COMPUTATION OF TRANSMISSION EIGENVALUES FOR ELASTIC WAVES

XIA JI, PEIJUN LI, AND JIGUANG SUN

Abstract. The goal of this paper is to develop numerical methods computing a few smallest elasticity transmission eigenvalues, which are of practical importance in inverse scattering theory. The problem is challenging since it is nonlinear, non-self-adjoint, and of fourth order. We construct a nonlinear function whose values are generalized eigenvalues of a series of self-adjoint fourth order problems. The roots of the function are the transmission eigenvalues. Using an $H^2$-conforming finite element for the self-adjoint fourth order eigenvalue problems, we employ a secant method to compute the roots of the nonlinear function. The convergence of the proposed method is proved. In addition, a mixed finite element method is developed for the purpose of verification. Numerical examples are presented to verify the theory and demonstrate the effectiveness of the two methods.

1. Introduction

Transmission eigenvalues have important applications in inverse scattering theory. For example, they can be used to obtain useful information on the physical properties of the scattering targets [8,25]. In this paper, we consider the interior transmission eigenvalue problem for elastic waves. Similar to the cases of acoustic and electromagnetic waves, the elasticity transmission eigenvalue (ETE) problem plays a critical role in the qualitative reconstruction methods for inhomogeneous media. There are only a few theoretical studies on the ETE problem [3,10,11]. It is shown in [3] that there exists a countable set of elasticity transmission eigenvalues under suitable conditions on elastic tensors and mass densities.

Numerical methods for the acoustic transmission eigenvalues have been developed by many researchers recently [1,9,12,10,22,21,25]. However, there exist much fewer papers [10,23,25] for the electromagnetic transmission eigenvalue problems. It is highly non-trivial to develop finite element methods for the transmission eigenvalue problems in general since the problem is nonlinear and non-self-adjoint [27]. Although out of the scope of the current paper, it is useful to point out that the finite element discretization usually leads to non-Hermitian matrix eigenvalue problems. It is challenging to compute (interior) generalized eigenvalues for non-Hermitian matrices. In particular, when the size of matrices is large and there is no spectrum information, classical methods in numerical linear algebra would fail. New methods have emerged to treat such difficult problems [17,18].

The goal of this paper is to develop effective numerical methods to compute a few smallest real transmission eigenvalues, which can be used to estimate material property of the elastic body (see, e.g., [25]). Unlike the classical Laplacian eigenvalue problem or the biharmonic eigenvalue problem, the transmission eigenvalue problem is nonlinear and non-self-adjoint. To overcome this issue, we reformulate the problem as a combination of a nonlinear function and a series of fourth order self-adjoint eigenvalue problems. Specifically, the ETE is first written as a nonlinear fourth order problem, which turns out to be a quadratic eigenvalue problem. To avoid dealing with the non-self-adjointness directly, we construct a nonlinear function whose roots are the elasticity transmission eigenvalues. The values of the nonlinear function are generalized eigenvalues of self-adjoint coercive fourth order problems, which can be treated using classical $H^2$-conforming finite elements. A secant based iterative method is adopted to compute the roots of the nonlinear function. In addition, we give a mixed method using the Lagrange elements for the purpose of verification.

The current paper, to the authors’ knowledge, is the first numerical study on the ETE. We hope that it can attract more numerical analysts to this interesting and challenging topic. The rest of the paper is

2000 Mathematics Subject Classification. 65N25, 65N30, 47B07.
Key words and phrases. Transmission eigenvalue problem, elastic wave equation, finite element method.
The research of X. Ji is partially supported by the National Natural Science Foundation of China with Grant Nos. 11271018 and 91630313, and National Center for Mathematics and Interdisciplinary Sciences, Chinese Academy of Sciences. The research of P. Li was supported in part by the NSF grant DMS-1151308. The research of J. Sun was supported in part by the NSF grant DMS-1521555.
organized as follows. In Section 2, we introduce the elasticity transmission eigenvalue problem and derive a quadratic eigenvalue problem based on a fourth order partial differential equation. To avoid direct treatment of the nonlinearity and nonself-adjointness, the problem is decomposed into a nonlinear function and a series of related linear self-adjoint fourth order eigenvalue problems. The values of the nonlinear function are generalized eigenvalues of the fourth order problems. The roots of the nonlinear function are transmission eigenvalues. \( H^2 \)-conforming Argyris element for the fourth order problems is presented in Section 3. A secant based iterative method is used in Section 4 to compute roots of the nonlinear function. Section 5 introduces a mixed finite element method for verification. Numerical experiments are presented in Section 6. The paper is concluded with some discussion and future works in Section 7.

2. The Elasticity Transmission Eigenvalue Problem

Let \( \mathbf{x} = (x, y)^T \in \mathbb{R}^2 \) and \( D \subset \mathbb{R}^2 \) be a bounded Lipschitz domain. Consider the two-dimensional elastic wave problem of finding \( \mathbf{u} \) with zero trace on the boundary of \( D \), i.e., \( \Gamma \), such that
\[
\nabla \cdot \sigma(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{0} \quad \text{in} \quad D \subset \mathbb{R}^2,
\]
where \( \mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))^T \) is the displacement vector of the wave field, \( \omega > 0 \) is the angular frequency, \( \rho \) is the mass density, and \( \sigma(\mathbf{u}) \) is the stress tensor given by the generalized Hooke law
\[
\sigma(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda \varepsilon(\mathbf{u}) I.
\]
Here the strain tensor \( \varepsilon(\mathbf{u}) \) is given by
\[
\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T),
\]
where the two constants \( \mu, \lambda \) are called the Lamé parameters satisfying \( \mu > 0, \lambda + \mu > 0, \) \( I \in \mathbb{R}^{2 \times 2} \) is the identity matrix, and \( \nabla \mathbf{u} \) is the displacement gradient tensor
\[
\nabla \mathbf{u} = \begin{bmatrix}
\partial_x u_1 & \partial_y u_1 \\
\partial_x u_2 & \partial_y u_2 
\end{bmatrix}.
\]
Explicitly, we have
\[
\sigma(\mathbf{u}) = \begin{bmatrix}
(\lambda + 2\mu)\partial_x u_1 + \lambda\partial_y u_2 & \mu(\partial_y u_1 + \partial_x u_2) \\
\mu(\partial_y u_1 + \partial_x u_2) & \lambda\partial_x u_1 + (\lambda + 2\mu)\partial_y u_2
\end{bmatrix}.
\]

Given \( \mathbf{u}, \mathbf{v} \in H_0^1(D)^2 \), it follows from the integration by parts that
\[
(\sigma(\mathbf{u}), \nabla \mathbf{v}) = \int_D \sigma(\mathbf{u}) : \nabla \mathbf{v} \, dx = \int_D (2\mu\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) + \lambda(\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v})) \, dx,
\]
where \( A : B = \text{tr}(AB^T) \) is the Frobenius inner product of square matrices \( A \) and \( B \). We recall the first Korn inequality [5 Corollary 11.2.25]: there exists a positive constant \( C \) such that
\[
\|\varepsilon(\mathbf{u})\|_{L^2} \geq C\|\mathbf{u}\|_{H^1}, \quad \text{for all} \quad \mathbf{u} \in H_0^1(D)^2,
\]
which guarantees the well-posedness of \( \mathbf{u}, \mathbf{v} \).

Let \( \mu_0, \lambda_0 \) be the Lamé parameters of the free space. Assume the domain \( D \) is filled with a homogeneous and isotropic elastic medium with Lamé constants \( \lambda_1 \) and \( \mu_1 \). The transmission eigenvalue problem for the elastic waves is to find values of \( \omega^2 \) such that there exists non-trivial solutions \( \mathbf{u}, \mathbf{v} \) satisfying
\[
\begin{align*}
\nabla \cdot \sigma_0(\mathbf{u}) + \omega^2 \rho_0 \mathbf{u} &= \mathbf{0} \quad \text{in} \quad D, \\
\nabla \cdot \sigma_1(\mathbf{v}) + \omega^2 \rho_1 \mathbf{v} &= \mathbf{0} \quad \text{in} \quad D, \\
\mathbf{u} &= \mathbf{v} \quad \text{on} \quad \Gamma, \\
\sigma_0(\mathbf{u}) \mathbf{\nu} &= \sigma_1(\mathbf{v}) \mathbf{\nu} \quad \text{on} \quad \Gamma,
\end{align*}
\]
where
\[
\sigma_i(\mathbf{u}) = \begin{bmatrix}
(\lambda_i + 2\mu_i)\partial_x u_1 + \lambda_i\partial_y u_2 & \mu_i(\partial_y u_1 + \partial_x u_2) \\
\mu_i(\partial_y u_1 + \partial_x u_2) & \lambda_i\partial_x u_1 + (\lambda_i + 2\mu_i)\partial_y u_2
\end{bmatrix}, \quad i = 0, 1,
\]
and \( \sigma \mathbf{\nu} \) denotes the matrix multiplication of the stress tensor \( \sigma \) and the normal vector \( \mathbf{\nu} \).

In this paper, we consider the case when \( \rho_0 \neq \rho_1, \sigma_0 = \sigma_1 = \sigma \), i.e., the case of equal elastic tensors [3]. In addition, we assume that the mass density distributions satisfy the following conditions
\[
p \leq \rho_0(\mathbf{x}) \leq P, \quad p_s \leq \rho_1(\mathbf{x}) \leq P_s, \quad \mathbf{x} \in D,
\]
where \( p, p_\ast \) and \( P, P_\ast \) are positive constants.

Define the Sobolev space
\[
V = \{ \phi \in H^2(D)^2 : \phi = 0 \text{ and } \sigma(\phi) \nu = 0 \text{ on } \Gamma \}.
\]

Let \( w = u - v \). The transmission eigenvalue problem can be formulated as follows: Find \( \omega^2 \) and \( w \neq 0 \) such that
\[
(\nabla \cdot \sigma + \omega^2 \rho_1)(\rho_1 - \rho_0)^{-1}(\nabla \cdot \sigma + \omega^2 \rho_0) w = 0.
\]

The corresponding weak formulation of (2.7) is to find \( \omega^2 \in \mathbb{C} \) and \( 0 \neq w \in V \) such that
\[
((\rho_1 - \rho_0)^{-1}(\nabla \cdot \sigma + \omega^2 \rho_0) w, (\nabla \cdot \sigma + \omega^2 \rho_1) \varphi) = 0 \quad \text{for all } \varphi \in V.
\]

Let \( \tau = \omega^2 \). We define two sesquilinear forms on \( V \times V \)
\[
\mathcal{A}_\tau(\phi, \varphi) = ((\rho_1 - \rho_0)^{-1}(\nabla \cdot \sigma + \tau \rho_0) \phi, (\nabla \cdot \sigma + \tau \rho_0) \varphi) + \tau^2(\rho_0 \phi, \varphi),
\]
\[
\mathcal{B}(\phi, \varphi) = (\sigma(\phi), \nabla \varphi).
\]

It is clear that \( \mathcal{A}_\tau \) is symmetric. Due to (2.3), \( \mathcal{B} \) is also symmetric.

The variational problem (2.8) can be written equivalently as follows: Find \( \tau \in \mathbb{C} \) and \( 0 \neq w \in V \) such that
\[
\mathcal{A}_\tau(w, \varphi) = \tau \mathcal{B}(w, \varphi) \quad \text{for all } \varphi \in V.
\]

This is a nonlinear problem since \( \tau \) appears on both sides of the equation. For a fixed \( \tau \), we consider an associated generalized eigenvalue problem
\[
\mathcal{A}_\tau(w, \varphi) = \gamma(\tau)\mathcal{B}(w, \varphi) \quad \text{for all } \varphi \in V.
\]

Formally, \( \tau \) is a transmission eigenvalue if \( \tau \) is a root of the nonlinear function
\[
f(\tau) := \gamma(\tau) - \tau.
\]

In the rest of this section, we study the generalized eigenvalue problem (2.10). It is shown in [15] that there exists \( \beta > 0 \) such that
\[
\|\nabla \cdot \sigma(\phi)\|^2 + \|\phi\|^2 \geq \beta \|\phi\|^2_{H^2(D)^2} \quad \text{for } \phi \in V.
\]

The following lemma is useful in the subsequent analysis. The proof can be found in [3].

**Lemma 2.1.** Assume that \( p_\ast \geq 1 \geq P \). Then \( \mathcal{A}_\tau \) is a coercive sesquilinear form on \( V \times V \), i.e., there exists a constant \( \alpha > 0 \) such that
\[
\mathcal{A}_\tau(\phi, \phi) \geq \alpha \|\phi\|^2 \quad \text{for all } \phi \in V.
\]

The source problem associated with (2.10) is to find \( u \in V \) such that, for \( f \in H^1(D)^2 \)
\[
\mathcal{A}_\tau(u, \phi) = (\sigma(f), \nabla \phi) \quad \text{for all } \phi \in V.
\]

The following theorem is a direct consequence of the Lax–Milgram Lemma.

**Theorem 2.2.** There exists a unique solution \( u \in V \) to (2.12). Furthermore, it holds that
\[
\|u\|_{H^2(D)^2} \leq C\|f\|_{H^1(D)^2}.
\]

**Proof.** It is easy to show that \( \mathcal{A}_\tau \) is bounded. The coercivity of \( \mathcal{A}_\tau \) follows Lemma 2.1. Let \( F \) be a linear functional on \( V \) such that
\[
F(\phi) := (\sigma(f), \nabla \phi),
\]
for all \( \phi \in V \). Then the Lax-Milgram Lemma implies that there exists a unique solution \( u \) to the problem
\[
\mathcal{A}_\tau(u, \phi) = F(\phi) \quad \text{for all } \phi \in V.
\]

Moreover, we have
\[
\|u\|_{H^2(D)^2} \leq C\|F\|_{V'},
\]
where \( V' \) represents the dual space of \( V \). Following from the definition of \( \sigma(f) \), we obtain from a simple calculation that
\[
\|F\|_{V'} \leq C_{\lambda, \mu}\|f\|_{H^1(D)^2},
\]
which shows the estimate (2.13) and completes the proof. \( \square \)
Remark 2.3. In the rest of the paper, we assume that the following regularity for $u$ holds
\begin{equation}
\|u\|_{H^{2+\epsilon}(\Omega)} \leq C\|f\|_{H^1(\Omega)^2}.
\end{equation}

Note that a similar regularity holds for the biharmonic equation [6-7,14] where the elliptic regularity $\xi \in (\frac{1}{2},1]$ is determined by the angles at the corners of $D$ and $\xi = 1$ if $D$ is convex.

It follows from Theorem 2.2 that there exists a solution operator $T : H^1(D)^2 \rightarrow V$ such that
\begin{equation}
u = T f.
\end{equation}

Clearly, the operator $T$ is self-adjoint since $A_\tau$ is symmetric; $T$ is also a compact operator due to the compact embedding of $H^2(D)^2$ into $H^1(D)^2$ (see, e.g., Theorem 1.2.1 of [27]). The generalized eigenvalue problem (2.10) has the following equivalent operator form
\begin{equation}
u = \eta T \nu, \quad \text{where } \eta = \gamma^{-1}.
\end{equation}

From classical spectral theory of compact self-adjoint operators, i.e., the Hilbert-Schmidt theory, $T$ has at most a countable set of real eigenvalues and 0 is the only possible accumulation point. Consequently, we have the following lemma for the generalized eigenvalue value problem (2.10).

Lemma 2.4. Let $\rho_0$ and $\rho_1$ satisfy (2.3) such that the condition in Lemma 2.1 is fulfilled. Then the generalized eigenvalue value problem (2.10) has at most a countable set of positive eigenvalues and $+\infty$ is the only possible accumulation point.

Roughly speaking, to compute real transmission eigenvalues, one needs to computes the roots of the nonlinear function $f$. The values of $f(\tau)$ are generalized eigenvalues of (2.10), which is approximated by the $H^2$-conforming Argyris element.

3. A CONFORMING FINITE ELEMENT METHOD

In this section, we propose a conforming finite element for (2.10). The convergence of the source problem (2.12) is established first. The theory of Babuška and Osborn [2] is then applied to obtain the convergence of the eigenvalue problem (2.10).

Let $\mathcal{T}$ be a regular triangular mesh for $D$ and $K \in \mathcal{T}$ be a triangle. We employ the $H^2$-conforming Argyris element, which uses $P_5$ - the set of polynomials of degree up to 5 on $K$, to discretize (2.10). Note that $\dim(P_5) = 21$. For $N = \{N_1, \ldots, N_{21}\}$, 3 degrees of freedom are the values at the vertices of $K$, 6 degrees of freedom are the values of the first order partial derivatives at the vertices of $K$, 9 degrees of freedom are the values of the second order derivatives at the vertices of $K$, and 3 degrees of freedom are the values of the normal derivatives at the midpoints of three edges of $K$ [5].

Note that the Argyris element does not belong to the affine families. This is due to the fact that normal derivatives are used as degrees of freedom. Fortunately, their interpolation properties are quite similar to those of affine families. Hence the Argyris element is referred to be almost-affine element. Let $v \in H^2(D)$ and $I_hv$ be the interpolation of $v$ by the Argyris element. For $v \in H^{1+\alpha}(D)$, $\alpha > 0$, the following interpolation result holds (see, e.g., [13])
\begin{equation}
\|v - I_hv\|_{H^2(D)} \leq Ch^{s-1}|v|_{H^{s+1}(D)},
\end{equation}

where $1 \leq s \leq \min\{5,1+\alpha\}$ depending regularity of $v$.

Let $V_h$ be the Argyris finite element space associated with $\mathcal{T}$. The discrete problem for (2.12) is to find $u_h \in V_h$ such that
\begin{equation}
A_\tau(u_h, \phi_h) = (f(\tau), \nabla \phi_h) \quad \text{for all } \phi_h \in V_h.
\end{equation}

The existence of a unique solution $u_h$ to (3.2) is the same as the continuous problem. As a consequence, there exists a discrete solution operator $T_h : H^1(D)^2 \rightarrow H^2(D)^2$ such that
\begin{equation}
u_h = T_h f.
\end{equation}

Theorem 3.1. Let $u$ and $u_h$ be the solutions of the continuous problem (2.12) and discrete problem (3.2), respectively. Then the following error estimate holds
\begin{equation}
\|u - u_h\|_{H^2(D)^2} \leq Ch^{2\alpha}\|f\|_{H^1(D)^2}.
\end{equation}
Proof. From Céa’s Lemma, the following error estimate holds
\[ \| u - u_h \|_{H^2(D)} \leq C \inf_{v_h \in V_h} \| u - v_h \|_{H^2(D)}, \]
for some constant C. Using (3.1) and (2.14), one has that
\[ \| u - u_h \|_{H^2(D)} \leq Ch^{s-1} |u|_{H^{s+1}(D)} = Ch^s |u|_{H^{s+1}(D)} \leq Ch^s \| f \|_{H^1(D)}. \]
For \( g \in H^1_0(D) \), let \( \phi_g \) be the unique solution of
\[ \mathcal{A}_r(\phi_g, \phi) = (\sigma(g), \nabla \phi) \quad \text{for all } \phi \in V. \]
The rest of the proof follows the Aubin–Nitsche Lemma (see, e.g., Theorem 3.2.4 of [27]) with suitable choices of Sobolev spaces. Let \( w := u - u_h \) and \( g \in H^1(D) \). Using the Galerkin orthogonality, we have for any \( v_h \in V_h \) that
\[ (\sigma(g), \nabla w) = \mathcal{A}_r(\phi_g, u - u_h) \]
\[ = \mathcal{A}_r(\phi_g - v_h, u - u_h) \]
\[ \leq C \| \phi_g - v_h \|_{H^2} \| u - u_h \|_{H^2}, \]
which yields
\[ (\sigma(g), \nabla w) \leq C \| u - u_h \|_{H^2} \inf_{v_h \in V_h} \| \phi_g - v_h \|_{H^2}. \]
Furthermore,
\[ \| u - u_h \|_{H^2}^{\alpha} = \sup_{g \in H^1(D)^2, g \neq 0} \frac{(u - u_h, g)}{\| g \|_{H^2}} \]
\[ \leq C \| u - u_h \|_{H^2} \sup_{g \in H^1(D)^2, g \neq 0} \left\{ \inf_{v_h \in V_h} \frac{\| \phi_g - v_h \|}{\| g \|_{H^2}} \right\}. \]
Consequently, we get
\[ \| u - u_h \|_{H^2(D)} \leq Ch^{2\alpha} \| f \|_{H^1(D)}. \]
which completes the proof. \qed

Using operators \( T \) and \( T_h \), we can rewrite the above error estimate as
\[ \| T f - T_h f \| \leq Ch^{2\alpha} \| f \|_{H^1(D)}. \]
Thus we have
\[ \| T - T_h \| \leq Ch^{2\alpha}. \]

Now we consider the discrete eigenvalue problem: Find \( \gamma_h \in \mathbb{R} \) such that
\[ \mathcal{A}_r(u_h, \phi_h) = \gamma_h (\sigma(u_h), \nabla \phi_h) \quad \text{for all } \phi_h \in V_h. \]

(3.3)
Since both \( \mathcal{A}_r \) and \( \mathcal{B} \) are symmetric, \( T \) is self-adjoint. Similarly, \( T_h \) is symmetric. The estimate of eigenvalue problem follows directly from the theory of Babuška and Osborn [2].

**Theorem 3.2.** Let \( \gamma \) be a generalized eigenvalue of (2.10) with algebraic multiplicity \( m \). Let \( \gamma_{1,h}, \ldots, \gamma_{m,h} \) be the \( m \) eigenvalues of (3.3) approximating \( \gamma \). Define \( \tilde{\gamma}_h = \frac{1}{m} \sum_{j=1}^m \gamma_{j,h} \). The following estimate holds
\[ | \gamma - \tilde{\gamma}_h | \leq Ch^{2\alpha}, \]
where \( C > 0 \) is a constant.

**Remark 3.3.** The boundary conditions for \( V \) defined in (2.6) need special treatment. The detail of how to impose the boundary conditions is shown in Appendix B.
4. AN ITERATIVE METHOD

Now we turn to the problem of how to compute the root(s) of the nonlinear function \( f(\tau) \) defined in (2.11). In this section, we assume that \( \rho_0 \) and \( \rho_1 \) are constants and consider the case when \( \gamma(\tau) \) is the smallest eigenvalue of (2.10). Similar result holds for other eigenvalues. The continuity of \( f \) is clear since the generalized eigenvalue \( \gamma(\tau) \) of (2.10) depends on \( \tau \) continuously. The following lemma is shown in [3]. It is written in a slightly different way to better serve the current paper.

Lemma 4.1. Let \( \tau_0 > 0 \) be small enough and \( \tau_1 > 0 \) be large enough. Then the nonlinear function \( f \) is continuous and has at least one root in \([\tau_0, \tau_1]\).

In fact, \( f \) is differentiable and the derivative is negative on an interval given in Theorem 4.2. We first recall the elasticity eigenvalue problem which will be used in the proof (see, e.g., [2]). Find non-trivial eigenpair \((\delta, u) \in \mathbb{R} \times H^1(\Omega)^2\) such that

\[
(\text{div } v, w) = \delta (v, w) \quad \text{for all } v \in H^1(\Omega)^2.
\]

Theorem 4.2. Let \( \delta_1 \) be the smallest elasticity eigenvalue. The function \( f(\tau) \) is differentiable. Furthermore, \( f(\tau) \) is a decreasing function on \([0, \frac{\delta_1(\rho_0 + \rho_1)}{2\rho_0 \rho_1}]\).

Proof. Let \( \gamma_1(\tau, \rho_0, \rho_1) \) be the first generalized eigenvalue of (2.10). The following Rayleigh quotient holds

\[
\gamma_1(\tau, \rho_0, \rho_1) = \inf_{w \in V} \frac{A_\tau(w, w)}{B(w, w)}
= \inf_{w \in V} \left( \frac{1}{\rho_1 - \rho_0} \left( \nabla \cdot \sigma + \tau \rho_0 \right) w, (\nabla - \sigma + \tau \rho_0) w \right) + \tau^2 (\rho_0 w, w)
= \inf_{w \in V} \left( \frac{1}{\rho_1 - \rho_0} \nabla \cdot \sigma w, \nabla \cdot \sigma w \right) + 2\tau \left( \frac{\rho_0}{\rho_1 - \rho_0} w, \nabla \cdot \sigma w \right) + \tau^2 \left( \frac{\rho_0 \rho_1}{\rho_1 - \rho_0} w, w \right).
\]

When \( \rho_0 \) and \( \rho_1 \) are constants, we have

\[
\gamma_1(\tau, \rho_0, \rho_1) = \inf_{w \in V} \frac{1}{\rho_1 - \rho_0} \left( \nabla \cdot \sigma w, \nabla \cdot \sigma w \right) + \tau^2 \frac{\rho_0 \rho_1}{\rho_1 - \rho_0} \left( w, w \right) - \frac{2\tau \rho_0}{\rho_1 - \rho_0}.
\]

Note that the sesquilinear form

\[
a(u, v) := (\sigma(u), \nabla v) = 2\mu \varepsilon(u) : \varepsilon(v) + \lambda(\nabla \cdot u)(\nabla \cdot v)
\]

is bounded, symmetric, and coercive. Hence

\[
\gamma_1(\tau, \rho_0, \rho_1) = \inf_{w \in V, a(w, w) = 1} \left\{ \frac{1}{\rho_1 - \rho_0} \left( \nabla \cdot \sigma w, \nabla \cdot \sigma w \right) + \tau^2 \frac{\rho_0 \rho_1}{\rho_1 - \rho_0} \left( w, w \right) \right\} - \frac{2\tau \rho_0}{\rho_1 - \rho_0}.
\]

Let \( \kappa := \tau^2 \frac{\rho_0 \rho_1}{\rho_1 - \rho_0} \). We define a new function

\[
s(\kappa) = \inf_{w \in V, a(w, w) = 1} \left\{ \frac{1}{\rho_1 - \rho_0} \| \nabla \cdot \sigma w \|^2 + \kappa \| w \|^2 \right\}.
\]

For a fixed \( \kappa \in (0, \infty) \), there exists a \( w_\kappa \) such that \( w_\kappa \in V, a(w_\kappa, w_\kappa) = 1 \), and

\[
s(\kappa) = \left\{ \frac{1}{\rho_1 - \rho_0} \| \nabla \cdot \sigma w_\kappa \|^2 + \kappa \| w_\kappa \|^2 \right\}.
\]

For a small enough positive \( h \),

\[
s(\kappa + h) - s(\kappa) \leq \left\{ \frac{1}{\rho_1 - \rho_0} \| \nabla \cdot \sigma w_\kappa \|^2 + (\kappa + h) \| w_\kappa \|^2 \right\} - \left\{ \frac{1}{\rho_1 - \rho_0} \| \nabla \cdot \sigma w_\kappa \|^2 + \kappa \| w_\kappa \|^2 \right\}
= h \| w_\kappa \|^2.
\]
On the other hand, we have
\[
\begin{align*}
  s(\kappa + h) - s(\kappa) & \geq \left\{ \frac{1}{\rho_1 - \rho_0} \| \nabla \cdot \sigma(w_{\kappa+h}) \|^2 + (\kappa + h) \| w_{\kappa+h} \|^2 \right\} \\
  & \quad - \left\{ \frac{1}{\rho_1 - \rho_0} \| \nabla \cdot \sigma(w_{\kappa+h}) \|^2 + \kappa \| w_{\kappa+h} \|^2 \right\} \\
  & = h \| w_{\kappa+h} \|^2.
\end{align*}
\]

Consequently,
\[
\| w_{\kappa+h} \|^2 \leq \frac{s(\kappa + h) - s(\kappa)}{h} \leq \| w_{\kappa} \|^2.
\]

The above inequality implies that \( \| w_{\kappa} \|^2 \) is monotonically decreasing and thus bounded. Note that \( a(w_{\kappa}, w_{\kappa}) = 1 \). Then the continuity of \( s \) and the compact embedding of \( V \) into \( L^2(D)^2 \) imply the existence of a \( \hat{w} \) such that \( w_{\kappa+h} \) converges in \( L^2(D)^2 \) strongly and \( w_{\kappa+h} \) converges in \( H^2(D)^2 \) weakly. In addition, \( w_{\kappa+h} \) satisfies
\[
\left( \frac{1}{\rho_1 - \rho_0} \nabla \cdot \sigma(w_{\kappa+h}), \nabla \cdot \phi \right) + (\kappa + h) ( w_{\kappa+h}, \phi ) = s(k + h) (\sigma(w), \nabla \phi),
\]
for all \( \phi \in V \). Taking \( h \to 0 \), we obtain
\[
\left( \frac{1}{\rho_1 - \rho_0} \nabla \cdot \sigma(w), \nabla \cdot \phi \right) + \kappa (w, \phi) = s(k) (\sigma(w), \nabla \phi),
\]
for all \( \phi \in V \). Thus \( \hat{w} = w_{\kappa} \). Consequently
\[
\| w_{\kappa+h} \|^2 \to \| w_{\kappa} \|^2, \quad h \to 0.
\]

Then the derivative of \( s(\kappa) \) is \( \| w_{\kappa} \|^2 \).

Combing the above estimates, we obtain
\[
\begin{align*}
  \frac{\partial f(\tau)}{\partial \tau} & = 2\tau \frac{\rho_0 \rho_1}{\rho_1 - \rho_0} \| w_{\kappa} \|^2 - \frac{2\rho_0}{\rho_1 - \rho_0} - 1 \\
  & = 2\tau \frac{\rho_0 \rho_1}{\rho_1 - \rho_0} \| w_{\kappa} \|^2 - \frac{\rho_1 + \rho_0}{\rho_1 - \rho_0}.
\end{align*}
\]

Let \( \delta_1 \) be the smallest elasticity eigenvalue. One has that
\[
\| w_{\kappa} \|^2 \leq \frac{1}{\delta_1} (\sigma(w_{\kappa}), \nabla w_{\kappa}) = \frac{1}{\delta_1},
\]
since \( (\sigma(w_{\kappa}), \nabla w_{\kappa}) = 1 \). This implies that
\[
(4.2) \quad \frac{\partial f(\tau)}{\partial \tau} \leq \frac{2\tau \rho_0 \rho_1}{\delta_1 (\rho_1 - \rho_0)} - \frac{\rho_1 + \rho_0}{\rho_1 - \rho_0}.
\]

In particular, \( f \) is decreasing, i.e.,
\[
\frac{\partial f(\tau)}{\partial \tau} \leq 0 \quad \text{if} \quad \tau < \frac{\delta_1 (\rho_0 + \rho_1)}{2\rho_0 \rho_1}.
\]

It is easy to see that \( f(\tau) > 0 \) if \( \tau \to 0 \) and \( f(\tau) < 0 \) if \( \tau \to \infty \).

Since we only have the finite element approximation for the values for \( f \), the nonlinear equation which we solve is in fact a discrete version of (2.11)
\[
(4.3) \quad f_h(\tau) := \gamma_h(\tau) - \tau.
\]

Let \( \epsilon > 0 \). From (4.2), there exists \( \eta > 0 \) such that
\[
(4.4) \quad \frac{\partial f(\tau)}{\partial \tau} \leq -\eta \quad \text{for} \quad \tau \in \left[ 0, \frac{\delta_1 (\rho_0 + \rho_1)}{2\rho_0 \rho_1} - \epsilon \right].
\]

**Theorem 4.3.** Assume that we apply the conforming Argyris finite element method for (2.10) on a regular mesh \( \mathcal{T} \) with mesh size \( h \). Let \( \tau_0 \) be the exact root of (2.11) and \( \tau_{0,h} \) be the root of (4.3) such that \( \tau_0, \tau_{0,h} \in \left[ 0, \frac{\delta_1 (\rho_0 + \rho_1)}{2\rho_0 \rho_1} - \epsilon \right] \). Then there exists \( h_0 \) such that for \( h < h_0 \)
\[
(4.5) \quad |\tau_{0,h} - \tau_0| \leq Ch^2/\eta.
\]
Proof. The assumption implies that \( \gamma(\tau_0) - \tau_0 \) and \( \gamma_h(\tau_0, h) = \tau_0, h \), i.e.,

\[
\gamma = \tau_0 \quad \text{and} \quad \gamma_h = \tau_0, h.
\]

By Theorem 5.2, there exist \( \delta < h_0 \) such that for a regular mesh with \( h < h_0 \), we have

\[
|\gamma_h - \tau_0| = |\gamma - \gamma_h| < Ch^{2\alpha}.
\]

Then direct application of (4.4) leads to (4.5). \( \square \)

Note that \( f_h(\gamma) \) is a nonlinear function. It is natural to use some iterative methods to compute the roots of \( f_h \). We choose to use the secant method which avoids using the derivatives of \( f_h(\gamma) \).

Given a regular triangular mesh \( T \) for \( D \), let \( x_0 \) and \( x_1 \) be two positive numbers close to 0 such that \( 0 < x_0 < x_1 \). Let \( N \) be the number of smallest transmission eigenvalues one wants to compute. Let \( tol \) and \( maxit \) be the preset precision and the maximum number of iteration of the secant method, respectively. The following algorithm uses a secant iteration to compute \( N \) smallest positive transmission eigenvalues.

**SMETE**

construct matrix \( B_h \) corresponding to \( B \) in (2.10)

for \( i = 1 : N \)

\[
\begin{align*}
    & \text{it} = 0 \\
    & \delta = \text{abs}(x_1 - x_0) \\
    & \tau = x_0 \quad \text{and construct the matrix } \quad A_{\tau,h} \\
    & \quad \text{compute the } ith \text{ generalized eigenvalue } \quad \gamma_0 \text{ of } \quad A_{\tau,h}x = \gamma B_h x \\
    & \tau = x_1 \quad \text{and construct matrix } \quad A_{\tau,h} \\
    & \quad \text{compute the } ith \text{ generalized eigenvalue } \quad \gamma_1 \text{ of } \quad A_{\tau,h}x = \gamma B_h x \\
    & \quad \text{while } \delta > tol \text{ and } it < maxit \\
    & \quad \quad \tau = x_1 - \gamma_1 \frac{\tau_1 - \tau_0}{\gamma_1 - \gamma_0} \\
    & \quad \quad \text{construct the matrix } \quad A_{\tau,h} \\
    & \quad \quad \text{compute the } ith \text{ smallest eigenvalue } \quad \gamma_\tau \text{ of } \quad A_{\tau,h}x = \gamma B_h x \\
    & \quad \quad \delta = \text{abs}(\gamma_\tau - \tau) \\
    & \quad \quad x_0 = x_1, x_1 = \tau \\
    & \quad \quad \gamma_0 = \gamma_1, \gamma_1 = \gamma_\tau \\
    & \quad \quad it = it + 1
\end{align*}
\]

end

**Remark 4.4.** Similar to the cases for acoustic and electromagnetic waves, the elasticity transmission eigenvalue problem is nonlinear and non-self-adjoint. Although not theoretically justified, numerical results indicate that there exist complex elasticity transmission eigenvalues, as we will see shortly. The above method can compute only real eigenvalues, which correspond to the frequencies of elasticity waves. The physical meaning of complex eigenvalues is not yet clear.

### 5. A Mixed Finite Element Method

The method proposed above needs to compute roots of nonlinear functions and many generalized eigenvalue eigenvalue problems (2.10). In addition, the \( H^2 \)-conforming Argyris element is used for discretization. In this section, we give a much simpler mixed method for (2.10), which also computes complex eigenvalues. It rewrites the fourth order problem into two second order equations. Then the simpler \( H^1 \)-conforming Lagrange element can be applied directly. The purpose of this section is to provide an alternative method for verification.

Recall that the fourth order formulation of ETE is to find \( \tau \) and \( 0 \neq w \in V \) such that

\[
(\nabla \cdot \sigma + \tau \rho_1)(\rho_1 - \rho_0)^{-1}(\nabla \cdot \sigma + \tau \rho_0)w = 0.
\]

Following [21], we introduce an auxiliary variable \( v \) such that

\[
v = (\rho_1 - \rho_0)^{-1}(\nabla \cdot \sigma + \tau \rho_0)w.
\]

The special form of \( v \) is chosen such that the quadratic eigenvalue problem (5.1) can be written as a linear eigenvalue system. Using \( w \) and the auxiliary variable \( v \), we obtain

\[
(\nabla \cdot \sigma + \tau \rho_0)w = (\rho_1 - \rho_0)v,
\]

\[
(\nabla \cdot \sigma + \tau \rho_1)v = 0,
\]
Table 1. The first generalized eigenvalue of (2.10) for three domains.

| $h$   | Unit square order | L-shape order | Circle order |
|-------|------------------|---------------|--------------|
| 0.1   | 1.97544798       | 4.254621      | 2.074928     |
| 0.05  | 1.97544304       | 4.244708      | 2.062945     |
| 0.025 | 1.97544109       | 1.533635      | 0.924724     |
| 0.0125| 1.97544043       | 1.979999      | 1.384228     |

The weak form is to find $(\tau, w, v) \in C \times H^1_0(D)^2 \times H^1_0(D)^2$ such that

$$((\rho_1 - \rho_0)v, \phi) + (\sigma(w), \nabla \phi) = \tau(\rho_0 w, \phi) \quad \text{for all } \phi \in H^1_0(D)^2,$$

$$-\nabla \cdot \sigma(v) = \tau(\rho_1 v, \psi) \quad \text{for all } \psi \in H^1_0(D)^2.$$
6.2. Result for \( f_h(\tau) \). We plot function \( f_h(\tau) \) for three domains using the meshes with \( h_3 \approx 0.025 \). For parameters given in (6.1), the first elasticity eigenvalue \( \delta_1 = 3.679328 \) for the disk, \( \delta_1 = 3.251402 \) for the unit square, \( \delta_1 = 4.325472 \) for the L-shaped domain, which are computed by linear finite elements. According to Theorem 4.2, \( f(\tau) \) is a decreasing function on \((0, \delta_1(\rho_0 + \rho_1)/2\rho_0\rho_1)\). Plugging the values for \( \delta_1, \rho_0, \rho_1 \), we have that \( f(\tau) \) is decreasing on
\[(6.3) \quad (0, 2.299580), \quad (0, 2.032126), \quad \text{and} \quad (0, 2.703420)\]
for the three domains, respectively. In Figure 1 we plot \( f_h(\tau) = \gamma_h(\tau) - \tau \) for several eigenvalues. The plots show that \( f_h \) is decreasing right to the origin. In fact, it can be seen that \( f_h(\tau) \) is decreasing on much larger intervals than those in (6.3) predicted by Theorem 4.2.

6.3. Real transmission eigenvalues. Next we present the results of the smallest transmission eigenvalues. Table 2 gives the computed eigenvalues and the convergence orders of the first transmission eigenvalue of three domains using the secant method. It can be seen that the convergence rate for the unite square is approximately 2 indicating that the associated eigenfunction \( u \in H^3(D) \). The convergence rate for the L-shaped domain is much lower, which is likely caused by the low regularity of the eigenfunction. Similar results can be observed for the biharmonic eigenvalue problem (see Chap. 4 of [27] or [7]).

Remark 6.1. When the domain is a disk, one can obtain transmission eigenvalues associated with radially-symmetric eigenfunctions using separation of variables. The detail of derivation of such eigenvalues is given in Appendix A.
In Table 2, we show the six smallest transmission eigenvalues computed by the secant method for the three domains with mesh size $h \approx 0.025$. The method converges in a few steps. In Table 3, we give the six smallest transmission eigenvalues computed by the secant method for the three domains using a different set of parameters $\mu = 1/5, \lambda = 1/5, \rho_0 = 1/20, \rho_1 = 3$. The iteration converges in a few steps as well.

### 6.4. The special case of disk.

From Appendix A, a radially-symmetric transmission eigenvalue of the disk is the first root of $Z_0$ defined in (7.3). Using some root finding technique, we find that $\omega = 3.554954$, i.e., $\tau = 12.637700$. However, it is not the smallest transmission eigenvalue of the disk. The secant method computes an approximate transmission eigenvalue $\tau = 12.662693$ with $h = 0.05$ and $\tau = 12.624538$ with $h = 0.025$. The mixed method also computes a transmission eigenvalue $\tau = 12.713678$ with $h = 0.025$. Figure 2 plots the eigenfunction $u$ associated with this eigenvalue, which appear to be radially-symmetric. Note that not all eigenfunctions are radially-symmetric. Figure 3 is the eigenfunction associated with the second eigenvalue. Clearly, it is not a radially-symmetric function.

### 6.5. Results by the mixed method.

For comparison and verification, we also computes the transmission eigenvalues using the mixed method. Table 5 shows the first transmission eigenvalue and convergence rates. Note that ETE is non-self-adjoint and the secant method only computes real transmission eigenvalues. The mixed method can compute complex eigenvalues. Table 6 gives the convergence orders of the first complex transmission eigenvalue.

### 7. Conclusions and Future Work

In this paper, we propose an iterative method to compute a few smallest transmission eigenvalues for elastic waves. The major advantage of this method is the accuracy and effectiveness since we only need to compute a few eigenvalues of Hermitian eigenvalue problems instead of computing the full eigensystem of a
Figure 2. A radially-symmetric eigenfunction. Left: $u_1$. Middle: $u_2$. Right: $|u = (u_1, u_2)|$.

Figure 3. Second eigenfunction. Left: $u_1$. Middle: $u_2$. Right: $|u = (u_1, u_2)|$.

| $h$   | Unit square order | L-shaped order | Circle order |
|-------|-------------------|----------------|--------------|
| 0.1   | 2.393618          | 7.117873       | 2.734748     |
| 0.05  | 2.040967          | 5.466449       | 2.221626     |
| 0.025 | 1.967283          | 2.273017       | 1.983289     |
| 0.0125| 1.948971          | 2.328781       | 2.055511     |

Table 5. The first real transmission eigenvalue of the mixed method.

| $h$   | Unit square order | L-shaped order | Circle order |
|-------|-------------------|----------------|--------------|
| 0.1   | 3.335717 - 3.171243i | 3.631454 - 3.333002i | 4.153116 - 0.863572i |
| 0.05  | 3.494072 - 1.117876i | 3.616318 - 3.116377i | 3.963564 - 1.126331i |
| 0.025 | 3.422905 - 1.097453i | 3.613011 - 3.061526i | 2.204651 - 1.131189i |
| 0.0125| 3.402856 - 1.090597i | 2.167773 - 3.047409i | 2.280984 - 1.126730i |

Table 6. The first complex transmission eigenvalue of the mixed method.

non-Hermitian eigenvalue problem. This fits the practical need in the sense that in general only the smallest transmission eigenvalues are needed for the estimation of the elastic properties of the material. We prove the convergence of the proposed method. The effectiveness of the method is verified by some numerical examples. We have also given a mixed method without proof. In future, we will consider the complex eigenvalues of the elastic waves. The analytic property of the function $f(\tau) = \gamma(\tau) - \tau$ is also an interesting topic.

Appendix A: Radially Symmetric Case on Disks

We derive the equation satisfied by a transmission eigenvalue whose associated eigenfunction is radially symmetric on a disk. Let $D \subset \mathbb{R}^2$ be a disk with radius $R$. Let $u = (w, v)^\top$. Writing the elasticity wave
The above equation can be written as

\[(7.1)\quad (2\mu + \lambda) \frac{\partial^2 w}{\partial x_1^2} + (\lambda + \mu) \frac{\partial^2 v}{\partial x_2 \partial x_1} + \mu \frac{\partial^2 w}{\partial x_2^2} + \omega^2 \rho w = 0,\]

\[(7.2)\quad (\mu + \lambda) \frac{\partial^2 w}{\partial x_2^2} + (\lambda + \mu) \frac{\partial^2 v}{\partial x_1^2} + (2\mu + \lambda) \frac{\partial^2 v}{\partial x_2^2} + \omega^2 \rho v = 0.\]

If we consider the solution in the form of radially-symmetric vector field \(u(x) = u(r)e_r\), where \(r = |x|\) and \(e_r = x/r\), \(w = u(r)\cos\theta\), \(v = u(r)\sin\theta\), \((7.1)\) can be written as

\[(\mu + \lambda) \left( \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2 \partial x_1} \right) + \mu \left( \frac{\partial^2 w}{\partial x_2^2} + \frac{\partial^2 v}{\partial x_1^2} \right) + \omega^2 \rho w = 0.\]

Using polar coordinate, \((7.1)\) becomes

\[(\lambda + \mu) \left( u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u \right) + \mu \left( u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u \right) + \omega^2 \rho u = 0,\]

i.e.,

\[(\lambda + 2\mu) \left( u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u \right) + \omega^2 \rho u = 0.\]

Similarly, \((7.2)\) is simply

\[(\mu + \lambda) \left( u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u \right) + \mu \left( u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u \right) + \omega^2 \rho u = 0,\]

i.e.,

\[(2\mu + \lambda) \left( u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u \right) + \omega^2 \rho u = 0.\]

The above equation can be written as

\[r^2 \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \left( r^2 \frac{\omega^2 \rho}{2\mu + \lambda} - 1 \right) u = 0.\]

The solution of the above equation is given by the Bessel function of order one \(J_1(ar)\), where

\[a = \omega \sqrt{\frac{\rho}{2\mu + \lambda}}.\]

Then we obtain that \(u = (w, v)^T := (J_1(ar)\cos\theta, J_1(ar)\sin\theta)^T\).

Next we look at the boundary condition involving \(\sigma(u)v\). For the transmission eigenvalue problem, we assume that

\[u = \begin{pmatrix} J_1(a_1 r) \cos \theta \\ J_1(a_1 r) \sin \theta \end{pmatrix}, \quad v = C \begin{pmatrix} J_1(a_2 r) \cos \theta \\ J_1(a_2 r) \sin \theta \end{pmatrix},\]

where \(C\) is a constant to be determined from \((2.1)\) and

\[a_1 = \omega \sqrt{\frac{\rho_1}{2\mu + \lambda}}, \quad a_2 = \omega \sqrt{\frac{\rho_2}{2\mu + \lambda}}.\]

Note that \(v = (v_1, v_2)^T = (\cos\theta, \sin\theta)^T\). It follows from \((2.2)\) that the first component of \(\sigma(u)v\) is

\[\lambda \left[ \frac{\partial}{\partial r} J(ar) + \frac{1}{r} J(ar) \right] \cos \theta + 2\mu \frac{\partial}{\partial r} J(ar) \cos \theta.\]

The second component of \(\sigma(u)v\) is

\[2\mu \frac{\partial}{\partial r} J(ar) \sin \theta + \lambda \left[ \frac{\partial}{\partial r} J(ar) + \frac{1}{r} J(ar) \right] \sin \theta.\]

Using the boundary conditions \((2.4)\), we obtain

\[C = \frac{J_1(a_1 R)}{J_1(a_2 R)}.\]
Appendix B: Imposing Boundary Conditions

The boundary condition for $V$ needs careful treatment for the Argyris element. On one boundary of a triangle $\gamma_e \subset T$ with unit outward normal $(n_x, n_y)$ and unit tangential vector $(\tau_x, \tau_y)$, we consider the case $n_x n_y \neq 0$. Otherwise it is easy to figure out. It is clear that

$$\tau_x n_x + \tau_y n_y = 0, \quad \tau_x^2 + \tau_y^2 = 1, \quad n_x^2 + n_y^2 = 1.$$  

On the boundary $\gamma_e$, $u = (u_1, u_2)^T = \mathbf{0}$, so the tangent derivatives are also $\mathbf{0}$, i.e.,

$$\frac{\partial u_1}{\partial x} \tau_x + \frac{\partial u_1}{\partial y} \tau_y = 0, \quad \frac{\partial u_2}{\partial x} \tau_x + \frac{\partial u_2}{\partial y} \tau_y = 0.$$  

The boundary condition $\sigma(u) \nu = 0$ means

$$((\lambda + 2\mu)\partial_x u_1 + \lambda \partial_y u_2)n_x + \mu(\partial_y u_1 + \partial_x u_2)n_y = 0,$$

$$\mu(\partial_x u_2 + \partial_y u_1)n_x + (\lambda \partial_x u_1 + (\lambda + 2\mu)\partial_y u_2)n_y = 0.$$  

Substituting

$$\frac{\partial u_1}{\partial y} = -\frac{\tau_x}{\tau_y} \frac{\partial u_1}{\partial x}, \quad \frac{\partial u_2}{\partial y} = -\frac{\tau_x}{\tau_y} \frac{\partial u_2}{\partial x},$$  

into above equations, we have

$$((\lambda + 2\mu)\partial_x u_1 + \lambda \partial_y u_2)n_x - \frac{\tau_x}{\tau_y} \mu \partial_x u_1 n_y + \mu \partial_x u_2 n_y = 0,$$

$$\mu \partial_x u_2 n_x - \frac{\tau_x}{\tau_y} \mu \partial_x u_1 n_y + \lambda \partial_x u_1 n_y - \frac{\tau_x}{\tau_y} (\lambda + 2\mu) \partial_y u_2 n_y = 0,$$

i.e.,

$$\partial_x u_1 [n_x(\lambda + 2\mu) - \frac{\tau_x}{\tau_y} n_y] + \partial_x u_2 [\mu n_y - \frac{\tau_x}{\tau_y} n_x] = 0,$$

$$\partial_x u_1 [\lambda n_y - \frac{\tau_x}{\tau_y} n_x] + \partial_x u_2 [\mu n_x - (\lambda + 2\mu) n_y \frac{\tau_x}{\tau_y}] = 0.$$  

Figure 4 is the contour plot of $|Z_0(\omega)|$ with $\mu = 1/16, \lambda = 1/4, \rho_0 = 1, \rho_1 = 4$. The centers of the circular curves indicate the locations of transmission eigenvalues.
which yields
\[
\begin{align*}
\partial_x u_1 [\tau_y n_x (\lambda + 2\mu) - \mu \tau_x n_y] - \partial_x u_2 (\lambda + \mu) n_x \tau_x &= 0, \\
- \partial_x u_1 (\lambda + \mu) n_x \tau_x + \partial_x u_2 [\mu n_x \tau_y - (\lambda + 2\mu) n_y \tau_x] &= 0.
\end{align*}
\]

Here we have used the fact that \(\tau_x n_x + \tau_y n_y = 0\). Using (7.4), the determination of the above equations is
\[
\begin{align*}
\mu (\lambda + 2\mu) n_x^2 (1 - n_y^2) + \mu (\lambda + 2\mu) \tau_x^2 (1 - n_y^2) + (\lambda + 2\mu)^2 n_x^2 \tau_x^2 + \mu^2 n_x^2 \tau_x^2 - (\lambda + \mu)^2 n_x^2 \tau_x^2
\end{align*}
\]
\[
= \mu (\lambda + 2\mu) (n_x^2 + \tau_x^2) + n_x^2 \tau_x^2 ((\lambda + 2\mu)^2 + \mu^2 - (\lambda + \mu)^2 - 2\mu (\lambda + 2\mu))
\]
\[
= \mu (\lambda + 2\mu) (n_x^2 + \tau_x^2) > 0.
\]

Consequently,
\[
\begin{align*}
\partial_x u_1 = \partial_y u_1 = \partial_x u_2 = \partial_y u_2 = 0.
\end{align*}
\]

The tangential derivatives of all the first-order derivatives are 0. Taking \(u_1\) for example, we have
\[
\begin{align*}
\frac{\partial^2 u_1}{\partial x^2} \tau_x + \frac{\partial^2 u_1}{\partial x \partial y} \tau_y &= 0, \\
\frac{\partial^2 u_1}{\partial x \partial y} \tau_x + \frac{\partial^2 u_1}{\partial y^2} \tau_y &= 0,
\end{align*}
\]

which implies
\[
\frac{\partial^2 u_1}{\partial x^2} = -\frac{\tau_y}{\tau_x} \frac{\partial^2 u_1}{\partial y^2}, \quad \frac{\partial^2 u_1}{\partial x \partial y} = -\frac{\tau_y}{\tau_x} \frac{\partial^2 u_1}{\partial y^2}.
\]

If a point belonging to two edges \(\gamma_1\) and \(\gamma_2\) satisfies (7.5) with different tangential derivatives, we have
\[
\begin{align*}
\frac{\partial^2 u_1}{\partial x^2} = \frac{\partial^2 u_1}{\partial y^2} = \frac{\partial^2 u_1}{\partial x \partial y} = 0.
\end{align*}
\]

References

[1] J. An and J. Shen, A Fourier-spectral-element method for transmission eigenvalue problems, J. Sci. Comput. 57 (2013), 670–688.
[2] I. Babuška and J. Osborn, Eigenvalue problems, in Handbook of Numerical Analysis II, P. Ciarlet and J. Lions, eds., North-Holland, Amsterdam, 1991, 641–787.
[3] C. Bellis, F. Cakoni, and B. Guzina, Nature of the transmission eigenvalue spectrum for elastic bodies, IMA J. Appl. Math. 78 (2013), 895–923.
[4] C. Bellis and B. Guzina, On the existence and uniqueness of a solution to the interior transmission problem for piecewise-homogeneous solids, J. Elasticity 101 (2010), 29–57.
[5] S. Brenner and L. Scott, The mathematical theory of finite elements methods, 2nd Edition, Texts in Applied Mathematics, Springer, 2002.
[6] H. Blum and R. Rannacher, On the boundary value problem of the biharmonic operator on domains with angular corners, Math. Methods Appl. Sci. 2 (1980), 556–581.
[7] S. Brenner, P. Monk, and J. Sun, \(C^0\) interior penalty Galerkin method for biharmonic eigenvalue problems, Spectral and High Order Methods for Partial Differential Equations, Lect. Notes Comput. Sci. Eng. 106 (2015), 3–15.
[8] F. Cakoni, D. Colton, P. Monk and J. Sun, The inverse electromagnetic scattering problem for anisotropic media, Inver. Probl. 26 (2010), 074004.
[9] F. Cakoni, P. Monk, and J. Sun, Error analysis of the finite element approximation of transmission eigenvalues, Comput. Methods Appl. Math. 14 (2014), 419–427.
[10] A. Charalambopoulos, On the interior transmission problem in nondissipative, inhomogeneous, anisotropic elasticity, J. Elasticity 67 (2002), 149–170.
[11] A. Charalambopoulos and K. Anagnostopoulos, On the spectrum of the interior transmission problem in isotropic elasticity, J. Elasticity 90 (2008), 295–313.
[12] D. Colton, P. Monk, and J. Sun, Analytical and computational methods for transmission eigenvalues, Inver. Probl. 26 (2010), 045011.
[13] P. Ciarlet, The finite element method for elliptic problems, Classics in Applied Mathematics, 40, SIAM, Philadelphia, 2002.
[14] P. Grisvard, Elliptic problems in nonsmooth domains, Pitman, Boston, 1985.
[15] J. Marsden and T. Hughes, Mathematical foundations of elasticity, Dover, 1994.
[16] T. Huang, W. Huang, and W. Lin, A robust numerical algorithm for computing Maxwell’s transmission eigenvalue problems, SIAM J. Sci. Comput. 37 (2015), A2403–A2423.
[17] R. Huang, A. Strubers, J. Sun, and R. Zhang, Recursive integral method for transmission eigenvalues, J. Comput. Phys. 327 (2016), 830–840.
[18] R. Huang, J. Sun and C. Yang, Recursive integral method with Cayley transformation, arXiv:1705.01646, 2017.
[19] A. Kileel, A numerical method to compute interior transmission eigenvalues, Inver. Probl. 29 (2013), 104012.
[20] T. Li, W. Huang, W. Lin, and J. Liu, *On spectral analysis and a novel algorithm for transmission eigenvalue problems*, J. Sci. Comput. 64 (2015), 83–108.

[21] X. Ji, J. Sun, and T. Turner, *A mixed finite element method for Helmholtz transmission eigenvalues*, ACM Trans. Math. Software 8 (2012), Algorithm 922.

[22] X. Ji, J. Sun, and H. Xie, *A multigrid method for Helmholtz transmission eigenvalue problem*, J. Sci. Comput. 60 (2014), 276–294.

[23] P. Monk and J. Sun, *Finite element methods of Maxwell transmission eigenvalues*, SIAM J. Sci. Comput. 34 (2012), B247–B264.

[24] J. Sun, *Iterative methods for transmission eigenvalues*, SIAM J. Numer. Anal. 49 (2011), 1860–1874.

[25] J. Sun, *Estimation of transmission eigenvalues and the index of refraction from Cauchy data*, Inver. Probl. 27 (2011), 015009.

[26] J. Sun and L. Xu, *Computation of the Maxwell’s transmission eigenvalues and its application in inverse medium problems*, Inver. Probl. 29 (2013), 104013.

[27] J. Sun and A. Zhou, *Finite element methods for eigenvalue problems*, CRC Press, Taylor & Francis Group, Boca Raton, London, New York, 2016.

[28] Y. Yang, H. Bi, H. Li, and J. Han, *Mixed methods for the Helmholtz transmission eigenvalues*, SIAM J. Sci. Comput. 38 (2016), A1383–A1403.

LSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing, 100190, China.

E-mail address: jixia@lsec.cc.ac.cn

Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA.

E-mail address: lipeijun@math.purdue.edu

Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931, USA.

E-mail address: jiguangs@mtu.edu