Q-Factor and Bandwidth of Periodic Antenna Arrays Over Ground Plane

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Abstract—In this letter, the Q-factor expression for periodic arrays over a ground plane is determined in terms of the electric current density within the array’s unit cell. The expression accounts for the exact shape of the array element. The Q-factor formula includes integration only over a volume of an element in a unit cell and can thus be efficiently implemented numerically. The examples show good agreement between the proposed Q-factor, the full-wave-calculated tuned fractional bandwidth, and the input-impedance-based formula by Yaghjian and Best (IEEE Trans. Antennas Propag., vol. 53, no. 4, pp. 1298–1324, Apr. 2005) for the array of tilted dipoles and the loops array.

Index Terms—Antenna Q, electromagnetic theory, periodic structures, quality factor, stored energies.

I. INTRODUCTION

The Q-factor gives accurate prediction of the fractional bandwidth, when an antenna is not too wideband (i.e., $Q \geq 5$) [1]. Thus, expressing the Q-factor in terms of physical quantities of antenna provides an understanding of the connection between bandwidth and those quantities. In particular, the Q-factor representation in terms of the electric current density [2]–[4] has been used to obtain Q-factor bounds for finite-sized antennas, subject to various constraints on, e.g., antenna shape, size, or radiation pattern [5]–[10]. As a step toward obtaining similar type of bounds for large array antennas, we have recently reported the Q-factor expression in terms of electric current density for unit cell representations of antenna arrays [11]. The primary goal of this letter is to extend the expression to a practically important case for arrays: the presence of a ground plane.

The unit-cell Q-factor of a strip dipole array was derived by Kwon and Pozar, where one propagating mode was assumed [12]. We generalized in [11] the free-space case to include arbitrary three-dimensional (3-D) geometries of array elements, and to permit arbitrarily directed currents and multiple propagating modes. Yaghjian and Best input-impedance-based formula [1], shown to be applicable to array structures in [11] and [12], provides a simple way to estimate the Q-factor from the input impedance and its frequency derivative; however, it does not give a direct connection to the electric current density. Such connection is instrumental in obtaining fundamental bounds.

This letter presents the derivation of the unit-cell Q-factor for arbitrarily shaped PEC arrays over the ground plane. We have used the method of images to express potentials and fields in order to evaluate stored energies. The resulting Q-factor expression is a quadratic form in terms of a current density at an array element. The numerical examples compare the Q-factor obtained by three different methods: the proposed Q-factor, the input-impedance $Q$ by Yaghjian and Best [1], and a $Q$-equivalent representing a full-wave-solver determined tuned bandwidth.

II. STORED ENERGIES FOR GROUND-PLANE CASE

Consider a phased array of PEC elements on a rectangular infinitely periodic grid over a ground plane. The array elements are finite, in general, 3-D, arbitrarily shaped, and regular enough to support a solution to Maxwell’s equations. An example of such a structure is shown in Fig. 1, where the metal regions are given by gray color. The blue column represents a unit-cell region where $U_d = \{(x, y, z) \in \mathbb{R}^3 : x \in [0, a], y \in [0, b], z \in [0, d]\}$, where $a, b > 0$ are the grid periods. The phased array configuration with a phase-shift vector $k_{t00}$ is imposed by the condition on the current density

$$J(r + \zeta_{mn}) = J(r)e^{ik_{t00}\zeta_{mn}}$$

where $r \in \mathbb{R}^3$ is a position vector and $\zeta_{mn} = am\hat{x} + bny; m, n \in \mathbb{Z}$. The goal here is to find an expression for a Q-factor of such an array represented by the current density $J$ in a unit cell. This letter is entirely in frequency domain, and the time-dependent phase factor $e^{j\omega t}$ is assumed but omitted. The tuned Q-factor is defined as [13]

$$Q = \max(Q_c, Q_m), \quad Q_{c/m} = \frac{2\omega W_{c/m}}{P_d}$$

Fig. 1. Example of an array over a ground plane with a unit-cell region represented by a blue column, and metal is shown by gray color.
where $W_{e/m}$ is electric/magnetic stored energy, and $P_d$ is the dissipated power. For lossless radiating structures, the dissipated power is equal to the radiated power $P_r$.

The ground-plane configuration is treated by the method of images. The total current in the image problem is given by [14]

$$J_{tot}(r) = J(r) - \overline{J}_z \cdot J(r_1)$$

(3)

where $\overline{J}_z = \hat{x}x + \hat{y}y - \hat{z}z$ is a dyadic that inverts the sign of $z$-component of a vector, and the image coordinate is $r_1 = \overline{r} \cdot r$.

The vector and scalar potentials associated with the total current are (the Lorenz gauge is assumed [15], [16])

$$A(r_1) = \mu \int_{\Omega} \left[ G(r_1, r_2) \overline{r} - G(r_1, r_2) \overline{r}_z \right] \cdot J(r_2) d\Omega$$

(4)

$$\phi(r_1) = -\frac{1}{\omega e} \int_{\Omega} \left( \nabla_2 \left[ G(r_1, r_2) - G(r_1, r_2) \right] \right) \cdot J(r_2) d\Omega$$

(5)

Here, $G(r_1, r_2)$ is a free-space 2-D periodic Green’s function (30), and $\mu$ and $e$ are the permeability and permittivity, respectively. The electric and magnetic fields are found from the potentials via

$$E = -\nabla \phi - j\omega A, \quad H = \frac{1}{\mu} \nabla \times A.$$ 

(6)

The electric and magnetic stored energies are given by subtracting the energy density of the propagating Floquet modes from the total energy density

$$W_e = \frac{\epsilon}{4} \int_{U_\infty} |E|^2 - |E_p|^2 d\nu$$

(7)

$$W_m = \frac{\mu}{4} \int_{U_\infty} |H|^2 - |H_p|^2 d\nu.$$ 

(8)

Here, $E_p$ and $H_p$ are electric and magnetic fields associated with propagating Floquet modes. They are obtained similarly to $E, H$ with the Green’s function $G(r_1, r_2)$ replaced by its propagating-modes part $G_p(r_1, r_2)$ [see (31)]. Note that the energy densities are integrated above the ground plane. A similar definition of the stored energies for the free-space case is used in our earlier work [11].

We follow the same approach as in the free-space case [11] to derive the stored energies. The electric stored energy is represented by the potentials

$$W_e = \frac{\epsilon}{4} \int_{U_\infty} \left( |\nabla \phi|^2 - k^2 |\phi|^2 \right) + \omega^2 (|A|^2 - |A_p|^2) - k^2 (|\phi|^2 - |\phi_p|^2) d\nu.$$

(9)

Each pair in the integrand is then addressed separately to derive the current density representation.

The representation of the first pair in terms of the current density is found from the scalar potential (5) and the continuity equation $j_0 \omega \phi = -\nabla \times J$ as

$$W_{e,1} = \frac{\epsilon}{4} \int_{U_\infty} |\nabla \phi|^2 - k^2 |\phi|^2 d\nu = \frac{1}{4} \text{Re} \left\{ \int_{\Omega} \phi \, d\nu \right\}$$

$$= \int_{\Omega} \int J^*(r_1) \cdot K_{e,1}(r_1, r_2) \cdot J(r_2) d\nu_1 d\nu_2$$

(10)

where the dyadic kernel is

$$K_{e,1}(r_1, r_2) = \frac{\mu}{4k^2} \text{Re} \left\{ \overline{\nabla}_1 \nabla_2 [G(r_1, r_2) - G(r_1, r_2)] \right\}.$$

(11)

Here, $\overline{\nabla}_1$ is a Jacobian with respect to $r_1$.

The vector-potential pair is evaluated by straightforward substitution of the magnetic vector potential (4) and the vector potential $A_p$, associated with the propagating modes and found by replacing $G$ with $G_p$ in (4)

$$W_{em,1} = \frac{\epsilon \omega^2}{4} \int_{U_\infty} (|A(r)|^2 - |A_p(r)|^2) d\nu$$

$$= \int_{\Omega} \int J^*(r_1) \cdot K_{e,1}(r_1, r_2) \cdot J(r_2) d\nu_1 d\nu_2$$

(12)

with

$$K_{em,1}(r_1, r_2) = \frac{\mu k^2}{4} \left( g(r_1, r_2) \overline{r} - g(r_1, r_2) \overline{r}_z \right).$$

(13)

Here, the function $g$ is defined by

$$g(r_1, r_2) = \int_{U_\infty} G^*(r, r_1) G(r, r_2) - G_p^*(r, r_1) G_p(r, r_2) + G^*(r, r_1) G(r, r_2) - G_p^*(r, r_1) G_p(r, r_2) d\nu.$$ 

(14)

Note here that the integration is performed for the unit-cell column at $z \geq 0$. Utilizing the $z$-symmetry of the Green’s function, we can rewrite the integral as

$$g(r_1, r_2) = \int_{U_\infty} G^*(r, r_1) G(r, r_2) - G_p^*(r, r_1) G_p(r, r_2) d\nu.$$ 

(15)

where $U_\infty$ denotes the unit-cell column at $z \leq 0$. We recognize the calculation of $g$ from [11] to find the analytic expression (32).

The last term for the scalar-potential pair is obtained analogously

$$W_{em,2} = \frac{\epsilon \omega^2}{4} \int_{U_\infty} (|\phi|^2 - |\phi_p|^2) d\nu$$

$$= \int_{\Omega} \int J^*(r_1) \cdot K_{em,2}(r_1, r_2) \cdot J(r_2) d\nu_1 d\nu_2$$

(16)

with

$$K_{em,2}(r_1, r_2) = \frac{\mu k^2}{4} \overline{\nabla}_1 \nabla_2 (g(r_1, r_2) - g(r_1, r_2)).$$

(17)

For lossless materials, Poynting theorem [15] gives

$$W_m = W_e + \frac{1}{2\omega} \text{Im} P_c$$

(18)

$$P_c = -\frac{1}{2} \int_{\Omega} E \cdot J^* d\nu.$$ 

(19)

is the complex power. Substitution of electric field (6) and potentials (4) and (5) into (19) gives

$$P_c = \int_{\Omega} \int J^*(r_1) \cdot K_p(r_1, r_2) \cdot J(r_2) d\nu_1 d\nu_2$$

(20)

where

$$K_p(r_1, r_2) = -\frac{\eta}{2k} \overline{\nabla}_1 \nabla_2 [G(r_1, r_2) - G(r_1, r_2)]$$
We introduce the magnetic contribution to stored energy as

$$W_{m,1} = \int_\Omega \int_\Omega J^*(r_1) \cdot \mathbf{K}_{m,1}(r_1, r_2) \cdot J(r_2) \, dv_2 \, dv_1$$

(22)

with

$$\mathbf{K}_{m,1}(r_1, r_2) = \frac{1}{4} \mathbb{I} \text{Re} \left\{ G(r_1, r_2) \bar{\mathbb{T}} - G(r_1, r_2) \bar{T}_z \right\}.$$  

(23)

Then, the imaginary part of (20) is identified as

$$\text{Im} \, P_e = 2\omega (-W_{e,1} + W_{m,1}).$$

(24)

The electric and magnetic stored energies, thus, are

$$W_e = W_{e,1} + W_{e,m,1} - W_{e,m,2}$$

(25)

$$W_m = W_{m,1} + W_{e,m,1} - W_{e,m,2}.$$  

(26)

The radiated power is found from the Poyntings theorem [15] as

$$P_t = \text{Re} \, P_e.$$  

(27)

### III. Numerical Examples

In this section, we compare the derived Q-factor for arrays over a ground plane with a tuned bandwidth, recalculated into an equivalent Q-factor $Q_{dB}$, as described in the following paragraph. We also compare our results with the input-impedance-based formula by Yaghjian and Best [1] $Q_{dB}$. In all the examples, the proposed Q-factors $Q_e$ and $Q_m$ are computed based on our in-house method of moments (MoM) code with Rao–Wilton–Glisson (RWG) basis functions on a triangular mesh.

The tuned fractional bandwidth $B(\omega)$ at each angular frequency $\omega$ is obtained as in [1, Sec. VI]. The input impedance from a full-wave simulation is matched for the frequency $\omega$ by a series lossless reactive element (inductor or capacitor). The fractional bandwidth is then estimated from the matched reflection coefficient $\Gamma$ for a given threshold $\Gamma_0$ ($-10$ dB in all numerical examples here). The tuned fractional bandwidth is then recalculated into the equivalent Q-factor by

$$Q_{dB} = \frac{2\Gamma_0}{B \sqrt{1 - \Gamma_0^2}}.$$  

(28)

In the examples here, to compute $Q_{dB}$, we used the input impedance from a unit-cell frequency-domain simulation in CST Microwave Studio.

The Q-factor by Yaghjian and Best [1], based on input impedance $(\mathbf{R} + j\mathbf{X})$, is calculated as

$$Q_Z(\omega) = \frac{\omega}{2R(\omega)} \sqrt{[R'(\omega)]^2 + [X'(\omega)]^2 + |X(\omega)|^2}.$$  

(29)

Here, $(\cdot)'$ is a derivative with respect to $\omega$.

#### A. Tilted Dipole Array

In the first example, we test the derived expression on an array of tilted dipoles. The length of each dipole element is $l$, the width $w = l/40$, and the unit-cell period is $p = 1.2l$ in both directions. The dipoles are tilted by an angle $\alpha = 20^\circ$ as compared with the ground plane, and their center is at the distance $d = 0.25l$ above the ground plane. Each element is fed in its center by a voltage-gap excitation, and the broadside radiation case is considered. The comparison of different Q-factor methods is shown in Fig. 2: our expression $\max(Q_e, Q_m)$ (dashed lines), the tuned bandwidth from CST simulation $Q_{dB}$ (magenta line), and the input-impedance-based Q-factor $Q_Z$ calculated from the impedance given by both our MoM code (black dotted curve) and CST simulation (green curve). In the following, for the first grating lobe, located at $kl \approx 5.2$, an overall good agreement between all the methods is observed; the proposed expression slightly overestimates the Q-factor at $kl \approx 3 - 4.5$. Above the electrical length $kl \approx 5.2$, all methods give somewhat different Q-factor values; however, the second grating lobe at $kl \approx 7.4$ is captured by all of the methods. The differences in the curves at the electric lengths between the two grating lobes are due to the limited validity of the Q-factor description at low Q-values, as $Q_{dB} \approx 5$ there. The Q-factor approximation of fractional bandwidth is less predictive for low Q-values [17].

#### B. Beam Scanning

We consider the beam scanning of the tilted dipoles array from the previous example at $kl \approx 2.7$. The beam-scanning angles, polar $\theta_0$ and azimuthal $\phi_0$, enter the Green’s function (30) via the phase-shift vector $k_{10} = k \sin \theta_0 \cos \phi_0 \hat{x} + k \sin \theta_0 \sin \phi_0 \hat{y}$. The Q-factor as a function of the polar angle $\theta_0$ is shown in Fig. 3 for E-plane ($\phi_0 = 0^\circ$, red curves) and H-plane ($\phi_0 = 90^\circ$, blue curves), where the azimuthal angle is counted from the positive x-semiaxis (see the inset of Fig. 2). The Q-factor $Q = \max(Q_e, Q_m)$ calculated by the proposed expression is shown by solid curves; the tuned bandwidth, recalculated in the equivalent Q-factor $Q_{dB}$, is given by dashed lines; and the input-impedance Q-factor $Q_Z$ is shown by dotted lines. All the Q-factor methods agree reasonably well. The slight discrepancies between the proposed expression and the other two methods are observed around $\theta_0 = \pm 60^\circ$ in both planes. The phased array blind spots at $\theta_0 = \pm 67^\circ$ are depicted by all the methods. The differences between the proposed Q-factor and $Q_{dB}$ are due to the grating lobe in frequency domain, perturbing the Q-factor model. Due to the symmetry of the unit cell with respect to the $xz$ plane, the H-plane Q-factor is symmetric in $\theta_0$. Although the unit cell is not symmetric with respect to the $yz$ plane, the asymmetry of the Q-factor in the E-plane is negligible.
C. Square Loop Array

The $Q$-factor for an array of rectangular loops is shown in Fig. 4. The loop size is $l/2 \times l$, strip width is $w = l/10$, unit cell period is $p = 1.2l$, and the distance between the elements and the ground plane is $d = 0.75l$, which corresponds to a quarter of the loops perimeter. The loops are fed by a voltage grating lobe. The distance $kl$ and the ground plane is cell period is $l/kl \approx 2.7$. The part of the $Q$-factor methods is related with the fact that low $Q$-factor has a larger uncertainty in estimating fractional bandwidth [17].

The derived $Q$-factor expression is suitable for current–density optimization to obtain lower bounds on array $Q$-factor. In that case, inclusion of, for example, constraint on conductive losses in the optimization problem is of practical interest. The losses in array can be accounted by a surface resistance model, similarly to nonperiodic case (see, e.g., [18]).

APPENDIX

The two-dimensionally periodic free-space Green’s function in the spectral form is [19]

\[
G(r_1, r_2) = \frac{1}{2iab} \sum_{(m,n) \in \mathbb{Z}^2} \frac{1}{k_{zmn}} e^{-jk_{tmn} \rho_1 \rho_2} e^{-jk_{zmn} z_1 z_2} \quad (30)
\]

with $k_{tmn} = k_{100} + 2\pi \frac{2}{l} \hat{x} + 2\pi \frac{m}{l} \hat{y}$, $k_{zmn} = \sqrt{k^2 - k_{tmn} \cdot k_{tmn}}$, and $z_i = x_i \cdot \hat{z}$, $\rho_i = x_i - z_i z_i$, $i = \{1, 2\}$. The part of the Green’s function that is associated with the propagating modes is

\[
G_D(r_1, r_2) = \frac{1}{2iab} \sum_{(m,n) \in \mathcal{D}} \frac{1}{k_{zmn}} e^{-jk_{tmn} \rho_1 \rho_2} e^{-jk_{zmn} z_1 z_2} \quad (31)
\]

where $\mathcal{D} = \{(m, n) : k^2 - k_{tmn} \cdot k_{tmn} \geq 0\}$ is the set of propagating modes.

The integral (15) reduces to [11]

\[
g(r_1, r_2) = \frac{1}{4iab} \sum_{(m,n) \in Z^2} \frac{1}{|k_{zmn}|^2} jk_{tmn} (\rho_1 - \rho_2) e^{-jk_{tmn} |z_2 - z_1|} \left( \frac{1}{|k_{zmn}| + |z_1 - z_2|} \right). \quad (32)
\]
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