A TOPOLOGICAL APPROACH TO PERIODIC OSCILLATIONS RELATED TO THE LIEBAU PHENOMENON

J. A. CID, G. INFANTE, M. TVRDÝ, AND M. ZIMA

ABSTRACT. We give some sufficient conditions for existence, non-existence and localization of positive solutions for a periodic boundary value problem related to the Liebau phenomenon. Our approach is of topological nature and relies on the Krasnosel’skii-Guo theorem on cone expansion and compression. Our results improve and complement earlier ones in the literature.

1. INTRODUCTION

In the 1950’s the physician G. Liebau developed some experiments dealing with a valveless pumping phenomenon arising on blood circulation and that has been known for a long time: roughly speaking, Liebau showed experimentally that a periodic compression made on an asymmetric part of a fluid-mechanical model could produce the circulation of the fluid without the necessity of a valve to ensure a preferential direction of the flow [1, 12, 13]. After his pioneering work this effect has been known as the Liebau phenomenon.

In [14] G. Propst, with the aim of contributing to the theoretical understanding of the Liebau phenomenon, presented some differential equations modeling a periodically forced flow through different pipe-tank configurations. He was able to prove the presence of pumping effects in some of them, but the apparently simplest model, the “one pipe-one tank” configuration, skipped his efforts due to a singularity in the corresponding differential equation model, namely

\begin{equation}
\begin{aligned}
&u''(t) + a u'(t) = \frac{1}{u} \left( e(t) - b u'(t)^2 \right) - c, \quad t \in [0, T], \\
u(0) = u(T), &\quad u'(0) = u'(T),
\end{aligned}
\end{equation}

being $a \geq 0$, $b > 1$, $c > 0$ and $e(t)$ continuous and $T$-periodic on $\mathbb{R}$.

The singular periodic problem (1) was studied in [4], where the authors gave general results for the existence and asymptotic stability of positive solutions by performing the change of variables $u = x^\mu$, where $\mu = \frac{1}{b+1}$, which transforms the singular problem (1) into the regular
\[
\begin{aligned}
&x''(t) + a x'(t) = \frac{e(t)}{\mu} x^{1-2\mu}(t) - \frac{c}{\mu} x^{1-\mu}(t), \quad t \in [0, T], \\
x(0) = x(T), & \quad x'(0) = x'(T).
\end{aligned}
\]

Then the existence and stability of positive solutions for (2) were obtained by means of the lower and upper solution technique [5] and using tricks analogous to those used in [15].

In this paper we deal with the existence of positive solutions for the following generalization of the problem (2)

\[
\begin{aligned}
&x''(t) + a x'(t) = r(t)x^\alpha(t) - s(t)x^\beta(t), \quad t \in [0, T], \\
x(0) = x(T), & \quad x'(0) = x'(T),
\end{aligned}
\]

where we assume

\( (H_0) \ a \geq 0, \ r, s \in C[0, T], \ 0 < \alpha < \beta < 1. \)

Note that, by defining \( r(t) = \frac{e(t)}{\mu}, \ s(t) = \frac{c}{\mu}, \ \alpha = 1 - 2\mu \) and \( \beta = 1 - \mu \), the problem (2) fits within (3).

Our approach is essentially of topological nature: in Section 2 we rewrite problem (3) as an equivalent fixed point problem suitable to be treated by means of the Krasnosel’skiĭ-Guo cone expansion/compression fixed point theorem. A careful analysis of the related Green’s function, necessary in our approach, is postponed to a final Appendix. In Section 3 we present our main results: existence, non-existence and localization criteria for positive solutions of the problem (3). Some corollaries with more ready-to-use results are also addressed. We point out that our results are valid not only for the more general problem (3), but also when applied to the singular model problem (1) we improve previous results of [4].

2. A FIXED POINT FORMULATION

First of all, by means of a shifting argument, we rewrite the problem (3) in the equivalent form

\[
\begin{aligned}
&x''(t) + a x'(t) + m^2 x(t) = r(t)x^\alpha(t) - s(t)x^\beta(t) + m^2x(t) := f_m(t, x(t)), \quad t \in [0, T], \\
x(0) = x(T), & \quad x'(0) = x'(T),
\end{aligned}
\]

with \( m \in \mathbb{R} \). A similar approach has been used, under a variety of boundary conditions in [7, 9, 16, 17].

We say that problem (4) is non-resonant if zero is the unique solution of the homogeneous linear problem

\[
\begin{aligned}
&x''(t) + a x'(t) + m^2 x(t) = 0, \quad t \in [0, T], \\
x(0) = x(T), & \quad x'(0) = x'(T).
\end{aligned}
\]
In this case the non-homogeneous linear problem

\[
\begin{align*}
\begin{cases}
x''(t) + ax'(t) + m^2 x(t) = h(t), & t \in [0, T], \\
x(0) = x(T), & x'(0) = x'(T),
\end{cases}
\end{align*}
\]

is also uniquely solvable and its unique solution is given by

\[
K h(t) = \int_0^T G_m(t, s) h(s) ds,
\]

where \(G_m(t, s)\) is the related Green’s function (see [2, 3]).

In particular, if \(m > 0\) and \(m^2 < \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2\) then the problem (4) is non-resonant and moreover \(G_m(t, s)\) satisfies the following properties (see the Appendix for the details):

i) \(G_m(t, s) > 0\), for all \((t, s) \in [0, T] \times [0, T]\).

ii) \(\int_0^T G_m(t, s) ds = \frac{1}{m^2}\).

iii) There exists a constant \(c_m \in (0, 1)\) such that

\[
G_m(t, s) \geq G_m(s, s) \geq c_m G_m(t, s), \quad \text{for all } (t, s) \in [0, T] \times [0, T].
\]

A \textit{cone} in a Banach space \(X\) is a closed, convex subset of \(X\) such that \(\lambda x \in K\) for \(x \in K\) and \(\lambda \geq 0\) and \(K \cap (-K) = \{0\}\). Here we work in the space \(X = C[0, T]\) endowed with the usual maximum norm \(\|x\| = \max\{|x(t)| : t \in [0, T]\}\) and use the cone

\[
P = \{x \in X : x(t) \geq c_m \|x\| \text{ on } [0, T]\},
\]

a type of cone firstly used by Krasnosel’skiii, see for example [10], and D. Guo, see for example [6]. The cone \(P\) is particularly useful when dealing with singular nonlinearities (see [11]), or for localizing the solutions (see for example [8]).

Let us define the operator \(F : P \to X\)

\[
Fx(t) = \int_0^T G_m(t, s) f_m(s, x(s)) ds.
\]

Note that a fixed point of \(F\) in \(P\) is a non-negative solution of the problem (4). In order to get such a fixed point we employ the following well-known Krasnosel’skiii-Guo cone compression/expansion theorem.

\textbf{Theorem 2.1.} [6] Let \(P\) be a cone in \(X\) and suppose that \(\Omega_1\) and \(\Omega_2\) are bounded open sets in \(X\) such that \(0 \in \Omega_1\) and \(\overline{\Omega}_1 \subset \Omega_2\). Let \(F : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P\) be a completely continuous operator such that one of the following conditions holds:

(i) \(\|Fx\| \geq \|x\|\) for \(x \in P \cap \partial \Omega_1\) and \(\|Fx\| \leq \|x\|\) for \(x \in P \cap \partial \Omega_2\),

(ii) \(\|Fx\| \leq \|x\|\) for \(x \in P \cap \partial \Omega_1\) and \(\|Fx\| \geq \|x\|\) for \(x \in P \cap \partial \Omega_2\).

Then \(F\) has a fixed point in the set \(P \cap (\overline{\Omega}_2 \setminus \Omega_1)\).
In the sequel, for a given continuous function \( h : [0, T] \to \mathbb{R} \) we use the notation
\[
h_\ast = \min \{ h(t) : t \in [0, T] \} \quad \text{and} \quad h^* = \max \{ h(t) : t \in [0, T] \}.
\]

3. Main results

Integrating on \([0, T]\) the differential equation in the problem (3), and taking into account the boundary conditions, we get a necessary condition for the existence of positive solutions, namely
\[
0 = \int_0^T [r(t)x^\alpha(t) - s(t)x^\beta(t)]dt.
\]
So it is easy to arrive to the following non-existence result.

**Theorem 3.1.** If one of the following conditions holds,

1. \( r_* \geq 0 \) and \( s^* < 0 \),
2. \( r_* > 0 \) and \( s^* \leq 0 \),
3. \( r^* \leq 0 \) and \( s_* > 0 \),
4. \( r^* < 0 \) and \( s_* \geq 0 \),

then problem (3) does not have positive solutions.

Our main result, concerning not only the existence but also the localization of positive solutions, is the following one.

**Theorem 3.2.** Assume that (H0) and the following condition hold:

(H1) There exist \( m > 0 \) and \( 0 < R_1 < R_2 \) such that \( m^2 < \left( \frac{\pi}{T} \right)^2 + \left( \frac{a}{2} \right)^2 \),

(7) \( f_m(t,x) = r(t)x^\alpha - s(t)x^\beta + m^2x \geq 0 \) for \( t \in [0, T] \) and \( x \in [c_mR_1, R_2] \),

(8) \( f_m(t,x) \geq m^2R_1 \) for \( t \in [0, T] \) and \( x \in [c_mR_1, R_1] \)

and

(9) \( f_m(t,x) \leq m^2R_2 \) for \( t \in [0, T] \) and \( x \in [c_mR_2, R_2] \).

Then, the problem (3) has a positive solution \( x(t) \) such that
\[
c_mR_1 \leq x(t) \leq R_2.
\]

**Proof.** The problem (4) with the value of \( m \) given by (H1) is non-resonant and its Green’s function \( G_m(t, s) \) satisfies the properties i), ii) and iii) stated in the Introduction. Let us define
\[
\Omega_j = \{ x \in X : \|x\| < R_j \} \quad j = 1, 2.
\]
Then by (7) we have $f_m(t, x(t)) \geq 0$ for $x \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Now, for every $x \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and for every $t \in [0, T]$, we have that
\[
\int_0^T G_m(s, s)f_m(s, x(s))ds \leq Fx(t) = \int_0^T G_m(t, s)f_m(s, x(s))ds \leq \int_0^T \frac{G_m(s, s)}{c_m} f_m(s, x(s))ds,
\]
and therefore $F(P \cap (\overline{\Omega}_2 \setminus \Omega_1)) \subset P$. A standard argument shows that $F$ is a completely continuous operator.

Now, we check that condition (i) in Theorem 2.1 is fulfilled.

Claim 1. $\|Fx\| \geq \|x\|$ for $x \in P \cap \partial \Omega_1$.

Let $x \in P \cap \partial \Omega_1$, that is, $x \in P$ and $\|x\| = R_1$. Then we have that $c_m R_1 \leq x(t) \leq R_1$ for all $t \in [0, T]$ and hence
\[
Fx(t) = \int_0^T G_m(t, s)f_m(s, x(s))ds \geq \int_0^T G_m(t, s)m^2R_1ds = m^2R_1 \int_0^T G_m(t, s)ds = R_1 = \|x\|.
\]

Claim 2. $\|Fx\| \leq \|x\|$ for $x \in P \cap \partial \Omega_2$.

Let $x \in P \cap \partial \Omega_2$, that is, $x \in P$ and $\|x\| = R_2$. Then $c_m R_2 \leq x(t) \leq R_2$ for all $t \in [0, T]$ and hence
\[
Fx(t) = \int_0^T G_m(t, s)f_m(s, x(s))ds \leq \int_0^T G_m(t, s)m^2R_2ds = m^2R_2 \int_0^T G_m(t, s)ds = R_2 = \|x\|.
\]

Therefore by Theorem 2.1 the existence of a solution of the problem (3) with the desired localization property immediately follows. \qed

The following result is a consequence of Theorem 3.2.

**Theorem 3.3.** Assume that (H0) and the following conditions hold:

(H2) There exists $m > 0$ such that $m^2 < \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2$ and $f_m(t, x) = r(t)x^\alpha - s(t)x^\beta + m^2x \geq 0$ for $t \in [0, T]$ and $x \geq 0$.

(H3) $r_* > 0$ and $s_* > 0$.

Then the problem (3) has a positive solution.

**Proof.** We are going to prove that assumption (H1) in Theorem 3.2 is satisfied. By (H0), (H3) and since $c_m \in (0, 1)$ we can choose $0 < R_1 < R_2$ such that
\[
(1 - c_m)m^2R_1^{1-\alpha} + s_*R_1^{\beta-\alpha} \leq r_* c_m^\alpha
\]
and
\[(11) \quad r^* \leq s_cR_2^{\beta - \alpha}.
\]
Note that the inequality (7) follows from (H2) so it only remains to check the inequalities (8) and (9).

If \( t \in [0, T] \) and \( x \in [c_mR_1, R_1] \) then, taking into account (H3) and (10), we have that
\[ f_m(t, x) \geq r_*(c_mR_1)^\alpha - s_1R_1^\beta + m^2c_mR_1 \geq m^2R_1, \]
so the inequality (8) is fulfilled.

If \( t \in [0, T] \) and \( x \in [c_mR_2, R_2] \) then, taking into account (H3) and (11), we have that
\[ f_m(t, x) \leq r^*R_2^\alpha - s_*(c_mR_2)\beta + m^2R_2 \leq m^2R_2, \]
so the inequality (9) is also fulfilled. \( \Box \)

**Remark 3.4.** Note that (H0) and (H2) imply that \( r_*>0 \), thus the first part of condition (H3) is redundant but we have included it for the sake of clarity. On the other hand, notice that condition (7) in Theorem 3.2 could be satisfied even if \( r(t) \) assumes negative values.

Next, we present an explicit condition sufficient in order to get the condition (H2) in Theorem 3.3.

**Corollary 3.5.** Assume that (H0) and (H3) hold and, moreover, that
\[(12) \quad s^* < \min\{\left(\frac{\pi}{T}\right)^2 + (\frac{a}{2})^2, r_*\}.
\]
Then the problem (3) has a positive solution.

**Proof.** It is enough to show that condition (12) implies (H2). Indeed, by (12) we can choose \( m > 0 \) such that
\[ s^* < m^2 < \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2. \]

Now, we are going to prove that for such \( m \) we have that \( f_m(t, x) \geq 0 \) for all \( t \in [0, T] \) and all \( x \geq 0 \).

**Step 1.** We show that \( f_m(t, x) \geq 0 \) for all \( t \in [0, T] \) and all \( x \in [0, 1] \).

Since \( 0 < \alpha < \beta < 1 \) we have \( x^\beta \leq x^\alpha \) for \( 0 \leq x \leq 1 \). Moreover \( s^* < r_* \) by (12) and then
\[ f_m(t, x) = r(t)x^\alpha - s(t)x^\beta + m^2x \geq r_*x^\alpha - s_*x^\beta + m^2x \]
\[ \geq r_*x^\beta - s_*x^\beta + m^2x = (r_* - s_*)x^\beta + m^2x > 0. \]

**Step 2.** We show that \( f_m(t, x) \geq 0 \) for all \( t \in [0, T] \) and all \( x \geq 1 \).

Since \( 0 < \beta < 1 \) we have \( x^\beta \leq x \) for \( x \geq 1 \) and therefore
\[ f_m(t, x) = r(t)x^\alpha - s(t)x^\beta + m^2x \geq r_*x^\alpha - s_*x^\beta + m^2x \]
\[ \geq r_*x^\alpha - s_*x^\beta + m^2x^\beta = r_*x^\alpha + (m^2 - s_*)x^\beta \geq 0. \]
In the particular case of problem (2) we recover [4, Theorem 1.8].

**Corollary 3.6.** Assume that \( a \geq 0, b > 1, c > 0 \) and \( e_* > 0 \). Then problem (2) has a positive solution provided that

\[
\frac{(b+1)c^2}{4e_*} < \left( \frac{\pi}{T} \right)^2 + \frac{a^2}{4}.
\]

**Proof.** We recall that the problem (2) is of the form (3) with

\[
\mu = \frac{1}{b+1}, \quad r(t) = \frac{e(t)}{\mu}, \quad s(t) = \frac{c}{\mu}, \quad \alpha = 1 - 2\mu, \quad \beta = 1 - \mu.
\]

From our assumptions it follows immediately that the conditions (H0) and (H3) of Theorem 3.3 hold. Thus it only remains to check the condition (H2). By (13) we can choose \( m > 0 \) such that

\[
\frac{(b+1)c^2}{4e_*} < m^2 < \left( \frac{\pi}{T} \right)^2 + \frac{a^2}{4}.
\]

Then for all \( t \in [0, T] \) and \( x \geq 0 \) we have

\[
f_m(t, x) = \frac{e(t)}{\mu} x^{1-2\mu} - \frac{c}{\mu} x^{1-\mu} + m^2 x \geq \frac{e_*}{\mu} x^{1-2\mu} - \frac{c}{\mu} x^{1-\mu} + m^2 x
\]

\[
= \frac{x^{1-2\mu}}{\mu} \left( e_* - cx^{\mu} + m^2 \mu x^{2\mu} \right) \geq 0,
\]

because \( e_* - cx^{\mu} + m^2 \mu x^{2\mu} > 0 \) (it is a parabola in the variable \( x^{\mu} \) which has by (14) negative discriminant \( c^2 - 4m^2\mu e_* < 0 \), so it has constant sign, and positive independent term \( e_* > 0 \)).

The following example shows that Theorem 3.2 is in fact more general than Corollary 3.6. Therefore, even in the case of the model problem (2), Theorem 3.2 is a true extension of [4, Theorem 1.8]. Moreover, notice that we also obtain information about the localization of the solution which was not provided in [4]. Some computations here were made with MAPLE.

**Example 3.7.** Consider the problem (2), that is,

\[
\begin{cases}
  x''(t) + ax'(t) = \frac{e(t)}{\mu} x^{1-2\mu} - \frac{c}{\mu} x^{1-\mu}, & t \in [0, T], \\
  x(0) = x(T), \quad x'(0) = x'(T),
\end{cases}
\]

with the parameter values \( a = 1.6, e(t) \equiv 1.54, \mu = 0.01, c = 1.49 \) and \( T = 1 \).

Therefore we have

\[
b = 99 \quad (\text{recall that } \mu := \frac{1}{b+1}), \quad r(t) := \frac{e(t)}{\mu} \equiv 154, \quad s(t) := \frac{c}{\mu} \equiv 149,
\]

\[
\alpha := 1 - 2\mu = 0.98 \quad \text{and} \quad \beta := 1 - \mu = 0.99.
\]
Thus, condition (13) in Corollary 3.6 is not satisfied because
\[
\frac{(b + 1) e_c^2}{4e_s} \approx 36.0406 > 10.5096 \approx \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}.
\]
Notice that neither condition (12) in Corollary 3.5 is satisfied since
\[
s^* = 149 > 10.5096 \approx \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}.
\]
However, the condition (H0) in Theorem 3.2 is clearly satisfied and if we define \(m := 0.7\), \(R_1 := 27\) and \(R_2 := 29\) then (H1) is also fulfilled (notice that \(c_m \approx 0.9414\) as it is given by (20)).

Thus, Theorem 3.2 ensures the existence of a solution \(x(t)\) of this problem such that
\[
25.4189 \approx c_m R_1 \leq x(t) \leq R_2 = 29.
\]
Observe that \(f_m\) is not positive for all \(x \geq 0\).
Remark 3.8. From a careful reading of the proof of [4, Theorem 1.8] it follows that the existence of a positive periodic solution to problem (2) is ensured if condition (13) in Corollary 3.6 is replaced by the more general one:

(H) There is $\delta_0 > 0$ such that $f(t, x) < 0$ for all $t \in [0, T]$ and all $x \in (0, \delta_0)$ and $\lambda^* x - f(t, x) > 0$ for all $t \in [0, T]$ and all $x > 0$, where $\lambda^* = \left(\frac{\pi}{T}\right)^2 + \frac{a^2}{4}$. 

However, condition (H) is neither satisfied for the problem presented in Example 3.7 because in that case

$$\lambda^* x - f(t, x) \approx 10.5096 x - 149 x^{0.99} + 154 x^{0.98},$$

which is a sign-changing nonlinearity on the positive real line. Indeed, dividing by $x^{0.98}$ and making the change $y = x^{0.01}$, it is clear that $\lambda^* x - f(t, x)$ is negative on $(r_1, r_2)$ and positive on $(0, r_1) \cup (r_2, \infty)$, being $r_1 \approx 103620.4135$ and $r_2 \approx 3.7845 \times 10^{11}$.

4. **Appendix: The Green’s function**

4.1. The case $0 < m < \frac{a}{2}$. The Green’s function related to problem (5) is given by

$$G_m(t, s) = \frac{1}{\lambda_2 - \lambda_1} \begin{cases} \frac{e^{\lambda_2(t-s)}}{1 - e^{\lambda_2 T}} - \frac{e^{\lambda_1(t-s)}}{1 - e^{\lambda_1 T}}, & 0 \leq s \leq t \leq T, \\ \frac{e^{\lambda_2(t-s+T)}}{1 - e^{\lambda_2 T}} - \frac{e^{\lambda_1(t-s+T)}}{1 - e^{\lambda_1 T}}, & 0 \leq t \leq s \leq T, \end{cases}$$

where $\lambda_1$ and $\lambda_2$ are the characteristic roots of the homogeneous equation

$$x''(t) + ax'(t) + m^2 x(t) = 0,$$

that is,

$$\lambda_1 = -a - \sqrt{a^2 - 4m^2}, \quad \lambda_2 = -a + \sqrt{a^2 - 4m^2}.$$ 

Note that $\lambda_1 < \lambda_2 < 0$.

**Remark 4.1.** $G_m(s, s)$ is constant for $s \in [0, T]$,

$$G_m(s, s) = \frac{1}{\lambda_2 - \lambda_1} \left( \frac{1}{1 - e^{\lambda_2 T}} - \frac{1}{1 - e^{\lambda_1 T}} \right),$$

and

$$\int_0^T G_m(t, s) ds = \frac{1}{\lambda_1 \lambda_2} = \frac{1}{m^2}.$$ 

The Green’s function (15) satisfies

$$G_m(t, s) \geq G_m(s, s) > 0,$$

and

$$G_m(s, s) \geq c_m G_m(t, s).$$
for all \((t, s) \in [0, T] \times [0, T]\), where

\[
\begin{align*}
(19) & \quad c_m = \frac{G_m(s, s)}{\max\{G_m(t, s) : t, s \in [0, T]\}}, \\
(20) & \quad c_m = -\frac{\lambda_2}{\lambda_2 - \lambda_1} \frac{e^{\lambda_2 T} - e^{\lambda_1 T}}{(1 - e^{\lambda_2 T})(1 - e^{\lambda_1 T})}\frac{\lambda_1}{\lambda_2 - \lambda_1}.
\end{align*}
\]

4.2. The case \(m = \frac{a}{2}\). The Green’s function related to the problem (5) is given by

\[
(21) \quad G_m(t, s) = \frac{1}{e^{mT} - 1} \begin{cases}
  e^{m(t-s)} \left[ \frac{T e^{mT}}{e^{mT} - 1} + s - t \right], & 0 \leq s \leq t \leq T, \\
  e^{m(T-s+t)} \left[ \frac{T}{e^{mT} - 1} + s - t \right], & 0 \leq t \leq s \leq T.
\end{cases}
\]

Then we have

\[
\int_0^T G_m(t, s) ds = \frac{1}{m^2},
\]

\[
G_m(t, s) \geq G_m(s, s) > 0,
\]

and

\[
G_m(s, s) \geq c_m G_m(t, s),
\]

for all \((t, s) \in [0, T] \times [0, T]\), where

\[
(22) \quad c_m = \frac{G_m(s, s)}{\max\{G_m(t, s) : t, s \in [0, T]\}},
\]

that is,

\[
c_m = \kappa e^{1-\kappa},
\]

with

\[
\kappa = \frac{mT}{e^{mT} - 1}.
\]

4.3. The case \(m > \frac{a}{2}\) and \(m^2 < \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2\). We use the following notation:

\[
\gamma = -\frac{a}{2}, \quad \delta = \sqrt{4m^2 - a^2}, \quad D = 1 - 2e^{-\gamma T} \cos(\delta T) + e^{2\gamma T}.
\]

The Green’s function related to the problem (5) is given by

\[
G_m(t, s) = \frac{1}{\delta D} \begin{cases}
  e^{\gamma(T+t-s)} \sin(\delta(T + s - t)) + e^{\gamma(t-s)} \sin(\delta(t - s)), & 0 \leq s \leq t \leq T, \\
  e^{\gamma(T+t-s)} \sin(\delta(T + t - s)) + e^{\gamma(2T+t-s)} \sin(\delta(s - t)), & 0 \leq t \leq s \leq T.
\end{cases}
\]
Then we have
\[ \int_0^T G_m(t, s) \, ds = \frac{1}{m^2}, \]
\[ G_m(t, s) \geq G_m(s, s) > 0, \]
and
\[ G_m(s, s) \geq c_m G_m(t, s), \]
for all \((t, s) \in [0, T] \times [0, T]\), where
\[ c_m = \frac{G_m(s, s)}{\max\{G_m(t, s) : t, s \in [0, T]\}} = \frac{1}{\delta D} e^{\gamma T} \sin(\delta T) \]
\[ \max\{G_m(t, s) : t, s \in [0, T]\}. \]

If \(a = 0\) then \(\gamma = 0, \delta = m, D = 2(1 - \cos mT)\) and it is known that \(\max\{G_m(t, s) : t, s \in [0, T]\} = \frac{1}{2m \sin \frac{mT}{2}}\), see [16]. Therefore we have \(c_m = \cos \frac{mT}{2}\).

On the other hand, if \(a > 0\) it is difficult to find exactly the maximum of \(G_m(t, s)\), but we are able to get an upper bound in the following way: since \(0 < \delta T < \pi\) we have for \(0 \leq s \leq t \leq T\) (the case \(0 \leq t \leq s \leq T\) is analogous)
\[ e^{\gamma(T+t-s)} \sin(\delta(T + s - t)) + e^{\gamma(t-s)} \sin(\delta(t - s)) \leq \sin(\delta(T + s - t)) + \sin(\delta(t - s)) \]
\[ = 2 \sin \frac{\delta T}{2} \cos \left(\delta \left(\frac{T}{2} - (t - s)\right)\right) \leq 2 \sin \frac{\delta T}{2}. \]
So, \(\max\{G_m(t, s) : t, s \in [0, T]\} \leq \frac{2}{\delta D} \sin \frac{\delta T}{2}\) which yields \(c_m \geq e^{\gamma T} \cos \frac{\delta T}{2}\).

**Remark 4.2.** If \(m^2 = \left(\frac{\pi}{T}\right)^2 + \left(\frac{a}{2}\right)^2\) then \(G_m(s, s) = 0\).

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José Ángel Cid, Departamento de Matemáticas, Universidade de Vigo, 32004, Pabellón 3, Campus de Ourense, Spain

E-mail address: angelcid@uvigo.es

Gennaro Infante, Dipartimento di Matematica e Informatica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy

E-mail address: gennaro.infante@unical.it

Milan Tvrdý, Mathematical Institute, Academy of Sciences of the Czech Republic, CZ 115 67 Praha 1, Žitná 25, Czech Republic

E-mail address: tvrdy@math.cas.cz

Mirosława Zima, Department of Differential Equations and Statistics, Faculty of Mathematics and Natural Sciences, University of Rzeszów, Pogonia 1, 35-959 Rzeszów, Poland.

E-mail address: mzima@ur.edu.pl