Adaptive Interpolation Wavelet and Homotopy Perturbation Method for Partial Differential Equations

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Abstract. The homotopy perturbation method proposed by Ji-Huan He has been developed to solve nonlinear matrix differential equations. This paper constructs an adaptive multilevel quasi-wavelet operator according to the interpolation wavelet theory, with which the nonlinear partial differential equations can be discretized adaptively in physical spaces as a matrix differential equation, its numerical solution can be obtained by using the homotopy perturbation method. Numerical results show that the homotopy perturbation method is not sensitive to the time step, so the arithmetic error mainly arises in the space step. Burgers equation is taken as examples to illustrate its effectiveness and convenience.

1. Introduction

As a new powerful tool in both of pure and applied mathematics, wavelet theory has being developed rapidly in recent years. Wavelet and multilevel systems are by now used very widely in many fields of science and technology such as signal analysis, data compression, pattern recognition and solutions of the partial differential equations. The major advantage of the wavelet analysis method is that it has excellent properties of localizations in both of space and frequency domains. For this reason, they seem particularly adaptable to approximate functions with local lack of regularity like the solutions of some partial differential equations. In recent years, many effective numerical method for PDEs based on wavelets have been proposed[1-3], such as the wavelet Galerkin method (WGM), the wavelet finite element method and the wavelet collocation method etc.

The homotopy perturbation method (HPM) proposed by He [4-5] is constantly being developed and applied to solve various nonlinear problems [6-11]. Unlike analytical perturbation methods, HPM doesn’t depend on small parameter which is difficult to find [12]. Furthermore, the computation precision of HPM in solving multi-degree-of-freedom nonlinear differential equations is better than traditional methods[9-10].

In this paper, we combined the interpolation wavelet transform with the construction method of adaptive interpolation operator together, and then a more efficiency multilevel interpolation wavelet collocation method was proposed. In this new method, the quasi-Shannon wavelet[13], instead of the auto-correlation function of Daubechies Scaling functions, was taken as an example. The compact support property of quasi-Shannon wavelet is better than Shannon wavelet. This property is useful for solving PDEs which have heavy gradient solutions. Furthermore, we combine HPM with the

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multilevel interpolation wavelet collocation methods together to obtain an efficient second-order time-
marching solver for time-dependent problems as long as a factorization of the differential operator is
available.

2. Construction of interpolation operator

The construction method of the interpolation operator in this paper is similar to that of Vasilyev et
al. [1-3], but interpolation wavelet transform is used in this paper.

Let \( \phi(x) \) denote an interpolation scaling function. A discrete point sequence of the scaling function
is defined as

\[
\phi_{j,k}(x) = \phi(2^{-j}x - k), \quad j \in Z, \quad k = 0,1,2,...,2^j.
\]

The subspaces of \( L^2(0,1) \) constructed from these discrete point sequence are defined as

\[
V_j := \text{span}\left\{ \phi_{j,k} \mid k = 0,1,2,...,2^j \right\} \subset L^2(0,1), \quad j \in Z
\]

\( \phi_{j,k}(x) \) has an interpolation property, such as \( \phi_{j,k}(2^{-j}x) = 2^j \delta_{n,k} \), so an interpolation operator
\( I_j : C^0(0,1) \rightarrow V_j \) can be defined as

\[
I_j f = \sum_{k=0}^{2^j} f(x_{j,k}) \phi_{j,k}, \quad x_{j,k} = k2^{-j}.
\]

The wavelet function subspace \( W_j \subset V_{j+1} \) is defined as

\[
W_j = \text{span}\left\{ \psi_{j,k} \mid k = 0,1,2,...,2^j - 1 \right\} \subset L^2(0,1), \quad j \in Z
\]

where

\[
\psi_{j,k} = 2^{j/2} \phi_{j+1,2k+1}
\]

Letting \( y_{jk} = x_{j+1,k+1} \), the interpolation property of wavelet function \( \psi_{j,k} \) can be expressed as

\[
\psi_{j,k}(y_{jk}) = 2^j \delta_{kn}
\]

\[
\psi_{j,k}(y_{j',n}) = 0, \quad \forall j' < j.
\]

It is easy to check that

\[
V_{j+1} = V_j \oplus W_j
\]

Such multi-resolution analysis was extensively investigated in [7], where several important
properties were stated. First of all, it is possible to define an interpolating wavelet transform mapping
any continuous function \( f \) into a sequence of its coefficients \( \{ \beta_{j,k} \}, \{ \alpha_{j,k} \}, \{ \alpha_{j+1,k} \}, ... \). Any
function \( f \in C^0(0,1) \) may be reconstructed from its transform by means of

\[
f \approx f_j = \sum_{k=0}^{2^j} \beta_{j,k} \phi_{j,k} + \sum_{j',k}^{j+1} \alpha_{j,k} \psi_{j,k}
\]

Under a minimal regularity assumption the coefficients \( \beta_{jk} \) and \( \alpha_{jk} \) have the following meaning[7]
This formula clearly shows that the wavelet coefficients $\alpha_{j,k}$ is the interpolation error. In other words, the size of the wavelet coefficients gives us information on the regularity of the analyzed function, analogously to what happens with orthonormal wavelets.

Based on the theory of interpolation wavelet transform and Vasilyev’s construction method of multilevel interpolation operator, the interpolating operator based on multilevel interpolating wavelet can be deduced as follows

$$I_i(x) = \sum_{k=0}^{2^j} R_{k,j}^{l,j}(x) \psi_{j,k}(x) + \sum_{j=1}^{J-1} \sum_{k \in \mathbb{Z}} C_{k,j}^{l,j} \psi_{j,k}(x),$$

where

$$Z_j = \{0,1,2,\ldots,2^j\},$$

$$C_{k,a}^{l,j} = R_{2^j+1,j} - \sum_{k=0}^{2^{j-1}} R_{k,a}^{l,j} \phi_{j,k}(x_{j+1,2^{j+1}}) - \sum_{k_{j-1}}^{j-1} \sum_{k_{j-1}}^{j-1} C_{k,a}^{l,j} \psi_{j,k}(x_{j+1,2^{j+1}}),$$

$$j = j_0 + 1, j_0 + 2,\ldots, J-1, \quad k = 0,1,2,\ldots,2^j-1.$$

The operator $C_{k,a}^{l,j}$ maps a set of functional values at the $J$-level of resolution into a set of wavelet coefficients at $j$-level. For $j=j_0$

$$C_{k,a}^{l,j} = R_{2^{j+1},j}^{l,j} - \sum_{k=0}^{2^{j-1}} R_{k,a}^{l,j} \theta_{j,a}(x_{j+1,2^{j+1}}),$$

where the operator $R_{k,a}^{l,j}$ is the restriction operator defined as[1]

$$R_{k,a}^{l,j} = \begin{cases} 1, & x_l = x_m, \\ 0, & \text{otherwise.} \end{cases}$$

Since the restriction operator is known, it is easy to calculate the numerical solution of the interpolation operator from equations (11)–(13). The $m$-th derivative of $I_i(x)$ is

$$D_{i}^{(m)}(x) = \sum_{k=0}^{2^j} R_{k,a}^{l,j} \theta_{j,a}^{(m)}(x) + \sum_{j=1}^{J-1} \sum_{k \in \mathbb{Z}} C_{k,a}^{l,j} \psi_{j,a}^{(m)}(x).$$

From the definition of the interpolating operator, it is easy to deduced the following relation:

$$SR(x) = \sum_{k=0}^{2^j} R_{k,a}^{l,j} \theta_{j,a}(x) = \begin{cases} \theta_{j,a}(2^{j-1}x), & \text{mod}(i, 2^{j-1}) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

For compact support function, the support interval is supposed as $[-L, L]$. If $j$ and $x$ is given, the function $\theta_{j+1,2^{j+1}}(x)$ is non-zero only when the parameter $k$ belongs to a certain interval $[N_1, N_2]$, that is

$$|x - x_{j+1,2^{j+1}}| < \frac{L}{2^{j+1}}, \quad j \in \mathbb{Z}.$$
\[ x_{j+1,2k+1} = a + (2k + 1) \frac{b - a}{2^{j+1}}. \]  

(18)

Substituting equation (18) into equation (17), we have the following relation:

\[ \frac{2^j (x - a)}{b - a} - \frac{L}{2(b - a)} - \frac{1}{2} \leq k \leq \frac{2^j (x - a)}{b - a} + \frac{L}{2(b - a)} - \frac{1}{2}. \]  

(19)

Let \( n_1 = \frac{2^j (x - a)}{b - a} - \frac{L}{2(b - a)} - \frac{1}{2} \) and \( n_2 = \frac{2^j (x - a)}{b - a} + \frac{L}{2(b - a)} - \frac{1}{2} \), and then the expression of \( N_1 \) and \( N_2 \) can be given as following:

\[
N_1 = \begin{cases} 
0, & n_1 \leq 0, \\
2^j - 1, & n_1 \geq 2^j - 1, \\
n_1, & \text{otherwise};
\end{cases}
\]  

(20)

\[
N_2 = \begin{cases} 
0, & n_2 \leq 0, \\
2^j - 1, & n_2 \geq 2^j - 1, \\
n_2, & \text{otherwise}.
\end{cases}
\]  

(21)

and equations (11)–(13) and (12) can be modified as

\[ I_j(x) = SR(x) + \sum_{j=1}^{J} \sum_{k=1}^{N_j} C_{k,n}^{j,j} \psi_{j,k}(x) \]  

(22)

where the expressions of \( N_1 \) and \( N_2 \) are the same as equations (20) and (21).

\[ C_{k,n}^{j,j} = R_{2k+1,n}^{j+1,j} - SR(x_{j+1,2k+1}) - \sum_{j=1}^{J} \sum_{k=1}^{N_j} C_{k,n}^{j,j} \psi_{j,k}(x_{j+1,2k+1}) \]  

(23)

where

\[ j = j_0 + 1, j_0 + 2, \ldots, J - 1. \]

\[ N_1 = \begin{cases} 
\frac{1}{2} \left( \frac{2k + 1}{2^{j-h}} - \frac{L}{b - a} - 1 \right), & \text{if } N_1 < 0, \quad N_1 = 0,
\end{cases} \]

\[ N_2 = \begin{cases} 
\frac{1}{2} \left( \frac{2k + 1}{2^{j-h}} + \frac{L}{b - a} - 1 \right), & \text{if } N_2 > 2^h, \quad N_2 = 2^h.
\end{cases} \]

(24)

(25)

For \( j=j_0 \), equations (13) and (15) can be written as:

\[ C_{k,d}^{j_0,j} = R_{2k+1,d}^{j+1,j} - SR(x_{j_0+1,2k+1}) \]  

(24)

\[ D_{i}^{(m)}(x) = SR(x) + \sum_{j=1}^{J} \sum_{k=1}^{N_j} C_{k,n}^{j,j} \psi_{j,k}^{(m)}(x) \]  

(25)

where the expressions of \( N_1 \) and \( N_2 \) are the same as equation (20) and (21).
3. Construction of adaptive interpolating operator based on interpolating wavelet
It is well known that a wavelet transform has the function for detecting singularities or sharp transitions of signal, that is to say, in those positions where the signal is singular or sharp transitions, the absolute value of any wavelet coefficient is bigger than that in other position; In addition, a wavelet function has compact support property, and so those sharp positions of signal can be detected. In solving partial differential equations, this property can be used to construct adaptive calculation method. For collocation method, it is necessary to setup a threshold \( \varepsilon \) so that the collocation points corresponding to wavelet coefficients less than \( \varepsilon \) can be neglected. For convenience, let us define two integer subsets of integers \( Z^J \) and \( Z^c \). \( Z^c \) is the set of the indices of collocation points whose wavelet coefficients are above a certain threshold at the \( J \) level of resolution, and \( Z^c \subset Z^J \) is the set of the indices of the ultimate set of collocation points, which are above a certain threshold used in the interpolation, that is

\[ Z^c = Z^1 \cup Z^2 \cup \cdots \cup Z^J \]

The detailed procedures have been proposed in [1]. In this paper, we give the adaptive interpolating operator and its \( m \)-th derivative corresponding to the interpolating wavelets as follows

\[ I_i(x) = SR(x) + \sum_{j=j_0(k \in Z^J)}^{J-1} \sum_{k \in Z^J} C_{j,k}^{i,j} \psi_{j,k}(x), \quad i \in Z^c \]  

(26)

where

\[ C_{j,k}^{i,j} = R_{j+1,j}^{k+1,k} - SR(x_{j+1,k+1}) - \sum_{j_{1}=j_{0}(k \in Z^J)}^{J-1} \sum_{k_{1} \in Z^J} C_{j_{1},k_{1}}^{i,j} \psi_{j_{1},k_{1}}(x_{j_{1}+1,k_{1}}), \]

(27)

If \( j=j_0 \), then

\[ C_{j_0,k}^{i,j} = R_{j+1,j}^{k+1,k} - SR(x_{j_0+1,k+1}), \quad k \in Z^{j_0}, i \in Z^c. \]

(28)

The \( m \)-th derivative of \( I_i(x) \) is

\[ D_i^{(m)}(x) = SR(x) + \sum_{j=j_0(k \in Z^J)}^{J-1} \sum_{k \in Z^J} C_{j,k}^{i,j} \psi_{j,k}^{(m)}(x), \quad i \in Z^c \]

(29)

4. Coupling technique of HPM and Wavelet method on Burgers equation
Consider the Burgers equation

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{1}{Re} \frac{\partial^2 u}{\partial x^2}, \quad x \in [0,2], \quad t \geq 0, \]

(30)

with the initial and boundary conditions

\[ u(x,0) = \sin(\pi x), \quad u(0,t) = u(2,t) = 0, \]

(31)

where \( t \) denotes time and \( Re \) denotes Reynolds number.

Following the classical collocation approach and the definition of the operator \( I_i(x) \) in (3), the approximating formulation \( u_j(x) \) of a function \( u(x) \) can be written as
\[ u_j(x) = \sum_{i \in \mathbb{Z}^n} I_i(x) u_{ji} \]  \hspace{1cm} (32)

where \( I_i(x) \) is the adaptive interpolating operator \((22)\). Substituting \((32)\) into \((30)\) lead to a system of nonlinear ordinary differential equations as follows:

\[ \sum_{i \in \mathbb{Z}^n} u_j(x_n,t) \left[ \frac{1}{\Re} D_n^{-1}(x_k) - u_j(x_k,t)D_n^n(x_k) \right] \frac{\partial u_j(x_k,t)}{\partial t} = \frac{\partial u_j(x_k,t)}{\partial t}, \]  \hspace{1cm} (33)

where \( k \in \mathbb{Z}^c \). The corresponding vector expression is

\[ \frac{\partial}{\partial t} V_j = M_0 V_j + M_1(V_j)V_j, \]  \hspace{1cm} (34)

where

\[ V_j = \left( u_j(x_0,t), u_j(x_1,t), \ldots, u_j(x_{2^n},t) \right)^T, \]

\[ M_0(k,n) = m_{e,n}^0 = \frac{1}{\Re} D_n^{-1}(x_k), \quad k, n \in \mathbb{Z}^c, \]

\[ M_1(k,n) = m_{e,n}^1 = -u_j(x_k,t)D_n^{-1}(x_k), \quad k, n \in \mathbb{Z}^c, \]  \hspace{1cm} (35)

According to HPM. To equation \((34)\), we can construct the linear homotopy function as follows

\[ \left[ \frac{dV_j}{dt} - M_0 V_j - M_1(A)V_j \right] + \varepsilon \left[ \frac{dV_j}{dt} - M_0 V_j - M_1(V_j)V_j - \left( \frac{dV_j}{dt} - M_0 V_j - M_1(A)V_j \right) \right] = 0, \]  \hspace{1cm} (36)

by simplification, which becomes:

\[ \frac{dV_j}{dt} = M_0 V_j + M_1(A)V_j - \varepsilon \left[ M_1(A)V_j - M_1(V_j)V_j \right] \]  \hspace{1cm} (37)

where, \( A \) is a known initial value, \( \varepsilon \in [0,1] \) is the homotopy parameter. According to the perturbation theory, equation \((38)\) can be expressed as series expansion of \( \varepsilon \),

\[ V_j = V_j^0 + \varepsilon V_j^1 + \varepsilon^2 V_j^2 + \cdots \]  \hspace{1cm} (38)

Substituting equation \((39)\) into equation \((40)\), and letting the coefficients of two power of \( \varepsilon \) which have same exponent between two side of the equation is equal, we can obtain equations as follows:

\[ \varepsilon^0: \]

\[ \frac{dV_j^0}{dt} = \left( M_0 + M_1(A) \right) V_j^0 \]  \hspace{1cm} (39)

\[ \varepsilon^1: \]

\[ \frac{dV_j^1}{dt} = \left[ M_0 + M_1(A) \right] V_j^1 + \left[ M_1(V_j^0) - M_1(A) \right] V_j^0 \]  \hspace{1cm} (40)
\( \varepsilon^2 \cdot \frac{dV_j^2}{dt} = \left[ M_0 + M_1(A) \right] V_j^2 + \left[ M_1(V_j^0) - M_1(A) \right] V_j^1 + M_1(V_j^1) V_j^0 \) \hspace{1cm} (42)

Equation (40) is the system of homogeneous linear ODEs and its general solution is:

\[ V_j^0(t) = e^{H^1} A \] \hspace{1cm} (43)

where \( H = M_0 + M_1(A) \). Equation (41) and equation (42) are the system of inhomogeneous linear ODEs, the general solutions are

\[ V_j^1(t) = e^{H^1} \left( H^{-1} r_0 \right) - H^{-1} r_0 \] \hspace{1cm} (44)

and

\[ V_j^2(t) = e^{H^1} \left( H^{-1} r_i \right) - H^{-1} r_i \] \hspace{1cm} (45)

respectively, where \( r_0 = (M_1(V_j^0) - M_1(A)) V_j^0 \), \( r_i = (M_1(V_j^0) - M_1(A)) V_j^1 + M_1(V_j^1) V_j^0 \). The matrix exponential function \( e^{H^1} \) can be calculated accurately in precise integration method. Substituting equation (43)–equation (45) into equation (39) and letting \( \varepsilon = 1 \), the numerical solution of equation (37) can be obtained.

To improve the computational accuracy, the time interval \([0, t]\) can be divided into sections uniformly by taking the time step as \( \tau \), and a series of time nodes can be expressed as:

\[ t_0 = 0, t_1 = \tau, \ldots, t_k = k \cdot \tau, \ldots \]

Taking the solution at time \( t_k \) instead of ‘\( A \)’ as the initial value in equation (43) and equation (44), the recurrence formula can be obtained.

5. Numerical result and discussion

In this section, the adaptive method which is proposed in this paper is used to calculate Burgers equation (30) with Reynolds number \( Re = 1000 \). Quasi-Shannon wavelet is used in calculation, the regular width parameter \( \sigma = 3 \triangle \), scale parameter \( j = 15 \), time step \( \tau = 0.02 \), threshold parameter \( \varepsilon = 0.005 \). The evolution of the solution of Burgers equation from the uniformly smooth distribution to the shock structure causes the growth of the wavelet coefficients corresponding to the smaller scales, which in turn results in the refinement of the grid. Figure 1 shows the progressive refinement of the irregular grid with the decreasing of the shock thickness.

6. Conclusion

Coupling technique of HPM and the dynamically adaptive interpolating wavelet collocation method based Vasilyev’s method was developed to solve partial differential equations in a finite domain. In this method, the adaptive interpolating operator must be reconstructed at different moment because the amount and the position of collocation points are different at different time. The constructing efficiency of the interpolating operator is crucial to the efficiency of the adaptive method. The time complexity of the method in this paper is less than the method in [2]. Quasi-Shannon wavelet [13] has analytical expression, and is helpful to improve calculation efficiency. Furthermore, the method becomes more efficiency and accurate by means of HPM. The numerical results for Burgers equation illustrate its effectiveness and convenience.

Acknowledgments

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Figure 1. Evolution of the solution and collocation points for the solution of the Burgers equation using the Quasi-Shannon wavelet.

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