Toric Geometry and String Theory Descriptions of Qudit Systems

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Abstract

In this paper, we propose a new way to approach qudit systems using toric geometry and related topics including the local mirror symmetry used in the string theory compactification. We refer to such systems as \((n,d)\) quantum systems where \(n\) and \(d\) denote the number of the qudits and the basis states respectively. Concretely, we first relate the \((n,d)\) quantum systems to the holomorphic sections of line bundles on \(n\) dimensional projective spaces \(\mathbb{CP}^n\) with degree \(n(d−1)\). These sections are in one-to-one correspondence with \(d^n\) integral points on a \(n\)-dimensional simplex. Then, we explore the local mirror map in the toric geometry language to establish a linkage between the \((n,d)\) quantum systems and type II D-branes placed at singularities of local Calabi-Yau manifolds. \((1,d)\) and \((2,d)\) are analyzed in some details and are found to be related to the mirror of the ALE space with the \(A_{d−1}\) singularity and a generalized conifold respectively.

Keywords: Qudits; toric geometry, local mirror symmetry; D-branes and string theory.
1 Introduction

Recently, many efforts have been devoted to study quantum computation and information theory using different approaches. These activities have brought new understanding of the fundamental physics associated with quantum theories \[1, 2, 3, 4\]. It is shown that the quantum bit or the qubit, which is based on a superposition of the two possible states, is considered as the most basic building block of such quantum theories. In fact, qubit and its generalizations are connected to many theories including string theory, D-branes, black holes, toric geometry and supermanifolds \[5, 6, 7, 8, 9\]. A particular emphasis put on the link with the black holes and branes obtained from supergravity models in high dimensions. More precisely, a nice interplay between the STU black holes having eight charges and three qubits have been proposed and developed in \[6, 7\].

More recently, many works have suggested extended models using different ways. In connection with supersymmetry, superqubits get developed using \(so(2|1)\) Lie superalgebra \[10, 11\]. On the other hand, the qudit, which is a \(d\)-level quantum system, emerge naturally in the extended models by considering more than two bosonic states on which the qubit is built. It is recalled that \(n\) qudits can be characterized by a couple \((n, d)\) where \(n\) and \(d\) indicate the number of the qudits and the states respectively. Here, we refer to these systems as \((n, d)\) quantum systems. Inspection shows that there have been some attempts to dealt with such quantum systems but unfortunately with partial results only.

The aim of the paper is to enrich these activities by proposing a new way to deal the qudits using toric geometry and its relation to type II D-branes placed at singularities of Calabi-Yau manifolds used in the string theory compactification. In particular, we relate the \((n, d)\) quantum systems to the holomorphic sections of the line bundles on \(n\) dimensional projective spaces \(\mathbb{CP}^n\) with degree \(n(d-1)\). Using toric geometry technics, these sections are in one-to-one correspondence with \(d^n\) integral points on \(n\)-dimensional simplex. To give an explicit analysis, a particular interest has been on lower dimensional examples. Then, we explore the local mirror map to elaborate a linkage between the \((n, d)\) quantum system and type II D-branes placed at toric singularities. \((1, d)\) and \((2, d)\) are analyzed in some details and are found to be related to the mirror of the ALE space with the \(A_{d-1}\) singularity and a generalized conifold respectively.

The present paper is organized as follows. In section 2, we give a short review on the qubit and qudit systems. Section 3 concerns a toric description of the \((1, d)\) quantum systems while the generalization to \((n, d)\) ones is discussed in section 4. In section 5, we propose a string theory interpretation in terms of type II D-branes placed at toric Calabi-Yau singularities using local mirror symmetry. Section 6 contains some including remarks.
2 Qudit quantum systems

Inspired by toric varieties and motivated by the existence of combinatorial calculations in quantum information, we use toric geometry to handle multiple qudit systems. Concretely, we elaborate a toric description in terms of holomorphic section of line bundles on the projective spaces $\mathbb{CP}^n$. Then, we propose a stringy interpretation using type II D-branes placed at mirror Calabi-Yau singularities used in the string theory compactification. To end, let us first recall the qubit which has been extensively studied using different physical and mathematical approaches [1,2,3,4].

2.1 Qubit quantum systems

The qubit is a two state system which can be realized, for instance, by the position of the electron in the hydrogen atom. The superposition state of a single qubit is generally given by the following Dirac notation

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle$$  \hspace{1cm} (2.1)

where $a_i$ are complex numbers satisfying the normalization condition

$$|a_0|^2 + |a_1|^2 = 1. \hspace{1cm} (2.2)$$

It is observed that this equation can be interpreted geometrically in terms of the so called Bloch sphere [1,2,3,4]. The analysis can be extended to more than one qubit which has been used to discuss entanglement states. In fact, the two qubits are four level systems. Using the usual notation $|ij\rangle = |i\rangle|j\rangle$, the corresponding state superposition can be expressed as

$$|\psi\rangle = a_{00}|00\rangle + a_{10}|10\rangle + a_{01}|01\rangle + a_{11}|11\rangle,$$  \hspace{1cm} (2.3)

where $a_{ij}$ are complex numbers verifying now the following normalization condition

$$|a_{00}|^2 + |a_{10}|^2 + |a_{01}|^2 + |a_{11}|^2 = 1. \hspace{1cm} (2.4)$$

$n$ qubits, in fact, are $2^n$ configuration states which can be represented by graphs sharing a strong resemblance with a particular class of Adinkras formed by $2^n$ nodes connected with $n$ colored lines [9]. This observation has been explored to show a new similarity between $n$ qubit systems and $2^n$ cycles embedded in $n$-dimensional torii $T^n$. In this way, a quantum state has been interpreted as the Poincaré dual of the real homology cycle in $T^n$ on which type II D-branes can wrap to generate black holes in the compactification of type IIA superstring on $T^n$. 

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2.2 Qudit systems

As in the case of qubits, the qudits represent, for instance, a physical system of spin $\frac{d-1}{2}$ particles with $d$ states. In the basis formed by the states $\{i\}$, $i = 0, 1, \ldots, d - 1$, the general state can be written as follows

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle + \ldots + a_{d-1}|d-1\rangle,$$

where $a_i$ are complex numbers satisfying now the extended normalization condition

$$\sum_{i=0}^{d-1} |a_i|^2 = 1.$$  

For the organization reason, one may be interested in characterizing the multiple of qudits. In fact, it should be evident to characterize them by a couple $(n, d)$ where $n$ and $d$ indicate the number of the qudits and the states respectively. Using a similar qubit notation, the general state can take the form

$$|\psi\rangle = \sum_{i_1 \ldots i_n = 0, 1, \ldots, d-1} a_{i_1 \ldots i_n} |i_1 \ldots i_n\rangle.$$  

In the handling of such physical quantities, it is remarked that there are some similarities between qudits and toric geometry. Here, thought, we will be concerned with them. Our main objective is to explore the toric geometry language, and related topics including string theory compactification, to deal with the $(n, d)$ quantum systems.

3 Toric geometry realization of a single qudit

In this section, we borrow the idea of toric geometry to discuss qudits and make contact with other physics including the compactification of higher dimensional theories. We thus are considering a basic correspondence by replacing states by holomorphic sections of line bundles on toric varieties. Before going ahead, we first start by giving some basic facts on toric geometry being one of the most useful mathematical tools used in superstrings, M and F-theories [12, 13, 14, 15]. Indeed, a $n$-dimensional toric variety $V^n$ is a complex manifold which can be represented by a toric graph known by polytope $\Delta(V^n)$ consisting of $n + r$ vertices $v_i$ in an $Z^n$ lattice satisfying

$$\sum_{i=0}^{n+r-1} q_i^a v_i = 0, \quad a = 1, \ldots, r$$

where $q_i^a$ are called Mori vectors. It is observed that $V^n$ can be fixed by $\{q_i^a, v_i\}$ toric data. It is interesting to note, in passing, that the local Calabi-Yau condition is satisfied by the following relation

$$\sum_{i=0}^{n+r-1} q_i^a = 0.$$  

Familiar examples of toric varieties, which have many applications in both mathematics and quantum physics, are the complex projective spaces $\mathbb{CP}^n$ [16, 17]. For $\mathbb{CP}^n$, $q_i^a$ reduce to a simple vector $q_i = (1, \ldots, 1)$ with $a = 1$ and the equation (3.1) becomes a simple relation given by

$$v_0 + \ldots + v_n = 0$$

(3.3)

forming a $n$-dimensional simplex which will be explored to discuss holomorphic sections of the line bundles on $\mathbb{CP}^n$.

Having introduced the mathematical backgrounds, we move now to present a toric description of a single qudit. For a sake of simplicity, we consider quantum trit or qutrit associated with the couple $(n, d) = (1, 3)$. It is a three level system which can be simulated by a physical system dealing with spin-1 particles. It can be also realized by the degree of freedom of a photon living in five dimensional space-time, which can be obtained from the string theory compactification on a 5-dimensional compact space. In this way, a superposition state is a vector of a three dimensional Hilbert space, given by

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle,$$

(3.4)

where $|i\rangle$, $(i = 0, 1, 2)$, denote its basis. $a_i$ are complex coefficients satisfying the normalization condition

$$|a_0|^2 + |a_1|^2 + |a_2|^2 = 1,$$

(3.5)

where $|a_i|^2$ is the probability of measuring the qutrit in the state $|i\rangle$. Equation (3.5) plays the same role as the Bloch sphere in the single qubit model associated with the $(n, d) = (1, 2)$ quantum system. To give a toric description of the qudit, the state $|i\rangle$ is replaced by a Laurent monomial $z^i$ as follows

$$|i\rangle \rightarrow z^i, \quad i = 1, 2, 3.$$  

(3.6)

We naturally requires that (3.4) takes the following form

$$|\psi\rangle \rightarrow P(z) = a_0 + a_1z + a_2z^2,$$

(3.7)

where $z$ will be interpreted as the complex coordinate of one dimensional projective space $\mathbb{CP}^1$. It recalled that $\mathbb{CP}^1$ is the simplest example in toric geometry which turns out to play a primordial role in the elaboration of our basic correspondence. It is considered a building block of higher dimensional toric manifolds explored in string theory compactification in particular in the blow up of ADE singularities of Calabi-Yau manifolds producing four dimensional quantum field models [12, 13, 14, 15]. It is known that $\mathbb{CP}^1$ has a $U(1)$ toric action having two fixed points $p_1$ and $p_2$ describing respectively north and south poles of the real two sphere, identified with $\mathbb{CP}^1$. The corresponding toric graph is one dimensional polytope which is just the 1-dimensional simplex identified with the interval $[p_1, p_2]$. Using Laurent polynomials, (3.7) can
be associated with holomorphic sections of line bundles on $\mathbb{CP}^1$. It is observed that such line bundles are associated with integral points on the interval $[17]$. In this way, the number of holomorphic sections of the bundle corresponds to the number of integral points on the interval. Indeed, let us assume that the interval $[p_1, p_2]$ goes from 0 to 2. The holomorphic sections can be identified now with the terms $z^i$, where $i = 0, 1, 2$. Each integral point, specified by $i$ on the interval $[0, 2]$, corresponds to a section of the line bundle. The state $|i\rangle$, appearing in the $(1, 3)$ quantum system, is associated then with such a holomorphic section.

It should be evident that this extends to the $(1, d)$ quantum system. In this case, the general state can be represented by the following polynomial

$$|\psi\rangle \rightarrow P(z) = a_0 + a_1 z + \ldots + a_{d-1} z^{d-1}. \quad (3.8)$$

This should be associated with a 1-dimensional simplex involving $d$ integral points. With this analogy, we are considering a similarity between holomorphic sections of line bundles on $\mathbb{CP}^1$ with degree $d - 1$ and $(1, d)$ quantum system. This correspondence can be illustrated in figure 1.

![Figure 1: Toric realization of the (1, d) quantum system.](image)

4 Multiple qudit systems

We can also discuss, from the perspective of toric geometry, the more general case associated with the $(n, d)$ quantum system. The analysis requires the incorporation of the $n$ dimensional projective space $\mathbb{CP}^n$ and holomorphic sections of line bundles on it. These sections will be controlled by $d^n$ integral points placed at a $n$-dimensional simplex representing the toric description of the line bundles on $\mathbb{CP}^n$ with degree $n(d - 1)$.

The general graphic representation is beyond the scope of the present work, though we will consider an explicit example corresponding to $(n, d) = (2, d)$. Then, we give some remarks on the general case. For $(n, d) = (2, d)$, (2.7) reduces to

$$|\psi\rangle = \sum_{i_1 i_2=0,1,\ldots,d-1} a_{i_1 i_2} |i_1 i_2\rangle. \quad (4.1)$$
Using the above basic correspondence, this state can be represented by the following polynomial with two complex variables $z_1$ and $z_2$

$$|\psi\rangle \rightarrow P(z_1, z_2) = \sum_{i_1, i_2 = 0,1, \ldots, d-1} a_{i_1 i_2} z_1^{i_1} z_2^{i_2}. \quad (4.2)$$

Here $z_1$ and $z_2$ can be considered as local complex variables defining $\mathbb{CP}^2$. It is worth noting that the number of the complex variables is exactly the number of qudits (here $n = 2$). Roughly speaking, $\mathbb{CP}^2$ is a complex two-dimensional toric manifold with an $U(1)^2$ toric action exhibiting three fixed points $p_1$, $p_2$, and $p_3$. The corresponding polytope is a 2-dimensional simplex belonging to the $Z^2$ square lattice. It is obtained from the intersection of three $\mathbb{CP}^1$ complex lines defining a triangle $(p_1, p_2, p_3)$ in toric geometry language. The holomorphic sections of the line bundles on $\mathbb{CP}^2$ are characterized by integral points on the triangle $(p_1, p_2, p_3)$.

To make contact with toric geometry, we should regard $(i_1, i_2)$ as integral points placed on a 2-dimensional simplex. Then, we assign them to a monomial $z_1^{i_1} z_2^{i_2}$. In toric geometry language, this monomial can be interpreted as a holomorphic section of the line bundles on $\mathbb{CP}^2$ [17]. Let us illustrate this model for $(n, d) = (2, 3)$. The corresponding general state of this qudit reads

$$|\psi\rangle = a_{00}|00\rangle + a_{10}|10\rangle + a_{01}|01\rangle + c_{11}|11\rangle + a_{20}|20\rangle + a_{02}|02\rangle + a_{21}|21\rangle + a_{12}|12\rangle + a_{22}|22\rangle \quad (4.3)$$

which can be replaced by the following Laurent polynomial

$$|\psi\rangle \rightarrow P(z_1, z_2) = \sum_{i_1, i_2 = 0,1,2} a_{i_1 i_2} z_1^{i_1} z_2^{i_2}. \quad (4.4)$$

The integers $i_1$ and $i_2$ should satisfy the following constrains

$$0 \leq i_1 \leq 2, \quad 0 \leq i_2 \leq 2, \quad 0 \leq i_1 + i_2 \leq 4, \quad (4.5)$$

producing holomorphic sections on the line bundles on $\mathbb{CP}^2$ associated with the $(2, 3)$ quantum system. They represent a collection of $3^2 = 9$ integral points in $Z^2$ given by

$$\{ (0, 0), (1, 0), (0, 1), (1, 1), (2, 0), (0, 2), (1, 2), (2, 1), (2, 2) \}. \quad (4.6)$$

belonging to a 2-dimensional simplex representing a toric realization of the line bundles on $\mathbb{CP}^2$ with degree 4, as illustrated in figure 2.

For a physical system with $d$ states, the number $i_1$ and $i_2$ have to be lower than a maximal value in terms of $d$. These numbers are seen to satisfy this condition

$$0 \leq i_1 \leq d - 1, \quad 0 \leq i_2 \leq d - 1 \quad (4.7)$$
with the extra constraint
\[ 0 \leq i_1 + i_2 \leq 2(d - 1). \] (4.8)

The pairs \((i_1, i_2)\) generate now a toric description of holomorphic sections of the line bundles \(\mathbb{CP}^2\) with degree \(2(d - 1)\) associated with the \((2, d)\) quantum system. It is represented by \(d^2\) integral points on a 2-dimensional simplex.

These results can be extended to higher orders of qudits. More precisely, we can do something similar for the more general case with \(n > 2\) corresponding to the \((n, d)\) quantum system using \(n\) local complex variables defining \(\mathbb{CP}^n\). The main point is to find all possible integral points of the \(n\)-dimensional lattice \(\mathbb{Z}^n\) corresponding to the monomials
\[ z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}. \] (4.9)

Quantum information requires that \((i_1, \ldots, i_n)\) should satisfy
\[ 0 \leq i_\ell \leq d - 1, \quad i_\ell = 1, \ldots, n \] (4.10)
together with the extra constraint
\[ 0 \leq \sum_{\ell=1}^{n} i_\ell \leq n(d - 1) \] (4.11)
producing holomorphic sections on the line bundles on a \(\mathbb{CP}^n\) with degree \(n(d - 1)\). These \(d^n\) integral points give a toric description of the \((n, d)\) quantum system on a \(n\)-dimensional simplex.
5 String theory interpretation

Toric geometry has been considered as a powerful tool to study mirror symmetry in the context of supersrtring theory compactification on global and local Calabi-Yau manifolds with singularities in the presence of type II D-branes [18, 19]. On the basis on these activities, we bridge qudits to brane configurations which are mirror to the singularities of local Calabi-Yau manifolds having toric representations. In fact, it has been suggested that toric geometry can encode information concerning D-brane configurations in type II superstrings [17]. This can be done by exploring the toric graph to encode the corresponding physical data. We will see that this may produce a new link between n qudits and D-branes moving on local Calabi-Yau geometries. Given a toric a toric manifold, the local mirror symmetry maps the toric data \( \{q_i^a, v_i\} \) to an algebraic equation describing the mirror geometry given by

\[
\sum_{i=0}^{n-1} a_i y_i = 0, \tag{5.1}
\]

where \( a_i \) are complex coefficients and \( y_i \) monomials satisfying the famous mirror constraint relations

\[
\prod_{i=0}^{n-1} y_i^{q_i^a} = 1, \quad a = 1, \ldots, r \tag{5.2}
\]

There are many ways to solve these constraint equations giving the type II local mirror geometries [18, 19]. These geometries have been extensively studied for bosonic and fermionic Calabi-Yau manifolds in connection with sigma model with four supercharges [20, 21].

In what follows, we will show that the \((n,d)\) quantum system can be related with the local mirror geometry equations. For the simplicity, we consider lower dimensional models but we expect that possible generalizations to higher dimensional cases could be done using a similar method. Indeed, we consider the case of \((n,d) = (1,d)\). In fact, this model can be related to the mirror symmetry of M-theory on ALE space with \(A_{d-1}\) singularity given by the following algebraic equation

\[
x_1 x_2 = x_3^d \tag{5.3}
\]

where \(x_1, x_2\) and \(x_3\) are complex variables [12]. This equation involves a singularity located at the \(C^3\) origin \(x_1 = x_2 = x_3 = 0\). The latter can be resolved in two ways either by deforming the complex structure of \((5.3)\) or varying its Kahler structure. These two deformations are equivalent due to the self mirror property of the ALE spaces considered as local versions of the K3 surfaces [12]. Using sigma model approach, it is shown that the Kahler deformation consists on replacing the singular point by a collection of \(\mathbb{CP}^1\) according to the Dynkin diagram of the \(A_{d-1}\) finite Lie algebra. This nice connection between toric geometry and Lie algebras will be explored to build a bridge between local mirror geometries and the \((n,d)\) quantum systems.
Roughly speaking, the above model is dual to the mirror of type IIA superstring with \( d \) D6-branes. Identifying \( q_i^a \), up some details, with the Cartan matrix of the finite Lie algebra \( A_{d-1} \), the mirror geometry equations of the ALE space with \( A_{d-1} \) singularity read as

\[
y_i y_{i+2} = y_{i+1}^2, \quad i = 0, \ldots, d - 1.
\]  

These equations can be solved by the following monomials

\[
y_i = z^i
\]

which can be identified with the holomorphic sections of the line bundles on \( \mathbb{CP}^1 \) as we have discussed previously. Using mirror equations (5.1), the polynomial representation of the \((1, d)\) quantum information system can be identified with the mirror geometry solution given by

\[
\sum_{i=0}^{d-1} a_i z^i = uv
\]

where \( u \) and \( v \) are auxiliary variables which have been introduced to recover the right dimension. It is worth noting that the quadratic term \( uv \) has no physical importance since it does not affect the moduli space of the mirror geometry. The relevant physical quantities are \( a_i \) specifying the D6-brane locations associated with \( d \) integral points on a 1-dimensional simplex as illustrated in figure 1. In this scenario, we associate the holomorphic sections of the line bundles on \( \mathbb{CP}^1 \) with D6-branes in type IIA superstring.

The second example we want to discuss is the \((2, d)\) quantum system. In this case, the \((i_1, i_2)\) can be associated with the following mirror geometry equation

\[
\sum_{i_1, i_2=0,1,...,d-1} a_{i_1i_2} z_1^{i_1} z_2^{i_2} = uv
\]

where \((i_1, i_2)\) have been explored to solve the mirror equations. A close inspection shows that this equation describes the local mirror geometry of a generalized conifold given by

\[
x_1 x_2 = x_3^d x_4^d,
\]

where \( x_1, x_2, x_3 \) and \( x_4 \) are complex variables. This geometry has been dealt with in connection with D-branes placed at toric singularities of local Calabi-Yau manifolds as developed in [22, 23]. Inspired by these activities, the \((2, d)\) quantum system can be related to the D3-branes probing such a geometry. Indeed, the complex parameters \( a_{i_1i_2} \) can be interpreted as the moduli space of the gauge theory living on the world volume of D3-branes. In this way, \((i_1, i_2)\) give the locations of the D3-branes as illustrated in figure 2. According to [22], this model can be T-dual with D4-branes wrapping \( S^1 \) and stretched between NS-branes placed at various points on \( S^1 \). We believe that this connection deserves more deeper study. We hope to come back to this issue in future.
6 Conclusion

In this paper, we have approached qudit systems using toric geometry language and its relation with type II D-branes placed at Calabi-Yau singularities explored in string theory compactification. We have refereed to these systems as \((n, d)\) quantum systems where \(n\) and \(d\) indicate the number of the qudits and the states respectively. Concretely, we have shown that the \((n, d)\) quantum systems can be related to the holomorphic sections of the line bundles on \(n\)-dimensional projective spaces \(\mathbb{CP}^n\). These sections are in one-to-one correspondence with \(d^n\) integral points on \(n\)-dimensional simplex associated with the toric realization of holomorphic section of line bundles with degree \(n(d−1)\). Using the local mirror map in the toric language, we have established a linkage between the \((n, d)\) quantum systems and type D-branes placed at singularities of local Calabi-Yau manifolds used in the type II superstring compactification. It has been found that \((1, d)\) and \((2, d)\) can be linked with the mirror of the ALE spaces with \(A_{d−1}\) singularity and a generalized conifold respectively.

We expect that our approach can be adaptable to a broad variety of geometries represented by non trivial polytopes going beyond the simplex geometry. We anticipate that other concepts used in quantum information could be discussed by implementing extra constraints on the integral points belonging on such polytopes.

We believe that the present study could be considered as a first steep for developing such concepts using toric geometry technics used in the string theory compactification. Concretely, we intend to discuss elsewhere gates and non separated states in future works.

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