ON THE BOUNDS OF COEFFICIENTS OF DAUBECHIES ORTHONORMAL WAVELETS

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Abstract. This article is a continuation of the studies published in [1]. In the present work we provide the bounds for Daubechies orthonormal wavelet coefficients for function spaces $A^p_k := \{ f : \| (i\omega)^k \hat{f}(\omega) \|_p < \infty \}, k \in \mathbb{N} \cup \{0\}, p \in (1, \infty)$.

1. Introduction and motivations

A function $\psi$ is called a wavelet if there exists a dual function $\tilde{\psi}$, such that any function $f \in L_2(\mathbb{R})$ can be expressed in the form

$$f(t) = \sum_{j \in \mathbb{Z}} \sum_{\nu \in \mathbb{Z}} \langle f, \tilde{\psi}_{j,\nu} \rangle \psi_{j,\nu}(t).$$

The development of wavelets goes back to A. Haar’s work in early 20th century and to D. Gabor’s work (1946), who constructed functions similar to wavelets. Notable contributions to wavelet theory can be attributed to G. Zweig’s discovery of the continuous wavelet transform in 1975; D. Goupilland, A. Grossmann and J. Morlet’s formulation of the cosine wavelet transform (CWT) in 1982; J. Strömberg’s work on discrete wavelets (1983); I. Daubechies’ orthogonal wavelets with compact support (1988); Y. Meyer’s orthonormal basis of wavelets (1989); S. Mallat’s multiresolution framework (1989); and many others.

Wavelets are used in signal analysis, molecular dynamics, density-matrix localisation, optics, quantum mechanics, image processing, DNA analysis, speech recognition, to name few. Wavelets have such a wide variety of applications mainly because of their ability to encode a signal using only a few of the larger coefficients. The numbers of large coefficients depends on

- the size of the support of the signal: the shorter support the better;

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- the number of vanishing moments: the more vanishing moments a wavelet has, the more it oscillates. (The number of vanishing moments determines what the wavelet does not see).
- regularity (smoothness) of the signal: the number of continuous derivatives.

In general the Daubechies wavelets are chosen to have highest number $m$ of vanishing moments for a given support width $2m - 1$. Let $m \in \mathbb{N}$. The trigonometric polynomials

$$H_m(\omega) = 2^{-1/2} \sum_{\ell=0}^{2m-1} h_m(\ell)e^{i\ell\omega}, \; h_m(\ell) \in \mathbb{R},$$

which satisfy the equalities

$$|H_m(\omega)|^2 = \left(\cos^2 \frac{\omega}{2}\right)^m P_{m-1} \left(\sin^2 \frac{\omega}{2}\right),$$

where

$$P_{m-1}(x) = \sum_{k=0}^{m-1} \binom{m-1+k}{k} x^k,$$

are called Daubechies filters (see e.g. [4]).

The Fourier transform, $\hat{f}$, of function $f \in L_1(\mathbb{R})$ is defined to be

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\omega x} \, dx.$$

A function $\varphi_m^D$ whose Fourier transform has the form

$$(\varphi_m^D)(\omega) = \frac{1}{\sqrt{2\pi}} \prod_{\ell=1}^{\infty} H_m(2^{-\ell})$$

is the orthogonal scaling function. A function whose Fourier transform has the form

$$(\psi_m^D)(\omega) = e^{-\frac{i\pi}{2} H_m \left(\frac{\omega}{2} + \pi\right)} (\varphi_m^D)(\omega)$$

is called an orthogonal Daubechies wavelet $\psi_m^D$.

Note (see e.g. [8]),

$$|\varphi_m^D(\omega)|^2 \leq |H_m \left(\frac{\omega}{2} + \pi\right)|^2 \left|\frac{\omega}{2}\right|^2 = \frac{1}{2\pi} \left|H_m \left(\frac{\omega}{2} + \pi\right)\right|^2 \prod_{\ell=1}^{\infty} \left|H_m \left(2^{-\ell-1}\omega\right)\right|^2$$

and

$$|H_m(\omega)|^2 = 1 - \frac{\Gamma(m+1/2)}{\sqrt{\pi\Gamma(m)}} \int_0^t \sin^{2m-1}\omega \, d\omega.$$
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Many wavelet applications, for example, image/signal compression, denoising, inpainting, compressive sensing, and so on, are based on investigation of the wavelet coefficients \( \langle f, \psi_{j,\nu} \rangle \) for \( j,\nu \in \mathbb{Z} \), where \( \langle f, g \rangle := \int_{\mathbb{R}} f(x) g(x) \, dx \) and \( \varphi_{j,\nu} := 2^{j/2} \varphi(2^j \cdot -\nu), \psi_{j,\nu} := 2^{j/2} \psi(2^j \cdot -\nu) \). The magnitude of the wavelet coefficients depends on both the smoothness of the function \( f \) and the wavelet \( \psi \). In this paper, we shall investigate the quantity

\[
C_{k,p}(\psi) = \sup_{f \in A_{k}^{p'}} \frac{|\langle f, \psi \rangle|}{\|\hat{\psi}\|_p},
\]

where \( 1 < p, p' < \infty, 1/p' + 1/p = 1, k \in \mathbb{N} \cup \{0\} \), and \( A_{k}^{p'} := \{ f \in L_{p'}(\mathbb{R}) : \| (i\omega)^{k} \hat{f}(\omega) \|_{p'} \leq 1 \} \).

Let us note here, that the quantity \( C_{k,p}(\psi) \) in (1.1) is the best possible constant in the following Bernstein type inequality

\[
|\langle f, \psi_{j,\nu} \rangle| \leq C_{k,p}(\psi) 2^{-j(k+1/p'-1/2)} \| \psi \|_p \|(i\omega)^{k} \hat{f}(\omega)\|_{p'}
\]

Such type of inequalities plays an important role in wavelet algorithms for the numerical solution of integral equations (see e.g. [2]) where wavelet coefficients arise by applying an integral operator to a wavelet and bound of the type (1.2) gives priori information on the size of the wavelet coefficients. We refer to [1] for the more explanations about relation of \( C_{k,p}(\psi) \) to the estimates of the expansion coefficients for the Daubechie’s wavelets.

Note here, that

\[
C_{k,p}(\psi) = \sup_{f \in A_{k}^{p'}} \frac{|\langle f, \psi \rangle|}{\|\hat{\psi}\|_p} = \sup_{f \in A_{k}^{p'}} \frac{|\langle \hat{f}, \hat{\psi} \rangle|}{\|\hat{\psi}\|_p} = \frac{\|\hat{k}\psi\|_p}{\|\hat{\psi}\|_p},
\]

where for a function \( f \in L_1(\mathbb{R}) \), \( kf \) is defined to be the function such that

\[
\hat{k}f(\omega) = (i\omega)^{-k} \hat{f}(\omega), \quad \omega \in \mathbb{R}.
\]

2. Main Results

**Theorem 1.** Let \( k \geq 0 \) be a nonnegative integer. Let \( \psi_{m} \) be the Daubechies orthonormal wavelet of order \( m \), \( m > k \). Then, for \( p \in (1, \infty), C,c > 0, \varepsilon \in (0,\pi), \]

\[
B \leq \| k(\psi_{m}^{D}) \|_p \leq A,
\]
where

\[ A = A(p, k, m, \tilde{C}, c) = \frac{(2\pi)^{1/p-1/2}}{\pi^k} \left( 1 - \frac{2^{1-p}k}{p^k - 1} \right)^{1/p} \]

\[ + \left[ 2^{1-p} \left( 2m + \frac{1}{2} \right) \pi^{p(m-k-\frac{1}{2})+1} \left( \frac{(2m)!}{m!(m-1)!} \right)^{p/2} \right. \]

\[ + \left. \frac{1}{(2\pi)^{p\log m-2}} + \frac{1}{2^{p/2-1} \pi^{p(k+\frac{1}{2})-1}} \right]^{1/p} \]

and

\[ B = B(p, k, m, \tilde{C}, c) = \frac{(2\pi)^{1/p-1/2}}{\pi^k} \left( 1 - \frac{2^{1-p}k}{p^k - 1} \right)^{1/p} \]

\[ - \left[ 2^{1-p} \left( 2m + \frac{1}{2} \right) \pi^{p(m-k-\frac{1}{2})+1} \left( \frac{(2m)!}{m!(m-1)!} \right)^{p/2} \right. \]

\[ + \left. \frac{1}{(2\pi)^{p\log m-2}} + \frac{1}{2^{p/2-1} \pi^{p(k+\frac{1}{2})-1}} \right]^{1/p} \]

**Proof.** The first part of the proof is going along the lines of the asymptotic version of this theorem in [1].

Let \( \hat{\Psi} = \frac{1}{\sqrt{2\pi}} \left( \chi_{[-2\pi, -\pi]} + \chi_{[\pi, 2\pi]} \right) \), where \( \chi_I \) is the characteristic function of the interval \( I \).

Using Minkowski inequality, we obtain,

\[ I := \| \hat{k}(\hat{\psi}^D_m) \|_p = \left( \int_{\mathbb{R}} |(i\omega)^{-k}|^p |\hat{\psi}^D_m|^p d\omega \right)^{1/p} \]

\[ = \left( \int_{\mathbb{R}} |\omega|^{-pk} |\hat{\Psi}(\omega) - (\hat{\Psi}(\omega) - \hat{\psi}^D_m)|^p d\omega \right)^{1/p} \]

\[ \leq \left( \int_{\mathbb{R}} |\omega|^{-pk} |\hat{\Psi}(\omega)|^p d\omega \right)^{1/p} + \left( \int_{\mathbb{R}} |\omega|^{-pk} |\hat{\psi}^D_m|^p d\omega \right)^{1/p} \]

\[ := I_1 + I_2. \]

On the other hand,

\[ I := \| \hat{k}(\hat{\psi}^D_m) \|_p = \left( \int_{\mathbb{R}} |\omega|^{-pk} |\hat{\Psi}(\omega) - (\hat{\Psi}(\omega) - \hat{\psi}^D_m)|^p d\omega \right)^{1/p} \]

\[ \geq \left( \int_{\mathbb{R}} |\omega|^{-pk} |\hat{\Psi}(\omega)|^p d\omega \right)^{1/p} - \left( \int_{\mathbb{R}} |\omega|^{-pk} |\hat{\psi}^D_m|^p d\omega \right)^{1/p} \]

\[ := I_1 - I_2. \]
Consider separately

\[ I_1 = \left( \int_{\mathbb{R}} |\omega|^{-pk} |\hat{\Psi}(\omega)|^p \, d\omega \right)^{1/p} \]

\[ = \frac{1}{\sqrt{2\pi}} \left( \int_{-\pi}^{-2\pi} \omega^{-pk} \, d\omega + \int_{\pi}^{2\pi} \omega^{-pk} \, d\omega \right)^{1/p} \]

\[ = (2\pi)^{1/p-1/2} \left( \frac{1 - 2^{1-pk}}{pk - 1} \right)^{1/p}. \]

It is only left to find an upper bound for

\[ I_2 := \left( \int_{\mathbb{R}} |\omega|^{-pk} |\hat{\Psi}(\omega) - \hat{\varphi}_m^D| |\omega|^{-pk} \, d\omega \right)^{1/p}. \]

Fix \( \varepsilon \in (0, \pi) \). Then,

\[ I_2^p = \int_{|\omega|<\varepsilon} |\omega|^{-pk} |\hat{\varphi}_m^D(\omega)|^p \, d\omega + \int_{|\omega|>\varepsilon} |\omega|^{-pk} |\hat{\varphi}_m^D(\omega) - \hat{\Psi}(\omega)|^p \, d\omega := I_{21} + I_{22}. \]

Since \( \hat{\Psi}(\omega) = 0 \) for all \( \omega \in (-\pi, \pi) \), we have

\[ I_{21} := \int_{|\omega|<\varepsilon} |\omega|^{-pk} |\hat{\varphi}_m^D(\omega)|^p \, d\omega \]

\[ = \int_{|\omega|<\varepsilon} |\omega|^{-pk} |\hat{\varphi}_m^D(\omega)|^p \, d\omega \]

\[ \leq \left( \frac{1}{2\pi} \right)^{p/2} \int_{|\omega|<\varepsilon} |\omega|^{-pk} \left| H_m \left( \frac{\omega}{2} + \pi \right) \right|^p \, d\omega \]

\[ \leq \left( \frac{\Gamma(m + 1/2)}{2\pi^{3/2}\Gamma(m)} \right)^{p/2} \int_{|\omega|<\varepsilon} |\omega|^{-pk} \left( \int_0^{|\omega|/2} \sin^{2m-1} t \, dt \right)^{p/2} \, d\omega. \]
Using the fact that \( \left( \frac{|\omega|}{2} \right)^{-1} \sin \left( \frac{|\omega|}{2} \right) \leq 1 \) and that \( k < m \), we obtain
\[
I_{21} \leq 2^{-p(1/2+k)} \left( \frac{\Gamma(m + 1/2)}{2 \pi^{3/2} \Gamma(m)} \right)^{p/2} \int_{|\omega| < \varepsilon} \left( \frac{|\omega|}{2} \right)^{-p(1-k/2)} \left( \sin^{2m-1} \frac{|\omega|}{2} \right)^{p/2} d\omega
\]
\[
\leq 2^{-p(1/2+k)} \left( \frac{\Gamma(m + 1/2)}{2 \pi^{3/2} \Gamma(m)} \right)^{p/2} \int_{|\omega| < \varepsilon} \sin^{p(m-k)} \frac{|\omega|}{2} d\omega
\]
\[
\leq 2^{-p(1/2+k)} \left( \frac{\Gamma(m + 1/2)}{2 \pi^{3/2} \Gamma(m)} \right)^{p/2} (\varepsilon)^{p(m-k)+1}
\]
\[
\leq 2^{-p(1/2+k)} \left( \frac{\Gamma(m + 1/2)}{2 \pi^{3/2} \Gamma(m)} \right)^{p/2} (\pi)^{p(m-k)+1}
\]
\[
= 2^{-p(1/2+2m)+1} \pi^{p(m-k-1/2)+1} \left( \frac{(2m)!}{m!(m-1)!} \right)^{p/2}.
\]

In order to bound \( I_{21} := \int_{|\omega| > \varepsilon} |\omega|^{-pk} |\hat{\psi}_m^D(\omega) - \hat{\Psi}(\omega)|^p d\omega \), we notice that \( \hat{\Psi}(\omega) = 0 \) for all \( |\omega| > 2\pi \). We break \( I_{22} \) into three integrals which we have to bound from above.
\[
I_{21} := \int_{-2\pi < \omega < \pi} |\omega|^{-pk} |\hat{\psi}_m^D(\omega) - \hat{\Psi}(\omega)|^p d\omega
\]
\[
+ \int_{\pi < \omega < 2\pi} |\omega|^{-pk} |\hat{\psi}_m^D(\omega) - \hat{\Psi}(\omega)|^p d\omega
\]
\[
+ \int_{|\omega| > \pi} |\omega|^{-pk} |\hat{\psi}_m^D(\omega) - \hat{\Psi}(\omega)|^p d\omega
\]
\[
= I_{31} + I_{32} + I_{33}.
\]

In order to bound \( I_{33} \) we use result from [7] (Sec. 2.4.26), that there are constants \( \tilde{C}, c > 0 \), such that for any \( \omega > 2\pi \),
\[
|\hat{\psi}_m^D(\omega)| \leq \tilde{C} |\omega|^{-c \log m}.
\]
Thus,
\[
I_{33} \leq (2\pi)^{-cp \log m + 2} \int_{|\omega| > 2\pi} \omega^{-2} d\omega \leq (2\pi)^{-cp \log m + 2}.
\]

Integrals \( I_{31} \) and \( I_{32} \) have the same bound
\[
I_{32} \leq \pi^{-pk} \int_{\pi}^{2\pi} \left| \hat{\psi}_m^D - \frac{1}{\sqrt{2\pi}} \chi[\pi,2\pi] \right|^p d\omega \leq 2^{-p/2} \pi^{1-p(k+1/2)}.
\]
Analogically, \( I_{31} \leq 2^{-p/2} \pi^{1-p(k+1/2)} \).
We obtained the following bound for
\[ I_p^2 \leq 2^{1-p(2m+\frac{3}{2})} \pi^{p(m-k-\frac{1}{2})+1} \left( \frac{(2m)!}{m!(m-1)!} \right)^{p/2} + \frac{1}{(2\pi)^{cp\log m}} + \frac{1}{2^p 2^{p/2-1} \pi^{p(k+\frac{1}{2})}}. \]

\[ \square \]

**Corollary 1.** Let \( \psi_m^D \) be the Daubechies orthonormal wavelet of order \( m \). Then, for \( p \in (1, \infty) \),
\[ (2.6) \quad E \leq \| (\psi_m^D)^\wedge \|_p \leq D, \]
where
\[ D = 2(2\pi)^{1/p-1/2} + 2^{1/2-2m} \pi^{m+1/2} \left( \frac{(2m)!}{m!(m-1)!} \right)^{1/2} + (2\pi)^{2-c\log m} \]
and
\[ E = (2\pi)^{1/p-1/2} - \left[ 2^{-p(1/2+2m)+1} \pi^{p(m-1/2)+1} \left( \frac{(2m)!}{m!(m-1)!} \right)^{p/2} \right. \\
\left. + (2\pi)^{2-cp\log m} + (2\pi)^{1-p/2} \right]^{1/2}. \]

**Corollary 2.** Let \( \psi_m^D \) be the Daubechies orthonormal wavelet of order \( m \). Then, for \( p \in (1, \infty) \),
\[ (2.7) \quad \frac{B}{D} \leq C_{k,p}(\psi_m^D) = \frac{\| \hat{\psi}_m^D \|_p}{\| \psi_m^D \|_p} \leq \frac{A}{E}. \]

We provide now a different method of bounding \( \| (\psi_m^D)^\wedge \|_p \) for special case when \( k = m \). This method gives us more accurate bounds.

**Theorem 2.** Let \( \psi_m^D \) be the Daubechies orthonormal wavelet of order \( m \). Then, for \( p \in (1, \infty), m \) even,
\[ (2.8) \quad G \leq \| m(\psi_m^D)^\wedge \|_p \leq F, \]
where
\[ F^p = \frac{2^{1-p/2}}{\pi^{p(m+1/2)-1}(mp-1)} + \frac{2^{1-p(2m-1)}}{\pi^{p/2-1}(mp-1)!} \sum_{i=0}^{\lfloor mp \rfloor} (-1)^i \frac{(mp)!}{(mp-i)!i!} (mp-2i)^{mp-1} \]
and
\[ G^p = \frac{(1 - o(1))2^{1-2mp}(2m)!}{\pi^{p/2-1}mp/23^{mp}m!(m-1)!}. \]
Proof. The first part of the proof follows along the lines from the asymptotic variant of this theorem in [9] (see Theorem 3 there). We provide details for the reader convenience.

\[ \| \hat{\psi}_m \|_p^p = \int_{\mathbb{R}} |\omega|^{-mp} \left| \hat{\psi}_m(\omega) \right|^p d\omega = \int_{|\omega| \leq \pi} |\omega|^{-mp} \left| \hat{\psi}_m(\omega) \right|^p d\omega + \int_{|\omega| > \pi} |\omega|^{-mp} \left| \hat{\psi}_m(\omega) \right|^p d\omega =: I_1 + I_2. \]

We first estimate \( I_2 \). Since \( |H_m(t)| \leq 1 \)

\[ 0 \leq I_2 \leq \frac{2}{(\sqrt{2\pi})^p} \int_{\pi}^{\infty} \omega^{-mp} d\omega \leq \frac{2}{(\sqrt{2\pi})^p} \cdot \frac{1}{mp - 1} \left( \frac{1}{\pi} \right)^{mp-1}, \quad mp > 1. \]

Consider now \( I_1 \):

\[ I_1 = \frac{1}{(\sqrt{2\pi})^p} \int_{|\omega| \leq \pi} |\omega|^{-mp} \left[ |H_m(\omega/2 + \pi)|^2 |H_m(\omega/4)|^2 |H_m(\omega/8)|^2 \right]^{p/2} \prod_{l=1}^{\infty} \left| H_m(2^{-l-3}\omega) \right|^2 d\omega \]

\[ \geq (1 - o(1)) \frac{1}{(\sqrt{2\pi})^p} \int_{|\omega| \leq \pi} |\omega|^{-mp} |H_m(\omega/2 + \pi)|^p d\omega. \]

Obviously,

\[ I_1 \leq \frac{1}{(\sqrt{2\pi})^p} \int_{|\omega| \leq \pi} |\omega|^{-mp} |H_m(\omega/2 + \pi)|^p d\omega. \]

Now, we use the property of \( H_m \) to deduce the bounds for of

\[ I_{11} := \int_{|\omega| \leq \pi} |\omega|^{-mp} |H_m(\omega/2 + \pi)|^p d\omega. \]

Let \( u = \frac{\sin^2 t}{\sin^2 (\omega/2)} \). We have, with \( c_m = \frac{\Gamma(m + 1/2)}{\sqrt{\pi} \Gamma(m)} = \frac{(2m)! \sqrt{\pi}}{2^{2m} m!(m - 1)!} \)

\[ |H_m(\omega/2 + \pi)|^2 = c_m \int_0^{\omega/2} \sin^{2m-1} t dt \]

\[ = \frac{c_m}{2} \sin^{2m}(\omega/2) \int_0^1 u^{m-1} (1 - u \sin^2 (\omega/2))^{-1/2} du. \]

Since

\[ \frac{1}{m} = \int_0^1 u^{m-1} du \leq \int_0^1 u^{m-1}(1 - u \sin^2 (\omega/2))^{-1/2} du \leq \int_0^1 u^{m-1}(1 - u)^{-1/2} du = c_m^{-1} \]
and

\[ I_{11} = 2 \int_0^\pi |\omega|^{-mp} \left[ \frac{c_m}{2} \sin^{2m}(\omega/2) \int_0^1 u^{m-1}(1 - u \sin^2(\omega/2))^{-1/2}du \right]^{p/2} d\omega, \]

we obtain

\[ \left( \frac{c_m}{2m} \right)^{p/2} 2^{-mp} \int_0^\pi \left( \frac{\sin(\omega/2)}{\omega/2} \right)^{mp} d\omega \leq \frac{1}{2} I_{11} \leq \left( \frac{1}{2} \right)^{-2^{-mp}} \int_0^\pi \left( \frac{\sin(\omega/2)}{\omega/2} \right)^{mp} d\omega. \]

Making change of variables, \( t = \omega/2 \), we obtain:

\[ \left( \frac{c_m}{m} \right)^{p/2} 2^{2-p(m+1/2)} \int_0^\pi \left( \frac{\sin t}{t} \right)^{mp} dt \leq I_{11} \leq 2^{1-p(m-1/2)} \int_0^\pi \left( \frac{\sin t}{t} \right)^{mp} dt. \]

In order to bound \( \int_0^{\pi/2} \left( \frac{\sin t}{t} \right)^{mp} dt \) from above, we make restriction that \( mp \) is even. We have

\[ \int_0^{\pi/2} \left( \frac{\sin t}{t} \right)^{mp} dt \leq \int_0^\infty \left( \frac{\sin t}{t} \right)^{mp} dt = \frac{\pi}{2mp(mp - 1)!} \sum_{i=0}^{\left\lfloor mp/2 \right\rfloor} (-1)^i \binom{mp}{i} (mp - 2i)^{mp-1}, \]

where last formula can be found, for example, in [3] (see example 22, p. 518 there).

Since for all \( x \in \mathbb{R} \), one has \( \frac{\sin t}{t} \geq 1 - \frac{t^2}{6} \) (see e.g., [5] or [6]), we get

\[ \left( \frac{\sin t}{t} \right)^{mp} dt \geq \left( 1 - \frac{t^2}{6} \right)^{mp} \geq \left( 1 - \frac{(\pi/2)^2}{6} \right)^{mp} \geq \left( \frac{1}{3} \right)^{mp} dt = \frac{\pi 3^{-mp}}{2} \]

and

\[ \int_0^{\pi/2} \left( \frac{\sin t}{t} \right)^{mp} dt \geq \int_0^{\pi/2} \left( \frac{1}{3} \right)^{mp} dt = \frac{\pi 3^{-mp}}{2} \]

With that we have obtained bounds for \( I_1 \)

\[ \frac{(1 - o(1)) 2^{1-p(2m+1/2)} (2m)!}{(2\pi)^{p/2} 3^{mp} m^{p/2} m!(m-1)!} \leq I_1^p \]

\[ \leq \frac{2^{1-p(2m-1/2)}}{(2\pi)^{p/2} (mp - 1)!} \sum_{i=0}^{\left\lfloor mp/2 \right\rfloor} \frac{(-1)^i (mp)!}{i!(mp - i)!} (mp - 2i)^{mp-1}, \]

which is completes the proof. \( \square \)

**Corollary 3.** Let \( \psi_m^D \) be the Daubechies orthonormal wavelet of order \( m \). Then, for \( p \in (1, \infty) \), \( mp \) even,

\[ (2.9) \quad G \leq C_{m,p}(\psi_m^D) = \frac{\| \hat{m}(\psi_m^D) \|_p}{\| \psi_m^D \|_p} \leq \frac{F}{E}. \]
REFERENCES

[1] V. F. Babenko and S. A. Spektor, *Estimates for wavelet coefficients on some classes of functions*, Ukrainian Mathematical Journal. Vol. 59, 12 (2007), 1791–1799.

[2] G. Beylkin, R. Coifman and V. Rokhlin, *Fast wavelet transforms and numerical algorithms I*, Comm. Pure Appl. Math., 44 (1991), 141–183.

[3] T.J. Bronwich, *Theory of infinite series*, First Edition 1908, Second Edition 1926, Blackie & Sons, Glasgow.

[4] I. Daubechies, Ten Lectures on Wavelets, SIAM, CBMS Series, 1992.

[5] G. Beylkin, R. Coifman and V. Rokhlin, *Fast wavelet transforms and numerical algorithms I*, Comm. Pure Appl. Math., 44 (1991), 141–183.

[6] I.E. Leonard, J. Duemmel, *Moore - and - moore power series without Taylor’s theorem*, The America Mathematical Monthly, 92, 1985, 588–589.

[7] A.K. Louis, P. Maab and A. Rieder, *Wavelets: Theory and Applications*, Wiley, Chichester, 1997. 141–183.

[8] G. Strang, T. Nguyen *Wavelets and filter banks*, Cambridge Press, Wellesley, 1996.

[9] S. Spektor, X. Zhuang, *Asymptotic Bernstein Type Inequalities and Estimation of Wavelet Coefficients*, Methods and Applications of Analysis, Vol. 19, No. 3 (2012), 289-312.

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