Effective interactions for light nuclei: an effective (field theory) approach

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Abstract. One of the central open problems in nuclear physics is the construction of effective interactions suitable for many-body calculations. We discuss a recently developed approach to this problem, where one starts with an effective field theory containing only fermion fields and formulated directly in a no-core shell-model space. We present applications to light nuclei and to systems of a few atoms in a harmonic-oscillator trap. Future applications and extensions, as well as challenges, are also considered.
1. Introduction

How does the complexity of nuclear physics arise from the relatively simple QCD Lagrangian? Of the open questions in nuclear structure, this is one of the broadest and most fundamental. The long-range goal of nuclear-structure theory is indeed to calculate the properties of finite nuclei starting from the strong-interaction physics of QCD. Although this goal is still several years away, a confluence of nearly parallel conceptual and computational developments is stirring unprecedented excitement about our ability to reach it.

Tremendous progress has been made in the last decade \[1, 2\] in our understanding of how nuclear structure arises from the properties of the interactions among nucleons inside a nucleus, even though these interactions are often modeled only in terms of ad hoc potentials. In the previous twenty years a number of many-body techniques has been developed for exactly solving the nuclear few-body problem \[3, 4\]. The application of these approaches to nuclei ranging in mass from \(A = 2\) to \(A = 15\) has clearly demonstrated how the structure of these light nuclei, \(i.e.,\) binding energies, excitation spectra, electromagnetic moments, \(etc.,\) arise directly from the properties of nucleon-nucleon (NN), smaller three-nucleon (NNN), and perhaps some tiny four-nucleon (NNNN) interactions. One of the most tantalizing current problems is how to extend the many-body approaches for light nuclei to medium- and heavy-mass systems.

Independently, a framework has been developed based on effective field theories \[5, 6\] to construct nuclear potentials that respect the symmetry pattern of QCD and produce observables in a systematic and controlled expansion in powers of momentum. These ingredients restrict the shape of nuclear interactions, particularly in the range of pion exchange, and encapsulate the complicated short-range physics into a number of “low-energy constants”. Now, the first results are emerging \[7\], where the low-energy constants, so far simply fitted to data, can be calculated in full lattice QCD simulations. How can one link these results to nuclear properties?

In this paper we map some of the landmarks and crucial crossings in the long road from QCD to nuclear structure.

2. Effective Interactions and Operators

The general goal of microscopic nuclear-structure theory is to begin with the free internucleon interaction and determine, using many-body quantum mechanics, the properties of finite nuclei. In principle, one would like to solve the many-body Schrödinger equation for all \(A\) nucleons,

\[
H |\Psi_\alpha\rangle = E_\alpha |\Psi_\alpha\rangle, \tag{1}
\]

in the “full” Hilbert space \(S\). Here the Hamiltonian is

\[
H = \sum_{i=1}^{A} t_i + \sum_{i \leq j}^{A} v_{ij} + \sum_{i \leq j \leq k}^{A} v_{ijk} + \ldots, \tag{2}
\]
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where the first term involves the kinetic energies $t_i$ and the subsequent terms, the potentials among an increasing number of nucleons. It used to be assumed that only accurate NN interactions were required, although evidence has now accumulated for the need for NNN interactions \[8\] (and perhaps NNNN interactions). Relativistic effects do not play a significant role in understanding low-energy nuclear structure, and can be included as corrections around the non-relativistic limit.

In general, Eq. (1) cannot be solved in the full Hilbert space $S$, because of the infinite number of configurations, so the problem must be truncated to a smaller Hilbert space $S'$ of dimension $d$ (called the shell-model basis space or simply the model space). Because the full space has been truncated, full space operators cannot be used, but must be replaced with effective operators appropriate for the given size of the model space. In this case, $H$ must be replace by the effective Hamiltonian $H'$, such that

$$H'\Phi_\beta = E_\beta \Phi_\beta,$$

(3)

where $\Phi_\beta = P|\Psi_\beta\rangle$, $P$ is a projection operator from $S$ into $S'$, and the $E_\beta$ are a subset of dimension $d$ of the exact eigenvalues in Eq. (1). The projections $\Phi_\beta$ are usually not orthogonal, so one must construct the biorthogonals $\tilde{\Phi}_\gamma$, such that

$$\langle \tilde{\Phi}_\gamma | \Phi_\beta \rangle = \delta_{\gamma\beta}.$$

(4)

Using the biorthogonals, one can easily obtain the $H'$ that satisfies Eq. (3), i.e.,

$$H' = \sum_{\beta \in S} |\Phi_\beta\rangle E_\beta \langle \tilde{\Phi}_\beta |.$$

(5)

It should be noted that $H'$ will usually be non-Hermitian, because of the non-orthogonality of the $|\Phi_\beta\rangle$. All other physical operators relevant to the nuclear system being investigated, e.g., the rms radius operator, the electromagnetic-moment operators, the transition operators, etc., must be renormalized in a similar manner for use in the given model space.

In heavy nuclei, a drastic projection is needed in the standard nuclear shell model (SNSM), consisting of an inert, closed-shell core and a few valence nucleons. The theoretical construction of SNSM effective interactions and operators, both empirically and microscopically, has an extremely long and large history in nuclear physics \[9, 10\]. The microscopic approach has been more-or-less unsuccessful due to problems connected with the convergence of the perturbation-theory expansion for the effective operators and because of the complexity of the calculations.

A response to these difficulties was found in the no-core shell model (NCSM) \[1, 2, 11\], in which all $A$ nucleons in the nucleus are active. In its usual formulation, the NCSM employs a harmonic-oscillator (HO) single-particle basis of frequency $\omega$, and nucleons are allowed to share a maximum number $N_{max}$ of oscillator quanta (the maximum principal quantum number of the wave functions) above the minimum-energy configurations. It is important to include all configurations up to some total energy. This guarantees that all spurious center-of-mass components will be projected from the final wave functions, when using the Lawson procedure \[12\]. The need for effective
interactions and operators, in order to calculate nuclear properties, exists whether one is performing SNSM or NCSM calculations. However, the physical nature of the required effective operators will be quite different in the two cases. In the simpler NCSM, it involves only unitary transformations based on Eq. (5) [11, 13].

Although the determination of the effective operators for NCSM calculations is more straightforward, the generated effective operators are still dependent upon the nature of the inter-nucleon interactions employed in the calculations [14], a low-body (usually only two-body) cluster approximation, and the size of the model space. Thus, it is highly desirable to find a better method for calculating these NCSM effective operators, which is not potential dependent, yet is directly related to the underlying QCD symmetries and can always be employed in small model spaces.

3. Effective Field Theory and the NCSM

The framework of effective field theory (EFT) emerged in the late 1970s [15] from the realization that, because in a quantum field theory any physics problem is a many-body problem, one never has access to the “full” Hilbert space $S$. It is always necessary to truncate the Hilbert space so as to exclude states associated with energies beyond those we can confidently access experimentally and theoretically. This paradigm shift implies that interactions are only defined in the context of a model space: the Platonic concept of “the” Hamiltonian is not particularly useful in physics.

What is needed is a method to construct effective interactions and operators whether or not the underlying, higher-energy physics is known. Since relativistic quantum effects of arbitrary complexity exist unless they are forbidden by a symmetry, any effective interaction or operator with assumed symmetries should be included. Still, the intrinsic connection between Hamiltonian and model space would be problematic without a way to ensure that the arbitrariness in the choice of model space does not contaminate observables. The projector $P$ always contains at least one dimensionful parameter, the ultraviolet (UV) cutoff $\Lambda$, for definiteness taken here to be a momentum. One thus requires that the Hamiltonian $H'$ depend on $\Lambda$ in such a way that observables at momenta $Q \ll \Lambda$ are independent of how $P$ is chosen, and in particular, independent of $\Lambda$. This is termed renormalization-group (RG) invariance.

Nuclear physics is a great arena for these ideas, because it is not easy to find solutions for QCD at momenta $Q < M_{QCD} \sim 1 \text{ GeV}$. Yet we know the symmetries of the QCD Lagrangian. In particular, chiral symmetry plays an important role, even at low energies, thanks to the appearance of light pions, due to spontaneous and small explicit breaking. It is, thus, not difficult to write the most general Hamiltonian with the appropriate degrees of freedom and symmetries [5], which is a generalization to systems with more than one nucleon of the Lagrangian used in chiral perturbation theory. Interactions among nucleons consist of pion exchanges and contact interactions, which subsume short-range dynamics (say, exchange of heavier mesons). At very low energies even pion exchange can be treated as short ranged, leaving only contact interactions in
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the theory.

The real challenge is ordering the infinite number of interactions so that observables can be calculated in an expansion in powers of $Q/M_{QCD}$. A guide is provided by the RG: a truncation of this expansion at any given order must respect RG invariance except for small errors contained in higher orders. This approach had mostly been applied in particle physics to systems where unitarity could be accounted for perturbatively. In nuclear physics, the leading order (LO) must contain non-perturbative physics to generate nuclear bound states and resonances. Only subleading-order corrections, if truly corrections, should be treated in perturbation theory.

Most applications of EFT in nuclear physics have been carried out using a continuum free-particle basis [6]. This facilitates the study of reactions, in addition to structure, but unfortunately, with foreseeable computational resources, is limited to $A \leq 4$. To increase $A$, we need to limit the number of accessible one-body states by introducing an additional, infrared (IR) momentum cutoff $\lambda$, which discretizes momentum. This is a traditional method in QCD itself, where simulations are carried out on lattices, and in addition to an UV cutoff $\Lambda \sim 1/a$, with $a$ the lattice spacing, there is also an IR cutoff $\lambda \sim 1/L$, with $L$ the lattice size. Lattice regularization can be, and has been, applied to nuclear EFT as well [16, 17, 18], where it is particularly suited to the study of nuclear matter at finite temperature.

The successes of the SNSM suggest, however, that formulating the nuclear EFT in an HO basis might be an efficient way to reach larger, finite nuclei. An UV cutoff equivalent to the cutoff used in a free-particle basis can be defined in an HO basis: if $\mu$ is the two-body reduced mass, $\Lambda = \sqrt{2\mu N_{max} + 3/2}\omega$ is the momentum associated with the energy of the highest HO shell included explicitly in the two-body system in the center-of-mass frame. The HO frequency $\omega$ defines an additional IR cutoff $\lambda = \sqrt{2}/b$, where $b = 1/\sqrt{\mu\omega}$ is the HO length, which plays a role analogous to the box size $L$ in a lattice discretization [17].

EFT in an HO basis is essentially the NCSM formulated directly within the model space: EFT provides the form of the effective interactions and operators needed in the NCSM. Since NCSM is a full diagonalization approach in a basis constructed with HO wave functions, each model space is determined by two parameters: $N_{max}$ and $\omega$, or alternatively, $\Lambda$ and $\lambda$. Consequently, all low-energy constants are functions of these two parameters. As in any EFT, they need to be determined either from some experimental data or, eventually, from QCD itself. Enormous progress in lattice QCD has already produced [7] NN scattering lengths, albeit at unphysical values of the pion mass. It is reasonable to expect that the next few years will see scattering lengths at lower pion masses, effective ranges, other NN quantities, and perhaps even observables involving more nucleons. A key aspect of this work is the use of Lüscher’s formula [19], where the energy levels inside a box are linked to scattering parameters.

The drawback of formulating the EFT in an HO basis is that the connection to scattering states becomes less obvious than when using a continuum free-particle basis. As we show in the rest of this paper, this is not an impassable roadblock, for two reasons.
First, one can always determine the low-energy constants from bound-state data (e.g., binding energies). We illustrate the method in Sec. 4 where we use data from a few bound states to fix the low-energy constants and then many-body theory in the form of the NCSM to calculate properties of larger systems [20]. It is true, however, that as the EFT order increases, the low-energy constants multiply, and it is desirable to determine them using the more abundant scattering (experimental or lattice) data.

Second, it is possible to connect the energies inside an HO well to scattering parameters in a way [21] very similar to Lüscher’s formula. We show how this can be done in Sec. 5 in the simpler problem of spin-1/2 fermions without isospin [22], which is relevant for trapped two-state atoms close to a Feshbach resonance.

4. Light Nuclei

The first direct application of EFT principles to the derivation of effective interactions in finite NCSM model spaces was presented in Ref. [20]. In that work, we have opted for a theory without explicit pions for two reasons: (i) for very-low-energy processes (involving momenta less than the pion mass) the formalism becomes very simple, and (ii) the same techniques would be readily applicable to the pionful EFT.

In pionless EFT at LO, the Hamiltonian \( H' \) in Eq. (3) can be written as a sum of the relative kinetic energy, two contact interactions in the \( ^3S_1 \) and \( ^1S_0 \) NN channels, with corresponding parameters \( C_{0}^1 \) and \( C_{0}^0 \), and a contact three-body interaction in the \( ^2S_1/2 \) NNN channel, with parameter \( D_{0} \):

\[
H' = \frac{1}{4\mu A} \sum_{[i<j]} (\vec{p}_i - \vec{p}_j)^2 + C_{0}^1 \sum_{[i<j]^1} \delta(\vec{r}_i - \vec{r}_j) + C_{0}^0 \sum_{[i<j]^0} \delta(\vec{r}_i - \vec{r}_j) + D_{0} \sum_{[i<j<k]} \delta(\vec{r}_i - \vec{r}_j) \delta(\vec{r}_j - \vec{r}_k),
\]

where \([i < j]\) denotes all pairs of particles, \([i < j]^s\) pairs in the S-wave NN channel of spin \( s \) and \([i < j < k]\) triplets in the spin-1/2 \( S\)-wave NNN channel. Unlike in the conventional NCSM approach based on a unitary transformation, in this approach the interaction in each model space has the same structure, i.e., matrix elements of the contact two- and three-body interactions. Only the coupling constants differ in each model space; they depend on \( N_{max} \) and \( \omega \): \( C_{0}^0(N_{max}, \omega), D_{0}(N_{max}, \omega) \). The three coupling constants are fixed for each model space so that three observables, the deuteron, triton, and \(^4\)He binding energies, are simultaneously reproduced. With the Hamiltonian, thus, defined, we have investigated the energy of the first \((0^+, 0)\) excited state of \(^4\)He as well as the \(^6\)Li ground-state energy.

Because the errors associated with the terms neglected in the LO Hamiltonian \( H' \) decrease with increasing the ultraviolet cutoff, large values of \( \Lambda \) are desirable. While increasing \( N_{max} \) makes the calculation impractical even for a small number of particles, increasing \( \omega \) is not a good option either, as this would increase the errors associated with the infrared cutoff. Instead, for fixed \( \omega \), we have calculated the energies of interest in the four- and six-nucleon systems for the maximum \( N_{max} \) we were able
to handle, extrapolating to other values of $\Lambda$ using $E_0(\omega) + A(\omega)/\Lambda$, an expression motivated by a similar running in the two-body system in the continuum \[4\]. Then for fixed $\Lambda$, we extrapolate energies to $\omega \to 0$, which eliminates infrared errors, using a quadratic formula. Using this approach, we have predicted that the energy of the first $(0^+, 0)$ excited state of $^4$He is 18.5 MeV, in very good agreement (within 10%) with the experimental value. (The agreement can be understood if we consider that this state is very close to the four-nucleon continuum threshold, a regime well within the limits of applicability of the theory.) A similar analysis for the $^6$Li ground state also produced results within 30% of the experimental level, with an underbinding of about 9 MeV. Better results can be obtained if one includes an extra term to the running, i.e., $E_0(\omega) + A(\omega)/\Lambda + \log(B(\omega)\Lambda)/\Lambda$. While the extra term essentially does not change the result in the four-body system, for $^6$Li the new results overbind the system by only 15%. In Fig. \[1\] we present the results for $^6$Li; they include the log term in the fit. Obviously, more investigations are necessary to pin down the running of the observables in the many-body system.

5. Few-Atom Systems in Traps

Beyond LO (or even at LO in the theory with pions \[23\]), the procedure discussed in the previous section to determine low-energy constants becomes impractical: the number of required input observables increases rapidly, and an ever larger number of states in light nuclei is needed to adjust the interaction. This motivated us to devise another approach, in which the two-body renormalization is realized at the two-body level \[22\]. In such an approach, the low-energy scattering properties in free space are related to the energy spectrum of two interacting particles in a harmonic trap.

The perfect testing grounds for such an approach are systems of two-component fermions with a large two-body scattering length $a_2$ in an external harmonic trap, which can be realized experimentally with atoms trapped by lasers in variable magnetic fields \[24\]. In this case, there is no need to take the limit $\omega \to 0$, since $\omega$, or equivalently the trap length $b$, is given by the trapping laser. In the unitary limit $b/a_2 \to 0$, the spectrum for three particles is known \[25\].

In this case, the Hamiltonian is given in LO by

$$H' = \frac{\omega^2}{2} \sum_i \left[ \frac{(b\vec{p}_i)^2}{2} + 2 \frac{(\vec{r}_i)^2}{b^2} \right] + 2\mu b^2 C_0 \sum_{i<j} \delta(\vec{r}_i - \vec{r}_j). \tag{7}$$

There is just one two-body coupling constant $C_0(N_{\text{max}}, \omega)$ and three-body forces appear only at high orders.

In order to renormalize the two-body interaction in a finite model space, we first consider the two-body system in the center-of-mass frame, where the relative position is denoted $\vec{r}$. Since the contact interaction connects $S$ states only, higher angular-momentum states are undisturbed from HO ones. It is sufficient to consider the two-body wave function described by a superposition of all $S$ states with the HO quantum
number $N = 2n \leq N_{\text{max}}$, 
\[ \psi(\vec{r}) = \sum_{n=0}^{N_{\text{max}}/2} A_n \phi_n(\vec{r}), \] (8)

with $\phi_n(\vec{r})$ the S-wave HO state with radial quantum number $n$ and $A_n$ a set of complex coefficients. The $N_{\text{max}}/2 + 1$ unknown coefficients $A_n$ and the energy spectrum can be found by solving the Schrödinger equation for the relative motion, once $C_0(N_{\text{max}}, \omega)$ is determined.

The coupling $C_0(N_{\text{max}}, \omega)$ can be determined by the following procedure. In order for the Schrödinger equation with the delta function to be well defined, $C_0(N_{\text{max}}, \omega)$ and the energies $\varepsilon(N_{\text{max}}, \omega)$ (in units of $\omega$) have to satisfy an RG condition [22],
\[ \frac{1}{C_0(N_{\text{max}}, \omega)} = -\frac{\mu}{\pi^{3/2}b^2} \sum_{n=0}^{N_{\text{max}}/2} \frac{I_n^{(1/2)}(0)}{(2n + 3/2) - \varepsilon(N_{\text{max}}, \omega)}, \] (9)
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Figure 2. First excited-state energy of the two-body system, \( \varepsilon \), in units of \( \omega \), as a function of the UV cutoff \( \Lambda \), in units of the inverse of the trap length \( b \). Three interaction strengths determined by the indicated values of \( b/a_2 \) are shown. The ground-state energy given by Eq. (10) is used in each case to fix the coupling constant. The exact values for the first excited-state energy are displayed with dotted lines.

where \( L_n^{(1/2)}(0) \) is a generalized Laguerre polynomial at the origin. In the \( \Lambda \to \infty \) limit, one can show \cite{26} that this condition leads to

\[
\frac{\Gamma(3/4 - \varepsilon(\infty, \omega)/2)}{\Gamma(1/4 - \varepsilon(\infty, \omega)/2)} = \frac{b}{2a_2}. \tag{10}
\]

That is, the energy spectrum inside the trap depends only upon the scattering length in units of the trap length. This formula, the HO oscillator counterpart of the Lüscher formula for a box with sharp boundaries, was first derived in Ref. \cite{21}, using a pseudopotential. The \( C_0(N_{\text{max}}, \omega) \) can then be fixed so that in each model space one of the states obtained by the exact diagonalization of the Schrödinger equation reproduces the corresponding value given by Eq. (10). For simplicity, we match the lowest state. Because just one coupling constant needs to be fixed in LO, all the other levels can be calculated. They satisfy Eq. (9) and deviate from the exact value given by Eq. (10), but the error decreases with increasing the size of the model space. As an illustration, in Fig. 2 we present the running of the first excited state of the two-body system as a function of the dimensionless quantity \( \Lambda b \) for three selected values of the \( b/a_2 \) ratio. In all cases, the energy goes to the exact value with errors that decrease as \( 1/\Lambda b \). Faster convergence can be achieved by considering terms beyond leading order, that is, corrections that involve derivatives of the contact interaction and account for the effective range, the shape parameter, etc., introduced by the model-space truncation \cite{26}.

With the two-body interaction fixed in a truncated two-body model space, we turn now to the few-body problem, considering the unitary regime as well as a general non-vanishing \( b/a_2 \) value, for both positive and negative scattering lengths. The three-body solutions are obtained by a diagonalization in a finite model space, constructed as anti-symmetrized three-body states of HO wave functions (for details on the basis construction in each model space, see Refs. \cite{20,22}). In Fig. 3 we present the running of the energy of the lowest three-body state (which has orbital angular momentum
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Figure 3. Ground-state energy of the three-body system in units of $\omega$ as a function of the UV cutoff $\Lambda$, in units of the inverse of the trap length $b$. Three interaction strengths determined by the indicated values of $b/a_2$ are shown. The two-body ground-state energy given by Eq. (10) is used in each case to fix the coupling constant. The exact value at unitarity [25] is displayed with a dotted line.

$L = 1$, at unitarity and at $b/a_2 = \pm 1$. At unitarity, where a semi-analytical result exists [25], our results are within a few percent of the exact results even for relatively modest model-space sizes. However, it is worth noting that the running is rather slow, having in general a $1/(\Lambda b)^\alpha$ dependence, with $\alpha$ a power that is state dependent. For the ground state at unitarity, we have shown by direct fit that $\alpha = 1$ [22], and in the limit $\Lambda b \to \infty$, we obtain the exact result [25]. Faster convergence, although not necessarily faster running (that is, same power $\alpha$, but larger coefficients in front of $1/(\Lambda b)^\alpha$), can be achieved if one introduces higher-order terms [27].

In the non-interacting limit ($b/a_2 \to -\infty$), the lowest state of three two-component fermions has $L = 1$ (negative parity). As $b/a_2$ increases, this state persists as the ground state beyond unitarity. However, around $b/a_2 \approx 1.5$, the lowest state becomes an $L = 0$ state [22], which remains lowest as $b/a_2$ increases further. In the limit of large $b/a_2$, which approaches the untrapped case, the energy of the three-body ground state becomes $E_3 \approx -1/2\mu a_2^2$. This result suggests that the system of three fermions is near the threshold of the $S$-wave scattering of one fermion on the bound state of the other two.

Further investigation has shown that the known results in the four-body system [28] are also reproduced, although with less accuracy because of the complexity of solving the four-body problem in larger model spaces. Nevertheless, the errors can be reduced by introducing higher-order corrections, and preliminary calculations show very nicely this trend [22, 27].
6. Future Applications and Challenges

As discussed in Sec. 3, our goal is to obtain effective interactions and operators in truncated model spaces for solving the few-body problem (for fermions and bosons), based on the underlying assumed QCD symmetries, thus bypassing, for example, the need for a phenomenological potential. As we have shown, this approach works reasonably well for light nuclei (Sec. 4) and even better for few-fermion systems in an HO trap (Sec. 5).

Now that we have achieved good results linking few-fermion energies in an HO trap to scattering parameters, we can pursue our original idea of applying this method to nuclei. Doing this is, indeed, a great challenge, as we have indicated in Secs. 3 and 4: first, because the number of low-energy constants increases significantly as the EFT order increases; and, second, because the many-body calculations (i.e., within the NCSM) become more and more difficult as the model-space size, as defined by $N_{\text{max}}$, increases for larger values of $A$. For given computational resources, it is, thus, important to obtain better converged results in smaller model spaces for NCSM calculations of heavier nuclei.

Obviously, we want to extend this approach to higher orders and include the effective range. In preliminary calculations for few-fermion systems in a trap [26, 27] we have included corrections to the potential up to $N^2$LO and observed an acceleration of the convergence for energies. Corrections beyond LO are treated as perturbations and these preliminary results show an excellent agreement at unitarity with known results for the three-fermion system [25] and with other methods for the four-fermion system [28]. One finds that this perturbative treatment of subleading interactions, demanded by RG consistency, gives faster convergence than a non-perturbative treatment.

Another new idea deals with making sure that the available energy in the many-body system is larger than the maximum two-body energy employed in determining the two-body interactions. This is achieved by introducing a different total number of oscillator quanta $N_{\text{max}}^{(n)}$ for each $n$-body space. Preliminary calculations [27] for the problem of few-fermion systems in an HO trap indicate that one obtains significant gains in the converged results in smaller model spaces, when for a fixed value of $N_{\text{max}}^{(2)}$, the many-body model space is increased until convergence. The next step is to apply this approach to few nucleon systems.

Although we have so far concentrated on binding energies, other bound-state observables can be calculated with our method. Using similar techniques, we can construct other operators for describing observables of interest. It will be instructive to compare with results obtained in the traditional NCSM [14].

Finally, an important long-term goal of this program is to eventually complete the link from QCD to nuclear-structure observables. In the direction of QCD, we should determine the low-energy constants of the EFT expansion from lattice simulations. In the direction of heavier nuclei, we need to understand the limits of the pionless theory, and presumably go beyond it. The pionless EFT is valid for low momenta ($Q < m_\pi$),
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and so is Eq. (10). One of the challenges of future applications to nuclear systems is to extend the equation that determines the spectra of two trapped particles for momenta large enough to discern pions.

7. Summary and Conclusions

We have presented a discussion of the first implementation of EFT principles directly into a many-body method, the NCSM. We have proposed that using EFT to construct effective interactions in restricted model spaces used in NCSM calculations might provide an important step in the long road from QCD to nuclear structure.

In one implementation, we have determined the low-energy constants by a direct fit to ground-state binding energies in two-, three- and four-nucleon systems, and predicted an excited state in $^4$He, as well as the ground-state energy of $^6$Li [20]. We have presented an improved extrapolation for the latter in Sec. 4. For both states, results are within the expected errors of the pionless EFT. Such an application becomes quickly impractical if one considers the subleading orders in the pionless EFT, or the pionful theory, as the number of low-energy constants that have to be determined by bound states increases considerably.

In a second implementation, we have considered a system of trapped fermions [22]. Such an approach would allow the determination of low-energy constants in the two-body system alone, as shown in Sec. 5. We have illustrated the application of the two-body renormalization by computing the spectrum of three spin-1/2 fermions in a trap at unitarity, where a semi-analytical calculation exists [23]. Thus, we have shown that in LO our results converge to the exact ones, in the limit of large UV cutoffs. The advantage of our approach is that it can be extended to finite two-body scattering lengths, the convergence of which we examined here, and to include other scattering parameters [26, 27].

Despite the different underlying physics, the systems of trapped cold atoms near Feshbach resonances and of nucleons at low energies are quite similar. One can hope that the same procedures can be transferred to the nuclear many-body problem, as we discussed in Sec. 6, thus providing a QCD-based solution to the open problem of constructing nuclear effective interactions. Work in this direction is in progress.

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References

[1] P. Navrátil, J.P. Vary, and B.R. Barrett, Phys. Rev. Lett. 84, 5728 (2000); Phys. Rev. C 62, 054311 (2000).
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[2] P. Navrátil, S. Quaglioni, I. Stetcu, and B.R. Barrett, J. Phys. G 36, 083101 (2009).
[3] D.J. Dean, Physics Today 60, no. 11, 48 (2007).
[4] B.R. Barrett, B. Mihaila, S.C. Pieper, and R.B. Wiringa, Nuclear Physics News 13, no. 1, 17 (2003).
[5] S. Weinberg, Phys. Lett. B251 288 (1990); Nucl. Phys. B363 3 (1991); C. Ordóñez and U. van Kolck, Phys. Lett. B291 (1992) 459.
[6] P.F. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. 52, 339 (2002).
[7] S.R. Beane, P.F. Bedaque, K. Orginos, and M.J. Savage, Phys. Rev. Lett. 97, 012001 (2006).
[8] H. Kamada et al., Phys. Rev. C 64, 044001 (2001).
[9] I. Talmi, Adv. Nucl. Phys. 27, 1 (2003) and references therein.
[10] B.R. Barrett, Czech. J. Phys. 49, 1 (1998).
[11] P. Navrátil and B.R. Barrett, Phys. Rev. C 54, 2986 (1996).
[12] R.D. Lawson, Theory of the Nuclear Shell Model, (Clarendon Press, Oxford, 1980).
[13] A.F. Lisetskiy, B.R. Barrett, M.K.G. Kruse, P. Navrátil, I. Stetcu, and J.P. Vary, Phys. Rev. C 78, 044302 (2008).
[14] I. Stetcu, B.R. Barrett, P. Navrátil, and J.P. Vary, Phys. Rev. C 71, 044325 (2005).
[15] S. Weinberg, Physica 96A, 327 (1979); Rev. Mod. Phys. 52, 515 (1980).
[16] H.M. Müller, S.E. Koonin, R. Seki, and U. van Kolck, Phys. Rev. C 61, 044320 (2000); D. Lee, B. Borasoy, and T. Schäfer, Phys. Rev. C 70, 014007 (2004).
[17] R. Seki and U. van Kolck, Phys. Rev. C 73, 044006 (2006).
[18] D. Lee, Prog. Part. Nucl. Phys. 63, 117 (2009).
[19] M. Lüscher, Nucl. Phys. B 354, 531 (1991).
[20] I. Stetcu, B.R. Barrett, and U. van Kolck, Phys. Lett. B 653, 358 (2007).
[21] T. Busch, B.-G. Englert, K. Rzążewski, and M. Wilkens, Found. Phys. 28, 549 (1998); S. Jonsell, Few-Body Syst. 31, 255 (2002).
[22] I. Stetcu, B.R. Barrett, U. van Kolck, and J.P. Vary, Phys. Rev. A 76, 063613 (2007).
[23] A. Nogga, R.G.E. Timmermans, and U. van Kolck, Phys. Rev. C 72, 054006 (2005).
[24] T. Stöferle, H. Moritz, K. Günter, M. Köhl, and T. Esslinger, Phys. Rev. Lett. 96, 030401 (2006).
[25] F. Werner and Y. Castin, Phys. Rev. Lett. 97, 150401 (2006).
[26] I. Stetcu, J. Rotureau, B.R. Barrett, and U. van Kolck, in preparation.
[27] J. Rotureau, I. Stetcu, B.R. Barrett, M. Birse, and U. van Kolck, in preparation.
[28] S.Y. Chang and G.F. Bertsch, Phys. Rev. A 76, 021603 (2007); J. von Stecher, C.H. Greene, and D. Blume, Phys. Rev. A 76, 053613 (2007); Y. Alhassid, G.F. Bertsch, and L. Fang, Phys. Rev. Lett. 100, 230401 (2008).