subsequence $\{\mu_{n_k}\}_{k \in \mathbb{N}}$ weakly convergent to a measure $\mu \in \Lambda$ (respectively to a measure $\mu$ on $(E, \mathcal{B}(E))$).

Theorem D.3 (Prokhorov Theorem). The family $\Lambda$ of probability measures on $(E, \mathcal{B}(E))$ is relatively compact if and only if it is tight.
**Remark D.4.** Let $X$, $Y$ be two topological spaces and let $(\Omega, \mathcal{F}, P)$, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be two probability spaces. Also assume that

$$z_n : \Omega \to X, \quad v_n : \Omega \to Y;$$

$$\tilde{z}_n : \tilde{\Omega} \to X, \quad \tilde{v}_n : \tilde{\Omega} \to Y$$

are Borel-measurable maps. Let $F : X \times Y \ni (z, v) \mapsto F(z, v) \in \mathbb{R}$ is a Borel-measurable map. If $\mathcal{L}((z_n, v_n)) = \mu = \tilde{\mathcal{L}}((\tilde{z}_n, \tilde{v}_n))$ on $\mathcal{B}(X \times Y)$, then

$$\int_{\Omega} F(z_n, v_n)(\omega) \, dP(\omega) = \int_{\Omega} F(\tilde{z}_n, \tilde{v}_n)(\omega) \, d\tilde{P}(\omega). \quad (D.1)$$

**Proof.** Starting from the LHS of (D.1), using the definition of law of a random variable and equivalence of laws on $\mathcal{B}(X \times Y)$, we get

$$\int_{\Omega} F(z_n, v_n)(\omega) \, dP(\omega) = \int_{\Omega} F(z_n(\omega), v_n(\omega)) \, dP(\omega) = \int_{X \times Y} F(x, y) \, d\mathcal{L}(x, y)$$

$$= \int_{X \times Y} F(x, y) \, d\mu(x, y) = \int_{X \times Y} F(\tilde{x}_n(\omega), \tilde{v}_n(\omega)) \, d\tilde{\mathcal{L}}(\omega) \, d\tilde{P}(\omega).$$

The approach to establish Hölder continuity of a stochastic integral then relies on the Kolmogorov continuity theorem, which statement and proof follow directly from \[23, Theorem 3.3\].

**Theorem D.5** (Kuratowski Theorem). Assume that $X_1$, $X_2$ are two Polish spaces with their Borel $\sigma$-fields denoted respectively by $\mathcal{B}(X_1)$, $\mathcal{B}(X_2)$. If $\phi : X_1 \to X_2$ is an injective Borel measurable map, then for any $E_1 \in \mathcal{B}(X_1)$, $E_2 := \phi(E_1) \in \mathcal{B}(X_2)$.

We may also need the following corollary \[20, Proposition C.2\] of the above result.

**Corollary D.6.** Suppose that $X_1$, $X_2$ are two topological spaces with their Borel $\sigma$-fields denoted respectively by $\mathcal{B}(X_1)$, $\mathcal{B}(X_2)$. Suppose that $\phi : X_1 \to X_2$ is an injective Borel measurable map such that for any $E_1 \in \mathcal{B}(X_1)$, $E_2 := \phi(E_1) \in \mathcal{B}(X_2)$. Then, if $g : X_1 \to \mathbb{R}$ is a Borel measurable map then a function $f : X_2 \to \mathbb{R}$ defined by

$$f(x_2) = \begin{cases} g(\phi^{-1}(x_2)), & \text{if } x_2 \in \phi(X_1), \\ \infty, & \text{if } x_2 \notin \phi(X_1), \end{cases} \quad (D.2)$$

is also Borel measurable.

**Theorem D.7** (Kolmogorov Continuity Theorem). Let $X$ be a stochastic process taking values in a separable Banach space $U$. Assume that there exist constant $K > 0$, $\nu \geq 1$, $\sigma > 0$ such that $X(0) = 0$ and

$$\mathbb{E}\|X(t) - X(s)\|_U^\nu \leq K|t-s|^{1+\sigma}, \text{ for all } s, t \in [0, T]. \quad (D.3)$$

Then there exists $Y$, a modification of $X$, which has $\mathbb{P}$-a.s. Hölder continuous trajectories with exponent $\gamma$ for every $\kappa \in (0, \frac{\nu}{p})$. In addition, we have

$$\mathbb{E}\|Y\|_{C^\kappa([0,T];U)}^\nu \lesssim K, \quad (D.4)$$

where the proportional constant depends only on $T$.  

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We use the following consequence of the Vitali Convergence Theorem [40, Theorem C.4].

**Theorem D.8 (Vitali).** Let \((a_N)\) be a sequence of integrable functions on a probability space \((\Omega, \mathcal{B}(\Omega), \mathbb{P})\) such that \(a_N \rightarrow a\) a.e. as \(N \rightarrow \infty\) (or \(a_N \rightarrow a\) in measure) for some integrable function \(a\) and there exists \(r > 1\) and a constant \(C > 0\) such that

\[
\mathbb{E}|a_N|^r \leq C \quad \text{for all } N \in \mathbb{N}.
\]

Then \(\mathbb{E}|a_N| \rightarrow \mathbb{E}|a|\) as \(N \rightarrow \infty\).

### D.1 Tightness Preliminaries

Let \(E\) be a Hausdorff topological space with the Borel sigma-field denoted by \(\mathcal{B}(E)\).

**Definition D.9.** The family \(\Lambda\) of probability measures on \((E, \mathcal{B}(E))\) is said to be tight if for arbitrary \(\varepsilon > 0\) there exists a compact set \(K_\varepsilon \subset E\) such that

\[
\nu(K_\varepsilon) \geq 1 - \varepsilon \quad \forall \nu \in \Lambda.
\]

We use the following two lemmas, see [43], to establish the tightness of laws for various processes.

**Lemma D.10.** Let \(X, B, Y\) be Banach spaces such that the embeddings

\[
X \hookrightarrow B \hookrightarrow Y\]

are dense and continuous and the embedding

\[
X \hookrightarrow B\]

is compact. Assume that \(r > 1\). Let \(F \subset L^\infty(0,T;X)\) be a bounded set such that the set

\[
\{\frac{\partial u}{\partial t} : u \in F\} \subset L^r(0,T;Y) \quad \text{and is bounded.}
\]

Then \(F\) is relatively compact in \(C([0,T];B)\). In other words, the embedding

\[
W^{1,r}(0,T;Y) \cap L^r(0,T;Y) \hookrightarrow C([0,T];B)
\]

is compact.

**Lemma D.11.** Let \(X, B, Y\) be Banach spaces such that the embeddings

\[
X \hookrightarrow B \hookrightarrow Y\]

are dense and continuous and the embedding

\[
X \hookrightarrow B\]

is compact. Assume that \(1 \leq p < q \leq \infty\) and

1. \(F\) is uniformly bounded in \(L^q(0,T;B) \cap L^1(0,T;X)\);
2. \(\forall 0 < t_1 < t_2 < T, \lim_{h \to 0} \sup_{f \in F} \|\tau_n f - f\|_{L^1(t_1,t_2;Y)} = 0\).

Then \(F\) is relatively compact in \(L^p(0,T;B)\).

**Definition D.12.** Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of stochastic processes indexed by \(t \in [0,T]\) and taking values in a Banach space \(E\). We say that \((X_n)_{n \in \mathbb{N}}\) satisfies the Aldous condition \([A]\) iff for every \(\epsilon > 0\) and \(\eta > 0\) there exists a \(\theta > 0\) such that for every sequence \((\tau_n)_{n \in \mathbb{N}}\) of \([0,T]\)-valued stopping times one has

\[
\sup_{n \in \mathbb{N}} \sup_{0 \leq h \leq \theta} \mathbb{P}\{\|X_n((\tau_n + h) \wedge T) - X_n(\tau_n)\|_E \geq \eta\} \leq \epsilon.
\]
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