Floating Bodies of Equilibrium
at Density 1/2
in Arbitrary Dimensions

Franz Wegner, Institut für Theoretische Physik
Ruprecht-Karls-Universität Heidelberg
Philosophenweg 19, D-69120 Heidelberg
Email: wegner@tphys.uni-heidelberg.de

Abstract Bodies of density one half (of the fluid in which they are immersed) that can float in all orientations are investigated. It is shown that expansions starting from and deforming the (hyper)sphere are possible in arbitrary dimensions and allow for a large manifold of solutions: One may either (i) expand $r(n) + r(-n)$ in powers of a given difference $r(u) - r(-u)$, ($r(n)$ denoting the distance from the origin in direction $n$). Or (ii) the envelope of the water planes (for fixed body and varying direction of gravitation) may be given. Equivalently $r(n)$ can be expanded in powers of the distance $h(u)$ of the water planes from the origin perpendicular to $u$.

1 Introduction and Results

A long standing problem asked by Stanislaw Ulam in the Scottish Book[1] (problem 19) is, whether a sphere is the only solid of uniform density which will float in water in any position. Such a solid is called a floating body of equilibrium. It will be in indifferent equilibrium in all orientations.

The simpler, two-dimensional, problem to find non circular cross-sections of a long cylindrical log which floats without tending to rotate (the axis of the log is assumed to be parallel to the water surface.) was solved for relative density $\rho_d = 1/2$ in 1938 by Auerbach [2]. He found a large class of solutions. This is in contrast to the solutions found for $\rho_d \neq 1/2$ in two dimensions[3][4][5] and for central symmetric bodies in three dimensions[6], where the variety of shapes is much more restricted, provided one restricts to star-shaped bodies.

Here it will be shown that also in dimensions $d > 2$ there is a large variety of bodies which can float in arbitrary orientation at $\rho_d = 1/2$. More precisely: Denote the distance from the origin to the boundary of the body by $r(n)$, with the unit vector $n$ pointing into the direction. Starting from the hypersphere
\( r(\mathbf{n}) = r_0 \) a deformation governed by an expansion parameter \( \epsilon \) will be considered
\[
r(\mathbf{n}) = r_0 + \sum_{k=1}^{\infty} \epsilon^k r_k(\mathbf{n}).
\] (1)

Due to Archimedes’ principle one half of the body is below the water, the other half above. Thus the water plane cuts the body into two halves. Denote the height of this plane above the origin by \( h(\mathbf{n}) \) with \( \mathbf{n} \) perpendicular to the plane, then \( h(-\mathbf{n}) = -h(\mathbf{n}) \). A similar expansion will be performed for \( h(\mathbf{n}) \),
\[
h(\mathbf{n}) = \sum_{k=1}^{\infty} \epsilon^k h_k(\mathbf{n}).
\] (2)

For practical reasons restriction will be made to solutions which obey
\[
r_k(-\mathbf{n}) = (-)^k r_k(\mathbf{n}) \quad \text{and} \quad h_{2k+1}(-\mathbf{n}) = -h_{2k+1}(\mathbf{n}), \quad h_{2k}(\mathbf{n}) = 0.
\] (3)

In each order in \( \epsilon \) there are two conditions on \( r_k \) and \( h_k \). One condition guarantees that the volumes above and below the water level are equal (V-condition, eq. 71), the other guarantees that the potential energy does not depend on the orientation (Z-condition, eq. 70).

With the restrictions (3, 4) one finds in odd orders \( \epsilon^{2k+1} \) that the Z-condition is identically fulfilled. The V-condition yields that a linear combination of \( r_{2k+1} \) and \( h_{2k+1} \) is determined by \( r_n \) and \( h_n \) with \( n < 2k + 1 \). In even orders \( \epsilon^{2k} \) the V-condition is identically fulfilled. The Z-condition determines \( r_{2k} \).

This may be used in the following ways:

**Given** \( r \). If \( r_1(\mathbf{n}) \) is given, which obeys \( r_1(-\mathbf{n}) = -r_1(\mathbf{n}) \) and \( r_{2k+1} = 0 \) for \( k > 0 \), then the conditions yield in odd order \( h_k \) and in even order \( r_k \). If one chooses also \( r_1 = 0 \), then the \( r_{2k} \) will also vanish (except \( \mathbf{n} \) independent contributions). This is in agreement with the theorem by Schneider[7, 8] and Falconer[9], also referred to by Hensley in the Scottish book[10], compare[6]: For arbitrary dimension \( d \) and density \( 1/2 \), if the body is star-shaped, symmetric, bounded and measurable, then it differs from a ball by a set of measure 0.

**Given** \( h \). If \( h_1(\mathbf{n}) \) is given, which obeys \( h_1(-\mathbf{n}) = -h_1(\mathbf{n}) \) and \( h_k = 0 \) for \( k > 1 \), then the conditions yield \( r_k \) in all orders. Suppose the envelope of the water planes is given. The distance of the planes tangent to this envelope yield \( h(\mathbf{n}) \). Thus to a given envelope one may construct the surface of the floating body. If one rescales \( r \) and \( h \) by a factor \( 1/\epsilon \), then one obtains
\[
r(\mathbf{n}) = \frac{r_0}{\epsilon} + \sum_{k=1}^{\infty} \epsilon^{k-1} r_k(\mathbf{n}), \quad h(\mathbf{n}) = h_1(\mathbf{n}).
\] (5)

Thus for a given envelope one obtains a one-parameter manifold parametrized by \( \epsilon \). These envelopes have the property that there is exactly one tangent hyperplane to them parallel to any given plane. As a consequence these envelopes
have cusps and are quite different from surfaces of convex bodies, which have two tangent hyperplanes parallel to any given plane. A one-parameter manifold for a given envelope was obtained in dimension $d = 2$ by Auerbach [2]; see appendix A.1. From given $h(n)$ one can determine the envelope (appendix A.2).

Basically the expansion scheme of ref. [6] will be used. In subsection 2.1 the two conditions for equilibrium is given. Two expansions are used: the expansion in $\epsilon$, which describes the deviation from the hypersphere (2.2) and the expansion in ultraspherical harmonics (2.3). In subsection 2.4 the expansion of $V$ and $Z$ in ultraspherical harmonics is completed. The calculation in first and second and order in $\epsilon$ is given in subsections 2.5 and 2.6. The general discussion in higher orders is presented in 2.7. The paper is closed by a remark on reparametrization with respect to the expansion in $\epsilon$ (2.8). Although the presented arguments hold to arbitrary order in $\epsilon$, no solution in closed form is presented nor even a proof of convergency for the $\epsilon$-expansion.

## 2 Equilibrium and Expansion

### 2.1 Equilibrium

Due to Archimedes’ principle the body obeys

$$V_a = V_b = \frac{1}{2}V_t,$$

(6)

where $V_a$ is the volume above, $V_b$ the volume below the water plane and $V_t$ the total volume. Condition (6) determines the height of the water plane.

The potential energy $V$ of the body is given by

$$V = \frac{1}{2}mg(z_a - z_b) = \frac{1}{2}\rho gV_t(z_a - z_b) = \rho g(Z_a - Z_b) = 2\rho g(Z_a - \frac{1}{2}Z),$$

(7)

with $m$ and $\rho$ mass and density of the body, $g$ the gravitational acceleration, $z_a$ and $z_b$ the $z$-coordinates of the part of the body above and below the water plane, (the $z$-coordinate measured in direction of $n$, $Z_a$, $Z_b$, and $Z_t$ the integral of $z$ over the volume above, below the water plane and over the total volume.

In order that the body can float freely in all orientations the potential energy has to be independent of the orientation. Thus $Z_a - \frac{1}{2}Z_t$ has to be constant.

### 2.2 Expansion

In the following a unit vector $u$ in $d$ dimensions is decomposed into the component upward and a vector $u'$ in the $d - 1$-dimensional space parallel to the water plane

$$u = \cos \theta n + \sin \theta u'$$

(8)
Then the element \(d\Omega_\theta\) of the solid angle in \(d\) dimensions can be written
\[
d\Omega_u = d\Omega_u' \, d\theta \, (\sin \theta)^{d-2}.
\] (9)
The angle \(\Theta\), at which the water plane intersects the surface of the body as a function of \(u'\) has to be determined from

\[
h = r \cos \Theta.
\] (10)

Expanding
\[
\Theta = \frac{\pi}{2} + \delta \Theta, \quad r = r_0 + \delta r, \quad \delta r(\Theta) = \sum_{j=0}^{\infty} \frac{(\delta \Theta)^j}{j!} \delta r^{(j)}
\] (11)
(the \(\delta r^{(j)}\) are the coefficients of a Taylor expansion of \(\delta r\) in \(\delta \Theta\)) one obtains the equation
\[
h = -(r_0 + \sum_{j=0}^{\infty} \frac{(\delta \Theta)^j}{j!} \delta r^{(j)}) \sin \delta \Theta,
\] (12)
which has to be solved for \(\delta \Theta\) for arbitrary \(u'\), which is not indicated explicitly.

This eq. is invariant against the transformation \(h \rightarrow -h, \delta \Theta \rightarrow -\delta \Theta, \delta r^{(j)} \rightarrow (-)^j \delta r^{(j)}\), thus \(\delta \Theta\) obeys the symmetry relation
\[
\delta \Theta_+ = -\delta \Theta_-, \quad \delta \Theta_+ := \delta \Theta(h, \{\delta r^{(j)}\}), \quad \delta \Theta_- := \delta \Theta(-h, \{(-)^j \delta r^{(j)}\}).
\] (13)

The volume \(V_a\) and \(Z_a\) are obtained by realizing that the volume, which is a hypersegment, can be divided into a hypersector (first integral) and a hypercone (second integral)

\[
V_a = V_a^{(1)} + V_a^{(2)}, \quad Z_a = Z_a^{(1)} + Z_a^{(2)},
\] (17)

\[
V_a^{(1)} = \frac{1}{d} \int d\Omega_u' \int_0^{\Theta(u')} d\theta \, r^d(\theta, u')(\sin \theta)^{d-2},
\] (18)

\[
Z_a^{(1)} = \frac{1}{d+1} \int d\Omega_u' \int_0^{\pi/2} d\theta \, r^{d+1}(\theta, u') \cos \theta(\sin \theta)^{d-2},
\] (19)

\[
V_a^{(2)} = \int d\Omega_u' \, g, \quad Z_a^{(2)} = \int d\Omega_u' \, g^z.
\] (20)
The quantities under the $\Omega$ Expansion in ultraspherical harmonics perpendicular to $n$ Similarly one shows relation so that the relations one observes that the second terms in (21) and (23) obey the same symmetry

\[ I_g(h, \{\delta r^{(j)}\}) = \frac{1}{d} \int_0^{\delta \Theta_+} d\theta' r^d \left( \frac{\pi}{2} + \theta' \right) \cos \delta \Theta_+(u') \cos^d \theta' \cos \theta', \quad (22) \]

\[ g^2(h, \{\delta r^{(j)}\}) = I_g - \frac{h^2}{d} \int_0^{\delta \Theta_+} d\theta' r^d \left( \frac{\pi}{2} + \frac{\delta \Theta_+(u')}{2} \right) \cos \delta \Theta_+(u') \cos^d \theta' \cos^2 \theta' \cos \theta', \quad (23) \]

Insertion of (13) yields for the integral

\[ I_g(h, \{\delta r^{(j)}\}) = -I_g(h, \{(-)^{j} \delta r^{(j)}\}). \quad (27) \]

Similarly one shows

\[ I_g^2(h, \{\delta r^{(j)}\}) = I_g^2(h, \{(-)^{j} \delta r^{(j)}\}). \quad (28) \]

Since

\[ r^d \left( \frac{\pi}{2} + \delta \Theta_+, \{\delta r^{(j)}\} \right) = r^d \left( \frac{\pi}{2} + \delta \Theta_-, \{(-)^{j} \delta r^{(j)}\} \right) = r^d \left( \frac{\pi}{2} + \delta \Theta_+ \right) \left( \{\delta r^{(j)}\} \right) \quad (29) \]

one observes that the second terms in (21) and (23) obey the same symmetry relation so that the relations

\[ g(h, \{\delta r^{(j)}\}) = -g(h, \{(-)^{k} \delta r^{(j)}\}), \quad (30) \]

\[ g^2(h, \{\delta r^{(j)}\}) = g^2(h, \{(-)^{j} \delta r^{(j)}\}) \quad (31) \]

hold. In performing these calculations $h$ depends on $n$, whereas $\delta r$ depends on $u'$. Since $\delta r$ is expanded around $\theta = \pi/2$ the $\delta r^{(j)}$ are to be taken at $u'$ perpendicular to $n$. Thus $g$ and $g^2$ depend on

\[ h = h(n), \quad \delta r^{(j)} = \delta r^{(j)}(u'). \quad (32) \]

### 2.3 Expansion in ultraspherical harmonics

The quantities under the $\Omega_{u'}$ integrals are expanded in ultraspherical harmonics

\[ f(u) = f(\cos \theta n + \sin \theta u') = \sum_{l=0}^{\infty} f_{l}(u), \quad (33) \]
where the ultraspherical harmonics are eigenfunctions of the Laplacian on the unit sphere
\[ \Delta u_{l} f_{l} (u) = -l (l + d - 2) f_{l} (u). \] (34)

To distinguish this expansion from that in \( \epsilon \) the index \( l \) is always preceded by a semicolon. For given \( d \) and \( l \) there are
\[ \#_{d}^{l} = (2l + d - 2) \frac{(l + d - 3)!}{(d - 2)! l!} \] (35)

linearly independent ultraspherical harmonics.

The integral over \( \Omega_{\nu'} \) yields a function depending only on \( \cos \theta \). The integral over \( f_{l} \) yields
\[ \int d\Omega_{\nu'} f_{l} (u) = \Omega_{d-1} f_{l} (n) \frac{C_{l}^{(d/2-1)} (\cos \theta)}{C_{l}^{(d/2-1)} (1)} \] (36)

with the ultraspherical (Gegenbauer) polynomial \( C_{l}^{(d/2-1)} (x) \), which is the only function \( f_{l} \) independent of \( u' \). The factor in front is obtained, since the integral is the factor \( \Omega_{d-1} \) times the average over \( u' \), which for \( \theta \to 0 \) approaches \( f_{l} (n) \).

In appendix A.3 a few formula for these polynomials are listed. See also [10, 11].

In evaluating the integrals for \( V_{a}^{(2)} \) and \( Z_{a}^{(2)} \) (20) one has to integrate over \( u' \) at \( \theta = \frac{\pi}{2} \). Thus
\[ \int d\Omega_{\nu'} f_{l} (u') = \Omega_{d-1} \gamma_{l}^{(d)} f_{l} (n), \] (37)

\[ \gamma_{l}^{(d)} = \frac{C_{l}^{(d/2-1)} (0)}{C_{l}^{(d/2-1)} (1)} = \begin{cases} (-l!)^{\frac{d}{2} \frac{d-1}{2}} \frac{\Gamma (\frac{d}{2} \frac{d-1}{2})}{\sqrt{\pi} \Gamma (\frac{d-2}{2})} & \text{even } l \\ \frac{d}{2} \frac{d-1}{2} & \text{odd } l \end{cases} \] (38)

These quantities are listed in appendix A.4 for \( d = 2 \) and 3.

### 2.4 \( V_{t}, Z_{t}, V_{a}^{(1)}, \text{ and } Z_{a}^{(1)} \)

The total volume is given by
\[ V_{t} = \frac{1}{d} \int d\Omega' \int_{0}^{\pi} d\theta r^{d} (\theta, u') \sin^{d-2} \theta \\
= \frac{1}{d} \Omega_{d-1} \sum_{l} (r^{d})_{l} (n) \int_{0}^{\pi} d\theta C_{l}^{(d/2-1)} (\cos \theta) \sin^{d-2} \theta. \] (39)

The integral over \( \theta \) vanishes for all \( l \) except \( l = 0 \),
\[ V_{t} = \frac{1}{d} \Omega_{d} (r^{d})_{0} (n). \] (40)

There is only one ultraspherical harmonic \( C_{0}^{(d/2-1)} (x) = 1 \) for \( l = 0 \). \( (r^{d})_{0} \) is a scalar and does not depend on \( n \).
The integral over $\theta$ vanishes for all $l$ except $l = 1$,

$$Z_t = \frac{1}{d(d + 1)} \Omega_d (r^{d+1})_1(n).$$

There are $d$ ultraspherical harmonics for $l = 1$. They transform under rotations like the components of a vector. $(r^{d+1})_1(n)$ is the projection of volume times the vector to the centre of gravity onto $n$.

One obtains

$$\frac{1}{\Omega_{d-1}} (V_a^{(1)} - \frac{1}{2} V_1) = \frac{1}{d} \sum_l I^{(v)}_l (r^d)_l(n)$$

with

$$I^{(v)}_l = \frac{(d - 2) C_{l-1}^{(d/2)}(0)}{l(l + d - 2) C_l^{(d/2-1)}(1)} = \begin{cases} 0 & \text{even } l \\ \frac{(-1)^{(l-1)/2} \Gamma(l+1) \Gamma(l+1)}{2^{l+1} \Gamma(l+1/2)} & \text{odd } l \end{cases}$$

and

$$\frac{1}{\Omega_{d-1}} (Z_a^{(1)} - \frac{1}{2} Z_t) = \frac{1}{d + 1} \sum_l I^{(z)}_l (r^{d+1})_l(n)$$

with

$$I^{(z)}_l = \frac{d - 2}{(2l + d - 2) C_l^{(d/2-1)}(1)} \left( \frac{C_l^{(d/2)}(0)}{l + d - 1} + \frac{C_{l-2}^{(d/2)}(0)}{l - 1} \right)$$

$$= \begin{cases} \frac{(-1)^{(l-2)/2} \Gamma(l+1) \Gamma(l+1)}{2^{l+1} \Gamma(l+1/2)} & \text{even } l \\ 0 & \text{odd } l \end{cases}$$

### 2.5 First Order in $\epsilon$

Eq. (12) yields in first order in $\epsilon$

$$\delta \Theta_1 = -\frac{h_1}{r_0}.$$ 

One obtains

$$\frac{1}{\Omega_{d-1}} (Z_{a,1}^{(1)} - \frac{1}{2} Z_{t,1}) = r_0^d \sum_l I^{(z)}_l r_{1,l}(n),$$

$$I_{g^*,1} = 0, \quad g^*_1 = 0, \quad Z_{a,1}^{(2)} = 0.$$
Thus

\[ I_{l}^{(z)} r_{1;l}(n) = 0 \text{ for } l \neq 0 \]  

(50)

has to be fulfilled. Since \( I_{l}^{(z)} \) vanishes for odd \( l \) one can choose \( r_{1;l} \) arbitrarily for odd \( l \), whereas \( r_{1;l} = 0 \) for even \( l \) with the exception of \( l = 0 \). \( r_{1;0} \neq 0 \) would change the radius. For simplicities’ sake \( r_{1;0} = 0 \) is chosen.

Further one obtains

\[ \frac{1}{\Omega_{d-1}} (V_{a,1}^{(z)} - \frac{1}{2} V_{t,1}) = r_{0}^{d-1} \sum_{l} I_{l}^{(e)} r_{1;l}(n), \]  

(51)

\[ I_{g,1} = \frac{h_{1}}{d} r_{0}^{d-1}, \quad g_{1} = -\frac{h_{1}}{d-1} r_{0}^{d-1}, \]  

(52)

\[ \frac{1}{\Omega_{d-1}} V_{a,1}^{(z)} = -\frac{1}{d-1} h_{1;l} r_{0}^{d-1}, \]  

(53)

which yields

\[ h_{l;l} = (d-1) I_{l}^{(e)} r_{1;l}. \]  

(54)

Thus one obtains contributions \( h_{l;l} \) proportional to \( r_{1;l} \) for odd \( l \), whereas \( h_{l;l} \) vanishes for even \( l \), since both \( r_{1;l} \) and \( I_{l}^{(e)} \) vanish for even \( l \).

### 2.6 Second Order in \( \epsilon \)

In second order in \( \epsilon \) one obtains

\[ \delta \Theta_{2} = -\frac{h_{2}}{r_{0}} + \frac{r_{1} h_{1}}{r_{0}^{2}}, \]  

(55)

\[ \frac{1}{\Omega_{d-1}} (Z_{a,2}^{(1)} - \frac{1}{2} Z_{a,2}) = r_{0}^{d-1} \sum_{l} I_{l}^{(z)} (r_{0} r_{2} + \frac{d}{2} r_{1}^{2})_{l}, \]  

(56)

\[ I_{g_{2}} = \frac{h_{2}}{2 (d+1)} h_{1;l}^{2}, \]  

(57)

\[ g_{2}^{z} = -\frac{1}{2 (d-1)} h_{1;l}^{2}, \]  

(58)

\[ \frac{1}{\Omega_{d-1}} Z_{a,2}^{(2)} = -\frac{1}{2 (d-1)} h_{1;l}^{2}. \]  

(59)

This yields the equation

\[ I_{l}^{(z)} (r_{0} r_{2} + \frac{d}{2} r_{1}^{2})_{l} - \frac{1}{2(d-1)} (h_{1;l}^{2})_{l} = 0. \]  

(60)

For odd \( l \) one obtains \( I_{l}^{(z)} = 0 \), \( (r_{1}^{2})_{l} = 0 \), \( (h_{1;l}^{2})_{l} = 0 \). Thus \( r_{2;l} \) can be chosen arbitrarily, but in conformity with (3), the choice \( r_{2;l} = 0 \) is made for odd \( l \). \( r_{2;l} \) is determined for even \( l \) by

\[ r_{2;l} = -\frac{d(r_{1}^{2})_{l}}{2 r_{0}} + \frac{1}{2(d-1) I_{l}^{(z)}} (h_{1;l}^{2})_{l}. \]  

(61)
Thus only terms with even \( \sum_k \) for odd \( k \). Except for linear terms in the polynomials \( g \), use of (3) yields \( \epsilon g \) for \( \sum_k \).

One obtains for \( V \)

\[
\frac{1}{\Omega_{d-1}} (V^{(1)}_{a,2} - \frac{1}{2} V_{1,2}) = r_0^{d-2} \sum_l t_l^{(v)} (r_0 r_2 + \frac{d-1}{2} r_1^2)_l, \tag{62}
\]

\[
I_{g,2} = -\frac{1}{2d} r_0^{d-1} h_2(n) - \frac{2d-1}{2d} r_1(u')h_1(n)r_0^{d-2}, \tag{63}
\]

\[
g_2 = -\frac{d+1}{2d(d-1)} r_0^{d-1} h_2(n) - \frac{2d+1}{2d} r_1(u')h_1(n) \tag{64}
\]

\[
\frac{1}{\Omega_{d-1}} V^{(2)}_{a,2} = \frac{d+1}{2d(d-1)} r_0^{d-1} h_2(n)
- \sum_l 2d + \frac{d+1}{2d} r_0^{d-2} (r_1)_l h_1(n) \tag{65}
\]

\( V^{(1)}_{a,2} - \frac{1}{2} V_{1,2} \) vanishes, since \( (r_0 r_2 + \frac{d-1}{2} r_1^2)_l \) vanishes for odd \( l \) and \( t_l^{(v)} \) for even \( l \). The second term in the expression for \( V^{(2)}_{a,2} \) vanishes, since \( (r_1)_l \) vanishes for even \( l \) and \( C^{(d/2-1)}_l (0) \) for odd \( l \). Thus the condition \( (V^{(1)}_{a,2} + V^{(2)}_{a,2} - \frac{1}{2} V_{1,2})_l = 0 \) reduces to \( h_2(n) = 0 \).

Thus it has been shown that (1) and (2) hold up to second order in \( \epsilon \). Next it will be shown by complete induction that they hold in arbitrary order.

### 2.7 Higher Orders in \( \epsilon \)

\( g_k \) and \( g_k^2 \) are polynomials in \( h_q \) and \( \delta r_p^{(j)} \). They are linear combinations of terms

\[
\prod_m h_{q_m} \prod_m \delta r_p^{(j_m)} \tag{66}
\]

In order \( \epsilon^k \) contribute terms with

\[
k = \sum q_m + \sum p_m. \tag{67}
\]

Except for linear terms in the polynomials \( g_k \) and \( g_k^2 \), which have to be determined, use of (3) yields

\[
\prod_m \delta r_p^{(j_m)} (u') = (-)^{\sum q_m + \sum p_m} \prod_m \delta r_p^{(j_m)} (u'). \tag{68}
\]

Thus only terms with even \( \sum_m j_m + \sum_m p_m \) contribute to \( V^{(2)}_{a,k} \) and \( Z^{(2)}_{a,k} \), eq. (20). From eqs. (60) and (31) one deduces \( \sum q_m + \sum j_m \) is odd for \( g_k \) and even for \( g_k^2 \). Due to (4) only \( h_q \) with odd \( q \) can differ from zero. Thus all non-linear terms (n.l.t.) to \( V^{(2)}_{k} \) vanish for even \( k \) and such contributions to \( Z^{(2)}_{k} \) vanish for odd \( k \).

Next consider the contributions \( V^{(1)}_{a,k} - \frac{1}{2} V_{k} \) and \( Z^{(1)}_{a,k} - \frac{1}{2} Z_{k} \). Eq. (1) yields

\[
(r^d)_k (u) = (-)^k (r^d)_k (u), \quad (r^{d+1})_k (u) = (-)^k (r^{d+1})_k (u). \tag{69}
\]
Since \( f_d(-u) = (-)^{d} f_d(u) \) there are only contributions \( (r^d)_{k;l} \) and \( (r^{d+1})_{k;l} \) with even \( k-l \).

Extracting the terms linear in \( r_k \) and \( h_k \) one obtains

\[
\frac{1}{\Omega_{d-1}}(Z_a - \frac{1}{2} Z_t)_{k} = r^d_0 \sum_{l} I^{(z)}_l r_{k;l} + \text{n.l.t.,} \tag{70}
\]

\[
\frac{1}{\Omega_{d-1}}(V_a - \frac{1}{2} V_t)_{k} = r^{d-1} \sum_{l} (I^{(v)}_l r_{k;l} - \frac{1}{d-1} h_{k;l}) + \text{n.l.t..} \tag{71}
\]

Consider odd \( k \). \( Z_{a,k}^{(2)} \) vanishes, the n.l.t. in \( Z_{a,k}^{(1)} \) vanish, too, since \( I^{(z)}_l = 0 \) for odd \( l \) and all n.l.t. contribute to odd \( l \). One is free to choose \( r_{k;l} \) with odd \( l \). One obtains \( h_{k;l} \) from (71). Since all n.l.t. contribute to odd \( l \), only \( h_{k;l} \) with odd \( l \) can differ from zero.

Consider even \( k \). Then all n.l.t. from \( Z \) contribute to even \( l \). They determine \( r_{k;l} \) with even \( l \). For odd \( l \) the n.l.t. vanish and since \( I^{(z)}_l = 0 \) for odd \( l \) one could choose non-zero \( r_{k;l} \). However, choose \( r_{k;l} = 0 \) for odd \( l \) and even \( k \) in agreement with (3). All n.l.t. to \( V_{a,k} \) vanish. Moreover \( I^{(v)}_l r_{k;l} = 0 \), since the first factor vanishes for even \( l \), the second for odd \( l \). Thus \( h_{k} = 0 \) for even \( k \).

This completes the proof.

### 2.8 Reparametrization

In each order \( \epsilon^k \) it is possible to choose \( r_{k;l} \) for odd \( l \) freely. We restricted ourselves to do this for odd \( k \) only. Basically it is sufficient to do this for \( k = 1 \) only by reparametrizing

\[
r'_{1;l} = \sum_{k=1}^{\infty} \epsilon^{2k} r_{2k+1;l} \tag{72}
\]

and \( r'_{2k+1;l} = 0 \) for \( k > 0 \), since it will yield the same result \( r(n) \).

If one requires that the centre of gravity is located at the origin, then one has to adjust \( r_{k;1} \) in all odd orders \( k \). Then in first order it has to be \( r_{1;1} = 0 \). Thus the restriction (3) does not restrict the manifold of solutions.

It is not necessary to introduce an expansion in \( \epsilon \). Alternatively one expands \( r(u) + r(-u) - 2r_0 \) in powers of \( r(u) - r(-u) \).

### A Appendices

#### A.1 Auerbach’s Solution

Basically Auerbach\[2\] starts from the envelope of the water line. The length \( 2\ell \) of the water line is constant. The line touches the envelope in the middle. Thus the envelope may be represented by

\[
x_e(\phi) = x_0 + \int_0^{x} d\phi' s(\phi') \cos(\phi'), \quad y_e(\phi) = y_0 + \int_0^{x} d\phi' s(\phi') \sin(\phi'). \tag{73}
\]
The water line and thus the point, at which the envelope touches the water line has to be the same after $\phi$ has increased by $\pi$. Thus
\[ x_e(\pi) = x_e(0), \quad y_e(\pi) = y_e(0), \quad s(\pi + \phi) = -s(\phi) \tag{74} \]
is required, which yields
\[ x_e(\phi + \pi) = x_e(\phi), \quad y_e(\phi + \pi) = y_e(\phi). \tag{75} \]
Then the boundary of the body is given by
\[ x(\phi) = x_e(\phi) + \ell \cos \phi, \quad y(\phi) = y_e(\phi) + \ell \sin \phi. \tag{76} \]
The scheme presented here is less elegant than the procedure used by Auerbach, however the present scheme applies to general dimensions. It would be desirable to have an elegant procedure which does not refer to a perturbation expansion for the construction of such bodies in $d > 2$ dimensions.

A.2 Envelope from $h(n)$
For given $h(n)$ one can determine the corresponding point $r_e(n)$ on the envelope. It obeys
\[ (n + \delta n) \cdot r_e(n) = h(n + \delta n) \tag{77} \]
to first order in $\delta n$, which is orthogonal to $n$. Expanding both sides
\[ n \cdot r_e + \delta n \cdot r_e = h(n) + \delta n \cdot \nabla h(n) \tag{78} \]
yields
\[ r_e(n) = nh(n) + \nabla h(n). \tag{79} \]

A.3 Ultraspherical (Gegenbauer) Polynomials
A few relations on ultraspherical (Gegenbauer) polynomials are given. Rodrigues’ formula
\[ C_n^\alpha(x) = (-)^n \frac{\Gamma(n+1/2)\Gamma(n+2\alpha)}{2^n n! \Gamma(2\alpha)\Gamma(n+\alpha+1/2)} (1-x^2)^{-\alpha+1/2} \frac{d^n}{dx^n} (1-x^2)^{n+\alpha-1/2} \tag{80} \]
Recurrence relation
\[ 2(n+\alpha)x C_n^\alpha(x) = (n+1) C_{n+1}^\alpha(x) + (n+2\alpha-1) C_{n-1}^\alpha(x). \tag{81} \]
Special values
\[ C_n^\alpha(\pm 1) = (\pm 1)^n \binom{n+2\alpha-1}{n}, \quad C_{2m}^\alpha(0) = (-)^m \binom{m+\alpha-1}{m}, \quad C_{2m+1}^\alpha(0) = 0. \tag{82} \]
Integrals

\[ \int d\theta (\sin \theta)^{d-2} C_{l}^{(d/2-1)}(\cos \theta) d\theta \]
\[ = \frac{d - 2}{l(l + d - 2)} (\sin \theta)^{d-1} C_{l-1}^{(d/2)}(\cos \theta), \quad l > 0 \] (84)
\[ \int d\theta \cos \theta (\sin \theta)^{d-2} C_{l}^{(d/2-1)}(\cos \theta) d\theta \]
\[ = \begin{cases} \frac{d - 2}{2 + d - 2} (\sin \theta)^{d-1} \left( \frac{1}{l + d - 1} C_{l}^{(d/2)}(\cos \theta) + \frac{1}{l - 1} C_{l-2}^{(d/2)}(\cos \theta) \right), & l > 1 \\ \frac{1}{l - 1} (\sin \theta)^{d-1}, & l = 0 \end{cases} \] (85)

The full solid angle is obtained from

\[ \Omega_d = \Omega_{d-1} \int_0^\pi d\theta \sin^{d-2} \theta = \Omega_{d-1} \frac{\sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \] (86)

with \( \Omega_1 = 2 \) for the two directions in one dimension, which yields

\[ \Omega_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \] (87)

The duplication formula for the \( \Gamma \)-function reads

\[ \Gamma(2z) = \frac{1}{\sqrt{\pi}} 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right). \] (88)

A.4 Dimensions two and three

A.4.1 The limit \( d \to 2 \)

The normalization of \( C_{l}^{(d/2-1)} \), (80) is not convenient for the limit \( d \to 2 \), since \( C \) approaches 0 in this limit for \( l \neq 0 \). One obtains the limit

\[ \lim_{d \to 2} \frac{C_{l}^{(d/2-1)}(\cos \theta)}{C_{l}^{(d/2-1)}(1)} = T_l(\cos \theta) = \cos(l \theta), \] (89)

where \( T_l \) denotes the Chebyshev polynomials. That this normalization is not convenient, can also be seen from the factors \( (d - 2) \) in (44, 46). However, one finds in this limit

\[
\begin{array}{c|cc}
\gamma_l^{(2)} & \text{even } l & \text{odd } l \\
I_l^{(v)} & (-1)^l & 0 \\
I_l^{(z)} & 0 & (-1)^{(l-1)/2} \\
\end{array}
\] (90)
A.4.2  Three Dimensions

In three dimensions one obtains

\[ C_l^{1/2}(\cos \theta) = P_l(\cos \theta) \]  \hspace{1cm} (91)

with the Legendre polynomials \( P_l \). One obtains

| \( \gamma_l^{(3)} \) | even \( l \) | odd \( l \) |
|----------------|----------------|----------------|
| \( f_{l}^{(w)} \) | \((-)^{l} \frac{l}{2((\frac{l}{2})!)^2} \) | 0 |
| \( f_{l}^{(z)} \) | \((-)^{(l-2)/2} \frac{l}{2((l-1)(l+2)(\frac{l}{2})!)^2} \) | \((-)^{(l-1)/2} \frac{(l-1)!}{2((\frac{l}{2})!)^2(l-1)!} \) |

References

[1] R.D. Mauldin (ed.), The Scottish Book, Birkhäuser Boston 1981
[2] H. Auerbach, Sur un probleme de M. Ulam concernant l’équilibre des corps flottant, Studia Math. 7 (1938) 121-142
[3] F. Wegner, Floating Bodies of Equilibrium Studies of Applied mathematics 111 (2003) 167-183
[4] F. Wegner, Floating Bodies of Equilibrium. Explicit Solution, e-print archive physics/0603160
[5] F. Wegner, Floating Bodies of Equilibrium in 2D, the Tire Track Problem and Electrons in an Inhomogeneous Magnetic Field, e-print archive physics/0701241
[6] F. Wegner, Floating Bodies of Equilibrium in Three Dimensions. The central symmetric case, e-print archive physics/0803.1043
[7] R. Schneider, Über eine Integralgleichung in der Theorie konvexer Körper, Math. Nachr. 44 (1970) 55-75
[8] R. Schneider, Functional Equations Connected with Rotations and Their Geometrical Applications, L’Enseignement Math. 16 (1970) 297-305
[9] K.J. Falconer, Applications of a Result on Spherical Integration to the Theory of Convex Sets, Amer. Math. Monthly 90 (1983) 690-693
[10] M. Abramowitz, I.A. Stegun (eds.), Handbook of Mathematical Functions, chapter 22, Dover
[11] A. Erdelyi (ed.), Bateman Manuscript Project: Higher Transcendental Functions, volume I, section 3.15, McGraw-Hill