Localization in a quasiperiodic model on quantum graphs

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Dedicated to the memory of Vladimir Geyler (1943–2007)

Abstract. We show the presence of a dense pure point spectrum on quantum graphs with Maryland-type quasiperiodic Kirchhoff coupling constants at the vertices.

Introduction

In the present contribution we are going to show how quantum graphs can be used to construct exactly solvable quasiperiodic models showing the Anderson localization at all energies.

In [GFP82] it was shown that the one-dimensional quasiperiodic difference Hamiltonian

\[ L\psi(n) = \psi(n+1) + \psi(n-1) + \lambda \tan(\omega n - \alpha)\psi(n), \quad \psi \in \ell^2(\mathbb{Z}), \quad \lambda, \alpha, \omega \in \mathbb{R}, \]

has a pure point spectrum dense everywhere under some arithmetic conditions for \( \omega \) and \( \alpha \); this operator is often referred to as the Maryland model. Later the class of such Hamiltonians was considerably extended in several directions, e.g. to the multidimensional case with more general coefficients [BLS83, FP84], see also [BHP05] for a recent review. It is a rather interesting problem to construct continuous quasiperiodic operators where one can describe the dense point spectrum in a more or less explicit way. An example of such models was proposed in [GM87], where Schrödinger operators with tan-like quasiperiodic point perturbations were studied and the Anderson localization in the gaps of the unperturbed Hamiltonians was shown. In the paper [Ex97] a comb-shaped quantum graph was proposed whose spectral study reduces at the formal level to operators of the form (1); however, the machinery used does not allow to prove rigorously the presence of a dense pure point spectrum in that case. We are going to show, using some modification of the constructions from [FP84] and [GM87] and the machinery of self-adjoint extensions [BGP06], that the Anderson localization at all energies can be achieved by placing tan-like quasiperiodic \( \delta \)-interactions at the vertices of quantum graph lattices. We consider this case as the most illustrative one in many aspects, and we plan to treat much more general quasiperiodic interactions in subsequent works.
1. Schrödinger operator on a quantum graph

Below we describe some basic constructions for quantum graphs; a detailed discussion can be found e.g. in [GS06, Ku04, Ku05]. There are many approaches to the study of the spectra of quantum graphs, we use the one from [BGP06, P06] based on the theory of self-adjoint extensions.

We consider a quantum graph whose set of vertices is identified with $\mathbb{Z}^d$, $d \geq 1$. By $h_j$, $j = 1, \ldots, d$, we denote the standard basis vectors of $\mathbb{Z}^d$. Two vertices $m, m'$ are connected by an oriented edge $m \to m'$ iff $m' = m + h_j$ for some $j \in \{1, \ldots, d\}$; this edge is denoted as $(m, j)$ and one says that $m$ is the initial vertex and $m' \equiv m + h_j$ is the terminal vertex.

Fix some $l_j > 0$, $j \in \{1, \ldots, d\}$, and replace each edge $(m, j)$ by a copy of the segment $[0, l_j]$ in such a way that 0 is identified with $m$ and $l_j$ is identified with $m + h_j$. In this way we arrive at a certain topological set carrying a natural metric structure.

The quantum state space of the system is

$$ \mathcal{H} := \bigoplus_{(m,j) \in \mathbb{Z}^d \times \{1, \ldots, d\}} \mathcal{H}_{m,j}, \quad \mathcal{H}_{m,j} = L^2[0, l_j] $$

and vectors $f \in \mathcal{H}$ will be denoted as $f = (f_{m,j})$, $f_{m,j} \in \mathcal{H}_{m,j}$, $m \in \mathbb{Z}^d$, $j = 1, \ldots, d$. Let us describe the Schrödinger operator acting in $\mathcal{H}$. Fix some real-valued functions (potentials) $U_j \in L^2[0, l_j]$, $j = 1, \ldots, d$, and some real constants $\alpha(m)$, $m \in \mathbb{Z}^d$. Set $A := \text{diag}(\alpha(m))$; this is a self-adjoint operator in $L^2(\mathbb{Z}^d)$. Denote by $H_A$ an operator acting as

$$ (f_{m,j}) \mapsto \left( (-D^2 + U_j) f_{m,j} \right), \quad D f_{m,j} = f'_{m,j}, $$

on functions $f = (f_{m,j}) \in \bigoplus_{m,j} H^2[0, l_j]$ satisfying the following boundary conditions:

$$ f_{m,j}(0) = f_{m-h_j,k}(l_k) := f(m), \quad j, k = 1, \ldots, d, \quad m \in \mathbb{Z}^d, $$

(which means the continuity at all vertices) and

$$ f'(m) = \alpha(m) f(m), \quad m \in \mathbb{Z}^d, $$

where

$$ f'(m) := \sum_{j=1}^d f'_{m,j}(0) - \sum_{j=1}^d f'_{m-h_j,j}(l_j). $$

The constants $\alpha(m)$ are usually referred to as Kirchhoff coupling constants and interpreted as the strengths of zero-range impurity potentials at the corresponding vertices [Ex96]. The zero coupling constants correspond hence to the ideal couplings and are usually referred to as the standard boundary conditions.

Denote by $S$ the operator acting as (2a) on the functions $f$ satisfying only the boundary conditions [25]. On the domain of $S$ one can define linear maps

$$ f \mapsto \Gamma f := (f(m))_{m \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d), \quad f \mapsto \Gamma' f := (f'(m))_{m \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d). $$

By the Sobolev embedding theorem, $\Gamma, \Gamma'$ are well-defined, and the joint map $(\Gamma, \Gamma') : \text{dom} \, S \to \ell^2(\mathbb{Z}^d) \times \ell^2(\mathbb{Z}^d)$ is surjective. Moreover, by a simple algebra, for any $f, g \in \text{dom} \, S$ one has $\langle f, S g \rangle - \langle S f, g \rangle = \langle \Gamma f, \Gamma' g \rangle - \langle \Gamma' f, \Gamma g \rangle$ (see e.g. proposition 1 in [P06]). In the abstract language, $(\mathbb{Z}^d, \Gamma, \Gamma')$ form a boundary
triple for $S$. This permits to write a useful formula for the resolvent of $H_A$, which will play a crucial role below.

First, denote by $H^0$ the restriction of $S$ to ker $\Gamma$. Clearly, $H^0$ acts as \([2a]\) on functions $(f_{m,j})$ with $f_{m,j} \in H^2[0, l_j]$ satisfying the Dirichlet boundary conditions, $f_{m,j}(0) = f_{m,j}(l_j) = 0$ for all $m, j$, and the spectrum of $H^0$ is just the union of the Dirichlet spectra of the operators $-\frac{d^2}{dt^2} + U_j$ on the segments $[0, l_j]$. We will refer to spec $H^0$ as to the Dirichlet spectrum of the graph.

Denote by $s_j$ and $c_j$ the solutions to $-y'' + U_j y = zy$ satisfying $s_j(0; z) = c_j'(0; z) = 0$ and $s_j'(0; z) = c_j(0; z) = 1$, $z \in \mathbb{C}$, $j = 1, \ldots, d$.

For $z$ outside spec $H^0$ consider operators $\gamma(z) : \ell^2(\mathbb{Z}^d) \to \mathcal{H}$ defined as follows. For $\xi \in \ell^2(\mathbb{Z}^d)$, $\gamma(z)\xi$ is the unique solution to $(S - z)\xi = 0$ with $\Gamma\xi = \xi$. For each $z$, $\gamma(z)$ is a linear topological isomorphism between $\ell^2(\mathbb{Z}^d)$ and ker$(S - z)$. Clearly, in terms of the functions $s_j$ and $c_j$ introduced above, one has

$$
(\gamma(z)\xi)_{m,j}(t) = \frac{1}{s_j(l_j; z)} \left( \xi(m + h_j)s_j(t; z) + \xi(m)(s_j(l_j; z)c_j(t; z) - c_j(l_j; z)s_j(t; z)) \right),
$$

$t \in [0, l_j]$, $(m, j) \in \mathbb{Z}^d \times \{1, \ldots, d\}$.

Furthermore, for the same $z$’s define an operator $M(z) : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ by $M(z) := \Gamma'\gamma(z)$. In our case, $M(z)\xi(m) = \sum_{j=1}^{d} \frac{1}{s_j(l_j; z)} \cdot \left( \xi(m - h_j) + \xi(m + h_j) - \eta_j(z) \xi(m) \right)$, $\xi \in \ell^2(\mathbb{Z}^d)$, where $\eta_j(z) := c_j(l_j; z) + s_j'(l_j; z)$ is the Hill discriminant associated with $U_j$. The maps $\gamma$ and $M$ satisfy a number of important properties. In particular, $\gamma$ and $M$ depend analytically on their argument (outside of spec $H^0$), $M(z)$ is self-adjoint for real $z$,

(3) for any non-real $z$ there is $c_z > 0$ with $\frac{3M(z)}{3z} \geq c_z$, and

(4) $M'(\lambda) = \gamma^*(\lambda)\gamma(\lambda) > 0$ for $\lambda \in \mathbb{R} \setminus \text{spec } H^0$.

Furthermore,

(5) $\gamma^*(z)f = 0$ for any $f \in \ker(S - z)^\perp \equiv \gamma(z)(\ell^2(\mathbb{Z}^d))^\perp$,

see \[BGP06\] Section 1].

**Proposition 1.** The resolvents of $H^0$ and $H_A$ are related by the Krein resolvent formula,

(6) $(H_A - z)^{-1} = (H^0 - z)^{-1} - \gamma(z)(M(z) - A)^{-1}\gamma^*(z)$, $z \notin \text{spec } H_A \cup \text{spec } H^0$,

and the set spec $H_A \setminus \text{spec } H^0$ coincides with $\{ z \notin \text{spec } H^0 : 0 \in \text{spec } (M(z) - A) \}$. For any $z \notin \text{spec } H^0$ there holds ker$(H_A - z) = \gamma(z)\ker(M(z) - A)$, i.e. $z$ is an eigenvalue of $H_A$ iff $0$ is an eigenvalue of $M(z) - A$, and $\gamma(z)$ is an isomorphism of the corresponding eigensubspaces.
2. Eigenvalues for Maryland-type coupling constants

We are going to study the above operator $H_A$ for a special choice of the coefficients $\alpha(m)$. Namely, pick $g > 0$, $\omega \in \mathbb{R}^d$, $\varphi \in \mathbb{R}$ with

$$\varphi \neq \pi/2 - \pi \langle \omega, m \rangle \mod \pi$$

and set

$$\alpha(m) := -g \tan \left( \pi \langle \omega, m \rangle + \varphi \right), \quad m \in \mathbb{Z}^d.$$  

We assume additionally that $\omega$ satisfies the following Diophantine condition:

$$| \langle \omega, m \rangle - r| \geq C|m|^{-\beta}$$

for all $m \in \mathbb{Z}^d \setminus \{0\}$, $r \in \mathbb{Z}$. Clearly, (8) implies $\omega \notin \mathbb{Q}^d$.

The operator $H_A$ corresponding to the above choice of the coupling constants will be denoted simply by $H$, and this will be our main object of study. Our main result concerning the spectrum of $H$ is the following theorem.

**Theorem 2.** The spectrum of $H$ is pure point and coincides with $\mathbb{R}$.

We will repeat first some algebraic manipulations in the spirit of [FP84]. Let $U$ be the multiplication by the sequence $(e^{2\pi i \langle \omega, m \rangle})$ in $\ell^2(\mathbb{Z}^d)$, $B(z) := (M(z) - ig)^{-1}$, $C(z) := -(M(z) + ig)(M(z) - ig)^{-1}$. The operators $B(z)$ and $C(z)$ are defined at least for $z$ with $\Re z \notin \text{spec} H^0$ and $|\Im z|$ sufficiently small. Denoting $\chi := e^{2i\varphi}$ one can write for such $z$ the identity

$$M(z) - A = B(z)^{-1}(1 - \chi C(z)U)(1 + \chi U)^{-1}.$$  

In what follows we denote $S^1 := \{ z \in \mathbb{C} : |z| = 1 \}$ and $T^d := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \subset \mathbb{C}^d$.

For $\theta = (\theta_1, \ldots, \theta_d) \subset \mathbb{C}^d$ and $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ we write $\theta^m := \theta_1^{m_1} \cdots \theta_d^{m_d}$, and in this context $k \in \mathbb{Z}$ will be identified with the vector $(k, \ldots, k) \in \mathbb{Z}^d$, i.e. $\theta^{-1} := \theta_1^{-1} \cdots \theta_d^{-1}$ etc.

Denote by $F$ the Fourier transform carrying $\ell^2(\mathbb{Z}^d)$ to $L^2(\mathbb{T}^d)$,

$$F\psi(\theta) = \sum_{m \in \mathbb{Z}^d} \psi(m) \theta^m, \quad F^{-1}f(m) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\theta)\theta^{-m-1}d\theta.$$  

Under this transformation $M(z)$ becomes the multiplication by a function $M(z, \theta)$,

$$M(z, \theta) := \sum_{j=1}^d \frac{1}{s_j(l_j; E)} (\theta_j + \theta_j^{-1}) - \sum_{j=1}^d \frac{\eta_j(E)}{s_j(l_j; E)},$$  

the operators $B(z)$ and $C(z)$ become the multiplications by $B(z, \theta) := (M(z, \theta) - ig)^{-1}$ by $C(z, \theta) := -(M(z, \theta) + ig)(M(z, \theta) - ig)^{-1}$, respectively, and $U$ becomes a shift operator, $Uk(\theta_1, \ldots, \theta_d) = k(e^{2\pi i \omega_1 \theta_1} \cdots, e^{2\pi i \omega_d \theta_d})$.

Consider an arbitrary segment $[a, b] \subset \mathbb{R} \setminus \text{spec } H^0$. Eq. (8), the analyticity of $\gamma$, and the self-adjointness of $M(z)$ for real $z$ imply the existence of $\delta' > 0$ such that $|\Im M(z)| \leq g/2$ for $z \in Z := \{ z \in \mathbb{C} : |\Im z| \leq \delta' \}$. At the same time, this means that $|\Im M(z, \theta)| \leq g/2$ for $z \in Z$. As follows from (10), $M(z, \theta)$ can be continued to an analytic function in $Z \times \Theta$, $\Theta := \{ (\theta_1, \ldots, \theta_d) \subset \mathbb{C}^d : r < |\theta_j| < R \}$,
$0 < r < 1 < R < \infty$. Choosing $r$ and $R$ sufficiently close to 1 one immediately sees that the function
\[
C(z, \theta) := \frac{g^2 - (\Im M(z, \theta))^2 - (\Re M(z, \theta))^2 - 2ig\Re M(z, \theta)}{|M(z, \theta) - ig|^2}
\]
does not take values in $(-\infty, 0)$ for $(z, \theta) \in Z \times \Theta$. Therefore, the function $f(z, \theta) := \log C(z, \theta)$ is well-defined and analytic in $Z \times \Theta$, where $\log$ denotes the principal branch of the logarithm. We will use the following assertion [FP84, Lemma 3.2] implied by the Diophantine property [8]:

**Lemma 3.** The operator $1 - U$ is a bijection on the set of functions $v$ analytic in $\Theta$ with $\int_{T^d} v(\theta)\theta^{-1}d\theta = 0$.

By lemma 3, the function $t(z, \theta) := (1 - U)^{-1}\left(f(z, \theta) - f_0(z)\right)$ is well-defined and analytic in $Z \times \Theta$, where

\[
f_0(z) := \frac{1}{(2\pi i)^d} \int_{T^d} f(z, \theta)\theta^{-1}d\theta.
\]

**Lemma 4.** The function $f_0$ is analytic in $Z$,

\[
\Re f_0(z) < 0 \quad \text{for} \quad \Im z > 0,
\]

\[
\Re f(z, \theta) = \Re t(z, \theta) = \Re f_0(z) = 0 \quad \text{for} \quad \Im z = 0.
\]

For real $\lambda$ one has $f_0(\lambda) = 2i\sigma(\lambda)$, where

\[
\sigma(\lambda) = \frac{1}{(2\pi i)^d} \int_{T^d} \arctan \frac{M(\lambda, \theta)}{g} \theta^{-1}d\theta.
\]

The function $\sigma$ is real-valued, strictly increasing, and continuously differentiable on $[a, b]$.

**Proof.** The analyticity of $f_0$ follows from its integral representation. Eq. (12) follows from (11) if one takes into account the inequalities $\Im M(z, \theta) > 0$ for $\Im z > 0$ and $\Re \log z < 0$ for $|z| < 1$. Equalities (12) follows from from (11) and the real-valuedness of $M(z, \theta)$ for real $z$.

By elementary calculations, for $x \in \mathbb{R}$ and $y > 0$ one has

\[
g_1(x) := \frac{1}{2i} \log \frac{iy + x}{iy - x} \equiv \arctan \frac{x}{y} =: g_2(x).
\]

In fact, this follows from

\[
g'_1(x) = g'_2(x) = \frac{y}{x^2 + y^2}
\]

and $g'_1(0) = g'_2(0) = 0$. Eq. (14) obviously implies $f_0(\lambda) = 2i\sigma(\lambda)$ for $\lambda \in \mathbb{R}$. Furthermore, as follows from (15),

\[
\sigma'(\lambda) = \frac{1}{(2\pi i)^d} \int_{T^d} \frac{gM'_1(\lambda, \theta)}{M(\lambda, \theta)^2 + g^2} \theta^{-1}d\theta,
\]

and, by (14), $\sigma'(\lambda) > 0$. □

An immediate corollary of the analyticity of $f_0$ and of (12) is

**Lemma 5.** There exists $\epsilon_0 > 0$ such that $|e^{f_0(\lambda)} \xi - 1| \leq 2|e^{f_0(\lambda + i\epsilon)} \xi - 1|$ for all $\xi \in S^1$, $\lambda \in [a, b]$, and $\epsilon \in [0, \epsilon_0]$.
Denote by \( t(z) \) and \( f(z) \) the multiplication operators by \( t(z, \theta) \) and \( f(z, \theta) \) in \( L^2(\mathbb{T}^d) \), respectively. By definition of \( t(z, \theta) \) for any \( \varphi \in L^2(\mathbb{T}^d) \)
\[
\begin{align*}
(16) \quad e^{t(z)} e^{f_0(z)} U e^{-t(z)} \varphi(\theta) &= e^{t(z, \theta)} e^{f_0(z, \theta)} \exp \left( -t(z, e^{2\pi i \omega_1 \theta_1}, e^{2\pi i \omega_4 \theta_4}) \right) U \varphi(\theta) \\
&= \exp \left( t(z, \theta) - U t(z, \theta) + f_0(z, \theta) \right) U \varphi(\theta) = e^{f(z)} U \varphi(\theta) = C(z) U \varphi(\theta).
\end{align*}
\]
Therefore, one can rewrite Eq. (9) as
\[
(17) \quad M(z) - A = B(z)^{-1} e^{t(z)} (1 - e^{f_0(z)} \chi U) e^{-t(z)} (1 + \chi U)^{-1}.
\]

**Proposition 6.** The set of the eigenvalues of \( H \) in \([a, b]\) is dense and coincides with the set of solutions \( \lambda \) to
\[
\begin{align*}
(18) \quad \sigma(\lambda) + \varphi + \pi(\omega, m) &= 0 \mod \pi, \quad m \in \mathbb{Z}^d.
\end{align*}
\]
Each of these eigenvalues is simple, and for any fixed \( m \in \mathbb{Z}^d \) Eq. (18) has at most one solution \( \lambda(m) \), and \( \lambda(m) \neq \lambda(m') \) for \( m \neq m' \).

**Proof.** As follows from proposition 1, the eigenvalues \( \lambda \) of \( H \) outside \( \text{spec} \, H^0 \) are determined by the condition \( \ker (M(\lambda) - A) \neq 0 \), an their multiplicity coincides with the dimension of the corresponding kernels. Eq. (19) shows that the condition \( (M(\lambda) - A) u = 0 \) is equivalent to \( (1 - e^{f_0(\lambda)} \chi U) e^{-\xi t(\lambda)} (1 + \chi U)^{-1} u = 0 \) or, denoting \( v := e^{-\xi t(\lambda)} (1 + \chi U)^{-1} u, (1 - e^{f_0(\lambda)} \chi U) v = 0 \), which can be rewritten as
\[
(19) \quad U v = \chi e^{-f_0(\lambda)} v, \quad v \neq 0.
\]
As \( U \) has the simple eigenvalues \( e^{2\pi i \omega, m} \), \( m \in \mathbb{Z}^d \), and the corresponding eigenvectors form a basis, Eq. (19) implies (18) if one takes into account the identity \( f_0(\lambda) = 2i(\sigma(\lambda) \) proved in lemma 1. The rest follows from the monotonicity of \( \sigma \), the inclusion \( \text{ran} \, \sigma \subset (-\pi/2, \pi/2) \), and the arithmetic properties (7) and (3). \( \square \)

As \([a, b]\) was an arbitrary interval from \( \mathbb{R} \setminus \text{spec} \, H^0 \) and \( \text{spec} \, H^0 \) is a discrete set, one has an immediate corollary

**Proposition 7.** The pure point spectrum of \( H \) is dense in \( \mathbb{R} \).

We note that propositions 6 and 7 automatically imply \( \text{spec} \, H^0 \subset \text{spec} \, H \) (as \( \text{spec} \, H = \mathbb{R} \)), as \( \text{spec} \, H^0 \) is discrete and lies in the closure of the set of the eigenvalues given by (18). We cannot say in general if the Dirichlet eigenvalues are eigenvalues of \( H \) and, if it is the case, if they are simple, this depends on the edge lengths \( l_j \) and the edge potentials \( U_j \).

### 3. Estimates for spectral measures

Take some \( \alpha > 0 \). For any \( \delta > 0 \) we denote
\[
S_1^\delta = \bigcup_{m \in \mathbb{Z}^d} \left\{ \xi \in S^1 : |\text{Arg} \, \xi - \text{Arg} \, e^{-2\pi i (\omega, m)}| \leq \delta (1 + |m|)^{-d-\alpha} \right\}, \quad \bar{S}_1^\delta := S^1 \setminus S_1^\delta.
\]
Clearly, there holds
\[
(20) \quad |1 - \xi e^{2\pi i (\omega, m)}| \geq 2\pi^{-1} \delta (1 + |m|)^{-d-\alpha}, \quad \xi \in \bar{S}_1^\delta, \quad m \in \mathbb{Z}^d.
\]
Let \( \Delta \subset [a, b] \) be an interval whose ends are not eigenvalues of \( H \). Consider the mapping \( h : \lambda \mapsto \chi e^{f_0(\lambda)} \). By lemma 4 \( h \) is a diffeomorphism between \( \Delta \) and
By proposition 5 one has $h(\lambda(m)) = e^{-2\pi i(\omega, m)}$. Take an arbitrary $\delta > 0$ and denote

$$\Delta_\delta := \Delta \cap h^{-1}(S_\delta^1), \quad \Delta_\delta^c := \Delta \cap h^{-1}(S_\delta^e) \equiv \Delta \setminus \Delta_\delta.$$  

Clearly, $\Delta_\delta$ is a countable union of intervals, and the limit set $\bigcap_{\delta > 0} \Delta_\delta$ coincides with the set of all the eigenvalues $\bigcup_m \{\lambda(m)\}$.

**Lemma 8.** There exists $\varepsilon_0 > 0$ such that for any $\delta > 0$ and any $n \in \mathbb{Z}^d$ there exists $C > 0$ such that

$$\| (M(\lambda + i\varepsilon) - A)^n \| \leq C$$

for all $\lambda \in \Delta_\delta$, and $\varepsilon \in (0, \varepsilon_0)$.

**Proof.** Rewrite Eq. (17) in the form

$$\left(M(z) - A\right)^{-1} = (1 + \chi U)e^{t(z)}(1 - e^{f_0(z)}\chi U)^{-1}e^{-t(z)}B(z).$$

Take an arbitrary $n \in \mathbb{Z}^d$ and denote $\Psi(z, \theta) := e^{-t(z, \theta)}B(z, \theta)\theta^n$. Due to the analyticity one can estimate uniformly in $\mathbb{Z}^d$:

$$|\psi_z(m)| \leq C' e^{-\rho|m|}, \quad C', \rho > 0, \quad \psi_z := F^{-1}\Psi, \quad \| (1 + \chi U)e^{t(z)} \| \leq C'.$$

Therefore, (21) follows from the inequality

$$\| (1 - e^{f_0(\lambda + i\varepsilon)\chi U})^{-1}\Psi \| \leq C.$$

Assume that $\varepsilon_0$ satisfies the conditions of lemma 5 then uniformly for $\lambda \in \Delta$ and $\varepsilon \in (0, \varepsilon_0)$ one has

$$\| (F^{-1}(1 - e^{f_0(\lambda + i\varepsilon)\chi U})^{-1}\Psi)(m) \| = \| (1 - e^{f_0(\lambda + i\varepsilon)\chi e^{2\pi i(\omega, m)}})^{-1}\psi_{\lambda + i\varepsilon}(m) \| \leq 2\| (1 - e^{f_0(\lambda)\chi e^{2\pi i(\omega, m)}})^{-1} \cdot |\psi_{\lambda + i\varepsilon}(m) |\|.$$

As in our case $h(\lambda) \equiv \chi e^{f_0(\lambda)} \in \mathbb{S}_1^1$, due to (20) we have

$$\| (1 - e^{f_0(\lambda)\chi e^{2\pi i(\omega, m)}})^{-1} \| \leq \frac{\pi}{2\rho} (1 + |m|)^{d + \alpha}.$$

Finally,

$$\| (1 - e^{f_0(\lambda + i\varepsilon)\chi U})^{-1}\Psi \|^2 = \sum_{m \in \mathbb{Z}^d} \| (F^{-1}(1 - e^{f_0(\lambda + i\varepsilon)\chi U})^{-1}\Psi)(m) \|^2 \leq \left( \frac{\pi C'}{\delta} \right)^2 \sum_{m \in \mathbb{Z}^d} (1 + |m|)^{2(d + \alpha)} e^{-2\rho|m|} < \infty,$$

and (22) is proved.

Now we are able to estimate the spectral projections corresponding to $H$.

**Lemma 9.** For any $f \in \mathcal{H}$ and any $\delta > 0$ one has

$$\lim_{\varepsilon \to 0^+} \int_{\Delta_\delta} \| (H - \lambda - i\varepsilon)^{-1} f \|^2 d\lambda = 0.$$
Proof. Here we are going to use proposition [1]. First note that due to \( \tilde{\Delta}_\delta \subset \mathbb{R} \setminus \sigma(H^0) \) one has
\[
\lim_{\varepsilon \to 0} \varepsilon \int_{\tilde{\Delta}_\delta} \|(H^0 - \lambda - i\varepsilon)^{-1} f\|^2 d\lambda = 0 \quad \text{for any } f \in \mathcal{H}.
\]
Represent \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \), where
\[
\mathcal{H}_0 := \left( \bigcup_{\Im z \neq 0} \gamma(z)(\ell^2(Z^d)) \right) \perp, \quad \mathcal{H}_1 := \mathcal{H}_0^\perp;
\]
in other words, \( \mathcal{H}_1 \) is the closure of the linear hull of the set \( \{ \gamma(z) : \Im z \neq 0, \varphi \in \ell^2(Z^d) \} \).

By (5), for any \( f \in \mathcal{H}_0 \) and any \( z \) with \( \Im z \neq 0 \) one has \( \gamma^*(z) f = 0 \). Hence, by (6), there holds \( (H - z)^{-1} f = (H_0 - z)^{-1} \), and (24) implies (23) for \( f \in \mathcal{H}_0 \).

Now it is sufficient to show (24) for vectors \( f = \gamma(z) h \) for \( h = (M(z) - A)^{-1} \theta^m \), \( m \in Z^d \), \( \Im z \neq 0 \). The operators \( (M(z) - A)^{-1} \) have dense range (coinciding with \( \text{dom} \ A \)), hence the linear hull of such vectors \( f \) is dense in \( \mathcal{H}_1 \). By elementary calculations (see e.g. section 3 in [BGP06]) one rewrites Eq. (6) as
\[
\lim_{\varepsilon \to 0} \varepsilon \int_{\tilde{\Delta}_\delta} \|(H^0 - \lambda - i\varepsilon)^{-1} f\|^2 d\lambda = 0 \quad \text{for any } f \in \mathcal{H}.
\]
Due to lemma [5] we have \( \| (M(z) - A)^{-1} \theta^m \| \leq C \) with some \( C > 0 \), for all \( \lambda \in \tilde{\Delta}_\delta \) and sufficiently small \( \varepsilon \), and (25) implies
\[
\|H - \lambda - i\varepsilon\|^{-1} f \leq \frac{\|f\| + C \|\gamma(\lambda + i\varepsilon)\|}{|\zeta - \lambda - i\varepsilon|},
\]
and due to the analyticity of \( \gamma \), one can estimate \( \| (H - \lambda - i\varepsilon)^{-1} f \| \leq C' \) with some \( C' > 0 \) for all \( \lambda \in \tilde{\Delta}_\delta \) and sufficiently small \( \varepsilon \). This obviously implies (23).

Using the above estimates we are now completing the proof of the main result.

Proof of theorem [2]. The denseness of the pure point spectrum is shown already (proposition [7]). We are going to show that for any \( f \in \mathcal{H} \) and any interval \( \Delta \subset \mathbb{R} \setminus \sigma(H^0) \) the spectral measure \( \mu_f \) associated with \( H \) and \( f \) satisfies \( \mu_f(\Delta) = \mu_f(\Delta \cap \bigcup_m \{ \lambda(m) \}) \); this proves that all the spectral measures are pure point.

By the Stone formula, for any set \( X \) which is a countable union of intervals whose ends are not eigenvalues of \( H \) one has
\[
\mu_f(X) = \lim_{\varepsilon \to 0+} \frac{\varepsilon}{\pi} \int_X \|(H - \lambda - i\varepsilon)f\|^2 d\lambda.
\]
Using lemma [3] for any \( \delta > 0 \) we estimate
\[
\mu_f(\Delta) = \lim_{\varepsilon \to 0+} \frac{\varepsilon}{\pi} \int_\Delta \|(H - \lambda - i\varepsilon)f\|^2 d\lambda
\]
\[
= \lim_{\varepsilon \to 0+} \frac{\varepsilon}{\pi} \int_{\tilde{\Delta}_\delta} \|(H - \lambda - i\varepsilon)f\|^2 d\lambda + \lim_{\varepsilon \to 0+} \frac{\varepsilon}{\pi} \int_{\tilde{\Delta}_\delta} \|(H - \lambda - i\varepsilon)f\|^2 d\lambda
\]
\[
= \lim_{\varepsilon \to 0+} \frac{\varepsilon}{\pi} \int_{\tilde{\Delta}_\delta} \|(H - \lambda - i\varepsilon)f\|^2 d\lambda = \mu_f(\tilde{\Delta}_\delta).
\]
As \( \delta \) is arbitrary and \( \bigcap_{\delta > 0} \tilde{\Delta}_\delta = \bigcup_m \{ \lambda(m) \} \), the theorem is proved. \( \square \)
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