ON ASYMPTOTIC BASE LOCI
OF RELATIVE ANTI-CANONICAL DIVISORS
OF ALGEBRAIC FIBER SPACES

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ABSTRACT. In this paper, we study the relative anti-canonical divisor $-K_{X/Y}$ of an algebraic fiber space $\phi : X \to Y$, and we reveal relations among positivity conditions of $-K_{X/Y}$, certain flatness of direct image sheaves, and variants of the base loci including the stable (augmented, restricted) base loci and upper level sets of Lelong numbers. This paper contains three main results: The first result says that all the above base loci are located in the horizontal direction unless they are empty. The second result is an algebraic proof for Campana–Cao–Matsumura’s equality on Hacon–McKernan’s question, whose original proof depends on analytic methods. The third result partially solves the question which asks whether algebraic fiber spaces with semi-ample relative anti-canonical divisor actually have a product structure via the base change by an appropriate finite étale cover of $Y$. Our proof is based on algebraic as well as analytic methods for positivity of direct image sheaves.

CONTENTS

1. Introduction 2
1.1. Relative anti-canonical divisors 2
1.2. On augmented base loci 2
1.3. On restricted base loci and stable base loci 3
1.4. On invariants of nef relative anti-canonical divisors 3
1.5. On the structure of semi-ample relative anti-canonical bundles 4
Acknowledgements 5
2. Preliminaries 5
2.1. Notations and conventions 5
2.2. Asymptotic variants of base loci 6
3. Augmented and restricted base loci for lc pairs 7
3.1. On augmented and restricted base loci for lc pairs 7
3.2. Algebraic proof of Campana–Cao–Matsumura’s equality 11
4. On asymptotic variants of base loci 13
4.1. Extension of Cao-Höring’s work for nef anti-canonical divisors 13
4.2. Variants of base loci and flatness of direct images 17

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1. Introduction

1.1. Relative anti-canonical divisors. This paper studies an algebraic fiber space \( \phi : X \to Y \) between projective varieties over the complex number field (that is, a surjective morphism with connected fibers) and its relative anti-canonical divisor \( -K_{X/Y} := -K_X + \phi^*K_Y \). The total space \( X \) and the base \( Y \) are assumed to be smooth in this section, but the case where they are singular is also treated in this paper. The geometric structure of \( \phi : X \to Y \) is known to be deeply connected with several positivity conditions of \( -K_{X/Y} \). Positivity conditions of \( -K_{X/Y} \) in algebraic geometry can be measured by the asymptotic base loci of \( -K_{X/Y} \). Here the asymptotic base loci mean the stable base locus \( B(-K_{X/Y}) \) and its approximative variants introduced by [ELMNP1]: the augmented base locus \( B_+( -K_{X/Y} ) \) and the restricted base locus \( B_-( -K_{X/Y} ) \). The semi-ampleness (resp. ampleness, nefness) is equivalent to the condition \( B_-( \bullet ) = \emptyset \) (resp. \( B_+( \bullet ) = \emptyset \), \( B_-( \bullet ) = \emptyset \)).

Our interest is to understand how restricted the geometric structure of \( \phi : X \to Y \) is when \( -K_{X/Y} \) satisfies certain positivity, in other words, to reveal relations between the asymptotic base loci and the geometric structure of \( \phi : X \to Y \). Regarding this, there are several known results. When \( -K_{X/Y} \) is nef, an argument due to Cao–Höring ([CH19]) determines the detailed geometric structure of \( \phi : X \to Y \) including the local triviality (cf. [PZ19, §A]). The celebrated work due to Kollár–Miyaoka–Mori ([KoMM92, Corollary 2.8]) tells us that, in the case when \( Y \) is not one point, \( -K_{X/Y} \) is not ample (i.e., \( B_+( -K_{X/Y} ) \neq \emptyset \)). In the same case, Deng’s result ([Den17, Theorem E]) further shows that \( B_+( -K_{X/Y} ) \) is dominant over \( Y \). All the above mentioned results can be generalized to klt pairs. As a natural question, it arises the problem of studying the case of lc pairs or the case when \( B_-( -K_{X/Y} ) \neq \emptyset \).

The aim of this paper is to systematically understand such problems and deeper relations between the geometric structure of \( \phi \) and positivity conditions on \( -K_{X/Y} \). To this end, we focus on the various base loci of \( -K_{X/Y} \) and apply the theory of positivity of direct image sheaves.

1.2. On augmented base loci. This subsection introduces the results related to the augmented base locus \( B_+( -K_{X/Y} ) \). The following theorem was firstly proved by Deng ([Den17, Theorem E]).

**Theorem 1.1** (A special case of Corollary 3.7). If \( B_+( -K_{X/Y} ) \) is not dominant over \( Y \), then \( Y \) is a point. In particular, if \( \dim Y > 0 \), then \( \dim B_+( -K_{X/Y} ) \geq \dim Y \).

Note that Deng’s proof uses an analytic method, but we give an algebraic proof of this theorem. Furthermore, the full statement of Theorem 1.1 includes the case where \( X \) has at worst lc singularities (see Corollary 3.7), which does not follow
from [Den17, Theorem E]. The behavior of $\mathcal{B}_+(-K_{X/Y} - \Delta)$ differs according to the singularity of the pair $(X, \Delta)$ (see Example 6.3). Hence Theorem 1.2 seems to be an extremely generalized result.

Theorem 1.1 is one of corollaries to the theorem below.

**Theorem 1.2** (Theorem 3.3). Fix a smooth fiber $F$ of smallest dimension. Let $D$ be a divisor on $X$ such that $\mathcal{O}_F(D)$ is nef (resp. ample). If $\mathcal{B}_-(D)$ (resp. $\mathcal{B}_+(D)$) intersects $F$, then so does $\mathcal{B}_-(rD - K_{X/Y})$ (resp. $\mathcal{B}_+(rD - K_{X/Y})$) for any positive rational number $r$.

Using this theorem with $D = K_{X/Y}$ or $K_X$, one obtains the following corollary:

**Corollary 1.3** ([Pat14, Corollary 1.3]). Fix a smooth fiber $F$ of smallest dimension. Assume that $K_F$ is nef. Then $F$ and $\mathcal{B}_-(K_{X/Y})$ do not intersect. Suppose further that $K_Y$ is nef. Then any proper curve $C \subset X$ with $K_X \cdot C < 0$ does not intersect $F$.

1.3. **On restricted base loci and stable base loci.** It is natural to consider the restricted base locus $\mathcal{B}_-(K_{X/Y})$ and other base loci, as the next problem of Cao–Höring’s critical work for the case $\mathcal{B}_-(K_{X/Y}) = \emptyset$. The following result can be seen as an analogue of Theorem 1.1 to various base loci (restricted base loci, stable base loci, and upper level sets of Lelong numbers).

**Theorem 1.4** (Theorem 4.7, Theorem 4.8, Theorem 4.9). Let $X$ and $Y$ be projective manifolds and $\phi : X \to Y$ be a surjective morphism with connected fibers. Then we have:

- If $\mathcal{B}_-(K_{X/Y})$ is not dominant over $Y$, then $\mathcal{B}_-(K_{X/Y})$ is empty (that is, $-K_{X/Y}$ is nef).
- If $\mathcal{B}(-K_{X/Y})$ is not dominant over $Y$, then $\mathcal{B}(-K_{X/Y})$ is empty (that is, $-K_{X/Y}$ is semi-ample).
- Let $h$ be a singular hermitian metric on $-K_{X/Y}$ with semipositive curvature and let $P(h)$ denote the set of points at which $h$ has positive Lelong number. If $P(h)$ is not dominant over $Y$, then $P(h)$ is empty (that is, the Lelong number of $\sqrt{-1}\Theta_h$ is zero everywhere).

The direct image $\phi_*(\tilde{A})$ of an appropriate relatively ample divisor $\tilde{A}$ on $X$ satisfies numerical flatness if $-K_{X/Y}$ is nef by [CH19, CCM19]. For the proof of Theorem 1.4, we confirm that the same conclusions hold under the slightly weaker assumption that the restricted base locus $\mathcal{B}_-(K_{X/Y})$ is not dominant over $Y$ (see Theorem 4.6). In this process, we can find a relation between positivity conditions of $-K_{X/Y}$ and certain flatness of the direct image sheaves, e.g. numerical flatness, hermitian flatness, and étale trivializability. Further we clarify that the certain flatness recovers the nefness, semi-ampleness, or the property of $P(h) = \emptyset$.

1.4. **On invariants of nef relative anti-canonical divisors.** Motivated by Hacon–McmKernan’s question ([HM07]), the Iitaka–Kodaira dimension $\kappa(-K_{X/Y})$ and the numerical Kodaira dimension $\text{nd}(-K_{X/Y})$ of nef relative anti-canonical divisors
−\(K_{X/Y}\) have been studied, and we now have the following relations:

\[
\kappa(-K_{X/Y}) \leq \text{nd}(-K_{X/Y}) \leq \dim X \quad (1)
\]

\[
\kappa(-K_F) \leq \text{nd}(-K_F) \leq \dim F \quad (2)
\]

The inequality (1) was showed by Ejiri–Gongyo ([EG19]). The equality (2) was proved by Campana–Cao–Matsumura ([CCM19]), whose methods heavily depends on analytic methods based on metric positivity of direct image sheaves ([BP08, HPS18, PT18], and references therein).

This paper gives an algebraic proof of Campana–Cao–Matsumura’s equality (2) and slightly generalize it to the case where \(Y\) has only canonical singularities.

**Theorem 1.5** (Theorem 3.9). Let \(X\) be a normal projective variety and let \(\Delta\) be an effective \(\mathbb{Q}\)-divisor on \(X\) such that \((X, \Delta)\) is klt. Let \(\phi : X \to Y\) be a morphism with connected fibers onto a \(\mathbb{Q}\)-Gorenstein normal projective variety \(Y\) with only canonical singularities. If \(- (K_{X/Y} + \Delta)\) is nef, then we have

\[
\text{nd}(-(K_{X/Y} + \Delta)) = \text{nd}(-(K_F + \Delta|_F))
\]

for a general fiber \(F\).

### 1.5. On the structure of semi-ample relative anti-canonical bundles.

The situation where \(-K_{X/Y}\) is semi-ample can be expected to occur only in a more restricted case than the case of \(\phi\) being locally trivial. Here we propose the following question:

**Question 1.6.** If \(-K_{X/Y}\) is semi-ample, then does there exist a finite étale cover \(Y' \to Y\) such that the morphism \(F \times_C Y' \to Y\) induced by the second projection factors through \(\phi : X \to Y\)?

\[
\begin{array}{ccc}
F \times_C Y' & \longrightarrow & X \\
\pr_1 \downarrow & \circ & \downarrow \phi \\
Y' & \longrightarrow & Y.
\end{array}
\]

Here \(F\) is a fiber of \(\phi\).

For example, it follows that the above question is affirmative when \(-K_{X/Y}\) is trivial and the irregularity of \(F\) is zero from [LPT18, Theorem 5.8] and [Dru17, Lemma 6.4]. Note that Example 6.1 tells us that \(\phi : X \to Y\) itself is not necessarily a product without taking an étale cover \(Y' \to Y\). We give a partial affirmative answer to Question 1.6:

**Theorem 1.7.** Suppose that \(-K_{X/Y}\) is semi-ample. Then the direct image

\[
\phi_*( -mK_{X/Y})
\]

is étale trivializable for every \(m \in \mathbb{Z}_{>0}\). More precisely, there exists a finite étale cover \(Y' \to Y\) such that the pull-back of \(\phi_*(-mK_{X/Y})\) to \(Y'\) is a trivial vector bundle. Moreover, the following hold:
(1) If the anti-canonical bundle of $-K_F$ on a fiber $F$ is ample, then there exists a finite étale cover $Y' \to Y$ such that the fiber product $X' := Y' \times_Y X$ is isomorphic to the product of $Y' \times_Y F$ as a $Y'$-scheme.

(2) If the irregularity of a general fiber of the Iitaka fibration associated to $-K_F$ is equal to zero, then the same conclusion as in (1) holds.

Our proof is based on a geometric comparison among the original morphism $\phi : X \to Y$ and the Iitaka fibrations of $-K_{X/Y}$ and $-K_F$, which reduces the general case to the extremal case where $-K_F$ is ample or $K_{X/Y}$ is trivial via the direct image $\phi_* (-mK_{X/Y})$.

**Corollary 1.8.** Let $X := \mathbb{P}(E) \to Y$ be the projective space bundle over a projective manifold $Y$. If $-K_{X/Y}$ is semi-ample, then there exists a finite étale cover $Y' \to Y$ such that the fiber product of $X$ and $Y'$ over $Y$ is isomorphic to the product of the projective space and $Y'$.

This paper is organized as follows: In Section 2, we recall the definition and properties of the asymptotic base loci. Section 3, 4, and 5 are respectively devoted to topics explained in Section 1.2, 1.3, and 1.5. In Section 6, we collect examples, which help us to understand our results. In Section A, we give an analytic proof for some results in Section 3 and 4.

After we have finished to write this paper, Yoshinori Gongyo informed the authors that Theorem 1.7 follows from [Amb05, Proposition 4.4, Theorem 4.7]. However, our argument is quite different from that of Ambro and gives more geometric proof, and thus we believe that it is worth to displaying it in this paper.

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### 2. Preliminaries

#### 2.1. Notations and conventions.

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers. We use the following conventions: A *variety* is an integral separated scheme of finite type over $\mathbb{C}$, a *curve* is a 1-dimensional variety, and a *manifold* is a smooth variety. We interchangeably use the words “Cartier divisors”, “invertible sheaves”, and “line bundles”.

2.2. **Asymptotic variants of base loci.** Let \( D \) be a \( \mathbb{Q} \)-Cartier divisor on a projective variety \( X \). We recall the definitions and several properties of the stable base locus \( \mathbb{B}(D) \) of \( D \) and its approximative variants: the augmented base locus \( \mathbb{B}_+(D) \) and restricted base locus \( \mathbb{B}_-(D) \).

**Definition 2.1.** Let \( D \) be a \( \mathbb{Q} \)-Cartier divisor on a projective variety \( X \) and take an integer \( i > 0 \) so that \( iD \) is Cartier. Let \( \text{sp}(X) \) denote the underlying Zariski-topological subset of \( X \). (1) The **stable base locus** \( \mathbb{B}(D) \) is defined by
\[
\mathbb{B}(D) := \bigcap_{m \geq 1} \text{Bs}(imD)_{\text{red}} \subseteq \text{sp}(X).
\]
(2) The **restricted base locus** (or **diminished base locus**) \( \mathbb{B}_-(D) \) and the **augmented base locus** \( \mathbb{B}_+(D) \) are defined by
\[
\mathbb{B}_-(D) := \bigcup_A \mathbb{B}(D + A) \subseteq \text{sp}(X) \quad \text{and} \quad \mathbb{B}_+(D) := \bigcap_A \mathbb{B}(D - A) \subseteq \text{sp}(X),
\]
where \( A \) runs over all the \( \mathbb{Q} \)-Cartier ample divisors on \( X \). (3) Let \( X \) be a projective manifold and \((L,h)\) be a line bundle with a singular hermitian metric with semipositive curvature current. The **Lelong number** \( \nu(h,x) \) of \( h \) at \( x \in X \) is defined by
\[
\nu(h,x) := \liminf_{z \to x} \frac{\varphi(z)}{\log|z - x|},
\]
where \( \varphi \) is a local weight of \( h \). The **upper level set of positive Lelong numbers** \( P(h) \) is defined by
\[
P(h) := \{ x \in X \mid \nu(h,x) > 0 \}.
\]

In general we have the following inclusions
\[
\mathbb{B}_-(L) \subseteq P(h) \quad \text{and} \quad \mathbb{B}_-(L) \subseteq \mathbb{B}(L) \subseteq \mathbb{B}_+(L)
\]
for any singular hermitian line bundle \((L,h)\) on a projective manifold. We remark that \( P(h) \) and \( \mathbb{B}_-(L) \) is a countable union of Zariski-closed sets.

The following preliminary lemma compares the restricted (and augmented) base locus of a divisor with that of its pullback via a surjective morphism.

**Lemma 2.2.** Let \( \phi : X \to Y \) be a surjective morphism from a projective variety \( X \) to a normal projective variety \( Y \). Let \( D \) be a \( \mathbb{Q} \)-Cartier divisor on \( Y \) and \( V \) be a Zariski-open subset of \( Y \) such that \( U := \phi^{-1}(V) \to V \) is flat over \( V \). Then the following holds:
1. \( V \cap \mathbb{B}_-(D) = V \cap \phi(\mathbb{B}_-(\phi^*D)) \).
2. \( V \cap \mathbb{B}_+(D) \subseteq V \cap \phi(\mathbb{B}_+(\phi^*D)) \).

**Proof.** We prove (1). We first consider the inclusion “\( \supseteq \)”.

Let \( A \) be an ample \( \mathbb{Q} \)-Cartier divisor on \( X \). Take an ample \( \mathbb{Q} \)-Cartier divisor \( H \) on \( Y \) so that \( A - \phi^*H \) is ample. Then we have
\[
\mathbb{B}(\phi^*D + A) = \mathbb{B}(\phi^*D + \phi^*H + A - \phi^*H) \subseteq \mathbb{B}(\phi^*(D + H)) \subseteq \phi^{-1}(\mathbb{B}(D + H)) \subseteq \phi^{-1}(\mathbb{B}_-(D)),
\]
and thus $\mathcal{B}_-(\phi^*D) = \bigcup_A \mathcal{B}(\phi^*D + A) \subseteq \phi^{-1}(\mathcal{B}_-(D))$. The assertion can be obtained from the surjectivity of $\phi$. We can prove (2) by a similar argument, and thus we omit it.

Next we show the converse inclusion “$\subseteq$”. For a fixed $v \in V \setminus \phi(\mathcal{B}_-(\phi^*D))$, we check that $v \notin \mathcal{B}_-(D)$. Take a closed subvariety $X' \subseteq X$ so that $\dim X' = \dim Y$ and that the induced morphism $\phi' : X' \to Y$ is flat over $v$. We have $\mathcal{B}_-(\phi^*D) \subseteq X' \cap \mathcal{B}_-(\phi^*D)$ by the above argument. Replacing $X$ with $X'$, we may assume that $\phi$ is generically finite. Let $A$ be an ample Cartier divisor on $X$ and $H$ an ample Cartier divisor on $Y$ such that $\phi^{-1}(v) \cap \text{Supp}(\phi^*H - A) = \emptyset$. Such an $H$ exists, as $\phi$ is finite over a neighborhood of $v$. Take $m \in \mathbb{Z}_{>0}$. Then

$$X \setminus \phi^{-1}(v) \supseteq \mathcal{B}_-(\phi^*D) \supseteq \mathcal{B}(m\phi^*D + A) = \text{Bs}(lm\phi^*D + lA)$$

for some $l > 0$, so $\phi^{-1}(v) \cap \text{Bs}(l\phi^*(mD + H))$, and thus there is a morphism $\alpha : \bigoplus \deg \phi \mathcal{O}_X \to \mathcal{O}_X(l\phi^*(mD + H))$ that is surjective over $\phi^{-1}(v)$. Since $\phi$ is affine over a neighborhood of $v$, the composite $\beta$ of the morphisms

$$\bigoplus \phi_*\mathcal{O}_X \xrightarrow{\phi_*(\alpha)} \phi_*\mathcal{O}_X(l\phi^*(mD + H)) \cong (\phi_*\mathcal{O}_X) \otimes \mathcal{O}_Y(l(mD + H))$$

is surjective over $v$. Furthermore, since $\phi_*\mathcal{O}_X$ is free at $v$, the natural morphism

$$\gamma : (\phi_*\mathcal{O}_X)^\vee \otimes \phi_*\mathcal{O}_X \to \mathcal{O}_Y$$

is surjective over $v$. Take $n \in \mathbb{Z}_{>0}$ so that $((\phi_*\mathcal{O}_X)^\vee \otimes \phi_*\mathcal{O}_X)(nH)$ is globally generated. Then we have the following morphisms:

$$\bigoplus ((\phi_*\mathcal{O}_X)^\vee \otimes \phi_*\mathcal{O}_X)(nH) \cong (\phi_*\mathcal{O}_X)^\vee \otimes \left(\bigoplus \phi_*\mathcal{O}_X\right)(nH)$$

by $\beta$$

$$(\phi_*\mathcal{O}_X)^\vee \otimes \phi_*\mathcal{O}_X)(lmD + (l + n)H)$$

by $\gamma$$

$\mathcal{O}_Y(lmD + (l + n)H)$. These morphisms are surjective over $v$, so $v \notin \mathcal{B}(lmD + (l + n)H) = \mathcal{B}(D + \frac{l+n}{lm}H)$. Then it follows that $v \notin \mathcal{B}_-(D)$ since $m$ can be chosen as large as we want. $\square$

3. Augmented and restricted base loci for lc pairs

3.1. On augmented and restricted base loci for lc pairs. The aim of this subsection is to prove Theorem 3.3 and its corollaries (including Theorem 1.1 and 1.2). We emphasize that the argument in this subsection works not only for non-singular cases (or klt pairs) but also for lc pairs. We use the following notation:

Notation 3.1. Let $\phi : X \to Y$ be a surjective morphism with connected fibers from a normal projective variety $X$ to a normal $\mathbb{Q}$-Gorenstein (that is, $K_Y$ is $\mathbb{Q}$-Cartier) projective variety $Y$ with only canonical singularities. Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We denote by $V(\phi, \Delta)$ the subset of $Y$ consisting of regular points $v$ with the following properties:

(1) The pair $(X, \Delta)$ is log canonical at every point in $\phi^{-1}(v)$.

(2) There exists a birational morphism $\pi : X' \to X$ from a normal projective variety
$X'$ such that $\phi' := \phi \circ \pi : X' \to Y$ is smooth over a neighborhood of $v$.

(3) There exists an effective $\pi$-exceptional reduced Weil divisor $E$ on $X'$ such that
- $\pi_*^{-1}\Delta + E \geq \pi^*(K_X + \Delta) - K_{X'}$, and
- $\phi'^{-1}(v)$ intersects transversally with each component of $\pi_*^{-1}\Delta + E$.

Remark 3.2. (1) The set $V(\phi, \Delta)$ is a Zariski-open set in $Y$.
(2) The pair $(X, \Delta)$ has at worst log canonical singularities at every point in $\phi^{-1}(V(\phi, \Delta))$.
Furthermore, if the non-log canonical locus of $(X, \Delta)$ is not dominant over $Y$, then $V(\phi, \Delta)$ is automatically non-empty.
(3) The morphism $\phi : X \to Y$ is flat over $V(\phi, \Delta)$.

The following is one of our main results:

**Theorem 3.3.** Let the notation be as in Notation 3.1. Fix a point $v$ in $V(\phi, \Delta)$.
Let $D$ be a $\mathbb{Q}$-Cartier divisor on $X$ and let $i \in \mathbb{Z}_{>0}$ be an integer such that $iD$ is Cartier. Then we have:

1. Suppose that $\mathcal{O}_{X_v}(iD)$ is nef. If $B_-(D)$ intersects $\phi^{-1}(v)$, then so does $B_-(D - (K_{X/Y} + \Delta))$.
2. Suppose that $\mathcal{O}_{X_v}(iD)$ is ample. If $B_+(D)$ intersects $\phi^{-1}(v)$, then so does $B_+(D - (K_{X/Y} + \Delta))$.

To prove Theorem 3.3, we need Proposition 3.4, which is an application of weak positivity theorems developed by [Fuj78, Kaw81, Vie83, Cam04, Fuj17]. We employ [DM19, Theorem E], since it deals with log canonical pairs and gives explicitly a Zariski-open subset over which direct images of relative log pluricanonical bundles are weakly positive.

**Proposition 3.4.** Let the notation be as in Notation 3.1. Let $V$ be a Zariski-open subset in $V(\phi, \Delta)$ and put $U := \phi^{-1}(V)$. Suppose that $K_U + \Delta|_U$ is relatively ample over $V$. For an ample $\mathbb{Q}$-Cartier divisor $H$ on $Y$, we have

$$B(K_{X/Y} + \Delta + \phi^*H) \cap U = \emptyset.$$ 

**Proof of Proposition 3.4.** Fix $v \in V$. We prove $B(K_{X/Y} + \Delta + \phi^*H) \cap f^{-1}(v) = \emptyset$.

**Step 1.** We set up the notation used in the proof. Let $\tau : Y^b \to Y$ be a resolution of singularities passing through the flattening of $\phi$. Let $X^b$ be the normalization of the main component of $X \times_Y Y^b$. Let $\pi : X^z \to X$ be a log resolution of $(X, \Delta)$ passing through $X^b \to X$. We define the notation by the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\rho} & X^b \\
\downarrow{\phi} & & \downarrow{\phi} \\
Y & \xrightarrow{\tau} & Y^b.
\end{array}
\]

Let $E$ and $F$ be two effective $\pi$-exceptional $\mathbb{Q}$-divisors on $X^z$ with no common components such that

$$K_{X^z} + \pi_*^{-1}\Delta + E = \pi^*(K_X + \Delta) + F.$$
It follows that $E' := K_{X/Y} - \tau^*K_Y \geq 0$ since $Y$ has only canonical singularities. Put $\Delta^x := \pi_x^{-1}\Delta + E + \phi^*E'$. Then we have

$$K_{X^1/Y^1} + \Delta^x = \tau^*(K_{X/Y} + \Delta) + F.$$ 

Let $k$ be an integer large and divisible enough. Set $P^x := k(K_{X^1/Y^1} + \Delta^x)$. Put $V^\# := \tau^{-1}(V)$ and $U^\# := \rho^{-1}(U) = (\phi^\#)^{-1}(V^\#)$.

We may assume the following conditions.

- The morphism $\tau$ (resp. $\rho$) is an isomorphism over $V$ (resp. $U$).
- The divisor $\Delta^x$ is simple normal crossing.
- The morphism $\phi^\#$ is smooth over a neighborhood of $\tau^{-1}(v)$.
- The divisor $\Delta^x|_{X^1}$ is simple normal crossing and its each coefficient is at most one, by the definition of $V(\phi, \Delta)$ in Notation 3.1;
- In particular, each coefficient of horizontal components (that is, components dominating $Y$) in $\Delta^x$ is at most one.

**Step 2.** We prove that $\phi^\#_sO_{X^1}(P^x)$ is locally free over $V^\#$. Since

$$P^x = k\pi^*(K_{X/Y} + \Delta) + kF$$

and $F$ is $\pi$-exceptional, we have $\pi_*O_{X^1}(P^x) \cong \mathcal{O}_{X/k(K_{X/Y} + \Delta)}$. Hence the claim of this step is equivalent to saying the local freeness of

$$\tau_*\phi^\#_sO_{X^1}(P^x) \cong \phi_*\pi_*O_{X^1}(P^x) \cong \phi_*\mathcal{O}_{X/k(K_{X/Y} + \Delta)}$$

over $V$, since $\tau$ is an isomorphism over $V$. This follows from the flatness of $\phi$ over $V$ and the relative ampleness of $(K_{X/Y} + \Delta)|_U$ over $V$.

**Step 3.** Shrinking $V$ if necessarily, we show that the sheaf

$$(\phi^\#_sO_{X^1}(P^x))^{[s]} \otimes \mathcal{O}_{Y^\#}(ks\tau^*H)$$

is globally generated on $V^\#$ for $s \gg 1$. Here $(\bullet)^{[s]}$ denotes the double dual ($(\bullet)^{\otimes s})^{\vee\vee}$ of the $s$-times tensor. Let $H^\#$ be a very ample divisor on $Y^\#$. Thanks to [DM19, Theorem E], we find a neighborhood of $\rho^{-1}(v)$ on which

$$(2) \quad (\phi^\#_sO_{X^1}(P^x))^{[s]} \otimes \mathcal{O}_{Y^\#}(kK_{Y^\#} + k(\dim Y + 1)H^\#)$$

is globally generated for any $s \geq 1$. Shrinking $V$ again, we may assume that these sheaves are globally generated over $V^\#$. If $s \gg 0$, then the sheaf

$$(3) \quad \mathcal{O}_{Y^\#}(ks\tau^*H - kK_{Y^\#} - k(\dim Y + 1)H^\#)$$

is globally generated on $V^\#$, since $\tau$ is an isomorphism over $V$. By taking the tensor product of (2) and (3), we obtain the claim of this step.

**Step 4.** Put $P^\# := \sigma_\#P^x = k\rho^*(K_{X/Y} + \Delta) + k\sigma_*F$. Note that $P^\#$ is a Weil divisor that is not necessarily $\mathbb{Q}$-Cartier. We prove that $\mathcal{O}_{X^1}(sP^\# + ks\rho^*\phi^\#H)$ is globally generated on $U^\#$. Let $W \subseteq Y^\#$ be the largest Zariski-open subset on which $\phi^\#O_{X^1}(P^x)$ is locally free. Set $T := (\phi^\#)^{-1}(W)$ and $h := \phi^\#|_T : T \to W$. Then, by Step 2, we have $V^\# \subseteq W$ and $U^\# \subseteq T$. Since $\phi^\#$ is an equi-dimensional morphism, we have

$$\text{codim}_{X^\#}(X^\# \setminus T) = \text{codim}_{Y^\#}(Y^\# \setminus W) \geq 2,$$
and thus the claim of this step is equivalent to saying that $\mathcal{O}_T((sP^\rho + k\rho^s\phi^sH)|_T)$ is globally generated over $U^\rho$. Since $P^\rho \sim k\rho^s(K_{X/Y} + \Delta) + k\sigma F$ and $\sigma F$ is $\rho$-exceptional, it follows that $P^\rho|_{U^\rho} \sim k\rho^s(K_{X/Y} + \Delta)|_{U^\rho}$, and so the natural inclusion

$$\sigma_*\mathcal{O}_{X^\rho}(sP^\rho + k\rho^s\phi^sH) \hookrightarrow \mathcal{O}_{X^\rho}(sP^\rho + k\rho^s\phi^sH)$$

is an isomorphism over $U^\rho$. Note that $\rho$ is an isomorphism over $U$. Hence, we show that $(\sigma_*\mathcal{O}_{X^\rho}(sP^\rho + k\rho^s\phi^sH))|_T$ is globally generated over $U^\rho$. Put $\mathcal{M} := (\sigma_*\mathcal{O}_{X^\rho}(P^\rho))|_T$. By Step 3, we see that

$$\mathcal{G} := (h_*\mathcal{M})^* \otimes \mathcal{O}_W(k\sigma^sH)|_W \cong \left(\left(\sigma^s\mathcal{O}_{X^\rho}(P^\rho)\right)^{[s]} \otimes \mathcal{O}_{Y^\rho}(k\sigma^sH)\right)|_W$$

is globally generated over $V^\rho$. Since $\sigma_*\mathcal{O}_{X^\rho}(P^\rho) \xrightarrow{\alpha} \mathcal{O}_{X^\rho}(P^\rho)$ is an isomorphism on $U^\rho$ and $P^\rho|_{U^\rho}$ is a Cartier divisor that is relatively very ample over $V^\rho$, we see that $\mathcal{M}|_{U^\rho}$ is a line bundle that is relatively very ample over $V^\rho$, so the natural morphisms

$$h^*(h_*\mathcal{M})^* \otimes \mathcal{O}_W(k\sigma^sH)|_W \cong \left(\left(\sigma^s\mathcal{O}_{X^\rho}(P^\rho)\right)^{[s]} \otimes \mathcal{O}_{Y^\rho}(k\sigma^sH)\right)|_W$$

are surjective over $U^\rho$, which induce the morphism

$$h^*\mathcal{G} \rightarrow \mathcal{O}_T(sP^\rho|_T + ksh^*(\tau^sH)|_W)$$

that is surjective over $U^\rho$, and thus the claim of this step follows from $h^*(\tau^sH)|_W = (\rho^s\phi^sH)|_T$.

**Step 5.** We complete the proof. By the projection formula, we have

$$\mathcal{O}_X(k\sigma^s(K_{X/Y} + \Delta + \phi^sH)) \cong (d_\mathcal{O}_X(sP + k\rho^s\phi^sH)).$$

By Step 4, the right-hand side is globally generated over $U \supseteq X_\nu$, since $\rho$ is an isomorphism over $U$ and $d_\mathcal{O}_X \cong \mathcal{O}_X$. Thus $\mathcal{B}(K_{X/Y} + \Delta + \phi^sH) \cap X_\nu = \emptyset$.

\[ \square \]

**Proof of Theorem 3.3.** Put $L := K_{X/Y} + \Delta$. We first prove (1). Under the assumption of $\mathbb{B}_+(D - L) \cap X_\nu = \emptyset$, we show that $\mathbb{B}_-(D) \cap X_\nu = \emptyset$. Let $A$ be an ample $\mathbb{Q}$-Cartier divisor on $X$ such that $\mathbb{B}(D - L + A) \cap X_\nu = \emptyset$. It is enough to show that $\mathbb{B}(D + 2A) \cap X_\nu = \emptyset$.

By the choice of $A$, we can find a $\mathbb{Q}$-Cartier divisor $\Gamma \geq 0$ on $X$ such that

- $\Gamma \sim Q D - L + A$, and
- $V(\phi, \Delta')$ defined in Notation 3.1 contains $\nu$, where $\Delta' := \Delta + \Gamma$.

By the assumption, shrinking $V$ if necessary, we may assume that $(A + D)|_U$ is ample over $V$. It follows from

$$K_{X/Y} + \Delta' \sim_Q L + \Gamma \sim_Q A + D$$

that $K_{U|V} + \Delta'|_U$ is also ample over $V$. Take an ample $\mathbb{Q}$-Cartier divisor $H$ on $Y$ so that $A - f^*H$ is ample. Then Proposition 3.4 implies that

$$\mathbb{B}(2A + D) \cap X_\nu \subseteq \mathbb{B}(A + D + \phi^sH) \cap X_\nu = \mathbb{B}(K_{X/Y} + \Delta' + \phi^sH) \cap X_\nu = \emptyset.$$

Note that $X_\nu \subseteq U$.

Now we prove (2). Assuming $\mathbb{B}_+(D - L) \cap X_\nu = \emptyset$, we show $\mathbb{B}_+(D) \cap X_\nu = \emptyset$. Take an ample $\mathbb{Q}$-Cartier divisor $A$ on $X$ so that

- $\Gamma \sim Q D - L + A$, and
- $V(\phi, \Delta')$ defined in Notation 3.1 contains $\nu$, where $\Delta' := \Delta + \Gamma$.

By the assumption, shrinking $V$ if necessary, we may assume that $(A + D)|_U$ is ample over $V$. It follows from

$$K_{X/Y} + \Delta' \sim_Q L + \Gamma \sim_Q A + D$$

that $K_{U|V} + \Delta'|_U$ is also ample over $V$. Take an ample $\mathbb{Q}$-Cartier divisor $H$ on $Y$ so that $A - f^*H$ is ample. Then Proposition 3.4 implies that

$$\mathbb{B}(2A + D) \cap X_\nu \subseteq \mathbb{B}(A + D + \phi^sH) \cap X_\nu = \mathbb{B}(K_{X/Y} + \Delta' + \phi^sH) \cap X_\nu = \emptyset.$$

Note that $X_\nu \subseteq U$.
• $\mathbb{B}_+(D - L) = \mathbb{B}(D - L - A) \supseteq \mathbb{B}_-(D - L - A)$, and
• $\mathcal{O}_{X_v}(j(D - A))$ is nef for some $j \geq 1$ divisible enough.

Then $\mathbb{B}_-(D - L - A) \cap X_v = \emptyset$ by the assumption, and thus $\mathbb{B}_-(D - A) \cap X_v = \emptyset$ by (1). Then the assertion follows from $\mathbb{B}_+(D) \subseteq \mathbb{B}_-(D - A)$. \hfill $\Box$

**Corollary 3.5.** Let the notation be as in Notation 3.1. Let $E$ be a $\mathbb{Q}$-Cartier divisor on $Y$. Then we have:

1. $V(\phi, \Delta) \cap \mathbb{B}_-(E) \subseteq \phi(\mathbb{B}_-(\phi^*E - (K_{X/Y} + \Delta)))$, and
2. $V(\phi, \Delta) \cap \mathbb{B}_+(E) \subseteq \phi(\mathbb{B}_+(\phi^*E - (K_{X/Y} + \Delta)))$.

**Remark 3.6.** By putting $E := -K_Y$ in the above corollary, we can recover [KoMM92, Corollary 2.9] (in characteristic zero), [Den17, Theorems D and E], and [FG12, Theorem 1.1].

**Proof of Corollary 3.5.** Put $V := V(\phi, \Delta)$. Since we have $V \cap \mathbb{B}_-(E) = V \cap \phi(\mathbb{B}_-(\phi^*E))$ by Lemma 2.2 (1), we get (1) of the corollary by applying Theorem 3.3 (1) for $D = \phi^*E$. We show (2). Put $L := K_{X/Y} + \Delta$. Pick an ample $\mathbb{Q}$-Cartier divisor $A$ on $X$ such that $\mathbb{B}_+(\phi^*E - L) = \mathbb{B}(\phi^*E - L - A)$. Take an ample $\mathbb{Q}$-Cartier divisor $H$ on $Y$ so that $\mathbb{B}_+(E) \subseteq \mathbb{B}_-(E - H)$ and $A - \phi^*H$ is ample. Then we have

$$V \cap \mathbb{B}_+(E) \subseteq \mathbb{B}_-(E - H) \subseteq V \cap \phi(\mathbb{B}(\phi^*E - \phi^*H - L)),$$

and thus the assertion follows from $\mathbb{B}(\phi^*E - \phi^*H - L) \subseteq \mathbb{B}(\phi^*E - L - A) = \mathbb{B}_+(\phi^*E - L)$. \hfill $\Box$

**Corollary 3.7.** Let the notation be as in Notation 3.1. Then $\mathbb{B}_+(-(K_{X/Y} + \Delta))$ and $\mathbb{B}_-(-(K_{X/Y} + \Delta + \phi^*H))$ are dominant over $Y$ for any ample $\mathbb{Q}$-Cartier divisor $H$ on $Y$ unless $Y$ is one point.

**Proof.** This follows from Corollary 3.5 by putting $E = 0$ or $E = -H$. \hfill $\Box$

**Corollary 3.8.** Let the notation be as in Notation 3.1. Suppose that $\mathcal{O}_{X_v}(i(K_X + \Delta))$ is nef for $i \in \mathbb{Z}_{>0}$ divisible enough and a point $v \in V(\phi, \Delta)$. Then $\mathbb{B}_-(K_{X/Y} + \Delta) \cap X_v = \emptyset$. In particular, $K_{X/Y} + \Delta$ is pseudoeffective.

**Proof.** Set $D := K_{X/Y} + \Delta$. Then we have $\mathbb{B}_-((D - (K_{X/Y} + \Delta)) = \emptyset$, and thus the assertion follows from Theorem 3.3 (1). \hfill $\Box$

3.2. **Algebraic proof of Campana–Cao–Matsumura’s equality.** The following theorem is a slight extension of [CCM19, Theorem 1.2], which generalizes Hacon–MëKernan’s question proved in [EG19]. In this subsection, we give an algebraic proof for this equality.

**Theorem 3.9.** Let $X$ be a normal projective variety and let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, \Delta)$ is klt. Let $\phi : X \to Y$ be a morphism with connected fibers onto a $\mathbb{Q}$-Gorenstein normal projective variety $Y$ with only canonical singularities. If $-(K_{X/Y} + \Delta)$ is nef, then

$$\text{nd}(-(K_{X/Y} + \Delta)) = \text{nd}(-(K_F + \Delta|_F))$$

for a general fiber $F$.

To prove this theorem, we need the following proposition:
Proposition 3.10. Let the notation be as in Notation 3.1. Let $V$ be a Zariski-open subset of $V(\phi, \Delta)$ and put $U := \phi^{-1}(V)$. Suppose that

- $(U, \Delta|_U)$ is klt, and
- $-(K_{X/Y} + \Delta)$ is nef.

Let $D$ be a $\mathbb{Q}$-Cartier divisor on $X$ such that $D|_U$ is relatively nef over $V$. Fix a smooth closed subvariety $Z \subset Y$ that is properly contained in $V$. Then there exists a constant $C$ such that

$$\text{mult}_{\phi^{-1}(Z)}(\Gamma) \leq C$$

for every $\mathbb{Q}$-Cartier divisor $\Gamma$ with $0 \leq \Gamma \equiv D - \lambda(K_{X/Y} + \Delta)$ for some $0 \leq \lambda \in \mathbb{Q}$. Here $\text{mult}_{\phi^{-1}(Z)}(\Gamma)$ is the multiplicity of $\Gamma$ along $\phi^{-1}(Z)$.

Note that it follows from the definition of $V(\phi, \Delta)$ that $\phi^{-1}(Z)$ is irreducible.

Proof. Let $Y'$ (resp. $X'$) be the blow-up of $Y$ (resp. $X$) along $Z$ (resp. $\phi^{-1}(Z)$), and let $\phi' : X' \to Y'$ denote the induced morphism. Then we have the following cartesian diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\phi} & X' \\
\downarrow \phi & & \downarrow \phi' \\
Y & \xrightarrow{\tau} & Y'.
\end{array}
$$

It is easy to check that the following hold:

- $\sigma^*(K_{X/Y} + \Delta) \sim K_{X'/Y'} + \Delta'$, where $\Delta' := \sigma^{-1}_* \Delta$.
- $(\sigma^{-1}(U), \Delta'|_{\sigma^{-1}(U)})$ and $\tau^{-1}(V)$ respectively satisfy the conditions on $(U, \Delta|_U)$ and $V$ in Theorem 3.3.

Hence by respectively replacing $\phi : X \to Y$ and $D$ with $\phi' : X' \to Y'$ and $\sigma^* D$, we may assume that $Z$ is a smooth prime divisor on $Y$.

Put $L := K_{X/Y} + \Delta$. Take an ample $\mathbb{Q}$-Cartier divisor $A$ on $X$. Fix $\lambda \in \mathbb{Q}_{\geq 0}$ and $0 \leq \Gamma \equiv D - \lambda L$. Then $\Gamma$ can be written as $\Gamma = \alpha \phi^* Z + E$ with $\text{mult}_{\phi^* Z}(E) = 0$, where $\alpha := \text{mult}_{\phi^* Z}(\Gamma)$. Since $(U, \Delta|_U)$ is klt, the pair $(U, (\Delta + (\lambda + m)^{-1} E)|_U)$ is also klt for some $m \gg 0$. Since $A - mL$ is ample, there is $0 \leq G \sim_{\mathbb{Q}} A - mL$ such that $(U, B|_U)$ is klt, where $B := \Delta + (\lambda + m)^{-1} (E + G)$. Then we have

$$D - \lambda(K_{X/Y} + \Delta) \equiv \alpha \phi^* Z + E \quad \text{and} \quad A - m(K_{X/Y} + \Delta) \equiv G,$$

and thus we can see that

$$\frac{1}{\lambda + m} (A + D - \alpha \phi^* Z) \equiv K_{X/Y} + \Delta + \frac{1}{\lambda + m} (E + G) = K_{X/Y} + B.$$

The divisor $(K_{X/Y} + B)|_U$ is relatively nef over $V$ by the assumption, and thus it follows that $K_{X/Y} + B$ is pseudo-effective from Corollary 3.8. This implies that $A + D - \alpha \phi^* Z$ is also pseudo-effective. Then we obtain

$$C := \frac{(A + D) \cdot A^{\dim X - 1}}{\phi^* Z \cdot A^{\dim X - 1}} \geq \alpha = \text{mult}_{\phi^* Z}(\Gamma),$$

which completes the proof. \qed
ON ASYMPTOTIC BASE LOCI OF RELATIVE ANTI-CANONICAL DIVISORS

Proof of Theorem 3.9. The inequality “≥” follows from the well-known fact that the numerical Kodaira dimension of a nef \( \mathbb{Q} \)-Cartier divisor \( D \) on a projective variety \( V \) coincides with

\[
\max \{ \dim W | \text{closed subvariety } W \subseteq V \text{ with } W \cdot D = 0 \}.
\]

We prove the converse inequality “≤”. Put \( L := K_X + \Delta \). Take an ample Cartier divisor \( A \) on \( X \). Let \( I \) denote the ideal defining \( F \) and set \( G_m := I^m/I^{m+1} \) for each \( m \in \mathbb{Z}_{\geq 0} \). By the exact sequence

\[
0 \to I^{m+1}(A - lL) \to I^m(A - lL) \to G_m(A - lL) \to 0,
\]

we get

\[
(8) \quad h^0(X, I^m(A - lL)) \leq h^0(X, I^{m+1}(A - lL)) + h^0(F, G_m(A - lL)).
\]

Thanks to Proposition 3.10, we can find \( m_0 \in \mathbb{Z}_{> 0} \) such that \( h^0(X, I^{m_0}(A - lL)) = 0 \) for each \( l \in \mathbb{Z}_{\geq 0} \), and thus from \( (8) \) we obtain that

\[
h^0(X, \mathcal{O}_X(A - lL)) \leq \sum_{0 \leq m < m_0} h^0(F, G_m(A - lL))
\]

for each \( l \in \mathbb{Z}_{\geq 0} \). Put \( \nu := \text{nd}(-L|_F) \). Since each \( G_m \) is a torsion-free \( O_F \)-module, it is isomorphic to a subsheaf of the direct sum of ample divisors, and thus there is a constant \( C \) such that

\[
\frac{h^0(X, \mathcal{O}_X(A - lL))}{\nu} \leq \sum_{0 \leq m < m_0} \frac{h^0(F, G_m(A - lL))}{\nu} \leq C.
\]

This means that \( \text{nd}(-L) \leq \nu = \text{nd}(-L|_F) \). \( \Box \)

4. ON ASYMPTOTIC VARIANTS OF BASE LOCI

The aim of this section is to prove Theorem 1.4. For this purpose, we need Proposition 4.4, Theorem 4.5, and Theorem 4.6, which have been essentially proved in the case \( -(K_X + \Delta) \) being nef by [CH19] and [CCM19], but we actually need them under the weaker assumption that its restricted base locus is not dominant over \( Y \). In subsection 4.1, for reader’s convenience, we give a proof and an explanation for them with slightly generalized form.

Throughout this section, let \( \phi : X \to Y \) be a surjective morphism with connected fiber between projective manifolds. Let \( \Delta \) be an effective \( \mathbb{Q} \)-divisor on \( X \) and fix \( N \in \mathbb{Z}_{> 0} \) such that \( N\Delta \) is a Cartier divisor.

4.1. Extension of Cao-Höring’s work for nef anti-canonical divisors. We first begin with the following proposition:

Proposition 4.1. If \( (X, \Delta) \) is an lc pair and \( \mathbb{B}_-(-(K_X + \Delta)) \) is not dominant over \( Y \), then \( \phi \) is flat and semistable (that is, the fiber \( \phi^{-1}(y) \) has the same dimension and it is reduced for any \( y \in Y \)).

Proof of flatness. Our proof follows an argument in [PZ19] and the basic idea is the same as in [LTZZ10]. Let \( y \in Y \) be a point such that \( X_y \) has the largest dimension among all the closed fibers. If \( \text{codim}_X(X_y) = 0 \), then \( Y = \{ y \} \), and thus \( f \) is flat. We may assume that \( \text{codim}_X(X_y) \geq 1 \).
Step 1. We first prove the assertion in the case where \( \text{codim}_X(X_y) = 1 \). Let \( \rho : Y^b \to Y \) be a resolution passing through the flattening of \( \phi \). Let \( X^b \) be a resolution of the main component of \( X \times_Y Y^b \). We have the following commutative diagram:

\[
\begin{array}{ccc}
X^b & \xrightarrow{\pi} & X \\
\phi \downarrow & & \downarrow \phi \\
Y^b & \xrightarrow{\rho} & Y.
\end{array}
\]

Then we have:

- all \( \phi^b \)-exceptional divisors are \( \pi \)-exceptional,
- the \( \rho \)-exceptional divisor \( D := K_{Y^b} - \rho^*K_Y \) on \( Y^b \) is effective, and
- \( \mathbb{B}_-(\rho^*(K_{X/Y} + \Delta)) \) is not dominant over \( Y^b \).

Set \( E := K_{X^b} + \pi_*^{-1}\Delta - \pi^*(K_X + \Delta) \). Then \( E \) is \( \pi \)-exceptional, and

\[
K_{X^b/Y^b} + \pi_*^{-1}\Delta - \pi^*(K_{X/Y} + \Delta) = E - \phi^b D.
\]

We show that \( K_{X^b/Y^b} + \pi_*^{-1}\Delta - \pi^*(K_{X/Y} + \Delta) \) is pseudo-effective. Set \( L := -\pi^*(K_{X/Y} + \Delta) \) and \( M := K_{X^b/Y^b} + \pi_*^{-1}\Delta + L \). Let \( F \) be a very general fiber of \( \phi^b \). Then \( \mathbb{B}_-(L) \) does not intersect with \( F \). Hence Theorem 3.3 (1) implies that \( \mathbb{B}_-(M) \) does not intersect \( F \), and thus \( M \) is pseudo-effective.

Let \( A \) be a very ample divisor on \( X \) and \( \alpha := \pi^*A^{\dim X - 1} \). Since \( E \) is \( \pi \)-exceptional, we have

\[
0 \geq -(E - \phi^b D) \cdot \alpha = (\phi^b D) \cdot \alpha.
\]

On the other hand, if \( \phi \) is not flat at \( y \), we see that \( \rho(\text{Supp}(D)) \ni y \), and thus \( \pi_*\phi^b D \geq (X_y)_{\text{red}} \). Then we get \( (\phi^b D) \cdot \alpha > 0 \), which is a contradiction.

Remark 4.2. The argument in this step works when \( K_X + \Delta = \mathbb{Q} \)-Cartier and the non-canonical locus of \( (X, \Delta) \) is not dominant over \( Y \).

Step 2. Take \( r \geq 2 \). Assuming to the assertion holds if \( \text{codim}_X(X_y) < r \), we prove it in the case when \( \text{codim}_X(X_y) = r \). Let \( Z \subseteq Y \) be the closed subset consisting of points over which \( \phi \) is not flat, and \( Z' \subseteq Z \) is the Zariski-closure of \( Z \setminus \{y\} \). Let \( H_0 \) be a very ample divisor on \( Y \) and \( \mathfrak{d} \subseteq |H_0| \) be the linear system of members containing \( y \). Take a very general member \( H \in \mathfrak{d} \). Then \( H \) is smooth and \( H \cap Z'' \) is properly contained in \( Z'' \) for each irreducible component \( Z'' \) of \( Z' \). This implies that \( \phi^{-1}(H) \cap W \) is also properly contained in \( W \) for each irreducible component \( W \) of \( \phi^{-1}(Z') \), and thus we have

\[
\dim(\phi^{-1}(H \cap Z')) \leq \dim \phi^{-1}(Z') - 1 \leq \dim X - 2.
\]

Then it follows that \( \dim(\phi^{-1}(H \cap Z)) \leq \dim X - 2 \) from \( Z = Z' \cup \{y\} \) and \( \text{codim}_X(X_y) \geq 2 \). Let \( G \) be an irreducible component of \( \phi^{-1}(H) \). Then \( \dim G = \dim X - 1 \), and thus \( G \not\subseteq \phi^{-1}(Z) \). Hence \( \phi \) is flat at a point in \( G \), and thus so is the induced morphism \( \psi : \phi^{-1}(H) \to H \). In particular \( G \) dominates \( H \), from which we see that \( \phi^{-1}(H) \) is irreducible, since a general fiber of \( \psi \) is a smooth variety. We also see that \( \phi^*H = G \) as divisors. Let \( \nu : X' \to G \) be the normalization and \( C \geq 0 \).
be the Weil divisor on $X'$ corresponding the conductor ideal. Then $\text{Supp}(C)$ is not dominant over $Y$. Put $\Delta' := C + \nu^*(\Delta|_G)$. Then we have

$$K_{X'/H} + \Delta' \sim_{Q} \nu^*(K_G + \Delta|_G - \psi^*K_H)$$

$$\sim_{Q} \nu^*(K_X + G + \Delta - \phi^*(K_Y + H))|_G \sim_{Q} \nu^*(K_{X/Y} + \Delta)|_G,$$

and thus we can see that $\phi\left(\mathbb{B}_-(-(K_{X'/H} + \Delta'))\right)$ is properly contained in $H$. Since $\mathfrak{a}$ is free on $Y \setminus \{y\}$, the non-lc locus of $(G, \Delta|_G)$ is not dominant over $Y$, so the same property holds for $(X', \Delta')$. Then, by the assumption, $X'$ is flat over $H$, we obtain

$$\text{codim}_X(X_y) - 1 = \text{codim}_{X'}(X'_y) = \dim H = \dim Y - 1,$$

and thus $\text{codim}_X(X_y) = \dim Y$. This means that $\phi$ is flat.

\[\square\]

**Proof of semistability.** Assume there exists a point $y \in Y$ such that $\phi^{-1}(y)$ is not reduced. We can take a smooth curve $C$ in $Y$ such that $y \in C$ and $\mathbb{B}_-(-(K_{X'/H} + \Delta))$ is not dominant over $C$. Set $T := \phi^{-1}(C)$ and $\Delta_T := \Delta|_T$. By the assumption, there exists a prime divisor $W$ on $T$ such that $e := \text{ord}_W(\phi|_T^*(y)) > 1$. Take $\pi^2 : T^2 \to T$ be a log resolution of $(T, \Delta_T)$ and set $\phi^2 := \phi|_T \circ \pi^2$. Then we have the following diagram:

$$\begin{array}{ccc}
T^2 & \xrightarrow{\phi^2} & T \\
\downarrow & & \downarrow \phi \\
C & \xrightarrow{\phi} & Y.
\end{array}$$

Let $\Delta^2$ and $G$ be the effective divisors such that $K_{T^2} + \Delta^2 = \pi^2^*(K_T + \Delta_T) + G$. Then $(T^2, \Delta^2)$ is log canonical, the divisor $G$ is $\pi^2$-exceptional, and $\mathbb{B}_-(-\pi^2^*(K_{T^2/C} + \Delta_T))$ is not dominant over $C$.

Now we define an $mN$-Bergman metric $H_{mN}$ on $mNG + 2A^2$ for any $m \in \mathbb{Z}_{>0}$ and an ample divisor $A^2$ on $T^2$ such that $N\Delta^2 + A^2$ is ample. We consider the equality

$$mNG + 2A^2$$

$$= mN\Delta^2 + (\text{with } h_1) \quad \text{with } h_1$$

$$\text{with } h_2$$

$$\text{with } h_3^m$$

and the singular hermitian metrics $h_1$, $h_2$, and $h_3$ with semipositive curvature satisfying the following:

- $h_1$ is a singular hermitian metric on $-mN\pi^2^*(K_{T^2/C} + \Delta_T) + A^2$ such that $h_1|_{T^2_w}$ is smooth on $T^2_w$ for a general point $w \in C$, which is obtained from the property that $\mathbb{B}_-(-\pi^2^*(K_{T^2/C} + \Delta_T))$ is not dominant over $C$.
- $h_2$ is a smooth hermitian metric on the ample divisor $N\Delta^2 + A^2$ with positive curvature.
- $h_3$ is a singular hermitian metric on $(1 - 1/m)\Delta^2$ such that $\mathcal{J}(h_3|_{T^2_w}) = \mathcal{O}_{T^2_w}$ holds for a general point $w \in C$, which is obtained from the klt condition of $(T, (1 - 1/m)\Delta^2)$. 


Then the singular hermitian metric \( h := h_1 h_2 h_3^{mN} \) has semipositive curvature current and satisfies that \( J(h^{1\over h_1^*}|_{T_w}) = O_{T_w} \) for a general point \( w \in C \). Then, by \( \phi^* (2A^2 + mNG) \neq 0 \) and the above properties, we can see that the induced \( mN \)-Bergman metric \( H_{mN} \) on \( 2A^2 + mNG \) has semipositive curvature (see [BP10, A.2]). Then, by [CP17, Remark 2.5] or [Wang19, Proposition 2.5], we obtain

\[
\sqrt{-1} \Theta_{H_{mN}} \geq mN(e-1)[\pi^\sharp W].
\]

This means that \( (2A^2 + mNG) - mN(e-1)\pi^\sharp W \) is pseudo-effective for any \( m \in \mathbb{Z}_{>0} \), and thus so is \( G - (e-1)\pi^\sharp W \). This is a contradiction to the fact that \( G \) is \( \pi^\sharp \)-exceptional if \( e > 1 \). \( \square \)

The proof of the following proposition is essentially the same as in [CCM19, Section 4.3].

**Proposition 4.3.** (cf. [CCM19, Section 4.3]) Under the same assumption as in Proposition 4.1, the vertical part of \( \Delta \) is 0.

**Proof.** Let \( \Delta^v \) (resp. \( \Delta^h \)) be the vertical part (resp. the horizontal part) of \( \Delta \). We take an ample divisor \( A \) of \( X \) such that \( N\Delta^h + A \) is ample. It is sufficient to prove that \( -mN\Delta^v + 2A \) is pseudo-effective for any \( m \in \mathbb{Z}_{>0} \). By the same way as in the proof of Proposition 4.1, we can see that \( -mN\Delta^v + 2A \) is pseudo-effective since we have

\[
-mN\Delta^v + 2A = mNK_{X/Y} + (-mN(K_{X/Y} + \Delta) + A) + (N\Delta^h + A) + mN(1 - 1/m)\Delta^h.
\]

\( \square \)

As we confirmed in the proof of Proposition 4.1, the assumption of \(- (K_{X/Y} + \Delta) \) being nef can be replaced with our weaker assumption that \( \mathbb{B}_{-}(- (K_{X/Y} + \Delta)) \) is not dominant over \( Y \) in the situation where we use positivity of direct images. Hence, by the same proof as in [CH19, CCM19], we can obtain Proposition 4.4, Theorem 4.5 and 4.6 by paying attention to the flatness and semistability of \( \phi \). For this reason, we put the proof of them in Appendix.

**Proposition 4.4** (cf. [CH19, 2.8 Proposition], [CCM19, Theorem 2.2(1)]). Let \( L \) be a \( \phi \)-big divisor on \( X \) admitting a singular hermitian metric \( h_L \) such that \( \sqrt{-1} \Theta_{h_L} \geq \phi^* \theta \) for some smooth \((1,1)\)-form \( \theta \) on \( Y \). If \( (X, \Delta) \) is a klt pair and \( \mathbb{B}_{-}(- (K_{X/Y} + \Delta)) \) is not dominant over \( Y \), then for any \( \varepsilon > 0 \) and any \( p, q \in \mathbb{Z} \) with \( \phi_*(pK_{X/Y} + qN\Delta + L) \neq 0 \), there exists a singular hermitian metric \( H \) on \( \phi_*(pK_{X/Y} + qN\Delta + L) \) such that

\[
\sqrt{-1} \Theta_H \geq -\varepsilon \omega_Y \otimes \text{id} + (1 - \varepsilon) \theta \otimes \text{id}.
\]

Here \( \omega_Y \) is a Kähler form on \( Y \).

In particular, if \( L \) is a pseudo-effective and \( \phi \)-big divisor of \( X \), then \( \phi_*(pK_{X/Y} + qN\Delta + L) \) is weakly positively curved.

**Theorem 4.5** (cf. [CH19, 3.5 Lemma], [Cao19, Proposition 3.15], [CCM19, Proposition 3.5]). Let \( L \) be a \( \phi \)-big divisor on \( X \) and \( r \) be the rank of \( \phi_*L \). If \( (X, \Delta) \) is a klt pair and...
\( \mathbb{B}_-(-K_{X/Y} + \Delta) \) is not dominant over \( Y \), then there exists a \( \phi \)-exceptional effective divisor \( E \) on \( X \) such that

\[
L + E - \frac{1}{r} \phi^* (\det(\phi_* L))
\]

is pseudo-effective. Moreover \( E \) can be taken as the zero divisor if \( \phi_* L \) is locally free.

**Theorem 4.6** (cf. [CCM19, Theorem 1.3], [Cao, Remarque 2.10]). If \( (X, \Delta) \) is a klt pair and \( \mathbb{B}_-(-K_{X/Y} + \Delta) \) is not dominant over \( Y \), then \( \phi \) is locally trivial with respect to \( (X, \Delta) \). Further the fiber product \( (\tilde{X}, \tilde{\Delta}) := (X, \Delta) \times_Y \tilde{Y} \) by the universal cover \( \pi : \tilde{Y} \to Y \) admits the following splitting:

\[
(\tilde{X}, \tilde{\Delta}) \cong \tilde{Y} \times (F, \Delta |_F)
\]

where \( F \) is the fibre of \( \phi \). Moreover there exists a homomorphism \( \rho : \pi_1(Y) \to \text{Aut}(F) \) such that \( X \) is isomorphic to the quotient of \( \tilde{Y} \times F \) by the action of \( \pi_1(Y) \) and \( \rho \).

### 4.2. Variants of base loci and flatness of direct images.

In this subsection, we prove Theorem 4.7, 4.8, and 4.9, which lead to Theorem 1.4. In [Cao19, CH19, CCM19], it was proven that nefness of \(-K_{X/Y}\) implies numerical flatness of the direct image sheaves, however, in this subsection, we study the converse implication. This paper reveals a relation between positivity conditions (nefness, semi-ampleness, vanishing Lelong number) and certain flatness of the direct image sheaves.

**Theorem 4.7.** If \( \mathbb{B}_-(-K_{X/Y}) \) is not dominant over \( Y \), then it is empty (that is, \(-K_{X/Y}\) is nef).

**Proof.** Note that \( \phi : X \to Y \) is automatically locally trivial by Theorem 4.6. For a sufficiently ample divisor \( A \) on \( X \), we define the divisor \( \tilde{A} \) on \( X \) by

\[
\tilde{A} := A - \frac{1}{r} \phi^* \det \phi_* A
\]

by the same way as in Theorem 4.5. Also, for every \( m \in \mathbb{Z}_{>0} \), we define the coherent sheaf \( \mathcal{V}_m \) by

\[
\mathcal{V}_m := \phi_* (-mK_{X/Y} + \tilde{A}).
\]

The sheaf \( \mathcal{V}_m \) is reflexive by the flatness of \( \phi \), and thus it is actually locally free and numerically flat by Theorem 4.1, 4.4, and [CCM19, Proposition 2.7] (see also [HIM19, Theorem 3.4]). Then we find an ample divisor \( B \) (independent of \( m \)) such that \( \mathcal{V}_m \otimes B \) is globally generated on \( X \), by the argument of the Castelnuovo-Mumford regularity. Indeed, the Kodaira type vanishing theorem holds for numerically flat vector bundles by Lemma [CCM19, Lemma 2.10], and thus \( \mathcal{V}_m \) is \((n+1)\)-regular with respect to \( A_Y \), namely, \( H^q(Y, \mathcal{V}_m \otimes A_Y^{\otimes(n+1-q)}) = 0 \) holds for any \( q > 0 \). Here \( A_Y \) is a very ample divisor on \( Y \) such that \( A_Y - K_Y \) is ample and \( n \) is the dimension of \( Y \). This implies that \( B := A_Y^{\otimes n+1} \) satisfies the desired property (for example see [Laz04, Theorem 1.8.5]).

For an arbitrary point \( y \in Y \), the natural map

\[
H^0(X, -mK_{X/Y} + \tilde{A} + \phi^* B) = H^0(Y, \mathcal{V}_m \otimes B) \longrightarrow (\mathcal{V}_m \otimes B)_y
\]
is surjective since $V_m \otimes B$ is globally generated. On the other hand, there exists $y \in Y$ such that $-K_{X_y} = -K_{X/Y}|_{V_y}$ is nef since $\mathbb{B}(-K_{X/Y})$ is properly contained in $Y$. Then if follows that $K_{X_y}$ is nef for any $y \in Y$ since all the fibers of $\phi$ are isomorphic each other by Theorem 4.6. Hence we may assume that $-mK_{X_y} + A|_{X_y}$ is positively curved metric by $[CP17]$ and $[HIM19, \text{Lemma 3.5}]$. To get a $Y$ in $\mathbb{P}^1$ which is nef with semipositive curvature by Theorem 4.5. Then, by Skoda’s lemma and Hölder’s inequality, we can easily check that

$$I(h^{(1+m/p)}g^{1/p}) = \mathcal{O}_X \text{ on } X \setminus \phi^{-1}(\phi(P(h))) \text{ for } p \gg 1$$

since the singularities of $g$ can be killed by taking a sufficiently large $p$. Note that the above property holds for any $m \in \mathbb{Z}_{>0}$ by the assumption. Hence, by [CCM19, Theorem 2.2], we can obtain the singular hermitian metric $H_{m,p}$ on $V_m$ induced by the metric $h^{(p+m)}g$ on $-(p+m)K_{X/Y} + \tilde{A}$ and the equality

$$V_m = \phi_* (pK_{X/Y} - (p+m)K_{X/Y} + \tilde{A}).$$

For a contradiction, we assume that there exists a point $x_0 \in X$ with $\nu(h, x_0) > 0$. We can take a section $t$ of $-(mK_{X/Y} + \tilde{A})|_{X_y}$ whose valued at $x_0$ is non-zero since we may assume that it is globally generated on $X_y$ for any $m \geq 0$. By locally extending the section $t$, we can find a local section $s(y)$ of $V_m$ defined on a (Euclidean) open neighborhood of $y_0 := \phi(x_0)$ such that $s(y_0) = t$. The norm $|s(y)|_{H_{m,p}}$ is a smooth function (in particular, it is bounded) since $H_{m,p}$ is automatically smooth. On the other hand, by the construction of $H_{m,p}$, we have

$$|s(y)|^2_{H_{m,p}} = \int_{X_y} \frac{|s(y)|^2}{|s(y)|_{B_p^{1/(1+m/p)}g^{1/p}}} \text{,}$$

where $B_p$ is the $p$-Bergman kernel on $pK_F + (-(p+m)K_F + \tilde{A})$. The section $s(y_0) = t$ must be zero at $x_0$ for a sufficiently large $m$, since the above norm is smooth and thus bounded. This is a contradiction to the choice of $t$. □

**Theorem 4.9.** If $\mathbb{B}(-K_{X/Y})$ is not dominant over $Y$, then it is empty (that is, $-K_{X/Y}$ is semi-ample).
Proof. The stable base locus $\mathcal{B}(-K_{X/Y})$ coincides with the base locus of $-m_0K_{X/Y}$ for some $m_0 \in \mathbb{Z}_{>0}$. It is easy to see that $\nu(\varphi, x) > 0$ if and only if $\varphi(x) = \infty$ in the case where a (quasi)-psh function $\varphi$ has analytic singularities. Hence the singular hermitian metric $h$ on $-m_0K_{X/Y}$ defined by a basis of $H^0(X, -m_0K_{X/Y})$ satisfies that

$$\mathcal{B}(-K_{X/Y}) = \text{Bs}(-m_0K_{X/Y}) = P(h^{1/m}).$$

Then the assertion follows from Theorem 4.8. \hfill \square

Remark 4.10. (1) In Theorem 4.9, we do not mention of the direct images, but we show that $\phi_*(-mK_{X/Y})$ is étale trivializable when $-K_{X/Y}$ is semi-ample in Section 5.

(2) Theorem 1.4 holds for klt pairs $(X, \Delta)$ by the same argument.

5. ALGEBRAIC FIBER SPACES WITH SEMI-AMPLE ANTI-CANONICAL DIVISORS

This section is devoted to the proof of Theorem 1.7.

Proof of Theorem 1.7. Let $\phi : X \to Y$ be a morphism with connected fibers between smooth projective varieties $X$ and $Y$ such that the relative anti-canonical divisor $-K_{X/Y}$ is semi-ample. For simplicity, we put $L := -K_{X/Y}$. The proof can be divided into six steps.

Step 1 (Explanation of our strategy). By the main result of [CH19, Cao19], the fiber product of $X$ by the universal cover $\bar{Y} := Y_{\text{univ}} \to Y$ is isomorphic to the product $\bar{Y} \times F$. Furthermore, there exists a homomorphism $\rho : \pi_1(Y) \to \text{Aut}(F)$ of groups so that $X$ can be constructed from the quotient $\bar{Y} \times F \cong \bar{X}$ by $\pi_1(Y)$ and $\rho$. Here $\pi_1(Y)$ naturally acts on $\bar{Y}$ and also acts on $\bar{Y} \times F$ by $(y, p) \mapsto (\gamma \cdot y, \rho(\gamma) \cdot p)$, where $(y, p) \in \bar{Y} \times F$ and $\gamma \in \pi_1(Y)$. We will denote by $\bar{Y}/N$ (resp. $\bar{Y} \times F/N$) the quotient of $\bar{Y}$ (resp. $\bar{Y} \times F$) corresponding to a subgroup $N \subseteq \pi_1(Y)$. Our situation can be summarized by the following commutative diagram and isomorphisms:

$$
\begin{array}{cccc}
\bar{Y} \times F \cong \bar{X} := \bar{Y} \times_Y X & \longrightarrow & X' := \bar{Y} \times F/N & \longrightarrow & X \cong \bar{Y} \times F/\pi_1(Y) \\
\downarrow & & \downarrow & & \downarrow \\
\bar{Y} := Y_{\text{univ}} & \cong & \bar{Y}' = \bar{Y}/N & \cong & \bar{Y}/\pi_1(Y).
\end{array}
$$

The following claim based on an elemental observation gives our basic strategy.

Claim 1. Let $N$ be a normal subgroup in $\pi_1(Y)$. Then the following conditions are equivalent:

- The quotient group $\pi_1(Y)/N$ is a finite group and $N$ is contained in $\text{Ker}(\rho)$.
- The quotient $X' := \bar{Y} \times F/N \to X$ corresponding to $N$ is a finite étale cover and $X'$ is isomorphic to the product $Y' \times F$, where $Y'$ is the quotient of $Y$ corresponding to $N$.

Proof. It is easy to check that $X' = \bar{Y} \times F/N \to X$ is a finite morphism if and only if $\pi_1(Y)/N$ is a finite group. In the case of $N \subseteq \text{Ker}(\rho)$, the action $g := \rho(\gamma) : F \to F$ on $F$ induced by $\rho$ and an arbitrary loop $\gamma \in N$ in $Y$ is trivial (that is, $g = \text{id}_F$), and thus the corresponding quotient $\bar{Y} \times F/N \to X$ is the product $Y/N \times F$. Conversely,
when the quotient $\tilde{Y} \times F/N$ is isomorphic to the product, an arbitrary loop $\gamma \in N$ trivially acts on $F$.

Our basic strategy is to show that $\pi_1(Y)/\text{Ker}(\rho) \cong \text{Im}(\rho) \subseteq \text{Aut}(F)$ is a finite subgroup.

Step 2 (Comparison of semi-ample fibrations). Let $H$ (resp. $G_y$) be the image of the semi-ample fibration associated to $mL$ (resp. the restriction $L|_{X_y}$ to a fiber $X_y$). For a fixed sufficiently large $m > 0$, we have the morphisms with connected fibers

$$\Phi := \Phi|_{mL} : X \to H \quad \text{and} \quad \varphi := \Phi|_{mL|_{X_y}} : X_y \to G_y.$$ 

All the fibers $X_y$ are isomorphic to $F$ in our situation, but we will use the notations $X_y$ and $G_y$ (which looks like depending on $y$) to carefully treat the automorphisms of $F$. In this step, we will compare the above semi-ample fibrations and prove that the Stein factorization of the restriction of $\Phi$ to $X_y$ actually coincides with $\varphi$ by using the solution of Hacon-McKernan’s question proved by Ejiri-Gongyo (see [HM07, EG19] and see [CCM19] for its generalization).

The main result of [EG19] asserts that

$$H^0(X, mL) \to H^0(X_y, mL|_{X_y})$$

is injective. Hence, for a basis $\{s_i\}_{i=1}^k$ of $H^0(X, mL)$, we can take sections $\{t_j\}_{j=1}^\ell$ of $H^0(X_y, mL|_{X_y})$ such that $s_i|_{X_y}$, $t_j$ is a basis $H^0(X_y, mL|_{X_y})$. We may assume that $H$ (resp. $G_y$) is the image of the semi-ample fibration $\Phi$ (resp. $\varphi$) defined by $[s_1 : \cdots : s_k]$ (resp. $[s_1 : \cdots : s_k, t_1 : \cdots : t_\ell]$). Now we consider the linear projection $p : \mathbb{P}^{k+\ell-1} \to \mathbb{P}^{k-1}$ to the first $k$ components and its restriction to $G_y \subseteq \mathbb{P}^{k+\ell-1}$. We remark that the linear projection is a rational map, but its restriction to $G_y \subseteq \mathbb{P}^{k+\ell-1}$ determines the morphism. We have the following commutative diagram:

$$\begin{array}{ccc}
X & \xrightarrow{\Phi} & H \\
\cup & & \downarrow \\
X_y & \xrightarrow{\varphi} & G_y \subseteq \mathbb{P}^{k+\ell-1}.
\end{array}$$

Let $A$ (resp. $B$) be the ample divisor on $H$ (resp. $G_y$) defined by the restriction of the hyperplane bundle of $\mathbb{P}^{k-1}$ (resp. $\mathbb{P}^{k+\ell-1}$). By the definition, we have $B = p^*A$. On the other hand, we have

$$mL = \Phi^* A \quad \text{and} \quad mL|_{X_y} = \varphi^* B$$

by the construction of $\Phi$ and $\varphi$. Hence, by considering the restriction of the first equality to $X_y$, we obtain

$$\varphi^*(p^* A) = mL|_{X_y} = \varphi^* B.$$ 

By the projection formula, we can obtain $p^*A = B$ since the morphism $\varphi$ is algebraic fiber space (that is, it is proper and its fibers are connected). It follows that $p$ must be a finite morphism since the pull-back of an ample divisor is ample again. This implies that $\varphi : X_y \to G_y$ and $G_y \to H$ gives the Stein factorization of $\Phi|_{X_y} : X_y \to H$.

If $p : G_y \to H$ is injective (that is, $p$ is isomorphic), the morphism defined by $p \mapsto (\phi(p), \Phi(p))$ determines the isomorphism $X \to Y \times H$ (which finishes the
proof). However \( p : G_y \to H \) is finite but not always one to one mapping (see Example 6.1). For this reason, we have to construct an appropriate a finite étale cover of \( Y \) (and also \( X \)) so that \( p : G_y \to H \) is isomorphic. The key point here is the space of sections of \( mL|_{X_y} \) is invariant, but the space of sections of \( X \) is expanded, by taking the fiber product by étale covers of \( Y \) by taking the fiber product by étale covers of \( Y \) (see [CCM19, Theorem 1.3]). This step is the core of Theorem 1.7.

**Step 3 (Relations between \( \rho : \pi_1(Y) \to \text{Aut}(F) \) and sections of \( -K_{X/Y} \)).** In this step, we reveal relations of \( \rho : \pi_1(Y) \to \text{Aut}(F) \) and sections of \( -K_{X/Y} \) by applying the structure theorem of klt pairs with nef anticanonical bundles (see [CCM19, Theorem 1.3]). This step is the core of Theorem 1.7.

Let \( D \) be an effective divisor \( D \) such that \( D \) is \( \mathbb{Q} \)-linearly equivalent to \( L \). Then, for a sufficiently small \( \delta > 0 \), we can easily check that \( (X, \delta D) \) is a klt pair and \(- (K_{X/Y} + \delta D) \sim \mathbb{Q} -(1 - \delta) K_{X/Y} \) is nef. By applying [CCM19, Theorem 1.3] to the pair \((X, \delta D)\), we can conclude that not only the manifold \( X \) but also the pair \((\bar{X}, \delta \bar{D})\) has the structure of the product, where \( \bar{D} \) is the pull-back \( \bar{D} := \pi_X^* D \) and \( \pi_X : \bar{X} \to X \). Hence, for a section \( s \in H^0(X, mL) \), we can find a section \( s_F \in H^0(F, mL|_F) \) such that \[ \pi_X^* s = pr_2^* s_F, \] where \( pr_2 : \bar{X} = Y \times F \to F \) is the second projection. Then it can be shown that the composite morphism \( \Phi \circ \pi_X : \bar{X} \to H \), which is defined by sections \( \{ \pi_X^* s_i \}_{i=1}^k \), is actually determined by sections in \( H^0(F, mL|_F) \). This implies that the restriction of \( \Phi \circ \pi_X \) to the fiber \( \{y\} \times F \) does not depend on \( y \in \bar{Y} \). In other words, the composite morphism \( \Phi \circ \pi_X \) factors into the second projection \( pr_2 : \bar{X} = Y \times F \to F \) and the composite morphism \( \psi := p \circ \varphi : F(\cong X_y) \to H \):

\[
\begin{array}{ccc}
\bar{X} & \xrightarrow{\pi_X} & X \\
| & \downarrow{\Phi} & \downarrow{p} \\
F & \xrightarrow{\cong} & X_y \\
| & \downarrow{\varphi} & \\
G_y & \xrightarrow{\psi} & H \\
& \downarrow{\varphi} & \\
& F & \xleftarrow{\psi} & G_y.
\end{array}
\]

By the above argument, for an arbitrary loop \( \gamma \) in \( Y \), the automorphism \( g := \rho(\gamma) \in \text{Aut}(F) \) induced by \( \rho : \pi_1(Y) \to \text{Aut}(F) \) always satisfies the following commutative diagram:

\[
\begin{array}{ccc}
F & \xrightarrow{g} & F \\
\downarrow{\varphi} & \quad & \downarrow{\varphi} \\
G_y & \xleftarrow{\psi} & H \\
& \downarrow{\psi} & \\
& F & \xrightarrow{g} & F.
\end{array}
\]

**Step 4 (Proof of Theorem 1.7 (1)).** In this step, we will prove Theorem 1.7 (1) under the assumption that \( L|_F = -K_F \) is ample. It is sufficient for this purpose to prove that \( \text{Im}(\rho) \subseteq \text{Aut}(F) \) is a finite subgroup by Claim 1.

We may assume that \( \varphi : F \to G_y \) is isomorphic since \( L|_F = -K_F \) is ample. Then \( \psi = \varphi \circ p \) is finite since \( p \) is so. Hence, by considering an unramified point of \( \psi \), we can find a (Euclidean) open subset \( U_H \subseteq H \) such that \( \psi : V_\lambda \to U_H \) is isomorphic for any \( \lambda \in \Lambda \), where \( \psi^{-1}(U_H) = \coprod_{\lambda \in \Lambda} V_\lambda \) and \( \Lambda \) is a finite set with \( |\Lambda| = \text{deg } \psi \). It follows that \( g \in \text{Im}(\rho) \) is an isomorphism over \( H \) (that it, it preserves
the morphism \( \psi : F \to H \) from the commutative diagram (9). Hence we obtain the homomorphism

\[
\text{Im}(\rho) \subseteq \text{Aut}(F/H) \to \text{Aut}(\psi^{-1}(U_H)/U_H),
\]

where \( \text{Aut}(F/H) \) (resp. \( \text{Aut}(\psi^{-1}(U_H)/U_H) \)) is the set of automorphisms of \( F \) (resp. \( \psi^{-1}(U_H) \)) preserving \( \psi : F \to H \) (resp. \( \psi : \psi^{-1}(U_H) \to U_H ) \). By \( \psi^{-1}(U_H) = \coprod_{\lambda \in \Lambda} V_\lambda \), we can show that the automorphisms \( \text{Aut}(\coprod_{\lambda \in \Lambda} V_\lambda/U_H) \) are just the permutations of \( \Lambda \), and thus we have \( |\text{Aut}(\psi^{-1}(U_H)/U_H)| \leq |\Lambda| = (\deg p)! \). On the other hand, we can see that the above homomorphism is injective by an observation on the graph of \( g \in \text{Im}(\rho) \) and its irreducibility. Hence the conclusion (1) can be obtained by Claim 1 and the finiteness of \( \pi_1(Y)/\text{Ker}(\rho) \).

**Step 5 (Proof of étale trivializability of direct images).** In this step, we will prove that the direct image \( \phi_*(-mK_{X/Y}) \) is a trivial vector bundle after we take an appropriate finite étale cover \( Y' \to Y \) and its base change. We go back to the commutative diagram (9). The morphism \( \varphi \) with connected fibers and the finite morphism \( p \) give the Stein factorization not only for \( \psi \) but also for \( \psi \circ g \). By the uniqueness of the Stein factorization, the automorphism \( g \in \text{Im}(\rho) \) induces the automorphism of \( G_y \) (which we denote by \( \bar{g} \)). This determines the homomorphism

\[
\bar{\rho} : \pi_1(Y) \longrightarrow \text{Aut}(F) \longrightarrow \text{Aut}(G_y).
\]

By the same argument as in Step 4, we can conclude that \( \pi_1(Y)/\text{Ker}(\bar{\rho}) \cong \text{Im}(\bar{\rho}) \) is a finite group since \( p \) is a finite morphism. Now we define \( Y' \) (resp. \( X' \)) by the quotient of \( \bar{Y} \) (resp. \( \bar{X} \)) corresponding to \( \text{Ker}(\bar{\rho}) \). Then we have the following diagram:

\[
\begin{array}{ccc}
\bar{Y} \times F & \cong & \bar{X} \\
\downarrow & & \downarrow \\
Y' & = & \bar{Y}/\text{Ker}(\bar{\rho}) \longrightarrow Y.
\end{array}
\]

By the construction, the morphism \( Y' \to Y \) is finite and the induced morphism \( \phi' : X' \to Y' \) is the fiber product of \( X \) by \( Y' \to Y \) (see the proof of Claim 1). Hence, by replacing \( \phi : X \to Y \) with \( \phi' : X' \to Y' \), we may assume that any loop in \( Y \) trivially acts on \( G_y \) (that is, \( \text{Im}(\bar{\rho}) = \{ \text{id}_{G_y} \} \)).

By applying the result of [EG19] to our new morphism \( \phi : X \to Y \) again (see Step 2), we can see that the restriction map of sections of \( -mK_{X/Y} \) to \( X_y \) is injective. Hence it is sufficient to prove that any section in \( H^0(X_y, -mK_{X_y}) \) can be extended to a section in \( H^0(X, -mK_{X/Y}) \). For a given section \( t \in H^0(X_y, -mK_{X_y}) \), we can obtain the extended section \( \bar{t} = \text{pr}_2^* t \in H^0(\bar{X}, -mK_{\bar{X}/Y}) \) by identifying \( t \) with the section in \( H^0(F, -mK_F) \). On the other hand, we can take \( u \in H^0(G_y, B) \) such that \( t = \varphi^* u \) by the definition of \( \varphi \) and \( B \). If a loop \( \gamma \) in \( Y \) is in \( \text{Ker}(\bar{\rho}) \), the induced automorphism \( \bar{g} \) trivially acts on \( G_y \), where \( g := \rho(\gamma) \). Hence \( \bar{t} = \text{pr}_2^* t = \text{pr}_2^* \varphi^* u \) is invariant under the actions of \( \pi_1(Y) \). This implies that the extended section \( \bar{t} \) can be descended to the section in \( H^0(X, -mK_{X/Y}) \). Therefore \( \phi : X \to Y \) satisfies that \( \phi_*(mL) = Y \times H^0(F, -mK_F) \).
Step 6 (Proof of Theorem 1.7 (2)). In this step, we finally give a proof of Theorem 1.7 (2). Our strategy is to reduce our situation to the case of $K_{X/Y} = \mathcal{O}_X$, in which the assertion holds by [LPT18, Theorem 5.8] (see [Dru17]).

By Step 5, we may assume that $\phi_*(mL)$ is a trivial vector bundle. In this case, since the morphism $p : G_y \to H$ is isomorphic, we identify $\varphi : X_y \to G_y$ with the restriction of $\Phi : X \to H$ to the fiber $X_y$. For the fiber $P_h := \Phi^{-1}(h)$ at a point $h \in H$, the intersection $P_h \cap X_y$ coincides with the fiber of $\varphi : X_y \to G_y$ at $h \in H$. In particular, for a general point $h \in H$, the fiber $P_h$ is a smooth projective variety and the restriction $\phi|_{P_h} : P_h \to Y$ is also a surjective morphism with connected fibers. Note that the irregularity of a general fiber $\phi|_{P_h} : P_h \to Y$ is equal to zero by the assumption of Theorem 1.7 (2). Further its relative canonical bundle $mK_{P_h/Y}$ is trivial. Indeed, since $P_h$ is a fiber of the semi-ample fibration associated to $mL$, we have $mL|_{P_h} = \mathcal{O}_{P_h}$ and $K_X|_{P_h} = K_{P_h}$, and thus we obtain $mK_{P_h/Y} = \mathcal{O}_{P_h}$. Hence, for a general point $h \in H$, the algebraic fiber space $P_h \to Y$ has the structure of the product by taking the base change by a finite étale cover $Y' \to Y$ since the restriction $\phi|_{P_h} : P_h \to Y$ satisfies the assumptions of [LPT18, Theorem 5.8].

On the other hand, for an automorphism $g = \rho(\gamma) \in \text{Im}(\rho)$, the induced automorphism $\tilde{g} \in \text{Im}(\rho)$ trivially acts on $H$ in our case, and thus $g$ preserves the fiber $P_h \cap F$. This implies that the restriction $g$ to $P_h \cap F$ gives the automorphism $g_h := g|_{P_h \cap F} \in \text{Aut}(P_h \cap F)$. This determines the homomorphism

$$\rho_h : \pi_1(Y) \xrightarrow{\rho} \text{Im}(\rho) \longrightarrow \text{Aut}(P_h \cap F)$$

Our goal is to prove that $\text{Im}(\rho)$ is a finite group (see Claim 1). By the above argument with [LPT18, Theorem 5.8] and Claim 1, we have already shown that $\text{Im}(\rho_h)$ is a finite subgroup in $\text{Aut}(P_h \cap F)$ for a general point $h$. Therefore it is sufficient for our goal to show that $\text{Im}(\rho) \to \text{Aut}(P_h \cap F)$ is injective for some $h \in H$. This is proved by the following claim:

Claim 2. For an automorphism $g \in \text{Im}(\rho)$ induced by a loop in $Y$ and $\rho$, the subset $A_g \subseteq H$ defined by

$$A_g := \{ h \in H \mid g_h = \text{id}_{P_h \cap F} \}.$$

is a Zariski-closed subset in $H$. Further, if $g \neq \text{id}_F$, then $A_g$ is a proper Zariski-closed subset in $H$.

Proof. It is easy to check the latter conclusion. Indeed, in the case of $A_g = H$, we have $g_h = g|_{P_h \cap F}$ for any $h \in H$, and thus $g$ trivially acts on $F$ (that is, $g = \text{id}_F$).

For a fixed $g \in \text{Im}(\rho)$, we define the (not necessarily irreducible) subvariety $\mathcal{B}$ by

$$\mathcal{B} := \{ x \in F \mid g(x) = x \} \subseteq F.$$ 

Let $m$ be the relative dimension of $\varphi : F \to H$. Now we consider the morphism $\varphi|_{\mathcal{B}} : \mathcal{B} \to H$. We remark that the condition of $h \in A_g$ is equivalent to the condition of $\dim(\mathcal{B}_h) = m$, where $\mathcal{B}_h$ is the fiber at $h \in H$. On the other hand, we have

$$m \geq \dim(\mathcal{B} \cap F_h) = \dim(\mathcal{B} \cap P_h \cap F) \quad \text{and} \quad \mathcal{B}_h = \mathcal{B} \cap P_h \cap F$$

for a general $h \in H$. Further it can be shown that

$$\{ h \in H \mid \dim(\mathcal{B}_h) \geq m \}$$
is a Zariski-closed set by the upper semi-continuity of fiber dimensions. This completes the proof.

The above claim finishes the proof of Theorem 1.7 (2). Indeed, let $A_0$ be a (proper) Zariski-closed set in $H$ such that the fiber $X_h$ at $h \notin A_0$ is a projective manifold. Note that $\pi_1(Y)$ is an (at most) countable set since any loop can be approximated by polygonal lines. Then, for a very general point $h$ with $h \notin A_0 \cup \bigcup_{d \neq g \in \text{Im}(\rho)} A_g$, the map $\text{Im}(\rho) \rightarrow \text{Aut}(P_h \cap F)$ is injective by the definition of $A_g$. Therefore $\text{Im}(\rho_h)$ is a finite subgroup, and thus so is $\text{Im}(\rho)$.

\[ \square \]

6. Examples

In this section, we construct examples of algebraic fiber spaces having nef (semi-positive, or semi-ample) relative anti-canonical divisors, which helps us to understand this paper.

Example 6.1 (Algebraic fiber spaces with semi-ample $-K_{X/Y}$). This example tells us that Theorem 1.7 does not hold without taking étale covers.

Let $Y$ be a smooth projective curve of genus at least one, and let $D$ be a divisor on $Y$ such that $\deg D \geq 0$ and $D \not\sim 0$. Set $\mathcal{E} := \mathcal{O}_Y \oplus \mathcal{O}_Y(-D)$ and $X := \mathbb{P}(\mathcal{E})$. Let $\phi : X = \mathbb{P}(\mathcal{E}) \rightarrow Y$ denote the natural projection. Let $C_0 \subseteq X$ be the section corresponding to the quotient $\mathcal{E} \rightarrow \mathcal{O}_Y(-D)$. Then $-K_{X/Y} \sim 2C_0 + \phi^*D$ (cf. [Har77, V, Lemma 2.10]), and thus it follows from the projection formula that, for each $m \geq 0$,

\[ \phi_*\mathcal{O}_X(-mK_{X/Y}) \cong \phi_*\mathcal{O}_X(2mC_0) \otimes \mathcal{O}_Y(mD) \cong S^{2m}(\mathcal{E}) \otimes \mathcal{O}_Y(mD) \]

\[ \cong \left( \bigoplus_{i=0}^{2m} \mathcal{O}_Y(-iD) \right) \otimes \mathcal{O}_Y(mD) \cong \bigoplus_{i=0}^{2m} \mathcal{O}_Y((m-i)D). \]

(10)

If $D$ is a torsion element in $\text{Cl}(X) = \text{Div}(X)/\sim_{\text{lin}}$ of order $n$, then we have

(i) $-K_{X/Y}$ is nef, since $\phi^*D$ is nef and $C_0^2 = -\deg D = 0$,

(ii) $2nC_0 \in |-nK_{X/Y}|$, and

(iii) there is $E \in |-nK_{X/Y}|$ with $E \neq nC_0$, since $\dim |-nK_{X/Y}| = 2$ by (10).

This implies that $|-nK_{X/Y}|$ is free, and thus $-K_{X/Y}$ is semi-ample. However $\phi : X = \mathbb{P}(\mathcal{E}) \rightarrow Y$ itself is not a product and the direct images $\phi_* -mK_{X/Y}$ are not trivial.

We can directly check the assertion of Theorem 1.7 for this example. Indeed, the $n$-torsion divisor $D$ defines an étale cyclic cover $\pi : Z \rightarrow Y$ such that $\pi^*D \sim 0$. Hence $\pi^*\mathcal{E} \cong \mathcal{O}_Z \oplus \mathcal{O}_Z$, and $X \times_Y Z$ is isomorphic to $\mathbb{P}^1 \times_{\mathbb{C}} Z$.

Example 6.2 (Algebraic fiber spaces with semipositive but not semi-ample $-K_{X/Y}$). Let the notation be as in Example 6.1. If $D$ is a divisor such that $\deg D = 0$ and that $D$ is not a torsion element in $\text{Cl}(X)$, then we see that $-K_{X/Y}$ is semipositive (that is, it admits a smooth hermitian metric with semipositive curvature), but it is not semi-ample by

\[ 0 \overset{(10)}{=} \kappa(-K_{X/Y}) < nd(-K_{X/Y}) = 1. \]
Then it is easy to see that the direct images
\[ \phi_*(-mK_{X/Y}) \cong \bigoplus_{i=0}^{2m} \mathcal{O}_Y((m-i)D) \]
are hermitian flat by \( c_1(\mathcal{O}_Y(D)) = 0 \).

**Example 6.3** (Lc pairs \((X, D)\) with semi-ample and big \(- (K_{X/Y} + D)\)). This example says that Theorem 1.7 does not hold for an lc pair \((X, D)\) even if \(- (K_{X/Y} + D)\) is semi-ample and big. It is known that \( B_+(- (K_{X/Y} + D)) = X \) holds if \((X, D)\) is klt and if \(- (K_{X/Y} + D)\) is nef, but this example also says that it is not true for lc pairs. However the statement of Corollary 3.7 (that is, \( B_+(- (K_{X/Y} + D)) \) is dominant over \( Y \)) is still true for lc pairs.

Let the notation be as in Example 6.1. Let \( C_1 \) be the section of \( \phi \) corresponding to the quotient \( E \twoheadrightarrow \mathcal{O}_Y \). Then \( C_1 \sim C_0 + \phi^* D \) and \( C_0 \cdot C_1 = 0 \) (cf. [Har77, V, Proposition 2.9]). We consider the lc pair \((X, C_1)\). Put \( L := K_{X/Y} + C_1 \). Then we have
\[ -L \sim 2C_0 + \phi^* D - C_1 \sim C_0, \]
and thus \(-L\) is nef. Assume that \( \text{deg } D > 0 \). Then \( C_0^2 > 0 \), so \(-L\) is big. We have \( B_+(-L) = C_1 \) thanks to [ELMNP2, Theorem C]. Further \(-L\) is semi-ample by
\[ B(-L) \subseteq C_0 \cap B_+(-L) = C_0 \cap C_1 = \emptyset. \]
Note that there is no finite cover \( \pi : Z \to Y \) of \( Y \) such that \( X \times_Y Z \) is isomorphic to \( \mathbb{P}^1 \times_\mathbb{C} Z \) as \( Z \)-schemes.

**Example 6.4.** This example tells us that Theorem 1.7 does not hold for Kähler manifolds.

Let \((z, w)\) be the standard coordinate of \( \mathbb{C}^2 \). We consider the compact complex torus \( X := \mathbb{C}^2/\Gamma \) defined by the lattice generated by \( (0, 1), (0, \tau), (\tau, 0), (1, \alpha) \).

Here \( \alpha := a + \tau b \), \( \tau \) is a complex number whose imaginary part is non-zero, and \( a, b \) are real numbers with \( 0 \leq a, b \leq 1 \). Note that \( X \) is always Kähler, but not projective for general \( a, b \). Then the natural first projection \( X \to Y \) to the elliptic curve \( Y := \mathbb{C}/(1, \tau) \) defined by 1 and \( \tau \) is a locally trivial morphism with the fiber that is isomorphic to \( \mathbb{C}/(1, \tau) \). Further its relative canonical bundle is trivial. In particular, the morphism \( X \to Y \) satisfies the assumptions of [LPT18, Theorem 5.8] (see [Dru17]). However \( X \) is projective if \( X \) is a product up to finite étale covers. This is a contradiction to the case of \( a, b \) being general.

Theorem 1.7 was proved for morphisms with trivial relative canonical bundle in [Dru17, Lemma6.4], in which the key point is the existence of fine moduli of polarized abelian varieties with level structures. The lack of fine modulus causes this example.

As in [HM07, CH19, EG19, CCM19], it is natural to consider an application of our argument to maximally rationally connected (MRC) fibrations, but we do not deal with it, because this paper only treat morphisms. Recall that, by definition, the MRC fibrations associated to a variety are almost holomorphic maps, which may not be represented by a morphism. In fact, the authors leaned from Kento Fujita that there exists a normal projective variety whose MRC fibrations cannot be
represented by a morphism (Example 6.5). The authors do not know such varieties except this example. Fujita constructed a normal projective 3-fold \( X' \) with Picard number one that is uniruled but not rationally connected.

**Example 6.5.** Let \( S \) be a smooth projective surface such that

- \( \rho(S) = 1 \) and
- the MRC fibration associated to \( S \) is represented by \( \text{id} : S \rightarrow S \).

For example, a very general Abelian surface or hypersurface of degree \( d \geq 4 \) in \( \mathbb{P}^3 \) satisfies the above conditions. Take a very ample divisor \( \mathcal{L} \) on \( S \) and a smooth curve \( C \) in \( |\mathcal{L}| \). We define the notation by the following commutative diagram whose all squares denote base changes:

\[
\begin{array}{c}
C_X & \rightarrow & B_X \\
\sigma \cong & & \\
F_X & \rightarrow & X & \rightarrow & \mathbb{P}^1 \\
\downarrow & & \downarrow & & \downarrow \\
C & \rightarrow & S & \rightarrow & \text{Spec} \mathbb{C}
\end{array}
\]

Here, \( B_X \) is a general member in the complete linear system of \( \mathcal{M}_k := p_1^* \mathcal{L}^k \otimes p_2^* \mathcal{O}(1) \) for \( k \geq 2 \). We have \( \mathcal{N}_{C_X/X} \cong \mathcal{M}_{k-1} \oplus \sigma^* \mathcal{L} \).

Let \( \pi : Y \rightarrow X \) be the blow-up along \( C_X \) and consider the elemental transformation

\[
\begin{array}{c}
Y \\
\pi \rightarrow \\
X \\
p_1 \rightarrow \\
S.
\end{array}
\]

Let \( \tau : Z \rightarrow X' \) be the blow-up along \( \tau \) and consider the elemental transformation

\[
\begin{array}{c}
Y \\
\tau \rightarrow \\
X \\
p_2 \rightarrow \\
Z \\
\downarrow & & \downarrow & & \downarrow \\
S.
\end{array}
\]

Let \( E \) denote the exceptional divisor of \( \pi \). Set \( B := \pi^{-1}_X B_X \) and \( F := \pi^{-1}_X F_X \). Since

\[
B - F \sim \pi^*(B_X - F_X) \in \pi^*|\mathcal{M}_{k-1}|
\]

and \( B \cap F = \emptyset \), we see that \( |B| \) is free, and thus it defines a morphism \( \varphi : Y \rightarrow X' \) with \( \varphi_* \mathcal{O}_Y \cong \mathcal{O}_{X'} \), which contracts \( F \) to a point.

We show that \( \varphi \) induces an isomorphism from \( Y \setminus F \) to \( X' \setminus \varphi(F) \). Suppose that there is a curve \( \gamma \subseteq Y \) such that \( \gamma \not\subseteq F \) and \( \varphi(\gamma) \) is a point. Then we have

\[
0 \geq -\gamma \cdot F = \gamma \cdot (B - F) = \gamma \cdot \pi^* \mathcal{M}_{k-1} = \pi(\gamma) \cdot \mathcal{M}_{k-1}.
\]

Then it follows that \( \pi(\gamma) \) is one point since \( \mathcal{M}_{k-1} \) is ample. Therefore \( \gamma \subseteq E \cong \mathbb{P}(\mathcal{N}_{C_X/X}^*), \) and the choice of \( \gamma \) implies that it is contained in the section \( F \cap E \cong \mathbb{P}(\sigma^* \mathcal{L}^*) \subseteq \mathbb{P}(\mathcal{N}_{C_X/X}^*), \) so \( \gamma \subseteq F \), which is a contradiction.

Hence \( \varphi : Y \rightarrow X' \) contracts every fiber of \( \tau : Y \rightarrow Z \) to a point, and thus \( \varphi \) factors through \( Z \). The induced morphism \( \psi : Z \rightarrow X' \) is not an isomorphism, since
\(\tau(F)\) is not a point. Then we have
\[
\rho(X') \leq \rho(Z) - 1 = \rho(X) - 1 = 2 - 1 = 1,
\]
so we obtain \(\rho(X') = 1\). The MRC fibration associated to \(X'\) is the composite of \(X' \rightarrow X \xrightarrow{\rho} S\).

\section*{A. Appendix}

\subsection*{A.1. Proof of the statements in subsection 4.1.}
In this subsection, we give a proof for Proposition 4.4, Theorem 4.5, and 4.6.

**Proof of Proposition 4.4.** We consider the direct image of the divisor
\[
pK_{X/Y} + qN\Delta + L = (mN + p)K_{X/Y} - mN(K_{X/Y} + \Delta) + (m + q)N\Delta + \frac{L}{mN} \quad \text{with } g^{\rho}_{\varepsilon}\quad \text{with } h^{(m+q)N}_\Delta \quad \text{with } h^{1-\varepsilon}_L \text{H}^\varepsilon
\]
equipped with singular metrics defined as follows: The metric \(g_{\varepsilon}\), which is obtained from the assumption of the restricted base locus, is a singular metric on \(\rho(N(K_{X/Y} + \Delta))\) such that \(\sqrt{-1}\Theta_{g_{\varepsilon}} \geq -\varepsilon\omega_X\) and \(g_{\varepsilon}|_{X_y}\) is smooth on \(X_y\) for a general point \(y \in Y\). The metric \(h_\Delta\) is the canonical singular metric on \(\Delta\). The metric \(H_L\), which is obtained from the \(\phi\)-bigness of \(L\), is a singular metric on \(L\) so that \(\sqrt{-1}\Theta_{H_L} + k\phi^*\omega_Y \geq \delta\omega_X\) for some \(k\) and \(\delta > 0\). Then \(h := g^{\rho}_{\varepsilon}h^{(m+q)N}_\Delta h^{1-\varepsilon}_L \text{H}^\varepsilon\) satisfies that
\[
\sqrt{-1}\Theta_h = -\varepsilon' m\omega_X + (1 - \varepsilon)\phi^*\theta + \varepsilon(\delta\omega_X - k\phi^*\omega_Y) \geq (1 - \varepsilon)\phi^*\theta - \varepsilon k\phi^*\omega_Y
\]
for \(\varepsilon \gg \varepsilon' > 0\). On the other hand, for a sufficiently large \(m \in \mathbb{Z}_{>0}\), we can see that \(\mathcal{I}(h^{1/(mN+p)}_\Delta)|_y = \mathcal{O}_{X_y}\) for a general \(y \in Y\), since we have \(\mathcal{I}(h^{1/mN+q}_L) = \mathcal{I}(h^{1/mN+q}_L) = \mathcal{O}_{X_y},\ g_{\varepsilon'}\) is smooth on \(X_y\), and \((X, \frac{(m+q)N(1+mN+p)}{(mN+q)}\Delta)\) is a klt pair. The induced metric on the direct image sheaf satisfies the desired conclusion (see [CP17, Lemma 5.25], [PT18, BP08].)

**Proof of Theorem 4.6.** By [CCM19, Proposition 2.7, Theorem 2.8] and Proposition 4.3, it is sufficient to show that \(\phi_*(p\tilde{A})\) and \(\phi_*(qN\Delta+p\tilde{A})\) are numerically flat vector bundles for any \(p \in \mathbb{Z}_{>0}\) and any \(q \in \mathbb{Z}_{>0}\). The proof of the numerical flatness is the same as in [Cao19, CH19, CCM19], but we give the proof here for reader’s convenience.

Let \(A\) be a sufficiently ample divisor on \(X\). Set \(r := \text{rank}(c_1(\phi_*(A)))\) and
\[
\tilde{A} := A - \frac{1}{r}\phi^*\left(\det \phi_*(A)\right).
\]
Note that we may assume that \(\phi_*(A)\) is locally free by Proposition 4.1 and \(\tilde{A}\) is a Cartier divisor by [CH19, Prop 3.9].

Now we prove that
\[
\mathcal{V}_{m,q,p} := \phi_*(-mN(K_{X/Y} + \Delta) + qN\Delta + p\tilde{A})
\]
is numerically flat for any \(m, q \in \mathbb{Z}_{>0}\) and any \(p \in \mathbb{Z}_{>0}\), by applying Theorem 4.5 which is proved later. It follows that \(\mathcal{V}_{m,q,p}\) is weakly positively curved (and thus
$c_1(\mathcal{V}_{m,q,p})$ is pseudo-effective. By Theorem 4.5 and Proposition 4.4. By Theorem 4.5, there exists a $\phi$-exceptional effective divisor $E$ such that

$$-mN(K_{X/Y} + \Delta) + qN\Delta + p\tilde{A} + E - \frac{1}{r_{m,q,p}}\phi^*(\det \mathcal{V}_{m,q,p})$$

is pseudo-effective. By applying Proposition 4.4 to $\theta$ representing the first Chern class $c_1(\mathcal{V}_{m,q,p})$ and the divisor

$$p\tilde{A} + E = mNK_{X/Y} + (m-q)N\Delta + (-mN(K_{X/Y} + \Delta) + qN\Delta + p\tilde{A} + E),$$

we obtain $c_1(\phi_*(p\tilde{A})) \geq \frac{r_{m,q,p}}{r_{m,q,p}}c_1(\mathcal{V}_{m,q,p})$. This implies that $c_1(\mathcal{V}_{m,q,p}) = 0$ from $0 = c_1(\phi_*(\tilde{A})) \geq \frac{1}{r_{m,q,p}}c_1(\mathcal{V}_{0,0,p})$ by Theorem 4.5 and Proposition 4.4.

\[\square\]

**Proof of Theorem 4.5.** The basic idea is the same as in [Cao19, Proposition 3.15]. Note that $\phi$ is smooth in codimension 1 by Proposition 4.1. Hence there exists a Zariski-closed set $Z \subseteq X$ such that $\text{codim} Z \geq 2$ and $\phi|_{X\setminus Z} : X \setminus Z \to Y \setminus \phi(Z)$ is smooth.

Let $X^{(r)}$ be a desingularization of the $r$-times fiber product $X \times_Y X \times_Y \cdots \times_Y X$. Let $pr_i : X^{(r)} \to X$ be the $i$-th projection and $\phi^{(r)} : X^{(r)} \to Y$ be the induced morphism. Set $V := X^{(r)} \setminus (\cup pr_i^{-1}(X \setminus Z))$.

By the same argument as in [Cao19, Proposition 3.15], there exist effective divisors $D_1, D_2$ supported in $V$ such that

$$K_{X^{(r)}/Y} = \sum_{i=1}^{r} pr_i^*K_{X/Y} + D_1 - D_2.$$

From the assumption on the restricted base locus, we obtain

$$\phi^{(r)}(\mathbb{B}_-)\left(\sum_{i=1}^{r} pr_i^*(K_{X/Y} + \Delta)\right) \subseteq \cup_{i=1}^{r} \phi^{(r)}(\mathbb{B}_-(-pr_i^*(K_{X/Y} + \Delta))) \neq Y.$$

For the divisor $L := \sum_{i=1}^{r} pr_i^*L$, we have the canonical map

$$\det(\phi_*(L)) \to \otimes^r \phi_*(L) \cong \phi^{(r)}_*(L_r)$$

on $Y_L$, where $Y_L$ be the maximum Zariski-open set such that $\phi_*(L)$ is locally free. Hence we have the nonzero section of $\phi^{(r)}_*(L_r) \otimes \det(\phi_*(L))^\vee$ on $Y_L$, and thus, by taking double dual, we have

$$0 \neq H^0(Y_L, \phi^{(r)}_*(L_r)^{\vee\vee} \otimes \det(\phi_*(L))^\vee) = H^0(Y, \phi^{(r)}_*(L_r)^{\vee\vee} \otimes \det(\phi_*(L))^\vee).$$

By [Nak04, III. Lemma 5.10], there exists a $\phi^{(r)}$-exceptional effective divisor $E_r$ such that

$$\phi^{(r)}_*(L_r)^{\vee\vee} = \phi^{(r)}_*(L_r + E_r).$$

Thus $L' := L_r + E_r - \phi^{(r)*}(\det(\phi_*(L)))$ is an effective divisor of $X^{(r)}$. Since $L'$ is also $\phi^{(r)}$-big, we have:

- $L'$ has a singular hermitian metric $h_{L'}$ with semipositive curvature current.
- $L'$ has a singular hermitian metric $H_{L'}$ such that $\sqrt{-1}\Theta_{L',H_{L'}} \geq \omega_{X^{(r)}} - \phi^*\omega_Y$ for some Kähler form $\omega_{X^{(r)}}$ (resp. $\omega_Y$) on $X^{(r)}$ (resp. $Y$).
We take a sufficiently large \( m \in \mathbb{Z}_{>0} \) so that \( \mathcal{J}(h_{L'}^m|_{X_y^{(r)}}) = \mathcal{J}(H_{L'}^m|_{X_y^{(r)}}) = \mathcal{O}_{X_y^{(r)}} \) for a general point \( y \in Y \). Then we have:

**Claim 3.** There exists an ample divisor \( A_Y \) of \( Y \) such that

\[
H^0(X^{(r)}, mpD_1 + pL' + \phi^{(r)*}A_Y) \rightarrow H^0(X_y^{(r)}, mpD_1 + pL' + \phi^{(r)*}A_Y|_{X_y^{(r)}})
\]

is surjective for any \( p \in \mathbb{Z}_{>0} \) and a general point \( y \in Y \).

**Proof.** We first take an ample divisor \( A \) of \( Y \) so that \( A \) admits a smooth hermitian metric \( h_A \) such that \( \sqrt{-1} \Theta h_A - \omega_Y > 0 \). Set \( A_Y := 2A \). For any \( p \in \mathbb{Z}_{>0} \), we have

\[
mpD_1 + pL' + \phi^{(r)*}A_Y = mpK_{X^{(r)}/Y} + \left( -mp \sum_{i=1}^r \text{pr}_i^*(K_{X/Y} + \Delta) + mp \sum_{i=1}^r \text{pr}_i^*\Delta + mp \right) \notag
\]

with \( h_\varepsilon \) and \( H_\Delta \) by the following way:

- \(-mp \sum_{i=1}^r \text{pr}_i^*(K_{X/Y} + \Delta) \) admits a singular hermitian metric \( h_\varepsilon \) such that \( \sqrt{-1} \Theta h_\varepsilon \geq -\varepsilon \omega_{X^{(r)}} \) and \( h_\varepsilon|_{X_y^{(r)}} \) is smooth on \( X_y^{(r)} \) for a general point \( y \in Y \) by the assumption on the restricted base locus.

- \( \sum_{i=1}^r \text{pr}_i^*\Delta \) admits the singular hermitian metric \( H_\Delta := \prod_{i=1}^r \text{pr}_i^*h_\Delta \), where the canonical singular hermitian metric \( h_\Delta \) on the effective divisor \( \Delta \). Then we have \( \mathcal{J}(H_\Delta|_{X_y^{(r)}}) = \mathcal{O}_{X_y^{(r)}} \) for a general point \( y \in Y \) by the klt condition.

- \( D_2 \) admits the canonical singular hermitian metric \( h_{D_2} \). Then it follows that \( h_{D_2}|_{X_y^{(r)}} \) is smooth on \( X_y^{(r)} \) for a general point \( y \in Y \), since \( D_2 \) is supported in \( V \).

Then \( h = h_1H_{\Delta}^mph_{D_2}^mhp_{L'}^pH_{L'}(\phi^{(r)*}h_A) \) satisfies the desired condition, that is,

\[
\sqrt{-1} \Theta h \geq -\omega_{X^{(r)}} + \omega_{X^{(r)}} - \phi^{(r)*}\omega_Y + \sqrt{-1} \Theta \phi^{(r)*}h_A \geq 0
\]

and \( \mathcal{J}(h_{mp}|_{X_y^{(r)}}) = \mathcal{O}_{X_y^{(r)}} \) for a general point \( y \in Y \). This completes the proof. \( \square \)

We may regard \( X \setminus Z \) as a subset of \( X^{(r)} \) with the diagonal map \( j : X \setminus Z \rightarrow X^{(r)} \). We have \( H^0(X_y, pL' + \phi^{(r)*}A_Y|_{X_y}) \neq 0 \) for a general point \( y \in Y \) and sufficiently large \( p \in \mathbb{Z}_{>0} \), and thus we can conclude that \( H^0(X \setminus Z, pL' + \phi^{(r)*}A_Y|_{X \setminus Z}) \neq 0 \) by Claim 3 and the following diagram:

\[
\begin{array}{ccc}
H^0(X^{(r)}, mpD_1 + pL' + \phi^{(r)*}A_Y) & \xrightarrow{j^*} & H^0(X_y^{(r)}, pL' + \phi^{(r)*}A_Y|_{X_y^{(r)}}) \\
\downarrow & & \downarrow \\
H^0(X \setminus Z, pL' + \phi^{(r)*}A_Y|_{X \setminus Z}) & \xrightarrow{j^*} & H^0(X_y, pL' + \phi^{(r)*}A_Y|_{X_y})
\end{array}
\]
On the other hand, from \( \text{codim} Z \geq 2 \), it follows that
\[
0 \neq H^0(X \setminus Z, pL + \phi r + \phi (r)^* A_Y |_{X \setminus Z}) = H^0(X, prL + pE - p\phi^* \det(\phi_*(L)) + \phi^* A_Y).
\]
Here \( E \) is the \( \phi \)-exceptional effective divisor of \( X \) such that \( E|_{X \setminus Z} = E_r|_{X \setminus Z} \). We can see that \( rL + E - \phi^* \det(\phi_*(L)) \) is pseudo-effective by taking \( p \to +\infty \).

If \( \phi_*(L) \) is locally free, \( E \) can be taken as a zero divisor from \( E_r = 0 \) in Equation (15). \( \square \)

A.2. Analytic proof for Corollary 3.7. In this subsection, we give an analytic proof for Corollary 3.7. Let \( \phi : X \to Y \) be a surjective morphism with connected fiber between projective manifolds over \( \mathbb{C} \). Let \( \Delta \) be an effective \( \mathbb{Q} \)-divisor on \( X \).

**Theorem A.1.** If \( (X, \Delta) \) is an lc pair, then \( \mathbb{B}_+(-(K_{X/Y} + \Delta)) \) and \( \mathbb{B}_-(-(K_{X/Y} + \Delta + \phi^* L)) \) are dominant over \( Y \) for any ample divisor \( L \) on \( Y \) unless \( Y \) is one point.

**Proof.** Let \( \omega_X \) be a Kähler form of \( X \) and \( \omega_Y \) be a Kähler form of \( Y \) such that \( \omega_X \geq \phi^* \omega_Y \).

We assume that \( \mathbb{B}_+(-(K_{X/Y} + \Delta)) \) is not dominant over \( Y \). Take \( N \in \mathbb{Z}_{>0} \) such that \( N \Delta \) is Cartier and an ample divisor \( A \) of \( X \) such that \( N \Delta + A \) is ample. Fix \( m \in \mathbb{Z}_{>0} \) such that \( \mathbb{B}_+(-(K_{X/Y} + \Delta)) = \mathbb{B}(-mN(K_{X/Y} + \Delta) - A) \). Then we consider
\[
O_Y = \phi_*(mNK_{X/Y} + (mN(K_{X/Y} + \Delta) - A) + (N \Delta + A) + mN(1 - 1/m) \Delta)
\]
and singular hermitian metrics defined as follows:

- \( h_1 \) is a singular hermitian metric on \( -mN(K_{X/Y} + \Delta) - A \) such that \( h_1|_{X_Y} \) is smooth for a general point \( y \in Y \).
- \( h_2 \) is a smooth hermitian metric on the ample divisor \( N \Delta + A \) with \( \sqrt{-1} \Theta h_2 \geq \varepsilon \omega_X \) for some \( \varepsilon > 0 \).
- \( h_3 \) is the canonical singular metric on \( (1 - 1/m) \Delta \).

Then \( h := h_1 h_2 h_3^{mN} \) satisfies that \( \sqrt{-1} \Theta h \geq \varepsilon \omega_X \geq \varepsilon \phi^* \omega_Y \) and \( J(h^{mN}|_{X_Y}) = O_{X_Y} \) for a general point \( y \in Y \). By [CCM19, Theorem 2.2(1)], the induced metric \( H \) on \( O_Y \) satisfies that \( \sqrt{-1} \Theta_H \geq \varepsilon \omega_Y \), which is a contradiction.

Now we assume that \( \mathbb{B}_-(-(K_{X/Y} + \Delta + \phi^* L)) \) is not dominant over \( Y \). Let \( h_L \) be a smooth hermitian metric on \( L \) with positive curvature. By the same way as in the above argument, we can see that the direct image of
\[
2A = mNK_{X/Y} + (mN(K_{X/Y} + \Delta + \phi^* L) + A) + mN(1 - 1/m) \Delta
\]
amits a singular hermitian metric \( H_m \) such that \( \sqrt{-1} \Theta_{H_m} \geq mN \sqrt{-1} \Theta h_L \otimes \text{id} \).

This implies that \( \det(\phi_*(2A)) - mNL \) is pseudo-effective for any \( m \in \mathbb{Z}_{>0} \), which is a contradiction. \( \square \)

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