OPTIMAL STABLE ORNSTEIN-UHLENBECK REGRESSION

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Abstract. We prove some efficient inference results concerning an Ornstein-Uhlenbeck regression model driven by a non-Gaussian stable Lévy process, where the output process is observed at high-frequency over a fixed time period. Local asymptotics for the likelihood function is presented, followed by a way to construct an asymptotically efficient estimator through a suboptimal yet simple preliminary estimator, which enables us to bypass not only numerical optimization of the likelihood function, but also the multiple-root problem.

1. Introduction

The Ornstein-Uhlenbeck (OU) models has been used in a wide variety of applications, such as electric consumption modeling [15], [1], [20], and [21], ecology [9], and protein dynamics modeling [3], to mention a few. In this paper, we consider the following OU regression model

\[ Y_t = Y_0 + \int_0^t (\mu \cdot X_s - \lambda Y_s) \, ds + \sigma J_t, \quad t \in [0, T], \]

where \( J \) is the symmetric \( \beta \)-stable (càdlàg) Lévy process characterized by

\[ \mathbb{E}(e^{i u J_t}) = \exp(-t|u|^\beta), \quad t \geq 0, \ u \in \mathbb{R}, \]

where \( X = (X_t)_{t \in [0,T]} \) is an \( \mathbb{R}^q \)-valued non-random càdlàg function, and where the driving noise \( J \) is stochastically independent of the initial variable \( Y_0 \). Throughout, the terminal sampling time \( T > 0 \) is fixed (see, however, Remark 2.3). Let

\[ \theta := (\lambda, \mu, \beta, \sigma) \in \Theta, \]

where \( \Theta \subset \mathbb{R}^p \ (p := q + 3) \) is a bounded convex domain such that its closure \( \overline{\Theta} \subset \mathbb{R} \times \mathbb{R}^q \times (0,2) \times (0,\infty) \). The objective of this paper is to study asymptotically efficient estimation of the parameter \( \theta \), assuming that the model is correctly specified. We denote by \( (\mathbb{P}_\theta)_{\theta \in \Theta} \) a family of induced image measures of \((J,Y)\) on some probability space, and assume that there exists a true value \( \theta_0 \in \Theta \) inducing the distribution \( \mathcal{L}(J,Y) \). In the likelihood asymptotics, we assume that available data is \( (X_t, Y_t)_{t \in [0,T]} \) and \( (Y_t^j)_{j=1}^n \), where \( t_j^j := jh \) with \( h := h_n := T/n \). The model (1.1) thus described can be seen as a continuous-time counterpart of the simple first-order autoregressive exogenous (ARX) model. Nevertheless, any proper form of the efficient estimation result has been missing in the literature, due to the lack of background theory which can deal with estimation of all the parameters involved under the bounded-domain infill asymptotics.

Let us recall that, when \( J \) is a Wiener process (\( \beta = 2 \)), the drift parameters are consistently estimable only when the terminal sampling time tends to infinity, and the associated statistical experiments are known to possess essentially different properties according to the sign of \( \lambda \). That is to say, the model is:

- locally asymptotically normal for \( \lambda > 0 \) (ergodic case);
- locally asymptotically Brownian functional for \( \lambda = 0 \) (unit-root case);
- locally asymptotically mixed-normal (LAMN) for \( \lambda < 0 \) (non-ergodic (explosive) case).

In the present stable-noise setup, we will show that the model is uniformly locally asymptotically mixed normal (LAMN), and also that the likelihood equation has a root which is asymptotically efficient in the classical sense. Besides, we will provide a way to provide an asymptotically efficient estimator through a suboptimal yet very simple preliminary estimator, which enables us to bypass computationally demanding numerical optimization of the likelihood function involving the \( \beta \)-stable density.

We will show that, unlike the case of ARX time-series models and the Gaussian OU models, it does not matter that if the model is ergodic or not. The asymptotic results presented here are uniformly valid in a single manner over any compact subset of the parameter space \( \Theta \). In particular, the sign of the
autoregressive parameter \( \lambda_0 \) is no longer important, which in turn implies that the conventional unit-root problem (see [16] and the references therein) is not relevant here at all.

Here is a summary of some basic notation. To investigate asymptotic behavior of \( \ell_n(\theta) \) under \( P_{\theta} \) uniformly in \( \theta \), we introduce further notation. We denote by \( \to_u \) the uniform convergence of non-random quantities with respect to \( \theta \) over \( \Theta \), and use the order symbols \( a_n(\cdot) \) and \( O_u(\cdot) \) for \( n \to \infty \) (hence for \( h \to 0 \)) for sequences being \( o(\cdot) \) and \( O(\cdot) \) uniformly in \( \theta \) over \( \Theta \). We write \( a_n \lesssim b_n \) when there exists a universal constant \( C \) such that \( a_n \leq C b_n \) for every \( n \) large enough. For positive functions \( a_n(\theta) \) and \( b_n(\theta) \), we denote \( a_n(\theta) \lesssim_u b_n(\theta) \) if \( \sup_{\theta} a_n(\theta)/b_n(\theta) \leq 1 \); here and in what follows, the supremum sign \( \sup_{\theta} \) is always taken over the compact set \( \Theta \). We write \( A_n(\theta) \sim_u B_n(\theta) \) if \( \sup_{\theta} |A_n(\theta)/B_n(\theta) - 1| \lesssim_u 0 \).

For a matrix \( A \) we write \( A^{\otimes 2} = AA^\top \), with \( \top \) denoting the transposition. We will simply write \( \int_{j}^{j+1} \) instead of \( \int_{j-1}^{j} \). Given continuous random functions \( \zeta_0(\theta) \) and \( \zeta_n(\theta), n \geq 1 \), we write: \( \zeta_n(\theta) \lesssim_u \zeta_0(\theta) \) if \( \sup_{\theta} [P_{\theta} \zeta_n(\theta)/P_{\theta} \zeta_0(\theta)] \to_u 0 \) for each bounded continuous function \( f \), where \( P^\beta \) denotes the distribution of \( \zeta \) under \( P_{\theta} \); also, \( \zeta_n(\theta) \lesssim_{P_{\theta}} \zeta_0(\theta) \) if the joint distribution of \( \zeta_n(\theta) \) and \( \zeta_0(\theta) \) is well-defined under \( P_{\theta} \) and if \( \sup_{\theta} \{ |\zeta_n(\theta) - \zeta_0(\theta)| > \epsilon \} \to_u 0 \) for every \( \epsilon > 0 \) as \( n \to \infty \). Additionally, for a sequence \( a_n > 0 \) we write \( \zeta_n(\theta) = O_{\theta}(a_n) \) if \( \sup_n \zeta_n(\theta) \lesssim_{P_{\theta}} a_n \) if \( \inf_n \zeta_n(\theta) > 0 \) and \( \zeta_n(\theta) = o_{\theta}(b_n) \) if \( \sup_n \zeta_n(\theta) \lesssim_{P_{\theta}} b_n \) for every \( \epsilon > 0 \) there exists a constant \( K > 0 \) for which \( \sup_n P_{\theta} [\zeta_n(\theta) > K] < \epsilon \). Similarly, for any random functions \( \chi_n(\theta) \) doubly indexed by \( n \) and \( j \leq n \), we write \( \chi_n(j)(\theta) = O_{\theta}(a_n) \) if \( \sup_n \chi_n(j)(\theta) \lesssim_{P_{\theta}} a_n \).

for any \( K > 0 \); likewise, we will use the symbol \( O^*_\theta(a_n) \) for a non-random quantities \( \chi_n(\theta) \) such that \( \sup_n \chi_n(j)(\theta) \lesssim_{P_{\theta}} a_n \).

2. Likelihood asymptotics

Denote by \( \phi_{\beta} \) the density of the distribution \( \mathcal{L}(J_1): \ P_{\theta}(J_1 \in dy) = \phi_{\beta}(y) dy \) for each \( \theta \). It is known that \( \phi_{\beta}(y) > 0 \) for each \( y \in \mathbb{R} \), that \( \phi_{\beta} \) is smooth in \( (y, \beta) \in \mathbb{R} \times (0, 2) \), and from [5] that

\[
\sup_{y \in \mathbb{R}} \frac{|y|^{\beta+1+k}}{\log (1+|y|)} |\partial^k \partial^\beta \phi_{\beta}(y)| < \infty, \quad k, l \in \mathbb{Z}_+.
\]

Here we wrote \( \partial^k \partial^\beta \phi_{\beta}(y) := (\partial^k/\partial y^k)(\partial^\beta/\partial \beta^l) \phi_{\beta}(y) \); analogous notation for the partial derivatives will be used in the sequel. Integrating by parts applied to \( t \mapsto e^{\lambda t} Y_t \) provides us with the explicit càdlàg solution process: under \( P_{\theta} \),

\[
Y_t = e^{-\lambda(t-s)} Y_s + \mu \cdot \int_s^t e^{-\lambda(t-s)} X_s ds + \sigma \int_s^t e^{-\lambda(t-s)} dJ_s, \quad t > s.
\]

For \( x, \lambda \in \mathbb{R} \), we write

\[
\eta(x) = \frac{1}{x}(1 - e^{-x}),
\]

\[
\zeta_j(\lambda) = \int_0^\lambda e^{-\lambda(t-s)} X_s ds.
\]

By the property of the Lévy integral, we have

\[
\log E_{\theta} \left\{ \exp \left( iu \sigma \int_{t_{j-1}}^{t_j} e^{-\lambda(t_j-s)} dJ_s \right) \right\} = -|\sigma u|^\beta \int_{t_{j-1}}^{t_j} e^{-\lambda s} dJ_s.
\]

Hence

\[
\epsilon_j(\lambda) := \frac{Y_{t_j} - e^{-\lambda t_j} Y_{t_{j-1}} - \mu \cdot \zeta_j(\lambda)}{\sigma h^{1/\beta} \eta(\lambda \beta h)^{1/\beta}} \lesssim \text{i.i.d.} \ \mathcal{L}(J_1).
\]

Then, the exact log-likelihood function \( \ell_n(\theta) = \ell_n \begin{pmatrix} \theta; (X_t)_{t \in [0, T]}, (Y_t)_{j=0}^n \end{pmatrix} \) is given by

\[
\ell_n(\theta) = \sum_{j=1}^n \log \left( \frac{1}{\sigma h^{1/\beta} \eta(\lambda \beta h)^{1/\beta}} \phi_{\beta}(\epsilon_j(\theta)) \right)
\]
We introduce the non-random \((p + 3) \times (p + 3)\)-matrix
\[
\varphi_n = \varphi_n(\theta) := \text{diag}\left\{ \frac{1}{\sqrt{n}}h^{1-\beta}, I_{1+p}, \frac{1}{\sqrt{n}} \left( \varphi_{11,n}(\theta), \varphi_{12,n}(\theta) \right) \right\},
\]
where the real entries \(\varphi_{kl,n} = \varphi_{kl,n}(\theta)\) are assumed to be continuous in \(\theta\) and to satisfy the following conditions for some finite values \(\varphi_{kl} = \varphi_{kl}(\theta)\):
\[
\begin{align*}
\varphi_{11,n}(\theta) &\to u \varphi_{11}(\theta), \\
\varphi_{12,n}(\theta) &\to u \varphi_{12}(\theta), \\
\varphi_{21,n}(\theta) &:= \beta^{-2} \log(1/n) \varphi_{11,n}(\theta) + \sigma^{-1} \varphi_{21,n}(\theta) - u \varphi_{21}(\theta), \\
\varphi_{22,n}(\theta) &:= \beta^{-2} \log(1/n) \varphi_{12,n}(\theta) + \sigma^{-1} \varphi_{22,n}(\theta) - u \varphi_{22}(\theta), \\
\inf_{\theta} |\varphi_{11}(\theta)| &> 0, \\
\max_{(k,l)} |\partial_{(\beta,\sigma)} \varphi_{kl,n}(\theta)| &\lesssim u \log^2(1/h).
\end{align*}
\]
The matrix \(\varphi_n(\theta)\) will turn out to be the right norming with which \(u \to \ell_n(\theta + \varphi_n(\theta)u) - \ell_n(\theta)\) under \(P_\theta\) has an asymptotically quadratic structure in \(\theta\); see \([3,\text{page 292}]\) for previous related studies. Note that \(\sqrt{n}h^{1-\beta} \to u \infty\) and \(\|\varphi_{21,n}(\theta)\| \lor \|\varphi_{22,n}(\theta)\| \lesssim \log(1/h)\). By the same reasoning as in \([2,\text{page 292}]\), we also have \(\inf_{\theta} |\varphi_{21,n}(\theta)|, |\varphi_{22,n}(\theta)| \gtrsim 1\) and \(|\varphi_n(\theta)| \to u 0\).

Let
\[
f_\beta(y) := \frac{\partial \phi_\beta}{\phi_\beta}(y), \quad g_\beta(y) := \frac{\partial \phi_\beta}{\phi_\beta}(y).
\]
We define the block-diagonal random matrix
\[
\mathcal{I}(\theta) = \text{diag}\{\mathcal{I}_{\lambda,\mu}(\theta), \mathcal{I}_{\beta,\sigma}(\theta)\},
\]
where, for a random variable \(\epsilon\) such that \(\mathcal{L}(\epsilon) = \phi(\beta)(y)\)
\[
\begin{align*}
\mathcal{I}_{\lambda,\mu}(\theta) &:= \frac{1}{\sigma^2} \mathbb{E}_{\theta}\{g_\beta(\epsilon)^2\} \frac{1}{T} \int_0^T \left( \begin{array}{cc}
Y_t^2 - Y_t X_t \otimes 2 \\
X_t^2
\end{array} \right) dt, \\
\mathcal{I}_{\beta,\sigma}(\theta) &:= \left( \begin{array}{cc}
\varphi_{11} & \varphi_{12} \\
\varphi_{21} & \varphi_{22}
\end{array} \right) \begin{pmatrix}
\mathbb{E}_\theta\{f_\beta(\epsilon)^2\} \\
\mathbb{E}_\theta\{f_\beta(\epsilon)g_\beta(\epsilon)\} \\
\mathbb{E}_\theta\{(1 + \epsilon g_\beta(\epsilon))^2\}
\end{pmatrix}
\begin{pmatrix}
\varphi_{11} & \varphi_{12} \\
\varphi_{21} & \varphi_{22}
\end{pmatrix}.
\end{align*}
\]
Note that \(\mathcal{I}(\theta)\) depends on the choice of \(\varphi(\theta) = \{\varphi_{kl}(\theta)\}\); if \(\varphi(\theta)\) is free from \((\lambda, \mu)\), then so is \(\mathcal{I}(\theta)\). In what follows, we assume that
\[
\int_0^T X_t^2 dt > 0 \quad (\text{positive definite}),
\]
Then we have
\[
\mathcal{I}(\theta) > 0, \quad P_\theta\text{-a.s.}
\]
Indeed, \(\mathcal{I}_{\beta,\sigma}(\theta) > 0 \text{ a.s.}\) was verified in \([2,\text{Theorem 1}]\). To deduce that \(\mathcal{I}_{\lambda,\mu}(\theta) > 0 \text{ a.s.}\), we note that
\[
\int_0^T Y_t^2 dt > 0 \text{ a.s.}
\]
and that
\[
\begin{align*}
u^T \left( \int_0^T X_t^2 dt - \left( \int_0^T Y_t X_t dt \right) \left( \int_0^T Y_t^2 dt \right)^{-1} \left( \int_0^T Y_t X_t dt \right)^T \right) u \\
= \int_0^T (u \cdot X_t)^2 dt - \left( \int_0^T Y_t^2 dt \right)^{-1} \left( \int_0^T Y_t (u \cdot X_t) dt \right)^2 > 0
\end{align*}
\]
for every nonzero \(u \in \mathbb{R}^q\), by means of Schwarz’s inequality; for any constant real \(\xi\), the identity \(Y = (u \cdot X)\xi\) on \([0, T]\) does not hold with positive probability. Now apply the identity \(\det(A) \det(C - BA^{-1}BT) = \det(\det(C - BA^{-1}BT)\) to conclude \([2,\text{Theorem 1}]\).

The normalized score function \(\Delta_n(\theta_0)\) and the normalized observed information matrix \(I_n(\theta_0)\) are given by
\[
\Delta_n(\theta) := \varphi_n(\theta)^T \partial_\theta \ell_n(\theta),
\]
\[ I_n(\theta) := -\varphi_n(\theta)^T \frac{1}{\theta} \ell_n(\theta) \varphi_n(\theta), \]
respectively. Let \( MN_{p,\theta}(0, I(\theta)^{-1}) \) denote the covariance mixture of \( p \)-dimensional normal distribution, corresponding to the characteristic function \( u \mapsto \mathbb{E}_\theta \{ \exp(-u^T I(\theta)^{-1} u/2) \} \). Here is the main claim of this section.

**Theorem 2.1.** Under the aforementioned setup, the following statements hold.

1. The uniform local asymptotically mixed normal (LAMN) property holds: for any bounded sequence \( (v_n) \subset \mathbb{R}^p \), it holds that

\[ \ell_n(\theta + \varphi_n(\theta) v_n) - \ell_n(\theta) - \left( \Delta_n(\theta)[v_n] - \frac{1}{2} I_n(\theta)[v_n, v_n] \right) \overset{P}{\to} 0, \]

where \( (\Delta_n(\theta), I_n(\theta)) \overset{L}{\to} (I(\theta)^{1/2} Z, I(\theta)) \) with \( Z \sim N_p(0, I) \) being independent of \( (Y_0, J) \), defined on an extended probability space on which \( Z \) is defined.

2. There exists a local maximum point \( \hat{\theta}_n \) of \( \ell_n \) with probability tending to 1 such that

\[ \varphi_n(\theta)^{-1} (\hat{\theta}_n - \theta) \overset{P}{\to} MN_{p,\theta}(0, I(\theta)^{-1}). \]

The terminology “uniform” in the claim (1) above refers not to the local parameter (here \( v = v_n \)) but to the uniformity in the parameter \( \theta \) over \( \Theta \). We simply call the good maximum point \( \hat{\theta}_n \) in Theorem 2.1 an asymptotically efficient maximum likelihood estimator (MLE). As in [2], the particular non-diagonal form of \( A_n(\theta) \) is inevitable in the maximum likelihood estimation.

The LAMN property ensures the asymptotic optimality property of the asymptotically efficient MLE [8, Theorem 8]. From a similar point of view, under the assumptions of Theorem 2.1, the result of [15] ensures the following uniform asymptotic maximum concentration: for any estimator \( \hat{\theta}_n \) such that \( \varphi_n(\theta)^{-1} (\hat{\theta}_n - \theta) \overset{P}{\to} F_\theta \) for some distribution \( F_\theta \) and for any convex symmetric Borel set \( A \subset \mathbb{R}^{p+3} \), we have

\[ F_\theta(A) \leq \lim_{n \to \infty} \mathbb{P}_\theta \left( \varphi_n(\theta)^{-1} (\hat{\theta}_n - \theta) \in A \right); \]

if further \( \mathbb{E}_\theta \{ I(\theta)^{-1} \} \) exists and is finite, then \( \int x^{2}\mathbb{P}_\theta(dx) - \mathbb{E}_\theta \{ I(\theta)^{-1} \} \) is nonnegative definite.

**Remark 2.2.** The present framework allows us to do unit-period wise (for example, day-by-day) inference for both trend and scale structures, providing a sequence of period-wise estimates with theoretically valid approximate confidence sets. This would suggest an aspect of change-point analysis in high-frequency data: if we have high-frequency sample over \( [k-1, k] \) for \( k = 1, \ldots, T \), then we can construct a sequence of estimators \( \{ \hat{\theta}_n(k) \}_{k=1}^T \); then it would be possible in some way to reject the constancy of \( \theta \) over \( [0, T] \) if \( k \to \hat{\theta}_n(k) \) \( (k = 1, \ldots, T) \) is not likely to stay unchanged.

**Remark 2.3.** Here are some comments on the model time scale.

1. We are fixing the terminal sampling time \( T \), so that \( \sqrt{nh^{1-1/\beta}} = h^{1/\beta-1/2} T^{1-1/\beta} = O(h^{1/\beta-1/2}). \)
   If \( \beta > 1 \) (resp. \( \beta < 1 \)), then a longer period would lead to a better (resp. worse) performance of estimating \( (\lambda, \mu) \). The Cauchy case \( \beta = 1 \), where the two rates of convergence coincide, is exceptional.

2. We can explicitly associate change of \( T \) with those of the components of \( \theta \). Specifically, changing the model time scale from \( t \) to \( t/T \) in \( \mathbb{1} \), we see that the process

\[ Y^T = (Y^T_t)_{t \in [0,1]} := (Y_{tT})_{t \in [0,1]} \]

satisfies exactly the same integral equation as in \( \mathbb{1} \) except that \( \theta = (\lambda, \mu, \beta, \sigma) \) is replaced by

\[ \theta_T = (\lambda_T, \mu_T, \beta_T, \sigma_T) := (T\lambda, T\mu, \beta, T^{1/\beta}\sigma) \]

(\( \beta \) is unchanged), \( X_t \) by \( X^T_t := X_{tT} \), and \( J_t \) by \( J^T_t := T^{-1/\beta} J_{tT} \):

\[ Y^T_t = Y^0_T + \int_0^t (\mu_T \cdot X^T_s - \lambda_T Y^T_s) ds + \sigma_T J^T_s, \quad t \in [0,1]. \]

Note that \( (J^T_t)_{t \in [0,1]} \) defines the standard \( \beta \)-stable Lévy process. This indeed shows that we may set \( T \equiv 1 \) in the virtual (model) world without loss of generality. Of course, this is impossible for diffusion type (Wiener driven) models where we cannot consistently estimate the drift coefficient unless we let the terminal sampling time \( T \) tend to infinity.
Proof of Theorem 2.7. The proof is analogous to that of [2] Theorem 1, the argument of which essentially makes use of [17]. Although the Fisher information matrix \( \mathcal{I}(\theta) \) is random, we do not need to take care about the stable convergence in law for the normalized score function, which is quite often crucial when concerned with a high-frequency sampling for a process with dependent increments.

We have \( \sup_{t \in [0,T]} |X_t| < \infty \) since \( X : [0,T] \to \mathbb{R}^q \) is assumed to be càdlàg. By means of the localization procedure, we may and do suppose that the driving stable Lévy process does not have jumps of size greater than some fixed threshold (see [14] Section 6.1 for a concise account); in particular, we may suppose that

\[
\sup_{\theta \in \Theta} \mathbb{E}_\theta (|J_t|^K) < \infty
\]

for any \( K > 0 \) and \( t \in [0,T] \). Also to be noted is that, due to the symmetry of \( \mathcal{L}(J_1) \), the removal of large jumps does not change the parametric form of the drift coefficient.

Let

\[ \mathfrak{M}_n(c;\theta) := \{ \theta' \in \Theta : |\varphi_n(\theta)^{-1}(\theta' - \theta)| \leq c \}. \]

We will complete the proof of Theorem 2.1 by verifying the three statements corresponding to the conditions (12), (13), and (14) in [2], which here read

\[
\mathcal{I}_n(\theta) \xrightarrow{p} \mathcal{I}(\theta),
\]

\[
\sup_{\nu' \in \mathfrak{M}_n(c;\theta)} |\varphi_n(\theta)^{-1} \varphi_n(\theta) - I_n| \to_u 0,
\]

\[
\sup_{\theta' \in \mathfrak{M}_n(c;\theta)} |\varphi_n(\theta)^T \{ \partial^2_\theta \ell_n(\theta^1, \ldots, \theta^p) - \partial^2_\theta \ell_n(\theta) \} \varphi_n(\theta)| \to_n 0,
\]

respectively, where (2.10) and (2.11) should hold for all \( c > 0 \) and where \( \partial^2_\theta \ell_n(\theta^1, \ldots, \theta^p) \), \( \theta^p \in \Theta \), denotes the \( p \times p \) Hessian matrix, whose \((k,l)\)th element is given by \( \partial_{\theta_k} \partial_{\theta_l} \ell_n(\theta^k) \); we refer to [13] Section 2.3.1 for a little bit more details. We can verify (2.10) exactly as in [2], so will look at (2.9) and (2.11) below in this order.

For (2.9), we first recall the expression (2.2). For looking at the entries of \( \partial^2_\theta \ell_n(\theta) \), we introduce several shorthands. Let us omit the subscript \( \beta \) and the argument \( \epsilon_j \) of the aforementioned notation, such as \( g := g_\beta(\epsilon_j) \) and so on. For brevity we also write

\[ l'_n = \log(1/h), \quad c = \eta^{1/\beta}(\lambda \beta h), \quad \epsilon = \epsilon_j(\theta), \]

so that (2.2) becomes

\[ \ell_n(\theta) = \sum_{j=1}^n \left( -\log \sigma + \frac{1}{\beta} l'_n + \log c + \log \phi(\epsilon) \right). \]

Further, the partial differentiation with respect to a variable will be denoted by the braced subscript such as \( \epsilon_{(a)} := \partial_a \epsilon_j(\theta) \) and \( \epsilon_{(a,b)} := \partial_a \partial_b \epsilon_j(\theta) \). Then, direct computations give the first-order partial derivatives

\[
\partial_\mu \ell_n(\theta) = \sum_{j=1}^n \left( (\log c)_{(\mu)} + g \epsilon_{(\lambda)} \right),
\]

\[
\partial_\lambda \ell_n(\theta) = \sum_{j=1}^n g \epsilon_{(\mu)},
\]

\[
\partial_\beta \ell_n(\theta) = \sum_{j=1}^n \left( -\beta^{-2} l'_n + (\log c)_{(\beta)} + g \epsilon_{(\beta)} + f \right),
\]

\[
\partial_\sigma \ell_n(\theta) = \sum_{j=1}^n \left( -\sigma^{-1} + g \epsilon_{(\sigma)} \right),
\]

followed by the second-order ones:

\[
\partial^2_\mu \ell_n(\theta) = \sum_{j=1}^n \left\{ (\partial g) (\epsilon_{(\mu)})^2 + g \epsilon_{(\lambda,\lambda)} + (\log c)_{(\lambda,\lambda)} \right\},
\]

\[
\partial^2_\mu \ell_n(\theta) = \sum_{j=1}^n (\partial g) (\epsilon_{(\mu)})^2,
\]
\[ \frac{\partial^2 \ell_n(\theta)}{\partial \beta_n(\theta)} = \sum_{j=1}^{n} \{ g(\beta_j) \epsilon(\beta_j) + g(\beta_j) \epsilon(\beta_j) + (\partial g) \epsilon(\beta_j)^2 + f(\beta_j) + (\partial f) \epsilon(\beta_j) \}, \]

\[ \frac{\partial^2 \ell_n(\theta)}{\partial \sigma_n(\theta)} = \sum_{j=1}^{n} \{ \sigma^{-2} + (\partial g) \epsilon(\sigma_j)^2 + g(\epsilon(\sigma_j)) \}, \]

\[ \frac{\partial \ell_n(\theta)}{\partial \beta_n(\theta)} \] and \[ \frac{\partial \ell_n(\theta)}{\partial \sigma_n(\theta)} \] will show the details here, but for later reference mention a few of the points:

- The details here, but for later reference mention a few of the points:

- For any continuous \( \epsilon(x, y; \theta) \) and for \( U(\epsilon(\theta)) \) such that \( \mathbb{E} \{ U(\epsilon(\theta)) \} = 0 \) \( \theta \in \Theta \) and that the left-hand side is continuous over \( \theta \in \Theta \), where \( \pi_{j-1}(\theta) := \pi(X_{t_{j-1}}, Y_{t_{j-1}}; \theta) \); this is a basic device which we will make use of several times below without mention. Also, note that the right continuity of \( t \mapsto X_t \) implies that

- For convenience, we denote any random sequence \( p_n \) by \( 1_{u, p} \). Then, direct computations give the following expressions for the components of \( I_{1,n}(\theta) = -r_n^{-2} \partial^2_{\alpha_{\mu}} \ell_n(\theta) \):

\[ -\frac{1}{r_n^2} \partial^2 \ell_n(\theta) = -\frac{1}{n} \sum_{j=1}^{n} \sigma^{-2} (\partial g) \epsilon(\lambda_j)^2 + c_{u, p}(1). \]
\[ \frac{1}{n} \sum_{j=1}^{n} \sigma^{-2} g^2 X_{j-1}^{\otimes 2} + o_{u,p}(1), \]

\[ \frac{1}{r_n^2} \partial_n^2 \ell_n(\theta) = \frac{1}{n} \sum_{j=1}^{n} \sigma^{-2} (\partial g) Y_{j-1}^{2} 1_{u,p} + o_{u,p}(1) + O_{u,p}(h^{1/\beta}) \]

\[ = \frac{1}{n} \sum_{j=1}^{n} \sigma^{-2} g^2 Y_{j-1}^{2} + o_{u,p}(1), \]

\[ \frac{1}{r_n^2} \partial_n \partial_n^2 \ell_n(\theta) = \frac{1}{n} \sum_{j=1}^{n} \left\{ (\partial g) \left( 1_{u,p} \sigma^{-1} Y_{j-1} \zeta_j(\lambda) + O_{u,p}(h^{1/\beta}) \right) (-\sigma^{-1} 1_{u,p}) + g O_{u,p}(h^{1/\beta}) \right\} \]

\[ = 1_{u,p} \left( \frac{1}{n} \sum_{j=1}^{n} \sigma^{-2} (\partial g) Y_{j-1} \zeta_j(\lambda) + o_{u,p}(1) \right) + o_{u,p}(1) \]

\[ = \frac{1}{n} \sum_{j=1}^{n} \sigma^{-2} g^2 Y_{j-1} X_{j-1} + o_{u,p}(1). \]

Now we can deduce that \( I_{11,n}(\theta) \overset{L_n}{\to} I_{\lambda,\mu}(\theta) \) as follows:

- First, noting that \( \epsilon_j = \epsilon_j(\theta) \sim i.i.d. \mathcal{L}(J_1) \) under \( \mathbb{P}_\theta \) we make the compensation \( g^2 = \mathbb{E}_\theta(g^2) + (g^2 - \mathbb{E}_\theta(g^2)) \) in the summation in rightmost sides of the last three displays;

- Then, we apply the law of large numbers

\[ (2.12) \]

\[ \frac{1}{n} \sum_{j=1}^{n} \psi(X_{j-1}, Y_{j-1}) \overset{L}{\to} \frac{1}{T} \int_{0}^{T} \psi(X_t, Y_t) dt, \quad n \to \infty, \]

which is valid for any \( \psi \in \mathcal{C}^2(\mathbb{R}^3 \times \mathbb{R}) \) with \( |(\partial_{x,y}) \psi(x,y)| \lesssim (1+|x|+|y|)^C \) under the a.s. Riemann integrability of \( t \mapsto (X_t(\omega), Y_t(\omega)) \).

Indeed, for the case of \( -r_n^{-2} \partial_n^2 \ell_n(\theta) \) we have

\[ \frac{1}{r_n^2} \partial_n^2 \ell_n(\theta) = \frac{1}{n} \sum_{j=1}^{n} \sigma^{-2} \mathbb{E}_\theta(g^2) X_{j-1}^{\otimes 2} + o_{u,p}(1) \overset{L}{\to} I_{\lambda,\mu,22}(\theta), \]

with \( I_{\lambda,\mu,22}(\theta) \) denoting the lower left \( q \times q \) component of \( I_{\lambda,\mu}(\theta) \). The other two analogously can be handled analogously.

Next we turn to looking at \( I_{12,n}(\theta) = \{ I_{12,n}(\theta) \}_{k,l} \):

\[ I_{11,n}(\theta) = \varphi_{11,n}(\theta) \partial_n \partial_n \ell_n(\theta) + \varphi_{21,n}(\theta) \partial_n \partial_n \ell_n(\theta), \]

\[ I_{12,n}(\theta) = \varphi_{12,n}(\theta) \partial_n \partial_n \ell_n(\theta) + \varphi_{22,n}(\theta) \partial_n \partial_n \ell_n(\theta), \]

\[ I_{12,n}(\theta) = \varphi_{12,n}(\theta) \partial_n \partial_n \ell_n(\theta) + \varphi_{22,n}(\theta) \partial_n \partial_n \ell_n(\theta). \]

However, we can deduce that \( I_{12,n}(\theta) \overset{L_n}{\to} 0 \) just by inspecting the four component separately in a similar way that we managed \( I_{11,n}(\theta) \). Let us only mention the lower-left \( q \times 1 \) component: recalling that \( |\varphi_{22,n}| \lesssim u \ell_n^l \),

\[ I_{12,n}(\theta) = -\frac{h^{1-1/\beta}}{n} \sum_{j=1}^{n} \left( \varphi_{12,n} (\log c)_{(\lambda,\beta)} + \varphi_{12,n} g(\beta) \epsilon(\lambda) + \varphi_{12,n} (\partial g) \epsilon(\lambda) \epsilon(\beta) \right) \]

\[ + \varphi_{12,n} g \epsilon(\lambda,\beta) + \varphi_{22,n} g(\beta) \epsilon(\mu) + \varphi_{22,n} (\partial g) \epsilon(\beta) \epsilon(\mu) + \varphi_{22,n} g \epsilon(\beta,\mu) \]

\[ = O(h^{1+1/\beta}) + O_{u,p}(n^{-1/2} \vee h^{1/\beta}) + O_{u,p} \left( n^{-1/2} \vee h^{1/\beta} \ell_n^l \right) \]

\[ + O_{u,p}(n^{-1/2}) + O_{u,p} \left( n^{-1/2} \ell_n^l \right)^2 + O_{u,p} \left( n^{-1/2} (\ell_n^l)^2 \right) \overset{L_n}{\to} 0. \]

Thus we have obtained \( (2.9) \).

Turning to verification of \( (2.11) \), we note that

\[ |\epsilon_j(\theta')| \lesssim u |\epsilon_j(\theta)| + o_{u,p}(1), \quad \theta' \in \mathcal{R}_n(c; \theta). \]
Hence, for each \( k, l, m \in \mathbb{Z}_+ \), we have

\[
\frac{1}{n} \left| \frac{\partial^k \partial^l \varphi_n}{\partial \theta^m} \right| \lesssim_n \frac{1}{n} \sum_{j=1}^n (1 + |Y_{ij}|)^2 \left( 1 + \log \left( 1 + |\varphi_n(\theta)| \right) \right)^k.
\]

As in the proof of Eq. (14) in [2], for each \( c > 0 \) we can find a constant \( R = R(c) > 0 \) such that

\[
\sup_{\theta \in \mathbb{R}} \left| \varphi_n(\theta) \right| \lesssim_n \frac{1}{n} \sum_{j=1}^n (1 + |\varphi_n(\theta)|)^3 \lesssim C_n (\frac{\sqrt{\varphi_n}}{\sqrt{n}}) \mathbb{P} \left( \frac{\sqrt{n}}{\sqrt{\varphi_n}} \right).
\]

where \( B(\beta; R/n) \) denotes the closed ball with center \( \beta \) and radius \( R/n \). This shows (2.11). The proof of Theorem 2.1 is complete. \( \square \)

3. One-step improvement

The well-known shortcoming of the classical Cramér-type argument is its local character: the result just tells us the existence of an asymptotically nicely behaving root of the likelihood equation, but does not provide us with information about which local maxima is the one when there are multiple local maxima, equivalently multiple roots for the likelihood equations [10] Section 7.3. Indeed, the log-likelihood function \( \ell_n \) is highly nonlinear and non-concave. In this section, we consider removing the locality by a one-step improvement, which in our case will not only remedy the aforementioned inconveniency about the multiple-root problem, but also enable us to bypass the numerical optimization involving the stable density \( \phi_\beta \).

In [2] Section 3], for the \( \beta \)-stable Lévy process (the special case of [11] with \( \lambda = 0 \) and \( X \equiv 1 \), we provided an initial estimator based on the sample median and the method of moments associated with logarithm and/or lower-order fractional moments. In that paper it was crucial that the model was a Lévy process, for which we could apply the median-adjusted central limit theorem; moreover, it should be noted that the initial estimator considered in [2] Section 3.2] possesses the optimal (non-diagonal and asymmetric) rate. In the present case, we have to take a different route.

3.1. Initial rates of convergence. First we prove a basic result about the classical one-step estimator. Recall the definition (2.3) of the non-diagonal matrix \( \varphi_n = \varphi_n(\theta) \). We have seen that the likelihood equation \( \partial_{\theta} \varphi_n(\theta) = 0 \) admits an asymptotically efficient MLE \( \hat{\theta} \) as its root: \( \varphi_n(\theta)^{-1}(\hat{\theta} - \theta) \) is uniformly asymptotically mixed-normal in the sense of Theorem 2.1.

We introduce the diagonal matrix \( \varphi_n^0 = \varphi_n^{0}(\theta) \) by

\[
\varphi_n^0 = \text{diag}(\varphi_{0,1}, \ldots, \varphi_{p,n}) := \text{diag} \left( \frac{1}{\sqrt{nh^{1-\lambda}}} \mathcal{I}_{1+q}, \frac{1}{\sqrt{n}} \frac{\log(1/h)}{\sqrt{n}} \right).
\]

Suppose that there exists an estimator \( \hat{\theta}_n^0 = (\hat{\lambda}_n^0, \hat{\mu}_n^0, \hat{\beta}_n^0, \hat{\sigma}_n^0) \) such that \( (\varphi_n^0)^{-1}(\hat{\theta}_n^0 - \theta) = O_{u,p}(1) \):

\[
\left( \sqrt{nh^{1-\lambda}} (\hat{\lambda}_n^0 - \lambda), \sqrt{nh^{1-\beta}} (\hat{\beta}_n^0 - \beta), \sqrt{n} (\hat{\sigma}_n^0 - \sigma) \right) = O_{u,p}(1).
\]

Then, we define the one-step estimator \( \hat{\theta}_n^1 = (\hat{\lambda}_n^1, \hat{\mu}_n^1, \hat{\beta}_n^1, \hat{\sigma}_n^1) \) starting from \( \hat{\theta}_n^0 \) toward \( \hat{\theta}_n \) by

\[
\hat{\theta}_n^1 = \hat{\theta}_n^0 - \left( \partial_{\theta} \varphi_n(\hat{\theta}_n^0) \right)^{-1} \partial_{\theta} \varphi_n(\hat{\theta}_n^0)
\]

\[
= \hat{\theta}_n^0 + \varphi_n(\hat{\theta}_n^0) \mathcal{I}_n(\hat{\theta}_n^0)^{-1} \Delta_n(\hat{\theta}_n^0),
\]

which is well-defined with probability tending to 1 for \( n \to \infty \) since (recall the property (2.11))

\[
\mathcal{I}_n(\hat{\theta}_n^0) \mathbb{P} \left( \mathcal{I}(\theta) \right)
\]

with the limit being a.s. positive definite; we could slightly modify the definition of \( \hat{\theta}_n^1 \) through an appropriate indicator function in order to avoid saying “with probability tending to 1”, but it is not made here just for brevity.
Theorem 3.1. For any $\hat{\theta}_n^0$ satisfying (3.1), the one-step estimator $\hat{\theta}_n^1$ defined by (3.2) is uniformly asymptotically equivalent to the asymptotically efficient MLE $\hat{\theta}_n$ in the sense that $\varphi_n(\theta)^{-1}(\hat{\theta}_n - \hat{\theta}_n^1) = o_u,p(1)$, hence in particular

$$\varphi_n(\hat{\theta}_n^1 - \theta) \xrightarrow{L_n} M_{N_p,\theta}(0, I(\theta)^{-1}).$$

In (3.3), we could replace the term $\varphi_n(\hat{\theta}_n^1 - \theta)$ by $\varphi_n(\hat{\theta}_n^0 - \theta)$ since $|\varphi_n(\hat{\theta}_n^1 - \varphi_n(\hat{\theta}_n^0)) = I_{P+1}| \xrightarrow{L_n} 0$.

Remark 3.2. (1) In Fisher’s scoring, we could replace the definition of $\hat{\theta}_n^1$ by

$$\hat{\theta}_n^1 = \hat{\theta}_n^0 + \varphi_n(\hat{\theta}_n^0)^{-1}H_n(\hat{\theta}_n^0)^{-1}\Delta_n(\hat{\theta}_n^0),$$

for any random function $H_n(\theta)$ for which

$$H_n(\theta) \xrightarrow{D_n} u, I(\theta).$$

This is easily seen by inspecting the proof above: $\delta_{\hat{\theta}_n^1, \theta}$ is no longer valid, while we have $\delta_{u, \hat{\theta}_n^1, \theta} = 0$.

(2) We may repeatedly apply the above argument to obtain a $k$-step estimator. In the literature, one may need a refined one-step estimator when rate of convergence of an initial estimator is “much” slower than the target one, see [11]; although the initial rate of convergence (3.1) is not the optimal one, just one step is enough to achieve the best.

Remark 3.3. Having $\hat{\theta}_n^0$ in hand, we want to have consistent estimators $\hat{\theta}_{\lambda,\mu,\nu} \xrightarrow{P} \lambda, \mu, \nu$ and $\hat{\theta}_{\beta,\sigma,\nu} \xrightarrow{P} \hat{\theta}_{\beta,\sigma,\nu}(\theta)$, so that

$$\hat{\theta}_{\lambda,\mu,\nu} \xrightarrow{P} \lambda, \mu, \nu$$

$$\hat{\theta}_{\beta,\sigma,\nu} \xrightarrow{P} \hat{\theta}_{\beta,\sigma,\nu}(\theta).$$

where $n^{-1/2}\varphi_n(\theta)$ denotes the lower-right 2 × 2-part of $\varphi_n(\theta)$; (3.4) can be used for goodness-of-fit testing, in particular variable selection among the components of $X$.

Proof of Theorem 3.1. We only need to show

$$\varphi_n^{-1}(\hat{\theta}_n - \hat{\theta}_n^1) = o_u,p(1),$$

from which (3.3) is trivial. Below we will use the shorthand “hat” for plugging-in $\hat{\theta}_n^0$: $\hat{x}_n = \hat{\theta}_n(\hat{\theta}_n^0)$, $\hat{\varphi}_n = \varphi_n(\hat{\theta}_n^0)$, $\hat{\theta}_n = \hat{\theta}_n(\hat{\theta}_n^0) = \hat{\varphi}_n(\hat{\theta}_n^0)$, and so on.

Note that (2.4) and (3.1) imply

$$\varphi_n^{-1}\varphi_n - I_p \equiv 0,$$

so that it suffices to show $\varphi_n^{-1}(\hat{\theta}_n^1 - \hat{\theta}_n) = o_u,p(1)$.

Applying the second-order Taylor expansion of $\partial_0\ell_n(\hat{\theta}_n^0)$ around $\hat{\theta}_n$ in (3.2), we have for some random point $\xi_n \in [0, 1]$,

$$\varphi_n^{-1}(\hat{\theta}_n^1 - \hat{\theta}_n) = \hat{\varphi}_n^{-1}(\hat{\theta}_n - \hat{\theta}_n) + \hat{x}_n^{-1}\varphi_n^\top \partial_0\ell_n(\hat{\theta}_n) + \hat{x}_n^{-1}\varphi_n\partial_0^2\ell_n(\hat{\theta}_n + \xi_n(\hat{\theta}_n - \hat{\theta}_n))(\hat{\theta}_n - \hat{\theta}_n)$$

$$= \hat{x}_n^{-1}\varphi_n^\top \partial_0\ell_n(\hat{\theta}_n) + \hat{x}_n^{-1}\varphi_n\partial_0^2\ell_n(\hat{\theta}_n + \xi_n(\hat{\theta}_n - \hat{\theta}_n))(\hat{\theta}_n - \hat{\theta}_n)$$

$$\varphi_n^{-1}(\hat{\theta}_n^1 - \hat{\theta}_n) = \hat{x}_n^{-1}\varphi_n^\top \partial_0\ell_n(\hat{\theta}_n) + \hat{x}_n^{-1}\varphi_n\partial_0^2\ell_n(\hat{\theta}_n + \xi_n(\hat{\theta}_n - \hat{\theta}_n))(\hat{\theta}_n - \hat{\theta}_n).$$
This and (3.7) combined with the conventional convexity argument will make the subsequent exposition leave as the aspects in the present situation are quite analogous to those of [12], we will only mention an outline. In [12] in the context of drift estimation of the ergodic locally stable OU process. Because the technical δ

For technical convenience, we further assume that we observe \( \{ \mathbb{I} \}_{t=1}^{n} \) and note the approximation of the Riemann integral:

\[
\int_{1}^{t_{j}} e^{-\lambda(t_{j} - s)} \, ds, \quad \text{and let the approximation of the Riemann integral:}
\]

\[
\int_{1}^{t_{j}} e^{-\lambda(t_{j} - s)} \, ds = \int_{1}^{t_{j}} X_{s} \, ds + O_{n}(\lambda^{2}).
\]

This and (3.7) combined with the conventional convexity argument will make the subsequent exposition much simpler.

We define the LAD estimator \((\hat{\lambda}^{0}_{n}, \hat{\mu}^{0}_{n})\) by a minimizer of the random function

\[
(\lambda, \mu) \mapsto \sum_{j=1}^{n} \left| Y_{t_{j}} - e^{-\lambda Y_{t_{j-1}}} - \mu \cdot \int_{1}^{t_{j}} X_{s} \, ds \right|,
\]

leaving \(\sigma\) unknown. This is a slight modification of the previously studied approximate LAD estimator in [12] in the context of drift estimation of the ergodic locally stable OU process. Because the technical aspects in the present situation are quite analogous to those of [12], we will only mention an outline.

Let

\[
\epsilon'_{j} = \epsilon'_{j}(\theta) := h^{-1/3} \left( Y_{t_{j}} - e^{-\lambda Y_{t_{j-1}}} - \mu \cdot \int_{1}^{t_{j}} X_{s} \, ds \right).
\]

Then, by (3.30) we have

\[
\epsilon'_{j} = \epsilon_{j}(\theta)\sigma \eta(\lambda \beta h)^{1/3} + O_{n}(h^{2-1/3}),
\]

which will later enable us to approximate as \(\epsilon'_{j}(\theta) \approx \text{i.i.d.} \mathcal{L}(\sigma \eta(\lambda \beta h)^{1/3} J_{1}) \approx \mathcal{L}(\sigma J_{1})\).

Let \(x_{j-1} := (-Y_{t_{j-1}}, \int_{1}^{t_{j}} X_{s} \, ds)\). Introduce the convex random fields on \(\mathbb{R}^{1+q}\):

\[
\Lambda'_{n}(v) := \sum_{j=1}^{n} \left( \left| \epsilon'_{j} - \frac{1}{\sqrt{n}} v \cdot x_{j-1} \right| - |\epsilon'_{j}| \right),
\]

As a concrete example satisfying (3.1), borrowing some existing results partly with slight modifications, we will propose to make use of the least absolute deviation (LAD) estimator and the power-varying based estimator for \((\lambda, \mu)\) and \((\beta, \sigma)\), respectively, under the assumption that

\[
(\varphi^{-1}_{n})_{\sigma} = O_{u,p}(1),
\]

to conclude \(\delta_{3,n} \rightarrow 0\) if it suffices to show that

\[
(\varphi^{-1}_{n})_{\delta} \rightarrow 0,
\]

\[
\forall c > 0, \quad \sup_{1 \leq k \leq p, \varphi \in \mathcal{F}_{c} \sup_{(\varphi^{-1}_{n})_{\delta} \leq c} \left| (\varphi^{-1}_{n})_{\sigma} \delta_{k,n} - \delta_{k,n} \right| = O_{u,p}(\epsilon_{n}).
\]

\[
\varphi^{-1}_{n} \rightarrow 0,
\]

\[
\forall c > 0, \quad \sup_{1 \leq k \leq p, \varphi \in \mathcal{F}_{c} \sup_{(\varphi^{-1}_{n})_{\delta} \leq c} \left| (\varphi^{-1}_{n})_{\sigma} \delta_{k,n} - \delta_{k,n} \right| = O_{u,p}(\epsilon_{n}).
\]

\[
\beta < \left( \frac{2}{3} \right).
\]

For technical convenience, we further assume that we observe \(\{(f_{t_{j-1}}^{t_{j}} X_{s} \, ds, Y_{t_{j}})\}_{j=1}^{n}\) in the sequel.

3.2. LAD estimator. Let us recall the expression

\[
Y_{t_{j}} = e^{-\lambda h} Y_{t_{j-1}} + \mu \cdot \zeta_{j}(\lambda) h + \sigma \int_{1}^{t_{j}} e^{-\lambda(t_{j} - s)} \, ds J_{s},
\]

and note the approximation of the Riemann integral:

\[
\zeta_{j}(\lambda) h = \int_{1}^{t_{j}} e^{-\lambda(t_{j} - s)} X_{s} \, ds = \int_{1}^{t_{j}} X_{s} \, ds + O_{n}(\lambda^{2}).
\]

\[
\zeta_{j}(\lambda) h = \int_{1}^{t_{j}} e^{-\lambda(t_{j} - s)} X_{s} \, ds = \int_{1}^{t_{j}} X_{s} \, ds + O_{n}(\lambda^{2}).
\]

\[
\int_{1}^{t_{j}} e^{-\lambda(t_{j} - s)} X_{s} \, ds = \int_{1}^{t_{j}} X_{s} \, ds + O(\lambda^{2}).
\]

\[
\int_{1}^{t_{j}} e^{-\lambda(t_{j} - s)} X_{s} \, ds = \int_{1}^{t_{j}} X_{s} \, ds + O(\lambda^{2}).
\]
\[ \Lambda_n(v) := \sum_{j=1}^n \left( \epsilon_j(\theta) \sigma(\lambda h)^{1/\beta} \right. \left| \frac{1}{\sqrt{n}} v \cdot x_{j-1} \right| - \left| \epsilon_j(\theta) \sigma(\lambda h)^{1/\beta} \right| \).

Define a random variable \( \hat{v}_n = (\hat{v}_{1,n}, \hat{v}_{2,n}) \in \mathbb{R} \times \mathbb{R} \) by
\[
\hat{v}_{1,n} = \sqrt{nh^{-1/\beta}} (e^{-\lambda h} - e^{-h\mu_0}), \\
\hat{v}_{2,n} = \sqrt{nh^{-1/\beta}} (\mu_0 - \mu).
\]

Then, according to the definition of \((\hat{\lambda}_n^0, \hat{\mu}_n^0)\), \( \hat{v}_n \) is a minimizer of \( \Lambda_n^* \). Further, it follows from the triangular inequality and (3.5) that for each \( v \),
\[ |\Lambda_n(v) - \Lambda_n^*(v)| \lesssim |v| O_{n,p}(\sqrt{nh}^{2-1/\beta}) \lesssim |v| O_{n,p}(h^{3/2-1/\beta}) = |v| o_{n,p}(1) \]
Next we introduce the quadratic random field
\[ \Lambda_n^*(v) := \Delta_n^0[v] - \frac{1}{2} \Gamma_n^0[v, v], \]
where
\[
\Delta_n^0 := \sum_{j=1}^n \frac{1}{\sqrt{n}} x_{j-1} \text{sgn}(\epsilon_j), \\
\Gamma_n^0 := 2 \phi_{\sigma, \beta}(0) \frac{1}{\sqrt{n}} \sum_{j=1}^n \begin{pmatrix} -Y_{t_{j-1}}^2 X_{t_{j-1}}^2 & -Y_{t_{j-1}} X_{t_{j-1}}^2 X_{t_{j-1}}^2 \\ -Y_{t_{j-1}} X_{t_{j-1}} & X_{t_{j-1}}^2 \end{pmatrix},
\]
with \( \phi_{\sigma, \beta} \) denoting the density of \( \mathcal{L}(\sigma J_1) \). The a.s. positive definiteness of \( \Gamma_0 \) implies that \argmin \Lambda_n^* a.s. consists of the single point \( \hat{v}_n := \Gamma_0^{-1} \Delta_n^0 \). We have
\[ \Gamma_n^0 \overset{p}{\to} \Gamma_0 := 2 \phi_{\sigma, \beta}(0) \frac{1}{\sqrt{T}} \int_0^T \begin{pmatrix} Y_t^2 & -Y_t X_t^2 X_t^2 \\ -Y_t X_t & X_t^2 \end{pmatrix} dt. \]
Then, we can deduce the locally asymptotically quadratic structure:
\[ \Lambda_n(v) - \Lambda_n^*(v) = o_{n,p}(1) \]
uniformly in \( v \in \mathbb{R}^{1+q} \) over compact sets; the proof can be achieved in a quite similar fashion to that of [12] Theorem 2.1, hence omitted.

We now verify the representation
\[ \hat{v}_n = \hat{v}_n^\delta + o_{n,p}(1), \]
from which the desired \( \sqrt{nh}^{1-1/\beta} \)-consistency of the LAD estimator \((\hat{\lambda}_n^0, \hat{\mu}_n^0)\) follows. Fix any \( \epsilon > 0 \) and let \( \delta_n(v) := \Lambda_n(v) - \Lambda_n^*(v) \). Making use of the convexity of \( \Lambda_n \), (3.11), and [6, Lemma 2], we have
\[ \mathbb{P}_\theta \left( |\hat{v}_n - \hat{v}_n^\delta| \geq \epsilon \right) \leq \mathbb{P}_\theta \left( \sup_{v : |v - \hat{v}_n^\delta| \leq \epsilon} |\delta_n(v)| \geq \frac{1}{2} \inf_{\epsilon = \epsilon_\delta + v} \Lambda_n^*(v) \right) \]
\[ \leq \mathbb{P}_\theta \left( \sup_{v : |v - \hat{v}_n^\delta| \leq \epsilon} |\delta_n(v)| \geq \frac{\epsilon^2}{2} \lambda_{\min} \right) \]
for each \( n \in \mathbb{N} \), where \( \lambda_{\min} > 0 \) a.s. denotes the minimum eigenvalue of \( \Gamma_0 \). Given any \( K > 0 \), the upper bound in (3.13) is bounded by
\[ \mathbb{P}_\theta \left( \sup_{v : |v| \leq K + \epsilon} |\delta_n(v)| \geq \frac{\epsilon^2}{2} \lambda_{\min} \right) + \mathbb{P}_\theta \left( |\hat{v}_n^\delta| \geq K \right). \]
By (3.11), the first term \( \to 0 \) \((n \to \infty)\) for each \( K > 0 \), hence (3.12) follows on showing \( \hat{v}_n^\delta = O_{n,p}(1) \).
But the latter can be seen first by applying the Lenglart inequality (e.g. [7, I.3.31]): for any \( K', K'' > 1 \),
\[ \sup_n \mathbb{P}_\theta(|\Delta_n| \geq K) \lesssim \frac{K' K''}{K} + \frac{K''}{K} + \sup_n \mathbb{P}_\theta \left( \frac{1}{n} \sum_{j=1}^n (1 + Y_{t_{j-1}}^2) \gtrsim \frac{K''}{1 + \sup_{t \in [0,T]} |X_t|^2} \right), \]
and then by noting that \( n^{-1} \sum_{j=1}^n Y_{t_{j-1}}^2 = O_{n,p}(1) \).
Remark 3.4. If there exists a constant $\kappa > 0$ such that $\int_{J}(X_s - X_{t_{j-1}})ds = O^*(h^{1+\kappa})$, then (3.9) becomes

$$\zeta_j(\lambda)h = \int_{J}e^{-\lambda h(s)}X_sds = X_{t_{j-1}}h + O^*_n(h^{1+\kappa} \vee h^2).$$

If in particular we can take $\kappa = 1$, then the LAD objective function (3.10) could be further simplified as

$$(\lambda, \mu) \mapsto \sum_{j=1}^{n} |Y_{t_j} - e^{-\lambda h}Y_{t_{j-1}} - \mu \cdot X_{t_{j-1}}h|.$$ 

3.3. Power variation based estimator of $$(\beta, \sigma)$$. The remaining objective is to provide an initial estimator $(\hat{\beta}_n^0, \hat{\sigma}_n^0)$ of $(\beta, \sigma)$ such that

$$\left(\sqrt{n}(\hat{\beta}_n^0 - \beta), \frac{\sqrt{n}}{\log(1/h)}(\hat{\sigma}_n^0 - \sigma)\right) = O_{u,p}(1).$$

For this purpose, we can make use of the limit theorems for the power-variation statistics via second-order increments [19]; note that we keep assuming (3.7). Pick an $r \in (\frac{\beta-1}{2(\beta+1)}, \beta/2) \subseteq [0, 1)$, and let

$$V'_n(r) := \sum_{j=2}^{n} |\Delta_j Y - \Delta_{j-1} Y|^r,$$

$$V''_n(r) := \sum_{j=4}^{n} |\Delta_j Y - \Delta_{j-1} Y + \Delta_{j-2} Y - \Delta_{j-3} Y|^r.$$

Then, we introduce the following estimators of $\beta$ and $\sigma$:

$$\hat{\beta}_n^0(r) := r \log(2)/\log \left(V''_n(r)/V'_n(r)\right),$$

$$\hat{\sigma}_n^0(r) := T^{-1/\beta(r)} \left\{ n^{r/\beta(r)-1}V'_n(r)/m(r, \hat{\beta}_n^0(r)) \right\}^{1/r},$$

where, with $J'$ and $J''$ denoting i.i.d. copies of $J_1$,

$$m(r, \beta) := E_\theta(|J' - J''|^r) = 2^{r/\beta} 2^{-\Gamma((r+1)/2)} \Gamma(1-r/\beta),$$

By [19] Theorem 3] we have both $\sqrt{n}(\hat{\beta}_n^0(r) - \beta) = O_{u,p}(1)$ and $\frac{\sqrt{n}}{\log(1/h)}(\hat{\sigma}_n^0(r) - \sigma) = O_{u,p}(1)$.

The annoying point of this approach is that we need to pick an $r \in (\frac{\beta-1}{2(\beta+1)}, \beta/2)$ without knowing $\beta$. If we beforehand know that the true value of $\beta$ lies in (1, 2), then the choice $r = 1$ does work.

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