This paper aims at extending the criterion that the quasi-stability of a polynomial is equivalent to the total nonnegativity of its Hurwitz matrix. We give a complete description of functions generating doubly infinite series with totally nonnegative Hurwitz and Hurwitz-type matrices (in a Hurwitz-type matrix odd and even rows come from two distinct power series). The corresponding result for singly infinite series is known: it is based on a certain factorization of Hurwitz-type matrices, which is absent in the doubly infinite case. A necessary condition for total nonnegativity of generalized Hurwitz matrices follows as an application.

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Keywords: Total positivity · Pólya frequency sequence · Hurwitz matrix · Generalized Hurwitz matrix · Doubly infinite series.

1 Introduction

Definition. A doubly (i.e. two-way) infinite sequence \((f_n)_{n=-\infty}^{\infty}\) is called totally positive if all minors of the (four-way infinite) Toeplitz matrix

\[
\begin{pmatrix}
... & \vdots & \vdots & \vdots & \vdots & \ddots \\
... & f_0 & f_1 & f_2 & f_3 & f_4 & \ldots \\
... & f_{-1} & f_0 & f_1 & f_2 & f_3 & \ldots \\
... & f_{-2} & f_{-1} & f_0 & f_1 & f_2 & \ldots \\
... & f_{-3} & f_{-2} & f_{-1} & f_0 & f_1 & \ldots \\
... & f_{-4} & f_{-3} & f_{-2} & f_{-1} & f_0 & \ldots \\
... & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
=: T(f), \quad \text{where} \quad f(z) := \sum_{n=-\infty}^{\infty} f_n z^n
\]

are nonnegative (i.e. the matrix is totally nonnegative).

Note that the indexation of four-way infinite matrices affects the multiplication. Here we adopt the following convention: the uppermost row and the leftmost column, which appear in representations of such matrices, have the index 1 unless another is stated explicitly.

The total nonnegativity of the corresponding Toeplitz matrices is a characteristic property of power series converging to functions of a very specific form:
Theorem 1 (Edrei [Edr53]). Let a non-trivial sequence \( (f_n)_{n=-\infty}^{\infty} \) be totally positive. Then, unless \( f_n = f_0^{-1-n} f_1^n \) for every \( n \in \mathbb{Z} \), the series \( f(z) \) converges in some annulus to a function with the following representation

\[
C z^j e^{A z} \frac{\prod_{\mu>0} \left( 1 + \frac{z}{\beta_\mu} \right)}{\prod_{\nu>0} \left( 1 - \frac{z}{\alpha_\nu} \right)} \prod_{\mu<0} \left( 1 + \frac{z^{-1}}{\beta_\mu} \right) \frac{\prod_{\nu<0} \left( 1 - \frac{z^{-1}}{\alpha_\nu} \right)}{\prod_{\nu>0} \left( 1 - \frac{z^{-1}}{\delta_\nu} \right)},
\]

where the products converge absolutely, \( j \) is integer and the coefficients satisfy \( A, A_0 \geq 0 \), \( C, \beta_\mu, \delta_\nu > 0 \) for all \( \mu, \nu \).

The converse is also true: every function of this form generates (i.e. its Laurent coefficients give) a doubly infinite totally positive sequence.

Recent publications [HT2012, Dy2014] have shown that a relevant criterion holds for the so-called Hurwitz-type matrices, which are built from two Toeplitz matrices and have applications to questions of stability.

Definition. The Hurwitz-type matrix is a matrix of the form

\[
H(p, q) = \begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\vdots & a_0 & a_1 & a_2 & a_3 & a_4 & \ddots \\
\vdots & b_0 & b_1 & b_2 & b_3 & b_4 & \ddots \\
\vdots & a_{-1} & a_0 & a_1 & a_2 & a_3 & \ddots \\
\vdots & b_{-1} & b_0 & b_1 & b_2 & b_3 & \ddots \\
& \ddots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

where \( p(z) = \sum_{k=-\infty}^{\infty} a_k z^k \) and \( q(z) = \sum_{k=-\infty}^{\infty} b_k z^k \) are formal power series.

Definition. The Hurwitz matrix corresponding to a power series \( f(z) = \sum_{k=-\infty}^{\infty} f_k z^k \) is the Hurwitz-type matrix \( H(p, q) \) in which the series \( p(z) \) and \( q(z) \) are defined by \( f(z) = q(z^2) + zp(z^2) \).

The main goal of the present study is to determine conditions on the power series \( p(z) \) and \( q(z) \) necessary and sufficient for total nonnegativity of the matrix \( H(p, q) \): like in the case of singly infinite series, one of the conditions is that the ratio \( \frac{d(z)}{p(z)} \) maps the upper half-plane \( \mathbb{C}_+ := \{ z \in \mathbb{C} : \text{Im} z > 0 \} \) into itself. To give a more precise statement, let us introduce the following class of functions:

Definition. A function \( F(z) \) is called an \( \mathcal{S} \)-function if it is holomorphic and satisfies \( \text{Im} z \cdot \text{Im} F(z) > 0 \) for all \( z \neq 0 \) and if additionally \( F(z) \neq 0 \) wherever \( z > 0 \).

The straightforward corollary of the definition is that \( F(\mathbb{Z}) = F(\mathbb{Z}) \) for each \( \mathcal{S} \)-function \( F(z) \) wherever it is regular. We need a subclass of \( \mathcal{S} \)-functions introduced in the following lemma.

Lemma 2. Let \( p(z) \) and \( q(z) \) be two functions of the form \( (1) \), then their ratio \( F(z) = \frac{q(z)}{p(z)} \) is an \( \mathcal{S} \)-function if and only if there exists a function \( g(z) \) of the form \( (1) \), such that

\[
\frac{p(z)}{g(z)} = a_0 \prod_{\nu>0} \left( 1 + \frac{z}{\alpha_\nu} \right) \prod_{\nu<0} \left( 1 + \frac{z^{-1}}{\alpha_\nu} \right), \quad \frac{q(z)}{g(z)} = b_0 \prod_{\mu>0} \left( 1 + \frac{z}{\beta_\mu} \right) \prod_{\mu<0} \left( 1 + \frac{z^{-1}}{\beta_\mu} \right)
\]

and

\[
0 < \cdots < \alpha_{-2}^{-1} < \beta_{-1}^{-1} < \alpha_{-1}^{-1} < \beta_{-1}^{-1} < \alpha_{1} < \beta_{1} < \alpha_{2} < \cdots;
\]

if the sequence of \( \mu \) terminates on the left at \( \mu_0 \), then \( \beta_{\mu_0} \) can be positive or zero\(^1\) and the sequence of \( \nu \) also

\(^1\)An earlier publication [AESW51] studies the singly infinite case. Under additional conditions, for example \( \beta_{-1}^{-1} < \beta_{\mu} \) for each \( \mu > 0 \), the representation \( (1) \) is unique in the annulus of convergence, see e.g. [DG2010, Theorem 4].

\(^2\)When \( \beta_{\mu_0} = 0 \), the corresponding factor \( 1 + \frac{z}{\beta_{\mu_0}} \) needs to be replaced by the factor \( z \).
terminates on the left at $\mu_0$.

We prove this lemma in the end of Section \[\text{2}\] it is an analogue of a theorem due to Krein, see \[\text{Lev64},\ p.\ 508\]. In other words, under the conditions of Lemma \[\text{2}\], the function $F(z)$ can be expressed as in \[(4)\] or \[(5)\] below. The chain inequality means that zeros of $\frac{p(\zeta)}{g(\zeta)}$ and $\frac{q(\zeta)}{g(\zeta)}$ are interlacing, that is all zeros of each of the functions are real and separated by zeros of another. Lemma \[\text{2}\] provides an alternative reformulation of the item \[(a)\] in our main result:

**Theorem 3.** If $a_0 \neq 0$, then the following conditions are equivalent:

(a) The series $p(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ and $q(z) = \sum_{k=-\infty}^{\infty} b_k z^k$ converge in some common annulus to functions of the form \[(1)\] and their ratio $F(z) = \frac{q(z)}{p(z)}$ is an $\mathcal{H}$-function.

(b) The matrices $T(Ap + Bq)$ and $T(Aq + B\bar{p})$, where $\bar{p}(z) := zp(z)$, are totally nonnegative for every choice of the numbers $A, B \geq 0$, and $T(p)$ has a nonzero minor of order 2.

(c) The matrix $H(p, q)$ is totally nonnegative and has a nonzero minor of order 2.

**Remark 4.** Let $a_0 \neq 0$. Then the totally nonnegative matrix $H(p, q)$ has only zero minors of order 2 if and only if $a_k = a_0^{k-1} a_1 \neq 0$ and $b_k = b_0^k a_k$ for all $k$, as is stated in Corollary \[\text{14}\]. This case is excluded from Theorem \[\text{3}\], as corresponding to the divergence of the power series $p(z)$, see Theorem \[\text{1}\].

Both earlier works \[\text{HT2012, Dy2014}\] exploit a relation to the matching moment problem through the Hurwitz transform (see e.g. \[\text{ChM49}, p.\ 44\] or \[\text{HT2012}, p.\ 427\]). In turn, doubly infinite series do not allow conducting the same procedure due to the lack of the matching moment problem. Accordingly, the corresponding Hurwitz-type matrices have no induced factorizations. To get around this difficulty, we first obtain the implication \[(c) \Rightarrow (b)\] of Theorem \[\text{3}\]. Then Theorem \[\text{1}\] allows us to reduce the problem to studying ratios of functions of the form \[(1)\]. The inclusion of Item \[(b)\] in Theorem \[\text{3}\] yields a generalization of a fact known for polynomials, see e.g. \[\text{Wa2000, Lemma 3.4}\]. In the related publication \[\text{Dy2016a}\], we aim at deriving properties directly from estimates of minors of Hurwitz-type matrices.

By definition, a polynomial is quasi-stable if it has no zeros in the right half of the complex plane. It is known \[\text{Asn70, Kem82}\], that Hurwitz matrices of quasi-stable polynomials with positive leading coefficients are totally nonnegative. The relevant criterion of total nonnegativity of Hurwitz matrices follows from Theorem \[\text{3}\], it is an extension of the recent result \[\text{Dy2014, Theorem 1.1}\] to doubly infinite series:

**Theorem 5.** A non-trivial two-way series $f(z) = \sum_{k=-\infty}^{\infty} f_k z^k = q(z^2) + zp(z^2)$ converges to a function of the form

$$g(z^2) \cdot z^r e^{Bz} + \frac{1}{\prod_{\lambda \neq 0} \left(1 + \frac{z^{\text{sign } \lambda}}{\xi_\lambda} \right) \cdot \prod_{\nu \neq 0} \left(1 + \frac{z^{\text{sign } \nu}}{\gamma_\nu} \right) \cdot \left(1 + \frac{z^{\text{sign } \nu}}{\gamma_\nu} \right)}.$$

where $\xi_\lambda, \text{Im } \gamma_\nu > 0$ and $B, B_0 \geq 0$ for all $\lambda, \nu \neq 0$, the function $g(z)$ can be represented as in \[(1)\] and $r$ is an integer, if and only if the corresponding Hurwitz matrix $H(p, q)$ is totally nonnegative and has a nonzero minor of order at least two.$^3$

Note that the expression \[(3)\] can be rewritten as $q_1(z) \cdot q_2 \left(\frac{1}{2}\right)$, where both $q_1(z)$ and $q_2(z)$ can be represented in the form

$$Cz^r e^{A\mu} + Bz \cdot \prod_{\lambda > 0} \left(1 + \frac{z}{\xi_\lambda} \right) \cdot \prod_{\nu > 0} \left(1 + \frac{z}{\gamma_\nu} \right) \cdot \prod_{\mu > 0} \frac{1}{1 - \frac{z}{\delta_\mu}}$$

with the same conditions on the coefficients, except that $\text{Re } \gamma_\nu > 0$ and additionally $A \geq 0$ and $C, c_\mu > 0$ for all $\mu > 0$. Theorem 1.1 of \[\text{Dy2014}\] says that the Hurwitz matrices generated by the involved functions $q_1(z)$

---

$^3$When all minors of the matrix $H(p, q)$ of order two turn to zero, the series $f(z)$ must be trivial (i.e. all its coefficients are zero) or divergent, see Remark \[\text{2}\].
and \(q_2(z)\) are totally nonnegative; the annulus of convergence of \(f(z)\) is the domain, where both power series for \(q_1(z)\) and \(q_2\left(\frac{1}{z}\right)\) converge. In other words, the relations between limits of singly and doubly infinite series corresponding to totally nonnegative Hurwitz and Toeplitz matrices are akin.

Another outcome of Theorem 5 is an extension of \cite[Theorem 4]{HKK2016} on total nonnegativity of the generalized Hurwitz matrices, i.e. the matrices defined by

\[
(f_{jM-i+1})_{i,j=-\infty}^{\infty} = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & f_M & f_{2M} & f_{3M} & f_{4M} & \cdot \\
\cdot & f_{M-1} & f_{2M-1} & f_{3M-1} & f_{4M-1} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & f_0 & f_M & f_{2M} & f_{3M} & \cdot \\
\cdot & f-1 & f_{M-1} & f_{2M-1} & f_{3M-1} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot 
\end{pmatrix},
\]

where \(M = 1, 2, \ldots\) and \(f_k\) is the \(k\)th coefficient of a power series. (The extension seems to be new even for the singly infinite case.) Let \(\arg z\) denote the principal value of the argument of a complex number \(z \neq 0\). Given a doubly infinite series \(f(z) = \sum_{k=-\infty}^{\infty} f_k z^k = \sum_{n=0}^{M-1} z^n p_n(z^M)\) assume that the corresponding generalized Hurwitz matrix is totally nonnegative. Then all the Hurwitz-type matrices \(H(p_m, p_n), n < m,\) are totally nonnegative as submatrices of \((f_{jM-i})_{i,j=-\infty}^{\infty}\); this weaker property ensures that the function represented by \(f(z)\) does not vanish in a certain sector of the complex plane:

**Theorem 6.** If all Hurwitz-type matrices \(H(p_m, p_n)\) are totally nonnegative for \(0 \leq n < m < M\) and at least one of them contains a nonzero minor of order two, then the series \(f(z) = \sum_{n=0}^{M-1} z^n p_n(z^M)\) converges to a function \(g(z^M) \cdot q_1(z) \cdot q_2\left(\frac{1}{z}\right)\) with no zeros in the sector \(C_M := \{z \in \mathbb{C} : \arg z < \frac{\pi}{M}\}\), where both \(q_1(z)\) and \(q_2(z)\) are entire functions of genus at most \(M - 1\) and \(g(z)\) has the form \(\{1\}\).

Already for polynomials, the absence of zeros in the sector \(C_M\) for \(M > 2\) is necessary but not sufficient for the total nonnegativity of the corresponding generalized Hurwitz matrix: see \cite[Example 35]{HKK2016}. The question of sufficient conditions is opened. Following \cite{HKK2016}, we only remark here that total nonnegativity of the matrix \((f_{jM-i})_{i,j=-\infty}^{\infty}\) implies total nonnegativity of its submatrix \((f_{jkm-i})_{i,j=-\infty}^{\infty}\) for each \(k = 1, 2, \ldots\).

In the products and sums with inequalities in limits, we assume that the indexing variable changes in \(Z\) or in some finite or infinite subinterval of \(Z\), and that it additionally satisfies the indicated inequalities. Accordingly, a product or sum can be empty, finite or infinite. By writing that a function has one of the above representations, we assume that the involved products are locally uniformly convergent unless the converse is stated explicitly. In the above theorems, the convergence follows from the total nonnegativity of the involved matrices. The condition of convergence is well-known and can be expressed as the following theorem, which we apply in the settings \(\zeta = z^{\pm 1}\) or \(\zeta = z^2\).

**Theorem 7** (see e.g. \cite[pp. 7–13, 21]{Lev64}). The infinite product \(\Pi_{v=0}^{\infty} \left(1 + \frac{\zeta}{\alpha_v}\right)\) converges uniformly in \(\zeta\) varying in compact subsets of \(C\) if and only if the series \(\sum_{v=0}^{\infty} \frac{1}{|\alpha_v|}\) converges. If so, then for any \(\epsilon > 0\) the estimates \(\Pi_{v=0}^{\infty} \left|1 + \frac{\zeta}{\alpha_v}\right| < Ce^{\epsilon R}\) and, outside exceptional disks with an arbitrarily small sum of radii, \(\Pi_{v=0}^{\infty} \left|1 + \frac{\zeta}{\alpha_v}\right| > Ce^{-\epsilon R}\) provided that \(|\zeta| \leq R\) and the positive numbers \(R\) and \(C\) are big enough.

For example, by this theorem the convergence (locally uniform in some annulus centred at the origin) of a series to a function of the form \((3)\) implies the condition

\[
\sum_{\lambda \neq 0} \frac{1}{|\lambda|} + \sum_{v \neq 0} \frac{1}{|\Re \gamma_v|} + \sum_{v \neq 0} \frac{1}{|\gamma_v|^2} < \infty.
\]
Remark 8. The matrix $H(p', p)$ is totally nonnegative if and only if $p(z)$ represents an entire function generating a totally positive sequence (see [Dy2014, Theorem 1.2]). If so, then $p'(z)$ also generates a totally positive sequence. Considering doubly infinite series does not change the picture: the Toeplitz matrix $T(p')$ contains negative entries provided that the series $p(z)$ has a positive coefficient at a negative power of $z$. It can be shown that $\frac{p(z)}{p'(z)}$ is not a mapping of $\mathbb{C}_+$ into itself (cf. Theorem 3) provided that the function $p(z)$ is not entire and generates a totally positive sequence. However, if $p(z)$ is meromorphic in $\mathbb{C} \setminus \{0\}$, then the ratio $\frac{z^p(z)}{p'(z)}$ is a mapping of the upper half of the complex plane into itself exactly when $p(z)$ generates a totally positive sequence. The further details can be found in [Dy2016, Chapter 5] and [TD2016].

2 $\mathcal{I}$-functions

Lemma 9. The product

$$\Pi_{\mu>0} \frac{1 + \frac{z}{\beta_{\mu}}}{1 + \frac{1}{\beta_{\mu}}}, \quad \Pi_{\nu>0} \frac{1 + \frac{z}{\alpha_{\nu}}}{1 + \frac{1}{\alpha_{\nu}}}, \quad \text{where } C > 0, \text{ and the numbers}$$

$$0 < \ldots < \alpha_1^{-1} < \beta_1^{-1} < \alpha_2^{-1} < \beta_2 < \alpha_2 < \ldots$$

satisfy $\sum_{\nu \neq 0} (\alpha_{\nu}^{-1} + \beta_{\nu}^{-1}) < \infty$, determines an $\mathcal{I}$-function. Analogously,

$$C \frac{z + \beta_0}{z + \alpha_0} \cdot \frac{\Pi_{\mu>0} \left(1 + \frac{z}{\beta_{\mu}}\right)}{\Pi_{\nu>0} \left(1 + \frac{z}{\alpha_{\nu}}\right)},$$

where $C > 0$ and the numbers $0 < \beta_0 < \alpha_0 < \beta_1 < \alpha_1 < \ldots$ satisfy $\sum_{\nu > 0} (\alpha_{\nu}^{-1} + \beta_{\nu}^{-1}) < \infty$ is a meromorphic $\mathcal{I}$-function. Products over $\mu$ and $\nu$ in (4) or (5) can be terminating, in which case the numerator and the denominator retain to have interlacing zeros.

Proof. Suppose that $F(z)$ has the form (4) and denote

$$F_n(z) := C \frac{q_n(z)}{p_n(z)}, \quad \text{where } q_n(z) = \prod_{\nu=1}^{n} \left(1 + \frac{z}{\beta_{\nu}}\right)^{1 + \frac{1}{\beta_{\nu}}} \cdot \frac{z + \beta_0}{z + \alpha_0}, \quad p_n(z) = \prod_{\nu=1}^{n} \left(1 + \frac{z}{\alpha_{\nu}}\right)^{1 + \frac{1}{\alpha_{\nu}}},$$

(6)

Note that the product $\prod_{\nu=-n}^{-1} \frac{1}{\beta_{\nu}} \cdot \prod_{\nu=-n}^{-1} \frac{1}{\alpha_{\nu}} < 1$ is bounded. For each $n \in \mathbb{Z}_{>0}$, the rational function

$$F_n(z) = C \cdot \prod_{\nu=-n}^{-1} \frac{\alpha_{\nu}}{\beta_{\nu}} \cdot \sum_{\nu=1}^{n} \left( A_{\nu,n} \frac{z + \alpha_{\nu}}{z + \alpha_{\nu}} + A_{-\nu,n} \frac{1}{z + \beta_{\nu}} \right), \quad \text{where } A_{\nu,n} = C \frac{q_n(z)}{zp_n'(z)} \Big|_{z=-\alpha_{\nu}},$$

(7)

is an $\mathcal{I}$-function as each of its partial fractions is such. The condition $\sum_{\nu \neq 0} (\alpha_{\nu}^{-1} + \beta_{\nu}^{-1}) < \infty$ implies the locally uniform convergence of each product in (4) (see Theorem 7) and, therefore, of the numerator $q_n(z)$ and the denominator $p_n(z)$ as $n \to \infty$. Since the denominator is nonzero for $z \neq 0$, the function $F(z)$ is the limit of $F_n(z)$ as $n \to \infty$ uniform on compact subsets of $\mathbb{C} \setminus (-\infty, 0]$. Moreover,

$$\text{Im } F(z) \cdot \text{Im } z = \lim_{n \to \infty} F_n(z) \cdot \text{Im } z \geq 0;$$

the inequality is strict outside the real line due to the maximum principle for the harmonic function $\text{Im } F(z)$.

The assertion that the expression (5) represents an $\mathcal{I}$-function follows by omitting from (7) terms that
correspond to absent poles.

Lemma 10. Let $F(z)$ be a function of the form (4) or (5) and let real numbers $A, A_0, p$ be such that $A^2 + A_0^2 > 0$. Then $G(z) := e^{A z^p} z^p F(z)$ is not a mapping of $\mathbb{C}_+$ into itself and $G(z_0) < 0$ for some $z_0 \in \mathbb{R}$.

\textbf{Proof.} Denote the multivalued argument function by Arg and its principal value by arg. In the special case when $F(z)$ is a real constant $A g F(z)$ does not depend on $z$. Otherwise $0 < \arg F(z) < \pi$ wherever $\Im z > 0$ by Lemma 9, and therefore $|\arg F(i r_1) - \arg F(i r_2)| < 2 \pi$ for any $r_2 > r_1 > 0$. Furthermore, $\arg(i r_1)^p = \arg(i r_2)^p$ and $e^{i Ar - i \frac{\arg}{\pi}} = 1 \frac{1}{\pi} \ln e^{i Ar - i \frac{\arg}{\pi}}$, where $\ln$ is the multivalued logarithm and $r > 0$. Branches of the logarithm differ by a constant, thus

$$\frac{d}{dr} \arg e^{i Ar - i \frac{\arg}{\pi}} = \frac{1}{i} \left( i Ar - \frac{A_0}{r} \right) = A + \frac{A_0}{r^2}$$

and (due to $A^2 + A_0^2 > 0$) the integral of this expression over $(r_1, r_2)$ can be made arbitrarily big in absolute value through the choice of positive numbers $r_1$ and $r_2$. More specifically, we always can chose $r_2 > r_1 > 0$ so that

$$\left| \arg e^{i Ar_2 - i \frac{\arg}{\pi}} - \arg e^{i Ar_1 - i \frac{\arg}{\pi}} \right| = \left| \int_{r_1}^{r_2} \frac{d}{dr} \arg e^{i Ar - i \frac{\arg}{\pi}} \, dr \right| \geq 4 \pi, \quad \text{and hence}$$

$$\left| \arg G(i r_2) - \arg G(i r_1) \right| > \left| \arg e^{i Ar_2 - i \frac{\arg}{\pi}} - \arg e^{i Ar_1 - i \frac{\arg}{\pi}} \right| = \left| \arg F(i r_2) - \arg F(i r_1) \right| > 2 \pi.$$

In other words, the interval $(r_1, r_2)$ contains at least one point $r$ such that $\arg G(i r) = \pi$, that is $G(z_0) < 0$ with $z_0 = i r$.

Lemma 11. Given $F(z)$ of the form (4) or (5) not equal identically to $C z$ or $C$, the function $z^p F(z)$ with any real $p \in (-1, 0)$ and $A, A_0 \in \mathbb{R}$ cannot be a mapping of $\mathbb{C}_+$ into itself; the function $\frac{F(z)}{z^p}$ is an $\mathcal{F}$-function of the form (4) or (5). Moreover, under the additional condition $p \neq -1$ there exist a point $z_0 \in \mathbb{R}$ such that $z_0^p F(z_0) < 0$.

\textbf{Proof.} The reciprocal of the product (4) can be expressed as

$$\frac{1}{F(z)} = C \frac{\Pi_{\nu > 0} \left( 1 + \frac{\bar{z}}{\nu} \right) \Pi_{\nu < 0} \left( 1 + \frac{\bar{z}}{\nu} \right)}{\Pi_{\mu > 0} \left( 1 + \frac{\alpha}{\mu} \right) \Pi_{\mu < 0} \left( 1 + \frac{\alpha}{\mu} \right)} = C \frac{(z \alpha_{-1} + 1) \Pi_{\nu > 0} \left( 1 + \frac{\bar{z}}{\nu} \right) \Pi_{\nu < 0} \left( 1 + \frac{\bar{z}}{\nu} \right)}{z \alpha_{-1} \Pi_{\mu > 0} \left( 1 + \frac{\alpha}{\mu} \right) \Pi_{\mu < 0} \left( 1 + \frac{\alpha}{\mu} \right)}.$$ 

(8)

Therefore, relabelling the $\beta_\mu \rightarrow \bar{\alpha}_\mu$ for all $\mu \neq 0$; $\alpha_{-1} \rightarrow \bar{\beta}_1$ and $\alpha_{\nu} \rightarrow \bar{\beta}_{\nu-1}$ for all $\nu \in (0, 1)$ yields that $\frac{F(z)}{z^p}$ has the form (4). An analogous reasoning works for the reciprocal of (5).

Now, let $p > 0$. Non-constant functions of the form (4) or (5) have at least one negative simple zero. Therefore, there exists $r > 0$ such that $F(-r) < 0$. On the semicircle $\{ z \in \mathbb{C}_+ : |z| = r \}$, we have the following conditions:

$$0 < \arg F(z) - \arg F(-r) < \arg F(-r) - \arg F(r) = \pi \quad \text{and} \quad \arg z^p - \arg r^p = p \arg z < p \pi.$$

The above inequalities yield that

$$\arg (-r)^p F(-r) - \arg r^p F(r) = p \pi + \pi > \pi,$$

so the increment $\arg z^p F(z) - \arg z^p F(z)$ equals to $\pi$ at least at one point $z_0 \in \mathbb{C}_+$ satisfying $|z_0| = r$. In particular, $F(z_0) < 0$. If $p < -1$, then the previous reasoning implies that the function $z^{-p-1} \frac{F(z)}{z^p}$ is negative at some point $z_0$ of the upper half-plane; therefore, $z_0^p F(z) = z_0^{p+1} \frac{F(z)}{z_0^p} < 0$ as well.

The remaining case of $p = -1$ follows from the identity $\frac{F(z)}{z} = \left( \frac{z}{F(z)} \right) \frac{|F(z)|^2}{|z|^2}$ because $\Im \frac{z}{F(z)} > 0$ wherever $\Im z > 0$. 

\qed
Proof of Lemma 2. On account of Lemma 9, it is enough to prove that the function \( F(z) \) is an \( \mathcal{S} \)-function only if has the form (4) or (5). Since \( F(z) \) is regular for \( z > 0 \), the poles of \( p(z) \) and \( q(z) \) coincide and have the same orders. Therefore, the function \( F(z) \) is positive when \( z > 0 \). As is shown in Lemma 10, \( F(z) \) does not have exponential factors; that is, the exponential factors in the representations (1) of the functions \( p(z) \) and \( q(z) \) coincide. As a result, \( F(z) = z^p G(z) \), where \( p \) is integer and \( G(z) \) is an \( \mathcal{S} \)-function. The exponent \( p \) must then be zero by Lemma 11. \( \square \)

3 Total nonnegativity and interlacing zeros

Assume in this section that \( p(z) := \sum_{k=-\infty}^{\infty} a_k z^k \), \( q(z) := \sum_{k=-\infty}^{\infty} b_k z^k \) and \( \tilde{p}(z) := z p(z) = \sum_{k=-\infty}^{\infty} a_k z^{k+1} \).

Lemma 12. If the matrix \( H(p, q) \) is totally nonnegative, then for arbitrarily taken nonnegative numbers \( A \) and \( B \) both matrices \( T(Ap + Bq) \) and \( T(Aq + B\tilde{p}) \) are totally nonnegative.

Proof. The matrices \( H(p, q) \) and

\[
H(q, \tilde{p}) = \begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\vdots & b_0 & b_1 & b_2 & b_3 & b_4 & \ddots \\
\vdots & a_0 & a_1 & a_2 & a_3 & a_4 & \ddots \\
\vdots & b_1 & b_0 & b_1 & b_2 & b_3 & \ddots \\
\vdots & a_1 & a_0 & a_1 & a_2 & a_3 & \ddots \\
\vdots & b_2 & b_1 & b_0 & b_1 & b_2 & \ddots \\
\vdots & a_2 & a_1 & a_0 & a_1 & a_2 & \ddots \\
\ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]

coincide up to a shift in indexation; that is, \( H(q, \tilde{p}) \) can be obtained by increasing the indices of rows in \( H(p, q) \) by 1. In particular, the matrix \( H(q, \tilde{p}) \) is totally nonnegative.

Observe that

\[
T(Ap + Bq) = H^T(A, B) H(p, q) \quad \text{and} \quad T(Aq + B\tilde{p}) = H^T(A, B) H(q, \tilde{p}),
\]

where the auxiliary totally nonnegative matrix \( H^T(A, B) \) is the transpose of \( H(A, B) \):

\[
H^T(A, B) = \begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\vdots & A & B & 0 & 0 & 0 & \ddots \\
\vdots & 0 & A & B & 0 & 0 & \ddots \\
\vdots & 0 & 0 & A & B & \ddots & \vdots \\
\ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix} = \{h_{ij}\}_{i,j=-\infty}^{\infty}, \quad \text{where} \quad h_{ij} = \begin{cases} A & \text{if } j = 2i - 1, \\
B & \text{if } j = 2i, \\
0 & \text{otherwise.}
\end{cases}
\]

Therefore, applying the Cauchy-Binet formula to the expressions (9) yields that all minors of the matrices \( T(Ap + Bq) \) and \( T(Aq + B\tilde{p}) \) must be nonnegative. \( \square \)

Lemma 13. Let the matrix \( T(Ap + Bq) \) be totally nonnegative for every choice of the numbers \( A, B > 0 \). If \( T(p) \) has a nonzero minor of order 2, then there exists a nonempty annulus \( \mathbb{A} \) centred at the origin where both series \( \phi(z; A, B) := Ap(z) + Bq(z) \) and \( \psi(z; A, B) := Aq(z) + Bzp(z) \) with any choice of \( A, B > 0 \) converge absolutely. If all minors of \( T(p) \) of order 2 are equal to zero, then \( b_0p(z) = a_0q(z) \) coefficient-wise.
Proof. Given a real number $a$ let $\lfloor a \rfloor$ denote the maximal integer less than or equal to $a$. The straightforward consequence of the total nonnegativity of $T(p)$ is $a_{n+1}a_n \leq a_{n+1}a_m$ for all indices $n < m$. If $a_n a_k > 0$ for some $k > n$, then we therefore have $0 < a_n a_k \leq a_{n+1}a_k-1 \leq \cdots \leq a_{n+k-1}a_{n+k-1}$, which implies $a_m > 0$ whenever $m = n+1, n+2, \ldots, k-1$; in other words, the series $p(z)$ has no gaps. Moreover,

$$0 \leq \frac{a_n}{a_{n+1}} \leq \frac{a_m}{a_{m+1}}$$

provided that $n < m$ and the denominators are nonzero. Accordingly, $p(z)$ converges in the annulus $0 \leq r < z < R \leq +\infty$ by the ratio test (unless $r = R$), where

$$R := \lim_{k \to +\infty} \frac{a_k}{a_{k+1}}, \quad r := \lim_{k \to +\infty} \frac{a_k}{a_{k+1}}$$

If the totally nonnegative matrix $\psi$ and $\phi$ are nonnegative and $\phi$ is nonempty, Theorem 1 implies that the annulus $r < z < R$ for the series $p(z)$ is not empty.

Suppose that the series $q(z)$ has an empty annulus of convergence; then Theorem 1 implies $b_n^2 = b_{n+1}b_{n-1} \neq 0$ for any integer $n$. The estimate

$$0 \leq \frac{a_n + B b_n}{a_{n-1} + B b_{n-1}} \leq \frac{a_n + B b_{n+1}}{a_{n+1} + B b_{n+1}} = (2a_n b_n - a_{n+1}b_{n-1} - a_{n-1}b_{n+1})b + a_n^2 - a_{n-1}a_{n+1}$$

holds true for every $B > 0$, and hence $0 \leq 2a_n b_n - a_{n+1}b_{n-1} - a_{n-1}b_{n+1}$. Since the ratio $C := \frac{b_{n+1}}{b_n}$ is independent of $n$, the inequality

$$0 \leq 2a_n C_n - a_{n+1}C b_{n+1} - a_{n-1} = -a_{n+1}C^2 + 2a_n C - a_{n-1}$$

must be satisfied for each $n$. Nevertheless, the condition $a_n = 0 \neq a_{n-1} + a_{n+1}$ for some $n$ gives the contradiction $0 \leq -a_{n+1}C^2 - a_{n-1} < 0$. Thus, all coefficients of the series $p(z)$ are nonzero and the estimate (10) is equivalent to

$$\left( C - \frac{a_n}{a_{n+1}} \right)^2 \leq \left( \frac{a_n}{a_{n+1}} \right)^2 - \frac{a_{n-1}}{a_{n+1}} = \left( \frac{a_n}{a_{n+1}} - \frac{a_{n-1}}{a_n} \right) a_n.$$

Taking limits as $n \to \pm\infty$ then yields the equality $r = C = R$.

Suppose that $T(p)$ has a nonzero minor of order 2. Theorem 1 then implies that the series $p(z)$ converges; that is we have the contradiction $r < R$ unless $q(z)$ converges in some annulus. Analogously, the sequences $\varphi(z; 1, 1) = p(z) + q(z)$ has a nonempty annulus $\mathcal{A}$ of convergence as well (otherwise the above reasoning for $\varphi(z; 1, 1)$ instead of $q(z)$ would imply that $r = R$). Moreover, both series $\varphi(z; A, B) = Ap(z) + Bq(z)$ and $\psi(z; A, B) = Ap(z) + Bq(z)$ with any choice of $A, B \geq 0$ are absolutely convergent in $\mathcal{A}$ since all coefficients of the involved series are nonnegative.

Suppose that $T(p)$ has no nonzero minors of order 2. If $p(z) = 0$ or $q(z) = 0$, then $b_0 p(z) \equiv a_0 q(z) = 0$ which implies the lemma in this case. Otherwise, the series $p(z)$ diverges by Theorem 1 and $r = R = \frac{a_n}{a_{n+1}}$ for all $n$. Thus, the above part of the proof with the exchanged roles of $p(z)$ and $q(z)$ yields that $\frac{b_n}{b_{n+1}} = r$ whenever $n \in \mathbb{Z}$. Consequently, the equality $a_0 q(z) = b_0 p(z)$ is satisfied in the sense of formal power series, i.e. coefficient-wise.

**Corollary 14.** If the totally nonnegative matrix $H(p, q)$ has a nonzero minor of order 2 and $p(z) \neq 0$, then $T(p)$ has a nonzero minor of order 2 as well.
Proof. Indeed, Lemma 12 implies that all minors of the matrix $T(Ap + Bq)$ are nonnegative for every choice of the numbers $A, B \geq 0$, so we can apply Lemma 13. We have two possibilities: the first is that the entries involved in the nonzero minor of $H(p, q)$ only come from one of the involved series. That is, this minor is actually a minor of $T(p)$, or of $T(q)$ and hence $T(p)$ also has a nonzero minor of order 2 by Lemma 13. Another possibility is that the terms of the nonzero minor come from both involved series: $a_m b_{n+k} > a_{m+k} b_n$ or $a_{n+k} b_{m+1} > a_n b_{m+1+k}$ for some integers $k > 0$ and $m \geq n$, so automatically $p(z) \neq 0$ and $q(z) \neq 0$. In this case, the assumption that all minors of $T(p)$ or $T(q)$ or order 2 are zero yields a contradiction due to $a \neq b$.

Lemma 15. Let the matrices $T(Ap + Bq)$ and $T(Aq + Bp)$ be totally nonnegative for all $A, B \geq 0$, and let $T(p)$ have a nonzero minor of order 2. Then the ratio $F(z) := \frac{q(z)}{p(z)}$ is an $A$-function.

Proof. By Lemma 13 the series $\phi(z; A, B) = Ap(z) + Bq(z)$ and $\psi(z; A, B) = Aq(z) + Bzp(z)$ converge in some common annulus. The Toeplitz matrices constructed from the coefficients of $\phi(z; A, B)$ and $\psi(z; A, B)$ are totally nonnegative, and hence the analytic continuations of $\phi(z; A, B)$ and $\psi(z; A, B)$ have the form (I). In particular, all zeros of the functions $\phi(z; A, B)$ and $\psi(z; A, B)$ lie in $(-\infty, 0]$. If $z_0$ is such that $F(z_0) = \frac{q(z_0)}{p(z_0)} \leq 0$, then $\phi(z_0) - F(z_0)1 = - \frac{q(z_0)}{p(z_0)} p(z_0) + q(z_0) = 0$. Since for each $A \geq 0$ the function $\phi(z; A, 1)$ does not vanish outside $(-\infty, 0]$, the inequality $z_0 \leq 0$ must be true. Analogously, if $z_1$ is such that $\frac{z_1}{F(z_1)} = \frac{z_1 p(z_1)}{q(z_1)} \leq 0$, then $\frac{\phi(z_1) - F(z_1)1}{p(z_1)} = 0$. Since for each $A \geq 0$ the function $\psi(z; A, 1)$ is nonzero outside $(-\infty, 0]$, we obtain $z_1 \leq 0$. In particular, the function $F(z)$ has no positive poles or zeros; thus, it is positive and holomorphic in $(0, +\infty)$. In other words, all positive poles of the functions $p(z)$ and $q(z)$ coincide with orders.

Fact 1 (Details can be found in e.g. [Du 2004, p. 19]). Let $h(z)$ be a real function holomorphic in a neighbourhood of a real point $x$ and such that $h(z) \leq 0$ for a complex $z$ implies $z \leq 0$. Then the expression $h(z) - h(x)$ has a zero at $x$ of some multiplicity $r \geq 1$. Therefore, $h(z) - h(x) \sim (z - x)^r$ as $z$ is close to $x$ in a small enough neighbourhood of $x$, and we have $\text{Im} \ h(z) = 0$ on the union of $r$ arcs meeting in this neighbourhood only at $x$; one of these arcs is a subinterval of the real line due to the reality of $h(z)$. Furthermore, the half of (if $r$ is even) or all (if $r$ is odd) the arcs contain an interval where $h(z) \leq h(x)$. In particular, the condition $h(x) = 0$ implies $r \leq 2$, and the condition $h(x) < 0$ implies $r = 1$.

Fact 1 with $h(z) := F(z)$ implies that $F(z)$ can have at most double zeros. It is possible that the function $F(z)$ is holomorphic at the origin and equal to zero there. The assumption that the point $x = 0$ can be a double zero of $F(z)$ is contradictory: Fact 1 implies that $F(z)$ is negative for all real $z \neq 0$ small enough, which is impossible for $z > 0$. Suppose that $x < 0$ is a double zero of $F(z)$, that is $F(x) = F'(x) = 0 \neq F''(x)$. Then $F(z) < 0$ and, therefore, $z^{-1}F(z) > 0$ for all real $z$ in a sufficiently small punctured neighbourhood of $x$. At the point $x$, the function $z^{-1}F(z)$ has a double zero:

$$\frac{F(x)}{x} = \frac{F(x) - xF'(x)}{x^2} = 0 \neq \frac{2xF(x) -xF'(x) -x^3F''(x)}{x^4}.$$

Putting $h(z) := z^{-1}F(z)$ in Fact 1 then yields a contradiction, since the inequality $z^{-1}F(z) \leq 0$ must be satisfied for all real $z$ which are close enough to $x$. Consequently, the only possible case is $r = 1$, that is, all zeros of $F(z)$ are simple. Considering in the same way $h(z) = \frac{x}{F(z)}$ and $h(z) = \frac{1}{F(z)}$ shows that all poles of $\frac{F(z)}{x}$ are simple. In particular, $F(z)$ cannot have a pole at the origin.

Now, let us prove that zeros and poles of $F(z)$ are interlacing. Suppose that $x_1 < x_2 \leq 0$ are two consecutive zeros of the function $F(z)$, such that the interval $(x_1, x_2)$ contains no poles of $F(z)$. The ratio $z^{-1}F(z)$ also vanishes at $x_1$ and $x_2$ unless $x_2 = 0$; therefore, Rolle’s theorem gives the points $\xi_1, \xi_2 \in (x_1, x_2)$ such
that \( F'(\xi_1) = \xi_2^2 (F'(\xi_2) - \xi_2 F'(\xi_2)) = 0 \). Let \( h(z) = F(z) \) and \( x := \xi_1 \) if \( F(\xi_1) < 0 \), or \( h(z) = z^{-1} F(z) \) and \( x := \xi_2 \) if \( F(\xi_1) > 0 \). In the special case \( x_2 = 0 \), the function \( F(z) \) is negative in \((x_1, x_2)\), so we put \( h(z) = F(z) \) and denote a zero of \( h'(z) \) in this interval by \( x \). Fact \( \| \) implies \( h'(x) \neq 0 \) in the whole interval \( x_1 < z < x_2 \leq 0 \) including \( z = x \), which contradicts to our choice of \( x \). This shows that the function \( F(z) \) has at least one pole between each pair of its zeros. The same argumentation for \( \frac{z}{1 - z^2} \) instead of \( F(z) \) yields that \( F(z) \) has a zero between each pair of its poles. As a result, zeros and poles of \( F(z) \) are interlacing.

Recall that the functions \( p(z) \) and \( q(z) \) can be represented as in (1) and that their poles coincide with orders. These functions cannot have distinct exponential factors: otherwise \( F(z) \) gets the corresponding exponential factor, so Lemma \([10]\) implies that \( F(z_0) < 0 \) for some \( z_0 \) outside the real line. Therefore, \( F(z) = z^p G(z) \), where \( p \) is an integer and \( G(z) \) has the form (4) or (5). The case when \( F(z) \) has the form \( Cz^r \) with \( r \in \mathbb{Z} \) yields \( r = 1 \) or \( r = 0 \) because \( F(z) \) has no poles at the origin and the possible zero at the origin can only be simple. In the case \( F(z) \neq Cz^r \), Lemma \([11]\) implies \( p = 0 \) since both functions \( F(z) \) and \( z^{-1} F(z) \) can attain negative values only on the real line.

**Lemma 16.** If functions \( p(z) \) and \( q(z) \) have the form (1) and their ratio \( F(z) = \frac{p(z)}{q(z)} \) can be represented as in (4) or (5), then the matrix \( H(p, q) \) is totally nonnegative.

**Proof.** Indeed, denote by \( p_*(z) := \frac{p(z)}{q(z)} \neq 0 \) and \( q_*(z) := \frac{q(z)}{q(z)} \) the denominator and numerator of the function \( F(z) \) given in (4). This means that \( p_*(z) \) and \( q_*(z) \) have no common zeros, no poles and no exponential factors; the function \( g(z) \) has the form (1). The function \( C^n p_n(z) = F_n(z) \) introduced in (6) maps the upper half-plane into itself for each positive integer \( n \). According to Theorem 3.44 of [HT2012] (see also Theorem 1.4 of [Dy2013] where the notation is closer to the current paper) the matrix \( H(p_n, q_n) \) is totally nonnegative. Since \( p_n(z) \) and \( q_n(z) \) converge in \( \mathbb{C} \setminus \{0\} \) locally uniformly to \( p_*(z) \) and \( q_*(z) \) respectively, their Laurent coefficients converge as well. Therefore, the matrix \( H(p_*, q_*) \) is totally nonnegative as an entry-wise limit of totally nonnegative matrices. Then the Cauchy-Binet formula implies the total non-negativity of the matrix \( H(p, q) = H(p_*, g, q_*, g) = H(p_*, q_*) \cdot T(g) \), because \( T(g) \) is totally nonnegative by Theorem \([2]\).

**Proof of Theorem 5.** Lemma \([2]\) shows that the implications \((b) \implies (a)\) and \((c) \implies (d)\) follow, respectively, from Lemma \([15]\) and Lemma \([16]\). The implication \((c) \implies (b)\) follows from Lemma \([12]\) and Corollary \([14]\).

### 4 Proofs of Theorems 5 and 6

There are well-known relations between stable entire functions (more specifically, strongly stable — of the class \( \mathcal{HE} \) up to a change of the variable) and mappings of the upper half of the complex plane into itself, see e.g. [CHM49, Lev64]. In this section, we adapt these relations to suit our problem; some simplifications arise since we only consider the real case.

**Lemma 17 (cf. [Lev64, pp. 307–308]).** Let \( q(z^2) + z p(z^2) \) be a non-trivial two-way infinite series. If the Hurwitz matrix \( H(p, q) \) is totally nonnegative and has a nonzero minor of order two, then this series converges in some annulus to a function \( g(z^2)h(z) \), where \( g(z) \) generates a totally positive sequence, \( h(z) \) is holomorphic for \( z \neq 0 \) and (unless it is equal identically to a constant) satisfies \( |h(z)| > |h(-z)| \) wherever \( \text{Re} \, z > 0 \).

Note that the statement of this lemma implies that the function \( h(z) \) is real (i.e \( h(\mathbb{R}) = h(z) \) for all \( z \)) and that the function \( g(z^2) \) can only have real poles and purely imaginary zeros. The converse of Lemma \([17]\) to be true requires a more delicate characterization of the function \( h(z) \), which is introduced in Theorem \([5]\).
Proof. By Theorem 3, total nonnegativity of $H(p, q)$ implies that both $p(z)$ and $q(z)$ are of the form (1) and their ratio $\frac{q(z)}{p(z)}$ is an $\mathcal{F}$-function. By Lemma 11, the ratio $\frac{z p(z)}{q(z)}$ is an $\mathcal{F}$-function as well. If $\arg$ denotes the principle branch of the argument, then the implication

$$0 \leq \arg z < \frac{\pi}{2} \implies -\pi < \arg \frac{p(z^2)}{q(z^2)} \leq 0 \quad \text{and} \quad 0 \leq \arg \frac{z^2 p(z^2)}{q(z^2)} < \pi$$

yields that the product $\frac{p(z^2)}{q(z^2)} \cdot \frac{z^2 p(z^2)}{q(z^2)}$ cannot be negative; the product cannot be zero or infinite by the definition of $\mathcal{F}$-functions. Therefore, the principal value of its square root satisfies

$$-\frac{\pi}{2} < \arg \sqrt{\frac{z^2 p(z^2)}{q(z^2)}} = \arg z < \frac{\pi}{2}, \quad \text{where} \quad \arg z := \frac{z p(z^2)}{q(z^2)},$$

on condition that $0 \leq \arg z < \frac{\pi}{2}$; the same inequality for $-\frac{\pi}{2} < \arg z < 0$ follows by complex conjugation. In other words, the function $\arg z$ maps the right half of the complex plane into itself. (Up to a change of the variable, we got an adaptation of [KaKr68, Lemma 5.1].) Taking a linear-fractional transform of the right half-plane onto the unit disk gives

$$1 > \frac{1 - \arg z}{1 + \arg z} = \frac{|q(z^2) - z p(z^2)|}{|q(z^2) + z p(z^2)|} = \frac{|h(-z)|}{h(z)},$$

as $\Re z > 0$.

If $h(z)$ is analytic in $\mathbb{C} \setminus \{0\}$ and satisfies to $|h(z)| > |h(-z)|$ as $\Re z > 0$, then $h(z)$ clearly have no zeros with positive real parts. This conclusion can be strengthened with the help of the Carleman formula.

**Lemma 18.** Suppose that $h(z)$ is analytic in $\mathbb{C} \setminus \{0\}$ and satisfies $|h(z)| > |h(-z)|$ as $\Re z > 0$. If $I$ is a subinterval of $\mathbb{Z}$ and $(\gamma_v)_{v \in I}$ is a sequence of all zeros of $h(z)$ ordered so that $v, v + 1 \in I \implies \Re \gamma_v \geq \Re \gamma_{v+1}$ (counting with multiplicities), then

$$\sum_{v \in I, v \geq 0} \left| \Re \frac{1}{\gamma_v} \right| + \sum_{v \in I, v < 0} \left| \Re \gamma_v \right| < \infty.$$

**Proof.** Let $0 < \xi < R$ be such that a function $f(z)$ is analytic in the semi-annulus $\xi \leq |z| \leq R$, $\Im z \geq 0$ and nonzero on its boundary. The Carleman formula (see e.g. [Lev64, p. 224] or [ChM49, p. 153]) for the function $f(z)$ can be written as

$$\sum_{\nu \in J} \left( \frac{1}{r_v} - \frac{r_v}{R^2} \right) \sin \theta_v = \frac{1}{\pi R} \int_0^\pi \ln |f(Re^{i\theta})| \sin \theta d\theta + \frac{1}{\pi R} \int_{|z|^2}^R \left( \frac{1}{x^2} - \frac{1}{R^2} \right) \ln |f(z)| f(-z) |dz + A_{\xi,R},$$

where $(r_v e^{i\theta_v})_{v \in J}$ is the set of all zeros counted with multiplicities, which $f(z)$ has in this semi-annulus; the number $A_{\xi,R}$ is defined as

$$A_{\xi,R} = \Im \frac{1}{\pi R} \int_{\pi}^0 \ln \left| f(\xi e^{i\theta}) \right| \left( \frac{1}{R^2} - \frac{e^{-2i\theta}}{\xi^2} \right) \xi e^{i\theta} d\theta.$$

Note that $A_{\xi,R}$ remains bounded if $R$ grows to infinity. So, if $f(z)$ is bounded, then putting $R \to +\infty$ in the Carleman formula yields that the series

$$\sum_{r_v > \xi} \left| \frac{\sin \theta_v}{r_v} - \frac{r_v \sin \theta_v}{R^2} \right|$$

and, hence,

$$\sum_{r_v > \xi} \left| \frac{\sin \theta_v}{r_v} \right| = \sum_{r_v > \xi} \left| \Im \frac{1}{r_v e^{i\theta_v}} \right|$$

must be convergent.
Chose ζ so that |γ_v| ≠ ζ for all v ∈ I. Within the settings \( f(iz) := \frac{h(iz)}{g(iz)} \) or \( f(iz^{-1}) := \frac{h(iz)}{g(iz)} \), we are getting \( |f(z)| ≤ 1 \) if \( \text{Im} \ z > 0 \); thus, the above formula implies the convergence of the series \( \sum_{v∈I, v>0} |\text{Re} \ γ_v| \) and \( \sum_{v∈I, v<0} |\text{Re} \ γ_v| \), respectively.

**Proof of Theorem [5]** Assume that \( H(p, q) \) is totally nonnegative; by Lemma [2] and Theorem [5], there is a function \( g(z) \) of the form [1] such that the ratios \( \frac{p(z)}{g(z)} \) and \( \frac{q(z)}{g(z)} \) represent functions with no common zeros. Moreover, according to Theorem [7], the estimate

\[
\max_{|z| ≤ R} \left( \frac{p(z)}{g(z)} + \frac{p(1)}{g(1)} \right) + \max_{|z| ≤ R} \left( \frac{q(z)}{g(z)} + \frac{q(1)}{g(1)} \right) < e^{R^2}
\]

holds true for an arbitrary \( ε > 0 \) and for \( R > 1 \) big enough; therefore,

\[
\max_{R ≤ |z| ≤ R^2} \left| \frac{f(z)}{g(z)} \right| < e^{R^2}.
\] (11)

Unless \( h(z) := \frac{f(z)}{g(z)} \) is a constant, by Lemma [17] it satisfies \( |h(z)| > |h(−z)| \) in the right half of the complex plane. Since \( \frac{p(z)}{g(z)} \) and \( \frac{q(z)}{g(z)} \) do not vanish simultaneously, the function \( h(z) = \frac{q(z)}{g(z)} + \frac{p(z)}{g(z)} \) has no purely imaginary zeros. Thus, the estimate (11) implies the following representation:

\[
h(z) = z^r e^{Bz+\frac{B_0}{2}} \prod_{\lambda > 0} \left( 1 + \frac{z}{\xi_\lambda} \right) \prod_{\nu > 0} \left( 1 + \frac{z}{\gamma_\nu} \right) \prod_{\lambda < 0} \left( 1 + \frac{z^{-1}}{\xi_\lambda} \right) \prod_{\nu < 0} \left( 1 + \frac{z^{-1}}{\gamma_\nu} \right) \left( 1 + \frac{z^{-1}}{\gamma_\mu} \right)
\] (12)

where \( B, B_0 ∈ \mathbb{R} \): the involved products are convergent since the sums \( \sum_{\nu > 0} |\text{Re} \ γ_\nu| \), \( \sum_{\nu < 0} |\text{Re} \ γ_\nu| \), \( \sum_{\lambda > 0} |\text{Re} \ xi_\lambda| \) and \( \sum_{\lambda < 0} |\text{Re} \ xi_\lambda| \) are finite by Lemma [18]. In particular, for \( x > 0 \) we have

\[
h(x) = x^r e^{Bx+\frac{B_0}{2}} \prod_{\lambda > 0} \left( 1 + \frac{x}{\xi_\lambda} \right) \prod_{\nu > 0} \left( 1 + \frac{x}{\gamma_\nu} \right) \prod_{\lambda < 0} \left( 1 + \frac{x^{-1}}{\xi_\lambda} \right) \prod_{\nu < 0} \left( 1 + \frac{x^{-1}}{\gamma_\nu} \right) \left( 1 + \frac{x^{-1}}{\gamma_\mu} \right)
\]

On the one hand, Theorem [7] for each \( ε > 0 \) implies

\[
|x|^r \prod_{\lambda > 0} \left( 1 + \frac{x}{\xi_\lambda} \right) \prod_{\nu > 0} \left( 1 + \frac{x}{\gamma_\nu} \right) \prod_{\lambda < 0} \left( 1 + \frac{x^{-1}}{\xi_\lambda} \right) \prod_{\nu < 0} \left( 1 + \frac{x^{-1}}{\gamma_\nu} \right) \left( 1 + \frac{x^{-1}}{\gamma_\mu} \right) < e^{2ε|x + \frac{1}{2}|}
\]

when \( |x + \frac{1}{2}| \) is big enough; so, the ratio \( \frac{h(−x)}{h(x)} \sim e^{−2Bx−\frac{2B_0}{2}} \) grows to infinity as \( x → +∞ \) or as \( x → 0+ \) unless both conditions \( B ≥ 0 \) and \( B_0 ≥ 0 \) are satisfied. On the other hand, \( \frac{h(−x)}{h(x)} \) < 1 for any \( x > 0 \) by Lemma [17]. As a result, the only consistent case is that \( f(z) \) can be represented as in [5].

Conversely, let \( f(z) \) have the form [5]. Then there exists a function \( g(z) \) of the form [1] such that the ratio \( \frac{f(z)}{g(z)} \) satisfies (12) with \( B, B_0 ≥ 0 \). The polynomials

\[
h_n(z) = \left( 1 + \frac{Bz}{n} \right)^n \prod_{0 < \lambda ≤ n} \left( 1 + \frac{z}{\xi_\lambda} \right) \prod_{0 < \nu ≤ n} \left( 1 + \frac{z}{\gamma_\nu} \right) \left( 1 + \frac{z}{\gamma_\mu} \right)
\]

\[
\prod_{0 > \lambda ≥ −n} \left( 1 + \frac{1}{\xi_\lambda} \right) \prod_{0 > \nu ≥ −n} \left( 1 + \frac{1}{\gamma_\nu} \right)
\]

are stable for each positive integer \( n \); if \( P_n(z) \) and \( Q_n(z) \) are defined by \( P_n(z^2) := \frac{1}{2}(h_n(z) + h_n(−z)) \) and \( Q_n(z^2) := \frac{1}{2}(h_n(z) + h_n(−z)) \), then the four-way infinite Hurwitz matrix \( H(P_n, Q_n) \) corresponding to \( h_n(z) \)
Without loss of generality suppose that all ratios behave at most subexponentially. Since these rational functions converge to \( h(z) \) in any annulus centred at the origin as \( n \to \infty \), their Laurent coefficients converge to the coefficients of \( h(z) \). Thus, each minor of the matrix \( H(P,Q) \), where \( P(z^2) := \frac{1}{\pi z} (h(z) - h(-z)) \) and \( Q(z^2) := \frac{1}{2} (h(z) + h(-z)) \), is nonnegative as a limit of nonnegative minors of \( H(P_n,Q_n) \). Recall that all minors of \( T(g) \) are nonnegative by Theorem 1, so the Cauchy-Binet formula and the identity

\[
H(p,q) = H(P \cdot g, Q \cdot g) = H(P,Q) \cdot T(g)
\]

imply the total nonnegativity of \( H(p,q) \).

**Proof of Theorem 6.** We reproduce the original proof of [HKK2016, Theorem 4] with minimal changes. Without loss of generality suppose that \( p_0(z) \neq 0 \) and that \( H(p_m,p_0) \) has a nonzero minor for some \( m > 0 \), which can be achieved by multiplying \( f(z) \) by some power of \( z \). By Theorem 3, \( p_0(z) \) must be a non-trivial series convergent in some non-empty annulus and all other series \( p_1(z), \ldots, p_{M-1}(z) \) converge in the same annulus (or trivial) since the corresponding matrices \( H(p_1,p_0), \ldots, H(p_{M-1},p_0) \) are totally nonnegative. Let us keep the notation \( p_0(z), p_1(z), \ldots, p_{M-1}(z) \) for functions represented by the same-name series. Then Theorem 3 additionally implies that all poles and exponential factors of the functions \( p_0(z), p_1(z), \ldots, p_{M-1}(z) \) coincide. In other words, there exists a function \( g(z) \) of the form \( \{1\} \) such that the ratios \( \frac{p_0(z)}{g(z)}, \ldots, \frac{p_{M-1}(z)}{g(z)} \) can be represented as

\[
a_0 \prod_{\nu > 0} \left( 1 + \frac{z}{\alpha_{\nu}} \right) \prod_{\nu < 0} \left( 1 + \frac{z^{-1}}{\alpha_{\nu}} \right) \quad \text{or} \quad C(z + a_0) \prod_{\nu > 0} \left( 1 + \frac{z}{\alpha_{\nu}} \right)
\]

with positive coefficients (see Lemma 2). In particular, all these ratios behave at most subexponentially as \( z \to 0^+ \) and as \( z \to +\infty \) by Theorem 7, therefore, we can chose two entire functions \( q_1(z) \) and \( q_2(z) \) of genus less than \( M \), so that \( f(z) \) has the required factorization \( g(z^M) \cdot q_1(z) \cdot q_2 \left( \frac{1}{z^M} \right) \).

Let us prove that the sector \( C_M \) contains no zeros of \( f(z) \). On the one hand, for any \( z \in \{ \xi \in \mathbb{C} : 0 < \arg \xi < \frac{\pi}{M} \} \) and integers \( n, m \) satisfying \( 0 \leq n < m < M \), Theorem 3 yields the inequality

\[
0 \leq \arg \frac{p_n(z^M)}{p_m(z^M)} < \pi, \quad \text{that is} \quad -\pi < \arg \frac{p_m(z^M)}{p_n(z^M)} \leq 0.
\]

Since

\[
0 \leq \arg z^{m-n} = (m-n) \arg z < \frac{m-n}{M} \pi < \pi,
\]

the argument (its principal value) of the product \( z^{m-n} \cdot \frac{p_m(z^M)}{p_n(z^M)} \) is equal to the sum of arguments of the factors, and thus

\[
(m-n) \arg z - \pi < \arg \frac{z^{m-n} p_m(z^M)}{p_n(z^M)} \leq (m-n) \arg z \quad \text{if} \quad m > n.
\]

(15)
On the other hand, suppose that the condition \(0 = f(z) = p_0(z^M) + z p_1(z^M) + \cdots + z^{M-1} p_{M-1}(z^M)\) holds true for some \(z\) varying in the chosen sector, which is is equivalent to

\[
\sum_{n=1}^{M-1} \frac{z^n p_n(z^M)}{p_0(z^M)} = -1
\]  

(14)
due to \(p_0(z^M) \neq 0\). There are at least two nonzero summands on the left-hand side: otherwise this equality would contradict to \((13)\). Let \(m\) and \(n\) be the indices of the nonzero summands with maximal and minimal arguments, respectively. Among the inequalities \((13)\), we have

\[
m \arg z - \pi < \arg \frac{z^m p_m(z^M)}{p_0(z^M)} \leq m \arg z \quad \text{and} \quad n \arg z - \pi < \arg \frac{z^n p_n(z^M)}{p_0(z^M)} \leq n \arg z,
\]

and therefore

\[
0 = \max \{0, (m - n) \arg z - \pi\} \leq \arg \frac{z^m p_m(z^M)}{p_0(z^M)} - \arg \frac{z^n p_n(z^M)}{p_0(z^M)} < (m - n) \arg z + \pi.
\]

Then the inequality

\[
\pi < \arg \frac{z^m p_m(z^M)}{p_0(z^M)} - \arg \frac{z^n p_n(z^M)}{p_0(z^M)}
\]

implies \(m > n\) and

\[
\pi < \arg \frac{z^{m-n} p_m(z^M)}{p_n(z^M)} + 2\pi < (m - n) \arg z + \pi, \quad \text{that is} \quad -\pi < \arg \frac{z^{m-n} p_m(z^M)}{p_n(z^M)} < (m - n) \arg z - \pi,
\]

which is inconsistent with \((13)\); the reverse inequality

\[
\pi \geq \arg \frac{z^m p_m(z^M)}{p_0(z^M)} - \arg \frac{z^n p_n(z^M)}{p_0(z^M)}
\]

implies that the cone \([0] \cup \{\zeta \in \mathbb{C} \setminus \{0\} : \arg \frac{z^m p_m(z^M)}{p_0(z^M)} \geq \arg \zeta \geq \arg \frac{z^n p_n(z^M)}{p_0(z^M)}\}\) is convex. This cone contains all the ratios \(z^k p_k(z^M)/p_0(z^M)\), where \(k = 1, \ldots, M - 1\), and hence

\[
\pi > \arg \frac{z^m p_m(z^M)}{p_0(z^M)} \geq \arg \sum_{k=1}^{M-1} z^k p_k(z^M)/p_0(z^M) \geq \arg \frac{z^n p_n(z^M)}{p_0(z^M)} > -\pi,
\]

which contradicts to \((14)\). Consequently, there are no points \(z\) in the sector \(C_M \cap |\text{Im } z| > 0\) such that \(f(z) = 0\). Complex conjugation gives that solutions to \(f(z) = 0\) cannot belong to \(C_M \cap |\text{Im } z| < 0\) as well.

\[
\square
\]

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