On the optimality of a $\ell_1/\ell_1$ solver for sparse signal recovery from sparsely corrupted compressive measurements.

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Abstract

This short note proves the $\ell_2 - \ell_1$ instance optimality of a $\ell_1/\ell_1$ solver, i.e., a variant of basis pursuit denoising with a $\ell_1$ fidelity constraint, when applied to the estimation of sparse (or compressible) signals observed by sparsely corrupted compressive measurements. The approach simply combines two known results due to Y. Plan, R. Vershynin and E. Candès.

Conventions: Most of domain dimensions (e.g., $M$, $N$) are denoted by capital roman letters. Vectors and matrices are associated to bold symbols while lowercase light letters are associated to scalar values. The $i^{th}$ component of a vector $u$ is $u_i$ or $(u)_i$. The identity matrix is $\text{Id}$. The set of indices in $\mathbb{R}^D$ is $[D] = \{1, \cdots, D\}$. Scalar product between two vectors $u, v \in \mathbb{R}^D$ reads $u^*v = \langle u, v \rangle$ (using the transposition $(\cdot)^*$). For any $p \geq 1$, $\| \cdot \|_p$ represents the $\ell_p$-norm such that $\|u\|_p^p = \sum_i |u_i|^p$ with $\|u\| = \|u\|_2$ and $\|u\|_\infty = \max_i |u_i|$. The $\ell_0$ “norm” is $\|u\|_0 = \#\text{supp } u$, where $\#$ is the cardinality operator and $\text{supp } u = \{i : u_i \neq 0\} \subseteq [D]$. For $S \subseteq [D]$, $u_S \in \mathbb{R}^\#S$ (or $\Phi_S$) denotes the vector (resp. the matrix) obtained by retaining the components (resp. columns) of $u \in \mathbb{R}^D$ (resp. $\Phi \in \mathbb{R}^{D \times D}$) belonging to $S \subseteq [D]$. The operator $H_K$ is the hard thresholding operator setting all the coefficients of a vector to 0 but those having the $K$ strongest amplitudes. The set of canonical $K$-sparse signals in $\mathbb{R}^N$ is $\Sigma_K = \{v \in \mathbb{R}^N : \|v\|_0 \leq K\}$. $B_2^N$ and $S_{N-1}$ are the $\ell_2$ ball and $(N - 1)$-sphere in $\mathbb{R}^N$, respectively. Finally, the operator $\text{sign } \lambda$, which equals to 1 if $\lambda$ is positive and $-1$ otherwise, is applied component wise onto vectors.

1 Introduction

Let us consider the case where a sparse (or compressible) signal $x \in \mathbb{R}^N$ is observed with a random Gaussian matrix $\Phi \sim \mathcal{N}^{M \times N}(0, 1)$,

$$y = \Phi x + n,$$

with a sparse (or Laplacian) noise $n$ of bounded $\ell_1$-power, i.e., there exists a bound $\epsilon > 0$ such that $\|n\|_1 \leq \epsilon$ with high (and controlled) probability.

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In this short note, we prove the stability of a variant of the basis pursuit denoising program, namely
\[
\arg\min_{u \in \mathbb{R}^N} \|u\|_1 \quad \text{s.t.} \quad \|y - \Phi u\|_1 \leq \epsilon, \tag{BPDN-\ell_1}
\]
in estimating \(x\) from \(y\) under an \(\ell_1\)-fidelity constraint. The mathematical tools we are going to use are those developed in the recent work of Y. Plan and R. Vershynin in the context of 1-bit compressed sensing \[1\] combined with Candès’ simplified proof of basis pursuit denoising \(\ell_2 - \ell_1\)-instance optimality \[2\]. No elements are specially new except their combination. In particular, it is interesting to see how these two pieces of works fit nicely in order to reach the announced objective.

## 2 BPDN-\(\ell_1\) instance optimality

Here is the main result of this note.

**Theorem 1.** Let \(\Phi \in \mathbb{R}^{M \times N}\) be a sensing matrix used in (1) and assume that there exist 3 constants \(\delta_{2K}, \delta_{3K} \in (0, 1)\) and \(\nu > 0\) such that, for all \(u \in \Sigma_{2K}\) and \(v \in \Sigma_K\) with \(\langle u, v \rangle = 0\),
\[
\left| \frac{1}{\pi} \| \Phi u \|_1 - \nu \| u \| \right| \leq \delta_{2K} \| u \|, \tag{2}
\]
\[
\left| \frac{1}{\pi} \langle \text{sign}(\Phi u), \Phi v \rangle \right| \leq \delta_{3K} \| v \|. \tag{3}
\]

Then, if \(\delta_{2K} + \delta_{3K} \leq \nu - \frac{1}{2}\), the solution \(x^*\) of BPDN-\(\ell_1\) respects
\[
\|x^* - x\| \leq 8 \frac{\epsilon}{M} + 12 e_0(K),
\]
with \(e_0(K) = \|x - x_K\|_1 / \sqrt{K}\).

Before to prove this theorem, the following lemma (mainly a rewriting of a result given in \[1\]) assures us on the feasibility of the conditions (2) and (3).

**Lemma 1.** Let \(N, M, K \in \mathbb{N}\) and \(\delta \in [0, 1]\). There exist two constants \(C, c > 0\) such that, for
\[
M \geq C\delta^{-6}K \log(2N/K) \tag{4}
\]
and \(\Phi \sim \mathcal{N}^{M \times N}(0, 1)\), we have, with a probability at least \(1 - 8 \exp(-c\delta^2M)\),
\[
\left| \frac{1}{\pi} \| \Phi u \|_1 - \sqrt{\frac{2}{\pi}} \| u \| \right| \leq \delta \| u \|, \tag{5}
\]
\[
\left| \frac{1}{\pi} \langle \text{sign}(\Phi u), \Phi v \rangle \right| \leq \delta \| v \|, \tag{6}
\]
for all \(u, v \in \Sigma_K\) with \(\langle u, v \rangle = 0\).

**Proof.** Let us write \(K = \Sigma_K \cap B^N_2\) and \(K^* = \Sigma_K \cap S^{N-1}\). Using \[1\], Prop. 4.3 with \(\tau = 0\), we know that there exist two constants \(C, c > 0\) such that if
\[
M \geq C\delta^{-6}K \log(2N/K)
\]
and if \( \Phi = (\varphi_1, \cdots, \varphi_M)^T \sim \mathcal{N}^{M \times N}(0, 1) \) with \( \varphi_i \in \mathbb{R}^N \) (1 \( \leq i \leq M \)), then, with probability at least \( 1 - 8 \exp(-c\delta^2 M) \),

\[
\sup_{a \in \mathcal{K}^*, b \in \mathcal{K} - \mathcal{K}} | f_a(b) - \mathbb{E} f_a(b) | \leq \delta,
\]

where \( f_a(b) := \frac{1}{M} \sum_j \text{sign} (\langle \varphi_j, a \rangle / \langle \varphi_j, b \rangle) \). Knowing that \( \mathbb{E} f_a(b) = \sqrt{\frac{2}{\pi}} \langle a, b \rangle \), this means that, under the same conditions,

\[
\sup_{a \in \mathcal{K}^*, b \in \mathcal{K} - \mathcal{K}} \left| \frac{1}{M} (\text{sign} (\Phi a), \Phi b) - \sqrt{\frac{2}{\pi}} \langle a, b \rangle \right| \leq \delta.
\]

In particular, for any \( u, v \in \Sigma_{\mathcal{K}} \), since \( u/\|u\| \in \mathcal{K}^* \) and \( v/\|v\| \in \mathcal{K}^* \subset \mathcal{K} - \mathcal{K} \), we have

\[
\left| \frac{1}{M} \langle \text{sign} (\Phi u), \Phi v \rangle - \sqrt{\frac{2}{\pi}} \|u\|^{-1} \langle u, v \rangle \right| \leq \delta \|v\|.
\]

Therefore, if \( \langle u, v \rangle = 0 \), \( \frac{1}{M} | \langle \text{sign} (\Phi u), \Phi v \rangle | \leq \delta \|v\| \), while taking \( u = v \) leads to

\[
\left| \frac{1}{M} \| \Phi u \| - \sqrt{\frac{2}{\pi}} \|u\| \right| \leq \delta \|u\|.
\]

\( \square \)

**Remarks on \( \delta \):** The dependency in \( \delta^{-6} \) in (4) is probably not optimal and could be improved. This is actually due to the fact that this lemma is extendable to much more general sets than \( \mathcal{K} \) (e.g., compressible signals) [1]. For having only (5), [3, Lemma 5.3] shows that a dependency in \( \delta \) is allowed. Moreover, [4] shows that (5) holds of \( M \geq M_0 \) with \( M_0 = O(\delta^{-2} K \log N/K) \). Proving that (6) is respected from the same number of measurements is an open problem.

**Proof of Theorem 1.** We follow partially the procedure given in [2] with an adaption due to the \( \ell_1 \)-norm fidelity of BPDN-\( \ell_1 \). Let us write \( x^* \) the solution of BPDN-\( \ell_1 \) and \( x^* = x + h \). In order to bound the reconstruction error of BPDN-\( \ell_1 \), we have to characterize the behavior of \( \| x^* - x \| = \| h \| \).

We define \( T_0 = \text{supp} x_K \) and a partition \( \{ T_k : 1 \leq k \leq \lceil (N - K)/K \rceil \} \) of the support of \( h_{T_0} \). This partition is determined by ordering elements of \( h \) off of the support of \( x_K \) in decreasing absolute value. We have \( |T_k| = K \) for all \( k \geq 1 \), \( T_k \cap T_{k'} = \emptyset \) for \( k \neq k' \), and crucially that \( |h_j| \leq |h_i| \) for all \( j \in T_{k+1} \) and \( i \in T_k \).

We start from

\[
\| h \| \leq \| h_{T_0} \| + \| h_{T_0}^c \|, \tag{7}
\]

with \( T_{01} = T_0 \cup T_1 \), and we are going to bound separately the two terms of the RHS. In [2], it is proved that

\[
\| h_{T_0} \| \leq \sum_{k \geq 2} \| h_{T_k} \| \leq \| h_{T_{01}} \| + 2e_0(K), \tag{8}
\]

with \( e_0(K) = \frac{1}{\sqrt{K}} \| x_{T_0} \|_1 \). Therefore,

\[
\| h \| \leq 2 \| h_{T_{01}} \| + 2e_0(K). \tag{9}
\]

Let us bound now \( \| h_{T_{01}} \| \). We have

\[
\| \Phi h_{T_{01}} \|_1 = \langle \text{sign} (\Phi h_{T_{01}}), \Phi h_{T_{01}} \rangle = \langle \text{sign} (\Phi h_{T_{01}}), \Phi h \rangle - \sum_{k \geq 2} \langle \text{sign} (\Phi h_{T_{01}}), \Phi h_{T_k} \rangle.
\]
By Hölder inequality,
\[
\langle \text{sign}(\Phi h_{T_0}), \Phi h \rangle \leq \|\Phi h\|_1 \leq \|\Phi x - y\|_1 + \|\Phi x - y\|_1 \leq 2\epsilon.
\]

For any \(k \geq 2\), since \(h_{T_0}\) and \(h_{T_k}\) are \(2K\)- and \(K\)-sparse, respectively, with \(\langle h_{T_0}, h_{T_k} \rangle = 0\), we know from (3) that
\[
|\langle \text{sign}(\Phi h_{T_0}), \Phi h_{T_k} \rangle| \leq M \delta_3 K \|h_{T_k}\|.
\]

Therefore, using (2) and (8),
\[
M(\nu - \delta_2 K) \|h_{T_0}\| \leq \|\Phi h_{T_0}\|_1 \leq 2\epsilon + M \delta_3 K \sum_{k \geq 2} \|h_{T_k}\|
\leq 2\epsilon + M \delta_3 K (\|h_{T_0}\| + 2\epsilon_0(K)),
\]
or equivalently,
\[
\|h_{T_0}\| \leq \frac{2}{\nu - (\delta_2 K + \delta_3 K)} \left( \frac{\epsilon}{M} + \delta_3 K \epsilon_0(K) \right).
\]

Using (9), we find,
\[
\|h\| \leq \frac{4}{\nu - (\delta_2 K + \delta_3 K)} \frac{\epsilon}{M} + 4 \frac{\nu + \delta_2 K - \delta_2 K}{\nu - (\delta_2 K + \delta_3 K)} \epsilon_0(K).
\]

Finally, taking \(\delta_2 K + \delta_3 K \leq \nu - \frac{1}{2}\) provides the result. \(\square\)

References

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