Universal critical behaviour of 3-level atoms interacting dipolarly with radiation

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Abstract. A system of 3-level atoms interacting with 2-modes of electromagnetic radiation is studied by means of a variational procedure. This allows us to establish the corresponding quantum phase diagram and demonstrate that the super-radiant region is divided in monochromatic sub-regions where only one coupling strength $\mu_{ij}$ and one radiation field mode rule the behaviour of the system. In this contribution we prove that the variational coherent states, the symmetry-adapted states, and the exact quantum solution exhibit a universal parametric curve associated essentially with the two-level population and the number of photons exciting these levels.

1. Introduction

Studies of quantum phase transitions have attracted the attention of physicists of the nuclear, molecular, quantum optics, and condense matter communities. The concept of quantum phase transitions was introduced in the studies of quantum critical phenomena in spin lattice systems at zero temperature [1]. They result from competing ground state phases, and revisions of the field in condensed matter are given in [2, 3]. In nuclear physics the phase transitions were studied using an algorithm introduced at the end of the seventies by Gilmore [4]. Applications of the procedure to describe the shape transitions in nuclei can be found in [5, 6].

The Gilmore algorithm has been applied to many body systems in quantum mechanics: As examples we can mention the Lipkin-Meshkov-Glick (LMG) collective model to describe $N$ identical nucleons which can occupy two possible states separated by a given energy difference; and the Dicke model, considered a key piece to study the field-matter interactions, which describes the interaction between a low-density gas of two-level atoms with one mode of an electromagnetic field in the long wave approximation [7].

A study of the critical behaviour of the nucleus within the LMG model or, equivalently, to the infinitely coordinated ferromagnetic Ising model, was carried out at the beginning of the eighties [8, 9]. These systems have been used to illustrate the theory of phase transitions, where the case with $T = 0$ is included [10].

The quantum phase transition in the Dicke model was determined since the early seventies by Hepp and Lieb [11] and the free energy of the system in the thermodynamic limit was calculated analytically by Wang and Hioe [12]. Experimentally, this phase transition has been realised in an open system formed by a Bose-Einstein condensate coupled to an optical cavity, observing a superradiant phase and a quantitative description is obtained using the Dicke model [13].
Generalisations of the Dicke model to 3-level atomic systems were introduced in the middle of the eighties and a review of the dynamics of an atom interacting with radiation can be found in [14] and references therein.

The superradiant transition has been thoroughly studied and a recent review is found in [15]. Lately, it has been established that symmetry-adapted coherent states can describe very well the superradiant region [16]. A nice characterisation of the quantum phase transition can be obtained by means of the Husimi function, Rényi-Wehrl entropies, and second moment of the Husimi function [17, 18]. The critical points in the Dicke model, with a finite number of particles, has been determined by means of semiclassical and exact numerical solutions [16]. In the semiclassical approach we used a variational method with coherent and symmetry adapted coherent states, whereas for the exact quantum case we use the concept of fidelity introduced in quantum information theory. For the coherent states (CS), symmetry-adapted coherent states (SAS), and the quantum exact solutions, we have shown the existence of a universal parametric curve for the expectation values of the number of photons and the relative population as function of the matter-field coupling strength [19].

Generalisations of the quantum phase diagram to 3-level atoms interacting dipolarly with 2-modes of an electromagnetic field has also been established in [20], where the phase diagram is divided into two regions: (i) the normal region where all the atoms are in their lowest energy level and the state has zero photons, and (ii) the superradiant region, where there is an enhancement in the emission rates of the excited atoms. The latter region divides itself into subregions where only one coupling strength \( \mu_{ij} \) and one radiation field mode \( \Omega_{ij} \) rule the behaviour of the system. This behaviour has been explicitly studied for the \( \lambda \) and \( N \) atomic configurations of 4-level atoms. As an example, for the \( \lambda \) case we determine three superradiant subregions associated to two electromagnetic modes denoted by \( S_{\Lambda}, S_{23}, \text{and } S_{34} \) [21].

These studies allow us to generalise the result for \( n \)-level atoms interacting dipolarly with \( \ell \)-modes of an electromagnetic field in a cavity [22]. The procedure to determine its quantum phase diagram is the following: A test function is constructed from the tensorial product of coherent states associated to the matter and field sectors. Then the expectation value of the Hamiltonian with respect to the variational state is calculated defining the so-called energy surface. The critical points of this energy surface then determine, in the parameter space of the dipolar matter-field couplings, the locus where the ground state of the system suffers quantum phase transitions.

In this contribution we show that the CS, SAS, and the exact quantum solutions for the 3-level atomic \( V \)-configuration follow the Universal curve established for the Dicke model in the different superradiant subregions of the generalised Dicke model. Thus, for the region \( S_{12} \) the expectation values of the number of photons in the electromagnetic mode \( \Omega_{12} \) and the relative population between the atomic levels with energies proportional to \( \omega_2 \) and \( \omega_1 \), as function of \( \mu_{12} \) fall on a Universal parametric curve. A similar result is obtained for the region \( S_{13} \), where the relevant parameters are \( \Omega_{13} \), for the photon field, and \( \omega_3 \) and \( \omega_1 \) for the matter part with small values of \( \mu_{12} \).

This contribution is organised as follows. In section 2 the generalised Dicke model for 3-level atoms interacting with electromagnetic radiation is presented. The energy surface is calculated with respect to the tensorial product of coherent states of the matter and field sectors. The stability properties of the energy surface are determined together with the variational ground state energies and eigenfunctions in section 3. These results allow us to calculate the quantum phase diagram of the atomic \( V \)-configuration and contour plots of the energy surface are shown to illustrate its behaviour when a quantum phase transition is crossed. In section 4 two independent constants of motion of the Hamiltonian for the \( V \) quantum system, in the rotating wave approximation (RWA), are obtained, which allow to divide the Hilbert space into four orthogonal subspaces. The corresponding quantum phase diagram together with the
symmetries of the Hamiltonian suggest a truncation of the Hilbert space into the basis states of two subgroups, viz., the SU$_{12}(2)$ and SU$_{13}(2)$ subgroups of the three dimensional unitary group of the 3-level system, U(3). We present the calculation of the Universal parametric curve for the Dicke model in section 5 together with the variational coherent state solution and the exact quantum calculation for a small number of particles. Additionally, the corresponding quantum phase diagram for a finite number of atoms is calculated. In section 6 a summary of the results is presented together with additional remarks.

2. Generalised Dicke model (GDM)

We are considering a low density gas of 3-level atoms interacting dipolarly with 2-modes of electromagnetic radiation in the long wave approximation, that is, neglecting the position dependence of the atoms [20].

The Hamiltonian of the 3-level atoms is given by

$$H_M = \hbar \omega_1 A_{11} + \hbar \omega_2 A_{22} + \hbar \omega_3 A_{33},$$

with $\omega_1 \leq \omega_2 \leq \omega_3$ and $A_{jj}$ indicating the number operator of particles in level $j$. The 2-mode electromagnetic field is described by the Hamiltonian

$$H_F = \hbar \Omega_{12} a_{12}^\dagger a_{12} + \hbar \Omega_{13} a_{13}^\dagger a_{13},$$

with $\Omega_{jk}$ denoting the frequency of the photon creation and annihilation operators $(a_{jk}^\dagger, a_{jk})$. These are boson operators satisfying the commutation relations $[a_{jk}, a_{mn}^\dagger] = \delta_{jm} \delta_{kn}$ and $[a_{jk}, a_{mn}] = [a_{jk}^\dagger, a_{mn}^\dagger] = 0$. The energy separation between the $(i,j)$ levels satisfies the expression $\omega_j - \omega_i = \Omega_{ji} + \Delta_{ij}$, with $\Delta_{ij}$ defining the corresponding detuning parameter.

In this model there is no direct interaction between the atoms, the atoms only interact through the quantised electromagnetic field. The coupling interaction between the matter and field is constituted by two terms called rotating $O_R$ and counterrotating terms $O_{CR}$, given by

$$H_{FM} = O_R + O_{CR}$$

$$= -\frac{\hbar}{\sqrt{N_a}} \left\{ \mu_{12}(A_{12} a_{12}^\dagger + A_{21} a_{12}) + \mu_{13}(A_{13} a_{13}^\dagger + A_{31} a_{13}) \right\}$$

$$-\frac{\hbar}{\sqrt{N_a}} \left\{ \mu_{12}(A_{21} a_{12}^\dagger + A_{12} a_{12}) + \mu_{13}(A_{31} a_{13}^\dagger + A_{13} a_{13}) \right\},$$

with $\mu_{jk} = (\omega_j - \omega_k) d_{jk} \sqrt{\frac{2 \pi \rho_m}{M_{jk}}} \delta_{ij}$ denoting the strength of the interaction proportional to the dipolar interaction $d_{ij}$ between the atomic levels $i$ and $j$. Here, $N_a$ is the total number of atoms and $\rho_m$ is the density of atoms in the quantisation volume.

The collective (many-body) operators $A_{jk} = \sum_{s=1}^{N_a} |j s\rangle \langle k s| \equiv b_{j}^\dagger b_k$ satisfy the commutation relations of a unitary algebra in three dimensions $u(3)$,

$$[A_{jk}, A_{mn}] = A_{jn} \delta_{mk} - A_{mk} \delta_{jn}. \quad (4)$$

Next we use as variational state a condensate formed by the tensorial product of coherent states for the matter and field parts because they survey properly the available Hilbert space of the system.
3. Energy surface of the GDM

The coherent states have played a central importance in quantum mechanics and quantum optics [23]. In what follows we define the coherent states of the matter and field components of the condensate.

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For the matter sector we consider the coherent states associated to the totally symmetric representation of $u(3)$ algebra [27]. These coherent states have been used in nuclear physics to describe the shapes within the interacting boson model [5]. In our case we have two complex parameters $\alpha_1$ and $\alpha_2$, as

$$|\alpha_2, \alpha_3\rangle = |\alpha_2\rangle \otimes |\alpha_3\rangle = e^{-\frac{i\alpha_2^* e^{i\theta_2} + \alpha_2^* e^{i\theta_2}}{2} \sum_{k_1,k_2} (\alpha_2)^{k_1} (\alpha_3)^{k_2} |k_1,k_2\rangle},$$

where $|k_1,k_2\rangle$ define eigenstates of the photon number operators, $\nu_2$ and $\nu_3$. These states minimise the uncertainty relation between the quadratures components of the 2-mode electromagnetic field.

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$$|\gamma_2, \gamma_3\rangle = \frac{1}{\sqrt{N_a!}} \Gamma^\dagger N_a |0\rangle,$$

where

$$\Gamma^\dagger = \frac{1}{\sqrt{1 + |\gamma_2|^2 + |\gamma_3|^2}} (b_1^\dagger + \gamma_2 b_2^\dagger + \gamma_3 b_3^\dagger).$$

with $b_k|0\rangle = 0$, $k = 1, 2, 3$. Notice that $\Gamma$ and $\Gamma^\dagger$ satisfy the boson commutation relations.

The energy surface of the system is calculated taking the expectation value of the Hamiltonian with respect to the test function and dividing the result by the number of atoms to have an intensive quantity. Then, by means of the expectation values: (i) For the matter part,

$$\langle \gamma_2 \gamma_3 | A_{1k} | \gamma_2 \gamma_3 \rangle = \frac{N_a \gamma_k}{1 + |\gamma_2|^2 + |\gamma_3|^2}, \quad \langle \gamma_2 \gamma_3 | A_{2k} | \gamma_2 \gamma_3 \rangle = \frac{N_a \gamma_k^*}{1 + |\gamma_2|^2 + |\gamma_3|^2},$$

$$\langle \gamma_2 \gamma_3 | A_{3k} | \gamma_2 \gamma_3 \rangle = \frac{N_a \gamma_k^* \gamma_3}{1 + |\gamma_2|^2 + |\gamma_3|^2}, \quad \langle \gamma_2 \gamma_3 | A_{11} | \gamma_2 \gamma_3 \rangle = \frac{N_a}{1 + |\gamma_2|^2 + |\gamma_3|^2}. \quad (7)$$

with $k = 2, 3$ and (ii) for the electromagnetic field,

$$\langle \alpha_2 \alpha_3 | a_{12} | \alpha_2 \alpha_3 \rangle = \sqrt{N_a r_{12} e^{i\theta_2}}, \quad \langle \alpha_2 \alpha_3 | a_{13} | \alpha_1 \alpha_3 \rangle = \sqrt{N_a r_{13} e^{i\theta_3}}. \quad (8)$$

where $\alpha = \sqrt{N_a r e^{i\theta}}$, with $r \in \mathbb{R}$ for each mode field, one arrives to the energy surface of the generalised Dicke model (GDM),

$$E = \Omega_{12} r_{12}^2 + \Omega_{13} r_{13}^2 + \frac{\omega_1 + \omega_2 \rho_2^2 + \omega_3 \rho_3^2}{1 + \rho_2^2 + \rho_3^2} \quad - \frac{4(\mu_{12} r_{12} \rho_2 \cos \theta_{12} \cos \phi_2 + \mu_{13} r_{13} \rho_3 \cos \theta_{13} \cos \phi_3)}{1 + \rho_2^2 + \rho_3^2}. \quad (9)$$
with $\gamma_2 = \rho_2 e^{i\phi_2}$ and $\gamma_3 = \rho_3 e^{i\phi_3}$. Notice that $|\alpha_{12}|^2$ and $|\alpha_{13}|^2$ measure the intensity of the electromagnetic modes (equivalently to the number of photons) while $|\gamma_2|^2 = (n_2/n_1)$ and $|\gamma_3|^2 = (n_3/n_1)$ measure the ratio of the populations $n_k$, with $k = 1, 2, 3$, of the atomic levels and $E$ is an scalar intensive quantity.

3.1. Critical points

The critical points of the variational energy surface are obtained by solving the set of coupled equations

$$
\left( \frac{\partial E}{\partial \theta_{1k}} , \frac{\partial E}{\partial \phi_k} , \frac{\partial E}{\partial r_{1k}} , \frac{\partial E}{\partial \rho_k} \right) = 0 ,
$$

with $k = 2, 3$.

From expression (10), one determines the critical points for the angle variables $\theta_{12}^c = 0, \pi$, $\theta_{13}^c = 0, \pi$, $\phi_2^c = 0, \pi$, and $\phi_3^c = 0, \pi$. These critical points provide a minimum value of the energy surface when $\mu_{jk} \cos \theta_{jk}^c \cos \phi_j^c \geq 0$, which under the assumption that the coupling interactions $\mu_{jk} \geq 0$, implies $\theta_{jk}^c = \phi_j^c$.

The critical points $r_{12}^c$ and $r_{13}^c$ that satisfy the expression (10) are given by

$$
r_{12}^c = \frac{2 \mu_{12} \rho_2}{\Omega_{12}(1 + \rho_2^2 + \rho_3^2)} , \quad r_{13}^c = \frac{2 \mu_{13} \rho_3}{\Omega_{13}(1 + \rho_2^2 + \rho_3^2)} .
$$

They determine the expectation value of the number of photons in each mode of the electromagnetic field.

The variables $\rho_2^c$ and $\rho_3^c$ are also obtained from expression (10), it is straightforward to show that $\rho_2^c = \rho_3^c = 0$ is a solution and the corresponding energy surface is $E_N = 0$, where we take without loss of generality $\omega_1 = 0$. This solution defines the normal region, which has all the atoms in their lowest level and zero photons. Therefore, the variational solution is given by the state

$$
|\psi\rangle_N = |0,0\rangle_F \otimes |N_a,0,0\rangle_M , \quad \text{with } E_N = 0 .
$$

Considering $\rho_3^c = 0$ one has solutions for $\rho_2^c$ and the corresponding energy

$$
\rho_2^c = \sqrt{\frac{4 \mu_{12}^2 - \Omega_{12} \omega_2}{4 \mu_{12}^2 + \Omega_{12} \omega_2}} , \quad E_{12} = -\frac{(4 \mu_{12}^2 - \Omega_{12} \omega_2)^2}{16 \mu_{12}^2 \Omega_{12}} ,
$$

which define the superradiant subregion $S_{12}$. In this region one has $\mu_{12} \geq \sqrt{\Omega_{12} \omega_2}/2$, and the expectation value of the photons of the mode $\Omega_{13}$ is zero. In consequence the test function takes the form

$$
|\psi\rangle_{S_{12}} = e^{-\frac{n_a \rho_2^c 2}{(1 + \rho_2^2)^2 \omega_2} \sum_{n \nu} \left( \frac{N_a}{n} \right)^{1/2} \left( \frac{\sqrt{N_a} \rho_2^c}{\sqrt{n}} \right)^{\nu} |\nu,0\rangle_F \otimes |N_a - n, n, 0\rangle_M .
$$

Observe that the the expectation value of the number of atoms in the lowest energy level is bounded by $1/2 \leq \langle A_{11} \rangle / N_a \leq 1$, indicating also that $0 \leq \langle A_{22} \rangle / N_a \leq 1/2$ and there are no atoms in the energy level $\omega_2$.

Now we consider the opposite case, that is, $\rho_2^c = 0$ which implies the solutions for $\rho_3^c$ and its associated energy as follows

$$
\rho_3^c = \sqrt{\frac{4 \mu_{13}^2 - \Omega_{13} \omega_3}{4 \mu_{13}^2 + \Omega_{13} \omega_3}} , \quad E_{13} = -\frac{(4 \mu_{13}^2 - \Omega_{13} \omega_3)^2}{16 \mu_{13}^2 \Omega_{13}} ;
$$

$$
\rho_3^c = 2 \pi \mu_{13}^2 \rho_3 = 0 , \quad \text{with } E_{13} = 0 .
$$
Figure 1. Contour plots for the energy surface of the atomic $V$-configuration. The plots are for $(\mu_{12} = 1/2, \mu_{13} = 4/5)$ at left in the region $S_{12}$, $(\mu_{12} = 1/5, \mu_{13} = 1)$ at the middle in the region $S_{13}$, and $(\mu_{12} = 1/10, \mu_{13} = 1/3)$ at the right in the normal region. The red dots indicate the corresponding critical minimum values. We use the parameters $\Omega_{13} = 1, \Omega_{12} = 0.8, \omega_2 = 0.8$ and $\omega_3 = 1$.

This solution determines the other superradiant subregion $S_{13}$ of the atomic $V$-configuration. This region exists for coupling interactions satisfying the condition $\mu_{13} \geq \sqrt{\Omega_{13} \omega_3}/2$, the expectation value of $\nu_{12}$ is zero, and the expectation value of the number of atoms occupying $\omega_1$ is again bounded by $1/2 \leq \langle A_{11}/N \rangle \leq 1$, and thus $0 \leq \langle A_{33}/N \rangle \leq 1/2$ and there are no atoms in energy level with $\omega_2$. In this region the variational solution takes the form

$$|\psi\rangle_{S_{13}} = e^{-\frac{N a_{13}^2}{2}} \sum_{n, \nu} \left( \begin{array}{c} N_a \\ n \end{array} \right)^{1/2} \left( \sqrt{N a_{13}^c} \right)^\nu \frac{\rho_3^n}{\sqrt{n!}} |0, \nu\rangle_F \otimes |N_a - n, 0, n\rangle_M. \quad (16)$$

Next we establish explicitly the quantum phase diagram for the atomic $V$-configuration, where the previous results play a fundamental role. Contour plots for the energy surface of the system in terms of the variables $\rho_2$ and $\rho_3$ are shown in Fig. 1, we consider dipolar interactions of the $V$-configuration. The plots are for monochromatic superradiant subregions at two critical atom-field coupling $\mu_{12}$ is zero, and the expectation value of the number of atoms occupying $\omega_1$ is again bounded by $1/2 \leq \langle A_{11}/N \rangle \leq 1$, and thus $0 \leq \langle A_{33}/N \rangle \leq 1/2$ and there are no atoms in energy level with $\omega_2$. In this region the variational solution takes the form

$$|\psi\rangle_{S_{13}} = e^{-\frac{N a_{13}^2}{2}} \sum_{n, \nu} \left( \begin{array}{c} N_a \\ n \end{array} \right)^{1/2} \left( \sqrt{N a_{13}^c} \right)^\nu \frac{\rho_3^n}{\sqrt{n!}} |0, \nu\rangle_F \otimes |N_a - n, 0, n\rangle_M. \quad (16)$$

3.2. Quantum phase diagram

We have found that the atomic $V$-configuration exhibits a quantum phase transition from the no photons present to monochromatic superradiant subregions at two critical atom-field coupling interactions $\mu_{12}^c = \sqrt{\Omega_{12} \omega_2}/2$ and $\mu_{13}^c = \sqrt{\Omega_{13} \omega_3}/2$. This is equivalent to consider the equalities of the energies,

$$E_N = E_{12} \Rightarrow 4 \mu_{12}^2 = \Omega_{12} \omega_2, \quad E_N = E_{13} \Rightarrow 4 \mu_{13}^2 = \Omega_{13} \omega_3. \quad (17)$$

At these values the critical points of the superradiant subregions coalesce to $(\rho_2^c, \rho_3^c) = (0, 0)$. Then, from the catastrophe formalism point of view, the borders of the normal to the monochromatic regions are both bifurcations and Maxwell sets [28].

Taking $E_{12} = E_{13}$, one finds

$$\mu_{12}^2 = \frac{\Omega_{12} \left( \mu_{13}^2 - \frac{\Omega_{13} \omega_3}{4} \right)^2}{2 \Omega_{13} \mu_{13}^2} \left( 1 + \frac{\mu_{13}^2 \Omega_{13} \omega_3}{2 (\mu_{13}^2 - \frac{\Omega_{13} \omega_3}{4})^2} \right) \left( 1 + \frac{\mu_{13}^2 \Omega_{13} \omega_2}{(\mu_{13}^2 - \frac{\Omega_{13} \omega_3}{4})^2} \right), \quad (18)$$
Figure 2. Energy and phase diagram plots for the atomic V-configuration are shown at the top left. The yellow arrows indicate second order phase transitions while the red arrow specify first order phase transitions. Contour plots of the energy surface are displayed at top right for the region \(S_{12}\), at the left side of bottom left on the separatrix of the superradiant subregions, and at the bottom right for the region \(S_{13}\).

which must satisfy the conditions associated to the existence of the critical points \(4\mu_{12}^2 \geq \Omega_{12}\omega_2\), and \(4\mu_{13}^2 \geq \Omega_{13}\omega_3\). This curve separates the subregion \(S_{12}\), where the photons of the mode \(\Omega_{12}\) dominate, from the sector \(S_{13}\), where the photons of the mode \(\Omega_{13}\) dominate.

For the parameters \(\Omega_{13} = 1\), \(\Omega_{12} = 0.8\), \(\omega_2 = 0.8\), and \(\omega_3 = 1\) we display in Fig. 2 the minimum energy as function of the matter-field dipolar intensities \(\mu_{12}\) and \(\mu_{13}\). The energy scale is shown at the right side of the figure. The normal region is indicated by \(N\) (black region) while the superradiant subregions by \(S_{12}\) and \(S_{13}\). The quantum phase diagram of the 3 level system is also shown (see white lines) together with the order of the transitions. These are established by means of the Ehrenfest classification [28].

The behaviour of the energy surface, when there is a crossing between the superradiant subregions \(S_{12}\) and \(S_{13}\), as function of the variables \(\rho_2\) and \(\rho_3\) is also exhibited in Fig. 2. Notice that the crossing corresponds to a saddle point, i.e., a critical point unstable under a small perturbation, going to the minima in the variable \(\rho_2\) or \(\rho_3\). The parameters used for the different contour plots of the energy surface at sector \(S_{12}\), at the separatrix, and at \(S_{13}\), are \((\mu_{12},\mu_{13}) = \{(0.18763, 2/3), (0.21763, 2/3), (0.24763, 2/3)\}\), respectively.

4. Symmetries of the atomic V-configuration Hamiltonian

The symmetries of the GDM can be established by considering first the rotating wave approximation (RWA). Then we propose as constant of the motion the operator

\[
K = \eta_1 \nu_{12} + \eta_2 \nu_{13} + \lambda_1 A_{11} + \lambda_2 A_{22} + \lambda_3 A_{33},
\]
where the set of real parameters \( \eta_k \) and \( \lambda_j \), with \( k = 1, 2 \) and \( j = 1, 2, 3 \), will be determined by asking that the commutation relation \([K, O_R] = 0\) holds. One finds

\[
K = \eta_1 (\nu_{12} + A_{22}) + \eta_2 (\nu_{13} + A_{33}) + \lambda_1 (A_{11} + A_{22} + A_{33}),
\]

which implies three independent constants of motion,

\[
K_1 = A_{11} - \nu_{12} - \nu_{13}, \quad K_2 = A_{22} + \nu_{12}, \quad K_3 = A_{33} + \nu_{13}.
\]

Observe that the number of atoms is also a constant of the motion, which can be obtained by summing the previous invariants.

For the counter-rotating term one finds that the parity of the constants of motion is an invariant, that is, \( e^{i\pi K_j} O_{CR} e^{-i\pi K_j} = O_{CR} \) because

\[
e^{i\pi K_j} a_{jk} e^{-i\pi K_j} = e^{-i\pi} a_{jk}, \quad e^{i\pi K_j} A_{jk} e^{-i\pi K_j} = e^{-i\pi} A_{jk},
\]

for all possible values of \( j, k \) and \( \ell \). The basis for the atomic \( V \)-configuration Hamiltonian can be defined in terms of the infinite dimensional set of Fock states: \(|\nu_{12}, \nu_{13}, n_1, n_2, n_3\rangle\), two of them describing the photons and three related with the populations of the atomic levels.

The quantum labels for the field part can be replaced by the eigenvalues of the constants of motion \( K_2 \) and \( K_3 \) while the total number of atoms in the lowest level in terms of number of atoms of the system, \( N_a = n_1 + n_2 + n_3 \). Thus, the infinite dimensional Hilbert space is divided into four orthogonal subspaces related with the parities of the eigenvalues \( K_2 \) and \( K_3 \),

- \( \mathcal{H}_{ee} : \{ |\nu_{12}, \nu_{13}; n_1, n_2, n_3\rangle \} \) with \( \nu_{12} + n_2 \) and \( \nu_{13} + n_3 \) even values.
- \( \mathcal{H}_{eo} : \{ |\nu_{12}, \nu_{13}; n_1, n_2, n_3\rangle \} \) with \( \nu_{12} + n_2 \) even and \( \nu_{13} + n_3 \) odd values.
- \( \mathcal{H}_{oe} : \{ |\nu_{12}, \nu_{13}; n_1, n_2, n_3\rangle \} \) with \( \nu_{12} + n_2 \) odd and \( \nu_{13} + n_3 \) even values.
- \( \mathcal{H}_{oo} : \{ |\nu_{12}, \nu_{13}; n_1, n_2, n_3\rangle \} \) with \( \nu_{12} + n_2 \) and \( \nu_{13} + n_3 \) odd values.

Therefore, from the variational solution we are able to get four independent energy bands of the system, associated to the four Hilbert subspaces. We have found that the ground state is always determined within the Hilbert space \( \mathcal{H}_{ee} \).

The calculation of the eigensystem of the Hamiltonian requires a truncation of the infinite dimensional Hilbert space. This is done by establishing the maxima values of \( K_{2\text{Max}} \) and \( K_{3\text{Max}} \) for a given interval of the dipolar interactions \( \mu_{12} \) and \( \mu_{13} \) (details will be published elsewhere).

Then, to construct the matrix hamiltonian we propose the finite dimensional set of states,

\[
\mathcal{H}_{ee} : \cup_{N_2=0}^{N_2} \cup_{s=0}^{N_3} \cup_{n_2,n_3=0}^{N_3} |2q - n_2, 2s - n_3; N_a - n_2 - n_3, n_2, n_3\rangle
\]

(20)

with \( N_2 \equiv [K_{2\text{Max}}/2], \ N_3 \equiv [K_{3\text{Max}}/2] \) together with the conditions \( 2q - n_2 \geq 0 \) and \( 2s - n_3 \geq 0 \).

For \( \mu_{13} = 0 \), the Hilbert space \( \mathcal{H}_{ee}^{(12)} \) can be generated by (20) with \( s = n_3 = 0 \), which is equivalent to the Hilbert space of the Dicke model. For \( \mu_{12} = 0 \), we have an analogous result, i.e., the Hilbert space \( \mathcal{H}_{ee}^{(13)} \) is obtained from (20) with \( q = n_2 = 0 \).

For \( (\mu_{13}, \mu_{12}) \neq 0 \), the symmetries of the Hamiltonian and the variational ground state result lead us to propose as zero order approximation a Hilbert space of the form

\[
\mathcal{H}_{ee}(0) = \mathcal{H}_{ee}^{(12)}(k) \oplus \mathcal{H}_{ee}^{(13)}(k).
\]

The approximation can be improved by considering the union of the Hilbert space \( \mathcal{H}_{ee} \) with \( s = 0, 1, \ldots k \), which we denote by \( \mathcal{H}_{ee}^{(12)}(k) \) and \( \mathcal{H}_{ee} \) with \( q = 0, 1, \ldots k \), which we denote by \( \mathcal{H}_{ee}^{(13)}(k) \). Therefore a truncated basis of \( k \)-th-order can be written as follows

\[
\mathcal{H}_{ee}(k) = \mathcal{H}_{ee}^{(12)}(k) \oplus \mathcal{H}_{ee}^{(13)}(k).
\]

(21)
5. Universal curves

We will extend the result obtained for the Dicke model in relation with the determination of the Universal parametric curve for the expectation value of the number of photons of the electromagnetic field and the relative atomic population of the levels as implicit function of the dipolar interaction [16].

5.1. Universal parametric curve for the Dicke model

We give here a brief review of the determination of the variational ground state for the Dicke model. Considering the same notation as for the 3-level case, the Dicke Hamiltonian takes the form

\[ H = \hbar \omega_1 A_{11} + \hbar \omega_2 A_{22} + \hbar \Omega a^\dagger a - \frac{\hbar \mu}{\sqrt{N_a}} (A_{12} + A_{21})(a^\dagger + a). \]  

The expectation value of the Hamiltonian with respect to the corresponding tensorial product of the matter and field coherent states \(|\alpha \gamma\rangle\) determines the energy surface,

\[ E = \frac{\langle \alpha \gamma | H | \alpha \gamma \rangle}{\hbar N_a} = \Omega r^2 + \omega_1 + \omega_2 \rho^2 - \frac{4 \mu \rho \cos \theta \rho \cos \phi_\gamma}{1 + \rho^2}, \]  

with \(\alpha = \sqrt{N_a} r e^{i \theta_\rho}\), and \(\gamma = \rho e^{i \phi_\gamma}\). Notice that in this two-level problem the coherent states of matter are equivalent to those introduced by Radcliffe related with the so-called spin coherent states [29].

The critical points of the energy surface are obtained by solving the set of coupled equations

\[ \left( \frac{\partial E}{\partial \theta_\rho}, \frac{\partial E}{\partial \phi_\gamma}, \frac{\partial E}{\partial r}, \frac{\partial E}{\partial \rho} \right) = 0. \]

The first three equations are solved immediately

\[ \theta_\rho^c = 0, \pi, \quad \phi_\gamma^c = 0, \pi, \quad r^c = \frac{2 \mu \rho}{\Omega (1 + \rho^2)}, \]  

which provide a minimum value for the energy surface when \(\mu \cos \theta_\rho \cos \phi_\gamma \geq 0\). Then, by assuming \(\mu \geq 0\), the above condition implies \(\theta_\rho^c = \phi_\gamma^c\).

From \(\frac{\partial E}{\partial \rho} = 0\), one finds the following solutions: (i) \(\rho^c = 0\), which yields \(E_N = 0\), where we take \(\omega_1 = 0\) without lose of generality. (ii) Considering \(\rho^c \neq 0\), one gets

\[ \rho^c = \sqrt{\frac{4 \mu^2 - 2 \omega_2}{4 \mu^2 + \Omega \omega_2}}, \quad E = -\frac{(4 \mu^2 - \Omega \omega_2)^2}{16 \mu^2 \Omega}. \]  

Notice that this minimum critical point exists for \(\mu \geq \sqrt{\Omega \omega_2}/2\). Therefore there is a quantum phase transition between the zero field with all the atoms in the lowest energy level to the superradiant region when the dipolar coupling parameter is larger than

\[ \mu^c = \sqrt{\Omega \omega_2}/2 \]  

in agreement with the result for the superradiant subregions of the phase diagram of the 3-level system.

The symmetry preserved by the Hamiltonian (22) is the parity of the constant of motion in the RWA, i.e., \(M = A_{22} + a^\dagger a\), because we have

\[ e^{i \pi M} a e^{-i \pi M} = e^{-i \pi} a, \quad e^{i \pi M} A_{12} e^{-i \pi M} = e^{-i \pi} A_{12}. \]
The minima critical points for the even and odd variational coherent states for the following cases: 2 particles (blue); 4 particles (orange); 10 particles (green); 200 particles (gray), and for the coherent states (red). We use $\Omega = 1$, $\omega_2 = 1$, and $\omega_1 = 0$.

The Hamiltonian symmetry is restored on the variational states by means of the projectors $P_\pm = \frac{1}{\sqrt{2}} (1 \pm e^{i \pi M})$. Then the symmetry adapted coherent states (SAS) are given by

$$|\alpha \gamma\rangle_\pm = N_\pm (|\alpha\rangle \otimes |\gamma\rangle \pm | -\alpha\rangle \otimes | -\gamma\rangle) ,$$

where the normalisation constant is given by

$$N_\pm = \frac{1}{\sqrt{2}} \left( 1 \pm e^{-2N_a r^2} \left( \frac{1 - \rho^2}{1 + \rho^2} \right)^{N_a} \right)^{-\frac{1}{2}} .$$

The expectation value of the Hamiltonian with respect to the SAS can be calculated and thus the corresponding energy surface takes the form

$$E_\pm = \Omega r^2 \left( \frac{1 \pm \frac{1 - \rho^2}{1 + \rho^2}}{1 + \frac{1 - \rho^2}{1 + \rho^2}} \right) - \frac{1}{2} \left( \frac{1 - \rho^2}{1 + \rho^2} \right)^{\frac{1}{2}} \left( \frac{1 \pm \frac{1 - \rho^2}{1 + \rho^2}}{1 \pm \frac{1 - \rho^2}{1 + \rho^2}} \right)$$

$$- \left( \frac{4 \mu r \rho}{1 + \rho^2} \right) \frac{1 - \frac{\mu^2}{1 + \rho^2}}{1 + \frac{\mu^2}{1 + \rho^2}} \frac{1}{e^{2N_a r^2}} \pm 1 .$$

Firstly, one finds, in the thermodynamic limit,

$$\lim_{N_a \to \infty} E_\pm = \Omega r^2 - (\omega_2 - \omega_1) \frac{1 - \rho^2}{1 + \rho^2} - \frac{4 r \mu \rho}{1 + \rho^2} ,$$

Figure 3.
Figure 4. Even (left) and odd (right) results for: 2 particles (blue); 4 particles (orange); 10 particles (green); 200 particles (gray), and for the coherent states (red). We use $\Omega = 1$, $\omega_2 = 1$, and $\omega_1 = 0$.

which corresponds exactly to the energy surface obtained for the variational coherent state. Next we show that the minima critical points of the energy surface given in (27), coincide with those of the expression (28) in the thermodynamic limit.

In Fig. 3 the minima critical points of $E_\pm$ are displayed as functions of the dipolar coupling interaction for the following number of particles: $N_a = 2, 4, 10, 200$. One can observe that the critical points approximate the results obtained with the coherent state variational function when the number of atoms is large. Notice that the normal region shows more differences than the superradiant one, except in the vicinity of the quantum phase transition.

The variables $r^c$ and $\theta^c$, in the superradiant region, can be written as functions of the dipolar coupling interaction $\mu$,

$$r^c = \frac{\mu}{\Omega} \sqrt{1 - \left(\frac{\mu^c}{\mu}\right)^4}, \quad \theta^c = \text{ArcCos}(\mu^c/\mu)^2, \quad (29)$$

where $\theta^c$ defines the critical polar angle on the Bloch sphere [16]. In the normal region they are given by $r^c = 0$ and $\theta^c = 0$. The critical value of the coupling constant is given in expression (26). Then one can make a parametric plot of $-r^c$ vs. $\theta^c$ as functions of $\mu$. The whole of the normal region is located at the origin and the expression (29) is valid for $\mu > \mu^c$ and for $N_a \geq 2$. The result is indicated by the dashed red line at the left and right sides of Fig. 4.

The same parametric plot can be done for the results obtained with the SAS states by means of a numerical determination of the minima critical points of the energy surface (27), after the substitution of $r^c = \frac{2\mu\rho}{\Omega(1+\rho^2)}$.

We have displayed in Fig. 4 the parametric plot of $-r_c$ vs. $\theta_c = \text{ArcCos}\left(-\frac{\mu^c}{\mu}\right)$ for the even and odd coherent states, with different number of particles. Notice that they are almost indistinguishable under the change in the number of particles, besides of having the same result for the even and odd cases.

The quantum calculation is done through the following identification

$$-r_c \rightarrow -\sqrt{\frac{\langle a^\dagger a \rangle}{N_a}}, \quad \theta_c \rightarrow \text{ArcCos}\left(-\frac{(A_{22} - A_{11})}{N_a}\right). \quad (30)$$

The calculation of the expectation values in numerical form satisfies the mentioned Universal parametric plot as function of $\mu$, and it is valid for any number of particles $N_a \geq 2$. 

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Figure 5. At left we display the universal parametric plot for \( N_a = 12 \). We use \( \mu_{13} = 0 \) (gray), \( \mu_{13} = 0.1 \) (red), \( \mu_{13} = 0.3 \) (green), and \( \mu_{13} = 0.5 \) (blue), and the 2-level case (red-dashed line). At right the quantum phase diagram for a finite number of atoms as function of the dipolar interactions is plotted. The continuous blue line defines the case when \( N_a \to \infty \). In both panels we use the parameters \( \Omega_{13} = 1 \), \( \Omega_{12} = 0.25 \), \( \omega_2 = 0.25 \) and \( \omega_3 = 1 \).

5.2. Universal parametric curve for the GDM
We study the existence of a Universal curve of the type discussed in the previous subsection. This is motivated by the quantum phase diagram that we have obtained for the 3-level atomic \( V \)-configuration, i.e., the superradiant region is divided into sectors where there is dominance of only one photon mode of the electromagnetic field. Therefore, essentially they behave as 2-level systems. This can be seen in the expressions for the energy and corresponding eigenstates of the ground state of the system. For the normal region they are given in (12) while for the superradiant sectors \( S_{12} \) in (13) and (14), and for \( S_{13} \) in (15) and (16).

The numerical quantum calculations are done diagonalising the Hamiltonian matrix with the zero order basis \( \mathcal{H}_{am}(0) \). The universal parametric plot as a function of the dipolar strength \( \mu_{12} \) was calculated for several values of the dipolar interaction \( \mu_{13} \), which are indicated in the left side of Fig. 5. All the calculated points curves fall approximately along the same curve together with the universal curve for the Dicke model indicated by a dashed red line. We have found that this agreement improves when the number of atoms is large; in the plot we consider \( N_a = 12 \).

The determination of the finite phase diagram is done by calculating the minima of the fidelity, a concept introduced in quantum information theory [30, 31]. The fidelity is a measure of the similarity of two probability distributions, which for pure states \( |\xi\rangle \) and \( |\phi\rangle \) reduces to scalar product, i.e., \( F = |\langle \xi | \phi \rangle|^2 \). Here we use the fidelity to determine when the ground state exhibits a sudden change as function of the control parameters \( \mu_{12} \) and \( \mu_{13} \). We keep \( \mu_{13} \) fixed and get the minimum value of \( F \) when \( \mu_{12} \) is suffering infinitesimal changes (see the dotted lines of the right side of Fig. 5) and vice versa (see the continuous lines at the right side of the figure). In this way, in Fig. 5 we have displayed the calculated phase diagrams for the atomic \( V \)-configuration and different number of atoms \( N_a = 1, 2, 4, 20 \) as indicated in the plot.

6. Conclusions
For the GDM the energy surface is determined together with its stability properties, which allows us to establish the quantum phase diagram. We have explicitly studied the quantum phase diagram for the 3-level atomic \( V \)-configuration, where there are two regions: the normal one, characterised by a zero-field and all the atoms in the lowest energy level, and the superradiant, which it is divided into two sectors with monochromatic dominance of one-mode photons.

Additionally, the contour plots of the energy surface are exhibited where there is a crossing between the two superradiant sectors \( S_{12} \) and \( S_{13} \).
For the full Hamiltonian, without the RWA approximation, the constants of motion (parities) divide the Hilbert space into 4 orthogonal subspaces. Here the quantum numerical calculation is done for the even-even Hilbert space, that is, for even-even eigenvalues of the parity operators of $K_2$ and $K_3$. A truncated basis suggested by the symmetries of the Hamiltonian together with the quantum phase diagram allows us to determine the finite quantum crossovers of the GDM. The analogous quantum crossovers were also determined for a finite number of atoms, and they approach the quantum phase diagram for a large number of particles. This was done by choosing properly the basis of the Hilbert space because the dimension grows very rapidly with the number of particles.

The behaviour of the critical points of the energy surface for the even and odd variational coherent states of the Dicke model were compared with the critical points of the energy surface calculated with the coherent states, as functions of the dipolar strength.

We have reviewed the determination of the universal parametric curve of the Dicke model in terms of the number of photons and the relative atomic population between the levels. This universal curve exists also for the 3-level atomic $V$-configuration in the superradiant sectors, although it is perturbed for large values of the dipolar strength, i.e., for the subregion $S_{12}$ troubled by $\mu_{13}$ while for $S_{13}$ bothered by $\mu_{12}$.

The variational state for the Dicke and GDM models does not exhibit entanglement and does not possess the symmetries of the Hamiltonian. This is, however, not so for the SAS states. We have demonstrated that the phase diagram obtained in the thermodynamic limit does not depend on the variational state chosen. As the ground state function must satisfy the symmetry properties of the Hamiltonian, we may conclude implying that there is entanglement between the matter and field components.

From the results presented here, we propose the existence of the universal parametric curve for the atomic $\Xi$- and $\Lambda$-configurations of 3-level systems, as well as for matter-field systems of $n$-levels and $\ell$-modes of the electromagnetic field under dipolar interactions.

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References
[1] Hertz J A 1976 Phys. Rev. B 14 1165.
[2] Sachdev S 2011 Quantum Phase Transitions Cambridge University Press 2nd. ed.
[3] Vojta M, 2003 Rep. Prog. Phys. 66 2009.
[4] Gilmore R 1979 J. Math. Phys. 20 891.
[5] Iachello F, and Arima A 1987 The interacting boson model Cambridge University Press.
[6] Cejnar P, Jolie J, and Casten R F 2010 Rev. Mod. Phys. 82 2155.
[7] Dicke R H 1954 Phys. Rev. 93 99.
[8] Botet R, Jullien R, and Pleuty P 1982 Phys. Rev. Lett. 49 478.
[9] Botet R and Jullien R 1983 Phys. Rev. B 28 3995.
[10] Kittel C and Shore H 1965 Phys. Rev. 138 1165.
[11] Hepp K and Lieb E H 1973 Ann. of Phys. 76 360.
[12] Wang Y K and Hioe F T 1973 Phys. Rev A 7 831.
[13] Baumann K, Guerlin C, Brennecke F, and Esslinger T 2010 Nature (London) 464, 1301.
[14] Yoo H I and Eberly J H 1985 Phys. Rep. 118 239.
[15] B. M. Garraway 2011 Philos. Trans. R. Soc. A 369 1137.
[16] Castaños O, Nahmad-Achar E, López-Peña, and Hirsch J G 2011 Phys. Rev A 84 013819.
[17] Romera E, del Real R, and Calixto M 2012 Phys. Rev. A 85 053831.
[18] Romera E, Calixto M, and Nagy A 2012 Europhys. Lett. 97 20011.
[19] Castaños O, Nahmad-Achar E, López-Peña R, and Hirsch J G 2011 Phys. Rev A 86 023814.
[20] Cordero S et al 2015 Phys. Rev. A 92 053843.
[21] Cordero S et al 2016 Phys. Rev. A 94 013802.
[22] Cordero S, Nahmad-Achar E, Castaños O, and López-Peña R, 2017 Phys. Scr. 92 044004.
[23] Schleich W P 2001 Quantum Optics in Phase Space Wiley-VCH.
[24] Glauber R J 1963 Phys. Rev. Lett. 10 84.
[25] Sudarshan E C G 1963 Phys. Rev. Lett. 10 277.
[26] Klauder J R 1963 J. Math. Phys. 4 1055.
[27] Perelomov A 1986 Generalised coherent states and their applications Springer-Verlag.
[28] Gilmore R 1993 Catastrophe theory for scientists and engineers Dover Publications.
[29] Radcliffe J M 1971 J. Phys. A 4 313.
[30] Uhlmann A 1976 Rep. Math. Phys. 9 273.
[31] Zanardi P and Paunkovic N 2006 Phys. Rev E 74 031123.