Zero energy correction method for non-Hermitian Harmonic oscillator
with simultaneous transformation of co-ordinate and momentum

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We propose zero energy correction method for non-Hermitian Harmonic oscillator under simultaneous transformation of co-ordinate \((x \rightarrow \frac{x+i\lambda p}{\sqrt{1+\beta \lambda}})\) and momentum \((p \rightarrow \frac{p+i\beta x}{\sqrt{1+\beta \lambda}})\) for getting energy eigenvalue in place of extending the idea of gaugelike transformation proposed earlier in momentum transformation \((p \rightarrow p + i\beta x)\) by Z. Ahmed [Phys.Lett A294, 287 (2002)]. Further energy of the non-Hermitian Harmonic oscillator remains the same as that of Hermitian Harmonic oscillator.

PACS 03.65Db

Key words- Non-Hermitian Harmonic oscillator, Perturbation theory, Energy level

I.INTRODUCTION

The physics of Harmonic oscillator is not only interesting because of its wide range of applicability but also its exact solvability behaviour [1,2,3]. It plays a major role in understanding the basic approximation methods like; variational method, W.K.B method, Bohr-Sommerfeld quantization relation and so also the perturbation theory[1-3]. In all most all text books on Quantum Mechanics give details of of the Harmonic oscillator problem because it is the simplest hermitian operator in Quantum Mechanics. The Hermiticity of a Hamiltonian is supposed to yield real energy spectrum as per the understanding of basic quantum mechanics. A deviation to this was proposed by Bender et al. [4] stating that non-Hermitian Hamiltonian can yield real energy spectrum. Inspired by the work of Bender et al. [4], Hatano and Nelson
[5] and Mostafazeh [6], Ahmed [7] proposed that linear harmonic oscillator having non-Hermitian transformation of momentum i.e. \( p \rightarrow p + i\beta x \) can be easily solved by gauge-like transformation as

\[
e^{\frac{f(x)}{2}} H_\beta(p \rightarrow p + i\beta x)e^{-\frac{f(x)}{2}} = H_{SHO} = \frac{p^2}{2} + \frac{(\alpha^2 + \beta^2)x^2}{2}
\]

(1a)

where

\[
f(x) = -2\beta \int xdx = -\beta x^2
\]

(1b)

Explicitly,

\[
H_\beta = \frac{(p + i\beta x)^2}{2} + \frac{(\alpha^2 + \beta^2)x^2}{2}
\]

(2)

and

\[
H_{SHO} = \frac{p^2}{2} + \frac{(\alpha^2 + \beta^2)x^2}{2}
\]

(3)

From the above, it is clear that the energy levels of \( H_{SHO} \) and \( H_\beta \) are the same i.e.

\[
E_n = (n + \frac{1}{2})\sqrt{(\alpha^2 + \beta^2)}
\]

(4)

Now question arises when \( p \) can be transformed as \( p \rightarrow p + i\beta x \), \( x \) can also be transformed as \( x \rightarrow x + i\lambda p \). **Under the simultaneous transformation of \( x \) and \( p \), it is not possible to get a suitable gauge like transformation (as in Eqn. (1a and 1b)).** However we feel that the standard perturbation theory [1,2,3] can be used to get the exact energy level as in Eqn. (4) easily rather than gauge like transformation suggested earlier [7]. Hence the aim of this paper is to show that how the perturbation theory can be used easily to get the energy eigenvalues under simultaneous non-hermitian transformation of co-ordinate (\( x \)) and momentum (\( p \)).

**II. NON-HERMITIAN TRANSFORMATION OF CO-ORDINATE AND MOMENTUM**

Let us consider the Harmonic oscillator Hamiltonian

\[
H_{HO} = \frac{p^2}{2} + \frac{x^2}{2}
\]

(5a)
whose exact energy eigenvalues are

\[ E_n = (n + \frac{1}{2}) \]  

In the above, \( x \) and \( p \) satisfy the commutation relation

\[ [x, p] = i \]  

Now use the transformation as

\[ x \rightarrow x + i\lambda p \sqrt{1 + \beta \lambda} \]  

and

\[ p \rightarrow p + i\beta x \sqrt{1 + \beta \lambda} \]  

Hence the new Hamiltonian becomes non-Hermitian in nature and is

\[ H = \frac{(p + i\beta x)^2}{2(1 + \lambda \beta)} + \frac{(x + i\lambda p)^2}{2(1 + \lambda \beta)} \]  

III. SECOND QUANTIZATION AND HAMILTONIAN

In order to solve the above Hamiltonian (Eqn. (8)), we use the second quantization formalism as

\[ x = \frac{(a + a^+)}{\sqrt{2\omega}} \]  

and

\[ p = i\sqrt{\frac{\omega}{2}}(a^+ - a) \]  

where the creation operator, \( a^+ \) and annihilation operator \( a \) satisfy the commutation relation

\[ [a, a^+] = 1 \]  

and \( \omega \) is an unknown parameter. The Hamiltonian in Eqn. (8) can be written as

\[ H = H_D + H_N \]
where
\[ H_D = [(1 - \lambda^2)\omega + \frac{(1 - \beta^2)}{\omega}] \frac{(2a^+a + 1)}{4(1 + \lambda\beta)} \] (12)
and
\[ H_N = U \frac{a^2}{4(1 + \lambda\beta)} + V \frac{(a^+)^2}{4(1 + \lambda\beta)} \] (13)
\[ V = [-\omega(1 - \lambda^2) + \frac{(1 - \beta^2)}{\omega} - 2(\lambda + \beta)] \] (13a)
\[ U = [-\omega(1 - \lambda^2) + \frac{(1 - \beta^2)}{\omega} + 2(\lambda + \beta)] \] (13b)

IV. ZERO ENERGY CORRECTION METHOD

Now we solve the eigenvalue relation:
\[ H\psi_n(x) = \epsilon_n\psi_n(x) \] (14)
using perturbation theory as follows. Here we express
\[ \epsilon_n = \epsilon_n^{(0)} + \sum_{m=1}^{k} \epsilon_n^{(m)} \] (15)
The zeroth order energy \(\epsilon_n^{(0)}\) satisfies the following eigenvalue relation
\[ H_D|\psi_n^{(0)}\rangle = H_D|n\rangle = \epsilon_n^{(0)}|n\rangle \] (16a)
where \(\psi_n^{(0)}\) is the zeroth order wave function and \(\epsilon_n^{(m)}\) is the mth order perturbation correction.
\[ \epsilon_n^{(0)} = \frac{(2n + 1)}{4(1 + \lambda\beta)} \left[(1 - \lambda^2)\omega + \frac{(1 - \beta^2)}{\omega}\right] \] (17a)
and
\[ \sum_{m=1}^{k} \epsilon_n^{(m)} = \epsilon_n^{(1)} + \epsilon_n^{(2)} + \epsilon_n^{(3)} + \ldots \] (17b)
The energy correction terms will give zero contribution if the parameter is determined from non-diagonal terms of \(H_N\) [8,9].

Case-I
Let the coefficient of $a^2$ is zero i.e.

$$U = [-\omega(1 - \lambda^2) + \frac{(1 - \beta^2)}{\omega} + 2(\lambda + \beta)] = 0 \quad (18)$$

which leads to (considering positive sign)

$$\omega = \frac{(1 + \beta)}{(1 - \lambda)} \quad (19)$$

In this case,

$$\epsilon_n^{(0)} = (n + \frac{1}{2}) \quad (20)$$

Now the perturbation correction term is

$$H_N = V \frac{(a^+)^2}{4(1 + \lambda\beta)} \quad (21)$$

In this case one will notice that

$$< n|H_N|n-2 > = V \frac{\sqrt{n(n-1)}}{4(1 + \lambda\beta)} \quad (22a)$$

$$< n-2|H_N|n > = 0 \quad (22b)$$

Hence it is easy to note that all orders of energy corrections will be zero. Let us consider explicitly corrections up to third order using standard perturbation series given in literature[1,3,10,11], which can be written as

$$\epsilon_n^{(1)} = < n|H_N|n > = 0 \quad (23a)$$

$$\epsilon_n^{(2)} = \sum_{k \neq n} \frac{< n|H_N|k >< k|H_N|n >}{(\epsilon_n^{(0)} - \epsilon_k^{(0)})} \quad (23b)$$

$$= \frac{< n|H_N|n-2 >< n-2|H_N|n >}{(\epsilon_n^{(0)} - \epsilon_{n-2}^{(0)})} = 0 \quad (23b)$$

$$\epsilon_n^{(3)} = \sum_{p,q} \frac{< n|H_N|p >< p|H_N|q >< q|H_N|n >}{(\epsilon_n^{(0)} - \epsilon_p^{(0)})(\epsilon_n^{(0)} - \epsilon_q^{(0)})} \quad (23c)$$

$$= \frac{< n|H_N|n-2 >< n-2|H_N|n-4 >< n-4|H_N|n >}{(\epsilon_n^{(0)} - \epsilon_{n-2}^{(0)})(\epsilon_n^{(0)} - \epsilon_{n-4}^{(0)})} = 0 \quad (23c)$$
Here second order correction is zero due to \( < n - 2 | H_N | n > = \delta_{n-2,n+2} \) and third order correction is zero due to \( < n - 4 | H_N | n > = \delta_{n-4,n+2} \). Similarly one can notice all correction terms \( \epsilon^{(m)}_n \) will be zero. Hence

\[
\epsilon_n = \epsilon^{(0)}_n = E^{(0)}_n = (n + \frac{1}{2})
\]  

(24)

which is the same as the energy level of harmonic oscillator as given in Eq(5b).

**Case-II**

Let the coefficient of \((a^+)^2\) is zero i.e.

\[
V = \left[ -\omega (1 - \lambda^2) + \frac{(1 - \beta^2)}{\omega} - 2(\lambda + \beta) \right] = 0
\]  

(25)

which leads to

\[
\omega = \frac{(1 - \beta)}{(1 + \lambda)}
\]  

(26)

Now the perturbation term becomes

\[
H_N = \frac{a^2}{4(1 + \lambda \beta)}
\]  

(27)

In this case one will notice that

\[
< n | H_N | n + 2 > = U \frac{\sqrt{(n+1)(n+2)}}{4(1 + \lambda \beta)}
\]  

(28a)

\[
< n + 2 | H_N | n > = 0
\]  

(28b)

Hence we have

\[
\epsilon^{(1)}_n = < n | H_N | n > = 0
\]  

(29a)

\[
\epsilon^{(2)}_n = \sum_{k \neq n} \frac{< n | H_N | k > < k | H_N | n >}{(\epsilon^{(0)}_n - \epsilon^{(0)}_k)} = 0
\]  

(29b)

\[
\epsilon^{(3)}_n = \sum_{p,q} \frac{< n | H_N | p > < p | H_N | q > < q | H_N | n >}{(\epsilon^{(0)}_n - \epsilon^{(0)}_p)(\epsilon^{(0)}_n - \epsilon^{(0)}_q)} = 0
\]  

(29c)
Here second order correction is zero due to $< n + 2| H_N | n > = \delta_{n-2,n+2}$ and third order correction is zero due to $< n + 4| H_N | n > = \delta_{n+4,n-2}$. Similarly one can notice all correction terms $\epsilon_n^{(m)}$ will be zero. Hence

$$\epsilon_n = \epsilon_n^{(0)} = E_n^{(0)} = (n + \frac{1}{2})$$

which is the same as the energy level of harmonic oscillator as given in Eq(5b).

V. MOMENTUM TRANSFORMATION A special case: $\lambda = 0$ The Hamiltonian (Eqn. (8)) with $\lambda = 0$ now reduced to

$$H_T = \frac{(p + i\beta x)^2}{2} + \frac{x^2}{2}$$

(30)

This Hamiltonian is similar one considered earlier by Ahmed [7] in which the author has used gauge like transformation to show the invariance principle of energy of the Harmonic oscillator. Following the above procedure, one can write the Hamiltonian (Eqn. 30) as

$$H_T = \frac{p^2}{2} + \frac{(1 - \beta^2)x^2}{2} + \frac{i\beta(xp + px)}{2}$$

(31)

Using second quantization notation as used earlier, one can write the Hamiltonian as

$$H_T = H_D + H_N$$

Here

$$H_D = \frac{(2a^+a + 1)}{4}[\omega + \frac{(1 - \beta^2)}{\omega}]$$

(32)

and

$$H_N = U\frac{a^2}{4} + V\frac{(a^+)^2}{4}$$

(33)

where

$$U = [-\omega + \frac{(1 - \beta^2)}{\omega} + 2\beta]$$

(34)

and

$$V = [-\omega + \frac{(1 - \beta^2)}{\omega} - 2\beta]$$

(35)
Following perturbation method as used above, one can use either $U = 0$ or $V = 0$ to find out the energy levels. If $U = 0$ (ref Eq(19)) then

$$\omega = 1 + \beta \quad (36)$$

In this case, the total energy is

$$\epsilon_n = \epsilon_n^{(0)} = E_n^{(0)} = (n + \frac{1}{2})$$

Similarly, one can $V = 0$ and find the same energy level as given above.

VI. CO-ORDINATE TRANSFORMATION A special case $\beta = 0$

The Hamiltonian (Eqn. (8)) with $\beta = 0$ is now reduced to the Hamiltonian (considered by us in a recent work [13]) as

$$H_\lambda = \frac{p^2}{2} + \frac{(x + i\lambda p)^2}{2} \quad (37)$$

whose energy levels are

$$\epsilon_n^{(0)} = (2n + 1)[\omega(1 - \lambda^2) + \frac{1}{\omega}] \quad (38)$$

with (ref Eq(26))

$$\omega = \frac{1}{(1 + \lambda)} \quad (38)$$

The final expression for energy remains the same as that of Harmonic Oscillator (Eq(5b)) i.e

$$\epsilon_n = \epsilon_n^{(0)} = E_n^{(0)} = (n + \frac{1}{2})$$

VII. Conclusion

In this paper, we suggest a simpler procedure for calculating energy levels of the non-Hermitian harmonic oscillator under simultaneous transformation of co-ordinate and momentum. We also show that under special circumstances, the Hamiltonian
reduces to the case of either momentum transformation or co-ordinate transformation. In all the cases, we show that the energy levels remain the same as that of simple Harmonic oscillator. Here we would like to state that if the parameter $\omega$ is determined using variational principle [14, 15] i.e. $\frac{d\epsilon_n^{(0)}}{d\omega} = 0$, then one will calculate all orders of perturbation corrections i.e. $\sum_{m=2}^{k} \epsilon_n^{(m)}$ because $\langle n|H_N|n + 2 \rangle \neq 0$ so also $\langle n + 2|H_N|n \rangle \neq 0$ and it will be a cumbersome process. However, if the parameter is determined either using $\langle n|H_N|n + 2 \rangle$ or $\langle n + 2|H_N|n \rangle$ as done in above procedure then the energy levels are obtained easily. Lastly, one can use nonlinear perturbation series [16] and easily get convinced that the parameter $\omega$ determined using the condition $\langle n|H_N|n + 2 \rangle$ or $\langle n + 2|H_N|n \rangle$, the calculations become very easy to get the desired result.

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