Beurling moving averages and approximate homomorphisms
by
N. H. Bingham and A. J. Ostaszewski

Abstract. The theory of regular variation, in its Karamata and Bojanić-Karamata/de Haan forms, is long established and makes essential use of homomorphisms. Both forms are subsumed within the recent theory of Beurling regular variation, developed further here, especially certain moving averages occurring there. Extensive use of group structures leads to an algebraization not previously encountered here, and to the approximate homomorphisms of the title. Dichotomy results are obtained: things are either very nice or very nasty. Quantifier weakening is extended, and the degradation resulting from working with limsup and liminf, rather than assuming limits exist, is studied.

Key words: Beurling regular variation, Beurling’s equation, self-neglecting functions, self-equivarying functions, uniform convergence theorem, category-measure duality, Bloom dichotomy, Golab-Schinzel functional equation.

Mathematics Subject Classification (2000): Primary 26A03; 39B62; 33B99, 39B22, 34D05; 39A20

CONTENTS

1. Introduction
2. From Beurling to Karamata
3. Popa groups – the Popa-Javor Theorem
4. Extensions to Beurling’s Tauberian Theorem
5. Uniformity, semicontinuity
6. Dichotomy
7. Quantifier weakening
8. Representation
9. Divided difference and double sweep
10. Uniform Boundedness Theorem
11. Character degradation from limsup
12. Appendix

1 Introduction

This work is a sequel to our recent papers [BinO12], [BinO13], [BinO14] together with the related paper [Ost3] by the second author, reexamined in the light of two much earlier works [BinG2] and [BinG3] by the first author and Goldie. Our title Beurling moving averages addresses both the Beurling slow and regular
variation in [BinO12] (to which we refer for background), and [BinG2,3], the motivation for which is strong laws of large numbers in probability theory.

Beurling regular variation is closely linked with Karamata regular variation (the standard work on which is [BinGT], BGT below, to which we refer for background). In [BinO12], it emerged that Beurling regular variation in fact subsumes the traditional (and very widely used) Karamata regular variation, together with its Bojanić-Karamata/de Haan relative – BGT Ch. 1-3; [BojK], [deH]. Whereas the traditional approach is to develop the measurable and Baire-property cases in parallel, measure being regarded as primary, it is now clear both that one can subsume both cases together and that it is in fact the Baire case that is primary; this is the theory of topological regular variation, for which see [BinO6] [BinO1], [BinO5], [Ost2] – this informs our approach in §10.

Beurling slow variation was introduced by Beurling in 1957 (unpublished) for use in Beurling’s Tauberian Theorem ([Kor, IV §11]; [BinO12] and §4 below), which gives a little known but very useful extension of Wiener’s Tauberian Theorem, to which it reduces in the special case $\phi(x) \equiv 1$ (see below).

It is convenient to work both multiplicatively in $\mathbb{R}^+ := (0, \infty)$ and additively in $\mathbb{R}$. A self-map $f$ of $\mathbb{R}$ or $h$ of $\mathbb{R}$ is Beurling $\phi$-slowly varying if, according to context,

$$f(x + t\phi(x))/f(x) \to 1, \text{ or } h(x + u\phi(x)) - h(x) \to 0, \quad (BSV/BSV_+)$$

as $x \to \infty$, where $\phi$ is a self-map of $\mathbb{R}$ and is self-neglecting ($\phi \in SN$), so that

$$\phi(x + t\phi(x))/\phi(x) \to 1 \text{ locally uniformly in } t \text{ for all } t \in \mathbb{R}, \quad (SN)$$

and $\phi(x) = O(x)$. This traditional restriction may be usefully relaxed in two ways, as in [Ost3]: firstly, in imposing the weaker order condition $\phi(x) = O(x)$, and secondly by replacing the limit 1 by a general limit function $\eta$, so that

$$\phi(x + t\phi(x))/\phi(x) \to \eta(t) \text{ locally uniformly at } t \text{ for all } t \in \mathbb{R}. \quad (SE)$$

Such a $\phi$ is called self-equivarying in [Ost3], and the limit function $\eta = \eta^2$ necessarily satisfies the equation

$$\eta(u + v\eta(u)) = \eta(u)\eta(v) \text{ for } u, v \in \mathbb{R} \quad (BFE)$$

(this is a special case of the Gołąb-Schinzel equation – see also e.g. [Brz1], or [BinO13], where the equation above is termed the Beurling functional equation). As $\eta \geq 0$, imposing the natural condition $\eta > 0$ (on $\mathbb{R}$+) implies that it is continuous and of the form

$$\eta(t) = 1 + \rho t, \quad \text{for some } \rho \geq 0$$

\footnote{Note that we have changed the original notation $\lambda_\phi$ for this context, both to free up the use of $\lambda$ for other conventional uses, and to reflect the connection to the function $H_\rho$ below (as $H$ denotes the Greek capital 'eta').}
a sequence \( u \). For behaviour at infinity, we lose nothing by assuming here and elsewhere that
\[ \lim_{x \to \infty} f(x)/g(x) = 1, \]
as \( x \to \infty \) and the second factor is in \( SN \) (see [BinO12, Th. 9], [Ost3]).

For \( \varphi \in SE \), a self-map \( f \) of \( \mathbb{R}_+ \) or \( h \) of \( \mathbb{R} \) is Beurling \( \varphi \)-regularly varying if, according to context, the limits below exist:
\[ f(t + k(t)) = f(t), \quad h(x + k(x)) = h(x) \to k(t). \]

For \( \varphi \in SN \) and \( f \) Baire/measurable, the limit \( g(t) \) is necessarily an exponential function \( e^{\gamma t} \) (provided \( g > 0 \) on a non-negligible set), equivalently \( k \) is linear; \( \gamma t \), convergence is locally uniform, and the function \( f \) is characterized (see [BinO12]) via the representation
\[ f(x) = \exp(\gamma \cdot \varphi(x)) f(x), \quad h(x) = \gamma \cdot \varphi(x) + \tilde{h}(x), \]
for \( \tau_{\varphi}(x) := \int_0^x du/\varphi(u) \),
and \( \tilde{f} \) (respectively \( \tilde{h} \)) a \( \varphi \)-slowly varying function, as above (as we are interested in behaviour at infinity, we lose nothing by assuming here and elsewhere that \( 1/\varphi \) is locally integrable, e.g. by modifying \( \varphi \) near 0). Here \( \gamma \) is the \( \varphi \)-index of Beurling variation, or Beurling \( \varphi \)-index for short. For \( \varphi \in SE \) with \( \varphi \)-index \( \rho > 0 \), the situation is altered from \( g(t) = e^{\gamma t} \) so that (see [Ost3, Th. 1’])
\[ g(t) = (1 + \rho t)^\gamma, \quad k(t) = \gamma \log(1 + \rho t). \]

Motivated by a study of the ‘moving average under \( \varphi \)’, for \( \varphi \in SN/SE \), of a sequence \( u = \{u_n\} \) defined as
\[ K_u(t; x) := \frac{1}{\varphi(x)} \sum \{u_n : x < n \leq x + t\varphi(x)\} \to c_u t, \quad (x \to \infty) \]
we study its limit function \( K_u(t) := \lim_{x \to \infty} K_u(t; x) \), when that exists. Below, functions \( U \) rather than sequences \( u \) are more convenient, with moving average
\[ K_U(t; x) := \frac{U(x + t\varphi(x)) - U(x)}{\varphi(x)}, \]
which specializes to (*) for the partial-sum function \( U(x) := \sum \{u_n : n \leq x\} \). This leads us to the question of existence and additivity properties of the limit functions below:
\[ K_F(t) := \lim \Delta_t^\varphi F(x)/\varphi(x), \quad K_F^+(t) := \limsup \Delta_t^\varphi F(x)/\varphi(x), \]
with \( \Delta_t^\varphi \) the difference operator
\[ \Delta_t^\varphi F(x) := F(x + t\varphi(x)) - F(x), \]
and local uniform convergence assumed (unless otherwise stated). For \( \varphi(x) \equiv 1 \) this reduces to the usual difference operator \( \Delta_t \). Motivated by classical analysis, we introduce a more general auxiliary function \( \psi(x) \) in the denominator:

\[
K_F(t) := \lim \Delta^t F(x) / \psi(x), \quad K^*_F(t) := \limsup \Delta^t F(x) / \psi(x).
\]

If \( K_F \) is defined at \( u \) and \( v \), then

\[ K_F(u + vh(u)) = K_F(u)g(v) + K_F(v), \]

provided

\[ h(u) := \lim \varphi(x + u\varphi(x)) / \varphi(x) \quad \text{and} \quad g(v) := \psi(x + v\varphi(x)) / \psi(x) \]

exist (and convergence to \( K_F \) is locally uniform), which will be the case when \( \varphi \in SE \) (so that \( h = h_\rho \)) and \( \psi \) is \( \varphi \)-regularly varying (so that either \( \rho = 0 \) and \( g = e^{\gamma t} \), or \( \rho > 0 \) and \( g \equiv (1 + \rho \cdot \gamma)^t \), by \( \rho-BR_\gamma \) above). The related functional equation – the extended Goldie-Beurling (Pexiderized\(^\text{2}\)) equation,

\[
K(v + uh(v)) = \kappa(u)g(v) + K(v), \quad \text{for } h, \kappa \text{ positive – is studied in } [\text{BinO13, Th. 9 and 10}].
\]

Its solutions \( K \), necessarily continuous, are there characterized (subject to \( K(0) = 0 \)) as

\[ K(x) \equiv c \cdot \tau_f(x) \text{ with } f := h/g \text{ and } \tau_f(x) := \int_0^x dw/f(w), \]

as before (but with \( f \) for \( \varphi \)), an ‘occupation time measure’ (of the interval \([0,x]\); \( \S2 \)); the ‘relative flow rate’ \( f \) satisfies the Cauchy-Beurling exponential equation:

\[
f(v \circ_h u) = f(u) f(v), \quad \text{for } h, \kappa \text{ positive.}
\]

(cf. [Ost4]. Here \( \circ_h \) denotes Popa’s binary operation ([Pop], cf. [Jav], \( \S3 \) below)

\[ v \circ_h u = v + uh(v), \]

so that \( h = h_\rho \) itself also satisfies \( \text{(CBE)} \); this confers a group structure, turning certain subsets of \( \mathbb{R} \) into groups, called Popa groups in \( \S3 \); furthermore, necessarily \( \kappa = K \). Solving \( \text{(GBE-P)} \) may be expressed as an equivalent Popa homomorphism problem of finding \( k, h \in GS \)

\[ K(v \circ_h u) = K(u) \circ_k K(v) \quad \text{(GCBE)} \]

(cf. [Brz2], [Mu]), where

\[ k(u) = g(K^{-1}(u)). \]

This observation is new even for the classical context \( h \equiv 1 \); here \( f = e^{-\gamma t} \), so

\[ \tau_f(x) \equiv H_\gamma(x) := (e^{\gamma x} - 1) / \gamma \text{ with } H_0(x) \equiv 0. \]

\(^2\)After Pexider’s equation: \( f(xy) = g(x) + h(y) \) in three unknown functions and its generalizations – cf. [Kuc, 13.3], [Brz1,2], [Jab].
For $\eta \equiv \eta_\rho$ with $\rho > 0$, $g \equiv (1 + \rho \cdot \gamma)$, by $(\rho-BR_\gamma)$ above, $f(x) = (1 + \rho x)^{1-\gamma}$, so

$$K \equiv c \cdot \tau_f = c \cdot K_{\rho_\gamma},$$

where $K_{\rho_\gamma}(x) := \int_0^x (1 + \rho w)^{\gamma-1} dw = ((1 + \rho x)^{\gamma} - 1)/\rho \gamma$

(linear for $\gamma = 1$). The ‘slow case’ $\gamma = 0$ may also be handled via

$$\lim_{\gamma \to 0} K_{\rho_\gamma}(x) = \log(1 + \rho x)/\rho.$$

When $\varphi(x) \equiv 1$, the moving averages reduce to classical Bojanić-Karamata/de Haan limits (BGT Ch. 3), for which the auxiliary $\psi(x)$ is necessarily Karamata regularly varying, so just as before (trivially, since $\varphi \in SE$) has exponential limit function, $g \equiv e^{\gamma}$ say, and then $(\text{GBE-P})$ simplifies to the original Goldie functional equation:

$$K(u + v) = e^{\gamma t} K(v) + K(u),$$

(GFE)

with solution $K(u) \equiv c \cdot H_\gamma(u)$, as before. The latter function plays a crucial role in the Bojanić-Karamata/de Haan theory of regular variation. Here, and in the general case, if $\Delta_\varphi F/\psi$ has a limiting moving average $K_F$, then for some $c_F \in \mathbb{R}$, as above (cf. [BinO13, Th. 3, 9, 10]),

$$K_F(u) = c_F \cdot H_\gamma(u),$$

with $c_F$ the $\psi$-index of $F$ (for $\psi$ which is $\varphi$-regularly varying), while $\psi$ has Beurling $\varphi$-index $\rho$.

In the classical context, one works also with $K_F^*$, abbreviated to $K^*$, and with $K_*$. Here the equations (GFE) give way to functional inequalities, so that for instance

$$K^*(u + v) \leq e^{\gamma t} K^*(v) + K^*(u)$$

(GFI)

(BGT (3.2.5)), which we summarize by saying that $K^*$ is $\text{exp-subadditive}$. Equivalently, this may be re-expressed symmetrically here as group sub-additivity:

$$K^*(x + y) \leq K^*(x) \circ_k K^*(x)$$

with $k$ as above, and in the more general Beurling case correspondingly to $(\text{GCBE})$ as

$$K^*(x \circ_h y) \leq K^*(x) \circ_k K^*(x).$$

For $\psi$ regularly varying, the set

$$A := \{ t : \lim_\Delta_\varphi F(x)/\psi(x) \text{ exists and is finite} \},$$

for which see e.g. BGT Th. 3.2.5 (proof) and §§4,5 below, constitutes the domain of the function

$$K_F(a) := \lim_{x \to \infty} \Delta_\varphi F(x)/\psi(x) \quad (a \in A);$$

(ker)

we refer to $K_F$ as the regular kernel of $F$ – the homomorphism approximating $F$ of our title. In [BinO13] (and in [BinO14] for the case $\rho = 0$), we study
conditions on $K^*$ implying that $K^*_F$ exists, i.e. that the inequality becomes an equation, by imposing ‘Heiberg-Seneta’ side-conditions, and density of $\kappa$ – again cf. BGT Ch. 3, especially the crucial Theorem 3.2.5. Below these findings are extended to the Beurling context.

Alternative conditions are developed in BGT Ch. 3 based on the function

$$\Omega_F(\lambda) := \limsup_x \sup\{(F(tx) - F(x))/\psi(x) : s \in [1, \lambda]\}.$$  

This on account of its multiplicative formulation leads to the study of power-subadditive functions, i.e. those satisfying

$$\Omega(\lambda\mu) \leq \mu^\rho \Omega(\lambda) + \Omega(\mu)$$

(cf. [AczG]), and consequences of the existence of $\Omega'(1) = \lim_{\lambda \to 1} \Omega(\lambda)/(\lambda - 1)$, for which see [BinG1, §2], [BojK, §3], or the related BGT Th. 3.3.3. It will be more convenient here to work with its additive version (obtained by writing $\lambda = e^u, x = e^\xi$, etc.):

$$\omega_F(u) := \limsup_\xi \sup\{(F(\tau + \xi) - F(\xi))/g(\xi) : \tau \in [0, u]\},$$

which is exp-subadditive; here $\lim_{u \to 0} \omega(u)/(e^u - 1) = \omega_F'(0+)$. Our general aim is to extend, simplify, unify and so clarify the classical theory of moving averages for regularly varying functions via the wider, Beurling, regular variation. This captures more than just the sum of the extant additive and multiplicative Karamata variants, embracing new types of regular kernels.

2 From Beurling to Karamata

The function $H_\rho$ (of §1) satisfies

$$dH_\rho/\,dx = e^{\rho x} = 1 + \rho H_\rho(x) = \eta_\rho(H_\rho(x)),$$

and solves the Goldie equation ($GFE$), in which the auxiliary function $g$, which is necessarily exponential for $K$ Baire/measurable, takes the form $g(x) = e^{\rho x}$ – again see [BinO13, Th. 1]. Regarding $\varphi, \eta \in SE$ as generating (velocity) flows as in [BinO12], their occupation ‘times’ (on $[0, x]$) are (cf. [Bec, p.153]):

$$\tau_\varphi(x) := \int_0^x \frac{dw}{\varphi(w)} \text{ and } \tau_\eta(x) := \int_0^x \frac{dw}{\eta(w)};$$

both strictly increasing. (For present needs this notation is more symmetrical than that of [BinG1] with $\Phi$ for $\tau_\varphi$, and of BGT 2.12.29, which we mention for purposes of comparison.) For $\rho > 0$ and $\eta = \eta_\rho \in GS \subseteq SE$

$$\tau_\eta(x) := \int_0^x \frac{dw}{1 + \rho w} = \frac{1}{\rho} \log(1 + \rho x),$$

so

$$\tau_\eta^{-1}(t) = H_\rho(t) = (e^{\rho t} - 1)/\rho.$$
In particular, the trajectory \( w(t) := \tau^{-1}_\eta(t) \) satisfies the equation \( dw(t)/dt = e^{ho t} = 1 + \rho w(t) = \eta(w(t)) \) with \( w(0) = 0 \). Necessarily, working with the (inverse) re-parametrization \( dt(w)/dw = e^{-\rho t} = \psi(t) \in SN \) gives \( \tau_\psi(x) = H_{\rho}(x) \), again an occupation time measure.

We now generalize a theorem of Bingham and Goldie \cite[Th. 2]{BinG2}. This recovers their theorem when \( \rho_\eta = 0 \) and \( \varphi(x) = o(x) \), as then \( \varphi \in SN \). The result may be interpreted as a local `chain rule', for \( V(s) = U(s(t)) \) where the trajectory \( s(t) := \tau^{-1}_\varphi(t) \) satisfies \( ds(t)/dt = \varphi(s(t)) = \varphi(\tau^{-1}_\varphi(t)) = g(t) \) (with \( \varphi \in SE \), a `self-equivarying flow').

**Theorem 0 (Time-change Equivalence Theorem for Moving Averages).** For positive \( \varphi \in SE \) with \( 1/\varphi \) locally integrable, \( U \) satisfies

\[
\frac{U(x + t\varphi(x)) - U(x)}{\varphi(x)} \to c_U t \quad \text{as} \quad x \to \infty, \quad \forall t \in \mathbb{R} \quad (BMA_{\varphi})
\]

iff its **time-changed version** \( V := U \circ \tau^{-1}_{\varphi} \) satisfies, for \( g(y) := \varphi(\tau^{-1}_\varphi(y)) \),

\[
\frac{V(y + s) - V(y)}{g(y)} \to c_U H_{\rho_\varphi}(s) \quad \text{as} \quad y \to \infty, \quad \forall s \in \mathbb{R} \quad (KMA_{\varphi})
\]

This is proved exactly as in \cite[Th. 2]{BinG2}, using the following.

**Proposition 1.** For \( \varphi \in SE \) and \( \eta = \eta^\varphi \), uniformly in \( s \)

\[
\lim [\tau_\varphi(x + s\varphi(x)) - \tau_\varphi(x)] = \tau_\eta(s).
\]

In particular, this is so for \( \varphi \in SN \), where \( \tau_\eta(s) \equiv s \).

**Proof.** Let \( \rho \) be the \( \eta^\varphi \)-index. Fix \( u > 0 \), then for \( t \in [0, s] \) uniformly in \( t \)

\[
\varepsilon(x,t) := \varphi(x)/\varphi(x + t\varphi(x)) - 1/\eta_\rho(t) \to 0, \quad \text{so} \quad e(x,s) := \int_0^s \varepsilon(x,t)dt \to 0.
\]

Then, as in \cite[Th. 2]{BinG2}, using the substitution \( w = x + t\varphi(x) \)

\[
\tau_\varphi(x + s\varphi(x)) - \tau_\varphi(x) = \int_x^{x+s\varphi(x)} \frac{dw}{\varphi(w)} = \int_0^s \frac{\varphi(w)dt}{\varphi(x + t\varphi(x))} = \int_0^s \left( \frac{1}{\eta_\rho(t)} + \varepsilon(x,t) \right) dt = \tau_\eta(s) + e(x,u).
\]

If \( \varphi \in SN \), then \( \tau_\eta(s) \equiv s \), as \( \eta^\varphi \equiv 1 \). □

Our first corollary characterizes \( SE \) in terms of a **multiplicative** Karamata index via its time-changed version \( g \); this is a **consistency** result in view of the characterization from \cite{Ost3} of \( \varphi \in SE \) as the product of \( \eta^\varphi \psi \) with \( \psi \in SN \). The latter identifies \( \varphi \) itself as having **additive** Karamata index \( \rho_\varphi \).
**Corollary 1.** \( \varphi \in SE \) iff \( g = \varphi \circ \tau_{\varphi}^{-1} \) is regularly varying with multiplicative Karamata index \( \rho_{\varphi} \). In particular, \( \varphi \in SN \) iff \( g = \varphi \circ \tau_{\varphi}^{-1} \) is regularly varying with multiplicative Karamata index \( \rho_{\varphi} = 0 \).

**Proof.** Since
\[
(\varphi(x + t\varphi(x)) - \varphi(x))/\varphi(x) = \varphi(x + t\varphi(x))/\varphi(x) - 1 \to \rho_{\varphi}t,
\]
we may apply Th. 0 to \( U = \varphi \) so that \( V := \varphi \circ \tau_{\varphi}^{-1} = g \); then by \((KMA)\)
\[
g(y + s)/g(y) - 1 = (g(y + s) - g(y))/g(y) \to (e^{\rho_{\varphi}x} - 1): \quad g(y + s)/g(y) \to e^{\rho_{\varphi}x},
\]
and conversely. □

We now show that \( K_V \) satisfies a Goldie equation, from which its format can be read off, as in the Equivalence Theorem above.

**Corollary 2.** For \( \varphi \in SE, \) so that \( g = \varphi \circ \tau_{\varphi}^{-1} \) is regularly varying with multiplicative Karamata index \( \rho_{\varphi} \):
if \( KMA_g \) holds – or equivalently \( BMA_{\varphi} \) holds – then for \( K_V(u) \), as above,
\[
K_V(s + t) = K_V(s)e^{\rho_{\varphi}t} + K_V(t),
\]
and so for some \( c \)
\[
K_V(s) = cH_{\rho_{\varphi}}(s).
\]
**Proof.** The Goldie equation follows from Corollary 1, since
\[
\frac{V(y + s + t) - V(y)}{g(y)} = \frac{V(y + s + t) - V(y + t)}{g(y + t)} \frac{g(y + t)}{g(y)} + \frac{V(y + t) - V(y)}{g(y)}.
\]
Now apply Theorem 1 of [BinO13] to deduce the form of \( K_V \). □

## 3 Popa groups – the Popa-Javor Theorem

Recall from Popa [Pop], for \( h : \mathbb{R} \to \mathbb{R} \), the **Popa operation** \( \circ_h \) and its **Popa domain** \( \mathbb{G}_h \) (our terminology) defined by:
\[
a \circ_h b := a + bh(a), \quad \mathbb{G}_h := \{g : h(g) \neq 0\}.
\]
We recall also, from Javor [Jav] (in the broader context of \( h : E \to F \), with \( E \) a vector space over a commutative field \( F \)), that \( \circ_h \) is associative iff \( h \) satisfies the Golab-Schinzel equation, briefly \( h \in GS \):
\[
h(x + yh(x)) = h(x)h(y). \quad (x, y \in \mathbb{G}_\varphi) \quad (GS)
\]
Their role below is fundamental; first, \( GS \subseteq SE \), and for \( \varphi \in SE \) the Popa operation \( x \circ_{\varphi} t = x + t\varphi(x) \) compactly expresses the Beurling transformation.
In general, if\( G \) carries \( \rho \) for \( G_0 \).

For Proposition 2 (Popa-Javor Theorem, \[\text{cf. \cite[Prop. 2]{Pop}}\]), we collect relevant facts in the following.

Commutative, and \( \phi \) is symmetric on the right-hand side. Commutativity of \( (\varphi \cdot \eta) \) follows directly from \( \eta \in GS \) – see \[\text{equation (3.1)}\] for background. The fact that \( \eta \in GS \) is proved in \[\text{Ost3} - \text{see \S 1}\]; solutions of \( (GS) \) that are \textit{positive} on \( \mathbb{R}_+ := (0, \infty) \) are key here, being of the form \( \eta_\rho(x) := 1 + \rho x \) with \( \rho > 0 \). The case \( \rho = 0 \) corresponds to the classical Karamata setting, and \( \rho > 0 \) to the recently established, general, theory of Beurling regular variation \[\text{BinO12}]. For the corresponding \textit{Popa groups} we write \( \varphi_\rho \) (when \( h = \eta_\rho \)), or even \( \circ \), omitting subscripts both on \( \circ \) and on \( \eta \), if context permits.

So here we return to \( GS \).

The appearance of a group structure ‘in the limit’ is not accidental – see \[\text{Ost4} \text{for background}. The fact that \( \eta \in GS \) is proved in \[\text{Ost3} - \text{see \S 1}\]; solutions of \( (GS) \) that are \textit{positive} on \( \mathbb{R}_+ := (0, \infty) \) are key here, being of the form \( \eta_\rho(x) := 1 + \rho x \) with \( \rho > 0 \). The case \( \rho = 0 \) corresponds to the classical Karamata setting, and \( \rho > 0 \) to the recently established, general, theory of Beurling regular variation \[\text{BinO12}. For the corresponding \textit{Popa groups} we write \( \varphi_\rho \) (when \( h = \eta_\rho \)), or even \( \circ \), omitting subscripts both on \( \circ \) and on \( \eta \), if context permits.

To prevent confusion, \( u_\rho^{-1} \) denotes the relevant group inverse. Furthermore, we employ the notation:

\[
G_+^\rho := \mathbb{R}\backslash\{\rho^{-1}\}, \quad G_0^\rho := (-1/\rho, \infty), \quad (\rho \neq 0), \quad G_\infty^\rho := \mathbb{R}\backslash\{0\} = \mathbb{R}^*, \quad G_\rho^0 := \mathbb{R}, \\
\eta_\rho^\rho(x) := \eta_\rho(x) (\rho \neq 0); \quad \eta_\rho^0(x) := e^x.
\]

We collect relevant facts in the following.

**Proposition 2 (Popa-Javor Theorem, \[\text{Pop, Prop. 2}; \text{cf. \cite[Lemma 1.2]{Jav}}\]).**

For \( 0 \neq \varphi \in GS, (G_\varphi, \circ_\varphi) \) is a group. If \( \varphi \) is injective on \( G_\varphi \), then \( \circ_\varphi \) is commutative, and \( \varphi \) maps homomorphically into \( (\mathbb{R}^*, \cdot) \):

\[
\varphi(x \circ_\varphi y) = \varphi(x) \varphi(y).
\]

In particular, \( G = G^0 := (G_\varphi^\rho, \circ_\rho) \) is an abelian group with \( 1_G = 0 \) and inverse

\[
u_\rho^{-1} = -v/\eta(v).
\]

\( G^0 := (\mathbb{R}, \circ) \) is \( (\mathbb{R}, +) \) for \( \rho = 0 \), so that \( G^0 \) is isomorphic under \( \eta_\rho \) to \( (\mathbb{R}^*, \cdot) \) for \( \rho > 0 \). Furthermore, inversion carries \( G^\rho_+ \) into itself: \( (G^\rho_+)^{-1} = G^\rho_- \) and \( \eta_\rho^{-1} \) carries \( G^\rho_+ \) onto \( \mathbb{R}_+ \).

**Proof.** In general, if \( \varphi \) is injective on \( G_\varphi \), then \( \circ_\varphi \) is commutative, as \( (GS) \) is symmetric on the right-hand side. Commutativity of \( \circ_\varphi \) follows directly from

\[
v_\rho^{-1} = -v/\eta(v) = -v/(1 + \rho v) \quad \text{for} \quad v \neq -1/\rho.
\]

9
Isomorphic maps of $G$ are provided for $\rho = 0$ by $\iota : x \rightarrow x$ onto $(\mathbb{R}, +)$, and for $\rho > 0$ by $\eta : x \rightarrow 1 + \rho x$ onto $(\mathbb{R}^+, \cdot)$, since
\[ \eta(u) \eta(v) = (1 + \rho u)(1 + \rho v) = 1 + \rho [v + u(1 + \rho v)] = \eta(u \circ \eta v). \]

The rest follows since $\rho > 0$ and $x > -1/\rho$ imply $\eta(x) > 0$. □

Remarks. 1. For $\rho \neq 0$, $G_\rho$ is typified (rescaling its domain) by the case $\rho = 1$, where
\[ a \circ_1 b = (1 + a)(1 + b) - 1 : \quad (G_1, \circ_1) = (\mathbb{R}^*, \cdot) - 1, \]
and the isomorphism is a shift (cf. [Pop, §3]).

2. For $\rho > 0$, note that $u \in G_\rho^+ \cap (0, \infty)$ has $u_{\rho}^{-1} \in G_\rho^+ \cap (-1/\rho, 0)$.

3. The convolution $v * t := v \circ t^{-1} = (v - t)/\eta(t)$ is the asymptotic form of the Beurling convolution $(v - t)/\varphi(t)$ occurring in the Beurling Tauberian Theorem (§4) for $\varphi \in SN$.

4. For $\rho > 0$, the inverse $\eta^{-1}(y) = (y - 1)/\rho$ maps $(0, \infty)$ onto $G$; moreover, $\eta^{-1}$ is super-additive on $(1, \infty)$, i.e. for $x, y > 1 = 1_{\mathbb{R}^*}$,
\[ \eta^{-1}(x) + \eta^{-1}(y) \leq \eta^{-1}(xy), \]
as
\[ 0 \leq \rho^2 \eta^{-1}(x) \eta^{-1}(y) = (xy - 1) - (x - 1) - (y - 1) = \rho \eta^{-1}(xy) - \rho \eta^{-1}(x) - \rho \eta^{-1}(y). \]
It is also super-additive on $(0, 1)$.

Below we list further useful arithmetic facts including the iterates $a_{\varphi x}^{n+1} = a_{\varphi x}^n \circ \varphi, x$ a with $a_{\varphi x}^1 = a$ (cf. Appendix).

**Proposition 3** (Arithmetic of Popa operations).

i) \[ 1_{\varphi x} = 0 \quad ; \quad a \circ_\varphi a_{\varphi x}^{-1} = 0 \quad \text{for} \quad a_{\varphi x}^{-1} := (-a)/\eta_x(c(a)), \]
ii) \[ x \circ_\varphi (b \circ_\varphi x a) = y \circ_\varphi a, \quad \text{for} \quad y := x \circ_\varphi b, \]
iii) \[ x \circ_\varphi (b \circ_\varphi x a) = y \circ_\varphi b, \quad \text{for} \quad y := x \circ_\varphi b, \]
iv) \[ x = y \circ_\varphi b_{\varphi x}^{-1} \quad \text{for} \quad y := x \circ_\varphi b, \]
v) \[ \eta_x(a_{\varphi x}^m) = \prod_{i=1}^m \eta_x(a), \quad \text{for the iterates} \quad a_{\varphi x}^n \quad \text{and} \quad y_i = x \circ a_{\varphi x}^{m-i}, \quad (i = 1, \ldots, m). \]

**Proof.** (i)
\[ a \circ_\varphi a_{\varphi x}^{-1} = a + a_{\varphi x}^{-1} \eta_x(c(a)) = 0. \]

(ii) For $y = x \circ_\varphi b$,
\[ x \circ_\varphi (b \circ_\varphi x a) = x \circ_\varphi (b + a_{\varphi x}^{-1}(b)) = x + b(x) + a_{\varphi x}(x + b_{\varphi x}(x)) = y \circ_\varphi a. \]

(iii) Follows from (ii) by writing $a_{\varphi x}(b)/\eta_x(b)$ in place of $a$, as then $b \circ_\varphi x a_{\varphi x}(b)/\eta_x(b) = b \circ_\varphi a_x$.

(iv) For $y = x \circ_\varphi b$, using $b_{\varphi x}^{-1} = -b/\eta_x(b)$ from (i),
\[ x = y - b_{\varphi x}(x) = y - [b_{\varphi x}(x)/\eta_x(b)]y = y \circ_\varphi t_{\varphi x}^{-1}. \]
(v) For \( y_i = x \circ a_{\varphi x}^{m-i} (i = 1, \ldots, m) \) and \( a_{\varphi x}^{n+1} = a_{\varphi x}^n \circ \varphi x, \) by (ii),
\[
x \circ \varphi a_{\varphi x}^{n+1} = x \circ \varphi (a_{\varphi x}^n \circ \varphi x) = (x \circ \varphi a_{\varphi x}^n) \circ \varphi x.
\]
So
\[
\eta_x(a_{\varphi x}^{m}) = \frac{\varphi(x \circ \varphi a_{\varphi x}^{m})}{\varphi(x)} = \frac{\varphi((x \circ \varphi a_{\varphi x}^{m-1}) \circ \varphi x)}{\varphi(x \circ \varphi a_{\varphi x}^{m-1})} \frac{\varphi(x \circ \varphi a_{\varphi x}^{m-1})}{\varphi(x)} = \ldots = \frac{\varphi((x \circ \varphi a_{\varphi x}^{m-2}) \circ \varphi x)}{\varphi(x \circ \varphi a_{\varphi x}^{m-2})} \frac{\varphi(x \circ \varphi a_{\varphi x}^{m-2})}{\varphi(x)}. \]
\[ \Box \]

4 Extension to Beurling’s Tauberian Theorem

Theorem 1 below extends one proved by Beurling in lectures in 1957; it was published in the papers of Moh [Moh] and Peterson [Pet]; [BinG] extended Beurling’s result by replacing the Lebesgue integrator \( H(y)dy \) below by a suitable Lebesgue-Stieltjes integrator \( dU(y) \), and demanding more of the Wiener kernel (than just non-vanishing of its Fourier transform), and gave a corollary for Beurling moving averages.

Here we extend the class of Beurling convolutions applied in the other term of the integrand, replacing \( \varphi \in BSV \) by \( \varphi \in SE \), so widening the application to moving averages, as we note below. With the following ‘Beurling notation’ for Lebesgue and Stieltjes integrators
\[
F \ast \varphi H(x) : = \int F\left(\frac{x-u}{\varphi(x)}\right)H(u)\frac{du}{\varphi(x)} = \int F(t)H(x+t\varphi(x))dt,
\]
\[
F \ast \varphi dU(x) : = \int F\left(\frac{x-t}{\varphi(x)}\right)dU(t)\frac{dt}{\varphi(x)} = \int F(t)dU(x+t\varphi(x))dt,
\]
reducing for \( \varphi \equiv 1 \) to their classical counterparts
\[
F \ast H(x) = \int F(x-t)H(t)dt, \quad F \ast dU(x) = \int F(x-t)dU(t),
\]
we recall Wiener’s theorem for the Lebesgue and the Lebesgue-Stieltjes integrals. The latter uses the class \( M \) of continuous functions (see Widder [Wid, V.12]; cf. [Wien, II.10]) with norm:
\[
||f|| := \sup_{y \in \mathbb{R}} \sup_{n \in \mathbb{Z}} \sup_{x \in [0,1]} |f(x + y + n)| < \infty,
\]
and places a uniform bounded-variation restriction on the integrator \( U \) as follows. Denote by \( |\mu_x| \) the usual norm of the charge (signed measure) generated from the function \( y \rightarrow U_x(x \circ \varphi y) / \varphi(x) \); then there should exists \( \delta > 0 \) and \( M < \infty \) with
\[
\sup_{x,y \in \mathbb{R}} |\mu_x||(I^+_y(y))| \leq M, \quad (BV)
\]
where $I_\delta(y) := [y, y+\delta)$. It will be convenient to refer to the following conditions as $x \to \infty$ with or without the subscript $\varphi$ (the latter when $\varphi \equiv 1$):

$$K * \varphi H(x) \to c \int K(y)H(y)dy \quad \text{and} \quad K * \varphi dU(x) \to c \int K(y)dy.$$  

**Theorem W (Wiener’s Tauberian Theorem).** For $K \in L^1(\mathbb{R})$ (resp. $K \in M$) with $\hat{K}$ non-zero on $\mathbb{R}$, if $H$ is bounded (resp. $H \in M$), and $(K * \varphi H)$, resp. $(K * \varphi dU)$, holds, then for all $F \in L^1(\mathbb{R})$ (resp. $F \in M$),

$$F * H(x), \text{ resp. } F * \varphi H(x) \to c \int F(t)dt \quad (x \to \infty).$$

**Theorem B (Beurling’s Tauberian theorem).** For $K \in L^1(\mathbb{R})$ with $\hat{K}$ non-zero on $\mathbb{R}$, and $\varphi$ ‘Beurling slowly varying’:

$$\varphi(x + t\varphi(x))/\varphi(x) \to 1, \quad (x \to \infty) \quad (t \in \mathbb{R}) \quad (BSV)$$

if $H$ is bounded, and $(K * \varphi H)$ holds, then for all $F \in L^1(\mathbb{R})$

$$F * \varphi H(x) \to c \int F(y)dy \quad (x \to \infty).$$

We recommend the much later, slick, and elegant proof in [Kor, IV.11].

**Theorem BG1 (LS-Extension to Beurling’s Tauberian theorem, [BinG, Th. 8]).** If $\varphi \in BSV$, $K \in M$ with $\hat{K}$ non-zero on $\mathbb{R}$, $U$ satisfies $(BV)$ and $(K * \varphi dU)$ holds

– then for all $G \in M$,

$$G * \varphi dU(x) \to c \int G(y)dy \quad (x \to \infty).$$

We show how to amend the [BinG] proof of Th. BG (similar in essence to that cited above in [Kor, IV.11]) to obtain the following.

**Theorem 1 (Extension to Beurling’s Tauberian theorem).** If $\varphi \in SE$, i.e. locally uniformly in $t$

$$\varphi(x + t\varphi(x))/\varphi(x) \to \eta(t) \in GS, \quad (x \to \infty) \quad (t \in \mathbb{R}) \quad (SE)$$

$K \in L^1(\mathbb{R})$ (resp. $K \in M$ ) with $\hat{K}$ non-zero on $\mathbb{R}$, $H$ is bounded (resp.$U$ satisfies $(BV)$) and $(K * \varphi H)$, resp. $(K * \varphi dU)$, holds

– then for all $G \in L^1(\mathbb{R})$ (resp. $G \in M$)

$$G * \varphi H(x) \to c \int G(y)dy \quad \text{resp. } G * \varphi dU(x) \to c \int G(y)dy \quad (x \to \infty).$$
Proof. In view of the amendments needed, it suffices to consider the Lebesgue-Stieltjes case. For fixed $a$ and with $K$ as in the Theorem, set $K_a(s) := K(s - a)$, and take

$$t := (s - a)/\eta_x(a)$$

$dt = ds/\eta_x(a)$ and $s = a + t\eta_x(a) = a \circ \varphi, x t$.

Then for $y = x + b\varphi(x)$, by Prop. 3(ii), $x \circ \varphi(a \circ \varphi, x t) = y \circ \varphi a$ and so

$$K_a(s)U(x \circ \varphi s) = K(t\eta_x(a))U(x \circ \varphi(a \circ \varphi, x t)) = K(t\eta_x(a))U(y \circ \varphi t).$$

So, as in [BinG], for $K$ continuous ($K \in \mathcal{M}$),

$$\int K_a(s)dU(x \circ \varphi s) = \eta_x(a) \int K(t\eta_x(a))dU(y \circ \varphi t) \rightarrow A \int K(t\eta(a))\eta(a)dt$$

$$= A \int K(u)du, \text{ for } u := t\eta(a).$$

Now continue with the proof verbatim as in [BinG]. □

Corollary 3 ([BinG, §5 Cor. 2] for $\varphi \in SN$). For $\varphi \in SE$, if $U$ is non-decreasing and for some $\delta > 0$

$$\sup_{x,y \in \mathbb{R}} \left[U_x(x \circ \varphi (y + \delta)) - U_x(x \circ \varphi (y))\right]/\varphi(x) < \infty$$

then ($K * \varphi dU$) holds for some $c$ and Wiener kernel $K \in \mathcal{M}$ iff for some $c_U$ either of the following holds:

$$(\Delta^f U/\varphi)(x) \equiv \left[U(x \circ \varphi t) - U(x)\right]/\varphi(x) \rightarrow c_U t \quad (x \rightarrow \infty) \quad (t > 0)$$

$$(\Delta^\varphi U/\varphi)(x) \rightarrow c_U t \quad (x \rightarrow \infty) \quad \text{for two incommensurable } t.$$

Proof. Repeat verbatim the proof in [BinG, §5 Cor 2], using $H(x) = t^{-1}1_{[0, t]}(x)$, with $1_{[0, t]}$ the indicator function of the interval $[0, t]$. □

5 Uniformity, semicontinuity

To motivate our results below on limsup convergence type, we first recall that $f_n \rightarrow f$ uniformly near $t$ if for every $\varepsilon > 0$ there is $\delta > 0$ and $m \in \mathbb{N}$ such that

$$f(t) - \varepsilon < f_n(s) < f(t) + \varepsilon \text{ for } n > m \text{ and } s \in I_\delta(t),$$

where $I_\delta(t) := (t - \delta, t + \delta)$. This may be equivalently stated in limsup language, as follows, bringing to the fore the underlying uniform upper and lower semicontinuity.
**Proposition 4 (Uniform semicontinuity).** If $f_n \to f$ pointwise, then $f_n \to f$ converges locally uniformly near $t$ iff

$$f(t) = \lim_{\delta \downarrow 0} \limsup_n \{ f_n(s) : s \in I_\delta(t) \} = \lim_{\delta \downarrow 0} \liminf_n \{ f_n(s) : s \in I_\delta(t) \}.$$

Putting $I^+_\delta(t) := [t, t + \delta)$, we may now consider the one-sided limsup-sup condition:

$$f(t) = f_+(t) \text{ with } f_+(t) := \lim_{\delta \downarrow 0} \limsup_n \{ f_n(s) : s \in I^+_\delta(t) \}. \tag{1}$$

The next result is akin to the Dini/Pólya-Szegö monotone convergence theorems (respectively [Rud1, 7.13], for monotone convergence of continuous functions, and [PolS], Vol. 1 p.63, 225, Problems II 126, 127, or Boas [Boa], §17, p. 104-5, when the functions are monotone); here we start with one-sided assumptions on the domain and range, and conclude by improving to a two-sided condition.

**Proposition 5 (Uniform Upper semicontinuity).** If $f_n$ converges pointwise to an upper semi-continuous limit $f$ satisfying (1) quasi everywhere in the domain, then quasi everywhere $f$ is uniformly upper semicontinuous:

$$f(t) = \lim_{\delta \downarrow 0} \limsup_n \{ f_n(s) : s \in I_\delta(t) \}.$$

**Proof.** Put

$$G(\varepsilon) := \bigcup_{q \in \mathbb{Q}} \bigcup_{\delta > 0} \bigcup_{m \in \mathbb{N}} \{ I^+_\delta(q) : f_n(s) < f(q) + \varepsilon \ (\forall n > m \ & \ \forall s \in I^+_\delta(q)) \},$$

which is open. It is also dense: for rational $q$ there exists $\delta > 0$ and $N_q$ such that

$$f_n(s) < f(q) + \varepsilon \text{ for } n > N_q \text{ and } s \in I^+_\delta(q);$$

so $q \in G(\varepsilon)$. Consider $T := \bigcap_{\varepsilon \in \mathbb{Q}} G(\varepsilon)$; then, by Baire’s Theorem, $T$ is comeagre, so we may assume w.l.o.g. that the one-sided uniformity condition (1) holds on $T$ and that $f$ is upper semi-continuous on $T$.

Given $\varepsilon > 0$ and $t \in T$, by semi-continuity of $f$, pick $\rho > 0$ such that $f(u) < f(t) + \varepsilon$ for all $u \in I_\rho(t) \cap T$. Now, as $t \in G(\varepsilon)$, which is dense open, we may pick $q \in I_\rho(t)$, $\delta > 0$, $m \in \mathbb{N}$ such that $t \in I^+_\delta(q)$ and

$$f_n(s) < f(q) + \varepsilon \text{ for } n > m \text{ and } s \in I^+_\delta(q).$$

Now choose $\eta > 0$ such that $I_\eta(t) \subseteq I_\delta(q)$. Then for $n > m$ and $s \in I_\eta(t)$

$$f_n(s) < f(q) + \varepsilon < f(t) + 2\varepsilon.$$

As $\varepsilon > 0$ was arbitrary,

$$f(t) = \lim_{\delta \downarrow 0} \limsup_n \{ f_n(s) : s \in I_\delta(t) \} \text{ for } t \in T. \quad \Box$$
Definitions. Recalling that
\[ \Delta^\varphi_t h(x) := h(x + t\varphi(x)) - h(x), \]
put
\[ A = A^\varphi := \{ t : \Delta^\varphi_t h \text{ converges to a finite limit} \}, \]
\[ \mathcal{A}_u = A^\varphi_u := \{ t : \Delta^\varphi_t h \text{ converges to a finite limit locally uniformly at } t \}. \]
So \( 0 \in A^\varphi \), but we cannot yet assume either that \( A^\varphi \) is a subgroup, or that \( 0 \in \mathcal{A}_u \), a critical point in Proposition 7 below. In the Karamata case \( \varphi \equiv 1 \), \( A^\varphi = A^1 \) is indeed a subgroup (see [BinO14, Prop. 1]).

Proposition 6. \( h(x + t\varphi(x)) - h(x) \) converges locally (right-sidedly) uniformly to \( K(t) \) at \( t \neq 0 \), iff for each divergent \( x_n \) and any \( c_n \to 1 \) \( (c_n \downarrow 1) \)
\[ h(x_n \circ \varphi c_n t) - h(x_n) \to K(t), \]
in which case
\[ K(t) := \sup_{c_n \downarrow 1, x_n \to \infty} \{ \limsup h(x_n \circ \varphi c_n t) - h(x_n) \}. \]

Proposition 7. For \( \varphi \in SE \), \( \mathcal{A}_u \) is a subgroup of \( G \) iff \( 0 \in \mathcal{A}_u \), in which case, \( K : (\mathcal{A}_u, \circ) \to (\mathbb{R}, +) \), defined by \( K \) above, is a homomorphism.

Proof. We show that \( \mathcal{A}_u \) is closed under \( \circ \) and inverses, so it is a subgroup of \( G \) iff \( 1_G \in \mathcal{A}_u \). For \( u, v \in \mathcal{A}_u \), since \( \eta_u(v) = \eta(v) \),
\[ u_v := u\eta(v)/\eta_u(v) \to u, \]
so with \( y = x \circ \varphi v \), since by Prop. 3(iii) \( x \circ \varphi (v \circ \eta u) = y \circ \varphi y u_v \),
\[ h(x \circ \varphi (v \circ \eta u)) - h(x) = [h(y \circ \varphi y u_v) - h(y)] + [h(x \circ \varphi v) - h(x)] \to K(u) + K(v), \]
i.e.
\[ K(v \circ \eta u) = \lim [h(x \circ \varphi (v \circ \eta u)) - h(x)] = K(u) + K(v). \]
As the convergence at \( u, v \) on the right occurs locally uniformly, this is locally uniform at \( v \circ u \), using Prop. 6.
For non-zero \( t \in \mathbb{N} \), this time put \( y := x \circ t \); then, by Prop. 3(iv), \( x = y \circ t^{-1} \), so
\[
h(y \circ t^{-1}) - h(y) = [h(x) - h(y)] = -[h(x \circ t) - h(x)] \to -K(t).
\]
So, since \( t^{-1} = -t/\eta(x)(t) \to -t/\eta(t) \),
\[
K(t^{-1}_o) = K(-t/\eta(t)) = \lim[h(y \circ t^{-1}) - h(y)] = -K(t).
\]
That is \( t^{-1}_o \in \mathbb{N} \) (and \( K(t^{-1}_o) = -K(t) \)); again this is locally uniform at \( t \neq 0 \), using Prop. 6. \( \square \)

The following result extends the Uniformity Lemma of [BinO12, Lemma 3]. Although the proof parallels the original, the current one-sided context demands the closer scrutiny offered here. To describe more accurately the convergence in \((K)\) above, we write
\[
\Delta^\eta h(x) \to K_+(t) \text{ if loc. uniform at } t \text{ on the right,}
\]
\[
\Delta^\eta h(x) \to K_-(t) \text{ if loc. uniform at } t \text{ on the left,}
\]
\[
\Delta^\eta h(x) \to K_\pm(t) \text{ if loc. uniform at } t.
\]

Lemma 0. (i) For \( \varphi \in \mathbb{S} \): (a) if the convergence in \((K)\) is uniform (resp. right-sidedly uniform) at \( t = 0 \), then it is uniform (resp. right-sidedly uniform) everywhere in \( \mathbb{N} \cap \mathbb{G}_+ \) and for \( u \in \mathbb{N} \cap \mathbb{G}_+ \)
\[
K_+(u) = K(u) + K_+(0);
\]
(b) if the convergence in \((K)\) is uniform at \( t = u \in \mathbb{N} \cap (\mathbb{N})^{-1} \cap \mathbb{G}_+ \), then it is uniform at \( t = 0 \)
\[
K_+(0) = K_+(u) + K(u^{-1});
\]
(ii) if \( \rho = 0 \) and \( \varphi \in \mathbb{S} \) is monotonic increasing, and the convergence in \((K)\) is right-sidedly uniform at \( t = u \in \mathbb{N} \cap \mathbb{G}_+ \), then it is right-sidedly uniform at \( t = 0 \)
\[
K_+(0) = K_+(u) + K(u^{-1}).
\]

Proof. (i) Suppose \((K)\) holds locally right-sidedly uniformly (uniformly) at \( t = 0 \). Let \( u \in \mathbb{G}_+ \cap \mathbb{A}^\varphi \) and \( z_n \downarrow 0 \) (resp. \( z_n \to 0 \)). For \( x_n \) divergent \((x_n \to \infty)\), \( y_n := x_n \circ u = x_n(1 + u\varphi(x_n)/x_n) \) is divergent and
\[
h(x_n \circ (u + z_n)) - h(x_n) = h(x_n \circ u) - h(x_n) + h(y_n \circ u \circ z_n) - h(y_n)
\]
where
\[
\eta_{x(n)}(u) \to \eta(u) > 0.
\]
Since \( z_n/\eta_{x(n)}(u) \downarrow 0 \) (resp. \( z_n/\eta_{x(n)}(u) \to 0 \)), from \( h(x_n \circ u) - h(x_n) \to K(u) \), and the assumed uniform behaviour at the origin, there is right-sidedly uniform (uniform) behaviour at \( u \).
(ii) Conversely, suppose uniformity holds at \( u \in A^\varphi \cap (A^\varphi)^{-1} \cap \mathbb{G}^+ \cap (0, \infty) \); then \( v := u_0^{-1} = -u/\eta(u) \in A^\varphi \cap \mathbb{G}^+ \cap (-\rho^{-1}, 0) \). Let \( z_n \to 0 \); then \( z'_n := z_n/\eta_{x(n)}(v) \to 0 \), as \( \eta_{x(n)}(v) \to \eta(v) \). Also \( (-v)/\eta_{x(n)}(v) \to (-v)/\eta(v) = v_0^{-1} = u \), so
\[
\lim(-v + z_n)/\eta_{x(n)}(v) = u + 0.
\]
Taking \( y_n := x_n \circ \varphi v \) (< \( x_n \) for \( v < 0 \)),
\[
x_n \circ \varphi z_n = (x_n \circ \varphi v) \circ \varphi (-v + z_n)/\eta_{x(n)}(v)
\]
and
\[
h(x_n \circ \varphi z_n) - h(x_n) = h(y_n + \circ \varphi (-v + z_n)/\eta_{x(n)}(v)) - h(y_n) + h(x_n \circ \varphi v) - h(x_n)
\]
\[
\to K(u) + K(v) = K(u) + K(u_0^{-1}),
\]
where the convergence on the right is uniform in the first term and pointwise in the second term.

(iii) When \( \varphi \in SN \) is monotone, the argument in (ii) above may be amended to deal with right-sided convergence, as \( 1/\eta_{x(n)}(v) = \varphi(x_n)/\varphi(y_n) \geq 1 \) (for \( v < 0 \)) and so \( 1/\eta_{x(n)}(v) \downarrow 1 \), as \( \varphi = 0 \). Also \( z'_n = z_n \), so if \( z_n \downarrow 0 \), then
\[
z_n \varphi(x_n)/\varphi(y_n) \downarrow 0 \quad \text{since} \quad d_n \uparrow 0
\]
and
\[
(-v + z_n)/\eta_{x(n)}(v) \downarrow u,
\]
as \( (-v)/\eta_{x(n)}(v) \downarrow (-v) = u > 0 \). From here the argument is valid when ‘uniform’ is replaced by ‘right-sidedly uniform’. □

Remark. Write \( \varphi \in SE^\pm \) (for \( u > 0 \)) according as
\[
\varphi(x + u\varphi(x))/\varphi(x) \downarrow \eta(u) \quad \text{or} \quad \varphi(x + u\varphi(x))/\varphi(x) \uparrow \eta(u).
\]
So if \( \varphi \in SN \) and \( \varphi \) is increasing, then \( \varphi \in SN^- \), since \( \varphi(x + u\varphi(x)) > \varphi(x) \) for \( u > 0 \) so
\[
\varphi(x + u\varphi(x))/\varphi(x) \downarrow 1.
\]
This was used in (iii) above, and extends to \( SE \). Of course \( \eta \in SE^+ \cap SE^- \).

The next result leads from a one-sided condition to a two-sided conclusion. This is the prototype of further such results, which will be useful in later sections.

Theorem 2. If the pointwise convergence (K) holds with the limit function \( K \) upper semicontinuous on a co-meagre set, and the one-sided condition
\[
K(t) = \lim \limsup_{s \uparrow 0} \sup_{x \in I_0^+(t)} \{h(x + s\varphi(x)) - h(x) : s \in I_0^+(t)\} \quad (UNIF^+)
\]
holds at the origin – then two-sided limsup convergence holds everywhere:
\[
A^\varphi = A_u = \mathbb{R}.
\]
Proof. The pointwise convergence assumption says $\mathbb{A}^\varphi$ is co-meagre; w.l.o.g. $\mathbb{A}^\varphi = (\mathbb{A}^\varphi)^{-1}$, otherwise work below with the co-meagre set $\mathbb{A}^\varphi \cap (\mathbb{A}^\varphi)^{-1}$. Take $f(t) := K(t)$; then $f_n(t) := h(x_n \circ \varphi t) - h(x_n) \to f(t)$ holds pointwise quasi everywhere on $\mathbb{A}^\varphi$. Since $(UNIF^+)$ holds at $t = 0$, by Lemma 0(i)(a), it holds everywhere in $\mathbb{A}^\varphi$ and so quasi everywhere. By Proposition 5, its two-sided limsup version holds quasi everywhere, and so at some point $u \in \mathbb{A}^\varphi \cap (\mathbb{A}^\varphi)^{-1} \cap \mathbb{G}^+_p$. Then by Lemma 0(i)(b) the two-sided limsup version holds at 0, and so by Lemma 0(i)(a) it holds everywhere in $\mathbb{A}^\varphi$. It now follows that 0 $\in \mathbb{A}^\varphi = \mathbb{A}_u$ and so $\mathbb{A}_u$ is a co-meagre subgroup of $\mathbb{G}$; so, by the Steinhaus subgroup theorem (see [BinO11]), which applies here by Prop. 6, $\mathbb{A}_u = \mathbb{G} = \mathbb{R}$. □

6 Dichotomy

We continue with the setting of §3, but here we assume less about $\mathbb{A}^\varphi$ – in place of being co-meagre we ask that it contains a Baire subset $S$ that is non-meagre. This is a local version of the situation in §5 in that (i) $S$ is locally co-meagre and (ii) $\mathbb{A}^\varphi$ is non-meagre and contains a Baire subset to witness this. For general $h$ and $\varphi$ we cannot assume this happens. However, under certain axioms of set-theory this will be guaranteed; see §11. Now $\langle S \rangle$, the additive subgroup generated by $S$, will of course be $\mathbb{R}$, again by the Steinhaus Subgroup Theorem, as in Theorem 2. So our aim here is to verify that $\mathbb{A}^\varphi$ is a subgroup by checking that $\mathbb{R} = \langle S \rangle \subseteq \mathbb{A}_u \subseteq \mathbb{A}^\varphi$.

Theorem 3. For $\varphi \in SE$, if $\mathbb{A}^\varphi$ contains a non-meagre Baire subset, then $\mathbb{A}^\varphi = \mathbb{R}$ and $K$ is linear: $K(u) = cu$.

Given our opening remarks, this reads as an extension of the Fréchet-Banach Theorem on the continuity of Baire/measurable additive functions – for background see [BinO11]. The proof parallels Prop. 1 of [BinO14], extending the cited result from the Karamata to the Beurling setting, but here we need the Baire property to employ uniformity arguments needed in the current context.

Proposition 8 extends Theorem 6 (UCT) of [BinO12] and is crucial here.

Proposition 8 (Uniformity). Suppose $S \subseteq \mathbb{A}^\varphi$ for some Baire non-meagre $S$. Then the convergence in $(K)$ is uniform near $u = 0$ and so also near $u = t$ for $t \in S$, i.e. $S \subseteq \mathbb{A}_u \subseteq \mathbb{A}^\varphi$.

Proof. For each $n$, define on $\mathbb{R}$ the function $k_n(t) := H(n \circ \varphi t) - H(n)$, which is Baire; for $x \in \mathbb{A}^\varphi$, then

$$K(t) = \lim k_n(t),$$

and so $k = K|S$ is a Baire function with non-meagre domain. Now apply the argument of Theorem 6 of [BinO12] to $S$ and $k$ as defined here (so that Baire’s Continuity Theorem applies to the Baire function $k$), giving uniform convergence near $u = 0$, so uniform convergence near any $u \in S$, by Lemma 0(i)(a). □
Corollary 4. If $S \subseteq \mathbb{A}^\varphi$ with $S \subseteq \mathbb{R}_+$ Baire and non-meagre, then
(i) $S^{-1}_0 = \{-s/(1 + \rho s) : s \in S\} \subseteq \mathbb{A}^\varphi$;
(ii) $S \circ S = \{s + t\eta(s) : s, t \in S\} \subseteq \mathbb{A}^\varphi$.

Proof. (i) As $S^{-1}$ is Baire and non-meagre, Prop. 8 applies and $S^{-1} \subseteq \mathbb{A}_u \subseteq \mathbb{A}^\varphi$.
(ii) By Prop. 2, $S \circ S$ is isomorphic either to $S + S$ (for $\rho = 0$) or to $\eta(S)\eta(S)$ (for $\rho > 0$) and so is Baire and non-meagre, by the Steinhaus Sum-Theorem ([BinO11]); again Prop. 8 applies and $S \circ S \subseteq \mathbb{A}_u \subseteq \mathbb{A}^\varphi$. $\square$

Proof of Theorem 3. Replacing $S$ by $S \cup (S^{-1}_0)$ if necessary, we may assume by Cor. 4 that $S$ is symmetric ($S = S^{-1}_0$), and w.l.o.g. $0 = 1 \in S$, by Prop. 8.
Applying Cor. 4(ii) inductively, we deduce that $S^*_n := \bigcup_n (n) \circ S \subseteq \mathbb{A}^\varphi$,
where $(n) \circ S$ denotes $S \circ \eta \cdots \eta S$ to $n$ terms. So $S^*$ is symmetric, and a semigroup: if $s \in (n) \circ S$ and $s' \in (m) \circ S$, then $s \circ s' \in (n + m) \circ S \subseteq S^*$. So $\mathbb{A}^\varphi$ contains $S^*$. As $0 \in S^*$ (as above), $S^*$ is a subgroup and hence all of $\mathbb{R}$. So $S^* = \mathbb{R} = \mathbb{A}_u = \mathbb{A}^\varphi$. By Prop. 7 $K$ is additive on $\mathbb{R}$, and by Prop. 8 is uniformly continuous at $u = 0$ and, being additive, is linear; see e.g. BGT, [Kuc], [BinO13], [BinO11]. $\square$

7 Quantifier weakening

Here we drop the assumption that $\mathbb{A}^\varphi$ is co-meagre; instead we will impose a density assumption, and employ a subadditivity argument developed in [BinO14]. To motivate this, we recall the following decomposition theorem of a function, with a one-sided finiteness condition, into two parts, one decreasing, one with suitable limiting behaviour.

Theorem BG 2 ([BinG2, Th. 7]). The following are equivalent:
(i) The function $U$ has the decomposition
$$U(x) = V(x) + W(x),$$
where $V$ has linear limiting moving average $K_V$ as in §1, and $W(x)$ is non-increasing;
(ii) the following limit is finite:
$$\lim_{\delta \downarrow 0} \lim_{x} \sup \sup \left\{ \frac{U(x \circ \varphi t) - U(x)}{\delta \varphi(x)} : t \in I^+_\delta(0) \right\} < \infty.$$

Definitions.
$$H_1(t) := \lim_{x} \sup \sup \left\{ h(x \circ \varphi t) - h(x) : t \in I^+_\delta(s) \right\},$$
$$\mathbb{A}^1_u := \{ t : H_1(t) < \infty \}.$$
So $A_n \subseteq A_u$, as $H^+ (t) = K(t)$ on $A_u$. In Theorem 4 below we apply the techniques of [BinO14] and [BinO13]; a first step for this is the following. Here it is again convenient to rely on Prop. 6.

**Proposition 9.** For $\varphi \in SE$ and $\eta = \eta^\varphi$, $H^+$ is subadditive on $A_u^+$:

$$H^+ (s \circ t) \leq H^+ (s) + H^+ (t) \ (s, t \in A_u^+).$$

**Proof.** For $c = \{c_n\} \to 1$ and $x = \{x_n\}$ divergent, put

$$H(t; x, c) := \lim sup h(x_n \circ \varphi c_n t) - h(x_n).$$

As in Prop. 7, for a given $c_n \to 1$ and divergent $x_n$, take $y_n := x_n \circ \varphi c_n s$, $d_n := \varphi(x_n)/\varphi(y_n) \to \eta(s)^{-1}$. Now

$$h(x_n \circ \varphi c_n (s + t\eta(s))) - h(x_n) = h(y_n \circ \varphi d_n t\eta(s)) - h(y_n) + h(x_n \circ \varphi c_n s) - h(x_n),$$

so

$$H(s + t\eta(s); c, x) \leq H(t; d, y) + H(s; c, y).$$

Now take suprema. □

Our next result clarifies the role of the Heiberg-Seneta condition, for which see BGT §3.2.1 and [BinO14].

**Proposition 10.** For $\varphi \in SE$, the following are equivalent:

(i) $0 \in A_u$ (i.e. $A_u \neq \emptyset$ and so a subgroup);

(ii) $\lim_{x \to -\infty} [h(x + u\varphi(x)) - h(x)] = 0$ uniformly as $u \to 0$;

(iii) $H^+(t)$ satisfies the two-sided Heiberg-Seneta condition:

$$\lim_{u \to 0} \sup_u H^+(u) < 0. \quad (HS_u (H^+))$$

**Proof.** It is immediate that (i) and (ii) are equivalent. As to their equivalence with the Heiberg-Seneta condition, $HS_u (H^+)$ requires that for each $\varepsilon > 0$ there is $\delta > 0$ such that for $0 < |t| < \delta$ one has for all large enough $n$

$$h(x_n + c_n t \varphi(x_n)) - h(x_n) < \varepsilon,$$

for every $c_n \to 1$ and $x_n \to \infty$. Equivalently (halving $\delta$ if necessary), for $|u| < \delta$ and large enough $n$

$$h(x_n + u \varphi(x_n)) - h(x_n) < \varepsilon.$$

Take $y_n = x_n \circ \varphi u$; then $x_n = y_n \circ \varphi (-u)/\eta(x_n)(u)$ and $u/\eta(x_n)(u) \to u$. So

$$-\varepsilon < h(y_n \circ \varphi (-u)/\eta(x_n)(u)) - h(y_n).$$

So the Heiberg-Seneta condition also requires

$$\lim [h(x + u \varphi(x)) - h(x)] = 0 \text{ uniformly as } u \to 0 : \ 0 \in A_u. \quad \square$$
The final result of this section is the Beurling version of a theorem proved in the Karamata framework of [BinO14]. However, uniformity plays no role there, whereas here it is critical. The result shows that weakening the quantifier in the definition of additivity to range over a dense subgroup (rather than $\mathbb{R}$), determined by locally uniform limits, yields linearity of $H^1$.

**Theorem 4 (Quantifier Weakening from Uniformity).** If $A_u$ is dense – equivalently, $H^1$ is additive on $A_u$ – then $A_u = \mathbb{R}$ and $H^1$ is linear:

$$H^1(t) = ct.$$  

**Proof.** Since as $H^1(t) = K(t)$ on $A_u$, this follows from Propositions 8, 9 and 10 by Theorem 1 of [BinO14]. □

## 8 Representation

We begin by identifying the limiting moving average $K_F$ of §1.

**Lemma 1.** If $\varphi \in SE$ is increasing and the following limit exists for $F : \mathbb{R} \to \mathbb{R}$:

$$K_F(u) := \lim_{x \to \infty} \frac{F(x \circ \varphi u) - F(x)}{\varphi(x)}, \quad (u \in \mathbb{R})$$

then $K_F$ as above satisfies for $\eta = \eta^\varphi$

$$K_F(u \circ v) = K_F(u) + K_F(v)\eta(u);$$

if $F$ is Baire/measurable, then $K_F$ and $\eta^\varphi$ are of the form

$$K_F(u) = c_F u, \quad \eta^\varphi(u) = 1 + \rho\varphi u.$$

**Proof.** Write $y = x + u\varphi(x)$; then $\varphi(y)/\varphi(x) \to \eta^\varphi(u)$. Now

$$\frac{F(x \circ \varphi [u + v]) - F(x)}{\varphi(x)} = \frac{F(y \circ \varphi [v\varphi(x)/\varphi(y)]) - F(y) \varphi(y)}{\varphi(y)} \varphi(x) + \frac{F(x \circ \varphi u) - F(x)}{\varphi(x)}.$$

Write $w := v/\eta^\varphi(u)$; then, taking limits above, gives

$$K_F(u + w\eta^\varphi(u)) = K_F(w)\eta^\varphi(u) + K(u).$$

Assuming $F$ is Baire/measurable, $K_F(t) = \lim_{n \to \infty}[(F(u \circ \varphi u) - F(x))/\varphi(n)]$ is Baire/measurable (as in Prop. 8). By [BinO13, Th. 9.10] $K_F(x) = c_F H_0(x)$, where $H_0(x) := x$. So $K_F(u) = c_F u$, for some $c_F$. □
The result above formally extends to the Beurling framework and to the class $SE$ the notion of $\Pi_\varphi$-class, due to Bojanić-Karamata/de Haan, for which see BGT Ch. 3, since just as there

\[(i) \quad \frac{F(x \circ \varphi u) - F(x)}{\varphi(x)} \sim c_F H_0(u) : \quad (ii) \quad \frac{F(x \circ \varphi u) - F(x)}{u \varphi(x)} \to c_F. \quad (\Pi_\varphi)\]

**Definition.** Say that $F$ is of Beurling $\Pi_\varphi$-class with $\varphi$-index $c$ (cf. BGT Ch. 3) if the convergence in $(\Pi_\varphi(ii))$ is locally uniform in $u$.

This should be compared with Theorem BG 2 in §7. We now use a Goldie-type argument (see [BinO13]) to establish the representation below for the class $\Pi_\varphi$.

**Theorem 5 (Representation for Beurling $\Pi_\varphi$-class with $\varphi$-index $c$).** For $F$ Baire/measurable, $F$ is of additive Beurling $\Pi_\varphi$-class with $\varphi$-index $c$ iff

\[F(x) = b + cx + \int_1^x e(t)dt, \quad b \in \mathbb{R} \text{ and } e \to 0.\]

**Proof.** As above, by the $\lambda$-UCT of [Ost3, Th. 1], there exists $X$ such that for all $x \geq X$ and all $u$ with $|u| \leq 1$

\[\frac{F(x \circ \varphi u) - F(x)}{u \varphi(x)} = c + \varepsilon(x; u),\]

with

\[\varepsilon(x; u) \to 0 \text{ uniformly for } |u| \leq 1 \text{ as } x \to \infty.\]

Put

\[e(x) = \sup\{\varepsilon(x, u) : |u| \leq 1\},\]

then $e(x) \to 0$ as $x \to \infty$.

Using a Beck sequence ([BinO13, §3]; cf. Bloom [Blo], BGT Lemma 2.11.2) starting at $X$ and ending at $x(u) \leq x$ with $x \leq x(u) \circ \varphi u$ (so $x_{n+1} = x_n \circ \varphi u$) yields

\[F(x(u)) - F(X) = \sum F(x_{n+1}) - F(x_n) = \sum (c + \varepsilon(x_n; u))u \varphi(x_n)\]

\[= \sum (c + \varepsilon(x_n; u))(x_n + u \varphi(x_n) - x_n)\]

\[= c \sum (x_{n+1} - x_n) + \sum \varepsilon(x_n; u)u \varphi(x_n)\]

\[= c(x(u) - X) + \sum \varepsilon(x_n; u)u \varphi(x_n).\]

Since $F$ is Baire/measurable we may restrict attention to points $x$ where $F$ is continuous. Note that $u \varphi(x_n) \leq u \varphi(x) \to 0$ as $u \to 0$, so $x(u) \to x$; taking limsup as $u \to 0$,

\[F(x) = F(X) + c(x - X) + \int_1^x e(t)dt,

22
with \( e(x) \to 0 \), as above. Now
\[
\frac{F(x + u\varphi(x)) - F(x)}{u\varphi(x)} = c + \frac{1}{u\varphi(x)} \int_x^{x+u\varphi(x)} c(t)dt \to c.
\]
So \( F \) is Beurling \( \Pi_\varphi \)-class with \( \varphi \)-index \( c \) iff it has the representation stated. \( \square \)

We note also a generalization of Prop. 9 and Lemma 1, for which we need notation (similar to that in §7) analogous to the Karamata \( \Omega_F \) (of §1).

Definitions.
\[
\Omega^+_\eta(t) := \lim_{\delta \to 0} \limsup_{x \to t} \sup \{ h(x \circ \varphi) - h(x) : s \in I^+_\delta(t) \},
\]
\[
\mathbb{A}^+_{11} := \{ t : \Omega^+_\eta(t) < \infty \}.
\]

**Proposition 9'**. For \( \varphi \in SE \) and \( \eta = \eta^\varphi \), \( \Omega^+_\eta \) is \( \eta \)-subadditive on \( \mathbb{A}^+_{11} \):
\[
\Omega^+_\eta(s \circ t) \leq \Omega^+_\eta(s) + \Omega^+_\eta(t), \quad (s, t \in \mathbb{A}^+_{11}).
\]

**Proof.** For \( c = \{ c_n \} \to 1 \) and \( x = \{ x_n \} \) divergent, put
\[
\Omega^+_\eta(t; x, c) := \limsup_{n \to \infty} [h(x_n \circ \varphi c_n t) - h(x_n)]/\varphi(x_n).
\]
As in Prop. 7, for a given \( c_n \to 1 \) and divergent \( x_n \), take
\[
y_n = x_n \circ \varphi c_n s, \quad d_n := 1/\eta_{x(n)}(s) \to \eta(s)^{-1}.
\]
Since
\[
[h(x_n \circ \varphi c_n (s \circ \eta t)) - h(x_n)]/\varphi(x_n)
= [h(y_n \circ \varphi d_n t \eta(s)) - h(y_n)]/\varphi(y_n) \cdot \eta_{x(n)}(c_n s) + [h(x_n \circ \varphi c_n s) - h(x_n)]/\varphi(x_n),
\]
\[
\Omega^+_\eta(s + t \eta(s); c, x) \leq \Omega^+_\eta(t; d, y) \eta(s) + \Omega^+_\eta(s; c, y).
\]
Now take suprema. \( \square \)

We note an extension of [BinG3, Th. 1] – cf. the more recent [Bin].

**Theorem BG 3.** If \( \varphi \in SE \) and \( \varphi \uparrow \infty \), then \( U \) has a limiting moving average \( K_U(x) = cu \) iff
\[
\frac{1}{\lambda(x)} \int_0^x U(y)d\lambda(y) \to cu,
\]
where \( \lambda(x) := \varphi(x) \exp \tau_{\varphi}(x) \).

**Corollary 5.** For \( \varphi \in SE \) and \( \varphi \uparrow \infty \), and with \( \lambda \) as previously, if \( F \) is of additive Beurling \( \Pi_\varphi \)-class with \( \varphi \)-index \( c \), then
\[
\frac{1}{\lambda(x)} \int_0^x F(y)d\lambda(y) \to c.
\]
9 Divided difference and double sweep

The concern of previous sections was the asymptotics of differences: $\Delta_t \varphi$ in the Beurling theory, and exceptionally in §8 moving averages $\Delta_t \varphi / \varphi$ in the Beurling version of the Bojanić-Karamata/de Haan theory. Introducing an appropriate general denominator $\psi$ carries the same advantage as in BGT (e.g. 3.13.1) of “double sweep”: capturing the former theory via $\psi \equiv 1$ and the latter via $\psi \equiv \varphi$, embracing both through a common generalization – see Prop. 9′ above for a first hint of such possibilities. The work of this section is mostly to identify how earlier results generalize, much of it focussed on §3, to which we refer for group-theoretic notation; in particular $\mathbb{G}$ denotes an unspecified Popa group, i.e. $\mathbb{G}^\rho$ for some $\rho$.

Let $\varphi \in SE$; fix a $\varphi$-regularly varying $\psi$ with $\varphi$-index $\gamma$ and limit function $g$, i.e.

$$\psi(x + t\varphi(x))/\psi(x) \to g(t) \text{ loc. uniformly at } t \quad (t \in \mathbb{R}),$$

and, since $g(t)$ is a homomorphism, it is either $e^{\gamma t}$ $(\rho = 0)$, or else $\eta(t)^\gamma$ (see [Ost4]). Recalling the notation $\Delta_t \varphi$ from §1, we also write $(\Delta_t \varphi / \psi)$ to mean $(\Delta_t \varphi(x)/\psi(x))$. We are concerned below with

$$H^*(t) := \limsup_x |\Delta_t \varphi / \psi|,$$

whenever this exists, and with the nature of the convergence:

$$\lim_x |\Delta_t \varphi / \psi| \to H^*(t).$$

$$(H^*)$$

To specify whenever a case below of convergence arises, we write

$$\lim \limsup \sup_{x \delta} \{\Delta_s \varphi / \psi : s \in I^+_{\delta}(t)\} \to H^+_\delta(t),$$

$$\lim \limsup \sup_{x \delta} \{\Delta_s \varphi / \psi : s \in I^-_{\delta}(t)\} \to H^-_\delta(t),$$

$$\lim \limsup \sup_{x \delta} \{\Delta_s \varphi / \psi : s \in I_0(t)\} \to H^0_\delta(t).$$

We begin with an extension of Lemma 0. The proofs are almost identical – so are omitted.

Lemma 0′. (i) If $\varphi \in SE$ and $(G)$ holds – then:

(a) if the convergence in $(H^*)$ is uniform (resp. right-sidedly uniform) at $t = 0$, then it is uniform (resp. right-sidedly uniform) everywhere in $\mathbb{A}_1^\delta \cap \mathbb{G}_+$ and

$$H^+_\delta(u) \leq H^*(u)g(u) + H^*_\delta(0);$$

(b) if the convergence in $(H^*)$ is uniform at $t = u \in \mathbb{A}_1^\delta \cap \mathbb{G}_+$, then it is uniform at $t = 0$:

$$H^+_\delta(0) \leq H^*_\delta(u)g(u^{-1}) + H^*(u^{-1}).$$
(ii) if $\rho = 0$ and $\varphi \in SN$ is monotonic increasing and the convergence in $(H^*)$ is right-sidedly uniform at $t = u \in \mathbb{A}_\Omega^\dagger \cap G_+$, then it is right-sidedly uniform at $t = 0$:

$$H^*_+(0) \leq H^*_+(u)g(u_{\omega}^{-1}) + H^*(u_{\omega}^{-1}).$$

Recall that the terms below $\Omega^\dagger_h(t), \mathbb{A}_\Omega^\dagger$ were defined in §8.

**Proposition 11.** (i) With $g$ as in (G) above, $g(u \circ v) = g(u)g(v)$, so that

$$K(u \circ v \circ w) \leq K(u)g(v \circ w) + K(v)g(w) + K(w),$$

and furthermore

$$H^*(s \circ t) \leq H^*(s)g(t) + H^*(t), \ (s, t \in \mathbb{A}_\Omega^\dagger).$$

(ii) If

(a) $H^*(t) > -\infty$ for $t$ in a subset $\Sigma$ that is unbounded below;

(b) the Heiberg-Seneta condition $\Omega^\dagger_h(0+) \leq 0$ holds

then $H^*(t)$ is finite and $H^*(0+) = 0$.

Moreover, for $\mathbb{A}_\Omega^\dagger$ is dense,

$$H^*(u \circ v) = K(u)g(v) + H^*(u) \ (u \in \mathbb{A}^\forall, v \in \mathbb{R}).$$

**Proof.** (i) The first assertion is a restatement of the Cauchy exponential equation for $e^{\gamma x}$ when $\rho = 0$ and for $\eta(x)\gamma$ for $\rho > 0$, and so implies the second. As for the third assertion, argue as in Prop. 10 above.

(ii) Part (a) is proved as in [BinO14, Prop. 6], and part (b) as in [BinO14, Prop. 8]—cf. BGT Th. 3.2.5; the latter uses part (i) and the two facts that $g(u \circ v) = g(u)g(v)$ and $g(u) \geq 1$ for $u > 0$.

As a corollary, since $H^*$ is $g$-subadditive, we have the analogue of Th. 1 of [BinO14].

**Theorem 6.** In the setting of Proposition 11, if $\mathbb{A}_\Omega^\dagger$ is dense, then $\mathbb{A}_\Omega^\dagger = \mathbb{R}$ and for some $c, \gamma, \rho \in \mathbb{R}$:

either (i) $\rho = 0$ and $H^*(u) \equiv cH(-\gamma)(u) = c(1 - e^{-\gamma u})/\gamma \ (u \in \mathbb{R})$,

or (ii) $\rho > 0$ and $H^*(u) \equiv [(1 + \rho x)^{\gamma+1} - 1]/\rho(1 + \gamma) \ (u \in \mathbb{R})$.

**Proof.** As in Prop. 6 above, $(\mathbb{A}_\Omega^\dagger, \circ)$ is a subgroup. Now use Prop. 11, the Popa-Javor Theorem, and Th. 3 of [BinO13].
10 Uniform Boundedness Theorem

As above, let \( h \) be Baire and \( \varphi \in SE \) on \( \mathbb{R} \) be positive. Thus for all divergent \( x_n \) (i.e. divergent to \(+\infty\)),

\[
\varphi(x_n \circ \varphi t)/\varphi(x_n) \to \eta \text{ for all } t \in \mathbb{R} \text{ and } \varphi(x) = O(x).
\]

So \( y_n = x_n \circ \varphi t = x_n (1 + t\varphi(x_n)/x_n) \) is divergent if \( x_n \) is.

We work additively, and recall that

\[
H^*(t) = \limsup h(x \circ \varphi t) - h(x), \quad H_*(t) = \liminf h(x \circ \varphi t) - h(x).
\]

If \( x_n \to \infty \) and \( H^*(t) < \infty \), then for all large enough \( n \)

\[
h(x_n \circ \varphi t) - h(x_n) < n.
\]

Likewise if \( H_*(t) > -\infty \), then for all large enough \( n \)

\[
h(x_n) - h(x_n \circ \varphi t) < n.
\]

In the theorem below we need to assume finiteness of both \( H^* \) and \( H_* \); we recall that in the Karamata case, substituting \( y \) for \( u + x \), one has

\[
h^*(u) = \limsup [h(u + x) - h(x)] = -\liminf [h(y - u) - h(y)] = -h_*(-u).
\]

This relationship is used implicitly in the standard development of the Karamata theory – see e.g. [BGT, §2.1]. Theorem 7 below extends [BinO10, Th 8].

**Theorem 7 (Uniform Boundedness Theorem; cf. [Ost1]).** For \( \varphi \in SE \), suppose that \( -\infty < H_*(t) \leq H^*(t) < \infty \) for \( t \in S \) with \( S \) a non-meagre Baire set. Then for compact \( K \subseteq S \)

\[
\limsup_{x \to \infty} \left( \sup_{u \in K} h(x \circ \varphi u) - h(x) \right) < \infty.
\]

**Proof.** Suppose otherwise, and w.l.o.g. that for some \( x_n \to \infty \) and \( z_n \to 0 \)

\[
h(x_n \circ \varphi z_n) - h(x_n) > 3n. \quad (2)
\]

Put \( y_n := x_n \circ \varphi z_n \). As \( \varphi \in SE \),

\[
c_n := \varphi(x_n \circ \varphi z_n)/\varphi(x_n) \to 1.
\]

Write \( \gamma_n(s) := c_n s + z_n \). Put

\[
\begin{align*}
V_n &:= \{ s \in S : h(x_n \circ \varphi s) - h(x_n) < n \}, \quad H_k^+ := \bigcap_{n \geq k} V_n, \\
W_n &:= \{ s \in S : h(y_n) - h(y_n \circ \varphi s) < n \}, \quad H_k^- := \bigcap_{n \geq k} W_n.
\end{align*}
\]

26
These are Baire sets, and since \(-\infty < H_\ast(t) < H_\ast'(t) < \infty\) on \(S\),
\[
S = \bigcup_k H_k^+ = \bigcup_k H_k^-.
\]
(3)
The increasing sequence of sets \(\{H_k^+\}\) covers \(S\). So for some \(k\) the set \(H_k^+\) is non-negligible. As \(H_k^+\) is non-negligible, by \(3\), for some \(l\) the set
\[
B := H_k^+ \cap H_l^-
\]
is also non-negligible. Take \(A := H_k^+\); then \(B \subseteq H_l^-\) and \(B \subseteq A\) with \(A, B\) non-negligible. Applying the Affine Two-sets Lemma [BinO12, Lemma 2] to the maps \(\gamma_m(s) = c_n s + z_n\) with \(c = \lim_n c_n = 1\), there exist \(b \in B\) and an infinite set \(\mathcal{M}\) with
\[
\{c_m b + z_m : m \in \mathcal{M}\} \subseteq A = H_k^+.
\]
That is, as \(B \subseteq H_l^-\), there exist \(t \in H_l^-\) and an infinite \(\mathcal{M}_l\) with
\[
\{\gamma_m(t) = c_m t + z_m : m \in \mathcal{M}_l\} \subseteq H_k^+.
\]
In particular, for this \(t\) and \(m \in \mathcal{M}_l\) with \(m > k, l\),
\[
t \in W_m\text{ and }\gamma_m(t) \in V_m.
\]
As \(\gamma_m(t) \in V_m\),
\[
h(x_m \circ \varphi \gamma_m(t)) - h(x_m) < m.
\]
(4)
But \(\gamma_m(t) = z_m + c_m t = z_m + t \varphi(y_m)/\varphi(x_m)\), so
\[
x_m \circ \varphi \gamma_m(t) = x_m + z_m \varphi(x_m) + t \varphi(y_m) = y_m \circ \varphi t.
\]
So, by \(\text{4}\),
\[
h(y_m \circ \varphi t) - h(x_m) < m.
\]
But \(t \in W_m\), so
\[
h(y_m) - h(y_m \circ \varphi t) < m.
\]
Combining these with \(\text{3}\) and \(\text{2}\),
\[
3m < h(y_m) - h(x_m) \leq \{h(y_m) - h(y_m \circ \varphi t)\} + \{h(y_m \circ \varphi t) - h(x_m)\} \leq 2m,
\]
a contradiction. \(\Box\)

As in the classical Karamata case, this result implies global bounds on \(h\) – see BGT of Th. 2.0.1.

**Theorem 8.** In the setting of Theorem 7, for \(\varphi \in SE\), if the set \(S\) on which \(H_\ast(t)\) and \(H_\ast'(t)\) are finite contains a half-interval \([a_0, \infty)\) with \(a_0 > 0\) – then there is a constant \(K > 0\) such that for all large enough \(x\) and \(u\)
\[
h(u \varphi(x) + x) - h(x) \leq K \log u.
\]
The proof parallels the tail end of the proof in BGT of Th. 2.0.1, but is technically more demanding, as it uses in place of the usual sequence of powers \( a^n \), a Popa-style generalization (cf. Prop. 3(v)):

\[
a_{\varphi x}^{n+1} := a^n_{\varphi x} \circ \varphi x a = a^n_{\varphi x} \eta_x(a_{\varphi x}^n) \quad \text{with} \quad a^1_{\varphi x} = a,
\]

and relies on estimation results for \( a_m^{\varphi x} \) that are uniform in \( m \) (this only needs \( \eta_x \to \eta \rho \) pointwise):

**Proposition 12.** If \( \varphi \in SE \) with \( \rho = \rho_{\varphi} > 0 \), then for any \( a > 1 \), \( 0 < \varepsilon < 1 \),

(i) \( (a_m^{\varphi x}) \)-estimates under \( \eta_x^\rho \) for all large enough \( x \):

\[
(1 - \varepsilon) \leq \eta_x^\rho(a_m^{\varphi x})^{1/m} / \eta_\rho(a) \leq (1 + \varepsilon), \quad (m \in \mathbb{N})
\]

(ii) \( (a_m^{\varphi x}) \)-estimates under \( \eta_\rho \) for all large enough \( x \):

\[
\frac{\eta_\rho(a(1 - \varepsilon))^m}{1 - \varepsilon} - \frac{\varepsilon}{1 - \varepsilon} \leq \eta_\rho(a_m^{\varphi x}) \leq \frac{\eta_\rho(a(1 + \varepsilon))^m}{1 + \varepsilon} + \frac{\varepsilon}{1 + \varepsilon}, \quad (m \in \mathbb{N})
\]

(iii) \( a_m^{\varphi x} \to \infty \), and

(iv) there are \( C_\pm = C(\rho, a, \varepsilon) > 0 \) such that for all large enough \( x \) and \( u \):

\[
a_m^{\varphi x} \leq u < a_{\varphi x}^{n+1} \implies m C_- \leq \log u \leq (m + 1) C_+.
\]

**Proofs.** See the Appendix. □

11 Character degradation from limsup

We refer the reader to [BinO10] for a discussion, from the perspective of the practising analyst (employing ‘naive’ set theory), of the broader set-theoretic context below; for convenience we repeat part of the commentary there. As there so too here, our interest in the complexities induced by the limsup operation points us in the direction of definability and descriptive set theory because of the question of whether certain specific sets, encountered in the course of the analysis, have the Baire property. The answer depends on what further axioms one admits. For us there are two alternatives yielding the kind of decidability we seek: Gödel’s Axiom of Constructibility \( V = L \), as an appropriate *strengthening* of the Axiom of Choice (AC) which creates definable sets without the Baire property (without measurability), or, at the opposite pole, the Axiom of Projective Determinacy, \( PD \) (see [MySw], or [Kech] 5.38.C), an *alternative* to AC which guarantees the Baire property in the kind of definable sets we encounter. Thus to decide whether sets of the kind we encounter below have the Baire property, or are measurable, the answer is: it depends on the axioms of set theory that one adopts.

To formulate our results we need the language of descriptive set theory, for which see e.g. [JayR], [Kech], [Mos]. Within such an approach we will regard a function as a set, namely its *graph*; formulas written in naive set-theoretic
notation then need a certain amount of formalization – for quick approach to such matters refer to [Dra, Ch. 1,2] or the very brief discussion in [Kun, §1.2]. We need the beginning of the projective hierarchy in Euclidean space (see [Kech] S. 37.A), in particular the following classes:

- the *analytic* sets $\Sigma_1^1$;
- their complements, the *co-analytic* sets $\Pi_1^1$;
- the common part of the previous two classes, the ambiguous class $\Delta_1^1 := \Sigma_1^1 \cap \Pi_1^1$, that is, by Souslin’s Theorem ([JayR], p. 5, and [MaKe] p.407 or [Kec] 14. C) the *Borel* sets;

The notation reflects the fact that the canonical expression of the logical structure of their definitions, that is with the quantifiers (ranging over the reals, hence the superscript 1, as reals are type 1 objects - integers are of type 0) all at the front, is determined by a string of alternating quantifiers starting with an existential or universal quantifier (resp. $\Sigma$ or $\Pi$). Here the subscript accounts for the number of alternations.

Interest in the character of a function $H$ is motivated by an interest within the theory of regular variation in the character of the level sets

$$H^k := \{ s : |H(s)| < k \} = \{ s : (\exists t)[(s,t) \in H \land |t| < k] \},$$

for $k \in \mathbb{N}$ (where as above $H$ is identified with its graph). The set $H^k$ is thus the projection of $H \cap (\mathbb{R} \times [0,k])$ and hence is $\Sigma_n^1$ if $H$ is $\Sigma_n^1$, e.g. it is $\Sigma_1^1$, i.e. analytic, if $H$ is analytic (in particular, Borel). Also

$$H^k = \{ s : (\forall t)[(s,t) \in H \implies |t| \leq k] \} = \{ s : (\forall t)[(s,t) \notin H \lor |t| \leq k] \},$$

and so this is also $\Pi_n^1$ if $H$ is $\Sigma_n^1$. Thus if $H$ is $\Sigma_n^1$ then $H^k$ is $\Delta_n^1$. So if $\Delta_n^1$ sets are Baire, for some $k$ the set $H^k$ is Baire non-null, and hence subuniversal, as

$$\mathbb{R} = \bigcup_{k \in \omega} H^k.$$

With this in mind, it suffices to consider upper limits; as before, we prefer to work with the additive formulation. Consider the definition:

$$H^*_\varphi(x) := \lim_{t \to \infty} \sup_{t \to \infty} [h(t + x\varphi(t)) - h(t)].$$

Thus in general $H^*_\varphi$ takes values in the extended real line. The problem is that the function $H^*_\varphi$ is in general less well behaved than the function $h$ – for example, if $h$ is measurable/Baire, $H^*_\varphi$ need not be. The problem we address here is the extent of this degradation – saying exactly how much less regular
than \( h \) the \( \limsup H^*_\varphi \) may be. The nub is the set \( S \) on which \( H^*_\varphi \) is finite. This set \( S \) is an additive semi-group on which the function \( H^*_\varphi \) is subadditive (see [BinO9]) – or additive, if limits exist (see [BinO8]). Furthermore, if \( H \) has Borel graph then \( H^*_\varphi \) has \( \Delta^1_2 \) graph (see below). But in the presence of certain axioms of set-theory (for which see below) the \( \Delta^1_2 \) sets have the Baire property and are measurable; hence if \( S \) is large in either of these two senses, then in fact \( S \) contains a half-line.

The extent of the degradation in passing from \( h \) to \( H^*_\varphi \) is addressed in the following result, which we call the First Character Theorem, and then contrast it with two alternatives. These extend corresponding results established in the Karamata context as follows and differ from the former merely by duplicating assumptions previously made only on \( h \) there to identical ones on \( \varphi \).

**Theorem 9 (First Character Theorem).**

(i) If \( h \) and \( \varphi \) are Borel (have Borel graph), then the graph of the function

\[
H^*(x) = \limsup_{t \to \infty} [h(t + x\varphi(t)) - h(t)]
\]

is a difference of two analytic sets, hence is measurable and \( \Delta^1_2 \). If the graphs of \( h \) and \( \varphi \) are \( F_\sigma \), then the graph of \( H^*(x) \) is Borel.

(ii) If \( h \) and \( \varphi \) are analytic (have analytic graph), then the graph of the function \( H^*(x) \) is \( \Pi^1_3 \).

(iii) If \( h \) and \( \varphi \) are co-analytic (have co-analytic graph), then the graph of the function \( H^*(x) \) is \( \Pi^1_3 \).

The next theorem assumes much more than the First Character Theorem.

**Theorem 10 (Second Character Theorem).** If the following limit exists:

\[
\partial^\varphi h(x) := \lim_{t \to \infty} [h(t + x\varphi(t)) - h(t)],
\]

and \( h, \varphi \in \Delta^1_2 \) – then the graph of \( \partial^\varphi h \) is \( \Delta^1_2 \).

**Theorem 11 (Third Character Theorem).** If the function \( h \) and the ultra-filter \( \mathcal{U} \) (both on \( \omega \)) are of class \( \Delta^1_2 \) – then so is:

\[
\partial^\mathcal{U} h(x) := \mathcal{U}-\lim_{n}[h(n + t\varphi(n)) - h(n)].
\]

The proofs of all three character theorems closely follow the proofs of the Karamata special case in [BinO10, §4], by using just two amendment procedures. Firstly, apply a replacement rule: all uses of the formula \( y = h(x, t) := h(x + t) - h(t) \) (h as there) be replaced by a formalized conjunction of \( y = h(x, s, t) := h(x + ts) - h(t) \) and \( s = \varphi(x) \), as follows. Translate these two formulas to ‘\((x, s, t, y) \in h \& (x, s) \in \varphi\)' (interpreting \( h \) and \( \varphi \) as naming the graphs of the two functions), and replace each \((x, t, y) \in h \) there by the the translate just indicated here above. Secondly, apply an insertion rule: insert the variable
$s$ everywhere to precede the variable $w$. An example of the translation will suffice; here is a sample amendment:

$$y = h(t + xs) - h(t) \Leftrightarrow (\exists s, u, v, w \in \mathbb{R}) r(x, t, y, s, u, v, w),$$

where $r(x, t, y, s, u, v, w)$ stands for:

$$[y = u - v \& w = t + xs \& (w, u) \in h \& (t, v) \in h \& (x, s) \in \varphi]. \tag{5}$$

**Comment 1.** The last of the three theorems applies under the assumption of Gödel’s Axiom $V = L$ (see [Dev, §B.5, 453-489]), under which $\Delta^1_2$ ultrafilters exist on $\omega$ (e.g. for Ramsey ultrafilters – see [Z]). Above sets of natural numbers are identified with real numbers (via indicator functions), and so ultrafilters are subsets of $\mathbb{R}$ – for background see [C-Ne], or [HS]. Th. 11 offers a midway position between the First and Second Character Theorems.

In Th. 11 $\partial^2_t h(t)$ is additive, whereas in Th. 9 one has only sub-additivity (cf. BGT p. 62 equation (2.0.3)).

**Comment 2.** Replacing $h(n + t \varphi(n)) - h(n)$ by $h(x(n) + t \varphi(x(n))) - h(x(n))$, as in the Equivalence Theorem of [BinO3], to take limits along a specified sequence $x : \omega \to \omega^\omega$, gives an ‘effective’ version of the character theorems – given an effective descriptive character of $x$.

**References**

[AczG] J. Aczél and S. Gołąb, Remarks on one-parameter subsemigroups of the affine group and their homo- and isomorphisms, *Aequat. Math.*, **4** (1970), 1-10.

[Bec] A. Beck, *Continuous flows on the plane*, Grundl. math. Wiss. **201**, Springer, 1974.

[Bin] N. H. Bingham, Riesz means and Beurling moving averages, submitted.

[BinG1] N. H. Bingham, C. M. Goldie, Extensions of regular variation. II. Representations and indices. *Proc. London Math. Soc.* (3) **44** (1982), 497–534.

[BinG2] N. H. Bingham, C. M. Goldie, On one-sided Tauberian conditions. *Analysis* **3** (1983), 159–188.

[BinG3] N. H. Bingham, C. M. Goldie, Riesz means and self-neglecting functions. *Math. Z.*, **199** (1988), 443–454.

[BinGT] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular variation*, 2nd ed., Cambridge University Press, 1989 (1st ed. 1987).

[BinO1] N. H. Bingham and A. J. Ostaszewski, Beyond Lebesgue and Baire: generic regular variation. *Coll. Math.* **116** (2009), 119-138.

[BinO2] N. H. Bingham and A. J. Ostaszewski, The index theorem of topological regular variation and its applications. *J. Math. Anal. Appl.* **358** (2009), 238-248.

[BinO3] N. H. Bingham and A. J. Ostaszewski, Infinite combinatorics and the foundations of regular variation, *J. Math. Anal. Appl.* **360** (2009), 518-529.

[BinO4] N. H. Bingham and A. J. Ostaszewski, Infinite combinatorics in function spaces: category methods, *Publ. Inst. Math. (Beograd)* (N.S.) **86** (100) (2009), 55–73.

[BinO5] N. H. Bingham and A. J. Ostaszewski, Beyond Lebesgue and Baire II:
Bitopology and measure-category duality. *Coll. Math.*, 121 (2010), 225-238.

[BinO6] N. H. Bingham and A. J. Ostaszewski, Topological regular variation. I: Slow variation; II: The fundamental theorems; III: Regular variation. *Topology and its Applications* **157** (2010), 1999-2013, 2014-2023, 2024-2037.

[BinO7] N. H. Bingham and A. J. Ostaszewski, Beyond Lebesgue and Baire II: bitopology and measure-category duality. *Colloq. Math.* **121** (2010), 225-238.

[BinO8] N. H. Bingham and A. J. Ostaszewski, Normed groups: Dichotomy and duality. *Dissertationes Math.* **472** (2010), 138p.

[BinO9] N. H. Bingham and A. J. Ostaszewski, Kingman, category and combinatorics. *Probability and Mathematical Genetics* (Sir John Kingman Festschrift, ed. N. H. Bingham and C. M. Goldie), 135-168, London Math. Soc. Lecture Notes in Mathematics **378**, CUP, 2010.

[BinO10] N. H. Bingham and A. J. Ostaszewski: Regular variation without limits, *J. Math. Anal. Appl.*, 370 (2010), 322-338.

[BinO11] N. H. Bingham and A. J. Ostaszewski, Dichotomy and infinite combinatorics: the theorems of Steinhaus and Ostrowski. *Math. Proc. Camb. Phil. Soc.* **150** (2011), 1-22.

[BinO12] N. H. Bingham and A. J. Ostaszewski: Beurling slow and regular variation. *Trans. London. Math. Soc.*, to appear (see also Part I: arXiv:1301.5894 Part II: arXiv:1307.5305).

[BinO13] N. H. Bingham and A. J. Ostaszewski, Cauchy’s functional equation and extensions: Goldie’s equation and inequality, the Gółąb-Schinzel equation and Beurling’s equation, arXiv1405.3947.

[BinO14] N. H. Bingham and A. J. Ostaszewski: Additivity, subadditivity and linearity: automatic continuity and quantifier weakening, arXiv.1405.3948.

[Blo] S. Bloom, A characterization of B-slowly varying functions. *Proc. Amer. Math. Soc.* **54** (1976), 243-250.

[Boa] R. P. Boas, *A primer of real functions*. 3rd ed. Carus Math. Monographs 13, Math. Assoc. America, 1981.

[BojK] R. Bojanić and J. Karamata, On a class of functions of regular asymptotic behavior. Math. Research Center Tech. Report 436, Madison, Wis. 1963; reprinted in *Selected papers of Jovan Karamata* (ed. V. Marić, Zevod Udžbenika, Beograd, 2009), 545-569.

[Brz1] J. Brzdęk, The Gółąb-Schinzel equation and its generalizations, *Aequat. Math.* **70** (2005), 14-24.

[Brz2] J. Brzdęk, A remark on solutions of a generalization of the addition formulae, *Aequationes Math.*, **71** (2006), 288–293.

[CoN] W.W. Comfort, S. Negrepontis, *The theory of ultrafilters*. Die Grundlehren der mathematischen Wissenschaften, Band 211. Springer-Verlag, New York-Heidelberg, 1974.

[Dra] F. R. Drake, D. Singh, *Intermediate set theory*. Wiley, 1996

[Dev] K. J. Devlin, *Constructibility*. Springer 1984.

[dH] L. de Haan, On regular variation and its applications to the weak convergence of sample extremes. Math. Centre Tracts 32, Amsterdam 1970.

[HinS] N. Hindman, D. Strauss, *Algebra in the Stone-Čech compactification. Theory and applications*. 2nd rev. ed., de Gruyter, 2012.
[JayR] J. Jayne and C. A. Rogers, *K-analytic sets*, Part 1 (p.1-181) in [Rog].

[Jay] P. Javor, On the general solution of the functional equation \( f(x + yf(x)) = f(x)f(y) \). *Aequat. Math.* 1 (1968), 235-238.

[Kech] A. S. Kechris: *Classical Descriptive Set Theory*. Grad. Texts in Math. 156, Springer, 1995.

[Kor] J. Korevaar, *Tauberian theorems: A century of development*. Grundl. math. Wiss. 329, Springer, 2004.

[Kuc] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy’s equation and Jensen’s inequality*. 2nd ed., Birkhäuser, 2009 [1st ed. PWN, Warszawa, 1985].

[Kun] K. Kunen, *Set theory. An introduction to independence proofs*. Reprint of the 1980 original. Studies in Logic and the Foundations of Mathematics, 102. North-Holland, 1983.

[MaKe] D. A. Martin, A.S. Kechris, Infinite games and effective descriptive set theory, Part 4 (p. 403-470) in [Rog].

[Mos] Y. N. Moschovakis, *Descriptive set theory*, Studies in Logic and the Foundations of Math. 100. North-Holland, Amsterdam, 1980.

[Mur] A. Muréńko, On the general solution of a generalization of the Golab-Schinzel equation, *Aequat. Math.*, 77 (2009), 107-118.

[MySw] J. Mycielski and S. Świerczkowski, *On the Lebesgue measurability and the axiom of determinateness*, Fund. Math. 54 (1964), 67–71.

[Ost1] A. J. Ostaszewski, Beyond Lebesgue and Baire III: Steinhaus’ Theorem and its descendants. *Top. and its App.* 160 (2013), 1144-1154.

[Ost2] A. J. Ostaszewski, Regular variation, topological dynamics, and the Uniform Boundedness Theorem, *Top. Proc.*, 36 (2010), 305-336.

[Ost3] A.J. Ostaszewski, Beurling regular variation, Bloom dichotomy, and the Golab-Schinzel functional equation, *Aequat. Math.*, to appear.

[Ost4] A. J. Ostaszewski, Homomorphisms from Functional Equations, preprint.

[PolS] G. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis* Vol. I. Grundl. math. Wiss. XIX, Springer, 1925.

[Pop] C. G. Popa, Sur l’équation fonctionelle \( f[x + yf(x)] = f(x)f(y) \), *Ann. Polon. Math.* 17 (1965), 193-198.

[Rog] C. A. Rogers, J. Jayne, C. Dellacherie, F. Topsøe, J. Hoffmann-Jørgensen, D. A. Martin, A. S. Kechris, A. H. Stone, *Analytic sets*, Academic Press,1980.

[Rud] W. Rudin, *Principles of Mathematical Analysis*. 3rd ed. McGraw-Hill, 1976 (1st ed. 1953).

[Wie] N. Wiener, *The Fourier integral & certain of its applications*, CUP, 1988.

[Wid] D. V. Widder, *The Laplace Transform*, Princeton, 1972.

[Z] J. Zapletal, Terminal Notions, *Bull. Symbolic Logic* 5 (1999), 470-478.
12 Appendix: Global bounds

Below we need Bloom’s [Blo] result that for $x$ large enough the Beck sequence $x_n^u$ defined recursively by its starting value $x$ and the step-size $u$:

$$x_{n+1}^u = x_n \circ u = x_n + u \varphi(x_n), \text{ with } x_0^u = x, \ x_1^u = x + u \varphi(x)$$

is divergent (see [BinO-B, §9] and compare [Ost-B, §6]). Say for $x \geq x_0$.

We briefly review a number of examples of Beck sequences; Example 2 is crucial.

Example 1. $a_n^\varphi = a_n$, so that $a_n^{\varphi+1} = a_n^\varphi \circ \varphi = a_n + a \varphi(a_n^\varphi)$. Performing the recurrence the other way about, $u_{n+1} = u \circ u_n = u + u_n \varphi(u)$, generates a GP:

$$u_n = (1 - \varphi(u)^n+1) \cdot u/(1 - \varphi(u)),$$

with

$$u_{n+1} - u_n = (u_n - u_{n-1}) \varphi(u) = ... = u \varphi(u)^n.$$

For $\varphi \in GS$ the two are the same. They are not altogether dissimilar, as the other one has

$$a_k^\varphi = a[1 + \varphi(a) + \varphi(a^2) + ... + \varphi(a^{k-1})],$$

and, assuming divergence, the term-on-term growth is

$$\varphi(a_k^\varphi)/\varphi(a_{k-1}^\varphi) = \varphi(a_k^\varphi + a \varphi(a_k^\varphi))/\varphi(a_{k-1}^\varphi) \to \eta^\varphi(a),$$

so the series behaves, up to a multiplier $\varphi(a_k^\varphi)$, eventually like

$$\sum_{j<k} \eta^\varphi(a)^j = (1 - \eta^\varphi(a_n^\varphi))/(1 - \eta^\varphi(a)).$$

Example 2. Consider the sequence

$$a_{n+1}^\varphi_x := a_n^\varphi \circ \varphi_x a = a_n^\varphi + a \eta_x^\varphi(a_n^\varphi) \text{ with } a_1^\varphi_x = a,$$

where $a$ is fixed; on the back of Example 1 we guess that since uniformly in $x$

$$\eta_x^\varphi(a) \to \eta^\varphi(a),$$

this $a_n^\varphi_x$ is a divergent sequence for $x$ large enough, say $x > x_a$. Indeed, it is – see the proof of Prop. 12; this is to be expected from the related iteration

$$a_{n+1}^\eta \circ a = a_n^\eta + a \eta^\varphi(a_n^\eta) \text{ with } a_1^\eta = a,$$

where for $\rho = 0$ growth is linear: $\eta(a_n^\eta) = na$, whereas for $\rho > 0$ it is exponential:

$$\eta(a_n^\eta) = \eta(a_n^\eta - 1 \circ a) = \eta(a_n^{\eta-1})\eta(a) = ... = \eta^\eta(a)^n = (1 + \rho a)^n.$$

Below we need the solution of a recurrence; we present this as a lemma, delaying the calculation to the end.
Lemma 2. The solution of $bv_{n+1} - v_n = r^n$ for $b \neq 1$ is

$$v_n = r^n/(br - 1) + b^{1-n}(v_1 - r/(br - 1)).$$

(sln)

If $b = \eta_p(a)$ with $\rho > 0$, $v_1 = 1/\rho a$, $r = 1+\delta$, with $\delta = \varepsilon/\eta_p(a)$ and $0 < \varepsilon < 1$, then

$$v_1 - r/(br - 1) = \varepsilon/\eta_p(a)\rho a\quad \text{or} \quad -\varepsilon/\eta_p(a)\rho a.$$

We now proceed to verify the details of Prop. 12 in §10.

Proof of Prop. 12. Fix $a, \rho > 0$ and $0 < \varepsilon < 1$. Taking $\delta := \varepsilon/\rho a/\eta(a)$,

$$\eta(a) \pm \rho a \varepsilon = (1 + \rho a(1 \pm \varepsilon)) = \eta(a(1 \pm \varepsilon)) = \eta(a)(1 \pm \delta).$$

In particular, $\eta(a)(1 - \delta) = \eta(a(1 - \varepsilon)) > 1$, since $\varepsilon < 1$. Since $\eta_x(a) \rightarrow \eta(a)$, there is $X = X_{a,\varepsilon}$ with

$$|\eta(a) - \eta_x(a)| < \rho a \varepsilon : \quad \eta(a)(1 - \delta) < \eta_x(a) < \eta(a)(1 + \delta). \quad (x > X)$$

(i) By Prop. 3(v), for $y_i$ running through $x \circ a_{\varphi x}^{m-1}, x \circ a_{\varphi x}^{m-2}, ..., x > X$,

$$\eta_x(a_{\varphi x}^m) = \prod_{i=1}^m \eta_y(a),$$

(prod)

so that, by $(\delta$-bd),

$$\eta(a(1 - \varepsilon)) \leq \eta_x(a_{\varphi x}^m)^{1/m} \leq \eta(a(1 + \varepsilon)).$$

(ii) As $\eta \in GS$, $\eta(a_{\varphi x}^n) = \eta(a_{\varphi x}^n + a_{\varphi x}^n) = \eta(a_{\varphi x}^n)\eta(a_{\varphi x}^n)/\eta(a_{\varphi x}^n))$. So

$$\eta(a_{\varphi x}^{n+1})/\eta(a_{\varphi x}^n) = 1 + \rho a \eta_x(a_{\varphi x}^n)/\eta(a_{\varphi x}^n) : \quad \eta(a_{\varphi x}^{n+1}) - \eta(a_{\varphi x}^n) = \rho a \eta_x(a_{\varphi x}^n).$$

Putting $u_n := \eta(a_{\varphi x}^n)/\rho a \eta(a)^n$, so that $u_1 = 1/\rho a$, and using $(\delta$-bd) again,

$$(1 - \delta)^n \leq \frac{\eta(a_{\varphi x}^{n+1}) - \eta(a_{\varphi x}^n)}{\rho a \eta(a)^n} = \eta(a)u_{n+1} - u_n \leq (1 + \delta)^n.$$

As $\eta(a)(1 \pm \delta) \neq 1$, apply Lemma 2 to $b = \eta(a)$ and $r = 1 \pm \delta$; then

$$\frac{(1 - \delta)^n \eta(a)^n}{1 - \varepsilon} - \frac{\varepsilon}{1 - \varepsilon} \leq \eta(a_{\varphi x}^n) \leq \frac{(1 + \delta)^n \eta(a)^n}{1 + \varepsilon} + \frac{\varepsilon}{1 + \varepsilon}.$$

(iii) As $\eta(a)(1 - \delta) > 1$, the left inequality implies $a_{\varphi x}^m$ is divergent.

(iv) If $a_{\varphi x}^m \leq u < a_{\varphi x}^{m+1}$, then (as $\rho$ is monotone), $\eta(a_{\varphi x}^m) \leq 1 + \rho u \leq \eta(a_{\varphi x}^{m+1})$; so, for $x > X_x$

$$\frac{\eta(a(1 - \varepsilon))^m}{(1 - \varepsilon)} - \frac{1}{1 - \varepsilon} \leq \rho u < \frac{\eta(a(1 + \varepsilon))^{m+1}}{1 + \varepsilon} - \frac{1}{1 + \varepsilon}.$$

35
So for $\varepsilon < 1/2$
\[
\frac{\eta(a(1 - \varepsilon))}{1 - \varepsilon} - 2 < \eta(u) < \frac{\eta(a(1 + \varepsilon))}{1 + \varepsilon},
\]
implies and so for $u > 1$
\[
\frac{\eta(a(1 - \varepsilon))}{1 - \varepsilon}(2 + \rho) \leq u < \frac{\eta(a(1 + \varepsilon))}{1 + \varepsilon},
\]
where $\log \eta(a(1 - \varepsilon)) > 0$. Taking
\[
C_-(\rho, a, \varepsilon) := \log(\eta(a(1 - \varepsilon))/(\rho + 2)(1 - \varepsilon)), \quad C_+(\rho, a, \varepsilon) := \log(\eta(a(1 + \varepsilon))/\rho(1 + \varepsilon)),
\]
and
\[
mC_\leq \log u < (m + 1)C_+ \quad (u \geq a > 1 \& x \geq X_{a, \varepsilon}) \quad \square
\]

We are now ready to prove Th. 8 of §10.

**Proof of Theorem 8.** (This parallels the tail end of the proof in BGT of Th. 2.0.1.) W.l.o.g. we assume that $\eta^x(x) = 1 + \rho x$ with $\rho > 0$, as the case $\rho = 0$ is already known. By Theorem 7 (UBT) in §10, for any $a \geq a_0$
\[
\limsup_{x \to \infty} \left( \sup_{u \leq u \leq 2\eta(a)} h(x \circ \varphi u) - h(x) \right) < \infty.
\]
So there is $C_a$ such that
\[
\sup_{a \leq u \leq 2\eta(a)} h(x \circ \varphi u) - h(x) < C_a
\]
for all large enough $x$, say for $x > x_a$. Choose $a > \max\{a_0, x_a\}$ and fix $u \geq a = a_{\varphi x}^1$. Then, By Prop. 12(iii), we may choose $m = m_x(u)$ such that
\[
a_{\varphi x}^{m-1} < a_{\varphi x}^m \leq u \leq a_{\varphi x}^{m+1}.
\]
This time put $\delta := (u - a_{\varphi x}^{m-1})/\eta_x(a_{\varphi x}^{m-1})$, so that $u = a_{\varphi x}^{m-1} \circ \varphi x \delta$; then
\[
x \circ \varphi u = [x + a_{\varphi x}^{m-1} \varphi(x)] + \delta \varphi(x + a_{\varphi x}^{m-1} \varphi(x)) = y \circ \varphi \delta,
\]
with $y = x \circ \varphi a_{\varphi x}^{m-1}$; referring to $a_{\varphi x}^m - a_{\varphi x}^{m-1}$ and $a_{\varphi x}^{m+1} - a_{\varphi x}^m$,
\[
an_x(a_{\varphi x}^{m-1}) \leq \eta_x(a_{\varphi x}^{m-1}) < an_x(a_{\varphi x}^m) = an_x(a_{\varphi x}^{m-1}) + a_{\varphi x}(a_{\varphi x}^m) = an_x(a_{\varphi x}^{m-1}) + an_x(a_{\varphi x}^m),
\]
as in Prop. 3(v). But by Prop. 12, since $y > x > x_a$,
\[
a \leq \delta < a(1 + \eta_y(a)) < a(1 + \eta(a)(1 + \delta)) < a(1 + \rho a + \eta(a)) = 2\eta(a).
\]
So by choice of $C_a$,
\[
h(x \circ \varphi u) - h(x \circ \varphi a_{\varphi x}^{m-1}) = h(y \circ \varphi \delta) - h(y) < C_a.
\]
as \( \delta \in [a, 2a\eta(a)] \). As in Prop. 12,
\[
x \circ \varphi a_{\varphi x}^{n+1} = x \circ \varphi \left( a_{\varphi x}^n \circ \varphi a \right) = (x \circ \varphi a_{\varphi x}^n) \circ \varphi a,
\]
and, setting \( y_k = x \circ \varphi a_{\varphi x}^k \) for \( k = 1, ..., m - 1 \),
\[
h(x \circ \varphi a_{\varphi x}^{k+1}) - h(x \circ \varphi a_{\varphi x}^k) = h((x \circ \varphi a_{\varphi x}^k) \circ \varphi a) - h(x \circ \varphi a_{\varphi x}^k) = h(y_k \circ \varphi a) - h(y_k) < C_a,
\]
since \( y_k > x > X \). So for \( x > x_a \)
\[
h(x \circ \varphi a_{\varphi x}^{k+1}) - h(x \circ \varphi a_{\varphi x}^k) = h(x \circ \varphi a_{\varphi x}^{m-1}) + \sum_{k=1}^{m-1} h(x \circ \varphi a_{\varphi x}^k) - h(x \circ \varphi a_{\varphi x}^{k-1}) < mC_a.
\]
Again by Prop. 12, there is a constant \( C \) such that
\[
m \leq C \log u.
\]
Taking \( K = C_a C \) yields the desired inequality. \( \Box \)

**Proof of Lemma 2.** A particular solution is \( r^n/(br - 1) \), \( bw_{n+1} - w_n = 0 \) for \( w_n = v_n - r^n/(br - 1) \) and \( w_1 = w_1 b^{1-n} \), where \( w_1 = v_1 - r/(br - 1) \).

For \( b = \eta_p(a) \), \( v_1 = 1/\rho a \) and \( r = 1 \pm \delta \), we calculate that
\[
\rho a w_1 = \frac{[\eta(a)](1 \pm \delta) - \rho a(1 \pm \delta)}{(1 + \rho a)(1 \pm \delta) - 1} = \frac{[(1 + \rho a)(1 \pm \delta) - 1] - \rho a(1 \pm \delta)}{\rho a + \eta(a)(\pm \delta)}
\]
\[
= \pm \frac{\delta}{\rho a + \eta(a)(\pm \delta)} = \pm \frac{\varepsilon/\eta(a)}{(1 + (\pm 1)\varepsilon)} = \frac{\varepsilon/\eta(a)}{(1 + \varepsilon)} \text{, or } - \frac{\varepsilon/\eta(a)}{(1 - \varepsilon)} \text{.}
\]