On the norms and roots of orthogonal polynomials in the plane and $L^p$-optimal polynomials with respect to varying weights

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Abstract

For a measure on a subset of the complex plane we consider $L^p$-optimal weighted polynomials, namely, monic polynomials of degree $n$ with a varying weight of the form $w^n = e^{-nV}$ which minimize the $L^p$-norms, $1 \leq p \leq \infty$. It is shown that eventually all but a uniformly bounded number of the roots of the $L^p$-optimal polynomials lie within a small neighborhood of the support of a certain equilibrium measure; asymptotics for the $n$th roots of the $L^p$ norms are also provided. The case $p = \infty$ is well known and corresponds to weighted Chebyshev polynomials; the case $p = 2$ corresponding to orthogonal polynomials as well as any other $1 \leq p < \infty$ is our contribution.

1 Introduction, background and results

In approximation theory an important role is played by the so-called Chebyshev polynomials associated to a compact set $K \subseteq \mathbb{C}$, namely monic polynomials of degree $n$ that minimize the supremum norm over $K$. As a natural generalization, one can consider weighted Chebyshev polynomials with respect to a varying weight of the form $w^n$ on some $\Sigma \subseteq \mathbb{C}$ that are minimizing the supremum norm of weighted polynomials $Q_n w^n$ over $\Sigma$, where $Q_n$ is a monic polynomial of degree $n$ (the weight function $w$ is assumed to satisfy certain standard admissibility conditions that make the extremal problem well-posed $\spadesuit$).

Along the same lines, given a positive Borel measure $\sigma$ on $\Sigma \subseteq \mathbb{C}$, one can consider optimal weighted polynomials in the $L^2(\sigma)$-sense; provided that the integrals below are finite, it is easy to see that there is a unique monic polynomial $P_n$ for which the weighted polynomial $P_n w^n$ minimizes the $L^2(\sigma)$-norm

$$\|P_n w^n\|_{L^2(\sigma)} := \left( \int_\Sigma |P_n|^2 w^{2n} d\sigma \right)^{\frac{1}{2}}$$

(1-1)
among all monic weighted polynomials of degree $n$. This polynomial may be characterized as the $n$th monic orthogonal polynomial with respect to the varying measure $w^2n \, d\sigma$, satisfying
\[
\int \sum P_n(z)z^k w^2n(z) d\sigma(z) = \delta_{kn} h_n \quad 0 \leq k \leq n ,
\]
where
\[
h_n = \inf \left\{ \|Q_n w^n\|_{L^2(\sigma)} : Q_n(z) \text{ monic polynomial of degree } n \right\} .
\]

Orthogonal polynomial sequences for varying measures of the form $w^2n \, d\sigma$ appear naturally in the context of random matrix models [2, 3]: on the space of $n \times n$ Hermitian matrices $H_n$, probability distributions of the form
\[
\rho_n(M) dM = \frac{1}{Z_n} \exp(-nTr(V(M))) dM , \quad Z_n = \int_{H_n} \exp(-nTr(V(M))) dM
\]
are considered where the potential function $V(x)$ grows sufficiently fast as $|x| \to \infty$ to make the integral in (1-4) finite ($dM$ stands for the Lebesgue measure on $H_n$). The apparent unitary invariance of (1-4) implies that the analysis of statistical observables of $M$ may be reduced to that of the random eigenvalues $\lambda_1, \ldots, \lambda_n$ with probability distribution
\[
p_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} \prod_{1 \leq k < l \leq n} (\lambda_k - \lambda_l)^2 e^{-n \sum_{k=1}^n V(\lambda_k)} ,
\]
\[
Z_n = \int_{\mathbb{R}^n} \prod_{1 \leq k < l \leq n} (\lambda_k - \lambda_l)^2 e^{-n \sum_{k=1}^n V(\lambda_k)} d\lambda_1 \cdots d\lambda_n .
\]
The marginal distributions of $p_n$ (referred to as correlation functions) are expressible as determinants of the weighted polynomials $p_n(x) e^{-nV(x)/2}$ where the $p_n$ satisfies the orthogonality relation
\[
\int_{\mathbb{R}} p_n(x) x^k e^{-nV(x)} dx = \delta_{kn} h_n \quad k = 0, \ldots, n .
\]
Therefore the asymptotic analysis of the correlation functions reduces to the study of the corresponding orthogonal polynomials. On the real line, the asymptotic analysis is done effectively by the so-called Riemann–Hilbert method [3]: however, for the so–called normal matrix models [4, 5], for which the eigenvalues may fill regions of the complex plane, much less is known in general. While random matrix theory was the original impetus behind our interest, the paper will not draw any conclusions on these important connections.

Following instead a more approximation-theoretical spirit, it is also natural to consider $L^p$-optimal weighted polynomials [6, 7, 8, 9] with respect to the varying weight $w^n$ and the measure $\sigma$, i.e. to minimize the $L^p$-norm
\[
\|P_n w^n\|_{L^p(\sigma)} := \left( \int |P_n w^n|^p d\sigma \right)^{\frac{1}{p}}
\]
over all monic polynomials of degree $n$. The paper addresses two questions; the first concerns the location of the roots of $L^p$-optimal polynomials or rather where the roots cannot be. We find that eventually (i.e. for sufficiently large $n$) all roots fall in an arbitrary neighborhood of the convex hull of the support $S_w$ of the relevant equilibrium measure $\mu_w$ (whose definition is recalled in Sect. 1.1): this is accomplished in Prop. 2.1 (with a more precise statement).

If the support is not convex (possibly with holes and several disjoint connected components) then we can state that (Prop. 2.2) all but a finite (and uniformly bounded) number of roots falls within any arbitrary neighborhood of the polynomially convex hull of the support. A consequence of the above is that

$$\lim_{n \to \infty} \frac{1}{n} \ln P_n(z) = \int \ln |z - t| d\mu_w(t)$$

where the quotation marks indicate that the statement is imprecise (see Thm. 2.1 for the precise one); the convergence is uniform over closed subsets of the unbounded component $\mathbb{C} \setminus S_w$. If $K$ does not contain roots of $P_n$ (eventually) then we can remove the quotations and the statement is correct (for example, if $K$ is disjoint from the convex hull of $S_w$).

The second question deals with the leading order behaviour of the $L^p$ norms of the $p$–optimal polynomials and we show that – in fact – they all have the exact same asymptotic behaviour

$$\lim_{n \to \infty} \left( \|P_n w^n\|_{L^p(\sigma)} \right)^{1/n} = \exp(-F_w)$$

where $F_w$ is the modified Robin’s constant of the equilibrium measure $\mu_w$, and this limit is independent of $1 \leq p \leq \infty$.

The case of $p = \infty$ of the above statements is known in the literature ([10] for the unweighted case, and [1] for the weighted one) even in the varying weight case. It seems to be new for $p \neq \infty$.

1.1 Potential-theoretic background

We will consider polynomials on a closed set $\Sigma \subseteq \mathbb{C}$, called a condenser; on this set a reference measure $\sigma$ is supposed to be given. Since we are not seeking the greatest generality (at cost of simplicity) we will restrict ourselves to the following situations:

- $\Sigma$ is a finite collection of Jordan curves, with typical example the real axis or union of intervals thereof. In this case the measure $\sigma$ is simply the arc-length,

- $\Sigma$ is a finite union of regions of the plane, with the area measure $d\sigma = dA$,

- $\Sigma$ is a finite union of elements of both types above.

The main focus will be $\Sigma = \mathbb{C}$ or $\Sigma = \mathbb{R}$ or $\Sigma = \gamma$ a smooth curve in $\mathbb{C}$.

The weight function $w : \Sigma \to [0, \infty)$ introduced above is assumed to satisfy the following standard admissibility conditions ([1]):
• \(w\) is upper semi-continuous,
• \(\text{cap}\left(\{z : w(z) > 0\}\right)\) has positive capacity,
• \(|z|w(z) \to 0\) as \(|z| \to \infty\) in \(\Sigma\).

The potential \(V(z)\) is the function for which \(w(z) = \exp(-V(z))\) and it inherits the corresponding admissibility conditions. The weighted energy functional is defined as follows; for a probability measure \(\mu\) on \(\Sigma\) we define

\[
\mathcal{I}_w[\mu] := \int_{\Sigma} \ln \frac{1}{|z - w|} d\mu(z) d\mu(w) + 2 \int V(z) d\mu(z) .
\]  

(1-10)

It is well known in potential theory \([1]\) that there exists a unique measure \(\mu_w\) that realizes the minimum of \(\mathcal{I}_w\); such a measure is referred to as the equilibrium measure. Its support \(S_w = \text{supp}(\mu_w)\) is a compact set.

Although it will not be used directly we recall the following indirect characterization of \(\mu_w\): if we denote with

\[
U^\mu(z) := \ln \frac{1}{|z - w|} d\mu(w)
\]

(1-11)

the logarithmic potential of a probability measure \(\mu\) then \(\mu_w\) is uniquely characterized as follows. There exists a constant \(F_w\) called the modified Robin’s constant such that the effective potential

\[
\Phi(z) := U^{\mu_w}(z) + V(z) - F_w
\]

(1-12)

satisfies

\[
\begin{cases}
\Phi(z) \leq 0 & z \in S_w \\
\text{and} & \\
\Phi(z) \geq 0 & z \in \Sigma \text{ q. e.}
\end{cases}
\]

(1-13)

\(\Rightarrow \Phi(z) = 0 \text{ z } \in S_w \text{ q. e.}\)

where 'q. e.' stands for “quasi-everywhere”, namely up to sets of zero logarithmic capacity.

### 2 Where the roots are not

Let \(P_n(z)\) be any sequence of polynomials of degree \(\leq n\), \(S_w = \text{supp}(\mu_w)\) and let \(\mathcal{N} \supset S_w\) be an open bounded set containing \(S_w\).

In \([1]\) III.6 (eq. 6.4) it is shown in general (under certain assumptions on \(\Sigma, w\) and \(\sigma\)) that if \(P_n\) is any sequence of polynomials of degree \(\leq n\) we have

\[
\|P_n w^n\|_{L^p(\sigma)}^p = \int_\Sigma |P_n w^n|^p d\sigma \leq (1 + C e^{-cn}) \int_\mathcal{N} |P_n w^n|^p d\sigma
\]

(2-1)

where the constants \(c > 0\) and \(C\) do not depend on the polynomial sequence under consideration (they depend –of course– on \(w, p\) and \(\mathcal{N}\)).

The inequality (2-1) can be rewritten or equivalently (\(\chi_\mathcal{N}\) denotes the indicator function of the set \(\mathcal{N}\))

\[
1 \leq \frac{\|P_n w^n\|_p}{\|P_n w^n \chi_\mathcal{N}\|_p} \leq 1 + C e^{-cn} .
\]

(2-2)
The inequality (2-1) shows that the norm of $P_n w^n$ lives in a small neighborhood of $S_w$; this will be the main tool in what follows. The ideas follow very closely similar steps for the so-called weighted Chebyshev polynomials in III.3 of [1].

For any set $X \subset \mathbb{C}$ we will denote by $\text{Co}(X)$ the (closed) convex hull of said set.

Let, as before $\mathcal{N} \supset S_w$ be an open, bounded neighborhood of $S_w$. We start from the

**Lemma 2.1** Let $X \subset \mathbb{C}$ be compact that is not a singleton and $w \in \mathbb{C}$ be such that $\text{dist}(w, \text{Co}(X)) = \delta > 0$. Then

$$\frac{|z - z_w|}{|z - w|} \leq \frac{D}{\sqrt{D^2 + \delta^2}} < 1, \quad D := \text{diam}(\text{Co}(X))$$

where $z_w \in \text{Co}(X)$ is the (unique) closest point to $w$.

**Proof.** The set $\text{Co}(X)$ lies entirely on one half-plane passing through $z_w$ and perpendicular to the line segment $[z_w, w]$. Let $\theta_w$ the smallest angle such that $\text{Co}(X)$ is entirely contained in a $\theta_w$ sector centered at $w$; by the convexity and compactness of $\text{Co}(X)$, $\theta_w < \pi$. In fact we can estimate the upper bound on $w$ of such $\theta_w$ as

$$\theta_w \leq \arctan \left( \frac{\delta}{D} \right), \quad D = \text{diam}(\text{Co}(X)) .$$

from which (2-3) follows (see Fig. 1). Q.E.D.

**Proposition 2.1** Let $K$ be a closed subset in $\mathbb{C} \setminus \text{Co}(S_w)$. Then eventually there are no roots of $P_n$ belonging to $K$. In particular, for any $\epsilon > 0$ there is a $n_0 \in \mathbb{N}$ such that $\forall n > n_0$ all roots of $P_n$ are within distance $\epsilon$ from the convex hull.

**Proof.** Since $K$ is closed and has no intersection with $\text{Co}(S_w)$ we have $\text{dist}(K, \text{Co}(S_w)) = 2\delta > 0$; Let $\mathcal{N}$ be the $\delta$-fattening of $\text{Co}(S_w)$, namely

$$\mathcal{N} := \{z \in \mathbb{C} : \text{dist}(z, \text{Co}(S_w)) \leq \delta\}$$

It is easy to see that $\mathcal{N}$ is convex as well.

Now consider the $p$-optimal polynomial $P_n(z)$ and let us decompose it as $P_n(z) = R_n(z)Q_n(z)$ where $R_n(z)$ is the factor of all roots within $K$; note that each of these roots is at distance $\geq \delta$ from $\mathcal{N}$. 

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For each root $z_j$ of $R_n(z)$ we can find the closest point $\tilde{z}_j \in \mathcal{N}$; hence we will define $\tilde{R}_n(z)$ as the “proximal substitute” of $R_n$, where each root of $R_n$ has been replaced by its proximal point in $\mathcal{N}$. Then for all $z \in \mathcal{N}$ we have $|\tilde{R}_n(z)| \leq \rho^{r_n} |R_n(z)|$ where $r_n = \text{deg}(R_n)$. Indeed, by Lemma 2.1

$$|\tilde{R}_n(z)| = \prod_{j=1}^{r_n} |z - \tilde{z}_j| \leq \rho^{r_n} \prod_{j=1}^{r_n} |z - z_j| = \rho^{r_n} |R_n(z)|$$

Thus pointwise

$$|\tilde{P}_n(z)| \leq \rho^{r_n} |P_n(z)|$$

where in the second inequality we have used (2-1) on the sequence of polynomials $\tilde{P}_n$. Inequalities (2-9) amount to

$$1 \leq (1 + Ce^{-cn}) \rho^{pr_n}$$

and recall that $\rho < 1$. This inequality implies at once that $\limsup r_n = 0$, and hence the sequence of natural numbers $r_n$ must eventually be identically zero. The second statement in the theorem is simply obtained by taking for $K$ the complement of the $\epsilon$–fattening of $\text{Co}(S_w)$. Q.E.D.

Having established that there are no roots (eventually) “outside” the convex hull, we get some further information about what happens in general.

We borrow the following nice Lemma 2.2 (Lemma III.3.5 in [1], originally in [11])

**Lemma 2.2** If $S$ and $K$ are compact sets such that $Pc(S) \cap K = \emptyset$ then there is a positive integer $m = m(K)$ and a constant $0 < \alpha(K) < 1$ such that for all $(z_1, \ldots, z_m) \in K^m$ there are points $\tilde{z}_1, \ldots, \tilde{z}_m$ such that the rational function

$$r(z) := \frac{\prod_{j=1}^{m(K)} (z - \tilde{z}_j)}{\prod_{j=1}^{m(K)} (z - z_j)}$$

satisfies

$$\sup_{z \in S} |r(z)| \leq \alpha(K) .$$

Lemma 2.2 allows us to prove
Proposition 2.2 For any compact set $K$ contained in the unbounded component of $\mathbb{C} \setminus S_w$ the number of roots of the $p$–optimal polynomials $P_n$ within $K$ is bounded. In particular $\forall \epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\forall n > n_0$ all but a finite number (uniformly bounded) roots of $P_n$ lie within distance $\epsilon$ from the polynomial convex hull of $S_w$ (i.e. $\mathbb{C} \setminus \Omega$, where $\Omega$ is the unbounded component of $\mathbb{C} \setminus S_w$).

Proof. In parallel with the proof of Prop. 2.1 let $2\delta = \text{dist}(K, S_w)$ and let $N$ be the $\delta$-fattening of $S_w$. We decompose $P_n = R_n Q_n$ where $R_n$ has $r_n$ roots (counted with multiplicity) within $K$. We will prove that $r_n < m(K)$ eventually, where $m(K)$ is the number of poles in Lemma 2.2 for $S = N$ and $K$. Proceeding by contradiction, there would be a subsequence where $r_n \geq m(K)$; but then we can use Lemma 2.2 to find a polynomial $\tilde{R}_n$ such that

$$|\tilde{R}_n(z)| \leq \alpha(K)|R_n(z)|, \quad z \in N \Rightarrow |\tilde{P}_n(z)| \leq \alpha(K)|P_n(z)|, \quad z \in N. \quad (2-13)$$

At this point we proceed exactly as in the proof of Prop. 2.1 starting from (2-8) with $\rho^{pr}n \mapsto \alpha(K)$, namely,

$$\left\| \tilde{P}_n w^n \chi_N \right\|_p^p \leq \alpha(K)^p \left\| P_n w^n \chi_N \right\|_p^p \leq \alpha(K)^p \left\| P_n w^n \right\|_p^p \quad (2-14)$$

By the $p$–optimality of the polynomial $P_n$ we must have

$$1 \leq \frac{\left\| \tilde{P}_n w^n \right\|_p^p}{\left\| P_n w^n \right\|_p^p} \leq \frac{\left\| \tilde{P}_n w^n \chi_N \right\|_p^p}{\left\| P_n w^n \chi_N \right\|_p^p} \leq \frac{\left\| \tilde{P}_n w^n \chi_N \right\|_p^p}{\left\| P_n w^n \chi_N \right\|_p^p} \leq 1 + C e^{-cn} \alpha(K)^p. \quad (2-15)$$

It is clear that the last expression in (2-15) is eventually less than one (since $\alpha(K) < 1$), which leads to a contradiction with the assumption that there were $\geq m(K)$ roots in $K$. The last statement follows from the fact that there are no roots outside the convex hull by Prop. 2.1 together with the above. Q.E.D.

Example 2.1 Suppose that the support of the equilibrium measure consists of intervals in the real axis, as in the case of ordinary orthogonal polynomials. It is an exercise to see that for any gap the number $m(K) = 2$ and hence there can be at most one zero within each gap.

We next prove

**Theorem 2.1** Let $\Omega$ be the unbounded connected component of $\mathbb{C} \setminus S_w$ and $K \subset \Omega$ a compact subset. Let $\zeta_{\ell,n}(K)$ be the roots of $P_n$ belonging to $K$, $\ell = 1, \ldots, m_n(K)$. Then, uniformly in $K$ we have

$$\lim_{n \to \infty} \frac{1}{n} \ln |P_n(z)| + \frac{1}{n} \sum_{j=1}^{m_n(K)} G\Omega(z, \zeta_{\ell,n}) = \int \ln |z - t| d\mu(t), \quad (2-16)$$
where \( G_\Omega(z, w) \) is the Green’s function of \( \Omega \), namely the function such that

\[
\begin{align*}
\Delta z G_\Omega(z, w) &\equiv 0, \quad z \in \Omega \setminus \{w\} \\
G_\Omega(z, w) &\equiv 0, \quad z \in \partial \Omega \\
G(z, w) &\geq 0, \quad z, w \in \Omega \\
G_\Omega(z, w) &\equiv -\frac{1}{|z - w|} + O(1) \quad z \to w
\end{align*}
\]  

Additionally, if \( K \) is closed and does not contain (eventually) any roots, then, uniformly,

\[
\lim_{n \to \infty} \frac{1}{n} \ln |P_n(z)| = \int \ln |z - t| d\mu(t)
\]

**Proof.** We reason on the functions

\[
f_n(z) := \frac{1}{n} \ln |P_n(z)| + \frac{1}{n} \sum_{j=1}^{m_n(K)} G_\Omega(z, z_{\ell,n}) - \int \ln |z - t| d\mu(t)
\]

We will see in Prop. 3.1 together with Corollary 3.1 that \( \forall \epsilon > 0 \ \exists n_0 : \ n \geq n_0 \)

\[
\frac{1}{n} \ln |P_n(z)w^n(z)| \leq -F_w + \epsilon, \quad \forall z \in \mathbb{C}
\]

Additionally, the \( f_n(z) \)'s are subharmonic in \( \Omega \) and harmonic in a neighborhood of \( z = \infty \): indeed all roots are uniformly bounded (from Prop. 2.1) and the Green’s function \( G_\Omega(z, w) \) is harmonic away from the singularity \( z = w \) (in a neighborhood of which it is superharmonic) and in the neighborhoods of \( z_{\ell,n} \) the \( f_n \)'s are actually harmonic because the singularities coming from \( P_n \)'s cancel out exactly those coming from the Green’s functions.

For \( z \in \partial \Omega \) and \( \forall \epsilon > 0 \) we have eventually (recall that \( G_\Omega(z, w) = 0 \) for \( z \in \partial \Omega \))

\[
f_n(z) \leq V(z) + U^w(z) - F_w + \epsilon \leq \epsilon, \quad z \in \partial \Omega.
\]

Since \( f_n(z) \) are subharmonic, they cannot have isolated maxima in the interior of \( \Omega \) and hence we conclude that \( f_n(z) \leq \epsilon \) throughout \( \Omega \) (including \( z = \infty \)).

Let \( f_\infty(z) = \limsup_{n \to \infty} f_n(z) \); then \( \forall \epsilon > 0 \)

\[
f_\infty(z) = \limsup_{n \to \infty} f_n(z) \leq \epsilon \Rightarrow f_\infty(z) \leq 0, \quad z \in \mathbb{C}
\]

Let \( \text{Pe}(S_w) = \mathbb{C} \setminus \Omega \) be the polynomial convex hull of \( S_w \) and let now \( K \) be a compact set \( K \subset \Omega \).

We next analyze the \( \liminf \); let \( z_0 \in K \) and set

\[
L_{z_0} := \liminf_{n \to \infty} f_n(z_0) \leq f_\infty(z_0) \leq 0.
\]

where \( z_0 \in K \) is some (arbitrary but fixed) point. There is a subsequence \( n_k \) of the numbers \( f_n(z_0) \)'s which converges to this limit; out of it, we can extract another subsequence (which we denote again \( n_k \) for
brevity) such that the counting measures \( \sigma_{n_k} \) have a weak* limit (since they are all compactly supported) which we denote by \( \sigma_{z_0} \) (note that both the subsequence and this limiting distribution may depend on \( z_0 \)). Prop. 2.2 implies that its support of \( \sigma_{z_0} \) lies in the polynomial convex hull of \( S_w \); in particular the function \( \ln |z - \bullet| \) is harmonic on \( \text{supp}(\sigma_{z_0}) \) for any \( z \in K \). Let \( \widehat{\sigma}_{n_k} \) be the restriction of \( \sigma_{n_k} \) to those atoms outside of \( K \); we know that it differs from \( \sigma_{n_k} \) by a finite number \( m_n(K) \) of atoms (uniformly bounded in \( n \)) and hence it obviously has the same weak* limit. Now, for any \( z \in K \) along the chosen subsequence we have

\[
0 \geq f_\infty(z) \geq \lim_{k \to \infty} f_{n_k}(z) = \lim_{k \to \infty} \int \ln |z - t| d\widehat{\sigma}_{n_k}(t) - \int \ln |z - t| d\mu_w(t) + \frac{1}{n_k} \sum_{k=1}^{m_n(K)} (G_{n_k}(z, z_{\ell,n_k}) + \ln |z - z_{\ell,n_k}|)
\]  

(2-27)

Since \( G_{n_k}(z, w) + \ln |z - w| \) is jointly continuous in \( z, w \) for \( z, w \in \Omega \), it is also (jointly) bounded on compact sets; we know already that \( z_{\ell,n_k} \) all are uniformly bounded, hence the last term in (2-27) tends to zero. We thus have

\[
\lim_{k \to \infty} f_{n_k}(z) = \lim_{k \to \infty} \int \ln |z - t| d\widehat{\sigma}_{n_k}(t) - \int \ln |z - t| d\mu_w(t) = \int \ln |z - t| d\widehat{\sigma}_{z_0}(t) - \int \ln |z - t| d\mu_w(t)
\]  

(2-28)

The right hand side of (2-28) is harmonic in \( \Omega \) (by inspection) and by (2-25) it is \( \leq 0 \); on the other hand at \( z = \infty \) it vanishes (since both measures are probability measures) and hence it must be identically zero. Evaluating it at \( z = z_0 \) yields that \( L_{z_0} = \lim \inf_{n \to \infty} f_n(z_0) = 0 \); since \( z_0 \) was arbitrary, this shows that \( \lim_{n \to \infty} f_n(z) = 0 \); the uniformity of the convergence follows from the fact that the sequence of functions

\[
h_n(z) := \int \ln |z - t| d\widehat{\sigma}_n(t)
\]  

(2-29)

are equicontinuous for \( z \in K \) and hence the Arzela–Ascoli Theorem 12 guarantees uniform convergence. To see equicontinuity we compute

\[
|h_n(z) - h_n(z')| = \int \ln \left| \frac{z - t}{z' - t} \right| d\widehat{\sigma}_n(t) = \int \ln \left| 1 + \frac{z - z'}{z' - t} \right| d\widehat{\sigma}_n(t) \leq \int \frac{|z - z'|}{|z' - t|} d\widehat{\sigma}_n(t) \leq \frac{|z - z'|}{\text{dist}(K, S_w)}
\]  

(2-30)

Note that the above chain of inequalities applies more generally for any closed \( K \subset \Omega \) and also says that the sequence is uniformly Lipschitz.

To prove (2-21) we note that we have used compactness only after (2-27), but if \( m_n(K) \equiv 0 \) (eventually) then the same arguments prove uniform convergence without having to use compactness. Q.E.D.

Theorem 2.1 says loosely speaking that \( \frac{1}{n} \ln |P_n| \) converges to the logarithmic transform of the equilibrium measure as uniformly as it is possible on the “outside” of the support, given that there are possibly
some stray roots; if we restrict to the outside of the convex hull of \( S_w \), then this convergence is truly uniform (over closed subsets) because –eventually– there are no roots at all (Prop. 2.1).

Theorem 2.1 has an interesting corollary

**Corollary 2.1** Let \( h(z) \) be any harmonic function on a neighborhood of \( \text{Pc}(S_w) \) and let \( \sigma \) be a weak\(^*\) limit point of the counting measures of the \( L^p \)-optimal polynomials. Then

\[
\int h(z) d\sigma(z) = \int h(z) d\mu_w(z)
\]

(2-31)

**Proof.** By Mergelyan’s theorem it suffices to verify it for the monomials \( z^j \); we have seen in the proof of Thm. 2.1 (2-27 and discussion thereafter) that

\[
\int \ln |z - t| d\sigma(t) - \int \ln |z - t| d\mu_w(t) \equiv 0
\]

(2-32)

for \( z \in \Omega \) (the complement of the polynomial convex hull of \( S_w \)). Taking the large \( z \) expansion we have easily the statement Q.E.D.

**Remark 2.1** The Theorem 2.1 and Corollary 2.1 assert that whatever limiting distribution the roots of the \( p \)-optimal polynomials may have, it must be a balayage of the equilibrium measure onto the support of this limiting distribution. In order not to swindle the reader, we should point out that it falls short of saying that there is a unique limiting distribution, and even further away from any statement about what distribution that should be.

### 3 Norm estimates

#### 3.1 Upper estimate for the norms

The aim of this section is twofold: first we will prove that if \( P_n w^n \) are the \( p \)-optimal weighted polynomials then

\[
\lim_{n \to \infty} \frac{1}{n} \ln \| P_n w^n \|_p = -F_w \iff \| P_n w^n \|_p = e^{-nF_w + o(n)}
\]

(3-1)

*En route* we will see that the \( L^p \) norms of the wave-functions \( P_n w^n \) are asymptotically equal to the \( L^\infty \) ones. In particular this implies that the \( n \)-th root of the wave functions is uniformly bounded.

**Proposition 3.1** Let \( P_n w^n \) be the \( p \)-optimal weighted polynomial; then

\[
\limsup_{n \to \infty} \frac{1}{n} \ln \| P_n w^n \|_p \leq -F_w,
\]

(3-2)

where \( \ell \) is the Robin constant for the equilibrium measure.
Proof. We compare the $L^p$ norms of the $P_n w^n$'s with the weighted Fekete polynomials $F_n w^n$. Let $\mathcal{N}$ be a bounded open neighborhood of $S_w$. Then

$$\|P_n w^n\|_p \overset{\text{by optimality}}{\leq} \|F_n w^n\|_p \leq (1 + Ce^{-cn})\|F_n w^n\|_\infty \text{Area}(\mathcal{N})^{\frac{1}{p}} \quad (3-3)$$

Now taking $\frac{1}{n} \ln(\cdot)$ of both sides gives

$$\frac{1}{n} \ln (\|P_n w^n\|_p) \leq \frac{1}{n} \ln (\|F_n w^n\|_p) \leq \frac{1}{n} \ln (\|F_n w^n\|_\infty) + O(n^{-1}) \quad (3-4)$$

Since

$$\lim_{n \to \infty} \frac{1}{n} \ln (\|F_n w^n\|_\infty) = -F_w \quad (3-5)$$

(see Thm. III.1.9 in [1]) we have

$$\limsup_{n \to \infty} \frac{1}{n} \ln (\|P_n w^n\|_p) \leq \limsup_{n \to \infty} \frac{1}{n} \ln (\|F_n w^n\|_p) \leq \limsup_{n \to \infty} \frac{1}{n} \ln (\|F_n w^n\|_\infty) \leq -F_w. \quad (3-6)$$

Q.E.D.

Remark 3.1 It may be of some importance to note that the above proof can be used to show

$$\limsup_{n \to \infty} \ln \left(\frac{\|P_n w^n\|_p}{\|F_n w^n\|_\infty}\right) \leq \sqrt{\text{Area}(S_w)} \quad (3-7)$$

3.2 Lower estimate for the norms

We follow the idea in [1], pp 182.

Lemma 3.1 Let $P_n(z)$ be a sequence of polynomials of degree at most $n$. Assume further that the potential $V$ is twice continuously differentiable. Then there is a constant $D > 0$ and $d_\Sigma$ (the Hausdorff dimension of $\Sigma$, which for us is either 2 or 1) such that

$$\frac{\|P_n w^n\|_p}{\|P_n w^n\|_\infty} \geq Dn^{-\frac{d_\Sigma}{p}} \quad (3-8)$$

In particular

$$\liminf_{n \to \infty} \frac{\|P_n w^n\|_p^{\frac{1}{p}}}{\|P_n w^n\|_\infty^{\frac{1}{p}}} \geq \liminf_{n \to \infty} \|P_n w^n\|_\infty^{\frac{1}{p}} \quad (3-9)$$

$$\limsup_{n \to \infty} \frac{\|P_n w^n\|_p^{\frac{1}{p}}}{\|P_n w^n\|_\infty^{\frac{1}{p}}} \geq \limsup_{n \to \infty} \|P_n w^n\|_\infty^{\frac{1}{p}}. \quad (3-10)$$

Proof. We work with the normalized polynomials

$$Q_n(z) := \frac{1}{\|P_n w^n\|_\infty} P_n(z). \quad (3-11)$$
Let $z_0$ be a point where $|Q_n(z)w^n(z)|$ achieves its maximum value 1 (such a point exists by the assumed admissibility conditions on $w$). We claim that

$$\exists C > 0 : |z - z_0| \leq \frac{1}{2eC_n} \implies |Q_n(z)|e^{-nV(z)} \geq \frac{1}{2e} \quad (3-12)$$

Since $|Q(z_0)|e^{-nV(z_0)} = 1$ the inequality can be rewritten

$$|Q_n(z)| \leq |Q_n(z_0)|e^{n(V(z)-V(z_0))} , \quad \forall z \in \mathbb{C}. \quad (3-13)$$

Let $\delta > 0$ and set $C_\delta(z_0) := \sup_{|z-z_0|<\delta} |V(z) - V(z_0)|$; since we are assuming $V(z)$ to be twice continuously differentiable, $z_0 \in S_w$ and $S_w$ is compact, we see that a simple argument shows $C_\delta(z_0) < C\delta$ for some constant $C > 0$ (independent of $z_0 \in S_w$). Let $|z - z_0| < \frac{1}{2}\delta$; the formula of Cauchy for the derivative implies

$$|Q'_n(z)| \leq |Q_n(z_0)|\frac{2e^{nC\delta}}{\delta} , \quad |z - z_0| \leq \frac{1}{2}\delta. \quad (3-14)$$

On the even smaller disk $|z - z_0| < \frac{1}{4e}\delta$ we have

$$|Q_n(z) - Q_n(z_0)| \leq \int_{z_0}^z |Q'_n(t)||dt| \leq |Q_n(z_0)|\frac{2e^{nC\delta}}{\delta} \frac{|z - z_0|}{\delta} \leq \frac{1}{2}|Q_n(z_0)|e^{nC\delta-1} \quad (3-15)$$

If we choose $\delta = \frac{1}{Cn}$ and hence $|z - z_0| < \frac{\delta}{4e} = \frac{1}{4eCn}$ we have

$$|Q_n(z) - Q_n(z_0)| \leq \frac{1}{2}|Q_n(z_0)| \implies |Q_n(z)| \geq \frac{1}{2}|Q_n(z_0)|. \quad (3-16)$$

Multiplying both sides

$$|Q_n(z)|e^{-n(V(z)-V(z_0))} \geq \frac{1}{2}|Q_n(z_0)|e^{-nV(z)} \geq \frac{1}{2}|Q_n(z_0)|e^{-nC\delta} = \frac{|Q_n(z_0)|}{2e} \implies |Q_n(z)|e^{-nV(z)} \geq \frac{1}{2e}|Q_n(z_0)|e^{-nV(z)} = \frac{1}{2e}. \quad (3-17)$$

Integrating the inequality (3-12)

$$\left(\int_{\Sigma} |Q_n(z)w^n|^p d\Sigma z\right) \geq \left(\int_{|z-z_0|<\frac{\delta}{4e}} |Q_n(z_0)w^n|^p d\Sigma z\right) \geq \frac{1}{2e}\left[B_{\Sigma} \left(\frac{1}{4nCe}\right)\right]^{\frac{1}{p}} \quad (3-18)$$

Here $B_{\Sigma}(\delta)$ is the $d\sigma$ volume of the ball of radius $\delta$ centered at $z_0$ in $\Sigma$: in the case $\Sigma = \mathbb{C}$ this is simply $\pi\delta^2$, in the case $\Sigma$ is a smooth curve then $B_{\Sigma}(\delta) \geq c\delta$ for some $c > 0$. The only important fact for us below is that $B_{\Sigma}(\delta)$ is bounded below by some positive power of $\delta$. Therefore, recalling that $Q_n(z) = P_n(z)/\|P_nw^n\|_{\infty}$ the inequality (3-19) reads

$$\|P_n(z)w^n\|_p \geq \frac{1}{2e}\left[B_{\Sigma} \left(\frac{1}{4nCe}\right)\right]^{\frac{1}{p}} \|P_nw^n\|_{\infty} \quad (3-20)$$
Summarizing, there are constants $D > 0$ and $d_{\Sigma}$ (the “dimension” of $\Sigma$, which for us is either 2 or 1) such that
\[
\frac{\|P_n(z)w^n\|_p}{\|P_n w^n\|_{\infty}} \geq Dn^{-\frac{d_{\Sigma}}{p}} .
\]
(3-21)

Q.E.D.

Before proceeding we recall

Theorem 3.1 (Thm. I.3.6 in [1]) Let $P_n$ be any sequence of monic polynomials of degree $n$. Then
\[
\liminf_{n \to \infty} (\|P_n w^n\|_{\infty})^{\frac{1}{n}} \geq \exp(-F_w) .
\]
(3-22)

As a corollary of Thm. 3.1 and Prop. 3.1 we have

Corollary 3.1 The norms of the $p$-optimal polynomials satisfy
\[
\sqrt[n]{\|P_n w^n\|_p} \to e^{-F_w}, \quad n \to \infty .
\]
(3-23)

Proof. Using Lemma 3.1 and (3-9) together with Thm. 3.1 we have that the lim inf of the left hand side cannot be less than $e^{-F_w}$:
\[
-F_w \geq \limsup_{n \to \infty} \frac{1}{n} \ln \|P_n w^n\|_p \geq \liminf_{n \to \infty} \frac{1}{n} \ln \|P_n w^n\|_p \geq \liminf_{n \to \infty} \frac{1}{n} \ln \|P_n w^n\|_{\infty} \geq -F_w .
\]
Q.E.D.

References

[1] Edward B. Saff and Vilmos Totik. Logarithmic potentials with external fields, volume 316 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1997. Appendix B by Thomas Bloom.

[2] Madan Lal Mehta. Random matrices, volume 142 of Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, third edition, 2004.

[3] P. A. Deift. Orthogonal polynomials and random matrices: a Riemann-Hilbert approach, volume 3 of Courant Lecture Notes in Mathematics. New York University Courant Institute of Mathematical Sciences, New York, 1999.

[4] A. Zabrodin. Matrix models and growth processes: from viscous flows to the quantum Hall effect. In Applications of random matrices in physics, volume 221 of NATO Sci. Ser. II Math. Phys. Chem., pages 261–318. Springer, Dordrecht, 2006.

[5] Peter Elbau and Giovanni Felder. Density of eigenvalues of random normal matrices. Comm. Math. Phys., 259(2):433–450, 2005.
[6] Harold Widom. Extremal polynomials associated with a system of curves in the complex plane. 
Advances in Math., 3:127–232 (1969), 1969.

[7] D. S. Lubinsky and E. B. Saff. Strong asymptotics for $L_p$ extremal polynomials ($1 < p \leq \infty$) associated with weights on $[-1,1]$. In Approximation theory, Tampa (Tampa, Fla., 1985–1986), volume 1287 of Lecture Notes in Math., pages 83–104. Springer, Berlin, 1987.

[8] H. N. Mhaskar and E. B. Saff. The distribution of zeros of asymptotically extremal polynomials. J. Approx. Theory, 65(3):279–300, 1991.

[9] H. N. Mhaskar and E. B. Saff. Where does the $L^p$-norm of a weighted polynomial live? Trans. Amer. Math. Soc., 303(1):109–124, 1987.

[10] Leopold Fejér. Über die Lage der Nullstellen von Polynomen, die aus Minimumforderungen gewisser Art entspringen. Math. Ann., 85(1):41–48, 1922.

[11] Harold Widom. Polynomials associated with measures in the complex plane. J. Math. Mech., 16:997–1013, 1967.

[12] Walter Rudin. Real and complex analysis. McGraw-Hill Book Co., New York, third edition, 1987.