Investigation of paraxial and nonparaxial self-focusing of Gaussian beam in chalcogenide glass medium using NLSE

Chironjit Hazarika1,*, Abhijeet Das2 and Subrata Hazarika2

1Department of Physics, Mariani College, Jorhat, Assam, India. 
2Department of Physics, Assam University, Diphu Campus, Karbi Anglong, Assam, India.

*E-mail: chironjithazarika@gmail.com

Abstract. Variational method is used for the investigation of paraxial and nonparaxial self-focusing of Gaussian beam in bulk chalcogenide glass (Kerr media). Stationary self-focusing for paraxial Gaussian beam with distinct singularity and focusing-defocusing cycles for nonparaxial propagation of Gaussian beam is observed. Emphasis is laid on the study of variation in beam width and intensity of the beam with propagation distance.

1. Introduction
Self-focusing of laser in nonlinear optical media has been a very important research topic since sixties of the last century because of its application in optical communication, signal processing, optical switching, controlled laser fusion etc [1-5]. Investigation of self-focusing of laser beam can be studied using nonlinear Schrödinger equation which is the model equation for the optical self-action processes [6]. In such studies, usually the paraxial ray approximation is adopted [7]. However, the paraxial ray approximation predicts formation of singularities at finite propagation distance when the power of the input laser beam is above a certain critical power [8, 9]. Since singularity formation is unphysical, from a mathematical point of view, the validity of the paraxial ray approximation is highly questionable. Nevertheless, in most cases in optics, the paraxial ray approximation leads to results that are in excellent agreement with experiments. In NLS model the scenario of self-focusing is much more complicated and the description of physical self-focusing near and beyond the singularity should include additional stabilizing mechanism. According to Fiet and Fleck, no singularity will form if beam nonparaxiality is included in the NLS model [10]. Beyond the paraxial approximation, the beam does not collapse as a whole and further it can be shown that both non-paraxial and the vectorial effect can arrest catastrophic collapse of the beam leading to periodic focusing and defocusing oscillation. The results of numerical simulations of NLSE show that self-focusing is arrested before the beam diameter goes below the order of one wavelength, followed by several focusing and defocusing cycles.

In this paper, variational method is used to investigate the nonlinear paraxial and non-paraxial propagation of Gaussian beam in chalcogenide glass medium. Dynamical equations derived from the variational analysis are used for the comparative study of the various beam parameters like intensity and beam width in the paraxial and nonparaxial propagation of Gaussian beam.

The present study bears significance as analytical methods for the study of the phenomenon of collapse-arresting are restricted to beams in which input powers are only slightly higher than critical power for self-focusing [11]. Also, in chalcogenide glass non-linear effects can be exploited at low optical powers due to large intensity dependent refractive index and their low linear absorption at optical
communication wavelength makes these glasses attractive materials for use in a variety of optical devices [12].

2. Basic formulation

The scalar Helmholtz equation for the propagation of a beam through a Kerr medium is given by

\[
\frac{\partial^2 E}{\partial z^2} + \nabla^2 E + \frac{\omega_0^2}{c^2} n^2 E = 0
\]

(1)

here \( E \) is the scalar optical field of frequency \( \omega_0 \) and \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the transverse Laplacian.

The total refractive index is \( n = n_0 + n_2 |E|^2 \), \( n_0 \) and \( n_2 \) being the linear and nonlinear refractive index respectively and \( c \) is the speed of light in vacuum. In the slowly varying envelope approximation, the field can be written as

\[ E(x, y, z) = \epsilon(x, y, z) \exp(ikz) \]

Now, for \( n_0 \gg n_2 \) and neglecting higher order terms, the resulting equation for the complex envelope \( \epsilon(x, y, z) \) is the non-paraxial nonlinear Schrödinger equation

\[
\frac{\partial \epsilon}{\partial z} = \frac{i}{2k} \nabla^2 \epsilon + i \frac{n_2}{n_0} k |\epsilon|^2 \epsilon + \frac{ik^2}{2k} \frac{\partial^2 \epsilon}{\partial z^2}
\]

(2)

where \( k = \frac{n_0 \omega_0}{c} \). The parameter \( f \) is introduced so that when \( f = 1 \) the term accounting for nonparaxial effect is incorporated and for \( f = 0 \), this equation reduces to well-known NLSE, which accounts for paraxial beam self-focusing.

The Lagrangian density, \( L \) from equation (2) can be written as

\[
L = |\epsilon_x|^2 + |\epsilon_y|^2 + \left( \epsilon \epsilon^* \frac{\partial \epsilon^*}{\partial z} - \epsilon^* \frac{\partial \epsilon}{\partial z} \right) \frac{n_2 k^2}{n_0} |\epsilon|^4 + f |\epsilon|^2
\]

(3)

The solution of the variational problem

\[ \delta \int L \, dx dy dz = 0 \]

(4)

also solves the nonlinear Schrodinger equation (2).

2.1. Paraxial analysis

Now, for \( f = 0 \) equation (3) becomes

\[
L = |\frac{\partial \epsilon}{\partial x}|^2 + |\frac{\partial \epsilon}{\partial y}|^2 + \left( \epsilon \epsilon^* \frac{\partial \epsilon^*}{\partial z} - \epsilon^* \frac{\partial \epsilon}{\partial z} \right) \frac{n_2 k^2}{n_0} |\epsilon|^4
\]

(5)

As a trial solution, the Gaussian field distribution function is considered for the procedure of the variational approach

\[
E(x, y, z) = A_1(z) \exp \left[ -\frac{x^2}{2\alpha^2} - \frac{y^2}{2\beta^2} \right] \exp \left[ \left\{ x^2 c_1(z) + y^2 c_2(z) + \phi(z) \right\} \right]
\]

(6)
where \( A_i(z) \) and \( \phi(z) \) are the amplitude and phase of the complex field \( E(x,y,z) \), \( \alpha \) and \( \beta \) are the width parameters of the Gaussian beam in \( x \) and \( y \) directions respectively, \( c_1(z) \) and \( c_2(z) \) are the curvature parameters in \( x \) and \( y \) directions.

Inserting the trial function into equation (5), and integration with respect to \( x \) and \( y \) yield the effective Lagrangian

\[
\langle L \rangle = \int_{-\infty}^{\infty} L \, dx \, dy
\]  

(7)

The effective Lagrangian can be written as,

\[
\langle L \rangle = 2\pi a^3 \beta A_i^2(z) \frac{d}{dz} c(z) + 2\pi a^3 \beta \frac{d}{dz} A_i^2(z) + \frac{1}{2\alpha} \sigma A_i^2(z) + \frac{1}{2\beta} \pi a A_i^2(z) + \\
2\pi a \beta A_i(z) \left[ \frac{\alpha^2}{2} \frac{\partial^2 c}{\partial z^2} + \frac{\beta^2}{2} \frac{\partial^2 c}{\partial z^2} + \frac{\partial \phi}{\partial z} \right] - \frac{\pi^2 a^2 k^2}{n_0} \alpha^2 \beta^2 A_i^2(z).
\]  

(8)

Now from the reduced variational principle, \( \delta \langle L \rangle \, dz = 0 \) i.e. considering the beam propagation along \( z \) direction, it is possible to derive variational equations with respect to \( \alpha, \beta, A_i(z), c_1(z), c_2(z), \phi(z) \) etc. Using the procedure of [13, 14] and after some rearrangement, the following differential equations for \( \alpha, \beta, c_1(z) \) and \( c_2(z) \) are obtained, which describe the evolution of the Gaussian beam in Kerr medium.

\[
c_1(z) = \frac{k}{2\alpha} \frac{d\alpha}{dz}
\]  

(9)

\[
c_2(z) = \frac{k}{2\beta} \frac{d\beta}{dz}
\]  

(10)

\[
\frac{\partial c_1}{\partial z} = -\frac{k}{2a^2} \left( \frac{d\alpha}{dz} \right)^2 + \frac{kH_0}{\alpha^2} + \frac{kH_0}{\alpha^2}
\]  

(11)

\[
\frac{\partial c_2}{\partial z} = -\frac{k}{2\beta^2} \left( \frac{d\beta}{dz} \right)^2 + \frac{kH_0}{\beta^2} + \frac{kH_0}{\beta^2}
\]  

(12)

where, \( H_0 = \frac{m_2 a \beta A_i^2(z)}{n_0} \)

(13)

\( c_1 \) and \( c_2 \) only depends on \( z \), so we can write \( \frac{\partial c_1}{\partial z} = \frac{dc_1}{dz} \) and \( \frac{\partial c_2}{\partial z} = \frac{dc_2}{dz} \).

Differentiating equation (9) and (10) and putting the values of \( \frac{dc_1}{dz} \) and \( \frac{dc_2}{dz} \) in equation (11) and (12) yield,
Equations (14) and (15) will be used for numerical analysis for paraxial wave propagation.

2.2. Nonparaxial analysis

As a trial solution of equation (3), the Gaussian field distribution function is considered for the procedure of the variational approach

\[ E(x, y, z) = A_1(z) \exp \left[ -\frac{x^2}{2\alpha^2} - \frac{y^2}{2\beta^2} \right] \exp \left[ \frac{1}{2} \left( c_1(z) + c_2(z) + \phi(z) \right) \right] \] (16)

where \( A_1(z), c_1(z) \) and \( c_2(z) \) are the amplitude and curvature of the complex field \( E(x, y, z) \) in \( x \) and \( y \) direction, \( \alpha \) and \( \beta \) are the width parameters of the Gaussian beam in \( x \) and \( y \) directions respectively.

Inserting the trial function into equation (3), and integrating with respect to \( x \) and \( y \) yield the effective Lagrangian \( \langle L \rangle \).

\[ \langle L \rangle = \int_{-\infty}^{\infty} L \, dx \, dy \] (17)

The effective Lagrangian can be written as

\[
\langle L \rangle = 2\pi^3 \beta A_1^2(z) \left( \frac{\alpha^2}{2} \frac{\partial c_1}{\partial z} + \frac{\beta^2}{2} \frac{\partial c_2}{\partial z} + \frac{\partial \phi}{\partial z} \right) - \frac{\pi^2 n_0 k^2}{\alpha^2} A_1^4(z) + \\
\left( \frac{3}{4} \left( A_1 \right) \alpha^5 \beta \frac{\partial^2 c_1}{\partial z^2} + \frac{3}{4} \left( A_1 \right) \alpha^3 \beta^3 \frac{\partial^2 c_2}{\partial z^2} + \frac{1}{2} \left( A_1 \right) \alpha \beta^2 \frac{\partial^2 \phi}{\partial z^2} \right) A_1^2(z) + \\
\left( \frac{3}{4} \left( A_1 \right) \alpha^5 \beta \frac{\partial c_1}{\partial z} \frac{\partial \phi}{\partial z} + \frac{3}{4} \left( A_1 \right) \alpha^3 \beta^3 \frac{\partial c_2}{\partial z} \frac{\partial \phi}{\partial z} + \frac{1}{2} \left( A_1 \right) \alpha \beta^2 \frac{\partial A_1}{\partial z} \frac{\partial c_1}{\partial z} \right) \frac{\partial^2 c_1}{\partial z^2} + \\
\left( \frac{3}{4} \left( A_1 \right) \alpha^5 \beta \frac{\partial c_1}{\partial z} \frac{\partial \phi}{\partial z} + \frac{3}{4} \left( A_1 \right) \alpha^3 \beta^3 \frac{\partial c_2}{\partial z} \frac{\partial \phi}{\partial z} + \frac{1}{2} \left( A_1 \right) \alpha \beta^2 \frac{\partial A_1}{\partial z} \frac{\partial c_1}{\partial z} \right) \frac{\partial^2 c_2}{\partial z^2} + \\
\left( \frac{3}{4} \left( A_1 \right) \alpha^5 \beta \frac{\partial c_1}{\partial z} \frac{\partial \phi}{\partial z} + \frac{3}{4} \left( A_1 \right) \alpha^3 \beta^3 \frac{\partial c_2}{\partial z} \frac{\partial \phi}{\partial z} + \frac{1}{2} \left( A_1 \right) \alpha \beta^2 \frac{\partial A_1}{\partial z} \frac{\partial c_1}{\partial z} \right) \frac{\partial^2 \phi}{\partial z^2}. \] (18)

Now from the reduced variational principle, \( \delta \int \langle L \rangle \, dz = 0 \) i.e. considering beam propagation along \( z \) direction, it is possible to derive variational equations with respect to \( A_1(z), c_1(z) \) and \( c_2(z) \). Using the procedure of [13, 14] the following differential equations are obtained, which describe the evolution of the Gaussian beam in Kerr medium.
\(6\alpha_1^2(z) - \frac{1}{2\alpha^3} + \frac{2\beta^2}{\alpha} c_2^2(z) + \frac{1}{2\alpha^2} c_1^2(z) + 3k \beta^2 \frac{d c_1}{dz} + k \beta^2 \frac{d c_2}{dz} + \frac{2k}{\alpha} \frac{\partial \phi}{\partial z} - 2H_0 k^2 \beta A_1^2(z) +
\]

\[
\begin{align*}
\left[ \frac{15}{4} \alpha^3 \frac{d^2 c_1}{dz^2} + \frac{3 \beta^4}{4 \alpha} \frac{d^2 c_2}{dz^2} + \frac{3 \beta^2}{2} \frac{d c_1}{dz} \frac{d c_2}{dz} + \frac{3 \alpha^2}{2} \frac{\partial c_1}{\partial z} \frac{\partial c_2}{\partial z} + \frac{3 \beta}{\alpha} \frac{\partial c_1}{\partial z} \frac{\partial c_2}{\partial z} + \beta \frac{\partial c_1}{\partial z} \frac{\partial \phi}{\partial z} + \frac{1}{\alpha A_1^2(z)} \frac{d^2 A_1}{dz^2} + \frac{1}{\alpha A_1^2(z)} \frac{\partial \phi}{\partial z} + \frac{1}{\beta^2} \frac{\partial c_1}{\partial z} \frac{\partial c_2}{\partial z} + \frac{1}{\beta^2} \frac{\partial c_1}{\partial z} \frac{\partial \phi}{\partial z} + \frac{1}{\beta^2} \frac{\partial c_2}{\partial z} \frac{\partial \phi}{\partial z} \right] = 0
\end{align*}
\]

(19)

\[
\begin{align*}
\left[ \frac{15}{4} \beta^3 \frac{d^2 c_2}{dz^2} + \frac{3 \alpha^4}{4 \beta} \frac{d^2 c_1}{dz^2} + \frac{3 \alpha^2}{2} \frac{d c_1}{dz} \frac{d c_2}{dz} + \frac{3 \beta}{\alpha} \frac{\partial c_1}{\partial z} \frac{\partial c_2}{\partial z} + \frac{1}{\beta^2} \frac{\partial c_1}{\partial z} \frac{\partial \phi}{\partial z} + \frac{1}{\beta^2} \frac{\partial c_2}{\partial z} \frac{\partial \phi}{\partial z} \right] = 0
\end{align*}
\]

(20)

where, \(H_0 = \frac{m_2 \alpha \beta A_1^2(z)}{n_0} \).

Equation (19) and (20) can be used for numerical analysis of nonparaxial propagation of Gaussian beam. Here \(\phi(z)\) is a slowly varying term with \(z\), hence we may suppress the variation of \(\phi\) with \(z\) in our analysis.

3. Result and discussion

The finite difference method is used to obtain numerical solutions of equations (14) and (15) that describe the spatial evolution of beam width. Here the critical power is calculated using the relation [15],

\[P_{cr} = \frac{\pi \lambda^2 (61)^2}{8n_0^2 n_2^2}\]

The solutions are plotted in figure 1 and are for a beam propagating in chalcogenide glass of refractive index \((n_0)\) 2.8 and non-linear refractive index \((n_2)\) of 13x10^{-14} cm^2/W at three input powers viz.1.5, 3 and 10 times critical power \(P_{cr}\) which is 9.6 kw for the wavelength \(\lambda= 1550\) nm. Using the boundary condition \(\alpha = \alpha_0\) and \(\beta = \beta_0\) at \(z=0\) for values of \(\alpha_0 = 0.02\) cm and \(\beta_0 = 0.015\) cm the result predict an exponential decrease in \(\alpha\) and \(\beta\) which approaches zero over a finite propagation distance i.e. when \(P_{in} > P_{cr}\) the width of an elliptical beam collapses over a finite propagation distance which for 1.5, 3 and 10 times \(P_{cr}\) is 8.3cm, 6.3cm and 4 cm, respectively. However, from equations (14) and (15) a necessary condition for self-focusing can be written as

\[\frac{d^2 \alpha}{dz^2} = 0 = \frac{d^2 \beta}{dz^2} \text{ at } z = z_{sf}\]

(21)

where \(z_{sf}\) is self-focusing length. It implies that at \(z_{sf}\), \(\alpha = \beta\) which is a condition at self-focusing. The values of \(z_{sf}\) thus determined from the derived condition are 7.1 cm, 5.6 cm and 3.0 cm for 1.5, 3
and 10 times $P_{cr}$, respectively. In these solutions value of $k$ was taken as $1.4 \times 10^3 \text{cm}^{-1}$ and the wavelength of propagation as 1550 nm — the ‘eye-safe region’ wavelength in optical communication.

Figure 1. Variation of beam widths $\alpha$ and $\beta$ with propagation distance for input power 1.5, 3, 10 times the critical power for $\alpha_0 = 0.02 \text{ cm}$, $\beta_0 = 0.015 \text{ cm}$.

It is observed from figure 1 that the beam width collapses more rapidly with input power and the collapse distance decreases with increase in input power of the Gaussian beam. The collapse distance scales as $\frac{1}{\sqrt{\beta}}$ for $p > p_{cr}$ but, at higher powers the collapse distance scales as $\frac{1}{p}$ [16]. Collapse occurs when the whole beam shrinks to a point. Further, a collapsing beam exhibits sharp rise in intensity [17]. The intensity of the propagating beam at three input powers in the present case increases on reaching $z_{sf}$. The variation of intensity with propagation distance is presented in figure 2 and is plotted using equation (13). The values of $H_0$ for the three input powers are $0.1974 \times 10^{-10}$, $0.3948 \times 10^{-10}$ and $0.1316 \times 10^{-9}$ (in c. g. s).

The threshold power for collapse of a beam for an elliptical beam is approximated as $P_{th}(e) = \left[ \frac{e + \frac{1}{2} c_1 + 0.6}{2} \right] \times \text{threshold power for circular beam}, \; P_{th}(e = 1)$. Here $e = \beta / \alpha$ [18] is the beam ellipticity. The threshold power for collapse increases with input beam ellipticity and the threshold power for collapse is above the critical power [19].

Again using the boundary condition at $z = 0$, for $\alpha(0) = 0.02 \text{ cm}$ and for $\beta(0) = 0.015 \text{ cm}$, $A_1(0) = 1$, $c_1(0) = 1$, $c_2(0) = 1$ the variation of nonparaxial intensity with propagation distance is also presented in figure 2 and is plotted using equations (19) and (20).

For input power slightly above the critical power $3P_{cr}$, it is observed that intensity of paraxial curve reaches maximum at $z = 6.5 \text{ cm}$, indicates the formation of singularity at which solution of NLSE collapses and beam intensity becomes infinite. The formation of singularity prevents self-focusing behaviour beyond the self-focusing point. In the nonparaxial curve, intensity reaches maximum at $z = 7.5 \text{ cm}$, which passes through focus and begins to defocus as shown in figure 2. The onset of focusing-defocusing cycles is at a value of $z > z_{sf}$ for the paraxial beam.
Figure 2. Variation of beam intensity with propagation distance for paraxial and nonparaxial NLSE.

4. Conclusion
The evolution of the beam width and amplitude was analysed for paraxial and nonparaxial propagation by variational approach. The solutions obtained using finite difference method predict stationary self-focussing for paraxial Gaussian beam propagating in chalcogenide glass with a distinct singularity evident at $z = 6.5\,\text{cm}$ and $3\,P_c$ input power. In case of nonparaxial propagation of a Gaussian beam at input power $3\,P_c$ that intensity reaches maximum at $z = 7.5\,\text{cm}$, which focuses and defocuses.

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