Rotation and Pseudo-Rotation

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Eigenvectors of stress-energy tensor (the source in Einstein’s equations) form privileged bases in description of the corresponding space-times. When one or more of these vector fields are rotating (the property well determined in differential geometry), one says that the space-time executes this rotation. Though the rotation in its proper sense is understood as that of a timelike congruence (vector field), the rotation of a spacelike congruence is not a less objective property if it corresponds to a canonical proper basis built of the just mentioned eigenvectors. In this last case, we propose to speak on pseudo-rotation. Both properties of metric, its material sources, and space-time symmetries are considered in this paper.

KEY WORDS: Rotation; Killing vectors; r-forms; proper basis

{ See the (mixed) English–Spanish–Russian poem dedicated to Alberto A. García Díaz in [15] as it was published in Gen. Relat. and Gravit.}

1. INTRODUCTION

One seems to know quite well what is the rotation incorporated into the metric tensor. But there is an alternative way, sometimes used in describing, for example, the Kerr metric in a frame “without rotation”, but with an alternative combination of \(d\phi\) and \(dt\) (now in this succession), see, e.g., the textbook by Misner, Thorne, and Wheeler [7], Exercises 33.3 and 33.4. We call this choice of frame as that with ‘pseudo-rotation’, being studied below,

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alongside with rotation, using some typical examples of non-empty space-times (the Kerr space-time is not so much appropriate since it corresponds to a vacuum, thus in the absence of material repères, with Killing vectors only remaining as a possible tool). This non-emptiness means that the space-time is filled by some sort(s) of matter (electromagnetic field, fluids, etc.), in other words, there actually is a stress-energy tensor as a source in Einstein’s equations. The concrete structure of this tensor not only determines the nature of matter under consideration, but also (via its eigenvectors) influences upon the properties of space-time, for example, through a rotation.

In this paper (published in [15]) we consider rotation and its counterpart, pseudo-rotation. The first one corresponds to rotation of a timelike congruence, and the second (whose importance is widely underestimated), to rotation of a spacelike congruence. When these congruences (equivalently, vector fields) reflect objective properties of a physical system (e.g., they are Killing vectors of the space-time, or eigenvectors of the stress-energy tensor), they obviously have equal logical footing and importance. Any approach based on Killing vectors is however quite restrictive: if a space-time possessing isometries is superimposed with even small exterior perturbation (say, a gravitational wave comes from a faraway source), its symmetries disappear, although this cannot mean that the rotation so abruptly ceases to exist. This vulnerability of exact mathematical symmetries calls for extreme care when one intends to draw from them a working physical definition.

In the next Section we give prominence to rotation of perfect fluids using their field-theoretic description. In this case, the rotation gains on pseudo-rotation, since the former is so easily visualizable as a rotation about a spatial axis (the axis of a pseudo-rotation may be timelike, not only spacelike). Moreover, the four-velocity of a perfect fluid is equivalent not only to the timelike basis vector of the proper tetrad, but also to the intensity of the 2-form field, thus, via its field equation, it should be closely related to the specific mechanism which introduces rotation in the theory (we emphasize that rotation plays in the theory of the 2-form field, at least formally, a similar rôle to that of the sources in the electromagnetic and gravitational theories). Nevertheless, there are nice and important exact solutions in general relativity (also Einstein–Maxwell fields) with pseudo-rotation and a combination of rotation and pseudo-rotation, and one cannot ignore them, especially because of their fundamental simplicity and symmetric structure. In Sections 3 and 5 we consider two such examples, while in Section 4 we find that pseudo-rotation automatically appears in the well-known Kerr–Newman space-time between the event and Cauchy horizons.

Below we are working in four space-time dimensions with signature $(+,−,−,−)$, Greek indices being four-dimensional. The Ricci tensor is $R_{\mu\nu} = R^{\alpha}_{\mu\nu\alpha}$, thus Einstein’s equations take the form $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = - \kappa T_{\mu\nu}$. 
2. ROTATING FLUIDS IN FIELD-THEORETIC DESCRIPTION

Perfect fluids can be conveniently described with use of the Lagrangian formalism, especially in the absence of rotation [9, 10]. In this case they are represented via the 2-form field potential $B = \frac{1}{2} B_{\mu \nu} dx^\mu \wedge dx^\nu$, the respective field intensity being $G = dB = \frac{1}{2} B_{\mu \nu; \lambda} dx^\mu \wedge dx^\nu \wedge dx^\nu$ (where $B_{\mu \nu; \lambda} = B_{\mu \nu, \lambda}$), and $G_{\lambda \mu \nu} = B_{\lambda \mu, \nu} + B_{\mu \nu, \lambda} + B_{\nu \lambda, \mu}$ whose invariant $J = *(G \wedge G)$ (we shall also denote $*G = \tilde{G}$) is used in constructing the fluid Lagrangian density $L = \sqrt{-g} L(J)$. Here the Hodge star * denotes, as usual, a generalization of the dual conjugation applied to Cartan exterior forms: with an $r$-form $\alpha = \alpha_{\nu_1 \ldots \nu_r} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_r}$, it yields a $(4 - r)$-form $*\alpha$ with the components $(*\alpha)_{\nu_1 \ldots \nu_{4-r}} = \frac{1}{r!} E_{\nu_1 \ldots \nu_r \nu_{4-r} \ldots \nu_4} \alpha_{\nu_1 \ldots \nu_r}$ where $E_{\kappa \lambda \mu \nu} = \sqrt{-g} \epsilon_{\kappa \lambda \mu \nu}$ and $E_{\kappa \lambda \mu \nu} = - (1/\sqrt{-g}) \epsilon_{\kappa \lambda \mu \nu}$ are co- and contravariant axial skew rank-4 tensors, $\epsilon_{\kappa \lambda \mu \nu} = \epsilon_{[\kappa \lambda \mu \nu]}$, $\epsilon_{0123} = +1$ being the Levi-Civita symbol (cf. somewhat other notations in [8]).

The reason why this description of perfect fluids is valid, is simply the fact that the stress-energy tensor of a 2-form field is

$$T_{\alpha \beta}^\beta = 2 \frac{dL}{dJ} b_{\alpha \beta} - L \delta_{\alpha \beta}$$

(1)

where

$$b_\alpha^\beta = \delta_\alpha^\beta - u_\alpha u^\beta, b_\alpha^\beta u_\beta = 0 = b_\alpha^\beta u_\beta, u = J^{-1/2} \tilde{G}.$$  

(2)

When $u$ is timelike ($u \cdot u = +1$, as we above supposed it to be), we come to the usual perfect fluid whose (arbitrary) equation of state is determined by the dependence of $L$ on its only argument, $J$ (see [9], [10]) [however when it is spacelike, the ‘fluid’ is tachyonic (see for some details [12], subsection 3.2)]. Since $b_\alpha^\beta$ is the projector on the (local) subspace orthogonal to the congruence of $u$, the latter is an eigenvector of the stress-energy tensor with the eigenvalue $-L$, $T_{\alpha}^\beta u^\alpha = -Lu^\beta$, while any vector orthogonal to $u$ is also eigenvector, now with the three-fold eigenvalue $2J \frac{dL}{dJ} - L$. This is the property of the stress-energy tensor of a perfect fluid possessing the proper mass density $\mu$ and pressure $p$ (in its local rest frame):

$$\mu = -L, \ p = L - 2J \frac{dL}{dJ}.$$  

(3)

Below we consider perfect fluids characterized by the simplest equation of state

$$p = (2k - 1)\mu$$

(4)

(the frequently used notation is $2k = \gamma$) which correspond to the Lagrangian $L = -\sigma J^k$, $\sigma > 0$. In a four-dimensional spacetime, the important special cases are: the incoherent dust ($p = 0$) for $k = 1/2$, intrinsically relativistic
incoherent radiation \((p = \mu/3)\) for \(k = 2/3\), and hyperrelativistic stiff matter \((p = \mu)\) for \(k = 1\).

However the 2-form field equation which follows from the above Lagrangian,

\[
\left( \sqrt{-g} \frac{dL}{dJ} G^{\lambda\mu} \right)_{,\nu} = 0 \iff \frac{d}{dJ} \left( J^{1/2} \frac{dL}{dJ} u \right) = 0,
\]

(5)

only means that the \(\vec{G}\) (equivalently, \(u\)) congruence is non-rotating. To describe a rotating fluid, one has to introduce in (5) a non-zero right-hand side. This, in a sharp contrast to the usual equations of mathematical physics (cf., for example, electrodynamics), cannot then be interpreted as a usual source term (this was stressed in [12]): its meaning essentially is to indicate the presence of rotation \((u \wedge du \neq 0)\). To this end it is necessary to consider one more field which we call the Machian one, a 3-form field \(C\) with the intensity \(W = dC\) (see [9, 10]). In terms of \(L(K)\), \(K = -(1/4!)W_{\kappa\lambda\mu\nu}W^{\kappa\lambda\mu\nu} = \tilde{W}^2\), its equations reduce to

\[
\left( \sqrt{-g} \frac{dL}{dK} W_{\kappa\lambda\mu\nu} \right)_{,\nu} = 0 \Rightarrow K^{1/2} \frac{dL}{dK} = \text{const.}
\]

(6)

We use also the duality relations \(B_{\mu\nu}^* = \frac{1}{2} E_{\mu\alpha\nu\beta} B_{\alpha\beta}^\kappa, G_{\lambda\mu\nu} = \tilde{G}^\kappa E_{\kappa\lambda\mu\nu},\), \(W_{\kappa\lambda\mu\nu} = \tilde{W} E_{\kappa\lambda\mu\nu}\). Moreover, \(B_{\mu\nu}^* \equiv -(**G)^\mu\).

Since we were confronted with the no rotation property of perfect fluid when the rank 2 field was considered to be free, the only remedy now is to introduce a non-trivial “source” term in the \(r = 2\) field equations, thus to consider the non-free field case or, at least, to include in the Lagrangian a dependence on the rank 2 field potential \(B\). The simplest way to do this is to introduce in the Lagrangian density dependence on a new invariant \(J_1 = -B_{[\kappa\lambda}B_{\mu\nu]}B^{[\kappa\lambda}B^{\mu\nu]}\) which does not spoil the structure of stress-energy tensor, simultaneously yielding a “source” term (thus permitting to destroy the no rotation property) without changing the divergence term in the \(r = 2\) field equations. We shall use below three invariants: the obvious ones, \(J\) and \(K\), and the just introduced invariant of the \(r = 2\) field potential, \(J_1\). Then

\[
B_{[\kappa\lambda}B_{\mu\nu]} = -\frac{2}{4!} B_{\alpha\beta}B^\alpha_B^\beta E_{\kappa\lambda\mu\nu}.\]

(7)

Thus \(J_1^{1/2} = 6^{-1/2}B_{\alpha\beta}B^\alpha_B^\beta\). In fact, \(J_1 = 0\), if \(B\) is a simple bivector \((B = a \wedge b, a\) and \(b\) being 1-forms). This cannot however annul the expression which this invariant contributes to the \(r = 2\) field equations: up to a factor, it is equal to \(\partial J_1^{1/2}/\partial B_{\mu\nu} \neq 0\). Thus let the Lagrangian density be

\[
\mathcal{L} = \sqrt{-g} \left( L(J) + M(K)J_1^{1/2} \right),
\]

(8)
so that the \( r = 2 \) field equations take the form (cf. (5))

\[
d\left(\frac{dL}{dJ} \tilde{G}\right) = \sqrt{\frac{2}{3} M(K) B} \iff \left(\sqrt{-g} \frac{dL}{dJ} G^{\alpha \beta \nu}\right)_{,\nu} = \sqrt{-g} \sqrt{\frac{2}{3} M(K) B}^{\alpha \beta}.
\]

In their turn, the \( r = 3 \) field equations (cf. (6)) yield the first integral

\[
J_1^{1/2} K^{1/2} \frac{dM}{dK} = \text{const} \equiv 0 \tag{10}
\]

(since \( J_1 = 0 \)). We know from [9, 10] that \( K \) (hence, \( M \)) \textit{arbitrarily} depends on the space-time coordinates, if only the \( r = 3 \) field equations are taken into account, and the Machian field \( K \) has to be essentially non-constant.

The stress-energy tensor which corresponds to the new Lagrangian density (8), automatically coincides with its previous form (1), since \( J_1 = 0 \). For a perfect fluid with the equation of state \( p = (2k - 1)\mu \), one finds \( L = -\sigma J^k \), thus \( T_{\alpha \beta} = -2k L u_{\alpha} w^\beta + (2k - 1) L \delta_{\alpha}^\beta \). Then the traditional perfect fluid language is obviously related with that of the \( r = 2 \) and \( r = 3 \) fields:

\[
\begin{align*}
\mu &= -L = \sigma J^k, \quad \tilde{G}^\mu = \Xi \delta_\mu^r, \quad \Xi &= \frac{1}{\sqrt{g_{00}}} \left( \frac{\mu}{\sigma} \right)^{1/2k}, \\
G &= dB = d \left( \sqrt{\frac{3/2}{M(K)}} \right) \land d \left( \frac{dL}{dJ} \tilde{G} \right)
\end{align*}
\]

(cf. (9)). The function \( M \) depends arbitrarily on coordinates; thus one can choose its adequate form using the last relation without coming into contradiction with the dynamical Einstein–Euler equations.

When one describes a fluid in its proper basis, \( u = J^{-1/2} \tilde{G} = \theta^{(0)} \), the rotation of the fluid’s co-moving reference frame is defined as \( \omega = * (\theta^{(0)} \land d\theta^{(0)}) = J^{-1} * (\tilde{G} \land d\tilde{G}) \). Let us assume \( \theta^{(0)} = e^{\alpha} (dt + f d\phi) \) where \( \alpha \) and \( f \) are functions of coordinates (usually determined \textit{via} Einstein’s equations), cf. the examples of metrics considered in the next Sections, though in these examples are treated Einstein–Maxwell fields and still not the perfect fluid solutions. It is inevitable to conclude that the field theoretic approach to perfect fluids automatically gives hints and even concrete relations (often having a simple algebraic form) imposed upon these and other functions characterizing the metric tensor and the 2-form field, as well as the Machian one. This makes it possible to substantially simplify the treatment of Einstein’s equations. The purpose of this paper is not to come into further details of such calculations, and we shall return to them in other publications (for some simple examples see [12]).
3. A SIMPLE ELECTROVACUUM SPACETIME WITH ROTATION AND PSEUDOROTATION

We now consider a special case ($\Phi(u) = C/\sqrt{\kappa} = \text{const.}$) of the conformally flat null Einstein–Maxwell field (32.103) in [4], whose metric obviously takes the Kerr–Schild form

$$ds^2 = \left(dt + \frac{C}{2} \rho^2 d\varphi\right)^2 - d\rho^2 - \rho^2 d\varphi^2 - \left(dz + \frac{C}{2} \rho^2 d\varphi\right)^2,$$

as well as the cylindrically symmetric forms with both rotation and pseudo-rotation

$$ds^2 = (d\tilde{t} + C \rho d\varphi)^2 - d\rho^2 - \rho^2 d\varphi^2 - (d\tilde{z} + C \rho d\varphi)^2.$$

The corresponding natural orthonormal tetrads are: for (12),

$$\theta^{(0)} = dt + \frac{C}{2} \rho^2 d\varphi, \quad \theta^{(1)} = d\rho, \quad \theta^{(2)} = \rho d\varphi, \quad \theta^{(3)} = dz + \frac{C}{2} \rho^2 d\varphi,$$

and for (13),

$$\tilde{\theta}^{(0)} = d\tilde{t} + C \rho d\varphi, \quad \tilde{\theta}^{(1)} = dx, \quad \tilde{\theta}^{(2)} = dy, \quad \tilde{\theta}^{(3)} = d\tilde{z} + C \rho d\varphi.$$

The relations between coordinates (those with a tilde and without it, as well as $\rho$, $\phi$ and $x$, $y$) are obvious.

It is remarkable that this space-time admits seven independent Killing vectors given here in the coordinates of (12), but in the basis (14):

$$\xi[1] = \theta^{(0)}, \quad \xi[2] = -\theta^{(3)}, \quad \xi[3] = \frac{C}{2} \rho^2 \left(\theta^{(0)} - \theta^{(3)}\right) - \rho \theta^{(2)},$$

and for (13),

$$\tilde{\xi}[1] = \theta^{(0)}, \quad \tilde{\xi}[2] = dx, \quad \tilde{\xi}[3] = dy, \quad \tilde{\xi}[4] = \rho \theta^{(2)}.$$

Contravariant Killing vectors in the coordinate frame satisfy the following non-trivial commutation relations:

$$[\xi[1], \xi[6]] = -C \xi[7], \quad [\xi[3], \xi[5]] = -C \xi[4],$$

$$[\xi[1], \xi[7]] = C \xi[6], \quad [\xi[3], \xi[6]] = C \xi[5],$$

$$[\xi[2], \xi[6]] = C \xi[7], \quad [\xi[3], \xi[7]] = -C \xi[6],$$

$$[\xi[2], \xi[7]] = -C \xi[6], \quad [\xi[4], \xi[5]] = C(\xi[1] + \xi[2]),$$

$$[\xi[3], \xi[4]] = -C \xi[5], \quad [\xi[6], \xi[7]] = C(\xi[1] + \xi[2]).$$
while $\xi_1 \cdot \xi_3 = 1$, $\xi_3 \cdot \xi_3 = -\rho^2$, and the five other spacelike Killing vectors are unitary ($\xi \cdot \xi = 1$). It is worth mentioning that $\xi_{[1]}$ and $\xi_{[2]}$ are rotating and pseudo-rotating respectively, the first around the axis $\theta^{(3)}$ and another, ‘around’ $\theta^{(0)}$, with one and the same magnitude of ‘angular velocity’,

$$\omega = \frac{1}{2} * (\xi_{[1]} \wedge d\xi_{[1]}) = \frac{C}{2} \tilde{\theta}^{(3)}, \quad \varpi = \frac{1}{2} * (\xi_{[2]} \wedge d\xi_{[2]}) = \frac{C}{2} \tilde{\theta}^{(0)}.$$ 

Another remarkable property of the space-time under consideration is that its metric can be expressed exclusively in terms of the Killing covectors:

$$ds^2 = \xi_{[1]}\xi_{[1]} - \xi_{[2]}\xi_{[2]} - \xi_{[6]}\xi_{[6]} - \xi_{[7]}\xi_{[7]}.$$  

(21)

It might seem that the explicit form of the Maxwell field as the source of the gravitational field of this simple electrovacuum space-time is already known being a special case of the more general solution given in [4], [13], but this is not exactly the case. We show here that there is a multitude of electromagnetic fields (in the sense of the field tensor and, of course, not only of the potential) which yield one and the same stress-energy tensor in the fixed four-geometry under consideration, and this is a perfectly special case in general relativity completely beyond the framework of the well known invariance of the stress-energy tensor with respect to the dual conjugation of the field tensor $F_{\mu\nu}$. Moreover, some of the seven Killing vectors being at our disposal, when multiplied by a suitable constant coefficient, not only satisfy the vacuum Maxwell equations in this space-time (which is only natural due to the well known Wald theorem, see [6] and — for applications to the case of test electromagnetic fields — [2]), but give together with the geometry (the gravitational field) of the space-time, self-consistent solutions of the Einstein–Maxwell equations, and this is not only one solution, but a multitude of self-consistent solutions in one and the same space-time. One of us (N. M.), in collaboration with J. Horský, developed and applied a new method of purposeful constructing exact self-consistent Einstein–Maxwell fields using Killing vectors of seed space-times [3]. Naturally, this method led always to generalizations of these seed space-times, the Killing vector having generated exact perturbations of seed geometries. Now we find that in this new special case, the Killing vector (in fact, four of them simultaneously), up to a constant factor directly related to the parameter in the metric tensor, already represents the electromagnetic four-potential of this self-consistent solution. And different Killing vectors (of these four) form different self-consistent solutions whose four-geometry, however, is exactly one and the same. Quite naturally, we came to this conclusion without any intention to find such a clear example or even look for it at all. Of course, since the geometries created by these different fields, exactly coincide, and Maxwell’s equations are linear with respect to the electromagnetic field, a superposition of these fields becomes automatic, without any apparent interaction between such electromagnetic fields. Only the constant parameter in the metric has
to be built by additive contributions of the coefficients by individual Killing vectors.

Let an electromagnetic four-potential be proportional to a Killing covector, \( A = k \xi \). The corresponding electromagnetic field tensor then is \( F_{\mu \nu} = k(\xi_{\nu \mu} - \xi_{\mu \nu}) = 2k\xi_{\nu \mu} \). The identity \( \xi_{\alpha \beta ; \gamma} - \xi_{\alpha \beta ; \gamma} = \xi^\delta R_{\alpha \beta \gamma \delta} \) and the Killing equation yield

\[
\xi_{\alpha \beta ; \gamma} = \xi^\delta R_{\alpha \beta \gamma \delta}.
\]  

Due to the structure of \( F_{\mu \nu} \) and (22), Maxwell’s equations take the form \( F_{\mu \nu ; \nu} = -2k\xi_{\nu} R_{\mu \nu} \), while \( F_{\mu \nu}^* ; \nu = 0 \) follows from the Ricci identities. Now it is clear that we have to consider only such Killing vectors which are orthogonal to the Ricci tensor (the electrovacuum case) which for (14) has components \( R_{\mu (\nu)} = C_{\mu}^{\alpha} (\delta_{\alpha 0} + \delta_{\alpha 3}) (\delta_{\nu 0} + \delta_{\nu 3}) \) (hence it is clear that the scalar curvature \( R = 0 \)). Since \( (\delta_{\mu 0} + \delta_{\mu 3}) \) (here the tetrad basis is used) is orthogonal to the five Killing vectors \( \xi_{[3]} \), \( \xi_{[4]} \), \( \xi_{[5]} \), \( \xi_{[6]} \), and \( \xi_{[7]} \) (the combination \( \xi_{[3]} - \xi_{[2]} \) is excluded since this is an exact form thus not producing any electromagnetic field), we have five candidates for four-potentials (1-forms).

This final proof is based on the desired form

\[
T = \frac{C^2}{2\pi} (\theta^{(0)} - \theta^{(3)}) \odot (\theta^{(0)} - \theta^{(3)})
\]

of the electromagnetic stress-energy tensor (here it is worth being mentioned that (23) has the standard canonical structure for a null electromagnetic field (cf. [5], [14]). In fact, only the Killing vector \( \xi_{[3]} \) is not successful in yielding the form (23), thus merely describing a test electromagnetic field in this space-time; all other four Killing vectors do indeed pertain to self-consistent Einstein–Maxwell solutions involving the space-time under consideration. Some pairs of them describe dually conjugated electromagnetic situations, and their linear combinations (with appropriate constant coefficients) correspond to ‘dual rotations’, but there are also completely different ones two of which we shall consider below.

In order to determine the electric and magnetic field vectors we introduce a reference frame described by the monad (see [8]) which we choose to be

\[
\tau = \theta^{(0)} = \xi_{[1]}
\]

in order to correspond to the rotating-pseudo-rotating basis (12). This reference frame is rotating, \( \omega = \frac{1}{2} \star (\tau \wedge d\tau) = \xi_{[3]} \), but has neither acceleration, \( G = -\star (\tau \wedge \star d\tau) = 0 \), nor expansion and shear since the rate-of-strain tensor vanishes, \( D_{\mu \nu} = \frac{1}{2} \mathcal{L}_\tau b_{\mu \nu} = 0 \) (cf. [8]). Here \( b_{\mu \nu} = g_{\mu \nu} - \tau_{\mu} \tau_{\nu} \) is the three-metric in the local subspace orthogonal to the monad and \( \mathcal{L} \) denotes the Lie derivative. With respect to this reference frame we split the electromagnetic field tensor in the electric and magnetic (co)vectors

\[
E = \star (\tau \wedge \star F), \quad B = \star (\tau \wedge F).
\]
The fourth Killing vector case. The electromagnetic four-potential, field tensor, and electric and magnetic (co)vectors are

\[ A_{[4]} = \sqrt{\frac{2\pi}{\kappa}} C x (dt - dz), \quad F_{[4]} = \sqrt{\frac{2\pi}{\kappa}} C \theta^{(1)} \wedge \left( \theta^{(0)} - \theta^{(3)} \right), \]
\[ E_{[4]} = \sqrt{\frac{2\pi}{\kappa}} C \theta^{(1)}, \quad B_{[4]} = \sqrt{\frac{2\pi}{\kappa}} C \theta^{(2)}. \]

Here we have constant mutually orthogonal electric and magnetic fields with equal magnitudes (a static pure null field). Formally, one may say that this solution contains an electromagnetic wave whose frequency is equal to zero.

The sixth Killing vector case. The electromagnetic four-potential, field tensor, and electric and magnetic (co)vectors are

\[ A_{[6]} = \sqrt{\frac{2\pi}{\kappa}} \left\{ \cos[C(t - z)] dx + \sin[C(t - z)] dy \right\}, \]
\[ F_{[6]} = \sqrt{\frac{2\pi}{\kappa}} C \left\{ \sin[C(t - z)] \theta^{(1)} - \cos[C(t - z)] \theta^{(2)} \right\} \wedge \left( \theta^{(0)} - \theta^{(3)} \right), \]
\[ E_{[6]} = \sqrt{\frac{2\pi}{\kappa}} C \left\{ \sin[C(t - z)] \theta^{(1)} - \cos[C(t - z)] \theta^{(2)} \right\}, \]
\[ B_{[6]} = -\sqrt{\frac{2\pi}{\kappa}} C \left\{ \cos[C(t - z)] \theta^{(1)} - \sin[C(t - z)] \theta^{(2)} \right\}. \]

When \( C > 0 \), this pure null electromagnetic field represents a left circularly polarized (positive helicity) plane monochromatic wave with frequency \( C \).

In all cases, the electromagnetic linear momentum density (coinciding with the Poynting covector in our units) is equal to

\[ S = \frac{1}{4\pi} \star (E \wedge \tau \wedge B) = -\frac{C^2}{2\pi C^{(3)}} \tag{26} \]

(see [8]); it is constant, directed along the positive \( z \) axis and does not depend on the sign of \( C \). The plane electromagnetic wave has its spin angular momentum in an opposite direction to that of the angular velocity of the reference frame. If \( C \) changes its sign, then the plane electromagnetic wave acquires negative helicity, and the relative situation continues to be as before.

All these solutions possess a semi-cylindrical symmetry (à la Wils), since the Lie derivatives with respect to the Killing vectors of the space-time, \( \mathcal{L}_\xi \), in general do not annihilate the electromagnetic field tensor: this property holds in all cases with respect to \( \xi^{[3]} \); moreover, for \( F_{[6]} \) and \( F_{[7]} \) this also occurs with respect to \( \xi^{[1]} \) and \( \xi^{[2]} \).

4. THE KERR–NEWMAN SPACE-TIME

We consider in this Section the well-known rotating space-time filled with
electromagnetic field which has well determined proper directions (eigenvectors) rigidly connected with the field distribution. Thus we can trace interrelations between the material properties (the electromagnetic field visualized via its stress-energy tensor) and their four-geometric description (the behavior of a properly chosen tetrad).

The rotating frame. The usual rotating tetrad in the Boyer–Lindquist coordinates is

\[
\theta^{(0)} = e^\alpha (dt + af d\phi), \quad \theta^{(1)} = e^\beta dr, \quad \theta^{(2)} = e^\gamma d\vartheta, \quad \theta^{(3)} = e^\delta \sin \vartheta d\phi \quad (27)
\]

where

\[
e^{2\alpha} = \frac{\Delta - a^2 \sin^2 \vartheta}{\rho^2}, \quad e^{2\beta} = \frac{\rho^2}{\Delta}, \quad e^{2\gamma} = \rho^2,
\]

\[
e^{2\delta} = \frac{\rho^2}{\Delta - a^2 \sin^2 \vartheta}, \quad f = \frac{a r^2 + a^2 - \Delta}{\Delta - a^2 \sin^2 \vartheta} \sin^2 \vartheta,
\]

\[
\Delta(r) = r^2 - 2Mr + a^2 + Q^2, \quad \rho(r, \vartheta) = r^2 + a^2 \cos^2 \vartheta.
\]

Hence, \(e^{2(\beta-\gamma)} = \Delta^{-1}, e^{2(\alpha+\delta)} = \Delta\), so that \(\sqrt{-g} = \rho^2 \sin \vartheta\). The \(\theta^{(0)}\) congruence obviously rotates: this can be seen as non-vanishing of \(\theta^{(0)} \wedge d\theta^{(0)}\).

One has to keep in mind that physically the rotation property of a frame of reference is related to a timelike congruence whose unit tangent vector is the monad \(\tau\) (see [8]) denoted here by \(\theta^{(0)}\), but in the Kerr–Newman case, as this can be seen from the above expressions, its square changes sign, at least with \(\phi\) remaining constant, when \(\Delta = a^2 \sin \vartheta\) (the well-known static limit), thus on this surface the tetrad is inadmissible. The situation is then similar to that of inadmissibility of the Boyer–Lindquist coordinates on the horizons (for simplicity, we speak on the exterior horizon only). However, the tetrad given above is inadmissible already on the surface of the static limit (which suggests the interpretation of the latter). Below this limit all four tetrad (co)vectors are spacelike, and only under the horizon the tetrad covector \(\theta^{(1)}\) can play the role of timelike congruence which is however non-rotating (instead we observe pseudo-rotation of \(\theta^{(0)}\), now being spacelike). Of course, in the region between the surfaces of static limit and horizon (excluding the horizon itself) one can still use easily normalizable timelike combinations of \(\theta^{(0)}\) and \(\theta^{(3)}\) as the new 0th tetrad covector, though it always serves only in a final radial region (the so-called ‘local stationarity’ of the Kerr–Newman space-time in the ergosphere). Thus it seems that there is an abrupt change from pseudo-rotation to pseudo-rotation when the horizon is being crossed, but this is not exactly the case: since the Boyer–Lindquist coordinates at the horizon are inadmissible, the exterior and interior (with respect to horizon) space-time regions are absolutely disjoint. The only way to deal with this problem is to introduce a system of synchronous coordinates which, however, does not rotate per se.

The pseudo-rotating frame. Another combination of terms in the Kerr metric in Boyer–Lindquist coordinates yields the pseudo-rotating (but not
rotating in the sense of the timelike congruence of $\theta^{(0)}$ orthonormal basis (the notations are now changed in all cases, essentially with the exception of $\Delta$ and $\rho$)

$$
\begin{align*}
\theta^{(0)} &= e^\alpha dt, \quad \theta^{(1)} = e^\beta dr, \quad \theta^{(2)} = e^\gamma d\vartheta, \quad \theta^{(3)} = e^\delta \sin \vartheta (d\varphi + aFdt), \\
e^{2\alpha} &= \frac{\rho^2 \Delta}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \vartheta}, \quad e^{2\beta} = \frac{\rho^2}{\Delta}, \quad e^{2\gamma} = \rho^2, \\
e^{2\delta} &= \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \vartheta}{\rho^2}.
\end{align*}
$$

Here

$$F = \frac{a(r^2 + a^2 - \Delta)}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \vartheta}$$

is a new function, though similar to $f(r, \vartheta)$ of the preceding basis choice. At $r = Q^2/2M$ the pseudo-rotation vanishes (it changes direction when crossing this sphere); the same occurs with the rotation in the preceding basis. The equation $(\rho^2 + a^2)^2 - \Delta a^2 \sin^2 \vartheta = 0$ has no real solutions for realistic values of the charge $Q$. This means that the only singularity of the pseudo-rotating tetrad occurs at the horizon ($\Delta = 0$), but this is a singularity of the basis (and of the system of coordinates) only.

The Kerr–Newman electromagnetic stress-energy tensor is related to the rotating tetrad $\theta^{(\alpha)}$ as to its proper basis; the pseudo-rotating tetrad is not built of eigenvectors of the electromagnetic field.

5. A PSEUDO-ROTATING SPACE-TIME

An example of a pseudo-rotating space-time was found in the electrovacuum case in [1] (see also [4], p. 222; but this result was not included in the new edition [13]). In this ‘static’ space-time the orthonormal covector basis is

$$
\begin{align*}
\theta^{(0)} &= b\rho^{-2/9} e^{(1/2)\alpha^2} \rho^{2/3} dt, \quad \theta^{(1)} = b\rho^{-2/9} e^{(1/2)\alpha^2} \rho^{2/3} d\rho, \\
\theta^{(2)} &= \rho^{2/3} d\vartheta, \quad \theta^{(3)} = \rho^{1/3} (dz + a\rho^{2/3} d\varphi),
\end{align*}
$$

being accompanied by the sourceless (outside the symmetry axis) electromagnetic field with the potential 1-form

$$
A = \frac{4a}{3} \sqrt{\pi} \phi dt = -\frac{4a}{3} \sqrt{\pi} td\varphi + \text{exact form}
$$

and the stress-energy tensor

$$
T = \frac{2a^2}{9\pi\rho^2} e^{-8/9} \rho^{2/3} \left( \theta^{(0)} \otimes \theta^{(0)} + \theta^{(1)} \otimes \theta^{(1)} - \theta^{(2)} \otimes \theta^{(2)} + \theta^{(3)} \otimes \theta^{(3)} \right).
$$
The tetrad basis (29) obviously is a proper one for the electromagnetic field stress-energy tensor (31), thus this source well matches the property of \( \theta^{(3)} \)-pseudo-rotation and *vice versa*. The electromagnetic field is either of purely electric or purely magnetic type (one case is merely the dual conjugate of the other), the latter permitting a more natural physical interpretation. Then the magnetic covector is the only non-trivial one in the above basis and directed along \( \phi \):

\[
B = \frac{4\sqrt{\pi}a \exp(-\frac{a^2}{2}\rho^{2/3})}{3b\sqrt{\pi}\rho^{10/9}} \theta^{(2)},
\]  

(32)

7. CONCLUDING REMARKS

In this paper we have seen that the rotation phenomenon has very different sides: (a) in general, it separates in two alternative cases, proper rotation and pseudo-rotation, or their combination, rotation being related to timelike vectors, and pseudo-rotation, to spacelike ones; (b) this phenomenon can be related to rotating congruence(s) and rotating tetrad(s), leaving its mark primarily upon metric; (c) from the viewpoint of the material contents of the space-time, it corresponds to rotation of eigenvector(s) of the stress-energy tensor; (d) if isometries are taken into account, rotating Killing vector field(s) should be considered, and this is the only method to locally deal with this phenomenon in a vacuum; (e) in a perfect fluid, the rotation is considered as that of the fluid’s four-velocity vector field, and in the \( r \)-form field theoretic description of fluids it is equivalent to a presence of inhomogeneity term in the dynamic field equation (which however cannot be interpreted as a source term, in contrast to the traditional treatment of such terms in the gravitational and electromagnetic equations). It is interesting that more than one timelike or spacelike Killing vectors with rotation or pseudo-rotation, respectively, may exist simultaneously (for example, there are even four rotating independent timelike Killing vectors in the Gödel space-time, see [11]) .

In the concrete examples of rotation and pseudo-rotation, we used here the electromagnetic field as a material contents of space-time, since in general relativity this field proved to be much richer of sufficiently simple and informative exact self-consistent solutions than any other type of distributed sources. Considering these examples, we not only illustrated the different sides of the rotation phenomenon, in particular, showing that pseudo-rotation frequently is an indispensable counterpart of rotation, but we also have drawn some new conclusions about the geometrical and physical properties of a specific choice of exact solutions: 1. The special case of conformally flat null Einstein–Maxwell field (Section 3) admits seven independent
Killing vectors, exclusively in terms of four of which its metric can be expressed. 2. This is in fact a multitude of self-consistent exact solutions with radically different null electromagnetic fields, but with one and the same space-time geometry, while several Killing vectors of this space-time serve as four-potentials for these electromagnetic fields (not merely test ones, as this could be normally expected). The symmetry group of the electromagnetic field is more restricted that that of the resulting space-time (semi-cylindrical symmetry à la Wils). 3. In the Kerr–Newman space-time (Section 4) it was shown that the (usual) rotating tetrad becomes pseudo-rotating inside the event horizon, still being built of eigenvectors of the electromagnetic stress-energy tensor, and the properties of the pseudo-rotating (but not rotating) tetrad outside the horizon were studied.

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