A higher-dimensional generalization of the Lozi map: bifurcations and dynamics

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ABSTRACT

We generalize the two-dimensional Lozi map in order to systematically obtain piecewise continuous maps in three and higher dimensions. Similar to higher dimensional generalizations of the related Hénon map, these higher dimensional Lozi maps support hyperchaotic dynamics. We carry out a bifurcation analysis and investigate the dynamics through both numerical and analytical means. The analysis is extended to a sequence of approximations that smooth the discontinuity of the derivatives in the Lozi map.

1. Introduction

The behaviour of low-dimensional nonlinear iterative maps and flows has been extensively studied and characterized over the past few decades, particularly with reference to the creation of chaotic dynamics [5,11,13–15]. The various scenarios or routes to chaos in such systems are by now fairly well known [2,15,16,18]. Similar exploration of the properties of higher dimensional dynamical systems – for instance, the dynamics of attractors with more than one positive Lyapunov exponent and the bifurcations through which they have been created – has not been studied in as much detail even in relatively simple systems [1,4,9,14,21].

Piecewise smooth mappings occur in applications ranging from power electronics to medicine [5,9]. Yet the linear and piecewise-linear mappings are among the simplest examples of iterative dynamical systems exhibiting complex dynamics. The so-called Lozi map [14] is analogous to the quadratic Hénon mapping [11] but has the advantage that more extensive analysis is possible [7]. The mapping itself is only piecewise continuous, and this introduces some additional features that need to be understood more clearly [8,12,19,20]. Indeed, specific bifurcation phenomenon such as border-collision bifurcations can only occur in piecewise smooth dynamical systems [8,20].
Our interest in the present paper is the generalization of the Lozi map to higher dimensions. One motivation is to compare this piecewise continuous system to a similar high-dimensional Hénon mapping [6]. Of the different ways in which this can be done, we choose to extend the map to \(d\)-dimensions by incorporating time-delay feedback while ensuring that the system remains an endomorphism in the absence of dissipation. The dissipation is introduced at the \(k\)th step, \(k < d\), and this also ensures that the map is a diffeomorphism. The system can therefore have \(k\)-positive Lyapunov exponents, and we examine the transition to high-dimensional chaos as a function of parameters, characterizing the different bifurcations that can occur. An intermediate ‘smooth’ approximation [3] of the piecewise map is also investigated vis-à-vis bifurcations for comparison.

In the next Section, the generalized Lozi map is described and a detailed analysis of the local bifurcations of the elementary fixed points is presented. The emergence of chaotic and hyperchaotic attractors and the global bifurcations that arise are discussed in Section 3. Section 4 is devoted to the analysis of smooth approximations to the map. The paper concludes with a discussion and summary in Section 5.

2. The generalized Lozi map

The two-dimensional Lozi map [14] is given by

\[
\begin{align*}
x_{n+1} &= 1 - (1 - \nu)y_n - a|x_n| \\
y_{n+1} &= x_n.
\end{align*}
\] (1)

This map is a modification of the quadratic Hénon mapping, with the parameters \(\nu\) and \(a\) tuning the dissipation and nonlinearity, respectively. Since the map is piecewise linear, it lends itself to extensive analysis, some of which has been recently summarized [9].

Rewriting Equation (1) as a difference delay equation, one has

\[
x_n = 1 - a|x_{n-1}| - (1 - \nu)x_{n-2},
\] (2)

which suggests a natural generalization to higher dimensions,

\[
x_n = 1 - a|x_{n-k}| - (1 - \nu)x_{n-d}.
\] (3)

Here \(d\) and \(k\) are integers such that \(k < d\), and we take \(0 \leq \nu \leq 2\). The mapping is conservative when \(\nu\) is 0 or 2 and is dissipative otherwise. For \(\nu = 1\), the map reduces to a \(k\)-dimensional endomorphism, while for \(\nu \neq 1\) the map is a \(d\)-dimensional diffeomorphism. In the next section, we analyse the implications of different choices of \(d\) and \(k\) for this map.

2.1. The base maps and \(q\)-degeneracy

The integers \(d\) and \(k\) are either co-prime or share a common factor \(q\). When \(k\) and \(d\) have a common factor \(q\), it is easy to see that all the eigenvalues of the Jacobian are \(q\)-fold degenerate, and this leads to a \(q\)-fold degeneracy in the Lyapunov exponents. It, therefore, suffices to examine the co-prime pair \((d, k)\) since these form the base for all other values of \(k < d\),
leading to what we call as the base-maps, as can be seen by the following argument. For the \( n = qm \)th iterate, the substitution \( x_q \) by \( \xi \) gives

\[
x_{qm} = 1 - a|x_{qm-1}q' - (1 - \nu)x_{qm-qd'}
\]

\[\downarrow\]

\[
\xi_m = 1 - a|\xi_m-k'| - (1 - \nu)\xi_{m-d'}.
\]  

The maps Equation (3) and Equation (4) differ in that there are \( q \) hidden variables within each \( \xi_m \). Thus the Jacobian can be separated into \( q \) identical blocks, giving rise to a \( q \)-degenerate \((d', k')\) map.

### 2.2. Fixed points: stability

The generalized map, Equation (3), can be rewritten as a \( d \)-dimensional iteration,

\[
x_{n+1}^{(1)} = 1 - a|x_n^{(k)}| - (1 - \nu)x_n^{(d)}
\]

\[
x_{n+1}^{(2)} = x_n^{(1)}
\]

\[
\vdots
\]

\[
x_{n+1}^{(d)} = x_n^{(d-1)}
\]  

The fixed points of the mapping are those for which \( \{x_{n+1}^{(1)}, \ldots, x_{n+1}^{(d)}\} = \{x_n^{(1)}, \ldots, x_n^{(d)}\} \).

Solving, we find

\[
x^{(1)*} = x^{(2)*} = \ldots = x^{(d)*} = x_\pm
\]

with \( x_\pm = \frac{1}{2 - \nu \pm a} \),

of these two fixed points \( x_\pm, x_- \) is always unstable. The matrix elements of the Jacobian of the map in Equation (3) are given by

\[
[J]_{ij} = \begin{cases}
1 & j = i - 1, \quad d \geq i \geq 2 \\
\pm a & i = 1, \quad j = k \\
-(1 - \nu) & i = 1, \quad j = d \\
0 & \text{otherwise}
\end{cases}
\]

and it is straightforward to obtain the stability conditions on the fixed point \( x_+ \) (for arbitrary \( d \) and \( k \)) from the characteristic polynomial \( P(\lambda) \),

\[
P_{1,2}(\lambda) = \lambda^d \pm a\lambda^{d-k} + (1 - \nu).
\]  

If the number of real roots of the polynomials \( P_{1,2} \) with real part greater than \(+1\) (or smaller than \(-1\)) is \( \sigma_{1,2}^+ (\sigma_{1,2}^-) \), then according to Feigin’s classification of border-collision bifurcations in piecewise smooth maps [8], a non-smooth fold bifurcation occurs when \( \sigma_1^+ + \sigma_2^- \) is odd. If \( \sigma_1^- + \sigma_2^- \) is odd, on the other hand, a non-smooth flip bifurcation occurs [8].
The characteristic of the Neimark–Sacker bifurcation in smooth dynamical systems is that a pair of complex eigenvalues cross the unit circle. This theory has been extended to piecewise smooth maps only recently [19] and although some of the features are common, there are major differences [19]. We find from numerical estimation of eigenvalues of $P_{1,2}$ for the base maps that a Neimark–Sacker-like [19] bifurcation occurs via a border-collision if an odd number of pairs of complex eigenvalues cross the unit circle. In particular, for the base map with $d = 3, k = 2$ we find that the stability region of the fixed point $x_+$ is bounded by the curves as shown in Figure 1(a):

$$L^{ns}\text{-fold}(d) : a = -2 + \nu$$
$$L^{NSI}(d = 3) : a = \nu(2 - \nu).$$
$$L^{ns}\text{-flip} : a = -\nu.$$

If the polynomials produce their largest roots with absolute values less than one, the fixed point $x_+$ is stable and contributes to the period-one region in the $a-\nu$ parameter space. The stability of fixed point is lost as either a high period/quasi-periodic or a period-two orbit is stabilized via a bisecting bifurcation, where a continuum of orbits exists at the bifurcation point [7]. Although the equivalence between a border collision bifurcation and
bisection bifurcation has been argued [7] at the period-1–period-2 and period-2–period-4 boundary [7], similar results are not available for the generalized Lozi map studied here. However, we concur based on numerical evidence that bisecting bifurcations are the norm in this family of maps even for bifurcation from fixed point to quasiperiodic/high periodic orbits. Since these bifurcation clearly show that high period orbits hit the border $x = 0$ at bifurcation points, we attribute this distinct feature of the generalized version of the Lozi map (3) to bisecting bifurcations accompanied by a border-collisions in the map.

2.3. Bifurcations diagrams

Bifurcation diagram in the two-dimensional $a–\nu$ parameter space for $d = 3, k = 2$ is shown in Figure 1(b). A typical feature of $a–\nu$ parameter space is the existence of hyper-chaotic (HC), chaotic (C), high-period (HP) (which could include quasiperiodic curves in the parameter space), and periodic regions (tounge like features). These features are shared by other members (with different $d$ and $k$ values) of this generalized Lozi map. Representative bifurcation diagrams as a function of $a$ for different $(d,k)$ combinations are shown in Figure 2 and the corresponding orbital characteristics are shown via Lyapunov exponents in Figure 3. In each case, $\nu$ is different but fixed. An interesting feature of these bifurcation diagrams is that for $0 < \nu < 1$ some base maps exhibit bounded dynamics as the parameter $a$ is decreased to the left of the period one boundary. Typically these were found to be a non-smooth flip bifurcation below $a < -\nu$ when $d = 3, k = 2$ and non-smooth Neimark–Sacker-like bifurcations for $d = 4, k = 3$. Such feature are typical

![Figure 2](image-url). Bifurcation diagrams as a function of the nonlinearity $a$ for different base maps (a) $d = 3, k = 1, \nu = 0.3$ (b) $d = 4, k = 1, \nu = 0.5$, (c) $d = 3, k = 2, \nu = 0.6$, (d) $d = 4, k = 3, \nu = 0.6$. The boundaries of stable period-1 dynamics are where high period orbits are created/destroyed via the bisection bifurcation that accompanied by border-collisions. The red line indicates the border at $x = 0$. 
of these maps even across different combination of dimensionality parameters \((d,k)\). In these bifurcations, numerical evidence suggests that a form of bisecting bifurcation takes place, where the high period orbit hits the border at \(x = 0\). We should mention that such phenomenon was not found for a similarly generalized Hénon map \([6]\) and appears to be a result of border-collision bifurcations. It is important to note that the theory of bifurcations in smooth dynamical systems does not explain these features \([12]\).

### 3. High-dimensional dynamics

In this section, we examine the bifurcations starting from the fixed point as a function of nonlinearity parameter for different embedding dimensions \(d\) and the endomorphism dimension \(k\).

#### 3.1. Bounded dynamics

In the limit \(d \to \infty\), the stability region of period-1 motion converges to a region shown in Figure 1(c): the boundaries are the following curves,

\[
L^{TL} : a = -2 + \nu \\
L^{TR} : a = 2 - \nu
\]
in the $a - v$ parameter space. These curves can be understood from the properties of the characteristic polynomials Equation (8) in the limit of $d \to \infty$. The distance $r_1$ between the leading root of the characteristic polynomial Equation (8) and the unit circle on the curve $L^{TL}$, in the extreme case of $k = 1$ and $k = d - 1$, are shown in Figure 1(d): $r_1$ approaches zero as the dimension $d$ is increased. Similar behaviour of $r_1$ is also observed on $L^{BR}, L^{BL}, T^{LR}$ and for $1 < k < d - 1$.

3.2. Hyperchaos

The map Equation (3) exhibits at most $k$ positive Lyapunov exponents as nonlinearity parameter $a$ is varied: this is due to the fact that the piecewise-smoothness in the map occurs at the $k$th previous iteration step (therefore the map remains piecewise-smooth). The stretching and folding that is responsible for introducing sensitivity to initial conditions in the map [17], occurs in $k$ directions, and this results in the maximum $k$ number of possible positive Lyapunov exponents; see Figure 4(a,b). Additionally, for $v = 1$, the map is a $k$-dimensional endomorphism (and is not invertible) with $k$-fold degenerate Lyapunov exponents, i.e. all of these LEs are identical and become positive at the same value of the nonlinearity parameter $a$ as can be seen in Figure 4(c). Embedding the $k$-dimensional endomorphism in a $d$-dimensional space does not change this behaviour. However, when the map is made diffeomorphic by enabling the contraction/dissipation parameter ($v \neq 1 \& 0 \leq v \leq 2$) the degeneracy in the Lyapunov exponents is lifted, although the maximum possible number of positive Lyapunov exponents is still $k$ as can be seen in Figure 4(d).

Route to chaos is observed via the quasiperiodic and also via finite period–doubling route, as seen in the Lyapunov spectra Figure 3 and Figure 4. The period-doubling cascade terminates after a few doublings, leading to chaos. On the other hand, chaos and hyperchaos transition is smooth, since the first $k$-largest Lyapunov exponents behave smoothly as they hierarchically become positive at different values of the nonlinear parameter $a$. This typically means that the map Equation (3) can be written as a hierarchy of chaotically driven maps at subsequent transitions to higher chaos [10], this is similar to the chaos hyperchaos transition in the generalized Hénon map [6].

4. Smooth approximations

In this section, we analyse a smooth approximation of the generalized Lozi map (3). We replace the modulus function $|\cdot|$ in Equation (2) with a smooth function $S_\epsilon(\cdot)$:

\[
L^{BL} : a = -v \\
L^{BR} : a = v,
\]

(9)
Figure 4. The Lyapunov exponent spectrum \((\lambda_i, i = 1, \ldots, k + 1)\) as a function of nonlinearity parameter for \(d = 5\). (a) and (b) demonstrate that for a given dimension \(d\) the number of positive Lyapunov exponents are governed by the parameter \(k\). We keep \(\nu = 1.4\) and plot all the Lyapunov exponents for \(k = 2\) in (a) and \(k = 4\) in (b). (c) and (d) demonstrate the breaking of degeneracy in the Lyapunov spectrum observed in the endomorphism \(\nu = 1\) by making the map diffeomorphic \(\nu = 1.1\) (see text for explanation). The largest Lyapunov exponent is coloured black, the second largest in red and the third largest in green (applicable in (a) and (b)).

where \(0 < \epsilon < 1\). The function \(S_\epsilon (x_{n-k})\) extends the smooth approximation applied to the two-dimensional Lozi map \([3]\) to our high-dimensional generalization of the Lozi map and removes the discontinuity in the slope at \(x_{n-k} = 0\).

The fixed points of the new map in Equation (10) are given by:

\[
x_\pm = \begin{cases} 
\frac{1}{2 - \nu \pm a} & \text{if } |x| > \epsilon \\
\frac{\epsilon}{a} (-(2 - \nu)) & \text{if } |x| \leq \epsilon \\
\pm \sqrt{(2 - \nu)^2 + \frac{2a}{\epsilon} - a^2} & \text{if } |x| \leq \epsilon
\end{cases}
\]

the first set of these fixed points are similar to those of Lozi map (see Equation (6)) and lose stability by colliding with one of the borders located at \(\pm \epsilon\) as nonlinearity parameter \(a\) is varied. The new orbits that appear following this border-collision bifurcation depend on the delay parameters \(d\) and \(k\), although the maximum number of positive Lyapunov exponents is still limited to \(k\). Assuming fixed values of the dissipation parameter and \(\epsilon\), the variation in nonlinearity parameter can take iterations of the map also inside the region \(|x| < \epsilon\) then subsequent bifurcations are no longer only due to border collision: borders
at $\pm \epsilon$ have well defined first derivatives and once an orbit enters the region $|x| < \epsilon$ the dynamics is also governed by the smooth approximation.

The case of $d = 2$ and $k = 1$ was illustrated in [3], where it was observed that the period-doubling route to chaos is achieved for finite values of $\epsilon$ (note that period-doubling route to chaos is absent for $\epsilon = 0$).

For $d > 2$ bifurcation scenarios easily understood by writing the smoothness parameter as $\epsilon = 1/2^n \epsilon$, $n \epsilon \geq 0$. For the case of $d = 3$, $k = 1$ the loss of stability of the fixed point as nonlinearity parameter is varied is dependent on $n \epsilon$: border-collision bifurcations gives way to period–doubling bifurcations as $n \epsilon$ decreases Figure 5(a–d). Similar observations are made when a Neimark–Sacker type bifurcation is involved in the Lozi map with $d = 3$, $k = 2$, where border-collision bifurcations Figure 6(a) give way to smooth bifurcations Figure 6(d). In the Hénon maps the chaotic dynamics follows directly after the Neimark–Sacker bifurcation via a crisis-like transition [6]. Thus the effect of $\epsilon > 0$ is to introduce bifurcations which are a mix of border collisions and smooth bifurcations.

5. Discussion and summary

In this paper, we have introduced a generalized time-delayed Lozi map with nonlinear feedback from $k$ earlier steps and linear feedback from $d$ earlier steps. This simple feedback process governs the dimensionality of the maps. The parameter $k (< d)$ determines the number of positive Lyapunov exponents. Furthermore, the family of maps formed as a result of different combinations of dimensionality parameters are classified into base maps when $d$ and $k$ are co-prime. All other maps reducible to these base maps exhibit
Figure 6. The bifurcation diagrams for \( d = 3, k = 2 \) for different values of \( \epsilon \). As \( \epsilon \) is increased from (a) to (d) characteristics of smooth bifurcations emerge in the form of period-doubling. The figures (d)–(a) also indicate emergence of border-collision bifurcation beginning from smooth period-doubling bifurcations in (d).

A \( q \)-fold degenerate Lyapunov spectrum when \( d \) and \( k \) share a common factor \( q \). Bifurcation analysis was performed in a limited region of the parameter space. In particular, fixed point dynamics is stabilized in regions of the parameters space surrounded by other invariant sets created via the non-smooth fold, non-smooth flip and the non-smooth Neimark–Sacker-like bifurcations which are often accompanied by border collisions. Analytic forms were determined for some of these boundaries, and the non-smooth flip and NS-like bifurcation curves were found to depend on the dimension \( d \). With increasing dimension, the region of period-one dynamics was found to converge in the parameter plane.

The dynamics evolve abruptly from regular to chaos due to piecewise nature of the map. Subsequent transitions from chaos to hyperchaos, however, are smooth as indicated by Lyapunov spectrum: the dimension of the attractor changes smoothly if there are no abrupt transitions in Lyapunov spectrum [6,10].

A smooth approximation of the map enabled the analysis of the bifurcations vis-à-vis further comparing some of the bifurcations to the generalized the Hénon map. It showed that some of the bifurcations observed persist on both the piecewise Lozi and Hénon map. Further exploration of a more general unified mapping is a project for future work. Another possible project for a future work is the analysis of conservative limit in these class of maps: orbits in the conservative limit are only possible when \( d = 2k \); therefore, in the conservative limit, hyperchaotic orbits are indeed possible with \( k \)—positive Lyapunov exponents. The exploration of the conservative limit of this map could be a task for future work.
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Disclosure statement

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