A compatible embedded-hybridized discontinuous Galerkin method for the Stokes–Darcy-transport problem

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Abstract

We present a stability and error analysis of an embedded-hybridized discontinuous Galerkin (EDG-HDG) finite element method for coupled Stokes–Darcy flow and transport. The flow problem, governed by the Stokes–Darcy equations, is discretized by a recently introduced exactly mass conserving EDG-HDG method while an embedded discontinuous Galerkin (EDG) method is used to discretize the transport equation. We show that the coupled flow and transport discretization is compatible and stable. Furthermore, we show existence and uniqueness of the semi-discrete transport problem and develop optimal a priori error estimates. We provide numerical examples illustrating the theoretical results. In particular, we compare the compatible EDG-HDG discretization to a discretization of the coupled Stokes–Darcy and transport problem that is not compatible. We demonstrate that where the incompatible discretization may result in spurious oscillations in the solution to the transport problem, the compatible discretization is free of oscillations. An additional numerical example with realistic parameters is also presented.

Keywords: Stokes–Darcy flow, Coupled flow and transport, Beavers–Joseph–Saffman, Embedded and Hybridized methods, Discontinuous Galerkin, Multiphysics.

1 Introduction

The coupled Stokes–Darcy equations describe the interaction between free flow and flow in porous media. To model the transport of chemicals and contaminants in, for example, surface/subsurface flows, biochemical transport, or vascular hemodynamics problems, the Stokes–Darcy equations are coupled to a transport equation [14]. In this paper we consider one way coupling in which the velocity solution to the Stokes–Darcy flow problem is used in the transport equation to advect and diffuse a contaminant.

Many different finite element and mixed finite element [17, 28, 5, 7, 22, 6, 30], discontinuous Galerkin (DG) [8, 23, 24, 29, 34, 36], and hybridizable discontinuous Galerkin (HDG) [19, 20, 21, 27] methods have been proposed to discretize the Stokes–Darcy problem. Likewise, many different finite element methods have been proposed to discretize the transport equation, such as streamline diffusion, DG, and HDG methods [4, 11, 25, 26, 31, 41]. However, stability and accuracy of a discretization for the Stokes–Darcy problem and separately a discretization for the transport problem, does not guarantee that the coupled discretization for the Stokes–Darcy-transport problem will be stable and accurate. Examples of discontinuous Galerkin methods for the Stokes–Darcy-transport problem have been proposed in [35, 40].

To address the issue of coupling the discretization of a flow problem to the discretization of a transport problem, Dawson et al. [16] introduced the concept of compatibility; a discretization for flow and transport is compatible if it is globally conservative and zeroth-order accurate. They showed that loss of accuracy and/or loss of global conservation may occur if the discretization is not compatible. They furthermore showed that, for discontinuous Galerkin methods, compatibility is a stronger statement than local conservation of the flow field.

In this paper we discretize the Stokes–Darcy problem by an embedded-hybridized discontinuous Galerkin (EDG-HDG) finite element method [9]. The two main reasons to consider this discretization are: (i) mass is conserved exactly, i.e., the velocity field is divergence-conforming on the whole domain and mass is...
conserved point-wise; and (ii) the EDG-HDG discretization has fewer globally coupled degrees of freedom than traditional DG and HDG methods and is generally better suited to fast iterative solvers than HDG methods, as shown in [33]. We discretize the transport equation by an embedded discontinuous Galerkin (EDG) method [41], the main motivation being that EDG discretizations are generally computationally more efficient than DG and HDG discretizations. We will show that the EDG-HDG flow discretization is compatible with the EDG discretization of the transport equation. We prove well-posedness of the discrete transport problem and present optimal error estimates.

The outline for the remainder of this paper is as follows. In section 2 we describe the Stokes–Darcy flow system and its coupling to the transport problem. The compatible EDG-HDG discretization for the Stokes–Darcy-transport problem, together with some of its properties, is introduced in section 3. In section 4 we discuss useful inequalities that will be used to prove existence and uniqueness of a solution to the semi-discrete transport problem in section 5. Error estimates are developed for the semi-discrete transport scheme in section 6. In section 7 we verify the analysis by numerical experiments while conclusions are drawn in section 8.

2 The Stokes–Darcy system coupled to transport

Let \( \Omega \subset \mathbb{R}^{\dim} \), \( \dim = 2,3 \), be a bounded polygonal domain with boundary \( \partial \Omega \) and boundary outward unit normal vector \( n \). We assume the domain is divided into two non-overlapping polygonal subdomains, \( \Omega^s \) and \( \Omega^d \), such that \( \Omega = \Omega^s \cup \Omega^d \). The interface separating these subdomains is denoted by \( \Gamma^I \). Furthermore, for \( j = s,d \), the outward unit normal vector of \( \Omega^j \) is denoted by \( n^j \) and the exterior boundary of \( \Omega^j \) is denoted by \( \Gamma^j = \partial \Omega \cap \Omega^j \). The Stokes–Darcy system for the velocity \( u : \Omega \rightarrow \mathbb{R}^{\dim} \) and pressure \( p : \Omega \rightarrow \mathbb{R} \) is given by

\[
\begin{align*}
-\nabla \cdot 2\mu \varepsilon(u) + \nabla p &= f^s \quad \text{in } \Omega^s, \\
\kappa^{-1}u + \nabla p &= 0 \quad \text{in } \Omega^d, \\
-\nabla \cdot u &= \chi^d f^d \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma^s, \\
u \cdot n &= 0 \quad \text{on } \Gamma^d,
\end{align*}
\]

where \( \varepsilon(u) = (\nabla u + \nabla u^T)/2 \) is the strain rate tensor, \( \mu > 0 \) is the constant kinematic viscosity, \( \kappa > 0 \) is the permeability constant, \( f^s : \Omega^s \rightarrow \mathbb{R}^{\dim} \) is a forcing term, \( f^d : \Omega^d \rightarrow \mathbb{R} \) is a source/sink term, and \( \chi^d \) is the characteristic function of \( \Omega^d \). In the following we will denote the restriction of \( u \) and \( p \) on \( \Omega^j \) by \( u^j \) and \( p^j \), respectively, for \( j = s,d \).

Let \( n \) denote the unit normal vector on \( \Gamma^I \) pointing outwards from \( \Omega^s \), that is, \( n = n^s = -n^d \). We denote the tangential component of a vector \( w \) by \( (w)^I := w - (w \cdot n)n \). On the interface we then prescribe the following transmission conditions:

\[
\begin{align*}
u^s \cdot n &= u^d \cdot n \quad \text{on } \Gamma^I, \\
p^s - 2\mu \varepsilon(u^s)n \cdot n &= p^d \quad \text{on } \Gamma^I, \\
-2\mu (\varepsilon(u^s)n)^I &= \alpha \kappa^{-1/2}(u^s)^I \quad \text{on } \Gamma^I,
\end{align*}
\]

where \( \alpha > 0 \) is an experimentally determined constant. Equations (2a) and (2b) denote balance of flux and normal stress, and eq. (2c) is the Beavers–Joseph–Saffman interface condition [2, 37].

The Stokes–Darcy system described above is coupled to a transport equation over the time interval of interest \( I = [0,T] \). Given a porosity constant \( \phi \) such that \( 0 < \phi \leq 1 \) in \( \Omega^d \) and \( \phi = 1 \) in \( \Omega^s \), velocity field \( u : \Omega \rightarrow \mathbb{R}^{\dim} \), and source/sink term \( f : \Omega \times I \rightarrow \mathbb{R} \), the transport equation for the concentration
global conservation: transport system. First, integrating the transport equation eq. (3a) over \( \Omega \)

\[
\phi \partial_t c + \nabla \cdot (cu - D(u)\nabla c) = f \quad \text{in } \Omega \times I, \tag{3a}
\]

\[
D(u)\nabla c \cdot n = 0 \quad \text{on } \partial \Omega \times I, \tag{3b}
\]

\[
c(x, 0) = c_0(x) \quad \text{in } \Omega, \tag{3c}
\]

where \( c_0 : \Omega \to \mathbb{R} \) is a suitably smooth initial condition, and \( D(u) \) is the diffusion/dispersion tensor. Let \( |\cdot| \) denote the Euclidean norm. We will assume that \( D(u) \) satisfies the following conditions for \( u, v \in \mathbb{R}^{\dim} \).

\[
D_{\min} |x|^2 \leq D(u)x \cdot x \quad \forall x \in \mathbb{R}^{\dim}, \tag{4a}
\]

\[
|D(u)| \leq C(1 + |u|), \tag{4b}
\]

\[
|D(u) - D(v)| \leq C|u - v|, \tag{4c}
\]

where \( D_{\min} > 0 \) and \( C > 0 \) a constant. For the remainder of this paper, \( C > 0 \) will always denote a generic constant.

## 3 The numerical method

Before introducing the numerical method we consider two properties of the coupled Stokes–Darcy flow and transport system. First, integrating the transport equation eq. (3a) over \( \Omega \times (0, t) \), applying the boundary conditions eqs. (1d), (1e) and (3b), and the initial condition eq. (3c), results in the following expression of global conservation:

\[
\int_{\Omega} \phi c(x, t) \, dx = \int_{\Omega} \phi c_0(x) \, dx + \int_0^t \int_{\Omega} f \, dx \, ds.
\]

Second, if the initial condition in eq. (3c) is set as \( c_0(x) = \tilde{c} \), with \( \tilde{c} \) a constant, and the source/sink term \( f \) in eq. (3a) takes the form \( f = -\chi^d f^d \tilde{c} \), then \( c(x, t) = \tilde{c} \) for \( t > 0 \).

A discretization of Stokes–Darcy flow eqs. (1) and (2) coupled to transport eq. (3) is called compatible if it satisfies these two properties at the discrete level [16]. Compatibility was shown in [16] to be a desirable property of a discretization that couples a flow model to transport for reasons of accuracy and/or stability. They furthermore showed that for discontinuous Galerkin discretizations, compatibility is a stronger statement than local conservation of the flow field.

In this section we present a compatible embedded-hybridized discontinuous Galerkin method for the Stokes–Darcy flow coupled to transport eqs. (1) to (3).

### 3.1 Notation

For \( j = s, d \), let \( \mathcal{T}^j := \{ K \} \) be a shape-regular triangulation of \( \Omega^j \) consisting of non-overlapping elements \( K \) and such that meshes match at the interface \( \Gamma^I \). Furthermore, let \( \mathcal{T} = \mathcal{T}^s \cup \mathcal{T}^d \). We denote the diameter of an element \( K \) by \( h_K \) and set \( h := \max_K h_K \). The boundary of an element is denoted by \( \partial K \) and the element boundary outward unit normal vector is denoted by \( n \).

An interior facet is a facet that is shared by two adjacent elements. A boundary facet is a facet of \( \partial K \) that lies on \( \partial \Omega \). The set and union of all facets are denoted by, respectively, \( F = \{ F \} \) and \( \Gamma_0 \). Furthermore, the set of all facets that lie on the interface \( \Gamma^I \) is denoted by \( F^I \), and by \( F^j \) and \( \Gamma_0^j \) \((j = s, d)\) we denote, respectively, the set and union of all facets in \( \overline{\mathcal{T}^j} \).

We use \( \langle v, w \rangle_D \) to denote the \( L^2 \)-inner product of two functions \( v \) and \( w \) defined on \( D \subset \mathbb{R}^{\dim} \). By \( \langle v, w \rangle_D \) we denote the \( L^2 \)-inner product of two functions \( v \) and \( w \) defined on \( D \subset \mathbb{R}^{\dim-1} \). For \( D \subset \mathbb{R}^{\dim} \) or \( D \subset \mathbb{R}^{\dim-1} \) we denote the standard norm on the Sobolev space \( W^{s,p}(D) \) by \( \| \cdot \|_{s,p,D} \) and the semi-norm on \( W^{s,p}(D) \) by \( \| \cdot \|_{s,p,D} \). When \( p = 2 \), we drop the subscript \( p \) and write for the norm and semi-norm \( \| \cdot \|_{s,D} \) and \( \| \cdot \|_{s,D} \), respectively. For the \( L^2 \)-norm we drop the subscript \( s = 0 \) and write \( \| \cdot \|_D \).
3.2 The embedded-hybridized DG method for the Stokes–Darcy equations

We discretize the Stokes–Darcy flow problem eqs. (1) and (2) by an exactly mass conserving embedded-hybridized discontinuous Galerkin (EDG-HDG) method [9]. For this we consider the following discontinuous Galerkin finite element function spaces on $\Omega$,

$$
V_h := \{ v_h \in [L^2(\Omega)]^\text{dim} : v_h \in [P_k(K)]^\text{dim} \forall K \in T \}, \\
Q_h := \{ q_h \in L^2(\Omega) : q_h \in P_{k-1}(K) \forall K \in T \} \cap L_0^2(\Omega), \\
Q_h^j := \{ q_h \in L^2(\Omega^j) : q_h \in P_{k-1}(K) \forall K \in T^j \}, \quad j = s,d,
$$

where $P_k(D)$ denotes the space of polynomials of degree $k$ on domain $D$ and $L_0^2(\Omega) := \{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \}$. On $\Gamma_0^s$ and $\Gamma_0^d$, we consider the finite element spaces:

$$
\bar{V}_h := \{ \bar{v}_h \in [L^2(\Gamma_0^s)]^\text{dim} : \bar{v}_h \in [P_k(F)]^\text{dim} \forall F \in \mathcal{F}^s, \quad \bar{v}_h = 0 \text{ on } \Gamma^s \} \cap [C^0(\Gamma_0^s)]^\text{dim}, \\
\bar{Q}_h^j := \{ \bar{q}_h^j \in L^2(\Gamma_0^j) : \bar{q}_h^j \in P_k(F) \forall F \in \mathcal{F}^j \}, \quad j = s,d.
$$

Note that functions in $\bar{V}_h$ are continuous on $\Gamma_0^s$, while functions in $\bar{Q}_h^j$ are discontinuous on $\Gamma_0^j$, for $j = s,d$.

Following the notation of [9], we introduce the spaces $V_h := V_h \times \bar{V}_h$, $Q_h := Q_h \times \bar{Q}_h^s$, and $Q_h^j := Q_h^j \times \bar{Q}_h^j$ for $j = s,d$. We denote function pairs in $V_h$, $Q_h$ and $Q_h^j$, for $j = s,d$, by $(\mathbf{v}_h, \bar{v}_h) \in V_h$, $q_h := (q_h, \bar{q}_h^s, \bar{q}_h^d) \in Q_h$ and $q_h^j := (q_h, \bar{q}_h^j)$ in $Q_h^j$. Finally, we set $X_h := V_h \times Q_h$.

The EDG-HDG method for the Stokes–Darcy flow problem now reads: find $(\mathbf{u}_h, \mathbf{p}_h) \in X_h$ such that

$$
F_h((\mathbf{u}_h, \mathbf{p}_h), (\mathbf{v}, q)) = (f_s, v_h)_{\Omega^s} + (f_d, q_h)_{\Omega^d} \quad \forall (\mathbf{v}_h, q_h) \in X_h,
$$

where

$$
F_h((\mathbf{u}, \mathbf{p}), (\mathbf{v}, q)) = a_h(\mathbf{u}, \mathbf{v}) + b_h(\mathbf{p}, \mathbf{v}) + b_h(q, \mathbf{u}).
$$

Here the bi-linear form $a_h(\cdot, \cdot)$ is defined as

$$
a_h(\mathbf{u}, \mathbf{v}) = \sum_{K \in T^s} (2\mu \varepsilon(u), \varepsilon(v))_K + \sum_{K \in T^s} \langle \frac{2\beta_f \mu}{\kappa K} (u - \bar{u}) - \bar{v} \rangle_{\partial K}
$$

$$
- \sum_{K \in T^s} (2\mu \varepsilon(u)n^s, v - \bar{v})_{\partial K} - \sum_{K \in T^s} \langle 2\mu \varepsilon(v)n^s, u - \bar{u} \rangle_{\partial K}
$$

$$
+ (\kappa^{-1} u, v)_{\Omega^d} + \langle \alpha \kappa^{-1/2} \bar{u}, \bar{v} \rangle_{\Gamma^d},
$$

where $\beta_f > 0$ is a penalty parameter. The bi-linear form $b_h(\cdot, \cdot)$ is defined as

$$
b_h(\mathbf{p}, \mathbf{v}) = - \sum_{K \in T} (p, \nabla \cdot \mathbf{v})_K + \sum_{j=s,d} \sum_{K \in T^j} \langle \bar{p}^j, \bar{v}^j \rangle_{\partial K} - \langle \bar{p}^d - \bar{p}^s, \bar{n} \rangle_{\Gamma^d}.
$$

The following results are from [9] and will be used in the analysis. For sufficiently large $\beta_f$, there exists a unique solution $(\mathbf{u}_h, \mathbf{p}_h) \in X_h$ to eq. (5) (see [9, Proposition 1]). An a priori error analysis showed that if the velocity solution $u$ to eqs. (1) and (2) satisfies $u^s \in H^{k+1}(\Omega^s)$ and $u^d \in H^{k+1}(\Omega^d)$ with $k \geq 1$, then [9, Theorem 3]

$$
\| u - u_h \|_{\Omega} \leq C h^{k+1},
$$

where $C$ is a generic constant that depends on the regularity of $u^s$ and $u^d$. Furthermore, the EDG-HDG method for the Stokes–Darcy system is exactly mass conserving, divergence-conforming, and satisfies eq. (2a), i.e.,

$$
- \nabla \cdot u_h = \chi_d \Pi Q f^d \quad \forall x \in K, \forall K \in T, \\
\llbracket u_h \cdot n \rrbracket = 0 \quad \forall x \in F, \forall F \in \mathcal{F}, \\
u_h \cdot n = \bar{u}_h \cdot n \quad \forall x \in F, \forall F \in \mathcal{F}^d,
$$

where $\Pi Q$ is the standard $L^2$-projection into $Q_h$ and $\llbracket \cdot \rrbracket$ is the usual jump operator.
3.3 The embedded DG method for the transport equation

Before introducing the embedded discontinuous Galerkin (EDG) method for the transport equation, we first replace the exact velocity \( u \) in eq. (3) by the discrete velocity \( u_h \):

\[
\phi \partial_t c + \nabla \cdot (c u_h - D(u_h) \nabla c) = f \quad \text{in } \Omega \times I, \tag{8a}
\]

\[
D(u_h) \nabla c \cdot n = 0 \quad \text{on } \partial \Omega \times I, \tag{8b}
\]

\[
c(x, 0) = c_0(x) \quad \text{in } \Omega. \tag{8c}
\]

We will introduce the EDG method for eq. (8). For this we require the following discrete spaces:

\[
C_h = \{ c_h \in L^2(\Omega) : c_h \in P_\ell(K), \forall K \in \mathcal{T} \},
\]

\[
\bar{C}_h = \{ \bar{c}_h \in L^2(\Gamma_0) : \bar{c}_h \in P_\ell(F) \forall F \in \mathcal{F} \cap C^0(\Gamma_0),
\]

where the choice of \( \ell \) will be discussed in section 3.4. For notational purposes, we introduce \( C_h = C_h \times \bar{C}_h \) and \( c_h = (c_h, \bar{c}_h) \in C_h \).

The semi-discrete EDG method for the transport problem eq. (8) is now given by: For each \( t > 0 \), find \( c_h(t) \in C_h \) such that

\[
\sum_{K \in \mathcal{T}} (\phi \partial_t c_h(t), w_h)_K + B_h(u_h; c_h(t), w_h) = \sum_{K \in \mathcal{T}} (f(t), w_h)_K \quad \forall w_h \in C_h, \tag{10}
\]

where

\[
B_h(u; c(t), w) = B^a_h(u; c(t), w) + B^d_h(u; c(t), w).
\]

Here \( B^a_h(u; c(t), w) \) and \( B^d_h(u; c(t), w) \) represent, respectively the advective and diffusive parts of the bi-linear form. They are defined as:

\[
B^a_h(u; c(t), w) = -\sum_{K \in \mathcal{T}} (c(t) u, \nabla w)_K + \sum_{K \in \mathcal{T}} (c(t) u \cdot n, w - \bar{w})_{\partial K} - \sum_{K \in \mathcal{T}} (u \cdot n (c(t) - \bar{c}(t)), w - \bar{w})_{\partial K^{in}}, \tag{12}
\]

where \( \partial K^{in} \) denotes the portion of the boundary where \( u_h \cdot n < 0 \), and

\[
B^d_h(u; c(t), w) = \sum_{K \in \mathcal{T}} (D(u) \nabla c(t), \nabla w)_K - \sum_{K \in \mathcal{T}} ([D(u) \nabla c(t)] \cdot n, w - \bar{w})_{\partial K}
\]

\[
+ \sum_{K \in \mathcal{T}} \beta_c \kappa ([D(u) n](c(t) - \bar{c}(t)), (w - \bar{w}) n)_{\partial K}
\]

\[
- \sum_{K \in \mathcal{T}} ([D(u) \nabla w] \cdot n, c(t) - \bar{c}(t))_{\partial K},
\]

where \( \beta_c > 0 \) is a penalty parameter.

To complete the discretization, we impose the initial condition eq. (3c) by an \( L^2 \)-projection of \( c_0 \) into \( C_h \).

3.4 Compatibility

In this section we show that eq. (5) and eq. (10) describe a compatible discretization of the coupled Stokes–Darcy flow and transport problem provided \( \ell \) in eq. (9) is suitably chosen. Since global conservation of the EDG method for the transport equation was shown in [41], we only show that eq. (10) is able to preserve the constant solution when \( f = -\chi^d f^d \bar{c} \).

The constant \( c_h = (\bar{c}, \bar{c}) \) is preserved by eq. (10) if and only if

\[
B_h(u_h; (\bar{c}, \bar{c}), w_h) = -\sum_{K \in \mathcal{T}}^d (f^d \bar{c}, w_h)_K \quad \forall w_h \in C_h. \tag{13}
\]
We observe that eq. (13) is equivalent to
\[
- \sum_{K \in \mathcal{T}} (u_h, \nabla w_h)_K + \sum_{K \in \mathcal{T}} \langle u_h \cdot n, w_h - \bar{w}_h \rangle_{\partial K} = - \sum_{K \in \mathcal{T}^d} (f^d, w_h)_K \quad \forall w_h \in C_h.
\] (14)

Using that \( \nabla \cdot (u_h w_h) = u_h \cdot \nabla w_h + w_h \nabla \cdot u_h \) on each element \( K \), integration by parts, single-valuedness of \( \bar{w}_h \) and \( u_h \cdot n \) on interior facets (by eq. (7b)), and that \( u_h \cdot n = 0 \) on the boundary of the domain, eq. (14) simplifies to
\[
\sum_{K \in \mathcal{T}} (\nabla \cdot u_h, w_h)_K = - \sum_{K \in \mathcal{T}^d} (f^d, w_h)_K \quad \forall w_h \in C_h,
\]
and so, using eq. (7a), the constant \( c_h = (\tilde{c}, \tilde{c}) \) is preserved by eq. (10) if and only if
\[
\sum_{K \in \mathcal{T}^d} (\Pi_Q f^d, w_h)_K = \sum_{K \in \mathcal{T}^d} (f^d, w_h)_K \quad \forall w_h \in C_h.
\]

This statement implies that if \( f^d \neq 0 \) we must choose \( \ell = k - 1 \) in eq. (9) for the discretization defined by eq. (5) and eq. (10) to be a compatible discretization of the coupled Stokes–Darcy flow and transport problem. If \( f^d = 0 \) then \( \ell \) can be chosen independent of \( k \).

4 Useful inequalities

In subsequent sections, extensive use will be made of the following continuous trace inequalities [3, Theorem 1.6.6]:
\[
\|v\|^2_{\partial K} \leq C \left( h_K^{-1} \|v\|^2_K + h_K |v|_{1,K}^2 \right) \quad \forall v \in H^1(K),
\] (15)
\[
\|v\|_{0,\infty,\partial K} \leq C \|v\|_{0,\infty,K} \quad \forall v \in W^{1,\infty}(K),
\] (16)
as well as the following discrete inverse and trace inequalities [32, Lemma 1.50, Lemma 1.52]
\[
\|v_h\|_{0,\infty,K} \leq Ch_K^{-\dim/2} \|v_h\|_K \quad \forall v_h \in P_k(K),
\] (17)
\[
\|v_h\|_{\partial K} \leq Ch_K^{-1/2} \|v_h\|_K \quad \forall v_h \in P_k(K).
\] (18)

The inequalities (15)–(18) hold also for \( \dim \)-dimensional vector functions.

An immediate consequence of eqs. (6), (15) and (18) is the following trace inequality that holds for the velocity solution \( u \) to eqs. (1) and (2) and velocity solution \( u_h \) to eq. (5): For \( u^s \in [H^{k+1}(\Omega^s)]^{\dim} \) and \( u^d \in [H^{k+1}(\Omega^d)]^{\dim} \) with \( k \geq 1 \),
\[
\|u - u_h\|_{\partial K} \leq Ch_K^{k+1/2}.
\] (19)

By \( \Pi_V \) we denote the \( L^2 \)-projection onto \( V_h \) and recall that for all \( 0 \leq s \leq k + 1 \) and \( u \in [W^{s,p}(K)]^{\dim} \), \( k \geq 0 \) [10, Proposition 1.135]:
\[
\|u - \Pi_V u\|_{0,p,K} \leq Ch_K^s \|u\|_{s,p,K}, \quad 1 \leq p \leq \infty.
\] (20)

We also require a bound on \( \|u_h\|_{0,\infty,\Omega} \) which we prove next.

**Lemma 4.1.** Let \( u \) denote the velocity solution to eqs. (1) and (2) and assume \( u \in [L^\infty(\Omega)]^{\dim} \) such that \( u^s \in [H^{k+1}(\Omega^s)]^{\dim} \) and \( u^d \in [H^{k+1}(\Omega^d)]^{\dim} \) with \( k \geq 1 \). Then the velocity solution \( u_h \) to eq. (5) satisfies
\[
\|u_h\|_{0,\infty,\Omega} \leq C.
\] (21)
Lemma 5.1. Let

We first prove useful properties of the advective and diffusive parts of the bi-linear form eq. (11).

In this section we show stability and existence of the solution to the semi-discrete EDG method of the transport equation eq. (10) as well as consistency of the method. For this we define the following semi-norm

Finally, by eq. (4b), we note that for \(v \in [L^\infty(\Omega)]^{\text{dim}}\),

The result follows by taking the maximum over \(K \in \mathcal{T}\).

Finally, by eq. (4b), we note that for \(v \in [W^{1,\infty}(K)]^{\text{dim}}, K \in \mathcal{T},\)

\[
\|D(v)\|_{0,\infty,\partial K} \leq C(1 + \|v\|_{0,\infty,\partial K}) \leq C(1 + \|v\|_{0,\infty,K}).
\]

5 Stability, existence and consistency of the EDG method for the transport equation

In this section we show stability and existence of the solution to the semi-discrete EDG method of the transport equation eq. (10) as well as consistency of the method. For this we define the following semi-norm on \(C_h:\)

\[
\|w_h\|_c^2 = \sum_{K \in \mathcal{T}} \Big( \|\nabla w_h\|_K^2 + h_K^{-1} \|w_h - \bar{w}_h\|_{\partial K}^2 \Big).
\]

We first prove useful properties of the advective and diffusive parts of the bi-linear form eq. (11).

Lemma 5.1. Let \(u_h \in V_h\) be the velocity solution to eq. (5). Then for all \(w_h \in C_h,\)

\[
B^a_h(u_h; w_h, w_h) = -\frac{1}{2} \sum_{K \in \mathcal{T}^d} (w_h, u_h \cdot \nabla w_h)_K + \sum_{K \in \mathcal{T}} \langle (u_h \cdot n) w_h, w_h - \bar{w}_h \rangle_{\partial K}
\]

Proof. By definition of \(B^a_h\) eq. (12),

\[
B^a_h(u_h; w_h, w_h) = -\sum_{K \in \mathcal{T}} (w_h, u_h \cdot \nabla w_h)_K + \sum_{K \in \mathcal{T}} \langle (u_h \cdot n) w_h, w_h - \bar{w}_h \rangle_{\partial K}
\]

Using integration by parts for the first term on the right hand side, the algebraic identity \(a(a - b) = \frac{1}{2}(a^2 - b^2 + (a - b)^2),\) and eq. (7), we find

\[
B^a_h(u_h; w_h, w_h) = -\frac{1}{2} \sum_{K \in \mathcal{T}} \langle w_h^2, u_h \cdot n \rangle_{\partial K} + \frac{1}{2} \sum_{K \in \mathcal{T}} \langle \nabla \cdot u_h, w_h^2 \rangle_K
\]

The result follows using the single valuedness of \(\bar{w}_h\) and \(u_h \cdot n\) on element boundaries, and that \(|u_h \cdot n| = u_h \cdot n\) on \(\partial K \backslash \partial K^{\text{in}}\) and \(|u_h \cdot n| = -u_h \cdot n\) on \(\partial K^{\text{in}}\).
Lemma 5.2 (coercivity of $B^d_h$). Let $u$ denote the velocity solution to eqs. (1) and (2) and assume that
$u \in [W^{1,\infty}(\Omega)]^\text{dim}$ such that $u^s \in [H^{k+1}(\Omega^s)]^\text{dim}$ and $u^d \in [H^{k+1}(\Omega^d)]^\text{dim}$ with $k \geq 1$. Let $u_h \in V_h$ be the velocity solution to eq. (5). There exists a constant $\beta_{c,0} > 0$ such that if $\beta_c > \beta_{c,0}$, then for all $w_h \in C_h$

$$B^d_h(u_h; w_h, w_h) \geq C\|w_h\|^2_c.$$

Proof. The proof is similar to [41, Lemma 5.2] but in addition using eq. (4a), eq. (21), eq. (22) and eq. (23) to bound the terms involving $D(u_h)$.

By lemmas 5.1 and 5.2 we may now conclude that when the velocity solution to eqs. (1) and (2) satisfies
$u \in [W^{1,\infty}(\Omega)]^\text{dim}$ such that $u^s \in [H^2(\Omega^s)]^\text{dim}$ and $u^d \in [H^2(\Omega^d)]^\text{dim}$, and when $u_h \in V_h$ is the solution to eq. (5), then

$$B^d_h(u_h; w_h, w_h) \geq C\|w_h\|^2_c + \frac{1}{2} \sum_{K \in T^d} (\nabla \cdot u_h, w^2_h)_K. \tag{24}$$

In what follows we will require the following modification of the classical Grönwall’s lemma [1, Corollary 1.2].

Lemma 5.3. Let $f(t), g(t), h(t)$ be continuous functions defined on $[0, T]$ such that $g(t)$ is non-negative and $h(t)$ is non-decreasing in $[0, T]$. Let $k$ be a non-negative constant. If

$$f(t) + \int_0^t g(s) \, ds \leq h(t) + \int_0^t k f(s) \, ds \quad \forall t \in [0, T],$$

then

$$f(t) + \int_0^t g(s) \, ds \leq h(t)e^{kt} \quad \forall t \in [0, T].$$

Lemma 5.4 (stability). Assume that $u$, the velocity solution to eqs. (1) and (2), satisfies $u \in [W^{1,\infty}(\Omega)]^\text{dim}$ such that $u^s \in [H^{k+1}(\Omega^s)]^\text{dim}$, $u^d \in [H^{k+1}(\Omega^d)]^\text{dim}$, $k \geq 1$ and $\nabla \cdot u^d \in L^\infty(\Omega^d)$. Let $c_0 \in L^2(\Omega)$. Then the solution $c_h \in C_h$ to eq. (10) satisfies

$$\|c_h(t)\|^2_\Omega + \int_0^t \|c_h(s)\|^2_c \, ds \leq C \left( \|c_0\|^2_\Omega + \int_0^t \|f(s)\|^2_\Omega \, ds \right) \quad \forall t \in [0, T].$$

Proof. Take $w_h = c_h(t)$ in eq. (10). Then by eq. (24) and the Cauchy–Schwarz inequality,

$$\frac{\phi}{2} \frac{d}{dt} \|c_h(t)\|^2_\Omega + C \|c_h(t)\|^2_c \leq \sum_{K \in T^d} (f(t), c_h(t))_K - \frac{1}{2} \sum_{K \in T^d} (\nabla \cdot u_h, c^2_h(t))_K \leq \|f(t)\|_\Omega \|c_h(t)\|_\Omega + \frac{1}{2} \|\nabla \cdot u_h\|_{0,\infty,\Omega^d} \|c_h\|^2_\Omega. \tag{25}$$

Note that by eq. (7), $\|\nabla \cdot u_h\|_{0,\infty,\Omega^d} = \|\Pi_Q f^d\|_{0,\infty,\Omega^d} \leq C \|f^d\|_{0,\infty,\Omega^d}$. Combining with eq. (25), integrating from 0 to $t$ for some $0 < t \leq T$, and applying the Cauchy–Schwarz and Young’s inequalities,

$$\frac{\phi}{2} \|c_h(t)\|^2_\Omega + C \int_0^t \|c_h(s)\|^2_c \, ds \leq \frac{\phi}{2} \|c_h(0)\|^2_\Omega + \frac{1}{2} \int_0^t \|f(s)\|^2_\Omega \, ds + \frac{1}{2} \int_0^t (1 + C \|f^d\|_{0,\infty,\Omega^d}) \|c_h(s)\|^2_\Omega \, ds.$$

The result follows by lemma 5.3 and recalling that $c_h(0)$ is the $L^2$-projection of $c_0$ into $C_h$. \qed

A consequence of this stability result is existence and uniqueness, which can be obtained by setting $c_0 = 0$ and $f = 0$. 

8
Theorem 5.1 (existence and uniqueness). The semi-discrete scheme eq. (10) has a unique solution \( c_h \in C_h \).

We next prove consistency of the method.

Lemma 5.5 (consistency). If \( c \in L^2(0,T;H^2(\Omega)) \) solves eq. (3) and \( u \) is the velocity solution to eqs. (1) and (2), then for all \( t \in (0,T) \),

\[
\sum_{K \in T} (\phi \partial_t c(t), w_h)_K + B_h(u; c(t), w_h) = \sum_{K \in T} (f(t), w_h)_K \quad \forall w_h \in C_h,
\]

where \( c = (c, \bar{c}) \) with \( \bar{c} \) the restriction of \( c \) to \( \Gamma^0 \).

Proof. The result follows by integrating by parts, noting that \( c = \bar{c} \) on \( \partial K \), the boundary conditions eqs. (1d) and (1e), and eqs. (3a) and (3b). \qed

6 Error analysis of the semi-discrete scheme

Given the velocity solution \( u_h \in V_h \) to eq. (5) we prove that the solution \( c_h \in C_h \) to the semi-discrete scheme eq. (10) converges to the solution of eq. (3) in the energy norm and in the \( L^2 \)-norm. We consider only the analysis of a compatible discretization for the general case when \( f^d \neq 0 \) in eq. (1c), i.e., we take \( k > 1 \) in \( X_h \) and \( l = k - 1 > 0 \) in eq. (9).

6.1 Error estimate of the concentration in the energy norm

To prove convergence in the energy norm we will use the continuous interpolant \( Ic \in C_h \cap C^0(\Omega) \) of \( c \) [3] and we set \( \tilde{I}c(t) = Ic|_{\Gamma^0}(t) \in \tilde{C}_h \). Denoting the restriction of \( c \) to \( \Gamma^0 \) by \( \bar{c} \), we split the approximation errors as follows:

\[
c - c_h = \xi_c - \zeta_c \quad \text{and} \quad \bar{c} - \bar{c}_h = \xi_{\bar{c}} - \zeta_{\bar{c}},
\]

where

\[
\xi_c = c - \tilde{I}c, \quad \zeta_c = c_h - \tilde{I}c, \quad \xi_{\bar{c}} = \bar{c} - \tilde{I}c, \quad \zeta_{\bar{c}} = \bar{c}_h - \tilde{I}c, \quad \xi_c = (\xi_c, \xi_{\bar{c}}), \quad \zeta_c = (\zeta_c, \zeta_{\bar{c}}).
\]

The following interpolation estimates hold [3, Chapter 4]:

\[
\|\xi_c\|_{r,K} \leq Ch^{k-r}_K\|c\|_{s,K}, \quad r = 0, 1, \quad 2 \leq s \leq k,
\]

and

\[
\|\xi_{\bar{c}}\|_{r,K} \leq Ch^{k-r}_K\|c\|_{s,K}, \quad r = 0, 1, \quad 1 \leq s \leq k.
\]

A straightforward consequence is

\[
\|\xi_c(t)\|_c = \left( \sum_{K \in T} \|\nabla \xi_c(t)\|_K^2 \right)^{1/2} \leq Ch^{s-1}_K\|c(t)\|_{s,\Omega}, \quad 2 \leq s \leq k.
\]

We will also require the following continuity results.

Lemma 6.1. Let \( u \in [W^{1,\infty}(\Omega)]^{\dim} \) be the velocity solution to eqs. (1) and (2) such that \( u^s \in [H^{k+1}(\Omega^s)]^{\dim} \) and \( u^d \in [H^{k+1}(\Omega^d)]^{\dim} \), let \( c \in L^2(0,T;H^k(\Omega)) \) be the solution to eq. (3) and let \( u_h \in V_h \) be the velocity solution to eq. (5). Then for any \( 0 \leq t \leq T \),

\[
B_h(u_h; \xi_c(t), w_h) \leq Ch^{k-1}_K\|c(t)\|_{k,\Omega} \|w_h\|_c,
\]

for all \( w_h \in C_h \) and \( k > 1 \).
Proof. By definition of $B_h$ eq. (11),
\[
B_h(u_h; \xi_c, w_h) = B^c_h(u_h; \xi_c, w_h) + B^d_h(u_h; \xi_c, w_h).
\]
We will first bound $B^c_h$. Noting that $\xi_c - \bar{\xi}_c$ vanishes on facets, we have by eq. (12)
\[
B^c_h(u_h; \xi_c, w_h) = -\sum_{K \in T} (\xi_c, u_h \cdot \nabla w_h)_K + \sum_{K \in T} (\xi_c, u_h \cdot n, w_h - \bar{w}_h)_{\partial K} = J_1 + J_2.
\]

We start by bounding $J_1$. By eq. (27), we have
\[
J_1 \leq C \|u_h\|_{0, \infty, \Omega} \|\xi_c\|_{\Omega} \|\nabla w_h\| \leq C h^k \|u_h\|_{0, \infty, \Omega} \|c\|_{k, \Omega} \|w_h\|_c.
\]
By eqs. (15), (16) and (27),
\[
J_2 \leq C \|u_h\|_{0, \infty, \Omega} \left( \sum_{K \in T} h_K \|\xi_c\|_{\partial K}^2 \right)^{1/2} \left( \sum_{K \in T} h_K^{-1} \|w_h - \bar{w}_h\|_{\partial K}^2 \right)^{1/2}
\leq C h^k \|u_h\|_{0, \infty, \Omega} \|c\|_{k, \Omega} \|w_h\|_c.
\]
Combining the bounds for $J_1$ and $J_2$ and using eq. (21) we obtain
\[
B^c_h(u_h; \xi_c, w_h) \leq C h^k \|c\|_{k, \Omega} \|w_h\|_c. \tag{31}
\]
We next bound $B^d_h$. Since $u_h \in \left[L^\infty(\Omega)\right]^{\dim}$ by eq. (21), we can use eq. (22) to bound $D(u_h)$. Then, noting that $\xi_c - \bar{\xi}_c$ vanishes on facets, and using eq. (15) and eq. (27),
\[
B^d_h(u_h; \xi_c, w_h) \leq D_{\max} \|\nabla \xi_c\|_\Omega \|w_h\|_c
+ D_{\max} \left( \sum_{K \in T} h_K \|\nabla \xi_c\|_{\partial K}^2 \right)^{1/2} \left( \sum_{K \in T} h_K^{-1} \|w_h - \bar{w}_h\|_{\partial K}^2 \right)^{1/2}
\leq C h^{k-1} \|c\|_{k, \Omega} \|w_h\|_c. \tag{32}
\]
The result follows after combining eqs. (31) and (32). \qed

Lemma 6.2. Let $u \in \left[W^{1, \infty}(\Omega)\right]^{\dim}$ be the velocity solution to eqs. (1) and (2) such that $u^s \in \left[H^{k+1}(\Omega^s)\right]^{\dim}$ and $u^d \in \left[H^{k+1}(\Omega^d)\right]^{\dim}$, $c \in L^2(0, T; H^k(\Omega)) \cap W^{1, \infty}(\Omega)$ be the solution to eq. (3) and let $u_h \in V_h$ be the velocity solution to eq. (5). Then for any $0 \leq t \leq T$,
\[
B_h(u_h; c(t), w_h) - B_h(u; c(t), w_h) \leq C h^{k-1} \left( \|c(t)\|_{1, \infty, \Omega} + \|c(t)\|_{k, \Omega} \right) \|w_h\|_c, \tag{33}
\]
for all $w_h \in C_h$, where $c = (c, \bar{c})$ and $k > 1$.

Proof. We first note that
\[
B_h(u_h; c, w_h) - B_h(u; c, w_h) = B^c_h(u_h - u; c, w_h) + [B^d_h(u_h; c, w_h) - B^d_h(u; c, w_h)].
\]
Since $c = \bar{c}$ on element boundaries,
\[
B^c_h(u_h - u; c, w_h) = -\sum_{K \in T_h} (c(u_h - u), \nabla w_h)_K + \sum_{K \in T_h} (c(u_h - u) \cdot n, w_h - \bar{w}_h)_{\partial K}
= G_1 + G_2.
\]
By eq. (6),
\[
G_1 \leq \|c\|_{0, \infty, \Omega} \|u - u_h\|_{\Omega} \|\nabla w_h\|_{\Omega} \leq C h^{k+1} \|c\|_{0, \infty, \Omega} \|w_h\|_c.
\]
Furthermore, by eqs. (16) and (19),

\[ G_2 \leq C \|c\|_{0, \infty, \Omega} \left( \sum_{K \in T_h} h_K \|u - u_h\|_{\partial K}^2 \right)^{1/2} \left( \sum_{K \in T_h} h_K^{-1} \|w - \bar{w}\|_{\partial K}^2 \right)^{1/2} \]

\[ \leq Ch^{k+1} \|c\|_{0, \infty, \Omega} \|w_h\|_c. \]

It follows that \( B_h(u_h - u; c, w_h) \leq Ch^{k+1} \|c\|_{0, \infty, \Omega} \|w_h\|_c. \)

Consider now \( B_h(u_h; c, w_h) - B_h^d(u; c, w_h) \). Since \( c = \bar{c} \) on \( \partial K \),

\[ B_h^d(u_h; c, w_h) - B_h(u; c, w_h) = \sum_{K \in T} ((D(u_h) - D(u)) \nabla c, \nabla w_h)_K \]

\[ - \sum_{K \in T} \langle [(D(u_h) - D(u)) \nabla c] \cdot n, w_h - \bar{w}_h \rangle_{\partial K} \]

\[ = H_1 + H_2. \]

Using eqs. (4c) and (6),

\[ H_1 \leq C \|u_h - u\|_{\Omega} \|\nabla c\|_{0, \infty, \Omega} \|\nabla w_h\|_{\Omega} \leq Ch^{k+1} \|c\|_{1, \infty, \Omega} \|w_h\|_c. \]

To bound \( H_2 \), we first rewrite it as follows:

\[ H_2 = \sum_{K \in T} \langle [(D(u_h) - D(u)) \nabla Ic] \cdot n, w_h - \bar{w}_h \rangle_{\partial K} \]

\[ + \sum_{K \in T} \langle [D(u) \nabla (Ic - c)] \cdot n, w_h - \bar{w}_h \rangle_{\partial K} \]

\[ - \sum_{K \in T} \langle [D(u_h) \nabla (Ic - c)] \cdot n, w_h - \bar{w}_h \rangle_{\partial K} \]

\[ = H_{21} + H_{22} + H_{23}. \]

By eq. (19), eq. (4c), and eq. (28),

\[ H_{21} \leq \sum_{K \in T} C \|u_h - u\|_{\partial K} \|\nabla Ic\|_{0, \infty, K} \|w_h - \bar{w}_h\|_{\partial K} \]

\[ \leq C \|c\|_{1, \infty, \Omega} \left( \sum_{K \in T} h_K \|u_h - u\|_{\partial K}^2 \right)^{1/2} \left( \sum_{K \in T} h_K^{-1} \|w_h - \bar{w}_h\|_{\partial K}^2 \right)^{1/2} \]

\[ \leq Ch^{k+1} \|c\|_{1, \infty, \Omega} \|w_h\|_c. \]

Using eq. (23), eq. (15), and eq. (27), we obtain

\[ H_{22} \leq C(1 + \|u\|_{0, \infty, \Omega}) \left( \sum_{K \in T} h_K \|\nabla (Ic - c)\|_{\partial K}^2 \right)^{1/2} \left( \sum_{K \in T} h_K^{-1} \|w_h - \bar{w}_h\|_{\partial K}^2 \right)^{1/2} \]

\[ \leq C(1 + \|u\|_{0, \infty, \Omega})h^{-1} \|c\|_{k, \Omega} \|w_h\|_c. \]

Following similar steps as above, this time using eqs. (16) and (22), we obtain

\[ H_{23} \leq D_{max} h^{-1} \|c\|_{k, \Omega} \|w_h\|_c. \]

The result follows.

We next find an estimate of the concentration approximation error in the energy norm.
Lemma 6.3. Let $u \in [W^{1,\infty}(\Omega)]^{\dim}$ be the velocity solution to eqs. (1) and (2) such that $u^s \in [H^{k+1}(\Omega^s)]^{\dim}$ and $u^d \in [H^{k+1}(\Omega^d)]^{\dim}$ with $k > 1$, and let $c \in L^2(0,T;H^k(\Omega) \cap W^{1,\infty}(\Omega))$ be the solution to eq. (3) such that $\partial_t c \in L^2(0,T;H^k(\Omega))$ and $c_0 \in H^k(\Omega)$. Furthermore, let $u_h \in V_h$ be the velocity solution to eq. (5). Then,

$$\|\zeta_c(t)\|_\Omega^2 + \int_0^t \|\zeta_c(s)\|_c^2 \, ds \leq Ch^{2(k-1)} \quad \forall t \in [0,T].$$

Proof. Subtracting eq. (26) from eq. (10) and splitting the error,

$$\sum_{K \in T} (\phi \partial_t \zeta_c, w_h)_K + B_h(u_h; \zeta_c, w_h) = \sum_{K \in T} (\phi \partial_t \zeta_c, \zeta_c)_K + B_h(u_h; \zeta_c, w_h) - [B_h(u_h; c, w_h) - B_h(u; c, w_h)].$$

Setting $w_h = \zeta_c(t)$, using coercivity eq. (24), and integrating from 0 to $t$, where $0 \leq t \leq T$, we obtain

$$\frac{\phi}{2} \|\zeta_c(t)\|_\Omega^2 + C \int_0^t \|\zeta_c(s)\|_c^2 \, ds \leq \frac{\phi}{2} \|\zeta_c(0)\|_\Omega^2 + \int_0^t \sum_{K \in T} (\phi \partial_t \zeta_c(s), \zeta_c(s))_K \, ds$$

$$+ \int_0^t B_h(u_h; \zeta_c(s), \zeta_c(s)) \, ds - \int_0^t B_h(u_h; c(s), \zeta_c(s)) \, ds + \int_0^t B_h(u; c(s), \zeta_c(s)) \, ds$$

$$- \int_0^t \frac{1}{2} \sum_{K \in T^d} (\nabla \cdot u_h, \zeta_c^2(s))_K \, ds =: I_1 + \ldots + I_6.$$ 

Consider first $I_1$. Observe that by eq. (27)

$$\|\zeta_c(0)\|_\Omega \leq \|c_h(0) - c_0\| + \|c_0 - Ic_0\| \leq Ch^k \|c_0\|_{k,\Omega},$$

since $c_h(0)$ is the $L^2$-projection of $c_0$ into $C_h$. Therefore $I_1 \leq C\frac{\phi}{2} h^{2(k)}\|c_0\|_{k,\Omega}^2$.

Using eq. (27) and Cauchy–Schwarz and Young’s inequalities,

$$I_2 \leq C \frac{h^{2k}}{2}\|\partial_t c\|_{L^2(0,t;H^k(\Omega))}^2 + C \int_0^t \|\zeta_c(s)\|_c^2 \, ds.$$

Next, by eq. (30) and Young’s inequality,

$$I_3 \leq Ch^{2(k-1)}\|c\|_{L^2(0,t;H^k(\Omega))}^2 + \delta \int_0^t \|\zeta_c(s)\|_c^2 \, ds,$$

with $\delta > 0$ a constant. Similarly, using eq. (33),

$$I_4 + I_5 \leq Ch^{2(k-1)}(\|c\|_{L^2(0,t;W^{1,\infty}(\Omega))}^2 + \|c\|_{L^2(0,t;H^k(\Omega))}^2) + \delta \int_0^t \|\zeta_c(s)\|_c^2 \, ds.$$

For the final term, as in the proof of Lemma 5.4, we have

$$I_6 \leq \int_0^t C \|f^d\|_{0,\infty,\Omega^d} \|\zeta_c(s)\|_\Omega^2 \, ds.$$

Choosing $\delta$ small enough and combining the above bounds, we obtain

$$\|\zeta_c(t)\|_\Omega^2 + \int_0^t \|\zeta_c(s)\|_c^2 \, ds$$

$$\leq Ch^{2(k-1)} \left(\|c_0\|_\Omega^2 + \|\partial_t c\|_{L^2(0,t;H^k(\Omega))}^2 + \|c\|_{L^2(0,t;H^k(\Omega))}^2 + \|c\|_{L^2(0,t;W^{1,\infty}(\Omega))}^2 \right)$$

$$+ \int_0^t C \left(1 + \|f^d\|_{0,\infty,\Omega^d}\right) \|\zeta_c(s)\|_\Omega^2 \, ds.$$ 

The result follows by lemma 5.3. □
A consequence of eqs. (27) and (29) and lemma 6.3 is the following error estimate for the concentration in the energy norm.

**Corollary 6.1.** Let \( u, c \) and \( c_0 \) satisfy the assumptions of lemma 6.3. Then,

\[
\|c(t) - c_h(t)\|^2_{\Omega} + \int_0^t \|c(s) - c_h(s)\|^2_c \, ds \leq Ch^{2(k-1)} \quad \forall t \in [0, T].
\]

### 6.2 Error estimate of the concentration in the \( L^2 \)-norm

To obtain the \( L^2 \)-norm estimate for the concentration we consider the following dual problem for \( 0 < t < T \) [15]:

\[
\begin{align*}
\phi \partial_t \sigma + u \cdot \nabla \sigma + \nabla \cdot [D(u)\nabla \sigma] &= \Psi & \text{in } \Omega, \quad (34a) \\
D(u)\nabla \sigma \cdot n &= 0 & \text{on } \partial \Omega, \quad (34b) \\
\sigma(\cdot, T) &= 0 & \text{in } \Omega. \quad (34c)
\end{align*}
\]

We will assume that \( \sigma \in W^{2,\infty}(\Omega) \) and that

\[
\max_{0 \leq t \leq T} \|\sigma(\cdot, t)\|^2_{L^2(\Omega)} + \int_0^T \|\sigma(t)\|^2_{L^2(\Omega)} \, dt \leq C \|\Psi\|^2_{L^2(0, T; L^2(\Omega))}. \quad (35)
\]

For the analysis in this section we require \( \Pi_C \) and \( \Pi_C \), the \( L^2 \)-projections into \( C_h \) and \( \bar{C}_h \), respectively. We will use the following standard results of approximation theory [10, Chapter 3]:

\[
\begin{align*}
\|w - \Pi_C w\|_{r, K} &\leq Ch^{k-r} \|w\|_{s, K}, & 0 \leq s \leq k, & r = 0, 1, \quad (36) \\
\|\Pi_C w - \bar{\Pi}_C w\|_{\partial K} &\leq Ch^{k-1/2} \|w\|_{s, K}, & 1 \leq s \leq k, \quad (37)
\end{align*}
\]

for all \( w \in H^k(\Omega) \). We will now prove an \( L^2 \)-norm estimate for the concentration.

**Theorem 6.1** (Concentration error estimate in the \( L^2 \)-norm.). Suppose that \( c, u \) and \( u_h \) satisfy the assumptions of Lemma 6.3 and \( c_h \) is the solution to eq. (10). Then

\[
\|c - c_h\|_{L^2(0, T; L^2(\Omega))} \leq Ch^k.
\]

**Proof.** Set \( \Psi = c - c_h \) in eq. (34). We multiply eq. (34a) by \( c - c_h \), integrate over \( K \in \mathcal{T} \), integrate by parts the diffusion term over an element \( K \in \mathcal{T} \), sum over the elements, and integrate from 0 to \( T \). Then, integrating by parts in time and using eq. (34c),

\[
-\int_0^T \sum_{K \in \mathcal{T}} (\phi \sigma, \partial_t (c - c_h))_K + \int_0^T \sum_{K \in \mathcal{T}} (\nabla \sigma, u(c - c_h))_K \, dt \\
-\int_0^T \sum_{K \in \mathcal{T}} (D(u)\nabla \sigma \cdot \nabla (c - c_h))_K \, dt + \int_0^T \sum_{K \in \mathcal{T}} (D(u)\nabla \sigma \cdot n, c - c_h)_{\partial K} \, dt \\
-\sum_{K \in \mathcal{T}} (\phi \sigma(0), c_0 - c_h(0))_K = \|c - c_h\|^2_{L^2(0, T; L^2(\Omega))}. \quad (38)
\]

Subtracting eq. (10) from eq. (26), choosing \( w_h = \Pi_C \sigma = (\Pi_C \sigma, \bar{\Pi}_C \sigma) \) and integrating with respect to time,

\[
\int_0^T \sum_{K \in \mathcal{T}} \left\{ (\phi \partial_t (c - c_h), \Pi_C \sigma)_K + \left[ B_h(u; c, \Pi_C \sigma) - B_h(u_h; c_h, \Pi_C \sigma) \right] \right\} \, dt = 0. \quad (39)
\]
Adding eqs. (38) and (39), and manipulating the integrals, we obtain

\[
\|c - c_h\|_{L^2(0,T;L^2(\Omega))}^2 = - \int_0^T \sum_{K \in T} (\phi \partial_t (c - c_h), \sigma - \Pi_C \sigma)_K \, dt \\
+ \int_0^T \sum_{K \in T} (\nabla (\sigma - \Pi_C \sigma), u(c - c_h))_K \, dt - \int_0^T \sum_{K \in T} (D(u) \nabla (\sigma - \Pi_C \sigma), \nabla (c - c_h))_K \, dt \\
+ \int_0^T \sum_{K \in T} ([D(u) \nabla \sigma] \cdot n, c - c_h)_{\partial K} \, dt + \int_0^T \sum_{K \in T} ([D(u_h) \nabla \Pi_C \sigma] \cdot n, c_h - \bar{c}_h)_{\partial K} \, dt \\
- \int_0^T \sum_{K \in T} (c_h (u_h, \nabla \Pi_C \sigma))_K \, dt + \int_0^T \sum_{K \in T} (\nabla c_h, [D(u) - D(u_h)] \nabla \Pi_C \sigma)_K \, dt \\
- \sum_{K \in T} (\phi \sigma(0), c_0 - c_h(0))_K + \int_0^T \sum_{K \in T} (c - c_h) u \cdot n, \Pi_C \sigma - \bar{\Pi}_C \sigma)_{\partial K} \, dt \\
+ \int_0^T \sum_{K \in T} (c_h (u - u_h) \cdot n, \Pi_C \sigma - \bar{\Pi}_C \sigma)_{\partial K} \, dt \\
- \int_0^T \sum_{K \in T} (\sigma \nabla (c - c_h)) \cdot n, \Pi_C \sigma - \bar{\Pi}_C \sigma)_{\partial K} \, dt \\
- \int_0^T \sum_{K \in T} (\sigma \nabla (c - c_h)) \cdot n, \Pi_C \sigma - \bar{\Pi}_C \sigma)_{\partial K} \, dt \\
+ \int_0^T \sum_{K \in T} (u_h \cdot (c_h - \bar{c}_h), \Pi_C \sigma - \bar{\Pi}_C \sigma)_{\partial K} \, dt \\
- \int_0^T \sum_{K \in T} \frac{\beta_c}{h_K} ([D(u_h) n] (c_h - \bar{c}_h), (\Pi_C \sigma - \bar{\Pi}_C \sigma) n)_{\partial K} \, dt \\
= T_1 + \ldots + T_{14}.
\]

Observe that since \(c_h, \Pi_C C \in C_h, (\phi \partial_t c_h, \sigma - \Pi_C \sigma)_K = 0\) and \((\phi \partial_t C c, \sigma - \Pi_C \sigma)_K = 0\). Then, by eq. (36), the Cauchy–Schwarz inequality, eq. (35), and Young’s inequality,

\[
T_1 = - \int_0^T \sum_{K \in T} (\phi \partial_t (c - \Pi_C c), \sigma - \Pi_C \sigma)_K \, dt \\
\leq \int_0^T \sum_{K \in T} C \phi h^k \|\partial_t c\|_{L^2(0,T)} \|\sigma\|_{L^2(0,T)} \, dt \\
\leq C \phi h^k \|\partial_t c\|_{L^2(0,T;H^k(\Omega))} \left( \int_0^T \|\sigma\|_{L^2(\Omega)}^2 \, dt \right)^{1/2} \\
\leq C h^2 k \|\partial_t c\|_{L^2(0,T;H^k(\Omega))}^2 + \delta \|c - c_h\|_{L^2(0,T)}^2,
\]

where \(\delta > 0\).

By corollary 6.1, eqs. (35) and (36),

\[
T_2 \leq \|u\|_{0,\infty,\Omega} \left( \int_0^T \|c - c_h\|_{L^2(\Omega)} \, dt \right)^{1/2} \left( \int_0^T \|\nabla (\sigma - \Pi_C \sigma)\|_{L^2(\Omega)} \, dt \right)^{1/2} \\
\leq \|u\|_{0,\infty,\Omega} \left( \int_0^T C h^2 (k-1) \, dt \right)^{1/2} \left( \int_0^T \|\sigma\|_{L^2(\Omega)}^2 \, dt \right)^{1/2} \\
\leq C h^k T^{1/2} \|u\|_{0,\infty,\Omega} \|c - c_h\|_{L^2(0,T;L^2(\Omega))} \\
\leq \delta \|c - c_h\|_{L^2(0,T;L^2(\Omega))} + C T \|u\|_{0,\infty,\Omega}^2 h^{2k}.
\]
Following similar steps as in the bound for $T_2$, using eq. (22) and corollary 6.1, we find that
\[ T_3 \leq \delta \|c - c_h\|_{L^2(0,T;L^2(\Omega))}^2 + CD_{\text{max}}^2 h^{2k}. \]

Next, observe that by smoothness of $u$, $c$ and $\sigma$, eq. (34b) and single valuedness of $\bar{c}_h$, we can write
\[ \sum_{K \in \mathcal{T}} \langle D(u) \nabla \sigma \cdot n, c - c_h \rangle_{\partial K} = \sum_{K \in \mathcal{T}} \langle D(u) \nabla \sigma \cdot n, \bar{c}_h - c_h \rangle_{\partial K}. \]

Therefore,
\[ T_4 + T_5 = \int_0^T \sum_{K \in \mathcal{T}} \langle D(u) \nabla (\sigma - \Pi_C \sigma) \cdot n, \bar{c}_h - c_h \rangle_{\partial K} \, dt \]
\[ + \int_0^T \sum_{K \in \mathcal{T}} \langle (D(u_h) - D(u)) \nabla \Pi_C \sigma \cdot n, c_h - \bar{c}_h \rangle_{\partial K} \, dt = T_{451} + T_{452}. \]

Since $\bar{c}_h - c_h = (c - c_h) - (\bar{c} - \bar{c}_h)$ on $\partial K$, using eqs. (15), (23) and (36),
\[ T_{451} \leq C(1 + \|u\|_{0,\infty,\Omega}) \int_0^T \left( \sum_{K \in \mathcal{T}} h_K \|\nabla (\sigma - \Pi_C \sigma)\|_{\partial K}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}} h_K^{-1} \|\bar{c}_h - c_h\|_{\partial K}^2 \right)^{1/2} \, dt \]
\[ \leq C h(1 + \|u\|_{0,\infty,\Omega}) \left( \int_0^T \|\sigma\|_{L^2(\Omega)}^2 \, dt \right)^{1/2} \left( \int_0^T \|c - c_h\|_{\bar{c}_h}^2 \, dt \right)^{1/2}. \]

Next, apply eq. (4c), the trace inequality eq. (16), eq. (19), the inverse inequality eq. (17), and eq. (36),
\[ T_{452} \leq C h^{k+1/2} \int_0^T \sum_{K \in \mathcal{T}} \|\nabla \Pi_C \sigma\|_{0,\infty,K} \|c_h - \bar{c}_h\|_{\partial K} \, dt \]
\[ \leq C h^{k+1-\text{dim}/2} \int_0^T \sum_{K \in \mathcal{T}} \|\nabla \Pi_C \sigma\|_{K} h_K^{-1/2} \|c_h - \bar{c}_h\|_{\partial K} \, dt \]
\[ \leq C \left( \int_0^T |\sigma|_{L^2(\Omega)}^2 \, dt \right)^{1/2} \left( \int_0^T \|c - c_h\|_{\bar{c}_h}^2 \, dt \right)^{1/2}. \]

Therefore, by eq. (35) and corollary 6.1,
\[ T_4 + T_5 \leq C h^k \|c - c_h\|_{L^2(0,T;L^2(\Omega))}^2 \leq \delta \|c - c_h\|_{L^2(0,T;L^2(\Omega))}^2 + C h^{2k}. \]

To bound $T_6$, note that
\[ T_6 = \int_0^T \sum_{K \in \mathcal{T}} ((c - c_h)(u - u_h), \nabla \Pi_C \sigma)_K \, dt - \int_0^T \sum_{K \in \mathcal{T}} (c(u - u_h), \nabla \Pi_C \sigma)_K \, dt \]
\[ =: T_{61} + T_{62}. \]

Using the inverse inequality eq. (17) and eq. (6),
\[ T_{61} \leq \int_0^T \sum_{K \in \mathcal{T}} \|c_h - c\|_K \|u - u_h\|_K \|\nabla \Pi_C \sigma\|_{0,\infty,K} \, dt \]
\[ \leq C h^{k+1-\text{dim}/2} \int_0^T \|c_h - c\|_\Omega \|\nabla \sigma\|_\Omega \, dt \leq C h^k T^{1/2} \left( \int_0^T \|\nabla \sigma\|_{\Omega}^2 \, dt \right)^{1/2}, \]

where we used corollary 6.1 and the assumption that $k > 1$. Therefore, by eq. (35) and Young’s inequality,
\[ T_{61} \leq C h^k T^{1/2} \|c - c_h\|_{L^2(0,T;L^2(\Omega))} \leq C h^{2k} + \delta \|c - c_h\|_{L^2(0,T;L^2(\Omega))}^2. \]
By eq. (36), Hölder’s inequality and eq. (6),
\[ T_{62} \leq C\|u - u_h\|_\Omega \int_0^T \|c\|_{0,\infty,\Omega} \|\sigma\|_{1,\Omega} \, dt \]
\[ \leq C h^{k+1} \|c\|_{L^2(0,T;L^\infty(\Omega))} \left( \int_0^T \|\sigma\|_{2,\Omega}^2 \, dt \right)^{1/2} \leq C h^{2k+2} + \delta \|c - c_h\|^2_{L^2(0,T;L^2(\Omega))}, \]
where we applied eq. (35), and Young’s inequality. Therefore,
\[ T_6 \leq C h^{2k} + \delta \|c - c_h\|^2_{L^2(0,T;L^2(\Omega))}. \]

Similarly, this time using eq. (4c),
\[ T_7 \leq C h^{2k} + \delta \|c - c_h\|^2_{L^2(0,T;L^2(\Omega))}. \]
By eq. (35), Young’s inequality and since \(c_h(0)\) is the \(L^2\)-projection of \(c_0\),
\[ T_8 \leq \phi \|\sigma(0)\|_\Omega \|c_0 - c_h(0)\|_\Omega \leq \delta \|c - c_h\|^2_{L^2(0,T;L^2(\Omega))} + C h^{2k} \|c_0\|^2_{k,\Omega}. \]

Using eq. (37), eq. (15), eq. (16), corollary 6.1, and eq. (35),
\[ T_9 \leq \|u\|_{0,\infty,\Omega} \left( \int_0^T \sum_{K \in T} \|c - c_h\|^2_{\partial K} \right)^{1/2} \left( \int_0^T \sum_{K \in T} \|\Pi_C \sigma - \Pi_C \sigma\|^2_{\partial K} \right)^{1/2} \]
\[ \leq C h^{3/2} \|u\|_{0,\infty,\Omega} \left( \int_0^T \sum_{K \in T} \left( h_k^{-1} \|c - c_h\|^2_{K} + h_k \|c - c_h\|^2_{1,K} \right) \right)^{1/2} \left( \int_0^T \|\sigma\|^2_{2,\Omega} \, dt \right)^{1/2} \]
\[ \leq C h^{3/2} \|u\|_{0,\infty,\Omega} \left( T^{1/2} h^{k-3/2} + h^{k-1/2} \right) \left( \int_0^T \|\sigma\|^2_{2,\Omega} \, dt \right)^{1/2} \]
\[ \leq C h^{2k} + \delta \|c - c_h\|^2_{L^2(0,T;L^2(\Omega))}. \]

By eq. (16), eq. (19), eq. (17), eq. (37), Hölder’s inequality, lemma 5.4, that \(k + 2 - \dim/2 \geq k\) and eq. (35),
\[ T_{10} \leq C h^{k+1/2} \int_0^T \sum_{K \in T} \|c_h\|_{0,\infty,K} \|\Pi_C \sigma - \Pi_C \sigma\|_{\partial K} \, dt \]
\[ \leq C h^{k+2 - \dim/2} \int_0^T \|c_h\|_{\Omega} \|\sigma\|_{2,\Omega} \, dt \]
\[ \leq C h^{k+2 - \dim/2} \max_{0 \leq t \leq T} \|c_h\|_{\Omega} T^{1/2} \|c - c_h\|_{L^2(0,T;L^2(\Omega))} \]
\[ \leq C h^{2k} + \delta \|c - c_h\|^2_{L^2(0,T;L^2(\Omega))}. \]

Using eq. (23) and eq. (37),
\[ T_{11} \leq C (1 + \|u\|_{0,\infty,\Omega}) \int_0^T \sum_{K \in T} \left( \|\nabla \xi_c\|_{\partial K} + \|\nabla \zeta_c\|_{\partial K} \right) h_k^{3/2} \|\sigma\|^2_{2,K} \, dt. \]

By the trace inequalities eqs. (15) and (18), the interpolation estimate eq. (27), and the Cauchy–Schwarz and Young’s inequalities,
\[ T_{11} \leq C (1 + \|u\|_{0,\infty,\Omega}) \left[ h^{k+1/2} \|c\|_{L^2(0,T;H^k(\Omega))} + h \left( \int_0^T \|\zeta_c\|^2_{c} \, dt \right)^{1/2} \right] \left( \int_0^T \|\sigma\|^2_{2,\Omega} \, dt \right)^{1/2} \]
\[ \leq C h^{2k} + \delta \|c - c_h\|^2_{L^2(0,T;L^2(\Omega))}. \]
By eq. (4c), eq. (19), eq. (37), eq. (16), eq. (17), lemma 5.4 and eq. (35),

\[ T_{12} \leq C \int_0^T \sum_{K \in T} \| u - u_h \|_{\partial K} \| \nabla c_h \|_{0, \infty, K} h_{3/2}^6 \| \sigma \|_{2, K} \, dt \]

\[ \leq C h^{k+2-\dim/2} \int_0^T \| \nabla c_h \|_{\Omega} \| \sigma \|_{2, \Omega} \, dt \leq C h^{2k} + \delta \| c - c_h \|_{L^2(0,T;L^2(\Omega))}^2. \]

Using eq. (16), eq. (37), corollary 6.1, eq. (35) and eq. (21),

\[ T_{13} \leq C \| u_h \|_{0, \infty, \Omega} \int_0^T \sum_{K \in T} \| c_h - c - (\bar{c}_h - \bar{c}) \|_{\partial K} h_{3/2}^6 \| \sigma \|_{2, K} \, dt \]

\[ \leq C h^{2} \| u_h \|_{0, \infty, \Omega} \left( \int_0^T \| c - c_h \|^2 \, dt \right)^{1/2} \left( \int_0^T \| \sigma \|^2 \, dt \right)^{1/2} \]

\[ \leq C h^{2(k+1)} + \delta \| c - c_h \|_{L^2(0,T;L^2(\Omega))}^2. \]

Finally we consider \( T_{14} \). By eq. (22), eq. (16), eq. (37), corollary 6.1, eq. (35)

\[ T_{14} \leq C \int_0^T \sum_{K \in T} \frac{\beta_c}{h_K} \| D(u_h) \|_{0, \infty, \partial K} \| c_h - \bar{c}_h \|_{\partial K} h_{3/2}^6 \| \sigma \|_{2, K} \, dt \]

\[ \leq C \beta c h \| D(u_h) \|_{0, \infty, \Omega} \int_0^T \sum_{K \in T} h_{1/2}^{-k} \| c_h - \bar{c}_h \|_{\partial K} \| \sigma \|_{2, K} \, dt \]

\[ \leq C \beta c D_{\max} h \left( \int_0^T \| c - c_h \|^2 \, dt \right)^{1/2} \left( \| \sigma \|^2 \, dt \right)^{1/2} \]

\[ \leq C h^{2k} + \delta \| c - c_h \|_{L^2(0,T;L^2(\Omega))}^2. \]

Therefore, combining all bounds and picking \( \delta > 0 \) small enough the result follows. \( \square \)

Theorem 6.1 shows optimal convergence for the concentration in the \( L^2 \)-norm for sufficiently smooth \( c \), \( c_0 \), and \( u \).

7 Numerical Experiments

In this section we demonstrate performance of the method. We will first verify theorem 6.1, i.e., that the concentration converges optimally in the \( L^2 \)-norm, after which we verify compatibility of the coupled discretization. As final test case we consider a more realistic test case of contaminant transport and compare our results to those obtained in literature.

All simulations have been implemented in the finite element library NGSolve [38]. We use unstructured simplicial meshes to discretize the domain \( \Omega \) and use Crank–Nicolson time stepping. Furthermore, the penalty parameters in eqs. (5) and (10) are set to \( \beta_f = 10k^2 \) and \( \beta_c = 6k^2 \).

7.1 Rates of convergence in the \( L^2 \)-norm

Let \( \Omega = [0,1]^2 \) with \( \Omega^d = [0,1] \times [0, 0.5] \) and \( \Omega^s = [0,1] \times [0.5,1] \). The source terms and boundary conditions for the Stokes–Darcy problem eqs. (1) and (2) are chosen such that the exact solution is given by

\[ u|_{\Omega^s} = \begin{bmatrix} -\sin(\pi x_1) \exp(x_2/2)/(2\pi^2) \\ \cos(\pi x_1) \exp(x_2/2)/\pi \end{bmatrix}, \quad p|_{\Omega^s} = \frac{\kappa \mu - 2}{\kappa \pi} \cos(\pi x_1) \exp(x_2/2), \]

\[ u|_{\Omega^D} = \begin{bmatrix} -2\sin(\pi x_1) \exp(x_2/2) \\ \cos(\pi x_1) \exp(x_2/2)/\pi \end{bmatrix}, \quad p|_{\Omega^D} = \frac{2}{\kappa \pi} \cos(\pi x_1) \exp(x_2/2), \]

(40)
\[ \ell = 1 \quad |c - c_h|_\Omega \quad \text{Rate} \quad \ell = 2 \quad |c - c_h|_\Omega \quad \text{Rate} \]

| Elements | \( |c - c_h|_\Omega \) | Rate | Elements | \( |c - c_h|_\Omega \) | Rate |
|----------|----------------|------|----------|----------------|------|
| 28       | 2.2e-1         | -    | 8        | 3.6e-1        | -    |
| 152      | 3.1e-2         | 2.8  | 28       | 4.7e-2        | 3.0  |
| 578      | 8.5e-3         | 1.9  | 152      | 3.2e-3        | 3.9  |
| 2416     | 2.0e-3         | 2.1  | 578      | 3.3e-4        | 3.3  |
| 9584     | 4.5e-4         | 2.1  | 2416     | 3.3e-5        | 3.3  |

Table 1: Errors and rates of convergence in \( \Omega \) for the solution \( c_h \) of the EDG discretization of the transport equation eq. (10) using an approximate velocity \( u_h \) computed by the EDG-HDG method for the Stokes–Darcy system eq. (5). The test case is described in section 7.1.

with \( \alpha = \mu \kappa^{1/2}(1 + 4\pi^2)/2 \) as considered also in [9, 13]. We take \( \mu = 1 \) and \( \kappa = 1 \).

We solve the Stokes–Darcy problem using the EDG-HDG discretization eq. (5). We then replace the exact velocity \( u \) in the transport problem eq. (3) by \( u_h \), i.e., the discrete velocity solution to eq. (5). Other parameters in eq. (3) are set as: \( \phi = 1 \) and

\[
D = \begin{bmatrix}
0.01 & 0.005 \\
0.005 & 0.02
\end{bmatrix}.
\] (41)

We set the source and boundary terms such that the exact solution to eq. (3) is given by \( c(x,t) = \sin(2\pi(x_1 - t)) \cos(2\pi(x_2 - t)) \).

We compute the rates of convergence of the contaminant \( c \) for \( \ell = k - 1 = 1 \) and \( \ell = k - 1 = 2 \) with the time step small enough so that spatial errors dominate over temporal errors. The error in the \( L^2 \)-norm and rates of convergence at final time \( t = 1 \) are presented in table 1. We obtain optimal rates of convergence for \( c_h \), verifying theorem 6.1.

7.2 Compatibility of the Stokes–Darcy and transport discretization

To verify compatibility of the Stokes–Darcy and transport discretization eqs. (5) and (10), we need to verify that eq. (10) is able to preserve the constant solution when \( f = -\chi^d f^d \tilde{c} \) where \( \tilde{c} \) is a constant.

For this test case we first solve the discrete Stokes–Darcy flow problem eq. (5). We use the same setup as in section 7.1 and compute the discrete velocity \( u_h \) and discrete pressure \( p_h \) solutions on a grid consisting of 578 elements and using \( k = 2 \).

We then solve the discrete transport problem eq. (10). We set \( u = u_h \), \( \phi = 1 \), \( \ell = k - 1 = 1 \) and use the diffusion tensor given by eq. (41) and time step \( \Delta t = 10^{-3} \). We choose the initial and boundary conditions such that the exact solution is given by \( c = \tilde{c} = 1 \). At final time \( t = 1 \) we compute \( |1 - c_h|_\Omega = 1.5 \cdot 10^{-13} \), verifying compatibility of the discretization eqs. (5) and (10).

To compare, we now consider a discretization of the Stokes–Darcy problem that is not compatible with the EDG discretization eq. (10) of the transport problem. The setup is the same as above except that we discretize the Stokes–Darcy problem by an embedded discontinuous Galerkin method on \( \Omega^s \) for the Stokes equations [33], and a standard L-HDG method on \( \Omega^d \) for the Darcy equations [12]. We remark that this discretization of the Stokes–Darcy problem is not exactly mass conserving, i.e., the discrete velocity solution \( u_h \) does not satisfy the properties described in eq. (7). As a result, this discretization of the Stokes–Darcy problem cannot be proven to be compatible with the EDG discretization eq. (10) of the transport problem. Indeed, computing the error in the concentration at final time \( t = 1 \) we find that \( |1 - c_h|_\Omega = 2.4 \cdot 10^{-4} \), i.e., this incompatible discretization is not able to preserve the constant solution.

In fig. 1 we compare the solution of the concentration at final time \( t \) found using the compatible EDG-HDG discretization with the solution found using the incompatible discretization. It is clear that the incompatible discretization does not preserve the constant \( c = 1 \).
Figure 1: Test case from section 7.2. The solution to the transport equation at $t = 1$. Left: the solution $c = 1$ is approximated up to machine precision using the compatible EDG-HDG discretization. Right: an incompatible flow-transport discretization cannot preserve the constant solution $c = 1$. Here $c_h \in [0.99927, 1.00019]$.

7.3 Contaminant transport

We finally consider a simulation of coupled surface/subsurface flow and contaminant transport. This test case is similar to that proposed in [40, Example 7.2].

For the Stokes–Darcy problem we consider the same setup as in [9, Section 6.2]; we consider the unit square domain $\Omega = (0, 1)^2$ which is divided into a Stokes region $\Omega^s = (0, 1) \times (0.5, 1)$ that represents a lake or a river, and a Darcy region $\Omega^d = \Omega \setminus \Omega^s$ representing an aquifer. The domain is divided into 14900 simplicial elements. The mesh is such that $T^s$ is an exact triangulation of $\Omega^s$, $T^d$ is an exact triangulation of $\Omega^d$, and element boundaries match on the interface $\Omega^s \cap \Omega^d$. We use a time step of $\Delta t = 10^{-3}$ and set $\ell = k - 1 = 2$.

Let the boundary of the Stokes region be partitioned as $\Gamma^s = \Gamma^s_1 \cup \Gamma^s_2 \cup \Gamma^s_3$ where $\Gamma^s_1 := \{ x \in \Gamma^s : x_1 = 0 \}$, $\Gamma^s_2 := \{ x \in \Gamma^s : x_1 = 1 \}$ and $\Gamma^s_3 := \{ x \in \Gamma^s : x_2 = 1 \}$. Similarly, let $\Gamma^d = \Gamma^d_1 \cup \Gamma^d_2$ where $\Gamma^d_1 := \{ x \in \Gamma^d : x_1 = 0 \text{ or } x_1 = 1 \}$ and $\Gamma^d_2 := \{ x \in \Gamma^d : x_2 = 0 \}$. We impose the following boundary conditions:

$$u = (x_2(3/2 - x_2)/5, 0)$$  \text{ on } \Gamma^s_1,$$

$$(-2\mu \varepsilon(u) + p I) n = 0$$  \text{ on } \Gamma^s_2,$$

$$u \cdot n = 0$$  \text{ on } \Gamma^s_3,$$

$$(-2\mu \varepsilon(u) + p I)^t = 0$$  \text{ on } \Gamma^d_1,$$

$$u \cdot n = 0$$  \text{ on } \Gamma^d_2,$$

$$p = -0.05$$  \text{ on } \Gamma^d_2.$$

We set the permeability to

$$\kappa(x) = 700(1 + 0.5(\sin(10\pi x_1) \cos(20\pi x_2^2) + \cos^2(6.4\pi x_1) \sin(9.2\pi x_2))) + 100.$$

Other parameters in eqs. (1) and (2) are set as $\mu = 0.1$, $\alpha = 0.5$, $f^s = 0$, and $f^d = 0.$
For the transport equation eq. (3) the diffusion tensor is set to

\[
D(u_h) = \begin{cases} 
\delta I, & \text{in } \Omega^s, \\
\phi d_m \mathbb{I} + d_l |u_h| T + d_t |u_h|(I - T), & \text{in } \Omega^d,
\end{cases}
\]

where \(u_h\) is the velocity solution to eq. (5), \(\delta > 0, d_l, d_t \geq 0\) are longitudinal and transverse dispersivities and \(d_m > 0\) is the molecular diffusivity, \(\mathbb{I}\) denotes the identity matrix, \(T = u_h u_h^T / |u_h|^2\) and \(u_h^T\) is the transpose of the vector \(u_h\). This diffusion tensor satisfies eqs. (4a), (4b) and (22) (assuming \(d_l \geq d_t\), which is usually the case) and eq. (4c) [18, 39]. In our numerical example, we choose \(\delta = 10^{-6}\), \(\phi = 1\) on \(\Omega^s\) and \(\phi = 0.4\) on \(\Omega^d\) and \(d_m = d_l = d_t = 10^{-5}\). The initial condition for the plume of contaminant is given by

\[
c_0(x) = \begin{cases} 
0.95 & \text{if } \sqrt{(x_1 - 0.2)^2 + (x_2 - 0.7)^2} < 0.1, \\
0.05 & \text{otherwise.}
\end{cases}
\]

In fig. 2 we show the permeability and the computed velocity field (which are identical to [9, Figure 2]). In fig. 3 we show the plume of contaminant spreading through the surface water region and penetrating into the porous medium. As observed also in [40, Example 7.2], the contaminant plume stays compact while in the surface water region but spreads out in the groundwater region due to the heterogeneity of the porous media.

8 Conclusions

We have analyzed a compatible embedded-hybridized discontinuous Galerkin discretization for the one-way coupling between Stokes–Darcy flow and transport and proved existence and uniqueness and optimal convergence rates for the discrete transport problem. These results complement our previous work in which we proved optimal and pressure-robust error estimates for the EDG-HDG discretization of the Stokes–Darcy system. We verified our theory by numerical examples. We furthermore demonstrated that an incompatible discretization of the coupled Stokes–Darcy and transport problem can result in small oscillations in the solution to the transport equation. This shows the importance of compatible discretizations for coupled flow and transport problems.
Figure 3: The plume of contaminant spreading through the surface water region and penetrating into the porous medium with snapshots at different instances in time. The test case is described in section 7.3.
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