Perturbation theory in the invariant subspace

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Abstract

The unitary transformation of path-integral differential measure is described. The main properties of perturbation theory in the phase space of action-angle, energy-time variables are investigated. The measure in cylindrical coordinates is derived also. The dependence of perturbation theory contributions from global (topological) properties of corresponding phase spaces is shown.

1 Introduction

The main purpose of this article is to describe a quantum system with nontrivial phase-space topology. The approach will be illustrated, risking to lose generality, by simplest quantum-mechanical examples of particle motion in the potential hole \( v(x) \) with one nondegenerate minimum at \( x = 0 \).

There is a definite hope that offered below formalism will be useful for quantization of nonlinear waves. Indeed, the last problem was considered previously by many authors, e.g. [1, 2], introducing the convenient variables (collective coordinates) in order that reduce the quantum soliton-like excitations problem to quantum-mechanical one.

Quantitatively the problem looks as follows. It is not difficult to describe one-particles dynamics in the quasiclassical approximation since corresponding equation for trajectory \( x_c \) always can be solved. But, beyond this approximation, to use the ordinary WKB expansion of path integrals one should solve the equation for Green function \( G \):

\[
(\partial^2 + v''(x_c))_t G(t, t') = \delta(t - t').
\]  

(1.1)

Just eq. (1.1) offers a difficulty: it is impossible to find a strict solution of this equation since \( x_c = x_c(t) \) is the nontrivial function and, therefore, \( G(t, t') \) is not translationally invariant describing propagation of a ‘particle’ in the time-dependent potential \( v''(x_c) \). One can hope to avoid this difficulty introducing the convenient dynamical variables. Demonstration of the way as this program can be realized is the aim of this article.

The main formal difficulty, e.g. [3], of this program consisting in transformation of the path-integral measure was solved in [4]. It was shown in this paper that the phase-space \((x, p)\) path integrals differential measure \( DM(x, p) \) of probability-like quantities \( R = |A|^2 \), where \( A \) is the amplitude, is \( \delta \)-like:

\[
DM(x, p) = \prod_t dx(t)dp(t)\delta(\dot{x} + \frac{\partial H}{\partial p})\delta(\dot{p} - \frac{\partial H}{\partial x}),
\]  

(1.2)
where the Hamiltonian

\[ H_j = \frac{1}{2}p^2 + v(x) - jx \]

includes the energy of quantum fluctuations \( jx \), with the provoking quantum excitations force \( j = j(t) \). The dynamical equilibrium between ordinary mechanical forces (kinetic \( \dot{p}(t) \) plus potential \( v'(x) \)) and quantum force \( (j(t)) \) fixed by \( \delta \)-like measure (1.2) allows to perform an arbitrary transformation of quantum measure caused by transformation of classical forces, i.e. of \( x \) and \( p \).

We will use this property introducing the ‘motion’ on the cotangent bundle \((\theta, h)\), where \( h \) is the bundles parameter and \( \theta \) is the coordinate on it. For definiteness, let \( h \) be the conserved on the classical trajectory energy and \( \theta \) is the conjugate to \( h \) ‘time’. (The transformation to action-angle variables will be described also.) The mapping \((x, p) \rightarrow (\theta, h)\) is canonical and the corresponding equations of motion on the cotangent bundle should have the form:

\[
\begin{align*}
\dot{h} &= -\frac{\partial H_c}{\partial \theta} = +j \frac{\partial x_c(\theta, h)}{\partial \theta}, \\
\dot{\theta} &= +\frac{\partial H_c}{\partial h} = 1 - j \frac{\partial x_c(\theta, h)}{\partial h},
\end{align*}
\]

where

\[ H_c = h - jx_c(\theta, h) \]

is the transformed Hamiltonian and \( x_c(\theta, h) \) is the classical trajectory in the \((\theta, h)\) terms. The Green function of the eq.(1.3) \( g(t, t') \) is translationally invariant since classically (at \( j = 0 \)) the cotangent bundle is the time-independent manifold.

Above example shows that the mapping is constructive iff the bundle parameters are generators of (sub)group violated by \( x_c \). Corresponding phase space is the invariant subspace [5]. For some problems the mapping on the group manifolds is useful, see, e.g., [6]. We will demonstrate the noncanonical transformation to cylindrical coordinates.

The perturbation theory with measure (1.2) on the cotangent bundles has unusual properties [7, 8] and in Sec.3 few proposition concerning perturbation theory on the invariant subspaces will be offered. The main of them demonstrate that each order of perturbation theory can be reduced to the total derivative over global coordinate of the invariant subspace. The application of derived formalism in quantum mechanics and field theories will be published later. For completeness in the following Sec.2 the origin of approach [4] will be shown.

2 Unitarity condition

It is known that the transformations of the path-integral measure deform perturbation theory contributions uncontrollably because of the stochastic nature of quantum trajectories [4]. Purpose of this Section is to show how the \( S \)-matrix unitarity condition can be introduced into the path-integral formalism to find measure (1.2) (preliminaries were given in [4]).
The unitarity condition for the \( S \)-matrix \( SS^+ = S^+S = I \) presents the infinite set of nonlinear equalities:

\[
iAA^* = A - A^*,
\]

(2.1)

where \( A \) is the amplitude, \( S = I + iA \). Expressing the amplitude by the path integral one can see that the l.h.s. of this equality offers the double integral and, at the same time, the r.h.s. is the linear combination of integrals. Let us consider what this linearization of product \( AA^* \) gives.

Using the spectral representation of one-particle amplitude:

\[
A(x_1, x_2; E) = \sum_n \frac{\Psi_n^*(x_2)\Psi_n(x_1)}{E - E_n + i\varepsilon}, \quad \varepsilon \to +0,
\]

(2.2)

let us calculate

\[
R(E) = \int dx_1 dx_2 A(x_1, x_2; E)A^*(x_1, x_2; E).
\]

(2.3)

The integration over end points \( x_1 \) and \( x_2 \) is performed for sake of simplicity only. Using ortho-normalizability of the wave functions \( \Psi_n(x) \) we find that

\[
R(E) = \sum_n \left| \frac{1}{E - E_n + i\varepsilon} \right|^2 = \frac{\pi}{\varepsilon} \sum_n \delta(E - E_n).
\]

(2.4)

Certainly, the last equality is nothing new but it is important to note that \( R(E) \equiv 0 \) for all \( E \neq E_n \), i.e. that all unnecessary contributions with \( E \neq E_n \) were canceled by difference in the r.h.s. of eq.(2.1). We will put this phenomena in the basis of the approach.

We will build the perturbation theory for \( R(E) \) using the path-integral definition of amplitudes \([4]\). It leads to loss of some information since the amplitudes can be restored in such formulation with phase accuracy only \([4]\). Yet, it is sufficient for calculation of the energy spectrum. We would consider this quantity to demonstrate following statement:

**S1.** The unitarity condition unambiguously determines contributions in the path integrals for \( R(E) \).

This statement looks like a tautology since \( \exp\{iS(x)\} \), where \( S(x) \) is the quantum-mechanical action, is the unitary operator which shifts a system along the trajectory\([4]\). I.e. the unitarity is already fixed in the path-integrals. But the general path-integral solution contains unnecessary degrees of freedom (unobservable states with \( E \neq E_n \) in our example). We want define the quantum measure \( DM \) in such a way that the condition of absence of unnecessary contributions in the final (measurable) result be loaded from the very beginning. Just in this sense the unitarity looks like the necessary and sufficient condition unambiguously determining the complete set of contributions. Solution is simple: one should search, as it follows from(2.4), the linear path-integral representation for \( R(E) \) to introduce this condition into the formalism.

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1 One can use the dispersion relations to reconstruct the amplitude from \( R(E) \).

2 It is well known that this unitary transformation is the analogy of tangent transformations of classical mechanics \([4]\).
Indeed, to see the integral form of our approach, let us use the proper-time representation:

\[ A(x_1, x_2; E) = \sum_n \Psi_n(x_1) \Psi_n^*(x_2) i \int_0^\infty dT e^{i(E - E_n + i\epsilon)T} \tag{2.5} \]

and insert it into (2.3):

\[ R(E) = \sum_n \int_0^\infty dT_+ dT_- e^{-(T_+ + T_-)\epsilon} e^{i(E - E_n)(T_+ - T_-)} \tag{2.6} \]

We will introduce new time variables instead of \( T_\pm \):

\[ T_\pm = T \pm \tau, \tag{2.7} \]

where, it follows from Jacobian of the transformation, \(|\tau| \leq T, 0 \leq T \leq \infty \). But we can put \(|\tau| \leq \infty \) since \( T \sim 1/\epsilon \rightarrow \infty \) is essential in integral over \( T \). In result,

\[ R(E) = 2\pi \sum_n \int_0^\infty dT e^{-2\epsilon T_\tau} \int_{-\infty}^{+\infty} \frac{d\tau}{\pi} e^{2i(E - E_n)\tau}. \tag{2.8} \]

In the last integral all contributions with \( E \neq E_n \) are canceled. Note that the product of amplitudes \( AA^* \) was ‘linearized’ after introduction of ‘virtual’ time \( \tau = (T_+ - T_-)/2 \). The physical meaning of such variables will be discussed, see also [9].

We will consider following path-integral:

\[ A(x_1, x_2; E) = i \int_0^\infty dT e^{iET} \int Dx e^{iS_{C_+}(x)} \delta(x_1 - x(0)) \delta(x_2 - x(T)), \tag{2.9} \]

where

\[ C_+ : t \rightarrow t + i\epsilon, \epsilon \rightarrow +0, \]

is the Mills complex time contour [10]. Calculating the probability to find a particle with energy \( E \) (\( Im \ E \) will not be mentioned for sake of simplicity) we have:

\[ R(E) = \int dx_1 dx_2 |A|^2 = \int_0^\infty dT_+ dT_- e^{iE(T_+ - T_-)} \int DC_+ x_+ DC_- x_- \times \delta(x_+(0) - x_-(0)) \delta(x_+(T_+) - x_-(T_-)) e^{iS_{C_+}(T_+)(x_+)} e^{-iS_{C_-}(T_-)(x_-)}, \tag{2.10} \]

where \( C_-(T) = C_+^*(T) \). Note that the total action in (2.10) \( S_{C_+}(T_+)(x_+) - S_{C_-}(T_-)(x_-) \) describes the closed-path motion by definition.

New time variables \( T \) and \( \tau \) will be used: \( T_\pm = T \pm \tau \). If \( Im \ E \rightarrow +0 \) then \( T \) and \( \tau \) can be considered as the independent variables: \( 0 \leq T \leq \infty, -\infty \leq \tau \leq \infty \). We will introduce also the mean trajectory \( x(t) = (x_+(t) + x_-(t))/2 \) and the deviation \( e(t) \) from it: \( x_\pm(t) = x(t) \pm e(t) \). Note that one can do surely this linear transformations in the path integrals if the space is flat. We will consider \( e(t) \) and \( \tau \) as the fluctuating, virtual, quantities and calculate the integrals over them perturbatively. In zero order over \( e \) and \( \tau \), i.e. in the quasiclassical approximation, \( x \) is the classical path and \( T \) is the total time of classical motion.
The boundary conditions (see (2.10)) should fix the closed-path motion and therefore we have the boundary conditions for $e(t)$ only:

$$e(0) = e(T) = 0.$$  \hspace{1cm} (2.11)

Note the uniqueness of this solution if the integral over $\tau$ is calculated perturbatively.

Extracting the linear over $e$ and $\tau$ terms from the closed-path action $S_{C_+(T_+)}(x_+) - S_{C_-(T_-)}(x_-)$ and expanding over $e$ and $\tau$ the remainder terms:

$$-\tilde{H}_T(x; \tau) = S_{C_+(T+\tau)}(x) - S_{C_-(T-\tau)}(x) + 2\tau H_T(x),$$  \hspace{1cm} (2.12)

where $H_T(x)$ is the Hamiltonian at the time moment $T$, and

$$-V_T(x, e) = S_{C_+(T)}(x + e) - S_{C_-(T)}(x - e) + 2\Re \int_{C^+} dt e(\ddot{x} + v'(x))$$  \hspace{1cm} (2.13)

we find that

$$R(E) = 2\pi \int^\infty_0 dTe^{-i\tilde{K}(\omega, \tau; j, e)} \int DM(x)e^{-i\tilde{H}_T(x; \tau) - iV_T(x, e)}. $$  \hspace{1cm} (2.14)

The expansion over differential operators:

$$\tilde{K}(\omega, \tau; j, e) = \frac{1}{2} \left( \frac{\partial}{\partial \omega} \frac{\partial}{\partial \tau} + \Re \int_{C^+(T)} dt \frac{\delta}{\delta j(t)} \frac{\delta}{\delta e(t)} \right)$$  \hspace{1cm} (2.15)

will generate the perturbation series. We propose that it exist in Borel sense.

In (2.14) the functional measure

$$DM(x) = \delta(E - \omega - H_T(x)) \prod_t dx(t) \delta(\ddot{x} + v'(x) - j)$$  \hspace{1cm} (2.16)

unambiguously defines the complete set of contributions in the path integral. The functional $\delta$-function is defined as follows:

$$\prod_t \delta(\ddot{x} + v'(x) - j) = (2\pi)^2 \int \prod_t \frac{de(t)}{\pi} \delta(e(0)) \delta(e(T))e^{-2i\Re \int_{C^+_+} dt e(\ddot{x} + v'(x) - j)} = \prod_{t \in C^+_+} \delta(Re(\ddot{x} + v'(x) - j)) \delta(Im(\ddot{x} + v'(x) - j))$$  \hspace{1cm} (2.17)

The physical meaning of this $\delta$-function is following. We can consider $(\ddot{x} + v'(x) - j)$ as the total force and $e(t)$ as the virtual deviation from true trajectory $x(t)$. In classical mechanics the virtual work must be equal to zero: $(\ddot{x} + v'(x) - j)e(t) = 0$ (d’Alembert) since the motion is time reversible. From this evident dynamical principle one can find the ‘classical’ equation of motion:

$$\ddot{x} + v'(x) = j,$$  \hspace{1cm} (2.18)

since $e(t)$ is arbitrary.
Generally, in quantum theories the virtual work is not equal to zero, i.e. the quantum motion is not time reversible since the quantum corrections can shift the energy levels. But integration over \( e(t) \), with boundary conditions (2.11), leads to the same result. So, in quantum theories the unitarity condition (i.e. the destructive interference among two exponents in product \( AA^* \)) play the same role as the d’Alembert’s variational principle in classical mechanics. We can conclude, the unitarity condition as the dynamical principle establish the local equilibrium between classical (r.h.s. of (2.18)) and quantum (l.h.s. of (2.18)) forces.

3 Perturbation theory

Now let us consider representation (2.14). It is not hard to show that

\[ S_2. \text{ Eq.}(2.14) \text{ restores the perturbation theory of stationary phase method.} \]

For this purpose it is enough to consider the ordinary integral:

\[
A(a, b) = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} e^{i\left(\frac{1}{2}ax^2 + \frac{1}{3}bx^3\right)}, \tag{3.1}
\]

with \( Im \ a \to +0 \) and \( b > 0 \). Computing the ‘probability’ \( R = |A|^2 \) we find:

\[
R(a, b) = e^{\frac{1}{2}j\hat{e}} \int_{-\infty}^{+\infty} dx e^{-2(x^2 + e^2)Im \ a} e^{2i\frac{\hat{e}}{3}x^3} \delta(Re \ ax + bx^2 + j). \tag{3.2}
\]

Performing the trivial transformation \( e \to ie, \ \hat{e} \to -i\hat{e} \) of auxiliary variable we find at the limit \( Im \ a = 0 \) that the contribution of \( x = 0 \) extremum (minimum) gives expression:

\[
R(a, b) = \frac{1}{a} e^{-\frac{1}{2}j\hat{e}}(1 - 4bj/a^2)^{-1/2} e^{2j\frac{\hat{e}}{3}x^3} \tag{3.3}
\]

and the expansion of operator exponent gives the asymptotic series:

\[
R(a, b) = \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \frac{(6n - 1)!!}{n!} \left(\frac{2b^3}{3a^6}\right)^n, \quad (-1)!! = 0!! = 1. \tag{3.4}
\]

This series is convergent in Borel sense.

Eq.\((3.2)\) can be considered as the definition of integral \((3.1)\). By this reason one may put \( Im \ a = 0 \) from the very beginning. We will use this property.

Let us calculate now \( R \) using the stationary phase method. Contribution from the minimum \( x = 0 \) gives \((Im \ a = 0)\):

\[
A(a, b) = e^{-\frac{1}{2}j\hat{e}} e^{-\frac{1}{3}j^2} e^{i\frac{1}{3}x^3} \left(\frac{i}{a}\right)^{1/2}.
\]

The corresponding ‘probability’ is

\[
R(a, b) = \frac{1}{a} e^{-\frac{1}{2}j\hat{e}} e^{2j\frac{\hat{e}}{3}x^3} e^{2bj^2} \tag{3.5}
\]

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This expression does not coincide with (3.3) but it leads to the same asymptotic series (3.4).

The solution \( x_j(t) \) of eq. (2.18) we would search expanding it over \( j(t) \):

\[
x_j(t) = x_c(t) + \int dt_1 G(t, t_1) j(t_1) + ...
\]

This is sufficient since \( j(t) \) is the auxiliary variable. In this decomposition \( x_c(t) \) is the strict solution of unperturbated equation \( \ddot{x} + v'(x) = 0 \) and \( G(t, t') \) must obey eq. (1.1). Note that the functional \( \delta \)-function in (2.16) does not contain the end-point values of time \( t = 0 \) and \( t = T \). This means that the initial conditions to the eq. (2.18) are not fixed and the integration over them is assumed because of the definition of \( R \).

The \( \delta \)-likeness of measure allows to conclude:

**S3.** All strict regular solutions (including trivial) of classical (unperturbed by \( j \) equation(s) of motion must be taken into account.

We must consider only ‘strict’ solutions because of strict cancellation of needless contributions. The \( \delta \)-likeness of measure means that the probability \( R(E) \) should contain sum over all discussed solutions. This is the main distinction of our unitary method of quantization from stationary phase method: even having few solutions there is not interference terms in the sum over them in \( R \).

Note that the interference terms are absent independently from solutions ‘nearness’ in the functional space. This reflects the orthogonality of Hilbert spaces builded on the various \( x_c \) and is the consequence of unitarity condition.

Summation over all solutions of classical equation of motion means necessity to take into account all topologically-equivalent orbits \( x_c \) also. This naturally gives integration over zero-mode degrees of freedom. The corresponding measure will be defined by mapping on the cotangent bundle (see S4).

It is evident that in the sum over contributions of various \( x_c \) we must leave largest, i.e. with maximal number of zero modes. This selection rule is equivalent of definition of the vacuum in the canonical quantization scheme.

The solutions must be regular since the singular \( x_c \) gives zero contribution on \( \delta \)-like measure. •

It is evident that

**S4.** The measure (2.16) admits the canonical transformations.

This evidently follows from \( \delta \)-likeness of measure. In the phase space we have:

\[
DM(x, p) = \delta(E + \omega - H_T(x)) \prod_t dx dp \delta(\dot{x} - \frac{\partial H_j}{\partial p}) \delta(\dot{p} + \frac{\partial H_j}{\partial x}), \tag{3.6}
\]

where

\[
H_j = \frac{1}{2} p^2 + v(x) - j x \tag{3.7}
\]

is the total Hamiltonian which is time dependent through \( j(t) \).

Instead of pare \((x, p)\) we can introduce new pare \((\theta, h)\) inserting

\[
1 = \int D\theta Dh \prod_t \delta(h - \frac{1}{2} p^2 - v(x)) \delta(\theta - \int^x dx (2(h - v(x)))^{-1/2}). \tag{3.8}
\]
It is important that both differential measures in (3.8) and (3.6) are $\delta$-like. This allows to change the order of integration surely and firstly integrate over $(x, p)$. Calculating result one can use $\delta$-functions of (3.6). In this case $\delta$-functions of (3.8) will define the constraints. But if we will use $\delta$-functions of (3.8) the mapping $(x, p) \rightarrow (\theta, h)$ is performed. We conclude that our transformation takes into account the constraints since both ways must give the same result.

We find:

$$DM(\theta, h) = \delta(E + \omega - h(T)) \prod_t \delta(\dot{\theta} - \frac{\partial H_c}{\partial h}) \delta(\dot{h} + \frac{\partial H_c}{\partial \theta}) , \quad (3.9)$$

since considered transformation is canonical, $\{h(x, p), \theta(x, p)\} = 1$, where

$$H_c = h - j x_c(h, \theta) \quad (3.10)$$
is the transformed Hamiltonian and $x_c(\theta, h)$ is the classical trajectory parametrized by $h$ and $\theta$.

So, on the cotangent bundle we must solve following equations of motion:

$$\dot{h} = j \frac{\partial x_c}{\partial \theta}, \quad \dot{\theta} = 1 - j \frac{\partial x_c}{\partial h} , \quad (3.11)$$

which have a simple structure:

\textit{S5. The Green function on the cotangent bundle is simple $\Theta$-function.}

Indeed, expanding solutions of eqs. (3.11) over $j$ in the zero order we have $\theta_0 = t_0 + t$ and $h_0 = \text{const}$. In the first order we have equation for Green function $g(t, t')$:

$$\dot{g}(t, t') = \delta(t - t') . \quad (3.12)$$

Noting that the $S$-matrix problem with definite boundary conditions $x(0) = x_1$ and $x(T) = x_2$ is solved and \( \delta \)-like measure was arise in result of destructive interference between expanding and converging waves in $AA^*$, i.e. wishing to describe the quantum motion from $x(0) = x_1$ to $x(T) = x_2$ we without fail introduce the time ‘irreversibility’:

$$g(t, t') = \Theta(t - t') , \quad (3.13)$$
in opposite to causal particles propagator $G(t, t')$, which contains both retarded and advanced parts. But, as will be seen below, see $S10$, the perturbation theory with Green function (3.13) is time reversible. Note also, that the solution (3.13) is the unique and is the direct consequence of usual in the quantum theories $i\varepsilon$-prescription.

The uncertainty is contained in the boundary value $g(0)$. We will see that $g(0) = 0$ excludes some quantum corrections. By this reason one should consider $g(0) \neq 0$. We will assume that

$$g(0) = 1 \quad (3.14)$$

\(^3\text{I.e., the argument of second $\delta$-function in (3.8) becomes equivalent to zero at $x = x_c$.}\)
since this boundary condition to eq.(3.12) is natural for local theories. We will use also following formal equalities:

\[ g(t, t')g(t', t) = 0, \quad 1 = g(t, t') + g(t', t) \]  (3.15)

since \( g(t, t') \) is introduced for definition of boundaries of time integrals. It is important to note that \( \text{Im} \ g(t) = 0 \) on the real time axis. This allows to conclude that

**S6.** The perturbation theory on the \((h, \theta)\) bundle can be constructed on the real-time axis.

Indeed, the \(i\varepsilon\)-prescription is not necessary since, as was mentioned above in S2, the \(\delta\)-functional measure defines a complete set of contributions. But for more confidence one may introduce the \(i\varepsilon\)-prescription and, extracting the \(\delta\)-function in the measure, one can put \(\varepsilon = 0\) if the contributions are regular at this limit.

One can point out the examples when \(\varepsilon = 0\) is the singular point.

(i) The Green function \(G(t, t')\) is singular at \(\varepsilon = 0\). The \(i\varepsilon\)-prescription introduces the wave damping in this case.

(ii) The terms of perturbation theory are singular at \(\varepsilon = 0\) even if the Green functions are regular. This singularities are connected with light-cone singularities of the real-time theories.

(iii) There is the tunneling phenomena. The \(i\varepsilon\)-prescription is necessary to define a theory in the turning points (it is the usual WKB prescription).

(iv) The extremum of the action is unstable (is the maximum). In this case the \(\varepsilon = 0\) limit is absent for corresponding contribution and one should omit it (as in example considered in S2).

Considered in this paper examples allows the shift on the real-time axis.

Note now that \(\partial x_c/\partial \theta\) and \(\partial x_c/\partial h\) in the r.h.s. of (3.11) can be considered as the sources. This allows to offer the statement:

**S7.** The mapping on the cotangent bundle splits ‘Lagrange’ quantum force \(j\) on a set of quantum forces individual to each independent degree of freedom, i.e. to each independent local coordinate of the cotangent manifold.

To show the splitting mechanism let us consider the action of the perturbation-generating operators:

\[
e^{-\frac{i}{2}Re \int_{C^+} dt \dot{j} e^{i(T(x_c, e_c))}} e^{-iV_T(x_c, e_c)} \prod_t \delta(\dot{h} - j \frac{\partial x_c}{\partial \theta}) \delta(\dot{\theta} - 1 + j \frac{\partial x_c}{\partial h}) = \\
= \int De_h De_\theta e^{2Re \int_{C^+} dt (e_h h + e_\theta (\theta - 1))} e^{-iV_T(x_c, e_c)}, \]  (3.16)

where

\[
e_c = e_h \frac{\partial x_c}{\partial \theta} - e_\theta \frac{\partial x_c}{\partial h} \]  (3.17)
The integrals over \((e_h, e_\theta)\) will be calculated perturbatively:

\[
e^{-iVT(x_c, e_c)} = \sum_{n_h, n_\theta = 0}^{\infty} \frac{1}{n_h! n_\theta!} \int \prod_{k=1}^{n_h} (dt_k e_h(t_k)) \prod_{k=1}^{n_\theta} (dt'_k e_\theta(t'_k)) P_{n_h, n_\theta}(x_c, t_1, ..., t_{n_h}, t'_1, ..., t_{n_\theta}),
\]

where

\[
P_{n_h, n_\theta}(x_c, t_1, ..., t_{n_h}, t'_1, ..., t_{n_\theta}) = \prod_{k=1}^{n_h} \hat{e}_h(t_k) \prod_{k=1}^{n_\theta} \hat{e}_\theta(t'_k) e^{-iVT(x_c, e'_c)}
\]

with \(e'_c \equiv e_c(e'_h, e'_\theta)\) and the derivatives in this equality are calculated at \(e'_h = 0, e'_\theta = 0\). At the same time,

\[
\prod_{k=1}^{n_h} \hat{e}_h(t_k) \prod_{k=1}^{n_\theta} \hat{e}_\theta(t'_k) = \prod_{k=1}^{n_h} (i\hat{j}_h(t_k)) \prod_{k=1}^{n_\theta} (i\hat{j}_\theta(t'_k)) e^{-2iRe \int_{C^+} dt (\hat{j}_h(t)\hat{e}_h(t) + \hat{j}_\theta(t)\hat{e}_\theta(t))}.
\]

The limit \((j_h, j_\theta) = 0\) is assumed. Inserting (3.11), (3.20) into (3.16) we find new representation for \(R(E)\):

\[
R(E) = 2\pi \int_0^{\infty} dT \exp\left\{ \frac{1}{2i} (\hat{\omega} \hat{\tau} + Re \int_{C^+} dt (\hat{j}_h(t)\hat{e}_h(t) + \hat{j}_\theta(t)\hat{e}_\theta(t))) \right\} \times \int DhD\theta e^{-i\hat{H}(x_c; \tau) - iVT(x_c, e_c)} \delta(E + \omega - h(T)) \prod_t \delta(\hat{h} - j_h) \delta(\hat{\theta} - 1 - j_\theta),
\]

in which the ‘energy’ and the ‘time’ quantum degrees of freedom are splitting.

Therefore, splitting \(j \rightarrow (j_h, j_\theta)\) we must change \(e \rightarrow e_c\), where \(e_c\) carry the simplexic structure of Hamilton’s equations of motion, see (3.17), i.e. \(e_c\) is the invariant of canonical transformations. This quantity describes the flow \(\delta_h x_c \wedge \delta_\theta p_c\) generated by quantum perturbations through the bundles elementary cell.

Hiding the \(x_c(t)\) dependence in \(e_c\) we had solve the problem of the functional determinants and simplify the equation of motion as much as possible:

\[
DM(h, \theta) = \delta(E + \omega - h(T)) \prod_t dh(t) d\theta(t) \delta(\dot{h}(t)) \delta(\dot{\theta}(t) - 1)
\]

and the perturbations generating operator

\[
\hat{K} = \frac{1}{2} (\hat{\omega} \hat{\tau} + \int_0^T dt_1 dt_2 \Theta(t_1 - t_2) (\hat{e}_h(t_1) \hat{h}'(t_2) + \hat{e}_\theta(t_1) \hat{\theta}'(t_2)).
\]

In \(V_T(x_c, e_c)\) we must change \(h \rightarrow (h + h')\) and \(\theta \rightarrow (\theta + \theta')\). 

To describe quantum dynamics on the group manifolds we can consider also the coordinate transformations. For instance, the two dimensional model with potential \(v = v((x_1^2 + x_2^2)^{1/2})\) should be considered in the cylindrical coordinates \(x_1 = r \cos \phi\), \(x_2 = r \sin \phi\). Note that this transformation is not canonical. In result we will see that

\(S8. The transformed measure can not be deduced from direct transformations of path integral (2.9).\)
Indeed, the measure in the cylindrical coordinates

\[ D^{(2)} M(r, \phi) = \delta(E + \omega - H_T(x)) \prod_t dr d\phi r^2(t) \delta(\dot{r} - \dot{\phi}^2 r + v'(r) - \dot{j}_r) \delta(\dot{\phi} - \dot{j}_\phi), \tag{3.24} \]

where \( v'(r) = \partial v(r) / \partial r \) and \( j_r, j_\phi \) are the components of \( \vec{j} \) in the cylindrical coordinates.

The perturbation generating operator has the form:

\[ \hat{K} = \frac{1}{2} \hat{\omega} \hat{\tau} + \int_{C(T)} dt (\dot{j}_r(t) \dot{e}_r(t) + \dot{j}_\phi(t) \dot{e}_\phi(t))) \tag{3.25} \]

and in \( V_T(x, \vec{e}) \) we must change \( e_1, e_2 \) on \( e_{C,1}, e_{C,2} \), where

\[ e_{C,1} = e_r \cos \phi - re_\phi \sin \phi, \quad e_{C,2} = e_r \sin \phi + re_\phi \cos \phi. \tag{3.26} \]

The transformation looks quite classically but it can not be derived from naive coordinate transformation of initial path integral for amplitude: transformed representation for \( R(E) \) can not be written in the product form \( AA^* \) of two functional integrals, i.e. has not the factorization property because of the mixing of various quantum degrees of freedom when the transformation is performed.

We can introduce the motion in the phase space with Hamiltonian

\[ H_j = \frac{1}{2} p^2 + \frac{l^2}{2r^2} + v(r) - j_r r - j_\phi \phi \]

. The Dirac’s measure becomes four dimensional:

\[ D^{(4)} M(r, \phi, p, l) = \delta(E + \omega - H_T(r, \phi, p, l)) \prod_t dr d\phi dp dl \times \delta(\dot{r} - \frac{\partial H_j}{\partial p}) \delta(\dot{\phi} - \frac{\partial H_j}{\partial l}) \delta(\dot{p} + \frac{\partial H_j}{\partial r}) \delta(\dot{l} + \frac{\partial H_j}{\partial \phi}) \tag{3.27} \]

Note absence of the coefficient \( \prod_t r^2(t) \) in this expression. •

The above example allows to note following general property of considered formalism:

\[ S9. \text{Arbitrary transformation of measure } DM \text{ is ‘host-free’}. \]

Indeed, the transformation of measure

\[ DM(x) = \prod_t dx(t) \delta(\dot{x} + v'(x) - j) \tag{3.28} \]

can be performed inserting

\[ 1 = \int \prod_t dy(t) \delta(y - Y(x)). \]

In result,

\[ DM(y) = \prod_t dy(t) \delta(\dot{y} + Y'(X)X''(y)\dot{y}^2 + Y'(X)v'(X) - Y'(X)j) \tag{3.29} \]

since

\[ Y'(X)X'(y) \equiv 1 \tag{3.30} \]

considering \( X = X(y) \) as the inverse to \( Y(x) \) function. •
4 Topology properties

Let us consider motion in the action-angle phase space. Corresponding operator has the form:

$$\hat{K} = \frac{1}{2} \int_0^T dt dt' \Theta(t' - t) (\hat{I}(t) \hat{e}_I(t') + \hat{\phi}(t) \hat{e}_\phi(t')) \equiv \hat{K}_I + \hat{K}_\phi. \quad (4.1)$$

The result of integration using last $\delta$-function is

$$R(E) = 2\pi \int_0^\infty dT e^{-i\hat{K} \int_0^{2\pi} d\phi_0 \Omega(E) e^{-iV_T(x_c,e_c)}}, \quad (4.2)$$

where

$$\Omega = \frac{\partial h(I_0)}{\partial I_0}$$

with $I_0 = I_0(E)$ defined by algebraic equation:

$$E = h(I).$$

The classical trajectory

$$x_c(t) = x_c(I_0(E) + I(t) - I(T), \phi_0 + \tilde{\Omega}t + \phi(t)), \quad (4.3)$$

where

$$\tilde{\Omega} = \frac{1}{t} \int dt' g(t, t') \Omega(I_0 + I(t'))$$

. The interaction ‘potential’ $V_T$ depends from

$$e_c = e_\phi \frac{\partial x_c}{\partial I} - e_I \frac{\partial x_c}{\partial \phi}. \quad (4.4)$$

One can note that eq.(3.25) contains unnecessary contributions. Indeed, action of the operator

$$\int_0^T dt dt' \Theta(t' - t) \hat{e}_I(t') \hat{I}(t)$$

on $\tilde{H}(x_c; \tau)$, defined in (2.12), leads to the time integrals with zero integration range:

$$\int_0^T dt \Theta(T - t) \Theta(t - T) = 0.$$ 

This simplification was used in (4.1) and (4.2).

One can easily compute action of the operator $\exp(-i\hat{K})$ since $\hat{K}$ is linear over $\hat{e}_\phi$, $\hat{e}_I$.

The result can be written in the form:

$$R(E) = 2\pi \int_0^\infty dT \int_0^{2\pi} d\phi_0 \frac{\Omega(E)}{\Omega(E)} : e^{-iV_T(x_c,e_c/2i)} :,$$

where

$$\hat{e}_c = \hat{j}_\phi \frac{\partial x_c}{\partial I} - \hat{j}_I \frac{\partial x_c}{\partial \phi} \quad (4.6)$$
\[ \hat{j}_X(t) = \int_0^T dt' \theta(t - t') \hat{X}(t'), \quad X = \phi, I. \] (4.7)

The colons in (4.5) means ‘normal product’: in expansion over \( \hat{j}_X(t) \) the differential operators must stay to the left of functions.

Now we are ready to offer the important statement:

**S10.** Each term of perturbation theory in the invariant subspace can be represented as the total derivatives over one of the cotangent manifolds coordinate.

This statement directly follows from definition of perturbation generating operator \( \hat{K} \) on the cotangent bundle (3.23) and of translationally invariance of the cotangent manifold in the classical approximation.

By definition \( V_T \) is the odd over \( \hat{e}_c \) local functional:

\[ V_T(x_c, \hat{e}_c) = 2 \int_0^T \sum_{n=1}^{\infty} (\hat{e}_c(t)/2i)^{2n+1} v_n(x_c), \] (4.8)

where \( v_n(x_c) \) is some function of \( x_c \). Inserting (4.6) we find:

\[ : e^{-iV_T(x_c, \hat{e}_c)} : = \prod_{n=1}^{\infty} \prod_{k=0}^{2n+1} : e^{-iV_{k,n}(\hat{j}, x_c)} : , \] (4.9)

where

\[ V_{k,n}(\hat{j}, x_c) = \int_0^T dt (\hat{\phi}(t))^{2n-k+1}(\hat{j}_1(t))^k b_{k,n}(x_c). \] (4.10)

Explicit form of the function \( b_{k,n}(x_c) \) is not important.

Using definition (4.7) it easy to find:

\[ \hat{j}(t_1)b_{k,n}(x_c(t_2)) = \Theta(t_1 - t_2) \partial b_{k,n}(x_c)/\partial X_0 \]

since \( x_c = x_c(X(t) + X_0) \), see (4.3), or

\[ \hat{j}_{X,1}b_2 = \Theta_{12} \partial_{X_0} b_2 \] (4.11)

since indices \( (k, n) \) are not important.

Let as start consideration from the first term with \( k = 0 \). Then expanding \( \hat{V}_{0,n} \) we describe the angular quantum fluctuations only. Noting that \( \partial_{X_0} \) and \( \hat{j} \) commute we can consider lowest orders over \( \hat{j} \). The typical term of this expansion is (omitting index \( \phi \))

\[ \hat{j}_1 \hat{j}_2 \ldots \hat{j}_m b_1 b_2 \ldots b_m. \] (4.12)

It is enough to show that this quantity is the total derivative over \( \phi_0 \). The number \( m \) counts an order of perturbation, i.e. in \( m \)-th order we have \( (\hat{V}_{0,n})^m \).

\( m = 1 \). In this approximation we have, see (1.11),

\[ \hat{j}_1 b_1 = \Theta_{11} \partial_0 b_1 = \partial_0 b_1 \neq 0. \] (4.13)
Here the definition (3.12) was used.

$m = 2$. This order is less trivial:

\[
\hat{j}_1 \hat{j}_2 b_1 b_2 = \Theta_{21} b_1^2 b_2 + b_1^1 b_2^1 + \Theta_{12} b_1 b_2^2,
\]

(4.14)

where

\[
b^n_i \equiv \partial^n b_i.
\]

(4.15)

Deriving (4.14) the first equality in (3.15) was used. At first glance (4.14) is not the total derivative. But inserting

\[1 = \Theta_{12} + \Theta_{21},\]

(see the second equality in (3.15)) we can symmetrize it:

\[
\hat{j}_1 \hat{j}_2 b_1 b_2 = \Theta_{21} (b_2^2 b_1 + b_1^1 b_2^1) = \partial (\Theta_{21} b_1 b_2^2 + \Theta_{12} b_1^1 b_2) \equiv \partial (b_1 \to b_2 + b_2 \to b_1)
\]

(4.16)

since the explicit form of function $b$ is not important. So, the second order term can be reduced to the total derivative also. Note, that the contribution (4.16) contains the sum of all permutations. This shows the ‘time reversibility’ of the constructed perturbation theory.

$m = 3$. In this order we have:

\[
\hat{j}_1 \hat{j}_2 \hat{j}_3 b_1 b_2 b_3 = \partial_0 \left\{ \sum_{i \neq j \neq k=1}^3 (i^2 \to j \to k + i^1 \to j^1 \to k) \right\}
\]

(4.17)

since

\[
\Theta_{ij} \Theta_{jk} \Theta_{ki} = 0, \quad \Theta_{ij} \Theta_{jk} \Theta_{ik} = \Theta_{ij} \Theta_{jk}
\]

(4.18)

as the natural generalization of (3.15).

So, $m$-th order contribution is following total derivative:

\[
\hat{j}_1 \hat{j}_2 \cdots \hat{j}_m b_1 b_2 \cdots b_m = \partial_0 \left\{ \sum_{i_1 \neq i_2 \neq \ldots \neq i_m=1}^m (i_1^m \to i_2 \to i_3 \to \cdots \to i_m + i_1^1 \to i_2 \to i_3 \to \cdots \to i_m + \cdots + i_1^1 \to i_2 \to i_3 \to \cdots \to i_{m-1}^1 \to i_m) \right\}
\]

(4.19)

Let us consider now expansion over $\hat{\nu}_{k,m}$, $k \neq 0$. The typical term in this case is

\[
\hat{j}_1 \hat{j}_2 \cdots \hat{j}_l \hat{j}_{l+1} \hat{j}_{l+2} \cdots \hat{j}_m b_1 b_2 \cdots b_m, \quad 0 < l < m,
\]

(4.20)

where, for instance,

\[
\hat{j}_k^1 \equiv \hat{j}_l(t_k), \quad \hat{j}_k^2 \equiv \hat{j}_\phi(t_k)
\]

and

\[
\hat{j}_k^1 b_2 = \Theta_{12} \partial_0^b b_2
\]

(4.21)
instead of \((1.11)\).

\[ m = 2, \ l = 1. \] We have in this case:

\[
\hat{j}^2_1 b_1 b_2 = \Theta_{21} (b_2 \partial_0^1 \partial_0^1 b_1 + (\partial_0^1 b_2)(\partial_0^1 \partial_0^1 b_1)) + \Theta_{12} (b_1 \partial_0^1 \partial_0^1 b_2 + (\partial_0^1 b_2)(\partial_0^1 \partial_0^1 b_1)) = \\
= \partial_0^1 (\Theta_{21} b_2 \partial_0^1 b_1 + \Theta_{12} b_1 \partial_0^1 b_2) + \Theta_{12} b_1 \partial_0^1 (\Theta_{21} b_2 \partial_0^1 b_1 + \Theta_{12} b_1 \partial_0^1 b_2). \tag{4.22}
\]

Therefore, we have the total-derivative structure yet. This property is conserved in arbitrary order over \(m\) and \(l\) since the time-ordered structure does not depend from upper index of \(\hat{j}\), see \((4.21)\).

We can conclude, contributions are defined by boundary values of classical trajectory \(x_c\) in the invariant subspace since the integration over \(X_0\) is assumed, see \((4.2)\), and since contributions are the total derivatives over \(X_0\). One can say that the contributions are defined by topology (boundary) properties of invariant subspace. For instance, the new phenomena for quantum theories follows from \(S_{10}\):

\[ S_{11}. \] The quantum fluctuations of angular variables are canceled if the classical motion is periodic.

If \(x_c\) is the periodic function:

\[
x_c (I_0 (E) + I (t) - I (T), (\phi_0 + 2 \pi) + \tilde{\Omega} t + \phi (t)) = x_c (I_0 (E) + I (t) - I (T), \phi_0 + \tilde{\Omega} t + \phi (t)). \tag{4.23}
\]

this statement is elementary consequence of \(S_{10}\) and is the result of averaging over \(\phi_0\), see eq.\((4.2)\).

This cancelation mechanism can be used for the path-integral explanation of quantum-mechanical systems integrability phenomena. The quantum problem can be quasiclassical over the part of the degrees of freedom and quantum over another ones. The transformation to the action-angle variables maps the \(N\)-dimensional Lagrange problem on the \(2N\)-dimensional phase-space torus. If the winding number on this hypertorus is a constant (i.e. the topological charge is conserved) one can expect the same cancellations.

In the classical mechanics following approximated method of calculations is used \[11\].

The canonical equations of motion:

\[
\dot{I} = a (I, \phi), \quad \dot{\phi} = b (I, \phi) \tag{4.24}
\]

are changed by the averaged equations:

\[
\dot{J} = \frac{1}{2 \pi} \int_0^{2 \pi} d\phi a(J, \phi), \quad \dot{\phi} = b(J, \phi), \tag{4.25}
\]

This is possible if the periodic oscillations can be extracted from the systematic evolution.

In our case

\[
a (I, \phi) = j \partial x_c / \partial \phi, \quad b (I, \phi) = \Omega (I) - j \partial x_c / \partial I. \tag{4.26}
\]

Inserting this definitions into \((4.23)\) we find evidently wrong result since in this approximation the problem looks like pure quasiclassical for the case of periodic motion:

\[
\dot{J} = 0, \quad \dot{\phi} = \Omega (J). \tag{4.27}
\]

15
This shows that the procedure of extraction of the periodic oscillations from the systematic evolution is not trivial for quantum theories and this method should be used carefully in the quantum theories. (This approximation of dynamics is ‘good’ on the time intervals $\sim 1/|a|$, i.e. for higher energy levels.)

## 5 Conclusion

Some remarks would be useful in conclusion.

- **Splitting of quantum excitations.**
  In result of described splitting $j \rightarrow (j_h, j_{\theta})$ we obtain a possibility to count the quantum excitations of each classical degree of freedom independently. It is the truly Hamilton’s description. It allows distinguish ‘radial’ and ‘angular’ quantum excitations, i.e. allows to consider the quantum fluctuations of bundles parameters and coordinates on it independently.

- **Cancelation of quantum corrections.**
  Investigation of integrable quantum-mechanical problems (of the Poschle-Teller model, of the rigid rotator model) allows conclude that the reason of total integrability is cancelation of quantum corrections. In this sense the totally integrable quantum systems are trivial, quasiclassical by theirs nature.

  Note, the ‘level’ of integrability of the problem shifts position of singularities over the interaction constants. Indeed, for the nonintegrable case the singularities are located at the origin, e.g. [12]. In the semi-integrable case the singularities at origin are canceled [13] and the main (rightist) singularities are located at the finite negative values. And, at the end, the singularities of the integrable systems are located infinitely far from the origin.

  It is known that the case of totally integrable systems is very rear in the Nature. But our secondary result is the observation that a quantum problem can be integrable over part of the degrees of freedom. This fact would have important consequences in the field theory.

- **Mapping on the cotangent bundle solves the nonlinear waves quantization problem.**
  The canonical transformation to the inverse scattering problems data for sin-Gordon model is known (e.g. [3]). Half of them (solitons momenta) are the parameters of bundle and others (solitons coordinate) are coordinates on the bundle. The problem of solitons quantization would be solved by mapping of functional measure on the bundle (see also [2]). In result we get to $2N$-dimensional quantum-mechanical problem quantizing $N$-soliton configuration and all above offered statements becomes applicable for this field-theoretical problem also.

  In our terms the quantum sin-Gordon model is totally integrable since the topological charge is conserved, i.e. the winding number on the compact cotangent bundle is a constant. In result the quantum corrections are canceled and sin-Gordon model is pure quasiclassical. This fact for sin-Gordon model was noted firstly in [14]. Our prove of this result will be published later [15, 16].
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