An improved Trapezoidal rule for numerical integration

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Abstract—Numerical Methods have attracted of research community for solving engineering problems. This interest is due to its practicality and the improvement of highspeed calculations done on current century processors. The increase in numerical method tools in engineering software, such as Matlab, is an example of the increased interest. In this paper, we are present a new improved numerical integration method, that is based on the well-known trapezoidal rule. The proposed method gives a great enhancement to the trapezoidal rule and overcomes the issue of the error value when dealing with some higher order functions even when solving for a single interval. After literature review, the proposed system is mathematically explained along with error analysis. Few examples are illustrated to prove improved accuracy of the proposed method over traditional trapezoidal method.

Keywords—Numerical methods; Numerical integration; Trapezoidal rule; Improved Trapezoidal rule

1. Introduction
Mathematical integration is a kind of summation, even the symbol used for integration is a capital ‘S’ stylized to show that integration and summation are close to each other [1]. For numerical data or complex analytical solutions, the numerical methods, for example, for integration were developed to solve problems for some practical engineering problems. Many numerical methods have been developed to help make life easier in solving these complex problems. For integration, one of the most common methods is the trapezoidal rule, which is a very common and simple method due to its certain level of accuracy. Unfortunately, in most of the cases, the accuracy of this method is not achieved except that the integration interval is divided into many small intervals, which may not be possible in some cases, otherwise the result will be far from the actual solution. To achieve relatively better accuracy for the smaller number of intervals, an improved numerical method is proposed in this paper based on the trapezoidal rule concept, with slightly more complexity, but with much more accurate results even with a single (whole integration) interval period. In the next section, literature review is presented about this topic, followed by proposed method in section three, where error value is derived. This section also contains experimental calculations conducted to show results compared with ordinary trapezoidal rule. The conclusions are presented in section four.

2. Literature review
Numerical methods topic is an essential subject, that all engineers study as part of their degree in higher education, and the book [1] is one of the commonly used textbooks that are widely preferred in teaching numerical methods to students in many universities. It covers several methods for integration
such as the Trapezoidal rule, Simpson rules, Romberg, etc. and, is considered a great tool for students and engineers to learn, practice and solve engineering problems. In this paper, the equations and examples related to the Trapezoidal rule are referenced to this book for comparative purposes. In [2], the authors tried to solve the Runge phenomena for getting higher order accuracy compared to the ones in the traditional quadrature methods which are limited to 4th order only. Their proposed scheme is capable of utilizing the trapezoidal rule spectral accuracy over the interior of the intervals. In another work [3], Trefethen and Weideman reviewed the trapezoidal rule both mathematically and historically, and they have shown that it is linked with the computational methods all across scientific computing. It was shown that it is linked to algorithms related to inverse Laplace transform, complex analysis, special functions, integral equations, rational approximation, and the computation of eigenvalues of matrices and operators. Similarly, the authors in [4] did a survey on the methods for highly accurate numerical integration and reviewed many methods from 1980 till recent years, and they highlighted some of the techniques that when combined with powerful integer detection methods have resulted in the analytical evaluation of many “impossible” integrands before finding these techniques.

As far as applications in engineering disciplines are concerned, the authors in [5], discuss a method to improve the contrast of the images with low contrast. The method defines a novel contrast enhancement algorithm, which is based on discrete integration method using Trapezoidal rule. The work in [6] describes the design of half sample delay recursive digital integrators, in which a half sample delay is applied on trapezoidal integration rule, after that, a modified finite impulse response (FIR) fractional delay filter is used to design recursive digital integrators. That modified FIR fractional delay filter is less complex but more efficient than original one, which leads to the design of lower order recursive digital integrators that have been compared with the existing half sample delay and the conventional recursive digital integrators. The results of the proposed method show effectiveness with considerably lower percentage absolute magnitude relative error (PARE), and linear phase response over almost 0% - 80% of the total Nyquist frequency range.

The work in [7] discusses the classical composite trapezoidal rule for computing two dimensional singular integrals, to obtain the convergence results $O(h^2)$ that is the same as the Riemann integral convergence rate. The respective authors were able, at specific points of the classical composite trapezoidal rule, to get the upper convergence phenomenon when the special function used in error function equates to zero. The authors in [8] propose a new method derived from trapezoidal rule, which is capable of computing fractional integrals and derivatives to within a given error tolerance.

In this paper, we present a new and simpler method that can, significantly, improve the ordinary trapezoidal rule method. The proposed method is shown mathematically along with experimental results on some engineering examples for comparative purposes.

3. Proposed method

In this section, we explain the method that we are proposing and solve some examples to show its effectiveness, and later will compare the results with the conventional trapezoidal rule method results.

3.1 Concept of Proposed Method

Inspired by the trapezoidal rule, which is easy to understand, easy to implement method, but unfortunately, the error value produced is substantial in some cases. In trapezoidal rule, the area under the curve is replaced by the area under the line to simplify the solution. The line here is in between the first point and the last point on the curve, known as “the integration limits”. This forms with the x-axis, a half trapezoid and its area can be calculated using the rule:

$$\text{Trapezoid Area} = \text{Base} \times \text{Average Height}$$  \hspace{1cm} (1)

In Figure 1, the $f(a)$-$f(b)$ line form with the x axis for the area that we are interested in calculating. In order to see the weakness of this method, we can observe that the line $f(a)$-$f(b)$ falls under a big curve, thus causes a significant error, as we will see in the examples later. On the other hand, the new method attempts to reduce this error by dividing the quadrilateral located between the trapezoid line and the
parallel tangent line at the far edge of the curve (red line in Figure 1), into two and adding one half of this rectangle (green line in Figure 1 below) to the calculated area “the integration value”.

Figure 1: Trapezoidal rule and new method

To do so mathematically, the method undergoes the following steps:

- Calculate the slope of the black line \([f(a)-f(b)]\), using the formula:
  \[
m = \frac{f(b) - f(a)}{x_b - x_a}
  \]  
  (2)

- Find the derivative of the original function \(f(x)\), then make it equal to the slope ‘\(m\)’ obtained as above.
  \[f'(x) = m\]
  This will give us the value of \(x\) at the tangency point.

- Now find the value of \(f(x)\) at the tangency point using the original equation. Once we get the slope and a single point coordinates of the line, then we can find its equation, using the formula:
  \[f_1(x) = mx + c\]
  All are known except \(c\). Once we get \(c\) value, we have the line equation that is parallel to the trapezoid top line (red line in Figure 1).

- Now, we find \(f_1(a)\) by putting the given value of “\(x = a\)”, in the new line equation we found above.
  Then, we go back to the integration method. The trapezoidal method says:
  \[I = (b - a) \times \frac{f(a) + f(b)}{2}\]  
  (3)

The new method will replace \(f(a)\) by the average of \(f(a)\) and \(f_1(a)\), also average of \(f(b)\) and \(f_1(b)\).

Thus, the new formula will be:

\[I = (b - a) \times \frac{f(a) + f_1(a) + f(b) + f_1(b)}{4}\]  
(4)

And since

\[f_1(a) - f(a) = f_1(b) - f(b) = k\]

because they are parallel lines, then we can also say:

\[I = (b - a) \times \frac{f(a) + f(b)}{2} + \frac{f_1(a) - f(a)}{2}\]

which also equals:

\[I = (b - a) \times \frac{f(b)}{2} + \frac{f_1(a)}{2}\]  
(5)

And this is the easiest formula to calculate the area under the green line as shown in Figure 2.
3.2 Error Estimation

To derive the error for the proposed method, same approach was used as performed in the calculation of the trapezoidal rule error method. For trapezoidal rule, the error is:

\[ \text{Error} = \text{Area under the curve} - \text{Area under the trapezoid line}. \]

Which is equal to:

\[ E_{\text{error}} = \int_a^b f(x) \, dx - \int_a^b (f(b) - f(a)) \frac{b - a}{2} \, dx \]

Thus, according to [1], the integration results in:

\[ I = h \cdot \frac{f(a) + f(b)}{2} - \frac{1}{12} f''(\xi) h^3 \]  \hspace{1cm} (6)

The first term is the trapezoidal rule, and the second represents the truncation error. In the improved method, the area of the trapezoidal rule is modified to include the average of the area in between the two lines, as shown in equation (4). In another way, if we consider the center line (with additional height which is \( k/2 \)) from Figure 1, then the area will be:

\[ I_{\text{improved}} = h \cdot \frac{f(a) + f(b) + k}{2} \]

If we consider \( I \) as fixed at all times for the same function, then the new error will be:

\[ E_{\text{improved}} = - \frac{1}{12} f''(\xi) h^3 - h \cdot k/2 \]  \hspace{1cm} (7)

3.3 Experimental work for validation

To prove our method, we have attempted some examples that are already presented in [1] and solved using the normal trapezoidal rule. Below are some of the examples:

Example 1:
This is a single application of trapezoidal rule. In this example (Figure 3), data given is such that \( a=0 \), and \( b=0.8 \), with exact analytical solution for the integration of the function as 1.640533, while the normal trapezoidal rule solution was 0.1728.

\[ f(x)=0.2+25x-200x^2+675x^3-900x^4+400x^5 \]

The solution using proposed method is explained below:

Given in the question,

\( a=0, \ b=0.8, \ f(a)=0.2, \ f(b)=0.232 \)
Figure 3: Example 1 with trapezoidal integration

slope \( m = 0.032 / 0.8 = 0.004 \)

\( f'(x) = m \) yields

\[ 25 - 400x + 2025x^2 - 3600x^3 + 2000x^4 = 0.04 \]

\[ \Rightarrow x = 0.564417 \]

\( y = f(0.564417) = 3.54 \)

The tangency point is (0.564, 3.54), so the tangent line is:

\[ y = mx + c, \Rightarrow c = y - mx = 3.54 - 0.004(0.564), \Rightarrow c = 3.5377 \]

The new line is:

\[ y = 0.004x + 3.5377; \text{ and } f_1(a) = f_1(0) = 3.5377 \]

We apply the rule again to get the below:

\[ I \sim (0.8 - 0) \times (3.5377 + 0.232) / 2 \]

\[ I \sim 1.50788 \]

which is very much closer to the true solution than the estimated value using normal trapezoidal rule, see Figure 4 for the area under the green line.

Figure 4: Example 2 with improved trapezoidal rule

Example 2:

This example, Figure 5, is similar to the previous one, but with dividing the interval into two segments. The true solution = 1.640533, and the normal 2-segment trapezoidal rule yields 1.0688.

Using proposed method, we get:

\[ a = 0, b = 0.8, n = 2, h = 0.4, f(0) = 0.2, f(0.4) = 2.456, f(0.8) = 0.232 \]
Figure 5: Example 2, 2-segment trapezoidal integration

slope 1 $m_1 = \frac{2.256}{0.4} = 5.64$

slope 2 $m_2 = \frac{-2.224}{0.4} = -5.56$

$f'(x) = m$ yields:

$25 - 400x + 2025x^2 - 3600x^3 + 2000x^4 = m$

For $m_1 = > x = 0.0705$ or 0.2898 or 0.5078 or 0.9317 (only first 2 are between 0 and 0.4)

$y = f'(0.0705 \ or \ 0.2898) = 1.1834 \ or \ 1.5463$

The tangency points are (0.0705, 1.1834) and (0.2898, 1.5463)

So, the tangent line is

$y = mx + c$; $=> c = y - mx = 1.1834 - 5.64(0.0705)$

$=> c = 0.78578 \ or \ 1.5463 - 5.64(0.2898)$; $=> c = 0.08817$

The new line is:

$f_1 = y = 5.64x + 0.78578 \ or \ y = 5.64x - 0.8817$

$f_1(a) = f_1(0) = 0.7857 \ or \ -0.0881$

(we select the farthest from $f(0) = 0.2$, so it is $f_1(0) = 0.7857)$

We apply the rule again to get:

$I_1 \approx (0.4 - 0) \ * (0.7857 + 2.456)/2$

$I_1 = 0.64834$, For $m_2$

$x = 0.6088$ or 0.9041,

only first one is in the range between (0.4 and 0.8). Thus,

$y = f(0.6088) = 3.4202$

The tangency point is (0.6088, 3.4202), so the tangent line is:

$y = mx + c$; $=> c = y - mx$

$c = 3.4202 - (-5.56) * (0.6088); \Rightarrow c = 6.8051$

The new line is:

$f_2 = y = -5.56x + 6.8051$

$f_2(a) = f_2(0.4) = 4.5811$

We apply the rule again to get the below:

$I_2 \approx (0.8 - 0.4) \ * (4.5811 + 0.232)/2$

$I_1 \approx 0.96262$, So, total $I$
\[ I = \left(0.64834+0.96262\right) = 1.61096 \]

which is very much closer to the true value than the estimated normal trapezoidal value which is 1.0688.

![Figure 6: Example 2, improved trapezoidal integration](image)

**Example 3:**
This is a single application of trapezoidal rule. In this example, Figure 7, given that \(a=0\), and \(b=3\), given that the exact solution for the integration of the function (shown below) “analytically” is 15.42336, while the normal trapezoidal rule solution yields 15.04503.

\[ f(x) = 5+3 \cos(x) \]

Using proposed method, the solution is explained below:

Given in the question:
\(a=0\), \(b=3\), \(f(a)=8\), \(f(b)=2.030023\)
\[ \text{slope } m = -1.98999 \]
\[ f'(x)=m \text{ yields} \]
\[ -3 \cos(x)=-1.98999 \quad \Rightarrow x=0.845535 \]
\[ y = f(0.845535) = 6.989992 \]

The tangency point is \((0.8455, 6.9899)\), so the tangent line is:
\[ y = mx + c \]
\[ c = y - mx = 6.989992 - (-1.9899) * (0.845535) \]
\[ \Rightarrow c = 8.6726 \]

The new line is: \(y = -1.9899x + 8.6726\)

![Figure 7: Trapezoidal integration vs. proposed method](image)
\[ f(a) = f(0) = 8.6726 \]

We apply the rule again to get:

\[ I \approx (3-0) \times (8.6726 + 2.03)/2 \]

\[ I \approx 16.05394 \]

which is close but little farther from the true solution than the normal trapezoidal rule estimated value (see Figure 7) for the area under the green line. This is because the function we have is oscillating around the trapezoid line, that is why, the trapezoidal method gave close result. Next, we see the results, when we divide the interval into two segments.

Example 4:

This example, Figure 8, is similar to the previous one, but with dividing the interval into two segments. The true solution = 15.42336, and the normal 2-segment trapezoidal value = 15.3408.

Using proposed method, the solution is as below:

\[ a=0, b=3, n=2, h=1.5, f(0) = 8, f(1.5) = 5.212212, f(3) = 2.030023 \]

slope1 \( m_1 = (5.2122-8)/1.5 = -1.85853 \)

slope2 \( m_2 = (2.0300-5.2122)/1.5 = -2.12147 \)

![Figure 8: Trapezoidal integration (2-segments)](image)

\[ f'(x) = m \text{ yields } -3 \times \sin (x) = m \]

For \( m_1 \Rightarrow x = 0.66812 \) (between 0 and 1.5)

\[ y = f(0.66812) = 7.354964 \]

The tangency point is (0.6681,7.3549)

So, the tangent line is

\[ y = mx + c \Rightarrow c = y - mx \]

\[ c = 7.3549 - (-1.8585) \times 0.6681; \Rightarrow c = 8.596686 \]

The new line is:

\[ f_1 = y = -1.8585x + 8.5966 \]

\[ f_1(a) = f_1(0) = 8.5966 \]

We apply the rule again to get the below:

\[ I_1 \approx (1.5-0) \times (8.5966+5.2122)/2 \]

\[ I_1 \approx 10.3567, \]

For \( m_2, x = 0.78546 \) or 2.356125, we select the second, which is in the range between (1.5 and 3):

\[ y = f(2.356125) = 2.878827 \]

The tangency point is (2.3561,2.8788), so the tangent line is:

\[ y = mx + c \Rightarrow c = y - mx \]
\[ c = 2.8788 - (-2.1214) \times (2.3561) = 7.877268 \]

The new line is:
\[ f_i = y = -2.1214x + 7.8772 \]
\[ f_i(a) = f_i(1.5) = 4.695068 \]

We apply the rule again to get:
\[ I = (3-1.5) \times (2.0300+4.6950)/2 \]
\[ I_2 \approx 5.043818, \]

So, total \( I \)
\[ I \approx (10.35667+5.043818) = 15.05174 \]

which is close to the true value but less than the normal trapezoidal estimated value, as can be seen in Table 1. The table also shows results achieved when Simpson’s 1/3 rule is applied, for comparison purposes.

### Table 1: Comparing results of different methods

| Ex. No. | \((\text{n (number of intervals)})\) | \(\text{Trapezoidal}\) | \(\text{Simpson 1/3}\) | \(\text{Proposed method}\) | \(\text{True solution}\) |
|---------|----------------------------------|-----------------|-----------------|------------------|-----------------|
| 1       | 1                                | 0.1728          | -               | 1.5078           | 1.6405          |
| 2       | 2                                | 1.0688          | 1.3674          | 1.6109           | 1.6405          |
| 3       | 1                                | 15.0450         | -               | 16.0539          | 15.4233         |
| 4       | 2                                | 15.3408         | 15.4394         | 15.0517          | 15.4233         |

### 4. Conclusions

We presented a new method for numerical integration that is based on trapezoidal rule. We have shown with examples that the new method is much more accurate than the normal trapezoidal rule when the function curve is above and far from the trapezoid line, but in some cases when the function curve is equally oscillating around the trapezoid line, the normal trapezoidal rule gave better results, though the new method was not far from the true solution. Despite that the new method involves little more steps to implement as compared to the classical trapezoidal rule, we recommend that this method to be practiced by engineering and mathematics students as it, in general, results in relatively higher accuracy than traditional trapezoidal approach in solving problems. Likewise, the proposed approach can also be applied in situations, where multiple segments are involved.

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