ON INDEX FORMULAS FOR MANIFOLDS WITH METRIC HORNS

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Abstract
In this paper we discuss the index problem for geometric differential operators (Spin–Dirac operator, Gauß-Bonnet operator, Signature operator) on manifolds with metric horns. On singular manifolds these operators in general do not have unique closed extensions. But there always exist two extremal extensions $D_{\min} \text{ and } D_{\max}$. We describe the quotient $D(D_{\max})/D(D_{\min})$ explicitely in geometric resp. topological terms of the base manifolds of the metric horns. We derive index formulas for the Spin–Dirac and Gauß-Bonnet operator. For the Signature operator we present a partial result.

1. Introduction

It is well known, that an elliptic differential operator

$$D : C_0^\infty(M, E) \to C_0^\infty(M, F)$$

between sections of vector bundles $E$ and $F$ over a compact, closed manifold $M$ has a unique closed extension which is a Fredholm operator. In the more general case of an open manifold

$$M = M_0 \cup U$$
with a compact part $M_\epsilon$ (with nonempty boundary $N$) and an open part $U$ which we consider as a punctured neighborhood of the “singularities” the situation is more complicated: There may be many closed extensions between two extremal extensions $D_{\min}$ and $D_{\max}$. These extensions can be parametrized by closed subspaces $V$ of the quotient $\mathcal{D}(D_{\max})/\mathcal{D}(D_{\min})$ ($\mathcal{D}$ denotes the domain of the operator). It is natural to ask for a characterization of $\mathcal{D}(D_{\max})/\mathcal{D}(D_{\min})$ for special “singular” manifolds as well as for the Fredholm property and explicit index formulas of the assigned closed extensions $D_V$ of a given operator $D$. The complete answers to these and related questions for the Gauß-Bonnet operator, the Signature operator and the Spin–Dirac operator (which we refer to as the “geometric operators”) on manifolds with cone like singularities are given in the papers [9], [10], [11], [13], [8], [2]. In this paper we consider manifolds with metric horns in the restricted sense of [10]. This means that $U \cong (0, \epsilon) \times N$ with warped product metric
\[
g = dx^2 + h(x)^2g_N,
\]
and $h(x) = x^\alpha$, $\alpha > 1$. On these manifolds all the geometric operators have a common normal form on $U$
\[
D|U \cong \left( \frac{d}{dx} + h'(x)S \right) \oplus \left( \frac{d}{dx} + \frac{1}{h(x)}\tilde{S} + \frac{h'(x)}{h(x)}\tilde{A}(x) \right).
\]
(1.1)

This will be discussed in detail in Section 2. Section 3 investigates the nature of $\mathcal{D}(D_{\max})/\mathcal{D}(D_{\min})$ and the Fredholm property. In Section 4 we elaborate on the index problem. The key tool is a homotopy transforming the operator of interest into a so called regular singular operator. Regular singular operators turn up in the conic situation, for which index formulas exist (cf. the papers mentioned above). With the help of a result in [12] we will show that the index remains constant during the homotopy. By this procedure we bypass the need of an explicit functional calculus on metric horns, which seems to be very difficult and not yet developed to the knowledge of the authors. (Cf. also the remark in [10, pp.138]: “At the level of an explicit functional calculus the more difficult case of metric horns must be distinguished from the special case of metric cones...”) The last two sections are devoted to the geometric operators as concrete examples.

We point out as a remarkable fact that the situation of metric horns turns out to be much simpler than the case of metric cones. For example, by looking at the space $\mathcal{D}(D_{\max})/\mathcal{D}(D_{\min})$ describing the variety of closed extensions, we prove the following characterizations for the geometric operators on manifolds $M$ with metric horns:
$D(D_{\text{max}})/D(D_{\text{min}}) \cong \begin{cases} 
\{0\} & \text{for the Signature operator}, \\
\{0\} & \text{for the Gauß-Bonnet operator (}m\text{ even)}, \\
\mathcal{H}^{n/2}(N) & \text{for the Gauß-Bonnet operator (}m\text{ odd)}, \\
\ker D_N & \text{for the Spin–Dirac operator (}m\text{ even)}, 
\end{cases}$

where $m = \dim M$, $n = \dim N$ and $D_N$ denotes the Spin–Dirac operator on the base manifold $N$ of the horns. Let us emphasize, that the nature of $D(D_{\text{max}})/D(D_{\text{min}})$ depends only on the topology of $N$ for the first two operators. It is an interesting question, whether there exists a canonical 1:1 correspondence between ideal boundary conditions and closed extensions of the Gauß-Bonnet operator in this particular situation (cf. [6, Theorem 3.8 a]).

For the Spin–Dirac operator $\ker D_N$ is not a topological invariant (cf. [15]). However, we have at least independence of $\dim D(D_{\text{max}})/D(D_{\text{min}})$ under conformal changes of the metric on $N$ for this operator. This yields a remarkable contrast to the cone like case, where always small eigenvalues (in the interval $(-1/2, 1/2)$) are involved. They are obviously not stable under even such an easy deformation as multiplying the metric of $N$ by a constant factor.

In Section 5 we derive an index formula for the Spin–Dirac operator:

$$\text{ind } D^+_{\text{max/\min}} = \int_M \hat{A} - \frac{\eta(0)}{2} \pm \frac{b}{2}$$

with $b := \dim \ker D_N$.

The index formula for the Gauß-Bonnet operator on manifolds with horns developed in Section 6 becomes simply

$$\text{ind } D_{\text{GB}} = \int_M e + \frac{1}{2} \left( \sum_{k=0}^{n-1} (1)^k \dim H^k(N) - \sum_{k=0}^{n} (-1)^k \dim H^k(N) \right).$$

We emphasize that this formula is not based on our analysis but is derived by reinterpretation of results of J. Cheeger. In contrast to the cone like case there occurs no additional spectral data of $N$ and we conclude in particular $\int_M e \in \mathbb{Z}$.

At this point we would like to make some historical remarks: A detailed analytic study (Hodge theory, functional calculus, index formulas, ...) of inductively defined singular spaces based on manifolds with metric cones was first carried out by J. Cheeger in the papers [9, 10, 11]. In the same spirit [13] considered the Spin–Dirac operator, stressing the importance of small eigenvalues of the operator on the base manifold of the cones. The paper [8] derived an index formula for differential operators with a special normal form. The framework of [8] covers the geometric operators on manifolds with asymptotic conic singularities and motivated the present paper to a great extend. In the
meantime the regular singular case was generalized in [2] and [3] and became applicable to singular algebraic curves (cf. [4]) and to manifolds with finitely many ends and some further geometric conditions. Another generalization of the regular singular case to operators of Fuchsian type is part of [7]. Homotopy arguments based on the result of [12, Theorem 4.1.] and similar to those in this article have been carried out in [21], [20] and [3].

Part of this article (concerning the Spin–Dirac operator on metric horns) is one of the main results of the thesis of the second named author.

List of Notations

\begin{align*}
\mathcal{D}(D) & \quad \text{domain of the (unbounded) operator } D \\
\mathcal{D}_{\text{min}}, \mathcal{D}_{\text{max}} & \quad \text{minimal/maximal closed extension of a differential operator} \\
D^\dagger & \quad \text{formal adjoint of the differential operator } D \\
\mathcal{L}(H) & \quad \text{algebra of bounded operators on the Hilbert space } H \\
X & \quad \text{operator of multiplication by } x \\
\Rightarrow & \quad \text{uniform convergence} \\
\| \cdot \|_{\text{HS}} & \quad \text{Hilbert–Schmidt norm} \\
\sigma(S) & \quad \text{spectrum of the self–adjoint operator } S \\
P^* & \quad \text{Hilbert space adjoint of the (possibly unbounded) operator } P
\end{align*}

2. A generalization of regular singular operators

We consider the following situation:

Let \((M, g_M)\) be a (singular) Riemannian manifold of dimension \(m\), \(E, F\) two Riemannian resp. hermitian vector bundles over \(M\) and \(D : C_0^\infty(E) \to C_0^\infty(F)\) a first order elliptic differential operator. Let \(U\) be a neighborhood of the singularities, such that \(M\setminus U\) is compact with smooth compact boundary \(N\). Furthermore, let \(G\) be a Riemannian resp. hermitian vector bundle over \(N\), such that we obtain the following identifications by a suitable unitary separation of variables (cf. [8]):

\[
\begin{align*}
U & \cong (0, \epsilon) \times N \quad \text{with } 0 < \epsilon \leq 1, \\
D & \cong \frac{d}{dx} + \frac{a}{x} S \quad \text{on the open set } U.
\end{align*}
\]

\(S : C^\infty(G) \to C^\infty(G)\) is supposed to be a first order symmetric elliptic differential operator on \(N\). By a slight abuse of notation, there exists an orthonormal basis of \(L^2(G), \{e_s\}\), of eigenfunctions of \(S\).
At this point the meaning of the constant $\alpha \geq 1$ is not apparent because we could write $S$ instead of $\alpha S$. However, the parameter $\alpha$ will play a crucial role in the case of a metric horn. To avoid notational confusion, it is introduced here.

This is the easiest example of a so called regular singular operator. More general operators of regular singular type are treated in [8], [2], [3] or [17]. The example above serves as a starting point for a general framework, which applies to the “geometric operators” (i.e. Spin–Dirac operator, Gauß-Bonnet operator and Signature operator) on manifolds with metric horns. However, in order to treat the case of horns we will have to leave the regular singular situation.

Before doing so, let us introduce some results of [8] and [2] which we shall need:

1. Let $H_{\frac{1}{2\alpha}} := \bigoplus_{|s| < 1/(2\alpha)} \ker (S - s)$ and consider the Hilbert spaces $\mathcal{D}(D_{\text{max}})$ and $\mathcal{D}(D_{\text{max}}^t)$ equipped with graph-norms. Then the linear functionals

$$\Phi : \begin{cases} \mathcal{D}(D_{\text{max}}) \to H_{\frac{1}{2\alpha}} \\ f \mapsto \bigoplus_{|s| < 1/(2\alpha)} \lim_{x \to 0} x^\alpha \Pr_{\ker (S - s)} f(x), \end{cases}$$

$$\Phi' : \begin{cases} \mathcal{D}(D_{\text{max}}^t) \to H_{\frac{1}{2\alpha}} \\ f \mapsto \bigoplus_{|s| < 1/(2\alpha)} \lim_{x \to 0} x^{-\alpha} \Pr_{\ker (S - s)} f(x), \end{cases}$$

are well defined and continuous. Here $\Pr_W : L^2(G) \to W$ denotes the orthogonal projection onto $W \subset H_{\frac{1}{2\alpha}}$.

2. For all $f \in \mathcal{D}(D_{\text{max}})$ and $g \in \mathcal{D}(D_{\text{max}}^t)$

$$(D_{\text{max}} f, g) = (f, D_{\text{max}}^t g) - (\Phi f, \Phi' g)_{H_{\frac{1}{2\alpha}}}.$$ 

3. All closed extensions of $D$ are Fredholm. These extensions are in 1:1 correspondence to subspaces of $H_{\frac{1}{2\alpha}}$. The corresponding extension to a subspace $W \subset H_{\frac{1}{2\alpha}}$ is given by

$$\mathcal{D}(D_W) := \{ f \in \mathcal{D}(D_{\text{max}}) \mid \Phi f \in W \}.$$

Any orthogonal decomposition $H_{\frac{1}{2\alpha}} = V \perp W$ implies: $D_{\text{max}}^t = (D_W)^*$.

We now describe the generalizations which will be applicable to metric horns. We consider again the situation described at the beginning of this section, but now with a more general

$$D : C_0^\infty(E) \to C_0^\infty(F).$$

(2.1)
We assume that after a suitable choice of isometries of Hilbert spaces
\[
L^2(E|U) \cong L^2(0, \epsilon), L^2(G),
\]
\[
L^2(F|U) \cong L^2((0, \epsilon), L^2(G))
\]
and a suitable splitting \(L^2(G) = H_{tot} = H \oplus \tilde{H}\) (\(H, \tilde{H}\) Hilbert spaces), \(D|U\) decomposes into \(T \oplus \tilde{T}\) with
\[
T = \frac{d}{dx} + \frac{h'(x)}{h(x)} S,
\]
\[
\tilde{T} = \frac{d}{dx} + \frac{1}{h(x)} \tilde{S} + \frac{h'(x)}{h(x)} \tilde{A}(x),
\]
and \(h(x) = x^\alpha\) for \(\alpha > 1\). The operator \(T : C^\infty_0((0, \epsilon), H)\) is the same as introduced at the beginning since \(\frac{h'}{h} = \frac{\alpha}{x}\). So \(S\) is supposed to be a self-adjoint operator on \(D(S)\) with an appropriate family of eigenfunctions \(e_s\). In the case of geometric differential operators on metric horns it turns out that \(H\) is of finite dimension which makes the regular singular component of the operator \(D|U\) particularly easy to handle.

Let us turn over to the properties of \(\tilde{T}\). For notational convenience we put a tilde \(\tilde{}\) over all notions connected with this operator. \(\tilde{S}\) is a symmetric operator defined on a dense subspace \(D(\tilde{S})\) of \(\tilde{H}\). As before, let \(\{\tilde{e}_s\}\) be an orthonormal basis of \(\tilde{H}\) consisting of eigenfunctions of \(\tilde{S}\). Additionally, we assume \(\text{ker } \tilde{S} = \{0\}\). \(\tilde{A}(\cdot)\) is a smooth family of bounded operators on \(\tilde{H}\): \(\tilde{A} \in C^\infty((0, \epsilon), L(H))\). Moreover, there exists a constant \(C > 0\) with
\[
\|\tilde{A}(x)\| \leq C \quad \text{for all } x \in (0, \epsilon).
\]
In the sequel we consider the term \(\frac{h'}{h} \tilde{A}\) as a perturbation and denote by
\[
\tilde{T}_0 = \frac{d}{dx} + \frac{1}{h(x)} \tilde{S}
\]
the “unperturbed part of \(\tilde{T}\)”. This point of view turns out to make sense as long as \(\alpha > 1\).

The geometric operators on manifolds with metric horns are examples of this general framework. Let us discuss this in more detail:

**Spin–Dirac operator** Let \(M\) be an open spin manifold of even dimension with metric \(g_M\), where \(U\) and \(N\) are as in the beginning of this section. Assume the existence of a fixed metric \(g_N\) with
\[
g_M|U \cong dx^2 + h(x)^2 g_N.
\]
Denote by $S(M) \to M$ an irreducible spin bundle and $D : C_0^\infty(S(M)) \to C_0^\infty(S(M))$ the corresponding Spin–Dirac operator. As it is well known, $D$ splits into

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

with respect to the eigenbundles $S^+(M), S^-(M)$ of the multiplication with the complex volume element. By a suitable unitary separation of variables $D^+$ can be transformed into (see [20, p. 652])

$$D^+ \cong \frac{d}{dx} + \frac{1}{h(x)} D_N$$

on $U$, (2.6)

where $D_N : C^\infty(S(N)) \to C^\infty(S(N))$ denotes the Spin–Dirac operator on $N$ with the induced spin structure. (We use the orientation of [1] which is opposite to that in [20].) Consequently, we choose $H := \ker D_N$, $\tilde{H}$ its orthogonal complement in $L^2(S(N))$ and $S = 0$, $\tilde{S} := D_N|\tilde{H}$, $\tilde{A} = 0$.

**Gauß-Bonnet operator** This operator is given by

$$D_{GB} := d_M + d_M^t : \Omega^{even}_0(M) \to \Omega^{odd}_0(M).$$

Let $M$ be an open Riemannian manifold and $U, N, g_M$ and $g_N$ etc. as above. The calculations in [8, pp. 696] yield:

$$D_{GB}|U \cong T_{GB} : C^\infty((0, \epsilon), \bigoplus_{j=0}^n \Omega^j(N)) \to C^\infty((0, \epsilon), \bigoplus_{j=0}^n \Omega^j(N))$$

with $n = \dim N$, $T_{GB} = \frac{d}{dx} + \frac{k(x)}{h(x)} S_1 + \frac{1}{h(x)} S_2$ and

$$S_1 = \text{diag}(c_j)_{0 \leq j \leq n}, \quad c_j := (-1)^j(\epsilon - \frac{n}{2}),$$

$$S_2 = d_N + d_N^t = \begin{pmatrix} 0 & d_N^t \\ d_N & \ddots & \ddots \\ & \ddots & \ddots & d_N^t \\ & & \ddots & 0 \end{pmatrix}.$$ 

Obviously, $S_2$ is a square root of the Laplacian on forms. (For metric collars, i.e. $h = 1$, the situation simplifies to $D_{GB}|U \cong \frac{d}{dx} + S_2$.)

In the case of metric horns, i.e. $h(x) = x^a$, we choose $H$ to be the space $\mathcal{H}(N)$ of harmonic forms and $\tilde{H}$ its orthogonal complement in $L^2(\Lambda^*_T N)$. $H$ and $\tilde{H}$ are obviously invariant subspaces of $S_1$ and $S_2$ and $S_2|H = 0$. Moreover, we choose

$$S := S_1|H, \quad \tilde{S} := S_2|\tilde{H}, \quad \tilde{A} := S_1|\tilde{H}.$$
**Signature operator** We assume $M$ to be an oriented, open Riemannian manifold of dimension $m = 4k$, $U$ and $N$ as above and that (2.5) holds. The involution

$$\tau : \left\{ \begin{array}{c} \Omega(M) \to \Omega(M), \\
\tau \alpha := i^{2k+j(j+1)} \alpha \quad \text{for} \ \alpha \in \Omega^j(M) 
\end{array} \right.$$  

anti-commutes with the operator $d_M + d_M^t : \Omega_0(M) \to \Omega_0(M)$. Hence $d_M + d_M^t$ interchanges the $\pm 1$-eigenspaces $\Omega^\pm_0(M)$ of $\tau$. The restriction

$$D_S : d_M + d_M^t : \Omega^+_0(M) \to \Omega^-_0(M)$$

is called the Signature operator. Unitary separation of variables yields according to [8, pp.707]:

$$D_S|U \cong T_S = \frac{d}{dx} + \frac{h'(x)}{h(x)} S_1 + \frac{1}{h(x)} S_2$$

with $S_1 = \text{diag}(b_j)_{0 \leq j \leq n}$, $n = \dim N$, $b_j = \frac{n}{2} - j$ and

$$S_2|\Omega^j(N) = (-1)^{k+1+\left[\frac{j+1}{2}\right]}((-1)^j \ast_N d_N - d_N \ast_N).$$

$S_2$ is symmetric and $S_2^2 = \Delta_N$. Consequently, $H := \mathcal{H}(N)$ and $\tilde{H} := \mathcal{H}(N)^\perp$ are invariant subspaces of $S_1$ and $S_2$. Choose

$$S := S_1|H, \quad \tilde{S} := S_2|\tilde{H}, \quad \tilde{A} := S_1|\tilde{H}.$$

It was pointed out by the referee that the metric horn case can be ‘reduced’ to the conic case by a logarithmic change of variables. Namely, given the metric

$$g = dx^2 + x^{2\alpha} g_N, \quad \alpha > 1.$$  

Putting $x(t) = ((1 - \alpha) \log t)^{\frac{1}{1-\alpha}}$ we find

$$g = \frac{x(t)^{2\alpha}}{t^2} (dt^2 + t^2 g_N) =: \rho(t)^2 (dt^2 + t^2 g_N),$$

which is a conic metric, however with a singular conformal factor. Applying the standard unitary separation of variables (cf. [8, p.443]) to the Gauss-Bonnet operator, one finds

$$D_{GB} = \frac{1}{\rho} \left( \frac{d}{dt} + \frac{1}{t} (S_1 + S_2) + \frac{\rho'}{\rho} \tilde{S}_3 \right),$$
where $S_1, S_2$ are as in the above description of the Gauß-Bonnet operator and $S_3 = \text{diag}(\tilde{c}_j)_{0 \leq j \leq n}, \tilde{c}_j := (-1)^j (j - \frac{n}{2}) - \frac{1}{2}$. However, $1/\rho, \rho'/\rho^2$ are singular at 0 and the usual regular singular analysis cannot be applied directly.

We turn back to our general $D$ described in (2.1)–(2.3). In Section 3 we will characterize the $L^2$–closed extensions of $D$ and show that these are Fredholm. Section 4 deals with a particular family $\{D_\beta\}_{\beta \in [\beta_1, \beta_2]}$ of operators and we will show that the index of corresponding $L^2$-closed extensions remains constant under variation of $\beta$. In both sections we need to construct parametrices and to prove certain properties about them. The construction is done by patching together an interior pseudo–differential parametrix and a suitable boundary parametrix. The boundary parametrix will be obtained by considering $D|U$ as an infinite sum of one–dimensional differential operators (cf. (2.10) below).

For the construction of the boundary parametrix we have to introduce some integral operators:

Similar to [18, 4.5/4.6], to a given function $F \in C^\infty(0,1)$ and $s \in \mathbb{R}$ we introduce (whenever these integrals exist)

$$
P_{0,s}^F f(x) := \int_0^x \exp\left(-\int_y^x sF(t)dt\right) f(y)dy,
$$

$$
P_{1,s}^F f(x) := \int_1^x \exp\left(-\int_y^x sF(t)dt\right) f(y)dy,
$$

for all $f \in L^2(0,1)$.

In the particular case $F(x) = 1/x$ these operators become

$$
P_{0,s}^F f(x) := \int_0^x \left(\frac{y}{x}\right)^s f(y)dy \quad \text{for } s > -\frac{1}{2},
$$

$$
P_{1,s}^F f(x) := \int_1^x \left(\frac{y}{x}\right)^s f(y)dy \quad \text{for all } s.
$$

These operators were already used in [18, (2.3), (2.4)].

One easily checks

$$
\left(\frac{d}{dx} + Fs\right) P_{j,s}^F f = f, \quad \text{for } f \in L^2(0,1),
$$

$$
P_{j,s}^F \left(\frac{d}{dx} + Fs\right) f = f, \quad \text{for } f \in C_0^\infty(0,1),
$$

$$
(P_{0,s}^F)^* = -P_{1,-s}^F.
$$

We introduce the unperturbed operators

$$
\tilde{T}_0 := \frac{d}{dx} + \frac{1}{h(x)}\tilde{S}, \quad D_0 := T \oplus \tilde{T}_0,
$$

where $D_0$ lives on $U$. 
Furthermore, we denote by $\Pi, \tilde{\Pi}$ the orthogonal projections from $H_{\text{tot}} := H \oplus \tilde{H}$ onto $H$ resp. $\tilde{H}$.

Via the spectral decomposition of $S, \tilde{S}$ the operators $T, \tilde{T}_0$ can be viewed as the infinite direct sum

$$
\oplus_{s \in \text{spec } S} \left( \frac{d}{dx} + \frac{h'}{h}s \right) \oplus \oplus_{s \in \text{spec } \tilde{S}} \left( \frac{d}{dx} + \frac{1}{h}s \right).
$$

(2.10)

Thus it seems natural to construct parametrices for $T, \tilde{T}$ as follows: Put

$$
P := P(T) := \left( \oplus_{s < s_1} P_{1,s}^{h'/h} \right) \oplus \left( \oplus_{s > s_1} P_{0,s}^{h'/h} \right),
$$

(2.11)

where $s_1 \in [-1/(2\alpha), 1/(2\alpha)] \setminus \text{spec } (S)$. The choice of $s_1$ corresponds to a particular choice of a $L^2$-closed extension of $D$. Furthermore, the parametrix for $\tilde{T}$ is:

$$
\tilde{P} := \tilde{P}(\tilde{T}) := \left( \oplus_{s < 0} P_{1,s}^{1/h} \right) \oplus \left( \oplus_{s > 0} P_{0,s}^{1/h} \right).
$$

(2.12)

The boundary parametrix is then given by $P_{\text{bd}} = P \oplus \tilde{P}$ and all operators $P_{j,s}^{h'/h}, P_{j,s}^{1/h}$ in this construction are Hilbert-Schmidt. We will use the following notation for particular choices of $s_1$ in (2.11):

$$
P^{\text{max}} := P^{\text{max}}(T) := \left( \oplus_{s < 1/(2\alpha)} P_{1,s}^{h'/h} \right) \oplus \left( \oplus_{s \geq 1/(2\alpha)} P_{0,s}^{h'/h} \right),
$$

$$
P^{\text{min}} := P^{\text{min}}(T) := \left( \oplus_{s \leq -1/(2\alpha)} P_{1,s}^{h'/h} \right) \oplus \left( \oplus_{s > -1/(2\alpha)} P_{0,s}^{h'/h} \right),
$$

(2.13)

$$
P_{\delta} := P_{\delta}(T) := \left( \oplus_{s < 0} P_{1,s}^{h'/h} \right) \oplus \left( \oplus_{s \geq 0} P_{0,s}^{h'/h} \right).
$$

In view of (2.8) we have

$$
P^{\text{max/min}}(T) = -P^{\text{min/max}}(-T^t), \quad \tilde{P}(\tilde{T}) = -\tilde{P}(-\tilde{T}^t).
$$

(2.14)
3. The $L^2$–closed extensions of $D$

The goal of this section is to classify the $L^2$–closed extensions of $D : C_0^\infty(E) \to C_0^\infty(F)$ and to prove their Fredholm property. In doing so, we follow step by step the scheme of [8]. It turns out, that the calculations for the second component $\tilde{T}$ can be carried out in a very similar way as in the regular singular case. Most of the calculations are even more simple than in the regular singular case.

In this section we relax the axioms (2.3) for $D$ slightly. Namely, we assume that $D|U$ decomposes into $T \oplus \tilde{T}$ with

$$T = \frac{d}{dx} + \frac{\alpha}{x}S,$$

$$\tilde{T} = \frac{d}{dx} + \frac{1}{h(x)}\tilde{S} + \frac{1}{x}\tilde{A}(x),$$

where $h$ is a positive increasing function with $h(x) \leq x^\alpha$ for some $\alpha > 1$. Moreover, we assume that $(0, 1) \ni x \mapsto (I + |\tilde{S}|)^{-1}\tilde{A}(x)$ and $(0, 1) \ni x \mapsto \tilde{A}(x)(I + |\tilde{S}|)^{-1}$ are smooth maps into $L(\tilde{H})$ and that

$$\sup_{0 < x < 1} \|(I + |\tilde{S}|)^{-1}\tilde{A}(x)\| + \|\tilde{A}(x)(I + |\tilde{S}|)^{-1}\| \leq C. \quad (3.1)$$

We will use the abbreviation

$$\mu(x) := \int_x^1 \frac{1}{h(y)}dy. \quad (3.2)$$

Note that

$$\lim_{x \to 0^+} \mu(x) = +\infty. \quad (3.3)$$

**Lemma 3.1** (cf. [8, Lemma 4.1]) For $f \in L^2(0, 1)$ we have the following estimates:

(i) $|P_{0,s}^{1/h}f(x)| \leq \frac{1}{\sqrt{2s}}\sqrt{h(x)}\|f\|_{L^2}, \quad s > 0,$

(ii) $|P_{1,s}^{1/h}f(x)| \leq \frac{c(h, s_0)}{\sqrt{|s|}}x^{\alpha/2}\|f\|_{L^2}, \quad s \leq s_0 < 0.$

**Proof** We use the Cauchy–Schwarz to obtain the estimate

$$|P_{i,s}^{1/h}f(x)| \leq e^{s\mu(x)}\left|\int_x^\infty e^{-2s\mu(y)}dy\right|^{1/2}\|f\|_{L^2}, \quad j = 0, 1. \quad (3.4)$$
Because of the monotonicity of $h$ this implies for $j = 0$

$$|P_{0,s}^{1/h}f(x)| \leq \sqrt{h(x)}e^{s\mu(x)} \left( \int_0^x \frac{1}{h(x)}e^{-2s\mu(y)}dy \right)^{1/2} \|f\|_{L^2}$$

$$= \frac{\sqrt{h(x)}}{\sqrt{2s}}\|f\|_{L^2},$$

and we have proved the first inequality.

In view of (3.4) to prove (ii) it suffices to prove

$$e^{2s\mu(x)} \int_x^1 e^{-2s\mu(y)}dy \leq \frac{c(h,s_0)}{|s|} x^\alpha.$$

To do this we split the integral:

$$e^{2s\mu(x)} \int_x^1 e^{-2s\mu(y)}dy = e^{2s\mu(x)} \int_x^{2x} e^{2s\mu(y)} \frac{1}{h(y)}e^{-2s\mu(y)}dy$$

$$\leq \frac{h(2x)}{2|s|}(1 - e^{2|s|(\mu(2x) - \mu(x)))})$$

$$\leq \frac{h(2x)}{2|s|} \leq \frac{2^{\alpha-1}}{|s|} x^\alpha.$$

Similarly we find the estimate

$$e^{2s\mu(x)} \int_{2x}^1 e^{-2s\mu(y)}dy \leq \frac{h(1)}{2|s|} \left(e^{2|s|(\mu(2x) - \mu(x))} - e^{2|s|\mu(x)} \right)$$

$$\leq \frac{c(h)}{e^{2|s|}f_x^2 \frac{dy}{h(y)}}$$

$$\leq \frac{c(h,s_0)}{|s|} x^\alpha.$$

Here we have used the inequality

$$e^{-|s|} \int_x^{2x} \frac{dy}{h(y)} \leq c(h,s_0,\gamma)x^\gamma, \quad |s| \geq s_0$$

for any $\gamma \geq 0$. This inequality is an easy consequence of $h(x) \leq x^\alpha$. \hfill \Box

**Lemma 3.2** Let $\beta > -\alpha$ and $0 < \epsilon \leq 1$. Then in $L^2(0,\epsilon)$ we have

$$\|X^\beta P_{0,s}^{1/h}\| + \|P_{1,-s}^{1/h}X^\beta\| \leq \frac{1}{s} \epsilon^{\alpha+\beta}, \quad s > 0,$$

$$\|X^\beta P_{0,s}^{1/h}\| + \|P_{0,-s}^{1/h}X^\beta\| \leq \frac{c(h,\beta,s_0)}{|s|} \epsilon^{\alpha+\beta}, \quad s \leq s_0 < 0.$$
Proof Since \((X^β P_{1/h}^{1/s})^* = P_{1,s}^1 X^β, (X^β P_{1/h}^{1/s})^* = P_{0,s}^1 X^β\), it suffices to estimate \(\|X^β P_{1/h}^{1/s}\|\) for \(s > 0\) and \(\|P_{1,s}^1 X^β\|\) for \(s ≥ |s_0| > 0\). Since the operators are integral operators with non-negative kernels, we may apply Schur’s test [16, p. 22].

The kernel of \(X^β P_{0,s}^{1/h}\) is given by

\[
k_1(x, y) = x^β e^{s(μ(x) - μ(y))}, \quad x > y.
\]

We have

\[
\int_0^x k_1(x, y) dy \leq x^β e^{sμ(x)} \int_0^x y^α \frac{1}{h(y)} e^{-sμ(y)} dy \\
\leq \frac{x^α + 1}{s} \leq \frac{ε^{α + β}}{s},
\]

and

\[
\int_0^x k_1(x, y) dx \leq e^{-sμ(y)} \int_y^ε x^α \frac{1}{h(x)} e^{sμ(x)} dx \\
\leq \frac{ε^{α + β}}{s} (1 - e^{s(μ(ε) - μ(y))}) \\
\leq \frac{ε^{α + β}}{s},
\]

hence Schur’s test implies \(\|X^β P_{0,s}^{1/h}\| ≤ \frac{ε^{α + β}}{s}\).

Next let \(s ≥ |s_0| > 0\) and consider the kernel of \(P_{0,s}^1 X^β\),

\[
k_2(x, y) = y^β e^{s(μ(x) - μ(y))}, \quad x > y.
\]

Then similarly as before

\[
\int_0^ε k_2(x, y) dy \leq \frac{ε^{α + β}}{s},
\]

and similar as in the proof of Lemma [3.1] we estimate

\[
\int_0^ε k_2(x, y) dx \leq y^β e^{-sμ(y)} \int_y^ε x^α \frac{1}{h(x)} e^{sμ(x)} dx \\
\leq y^β e^{-sμ(y)} \left( 2^α y^{α}(e^{sμ(y)} - e^{sμ(2y)}) + ε^α(e^{sμ(2y)} - e^{sμ(ε)}) \right) \\
\leq \frac{c(h, β, s_0)ε^{α + β}}{s},
\]

and invoking again Schur’s test we reach the conclusion. □
Now we introduce the operators
\[ P_{\text{bd}}^{\max/\min} := P_{\text{max}}^{\max/\min} \oplus \tilde{P}. \] (3.5)

For the next lemma recall the notation \( H_{\text{tot}} = H \oplus \tilde{H} \). \( \Pi \) and \( \tilde{\Pi} \) are the corresponding orthogonal projections.

**Lemma 3.3** 1. For \( f \in L^2((0,1), H_{\text{tot}}) \) we have the estimates
\[
\|\Pi(I + |S|) f(x)\| \leq c \sqrt{x} \|f\|,
\]
\[
\|\tilde{\Pi}(I + |\tilde{S}|) f(x)\| \leq c x^{\alpha/2} \|f\|.
\] (3.6)

2. For \( 0 < \varepsilon \leq 1 \) we have
\[
\|X^{-1} \tilde{A} P_{\text{bd}}^{\min/\max} \|_{L^2((0,\varepsilon), H_{\text{tot}})} + \|P_{\text{bd}}^{\min/\max} X^{-1} \tilde{A}\|_{L^2((0,\varepsilon), H_{\text{tot}})} \leq c \varepsilon^{(\alpha-1)}.
\]

3. \( P_{\text{bd}}^{\max} \) maps \( L^2((0,1), H_{\text{tot}}) \) into \( \mathcal{D}(D_{\text{max}}) \).

**Proof** 1. The first inequality follows from \( \ref{5} \), Lemma 2.1, the second is an immediate consequence of Lemma 3.1.

2. This is a consequence of Lemma 3.2 and (3.1).

3. This is an immediate consequence of (2.8) and Lemma 3.1 \( \square \)

**Lemma 3.4** 1. (cf. \( \ref{2} \), Lemma 3.6) Let \( \varphi \in C_0^\infty([0,1)) \) with \( \varphi = 1 \) near 0. Then we have for \( f \in \mathcal{D}(D_{\text{max}}) \)
\[
P_{\text{bd}}^{\max} D_{\text{max}} \varphi f = \varphi f + (P_{\text{bd}}^{\max} X^{-1} \tilde{A}) \varphi f.
\]

2. (cf. \( \ref{4} \), Lemma 3.4) There exists an \( \varepsilon > 0 \), such that for \( \varphi, \omega \in C_0^\infty([0,\varepsilon)), \varphi, \omega = 1 \) near 0 and \( \omega \varphi = \varphi \), we have for \( f \in \mathcal{D}(D_{\text{max}}) \)
\[
\varphi f = \omega P_{\text{bd}}^{\max} V D_{\text{max}} \varphi f
\]
with some bounded operator \( V \).

**Proof** 1. Since the projections \( \Pi, \tilde{\Pi} \) commute with \( P_{\text{bd}}^{\max}, D_{\text{max}} \) it follows from \( \ref{2} \), Sec. 3] that
\[
\Pi P_{\text{bd}}^{\max} D_{\text{max}} \varphi f = \varphi \Pi f.
\]

Now we put
\[
g := \varphi f + P_{\text{bd}}^{\max} X^{-1} \tilde{A} \varphi f.
\]
and $g_s(x) = (g(x), \tilde{e}_s)$ where $\tilde{e}_s$ is an eigenvector of $\tilde{S}$, $\tilde{S}\tilde{e}_s = s\tilde{e}_s$. We have $g_s \in L^2(0, 1)$ and, moreover,

$$g'_s(x) + \frac{s}{h(x)} g_s(x) = ((D_{\text{max}} \varphi f)(x), \tilde{e}_s).$$

Thus $g_s$ and $((P_{\text{bd}}^{\text{max}} D_{\text{max}} \varphi f)(x), \tilde{e}_s)$ satisfy the same first order differential equation. If $s < 0$, then

$$g_s(1) = \left( P_{1,s}^{1/h} (X^{-1} \tilde{A} \varphi f(\cdot), \tilde{e}_s) \right)(1) = 0$$

and

$$((P_{\text{bd}}^{\text{max}} D_{\text{max}} \varphi f)(1), \tilde{e}_s) = P_{1,s}^{1/h} (D_{\text{max}} \varphi f, \tilde{e}_s)(1) = 0$$

hence we have equality in this case.

If $s > 0$, then the $g_s$ and $((P_{\text{bd}}^{\text{max}} D_{\text{max}} \varphi f)(x), \tilde{e}_s)$ differ by $ce^s$, which is square integrable iff $c = 0$. Since $g_s$ and $((P_{\text{bd}}^{\text{max}} D_{\text{max}} \varphi f)(x), \tilde{e}_s)$ are square integrable they must be equal.

2. In view of Lemma $3.3$, we choose $\varepsilon > 0$ such that

$$\|X^{-1} \tilde{A} P_{\text{bd}}^{\text{min/max}}\|_{L^2((0, \varepsilon), H_{\text{tot}})} + \|P_{\text{bd}}^{\text{min/max}} X^{-1} \tilde{A}\|_{L^2((0, \varepsilon), H_{\text{tot}})} < \frac{1}{2}.$$ 

Furthermore, let $\omega, \psi, \varphi \in C^\infty_0((0, \varepsilon), \omega = \psi = \varphi = 1$ near 0 and $\omega \psi = \psi, \psi \varphi = \varphi$.

By 1.

$$(\omega P_{\text{bd}}^{\text{max}} D_{\text{max}} \psi) \varphi f = \varphi f + (\omega P_{\text{bd}}^{\text{max}} X^{-1} \tilde{A} \psi) \varphi f =: (I + R) \varphi f$$

and $\|R\| < 1/2$.

Introducing $R_1 := \psi X^{-1} \tilde{A} P_{\text{bd}}^{\text{max}} \omega$, we also have $\|R_1\| < 1/2$. By induction, one easily finds

$$R^n (\omega P_{\text{bd}}^{\text{max}} D_{\text{max}} \psi) \varphi f = \omega P_{\text{bd}}^{\text{max}} R^n D_{\text{max}} \varphi f.$$ 

Thus we conclude

$$\varphi f = (I + R)^{-1}(\omega P_{\text{bd}}^{\text{max}} D_{\text{max}} \psi) \varphi f$$

$$= \sum_{n=0}^\infty (-1)^n R^n (\omega P_{\text{bd}}^{\text{max}} D_{\text{max}} \psi) \varphi f$$

$$= \omega P_{\text{bd}}^{\text{max}} D_{\text{max}} \varphi f + \omega P_{\text{bd}}^{\text{max}} \sum_{n=1}^\infty (-1)^n R_1^n D_{\text{max}} \varphi f$$

$$=: \omega P_{\text{bd}}^{\text{max}} V D_{\text{max}} \varphi f$$

and we are done. \qed
Corollary 3.5 Let \( \varphi \) be as before.
1. We have \( \varphi \mathcal{D}(D_{\max/min}) = \varphi \mathcal{D}(D_{0,max/min}) \).
2. For \( f \in \mathcal{D}(D_{\max}) \) we have \( \|\tilde{\Pi}_f(x)\| = O(x^{\alpha/2}), \ x \to 0 \).
3. The maps \( \Phi \) and \( \Phi' \) are well-defined on \( \mathcal{D}(D_{\max}) \) and \( \mathcal{D}(D_t^{\max}) \) and for \( f \in \mathcal{D}(D_{\max}), g \in \mathcal{D}(D_t^{\max}) \) we have
   \[
   (D_{\max} \varphi f, \varphi g) - (\varphi f, D_t^{\max} \varphi g) = (\Phi f, \Phi' g)_{H^{1/2}}.
   \]

Proof 1. If \( f \in \mathcal{D}(D_{\max}) \), then Lemma 3.4, 2. shows that \( \varphi f \in \text{Im} \varphi P_{\text{bd}}^{\max} \subset \mathcal{D}(D_{0,max}) \) by Lemma 3.3, 2. On the other hand, if \( f \in \mathcal{D}(D_{0,max}) \), then \( \varphi f \in \mathcal{D}(D_{\max}) \).

On \( C_0^\infty((0,\varepsilon), \tilde{H}) \) we have, for \( \varepsilon \) small enough
\[
\|X^{-1}\tilde{A}f\| \leq \|X^{-1}\tilde{A}P_{\text{bd}}^{\min} D_{0,min} f\| < \frac{1}{2} \|D_{0,min} f\|,
\]

hence \( X^{-1}\tilde{A} \) is \( D_0 \)-bounded with bound \( < 1/2 \) which easily implies \( \varphi \mathcal{D}(D_{\min}) = \varphi \mathcal{D}(D_{0,\min}) \).

2. This is an immediate consequence of Lemma 3.4 and Lemma 3.3.

3. In view of 1. and 2. it suffices to prove this for \( D_0 \).

We use the decomposition \( \varphi f = f_0 + \tilde{f}_0 \) and \( \varphi g = g_0 + \tilde{g}_0 \). As a consequence of Corollary 3.3, 2.
\[
(\tilde{T} f_0, \tilde{g}_0) - (\tilde{f}_0, \tilde{T}^t \tilde{g}_0) = \int_0^\varepsilon \partial_x (\tilde{f}_0(x), \tilde{g}_0(x)) \tilde{H} \ dx
\]
\[
= - \lim_{x \to 0} (\tilde{f}_0(x), \tilde{g}_0(x)) \tilde{H}
\]
\[
= 0. \quad (3.7)
\]

This implies together with result 2 of Section 2 and the identity \( D_{\max}^t = D_{\min}^* \)
\[
(\tilde{T} f_0, \tilde{g}_0) + (T f_0, g_0) = (\varphi f, D_{\max}^t \varphi g) - (\Phi f, \Phi' g)_{H^{1/2}}. \quad \Box
\]

Theorem 3.6 The closed extensions of \( D \) are in \( 1:1 \) correspondence to subspaces of \( H_{1/2}^{\alpha} \). For a given subspace \( W \subset H_{1/2}^{\alpha} \) the corresponding extension is given by
\[
\mathcal{D}(D_W) := \{ f \in \mathcal{D}(D_{\max}) | \Phi f \in W \}.
\]

Any orthogonal decomposition \( H_{1/2}^{\alpha} = V \perp W \) implies \( D_V^t = (D_W)^* \).
Proof  Obviously, every closed extension $\bar{D}$ corresponds to a closed vector space $\tilde{W} = \mathcal{D}(\bar{D})$ between $\mathcal{D}(D_{\text{min}})$ and $\mathcal{D}(D_{\text{max}})$ with respect to the graph norm. So we only have to show, that

$$\Phi : \mathcal{D}(D_{\text{max}}) \to H^{1/2}$$

is a continuous epimorphism with kernel $\mathcal{D}(D_{\text{min}})$. To prove surjectivity we consider $f(x) := \varphi(x)x^{-\alpha}s e_s$ for any $s \in \sigma(S)$ with $|s| < \frac{1}{2\alpha}$. $f$ extends trivially to a section $f \in L^2(E)$ with $D_{\text{max}}f = \varphi'x^{-\alpha}s e_s \in L^2(F)$ and $\Phi f = e_s$.

Corollary 3.5, 3. implies immediately ker $\Phi = \mathcal{D}(D_{\text{min}})$ and $D^d_V = D^d_W^*$ for any orthogonal decomposition $H^{1/2} = V \perp W$.

Remark  An immediate consequence of the preceding proof is

$$\mathcal{D}(D_{\text{max}})/\mathcal{D}(D_{\text{min}}) \cong H^{1/2}.$$

Theorem 3.7  All $L^2$-closed extensions of $D$ are Fredholm and

$$\text{ind } D_W = \text{ind } D_{\text{min}} + \dim W$$

for all subspaces $W \subset H^{1/2}$.

Proof  First, we construct a right parametrix. We choose $\omega, \psi$ as before with $\omega \psi = \psi$, such that for $R := \omega X^{-1}AP_{\text{bd}}^{\text{min}}\psi$ we have $\|R\| < \frac{1}{2}$.

Since $D$ is an elliptic operator, there exists an interior parametrix, $P_i$, such that

$$DP_i = I - \psi + K$$

and $K$ is a compact pseudodifferential operator with compact support. We put

$$Q := P_i + \omega P_{\text{bd}}^{\text{min}}\psi$$

and find

$$DQ = I - \psi + K + \omega' P_{\text{bd}}^{\text{min}}\psi + \omega DP_{\text{bd}}^{\text{min}}\psi = I + R + K + \omega' P_{\text{bd}}^{\text{min}}\psi.$$

Now, $\omega' P_{\text{bd}}^{\text{min}}\psi$ is a continuous operator with Im $\omega' P_{\text{bd}}^{\text{min}}\psi \subset H^1_{\text{comp}}(M,E)$, the Sobolev space with compact support in $M$. By the closed graph theorem,

$$\omega' P_{\text{bd}}^{\text{min}}\psi : L^2(M,F) \to H^1_{\text{comp}}(M,E)$$
is continuous and hence $\omega' P_{\text{bd}}^{\text{min}} \psi$ is a compact operator $L^2(M, F) \to L^2(M, E)$. Finally, $Q(I + R)^{-1}$ is a right parametrix for $D_{\text{max}/\text{min}}$.

In the same way we find a right parametrix for $D_t^{\text{max}/\text{min}}$. The adjoint of this parametrix will then be a left parametrix for $D_{\text{max}/\text{min}}$ and we obtain the Fredholm property of $D_{\text{max}/\text{min}}$.

Since the inclusion map $\iota : D(D_{\text{min}}) \to D(D_W)$ is Fredholm with index $-\dim W$, we obtain by the logarithmic law

$$\text{ind } D_{\text{min}} = \text{ind } D_W + \text{ind } \iota = \text{ind } D_W - \dim W. \quad \square$$

### 4. Index considerations

In this section we assume $D : C^\infty_0(E) \to C^\infty_0(F)$ to be of the generalized form described in Section 2 with the difference, that $h(x) = x^\alpha$ ($\alpha > 1$) only on a subinterval $(0, \epsilon_0)$ of $(0, \epsilon)$.

We restrict ourselves to the particular extension $D_\delta := D_W$ (see Theorem 3.6) corresponding to the subspace $W = \bigoplus_{-\frac{1}{2\alpha} < s < 0} \ker (S - s)$ and denote this extension by $D_\delta$. Since we consider in this section only this particular extension we will often omit the index $\delta$ and simply write $D$. Furthermore we will use the notations

$$U_\epsilon := (0, \epsilon) \times N \quad \text{and} \quad M_\epsilon := M - ((0, \epsilon] \times N).$$

The following proposition states that we can remove the perturbation $\frac{h'}{h} \tilde{A}$ in the operator of consideration without changing the index.

**Proposition 4.1** Let $\alpha > 1$ and $\psi_0 \in C^\infty(0, \epsilon)$ with $\psi_0(\epsilon) = 0$ and $\psi_0(0, \epsilon_0) = 1$. Let $D_0$ be an “unperturbed version of $D$”, i.e. $D_0$ coincides with $D$ on $M_\epsilon$ and

$$D_0|U \cong \left( \frac{d}{dx} + \frac{h'}{h} S \right) \oplus \left( \frac{d}{dx} + \frac{1}{h} \tilde{S} + (1 - \psi_0) \frac{h'}{h} \tilde{A} \right).$$

Then $\text{ind } D_\delta = \text{ind } D_{0,\delta}$. \[\square\]

**Proof** From Section 3 we conclude $D(D_\delta) = D(D_{0,\delta})$ and $\tilde{A}_{\psi_0} := \psi_0 \frac{h'}{h} \tilde{A} \in \mathcal{L}(D(D_\delta), L^2(F))$. Here we think of $\tilde{A}_{\psi_0}$ as an operator on the whole manifold $M$ by using the unitary separation of variables and trivial extension to $M$. Consequently

$$[0, 1] \ni \mu \to D_{0,\delta} + \mu \tilde{A}_{\psi_0} \in \mathcal{L}(D(D_\delta), L^2(F))$$

is a continuous curve in the space of Fredholm operators connecting $D_{0,\delta}$ with $D_\delta$. \[\square\]
Next we consider a family \( \{D^{\beta}\}_{\beta \in [\beta_1, \beta_2]} \) of unperturbed operators coinciding on \( M_\epsilon \) and
\[
D^{\beta}|U \cong \left( \frac{d}{dx} + \frac{h_{\beta}'}{h_{\beta}} S \right) \oplus \left( \frac{d}{dx} + \frac{1}{h_{\beta}} \tilde{S} \right)
\] (4.1)
with \( h_{\beta}(0, \epsilon_0) = x_{\beta} \) and \( 1 \leq \beta_1 < \beta_2 \). Later, in our examples we will consider a particular geometric operator on a fixed manifold \( M \) with continuously changing metrics \( g_{M_{\beta}} \) on \( U \). The resulting identifications (4.1) will be based on different separations of variables for each \( \beta \in [\beta_1, \beta_2] \). After these identifications we are independent of metric considerations on \( U \) and can work entirely in the fixed Hilbert spaces
\[
\mathcal{H}_1 := L^2((0, \epsilon), H_{\text{tot}}) \oplus L^2(E|M_\epsilon),
\]
\[
\mathcal{H}_2 := L^2((0, \epsilon), H_{\text{tot}}) \oplus L^2(F|M_\epsilon)
\] (4.2)
with \( H_{\text{tot}} = H \oplus \tilde{H} \). Note, that we do not exclude the case \( \beta_1 = 1 \). In this case, however, both components \( T^{\beta_1} \) and \( \tilde{T}_{0}^{\beta_1} \) (see (2.3) and (2.4) for the definition) are regular singular and we choose \( D_{\delta}^{\beta_1} \) to be the extension corresponding to the subspace
\[
W_1 := \bigoplus_{-\frac{1}{2} < s < 0} \ker (S - s) \oplus \bigoplus_{-\frac{1}{2} < s < 0} \ker (\tilde{S} - s).
\]
This convention coincides with the operator \( D_{\delta} \) introduced in [8].

Our goal is to prove \( \text{ind} D^{\beta_1} = \text{ind} D^{\beta_2} \) under weak additional conditions. Let \( F_{\beta} := \frac{h_{\beta}'}{h_{\beta}} \) and \( \tilde{F}_{\beta} := \frac{1}{h_{\beta}} \) for notational convenience. Then
\[
D^{\beta}|U \cong T^{\beta} \oplus \tilde{T}_{0}^{\beta} = \left( \frac{d}{dx} + F_{\beta} S \right) \oplus \left( \frac{d}{dx} + \tilde{F}_{\beta} \tilde{S} \right).
\]
We assume the existence of a constant \( C > 0 \) with
\[
F_{\beta}(x), \tilde{F}_{\beta}(x) \geq C \quad \text{for all} \quad \beta \in [\beta_1, \beta_2] \quad \text{and} \quad x \in (0, \epsilon)
\] (4.3)
and uniform convergence \( F_{\beta} \to F_{\gamma} \) and \( \tilde{F}_{\beta} \to \tilde{F}_{\gamma} \) on compact intervals \( [x_1, x_2] \subset (0, \epsilon) \) for \( \beta \to \gamma \in [\beta_1, \beta_2] \). These are the only conditions we need for the proof, that \( \text{ind} D^{\beta} \) is independent of \( \beta \).

The corresponding boundary parametrices \( P_{\text{bd}, \beta} \) are given by
\[
P_{\text{bd}, \beta} := P_s(T^{\beta}) \oplus \tilde{P}(\tilde{T}^{\beta}),
\] (4.4)
using the notation (2.13) and (2.13).

Let \( 0 < \epsilon_0 < \epsilon_1 < \epsilon \) and \( \varphi, \psi \in C^\infty(0, \epsilon) \) with \( \varphi + \psi = 1 \). Moreover, \( \varphi(x) = 0 \) for \( x \) near 0 and \( \psi(x) = 0 \) for \( x \) near \( \epsilon \). Let \( \varphi_2, \psi_2 \in C^\infty(0, \epsilon) \).
with the same properties near \( x = 0 \) and \( x = \epsilon \) and \( \varphi_2|_{[0, \epsilon_1]} = 0 \), \( \varphi \varphi_2 = \varphi \), \( \psi \psi_2 = \psi \). We extend these functions in an obvious manner to the manifold \( M \).

Analogously to [8, Lemma 2.3. and (2.12)] we conclude

\[
\text{im} \, \psi_2 P_{\text{bd}, \beta} \psi \subset D(D^\beta)
\]

and

\[
D^\beta(\psi_2 P_{\text{bd}, \beta}(\psi f)) = \psi_2' P_{\text{bd}, \beta}(\psi f) + \psi f.
\]

Let \( P_i : \mathcal{H}_2 \to H^1_{\text{loc}}(E) \) be an interior parametrix, i.e.

\[
\varphi_2 P_i(\varphi D^\beta f) = \varphi f + L_i f \quad \text{for all } f \in D(D^\beta),
\]

\[
D^\beta(\varphi_2 P_i(\varphi g)) = \varphi g + R_i g \quad \text{for all } g \in \mathcal{H}_2.
\]

\( P_i \) does not dependent on \( \beta \) and \( L_i, R_i \) as well as their adjoints are assumed to be infinitely smoothing and compact. Obviously \( L_i f|_{U_{\epsilon_1}} = 0 \) for any \( f \in \mathcal{H}_1 \). The same holds true for \( R_i, L_i^*, R_i^* \). The previous remarks together with the explicit description of the adjoints in Theorem 3.6 and (4.1) imply easily (see (4.2) for the notation \( H^1, H^2 \))

Let \( Q_\beta := \psi_2 P_{\text{bd}, \beta} + \varphi_2 P_i \varphi \). \( Q_\beta \) resp. \( Q_\beta^* \) are maps into \( D(D^\beta) \) resp. \( D((D^\beta)^*) \) and

\[
D^\beta(Q_\beta f) = f + \psi_2' P_{\text{bd}, \beta}(\psi f) + R_i f \quad \text{for } f \in \mathcal{H}_2,
\]

\[
(D^\beta)^*(Q_\beta^* g) = g + \psi_2' P_{\text{bd}, \beta}^*(\psi_2 g) + L_i^* g \quad \text{for } g \in \mathcal{H}_1.
\]

To prove compactness of \( Q^\beta \) we will use the following special case of Lemma 3.2.

**Lemma 4.2** Let \( Q_\beta := \psi_2 P_{\text{bd}, \beta} + \varphi_2 P_i \varphi \). \( Q_\beta \) resp. \( Q_\beta^* \) are maps into \( D(D^\beta) \) resp. \( D((D^\beta)^*) \) and

\[
\|P \| F_j,s \leq C_0 \quad \text{for all } \epsilon = (-1)^j s > 0.
\]

**Proposition 4.4** \( Q^\beta \) and \( (Q^\beta)^* \) are compact operators.

**Proof** We prove compactness of \( \varphi_2 P_i \varphi \) and \( \psi_2 P_{\text{bd}, \beta} \psi \) separately. The ellipticity of \( D^\beta \) implies \( D(D^\beta) \subset H^1_{\text{loc}}(E) \), so compactness of \( \varphi_2 P_i \varphi \) follows from the Lemma of Rellich.

From (4.3) we conclude with Lemma 3.3

\[
\|P_{j,s}\| \to 0 \quad \text{as } s \to (-1)^j \infty. \quad (4.5)
\]
Let temporarily $\Pr_t$ denote the orthogonal projection of $H_{\text{tot}} = H \oplus \tilde{H}$ onto the space spanned by all eigenfunctions $e_s$ and $\tilde{e}_s$ with $|s| < t$. Since the operators $P_{j,s}^{\beta}$, $P_{\tilde{j},s}^{\tilde{\beta}}$ are Hilbert-Schmidt, $\psi_2(\Pr_t \circ \Pr_t)\psi$ is compact for all $t > 0$. (4.5) implies that this operators converge to $\psi_2 \Pr_t \psi$ as $t \to \infty$ in the operator norm. This proves compactness of $\psi_2 \Pr_t \psi$. \qed

Next we introduce the graphs
\[ G(D^\beta) := \{(f, D^\beta f) \mid f \in \mathcal{D}(D^\beta)\} \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \]
and the following metric $d$ between two closed subspaces $V$ and $W$ of $\mathcal{H}_1 \oplus \mathcal{H}_2$:
\[ d(V, W) = \sup_{x \in V, \|x\|=1} d(x, W) + \sup_{y \in W, \|y\|=1} d(y, V) \]
with $d(x, W) := \inf_{y \in W} \|x - y\|_{\mathcal{H}_1 \oplus \mathcal{H}_2}$. This metric induces a topology on the closed subspaces. Theorem 4.1. of [12] implies, that $\text{ind } D^\beta$ remains constant whenever the mapping $[\beta_1, \beta_2] \ni \beta \mapsto G(D^\beta)$ (4.6) is continuous.

The next lemma is exactly the note at the end of Section 2 in [21] which was given without a proof. For the sake of completeness, we give a short outline of the proof.

Lemma 4.5 We define
\[ E^\beta := \begin{pmatrix} I & -(D^\beta)^* \\ D^\beta & I \end{pmatrix} : \mathcal{D}(D^\beta) \oplus \mathcal{D}((D^\beta)^*) \to \mathcal{H}_1 \oplus \mathcal{H}_2. \]
$E^{-1}_\beta$ is a bounded operator, and continuity of (4.6) is equivalent to the continuity of
\[ [\beta_1, \beta_2] \ni \beta \mapsto E^{-1}_\beta \in \mathcal{L} (\mathcal{H}_1 \oplus \mathcal{H}_2). \]

Proof Obviously, $E_\beta$ is closed and $E_\beta \geq I$. The same holds for $E^*_\beta$. From this we conclude bijectivity of $E_\beta$. Hence $E^{-1}_\beta$ is bounded by the closed graph theorem.

According to [12], $\mathcal{G}(D^\beta) \to \mathcal{G}(D^\gamma)$ is equivalent to $\mathcal{G}((D^\beta)^*) \to \mathcal{G}((D^\gamma)^*)$. The equivalence to $\mathcal{G}(E_\beta) \to \mathcal{G}(E_\gamma)$ follows from explicit formulas for the metric, i.e. the fact that
\[ \|(f, \tilde{f}) - (g, \tilde{g})\| + \|\langle f - D^\beta \tilde{f}, D^\beta f - \tilde{f} \rangle - \langle g - D^\gamma \tilde{g}, D^\gamma g - \tilde{g} \rangle\| \]
can be estimated from below and above by multiples of
\[ \|f - g\| + \|D^\beta f - D^\gamma g\| + \|\tilde{f} - \tilde{g}\| + \|(D^\beta)^* \tilde{g} - (D^\gamma)^* \tilde{f}\|. \]
According to [12], $\mathcal{G}(E_\beta) \to \mathcal{G}(E_\gamma)$ is equivalent to $\mathcal{G}(E^{-1}_\beta) \to \mathcal{G}(E^{-1}_\gamma)$, which finally is equivalent to the convergence in norm $E^{-1}_\beta \to E^{-1}_\gamma$ by the addendum of [12]. \qed
The uniform convergence of $F_\beta$ and $\tilde{F}_\beta$ on compact intervals is needed in the following lemma which is in some sense the heart of the proof of $\text{ind } D^{\beta_1} = \text{ind } D^{\beta_2}$.

**Lemma 4.6** The mapping 

$$[\beta_1, \beta_2] \ni \beta \mapsto P_{\text{bd}, \beta} \in \mathcal{L}(L^2((0, \epsilon), H_{\text{tot}}))$$

is continuous.

**Proof** It is enough to prove that, for each $\delta_0 > 0$, there exists a $\mu > 0$ with

$$\|P_{0,s}^{F_\beta} - P_{0,s}^{F_\gamma}\|_{\text{HS}} < \delta_0 \quad \text{and} \quad \|P_{0,s}^{\tilde{F}_\beta} - P_{0,s}^{\tilde{F}_\gamma}\|_{\text{HS}} < \delta_0$$

for all $|\beta - \gamma| < \mu$. The corresponding statement for $P_{1,s}^{F_\beta}, P_{1,s}^{\tilde{F}_\beta}$ follows by taking adjoints. The proof proceeds in two steps. First we prove convergence of the Hilbert-Schmidt norms as $\beta \to \gamma$ for each $s$-value individually. A **contraction property** allows us in the second step to conclude uniform convergence for all $s$-values by considering only finitely many $s$-values.

(i) The uniform convergence on compact intervals implies for $\beta \to \gamma$

$$k_\beta(x, y) := \exp(-\int_x^y sF_\beta(t) \, dt) \longrightarrow k_\gamma(x, y) := \exp(-\int_y^x sF_\gamma(t) \, dt)$$

pointwise for all $0 < y \leq x < \epsilon$. From $F_\beta \geq 0$ we conclude

$$|k_\beta(x, y) - k_\gamma(x, y)| \leq 1,$$

and thus by Lebesgue’s dominated convergence theorem

$$\|P_{0,s}^{F_\beta} - P_{0,s}^{F_\gamma}\|_{\text{HS}} \to 0.$$

Similarly, we obtain $\|P_{0,s}^{\tilde{F}_\beta} - P_{0,s}^{\tilde{F}_\gamma}\|_{\text{HS}} \to 0$.

(ii) Let $\gamma \in [\beta_1, \beta_2]$ and $\delta_0 > 0$ be given. Choose a function $F \in C^\infty((0, \epsilon))$ with $F(x) \geq \frac{1}{x}$ near $x = 0$, $F$ being bounded away from $0$ and

$$F \leq F_\beta \quad \text{for all } \beta \in [\beta_1, \beta_2].$$

Let $k(x, y) := \exp(-\int_y^x F(t) \, dt)$ and $k_\beta(x, y) := \exp(-\int_y^x F_\beta(t) \, dt)$. (Differently to [LS] this time there is no ‘$s$’ in the definition of $k_\beta$!)

Figure 1 illustrates the behaviour of $k$ over the set $\Delta := \{(x, y) \mid 0 \leq y \leq x \leq \epsilon\}$. Since $k$ is continuous on $\Delta - \{(0, 0)\}$, there exists a $w > 0$ with

$$\text{vol}\left(A(w) := \{(x, y) \mid k(x, y) > 1 - w\}\right) < \frac{\delta_0}{2}.$$
Furthermore, there exists an $s_w > 0$, such that for all $s \geq s_w$ the mappings

$$\Lambda_s : \left\{ \begin{array}{cl}
[0, 1 - w] & \rightarrow [0, 1 - w], \\
x & \mapsto x^s
\end{array} \right.$$

are contractions. From this we conclude

$$\|P_{0,s}^{F_\beta} - P_{0,s}^{F_\gamma}\|_{\text{HS}}^2 = \int_\Delta \frac{|\Lambda_s(k_\beta(x,y)) - \Lambda_s(k_\gamma(x,y))|^2 d(x,y)}{\leq 1}$$

$$\leq \int_{\Delta - A(w)} |k_\beta(x,y) - k_\gamma(x,y)|^2 d(x,y) + \text{vol}(A(w))$$

$$< \|P_{0,1}^{F_\beta} - P_{0,1}^{F_\gamma}\|_{\text{HS}}^2 + \frac{\delta_0}{2}$$

for all $s \geq s_w$. By (i) we can obviously find a $\mu$ with $\|P_{0,1}^{F_\beta} - P_{0,1}^{F_\gamma}\|_{\text{HS}}^2 < \frac{\delta_0}{2}$ for $|\beta - \gamma| < \mu$. This implies $\|P_{0,s}^{F_\beta} - P_{0,s}^{F_\gamma}\|_{\text{HS}}^2 < \delta_0$ for all $|\beta - \gamma| < \mu$ and $s \geq s_w$. Applying (i) again to the finitely many $s$-values below $s_w$ finishes the proof. The same holds true for the functions $F_\beta$.

The remaining arguments follow exactly the scheme of [3, pp.285]. To keep the notation of loc. cit., we denote by $F_\beta$ the following operator

$$F_\beta := \begin{pmatrix}
0 & Q_\beta \\
-Q_\beta^* & 0
\end{pmatrix}.$$

(To avoid notational misunderstandings let us remark, that the functions $F_\beta$ are no longer used in this section.) $Z_\beta$ denotes the orthogonal projection onto
ker $F^\beta$. Then

$$E^\beta F^\beta = I + F^\beta + G^\beta$$

with $G^\beta := \begin{pmatrix} L_i^* + \psi' P_{bd,\beta}^* \psi_2 & 0 \\ 0 & R_i + \psi' P_{bd,\beta} \psi \end{pmatrix}$.

The compactness of $G^\beta$ follows easily, if we replace $\psi_2 P_{bd,\beta} \psi$ by $\psi_2' P_{bd,\beta} \psi'$ resp. $\psi_2 P_{bd,\beta} \psi$ in the proof of Proposition 4.4.

**Lemma 4.7** $Z^\beta$ has the following properties:

(i) $F^\beta + Z^\beta : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ is injective,

(ii) $Z^\beta$ is of finite rank and hence compact,

(iii) $Z^\beta = -G^\beta Z^\beta$,

(iv) for all $\gamma \in [\beta_1, \beta_2]$ we have $\text{im} \ Z^\beta \subset \mathcal{D}(E^\gamma)$, and $E^\gamma Z^\beta$ is compact and independent of $\gamma$, i.e.

$$E^\gamma Z^\beta = E^\beta Z^\beta.$$

**Proof** For (i), (ii), (iii) cf. [3]. For (iv), we first prove $\text{im} \ G^\beta \subset \mathcal{D}(E^\gamma)$. $f \in \text{im} \ (R_i + \psi_2' P_{bd,\beta} \psi)$ implies $f \in \mathcal{D}((D^\gamma)^*)$ by using the following facts:

- $R_i$ is infinitely smoothing,
- $\psi_2' P_{bd,\beta} \psi$ maps $\mathcal{H}_2$ into $\mathcal{D}(D^\beta)$,
- $f \in \text{im} \ (R_i + \psi_2' P_{bd,\beta} \psi)$ implies $f|_{(0, \epsilon_1)} = 0$,
- $D^\beta$ and $(D^\gamma)^*$ are both closed extensions of first order elliptic differential operators.

Analogously, we obtain $\text{im} \ (L_i^* + \psi' P_{bd,\beta}^* \psi_2) \subset \mathcal{D}(D^\gamma)$. Thus we conclude with (iii):

$$\text{im} \ Z^\beta \subset \text{im} \ G^\beta \subset \mathcal{D}(E^\gamma).$$

Obviously, the considerations above imply also $E^\gamma Z^\beta = E^\beta Z^\beta$. $E^\beta Z^\beta$ is compact, since the restriction of $E^\beta$ to the finite dimensional vector space $\text{im} \ Z^\beta$ is bounded.

Now, $E^\gamma(F^\gamma + Z^\gamma) = I + F^\gamma + G^\gamma + E^\beta Z^\beta$ is a Fredholm operator with index 0. This implies together with Lemma 4.7 (i), that $E^\beta(F^\beta + Z^\beta)$ is bijective and hence invertible. Lemma 4.6 guarantees the continuity of

$$[\beta_1, \beta_2] \ni \gamma \mapsto E^\gamma(F^\gamma + Z^\gamma) = I + F^\gamma + G^\gamma + E^\beta Z^\beta,$$
hence \((E^\gamma(F^\gamma + Z^\beta))^{-1}\) is well defined for \(\gamma\) close to \(\beta\) and continuous in \(\gamma\). This finally implies continuity of the map

\[
\gamma \mapsto (E^\gamma)^{-1} = (F^\gamma + Z^\beta)(E^\gamma(F^\gamma + Z^\beta))^{-1}
\]

and finishes the proof of \(\text{ind } D^{\beta_1} = \text{ind } D^{\beta_2}\). Let us state this result in a theorem:

**Theorem 4.8** Let \(1 \leq \beta_1 < \beta_2\) and \(\{D^\beta\}_{\beta \in [\beta_1, \beta_2]}\) a family of unperturbed operators which are of the type (4.1) and coincide on \(M\). Moreover, we assume \(h_\beta\) to be strictly increasing, \(h_\beta \mid (0, \epsilon_0) = x^\beta\) and \(h_\beta\) converge uniformly on compact intervals for \(\beta \to \gamma\). Then

\[
\text{ind } D_\delta^{\beta_1} = \text{ind } D_\delta^{\beta_2}.
\]

In the sequel we will apply the results of this and the previous section to geometric operators. In the case of the Spin–Dirac operator we will prove that the index does not change by deforming metric horns into metric cones. However, in the case of the Gauß-Bonnet or the Signature operator we may only prove constance of the index for deformations from horns with warping function \(x^{\beta_2}\) to horns with warping function \(x^{\beta_1}\) (both \(\beta_1, \beta_2 > 1\)). This is due to the fact, that there occurs a “perturbation” in the latter two operators which can only be removed for \(\beta > 1\) (cf. Proposition 4.1). Fortunately, we can derive a Gauß-Bonnet theorem for metric horns by \(L^2\)-cohomology arguments of J. Cheeger.

### 5. Application to the Spin–Dirac operator

Let \(M\) be an open spin manifold of even dimension \(m\) and let

\[
h(x), D, D^+, S(M) \text{ and } S(N)
\]

be as described in Section 2 in the example of the Spin–Dirac operator. We are interested in a classification of all closed extensions \(\bar{D}^+\) of \(D^+\) and a formula for \(\text{ind } \bar{D}^+\) in the spirit of [4]. The answers to these questions in the regular singular case, i.e. \(h(x) = x\) on \((0, \epsilon_0)\) are due to A.W. Chou, which we recall for the sake of completeness. Here we use the terminology of [4]:

**Theorem 5.1** (see [13, Theorems (3.2) and (5.23)]) Let \(h(x) = x\) on \((0, \epsilon_0)\). The closed extensions of \(D^+\) are in \(1 : 1\) correspondence to subspaces \(W\) of

\[
\bigoplus_{|s| < \frac{1}{2}} \ker (D_N - s)
\]
and
\[
\text{ind } D_W^+ = \int_M \hat{A} - \frac{1}{2}(\eta(0) + b) + \dim W - \sum_{s < 0} \dim E_s, \quad (5.1)
\]

where \( \hat{A} \) is the Hirzebruch \( \hat{A} \)-polynomial, \( \eta(0) \) the eta-invariant of \( D_N \), \( E_s \) the eigenspace of \( D_N \) to the eigenvalue \( s \) and \( b := \dim \ker D_N \). For the particular extension \( D_N^+ \) as defined in [8], the term \((\ast)\) vanishes.

In Theorem 5.1, we chose the orientation of [1] which is opposite to the orientation of [13]. This causes different signs of \( \eta(0) \) in the index formulas.

The index formula of Chou can be rederived from the index formula in Theorem 4.1. of [8] by using the identification (2.6). The vanishing of the residua in (4.52) of [8] is guaranteed by [1, Theorem (4.2)].

Now we turn over to the case of metric horns and assume \( h(x) = x^\alpha \) on \((0, \epsilon_0)\) with \( \alpha > 1 \). The idea is to establish a connection between horns and cones and to transfer the known results for cones to horns. The following technical lemma is useful to set up this connection:

**Lemma 5.2** Let \( 0 < \epsilon \leq 1, 1 \leq \beta_1 < \beta_2 \) and \( h_{\beta_1}, h_{\beta_2} \in C^\infty((0, \epsilon)) \) with the following properties:

- \( h_{\beta_1}(x) = x^{\beta_1} \) on \((0, \epsilon_0)\),
- \( h'_{\beta_1}(x) \geq c > 0 \) on \([\epsilon_0, \epsilon_1]\),
- \( h_{\beta_1}(x) = h_{\beta_2}(x) \) on \((\epsilon_1, \epsilon)\).

Then there exists a family \( \{h_\beta\}_{\beta \in [\beta_1, \beta_2]} \subset C^\infty(0, \epsilon) \) and an \( \bar{\epsilon} \in (0, \epsilon_0) \) such that

- \( h_\beta(x) = x^\beta \) on \((0, \bar{\epsilon})\),
- \( h'_\beta(x) \geq c_1 > 0 \) on \([\bar{\epsilon}, \epsilon_1]\),
- \( h_\beta(x) \) independent of \( \beta \) on \((\epsilon_1, \epsilon)\),
- \( h_\beta \Rightarrow h_\gamma \) and \( h'_\beta \Rightarrow h'_\gamma \) on compact intervals as \( \beta \to \gamma \).

**Proof** \( h_\beta \) is given by
\[
h_\beta(x) := \phi(x)x^\beta + (1 - \phi(x))(a_\beta h_{\beta_1}(x) + (1 - a_\beta)h_{\beta_2}(x)),
\]
where \( \phi \in C^\infty(0, \epsilon) \) is monotonic decreasing, \( \phi|((0, \bar{\epsilon}) \equiv 1, \phi|((\epsilon_0, \epsilon) \equiv 0 \) and
\[
a_\beta := \frac{\epsilon_0^\beta - \epsilon_0^{\beta_2}}{\epsilon_0^{\beta_1} - \epsilon_0^{\beta_2}}.
\]

\( \square \)
Let \((M, g^\alpha)\) be a spin manifold with a metric horn. The previous lemma enables us to establish a metric homotopy \((M, g^\beta)\) from a horn \((\beta = \alpha)\) to a cone \((\beta = 1)\) by assuming
\[
g^\beta|U \cong dx^2 + h_\beta^2 g_N.
\]
Obviously all conditions are fulfilled to apply Theorem 4.8 with \((\beta_1, \beta_2) = (1, \alpha)\). Hence we conclude from Chou’s index formula:
\[
\text{ind } D^+_{\delta^\alpha} = \text{ind } D^+_{\delta^1} = \int_{M_e} \hat{\mathcal{A}} + \int_{(U, g_1)} \hat{\mathcal{A}} - \frac{1}{2}(\eta(0) + b).
\]
The same holds true for manifolds with more than one metric horn and different warping exponents \(\alpha_j > 1\) for each horn. A drawback of this index formula yet is the occurrence of the term \(\int_{(U, g_1)} \hat{\mathcal{A}}\). Fortunately, the \(O(m)\)-invariance of \(\hat{\mathcal{A}}\) implies \(\hat{\mathcal{A}}_{m/2} \equiv 0\) on any warped product \(dx^2 + h(x)^2 g_N\). Here, \(\hat{\mathcal{A}}_{m/2}\) denotes the homogeneous component of \(\hat{\mathcal{A}}\) of degree \(m/2\). Using these facts we conclude from the last two sections (note, that for the Spin–Dirac operator \(H_{\frac{1}{2}} = H = \ker D_N\)):

**Theorem 5.3** Let \(M\) be a singular spin manifold with metric horns. The closed extensions of the Spin–Dirac operator \(D^+\) are in \(1:1\) correspondence to subspaces \(W \subset \ker (D_N)\) and
\[
\text{ind } D^+_W = \int_M \hat{\mathcal{A}} - \frac{\eta(0)}{2} + (\dim W - \frac{b}{2}).
\]
In particular, for the minimal and maximal extension we have
\[
\text{ind } D^+_{\max/min} = \int_M \hat{\mathcal{A}} - \frac{\eta(0)}{2} \pm \frac{b}{2}.
\]

An easy consequence of the Theorem of Lichnerowicz (cf. [19, Corollary II.8.9.]) is

**Corollary 5.4** Let \(M\) be as in Theorem 5.3 and the scalar curvature of \(g_N\) be positive everywhere. Then the Spin–Dirac operator on \(M\) has a unique closed extension.

These results allow the conclusion, that the case of metric horns is considerably easier than the case of metric cones: No longer all eigenvalues in \((-\frac{1}{2}, \frac{1}{2})\) of \(D_N\) are of importance but only its kernel. The latter is independent of conformal changes of the metric \(g_N\) (see [15]) whereas the small eigenvalues are not stable under even such an easy deformation as multiplying the metric \(g_N\) by a positive constant.
6. Gauß-Bonnet and Signature operator

According to Section 3, the closed extensions of the Gauß-Bonnet and the Signature operator on manifolds with metric horns are characterized by subspaces of \( H_{\frac{1}{2}} = \oplus_{|\alpha|<1/2} \ker (S - s) \). The examples in Section 2 describe the concrete choices of \( S, H, \tilde{S}, \tilde{H} \) and \( \tilde{A} \) for both operators. In either case we choose \( H \) equals the space of harmonic forms \( \mathcal{H}(N) \) and \( S \) to be a diagonal matrix operator with diagonal entries given by \( c_j := (-1)^j(j - \frac{n}{2}) \) or \( b_j := j - \frac{n}{2} \), respectively. From this we conclude easily

**Theorem 6.1** Let \( M \) be an oriented singular Riemannian manifold with metric horns and \( D_{GB} \) and \( D_S \) the Gauß-Bonnet operator and the Signature operator on \( M \) (for the latter we assume \( \dim M = 4k \)).

1. If \( \dim M \) is even, \( D_{GB} \) has a unique closed extension.

2. If \( \dim M \) is odd, the closed extensions of \( D_{GB} \) can be characterized by the subspaces of \( \mathcal{H}^{n/2}(N) \), where \( N \) is the cross section of the horns and \( n = \dim N \).

3. \( D_S \) has a unique closed extension.

Moreover, all closed extensions are Fredholm.

Now, let us consider the indices of \( D_{GB} \) and \( D_S \) on metric horns. Index formulas for the Gauß-Bonnet and the Signature operator on manifolds with metric cones were first obtained by J. Cheeger (see [11]). Similar formulas for the asymptotic conic case are derived in [8] (see Theorem 5.1. and 5.2.). From the results in Section 4 we may deduce

**Theorem 6.2** Let \( M \) be an even dimensional oriented open manifold. Let \( g_1 \) and \( g_2 \) be two metrics on \( M \) which induce the structure of a singular Riemannian manifold with a metric horn for the same choice of \( U \) and identification \( U \cong (0, \epsilon) \times N \). Moreover, we assume \( g_j|U \cong dx^2 + h_{\beta_j}^2 g_N \) and \( g_1|M_{\epsilon_1} = g_2|M_{\epsilon_1} \) for a suitable \( \epsilon_1 \in (0, \epsilon) \) and the following properties of the warping functions \( h_{\beta_j} \):

- \( h_{\beta_j}(x) = x^{\beta_j} \) near the singularity and \( \beta_1, \beta_2 > 1 \),
- \( h_{\beta_j} \) is strictly increasing on \((0, \epsilon_1)\).

Then the unique closed extensions of the Gauß-Bonnet resp. Signature operators \( D_{\beta_j}^{GB/S} \) have the same index:

\[
\text{ind } D_{\beta_1}^{GB/S} = \text{ind } D_{\beta_2}^{GB/S}.
\]

Of course, for the Signature operator we have additionally to assume \( \dim M = 4k \).
This theorem generalizes to manifolds with several horns in an obvious manner. However, in contrast to the Spin–Dirac operator, our method does not allow to conclude coincidence of the indices for metric horns and metric cones. This lack is due to the occurrence of a perturbation \( \tilde{A} \neq 0 \) in these two operators.

**Proof**  
As in the last section, Lemma 5.2 implies the existence of metrics \( \{ g^\beta \}_{\beta \in [\beta_1, \beta_2]} \) on \( M \) being a homotopy connecting \( g_1 \) with \( g_2 \). By Proposition 4.1 we conclude for each \( \beta \in [\beta_1, \beta_2] \):

\[
D(\hat{D}_{GB,0}) = D(D_{GB}) \quad \text{and} \quad \text{ind} \ D_{GB,0} = \text{ind} \ D_{GB}
\]  
(6.1)

where \( D_{GB,0} \) is the “unperturbed version” of \( D_{GB} \). For the unperturbed operators we conclude with Theorem 4.8:

\[
\text{ind} \ D_{GB,0} = \text{ind} \ D_{GB,0}.
\]  
(6.2)

(6.1) and (6.2) together imply the statement of the theorem for the Gauß-Bonnet operator and analogously for the Signature operator. Note that in contrast to (6.1) the domains of the two operators in (6.2) generally do not coincide.

\( \square \)

In the case of the Gauß-Bonnet operator one may easily conclude a stronger result from \( L^2 \)-cohomology considerations of J. Cheeger. These \( L^2 \)-cohomology considerations imply for example, that the statement of Theorem 6.2 holds also for \( \beta_1 = 1 \). Let us explain this in more detail: Let \( M, g_1 \) and \( g_2 \) be as in Theorem 6.2 with the only difference \( \beta_1 = 1 \). Using the terminology of *Hilbert complexes* (see [3] for a detailed treatment of this notion) we may conclude for both metrics, that each \( d_j : \Omega_j^0(M) \to \Omega_j^{j+1}(M) \) has a unique closed extension \( D_j : D(D_j) \to L^2(\Lambda^{j+1}T^*M) \) and that

\[
0 \to D(D_0) \xrightarrow{D_0} D(D_1) \xrightarrow{D_1} \cdots \xrightarrow{D_{m-1}} D(D_m) \xrightarrow{D_m} L^2(\Lambda^mT^*M) \to 0
\]

is a Fredholm complex (cf. e.g. [3, Theorem 3.7.(a)] for the metric \( g_1 \)). In the case of the metric \( g_2 \) this follows easily from uniqueness and the Fredholm property of the closed extension of \( D_{GB,0}^\beta \). Theorem 2.1. of [10] describes the \( L^2 \)-cohomology groups \( \ker D_j / \text{im} D_{j-1} \) in terms of relative cohomology groups of \((M, N)\). Being stated for manifolds with metric cones, Theorem 2.1. also holds true for horns since the only necessary tools (\( L^2 \)-versions of Poincaré lemma and Mayer Vietoris developed in [9]) are valid for both situations. This implies that corresponding \( L^2 \)-Betti numbers are exactly the same for both metrics \( g_1 \) and \( g_2 \). In particular, the \( L^2 \)-Euler characteristics coincide. Denoting by \( D_{GB}^{\beta_1} \) and \( D_{GB}^{\beta_2} \) the Gauß-Bonnet operators corresponding to the metrics \( g_1 \) and \( g_2 \), we conclude with [8, Theorem 2.4] and [8, Theorem 3.7(d)]:

\[
\text{ind} \ D_{GB}^{\beta_2} = \begin{cases} 
\text{ind} \ D_{GB,\max}^{\beta_1} & \text{if } \frac{m}{2} \text{ is even}, \\
\text{ind} \ D_{GB,\min}^{\beta_1} & \text{if } \frac{m}{2} \text{ is odd}.
\end{cases}
\]  
(6.3)
Note, that the distinction between $D_{\text{GB},\text{min}}$ and $D_{\text{GB},\text{max}}$ for the conic metric $g_1$ is necessary: though there is a unique ideal boundary condition there may be many closed extensions of the Gauß-Bonnet operator $D_{\text{GB}} : \Omega_0^{\text{even}} \rightarrow \Omega_0^{\text{odd}}$.

(6.3) establishes a connection between metric horns and cones and allows to proceed in the same way as in the previous section. Henceforth we change our notations from $\beta_2$ to $\alpha$ and from $g^2$ to $g^\alpha$. Moreover, choose $\epsilon_0 \in (0, \epsilon)$ such that $h_1$ resp. $h_\alpha$ coincide with $x$ resp. $x^\alpha$ on $(0, \epsilon_0)$. With the help of the Gauß-Bonnet formula for metric cones (see [10, [6.1]] or [8, Theorem 5.1]) we conclude:

$$\text{ind } D_{\text{GB}}^\alpha = \int_{\langle M_0, g^1 \rangle} e + \int_{\langle U_0, g^1 \rangle} e \left. \right|_{\epsilon_0}$$

$$+ \frac{1}{2} \left( \sum_{j=0}^{n-1} (-1)^j b_j(N) - \sum_{j=\frac{n}{2}}^n (-1)^j b_j(N) \right) \left( \sum_{p \geq 1} \alpha_p \text{ Res}_1 \eta_{S_0}(2p) - \frac{\eta_3(0)}{2} \right)$$

$$=: \Xi(N) \tag{6.4}$$

Here $e$ denotes the Euler-class and $b_j(N)$ the Betti numbers of the cross section $N$. The ingredients $\eta_{S_0}$ and $\eta_3(0)$ are defined as in [8]. We emphasize that $\Xi(N)$ is made up of spectral data on the cross section $N$, which generally are difficult to calculate.

The following two lemmas allow us to express the right-hand side of (6.4) completely in terms of the Riemannian manifold $(M, g^\alpha)$.

**Lemma 6.3** Let $(N, g_N)$ be an odd dimensional compact manifold, $n = \dim N$ and $U_{\delta\epsilon}$ the warped collar $[\delta, \epsilon] \times N$ with metric $g_{U_{\delta\epsilon}} = dx^2 + h(x)^2 g_N$. Then there exist differential forms $\alpha_k$ ($k = 1, 2, \ldots, \frac{n-1}{2}$) on $N$ depending only on the intrinsic metric $g_N$, such that

$$\int_{U_{\delta\epsilon}} e = \sum_{k=0}^{\frac{n-1}{2}} \left( (h'(\epsilon))^{2k+1} - (h'(|\delta|))^{2k+1} \right) \int_N \alpha_k.$$

Consequently, $\int_{U_{\delta\epsilon}} e = 0$ for linear warping functions $h$.

**Proof** The Chern-Gauß-Bonnet theorem for manifolds with boundary yields

$$\int_{U_{\delta\epsilon}} e = \chi([\delta, \epsilon] \times N) - \int_{\partial U_{\delta\epsilon}} Se$$
where $S_e$ is a $SO$-invariant form defined near the boundary of $U_\delta$. According to [14, pp. 252], $S_e$ can be written locally with respect to an oriented orthonormal frame $\{v_j\}$ of $TU_\delta$, $v_{n+1} = \frac{\partial}{\partial x}$ as

$$S_e = \sum_{k=0}^{n-1} c_{k,m} \sum_{\sigma \in S_n} \text{sgn}(\delta) \Omega^{U}_{\sigma_1 \sigma_2} \wedge \ldots \wedge \Omega^{U}_{\sigma_{2k-1} \sigma_{2k}} \wedge \omega^{U}_{\sigma_{2k+1} n+1} \wedge \ldots \wedge \omega^{U}_{\sigma_{n} n+1}. \quad (6.5)$$

$c_{k,m}$ are suitable chosen constants. Let $\omega^N, \omega^U$ and $\Omega^N, \Omega^U$ be connection and curvature forms w.r.t. orthonormal frames $e_1, \ldots, e_n$ and $\frac{1}{n} e_1, \ldots, \frac{1}{n} e_n, \frac{\partial}{\partial x}$ of $N$ and $U$. A standard calculation yields:

$$\omega^{U}_{jk} = \begin{cases} 
\omega^{N}_{jk} & \text{for } 1 \leq j, k \leq n, \\
-h' e^j & \text{for } k = n+1 \text{ and } 1 \leq j \leq n,
\end{cases}$$

$$\Omega^{U}_{jk} = \Omega^{N}_{jk} - (h')^2 e^j \wedge e^k \quad \text{for } 1 \leq j, k \leq n.$$ 

Inserting these identities in (6.5) one easily deduces how to choose the intrinsic defined forms $\alpha_k$ on $N$. \hfill \Box

**Lemma 6.4 ([14, p. 607])** Using the notations above, we have

$$\Xi(N) = \int_N \sum_{k=0}^{n-1} (h'_1(e_0))^2 e^{2k-1} \alpha_k. \quad (6.6)$$

Cheeger proved this identity between the spectral invariant $\Xi(N)$ and the boundary integral by comparing his Gauß-Bonnet formula for metric cones with the classical Gauß-Bonnet formula for manifolds with boundary. Using the notions of [8], $\Xi(N)$ has the form

$$\Xi(N) = \sum_{p} \left( \alpha_p \text{Res}_1 \eta_{S_0}(2p) - \frac{\beta_p}{2} \sum_{k} (-1)^k \text{Res}_1 \zeta_k(2p+1) \right)_{(*)}$$

$$+ \frac{1}{2} \sum_{k} (-1)^{k+1} \text{Res}_1 \zeta_k(1).$$

Actually, Cheeger states the identity (6.6) without the term $(*)$ in $\Xi(N)$, which is most likely equal 0. However, since our considerations are based on [8] we include this term to establish consistency. Let us sketch Cheeger’s proof.

**Proof** $L^2$-cohomology arguments imply

$$\chi(2)(M) = \chi(M_\delta) + \frac{1}{2} \chi(2)(CN) + \frac{1}{2} \chi(2)(CN, N), \quad (6.7)$$
where $CN$ denotes the metric cone $\{0, \epsilon_0\} \times N$, and
\[ \chi(2)(CN) = \sum_{j=0}^{m-1} (-1)^j b_j(N), \quad \chi(2)(CN, N) = - \sum_{j=m/2}^{n} (-1)^j b_j(N). \quad (6.8) \]

Since $\dim M_\epsilon$ is even we do not have to bother about absolute or relative boundary conditions for the corresponding Euler characteristics are the same (see [14, Theorem 4.2.7]). Using (6.7), (6.8) together with the Gauß-Bonnet formula for cones in the terminology of [8] we conclude
\[ \chi(M_\epsilon) = \int_{(M_\epsilon, \gamma^1)} e + \Xi(N). \]

On the other hand the classical Chern-Gauß-Bonnet formula for $M_\epsilon$ reads as
\[ \chi(M_\epsilon) = \int_{(M_\epsilon, \gamma^1)} e + \int_{\partial M_\epsilon} Se. \]

Similarly to the previous proof we deduce
\[ \int_{\partial M_\epsilon} Se = \int_N \sum_{k=0}^{n-1} (h_1'(\delta))^{2k+1} \alpha_k, \]
which finishes the proof. \hfill \Box

With all these identities at hand we conclude

**Theorem 6.5** (Gauß-Bonnet formula for metric horns) *Let $M$ be a singular manifold with metric horns of even dimension $m = n + 1$. There exists a unique closed extension of $D_{GB} : \Omega^\text{even}_0(M) \to \Omega^\text{odd}_0(M)$ which is Fredholm and its index is given by*
\[ \text{ind} \ D_{GB} = \int_M e + \frac{1}{2} \left( \sum_{j=0}^{m-1} (-1)^j b_j(N) - \sum_{j=m/2}^{n} (-1)^j b_j(N) \right). \]

**Proof** Let $\delta \in (0, \epsilon_0)$ be arbitrary. Using Lemma [6.3] twice and the fact that $h_\alpha$ and $h_1$ agree near $x = \epsilon$ we obtain:
\[ \int_{(U_{\delta}, \gamma^1)} e = \int_{(U_{\delta}, \gamma^\alpha)} e + \int_N \sum_{k} ((h'_\alpha(\delta))^{2k+1} - (h_1'(\delta))^{2k+1}) \alpha_k. \quad (6.9) \]

Since $h_1$ is linear on $(0, \epsilon_0)$: $\int_{(U_{\delta}, \gamma^1)} e = 0$. Together with Lemma [6.4], [6.4], and (6.9), this implies
\[ \text{ind} \ D_{GB}^\alpha = \underbrace{\int_{(M_\delta, \gamma^\alpha)} e + \int_N (\sum_k (h'_\alpha(\delta))^{2k+1} \alpha_k)}_{(*)} + \frac{1}{2} \left( \sum_{j=0}^{m-1} (-1)^j b_j(N) - \sum_{j=m/2}^{n} (-1)^j b_j(N) \right). \]
Using the fact that $h'_\alpha(\delta) \to 0$ as $\delta \to 0$ we see that (*) coincides with $\int_{(M,g^\alpha)} e$. 

Let us consider the following situation as an application of the previous results: Let $(M, g^1)$ be an oriented, compact closed manifold of even dimension $m$ and $p_1, \ldots, p_k$ arbitrary points of $M$. Without loss of generality we may assume, that in the neighborhood of each point $p_j$ the metric $g^1$ is of the form $dx^2 + x^2 g_{N_j}$ with $N_j$ diffeomorphic to the sphere $S^{m-1}$. We may consider $M_0 := M - \{p_0, \ldots, p_k\}$ as well as a singular manifold with metric cones. Now, let the metric change continuously so that the neighborhoods of each $p_j$ become metric horns. We denote the family of metrics on $M_0$ again by $\{g^\beta\}_{\beta \in [1,\alpha]}$. Whereas the index of the Gauß-Bonnet operator (corresponding to the unique ideal boundary condition) does not change, the map

$$[1, \alpha] \ni \beta \to \int_{(M_0,g^\beta)} e \in \mathbb{Z}$$

skips from $\chi(M)$ to $\chi(M) - k$ as soon as $\beta$ becomes greater than 1. This is somewhat surprising since the metric of $M_0$ changes smoothly. Such a phenomenon does not occur for the $O(m)$-invariant forms $\hat{A}$ or $L_k$ (Hirzebruch’s L-polynomial with $k := m/4$). This is an easy consequence of the fact that the integral of $O(m)$-invariant forms over warped products vanishes as shown in the last section.

Unfortunately, our method only partially applies to the Signature operator. By Theorem 6.1 the Signature operator has a unique closed extension which is Fredholm. However, we are not able to prove an index theorem for the Signature operator. By analogy the following result is conceivable. It was stated as a conjecture in an earlier version of this paper. Now it is a Theorem since a proof has been announced by J. Brüning [4].

**Theorem 6.6** (Signature formula for metric horns) Let $M$ be a $4k$-dimensional singular manifold with metric horns. There exists a unique closed extension of $D_S : \Omega^+_0(M) \to \Omega^-_0(M)$ which is Fredholm and its index is given by

$$\text{ind } D_S = \int_M L_k - \eta(N),$$

where $L_k$ is the $k$-th Hirzebruch L-polynomial and $\eta(N)$ is the eta-invariant of the operator

$$\alpha \mapsto (-1)^{k+j+1}(\ast_N d_N - d_N \ast_N) \quad \text{for } \alpha \in \Omega^{2j}(N),$$

and $N$ is the cross section of the horns.
Brüning’s method is different from ours. He announces a heat trace asymptotics for metric horns and hence he does not reduce the problem to the conic case. However, he needs our Theorem 6.1.

It may be useful to note, that the Signature operator on manifolds with horns decomposes into the infinite direct sum of operators acting on one and two dimensional subspaces. It was our hope that this could be used to prove the graph continuity (cf. Theorem 4.8) for \( \beta \to 1 \) for the Signature operator, too. Unfortunately, we did not succeed. However, this decomposition seems to be of some interest in its own. An analogous decomposition of the Gauß-Bonnet operator was proven in [3, Lemma 2.2].

We decompose \( L^2(\Lambda^* T^* N) \) into

\[
L^2(\Lambda^* T^* N) = \mathcal{H} \oplus H_1 \oplus H_2 \oplus H_3 \oplus H_4 \oplus H_5 \oplus H_6, \tag{6.10}
\]

where \( \mathcal{H} \) denotes the space of harmonic forms on \( N \) and \( H_l \) are chosen as follows (the matrices \( M_l \) will be used below):

| \( l \) | \( H_l \) | \( M_l \) |
|---|---|---|
| 1 | \( \bigoplus_{0 \leq j \leq k-2 \atop \lambda \geq 0} \left( E_{\lambda,cl}^{2k-2j-2} \oplus E_{\lambda,cl}^{2k+2j+2} \right) \) | \( \left( 2j + \frac{3}{2} h' \right) \left( -1 \right)^{j+1} \frac{1}{\sqrt{\lambda}} \left( \sqrt{1 - \frac{4}{\lambda}} \right) \) |
| 2 | \( \bigoplus_{0 \leq j \leq k-2 \atop \lambda \geq 0} \left( E_{\lambda,cl}^{2k-2j-3} \oplus E_{\lambda,cl}^{2k+2j+1} \right) \) | \( \left( 2j + \frac{5}{2} h' \right) \left( -1 \right)^{j+1} \frac{1}{\sqrt{\lambda}} \left( \sqrt{1 - \frac{4}{\lambda}} \right) \) |
| 3 | \( \bigoplus_{0 \leq j \leq k-1 \atop \lambda \geq 0} \left( E_{\lambda,cl}^{2k-2j-1} \oplus E_{\lambda,cl}^{2k+2j+1} \right) \) | \( \left( 2j + \frac{1}{2} h' \right) \left( -1 \right)^{j} \frac{1}{\sqrt{\lambda}} \left( \sqrt{1 - \frac{4}{\lambda}} \right) \) |
| 4 | \( \bigoplus_{0 \leq j \leq k-1 \atop \lambda \geq 0} \left( E_{\lambda,ccl}^{2k-2j-2} \oplus E_{\lambda,ccl}^{2k+2j} \right) \) | \( \left( 2j + \frac{3}{2} h' \right) \left( -1 \right)^{j} \frac{1}{\sqrt{\lambda}} \left( \sqrt{1 - \frac{4}{\lambda}} \right) \) |
| 5 | \( \bigoplus_{\lambda > 0} E_{\lambda,cl}^{2k} \) |  |
| 6 | \( \bigoplus_{\lambda > 0} E_{\lambda,ccl}^{2k-1} \) |  |

\( E_{\lambda,cl/ccl}^j \) denotes the space of closed resp. coclosed \( j \)-eigenforms of the Laplacian on \( N \) corresponding to the eigenvalue \( \lambda \).

\( E_{\lambda,cl}^{2k} \) resp. \( E_{\lambda,ccl}^{2k-1} \) admit a further decomposition (into \( \pm 1 \)-eigenspaces) via the involutions \( \frac{1}{\sqrt{\lambda}} d_{N \times N} \) resp. \( \frac{1}{\sqrt{\lambda}} d_{N} \):

\[
E_{\lambda,cl}^{2k} = E_{\lambda,cl}^{2k,+} \oplus E_{\lambda,cl}^{2k,-} \quad \text{and} \quad E_{\lambda,ccl}^{2k-1} = E_{\lambda,ccl}^{2k-1,+} \oplus E_{\lambda,ccl}^{2k-1,-}. \tag{6.11}
\]

By arguments analogously to [3] we conclude
Lemma 6.7 The operator $D_S$ reduces with respect to the decompositions (6.10), (6.11) in the neighborhood of a metric horn into

$$D_S|H \cong \partial_x + \frac{h'}{h} S_0 = \bigoplus_{j=0}^{n} \bigoplus_{\dim H} (\partial_x + \frac{1}{h} b_j),$$

$$D_S|V \cong \partial_x + \frac{1}{h} M_l \text{ w.r.t. } \{\eta_1, \eta_2\}$$

$$D_S|E_{\lambda,cl}^{2k,\pm} \cong \bigoplus_{m^\pm_\lambda} (\partial_x + \frac{1}{h} h' \pm \sqrt{\lambda} h),$$

$$D_S|E_{\lambda,ccl}^{2k-1,\pm} \cong \bigoplus_{m^\pm_\lambda} (\partial_x + \frac{1}{h} h' \pm \sqrt{\lambda} h),$$

where $m^\pm_\lambda := \dim E_{\lambda,cl}^{2k,\pm} = \dim E_{\lambda,ccl}^{2k-1,\pm}$, the matrices $M_l$ are as in the table above, $V := \langle \eta_1, \eta_2 \rangle \subset H_l$ for $1 \leq l \leq 4$, and $\eta_1$ and $\eta_2$ are chosen as follows:

| $l$ | $\eta_1 \in$ | $\eta_2 =$ |
|-----|----------------|----------------|
| 1   | $E_{\lambda,cl}^{2k-2j-2}$ | $\frac{1}{\sqrt{\lambda}} d_N * N \eta_1$ |
| 2   | $E_{\lambda,ccl}^{2k-2j-3}$ | $\frac{1}{\sqrt{\lambda}} d_N * N \eta_1$ |
| 3   | $E_{\lambda,cl}^{2k-2j-1}$ | $\frac{1}{\sqrt{\lambda}} d_N * N \eta_1$ |
| 4   | $E_{\lambda,ccl}^{2k-2j-2}$ | $\frac{1}{\sqrt{\lambda}} d_N * N \eta_1$ |

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