Familywise error rate (FWER) has been one of the most prominent frequentist approaches in simultaneous inference for decades, and stepwise procedures represent the most successful and ingenious attack on FWER control. A recent article proved that the FWER for the Bonferroni method asymptotically (i.e., when the number of hypotheses goes to infinity) goes to zero under any positively equicorrelated multivariate normal distribution. However, similar results for the limiting behaviors of FWER of general stepwise procedures are nonexistent. The present work addresses this problem by studying the asymptotic behavior of the FWER of step-down and step-up procedures under equicorrelated and general normality. Specifically, we show that the FWER of any step-down procedure (e.g., Holm’s method) goes to zero asymptotically for a broad class of correlated normal distributions. We also establish similar results on limiting FWER for other commonly used multiple testing procedures.

1. Introduction. Large-scale multiple testing problems arising in various disciplines often analyze related variables simultaneously. For example, in microRNA expression data, several genes may cluster into groups through their transcription processes and exhibit high correlation. In public health studies, the observed data from different time periods and locations are generally serially or spatially correlated. In spatial data with close geographical locations, the test statistics corresponding to different hypotheses are often influenced by each other. Multistage clinical trials and functional magnetic resonance imaging studies also concern variables involving complex and unknown dependence structures. Consequently, the study of the correlation effect of dependent test statistics in large-scale simultaneous inference problems has attracted considerable attention recently.

Benjamini and Yekutieli [5] proved that the Benjamini-Hochberg procedure controls the FDR at the desired level under positive regression dependency. Storey and Tibshirani [24] proposed methodologies of estimating FDR when the test statistics are dependent, with applications to gene expression data. Multiple testing procedures under dependence have been discussed by Sun and Cai [25], Efron [11], Liu, Zhang and Page [21] among others. Efron [12] mentions that the correlation penalty on the summary statistics depends on the root mean square (RMS) of the correlations. Efron [13] contains an excellent review of the relevant literature. Fan et al. [14] proposed a method of dealing with correlated test statistics with known covariance structure. Their method tackles the association between correlated statistics using the principal eigenvalues of the covariance matrix. Fan and Han [15]
extended this work when the underlying dependence structure is unknown. Qiu et al. [22] showed that dependence among test statistics significantly affects the power of many FDR controlling procedures.

There is relatively few literature on the performance of FWER controlling procedures under dependence. Das and Bhandari [7] have shown that under the equicorrelated multivariate normal setup, FWER(\(\rho\)) of Bonferroni procedure is a convex function in correlation \(\rho\) as the number of hypotheses grows to infinity. Consequently, they prove that the FWER of the Bonferroni procedure is bounded by \(\alpha(1 - \rho)\) where \(\alpha\) is the desired level. Dey and Bhandari [8] have improved this result by showing that the Bonferroni FWER asymptotically goes to zero for any positive equicorrelation. They have also extended this to the arbitrarily correlated setup where the limiting infimum of the correlations is strictly positive. Dey [9] has obtained upper bounds on FWER of the Bonferroni method in the equicorrelated and arbitrarily correlated setups with small and moderate dimensions. Finner and Roters [16] derived explicit formulas for the distribution of the number of Type I errors (i.e false rejections) in single-step, step-down and step-up procedures based on independent \(p\)-values. However, theoretical results concerning the limiting behavior of the FWER for stepwise procedures based on dependent test statistics do not seem to exist in the literature.

The present work addresses this problem by establishing results on the limiting FWER values of general step-down procedures under the correlated normal setup. These results provide some general insights into the behavior of step-down multiple comparison procedures. While we have not been able to derive corresponding limits for general step-up procedures, we have obtained the limiting performance for some of them.

The rest of this paper is organized as follows. In Section 2 we first introduce some notation, set up the framework formally and summarize some results on the limiting behavior of the Bonferroni procedure. Section 3 is concerned with theoretical results about the limiting behavior of the FWER of step-down procedures in equicorrelated and arbitrarily correlated normal setups. Section 4 presents similar results on Hochberg’s procedure. Hommel’s stepwise procedure is studied in Section 5. We conclude with a brief discussion in Section 6.

2. Preliminaries.

2.1. Testing Framework. We consider a Gaussian sequence model ([7], [8]):

\[ X_i \sim \mathcal{N}(\mu_i, 1), \quad i = 1, \ldots, n, \]

where \(X_i\)'s are correlated. We consider unit variance since the variance of \(X_i\) is often assumed to be known in the literature on the asymptotic properties of multiple testing procedures (see, e.g., [1], [6], [7], [8], [10]).

Here we are interested in the following multiple testing problem:

\[ H_{0i} : \mu_i = 0 \quad 1 \leq i \leq n. \]

The global null \(H_0 = \bigcap_{i=1}^n H_{0i}\) asserts that all means \(\mu_i = 0\), while under the alternative, some \(\mu_i\) is strictly positive. Throughout this work, \(\alpha \in (0, 1)\) denotes the desired level of FWER control and \(\Phi(\cdot)\) denotes the cumulative density function of the standard normal distribution.

Let \(V_n(T)\) be the number of false rejections of a multiple testing procedure (MTP henceforth) \(T\). The FWER of procedure \(T\) under the global null, is defined as

\[ FWER_T = \mathbb{P}_{H_0} (V_n(T) \geq 1). \]

The present work studies the limiting behaviors of \(FWER_T\) for \(T\) belonging to a broad class of MTPs under two dependent setups:
1. The Equicorrelated setup:
   \[ \text{Corr} \left( X_i, X_j \right) = \rho \quad \forall i \neq j \quad (\rho \geq 0). \]

2. The Arbitrarily Correlated setup:
   \[ \text{Corr} \left( X_i, X_j \right) = \rho_{ij} \quad \forall i \neq j \quad (\rho_{ij} \geq 0). \]

Although the first setup is a special case of the second one, we are considering them separately since the proofs of the results in the general case are based on the corresponding results in the equicorrelated case. It is also mention-worthy that most of the existing literature on the performance of FWER procedures under dependence consider the equicorrelated case. However, many scientific disciplines involve variables with more complex dependence structure (e.g., multistage clinical trials and functional magnetic resonance imaging studies). These general dependence scenarios need to be addressed with more general correlation matrices. Thus, the study of the limiting behavior of FWER controlling procedures in arbitrarily correlated normal setups becomes crucial.

Throughout this work, \( M_n(\rho) \) denotes the \( n \times n \) matrix with each diagonal entry equal to 1 and each off-diagonal entry equal to \( \rho \). Also, \( \Sigma_n \) denotes the \( n \times n \) correlation matrix with \((i, j)\)’th entry equal to \( \rho_{ij} \).

2.2. The Bonferroni Procedure. The classic Bonferroni procedure is the best-known and one of the most frequently used MTP for controlling FWER. This single-step method sets a same cut-off for each of the \( n \) hypotheses. In the one-sided setting, it rejects \( H_0 \) if \( X_i > \Phi^{-1}(1 - \alpha/n) (= c_{\alpha,n}, \text{say}) \). In other words, the FWER of Bonferroni’s method under the equicorrelated setup is given by

\[
FWER_{\text{Bon}}(n, \alpha, \rho) = \mathbb{P}(X_i > c_{\alpha,n} \text{ for some } i) = \mathbb{P}\left( \bigcup_{i=1}^{n} \{X_i > c_{\alpha,n}\} \right).
\]

Das and Bhandari [7] consider the above equicorrelated normal framework and establish the following:

**Theorem 2.1.** Given any \( \alpha \in (0, 1) \) and \( \rho \in [0, 1] \), \( FWER_{\text{Bon}}(n, \alpha, \rho) \) is asymptotically bounded by \( \alpha(1 - \rho) \) under the global null hypothesis.

Dey and Bhandari [8] improve this result as follows:

**Theorem 2.2.** Given any \( \alpha \in (0, 1) \) and \( \rho \in [0, 1] \), \( \lim_{n \to \infty} FWER_{\text{Bon}}(n, \alpha, \rho) = 0. \)

The proofs of Theorem 2.1 and 2.2 exploit an well known result on equicorrelated multivariate normal variables with equal marginal variances. The sequence \( \{X_i\}_{i \geq 1} \) is exchangeable under \( H_0 \) under the equicorrelated normal set-up. In other words,

\[
(X_{i_1}, \ldots, X_{i_k}) \sim N_k \left( 0_k, (1 - \rho)I_k + \rho J_k \right)
\]

where \( J_k \) is the \( k \times k \) matrix of all ones. Thus, for each \( i \geq 1 \), \( X_i = \theta + Z_i \) where \( \theta \) has a normal distribution with mean 0, independent of \( \{Z_n\}_{n \geq 1} \) and \( Z_i \)’s are i.i.d normal random variables. \( \text{Cov} \left( X_i, X_j \right) = \rho \) gives \( \text{Var}(\theta) = \rho \). Hence, \( \theta \sim \mathcal{N}(0, \rho) \) and \( Z_i \sim \mathcal{N}(0, 1 - \rho) \) for each \( i \geq 1 \).

The authors in [8] also extend Theorem 2.2 to arbitrarily correlated normal setups:
Theorem 2.3. Let $\Sigma_n$ be the correlation matrix of $X_1, \ldots, X_n$ with $(i, j)$'th entry $\rho_{ij}$ such that $\liminf \rho_{ij} = \delta > 0$. Then, for any $\alpha \in (0, 1)$,
\[
\lim_{n \to \infty} \text{FW ER}_{\text{Bon}}(n, \alpha, \Sigma_n) = 0.
\]

Theorem 2.3, a much stronger result than Theorem 2.1, highlights the fundamental problem of using Bonferroni method in a simultaneous testing problem. Note that Theorem 2.2 and Theorem 2.3 are valid under any configuration of true and false null hypotheses. The authors in [8] establish Theorem 2.3 using a famous inequality due to Slepian [23]. As some of our proofs also use Slepian’s inequality, we state it here:

Theorem 2.4 (Slepian). Let $X$ be distributed according to $N(0, \Sigma)$, where $\Sigma$ is a correlation matrix. For an arbitrary but fixed real vector $a = (a_1, \ldots, a_k)'$, consider the quadrant probability
\[
g(k, a, \Sigma) = P_{\Sigma} \left( \bigcap_{i=1}^{k} \{ X_i \leq a_i \} \right).
\]

Let $R = (\rho_{ij})$ and $T = (\tau_{ij})$ be two positive semidefinite correlation matrices. If $\rho_{ij} \geq \tau_{ij}$ holds for all $i, j$, then $g(k, a, R) \geq g(k, a, T)$, i.e
\[
P_{\Sigma=R} \left( \bigcap_{i=1}^{k} \{ X_i \leq a_i \} \right) \geq P_{\Sigma=T} \left( \bigcap_{i=1}^{k} \{ X_i \leq a_i \} \right)
\]
holds for all $a = (a_1, \ldots, a_k)'$. Furthermore, the inequality is strict if $R, T$ are positive definite and if the strict inequality $\rho_{ij} > \tau_{ij}$ holds for some $i, j$.

Throughout this work, $P_i$ denotes the $p$-value corresponding to the $i$-th null hypothesis $H_{0i}, 1 \leq i \leq n$. Also, let $P_{(1)} \leq \ldots \leq P_{(n)}$ be the ordered $p$-values. Let the null hypothesis corresponding to the $p$-value $P_{(i)}$ be denoted as $H_{(0i)}, 1 \leq i \leq n$. Now,
\[
\text{FW ER}_{\text{Bon}}(n, \alpha, \Sigma_n) = P_{\Sigma_n} \left( \bigcup_{i=1}^{n} \{ X_i > c_{\alpha,n} \} \right) = P_{\Sigma_n} \left( X_{(n)} > c_{\alpha,n} \right)
\]
\[
= P_{\Sigma_n} \left( \Phi(X_{(n)}) > 1 - \alpha/n \right)
\]
\[
= P_{\Sigma_n} \left( 1 - \Phi(X_{(n)}) < \alpha/n \right)
\]
\[
= P_{\Sigma_n} \left( P_{(1)} < \alpha/n \right).
\]

Therefore, we have the following restatement of Theorem 2.3:

Corollary 2.1. Let $\Sigma_n$ be the correlation matrix of $X_1, \ldots, X_n$ with $(i, j)$'th entry $\rho_{ij}$ such that $\liminf \rho_{ij} = \delta > 0$. Then, for any $\alpha \in (0, 1)$,
\[
\lim_{n \to \infty} P_{\Sigma_n} \left( P_{(1)} < \alpha/n \right) = 0.
\]

We give a brief description of step-down and step-up procedures in terms of $p$-values in the next subsection.
2.3. Step-down and Step-up Procedures. Single-step MTPs (e.g., Bonferroni’s method) compare individual test statistics to their critical values simultaneously, and after this simultaneous ‘joint’ comparison, the MTP stops. Often single-step methods can be improved in terms of power via stepwise methods, while still controlling FWER (or, in general, the error rate of interest) at the desired level.

Consider the simplex

\[ \text{Simp}^n = \{ t = (t_1, \ldots, t_n) \in \mathbb{R}^n : 0 \leq t_1 \leq \cdots \leq t_n \leq 1 \} . \]

A step-down procedure based on \( p \)-values uses a set of critical values \( u = (u_1, \ldots, u_n) \in \text{Simp}^n \), and works as follows. A hypothesis \( H(j) \) is rejected if and only if \( P(j) \leq u_j \) for all \( j \leq i \), otherwise it cannot be rejected. In other words, the step-down procedure starts with the most significant \( p \)-value by comparing it with the smallest \( u \)-value and so on. This step-up procedure can also be formally described as follows. Let

\[ m_1 = \max \{ i : P(j) \leq u_j \text{ for all } j = 1, \ldots, i \} . \]

Then the step-down procedure based on critical values \( u \) rejects \( H(1), \ldots, H(m_1) \).

**Example.** The Bonferroni method is a step-down procedure with \( u_i = \alpha/n \), \( i = 1, \ldots, n \).

**Example.** The Sidak method is a step-down procedure with \( u_i = 1 - (1 - \alpha)^{1/n} \), \( i = 1, \ldots, n \).

**Example.** The Holm [19] method is a popular step-down procedure with \( u_i = \alpha/(n - i + 1) \), \( i = 1, \ldots, n \).

**Example.** The procedure introduced by Benjamini and Liu [3] is a step-down procedure with

\[ u_i = \min \left( 1, \frac{1}{(n - i + 1)^2} \right), \quad 1 \leq i \leq n \quad (0 < q < 1) . \]

**Example.** Benjamini and Liu proposed another step-down procedure in [4] with the critical values

\[ u_i = 1 - \left[ 1 - \min \left( 1, \frac{1}{n - i + 1} \right) \right]^{1/(n-i+1)}, \quad 1 \leq i \leq n \quad (0 < q < 1) . \]

**Example.** Benjamini and Liu mentioned in [4] a Holm-type procedure with the critical values

\[ u_i = 1 - (1 - q)^{1/(n-i+1)}, \quad 1 \leq i \leq n \quad (0 < q < 1) . \]

The step-up procedure is also based on a set of critical values, say \( u = (u_1, \ldots, u_n) \in \text{Simp}^n \). But the step-up method is inherently different from the step-down method in the sense that it starts by comparing the least significant \( p \)-value with the largest \( u \)-value and so on. More precisely, the step-up procedure based on critical values \( u \) rejects \( H(1), \ldots, H(m_2) \), where

\[ m_2 = \max \{ i : P(i) \leq u_i \} . \]

**Example.** The Bonferroni procedure is also a step-up procedure, where \( u_i = \alpha/n \), \( i = 1, 2, \ldots, n \).
EXAMPLE. The Sidak method is also a step-up procedure with \( u_i = 1 - (1 - \alpha)^{1/n} \), \( i = 1, \ldots, n \).

EXAMPLE. The Hochberg [18] procedure is a popular step-up procedure with \( u_i = \alpha/(n - i + 1) \).

EXAMPLE. The classic Benjamini and Hochberg [2] procedure is a step-up procedure with \( u_i = i\alpha/n \).

3. Limiting FWER of Step-down Procedures. The authors in [8] show the following result on the asymptotic FWER of Holm’s procedure [19] under the equicorrelated normal setup:

Theorem 3.1. Suppose \( \mu^* = \sup \mu_i < \infty \). Then, \( \lim_{n \to \infty} \text{FWER}_{\text{Holm}}(n, \alpha, \rho) = 0 \) for all \( \alpha \in (0, 1) \) and \( \rho \in (0, 1] \).

Note that Theorem 3.1 is valid under any configuration of true and false null hypotheses. We extend this result for arbitrary correlated setups:

Theorem 3.2. Let \( \Sigma_n \) be the correlation matrix of \( X_1, \ldots, X_n \) with \( (i, j) \)'th entry \( \rho_{ij} \) such that \( \lim \inf \rho_{ij} = \delta > 0 \). Suppose \( \mu^* = \sup \mu_i < \infty \). Then, for any \( \alpha \in (0, 1) \),

\[
\lim_{n \to \infty} \text{FWER}_{\text{Holm}}(n, \alpha, \Sigma_n) = 0.
\]

Proof of Theorem 3.2. Let \( R_n(H) \) and \( V_n(H) \) denote the number of rejections and false rejections of Holm’s procedure respectively. Then,

\[
\text{FWER}_{\text{Holm}}(n, \alpha, \Sigma_n) = \mathbb{P}_{\Sigma_n}(V_n(H) \geq 1) \leq \mathbb{P}_{\Sigma_n}(R_n(H) \geq 1) = \mathbb{P}_{\Sigma_n}(P_1 \leq \alpha/n) = \mathbb{P}_{\Sigma_n}(X_{(n)} \geq c_{\alpha,n}).
\]

Theorem 2.4 gives \( \mathbb{P}_{\Sigma_n}(X_{(n)} \geq c_{\alpha,n}) \leq \mathbb{P}_{M_n(\rho)}(X_{(n)} \geq c_{\alpha,n}) \). Without loss of generality, we may assume \( X_i \sim N(\mu_i, 1) \) (\( \mu_i > 0 \)) for \( 1 \leq i \leq n_1 \) and for \( n_1 < i \leq n \), \( X_i \sim N(0, 1) \). Thus,

\[
\text{FWER}_{\text{Holm}}(n, \alpha, \Sigma_n) \leq \mathbb{P}_{M_n(\rho)}(X_{(n)} \geq c_{\alpha,n}) = 1 - \mathbb{P}_{M_n(\rho)}(X_{(n)} \leq c_{\alpha,n})
= 1 - \mathbb{P}_{M_n(\rho)}(X_i \leq c_{\alpha,n} \quad \forall i = 1, 2, \ldots, n)
= 1 - \mathbb{P}_{M_n(\rho)} \left[ \bigcap_{i=1}^{n} \{ \theta + Z_i + \mu_i \leq c_{\alpha,n} \} \cap \bigcap_{i=n_1+1}^{n} \{ \theta + Z_i \leq c_{\alpha,n} \} \right]
= 1 - \mathbb{E}_\theta \left[ \prod_{i=1}^{n_1} \Phi \left( \frac{c_{\alpha,n} - \theta - \mu_i}{\sqrt{1-\rho}} \right) \Phi^{n-n_1} \left( \frac{c_{\alpha,n} - \theta}{\sqrt{1-\rho}} \right) \right]
\leq 1 - \mathbb{E}_\theta \left[ \Phi^\rho \left( \frac{c_{\alpha,n} - \theta - \mu^*}{\sqrt{1-\rho}} \right) \right]
\to 1 - 1 = 0 \text{ as } n \to \infty \quad \text{(since } \mu^* < \infty) \].
We extend Theorem 3.2 to any step-down MTP below:

**Theorem 3.3.** Let $\Sigma_n$ be the correlation matrix of $X_1, \ldots, X_n$ with $(i, j)$'th entry $\rho_{ij}$ such that $\lim \inf \rho_{ij} = \delta > 0$. Suppose $\mu^* = \sup \mu_i < \infty$ and $T$ is any step-down MTP. Then, for any $\alpha \in (0, 1)$,

$$\lim_{n \to \infty} FWER_T(n, \alpha, \Sigma_n) = 0.$$ 

Theorem 3.3 can be established using the following result due to Gordon and Salzman [17].

**Theorem 3.4.** Let $T$ be a step-down MTP based on the set of cut-offs $u \in \text{Simp}_n$. If $FWER_T \leq \alpha < 1$, then $u_i \leq \alpha/(n - i + 1)$, $i = 1, \ldots, n$.

**Proof of Theorem 3.3.**

$$FWER_T(n, \alpha, \Sigma_n) = \Pr_{\Sigma_n}(V_n(T) \geq 1) \leq \Pr_{\Sigma_n}(R_n(T) \geq 1) = \Pr_{\Sigma_n}(P_1 \leq u_1) \leq \Pr_{\Sigma_n}(P_1 \leq \alpha/n) \quad \text{(since } u_1 \leq \alpha/n \text{ from Theorem } 3.4)$$

$\to 0$ as $n \to \infty$ (since $\mu^* < \infty$).

\[\square\]

4. Limiting FWER of Step-up Procedures.

4.1. Hochberg’s Procedure. The following two results depict the asymptotic behaviour of the FWER of Hochberg’s procedure under the independent normal setup and under the positively equicorrelated normal setup, respectively.

**Theorem 4.1.** Under the independent normal setup,

$$\lim_{n \to \infty} FWER_{Hochberg}(n, \alpha, 0) \in [1 - e^{-\alpha}, \alpha]$$

under the global null hypothesis.

**Theorem 4.2.** Under the equicorrelated normal setup with correlation $\rho \in (0, 1)$,

$$\lim_{n \to \infty} FWER_{Hochberg}(n, \alpha, \rho) = 0$$

for any $\alpha \in (0, 1/2)$.

**Proof of Theorem 4.1.** We have,

$$FWER_{Hochberg}(0) = \Pr \left[ \bigcup_{i=1}^{n} \left\{ P(i) \leq \frac{\alpha}{n - i + 1} \right\} \right] \geq \Pr \left[ P(1) \leq \frac{\alpha}{n} \right] \to 1 - e^{-\alpha}.$$ 

On the other hand, Hochberg’s procedure controls FWER at level $\alpha$. So, $FWER_{Hochberg}(0) \leq \alpha$. So, $1 - e^{-\alpha} \leq \lim_{n \to \infty} FWER_{Hochberg}(0) \leq \alpha$. Also, $\lim_{\alpha \to 0} \frac{1 - e^{-\alpha}}{\alpha} = 1$. thus, we have, as $\alpha \to 0$, $\lim_{n \to \infty} \frac{FWER_{Hochberg}(0)}{\alpha} = 1$. \[\square\]
Proof of Theorem 4.2. We have,

\[ P_{(i)} \leq \frac{\alpha}{n-i+1} \]

\[ \iff 1 - \frac{\alpha}{n-i+1} \leq \Phi \left( X_{(n-i+1)} \right) \]

\[ \iff \Phi^{-1} \left( 1 - \frac{\alpha}{n-i+1} \right) \leq U + Z_{(n-i+1)} \]

\[ \iff \Phi^{-1} \left( 1 - \frac{\alpha}{n-i+1} \right) \leq U + \sqrt{1 - \rho} \cdot \Phi^{-1} \left( 1 - \frac{i}{n} \right) \quad \text{(for sufficiently large values of } n). \]

Therefore, for sufficiently large \( n \), we have

\[ P_{(i)} \leq \frac{\alpha}{n-i+1} \]

\[ \iff -U - \sqrt{1 - \rho} \cdot \Phi^{-1} \left( 1 - \frac{i}{n} \right) \leq -\Phi^{-1} \left( 1 - \frac{\alpha}{n-i+1} \right) \]

\[ \iff -U + \sqrt{1 - \rho} \cdot \Phi^{-1} \left( \frac{i}{n} \right) \leq \Phi^{-1} \left( \frac{\alpha}{n-i+1} \right) \]

\[ \iff \frac{-U}{\Phi^{-1} \left( \frac{\alpha}{n-i+1} \right)} + \sqrt{1 - \rho} \cdot \Phi^{-1} \left( \frac{i}{n} \right) \geq 1 \quad \text{(since } \alpha \in (0, 1/2) \text{)} \]

\[ \iff \lim_{n \to \infty} \frac{\Phi^{-1} \left( \frac{i}{n} \right)}{\Phi^{-1} \left( \frac{\alpha}{n-i+1} \right)} \geq \frac{1}{\sqrt{1 - \rho}}. \]

Thus, we have \( i/n < 1/2 \), because otherwise the limiting ratio of \( \Phi^{-1} \left( \frac{i}{n} \right) \) and \( \Phi^{-1} \left( \frac{\alpha}{n-i+1} \right) \) can not be positive. So, we have

\[ \frac{i}{n} < \frac{\alpha}{n-i+1} < 1/2. \]

This implies \( i(n-i+1) < \alpha \cdot n \). But this is not valid for any value of \( i \) in \( \{1, \ldots, n\} \). Consequently, the limiting FWER is zero. \( \square \)

We now generalize Theorem 4.2 to general correlated normal setups:

Theorem 4.3. Let \( \Sigma_n \) be the correlation matrix of \( X_1, \ldots, X_n \) with \((i, j)\)th entry \( \rho_{ij} \) such that \( \lim inf \rho_{ij} = \delta > 0 \). Suppose \( \mu^* = \sup \mu_i < \infty \). Then, for any \( \alpha \in (0, 1) \),

\[ \lim_{n \to \infty} \text{FWER}_{Hochberg}(n, \alpha, \Sigma_n) = 0. \]

Proof of Theorem 4.3. Let \( (Y_1, \ldots, Y_n) \) follow a multivariate normal distribution with mean zero and covariance matrix \( M_n(\delta) \). Let \( Q_{(i)} \) and \( R_{(i)} \) denote the \( i \)th smallest \( p \)-values corresponding to \( (X_1, \ldots, X_n) \) and \( (Y_1, \ldots, Y_n) \) respectively. Suppose,

\[ I_Q = \max \left\{ 1 \leq i \leq n : Q_{(i)} \leq \frac{\alpha}{n-i+1} \right\} \quad \text{and} \quad I_R = \max \left\{ 1 \leq i \leq n : R_{(i)} \leq \frac{\alpha}{n-i+1} \right\}. \]

We note that, for each \( i \in \{1, \ldots, n\} \), the order statistic \( X_{(n-i+1)} \) is an increasing function of \( (X_1, \ldots, X_n) \). Since increasing functions preserve stochastic orders, using Theorem 2.4 we obtain that \( Y_{(n-i+1)} \) is stochastically larger than \( X_{(n-i+1)} \). This gives, \( 1 - \Phi(Y_{(n-i+1)}) \)
is stochastically smaller than $1 - \Phi(X_{(n-i+1)})$. In other words, $R_{(i)}$ is stochastically smaller than $Q_{(i)}$. Consequently, $I_R$ is stochastically larger than $I_Q$. This implies,

$$\text{FWER}_{Hochberg}(n, \alpha, \Sigma_n) \leq \text{FWER}_{Hochberg}(n, \alpha, M_n(\delta)).$$

The rest follows from Theorem 4.2.

5. Hommel’s Procedure. We have focused on step-down and step-up procedures so far. However, many powerful multiple testing procedures proposed in the literature do not fall under the step-down or step-up categories. The Hommel [20] procedure is such a $p$-value based multiple testing procedure that controls the FWER.

The decisions for the individual hypotheses are performed in the following simple way:

Step 1. Compute $j = \max \{i \in \{1, \ldots, n\} : P_{(n-i+k)} > \frac{k\alpha}{i} \text{ for } k = 1, \ldots, i\}$.

Step 2. If the maximum does not exist in Step 1, reject all the hypotheses. Otherwise reject all $H_i$ with $P_i \leq \frac{\alpha}{j}$.

Although the Hommel’s procedure does not have the simple stepwise structure of the Holm [19] or the Hochberg [18] procedures, it is uniformly more powerful than the methods of Bonferroni (Dunn, 1961), Holm, and Hochberg. The following two results depict the asymptotic behaviour of the FWER of Hommel’s procedure under the independent normal setup and under the positively equicorrelated normal setup, respectively.

**Theorem 5.1.** Under the independent normal setup,

$$\lim_{n \to \infty} \text{FWER}_{Hommel}(n, \alpha, 0) = 1 - e^{-\alpha}$$

under the global null hypothesis.

**Theorem 5.2.** Under the equicorrelated normal setup with correlation $\rho \in (0, 1)$,

$$\lim_{n \to \infty} \text{FWER}_{Hommel}(n, \alpha, \rho) = 0$$

with probability one under the global null hypothesis.

**Proof of Theorem 5.1.** For $1 \leq i \leq n$, we have $P_{(i)} = 1 - \Phi(X_{(n-i+1)})$. Putting $i = n - j + k$ (here $1 \leq j \leq n$ and $1 \leq k \leq j$) gives $P_{(n-j+k)} = 1 - \Phi(X_{(j-k+1)})$ for $k \leq j$. Now,

$$P_{(n-j+k)} \frac{k\alpha}{j} \iff 1 - \Phi(X_{(j-k+1)}) \frac{k\alpha}{j} \iff \Phi^{-1} \left(1 - \frac{k\alpha}{j}\right) > X_{(j-k+1)}$$

$$\iff \Phi^{-1} \left(1 - \frac{s\alpha}{t}\right) > X_{(n(t-s)+1)} \text{ where } s = k/n \text{ and } t = j/n.$$

For any $r \in (0, 1)$, $X_{(nr)}$ converges in probability to $r$’th quantile of the distribution of $X_1$ as $n \to \infty$. This implies, $X_{(n(t-s)+1)}$ converges in probability to $\Phi^{-1}(t - s)$ as $n \to \infty$. Thus, as $n \to \infty$,

$$P_{(n-j+k)} \frac{k\alpha}{j} \iff 1 - \frac{s\alpha}{t} > t - s \iff t - s > (\alpha - t) \iff t(1 - t) > s(\alpha - t).$$
We have \( t \geq s \) and \( 1 > \alpha \). So, \( t(1-t) > s(\alpha - t) \) always holds. This means that the largest \( t \) for which \( t(1-t) > s(\alpha - t) \) holds for each \( s \in (0, \hat{t}] \) is 1. This in turn implies that, as \( n \to \infty \), the largest integer \( j \leq n \) satisfying \( P_{n-j+k} > \frac{k\alpha}{j} \) for all \( k \in \{1, \ldots, j\} \) is \( n \) with probability one. Thus, the Hommel’s procedure is same as the Bonferroni’s procedure as \( n \to \infty \). Hence,

\[
\lim_{n \to \infty} FWER_{\text{Hommel}}(n, \alpha, 0) = 1 - e^{-\alpha}.
\]

\[ \square \]

**Proof of Theorem 5.2.** Under the equicorrelated normal setup with correlation \( \rho \in (0, 1) \), for each \( i \geq 1 \), we have \( X_i = U + Z_i \) where \( \theta \) is a normal random variable having mean 0, independent of \( \{Z_n\}_{n \geq 1} \) and \( Z_i \)’s are i.i.d normal random variables. Now, similar to the previous proof, we have

\[
P_{n-j+k} > \frac{k\alpha}{j} \iff 1 - \Phi(X_{j-k+1}) > \frac{k\alpha}{j}
\]

\[
\iff \Phi^{-1}\left(1 - \frac{k\alpha}{j}\right) > X_{j-k+1}
\]

\[
\iff \Phi^{-1}\left(1 - \frac{k\alpha}{j}\right) > U + Z_{j-k+1}
\]

\[
\iff \Phi^{-1}\left(1 - \frac{s\alpha}{t}\right) > U + Z_{(n(t-s)+1)} \quad \text{where} \quad s = k/n \quad \text{and} \quad t = j/n.
\]

For any \( r \in (0, 1) \), \( Z_{(nr)} \) converges in probability to \( r \)’th quantile of the distribution of \( Z_1 \) as \( n \to \infty \). This implies, \( Z_{(n(t-s)+1)} \) converges in probability to \( \sqrt{1-\rho} \cdot \Phi^{-1}(t-s) \) as \( n \to \infty \). Thus, as \( n \to \infty \),

\[
P_{n-j+k} > \frac{k\alpha}{j} \iff U < \Phi^{-1}\left(1 - \frac{s\alpha}{t}\right) - \sqrt{1-\rho} \cdot \Phi^{-1}(t-s).
\]

Let us denote the quantity \( \Phi^{-1}\left(1 - \frac{s\alpha}{t}\right) - \sqrt{1-\rho} \cdot \Phi^{-1}(t-s) \) as \( f(s) \) for fixed \( t \). This means, as \( n \to \infty \),

\[
P_{n-j+k} > \frac{k\alpha}{j} \quad \text{for all} \quad k = 1, \ldots, j \iff U < \min_{0 < s < t} f(s).
\]

Now, \( t > t - s \) as \( s > 0 \). This implies \( \Phi^{-1}(t) > \Phi^{-1}(t-s) \). Consequently, for each \( s > 0 \), \( f(s) > g(s) \) where \( g(s) = \Phi^{-1}\left(1 - \frac{s\alpha}{t}\right) - \Phi^{-1}(t) \). Thus,

\[
g(s) > U \iff f(s) > U.
\]
Now,

\[ g(s) > U \]
\[ \iff \Phi^{-1} \left( 1 - \frac{s\alpha}{t} \right) - \sqrt{1 - \rho} \cdot \Phi^{-1}(t) > U \]
\[ \iff \Phi^{-1} \left( 1 - \frac{s\alpha}{t} \right) > U + \sqrt{1 - \rho} \cdot \Phi^{-1}(t) \]
\[ \iff 1 - \frac{s\alpha}{t} > \Phi \left( U + \sqrt{1 - \rho} \cdot \Phi^{-1}(t) \right) \]
\[ \iff \Phi \left( -U - \sqrt{1 - \rho} \cdot \Phi^{-1}(t) \right) > s\alpha \]
\[ \iff \Phi \left( -U - \sqrt{1 - \rho} \cdot \Phi^{-1}(t) \right) > t. \]

Therefore, if \( \frac{\Phi(-U - \sqrt{1 - \rho} \cdot \Phi^{-1}(t))}{\alpha} > 1 \) then \( \forall s \in (0, t), \ g(s) > U \). Hence, \( \frac{\Phi(-U - \sqrt{1 - \rho} \cdot \Phi^{-1}(t))}{\alpha} > 1 \) implies \( f(s) > U \) for all \( s \in (0, t) \).

Now,

\[ \frac{\Phi(-U - \sqrt{1 - \rho} \cdot \Phi^{-1}(t))}{\alpha} > 1 \]
\[ \iff -U - \sqrt{1 - \rho} \cdot \Phi^{-1}(t) > \Phi^{-1}(\alpha) \]
\[ \iff \Phi \left( -U - \Phi^{-1}(\alpha) \right) > t. \]

Therefore, we have established the following:

\[ \Phi \left( \frac{-U - \Phi^{-1}(\alpha)}{\sqrt{1 - \rho}} \right) > t \implies U < \min_{0<s<t} f(s). \]

Thus,

\[ t_0 := \max_t \left\{ t \in (0, 1) : \min_{0<s<t} f(s) > U \right\} \]
\[ \geq \max_t \left\{ t \in (0, 1) : \Phi \left( \frac{-U - \Phi^{-1}(\alpha)}{\sqrt{1 - \rho}} \right) > t \right\}. \]

Now, \(|U| < n\) implies \( t_0 \geq \varepsilon_n \) where

\[ \varepsilon_n = \Phi \left( \frac{-n - \Phi^{-1}(\alpha)}{\sqrt{1 - \rho}} \right). \]

So, for every \( n \in \mathbb{N} \), there exists \( \varepsilon_n > 0 \) such that \( t_0 > \varepsilon_n \) if \(|U| < n\). In other words, there is \( \varepsilon_n \) such that \( t_0 > \varepsilon_n > 0 \) with probability at least \( \mathbb{P}(|U| < n) \). This implies, \( t_0 \) is bounded away from zero with probability one. Now, let

\[ j = \max_{1 \leq j \leq n} \left\{ P_{(n-j+k)} > \frac{k\alpha}{j} \text{ for all } k = 1, \ldots, j \right\}. \]
Evidently, \( j = nt_0 \). Consequently, under the global null,
\[
FWER_{Hommel}(n, \alpha, \rho) = \mathbb{P} \left[ \bigcup_{j=1}^{n} \left\{ P_i \leq \frac{\alpha}{j} \right\} \right] \\
= \mathbb{P} \left[ \bigcup_{j=1}^{n} \left\{ P_i \leq \frac{\alpha}{nt_0} \right\} \right] \\
= \mathbb{P} \left[ P_{(1)} \leq \frac{1}{t_0} \cdot \frac{\alpha}{n} \right] \\
\rightarrow 0 \text{ as } n \rightarrow \infty, \text{ with probability one using Corollary 2.1.}
\]

The following result extends Theorem 5.2 to arbitrarily correlated normal set-ups:

**Theorem 5.3.** Let \( \Sigma_n \) be the correlation matrix of \( X_1, \ldots, X_n \) with \((i,j)\)th entry \( \rho_{ij} \) such that \( \lim \inf \rho_{ij} = \delta > 0 \). Suppose \( \mu^* = \sup \mu_i < \infty \). Then, for any \( \alpha \in (0, 1) \),
\[
\lim_{n \rightarrow \infty} FWER_{Hommel}(n, \alpha, \Sigma_n) = 0
\]
with probability one under the global null hypothesis.

**Proof of Theorem 5.3.** We proceed similar to the proof of Theorem 4.3. Let \( (Y_1, \ldots, Y_n) \) follow a multivariate normal distribution with mean zero and covariance matrix \( M_n(\delta) \). Let \( Q^{(i)} \) and \( R^{(i)} \) denote the \( i \)'th smallest \( p \)-values corresponding to \( (X_1, \ldots, X_n) \) and \( (Y_1, \ldots, Y_n) \) respectively. Suppose,
\[
J_Q = \max_{1 \leq j \leq n} \left\{ Q_{(n-j+k)} > \frac{k \alpha}{j} \text{ for all } k = 1, \ldots, j \right\} \quad \text{and}
\]
\[
J_R = \max_{1 \leq j \leq n} \left\{ R_{(n-j+k)} > \frac{k \alpha}{j} \text{ for all } k = 1, \ldots, j \right\}.
\]

Using the similar arguments as in the proof of Theorem 4.3, we obtain \( R^{(i)} \) is stochastically smaller than \( Q^{(i)} \). Consequently, \( J_R \) is stochastically smaller than \( J_Q \). This implies,
\[
FWER_{Hommel}(n, \alpha, \Sigma_n) = \mathbb{P} \left[ \bigcup_{i=1}^{n} \left\{ Q_i \leq \frac{\alpha}{J_Q} \right\} \right] \\
\leq \mathbb{P} \left[ \bigcup_{i=1}^{n} \left\{ R_i \leq \frac{\alpha}{J_R} \right\} \right] \\
= FWER_{Hommel}(n, \alpha, M_n(\delta)).
\]
The rest follows from Theorem 5.2.

**6. Concluding Remarks.** In recent years, substantial efforts have been made to understand the properties of multiple testing procedures under dependence. The article [8] sheds light on the extent of the conservativeness of the Bonferroni method under dependent setups. However, there is little literature studying the effect of correlation on general step-down or step-up procedures. This paper addresses this gap in a unified manner by
investigating the limiting behaviors of several testing rules under the correlated Gaussian sequence model. We have proved asymptotic zero results for some popular multiple testing procedures controlling FWER at a pre-specified level. Specifically, we have shown that the limiting FWER approaches zero for any step-down rule provided the infimum of the correlations is strictly positive. Within our chosen asymptotic framework, we have also established limiting behaviors of other popular multiple testing rules, e.g., Hochberg’s and Hommel’s procedures.

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