GEOMETRIC STATISTICS OF FORD CIRCLES

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Abstract. We compute the distributions and moments of certain statistics of geometric quantities associated to Ford circles. Our methods to compute distributions use the equidistribution of periodic orbits of the BCZ map, while the methods to compute moments are based on analytic number theory.

1. Introduction and Main Results

Ford circles have been studied from many different perspectives, primarily number theory and hyperbolic geometry. We study some statistics associated to both the Euclidean and hyperbolic geometry of Ford circles: in particular distances between centers and angles. We use dynamical methods to compute limiting distributions, and analytic number theory to compute asymptotic behavior of moments for the hyperbolic distance between centers.

Below, we define Ford circles, describe the statistics that we study and state our main theorems. In §3 we prove our results on moments, and in §4 we prove our distribution results.

1.1. Ford Circles. The Ford circles $\mathcal{F}$ are the collection of circles in the upper half-plane $\mathbb{H}$ tangent to the real axis, consisting of the circles $C_{p/q}$ tangent at the point $(p/q,0)$ with diameter $1/q^2$ (here and in what follows we will assume $p$ and $q$ are relatively prime). These circles can be viewed as the image of the set $\{(x,y) \in \mathbb{H} : y \geq 1\}$ under fractional linear transformations with integer coefficients, that is, the modular group $SL(2,\mathbb{Z})$.

Given $\delta > 0$ and $I = [\alpha, \beta] \subseteq [0,1]$, we consider the subset of Ford circles with centers above the line $y = \delta$ and with $p/q \in I$, which we denote $\mathcal{F}_{I,\delta}$. Writing $Q = \lceil (2\delta^{-1/2}) \rceil$,
we have that the set of tangency points to the $x$-axis is $F(Q) \cap I$, where $F(Q)$ denotes the usual Farey fractions of order $Q$. We write

\[ F(Q) \cap I := \{ p_1/q_1 < p_2/q_2 \ldots < p_{N_I(Q)}/q_{N_I(Q)} \}, \]

where $N_I(Q)$ is the cardinality of the set (which grows quadratically in $Q$, so linearly in $\delta^{-1}$). We denote the circle based at $p_j/q_j$ by $C_j$.

1.2. Geometric Statistics. Given consecutive circles $C_j$ and $C_{j+1}$ in $F_{I,\delta}$, let

\[ x_j := \frac{q_j}{Q}, \quad y_j := \frac{q_{j+1}}{Q}. \]

For $1 \leq j \leq N_I(Q)$, we have $0 < x_j, y_j \leq 1$ and $x_j + y_j > 1$. Let

\[ \Omega := \{ (x, y) \in (0, 1]^2 : x + y > 1 \}, \]

so we have $(x_j, y_j) \in \Omega$ for $1 \leq j \leq N_I(Q)$. A geometric statistic $F$ for Ford circles is a measurable function $F : \Omega \to \mathbb{R}$. Given a geometric statistic, we define its limiting distribution $G_F(t)$ (if it exists) by the limiting proportion of Ford circles for which the statistic $F$ exceeds $t$, that is,

\[ G_F(t) := \lim_{\delta \to 0} \frac{\# \{ 1 \leq j \leq N_I(Q) : F(x_j, y_j) \geq t \}}{N_I(Q)}. \]

Similarly, we define the moment of $F$ by

\[ M_{F,I,\delta}(F) := \frac{1}{|I|} \sum_{j=1}^{N_I(Q)} F(x_j, y_j). \]  

(1.1)

We describe how various natural geometric properties of Ford circles can be studied using this framework.

1.2.1. Euclidean distance. Given consecutive circles $C_j$ and $C_{j+1}$ in $F_{I,\delta}$, we consider the Euclidean distance $d_j$ between their centers $O_j$ and $O_{j+1}$. Note that since consecutive circles are mutually tangent, the Euclidean geodesic connecting the centers of the circles passes through the point of tangency, and has length equal to the sum of the radii, that is,

\[ d_j := \frac{1}{2q_j^2} + \frac{1}{2q_{j+1}^2}. \]

Therefore, we have

\[ Q^2d_j = \frac{1}{2x_j^2} + \frac{1}{2y_j^2}. \]

Thus, by letting $F(x, y) = \frac{1}{2x^2} + \frac{1}{2y^2}$, we have expressed the Euclidean distance (appropriately normalized) as a geometric statistic.

In [4], the latter three authors computed averages of first and higher moments of $M_{E,I,\delta}$ for the Euclidean distance.
1.2.2. *Hyperbolic distance.* Given consecutive circles in $\mathcal{F}_{1,\delta}$ we let $\rho_j$ denote the hyperbolic distance between their centers $O_j$ and $O_{j+1}$, where we use the standard hyperbolic metric on $\mathbb{H}$ defined as

$$ds := \sqrt{dx^2 + dy^2}.$$ 

A direct computation shows that

$$\sinh \frac{\rho_j}{2} = \sqrt{\frac{q_j}{2q_{j+1}} + \frac{q_{j+1}}{2q_j} + \frac{x^2}{2} - \frac{y^2}{x}} = \frac{2x}{2x}.$$ 

Letting

$$H(x, y) = \frac{1}{2} \left( \frac{x}{y} + \frac{y}{x} \right),$$

we have expressed (sinh of half of) the hyperbolic distance as a geometric statistic.

1.2.3. *Angles.* Let $\theta_j$ be the angle between the line joining $O_j$ with $(p_j/q_j, 0)$ and the line joining $O_j$ with the point of tangency of the circles $C_j$ and $C_{j+1}$. In order to compute $\theta_j$, we complete the right angled triangle with vertices $O_j$, $O_{j+1}$ and the point lying on the line joining $O_j$ and $(p_j/q_j, 0)$. Then,

$$\theta_j = \arctan \frac{2x_j}{x_{j+1} - x_j} = \arctan \frac{2x_j}{x_{j+1} - x_j}.$$ 

Thus, the geometric statistic associated to $\theta_j$ is given by

$$\Theta(x, y) = \arctan \frac{2xy}{y^2 - x^2}.$$ 

1.3. *Distributions.* Our main theorem yields information on not only the distribution of individual geometric statistics, but also for the joint distribution of any finite family of shifted geometric statistics. For simplicity, we first state a theorem on individual statistics. Let $m$ denote the Lebesgue probability measure on $\Omega$.

**Theorem 1.1.** Let $F : \Omega \rightarrow \mathbb{R}^+$ be a geometric statistic. Then for any $I \subset [0, 1]$, the limiting distribution $G_F(t)$ exists, and is given by

$$G_F(t) = m \left( F^{-1}(t, \infty) \right).$$

1.3.1. *Joint distributions.* For our results on joint distributions, we will need the following notation. Let $T : \Omega \rightarrow \Omega$ be the BCZ map,

$$T(x, y) = \left( -x + \left\lfloor \frac{1 + x}{y} \right\rfloor y \right).$$ 

A crucial observation, due to Boca, Cobeli and one of the authors [3] is that

$$T(x_j, y_j) = (x_{j+1}, y_{j+1}).$$
Let \( F = \{ F_1, \ldots, F_k \} \) denote a finite collection of geometric statistics, and let \( n = (n_1, \ldots, n_k) \in \mathbb{Z}^k \) and \( t = (t_1, \ldots, t_k) \in \mathbb{R}^k \). Define \( G_{F,n}(t) \), the \((F,n)\)-limiting distribution for Ford circles, by the limit (if it exists),

\[
G_{F,n}(t) := \lim_{\delta \to 0} \frac{\#\{1 \leq j \leq N_I(Q) : F_i(x_{j+n_j}, y_{j+n_j}) \geq t_i, 1 \leq i \leq k\}}{N_I(Q)},
\]

where the subscripts are viewed modulo \( N_I(Q) \). We define

\[
(F \circ T)^{-n}(t_1, \ldots, t_k) := \bigcap_{i=1}^k (F_i \circ T^{n_i})^{-1}(t_i, \infty) .
\] (1.2)

Theorem 1.1 is a special \((k = 1, n = 0)\) case of the following.

**Theorem 1.2.** For any finite collection of geometric statistics \( F = \{ F_1, \ldots, F_k \} \), \( n = (n_1, \ldots, n_k) \in \mathbb{Z}^k \), \( t = (t_1, \ldots, t_k) \in \mathbb{R}^k \) the limiting distribution \( G_{F,n}(t) \) exists, and is given by

\[
G_{F,n}(t) = m \left( (F \circ T)^{-n}(t_1, \ldots, t_k) \right),
\]

where \((F \circ T)^{-n}\) is defined as in (1.2).

1.3.2. **Tails.** For the above statistics, one can also compute tail behavior of the limiting distributions, that is, the behavior of \( G_F(t) \) as \( t \to \infty \). We use the notation \( f(x) \sim g(x) \) as \( x \to \infty \) to mean that

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.
\]

We have the following result for the tail behaviors of limiting distributions for Euclidean and hyperbolic distances.

**Theorem 1.3.** For the geometric statistic \( F(x,y) = \frac{1}{2x^2} + \frac{1}{2y^2} \) representing the Euclidean distance, the limiting distribution has the following tail behavior

\[
G_F(t) \sim \frac{1}{t}.
\]

When \( F(x,y) = \frac{1}{2} \left( \frac{y}{x} + \frac{x}{y} \right) \), the tail of the hyperbolic distance limiting distribution is given by

\[
G_F(t) \sim \frac{1}{2t^2}.
\]

1.4. **Moments.** The following results concern the growth of the moments \( M_{E,I,\delta}, M_{H,I,\delta}, \) and \( M_{\theta,I,\delta} \).

**Theorem 1.4.** For any real number \( \delta > 0 \), and any interval \( I = [\alpha, \beta] \subseteq [0, 1] \), with \( \alpha, \beta \in \mathbb{Q} \),

\[
M_{E,I,\delta}(\mathcal{F}) = \frac{6}{\pi^2} \log Q + D_I + O_I \left( \frac{\log Q}{Q} \right),
\]

where \( M_{E,I,\delta}(\mathcal{F}) \) is defined as in (1.1), and \( D_I \) is a constant depending only on the interval \( I \).
Theorem 1.5. For any real number $\delta > 0$, and any interval $I = [\alpha, \beta] \subseteq [0, 1]$, 
\[
M_{H,I,\delta}(\mathcal{F}) = \frac{9}{2\pi^2}Q^2 + O_{I,\epsilon}(Q^{7/4+\epsilon}),
\]
where $M_{H,I,\delta}(\mathcal{F})$ is defined as in (1.1), $C_I$ is a constant depending only on the interval $I$, $\epsilon$ is any positive real number, and $Q$ is the integer part of $\frac{1}{\sqrt{2}\delta}$.

Theorem 1.6. For any real number $\delta > 0$, and any interval $I = [\alpha, \beta] \subseteq [0, 1]$, 
\[
M_{\theta,I,\delta}(\mathcal{F}) = \frac{12}{\pi}Q^2 + O_I(Q \log Q),
\]
where $M_{\theta,I,\delta}(\mathcal{F})$ is defined as in (1.1), and $Q$ is the integer part of $\frac{1}{\sqrt{2}\delta}$.

2. Moments for euclidean distance

In this section, we prove Theorem 1.4. Recall that the euclidean distance $d_j$ between the centers of two consecutive Ford circles $C_j$ and $C_{j+1}$ is given by 
\[
d_j = \frac{1}{2q_j^2} + \frac{1}{2q_{j+1}^2}.
\]
This gives 
\[
M_{E,I,\delta}(\mathcal{F}) = \frac{1}{|I|} \sum_{j=1}^{N_I(Q)-1} \left( \frac{1}{2q_j^2} + \frac{1}{2q_{j+1}^2} \right).
\]

Proof of Theorem 1.4.

\[
|M_{E,I,\delta}(\mathcal{F})| = \sum_{j=2}^{N_I(Q)} \frac{1}{q_j^2} + \frac{1}{2q_j^2} - \frac{1}{2q_{N_I(Q)}^2} = \frac{1}{2q_1^2} - \sum_{1 \leq q \leq Q} \frac{1}{q^2} \sum_{\alpha q < a \leq \beta q} \mu(d) \sum_{d|a,q} \\
= \frac{1}{2q_1^2} - \sum_{1 \leq q \leq Q} \frac{1}{q^2} \sum_{\alpha q < a \leq \beta q} \mu(d) \sum_{d|a,q} \\
= \frac{1}{2q_1^2} - \sum_{1 \leq q \leq Q} \frac{1}{q^2} \left( \sum_{d|q} \mu(d) \frac{(\beta - \alpha)q}{d} - \sum_{d|q} \mu(d) \left( \left\{ \frac{\beta q}{d} \right\} - \left\{ \frac{\alpha q}{d} \right\} \right) \right) \\
= |I| \sum_{q \leq Q} \frac{\phi(q)}{q^2} - \sum_{d \leq Q} \frac{\mu(d)}{d^2} \sum_{m \leq \frac{Q}{d}} \left( \frac{\{\beta m\} - \{\alpha m\}}{m^2} \right) + \left( \frac{1}{2q_1^2} - \frac{1}{2q_{N_I(Q)}^2} \right). 
\]
Note that for large $Q$, since $\alpha, \beta \in \mathbb{Q}$, one can assume that $\alpha$ and $\beta$ are Farey fractions of order $Q$. Therefore, the quantity \( \frac{1}{2q_1^2} - \frac{1}{2q_{N_1}(Q)} \) is a constant. Now,

\[
\sum_{m \leq \frac{Q}{4}} \left( \frac{\{bm\} - \{am\}}{m^2} \right) = \sum_{m=1}^{\infty} \left( \frac{\{bm\} - \{am\}}{m^2} \right) - \sum_{m > \frac{Q}{4}} \left( \frac{\{bm\} - \{am\}}{m^2} \right) = C_I + O_I \left( \frac{d}{Q} \right).
\]

This gives

\[
\sum_{d \leq Q} \frac{\mu(d)}{d^2} \left( C_I + O_I \left( \frac{d}{Q} \right) \right) = \frac{C_I}{\zeta(2)} + O_I \left( \frac{\log Q}{Q} \right).
\]

Also, from [1, p. 71],

\[
\sum_{q \leq Q} \frac{\phi(q)}{q^2} = \frac{6}{\pi^2} \log Q + \frac{6\gamma}{\pi^2} - A + O_I \left( \frac{\log Q}{Q} \right),
\]

where $\gamma$ is Euler constant, $A = \sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n^2}$. Combining these together in (2.1), we obtain

\[
M_{E,I,\delta}(\mathcal{F}) = \frac{6 \log Q}{\pi^2} + D_I + O_I \left( \frac{\log Q}{Q} \right).
\]

This completes the proof of Theorem 1.4. \hfill \Box

3. Moments for hyperbolic distance and angles

3.1. Hyperbolic distance. In this section, we prove Theorem 1.5. Recall that $\rho_j$ denotes the hyperbolic distance between consecutive centers of Ford circles $O_j$ and $O_{j+1}$, and

\[
\sinh \frac{\rho_j}{2} = \frac{q_j}{2q_{j+1}} + \frac{q_{j+1}}{2q_j}.
\]

In order to compute the total sum of the distances $\sinh \frac{\rho_j}{2}$,

\[
M_{H,I,\delta}(\mathcal{F}) = \frac{1}{|I|} \sum_j \sinh \frac{\rho_j}{2} = \frac{1}{|I|} \sum_{q,q' \text{ neighbours in } \mathcal{F}_I(Q)} \left( \frac{q}{2q'} + \frac{q'}{2q} \right),
\]

we use the following key analytic lemma.

**Lemma 3.1.** [3, Lemma 2.3] Suppose that $0 < a < b$ are two real numbers, and $q$ is a positive integer. Let $f$ be a piecewise $C^1$ function on $[a, b]$. Then

\[
\sum_{a < k \leq b} \frac{\phi(k)}{k} f(k) = \frac{6}{\pi^2} \int_a^b f(x) \, dx + O_I \left( \log b \left( ||f||_\infty + \int_a^b |f'(x)| \, dx \right) \right).
\]
We also use the Abel summation formula,

$$
\sum_{x<n \leq y} a(n)f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(t)f'(t) \, dt,
$$  \hspace{1cm} (3.3)

where $a : \mathbb{N} \to \mathbb{C}$ is an arithmetic function, $0 < x < y$ are real numbers, $f : [x, y] \to \mathbb{C}$ is a function with continuous derivative on $[x, y]$, and $A(t) := \sum_{1 \leq n \leq t} a(n)$.

**Proof of Theorem 1.5.** Let $\tilde{q}'$ denote the unique multiplicative inverse of $q'$ modulo $q$ in the interval $[1, q]$. Define

$$
a_q(q') = \begin{cases} 
1 & \text{if } (q, q') = 1 \text{ and } \tilde{q}' \in [q\alpha, q\beta], \\
0 & \text{otherwise}, 
\end{cases}
$$

Using Weil type estimates ([5], [9], [14]) for Kloosterman sums, the partial sum of $a_q(q')$ was estimated in [3]. More precisely, by [3, Lemma 1.7], for any $\epsilon > 0$,

$$
A_q(t) := \sum_{1 \leq q' \leq t} a_q(q') = \frac{\phi(q)}{q}(t - 1)|I| + O_{I, \epsilon} \left( q^{1/2+\epsilon} \right). \hspace{1cm} (3.4)
$$

In order to estimate the sum in (3.2), we write it as

$$
2|I|M_{H, I, \delta}(\mathcal{F}) = \sum_{q, q' \text{ neighbours in } \mathcal{F}_I(Q)} \left( \frac{q}{q'} + \frac{q'}{q} \right) + \sum_{q, q' \text{ neighbours in } \mathcal{F}_I(Q)} \left( \frac{q}{q'} + \frac{q'}{q} \right). \hspace{1cm} (3.5)
$$

Note that it suffices to estimate the first sum above because of the symmetry in $q$ and $q'$. And for the first sum, one has

$$
\sum_{q, q' \text{ neighbours in } \mathcal{F}_I(Q)} \left( \frac{q}{q'} + \frac{q'}{q} \right) = \sum_{Q/2 < q \leq Q-L \atop Q-k q \leq \leq} a_q(q') \left( \frac{q}{q'} + \frac{q'}{q} \right) + \sum_{Q-L < q \leq Q \atop L < q' \leq} a_q(q') \left( \frac{q}{q'} + \frac{q'}{q} \right)
$$

$$
+ \sum_{Q-L < q \leq Q \atop Q-k q \leq \leq} a_q(q') \left( \frac{q}{q'} + \frac{q'}{q} \right)
$$

$$
=: S_1 + S_2 + S_3. \hspace{1cm} (3.6)
$$

We begin with estimating the sum $S_1$.

$$
S_1 = \sum_{Q/2 < q \leq Q-L \atop Q-k q \leq \leq} \sum_{q, q' \text{ neighbours in } \mathcal{F}_I(Q)} a_q(q') \left( \frac{q}{q'} + \frac{q'}{q} \right) =: \sum_{Q/2 < q \leq Q-L} \Sigma_1.
$$
Employing (3.4), (3.3) and the fact that \( q > Q - q \), for the inner sum \( \Sigma_1 \) in \( S_1 \), we have

\[
\Sigma_1 = A_q(q) \left( \frac{q}{q} + \frac{q}{q} \right) - A_q(Q - q) \left( \frac{Q - q}{q} + \frac{q}{Q - q} \right) - \int_{Q - q}^{q} A_q(t) \left( \frac{1}{q} - \frac{q}{t^2} \right) \, dt
\]

\[
= 2 \left( \frac{\phi(q)}{q} (q - 1)|I| + O_{I, \epsilon} \left( q^{1/2 + \epsilon} \right) \right) - \left( \frac{\phi(q)}{q} (Q - q - 1)|I| + O_{I, \epsilon} \left( q^{1/2 + \epsilon} \right) \right)
\]

\[
\left( \frac{Q - q}{q} + \frac{q}{Q - q} \right) - \int_{Q - q}^{q} \left( \frac{\phi(q)}{q} (t - 1)|I| + O_{I, \epsilon} \left( q^{1/2 + \epsilon} \right) \right) \left( \frac{1}{q} - \frac{q}{t^2} \right) \, dt
\]

\[
= |I| \frac{\phi(q)}{q} \left( 2(q - 1) - (Q - q - 1) \left( \frac{Q - q}{q} + \frac{q}{Q - q} \right) \right) - \frac{q}{Q - q} - q \log(Q - q) + q \log Q + O_{I, \epsilon} \left( \frac{q^{3/2 + \epsilon}}{Q - q} \right)
\]

\[
= -\frac{Q^2}{2q} - q \log(Q - q) + q \log q + Q + O_{I, \epsilon} \left( \frac{q^{3/2 + \epsilon}}{Q - q} \right)
\]

(3.7)

Therefore,

\[
S_1 = |I| \sum_{\frac{q}{2} < q \leq Q - L} \frac{\phi(q)}{q} g(q) + O_{I, \epsilon} \left( \frac{Q^{5/2 + \epsilon}}{L} \right),
\]

where \( g(q) := -\frac{Q^2}{2q} - q \log(Q - q) + q \log q + Q \). Also, from (3.7) and Lemma 3.1, we obtain

\[
S_1 = \frac{6|I|}{\pi^2} \int_{Q/2}^{Q - L} g(x) \, dx + O_{I, \epsilon} \left( \log Q \left( \|g\|_{\infty} + \int_{Q/2}^{Q - L} |g'(x)| \, dx \right) \right) + O_{I, \epsilon} \left( \frac{Q^{5/2 + \epsilon}}{L} \right).
\]

(3.8)

And,

\[
\int_{Q/2}^{Q - L} g(q) = \frac{Q^2 - (Q - L)^2}{2} \log \left( \frac{L}{Q - L} \right) + \frac{3(Q - L)Q}{2} - \frac{3Q^2}{4};
\]

(3.9)

\[
\|g\|_{\infty} = O_I (Q \log Q) = \int_{Q/2}^{Q - L} |g'(x)| \, dx.
\]

(3.10)

Therefore, from (3.8), (3.9), (3.10) and Lemma 3.1

\[
S_1 = \frac{6|I|}{\pi^2} \left( \frac{Q^2 - (Q - L)^2}{2} \log \left( \frac{L}{Q - L} \right) + \frac{3(Q - L)Q}{2} - \frac{3Q^2}{4} \right) + O_{I, \epsilon} \left( \frac{Q^{5/2 + \epsilon}}{L} \right)
\]

(3.11)

+ O_I (Q \log Q).
Next, we estimate the sum $S_2$ in (3.10). Recall that since $q' \leq q$, we have
\[
S_2 = \sum_{Q-L<q\leq Q} \sum_{L<q'\leq q} a_q(q') \left( \frac{q}{q'} + \frac{q'}{q} \right) = O_I \left( \sum_{Q-L<q\leq Q} \sum_{L<q'\leq q} \frac{q}{q'} \right)
\]
\[
= O_I \left( \sum_{1 \leq q' \leq L} \sum_{Q-q'\leq L} \frac{q}{q'} \right) = O_I (QL \log Q).
\]
Lastly,
\[
S_3 = \sum_{Q-L<q\leq Q} \sum_{L<q'\leq q} a_q(q') \left( \frac{q}{q'} + \frac{q'}{q} \right)
\]
\[
= O_I \left( \sum_{1 \leq q' \leq L} \sum_{Q-q'\leq L} \frac{q}{q'} \right) = O_I (QL),
\]
Setting $L = Q^{3/4}$, and from (3.11), (3.13), (3.14), and (3.6), we obtain the first moment for hyperbolic distances as
\[
M_{H,I,\delta}(F) = \frac{9}{2\pi^2} Q^2 + O_I (Q^{7/4+\epsilon}).
\]

3.2. Angles. First we consider the case when $q_{j+1}$ is smaller than $q_j$. Let $\theta_j$ denote the angle for the circle $C_j$. Then, the sum of the angles for both circles is given by $\theta_j + \frac{\pi}{2} - \theta_{j+1} + \frac{\pi}{2} = \pi$. Similarly, in the case $q_j < q_{j+1}$, the sum of the angles is $\pi$. This gives
\[
M_{\theta,I,\delta}(F) = \frac{1}{|I|} \sum_{j=1}^{N_I(Q)} (\theta_j + \theta_{j+1}) = \frac{2\pi}{|I|} N_I(Q) = \frac{12}{\pi} Q^2 + O_I (\log Q).
\]

4. Dynamical Methods

In this section, we prove Theorems 1.1 and 1.2. They are both consequences of the equidistribution of a certain family of measures on the region
\[
\Omega = \{(x,y) \in (0,1]^2 : x+y > 1\}.
\]
Recall that $(x_j, y_j) = \left( \frac{q_j}{Q}, \frac{q_j+1}{Q} \right) \in \Omega$. Now let
\[
\rho_{Q,I} := \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_j,y_j)}
\]
be the probability measure supported on the set $\{(x_j, y_j) : 1 \leq j \leq N\}$. We have [2, Theorem 1.3] (see also [10], [11]):
Theorem 4.1. The measures $\rho_{Q,I}$ equidistribute as $Q \to \infty$. That is,

$$\lim_{Q \to \infty} \rho_{Q,I} = m,$$

where $dm = 2dx dy$ is the Lebesgue probability measure on $\Omega$ and the convergence is in the weak-* topology.

4.1. Distribution results. To prove Theorems 1.1 and 1.2, we write (with notation as in §1.3)

$$\frac{\# \{1 \leq j \leq N_I(Q) : F(x_j, y_j) \geq t \}}{N_I(Q)} = \rho_{Q,I} \left( F^{-1}(t, \infty) \right)$$

$$\frac{\# \{1 \leq j \leq N_I(Q) : F_i(x_{j+n_i}, y_{j+n_i}) \geq t_i, 1 \leq i \leq k \}}{N_I(Q)} = \rho_{Q,I} \left( (F \circ T)^{-n}(t_1, \ldots, t_k) \right)$$

Now the results follow from applying Theorem 4.1 to the above expressions. □

4.1.1. Shrinking intervals. Versions of Theorems 1.1 and 1.2 for shrinking intervals $I_Q = [\alpha_Q, \beta_Q]$ where the difference $\beta_Q - \alpha_Q$ is permitted to tend to zero with $Q$ can be obtained by replacing Theorem 4.1 with appropriate discretizations of results of Hejhal [8] and Strömbergsson [13], see [2, Remark 1].

5. Computations of the Tail Behavior of Geometric Statistics

In this section, we compute the tails of our geometric statistics (§1.3.2).

5.0.2. Euclidean distance. For $F(x, y) = \frac{1}{2x^2} + \frac{1}{2y^2}$, we have

$$G_F(t) = m(F^{-1}(t, \infty)) = m \left( \left\{ (x, y) \in \Omega, \frac{1}{x^2} + \frac{1}{y^2} \geq 2t \right\} \right).$$

Define

$$B_t := \left\{ (x, y) \in \Omega, \frac{1}{x^2} + \frac{1}{y^2} \geq 2t \right\}.$$

Therefore,

$$G_F(t) = \int_{B_t} dm = 2 \int \int_{B_t} dx dy.$$

Note that if $y < \frac{1}{\sqrt{2t}}$, the inequality $\frac{1}{x^2} + \frac{1}{y^2} \geq 2t$ automatically holds true. The region bounded by lines $0 < y < 1/\sqrt{2t}$ and $1 - y < x < 1$ is exactly an isosceles right triangle with area $\frac{1}{4t}$. Let us assume in what follows that $y > \frac{1}{\sqrt{2t}}$. Then,

$$x \leq \frac{1}{\sqrt{2t - \frac{1}{y^2}}}.$$
This yields

\[ 1 - y < x < \min \left\{ 1, \frac{1}{\sqrt{2t - \frac{1}{y^2}}} \right\}. \]

Therefore, \( 1 - y < \frac{1}{\sqrt{2t - \frac{1}{y^2}}} \), which implies either

\[
\alpha_1 := \frac{t - \sqrt{t^2 + 2t + 2t(\sqrt{1 + 2t})}}{2t} < y < \alpha_2 := \frac{t - \sqrt{t^2 + 2t - 2t(\sqrt{1 + 2t})}}{2t},
\]

or

\[
\alpha_3 := \frac{t + \sqrt{t^2 + 2t - 2t(\sqrt{1 + 2t})}}{2t} < y < \alpha_4 := \frac{t + \sqrt{t^2 + 2t + 2t(\sqrt{1 + 2t})}}{2t}.
\]

Also,

\[
\alpha_1 < 0 < \frac{1}{\sqrt{2t}} < \frac{1}{\sqrt{2t - 1}} < \alpha_2 < \alpha_3 < 1 < \alpha_4. \tag{5.1}
\]

Hence a point \((x, y)\) with \( y > 1/\sqrt{2t} \) belongs to \( B_t \) if and only if

\[ 1 - y < x < \min \left\{ 1, \frac{1}{\sqrt{2t - \frac{1}{y^2}}} \right\} \quad \text{and} \quad y \in \left( \frac{1}{\sqrt{2t}}, \alpha_2 \right) \cup (\alpha_3, 1). \tag{5.2} \]

Now for the measure of the set \( B_t \) minus the isosceles right triangle already discussed above, we consider the following two cases.

**Case I.** \( \min \left\{ 1, \frac{1}{\sqrt{2t - \frac{1}{y^2}}} \right\} = 1 \) i.e. \( y \leq \frac{1}{\sqrt{2t - 1}} \). Employing (5.1) and (5.2), we have that \( 1 - y < x < 1 \) and \( \frac{1}{\sqrt{2t}} < y \leq \frac{1}{\sqrt{2t - 1}} \).

**Case II.** \( \min \left\{ 1, \frac{1}{\sqrt{2t - \frac{1}{y^2}}} \right\} = \frac{1}{\sqrt{2t - \frac{1}{y^2}}} \) i.e. \( y \geq \frac{1}{\sqrt{2t - 1}} \). Then, using (5.1) and (5.2), we have that \( 1 - y < x < \frac{1}{\sqrt{2t - \frac{1}{y^2}}} \) and either \( \frac{1}{\sqrt{2t - 1}} < y < \alpha_2 \) or \( \alpha_3 < y < 1 \).
Combining the two cases above and adding the contribution from the isosceles right triangle, we obtain

\[
G_F(t) = \frac{1}{2t} + 2 \int_{\sqrt{2t}}^1 \int_{1-y}^1 \ dx \ dy + 2 \int_{\sqrt{2t}}^1 \int_{1-y}^1 \frac{1}{\sqrt{2t}} \ dx \ dy + 2 \int_{\alpha_3}^1 \int_{1-y}^1 \ dx \ dy
\]

\[
= \frac{4t - 1}{\sqrt{2t - 1}} - 1 - \frac{1}{t} \sqrt{\frac{1}{2t - 1}} + \frac{1}{t} \sqrt{t \left( t - 2\sqrt{2t + 1} + 2 \right)}
\]

\[
+ \frac{1}{t} \sqrt{t - \sqrt{2t + 1} - \sqrt{t \left( t - 2\sqrt{2t + 1} + 2 \right)}} - \frac{1}{t} \sqrt{t - \sqrt{2t + 1} + \sqrt{t \left( t - 2\sqrt{2t + 1} + 2 \right)}}.
\]

(5.3)

This gives

\[
G_F(t) \sim \frac{1}{t}.
\]

5.0.3. Hyperbolic distance. In this case \( F(x, y) = \frac{1}{2} \left( \frac{y}{x} + \frac{x}{y} \right) \) and

\[
G_F(t) = m(C_t), \text{ where } C_t := \left\{ (x, y) \in \Omega, \frac{1}{2} \left( \frac{y}{x} + \frac{x}{y} \right) \geq t \right\}.
\]

\[
\frac{1}{2} \left( \frac{y}{x} + \frac{x}{y} \right) \geq t \Rightarrow \text{ either } x \geq y(t + \sqrt{t^2 - 1}) \text{ or } x \leq y(t - \sqrt{t^2 - 1})
\]

Since \( x, y \in \Omega \), we note that either

\[
1 - y < x < \min \left\{ 1, y(t - \sqrt{t^2 - 1}) \right\}
\]

or

\[
\max \left\{ 1 - y, y(t + \sqrt{t^2 - 1}) \right\} < x < 1.
\]

Depending on the bounds of \( x \) and \( y \), we have the following cases:

*Case I.* \( 1 - y < x < \min \left\{ 1, y(t - \sqrt{t^2 - 1}) \right\} \).

As \( y \) and \( t - \sqrt{t^2 - 1} = \frac{1}{t+\sqrt{t^2-1}} < 1 \), we note that \( y(t - \sqrt{t^2 - 1}) < 1 \) and therefore

\[
\min \left\{ 1, y(t - \sqrt{t^2 - 1}) \right\} = y(t - \sqrt{t^2 - 1}).
\]

This implies \( y > \frac{1}{t+\sqrt{t^2-1}} \). Also, \( y(t - \sqrt{t^2 - 1}) < 1 \) implies \( y < (t - \sqrt{t^2 - 1})^{-1} = t + \sqrt{t^2 - 1} \), which always holds since \( y < 1 \) and \( t \gg 1 \). And so in this case the range for \( x \) and \( y \) is given by,

\[
\frac{1}{t + 1 - \sqrt{t^2 - 1}} < y < 1 \text{ and } 1 - y < x < y(t - \sqrt{t^2 - 1}).
\]

(5.4)
Case II. \( \max \{1 - y, y(t + \sqrt{t^2 - 1})\} < x < 1 \). Here, we consider two cases.

**Sub case I.** \( \max \{1 - y, y(t + \sqrt{t^2 - 1})\} = 1 - y \). Then

\[
y < \frac{1}{t + \sqrt{t^2 - 1} + 1}
\]

and the range for \( x \) and \( y \) is given by

\[
0 < y < \frac{1}{t + \sqrt{t^2 - 1} + 1} \quad \text{and} \quad 1 - y < x < 1.
\]

(5.5)

**Sub Case II.** When \( \max \{1 - y, y(t + \sqrt{t^2 - 1})\} = y(t + \sqrt{t^2 - 1}) \). This implies

\[
y > \frac{1}{t + \sqrt{t^2 - 1} + 1} \quad \text{and} \quad y(t + \sqrt{t^2 - 1}) < 1,
\]

and so we get,

\[
\frac{1}{t + \sqrt{t^2 - 1} + 1} < y < \frac{1}{(t + \sqrt{t^2 - 1})} \quad \text{and} \quad y(t + \sqrt{t^2 - 1}) < x < 1.
\]

(5.6)

Combining all the above cases, from (5.4), (5.5) and (5.6), we have

\[
G_F(t) = 2 \int_1^{\frac{1}{t+\sqrt{t^2-1}+1}} \int_{1-y}^{y(t+\sqrt{t^2-1})} dx \, dy + 2 \int_0^{\frac{1}{t+\sqrt{t^2-1}+1}} \int_{1-y}^{1} dx \, dy
\]

\[
+ 2 \int_{\frac{1}{t+\sqrt{t^2-1}+1}}^{\frac{1}{t+\sqrt{t^2-1}}} \int_{y(t+\sqrt{t^2-1})}^{1} dx \, dy
\]

\[
= \frac{1}{(t + \sqrt{t^2 - 1}) (t + 1 + \sqrt{t^2 - 1})} + \frac{1}{(t + \sqrt{t^2 - 1} + 1)^2}
\]

\[
+ \frac{1}{(t + \sqrt{t^2 - 1}) (t + \sqrt{t^2 - 1} + 1)^2}
\]

\[
= \frac{2}{(t + \sqrt{t^2 - 1}) (t + \sqrt{t^2 - 1} + 1)}.
\]

Thus,

\[
G_F(t) \sim \frac{1}{2t^2}.
\]

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