The Heisenberg antiferromagnet: 
an explicitly rotational invariant formulation

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Abstract

A simple derivation of an explicitly rotation invariant Lagrangian describing the dynamics of an antiferromagnetic spin system is presented. The scope of the derived Lagrangian is analysed in the context of schematic models. It is shown that the Lagrangian describes the behaviour spin systems from the anti-ferromagnetic to the ferromagnetic regimes.

1. Introduction

The theory of the two-dimensional Heisenberg antiferromagnet has been attracting much attention over the past several years in connection with the problem of spin fluctuations in copper oxides. For a review we refer the reader to the work of Manousakis [1]. In ref. [2], a Lagrangian for the antiferromagnetic spin system has been proposed which is explicitly invariant under rotations. Following the ideas of ref. [2], a new and simpler derivation of such a rotationally invariant Lagrangian is presented. Ideal systems made up of a small number of spins (two and four) are investigated in the framework of the derived Lagrangian. It is shown that the Lagrangian describes the behaviour these spin clusters from the anti-ferromagnetic to the ferromagnetic regimes.

2. General formalism

We consider a spin system described by the Heisenberg Hamiltonian

\[ \hat{H}_{Hei} = \frac{J}{2} \sum_l \sum_{l' \in (l)} \hat{S}_l \cdot \hat{S}_{l'}, \quad \hat{S}_l \cdot \hat{S}_l = s(s+1), \]  

where \( \hat{S}_l \) are spin operators, \( (l) \) stands for the set of the nearest neighbors of the site \( l \), the index \( l \) runs over the sites of a two-dimensional square lattice, \( J > 0 \) is the exchange constant corresponding to the anti-ferromagnetic (AF) spin interaction, and \( s \) is the magnitude of spin. In the study of spin systems, it is convenient to exploit the representation of the grand partition function (GPF) or the generating functional of the spin Green functions in the form of functional integrals over spin coherent states \( |z\rangle \) due to Klauder [3,4] or over spin coherent states parametrized by a unit vector \( \mathbf{n}, \mathbf{n}^2 = 1 \), proposed by Manousakis.
These approaches are equivalent. Here, we follow mostly the second alternative on account of the simplicity of the corresponding measure of integration. We have,

\[ Z = Tr \left[ \exp \left( -\beta \hat{H} \right) \right], \quad \beta = 1/T, \]

\[ Z = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} D\mu(n) \exp(A(n)), \]

\[ D\mu(n) = \prod_{\tau l} \frac{2s + 1}{2\pi} \delta(n_{\tau l}^2 - 1) dn_{\tau l} \]

where \( T \) is the temperature, \( \tau \) is the imaginary time, and \( A(n) \) is the action of the system. In the continuum approximation, which is valid in leading order in \( 1/2s \), the expression of the action \( A(n) \) simplifies and reads [3]

\[ A(n) = -\int_0^\beta \sum_L L_{tot}(\tau, l) d\tau, \quad L_{tot}(\tau, l) = L_{kin}(\tau, l) + H(\tau, l), \]

\[ L_{kin}(\tau, l) = is(1 - \cos \theta_{\tau l}) \dot{\phi}_{\tau l}, \quad H(\tau, l) = \frac{Js^2}{2} \sum_{\nu \in \langle l \rangle} n_{\tau \nu} \cdot n_{\tau l}, \]

where \( \theta, \phi \) are the Euler angles of the unit vector \( n = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \), and \( \dot{\phi}_{\tau l} \) is the time derivative of \( \phi_{\tau l} \). The kinetic part of the Lagrangian density \( L_{kin} \) is highly nonlinear and it is not clear how to proceed with it consistently. Essential steps were taken in this direction in [1,5,6,7] but we believe the issue is not definitely settled yet.

In this paper we use the idea of near AF order. Accordingly, we split our square lattice into two AF sublattices a and b. In the sublattice a the spins \( S \) are normally directed along some axis \( \Omega \), in the sublattice b they are normally directed in the opposite direction. In this way, we obtain a new square lattice with two spins a and b in the elementary cell with a volume \( 2a^2 \), where \( a \) is the space distance between neighbouring spins. The axes of this new lattice are rotated by 45 degrees with respect to the primary axes. We assume that this AF order is only defined locally and any global AF order is absent. Thus, the summations over the lattice sites \( l \) and \( l' \) can be expressed as summations over \( l \in a \) and \( l' \in b \) which specifies the space positions of the spins in the sublattices a and b. The Lagrangian density \( L_{kin} \) is expressed as a sum of two Lagrangian densities, one for the sublattice a and the other for the sublattice b, involving the vectors \( n_a(\tau, l) \) and \( n_b(\tau, l') \), respectively. The Hamiltonian \( H \) retains its form when the restrictions \( l \in a \) and \( l' \in b \) are imposed but the multiplier \( J/2 \) must be replaced by \( J \) because the double summation disappears. In this way we have two spins in each AF elementary cell which are defined in different space positions \( l \) and \( l' \). Since we are assuming \( J > 0 \), spins in sublattice a tend to align themselves in the opposite direction to the spins in the sublattice b, for low \( T \). However, magnetic order is not, in principle, excluded. It would occur for \( T > 0 \), if \( |T| \) is small enough, a condition which, physically, cannot be easily implemented.

3. Coherent states
Coherent states may be defined as

$$|n\rangle_s = e^{i\phi S_z} e^{i\theta S_y} e^{-i\phi' S_z} |s, s\rangle,$$

where $|s, s\rangle$ is such that

$$S_+ |s, s\rangle = 0, \quad S_- |s, s\rangle = s |s, s\rangle.$$  

(4)

The overlap between two coherent states reads

$$s \langle n|n'\rangle_s = \left( \cos(\theta/2) \cos(\theta'/2) + \sin(\theta/2) \sin(\theta'/2) e^{i(\phi - \phi')} \right)^{2s},$$

$$= \left( \frac{1 + \mathbf{k} \cdot (n + n') + n \cdot n' + i \mathbf{k} \cdot (n \times n')}{1 + \mathbf{k} \cdot (n + n') + n \cdot n' - i \mathbf{k} \cdot (n \times n')} \right)^s \left( \frac{1 + n \cdot n'}{2} \right)^s,$$

(5)

where $\mathbf{k}$ is an arbitrary unit vector. This expression lacks manifest rotational invariance since it involves the unit vector $\mathbf{k}$ which defines the quantization direction. In the continuum limit, this leads to the kinetic Lagrangian

$$L_{kin} = is \frac{\mathbf{k} \cdot (\mathbf{n} \times \dot{\mathbf{n}})}{1 + \mathbf{k} \cdot \mathbf{n}},$$

(6)

connected with our inforced coherent state dynamics. For a system of two spins, manifest rotational invariance may be implemented. To this end, we start by allowing for a time dependent quantization direction. We introduce the kets $|n'\rangle_s, |s, s\rangle'$, which are defined analogously to the previously introduced kets $|n\rangle_s, |s, s\rangle$, but refer to an arbitrary quantization direction which need not be fixed in time. More specifically, we write

$$|n'\rangle_s = e^{i\phi' S_z} e^{i\theta' S_y} e^{-i\phi' S_z} |s, s\rangle'$$

$$|s, s\rangle' = V |s, s\rangle = e^{i\alpha S_z} e^{i\beta S_y} e^{-i\alpha S_z} |s, s\rangle$$

$$e^{i\phi' S_z} e^{i\theta' S_y} e^{-i\phi' S_z} V = V e^{i\phi' S_z} e^{i\theta' S_y} e^{-i\phi' S_z} V^{-1}$$

$$|n'\rangle_s' = V e^{i\phi' S_z} e^{i\theta' S_y} e^{-i\phi' S_z} |s, s\rangle.$$  

(7)

Notice that $S_{2z} = VS_{2z} V^{-1}, S_{y'} = V S_{y} V^{-1}, S_z |s, s\rangle = s |s, s\rangle$ and $S_{2z} |s, s\rangle' = s |s, s\rangle'$. In order to compute the overlap we need the relation between $|n'\rangle_s'$ and $|s, s\rangle$. The required Euler angles $\tilde{\phi}, \tilde{\theta}, \tilde{\psi}$ characterizing the unitary operator $V e^{i\phi' S_z} e^{i\theta' S_y} e^{-i\phi' S_z} = e^{i\tilde{\phi} S_z} e^{i\tilde{\theta} S_y} e^{-i(\tilde{\phi} - \tilde{\psi}) S_z}$ are such that

$$\cos \frac{\tilde{\theta}}{2} e^{i(\tilde{\psi}/2)} = \cos \frac{\theta'}{2} \cos \frac{\beta}{2} - \sin \frac{\theta'}{2} \sin \frac{\beta}{2} e^{i(\alpha - \phi')}$$

$$\sin \frac{\tilde{\theta}}{2} e^{i(\tilde{\phi} - (\tilde{\psi}/2))} = \cos \frac{\beta}{2} \sin \frac{\theta'}{2} e^{i\phi'} + \cos \frac{\theta'}{2} \sin \frac{\beta}{2} e^{i\alpha}.$$  

(8)

so that

$$|n\rangle_s = \sqrt{(2s)!} \sum_{m=-s}^{+s} \frac{(\cos \frac{\theta}{2})^{s+m}(\sin \frac{\theta}{2} e^{i\phi})^{s-m}}{\sqrt{(s+m)!(s-m)!}} |s, m\rangle.$$  

(9)
\[ |n\rangle ' = e^{is\tilde{\psi}} \sqrt{(2s)!} \sum_{m=-s}^{+s} \frac{(\cos \frac{\theta}{2})^{s+m}(\sin \frac{\theta}{2}e^{i\tilde{\phi}})^{s-m}}{\sqrt{(s+m)!(s-m)!}} |s, m\rangle, \]  

(10)

\[ s\langle n|n\rangle ' = e^{is\tilde{\psi}} \left( \cos \frac{\theta}{2} \cos \frac{\tilde{\theta}}{2} + \sin \frac{\theta}{2} \sin \frac{\tilde{\theta}}{2} e^{i(\tilde{\phi} - \phi)} \right)^{2s} \]

\[ = e^{is\tilde{\psi}} \left( \frac{1 + k \cdot (n + n')} + n \cdot n' + i k \cdot (n \times n') }{1 + k \cdot (n + n') + n \cdot n' - i k \cdot (n \times n')} \right) ^{s} \left( \frac{1 + n \cdot n'}{2} \right) ^{s}, \]  

(11)

where \( e^{is\tilde{\psi}} \) is given by

\[ e^{i\tilde{\psi}} = \cos \frac{\beta}{2} \cos \frac{\theta'}{2} - \sin \frac{\beta}{2} \sin \frac{\theta'}{2} e^{i(\alpha - \phi')} \]

\[ = \frac{1 + k \cdot (k' + n')} + k' \cdot n' + i k \cdot (n' \times k') }{1 + k \cdot (k' + n') + k' \cdot n' - i k \cdot (n' \times k')} . \]  

(12)

In the continuum limit, this leads to the kinetic Lagrangian

\[ L_{kin} = is \frac{k \cdot (n \times \dot{n})} 1 + k \cdot n \]  

(13)

4. A simple example

In order to illustrate the meaning of eq. (13), it is convenient to consider the real time classical Lagrangian for the unit vector \( n \) parallel to the spin \( S \),

\[ L = \frac{k \cdot (n \times \dot{n})}{1 + k \cdot n} - \mathcal{H}(n) - \lambda((n)^2 - 1) \]  

(14)

where \( \lambda \) is a Lagrange multiplier. The equation of motion

\[ \dot{n} = n \times \frac{\partial \mathcal{H}}{\partial n} \]  

(15)

does not depend on \( k \). If this vector changes with time, the Lagrangian must be modified accordingly. Then, the Lagrangian becomes

\[ L = \frac{k \cdot (n \times \dot{n}) - n \cdot (k \times \dot{k})}{1 + k \cdot n} - \mathcal{H}(n) - \lambda((n)^2 - 1), \]  

(16)

but the equation of motion remains unchanged.

5. System of two interacting spins

We discuss now the system of two interacting spins. We denote by \( S_{c, j}, j = 1, 2, 3, c = a, b \) the hermitian generators of \( su(2) \), \([S_{c1}, S_{c2}] = iS_{c3}, \) etc., with \( S_{c1}^2 + S_{c2}^2 + S_{c3}^2 = s(s + 1) \). We consider the hamiltonian operator

\[ \mathcal{H} = JS_a \cdot S_b. \]  

(17)
and investigate the collective modes, free energy, etc.

Let \( S_t = S_a + S_b \). Then \( s_t = 0, 1, \cdots, 2s \) is the quantum number characterizing the magnitude of the total spin and \( S_a \cdot S_b = \frac{s}{2}(S_t \cdot S_t - 2s(s + 1)) \). The spectrum is

\[
J \left\{ 0 - s(s + 1), \frac{1}{2}2 - s(s + 1), \frac{1}{2}6 - s(s + 1), \cdots, \frac{1}{2}s_t(s_t + 1) - s(s + 1), \cdots, \frac{1}{2}2s(2s + 1) - s(s + 1) \right\}.
\]

The respective multiplicities are

\[
\{1, 3, 5, \cdots, 2s_t + 1, \cdots, 4s + 1\}.
\]

This is the spectrum of a rigid rotor with moment of inertia \( I = J^{-1} \) whose angular momentum is constrained to a maximum value \( 2s \).

We follow now a mean-field variational approach. Here, mean-field should be understood as an enforced coherent state dynamics of each individual spin of our 2 spins system, i.e., it is assumed that each spin is described by a coherent state \( |\hat{z}\rangle \) which essentially depends on time through the parameter \( z = z(t) \). Consider

\[
|z_1, z_2\rangle = e^{z_1 S_a + z_2 S_b}|0\rangle, \ S_a - |0\rangle = S_b - |0\rangle = 0, \ S_{a0}|0\rangle = S_{b0}|0\rangle = -s|0\rangle.
\]

Since \( S_a \cdot S_b = \frac{s}{2}(S_a + S_b - S_{a-}S_{b+}) + S_{a0}S_{b0} \), we have

\[
\frac{\langle z_1, z_2|S_a \cdot S_b|z_1, z_2\rangle}{\langle z_1, z_2|z_1, z_2\rangle} = s^2 \frac{2(z_1^* z_2 + z_2^* z_1) + (z_2^* z_2 - 1)(z_1^* z_1 - 1)}{(z_2^* z_2 + 1)(z_1^* z_1 + 1)},
\]

so that

\[-s^2 \leq \langle S_a \cdot S_b \rangle \leq s^2.\]

The lower bound, \(-s^2\), is attained for \( z_2 = -1/z_1^* \), (anti-parallel spins) while the upper bound, \( s^2\), is attained for \( z_2 = z_1 \) (parallel spins). The mean field-Lagrangian describing our system of two interacting spins may be written

\[
L = s \frac{k \cdot (n_a \times n_a)}{1 + k \cdot n_a} + s \frac{k \cdot (n_b \times n_b)}{1 + k \cdot n_b} - Js^2(n_a \cdot n_b) - \lambda_a((n_a)^2 - 1) - \lambda_b((n_b)^2 - 1) \tag{20}
\]

where \( \lambda_a, \lambda_b \) are Lagrange multipliers. Here, rotational invariance is not manifest. The equations of motion lead to

\[
\dot{n}_a = (n_a \times n_b)sJ, \quad \dot{n}_b = -(n_a \times n_b)sJ, \tag{21}
\]

or, with \( n = n_b - n_a, \ m = \frac{1}{2}(n_a + n_b), \)

\[
\dot{n} = 2(m \times n)sJ, \quad \dot{m} = 0. \tag{22}
\]
From these equations it is not apparent that $|\mathbf{n}|$ is constrained to a maximum value, $|\mathbf{n}| \leq 2s$. However, there is no incompatibility between these equations and such a constraint. Identifying, in Eq. (13), $k$ with $-n_b$ and $n$ with $n_a$, the previous Lagrangian may be replaced by

$$L = -s\frac{n_b \cdot (n_a \times \dot{n}_a) + n_a \cdot (n_b \times \dot{n}_b)}{1 - n_a \cdot n_b} - J s^2 (n_a \cdot n_b) - \lambda_a ((n_a)^2 - 1) - \lambda_b ((n_b)^2 - 1).$$

(23)

Here, rotational invariance is manifest. This Lagrangian is of the same form as the one implied by eq. (8) of ref. [2]. The equations of motion do not change when (22) is replaced by (23). It is convenient to introduce new variables

$$\frac{n_a + n_b}{2} = N, \quad \frac{n_a - n_b}{2} = \Omega \sqrt{1 - N^2},$$

(24)

We find

$$N \cdot (\Omega \times \dot{\Omega}) = \frac{-n_b \cdot (n_a \times n_a) - n_a \cdot (n_b \times \dot{n}_b)}{2(1 - n_a \cdot n_b)}.$$

(25)

In terms of the new variables the Lagrangian becomes

$$\mathcal{L} = 2sN \cdot (\Omega \times \dot{\Omega}) - J s^2 \left(2N^2 - 1\right) - \lambda(\Omega^2 - 1) - \mu(\Omega \cdot N)$$

or,

$$\mathcal{L} = sM \cdot \dot{\Omega} - J s^2 \left(\frac{1}{2}M^2 - 1\right) - \lambda(\Omega^2 - 1) - \mu(\Omega \cdot M),$$

(26)

where

$$M = 2N \times \Omega, \quad 0 \leq M \leq 2.$$ 

(27)

In terms of the variables $M, \Omega$, the path integral measure of integration becomes

$$\prod_\tau \frac{(2s + 1)^2}{8\pi^2} \delta(\Omega^2 - 1) \delta(\Omega \cdot M) d\Omega dM.$$ 

(28)

We observe that eq.(26) is the classical Lagrangian of a rigid rotor with moment of inertia $\mathcal{I} = J^{-1}$ whose angular momentum $sM$ is constrained to a maximum value $2s$. The equations of motion may be written

$$\dot{M} \cdot M = 0, \quad \dot{\Omega} - sJM = 0, \quad \ddot{M} + sJM^2\Omega = 0, \quad \dot{N} = 0, \quad N = \frac{1}{2} \Omega \times M.$$ 

(29)

The mean field frequencies so obtained are in agreement with the exact quantal results for the excitation energies in the whole range from the AF ($M = 0$) to the ferromagnetic ($M = 2$) regimes.

Classical partition function
In eq. (26), $\mathcal{L}$ is the classical Lagrangian of a rigid rotor with moment of inertia $I = J^{-1}$ whose angular momentum $s\mathbf{M}$ is constrained to a maximum value $2s$. The corresponding classical partition function may be written

$$Z = s^2 \int_{0 \leq p_\phi^2 + p^2_\theta \leq 4} \int_0^{\pi/2} d\phi d\theta dp_\phi dp_\theta e^{-\beta Js^2[(p_\phi^2 + p^2_\theta) / \sin^2 \theta] / 2 - 1}$$

$$= s^2 \int_{0 \leq p_\phi^2 + p^2 \leq 4} \int_0^{\pi/2} d\phi d\theta dp_\phi' \sin \theta dp_\theta e^{-\beta Js^2[(p_\phi^2 + p^2)/2] - 1}$$

$$= \frac{8\pi^2 s^2}{\beta Js^2} (e^{\beta Js^2} - e^{-\beta Js^2}).$$

(30)

The classical free energy, average energy and entropy, are

$$-\beta F_{cl} = \log 16\pi^2 s^2 + \log \frac{\sinh \beta Js^2}{\beta Js^2},$$

(31)

$$\beta E_{cl} = 1 - \beta Js^2 \coth \beta Js^2,$$

(32)

$$S_{cl} = \log 16\pi^2 s^2 + 1 + \log \frac{\sinh \beta Js^2}{\beta Js^2} - \beta Js^2 \coth \beta Js^2.$$  

(33)

These expressions are in good agreement with the corresponding quantal results even for not very large $s$. Indeed, if $s$ is not too small, the quantal partition function,

$$Z_q = \sum_{\sigma=0}^{2s} (2\sigma + 1)e^{-\beta J(\frac{1}{2}\sigma(\sigma+1) - s(s+1))}$$

is well approximated by the classical partition function (30).

**Path integral computation of the partition function**

From (26), integration over the field $\mathbf{M}$ after returning to complex time, leads to

$$\mathcal{L} = \frac{1}{2J} \dot{\Omega}^2 - \lambda(\Omega^2 - 1).$$

(34)

This is the continuum form associated with

$$Z = \int d\Omega_k d\lambda_k \exp \left[ - \sum_k \left( \frac{1}{2J\epsilon} (\Omega_k - \Omega_{k+1})^2 + \epsilon \lambda (\Omega_k^2 - 1) - \epsilon Js^2 \right) \right]$$

$$= K \exp \left( \frac{\beta Js^2}{\beta} \right).$$

(35)

This result is valid for $\beta > 0$. It involves the approximation of neglecting the constraint $|\Omega| = Js|\mathbf{M}| \leq 2Js$. 


Generator Coordinate Method and quantal fluctuations

For

\[ |z\rangle = e^{zs} |s, -s\rangle, \] (36)

where \(|s, -s\rangle\) is such that \(S_+|s, -s\rangle = 0, \quad S_-|s, -s\rangle = -s|s, -s\rangle\), we have

\[
\begin{align*}
\langle z|z' \rangle &= (1 + z'z^*)^{2s}, \\
\langle z|S_+|z' \rangle &= \frac{2sz^*}{1 + z'z^*}(1 + z'z^*)^{2s}, \\
\langle z|S_-|z' \rangle &= \frac{2sz'}{1 + z'z^*}(1 + z'z^*)^{2s}, \\
\langle z|S_0|z' \rangle &= -s \frac{1 - z'z^*}{1 + z'z^*}(1 + z'z^*)^{2s}.
\end{align*}
\] (37)

Having in mind investigating the expectation values of the operator \(S_a \cdot S_b\), we notice that

\[
\frac{\langle z|S|z' \rangle \cdot \langle -z^{*-1}|S| - z'^{-*} \rangle}{\langle z|z' \rangle \langle -z^{*-1}| - z'^{-*} \rangle} = -s^2 \frac{(1 + z^*z)(1 + z'^*z') + (z^* - z'^*)(z - z')}{(1 + z^*z')(1 + z'^*z)},
\]

\[
\langle z|z' \rangle \langle -z^{*-1}| - z'^{-*} \rangle = (1 + z'z^*)^{2s} \left(1 + \frac{1}{z'^*z}\right)^{2s}.
\] (38)

According to the mean field approximation, the expectation values of the operator \(S_a \cdot S_b\) lie between \(-s^2\) (value which is reached in (38) for \(z = z'\)) and \(s^2\). In order to correct the lower bound, it is instructive to apply the Generator Coordinate Method to this system. In the present context, this amounts to diagonalizing the Hamiltonian \(H\) in the subspace spanned by the states \(|z' \rangle \otimes | - z'^{-*} \rangle\), which are such that the spins of the coherent states \(|z' \rangle\) and \(| - z'^{-*} \rangle\) point in opposite directions. To this end, let us assume that \(z = \eta\) and \(z' = \eta'\) are real and the coherent states \(|\eta\rangle, |\eta'\rangle, | - \eta^{-1}\rangle, | - \eta'^{-1}\rangle\), are normalized. Then the overlaps may be approximated by

\[
\mathcal{H}(\tilde{\eta}, \tilde{\eta}') = \frac{\langle \eta \otimes (-\eta^{-1})|H|\eta' \rangle \otimes | - \eta'^{-1}\rangle}{\langle \eta \otimes (-\eta^{-1})|I|\eta' \rangle \otimes | - \eta'^{-1}\rangle} \approx -Js^2 \left(1 + 2\frac{(\eta - \eta')^2}{(1 + \eta^2)^2}\right)
\]

\[
\approx -Js^2 \left(1 + 2(\tilde{\eta} - \tilde{\eta}')^2\right)
\] (39)

\[
\mathcal{N}(\tilde{\eta}, \tilde{\eta}') = \langle \eta \otimes (-\eta^{-1})|I|\eta' \rangle \otimes | - \eta'^{-1}\rangle \approx \exp\left(-2s\frac{(\eta - \eta')^2}{(1 + \eta^2)^2}\right)
\]

\[
\approx \exp\left(-2s(\tilde{\eta} - \tilde{\eta}')^2\right)
\] (40)

where \(\tilde{\eta} = (\eta + \eta')/2\) and \(\tilde{\eta} = \int d\eta/(1 + \eta^2)\). Minimizing the expectation value of \(H\) for a state of the form

\[
|\Psi\rangle = \int d\tilde{\eta}' f(\tilde{\eta}')|\eta'\rangle \otimes | - \eta'^{-1}\rangle
\] (41)

it is found that the ground state energy is

\[
E_0 = \frac{\int d\tilde{\eta}' \mathcal{H}(\tilde{\eta}, \tilde{\eta}') \mathcal{N}(\tilde{\eta}, \tilde{\eta}')}{\int d\tilde{\eta}' \mathcal{N}(\tilde{\eta}, \tilde{\eta}')} = -Js^2 \left(1 + \frac{1}{s}\right).
\] (42)

It is also instructive to apply the Generator Coordinate Method to a linear combination of states of the form \(|\eta\rangle \otimes |\eta\rangle\), which are such that the spins of both coherent states point
in the same direction. This calculation confirms that the maximum energy eigenvalue is $Js^2$.

6. Extended system: AF regime

We return to our 2D Heisenberg antiferromagnet. In terms of the variables $\Omega$, $M$, defined for each antiferromagnet cell, the Lagrangian density (per AF elementary cell) reads

$$L_{\Omega M} = L_{kin} + H, \quad L_{kin} = is\dot{\Omega} \cdot M, \quad H = Js^2 \sum_{l' \in \langle l \rangle} H_{ll'},$$

$$H_{ll'} = -(\Omega \cdot \Omega') \sqrt{1 - \frac{1}{4} M^2} \sqrt{1 - \frac{1}{4} M'^2} + \frac{1}{4} (\Omega \cdot \Omega')(M \cdot M') - \frac{1}{4} (M' \cdot \Omega)(M \cdot \Omega') + \frac{1}{2} (\Omega \times \Omega') \cdot \left(M' \sqrt{1 - \frac{1}{4} M^2} + M \sqrt{1 - \frac{1}{4} M'^2}\right),$$

(43)

where $\Omega = \Omega_{\tau l}$, $\Omega' = \Omega_{\tau l'}$, $M = M_{\tau l}$, $M' = M_{\tau l'}$. Here, calligraphic symbols are used in order to stress that the Lagrangian and Hamiltonian densities, respectively $L$ and $H$, refer to a specific AF elementary cell $l$. The quantity $1 - (\Omega \cdot \Omega') = \frac{1}{2} (\Omega - \Omega')^2$ is quadratic in $|\Omega - \Omega'|$. Moreover, in the quadratic order in $|\Omega - \Omega'|$ and $M$, we have $(M' \cdot \Omega)(M \cdot \Omega') \approx 0$, $(\Omega \times \Omega') \cdot \left(M' \sqrt{1 - \frac{1}{4} M^2} + M \sqrt{1 - \frac{1}{4} M'^2}\right) \approx 0$. Thus, the physically interesting quadratic part of the Lagrangian density is

$$L^{(2)} = is\dot{\Omega} \cdot M + Js^2 \sum_{l' \in \langle l \rangle} \left[1 - (\Omega \cdot \Omega') + \frac{1}{2} (M \cdot M)\right].$$

(44)

7. Extended system: ferromagnetic regime

In the ferromagnetic regime, $M = 2$ so that $H_{ll'}$ reduces to

$$H_{ll'} = \frac{1}{4} (\Omega \cdot \Omega')(M \cdot M') - \frac{1}{4} (M' \cdot \Omega)(M \cdot \Omega') = N \cdot N'.$$

(45)

The Lagrangian density (per AF elementary cell) becomes, in this regime,

$$L_{ferro} = 2is\dot{Q} \cdot N + Js^2 \sum_{l' \in \langle l \rangle} (N \cdot N'),$$

(46)

where $\dot{Q} = N \times \dot{\Omega}$ and $N = 1$. If $J > 0$, this regime only occurs for negative temperatures.

8. System of four spins
The system we wish to consider now is a cluster of spins constituted by two antiferromagnetic cells such as is described by the Heisenberg Hamiltonian

\[ H = J(S_{a1} \cdot S_{b1} + S_{a2} \cdot S_{b2} + S_{a1} \cdot S_{b2} + S_{a2} \cdot S_{b1}). \]  

(47)

The geometrical arrangement of the spins in the cluster is a ring. Some of the formulae in this section are special cases of those given in section 6 and are only explicitly presented here for the sake of clarity. The classical Lagrangian corresponding to the Hamiltonian (47) reads

\[
L = s \frac{-n_{b1} \cdot (n_{a1} \times \dot{n}_{a1}) - n_{a1} \cdot (n_{b1} \times \dot{n}_{b1})}{(1 - n_{a1} \cdot n_{b1})} + s \frac{-n_{b2} \cdot (n_{a2} \times \dot{n}_{a2}) - n_{a2} \cdot (n_{b2} \times \dot{n}_{b2})}{(1 - n_{a2} \cdot n_{b2})}
- Js^2(n_{a1} \cdot n_{b1} + n_{a2} \cdot n_{b2} + n_{a1} \cdot n_{b2} + n_{a2} \cdot n_{b1})
\]  

(48)

If to each antiferromagnetic cell we associate the variables \( \mathbf{N}, \mathbf{\Omega} \) defined by eq. (24) we obtain

\[
\mathbf{n}_a = \mathbf{N} + \mathbf{\Omega} \sqrt{1 - \mathbf{N}^2}, \quad \mathbf{n}_b = \mathbf{N} - \mathbf{\Omega} \sqrt{1 - \mathbf{N}^2}.
\]

In terms of the new variables, the Lagrangian becomes

\[
L = 2s\mathbf{N}_1 \cdot (\mathbf{\Omega}_1 \times \dot{\mathbf{\Omega}}_1) + 2s\mathbf{N}_2 \cdot (\mathbf{\Omega}_2 \times \dot{\mathbf{\Omega}}_2)
- Js^2 \left(2\mathbf{N}_1^2 + 2\mathbf{N}_2^2 - 2 + 2\mathbf{N}_1 \cdot \mathbf{N}_2 - 2\mathbf{\Omega}_1 \cdot \mathbf{\Omega}_2 \sqrt{1 - \mathbf{N}_1^2} \sqrt{1 - \mathbf{N}_2^2}\right).
\]  

(49)

To this Lagrangian it is necessary to add the subsidiary conditions

\[
\Omega_1^2 - 1 = \Omega_2^2 - 1 = \mathbf{\Omega}_1 \cdot \mathbf{N}_1 = \mathbf{\Omega}_2 \cdot \mathbf{N}_2 = 0.
\]

For convenience, we have used here the variables \( \mathbf{N}, \mathbf{\Omega} \), instead of the variables \( \mathbf{M}, \mathbf{\Omega} \) used in in (43) for an extended antiferromagnet. However, it is clear that, for two AF elementary cells, (43) and (49) are equivalent. In the AF regime, \( \mathbf{N}_1, \mathbf{N}_2 \) and \( |\mathbf{\Omega}_1 - \mathbf{\Omega}_2| \) are small quantities which, in the classical limit, vanish for the ground state. The ground state energy (AF regime) is \(-4Js^2\). On the other hand, in the ferromagnetic regime, we have \( \mathbf{N}_1 = \mathbf{N}_2 = 1 \). The upper-state classical energy (ferromagnetic regime) is \( 4Js^2 \).

Low energy excitations (for which \( \mathbf{N}_1, \mathbf{N}_2 \) and \( |\mathbf{\Omega}_1 - \mathbf{\Omega}_2| \) are small) are described by the quadratic Lagrangian

\[
L^{(2)} = 2s\mathbf{N}_1 \cdot (\mathbf{\Omega}_1 \times \dot{\mathbf{\Omega}}_1) + 2s\mathbf{N}_2 \cdot (\mathbf{\Omega}_2 \times \dot{\mathbf{\Omega}}_2)
- Js^2 \left(3\mathbf{N}_1^2 + 3\mathbf{N}_2^2 + 2\mathbf{N}_1 \cdot \mathbf{N}_2 + (\mathbf{\Omega}_1 - \mathbf{\Omega}_2)^2\right)
\]  

(50)

It is convenient to consider the quantities

\[
\mathbf{M}_1 = 2\mathbf{N}_1 \times \mathbf{\Omega}_1, \quad \mathbf{M}_2 = 2\mathbf{N}_2 \times \mathbf{\Omega}_2
\]
in terms of which we have

\[ L^{(2)} = s \mathbf{M}_1 \cdot \dot{\mathbf{\Omega}}_1 + s \mathbf{M}_2 \cdot \dot{\mathbf{\Omega}}_2 - \frac{J}{4} s^2 \left( 3 M_1^2 + 3 M_2^2 + 2 \mathbf{M}_1 \cdot \mathbf{M}_2 + 4 (\mathbf{\Omega}_1 - \mathbf{\Omega}_2)^2 \right) \]

\[ = \frac{s}{2} (\mathbf{M}_1 + \mathbf{M}_2) \cdot (\dot{\mathbf{\Omega}}_1 + \dot{\mathbf{\Omega}}_2) + \frac{s}{2} (\mathbf{M}_1 - \mathbf{M}_2) \cdot (\dot{\mathbf{\Omega}}_1 - \dot{\mathbf{\Omega}}_2) \]

\[ - \frac{J}{4} s^2 \left( 2 (\mathbf{M}_1 + \mathbf{M}_2)^2 + (\mathbf{M}_1 - \mathbf{M}_2)^2 + 4 (\mathbf{\Omega}_1 - \mathbf{\Omega}_2)^2 \right). \tag{51} \]

It follows that \( \mathbf{M}_1 + \mathbf{M}_2 \) is a constant of motion. In order to present its interpretation we go back to (49), making \( \mathbf{\Omega}_1 = \mathbf{\Omega}_2 = \mathbf{\Omega} \), \( N_1 = N_2 = N \). This leads to the Lagrangian of a rigid rotor

\[ L' = 4 s N \cdot (\mathbf{\Omega} \times \dot{\mathbf{\Omega}}) - J s^2 \left( 8 N^2 - 4 \right) \]

\[ = s \mathbf{M}_t \cdot \dot{\mathbf{\Omega}} - \frac{J}{2} s^2 (M_t^2 - 8). \]

Here, \( \mathbf{M}_t = 4 \mathbf{N} \times \mathbf{\Omega}. \)

We investigate now the energy spectrum. The Hamiltonian associated with the Lagrangian (51), namely

\[ \mathcal{H} = \frac{J}{4} s^2 \left( 3 M_1^2 + 3 M_2^2 + 2 \mathbf{M}_1 \cdot \mathbf{M}_2 + 4 (\mathbf{\Omega}_1 - \mathbf{\Omega}_2)^2 \right), \]

is the sum of a term describing a rigid rotor

\[ \frac{J}{2} s^2 (M_t^2 - 8) \]

and another term describing a harmonic oscillator on a sphere,

\[ J s^2 \left( \left( \frac{\mathbf{M}_1 - \mathbf{M}_2}{2} \right)^2 + (\mathbf{\Omega}_1 - \mathbf{\Omega}_2)^2 \right), \]

which, for simplicity, we identify with a simple two dimensional harmonic oscillator. This is, admittedly, a rather drastic approximation. It is then natural to assume that the energy spectrum is given by

\[ E_{j, \nu_1, \nu_2} = J \left( -4 s (s + 1) + \frac{1}{2} j (j + 1) \right) + 2 J s \left( 1 + \nu_1 + \nu_2 \right). \tag{52} \]

with \( j = 0, 1, \ldots, 4s \) and \( \nu_1, \nu_2 = 0, 1, 2 \). The multiplicity associated with the quantum number \( j \) is \( 2j + 1 \). For \( s = \frac{1}{2} \), this expression predicts 4 energy levels, namely \(-2J, -J, 0, J\).

The eigenvalues and eigenvectors of the hamiltonian (47) are easily obtained and are shown in the Appendix. It is remarkable that the exact energy spectrum of (47) coincides with the one predicted by (52). Unfortunately, the interpretation of the multiplicities is not
so straightforward. The multiplicities 1, 3, 7, 5, determined by the exact diagonalization of the Hamiltonian (47), may be understood as follows. The lowest energy level $E = -2J$, corresponds to $j = \nu_1 = \nu_2 = 0$. The second energy level, $E = -J$, corresponds either to $j = 1$, or to $(\nu_1 = 1, \nu_2 = 0)$ or to $(\nu_2 = 1, \nu_1 = 0)$. The multiplicity 3 of this level may be explained assuming that the oscillator level $(\nu_1 = 1, \nu_2 = 0)$ coincides with the rotor level $(j = 1, m = 1)$ while the oscillator level $(\nu_1 = 0, \nu_2 = 1)$ coincides with the rotor level $(j = 1, m = -1)$. The third energy level $E = 0$, corresponds either to the $j = 1$ rotor levels combined with one of the harmonic oscillator levels $(\nu_1 = 1, \nu_2 = 0)$, or $(\nu_1 = 0, \nu_2 = 1)$, or to the rotor level $j = 0$ combined with $(\nu_1 = 2, \nu_2 = 0)$, or $(\nu_1 = 1, \nu_2 = 1)$, or $(\nu_1 = 0, \nu_2 = 2)$. The multiplicity 7 of this level may be explained assuming that the combination of $(j = 1, m = 1)$ with $(\nu_1 = 1, \nu_2 = 0)$ is equivalent to the combination of $j = 0$ with $(\nu_1 = 2, \nu_2 = 0)$ and the combination of $(j = 1, m = -1)$ with $(\nu_1 = 0, \nu_2 = 2)$ is equivalent to the combination of $j = 0$ with $(\nu_1 = 0, \nu_2 = 1)$. The fourth and highest energy level $E = J$ is simply made up of the 5 rotational states $j = 2$.

9. Concluding remarks

A simple derivation of an explicitly rotation invariant Lagrangian appropriate to a path integral description of the dynamics of an AF spin system was presented. The scope of the derived Lagrangian was analysed in the context of schematic models. In particular, it was found that the properties of clusters of two and four spins are well accounted for by the derived Lagrangian. It was shown that the Lagrangian obtained covers the whole region from the AF to the ferromagnetic regimes.

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Appendix

For completeness, the eigenvalues and eigenvectors of the Hamiltonian (47) are presented in the Appendix. They may be written

$$E = -2J; \quad |0\rangle = \frac{1}{2\sqrt{3}}(2|\uparrow\uparrow\downarrow\rangle + 2|\downarrow\uparrow\uparrow\rangle - |\downarrow\uparrow\downarrow\rangle),$$

$$E = -J; \quad |1,1\rangle = \frac{1}{2}(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle - |\downarrow\downarrow\uparrow\rangle),$$

$$E = 0; \quad |2,1\rangle = \frac{1}{2}(|\uparrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle - |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle),$$

$$E = J; \quad |3,1\rangle = |\uparrow\uparrow\uparrow\uparrow\rangle,$$

Here, $|0\rangle$ denotes the ground state vector and $|i,j\rangle$ denotes the state vector of the $j$th member of the $i$th excited energy eigenspace. Otherwise an obvious notation is used.