Comparisons of Conventional, SRI, EAS and Hybrid-Stress Plane Q4 and Q9 Finite Element Models in Displacement and Energy Norms

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Abstract. Advanced finite element models have been being pursued for several decades. For plane models, the popularly employed formulations include the selectively reduced integration, enhanced assumed strain, hybrid/mixed methods, stabilization methods, among others. The accuracies of the advanced finite element models with respect to the standard element models are often demonstrated by some popular numerical tests in which displacement and stress components at a few selected points are benchmarked. The more comprehensive but laborious means of assessing the accuracy in terms of the relative errors in energy norm (Sobolev norm of order zero) and energy norm are less popularly employed. In this paper, these norms of the standard and some advanced 4-node and 9-node quadrilateral finite element models for a number of tests are computed and reported. In contrast to the common perception, the advanced models are not always more accurate than the standard ones. The discussions in this paper can be helpful for the understanding and usage of the numerical benchmark tests.

1. Introduction

Tremendous efforts have been put on developing advanced finite element models which are expected to be more accurate than the standard finite element models. For plane models, the popularly employed formulations include but not restrict to selectively reduced integration (SRI) [1], enhanced assumed strain (EAS) [2], hybrid/mixed [3,4] and stabilization methods [5]. Note worthily, the accuracy of the advanced finite element models with respect to the standard element ones are traditionally demonstrated by popular numerical tests in which displacement and stress at a few selected points are compared [6]. On the other hand, the more comprehensive but laborious means of assessing the accuracy in terms of the following relative error in energy norm \( e_\sigma \) which is recommended for error estimate in some well-known textbooks such as [7,8], is less popularly employed to assess the advanced finite element models:

\[
\begin{align*}
\epsilon_\sigma &= \frac{\|\mathbf{u}^h - \mathbf{u}^{\text{exact}}\|_E}{\|\mathbf{u}^{\text{exact}}\|_E} = \left( \int_{\Omega} (\mathbf{\sigma}^h - \mathbf{\sigma}^{\text{exact}})^T \mathbf{C}^{-1} (\mathbf{\sigma}^h - \mathbf{\sigma}^{\text{exact}}) \, d\Omega \right)^{1/2} \\
&= \left( \int_{\Omega} (\mathbf{\sigma}^{\text{exact}})^T \mathbf{C}^{-1} \mathbf{\sigma}^{\text{exact}} \, d\Omega \right)^{1/2} \left( \int_{\Omega} (\mathbf{\sigma}^h - \mathbf{\sigma}^{\text{exact}})^T \mathbf{C}^{-1} (\mathbf{\sigma}^h - \mathbf{\sigma}^{\text{exact}}) \, d\Omega \right)^{1/2}
\end{align*}
\](1)
in which \( \Omega, \mathbf{u}, \sigma \) and \( \mathbf{C} \) denote the problem domain, displacement vector, vector of stress components and the material stiffness matrix, respectively. The superscripts “\( h \)” and “exact” refer to the numerical and exact solutions, respectively. One would naturally expect that the advanced models should be more accurate than the standard ones in terms of \( e_{\sigma} \).

What trigger us to prepare this paper is that the advanced models do not often meet the expectation. To achieve more understanding on the accuracy, we employ further the following relative error in displacement (Sobolev norm of order zero) norm \( e_u \):

\[
e_u = \frac{\| u - u^{\text{exact}} \|_0}{\| u^{\text{exact}} \|_0} = \frac{1}{2} \left( \int_\Omega (u - u^{\text{exact}})^T (u - u^{\text{exact}}) d\Omega \right)^{1/2}.
\]

Intuitively speaking, \( e_{\sigma} \) and \( e_u \) reflect the accuracies of the stress and displacement predictions, respectively.

In this paper, four-node quadrilateral (Q4) and 9-node quadrilateral (Q9) standard, SRI, EAS and HS (hybrid stress) models will be briefly reviewed first. Then, the numerical tests to be conducted and the convergence plots for the computed \( e_{\sigma} \) and \( e_u \) will be presented. From the results, it is found that while advanced models are more accurate than the standard model in terms of \( e_u \) in most cases, however, there are tests in which advanced models are less accurate than the standard ones in terms of both \( e_{\sigma} \) and \( e_u \).

2. The Standard Model

2.1 Four-Node Quadrilateral (Q4) Element

For Q4 element, its coordinate and displacement interpolations are:

\[
\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \sum_{i=1}^{4} N_i \mathbf{x}_i, \quad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \sum_{i=1}^{4} N_i \mathbf{u}_i = \mathbf{N} \mathbf{d}
\]

where \( \mathbf{x}_i \) is the nodal coordinate, \( \mathbf{u}_i \) is the nodal displacement, \( \mathbf{d} \) is the element displacement vector that contains all \( \mathbf{u}_s \), and \( \mathbf{N} \) is the standard Q4 displacement interpolation matrix [7, 8]. The interpolated displacement leads to the strain:

\[
\mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} u_x \\ v_y \\ u_x + v_y \end{bmatrix} = \mathbf{L} \mathbf{d} = \mathbf{L} \mathbf{N} \mathbf{d}
\]

where \( \mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \) is the strain-displacement operator. The total potential of the element is:

\[
\Pi' = \frac{1}{2} \int_{\Omega} (\mathbf{\varepsilon}' \mathbf{C} \mathbf{\varepsilon}) d\Omega - P^e = \frac{1}{2} \int_{\Omega} (\mathbf{L} \mathbf{N}' \mathbf{C} \mathbf{L} \mathbf{N}) \mathbf{d} d - P^e = \frac{1}{2} \int_{\Omega} \mathbf{d}' \mathbf{k}' \mathbf{d} - P^e
\]

in which

\[
\mathbf{k}' = \int_{\Omega} \mathbf{L} \mathbf{N}' \mathbf{C} \mathbf{L} \mathbf{N} d\Omega
\]

is the stiffness matrix and \( P^e \) is the work done by the external force. The standard Q4 element (Q4Std) is evaluated by the 2\(^{nd}\) order quadrature and is said to be fully integrated.

2.2 Nine-Node Quadrilateral (Q9) Element

For Q9 element, its coordinate and displacement interpolations are:

\[
\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \sum_{i=1}^{9} N_i \mathbf{x}_i, \quad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \sum_{i=1}^{9} N_i \mathbf{u}_i = \mathbf{N} \mathbf{d}
\]

where \( \mathbf{N} \) is the standard Q9 displacement interpolation matrix. The subsequent procedures to obtain the strain, the total potential and the element stiffness matrix are the same as those for Q4, Eqs. (4)-(5). The standard Q9 element (Q9Std) is evaluated by the 3\(^{rd}\) order quadrature and is said to be fully integrated.
3. Selectively Reduced Integration (SRI)

For plane strain isotropic materials, the elastic material stiffness matrix and the stress can be split into:

\[
C = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = C_\mu + C_\lambda, \quad \sigma = C \varepsilon = \mu \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \varepsilon + \lambda \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \varepsilon
\]

where \( \mu \) and \( \lambda \) are Lame’s constants, \( E \) is the elastic modulus, and \( \nu \) is the Poisson’s ratio. Thus, the total potential can be expressed as

\[
\Pi^\varepsilon = \frac{1}{2} \int_{\Omega} \varepsilon^T (C_\mu + C_\lambda) \varepsilon \, d\Omega \quad - P^e = \frac{1}{2} \int_{\Omega} (\varepsilon^T C_\mu \varepsilon + \varepsilon^T C_\lambda \varepsilon) \, d\Omega - P^e. \tag{8}
\]

In the above expression, the terms associated with \( C_\mu \) and \( C_\lambda \) correspond to the shear and volumetric energies, respectively. The corresponding stiffness matrix of the element can be written as:

\[
k^*_\mu = \int_{\Omega} (LN)^T C_\mu (LN) \, d\Omega + \int_{\Omega} (LN)^T C_\lambda (LN) \, d\Omega = k^\mu + k^\lambda
\]

in which \( k^\mu \) and \( k^\lambda \) are the stiffness matrices associated with \( C_\mu \) and \( C_\lambda \) respectively.

For nearly incompressible materials, the Poisson’s ratio \( \nu \) approaches to 0.5, \( \lambda \) becomes extremely large and the well-known dilatational locking may be induced. SRI alleviates the locking by evaluating \( k^\mu \) with full integration (the 2\text{nd} and 3\text{rd} order quadratures for Q4 and Q9, respectively), and \( k^\lambda \) with reduced integration (the 1\text{st} and 2\text{nd} order quadratures for Q4 and Q9, respectively). The SRI Q4 and Q9 elements are here denoted as Q4SRI and Q9SRI, respectively.

4. Enhanced Assumed Strain (EAS) Formulation

EAS is another commonly used method to alleviate the locking phenomenon. Following the idea of Simo & Rifai [2], the total strain in an element is split into the compatible strain \( \varepsilon^w \) and the enhanced strain \( \varepsilon^e \) as:

\[
\varepsilon = \varepsilon^w + \varepsilon^e \tag{10}
\]

where \( \varepsilon^w = Lu \) is the compatible part and \( \varepsilon^e \) is directly assumed in the EAS method as \( \varepsilon^e = E\alpha \) in which the matrix \( E \) to be defined for various elements contains the assumed shape functions and the vector \( \alpha \) contains the element internal parameters. Substituting Eq. (10) into the total potential yields:

\[
\Pi^e = \frac{1}{2} \int_{\Omega} \varepsilon^{wT} \begin{bmatrix} k_{ad} & k_{ae} \\ k_{ed} & k_{ee} \end{bmatrix} \varepsilon^{wT} \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \, d\Omega \quad - P^e = \frac{1}{2} \int_{\Omega} \varepsilon^{eT} \begin{bmatrix} k_{ad} & k_{ae} \\ k_{ed} & k_{ee} \end{bmatrix} \varepsilon^{eT} \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \, d\Omega. \tag{11}
\]

In which

\[
k_{ad} = \int_{\Omega} (LN)^T C(LN) \, d\Omega, \quad k_{ae} = \int_{\Omega} (LN)^T CEdL \, d\Omega, \quad k_{ed} = \int_{\Omega} E^T C(LN) \, d\Omega \quad \text{and} \quad k_{ee} = \int_{\Omega} E^T CEdL \, d\Omega.
\]

Variation of Eq.(13) with respect to \( \alpha \) leads to

\[
k_{ad}\alpha + k_{ae} = 0 \quad \text{and} \quad \alpha = -k^{-1}_{ae}k_{ad}\alpha.
\]

Substituting the last expression into Eq.(13), the total potential can be written as:

\[
\Pi^e = \frac{1}{2} \int_{\Omega} \varepsilon^{eT} \begin{bmatrix} k_{ad} - k_{ae} & k_{ae}^{-1}k_{ad} \end{bmatrix} \varepsilon^{eT} \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \, d\Omega. \tag{13}
\]

4.1 EAS Q4 Models

For Q4 models, the following sets of four and seven incompatible EAS modes are commonly used [9]:

\[
\begin{bmatrix} \beta_{1} \\ \beta_{2} \\ 2\beta_{3} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \xi & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \xi & \eta \end{bmatrix} \alpha, \quad \begin{bmatrix} \xi & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \xi & \eta \end{bmatrix} = \frac{1}{J} \begin{bmatrix} \beta_{1} \\ \beta_{2} \\ 2\beta_{3} \end{bmatrix} \alpha.
\]

where

\[
\mathbf{T}_x = \mathbf{T}_x|_{\nu=0}, \quad \mathbf{T}_x = \mathbf{T}_x^{\nu} = \begin{bmatrix} x_{x} & y_{x} & x_{y} & y_{y} & 2x_{x}y_{x} \\ y_{x} & y_{y} & y_{x} & y_{y} & 2y_{x}y_{y} \\ x_{x} & x_{y} & x_{y} & x_{x} & x_{x} + x_{y} \end{bmatrix}^{T}
\]
and $\mathbf{T}_\sigma$ is the stress transformation matrix. In the subsequent numerical examples, the Q4 elements using the four and seven EAS modes will be abbreviated as Q4E4 and Q4E7, respectively.

### 4.2 EAS Q9 Model

For Q9 element, the following set of eleven EAS modes were proposed by Bischoff & Ramm [10]:

$$
\begin{align*}
\mathbf{1}_0 & = \int_T \left[ f_\xi \eta f_\eta \eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta \right] \left[ \alpha_i \right] \left[ M \right] \\
\mathbf{1}_0 & = \int_T \left[ f_\xi \eta f_\eta \eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta \right] \left[ \alpha_i \right] \left[ M \right]
\end{align*}
$$

or

$$
\begin{align*}
\mathbf{1}_0 & = \int_T \left[ f_\xi \eta f_\eta \eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta \right] \left[ \alpha_i \right] \left[ M \right] \\
\mathbf{1}_0 & = \int_T \left[ f_\xi \eta f_\eta \eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta f_\eta \right] \left[ \alpha_i \right] \left[ M \right]
\end{align*}
$$

where $f_\xi = 3\xi^2 - 1$ and $f_\eta = 3\eta^2 - 1$. This model shall be abbreviated as Q9E11 in the subsequent numerical examples.

### 5. Hybrid Stress (HS) Formulation

In HS elements, the element-wise Hellinger-Reissner functional is expressed as:

$$
\Pi^\prime = \int_T \left( -\frac{1}{2} \mathbf{C}^{-1} \mathbf{e}^T \mathbf{u} + \mathbf{C}^{-1} (\mathbf{Lu}) - \mathbf{b}^T \mathbf{u} / d \Omega - P^\prime \right)
$$

in which the displacement $\mathbf{u}$ is again interpolated as in Eq.(3) or Eq.(6). The stress $\mathbf{e}$ is independently assumed from $\mathbf{u}$ and written as $\mathbf{e} = \mathbf{P}\beta$. With this stress and the interpolated $\mathbf{u}$, the functional Eq.(16) can be further manipulated to be

$$
\Pi^\prime = -\frac{1}{2} \mathbf{H}^T \mathbf{b} + \mathbf{P}^T \mathbf{d} - P^\prime
$$

where $\mathbf{H} = \int_T \mathbf{P}^T \mathbf{C}^{-1} \mathbf{P} / d \Omega$ and $\mathbf{G} = \int_T \mathbf{P}^T \mathbf{C}^{-1} \mathbf{L} / d \Omega$. Variation of Eq.(17) with respect to $\beta$ enforces:

$$
\mathbf{H} \beta = \mathbf{G} / d \Omega \text{ or } \beta = \mathbf{H}^+ \mathbf{G} / d \Omega.
$$

Substituting it into Eq.(17) yields:

$$
\Pi^\prime = \frac{1}{2} \mathbf{d}^T (\mathbf{G}^T \mathbf{H}^+ \mathbf{G}) / d \Omega - P^\prime.
$$

### 5.1 HS Q4 Model

For the HS Q4, the well-known 5$\beta$ element of Pian & Sumihara [3] is considered in this paper. In their element, the assumed stress is:

$$
\sigma = \left[ \begin{array}{c}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{array} \right] = \left[ \begin{array}{c}
\beta_1 \\
\beta_2 \\
\beta_3
\end{array} \right] + \mathbf{T}_\sigma \left[ \begin{array}{c}
\eta \\
\xi
\end{array} \right] \left[ \begin{array}{c}
\beta_4
\beta_5
\end{array} \right]
$$

where $\mathbf{T}_\sigma = \mathbf{T}|_{\xi=\eta=0}$. In the subsequent examples, the HS Q4 element shall be abbreviated as Q4H5.

### 5.2 HS Q9 Model

For the HS Q9, the hybrid stress model of Sze [4] is considered in this study. The assumed hybrid stress in the model is:

$$
\left[ \begin{array}{c}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{array} \right] = \left[ \begin{array}{c}
1_1, \xi 1_1, \eta 1_1, \xi \eta 1_1
\end{array} \right] \left[ \begin{array}{c}
\beta_1 \\
\beta_2 \\
\beta_3
\end{array} \right] + \mathbf{T}_\sigma \left[ \begin{array}{c}
\xi f_\eta \\
\eta f_\xi
\end{array} \right] \left[ \begin{array}{c}
\beta_4 \\
\beta_5 \\
\beta_6
\end{array} \right]
$$

In the subsequent examples, this model shall be abbreviated as Q9H14.

### 6. Numerical Tests

In this section, numerical tests will be conducted by using the afore-reviewed quadrilateral elements whereas the relative errors $e_\sigma$ and $e_\rho$ will be computed and compared. Both the full integration (FI) and the uniform reduced integration (URI) by using Gaussian quadrature are employed for computing $e_\sigma$ while nodal integrations (compounded trapezoidal rule for Q4 and compounded Simpson’s rule for Q9) are used for computing $e_\rho$. It is well-known that the SRI element can also be viewed as a hybrid element with an assumed pressure or volumetric strain field. In this regard, the strain will also be
interpolated/extrapolated at the reduced order integration points in the course of computing the stress $e_\sigma$ for the SRI elements. For elements of the same order, the relative computational efficiency is trivial among various formulations when compared to the computational cost of solving the global linear equation. To obtain the convergence plots, each element in the initial mesh shall be sub-divided into $n \times n$ elements where $n = 1, 2, 4, 8$ and $16$ for Q4 and $n = 1, 2, 4$ and $8$ for Q9. Thus, the maximum number of nodes used by Q4 and Q9 are the same.

6.1 Cantilever modelled by $2n \times n$ rectangular elements

Figure 1 shows a cantilever modelled by rectangular elements under plane strain condition and subjected to an end shear force $P$. For $\nu = 0.25$, the results predicted by Q4s and Q9s are plotted in Figure 2(a-c) and (d-f), respectively. The standard and SRI elements provide almost the same results for $e_\sigma$ (FI), $e_\sigma$ (URI) and $e_u$. Q4E4, Q4E7 and Q4H5 yield identical predictions in this case [11, 12] and they are significantly more accurate than Q4Std and Q4SRI except for $e_\sigma$ (URI) with only $2 \times 1$ elements. Q9E11 and Q9H14 produce the same $e_\sigma$ (FI) while their $e_\sigma$ (URI) and $e_u$ are also very close to each other. Q9E11 and Q9H14 are markedly more accurate than Q9Std and Q9SRI.

Figure 1. A cantilever modelled by $2 \times 1$ rectangular elements subjected to shear force.

Figure 2. The relative errors $e_\sigma$ and $e_u$ for the cantilever beam modelled by $2n \times n$ rectangular elements with Poisson’s ratio $\nu = 0.25$. (a)-(c): $e_\sigma$ (FI), $e_\sigma$ (URI) and $e_u$ for the Q4 models; (d)-(f): $e_\sigma$ (FI), $e_\sigma$ (URI) and $e_u$ for the Q9 models.

For $\nu = 0.4999$, the results are plotted in Figure 3. Q4Std seriously suffers from the dilatational locking while the Q4SRI, Q4E4, Q4E7 and Q4H5 are locking-free. It should be remarked that since SRI integrates $k_\lambda^2$ only at the reduced order Gaussian points, using the full integration to compute $e_\sigma$ appears to be unfair and the relative error $e_\sigma$ (URI) may be more pertinent. For Q9s, the dilatational locking is not noted in Q9Std whilst the SRI does not improve the predictions. Again, EAS and HS elements are the accurate elements in terms of both the displacement and energy norms.
Figure 3. The relative errors $e_\sigma$ and $e_u$ for the cantilever beam modelled by $2n \times n$ rectangular elements with Poisson’s ratio $\nu = 0.4999$. (a)-(c): $e_\sigma$ (FI), $e_\sigma$ (URI) and $e_u$ for the Q4 models; (d)-(f): $e_\sigma$ (FI), $e_\sigma$ (URI) and $e_u$ for the Q9 models.

6.2 Cantilever modelled by $5n \times n$ trapezoidal elements

Figure 4 considers again the cantilever problem but the beam is initially modelled by 5 trapezoidal elements. The computed error norms for $\nu = 0.25$ are plotted in Figure 5. The Q4 EAS and HS elements are the most accurate Q4s followed by Q4SRI which is more accurate than Q4Std in terms of $e_\sigma$ (URI) and $e_u$. However, it is interesting to note that Q9Std and Q9SRI are the most accurate Q9s in terms of $e_u$. All the Q9s yield close $e_\sigma$ and Q9E11 is a bit better.

Figure 4. A cantilever modelled by 5 trapezoidal elements subjected to shear force.

Figure 5. The relative errors $e_\sigma$ and $e_u$ for the cantilever beam modelled by $5n \times n$ trapezoidal elements with Poisson’s ratio $\nu = 0.25$. (a)-(c): $e_\sigma$ (FI), $e_\sigma$ (URI) and $e_u$ for the Q4 models; (d)-(f): $e_\sigma$ (FI), $e_\sigma$ (URI) and $e_u$ for the Q9 models.
When \( \nu = 0.4999 \), the standard elements are the least accurate ones due to the locking and the Q4 EAS and HS elements are again the most accurate Q4s. Thus, the results are not plotted here. However, it should be noted that Q9SRI is more accurate than the Q9E11 and Q9H14 in terms of \( e_u \).

### 6.3 Square panel with a circular cutout

This example considers a plane strain panel with a circular hole at the center. Owing to the symmetry, only one-quarter of the panel is modelled and the initial mesh contains 2 elements, see Figure 6. While the conditions of symmetry are applied along the \( x \)- and \( y \)-axes, the tractions along \( x = 4 \) and \( y = 4 \) are prescribed according to the analytical stress solution [13].

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{r} \left( \frac{3}{2} \cos 2\theta + \cos 4\theta \right) + \frac{3}{2r^2} \cos 4\theta \\
\frac{1}{r} \left( \frac{3}{2} \cos 2\theta - \cos 4\theta \right) - \frac{3}{2r^2} \cos 4\theta \\
\frac{1}{r} \left( \frac{1}{2} \sin 2\theta + \sin 4\theta \right) + \frac{3}{2r^2} \sin 4\theta
\end{bmatrix}
\]

The analytical displacement solution for this example can also be found in Ref.[13]:

Figure 6. Square panel with a circular cutout. (a) Geometry, (b) initial mesh.

When \( \nu = 0.25 \), the predictions of all Q4s are close in both the displacement and energy norms, see Figure 7(a-c). The SRI model is marginally better than the others in \( e_u \) and the EAS and HS elements are marginally better in \( e_\sigma \) (FI). It is interesting to note that Q9Std and Q9H14 are the most and least accurate elements in \( e_\sigma \), respectively, yet they are least and most accurate in \( e_u \), respectively, as shown in Figure 7(d-f).

Figure 7. The relative errors \( e_\sigma \) and \( e_u \) for the square panel with circular cutout modelled by \( 2n \times n \) elements with Poisson’s ratio \( \nu = 0.25 \). (a)-(c): \( e_\sigma \) (FI), \( e_\sigma \) (URI) and \( e_u \) for the Q4 models; (d)-(f): \( e_\sigma \) (FI), \( e_\sigma \) (URI) and \( e_u \) for the Q9 models.
For $\nu = 0.4999$, the standard models again suffer from the dilatational locking and the predictions are the poorest. Q4SRI is the most accurate Q4s in terms of $e_u$ and its $e_\sigma$ (URI) is also close to that of Q4H5. Q9H14 is the most accurate Q9 element in terms of $e_u$ while Q9E11 and Q9SRI are the most accurate ones in terms of $e_\sigma$.

6.4 Thick-wall cylinder
This example considers a thick-wall cylinder subjected to internal pressure $p = 1$. Owing to symmetry, only one-quarter of the cylinder is modelled. The computational domain is initially divided into 2 elements as shown in Figure 8. The analytical stress solution can be found in Ref.[13].

![Figure 8. A thick-wall cylinder under inner pressure. (a) Geometry and load, (b) initial.](image)

When $\nu = 0.25$, Q4SRI and Q4Std are the most accurate Q4 elements in terms of $e_u$ and $e_\sigma$ (URI), respectively, see Figure 9. The EAS and HS elements are only more accurate in terms of $e_\sigma$ (FI). Q9H14 is the most but least accurate Q9 element in terms of $e_u$ and $e_\sigma$. Q9E11 and Q9Std are the most accurate models in terms of $e_\sigma$ (FI) and $e_\sigma$ (URI), respectively.

![Figure 9. The relative errors $e_\sigma$ and $e_u$ for the thick-wall cylinder modelled by $2n\times n$ elements with Poisson’s ratio $\nu = 0.25$. (a)-(c): $e_\sigma$ (FI), $e_\sigma$ (URI) and $e_u$ for the Q4 models; (d)-(f): $e_\sigma$ (FI), $e_\sigma$ (URI) and $e_u$ for the Q9 models.](image)

For $\nu = 0.4999$, the standard models are again locked. Among the advanced Q4s, Q4SRI is the most accurate one in terms of $e_u$ and $e_\sigma$ (URI). Among the advanced Q9s, Q9H14 yields the best displacement prediction but is worse than Q9E11 in terms of $e_\sigma$ (FI). Q9SRI, Q9E11 and Q9H14 give almost the same $e_\sigma$ (URI).

6.5 Square panel loaded by a double-sinusoidal body force
This example considers a unit plane strain square panel subjected to the double-sinusoidal body force
The boundary conditions for the square panel are:

At $x = 0$: $u_x = 0; f_x = 0$; \quad At $y = 0$: $u_y = 0; f_y = 0$; \quad At $x = 1$: $u_x = 0; f_x = 0$; \quad At $y = 1$: $u_y = 0; f_y = 0$.

When the Poisson’s ratio approaches 0.5, $\lambda$ and thus the body force would be very large. To avoid the large force, only $\nu = 0.25$ is considered. The analytical displacement solution for this problem is:

$$\begin{bmatrix}
    u_x \\
    u_y 
\end{bmatrix} = \begin{bmatrix}
    \sin \left( \frac{3}{2} \pi x \right) \cos \left( \frac{3}{2} \pi y \right) \\
    \cos \left( \frac{3}{2} \pi x \right) \sin \left( \frac{3}{2} \pi y \right) 
\end{bmatrix}.$$ (24)

The square panel is initially modelled by 1 element and then is divided into $n \times n$ regular elements. The relative errors $e_{\sigma}$ and $e_u$ for the square panel loaded by a double-sinusoidal body force modelled by $n \times n$ elements with $\nu = 0.25$. (a)-(c): $e_{\sigma}$ (FI), $e_{\sigma}$ (URI) and $e_u$ for the Q4 models; (d)-(f): $e_{\sigma}$ (FI), $e_{\sigma}$ (URI) and $e_u$ for the Q9 models.

7. Discussion

From the above examples, the following features are remarked:

1) Since the EAS and hybrid models are devised with due consideration to the bending modes, they behave well in bending examples (Section 6.1- 6.2) and are more accurate than the standard ones in these cases in terms of both $e_u$ and $e_{\sigma}$.

2) In examples where the bending modes are not dominating, the advantages of the advanced models are not so obvious. For an example, the HS models usually give accurate predictions in terms of $e_u$ but its predictions are not so good in terms of $e_{\sigma}$. In Section 6.5, the standard elements even give more accurate predictions than those of the advanced elements in terms of both $e_u$ and $e_{\sigma}$.

3) The standard elements suffer from both the dilatational and trapezoidal locking. Thus, the superiority of the advanced elements is most obvious in problems prone to locking.

4) SRI is a simple scheme to improve the standard element. When bending deformation dominates, the present SRI scheme are less effective than EAS and HS. Otherwise, SRI are competitive among the advanced formulations. The bending response of the considered SRI can be further improved by computing the element strain in a local Cartesian coordinate and applying reduced integration to the shear energy defined with respect to the local coordinates.
5) The advanced formulations are more effective for improving the lower order elements. The accuracy improvements in the Q4 elements are more prominent than those in the Q9 elements.

8. Closure
In developing the advanced finite elements, they are often demonstrated to be more accurate than their standard counterparts by popular numerical tests in which displacement and stress predictions at a few selected points are compared. In this paper, a series of the numerical tests are conducted to assess not only the standard elements but also the advanced elements including those formulated by SRI, EAS and hybrid-stress methods in terms of the relative errors in displacement and energy norms. The results reveal that the common perception is not always true. While the advanced elements show significant improvement in accuracy over the standard ones in problems where the bending deformation and/or locking dominate, there are also examples in which the standard elements give better predictions than the advanced counterparts. Thus, more careful assessment may be necessary during developing novel advanced finite element models, especially when practical application scenarios need to be considered.

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