Time change for flows and thermodynamic formalism

Italo Cipriano¹ and Godofredo Iommi²

Facultad de Matemáticas, Pontificia Universidad Católica de Chile (PUC),
Avenida Vicuña Mackenna 4860, Santiago, Chile

E-mail: icipriano@gmail.com and giommi@mat.puc.cl

Received 6 July 2018, revised 19 March 2019
Accepted for publication 28 March 2019
Published 17 July 2019

Recommended by Professor Lorenzo J Diaz

Abstract
This paper is devoted to the study of how thermodynamic formalism properties vary for time changes of suspension flows defined over countable Markov shifts. We prove that in general no property is preserved. We also make a topological description of the space of suspension flows according to certain thermodynamic features. For example, we show that the set of suspension flows defined over the full shift on a countable alphabet having finite entropy is open. Of independent interest might be a set of analytic tools we use to construct examples with prescribed thermodynamic behavior.

Keywords: suspension flows, thermodynamic formalism, time changes
Mathematics Subject Classification numbers: Primary 05C80; Secondary 05C70, 05C63

(Some figures may appear in colour only in the online journal)

1. Introduction

A natural question that has attracted attention for quite some time is how ergodic properties of a flow vary with a time change. This question has been addressed under different regularity assumptions on the flow (and on the time change), and for a wide range of ergodic properties. For example, under certain measurability assumptions, it is known that ergodicity is preserved under time changes (see [To, theorem 5.1] or [P2, section 5]). On the other hand, mixing and weak-mixing are not necessarily preserved. Some examples have been extensively studied; for instance, it was shown by Marcus [Ma, section 3] that under mild differentiability...
assumptions, time changes of the horocycle flow are topologically mixing. Spectral properties of the time change of the horocycle flow have been studied recently in [FU, Ti]; for more results along these lines see [AFU]. In the context of Axiom A flows, Parry [P1, theorem 3] showed that there exists a time change such that the Sinai–Ruelle–Bowen measure coincides with the measure of maximal entropy, a property he called synchronization. More recently, Gelfert and Motter [GM], for a class of smooth flows, showed that several thermodynamic formalism quantities remained essentially unchanged by suitable time changes. The purpose of the present paper is to discuss how thermodynamic properties vary for time changes of suspension flows defined over countable Markov shifts. We stress that in this setting the phase space is no longer compact, and this yields a very different thermodynamic behavior.

The use of suspension flows over Markov shifts as a tool to study differentiable flows has a long history, probably starting with the study of geodesic flows over a modular surface [KU]. In the early 1970s, Bowen [Bo1] and Ratner [Ra] constructed Markov partitions for Axiom A flows. These were later used by Bowen and Ruelle [BR] to study thermodynamic formalism for such flows. In this setting, the symbolic model was a suspension flow defined over a subshift of finite type defined on a finite alphabet. Recently, Lima and Sarig [LS] have constructed symbolic codings for three-dimensional $C^\infty$ flows of positive entropy defined on compact manifolds. They showed that given $\chi > 0$, there is a suspension flow defined over a finite entropy countable Markov shift that captures every hyperbolic measure for which its positive Lyapunov exponent is larger than $\chi$. Other examples of flows that can be coded by suspension flows over countable Markov shifts are certain classes of geodesic flows defined over non-compact manifolds of variable pinched negative curvature [DP, IRV] and certain Teichmüller flows [BG, Ha].

In section 4.2, we provide a wide range of examples showing that, in strong contrast to the compact setting, no thermodynamic property is preserved in the context of suspension flows defined over countable Markov shifts. Most of these examples are constructed making use of analytic tools developed in the study of Dirichlet series (see section 2.3). We also establish conditions for the time change that ensure that some thermodynamic properties are actually preserved (theorem 4.16). Finally, in section 5, we topologize the space of suspension flows defined over a fixed countable Markov shift. We study topological properties of the sets of flows having finite (or infinite) entropy, those which have (or do not have) measures of maximal entropy, and other sets having particular ergodic properties.

2. Countable Markov shifts

This section is devoted, on the one hand, to recalling the necessary definitions and results from thermodynamic formalism for countable Markov shifts that will be used in the article. On the other hand, more interestingly, we provide tools to construct examples of potentials for which the pressure has prescribed properties. The tools come from the study of zeta functions and Dirichlet series in number theory.

2.1. Countable Markov shifts

Let $T = (t_{ij})_{i,j\in\mathbb{N}}$ be an infinite matrix of zeros and ones. Let

$$\Sigma = \{x \in N^N : t_{x,i+1} = 1 \text{ for every } i \in N\}.$$

The shift map $\sigma : \Sigma \to \Sigma$ is defined by $(\sigma(x))_i = x_{i+1}$. The space $\Sigma$ is equipped with the topology generated by the cylinder sets.
\[ C_{a_1 \ldots a_n} = \{ x \in \Sigma : x_i = a_i \text{ for } i = 1, \ldots, n \}. \]

In general, it is a non-compact space. The pair \((\Sigma, \sigma)\) is called countable Markov shift. In what follows, we will always assume it to be a topologically mixing dynamical system. Given a function \(\varphi : \Sigma \to \mathbb{R}\), let
\[ V_n(\varphi) := \sup \{|\varphi(x) - \varphi(y)| : x, y \in \Sigma, x_i = y_i \text{ for } i = 1, \ldots, n\}, \]
where \(x = (x_0, x_1, \ldots)\) and \(y = (y_0, y_1, \ldots)\). We say that \(\varphi\) has summable variation if \(\sum_{n=1}^{\infty} V_n(\varphi) < \infty\). We also say that \(\varphi\) is locally Hölder if there exist constants \(K > 0\) and \(\theta \in (0, 1)\) such that \(V_n(\varphi) \leq K\theta^n\) for all \(n \geq 1\). The following result summarizes work by Sarig [Sa1], which generalizes previous work by Gurevich [Gu1, Gu2] and by Mauldin and Urbanski [MU].

**Theorem 2.1.** Let \((\Sigma, \sigma)\) be a topologically mixing countable Markov shift and \(\varphi : \Sigma \to \mathbb{R}\) a function of summable variations.

\[
P_\sigma(\varphi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{x, \sigma^i x = x} \exp \left( \frac{\sum_{i=0}^{n-1} \varphi(\sigma^i x)}{n} \right) \chi_{C_{a_0}}(x) = \sup \left\{ h(\nu) + \int \varphi \, d\nu : \nu \in \mathcal{M}_\sigma \text{ and } -\int \varphi \, d\nu < \infty \right\}
\]

The functional \(P_\sigma\) is called Gurevich pressure of \(\varphi\). The symbol \(\chi_{C_{a_0}}(x)\) denotes the characteristic function of the cylinder \(C_{a_0} \subset \Sigma\). The space of \(\sigma\)-invariant probability measures is denoted by \(\mathcal{M}_\sigma\) and \(h(\nu)\) denotes the entropy of the measure \(\nu\). Finally, \(\mathcal{K} := \{ K \subset \Sigma : K \neq \emptyset \text{ compact and } \sigma\text{-invariant} \}\).

It is possible to show that the limit in the definition of \(P_\sigma\) always exists [Sa1]. Moreover, since \((\Sigma, \sigma)\) is topologically mixing, \(P_\sigma(\varphi)\) does not depend on \(a_0\). A measure \(\nu \in \mathcal{M}_\sigma\) such that \(P_\sigma(\varphi) = h(\nu) + \int \varphi \, d\nu\) is called equilibrium measure for \(\varphi\). Buzzi and Sarig [BS] proved that a potential of summable variations has at most one equilibrium measure.

**2.2. The BIP case**

We say that a countable Markov shift \((\Sigma, \sigma)\), defined by the transition matrix \(T = (t_{ij})_{i,j \in \mathbb{N}}\), satisfies the BIP condition if and only if there exists a finite set \(B \subset \mathbb{N}\) such that for every \(a \in \mathbb{N}\) there exist \(b, b' \in B\) with \(t_{0ab}t_{0ba'} = 1\). For this class of countable Markov shifts, introduced by Sarig [Sa2], the thermodynamic formalism is similar to that of subshifts of finite type defined on finite alphabets. The following theorem summarizes results proved by Sarig in [Sa3], and by Mauldin and Urbanski [MU].

**Theorem 2.2.** Let \((\Sigma, \sigma)\) be a topologically mixing countable Markov shift satisfying the BIP condition and \(\varphi : \Sigma \to \mathbb{R}\) a non-positive locally Hölder potential. Then, there exists \(s_\infty \in (0, \infty]\) such that pressure function \(t \to P_\sigma(t\varphi)\) has the following properties

\[
P_\sigma(t\varphi) = \begin{cases} 
\infty & \text{if } t < s_\infty; \\
\text{real analytic} & \text{if } t > s_\infty.
\end{cases}
\]

Moreover, if \(t > s_\infty\), there exists a unique equilibrium measure for \(t\varphi\).
We classify non-positive locally Hölder potentials according to the behavior of the pressure function $t \mapsto P_\sigma(t \varphi)$ at $t = s_\infty$.

**Definition 2.3.** Let $(\Sigma, \sigma)$ be a topologically mixing countable Markov shift satisfying the BIP condition and $\varphi : \Sigma \to \mathbb{R}$ a non-positive locally Hölder potential. We say that

(a) The potential $\varphi$ is of infinite type if, for every $t \in \mathbb{R}$, we have $P_\sigma(t \varphi) = \infty$.

(b) The potential $\varphi$ is of continuous type if there exists $t_0 \in \mathbb{R}$ such that $P_\sigma(t_0 \varphi) < \infty$, $P_\sigma(s_\infty \varphi) = \infty$, and $\lim_{t \to s_\infty^+} P_\sigma(t \varphi) = \infty$.

(c) The potential $\varphi$ is of discontinuous type if there exists $t_0 \in \mathbb{R}$ such that $P_\sigma(t_0 \varphi) < \infty$ and is not of continuous type, that is, $P_\sigma(s_\infty \varphi) < \infty$ or $\lim_{t \to s_\infty^+} P_\sigma(t \varphi) < \infty$.

In figure 1, we plot the typical graphs of the pressure function $t \mapsto P_\sigma(t \varphi)$ for a potential $\varphi$ of infinite, continuous, and discontinuous type, according to definition 2.3.

2.3. The full shift and analytic tools to construct examples

Let $(\Sigma, \sigma)$ be the full shift on countably many symbols, that is,

$$\Sigma := \{(x_n)_{n \in \mathbb{N}} : x_n \in \mathbb{N}\}.$$  

It clearly satisfies the BIP property. If $\varphi : \Sigma \to \mathbb{R}$ is a non-positive locally constant potential; that is, for every $n \in \mathbb{N}$ we have that $\varphi|_{C_n} := -\log \lambda_n$ for some $\lambda_n \in (1, \infty)$, then there is a simple formula for the pressure ([BI1, example 1])

$$e^{P_\sigma(t \varphi)} = \sum_{n=1}^{\infty} \lambda_n^{-t}.$$  \hspace{1cm} (2.1)

Note that, in this case, $s_\infty = \inf \{t : \sum_{m=1}^{\infty} \lambda_n^{-t} < \infty\}$. This formula allows for the use of analytic tools to construct examples of potentials having prescribed properties. But not only that, we will relate the continuity type of the non-positive potential $\varphi$ with the nature of the singularities of the meromorphic extensions of the Dirichlet series (2.1). Indeed, we observe that a potential $\varphi$ is of continuous type if (2.1) has meromorphic extension with maybe some poles, and $\varphi$ is of discontinuity type if (2.1) has meromorphic extension with only branch points. We now recall some theorems on meromorphic extensions of certain Dirichlet series that we will use later. The following result can be found in [Ei, main theorem I].

**Theorem 2.4.** Let $P(x) = \prod_{j=1}^{k}(x + \delta_j)$ be a polynomial with real coefficients and $\delta_j \in \mathbb{C}$ satisfying $\text{Re}(\delta_j) > -1$ for $j = 1, \ldots, k$. The associated Zeta function

\[ Z(s) = \prod_{j=1}^{k} \frac{1}{1 - \delta_j^{-s}} \]

is a meromorphic function of $s$ with simple poles at $s = \log \lambda_n$ for all $n \in \mathbb{N}$. The residues at these poles are given by

$$\text{Res}(Z(s), s = \log \lambda_n) = \frac{1}{\lambda_n}.$$  \hspace{1cm} (2.2

In particular, if $\varphi$ is of continuous type, then $Z(s)$ has no poles in the half-plane $\text{Re}(s) > 0$. If $\varphi$ is of discontinuous type, then $Z(s)$ has only branch points in the half-plane $\text{Re}(s) > 0$.
\[ Z(s) = \sum_{n=1}^{\infty} \frac{1}{P(n)^s} \]

is holomorphic for \( \text{Re}(s) > \frac{1}{p} \), and it has an analytic continuation in the whole complex plane with only possible poles at \( \frac{j}{k} \) for \( j = 1, 0, -1, -2, \ldots \) other than non-positive integers.

We recall the definitions of generalized Dirichlet series and asymptotic expansion.

**Definition 2.5.** A generalized Dirichlet series is an infinite series

\[ L(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}, \tag{2.2} \]

where \( s \in \mathbb{C} \), \( \{a_n\} \subset \mathbb{C} \), \( \{\lambda_n\} \subset \mathbb{R}^+ \) and such that \( 0 < \lambda_1 < \lambda_2 < \cdots \) and \( \lim_{n \to \infty} \lambda_n = \infty \).

**Definition 2.6.** We say that \( f(t) \) has asymptotic expansion

\[ f(t) \sim \sum_{n=0}^{\infty} a_n f_n(t) \text{ as } t \to 0, \]

if \( f(t) - \sum_{n=0}^{N-1} a_n f_n(t) \) is \( O(f_N(t)) \) as \( t \to 0 \) for any integer \( N \geq 0 \).

The next result appears in [Ze, p 313].

**Theorem 2.7.** Let \( L(s) \) be a generalized Dirichlet series as in (2.2). If the following conditions are satisfied:

(a) the sequence \( \{\lambda_n\} \) is growing at least as fast as some positive power of \( n \);
(b) the series \( L(s) \) is convergent at some \( s^* \in \mathbb{C} \); and
(c) \( f(t) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \) has asymptotic expansion

\[ f(t) \sim \sum_{n=1}^{\infty} b_n t^n \text{ as } t \to 0, \]

then \( L(s) \) has a meromorphic extension to all \( s \in \mathbb{C} \), with a simple pole of residue \( b_{-1} \) at \( s = 1 \) and no other singularities. Its values at non-positive integers are given by

\[ L(-n) = (-1)^n n! b_n \quad (n = 0, 1, 2, \ldots). \]

Note that the hypotheses of theorems 2.4 and 2.7 are not disjoint; indeed there are maps that satisfy both, for example \( L(s) = \sum_{n=1}^{\infty} \frac{1}{(n+1)^s} \). This special case corresponds to the Hurwitz zeta function.

**Theorem 2.8 (Hurwitz zeta function).** The formally defined series

\[ \zeta(s, q) = \sum_{n=0}^{\infty} (n + q)^{-s} \]

with \( \text{Re}(s) > 1 \) and \( \text{Re}(q) > 0 \) is absolutely convergent and can be extended by analytic continuation to a meromorphic function on all \( s \neq 1 \). At \( s = 1 \), it has a simple pole with residue 1. Moreover, the function has an integral representation in terms of the Mellin transform as
\[ \zeta(s, q) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-qt} \frac{dt}{1 - e^{-t}} \]

for \( \Re(s) > 1 \) and \( \Re(q) > 0 \).

**Remark 2.9.** We observe that if a Dirichlet series \( L(s) \) with positive terms in the sum has holomorphic extension to \( \Re(s) > p \), where \( p \) is a simple pole, then \( \lim_{s \to p^+} L(s) = +\infty \).

We will construct and describe potentials of continuous type. Indeed, motivated by theorem 2.4, one can prove directly the following proposition.

**Proposition 2.10.** Let \( P(x) \) be a real polynomial with degree \( k \) such that \( P > 0 \) in \((-1, \infty)\) and \( P > 1 \) in \( \mathbb{N} \). If \( \varphi \) is the locally constant potential defined by \( \varphi|C_i = -\log P(i) \) for \( i \in \mathbb{N} \), then \( s_\infty = 1 \) and \( \varphi \) is of continuous type.

Similarly, based on theorems 2.7 and 2.8, we obtain the following proposition.

**Proposition 2.11.** Let \((\lambda_n)_n\) be a strictly increasing sequence of real numbers with \( 1 \leq \lambda_1 \), growing at least as fast as some positive power of \( n \). Assume that there exists \( \varepsilon > 0 \) and a sequence of real numbers \((a_n)_n\) such that the function \( f : (0, \varepsilon) \to \mathbb{R} \) defined by

\[ f(t) := \sum_{m=1}^\infty e^{-\lambda_m t} \]

has asymptotic expansion

\[ f(t) \sim \sum_{n=-1}^\infty a_n t^n \]

as \( t \to 0 \).

If \( \varphi \) is the locally constant potential defined by \( \varphi|C_n := -\log \lambda_n \) for \( n \in \mathbb{N} \), then \( s_\infty = 1 \) and \( \varphi \) is of continuous type.

We now state a result on the meromorphic extension of certain Dirichlet series [GT, theorem 1] that is related to the construction of potentials of discontinuous type.

**Theorem 2.12.** Let \( \eta \) and \( \theta \) be real numbers; then the Dirichlet series

\[ L_{\eta, \theta}(s) = \sum_{k=2}^{\infty} \frac{(\log k)^\eta}{(k \log k)^\theta} s \]

admits an analytic continuation to the whole complex plane except at the line joining 1 with \(-\infty\). This line gives a branch cut of the function, whose nature depends on the parameters. The singular expansion of the function around \( s = 1 \) starts with

\[ \Gamma(\eta - \theta + 1)(s - 1)^{\theta - \eta - 1} \text{ for } \theta - \eta \notin \mathbb{N} \]

\[ \frac{(-1)^{m-1}}{(m-1)!}(s - 1)^{m-1} \log \frac{1}{s-1} \text{ for } \theta - \eta = m \in \mathbb{N} \]

**Remark 2.13.** We observe that if a Dirichlet series \( L(s) \) with positive terms in the sum admits an analytic continuation to the whole complex plane except the line joining 1 with \(-\infty\), then the map \([1, \infty) \ni s \to \hat{L}(s) \in \mathbb{R} \) is continuous, where

\[ \hat{L}(s) = \begin{cases} L(s) & \text{if } s > 1; \\ \lim_{s \to 1^+} L(s) & \text{if } s = 1. \end{cases} \]
From theorem 2.12 and remark 2.13, we obtain the following proposition, which allows us to construct and describe potentials of discontinuous type.

**Proposition 2.14.** Let $\theta > 0$. If $\varphi$ is the locally constant potential defined by $\varphi|_{C_n} := -\log \left( (n+1) \log^\theta (n+1) \right)$ for $n \in \mathbb{N}$, then $s_\infty = 1$ and $\varphi$ is of discontinuous type.

### 3. Suspension flows over countable Markov shifts

In this section, we recall definitions and properties of the thermodynamic formalism for suspension flows over countable Markov shifts.

#### 3.1. Suspension semi-flows and invariant measures

Let $(\Sigma, \sigma)$ be a countable Markov shift and let $\tau: \Sigma \to \mathbb{R}$ be a positive continuous function such that for every $x \in \Sigma$, we have $\sum_{n=0}^{\infty} \tau(\sigma^n x) = \infty$. Consider the space $Y = \{ (x, t) \in \Sigma \times \mathbb{R} : 0 \leq t \leq \tau(x) \}$ with the points $(x, \tau(x))$ and $(\sigma(x), 0)$ identified for each $x \in \Sigma$. The suspension semi-flow over $\sigma$ with roof function $\tau$ is the semi-flow $\Phi = (\varphi_t)_{t \geq 0}$ on $Y$ defined by $\varphi_t(x, s) = (x, s + t)$ whenever $s + t \in [0, \tau(x)]$.

A probability measure $\mu$ on $Y$ is $\Phi$-invariant if $\mu(\varphi_t^{-1} A) = \mu(A)$ for every $t \geq 0$ and every measurable set $A \subset Y$. Denote by $\mathcal{M}_\Phi$ the space of $\Phi$-invariant probability measures on $Y$.

Let $\mathcal{M}_\sigma(\tau) := \{ \nu \in \mathcal{M}_\sigma : \int \tau d\nu < \infty \}$. Note that if $\Sigma$ is compact, then $\mathcal{M}_\sigma(\tau) = \mathcal{M}_\sigma$. A result by Ambrose and Kakutani [AK] implies that if $m$ denotes the Lebesgue measure and $\nu \in \mathcal{M}_\sigma(\tau)$ then

$$
\frac{(\nu \times m)|_Y}{(\nu \times m)(Y)} \in \mathcal{M}_\Phi.
$$

Moreover, if $(\Sigma, \sigma)$ is a subshift of finite type defined over a finite alphabet, then equation (3.1) defines a bijection between $\mathcal{M}_\sigma$ and $\mathcal{M}_\Phi$. When $(\Sigma, \sigma)$ is a countable Markov shift and $\tau: \Sigma \to \mathbb{R}^+$ is bounded away from zero and not bounded above, then there is a bijection between $\mathcal{M}_\sigma(\tau)$ and $\mathcal{M}_\Phi$. Note though that there might be measure $\nu \in \mathcal{M}_\sigma$ such that $\int \tau d\nu = \infty$. In this case, the measure obtained in equation (3.1) is an infinite flow invariant measure. The more subtle case is when $(\Sigma, \sigma)$ is a countable Markov shift and $\tau: \Sigma \to \mathbb{R}^+$ is not bounded away from zero, since it is possible that for an infinite (sigma-finite) $\sigma$-invariant measure $\nu$, we have $\int \tau d\nu < \infty$. In this case, the measure $(\nu \times m)|_Y / (\nu \times m)(Y) \in \mathcal{M}_\Phi$.

#### 3.2. Thermodynamic formalism

The entropy of a flow with respect to an invariant measure, denoted $h_\Phi(\mu)$, can be defined as the entropy of the corresponding time one map. The entropy of the flow is related to the entropy of the shift. The following formula was obtained by Abramov [Ab] and later
generalized by Savchenko [Sav, theorem 1]. Let \( \mu \in \mathcal{M}_\Phi \) be an ergodic measure such that 
\( \mu = (\nu \times m)|_Y/(\nu \times m)(Y) \), where \( \nu \) is a sigma-finite (finite or infinite) invariant measure for the shift with \( \int \tau \, d\nu < \infty \). Then

\[
    h_\Phi(\mu) = \frac{h_\sigma(\nu)}{\int \tau \, d\nu}. \tag{3.2}
\]

It also possible to relate the integral of a potential on the flow to a corresponding one on the base. Indeed, given a continuous function \( g : Y \to \mathbb{R} \), we define the function \( \Delta g : \Sigma \to \mathbb{R} \) by

\[
    \Delta g(x) = \int_0^1 g(x, t) \, dt.
\]

The function \( \Delta g \) is also continuous; moreover, if \( \mu \in \mathcal{M}_\Phi \) is the normalization of \( \nu \times m \), then

\[
    \int_Y g \, d\mu = \frac{\int_\Sigma \Delta g \, d\nu}{\int_\Sigma \tau \, d\nu}.
\]

Thermodynamic formalism for suspension flows over countable Markov shifts has been stud-
ied by several people (see for example [BI1, IJ, IJT, JKL, Ke2, Sav]). The following result
summarizes some of the results that have been obtained on thermodynamic formalism for
suspension flows over countable Markov shifts.

**Theorem 3.1.** Let \( (\Sigma, \sigma) \) be a topologically mixing countable Markov shift and \( \tau : \Sigma \to \mathbb{R}^+ \) be a roof function of summable variations. Let \( (Y, \Phi) \) be the associated suspension semi-flow. Let \( g : Y \to \mathbb{R} \) be a function such that \( \Delta g : \Sigma \to \mathbb{R} \) is locally Hölder. Then the following equalities hold:

\[
    P_\Phi(g) := \lim_{t \to \infty} \frac{1}{t} \log \left( \sum_{\nu,(x,0)\in\mathcal{K},0<s\leq t} \exp \left( \int_0^s g(\varphi_k(x,0)) \, dk \right) \chi_{C_0}(x) \right) 
    = \inf\{ t \in \mathbb{R} : P_\sigma(\Delta g - t\tau) \leq 0 \} = \sup\{ t \in \mathbb{R} : P_\sigma(\Delta g - t\tau) \geq 0 \} 
    = \sup\{ P_{\sigma|K}(\varphi) : K \in \mathcal{K} \}, 
    = \sup \left\{ h_\mu(\Phi) + \int_Y g \, d\mu : \mu \in \mathcal{E}_\Phi \text{ and } -\int_Y g \, d\mu < \infty \right\},
\]

where \( \mathcal{K} \) is the set of all compact and \( \Phi \)-invariant sets, \( P_\sigma \) is the classical topological pressure of the potential \( \varphi \) restricted to the compact and \( \sigma \)-invariant set \( K \), and \( \mathcal{E}_\Phi \) is the set of ergodic \( \Phi \)-invariant measures.

**Corollary 3.2.** Let \( (\Sigma, \sigma) \) be a topologically mixing countable Markov shift and \( \tau : \Sigma \to \mathbb{R}^+ \) be a roof function of summable variations. Let \( (Y, \Phi) \) be the associated suspension flow. The topological entropy of the semi-flow is given by

\[
    h(\Phi) = \inf\{ t \in \mathbb{R} : P_\sigma(-t\tau) \leq 0 \}.
\]

Recall that a measure \( \mu \in \mathcal{E}_\Phi \) is called an *equilibrium measure* for \( g \) if \( P_\Phi(g) = h(\mu) + \int g \, d\mu \). The following result is a summary of [IJT, theorems 3.4 and 3.5],

**Theorem 3.3.** Let \( \Phi \) be a finite entropy suspension semi-flow on \( Y \) defined over a count-
able Markov shift \( (\Sigma, \sigma) \) and a locally Hölder roof function \( \tau \). Let \( g : Y \to \mathbb{R} \) be a continuous
function such that $\Delta_\epsilon$ is locally Hölder. In the following cases, there exists an equilibrium measure for $g$:

(a) If $P_\sigma(\Delta_\epsilon - P_\Phi(g)\tau) = 0$ and $\Delta_\epsilon - P_\Phi(g)\tau$ has an equilibrium measure $\nu_\epsilon$ satisfying $\int \tau \, d\nu_\epsilon < \infty$;

(b) If $P_\sigma(\Delta_\epsilon - P_\Phi(g)\tau) = 0$ and the potential $\Delta_\epsilon - P_\Phi(g)\tau$ has an infinite Ruelle–Perron–Frobenius measure $\nu_\epsilon$ (i.e. $\nu_\epsilon = hm$ where $h$ and $m$ are the density and the conformal measure provided by the Ruelle–Perron–Frobenius theorem with potential $g$) and $\int \tau \, d\nu_\epsilon < \infty$.

In any other case, the potential $g$ does not have an equilibrium measure. Moreover, every potential $g$, for which $\Delta_\epsilon$ is locally Hölder, has at most one equilibrium state.

A thorough account of the case in which $\Sigma$ is a compact subshift of finite type can be found in [PP].

4. Time change for flows

We begin this section showing that two suspension flows defined over the same base system are strongly related. Not only one is orbit equivalent to the other, but one is a time change of the other. Indeed, let $\Sigma$ be a fixed topologically mixing countable Markov shift, and let $\tau_1 : \Sigma \to \mathbb{R}$ and $\tau_2 : \Sigma \to \mathbb{R}$ be two positive roof functions. Denote by $(Y_1, \Phi_1)$ and $(Y_2, \Phi_2)$ the corresponding semi-flows.

**Definition 4.1.** The time change map $\pi : Y_1 \to Y_2$ is defined by

$$\pi(x, s) = \left( x, \frac{\tau_2(x)}{\tau_1(x)} s \right).$$

**Lemma 4.2.** The semi-flow $(Y_1, \Phi_1)$ is a time change of $(Y_2, \Phi_2)$.

**Proof.** In definition 4.1, we introduced the map $\pi : Y_1 \to Y_2$. This map preserves the orbit structure. It actually sends leaves to leaves and corresponds to the time change. □

It should be stressed though that the spaces $Y_1$ and $Y_2$ are different. With this time change map, we can relate potentials defined in $Y_1$ with those defined in $Y_2$. Let $\psi_2 : Y_2 \to \mathbb{R}$ and define $\psi_1 : Y_1 \to \mathbb{R}$ by $\psi_1(x, r) := \psi_2 \circ \pi(x, r)$. Note that

$$\Delta_{\psi_2}(x) = \int_0^{\tau_2(x)} \psi_2(x, r) \, dr$$

and

$$\Delta_{\psi_1}(x) = \int_0^{\tau_1(x)} \psi_2 \left( x, \frac{\tau_2(x)}{\tau_1(x)} r \right) \, dr.$$ 

Moreover, for every $x \in \Sigma$, we have

$$\Delta_{\psi_1}(x) = \frac{\tau_1(x)}{\tau_2(x)} \Delta_{\psi_2}(x). \quad (4.1)$$

4.1. The compact setting: preservation of thermodynamic properties

This subsection is devoted to proving that when the space is compact, thermodynamic properties are preserved by regular time changes. Indeed, we prove that the entropy changes at most by a factor that depends on the quotient of the roof functions. The closer this quotient is to 1,
the less the entropy changes. This result is obtained as a consequence of a more general result in which we describe the way in which the pressure varies. 

Note that the thermodynamic formalism for Axiom A flows was studied by Bowen and Ruelle [BR], making use of the fact that these flows can be modeled by suspension flows over subshifts of finite type with a Hölder roof function. Thus, the following result shows that thermodynamic properties are preserved by regular time changes not only at a symbolic level, but also in the differentiable category of Axiom A flows.

**Theorem 4.3.** Let \((\Sigma, \sigma)\) be a transitive subshift of finite type defined over a finite alphabet, and let \(\tau_1: \Sigma \to \mathbb{R}\) and \(\tau_2: \Sigma \to \mathbb{R}\) be two positive Hölder roof functions. Denote by \((Y_1, \Phi_1)\) and \((Y_2, \Phi_2)\) the corresponding semi-flows. Let \(\psi: Y_1 \to \mathbb{R}\) be a non-positive potential such that \(\Delta_{\psi_1}\) is a Hölder function and define \(\psi_1(x, r) := \psi_2 \circ \pi(x, r)\). For any \(C > 1\) such that

\[
\frac{1}{C} \leq \frac{\tau_1(x)}{\tau_2(x)} \leq C \text{ for every } x \in \Sigma, \tag{4.2}
\]

if \(P_{\Phi_1}(\psi_1) \geq 0\), then

\[
\frac{1}{C} P_{\Phi_2}(C\psi_2) \leq P_{\Phi_1}(\psi_1) \leq CP_{\Phi_2}\left(\frac{\psi_2}{C}\right)
\]

and if \(P_{\Phi_1}(\psi_1) < 0\), then

\[
CP_{\Phi_2}(C\psi_2) \leq P_{\Phi_1}(\psi_1) \leq \frac{1}{C} P_{\Phi_2}\left(\frac{\psi_2}{C}\right).
\]

Moreover, both \(\psi_1\) and \(\psi_2\) have a unique equilibrium state.

**Proof.** Since both \(\tau_1\) and \(\tau_2\) are positive continuous functions defined over a compact space, there exists \(C > 1\) such that \((4.2)\). Therefore, for every \(t \geq 0\) and every \(x \in \Sigma\), we have that

\[
-\frac{t}{C} \tau_2(x) \geq -t\tau_1(x) \geq -tC\tau_2(x). \tag{4.3}
\]

Assume that \(P_{\Phi_1}(\psi_1) \geq 0\); the other case will be considered later. It follows from equations \((4.2)\) and \((4.3)\), remark \((4.1)\) and \(\Delta_{\psi_2} \leq 0\), that for every \(t \geq 0\)

\[
\Delta_{\psi_1} - t\tau_1 \geq \frac{\tau_1}{\tau_2} \Delta_{\psi_2} - tC\tau_2 \geq C\Delta_{\psi_2} - tC\tau_2 = \Delta_{C\psi_2} - tC\tau_2. \tag{4.4}
\]

Evaluating \(P_{\sigma}(\Delta_{\psi_1} - t\tau_1)\) and \(P_{\sigma}(C\Delta_{\psi_2} - tC\tau_2)\) at \(t = P_{\Phi_1}(\psi_1) \geq 0\), we conclude from equation \((4.4)\) that

\[
0 = P_{\sigma}(\Delta_{\psi_1} - P_{\Phi_1}(\psi_1)\tau_1) \geq P_{\sigma}(C\Delta_{\psi_2} - P_{\Phi_1}(\psi_1)C\tau_2).
\]

Hence,

\[
P_{\Phi_2}(C\psi_2) \leq CP_{\Phi_1}(\psi_1). \tag{4.5}
\]

On the other hand, for every \(t \geq 0\), we have

\[
\Delta_{\psi_1} - t\tau_1 \leq \frac{\tau_1}{\tau_2} \Delta_{\psi_2} - \frac{t}{C} \tau_2 \leq \frac{1}{C} \Delta_{\psi_2} - \frac{t}{C} \tau_2 = \Delta_{\psi_2} - \frac{t}{C} \tau_2.
\]

2857
Therefore,
\[ 0 = P_\sigma (\Delta \psi_1 - P_{\Phi_1}(\psi_1) \tau_1) \leq P_\sigma \left( \Delta \frac{\psi}{C} \phi_1 - \frac{P_{\Phi_1}(\psi_1)}{C} \tau_2 \right). \]

Hence,
\[ \frac{P_{\Phi_1}(\psi_1)}{C} \leq P_{\Phi_2} \left( \frac{\psi_2}{C} \right). \tag{4.6} \]

Under the assumption that \( P_{\Phi_1}(\psi_1) \geq 0 \), the result follows from equations (4.5) and (4.6). Assume now that \( P_{\Phi_1}(\psi_1) < 0 \). It follows from equation (4.2) that for every \( t < 0 \) and every \( x \in \Sigma \),
\[ -\frac{t}{C} \tau_2 (x) < -t \tau_1 (x) < -t C \tau_2 (x). \tag{4.7} \]

Recalling that \( \Delta \psi_2 \leq 0 \) and applying remark (4.1), we conclude from equation (4.7) that for every \( t < 0 \)
\[ \Delta \psi_1 - t \tau_1 \geq \frac{\tau_1}{\tau_2} \Delta \phi_2 - \frac{t}{C} \tau_2 \geq C \Delta \psi_2 - \frac{t}{C} \tau_2 = \Delta \psi_2 - \frac{t}{C} \tau_2. \tag{4.8} \]

Evaluating \( P_\sigma (\Delta \psi_1 - t \tau_1) \) and \( P_\sigma (\Delta \psi_2 - \frac{t}{C} \tau_2) \) at \( t = P_{\Phi_1}(\psi_1) < 0 \), we conclude from equation (4.8) that
\[ 0 = P_\sigma (\Delta \psi_1 - P_{\Phi_1}(\psi_1) \tau_1) \geq P_\sigma \left( \Delta \psi_2 - \frac{P_{\Phi_1}(\psi_1)}{C} \tau_2 \right). \]

Hence,
\[ P_{\Phi_2}(C \psi_2) \leq \frac{1}{C} P_{\Phi_1}(\psi_1). \tag{4.9} \]

On the other hand, for every \( t < 0 \), we have
\[ \Delta \psi_1 - t \tau_1 \leq \frac{\tau_1}{\tau_2} \Delta \psi_2 - \frac{t}{C} \tau_2 \leq \frac{1}{C} \Delta \psi_2 - t C \tau_2 = \Delta \psi_2 - t C \tau_2. \]

Therefore,
\[ 0 = P_\sigma (\Delta \psi_1 - P_{\Phi_1}(\psi_1) \tau_1) \leq P_\sigma \left( \Delta \psi_2 - P_{\Phi_1}(\psi_1) C \tau_2 \right). \]

Hence,
\[ C P_{\Phi_1}(\psi_1) \leq P_{\Phi_2} \left( \frac{\psi_2}{C} \right). \tag{4.10} \]

Under the assumption \( P_{\Phi_1}(\psi_1) < 0 \), it follows from equations (4.9) and (4.10) that
\[ C P_{\Phi_2}(C \psi_2) \leq P_{\Phi_1}(\psi_1) \leq \frac{1}{C} P_{\Phi_2} \left( \frac{\psi_2}{C} \right). \]
The fact that both systems have unique equilibrium measures for Hölder potentials was proved by Bowen and Ruelle in [BR].

**Remark 4.4.** The assumption in theorem 4.3 that $\psi_2 \leq 0$ is not very restrictive since, if $\sup \psi_2 < \infty$, then the potential $\psi_2 - \sup \psi_2$ shares the same thermodynamic properties as $\psi_2$.

Since $P_\psi(0) = h(\Phi)$, theorem 4.3 has the following consequence for entropy.

**Corollary 4.5.** Let $(\Sigma, \sigma)$ be a transitive subshift of finite type defined over a finite alphabet. Let $\tau_1, \tau_2 : \Sigma \to \mathbb{R}^+$ be two roof functions and $\Phi_1 : Y_1 \to Y_1$ and $\Phi_2 : Y_2 \to Y_2$ the corresponding suspension semi-flows. Then, for any $C > 1$ satisfying (4.2), we have

$$\frac{h(\Phi_2)}{C} \leq h(\Phi_1) \leq Ch(\Phi_2).$$

### 4.2. The non compact setting: non-preservation of thermodynamic properties

This subsection is devoted to showing that if the base of the suspension flow is not assumed to be compact, then, despite the regularity assumed in the roof function or in the potentials considered, no natural thermodynamic property is necessarily preserved by time changes. We will exhibit explicit examples showing this. In order to construct such examples, we will mostly make use of the techniques developed in section 2.3. Indeed, we have the following results.

**Lemma 4.6.** Let $(\Sigma, \sigma)$ be the full shift on a countable alphabet. Let $a, b > 0$ and $\varphi_{(a,b)}$ be the family of locally constant potentials defined, for every $n \in \mathbb{N}$, by $\varphi_{(a,b)}|C_n := -b \log(n + a)$. Denote by $(Y, \Phi_{(a,b)})$ the suspension semi-flow with base $\Sigma$ and roof function $\tau = -\varphi_{(a,b)}$. This family of flows has the following properties.

(a) $s_\infty(\Phi_{(a,b)}) = \frac{1}{b}$ independently of $a > 0$.
(b) $P_\sigma(\varphi_{(a,b)}) = P_\sigma(b\varphi_{(a,1)})$, in particular $h(\Phi_{(a,b)}) = \frac{h(\Phi_{(a,1)})}{b}$.
(c) For every $b, c > 0$, there exists $a > 0$ such that $h(\Phi_{(a,b)}) > c$.
(d) For every $c > 1$, there exists a unique $a > 0$ such that $h(\Phi_{(a,1)}) = c$.
(e) For every $0 < c < 1 < d$, there exists a unique pair $(a, b)$ such that $s_\infty(\Phi_{(a,b)}) = c$ and $h(\Phi_{(a,b)}) = d$.

**Proof.** Part (a) follows from the observation that $e^{P_\sigma(\varphi_{(a,b)})} = \zeta(h, a + 1)$ and an application of theorem 2.8. Part (b) follows from definition; indeed:

$$P_\sigma(b\varphi_{(a,1)}) = \log \sum_{n=1}^{\infty} \exp \left( b\varphi_{(a,1)}|C_n \right) = \log \sum_{n=1}^{\infty} \exp \left( -b \log(n + a) \right)$$

$$= \log \sum_{n=1}^{\infty} \exp \left( \varphi_{(a,b)}|C_n \right) = P_\sigma(\varphi_{(a,b)}).$$

For part (c), it is sufficient to prove that for every $c > 1$ there exists $a > 0$ such that $h(\Phi_{(a,1)}) > c$, which is equivalent to

$$\inf \{ t \in \mathbb{R} : P_\sigma(t\varphi_{(a,1)}) \leq 0 \} > c.$$
We will prove that there exists $a > 0$ such that for $e^{P_{n}(t\phi(a,t))} = e^{P_{n}(\phi(a,t))} = \zeta(t,a + 1)$, we have that
\[ \zeta(t, a + 1) > 1 \text{ for all } t < c. \]

For any fixed $a > 0$, we have that $\zeta(t, a + 1)$ is decreasing in $t > 1$; therefore, it is enough to prove that there exists $a > 0$ such that
\[ \zeta(c, a + 1) > 1. \]

We notice that for every $c > 1$, $\zeta(c,1) > 1$, this, together with the fact that the map $a \mapsto \zeta(c,a + 1)$ is continuous, proves the result.

For part (d), let $a^* > 0$ such that $h(\Phi_{a^*}) > c$. By [Al, theorem 2.1], the map $a \mapsto \zeta(c,1 + a)$ is strictly decreasing in $[0, \infty]$, and it is clear that by taking $a$ big enough $\zeta(c,1 + a) < 1$, there exists a unique $a^{**} > a^*$ such that $\zeta(c,1 + a^{**}) = 1$. This implies that $h(\Phi_{a^{**}}) = c$, which completes the proof.

For part (e), let $a > 0$ such that $h(\Phi_{a}) = \varepsilon$. Then, $h(\Phi_{a}) = d$ and $s_{\infty}(\Phi_{a}) = c$.□

We state the following elementary lemma from calculus in order to justify the definition of certain potentials.

**Lemma 4.7.** Let $A(n, \theta) := \frac{1}{(d-1) \log^a r} \theta$ and $B(n, \gamma) := \sup \{ \theta > 1 : \gamma A(n, \theta) > 1 \}$ for $n \in \mathbb{N}, \theta > 1$ and $\gamma > 0$.

(a) For every $r = 2, 3, \ldots$
\[ A(r, \theta) < \sum_{k=r}^{\infty} \frac{1}{k \log^a k} < A(r - 1, \theta). \]

(b) For every $\varepsilon > 0$,
\[ 4 = \min \{ r \in \{ 2, 3, \ldots \} : A(r - 1, \theta) - A(r, \theta) \text{ is uniformly bounded for } \theta \in (1 + \varepsilon, \infty) \}. \]

(c) For every $r \geq 4$, $\gamma > 0$, $B(r, \gamma) < B(r - 1, \gamma)$, the map $(1, \infty) \ni \theta \mapsto \sum_{k=r}^{\infty} \frac{\gamma}{k \log^a k}$ is continuous strictly decreasing with
\[ \sum_{k=r}^{\infty} \frac{\gamma}{k \log^a k} = \begin{cases} < 1 & \text{if } \theta > B(r - 1, \gamma); \\ > 1 & \text{if } \theta < B(r, \gamma). \end{cases} \]

In figure 2, we plot the graphs of the functions $\theta \mapsto 5A(4, \theta)$ and $\theta \mapsto 5A(3, \theta)$ defined in lemma 4.7.

**Lemma 4.8.** Let $(\Sigma, \sigma)$ be the full shift on a countable alphabet. Let $\theta \in \mathbb{R}^+, k \in \mathbb{N}, 0 < \gamma \leq 2 + k$, and $\varphi_{\theta, k, \gamma}$ be the family of locally constant potentials defined for every $n \in \mathbb{N}$, by $\varphi_{\theta, k, \gamma}|C_n := -\log \left( \frac{(n+k+1) \log^a(n+k+1)}{\gamma} \right)$. If $\varphi_{\theta, k, \gamma} < 0$, denote by $(Y, \Phi_{\theta, k, \gamma})$ the suspension semiflow with base $\Sigma$ and roof function $\tau = -\varphi_{\theta, k, \gamma}$. This family of flows has the following properties.

(a) For every $\theta \in \mathbb{R}^+$ and $k \in \mathbb{N}, \varphi_{\theta, k, \gamma} < 0$ and $s_{\infty}(\Phi_{\theta, k, \gamma}) = 1$.

(b) The pressure satisfies
\[ P_{\sigma}(\varphi_{\theta,k,\gamma}) = \begin{cases} < \infty & \text{if } \theta > 1; \\ = \infty & \text{if } \theta \leq 1. \end{cases} \]

(c) If \( \theta > 1 \) and \( k \geq 2 \), \( P(\varphi_{\theta,k,\gamma}) = 0 \) for a unique \( \theta_{k,\gamma} \in (B(k + 2, \gamma), B(k + 1, \gamma)) \).

Moreover,

\[ P(\varphi_{\theta,k,\gamma}) = \begin{cases} \text{positive} & \text{if } \theta < \theta_{k,\gamma}; \\ \text{negative} & \text{if } \theta > \theta_{k,\gamma}, \end{cases} \]

and

\[ h_{\top}(\Phi_{\theta,k,\gamma}) = \begin{cases} > 1 & \text{if } \theta < \theta_{k,\gamma}; \\ = 1 & \text{if } \theta \geq \theta_{k,\gamma}. \end{cases} \]

(d) If \( \theta > 1 \), the right-derivative of the pressure satisfies

\[ \frac{d}{ds} P_{\sigma}(s \varphi_{\theta,k,\gamma})|_{s=1} = \begin{cases} > -\infty & \text{if } \theta > 2; \\ = -\infty & \text{if } 1 < \theta \leq 2. \end{cases} \]

**Proof.** To prove (a), we notice that for every \( \theta > 0, k \in \mathbb{N}, 0 < \gamma \leq 2 + k \)

\[ \varphi_{\theta,k,\gamma} < -\log \left( \frac{(2 + k) \log^{\theta}(2 + k)}{\gamma} \right) < -\log(\log^{\theta}(2 + k)) < 0, \]

and

\[ e^{P_{\sigma}(s \varphi_{\theta,k,\gamma})} = \gamma \mathcal{L}_{0,\theta}(s) - \sum_{n=2}^{k+1} \frac{\gamma}{(k(\log k)^\theta)^n} = \sum_{n=k+2}^{\infty} \frac{\gamma}{(k(\log k)^\theta)^n}, \quad (4.11) \]
in particular, an immediate consequence of theorem 2.12 is \( s_\infty(\Phi_{\theta,k,\gamma}) = 1 \). The proof of (b) follows from (4.11) and the equivalences \( P_\sigma(s_{\Phi_{\theta,k,\gamma}}) = \infty \) iff \( L_{0,\theta}(s) = \infty \) iff \( s < 1 \) or \( s = 1 \) and \( \theta \leq 1 \). The proof of (c) follows from lemma 4.7. Finally, the proof of (d) follows from right-differentiating in (4.11), indeed,

\[
\frac{d}{ds} P_\sigma(s_{\Phi_{\theta,k,\gamma}})|_{s=1} = \gamma \left( \frac{d}{ds} L_{0,\theta}(s)|_{s=1} - \frac{1}{\log 2} \sum_{n=2}^{k+1} (k(\log k)^n)|_{s=1} \right) e^{-P_\sigma(s_{\Phi_{\theta,k,\gamma}})}
\]

\[
= \left( \frac{d}{ds} L_{0,\theta}(s)|_{s=1} + O(1) \right) O(1),
\]

and

\[
\frac{d}{ds} L_{0,\theta}(s)|_{s=1} = \begin{cases} > -\infty & \text{if } \theta > 2; \\ = -\infty & \text{if } 1 < \theta \leq 2, \end{cases}
\]

which concludes the proof.

Motivated by lemmas 4.6 and 4.8, we construct examples showing that no natural thermodynamic property is necessarily preserved by time changes for suspension semi-flows which base the full shift on \( \mathbb{N} \).

Let \( \Sigma \) be the full shift on \( \mathbb{N} \). We will define in each example particular roof functions \( \tau_1, \tau_2 : \Sigma \to \mathbb{R}^+ \) constant on cylinder of length 1 and we will denote by \( (Y_1, \Phi_1) \) and \( (Y_2, \Phi_2) \) the suspension semi-flows with base \( \Sigma \) and roof functions \( \tau_1 \) and \( \tau_2 \), respectively.

**Example 4.9 (Time change of a flow with a unique maximal entropy measure such that the time-changed flow does not have a measure of maximal entropy).** Consider the flow \( \Phi_1 \) associated with \( \tau_1|_{C_0} := \log((n+1)(n+k)) \). Choose \( k \in \mathbb{N} \) such that

\[
\sum_{n=1}^{\infty} \frac{1}{(n+k) \log^2(n+k)} < 1
\]

and consider the flow \( \Phi_2 \) associated with \( \tau_2|_{C_0} := \log((n+k) \log^2(n+k)) \). Note that

\[
P_\sigma(-t\tau_1) = \log \sum_{n=1}^{\infty} \left( \frac{1}{n(n+1)} \right)^t.
\]

In particular, we have that the entropy of the flow is (see corollary 3.2)

\[
h(\Phi_1) := \inf \{ t \in \mathbb{R} : P_\sigma(-t\tau_1) \leq 0 \} = 1.
\]

Moreover, the normalization of the measure \( \mu_1 = \nu_1 \times m \), where \( \nu_1 \) is the equilibrium measure for \( -\tau_1 \) and \( m \) is the Lebesgue measure, is the measure of maximal entropy for \( \Phi_1 \). On the other hand,

\[
P_\sigma(-t\tau_2) = \log \sum_{n=1}^{\infty} \left( \frac{1}{(n+k) \log^2(n+k)} \right)^t.
\]

Thus,

\[
P_\sigma(-t\tau_2) = \begin{cases} \infty & \text{if } t < 1; \\ \text{negative} & \text{if } t \geq 1. \end{cases}
\]
Therefore, \( h(\Phi_2) := \inf \{ t \in \mathbb{R} : P_\sigma(-t\tau_2) \leq 0 \} = 1 \). But there is no measure of maximal entropy in virtue of theorem 3.3.

**Example 4.10 (Time change of a finite entropy flow such that the time-changed flow has infinite entropy).** The flow \( \Phi_1 \) associated with \( \tau_1 |_{C_n} := \log((n(n+1)) \) has entropy equal to 1 (see example 4.9), while the flow \( \Phi_2 \) associated with \( \tau_2 |_{C_n} := 1 \) has infinite entropy.

**Example 4.11 (Time-changed flows such that there is no bijection between the spaces of invariant probability measures).** Consider the flow \( \Phi_1 \) associated with \( \tau_1 |_{C_n} := \log((n(n+1)) \) has entropy equal to 1 (see example 4.9), this flow has entropy equal to 1 and the measure of maximal entropy is the normalization of the product measure \( \nu_1 = \nu_1 \times m \), where \( \nu_1 \) is the Gibbs (Bernoulli) measure given by

\[
\nu_1(C_n) = \frac{1}{n(n+1)}.
\]

Consider now the flow \( \Phi_2 \) associated with \( \tau_2 |_{C_n} := n \). Note that

\[
P_\sigma(-t\tau_2) = \log \sum_{n=1}^{\infty} (e^{-n})^n.
\]

We claim that the normalized (with respect to \( Y_2 \)) measure \( \nu_1 \times m \notin \mathcal{M}_\Phi \). Indeed, in order for \((\nu_1 \times m)/(\nu_1 \times m(Y_2)) \in \mathcal{M}_\Phi \), a necessary condition that needs to be satisfied is that

\[
\int \tau_2 \, d\nu_1 < \infty.
\]

However,

\[
\int \tau_2 \, d\nu_1 = \sum_{n=1}^{\infty} \frac{n}{n(n+1)} = \infty.
\]

In particular, the measure \( \nu_1 \times m \) is an infinite \( \Phi_2 \)-invariant measure.

**Remark 4.12.** In the next two examples, we use a tool mentioned in [BI1, section 4.3]. Given a suspension semi-flow \( \Phi \) on \( Y \) over \( \Sigma \), we can choose any locally Hölder function \( \varphi : \Sigma \to \mathbb{R} \), and, by a result in [BRW], there exists a continuous function \( \psi : Y \to \mathbb{R} \) such that \( \Delta \psi = \varphi \). In particular, we can implicitly define a continuous function \( \psi : Y \to \mathbb{R} \) by defining \( \Delta \psi |_{C_n} := a_n \), where \((a_n)_n \) is a sequence of real numbers. Let \( \psi_2 : Y_2 \to \mathbb{R} \) and \( \psi_1 : Y_1 \to \mathbb{R} \) defined by \( \psi_1 = \psi_2 \circ \pi \). It was noted in equation (4.1) that

\[
\Delta \psi_1(x) = \frac{\tau_1(x)}{\tau_2(x)} \Delta \psi_2(x). \tag{4.12}
\]

There exists \( \psi_2 : Y_2 \to \mathbb{R} \) such that \( \Delta \psi_2 = \tau_2 \). Therefore, we obtain

\[
\Delta \psi_2 = \tau_2 \quad \text{and} \quad \Delta \psi_1 = \tau_1,
\]

and \( P_{\Phi_i}(\psi_i) = 1 + h(\Phi_i) \) for \( i \in \{1, 2\} \). This remark allows for the construction of examples with prescribed thermodynamic behavior.
Example 4.13 (Time-changed flows such that a potential has an equilibrium measure in one flow and not in the other). Let $k \in \mathbb{N}$ such that $\log \sum_{n=1}^{\infty} ((n + k)^{2} \log(n + k))^{-1} < 0$. Let $\tau_1, \tau_2 : \Sigma \to \mathbb{R}$ be two locally constant roof functions defined by $\tau_1|C_n = \log(n(n + 1))$ and $\tau_2|C_n := \log \left( (n + k)^2 \log(n + k) \right)$, respectively. It follows from remark 4.12 that there exists a continuous function $\psi_2 : Y_2 \to \mathbb{R}$ such that $\Delta \psi_2 = \tau_2$, and that the continuous function $\psi_1 : Y_1 \to \mathbb{R}$ defined by $\psi_1 = \psi_2 \circ \pi$ satisfies $\Delta \psi_1 = \tau_1$ and $P_{\sigma}(\psi_1) = P_{\sigma}(\psi_2) = 2$. In this example, $\psi_1$ has an equilibrium measure whereas $\psi_2$ does not. Indeed, $P_{\sigma}(\Delta \psi_2 - 2\tau_2) < 0$ and $P_{\sigma}(\Delta \psi_1 - 2\tau_1) = 0$. Moreover, the potential $\Delta \psi_2 - \Delta \psi_1$ has equilibrium measures $\nu$ and $\int \tau_2 d\nu < \infty$.

Example 4.14 (Time-changed flows such that a potential has an equilibrium measure in one flow and the other has an infinite equilibrium measure). Let $\tau_2 : \Sigma \to \mathbb{R}$ be a locally constant roof function such that:

$$P_{\sigma}(-t\tau_2) = \begin{cases} \infty & \text{if } t < 1; \\ 0 & \text{if } t = 1; \\ \text{negative} & \text{if } t > 1. \end{cases}$$

Moreover, we will assume that $\lim_{t \to 1^+} P_{\sigma}(-t\tau) = \infty$. Such a function exists in virtue of lemma 4.8. Let $\tau_1 : \Sigma \to \mathbb{R}$ be a roof function such that $P_{\sigma}(-\tau_1) = 0$ and $-\tau_1$ has an equilibrium measure. Then, it follows from remark 4.12 that there exist potentials $\psi_i : Y_i \to \mathbb{R}$ with $\Delta \psi_i = \tau_i$ for every $i \in \{1, 2\}$. Therefore, the potential $\psi_2$ has an infinite equilibrium measure while $\psi_1$ has a (finite) equilibrium measure.

In the last example, we build a one-parameter family of suspension semi-flow $\Phi_t$ for $t \geq 0$, such that, for some $t^* > 0$, the map $t \mapsto h(\Phi_t)$ is constant in the interval $[0, t^*)$ and it is real analytic (and non-constant) in the interval $[t^*, \infty)$.

Example 4.15 (Non analytic entropy map). Choose $N \in \mathbb{N}$ such that

$$\sum_{n>N} \frac{1}{2n \log^2(2n)} < 1.$$ 

Consider the flow $\Phi$ associated with $\tau|C_n := \log(2(n + N) \log^2(n + N))$. Thus,

$$P_{\sigma}(-t\tau) = \log \sum_{n>N} \left( \frac{1}{2n \log^2(2n)} \right)^t = \begin{cases} \text{infinity} & t < 1; \\ \text{negative} & t > 1. \end{cases}$$

The entropy of the flow $\Phi$ is equal to 1 (moreover, it has no measure of maximal entropy). Consider now the one parameter family of roof functions given by $\tau_2(x) = \tau(x) + t$. Denote by $\Phi_t$ the associated suspension semi-flow. Then

$$h(\Phi_t) = \begin{cases} 1 & \text{if } P_{\sigma}(-\tau_1) \leq 0; \\ > 1 & \text{if } P_{\sigma}(-\tau_1) > 0. \end{cases}$$

The map $(0, \infty) \ni t \mapsto P_{\sigma}(-\tau_1) \in \mathbb{R}$ is continuous, strictly increasing, and negative at $t = 0$. Let $t^*$ be the unique zero of this map. For $t \geq t^*$, the function $t \mapsto h(\Phi_t)$ is real analytic (and non constant). Moreover, in that range, there exists a unique measure of maximal entropy (whereas for $t \in [0, t^*)$ there is no measure of maximal entropy).
4.3. Preservation of thermodynamic properties

We have seen that the thermodynamic formalism of two flows, one obtained from a time change of the other, can be completely different. The following result establishes conditions for some features of the thermodynamic formalism to be similar in a sense that would be made precise.

**Theorem 4.16.** Let \( \Sigma \) be a topologically mixing countable Markov shift. Let \( \tau_1, \tau_2 : \Sigma \to \mathbb{R}^+ \) be two roof functions and \( \Phi_1 : Y_1 \to Y_1 \) and \( \Phi_2 : Y_2 \to Y_2 \) the corresponding suspension semiflows. If there exists a positive constant \( C > 0 \) such that, for every \( x \in \Sigma \), we have

\[
\frac{1}{C} \leq \frac{\tau_2(x)}{\tau_1(x)} < C, \tag{4.13}
\]

then either both systems have finite entropy or both have infinite entropy. In the former case, we have that

\[
\frac{h(\Phi_2)}{C} \leq h(\Phi_1) \leq Ch(\Phi_2).
\]

While the proof of this result can be obtained as a consequence of corollary 4.5 and the approximation property of the Gurevich pressure (theorem 2.1), we provide a different proof.

**Proof.** It is a consequence of equation (4.13) that for every \( t > 0 \) and every \( x \in \Sigma \), we have

\[
-(tC)\tau_1(x) \leq -t\tau_2(x) \leq -\frac{t}{C}\tau_1(x). \tag{4.14}
\]

If \( h(\Phi_2) = \infty \) then, by the definition of entropy, \( P_\sigma(-t\tau_2) > 0 \) for every \( t \in \mathbb{R} \). Hence, from the inequalities in (4.14), we have that \( P_\sigma(-t\tau_1) > 0 \) for every \( t \in \mathbb{R} \). Thus \( h(\Phi_1) = \infty \).

Since the assumptions on the roof functions are symmetric in \( \tau_1 \) and \( \tau_2 \), we have proved that \( h(\Phi_1) = \infty \) if and only if \( h(\Phi_2) = \infty \).

Assume now that \( h(\Phi_2) < \infty \). That is, \( P_\sigma(-h(\Phi_2)\tau_2) \leq 0 \). From the monotone properties of the pressure and equation (4.14), we obtain that

\[
0 \geq P_\sigma(-h(\Phi_2)\tau_2) \geq P_\sigma(-Ch(\Phi_2)\tau_1).
\]

That is, \( h(\Phi_1) \leq Ch(\Phi_2) \). For every \( \varepsilon > 0 \), we have that \( P_\sigma(-(h(\Phi_2) - \varepsilon)\tau_2) > 0 \). It follows from equation (4.14) that

\[
0 < P_\sigma(-(h(\Phi_2) - \varepsilon)\tau_2) \leq P_\sigma\left(-\frac{h(\Phi_2) - \varepsilon}{C}\tau_1\right).
\]

Thus, \( h(\Phi_1) > \frac{h(\Phi_2) - \varepsilon}{C} \) for every \( \varepsilon > 0 \). Therefore \( \frac{h(\Phi_2)}{C} \leq h(\Phi_1) \), concluding the proof.

We can also consider, under the assumptions of theorem 4.16, the more subtle problem of stability of the pressure.

**Theorem 4.17.** Let \( \Sigma \) be a topologically mixing countable Markov shift. Let \( \tau_1, \tau_2 : \Sigma \to \mathbb{R}^+ \) be two roof functions and \( \Phi_1 : Y_1 \to Y_1 \) and \( \Phi_2 : Y_2 \to Y_2 \) the corresponding suspension semiflows. Consider the non-positive potentials \( \psi_2 : Y_2 \to \mathbb{R}_0^- \) and \( \psi_1 : Y_1 \to \mathbb{R} \) satisfying \( \psi_1(x, r) := \psi_2 \circ \pi(x, r) \). Assume that there exists a positive constant \( C > 0 \) such that for every \( x \in \Sigma \) we have
\[
\frac{1}{C} \leq \frac{\tau_2(x)}{\tau_1(x)} < C. \tag{4.15}
\]

If \( P_{\Phi_1}(\psi_1) \leq \infty \) and \( P_{\Phi_2}(1/C\psi_2) \leq \infty \) then:

if \( P_{\Phi_1}(\psi_1) \geq 0 \), then
\[
\frac{1}{C} P_{\Phi_2}(C\psi_2) \leq P_{\Phi_1}(\psi_1) \leq CP_{\Phi_2}\left(\frac{\psi_2}{C}\right)
\]

and if \( P_{\Phi_1}(\psi_1) < 0 \), then
\[
CP_{\Phi_2}(C\psi_2) \leq P_{\Phi_1}(\psi_1) \leq \frac{1}{C} P_{\Phi_2}\left(\frac{\psi_2}{C}\right).
\]

**Proof.** The proof is a consequence of the approximation by compact subshifts of the Gurevich pressure (see theorem 2.1) and the application of theorem 4.3 on each compact subshift.

\[\square\]

**Remark 4.18.** Note that if the suspension flows have finite entropy, which happens simultaneously (theorem 4.16), then for every \( t > 0 \), we have that \( P_{\Phi_1}(t\psi_1) \) and \( P_{\Phi_2}(t\psi_2) \) are both finite. In particular, the assumption \( P_{\Phi_1}(\psi_1) < \infty \) and \( P_{\Phi_2}(1/C\psi_2) < \infty \) in theorem 4.17 is not required.

The existence of equilibrium measures is a subtle matter, and conditions of the type given in equation (4.13) are not enough to guarantee that if one potential has an equilibrium measure so does the corresponding one in the time-changed flow.

**Example 4.19 (Two flows with roof functions with bounded quotient, the first has a measure of maximal entropy while the second does not).** Let \((\Sigma, \sigma)\) be the full shift. Consider the suspension semi-flow \( \Phi_1 \) with base \( \Sigma \) and roof function \( \tau_1|_{C_n} := \log(n(\log(2n))^3) \). This flow has entropy equal to 1 and \( P_\sigma(-\tau_1) = 0 \). If we denote by \( \nu_1 \) the equilibrium measure corresponding to \(-\tau_1\), we see that \( \mu_1 = \nu_1 \times m \) is the measure of maximal entropy for the flow \( \Phi_1 \). Let \( \Phi_2 \) be the suspension semi-flow with base \( \Sigma \) and roof function \( \tau_2|_{C_n} := \log((n + 8)(\log(2n))^2) \). This flow has entropy equal to 1 and \( P_\sigma(-\tau_2) < 0 \). Therefore, it does not have a measure of maximal entropy. Note that
\[
\lim_{n \to \infty} \frac{\tau_1|_{C_n}}{\tau_2|_{C_n}} = 1.
\]

In particular, the roof functions satisfy equation (4.2).

### 5. Stability results

This section is devoted to describing the space \( \mathcal{S} \) of all suspension flows defined over a countable Markov shift \( \Sigma \). We will show that the thermodynamic properties are stable or unstable depending upon the combinatorial structure of the shift space \( \Sigma \). Note that the space \( \mathcal{S} \) can clearly be identified with the space
\( \{ \tau : \Sigma \to \mathbb{R} : \tau \text{ is positive, locally Hölder and } \sum_{i=0}^{\infty} \tau(\sigma^i x) = \infty \} \).

This space can be made into a topological space in the following manner (see [CS, section 2.2]): fix an infinite sequence \( \omega = (\omega_n)_{n=0}^{\infty} \), with \( 0 \leq \omega_n \leq \infty \). For a potential \( \varphi : \Sigma \to \mathbb{R} \), we define

\[ \| \varphi \|_\omega := \sup |\varphi| + \sum_{n=1}^{\infty} \omega_n V_n(\varphi), \text{ where } 0 \cdot \infty = 0, \]

an \( \varepsilon \)-neighbourhood of \( \varphi \) is given by

\[ B(\varphi, \varepsilon) = \{ \varphi' \in S : \| \varphi - \varphi' \|_\omega < \varepsilon \}. \]

The \( \omega \)-topology is generated by \( B(\varphi, \varepsilon) \). If \( \omega = (0, 0, 0, \ldots) \), we obtain the sup norm; if \( \omega = (0, 1, 1, 1, \ldots) \), we obtain the summable variation norm; and if \( \omega = (0, \theta^{-1}, \theta^{-2}, \ldots) \), we obtain the Hölder norm. Note that if \( \varphi \) is a potential which is constant on cylinders of length 1, then for every \( \omega \) we have that \( \| \varphi \|_\omega = \sup |\varphi| \). Also note that if \( \| \varphi \|_\omega < \varepsilon \), then \( \sup |\varphi| < \varepsilon \). We study two cases, the full shift and the renewal shift, for which the pressure function is very well understood.

### 5.1. The full shift case

We begin studying the case in which \( \Sigma \) is the full shift on a countable alphabet. The behavior of the pressure function \( t \mapsto P_{\sigma}(-t \tau) \) is completely understood in this context and when finite is a real analytic function (see sections 2.2 and 2.3).

**Remark 5.1.** We stress that if \( \Sigma \) is the full shift and \( \varphi \) is such that \( P_{\sigma}(\varphi) < \infty \), then \( \| \varphi \|_\omega = \infty \) for any choice of \( \omega \).

The following theorem characterizes the behavior of the flow in terms of the finiteness (or not) of the entropy. Denote by \( F \subset S \) the flows with finite entropy and \( I \subset S \) those with infinite entropy.

**Theorem 5.2.** The sets \( F \) and \( I \) are both open with respect to any \( \omega \)-topology.

**Proof.** Let \((Y, \Phi)\) be a suspension semi-flow defined over the full shift with locally Hölder roof function \( \tau \) and with finite entropy. Since the system has finite entropy, there exists \( t_0 > 0 \) such that \( P_{\sigma}(-t_0 \tau) = B < 0 \). Recall that we are assuming that \( \sum_{i=0}^{\infty} \tau(\sigma^i x) = \infty \), Kempton [Ke1, p 40] showed that for the full shift, this condition implies that \( \tau \) is bounded away from zero. In particular, by ergodic optimization results (see [JMU, theorem 1]), we have that the asymptotic slope of the pressure is bounded away from zero. That is, there exists \( A < 0 \) such that

\[ \lim_{t \to \infty} P'_{\sigma}(-t \tau) < A < 0. \]

This implies that \( \lim_{t \to \infty} P_{\sigma}(-t \tau) = -\infty \). Let \( \varepsilon > 0 \) and consider \( \tau' \in S \) such that \( t_0 \| \tau - \tau' \|_\infty < \varepsilon \). For every every \( K \subset \Sigma \) compact and invariant, we have that (see [Wa, theorem 9.7 (iv)])

\[ |P_K(-t_0 \tau) - P_K(-t_0 \tau')| \leq t_0 \sup |\tau - \tau'| \leq \| t_0 \tau - t_0 \tau' \|_\infty < \varepsilon. \]
It is a direct consequence of the approximation property of the Gurevich pressure (see theorem 2.1) that

\[ |P_\sigma(-t_0\tau) - P_\sigma(-t_0\tau')| < \varepsilon. \]

This implies that there exists \( t_1 \in \mathbb{R} \) such that \( P_\sigma(-t_1\tau') < 0 \). In particular, the flow \((Y, \Phi')\) with base \((\Sigma, \sigma)\) and roof function \( \tau' \) has finite entropy.

Let us now consider a suspension semi-flow \((Y', \Phi)\) defined over the full shift with locally Hölder roof function \( \tau' \) with infinite entropy. Let \( \varepsilon > 0 \) and \( \tau_2 \in S \) such that \( \|\tau' - \tau_2\|_\omega < \varepsilon \). We will show that the suspension semi-flow based on the full shift and roof function \( \tau_2 \) has infinite entropy. Note that since \( \sum_{i=0}^{\infty} \tau'(\sigma^i x) = \infty \), we have that \( \tau' \) is bounded away from zero. Therefore, for every \( t \in \mathbb{R} \), we have that \( P_\sigma(-t\tau') = \infty \). Indeed, assume by way of contradiction that there exists \( t' > 0 \) such that \( P_\sigma(-t'\tau') < \infty \). Then, ergodic optimization results ([JMU, theorem 1]) imply that the asymptotic derivative of the pressure is a strictly negative constant. Thus, there exists \( t'' > 0 \) such that \( P_\sigma(-t''\tau') < 0 \). However, this would imply that the entropy of the corresponding semi-flow is not infinite. Therefore, in order to prove that the suspension semi-flow based on the full shift and with roof function \( \tau_2 \) has infinite entropy, it suffices to prove that for every \( t \in \mathbb{R} \), \( P_\sigma(-t\tau_2) = \infty \). Assume by way of contradiction that there exists \( t_1 \in \mathbb{R} \) such that \( P_\sigma(-t_1\tau_2) < \infty \). Let \( K \subset \Sigma \) be compact and invariant. Then

\[ |P_K(-t_1\tau') - P_K(-t_1\tau_2)| \leq |t_1| \sup |\tau' - \tau_2| \leq |t_1| \|\tau' - \tau_2\|_\omega \leq |t_1| \varepsilon. \quad (5.1) \]

It follows form the approximation property of the Gurevich pressure and the fact that \( P_\sigma(-t_1\tau_2) < \infty \) and \( P_\sigma(-t_1\tau') = \infty \) that the inequality in equation (5.1) cannot hold for an arbitrary compact invariant set \( K \). This contradiction proves that the set \( \mathcal{I} \) is open with respect to any \( \omega \)-topology.

Consider now the subset \( \mathcal{C} \subset \mathcal{F} \) of flows having measures of maximal entropy and \( \mathcal{N} \subset \mathcal{F} \) of those not having such a measure.

**Proposition 5.3.** Neither \( \mathcal{C} \) nor \( \mathcal{N} \) are open sets with respect to any \( \omega \)-topology.

**Proof.** There exists a locally constant (on cylinders of length 1), positive function \( \tau_1 : \Sigma \to \mathbb{R} \) such that

\[ P_\sigma(-t\tau_1) = \begin{cases} \infty & \text{if } t < t_0; \\ 0 & \text{if } t = t_0; \\ \text{negative} & \text{if } t > t_0 \end{cases} \]

and \( \lim_{t \to t_0^+} P_\sigma(-t\tau_1) = \infty \) (see lemma 4.8). The suspension semi-flow \( \Phi_1 \) with base \((\Sigma, \sigma)\) and roof function \( \tau_1 \) has entropy \( h(\Phi_1) = t_0 \), and the equilibrium measure \( \nu \) corresponding to \(-t_0\tau_1\) is such that \( \int \tau_1 d\nu < \infty \) (because of the assumption on the derivative of the pressure). Therefore, the measure

\[ \frac{\nu \times m}{\int \tau_1 d\nu}, \]

is the measure of maximal entropy for the flow \( \Phi_1 \). Let \( \tau_2 : \Sigma \to \mathbb{R}^+ \) be a locally constant potential (on cylinders of length 1) such that for every \( x \in \Sigma \), \( \tau_1(x) - \tau_2(x) = \varepsilon \). Since both potentials are constant on cylinders of length 1, we have that \( \|\tau_1 - \tau_2\|_\omega = \sup |\tau_1 - \tau_2| \leq \varepsilon \) and
Therefore, the suspension semi-flow $\Phi_2$ with base $(\Sigma, \sigma)$ and roof function $\tau_2$ has entropy $h(\Phi_2) = t_0$ and no measure of maximal entropy. This implies that the set $\mathcal{C}$ is not open with respect to any $\omega$-topology. The same argument shows that the set $\mathcal{N}$ is not open with respect to any $\omega$-topology. 

The number $s_\infty(\Phi)$ is a relevant dynamical quantity of the suspension flow. It was first used in [IJ] to construct examples of regular potentials exhibiting phase transitions. In [IRV], geodesic flows defined over non-compact manifolds of variable pinched negative curvature were studied. The manifolds considered are Hadamard manifold with an extended Schottky group (which contains parabolic elements). It turns out that these manifolds can be coded by suspension flows defined over countable Markov shifts satisfying the BIP property and with a regular roof function (see [IRV, section 4.2]). The number $s_\infty(\Phi)$ is related to the amount of entropy that measures escaping through the cups of the manifold can carry, and with the largest parabolic critical exponent of the extended Schottky group (see [IRV, theorem 4.12]). It is therefore of interest to understand how this quantity varies with time changes of the flow.

First note that from lemma 4.6 it follows that:

**Lemma 5.4.** Let $(\Sigma, \sigma)$ be the full shift on a countable alphabet. For every $\alpha > 0$ there exists a positive locally Hölder potential $\tau$ such that the suspension flow $(Y, \Phi)$ with base $\Sigma$ and roof function $\tau$ satisfies $s_\infty(\Phi) = \alpha$.

We will use the following notation: if $\tau \in \mathcal{S}$, we denote by $s_\infty(\Phi_\tau)$ the number $s_\infty$ for the suspension flow $(Y, \Phi)$ based on the full shift and roof function $\tau$. In order to study the properties of $s_\infty$ in $\mathcal{S}$, we define an equivalence relation. We say that $\tau, \tilde{\tau} \in \mathcal{S}$ are related if and only if $s_\infty(\Phi_\tau) = s_\infty(\Phi_{\tilde{\tau}})$. By lemma 5.4, we can identify each equivalence class $[\tau]$ with a number $\alpha > 0$, say $[\tau] = [\alpha]$ if and only if $s_\infty(\Phi_\tau) = \alpha$. We also define

$$[\infty] := \{ \tau \in \mathcal{S} : P_\sigma(-\tau) = \infty \text{ for every } t > 0 \}.$$ 

For any $\alpha > 0$, the class $[\alpha]$ can be partitioned into two sets

$$[\alpha] = \mathcal{A}_{\text{cont}}^{[\alpha]} \cup \mathcal{A}_{\text{disc}}^{[\alpha]},$$

where

$$\mathcal{A}_{\text{cont}}^{[\alpha]} := \{ \tau \in \mathcal{S} : s_\infty(\Phi_\tau) = \alpha \text{ and } -\tau \text{ is of continuous type} \}, \text{ and}$$

$$\mathcal{A}_{\text{disc}}^{[\alpha]} := \{ \tau \in \mathcal{S} : s_\infty(\Phi_\tau) = \alpha \text{ and } -\tau \text{ is of discontinuous type} \}.$$ 

The space of suspension flows over the full shift can be partitioned as:

$$\bigcup_{\alpha > 0 \vee \alpha = \infty} [\alpha] = [\infty] \cup \bigcup_{\alpha > 0} (\mathcal{A}_{\text{cont}}^{[\alpha]} \cup \mathcal{A}_{\text{disc}}^{[\alpha]}) = [\infty] \cup \mathcal{A}_{\text{cont}} \cup \mathcal{A}_{\text{disc}},$$

where $A_{\text{cont}} := \bigcup_{\alpha > 0} A_{\text{cont}}^{[\alpha]}$ and $A_{\text{disc}} := \bigcup_{\alpha > 0} A_{\text{disc}}^{[\alpha]}$. The topological properties of this partition will describe the behavior of the number $s_\infty$ in $\mathcal{S}$. The next result shows that if two suspension flows have different corresponding $s_\infty$, then their roof functions are at infinite distance with respect to any $\omega$-norm.

**Theorem 5.5.** If $\tau, \tilde{\tau} \in \bigcup_{\alpha \in \mathbb{R}} [\alpha]$ and $\| \tau - \tilde{\tau} \|_{\omega} < \infty$, then $[\tau] = [\tilde{\tau}]$. 

2869
and that $\tau \in [\alpha, \bar{\tau}]$, and $\alpha > \beta$. If $\|\tau - \bar{\tau}\|_\omega < \infty$, then $\sup |\tau - \bar{\tau}| < M < \infty$ for some $M$. Let $\varepsilon > 0$. For every $K \subset \Sigma$ compact and invariant set we have that

$$|P_K(-(\beta + \varepsilon)\tau) - P_K(-(\beta + \varepsilon)\bar{\tau})| \leq (\beta + \varepsilon)M.$$  

Because of the approximation property of the Gurevich pressure, we have that

$$|P_\alpha(-(\beta + \varepsilon)\tau)| \leq (\beta + \varepsilon)M + |P_\alpha(-(\beta + \varepsilon)\bar{\tau})| < \infty,$$

and this implies that $\alpha \leq \beta + \varepsilon$. □

From this result, we have the following consequences that show that the classification is stable under perturbations with respect to the norm $\| \cdot \|_\omega$.

**Corollary 5.6.** For every $\alpha > 0$, the set $[\alpha]$ is open with respect to any $\omega$-topology and $\text{diam}([\alpha]) = \infty$, where the diameter is computed with respect to the distance induced by the $\| \cdot \|_\omega$ norm.

**Corollary 5.7.** If $\alpha, \beta > 0$ and $\alpha \neq \beta$, then $\text{dist}([\alpha], [\beta]) = \infty$, with respect to the distance induced by the $\| \cdot \|_\omega$ norm.

Finally, from the last inequality in the proof of the theorem 5.5, we have that

**Corollary 5.8.** For every number $\alpha > 0$, the sets $A_{\text{cont}}^{[\alpha]}$ and $A_{\text{disc}}^{[\alpha]}$ are open with respect to any $\omega$-topology.

We conclude this subsection describing the topological properties of the sets having measures of maximal entropy $\mathcal{C}$ and not having such a measure $\mathcal{N}$ when restricted to the classes $A_{\text{cont}}$ and $A_{\text{disc}}$.

**Proposition 5.9.** With respect to any $\omega$-topology, we have that

(a) The set $\mathcal{C} \cap A_{\text{cont}}$ is open in $A_{\text{cont}}$ and $\mathcal{C} \cap A_{\text{disc}}$ is not open in $A_{\text{disc}}$.

(b) The set $\mathcal{N} \cap A_{\text{cont}}$ is empty and $\mathcal{N} \cap A_{\text{disc}}$ is not open in $A_{\text{disc}}$.

**Proof.** From the definition of $A_{\text{cont}}$, it is clear that $\mathcal{C} \cap A_{\text{cont}}$ is open in $A_{\text{cont}}$ and that $\mathcal{N} \cap A_{\text{cont}} = \emptyset$. The proofs that the set $\mathcal{C} \cap A_{\text{disc}}$ is not open in $A_{\text{disc}}$ and that the set $\mathcal{N} \cap A_{\text{disc}}$ is not open in $A_{\text{disc}}$ follow from lemma 4.8, and the observations that for every $\theta > 1, k \geq 2$, given $\varepsilon > 0$ small enough, we have that $\tau := -\varphi_{\theta, k} \geq 0$ and $\tau - \varepsilon > 0$, moreover $[\tau] = [\tau - \varepsilon] = [1]$, $P_\alpha(-\tau) = 0$, $P_\alpha((\tau - \varepsilon)) = -\varepsilon < 0$, and $\|\tau - (\tau - \varepsilon)\|_\omega = \varepsilon$. As $\tau \in \mathcal{C} \cap A_{\text{disc}}$ and $\tau - \varepsilon \in \mathcal{N} \cap A_{\text{disc}}$, this implies that there are elements of $\mathcal{N} \cap A_{\text{disc}}$ arbitrarily close to $\mathcal{C} \cap A_{\text{disc}}$, which concludes the proof. □

As observed in theorem 3.3, a suspension flow $(Y, \Phi)$ based on the full shift and roof function $\tau$ can fail to have a measure of maximal entropy for two reasons. One, which we have already studied, is that $P_\alpha(-h(\Phi)\tau) \neq 0$. The other is that despite the fact that $P_\alpha(-h(\Phi)\tau) = 0$, the Gibbs measure $\nu$ corresponding to $-h(\Phi)\tau$ is such that $\int h(\Phi)\tau d\nu = \infty$. This means that for any $\varepsilon > 0$, the derivative of the map $t \mapsto P_\alpha(-t\tau)$ is unbounded in the interval $(h(\Phi), h(\Phi) + \varepsilon]$. The roof functions studied in lemma 4.8 provide examples of this type. For example, when $k = 2$ and $\gamma = 1$, we have that $1 < \theta_{2,1} < 2$, and $\tau := -\varphi_{\theta_{2,1}} > 0$ satisfies $h_{\text{top}}(\Phi_\tau) = 1$, $P_\alpha(-\tau) = 0$, and there exists a unique equilibrium measure $\nu$ for $-\tau$ with $\int \tau d\nu = \infty$. 

2870
5.2. The renewal shift case

We now consider another countable Markov shift. For the alphabet \( \mathbb{N} \cup \{0\} \), consider the transition matrix \( T = (t_{ij})_{i,j \in \mathbb{N} \cup \{0\}} \) with \( t_{00} = t_{0n} = t_{n0} = 1 \) for each \( n \geq 1 \) and with all other entries equal to zero. The renewal shift is the Markov shift \((\Sigma_R, \sigma)\) defined by the transition matrix \( T \), that is, the shift map \( \sigma \) on the space
\[
\Sigma_R = \{ (x_i)_{i \geq 0} : x_i \in \mathbb{N} \cup \{0\} \text{ and } a_{ix_{i+1}} = 1 \text{ for each } i \geq 0 \}.
\]
This shift has entropy equal to \( \log 2 \). The pressure function in this context is also very well understood. Indeed, Sarig [Sa2] proved the following (the version of this result for every \( t \in \mathbb{R} \), appears in [B12, proposition 3]):

**Proposition 5.10.** Let \((\Sigma_R, \sigma)\) be the renewal shift. For each bounded \( \varphi \in \mathcal{R} \), there exists \( t^*_\varphi \in (0, +\infty) \) and such that:

(a) \( t \mapsto P_{\sigma}(t\varphi) \) is strictly convex and real analytic in \((0, t^*_\varphi)\).

(b) \( P_G(t\varphi) = Mt \) for \( t > t^*_\varphi \). \( M := \sup \{ \int_{\Sigma_R} \varphi \, d\nu : \nu \in \mathcal{M}_{\sigma} \} \).

(c) At \( t^*_\varphi \), the function \( q \mapsto P_{\sigma}(t\varphi) \) is continuous but not analytic.

(d) For each \( t \in (0, t^*_\varphi) \), there is a unique equilibrium measure \( \mu_t \) for \( t\varphi \).

(e) For each \( t > t^*_\varphi \) there is no equilibrium measure for \( t\varphi \).

We consider now the set \( \mathcal{S} \) of suspension flows with renewal shift for base. These flows are known as renewal flows, and have been studied in [IJT, section 6]. There exist renewal flows with infinite entropy (see [IJT, example 6.4]). Again, we call \( \mathcal{F} \subset \mathcal{S} \) the flows with finite entropy and \( \mathcal{I} \subset \mathcal{S} \) those with infinite entropy.

**Lemma 5.11.** The set \( \mathcal{F} \) contains an open set.

**Proof.** Let \( \tau \) be a roof function bounded away from zero. That is, there exists \( A > 0 \) such that for every \( x \in \Sigma_R \) we have \( \tau(x) > A > 0 \). Therefore, for every \( t \geq 0 \)
\[
P_{\sigma}(-t\tau) \leq \log(2) - tA.
\]
In particular, there exists \( h > 0 \) such that \( P_{\sigma}(-h\tau) = 0 \). Thus, the suspension flow with roof function \( \tau \) has finite entropy. This readily implies that the set \( \mathcal{F} \) contains an open set with respect to any \( \omega \)-topology. Indeed, for every \( \varepsilon > 0 \) such that \( A - \varepsilon > 0 \) and any roof function \( \tau_1 \) for which \( \|\tau - \tau_1\|_{\omega} < \varepsilon \), we have that \( \tau_1 > A - \varepsilon > 0 \). Indeed,
\[
\sup \{ |\tau(x) - \tau_1(x)| : x \in \Sigma_R \} \leq \|\tau - \tau_1\|_{\omega} < \varepsilon.
\]
Therefore, the suspension flow with roof function \( \tau_1 \) has finite entropy. \( \square \)

**Theorem 5.12.** Neither \( \mathcal{F} \) nor \( \mathcal{I} \) are open sets with respect to any \( \omega \)-topology.

**Proof.** Let \( \tau_1 \) be a roof function such that it is constant in cylinders of length 1. For every \( t \in \mathbb{R} \), we have that \( P_{\sigma}(-t\tau_1) > 0 \) and \( \lim_{n \to \infty} \tau_1|C_n = 0 \) (for an example, see [IJT, example 6.4]). Let \( \tau_2 \) be a roof function constant in cylinders of length 1 with
\[
P_{\sigma}(-t\tau_2) = \begin{cases} 
0 & \text{if } t \geq 1; \\
\text{positive} & \text{if } t < 1; 
\end{cases}
\]
and \( \lim_{n \to \infty} \tau_2|C_n| = 0 \). Examples of these types of functions were first constructed by Hofbauer [Ho] (other examples can be found in [IT, Sa2]). The suspension flow associated with \( \tau_1 \) belongs to \( \mathcal{I} \) and the one corresponding to \( \tau_2 \) to \( \mathcal{F} \).

Let \( \varepsilon > 0 \) and consider \( N \in \mathbb{N} \) such that for any \( n > N \) we have that, if \( x \in C_n \), then

\[
0 \leq \tau_i(x) < \frac{\varepsilon}{2} \quad \text{for} \quad i = 1, 2.
\]

Define a new roof function \( \tau : \Sigma_R \to \mathbb{R} \) by

\[
\tau(x) := \begin{cases} 
\tau_2(x) & \text{if } x \in C_n \text{ for } n \in \{0, 1, \ldots, N\}; \\
\tau_1(x) & \text{if } x \in C_n \text{ for } n > N.
\end{cases}
\]

Note that for every \( t \in \mathbb{R} \), we have that \( P_\sigma(-t\tau) > 0 \). Indeed, the tail of \( \tau \) determines the behavior of the pressure for large values of \( t \). Moreover,

\[
\|\tau - \tau_2\|_\omega < \varepsilon.
\]

Therefore, the set \( \mathcal{F} \) is not open. An analogous construction shows that \( \mathcal{I} \) is not open. \( \square \)

Acknowledgments

The authors would like to express their gratitude to Katrin Gelfert for interesting discussions and comments around the subject of this paper. Her ideas motivated part of this work. We also thank the referees for their careful reading of the manuscript and several relevant comments. GI was partially supported by Proyecto Fondecyt 1190194. GI and IC were partially supported by CONICYT PIA ACT172001.

References

[Ab] Abramov L M 1959 On the entropy of a flow Dokl. Akad. Nauk SSSR 128 873–75

[Al] Alzer H 2015 The Hurwitz zeta function: monotonicity, convexity and inequalities Aequ. Math. 89 1401–14

[AK] Ambrose W and Kakutani S 1942 Structure and continuity of measurable flows Duke Math. J. 9 25–42

[AFU] Avila A, Forni G and Ulcigrai C 2011 Mixing for time-changes of Heisenberg nilflows J. Differ. Geom. 89 369–410

[BI1] Barreira L and Iommi G 2006 Suspension flows over countable Markov shifts J. Stat. Phys. 124 207–30

[BI2] Barreira L and Iommi G 2011 Multifractal analysis and phase transitions for hyperbolic and parabolic horseshoes Israel J. Math. 181 347–79

[BRW] Barreira L, Radu L and Wolf C 2004 Dimension of measures for suspension flows Dyn. Syst. 19 89–107

[Bo1] Bowen R 1973 Symbolic dynamics for hyperbolic flows Am. J. Math. 95 429–60

[BR] Bowen R and Ruelle D 1975 The ergodic theory of Axiom A flows Inventory Math. 29 181–202

[BG] Bufetov A I and Gurevich B M 2008 On a measure with maximum entropy for the Teichmüller flow on the moduli space of Abelian differentials Funkt. Anal. Prilozhen. 42 75–7

Bufetov A I and Gurevich B M 2008 Funct. Anal. Pril. 42 224–6 (transl.)
[BS] Buzzi J and Sarig O 2003 Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps Er"{o}d. Theor. Dynam. Syst. 23 1383–400

[CS] Cyr V and Sarig O 2009 Spectral gap and transience for Ruelle operators on countable Markov shifts Commun. Math. Phys. 292 637–66

[DP] Dal'bo F and Peigné M 1998 Some negatively curved manifolds with cusps, mixing and counting J. Reine Angew. Math. 497 141–69

[Ei] Eie M 1990 On a Dirichlet series associated with a polynomial Proc. Am. Math. Soc. 110 583–90

[FU] Forni G and Ulcigrai C 2012 Time-changes of horocycle flows J. Mod. Dyn. 6 251–73

[GM] Gelfert K and Motter A E 2010 (Non)Invariance of dynamical quantities for orbit equivalent flows Commun. Math. Phys. 300 411–33

[GT] Grabner P J and Thuswaldner J M 1996 Analytic continuation of a class of Dirichlet series Abh. Math. Sem. Univ. Hamburg 66 281–7

[Gu1] Gurevič B M 1969 Topological entropy for denumerable Markov chains Dokl. Akad. Nauk SSSR 10 911–5

[Gu2] Gurevič B M 1970 Shift entropy and Markov measures in the path space of a denumerable graph Dokl. Akad. Nauk SSSR 11 744–7

[Ha] Hamenstädt U Symbolic dynamics for the Teichmüller flow (arXiv:1112.6107)

[Ho] Hofbauer F 1977 Examples for the nonuniqueness of the equilibrium state Trans. Am. Math. Soc. 228 223–41

[IJ] Iommi G and Jordan T 2013 Phase transitions for suspension flows Commun. Math. Phys. 320 475–98

[IJT] Iommi G, Jordan T and Todd M 2015 Recurrence and transience for suspension flows Israel J. Math. 209 547–92

[IT] Iommi G and Todd M 2013 Transience in dynamical systems Er"{o}d. Theor. Dynam. Syst. 33 1450–76

[IRV] Iommi G, Riquelme F and Velozo A 2018 Entropy in the cusp and phase transitions for geodesic flows Israel J. Math. 225 609–59

[JKL] Jaerisch J, Kesseböhmer M and Lamei S 2014 Induced topological pressure for countable state Markov shifts Stoch. Dyn. 14 1350016

[JMU] Jenkinson O, Mauldin R D and Urbański M 2005 Zero temperature limits of Gibbs-equilibrium states for countable alphabet subshifts of finite type J. Stat. Phys. 119 765–76

[KU] Katok S and Ugarcovici I 2007 Symbolic dynamics for the modular surface and beyond Bull. Am. Math. Soc. 44 87–132

[Ke1] Kempton T 2011 Thermodynamic formalism for symbolic dynamical systems PhD Thesis University of Warwick

[Ke2] Kempton T 2011 Thermodynamic formalism for suspension flows over countable Markov shifts Nonlinearity 24 2763–75

[LS] Lima Y and Sarig O Symbolic dynamics for three dimensional flows with positive topological entropy J. Eur. Math. Soc. 21 199–256

[Ma] Marcus B 1977 Ergodic properties of horocycle flows for surfaces of negative curvature Ann. Math. 105 81–105

[MU] Mauldin R and Urbański M 2003 Graph Directed Markov Systems: Geometry, Dynamics of Limit Sets (Cambridge Tracts in Mathematics vol 148) (Cambridge: Cambridge University Press)

[P1] Parry W 1986 Synchronisation of canonical measures for hyperbolic attractors Commun. Math. Phys. 106 267–75

[P2] Parry W 2004 Topics in Ergodic Theory (Cambridge Tracts in Mathematics vol 75) (Cambridge: Cambridge University Press) (Reprint of the 1981 original)

[PP] Parry W and Pollicott M 1990 Zeta functions and the periodic orbit structure of hyperbolic dynamics Ast"{e}risque 187–8 268

[Ra] Ratner M 1973 Markov partitions for Anosov flows on n-dimensional manifolds Israel J. Math. 15 92–114

[Sav] Savchenko S 1998 Special flows constructed from countable topological Markov chains Funct. Anal. Appl. 32 32–41
[Sa1] Sarig O 1999 Thermodynamic formalism for countable Markov shifts Ergod. Theor. Dynam. Syst. 19 1565–93
[Sa2] Sarig O 2001 Phase transitions for countable Markov shifts Commun. Math. Phys. 217 555–77
[Sa3] Sarig O 2003 Existence of Gibbs measures for countable Markov shifts Proc. Am. Math. Soc. 131 1751–8
[Ti] Tiedra De Aldecoa R 2012 Spectral analysis of time changes of horocycle flows J. Mod. Dyn. 6 275–85
[To] Totoki H 1966 Time changes of flows Mem. Fac. Sci. Kyushu Univ. A 20 27–55
[Wa] Walters P 1981 An Introduction to Ergodic Theory (Graduate Texts in Mathematics vol 79) (Berlin: Springer)
[Ze] Zeidler E 2006 Quantum Field Theory I: Basics in Mathematics and Physics. A Bridge Between Mathematicians and Physicists (Berlin: Springer)