APPLYING QUANTUM CALCULUS FOR THE EXISTENCE OF SOLUTION OF \( q \)-INTEGRO-DIFFERENTIAL EQUATIONS WITH THREE CRITERIA

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Abstract. Crisis intervention in natural disasters is significant to look at from many different slants. In the current study, we investigate the existence of solutions for \( q \)-integro-differential equation

\[
D^q_αu(t) + w\left(t, u(t), u'(t), \int_0^t f(r)u(r) \, dr, \varphi(u(t))\right) = 0,
\]

with three criteria and under some boundary conditions which therein we use the concept of Caputo fractional \( q \)-derivative and fractional Riemann-Liouville type \( q \)-integral. New existence results are obtained by applying \( α \)-admissible map. Lastly, we present two examples illustrating the primary effects.

1. Introduction. Fractional calculus and fractional \( q \)-calculus are the significant branches in mathematical analysis. Similarly, the subject of fractional differential equations ranges from the theoretical views of existence and uniqueness of solutions to the analytical and mathematical methods for finding solutions (for more details, consider [3, 9, 18, 19, 22, 29, 34]). It has been shown that these types of equations have numerous applications in diverse fields and thus have evolved into multidisciplinary subjects (for example, see [1, 2, 4, 8, 17, 30, 32, 33, 43] and references therein). Many researchers have been worked and have been published some works about the crisis which its use in different domains of science, such as science, economic and social (for more information, see [6, 11, 13]). We can model very well this phenomena by employing the problems fractional \( q \)-integro-differential equations, which have been discussed by many researchers (for example, see [28, 37, 39, 41, 42]).

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We recall some of the previous works briefly. In 1910, the subject of $q$-difference equations introduced by Jackson [20]. After that, at the beginning of the last century, studies on $q$-difference equation, appeared in so many works, especially in Carmichael [15], Adams [3]. In 2010, Agarwal et al. [5] by applying $\mu$, the functional of bounded variation with $\int_0^1 d\mu(s) < 1$, studied the problem

$$D^\alpha x(t) + w(t, x(t)) = 0,$$

and using Riemann-Stieltjes integral reviewed the system

$$D^\alpha x_i(t) + w_i(t, x_1(t), x_2(t)) = 0,$$

under conditions $x'(0) = \cdots = x^{(n-1)} = 0$ and

$$x(1) = \int_0^1 x(s) \, d\mu(s),$$

where $\alpha \in (n-1, n]$ with $n \geq 2$, $w$ may have singularity at $t = 0$ and under conditions $x_i(0) = x_i'(0) = 0$ and

$$x_i(1) = \int_0^1 x_i(t) \, d\eta(t),$$

for $i = 1, 2$, where $t, \alpha$ belong to $(0, 1)$, $(2, 3]$, $w_i \in C([0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$, respectively. In 2012, Cabada et al. [14] investigated the existence of positive solutions for the nonlinear fractional differential equation

$$D^\beta x(t) + h(t, x(t)) = 0,$$

with boundary conditions $x(0) = x''(1) = 0$ and

$$x(1) = \int_0^1 x(s) \, ds,$$

where $t \in J$, $2 < \beta < 3$, $0 < \eta < 2$, $D$ is the Caputo fractional derivative and $h : J \times [0, \infty) \to [0, \infty)$ is a continuous function. In 2013, Bai et al. [10] studied the singular problem

$$D^\alpha u + g_1(t, u, D^\gamma u, D^\mu u) + g_2(t, u, D^\gamma u, D^\mu u) = 0,$$

under conditions $u(0) = u'(0) = u''(0) = u'''(0) = 0$, where $\alpha, \gamma, \mu$ belong to $(3, 4)$, $(0, 1)$, $(1, 2)$, respectively. $D^\alpha$ is the Caputo fractional derivative and $g_i$, for $i = 1, 2$, is a Carathéodory function on $J \times (0, \infty)^3$. In 2014, Li reviewed the problem

$$D^\beta u(t) + h(t, u(t), D^\gamma u(t)) = 0,$$

under conditions $u(0) = u'(1) = 0$ and $u'(1) = D^\beta u(t)$, where $t \in J$, $\beta \in (2, 3)$, $0 < \gamma < 1$, the singular function $h : (0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, at $t = 0$, is a continuous and $D$ is the standard Caputo derivative [26]. In 2016, the multi-singular pointwise defined fractional integro-differential equation

$$D^\gamma u(t) + \theta(t, u(t), u'(t), D^\alpha u(t), I^\beta u(t)) = 0,$$

under conditions $u''(0) = u(\eta)$, $u^{(i)}(0) = 0$ for $i = 2, \ldots, [\gamma] - 1$ and

$$u(1) = \int_0^\nu u(r) \, dr,$$

was investigated, where $t \in J$, $\gamma \in [2, 3)$, $u \in \mathbb{R}$, $\alpha, \eta, \nu \in J$, $\beta > 1$, $D$ is Caputo fractional derivative and $g : J \times \mathbb{R}^4 \to \mathbb{R}$ is a function such that $h(t, \ldots, \ldots)$ is singular at some point $t \in J$ [37]. In 2016, Almeida et al. [7] published a paper
about modeling of some real phenomena by fractional differential equations. Also in the same year, they introduce a new operator entitled the infinite coefficient-symmetric Caputo-Fabrizio fractional derivative and applying its to investigate the approximate solutions for two infinite coefficient-symmetric Caputo-Fabrizio fractional derivative and applying its to investigate the singular fractional integro-differential equation

$$D^\beta u(t) + w(t, u(t), u'(t), D^\gamma u(t), \mu(u(t))) = 0,$$

where

$$\mu(u(t)) = \int_0^t f(r)u(r) \, dr,$$

under conditions \( u(0) = u'(0) \) and \( u(1) = D^\gamma u(a) \), where \( t \in J, u \in \mathbb{B}, \beta > 2, 0 < \gamma, a < 1, f \in \mathcal{L}, ||f||_1 = m, w(t, u_1, u_2, u_3, u_4) \) is singular at some points \( t \in J \) and \( D \) is the Caputo fractional derivative. Recently, the multi-singular pointwise defined fractional integro-differential equation

$$D^\mu u(t) + w(t, u(t), u'(t), D^\beta u(t), \rho u(t)) = 0,$$

under conditions \( u'(0) = u(\xi) \) and

$$u(1) = \int_0^\eta u(s) \, ds,$$

when \( \mu \in [2, 3) \) and \( w^{(j)}(0) = 0 \) for \( j = 2, \ldots, [\mu] \) when \( \mu \in [3, \infty) \) has been studied, where \( 0 < t \leq 1, u \in C^1[0, 1], \mu \in [2, \infty), \beta, \xi, \eta \in (0, 1), p > 1 \), \( D^\mu \) is the Caputo fractional derivative of order \( \mu \) and \( w : [0, 1] \times \mathbb{R}^5 \rightarrow \mathbb{R} \) is a function such that \( w(t, \ldots, u) \) is singular at some points \( t \in [0, 1] \) [37]. Then, in 2017, Zhou et al. [43] provide existence criteria for the solutions of \( p \)-Laplacian fractional Langevin differential equations with ansi-periodic boundary conditions:

$$\begin{cases}
D_0^\lambda \phi_{[D_0^\alpha + \lambda]x(t)] = f(t, x(t), D_0^\alpha x(t)), \\
x(0) = -x(1), \\
D_0^\alpha x(0) = -D_0^\alpha x(1),
\end{cases}$$

and

$$\begin{cases}
q D_0^\lambda \phi_{[D_0^\alpha + \lambda]x(t)] = g(t, x(t), D_0^\alpha x(t)), \\
x(0) = -x(1), \\
q D_0^\alpha x(0) = -q D_0^\alpha x(1),
\end{cases}$$

for all \( 0 \leq t \leq 1 \), where \( 0 < \alpha, \beta \leq 1 \), \( \lambda \) is more than or equal to zero, \( 1 < \alpha + \beta < 2 \), \( q \in (0, 1) \), and \( \phi(p)(s) = |s|^{p-2}s \), with \( p \in (1, 2] \). In 2019, Gowami et al. [18] showed some use of fractional equal width equations in describing hydro-magnetic waves in cold plasma. In 2019, Samei et al. [31] discussed the fractional hybrid \( q \)-differential inclusions

$$^{c}D_q^\alpha \left( \frac{x}{f(t, x, I_q^{\alpha_1}x, \ldots, I_q^{\alpha_n}x)} \right) \in F \left( t, x, I_q^{\beta_1}x, \ldots, I_q^{\beta_k}x \right),$$

with the boundary conditions \( x(0) = x_0 \) and \( x(1) = x_1 \), where \( 1 < \alpha \leq 2, q \in (0, 1), x_0, x_1 \in \mathbb{R}, \alpha_i > 0, \) for \( i = 1, 2, \ldots, n, \beta_j > 0 \), for \( j = 1, 2, \ldots, k, n, k \in \mathbb{N} \), \(^{c}D_q^\alpha \) denotes Caputo type \( q \)-derivative of order \( \alpha \), \( I_q^\beta \) denotes Riemann-Liouville type \( q \)-integral of order \( \beta \), \( f : J \times \mathbb{R}^n \rightarrow (0, \infty) \) is continuous and \( F : J \times \mathbb{R}^k \rightarrow P(\mathbb{R}) \) is multifunction. Also, Ntouyas et al. [28] by applying definition of the fractional \( q \)-derivative of the Caputo type and the fractional \( q \)-integral of the Riemann–Liouville
type, studied the existence and uniqueness of solutions for a multi-term nonlinear fractional $q$-integro-differential equations under some boundary conditions
\[ ^c D^q_x x(t) = w(t, x(t), (\varphi_1 x)(t), (\varphi_2 x)(t), ^c D^\beta_1 x(t), ^c D^\beta_2 x(t), \ldots, ^c D^\beta_n x(t)). \]

In 2020, Liang et al. [25] investigated the existence of solutions for a nonlinear problems regular and singular fractional $q$-differential equation
\[ ^c D^q_x f(t) = w(t, f(t), f'(t), ^c D^\beta f(t)), \]
with conditions $f(0) = c_1 f(1), f'(0) = c_2 ^c D^\beta f(1)$ and $f^{(k)}(0) = 0$ for $2 \leq k \leq n - 1$, here $n - 1 < \alpha < n$ with $n \geq 3, \beta, q, c_1 \in (0, 1), c_2 \in (0, \Gamma_q(2 - \beta))$, function $w$ is a $L^\infty$-Carathéodory, $w(t, x_1, x_2, x_3)$ may be singular and $^c D^q_x$ the fractional Caputo type $q$-derivative.

In this article, motivated by among these achievements, by using definition of the standard Caputo fractional $q$-derivative of order $\alpha$, we investigate the pointwise $q$-integro-differential boundary problem
\[ D^q_x u(t) + w\left(t, u(t), u'(t), D^\beta u(t), \int_0^t f(r) u(r) \, dr, \varphi(u(t))\right) = 0 \quad (1) \]
under conditions $u(0) = u''(0) = u^{(n)}(0) = u(1) = 0$, where $\alpha \geq 2, q, \lambda, \mu, \beta \in J = (0, 1)$, self map $\varphi$ on $\mathcal{X}$ satisfies
\[ \|\varphi(u) - \varphi(v)\| \leq a_1 \|u - v\| + a_2 \|u' - v'\|, \quad (2) \]
for some non-negative real numbers $a_1, a_2$ and all $u, v$ belonging to $[0, \infty)$ and $\mathcal{X}$, respectively, and $w$ defined by three criteria
\[ w(t, u_1(t), \ldots, u_5(t)) = \begin{cases} w_1(t, u_1(t), \ldots, u_5(t)), & t \in [0, \tau_1), \\ w_2(t, u_1(t), \ldots, u_5(t)), & t \in [\tau_1, \tau_2), \\ w_3(t, u_1(t), \ldots, u_5(t)), & t \in [\tau_2, 1], \end{cases} \quad (3) \]
here $w_1(t, \ldots, \ldots)$ and $w_3(t, \ldots, \ldots)$ are continuous on $[0, \tau_1)$ and $(\tau_2, 1]$, respectively, and $w_2(t, \ldots, \ldots)$ is multi-singular. In fact, the map $w$ is multi-singular when it is singular at more than one point $t$. Also, a real-valued and non-continuous function $f$ on the interval $I = [a, b]$ is said to be singular whenever $f$ is non-constant on $I$, there exists a set $S$ of measure 0 such that for all $x \in S$ the derivative $f'(x)$ exists and is zero, that is, the derivative of $f$ vanishes almost everywhere.

The rest of the paper is arranged as follows. In Section 2, we recall some preliminary concepts and fundamental results of $q$-calculus. Section 3 is devoted to the main results, while examples illustrating the obtained results and algorithm for the problems are presented in Section 4. Finally, the paper is concluded in section 5.

2. Preliminaries. First, we point out some of the materials on the fractional $q$-calculus and fundamental results of it which needed in the next sections (for more information, consider [9, 20, 21, 23, 27, 29, 40]). Then, some well-known theorems of fixed point theorem and definition are expressed.
Assume that \( q \in (0, 1) \) and \( a \in \mathbb{R} \). Define \([a]_q = \frac{1-q^a}{1-q}\) [20]. The power function 
\[
(x - y)^{(n)}_q = \prod_{k=0}^{n-1} (x - yq^k),
\]
for \( n \geq 1 \) and \((x - y)^{(0)}_q = 1\), where \( x \) and \( y \) are real numbers and \( \mathbb{N}_0 := \{0\} \cup \mathbb{N} \) [3]. Also, for \( \alpha \in \mathbb{R} \) and \( \alpha \neq 0 \), we have
\[
(x - y)^{(\alpha)}_q = x^\alpha \prod_{k=0}^{\infty} \frac{x - yq^k}{x - yq^{\alpha+k}}.
\]
If \( y = 0 \), then it is clear that \( x^{(\alpha)} = x^\alpha \) (Algorithm 1). The \( q \)-Gamma function is given by \( \Gamma_q(z) = (1-q)^{(z-1)}/(1-q)^{z-1} \), where \( z \in \mathbb{R}\setminus\{0, -1, -2, \cdots\} \) [20]. Note that, \( \Gamma_q(z + 1) = \lfloor z \rfloor \Gamma_q(z) \). The value of \( q \)-Gamma function, \( \Gamma_q(z) \), for input values \( q \) and \( z \) with counting the number of sentences \( n \) in summation by simplifying analysis. For this design, we prepare a pseudo-code description of the technique for estimating \( q \)-Gamma function of order \( n \) which show in Algorithm 2. The \( q \)-derivative of function \( f \), is defined by
\[
(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x},
\]
and \( (D_q f)(0) = \lim_{x \to 0}(D_q f)(x) \) which is shown in Algorithm 3 [3]. Also, the higher order \( q \)-derivative of a function \( f \) is defined by \( (D_q^n f)(x) = D_q(D_q^{n-1} f)(x) \) for all \( n \geq 1 \), where \( (D_q^n f)(x) = f(x) \) [3]. The \( q \)-integral of a function \( f \) defined on \([0, b]\) is defined by
\[
I_q f(x) = \int_0^x f(s) d_q s = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k)
\]
for \( 0 \leq x \leq b \), provided the series is absolutely converges [3]. If \( a \) in \([0, b]\), then
\[
\int_a^b f(u) d_q u = I_q f(b) - I_q f(a) = (1-q) \sum_{k=0}^{\infty} q^k \left[ b f(bq^k) - a f(aq^k) \right],
\]
whenever the series exists. The operator \( I_q^n \) is given by \((I_q^n h)(x) = h(x)\) and
\[
(I_q^n h)(x) = (I_q(I_q^{n-1} h))(x),
\]
for \( n \geq 1 \) and \( g \in C([0, b]) \) [3]. It has been proved that \( (D_q(I_q f))(x) = f(x) \) and \( (I_q(D_q f))(x) = f(x) - f(0) \) whenever \( f \) is continuous at \( x = 0 \) [3]. The fractional Riemann-Liouville type \( q \)-integral of the function \( f \) on \( J \) for \( \alpha \geq 0 \) is defined by \((I_q^\alpha f)(t) = f(t)\) and
\[
(I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s) d_q s,
\]
for \( t \in J \) and \( \alpha > 0 \) [16]. Also, the Caputo fractional \( q \)-derivative of a function \( f \) is defined by
\[
(^c D_q^\alpha f) (t) = \left( I_q^{[\alpha]-\alpha} (D_q^{[\alpha]} f) \right) (t)
= \frac{1}{\Gamma_q ([\alpha]-\alpha)} \int_0^t (t - qs)^{([\alpha]-\alpha-1)} (D_q^{[\alpha]} f)(s) d_q s, \tag{4}
\]
where \( t \in J \) and \( \alpha > 0 \) \([16]\). It has been proved that
\[
(I_q^\beta(I_q^\alpha f))(x) = (I_q^{\alpha+\beta} f)(x),
\]
and \((D_q^\alpha(I_q^\alpha f))(x) = f(x)\), where \( \alpha, \beta \geq 0 \) \([16]\). By using Algorithm 2, we can calculate \((I_q^\alpha f)(x)\) which is shown in Algorithm 4.

Now, we present some necessary notions. Throughout this article, we use the norms \(\|\cdot\|\), \(\|\cdot\|_1\) and \(\|u\|_s = \max\{\|u\|, \|u'\|\}\) for the spaces \(\mathcal{A} = C(\overline{J})\), \(\mathcal{L} = L^1(\overline{J})\) and \(\mathcal{B} = C^1(\overline{J})\), respectively. We say that, \(D_q^\alpha u(t) + g(t) = 0\) is a pointwise defined equation on \(J\) if there exists set \(S \subset J\) such that the measure of \(S^c\) is zero and the equation holds on \(S\) \([37]\). Let \(\Psi\) be the family of nondecreasing functions \(\psi : [0, \infty) \to [0, \infty)\) such that \(\sum_{n=1}^\infty \psi^n(t) < \infty\) for all \(t > 0\). One can check that \(\psi(t) < t\) for all \(t > 0\) \([35]\). The map \(T : X \to X\) is called an \(\alpha\)-admissible whenever \(\alpha(x_1, x_2) \geq 1\) implies \(\alpha(T(x_1), T(x_2)) \geq 1\) where \(\alpha\) maps \(X^2\) to \([0, \infty)\).

**Definition 2.1.** \([35]\) Let \((X, \rho)\) be a metric space, where \(\psi \in \Psi\) and \(\alpha : X^2 \to [0, \infty)\) is a map. A self-map \(T\) define on \(X\) is called an \(\alpha\)-\(\psi\)-contraction whenever
\[\alpha(x_1, x_2)\rho(T(x_1), T(x_2)) \leq \psi(\rho(x_1, x_2))\]
for all \(x_1, x_2 \in X\).

**Lemma 2.2.** \([35]\) Let \((X, \rho)\) be a complete metric space. If \(T : X \to X\) is continuous then \(T\) has a fixed point whenever there exists \(x_0 \in X\) such that \(\alpha(x_0, T(x_0)) \geq 1\), \(\psi \in \Phi\), \(\alpha : X^2 \to [0, \infty)\) a map and \(T : X \to X\) an \(\alpha\)-admissible \(\alpha\)-\(\psi\)-contraction.

**Lemma 2.3.** \([36, 41]\) If \(x \in \mathcal{A} \cap \mathcal{L}\) with \(D_q^\alpha x \in \mathcal{L} \cap \mathcal{L}\), then
\[I_q^\alpha D_q^\alpha x(t) = x(t) + \sum_{i=1}^n c_i t^{\alpha-i},\]
where \([\alpha] \leq n < [\alpha] + 1\) and \(c_i\) is some real number.

**Lemma 2.4.** \([24]\) Let \(X\) be a Banach space, \(P \subseteq X\) a cone and \(O_1, O_2\) two bounded open balls of \(X\) centered at the origin with \(\overline{O_1} \subset O_2\). Suppose that
\[\Omega : P \cap (\overline{O_2}\setminus O_1) \to P,\]
is a completely continuous operator such that either
\[i_1) \|\Omega(p)\| \leq \|p\| \text{ for all } p \in P \cap \partial O_1 \text{ and } \|\Omega(p)\| \geq \|p\| \text{ for all } p \in P \cap \partial O_2, \text{ or}\]
\[i_2) \|\Omega(p)\| \geq \|p\| \text{ for all } p \in P \cap \partial O_1 \text{ and } \|\Omega(p)\| \leq \|p\| \text{ for all } p \in P \cap \partial O_2,\]
holds. Then \(\Omega\) has a fixed point in \(P \cap (O_2\setminus O_1)\).

3. **Main results.** In this part, first we provide a lemma.

**Lemma 3.1.** Let \(n = [\alpha] + 1\) with \(\alpha \in [2, \infty)\), and \(g \in \mathcal{L}\). Then the solution of the pointwise defined equation \(D_q^\alpha u(t) + g(t) = 0\) involving boundary conditions \(u'(1) = u(0) = u''(0) = \cdots = u^{n-1}(0) = 0\) is a map \(f\) if and only if
\[f(t) = \int_0^1 G_q(t,s)g(s) \, ds\]
for all \( t \in \mathcal{J} \), where

\[
G_q(t, s) = \begin{cases} 
\frac{t(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)}, & t \leq s, \\
\frac{t(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} - \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}, & s \leq t,
\end{cases}
\]

(5)

for \( t \) and \( s \) belonging to \( \mathcal{J} \).

**Proof.** Let \( S \) be a subset of \( \mathcal{J} \) such that \( \mu(S^c) = 0 \) and \( D_q^\alpha u(t) + g(t) = 0 \) for \( t \in S \). Here, \( \mu \) is the Lebesgue measure on \( \mathbb{R} \). Note that, \( S \) is dense in \( \mathcal{J} \). Let \( g_0 \in \mathcal{J} \) be a function such that \( g_0 = g \) on \( S \). Then, we have

\[
I_q^\alpha(g(t)) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} g(s) \, dq_s
\]

for each \( t \in S \). Let \( t \in S^c \setminus \{0\} \). Choose a sequence \( \{t_n\}_{n \geq 1} \subset S \) such that \( t_n \to t^- \). Then,

\[
I_q^\alpha(g(t)) = \lim_{n \to \infty} \frac{1}{\Gamma_q(\alpha)} \int_0^{t_n} (t_n - qs)^{(\alpha-1)} g(s) \, dq_s
\]

\[
= \lim_{n \to \infty} I_q^\alpha(g(t_n))
\]

\[
= \lim_{n \to \infty} I_q^\alpha(g_0(t_n))
\]

\[
= \lim_{n \to \infty} \frac{1}{\Gamma_q(\alpha)} \int_0^{t_n} (t_n - sq)^{(\alpha-1)} g_0(s) \, dq_s
\]

\[
= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} g_0(s) \, dq_s = I_q^\alpha(g_0(t))
\]

If \( t = 0 \in S^c \) then \( I_q^\alpha(g(t)) = I_q^\alpha(g_0(t)) = 0 \). Therefore, \( I_q^\alpha(g(t)) = I_q^\alpha(g_0(t)) \) for all \( t \in \mathcal{J} \). Thus, the equation \( D_q^\alpha u(t) + g(t) = 0 \) equivalents to

\[
I_q^\alpha(D_q^\alpha u(t)) = I_q^\alpha(-f_0(t)),
\]

on \( \mathcal{J} \). The boundary condition and Lemma 2.2 imply that \( u(t) = -I_q^\alpha g(t) + c_1 t \) and so \( u'(t) = -I_q^{\alpha-1} g(t) + c_1 \). Hence, \( u'(1) = -I_q^{\alpha-1} g(1) + c_1 \). Since \( u'(1) = 0 \),
\[ c_1 = I_q^{\alpha-1}g(1). \]

Thus
\[ u(t) = -I_q^*g(t) + I_q^{\alpha-1}g(1) = \int_0^1 G_q(t, s)y(1) \, dq \, ds, \]

where
\[ G_q(t, s) = \frac{t(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)}, \]

whenever \( t \leq s \) and
\[ G_q(t, s) = \frac{t(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)} - \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}, \]

whenever \( s \leq t \), for \( t, s \in \mathcal{J} \). One can easily check that function
\[ f(t) = \int_0^1 G_q(t, s)y(s) \, dq \, ds \]

is a solution for the equation under the boundary conditions. This finishes the validation. \( \square \)

Note that, the \( q \)-Green function (5) satisfies
\[ 0 \leq (\alpha - 2)(t - qs)^{(\alpha-1)} \frac{G_q(t, s)}{\Gamma_q(\alpha)} \leq G_q(t, s) \leq \frac{t(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)}, \]

and
\[ 0 \leq \frac{\partial}{\partial t} G_q(t, s) \leq \frac{(1 - qs)^{(\alpha-2)}}{\Gamma_q(\alpha - 1)}, \]

for any \( t \) and \( s \) belonging to \( \mathcal{J} \). Also, \( G_q \) and \( \frac{\partial}{\partial t} G_q \) are continuous respect to first variable. Let \( \tau_1, \tau_2 \in \mathcal{J} \) with \( \tau_1 < \tau_2 \); functions \( w_1, w_3 \) are continuous respect to the first variable on \( [0, \tau_1] \times \mathcal{B}^3 \), \( [\tau_2, 1] \times \mathcal{B}^3 \), respectively, and the function \( w_2 \) define on \( (\tau_1, \tau_2) \times \mathcal{B}^3 \) is singular at some points \( t \in (\tau_1, \tau_2) \). Consider map \( w \) define on \( \mathcal{J} \times \mathcal{B}^3 \) such that
\[ w|_{[0, \tau_1] \times \mathcal{B}^3} = w_1, \quad w|_{(\tau_1, \tau_2) \times \mathcal{B}^3} = w_2, \quad w|_{[\tau_2, 1] \times \mathcal{B}^3} = w_3. \]

We denote this case briefly by \( [\tau_1, \tau_2, w = (w_1, w_2, w_3)] \). Define the map \( \Omega : \mathcal{B} \rightarrow \mathcal{B} \) by
\[ \Omega_u(t) = \int_0^1 G_q(t, s)w\left(s, u(s), u'(s), D_q^2u(s), \int_0^s f(r)u(r) \, dr, \varphi(u(s))\right) \, dq \, ds, \]

for all \( t \in \mathcal{J} \). We need to remind that, the singular pointwise defined equation (1) has a solution \( u_0 \in \mathcal{B} \) if and only if \( u_0 \) a fixed point of the map \( \Omega \).

**Theorem 3.2.** Let \( [\tau_1, \tau_2, w = (w_1, w_2, w_3)] \) with
\[ w_1(s, 0, 0, 0, 0, 0, 0) = w_3(t, 0, 0, 0, 0, 0, 0) = 0, \]

for all \( s \) and \( t \) belonging to \( [0, \tau_1] \) and \( [\tau_2, 1] \), respectively. Then the problem (1) has a solution whenever the following assumptions are hold.

1) There exist two maps \( \Theta : \mathcal{B} \rightarrow [0, \infty) \) and \( h : (\tau_1, \tau_2) \rightarrow [0, \infty) \) such that
\[ w_2(t, u_1, u_2, \ldots, u_5) \leq h(t)\Theta(u_1, u_2, \ldots, u_5), \]
for each \((u_1, \ldots, u_5) \in \mathbb{B}^5\) and almost all \(t \in (\tau_1, \tau_2)\), where \(\Theta : \mathbb{B}^5 \to [0, \infty)\) is a nondecreasing respect to all its components,

\[
\int_{\tau_1}^{\tau_2} (1 - q s)^{(\alpha - 1)} h(s) \, d q s < \infty
\]

and \(\lim_{y \to 0^+} \frac{\Theta(y, y, y, y, y)}{y} = 0\).

2) The map \(\mu\) which is defined by

\[
\mu(t) = \max\{\|u_1\|, \ldots, \|u_5\|\} \to \infty \max\{\|u_1\|, \ldots, \|u_5\|\}
\]

for each \(t \in (\tau_1, \tau_2)\) has this property that

\[
(\alpha - 2) \int_{\tau_1}^{\tau_2} (\tau_2 - q s)^{(\alpha - 2)} \mu(s) \, d q s > \Gamma_q(\alpha - 1).
\]

3) There exist nonnegative real numbers \(\eta_1, \ldots, \eta_5, \eta'_1, \ldots, \eta'_5\) and mappings

\[
h_1, \ldots, h_5 : (\tau_1, \tau_2) \to [0, \infty),
\]

and \(\theta_1, \ldots, \theta_5 : \mathbb{B}^5 \to [0, \infty)\) such that

\[
|w_1(t, u_1, \ldots, u_5) - w_1(t, v_1, \ldots, v_5)| \leq \sum_{i=1}^{5} \eta_i |u_i - v_i|,
\]

\[
|w_2(t, u_1, \ldots, u_5) - w_2(t, u_1, \ldots, u_5)| \leq \sum_{i=1}^{5} h_i(t) \theta_i(|u_1 - v_1|, \ldots, |u_5 - v_5|),
\]

\[
|w_3(t, u_1, \ldots, u_5) - w_3(t, v_1, \ldots, v_5)| \leq \sum_{i=1}^{5} \eta'_i |u_i - v_i|,
\]

for all \(t\) and \(u_1, \ldots, u_5, v_1, \ldots, v_5 \in \mathbb{B}\). Also,

\[
\lim_{y \to 0^+} \frac{\theta_i(y, y, y, y, y)}{y} = \gamma_i < \infty
\]

and

\[
[\ell(1 - (1 - \tau_1)^{\alpha - 1}) + \ell'(1 - \tau_2)^{\alpha - 1}] < \Gamma(\alpha),
\]

for \(i = 1, \ldots, 5\), where

\[
\ell = \eta_1 + \eta_2 + \frac{\eta_3}{\Gamma_q(2 - \beta)} + m_0 \eta_4 + a_1 \eta_5 + a_2 \eta_5,
\]

\[
\ell' = \eta'_1 + \eta'_2 + \frac{\eta'_3}{\Gamma_q(2 - \beta)} + m_0 \eta'_4 + a_1 \eta'_5 + a_2 \eta'_5
\]

(6)

and \(m_0 = \int_0^1 |f(r)| \, dr\).

Proof. Consider the closed cone \(P \subset \mathbb{B}\) of all \(u \in \mathbb{B}\) such that \(u(t)\) and \(u'(t)\) more than or equal to zero for each \(t \in J\). Also, consider the sequence \(\{u_n\}_{n \geq 1}\) in \(\mathbb{B}\) such that \(u_n \to u\). Let \(\epsilon > 0\) be given. Let us We choose natural number \(N\) such that \(\|u_n - u\| < \epsilon\) for each \(n \geq N\). Take \(\epsilon > 0\) such that

\[
\frac{\ell(1 - (1 - q \tau_1)^{\alpha - 1})}{\Gamma_q(\alpha)} + \frac{(\gamma_i + \epsilon) \epsilon}{\Gamma_q(\alpha - 1)} \sum_{i=1}^{5} L_i(\tau_1, \tau_2) + \frac{\ell'}{\Gamma_q(\alpha)} (1 - q \tau_2)^{\alpha - 1} < 1,
\]

(7)
for \( i = 1, \ldots, 5 \), where

\[
L_i(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} (1 - q s)^{(\alpha - 2)} h_i(s) \, dq s.
\]

It can be seen that,

\[
|\Omega_{\alpha_n}(t) - \Omega_{\alpha}(t)| \leq \int_0^{\tau_1} G_q(t, s)
\]

\[
\times \left| w_1 \left( s, u_n(s), D_q^{\alpha} u_n(s), \int_0^x f(r)u_n(r) \, dr, \varphi(u_n(s)) \right) 
- w_1 \left( s, u(s), D_q^{\alpha} u(s), \int_0^x f(r)u(r) \, dr, \varphi(u(s)) \right) \right| \, dq s
\]

\[
+ \int_{\tau_1}^{\tau_2} G_q(t, s)
\]

\[
\times \left| w_2 \left( s, u_n(s), u_n'(s), D_q^{\alpha} u_n(s), \int_0^x f(r)u_n(r) \, dr, \varphi(u_n(s)) \right) 
- w_2 \left( s, u(s), u'(s), D_q^{\alpha} u(s), \int_0^x f(r)u(r) \, dr, \varphi(u(s)) \right) \right| \, dq s,
\]

\[
+ \int_{\tau_2}^{1} G_q(t, s)
\]

\[
\times \left| w_3 \left( s, u_n(s), u_n'(s), D_q^{\alpha} u_n(s), \int_0^x f(r)u_n(r) \, dr, \varphi(u_n(s)) \right) 
- w_3 \left( s, u(s), u'(s), D_q^{\alpha} u(s), \int_0^x f(r)u(r) \, dr, \varphi(u(s)) \right) \right| \, dq s
\]

\[
\leq \int_0^{\tau_1} G_q(t, s)
\]

\[
\times \left[ \eta_1 |u_n(s) - u(s)| + \eta_2 |u_n'(s) - u'(s)| + \eta_3 |D_q^{\beta}(u_n - u)(s)|
\right.
\]

\[
+ \eta_4 \int_0^x |u_n(r) - u(r)| \, dr + \eta_5 |\varphi(u_n(s) - u(s))| \right]
\]

\[
+ \int_{\tau_1}^{\tau_2} G_q(t, s) h_1(s) \theta_1 \left( |u_n(s) - u(s)|, |u_n'(s) - u'(s)|, \right.
\]

\[
|D_q^{\beta}(u_n - u)(s)|, \int_0^x |u_n(r) - u(r)| \, dr, |\varphi(u_n(s) - u(s))| \right)
\]

\[
\left. + \cdots + h_5(s) \theta_5 \left( |u_n(s) - u(s)|, |u_n'(s) - u'(s)|, \right.
\]

\[
|D_q^{\beta}(u_n - u)(s)|, \int_0^x |u_n(\xi) - u(\xi)| \, d\xi, |\varphi(u_n(s) - u(s))| \right)
\]

\[
+ \int_{\tau_2}^{1} G_q(t, s)
\]

\[
\times \left[ \eta'_1 |u_n(s) - u(s)| + \eta'_2 |u_n'(s) - u'(s)| + \eta'_3 |D_q^{\beta}(u_n - u)(s)|
\right.
\]

\[
+ \eta'_4 \int_0^x |u_n(r) - u(r)| \, dr + \eta'_5 |\varphi(u_n(s) - u(s))| \right] \]
\[
\begin{align*}
&\leq \int_0^{\tau_1} G_q(t,s) \left[ \eta_1 \|u_n - u\| + \eta_2 \|u'_n - u'\| + \frac{\eta_3}{\Gamma_q(2-\beta)} \|u'_n - u'\| \\
&+ m_0 \eta_4 \|u_n - u\| + a_1 \eta_5 \|u_n - u\| + a_2 \eta_5 \|u'_n - u'\| \right] d_q s \\
&+ \int_{\tau_1}^1 G_q(t,s) h_1(s) \theta_1 \left( \|u_n - u\|, \|u'_n - u'\|, \frac{1}{\Gamma_q(2-\beta)} \|u'_n - u'\| \\
&+ m_0 \eta_4 \|u_n - u\| + a_1 \eta_5 \|u_n - u\| + a_2 \eta_5 \|u'_n - u'\| \right) d_q s \\
& + \int_{\tau_1}^1 G_q(t,s) \left[ \eta_1 + \eta_2 + \frac{\eta_3}{\Gamma_q(2-\beta)} \eta_4 m_0 + \eta_5 a_1 + \eta_5 a_2 \right] \\
& \times \|u_n - u\|_* \int_0^{\tau_1} G_q(t,s) d_q s \\
&+ \int_{\tau_1}^2 G_q(t,s) \left[ \sum_{i=1}^5 h_i(s) \theta_i \left( \eta \|u_n - u\|_* \eta \|u_n - u\|_*, \eta \|u_n - u\|_* \right) \\
&\eta \|u_n - u\|_* \eta \|u_n - u\|_* \eta \|u_n - u\|_* \right] d_q s \\
&+ \left[ \eta_1 + \eta_2 + \frac{\eta_3}{\Gamma_q(2-\beta)} \eta_4 m_0 + \eta_5 a_1 + \eta_5 a_2 \right] \\
& \times \|u_n - u\|_* \int_{\tau_1}^1 G_q(t,s) d_q s,
\end{align*}
\]

for \( t \in \mathcal{J} \), where

\[
\eta = \max \left\{ 1, \frac{1}{\Gamma_q(2-\beta)} m_0, a_1 + a_2 \right\}.
\]

Let \( \delta_i(\varepsilon) \in (1, \varepsilon^2) \) for \( 1 \leq i \leq 5 \), such that \( \theta_i(x, x, x, x, x) < x(\gamma_i + \varepsilon) \) for all \( x \in (0, \delta_i(\varepsilon)) \). Take \( \delta = \min_{1 \leq i \leq 5} \delta_i(\varepsilon) \). Then, we get

\[
\theta_i(\delta, \delta, \delta, \delta, \delta) < (\gamma_i + \varepsilon)\delta < (\gamma_i + \varepsilon)\varepsilon^2.
\]

We choose a natural number \( m_1 \) such that \( \eta \|u_n - u\|_* < \delta \) for all \( n \geq m_1 \). This implies for \( i = 1, \ldots, 5 \) that

\[
\theta_i(\eta \|u_n - u\|_*, \ldots, \eta \|u_n - u\|_*) < \theta_i(\delta, \delta, \delta, \delta, \delta) < (\gamma_i + \varepsilon)\varepsilon^2,
\]
for \( n \geq m_1 \). Thus,

\[
|\Omega_{u_n}(t) - \Omega_u(t)| \leq \ell \|u_n - u\|_* \int_0^{\tau_1} G(t, q s) \, dq \, ds \\
+ (\gamma_i + \varepsilon) \varepsilon^2 \int_{\tau_1}^{\tau_2} G_q(t, s) \sum_{i=1}^5 h_i(s) \, dq \, ds \\
+ \ell' \|u_n - u\|_* \int_{\tau_2}^1 G_q(t, s) \, dq \, ds
\]

for all \( n \geq \max\{N, m_1\} \). Therefore, we can conclude that

\[
|\Omega_{u_n}(t) - \Omega_u(t)| \leq \frac{\varepsilon \ell t}{\Gamma_q(\alpha - 1)} \int_0^{\tau_1} (1 - q s)^{(\alpha - 2)} \, dq \, ds \\
+ \frac{(\gamma_i + \varepsilon) \varepsilon^2 t}{\Gamma_q(\alpha - 1)} \sum_{i=1}^5 \int_{\tau_1}^{\tau_2} (1 - q s)^{(\alpha - 2)} h_i(s) \, dq \, ds \\
+ \frac{\ell' t}{\Gamma_q(\alpha)} \int_{\tau_2}^1 (1 - q s)^{(\alpha - 2)} \, dq \, ds \\
= \frac{\ell t (1 - (1 - \tau_1)^{(\alpha - 1)})}{\Gamma_q(\alpha)} + \frac{(\gamma_i + \varepsilon) \varepsilon^2 t}{\Gamma_q(\alpha - 1)} \sum_{i=1}^5 M_i(\tau_1, \tau_2) \\
+ \frac{\ell t}{\Gamma_q(\alpha)} (1 - \tau_2)^{\alpha - 1}
\]

for all \( n \geq \max\{N, m_1\} \) and \( t \in \mathcal{J} \). Indeed,

\[
\|\Omega_{u_n} - \Omega_u\| \leq \left[ \frac{\ell (1 - (1 - \tau_1)^{(\alpha - 1)})}{\Gamma_q(\alpha)} + \frac{(\gamma_i + \varepsilon) \varepsilon^2 t}{\Gamma_q(\alpha - 1)} \sum_{i=1}^5 M_i(\tau_1, \tau_2) \\
+ \frac{\ell t}{\Gamma_q(\alpha)} (1 - q \tau_2)^{(\alpha - 1)} \right] \varepsilon < \varepsilon.
\]

Now, by applying similar calculations, we obtain

\[
|\Omega_{u_n}'(t) - \Omega_u'(t)| \leq \int_{\tau_1}^{\tau_2} \frac{\partial G_q(t, s)}{\partial t} \, dq \, ds \\
\times \left| w_1 \left(s, u_n(s), u_n'(s), D_q^\alpha u_n(s), \int_0^s f(r) u_n(r) \, dr, \varphi(u_n(s)) \right) \\
- w_1 \left(s, u(s), u'(s), D_q^\alpha u(s), \int_0^s f(r) u(r) \, dr, \varphi(u(s)) \right) \right| \, dq \, ds \\
+ \int_{\tau_1}^{\tau_2} \frac{\partial G_q(t, s)}{\partial t} \, dq \, ds \\
\times \left| w_2 \left(s, u_n(s), u_n'(s), D_q^\alpha u_n(s), \int_0^s f(r) u_n(r) \, dr, \varphi(u_n(s)) \right) \\
- w_2 \left(s, u(s), u'(s), D_q^\alpha u(s), \int_0^s f(r) u(r) \, dr, \varphi(u(s)) \right) \right| \, dq \, ds \\
+ \int_{\tau_2}^1 \frac{\partial G_q(t, s)}{\partial t} \, dq \, ds
\]
\[
\begin{align*}
&\times \left| w_3\left(s, u_n(s), u'_n(s), D^\delta_q u_n(s), \int_0^s f(r)u_n(r)\, dr, \varphi(u_n(s))\right) \right| \, \mathrm{d}_q s \\
&- w_3\left(s, u(s), u'(s), D^\delta_q u(s), \int_0^s f(r)u(r)\, dr, \varphi(u(s))\right) \right| \, \mathrm{d}_q s \\
&\leq \int_{\tau_1}^{\tau_2} \frac{\partial G_q}{\partial t}(t, s) \, \mathrm{d}_q s \\
&\times \left[ \eta_1\|u_n - u\| + \eta_2\|u'_n - u'\| + \frac{\eta_3}{\Gamma_q(2 - \beta)}\|u'_n - u'\| \\
+ m_0\eta_4\|u_n - u\| + a_1\eta_5\|u_n - u\| + a_2\eta_5\|u'_n - u'\| \right) \, \mathrm{d}_q s \\
&+ \cdots + h_5(s)\theta_5\left(\|u_n - u\|, \|u'_n - u'\|, \frac{1}{\Gamma_q(2 - \beta)}\|u'_n - u'\|, \\
m_0\|u_n - u\|, a_1\eta_5\|u_n - u\| + a_2\eta_5\|u'_n - u'\| \right) \, \mathrm{d}_q s \\
&\leq \left[ \frac{\varepsilon(1 - (1 - q\tau_1)\gamma_i)}{\Gamma_q(\alpha)} + m_0\eta_5\|u_n - u\| + a_1\eta_5\|u'_n - u'\| \right] \, \mathrm{d}_q s \\
&+ \frac{\varepsilon^\alpha}{\Gamma_q(\alpha)(1 - q\mu)^{\gamma_i}} \, \mathrm{d}_q s,
\end{align*}
\]

for \( n \geq \max\{N, m_1\} \). Thus, \( \|\Omega'_{u_n} - \Omega'_u\| \leq \varepsilon \) for sufficiently large \( n \). Hence

\[
\|\Omega'_{u_n} - \Omega_u\|_* = \max\{\|\Omega'_{u_n} - \Omega_u\|, \|\Omega'_{u_n} - u'_x\|\} < \varepsilon,
\]

for sufficiently large \( n \). This implies that \( \Omega'_{u_n} \to \Omega_u \) in \( \mathcal{B} \). We choose \( r > 0 \) such that \( \|u\|_* < r \) for all \( u \in B \). Now, Let \( B \subset \mathcal{B} \) be a bounded set and \( u \in B \). Then

\[
|\Omega_u(t)| \leq \left| \int_{\tau_1}^{\tau_2} G_q(t, s)w_1\left(s, u(s), u'(s), D^\delta_q u(s), \int_0^s f(r)u(r)\, dr, \varphi(u(s))\right) \, \mathrm{d}_q s \\
+ \int_{\tau_1}^{\tau_2} G_q(t, s)w_2\left(s, u(s), u'(s), D^\delta_q u(s), \int_0^s f(r)u(r)\, du, \varphi(u(s))\right) \, \mathrm{d}_q s \\
+ \int_{\tau_1}^{\tau_2} G_q(t, s)w_3\left(s, u(s), u'(s), D^\delta_q u(s), \int_0^s f(r)u(r)\, dr, \varphi(u(s))\right) \, \mathrm{d}_q s \right|
\]
\[
\leq \int_0^{\tau_1} G_q(t, s) \left( w_1 \left( s, u(s), u'(s), D_q^2 u(s), \int_0^s f(r) u(r) \, dr, \varphi(u(s)) \right) + \eta_1 \| u \| + \eta_2 \| u' \| + \eta_3 \| D_q^2 u \| \right) \, ds \\
- \| w_1(s, 0, 0, 0, 0) \| \, ds + \int_0^{\tau_1} G_q(t, s) |w_1(s, 0, 0, 0, 0)| \, ds \\
+ \int_{\tau_1}^{\tau_2} G_q(t, s) h(s) \Theta \left( u(s), u'(s), D_q^2 u(s), \int_0^s f(r) u(r) \, dr, \varphi(u(s)) \right) \, ds \\
+ \int_{\tau_1}^{\tau_2} G_q(t, s) \left( w_3 \left( s, u(s), u'(s), D_q^2 u(s), \int_0^s f(r) u(r) \, dr, \varphi(u(s)) \right) \right) \, ds \\
- \| w_3(s, 0, 0, 0, 0) \| \, ds + \int_{\tau_1}^{1} G_q(t, s) \left( w_3(s, 0, 0, 0, 0) \right) \, ds
\]

\[
\leq \int_0^{\tau_1} G_q(t, s) \left( \eta_1 \| u \| + \eta_2 \| u' \| + \eta_3 \| D_q^2 u \| \\
+ \eta_4 \| u \| \int_0^s |f(r)| \, dr + \eta_5 \varphi(\| u \|) \right) \, ds \\
+ \Theta(\ell \| u \|, \ldots, \ell \| u \|) \int_{\tau_1}^{\tau_2} G_q(t, s) h(s) \\
+ \frac{\eta_1}{\Gamma_q(\alpha - 1)} \int_0^{\tau_1} (1 - qs)^{\alpha - 2} \left( \eta_1 \| u \| + \eta_2 \| u' \| \right) \, ds \\
+ \frac{\eta_3}{\Gamma_q(2 - \beta)} \| u' \| + \eta_4 m_0 \| u \| + \eta_5 a_1 \| u \| + \eta_5 a_2 \| u' \| \right) \, ds \\
+ \Theta(\eta \| u \|, \ldots, \eta \| u \|) t \int_{\tau_1}^{\tau_2} (1 - qs)^{\alpha - 2} h(s) \\
+ \frac{\eta_1}{\Gamma_q(\alpha - 1)} \int_{\tau_1}^{\tau_2} (1 - qs)^{\alpha - 2} \left( \eta_1 \| u \| + \eta_2 \| u' \| \right) \, ds \\
+ \frac{\eta_3}{\Gamma_q(2 - \beta)} \| u' \| + \eta_4 m_0 \| u \| + \eta_5 a_1 \| u \| + \eta_5 a_2 \| u' \| \right) \, ds \\
+ \frac{\eta_1}{\Gamma_q(\alpha - 1)} \| u \| \right) \, ds
\]

Indeed,

\[
\| \Omega u \| \leq \frac{\ell}{\Gamma_q(\alpha - 1)} \| u \| + \frac{\ell}{\Gamma_q(\alpha - 1)} \| u \| \\
+ \Theta(\ell \| u \|, \ldots, \ell \| u \|) \int_{\tau_1}^{\tau_2} (1 - qs)^{\alpha - 2} h(s) \, ds.
\]

By similar manner, we obtain

\[
\| \Omega u' \| \leq \frac{\ell}{\Gamma_q(\alpha - 1)} \| u \| + \frac{\ell}{\Gamma_q(\alpha - 1)} \| u \|.
\]
the Arzelà-Ascoli theorem, we get
\[ \Omega \]

Let \( \varepsilon \) be given. Choose \( \delta = \delta(\varepsilon) > 0 \) such that \( \|u\|_* < \delta \) implies
\[ \Theta(\|u\|_*, \ldots, \|\eta\|_*) < \varepsilon \]
and so \( \Theta(\|u\|_*, \ldots, \|\eta\|_*) < \varepsilon \|u\|_* \). Since
\[ \ell[1 - (1 - q\tau_1)^{(\alpha - 1)}] + \ell'(1 - q\tau_2)^{(\alpha - 1)} \leq \Gamma_q(\alpha), \]
there exists \( \varepsilon_0 > 0 \) such that
\[ \ell[1 - (1 - q\tau_1)^{(\alpha - 1)}] + \ell'(1 - q\tau_2)^{(\alpha - 1)} < \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha - 1)} + \frac{\varepsilon_0\|h\|_*}{\Gamma_q(\alpha - 1)} < 1, \]
where \( \|h\|_* = \int_{\tau_1}^{\tau_2} (1 - q\tau)^{(\alpha - 2)} h(s) ds \). Let \( \delta_0 = \delta(\varepsilon_0) \). Define \( O_1 \) by the set of all \( u \in \mathcal{B} \) somehow \( \|u\|_* < \delta \). Then,
\[ \|\Omega u(\tau)\| \leq \int_{\tau_1}^{\tau_2} G_q(t, s) \left| w_1 \left( t, u(s), u'(s), D_q^\beta u(s), \int_0^s f(r)u(r) \, dr, \varphi(u(s)) \right) \right| d_q s \]
\[ + \int_{\tau_1}^{\tau_2} G_q(t, s) \left| w_2 \left( t, u(s), u'(s), D_q^\beta u(s), \int_0^s f(r)u(r) \, dr, \varphi(u(s)) \right) \right| d_q s \]
\[ + \int_{\tau_1}^{\tau_2} G_q(t, s) \left| w_3 \left( t, u(s), u'(s), D_q^\beta u(s), \int_0^s f(r)u(r) \, dr, \varphi(u(s)) \right) \right| d_q s \]
\[ \leq \int_{\tau_1}^{\tau_2} G_q(t, s) \left( \eta_1 \|u\| + \eta_2 \|u'\| + \eta_3 \|u'\| + \eta_4 \|m_0\| + \eta_5 a_1 \|u\| + \eta_6 a_2 \|x'\| + \eta_7 a_3 \|x'\| \right) d_q s \]
\[ + \int_{\tau_1}^{\tau_2} G_q(t, s) h(s) \Theta \left( u(s), u'(s), D_q^\beta u(s), \int_0^s f(r)u(r) \, dr, \varphi(u(s)) \right) d_q s \]
\[ + \int_{\tau_1}^{\tau_2} G_q(t, s) \left( \eta_1' \|u\| + \eta_2' \|u'\| + \eta_3' \|u'\| + \eta_4' \|m_0\| + \eta_5' a_1 \|u\| + \eta_6' a_2 \|u'\| \right) d_q s \]
\[
\leq \frac{t}{\Gamma_q(\alpha-1)} \|u\|_* \int_{\tau_1}^{\tau_2} (1 - q\tau)^{(\alpha-2)} d_{q}\tau \\
+ \frac{t}{\Gamma_q(\alpha-1)} \|\eta\|_{\mathcal{B}} \int_{\tau_1}^{\tau_2} (1 - q\tau)^{(\alpha-2)} h(s) d_{q}\tau \\
+ \int_{\tau_2}^{\tau_1} \frac{t}{\Gamma_q(\alpha-1)} \|u\|_* \int_{0}^{1} (1 - q\tau)^{(\alpha-2)} d_{q}\tau
\]
for all \( u \in \mathcal{O}_1 \) and \( t \in \mathcal{T} \). Hence,
\[
\|\mathcal{O}_u\| \leq \left[ \frac{L_1(1 - q\tau_1)^{\alpha-1}}{\Gamma_q(\alpha)} + \frac{t^2}{\Gamma_q(\alpha-1)} + \frac{t}{\Gamma_q(\alpha-1)} \right] \|u\|_* \leq \|u\|_*.
\]
Similarly, we get \( \|\mathcal{O}_u'\| \leq \|u\|_* \) and so \( \|\mathcal{O}_u\|_* \leq \|u\|_* \). Since
\[
\lim_{\max \|u_i\| \to \infty} \frac{w_2(t, u_1, u_2, \ldots, u_5)}{\max \|u_i\|} = \gamma(t),
\]
there exists \( R = R(\varepsilon) > 0 \) such that \( \max \|u_i\| > R(\varepsilon) \) implies that
\[
\frac{w_2(t, u_1, u_2, \ldots, u_5)}{\max \|u_i\|} > \gamma(t) - \varepsilon
\]
and so \( w_2(t, u_1, u_2, \ldots, u_5) > (\max \|u_i\|)(\gamma(t) - \varepsilon) \). Recall that
\[
\frac{\alpha - 2}{\Gamma_q(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_1 - q\tau)^{(\alpha-1)} d_{q}\tau \leq \frac{\varepsilon_1 (\alpha - 2)(1 - q\tau_1)^{(\alpha-1)}}{\Gamma_q(\alpha + 1)} > 1.
\]
Choose \( R_1 = R(\varepsilon_1) > 0 \). Put \( \mathcal{O}_2 = \{ u \in \mathcal{B} : \|u\|_* < R_1 \} \). Then,
\[
\|\mathcal{O}_u\| = \sup_{t \in \mathcal{T}} |\mathcal{O}_u(t)| \geq |\mathcal{O}_u(\tau_2)|
\]
\[
\geq \int_{\tau_1}^{\tau_2} G_q(t, s) w_2 \left( s, u(s), u'(s), D^q_qu(s), \int_{0}^{s} f(r)u(r) \, dr, \varphi(u(s)) \right) d_{q}s
\]
\[
\geq \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - q\tau)^{(\alpha-1)}(\alpha - 2)}{\Gamma_q(\alpha)} \left( \gamma(s) - \varepsilon_1 \right)
\times \max \{ \|u\|, \|u'\|, \ldots, \|\varphi(u(s))\| \} d_{q}s
\]
\[
\geq \|u\|_* \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - q\tau)^{(\alpha-1)}(\alpha - 2)}{\Gamma_q(\alpha)} \left( \gamma(s) - \varepsilon_1 \right) d_{q}s
\]
\[
= \|u\|_* \left[ \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - q\tau)^{(\alpha-1)}(\alpha - 2)}{\Gamma_q(\alpha)} \gamma(s) d_{q}s \\
- \varepsilon_1 \int_{\tau_1}^{\tau_2} \frac{(\tau_2 - q\tau)^{(\alpha-1)}(\alpha - 2)}{\Gamma_q(\alpha)} d_{q}s \right]
\]
\[
= \|u\|_* \left[ \frac{\alpha - 2}{\Gamma_q(\alpha)} \int_{\tau_1}^{\tau_2} (\tau_2 - q\tau)^{(\alpha-1)} \gamma(s) d_{q}s \\
- \varepsilon_1 \frac{(\alpha - 2)(\tau_2 - q\tau_1)^{(\alpha-1)}}{\Gamma_q(\alpha + 1)} \right]
\]
\[
> \|u\|_*
\]

for all $u \in P \cap \partial \mathcal{O}_2$. Hence, $\|\Omega_u\| \geq \|u\|$ on $P \cap \partial \mathcal{O}_2$. At present, Lemma 2.4 implies that $\Omega : \mathcal{B} \to \mathcal{B}$ has a fixed point on $P \cap (\mathcal{O}_2 \backslash \mathcal{O}_1)$ which is a solution for the problem (1). \hfill \square

**Theorem 3.3.** Let $[\tau_1, \tau_2, w = (w_1, w_2, w_3)]$ with

$$w_1(s, 0, 0, 0, 0, 0) = w_3(t, 0, 0, 0, 0, 0) = 0,$$

for all $s \in [0, \tau_1]$ and $t \in [\tau_2, 1]$. Also, suppose that the following assumptions are hold.

1) There exist nonnegative functions $\mu_1, \mu_3$ and $1 + \mu_2, \ldots, 5 + \mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ with

$$\alpha \mu_2 := (1 - t)^{\alpha - 2} \mu_2(t),$$

for $i = 1, \ldots, 5$, in $L^1[\tau_1, \tau_2]$ respectively, such that

$$|w_1(t, u_1, \ldots, u_5) - w_1(t, v_1, \ldots, v_5)| \leq \mu_1(t) \sum_{i=1}^{5} \|u_i - v_i\|,$$

$$|w_2(t, u_1, \ldots, u_5) - w_2(t, v_1, \ldots, v_5)| \leq \sum_{i=1}^{5} \mu_2(t) \|u_i - v_i\|,$$

$$|w_3(t, u_1, \ldots, u_5) - w_3(t, v_1, \ldots, v_5)| \leq \mu_3(t) \sum_{i=1}^{5} \|u_i - v_i\|,$$

for all $u_1, \ldots, u_5, v_1, \ldots, v_5 \in \mathcal{B}$ and almost all $t \in J$.

2) There exist a natural number $n_0$ and nonnegative functions $\varphi_1, \ldots, \varphi_{n_0}$ with

$$\varphi_i := (1 - t)^{\alpha - 2} \varphi_i(t) \in L^1[\tau_1, \tau_2],$$

and nonnegative and nondecreasing respect to all components maps $\theta_1, \ldots, \theta_{n_0}$ $\mathcal{B}^5 \to [0, \infty) \setminus \{0\}$ with $\lim_{y \to 0^+} \theta_i(y, y, y, y, y) = 0$ such that

$$|w_2(t, u_1, \ldots, u_5)| \leq \sum_{i=1}^{n_0} \varphi_i \theta_i(u_1, \ldots, u_5),$$

for all $(u_1, \ldots, u_5) \in \mathcal{B}$ and almost all $t \in [\tau_1, \tau_2]$.

Then the problem (1) has a solution whenever

$$\left[2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2 \right]$$

$$\times \left[ \|\mu_1]\|_{[0, \tau_1]} + \sum_{i=1}^{5} \|\hat{b}_i\| + (1 - \tau_2)^{\alpha - 2}\|\mu_3\|_{[\tau_1, \tau_2]} \right] < \Gamma_q(\alpha - 1).$$

**Proof.** Let $u_1, u_2 \in \mathcal{B}$. Then,

$$|\Omega_{u_1}(t) - \Omega_{u_2}(t)| \leq \int_{0}^{\tau_1} G_q(t, s)$$

$$\times \left[ w_1\left(s, u_1(s), u_1'(s), D_0^b u_1(s), \int_{0}^{s} f(r) u_1(r) \, dr, \varphi(u_1(s)) \right) - w_1\left(s, u_2(s), u_2'(s), D_0^b u_2(s), \int_{0}^{s} f(r) u_2(r) \, dr, \varphi(u_2(s)) \right) \right] \, ds$$

$$+ \int_{\tau_1}^{\tau_2} G_q(t, s)$$
\begin{align*}
&\times \left[ w_2 \left( s, u_1(s), u_1'(s), D_q^\beta u_1(s), \int_0^s f(r)u_1(r)dr, \varphi(u_1(s)) \right) \right] d_q s \\
&- w_2 \left( s, u_2(s), u_2'(s), D_q^\beta u_2(s), \int_0^s f(r)u_2(r)dr, \varphi(u_2(s)) \right) \right| d_q s \\
&+ \int_{\tau_1}^1 G_q(t, s) \\
&\times \left[ w_3 \left( s, u_1(s), u_1'(s), D_q^\beta u_1(s), \int_0^s f(r)u_1(r)dr, \varphi(u_1(s)) \right) \right] d_q s \\
&- w_3 \left( s, u_2(s), u_2'(s), D_q^\beta u_2(s), \int_0^s f(r)u_2(r)dr, \varphi(u_2(s)) \right) \right| d_q s \\
&\leq \int_0^{\tau_1} \frac{(1 - q s)^{\alpha - 2}}{\Gamma_q(\alpha - 1)} \mu_1(s) \\
&\times \left[ |u_1(s) - u_2(s)| + |u_1'(s) - u_2'(s)| + |D_q^\beta (u_1 - u_2)(s)| \\
&+ \int_0^s |u_1(r) - u_2(r)|dr + |\varphi(u_1(s) - u_2(s))| \right] d_q s \\
&+ \int_{\tau_1}^{\tau_2} \frac{(1 - q s)^{\alpha - 2}}{\Gamma_q(\alpha - 1)} \left[ 1 \mu_2(s)|u_1(s) - u_2(s)| \\
&+ 2 \mu_2(s)|u_1'(s) - u_2'(s)| + 3 \mu_2(s)|D_q^\beta (u_1 - u_2)(s)| \\
&+ 4 \mu_2(s) \int_0^s |u_1(r) - u_2(r)|dr + 5 \mu_2(s)|\varphi(u_1(s) - u_2(s))| \right] d_q s \\
&+ \int_{\tau_2}^1 \frac{(1 - q s)^{\alpha - 2}}{\Gamma_q(\alpha - 1)} \mu_3(s) \\
&\times \left[ |u_1(s) - u_2(s)| + |u_1'(s) - u_2'(s)| + |D_q^\beta (u_1 - u_2)(s)| \\
&+ \int_0^s |u_1(\xi) - u_2(\xi)|dr + |\varphi(u_1(s) - u_2(s))| \right] d_q s \\
&\leq \frac{t}{\Gamma_q(\alpha - 1)} \int_0^{\tau_1} (1 - q s)^{\alpha - 2} \mu_1(s) \\
&\times \left[ \|u_1 - u_2\| + \|u_1' - u_2'\| + \left\| \frac{1}{\Gamma_q(2 - \beta)} (u_1 - u_2) \right\| \\
&+ m_0 \|u_1 - u_2\| + a_1 \|u_1 - u_2\| + a_2 \|u_1' - u_2'\| \right] d_q s \\
&+ \frac{t}{\Gamma_q(\alpha - 1)} \int_{\tau_1}^{\tau_2} (1 - q s)^{\alpha - 2} \\
&\times \left[ 1 \mu_2(s)|u_1 - u_2| + 2 \mu_2(s)|u_1' - u_2'| \\
&+ 3 \mu_2(s) \|u_1' - u_2'\| + 4 \mu_2(s) m_0 \|u_1 - u_2\| \\
&+ 5 \mu_2(s) (a_1 \|u_1 - u_2\| + a_2 \|u_1' - u_2'\|) \right] d_q s 
\end{align*}
Thus

\[ + \frac{t}{\Gamma_q(\alpha - 1)} \int_{\tau_2}^{1} (1 - q s)^{(\alpha - 2) \mu_3(s)} \]
\[ \times \left[ \| u_1 - u_2 \| + \| u'_1 - u'_2 \| + \| u'_1 - u'_2 \| \frac{1}{\Gamma_q(2 - \beta)} \right] \]
\[ + m_0 \| u_1 - u_2 \| + a_1 \| u_1 - u_2 \| + a_2 \| u'_1 - u'_2 \| \right] d_q s \]
\[ \leq \frac{t}{\Gamma_q(\alpha - 1)} \left[ 2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2 \right] \]
\[ \times \| u_1 - u_2 \| \sum_{i=1}^{5} \int_{\tau_1}^{\tau_2} (1 - q s)^{(\alpha - 2) \mu_2(s)} d_q s \]
\[ + \frac{t}{\Gamma_q(\alpha - 1)} \left[ 2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2 \right] \]
\[ \times \| u_1 - u_2 \| \int_{\tau_1}^{1} (1 - q s)^{(\alpha - 2) \mu_3(s)} d_q s \]
\[ \leq \frac{t}{\Gamma_q(\alpha - 1)} \left[ 2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2 \right] \]
\[ \| u_1 - u_2 \| \left[ \int_{0}^{\tau_1} \mu_1(s) d + q s \right. \]
\[ + \sum_{i=1}^{5} \int_{\tau_1}^{\tau_2} (1 - q s)^{(\alpha - 2) \mu_2(s)} d_q s + \int_{\tau_2}^{1} \mu_3(s) d_q s \right]. \]

Thus

\[ \| \Omega_{u_1} - \Omega_{u_2} \| \leq \frac{1}{\Gamma_q(\alpha - 1)} \left[ 2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2 \right] \]
\[ \times \left[ \| \mu_1 \|_{[0, \tau_1]} + \sum_{i=1}^{5} \| \mu_2 \|_{[\tau_1, \tau_2]} + \| \mu_3 \|_{[\tau_2, 1]} \right] \| u_1 - u_2 \|_{\ast}. \]

If we use a similar method, then we have

\[ |\Omega'_{u_1}(t) - \Omega'_{u_2}(t)| \leq \int_{0}^{\tau_1} \frac{\partial G_q(t, s)}{\partial t} \left| w_1 \left( s, u_1(s), u'_1(s), D_q^\alpha u_1(s), \right. \right. \]
\[ \left. \int_{0}^{s} f(r)u_1(r) dr, \varphi(u_1(s)) \right) + \left. \int_{0}^{s} f(r)u_2(r) dr, \varphi(u_2(s)) \right) \right| d_q s \]
\[ - w_1 \left( s, u_2(s), u'_2(s), D_q^\alpha u_2(s), \int_{0}^{s} f(r)u_2(r) dr, \varphi(u_2(s)) \right) \right| d_q s \]
\[ + \int_{\tau_1}^{\tau_2} \frac{\partial G_q(t, s)}{\partial t} \left| w_2 \left( s, u_1(s), u'_1(s), D_q^\alpha u_1(s), \right. \right. \]
\[ \left. \int_{0}^{s} f(r)u_1(r) dr, \varphi(u_1(s)) \right) + \left. \int_{0}^{s} f(r)u_2(r) dr, \varphi(u_2(s)) \right) \right| d_q s \]
\[-w_2\left(s, u_2(s), u_2'(s), D_q^2 u_2(s), \int_0^s f(r)u_2(r) \, dr, \varphi(u_2(s))\right) \bigg| d_q s \]
\[+ \int_{\tau_2}^t \frac{\partial G_q(t, s)}{\partial t} w_3\left(s, u_1(s), u_1'(s), D_q^2 u_1(s), \int_0^s f(r)u_1(r) \, dr, \varphi(u_1(s))\right) d_q s \]
\[-w_3\left(s, u_2(s), u_2'(s), D_q^2 u_2(s), \int_0^s f(r)u_2(r) \, dr, \varphi(u_2(s))\right) \bigg| d_q s \]
\[\leq \int_0^{\tau_1} \frac{t(1 - qs)^{(\alpha - 2)}}{\Gamma_q(\alpha - 1)} \mu_1(s) \left|u_1(s) - u_2(s)\right| \, d_q s \]
\[+ \int_0^{\tau_2} \frac{t(1 - qs)^{(\alpha - 2)}}{\Gamma_q(\alpha - 1)} \mu_2(s) \left|u_1(s) - u_2(s)\right| \, d_q s \]
\[+ 2 \mu_2(s) \left|u_1'(s) - u_2'(s)\right| + 3 \mu_2(s) \left|D_q^2(u_1 - u_2)(s)\right| \]
\[+ 4 \mu_2(s) \int_0^s \left|u_1(r) - u_2(r)\right| \, dr + 5 \mu_2(s) \left|\varphi(u_1(s) - u_2(s))\right| \, d_q s \]
\[+ \int_{\tau_2}^t \frac{1}{\Gamma_q(2 - \beta)} \left[2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2\right] \times \left\|u_1 - u_2\right\| \, d_q s \]
\[\times \left\|u_1 - u_2\right\| \leq \int_{\tau_1}^{\tau_2} (1 - qs)^{(\alpha - 2)} \mu_3(s) \, d_q s \]
\[\leq \int_0^{\tau_1} \frac{2}{\Gamma_q(2 - \beta)} \left[2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + \theta_0 + \theta_1\right] \times \left\|u_1 - u_2\right\| \, d_q s \]
\[\times \left\|u_1 - u_2\right\| \sum_{i=1}^5 \int_{\tau_1}^{\tau_2} (1 - qs)^{(\alpha - 2)} \mu_2(s) \, d_q s \]
\[+ \int_{\tau_2}^t \frac{1}{\Gamma_q(2 - \beta)} \left[2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + \theta_0 + \theta_1\right] \times \left\|u_1 - u_2\right\| \, d_q s \]
\[\leq \int_{\tau_1}^{\tau_2} (1 - qs)^{(\alpha - 2)} \mu_3(s) \, d_q s \]
Thus
\[ \|\Omega'_{u_1} - \Omega'_{u_2}\| \leq \frac{1}{\Gamma_q(\alpha - 1)} \left[ 2 + \frac{1}{\Gamma(2 - \beta)} + m_0 + a_1 + a_2 \right] \times \left[ \|\mu_1\|_{[0,\tau_1]} + \sum_{i=1}^{5} \|i\hat{\mu}_2\|_{[\tau_1,\tau_2]} + \|\mu_3\|_{[\tau_2,1]} \right] \|u_1 - u_2\|_{*}. \]
This implies that
\[ \|\Omega_{u_1} - \Omega_{u_2}\|_{*} \leq \frac{1}{\Gamma_q(\alpha - 1)} \left[ 2 + \frac{1}{\Gamma(2 - \beta)} + m_0 + a_1 + a_2 \right] \times \left[ \|\mu_1\|_{[0,\tau_1]} + \sum_{i=1}^{5} \|i\hat{\mu}_2\|_{[\tau_1,\tau_2]} + \|\mu_3\|_{[\tau_2,1]} \right] \|u_1 - u_2\|_{*}, \]
and so \( \Omega_{u_1} \to \Omega_{u_2} \) in \( \overline{E} \) as \( u_2 \to u_1 \). Thus, \( \Omega \) is continuous on \( \overline{E} \). Since
\[ \lim_{x \to 0^+} \frac{\theta_i(x, x, x, x, x)}{x} = 0, \]
\[ \lim_{x \to 0^+} \frac{\theta_i(\eta x, \eta x, \eta x, \eta x, \eta x)}{x} = 0, \]
where
\[ \eta = \max \left\{ 1, \frac{1}{\Gamma_q(2 - \beta)}, m_0, a_1 + a_2 \right\}. \]
Let \( \varepsilon > 0 \) be given. Choose \( \delta_i := \delta_i(\varepsilon) > 0 \) such that \( x \in (0, \delta_i] \) implies that
\[ \lim_{x \to 0^+} \frac{\theta_i(\eta x, \eta x, \eta x, \eta x, \eta x)}{x} < \varepsilon, \]
for \( 1 \leq i \leq n_0 \). Hence,
\[ \theta_i(\eta x, \eta x, \eta x, \eta x, \eta x) < \varepsilon \varepsilon, \]
for \( 0 < x \leq \delta_i \) and so \( \theta_i(\eta x, \eta x, \eta x, \eta x, \eta x) < \varepsilon \varepsilon \) for \( 1 \leq i \leq n_0 \) and \( x \in (0, \delta] \), where
\[ \delta := \delta(\varepsilon) = \min_{1 \leq i \leq n_0} \{ \delta_i \}. \]
Since
\[ \left[ 2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2 \right] \times \left[ \|\mu_1\|_{[0,\tau_1]} + (1 - q\tau_2)\|\mu_3\|_{[1,\tau_2]} \right] < \Gamma_q(\alpha - 1), \]
there exists \( \varepsilon_0 > 0 \) such that
\[ \left[ 2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2 \right] \times \left[ \|\mu_1\|_{[0,\tau_1]} + (1 - q\tau_2)\|\mu_3\|_{[1,\tau_2]} + \varepsilon_0 \sum_{i=1}^{n_0} \|\hat{\mu}_i\|_{[\tau_1,\tau_2]} \right] < \Gamma_q(\alpha - 1). \]
Let \( r = \delta(\varepsilon_0) \). Then, \( \theta_i(\eta x, \eta x, \eta x, \eta x, \eta x) < \varepsilon_0 x \) for all \( 1 \leq i \leq n_0 \) and for \( x \in (0, r] \). We take the set \( E \) of all \( x \in \overline{E} \) such that \( \|x\|_{*} \) less than to \( r \). Define
the map $\alpha : \overline{B}^2 \to [0, \infty)$ by $\alpha(u, v) = 1$ whenever $u, v \in E$ and $\alpha(u, v) = 0$ otherwise. We show that $\Omega$ is $\alpha$-admissible. We choose an elements $u, v \in \overline{B}$ such that $\alpha(u, v) \geq 1$. Then, $u, v \in E$, $\|u\|_* < r$ and $\|v\|_* < r$. Let $t \in J$. Then, we get

$$|\Omega(u(t))| \leq \int_{0}^{\tau_1} G_q(t, s) \left|w_1(s, u(s), u'(s), D_q^2 u(s), \int_{0}^{s} f(r)u(r) \, dr, \varphi(u(s)))\right| \, dq \, ds$$

$$+ \int_{\tau_1}^{\tau_2} G_q(t, s) \left|w_2(s, u(s), u'(s), D_q^3 u(s), \int_{0}^{s} f(r)u(r) \, dr, \varphi(u(s)))\right| \, dq \, ds$$

$$+ \int_{\tau_2}^{1} G_q(t, s) \left|w_3(s, u(s), u'(s), D_q^4 u(s), \int_{0}^{s} f(r)u(r) \, dr, \varphi(u(s)))\right| \, dq \, ds$$

$$\leq \frac{t}{\Gamma_q(\alpha - 1)} \int_{0}^{\tau_1} (1 - qs)^{(\alpha - 2)} \left|w_1(s, 0, 0, 0, 0)\right| \, dq \, ds$$

$$+ \frac{t}{\Gamma_q(\alpha - 1)} \int_{\tau_1}^{\tau_2} (1 - qs)^{(\alpha - 2)} \left|w_1(s, 0, 0, 0, 0)\right| \, dq \, ds$$

$$+ \frac{t}{\Gamma_q(\alpha - 1)} \int_{\tau_2}^{1} (1 - qs)^{(\alpha - 2)} \left|w_2(s, 0, 0, 0, 0)\right| \, dq \, ds$$

$$+ \frac{t}{\Gamma_q(\alpha - 1)} \int_{0}^{\tau_1} (1 - qs)^{(\alpha - 2)} \left|w_3(s, 0, 0, 0, 0)\right| \, dq \, ds$$

$$+ \frac{t}{\Gamma_q(\alpha - 1)} \int_{0}^{\tau_1} (1 - qs)^{(\alpha - 2)} \left|w_4(s, 0, 0, 0, 0)\right| \, dq \, ds$$

$$+ \frac{t}{\Gamma_q(\alpha - 1)} \int_{\tau_1}^{\tau_2} (1 - qs)^{(\alpha - 2)} \mu_1(s) \left[\|u\| + \|u'\| + \frac{\|u''\|}{\Gamma(2 - \beta)} + m_0 \|u\| + a_1 \|u\| + a_2 \|u\|\right] \, dq \, ds$$

$$+ \frac{t}{\Gamma_q(\alpha - 1)} \int_{\tau_2}^{1} (1 - qs)^{(\alpha - 2)} \, dq \, ds$$
Indeed, \(\|\Omega_u\| < r\). Similarly, we can prove that \(\|\Omega'_u\| < r\) and so \[\|\Omega_u\|_* = \max\{\|\Omega_u\|, \|\Omega'_u\|\} < r.\]

Hence, \(\Omega_u \in E\) and by same reason \(\Omega_v \in E\). This implies that \(\alpha(\Omega_u, \Omega_v) \geq 1\) and so \(\Omega\) is \(\alpha\)–admissible. Also, \(\alpha(u_0, \Omega_{m_0}) \geq 1\) for all \(u_0 \in E\) (note that, \(E\) is nonempty).

Let \(u, v \in \mathcal{B}\) and \(t \in \mathcal{I}\). Then, we have
\[
|\Omega_u(t) - \Omega_v(t)| \leq \int_0^{\tau_{1}} G_q(t, s) \\
\times \left|\sum_{i=1}^{n_0} \varphi_i(s)\theta_i\left(\frac{\|u\| + \|u'\|}{\Gamma_q(2 - \beta)} + m_0\|u\| + a_1\|u\| + a_2\|u\|\right)\right| d_q s \\
+ \frac{t}{\Gamma_q(\alpha - 1)} \int_{\tau_{2}}^{1} (1 - q s)^{(\alpha - 2)} \mu_3(s) \\
\times \left[\|u\| + \|u'\| + \frac{\|u'\|}{\Gamma_q(2 - \beta)} + m_0\|u\| + a_1\|u\| + a_2\|u\|\right] d_q s \\
\leq \frac{t}{\Gamma_q(\alpha - 1)} \left(\frac{2}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2\right) \\
\times \|u\|_* \int_{0}^{\tau_{1}} \sup(1 - q s)^{(\alpha - 2)} \mu_1(s) d_q s \\
+ \sum_{i=1}^{n_0} \int_{\tau_{1}}^{\tau_{2}} (1 - q s)^{(\alpha - 2)} \varphi_i(s)\theta_i(\|u\|_*, \|u\|_*, \|u\|_*, \|u\|_*, \|u\|_*) \right) d_q s \\
+ \left[2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2\right] \|x\|_* \int_{\mu}^{1} \sup(1 - q s)^{(\alpha - 2)} \mu_3(s) d_q s \\
\leq \frac{1}{\Gamma_q(\alpha - 1)} \left(\frac{2}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2\right) \\
\times \left[\|u\|_* + (1 - q \tau_{2})^{(\alpha - 2)} \|\mu_3\|_{\tau_{2}, 1}\right] \right) \\
+ \sum_{i=1}^{n_0} \theta_i(\eta \tau, \eta \tau, \eta \tau, \eta \tau, \eta \tau) \right) \\
= \frac{1}{\Gamma_q(\alpha - 1)} \left(\frac{2}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2\right) \\
\times \left[\|u\|_* + (1 - q \tau_{2})^{(\alpha - 2)} \|\mu_3\|_{\tau_{2}, 1}\right] \\
+ \sum_{i=1}^{n_0} \|\varphi_i\| \theta_i(\eta \tau, \eta \tau, \eta \tau, \eta \tau, \eta \tau) \right) \\
\leq \frac{r}{\Gamma_q(\alpha - 1)} \left(\frac{2}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2\right) \\
\times \left[\|u\|_* + (1 - q \tau_{2})^{(\alpha - 2)} \|\mu_3\|_{\tau_{2}, 1}\right] + \varepsilon_0 \sum_{i=1}^{n_0} \|\varphi_i\| \right) \\
< \frac{r}{\Gamma_q(\alpha - 1)} \Gamma_q(\alpha - 1) = r.
\]
\[
-w_1 \left( s, v(s), v'(s), D_q^\beta v(s), \int_0^s f(r) v(r) \, dr, \varphi(v(s)) \right) \bigg| \Gamma_q \int_s^1 \left| \int_0^1 \left( \left\| u - v \right\| + \left\| u' - v' \right\| + \left\| D_q^\beta (u - v) \right\| \\
+ \int_0^s |f(r)||u - v|| dr + \varphi(||u - v||) \right) \bigg| q \, s \\
+ \int_{\tau_1}^{\tau_2} G_q(t, s) \\
\times \left| w_2 \left( s, u(s), u'(s), D_q^\beta u(s), \int_0^s f(r) u(r) \, dr, \varphi(u(s)) \right) \right| q \, s \\
-w_2 \left( s, v(s), v'(s), D_q^\beta v(s), \int_0^s f(r) v(r) \, dr, \varphi(v(s)) \right) \bigg| q \, s \\
+ \int_{\tau_1}^{\tau_2} G_q(t, s) \\
\times \left| w_3 \left( s, u(s), u'(s), D_q^\beta u(s), \int_0^s f(r) u(r) \, dr, \varphi(u(s)) \right) \right| q \, s \\
-w_3 \left( s, v(s), v'(s), D_q^\beta v(s), \int_0^s f(r) v(r) \, dr, \varphi(v(s)) \right) \bigg| q \, s \\
\leq \frac{t}{\Gamma_q(\alpha - 1)} \int_0^{\tau_1} (1 - qs)^{(\alpha - 2)} \mu_1(s) \\
\times \left[ \left\| u - v \right\| + \left\| u' - v' \right\| + \left\| D_q^\beta (u - v) \right\| \\
+ \int_0^s |f(r)||u - v|| dr + \varphi(||u - v||) \right] q \, s \\
+ \frac{t}{\Gamma_q(\alpha - 1)} \int_{\tau_1}^{\tau_2} (1 - qs)^{(\alpha - 2)} \\
\times \left[ 1\mu_2(s)||u - v|| + 2\mu_2(s)||u' - v'|| + 3\mu_2(s)(D_q^\beta||u - v||) \\
+ 4\mu_2(s) \int_0^s |f(r)||u - v|| dr + 5\mu_2(s) \varphi(||u - v||) \right] q \, s \\
+ \frac{t}{\Gamma_q(\alpha - 1)} \int_{\tau_2}^{1} (1 - qs)^{(\alpha - 2)} \mu_3(s) \\
\times \left[ \left\| u - v \right\| + \left\| u' - v' \right\| + \left\| D_q^\beta (u - v) \right\| \\
+ \int_0^s |f(r)||u - v|| dr + \varphi(||u - v||) \right] q \, s \\
\leq \frac{t}{\Gamma_q(\alpha - 1)} \int_0^{\tau_1} (1 - qs)^{(\alpha - 2)} \mu_1(s) \\
\times \left[ \left\| u - v \right\| + \left\| u' - v' \right\| + \left\| u' - v' \right\| \\
+ \frac{m_0||u - v|| + a_1||u - v|| + a_2||u' - v'||}{\Gamma_q(2 - \beta)} \right] q \, s \\
+ \frac{t}{\Gamma_q(\alpha - 1)} \int_{\tau_1}^{\tau_2} (1 - qs)^{(\alpha - 2)} \\
\times \left[ 1\mu_2(s)||u - v|| + 2\mu_2(s)||u' - v'|| + 3\mu_2(s)\frac{||u' - v'||}{\Gamma_q(2 - \beta)} \right].
\]
can be executed to calculate the value of $x$. For smaller values of $q$, consider a pseudo-code description of the method for calculating $\psi$. Table 1 shows that when $q$ is constant, the $q$-Gamma function is an increasing function. Also, for smaller values of $x$, an approximate result is obtained with less values of $n$. It has been shown by underlined rows, Table 2 shows that the $q$-Gamma function for values $q$ near to one is obtained with more values of $n$ in comparison with $\alpha(u, v)\|\Omega_u - \Omega_v\|_* \leq \psi(d(u, v))$ for all $u, v \in \mathcal{F}$. Since

$$\frac{1}{\Gamma_q(\alpha - 1)} \left[ 2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2 \right] \times \left[ \|\mu_1\|_{[0, \tau_1]} + \sum_{i=1}^{5} \|\mu_2\| + (1 - q \tau_2)^{(\alpha-2)} \|\mu_3\|_{[1, \tau_2]} \right] < 1,$$

$\psi \in \Psi$. Therefore, Lemma 2.2 implies that $\Omega$ has a fixed point which one can check that it is a solution for the problem (1).

4. Examples and numerical check technique for the problems. In this part, we give a complete computational techniques for checking working to illustrate of the problem (1) in Theorems 3.2, such that it covers all the problems and present numerical examples which solve perfect. Foremost, we present a simplified analysis can be executed to calculate the value of $q$-Gamma function, $\Gamma_q(x)$, for input values $q$ and $x$ by counting the number of sentences $n$ in summation. To this aim, we consider a pseudo-code description of the method for calculated $q$-Gamma function of order $n$ in Algorithm 2 (for more details, see the following link [https://en.wikipedia.org/wiki/Q-gamma_function]).

Table 1 shows that when $q$ is constant, the $q$-Gamma function is an increasing function. Also, for smaller values of $x$, an approximate result is obtained with less values of $n$. It has been shown by underlined rows, Table 2 shows that the $q$-Gamma function for values $q$ near to one is obtained with more values of $n$ in comparison with $\alpha(u, v)\|\Omega_u - \Omega_v\|_* \leq \psi(d(u, v))$ for all $u, v \in \mathcal{F}$. Since

$$\frac{1}{\Gamma_q(\alpha - 1)} \left[ 2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2 \right] \times \left[ \|\mu_1\|_{[0, \tau_1]} + \sum_{i=1}^{5} \|\mu_2\| + (1 - q \tau_2)^{(\alpha-2)} \|\mu_3\|_{[1, \tau_2]} \right] < 1,$$

$\psi \in \Psi$. Therefore, Lemma 2.2 implies that $\Omega$ has a fixed point which one can check that it is a solution for the problem (1).
with other columns. They have been underlined in line 8 of the first column, line 17 of the second column and line 29 of third columns of Table 2. Also, Table 3 is the same as Table 2, but $x$ values increase in 3. Similarly, the $q$-Gamma function for values $q$ near to one is obtained with more values of $n$ in comparison with other columns.

Remark that, all routines are written in “Matlab” software with the “Digits” 16 (Digits environment variable controls the number of digits in MATLAB) and work on a PC with 2.90 GHz of Core 2 CPU and 4 GB of RAM.

Here, we provide two examples to illustrate our main result.

**Example 1.** We consider the problem similar to (1) as follows:

$$D_q^2 u(t) + w(t, u(t), u'(t), D_q^3 u(t), \int_0^t u(r) \, dr, D_q^4 u(t)) = 0,$$

where

$$w(t, u_1, u_2, u_3, u_4, u_5) = \begin{cases} \sum_{i=1}^5 u_i, & 0 \leq t < 0.2, \\ \mu(t)\Theta(u_1, u_2, u_3, u_4, u_5), & 0.2 \leq t \leq 0.7, \\ (1-t)\sum_{i=1}^5 u_i, & 0.7 < t \leq 1, \end{cases}$$

$$\Theta(u_1, u_2, u_3, u_4, u_5) = \sum_{i=1}^5 \frac{||u_i||^2}{1 + ||u_i||},$$

the map $\mu$ define on $[0.2, 0.7]$ by $\mu(t) = \frac{1}{d(t)}$, $\mu(t) = 15$ whenever $t$ belongs to $[0.2, 0.7] \cap \mathbb{Q}$, $[0.2, 0.7] \cap \mathbb{Q}^c$, respectively, here $d(t) = 0$ on $[0.2, 0.7] \cap \mathbb{Q}$. Put

$$w_1(t, u_1, u_2, u_3, u_4, u_5) = \sum_{i=1}^5 u_i,$n

$$w_2(t, u_1, u_2, u_3, u_4, u_5) = \mu(t)\Theta(u_1, u_2, u_3, u_4, u_5),$$

$$w_3(t, u_1, u_2, u_3, u_4, u_5) = (1-t)\sum_{i=1}^5 u_i.$$

Note that, $w_1(t, 0, 0, 0, 0, 0) = w_3(t, 0, 0, 0, 0, 0) = 0$,

$$w_1(t, u_1, u_2, u_3, u_4, u_5) - w_1(t, v_1, v_2, v_3, v_4, v_5) \leq t \sum_{i=1}^5 ||u_i - v_i|| \leq \frac{1}{5} \sum_{i=1}^5 ||u_i - v_i||,$n

$$|w_2(t, u_1, u_2, u_3, u_4, u_5) - w_2(t, v_1, v_2, v_3, v_4, v_5)|$$

$$= \mu(t)\sum_{i=1}^5 \frac{||u_i||^2}{1 + ||u_i||} - \frac{||v_i||^2}{1 + ||v_i||}$$

$$= \mu(t)\sum_{i=1}^5 \frac{||u_i||^2 - ||v_i||^2}{(1 + ||u_i||)(1 + ||v_i||)}$$

$$= \mu(t)\sum_{i=1}^5 \frac{||u_i||^2 - ||v_i||^2 + ||u_i||(||u_i|| - ||v_i||^2)}{(1 + ||u_i||)(1 + ||v_i||)}$$

$$= \mu(t)\sum_{i=1}^5 \frac{||u_i||^2 - ||v_i||^2 + ||u_i||(||u_i|| - ||v_i||^2)}{(1 + ||u_i||)(1 + ||v_i||)}$$
\[ \mu(t) \sum_{i=1}^{5} \left( \frac{\|u_i\| - \|v_i\|}{(1 + \|u_i\|)(1 + \|v_i\|)} \right) \]
\[ = \mu(t) \sum_{i=1}^{5} \left( \frac{\|u_i\|}{\|u_i\| + \|v_i\| + \|u_i\||\|v_i\|} \right) \]
\[ \leq \mu(t) \sum_{i=1}^{5} \left( \frac{\|u_i\|}{\|u_i\| + \|v_i\| + \|u_i\||\|v_i\|} \right) \]
\[ = \mu(t) \sum_{i=1}^{5} \left( \|u_i\| - \|v_i\| \right) \]
\[ \leq \mu(t) \sum_{i=1}^{5} \|u_i - v_i\| := \mu(t) \sum_{i=1}^{5} \theta_i(u_1 - v_1, \ldots, u_5 - v_5) \]

and
\[ w_5(t, u_1, u_2, u_3, u_4, u_5) - w_5(t, v_1, v_2, v_3, v_4, v_5) \leq \frac{1}{5} \sum_{i=1}^{5} \|u_i - v_i\|, \]

where \( \theta_i(u_1, \ldots, u_5) = \|u_i\| \) for \( i = 1, \ldots, 5 \). Note that,
\[ \ell = \eta_1 + \eta_2 + \frac{\eta_3}{\Gamma_q(2 - \beta)} + m_0 \eta_4 + a_1 \eta_5 + a_2 \eta_5 \]
\[ = 0.2 + 0.2 + \frac{0.2}{\Gamma_q(2 - \frac{1}{2})} + 0.2 + \frac{0.2}{\Gamma_q(2 - \frac{1}{4})} < 1.2, \]
\[ \ell' = \eta'_1 + \eta'_2 + \frac{\eta'_5}{\Gamma_q(2 - \beta)} + m_0 \eta'_4 + a_1 \eta'_5 + a_2 \eta'_5' \]
\[ = 0.2 + 0.2 + \frac{0.2}{\Gamma_q(2 - \frac{1}{2})} + 0.2 + \frac{0.2}{\Gamma_q(2 - \frac{1}{4})} < 1.2 \]

and \( \lim_{v \to 0^+} \frac{\theta_i(v, v, v, v, v)}{v} = 1 = \gamma_i \) for \( i = 1, \ldots, 5 \). Then, we have
\[ A_1(q) = \frac{1}{\Gamma_q(\alpha)} \left[ \ell[1 - (1 - \tau_1)^{(\alpha - 1)}] + \ell'[1 - (1 - \tau_2)^{(\alpha - 1)}] \right] \]
\[ < \frac{1}{\Gamma_q(\frac{3}{2})} \left[ 1.2[1 - (1 - 0.2)^{(\frac{3}{2})}] + 1.2(1 - 0.7)^{(\frac{3}{2})} \right] < 1 \quad (8) \]

and for almost all \( t \in \mathcal{T} \),
\[ \gamma(t) := \lim_{\max \|u_i\| \to \infty} \frac{w_5(t, u_1 u_2, \ldots, u_5)}{\max \|u_i\|} \]
\[ = \mu(t) \lim_{\max \|u_i\| \to \infty} \frac{\sum_{i=1}^{5} \|u_i\|^2}{\max \|u_i\|} \]
\[ \geq \mu(t) \lim_{\|u_i\| \to \infty} \frac{\|u_r\|^2}{(1 + \|u_r\|)} \]
\[ = \mu(t) \lim_{\|u_i\| \to \infty} \frac{\|u_r\|}{1 + \|u_r\|} = \mu(t), \]

where \( \|u_r\| = \max_{1 \leq i \leq 5} \|u_i\| \). Table 4 show that some numerical resuls of \( \ell \), \( \ell' \) and \( A_1(q) \) for three different values \( q = \frac{1}{4}, \frac{1}{2} \) and \( \frac{4}{5} \) which proves that inequality (8) is
hold. Thus, we obtain

\[ A_2(q) = \frac{\alpha - 2}{\Gamma_q(\alpha - 1)} \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha - 2)} \gamma(s) \, dq \, ds \]

\[ \geq \frac{3}{\Gamma_q(\frac{1}{2})} \int_{0.2}^{0.7} 15(0.7 - qs)^{\frac{1}{2}} \, dq \, ds > 1. \]  

(9)

At present, Theorem 3.2 implies that the problem has a solution.

**Example 2.** In this example, we consider the problem similar to (1) as follows:

\[ D_{\eta d}^{13} u(t) + w(t, u(t), u'(t), D_{\eta d}^{13} u(t), \int_0^t u(r) \, dr, I_{t_\eta}^\gamma u(t)) = 0, \]

where

\[
\begin{align*}
  w(t, u_1, \ldots, u_5) &= \begin{cases} 
    w_1(t, u_1, \ldots, u_5) = \sin t \left( \sum_{i=1}^5 \|u_i\| \right), & t \in [0, 0.3), \\
    w_2(t, u_1, \ldots, u_5) = \frac{0.3}{d(t)} \Theta(u_1, u_2, u_3, u_4, u_5), & t \in (0.3, 0.8], \\
    w_3(t, u_1, \ldots, u_5) = t \left( \sum_{i=1}^5 \|u_i\| \right), & t \in [0.8, 1],
  \end{cases}
\end{align*}
\]

\[ \Theta(u_1, u_2, u_3, u_4, u_5) = \frac{\sum_{i=1}^5 \|u_i\|^2}{1 + \|u_i\|}, \]

and the map \( d \) define by \( d(t) = 0 \) and \( d(t) = \sqrt{t} \) whenever \( t \) belongs to \([0.3, 0.8] \cap Q\) and \([0.3, 0.8] \cap Q^c\), respectively. Also, Take \( \mu_1(t) = \sin t \)

\[ 1 \mu_2(t) = \cdots = 5 \mu_2(t) = \frac{1}{d(t)}, \]

and \( \mu_3(t) = \frac{t}{2} \) for all \( t \). Note that,

\[ |w_1(t, u_1, \ldots, u_5) - w_1(t, v_1, \ldots, v_5)| = \sin t \left| \sum_{i=1}^5 \|u_i\| - \|v_i\| \right| \leq \sin t \left( \sum_{i=1}^5 \|u_i - v_i\| \right), \]

and

\[ |w_2(t, u_1, \ldots, u_5) - w_2(t, v_1, \ldots, v_5)| = \frac{0.3}{d(t)} \left| \sum_{i=1}^5 \|u_i\|^2 + \|u_i\|^2 \|v_i\| - \|u_i\| \|v_i\| - \|v_i\|^2 \right| \]

\[ = \frac{0.3}{d(t)} \left| \sum_{i=1}^5 \frac{\|u_i\|^2 + \|v_i\|^2}{1 + \|u_i\|} - \frac{\|v_i\|^2}{(1 + \|u_i\|)(1 + \|v_i\|)} \right| \]

\[ = \frac{0.3}{d(t)} \left| \sum_{i=1}^5 \frac{(\|u_i\| + \|v_i\|)(\|u_i\| - \|v_i\|) + \|u_i\|(\|u_i\| - \|v_i\|)\|v_i\|}{(1 + \|u_i\|)(1 + \|v_i\|)} \right| \]

\[ = \frac{0.3}{d(t)} \left| \sum_{i=1}^5 \frac{(\|u_i\| - \|v_i\|)(\|u_i\| + \|v_i\| + \|u_i\|\|v_i\|)}{(1 + \|u_i\| + \|v_i\| + \|u_i\||v_i|)} \right| \]

\[ \leq \frac{0.3}{d(t)} \left| \sum_{i=1}^5 \|u_i\| - \|v_i\| \right| \leq \frac{0.3}{d(t)} \sum_{i=1}^5 \|u_i - v_i\|. \]
Define
\[ \theta_i(u_1, \ldots, u_5) = \frac{\|u_i\|^2}{1 + \|u_i\|}, \]
for \( i = 1, \ldots, 5 \). Then, \( \lim_{v \to 0^+} \frac{\theta_i(v,v,v,v,v)}{v} = 0 \) for all \( i \). Put \( i \mu_2(t) = \psi_i(t) = \frac{0.3}{\alpha(t)} \) for all \( i \), \( n_0 = 5 \) and \( \beta = \frac{6}{\pi} \). Since
\[ \left| \int_0^t u(r) \, dr \right| \leq t\|u\| \leq \|u\|, \]
put \( m_0 = 1 \). In the other hand,
\[
\left| I_{\varphi}^{(\frac{1}{\pi})} u(t) \right| = \left| \frac{1}{\Gamma_q(\frac{1}{\pi})} \int_0^t (t - qs)^{(\frac{1}{\pi} - 1)} u(s) \, dq \right| 
\leq \frac{1}{\Gamma_q(\frac{1}{\pi})} \int_0^t (t - qs)^{(\frac{1}{\pi} - 1)} u(s) \, dq \leq \frac{1}{\Gamma_q(\frac{1}{\pi})} \|u\|, 
\]
put \( a_1 = \frac{1}{\Gamma_q(\frac{1}{\pi})} \) and \( a_2 = 0 \). Note that,
\[
\|\mu_1\|_{[0, \tau_1]} = \int_0^{0.3} \sin t \, dt \leq 0.05, \\
\|i \hat{\mu}_2\|_{[\tau_1, \tau_2]} = \int_{0.3}^{0.8} \frac{0.04}{\sqrt{t}} \, dt \leq 0.03, \\
\|\mu_3\|_{[\tau_2, 1]} = \int_{0.8}^1 \frac{t}{2} \, dt = 0.1 
\]
and
\[
A_3(q) = \left[ 2 + \frac{1}{\Gamma_q(2 - \beta)} + m_0 + a_1 + a_2 \right] \\
\times \left( \|\mu_1\|_{[0, \tau_1]} + \sum_{i=1}^{5} \|i \hat{\mu}_2\| + (1 - q \tau_2)^{(\alpha - 2)} \|\mu_3\|_{[\tau_2, 1]} \right) \\
\leq \left[ 2 + \frac{1}{\Gamma_q(\frac{3}{\pi})} + 1 + \frac{1}{\Gamma_q(\frac{1}{\pi})} \right] \\
\times \left( 0.05 + \sum_{i=1}^{5} 0.03 + (1 - 0.8)^{\frac{2}{\pi}} 0.2 \right) \\
< \Gamma_q \left( \frac{11}{2} \right) = \Gamma_q(\alpha - 1). 
\]
Table 5 show that some numerical results of \( A_3(q) \) for three different values \( q = \frac{1}{2}, \frac{1}{2} \) and \( \frac{8}{9} \) which proves that inequality (11) hold. As one can be seen, Theorem 3.3 implies that the problem has a solution.

5. Conclusions. The pointwise \( q \)-integro-differential boundary equations and their applications represent a matter of high interest in the area of fractional \( q \)-calculus and its applications in various areas of science and technology. \( q \)-integro-differential boundary value problems occur in the mathematical modeling of a variety of physical operations. The end of this article is to investigate a complicated case by utilizing
an appropriate basic theory. In this manner, we prove the existence of a solution for a new \(q\)-integro-differential equations under boundary conditions and show the perfect numerical effects for the problem which it confirmed our results.

**Algorithm 1** The proposed method for calculated \((a - b)^{(\alpha)}_q\)

**Require:** \(a, b, \alpha, n, q\)

1: \(s \leftarrow 1\)
2: if \(n = 0\) then
3: \(p \leftarrow 1\)
4: else
5: for \(k = 0\) to \(n\) do
6: \(s \leftarrow s \times (a - b \times a^k)/(a - b \times q^{n+k})\)
7: end for
8: \(p \leftarrow a^\alpha \times s\)
9: end if
**Ensure:** \((a - b)^{(\alpha)}_q\)

**Algorithm 2** The proposed method for calculated \(\Gamma_q(x)\)

**Require:** \(n, q \in (0, 1), x \in \mathbb{R}\backslash\{0, -1, 2, \cdots\}\)

1: \(p \leftarrow 1\)
2: for \(k = 0\) to \(n\) do
3: \(p \leftarrow p(1 - q^{k+1})(1 - q^{x+k})\)
4: end for
5: \(\Gamma_q(x) \leftarrow p/(1 - q)^{x-1}\)
**Ensure:** \(\Gamma_q(x)\)

**Algorithm 3** The proposed method for calculated \((D_qf)(x)\) that is the standard \(q\)-defivative of function \(f\)

**Require:** \(q \in (0, 1), f(x), x\)

1: syms \(z\)
2: if \(x = 0\) then
3: \(g \leftarrow \lim((f(z) - f(q \times z))/((1 - q)z), z, 0)\)
4: else
5: \(g \leftarrow (f(x) - f(q \times x))/((1 - q)x)\)
6: end if
**Ensure:** \((D_qf)(x)\)

**Algorithm 4** The proposed method for calculated \((I_q^\alpha f)(x)\) which is defined by Riemann-Liouville

**Require:** \(q \in (0, 1), \alpha, n, f(x), x\)

1: \(s \leftarrow 0\)
2: for \(i = 0\) to \(n\) do
3: \(pf \leftarrow (1 - q^{i+1})^{\alpha-1}\)
4: \(s \leftarrow s + pf \times q^i \times f(x \times q^i)\)
5: end for
6: \(g \leftarrow (x^\alpha \times (1 - q) \times s)/(\Gamma_q(x))\)
**Ensure:** \((I_q^\alpha f)(x)\)
Algorithm 5 The proposed method for calculating $\int_a^b f(r)q_d r$

Require: $q \in (0, 1)$, $a$, $n$, $f(x)$, $a$, $b$
1: $s \leftarrow 0$
2: for $i = 0 : n$ do
3: $s \leftarrow s + q^i \ast (b \ast f(b \ast q^i) - a \ast f(a \ast q^i))$
4: end for
5: $g \leftarrow (1 - q) \ast s$
Ensure: $\int_a^b f(r)q_d r$

| Table 1 | Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{8}$ which is constant, for $x = 9.5, 65, 110, 780$ in Algorithm 2. |
|---------|----------------------------------------------------------------------------------------------------------------------------------|
| $n$     | $x = 9.5$ | $x = 65$ | $x = 110$ | $x = 780$ |
| 1       | 2.679786 | 442.545844 | 1804225.634753 | 1.2890686980473E + 45 |
| 2       | 2.674552 | 442.388518 | 1800701.756560 | 1.2883678993206E + 45 |
| 3       | 2.673899 | 442.208467 | 1800262.132108 | 1.28807224237793E + 45 |
| 4       | 2.673818 | 442.673494 | 1800207.192468 | 1.28802393533064E + 45 |
| 5       | 2.673808 | 442.656623 | 1800200.325222 | 1.28802802007493E + 45 |
| 6       | 2.673808 | 442.634514 | 1800199.466820 | 1.288027460589531E + 45 |
| 7       | 2.673808 | 442.634250 | 1800199.359519 | 1.28802732912289E + 45 |
| 8       | 2.673808 | 442.634217 | 1800199.344107 | 1.28802731952634E + 45 |
| 9       | 2.673808 | 442.654213 | 1800199.344430 | 1.28802731836072E + 45 |
| 10      | 2.673808 | 442.654213 | 1800199.344221 | 1.2880273178638E + 45 |
| 11      | 2.673808 | 442.654212 | 1800199.344419 | 1.28802731815544E + 45 |
| 12      | 2.673808 | 442.654212 | 1800199.344419 | 1.28802731815544E + 45 |
| 13      | 2.673808 | 442.654212 | 1800199.344419 | 1.28802731815544E + 45 |
| 14      | 2.673808 | 442.654212 | 1800199.344419 | 1.28802731815544E + 45 |
| 15      | 2.673808 | 442.654212 | 1800199.344419 | 1.28802731815544E + 45 |
| 16      | 2.673808 | 442.654212 | 1800199.344419 | 1.28802731815544E + 45 |
| 17      | 2.673808 | 442.654212 | 1800199.344419 | 1.28802731815544E + 45 |
| 18      | 2.673808 | 442.654212 | 1800199.344419 | 1.28802731815544E + 45 |
| 19      | 2.673808 | 442.654212 | 1800199.344419 | 1.28802731815544E + 45 |

| Table 2 | Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{8}, \frac{1}{2}, \frac{5}{8}, \frac{8}{5}$ for $x = 9.5$ of Algorithm 2. |
|---------|----------------------------------------------------------------------------------------------------------------------------------|
| $n$     | $q = \frac{1}{8}$ | $q = \frac{1}{2}$ | $q = \frac{5}{8}$ | $q = \frac{8}{5}$ |
| 1       | 2.679786 | 136.646206 | 79062.158227 | 6301918.359883 |
| 2       | 2.674552 | 119.081545 | 41793.335091 | 2528395.395827 |
| 3       | 2.673899 | 111.658224 | 26290.733638 | 1232715.509371 |
| 4       | 2.673818 | 108.178242 | 18589.881264 | 689176.848061 |
| 5       | 2.673808 | 106.492553 | 14278.326587 | 426538.394173 |
| 6       | 2.673806 | 105.662861 | 11650.586796 | 285518.687713 |
| 7       | 2.673806 | 105.251251 | 9946.350893 | 203363.796571 |
|        |        |        |        | |
| 16      | 2.673806 | 104.841780 | 5522.283831 | 25842.863721 |
| 17      | 2.673806 | 104.841780 | 5513.202433 | 25230.317188 |
| 18      | 2.673806 | 104.841779 | 5505.949683 | 24699.649904 |
| 19      | 2.673806 | 104.841779 | 5500.155385 | 24238.446645 |
|        |        |        |        | |
| 106     | 2.673806 | 104.841779 | 5477.048234 | 20879.606269 |
| 107     | 2.673806 | 104.841779 | 5477.048234 | 20879.566792 |
| 108     | 2.673806 | 104.841779 | 5477.048234 | 20879.517020 |
|        |        |        |        | |
| 118     | 2.673806 | 104.841779 | 5477.048234 | 20879.337427 |
| 119     | 2.673806 | 104.841779 | 5477.048234 | 20879.327822 |
| 120     | 2.673806 | 104.841779 | 5477.048234 | 20879.319284 |
Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{8}, \frac{1}{2}, \frac{8}{9}$ for $x = 110$ of Algorithm 2.

| $n$  | $q = \frac{1}{8}$ | $q = \frac{1}{2}$ | $q = \frac{8}{9}$ |
|------|------------------|-------------------|------------------|
| 1    | 1.000000000000   | 1.000000000000   | 0.853744015784   |
| 2    | 1.000000000000   | 2.000000000000   | 1.585752212191   |
| 3    | 1.000000000000   | 3.000000000000   | 2.396459772216   |
| 4    | 1.000000000000   | 4.000000000000   | 3.207207037784   |
| 5    | 1.000000000000   | 5.000000000000   | 4.018050261704   |
| 6    | 1.000000000000   | 6.000000000000   | 4.828893470014   |
| 7    | 1.000000000000   | 7.000000000000   | 5.639736718740   |
| 8    | 1.000000000000   | 8.000000000000   | 6.450579969697   |
| 9    | 1.000000000000   | 9.000000000000   | 7.261423244181   |
| 10   | 1.000000000000   | 10.000000000000  | 8.072265575236   |
| 11   | 1.000000000000   | 11.000000000000  | 8.883108885762   |
| 12   | 1.000000000000   | 12.000000000000  | 9.693952221687   |

Figure 1. Numerical results of $A_1(q)$ where $q = \frac{1}{8}, \frac{1}{2}, \frac{8}{9}$ in Example 1.
APPLYING QUANTUM CALCULUS FOR THE EXISTENCE OF SOLUTION OF ...

Table 4. Some numerical results for calculation of $\ell$, $\ell'$ and $A_1(q)$ in Example 1 for $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$.

| $n$ | $q = \frac{1}{7}$ | $q = \frac{1}{2}$ | $q = \frac{8}{9}$ |
|-----|------------------|------------------|------------------|
| 1   | 1.0124 0.9946 0.8133 | 1.0132 0.1243 0.8462 0.0164 | 1.0133 0.1406 0.873 0.0376 |
| 2   | 1.0132 0.1243 0.2154 | 1.0132 0.2154 0.8462 0.0164 | 1.0133 0.1406 0.873 0.0376 |
| 3   | 1.0133 0.1406 0.3091 | 1.0224 0.3091 0.873 0.0376 | 1.0133 0.1406 0.873 0.0376 |
| 4   | 1.0133 0.1429 0.3539 | 1.0269 0.3539 0.8954 0.0621 | 1.0133 0.1429 0.8954 0.0621 |
| 5   | 1.0133 0.1433 0.3759 | 1.0291 0.3759 0.9143 0.0888 | 1.0133 0.1433 0.9143 0.0888 |
| 6   | 1.0133 0.1433 0.3868 | 1.0302 0.3868 0.9305 0.1165 | 1.0133 0.1433 0.9305 0.1165 |
| ... | ... | ... | ... |
| 13  | 1.0133 0.1433 0.3975 | 1.0314 0.3975 0.9972 0.2933 | 1.0133 0.1433 0.9972 0.2933 |
| 14  | 1.0133 0.1433 0.3976 | 1.0314 0.3976 1.0026 0.313 | 1.0133 0.1433 1.0026 0.313 |
| 15  | 1.0133 0.1433 0.3976 | 1.0314 0.3976 1.0073 0.3312 | 1.0133 0.1433 1.0073 0.3312 |
| 16  | 1.0133 0.1433 0.3976 | 1.0314 0.3976 1.0115 0.3478 | 1.0133 0.1433 1.0115 0.3478 |
| ... | ... | ... | ... |
| 80  | 1.0133 0.1433 0.3976 | 1.0314 0.3976 1.0442 0.4985 | 1.0133 0.1433 1.0442 0.4985 |
| 81  | 1.0133 0.1433 0.3976 | 1.0314 0.3976 1.0442 0.4985 | 1.0133 0.1433 1.0442 0.4985 |
| 82  | 1.0133 0.1433 0.3976 | 1.0314 0.3976 1.0442 0.4986 | 1.0133 0.1433 1.0442 0.4986 |
| 83  | 1.0133 0.1433 0.3976 | 1.0314 0.3976 1.0442 0.4986 | 1.0133 0.1433 1.0442 0.4986 |

Figure 2. Numerical results of $A_2(q)$ where $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$ in Example 2

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Figure 3. Numerical results of $\Gamma_q(\alpha - 1)$ where $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$ in Example 2

Table 5. Some numerical results for calculation of $A_3(q)$ in Example 2 for $q = \frac{1}{7}, \frac{1}{2}, \frac{8}{9}$. One can verify that $A_3(q) < \Gamma_q(\alpha - 1)$ for all $n$ when $q$ changes.

| n   | $A_3(q)$ | $\Gamma_q(\alpha - 1)$ | $A_3(q)$ | $\Gamma_q(\alpha - 1)$ | $A_3(q)$ | $\Gamma_q(\alpha - 1)$ |
|-----|----------|-------------------------|----------|-------------------------|----------|-------------------------|
| 1   | 1.3246   | 1.6802                  | 1.3085   | 8.774                   | 1.393    | 1799.5494               |
| 2   | 1.2731   | 1.6753                  | 1.1294   | 7.1999                  | 1.0942   | 913.1535               |
| 3   | 1.2666   | 1.6746                  | 1.0755   | 7.2574                  | 0.9855   | 542.3743               |
| 4   | 1.2655   | 1.6745                  | 1.0539   | 7.0403                  | 0.9363   | 358.4859               |
| ... | ...      | ...                     | ...      | ...                     | ...      | ...                    |
| 19  | 1.2655   | 1.6745                  | 1.0350   | 6.8321                  | 0.8635   | 46.4785               |
| 20  | 1.2655   | 1.6745                  | 1.0350   | 6.832                  | 0.8632   | 44.7825               |
| 21  | 1.2655   | 1.6745                  | 1.035   | 6.832                  | 0.8629   | 43.3383               |
| 22  | 1.2655   | 1.6745                  | 1.035   | 6.832                  | 0.8627   | 42.1023               |
| ... | ...      | ...                     | ...      | ...                     | ...      | ...                    |
| 113 | 1.2655   | 1.6745                  | 1.035   | 6.832                  | 0.8612   | 33.6244               |
| 114 | 1.2655   | 1.6745                  | 1.035   | 6.832                  | 0.8612   | 33.6243               |
| 115 | 1.2655   | 1.6745                  | 1.035   | 6.832                  | 0.8612   | 33.6243               |
| 116 | 1.2655   | 1.6745                  | 1.035   | 6.832                  | 0.8612   | 33.6243               |

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