Research Article

Giaccardi Inequality for Modified $h$-Convex Functions and Mean Value Theorems

Yonghong Liu 1, Wasim Iqbal 2, Atiq Ur Rehman 3, Ghulam Farid 3, and Kamsing Nonlaopon 4

1 School of Computer Science, Chengdu University, Chengdu, 610106, China
2 Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan
3 Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock, Pakistan
4 Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

Correspondence should be addressed to Kamsing Nonlaopon; nkamsi@kku.ac.th

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In this article, we consider the class of modified $h$-convex functions and derive the famous Giaccardi and Petrović’ type inequalities for this class of functions. The mean value theorems for the functionals due to Giaccardi and Petrović’ type inequalities are formulated. Some special cases are discussed by taking different examples of function $h$.

1. Introduction and Preliminaries

Convex functions have played an important role in the development of various fields of pure and applied sciences. Convexity theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics, and engineering sciences.

Convexity theory is developing rapidly in recent years by utilising fresh and inventive methodologies. Toader [1] developed $m$--convex functions, which seemed like a nice generalization of the convex functions. Varošanec [2] gave the definition of $h$--convex functions.

It is important to note that $m$--convex functions and $h$--convex functions are clearly two distinct types of convex functions. It is only reasonable to group these classes together. Özdemir et al. [3] used these facts to introduce $(h, m)$--convex functions and derive some Hermite-Hadamard type inequalities. Orlicz [4] introduced $\phi$--convex functions, which was used in the theory of Orlicz spaces. Motivated by this, Dragomir and Fitzpatrick (see [5, 6]) introduced the class of $s$--convex functions in the first and second sense.

Here we recall some basic definitions.

Definition 1. A function $\phi : \Omega \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is convex, if

$$\phi(\tau v + (1 - \tau)w) \leq \tau \phi(v) + (1 - \tau)\phi(w), \forall v, w \in \Omega, \tau \in (0, 1).$$  

(1)

Varošanec [2] gave the definition of $h$--convex function and derived several results by imposing the conditions on $h$, which seemed like a nice generalization of the convex functions.

Definition 2. Let $h : J \rightarrow \mathbb{R}$ be a nonnegative function such that $(0, 1) \subseteq J$. A function $\phi : \Omega \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is $h$--convex, if

$$\phi(\tau v + (1 - \tau)w) \leq h(\tau)\phi(v) + h(1 - \tau)\phi(w), \forall v, w \in \Omega, \tau \in (0, 1).$$  

(2)

Toader defined a new class of nonconvex functions, known as $(h, \lambda, \mu)$--convex functions. Toader looked into the fundamental features of this type of nonconvex function. Here, we recall the definition of modified $h$--convex, which
is basically a special case of \((h, \lambda, \mu)\) – convex functions defined by Toader in [7]. Many researchers and mathematicians have explored the modified \(h\) – convex functions in the literature in recent years. Noor et al. [8] generalized the Hermite-Hadamard inequality for modified \(h\) – convex functions. Zhao et al. [9] discussed Schur-type, Hermite-Hadamard-type, and Fejér-type inequalities for the class of generalized strongly modified \(h\) – convex functions.

**Definition 3.** Let \(h : \mathbb{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}\) be a nonnegative functions. A function \(\phi : \Omega \subseteq \mathbb{R} \rightarrow \mathbb{R}\) is modified \(h\) – convex, if

\[
\phi(rv + (1-r)w) \leq h(r)\phi(v) + (1-h(r))\phi(w),
\]

\(\forall v, w \in \Omega, r \in (0, 1)\). \hspace{1cm} (3)

Here, we discuss Definition 3 in some detail.

1. Substituting \(h\) with an identity function in (3), one gets the convex function.
2. By taking \(h(r) = r^a\) in (3), one gets \((a, 1)\) – convex function given in [8]

**Definition 4.** Let \(a \in [0, 1]\). A function \(\phi : \Omega \subseteq \mathbb{R} \rightarrow \mathbb{R}\) is \((a, 1)\) – convex in the first sense, if

\[
\phi(rv + (1-r)w) \leq r^a\phi(v) + (1-r^a)\phi(w), \forall v, w \in \Omega, r \in (0, 1).
\]

The most famous inequality given by Giaccardi is known as the Giaccardi inequality [10] given in the following theorem.

**Theorem 5.** Let \(\Omega \subseteq \mathbb{R}\) be an interval, \(v_0 \in \Omega(v_1, \cdots, v_n) \in \Omega^n\), and \((w_1, \cdots, w_n) \in \mathbb{R}^n_+ (n \geq 2)\) such that

\[
\bar{v}_n = \sum_{r=1}^{n} w_r v_r \in \Omega and (v_r - v_0)(\bar{v}_n - v_r) \geq 0 \text{ for } r = 1, \cdots, n.
\]

If \(\omega : \mathbb{R} \rightarrow \mathbb{R}\) is a convex function, then

\[
\sum_{r=1}^{n} w_r \phi(v_r) \leq \frac{\sum_{r=1}^{n} w_r(v_r - v_0)}{\bar{v}_n - v_0} \phi(\bar{v}_n) + \frac{\bar{v}_n - v_0}{\bar{v}_n - v_0} \phi(v_0).
\]

Many scholars have contributed to the understanding of Giaccardi inequality by publishing results linked to it. In [11] Pečarić and Perić derived an elegant method of producing \(n\) – exponentially and exponentially convex functions when the Giaccardi and Petrović’ differences are applied. Rehman et al. [12] generalized the Giaccardi inequality to coordinates in plane. Also, the authors defined the nonnegative linear functional due to Giaccardi inequality and find the mean value theorems related to that functional. For further information on the Giaccardi and Petrović’ inequalities, see [10, 13, 14].

Giaccardi inequality is generalization of famous Petrović’ inequality introduced by Petrović [15]. It is a particular case of (6) when \(v_0 = 0\). In [16], authors considered the following functional due to Petrović’s inequality:

\[
\mathcal{P}(\phi) = \phi \left( \sum_{r=1}^{m} w_r v_r \right) - \frac{1}{n-1} \sum_{r=1}^{m} w_r \phi(v_r) - \left( \frac{n}{n-1} \right) \phi(0),
\]

where \((v_1, \cdots, v_n) \in [0, b]\) and \((w_1, \cdots, w_n)\) be positive \(n\)-tuples such that

\[
\bar{v}_n = \sum_{r=1}^{n} w_r v_r \geq v_r \geq v_0 \text{ for } r = 1, \cdots, n.
\]

The following mean value theorems of Lagrange and Cauchy type for above functional was proved in [17].

**Theorem 6.** Consider a functional defined in (7), if \(\phi \in C^1[0, b]\), then there exists \(\eta \in (0, b)\), such that

\[
\mathcal{P}(\phi) = \frac{\phi'(\eta)(\eta - v_0) + \phi(v_0)}{(\eta - v_0)^2} \mathcal{P}(\phi),
\]

which provided that \(\mathcal{P}(\phi)\) is nonzero and \(v_0 \in (0, b)\), where \(\phi(v) = v^a\).

**Theorem 7.** Let \((v_1, \cdots, v_n) \in [0, b]\) and \((w_1, \cdots, w_n)\) be positive \(n\)-tuples such that the condition given in (8) is valid. If \(\phi_1, \phi_2 \in C^1[0, b]\), then there exists \(\eta \in (0, b)\), such that

\[
\frac{\mathcal{P}(\phi_1; v_0)}{\mathcal{P}(\phi_2; v_0)} = \frac{(\eta - v_0)\phi_1'(\eta) - \phi_1(\eta)}{(\eta - v_0)\phi_2'(\eta) - \phi_2(\eta)} + \frac{\phi_0}{\phi_2(\eta)},
\]

which provided that the denominators are nonzero and \(v_0 \in (0, b)\).

The special case of above functional for a certain class of convex function has been considered in [18]. Many properties of this functional including its particular cases have been discussed in [16, 18].

This paper is organized as follows: in the section 2, the authors give important lemmas for modified \(h\) – convex functions. With the help of these lemmas, the authors derive the Giaccardi and Petrović’ inequalities for modified \(h\) – convex functions. Some special cases are discussed. In the section 3, the authors define the nonnegative linear functional due to Giaccardi inequality for modified \(h\) – convex and \((a, 1)\) – convex. Also, define the nonnegative linear functional for the Petrovic’s inequality for modified \(h\) – convex and derive the mean value theorems related to these functionals.
2. Main Results

For convenience, we assume that \( h : (0, \infty) \rightarrow \mathbb{R} \) is a positive function in the rest of the paper.

Lemma 8. A function \( \phi \) is modified \( h \) – convex on an interval \( \Omega \) if and only if \( \Phi_{(\phi, h)}(v) \) is increasing for \( v > v_0 \), where

\[
\Phi_{(\phi, h)}(v) = \frac{\phi(v) - \phi(v_0)}{h(v - v_0)} \quad \text{for all } v, v_0 \in \Omega.
\]  

(11)

Proof. Assume that \( \phi \) is modified \( h \) – convex function and \( \eta > v > v_0 \) such that \( v = \tau \eta + (1 - \tau)v_0 \), where \( \tau \in (0, 1) \), then (11) gives us

\[
\Phi_{(\phi, h)}(v) = \frac{\phi(\tau \eta + (1 - \tau)v_0) - \phi(v_0)}{h(\tau \eta + (1 - \tau)v_0 - v_0)} \leq \frac{h(\tau \eta) \phi(\eta) + (1 - h(\tau)) \phi(v_0) - \phi(v_0)}{h(\tau \eta - v_0)}.
\]  

(12)

As \( h \) is multiplicative, so we have

\[
\Phi_{(\phi, h)}(v) \leq \frac{h(\tau) \phi(\eta) - h(\tau) \phi(v_0)}{h(\tau) h(\eta - v_0)}.
\]  

(13)

It follows that

\[
\Phi_{(\phi, h)}(v) \leq \frac{\phi(\eta) - \phi(v_0)}{h(\eta - v_0)} = \Phi_{(\phi, h)}(\eta).
\]  

(14)

This shows that \( \Phi_{(\phi, h)}(v) \) is increasing on \( \Omega \). Conversely, let \( v, \eta \in \Omega \) such that \( v < \eta \) and

\[
\Phi_{(\phi, h)}(v) \leq \Phi_{(\phi, h)}(\eta).
\]  

(15)

That is,

\[
\frac{\phi(v) - \phi(v_0)}{h(v - v_0)} \leq \frac{\phi(\eta) - \phi(v_0)}{h(\eta - v_0)}.
\]  

(16)

Take \( v = \tau \eta + (1 - \tau)v_0 \), where \( \tau \in (0, 1) \), and then one has

\[
\frac{\phi(\tau \eta + (1 - \tau)v_0) - \phi(v_0)}{h(\tau \eta - v_0)} \leq \frac{\phi(\eta) - \phi(v_0)}{h(\eta - v_0)}.
\]  

(17)

Using the fact that \( h \) is multiplicative and then simplifying, one gets the definition of modified \( h \) – convex functions.

\[\square\]

Remark 9. One can note that if the function \( \phi \) is modified \( h \) – convex function, then the mapping \( \Phi_{(\phi, h)}(v) \) defined in Lemma 8 is increasing if and only if \( \Phi_{(\phi, h)}(v) \geq 0 \), provided that the derivatives exist. This is equivalent to

\[
\phi'(v) \geq \frac{\phi(v) - \phi(v_0)}{h(v - v_0)} h'(v - v_0).
\]  

(18)

Substituting \( h \) with an identity function in (18), one gets the result for convex functions given in [19], p. 09.

The Giaccardi inequality for modified \( h \) – convex functions is given in the following theorem.

Theorem 10. Let \( \Omega \subseteq \mathbb{R} \) be an interval, \( v_0 \in \Omega \), \( (\omega_1, \cdots, \omega_n) \in \Omega^n \), and \( (w_1, \cdots, w_n) \in \mathbb{R}^n(n \geq 2) \) such that the condition (8) is valid. Also, let \( \phi : \Omega \rightarrow \mathbb{R} \) be a modified \( h \) – convex function with the condition that \( h \) is a multiplicative function. Then,

\[
\sum_{\tau = 0}^{n} w_\tau \phi(v_\tau) \leq A\phi(\bar{v}_n) + \left( \sum_{\tau = 1}^{n} w_\tau - A \right) \phi(v_0),
\]  

(19)

where

\[
A = \frac{\sum_{\tau = 1}^{n} w_\tau h(v_\tau - v_0)}{h(\bar{v}_n - v_0)}.
\]  

(20)

Proof. To prove the main result, first assume that \( \phi(v)/h \) \( (v - v_0) \) is increasing for \( v \in \Omega \) such that \( v > v_0 \). As we have given \( v_0 \leq \bar{v}_n \), so one has

\[
\frac{\phi(v_\tau)}{h(v_\tau - v_0)} \leq \frac{\phi(\bar{v}_n)}{h(\bar{v}_n - v_0)}.
\]  

(21)

This gives

\[
h(\bar{v}_n - v_0)\phi(v_\tau) \leq h(v_\tau - v_0)\phi(\bar{v}_n).
\]  

(22)

Multiplying above inequality by \( w_\tau \) and taking sum from \( \tau = 1, \cdots, n \), one has

\[
h(\bar{v}_n - v_0) \sum_{\tau = 1}^{n} w_\tau \phi(v_\tau) \leq \sum_{\tau = 1}^{n} w_\tau h(v_\tau - v_0)\phi(\bar{v}_n).
\]  

(23)

This leads to

\[
\sum_{\tau = 1}^{n} w_\tau \phi(v_\tau) \leq \frac{\sum_{\tau = 1}^{n} w_\tau h(v_\tau - v_0)}{h(\bar{v}_n - v_0)} \phi(\bar{v}_n).
\]  

(24)

Since \( \phi \) is modified \( h \) – convex function, so by Lemma 8,

\[
\Phi_{(\phi, h)}(v) = \frac{\phi(v) - \phi(v_0)}{h(v - v_0)}
\]  

(25)

is increasing for \( v > v_0 \). Substituting \( \phi(v) \) by \( \phi(v) - \phi(v_0) \) in (24), one has

\[
\sum_{\tau = 1}^{n} w_\tau \phi(v_\tau) - \phi(v_0) \leq \frac{\sum_{\tau = 1}^{n} w_\tau h(v_\tau - v_0)}{h(\bar{v}_n - v_0)} (\phi(\bar{v}_n) - \phi(v_0)).
\]  

(26)
This is equivalent to
\[
\sum_{r=1}^{n} w_r \phi(v_r) \leq \sum_{r=1}^{n} w_r \frac{h(v_r - v_0)}{h(v_n - v_0)} (\phi(\bar{v}_n) - \phi(0)) - \phi(0) \sum_{r=1}^{n} w_r.
\]  
(27)

From above inequality, one can deduce (19). □

Remark 11. By taking \( h(v) = v \) in Theorem 10, one gets Theorem 5.
A Giaccardi inequality for \((a,1)\)–convex functions is given in the following corollary.

**Corollary 12.** Let the conditions of Theorem 10 be valid. Also, let \( \phi : \Omega \rightarrow \mathbb{R} \) be a \((a,1)\)–convex function. Then,
\[
\sum_{r=1}^{n} w_r \phi(v_r) \leq \sum_{r=1}^{n} w_r \frac{(v_r - v_0)^a}{(v_n - v_0)^a} \phi(\bar{v}_n) + \sum_{r=1}^{n} w_r \left( 1 - \frac{(v_r - v_0)^a}{(v_n - v_0)^a} \right) \phi(0).
\]
(28)

**Proof.** A function \( h(r) = r^a \), \( a \in (0,1) \), satisfied all the condition of Theorem 10. So, substituting this value of \( h \) in Theorem 10 gives the required result.

A Petrović’s inequality for \((a,1)\)–convex functions is given in the following corollary. □

**Corollary 13.** Let the conditions of Theorem 10 be valid. Also, let \( \phi : \Omega \rightarrow \mathbb{R} \) be a \((a,1)\)–convex function. Then,
\[
\sum_{r=1}^{n} w_r \phi(v_r) \leq \sum_{r=1}^{n} w_r \frac{(v_r - v_0)^a}{v_n^a} \phi(\bar{v}_n) + \sum_{r=1}^{n} w_r \left( 1 - \frac{v_r^a}{v_n^a} \right) \phi(0).
\]
(29)

**Proof.** Take \( h(r) = r^a \) and \( v_0 = 0 \) in Theorem 10 with the restriction that \( \Omega = [0,a) \) to get the required result.

A Petrović’s inequality for modified \( h \)–convex functions is given in the following corollary. □

**Corollary 14.** Let the conditions of Theorem 10 be valid for \( \Omega = [0,a) \). Then,
\[
\sum_{r=1}^{n} w_r \phi(v_r) \leq \sum_{r=1}^{n} w_r \frac{h(v_r)}{h(\bar{v}_n)} \phi(\bar{v}_n) + \left( \sum_{r=1}^{n} w_r - \sum_{r=1}^{n} w_r \frac{h(v_r)}{h(\bar{v}_n)} \right) \phi(0).
\]
(30)

**Proof.** Take \( v_0 = 0 \) in Theorem 10 with the restriction that \( \Omega = [0,a) \) to get the required result.

Remark 15. If one take \( \Omega = [0,a) \), \( h(r) = r \), and \( r_0 = 0 \) in Theorem 10, one gets the result given by the Petrović’ for convex function in [15].

### 3. Mean Value Theorems

To give the mean value theorems (MVTs) for the nonnegative difference of inequality (19), we define the linear functional as follows:

Let \( \Omega \subseteq \mathbb{R} \) be a closed interval, \( v_0 \in \hat{\Omega} \), \( (v_1, \ldots, v_n) \in \hat{\Omega}^n \), and \( (w_1, \ldots, w_n) \in \mathbb{R}_+^n (n \geq 2) \) such that condition (8) is valid. Then, for \( \phi : \hat{\Omega} \rightarrow \mathbb{R} \) and multiplicative positive function \( h \), we define a functional.
\[
\mathfrak{F}(\phi; h, v_0) = \sum_{r=1}^{n} w_r \phi(v_r) + \left( \sum_{r=1}^{n} w_r - A \right) \phi(0),
\]
(31)

where \( A \) is defined in (20).

By taking \( h(r) = r^a \) in (31), one gets the linear functional for Giaccardi inequality for \((a,1)\)–convex function as follows:
\[
\mathfrak{F}(\phi; v_0) = \sum_{r=1}^{n} w_r \frac{(v_r - v_0)^a}{(v_n - v_0)^a} \phi(\bar{v}_n) - \sum_{r=1}^{n} w_r \phi(v_r) + \sum_{r=1}^{n} w_r \left( 1 - \frac{(v_r - v_0)^a}{(v_n - v_0)^a} \right) \phi(0).
\]
(32)

By taking \( v_0 = 0 \), in (31), one gets the linear functional for Petrović’s inequality for modified \( h \)–convex functions given as follows:
\[
\mathfrak{F}(\phi; h) = \sum_{r=1}^{n} w_r \frac{h(v_r)}{h(\bar{v}_n)} \phi(\bar{v}_n) - \sum_{r=1}^{n} w_r \phi(v_r) + \sum_{r=1}^{n} w_r \left( 1 - \frac{h(v_r)}{h(\bar{v}_n)} \right) \phi(0).
\]
(33)

In the following lemma, two modified \( h \)–convex functions are introduced under certain condition to prove MVT of Lagrange type.

**Lemma 16.** Let \( \phi : \hat{\Omega} \rightarrow \mathbb{R} \) and \( h : (0,\infty) \rightarrow \mathbb{R}_+^+ \) be differentiable functions such that
\[
\frac{n}{2} h(v-v_0) \phi'(v) - (\phi(v) - \phi(v_0))h'(v-v_0) \leq Nv, v_0 \in \hat{\Omega}.
\]
(34)

The functions \( \phi_1, \phi_2 : \hat{\Omega} \rightarrow \mathbb{R} \) are modified \( h \)–convex function on \( \hat{\Omega} \), if
\[
\psi_1(v) = Nv^2 - \phi(v),
\]
(35)
\[
\psi_2(v) = \phi(v) - nv^2.
\]
(36)
Proof. First consider that
\[
\Phi_{(\psi, h)}(v) = \frac{\psi_1(v) - \psi_1(v_0)}{h(v - v_0)} = \frac{N\nu^2 - \phi(v) - \nu_0^2 + \phi(v_0)}{h(v - v_0)}.
\]

After differentiating, one has
\[
\Phi_{(\psi, h)}'(v) = N \frac{h(v - v_0)2\nu - (\nu^2 - \nu_0^2)h'(v - v_0)}{h^2(v - v_0)} - \frac{h(v - v_0)\phi'(v) - (\phi(v) - \phi(v_0))h'(v - v_0)}{h^2(v - v_0)}.
\]

From (34), one has
\[
h(v - v_0)\phi'(v) - (\phi(v) - \phi(v_0))h'(v - v_0) \leq N \left(2\nu h(v - v_0) - (\nu^2 - \nu_0^2)h'(v - v_0)\right).
\]

This leads to
\[
\frac{h(v - v_0)\phi'(v) - (\phi(v) - \phi(v_0))h'(v - v_0)}{h^2(v - v_0)} \leq \frac{2\nu h(v - v_0) - (\nu^2 - \nu_0^2)h'(v - v_0)}{h^2(v - v_0)}.
\]

This implies
\[
\frac{2\nu h(v - v_0) - (\nu^2 - \nu_0^2)h'(v - v_0)}{h^2(v - v_0)} \geq 0.
\]

Hence, \(\Phi_{(\psi, h)}'(v) \geq 0\).

In a similar way, one can prove \(\Phi_{(\psi, h)}'(v) \geq 0\).

It means \(\Phi_{(\psi, h)}(v)\) and \(\Phi_{(\psi, h)}(v)\) are increasing for \(v > v_0\). Hence, by Lemma 8, \(\psi_1\) and \(\psi_2\) are modified \(h\)–convex functions.

**Theorem 17.** Consider a functional \(\Theta\) defined in (31). If \(\phi \in C^1(\bar{\Omega})\) and \(h, h'\) are bounded, then there exists \(\eta\) in the interior of \(\bar{\Omega}\) such that

\[
\Theta(\phi; h, v_0) = \frac{h(\eta - v_0)\phi'(\eta) - (\phi(\eta) - \phi(v_0))h'(\eta - v_0)}{2\nu h(\eta - v_0) - (\eta^2 - v_0^2)h'(\eta - v_0)} \cdot \Theta(\phi; h, v_0),
\]

where \(\varphi(v) = v^2\), provided that \(\Theta(\phi; h, v_0)\) is nonzero.

Proof. Since \(\phi \in C^1(\bar{\Omega})\) and \(h, h'\) are bounded, there exists real numbers \(n\) and \(N\) such that

\[
n \leq \frac{h(v - v_0)\phi'(v) - (\phi(v) - \phi(v_0))h'(v - v_0)}{2\nu h(v - v_0) - (\nu^2 - \nu_0^2)h'(v - v_0)} \leq N, \forall v, v_0 \in \Omega.
\]

Consider the function \(\psi_1\) defined in Lemma 16. As \(\psi_1\) is modified \(h\)–convex function on \(\Omega\), therefore,

\[
\Theta(\psi_1; h, v_0) \geq 0.
\]

That is,

\[
\Theta(N\nu^2 - \phi(v); h, v_0) \geq 0.
\]

This implies

\[
N\Theta(\phi; h, v_0) \geq \Theta(\phi; h, v_0).
\]

In similar way, if one consider the function \(\psi_2\) defined in Lemma 16, then

\[
n\Theta(\phi; h, v_0) \leq \Theta(\phi; h, v_0).
\]

Combining inequalities (46) and (47), one has

\[
n \leq \frac{\Theta(\phi; h, v_0)}{\Theta(\phi; h, v_0)} \leq N.
\]

So, there exists \(\eta\) in the interior of \(\bar{\Omega}\) such that

\[
\Theta(\phi; h, v_0) = \frac{h(\eta - v_0)\phi'(\eta) - (\phi(\eta) - \phi(v_0))h'(\eta - v_0)}{2\nu h(\eta - v_0) - (\eta^2 - v_0^2)h'(\eta - v_0)}.
\]

This is equivalent to (52).

In the following corollary, Largrange type MVT related to functional due to Giaccardi inequality for \((\alpha, 1)\)–convex functions is given.

**Corollary 18.** Consider a functional \(\Theta\) defined in (32). Also, let \(\phi \in C^1(\Omega)\) and \(\phi : \Omega \longrightarrow \mathbb{R}\) be \((\alpha, 1)\)–convex function. Then,
where \( \varphi(v) = v^2 \), provided that \( \mathfrak{J}(\varphi; v_0) \) is nonzero.

Proof. By taking \( h(r) = \sigma(r) = r^a \), where \( a \in [0, 1] \), in (31), one has

\[
\mathfrak{J}(\varphi; v_0) = \Theta(\varphi; \sigma; v_0).
\]

Using it in Theorem 17, one gets the required result.

Lagrange type MVT for functional due to Petrović’s inequality for modified \( h \) – convex function has been stated in the following corollary.

**Corollary 19.** Consider a functional \( \mathfrak{S} \) defined in (33). If \( \phi \in C^1(\Omega) \) and \( h \) and \( h' \) are bounded, then there exists \( \eta \) in the interior of \( \Omega \) such that

\[
\mathfrak{P}(\phi; h) = \frac{h(\eta)\phi'(\eta) - (\phi(\eta) - \phi(0))h'(\eta)}{2\eta h(\eta) - \eta^2 h'(\eta)} \mathfrak{P}(\varphi; h),
\]

where \( \varphi(v) = v^2 \), provided that \( \mathfrak{P}(\varphi; h) \) is nonzero.

Proof. It is a simple consequence of the fact that

\[
\mathfrak{P}(\phi; h) = \Theta(\varphi; h, 0),
\]

as stated in (33). Using this fact in Theorem 17 gives the required result.

**Remark 20.** By taking \( h(r) = r \) in Theorem 17, one gets the result for Giaccardi inequality for convex function. A similar result for Petrović’s inequality for convex function was given by Rehman et al. in [17, Corollary 13].

**Theorem 21.** Consider a functional \( \mathfrak{Q} \) defined in (31) If \( \phi_1, \phi_2 \in C^1(\Omega) \), and then there exists \( \eta \) in the interior of \( \Omega \) such that

\[
\Theta(\phi_2; h, v_0) = \frac{h(\eta - v_0)\phi_1(\eta) - \phi_1(\eta)h'(\eta - v_0) + \phi_1(v_0)h'(\eta - v_0)}{h(\eta - v_0)\phi_2(\eta) - \phi_2(\eta)h'(\eta - v_0) + \phi_2(v_0)h'(\eta - v_0)},
\]

provided that the denominators are nonzero.

Proof. Let \( \mathcal{F} \in C^1(\Omega) \) be a function, defined as

\[
\mathcal{F} = t_1\phi_1 - t_2\phi_2,
\]

where \( t_1 = \Theta(\phi_2; h, v_0) \) and \( t_2 = \Theta(\phi_1; h, v_0) \).

Replace \( \phi \) with \( \mathcal{F} \) in Theorem 17, then one has

\[
0 = h(\eta - v_0)((t_1\phi_1 - t_2\phi_2)(\eta)') - (t_1\phi_1 - t_2\phi_2)(\eta)h'(\eta - v_0) + (t_1\phi_1 - t_2\phi_2)(v_0)h'(\eta - v_0)
\]

\[
= h(\eta - v_0)(t_1\phi_1'(\eta) - t_2\phi_2'(\eta)) - t_1\phi_1(\eta)h'(\eta - v_0)
\]

\[
+ t_2\phi_2(\eta)h'(\eta - v_0) + t_1\phi_1(v_0)h'(\eta - v_0) - t_2\phi_2(v_0)h'(\eta - v_0)
\]

\[
= t_1\left\{h(\eta - v_0)\phi_1'(\eta) - \phi_1(\eta)h'(\eta - v_0) + \phi_1(v_0)h'(\eta - v_0)\right\}
\]

\[
- t_2\left\{h(\eta - v_0)\phi_2'(\eta) - \phi_2(\eta)h'(\eta - v_0) + \phi_2(v_0)h'(\eta - v_0)\right\}.
\]

This gives

\[
\frac{t_2}{t_1} = \frac{h(\eta - v_0)\phi_2(\eta) - \phi_2(\eta)h'(\eta - v_0) + \phi_2(v_0)h'(\eta - v_0)}{h(\eta - v_0)\phi_1(\eta) - \phi_1(\eta)h'(\eta - v_0) + \phi_1(v_0)h'(\eta - v_0)}.
\]

Putting the values of \( t_1 \) and \( t_2 \), one gets the required result.

In the following corollary, Cauchy type MVT related to functional due to Petrović’s inequality for modified \( h \) – convex functions is given.

**Corollary 22.** Let the conditions of Theorem 17 be valid. If \( \phi_1, \phi_2 \in C^1(\Omega) \), then there exists \( \eta \) in the interior of \( \Omega \) such that

\[
\mathfrak{P}(\phi_2; h) = \frac{h(\eta)\phi_1'(\eta) - \phi_1(\eta)h'(\eta - v_0) + \phi_1(v_0)h'(\eta - v_0)}{h(\eta)\phi_2'(\eta) - \phi_2(\eta)h'(\eta - v_0) + \phi_2(v_0)h'(\eta - v_0)},
\]

provided that the denominators are nonzero.

Proof. It is just a natural result of the fact that

\[
\mathfrak{P}(\phi; h) = \Theta(\varphi; h, 0),
\]

as stated in (33). The desired result is obtained by applying this fact to Theorem 17.

Cauchy type MVT related to functional due to Giaccardi inequality for \( (a, 1) \) – convex functions is given in the following corollary.

**Corollary 23.** Let the conditions given in Theorem 6 are valid and \( \phi : \Omega \to \mathbb{R} \) be \( (a, 1) \) – convex function. Then,

\[
\mathfrak{J}(\phi_1; v_0) = \frac{(\eta - v_0)\phi_1'(\eta) - a\phi_1(\eta) + a\phi_1(v_0)}{(\eta - v_0)\phi_2'(\eta) - a\phi_2(\eta) + a\phi_2(v_0)},
\]

provided that the denominators are nonzero.
Proof. If one take $h(x) = \sigma(x) = x^\alpha$, where $\alpha \in [0, 1]$, in (31), then
\[ \mathcal{F}(\phi; \sigma, \varphi, \nu_0) = \Theta(\varphi; \sigma, \nu_0). \] (61)

This information is used in Theorem 17 to get the required result.

Cauchy type MVT related to functional due to Petrović's inequality for $\alpha, 1 - \text{convex functions is given in the following corollary.}$

Corollary 24. Let the conditions given in Theoerm 6 be valid and $\phi : \Omega \to \mathbb{R}$ be $\alpha, 1 - \text{convex function. Then,}$
\[ \mathcal{F}(\phi_1) = \frac{\eta \phi_1'(\eta) - \alpha \phi_1(\eta)}{\eta \phi_2'(\eta) - \alpha \phi_2(\eta)} \] (62)

proved that the denominators are nonzero.

Proof. If one take $h(x) = \sigma(x) = x^\alpha$, where $\alpha \in [0, 1]$, in (27), then
\[ \mathcal{F}(\phi; \sigma, \nu_0) = \Theta(\varphi; \sigma, \nu_0). \] (63)

This information is used in Theorem 17 to get the required result.

4. Conclusion

In this paper, the authors considered the modified $h - \text{convex function and derived the most important Giaccardi and Petrović’ inequalities for this class of functions. A linear functional due to the newly defined inequalities is considered to give the MVTs of Lagrange and Cauchy type. It is shown that the results of this article for some examples of functions $h$ give us previously known results published in [16–18]. This is an interesting direction for future research.}

Data Availability

There is no external data used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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