Efficient Simulation for Portfolio Credit Risk in Normal Mixture Copula Models

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Abstract

This paper considers the problem of measuring the credit risk in portfolios of loans, bonds, and other instruments subject to possible default under multi-factor models. Due to the amount of the portfolio, the heterogeneous effect of obligors, and the phenomena that default events are rare and mutually dependent, it is difficult to calculate portfolio credit risk either by means of direct analysis or crude Monte Carlo under such models. To capture the extreme dependence among obligors, we provide an efficient simulation method for multi-factor models with a normal mixture copula that allows the multivariate defaults to have an asymmetric distribution, while most of the literature focuses on simulating one-dimensional cases. To this end, we first propose a general account of an importance sampling algorithm based on an unconventional two-parameter exponential embedding. Note that this innovative tilting device is more suitable for the multivariate normal mixture model than traditional one-parameter tilting methods and is of independent interest. Next, by utilizing a fast computational method for how the rare event occurs and the proposed importance sampling method, we provide an efficient simulation algorithm to estimate the probability that the portfolio incurs large losses under the normal mixture copula. Here the proposed simulation device is based on importance sampling for a joint probability other than the conditional probability used in previous studies. Theoretical investigations and simulation studies, which include an empirical example, are given to illustrate the method.

Subject classifications: portfolio, simulation, variance reduction, importance sampling, portfolio credit risk, Fourier method, Copula models, rare-event simulation.

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1 Introduction

Loss ensuing from the nonfulfillment of an obligor to make required payments, typically referred to as credit risk, is one of the prime concerns of financial institutions. Modern credit
risk management usually takes a portfolio approach to measure and manage this risk, in which the dependence among sources of credit risk (obligors) in the portfolio is modeled. With the potential default of obligors, the portfolio approach evaluates the impact of the dependence among them on the probability of multiple defaults and large losses. Moreover, the default event triggered by an obligor is generally captured via so-called threshold models, in which an obligor defaults when a latent variable associated with this obligor goes beyond (or falls below) a predetermined threshold. This important concept is shared among all models originating from Merton’s credit risk model (Merton, 1974); the latent variable associated with each obligor is modeled using multiple factors occurring as a result of factor analysis, and thus captures common macroeconomic or industry-specific effects.

To model the dependence structure that shapes the multivariate default distribution, copula functions have been widely adopted in the literature (e.g., Li, 2000; Glasserman and Li, 2005; Glasserman et al., 2007, 2008; Bassamboo et al., 2008; Chan and Kroese, 2010). One of the most widely used models is the normal copula model, which assumes that the latent variables follow a multivariate normal distribution and forms the basis of many risk-management systems such as J. P. Morgan’s CreditMetrics (Gupton et al., 1997). In spite of its popularity, some empirical studies suggest that strong dependence exhibited among financial variables is difficult to capture in the normal copula model (Mashal and Zeevi, 2002). Therefore, in view of this limitation, Bassamboo et al. (2008) present a t-copula model derived from the multivariate t-distribution to capture the relatively strong dependence structure of financial variables. Further studies can be found in Chan and Kroese (2010) and Scott and Metzler (2015).

Monte Carlo simulation is the most widely adopted computational technique when modeling the dependence among sources of credit risk; it however converges slowly, especially with low-probability events. To obtain rare-event probabilities more efficiently in simulation, one common approach is to shift the factor mean via importance sampling, a type of variance reduction (e.g., Asmussen and Glynn, 2007; Rubinstein and Kroese, 2011). Various importance sampling techniques for rare-event simulation for credit risk measurement have been studied in the literature (e.g., Glasserman and Li, 2005; Glasserman et al., 2007; Bassamboo et al., 2008; Botev et al., 2013; Scott and Metzler, 2015; Liu, 2015). For example, Glasserman and Li (2005) develop a two-step importance sampling approach for the normal copula model and derive the logarithmic limits for the tail of the loss distribution associated with single-factor homogeneous portfolios. Due to the difficulty of generalizing the approach to the general multi-factor model, Glasserman et al. (2007) derive the logarithmic asymptotics for the loss distribution under the multi-factor normal copula model, the results of which are later utilized in Glasserman et al. (2008) to develop importance sampling techniques to estimate the tail probabilities of large losses, considering different types of obligors. Others focus on importance sampling techniques for the t-copula model; for instance, Bassamboo et al. (2008) present two importance sampling algorithms to estimate the probability of large losses under the assumption of the multivariate t-distribution. Chan and Kroese (2010) propose two simple simulation algorithms based on conditional Monte Carlo, which utilizes the
asymptotic description of how the rare event occurs; later in Scott and Metzler (2015), the authors develop a novel importance sampling algorithm that requires less computational time with only slightly less accurate results than that in Chan and Kroese (2010). However, most proposed importance sampling techniques for both normal copula and \( t \)-copula models choose tilting parameters by minimizing the upper bound of the second moment of the importance sampling estimator based on large deviations theory, and in these papers they only focus on the one-dimensional case.\(^1\)

An alternative for choosing a tilting parameter is based on the criterion of minimizing the variance of the importance sampling estimator directly. For example, Do and Hall (1991) and Fuh and Hu (2004) study efficient simulations for bootstrapping the sampling distribution of statistics of interest. Su and Fu (2000) and Fu and Su (2002) minimize the variance under the original probability measure. Fuh et al. (2011) apply the importance sampling method for value-at-risk (VaR) computation under a multivariate \( t \)-distribution.

Along the line of minimizing the variance of the importance sampling estimator, in this paper we provide an efficient simulation method for the problem in portfolio risk under the normal mixture copula model, which includes the popular normal copula and \( t \)-copula models as special cases. Here, we consider grouped normal mixture copulas in our general model setting. An important idea of the normal mixture model is the inclusion of randomness into the covariance matrix of a multivariate normal distribution via a set of positive mixing variables, which could be interpreted as “shock variables” (see Bassamboo et al., 2008; McNeil et al., 2015) in the context of modeling portfolio credit risk. There have been various empirical and theoretical studies related to normal mixture models in the literature (e.g., Barndorff-Nielsen, 1978, 1997; Eberlein and Keller, 1995); these are popular in financial applications because such models appear to yield a good fit to financial return data and are consistent with continuous-time models (Eberlein and Keller, 1995).

There are two aspects in this study. To begin with, based on the criterion of minimizing the variance of the importance sampling estimator, we propose a general account of the importance sampling algorithm, which is based on an unconventional two-parameter exponential embedding. Here we call it two-parameter tilting because the tilting parameters can be the location and scale parameters in the multivariate normal distribution, for instance. Theoretical investigations and numerical studies are given to support our newly proposed importance sampling method. Note that the innovative tilting formula is more suitable for the grouped normal mixture copula model, and is of independent interest. It is worth mentioning that for normal, multivariate normal, and Gamma distributions, our simulation study shows that two-parameter exponential tilting performs 2 to 5 times better than the classical one-parameter exponential tilting for some simple rare event simulations. In particular, when applying two-parameter tilting in the normal mixture models, the tilting parameter can be either the shape or the rate parameter for the underlying Gamma distribution, which results

\(^1\)Although some papers have mentioned that they can handle \( d \)-factor models, the factors follow i.i.d. Gaussian variables and thus can be reduced to the one-dimensional case via Cholesky decomposition.
in a more efficient simulation.

Next, by utilizing a fast computational method for how the rare event occurs and the proposed importance sampling method, we provide an efficient simulation algorithm for a multi-factor model with the normal mixture copula model to estimate the probability that the portfolio incurs large losses. To be more precise, in this stage, we use the inverse Fourier transform to handle the distribution of total losses, i.e., the sum of \( n \) “weighted” independent, but non-identically distributed Bernoulli random variables. An automatic variant of Newton’s method is introduced to determine the optimal tilting parameters. Note that the proposed simulation device is based on importance sampling for a joint probability other than the conditional probability used in previous studies. Finally, to illustrate the applicability of our method, we provide numerical results of the proposed algorithm under various copula settings, and highlight some insights of the trade-off between the reduced variances and increased computational time. Moreover, we also give an empirical example which contains a set of parameters of a multi-factor model calibrated from data of the CDXIG index. In particular, our contributions are summarized below:

1. We propose a general account of an importance sampling algorithm based on an innovative two-parameter tilting device. Our proposed two-parameter exponential embedding essentially differs from the one-parameter tilting methods commonly adopted in the literature for both normal copula or \( t \)-copula models and is more suitable for the grouped normal mixture copula model. Theoretical investigations and numerical studies are given to support our method.

2. Based on the proposed two-parameter tilting method, an efficient simulation method is presented for multi-factor models with a normal mixture copula. Note that the proposed algorithm is a multi-dimensional method, whereas most of the previous literature focuses on simulating one-dimensional cases.

3. Extensive simulation studies attest the capability and performance of the proposed method. The relation between variance reduction factors and time consumption ratios suggests that the proposed algorithm achieves good performance and thus makes a practical contribution to measuring portfolio credit risk in normal mixture copula models.

The remainder of this paper is organized as follows. Section 2 formulates the problem of estimating large portfolio losses and presents the normal mixture copula model. Section 3 presents a general account of importance sampling based on the two-parameter exponential embedding. We then study the proposed optimal importance sampling for portfolio loss under the normal mixture copula model in Section 4. The performance of our methods is demonstrated with an extensive simulation study and an empirical example in Section 5 and Section 6, respectively. Section 7 concludes. The proofs are deferred to the Appendix.
2 Portfolio loss under the normal mixture copula model

Consider a portfolio of loans consisting of \( n \) obligors, each of whom has a small probability of default. We further assume that the loss resulting from the default of the \( k \)-th obligor, denoted as \( c_k \) (monetary units), is known. In copula-based credit models, dependence among default indicator for each obligor is introduced through a vector of latent variables \( X = (X_1, \cdots, X_n) \), where the \( k \)-th obligor defaults if \( X_k \) exceeds some chosen threshold \( \chi_k \). The total loss from defaults is then denoted by

\[
L_n = c_1 \mathbb{1}_{\{X_1 > \chi_1\}} + \cdots + c_n \mathbb{1}_{\{X_n > \chi_n\}},
\]

where \( \mathbb{1} \) is the indicator function. Particularly, the problem of interest is to estimate the probability of losses, \( P(L_n > \tau) \), especially at large values of \( \tau \).

As mentioned earlier in the introduction, the widely-used normal copula model might assign an inadequate probability to the event of many simultaneous defaults in a portfolio. In view of this, Bassamboo et al. (2008); Chan and Kroese (2010) set forth the \( t \)-copula model for modeling portfolio credit risk. In this paper, we further consider the normal mixture model (McNeil et al., 2015), including normal copula and \( t \)-copula model as special cases, for the generalized \( d \)-factor model of the form

\[
X_k = \rho_{k1} \sqrt{W_1} Z_1 + \cdots + \rho_{kd} \sqrt{W_d} Z_d + \rho_k \sqrt{W_{d+1}} \varepsilon_k, \quad k = 1, \ldots, n,
\]

in which

- \( Z = (Z_1, \ldots, Z_d)^\top \) follows a \( d \)-dimensional multivariate normal distribution with zero mean and covariance matrix \( \Sigma \), where \( \top \) denotes vector transpose;
- \( W = (W_1, \ldots, W_d, W_{d+1}) \) are non-negative scalar-valued random variables which are independent of \( Z \), and each \( W_j \) is a shock variable and independent from each other, for \( j = 1, \ldots, d + 1 \);
- \( \varepsilon_k \sim N(0, \sigma_k^2) \) is an idiosyncratic risk associated with the \( k \)-th obligor, for \( k = 1, \cdots, n \);
- \( \rho_{k1}, \ldots, \rho_{kd} \) are the factor loadings for the \( k \)-th obligor, and \( \rho_{k1}^2 + \cdots + \rho_{kd}^2 \leq 1 \);
- \( \rho_k = \sqrt{1 - (\rho_{k1}^2 + \cdots + \rho_{kd}^2)} \), for \( k = 1, \cdots, n \).

Model (2) is the so-called grouped normal mixture copula in McNeil et al. (2015). The above distributions are known as variance mixture models, which are constructed by drawing randomly from this set of component multivariate normals based on a set of “weights” controlled by the distribution of \( W \). Such distributions enable us to blend in multiplicative shocks via the variables \( W \), which could be interpreted as shocks that arise from new information. Note that with \( W_j^{-1} = Q_j / \nu_j \) and \( Q_j \sim \text{Gamma}(\nu_j/2, 1/2) \) (for \( j = 1, \ldots, d + 1 \)), the vector of latent variables, \( X = (X_1, \ldots, X_n) \), forms an \( n \)-dimensional multivariate \( t \)-distribution. In addition, we consider the case that \( W_j \), \( j = 1, \ldots, d + 1 \), in Equation (2)
to have a generalized inverse Gaussian (GIG) distribution. The GIG mixing distribution, a special case of the symmetric generalized hyperbolic (GH) distribution, is very flexible for modeling financial returns. Moreover, GH distributions also include the symmetric normal inverse Gaussian (NIG) distribution and a symmetric multivariate distribution with hyperbolic distribution for its one-dimensional marginal as interesting examples. Note that this class of distributions has become popular in the financial modeling literature. An important reason is their link to Lévy processes (like Brownian motion or the compound Poisson distribution) that are used to model price processes in continuous time. For example, the generalized hyperbolic distribution has been used to model financial returns in Eberlein and Keller (1995); Eberlein et al. (1998). The reader is referred to (McNeil et al., 2015) for more details.

We now define the tail probability of total portfolio losses conditional on $Z$ and $W$. Specifically, the tail probability of total portfolio losses conditional on the factors $Z$ and $W$, denoted as $\varrho(Z,W)$ is defined as

$$\varrho(Z,W) = P(L_n > \tau|(Z,W)).$$  \hfill (3)$$

The desired probability of losses can be represented as

$$P(L_n > \tau) = E[\varrho(Z,W)].$$  \hfill (4)$$

For an efficient Monte Carlo simulation of the probability of total portfolio losses (4), we apply importance sampling to the distributions of the factors $Z = (Z_1, \ldots, Z_d)^\top$ and $W = (W_1, \ldots, W_{d+1})^\top$ (see Equation (2)). In other words, we attempt to choose importance sampling distributions for both $Z$ and $W$ that reduce the variance in estimating the integral $E[\varrho(Z,W)]$ against the original densities of $Z$ and $W$.

As noted in Glasserman and Li (2005) for normal copula models, the simulation of (4) involves two rare events: the default event and the total portfolio loss event. For the normal mixture model (2), this makes the simulation of (4) even more challenging. For a general simulation algorithm for this type of problems, we simulate $P(L_n > \tau)$ as expected value of $\varrho(Z,W)$ in (4). Our device is based on a joint probability simulation rather than the conditional probability simulation considered in the literature. Moreover, we note that the simulated distributions – the multivariate normal distribution $Z^2$ and the commonly adopted Gamma distribution for $W$ – are both two-parameter distributions. This motivates us to study a two-parameter exponential tilting in the next section.

## 3 A general account of importance sampling

Let $(\Omega, \mathcal{F}, P)$ be a given probability space. Let $X = (X_1, \ldots, X_d)^\top$ be a $d$-dimensional random vector having $f(x) = f(x_1, \ldots, x_d)$ as a probability density function (pdf), with

\footnote{Here we treat the mean vector and variance-covariance matrix as two parameters.}
respect to the Lebesgue measure $\mathcal{L}$, under the probability measure $P$. Let $\varphi(\cdot)$ be a real-valued function from $\mathbb{R}^d$ to $\mathbb{R}$. The problem of interest is to calculate the expectation of $\varphi(X)$,

$$ m = E_P[\varphi(X)], \quad (5) $$

where $E_P[\cdot]$ is the expectation operator under the probability measure $P$.

To calculate the value of (5) using importance sampling, one selects a sampling probability measure $Q$ under which $X$ has a pdf $q(x) = q(x_1, \ldots, x_d)$ with respect to the Lebesgue measure $\mathcal{L}$. The probability measure $Q$ is assumed to be absolutely continuous with respect to the original probability measure $P$. Therefore, Equation (5) can be written as

$$ \int_{\mathbb{R}^d} \varphi(x)f(x)dx = \int_{\mathbb{R}^d} \varphi(x)\frac{f(x)}{q(x)}q(x)dx = E_Q \left[ \varphi(X)\frac{f(X)}{q(X)} \right], \quad (6) $$

where $E_Q[\cdot]$ is the expectation operator under which $X$ has a pdf $q(x)$ with respect to the Lebesgue measure $\mathcal{L}$. The ratio $f(x)/q(x)$ is called the importance sampling weight, the likelihood ratio, or the Radon-Nikodym derivative.

Here, we focus on the exponentially tilted probability measure of $P$. Instead of considering the commonly adopted one-parameter exponential tilting appeared in the literature (Asmussen and Glynn, 2007), we propose two-parameter exponential tilting. To the best of our knowledge, the use of two-parameter exponential embedding seems to be new in the literature. As will be seen in the following examples, the tilting probabilities for existent two-parameter distributions, such as Gamma distribution and normal distribution, can be obtained by solving simple formulas.

Let $Q_{\theta,\eta}$ be the tilting probability measure, where the subscript $\theta = (\theta_1, \ldots, \theta_p)^T \in \Theta \subset \mathbb{R}^p$ and $\eta = (\eta_1, \ldots, \eta_q)^T \in H \subset \mathbb{R}^q$ are the tilting parameters. Here $p$ and $q$ denote the number of parameters in $\Theta$ and $H$, respectively. Assume that the moment-generating function of $(X, h(X))$ exists and is denoted by $\Psi(\theta, \eta)$. Let $f_{\theta,\eta}(x)$ be the pdf of $X$ under the exponentially tilted probability measure $Q_{\theta,\eta}$, defined by

$$ f_{\theta,\eta}(x) = \frac{e^{\theta^T x + \eta^T h(x)}}{\Psi(\theta, \eta)} f(x) = e^{\theta^T x + \eta^T h(x) - \psi(\theta, \eta)} f(x), \quad (7) $$

where $\psi(\theta, \eta) = \ln \Psi(\theta, \eta)$ is the cumulant function and $h(\cdot)$ is a function from $\mathbb{R}^d$ to $\mathbb{R}^q$. Note that in (7), we present one type of parameterization which is suitable for our derivation. Explicit representations of $h(x)$ for specific distributions are given in the following examples, which include the normal and multivariate normal distributions, and the Gamma distribution.

Consider two-parameter exponential embedding. Equation (6) becomes

$$ \int_{\mathbb{R}^d} \varphi(x)f(x)dx = \int_{\mathbb{R}^d} \varphi(x)\frac{f(x)}{f_{\theta,\eta}(x)}f_{\theta,\eta}(x)dx = E_{Q_{\theta,\eta}} \left[ \varphi(X)e^{-(\theta^T X + \eta^T h(X)) + \psi(\theta, \eta)} \right]. $$

Because of the unbiasedness of the importance sampling estimator, its variance is

$$ Var_{Q_{\theta,\eta}} \left[ \varphi(X)e^{-(\theta^T X + \eta^T h(X)) + \psi(\theta, \eta)} \right] = E_{Q_{\theta,\eta}} \left[ \left( \varphi(X)e^{-(\theta^T X + \eta^T h(X)) + \psi(\theta, \eta)} \right)^2 \right] - m^2, \quad (8) $$
where \( m = \int_{\mathbb{R}^d} \phi(x)f(x)dx \). For simplicity, we assume that the variance of the importance sampling estimator exists.

Define the first term of the right-hand side (RHS) of (8) by \( G(\theta, \eta) \). Then minimizing \( \text{Var}_{Q_{\theta, \eta}} \left[ \psi(X)e^{-\langle \theta X + \eta h(X) \rangle + \psi(\theta, \eta)} \right] \) is equivalent to minimizing \( G(\theta, \eta) \). Standard algebra gives a simpler form of \( G(\theta, \eta) \):

\[
G(\theta, \eta) := E_{Q_{\theta, \eta}} \left[ \left( \psi(X)e^{-\langle \theta X + \eta h(X) \rangle + \psi(\theta, \eta)} \right)^2 \right] = E_P \left[ \psi^2(X)e^{-\langle \theta X + \eta h(X) \rangle + \psi(\theta, \eta)} \right],
\]

which is used to find the optimal tilting parameters. In the following theorem, we show that \( G(\theta, \eta) \) in (9) is a convex function in \( \theta \) and \( \eta \). This property ensures that there exists no multi-mode problem in the search stage when determining the optimal tilting parameters.

To minimize \( G(\theta, \eta) \), the first-order condition requires the solution of \( \theta^*, \eta^* \), to satisfy \( \nabla_{\theta} G(\theta, \eta) |_{\theta=\theta^*} = 0 \), and \( \nabla_{\eta} G(\theta, \eta) |_{\eta=\eta^*} = 0 \), where \( \nabla_{\theta} \) denotes the gradient with respect to \( \theta \) and \( \nabla_{\eta} \) denotes the gradient with respect to \( \eta \). Simple calculation yields

\[
\nabla_{\theta} G(\theta, \eta) = E_P \left[ \psi^2(X)(-X + \nabla_{\theta} \psi(\theta, \eta))e^{-\langle \theta X + \eta h(X) \rangle + \psi(\theta, \eta)} \right],
\]

\[
\nabla_{\eta} G(\theta, \eta) = E_P \left[ \psi^2(X)(-h(X) + \nabla_{\eta} \psi(\theta, \eta))e^{-\langle \theta X + \eta h(X) \rangle + \psi(\theta, \eta)} \right],
\]

and, therefore, \( (\theta^*, \eta^*) \) is the root of the following system of nonlinear equations,

\[
\nabla_{\theta} \psi(\theta, \eta) = \frac{E_P \left[ \psi^2(X)Xe^{\langle \theta X + \eta h(X) \rangle} \right]}{E_P \left[ \psi^2(X)e^{\langle \theta X + \eta h(X) \rangle} \right]}, \tag{10}
\]

\[
\nabla_{\eta} \psi(\theta, \eta) = \frac{E_P \left[ \psi^2(X)h(X)e^{\langle \theta X + \eta h(X) \rangle} \right]}{E_P \left[ \psi^2(X)e^{\langle \theta X + \eta h(X) \rangle} \right]]. \tag{11}
\]

To simplify the RHS of (10) and (11), we define the conjugate measure \( \tilde{Q}_{\theta, \eta} := \tilde{Q}_{\theta, \eta}^\circ \) of the measure \( Q \) with respect to the payoff function \( \psi \) as

\[
\frac{d\tilde{Q}_{\theta, \eta}}{dP} = \frac{\psi^2(X)e^{-\langle \theta X + \eta h(X) \rangle}}{E_P[\psi^2(X)e^{\langle \theta X + \eta h(X) \rangle}]} = \psi^2(X)e^{\langle \theta X + \eta h(X) \rangle - \tilde{\psi}(\theta, \eta)}, \tag{12}
\]

where \( \tilde{\psi}(\theta, \eta) \) is \( \log \tilde{\Psi}(\theta, \eta) \) with \( \tilde{\Psi}(\theta, \eta) = E_P[\psi^2(X)e^{\langle \theta X + \eta h(X) \rangle}] \). Then the RHS of (10) equals \( E_{Q_{\theta, \eta}}[X] \), the expected value of \( X \) under \( Q_{\theta, \eta} \), and the RHS of (11) equals \( E_{Q_{\theta, \eta}}[h(X)] \), the expected value of \( h(X) \) under \( Q_{\theta, \eta} \).

The following theorem establishes the existence, uniqueness, and characterization for the minimizer of (9). Before that, to ensure the finiteness of the moment-generating function \( \tilde{\Psi}(\theta, \eta) \), we add a condition that \( \tilde{\Psi}(\theta, \eta) \) is steep, cf. Asmussen and Glynn (2007). To define steepness, let \( \theta^- = (\theta_1, \cdots, \theta_i, \cdots, \theta_p) \in \Theta \) such that all \( \theta_k \) fixed for \( k = 1, \cdots, i-1, i+1, \cdots, p \) except the \( i \)-th component. Denote \( \eta^j \in H \) in the same way for \( j = 1, \cdots, q \). Now, let \( \theta_{i,\text{max}} := \sup\{\theta_i : \tilde{\Psi}(\theta^- i, \eta^j) < \infty\} \) for \( i = 1, \cdots, p \), and \( \eta_{j,\text{max}} := \sup\{\eta_j : \tilde{\Psi}(\theta^i, \eta^- j) < \infty\} \) for \( j = 1, \cdots, q \) (for light-tailed distributions, we have \( 0 < \theta_{i,\text{max}} \leq \infty \) for \( i = 1, \cdots, p \), and \( 0 < \eta_{j,\text{max}} \leq \infty \) for \( j = 1, \cdots, q \)). Then steepness means \( \tilde{\Psi}(\theta, \eta) \to \infty \) as \( \theta_i \to \theta_{i,\text{max}} \) for \( i = 1, \cdots, p \), or \( \eta_j \to \eta_{j,\text{max}} \) for \( j = 1, \cdots, q \).
The following conditions are used in Theorem 3.1.

i) \( \bar{\Psi}(\theta, \eta) \Psi(\theta, \eta) \to \infty \) as \( \theta_i \to \theta_{i,\max} \) for \( i = 1, \cdots, p \), or \( \eta_j \to \eta_{j,\max} \) for \( j = 1, \cdots, q \);

ii) \( G(\theta, \eta) \) is a continuous differentiable function on \( \Theta \times H \), and

\[
\sup_{i=1,\cdots,p, \ j=1,\cdots,q} \left\{ \lim_{\theta_i \to \theta_{i,\max}} \frac{\partial G(\theta, \eta)}{\partial \theta_i}, \lim_{\eta_j \to \eta_{j,\max}} \frac{\partial G(\theta, \eta)}{\partial \eta_j} \right\} > 0. \tag{13}
\]

Note that condition i) or ii) is used to guarantee the existence of the minimum point. More details can be found in the Appendix.

**Theorem 3.1** Suppose the moment-generating function \( \Psi(\theta, \eta) \) of \( (X, h(X)) \) exists for \( \theta \in \Theta \subset \mathbb{R}^p \) and \( \eta \in H \subset \mathbb{R}^q \). Assume \( \Psi(\theta, \eta) \) is steep. Furthermore, assume either i) or ii) holds. Then \( G(\theta, \eta) \) defined in (9) is a convex function in \((\theta, \eta)\), and there exists a unique minimizer of (9), which satisfies

\[
\nabla_{\theta} \psi(\theta, \eta) = E_{\bar{Q}_{\theta, \eta}}[X], \tag{14}
\]

\[
\nabla_{\eta} \psi(\theta, \eta) = E_{\bar{Q}_{\theta, \eta}}[h(X)]. \tag{15}
\]

The proof of Theorem 3.1 is given in the Appendix.

To illustrate the proposed two-parameter exponential tilting, we present two examples: multivariate normal and Gamma distributions. We choose these two distributions to indicate the location and scale properties of the exponential tilting used in our general framework. In these examples, by using a suitable re-parameterization, we obtain neat tilting formulas for each distribution. Our simulation studies also show that two-parameter exponential tilting performs 2 to 5 times better than the classical one-parameter exponential tilting for some simple rare events.

We here check the validity of applying Theorem 3.1 for each example. First, we note that both multivariate normal distribution and Gamma distribution are steep. Next, it is easy to see that the sufficient conditions \( \bar{\Psi}(\theta) \Psi(\theta) \to \infty \) as \( \theta_i \to \theta_{i,\max} \) for \( i = 1, \cdots, p \), or \( \eta_j \to \eta_{j,\max} \) for \( j = 1, \cdots, q \) in Theorem 3.1 hold in each example. For example, when \( X \sim N_d(0, \Sigma) \), then \( \Psi(\theta) = O(e^{\|\theta\|^2}) \) approaches \( \infty \) sufficiently quickly. Another simple example illustrated here is when \( \varphi(X) = 1_{\{X \in A\}} \) and \( A := [a_1, \infty) \times \cdots \times [a_d, \infty) \), with \( a_i > 0 \) for all \( i = 1, \cdots, d \), and \( X \) has a \( d \)-dimensional standard normal distribution, then it is easy to verify that the sufficient conditions in Theorem 3.1 hold.

**Example 3.2** Multivariate normal distribution.

To illustrate the concept of two-parameter exponential embedding, we first consider a one-dimensional normal distributed random variable as an example. Let \( X \) be a random variable with the standard normal distribution, denoted by \( N(0, 1) \), with probability density function (pdf) \( \frac{dP}{d\mathcal{L}} = e^{-x^2/2}/\sqrt{2\pi} \). By using the two-parameter exponential embedding in (7) with \( h(x) := x^2 \), we have

\[
\frac{dQ_{\theta, \eta}}{dP}/d\mathcal{L} = \frac{\exp\{\theta x + \eta h(x)\}}{E[\exp\{\theta X + \eta h(X)\}]} = \sqrt{1 - 2\eta}\exp\{\eta x^2 + \theta x - \theta^2/(2 - 4\eta)\}. \tag{16}
\]
In this case, the tilting probability measure $Q_{\theta, \eta}$ is $N(\theta/(1-2\eta), 1/(1-2\eta))$, with $\eta < 1/2$, a location-scale family.

For the event of $\phi(X) = \mathbb{1}_{\{X > a\}}$ for $a > 0$, define $\bar{Q}_{\theta, \eta}$ as $N(-\theta/(1+2\eta), 1/(1+2\eta))$ with $\eta > -1/2$. For easy presentation, we consider the standard parameterization by letting $\mu := \theta/(1-2\eta)$, $\sigma^2 = 1/(1-2\eta)$ and define $\bar{Q}_{\mu, \sigma^2}$ as $N(-\mu/(2\sigma^2 - 1), \sigma^2/(2\sigma^2 - 1))$ with $\sigma^2 > 1/2$. Applying Theorem 3.1, $(\mu^*, \sigma^*)$ is the root of

$$
\mu = E_{\bar{Q}_{\mu, \sigma^2}}[X | X > a] \quad \text{and} \quad \sigma^2 + \mu^2 = E_{\bar{Q}_{\mu, \sigma^2}}[X^2 | X > a].
$$

(17)

Take the one-parameter exponential embedding case, with $\sigma$ fixed. Standard calculation gives $\Psi(\theta) = e^{\theta^2/2}$, $\psi(\theta) = \theta^2/2$ and $\psi'(\theta) = \theta$. Using the fact that $X | \{X > a\}$ is a truncated normal distribution with minimum value $a$ under $\bar{Q}$, $\theta^*$ must satisfy $\theta = \frac{\phi(a+\theta)}{1 - \Phi(\theta + \theta)} - \theta$, cf. Fuh and Hu (2004).

Table 1 presents numerical results for the normal distribution. As demonstrated in the table, using two-parameter exponential tilting for the simple event, $\mathbb{1}_{\{X > a\}}$, yields performance 2 to 5 times better than the one-parameter exponential tilting in terms of variance reduction factors.

| $X \sim N(0, 1)$ | $\{X > a\}$ | Crude for $P(X > a)$ | $\mu^*$ | $\sigma^*$ | $(\mu^*, \sigma^*)$ |
|-------------------|--------------|----------------------|--------|--------|-------------------|
| 1                 | $1.566 \times 10^{-1}$ | 5 2 12       |        |        |                   |
| 2                 | $2.300 \times 10^{-2}$ | 19 4 60      |        |        |                   |
| 3                 | $1.370 \times 10^{-3}$ | 222 35 617   |        |        |                   |
| 4                 | $3.000 \times 10^{-5}$ | $7,094 \times 860$ | 32,552 |        |                   |

Table 1: Two-parameter importance sampling for normal distribution

We now proceed to a $d$-dimensional multivariate normal distribution. Let $X = (X_1, \ldots, X_d)^\top$ be a random vector with the standard multivariate normal distribution, denoted by $N(0, \mathbb{I})$, with pdf $\det(2\pi \mathbb{I})^{-1/2} e^{-(1/2)x\mathbb{I}^{-1}x}$, where $\mathbb{I}$ is the identity matrix. By using the two-parameter exponential embedding in (7), we have

$$
\frac{dQ_{\theta, \eta}}{d\mathbb{P}} = \frac{\exp\{\theta^\top x + X^\top Mx\}}{E[\exp\{\theta^\top x + X^\top Mx\}]} \frac{e^{\theta^\top x + x^\top Mx - \frac{1}{2}(\theta^\top (\mathbb{I} - 2M)^{-1}\theta)}}{\sqrt{|(\mathbb{I} - 2M)^{-1}|}},
$$

(18)

where $| \cdot |$ denotes the determinant of a matrix, and

$$
M = (a_{ij}) \in \mathbb{R}^{d \times d},
$$

with $a_{ij} = \eta_i$ for $i = j$ and $a_{ij} = \eta_{i+1}$ for $i \neq j$. In this case, the tilting probability measure $Q_{\theta, \eta}$ is $N((\mathbb{I} - 2M)^{-1}\theta, (\mathbb{I} - 2M)^{-1})$.

For the event of $\phi(X) = \mathbb{1}_{\{X \in A\}}$, define $\bar{Q}_{\theta, \eta}$ as $N((\mathbb{I} + 2M)^{-1}(-\theta), (\mathbb{I} + 2M)^{-1})$. Similar to the above one-dimensional normal distribution, we consider the standard parameterization by
letting $\mu := (I - 2M)^{-1}\theta$, $\Sigma := (I - 2M)^{-1}$, and define $\tilde{Q}_{\mu,\Sigma}$ as $N(-(2I - \Sigma)^{-1}(\Sigma^{-1})\mu, (2I - \Sigma^{-1})^{-1})$. Applying Theorem 3.1, $(\mu^*, \Sigma^*)$ is the root of

$$
\mu = E_{\tilde{Q}_{\mu,\Sigma}}[X|X \in A],
$$

$$
K(\mu, \Sigma) = E_{\tilde{Q}_{\mu,\Sigma}}[X^T(\nabla_{\eta} M)X|X \in A] \quad \text{for } i = 1, 2, \cdots, d + 1,
$$

where

$$
\nabla_{\eta} M = (b_{jk}) \in \mathbb{R}^{d \times d} \quad \text{for } i = 1, 2, \cdots, d + 1,
$$

$$
K(\mu, \Sigma) = \frac{1}{2} \text{Tr}(-((\nabla_{\eta} \Sigma^{-1})(\Sigma)) - \frac{1}{2} \mu^T(\nabla_{\eta} \Sigma^{-1})\mu.
$$

Here in Equation (21), Tr($A$) is the trace of matrix $A$; the value of $b_{jk}$ is defined as follows: for each $i = 1, \cdots, d$,

$$
b_{jk} = \begin{cases} 
1, & \text{if } i = j = k, \\
0, & \text{otherwise},
\end{cases}
$$

and for $i = d + 1$,

$$
b_{jk} = \begin{cases} 
1, & \text{if } i \neq k, \\
0, & \text{otherwise}.
\end{cases}
$$

Remark 1 The left-hand sides (LHSs) of (19) and (20) are derivatives of the cumulant function $\psi_{MN}(\theta, \eta) := \log(\sqrt{|(I - 2M)^{-1}|}) + \frac{1}{2}(\theta^T(\mathbb{1} - 2M)^{-1}\theta)$ for a multivariate normal with respect to the parameters $\theta$ and $\eta$, respectively. Note that in the RHS of (22), Jacobi’s formula is adopted for the derivative of the determinant of the matrix $(I - 2M)^{-1}$ and $\nabla_{\eta} \Sigma^{-1}$ is

$$
\nabla_{\eta} \Sigma^{-1} = \nabla_{\eta} (I - 2M) = (m_{jk}) \in \mathbb{R}^{d \times d} \quad \text{for } i = 1, 2, \cdots, d + 1,
$$

where for each $i = 1, \cdots, d$,

$$
m_{jk} = \begin{cases} 
-2, & \text{if } i = j = k, \\
0, & \text{otherwise},
\end{cases}
$$

and for $i = d + 1$,

$$
m_{jk} = \begin{cases} 
-2, & \text{if } i \neq k, \\
0, & \text{otherwise}.
\end{cases}
$$

Table 2 presents the numerical results for the standard bivariate normal distribution. As shown in the table, for three types of events $\mathbb{1}_{\{X_1 + X_2 > a\}}, \mathbb{1}_{\{X_1 > a, X_2 > a\}},$ and $\mathbb{1}_{\{X_1 X_2 > a, X_1 > 0, X_2 > 0\}}$, tilting different parameters results in different performance in variance reduction. Although sometimes tilting the variance parameter or the correlation parameter alone provides poor performance, combining them to the mean parameter tilting can yield 2 to 3 times better performance than one-parameter exponential tilting.
Example 3.3 Gamma distribution.

Let \( X \) be a random variable with a Gamma distribution, denoted by \( \text{Gamma}(\alpha, \beta) \), with pdf
\[
\frac{dP}{dL} = \left( \frac{\beta^\alpha}{\Gamma(\alpha)} \right) x^{\alpha-1} e^{-\beta x}.
\]
By using the two-parameter exponential embedding in (7) with \( h(x) := -\ln(\beta x) \), we have
\[
\frac{dQ_{\theta, \eta}}{dP} = dP \frac{\exp\{\theta x + \eta h(x)\}}{E[\exp\{\theta X + \eta h(X)\}]} = e^{\theta - \eta \ln(\beta x)} \frac{\Gamma(\alpha)}{1/\beta - \alpha(1/\beta - \theta) - \alpha + \eta \Gamma(\alpha - \eta)}.
\]
(23)

In this case, the tilting probability measure \( Q_{\theta, \eta} \) is \( \text{Gamma}(\alpha - \eta, \beta - \theta) \). For the event of \( \omega(X) = 1_{\{X > a\}} \) for \( a > 0 \), define \( \tilde{Q}_{\theta, \eta} \) as \( \text{Gamma}(\alpha + \eta, \beta + \theta) \). Applying Theorem 3.1, \( (\theta^*, \eta^*) \) is the root of
\[
\frac{\alpha - \eta}{\beta - \theta} = E_{\tilde{Q}_{\theta, \eta}} [X | X > a] \quad (24)
\]
\[
\ln \beta - \ln(\beta - \theta) + \Upsilon(\alpha - \eta) = E_{\tilde{Q}_{\theta, \eta}} [\ln(\beta X) | X > a], \quad (25)
\]
where \( \Upsilon \) is a digamma function equal to \( \Gamma'(\alpha - \eta)/\Gamma(\alpha - \eta) \).

Table 3 presents the numerical results for the Gamma distribution. Note that the commonly used one-parameter exponential tilting involves a change for the parameter \( \beta \) only (i.e., changing \( \beta \) to \( \beta - \theta^* \)) in the case of the Gamma distribution. However, observe that tilting the other parameter \( \alpha \) (i.e., \( \alpha \rightarrow \alpha - \eta^* \)) for some cases yields 2 to 3 times better performance than the one-parameter exponential tilting in terms of variance reduction factors. This is due to the constraint \( \eta < \alpha \) and \( \theta < \beta \). For instance, consider the case in which we only tilt one parameter, either \( \theta \) or \( \eta \) as follows. For the simple event \( 1_{\{X > a\}} \), we can either choose a parameter \( \theta \) such that \( 0 < \theta < \beta \) or a parameter \( \eta \) such that \( \eta < 0 \) to obtain a larger mean for the tilted Gamma distribution. In this case, it is easy to see that \( \eta \)-tilting yields a larger search space for parameters and thus achieves better performance than \( \theta \)-tilting. Note that event \( 1_{\{1/X > a\}} \) shows the opposite effect.

| \( X \sim N_2(0, I) \) | \( k \) | Crude | \( \mu^* \) | \( \sigma^* \) | \( \rho^* \) | \( (\mu^*, \sigma^*, \rho^*) \) |
|---|---|---|---|---|---|---|
| \( P(X_1 + X_2 > a) \) | 3 | \( 1.663 \times 10^{-2} \) | 24 | 2 | 2 | 43 |
| 4 | \( 2.400 \times 10^{-3} \) | 138 | 4 | 2 | 354 |
| 5 | \( 1.800 \times 10^{-4} \) | 1,064 | 8 | 4 | 4,036 |
| \( P(X_1 > a, X_2 > a) \) | 1 | \( 2.532 \times 10^{-2} \) | 9 | 1 | 2 | 16 |
| 1.5 | \( 4.600 \times 10^{-3} \) | 34 | 2 | 7 | 68 |
| 2 | \( 5.800 \times 10^{-4} \) | 227 | 5 | 18 | 504 |
| \( P(X_1X_2 > a, X_1 > 0, X_2 > 0) \) | 2 | \( 1.538 \times 10^{-2} \) | 21 | 2 | 3 | 46 |
| 3 | \( 4.800 \times 10^{-3} \) | 57 | 2 | 3 | 145 |
| 5 | \( 5.600 \times 10^{-4} \) | 425 | 5 | 3 | 1,213 |

Table 2: Two-parameter importance sampling for standard bivariate normal distribution.
4 Importance sampling for portfolio loss

For easy presentation, this section is divided into three parts. We first introduce two-parameter exponential tilting for normal mixture distributions in Section 4.1. Section 4.2 provides importance sampling of \( \varrho(Z,W) \) defined in (3). Section 4.3 summarizes the importance sampling algorithm for calculating the probability of the portfolio loss \( P(L_n > \tau) \).

## 4.1 Two-parameter tilting for normal mixture distributions

Recall that in Equation (2), the latent random vector \( X \) follows a multivariate normal mixture distribution. In this section, for simplicity, we consider a one-dimensional normal mixture distribution as an example to demonstrate the proposed two-parameter exponential tilting. Let \( X \) be a one-dimensional normal mixture random variable with only one factor (i.e., \( d = 1 \)) such that

\[
X = \xi \sqrt{W}Z, \tag{26}
\]

where \( \xi \in \mathbb{R}, Z \sim N(0,1) \), and \( W \) is a non-negative and scalar-valued random variable which is independent of \( Z \). Since the random variable \( W \) is independent of \( Z \), by using the two-parameter exponential embedding with \( h(z) \) for \( Z \) and \( h(w) \) for \( W \), we have

\[
\frac{dQ_{\theta_1, \eta_1, \theta_2, \eta_2}}{dP} \frac{d\mathcal{L}}{d\mathcal{L}} = \frac{\exp\{\theta_1 z + \eta_1 h(z)\}}{E[\exp\{\theta_1 Z + \eta_1 h(Z)\}]} \frac{\exp\{\theta_2 w + \eta_2 h(w)\}}{E[\exp\{\theta_2 W + \eta_2 h(W)\}]}, \tag{27}
\]

where \( \theta_1, \eta_1 \) are the tilting parameters for \( Z \) and \( \theta_2, \eta_2 \) are the tilting parameters for \( W \).

Now, we consider the standard parameterization by letting \( \mu := \theta_1/(1 - 2\eta_1), \sigma^2 := 1/(1 - 2\eta_1), \theta := \theta_2, \) and \( \eta := \eta_2 \) adopted in the examples of Section 3. If \( W \) follows a Gamma distribution, \( \text{Gamma}(\alpha, \beta) \), from Equations (16) and (23), Equation (27) becomes

\[
\frac{dQ_{\mu, \sigma, \theta, \eta}}{dP} \frac{d\mathcal{L}}{d\mathcal{L}} = \frac{1}{\sigma} \exp\{\mu z - \mu^2/2 + \frac{\sigma^2 - 1}{2\sigma^2}(z - \mu)^2\} \times e^{w\theta - \eta \ln(\beta w)} \frac{\Gamma(\alpha)}{(1/\beta)^{-\alpha}(\beta)(\beta - \theta)^{-\alpha + \eta} \Gamma(\alpha - \eta)}. \tag{28}
\]

The optimal tilting parameters \( \mu^*, \sigma^* \) for \( Z \) can be obtained by solving Equation (17), and \( \theta^* \) and \( \eta^* \) for \( W \) are the solutions of (24) and (25).

| \( a \) | Crude for \( P(X > a) \) | \( \theta^* \) | \( \eta^* \) | \( (\theta^*, \eta^*) \) | Variance reduction factors |
|---|---|---|---|---|---|
| 10 | \( 2.613 \times 10^{-1} \) | 2 | 3 | 3 | 0.2 | \( 2.438 \times 10^{-1} \) | 4 | 2 | 6 |
| 20 | \( 1.050 \times 10^{-2} \) | 24 | 46 | 47 | 0.5 | \( 1.864 \times 10^{-2} \) | 41 | 15 | 45 |
| 30 | \( 1.800 \times 10^{-4} \) | 567 | 1,288 | 1,307 | 1.5 | \( 3.100 \times 10^{-4} \) | 1,321 | 294 | 1,744 |
| 35 | \( 3.000 \times 10^{-5} \) | 4,788 | 11,788 | 12,226 | 2.5 | \( 6.000 \times 10^{-5} \) | 11,904 | 2,156 | 11,939 |

Table 3: Two-parameter importance sampling for the Gamma distribution
Table 4 tabulates the numerical results for this one-dimensional, one-factor normal mixture distribution. Since for a normal mixture random variable, the variance is associated with the random variable $W$, tilting the standard deviation $\sigma$ with the mean $\mu$ of the normal random variable $Z$ is relatively insignificant in comparison to tilting the parameters of $W$ (i.e., $\theta$ or $\eta$) with $\mu$. This is shown in Table 4. In addition, similar to the case demonstrated in Example 3.3, in this case, tilting $\eta$ with $\mu$ also yields better performance than tilting $\theta$ with $\mu$ in the sense of variance reduction.

| $P(\sqrt{WZ} > a)$ | a Crude | Variance reduction factors | $\mu^*$ $\sigma^*$ $(\mu^*,\sigma^*)$ $\theta^*$ $\eta^*$ $(\mu^*,\theta^*)$ $(\mu^*,\eta^*)$ |
|---------------------|--------|-----------------------------|---------------------------------|
| $2 \times 10^{-1}$ | 2.344 | 3 1 6 1 1 4 5 |
| $4 \times 10^{-2}$ | 2.808 | 7 2 10 1 2 13 17 |
| $8 \times 10^{-4}$ | 9.500 | 30 8 48 3 4 234 394 |
| $12 \times 10^{-5}$ | 2.400 | 70 28 199 4 11 5,054 9,619 |

Table 4: Two-parameter importance sampling for normal mixture distribution

Next, we summarize tilting for the event that the $k$-th obligor defaults if $X_k$ exceeds a given threshold $\chi_k$ as “ABC-event tilting”, which actually involves the calculation of tail event

$$\{ (A + B) C > \tau \}, \quad (29)$$

where $A$ denotes the normal distributed part of the systematic risk factors, $B$ denotes the idiosyncratic risk associated with each obligor, and $C$ denotes the non-negative and scalar-valued random variables which are independent of $A$ and $B$. For example, for the normal mixture copula model in (2), the $d$-dimensional multivariate normal random vectors $Z = (Z_1, \cdots, Z_d)$ are associated with $A$, the $\epsilon_k$ is associated with $B$, and the non-negative and scalar-valued random variables $W$ are associated with $C$. Table 5 summarizes the exponential tilting used in Glasserman and Li (2005); Bassamboo et al. (2008); Chan and Kroese (2010); Scott and Metzler (2015), and our paper. Note that Glasserman and Li (2005) consider the normal copula model so that there is no need for tilting $C$. It is worth mentioning that Bassamboo et al. (2008); Chan and Kroese (2010); Scott and Metzler (2015) only consider the one-dimensional $t$-distribution, while we consider the multi-dimensional normal mixture distribution. Moreover, except for the proposed model, the other four methods adopt the so-called one-parameter tilting. For example, even though Scott and Metzler (2015) consider the tilting of $A$ and $C$, which is same as our setting, only one parameter is tilted for each of the two distributions (i.e., mean for normal distribution and shape for Gamma distribution); in our method, however, the tilting parameter can be either the shape or the rate parameter for the underlying Gamma distribution, which results in a more efficient simulation.

$^3$Here we omit the coefficients before $A$, $B$, and $C$ for simplicity.
Table 5: ABC-event tilting

|                               | One-parameter tilting (traditional) | Two-parameter tilting (proposed) |
|-------------------------------|------------------------------------|---------------------------------|
|                               | Glasserman and Li (2005)           | Bassamboo et al. (2008)         |
|                               |                                    | Chan and Kroese (2010)          |
|                               |                                    | Scott and Metzler (2015)        |
|                               | Multivariate normal dist.          | t-dist.                         | Normal mixture dist.       |
| A                             | ✓                                  | ✓                               | ✓                            |
| B                             | ✓                                  | ✓                               | ✓                            |
| C                             | NA                                 | ✓                               | ✓                            |

4.2 Exponentially tilting for $\varrho(Z, W)$

In this subsection, we use the same notation as in Section 3. Let $Z = (Z_1, \ldots, Z_d)^\top$ be a $d$-dimensional multivariate normal random variable with zero mean and identity covariance matrix $\mathbb{I}$, and denote $W = (W_1, \ldots, W_{d+1})^\top$ as non-negative scalar-valued random variables, which are independent of $Z$. Under the probability measure $P$, let $f_1(z) = f_1(z_1, \ldots, z_d)$ and $f_2(w) = f_2(w_1, \ldots, w_{d+1})$ be the probability density functions of $Z$ and $W$, respectively, with respect to the Lebesgue measure $\mathcal{L}$. As alluded to earlier, our aim is to calculate the expectation of $\varrho(Z, W)$,

$$m = E_P [\varrho(Z, W)]$$

(30)

under the probability measure $P$.

To evaluate (30) via importance sampling, we choose a sampling probability measure $Q$, under which $Z$ and $W$ have the corresponding probability density functions $q_1(z) = q_1(z_1, \ldots, z_d)$ and $q_2(w) = q_2(w_1, \ldots, w_{d+1})$. Assume that $Q$ is absolutely continuous with respect to $P$; Equation (30) can then be written as

$$E_P [\varrho(Z, W)] = E_Q \left[ \varrho(Z, W) \frac{f_1(Z) f_2(W)}{q_1(Z) q_2(W)} \right].$$

(31)

Let $Q_{\mu, \Sigma, \theta, \eta}$ be the two-parameter exponentially tilted probability measure of $P$. Here the subscripts $\mu = (\mu_1, \ldots, \mu_d)^\top$, and $\Sigma$, constructed via $\rho$ and $\sigma = (\sigma_1, \ldots, \sigma_d)^\top$, are the tilting parameters for random vector $Z$, and $\theta = (\theta_1, \ldots, \theta_{d+1})^\top$ and $\eta = (\eta_1, \ldots, \eta_{d+1})^\top$ are the tilting parameters for $W$. Define the likelihood ratios

$$r_{1, \mu, \Sigma}(z) = \frac{f_1(z)}{q_{1, \mu, \Sigma}(z)}; \quad \text{and} \quad r_{2, \theta, \eta}(w) = \frac{f_2(w)}{q_{2, \theta, \eta}(w)},$$

(32)

where $q_{1, \mu, \Sigma}(z)$ and $q_{2, \theta, \eta}(w)$ denote the probability density functions corresponding to $q_1(z)$ with tilting parameters $\mu$ and $\Sigma$ and $q_2(w)$ with tilting parameters $\theta$ and $\eta$, respectively. Then, combined with (32), Equation (31) becomes

$$E_Q \left[ \varrho(Z, W) \frac{f_1(Z) f_2(W)}{q_1(Z) q_2(W)} \right] = E_{Q_{\mu, \Sigma, \theta, \eta}} [\varrho(Z, W)r_{1, \mu, \Sigma}(Z)r_{2, \theta, \eta}(W)].$$

(33)

---

4 Although here we consider the identity covariance matrix for simplicity, it is straightforward to extend this to any valid covariance matrix $\Sigma$. 

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15
Denote
\[ G(\mu, \Sigma, \theta, \eta) = E_P \left[ g^2(Z, W) r_{1,\mu,\Sigma}(Z) r_{2,\theta,\eta}(W) \right]. \] 
(34)

By using the same argument as that in Section 3, we minimize \( G(\mu, \Sigma, \theta, \eta) \) to get the tilting formula. That is, tilting parameters \( \mu^*, \Sigma^*, \theta^*, \) and \( \eta^* \) are chosen to satisfy
\[ \frac{\partial G(\mu, \Sigma, \theta, \eta)}{\partial \mu} = 0, \quad \frac{\partial G(\mu, \Sigma, \theta, \eta)}{\partial \Sigma} = 0, \quad (35) \]
\[ \frac{\partial G(\mu, \Sigma, \theta, \eta)}{\partial \theta} = 0, \quad \frac{\partial G(\mu, \Sigma, \theta, \eta)}{\partial \eta} = 0. \quad (36) \]

Note that to simulate the portfolio loss considered in (1), \( \varrho(z, w) = P(L_n > \tau \mid Z = z, W = w) \). Therefore, to get the optimal tilting parameters in (35) and (36), we must calculate the conditional default probability \( P(L_n > \chi_k \mid Z = z, W = w) \) given \( Z = (z_1, \ldots, z_d)^T \) and \( W = (w_1, \ldots, w_{d+1})^T \) becomes
\[ p_{z,w,k} = P \left( \epsilon_k > \frac{\chi_k - \sum_{i=1}^d \rho_{ki} \sqrt{W_i} Z_i}{\rho_k \sqrt{W_{d+1}}} \bigg| Z = z, W = w \right). \] 
(37)

The (fast) inverse Fourier transform for non-identical \( c_k \).

With non-identical \( c_k \), the distribution of the sum of \( n \) independent but non-identically distributed “weighted” Bernoulli random variables becomes difficult to evaluate. Here we adopt the inverse Fourier transform to calculate \( \varrho(z, w) \) (Oberhettinger, 2014). Recall that \( L_n \mid (Z = z, W = w) \) equals
\[ L_{n,z,w}^w = \sum_{\ell=1}^n c_\ell H_{\ell}^{z,w} \] 
(38)
where \( H_{\ell}^{z,w} \sim \text{Bernoulli}(p_{z,w,\ell}) \), and the support of \( L_{n,z,w}^w \) is a discrete set with finite number of values. Its Fourier transform is
\[ \phi_{L_{n,z,w}^w}(t) = E[e^{itL_{n,z,w}^w}] = E[e^{it(\sum_{\ell=1}^n c_\ell H_{\ell}^{z,w})}] = \prod_{\ell=1}^n E[e^{itc_\ell H_{\ell}^{z,w}}] = \prod_{\ell=1}^n \phi_{H_{\ell}^{z,w}}(tc_\ell), \] 
(39)
where \( \phi_{H_{\ell}^{z,w}}(s) = 1 - p_{z,w,\ell} + p_{z,w,\ell}e^s \). For random variable \( L_{n,z,w}^w \), we can recover \( q_k^{z,w} = P(L_{n,z,w}^w = k) \) by inverting the Fourier series:
\[ q_k^{z,w} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikt} \prod_{\ell=1}^n \phi_{H_{\ell}^{z,w}}(tc_\ell) dt, \quad (40) \]
where \( k = 1, 2, \ldots, \infty \).
A FFT algorithm computes the discrete Fourier transform (DFT) of a sequence, or its inverse. To reduce the computational time, this paper uses the FFT to approximate the probability in Equation (40). With Euler’s relation $e^{i\theta} = \cos \theta + i \sin \theta$, we can confirm that $\phi_{L_n^z,w}(t)$ has a period of $2\pi$; i.e., $\phi_{L_n^z,w}(t) = \phi_{L_n^z,w}(t + 2\pi)$ for all $t$, which is due to the fact that $e^{i(t+2\pi)k} = e^{itk}$. With this periodic property, we now evaluate the characteristic function $\phi_{L_n^z,w}$ at $N$ equally spaced values in the interval $[0, 2\pi]$ as

$$b_m^z,w = \phi_{L_n^z,w} \left( \frac{2\pi m}{N} \right), \quad m = 0, 1, \ldots, N - 1,$$

which defines the DFT of the sequence of probabilities $q_k^z,w$. By using the corresponding sequence of characteristic function values above, we can recover the sequence of probabilities; that is, we aim for the sequence of $b_m^z,w$’s from the sequence of $b_m^z,w$’s, which can be achieved by employing the inverse DFT operation

$$\tilde{q}_k^z,w = \frac{1}{N} \sum_{m=0}^{N-1} b_m^z,w e^{-i2\pi km/N}, \quad k = 0, 1, \ldots, N - 1. \quad (41)$$

Finally, the approximation of $\varrho(z,w)$ can be calculated as

$$\tilde{\varrho}(z,w) = 1 - P_{\text{FFT}}(L_n^{z,w} \leq \tau) = 1 - \sum_{\ell=0}^{\tau} \tilde{q}_\ell^z,w, \quad (42)$$

where $P_{\text{FFT}}(\cdot)$ denotes the probability approximated using a fast inverse Fourier transform.

Table 6 provides several examples showing the approximation performance and computational time of the inverse Fourier transform. In the table, we set the number of obligors $n = 250$ and assume $p_{z,w,\ell} = 0.1$ (i.e., $H_{\ell}^{z,w} \sim \text{Bernoulli}(0.1)$) for simplicity. To check the approximation performance, we first consider the case with equal $c_i = 1$, where the probability (denoted as $P_{\text{Binomial}}(\cdot)$) is evaluated analytically via the cumulative density function of the binomial distribution with parameters $n = 250$ and $p = 0.1$. Observe that the differences between the approximated probabilities ($P_{\text{FFT}}(\cdot)$) and the analytical ones ($P_{\text{Binomial}}(\cdot)$) are nanoscopic, i.e., the bias is extremely small. Moreover, we investigate the case with five different $c_i$, in which we compare the approximated probabilities with the ones generated via simulation with 500,000 samples; as shown in Table 6, the approximated probabilities all lie within the corresponding 95% confidence intervals. We note also that the computational time grows linearly with the number of different $c_i$.

### 4.3 Algorithms

This subsection summarizes the steps when we implement the proposed two-parameter importance sampling algorithm, which consists of two components: the tilting parameter

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5All of the experiments were obtained by running programs via Mathematica 11 on a MacBook Pro with a 2.6 GHz Intel Core i7 CPU.
where the Jacobian of g

Table 6: Approximation performance and computational time (seconds) of the inverse Fourier transform

| \(c_t = 1\) | \(c_t = (\lfloor 5t \rfloor/n)^2\) |
|----------------|-----------------------------|
| \(\tau\) | \(P_{\text{FFT}}(L_n \leq \tau)\) | \(P_{\text{FFT}}(L_n \leq \tau) - P_{\text{Binomial}}(L_n \leq \tau)\) | Time | \(\tau\) | \(P_{\text{FFT}}(L_n \leq \tau)\) | \(P_{\text{MC}}(L_n \leq \tau)\) (95% CI) | Time |
| 20 | 1.72\times10^{-1} | -7.19\times10^{-15} | 0.03 | 200 | 1.29\times10^{-1} | 1.29\times10^{-1} (1.28\times10^{-1}, 1.30\times10^{-1}) | 0.13 |
| 10 | 3.53\times10^{-4} | -7.69\times10^{-17} | 0.04 | 100 | 1.32\times10^{-3} | 1.31\times10^{-3} (1.21\times10^{-3}, 1.41\times10^{-3}) | 0.14 |
| 5 | 5.84\times10^{-7} | 1.11\times10^{-16} | 0.04 | 50 | 1.20\times10^{-5} | 1.00\times10^{-6} (1.24\times10^{-6}, 1.88\times10^{-5}) | 0.14 |

search and the tail probability calculation. The aim of the first component is to determine the optimal tilting parameters. We implement the search phase using the automatic Newton’s method (Teng et al., 2016). We here define the conjugate measures \(\bar{\theta}, \bar{\eta}\) and the results in (19), (20), (24), and (25), we define functions \(g_\mu(\mu), g_\Sigma(\Sigma), g_\theta(\theta), g_\eta(\eta)\) (see Equations (35) and (36)) as

\[
g_\mu(\mu) = \mu - E_{\bar{Q}_{\mu,\Sigma}}[Z \mid L_n > \tau], \tag{43}
g_\Sigma(\Sigma) = K(\mu, \Sigma) - E_{\bar{Q}_{\mu,\Sigma}}[Z^T(\nabla_{\eta} M)Z \mid L_n > \tau] \quad \text{for } i = 1, 2, \ldots, d + 1, \tag{44}
g_\theta(\theta) = \left[\frac{\alpha_{1} - \eta_{1}}{\beta_{1}} + \cdots + \frac{\alpha_{d+1} - \eta_{d+1}}{\beta_{d+1}}\right]^T - E_{\bar{Q}_{\theta,\eta}}[W \mid L_n > \tau], \tag{45}
g_\eta(\eta) = \ln(\beta_1 - \theta_1) + \cdots + \ln(\beta_{d+1} - \eta_{d+1}) + \frac{\gamma}{\beta_{d+1}} \ln(\theta_{d+1} - \eta_{d+1}) + \frac{1}{\beta_{d+1}} \sum_{j=1}^{d} \ln(\beta_j), \tag{46}
\]

where \(\nabla_{\eta} M\) in (44) is defined in (21). To find the optimal tilting parameters, we must find the roots of the above four equations. With Newton’s method, the roots of (43), (44), (45), and (46) are found iteratively by

\[
\delta^{(k)} = \delta^{(k-1)} - J_{\delta}^{-1}(\delta^{(k-1)})g_\delta(\delta^{(k-1)}), \tag{47}
\]

where the Jacobian of \(g_\delta(\delta)\) is defined as

\[
J_\delta[i, j] := \frac{\partial}{\partial \delta_j}g_\delta,i(\delta). \tag{48}
\]

In (47) and (48), \(\delta\) can be replaced to \(\mu, \Sigma, \theta,\) and \(\eta,\) and \(J_{\delta}^{-1}\) is the inverse of the matrix \(J_\delta.\)

In order to measure the precision of roots to the solutions in (43), (44), (45), and (46), we define the sum of the square error of \(g_\delta(\delta)\) as

\[
\|g_\delta(\delta)\| = g_\delta(\delta)g_\delta(\delta), \tag{49}
\]

and a \(\delta^{(n)}\) is accepted when \(\|g_\delta(\delta^{(n)})\|\) is less than a predetermined precision level \(\epsilon.\) The detailed procedures of the first component are described as follow:

- Determine optimal tilting parameters:
values becomes vital for finding optimal $\rho$.

Figure 1, neither simple event $X$ in which the optimal tilting parameters, $\mu$, $\sigma$, $\rho$ are presented in Section 5.1 and the last is presented in Section 5.2. First, for comparison

This section demonstrates the capability and performance of the proposed method via an extensive simulation study. We split the following discussion into four parts; the first three are presented in Section 5.1 and the last is presented in Section 5.2. First, for comparison
purposes, we illustrate the results of the special case of the normal mixture copula model, the \( t \)-copula model for single-factor homogeneous portfolios, the settings of which are similar to those in Bassamboo et al. (2008) and Chan and Kroese (2010).\(^6\) Second, we compare the performance of the proposed method with crude simulation, under 3-factor normal mixture models, in which the \( t \)-distribution for \( X_k \) is considered. (Note that we only compare the performance of our method with crude Monte Carlo simulation henceforth as most of the literature focuses on simulating one-dimensional cases.) Third, to evaluate the robustness of the proposed method, we also compare its performance with that of crude simulation, under 3-factor normal mixture models, in which a GIG distribution for \( X_k \) is considered. The cases with different losses resulting from default of the obligors are also investigated. Finally, in Section 5.2, we compare the computational time of the crude Monte Carlo simulation with the proposed importance sampling under several scenarios, as well as provide insight into the trade-off between reduced variance and increased computational time.

Except for the experiments in Table 12, which compares the computational time under different numbers of samples, we generate \( B_1 = 5,000 \) samples to locate the optimal tilting parameters and \( B_2 = 10,000 \) samples to calculate the probability of losses in all of the rest experiments. Following the settings in Bassamboo et al. (2008), variances under crude Monte Carlo simulation are estimated indirectly by exploiting the observation that for a Bernoulli random variable with success probability \( p \), the variance equals \( p(1-p) \). Note that as mentioned in Section 4.1, for normal mixture random variables, tilting the variance of the normal random variable \( Z \) is relatively insignificant in comparison to tilting the parameters of \( W \); therefore, in the experiments, we only conduct mean tilting for \( Z \) and \( \theta \)- or \( \eta \)-tilting for \( W \) for multi-factor normal mixture models. It is worth mentioning that even in this case, our two-parameter importance sampling method differs from previous studies, as only \( \theta \) is considered as the tilting parameter in their methods.

\(^6\)Note that as stated in Scott and Metzler (2015), since their algorithm requires less computational time but is slightly less accurate than Chan and Kroese (2010), we here only compare the performance with Bassamboo et al. (2008); Chan and Kroese (2010).
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
 & \multicolumn{2}{c|}{Bassamboo et al. (2008)} & \multicolumn{2}{c|}{Chan and Kroese (2010)} & \multicolumn{2}{c|}{Importance sampling (IS)} \\
\hline
$\nu$ & $P(L_n > \tau)$ & V.R. factor & $P(L_n > \tau)$ & V.R. factor & $P(L_n > \tau)$ & V.R. factor \\
\hline
4 & $8.13 \times 10^{-3}$ & 65 & 271 & 2.440 & $8.11 \times 10^{-3}$ & 338 & $8.09 \times 10^{-3}$ & 1.600 \\
8 & $2.42 \times 10^{-4}$ & 878 & 1.690 & 20.656 & $2.36 \times 10^{-4}$ & 6.212 & $2.47 \times 10^{-4}$ & 14.770 \\
12 & $1.67 \times 10^{-5}$ & 7,331 & 12,980 & 2.08 $\times 10^{3}$ & $1.04 \times 10^{-5}$ & 16,100 & $1.10 \times 10^{-5}$ & 1.57 $\times 10^{3}$ \\
16 & $6.16 \times 10^{-7}$ & 52,185 & 81,170 & 1.30 $\times 10^{6}$ & $6.34 \times 10^{-7}$ & 2.78 $\times 10^{5}$ & $6.20 \times 10^{-7}$ & 1.89 $\times 10^{6}$ \\
20 & $4.38 \times 10^{-8}$ & 301,000 & 4.19 $\times 10^{7}$ & 1.27 $\times 10^{7}$ & $4.12 \times 10^{-8}$ & 5.44 $\times 10^{6}$ & $4.14 \times 10^{-8}$ & 1.61 $\times 10^{7}$ \\
\hline
\end{tabular}
\caption{Performance of proposed algorithm with equal loss resulting from default of the obligors for a one-factor model ($t$-distribution)}
\end{table}

\subsection{Computation and numerical experiments on different model settings}

First, in Table 7 we compare the performance between our method and the methods proposed in Bassamboo et al. (2008) and Chan and Kroese (2010). For comparison purposes, we adopt the same sets of parameter values as those in Table 1 of Bassamboo et al. (2008), where the latent variables $X_k$ in Equation (2) follow a $t$-distribution, i.e., $W_1^{-1} = W_2^{-1} = Q_1/\nu_1$ and $Q_1 \sim \text{Gamma}(\nu_1/2, 1/2)$. The model parameters are chosen to be $n = 250$, $\rho_{11} = 0.25$, the default thresholds for each individual obligor $\chi_i = 0.5 \times \sqrt{n}$, each $c_i = 1$, $\tau = 250 \times b$, $b = 0.25$, and $\sigma_\epsilon = 3$. The table reports the results of the exponential change of measure (ECM) proposed in Bassamboo et al. (2008) and conditional Monte Carlo simulation without and with cross-entropy (CondMC and CondMC-CE, respectively) in Chan and Kroese (2010). Observed from the table, the proposed algorithm (the last four columns) offers substantial variance reduction compared with crude simulation, and in general, it compares favorably to the ECM and CondMC estimators. Moreover, in order to fairly compare with the results of CondMC-CE, we first follow the CondMC method by integrating out the shock variable analytically; then, instead of using the cross-entropy approach, we apply the proposed importance sampling method for variance reduction. Under this setting, the proposed method yields variance reductions comparable to those of CondMC-CE.

Second, Tables 8, 9, and 10 compare the performance of the proposed importance sampling method with crude simulation for a 3-factor $t$-copula model, a special case of normal mixture copula models, where the latent variables $X_k$ follow a multivariate $t$-distribution, i.e., $W_j^{-1} = Q_j/\nu_j$ and $Q_j \sim \text{Gamma}(\nu_j/2, 1/2)$ for $j = 1, \ldots, 4$. In the three tables, the results with equal losses (Table 8) and different losses (Tables 9 and 10) resulting from the default of obligors with various parameter settings are listed. Following the settings in Bassamboo et al. (2008) and Chan and Kroese (2010), the threshold for the $i$-th obligor $\chi_i$ is set to $0.5 \times \sqrt{n}$, the idiosyncratic risk $\epsilon_k$ is set to $N(0,9)$, and the total loss $\tau$ is set to $n \times b$. The results with base-case model parameters are reported in the gray-background cells in Tables 8, 9, and 10. For the equal loss scenario ($c_i = 1$), the model parameter $b$ for the base case is set to 0.3, while for $c_i = \lceil 2i/n \rceil^2$ and $c_i = \lceil 5i/n \rceil^2$, $b$ is set to...
3. and 2, respectively. In addition, the other parameters for the base case are listed as follows: $\bar{\nu} = (8 \, 6 \, 4 \, 4)$, $n = 250$, each $\rho_{ki} = 0.1$ (for $k = 1, \ldots, n$ and $i = 1, \ldots, d$), and the covariance matrix $\Sigma = (u_{ij}) \in \mathbb{R}^{3 \times 3}$ of the multivariate normal distribution $Z$ is set to $u_{ii} = \sigma_i^2$, $u_{ij} = u_{ji} = \hat{\rho} \sigma_i \sigma_j$, where $\sigma_1 = 1$, $\sigma_2 = 0.8$, $\sigma_3 = 0.5$, $\hat{\rho} = 0.5$.

As shown in Tables 8, 9, 10, our IS approach performs significantly better than crude simulation, especially when the loss threshold $\tau$ increases and the probability becomes smaller. Moreover, in contrast to the results listed in Table 2 of *Chan and Kroese (2010)*, which show that when $\rho_{ki}$ increases, the performance of CondMC deteriorates, the performance of our method is demonstrated to be stable. The reason for this phenomenon is due to the fact that as $\rho_{ki}$ increases, the factor $Z$ gains importance in determining the occurrence of the rare event; while CondMC simply ignores the contribution of $Z$, the proposed method twists the distributions of both $Z$ and $W$.

Third, in addition to the $t$-copula model, Table 11 shows the results for another type of 3-factor normal mixture copula model, where $W_j$ follows a special case of generalized inverse Gaussian (GIG) distributions, i.e., $W_j \sim \text{Gamma}(v_j/2, 1/2)$ for $j = 1, \ldots, 4$ and $\bar{\nu} = (8 \, 6 \, 4 \, 4)$; except for $b$ set to 0.28, 0.32, and 0.36, the other model parameters are the same as those for the base case in Table 8. From Table 11 we observe that our approach outperforms crude simulation, which attests the capability of the proposed algorithm for normal mixture copula

### Table 8: Performance of proposed algorithm with equal loss resulting from default of the obligors for a 3-factor model ($t$-distribution)

| $b$     | $P(L_n > \tau)$ | V.R. factor | $\tilde{\nu}$ | $P(L_n > \tau)$ | V.R. factor | $n$     | $P(L_n > \tau)$ | V.R. factor |
|---------|-----------------|-------------|----------------|-----------------|-------------|---------|-----------------|-------------|
| 0.3     | $3.08 \times 10^{-3}$ | 863         | (4,4,4)        | $3.09 \times 10^{-3}$ | 1,009       | 100     | $1.91 \times 10^{-2}$ | 416         |
| 0.4     | $2.39 \times 10^{-4}$ | 5,931       | (8,6,4,4)      | $3.08 \times 10^{-3}$ | 863         | 250     | $3.08 \times 10^{-3}$ | 863         |
| 0.5     | $2.13 \times 10^{-6}$ | 20,300      | (8,8,8,8)      | $2.97 \times 10^{-5}$ | 1,667       | 400     | $1.17 \times 10^{-3}$ | 563         |

### Table 9: Performance of proposed algorithm with 2 different losses resulting from default of the obligors for a 3-factor model with inverse FFT ($t$-distribution)

| $b$     | $P(L_n > \tau)$ | V.R. factor | $\tilde{\nu}$ | $P(L_n > \tau)$ | V.R. factor | $n$     | $P(L_n > \tau)$ | V.R. factor |
|---------|-----------------|-------------|----------------|-----------------|-------------|---------|-----------------|-------------|
| 0.7     | $4.79 \times 10^{-3}$ | 832         | (4,4,4)        | $4.78 \times 10^{-3}$ | 692         | 100     | $3.02 \times 10^{-2}$ | 325         |
| 1.0     | $2.91 \times 10^{-4}$ | 3,078       | (8,6,4,4)      | $4.79 \times 10^{-3}$ | 832         | 250     | $4.79 \times 10^{-3}$ | 832         |
| 1.2     | $1.20 \times 10^{-5}$ | 14,471      | (8,8,8,8)      | $6.84 \times 10^{-5}$ | 7,305       | 400     | $1.87 \times 10^{-3}$ | 739         |

| $\rho_{ki}$ | $P(L_n > \tau)$ | V.R. factor | $\tilde{\rho}$ | $P(L_n > \tau)$ | V.R. factor | $\sigma_1, \sigma_2, \sigma_3$ | $P(L_n > \tau)$ | V.R. factor |
|--------------|-----------------|-------------|----------------|-----------------|-------------|---------------------------------|-----------------|-------------|
| 0.1          | $4.79 \times 10^{-3}$ | 832         | -0.5           | $4.81 \times 10^{-3}$ | 1,055       | (0.6,0.4,0.1)                 | $4.84 \times 10^{-3}$ | 700         |
| 0.3          | $2.96 \times 10^{-3}$ | 876         | 0              | $4.80 \times 10^{-3}$ | 745         | (0.8,0.6,0.3)                 | $4.79 \times 10^{-3}$ | 582         |
| 0.5          | $4.27 \times 10^{-4}$ | 739         | 0.5            | $4.79 \times 10^{-3}$ | 832         | (1,0,8,0,5)                  | $4.79 \times 10^{-3}$ | 832         |
Five different losses \((c_i = (\lfloor 5i \rfloor/n)^2)\)

| \(b\) | \(P(L_n > \tau)\) | V.R. factor | \(\tilde{b}\) | \(P(L_n > \tau)\) | V.R. factor | \(n\) | \(P(L_n > \tau)\) | V.R. factor |
|---|---|---|---|---|---|---|---|---|
| 2 | 2.38x10^{-2} | 141 | (4,4,4) | 2.39x10^{-2} | 139 | 100 | 1.09x10^{-1} | 59 |
| 4 | 8.59x10^{-4} | 3414 | (8,6,4) | 2.38x10^{-2} | 141 | 250 | 2.38x10^{-2} | 141 |
| 6 | 4.15x10^{-7} | 81,498 | (8,8,8) | 1.84x10^{-3} | 1,131 | 400 | 9.77x10^{-3} | 264 |

Table 10: Performance of our algorithm with 5 different losses resulting from default of the obligors for a 3-factor model with inverse FFT \((-\text{distribution})\)

| \(\rho_{ki}\) | \(P(L_n > \tau)\) | V.R. factor | \(\hat{\rho}\) | \(P(L_n > \tau)\) | V.R. factor | \(\sigma_1, \sigma_2, \sigma_3\) | \(P(L_n > \tau)\) | V.R. factor |
|---|---|---|---|---|---|---|---|---|
| 0.1 | 2.38x10^{-2} | 141 | -0.5 | 2.39x10^{-2} | 152 | (0.6,0.4,0.1) | 2.38x10^{-2} | 149 |
| 0.3 | 1.51x10^{-2} | 183 | 0 | 2.35x10^{-2} | 125 | (0.8,0.6,0.3) | 2.40x10^{-2} | 148 |
| 0.5 | 2.34x10^{-3} | 143 | 0.5 | 2.38x10^{-2} | 141 | (1,0.8,0.5) | 2.38x10^{-2} | 141 |

Table 11: Performance of our algorithm with equal loss resulting from default of the obligors for a 3-factor model (symmetric generalized hyperbolic distribution)

models. Moreover, it is also worth noting that in this setting, tilting the other parameter \(\nu_j/2\) of the Gamma distribution (i.e., \(\nu_k/2 \leftrightarrow \nu_k/2 - \eta\)) yields 2 to 4 times better performance than the one-parameter exponential tilting (tilting \(\theta\)) in terms of the variance reduction factors.\(^7\)

5.2 Computational time of proposed importance sampling algorithm

We now proceed to compare the computational time of crude Monte Carlo simulation with the proposed importance sampling under several scenarios, and provide insight into the trade-off between reduced variances and increased computational time. Table 12 tabulates the computational time of crude simulation and our method for the three cases listed in the top-left corner of Table 8. From the table we observe that using more samples \((B_1)\) to determine optimal tilting parameters greatly improves the variance reduction performance, which however linearly increases the computational time to determine the parameters; note that the variance reduction factor grows nonlinearly with the number of samples \(B_1\), and our search algorithm generally takes only 7 to 9 iterations to achieve convergence.\(^8\) Despite the need for additional computational time to find suitable tilting parameters, we can use a mere \(B_2 = 1,000\) samples to obtain rather good estimates with greatly reduced variances,\(^9\)

\(^7\) The phenomenon is consistent with the case demonstrated in Example 3.3 and Section 4.1.
\(^8\) In all of the experiments reported here, the predetermined precision level \(\epsilon\) is set to \(10^{-4}\).
especially when the tail probability is small. In the last column of Table 12, we also report the ratio of the computational time consumed by the crude simulation generating a fair estimate (the value with a star symbol on the left-hand side) to that by our importance sampling algorithm, including the time for the parameter search and probability calculation. Observe from the table, for the case $b = 0.5$, the crude simulation with 10,000 and even 100,000 fails to generate the estimate, whereas our method yields a good estimate with a 7,411 variance reduction ratio, while the crude simulation requires 12.77 times more computational time than ours. The relation between the variance reduction factor and the time consumption ratio listed in the last two columns of Table 12 suggests that the proposed algorithm achieves good performance and thus makes a practical contribution to measuring portfolio credit risk in normal mixture copula models.

| $b$   | $B_2$ | Time (a) | $P(L_n > \tau)$ | Search parameters | Calculate probability ($B_2 = 1,000$) | IS | IS | IS |
|-------|-------|----------|-----------------|-------------------|---------------------------------------|----|----|----|
| 0.3   | 100   | 2        | –               | 100               | 200                                  | 191| 9  | 3  |
|       | 1,000 | 1.00×10^{-3} | 500           | 647               | 191                                  | 13.38×10^{-3} | 1.42×10^{-4} | 22 | 0.22|
| 10,000| *181  | 3.80×10^{-3} | 1,000          | 2,049             | 201                                  | 3.01×10^{-3} | 5.12×10^{-6} | 602| 0.08|
| 0.4   | 1,000 | 17       | –               | 500               | 183                                  | 2.43×10^{-4} | 1.37×10^{-6} | 175| 2.71|
|       | 10,000| *184   | 1.00×10^{-4}   | 1,060             | 220                                  | 2.32×10^{-4} | 8.08×10^{-7} | 296| 1.13|
|       | 100,000| *2,070 | 2.00×10^{-4}  | 2,000             | 276                                  | 2.29×10^{-4} | 2.83×10^{-7} | 844| 0.7 |
| 0.5   | 10,000| 204      | –               | 1,000             | 210                                  | 1.70×10^{-6} | 2.88×10^{-10} | 7,411| 12.77|
|       | 100,000| 1,812  | 2               | 2,000             | 209                                  | 1.98×10^{-6} | 2.59×10^{-10} | 8,244| 7.13 |
|       | 1,000,000| *18,896 | 2.00×10^{-6} | 5,000             | 228                                  | 1.90×10^{-6} | 1.07×10^{-10} | 19,845| 3.03|

Table 12: Computational times (seconds)

6 Efficient simulation for a multi-factor model on the CDXIG index

To demonstrate the capability of the proposed method, in this section, we apply the importance sampling algorithm to a set of parameters of a multi-factor model calibrated from data of the CDXIG index and consider the underlying portfolio of the CDXIG index, using data from March 31, 2006 (Rosen and Saunders, 2009, 2010). By using the same settings in Rosen and Saunders (2010), the multi-factor model is assumed with a single global factor $Z_G$ and a set of sector factors $Z_{S_j}$ for $j = 1, \cdots, 7$; the details of the seven industrial sectors, merged from 25 Fitch sectors, can be found in Rosen and Saunders (2009). For this eight-factor model, each obligor has two non-zero factor loadings, and the creditworthiness index of the

$^9$Note that we treat this example as an eight-factor model; the method of combining our two-parameter tilting and the method proposed in Glasserman et al. (2008) will be a future work.
Table 13: **Performance of empirical example**

| Crude | IS |
|-------|----|
| \(b\) | \(P(L_n > \tau)\) Variance | \(P(L_n > \tau)\) Variance V.R. factor |
| 0.01  | \(2.38 \times 10^{-2}\) 2.32 \(\times 10^{-2}\) | \(2.19 \times 10^{-2}\) 2.61 \(\times 10^{-4}\) 89 |
| 0.05  | \(6.60 \times 10^{-3}\) 6.56 \(\times 10^{-3}\) | \(6.43 \times 10^{-3}\) 4.61 \(\times 10^{-5}\) 142 |
| 0.2   | \(5.00 \times 10^{-4}\) 5.00 \(\times 10^{-4}\) | \(4.18 \times 10^{-4}\) 1.01 \(\times 10^{-6}\) 494 |

\(k\)-th obligor in sector \(S_j\) is thus given by

\[
X_k = \sqrt{\rho_G} \cdot Z_G + \sqrt{\rho_S - \rho_G} \cdot Z_{S_j} + \sqrt{1 - \rho_S} \cdot \epsilon_k, \quad \text{if } s(k) = S_j, \tag{50}
\]

where \(s(k)\) denotes the sector for the \(k\)-th obligor, and \(\rho_G = 0.17\) and \(\rho_S = 0.23\) are the same for all obligors.\(^{10}\) As mentioned in (McNeil et al., 2015), the latent variable \(X_k\) can have general interpretations, including asset value and creditworthiness. For the creditworthiness in this example, the total portfolio loss is thus defined as

\[
L_n = c_1 \mathbb{1}_{\{X_1 < \chi_1\}} + \cdots + c_n \mathbb{1}_{\{X_n < \chi_n\}}. \tag{51}
\]

Since the CDXIG index has 125 equally-weighted obligors, we have \(n = 125\) and set \(c_1 = c_2 = \cdots = c_{125} = 1.\(^{11}\)

Rather than using a conventional normal copula model in Rosen and Saunders (2010), here we consider this specification under a \(t\)-copula model. The rationale of using this model can be found in Frey and McNeil (2003); Bassamboo et al. (2008); McNeil et al. (2015), in which they claim that assuming a normal dependence structure may underestimate the probability of joint large movements of risk factors, while the \(t\)-copula model is better for modeling the effects of extreme dependence. For the \(t\)-copula model considered in (50), the idiosyncratic risk \(\epsilon_k\) is set to \(N(0,1)\), the threshold for the \(i\)-th obligor \(\chi_i\) is set to \(-0.55 \times \sqrt{n}.\(^{12}\) The total loss \(\tau\) is set to \(n \times b\), the same setting as in the previous subsection, and the latent variables \(X_k\) follow a multivariate \(t\)-distribution with a degree of freedom equal to 4, i.e., \(W_j^{-1} = Q_j/4\) and \(Q_j \sim \text{Gamma}(4/2, 1/2)\) for \(j = 1, \ldots, 4\).

Table 13 tabulates the performance of this empirical example in terms of variance reduction factors. As shown in the table, the crude simulation suffers from the high variances of the estimates, which results in extremely long simulation time to obtain a good estimate, especially for very small probabilities. For instance, for the case of \(b = 0.2\) in the table, we may need over 10,000 simulation paths to obtain a non-zero estimate for each simulation, resulting in a large variance of \(5.00 \times 10^{-4}\); hence to reduce variances and obtain a good estimate, such simulations are necessarily lengthy. For more efficient simulation schemes, although

\(^{10}\) These two values are chosen to match estimated correlations from Akhavein et al. (2005).

\(^{11}\) Note that here we follow the setting in Rosen and Saunders (2010) to define the loss in (51), which is different from that in (1); the simulation scheme is thus almost the same with a simple modification.

\(^{12}\) Since this multi-factor model is meant only to illustrate the methodology, for simplicity, the threshold for each obligor here is chosen to roughly match the average one-year default probability of the index, 0.19%.
some previous studies address this issue for the multi-factor model under normal copula (Glasserman et al., 2008) or for the single-factor model under $t$-copula (Bassamboo et al., 2008; Chan and Kroese, 2010), little work has been done on importance sampling methods for efficient simulation for multi-factor models under the more general normal mixture copula model, which includes the popular normal copula and $t$-copula models as special cases. In particular, existing importance sampling algorithms cannot be used for the portfolio loss in model (50) and (51). To remedy this shortcoming, we propose an importance sampling algorithm to estimate the probability that the portfolio incurs large losses under the normal mixture copula. It is worth mentioning again that the newly proposed two-parameter importance sampling is more suitable for the normal mixture model simulations. This is confirmed by the variance reduction performance of this method listed in Table 13 via the empirical example under an eight-factor model. Note that for this set of calibrated factor loadings, the idiosyncratic risk with the rather high weight $\sqrt{1 - \rho_S} \approx 0.877$ dominates the value of the latent variable $X_k$; in this case, tilting the last non-negative scalar-valued random variable $W_{d+1}$ becomes critical for the variance reduction performance.

For more applications, as future work we consider applying this method to estimate the expected shortfall in more general copula models, such as Archimedean copula models. Note that the demand for simulation is even more serious in real applications, in which the parameters in model (50) are unknown, and must be estimated from real data sets; see Chapter 11.5 of (McNeil et al., 2015). Moreover, when using the bootstrap method for accurate interval estimation for the unknown parameters, a thousand replications of the bootstrap algorithm are required. We believe that the proposed importance sampling with modification, cf. Fuh and Hu (2004), would be useful in this setting.

7 Conclusions

This paper studies a multi-factor model with a normal mixture copula that allows multivariate defaults to have an asymmetric distribution. Due to the amount of the portfolio, the heterogeneous effect of obligors, and the phenomena that default events are rare and mutually dependent, it is difficult to calculate portfolio credit risk either by means of direct analysis or crude Monte Carlo simulation. To address this problem, we first propose a general account of a two-parameter importance sampling algorithm, and then propose an efficient simulation algorithm to estimate the probability that the portfolio incurs large losses under the normal mixture copula. We also provide theoretical justifications of the proposed method and illustrate its effectiveness through both numerical results and an empirical example.

There are several possible future directions based on this model. To name a few, first, we may consider simulating portfolio loss under the meta-elliptical copula and/or the Archimedean copula models. Second, although in this paper the default time is fixed, the default time could be any time before a pre-fixed time $T$. To capture this phenomenon, we could consider a first-passage time model based on the factor model (2). In this case,
we would develop an importance sampling algorithm for stopping time events. Third, by developing the first-passage time model, it would be more practical to consider the firm value process with credit rating. In such case, an importance sampling for Markov chains is needed.

Appendix: Proof of Theorem 3.1

To prove Theorem 3.1, we require the following three propositions. Proposition 1 is taken from Theorem VI.3.4. of Ellis (1985), Proposition 2 is a standard result from convex analysis, and Proposition 3 is taken from Theorem 1 of Soriano (1994). Note that although the function domain is the whole space in Propositions 2 and 3, the results for a subspace still hold with similar proofs.

Proposition 1 $f(\theta)$ is differentiable at $\theta \in \text{int}(\Theta)$ if and only if the $d$ partial derivatives $\frac{\partial f(\theta)}{\partial \theta_i}$ for $i = 1, \cdots, d$ exist at $\theta \in \text{int}(\Theta)$ and are finite.

Proposition 2 Let $f : \mathbb{R}^d \to \mathbb{R}$ be continuous on all of $x \in \mathbb{R}^d$. If $f$ is coercive (in the sense that $f(x) \to \infty$ if $\|x\| \to \infty$), then $f$ has at least one global minimizer.

Proposition 3 Let $f : \mathbb{R}^d \to \mathbb{R}$, and $f$ is continuous differentiable function and convex, satisfies (13). Then a minimum point for $f$ exists.

<Proof of Theorem 3.1>

In the following, we first show that $G(\theta, \eta)$ is a strictly convex function. For any given $\lambda \in (0, 1)$, and $(\theta_1, \eta_1), (\theta_2, \eta_2) \in \Theta \times H$, by the convexity of $\psi(\cdot, \cdot)$, we have

$$\psi(\lambda \theta_1 + (1 - \lambda) \theta_2, \lambda \eta_1 + (1 - \lambda) \eta_2) = \psi(\lambda (\theta_1, \eta_1) + (1 - \lambda) (\theta_2, \eta_2)) \quad (52)$$

$$\leq \lambda \psi(\theta_1, \eta_1) + (1 - \lambda) \psi(\theta_2, \eta_2).$$
Then

\[ G(\lambda(\theta_1, \eta_1) + (1 - \lambda)(\theta_2, \eta_2)) = G(\lambda \theta_1 + (1 - \lambda) \theta_2, \lambda \eta_1 + (1 - \lambda) \eta_2) \]

\[ = E_p \left[ \varphi^2(X) \exp \left( -((\lambda \theta_1 + (1 - \lambda) \theta_2)^\top X + (\lambda \eta_1 + (1 - \lambda) \eta_2)^\top h(X)) 
+ \psi(\lambda \theta_1 + (1 - \lambda) \theta_2, \lambda \eta_1 + (1 - \lambda) \eta_2) \right) \right] \]

\[ \leq E_p \left[ \varphi^2(X) \exp \left( -((\lambda \theta_1 + (1 - \lambda) \theta_2)^\top X + (\lambda \eta_1 + (1 - \lambda) \eta_2)^\top h(X)) 
+ \lambda \psi(\theta_1, \eta_1) + (1 - \lambda) \psi(\theta_2, \eta_2) \right) \right] \text{ by (52)} \]

\[ = E_p \left[ \varphi^2(X) \exp \left( -\lambda(\theta_1^\top X + \eta_1^\top h(X)) + \lambda \psi(\theta_1, \eta_1) - (1 - \lambda)(\theta_2^\top X + \eta_2^\top h(X)) + (1 - \lambda) \psi(\theta_2, \eta_2) \right) \right] \]

\[ < E_p \left[ \lambda \varphi^2(X) e^{-((\theta_1^\top X + \eta_1^\top h(X)) + \psi(\theta_1, \eta_1))} + (1 - \lambda) \varphi^2(X) e^{-((\theta_2^\top X + \eta_2^\top h(X)) + \psi(\theta_2, \eta_2))} \right] \]

\[ = \lambda G(\theta_1, \eta_1) + (1 - \lambda) G(\theta_2, \eta_2). \]

Next, we prove the existence of \((\theta, \eta)\) in the optimization problem (9). To get the global minimum of \(G(\theta, \eta)\), we note that \(G(\theta, \eta)\) is strictly convex from the above argument, and \(\frac{\partial G(\theta, \eta)}{\partial \theta_i}\) and \(\frac{\partial G(\theta, \eta)}{\partial \eta_j}\) exists for \(i = 1, \ldots, p, \, j = 1, \ldots, q\). Proposition 1 establishes that \(G(\theta, \eta)\) is continuously differentiable for \((\theta, \eta) \in \Theta \times H\). By the definition of \(G(\theta, \eta)\) in (9), it is easy to see that condition i) implies that \(G(\theta)\) is coercive. Then by Proposition 2, \(G(\theta, \eta)\) has a unique minimizer. It is easy to see that ii) implies conditions in Proposition 3 hold.

To prove (14) and (14), we simplify the right-hand side of (10) and (11) under \(Q_{\theta, \eta}\). Standard algebra gives

\[ \frac{E_P \left[ \varphi^2(X)X e^{-((\theta^\top X + \eta^\top h(X))} \right]}{E_P \left[ \varphi^2(X) e^{-((\theta^\top X + \eta^\top h(X))} \right]} = E_{Q_{\theta, \eta}}[X], \]

\[ \frac{E_P \left[ \varphi^2(X)h(X) e^{-((\theta^\top X + \eta^\top h(X))} \right]}{E_P \left[ \varphi^2(X) e^{-((\theta^\top X + \eta^\top h(X))} \right]} = E_{Q_{\theta, \eta}}[h(X)] \]

for \(i = 1, \ldots, p, \, j = 1, \ldots, q\). This implies the desired result. \( \square \)

### References

Akhavein, J. D., A. E. Kocagil, and M. Neugebauer (2005). A comparative empirical study of asset correlations. *Technical Report, Fitch Ratings.*
Asmussen, S. and P. Glynn (2007). *Stochastic Simulation: Algorithms and Analysis*. New York: Springer-Verlag.

Barndorff-Nielsen, O. E. (1978). Hyperbolic distributions and distributions on hyperbolae. *Scandinavian Journal of Statistics* 5(3), 151–157.

Barndorff-Nielsen, O. E. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics* 24(1), 1–13.

Bassamboo, A., S. Juneja, and A. Zeevi (2008). Portfolio credit risk with extremal dependence: Asymptotic analysis and efficient simulation. *Operations Research* 56(3), 593–606.

Botev, Z. I., P. L’Ecuyer, and B. Tuffin (2013). Markov chain importance sampling with applications to rare event probability estimation. *Statistics and Computing* 23(2), 271–285.

Chan, J. C. and D. P. Kroese (2010). Efficient estimation of large portfolio loss probabilities in t-copula models. *European Journal of Operational Research* 205(2), 361–367.

Do, K.-A. and P. Hall (1991). On importance resampling for the bootstrap. *Biometrika* 78(1), 161–167.

Eberlein, E. and U. Keller (1995). Hyperbolic distributions in finance. *Bernoulli* 1(3), 281–299.

Eberlein, E., U. Keller, and K. Prause (1998). New insights into smile, mispricing, and value at risk: The hyperbolic model. *The Journal of Business* 71(3), 371–405.

Ellis, R. S. (1985). *Entropy, Large Deviations, and Statistical Mechanics*. New York: Springer.

Frey, R. and A. J. McNeil (2003). Dependent defaults in models of portfolio credit risk. *Journal of Risk* 6(1), 59–92.

Fu, M. and Y. Su (2002). Optimal importance sampling in securities pricing. *Journal of Computational Finance* 5, 27–50.

Fuh, C.-D. and I. Hu (2004). Efficient importance sampling for events of moderate deviations with applications. *Biometrika* 91(2), 471–490.

Fuh, C.-D., I. Hu, Y.-H. Hsu, and R.-H. Wang (2011). Efficient simulation of value at risk with heavy-tailed risk factors. *Operations Research* 59(6), 1395–1406.

Glasserman, P., W. Kang, and P. Shahabuddin (2007). Large deviations in multifactor portfolio credit risk. *Mathematical Finance* 17(3), 345–379.

Glasserman, P., W. Kang, and P. Shahabuddin (2008). Fast simulation of multifactor portfolio credit risk. *Operations Research* 56(5), 1200–1217.
Glasserman, P. and J. Li (2005). Importance sampling for portfolio credit risk. *Management Science* 51(11), 1643–1656.

Gupton, G. M., C. C. Finger, and M. Bhatia (1997). *Creditmetrics: Technical Document*. JP Morgan & Co.

Li, D. X. (2000). On default correlation: A copula function approach. *The Journal of Fixed Income* 9(4), 43–54.

Liu, G. (2015). Simulating risk contributions of credit portfolios. *Operations Research* 63(1), 104–121.

Mashal, R. and A. Zeevi (2002). Beyond correlation: Extreme co-movements between financial assets. *Working paper, Columbia University*.

McNeil, A. J., R. Frey, and P. Embrechts (2015). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press.

Merton, R. C. (1974). On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance* 29(2), 449–470.

Oberhettinger, F. (2014). *Fourier Transforms of Distributions and Their Inverses: A Collection of Tables*. Academic Press, INC.

Rosen, D. and D. Saunders (2009). Valuing cdos of bespoke portfolios with implied multifactor models. *The Journal of Credit Risk* 5(3), 3.

Rosen, D. and D. Saunders (2010). Risk factor contributions in portfolio credit risk models. *Journal of Banking & Finance* 34(2), 336–349.

Rubinstein, R. Y. and D. P. Kroese (2011). *Simulation and the Monte Carlo Method*, Volume 707. John Wiley & Sons.

Scott, A. and A. Metzler (2015). A general importance sampling algorithm for estimating portfolio loss probabilities in linear factor models. *Insurance: Mathematics and Economics* 64, 279–293.

Soriano, J. (1994). Extremum points of a convex function. *Applied Mathematics and Computation* 66(2-3), 261–266.

Su, Y. and M. C. Fu (2000). Importance sampling in derivative securities pricing. In *Proceedings of the 2000 Winter Simulation Conference*, pp. 587–596.

Teng, H.-W., C.-D. Fuh, and C.-C. Chen (2016). On an automatic and optimal importance sampling approach with applications in finance. *Quantitative Finance* 16(8), 1259–1271.