Nonlinear optics of matter waves

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We give a brief overview of the way atomic physics is now developing in a way reminiscent of the optics revolution of the 1960’s. Thanks in particular to recent developments in atomic trapping and cooling, the new field of atom optics is rapidly leading to exciting new developments such as nonlinear atom optics and quantum atom optics. We illustrate these developments with examples out of our own research.

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I. INTRODUCTION

Atom optics has witnessed considerable progress in the last few years. A number of optical elements, including atomic mirrors \cite{1} and gratings \cite{4}, have been demonstrated, and several types of atom interferometers have been built \cite{5,6}. In addition, advances in laser cooling and trapping of atoms \cite{6–11} have led to spectacular developments, including the observation of quantized atomic motion in optical lattices \cite{12,13}, and, in the last few years, the demonstration of Bose-Einstein condensation \cite{14–16} and the realization of a primitive atom laser \cite{17,18}. Atom optics using condensates as sources has now become reality, as illustrated, e.g., by experiments involving the interference of two condensates \cite{18}, the Kapitza-Dirac diffraction of a condensate off a standing-wave grating \cite{19}, or proposed experiments in matter-wave phase conjugation \cite{20} and matter-wave amplification \cite{21,22}.

The early experiments in atom optics considered low density samples, where atom-atom interactions are negligible and the atoms in the beam behave independently. We call this regime linear atom optics. When atom-atom interactions become important, the dynamics of a given atom in the sample becomes dependent on the presence of other atoms, and one reaches the regime of nonlinear atom optics \cite{23–25}, the matter waves analog of nonlinear optics, which was pioneered by Peter Franken in 1962 \cite{26}.

In addition to making a distinction between linear and nonlinear atom optics, it is important to also separate the ray optics from the wave optics regime: In the first case, it is sufficient to describe the atoms as point particles following classical trajectories influenced by light fields. Much of laser cooling belongs to that category. In contrast the wave atom optics regime, which is normally — but not exclusively — associated with ultracold atomic samples, requires a proper quantum mechanical description of the center-of-mass motion of the atoms, including the effects of matter-wave diffraction.

Finally, a last distinction of importance is between what might be called the quantum and classical regimes of atom optics, again in analogy with the electromagnetic case: the quantum regime of both atom optics and optics is that regime where the effects of quantum statistics play a significant role. Similarly to optics, where both the classical and the quantum field are governed by Maxwell’s equations, both the “classical” and the second-quantized Schrödinger fields are governed by the Schrödinger equation.

The quantum regime of matter-wave optics is conveniently described within the framework of second quantization of the matter wave field, which is described “classically” by the wave function \( \psi(\mathbf{r},t) \), so that \( \psi(\mathbf{r},t) \rightarrow \Psi(\mathbf{r},t) \). This is analogous to the optical case, where the electromagnetic field is quantized, \( E(\mathbf{r},t) \rightarrow \hat{E}(\mathbf{r},t) \). It should be remarked, however, that in contrast to optics, where the quantization of the field introduces new physics, such is not the case when second-quantizing the Schrödinger field at the low energies considered here. In this regime, the total number of particles is of course conserved, since effects such as pair creation are ignored. Hence, for our purpose the second quantization procedure is merely a convenient book-keeping mechanism which automatically accounts for particles being added and removed from a given state with the proper quantum statistics. Note however that we intentionally distinguish the quantum from the nonlinear regime of atom optics, since reaching the first one requires that a condition on phase-space density be met, while the second one is related to density only.

The goal of this paper is to illustrate the recent developments in quantum nonlinear atom optics with the help of three specific examples. We first recall in Sec. II how collisions are the main source of nonlinearities in atom optics, showing the parallels between this and the situation in nonlinear optics. Section III presents a matter-wave four-wave mixing process, the generation of a phase conjugate (time-reversed) matter wave in multicomponent condensates.

* Dedicated to Peter Franken on his 70th Birthday
Section IV briefly reviews a proposed scheme of an atom laser, and Sec. V discusses a parametric process taking place in matter waves, the so-called collective atomic recoil laser. Finally, Sec. VI is a summary and outlook.

II. THE ROLE OF COLLISIONS

The analogy between nonlinear atom optics and nonlinear optics is quite profound: In conventional optics, effective nonlinear equations for the optical fields result from the elimination of the medium dynamics, while in atom optics, nonlinear matter-wave dynamics results from collisions, which are in turn an effective manybody interaction resulting from the (partial) elimination of the electromagnetic field. Indeed, at the most fundamental level of quantum electrodynamics there is no such thing as two-body interactions between atoms; rather, the potentials that describe them are the result of a series of approximations whose validity depends on the precise situation at hand.

One simple way to illustrate how this works is to consider the near-resonant dipole-dipole interaction inside a cavity. Consider two two-level atoms of Bohr transition frequency $\omega_0$, and located at positions $\mathbf{x}_1$ and $\mathbf{x}_2$, with $|\mathbf{x}_1 - \mathbf{x}_2| = x$ along the axis of a ring cavity of length $L$. This system is described by the Hamiltonian

$$H = H_a + H_{af} + H_f,$$

where the atomic Hamiltonian $H_a$ is the sum of the individual atomic Hamiltonians

$$H_a = \sum_{i=1,2} H_a^{(i)} = \sum_{i=1,2} \left( \frac{p_i^2}{2M} + \hbar \omega_0 \sigma_+^{(i)} \sigma_-^{(i)} \right),$$

$p_i$ are the momenta of the atoms of mass $M$, with $[x_i, p_j] = i\hbar \delta_{ij}$, and $\sigma_+^{(i)}$ and $\sigma_-^{(i)}$ are the atomic raising and lowering pseudo-spin operators for the $i$-th atom. The field Hamiltonian is

$$H_f = \sum_\mu \hbar \omega_{\mu} a_\mu \dagger a_\mu,$$

where the composite index $\mu = \{k_n, \ell\}$ labels both the wave number $k_n$, with $k_n = \omega_n/c$, and polarization $\ell$ of the cavity mode of frequency $\omega_n$. The creation and annihilation operators $a_\mu \dagger$ and $a_\mu$ satisfy the usual Bose commutation relation $[a_\mu, a_\mu \dagger] = \delta_{\mu \mu'}$. Finally, the atom-field interaction Hamiltonian $H_{af}$ is the sum of the individual atom-field electric dipole interaction Hamiltonians

$$H_{af}^{(i)} = -\mathbf{d}_i \cdot \mathbf{E}(x_i),$$

where

$$\mathbf{d}_i = \mathbf{\hat{e}} d (\sigma_+^{(i)} + \sigma_-^{(i)})$$

is the atomic dipole operator, aligned along $\mathbf{\hat{e}}$ and of magnitude $d$, and

$$\mathbf{E}(\mathbf{r}) = i \sum_\mu \mathcal{E}_\mu [a_\mu e^{i\mathbf{k}_n \cdot \mathbf{r}} \mathbf{\hat{e}}_\ell - H.c.]$$

is the electric field per photon in mode $\mu$, whose explicit form depends upon the choice of quantization scheme [27,28]. For a running wave quantization scheme and periodic boundary conditions appropriate for a ring cavity we have in one dimension $\mathcal{E}_\mu = (\hbar \omega_\mu/(2\epsilon_0 L))^{1/2}$, where $\omega_n = 2\pi nc/L$. In the rotating-wave approximation, the atom-field Hamiltonian (3) reduces then to

$$H_{af}^{(i)} = -i\hbar \sum_\mu \left[ g_{\mu i}(x_i) a_\mu \sigma_+^{(i)} - g_{\mu i}^{*}(x_i) a_\mu \dagger \sigma_-^{(i)} \right],$$

where

$$g_{\mu i}(x_i) = (\mathcal{E}_\mu d/\hbar)(\mathbf{\hat{e}}_\ell \cdot \mathbf{\hat{e}}) \exp(i\mathbf{k}_n \cdot \mathbf{x}_i).$$

We ignore in this section the kinetic energy part in the Hamiltonian (2), and consider the situation where one of the atoms is initially excited, the other is in its ground state, and the radiation field is in the vacuum state,
\[ |\psi(0)\rangle = |eg0\rangle. \]  

In the rotating-wave approximation, the Hamiltonian (1) conserves the number of excitations in the system, so that at time \( t \) its state vector can be expressed as

\[ |\psi(t)\rangle = b_1(t)|eg0\rangle + b_2(t)|ge0\rangle + \sum_{\mu} b_{\mu}(t)|g1\mu\rangle, \]  

with \( b_1(0) = 1 \) and \( b_2(0) = b_{\mu}(0) = 0. \)

From the Schrödinger equation \( i\hbar \dot{|\psi(t)\rangle} = H|\psi(t)\rangle \), the equations of motion for the various probability amplitudes involved are readily found to be

\[ \dot{b}_1(t) = \sum_{\mu} g_{\mu 1} b_{\mu}(t), \]
\[ \dot{b}_{\mu}(t) = -i\Delta_{\mu} b_{\mu}(t) - g_{\mu 1}^{*} b_1(t) - g_{\mu 2}^{*} b_2(t), \]

where \( i = 1, 2 \) and \( \Delta_{\mu} \equiv \omega_{\mu} - \omega_a \) is the detuning of mode \( \mu \) from the atomic transition frequency.

The simplest way to solve the set of equations (11) is to neglect the effects of interatomic propagation. This is appropriate provided that \( x/c \ll \Gamma^{-1} \), where \( \Gamma \) is the single-atom free-space spontaneous decay rate and \( x \) is an interatomic separation. Assuming that the atoms interact with a broadband vacuum and that the Born-Markov approximation holds, one then finds for the probability amplitudes involving excited-state atoms [29]

\[ b_{1,2}(t) = \frac{1}{2} [C_+(t) \pm C_-(t)], \]

where

\[ C_{\pm}(t) = \exp \left\{ -\frac{1}{2}(\Omega_a \pm \Omega_{12}(x)) t \right\}. \]

Here we have introduced the single-atom and the two-body complex frequencies \( \Omega_a \) and \( \Omega_{12}(x) \). The explicit form of \( \Omega_a \) is

\[ \Omega_a = \Gamma + i\Delta_a, \]

where

\[ \Gamma = 2 \sum_{\mu} |g_{\mu 1}|^2 \delta(c|k| - \omega_a) = \frac{L}{2\pi \hbar \epsilon_0 \omega_a} \int_{-\infty}^{\infty} dk c|k| \pi \delta(c|k| - \omega_a) = \frac{d^2 \omega_a}{\hbar \epsilon_0 c} \]

is a one-dimensional version of the free-space spontaneous emission rate, and

\[ \Delta_a = -\frac{P}{\pi} \int dk \frac{\omega_a^2 \omega}{\omega_a - \omega_a} \]

is the one-dimensional, two-level atom version of a Lamb shift.

More interesting in the present context is the two-body complex frequency \( \Omega_{12}(x) \) is defined as

\[ \Omega_{12}(x) \equiv \Gamma_2(x) + iV_{dd}(x)/\hbar \]

and its real part \( \Gamma_2(x) \) accounts for the modulation of the spontaneous decay rate of one of the atoms due to the presence of a second atom at a distance \( x \), while its imaginary part is proportional to the dipole-dipole interaction potential \( V_{dd}(x) \). For the one-dimensional situation considered here, one finds explicitly

\[ \Omega_{12}(x) = \frac{d^2}{2\pi \hbar \epsilon_0} \int_{-\infty}^{\infty} dk c|k| e^{ikx} \left[ \pi \delta(c|k| - \omega_a) - iP \frac{1}{c|k| - \omega_a} \right], \]

so that

\[ \Gamma_2(x) = \Gamma \int_0^{\infty} d\omega \frac{\omega_a}{\omega_a} \delta(\omega - \omega_a) = \Gamma \cos(k_0 x) \]
and
\[ V_{dd}(x) = -\hbar \Gamma \frac{P}{\pi} \int_0^\infty d\omega \frac{\omega \cos(\omega x/c)}{\omega_\lambda - \omega} = -\hbar \Gamma \sin(k_\lambda x), \]  
(20)

where \( k = \omega/c \) and \( k_\lambda = \omega_\lambda/c = 2\pi/\lambda_\lambda \). For the more familiar three-dimensional case, the trigonometric functions appearing in these expressions are replaced by combinations of Bessel functions, and \( V_{dd}(r) \) falls off as \( 1/r \) for large interatomic separations, instead of being periodic \([30–32]\). Note also that as a result of the Born-Markov approximation, these expressions neglect propagation and are independent of time.

We see, then, that the two-body dipole-dipole interaction between atoms results as advertised from the adiabatic elimination of the dynamics of the electromagnetic field. This is analogous to the nonlinear optics situation, except that the roles of the atoms and the electromagnetic field are reversed.

In the following sections we discuss several examples of quantum nonlinear atom optics which show how the familiar ideas of nonlinear optics can be readily transposed to the new situation. We consider first four-wave mixing in a multicomponent Bose-Einstein condensate, which can lead, e.g., to matter-wave phase conjugation. We then discuss in Sec. IV a form of matter-wave “amplification” reminiscent of laser action in optics, and in Sec. V a parametric process, the low-temperature version of the collective atom recoil laser. Finally, Sec. VI is a conclusion and outlook.

### III. FOUR-WAVE MIXING: MATTER-WAVE PHASE CONJUGATION

A particularly close analogy can be established between the dynamics of a spin-1 multicomponent condensate, as recently realized in sodium experiments and the situation of degenerate four-wave mixing in optics. Specifically, in the zero-temperature limit a \( g \)-component condensate can be thought of as a \( g \)-mode system, whereby the various modes are coupled by two-body (and possibly higher-order) collisions which result in the exchange of particles between these modes.

In particular, for the case of a \( ^{23} \text{Na} \) condensate in an optical dipole trap one can achieve situations where the \( \ell = 0 \) state is macroscopically populated while the \( \ell = \pm 1 \) states are weakly excited. One can then think of the first state as a “pump” or “central” mode, while \( \ell = \pm 1 \) form side modes which are coupled via the pump. This leads to familiar effects such as degenerate four-wave mixing and matter-wave phase conjugation.

Consider then a condensate of \( ^{23} \text{Na} \) atoms in their \( F = 1 \) hyperfine ground state, with three internal atomic states \( |F = 1, m = -1\rangle, |F = 1, m = 0\rangle \) and \( |F = 1, m = 1\rangle \) of degenerate energies in the absence of magnetic fields. It is described by the three-component vector Schrödinger field
\[ \Psi(r, t) = \{\Psi_{-1}(r, t), \Psi_0(r, t), \Psi_1(r, t)\} \]  
(21)

which satisfies the bosonic commutation relations
\[ [\Psi_i(r, t), \Psi_j^\dagger(r', t)] = \delta_{ij}\delta(r - r'). \]  
(22)

Accounting for the possibility of two-body collisions, its dynamics is described by the second-quantized Hamiltonian
\[ \mathcal{H} = \int dr \Psi^\dagger(r, t) H_0 \Psi(r, t) + \int \{dr\} \Psi^\dagger(r_1, t) \Psi^\dagger(r_2, t) V(r_1 - r_2) \Psi(r_2, t) \Psi(r_1, t), \]  
(23)

where the single-particle Hamiltonian is
\[ H_0 = p^2/2M + U(r) \]  
(24)

and the dipole trap potential \( U(r) \) does not depend on the hyperfine magnetic state \( m \).

The general form of the two-body interaction \( V(r_1 - r_2) \) has been discussed in detail in Refs. [33,34]. Labeling the hyperfine states of the combined system of hyperfine spin \( F = F_1 + F_2 \) by \( |f, m\rangle \) with \( f = 0, 1, 2 \) and \( m = -f, \ldots, f \), it can be shown that in the shapeless approximation the two-body interaction is of the general form \([33]\)
\[ V(r_1 - r_2) = \delta(r_1 - r_2) \sum_{f=0}^2 \hbar g_f P_f, \]  
(25)

where
\[ g_f = 4\pi\hbar a_f/M, \]  
(26)
\( P_f = \sum_m |f,m\rangle \langle f,m| \) is the projection operator which projects the pair of atoms into a total hyperfine \( f \) state and \( a_f \) is the \( s \)-wave scattering length for the channel of total hyperfine spin \( f \). For bosonic atoms only even \( f \) states contribute, so that
\[
V(r_1 - r_2) = \hbar \delta(r_1 - r_2)(g_2 P_2 + g_0 P_0) = \frac{\hbar}{2} \delta(r_1 - r_2) (c_0 + c_2 F_1 \cdot F_2).
\]
In this expression,
\[
c_0 = 2(g_0 + 2g_2)/3, \quad c_2 = 2(g_2 - g_0)/3.
\]
Substituting this form of \( V(r_1 - r_2) \) into the second-quantized Hamiltonian leads to
\[
\mathcal{H} = \sum_m \int dr \Psi^\dagger_m (r,t) \left( \frac{p^2}{2M} + U(r) \right) \Psi_m (r,t) + \frac{\hbar}{2} \int dr \{ (c_0 + c_2)|\Psi^\dagger_1 \Psi_1 + \Psi^\dagger_{-1} \Psi_{-1}|^2 + 2\Psi^\dagger_0 \Psi_0 (\Psi^\dagger_1 \Psi_1 + \Psi^\dagger_{-1} \Psi_{-1}) \}.
\]
This form of the Hamiltonian is quite familiar in quantum optics, where it describes four-wave mixing between a pump beam and two side-modes, which are identified with the field operators \( \Psi_0 \) and \( \Psi_{\pm 1} \) in the present situation.

The three terms in the two-body Hamiltonian which are quartic in one of the field operators only, i.e. of the form \( \Psi_i^\dagger \Psi_j^\dagger \Psi_j \Psi_i \), can be readily interpreted as self-defocussing terms, corresponding to the fact that the two-body potential is, for a positive scattering length and a scalar field, analogous to a defocussing cubic nonlinearity in optics. The terms involving two “modes”, i.e. of the type \( \Psi_i^\dagger \Psi_j \Psi_j^\dagger \Psi_j \), conserve the individual mode populations of the modes and simply lead to phase shifts. Finally, the terms involving the central mode \( \Psi_0 \) and both side-modes are the contributions of interest to us. They correspond to a redistribution of atoms between the “pump” mode \( \Psi_0 \) and the side-modes \( \Psi_{\pm 1} \), e.g., by annihilating two atoms in the central mode and creating one atom each in the side-modes. This is the kind of interaction that leads to phase conjugation in quantum optics, except that in that case the modes in question are modes of the Maxwell field instead of the Schrödinger field. Note also that a similar mechanism is at the origin of amplification in the Collective Atom Recoil Laser (CARL) [53, 57], see Sec. V.

In the Hartree approximation, which is well justified for condensates at \( T = 0 \), the dynamics of a condensate described by the Hamiltonian is governed by the system of coupled nonlinear Schrödinger equations
\[
i \dot{\phi}_1(r, t) = \frac{1}{\hbar} \left( \frac{p^2}{2M} + U(r) \right) \phi_1 + N\{ c_2 \phi_0 \phi_0 \phi_1^* + [c_0 + c_2] (|\phi_1|^2 + |\phi_0|^2) + (c_0 - c_2) |\phi_1|^2 \} \phi_1 \}
\]
\[
i \dot{\phi}_0(r, t) = \frac{1}{\hbar} \left( \frac{p^2}{2M} + U(r) \right) \phi_0 + N\{ c_0 |\phi_0|^2 \phi_0 + (c_0 + c_2) (|\phi_1|^2 + |\phi_0|^2) \phi_0 + 2c_2 \phi_1 \phi_1 \phi_0^* \}
\]
\[
i \dot{\phi}_{-1}(r, t) = \frac{1}{\hbar} \left( \frac{p^2}{2M} + U(r) \right) \phi_{-1} + N\{ c_2 \phi_0 \phi_0 \phi_{-1}^* + [c_0 + c_2] (|\phi_1|^2 + |\phi_0|^2) + (c_0 - c_2) |\phi_{-1}|^2 \} \phi_{-1} \}. \tag{30}
\]
Consider for example a situation where the central mode, described by the Hartree wave function \( \phi_0 \), is initially strongly populated while the side-modes \( \phi_{\pm 1} \) are weakly excited. In other words, we consider the phase conjugation of a weak atomic beam from a large condensate. It is then appropriate to introduce the matter-wave optics equivalent of the undepleted pump approximation, whereby
\[
\dot{\phi}_0 \approx 0, \tag{31}
\]
and the problem reduces to a set of coupled mode equations for the two side-modes \( \phi_{\pm 1} \), the central mode acting as a catalyst for the coupling between them.

We take the trap potential \( U(r) \) to be of the harmonic form
\[
U(r) = M\omega_0^2(x^2 + y^2)/2 \tag{32}
\]
that is, we assume that the dipole trap confines the atoms in the transverse plane \( (x, y) \), but not in the longitudinal direction \( z \). This geometry allows one to consider side-modes propagating along that axis, rather than bouncing back and forth in an elongated trap. In case of tight confinement in the transverse direction, we can assume to a good approximation that the transverse structure of the condensate is not significantly altered by many-body interactions and is determined as the ground-state solution of the transverse potential.

Expressing the Hartree wave function associated with the hyperfine level \( m \) as
\( \phi_m(r, t) = \varphi_{\perp}(x, y) \varphi_m(z, t) e^{-i\omega_0 t}, \) (33)

we then have

\[
\hbar \omega_0 \varphi_{\perp}(x, y) = \left[ -\frac{\hbar^2}{2M} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{M\omega_0^2}{2}(x^2 + y^2) \right] \varphi_{\perp}(x, y).
\] (34)

The physical situation we have in mind is that of a weak “probe” in the hyperfine state \( m = -1 \) propagating toward a large condensate in state \( m = 0 \) and at rest in the dipole trap, and generating a backward-propagating conjugate matter wave in the hyperfine state \( m = +1 \). Hence we express the longitudinal component of the Hartree wave function as

\[
\varphi(z, t) = \begin{pmatrix} \varphi_{-1}(z, t) \\ \varphi_0(z, t) \\ \varphi_1(z, t) \end{pmatrix} = \begin{pmatrix} \psi_{-1}(z, t)e^{-ikz} \\ 2\psi_0\cos(kz) \\ \psi_1(z, t)e^{ikz} \end{pmatrix} e^{-i\omega t},
\]

where the slowly varying envelopes \( \psi_m \) of the Hartree wave function components \( m = \pm 1 \) satisfy the familiar inequalities

\[
\left| \frac{\partial^2}{\partial z^2} \psi_m \right| \ll k \left| \frac{\partial}{\partial z} \psi_m \right| \ll k^2 |\psi_m|.
\] (35)

The “pump” wave function \( \varphi_0 \) is described by a standing wave, a configuration that can be achieved for instance by interfering two condensates [13], in a grating matter-wave interferometer [19] or in the CARL [37]. To first order in the probe and signal fields, this geometry leads to a linearized system of two coupled-mode equations for the probe and condensate fields. In the stationary state and projecting out the transverse part of the wave function, they reduce to

\[
i\frac{\hbar k}{2M} \frac{\partial}{\partial z} \psi_{-1}(z) = -N\eta[2(c_0 + c_2) \rho_0 \psi_{-1}(z) + c_2 \psi_0^2 \psi^*_1(z)],
\]

\[
\psi_0(z) = -N\eta[2(c_0 + c_2) \rho_0 \psi_0(z) + c_2 \psi_0^2 \psi_{-1}(z)],
\] (36)

where \( \rho_0 = |\psi_0|^2 \) and

\[
\eta = \frac{\int \! dx dy |\varphi_{\perp}(x, y)|^4}{\int \! dx dy |\varphi_{\perp}(x, y)|^2}.
\] (37)

These equations are of course familiar from optical phase conjugation and their solution is well-known. The evolution of the phase conjugate wave \( \psi^*_1 \) contains a term proportional to the density \( \rho_0 \) of the condensate and the field itself. In the absence of the second term, it would simply lead to a phase shift of \( \psi^*_1 \). Physically, it results from the self-interaction of the conjugate field, catalyzed by the condensate (pump) component. Its origin can be traced back to the term proportional to \( \Psi_0^\dagger \Psi_1^\dagger \psi_0 \psi_1 \) in the Hamiltonian [22]. The second term, in contrast, couples the two side-modes via the condensate and is responsible for phase conjugation.

The general solution of Eqs. (36) reads [38]

\[
\psi_{-1}(z) = \frac{e^{i\alpha z}}{\cos(|\kappa|L)} \left( -ie^{-i\beta} \sin(|\kappa|z) \psi^*_1(L) + \cos(|\kappa|z - L) \psi_{-1}(0) \right),
\]

\[
\psi_1(z) = \frac{e^{i\alpha z}}{\cos(|\kappa|L)} \left( \cos(|\kappa|z) \psi_1(L) + i e^{-i\beta} \sin(|\kappa|z - L) \psi^*_{-1}(0) \right),
\] (38)

where \( \alpha = 2N\eta(c_0 + c_2) \rho_0, \kappa = \frac{N\eta c_2^2}{\hbar k/2M} \), and \( e^{i\beta} = \kappa/|\kappa| \).

For the probe \( \psi_{-1}(0) \) incident at \( z = 0 \) and no incoming conjugate signal \( \psi_1(L) = 0 \), the conjugate wave in the input plane \( z = 0 \) becomes

\[
\psi_1(0) = -ie^{-i\beta} \tan(|\kappa|L) \psi^*_{-1}(0),
\] (39)

which demonstrates that the interaction of the probe and the condensate results in the generation of a counterpropagating phase-conjugated signal.

In addition to its interest from a nonlinear atom optics point of view, matter-wave phase conjugation could also be used as a diagnostic tool for Bose-Einstein condensates. For instance, we noted that the parameter \( |\kappa|L \) is proportional to the difference in scattering lengths between the singlet and triplet states. Hence, this quantity could in principle be inferred from phase conjugation measurements.
As a second example illustrating the close analogy between nonlinear optics and nonlinear atom optics, we now consider the atom laser. There has been quite a bit of confusion about what is meant by an “atom laser” in the past, apparently associated with the fact that since the number of atoms is conserved, one cannot possibly achieve their amplification.\footnote{Another problem is that the L in the acronym for laser stands for Light. But this is history repeating itself: in the early days of lasers, they were commonly called “optical masers.”} But of course, this is not the point. The main purpose of an atom laser is to create a coherent, macroscopic population in a given center-of-mass mode of atomic motion, such as to create a coherent atomic beam. Clearly, the number of particles in a given mode needs not be conserved: it can readily be amplified by cleverly extracting atoms from a large reservoir.

A primitive atom laser has been demonstrated experimentally by the MIT group, which outcoupled a fraction of a sodium condensate from a magnetic trap via rf coupling between the weak-field-seeking $m_F = -1$ state and the strong-field-seeking $m_F = 1$ state \cite{1, 18}. We concentrate here on a different scheme, which has so far not been realized experimentally, the binary collision atom laser \cite{40–44}. In this system, the matter-wave resonator consists, e.g., of an optical dipole trap. In order to concentrate on the essential dynamics only three out of the multitude of trap modes are taken into account explicitly.

The atom laser then operates as follows: Bosonic atoms in their ground electronic state are incoherently pumped into a trap level of “intermediary” energy (mode 1). There they undergo binary collisions which take one of the atoms involved to the tightly bound laser mode 0, whereas the other one is transferred to the heavily damped loss mode 2. This latter atom leaves the resonator quickly, thereby providing the irreversibility of the pumping process. A macroscopic population of the laser mode can build up as soon as the influx of atoms due to pumping compensates for the losses induced by the damping.

In the description of this laser scheme one has to take into account that in addition to the pumping collisions other types of interatomic collisions can also occur. These considerations lead to an ansatz for the atom laser master equation of the form

$$\dot{W} = -\frac{i}{\hbar}[H_0 + H_{\text{col}}, W] + \kappa_0 D[c_0]W + \kappa_1 (N+1)D[c_1]W + \kappa_2 N D[c_2]W.$$ \hspace{1cm} (40)

In this equation, we use the second quantized formalism in which each center-of-mass atomic mode is associated with a bosonic annihilation operator $c_i$, and $W$ denotes the atomic density operator.\footnote{Since we consider ground state atoms only, they are fully described by their center-of-mass quantum numbers.} The free Hamiltonian is given by

$$H_0 = \sum_{i=0,1,2} \hbar \omega_i c_i^\dagger c_i,$$

$\omega_i$ being the mode frequencies. The general form of the collision Hamiltonian can be written as

$$H_{\text{col}} = \sum_{i,j,k,l} \hbar V_{ijkl} c_i^\dagger c_j^\dagger c_k c_l$$ \hspace{1cm} (41)

with $V_{ijkl}$ the matrix elements of the two-body interaction Hamiltonian responsible for the collisions. It is sufficient for the present purpose to restrict our attention to the reduced form

$$H_{\text{col}} = \hbar (V_{0211} c_0^\dagger c_2^\dagger c_1 c_1 + V_{1102} c_1^\dagger c_1^\dagger c_0 c_2 + V_{0000} c_0^\dagger c_0^\dagger c_0 c_0 + V_{0101} c_0^\dagger c_0^\dagger c_0 c_1 + V_{1111} c_1^\dagger c_1^\dagger c_1 c_1)$$ \hspace{1cm} (42)

in which (besides the pumping collisions) only those collisions are retained which are expected to have the most significant influence on the phase dynamics.

It is worth emphasizing that there is a fundamental difference in the way collisions are handled in conventional atomic physics and in the present situation. Usually, and for instance in the phase conjugation problem of the preceding section, collisions are handled in terms of scattering amplitudes between initial and final states. In matter-wave resonators, however, such an approach becomes meaningless. Rather, the collisions are seen to induce transitions between cavity modes. Hence, the collision Hamiltonian \cite{12} takes the form of a \textit{mode-coupling} Hamiltonian between three cavity modes. It is therefore mathematically closely related to that of Eq. \cite{22}, except that the coupling is now between center-of-mass modes, rather than magnetic sublevels. Another difference is of course that in phase
conjugation, we are interested in a coherent interaction, and neglect the effects of dissipation, which are known to be detrimental to the achievement of phase reversal. In contrast, the laser and the atom laser are open systems that rely explicitly on the presence of pump and probe mechanisms to achieve steady-state operation.

The damping rates of the cavity modes are given by the coefficients $\kappa_i$, and the strength of the external pumping of mode 1 is characterized by the parameter $N$, which is the mean number of atoms to which mode 1 would equilibrate in the absence of collisions. The superoperator $\mathcal{D}$ is defined by

$$\mathcal{D}[a]P = aPa^\dagger - \frac{1}{2} (a^\dagger aP + Pa^\dagger a)$$

with arbitrary operators $a$ and $P$.

In order to achieve a sufficiently high degree of irreversibility it is necessary that $\kappa_2$ is much larger than the damping rates of the other modes. This suggests adiabatically eliminating this mode, an approximation that leads to the simplified master equation \[10,42\]

$$\dot{\rho} = -\frac{i}{\hbar} [H_c, \rho] + \kappa_0 \mathcal{D}[c_0] \rho + \kappa_1 (N + 1) \mathcal{D}[c_1] \rho + \kappa_1 N \mathcal{D}[c_1^\dagger] \rho + \mathcal{D}[c_0^\dagger c_1^2] \rho. \tag{44}$$

Equation (44) is written in the interaction picture with respect to $H_0 = \hbar \omega_0 c_0^\dagger c_0 + \hbar \omega_1 c_1^\dagger c_1$, and the reduced density matrix $\rho$ is $\rho = \text{Tr}_\text{mode 2} [W]$. The reduced collision Hamiltonian $H_c$ is

$$H_c = \hbar (V_{0000} c_0^\dagger c_0^\dagger c_0 c_0 + V_{1111} c_1^\dagger c_1^\dagger c_1 c_1), \tag{45}$$

and $\Gamma = 4 |V_{0211}|^2 / \kappa$. Consistently with Ref. [1] we call the limiting cases $\Gamma \ll \kappa_0$ and $\Gamma \gg \kappa_0$ the weak and strong collision regimes, respectively. The reduced master equation (44) forms the basis of most studies of binary collision atom lasers.

The master equations (40) and (44) can easily be solved numerically using standard quantum Monte Carlo simulation techniques \[15,16\]. This is discussed in detail in Ref. [13]. Figure 1 shows the probability $P(n)$ of having $n$ atoms in the laser mode as a function of time for the case where only $V_{0211} = V_{1102}$ is taken to be non-zero. The parameters used are $\kappa_0 = 0.01 \kappa_1$, $N = 2$, $\kappa_2 = 10 \kappa_1$ and $V_{0211} = 2 \kappa_1$, the time is in units of $\kappa_1^{-1}$. We see that the population of the lasing mode builds up from zero to a state that suggests a Poissonian distribution with a mean number of atoms $n_0 = 40$. In contrast, the evolution of the pump and decay modes (not shown) shows no significant build-up of population in steady state.

Among the most important characteristics of a laser are the coherence properties of its output, in particular its linewidth. In order to obtain an analytical approximation for the laser linewidth in the two-mode system [14] a linearized fluctuation analysis can be performed \[28,17\]. Assuming as usual that the first-order correlation function $C_0(\tau) = \langle c_0^\dagger (\tau) c_0(0) \rangle$ is determined only by the phase fluctuations one obtains in this way

$$C_0(\tau) = \bar{n}_0 e^{-i \tilde{\phi}_0} \exp[-\frac{1}{2} \sigma_{\phi_0}(\tau)], \tag{46}$$

where the equilibrium populations of laser and pump mode are given by

$$\bar{n}_0 = \frac{1}{2} \frac{\kappa_1}{\kappa_0} (N - \bar{n}_1), \tag{47}$$

$$\bar{n}_1 = \sqrt{\kappa_0 / \Gamma}, \tag{48}$$

the threshold condition being $N > \sqrt{\kappa_0 / \Gamma}$. The time-dependent deterministic phase drift is indicated by $\tilde{\phi}_0$. This phase drift leads to a shift of the center of the power spectrum (the Fourier transform of the correlation function) by an amount $2 V_{0000} \bar{n}_0 + V_{0101} \bar{n}_1$ with respect to the collisionless case. The behavior of the correlation function $C_0(\tau)$ is thus essentially determined by the coefficient $\sigma_{\phi_0}(\tau)$ which is the covariance of the phase fluctuations in the laser mode.

This coefficient can be evaluated explicitly in a straightforward way, however the ensuing expression is rather complicated. In the following we restrict the discussion to two limiting cases which illustrate the essential aspects of the influence of the elastic collisions on the laser linewidth. (It should also be noted at this point that $C_0(\tau)$ does not depend on elastic collisions between pumping mode atoms, which are characterized by the parameter $V_{1111}$.)

(a) $V_{0101} = 0$. Equation (14) implies that the most important aspects of the correlation function can be inferred from the study of $\sigma_{\phi_0}(\tau)$ in the time interval where it is smaller than or of the order of unity. Examining the behavior of $\sigma_{\phi_0}(\tau)$ as a function of $V_{0000}$ (with all other parameters kept constant) one can distinguish between two different regimes. For small $V_{0000}$ the time evolution of $\sigma_{\phi_0}(\tau)$ relevant for $C_0(\tau)$ is well approximated by
\[ \sigma_{\phi_0}(\tau) \simeq \tau|w + \kappa_0/(2\bar{n}_0)|, \]  
\[ (49) \]

where
\[ w = V_{0000}^2 \frac{\kappa_1 N}{\kappa_0} \left( 2 + 2\sqrt{\kappa_0/\Gamma} + \frac{N}{N - \sqrt{\kappa_0/\Gamma}} \right). \]  
\[ (50) \]

For such values of \( V_{0000} \), \( C_0(\tau) \) decays therefore exponentially, and the power spectrum is Lorentzian. If \( w \gg \kappa_0/(2\bar{n}_0) \) the linewidth is proportional to \( V_{0000}^2\bar{n}_0 \). In contrast to the situation with conventional lasers, it increases linearly with the number of atoms in the laser mode.

In case \( V_{0000} \gg \sqrt{\frac{N - \kappa_0}{2w'}} \) with \( w' = w/V_{0000}^2 \) we find a different behavior of the correlation function. Under these circumstances it decays like a Gaussian, i.e.,
\[ \sigma_{\phi_0}(\tau) \simeq 4V_{0000}^2\bar{n}_0\tau^2. \]  
\[ (51) \]

The spectrum is thus also of Gaussian shape and its linewidth proportional to \( V_{0000}\sqrt{\bar{n}_0} \). The atom laser linewidth still increases with \( \bar{n}_0 \), albeit less dramatically than in the preceding case.

(b) \( V_{0000} = 0 \). In this case the expansion of \( \sigma_{\phi_0}(\tau) \) to leading order in \( \bar{n}_0 \) yields
\[ \sigma_{\phi_0}(\tau) = \left( \frac{V_{0101}^2}{2\bar{n}_0} + \frac{\kappa_0}{2n_0} \right) \tau \]  
\[ (52) \]

The correlation function thus decays exponentially for all values of \( V_{0101} \). A qualitative change in behavior as in the previous situation does not occur. The linewidth of the spectrum is now proportional to \( V_{0101}^2/\bar{n}_0 \), and becomes narrower when the population of the laser mode is increased, very much like the familiar Shawlow-Townes linewidth of conventional lasers. The analytical estimates of Eqs. (49), (51), and (52) are in good agreement with numerical quantum Monte Carlo simulations [44].

V. PARAMETRIC AMPLIFICATION: THE COLLECTIVE ATOM RECOIL LASER

As a final example of nonlinear atom optics in ultracold atomic systems, we briefly review some of the most salient aspects of the ultracold atoms operation of the Collective Atomic Recoil Laser, or CARL [35]. This device consists of three main components: (1) the active medium, which consists of a gas of two-level atoms, (2) a strong pump laser which drives the two-level atomic transition, and (3) a ring cavity which supports an electromagnetic mode (the probe) counterpropagating with respect to the pump. What makes the CARL interesting is that the initial state, consisting of a thermal cloud of atoms and no photons in the cavity, is exponentially unstable. Laser oscillations appear spontaneously in the probe mode correlated with the appearance of a density modulation in the atomic sample. The original purpose of the CARL, which operates then at room or higher temperatures, was the generation of a tunable coherent light field from atoms in a way similar to light amplification in free-electron lasers, i.e., gain correlated with bunching and in the absence of population inversion. But as we shall see, when operating at ultracold temperatures it can simultaneously parametrically amplify spatial side-modes of a Bose-Einstein condensate. This is the aspect of the CARL that we concentrate on in this section.

In the absence of collisions, the second-quantized Hamiltonian of a sample of two-level atoms interacting with a classical pump laser and a counterpropagating probe cavity mode is
\[ H = \sum_k H(k) + \hbar c_q A^\dagger A. \]  
\[ (53) \]

Here \( H(k) \) is given by
\[ H(k) = \frac{\hbar^2 k^2}{2m} c_q^\dagger(k)c_q(k) + \left( \frac{\hbar^2 k^2}{2m} + \hbar \omega_0 \right) c_e^\dagger(k)c_e(k) \]
\[ + \left[ \frac{\Omega_q^2}{2} e^{i\omega_L t} c_q(k + k_L)c_e(k) + i\hbar g A^\dagger c_q^\dagger(k - q)c_e(k) + H.c. \right], \]  
\[ (54) \]

where the bosonic matter wave operator \( c_q(k) \) annihilates a ground state atom of momentum \( \hbar k \), and \( c_e(k) \) annihilates an excited atom of momentum \( \hbar k \), with
all other matter-wave commutators being equal to zero. The pump is characterized by its frequency \(\omega_L\), its Rabi frequency \(\Omega_L\), and its wavenumber \(k_L\). The probe field operator \(A\) annihilates a photon with wavenumber \(q\), and the atom-probe coupling constant is \(g = \frac{d|c_q|^2/(2\hbar c_{\text{e}} L S)}{2}\), where \(d\) is the atomic dipole moment, and \(L S\) is the cavity volume. Note that the atomic recoil is explicitly included in the electric dipole interaction coupling the electronic ground and excited states.

The Hamiltonian (54) readily yields the Heisenberg equation of motion for the probe field mode as

\[
\frac{d}{dt} A = -icqA + g \sum_k c_g^\dagger(k - q)c_e(k).
\] (56)

Hence, all that is required to determine the field evolution are bilinear combinations of atomic creation and annihilation operators. Their equations of motion are in turn

\[
\frac{d}{dt} c_g^\dagger(k)c_e(k') = \frac{i}{\hbar} [H, c_g^\dagger(k)c_e(k')].
\] (57)

Because the Hamiltonian (54) is bilinear in the atomic operators, we observe readily that the equations of motion involve only bilinear combinations. This is a direct consequence of the fact that collisions are neglected in this model. In the language of manybody theory, this means that the BBGKY hierarchy is exactly truncated. If the probe field is treated classically, then Eq. (54) becomes a c-number equation involving only the matter-wave expectation values \(\langle c_g^\dagger(k - q)c_e(k)\rangle\), which are nothing but the single-particle atomic density matrix elements. Since these elements are coupled only to other single-particle density matrix elements, the system can be solved without any truncation scheme being required, albeit numerically in general. If instead of treating probe field classically we had chosen to treat the atomic single-particle density matrix classically, we would have reached the same result. Thus converting Eqs. (56) and (57) into c-number equations is equivalent to a classical field theory for both the atomic and optical fields. This approach is discussed in detail in Ref. [37], and we do not pursue it further here.

Since we are interested in the interaction between light and a Bose condensate, we must be careful that electromagnetic heating does not occur. Hence, the optical fields must be far off-resonant from any electronic transition, and the upper electronic levels can be adiabatically eliminated. The atoms are then described as a scalar field, since only their electronic ground state remains. In that case one finds the system is described by the effective Hamiltonian

\[
H = \frac{\hbar^2}{2m} \sum_k k^2 c_g^\dagger(k)c_g(k) + \hbar c_q A^\dagger A + i \frac{\hbar}{2\Delta_L} \sum_k \left[ g\Omega_L e^{-i\omega_L t} A^\dagger c_g^\dagger(k - 2k_0)c_g(k) - H.c. \right] + \frac{\hbar}{\Delta_L} \left( \frac{\Omega_L^2}{4} + |g|^2 A^\dagger A \right) \sum_k c_g^\dagger(k)c_g(k),
\] (58)

where we have introduced the recoil kick \(2k_0 = q + k_L\), and the pump detuning \(\Delta_L = \omega_L - \omega_a\).

If the atomic sample is initially a condensate at zero temperature, the dominant mode, at least for short times, is the \(\kappa = 0\) mode. It is macroscopically populated, while all other matter wave modes are in a vacuum. The equation of motion for the condensate mode is

\[
\frac{d}{dt} c_g(0) = \frac{g}{2\Delta_L} \left[ \Omega_L^* e^{-i\omega_L t} A^\dagger c_g(2k_0) - \Omega_L e^{i\omega_L t} A c_g(-2k_0) \right]
\] (59)

This equation indicates that the condensate mode is coupled via the electromagnetic field to the two side-modes at \(\pm 2k_0\). In addition, each side mode is coupled to its neighbors at intervals of \(\pm 2k_0\). This is mathematically similar to the situation of Sec. III, see Eq. (40), with, however, three major differences: (a) in the case of phase conjugation, the mode coupling is between different magnetic sublevels of the atomic field, while it is now between momentum states; (b) the coupling is now due to a Raman-like coupling induced by two counterpropagating fields, instead of ground-state collisions. Hence the mode coupling now involves an infinite manifold of matter-wave modes and two electromagnetic fields modes — one of them treated classically in the present example — instead of four matter-wave modes. This is of course a direct consequence of the fact that collisions are the result of a phenomenological approach, eliminating the electromagnetic vacuum as we have seen in Sec. II. (c) Instead of three magnetic sublevels, we now have an infinite hierarchy of center-of-mass modes of the matter-wave field to deal with.

This system is similarly related to the atom laser of Sec. IV, except that in that case, matter-wave modes are selected by the atomic resonator, as well as possibly by the selection rules of the collision Hamiltonian (54), whereas
where they are selected from the continuum of modes simply by conservation of momentum. In addition, there are no atomic pump and decay mechanisms in the present model, which describes a fully coherent interaction.

Let us now discuss this hierarchy in some more detail. We note that each mode \( k \) is directly coupled only to its neighboring modes \( k \pm 2k_0 \). But except for the condensate mode \( k = 0 \), all modes are initially empty, so in the early stages the dominant dynamics results from the coupling between the condensate mode and its two neighboring modes. Neglecting then the higher-order modes, and further treating the condensate mode as a constant c-number, a sort of undepleted pump approximation for a classical atom-laser field and an excellent approximation at \( T = 0 \) and for a sufficiently large condensate, we find that we have reduced the system to a linear three-mode problem. It is easily shown that this reduced problem can be described by the effective Hamiltonian

\[
H = 4\hbar \omega_R \left( c_+^\dagger c_- + c_+^\dagger c_+ - \Delta a^\dagger a + \chi \left[ a^\dagger c_- + a^\dagger c_+ + c_+^\dagger a + c_- a \right] \right),
\]

where \( \omega_R = \hbar k_0^2/2m \) is the atomic recoil frequency,

\[
c_\pm = e^{i\Omega_L|2t/4\Delta_L}c g(\pm 2k_0),
\]

\[
a = -i(g^* \Omega_L^2/|g||\Omega_L||\Delta_L|)e^{i\omega_L t} A,
\]

\[
\chi = |g||\Omega_L|/8\omega_R \Delta_L \sqrt{N},
\]

\[
\Delta = \omega_L - \omega/4\omega_R,
\]

\( \omega = cq - |g|^2N/\Delta_L \) and \( N \) is the mean number of atoms in the condensate. We see that \( \chi^2 \) is an intensity parameter, proportional to the product of the intensities of the pump laser and the initial condensate, and \( \Delta \) is simply the pump-probe detuning in units of \( 4\omega_R \). The Hamiltonian \([64]\) gives the full quantum field theory description of the zero temperature CARL, and is valid for all times short enough so that \( \langle c_\pm^\dagger c_\pm \rangle \ll N \) and \( \langle a^\dagger a \rangle \ll |\Omega_L|^2/|g|^2 \).

The presence of terms such as \( a^\dagger c_\pm^\dagger \) in Eq. \([66]\) immediately brings to mind the non-degenerate optical parametric amplifier \([53]\) which generates highly non-classical optical fields. These fields exhibit two-mode intensity correlations and squeezing, and have been extensively employed in the creation of EPR photon pairs for fundamental studies of the laws of quantum mechanics. Our system represents a generalization in that we now have three entangled quantum fields, and is especially interesting in that two of the fields are atomic rather than optical. A detailed analysis of quantum mode coupling in the CARL can be found in Ref. \([21]\). Here we focus on the intensity dynamics and fluctuations.

The dynamics of the system can be determined by solving the three coupled mode equations

\[
\frac{d}{d\tau} \begin{pmatrix} a(\tau) \\ c_\pm(\tau) \end{pmatrix} = i \begin{pmatrix} -\Delta & -\chi & -\chi \\ \chi & 1 & 0 \\ -\chi & 0 & -1 \end{pmatrix} \begin{pmatrix} a(\tau) \\ c_\pm(\tau) \end{pmatrix},
\]

where we have introduced the dimensionless time variable \( \tau = 4\omega_R t \).

The spectrum of \([64]\) has been studied in detail in \([71, 72]\), with the result that under certain threshold conditions the eigenvalues take the form \( \lambda_1 = \omega_1 + \omega_2 + \Omega + i\Gamma \), and \( \lambda_2 = \Omega - i\Gamma \), where \( \omega_1, \Omega \), and \( \Gamma \) are all real quantities. This means that the time-dependence of the operators will, after some transients, grow exponentially in time at the CARL growth rate \( \Gamma \).

If now consider an initial state where the atomic side modes are in the vacuum state and the probe mode has been injected with an initial field in a coherent state \( |\alpha\rangle \), we find that after the transients have died away the mean intensities of the three modes are given by

\[
I_A(\tau) \equiv \langle a^\dagger(\tau)a(\tau) \rangle \approx \langle |\alpha|^2 R_{11}^2 + R_{12}^2 \rangle e^{2\Gamma \tau},
\]

\[
I_\pm(\tau) \equiv \langle c_\pm^\dagger(\tau)c_\pm(\tau) \rangle \approx \langle |\alpha|^2 R_{21}^2 + R_{22}^2 \rangle e^{2\Gamma \tau},
\]

and
\[ I_\pm(\tau) \equiv \langle c_\pm^\dagger(\tau)c_\pm(\tau) \rangle \approx (|\alpha|^2 R_{31}^2 + R_{32}^2)e^{2\Gamma\tau}. \] (68)

Here the coefficients \( \{ R_{ij} \} \) are given by \( R_{ij} = |v_i v_j^{-1}|3 \), where the columns of the matrix \( v \) are the eigenvectors of the \( 3 \times 3 \) linear system described by Eq. (65). For a given set of control parameters \( \chi \) and \( \Delta \) these coefficients are simply constants, whose analytic expressions, while straightforward to derive, are too unwieldy to reproduce here. Thus we see that the intensities have a stimulated component, proportional to \( |\alpha|^2 \), and a spontaneous component, which is present even when all three modes begin in the vacuum state. In this case the system is triggered by quantum noise in the form of fluctuations in the atomic bunching.

The second-order equal-time intensity correlation function is defined, e.g. in the case of the probe field, as

\[ g^{[2]}(\tau) = \frac{\langle a^\dagger(\tau)a^\dagger(\tau)a(\tau)a(\tau) \rangle}{\langle a^\dagger(\tau)a(\tau) \rangle^2}. \] (69)

This gives a measure of the intensity fluctuations, and hence the coherence properties of the various modes. After the transient regime, the correlation functions for all three modes are given by

\[ g^{[2]}(\tau) \approx 2 - \frac{|\alpha|^4}{||\alpha|^2 + f(\chi, \Delta)|^2}, \] (70)

where the fluctuation function \( f(\chi, \Delta) = |v_{32}^{-1}/v_{31}^{-1}|3 \) is approximately unity near the region on the \( \Delta \) axis where the exponential growth rate \( \Gamma \) is maximized for fixed \( \chi \), and steadily increases as one moves away from this region. We see that in the spontaneous case \( (|\alpha|^2 = 0) \) the intensity correlation functions are equal to 2, which is characteristic of a thermal, or chaotic field. However, as \( |\alpha|^2 \) is increased, \( g^{[2]} \) approaches 1, which signifies a coherent, or Poissonian field. Thus we see that by varying readily adjustable experimental parameters, such as the injected probe field strength and frequency, the condensate atom number, and/or the pump intensity and detuning, we have the capability to vary the intensity fluctuations of the generated fields continuously between thermal and coherent limits.

VI. OUTLOOK

In summary, then, we see that atom optics is progressing along a path that closely parallels that of optics following the invention of the laser. In many situations, it is possible to understand an atom optical effect by simply reversing the roles of light and matter, and indeed, much of the inspiration leading to atom lasers and nonlinear atom optics results from such an approach. The pioneering work of Peter Franken and his colleagues play a central role in these developments. But what would probably intrigue Peter most is not so much the way in which atom optics and optics are similar, but rather those important aspects in which they differ, as results in particular from the fact that atoms have an internal structure, are massive, and are composite particles. Work along these lines will no doubt keep him amused for many years.

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FIG. 1. Probability $P(n_0)$ for having $n_0$ atoms in mode 0 as a function of time. The parameters for this plot are $\kappa_0 = 0.1$, $\kappa_1 = 1$, $N = 2$, $\kappa_2 = 10$, $V_{0211} = 2$, and $V_{0000} = V_{0101} = V_{1111} = 0$, all given in units of $\kappa_1^{-1}$. 

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