Asymptotic stabilization of underactuated surface vehicles with actuator saturation

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ABSTRACT
This paper investigates the problem of global asymptotic stabilization of underactuated surface vessels (USVs) with input saturation. A novel input transformation is presented, so that the USV system can be transformed to a cascade structure. For the obtained system, the improved fractional power control laws are proposed to ensure input signals do not exceed actuator constraints and enhance convergence rates. Finally, stabilization and parameter optimization algorithm of USVs are proposed. Simulations are given to demonstrate the effectiveness of the presented method.

Subjects Autonomous Systems, Robotics
Keywords Input saturation, Asymptotically stable, Globally stable, USVs

INTRODUCTION
Underactuated surface vehicles (USV) stabilization has many practical uses in engineering practice, such as monitoring, rescue, and fishing. Generally, the USV has no side thruster considered, whose reduction always causes by an actuator failure or a deliberate decision. For example, the limitation of the actuator number due to, e.g., cost and weight considerations. This paper establishes a feedback control law to stabilize three states containing both position and orientation with only two available control inputs.

The reference (Pettersen & Egeland, 1996) shows that USVs have no asymptotic stability on desired equilibrium points with time-invariant continuous control laws. The reason is that the dynamic models of USVs have nonholonomic constraints, so they do not meet Brockett’s necessary condition (Brockett & Millman, 1983). Furthermore, the existing control approaches designed for conventional nonholonomic systems cannot stabilize USVs directly since the dynamics of a USV are not drift-less (Oriolo & Nakamura, 1991). For these reasons, significant interest exist in stabilization of USVs as evidenced by various control schemes presented for such a problem (Reyhanoglu, 1996; Pettersen & Egeland, 1996; Ghommem et al., 2006; Ma, 2009; Pettersen & Nijmeijer, 2000; Zhang & Wu, 2014; Mazenc, Pettersen & Nijmeijer, 2002; Dong & Yi, 2005; Ma & Xie, 2013; Zhang & Wu, 2015; Xie & Ma, 2015). To name some, the references (Reyhanoglu, 1996; Pettersen & Egeland, 1996) proposed control laws that can stabilize the system to a small neighborhood of origin. Pettersen & Nijmeijer (2000) proposed a control law guaranteeing the semi-global asymptotic stability of USVs. Ghommem et al. (2006), Ma (2009), Zhang & Wu (2014), Mazenc, Pettersen & Nijmeijer (2002) and Dong & Yi (2005) proposed control methods that can globally asymptotically stabilize USVs. Other related researches on the
stabilization control of USVs include, but are not limited to (Ma & Xie, 2013; Zhang & Wu, 2015; Xie & Ma, 2015) and many references therein. To enhance convergence rates (Zhang & Guo, 2018), presents a fast stabilization approach with fractional power terms. Zhang & Guo (2020) proposes the fixed-time control method stabilizing USVs with dead-zones. Further (Guo & Zhang, 2020) considers the case that both yaw constraints and disturbances exist.

This paper is concerned with actuator saturation in the USV stabilization control problem as supplementary studies of the literature (Guo & Zhang, 2020). The above researches have one thing in common: the assumption that there is no limit on the inputs. However, the functional USV movement usually relies on motor propulsion limited by the bounded amount of torque. Thus signals of the control algorithm for USVs should always be constrained. Under such constraints, the stability of the closed system with conventional methods is impacted. Hence, with no processing, the input saturation may affect system accuracy and speed and even cause system instability. Such an issue leads to significant challenges for the USV controller design. In short, developing a stabilization scheme for a USV with input saturation is an arduous task yet has theoretical and practical significance. To this end, the input saturation problem is solved in tracking control of USVs (such as Huang et al. (2015)). However, due to the difference of dynamics between stabilization and tracking, USV stabilization control requires its own anti-input saturation control method. To the best of the authors’ knowledge, possible solutions to the issue of actuator saturation include model predictive control (MPC) (Li & Yan, 2016), which is restricted to the unsolvable or singular point problems.

Motivated by the above discussion, this paper aims to address the stabilization problem of USVs with input saturation. The main contributions of this paper are twofold:

1. For the USVs with input saturation, a novel input transformation method is proposed in this paper. Thus, the USV system with input saturation can be transformed to the cascade structure.
2. An improved fractional power control method is presented to avoid inputs violating actuator saturation constraints.

The organization of this paper is as follows. The 2nd section is to model the USVs and establish the objective. The 3rd section contains input and state transitions so that the USV stabilization problem is solved by stabilizing the obtained two cascaded subsystems. Then the control law of the original USV system is given. This section also provides proof of the global asymptotic stability and the method of parameter optimization. The last section shows the numerical simulations and discussions.

**SYSTEM MODELING AND THE OBJECTIVE**

**Kinematic model**

Consider a USV as depicted in Fig. 1 with an inertial frame \(O_G\) and a body-fixed frame \(O_E\). Denote \(x\) and \(y\) as positions of the USV in an inertial frame, and \(\psi\) as the yaw angle.
relative to the geographic north. The kinematics and of the USV with mismatched and matched disturbances are given as follows:

\[
\begin{align*}
\dot{x} &= u \cos \psi - v \sin \psi, \\
\dot{y} &= u \sin \psi + v \cos \psi, \\
\dot{\psi} &= r,
\end{align*}
\]

(1a)

\[
\begin{align*}
\dot{u} &= -\frac{d_{11}}{m_{11}} u + \frac{m_{22}}{m_{11}} ur + \frac{s_{\text{max}}(\tau_u)}{m_{11}}, \\
\dot{v} &= -\frac{d_{22}}{m_{22}} v - \frac{m_{11}}{m_{22}} ur, \\
\dot{r} &= -\frac{d_{33}}{m_{33}} r - \frac{m_{22} - m_{11}}{m_{33}} uv + \frac{s_{\text{max}}(\tau_r)}{m_{33}},
\end{align*}
\]

(1b)

where \( x, y \) denotes the coordinate of surface vessel mass center in the earth-fixed frame, \( \psi \) is the orientation of vessel. Variables \( u, v, r \) are the velocities in surge, sway and yaw respectively. Parameters \( m_{11}, m_{22}, m_{33} \) are the inertia coefficients, \( d_{11}, d_{22}, d_{33} \) are the damping coefficients of the vessel. \( \tau_u \) is the force of surge, \( \tau_r \) is the torques of yaw.

\[
m_1 = m - X_u, m_2 = m - Y_v, m_3 = m - N_r, d_1 = X_{u'}, d_2 = Y_{v'}, d_3 = N_{r'}, X_u, Y_v, N_r, X_{u'}, Y_{v'}, N_{r'} \]

are the coefficients from Taylor series, which are described in Fossen (1994). The saturation function is defined as follows:

\[
s_{\beta}(z) = \begin{cases} 
    z, & \text{if } |z| \leq \beta, \\
    \beta \cdot \text{sign}(z), & \text{if } |z| > \beta,
\end{cases}
\]

(2)
where $\beta$ and $z$ denote the saturation constant and arbitrary variable. In terms of Eqs. (1a) and (1b), we can then define that $\tau_{umax}$ and $\tau_{rmax}$ are, respectively, the maximum of input variables $\tau_u$ and $\tau_r$.

**The objective**

The objective of this paper is to design the control inputs $\tau_u$ and $\tau_r$ that can globally asymptotically stabilize the USV system as modeled in (1a) and (1b) subject to input saturation, namely, the following holds true for any initial conditions

$$\lim_{t \to +\infty} [x(t), y(t), \psi(t), u(t), v(t), r(t)] = 0.$$  

The saturation restricts the controllability of USV systems, thus some references (such as Reyhanoglu, 1996, Pettersen & Egeland, 1996, Ghomem et al., 2006, Ma, 2009, Pettersen & Nijmeijer, 2000, Mazenc, Pettersen & Nijmeijer, 2002, Dong & Yi, 2005) are unable to be followed in the presence of saturation making it quite challenging to stabilize system (1a) and (1b) to the origin. As a compensatory research, this paper emphasize the input saturation problem instead of disturbances, input dead-zones and output constraints. This is because the methods in Guo & Zhang (2020) can be directly applied to solve these problems with some saturation constraints. The only work that deals with stabilization of USVs with yaw constraints is Li & Yan (2016). However, it is based on MPC and relies on singular state transformations to ensure iterative feasibility.

**MODEL TRANSFORMATION**

Consider the system (1a) and (1b), similar to the previous literatures about stabilization of USVs, the state transformation is still utilized. However, in terms of the input saturation, existing input transformed methods can not be applied directly. Hence a novel input transformation is proposed in this section under the following assumption.

**Assumption 1:** The parameters $d_{11}$, $d_{22}$ and $d_{33}$ satisfy the following condition:

$$\left(\frac{d_{22}}{d_{11}}\right)^2 < m_{11}d_{11} \cdot \min\left\{\frac{d_{11}}{m_{11}}, \frac{2d_{22}}{m_{22}}, \frac{d_{33}}{m_{33}}\right\}.$$  

This assumption is relatively strong, and its specific meaning refers to a surface vehicle with a relatively small volume and a slight difference between the horizontal and vertical directions. To facilitate the control design, first the state transformation is cited from Pettersen & Egeland (1996). Define the vector $\vartheta = [\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6]^T$ with

$$\begin{align*}
\vartheta_1 &= x \cos \psi + y \sin \psi, \\
\vartheta_2 &= -x \sin \psi + y \cos \psi + \frac{m_{22}}{d_{22}}v, \\
\vartheta_3 &= \dot{\psi}, \vartheta_4 = \nu, \\
\vartheta_5 &= -\vartheta_1 - \frac{m_{11}}{d_{22}}u, \vartheta_6 = r.
\end{align*}$$  

According to the system (1a) and (1b), one has
\[
\begin{align*}
\dot{\theta}_1 &= -\frac{d_{22}}{m_{11}} \dot{\theta}_1 - \frac{d_{22}}{m_{11}} \dot{\theta}_5 + \dot{\theta}_2 \dot{\theta}_6 - \frac{m_{22}}{d_{22}} \dot{\theta}_4 \dot{\theta}_6, \\
\dot{\theta}_4 &= -\frac{d_{22}}{m_{22}} \dot{\theta}_4 + \frac{d_{22}}{m_{22}} \dot{\theta}_6 (\theta_1 + \theta_5), \\
\dot{\theta}_2 &= \theta_5 \dot{\theta}_6; \dot{\theta}_3 = \dot{\theta}_6, \\
\dot{\vartheta}_3 &= (1 - d_{11}d_{22}^{-1}) \vartheta_7 \vartheta_6 - d_{22}^{-1} \vartheta_6; \vartheta_6, \\
\dot{\vartheta}_6 &= -d_{33}m_{33}^{-1} \vartheta_6 - (m_{11} - m_{22})m_{33}^{-1} \vartheta_6 + m_{33}^{-1} \tau_{\text{max}} (\tau_r). 
\end{align*}
\] (4a)

For the transformed system, there are following results cited from previous works:

**Lemma 1** Stabilization of system (1a) and (1b) is equivalent to that of system (4a) and (4b).

**Lemma 2** If subsystem (4b) is globally asymptotically stable, then systems (4a) and (4b) are globally asymptotically stable.

According to the Lemmas 1 and 2, the global asymptotic stability of the USV system in (1a) and (1b) can be achieved by designing a global asymptotical control law for subsystem (4b). In addition, for the transformed system (4b), input transformations was usually introduced to further simplify calculations. However, for USVs with saturations, input transformations in previous works (such as Reyhanoglu, 1996; Pettersen & Egeland, 1996; Ghommam et al., 2006; Ma, 2009; Pettersen & Nijmeijer, 2000; Mazenc, Pettersen & Nijmeijer, 2002; Dong & Yi, 2005) cannot be used. Hence, in this paper we propose a novel input transformation for system (4b) to deal with input saturations.

Define parameters \(\lambda_1, \lambda_2, \lambda_3, \beta_u, \) and \(\beta_r\) satisfying the following conditions:

\[
\begin{align*}
-\frac{m_{11}}{m_{22}} \lambda_1 + \frac{m_{22}}{m_{11}} \lambda_2 - \frac{m_{22} - m_{11}}{m_{33}} \lambda_3 &= 0, \\
\beta_u &= \min \{\tau_{\text{max}}, \Delta_0, \Delta_1, \Delta_2\}, \\
\epsilon_1 - \epsilon_2 < \beta_r &= \min \{\gamma_1 \beta_u, \epsilon_1 + \epsilon_2, \tau_{\text{max}}\},
\end{align*}
\] (5)

where constant parameters \(\gamma_1, \gamma_2, \gamma_3, \epsilon_1, \epsilon_2, \Delta_0, \Delta_1, \Delta_2\) and \(\Delta_3\) are defined as:

\[
\begin{align*}
\gamma_1 &= \sqrt{\frac{m_{33}d_{33}\lambda_1 \gamma_3}{\lambda_3 (d_{22} - d_{11})^2}} \left[ 1 - \left(\frac{(d_{22} - d_{11})^2}{m_{11}d_{11} \gamma_3}\right) \right], \\
\gamma_2 &= \frac{2(m_{11} - m_{22})}{\sqrt{\lambda_1 \lambda_2}}; \gamma_3 = \min \left\{ \frac{d_{11}}{m_{11}}, \frac{2d_{22}}{m_{22}}, \frac{d_{33}}{m_{33}} \right\}, \epsilon_1 = \frac{m_{33}d_{33}\gamma_3}{\lambda_3 \gamma_2}, \\
\epsilon_2 &= \frac{\epsilon_1^2 - \lambda_1 m_{33}d_{33} \beta_u^2}{\lambda_3 m_{11}d_{11}}; \Delta_0 = \gamma_3 \sqrt{\frac{m_{11}m_{33}d_{33}d_{11}}{\lambda_1 \lambda_3}}, \Delta_1 = \frac{2\gamma_1 \gamma_3 m_{33}d_{33}}{\lambda_3 \gamma_2 \left(\gamma_1^2 + \frac{\lambda_1 m_{33}d_{33}}{\lambda_3 m_{11}d_{11}}\right)}, \\
\Delta_2 &= \frac{\lambda_3 m_{11}d_{11}}{\lambda_3 m_{33}d_{33}} \left[ \frac{\epsilon_1^2 - \text{sign}(\Delta_3) + 1}{2} \right]; \Delta_3 = \epsilon_1 - \tau_{\text{max}}.
\end{align*}
\]

According to the Assumption 1 and inequality (6), we can have

\[
1 - \frac{(d_{22} - d_{11})^2}{m_{11}d_{11} \gamma_3} > 0, \epsilon_1^2 - \frac{\lambda_1 m_{33}d_{33} \beta_u^2}{\lambda_3 m_{11}d_{11}} > 0, \epsilon_1 - \epsilon_2 < \gamma_1 \beta_u, \epsilon_1 - \epsilon_2 < \tau_{\text{max}}.
\]
inequalities indicate that $\gamma_1$ and $\varepsilon_2$ are real and the inequality (7) is hold. Then for system (4b), we have the following result:

**Lemma 3** For the USV system (1a) and (1b), if we choose

$$
\tau_u = s_{p_1}([d_{22} - d_{11}]u - d_{22}s_{p_1} (\omega_1)], \tau_r = s_{p_1}[(m_{11} - m_{22})uv + m_{33}s_{p_1} (\omega_2)].
$$

(8)

there must be positive constants $\beta_1$ and $\beta_2$ satisfying that $\beta_1 < \frac{\beta_u}{d_{22}} \frac{(d_{22} - d_{11}) |u|}{d_{22}}, \beta_2 < \frac{\beta_u}{m_{33}} \frac{(m_{11} - m_{22}) |uv|}{m_{33}}$, where $\beta_u < \tau_{\text{umax}}$ and $\beta_r < \tau_{\text{rmax}}$ are positive constants, $\omega_1$ and $\omega_2$ are virtual inputs.

**Proof:** Define the Lyapunov function as $V_0 = \frac{1}{2} u^2 + \frac{\lambda_2}{2} v^2 + \frac{\lambda_3}{2} r^2$, whose derivative is

$$
\dot{V}_0 = -\frac{d_{11}}{m_{11}} \lambda_1 u^2 - \frac{d_{22}}{m_{22}} \lambda_2 v^2 - \frac{d_{33}}{m_{33}} \lambda_3 r^2 + \frac{\tau_u}{m_{11}} \lambda_1 u + \frac{\tau_r}{m_{22}} \lambda_2 r.
$$

Together with the fact that $|\tau_u| \leq \beta_u$ and $|\tau_r| \leq \beta_r$:

$$
\dot{V}_0 \leq -\gamma_3 V_0 + \frac{\lambda_1 \beta_u^2}{2m_{11}d_{11}^3} + \frac{\lambda_3 \beta_r^2}{2m_{33}d_{33}^3}.
$$

(9)

According to the inequality in (9), we can have

$$
u \leq \sqrt{\frac{2\beta}{\lambda_1}}, v \leq \sqrt{\frac{2\beta}{\lambda_2}}, r \leq \sqrt{\frac{2\beta}{\lambda_3}}.
$$

(10)

where $\beta = \frac{\lambda_1 \beta_u^2}{2m_{11}d_{11}^3} + \frac{\lambda_3 \beta_r^2}{2m_{33}d_{33}^3}$. According to the Eq. (6), we can have

$$
\beta_u < \gamma_1 \beta_u, m_{33}d_{33}^3 \gamma_3 \lambda_3^{-3} \epsilon < \beta_r < \frac{m_{33}d_{33}^3 \gamma_3}{\lambda_3 \gamma_2} + \epsilon. \text{ The above inequality implies}
$$

$$
[1 - \frac{(d_{22} - d_{11})^2}{m_{11}d_{11}^3}]^2 \frac{\beta_u}{\lambda_1} \frac{\beta_r}{\lambda_2} \text{ and } \beta_r > \gamma_2 \left( \frac{\beta_u^2}{2m_{11}d_{11}^3} + \frac{\lambda_3 \beta_r^2}{2m_{33}d_{33}^3} \right). \text{ This means that } |(d_{22} - d_{11})u|^2 \leq |d_{22} - d_{11}|^2 \frac{2\beta}{\lambda_1} = |d_{22} - d_{11}|^2 \left( \frac{\beta_u^2}{m_{11}d_{11}^3} + \frac{\lambda_3 \beta_r^2}{m_{33}d_{33}^3} \right),
$$

$$
< \beta_u^2, \text{ and } |(m_{11} - m_{22})uv| \leq |(m_{11} - m_{22})| \frac{2\beta}{\lambda_1 \lambda_2} = \frac{2(m_{11} - m_{22})}{\sqrt{\lambda_1 \lambda_2}} \left( \frac{\beta_u^2}{2m_{11}d_{11}^3} + \frac{\lambda_3 \beta_r^2}{2m_{33}d_{33}^3} \right).
$$

(11)

Therefore, there must be constants $\beta_1$ and $\beta_2$ satisfying that $\beta_1 < \frac{\beta_u}{d_{22}} \frac{|(d_{22} - d_{11})u|}{d_{22}}, \beta_2 < \frac{\beta_r}{m_{33}} \frac{|(m_{11} - m_{22})uv|}{m_{33}}$. Hence we can have $\beta_u > |d_{22} - d_{11}||u| + d_{22} \beta_1$ and $\beta_r > |m_{11} - m_{22}||uv| + m_{33}\beta_2$. The proof is closed.

According to the Lemma 3, the facts $\beta_u < \tau_{\text{umax}}$ and $\beta_r < \tau_{\text{rmax}}$, finally, we can have the following input transformation:

$$
\tau_u = (d_{22} - d_{11})u - d_{22}s_{p_1} (\omega_1) \leq \beta_u < \tau_{\text{umax}}, \quad \tau_r = (m_{11} - m_{22})uv + m_{33}s_{p_1} (\omega_2) \leq \beta_r < \tau_{\text{rmax}}.
$$

(11)
Since \( \tau_u < \tau_{\text{umax}} \) and \( \tau_r < \tau_{\text{rmax}} \), we can have,
\[
s_{\tau_{\text{umax}}} = \tau_u, s_{\tau_{\text{rmax}}} = \tau_r.
\]

Then the system (4b) can be transformed to the following cascade system:
\[
\begin{align*}
\dot{\vartheta}_2 &= \vartheta_5 \vartheta_6, \\
\dot{\vartheta}_3 &= \vartheta_6, \\
\dot{\vartheta}_5 &= -\vartheta_2 \vartheta_6 + s_{\beta_1(\vartheta_5)}, \\
\dot{\vartheta}_6 &= -\gamma_4 \vartheta_6 + s_{\beta_2(\vartheta_2)}.
\end{align*}
\]
where \( \gamma_4 = \frac{d_{33}}{m_{33}} \)

**Remark 1** By the proposed input transformation, the system (4b) who has input saturations can be further simplified as (13). This is one innovation point of this paper.

**Remark 2** According to the inequalities (6) and (7), the values of \( \beta_u \) and \( \beta_r \) depend on the terms \( \frac{\gamma_3}{\gamma_2} \sqrt{\frac{m_{11}m_{22}d_{11}d_{33}}{\lambda_1\lambda_3}} \Delta_1, \Delta_2, \frac{m_{33}d_{33}\gamma_3}{\lambda_3\gamma_2} + \varepsilon \) and \( \gamma_1 \). According to definitions of these parameters, we can have
\[
\begin{align*}
\Delta_0 &= \gamma_3 \sqrt{\frac{m_{11}m_{22}d_{11}d_{33}}{2(m_{11} - m_{22})}} \frac{\lambda_2}{\lambda_3} \Delta_1 = \frac{(d_{22} - d_{11})}{\sqrt{(m_{11} - m_{22})}} \sqrt{\Delta_1^*} \frac{\lambda_2}{\lambda_3} \frac{m_{33}d_{33}\gamma_3}{\lambda_3\gamma_2} \varepsilon \\
+ \varepsilon &\geq \frac{m_{33}d_{33}\gamma_3}{m_{22} - m_{11}} \frac{\sqrt{\lambda_1\lambda_2}}{\lambda_3} \gamma_1 \beta_u = \beta_u \sqrt{\frac{m_{33}d_{33}\gamma_3}{(d_{22} - d_{11})^2}} \left[ 1 - \frac{(d_{22} - d_{11})^2}{m_{11}d_{11}\gamma_3} \right] \frac{\lambda_1}{\lambda_3}.
\end{align*}
\]
with \( \Delta_1^* = m_{33}d_{33}\gamma_3 \left[ 1 - \frac{(d_{22} - d_{11})^2}{m_{11}d_{11}\gamma_3} \right] \). Obviously, since \( -\frac{m_{11}}{m_{22}} \lambda_1 + \frac{m_{22}}{m_{11}} \lambda_2 \), \( -\frac{m_{22} - m_{11}}{m_{33}} \lambda_3 = 0 \), similar positive values of \( m_{11}m_{22}^{-1}\lambda_1 \) and \( m_{22}m_{11}^{-1}\lambda_2 \) can result in smaller \( \lambda_3 \) and larger \( \Delta_1, \Delta_2 \) and \( \frac{m_{33}d_{33}\gamma_3}{\lambda_3\gamma_2} + \varepsilon \). However, this implies that
\[
\frac{m_{33}d_{33}\gamma_3}{\lambda_3(m_{11} - m_{22})} \lambda_1 > \tau_{\text{rmax}}.\text{ Hence we can have } \Delta_2 \approx \sqrt{\frac{m_{11}d_{11}\tau_{\text{rmax}}\gamma_3}{m_{11} - m_{22}}} \left( \frac{\lambda_2}{\lambda_1} \right)^{\frac{1}{4}}.\text{ A large } \lambda_2 \text{ is useful to the large } \Delta_2 \text{ and contradictory with the requirement of similar } m_{11}m_{22}^{-1}\lambda_1 \text{ and } m_{22}m_{11}^{-1}\lambda_2.\text{ Therefore, the difficulty lies that there exists a trade-off in selecting these parameters } \lambda_1, \lambda_2 \text{ and } \lambda_3.\text{ The details are stated in Section Optimization.}

**STABILIZATION OF USVS**

Although the system is simplified as (13), there still be input saturation problem in the transformed system implying that the existing methodologies for (13) are not applicable here. To overcome the challenge, in this section, we propose a novel control law by combining the control law in Section “Stabilization of USVS” with the proposed input transformation in the Model Transformation section to solve the stabilization problem of the underactuated surface vessel.
Control of transformed system

Define $\beta_3, \beta_4, \beta_5, \beta_6$ and $\beta_7$ as arbitrary constants satisfying that

$$\beta_4 + \frac{1}{2} \lambda_3 \beta_3 < \beta_1, \beta_5 + \beta_6 + \beta_7 < \beta_2. \tag{14}$$

where $\lambda_3, \beta_1$ and $\beta_2$ are defined in (5) and (8). We cite the Barbalat Lemma which the results in this paper will base on:

**Lemma 4** Slotine & Li (1991) Let $z : R \rightarrow R$ be a uniformly continuous function on $[0, \infty)$. Suppose that $\lim_{t \rightarrow \infty} \int_0^t z(s)ds$ exists and is finite. Then, $\lim_{t \rightarrow \infty} z(t) = 0$.

Based on the above parameters, we can have the following control law for system (13):

$$\begin{align*}
\varpi_1 &= -\beta_4 \arctan[\kappa_1 \vartheta_2] \vartheta_6 - \beta_4 \arctan[\kappa_2 \vartheta_3], \\
\varpi_2 &= -\beta_5 \arctan(\kappa_3 \vartheta_3) - \beta_6 \arctan(\kappa_4 \vartheta_6) + \beta_7 \text{sign} \{\arctan[\kappa_1 \vartheta_2]\} \rho(t). \tag{15}
\end{align*}$$

Here, $\text{sign} \{\arctan[\kappa_1 \vartheta_2]\} = \text{sign} \{\arctan[\kappa_1 \vartheta_2]\} \cdot |\arctan[\kappa_1 \vartheta_2]|$, $\kappa_1, \kappa_2, \kappa_3$ and $\kappa_4$ are positive constants, $\rho(t) = \sin t$, $\xi = \frac{p}{q}$, where $p$ is positive odd and $q$ is positive even.

According to the Eq. (14), we can have $\varpi_1 \leq \beta_3 \sqrt{\frac{\lambda_1 \rho_5^2}{\lambda_3 m_1 d_1}} + \beta_4$ and $\varpi_2 \leq \beta_5 + \beta_6 + \beta_7 < \beta_2$. This implies that the transformed inputs $\tau_u$ and $\tau_r$ can be less that their saturation values $\beta_u$ and $\beta_r$. Then substituting controller (15) into the system in (13), we can have

$$\begin{align*}
\hat{\vartheta}_2 &= \dot{\vartheta}_2 \vartheta_6, \hat{\vartheta}_3 = \vartheta_6, \\
\hat{\vartheta}_5 &= -\gamma_4 \vartheta_6 - \beta_5 \arctan(\kappa_3 \vartheta_3) - \beta_6 \arctan(\kappa_4 \vartheta_6) + \beta_7 \text{sign} \{\arctan[\kappa_1 \vartheta_2]\} \rho(t). \tag{16}
\end{align*}$$

For the system (16), we have the following result:

**Lemma 5** The system (16) is globally asymptotically stable.

**Proof**: Consider the following Lyapunov function:

$$V_1 = \frac{1}{2} \dot{\vartheta}_2^2 + \beta_3 \arctan(\kappa_1 \vartheta_2) \vartheta_2 - \frac{\beta_5}{2 \kappa_1} \ln(1 + \kappa_1 \vartheta_2^2) + \frac{1}{2} \vartheta_5^2, \tag{17}$$

whose derivative satisfies that $\dot{V}_1 = -\beta_4 \arctan(\vartheta_5) \vartheta_5 \leq 0$. The Lyapunov function $V_1$ is monotonically decreasing for $\vartheta_5 \neq 0$ and bounded, there must be a minimum value of $V_1$. Since $\vartheta_2$ and $\vartheta_5$ are bounded and $\hat{\vartheta}_3 = \vartheta_6, \hat{\vartheta}_6 = -\gamma_4 \vartheta_6 - \beta_5 \arctan(\kappa_3 \vartheta_3) - \beta_6 \arctan(\kappa_4 \vartheta_6) + \beta_7 \text{sign} \{\arctan[\kappa_1 \vartheta_2]\} \rho(t)$, states $\vartheta_3$ and $\vartheta_6$ are bounded. Take the derivative of $\dot{V}_1$, $\ddot{V}_1 = -\beta_4 \arctan(\vartheta_5) + \frac{\vartheta_5}{\dot{\vartheta}_2} \{\dot{\vartheta}_2 \vartheta_6 - \beta_3 \arctan(\kappa_1 \vartheta_2) \vartheta_6 - \beta_4 \arctan(\kappa_2 \vartheta_5)\}$. Since states $\vartheta_2$ and $\vartheta_5$ are bounded, then $\dot{V}_1$ and $\ddot{V}_1$ are continuous and bounded. This implies that $\ddot{V}_1$ is uniformly continuous, according to the Lemma 4, we can have that $\lim_{t \rightarrow \infty} \vartheta_5 = 0$. Define $\Gamma_1 = \vartheta_5$, according to Appendix A, $\Gamma_1$ is uniformly continuous. Together with Lemma 4, we have
Since \( \lim_{t \to +\infty} \vartheta_5 = 0 \) and states \( \vartheta_2 \) and \( \vartheta_6 \) are bounded, the Eq. (18) indicates that \( \lim_{t \to +\infty} \arctan(\kappa_1 \vartheta_2) \vartheta_6 = 0 \). Due to \( \vartheta_2 \) is bounded, the above equation indicates that there must be a positive real number \( K = \xi + 2 \) satisfying that

\[
\lim_{t \to +\infty} [\arctan(\kappa_1 \vartheta_2)]^{K-\xi} \vartheta_6 = \lim_{t \to +\infty} [\arctan(\kappa_1 \vartheta_2)]^{K-\xi-1} \arctan(\kappa_1 \vartheta_2) \vartheta_6 = 0.
\]  

Define \( \Gamma_2 = [\arctan(\kappa_1 \vartheta_2)]^{K-\xi} \vartheta_6 \), according to Appendix B, \( \Gamma_2 \) is uniformly continuous, according to Lemma 4, we can then have

\[
\lim_{t \to +\infty} \Gamma_2 = \lim_{t \to +\infty} (K - \xi)[\arctan(\kappa_1 \vartheta_2)]^{K-\xi-1} \frac{\kappa_1 \vartheta_5 \vartheta_6^2}{1 + \kappa_1^2 \vartheta_2^2} + [\arctan(\kappa_1 \vartheta_2)]^{K-\xi} [\arctan(\kappa_1 \vartheta_2)]^K \rho(t).
\]

Due to the fact that \( \lim_{t \to +\infty} \vartheta_3 = 0 \), \( \lim_{t \to +\infty} [\arctan(\kappa_1 \vartheta_2)]^{K-\xi} \vartheta_6 = 0 \) and \( \vartheta_3, \vartheta_2, \vartheta_6 \) are bounded. The Eq. (20) means that

\[
\lim_{t \to +\infty} (\kappa_3 \beta_5 \arctan(\kappa_3 \vartheta_3)) [\arctan(\kappa_1 \vartheta_2)]^{K-\xi} - \beta_7 [\arctan(\kappa_1 \vartheta_2)]^{K} \rho(t) = 0.
\]

Define \( \Gamma_3 = \kappa_3 \beta_5 \arctan(\kappa_3 \vartheta_3)[\arctan(\kappa_1 \vartheta_2)]^{K-\xi} - \beta_7 \rho(t) \times [\arctan(\kappa_1 \vartheta_2)]^{K} \), according to the Appendix C, we can have that \( \Gamma_3 \) is uniformly continuous, and according to Lemma 4,

\[
\lim_{t \to +\infty} \Gamma_3(t) = \lim_{t \to +\infty} \frac{\kappa_3^2 \beta_5 \vartheta_6}{1 + \kappa_3^2 \vartheta_2^2} [\arctan(\kappa_1 \vartheta_2)]^{K-\xi} + \frac{\kappa_1 \kappa_3 \beta_5 \vartheta_5 \vartheta_6}{1 + \kappa_1^2 \vartheta_2^2} K \arctan(\kappa_3 \vartheta_3)
\]

\[
\times [\arctan(\kappa_1 \vartheta_2)]^{K-\xi-1} - \frac{\kappa_1 \kappa_3 \beta_5 \vartheta_5 \vartheta_6}{1 + \kappa_1^2 \vartheta_2^2} \xi \arctan(\kappa_3 \vartheta_3) [\arctan(\kappa_1 \vartheta_2)]^{K-\xi-1}
\]

\[
-\beta_7 K [\arctan(\kappa_1 \vartheta_2)]^{K-1} \frac{\kappa_1 \vartheta_5 \vartheta_6}{1 + \kappa_1^2 \vartheta_2^2} - \beta_7 [\arctan(\kappa_1 \vartheta_2)]^{K} \rho(t).
\]

Together with the fact that \( \lim_{t \to +\infty} \vartheta_5 = 0 \), \( \lim_{t \to +\infty} [\arctan(\kappa_1 \vartheta_2)]^{K-\xi} \times \vartheta_6 = 0 \) and \( \vartheta_3, \vartheta_2, \vartheta_6 \) are bounded, we can have \( \lim_{t \to +\infty} \beta_7 \rho(t) \times [\arctan(\kappa_1 \vartheta_2)]^{K} = 0 \). Obviously, the state \( \vartheta_2 \) converges to zero, according to (18), dynamics of \( \vartheta_3 \) and \( \vartheta_6 \) can be \( \dot{\vartheta}_3 = \vartheta_6, \dot{\vartheta}_6 = -\gamma_4 \vartheta_6 - \beta_3 \arctan(\kappa_3 \vartheta_3) - \beta_6 \arctan(\kappa_4 \vartheta_6). \)

Define \( V_2 = \beta_5 \arctan(\kappa_3 \vartheta_3) \vartheta_3 - \frac{\beta_5^2}{2 \kappa_3} \ln(1 + \kappa_3^2 \vartheta_3^2) \geq 0 \), whose derivative satisfies

\[
\dot{V}_2 = -\gamma_4 \vartheta_6^2 - \beta_6 \arctan(\kappa_4 \vartheta_6) \vartheta_6 \leq 0.
\]

Therefore, \( \vartheta_3 \) and \( \vartheta_6 \) can all converge to zero globally and asymptotically. The proof is closed.

**The control algorithm for USVs**

Based on the above discussions, we are now in a position to come back to the USV system in (1a) and (1b) and give the following controller,
\[
\tau_u = -d_{22}s_{\beta_1}(\beta_3 \arctan[\kappa_1\vartheta_2]\vartheta_6 - \beta_4 \arctan[\kappa_2\vartheta_3]) + (d_{22} - d_{11})u, \quad (21a)
\]
\[
\tau_r = (m_{11} - m_{22})uv + [-\beta_5 \arctan(\kappa_3\vartheta_3) - \beta_6 \arctan(\kappa_4\vartheta_6) + \beta_7 \arctan(\kappa_2\vartheta_2)]
\]
\[
+ \beta_7 \arctan(\kappa_1\vartheta_2)] \times m_{33}s_{\beta_1} \rho(t), \quad (21b)
\]

with \(\beta_1 < \beta_2, d_{22}^{-1} - |(d_{22} - d_{11})u|d_{22}^{-1}, \beta_2 < \beta_5 m_{33}^{-1} - |(m_{11} - m_{22})uv| m_{33}^{-1}, \beta_4 + \beta_5 < \beta_3 \), \(\beta_1, \beta_5, \beta_6 + \beta_7 < \beta_2\). Here \(\beta, \beta_r, \kappa, \lambda, \lambda_1, \lambda_2, \lambda_3\) are constants defined in the Eqs. (5)–(7).

We now have the following theorem, whose proof can be done along similar lines of Lemmas 1, 2 and 5.

**Theorem 1** For input saturation USV system (1a) and (1b), the fractional power controller (21a) and (21b) ensure states \(x, y, \psi, u, v, \) and \(r\) globally asymptotically converged to zero.

Next, to facilitate practical application and summarize the main results, we provide the following algorithms to implement the proposed methods and results in USV.

**Stabilization control algorithm of USVs with input saturation**

**Step 1.** Take measurements \(\tau_{u\text{max}}, \tau_{r\text{max}}, m_{11}, m_{22}, m_{33}, d_{11}, d_{22} \text{ and } d_{33} \) of the USV.

**Step 2.** Let \(\varepsilon = \min \left\{ \frac{d_{11}}{m_{11}}, \frac{2d_{22}}{m_{22}}, \frac{d_{33}}{m_{33}} \right\} \) and \(\gamma_4 = \frac{d_{33}}{m_{33}}\).

**Step 3.** The parameters \(\lambda_1, \lambda_2, \lambda_3, \beta_u, \text{ and } \beta_r\) are chosen as the Eqs. (5)–(7) and (24).

**Step 4.** Make state transformation \(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6\).

\[
\vartheta_1 = x\cos\psi + y\sin\psi, \quad \vartheta_2 = -x\sin\psi + y\cos\psi + \frac{m_{22}}{d_{22}}, \quad \vartheta_3 = \psi, \quad \vartheta_4 = v, \quad \vartheta_5 = -\frac{m_{11}}{d_{22}}u, \quad \vartheta_6 = r.
\]

**Step 5.** Let \(\kappa_1, \kappa_2\) are arbitrary positive constants, \(\kappa_3 = \beta_5 \varepsilon_1, \kappa_4 = \beta_5 \varepsilon_1, \lambda = p/q, \) where \(p\) is positive odd and \(q\) is positive even.

**Step 6.** Compute virtual control inputs \(\varpi_1\) and \(\varpi_2\).

\[
\varpi_1 = -\beta_1 \arctan[\kappa_1\vartheta_2]\vartheta_6 - \beta_4 \arctan[\kappa_2\vartheta_3], \quad \varpi_2 = -s_{\beta_1}(\kappa_3\vartheta_3) + \beta_7 \arctan(\kappa_2\vartheta_2) \rho(t) - s_{\beta_3}(\kappa_4\vartheta_6).
\]

**Step 7.** Compute real control inputs \(\tau_u\) and \(\tau_r\),

\[
\tau_u = (d_{22} - d_{11})u - d_2 s_{\beta_1}(\varpi_1), \quad \tau_r = (m_{11} - m_{22})uv + m_3 s_{\beta_1}(\varpi_2).
\]

**Parameter optimization**

In this subsection, the optimization algorithm of parameters \(\lambda_1, \lambda_2\) and \(\lambda_3\) for the stabilization of the USV subject to input constraints are introduced.

To facilitate the calculation, first, define positive constant \(s\) as \(\lambda_1 = s\lambda_2\) satisfying

\[
0 < s < \frac{b_1}{b_2}.
\]

Then we can have

\[
\lambda_3 = (b_1 - b_2s)\lambda_2, \quad (22)
\]

where \(b_1 = \frac{m_{22}m_{33}}{(m_{22} - m_{11})m_{11}}, b_2 = \frac{m_{11}m_{33}}{(m_{22} - m_{11})m_{22}}\) are positive constants. According to
the Remark 2, when \( s \) satisfies the case that \( 0 < \sqrt{s} \leq \frac{\sqrt{T_4^2 + 4b_1b_2s^2} - T_4}{2b_2\tau_{\max}} \), \( \Delta_0 \) and \( \Delta_1 \) can be small. Therefore, we consider only the case that
\[
\sqrt{s} > \frac{\sqrt{T_4^2 + 4b_1b_2s^2} - T_4}{2b_2\tau_{\max}}. 
\] (23)

According to Eqs. (5)–(7), we have
\[
\Delta_0 = \frac{T_1}{\sqrt{b_1 - b_2}}, \quad \Delta_1 = \frac{T_1}{\sqrt{b_1 - b_2}}, 
\]
\[
\Delta_2 = \frac{T_2}{\sqrt{T_4} - (b_1 - b_2)s\tau_{\max}}, \quad \text{where } T_1 = \frac{\gamma_3}{\sqrt{m_{11}m_{33}d_{11}d_{33}}},
\]
\[
T_2 = \frac{d_{22} - d_{11}}{m_{11} - m_{22}} \sqrt{m_{33}\gamma_3[1 - \frac{(d_{22} - d_{11})^2}{m_{11}d_{11}\gamma_3}]}, \quad T_3 = \frac{m_{11}d_{11}\gamma_3}{m_{33}d_{33}} \quad \text{and } T_4 = \frac{m_{33}d_{33}\gamma_3}{(m_{11} - m_{22})} \text{ are constants.}
\]

Therefore, the partial derivatives of \( \Delta_0, \Delta_1 \) and \( \Delta_2 \) about \( s \) can be
\[
\frac{\partial \Delta_0}{\partial s} = \frac{b_1T_1}{2(b_1 - b_2)^2} > 0, \quad \frac{\partial \Delta_1}{\partial s} = \frac{b_1T_1}{2(b_1 - b_2)^2} > 0, \quad \frac{\partial \Delta_2}{\partial s} = \frac{T_2 \sqrt{\gamma_3} + b_1\tau_{\max}}{4\sqrt{\gamma_3}s(b_1 - b_2)}.
\]

This means that \( \Delta_0 \) and \( \Delta_1 \) are increasing about \( s \) and the sign of \( \frac{\partial \Delta_2}{\partial s} \) is determined by the values of \( s \) and \( \tau_{\max} \). Next, relations of \( \frac{\partial \Delta_2}{\partial s} \) and the values of \( s \) and \( \tau_{\max} \) will be analyzed as different cases.

According to the formula (23), it is easy to know that
\[
\frac{2b_1b_2s^2 - T_4\sqrt{T_4^2 + 4b_1b_2s^2}}{2b_2\tau_{\max}} < s < \frac{b_1}{b_2}.
\]

Define \( \Delta_4 = -\frac{T_1\sqrt{T_4} + b_1\tau_{\max}}{2\sqrt{b_1b_2}} \), we can have \( b_1\tau_{\max} - \frac{T_1\sqrt{b_1b_2}}{2\sqrt{b_1b_2}} < \Delta_4 < \Delta_4 \).

Case 1: If \( \tau_{\max} > \frac{T_1}{\sqrt{b_1b_2}} \), then we can have \( \Delta_4 > 0 \) and \( \frac{\partial \Delta_4}{\partial s} > 0 \) meaning \( \Delta_0, \Delta_1 \) and \( \Delta_2 \) are increasing. Therefore, in order to obtain a large \( \beta_{\mu} \), we choose \( s = \frac{b_1}{b_2} - \kappa \), where \( \kappa \) is the small positive constant.

Case 2: If \( \tau_{\max} < \frac{T_1}{\sqrt{b_1b_2}} \), according to the value of \( s \), the case can be divided into two intervals.

Case 2.1: When the parameter \( s \) satisfies that
\[
\frac{2b_1b_2s^2 - T_4\sqrt{T_4^2 + 4b_1b_2s^2}}{2b_2\tau_{\max}} < s \leq \frac{4b_1}{T_4^2} \tau_{\max}^2, 
\]
we can have \( \Delta_4 > 0 \) and \( \frac{\partial \Delta_4}{\partial s} > 0 \) meaning \( \Delta_0, \Delta_1 \) and \( \Delta_2 \) are increasing. Therefore, in order to obtain a large \( \beta_{\mu} \), we choose \( s = \frac{4b_1}{T_4^2} \tau_{\max}^2 \).

Case 2.2: When \( \frac{4b_1}{T_4^2} \tau_{\max}^2 < s < \frac{b_1}{b_2} \), we can have \( \Delta_4 < 0 \). This indicates terms \( \Delta_0 \) and \( \Delta_1 \) are increasing, \( \Delta_2 \) is decreasing about \( s \). Hence, if the values of \( \min\{\Delta_0, \Delta_1\} \) and \( \Delta_3 \) are more approximative, the term \( \min\{\Delta_0, \Delta_1, \Delta_3\} \) can be smaller. Define the index function \( Q \) as
\[
Q = \frac{1}{2} \left(\min\{\Delta_0, \Delta_1\}\right)^2 - \frac{1}{2} \Delta_2^2 = \frac{1}{2} \Delta_5^2, \quad \text{where } \Delta_5 = \frac{T_1}{b_1 - b_2} - T_3 \left[ T_3 \left( \frac{T_1}{\sqrt{b_1 - b_2}} - (b_1 - b_2)\tau_{\max} \right) \right] \quad \text{and } T_5 = \min\{T_1, T_3\}.
\]

Then the partial derivative of \( Q \) about \( s \) is \( \frac{\partial Q}{\partial s} = \Delta_5 \cdot \Delta_6 \), where
\[
\Delta_6 = \frac{\partial \Delta_5}{\partial s} = \frac{b_1T_1}{(b_1 - b_2)^2} + \frac{T_1}{2\sqrt{\gamma_3} - b_1\tau_{\max}} > 0. \quad \text{Since } \frac{\partial \Delta_5}{\partial s} = \Delta_6 > 0, \text{ then } \Delta_5 \text{ is increasing.}
\]

Generally, \( T_2 < T_1 \) and \( T_5 = T_2 \), then we can have \( \Delta_5 = \frac{T_1}{b_1 - b_2} \Delta_7 \), where
\[
\Delta_7 = b_1^2\tau_{\max}s^2 + T_4b_2\frac{3}{2} + T_6s - T_4b_1\sqrt{s} + b_1^2\tau_{\max}. \quad T_6 = \frac{\gamma_3}{\sqrt{b_1b_2}} - 2b_1b_2\tau_{\max}.
\]

This means
\[
\lim_{s \to b_1b_2} \Delta_5(s) = +\infty > 0. \quad \text{Thus the minimum value of } Q \text{ is determined by different cases of the values of } \Delta_5 \text{ for } s = \frac{4b_1}{T_4^2} \tau_{\max}^2.
\]
Case 2.2.1: If $\Delta_5 < 0$ for $s = \frac{4b_1^2}{T_4^2} \tau_{\text{rmax}}$, there must exist $s = s^*$ subject to $\frac{4b_1^2}{T_4^2} \tau_{\text{rmax}} < s^* < \frac{b_1}{b_2}$ and $b_2^2 \tau_{\text{rmax}} s^* + T_4 b_2 s^* - T_4 S T s^* - T_4 b_1 \sqrt{s^*} + b_1^2 \tau_{\text{rmax}} = 0$. If $s > s^*$, then $\frac{\partial J_1}{\partial s} > 0$, similarly, if $s < s^*$ then $\frac{\partial J_1}{\partial s} < 0$. This implies that $J_1(s = s^*)$ is the minimum of $J_1$. Then, we choose $s = s^*$.

Case 2.2.2: If $\Delta_5 \geq 0$ for $s = \frac{4b_1^2}{T_4^2} \tau_{\text{rmax}}$, we can have $\Delta_5 > 0$. This indicates that $\frac{\partial J_1}{\partial s} > 0$, $J_1$ is increasing. In order to obtain the minimum of $J_1$, we choose $s = \frac{4b_1^2}{T_4^2} \tau_{\text{rmax}}$.

Finally, summarizing above conclusions, $\frac{\Delta J}{\Delta s} = s$ can be chosen as:

$$s = \begin{cases} \frac{b_1}{b_2} - \kappa, & \text{if } \tau_{\text{rmax}} \geq \frac{T_4}{2\sqrt{b_1 b_2}}, \\ s' = \frac{4b_1^2}{T_4^2} \tau_{\text{rmax}}, & \text{else if } \Delta_5 \left(s = \frac{4b_1^2}{T_4^2} \tau_{\text{rmax}}\right) < 0, \\ \frac{4b_1^2}{T_4^2} \tau_{\text{rmax}}, & \text{else if } \Delta_5 \left(s = \frac{4b_1^2}{T_4^2} \tau_{\text{rmax}}\right) \geq 0. \end{cases}$$

(24)

**Remark 3** In this subsection, we analyze the selection method of parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$. In fact, generally, $\tau_{\text{rmax}} \geq \frac{T_4}{2\sqrt{b_1 b_2}}$. Thus in most practical cases, we choose $s = \frac{b_1}{b_2} - \kappa$.

**NUMERICAL SIMULATIONS**

Numerical simulations are given to illustrate the effectiveness of the presented method. In this section, let the USV have the following initial conditions: $[x(0), y(0), \psi(0), u(0), v(0), r(0)]^T = [3, -3, 0, 0, 0, 0]^T$. The controller parameters in (21a) and (21b), the inertial matrix and damping matrix in the USV model are given as Huang et al. (2015): $m_{11} = 200$ kg, $m_{22} = 250$ kg, $m_{33} = 50$ kg, $d_{11} = 70$ kg/s, $d_{22} = 100$ kg/s, $d_{33} = 70$ kg/s. The maximum value of the absolute value of control input $\tau_u$ and $\tau_r$ are $\tau_{\text{umax}} = 200$ N and $\tau_{\text{rmax}} = 200$ N.

According to the Eq. (22), we can have that $b_1 = 1.25$, $b_2 = 0.8$, $T_1 = 207.0628$, $T_2 = 26.7261$, $T_3 = 74.8331$ and $T_4 = 17.5$. From the inequality (24), we choose $s = 1.562$. Substituting it into the Eq. (23), we can have $\Delta_0 = 1035.3$, $\Delta_1 = 1,336.3$ and $\Delta_2 = 277.45$. This means $\beta_u$ satisfies that $\beta_u < \min\{1035.3, 1336.3, 277.45, 500\}$.

In this paper, we choose $\beta_u = 250$. According to above results and formulas (7) and (2), one has $809.037 < \beta_r < \min\{6220.2, 5387, 1000\}$. We choose $\tau_r = \tau_{\text{rmax}} = 1,000$. Thus together with Lemma 3, we can have $\beta_1 < 1.4163, \beta_2 < 3.6929$ and choose $\beta_1 = 1.4$ and $\beta_2 = 3.6$. Finally, according to formulas (14), we choose $\beta_3 = 0.0018, \beta_4 = 1, \beta_5 = 1, \beta_6 = 0.6, \beta_7 = 2, \kappa_1 = 10,000, \kappa_2 = 1, \kappa_3 = 1$ and $\kappa_4 = 1$.

In Figs. 2 and 3, it is shown that states $x, y, \psi, u, v$ and $r$ can be converged to zero. Fig. 4 shows the torque $\tau_u$ and $\tau_r$. It is obvious that the maximum value of $|\tau_u|$ and $|\tau_r|$ are less than $\tau_{\text{umax}}$ and $\tau_{\text{rmax}}$ respectively.

As is known, there must be some trade-off in the input and convergence of states. In this paper, we consider the input saturation of USVs, meaning that the performance may be reduced. To investigate the performance of our controller with other methods, to be specific, consider the following performance index.
\[ Q = x^2 + y^2 + \psi^2 + u^2 + v^2 + r^2. \]  \hspace{1cm} (25)

Figure 5 shows that compared with the method in Zhang & Guo (2018), the convergence time is increased in this paper, which is caused by the input saturation term.

However, in Figs. 6 and 7, the maximum of \( \tau_u \) and \( \tau_r \) in this paper are far less than them in paper Zhang & Guo (2018). This implies that the anti-saturation control law in this
paper is at work and obviously reduces the burden of actuators. Thus we now focus on the comparison of consumption. Consider the following consumption index

\[ J = \int_0^t \left( \tau_u(s)^2 + \tau_r(s)^2 \right) ds. \]  

(26)
We show the energy cost in Fig. 8, from which we can see that the energy consumption is far less than that of Zhang & Guo (2018).

To further illustrate the advantages of the method in this paper, we give a comparison result with the MPC casadi-windows-matlabR2016a-v3.4.5 method in Figs. 9–11.

The figure shows that although the MPC method can obtain faster convergence performance than our method. However, its control input oscillation frequency is...
relatively high, which requires relatively high actuators. Thus, we do not emphasize that our approach is superior to the MPC method, but proposes a particular strategy for USV stabilization control to avoid the problems of high oscillation frequency, difficulty in solving, and increased requirements for computing power in the MPC method.
CONCLUSION AND FUTURE WORK

In this paper, by introducing a novel input, an improved transformation is proposed. For the transformed system, a fractional power control law is presented to realize the global
asymptotic stability of the USV with input saturation, and a parameter optimization method is given.

The output limitation caused by the environment and the task may have a meaningful impact on the transient behavior and even the system’s stability. The stability of the output-constrained USV is still an open issue. The energy consumption optimization and stability control of USV is another interesting problem for future research. At the same time, due to the extreme randomness of the environment faced by the sea task, some control methods for stochastic nonlinear systems, such as Zhang & Wang (2021), Yin et al. (2020), Zhang et al. (2016), Zhang, Hu & Gow (2020), should also be considered for application in the USV stabilization control in the future.

**APPENDIX**

**Appendix A**

Then, the proof of uniform continuity of \( \dot{\Gamma}_1 \) is shown. Take the 1st and 2nd derivative of \( \Gamma_1 \), we can have \( \dot{\Gamma}_1 = -\vartheta_3 \vartheta_6 - \beta_3 \arctan(\kappa_1 \vartheta_2) \vartheta_6 - \beta_4 \arctan(\kappa_2 \vartheta_5) \),

\[
\dot{\Gamma}_1 = \vartheta_2 \vartheta_5 \vartheta_6 - \vartheta_2 [-d_{33} m_{33}^1 \vartheta_6 - (m_{31} - m_{22}) m_{33}^1 yuv + m_{33}^1 \sigma_{\text{max}}(\tau_\epsilon)] - \\
\beta_5 \vartheta_5 \vartheta_6^2 + \frac{\kappa_4 \beta_4 \vartheta_6}{1 + \kappa_1^2 \vartheta_2^2} \beta_3 \arctan(\kappa_1 \vartheta_2) \vartheta_6 + \frac{\kappa_4 \beta_4 \vartheta_6}{1 + \kappa_2^2 \vartheta_3^2} \beta_4 \arctan(\kappa_2 \vartheta_5) \beta_3 \arctan(\kappa_1 \vartheta_2) \\
\times [-\gamma_4 \vartheta_6 - \beta_5 \arctan(\kappa_3 \vartheta_3) - \beta_6 \arctan(\kappa_4 \vartheta_6) + \beta_7 \sigma(\vartheta_6) \rho(t)].
\]

Due to the fact that states \( \vartheta_2, \vartheta_3, \vartheta_4 \) and \( \vartheta_6 \) are bounded, according to above equations, \( \dot{\Gamma}_1 \) and \( \ddot{\Gamma}_1 \) are continuous and bounded. This implies \( \dot{\Gamma}_1 \) is uniformly continuous.

**Appendix B**

Here, the proof of uniform continuity of \( \dot{\Gamma}_2 \) is shown. Take the 1st and 2nd derivative of \( \Gamma_2 \), we can have \( \dot{\Gamma}_2 = (K - \zeta) [\arctan(\kappa_1 \vartheta_2)]^{K-\zeta-1} \frac{\kappa_1 \vartheta_2 \vartheta_6^2}{1 + \kappa_1^2 \vartheta_2^2} [\arctan(\kappa_1 \vartheta_2)]^{K-\zeta} \\
[-\gamma_4 \vartheta_6 - \beta_5 \arctan(\kappa_3 \vartheta_3) - \beta_6 \arctan(\kappa_4 \vartheta_6) + \beta_7 \sigma(\vartheta_6) \rho(t)],
\]

\[
\dot{\Gamma}_2 = (K - \zeta) [\arctan(\kappa_1 \vartheta_2)]^{K-\zeta-1} \frac{\kappa_1 \vartheta_2 \vartheta_6^2}{1 + \kappa_1^2 \vartheta_2^2} [\arctan(\kappa_1 \vartheta_2)]^{K-\zeta} \\
+ \beta_7 \sigma(\vartheta_6) \rho(t)] + (K - \zeta) [K - \xi - 1] [\arctan(\kappa_1 \vartheta_2)]^{K-\zeta-2} \frac{\kappa_1^2 \vartheta_2 \vartheta_6^2 \vartheta_4}{1 + \kappa_1^2 \vartheta_2^2} \\
+ \kappa_1 \vartheta_2 \vartheta_6^2 [\arctan(\kappa_1 \vartheta_2)]^{K-\zeta-1} \frac{K - \xi}{1 + \kappa_1^2 \vartheta_2^2} \kappa_4 \beta_4 \vartheta_6 - \beta_3 \arctan(\kappa_1 \vartheta_2) \vartheta_6 - \beta_4 \arctan(\kappa_2 \vartheta_5) \\
\vartheta_6^2 + 2 \kappa_1 \vartheta_3 \vartheta_6 [-\gamma_4 \vartheta_6 - \beta_5 \arctan(\kappa_3 \vartheta_3) - \beta_6 \arctan(\kappa_4 \vartheta_6) + \beta_7 \sigma(\vartheta_6) \rho(t)] \\
- 2 \kappa_1 \vartheta_4 \vartheta_6 \vartheta_6 + [\arctan(\kappa_1 \vartheta_2)]^{K-\zeta} [-\gamma_4 [\arctan(\kappa_1 \vartheta_2)]^{K-\zeta} - \beta_3 \arctan(\kappa_1 \vartheta_2)] \rho(t) \\
- \beta_6 \arctan(\kappa_4 \vartheta_6) + \beta_7 \sigma(\vartheta_6) \rho(t)] - \frac{\kappa_1 \beta_6 \vartheta_6^2}{1 + \kappa_1^2 \vartheta_2^2} \kappa_4 \beta_4 \vartheta_6 - \beta_5 \arctan(\kappa_3 \vartheta_3) \beta_6 \arctan(\kappa_4 \vartheta_6) - \beta_7 \sigma(\vartheta_6) \rho(t)] \\
+ \beta_7 \sigma(\vartheta_6) \rho(t) \times [\arctan(\kappa_1 \vartheta_2)] + \beta_7 \sigma(\vartheta_6) \rho(t) \times [\arctan(\kappa_1 \vartheta_2)].
\]
Due to states $\vartheta_2$, $\vartheta_3$, $\vartheta_5$ and $\vartheta_6$ are bounded and continuous. $\bar{\Gamma}_2$ and $\bar{\Gamma}_2$ are also bounded and continuous. Hence $\bar{\Gamma}_2$ is uniformly continuous.

Appendix C

At last, the proof of uniform continuity of $\bar{\Gamma}_3$ is shown. Take the 1st and 2nd derivative of $\Gamma_3$, we can have $\Gamma_3 = \frac{\kappa_1 \kappa_3 \beta_3 \vartheta_5 \vartheta_6}{1 + \kappa_1^2 \varphi_2^2} \arctan(\kappa_3 \vartheta_3) A_2^{K-\xi} - \beta_7 K A_2^{K-1} \frac{\kappa_1 \vartheta_5 \vartheta_6}{1 + \kappa_1^2 \varphi_2^2} - \beta_7 A_2^K \dot{\vartheta}(t)$,

$\Gamma_3 = (K - \xi) \frac{\kappa_2^2 \beta_3 \vartheta_6}{1 + \kappa_3^2 \varphi_2^2} + \frac{\kappa_1 \kappa_3 \beta_3 \vartheta_5 \vartheta_6}{1 + \kappa_1^2 \varphi_2^2} \arctan(\kappa_3 \vartheta_3) A_2^{K-\xi}$

$[-\gamma_4 \vartheta_6 - \beta_5 \times \arctan(\kappa_3 \vartheta_3) - \beta_6 \arctan(\kappa_4 \vartheta_6) + \beta_7 \arctan(\kappa_3 \vartheta_3) \hat{\vartheta}_2 \vartheta_6] - \frac{2\kappa_3^2 \beta_5}{1 + \kappa_3^2 \varphi_2^2} \times A_2^{K-\xi} \kappa_3^2 \vartheta_3 \vartheta_6 + \frac{\kappa_1 \kappa_3 \beta_3 \vartheta_5 \vartheta_6}{1 + \kappa_1^2 \varphi_2^2} (K^2 - K \xi - K) \arctan(\kappa_3 \vartheta_3) A_2^{K-\xi-2}$

$\times A_2^{K-\xi} \kappa_3^2 \vartheta_3 \vartheta_6 + \frac{\kappa_1 \kappa_3 \beta_3 \vartheta_5 \vartheta_6}{1 + \kappa_1^2 \varphi_2^2} (K^2 - K \xi - K) \arctan(\kappa_3 \vartheta_3) A_2^{K-\xi-2}$

$\times K \frac{\kappa_1 \kappa_3 \beta_3 \vartheta_5 \vartheta_6}{1 + \kappa_1^2 \varphi_2^2} \times \vartheta_2 \vartheta_6 - 2 \frac{\kappa_1 \kappa_3 \beta_3 \vartheta_5 \vartheta_6}{1 + \kappa_1^2 \varphi_2^2} \kappa_3 \vartheta_3 \vartheta_6 \kappa_3 \vartheta_3 \vartheta_6$.

$A_2 = \arctan(\kappa_1 \vartheta_2)$. According to above equations, $\bar{\Gamma}_3$ and $\bar{\Gamma}_3$ are continuous and bounded meaning $\bar{\Gamma}_3$ is uniformly continuous.

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Author Contributions
- Pengfei Zhang conceived and designed the experiments, analyzed the data, performed the computation work, prepared figures and/or tables, authored or reviewed drafts of the paper, and approved the final draft.
- Tingting Yang conceived and designed the experiments, performed the experiments, analyzed the data, authored or reviewed drafts of the paper, and approved the final draft.

Data Availability
The following information was supplied regarding data availability:
The Simulink model is available as a Supplemental File.

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REFERENCES
Brockett RW, Millman RS. 1983. Asymptotic stability and feedback stabilization. Differential Geometric Control Theory 27(1):181–191 DOI 10.1.1.451.2214.

Dong WJ, Yi G. 2005. Global time-varying stabilization of underactuated surface vessel. International Journal of Advanced Robotic Systems 50(6):859–864 DOI 10.1109/TAC.2005.849248.

Fossen TI. 1994. Guidance and control of ocean vehicles. Vol. 1. Hoboken: John Wiley & Sons.

Ghomam J, Mnif F, Benali A, Derbel N. 2006. Asymptotic backstepping stabilization of an underactuated surface vessel. IEEE Transactions on Control Systems Technology 14(6):1150–1157 DOI 10.1109/TCST.2006.880220.

Guo G, Zhang P. 2020. Asymptotic stabilization of USVs with actuator dead-zones and yaw constraints based on fixed-time disturbance observer. IEEE Transactions on Vehicular Technology 69(1):302–316 DOI 10.1109/TVT.2019.2955020.

Huang JS, Wen C, Wang W, Song Y-D. 2015. Global stable tracking control of underactuated ships with input saturation. Systems & Control Letters 85:1–7 DOI 10.1016/j.sysconle.2015.07.002.

Li H, Yan W. 2016. Model predictive stabilization of constrained underactuated autonomous underwater vehicles with guaranteed feasibility and stability. IEEE/Asme Transactions on Mechatronics 22(3):1185–1194 DOI 10.1109/TMECH.2016.2587288.
Ma BL. 2009. Global κ exponential asymptotic stabilization of underactuated surface vessels. *Systems & Control Letters* 58(3):194–201 DOI 10.1016/j.sysconle.2008.10.011.

Ma BL, Xie WJ. 2013. Global asymptotic trajectory tracking and point stabilization of asymmetric underactuated ships with non-diagonal inertia/damping matrices. *International Journal of Advanced Robotic Systems* 10(9):336 DOI 10.5772/56671.

Mazenc F, Pettersen K, Nijmeijer H. 2002. Global uniform asymptotic stabilization of an underactuated surface vessel. *IEEE Transactions on Automatic Control* 47(10):1759–1762 DOI 10.1109/TAC.2002.803554.

Oriolo G, Nakamura Y. 1991. Control of mechanical systems with second-order nonholonomic constraints: Underactuated manipulators. In: *Proceedings of the 39th IEEE Conference on Decision and Control*. Piscataway: IEEE, 967–972.

Pettersen KY, Egeland O. 1996. Exponential stabilization of an underactuated surface vessel. In: *Proceedings of the 39th IEEE Conference on Decision and Control*. Piscataway: IEEE, 2144–2149.

Reyhanoglu M. 1996. Control and stabilization of an underactuated surface vessel. In: *Proceedings of the 39th IEEE Conference on Decision and Control*. Piscataway: IEEE, 2371–2376.

Slotine J, Li W. 1991. *Applied nonlinear control*. Vol. 1. Englewood Cliffs: Prentice Hall.

Xie WJ, Ma BL. 2015. Robust global uniform asymptotic stabilization of underactuated surface vessels with unknown model parameters. *International Journal of Robust and Nonlinear Control* 25(7):1037–1050 DOI 10.1002/rnc.3129.

Yin X, Zhang Q, Wang H, Ding Z. 2019. RBFNN-based minimum entropy filtering for a class of stochastic nonlinear systems. *IEEE Transactions on Automatic Control* 65(1):376–381 DOI 10.1109/TAC.2019.2914257.

Zhang Q, Zhou J, Wang H, Chai T. 2016. Output feedback stabilization for a class of multi-variable bilinear stochastic systems with stochastic coupling attenuation. *IEEE Transactions on Automatic Control* 62(6):2936–2942 DOI 10.1109/TAC.2016.2604683.

Zhang QC, Hu L, Gow J. 2020. Output feedback stabilization for mimo semi-linear stochastic systems with transient optimisation. *International Journal of Automation and Computing* 17(1):83–95 DOI 10.1007/s11633-019-1193-8.

Zhang P, Guo G. 2018. Stabilization of underactuated surface vessels: a continuous fractional power control method. *Applied Science* 8(7):1024 DOI 10.3390/app8071024.

Zhang P, Guo G. 2020. Fixed-time switching control of underactuated surface vessels with dead-zones: global exponential stabilization. *Journal of the Franklin Institute* 357(16):11217–11241 DOI 10.1016/j.jfranklin.2019.05.030.

Zhang Q, Wang H. 2021. A novel data-based stochastic distribution control for non-Gaussian stochastic systems. *IEEE Transactions on Automatic Control* DOI 10.1109/TAC.2021.3064991.

Zhang Z, Wu Y. 2015. Further results on global stabilisation and tracking control for underactuated surface vessels with non-diagonal inertia and damping matrices. *International Journal of Control* 88(9):1679–1692 DOI 10.1080/00207179.2015.1013061.

Zhang Z, Wu Y. 2014. Switching-based asymptotic stabilisation of underactuated ships with non-diagonal terms in their system matrices. *Control Theory & Applications IET* 9(6):972–980 DOI 10.1049/iet-cta.2014.0869.