On supersaturation and stability for generalized Turán problems

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Abstract
Fix a graph $F$. We say that a graph is $F$-free if it contains no copy of $F$ as a subgraph. Let $\text{ex}(n, H, F)$ denote the maximum number of copies of a graph $H$ in an $n$-vertex $F$-free graph. In this note, we will give a new general supersaturation result for $\text{ex}(n, H, F)$ in the case when $\chi(H) < \chi(F)$ as well as a new proof of a stability theorem for $\text{ex}(n, K_r, F)$.

KEYWORDS
stability, supersaturation, Turán number

1 | INTRODUCTION

Let $F$ be a graph. A graph $G$ is $F$-free if it contains no copy of $F$ as a subgraph. Denote the maximum number of copies of a graph $H$ in an $n$-vertex $F$-free graph by

$$\text{ex}(n, H, F).$$

After several sporadic results (a famous example is $\text{ex}(n, C_5, K_3)$; see eg, [13, 14, 16, 17]), the systematic study of the function $\text{ex}(n, H, F)$ was initiated by Alon and Shikhelman [1]. An overview of results on $\text{ex}(n, H, F)$ can be found in [1] and [12].

We denote the number of copies of a graph $H$ in a graph $G$ by $\Lambda(H, G)$. Recall that the Turán graph $T_{k-1}(n)$ is the $n$-vertex complete $(k - 1)$-partite graph with classes of size as close as possible (ie, classes differ by at most one vertex). Zykov’s “symmetrization” proof [25] of Turán’s theorem [23] gives the following generalization.

Theorem 1 (Zykov [25]). The Turán graph $T_{k-1}(n)$ is the unique $n$-vertex $K_k$-free graph with the maximum number of copies of $K_r$. Thus,
where \( \binom{n}{k-1} \) is a binomial coefficient.

Theorem 1 has been rediscovered and reproved several times (see eg, [1, 2, 6]). The Turán graph \( T_{k-1}(n) \) contains no \( k \)-chromatic graph \( F \), so we always have the trivial lower bound

\[
\mathcal{N}(H, T_{k-1}(n)) \leq \text{ex}(n, H, F).
\]

Erdős-Stone-type generalizations of Theorem 1 were given in [1] and [12]. We state them as a single theorem below.

**Theorem 2** (Alon-Shikhelman [1] and Gerbner-Palmer [12]). Let \( H \) be a graph and let \( F \) be a graph with chromatic number \( k \), then

\[
\text{ex}(n, H, F) \leq \text{ex}(n, H, K_k) + o(n^{V(H)}).
\]

Thus, when \( H = K_r \),

\[
\text{ex}(n, K_r, F) = \left( \binom{k-1}{r} \right) \left( \frac{n}{k-1} \right)^r + o(n^r).
\]

Note that the first part of Theorem 2 only gives a useful upper-bound if \( \text{ex}(n, H, K_k) = \Omega(n^{V(H)}) \), which happens if and only if \( K_k \) is not a subgraph of \( H \).

An important description of the degenerate case is given by Alon and Shikhelman [1]. The blow-up \( G[t] \) of a graph \( G \) is the graph resulting from replacing each vertex of \( G \) with \( t \) copies of itself.

**Proposition 3** (Alon-Shikhelman [1]). The function \( \text{ex}(n, H, F) = o(n^{V(H)}) \) if and only if \( F \) is a subgraph of a blow-up of \( H \). Otherwise, \( \text{ex}(n, H, F) = \Omega(n^{V(H)}) \).

The purpose of this paper is to establish a general supersaturation result and give a new proof of a stability theorem for the function \( \text{ex}(n, H, F) \). Both of the main results are proved using modifications of standard proofs of supersaturation and stability for the ordinary Turán function \( \text{ex}(n, F) \). Previous supersaturation results were given by Cutler et al [4] who (among other things) proved a structural supersaturation for Theorem 1 as well as a supersaturation result for the case when \( H \) is a complete graph and \( F \) is a star. Our first result is a supersaturation theorem for \( \text{ex}(n, H, F) \) in the case when \( \chi(H) < \chi(F) \).

**Theorem 4.** Fix graphs \( H \) and \( F \) on \( h \) and \( f \) vertices, respectively, such that \( \chi(H) < \chi(F) \). For \( c > 0 \), there exists \( c' > 0 \) such that if \( G \) is an \( n \)-vertex graph with

\[
\mathcal{N}(H, G) > \text{ex}(n, H, F) + cn^h,
\]

then \( \mathcal{N}(F, G) \geq c'n^f \).

Through a standard argument we can reprove Theorem 2 via Theorem 4. The previous proofs of Theorem 2 both employ the regularity lemma and as a result our argument holds for a lower threshold on \( n \). This and the proof of Theorem 4 will be given in Section 2.
A general stability theorem for \( \text{ex}(n, K_r, F) \) was given by Ma and Qiu [19]. A second proof is
due to Liu [18]. We give a new short proof of this theorem by making use of the standard
stability theorem. This will be discussed in Section 3.

**Theorem 5** (Ma-Qiu [19]). Fix positive integers \( k > r \) and let \( F \) be a graph with \( \chi(F) = k \).
If \( G \) is an \( n \)-vertex \( F \)-free graph with
\[
N(K_r, G) > \text{ex}(n, K_r, F) - o(n^r),
\]
then \( G \) can be obtained from \( T_{k-1}(n) \) by adding and removing \( o(n^2) \) edges.

**Notation**: Notation is standard and follows the monograph of Bollobás [3]. For a graph \( G \), let \( V(G) \) and \( E(G) \) denote the vertex set and edge set, respectively. Denote the number of edges of
\( G \) by \( e(G) = |E(G)| \). For a vertex \( x \), the neighborhood is \( N(x) \) and the degree is \( d(x) \).

### 2  SUPERSATURATION

We begin with a modification of a result of Katona et al [11].

**Lemma 6.** Let \( H \) and \( F \) be graphs. Then
\[
\frac{\text{ex}(n, H, F)}{\left(\frac{n}{|V(H)|}\right)}.
\]
is monotone decreasing as \( n \) increases.

**Proof.** Suppose \( G \) is an \( n \)-vertex \( F \)-free graph with the maximum number of copies of \( H \).
We double-count the pair \((H, v)\), where \( H \) is a copy of the graph \( H \) in \( G \) and \( v \) is a vertex
not in the copy \( H \). We can fix \( H \) in \( \text{ex}(n, H, F) \) ways and then choose \( v \) in \( n - |V(H)| \)
ways. On the other hand, there are \( n \) ways to fix \( v \) and on the remaining \( n - 1 \) vertices
there are at most \( \text{ex}(n - 1, H, F) \) copies of \( H \). Thus,
\[
(n - |V(H)|) \cdot \text{ex}(n, H, F) \leq n \cdot \text{ex}(n - 1, H, F).
\]
Solving for \( \text{ex}(n, H, F) \) and dividing both sides by \( \left(\frac{n}{|V(H)|}\right) \) gives
\[
\frac{\text{ex}(n, H, F)}{\left(\frac{n}{|V(H)|}\right)} \leq \frac{n}{n - |V(H)|} \cdot \frac{\text{ex}(n - 1, H, F)}{\left(\frac{n}{|V(H)|}\right)} = \frac{\text{ex}(n - 1, H, F)}{\left(\frac{n - 1}{|V(H)|}\right)}.
\]

Observe that
\[
\frac{\text{ex}(n, H, F)}{{n \choose |V(H)|}}
\]

is bounded below by a constant (e.g., it is always nonnegative). Therefore, Lemma 6 implies that the limit

\[
\pi(H, F) = \lim_{n \to \infty} \frac{\text{ex}(n, H, F)}{{n \choose |V(H)|}}
\]

exists.

We are now ready to prove supersaturation in the generalized setting. The argument is essentially the same as an averaging argument used to prove supersaturation for hypergraphs (see [10, 8]).

*Proof of Theorem 4.* Fix graphs \( H \) and \( F \) on \( h \) and \( f \) vertices, respectively such that \( \chi(H) < \chi(F) \). Let

\[
q = \pi(H, F) = \lim_{n \to \infty} \frac{\text{ex}(n, H, F)}{{n \choose h}}.
\]

Fix \( c > 0 \) and let \( m \) be a constant such that

\[
\text{ex}(m, H, F) \leq \left( q + \frac{c}{2} \right) {m \choose h}.
\]

Note that the statement of the theorem holds for \( n \) at most a constant \( m \), so let \( n > m \) and suppose that \( G \) is an \( n \)-vertex graph with

\[
\mathcal{N}(H, G) > \text{ex}(n, H, F) + cn^h \geq \left( q + c \right) {n \choose h}.
\]

Assume (for the sake of a contradiction) that there are less than \( \frac{c}{2 \cdot h!} {n \choose m} \) sets of \( m \) vertices each spanning more than \( \left( q + \frac{c}{2} \right) {m \choose h} \) copies of \( H \). Note that among \( m \) vertices there are at most \( {m \choose h} h! \) distinct copies of \( H \). Therefore,

\[
\sum_{S \in {V(G) \choose m}} \mathcal{N}(H, S) < \frac{c}{2 \cdot h!} {n \choose m} {m \choose h} h! + {n \choose m} \left( q + \frac{c}{2} \right) {m \choose h} = \left( q + c \right) {n \choose m} {m \choose h}.
\]

On the other hand, each copy of \( H \) in \( G \) is contained in \( {n-h \choose m-h} \) vertex sets of size \( m \), so
Combining these two estimates for $\sum \mathcal{N}(H, S)$ gives a contradiction. Therefore, there are at least \( \frac{c}{2 \cdot h!} \binom{n}{m} \) sets of \( m \) vertices each spanning more than \( \binom{n-f}{m-f} \) times in this way. Therefore, the number of copies of \( F \) in \( G \) is

\[
\mathcal{N}(F, G) \geq \frac{c}{2 \cdot h!} \binom{n}{m} \left( \binom{n-f}{m-f} \right)^{-1} \geq c'n^f
\]

for \( c' \) small enough.

We now give a new proof of Theorem 2.

**Proof of Theorem 2.** Let \( H \) be an \( h \)-vertex graph and let \( F \) be a graph with chromatic number \( k \). Fix \( c > 0 \) and suppose \( G \) is an \( n \)-vertex \( F \)-free graph with

\[
\mathcal{N}(H, G) = \text{ex}(n, H, K_k) + cn^h.
\]

Then Theorem 4 implies that \( G \) contains at least \( c'n^k \) copies of \( K_k \). Proposition 3 gives \( \text{ex}(n, K_k, K_k[t]) = o(n^k) \), where \( K_k[t] \) is a blow-up of \( K_k \). Therefore, \( G \) contains a copy of \( K_k[t] \). For \( t \) large enough, \( K_k[t] \) contains a copy of the \( k \)-chromatic graph \( F \). Thus

\[
\text{ex}(n, H, F) \leq \text{ex}(n, H, K_k) + o(n^h).
\]

\[ \square \]

## 3 | STABILITY

We begin with two lemmas. The first will demonstrate that a \( K_k \)-free graph with nearly the maximum number of copies of \( H \) contains a large subgraph in which every vertex is contained in many copies of \( H \). The lemma is based on an argument due to Norin [21] that is adjusted to the subgraph counting context.

**Lemma 7.** Fix positive integers \( k > r \). For \( \alpha > 0 \), there exists \( \beta > 0 \) and \( n_0 > 0 \) such that every \( K_k \)-free graph \( G \) with \( |V(G)| \geq n_0 \) and

\[
\mathcal{N}(H, G) > (1 - \beta) \pi(H, K_k) \frac{|V(G)|^r}{r!}
\]

contains either:

1. a subgraph \( G' \) with \( |V(G')| > (1 - \alpha)|V(G)| \) such that every vertex of \( G' \) is contained in more than
copies of $H$, or

(2) a subgraph $G'$ with $|V(G')| = |(1 - \alpha)|V(G)|$ and $N(H, G') > \pi(H, K_k)\frac{|V(G')|^r}{r!}$.

Proof. Choose $\beta$ so that $(1 - \beta)^2 > 1 - \frac{\alpha}{2}$ and $\beta < \frac{\alpha^2}{2}$. Choose $n_0$ so that $n' \geq (n - 1)^r + (1 - \beta)rn'^{-1}$ for all $n \geq (1 - \alpha)n_0$. If every vertex of $G$ belongs to more than $(1 - \alpha)\pi(H, K_k)\frac{|V(G)|^{r-1}}{(r-1)!}$ copies of $H$, then we are done. If not, delete a vertex contained in the minimum number of copies of $H$ to obtain a subgraph $G_1$ of $G$. Repeat this procedure to obtain subgraphs $G_1, G_2, G_3$, and so forth. If we reach a graph $G'$ that satisfies the lemma, then we are done. Therefore, suppose we have reached a graph $G_m$ such that $m = \lceil \alpha n \rceil$. We shall prove by induction on $\ell$ that

\[
N(H, G_{\ell}) > \left(1 - \frac{m - \ell}{m} \beta\right)\pi(H, K_k)\frac{|V(G_{\ell})|^r}{r!}.
\]

for $\ell \leq m$.

The base case $\ell = 0$ (for $G_0 = G$) follows from the hypotheses of the lemma. So put $0 < \ell \leq m$ and assume (1) holds for $\ell - 1$. For ease of notation put $n' = |V(G_{\ell-1})| = |V(G_{\ell})| + 1$. Now, by the induction hypothesis for $G_{\ell-1}$ and the choice of $n_0$ and $\beta$, we have

\[
\frac{N(H, G_{\ell})}{\pi(H, K_k)} \geq \frac{N(H, G_{\ell-1})}{\pi(H, K_k)} - (1 - \alpha)\frac{(n')^{r-1}}{(r-1)!} \\
\geq \left(1 - \frac{m - \ell + 1}{m} \beta\right)\frac{(n')^r}{r!} - (1 - \alpha)\frac{(n')^{r-1}}{(r-1)!} \\
\geq \left(1 - \frac{m - \ell + 1}{m} \beta\right)\frac{(n' - 1)^r}{r!} + (1 - \beta)\frac{(n')^{r-1}}{(r-1)!} - \frac{\alpha}{an} \frac{(n')^{r-1}}{2(r-1)!} \\
\geq \left(1 - \frac{m - \ell}{m} \beta\right)\frac{(n' - 1)^r}{r!} + \frac{\alpha}{2r} \frac{(n')^{r-1}}{(r-1)!} \\
> \left(1 - \frac{m - \ell}{m} \beta\right)\frac{|V(G_{\ell})|^r}{r!} \\
= \left(1 - \frac{m - \ell}{m} \beta\right)\frac{|V(G_{\ell})|^r}{r!}.
\]

Multiplying through by $\pi(H, K_k)$ proves (1). When $m = \ell$ the inequality (1) gives

\[
N(H, G_m) > \pi(H, K_k)\frac{|V(G_m)|^r}{r!}
\]

which completes the proof of the lemma.
Our second lemma gives a lower bound on vertex degrees in $K_k$-free graphs with many copies of $K_r$.

**Lemma 8.** Let $G$ be an $n$-vertex $K_k$-free graph and $x \in V(G)$. If

$$N(K_{r-1}, N(x)) \geq (1 - \alpha) r \left( \frac{k - 1}{r} \right) \left( \frac{1}{k - 1} \right)^{r-1},$$

then

$$d(x) \geq (1 - \alpha)^{1/(r-1)} \frac{k}{k - 1} \left( \frac{k - 2}{k - 1} \right) n - (k - 3).$$

**Proof.** The neighborhood $N(x)$ is $K_{k-1}$-free as $G$ is $K_k$-free. Therefore, by Theorem 1 we have

$$N(K_{r-1}, N(x)) \leq \text{ex}(|N(x)|, K_{r-1}, K_{k-1}) \leq \left( \frac{k - 2}{r - 1} \right) \frac{d(x)}{k - 2} \left( \frac{k - 2}{k - 1} \right)^{r-1}.$$

Combining the above estimate for $N(K_{r-1}, N(x))$ with (2) and solving for $d(x)$ completes the proof. 

Lemma 8 implies that if each vertex of $G$ is contained in at least $\text{ex}(n, K_r, K_k) \frac{r^2}{2} - o(n^{r-1})$ copies of $K_r$, then $e(G) \geq (1 - \frac{1}{k-1}) \frac{n^2}{2} - o(n^2)$. Finally, we will need a standard stability result for edges (see eg, [7, 22]).

**Theorem 9** (Erdős-Simonovits stability theorem [7, 22]). Let $G$ be an $n$-vertex $F$-free graph with

$$e(G) > \left( 1 - \frac{1}{\chi(F) - 1} \right) \frac{n^2}{2} - o(n^2).$$

Then $G$ can be obtained from the Turán graph $T_{\chi(F)-1}(n)$ by adding and removing $o(n^2)$ edges.

**Proof of Theorem 5.** Fix integers $k > r$ and let $F$ be a graph with $\chi(F) = k$. Let $G$ be an $n$-vertex $F$-free graph with

$$N(K_r, G) > \text{ex}(n, K_r, F) - o(n^r).$$

As $G$ is $F$-free with $\chi(F) = k$, a removal lemma due to Erdős et al [9] asserts that $G$ can be made $K_k$-free with the removal of $o(n^2)$ edges\(^1\). Removing $o(n^2)$ edges destroys at most

\(^1\)Note that when $F = K_k$ we may skip the use of the removal lemma.
o(n^2) \cdot n^{-2} = o(n') \text{ copies of } K_r. \text{ Let } G' \text{ be the resulting } K_k\text{-free subgraph of } G. \text{ Now}

\mathcal{N}(K_r, G') > \text{ex}(n, K_r, F) - o(n') \geq \text{ex}(n, K_r, K_k) - o(n').

Let us apply Lemma 7 to \( G' \). Observe that by Theorem 1 the second outcome of Lemma 7 is impossible here as it would imply that \( G' \) contains a subgraph \( K_k \). Therefore, the first outcome gives that \( G' \) contains a subgraph \( G'' \) on \( n', (1 - \alpha)n \) vertices such that each vertex of \( G'' \) is contained in \( n' - o(n') \text{ copies of } K_r \). Applying Lemma 8 to \( G'' \) gives that every degree in \( G'' \) is at least \( \left( 1 - \frac{1}{k-1} \right) n - o(n) \) and therefore

\[ e(G'') \geq \left( 1 - \frac{1}{k-1} \right) \frac{n^2}{2} - o(n^2). \]

As \( e(G) - e(G'') = o(n^2) \) we may apply Theorem 9 to \( G'' \) to complete the proof. \( \square \)

4 CONCLUDING REMARKS

As remarked in Section 2, the proof of Theorem 4 is similar to that of a proof of supersaturation for hypergraphs. However, hypergraph supersaturation does not immediately imply Theorem 4. Indeed, the obvious approach is to begin with a graph \( G \) with more than \( nHF + |V(H)| \) copies of \( H \) and construct a hypergraph \( \mathcal{G} \) on the vertex set of \( G \) where each copy of \( H \) is a hyperedge in \( \mathcal{G} \). Now it is possible to apply hypergraph supersaturation to \( \mathcal{G} \) to give many copies of some “forbidden hypergraph” \( F \) in \( G \). Unfortunately, it is not clear that we can choose an appropriate \( F \) to guarantee many copies of \( F \) in \( G \). However, in the case when \( H = K_r \), the conclusion of Theorem 4 can be derived from a theorem of Mubayi and Verstraëte [20] on the hypergraph extremal number of the expansion \( F^+ \) of \( F \) and a supersaturation theorem for hypergraphs (see Keevash [15]).

In this paper, we did not address possible supersaturation theorems in the degenerate case, that is, when \( \chi(H) \leq \chi(F) \). In this case, it is not always clear what the appropriate supersaturation statements should be. However, there are specific graphs \( H \) and \( F \) in the degenerate case where \( \text{ex}(n, H, F) \) is well-understood—this would be a good starting point. For example, the function \( \text{ex}(n, K_r, S_{k+1}) \) where \( S_{k+1} \) is the star on \( k + 1 \) vertices (i.e., the complete bipartite graph \( K_{1,k} \)) was described in [5, 24]. Later, [4] gave a supersaturation theorem for \( \text{ex}(n, K_r, S_{k+1}) \).

The proofs of the Stability Theorem (Theorem 5) appear only to work when the graph \( H \) that we are counting is a complete graph \( K_r \). It would be interesting to find a stability theorem from graphs \( H \) other than \( K_r \). However, when \( H \) is not a complete graph, the “extremal graph” for \( \text{ex}(n, H, F) \) is not always a Turán graph. For example, the maximum for \( \text{ex}(n, C_5, K_3) \) is achieved by the blow-up of a \( C_5 \) and the maximum for \( \text{ex}(n, K_r, S_{k+1}) \) is achieved by disjoint copies of the complete graph \( K_k \).

ACKNOWLEDGMENTS

We would like to thank the anonymous referees for their careful reading of the manuscript and for pointing out the alternate approach for supersaturation for \( H = K_r \) via hypergraph supersaturation.
DATA AVAILABILITY STATEMENT
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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How to cite this article: Halfpap A, Palmer C. On supersaturation and stability for generalized Turán problems. J Graph Theory. 2021;97:232–240.
https://doi.org/10.1002/jgt.22652