Research article

The Burr III-Topp-Leone-G family of distributions with applications

Fastel Chipepa*, Broderick Ouyede, Peter O. Peter

Botswana International University of Science and Technology, P. Bag 16, Palapye, Botswana

ARTICLE INFO

Keywords:
Burr III-G distribution
Topp-Leone-G distribution
Maximum likelihood estimation
Simulation
Likelihood ratio test

ABSTRACT

In this article, we present a new family of generalized distributions called the Burr III-Topp-Leone-G (BIII-TL-G). We further study in detail its structural properties including moments, probability weighted moments, distribution of order statistics, and entropy. The maximum likelihood estimation method is used to estimate the model parameters. Simulations are carried out to show the consistency and efficiency of parameter estimates and finally, real data sets are used to demonstrate the applicability of the proposed model.

1. Introduction

There is an increasing motivation in constructing new generalized families of univariate continuous distributions by adding shape parameters to a baseline distribution due to the desirable properties exhibited by the generated models. Some well-known generated families have been introduced before, such as the exponentiated-G by Gupta and Kundu [1], beta-G by Eugene et al. [2], odd Burr generalized-G by Alizadeh et al. [3], gamma-G (type I) by Zografos and Balakrishnan [4], Weibull-G by Bourguignon et al. [5], exponentiated half-logistic-G by Cordeiro et al. [6], type I half logistic-G family by Cordeiro et al. [7], exponentiated Weibull-G by Hassan et al. [8], type II half logistic-G by Hassan et al. [9] and gamma-G (type II) by Ristić and Balakrishnan [10], among others.

In this paper, we develop and study in detail a new family of generalized distributions called the Burr III Topp-Leone-G (BIII-TL-G) distribution. Other Topp-Leone extensions in the literature include the Topp-Leone-G by Hassan et al. [11], Topp-Leone odd log-logistic by Brito et al. [12], odd log-logistic Topp-Leone-G by Alizadeh et al. [13], transmuted Topp-Leone-G by Yousif et al. [14] and Topp-Leone-Marshall-Olkin-G distribution by Chipeta et al. [15]. The proposed distribution provides better fit and flexibility in modeling real lifetime data sets compared to other models.

We are motivated by the desirable properties exhibited by the new distribution. The new generalized family has a good tractability property and can be expressed as an infinite linear combination of the exponentiated-G (Exp-G) distribution. Furthermore, the new generalized family can be applied to data with monotonic or non-monotonic hazard rate functions. In addition, the new generalized family of distributions is also applicable to heavy-tailed and skewed data.

The paper is organized as follows: Section 2, presents the proposed model, series expansion of the probability density function and some statistical properties. We do inference in Section 3. Some special cases of the new family of distributions are presented in Section 4. A simulation study is presented in Section 5. Section 6 provides some demonstrations to show the usefulness of the new proposed model followed by concluding remarks in Section 7.

2. The model and statistical properties

In this paper, we make use of the method proposed by Alzaatreh et al. [16]. The method is referred to as the transformed-transformer or T-X generator for short. The T-X family of distributions has cumulative distribution function (cdf) defined by

\[ F(x) = \int_{h_1}^{W(G(x); \xi)} r(t) dt, \]

where \( r(t) \) is the probability density function (pdf) of a random variable \( T, T \in [h_1, h_2] \) for \(-\infty \leq h_1 < h_2 < \infty \) and \( W(G(x); \xi) \) is a function of the cdf of a random variable \( X \) and satisfies the conditions described by Alzaatreh et al. [16].

Moreover, Al-Shomrani et al. [17] developed a generalization of the Topp-Leone distribution which was first introduced by Topp and Leone [18]. The Topp-Leone generated family of distributions by Al-Shomrani et al. [17] has cdf and probability density function (pdf) given by

\[ F_{TL-G}(x; b, \xi) = \left[ 1 - G(x; \xi) \right]^b, \]

and
for $b, a, \beta > 0$, and $\xi$ a parameter vector. Consequently, the cdf, pdf and hazard rate function (hrf) of the Burr III-Topp-Leone-G (BIII-TLG) family of distributions are given by

$$F(x; a, \beta, b, \xi) = \left(1 + \left(-\log\left[1 - \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}\right]\right)^{-a}\right)^{-1}, \quad (3)$$

and

$$h(x; a, \beta, b, \xi) = \frac{2a\beta b g(x; \xi) \Xi(x; \xi) \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{b-1}}{\left[1 - \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}\right]^a \left(1 + \left(-\log\left[1 - \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}\right]\right)^{-a}\right)^{-1}} \times \left(1 + \left(-\log\left[1 - \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}\right]\right)^{-a}\right)^{-a$$

for $a, \beta, b > 0$ and $\xi$ is a parameter vector from the baseline distribution.

2.1. Quantile function

We derive the quantile function of the BIII-TLG distribution by inverting the function

$$\left(1 + \left(-\log\left[1 - \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}\right]\right)^{-a}\right)^{-1} = u,$$

for $0 < u < 1$. Note that

$$\left[1 - \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}\right]^{-1} = \exp\left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right).$$

so that

$$G(x; \xi) = \left(1 - \left(1 - \left(\exp\left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)\right)\right)^{\alpha}\right)^{-1}.$$

The expression further simplifies to

Therefore, we obtain the quantile values of the BIII-TLG family of distributions by solving the non-linear equation

$$Q_{x,u}(u) = G^{-1}\left[1 - \left(1 - \left(\exp\left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)\right)\right)^{\alpha}\right]. \quad (5)$$

using iterative methods in R, MATLAB or SAS software. The median is obtained by substituting $u = \frac{1}{2}$ in equation (5) so that

$$\text{Median}(M_\alpha) = G^{-1}\left[1 - \left(1 - \left(\exp\left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)\right)\right)^{\alpha}\right]$$

Furthermore, we obtain the lower and upper quartiles by substituting $u = \frac{1}{4}$ and $u = \frac{3}{4}$ in equation (5), respectively.

2.2. Expansion of density

We present the linear representation of the BIII-TLG family of distributions in this subsection. The linear representation will aid in the derivation of other important statistical properties, for example moments and moment generating function. Note that from equation (14) we can write

$$\left(1 + \left(-\log\left[1 - \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}\right]\right)^{-a}\right)^{-1} = \sum_{j=0}^{\infty} (-a - 1)$$

using the generalized binomial expansion, so that

$$f(x; a, \beta, b, \xi) = \sum_{j=0}^{\infty} \left(\frac{a - 1}{j} \right) g(x; \xi) G(x; \xi) \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{b-1} \times \left(-\log\left[1 - \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}\right]\right)^{-j-1}.$$

We set $y = \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}$, and with $-\log(1 - y) = \sum_{m=1}^{\infty} y^m$, we have

$$-\log(1 - y)^{\alpha} = y^{\alpha} \left(\sum_{m=0}^{\infty} \left(\frac{\alpha - 1}{m}\right) y^m \left(\sum_{k=0}^{m} \frac{y^k}{k+1}\right)^{\alpha}\right),$$

$$0 < y < 1.$$ 

Now, applying the result on power series raised to a positive integer, with $a_0 = (s + 1)^{-1}$, that is,

$$\left(\sum_{m=1}^{\infty} a_m y^m\right)^s = \sum_{m=0}^{\infty} b_{m,s} y^m,$$

where

$$b_{m,s} = (sa_0)^{-1} \sum_{l=1}^{s} |m(l + 1) - s| a_{m-l,s}, \quad \text{and} \quad b_{0,s} = a_0^s, \quad \text{(see 19)},$$

we get

$$\left(-\log\left[1 - \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}\right]\right)^{-j-1} = \sum_{m=0}^{\infty} \sum_{l=0}^{s} b_{m,s} \left(-\frac{j + 1}{m}\right) \left(\frac{m}{j + 1}\right) \left(\frac{1}{m+1}\right) \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{b-1} \times \left(-\log\left[1 - \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}\right]\right)^{-j-1}$$

so that

$$f(x; a, \beta, b, \xi) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{s} b_{m,s} \left(-\frac{j + 1}{m}\right) \left(\frac{m}{j + 1}\right) \left(\frac{1}{m+1}\right) \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{b-1} \times \left(-\log\left[1 - \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}\right]\right)^{-j-1}.$$

Also, by applying the following series expansions

$$\left[1 - \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{\alpha}\right]^{-1} = \sum_{l=0}^{\infty} \frac{\Gamma(l + 1)}{\Gamma(1 + \alpha)} \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{l},$$

and

$$\left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{-1} = \sum_{l=0}^{\infty} \frac{\Gamma(l + 1)}{\Gamma(1 + \alpha)} \left(1 - \frac{1}{\alpha} \left(x; \xi\right)^{\beta}\right)^{l},$$
\[
F(x; \alpha, \beta, b, \xi) = \int_0^{x} \left(1 - \Gamma(\alpha; \xi)\right)^{\beta} \frac{\Gamma(\alpha - 1; \xi) + \beta x}{\Gamma(\alpha)} dt
\]

\[
(1 - G^*(x; \xi))^k \beta^{-1} (1 + r^\beta)^{-1} - \sum_{i=0}^{\infty} (-1)^i G^*(x; \xi) \times \left(\frac{b(-j+1)+m+x}{i}\right)
\]

and

\[
G^{(i)}(x; \xi) = \sum_{p=0}^{\infty} (-1)^p \left(\frac{2i+1}{p}\right) G^*(x; \xi)
\]

yields

\[
I_p(v) = \frac{1}{1-v} \log \left( \sum_{j=1, z=p}^{\infty} (2q)^j \frac{(b(-j+v)+m+z-s-v)}{j} \times \left(\frac{2i+1}{p}\right) \left(-\frac{\alpha}{i} - 1\right) \left(\frac{b(-j+1)+2m+s-1}{i}\right) \times (p+1)g(x; \xi)G(x; \xi)\right)
\]

where \(g_{p+1}(x; \xi) = (p+1)g(x; \xi)G(x; \xi)\) is the exponentiated-G (Exp-G) distribution with power parameter \((p+1)\) and

\[
w_{p+1} = \sum_{j=1, z=p}^{\infty} (-1)^j 2q \frac{\beta(i+1)}{1} \Gamma\left(\frac{1}{\beta}(p+1)\right) \frac{(-\beta(j-1)+1)}{m} \left(\frac{a-1}{j}\right)
\]

It follows that the BIII-TL-G family of distributions can be expressed as an infinite linear combination of the Exp-G distribution. Therefore, other statistical properties of the BIII-TL-G family of distributions can be derived directly from those of the Exp-G distribution.

### 2.3. Moments, generating and characteristic functions

Let \(Y_{p+1} \sim \text{Exp} - G(p+1)\), then using equation [7] the \(r^\text{th}\) raw moment, \(\mu_r\), of the BIII-TL-G family of distributions is given by

\[
\mu_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx = \sum_{p=0}^{\infty} w_{p+1} E(Y_{p+1}^r)
\]

where \(E(Y_{p+1})\) is the \(r^\text{th}\) raw moment of the Exp-G distribution with power parameter \((p+1)\) and \(w_{p+1}\) is defined by equation ([8]). The moment generating function (mgf) of the BIII-TL-G family of distributions is given by

\[
M_t(t) = E(e^{tX}) = \sum_{p=0}^{\infty} w_{p+1} M_{Y_{p+1}, (t)}
\]

where \(M_{Y_{p+1}}(t)\) is the mgf of the Exp-G distribution with power parameter \((p+1)\). The characteristic function is given by \(\varphi(t) = E(e^{i\omega X})\), where \(i = \sqrt{-1}\), so that

\[
\varphi(t) = \sum_{p=0}^{\infty} w_{p+1} \varphi_{p+1}(t),
\]

where \(\varphi_{p+1}(t)\) is the characteristic function of Exp-G distribution and \(w_{p+1}\) is given by equation ([8]).

### 2.4. Distribution of order statistics

Let \(X_1, X_2, ..., X_n\) be a random sample from the BIII-TL-G family of distributions and suppose \(X_{(1)} < X_{(2)} < ... < X_{(n)}\) denote the corresponding order statistics. The pdf of the \(k^\text{th}\) order statistic is given by

\[
f_{x_k}(x) = \frac{n!}{(k-1)! (n-k)!} \prod_{l=0}^{n-k} \left(-x\right)^{b-1} \frac{1}{1-x}
\]

Substituting the pdf and cdf of the BIII-TL-G family of distributions in equation ([9]), we get

\[
f_{x_k}(x) \frac{F(x)}{F(x)^{k-1}} = \frac{2q \frac{\beta(i+1)}{1} \Gamma\left(\frac{1}{\beta}(p+1)\right) \frac{(-\beta(j-1)+1)}{m} \left(\frac{a-1}{j}\right)}{1-x}
\]

Using the generalized binomial expansion

\[
\left(1 + \left(- \log \left[1 - \left(1 - G^*(x; \xi)\right)\right]\right)\right)^{-a(k+1)} = \sum_{j=0}^{\infty} \left(-\frac{a(k+l)}{j}\right)
\]

we write

\[
f_{x_k}(x) \frac{F(x)^{k-1}}{F(x)^{k-1}} = \sum_{j=0}^{\infty} \left(-\frac{a(k+l)}{j}\right) \frac{2q \frac{\beta(i+1)}{1} \Gamma\left(\frac{1}{\beta}(p+1)\right) \frac{(-\beta(j-1)+1)}{m} \left(\frac{a-1}{j}\right)}{1-x}
\]

Furthermore, applying the results as in Section [2.2], we get

\[
f_{x_k}(x) \frac{F(x)^{k-1}}{F(x)^{k-1}} = \sum_{j=0}^{\infty} \left(-\frac{a(k+l)}{j}\right) \frac{2q \frac{\beta(i+1)}{1} \Gamma\left(\frac{1}{\beta}(p+1)\right) \frac{(-\beta(j-1)+1)}{m} \left(\frac{a-1}{j}\right)}{1-x}
\]

where \(\Gamma(z+1) = \frac{\Gamma(z)}{z}\) and \(\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt\).
Using equation (10), we can write

\[
\frac{\Gamma(x)}{\Gamma(z+1)} \frac{\Gamma(z+1)}{\Gamma(1)!} J_{m,n} \left( -\beta (j+1) - 1 \right) \left( \frac{2i + 1}{p} \right)
\]

Therefore, the distribution of the kth order statistic from the BIII-TL-G family of distributions is given by

\[
f_{k,n}(x) = \sum_{j=0}^{n-k} \binom{n-k}{j} \frac{n!}{(n-k)!(n-j)!} \sum_{m\geq j} \frac{n-k}{n-1} \left( -\beta (j+1) - 1 \right) \left( \frac{2i + 1}{p} \right)
\]

2.5. Probability Weighted Moments (PWMs)

Probability Weighted Moments (PWMs), say \(\varphi_j\) of \(X \sim\) BIII-TL-G (\(\alpha, \beta, b, \xi\)) distribution is given by

\[
\varphi_j = E[X^j F(X^j)] = \int_{-\infty}^{\infty} x f(x) F(x)^j dx.
\]

Using equation (10), we can write

\[
f(x) = \sum_{m,j,\alpha \geq 0} (-1)^{\alpha+j} 2 \alpha b \Gamma(z+1) \Gamma(z+1) J_{m,n} \left( -\beta (j+1) - 1 \right) \left( \frac{2i + 1}{p} \right)
\]

Hence,

\[
f(x) = \sum_{j=0}^{m} \sum_{k=0}^{m-j} \binom{m}{j} \binom{m-j}{k} \left( -\beta (j+1) - 1 \right) \left( \frac{2i + 1}{p} \right)
\]

and \(h_{p+1}(x; \xi) = (p+1) \Gamma(x; \xi) \Gamma(x; \xi)^p\) is an Exp-G distribution with power parameter \((p+1)\). Therefore, the PWM of the BIII-TL-G family of distributions is given by

\[
\varphi_j = \sum_{p=0}^{\infty} \eta_{p+1} \int_{-\infty}^{\infty} x h_{p+1}(x; \xi) dx = \sum_{p=0}^{\infty} \eta_{p+1} E[H_{p+1}^{(x)}],
\]

where \(H_{p+1}^{(x)}\) is the kth power of an Exp-G distributed random variable with power parameter \((p+1)\).

2.6. Rényi entropy

Entropy is a measure of variation of uncertainty for a random variable \(X\) with pdf \(f(x)\). There are two famous measures of entropy, namely Shannon entropy [20] and R’enyi entropy [21]. Rényi entropy is defined by

\[
I_\nu(u) = (1 - \nu)^{-1} \log \left[ \int_0^\infty f^\nu(x) dx \right],
\]

where \(\nu > 0\) and \(\nu \neq 1\). Note that

\[
\left( 1 - \mathcal{C}^\nu(x; \xi) \right)^{(1-\beta(j+1)+m+z+\nu)-1} = \sum_{j=0}^{\infty} (-1)\mathcal{C}^\nu(x; \xi)
\]

and applying the results presented in Section 2.2, we get

\[
I_\nu(u) = \frac{1}{1 - \nu} \log \left[ \sum_{j=0}^{\infty} \frac{\mathcal{C}^\nu(x; \xi)}{1 - \mathcal{C}^\nu(x; \xi)^{1-\beta(j+1)+m+z+\nu}} \right]
\]

Using the generalized binomial series expansion

\[
(1 + \log \left[ 1 - \mathcal{C}^\nu(x; \xi)^{1-\beta(j+1)+m+z+\nu} \right])^{-\nu} = \sum_{j=0}^{\infty} (-\nu a - \nu)
\]

and applying the results presented in Section 2.2, we get

\[
I_\nu(u) = \frac{1}{1 - \nu} \log \left[ \sum_{j=0}^{\infty} \frac{(2 \alpha b)^j \mathcal{C}(x; \xi)^{1-\beta(j+1)+m+z+\nu}}{1 - \mathcal{C}^\nu(x; \xi)^{1-\beta(j+1)+m+z+\nu}} \right]
\]

Also, by applying the following generalized binomial series expansions

\[
\left[ 1 - \mathcal{C}^\nu(x; \xi)^{1-\beta(j+1)+m+z+\nu} \right]^{-\nu} = \sum_{j=0}^{\infty} \frac{(2 \alpha b)^j \mathcal{C}(x; \xi)^{1-\beta(j+1)+m+z+\nu}}{1 - \mathcal{C}^\nu(x; \xi)^{1-\beta(j+1)+m+z+\nu}}
\]

and

\[
\mathcal{C}^{p+1}(x; \xi) = \sum_{p=0}^{\infty} \left( 2i + 1 \right)^p \mathcal{C}^p(x; \xi),
\]

yields
3. Estimation

We present in this section, several methods of parameter estimation including maximum likelihood, minimum distance, weighted least squares, ordinary least squares and maximum product of spacings. The maximum likelihood technique method is studied in detail and used to estimate model parameters in this paper.

3.1. Maximum likelihood estimation

Let \( X_i \sim \text{BIII} - \text{TL} - G(\alpha, \beta, b, \xi) \) with the parameter vector. The log-likelihood for a random sample of size \( n \) is given by

\[
\ell(\alpha, \beta, b, \xi) = n \ln(2\alpha b / \beta) + (b - 1) \sum_{i=1}^{n} \ln[1 - G(x_i; \xi)^{b}] - \sum_{i=1}^{n} \ln \left[ \frac{1}{1 - G(x_i; \xi)^{b}} \right] \\
+ \sum_{i=1}^{n} \ln \left( \frac{g(x_i; \xi)}{B(\alpha, \beta)} \right) - (1 - \beta) \sum_{i=1}^{n} \ln \left( -\log \left[ 1 - G(x_i; \xi) \right] \right) \\
+ (-\alpha - 1) \sum_{i=1}^{n} \ln \left( 1 + \left( -\log \left[ 1 - G(x_i; \xi) \right] \right)^{b-1} \right) + \sum_{i=1}^{n} \ln \left( \frac{\Gamma(\alpha)}{\Gamma(\alpha) \xi} \right)
\]

The elements of the score vector is given by

\[
\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha b} \sum_{i=1}^{n} \ln \left( -\log \left[ 1 - G(x_i; \xi) \right] \right) \\
\frac{\partial \ell}{\partial \beta} = n \sum_{i=1}^{n} \ln \left( -\log \left[ 1 - G(x_i; \xi) \right] \right) \\
\frac{\partial \ell}{\partial b} = \frac{n}{\beta} \sum_{i=1}^{n} \ln \left( -\log \left[ 1 - G(x_i; \xi) \right] \right) \\
\frac{\partial \ell}{\partial \xi} = \frac{n}{\xi} \sum_{i=1}^{n} \ln \left( -\log \left[ 1 - G(x_i; \xi) \right] \right)
\]

These functions are not in closed form and can be solved through iterative methods using relevant software. The maximum likelihood estimates of the parameters, denoted by \( \hat{\Delta} \) is obtained by solving the nonlinear equation \( \frac{\partial \ell}{\partial \alpha} = \frac{\partial \ell}{\partial \beta} = \frac{\partial \ell}{\partial b} = \frac{\partial \ell}{\partial \xi} = 0 \) using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution, where the mean vector and is the observed Fisher information matrix evaluated at, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

3.2. Minimum distance methods

The minimum distance methods include minimizing the Anderson-Darling and Cram'er-von Mises goodness-of-fit statistics. The methods are based on the difference between the cdf of the BIII-TL-G distribution and the corresponding empirical distribution function.

3.2.1. Anderson-Darling method

Model parameters are obtained by minimizing the function

\[
AD(\Delta) = -n - \frac{1}{n} \sum_{i=1}^{n} \log(G(y_i; \Delta) \left| 1 - G(y_i; \Delta) \right|)
\]  

(16)

with respect to the model parameters. Solutions to the non-linear equations
3.3. Weighted least squares method

We obtain weighted least squares parameter estimates for the BIII-TL-G distribution by minimizing the function

\[ W(\Delta y) = \sum_{i=1}^{n} \frac{1}{\text{Var}(G(y_{ni}|\Delta))} \left( G(y_{ni}|\Delta) - \frac{i}{n+1} \right)^2 \]

where \( w_i = \frac{|(x_{i+1} - x_{i})|^2}{\text{Var}(G(y_{ni}|\Delta))} \). The solution to the non-linear system of equations

\[ \left( \frac{\partial W(\Delta y)}{\partial \alpha}, \frac{\partial W(\Delta y)}{\partial \beta}, \frac{\partial W(\Delta y)}{\partial \delta} \right) = 0 \]

(20)

gives weighted least squares parameter estimates for the BIII-TL-G distribution.

3.4. Least squares method

Ordinary least squares parameter estimates are obtained by minimizing the function

\[ Q(\Delta y) = \sum_{i=1}^{n} \left( G(y_{ni}|\Delta) - \frac{i}{n+1} \right)^2 \]

(21)

The solutions to the nonlinear equations

\[ \left( \frac{\partial W(\Delta y)}{\partial \alpha}, \frac{\partial W(\Delta y)}{\partial \beta}, \frac{\partial W(\Delta y)}{\partial \delta} \right) = 0 \]

(22)

gives the least squares parameter estimates for the BIII-TL-G distribution.

3.5. Maximum product of spacings method

The method involves finding the values of that maximizes the geometric mean of the spacings

\[ GM(\Delta y) = \left( \prod_{i=1}^{n} D_i \right)^{\frac{1}{n}} \]

(23)

where, \( i = 1, 2, \ldots, n \). We obtain BIII-TL-G model parameters by solving the non-linear equations

\[ \left( \frac{\partial GM(\Delta y)}{\partial \alpha}, \frac{\partial GM(\Delta y)}{\partial \beta}, \frac{\partial GM(\Delta y)}{\partial \delta} \right) = 0 \]

(24)

using numerical methods.

4. Some special cases

We present some special cases for the BIII-TL-G family of distributions by taking the baseline distribution to be log-logistic, Weibull, and Lindley distributions, respectively. We provide plots of the density function and hazard rate function for each special case. We further provide graphical analysis of skewness and kurtosis for the special cases (parameter \( a \) was fixed and other parameters could vary).

4.1. Burr III-Topp-Leone-log-logistic (BIII-TL-LLoG) distribution

Consider the log-logistic as the baseline distribution with pdf and cdf given by \( g(x; \lambda) = \lambda x^{-\lambda}(1 + x^{-\lambda})^{-2} \) and \( G(x; \lambda) = 1 - (1 + x^{-\lambda})^{-1} \), respectively, for \( \lambda > 0 \). The cdf, pdf and hazard rate function of the BIII-TL-LLoG distribution are given by

(16a)
for $\alpha, \beta, b, \lambda > 0$.

Figure 1 shows the flexibility of the BIII-TL-LLoG distribution for selected parameter values. From this figure, we note that the BIII-TL-LLoG pdf takes various shapes that include reverse-J, uni-modal, left or right-skewed shapes. Furthermore, the BIII-TL-LLoG distribution exhibit decreasing, increasing, upside down bathtub and bathtub followed by upside down bathtub hazard rate shapes.

Plots of coefficient of skewness (sk) and kurtosis (ck) for the BIII-TL-LLoG distribution are shown in Figure 2. Plots of sk and ck versus the shape parameter $\beta$, clearly shows the dependence of these functions on the parameter $\beta$.

4.2. Burr III-Topp-Leone-Lindley (BIII-TL-L) distribution

Consider the Lindley distribution as the baseline distribution with pdf and cdf given by $g(x; \lambda) = \frac{\lambda^2}{\lambda+1} (1+x)e^{-\lambda x}$ and $G(x; \lambda) = 1 - (1 + \frac{\lambda x}{\lambda+1}) e^{-\lambda x}$, respectively, for $\lambda > 0$. The cdf, pdf and hazard rate function of the BIII-TL-L distribution are given by

$$F(x; \alpha, \beta, b, \lambda) = \left( 1 + \left( -\log \left[ 1 - \left( 1 - \left( \frac{\lambda x}{\lambda + 1} \right)^{\beta} \right)^a \right] \right)^b \right)^{-\alpha}$$

(18)

$$f(x; \alpha, \beta, b, \lambda) = \frac{2\alpha \beta b \lambda^2}{(1+x)(1+2\lambda)(1+4\lambda)} \left( \frac{2\lambda x}{1+\lambda} \right)^{\beta-1} \left[ 1 - \left( \frac{\lambda x}{1+\lambda} \right)^2 e^{-2\lambda x} \right]$$

$$\times -\log \left[ 1 - 1 + \frac{\lambda x}{1+\lambda} \left( \frac{2\lambda x}{1+\lambda} e^{-2\lambda x} \right)^{\beta} \left( \frac{2\lambda x}{1+\lambda} \right)^{\beta} \right]$$

$$\left( 1 + \frac{\lambda x}{1+\lambda} \right)^{\beta} e^{-2\lambda x} \left( 1 + \frac{\lambda x}{1+\lambda} \right)^{\beta} \left( 1 + \frac{\lambda x}{1+\lambda} \right)^{\beta} \left( 1 + \frac{\lambda x}{1+\lambda} \right)^{\beta}$$

$$\times \left( 1 + \frac{\lambda x}{1+\lambda} \right)^{\beta} e^{-2\lambda x} \left( 1 + \frac{\lambda x}{1+\lambda} \right)^{\beta} \left( 1 + \frac{\lambda x}{1+\lambda} \right)^{\beta} \left( 1 + \frac{\lambda x}{1+\lambda} \right)^{\beta}$$

and

Figure 1. Plots of the pdf and hrf for the BIII-TL-LLoG distribution.

Figure 2. Plots of skewness and kurtosis for the BIII-TL-LLoG distribution.
for $a, b, \lambda > 0$. Figure 3 demonstrate the applicability of the BIII-TL distribution for selected parameter values. The pdfs of the BIII-TL distribution takes different shapes that include reverse-J, uni-modal, left or right skewed shapes. In addition, the BIII-TL distribution exhibit decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub hazard rate shapes.

Plots of coefficient of skewness (sk) and kurtosis (ck) for the BIII-TL distribution are shown in Figure 4. Plots of sk and ck versus the shape parameter $b$, clearly shows the dependence of these functions on the parameter $b$.

### 4.3. Burr III-Topp-Leone-Weibull (BIII-TL-W) distribution

By taking the Weibull distribution as the baseline distribution with pdf and cdf given by $g(x; \lambda) = \lambda x^{\lambda-1} e^{-x^\lambda}$ and $G(x; \lambda) = 1 - e^{-x^\lambda}$, respectively, for $\lambda > 0$, we obtain the BIII-TL-W distribution with cdf, pdf and hazard rate function given by

\[
F(x; \alpha, \beta, b, \lambda) = -\log \left[ 1 - \left( 1 - e^{-2x^\beta} \right)^{b^{-1}} \right] - \log \left[ 1 - \left( 1 - e^{-2x^\beta} \right)^{b^{-1}} \right]^{-\alpha}
\]

\[f(x; \alpha, \beta, b, \lambda) = \frac{\alpha q b \lambda x^{\lambda-1} e^{-x^\lambda}}{\left[ 1 - \left( 1 - e^{-2x^\beta} \right)^{b^{-1}} \right]} \left( -\log \left[ 1 - \left( 1 - e^{-2x^\beta} \right)^{b^{-1}} \right] \right)^{-\alpha}
\]

and

\[
h(x; \alpha, \beta, b, \lambda) = \frac{2q b \lambda x^{\lambda-1} e^{-x^\lambda}}{\left[ 1 - \left( 1 - e^{-2x^\beta} \right)^{b^{-1}} \right]^{2}} \left( -\log \left[ 1 - \left( 1 - e^{-2x^\beta} \right)^{b^{-1}} \right] \right)^{-\alpha}
\]

for $a, b, \beta, \lambda > 0$.

![Figure 3. Plots of the pdf and hrf for the BIII-TL distribution.](image-url)
Figure 5 illustrate the flexibility of the BIII-TL-W distribution for selected parameter values in terms of its pdfs that takes various shapes including reverse-J, uni-modal, left or right-skewed. Also, the BIII-TL-W distribution exhibit decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub hazard rate shapes.

Plots of coefficient of skewness (sk) and kurtosis (ck) for the BIII-TL-W distribution are shown in Figure 6. Plots of sk and ck versus the shape parameter \( \lambda \), clearly shows the dependence of these functions on the parameter \( \lambda \).

5. Simulation study

In this section, a simulation study was conducted for the BIII-TL-W distribution to assess consistency of the maximum likelihood estimators. We generated \( N = 1000 \) samples for sample sizes \( n = 25, 50, 100, 200, 400, 800 \) and 1000. We fixed the values of \( \alpha = 1 \) and \( \beta = 2 \) for the following sets of parameters values (I: \( \beta = 2.0, b = 1.0, \lambda = 1.1 \)), (II: \( \beta = 2.0, b = 1.5, \lambda = 0.5 \)), (III: \( \beta = 2.0, b = 1.1, \lambda = 2.0 \)), (IV: \( \beta = 2.0, b = 1.1, \lambda = 1.5 \)), (V: \( \beta = 2.0, b = 0.5, \lambda = 1.1 \)) and (VI: \( \beta = 2.0, b = 1.1, \lambda = 0.5 \)). We estimate the mean, root mean square error (RMSE), and average bias. If the model performs better, we except the mean to approximate the true parameter values, the RMSE and bias to decay towards zero for an increase in sample size. From the results in Tables 1, 2, and 3, the mean values approximate the true parameter values, RMSE and bias decay towards zero for all the parameter values.

6. Applications

In this section, we demonstrate the applicability and usefulness of the BIII-TL-W distribution. We apply the model to three real data examples. We compare the BIII-TL-W distribution to nested and several non-nested models. Model performance was assessed using the following goodness-of-fit statistics: -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramér-von Mises (\( W^* \)), sum of squares (SS), and Andersen-Darling (\( A^* \)) (see Chen and Balakrishnan [22] for details), Kolmogorov-Smirnov (K-S) statistic (and its p-value). The model with the smallest values of the goodness-of-fit statistics and a bigger p-value for the K-S statistic is regarded as the best model.

Model parameters (standard errors in parentheses) and the goodness-of-fit statistics are presented in Tables 4, 5, and 6. Furthermore, we also provide fitted densities and probability plots (as described by Chambers et al. [23]) to demonstrate how well our model fits the selected data sets.

The non-nested models considered in this paper are: exponentiated Weibull (EW) by Pal et al. [24] Marshall-Olkin-inverse Weibull (MO-IW) by Pakungwati et al. [25] Kumaraswamy odd Lindley-logistic (KOL-LoG) by Chipepa et al. [26], Kumaraswamy-Weibull (KW) by Cordeiro et al. [27], beta odd Lindley-exponential (BOL-E) and beta odd Lindley-uniform by Chipepa et al. [28] the Topp-Leone-Weibull-Lomax (TLWLx) distributions by Jamal et al. [29] and the exponential Lindley odd log-logistic Weibull (ELOLLW) by Korkmaz et al. [30] distributions. The pdfs of the non-nested models are as follows:
Table 1. Monte Carlo simulation results for BIII-TL-W distribution: Mean, RMSE and average bias.

| n  | Mean   | RMSE   | Bias   | Mean   | RMSE   | Bias   |
|----|--------|--------|--------|--------|--------|--------|
| β  | 25     | 2.8934 | 3.1902 | 0.8934 | 3.0162 | 4.2433 | 1.0162 |
|    | 50     | 2.7121 | 1.4118 | 0.7121 | 2.4764 | 1.1606 | 0.4764 |
|    | 100    | 2.5364 | 1.1547 | 0.5364 | 2.4625 | 1.2612 | 0.4625 |
|    | 200    | 2.3988 | 0.8743 | 0.3988 | 2.3353 | 0.7139 | 0.3353 |
|    | 400    | 2.3889 | 0.8706 | 0.3889 | 2.3225 | 0.6147 | 0.3225 |
|    | 800    | 2.2632 | 0.6461 | 0.2632 | 2.3094 | 0.5371 | 0.3094 |
|    | 1000   | 2.1550 | 0.4679 | 0.1550 | 2.1389 | 0.2971 | 0.1389 |

| b  | 25     | 1.2562 | 0.6532 | 0.1562 | 1.3990 | 0.6454 | -0.1010 |
|    | 50     | 1.2749 | 0.5794 | 0.1749 | 1.4785 | 0.5050 | -0.0215 |
|    | 100    | 1.2382 | 0.4897 | 0.1382 | 1.5599 | 0.4071 | 0.0599  |
|    | 200    | 1.1951 | 0.4217 | 0.0951 | 1.6024 | 0.3515 | 0.1024  |
|    | 400    | 1.2612 | 0.3854 | 0.1612 | 1.6314 | 0.2372 | 0.1314  |
|    | 800    | 1.2064 | 0.2972 | 0.1064 | 1.6289 | 0.2324 | 0.1289  |
|    | 1000   | 1.1670 | 0.2154 | 0.0670 | 1.5556 | 0.1550 | 0.0556  |

Table 2. Monte Carlo simulation results for BIII-TL-W distribution: Mean, RMSE and average bias.

| N  | n=β=2.0, b=1.1, λ=1.1 | n=β=2.0, b=1.5, λ=0.5 |
|----|------------------------|------------------------|
| β  | Mean   | RMSE   | Bias   | Mean   | RMSE   | Bias   |
|    | 25     | 3.1823 | 5.8036 | 1.1823 | 3.0549 | 4.5594 | 1.0549 |
|    | 50     | 2.7396 | 1.4264 | 0.7396 | 2.7110 | 1.4072 | 0.7110 |
|    | 100    | 2.5741 | 1.1580 | 0.5741 | 2.5688 | 1.1571 | 0.5688 |
|    | 200    | 2.4066 | 0.9213 | 0.4066 | 2.3807 | 0.9121 | 0.3807 |
|    | 400    | 2.3717 | 0.8740 | 0.3717 | 2.3760 | 0.9041 | 0.3760 |
|    | 800    | 2.2801 | 0.6418 | 0.2801 | 2.2690 | 0.6472 | 0.2690 |
|    | 1000   | 2.1240 | 0.3861 | 0.1240 | 2.1059 | 0.3075 | 0.1059 |

| b  | Mean   | RMSE   | Bias   | Mean   | RMSE   | Bias   |
|    | 25     | 1.2894 | 0.6652 | 0.1894 | 1.2554 | 0.6627 | 0.1554 |
|    | 50     | 1.3066 | 0.5862 | 0.2066 | 1.2902 | 0.5810 | 0.1902 |
|    | 100    | 1.2784 | 0.5065 | 0.1784 | 1.2707 | 0.4999 | 0.1707 |
|    | 200    | 1.2371 | 0.4337 | 0.1371 | 1.2194 | 0.4243 | 0.1194 |
|    | 400    | 1.2475 | 0.3786 | 0.1475 | 1.2403 | 0.3834 | 0.1403 |
|    | 800    | 1.2200 | 0.2995 | 0.1200 | 1.2088 | 0.3000 | 0.1088 |
|    | 1000   | 1.1435 | 0.2022 | 0.0435 | 1.1374 | 0.1802 | 0.0374 |

| λ  | Mean   | RMSE   | Bias   | Mean   | RMSE   | Bias   |
|    | 25     | 2.5351 | 2.6951 | 0.5351 | 1.9968 | 2.0961 | 0.4968 |
|    | 50     | 2.0203 | 1.3843 | 0.0203 | 1.5442 | 1.0649 | 0.0442 |
|    | 100    | 1.9200 | 1.0268 | -0.0800 | 1.4386 | 0.7609 | -0.0614 |
|    | 200    | 1.9141 | 0.8166 | -0.0859 | 1.4470 | 0.5978 | -0.0530 |
|    | 400    | 1.8113 | 0.6200 | -0.1887 | 1.3647 | 0.4788 | -0.1353 |
|    | 800    | 1.7980 | 0.5075 | -0.2020 | 1.3598 | 0.3800 | -0.1402 |
|    | 1000   | 1.9187 | 0.3790 | -0.0813 | 1.4384 | 0.2739 | -0.0616 |

\[ f_{α,b}(x; a, b, λ) = a^α e^{-a^β x^λ} (1 - e^{-a^β x^λ})^β, \]

for \( a, b, λ \geq 0, \)

\[ f_{α, b, λ}(x; a, b, λ, c) = \frac{x^2 e^{-x^λ}}{(1 + λ)(1 + x^λ)} \exp\left\{-\frac{1}{(1 + x^λ)}\right\} \]

\[ \times \left\{ \frac{-1 + (1 + x^λ)^{-1}}{(1 + λ)(1 + x^λ)} \exp\left\{-\frac{1}{(1 + x^λ)}\right\}\right\} x^{-1} \]

\[ \times \left\{ 1 - \frac{1}{(1 + λ)(1 + x^λ)} \exp\left\{-\frac{1}{(1 + x^λ)}\right\}\right\}^{λ-1}, \]

for \( a, b, λ, c > 0, \)
Table 3. Monte Carlo simulation results for BIII-TL-W distribution: Mean, RMSE and average bias.

|  | μ | σ | λ | V: β = 2 | 0, b = 0.5, i = 1.1 |  | V: β = 2 | 0, b = 1.1, i = 0.5 |
|---|---|---|---|---|---|---|---|---|
| 25 | 2.9072 | 1.9720 | 0.9072 | 2.7863 | 1.6889 | 0.7863 |
| 50 | 2.5159 | 1.3356 | 0.5159 | 2.6559 | 1.3494 | 0.6559 |
| 100 | 2.1945 | 0.7258 | 0.1945 | 2.5370 | 1.1424 | 0.5370 |
| 200 | 2.0834 | 0.4820 | 0.0834 | 2.4272 | 0.9042 | 0.4272 |
| 400 | 2.0276 | 0.1765 | 0.0276 | 2.4242 | 0.8620 | 0.4242 |
| 800 | 2.0304 | 0.1164 | 0.0304 | 2.3088 | 0.6752 | 0.3088 |
| 1000 | 2.0107 | 0.0947 | 0.0107 | 2.1880 | 0.4878 | 0.1880 |
| 25 | 0.7583 | 0.6868 | 0.2583 | 1.1707 | 0.6517 | 0.0707 |
| 50 | 0.6452 | 0.5098 | 0.1452 | 1.2145 | 0.5706 | 0.1145 |
| 100 | 0.5422 | 0.2993 | 0.0422 | 1.2238 | 0.4836 | 0.1238 |
| 200 | 0.5120 | 0.2016 | 0.0120 | 1.2572 | 0.4445 | 0.1572 |
| 400 | 0.5009 | 0.0983 | 0.0009 | 1.2865 | 0.3881 | 0.1865 |
| 800 | 0.5008 | 0.0695 | 0.0008 | 1.2297 | 0.3093 | 0.1297 |
| 1000 | 0.4935 | 0.0546 | -0.0065 | 1.1827 | 0.2274 | 0.0827 |
| 25 | 1.4116 | 1.2874 | 0.3116 | 0.7175 | 0.2372 | 0.2175 |
| 50 | 1.2297 | 0.7620 | 0.1297 | 0.5373 | 0.3794 | 0.0373 |
| 100 | 1.1814 | 0.4484 | 0.0814 | 0.4790 | 0.2551 | 0.0210 |
| 200 | 1.1619 | 0.3406 | 0.0619 | 0.4519 | 0.2023 | 0.0481 |
| 400 | 1.1162 | 0.2023 | 0.0162 | 0.4221 | 0.1560 | -0.0779 |
| 800 | 1.0924 | 0.1384 | -0.0076 | 0.4404 | 0.1307 | -0.0596 |
| 1000 | 1.1058 | 0.1171 | 0.0058 | 0.4562 | 0.1004 | -0.0438 |

Figure 6. Plots of skewness and kurtosis for the BIII-TL-LoG distribution.
Table 4. Parameter estimates and goodness-of-fit statistics for various models fitted for 50 mm carbon fibres data set.

| Model          | Estimates               | Statistics              |
|----------------|-------------------------|-------------------------|
|                | $a$                     | $b$                     | $\lambda$   | -2 log L | AIC  | AICC | BIC  | W   | A   | K-S   | p-value |
| BIII-TL-W      | 0.2810 (0.0528)         | 315.7900 (7.3570 x 10^5) | 0.0186 (0.0019) | 169.9 | 177.9 | 178.6 | 186.7 | 0.0436 | 0.2746 | 0.0655 | 0.9397 |
| BIII-TL-W      | 1.3025 x 10^3 (1.6745 x 10^-9) | 3.1693 (9.0894 x 10^-5) | 0.0022 (0.0010) | 183.3 | 189.3 | 189.7 | 195.9 | 0.2594 | 1.3937 | 0.0940 | 0.6036 |
| BIII-TL-W      | 4.4797 (1.1085)         | 1.7917 (1.1008)         | 1.6324 (0.2677) | 246.3 | 252.3 | 252.6 | 258.8 | 0.8256 | 4.6362 | 0.3126 | 4.9910 x 10^6 |
| BIII-TL-W      | 3.09215 (2.6580)        | 0.7252 (4.4254)         | (0.0522)       | 238.0 | 242.0 | 242.2 | 246.4 | 0.8997 | 4.9689 | 0.2734 | 0.0001 |
| EW             | 1.3186 (0.4391)         | 14.2572 (7.3551)        | -             | 203.3 | 209.3 | 209.7 | 215.8 | 0.3958 | 2.1888 | 0.1794 | 0.0286 |
| MO-IW          | 3.2588 x 10^5 (4.2992 x 10^-6) | 4.8957 (4.5157)         | 4.9322 (1.3813) | -     | 183.3 | 189.3 | 189.7 | 195.9 | 0.2594 | 1.3938 | 0.0938 | 0.6076 |
| KOL-Log        | 0.6471 (0.6643)         | 8.2776 (0.0216)         | 0.0100 (0.0435) | 2.9691 | 170.7 | 178.7 | 179.4 | 187.5 | 0.0686 | 0.4107 | 0.0766 | 0.8340 |
| BOL-E          | 1.9173 (0.8770)         | 0.7603 (1.4267)         | 0.5773 (0.4193) | 171.1 | 179.1 | 179.7 | 187.8 | 0.0715 | 0.4397 | 0.0816 | 0.7719 |
| BOL-U          | 3.1542 (0.9488)         | 54.3899 (103.7123)      | 0.3438 (0.3682) | 8.3728 | 172.4 | 180.4 | 181.1 | 189.2 | 0.0946 | 0.5635 | 0.0884 | 0.6815 |
| ELOLW          | 0.3771 (0.2975)         | 0.8337 (0.3392)         | 0.1368 (0.0810) | 2.7796 | 171.0 | 179.0 | 179.7 | 187.8 | 0.0754 | 0.4345 | 0.0735 | 0.8682 |
| KW             | 151.4900 (1.6535 x 10^-7) | 3.1194 x 10^7 (1.0314 x 10^-9) | 1.1465 x 10^3 (8.9253 x 10^-9) | 0.1332 | 175.3 | 183.3 | 183.9 | 192.0 | 0.1407 | 0.7595 | 0.0978 | 0.5533 |
| TL-WLx         | 4.1796 (6.5430)         | 0.0482 (0.0809)         | 2.7916 (0.9700) | 1.0206 | 171.7 | 179.7 | 180.4 | 188.5 | 0.0832 | 0.4965 | 0.0820 | 0.7671 |
Table 5. Parameter estimates and goodness-of-fit statistics for various models fitted for silicon nitride data set.

| Model      | Estimates | Statistics |
|------------|-----------|------------|
|            | α         | β          | b         | i         | 2 log L | AIC | AICC | BIC | W | A | K-S | p-value |
| BIII-TL-W  | 0.2636    | 42.9734    | 6.0260    | 0.1614    | 335.7  | 343.7 | 344.0 | 354.8 | 0.0542 | 0.3273 | 0.0558 | 0.8518 |
|            | (0.0756)  | (0.0626)   | (0.7436)  | (0.0251)  |         |      |      |      |    |    |      |         |
| BIII-TL-W  | 1         | 829.4500   | 3.2043    | 0.0050    | 356.7  | 362.7 | 362.9 | 371.0 | 0.3566 | 2.2409 | 0.0014 | 0.4095 |
|            | (1.6370 × 10⁹) | (0.0011)   | (8.5385 × 10⁸) | |         |      |      |      |    |    |      |         |
| BIII-TL-W  | 10.5571   | 1          | 4.0364    | 1.3668    | 512.0  | 518.0 | 518.2 | 526.4 | 1.0326 | 5.9088 | 0.3513 | 3.5330 × 10⁻¹³ |
|            | (2.004)   | (1.6171)   | (0.1328)  | (0.0294)  |         |      |      |      |    |    |      |         |
| BIII-TL-W  | 397.4000  | 3.0250     | 164.0618  | 0.7217    | 454.2  | 458.2 | 458.3 | 463.8 | 1.6979 | 9.3934 | 0.2828 | 1.0890 × 10⁸ |
|            | (135.8649)| (0.1840)   | (25.7737) | (0.0294)  |         |      |      |      |    |    |      |         |
| EW         | 0.7015    | 1.1441     | 23.9388   |          | 381.5  | 387.5 | 387.7 | 395.8 | 0.6594 | 3.8923 | 0.1676 | 0.0025 |
|            | (0.2973)  | (0.2012)   | (12.5289) | -         |         |      |      |      |    |    |      |         |
| MO-IW      | 2.4077 × 10³ | 7.0579     | 0.6972    |          | 356.6  | 362.6 | 362.9 | 371.0 | 0.3597 | 2.2545 | 0.0805 | 0.4237 |
|            | (8.9459 × 10⁷) | (0.5495)   | (0.0630)  | -         |         |      |      |      |    |    |      |         |
| KOL-LLoG   | 1.0957    | 120.67     | 0.0037    | 2.4535    | 337.5  | 345.5 | 345.9 | 356.6 | 0.0870 | 0.5406 | 0.0706 | 0.5930 |
|            | (0.4297)  | (0.0018)   | (0.7661)  |          |         |      |      |      |    |    |      |         |
| BOL-E      | 2.3297    | 16.5497    | 0.1086    | 0.3806    | 337.1  | 345.1 | 345.5 | 356.2 | 0.0712 | 0.4408 | 0.0648 | 0.7001 |
|            | (1.0946)  | (0.0092)   | (0.0866)  | (0.1066)  |         |      |      |      |    |    |      |         |
| BOL-U      | 6.1194    | 52.1616    | 0.5308    | 12.8957   | 339.9  | 347.9 | 348.3 | 359.1 | 0.1413 | 0.8678 | 0.1009 | 0.1773 |
|            | (5.6613)  | (47.2416)  | (0.4513)  | (10.2622) |         |      |      |      |    |    |      |         |
| ELOLLW     | 9.0995    | 0.1612     | 5.4785    | 4.2260    | 336.4  | 344.4 | 344.8 | 355.6 | 0.0698 | 0.4309 | 0.0628 | 0.7363 |
|            | (2.6518)  | (0.0328)   | (4.4044)  | (0.5822)  |         |      |      |      |    |    |      |         |
| KW         | 0.8261    | 0.5375     | 0.2326    | 5.4234    | 337.1  | 345.1 | 345.4 | 356.2 | 0.0812 | 0.4914 | 0.0685 | 0.6310 |
|            | (0.6252)  | (0.6203)   | (1.2277)  |          |         |      |      |      |    |    |      |         |
| TL-WLx     | 1.4492    | 0.1120     | 4.8671    | 0.8940    | 337.1  | 345.1 | 345.5 | 356.3 | 0.0829 | 0.5027 | 0.0693 | 0.6167 |
toughness of silicon nitride measured in MPa $m^{1/2}$. The data observations are $5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 27.7, 27.0, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.40, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 5.30, 4.50, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30, 2.45, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.

The estimated variance-covariance matrix is given by

$$
\begin{bmatrix}
0.0057 & -0.0043 & -0.0521 & -0.0017 \\
-0.0043 & 0.0039 & 0.0465 & 0.0015 \\
-0.0521 & 0.0465 & 0.5529 & 0.0186 \\
-0.0017 & 0.0015 & 0.0186 & 0.0006
\end{bmatrix}
$$

and the 95% confidence intervals for the model parameters are given by

$$
\alpha \in [0.2636 \pm 0.1482], \beta \in [42.9735 \pm 0.1227], b \in [6.0260 \pm 1.4575] \text{ and } \lambda \in [0.1614 \pm 0.0493].
$$

Furthermore, the second example also affirms that the BII-TL-W model performs better than the selected models. The values of the goodness-of-fit statistics $A^2$, $W^2$, K-S and the p-value of the K-S statistic indicates that the BII-TL-W model fits the fracture toughness of silicon nitride better than the several models considered in this paper. The values of the SS also show that the BII-TL-W model performs better than the selected models as shown in Figure 7.

### 6.3. Turbocharger failure times data

The data set represents failure times ($10^3 h$) of turbocharger of one type of engine as report by Xu et al. [33]. The data are 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.5, 8.7, 8.8, 9.0.

The estimated variance-covariance matrix is given by

$$
\begin{bmatrix}
3.6070 \times 10^{-6} & 3.8199 \times 10^{-9} & -1.1435 \times 10^{-5} & -7.2012 \times 10^{-7} \\
3.8199 \times 10^{-9} & 1.4832 \times 10^{-10} & -4.6320 \times 10^{-6} & -4.3688 \times 10^{-3} \\
-1.1435 \times 10^{-5} & -4.6320 \times 10^{-6} & 0.1446 & 1.3651 \times 10^{-3} \\
-7.2012 \times 10^{-7} & -4.3688 \times 10^{-8} & 1.3651 \times 10^{-3} & 1.4766 \times 10^{-5}
\end{bmatrix}
$$

and the 95% confidence intervals for the model parameters are given by

$$
\alpha \in [0.0117 \pm 0.0037], \beta \in [266.5100 \pm 2.3870 \times 10^{-5}], b \in [13.7790 \pm 0.7455] \text{ and } \lambda \in [0.2440 \pm 0.0075].
$$

More so, from the third example, we conclude that the BII-TL-W model performs better than the selected models. The values of the
Table 6. Parameter estimates and goodness-of-fit statistics for various models fitted for turbocharger data set.

| Model          | Estimates | Statistics |
|----------------|-----------|------------|
|                | $\alpha$  | $\beta$    | $b$ | $\delta$ | $\lambda$ | $\gamma$ | $2\log L$ | AIC | AICC | BIC | W | A | K-S | p-value |
| BIII-TL-W      | 0.0117    | 4266.5100  | 13.7790 | 0.2440 | 155.1 | 163.1 | 164.3 | 169.9 | 0.0144 | 0.1123 | 0.0568 | 0.9995 |
|                | (0.0019)  | (1.2179 x 10^-5) | (0.3803) | (0.0038) |
| BIII-TL-W      | 1         | 1.1833 x 10^3 | 3.1843 | 0.0024 | 177.4 | 183.4 | 184.1 | 188.5 | 0.2143 | 1.4078 | 0.1440 | 0.3778 |
|                | (2.7152 x 10^-9) | (0.0014) | (7.6008 x 10^-5) |
| BIII-TL-W      | 0.2375    | 315.6306   | 0.4823 |         | 195.0 | 201.0 | 201.7 | 206.1 | 0.4458 | 2.6669 | 0.2380 | 0.0215 |
|                | (0.0876)  | (110.5283) | (0.0304) |         |
| BIII-TL-W      | 80.4692   | 1.9565     | 96.4767 | 0.5266 | 212.6 | 216.6 | 216.9 | 220.0 | 0.6714 | 3.7950 | 0.3481 | 0.0001 |
|                | (33.3439) | (0.1990)   | (0.0393) |         |
| EW             | 0.7549    | 0.8389     | 17.9156 | -      | 187.4 | 193.4 | 194.1 | 198.5 | 0.3195 | 1.9997 | 0.1672 | 0.2134 |
|                | (0.5062)  | (0.2517)   | (14.4923) |         |
| MO-IW          | 4.8365 x 10^7 | 4.8391   | 0.9275 | -      | 177.4 | 183.4 | 184.1 | 188.5 | 0.2153 | 1.4131 | 0.1438 | 0.3800 |
|                | (8.4139 x 10^6) | (0.6512) | (0.2303) |         |
|                | a | b | $\delta$ | $\lambda$ |
| KOL-LoG        | 0.7277    | 344.2800   | 0.0001 | 2.7055 | 164.9 | 172.9 | 174.0 | 179.6 | 0.0745 | 0.5557 | 0.1063 | 0.7564 |
|                | (0.0915)  | (6.2303 x 10^-7) | (0.0249) |         |
|                | a | b | $\delta$ | $\lambda$ |
| BOL-E          | 0.2105    | 0.3251     | 0.0003 | 1.9111 | 159.0 | 167.0 | 168.1 | 173.7 | 0.0343 | 0.2175 | 0.0832 | 0.9449 |
|                | (0.0371)  | (0.2603)   | (0.0002) |         |
| BOL-U          | 1.2762    | 34.0275    | 0.1087 | 10.9050 | 157.5 | 165.5 | 166.6 | 172.2 | 0.0271 | 0.1680 | 0.0682 | 0.9923 |
|                | (0.4957)  | (0.0049)   | (0.0513) |         |
|                | $\beta$ | $\lambda$ | $\theta$ | $\gamma$ |
| ELOLLW         | 3.2768    | 0.1209     | 2.9438 | 3.4997 | 163.8 | 171.8 | 172.9 | 178.5 | 0.0636 | 0.4815 | 0.1017 | 0.8027 |
|                | (1.7673)  | (0.0284)   | (1.9669) |         |
|                | $\beta$ | $\lambda$ | $\theta$ | $\gamma$ |
| KW             | 0.3778    | 21.9782    | 0.0630 | 9.9157 | 164.3 | 172.3 | 173.5 | 179.1 | 0.0727 | 0.5442 | 0.1091 | 0.7278 |
|                | (0.2433)  | (0.0873)   | (0.0067) |         |
|                | $\beta$ | $\lambda$ | $\theta$ | $\gamma$ |
| TL-WLx         | 0.6569    | 0.2080     | 16.4716 | 0.1692 | 157.1 | 165.1 | 166.2 | 171.8 | 0.0162 | 0.1306 | 0.0839 | 0.9410 |
|                | (0.1754)  | (0.0897)   | (6.2762) |         |
|                | $\beta$ | $\lambda$ | $\theta$ | $\gamma$ |
goodness-of-fit statistics $A^*$, $W^*$, K-S and the p-value of the K-S statistic indicates that the BIII-TL-W model fits the turbocharger failure times data set better than the several models considered in this paper. The values of the SS also show that the BIII-TL-W model performs better than the selected models as shown in Figure 9.

6.4. Likelihood ratio test results

Likelihood ratio test results to assess if the BIII-TL-W distribution fits the given data sets better than the nested models are presented in this subsection.

From the likelihood ratio test results (see Table 7 for details), we conclude that the BIII-TL-W distribution performs better than the nested models on the three data sets at 5% significance level.

7. Concluding remarks

We present a new family of distributions called the Burr III-Topp-Leone-G (BIII-TL-G). We further study in detail its structural properties including moments, probability weighted moments, distribution of order statistics, probability weighted moments, and entropy. The new model can be expressed as an infinite linear combination of the exponentiated-G distribution. We also derive the maximum likelihood estimates of the BIII-TL-G family of distributions. Three special cases of the proposed model were also presented. A simulation study was conducted to assess consistency of the maximum likelihood estimates. Three real data examples are presented to demonstrate the applicability and usefulness of the BIII-TL-G family using the Weibull distribution as a special case of the baseline distribution. The BIII-TL-W distribution performs better than the nested and several non-nested models considered in this paper.
P. O. Peter: Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

**Funding statement**

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

**Data availability statement**

Data included in article supplementary material/referenced in article.

**Declaration of interests statement**

The authors declare no conflict of interest.

**Additional information**

Supplementary content related to this article has been published online at https://doi.org/10.1016/j.heliyon.2021.e06534.

**References**

[1] R.D. Gupta, D. Kundu, Generalized exponential distributions, Australian and New Zealand. J. Stat. 41 (1999) 173–188.
[2] N. Eugene, C. Lee, F. Famoye, Beta-normal distribution and its applications, Commun. Stat. Theor. Methods 31 (2002) 497–512.
[3] M. Alizadeh, G. M. Cordeiro, A. D. C. Nascimento, M. Lima, O. E. M. M., Odd Burr generalized family of distributions with some applications, J. Stat. Comput. Simulat. 87(2).
[4] K. Zografos, N. Balakrishnan, On families of beta- and generalized gamma generated distributions and associated inference, Stat. Methodol. 6 (2009) 344–362.
[5] M. Bourguignon, S. R. B., C. G. M., The Weibull-G family of probability distributions, J. Data Sci. 12 (2014) 53–68.
[6] G.M. Cordeiro, M. Alizadeh, E. Ortega, The exponentiated half-logistic family of distributions: properties and applications, J. Probab. Stat. 2014 (2014) 21.
[7] G.M. Cordeiro, A. Gomes, C. da Silva, E.M.M. Ortega, The type I half-logistic family of distributions, J. Stat. Comput. Simulat. 38 (1) (2016) 114–138.
[8] A.S. Hassan, M. Elgarby, A new family of exponentiated Weibull-generated distributions, Int. J. Mach. Intell. Appl. 4 (1-3) (2016) 135–148.
[9] A.S. Hassan, M. Elgarby, S. M., Type II half logistic family of distributions with applications, Pak. J. Statistics Oper. Res. XIII (2) (2017) 245–264.
[10] M.M. Ristić, N. Balakrishnan, The gamma exponentiated exponential distribution, J. Stat. Comput. Simulat. 82 (2012) 1191–1206.
[11] S. Rezaei, B.B. Sadr, M. Alizadeh, S. Nadarajah, Topp-Leone generated family of distributions: properties and applications, Commun. Stat. Theor. Methods 46 (6) (2017) 2993–2999.
[12] E. Brito, G.M. Cordeiro, H.M. Yousof, M. Alizadeh, G.O. Silva, The Topp-Leone odd log-logistic family of distributions, J. Stat. Comput. Simulat. 87 (15) (2017) 3040–3058.
[13] M. Alizadeh, F. Lak, M. Rasekhi, H.M. Yousof, E. Altun, The odd log-logistic Topp-Leone G family of distributions: heteroscedastic regression models and applications, Comput. Stat. 33 (3) (2018) 1217–1244.
[14] H.M. Yousof, M. Alizadeh, S.M.A. Jahanshahi, T.G. Ramires, I. Ghosh, G.G. Hamedani, The transmuted Topp-Leone G family of distributions: theory, characterizations and applications, J. Data Sci. 15 (4) (2017) 723–740.
[15] F. Chipepa, B. Oluyede, M. Makubate, The Topp-Leone-Marshall-Okin-G family of distributions with applications, Int. J. Stat. Probab. 9(6) 15–32.
[16] A. Alizadeh, H. Hamedani, Odd Burr XII distribution, properties and applications, Pak. J. Statistics Oper. Res. XIII (2) (2017) 2909–2929.
[17] A. Alzaatreh, C. Lee, F. Famoye, A new method for generating families of continuous distributions, Metron 71 (1) (2013) 63–79.
[18] A. Al-Shomrani, A. Aarif, A. Shawky, S. Hanif, M.Q. Shabbah, Topp-Leone family of distributions: some properties and application, Pak. J. Statistics Oper. Res. 12 (3) (2016) 443–451.
[19] C.W. Topp, F.C. Leone, A family of J-shaped frequency functions, J. Am. Stat. Assoc. 150 (1955) 209–219.
[20] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press, San Diego, 2000.
[21] A. Renyi, On measures of entropy and information, Proc. Fourth Berkeley Symp. Math. Stat. Prob. 1 (1960) 547–561.
[22] C.E. Shannon, Prediction and entropy of printed English, Bell Syst. Techn. J. 30 (1951) 50–64.
[23] J. Chambers, W. Cleveland, B. Kleiner, J. Tukey, Graphical Methods for Data Analysis, Chapman and Hall, London, 1983.
[24] M. Pal, M.M. Ali, J. Woo, Exponentiated Weibull Distribution, STATISTICA Anno LXVI (2), 2006, pp. 139–147.
[25] A. Liu, H. Yousof, The Weibull-G family of distributions with applications, J. Probabil. Stat. Sci. 17 (1) (2019) 51–64.
[26] F. Chipepa, B. Oluyede, M. Makubate, A new generalized family of odd Lindley-G distributions with application, Int. J. Stat. Probab. 8(6) I-22.
[27] G.M. Cordeiro, E.M.M. Ortega, S. Nadarajah, The Kumaraswamy Weibull distribution with application to failure data, J. Franklin Inst. 347 (2010) 1399–1429.
[28] F. Chipepa, B. Oluyede, M. Makubate, A.F. Famoye, The beta odd Lindley-G family of distributions with applications, J. Probabil. Stat. Sci. 17 (1) (2019) 51–83.
[29] F. Jamal, M. A. Reyad, M. A. Nasir, C. Chesneau, M. A. A. Shah, S. O. Ahmed, Topp-Leone Weibull-Lomax Distribution: Properties, Regression Model and Applications, hal-02270561.
[30] F. Chipepa, B. Oluyede, M. Makubate, A. F. Famoye, The beta odd Lindley-G family of distributions with applications, J. Probabil. Stat. Sci. 17 (1) (2019) 51–83.
[31] S. Nadarajah, S. Kotz, On the alternative to Weibull function, J. Stat. Theo. Appl. 17 (3) (2018) 554–571.
[32] M.C. Korkmaz, H.M. Yousof, G.G. Hamedani, The exponential Lindley odd log-logistic-G family: properties, characterizations and applications, J. Theor. Appl. 30 (2017) 561–566.