Tracking the Variety of Interleavings

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Abstract
In topological data analysis persistence modules are used to distinguish the legitimate topological features of a finite data set from noise. Interleavings between persistence modules feature prominently in the analysis. One can show that for $\epsilon$ positive, the collection of $\epsilon$-interleavings between two persistence modules $M$ and $N$ has the structure of an affine variety, Thus, the smallest value of $\epsilon$ corresponding to a nonempty variety is the interleaving distance. With this in mind, it is natural to wonder how this variety changes with the value of $\epsilon$, and what information about $M$ and $N$ can be seen from just the knowledge of their varieties.

In this paper, we focus on the special case where $M$ and $N$ are interval modules. In this situation we classify all possible progressions of varieties, and determine what information about $M$ and $N$ is present in the progression.

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2 Introduction
In [3] it was shown that the collection of $\epsilon$-interleavings between two generalized persistence modules for a poset has the structure of an affine variety. In this paper, we consider the class of (truncated) one-dimensional persistence modules which arise from applying homology to a filtration of a finite data set. Thus, we confine our attention to persistence modules which are direct sums of (finite) interval modules of the form $[\alpha, \beta)$, for $\alpha < \beta$.

We will investigate what happens when one studies the full collection of interleavings between two persistence modules, as opposed to the interleaving distance between them. If $M, N$ are persistence modules and $\epsilon > 0$, the collection of $\epsilon$-interleavings between $M$ and $N$ has the structure of a variety $V^{\epsilon}_{M,N}$. The interleaving distance $D$ between $M$ and $N$, is the smallest value of $\epsilon$ associated to a nonempty variety (see Figure 1).

![Figure 1: The progression of varieties $V^{\epsilon}_{M,N}$ and the interleaving distance $D$](image-url)

0 $\epsilon_1$ $\epsilon_2$ $\epsilon_3$ ...

$\emptyset$ $V^{D}_{M,N} = V^1$ $V^{\epsilon_2}_{M,N} = V^2$ $V^{\epsilon_3}_{M,N} = V^3$
In this setting many questions come to mind. Which varieties can appear in this way? As $\epsilon$ varies, how do they change? Which ordered lists (progressions) of varieties are possible? Does the progression itself detect something about the interleaving distance between the persistence modules that give rise to it? In our main result, Theorem 1, we answer all these questions in the special case that both $M$ and $N$ are interval modules.

It follows from the completion of the isometry theorem (see [1]), that every $\epsilon$-interleaving comes from an $\epsilon$-matching. Moreover, from [4], [2], it’s clear that the interleaving distance $D$ between the interval modules $[a, b)$ and $[c, d)$ is given by the formula

$$D = \min\{m_1, m_2\}, \text{ where } m_1 = \max\{|a - c|, |b - d|\} \text{ and } m_2 = \max\{\frac{b-a}{2}, \frac{d-c}{2}\} \quad (1)$$

In Proposition 15 we give a physical interpretation for these numbers, showing for example that $m_1$ corresponds to the place where the last homomorphism between the two intervals in born. The numbers $m_1$ and $m_2$ are then used in Theorem 1 in our classification of the progression $V^\epsilon_{M,N}$ when $M$ and $N$ are both intervals.

This paper is organized as follows. In Section 3 we remind the reader of some preliminaries. In Section 4 we define the variety of interleavings between two persistence modules, and then in Section 5 we give some examples of progressions of varieties. Lastly, in Section 6 we show that our examples in Section 5 constitute an exhaustive list, and we prove our main results.

3 Preliminaries

In this Section, we give a brief review of one-dimensional persistence modules. For a more extensive introduction, see [5].

**Definition 1.** A persistence module $M$ is an assignment

of vector spaces $\{M(x)\}$, for $x \in [0, \infty)$ and linear maps $\{M(x \leq y)\}$, for $x \leq y$ satisfying;

1. $M(x \leq y) : M(x) \to M(y)$ and
2. $M(x \leq z) = M(y \leq z) \circ M(x \leq y)$ for all $x, y, z$ with $x \leq y \leq z$.

If $M$ and $N$ are two persistence modules, a morphism from $F : M \to N$ is a collection of linear maps $\{F(x) : M(x) \to N(x)\}$ indexed by $x \in [0, \infty)$ with the property that

$$F(y) \circ M(x \leq y) = N(x \leq y) \circ F(x) \text{ for } x \leq y. \quad (2)$$

Thus, a morphism from $M$ to $N$ is a family of linear maps that gives rise to the commutative diagram in Figure 2. One can view persistence modules as functors on a thin category, in which case morphisms correspond to natural transformations. This categorical perspective will not be necessary for our purposes.

The monoid $(\mathbb{R}_{\geq 0}, +)$ acts on the category of persistence modules on the by the formulas;

$$(M \cdot \epsilon)(x) = M(x + \epsilon), \text{ and } (F \cdot \epsilon)(x) = F(x + \epsilon) \quad (3)$$

In the latter identity, if $F : M \to N$, then $F \cdot \epsilon : M \cdot \epsilon \to N \cdot \epsilon$. Thus, the action shifts to the left by $\epsilon$. When there is no ambiguity, we suppress the dot and simply write $M\epsilon$ and $F\epsilon$ respectively.
If $\alpha < \beta$, the interval module $M_{[\alpha, \beta]}$ is the persistence module defined by the equations;

$$M_{[\alpha, \beta]}(x) = \begin{cases} \mathbb{R}, & x \in [\alpha, \beta) \\ 0, & x \notin [\alpha, \beta) \end{cases}$$

$$M_{[\alpha, \beta]}(x \leq y) = \begin{cases} Id, & x, y \in [\alpha, \beta) \\ 0, & \text{otherwise} \end{cases}$$

Thus, $M_{[\alpha, \beta]}$ is the persistence module whose support is given by the interval $[\alpha, \beta)$. Whenever there is no ambiguity, we write $[\alpha, \beta)$ for $M_{[\alpha, \beta]}$. If $M$ and $N$ are two persistence modules, then $H(M,N)$ is the collection of morphisms from $M$ to $N$. If $M, N$ are interval modules, $H(M,N)$ is either one-dimensional or identically zero.

**Lemma 2.** Let $\alpha < \beta$ and $\gamma < \delta$. Then $H([\alpha, \beta), [\gamma, \delta)) \neq \{0\} \iff \gamma \leq \alpha < \delta \leq \beta$.

This result is well-known, and follows immediately from the requirement that a quotient of $[\alpha, \beta)$ be isomorphic to a submodule of $[\gamma, \delta)$. One immediately checks that if indeed $\gamma \leq \alpha < \delta \leq \beta$, every morphism $G$ from $[\alpha, \beta)$ to $[\gamma, \delta)$ is given by

$$G(x) = \begin{cases} \lambda \cdot Id, & x \in [\gamma, \beta) \\ 0, & \text{otherwise} \end{cases}$$

for a unique choice of real number $\lambda$. Thus, when nonzero, the collection of morphisms between interval modules are parametrized by real numbers. In a slight abuse of notation, we sometimes write

$$H([\alpha, \beta), [\gamma, \delta)) = \lambda 1_{[\gamma, \delta)}$$

where we think of ”1” as the characteristic function on the specified interval.

**Lemma 3.** Let $\alpha < \beta, \gamma < \delta$. Then,

$$\{x | H([\alpha, \beta), [\gamma, \delta)x) \neq \{0\} \} = \begin{cases} [r, s) & \text{for some } r < s \\ \emptyset \end{cases}$$

Moreover, the latter occurs exactly when $\delta \leq \alpha$.

That is to say, when $M$ and $N$ are interval modules, $\{x | H(M, Nx) \neq \{0\} \}$ is an interval, which is degenerate only if $M$ and $N$ are disjoint, with $N$ below $M$. Moreover, $H([\alpha, \beta), [\alpha, \beta)x)$ is not identically $\{0\}$ on the interval $[0, \beta-\alpha/2)$. The number $W([\alpha, \beta)) = \beta-\alpha/2$ is the width function used in the bottleneck (or Wasserstein) distance on persistence modules. The set of values for $x$ where $H(M,Nx)$ is not zero will occur in what follows with enough frequency to justify some notation.

**Definition 4.** Let $M, N$ be two persistence modules. Then let

$$S_{M,N} = \{x \geq 0 | H(M,Nx) \neq \{0\} \}.$$ 

Thus, Lemma 3 says that if $M$ and $N$ are intervals modules, then $S_{M,N}$ is either an interval or empty.
The Variety of Interleavings

Definition 5. Let $M, N$ be two persistence modules, and say $\epsilon > 0$. An $\epsilon$-interleaving between $M$ and $N$ is a pair of morphism $\Phi : M \to N\epsilon$ and $\Psi : N \to M\epsilon$ satisfying

$$\Psi\epsilon \circ \Phi = \Pi^M_{M_{2\epsilon}}, \Phi\epsilon \circ \Psi = \Pi^N_{N_{2\epsilon}}$$

where for a persistence module $P$, and $\tau > 0$, the morphsim $\Pi^P_{\tau} : P \to P\tau$ is given by

$$\Pi^P_{\tau}(x) = P(x \leq x + \tau).$$

From the definition, it is clear that whether the morphisms $\Phi, \Psi$ constitute an $\epsilon$-interleaving depends only on the triple $(\Phi, \Psi, \epsilon)$ in the sense that the remainder of the interleaving diagram is forced. That is, given only $\Phi, \Psi$ and $\epsilon$ one fills in the remainder of the diagram and simply checks whether the conditions (5) hold.

![Image of an $\epsilon$ interleaving between $M$ and $N$]

Figure 3: An $\epsilon$ interleaving between $M$ and $N$

If $M = \bigoplus_i M_i$ and $N = \bigoplus_j N_j$ using the Krull-Schmidt property, we may decompose $\Phi$ and $\Psi$.

So, $\Phi = (\phi^i_j)$, $\Psi = (\psi^i_j)$, where, $\phi^i_j : M_j \to N_i\epsilon$ and $\psi^i_j : N_i \to M_j\epsilon$.

If the $M_i$ and $N_j$ are interval modules, by Lemma 2 any of $\phi^i_j, \psi^i_j$ which could be nonzero are given by a real parameter. Moreover, the value of the parameter is preserved under the action of $\mathbb{R}_{\geq 0}$ when one shifts by $\epsilon$. These will be the variables used for the coordinate ring for our variety.

We now define the variety. Let $K$ and $L$ be given by $K \in M_{n,m}(\mathbb{R})$, $K = (k^b_a)$ and $L \in M_{m,n}(\mathbb{R})$, $L = (\ell^b_a)$.

$K$ and $L$ will correspond to the decomposition of the morphisms $\Phi$ and $\Psi$ respectively. Thus, we identify the $k^b_a$ with $\phi^b_a$, and $\ell^b_a$ with $\psi^b_a$.

Clearly the matrix for $\Pi^M_{M_{2\epsilon}}$ is diagonal with the morphism $\Pi^M_{M_{2\epsilon}}$ constituting the $(i, i)$-entry. Moreover, it’s easy to see that if $M_i = [\alpha_i, \beta_i)$,

$$\Pi^M_{M_{2\epsilon}} = \begin{cases} 1_{[\alpha_i, \beta_i - 2\epsilon)}, & \epsilon < \frac{\beta_i - \alpha_i}{2} \\ 0, & \text{otherwise} \end{cases}$$

Thus, we have $\Pi^M_{M_{2\epsilon}} \neq 0 \iff H(M_i, M_{i2\epsilon}) \neq \{0\} \iff \epsilon < \frac{\beta_i - \alpha_i}{2} \iff 2\epsilon \in S_{M_i, M_i}$, with the analogous statement holding for the diagonal in $\Pi^N_{N_{2\epsilon}}$. Therefore, substituting into (5), our commutativity conditions become

$$L \cdot K = \Pi^M_{M_{2\epsilon}}, \, K \cdot L = \Pi^N_{N_{2\epsilon}},$$

(6)
subject to two qualifications. First, many of the entries of \(K\) and \(L\) must be zero simply because there are no such nonzero morphisms. Secondly, while there are \(m^2 + n^2\) equations involving the variables in \([6]\), only those corresponding to entries which could vary must be satisfied. For example, the equations
\[
\sum_b^n k_b^b \ell_1^b = 0 \text{ or } \sum_b^n \ell_3^b k_b^b = 1
\]
are only constraints on those variables which appear only when the sets \(H(N_1, N_2; 2\epsilon)\) and \(H(M_3, M_2; 2\epsilon)\) are not \(\{0\}\) respectively. With this in mind, let
\[
\mathcal{S} = \{X(a,b) | 2\epsilon \in S_{M_a,M_n}\} \cup \{Y(a,b) | 2\epsilon \in S_{N_b,N_a}\}; \mathcal{T} = \{k_b^a | \epsilon \notin S_{M_a,N_b}\} \cup \{\ell_b^a | \epsilon \notin S_{N_a,M_b}\},
\]
where \(X = L \cdot K - \Pi_{M_2}, Y = K \cdot L - \Pi_{N_2}^M\), and the subscript corresponds to the \((a,b)\)-entry of the matrix. Thus, \(\mathcal{S}\) is the collection of matrix conditions that the variables must satisfy, and \(\mathcal{T}\) corresponds to those variables which should be regarded as missing from the matrices \(K, L\), since they can only be zero.

Finally, we are ready to define the variety at \(\epsilon\).

**Definition 6.** Using the above notation, the variety of \(\epsilon\)-interleavings between \(M\) and \(N\), \(V_{M,N}\) is the affine variety whose coordinate ring given by
\[
A^\epsilon_{M,N} = \mathbb{R}[\{k_b^a \ell_1^b | 1 \leq a \leq m, 1 \leq b \leq n\}] / (\mathcal{S} \cup \mathcal{T}).
\]

We will now illustrate with some examples.

**Example 7.** Let \(M = M_1 \oplus M_2, N = N_1 \oplus N_2\), where \(M_1 = N_1 = [1, 4], M_2 = [1.2, 3.9]\) and \(N_2 = [.9, 4.1]\).

By the isometry theorem, the interleaving distance is \(2\) corresponding to the \(2\)-matching
\[
M_1 \leftrightarrow N_2 \text{ and } M_2 \leftrightarrow N_1.
\]

We will compute the variety at the interleaving distance \(\epsilon = .2\). First, by inspection
\[
S_{M_1,N_1} = [0, 3), S_{M_1,N_2} = [.1, 3.1), S_{M_2,N_1} = [.1, 2.8) \text{ and } S_{M_2,N_2} = [.1, 2.9). \quad (7)
\]
Also,
\[
S_{N_1,M_1} = [0, 3), S_{N_1,M_2} = [.2, 2.9), S_{N_2,M_1} = [.1, 3.1) \text{ and } S_{N_2,M_2} = [.3, 3). \quad (8)
\]
Since only \(S_{N_2,M_2}\) does not contain \(\epsilon = .2\) as an element,
\[
K = \begin{pmatrix} k_1^1 & k_1^2 \\ k_2^1 & k_2^2 \end{pmatrix}, \quad L = \begin{pmatrix} \ell_1^1 & \ell_1^2 \\ \ell_2^1 & 0 \end{pmatrix} \text{ and } \mathcal{T} = \{\ell_2^2\}.
\]
Moreover,
\[
S_{M_1,M_1} = [0, 3), S_{M_1,M_2} = [.2, 2.9), S_{M_2,M_1} = [.1, 2.8), S_{M_2,M_2} = [0, 2.7), \quad (9)
\]
\[
S_{N_1,N_1} = [0, 3), S_{N_1,N_2} = [.1, 3.1), S_{N_2,N_1} = [.1, 3.1) \text{ and } S_{N_2,N_2} = [0, 3.2). \quad (10)
\]
Thus, since \(A = 2\epsilon\) is in all these intervals, all matrix equations must be satisfied, and
\[
\Pi_{M}^{M_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Pi_{N}^{N_2}.
\]
One can check that the variety \(V_{M,N}^2\) therefore corresponds to the polynomial equations
\[
\ell_2^2, k_1^1, \ell_1^2 k_2^1 - 1, \ell_2^1 k_1^2 - 1, \ell_1^1 k_2^2 + \ell_2^2 k_1^1 + \ell_2^2 k_2^2
\]
Example 8. We now consider the same persistence modules as in Example 7, now with a different value of the parameter $\epsilon$. This time, by (3) and (4), $T = \emptyset$, since $\epsilon = 4$ is in all the intervals. Also, since $2\epsilon$ is in all the intervals in (5), (6), all matrix equations must be satisfied and

$$K = \begin{pmatrix} k_1^1 & k_2^1 \\ k_1^2 & k_2^2 \end{pmatrix}, \quad L = \begin{pmatrix} \ell_1^1 & \ell_2^1 \\ \ell_1^2 & \ell_2^2 \end{pmatrix}, \quad \Pi_M^{M,4} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Pi_N^{N,4}.$$ 

Thus, our variety of interleavings $V_{M,N}^5$ is now given by the equations

$$K \cdot L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = L \cdot K.$$

Thus, we see that the $A$-interleavings are parametrized by the set

$$\{(Z, Z^{-1}) | Z \in \text{Gl}_2(\mathbb{R})\} \subseteq \mathbb{R}^8.$$ 

Note that all $2$-interleavings remain $A$-interleavings, though from $2$ to $4$, the variety grows in dimension.

Example 9. We continue, considering the same pair of persistence modules. Now, let $\epsilon = 1.55$. As was Example 8, since $\epsilon = 1.55$ is in all the intervals in (3) and (4), $T = \emptyset$. Now, however, from (5), (6), we see that $2\epsilon = 3.1$ is only in $S_{N^2,N^2}$. Thus, we see that $V_{M,N}^{1,55}$ is given by

$$k_2^1 \ell_2^2 + k_2^2 \ell_2^2 - 1$$

where all other variables are free.

Example 10. One last example. Now let $\epsilon = 3$. Here the variety $V_{M,N}^3$ is given by equations

$$k_1^1, k_2^1, k_2^2, \ell_1^1, \ell_2^1, \ell_2^2$$

where $k_2^1, \ell_2^1$ are free. Thus, $V_{M,N}^3$ is a plane in 8-dimensional affine space.

5 Some Progressions

As we see in Examples 7-10, the interleavings between a fixed pair of persistence modules gives rise to a progression of varieties indexed by the parameter $\epsilon$. It’s clear that for a fixed $M$ and $N$ only finitely many different varieties actually appear in the range of the assignment

$$\epsilon \rightarrow V^\epsilon_{M,N}.$$ 

In this section, we provide examples showing how the variety of $\epsilon$-interleavings associated to two interval modules changes with the value of the parameter $\epsilon$. When $M$ and $N$ are clear from the context, we’ll write $V^i$ for the $i$th nonempty variety in the progression, suppressing $\epsilon$. Note that since $M$ and $N$ are intervals, the matrices $K, L, \Pi_M^{M,2\epsilon}$ and $\Pi_N^{N,2\epsilon}$ are all scalar matrices. Now, we’ll provide some examples of progressions.

Example 11. $V^1$ is the origin, $V^2$ is a coordinate axis, and $V^3$ is the origin.
That is, the complete progression is origin, axis, origin. First, let $M = [6, 8]$ and $N = [1, 2]$. Note that $S_{M,N} = \emptyset$, and $S_{N,M} = [6, 7], S_{M,M} = [0, 2], S_{N,N} = [0, 1]$.

Recall that the last two intervals tell us that $H(M, M 2\epsilon) = \begin{cases} \mathbb{R} & \epsilon \in [0, 1) \\ \{0\} & \epsilon \in [1, \infty) \end{cases}$ and $H(N, N 2\epsilon) = \begin{cases} \mathbb{R} & \epsilon \in [0, \frac{1}{2}) \\ \{0\} & \epsilon \in [\frac{1}{2}, \infty) \end{cases}$.

Now, for $0 \leq \epsilon < 1$, $K = (0), L = (0), \Pi_{M}^{M 2\epsilon} = (1), T = \{k, \ell\}$. Since $2\epsilon \in S_{M,M}$, our matrix condition must be satisfied. Thus, the variety is given by $k, \ell, k\ell - 1$.

so we have the empty variety. Note that this is exactly because $\epsilon$ is less than the interleaving distance of 1.

Next, for $1 \leq \epsilon < 6$

$K = (0), L = (0), \Pi_{M}^{M 2\epsilon} = (0), \Pi_{N}^{N 2\epsilon} = (0), T = \{k, \ell\}$, thus no matrix conditions need to be satisfied, and $V^1$ is $k, \ell$

That is, $V^1$ is the origin in $\mathbb{R}^2$.

For $6 \leq \epsilon \leq 7$

$K = (0), L = (\ell), \Pi_{M}^{M 2\epsilon} = (0), \Pi_{N}^{N 2\epsilon} = (0), T = \{k\}$, so no matrix conditions need to be satisfied, and $V^2$ is given by $k$

so $k = 0$ and $\ell$ is free. Thus, $V^2$ is the y-axis.

Lastly, for $\epsilon \geq 7$, once again

$K = (0), L = (0), \Pi_{M}^{M 2\epsilon} = (0), \Pi_{N}^{N 2\epsilon} = (0), T = \{k, \ell\}$. 

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Thus, $V^3$ is the origin again. Of course, it’s clear that by switching the roles of $M, N$ we may obtain the progression origin, $x$-axis, origin.

The next Examples are provided without justification. The reader can easily check that each progression matches the given pair of interval modules.

**Example 12.** $V^1$ is a coordinate axis, $V^2$ is the origin.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{example12.png}
\end{figure}

Let $M = [1,3), N = [0,2)$ and $M = [0,2), N = [1,3)$ respectively.

**Example 13.** $V^1$ is a hyperbola, $V^2$ is a plane, $V^3$ is a coordinate axis and $V^4$ is the origin.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{example13.png}
\end{figure}

Let $M = [1,2.1), N = [8,2.2)$ and $M = [8,2.2), N = [1,2.1)$ respectively.

**Example 14.** $V^1$ is a hyperbola, $V^2$ is the plane and $V^3$ is the origin.

Let $M = [.9,2.1), N = [1,2)$
In the next Section we will show that Examples 11, 12, 13 and 14 constitute a complete list of all possible progressions between two interval modules. We will also connect the first variety $V^1$ to $m_1$ and $m_2$ (see (1)), the two terms whose minimum is the interleaving distance. We now prove a useful Proposition which provides a physical interpretation to $m_1$.

Proposition 15. Let $M = [a, b)$ and $N = [c, d)$ and set

$$
\sigma = \begin{cases} 
\min(S_{M,N}), & \text{if } S_{M,N} \neq \emptyset \\
0, & \text{if } S_{M,N} = \emptyset
\end{cases} \quad \tau = \begin{cases} 
\min(S_{N,M}), & \text{if } S_{N,M} \neq \emptyset \\
0, & \text{if } S_{N,M} = \emptyset
\end{cases}
$$

and similarly

$$
\sigma' = \begin{cases} 
\sup(S_{M,N}), & \text{if } S_{M,N} \neq \emptyset \\
0, & \text{if } S_{M,N} = \emptyset
\end{cases} \quad \tau' = \begin{cases} 
\sup(S_{N,M}), & \text{if } S_{N,M} \neq \emptyset \\
0, & \text{if } S_{N,M} = \emptyset
\end{cases}
$$

Thus, $Hom(N, Mx) \neq 0 \iff \sigma \leq x < \sigma'$ and $S_{N,M} \neq \emptyset$ and similarly $Hom(M, Nx) \neq 0 \iff \tau \leq x < \tau'$ and $S_{M,N} \neq \emptyset$.

Then,

i. $\max\{|a-c|, |b-d|\} = \max\{\sigma, \tau\}$. So $\max\{|a-c|, |b-d|\}$ corresponds to the number where the last homomorphism between $M$ and $N$ is born.

ii. $\max\{|a-d|, |b-c|\} = \max\{\sigma', \tau'\}$. So $\max\{|a-d|, |b-c|\}$ corresponds to the number where the last homomorphism between $M$ and $N$ dies.

Proof. We proceed in three main cases, conditioning on the way the supports of $M$ and $N$ compare. In the proof, we use Lemmas 2, 3 repeatedly without explicit reference.

Case 1: The supports are disjoint. That is, $M \cap N = \emptyset$.

Without loss of generality, suppose that $a < b \leq c < d$. Then, $S_{N,M} = \emptyset$, so $\tau, \tau'$ are both identically zero. Therefore, $\max\{\sigma, \tau\} = \sigma$ and $\max\{\sigma', \tau'\} = \sigma'$. Thus, we will show that

$$
\sigma = \max\{|a-c|, |b-d|\} = \max\{c-a, d-b\} \quad \text{and} \quad \sigma' = \max\{|a-d|, |b-c|\} = d-a.
$$

First, say $c-a \geq d-b$. If $x = c-a$, then $x \in S_{M,N}$, since $H(M, Nx) = H([a, b), [a, d-c+a) \neq \emptyset$, as

$$
a \leq a < d-c+a \leq b.
$$

Thus, $\sigma \leq c-a$. But if $\sigma < c-a$, then for some $x < c-a$, $H([a, b), [c-x, d-x)) \neq \emptyset$. If this were the case, then $c-x \leq a$, a contradiction. Thus we have shown that $\sigma = c-a$. 


Now set \( x = d - a \). Then \( H([a,b],[c-x,d-x]) = H([a,b],[c-d+a,d-d+a]) = \{0\} \), since \( a \not\in a \). Thus, \( \sigma' \leq d - a \). However, if \( \sigma' < d - a \), then there exists \( x \in (c-a,d-a) \) with \( H([a,b],[c-x,d-x]) = \{0\} \). However, for any such \( x, b + x \geq b + c - a \geq b + d - b = d \). Therefore,

\[
c - x \leq a < d - x \leq b
\]

Thus \( i.i. \) are proven if \( c - a \geq d - b \).

Now, say \( d - b > c - a \), and let \( x = d - b \). Then \( x \in S_{M,N} \) since

\[
c - x \leq c - c + a = a < b = d - x \leq b.
\]

Thus \( \sigma \leq d - b \). But if \( \sigma < d - b \), then there exists \( x < d - b \) with \( x \in S_{M,N} \) This is a contradiction since \( b < d - x \). Therefore \( \sigma = d - b \).

Moreover, \( H([a,b],[c-d+a,a]) = \{0\} \), since \( a \not\in a \) so \( \sigma' < d - a \). However for \( x \) with \( c - a \leq d - b < x < d - a \), \( H([a,b],[c-x,d-x]) = \{0\} \), since \( c - x \leq a < d - x \leq b \). Thus, \( \sigma' = d - a \). Our result is therefore established in Case 1.

\textbf{Case 2: } The supports intersect, but neither one contains the other. Explicitly, \( M \cap N \neq \emptyset, M \nsubseteq N \) and \( N \nsubseteq M \).

Without loss of generality, suppose \( a \leq c < b \leq d \). In this situation, \( 0 \in S_{N,M} \), so \( \tau = 0 \) and \( S_{M,N} \neq \emptyset \). Thus, to prove \( i. \) we must show that \( \max\{c - a, d - b\} = \sigma \). First, suppose that \( c - a \geq d - b \). Then, \( H([a,b],[c - c + a,d - c + a]) = H([a,b],[a,a + d - c]) \neq \{0\} \) since

\[
a \leq a < a + d - c \leq b.
\]

This says that \( \sigma \leq c - a \). On the other hand, if \( \sigma < c - a \), then for some \( x < c - a, x \in S_{M,N} \). If this were the case, then

\[
c - x \leq a < d - x \leq b,
\]

a contradiction, since \( c - x \geq a \). Thus, \( i. \) is established, when \( c - a \geq d - b \).

Similarly, say \( d - b \geq c - a \). Then, \( H([a,b],[c-d+b,d-d+b]) = H([a,b],[b+c-d,b]) \neq \{0\} \), since

\[
b + c - d \leq a < b \leq b.
\]

Thus we have that \( \sigma \leq d - b \). If \( \sigma < d - b \), then for some \( x < d - b, H([a,b],[c-x,d-x]) \neq \{0\} \). If this were true, then we must have

\[
c - x \leq a < d - x \leq b,
\]

a contradiction, since \( x \geq d - b \). Thus, \( \sigma = \max\{c - a, d - b\} \), so \( i. \) holds.

To establish \( ii. \), we'll show that \( \tau' = b - c \) and \( \sigma' = d - a \). First, \( H([c,d],[a-b+c,b-b+c]) = H([c,d],[a-b+c,c]) = \{0\} \), since \( c \not\in c \), thus \( \tau' \leq b - c \). But if \( \tau' < b - c \), then for some \( x < b - c \) we have \( H([c,d],[a-x,b-x]) = \{0\} \). However, for any such \( x \),

\[
a - x < c < b - x \leq d,
\]

a contradiction, since \( x \in S_{N,M} \). Thus \( \tau' = b - c \). Lastly \( H([a,b],[c-d+a,d-d+a]) = H([a,b],[c-d+a,a]) = \{0\} \), since \( a \not\in a \). On the other hand, if \( \max\{c - a, d - b\} \leq x < d - a \), \( H([a,b],[c-x,d-x]) \neq \{0\} \), since

\[
c - x \leq a < d - x \leq b.
\]
Lemma 16. As above, let $D$ be the interleaving distance between $M$ and $N$, and again, let

$$m_1 = \max\{|a-c|, |b-d|\}, \quad m_2 = \max\left\{\frac{b-a}{2}, \frac{d-c}{2}\right\}$$

If $m_1 \geq m_2$, then only one homomorphism will appear. That is, exactly one of $S_{M,N}, S_{N,M}$ intersects $[D, \infty)$.

6 Main Results

In this Section we show that the Examples in Section 5 constitute an exhaustive list of all possible progressions. We will also show that the first variety in a progression $V^1$, detects which whether the interleaving distance comes from a matching between the intervals $M$ and $N$.

Case 3: One support contains the other. That is $N \subseteq M$ or $M \subseteq N$.

Without loss of generality, say $M \supseteq N$ and $a \leq c < d \leq b$. First say $c - a \leq b - d$. Then, $c - a + d - c \leq b - d + d - c \Rightarrow d - a \leq b - c$. We’ll show

$$S_{M,N} = [c - a, d - c], \quad S_{N,M} = [b - d, b - c].$$

Note that $H(([a,b],[c-a+d-c+a]) = H(([a,b],[a-a-d+c]) \neq \{0\}$, since

$$a \leq a < a + d - c \leq b.$$ 

Also, for $x < c - a, H(([a,b],[c-x-d-x]) = \{0\}$, since $c - x > a$. Thus $\sigma = c - a$. Moreover, $H(([a,b],[c-d+a,d-d+a]) = \{0\}$, since $a \neq a$. But for $x \in (c-a,d-a), H(([a,b],[x-d-x]) \neq \{0\}$

$$c - x \leq a < d - x < d - c + a < a + d - c - c - a = d < b,$$

so $S_{M,N} = [c - a, d - a].$

In addition, $H(([c,d],[a-b+d,b-b+d]) = H(([c,d],[a-b+d,d]) \neq \{0\}$, since

$$a - b + d \leq d + c - d = c < d \leq d.$$ 

If $x < b - d$, then $H(([c,d],[a-x,b-x]) = \{0\}$, since $b - x > d$. Thus, $\tau = b - d$. For $x = b - c$ we have $H(([c,d],[a-b+c,b-c+c]) = H(([c,d],[a-b+c,c]) \neq \{0\}$, since $c \neq c$. However, for $x \in [b-d,b-c)$, $H(([c,d],[a-x,b-x]) \neq \{0\}$ since

$$a - x \leq a - b + d \leq d + c - d = c < b - x \leq d.$$ 

Thus, $\tau' = b - c$ and $S_{N,M} = [b - d, b - c]$ as required.

The case where $b - d < c - a$ is similar.

We point out that the proof also shows us that the last homomorphism born is the last to die. When $M = [a,b]$ and $N = [c,d]$, from [1] we know that the interleaving distance $D$ between $M$ and $N$ is given by

$$D = \min\{\max\{|a-c|, |b-d|\}, \max\left\{\frac{b-a}{2}, \frac{d-c}{2}\right\}\} = \min\{m_1, m_2\}.$$ 

Thus, Proposition 15 and Lemma 3 give us another way of interpreting the terms appearing in this formula for the interleaving distance, since $m_1$ is where the last homomorphism is born, and $m_2$ is the smallest number such that both $\Pi_M^{M_2m_2}$ and $\Pi_N^{N_2m_2}$ are identically zero.
Proof. First, the conclusion is clear whenever $M \cap N = \emptyset$ independent of the relationship between $m_1$ and $m_2$, since in this situation exactly one of $S_{M,N}, S_{N,M}$ is not empty. In fact, if the supports of $M$ and $N$ are disjoint, then necessarily $m_1 > m_2$. Now, say $m_1 > m_2$.

First, suppose the support of $M$ and $N$ intersect, but neither contains the other. Without loss of generality, say $a < c < b < d$. Then,

$$S_{N,M} = [0, b - c), \text{ and } S_{M,N} = [m_1, d - a).$$

We must show that $D = m_2 \geq b - c$. For a contradiction, suppose that $\frac{b-a}{2}, \frac{d-c}{2} < b - c$. However $b - c > \frac{d-c}{2}$ implies $d - b < \frac{d-c}{2}$ and $b - c > \frac{b-a}{2}$ implies $c - a < \frac{b-a}{2}$ by the pigeonhole principle. This means $m_1 < m_2$, a contradiction.

Lastly, suppose that the support of one contains the other. Without loss of generality, say $M \supseteq N$, so $a \leq c < d \leq b$. Then,

$$S_{M,N} = [c - a, d - a), \text{ and } S_{N,M} = [b - d, b - c).$$

Since $m_1 \geq m_2$,

$$m_2 = \frac{b-a}{2} \leq \max\{c - a, b - d\}.$$

If the maximum is $c - a$, then $c - a \geq \frac{b-a}{2} \geq b - c$ by the pigeonhole principle. Similarly, if the maximum is $b - d$, then $b - d \geq \frac{b-a}{2} \geq d - a$, as well. Thus the result holds in all permissible cases.

We are now ready to prove our Main Result.

**Theorem 1.** The previous list of examples is exhaustive. Moreover, we have the following results:

i. $m_1 > m_2 \iff V^1$ is the origin $\iff$ the full progression is origin, axis, origin.

ii. If $m_1 = m_2 \iff V^1$ is an axis $\iff$ the full progression is axis, origin.

iii. If $m_1 < m_2 \iff V^1$ is a hyperbola $\iff$ the full progression is hyperbola, plane, axis, origin or hyperbola, plane, origin.

Thus, in particular, $V^1$ can detect whether the interleaving distance $\epsilon$ comes from $m_1$ or $m_2$. Note that i corresponds to Example [11] ii. corresponds to Example [12] and iii. to Examples [13], [14]

Proof. We will proceed in cases.

First, suppose $m_1 > m_2$. Then,

$$\max\{|a-c|, |b-d|\} > \max\{\frac{b-a}{2}, \frac{d-c}{2}\},$$

so the interleaving distance $\epsilon$ is $m_2$. Without loss of generality, assume $\epsilon = \frac{b-a}{2}$, so we have

$$\frac{d-c}{2} \leq \frac{b-a}{2}, \text{ and either } \frac{b-a}{2} < |a-c| \text{ or } \frac{b-a}{2} < |b-d|.$$

At $\epsilon = \frac{b-a}{2}$, we’ll first show that $V^1$ is the origin. First, clearly at $\epsilon$, $H(M, 2M\epsilon)$ and $H(N, N2\epsilon)$ are both $\{0\}$. Suppose $\frac{b-a}{2} < |a-c|$. Then, we need only show that $H(M, N\epsilon) = \{0\} = H(N, M\epsilon)$. But, if $c > a$, then $H(M, N\epsilon) = \{0\}$ since $c - \epsilon > a$. Moreover, $H(N, M\epsilon) = \{0\}$ since $b - \epsilon < a$.

On the other hand, if $c < a$, then $H(M, N\epsilon) = \{0\}$ since $d - \epsilon < a$ and $H(N, M\epsilon) = \{0\}$ since $a - \epsilon > c$. Similarly, if $\frac{b-a}{2} < |b-d|$, then $d > b$, so $H(M, N\epsilon) = \{0\}$ since $c - \epsilon > a$. Also, $H(N, M\epsilon) = \{0\}$ since $b - \epsilon < c$. Similarly, if $d < b$, then both homomorphisms are necessarily zero.
Thus, we have shown that at \( \epsilon \) the variety \( V^1 \) is the origin. We now continue within this case. By **Proposition 15**, \( m_1 = \max\{|a-c|,|b-d|\} \) is the value when the last homomorphism is born. By **Lemma 16** the variety \( V^2 \) which must be an axis, appears at \( m_1 \). But at \( \max\{|a-c|,|b-d|\} \), exactly one of \( \text{Hom}(M,N\epsilon) \), \( \text{Hom}(M,N\epsilon) \) is nonzero, thus at \( \max\{|a-c|,|b-d|\} \) the variety is an axis. By **Proposition 15** the last variety \( V^3 \) is zero, which occurs at \( \max\{|a-d|,|b-c|\} = \max\{\sigma', \tau'\} \).

Now, suppose that \( m_1 = m_2 \), so \( \max\{|a-c|,|b-d|\} = \max\{\frac{b-a}{2}, \frac{d-c}{2}\} \). By our comments in **Lemma 16** necessarily, the supports of \( M, N \) overlap, and neither contains the other. Without loss of generality, say \( a < c < b < d \). Then,

\[
S_{M,N} = [m_1, \max\{d-a, b-c\}] = [m_1, d-a), \quad \text{and} \quad S_{N,M} = [0, b-c).
\]

First, say \( c-a = m_1 = \max\{c-a, d-b\} \), so

\[
c - a = \frac{b-a}{2}, \quad \text{or} \quad c - a = \frac{d-c}{2}.
\]

If \( c = a + \frac{b-a}{2} \geq a + \frac{d-c}{2} \), then

\[
a \leq a < d - \frac{b-a}{2} \leq b, \quad \text{so} \quad H(M, N\epsilon) \neq \{0\}.
\]

However, \( H(N, M\epsilon) = \{0\} \), since

\[
a - \frac{b-a}{2} \leq a + \frac{b-a}{2} = b - \frac{b-a}{2} \leq d.
\]

If instead \( c = a + \frac{d-c}{2} \geq a + \frac{b-a}{2} \), a similar analysis shows that

\[
H(M, N\epsilon) \neq \{0\}, \quad \text{and} \quad H(N, M\epsilon) = \{0\}.
\]

Thus, at \( \epsilon \) equal to the interleaving distance, one homomorphism is born, and clearly both

\[
H(M, M2\epsilon) = \{0\}, \quad \text{and} \quad H(N, N2\epsilon) = \{0\}.
\]

Thus, we have that \( V^1 \) is given by an axis (in this case the x-axis). By **Lemma 16** the last variety \( V^2 \) is the origin, which occurs at \( \max\{|a-d|,|b-c|\} = \max\{\sigma', \tau'\} \), which is \( d-b \) in our case.

Lastly, suppose \( m_1 < m_2 \). Then, \( \max\{|a-c|,|b-d|\} < \max\{\frac{b-a}{2}, \frac{d-c}{2}\} \), and the Interleaving distance is \( \epsilon = \max\{|a-c|,|b-d|\} \). Clearly, at least one of

\[
H(M, M2\epsilon), H(N, N2\epsilon) \quad \text{will survive}.
\]

First, suppose the supports of \( M, N \) intersect, but neither contains the other. Without loss of generality, say \( a < c < b < d \). Then,

\[
S_{M,N} = [m_1, d-a), S_{N,M} = [0, b-c).
\]

We will show that both homomorphisms are alive at the interleaving distance \( \epsilon = m_1 \). But this is clear, since

\[
d - b \leq c - a < \frac{b-a}{2} \quad \Rightarrow \quad b - c > \frac{b-a}{2} > c - a, \quad \text{and} \quad d - b \leq c - a < \frac{d-c}{2} \quad \Rightarrow \quad b - c > \frac{d-c}{2} > c - a.
\]
Similarly,

\[
\begin{align*}
  c - a &\leq d - b < \frac{d - c}{2} \implies b - c > \frac{d - c}{2} > d - b, \\
  c - a &\leq d - b < \frac{b - a}{2} \implies b - c > \frac{b - a}{2} > d - b.
\end{align*}
\]

Therefore, we have established that in this case, the first variety $V^1$ is a hyperbola. Moreover, the above calculation shows that $m_2 < b - c$, thus $V^2$ is a plane begining at $\epsilon = m_2$. Continuing, since $b - a < d - a$, $V^3$ is an axis occurring at $\epsilon = b - a$ and $V^4$ is the origin, which starts at $\epsilon = d - a$.

Since disjoint supports are not possible when $m_1 < m_2$, it remains only to consider when, say $M \supseteq N$, so suppose $a \leq c < d \leq b$. Then,

\[
S_{M,N} = [c - a, d - a), \text{ and } S_{N,M} = [b - d, b - c).
\]

Thus, by inspection, at the interleaving distance $m_1 = \max\{c - a, b - d\}$, both homomorphisms are alive. Thus, since $H(M, M_2 M_1) \neq \{0\}$, the first variety $V_1$ is a hyperbola. Continuing, it follows from properties of interleavings that $m_2 = \frac{b - a}{2} < \max\{d - a, b - c\}$, but we one easily checks that

\[
\frac{b - a}{2} \geq d - a \implies b - d \geq \frac{b - a}{2}, \text{ and } \frac{b - a}{2} \geq b - c \implies c - a \geq \frac{b - a}{2}.
\]

Thus, $V^2$ is the plane, which occurs at $m_2 = \frac{b - a}{2}$. Now let

\[
r = \min\{d - a, b - c\}, \text{ and } s = \max\{d - a, b - c\}.
\]

If $r = s$, then both homomorphisms die at the same time, and the progression ends with $V^3$ equal to the origin. Alternatively, if $r < s$, then $V^3$ is an axis occurring at $r$, and $V^4$ is the origin beginning at $s$.

Since the cases,

\[
m_1 > m_2, m_1 = m_2, m_1 < m_2
\]

constitute a partition, the result holds. \qed

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