COEFFICIENT PROBLEMS ON THE CLASS $U(\lambda)$

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Abstract. For $0 < \lambda \leq 1$, let $U(\lambda)$ denote the family of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in the unit disk $\mathbb{D}$ satisfying the condition $\left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda$ in $\mathbb{D}$. Although functions in this family are known to be univalent in $\mathbb{D}$, the coefficient conjecture about $a_n$ for $n \geq 5$ remains an open problem. In this article, we shall first present a non-sharp bound for $|a_n|$. Some members of the family $U(\lambda)$ are given by $z f(z) = 1 - (1 + \lambda) \phi(z) + \lambda \phi(z)^2$ with $\phi(z) = e^{i\theta} z$, that solve many extremal problems in $U(\lambda)$. Secondly, we shall consider the following question: Do there exist functions $\phi$ analytic in $\mathbb{D}$ with $|\phi(z)| < 1$ that are not of the form $\phi(z) = e^{i\theta} z$ for which the corresponding functions $f$ of the above form are members of the family $U(\lambda)$? Finally, we shall solve the second coefficient ($a_2$) problem in an explicit form for $f \in U(\lambda)$ of the form $f(z) = \frac{z}{1 - a_2 z + \lambda z \int_0^z \omega(t) \, dt}$, where $\omega$ is analytic in $\mathbb{D}$ such that $|\omega(z)| \leq 1$ and $\omega(0) = a$, where $a \in \mathbb{D}$.

We denote the unit disk by $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$, and let $\mathcal{H}$ be the linear space of analytic functions defined on $\mathbb{D}$ endowed with the topology of locally uniform convergence and $\mathcal{A} = \{ f \in \mathcal{H} : f(0) = f'(0) - 1 = 0 \}$. The family $\mathcal{S}$ of univalent functions from $\mathcal{A}$ and many of its subfamilies, for which the image domains have special geometric properties, have been investigated in detail. Among them are convex, starlike, close-to-convex, spirallike and typically real mappings. For the general theory of univalent functions we refer the reader to the books [7, 8, 16]. However, the class $U(\lambda)$ defined below seems to have many interesting properties (cf. [14, 15]). For $0 < \lambda \leq 1$, we consider the family $U(\lambda) = \{ f \in \mathcal{A} : |U_f(z)| < \lambda \text{ in } \mathbb{D} \}$, where

$$U_f(z) = \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 = \frac{z}{f(z)} - z \left( \frac{z}{f(z)} \right)' - 1, \quad z \in \mathbb{D}. \quad (1)$$

Set $\mathcal{U} := U(1)$, and observe that $\mathcal{U} \subseteq \mathcal{S}$ (see [1, 2]). Recently, in [15], the present authors have presented a simpler proof of it in a general setting.

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More recently, a number of new and useful properties of the family $U(\lambda)$ were established in [12, 13, 14]. However, the coefficient problem for $U(\lambda)$ remains open. This article supplements the earlier investigations in this topic. See [12, 13, 14].

Let $B = \{\omega \in H : |\omega(z)| < 1 \text{ on } |z| < 1\}$ and $B_0 = \{\omega \in B : \omega(0) = 0\}$. In addition, for $f, g \in H$, we use the symbol $f(z) \prec g(z)$, or in short $f \prec g$, to mean that there exists an $\omega \in B_0$ such that $f(z) = g(\omega(z))$. We now recall the following results from [12] which we need in the sequel.

**Theorem A.** Suppose that $f \in U(\lambda)$ for some $\lambda \in (0, 1]$ and $a_2 = f''(0)/2$. Then we have the following:

(a) If $|a_2| = 1 + \lambda$, then $f$ must be of the form

$$f(z) = \frac{z}{(1 + e^{i\phi}z)(1 + \lambda e^{i\phi}z)}.$$

(b) $f(z) \prec (1 + 2\lambda z + \lambda z^2)$ and $f(z)/z \prec 1/(1 - z)(1 - \lambda z)$, $z \in \mathbb{D}$.

As an analog to Bieberbach conjecture for the univalent family $S$ proved by de Branges [5] (see also [3]), the following conjecture was proposed in [12].

**Conjecture 1.** Suppose that $f \in U(\lambda)$ for some $0 < \lambda \leq 1$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then $|a_n| \leq \sum_{k=0}^{n-1} \lambda^k$ for $n \geq 2$.

This conjecture has been verified for $n = 2$ first in [18] and a simpler proof was given in [12]. More recently, in [14], Obradović et al. proved the conjecture for $n = 3, 4$ with an alternate proof for the case $n = 2$, but it remains open for all $n \geq 5$. Because $U(1) \subseteq S$ and the Koebe function belongs to $U(1)$, this conjecture obviously holds for $\lambda = 1$, in view of the de Branges theorem. Since no bound has been obtained for $|a_n|$ for $n \geq 5$, it seems useful to obtain a reasonable estimate. This attempt gives the following theorem and at the same time the proof for the case $\lambda = 1$ does not require the use of de Branges theorem that $|a_n| \leq n$ for $f \in S$ with equality for the Koebe function and its rotation.

**Theorem 1.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belong to $U(\lambda)$ for some $0 < \lambda \leq 1$. Then

$$|a_n| \leq 1 + \sqrt{n-1} \sum_{k=0}^{n-2} \lambda^{2k}, \quad \text{for } n \geq 2.$$  

**Proof.** Let $f \in U(\lambda)$. Then the second subordination relation in Theorem A(b) shows that

$$f(z)/z \prec \frac{1}{1 - \mu z} \frac{1}{1 - \lambda z} = f_1(z) f_2(z), \quad z \in \mathbb{D}.$$  

Note that for

$$g_1(z) = \sum_{n=0}^{\infty} b_n z^n \prec f_1(z) = \frac{1}{1 - \mu z} \quad \text{and} \quad g_2(z) = \sum_{n=0}^{\infty} c_n z^n \prec f_2(z) = \frac{1}{1 - z},$$
where \( b_0 = c_0 = 1 \), Rogosinski’s theorems [17] (see also [7, Theorems 6.2 and 6.4]) give that

\[
\sum_{k=1}^{n} |b_k|^2 \leq \sum_{k=1}^{n} \lambda^{2k} \quad \text{and} \quad |c_n| \leq 1 \quad \text{for} \quad n \geq 1.
\]

Moreover, the relation \( \frac{f(z)}{z} = g_1(z)g_2(z) \) gives

\[
a_{n+1} = \sum_{k=0}^{n} b_k c_{n-k}.
\]

Consequently, by (2), it follows from the classical Cauchy-Schwarz inequality that

\[
|a_{n+1}| \leq 1 + \sqrt{n} \sum_{k=1}^{n} |b_k|^2 \leq 1 + \sqrt{n} \sum_{k=1}^{n} \lambda^{2k},
\]

which implies the desired assertion. \( \square \)

Suppose that \( f \in U(\lambda) \). Then the second subordination relation in Theorem A(b) shows that there exists a function \( \phi \in B_0 \) such that

\[
z f(z) = 1 - (1 + \lambda)\phi(z) + \lambda(\phi(z))^2, \quad z \in \mathbb{D}.
\]

From Theorem A(a), we see that there is a member in the family \( U(\lambda) \) in the above form with \( \phi(z) = e^{i\theta}z \). In this type of functions, we have \(|a_2| = 1 + \lambda\). A natural question is whether there exist functions \( \phi \in B_0 \) that are not of the form \( \phi(z) = e^{i\theta}z \) of the above type for which the corresponding \( f \) of the form (3) belongs to \( U(\lambda) \).

In order to prove the next result, we need the classical Julia lemma which is often quoted as Jack’s lemma [10, Lemma 1] or Clunie-Jack’s lemma [6] although this fact was known much before the work of Jack. See the article of Boas [4] for historical commentary and the application of Julia’s lemma.

**Lemma B.** (Julia’s lemma) Let \( |z_0| < 1 \) and \( r_0 = |z_0| \). Let \( f(z) = \sum_{k=n}^{\infty} a_k z^k \) be continuous on \( |z| \leq r_0 \) and analytic on \( \{z : |z| < r_0\}\cup\{z_0\} \) with \( f(z) \neq 0 \) and \( n \geq 1 \). If \( |f(z_0)| = \max_{|z| \leq r_0} |f(z)| \), then \( z_0 f'(z_0) / f(z_0) \) is a real number and \( z_0 f''(z_0) / f(z_0) \geq n \).

**Theorem 2.** Suppose that \( \phi \in B_0 \) that are not of the form \( \phi(z) = e^{i\theta}z \) of the above type (3) such that there exists a \( \theta_0 \) with \( \phi(e^{i\theta_0}) = -1 \). In addition we let \( \phi \) be analytic on the closed unit disk \( \overline{\mathbb{D}} \). Then \( f \) expressed by the relation (3) cannot be a member of the family \( U(\lambda) \).

**Proof.** We observe that \( f \in U(\lambda) \) if and only if

\[
\left| \frac{z}{f(z)} - z \left( \frac{z}{f(z)} \right)' - 1 \right| < \lambda, \quad z \in \mathbb{D},
\]

which according to (1) and (3) implies that there exists a function \( \phi \in B_0 \) such that
(4) \[ L(\phi)(z) = |-(1 + \lambda)(\phi(z) - z\phi'(z)) + \lambda\phi(z)(\phi(z) - 2z\phi'(z))| < \lambda, \quad z \in \mathbb{D}. \]

Note that we consider analytic functions \( \phi \) in \( \overline{\mathbb{D}} \) that are not of the form \( \phi(z) = e^{i\theta}z \) of the above type such that there exists a \( \theta_0 \) with \( \phi(e^{i\theta_0}) = -1 \). Examples of such functions are the Blaschke products with the above exception. From Julia’s lemma with \( n = 1 \), we know that \( z_0\phi'(z_0)\phi(z_0) = m(\theta_0) \geq 1 \), \( z_0 = e^{i\theta_0} \).

If we let \( \phi(z) = z\psi(z) \), then we see that \( \psi(\mathbb{D}) \subset \overline{\mathbb{D}} \) and \( \psi(e^{i\theta_0}) = -e^{-i\theta_0} \). Now, we assume that \( m(\theta_0) = 1 \). Since \( z\phi'(z)\phi(z) = 1 + z\psi'(z)\psi(z) \), this means that \( \psi'(e^{i\theta_0}) = 0 \). But then an angle with width \( \pi \) and vertex \( e^{i\theta_0} \) would be mapped by \( \psi \) onto an angle with width \( 2\pi \) or more and a vertex \( -e^{-i\theta_0} \). This contradicts the fact that \( \psi(\mathbb{D}) \subset \overline{\mathbb{D}} \). Hence, \( m(\theta_0) > 1 \). From the above we get

\[ e^{i\theta_0}\phi'(e^{i\theta_0}) = -m(\theta_0), \]

and therefore,

\[
L(\phi)(z_0) = \left| - (1 + \lambda)(\phi(z_0) - z_0\phi'(z_0)) + \lambda\phi(z_0)(\phi(z_0) - 2z_0\phi'(z_0)) \right|
\]

\[ = \lambda + (1 + 3\lambda)(m(\theta_0) - 1) \]

which shows that \( L(\phi)(z_0) > \lambda \). This contradicts (4) and hence, \( f \) cannot be a member of the family \( \mathcal{U}(\lambda) \). The proof is complete.

In [12, Theorem 5], under a mild restriction on \( f \in \mathcal{U}(\lambda) \), the region of variability of \( a_2 \) is established as in the following form.

**Theorem C.** Let \( f \in \mathcal{U}(\lambda) \) for some \( 0 < \lambda \leq 1 \), and such that

\[ \frac{z}{f(z)} \neq (1 - \lambda)(1 + z), \quad z \in \mathbb{D}. \]

Then, we have

\[ \frac{z}{f(z)} - (1 - \lambda)z < 1 + 2\lambda z + \lambda z^2 \]

and the estimate \( |a_2 - (1 - \lambda)| \leq 2\lambda \) holds. In particular, \( |a_2| \leq 1 + \lambda \) and the estimate is sharp as the function \( f_\lambda(z) = z/((1 + \lambda z)(1 + z)) \) shows.

Certainly, it was not unnatural to raise the question whether the condition (5) is necessary for a function \( f \) to belong to the family \( \mathcal{U}(\lambda) \). This question was indeed raised in [12]. In the next result, we show that the condition (5) cannot be removed from Theorem C. Before, we present the proof, it is worth recalling from [12] that if \( f \in \mathcal{U}(\lambda) \), then for each \( R \in (0, 1) \), the function \( f_R(z) = R^{-1}f(Rz) \) also belongs to \( \mathcal{U}(\lambda) \).
Theorem 3. Let \( f(z) = z / ((1 - z)(1 - \lambda z)) \) and for a fixed \( R \in (0, 1) \), let \( f_R(z) = R^{-1} f(Rz) \). Then we have

(a) For \( 0 < \lambda \leq 1/2 \) there exists, for any \( R \in (0, 1) \), an \( r \in (0, 1) \) such that \( F(R, r) = 0 \), where

\[
F(R, r) = \frac{r}{f_R(r)} - (1 - \lambda)(1 + r).
\]

(b) For \( 1/2 < \lambda < 1 \) there exists, for any

\[ 1 > R > \frac{1 + \lambda - \sqrt{(1 - \lambda)(1 + 7\lambda)}}{2\lambda}, \]

an \( r \in (0, 1) \) such that \( F(R, r) = 0 \).

Proof. We consider \( F(R, r) \) given by (7) and observe that

\[
F(R, r) = \lambda R^2 r^2 - r[R(1 + \lambda) + 1 - \lambda] + \lambda.
\]

We see that in the cases indicated in the statement of the theorem \( F(R, 0) = \lambda > 0 \) and \( F(R, 1) < 0 \). Indeed

\[
F(R, 1) = \lambda R^2 - R(1 + \lambda) + 2\lambda - 1 = -R[(1 - R)\lambda + 1] - (1 - 2\lambda)
\]

which is less than zero for any \( R \in (0, 1) \) and for \( 0 < \lambda \leq 1/2 \). Similarly, for the case \( 1/2 < \lambda < 1 \), one can compute the roots of the equation \( F(R, 1) = 0 \) and obtain the desired conclusion. This proves the assertion of Theorem 3.

Because of the characterization of functions in \( U(\lambda) \) via functions in \( B \), the following result is of independent interest. As pointed out in the introduction, it is known that if \( f \in U(\lambda) \), then \( |a_2| \leq 1 + \lambda \) with equality for \( f(z) = z / ((1 - z)(1 - \lambda z)) \) and its rotation.

Theorem 4. Let \( f \in U(\lambda) \), \( \lambda \in (0, 1) \), have the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n = \frac{z}{1 - a_2 z + \lambda z \int_0^z \omega(t) \, dt}
\]

for some \( \omega \in B \) such that \( \omega(0) = a \in \mathbb{D} \) and \( v(x) \) be defined by

\[
v(x) = \int_0^1 \frac{x + t}{1 + xt} \, dt = \frac{1}{x} - \frac{1 - x^2}{x^2} \log(1 + x) < 1 \quad \text{for} \quad 0 < x < 1,
\]

and \( v(0) = \lim_{x \to 0^+} v(x) = 1/2 \). Then \( |a_2| \leq 1 + \lambda v(|a|) \). The result is sharp.

Proof. Recall the fact that \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in U(\lambda) \) if and only if

\[
\frac{z}{f(z)} = 1 - a_2 z + \lambda z \int_0^z \omega(t) \, dt \neq 0, \quad z \in \mathbb{D},
\]

where \( \omega \in B \). By assumption \( \omega(0) = a \in \mathbb{D} \). As in the proof of [12, Theorem 1], assume on the contrary that

\[
|a_2| = \frac{1 + \lambda v(|a|)}{r}, \quad r \in (0, 1),
\]
and consider the function $F$ defined by

$$F(z) = \frac{1}{a_2} \left[ 1 + \lambda z \int_0^z \omega(t) \, dt \right], \quad z \in \mathbb{D}.$$ 

Then, according to the Schwarz-Pick lemma applied to $\omega \in \mathcal{B}$, we can easily obtain that

$$|\omega(z)| \leq \frac{|a| + |z|}{1 + |a|}, \quad z \in \mathbb{D},$$

and thus, as in the proof of [12, Theorem 2], it follows that

$$\left| \int_0^z \omega(t) \, dt \right| \leq v(|a|) < 1, \quad z \in \mathbb{D},$$

where $v(x)$ is defined as in the statement. Consequently, for $|z| \leq r$, we get by (10)

$$|F(z)| \leq \frac{1}{|a_2|} \left[ 1 + \lambda |z| \left| \int_0^z \omega(t) \, dt \right| \right] \leq \frac{1 + r \lambda v(|a|)}{|a_2|} = \frac{(1 + r \lambda v(|a|))r}{1 + \lambda v(|a|)} < r.$$

Hence $F$ is a mapping of the closed disk $\overline{\mathbb{D}}_r$ into itself, where $\mathbb{D}_r = \{ z : |z| < r \}$. Secondly, we have for $z_1$ and $z_2$ in $\overline{\mathbb{D}}_r$,

$$|F(z_1) - F(z_2)| = \frac{\lambda r}{1 + \lambda v(|a|)} \left| z_1 \int_0^{z_1} \omega(t) \, dt + (z_1 - z_2) \int_0^{z_2} \omega(t) \, dt \right|$$

$$\leq \frac{\lambda r}{1 + \lambda v(|a|)} \left( |z_1| \left| \int_0^{z_1} \omega(t) \, dt \right| + |z_1 - z_2| \left| \int_0^{z_2} \omega(t) \, dt \right| \right)$$

$$\leq \frac{\lambda r (r + v(|a|))}{1 + \lambda v(|a|)} |z_1 - z_2|$$

$$< r |z_1 - z_2|.$$ 

Therefore, $F$ is a contraction of the disk $\overline{\mathbb{D}}_r$ and according to Banach’s fixed point theorem, $F$ has a fixed point in $\mathbb{D}_r$. This implies that there exists a $z_0 \in \mathbb{D}_r$ such that $F(z_0) = z_0$ which contradicts (9) at $z_0 \in \mathbb{D}$ (and thus, (10) is not true for any $r \in (0, 1)$). Hence, we must have $|a_2| \leq 1 + \lambda v(|a|)$ for $f \in \mathcal{U}(\lambda)$.

To prove that the second coefficient inequality is sharp, we consider

$$(11) \quad \omega(z) = \frac{z + a}{1 + az}, \quad a \in (0, 1),$$

and we use that

$$v(a) = \int_0^1 \omega(t) \, dt.$$ 

Hence,

$$1 - (1 + \lambda v(a))z + \lambda z \int_0^z \omega(t) \, dt = 1 - z - \lambda z \int_z^1 \omega(t) \, dt =: G(z).$$
We claim that $G(z) \neq 0$ in $\mathbb{D}$. Since $G(0) = 1$, we may assume on the contrary that there exists a $z \in \mathbb{D} \setminus \{0\}$ such that $G(z) = 0$. This is equivalent to

$$\frac{1}{\lambda z} = \frac{1}{1 - z} \int_z^1 \omega(t) \, dt.$$  

As

$$\left| \frac{1}{\lambda z} \right| > 1 \text{ and } \left| \frac{1}{1 - z} \int_z^1 \omega(t) \, dt \right| \leq 1,$$

we have now proved that $G(z) \neq 0$ for $z \in \mathbb{D}$. In particular, this implies that the function $f$ defined by

$$f(z) = \frac{z}{1 - (1 + \lambda v(a)) z + \lambda z \int_0^z \omega(t) \, dt}$$

belongs to the family $U(\lambda)$, where $\omega$ is given by (11). This proves the sharpness. □

Moreover, one can show that a similar sharp inequality is sharp for any $\omega$ as above.

Since $|\int_{z_1}^{z_2} \omega(t) \, dt| \leq |z_1 - z_2|$, the function $\int_0^z \omega(t) \, dt$ is uniformly continuous in the open unit disk. Therefore this function can be extended continuously onto the closed unit disk. Hence, the function $v(\omega) := \max\{ |\int_0^z \omega(t) \, dt| : z \in \mathbb{D} \}$ is well defined. Suppose that $f \in \mathcal{U}(\lambda)$ is given by

$$f(z) = \frac{z}{1 - a_2 z + \lambda z \int_0^z \omega(t) \, dt}$$

for some $0 \leq \lambda < 1$, where $\omega \in \mathcal{B}$. Then

(12) \hspace{1cm} |a_2| \leq 1 + \lambda v(\omega),

is valid and this inequality is sharp.

In order to prove this inequality, we assume again that

$$|a_2| = \frac{1 + \lambda v(\omega)}{r}, \quad r \in (0, 1),$$

and do similar steps as in the proof of Theorem 4. The inequality (12) can be shown to be sharp in the following way: Consider

$$\tilde{\omega}(z) = e^{i\varphi} \omega(e^{i\theta} z),$$

where $\varphi, \theta \in [0, 2\pi)$ are chosen such that

$$v(\omega) = \int_0^1 \tilde{\omega}(t) \, dt.$$  

Next, we may proceed as before to complete the proof. However, we omit the details to avoid a repetition of the arguments.

A more detailed consideration of these cases can give more explicit bounds for $|a_2|$ as follows.
Theorem 5. Let $f \in \mathcal{U}(\lambda)$, $\lambda \in (0, 1)$, have the form (8) for some analytic function $\omega$ such that $|\omega(z)| \leq 1$ and $\omega(0) = a \in \mathbb{D}$. Let further

$$B_a(z) = \begin{cases} \frac{1}{a} - \frac{1-|a|^2}{a^2z} \log(1 + az) & \text{for } a \in \mathbb{D} \setminus \{0\}, \\ \frac{a}{z} & \text{for } |a| = 1, \\ \frac{a}{2} & \text{for } a = 0. \end{cases}$$

Then

$$|a_2| \leq 1 + \lambda \max \{|B_a(e^{it})| : t \in [0, 2\pi]\}.$$

The inequality is sharp.

Proof. The function $f$ considered here by (8) is a member of the class $\mathcal{U}(\lambda)$ if and only if $z/f(z) \neq 0$, which is equivalent to

$$a_2 \neq \frac{1}{z} + \lambda \int_0^z \omega(t) \, dt := C_\omega(z), \quad z \in \mathbb{D}.$$

Using the above argument, it is clear that the function $C_\omega$ can be extended continuously onto the boundary $\partial \mathbb{D}$. Moreover this function is univalent on $\overline{\mathbb{D}}$. The proof of this assertion is similar to the above arguments. Indeed if $C_\omega(z_1) = C_\omega(z_2)$ for some $z_1 \neq z_2$, $z_1, z_2 \in \overline{\mathbb{D}}$, then

$$\frac{\lambda}{z_1 - z_2} \int_{z_1}^{z_2} \omega(t) \, dt = \frac{1}{z_1 z_2}$$

which is not possible. Thus, $C_\omega$ is univalent on $\overline{\mathbb{D}}$ and therefore, for each $\omega$, the curve $C_\omega(e^{i\theta})$, $\theta \in [0, 2\pi]$, is a Jordan curve which divides the plane into two components. Let us call the bounded closed component $\overline{\mathbb{C}} \setminus C_\omega(\mathbb{D}) =: A_2(\omega)$. Obviously, the function $f$ is in the class $\mathcal{U}(\lambda)$ if and only if

$$a_2 \in \bigcup_{\omega(0) = a} A_2(\omega).$$

Now, we look at the curves $C_\omega(e^{i\theta})$, $\theta \in [0, 2\pi]$. Since $\omega(0) = a$, the modulus of the function

$$\frac{\omega(z) - a}{1 - \overline{a}\omega(z)}$$

is bounded by unity in the unit disk and this function vanishes at the origin. This means that $\omega$ can be represented in the form

$$\omega(z) = \frac{a + z\varphi(z)}{1 + \overline{a}z\varphi(z)},$$

where $\varphi$ is analytic in $\mathbb{D}$ and $|\varphi(z)| \leq 1$ for $z \in \mathbb{D}$. In other words, $\omega(z)$ is subordinated to $(a + z)/(1 + \overline{a}z)$, $z \in \mathbb{D}$. Since the function $(a + z)/(1 + \overline{a}z)$ maps the unit disk onto the unit disk, a convex domain, we may use now a theorem proved
by Hallenbeck and Ruscheweyh in [9] (compare with [11, Theorem 3.1b]). In our case this theorem implies that the function

$$\frac{1}{z} \int_0^z \omega(t) \, dt$$

is subordinated to the function

$$\frac{1}{z} \int_0^z \frac{a + t}{1 + at} \, dt = B_a(z).$$

Therefore, we get the representation

$$\int_0^z \omega(t) \, dt = \frac{1}{\varphi(z)} \int_0^{z_{\varphi(z)}} \frac{a + t}{1 + at} \, dt = zB_a(z\varphi(z)),$$

where $\varphi$ is analytic in $\mathbb{D}$ and $|\varphi(z)| \leq 1$ for $z \in \mathbb{D}$. Since $B_a$ is analytic in the closed unit disk, this representation together with the above considerations implies that

$$|a_2| \leq \sup_{z \in \mathbb{D}, \theta \in [0, 2\pi]} \left| e^{-i\theta} + \lambda e^{i\theta} B_a(z) \right| \leq 1 + \lambda \max\{|B_a(e^{it})| : t \in [0, 2\pi]\}.$$

Now, we have to prove the sharpness of the inequality. To that end, let $t_0$ be chosen such that

$$|B_a(e^{it_0})| = \max\{|B_a(e^{it})| : t \in [0, 2\pi]\}, \text{ and } B_a(e^{it_0}) = e^{i\alpha} |B_a(e^{it_0})|.$$  

We take $2\theta = -\alpha, \psi = t_0 - \theta$, consider the function

$$\omega(z) = \frac{a + ze^{i\psi}}{1 + az e^{i\psi}},$$

and let $a_2 = e^{-i\theta} + \lambda e^{i\theta} B_a(e^{it_0})$. Then we have

$$|a_2| = |e^{-2i\theta} + \lambda e^{i\alpha} |B_a(e^{it_0})|| = 1 + \lambda |B_a(e^{it_0})|.$$  

Further, we consider

$$D(z) = 1 - (e^{-i\theta} + \lambda e^{i\theta} B_a(e^{it_0})) z + \lambda z \int_0^z \frac{a + te^{i\psi}}{1 + at e^{i\psi}} \, dt.$$  

It is easily seen that in our case

$$D(z) = 1 - (e^{-i\theta} + \lambda e^{i\theta} B_a(e^{it_0})) z + \lambda z^2 B_a(z e^{i\psi}) \quad \text{and} \quad D(e^{i\theta}) = 0.$$  

The assumption, that there would exist a second zero $w$ of $D$ in the unit disk, leads to

$$\frac{1}{w} + \lambda \int_0^w \omega(t) \, dt = e^{-i\theta} + \lambda \int_0^{e^{i\theta}} \omega(t) \, dt,$$

which is impossible, because the right hand side of the last relation is seen to be $a_2$. This implies that the function $f(z) = z/D(z)$ is a member of the class $\mathcal{U}(\lambda)$. \qed
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