General Monogamy Relations of Quantum Entanglement for Multiqubit W-class States

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Entanglement monogamy is a fundamental property of multipartite entangled states. We investigate the monogamy relations for multiqubit generalized W-class states. Analytical monogamy inequalities are obtained for the concurrence of assistance, the entanglement of formation and the entanglement of assistance.

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I. INTRODUCTION

Quantum entanglement\textsuperscript{[1-6]} is an essential feature of quantum mechanics that distinguishes the quantum from the classical world. It is one of the fundamental differences between quantum entanglement and classical correlations that a quantum system entangled with one of the other systems limits its entanglement with the remaining others. This restriction of entanglement shareability among multi-party systems is known as the monogamy of entanglement. The monogamy relations give rise to the structures of entanglement in the multipartite setting. For a tripartite system \( A, B, \) and \( C, \) the monogamy of an entanglement measure \( \varepsilon \) implies that the entanglement between \( A \) and \( BC \) satisfies

\[ \varepsilon_{A|BC} \geq \varepsilon_{AB} + \varepsilon_{AC}. \]

In Ref.\textsuperscript{[7,8]} the monogamy of entanglement for multiqubit \( W \)-class states has been investigated, and the monogamy relations for tangle and the squared concurrence have been proved. In this paper, we show the general monogamy relations for the \( x \)-power of concurrence of assistance, the entanglement of formation, and the entanglement of assistance for generalized multiqubit \( W \)-class states.

II. MONOGAMY OF CONCURRENCE OF ASSISTANCE

For a bipartite pure state \(|\psi\rangle_{AB}\) in vector space \( H_A \otimes H_B \), the concurrence is given by \textsuperscript{[9-11]}

\[ C(|\psi\rangle_{AB}) = \sqrt{2[1 - Tr(\rho_A^2)]}, \quad (1) \]

where \( \rho_A \) is reduced density matrix by tracing over the subsystem \( B, \rho_A = Tr_B(|\psi\rangle_{AB}\langle\psi|). \) The concurrence is extended to mixed states \( \rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|, p_i \geq 0, \sum_i p_i = 1, \) by the convex roof construction,

\[ C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \quad (2) \]

where the minimum is taken over all possible pure state decompositions of \( \rho_{AB}. \)

For a tripartite state \(|\psi\rangle_{ABC}\), the concurrence of assistance (CoA) is defined by \textsuperscript{[12]}

\[ C_a(|\psi\rangle_{ABC}) = C_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \quad (3) \]

for all possible ensemble realizations of \( \rho_{AB} = Tr_C(|\psi\rangle_{ABC}\langle\psi|) = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|). \) When \( \rho_{AB} = |\psi\rangle_{AB}\langle\psi| \) is a pure state, then one has \( C(|\psi\rangle_{AB}) = C_a(\rho_{AB}). \)

For an \( N \)-qubit state \(|\psi\rangle_{AB_1...B_{N-1}} \in H_A \otimes H_{B_1} \otimes ... \otimes H_{B_{N-1}}, \) the concurrence \( C(|\psi\rangle_{A|B_1...B_{N-1}}) \) of the state \(|\psi\rangle_{A|B_1...B_{N-1}}, \) viewed as a bipartite with partitions \( A \) and \( B_1B_2...B_{N-1}, \) satisfies the following inequality \textsuperscript{[13]}

\[ C_a^\alpha_{A|B_1B_2...B_{N-1}} \geq C_{A|B_1}^\alpha + C_{A|B_2}^\alpha + ... + C_{A|B_{N-1}}^\alpha, \quad (4) \]

and

\[ C_{A|B_1B_2...B_{N-1}}^\beta < C_{A|B_1}^\beta + C_{A|B_2}^\beta + ... + C_{A|B_{N-1}}^\beta, \quad (5) \]
where \( \alpha \geq 2, \beta \leq 0, C_{AB} = C(\rho_{AB}) \) is the concurrence of \( \rho_{AB} = Tr_{B_{1-1}B_{i+1}...B_{N-1}}(\rho) \), \( C_{A|B_1B_2...B_{N-1}} = C(\langle \psi \rangle_{A|B_1B_2...B_{N-1}}) \). Due to the monogamy of concurrence, the generalized monogamy relation based on the concurrence of assistance has been proved in Ref. [14],

\[
C^2(\langle \psi \rangle_{A|B_1...B_{N-1}}) \leq \sum_{i=1}^{N-1} C^2(\rho_{AB_i}).
\] (6)

In the following we study the monogamy property of the concurrence of assistance for the \( n \)-qubit generalized W-class states \( |\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes ... \otimes H_{A_n} \) defined by

\[
|\psi\rangle = a|000...\rangle + b_1|01...0\rangle + ... + b_n|0001\rangle,
\] (7)

with \( |a|^2 + \sum_{i=1}^{n} |b_i|^2 = 1 \).

**Lemma 1** For \( n \)-qubit generalized W-class states [4], we have

\[
C(\rho_{A_1A_i}) = C_a(\rho_{A_1A_i}),
\] (8)

where \( \rho_{A_1A_i} = Tr_{A_2...A_{i-1}A_{i+1}...A_n}(|\psi\rangle\langle \psi|) \).

**Proof** It is direct to verify that [4], \( \rho_{A_1A_i} = |x\rangle_{A_1A_i}\langle x| + |y\rangle_{A_1A_i}\langle y| \), where

\[
|x\rangle_{A_1A_i} = a|00\rangle_{A_1A_i} + b_1|10\rangle_{A_1A_i} + b_i|01\rangle_{A_1A_i},
\]

\[
|y\rangle_{A_1A_i} = \sqrt{\sum_{k \neq i} |b_k|^2} |00\rangle_{A_1A_i}.
\]

From the Hughston- Jozsa-wootters theorem Ref. [7], for any pure-state decomposition of \( \rho_{A_1A_i} = \sum_{h=1}^r |\phi_h\rangle_{A_1A_i}\langle \phi_h| \), one has \( |\phi_h\rangle_{A_1A_i} = u_{h1}|x\rangle_{A_1A_i} + u_{h2}|y\rangle_{A_1A_i} \) for each \( h \). Consider the normalized state \( |\tilde{\phi}_h\rangle_{A_1A_i} = |\phi_h\rangle_{A_1A_i}/\sqrt{p_h} \) with \( p_h = |\langle \phi_h|\phi_h\rangle| \). One has the concurrence of each two-qubit pure \( |\tilde{\phi}_h\rangle_{A_1A_i} \),

\[
C^2(|\tilde{\phi}_h\rangle_{A_1A_i}) = \frac{4}{p_h^2} |u_{h1}|^4|b_1|^2|b_i|^2.
\]

Then for the two-qubit state \( \rho_{A_1A_i} \), we have

\[
\sum_h p_h C(|\tilde{\phi}_h\rangle_{A_1A_i}) = \sum_h p_h \frac{2}{p_h^2} |u_{h1}|^2|b_1||b_i| = 2|b_1||b_i|.
\]

Thus we obtain

\[
C(\rho_{A_1A_i}) = \min_{\{p_h, |\tilde{\phi}_h\rangle_{A_1A_i}\})} \sum_h p_h C(|\tilde{\phi}_h\rangle_{A_1A_i})
\]

\[
= \max_{\{p_h, |\tilde{\phi}_h\rangle_{A_1A_i}\})} \sum_h p_h C(|\tilde{\phi}_h\rangle_{A_1A_i})
\]

\[
= C_a(\rho_{A_1A_i}).
\]  

Specifically, in Ref. [5] the same result \( C(\rho_{A_1A_i}) = C_a(\rho_{A_1A_i}) \) has been proved for the generalized W-class states [7] with \( a = 0 \).

**Theorem 1** For the \( n \)-qubit generalized W-class states \( |\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes ... \otimes H_{A_n} \), the concurrence of assistance satisfies

\[
C_a^x(\rho_{A_1|A_{j_1}...A_{j_{m-1}}}) \geq \sum_{i=1}^{m-1} C_a^x(\rho_{A_1A_i}),
\] (9)

where \( x \geq 2 \) and \( \rho_{A_1A_j...A_{j_{m-1}}} \) is the \( m \)-qubit, \( 2 \leq m \leq n \), reduced density matrix of \( |\psi\rangle \)
Theorem 2 For the due to the monogamy of concurrence (4). The last equality is due to the Lemma 1.

Here we have used in the first inequality the inequality

\[ a^x \geq b^x \text{ for } a \geq b > 0 \text{ and } x \geq 0. \]

The second inequality is due to the monogamy of concurrence \[11\]. The last equality is due to the Lemma \[1\].

Theorem 2 For the \( n \)-qubit generalized \( W \)-class state \( |\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes \cdots \otimes H_{A_n} \) with \( C(\rho_{A_1 A_2 \cdots A_n}) \neq 0 \) for \( 1 \leq i \leq m-1 \), we have

\[
C_a^y(\rho_{A_1|A_{j_1} \cdots A_{j_{m-1}}}) \leq \sum_{i=1}^{m-1} C_a^y(\rho_{A_1 A_i}),
\]

where \( y \leq 0 \) and \( \rho_{A_1 A_{j_1} \cdots A_{j_{m-1}}} \) is the \( m \)-qubit reduced density matrix as in Theorem 1.

[Proof] For \( y \leq 0 \), we have

\[
C_a^y(\rho_{A_1|A_{j_1} \cdots A_{j_{m-1}}}) \leq \sum_{i=1}^{m-1} C_a^y(\rho_{A_1 A_i}),
\]

We have used in the first inequality the relation \( a^x \leq b^x \) for \( a \geq b > 0 \) and \( x \leq 0 \). The second inequality is due to the monogamy of concurrence \[11\]. The last equality is due to Lemma 1.

According to \[19\] and \[10\], we can also obtain the lower bounds of \( C_a(\rho_{A_1|A_{j_1} \cdots A_{j_{m-1}}}) \). As an example, consider the 5-qubit generalized \( W \)-class states \[12\] with \( a = b_2 = \frac{1}{\sqrt{10}}, b_1 = \frac{1}{\sqrt{15}}, b_3 = \sqrt{\frac{2}{15}}, b_4 = \frac{3}{5} \). We have

\[
C_a(\rho_{A_1 A_2 A_3}) \geq \frac{2}{\sqrt{15}} \sqrt{ \left( \frac{1}{\sqrt{10}} \right)^x + \left( \frac{2}{15} \right)^x }
\]

and

\[
C_a(\rho_{A_1 A_2 A_3 A_4}) \geq \frac{2}{\sqrt{15}} \sqrt{ \left( \frac{1}{\sqrt{10}} \right)^x + \left( \frac{2}{15} \right)^x + \left( \frac{3}{5} \right)^x }
\]

with \( x \geq 2 \). The optimal lower bounds can be obtained by varying the parameter \( x \), see Fig. 1, where for comparison the upper bounds are also presented by using the formula \( C_a(\rho_{A B}) \leq \sqrt{2(1-T(\rho_A))} \) \[12\], namely, \( C_a(\rho_{A_1 A_2 A_3}) \leq \frac{2}{\sqrt{15}} \) and \( C_a(\rho_{A_1 A_2 A_3 A_4}) \leq \frac{2}{\sqrt{15}} \). From Fig. 1, one gets that the optimal lower bounds of \( C_a(\rho_{A_1 A_2 A_3}) \) and \( C_a(\rho_{A_1 A_2 A_3 A_4}) \) are 0.249 and 0.471, respectively, attained at \( x = 2 \).

III. MONOGAMY OF ENTANGLEMENT OF FORMATION

The entanglement of formation of a pure state \( |\psi\rangle \in H_A \otimes H_B \) is defined by

\[
E(|\psi\rangle) = S(\rho_A),
\]

(11)
Fig. 1: Solid line is the lower bound of \( C(\rho_{A_1|A_2A_3}) \), dashed line is the lower bound of \( C(\rho_{A_1|A_2A_3A_4}) \) as functions of \( x \geq 2 \), and dotted line is the upper bound of \( C(\rho_{A_1|A_2A_3}) \) and \( C(\rho_{A_1|A_2A_3A_4}) \).

where \( \rho_A = Tr_B(\ket{\psi}\bra{\psi}) \) and \( S(\rho) = Tr(\rho \log_2 \rho) \). For a bipartite mixed state \( \rho_{AB} \in H_A \otimes H_B \), the entanglement of formation is given by

\[
E(\rho_{AB}) = \min_{\{p_i, \ket{\psi_i}\}} \sum_i p_i E(\ket{\psi_i}),
\]

with the infimum taking over all possible decompositions of \( \rho_{AB} \) in a mixture of pure states \( \rho_{AB} = \sum_i p_i \ket{\psi_i}\bra{\psi_i} \), where \( p_i \geq 0 \) and \( \sum_i p_i = 1 \).

It has been shown that the entanglement of formation does not satisfy the inequality \( E_{AB} + E_{AC} \leq E_{A|BC} \) \[^{16}\]. Rather it satisfies \[^{13}\],

\[
E^\alpha_{A|B_1B_2...B_{N-1}} \geq E^\alpha_{AB_1} + E^\alpha_{AB_2} + ... + E^\alpha_{AB_{N-1}},
\]

where \( \alpha \geq \sqrt{2} \).

The corresponding entanglement of assistance (EoA) \[^{17}\] is defined in terms of the entropy of entanglement \[^{18}\] for a tripartite pure state \( \ket{\psi}_{ABC} \),

\[
E_a(\ket{\psi}_{ABC}) \equiv E_a(\rho_{AB}) = \max_{\{p_i, \ket{\psi_i}\}} \sum_i p_i E(\ket{\psi_i}),
\]

which is maximized over all possible decompositions of \( \rho_{AB} = Tr_C(\ket{\psi}_{ABC}) = \sum_i p_i \ket{\psi_i}\bra{\psi_i} \), with \( p_i \geq 0 \) and \( \sum_i p_i = 1 \). For any \( N \)-qubit pure state \( \ket{\psi} \in H_A \otimes H_{B_1} \otimes \ldots \otimes H_{B_{N-1}} \), it has been shown that the entanglement of assistance satisfies \[^{13}\],

\[
E(\ket{\psi}_{A|B_1B_2...B_{N-1}}) \leq \sum_{i=1}^{N-1} E_a(\rho_{AB_i}).
\]

In fact, generally we can prove the following results for the \( n \)-qubit generalized W-class states about the entanglement of formation and the entanglement of assistance.

**Theorem 3** For the \( n \)-qubit generalized W-class states \( \ket{\psi} \in H_{A_1} \otimes H_{A_2} \otimes \ldots \otimes H_{A_n} \), we have

\[
E(\ket{\psi}_{A_1A_2...A_n}) \leq \sum_{i=2}^{n} E(\rho_{A_1A_i}),
\]

where \( \rho_{A_1A_i}, \ 2 \leq i \leq n, \) is the 2-qubit reduced density matrix of \( \ket{\psi} \).
[Proof] For the $n$-qubit generalized W-class states $|\psi\rangle$, we have
\[
E(|\psi\rangle_{A_1|A_2...A_n}) = f\left(C^2(|\psi\rangle_{A_1|A_2...A_n})\right) \\
= f\left(\sum_{i=2}^{n} C^2(\rho_{A_iA_i})\right) \\
\leq \sum_{i=2}^{n} f(C^2(\rho_{A_iA_i})) \\
= \sum_{i=2}^{n} E(\rho_{A_iA_i}),
\]
where for simplify, we have denoted $f(x) = h(\frac{1+x}{2})$ with $h(x) = -x \log_2(x) - (1 - x) \log_2(1 - x)$. We have used in the first and last equalities that the entanglement of formation obeys the relation $E(\rho) = f(C^2(\rho))$ for a bipartite $2 \otimes D$, $D \geq 2$, quantum state $\rho$. The second equality is due to the fact that $C^2(|\psi\rangle_{A_1...A_n}) = \sum_{i=2}^{n} C^2(\rho_{A_iA_i})$. The inequality is due to the fact $f(x + y) \leq f(x) + f(y)$.

As for the entanglement of assistance, we have the following conclusion.

**Theorem 4** For the $n$-qubit generalized W-class states $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes ... \otimes H_{A_n}$, we have
\[
E(\rho_{A_1|A_{j_1}...A_{j_{m-1}}}) \leq \sum_{i=1}^{m-1} E(\rho_{A_{j_i}A_{j_i}}), \tag{17}
\]
where $\rho_{A_1|A_{j_1}...A_{j_{m-1}}}$ is the $m$-qubit reduced density matrix of $|\psi\rangle$, $2 \leq m \leq n$.

[Proof] From the lemma 2 of Ref.[7], one has $\rho_{A_1|A_{j_1}...A_{j_{m-1}}}$ of $|\psi\rangle$ is a mixture of a generalized W class state and vacuum. Then, we have
\[
E(\rho_{A_1|A_{j_1}...A_{j_{m-1}}}) \leq \sum_{h} p_h E(|\psi^h_{A_1|A_{j_1}...A_{j_{m-1}}}) \\
\leq \sum_{h} p_h \sum_{i=1}^{m-1} E(\rho^h_{A_{j_i}A_{j_i}}) \\
= \sum_{i=1}^{m-1} \left[ \sum_{h} p_h E(\rho^h_{A_{j_i}A_{j_i}}) \right] \\
\leq \sum_{i=1}^{m-1} \left[ \sum_{h} p_h \left( \sum_{j} q_j E(|\psi^h_{j_{A_{j_i}}}\langle\psi_j|) \right) \right] \\
= \sum_{h} \sum_{j} p_h q_j E(|\psi^h_{j_{A_{j_i}}}\langle\psi_j|).
\]

We obtain the first inequality by noting that $|\psi^h_{A_1|A_{j_1}...A_{j_{m-1}}}$ is a generalized W class state or vacuum[7]. When $|\psi^h_{A_1|A_{j_1}...A_{j_{m-1}}}$ is a generalized W class state, then we have $E(|\psi^h_{A_1|A_{j_1}...A_{j_{m-1}}}) \leq \sum_{i=1}^{m-1} E(\rho^h_{A_{j_i}A_{j_i}})$; When $|\psi^h_{A_1|A_{j_1}...A_{j_{m-1}}}$ is a vacuum, then we have $E(|\psi^h_{A_1|A_{j_1}...A_{j_{m-1}}}) = 0 \leq \sum_{i=1}^{m-1} E(\rho^h_{A_{j_i}A_{j_i}})$. The second inequality is due to the definition of the entanglement of formation [12] for mixed quantum states. Since $\sum_{h} p_h q_j = 1$ and $\sum_{j} p_h q_j |\psi^h_{j_{A_{j_i}}}\langle\psi_j|$ is a pure decomposition of $\rho_{A_{j_i}A_{j_i}}$, we have [17].

### IV. CONCLUSIONS AND REMARKS

Entanglement monogamy is a fundamental property of multipartite entangled states. We have shown the monogamy for the $x$-power of concurrence of assistance $C_x(\rho_{A_1|A_{j_1}...A_{j_{m-1}}})$ of the $m$-qubit reduced density matrices, $2 \leq m \leq
n, for the n-qubit generalized $W-$class states. The monogamy relations for the entanglement of formation and the entanglement of assistance the monogamy relation for the n-qubit generalized W-class states have been also investigated. These relations give rise to the restrictions of entanglement distribution among the qubits in generalized $W-$class states.

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