Quantum differential systems and some applications to mirror symmetry

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May 5, 2014

Abstract

We consider mirror symmetry (A-side vs B-side) in the framework of quantum differential systems. We focus on the logarithmic non-resonant case, which describes the geometric situation and for which quantum differential systems are produced on the B-side by avatars of rescalings of regular tame functions. We show that quantum differentials systems provide a good framework in order to generalize the construction of the rational structure given by Katzarkov, Kontsevich and Pantev for the complex projective space. As an application, we give a description of the rational structure obtained in this way on the orbifold cohomology of weighted projective spaces and their Landau-Ginzburg models. The formula on the B-side is at first sight more complicated that the one on the A-side, depending on numerous combinatorial data which are rearranged after the mirror transformation. In order to complete the panorama, we also calculate a mirror partner of the Hirzebruch surface $\mathbb{F}_2$ in the setting of quantum differential systems.

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*Partially supported by ANR grant ANR-08-BLAN-0317-01
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1 Introduction

This paper deals with quantum differential systems, namely trivial bundles equipped with a flat meromorphic connection with prescribed poles together with a flat nondegenerate bilinear form. We will focus mainly on their relation with mirror symmetry.

Such systems already appear, more or less explicitly and under various names, in the work of a lot of people, essentially motivated by the construction of Frobenius manifolds, see f.i. [14], [15], [21], [27], [28], [40], [35], [11], [32]... They first arose in singularity theory (B-side, local version) thirty years ago in the work of K. Saito about the primitive forms [40], and took the form that we will use in the work of B. Malgrange [26]. In connection with the construction of Frobenius manifolds (for which another important ingredient are the primitive forms), a global version of these objects has been discussed in [11] where it is explained how a regular tame function on an affine manifold gives quantum differential systems, naturally produced by solutions of the Birkhoff problem for its Brieskorn lattice (the construction is outlined in the Appendix). The tameness condition is required for finiteness reasons.
As it follows from Dubrovin’s formalism [13], [14], [15] (see also [6] and the references therein), quantum differential systems also appear in quantum cohomology theory (A-side), giving an extension of the Dubrovin connexion as an absolute flat connection and encoding the quantum product and its basic properties, taking into account a supplementary homogeneity condition.

It is thus natural to investigate mirror symmetry through quantum differential systems (the step before Frobenius manifolds): two models will be mirror partners if their associated quantum differential systems are isomorphic, as bundles with connections. Notice that in this setting, Givental’s quantum differential operators are interpreted as minimal polynomials of suitable (primitive) sections.

Some motivations are in order:

• First, and the aim of this paper is to emphasize this point, such systems can be computed on the B-side, without any references to correlators (and hence to the quantum product), which are rather complicated objects. In this way, mirror symmetry can be useful in order to understand more clearly (and sometimes predicts) what happens on the A-side; in turn the A-side produces B-models that are interesting on their own (see the discussion below about the B-models of hypersurfaces in projective spaces). A step in this direction can be found in [10], which gives a counterpart of the computations carried in [4] for the small quantum (orbifold) cohomology of weighted projective spaces. A connected class of examples is given by rescalings of regular tame functions which, despite its trivial appearance, give a quite good picture of the situation (see section 7.1). We also discuss the case of the Hirzebruch surface $\mathbb{F}_2$ in section 11 where Givental’s mirror map [18] appears naturally as a function in flat coordinates where flatness has to be understood with respect to a flat residual connection, naturally produced by the quantum differential systems involved. This flat connection is a central object for our purpose because, in mirror symmetry, flat coordinates are: on the A-side, coordinates are indeed flat. More generally, these techniques could be used for instance in order to study hypersurfaces or complete intersections in weighted projective spaces [19], [22], [42]. On the B-side, this leads to study quantum differential systems associated with non tame functions (and this problem remains open), see remark 2.2.2.

• Also, a quantum differential system is a very flexible object: for instance, and as emphasized in different papers [14], [20], [20], it can be universally unfolded in some cases and we can modify accordingly its base space, which can be the affine space, a torus (algebraic setting) or a punctual germ (analytic setting). In other words, a whole quantum differential system can be, in some cases, determined by a restricted set of data and this observation is very useful in order to simplify the computations on the B-side, see [8], [9].

• Quantum differential systems fit very well with “quantizations” (i.e. small quantum cohomology) and it is a good setting in order to study “large radius limits”, using the classical techniques in differential equation theory. Notice that these limits (these are of course quantum differential systems on a point) always produce meromorphic connections with regular singularities. This is explained in section 6 and in section 7.

• Last, it is a natural framework in order to generalize the construction of the rational structure on the A-side given in [21] for $\mathbb{P}^n$.

Nevertheless, it should be noticed that, on the B-side, a given tame regular function can produce several quantum differential systems which can be difficult to compare. While the situation
is clear on the $A$-side (the cohomology gives naturally a flat basis and flat coordinates), we have
to fix, on the $B$-side, some choices: the general principle is that a canonical quantum differential
system is built from the canonical solution of the Birkhoff problem given by M. Saito’s method, for
which a substantial tool is Hodge theory (see [9], [11] Appendix B], [12] and Appendix). Anyway,
the general place of this geometric solution in mirror symmetry has to be further explored.

Let us now discuss more precisely these motivations. An aspect of mirror symmetry is the
following: given a projective manifold $X$, one can computes its Gromov-Witten invariants, or
more generally its correlators, with the help of Picard-Fuchs equations associated with some mirror
partner. This is classically used to express these correlators in terms of combinatorial data: this
is $f.i$ what gives Givental’s “$I=J$” mirror theorem (see [6] for an overview). We explain how this
can be achieved using quantum differential systems, in particular what should be the correlators
of a quantum differential system. Another and connected goal is to define the $J$-function of a
general quantum differential system. Again, we have to fix some choices: we are led to define
canonical fundamental solutions of the Dubrovin connection of a quantum differential system and
this is done using Dubrovin’s conformal and symmetric solutions, see section 5. The situation
is particularly nice when the quantum differential systems are logarithmic and non-resonant, see
section 6: prototypes of such systems are given by small quantum cohomology, thanks to the divisor
axiom, see also [32], [33]. Their main properties are given by theorem 6.3.4 and corollary 6.3.6.
In this case, the canonical solutions have an explicit description: they are uniquely determined by
a matrix of holomorphic functions, satisfying an initial condition. This matrix can be computed
algebraically, using a recursion relation, see relation (29). We will call its coefficients the correlators
of the given quantum differential system: they provide the usual correlators in the case of small
quantum cohomology, see section 10. In order to compute the correlators of a projective variety, it
is thus enough to identify the canonical fundamental solutions of the mirror quantum differential
system and this is reduced, on the $B$-side, to computations of algebra. This is emphasized in section
11.

Quantum differential systems provide also a good framework in order to generalize the con-
struction of the rational structure on the cohomology of the complex projective space $\mathbb{P}^n$ and their
Landau-Ginzburg models given in [24]. The strategy in loc. cit. is the following: the rational struc-
ture is first constructed on the $B$-side on the flat sections of the Gauss-Manin connection associated
(after quantization) with a suitable regular function, the Landau-Ginzburg model. At the begin-
ning, this rational structure is provided by the Lefschetz thimbles and then transferred to the flat
sections using oscillating integrals. In order to get first a precise formula for this rational structure
on the $B$-side, one needs an explicit description of these flat sections and this is done using the
quantum differential system produced by the Landau-Ginzburg model. The rational structure
that we get on the $B$-side is then shifted, taking the classical limit, on the cohomology ($A$-side) using
a mirror theorem which identifies the standard cohomology basis with suitable explicit differential
forms. Our purpose is to extend this method: again, this is done using quantum differential sys-
tems, their classical limits and their conformal Dubrovin’s solutions. Conformality is used here in
order to get a precise description of the flat sections: this is discussed in section 8 (see proposition
8.1.11). For our geometric setting, it is enough to consider logarithmic quantum differential systems
(see section 6). It turns out that only flat (in the sense of definition 6.2.2) logarithmic quantum
differential systems are only relevant. As an application of our method, we give in section 9 a
description of the rational structure obtained in this way on the orbifold cohomology of weighted
projective spaces (see corollary 9.2.3) and their Landau-Ginzburg models (see theorem 9.2.2). More precisely, we first give a closed formula on the $B$-side, for which the mirror quantum differential system is identified in [10]. This formula involves various numbers (depending on the combinatorics), produced by the computation of some relevant oscillating integrals whose integral kernel depend on the choice of suitable bases of differential forms. This rational structure on the $B$-side is, after [38, Theorem 4.10], an ingredient of a (variation of a pure, rational) non-commutative Hodge structure in the sense of [24, Definition 2.7], related with the "$Q$-structure axiom" (the link between Hodge theory and Lefschetz thimbles is a quite old and long story: see for instance [1, Chapitre III, paragraphes 12 et 14] and the references therein and also, closer from our concern, [11, 12, section 6] and [36]). In order to reach the $A$-side, we then use the explicit description of the mirror partner of the standard orbifold cohomology basis given in [10, Theorem 5.1.1]. Notice that the construction of such structures on the $A$-side is also considered in [23] for toric orbifolds using a completely different approach (in particular we will not make use of equivariant perturbations and localization arguments in this paper): up to a ramification (due to the fact that we have to consider flat bases with respect to a residual connection), we find at the end on the $A$-side Iritani’s formula [23, Theorem 4.11] for weighted projective spaces. A striking fact is that a part of the constants in the formula for the rational structure that we get on the $B$-side miraculously disappear when we apply the mirror theorem and that we finally get a very simple formula for the rational structure on the $A$-side, see corollary 9.2.3.

Last, and in order to complete the panorama, we compute a mirror partner of the Hirzebruch surface $F_2$ (the classical non-Fano example) using quantum differential systems. This example is very interesting because it produces some new, but also in some sense intermediate (between the ones produced by projective space and the ones produced by weighted projective spaces; see for instance section 11.4 where the construction of a logarithmic Frobenius manifold is also discussed), phenomena. We show how our method allow to find some well-known results, see f.i [6, Section 11.2] and [16, Example 5.4]. In particular, it is readily seen that the change of variables considered there in order to get the “correct” quantum product (in other words, the mirror map) is naturally given by flat coordinates. We also verify that the quantum differential system associated with the mirror partner of the projective space $P(1,1,2)$ is obtained as a classical limit of the one associated with the mirror of $F_2$, as it has been first checked in [5].

Last, let us emphasize the fact that the point of view developed here is in essence not so far from Givental’s theory of mirror symmetry and “quantum differential equations” [17], [18] (roughly speaking, we consider matrices instead of their characteristic polynomials) but the techniques used here in order to get a mirror theorem are somewhat different: our main objective was to show how solutions of the Birkhoff problem for the Brieskorn lattice of a regular tame function as defined in [11] should be naturally exploited in order to understand better (small) quantum cohomology. We were motivated by the reading of [24], [41] and [23] about rational structures.

This paper is organized as follows: we define quantum differential systems in 2 and discuss their relationship with mirror symmetry. In sections 3, 4 and 5, we define the canonical fundamental solutions and the canonical $J$-functions of a quantum differentials system with the help of Dubrovin’s conformal and symmetric solutions [15]. The case of the non-resonant, logarithmic systems is handled in section 6 and we give some examples in section 7. We apply the results obtained there in order to describe a rational structure on the orbifold cohomology of weighted projective
spaces and to compute correlators in sections \[8\] and \[10\]. We explain in section \[11\] how to get an explicit mirror quantum differential system to the small quantum cohomology of the Hirzebruch surface \(F_2\). We briefly recall in the Appendix how to construct a quantum differential system from a regular tame function.

We thank E. Mann for numerous discussions about the subject and H. Iritani for his precisions about the formulas in \[23\].

2 Quantum differential systems

We introduce here our main object, the *quantum differential systems*\(^1\). The basic definitions and properties are for instance compiled in C. Sabbah’s book \[35\], using B. Malgrange’s setting \[26\], \[27\]. We first list some of them.

2.1 Definitions

Let \(M\) be a complex analytic manifold, equipped with coordinates \(x = (x_0, \cdots, x_r)\). We will denote by \(U_0\) (resp. \(U_\infty\)) the chart of \(\mathbb{P}^1\) centered at 0 (resp. \(\infty\)) and by \(\theta\) (resp. \(\tau := \theta^{-1}\)) the coordinate on \(U_0\) (resp. \(U_\infty\)).

**Definition 2.1.1** A quantum differential system on \(M\) is a tuple\(^2\)

\[Q = (M, G, \nabla, S, d)\]

where

- \(d\) is an integer,
- \(G\) is a trivial bundle on \(\mathbb{P}^1 \times M\),
- \(\nabla\) is a flat meromorphic connexion on \(G\), with poles of order less or equal to 2 along \(\{0\} \times M\), logarithmic along \(\{\infty\} \times M\),
- \(S\) is a \(\nabla\)-flat, non-degenerate bilinear form \(S : \mathcal{O}(G) \times j^* \mathcal{O}(G) \to \theta^d \mathcal{O}_{\mathbb{P}^1 \times M}\), where

\[j : \mathbb{P}^1 \times M \to \mathbb{P}^1 \times M\]

is defined by \(j(\theta, x) = (-\theta, x)\).

In what follows, we will denote by

- \(\mu\) the rank of the bundle \(G\),
- \(G_0\) the restriction of \(G\) at \(U_0\): this is a free \(\mathcal{O}_M[\theta]\)-module of rank \(\mu\),

\(^1\)They are also called “tr.TLEP-structures” in the work of C. Hertling \[20\]

\(^2\)In “the fibers” of the projection \(\pi : G \simeq \pi^* \pi_* G\)
• \(G_\infty\) the restriction of \(G\) at \(U_\infty\): this is a free \(\mathcal{O}_M[\tau]\)-module of rank \(\mu\).

Let \(Q = (M, G, \nabla, S, d)\) be a quantum differential system, \(i_{\{\theta=0\}}\) (resp. \(i_{\{\theta=\infty\}}\)) be the zero (resp. infinity) section of \(M\) in \(\mathbb{P}^1 \times M\) and \(E := \pi_* G\) where \(\pi\) be the projection \(\pi: \mathbb{P}^1 \times M \to M\).

**Proposition 2.1.2**

1. We have the isomorphisms \(i_{\{\theta=\infty\}}^* G \simeq E \simeq i_{\{\theta=0\}}^* G\).
2. The connection \(\nabla\) takes the form
   \[
   \nabla = \nabla + \frac{\Phi}{\theta} + \frac{V_0}{\theta} + \frac{V_\infty}{\theta} \frac{d\theta}{\theta}
   \]
   where
   - \(\nabla\) is a connection on \(E\),
   - \(\Phi\) is a Higgs bundle, that is an \(\mathcal{O}_M\)-linear map \(\Phi: E \to E \otimes \Omega^1_M\), such that \(\Phi \wedge \Phi = 0\),
   - \(V_0\) and \(V_\infty\) are two \(\mathcal{O}_M\)-linear endorphisms of \(E\),

these objects satisfying
   \[
   \nabla^2 = 0, \ \Phi \wedge \Phi = 0, \ \nabla \Phi = 0, \ \nabla V_\infty = 0, \ [V_0, \Phi] = 0 \ \nabla (V_0) + \Phi = [\Phi, V_\infty]
   \]

In particular, the connection \(\nabla\) is flat.

**Proof.** Standard, see f.i. [35], but we outline it, due to its importance for what follows: the isomorphisms expected in (1) follow from the triviality of the bundle \(G\) (restriction of sections). The assumptions on the order of the poles show that, in a basis \(\omega = (\omega_0, \cdots, \omega_{\mu-1})\) of \(E\), the matrix of \(\nabla\) is

\[
\begin{pmatrix}
A_0(x) & D(x) \\
A_\infty(x) & C(x)
\end{pmatrix}
\]

where \(x = (x_0, \cdots, x_\mu) \in M\),

\[
C(x) = \sum_{i=1}^r C^{(i)}(x) dx_i \ \text{et} \ \ D(x) = \sum_{i=1}^r D^{(i)}(x) dx_i.
\]

The connection \(\nabla\) is first defined on \(i_{\{\theta=\infty\}}^* G\) as the restriction at \(\tau = 0\) of a flat connection. Its matrix in the basis \(\omega\) is \(C(x)\). The \(\mathcal{O}_M\)-linear homomorphism \(\Phi\) is first defined on \(i_{\{\theta=0\}}^* G\). Its matrix in the basis \(\omega\) is \(D(x)\). Relations (1) follow from the flatness of \(\nabla\). This shows (2). \(\square\)

Notice that the flat residual connection \(\nabla\) is not defined if we forget the “logarithmic” assumption on the poles at infinity.

**Definition 2.1.3** We will call equation (2) characteristic equation of the quantum differential system \(Q\).
Remark 2.1.4  (1) $S$ induces bilinear forms (also denoted by $S$)

\[ S : E \times j^* E \rightarrow \mathcal{O}_M \theta^d, \quad (3) \]

\[ S : G_0 \times j^* G_0 \rightarrow \mathcal{O}_M [\theta] \theta^d \quad (4) \]

and

\[ S : G_\infty \times j^* G_\infty \rightarrow \mathcal{O}_M [\tau] \tau^{-d}. \quad (5) \]

If $\eta$ and $\nu$ are two global sections of $G$, that is if $\eta, \nu \in E$, will write

\[ S(\eta, \nu) = g(\eta, \nu) \theta^d \in \mathcal{O}_M \theta^d \quad (6) \]

where $g$ is a non-degenerate $\mathcal{O}_M$-bilinear form on $E$ and we have, because $S$ is flat,

\[ V_\infty + V_\infty^* = dI, \quad V_0^* = V_0 \quad \text{and} \quad \Phi^* = \Phi \quad (7) \]

where $*$ denotes the adjoint with respect to $g$.

(2) If the basis $\omega$ of $E$ is $\nabla$-flat, that is if $C(x) \equiv 0$ in equation (2), we have $S(\omega_i, \omega_j) \in \mathbb{C} \theta^d$ for all $i$ and for all $j$, because $S$ is $\nabla$-flat. If moreover $S(\omega_i, \omega_j) = 0$ except for a unique index $i$, we will also put $\omega_i^* := \frac{1}{g(\omega_i, \omega_i)} \omega_i^*$ and we will call $\omega_i^*$ the dual of $\omega_i$. We will also say that $\omega$ is adapted to $S$.

Remark 2.1.5  (1) If $\mathcal{Q}$ is a quantum differential system on a point, the connection $\nabla$ takes the form

\[ \nabla = (V_0^\theta + V_\infty) \frac{d\theta}{\theta} \]

where $V_0$ and $V_\infty$ are endomorphisms of the finite dimensional $\mathbb{C}$-vector space $E$, and $g$ is a bilinear, symmetric and non-degenerate form on $E$ such that

\[ V_0^* = V_0, \quad V_\infty + V_\infty^* = dI \]

where, as above, $*$ denotes the adjoint with respect to $g$.

(2) We will also consider quantum differential systems on the affine space (resp. the torus...). In this algebraic setting, the objects $\nabla$, $V_0$, $V_\infty$, $\Phi$ and $g$ are modified accordingly (replace $\mathcal{O}_M$-linear by $\mathbb{C}[x]$-linear, resp. $\mathbb{C}[x, x^{-1}]$-linear...). In this situation, $E$ is a free $\mathbb{C}[x]$-module (resp. $\mathbb{C}[x, x^{-1}]$-module...), see [3], [9].

In some cases, we can refine the previous definitions and define the logarithmic quantum differential systems (see [32]):

**Definition 2.1.6** Let $D$ be a divisor in $M$. We will say that the quantum differential system $\mathcal{Q}$ has logarithmic poles along $D$ if moreover

\[ \Phi : E \rightarrow E \otimes \Omega^1_M(\log D) \]

and

\[ \nabla : E \rightarrow E \otimes \Omega^1_M(\log D) \]

where $\Omega^1_M(\log D)$ denotes the module of the differential forms with logarithmic poles along $D$.  

The following example shows that quantum cohomology produces naturally logarithmic quantum differential systems:

**Example 2.1.7 (A-side)**

One associates a quantum differential system to the (small) quantum cohomology of a smooth projective variety $X$ as follows (see for instance [6] and the references therein; we assume here that the quantum product $\circ$ is everywhere convergent): the trivial bundle is the one whose the fibers are $H^*(X, \mathbb{C})$, that is

$$\pi : \mathbb{P}^1 \times M \times H^*(X, \mathbb{C}) \to \mathbb{P}^1 \times M$$

where $M = H^*(X, \mathbb{C})$. Let $\{\phi_k\}_{k=0}^{\mu-1}$ be a homogeneous basis of $H^*(X)$ and $\{t_k\}_{k=0}^{\mu-1}$ be a dual coordinate system on $H^*(X, \mathbb{C})$. The connection $\nabla^A$ is defined by

$$\nabla^A_{\partial t_k} = \partial t_k + \frac{1}{\theta} \phi_k \circ \xi \quad \text{and} \quad \nabla^A_{\partial \theta} = \theta \partial \theta + \frac{1}{\theta} E \circ \xi + \mu$$

where $\circ$ denotes the quantum product, parametrized by $\xi \in M$, $E$ is the “Euler vector field”

$$E := c_1(TX) + \sum_{k=0}^{\mu-1} (1 - \frac{1}{2} \deg \phi_k) t_k \phi_k$$

and

$$\mu(\phi_k) := (\frac{1}{2} \deg \phi_k - n) \phi_k.$$  

Notice that, by definition, the sections $\phi_k$ are $\nabla$-flat. Flatness of $\nabla^A$ follows from the associativity and the commutativity of the quantum product. The metric $S$ is built with the help of the form $(a, b) = \int_X a \cup b$, where $a$ and $b$ cohomology classes. In the case of small quantum cohomology, we have

$$\nabla^A_{q_k, \partial q_k} = q_k \partial q_k + \frac{1}{\theta} \phi_k \circ \xi$$

where $q_k = e^{t_k}$ for $k = 1, \cdots, r$ and $r = \dim_{\mathbb{C}} H^2(X, \mathbb{C})$ and $t_k \in H^2(X, \mathbb{C})/2\pi H^2(X, \mathbb{Z})$, thanks to the divisor axiom. Analogous construction for orbifolds (see f.i [22], [11]).

### 2.2 Quantum differential systems associated with regular functions

One can attach a quantum differential system to any regular tame function, see [11], [12], [8], [9]: an overview of the construction is given in the Appendix. Let us emphasize the following facts:

1. a solution of the Birkhoff problem for the Brieskorn lattice of a regular tame function (see step 2 in the Appendix) produces a quantum differential system.

2. Two different solutions of the Birkhoff problem give a priori two different bundles (which can be difficult to compare) and, even if the maps $\Phi$ (resp. $V_0$) associated with two different quantum differential systems are conjugated, the endomorphisms $V_{\infty}$ (resp. the connection $\nabla$) will not in general. We thus have to pay attention to this crucial problem on the B-side: what is the/a “good” choice?  

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3Replace $M$ by $U$ in what follows if the quantum product converges only on $U \subset M$
3. It turns out that the solutions of the Birkhoff problem are in one-to-one correspondance with the opposite filtrations, stable under the action of the monodromy, to the Hodge filtration defined on the nearby cycles, see [11, Appendix B]. In [11], canonical solutions of the Birkhoff problem (hence canonical quantum differential systems) are defined (we require in addition that $V_\infty$ is semi-simple, its eigenvalues running through the spectrum at infinity of the function, as defined in [37]): the opposite filtrations alluded to are constructed with the help of Deligne’s $I^{pq}$, following an idea of M. Saito [39].

All these phenomena are also discussed in [12], [8] and [9].

Example 2.2.1 (B-side, after [10] and [12])

Let $(w_1, \ldots, w_n)$ be strictly positive integers, $U = (\mathbb{C}^*)^n$ and $M^B = \mathbb{C}^*$. We define $F : U \times M^B \to \mathbb{C}$ by

$$F(u_1, \ldots, u_n, x) = u_1 + \cdots + u_n + \frac{x}{u_1^{w_1} \cdots u_n^{w_n}}.$$ 

A distinguished solution $\omega = (\omega_0, \ldots, \omega_{\mu-1})$ of the Birkhoff problem for the Brieskorn lattice $G_0$ of $F$ (see Appendix), which is a free $\mathbb{C}[x, x^{-1}, \theta]$-module of rank $\mu := 1 + w_1 + \cdots + w_n$, is described in [10, Section 4] (see also section 9.1 below). It gives an extension of this lattice as a trivial bundle $G^B$ on $\mathbb{P}^1 \times M^B$, equipped with a connection $\nabla^B$ with the required poles: the matrix of the Gauss-Manin connection in the basis $\omega$ takes the form

$$\left(\frac{A_0(x)}{\theta} + A_\infty\right)\frac{d\theta}{\theta} + (R - \frac{A_0(x)}{\mu\theta})\frac{dx}{x}$$

(8)

where

$$A_0(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{x}{w^w} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

$$R = \text{diag}(c_0, \ldots, c_{\mu-1}), \ c_i \in [0, 1]$$ (see formula (63) in section 9.1 for a definition of the $c_i$’s) and $A_\infty = \text{diag}(\alpha_0, \ldots, \alpha_{\mu-1})$. The rational numbers $\alpha_i$ are defined by $\alpha_i = i - \mu c_i$ and run through the spectrum at infinity, as defined in [37] (see also [12]), of the function $f := F(\bullet, 1)$.

Define

$$S^B(\omega_k, \omega_\ell) = \begin{cases} \frac{1}{x^{w_1 \cdots w_n}} \frac{\theta^n}{x^{w_1 \cdots w_n}} & \text{if } 0 \leq k \leq n \text{ and } k + \ell = n, \\ \frac{x^{w_1 \cdots w_n}}{x^{w_1 \cdots w_n}} \theta^n & \text{if } n + 1 \leq k \leq \mu - 1 \text{ and } k + \ell = \mu + n, \\ 0 & \text{otherwise} \end{cases}$$

These formulas are extended to $G^B$ and give a bilinear form satisfying the properties of definition 2.1.1 with $d = n$.

Summarizing ([10, Theorem 4.4.1]) the tuple

$$Q^B = (M^B, G^B, \nabla^B, S, n)$$

We put $w^w = w_1^{w_1} \cdots w_n^{w_n}$.
is a quantum differential system on $M^B = \mathbb{C}^*$. The restriction of this quantum differential system at $x = 1$ is a quantum differential system on a point which is precisely, after [12], the canonical quantum differential system associated with the function $F(\bullet, 1)$ by the construction above.

**Remark 2.2.2** Keeping mirror symmetry in mind (see section 2.4 below), some generalizations of the previous example are expected. For instance:

1. the ones considered in [22] (see also [33]), where the function $F$ is the function
\[ u_0 + \cdots + u_n \] defined on
\[ U = \{(u_0, \ldots, u_n) \in \mathbb{C}^{n+1} | u_0^{w_0} \cdots u_n^{w_n} = x_1, \ldots, u_0^{w_0} \cdots u_n^{w_n} = x_r \} \]
where $(x_1, \ldots, x_r) \in (\mathbb{C}^*)^r$ and $(w_0^i, \ldots, w_n^i) \in \mathbb{Z}^n$ for $1 \leq i \leq r$;

2. the Hori-Vafa models (see [22], [19], [42]...) where the function $F$ is the function
\[ u_0 + \cdots + u_n \] defined on
\[ U = \{(u_0, \ldots, u_n) \in \mathbb{C}^{n+1} | u_0^{w_0} \cdots u_n^{w_n} = x, \sum_{j \in I_i} u_j = 1 \}. \]
Here $x \in \mathbb{C}^*$, $(w_0, \ldots, w_n) \in (\mathbb{N}^*)^n$ and $I_i, i = 1, \ldots, k$, are $k$ non-intersecting subsets of \{0, \ldots, n\}. The coordinate $u_j$ is assumed to have the degree $w_j$ and $\sum_{j \in I_i} u_j$ the degree $l_i := \sum_{j \in I_i} w_j$. This should be the mirror partners of (smooth) complete intersections in weighted projective spaces. These functions are Laurent polynomials (this is shown in [42]) but these ones, and this is an interesting phenomenon at least on the B-side, are not necessarily tame. For instance, the mirror partner of the hypersurface of degree 2 in $\mathbb{P}^3$ is given by
\[ f(x, y) = x + 1 + \frac{(y + 1)^2}{xy} \]
which has two critical values $t = -3$ and $t = 5$, but it is readily seen that the value $t = 1$ is also an atypical value, probably produced by some vanishing cycles “at infinity” (to be further explored).

*How to construct a quantum differential system from this function?*

Another slightly different source of examples is given by the rescalings of a tame regular function: we will come back to this in detail in section 7.3. In order to complete the panorama, see also section 11 for the mirror of the Hirzebruch surface $F_2$.

### 2.3 Reconstruction theorem and (pre-)primitive forms

An important point in the theory of quantum differential systems is that one can define unfoldings and even *universal* unfoldings of such objects, see [20, Definition 2.3]. In some cases, a finite set of initial data allows to construct a universal unfolding of a given quantum differential system (see [20, Theorem 2.5] and the references to B. Malgrange and B. Dubrovin therein): this is the starting point in [8] and [9] in order to construct *canonical* quantum differential systems and canonical Frobenius manifolds associated with Laurent polynomials.
Definition 2.3.1 Let $Q = (M, G, \nabla, S, d)$ be a quantum differential system on $M$, $\omega$ be a global section of $G$. The period map associated with $\omega$ is the map

$$\varphi_\omega : \Theta_M \to i_*^{(\partial=0)}G$$

defined by $\varphi_\omega(\xi) = -\Phi_\xi(\omega)$ where $\Theta_M$ denotes the sheaf of vector fields on $M$.

Let $M = (C^{r+1}, 0)$ and $m$ be the maximal ideal of $O_M$. The index $o$ will denote the operation "modulo $m". Hertling and Manin's theorem is the following:

Theorem 2.3.2 (Theorem 2.5 in [20]) Let $Q = (M, G, \nabla, S, d)$ be a quantum differential system on $M$. Assume that there exists a $\nabla$-flat section $\omega \in i_*^{(\partial=0)}G$ such that

1. (GC) $\omega^o$ and its images under iteration of the maps $R^0_\theta$ and $\Phi_\xi^o$, for all $\xi \in \Theta_M^o$, generate $i_*^{(\partial=0)}G^o$,

2. (IC) the period map $\varphi_\omega : \Theta_M \to i_*^{(\partial=0)}G^o$ defined is injective.

Then the quantum differential system $Q$ has a universal deformation.

Definition 2.3.3

1. A section $\omega$ satisfying the conditions (GC) and (IC) is called pre-primitive. If moreover the period map $\varphi_\omega$ is an isomorphism we will say that $\omega$ is primitive.

2. We will say that a pre-primitive section $\omega$ is canonical if it generates the eigenspace associated with the smallest eigenvalue of $V_\infty$.

Remark 2.3.4 (1) Condition (IC) is empty if $M = \{\text{point}\}$. Assume moreover that $R_0$ is regular, i.e its characteristic polynomial is equal to its minimal polynomial: there exists $\omega$ such that

$$\omega, R_0(\omega), \ldots, (R_0)^{\mu-1}(\omega)$$

is a basis of $i_*^{(\partial=0)}G$ over $C$ and $\omega$ is thus pre-primitive.

(2) Condition (GC) is satisfied if $\omega$ and its derivative $\theta \nabla_{X_1} \theta \nabla_{X_2} \ldots \theta \nabla_{X_\ell} \omega$ generate $i_*^{(\partial=0)}G$.

Theorem 2.3.2 is originally stated for a punctual germ $M$. More convenient (for our purpose) global versions of this result can be found in [9]: for instance, if $M = \mathbb{A}^{r+1}$, the period map attached to $\omega$ is now a $\mathbb{C}[\bar{x}]$-linear map, defined on the Weyl algebra $\mathbb{A}^r(\mathbb{C}) = \mathbb{C}[\bar{x}] < \partial_{\bar{x}}, $, by $\varphi_\omega(\xi) = -\Phi_\xi(\omega)$ (analogous construction if $M = (\mathbb{C}^*)^r$ is a torus).

Example 2.3.5 In the situation of example 2.2.7, the period map associated with the section $\omega_0$ is

$$\varphi_{\omega_0} : \mathbb{C}[x, x^{-1}] < \partial_x \to G_0/\theta G_0$$

where $\varphi_{\omega_0}(x \partial_x) = \omega_1$. The section $\omega_0$ is pre-primitive and canonical in the sense of definition 2.3.3. We also define a connection $\nabla^{\omega_0}$ on $\mathbb{C}[x, x^{-1}] < \partial_x$ by

$$\nabla^{\omega_0}_{x \partial_x}(x \partial_x) = \nabla_{x \partial_x}(\varphi_{\omega_0}(x \partial_x)) = \nabla_{x \partial_x}(\omega_1)$$

The vector field $x \partial_x$ is $\varphi_{\omega_0}$-flat because $\nabla_{x \partial_x}(\omega_1) = c_1 \omega_1$ and $c_1 = 0$ for $n \geq 1$ (see section 9.1 below): the coordinate $t$ defined by $x = e^t$ is thus flat. See also section 11.3.3 for another concrete computation of flat coordinates.
2.4 A motivation: mirror symmetry via quantum differential systems

2.4.1 Mirror symmetry and quantum differential systems

Another (connected) important point is that one can compare two quantum differential systems:

Definition 2.4.1 The quantum differential systems

\[(M^A, H^A, \nabla^A, S^A, n^A) \text{ and } (M^B, H^B, \nabla^B, S^B, n^B)\]

are isomorphic if there exists an isomorphism \((id, \nu) : \mathbb{P}^1 \times M^A \to \mathbb{P}^1 \times M^B\) and an isomorphism of vector bundles \(\gamma : H^A \to (id, \nu)^*H^B\) compatible with the connections and the metrics.

Definition 2.4.2 Two models are mirror partners if their quantum differential systems are isomorphic.

Notice that the map \(\nu\) measures “flatness”: if \((M^A, H^A, \nabla^A, S^A, n^A)\) is the quantum differential system associated with the small quantum cohomology by example 2.1.7, it is defined by flat coordinates on \(M^B\), because the ones used on \(M^A\) are so: see f.i theorem 2.4.2 and section 11.5 below.

2.4.2 Example: mirror symmetry for weighted projective spaces

A nice class of examples, which bring to light some unexpected phenomena (this is discussed with some details in [10]), is given by weighted projective spaces for which we have the following result: let \(p = c_1(\mathcal{O}(1)) \in H^2_{\text{orb}}(\mathbb{P}(w), \mathbb{C})\) and

\[(p^{\circ \text{tp}})^i = p \circ_{\text{tp}} \cdots \circ_{\text{tp}} p,\]

where the quantum product \(\circ\) is counted \(i\) times. We keep the notations of example 2.2.1.

Theorem 2.4.3 (Theorem 5.1.1 in [10]) The quantum differential system associated with the weighted projective space \(\mathbb{P}(1, w_1, \ldots, w_n)\) by example 2.1.7 is isomorphic to the one associated with the function \(F\) by example 2.2.1. We have \(M^A = M^B = \mathbb{C}^*\) and the isomorphism \(\gamma\) (resp. \(\nu\)) sends \((p^{\circ \text{tp}})^i\) onto \(\omega_i\) (resp. is the identity).

Notice that the coordinate \(x\) is flat (see example 2.3.5 above) and this explains why the map \(\nu\) is equal to the identity. Nevertheless, \((p^{\circ \text{tp}})^i\) is not a flat section of the residual connection \(\nabla^A\).

Remark 2.4.4 (1) As a consequence of the theorem, the matrix of the small quantum multiplication \(p_{\circ \text{tp}}\) in the basis \((p^{\circ \text{tp}})^i\) is equal to \(\lambda_0^F(e^\tau)\), which is the matrix of multiplication \(\omega_0\) on \(G^F_0/\theta G^F_0\) in the basis induced by \(\omega\). One gets in this way a correspondance between the products which allows to see the quantum product as a simple computation in algebra.

(2) The rational number \(\alpha_i\) is equal to half of the orbifold degree of \((p^{\circ \text{tp}})^i\): the latter are thus in correspondance with the “spectrum at infinity” of the fonction \(F(\bullet, 1)\).

---

5The case \(w_1 = \cdots = w_n = 1\) has been first considered in [2].
6\(G^F_0/\theta G^F_0\) is naturally equipped with a structure of ring, via \(\omega_0\), a (pre-)primitive section.
More generally, the function in remark 2.2.2 (1) should give the mirror (in the sense of definition 2.4.2) of toric orbifolds (see [24, 33]) and the one in remark 2.2.2 (2) should give, after [22, 19, 32], the mirror of complete intersections in the weighted projective space \( \mathbb{P}(w_0, \cdots, w_n) \). The computation of our mirror partner of the Hirzebruch surface \( \mathbb{F}_2 \) is done in section 11 (see theorem 11.5.2).

3 The Dubrovin connection and the quantum product of a quantum differential system

Let \( Q = (M, \mathcal{G}, \nabla, S, n) \) be a quantum differential system, \( \omega = (\omega_0, \cdots, \omega_{\mu-1}) \) be an \( \mathcal{O}_M \)-basis of \( E = \pi_* \mathcal{G} \). As above, we will denote by \( \underline{x} = (x_0, \cdots, x_r) \) the coordinates on \( M \).

3.1 Dubrovin connection of a quantum differential system

An important object for our purpose is the Dubrovin connection of the quantum differential system \( Q \), which encodes the “quantum product”, as defined in section 3.2 below. We rewrite here what is known in a slightly different settings, see [14] and [6, section 8.4].

Definition 3.1.1 The connection \( \nabla^d \) defined by

\[
\nabla^d := \nabla + \frac{\Phi}{\theta},
\]

is called the Dubrovin connexion of the quantum differential system \( Q \).

Proposition 3.1.2 The connection \( \nabla^d \) is flat.

Proof. Flatness of \( \nabla^d \) is equivalent to \( \nabla^2 = 0, \Phi \wedge \Phi = 0 \) and \( \nabla \Phi = 0 \). This is precisely what gives proposition 2.1.2. \( \square \)

Let us emphasize once again that the flatness of \( \nabla \), and hence the flatness of \( \nabla^d \), is a characteristic property of quantum differential systems which is lost if we drop the assumption on the poles at infinity of the connection \( \nabla \).

Remark 3.1.3 A quantum differential system on \( M \) produces naturally a variation of semi-infinite Hodge structures on \( M \), in the sense of Barannikov [3]. Recall that \( G_0 \) denotes the restriction of \( \mathcal{G} \) at \( U_0 \): it is a free \( \mathcal{O}_M[\theta] \)-module. We define an increasing filtration \( F_* \) of \( G := G_0[\theta^{-1}] \) by \( \mathcal{O}_M[\theta] \)-submodules, putting \( F_p G := \theta^{-p} G_0 \): we thus have \( \nabla^d(F_p) \subset F_{p+1} \) (Griffith’s transversality condition). Let us now define the “\( \theta \)-connection”

\[
\nabla^\theta : G_0 \to \Omega^1_M \otimes G_0
\]

by \( \nabla^\theta := \theta \nabla^d \). By definition it satisfies

\[
\nabla^\theta_X (f(\underline{x}, \theta) \eta) = (\theta X f(\underline{x}, \theta)) \eta + f(\underline{x}, \theta) \nabla_X^\theta \eta,
\]

The algebraic version is straightforward.
\[ [\nabla_X^\theta, \nabla_Y^\theta] = \theta \nabla_{[X,Y]}^\theta \]

and

\[ \theta X S(\eta, \nu) = S(\nabla_X^\theta \eta, \nu) - S(\eta, \nabla_X^\theta \nu) \]

where \( \eta \) and \( \nu \) (resp. \( f(x, \theta) \)) are sections of \( G_0 \) (resp. \( \mathcal{O}_M[\theta] \)) and \( X \) and \( Y \) are vector fields on \( M \). Summarizing, the tuple \( VSHS_Q := (M, G_0, \nabla_M^\theta, S) \) is a variation of semi-infinite Hodge structures on \( M \).

### 3.2 Quantum product of a quantum differential system

In what follows, we will assume that \( \Phi_{\partial_0} \) is the identity.\(^8\)

**Definition 3.2.1** We define an \( \mathcal{O}_M \)-bilinear map \(*\) on \( E \simeq G_0/\theta G_0 \) by

\[ \omega_i * \omega_j := [\Phi_{\partial_{x_i}}(\omega_j)] \]

for all \( 0 \leq i \leq r \) and \( 0 \leq j \leq \mu - 1 \), where \([\ ]\) denotes the class in \( E \).

This map defines a product but it doesn’t need to have any associativity and/or commutativity property and/or an identity. This is however sometimes the case:

**Proposition 3.2.2** Assume that the section \( \omega_0 \) is such that \( \omega_i = \Phi_{\partial_{x_i}}(\omega_0) \) for all \( 0 \leq i \leq r \). Then

1. \( \omega_i * \omega_0 = \omega_i \) for all \( 0 \leq i \leq r \),
2. \( \omega_i * \omega_j = \omega_j * \omega_i \) for all \( 0 \leq i, j \leq r \),
3. \( (\omega_i * \omega_j) * \omega_k = \omega_i * (\omega_j * \omega_k) \) for all \( 0 \leq i, j, k \leq r \), under the assumption that \( \omega_i * \omega_j \in \sum_{k=0}^{r} \mathcal{O}_M \omega_k \),
4. \( g(\omega_i * \omega_j, \omega_k) = g(\omega_j, \omega_i * \omega_k) \) for all \( 0 \leq i \leq r \) and \( 0 \leq j, k \leq \mu - 1 \)

where \( g \) is the bilinear form defined by formula \((7)\).

**Proof.** The first point is clear, thanks to the assumption on \( \omega_0 \). The second one follows from the formula \( \Phi \wedge \Phi = 0 \) (see proposition \( 2.1.2 \)) from which we get \( \Phi_{\partial_{x_i}} \Phi_{\partial_{x_j}} = \Phi_{\partial_{x_j}} \Phi_{\partial_{x_i}} \). For the third one, one writes

\[ \omega_i * (\omega_j * \omega_k) = \Phi_{\partial_{x_i}}(\omega_j * \omega_k) = \Phi_{\partial_{x_i}}(\Phi_{\partial_{x_j}}(\Phi_{\partial_{x_k}}(\omega_0))) \]

and, using the moreover second assertion,

\[ (\omega_i * \omega_j) * \omega_k = \omega_k * (\omega_i * \omega_j) = \Phi_{\partial_{x_k}}(\Phi_{\partial_{x_i}}(\Phi_{\partial_{x_j}}(\omega_0))) \]

We get the desired formula because \( \Phi_{\partial_{x_i}} \) and \( \Phi_{\partial_{x_j}} \) commute. Last, we have

\[ g(\omega_i * \omega_j, \omega_k) = g(\Phi_{\partial_{x_i}}(\omega_j), \omega_k) = g(\omega_j, \Phi_{\partial_{x_i}}(\omega_k)) = g(\omega_j, \omega_i * \omega_k), \]

\(^8\)On the \( A \)-side, that is in the setting of example \( 2.1.7 \), the coordinate \( x_0 \) is associated with the first cohomology group \( H^0 \) while on the \( B \)-side \( x_0 \) is the constant term in the unfoldings of functions.

\(^9\)This happens for instance if \( r = \mu - 1 \). If not, the left hand is not well defined.
where the second equality follows from $\Phi^* = \Phi$ (see remark 2.1.4).

Notice that a section $\omega_0$ as in proposition 3.2.2 defines an injective period map, see definition 2.3.3. On the $B$-side it happens that there exists such sections, for instance if the quantum differential system is associated with a subdiagram deformation of a convenient and non-degenerate Laurent polynomial, see [9]. See also example 2.3.5 and section 11.

**Definition 3.2.3** In the situation of proposition 3.2.2, we will say that the map $\ast$ is the quantum product and that $E$ is the quantum algebra of the quantum differential system $Q$.

**Remark 3.2.4** (1) The variation of semi-infinite Hodge structure $V_{SHS_Q}$, and hence the quantum differential system $Q$, gives a “quantization” on the $\theta$-axis of the quantum algebra $E$ (on the $B$-side, $E$ is a Jacobian ring): $G_0$ is an $\mathcal{O}_M[\theta]$-free module and $E \simeq i^*_\theta G_0$.

(2) Assume that $\omega_0$ is as in proposition 3.2.2. Assume moreover that we have an isomorphism of $\mathcal{O}_M$-modules $\delta : E \to H$ and let $\eta_i = \delta(\omega_i)$. This isomorphism defines a product $\circ$ on $H$ by

$$\eta_k \circ \eta_\ell := \delta (\delta^{-1}(\eta_k) * \delta^{-1}(\eta_\ell)),$$

(12)
a connection $\nabla^\delta$ on $H$ by

$$\nabla^\delta (\eta_j) = \delta \nabla (\delta^{-1}(\eta_j))$$

(13)
and a “metric” $g^\delta$ on $H$ by

$$g^\delta(\eta_i, \eta_j) = g(\delta^{-1}(\eta_i), \delta^{-1}(\eta_j)),$$

(14)
these objects being extended on $H$ by linearity. By definition, the product $\circ$ inherits all the properties of $*$ and the connection $\nabla^\delta$ defined on $H$ by

$$\nabla_{\partial x_i}^\delta \left( \sum_k a_k(x) \eta_k \right) = \nabla_{\partial x_i}^\delta \left( \sum_k a_k(x) \eta_k \right) + \tau \sum_k a_k(x) \eta_i \circ \eta_k$$

is flat. If $H = \Theta_M$ and if the period map $\varphi_{\omega_0}$ (see theorem 2.3.2) is an isomorphism, we get in this way a Frobenius manifold.

### 4 Fundamental solutions of a quantum differential system

Let $Q = (M, G, \nabla, S, n)$ be a quantum differential system $\dim_{\mathbb{C}} M = r + 1$. We have

$$\nabla = \nabla + \Phi \frac{\partial}{\partial \theta} + (V_0 \frac{\partial}{\partial \theta} + V_\infty) \frac{d\theta}{\theta}$$

by proposition 2.1.2. Until the end of this paper, we will assume that $V_\infty$ is semi-simple as it will be the case in our favorite situations ($A$-side and $B$-side). In what follows, $\omega = (\omega_0, \ldots, \omega_{r-1})$ will denote a basis of $\pi_\ast G$ over $\mathcal{O}_M$, fixed once for all. In the basis $1 \otimes \omega$ de $G$, the matrix of $\nabla$ is thus

$$\begin{pmatrix} A_0(x) \frac{\partial}{\partial \theta} + A_\infty(x) \frac{d\theta}{\theta} + C(x) + \frac{D(x)}{\theta} \end{pmatrix}$$

(15)
where $x = (x_0, \ldots, x_r) \in M$. Recall that $\tau := \theta^{-1}$. 

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4.1 The fundamental solutions of the Dubrovin connection

Recall the flat Dubrovin connection $\nabla^d$ of definition 3.1.1. It has flat sections. Let us precise this point. Again, the results of this section are in essence classical, see f.i [15], [6], and we detail them in our situation mainly to establish notations and to set the different objects that we will use later.

**Lemma 4.1.1** There exists a (non necessarily unique) formal power series (in $\tau$)

$$Q(x, \tau) = I + \sum_{i \geq 1} Q_i(x) \tau^i$$

where $I \in \mathcal{M}_{\mu \times \mu}(\mathbb{C})$ is the identity matrix and $Q_i(x) \in \mathcal{M}_{\mu \times \mu}(\mathcal{O}(M))$ such that

$$\nabla^d(\omega Q) = (\nabla \omega)Q \quad (16)$$

**Proof.** We have to show that there exists a matrix $Q$ such that

$$d_M Q(x, \tau) = -\tau D(x) Q(x, \tau)$$

where $d_M$ denotes the differential on $M$. The independant term of $\tau$ in this equality shows that $d_M Q_0(x) = 0$ thus $Q_0(x)$ is constant: we choose it equal to the identity $I$. The term of degree $r \geq 1$ in $\tau$ gives

$$d_M Q_r(x) = -D(x) Q_{r-1}(x).$$

For $r = 1$, this equation has a solution because $d_M D(x) = 0$ (this is what gives equation $\nabla \Phi = 0$) and $d_M Q_0(x) = 0$. It has also a solution for $r \geq 2$ because $d_M(D(x) Q_{r-1}(x)) = 0$, which is equivalent to

$$\partial_{x_j} (D(i)(x) Q_{r-1}(x)) = \partial_{x_i} (D(j)(x) Q_{r-1}(x))$$

for all $i$ and for all $j$. These equalities are shown by induction, using moreover the fact that $\Phi \wedge \Phi = 0$. $\square$

**Corollary 4.1.2** 1. There exists a matrix

$$P(x, \tau) = P_0(x) + \sum_{i \geq 1} P_i(x) \tau^i \in \mathcal{M}_{\mu \times \mu}(\mathcal{O}(\tilde{M})[[\tau]])$$

such that $\nabla^d((\omega_0, \ldots, \omega_{\mu-1})P) = 0$.

2. After the base change of matrix $P$, the matrix of $\nabla$ takes the form $R(\tau)\frac{d\tau}{\tau}$ where

$$R(\tau) = \sum_{k \geq 0} R_k \tau^k$$

is a formal power series in $\tau$ and the matrices $R_k$ are constant.\textsuperscript{10}

\textsuperscript{10}In particular, $R_0 = -P_0^{-1} A_{\infty} P_0$. 

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Proof. (1) Let \( P_0(x) \) such that \( \nabla((\omega_0, \cdots, \omega_{\mu-1})P_0(x)) = 0 \). Apply the previous lemma to the \( \nabla \)-flat basis \((\omega_0, \cdots, \omega_{\mu-1})P_0(x) \) (if the basis \( \omega \) is \( \nabla \)-flat from the beginning, \( P_0(x) = I \) and \( P = Q \)). (2) Follows from the flatness of \( \nabla \). \( \square \)

**Definition 4.1.3** The matrix \( P := P(x, \tau) \) is called fundamental solution of the Dubrovin connection \( \nabla^d \).

**Remark 4.1.4** Let \( P(x, \tau) \) be a fundamental solution. Then \( \tilde{P}(x, \tau) \) is a fundamental solution if and only if there exists an invertible matrix
\[
\alpha(\tau) = \alpha_0 + \sum_{k \geq 1} \alpha_k \tau^k
\]
where \( \alpha_i \in M_{\mu \times \mu}(\mathbb{C}) \), such that \( \tilde{P}(x, \tau) = P(x, \tau)\alpha(\tau) \), as it follows from the base change formula for a connection.

### 4.2 The \( J \)-functions of a quantum differential system

We define here the \( J \)-functions of a general quantum differential system. In the situation of example 2.1.7, our definition agrees with the usual object considered in classical mirror symmetry.

Let \( P \) be a fundamental solution and \((e_0, \cdots, e_{\mu-1}) := (\omega_0, \cdots, \omega_{\mu-1})P\). We define

\[
\mathcal{H}_P = \{(e_0, \cdots, e_{\mu-1}) \in \Gamma(U_\infty \times M, \mathcal{G}) \mid P \left( \begin{array}{c} y_0 \\ y_1 \\ \vdots \\ y_{\mu-1} \end{array} \right) \in (\mathcal{O}_M)^\mu \}
\]

If \( \eta = \sum_{i=0}^{\mu-1} s_i \omega_i \in \Gamma(\mathbb{P}^1 \times M, \mathcal{G}) \), we will write

\[
J^{P,\eta}_Q := P^{-1} \eta = (e_0, \cdots, e_{\mu-1}) \left( \begin{array}{c} y_0 \\ y_1 \\ \vdots \\ y_{\mu-1} \end{array} \right) \text{ where } P \left( \begin{array}{c} y_0 \\ y_1 \\ \vdots \\ y_{\mu-1} \end{array} \right) = \left( \begin{array}{c} s_0 \\ s_1 \\ \vdots \\ s_{\mu-1} \end{array} \right) \tag{17}
\]

By definition, the function \( J^{P,\eta}_Q \) is thus the section \( \eta \) expressed in the frame \((e_0, \cdots, e_{\mu-1})\).

**Lemma 4.2.1** 1. Let \( \nabla^d \) be the connection induced by \( \nabla^d \) on \( \mathcal{H}_P \). Then

\[
\nabla^d_X(y_0e_0 + \cdots + y_{\mu-1}e_{\mu-1}) = X(y_0)e_0 + \cdots + X(y_{\mu-1})e_{\mu-1}
\]

for any vector field \( X \) on \( M \).

2. We have

\[
\nabla^d_X(J^{P,\eta}_Q) = P^{-1}\nabla^d_X\eta
\]

for any vector field \( X \) on \( M \) and \( \eta \in \Gamma(\mathbb{P}^1 \times M, \mathcal{G}) \).
Proof. 1. Let $X$ be a vector field on $M$. We have

$$\nabla^d_X((e_0, \cdots, e_{\mu-1}) \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{\mu-1} \end{pmatrix}) = \nabla^d_X((\omega_0, \cdots, \omega_{\mu-1})P \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{\mu-1} \end{pmatrix})$$

where the second equality follows from the fact that $P$ is a fundamental solution.

2. We now have, keeping the previous notations,

$$\begin{pmatrix} X(y_0) \\ X(y_1) \\ \vdots \\ X(y_{\mu-1}) \end{pmatrix} = (e_0, \cdots, e_{\mu-1}) \begin{pmatrix} X(y_0) \\ X(y_1) \\ \vdots \\ X(y_{\mu-1}) \end{pmatrix}$$

where $\Omega$ is the matrix of $\nabla^d$ in the basis $\omega$. □

As suggested by formula (18), we will write $X(J^P)$ instead of $\nabla^d_X(J^P)$. Notice that $YX(J^P) = P^{-1} \nabla^d_Y \nabla^d_X \eta$ etc...

Keeping quantum product in mind, see section 3.2, it is natural to consider the section $J^P$ where $\omega_0$ satisfies the condition of proposition 3.2.2. If it happens to be the case, we have $\theta \partial_j J^P = P^{-1} \omega_j + \theta P^{-1} \nabla \partial_j \omega_0$. In particular,

$$\theta \partial_j J^P = P^{-1} \omega_j$$

if $\nabla \omega_0 = 0$.

Definition 4.2.2 Let $P$ be a fundamental solution of the Dubrovin connection and assume that the section $\omega_0$ is as in proposition 3.2.2. We will call $J^P$ a $J$-function of the quantum differential system $Q$.

The following result explains the link with the product $*$ defined in section 3.2. We will write, for $H$ a polynomial function of $2r+3$ variables, $H(\theta \partial_{x_0}, \cdots, \theta \partial_{x_r}, x_0, \cdots, x_r, \theta)$ and, for simplicity, $J$ instead of $J^P$.

Proposition 4.2.3 Let $J$ be a $J$-function of the quantum differential system $Q$.

1. We have $H(\ast, \omega_0, \theta) = 0$ if $H(\theta \partial_{x_0}, \cdots, \theta \partial_{x_r}, x_0, \cdots, x_r, \theta) = 0$.

2. Assume moreover that the basis $\omega$ is $\nabla$-flat. We have (operators of order 2),

$$\theta \partial_{\partial_{x_j}} J = \sum_k a^k_{\partial_{x_j}} J \theta \partial_{x_k} J$$

if and only if $\omega_j \ast \omega_i = \sum_k a^k_{j} \omega_k$. 

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Proof. 1. Let us observe that 
\[ H(\theta \partial_{\theta} \cdot \theta J) = 0 \] 
if and only if 
\[ H(\theta \nabla_{\partial_{\theta}} \cdot \theta J) \omega_0 = 0, \]
as it follows from formula (19). Now, using the definition of the quantum product,
\[ \theta \nabla_{\partial_{\theta}} \cdot \theta \omega_0 = \omega_i \ast \cdots \ast \omega_i + \theta A(\theta) \]
for a suitable formal power series A. For 2., we have, thanks to the flatness of \( \omega \), \( \theta^2 \partial_{x_j} \partial_{x_i} J = P^{-1} \omega_j \ast \omega_i \) and we use again \( \theta \partial_{x_k} J = P^{-1} \omega_k \).
\[ \blacksquare \]

Let us emphasize the fact that the function \( J_{P,\omega^0} \) depends on the pre-primitive section and on the fundamental solution \( P \). The aim of the next section is to define canonical \( J \)-functions. In the situation of example 2.1.7, we will see that Givental’s \( J \) function is such a canonical function, obtained by taking \( \omega_0 = 1 \) together with a canonical fundamental solution \( P \). In this case, we have
\[ J_{P,\omega^0} = e^{\tau Q \ln x} (\omega_0 + O(\tau)) \]
for a suitable constant matrix \( Q \). See corollary 6.3.7. We will see also that, for suitable fundamental solutions \( P \), \( J_{P,\omega^0} \) can be expressed with the help of the bilinear form of the quantum differential system \( Q \), see proposition 5.2.6.

5  Canonical fundamental solutions of a quantum differential system

The aim of this section is to define canonical fundamental solutions, and hence canonical \( J \)-functions (see section 5.3): this is done with the help of Dubrovin’s symmetric and conformal fundamental solutions, see [15, Lecture 2]. We keep the situation of the beginning of section 4.

5.1 A class of convergent fundamental solutions: conformal solutions (after [15])

We define here some convergent (in \( \tau \)) fundamental solutions. First, one can precise corollary 4.1.2 with the help of the following well-known lemma:

**Lemma 5.1.1** There exist a fundamental solution \( \tilde{P}(x, \tau) \) such that the matrix of the connection is, after the base change of matrix \( \tilde{P}(x, \tau) \),
\[ (\tilde{R}_0 + \tilde{R}_1 \tau + \cdots + \tilde{R}_{\delta} \tau^{\delta}) \frac{d\tau}{\tau} \]
the matrices \( \tilde{R}_k \) satisfying \( [\tilde{R}_0, \tilde{R}_k] = -k \tilde{R}_k \) for all \( k \geq 1 \) and \( \tilde{R}_0 = R_0 \) where \( R_0 \) is defined in corollary 4.1.2.

**Proof.** It is essentially the one giving the Levelt normal form (see for instance [35, Exercice II.2.20], [15, Lemma 2.5] and the references therein).

\[ \blacksquare \]
Remark 5.1.2 One has $\delta \leq \max |\lambda_i - \lambda_j|$, where the maximum is taken over the differences of the eigenvalues of $R_0$. In particular, $\tilde{R}_k = 0$ for all $k \geq 1$ if $R_0$ is non-resonant. On the $B$-side (with the previous notations, $R_0 = -A_\infty$), we thus have $\delta \leq \dim U$ if the quantum differential system is associated with a regular tame function $f : U \to \mathbb{C}$, see Appendix, because the eigenvalues of $A_\infty$ run through the spectrum of $f$ at infinity.

Definition 5.1.3 A fundamental solution which has the properties of lemma 5.1.1 is called conformal.

The following two results motivate the definition of conformal solutions:

**Proposition 5.1.4** A conformal fundamental solution is convergent (in $\tau$).

**Proof.** Let $\tilde{P}(x, \tau)$ be such a solution. then it satisfies

$$\tau \partial_\tau \tilde{P}(x, \tau) = \tilde{P}(x, \tau) \tilde{R}(\tau) + (\tau A_0(x) + A_\infty) \tilde{P}(x, \tau)$$ \hspace{1cm} (21)

The matrices $\tilde{R}(\tau)$ and $\tau A_0(x) + A_\infty$ are convergent, and we can conclude using a classical argument of regular singularity (see f.i. [35, Proposition II.2.18]). \hfill \square

**Proposition 5.1.5** Let $\tilde{P}(x, \tau)$ be a conformal fundamental solution and $\Lambda$ be the diagonal matrix whose eigenvalues are the integer parts of those of $\tilde{R}_0$. Assume moreover that $\tilde{R}_0$ is block diagonal, each block corresponding to an eigenvalue.

1. After the base change, meromorphic in $\tau$, of matrix $\tilde{P}(x, \tau)\tau^{-\Lambda}$, the matrix of $\nabla$ takes the form

$$(\tilde{R}_0 - \Lambda + \tilde{R}_1 + \cdots + \tilde{R}_\delta) \frac{d\tau}{\tau}.$$ 

2. A basis of flat sections is $(\omega_0, \cdots, \omega_{\mu-1})\tilde{P}(x, \tau)\tau^{-\tilde{R}_0} \tau^{-\tilde{R}_1} \cdots \tau^{-\tilde{R}_\delta}$.

**Proof.** 1. Indeed, $[\Lambda, \tilde{R}_k] = -k\tilde{R}_k$ if $[\tilde{R}_0, \tilde{R}_k] = -k\tilde{R}_k$ and 2. follows because $\tilde{R}_0 - \Lambda$ commutes with $\tilde{R}_1 + \cdots + \tilde{R}_\delta$ and $\Lambda$. \hfill \square

**Remark 5.1.6** We can compare conformal fundamental solutions, as in remark 4.1.4. We will say that the matrix $\alpha(\tau)$ is homogeneous if

$$\alpha(\tau) = \sum_{\ell \geq 0} \alpha^{(-\ell)} \tau^\ell$$

where $\alpha^{(-\ell)} \in \ker(AdR_0 + \ell I)$ for all $\ell \in \mathbb{Z}$. Let $P$ and $\tilde{P}$ be two fundamental solutions and assume that $P$ is conformal. Then $\tilde{P}$ is conformal if and only if there exists a homogeneous matrix $\alpha(\tau)$ such that $\tilde{P} = P\alpha(\tau)$, see also [13].
5.2 Symmetric solutions (after [15])

Let $P$ be a fundamental solution and

$$e = (e_0, \ldots, e_{\mu-1}) := (\omega_0, \ldots, \omega_{\mu-1}) P$$

Let us analyze the behaviour of $e$ with respect to $S$. By the proof of corollary 4.1.2 (1), we may assume that the basis $\omega = (\omega_0, \ldots, \omega_{\mu-1})$ is $\nabla$-flat.

**Lemma 5.2.1** We have $S(e_i, e_j) \in \mathbb{C}[\tau]^{-n}$ for all $i$ and for all $j$.

**Proof.** $S(e_i, e_j)$ depends only on $\tau$, because $P$ is a fundamental solution and because $S$ is $\nabla$-flat, and the result follows from formula (5).

The best that we can expect is $S(e_i, e_j) \in \mathbb{C}^{-n}$ and this happens for instance if $S(e_i, e_j) = S(\omega_i, \omega_j)$, because $S(\omega_i, \omega_j) \in \tau^{-n}$ by remark 2.1.4.

**Definition 5.2.2** Let $P$ be a fundamental solution. We will say that it is symmetric if

$$P^*(x, -\tau) P(x, \tau) = I$$

where $*$ denotes the adjoint with respect to $S$.

We will consider the following situation in section 6 (see theorem 6.3.4):

**Proposition 5.2.3** Let us assume that $M = \mathbb{C}^*$ and consider the fundamental solution $P(x_1, \tau) = H(x_1, \tau)e^{\tau D \ln x_1}$ where $H$ is a matrix of holomorphic functions on $\mathbb{C} \times \mathbb{C}$ such that $H(0, \tau) = I$ and $D$ is a constant matrix. Then $P$ is symmetric.

**Proof.** Recall that $S(\omega_i, \omega_j) \in \mathbb{C}^{-n}$ for all $i$ and for all $j$. We have

$$S(e_i, e_j) = S(H(0, \tau)e^{\tau D \ln x_1} \omega_i, H(0, \tau)e^{\tau D \ln x_1} \omega_j) + o(1)$$

$$= S(e^{\tau D \ln x_1} \omega_i, e^{\tau D \ln x_1} \omega_j) + o(1) = S(e^{-\tau D^* \ln x} e^{\tau D \ln x} \omega_i, \omega_j) + o(1)$$

as $x \to 0$. By lemma 5.2.1 we must have $e^{\tau D^* \ln x} = e^{\tau D \ln x}$ and $S(e_i, e_j) = S(\omega_i, \omega_j)$: $P$ is thus symmetric.

More generally, but we won’t directly use this result, one can show, as in [15, Lemma 2.5], that there exists conformal and symmetric fundamental solutions.

**Remark 5.2.4** It follows from remark 5.1.6 that a conformal, symmetric fundamental solution is unique up to right multiplication by homogeneous and symmetric (i.e satisfying $\alpha(\tau)^* \alpha(\tau) = I$) sections $\alpha(\tau)$.

Here are two consequences of the symmetry:

**Proposition 5.2.5** Let $P$ be a conformal, symmetric fundamental solution. Then we have $R_k^* = (-1)^{k+1} \hat{R}_k$ for $k \geq 1$ and $R_0^* = -n \text{Id} - R_0$ in formula (20).
Proof. Transpose equality (21) taking into account equations (7).

Last, if if $P$ is symmetric and if the basis $\omega$ is orthogonal with respect to $S$, the $J$-functions of definition 4.2.2 are expressed as follows (the dual $\omega^i$ of $\omega_i$ is defined in remark 2.1.4):

**Proposition 5.2.6** Let us assume that the fundamental solution $P$ is symmetric. Then, the functions $J_{P,\omega_0}^Q$ is defined by the formula

$$J_{P,\omega_0}^Q = \sum_{j=0}^{\mu-1} g(P(\omega_j),\omega_0)\omega_j$$

where $g$ is given by formula (6) and we have

$$\partial_{x_i} J_{P,\omega_0}^Q = \sum_{j=0}^{\mu-1} \partial_{x_i} (g(P(\omega_j),\omega_0))\omega_j$$

for all $i = 1, \cdots, r$.

Proof. The first equality follows from symmetry. For the second one, we have, using respectively the definition, the symmetry and the flatness of $g$, together with the fact that $P$ is a fundamental solution,

$$g(\omega_j, \partial_{x_i} J_{P,\omega_0}^Q) = g(\omega_j, P^{-1}\nabla_d \partial_{x_i} \omega_0) = g(P\omega_j, \nabla_d \partial_{x_i} \omega_0) = \partial_{x_i} (g(P(\omega_j),\omega_0))$$

and this gives the expected result.

\[\square\]

5.3 Canonical fundamental solutions and canonical $J$-functions

**Definition 5.3.1** We will say that the fundamental solution $P$ is canonical if it is conformal, symmetric and if moreover

$$R(\tau) = R_0 + R_1 \tau \quad (22)$$

$R(\tau) \frac{d\tau}{d\xi}$ denoting the matrix of $\nabla$ after the base change of matrix $P$.

**Remark 5.3.2** Let $P$ be a canonical fundamental solution. It follows from proposition 5.1.3 that a basis of flat sections takes the form $(\omega_0, \cdots, \omega_{\mu-1})\Psi$ where

$$\Psi(x, \tau) = P(x, \tau)\tau^{-R_0} \tau^{-R_1} \Psi_{\text{const}}$$

$\Psi_{\text{const}}$ denoting a constant matrix.

**Lemma 5.3.3** Let $P$ be a conformal, symmetric solution and

$$R(\tau) = R_0 + R_1 \tau + \cdots + R_\delta \tau^\delta$$

the matrix associated with it by lemma 5.1.1. Then there exists a canonical fundamental solution if and only if there exists a homogeneous matrix $\alpha(\tau)$ (see remark 5.1.6)

$$\alpha(\tau) = I + \sum_{k \geq 1} a_k^{(-k)} \tau^k$$

such that

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• $[\alpha_k^{(-k)}, R_1] = R_{k+1}$ for all $k \geq 1$,
• $\alpha^*(-_\tau)\alpha(\tau) = I$.

In this case, $\tilde{P} = P\alpha(\tau)$ is a canonical solution and the matrix $\tilde{R}(\tau)$ associated with $\tilde{P}$ by lemma 5.1.1 is $R_0 + R_1\tau$.

Proof. By remark $\tilde{P} = P\alpha(\tau)$ is a conformal and fundamental solution if and only if $\alpha(\tau) = \sum_{i \geq 0} \alpha_i^{(-i)}\tau^i$ is a homogeneous matrix. Without loss of generality (replace $\alpha$ by $\alpha_0^{-1}\alpha$ if necessary) we can assume that $\alpha_0 = I$. The matrix $\tilde{R}(\tau)$ attached to $\tilde{P}$ by lemma 5.1.1 is $\tilde{R}_0 + \tilde{R}_1\tau$ if and only if

$$\tau\alpha' = \alpha(\tilde{R}_0 + \tilde{R}_1\tau) - (R_0 + R_1\tau + \cdots + R_6\tau^5)\alpha \quad (23)$$

The constant term gives $\tilde{R}_0 = R_0$, the one of degree 1 (in $\tau$) gives

$$(\text{Ad}R_0 + I)(\alpha_1^{(-1)}) = \tilde{R}_1 - R_1$$

and more generally the one of degree $k$ in $\tau$ ($k \geq 2$)

$$(\text{Ad}R_0 + kI)(\alpha_k^{(-k)}) = [\alpha_{k-1}, \tilde{R}_1] - R_k.$$ 

The first assertion follows because $\alpha$ is homogeneous. follows that $\alpha_1 \in V^{(-1)}$ and then $\tilde{R}_1 = R_1$. The last assertion about symmetry is clear.

A canonical solution is thus unique up to multiplication by constant homogeneous matrices. More precisely,

**Corollary 5.3.4** Assume that $\tilde{P}$ and $\tilde{Q}$ are two canonical solutions. The matrices of the connection $\nabla$ in the bases $\omega\tilde{P}$ and $\omega\tilde{Q}$ are respectively $(R_0 + R_1\tau)\frac{d\tau}{\tau}$ and $\alpha_0^{-1}(R_0 + R_1\tau)\alpha_0\frac{d\tau}{\tau}$ where $\alpha_0$ is a homogeneous matrix of degree 0.

In particular, formula (22) is unique up to conjugation by a constant homogeneous matrix.

The following definition is now natural (recall the pre-primitive sections defined in 2.3.3):

**Definition 5.3.5** The canonical $J$-functions are $J_{\tilde{Q},\omega_0}^{P,\omega_0} = P^{-1}\omega_0$ where $P$ is a canonical fundamental solution and $\omega_0$ is a canonical pre-primitive section of the quantum differential system $\tilde{Q}$.

### 6 Non-resonant logarithmic quantum differential systems

The goal of this section is to show that there exist explicit canonical fundamental solutions under the assumption that the quantum differential systems are logarithmic and non-resonant. It happens to be the case for systems are associated with the small quantum cohomology described in example 2.1.7 thanks to the divisor axiom. Notice that logarithmic Frobenius manifolds have been defined by T. Reichelt [32].
6.1 Logarithmic quantum differential systems

Let \( Q = (M, \mathcal{G}, \nabla, S, d) \) be a quantum differential system. We will use the following version of definition 2.1.6.

**Definition 6.1.1** We will say that \( Q \) is logarithmic if \( M = (\mathbb{C})^{r+1} \) and if its characteristic equation (definition 2.1.3) has the form

\[
\sum_{i=0}^{r} M^{(i)}(x, \tau) \frac{dx_i}{x_i} + N(x, \tau) \frac{d\tau}{\tau}
\]  

(24)

where the matrices \( M^{(i)}(x, \tau) \) and \( N(x, \tau) \) are matrices of holomorphic functions on \( \mathbb{C}^{r+1} \times \mathbb{C} \) and \( x = (x_0, \cdots, x_r) \in M \).

One can of course replace \( M \) by an open neighbourhood of the origin in \( (\mathbb{C})^{r+1} \).

It follows from proposition 2.1.2 that

\[
M^{(i)}(x, \tau) = M^{(i)}_0(x) + M^{(i)}_1(x) \tau \quad \text{and} \quad N(x, \tau) = N_0(x) + N_1(x) \tau
\]  

(25)

if the quantum differential system \( Q \) is logarithmic.

**Example 6.1.2** The quantum differential systems considered in examples 2.2.1 and 2.1.7 are logarithmic. Rescalings (see section 7.1) provide other examples of such systems.

**Remark 6.1.3** The classical limit (as \( x \to 0 \)) of a logarithmic differential system \( Q \) on \( M = \mathbb{C}^* \) is a quantum differential system on a point, that is a tuple

\[
Q^{cl} = (G^{cl}, \nabla^{cl}, S^{cl}, d^{cl})
\]

satisfying the conditions of definition 2.1.4. See section 6.4 below for a discussion about this.

6.2 Non-resonant logarithmic quantum differential systems on curves

Let \( Q = (M, \mathcal{G}, \nabla, S, d) \) be a logarithmic quantum differential system on \( M = \mathbb{C} \) (definition 6.1.1 with \( r = 0 \)) with pole at the origin: its characteristic equation is, in the basis \( \omega = (\omega_0, \cdots, \omega_{\mu-1}) \),

\[
M(x, \tau) \frac{dx}{x} + N(x, \tau) \frac{d\tau}{\tau}
\]

(26)

where \( M(x, \tau) = M_0(x) + M_1(x) \tau \) and \( N(x, \tau) = N_0(x) + N_1(x) \tau \) are matrices of holomorphic functions on \( \mathbb{C} \times \mathbb{C} \) (it could also be on \( (\mathbb{C}, 0) \times \mathbb{C} \), see above). We will also write

\[
N(x, \tau) = -(A_0(x) \tau + A_\infty(x))
\]

to match with the notations of section 2.

**Definition 6.2.1** The logarithmic quantum differential system \( Q \) is non-resonant at \( \tau_0 \) if the eigenvalues of \( M(0, \tau_0) \) do not differ from a non-zero integer.
Note that the quantum differential system is non-resonant at \( \tau_0 = 0 \) if \( M_0(0) = 0 \), in particular if \( M_0(x) = 0 \) for all \( x \). This will be our favorite situation:

**Definition 6.2.2** We will say that the logarithmic quantum differential system \( Q \) is flat if \( M_0(x) = 0 \) for all \( x \).

The word “flat” recalls the flatness with respect to the residual connection \( \nabla \), see section 2.

### 6.3 Fundamental solutions of a non-resonant logarithmic quantum differential system

The main result of this section (theorem 6.3.4 below) is a variation of [15, Isomonodromicity Theorem]. Let \( Q \) be a logarithmic quantum differential system.

**Lemma 6.3.1** The eigenvalues of \( M(0, \tau) \) do not depend on \( \tau \).

*Proof.* Indeed, by isomonodromy (the connection is flat), the eigenvalues of the monodromy around \( x = 0 \) do not depend on \( \tau \). \( \square \)

In particular, it follows that \( M(0) \) is nilpotent if \( M_0(0) = 0 \).

**Lemma 6.3.2** Let us assume that \( Q \) is non-resonant at \( \tau_0 \).

1. The matrix \( M(0, \tau) \) is non-resonant for all \( \tau \in U := \mathbb{C} \).

2. There exists a matrix \( H(x, \tau) \) of holomorphic function on \( (\mathbb{C}, 0) \times U \), uniquely determined by the initial condition \( H(0, \tau) = I \), such that, after the base change of matrix \( H \), the matrix of the connection \( \nabla \) takes the form

\[
M(0, \tau) \frac{dx}{x} + V(\tau) \frac{d\tau}{\tau}
\]

\( V(\tau) \) being a matrix of holomorphic functions on \( U \). If moreover \( 0 \in U \) then

\[
V(\tau) = N(0, \tau) = N_0(0) + N_1(0)\tau.
\]

3. The matrix \( P(x, \tau) = H(x, \tau)e^{-M(0, \tau)\ln x} \) is a fundamental solution of the Dubrovin connection.

*Proof.* 1. follows from lemma 6.3.1. The proof of 2. is classical, but we give some details in order to set the notations and to write down explicitly the equations that we will use later (mainly equations (28), (29) and (30)): since the matrix \( M(0, \tau) \) is non-resonant for all \( \tau \in U \), there exists a unique matrix

\[
H(x, \tau) = I + \sum_{i \geq 1} H^i(\tau)x^i,
\]

defined on \( \mathbb{C} \times U \), such that

\[
x \frac{\partial H}{\partial x}(x, \tau) = H(x, \tau)M(0, \tau) - M(x, \tau)H(x, \tau)
\]

(28)
This can be shown for instance as in [35, Proposition 2.11], separating the degrees in $x$: equation (28) is then equivalent to

$$dH^d(\tau) = H^d(\tau)M(0, \tau) - M(0, \tau)H^d(\tau) - \sum_{i=1}^{d} M^i(\tau)H^{d-i}(\tau)$$

for $d \geq 1$ and $M(x, \tau) = M(0, \tau) + \sum_{i \geq 1} M^i(\tau)x^i$. The non-resonant assumption shows that these equations are solved in an unique way because $\text{Ad}M(0, \tau) + dI$ is invertible for all integer $d$. After the base change of matrix $H$, the matrix of the connection is

$$M(0, \tau)\frac{dx}{x} + V(x, \tau)\frac{d\tau}{\tau}.$$ 

Equation (28), together with a classical argument of regularity, shows that $H(x, \tau)$ holomorphic on $(\mathbb{C}, 0) \times U$, because $M(x, \tau)$ is holomorphic on $\mathbb{C} \times U$. In particular, $V(x, \tau)$ is also holomorphic on $(\mathbb{C}, 0) \times U$. Now, it follows from the flatness of the connection that

$$\tau \frac{\partial M}{\partial \tau}(0, \tau) - x \frac{\partial V}{\partial x}(x, \tau) = [M(0, \tau), V(x, \tau)].$$

This gives, putting $V(x, \tau) = V(\tau) + \sum_{k \geq 1} V^k(\tau)x^k$ and comparing the degrees in $x$,

$$\tau M_1(0) = [M_0(0) + \tau M_1(0), V(\tau)]$$

for $k = 0$ and

$$(\text{Ad}M(0, \tau) + kI)V^k(\tau) = 0$$

for $k \geq 1$ and for all $\tau \in U$. By assumption $\text{Ad}M(0, \tau) + kI$ is invertible for $k \geq 1$, and we finally get $V^k(\tau) = 0$ for all $\tau \in U$ and all $k \geq 1$. This shows that $V(x, \tau) = V(\tau)$ and (27) follows.

Let us show the last assertion and let us assume that $\tau = 0 \in U$. We also have

$$\tau \frac{\partial H}{\partial \tau}(x, \tau) = H(x, \tau)V(\tau) - N(x, \tau)H(x, \tau).$$

Writing $V(\tau) = \sum_{k \geq 0} V_k\tau^k$ and $H(x, \tau) = \sum_{i \geq 0} H_i(x)\tau^i$ (this is possible because $\tau = 0 \in U$), this equation gives, separating now the degrees in $\tau$,

$$kH_k(x) = H_0(x)V_k - N_k(x)H_0(x) + \Psi_k(x)$$

for all $k \geq 0$ (recall that $N_k(x) = 0$ for $k \geq 2$) with $\Psi_0(x) = 0$ and

$$\Psi_k(x) = \sum_{i=1}^{k} (H_i(x)V_{k-i} - N_{k-i}(x)H_i(x))$$

for $k \geq 1$. It follows that $V_k = N_k(0)$ for all $k \geq 0$ because $H_i(0) = 0$ for all $i \geq 1$ and $H_0(0) = I$, thanks to the initial condition $H(0, \tau) = I$. This completes the proof of 2. and 3. follows. □

**Corollary 6.3.3** Assume that the eigenvalues of $M_0(0)$ are contained in an interval of length strictly smaller than 1.
1. There exists a matrix $H(x, \tau)$ of holomorphic functions on $(\mathbb{C}, 0) \times \mathbb{C}$, uniquely determined by the initial condition $H(0, \tau) = I$, such that the matrix of $\nabla$ takes the form

$$(M_0(0) + M_1(0)\tau) \frac{dx}{x} + (N_0(0) + N_1(0)\tau) \frac{d\tau}{\tau}$$

after the base change of matrix $H$. The matrix $P(x, \tau) = H(x, \tau)e^{-(M_0(0)+\tau M_1(0))\ln x}$ is a fundamental solution of the Dubrovin connection.

2. We have the relations

$$[N_0(0), M_0(0)] = [N_1(0), M_1(0)] = 0 \text{ and } [N_0(0), M_1(0)] + [N_1(0), M_0(0)] = -M_1(0) \quad (33)$$

Proof. Follows from lemma [6.3.2] because the assumption on $M_0(0)$ shows that the quantum differential system is non-resonant at $\tau_0 = 0$. The commutation relations follows from formula (30).

Of course, the fundamental solution $P$ in corollary [6.3.3] does not need to be conformal, neither symmetric. If $M_0(x) = 0$ for all $x$\footnote{In this case $M_1(0)$ is nilpotent, see lemma [6.3.1]}, the situation becomes better:

**Theorem 6.3.4** Let $Q$ be a flat logarithmic quantum differential system.

1. There exists a matrix $H(x, \tau)$ of holomorphic functions on $(\mathbb{C}, 0) \times \mathbb{C}$, uniquely determined by the initial condition $H(0, \tau) = I$, such that the matrix $P(x, \tau) = H(x, \tau)e^{-(M_0(0)+\tau M_1(0))\ln x}$ is a fundamental solution of the Dubrovin connection.

2. Any fundamental solution takes the form $H(x, \tau)e^{-(M_1(0)\ln x)}P_{cl}(\tau)$ where $P_{cl}(\tau)$ is a matrix depending only on $\tau$.

3. The matrix of the connection takes the form

$$(N_0(0) + N_1(0)\tau) \frac{d\tau}{\tau} \quad (34)$$

after the base change of matrix $P$.

4. The fundamental solution $P$ is symmetric and we have

$$\tau \partial_\tau S(e_i, e_j) = S(\tau \nabla_{\partial_\tau} e_i, e_j) + S(e_i, \tau \nabla_{\partial_\tau} e_j)$$

if $(e_0, \cdots, e_{\mu-1}) = (\omega_0, \cdots, \omega_{\mu-1})P$.

5. Assume that $M_1(0) = c N_1(0)$ for some non-zero constant $c$. Then the fundamental solution $P$ is conformal.
Proof. 1. Follows from corollary \[6.3.3\] because \(M_0(0) = 0\) and 2. then follows from remark \[4.1.4\]. For 3. we can proceed as follows: after the base change of matrix \(P(x, \tau)\), the matrix of the connection takes the form

\[
[e^{\tau M_1(0) \ln x} (N_0(0) + \tau N_1(0)) e^{-\tau M_1(0) \ln x} - \tau M_1(0) \ln x] \frac{d\tau}{\tau}.
\]

Now, we have \([N_0(0), M_1(0)] = -M_1(0)\) by relations \(63\) because \(M_0(0) = 0\) and this gives

\[
e^{\tau M_1(0) \ln x} N_0(0) = N_0(0) e^{\tau M_1(0) \ln x} + \tau M_1(0) e^{\tau M_1(0) \ln x} \ln x
\]

and we get the expected formula using then relation \([N_1(0), M_1(0)] = 0\).

4. The first assertion thus follows from proposition \[5.2.3\] because \(S(\omega_i, \omega_j) \in \mathbb{C} \tau^{-n}\) for all \(i\) and for all \(j\) if \(M_0(x) = 0\) (see remark \[2.1.4\]). Since \(P\) is symmetric, we have

\[
\tau \partial_\tau S(e_i, e_j) = \tau \partial_\tau S(\omega_i, \omega_j) = S(\tau \nabla_{\partial_\tau} \omega_i, \omega_j) + S(\omega_i, \tau \nabla_{\partial_\tau} \omega_j)
\]

and, by formula \(33\), the right hand side is equal to \(S(\tau \nabla_{\partial_\tau} e_i, e_j) + S(e_i, \tau \nabla_{\partial_\tau} e_j) + O(1)\) as \(x\) tends to 0. We conclude using the fact that \(S(e_i, e_j)\) does not depend on \(x\) (see lemma \[5.2.1\]).

5. Indeed, we have \([N_0(0), M_1(0)] = -M_1(0)\) by relations \(63\) because \(M_0(0) = 0\). \(\square\)

Remark 6.3.5 The assumption in item 5 will be satisfied in the geometric situations considered below (small quantum cohomology and/or its mirror partner on the B-side), see section \[7\] below.

We will use mainly the following corollary:

Corollary 6.3.6 Let \(Q\) be a flat quantum differential system.

1. Any basis of flat sections of \(\nabla\) takes the form \((\omega_0, \cdots, \omega_{\mu-1})\) \(\Psi\) where

\[
\Psi(x, \tau) = H(x, \tau) e^{-\tau M_1(0) \ln x} \Psi^{cl}(\tau),
\]

the matrix \(\Psi^{cl}(\tau)\) satisfying

\[
\tau \partial_\tau \Psi^{cl}(\tau) = -(N_0(0) + \tau N_1(0)) \Psi^{cl}(\tau).
\]

2. If moreover \(P(x, \tau) = H(x, \tau) e^{-\tau M_1(0) \ln x}\) is conformal, we have

\[
\Psi^{cl}(\tau) = \tau^{-N_0(0) - N_1(0)} \Psi^{const}
\]

where \(\Psi^{const}\) is a constant matrix.

Proof. 1. Follows from theorem \[6.3.4\] (items 2. and 3). We then get 2. using proposition \[5.1.5\]. \(\square\)

Corollary 6.3.7 Let \(Q\) be a flat quantum differential system and \(P\) be a fundamental solution as in theorem \[6.3.4\] 1. Then

\[
J_{Q}^{\omega_0} = e^{\tau M_1(0) \ln x} (\omega_0 + O(\tau))
\]

On the A-side (example \[2.1.7\]) we even have the asymptotic expansion, for small quantum cohomology (and we assume here that the Picard group of \(X\) is of rank one),

\[
J_X = e^{\tau \cup_\mu \ln q (1 + o(\tau))}
\]

where \(\cup_p\) stands for the cup-product by the generator of \(H^2(X, \mathbb{C})\) and \(q = e^t\). The previous formula is important in order to show for instance Givental’s mirror formula \(I_X = J_X\) for \(X\) a hypersurface of degree \(\ell < n\) in \(\mathbb{P}^n\), see \[6\] Section 11.2].
6.4 Classical limit of a non-resonant quantum differential system

The classical limit (as \( x \to 0 \)) of a logarithmic differential system \( Q \) on \( M = \mathbb{C} \) is a quantum differential system on a point, that is a tuple

\[
Q^\text{cl} = (G^\text{cl}, \nabla^\text{cl}, S^\text{cl}, d)
\]

satisfying the conditions of definition 2.1.1, see also remark 2.1.5. We explain here why such a limit exists, how to compute it and why it produces a meromorphic connection with regular singularity at \( \theta = 0 \) in the geometric situation.

6.4.1 Flat case

The classical limit of a flat quantum differential system \( Q \) is the tuple

\[
Q^\text{cl} = (G^\text{cl}, \nabla^\text{cl}, S^\text{cl}, d)
\]

where

- \( G^\text{cl} \) is the \( \mathcal{O}_\mathbb{P}^1 \)-free module associated with the lattice \((G_0^\text{cl}, G_\infty^\text{cl})\) (see \[35, I, \text{proposition 4.15}\]) where \( G_0^\text{cl} \) (resp. \( G_\infty^\text{cl} \)) is the \( \mathbb{C}[\theta] \) (resp. 12 \( \mathbb{C}[\tau] \)) free-module generated by the classes \([\omega_i]\) of the sections \( \omega_i \) in \( L/xL \), \( L \) being the \( \mathbb{C}\{x\}[[\theta,\theta^{-1}]\)-module generated by \( \omega \),

- \( \nabla^\text{cl} \) is the connection whose matrix is

\[
(\frac{A_0(0)}{\theta} + A_\infty(0)) \frac{d\theta}{\theta}
\]

in the basis \([\omega]\) of \( G^\text{cl} \),

- \( S^\text{cl}([\omega_i],[\omega_j]) = S(\omega_i,\omega_j) \) for all \( i \) and for all \( j \).

The tuple \( Q^\text{cl} \) satisfies the conditions of definition 2.1.1: \( S^\text{cl} \) is \( \nabla^\text{cl} \)-flat because of theorem 6.3.3 (4). In the setting of example 2.1.7 \( G^\text{cl} \) is the trivial bundle on \( \mathbb{P}^1 \) whose fibers are \( H^\cdot(X,\mathbb{C}) \) and the limit metric \( S^\text{cl} \) is the usual cup-product.

Assume moreover that \([A_\infty(0), A_0(0)] = A_0(0) \) (conformality). Then, and after the base change of matrix \( \theta^{-D} \), where \( D \) is the diagonal matrix whose eigenvalues are the integral part of the ones of \( A_\infty \), system 36 becomes

\[
(A_\infty(0) - D + A_0(0)) \frac{d\theta}{\theta}
\]

In particular, the meromorphic connection \((G_0^\text{cl}[\theta^{-1}], \nabla^\text{cl})\) has a regular singularity at the origin.

6.4.2 General case

Assume now that \( Q \) is non-resonant at \( \tau_0 \) but \( M_0(0) \neq 0 \). In this case, the definition of the limiting quantum differential system is more complicated: one has to take into account a monodromy

\[\text{As usual, } \tau := \theta^{-1}.\]
phenomenon and to work in a graded module with respect to the $V$-filtration in order to reduce to the previous situation (the difficult point is to get a “limit” bilinear form), and this is in fact what we do in example 9.1.2.

In order to make the link with section 6.4.1, assume that $M_0(0) = 0$. Let us first notice that, by lemma 6.3.1, the eigenvalues of $M_1(0)$ are all equal to zero. Let $V$ be the Malgrange-Kashiwara $V$-filtration of the Gauss-Manin system $G$ at $x = 0$ (see Appendix): we thus have $V^\alpha = V^0$ (which is, by definition the $C\{x\}[\theta, \theta^{-1}]$-module generated by $\omega$) for $\alpha \leq 0$ and $V^\alpha = xV^{\alpha - 1}$ for $\alpha > 0$. Finally, $L/xL = V^0G \cap G_0/V^1G \cap G_0$.

This construction is explained in detail in the case of the re scalings in section 7.1, and this will be after all the model for small quantum cohomology (see also [10] in the case of the small quantum cohomology of weighted projective spaces).

### 6.5 The $J$-functions of a non-resonant quantum differential system

We are now able to define the $J$-functions of a non-resonant quantum differential system, removing the ambiguity on the choice of the fundamental solution:

**Definition 6.5.1** Let $Q$ be a non-resonant quantum differential system $\tau_0$ and $\omega_0$ be a pre-primitive section. The $J$-function $J_Q^{\omega_0}$ of $Q$ is the function defined on $\mathbb{C} \times U$ by

$$J_Q^{\omega_0} = P^{-1}\omega_0$$

where $P = H(x, \tau)e^{-M(0,\tau)\ln x}$ is the fundamental solution given by lemma 6.3.2.

The function $J_Q^{\omega_0}$ is characterized by the initial condition $H(0, \tau) = I$ (notice then that $J_Q^{\omega_0} \sim e^{M(0,\tau)\ln x}\omega_0$ as $x \to 0$) and depends only on the chosen pre-primitive section. It other words, under the assumptions of theorem 6.3.4, $J_Q^{\omega_0}$ will be canonical if $\omega_0$ is so. This is what happens in examples 2.2.1 and 2.1.7 in the situation of example 2.1.7: Givental’s $J$-function is the canonical $J$-function in the previous sense, taking the section 1 as canonical pre-primitive section (see example 2.3.5).

### 6.6 Higher dimensional case

If $M = \mathbb{C}^{r+1}$ with $r$ greater or equal to 2 (in the geometric situations alluded above, this happens if the rank of the cohomology group $H^2$ is greater or equal to 2), one has analogous results in the case of a logarithmic quantum differential system $Q$ having a characteristic equation of the form

$$M^{(0)}(\underline{x}, \tau)\frac{dx_0}{x_0} + \cdots + M^{(r)}(\underline{x}, \tau)\frac{dx_r}{x_r} + N(\underline{x}, \tau)\frac{d\tau}{\tau}$$

where we put as above $\underline{x} = (x_1, \cdots, x_r)$ (this kind of quantum differential system is produced by the functions considered in 2.2.2(1); see also section 11 below). It follows from equations (25) that

$$M^{(i)}(\underline{x}, \tau) = M_0^{(i)}(\underline{x}) + M_1^{(i)}(\underline{x})\tau$$

and $N(\underline{x}, \tau) = N_0(\underline{x}) + N_1(\underline{x})\tau$

where the matrices involved are matrices of holomorphic functions. We will say that the quantum logarithmic differential system $Q$ is flat if $M_0^{(i)}(\underline{x})$ is identically equal to 0 for all $i = 0, \cdots, r$. 

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Lemma 6.6.1 Assume that the quantum logarithmic differential system $Q$ is flat\textsuperscript{13}.

1. There exists an invertible matrix $H(x, \tau)$, characterized by the initial condition $H(0, \tau) = I$, such that the matrix of $\nabla$ takes the form, after the base change of matrix $H(x, \tau)$,

$$
\tau M_1^{(1)}(0) \frac{dx_1}{x_1} + \cdots + \tau M_1^{(r)}(0) \frac{dx_r}{x_r} + V(\tau) \frac{d\tau}{\tau}
$$

(40)

where $V(\tau) = N_0(0) + N_1(0)\tau$.

2. Every fundamental solution of the Dubrovin connection takes the form

$$
H(x, \tau)e^{-\tau \sum_i M_1^{(i)}(0) \ln x_i} P_d(\tau)
$$

where $P_d(\tau)$ is a matrix depending only on $\tau$.

Proof. The eigenvalues of the residue matrices along $\{x_i = 0\}, i = 0, \cdots, r$, are constant and the assumption of the lemma shows that they are all equal to 0. In particular they do not differ from a non-zero integer. Notice that, due to the flatness of the connection, the matrices $M_1^{(i)}(0)$ and $M_1^{(j)}(0)$ commute. \hfill $\square$

Definition 6.6.2

1. The fundamental solution $H(x, \tau)e^{-\tau \sum_i M_1^{(i)}(0) \ln x_i}$ is called a canonical fundamental solution of the Dubrovin connection of the quantum differential system $Q$.

2. Let $\omega_0$ be pre-primitive. The $J$-functions of the quantum differential system $Q$ are the sections

$$
J_{\omega_0}^\omega = P^{-1}\omega_0
$$

where $P$ is a canonical fundamental solution of the Dubrovin connection.

7 Examples of non-resonant quantum differential systems and canonical fundamental solutions

We discuss here the existence of quantum differential systems and canonical fundamental solutions in the geometric setting.

7.1 Rescalings

Mirrors models for small quantum cohomology are in essence produced (at least in the toric case) by avatars of rescalings, see remark 7.1.1. Also, these rescalings give a quite good picture of what happens in general and this is why we first focus on them.

\textsuperscript{13}The condition $M_0^{(i)}(0) = 0$ for all $1 \leq i \leq r$ would be enough
7.1.1 Definitions

We will use notations and definitions of the Appendix. Let \( f : U \to \mathbb{C} \) be a regular tame function on the affine manifold \( U \), equipped with the coordinates \( \mathfrak{u} \), \( G^0_\mathfrak{u} \) (resp. \( G^0 \)) be the Brieskorn lattice (resp. the Gauss-Manin system) of \( f \). The Gauss-Manin system \( G^0 \) is equipped with a flat meromorphic connection \( \nabla^0 \) and \( G^0_0 \) is stable under \( \theta^2 \nabla_{\theta^0}^0 \).

Let us define

\[
F : U \times M \to \mathbb{C}
\]

where \( M = \mathbb{C}^* \) and \( F(\mathfrak{u}, x) = xf(\mathfrak{u}) \). This is a rescaling of \( f \). Let \( G_0 \) (resp. \( G \)) be the Brieskorn lattice (resp. the Gauss-Manin system) of \( F \):

\[
G_0 := \frac{\Omega^n(U)[x, x^{-1}, \theta]}{(\theta d_u - d_u F)} \wedge \Omega^{n-1}(U)[x, x^{-1}, \theta] \quad \text{and} \quad G := \frac{\Omega^n(U)[x, x^{-1}, \theta, \theta^{-1}]}{(\theta d_u - d_u F) \wedge \Omega^{n-1}(U)[x, x^{-1}, \theta, \theta^{-1}]}
\]

where \( d_u \) means that the differential is taken with respect to \( \mathfrak{u} \) only. We have \( G_0 = \pi^+ G^0_0 = \mathbb{C}[x, x^{-1}, \theta] \otimes G^0_0 \).

where \( \pi : (\theta, x) \mapsto \frac{\theta}{x} \). The Gauss-Manin system \( G \) is a free \( \mathbb{C}[x, x^{-1}, \theta, \theta^{-1}] \)-module, equipped with a meromorphic flat connection \( \nabla \): if \( \Omega^0 \) denotes the matrix of \( \nabla^0 \) in the basis \( \omega^0 \) then \( \pi^* \Omega^0 \) will be the one of \( \nabla \) in the basis \( \omega := 1 \otimes \omega^0 \). In general, and because \( \theta^2 \nabla_{\theta^0}^0 G^0_0 \subset G^0_0 \), the matrix of \( \nabla^0 \) in the basis \( \omega^0 \) is

\[
\Omega^0 = \left( \frac{A_0}{\theta} + A_\infty + A_1 \theta + \cdots + A_r \theta^r \right) \frac{d\theta}{\theta}
\]

for some \( r \geq 0 \) and the matrix of \( \nabla \) in the basis \( \omega \) will be

\[
\Omega = \left( \frac{xA_0}{\theta} + A_\infty + A_1 \frac{\theta}{x} + \cdots + A_r \frac{\theta^r}{x^r} \right) \left( \frac{d\theta}{\theta} - \frac{dx}{x} \right).
\]

Of course, the case \( r = 0 \) will give the connection of a logarithmic quantum differential system.

Remark 7.1.1 Given positive integers \( d_1, \cdots, d_n \) and \( (i_1, \cdots, i_n) \in \mathbb{Z}^n \), consider the function

\[
F(\mathfrak{u}, x) = u_1^{d_1} + \cdots + u_n^{d_n} + xu_1^{i_1} \cdots u_n^{i_n}
\]

where \( x \) is a non-zero complex parameter. Assume that \( \frac{i_1}{d_1} + \cdots + \frac{i_n}{d_n} \neq 1 \) and consider the change of variables

\[
\mathfrak{v} = (v_1, \cdots, v_n) = (x^{-r/d_1}u_1, \cdots, x^{-r/d_n}u_n)
\]

where \( r = \frac{1}{1 - (\frac{i_1}{d_1} + \cdots + \frac{i_n}{d_n})} \). Then,

\[
F(\mathfrak{v}, x) = x^r f(\mathfrak{v})
\]

with \( f(\mathfrak{v}) = v_1^{d_1} + \cdots + v_n^{d_n} + v_1^{i_1} \cdots v_n^{i_n} \). In other words, \( F \) can be expressed as a rescaling of \( f \). This applies in the situation of example 2.2.7 where \( F(\mathfrak{u}, x) = u_1 + \cdots + u_n + \frac{x}{u_1 \cdots u_n} \): we have \( F(\mathfrak{v}, \lambda) = x^\frac{1}{\mu} f(\mathfrak{v}) \) where \( f(\mathfrak{v}) = v_1 + \cdots + v_n + \frac{1}{v_1 \cdots v_n} \), \( \mathfrak{v} = (v_1, \cdots, v_n) = x^{-1/\mu}(u_1, \cdots, u_n) \) and \( \mu = 1 + w_1 + \cdots + w_n \).

\[\textit{14} \ G^0 \text{ is associated with the kernel } e^{-f/\theta} \text{ while } G \text{ is associated with the kernel } e^{-x f/\theta} \]
7.1.2 The quantum differential system associated with a rescaling

Let \( \omega^o = (\omega_0^o, \ldots, \omega_{\mu-1}^o) \) be the canonical solution of the Birkhoff problem for the Brieskorn lattice of \( f \) defined in the Appendix. Let us recall the two main properties of this solution: the matrix of \( \nabla^o \) takes the form

\[
(\frac{A_0^o}{\theta} + A_\infty) \frac{d\theta}{\theta}.
\]

in this basis and

\[
S^o(\omega_i^o, \omega_j^o) \in \mathbb{C}^{\theta_n}\tag{42}
\]

for all \( i, j \) where \( S^o : G_0^o \times j^*G_0^o \to \mathbb{C}[\theta]^{\theta^n} \) is the non-degenerate, \( \nabla^o \)-flat, symmetric bilinear form defined in step 3 of the Appendix (the involution \( j \) is defined in definition 2.1.1).

The matrix \( A_\infty \) is diagonal,

\[
A_\infty = \text{diag}(\alpha_0, \ldots, \alpha_{\mu-1})
\]

where \( \alpha_0 \leq \cdots \leq \alpha_{\mu-1} \) is the ordered (unless specified) spectrum at infinity (the spectrum of a limit mixed Hodge structure, see [37]) of the function \( f \). Due to the \( \nabla^o \)-flatness of \( S^o \) and formula (42), we can arrange the \( \alpha_i \)s in such a way that

\[
\alpha_\ell + \alpha_{\mu-1-\ell} = n\tag{43}
\]

for \( \ell = 0, \ldots, \mu-1 \). It should be emphasized that the matrix \( A_0^o \) is not nilpotent in general: this happens for instance if \( f \) has \( \mu \) distinct critical values and \( \nabla^o \) has then an irregular singularity at \( \theta = 0 \).

By construction, the basis \( \omega^o \) is adapted to the Kashiwara-Malgrange \( V \)-filtration \( V^\tau_\bullet \) at \( \tau = 0 \), that is

\[
V^\tau_\alpha G^o = \sum_{k_i \in \mathbb{Z}, \alpha_i - k_i \leq \alpha} \tau^{k_i} \omega_i^o
\]

for all \( \alpha \in \mathbb{Q} \). In particular, we have, and this is a key observation,

\[
A_0^o(\omega_i^o) \in \sum_{\alpha_j \leq \alpha_i + 1} \mathbb{C} \omega_j^o
\]

where \( \alpha_k \) denotes the \( V \)-order of \( \omega_k^o \), because \( \partial_\tau V^\tau_\alpha G^o \subset V^\tau_{\alpha+1} G^o \).

Let \( D = (d_0, \ldots, d_{\mu-1}) \) be the diagonal matrix whose entries are the integral part of the eigenvalues of \( A_\infty \) and

\[
\omega^{Del} = (\omega_1^{Del}, \ldots, \omega_\mu^{Del}) = (\omega_1, \ldots, \omega_\mu)x^D
\]

where, as above, \( \omega = (\omega_1, \ldots, \omega_\mu) \) is the basis of \( G_0 \) induced by \( \omega^o \). Then:

**Lemma 7.1.2** The matrix of the connection \( \nabla \) in the basis \( \omega^{Del} \) is

\[
(\sum_{i=0}^{n+1} \frac{x^i A_i^{Del}}{\theta} + A_\infty) \frac{d\theta}{\theta} - \sum_{i=0}^{n+1} \frac{x^i A_i^{Del}}{\theta} + A_\infty - D) \frac{dx}{x}\tag{45}
\]

where the constant matrices \( A_i^{Del} \) satisfy \( [D, A_i^{Del}] = -(i - 1)A_i^{Del} \) for \( i = 0 \cdots n+1 \).
Proof. (1) The matrix of $x \nabla_{\partial_x}$ in the basis $\omega^Delt$ is
\[- \frac{x^{-D}(xA_0^\alpha) x^D}{\partial} - (A_\infty - D)\]
and we have $(x^{-D}(xA_0^\alpha) x^D)_{ij} = x^{d_j-d_i+1}(A_0^\alpha)_{ij}$. By condition (43), $(A_0^\alpha)_{ij} \neq 0$ implies $\alpha_i \leq \alpha_j + 1$ hence $d_i \leq d_j + 1$. Since the $d_i$'s are contained in $[0, n]$ (because the $\alpha_i$'s are so) we get
\[x^{-D}(xA_0^\alpha) x^D = \sum_{i=0}^{n+1} x^i A_i^Delt\]
with $[D, A_i^Delt] = -(i-1)A_i^Delt$. \hfill \Box

Define $S := \pi^* S^\alpha$, where $S^\alpha$ is defined in formula (42): $S$ is a non-degenerate, $\nabla$-flat, symmetric bilinear form
\[S : G_0 \times j^*G_0 \to \mathbb{C}[x, x^{-1}, \theta] \theta^n \quad (46)\]
By definition we have $S(\omega_i, \omega_j) \in \mathbb{C} x^{-n} \theta^n$ and thus
\[S(\omega_i^Delt, \omega_j^Delt) = \begin{cases} 
-x^{-1}S^\alpha(\omega_i^\alpha, \omega_j^\alpha) & \text{if } \alpha_i + \alpha_j = n \text{ and } \alpha_i \text{ is not an integer} \\
S^\alpha(\omega_i^\alpha, \omega_j^\alpha) & \text{if } \alpha_i + \alpha_j = n \text{ and } \alpha_i \text{ is an integer} \\
0 & \text{otherwise} 
\end{cases} \quad (47)\]

We are now ready to describe the expected quantum differential system. Let $M = \mathbb{C}^*$ and $\mathcal{G}$ be the trivial bundle on $\mathbb{P}^1 \times M$ defined by the lattice $(G_0, G_\infty)$ where $G_0 = \sum_i \mathbb{C}[x, x^{-1}, \theta] \theta^{1} \omega_i^Delt$ and $G_\infty = \sum_i \mathbb{C}[x, x^{-1}, \theta^{-1}] \omega_i^Delt$.

Proposition 7.1.3 The tuple $\mathcal{Q} = (\mathcal{M}, \mathcal{G}, \nabla, S, n)$ is a quantum differential system on $M$.

In some cases, we get also a non-resonant logarithmic quantum differential system on $N = \mathbb{C}$ (see section 6.1): indeed, let $\mathcal{G}^{log}$ be the trivial bundle on $\mathbb{P}^1 \times N$ defined by the lattice $(G_0^{log}, G_\infty^{log})$ where $G_0^{log} = \sum_i \mathbb{C}[x, \theta] \theta^{1} \omega_i^{Delt}$ and $G_\infty^{log} = \sum_i \mathbb{C}[x, \theta^{-1}] \omega_i^{Delt}$. By equation (47), we get

Corollary 7.1.4 Assume that $\alpha_j$ is an integer for all $j$. The tuple $\mathcal{Q}^{log} = (N, \mathcal{G}^{log}, \nabla, S, n)$ is a non-resonant logarithmic quantum differential system on $N$, with pole at the origin of $N$.

7.1.3 Classical limit

We explain here how to construct the classical limit of $\mathcal{Q}$ at $x = 0$ using the theory of the Malgrange-Kashiwara $V$-filtration.

The $V$-filtration at $x = 0$. For $k = 0, \ldots, \mu - 1$, put $v(\omega_k^{Delt}) = d_k - \alpha_k \in [-1, 0]$. Define, for $-1 < \alpha \leq 0$,
\[V^\alpha G = \sum_{\alpha \leq v(\omega_k^{Delt})} \mathbb{C}[x][\theta, \theta^{-1}] \omega_k^{Delt} + x \sum_{\alpha > v(\omega_k^{Delt})} \mathbb{C}[x][\theta, \theta^{-1}] \omega_k^{Delt},\]
\[V^\alpha G = \sum_{\alpha < v(\omega_k^{Delt})} \mathbb{C}[x][\theta, \theta^{-1}] \omega_k^{Delt} + x \sum_{\alpha \geq v(\omega_k^{Delt})} \mathbb{C}[x][\theta, \theta^{-1}] \omega_k^{Delt}\]

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and $V^{\alpha+p}G = x^pV^\alpha G$ for $p \in \mathbb{Z}$ and $\alpha \in ]-1,0]$. This defines a decreasing filtration $V^\bullet$ of $G$ by $\mathbb{C}[x][\theta, \theta^{-1}]$-submodules. We will put $G_\alpha := V^\alpha G/V^{>\alpha}G$ and $\psi_x G := \oplus_{\alpha \in ]-1,0]} G_\alpha$. Notice that

$$V^{>-1}G = \sum_k \mathbb{C}[x][\theta, \theta^{-1}] \omega_{D_k}$$

By lemma 7.1.2 and the definition of the matrix $D$, it follows that $V^{>-1}G$ (resp. $V^{>k}G$, $k \in \mathbb{Z}$) is Deligne’s canonical extension of $G$ at $x = 0$ such that the eigenvalues of the residue are contained in $]-1,0]$ (resp. $]k,k+1]$).

Lemma 7.1.5 The filtration $V^\bullet$ is the Kashiwara-Malgrange filtration at $x = 0$.

Proof. It is directly checked that the filtration $V^\bullet$ satisfies all the characteristic properties of the Kashiwara-Malgrange filtration: the only point which is not completely obvious is the fact that $(x\nabla_{\partial_x} - \alpha)$ is nilpotent on $G_\alpha$, but this follows from lemma 7.1.2 because $\alpha_i \leq \alpha_j + 1$ and $d_i = d_j + 1$ imply $d_i - \alpha_i \geq d_j - \alpha_j$. □

Filtration $V^\bullet$ gives also an decreasing filtration $\psi_x G_0$ by $\mathbb{C}[\theta]$-submodules by $V^\alpha G_0 = V^\alpha G \cap G_0$ and $V^{>\alpha} G_0 = V^{>\alpha} G \cap G_0$.

We will write $G_{0,\alpha} = V^\alpha G_0/V^{>\alpha}G_0$ and $\psi_x G_0 := \oplus_{\alpha \in ]-1,0]} G_{0,\alpha}$. The metric $S$ in formula (46) induces, on each $V^{>-1}G_0$, a bilinear form

$$S : V^{>-1}G_0 \times V^{>-1}G_0 \to x^{2-1}\theta \mathbb{C}[x,\theta].$$

We have moreover

$$S : V^{>-1}G_0 \times V^{>-1}G_0 \to x^{2k}\theta \mathbb{C}[x,\theta]$$

if all the $\alpha_i$’s are integers.

The limit. We now construct a (the) limit of system (45) at $x = 0$, using the $V$-filtration at $x = 0$, and more precisely the nearby cycles. Let us define

$$\psi_x G_0 := \oplus_{\alpha \in ]-1,0]} V^\alpha G_0/V^{>\alpha}G_0.$$

It is a free $\mathbb{C}[\theta]$-module, equipped with a connection $\psi_x \nabla$. We will denote by $e^\psi$ the basis of $\psi_x G_0$ induced by $\omega^\text{Del}$.

Lemma 7.1.6 $x\nabla_{\partial_x}$ induces a map on $\psi_x G_0$ whose matrix, in the basis $e^\psi$, is

$$-\frac{\psi_x A_0}{\theta} - (A_\infty - D)$$

and the matrix of $\psi_x \nabla_{\partial_\theta}$ takes the form, in the same basis,

$$\frac{\psi_x A_0}{\theta^2} + \frac{A_\infty}{\theta}. \quad (48)$$

We have moreover $[A_\infty, \psi_x A_0] = \psi_x A_0$. 37
Proof. Follows from lemma 7.1.5: we have \((\psi_x A_0)_{ij} = (A_0^x)_{ij}\) if \(\alpha_i = \alpha_j + 1\), \((\psi_x A_0)_{ij} = 0\) otherwise: in other words \([A_\infty, \psi_x A_0] = \psi_x A_0\).

\[\text{Remark 7.1.7} \quad \text{As already quoted, the connection } \nabla^o \text{ has in general an irregular singularity at } \theta = 0 \text{ (see formula (41)) while our limit, that is system (48) has a regular singularity at } \theta = 0: \text{ indeed, after a base change of matrix } \theta^{-D}, \text{ it becomes} \]

\[\frac{1}{\theta}(A_\infty - D + \psi_x A_0).\]

\text{The construction of our limit thus gives a canonical "regularization" of system (41).}

We define now the limit metric on the \(\mathbb{C}[\theta]\)-free module \(\psi_x G_0\). Let \(\alpha \in [-1, 0]\). We have

\[S(V^\alpha G_0, V^{-1-\alpha} G_0) \subset x^{-1}\mathbb{C}[x, \theta]0^n\]

and this gives (compose the previous one with the residue at \(x = 0\)) a non-degenerate bilinear form

\[\psi_x S_\alpha : gr^0 G_0 \times j^* gr^0 G_0 \rightarrow \mathbb{C}[\theta]0^n\]

where \(\psi_x S_\alpha(e_i^\psi, e_j^\psi) = S^0(\omega_i^o, \omega_j^o)\). In the same way,

\[S(V^0 G_0, V^0 G_0) \subset \mathbb{C}[x, \theta]0^n\]

induces

\[\psi_x S_0 : gr^0 G_0 \times gr^0 G_0 \rightarrow \mathbb{C}[\theta]0^n\]

where \(\psi_x S_0(e_i^\psi, e_j^\psi) = S^0(\omega_i^o, \omega_j^o)\). All this gives the expected limit metric \(\psi_x S = \oplus_{\alpha \in [0, -1]} \psi_x S_\alpha\) on \(\psi_x G_0\).

\[\text{Lemma 7.1.8} \quad \text{The form } \psi_x S \text{ is } \psi_x \nabla\text{-flat.}\]

\[\text{Proof.} \quad \text{It is enough to show that } \psi_x A_0 \text{ is self-dual with respect to } \psi_x S. \text{ Because } (\psi_x A_0)_{ij} = (A_0^x)_{ij} \text{ if } \alpha_i = \alpha_j + 1 \text{ and } (\psi_x A_0)_{ij} = 0 \text{ otherwise, this follows from the following two facts: } A_0^x \text{ is self-adjoint with respect } S^0 \text{ and } \alpha_{\mu-k} = \alpha_i + 1 \text{ if and only if } \alpha_{\mu-i} = \alpha_k + 1 \text{ (by formula (43)).}\]

\[\text{Résumé (classical limit): By lemma 7.1.6, the basis } e^\psi \text{ gives an extension } G^{cl} \text{ of } \psi_x G_0 \text{ as a trivial bundle on } \mathbb{P}^1, \text{ equipped with a meromorphic connection with poles of rank less or equal to } 1 \text{ at } \theta = 0 \text{ and with logarithmic pole at } \theta = \infty \text{ (see formula (48)).}\]

\[\text{Proposition 7.1.9} \quad \text{The triple} \]

\[Q^{cl} = (G^{cl}, \psi_x \nabla, \psi_x S)\]

\[\text{is a quantum differential system. This is the classical limit of the quantum differential system } Q.\]

The quantum differential system \(Q^{cl}\) is the classical limit of the quantum differential system \(Q\).
Remark 7.1.10 One could also consider the free \( \mathbb{C}[\theta] \)-module of rank \( \mu \)

\[
\mathcal{L} := V^{-1}G_0/V^0G_0 = V^{-1}G_0/xV^{-1}G_0
\]

It is naturally equipped with a connection \( \nabla \) induced by \( \nabla \) whose matrix in the basis \( e \) induced by \( \omega^{\text{Del}} \) is

\[
\left( A_0^{\text{Del}} + A_\infty \right) \frac{d\theta}{\theta}
\]

We have also \( S(V^{-1}G_0, V^{-1}G_0) \subset x^{-1}\mathbb{C}[[x, \theta]]\theta^n \) and we thus get (composing with the residue at \( x = 0 \))

\[
\mathcal{S} : \mathcal{L} \times \mathcal{L} \to \mathbb{C}[\theta]\theta^n
\]

These data could also define a limit: the point is that \( \mathcal{S} \) is not always \( \nabla \)-flat. This is what happens for instance for \( f(u_1, u_2) = u_1 + u_2 + \frac{1}{u_1^5 u_2^5} \), in which case the matrix \( A_0^{\text{Del}} \) is not self-dual. Indeed, by example 2.2.1, we have in this situation \( \mu = 8 \) and

\[
(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (0, 1, 2, 7/5, 4/5, 1, 6/5, 3/5)
\]

The matrix \( A_0^{\text{Del}} \) is defined by

\[
A_0^{\text{Del}}(e_i) = \begin{cases} 
8e_{i+1} & \text{if } i = 0, 1 \text{ and } 4 \\
0 & \text{otherwise}
\end{cases}
\]

(49)

while the matrix \( \psi_x A_0 \) is defined by

\[
\psi_x A_0(e_i) = \begin{cases} 
8e_{i+1} & \text{if } i = 0 \text{ and } 1 \\
0 & \text{otherwise}
\end{cases}
\]

(50)

On the other hand, we have

\[
\mathcal{S}(A_0^{\text{Del}}(e_4), e_5) = 8/w^w \text{ and } \mathcal{S}(e_4, A_0^{\text{Del}}(e_5)) = 0.
\]

In particular, \( A_0^{\text{Del}} \) is not self-dual.

7.2 The small quantum cohomology of manifolds

Let \( X \) be a projective manifold with cohomology only in even degree. Let us assume that the rank of \( H^2(X, \mathbb{Z}) \) is equal to 1. Let \( \mathcal{Q} \) be the logarithmic quantum differential system on \( M = \mathbb{C} \) associated with the small quantum cohomology of \( X \) by example 2.1.7.

Proposition 7.2.1 1. The quantum differential system \( \mathcal{Q} \) is flat and logarithmic.

2. There exists a canonical fundamental solution \( P(x, \tau) = H(x, \tau)e^{-\tau M(0) \ln x} \) of the Dubrovin connection of the quantum differential system \( \mathcal{Q} \), uniquely characterized by the initial condition \( H(0, \tau) = I \).
Proof. 1. Follows from the definitions. 2. By 1. and theorem 6.3.4, the solution \( P \) is fundamental and symmetric. It remains to show “conformality”, and we keep the notations of section 6.2: the matrix \( M_1(0) \) is by definition the matrix of \( p\cup \) in a suitable basis of the cohomology algebra, while the matrix \( N_1(0) \) represents the multiplication (with respect to the cup-product) by \( E_{|x=0} \) in the same basis. Now, \( E_{|x=0} = c_1(TX) \) so that \( N_1(0) = r M_1(0) \) for some \( r \in \mathbb{Z} \) and we get the assertion using theorem 6.3.4.  \( \square \)

Remark 7.2.2 Denote by \( p^i := p \cup \cdots \cup p \) the iteration (i-time) by the usual cup-product \( \cup \). We will say that \( H^*(X) \) is \( H^2 \)-generated if \( p \) and its iterations \( p^i \) are a basis of it. In this case, the matrix \( M_1(0) \) is regular and the condition (GC) in theorem 2.3.2 is satisfied.

7.3 The small quantum orbifold cohomology of weighted projective spaces

We now come back to the quantum differential system of example 2.2.1. We thus have a basis (see section 9.1.2 below for a precise definition of this basis) of the Brieskorn lattice in which the matrix of the Gauss-Manin connection takes the form

\[
(M_0(x) + M_1(x)\tau)\frac{dx}{x} - (A_0(x)\tau + A_\infty)\frac{d\tau}{\tau}
\]

where

\[
M_1(x) = - \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & x \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

which is a \( \mu \times \mu \) matrix with \( \mu = 1 + w_1 + \cdots + w_n \),

\[
M_0(x) = diag(c_0, c_2, \cdots, c_{\mu-1}), \quad A_0(x) = -\mu M_1(x), \quad \text{and} \quad A_\infty = diag(\alpha_0, \alpha_1, \cdots, \alpha_{\mu-1}), \quad (52)
\]

the \( c_i \)'s being rational numbers contained in \([0, 1]\). This quantum differential system is thus non-resonant at \( \tau_0 = 0 \) but does not give directly a canonical fundamental solution, for a conformality reason: indeed, \([A_\infty, A_0(0)] \neq A_0(0) \) in general. Nevertheless, one can get a flat quantum differential system as follows: let us put

\[
r = \frac{1}{l.c.m(w_0, \cdots, w_n)}
\]

and \( \zeta = x^r \); the characteristic equation (51) takes the form, in the basis \( \tilde{\omega} := \omega x^{-R} \) of \( G_0[x^r] \),

\[
\tau \tilde{M}_1(\zeta) \frac{d\zeta}{\zeta} - (\tilde{A}_0(\zeta)\tau + A_\infty)\frac{d\tau}{\tau}
\]

where

\[
\tilde{A}_0(\zeta) = \mu \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & \zeta^{(1-c_{\mu-1})/r} \\
\zeta^{(c_{1-c_0})/r} & 0 & 0 & \cdots & 0 & \zeta^{(1-c_{\mu-1})/r} \\
0 & \zeta^{(c_{2-c_1})/r} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \zeta^{(c_{\mu-1-c_{\mu-2})/r}} \\
0 & 0 & \cdots & \cdots & \zeta^{(c_{\mu-1-c_{\mu-2})/r}} & 0
\end{pmatrix}
\]

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and $\tilde{M}_1(\zeta) = -\frac{1}{r_\mu} \tilde{A}_0(\zeta)$. The matrix $\tilde{A}_0(\zeta)$ has polynomial coefficients, see formula (63) below. The metric $S^B$ is defined, in the basis $\tilde{\omega}$, by

$$S^B(\tilde{\omega}_k, \tilde{\omega}_\ell) = \begin{cases} \frac{1}{w_0 \cdots w_n} \theta^n & \text{if } 0 \leq k \leq n \text{ and } k + \ell = n, \\ \frac{1}{w^* w_0 \cdots w_n} \theta^n & \text{if } n + 1 \leq k \leq \mu - 1 \text{ and } k + \ell = \mu + n, \\ 0 & \text{otherwise} \end{cases} \quad (54)$$

**Proposition 7.3.1** The matrix $P(\zeta, \tau) = H(\zeta, \tau)e^{-\tau \tilde{M}_1(0) \ln \zeta}$, where $H(\zeta, \tau)$ is the matrix associated with the characteristic equation (53) by lemma 6.3.2, is a canonical fundamental solution of the Dubrovin connection.

**Proof.** One uses assertions (1), (2) et (3) of theorem 6.3.4. Conformality follows from formulas $[A_\infty, \tilde{M}_1(0)] = \tilde{M}_1(0)$ and $\tilde{M}_1(0) = -\frac{1}{r_\mu} \tilde{A}_0(0)$ (see theorem 6.3.4 5.) and the symmetry follows from proposition 5.2.3. $\square$

### 8 Rational structures via quantum differential systems

We apply here the previous results in order to construct first a distinguished rational structure on the $B$-side. Then, we derive from this one a rational structure on the $A$-side. This provides a generalization of the method exposed in [24, Proposition 3.1] (see also [41]).

**8.1 Preamble: sketch of the method**

Let $Q = (M, \mathcal{G}, \nabla, S, d)$ be a flat logarithmic quantum differential system on $M = \mathbb{C}$. We will denote by

$$\tau M_1(x) \frac{dx}{x} - (A_\infty + \tau A_0(x)) \frac{d\tau}{\tau} \quad (55)$$

the matrix $A_\infty$ of the connection $\nabla$ in the basis $\omega = (\omega_0, \cdots, \omega_{\mu-1})$. Let us summarize the previous results: starting from the trivial bundle $\mathcal{G}$ on $\mathbb{P}^1 \times M$, one constructs a trivial bundle $\mathcal{G}^{cl}$ on $\mathbb{P}^1$ (the limiting bundle, see section 6.4) whose fiber $\mathcal{G}^{const} := \mathcal{G}^{cl}_{\{\tau = -1\}}$ at $\tau = -1$ is a finite dimensional vector space.

**Proposition 8.1.1** Let us denote $\mathcal{G}^{\nabla} = \ker \nabla$ and $\mathcal{G}^{cl, \nabla^{cl}} = \ker \nabla^{cl}$. Let us assume that the fundamental solution $P(\tau, x) = H(\tau, x)e^{-\tau \tilde{M}_1(0) \ln x}$ is canonical with $H(0, \tau) = I$. Then we have isomorphisms

$$\mathcal{G}^{\nabla}_{\Psi(\tau, x)} \rightarrow \mathcal{G}^{\nabla^{cl}}_{\Psi^{cl}(\tau)} \rightarrow \mathcal{G}^{const}_{\Psi^{const}}$$

where

- $\Psi(\tau, x) = H(\tau, x)e^{-\tau \tilde{M}_1(0) \ln x} \Psi^{cl}(\tau)$,
- $\Psi^{cl}(\tau) = (-\tau)^{A_\infty} (-\tau)^{A_0(0)} \Psi^{const}$.

---

15 Keeping in mind orbifold Poincaré duality, we choose the normalization $S^B(\omega_0, \omega_n) = \frac{1}{w_0 \cdots w_n} \theta^n$.

16 The matrix $A_\infty$ is constant because the quantum differential system is flat.
\( \Psi^{const} \) being a constant vector.

**Proof.** See corollary 6.3.6. \( \square \)

Using proposition 8.1.1, one can thus shift on \( G^{const} \) the natural structures of \( G^\nabla \) (and vice versa): this is one of the interest of the quantization. Assume for instance that one has a distinguished rational structure \( \Sigma^{quant}_Q \) on \( G^{quant} \): this structure shifts to a rational structure \( \Sigma^{const}_Q \) on \( G^{const} \), but also on its mirror partners (if any).

### 8.2 Rational structures via mirror symmetry and quantization: from the B-side to the A-side

Let us start from the B-side and assume that the quantum differential system

\[ Q^B = (M^B, G^B, \nabla^B, S^B, n) \]

is produced, as in section 2.2, by a function \( F \) on \( U \times M^B \) (\( n = \dim \mathbb{C} U \)) such that

- \( F(\bullet, x) \) is a tame regular function on \( U \) for all \( x \in M^B \),
- the global Milnor number of the function \( F(\bullet, x) \) does not depend on \( x \in M^B \) (we will denote by \( \mu \) this constant value),
- its Brieskorn lattice \( G_0 \) is free of rank \( \mu \) over \( \mathcal{O}_{M^B}(M^B)[\theta] \)

Typically, \( M^B = \mathbb{C}^* \), \( F(\bullet, x) \) is a convenient and nondegenerate Laurent polynomial for all \( x \in M^B \) (with the same Newton polyhedron at infinity for all \( x \in M^B \)) and \( G_0 \) is a free \( \mathbb{C}[x, x^{-1}, \theta] \)-module. We will consider this situation in section 9.

#### 8.2.1 Oscillating integrals

We will denote by \( \omega = (\omega_0, \ldots, \omega_{\mu - 1}) \) the basis of the Brieskorn lattice \( G_0 \), adapted to \( S \) (see formula (115)), in which the connection \( \nabla^B \) takes the form (55).

On the B-side, the relation between the basis \( \omega \) and the rational structure is given by the oscillating integrals

\[ I^{(i)}_\Gamma(\tau, x) = \int_{\Gamma} e^{\tau F} \omega_i \]

where \( \Gamma \) is a cycle with support on a “family of supports” \( \Phi \) as in [30] Section 1]. The integral depends only on the homology class of \( \Gamma \) in the nth homology group (with integral coefficients) with support in \( \Phi \). Let us be more precise about that: fix \( (\tau, x) \in \mathbb{C}^* \times M^B \). The homology group alluded to is \( H_n^{\Phi_{\tau,x}}(U, \mathbb{Z}) \) where \( \Phi_{\tau,x} \) is a family of supports \( A \subset U \) such that

\[ \text{Re}(\tau F(\bullet, x))|_A \to -\infty \]

as \( u \to +\infty \) or 0. We have

\[ H_n^{\Phi_{\tau,x}}(U, \mathbb{Z}) = H_n(U, \text{Re}(\tau F) < C; \mathbb{Z}) \]

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for \( C \ll 0 \) (see [31, Formula (1.0), p. 13]). Because \( F(\bullet, x) \) is tame, this is a free \( \mathbb{Z} \)-module of rank \( \mu \), the global Milnor number of \( F \). If the critical points of \( F(\bullet, x), x \in M \), are nondegenerate and the critical values are distinct, the cycles \( \Gamma \) are called Lefschetz thimbles [30, 1.5]. We will denote by \( H^{\Phi}_{n}(U, Q) \) (resp. \( H^{\Phi}_{n}(U, \mathbb{C}) \)) the \( Q \)-resp. \( \mathbb{C} \)-vector space generated by the linear combinations with rational coefficients of such cycles and we will assume that these vector spaces are organized into a local system \( H^{\Phi}_{n}(\mathbb{C}) \) (resp. \( H^{\Phi}_{n}(\mathbb{C}) \)) of \( Q \)-resp. \( \mathbb{C} \)-vector spaces on \( \mathbb{C}^* \times M^B \); this follows from the assumptions above and this will be the case in our situation, see section 9.1 below but also f.i [30, 1.5], [31, 4.1] and [23, Proposition 3.12].

### 8.2.2 Flat sections

We will denote by \( A^\top \) the transposed of a matrix \( A \). If \( \Omega(x, \tau) \) is the matrix of the connection \( \nabla^B \) in the basis \( \omega \), \( \nabla^{B,\ast} \) will denote the connection with matrix \(-\Omega^\top(x, -\tau)\) in the same basis.\footnote{The twist by the minus sign is explained by the fact that we consider the kernel \( e^{\tau f} \) instead of \( e^{-\tau f} \).}

**Lemma 8.2.1** The local system \( H^{\Phi}_{n}(\mathbb{C}) \) is identified with \( \mathcal{G}^{\nabla^{B,\ast}} := \ker \nabla^{B,\ast} \) via the map \( \Psi^* \) defined by

\[
\Psi^*(\Gamma)(\tau, x) = \sum_{i=0}^{\mu-1} I^{(i)}_\Gamma(\tau, x) \omega_i. \tag{56}
\]

*Proof.* Write \( I^\Gamma(\tau, x) = (I^{(0)}_\Gamma(\tau, x), \ldots, I^{(\mu-1)}_\Gamma(\tau, x)) \). We have

\[
dI^\Gamma(\tau, x)^\top = \Omega^\top(x, -\tau) I^\Gamma(\tau, x)^\top, \tag{57}
\]

where \( \Omega(x, \tau) \) is the matrix of \( \nabla^B \) in the basis \( \omega \) (this is precisely what \( \nabla^B \) is made for, see [31, 1ère partie, 6]) and this shows that the map is well defined. The fact that it is an isomorphism follows from a dimension argument: the assumption on the Brieskorn lattice shows that \( \mathcal{G}^B_{|\mathbb{C}^* \times M^B} \) is a connection and its solutions are therefore organized in a local system on \( \mathbb{C}^* \times M \) whose fiber at \((\tau, x)\) is, thanks to the tameness, \( H^{\Phi,\ast}_{n}(U, \mathbb{C}) \) (see [30, 37]). \( \square \)

In some cases, this general construction will give an identification between \( H^{\Phi}_{n}(\mathbb{C}) \) and \( \mathcal{G}^{\nabla^{B}} \):

**Corollary 8.2.2** Assume that the basis \( \omega \) is adapted to the bilinear form \( S^B \) and let \( \omega^j \) be the dual of \( \omega_j \) with respect to \( S^B \) (see remark 2.1.4). The map \( \Psi : H^{\Phi}_{n}(\mathbb{C}) \rightarrow \mathcal{G}^{\nabla^{B}} \) defined by

\[
\Psi(\Gamma)(\tau, x) = (-\tau)^n \sum_{i=0}^{\mu-1} I^{(i)}(\tau, x) \omega^i. \tag{58}
\]

is an isomorphism. In particular, \( \mathcal{O}_{\mathbb{C}^* \times M^B} \otimes H^{\Phi}_{n}(\mathbb{C}) \sim \mathcal{G}_{|\mathbb{C}^* \times M^B} \).

*Proof.* Because the matrices \( A_\infty \) and \( A_0(x) \) in equality (55) satisfy \( A_\infty + A_0^* = nI \) and \( A_0(x)^* = A_0(x) \) where \( \ast \) denotes the adjoint with respect to \( S^B \), see remark 2.1.4 \( \square \)

**Notation 8.2.3** From now on, we will write \( \Psi^\Gamma(\tau, x) \) instead of \( \Psi(\Gamma)(\tau, x) \).
8.2.3 Rational structures (A-side/ B-side)

As announced, we define:

**Definition 8.2.4** The image $\Sigma^{B,\text{quant}}_Q$ of $H^n_Q(\mathbb{Q})$ in $G^B := \ker \nabla^B$ under the isomorphism $\Psi$ is called the rational structure on $G^B$.

In other words, $\Sigma^{B,\text{quant}}_Q$ is the $\mathbb{Q}$-lattice generated by the solutions obtained after integration over cycles which are linear combinations, with rational coefficients, of the Lefschetz thimbles.

**Corollary 8.2.5**

1. The rational structure $\Sigma^{B,\text{quant}}_Q$ on $G^B$ provides a rational structure $\Sigma^{B,\text{const}}_Q$ on $G^B$ via the isomorphisms of proposition 8.1.1. Moreover, $\Sigma^{B,\text{quant}}_Q$ is completely determined by $\Sigma^{B,\text{const}}_Q$.

2. Let $Q^A = (M^A, S^A, \nabla^A, \psi^A, n)$ be a quantum differential system isomorphic to the quantum differential system $Q^B$ in the sense of definition 2.4.1. Then the rational structure $\Sigma^{B,\text{quant}}_Q$ on $G^B$ defines a rational structure $\Sigma^{A,\text{quant}}_Q$ on $G^A := \ker \nabla^A$ via this isomorphism.

**Proof.** Follows from proposition 8.1.1.

The challenge is then

- to understand $\Sigma^{B,\text{const}}_Q$,

- to describe the rational structure $\Sigma^{A,\text{const}}_Q$ on the cohomology of the mirror partner using a mirror theorem and to get a rational structure $\Sigma^{A,\text{quant}}_Q$ on $\ker \nabla^A$.

9 Application: rational structure for weighted projective spaces and their Landau-Ginzburg models.

We apply in this section the previous recipe for the weighted projective spaces $\mathbb{P}(1, w_1, \cdots, w_n)$ and their Landau-Ginzburg models. The main results of this section are theorem 9.1.3 (description of the rational structure on the $B$-side) and its corollary, theorem 9.2.2 (description of the rational structure on the $A$-side).

9.1 $B$-side

In what follows, we will use the notations of example 2.2.1.

9.1.1 The setting. Combinatorics.

Recall the function $F$ defined by

$$F(u_1, \cdots, u_n, x) = u_1 + \cdots + u_n + \frac{x}{u_1 \cdots u_n}.$$
on \( U \times \mathbb{C}^* \) where \( U = (\mathbb{C}^*)^n \) and \( w_1, \cdots, w_n \) are positive integers. For each \( x \in \mathbb{C}^* \), the function

\[
(u_1, \cdots, u_n) \mapsto F(u_1, \cdots, u_n, x)
\]

has \( \mu \) non-degenerate critical points\(^{18}\) with distinct critical values and satisfies the assumptions of the beginning of section 8.2.

We will need the following combinatorial tools: let

\[
\mathcal{F} = \{ \frac{i}{w_j} | 0 \leq i \leq w_j - 1, \ 0 \leq j \leq n \} = \{ f_0, \cdots, f_k \}
\]

(we put \( w_0 = 1 \)), the numbers \( f_\ell \) satisfying \( 0 = f_0 < \cdots < f_k < 1 \). Define

\[
I_\ell = \{ j \in [0, n], \ w_j f_\ell \in \mathbb{Z} \}
\]

and let \( d_\ell \) be its cardinal. We will write

\[
p_\ell = d_0 + \cdots + d_\ell
\]

and \( p_{-1} = 0 \). Last, let \( c_0, c_1, \cdots, c_{\mu-1} \) be the sequence

\[
\begin{array}{c}
\underbrace{f_0, \cdots, f_0,}^{d_0} \underbrace{f_1, \cdots, f_1,}^{d_1} \cdots \underbrace{f_k, \cdots, f_k,}^{d_k}
\end{array}
\]

arranged in increasing order. This sequence can be described as follows (see [12, p. 3]): define inductively the sequence \( (a(k), i(k)) \in \mathbb{N}^{n+1} \times \{ 0, \cdots, n \} \) by \( a(0) = (0, \cdots, 0) \), \( i(0) = 0 \) and

\[
a(k + 1) = a(k) + 1_{i(k)} \text{ where } i(k) := \min \{ i | a(k)_i / w_i = \min_j a(k)_j / w_j \}.
\]

Then we have

\[
c_k = a(k)_{i(k)} / w_{i(k)}.
\]

Notice that

- \( a(1) = (1, 0, \cdots, 0) \),
- \( a(n + 1) = (1, \cdots, 1) \),
- \( a(\mu) = (1, w_1, \cdots, w_n) \),
- \( \sum_{i=0}^{n} a(k)_i = k. \)

\(^{18}\)Recall that \( \mu = 1 + w_1 + \cdots + w_n. \)
9.1.2 A flat quantum differential system and a canonical fundamental solution

In order to apply the results of section 8, we need first a flat quantum differential system. It is provided by section 7.3. Let us recall the setting: example 2.2.1 provides a basis \( \omega = (\omega_0, \ldots, \omega_{\mu-1}) \) of the Brieskorn lattice in which the matrix of the Gauss-Manin connection takes the form (see equation (51))

\[
(R - \frac{1}{\mu} A_0(x) \frac{dx}{x} - (A_0(x) \tau + A_\infty) \frac{d\tau}{\tau})
\]

(64)

This basis is precisely defined as follows: we have, for \( k = 1, \ldots, \mu - 1 \),

\[
\omega_k = \frac{x}{w_1^{a(k)_1} \cdots w_n^{a(k)_n}} u_1^{a(k)_1 - w_1} \cdots u_n^{a(k)_n - w_n} \omega_0
\]

(65)

where \( g_0 \) denotes the class of \( g \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n} \) in the Gauss-Manin system \( G_0 \) of \( F \), see [10]. Notice that in this situation \( G \) is a free \( \mathbb{C}[x, x^{-1}, \tau, \tau^{-1}] \)-module equipped with a connection \( \partial_\tau \) and thus \( H^\bullet_n(\mathbb{C}) \) is a local system on \( \mathbb{C}^* \times M, M = (\mathbb{C})^* \). The flat quantum differential system alluded to is the following: put

\[
r = \frac{1}{l.c.m(w_1, \ldots, w_n)}
\]

(66)

and \( \zeta = x^r \); the characteristic equation (51) takes the form (see equation (53))

\[
\tau \tilde{M}_1(\zeta) \frac{d\zeta}{\zeta} - (\tilde{A}_0(\zeta) \tau + A_\infty) \frac{d\tau}{\tau}
\]

(67)

in the basis \( \tilde{\omega} := \omega x^{-R} \) of \( G_0[\zeta] \), where\(^{19}\)

\[
\tilde{A}_0(\zeta) = \mu
\]

\[
\begin{pmatrix}
\zeta^{(e_1 - c_0)/r} & 0 & 0 & \cdots & 0 & \zeta^{(1 - c_{\mu-1})/r} \\
0 & \zeta^{(e_2 - c_1)/r} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \zeta^{(e_{\mu-1} - c_{\mu-2})/r} & 0 & 0
\end{pmatrix}
\]

\(\tilde{M}_1(\zeta) = -\frac{1}{\mu \tau} \tilde{A}_0(\zeta) \). Notice that the matrix \( \tilde{M}_1(0) \) has \( k + 1 \) Jordan blocks \( B_0, \ldots, B_k \), all associated with the eigenvalue 0, of respective size \( d_\ell \) for \( \ell = 0, \ldots, k \).

By proposition 7.3.1, the matrix \( \tilde{P}(\zeta, \tau) = H(\zeta, \tau) e^{-\tau \tilde{M}_1(0) \ln \zeta} \), where \( H(\zeta, \tau) \) is the matrix associated with the characteristic equation (53) by lemma 6.3.2, is a canonical fundamental solution of the Dubrovin connection. We are now able to apply the results of section 8.

9.1.3 Definition of the rational structures \( \Sigma^{B, const}_Q \) and \( \Sigma^{B, quant}_Q \)

Let us define

\[
H^i(\Gamma, x) = \int_G e^{\tau x - c_i \tilde{\omega}_i}
\]

where \( \tilde{\omega}_i \) denotes a representative of \( \omega_i \) in \( \Omega^n((\mathbb{C}^*)^n)[\tau, \tau^{-1}] \) and \( \Gamma \in H^\bullet_n(\mathbb{Q}) \).

\(^{19}w^w = w_1^{w_1} \cdots w_n^{w_n}\)
We will denote by $\tilde{i}$ the index defined by
\begin{equation}
\tilde{i} = \begin{cases} n - i & \text{if } 0 \leq i \leq n, \\ \mu + n - i & \text{if } n + 1 \leq i \leq \mu - 1 \text{ and } k + \ell = \mu + n \end{cases}
\end{equation}

and
\begin{equation}
\tilde{I}^{(i)}(\tau, x) = \begin{cases} wI^{(i)}(\tau, x) & \text{if } 0 \leq i \leq n, \\ w^{i+1}I^{(i)}(\tau, x) & \text{if } n + 1 \leq i \leq \mu - 1 \end{cases}
\end{equation}
in order to take into account corollary 8.2.2 in the light of formula (54). We put $w := w_1 \cdots w_n$ and $w^{i+1} := w^{i+1}_1 \cdots w^{i+1}_n$.

Lemma 9.1.1
1. The section
\[(\omega_0 x^{-c_0}, \cdots, \omega_{\mu-1} x^{-c_{\mu-1}}) \Psi_{\Gamma}(x, \tau)^T,\]
where
\[
\Psi_{\Gamma}(\tau, x) = (-\tau)^{n}(\tilde{I}_{\Gamma}^{(0)}(\tau, x), \cdots, \tilde{I}_{\Gamma}^{(\mu-1)}(\tau, x))
\]
is a flat section of $\nabla^B$.

2. We have
\[
\Psi_{\Gamma}(\tau, \zeta)^T = H(\zeta, \tau) \exp(-\tau \tilde{M}_1(0) \ln \zeta) \Psi_{\Gamma}^{cl}(\tau)^T
\]
where $H(0, \tau) = I$ and
\[
\Psi_{\Gamma}^{cl}(\tau)^T = (-\tau)^{A_{\infty}(\tau)}(\Psi_{\Gamma}^{const})^T,
\]
$\Psi_{\Gamma}^{const}$ being a constant vector, the superscript $^T$ denoting the transpose vector.

Proof. The first assertion follows from corollary 8.2.2 and formula (54). The second one is then a consequence of proposition 8.1.1.

Corollary 9.1.2 The rational structure $\Sigma^{B, const}_Q$ is the $Q$-vector subspace of $G^{R, const}$ generated by the vectors $\Psi_{\Gamma}^{const}$, $\Gamma \in H^S_n(Q)$.

Proof. Use definition 8.2.4 and corollary 8.2.5.

Recall that $\Sigma^{B, const}_Q$ determines the rational structure $\Sigma^{B, quant}_Q$ of definition 8.2.4.

9.1.4 Description of $\Sigma^{B, const}_Q$

In order to get an explicit description of $\Sigma^{B, const}_Q$, we thus have to compute the vectors $\Psi_{\Gamma}^{const}$. First, the identification of the Lefschetz thimbles can be done as in [30, 1.5, p.323]: let
\[
\Gamma_0 = \mathbb{R}^*_+ \times \cdots \times \mathbb{R}^*_+ \subset (\mathbb{C}^n)^n.
\]
This is a Lefschetz thimble\(^{20}\) and other such cycles are \(\Gamma_\ell := e^\ell \Gamma_0\) where, as above, \(\epsilon\) is a \(\mu\)-th primitive root of 1. According to lemma 9.1.1, the section \(\Gamma_\ell\) determines a classical vector

\[
\Psi^c_{\Gamma_\ell} := (\Psi^{c,0}_{1,\ell}, \ldots, \Psi^{c,0}_{n+1,\ell}, \Psi^{c,1}_{1,\ell}, \ldots, \Psi^{c,1}_{d_1,\ell}, \ldots, \Psi^{c,k}_{1,\ell}, \ldots, \Psi^{c,k}_{d_k,\ell})
\]

and a constant vector

\[
\Psi^{\text{const}}_{\Gamma_\ell} := (\Psi^0_{1,\ell}, \ldots, \Psi^0_{n+1,\ell}, \Psi^1_{1,\ell}, \ldots, \Psi^1_{d_1,\ell}, \ldots, \Psi^k_{1,\ell}, \ldots, \Psi^k_{d_k,\ell}).
\]

In both cases, the superscripts recall the Jordan blocks. The description of \(\Sigma^B_{\text{const}}\) is then given by the following theorem: the first and the second part say that it is enough to compute the \(\Psi^c_{1,0}\)'s and this is done in the third part using a Mellin transform and a trick already used in \([24]\).

**Theorem 9.1.3**

1. The rational structure \(\Sigma^B_{\text{const}}\) is the \(\mathbb{Q}\)-vector subspace of \(\mathcal{G}^B_{\text{const}}\) generated by the vectors

\[
\Psi^c_{\Gamma_\ell} := (\Psi^{c,0}_{1,\ell}, \ldots, \Psi^{c,0}_{n+1,\ell}, \Psi^{c,1}_{1,\ell}, \ldots, \Psi^{c,1}_{d_1,\ell}, \ldots, \Psi^{c,k}_{1,\ell}, \ldots, \Psi^{c,k}_{d_k,\ell}),
\]

\(\ell = 0, \ldots, \mu - 1\).

2. We have

\[
\Psi^j_{i,\ell} = e^{2i\pi f_j} \sum_{m=0}^{i-1} \frac{(-2i\pi \ell)^m}{m!} \Psi^j_{i-m,0}
\]

for \(j = 0, \ldots, k, i = 1, \ldots, d_j\) and \(\ell = 0, \ldots, \mu - 1\).

3. We have, for \(j = 1, \ldots, k\),

\[
b_j\Gamma(rs + f_j) \prod_{m=1}^{n} \Gamma(w_m(rs - (1 - f_j)) + a(\mu + n - p_j + 1)m)
\]

\[
= \sum_{m=0}^{d_j-1} (rs)^{-m-1} \Psi^j_{d_j-m,0} + O(1)
\]

as \(s \to 0\), where the sequence \((a(k))\) is defined by formula \((62)\), the number \(r\) by formula \((66)\) and

\[
b_j = \frac{w_1^{w_1+1} \cdots w_n^{w_n+1}}{w_1^{a(\mu+n-p_j+1)+1} \cdots w_n^{a(\mu+n-p_j+1)n}}
\]

For \(j = 0\), we have

\[
w_1 \cdots w_n \prod_{m=0}^{n} \Gamma(w_mrs) = \sum_{m=0}^{n} (rs)^{-m-1} \Psi^0_{n+1-m,0} + O(1)
\]

as \(s \to 0\).

\(^{20}\)Strictly speaking, a section over \(\mathbb{R}_+ \times \mathbb{R}_+\) of such a cycle: notice that, for \(x > 0\), \(\Gamma_0\) contains one critical point, namely \((e^{x\mu}/\mu)^{1/\mu}(w_1, \ldots, w_n)\) and \(F|_{\Gamma_0}\) is proper and takes values in \([\mu(e^{x\mu})^{1/\mu}, +\infty[\).
Proof. 1. This is corollary 9.1.2.
2. Let $\epsilon$ be a $\mu$-th primitive root of 1. From the homogeneity condition $e^{-\ell}F(u, \zeta) = F(e^{-\ell}u, \zeta)$, we first get, using formula (65) and the fact that $\sum_{i=0}^{n} a(s)_i = s$,
\[
\int e^{\epsilon \tau F(u, \zeta)} \tilde{w}_s = \epsilon \int \Gamma(\tau, \zeta) e^{\epsilon \tau F(u, \zeta)} \tilde{w}_s
\]
for $s = 0, \ldots, \mu - 1$. Let $i = p_j - 1 + i - 1$, for $j \in \{0, \ldots, k \}$ and $i \in \{1, \ldots, d_j \}$. We thus have, using moreover equations (70) and (71),
\[
\sum_{m=0}^{i-1} \psi_{i-m, \ell}(\epsilon^{-\ell} \tau) \ln m \frac{\zeta(\epsilon^{-\ell} \tau)}{r} m = e^{-\epsilon} \sum_{m=0}^{i-1} \psi_{i-m, 0}(\tau) \ln m \frac{\zeta(\tau)}{r} m,
\]
see equation (68) for the definition of $T$. It follows that
\[
\psi_{i, \ell}^{(m, \ell)}(\epsilon^{-\ell} \tau) = e^{-\epsilon} \psi_{i, 0}^{(m, \ell)}(\tau)
\]
Now, the eigenvalue $\alpha_i$ of $A_\infty$ satisfies
\[
\alpha_i + \ell - n = \alpha_i - i = -\mu f_j
\]
if $0 \leq i \leq n$ and
\[
\alpha_i + \ell - n = \alpha_i - i + \mu = \mu(1 - f_j)
\]
otherwise. We deduce, using equation (72) and the fact that $\tilde{A}_0(0) = -\mu \tilde{M}_1(0)$ (see section 9.1.2),
\[
\sum_{m=0}^{i-1} \psi_{i-m, \ell}(\mu \ln(-\tau) + 2i\pi \ell) m = e^{2i\pi \ell f_j} \sum_{m=0}^{i-1} \psi_{i-m, 0}(\mu \ln m(-\tau) m,
\]
3. We have, for fixed indices $i$ and $\nu$, using again formula (65),
\[
(-\tau)^n \int_{0}^{+\infty} \int \Gamma(\tau, \zeta) \zeta^{-f_\nu/r} \tilde{w}_i \zeta^{a} d\zeta
\]
\[
= \frac{1}{w_1^{a(i)} \cdots w_n^{a(i)}} (-\tau)^{-\mu(r \nu - f_\nu) - i + n} \Gamma(r \nu + 1 - f_\nu) \prod_{m=1}^{n} \Gamma(w_m(r \nu - f_\nu) + a(i)_m).
\]
and we thus get, taking into account formulas (69), (70), (71) and using regularization,
\[
b_j(-\tau)^{-u(r \nu + 1 - f_j) + p_j - 1} \Gamma(r \nu + 1 + f_j) \prod_{m=1}^{n} \Gamma(w_m(r \nu - (1 - f_j)) + a(\mu + n - p_j + 1)_m)
\]
\[
= \sum_{m=0}^{d_j-1} (-\tau)^{-m} \psi_{d_j-m, 0}^{(m, \ell)}(\tau) + O(1)
\]
The expected equality follows putting $\tau = -1$. Similar computations for the case $j = 0$. Notice that we have used here the fact that the good differential forms to consider are the $\zeta^{-f_\nu/r} \tilde{w}_i$'s (and not only the $\tilde{w}_i$'s). \qed
Using an expansion in power series, we see that the numbers \( \Psi_{j,m,0}, j = 1, \ldots, k, m = 1, \ldots, d_j \), are determined by equation (75) while the numbers \( \Psi_{0,m,0}, j = 0, \ldots, k, m = 1, \ldots, n + 1 \), are determined by equation (77).

We have the following closed formula for the \( \Psi_{1,\ell} \)'s: let
\[
C_j = \{ m \in [1, n], w_m(1 - f_j) = a(\mu + n - p_j + 1)_m \}
\]
for \( j = 0, \ldots, k \).

**Corollary 9.1.4** We have
\[
\Psi_{1,0} = b_j \prod_{m \in C_j} \frac{1}{w_m} \prod_{m=0}^{n} \Gamma(1 - \{w_m f_j\})
\]
for \( j = 1, \ldots, k \) where the \( b_j \)'s are defined by formula (76) and \( \{r\} = [r] - r \). We have also \( \Psi_{1,0} = 1 \). In particular \( \Psi_{1,\ell} \neq 0 \) for \( j = 0, \ldots, k \) and \( \ell = 0, \ldots, \mu - 1 \).

**Proof.** The first formula follows from formula (75): because the cardinal of \( C_j \) is precisely equal to \( d_j \) for \( j = 1, \ldots, k \), by the very definition of \( d_j \) and formula (63), we first deduce that
\[
\Psi_{1,0} = b_j \Gamma(f_j) \prod_{m \notin C_j} \Gamma(w_m f_j - w_m + a(\mu + n - p_j + 1)_m) \prod_{m \in C_j} \frac{1}{w_m}
\]
for \( j = 1, \ldots, k \). Now, and by definition, we have
\[
\begin{align*}
[f_j w_m] &= w_m + 1 - a(\mu + n - p_j + 1)_m & \text{if } m \notin C_j, \\
[f_j w_m] &= w_m - a(\mu + n - p_j + 1)_m & \text{if } m \in C_j.
\end{align*}
\]
The assertion follows. For \( j = 0 \), use formula (77). Last, \( \Psi_{1,\ell} \neq 0 \) because \( \Psi_{1,\ell} = e^{2i\pi \ell f_j} \Psi_{1,0} \). □

**Example 9.1.5** Assume that \( w_1 = \cdots = w_n = 1 \). Equation (77) is
\[
\Gamma(s + 1)^{n+1} = \sum_{m=0}^{n} s^m \Psi_{m+1,0} + O(s^{n+1})
\]
and we get
\[
\Psi_{m+1,0} = \frac{1}{m!} \left[ \frac{d^m}{ds^m} \Gamma(s + 1)^{n+1} \right]_{s=0}.
\]
In particular \( \Psi_{1,0} = 1 \) and \( \Psi_{2,0} = -(n + 1)\gamma \) where \( \gamma \) is the Euler constant.

---

\[21\] Notice that, by the very definition, \(-w_m(1 - f_j) + a(\mu + n - p_j + 1)_m \geq 0\).
Example 9.1.6 (1) Let \( n = 2 \) and \((w_0, w_1, w_2) = (1, 2, 3)\). Then \( \mu = 6 \), \( f_0 = 0 \) and \( d_0 = 3 \), \( f_1 = \frac{1}{3} \) and \( d_1 = 1 \), \( f_2 = \frac{1}{2} \) and \( d_2 = 1 \), \( f_3 = \frac{2}{3} \) and \( d_3 = 1 \). We have

\[
\begin{align*}
a(0) &= (0, 0, 0), \quad a(1) = (1, 0, 0), \quad a(2) = (1, 1, 0), \quad a(3) = (1, 1, 1), \quad a(4) = (1, 1, 2), \quad a(5) = (1, 2, 2) \\
C_1 &= C_3 = \{2\}, \quad C_2 = \{1\} \\
b_1 &= 18, \quad b_2 = 36, \quad b_3 = 108
\end{align*}
\]

and

\[
\Psi_{1,0}^1 = 6 \Gamma(\frac{1}{3}) \Gamma(\frac{2}{3}), \quad \Psi_{1,0}^2 = 18 \Gamma(\frac{1}{2})^2, \quad \Psi_{1,0}^3 = 36 \Gamma(\frac{1}{3}) \Gamma(\frac{2}{3}).
\]

(2) Let \( n = 2 \) and \((w_0, w_1, w_2) = (1, 2, 2)\). Then \( \mu = 5 \), \( f_0 = 0 \) and \( d_0 = 3 \), \( f_1 = \frac{1}{2} \) and \( d_1 = 2 \). We have

\[
\begin{align*}
a(0) &= (0, 0, 0), \quad a(1) = (1, 0, 0), \quad a(2) = (1, 1, 0), \quad a(3) = (1, 1, 1), \quad a(4) = (1, 2, 1), \quad a(5) = (1, 2, 2) \\
C_1 &= \{1, 2\}
\end{align*}
\]

and

\[
\Psi_{1,0}^1 = 4 \Gamma(\frac{1}{2}).
\]

9.1.5 Conjugation

We now describe the conjugation on \( G^{B,\text{const}} \) defined by the rational structure \( \Sigma^{B,\text{const}}_Q \). We will denote by \( \overline{\eta} \) the conjugate of \( \eta \). Recall the set \( \mathcal{F} \) defined by formula (59). Notice first that \( 1 - f_j \notin \mathcal{F} \) if \( f_j \notin \mathcal{F} \), \( j \neq 0 \). For \( j = 1, \ldots, k \), let \( c(j) \) be the index such that \( 1 - f_j = f_{c(j)} \). For \( j = 0 \), we define \( c(0) = 0 \). We have \( d_{c(j)} = d_j \) for \( j = 0, \ldots, k \).

Corollary 9.1.7 We have, for \( j = 0, \ldots, k \) and \( m = 0, \ldots, d_j - 1 \),

\[
\sum_{i-1+m \leq d_j-1} \Psi_{i,0}^{j} \overline{\omega}_{p_j-1+i-1+m} = (-1)^m \sum_{i-1+m \leq d_j-1} \Psi_{i,0}^{c(j)} \overline{\omega}_{p_{c(j)}-1+i-1+m} \tag{81}
\]

In particular, the Jordan blocks \( B_j \) and \( B_{c(j)} \) are conjugate.

Proof. Use the relations \( \Psi_{\ell}^{\text{const}} = \Psi_{\ell}^{\text{const}} \) for \( \ell = 0, \ldots, \mu - 1 \) together with theorem 9.1.3 \( \square \)

Example 9.1.8 (Example 9.1.6 (1) continued)

Let \( n = 2 \) and \((w_0, w_1, w_2) = (1, 2, 3)\). Recall that \( \mu = 6 \), \( f_0 = 0 \) and \( d_0 = 3 \), \( f_1 = \frac{1}{3} \) and \( d_1 = 1 \), \( f_2 = \frac{1}{2} \) and \( d_2 = 1 \), \( f_3 = \frac{2}{3} \) and \( d_3 = 1 \). We have

\[
\begin{align*}
\Psi_{1,0}^1 \overline{\omega}_3 &= \Psi_{1,0}^3 \overline{\omega}_5, \\
\overline{\omega}_4 &= \omega_4, \\
\overline{\omega}_2 &= \omega_2, \\
\overline{\omega}_1 &= -\omega_1 - 2\Psi_{2,0}^0 \omega_2, \\
\overline{\omega}_0 &= \omega_0 + 2\Psi_{2,0}^0 \omega_1 + 2(\Psi_{2,0}^0)^2 \omega_2
\end{align*}
\]

\]
9.2 A-side
We give here a description of the rational structure \(\Sigma_{Q,\text{const}}^A\) on \(H^*_{\text{orb}}(\mathbb{P}(w), \mathbb{C})\) defined by the rational structure \(\Sigma_{Q,\text{const}}^B\) and the mirror theorem \[2.4.3\] and, as a by-product, a description of the rational structure \(\Sigma_{Q,\text{quant}}^A\) on \(\mathcal{G}^A\) given by corollary \[8.2.5\].

9.2.1 The rational structure
For any subset \(I = \{i_1, \ldots, i_r\} \subset \{0, \ldots, n\}\), we put
\[
\mathbb{P}(w_I) := \mathbb{P}(w_{i_1}, \ldots, w_{i_r}).
\]
Recall that we have the decomposition (as vector spaces)
\[
H^*_{\text{orb}}(\mathbb{P}(w), \mathbb{C}) = \bigoplus_{j=0}^k H^*([\mathbb{P}(w_I)], \mathbb{C})
\]
where \(I_j\) is defined by formula \[60\]. Each \(H^*([\mathbb{P}(w_I)], \mathbb{C})\) has a basis of the form
\[
1_{f_j}, 1_{f_j}p, \ldots, 1_{f_j}p^{d_j-1}
\]
where \(p \in H^2([\mathbb{P}(w)], \mathbb{C}) \subset H^2_{\text{orb}}(\mathbb{P}(w), \mathbb{C})\) is the Chern class of \(O(1)\) and \(1_{f_j} \in H^0([\mathbb{P}(w_I)], \mathbb{C}) \subset H^0_{\text{orb}}(\mathbb{P}(w), \mathbb{C})\). As usual, we will denote by \(1, p, \ldots, p^n\) the corresponding basis of \(H^*([\mathbb{P}(w_I)], \mathbb{C})\).

According to the discussion in section \[8.2.3\] we define:

**Definition 9.2.1** The rational structure \(\Sigma_{Q,\text{const}}^A\) on the orbifold cohomology of weighted projective spaces is the image of \(\Sigma_{Q,\text{const}}^B\) under the mirror isomorphism of theorem \[2.4.3\].

We then have the following explicit description of the rational structure \(\Sigma_{Q,\text{const}}^A\): define the rational numbers
\[
s_j = \prod_{r=0}^n w_r^{-\lceil c_j w_r \rceil} \quad \text{(82)}
\]
for \(j = 0, \ldots, k\).

**Theorem 9.2.2** The rational structure \(\Sigma_{Q,\text{const}}^A\) on the orbifold cohomology is the \(\mathbb{Q}\)-vector space generated by the vectors
\[
\Psi^\text{const}_{\Gamma_\ell^A} = \sum_{j=0}^k s_j \sum_{i=1}^{d_j} \Psi_{i,\ell}^j 1_{f_j}p^{i-1} \quad \text{(83)}
\]
for \(\ell = 0, \ldots, \mu - 1\), where the numbers \(\Psi_{i,\ell}^j\) are determined by equations \[74\], \[75\] and \[77\] and the numbers \(s_i\) are defined in \[82\].

**Proof.** We use the following mirror correspondence, see \[10\] Theorem 5.1.1 and Remark 5.1.3]: under the mirror theorem \[2.4.3\] the basis \((1_{f_j}p^i)\) of orbifold cohomology of \(\mathbb{P}(w)\) corresponds to the basis
\[
[\omega] = ([\omega_0], \ldots, [\omega_{\mu-1}])
\]
of $G^{B,\text{const}}$ induced by $(\omega_0 x^{-c_0}, \ldots, \omega_{\mu-1} x^{-c_{\mu-1}})$ as follows: the image of $1_{f_j} p^i, j = 0, \ldots, k, i = 0, \ldots, d_j - 1$ under this correspondence is $s_j^{-1} [\omega_{p_{j+1}^i}]$. Now, the theorem follows from theorem 9.1.3 (1).

\[ \sum \]

9.2.2 A description via characteristic classes

Inspired by [24] and [23], we now rewrite theorem 9.2.2 with the help of some characteristic classes. Among other things we will see that the constants $s_j$ and $b_j$ (see equation (75)) in formula (83) miraculously disappear. We use the notations of section 9.1.

Let us define, after formula (75) and (77),

- for $j = 1, \ldots, k$, the cohomology classes

$$\hat{\Gamma}_j = \prod_{m=0}^{n} \Gamma(r w_m 1_{f_j} p + 1 - \{w_m f_j\}) \quad (84)$$

where $\{x\} = [x] - x$ and $r$ is defined by formula (66),

- for $j = 0$, the cohomology class

$$\hat{\Gamma}_0 := \prod_{m=0}^{n} \Gamma(1 + r w_m p).$$

These definitions have to be understood in the following way: in order to calculate $\Gamma(a 1_{f_j} p + b)$ ($b > 0$) we expand in power series the function

$$s \mapsto \Gamma(as + b)$$

and we replace in this expansion $s^k$ by $1_{f_j} p^k$ keeping in mind that $1_{f_j} p^{d_j} = 0$.

**Corollary 9.2.3** The rational structure $\Sigma_{Q,\text{const}}^{A}$ is the $Q$-vector space generated in the orbifold cohomology by the vectors

$$\Psi_{\Gamma_\ell}^{\text{const}} = \sum_{j=0}^{k} e^{2i\pi \ell f_j} \exp(-2i\pi \ell p) \cup \hat{\Gamma}_{j} \quad (85)$$

for $\ell = 0, \ldots, \mu - 1$. Here $\cup$ denotes the cup-product on $H^*_\text{orb}(\mathbb{P}(w), \mathbb{C})$.

**Proof.** From theorem 9.1.3 and theorem 9.2.2 it first follows that

$$\Psi_{\Gamma_\ell}^{\text{const}} = \sum_{j=0}^{k} s_j e^{2i\pi \ell f_j} \exp(-2i\pi \ell p) \cup \Pi_{j}$$

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where
\[
\Pi_j = b_j \Gamma(r_1 f_j + f_j) \prod_{m \in C_j} \frac{1}{w_m} \Gamma(r w_m f_j + 1) \prod_{m \notin C_j} \Gamma(r w_m f_j + w_m f_j - w_m + a(\mu + n - p_j + 1)_m). 
\]

As already noticed, we have
\[
[f_j w_m] = w_m + 1 - a(\mu + n - p_j + 1)_m \text{ if } m \notin C_j,
\]
\[
f_j w_m = w_m - a(\mu + n - p_j + 1)_m \text{ if } m \in C_j.
\]

It follows that \( s_j b_j \prod_{m \in C_j} \frac{1}{w_m} = 1 \) and we get first
\[
\hat{\Gamma}_j = \Gamma(r_1 f_j + f_j) \prod_{m \in C_j} \Gamma(r w_m f_j + 1) \prod_{m \notin C_j} \Gamma(r w_m f_j + w_m f_j - w_m + a(\mu + n - p_j + 1)_m). 
\]

The assertion follows using again the formula for \( \lceil f_j w_m \rceil \) above. \( \Box \)

Up to the factor \( r \), formula (85) agrees with [23, Theorem 4.11].

**Remark 9.2.4** In the case of \( \mathbb{P}^n \), that is if \( w_0 = w_1 = \cdots = w_n = 1 \), we have
\[
\Sigma^{A,\text{const}}_{\mathbb{Q}} = \hat{\Gamma}_0 \cup \delta(H^*(\mathbb{P}^n, \mathbb{Q})) 
\]
where \( \delta(p^n) = (2i\pi)^n p^m \) and \( \hat{\Gamma}_0 = \Gamma(1 + p)^{n+1} \). Indeed, by corollary 9.2.3, the vectors
\[
\Psi^\text{const}_{\ell} = \hat{\Gamma}_0 \cup \delta(1) - \ell \hat{\Gamma}_0 \cup \delta(p) + \cdots + \left(\frac{(-\ell)^n}{n!}\right) \hat{\Gamma}_0 \cup \delta(p^n),
\]
\( \ell = 0, \cdots, n \) generate \( \Sigma^{A,\text{const}}_{\mathbb{Q}} \) over \( \mathbb{Q} \). It follows that the vectors \( \hat{\Gamma}_0 \cup \delta(p^i), i = 0, \cdots, n, \) also generate \( \Sigma^{A,\text{const}}_{\mathbb{Q}} \) over \( \mathbb{Q} \). This result can already be found in [24, Proposition 3.1].

**9.2.3 Conjugation**

We now describe the conjugation on \( H^*_{\text{orb}}(\mathbb{P}(w), \mathbb{C}) \) defined by the rational structure \( \Sigma^{A,\text{const}}_{\mathbb{Q}} \). We will denote by \( \bar{\eta} \) the conjugate of \( \eta \in H^*_{\text{orb}}(\mathbb{P}(w), \mathbb{C}) \). From corollary 9.2.4 we get, keeping the notations of section 9.1.5,

**Corollary 9.2.5** We have, for \( j = 0, \cdots, k \) and \( m = 0, \cdots, d_j - 1 \),
\[
\bar{\Gamma}_j \cup 1_{f_j p^m} = (-1)^m \hat{\Gamma}_{c(j)} \cup 1_{f_{c(j)} p^m} 
\]
In particular, the Jordan blocks \( B_j \) and \( B_{c(j)} \) are conjugate.
10 Correlators of a logarithmic quantum differential system

The aim of this section is to consider, in the light of quantum differential systems, the following question: how to compute the gravitational correlators from Picard-Fuchs equations? The correlators alluded to are defined on the $A$-side for instance in [6, Definition 10.1.1]. In general, we define the gravitational correlators (two points, genus 0) of a flat logarithmic quantum differential system to be the coefficients of the matrix $H$ defined in lemma 6.3.2.

Three remarks are in order:

- the correlators defined in this way by the quantum differential system of example 2.1.7 ($A$-side) are precisely the ones of algebraic geometry,
- given a flat quantum differential system, the matrix of correlators $H$ is calculated solving the recursion equations (29),
- the quantum differential system associated on the $B$-side with a regular tame function gives directly (i.e., without any reference to correlator) the matrix $H$ we are looking for.

In practice, a mirror theorem will thus give a way to compute the correlators of the mirror partner of a regular tame function. We apply the recipe in this section and we illustrate this by some simple examples.

10.1 Gravitational two-points correlators of a flat logarithmic quantum differential system

Let $Q$ be a flat logarithmic quantum differential system on $M = \mathbb{C}$ and

$$M(x, \tau) \frac{dx}{x} + N(x, \tau) \frac{d\tau}{\tau}$$

be the matrix of the connection $\nabla$ in the basis $\omega = (\omega_0, \ldots, \omega_{\mu-1})$. By lemma 6.3.2 there exists a unique matrix $H(x, \tau) = I + \sum_{d \geq 1} H^d(\tau) x^d$ of holomorphic functions such that, after the base change of matrix $H(x, \tau)$, the matrix of the connection $\nabla$ takes the form

$$M(0, \tau) \frac{dx}{x} + N(0, \tau) \frac{d\tau}{\tau}$$

Recall that the matrices $H^d$ (and thus the matrix $H$) are defined by the equations (29), that is

$$dH^d(\tau) = H^d(\tau) M(0, \tau) - M(0, \tau) H^d(\tau) - \sum_{i=1}^d M^i(\tau) H^{d-i}(\tau)$$

for $d \geq 1$ and $M(x, \tau) = M(0, \tau) + \sum_{i \geq 1} M^i(\tau) x^i$.

We will put $H^d(\tau) = \sum_{r \geq 0} H^{d,r} \tau^r$ and $H^d x = (H^{d,r}_{i,j})_{ij}$.

The fact that we can define only two points correlators from a flat logarithmic quantum differential system is not so surprising because such systems correspond on the $A$-side to the small quantum cohomology.
Definition 10.1.1 We will call the numbers

\[ < \tau_r \omega_a, \omega_j >_{0,2,d} := g(\omega_j, \omega_j) H^{d,r+1}_{j+1,a+1}, \]

\( r \geq 0, \ d \geq 1 \) and \( a, j = 0, \cdots, \mu - 1 \) (the integers \( j \) are defined in remark 2.1.4), gravitational two-points correlators in genus 0 of the quantum differential system \( Q \). The matrix \( H(x, \tau) \) is called the correlator matrix of \( Q \).

The previous definition can be extended to the case \( d = 0 \): keeping in mind that \( H(0) = I \), we define \( < \tau_r \omega_a, \omega_j >_{0,2,0} : e \) for all \( r \).

The link with the usual correlators \( < \tau_r \phi_a, \phi_j >_{0,2,d} \) of algebraic geometry as defined for instance in [6, Definition 10.1.1] is given by the following lemma, which explains the terminology:

Lemma 10.1.2 Let \( Q \) be the quantum differential system associated with the small quantum cohomology of a projective manifold \( X \) by example 2.1.7. Then we have

\[ < \tau_r \phi_a, \phi_j >_{0,2,d} = g(\phi_j, \phi_j) H^{d,r+1}_{j+1,a+1} \]

(90)

for all \( a, j = 0, \cdots, \mu - 1 \), \( r \geq 0 \) and \( d \geq 1 \). The Gromov-Witten invariants \( < \tau_0 \phi_i, \phi_j >_{0,2,d}, d \geq 1 \), are described by the coefficients of \( \tau \) in the matrix \( H^d(\tau) \).

Proof. We have

\[ H(\phi_a) = \phi_a + \sum_{d \geq 1} \sum_{j=0}^{\mu-1} H^{d,r+1}_{j+1,a+1} \phi_j \tau^{r+1} x^d = \phi_a + \sum_{d \geq 1} \sum_{j=0}^{\mu-1} H^{d,r+1}_{j+1,a+1} \phi_j \tau^{r+1} x^d \]

because \( \phi_j = g(\phi_j, \phi_j) \phi_j \) by definition and because \( H(0, \tau) = I \) and \( H^{d,0}_{j+1,a+1} = 0 \) for \( d \geq 1 \) (this follows from (29) because \( Q \) is flat). Define now, as in [6 section 10.2],

\[ \tilde{s}(\phi_a) = \phi_a + \sum_{d \geq 1} \sum_{j=0}^{\mu-1} \sum_{r \geq 0} \tau^{r+1} < \tau_r \phi_a, \phi_j >_{0,2,d} \phi_j q^d. \]

Under the correspondence \( x \leftrightarrow q \), we have \( H = \tilde{s} \): this follows from the unicity, once given the initial condition \( H(0, \tau) = I \), because, up to the factor \( e^{-\tau M_i(0) \ln x} \), \( H \) and \( \tilde{s} \) give fundamental solutions of the Dubrovin connection.

Let us emphasize once again that these correlators can be computed using the recursion relations (29).

10.2 Examples

We discuss here some very simple examples.
10.2.1 Projective space

Let us consider the quantum differential system of example \[22.1\] with \(w_1 = \cdots = w_n = 1\), the mirror \[23\] of the small quantum cohomology of \(X = \mathbb{P}^n\). We have \(M(x, \tau) = \tau M_1(x)\) where

\[
M_1(x) = -\begin{pmatrix}
0 & 0 & 0 & 0 & x \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

which is a \((n + 1) \times (n + 1)\) matrix. According to example \[7.2\] the correlator matrix is

\[
H(x, \tau) = I + \sum_{d \geq 1} H^d(\tau) x^d
\]

where the matrices \(H^d(\tau)\) are defined by the relations

\[
dH^d(\tau) = \tau[H^d(\tau) M_1(0) - M_1(0) H^d(\tau) + NH^{(d-1)}(\tau)]
\]

for all \(d \geq 1\) where \(N_{1,n+1} = 1\) and \(N_{i,j} = 0\) otherwise. Using definition \[10.1.1\] this gives some very well known results (see \textit{i.e.} \[6, Example 10.1.3.1\] and the references therein):

\textbf{Example 10.2.1} Let us assume that \(n = 1\). We have, for \(d \geq 1\),

- \(<\tau_{2d-1}\omega_0, \omega_1 >, 0,2,d >= \frac{1}{(d)!} - \frac{2d}{(d)!^2} (1 + \cdots + \frac{1}{d}) \text{ and } <\tau_\tau \omega_0, \omega_1 >, 0,2,d = 0 \text{ otherwise},\)
- \(<\tau_{2d-2}\omega_0, \omega_1 >, 0,2,d >= \frac{d}{(d)!} \text{ and } <\tau_\omega \omega_1, \omega_1 >, 0,2,d = 0 \text{ otherwise},\)
- \(<\tau_{2d}\omega_0, \omega_0 >, 0,2,d >= \frac{1}{(d)!} - \frac{2}{(d)!^2} (1 + \cdots + \frac{1}{d}) \text{ and } <\tau_\tau \omega_0, \omega_0 >, 0,2,d = 0 \text{ otherwise},\)
- \(<\tau_{2d-1}\omega_1, \omega_0 >, 0,2,d >= \frac{1}{(d)!} \text{ and } <\tau_\omega \omega_1, \omega_0 >, 0,2,d = 0 \text{ otherwise}.\)

Indeed, the recursion relation \[91\] gives

- \(H^d_{11}(\tau) = \frac{\tau^{2d}}{(d)!^2} - \frac{2d\tau^{2d}}{(d)!^3} (1 + \cdots + \frac{1}{d}) \text{ for } d \geq 1, 1 \text{ if } d = 0,\)
- \(H^d_{12}(\tau) = \frac{\tau^{2d-1}}{(d)!^2} \text{ for } d \geq 1, 0 \text{ if } d = 0,\)
- \(H^d_{21}(\tau) = \frac{2\tau^{2d+1}}{(d)!^2} (1 + \cdots + \frac{1}{d}) \text{ for } d \geq 1, 0 \text{ if } d = 0,\)
- \(H^d_{22}(\tau) = \frac{\tau^{2d}}{(d)!^2} \text{ for } d \geq 1, 1 \text{ if } d = 0\)

\textbf{Remark 10.2.2} Let us put \(\deg \omega_0 = 0\) and \(\deg \omega_1 = 2\). Then \(<\tau_\omega \omega_1, \omega_1 >, 0,2,d = 0 \text{ if } 2r + \deg \omega_i + \deg \omega_j \neq 4d\)

as predicted by the “degree axiom” \[6, page 192\].

\[23\] Strictly speaking, one should take into account the coordinate \(x_0\) of \(H^0(X, \mathbb{C})\): the relative part of the quantum differential system is

\[-\tau \frac{dx_0}{x_0} - \tau M_1(x_1) \frac{dx_1}{x_1}\]

and one has to twist the following results by \(x_0^{-\tau I}\).

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10.2.2 Weighted projective space

Let us continue with example 2.2.1 but now for general integers \( w_0, \ldots, w_n \), the mirror of the weighted projective spaces \( \mathbb{P}(w_0, \ldots, w_n) \). Recall the canonical fundamental solution

\[
P(\zeta, \tau) = H(\zeta, \tau) e^{-\tau \tilde{M}_1(0) \ln \zeta}
\]

defined in example 7.3. We write

\[
H(\zeta, \tau) = I + \sum_{d \geq 1} H^d(\tau) \zeta^d
\]

and \( H^d(\tau) = \sum_{r \geq 0} H^{d,r} \tau^r \). We then get the (orbifold) correlators

\[
< \tau \bar{\omega}_a, \bar{\omega}_j >_{0,2, d} := g(\bar{\omega}_j, \bar{\omega}_j) H^{d,r+1}_{j+1,a+1}
\]

where the sections \( \bar{\omega}_i \) are defined in example 7.3. Once again, these correlators can be computed in practise using formula (29).

**Example 10.2.3** Let \( n = 1, w_0 = 1 \) and \( w_1 = 2 \). We have, with the notations of section 9.1.2, \( \mu = 3, c_0 = c_1 = 0, c_2 = \frac{1}{2} \) and

\[
\tilde{M}_1(\zeta) = -2 \begin{pmatrix} 0 & 0 & \zeta/4 \\ 0 & 0 & 0 \\ 0 & \zeta & 0 \end{pmatrix}
\]

We have \( \mathcal{U} = 1 \) and \( g(\bar{\omega}_0, \bar{\omega}_1) = \frac{1}{2}, \mathcal{V} = 2 \) and \( g(\bar{\omega}_2, \bar{\omega}_2) = \frac{1}{5} \). The matrix \( H(\zeta, \tau) \) is determined by the relations

\[
dH^d(\tau) = 2\tau \left[ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} H^d(\tau) - H^d(\tau) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] \\
+ 2\tau \begin{pmatrix} 0 & 0 & 1/4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} H^{d-1}(\tau)
\]

for \( d \geq 1 \) and \( H^{(0)}(\tau) = I \). We have for instance

\[
H^{(1)}(\tau) = \begin{pmatrix} 0 & 0 & \frac{\tau}{2} \\ 0 & 0 & \tau^2 \\ -4\tau^2 & 2\tau & 0 \end{pmatrix} \quad \text{and} \quad H^{(2)}(\tau) = \begin{pmatrix} -\frac{3}{2}\tau^3 & \frac{\tau}{2} & 0 \\ -2\tau^4 & \frac{\tau}{2} & 0 \\ 0 & 0 & \tau^3 \end{pmatrix}
\]

This gives

- \( < \tau_2 \bar{\omega}_0, \bar{\omega}_1 >_{0,2,2} = -3/4, < \tau_1 \bar{\omega}_1, \bar{\omega}_1 >_{0,2,2} = 1/4, < \tau_3 \bar{\omega}_0, \bar{\omega}_0 >_{0,2,2} = -1, < \tau_2 \bar{\omega}_1, \bar{\omega}_0 >_{0,2,2} = 1/4, < \tau_2 \bar{\omega}_2, \bar{\omega}_2 >_{0,2,2} = 1/8 \) and \( < \tau, \bar{\omega}_1, \bar{\omega}_j >_{0,2,2} = 0 \) otherwise,

- \( < \tau_0 \bar{\omega}_2, \bar{\omega}_1 >_{0,2,1} = 1/4, < \tau_1 \bar{\omega}_0, \bar{\omega}_2 >_{0,2,1} = -1/2, < \tau_0 \bar{\omega}_1, \bar{\omega}_2 >_{0,2,1} = 1/4, < \tau_1 \bar{\omega}_2, \bar{\omega}_0 >_{0,2,1} = 1/2 \) and \( < \tau, \bar{\omega}_1, \bar{\omega}_j >_{0,2,1} = 0 \) otherwise.
Remark 10.2.4 In the previous example we have \(<\tau_i\omega_i, \omega_j> > 0_{0,2,d}=0\) if 
\[2r + \deg \omega_i + \deg \omega_j \neq 3d\]
where \(\deg \omega_0 = 0\), \(\deg \omega_1 = 2\) and \(\deg \omega_2 = 1\). More generally set now \(X = \mathbb{P}(w_0, \cdots, w_n)\). If 
\(<\tau_{k_0}\phi_0, \cdots, \tau_{k_{\ell-1}}\phi_{\ell-1}, \omega_i, \omega_j> > 0_{0,\ell,d}\) is not equal to zero then 
\[\sum_{i=0}^{\ell-1} (\deg orb \phi_i + 2k_i) = 2n + 2 < c_1(TX), d > + 2\ell - 6.\] (92)
We have \(c_1(TX) = \mu p\) where \(p := \frac{1}{\gcd(w_0, \cdots, w_n)} c_1(\mathcal{O}_X(\gcd(w_0, \cdots, w_n))) \in H^2(|X|, \mathbb{C})\). If moreover the numbers \(\mu := w_0 + \cdots + w_n\) and \(\frac{1}{r} := \text{lcm}(w_0, \cdots, w_n)\) are prime there exists a generator \(D_w\) of \(H_2(|X|, \mathbb{Z})\) such that \(\int_{D_w} p = r\). In these conditions, equation (92) becomes 
\[\sum_{i=0}^{\ell-1} (\alpha_i + k_i) = n + \mu rd + \ell - 3\] (93)
if \(R_{\infty}(\omega_i) = \alpha_i \omega_i\). This justifies the twist by \(r\) above.

11 A mirror partner of the Hirzebruch surface \(\mathbb{F}_2\) via quantum differential systems and its classical limit

In this section we compute a mirror partner of the small quantum cohomology of the Hirzebruch surface \(\mathbb{F}_2\) using quantum differential systems, as explained in section 2.4. This case is interesting because it brings to light some new phenomena:

- for tameness reasons, the base space of the expected quantum differential systems is not the whole \((\mathbb{C}^*)^2\);

- the mirror map \(\nu\) in definition 2.4.1 needs not to be the identity, being described by flat coordinates.

Of course, the second point is well known in the classical setting of mirror symmetry, as a consequence among others of the fact that \(\mathbb{F}_2\) is not Fano, but we believe that the theory of quantum differential systems is useful in order to understand better what happens. For instance, the description of the mirror map, for which the main step is the computation of flat coordinates with respect to a flat residual connection, is very natural in this setting. Notice that several normalizations of such flat coordinates are possible, and this is essentially due to the fact that the rank of the Picard group is greater than one: keeping in mind mirror symmetry: this ambiguity is finally set by the metric.

We describe briefly the setting in section 11.2 and we calculate, on the B-side, the mirror flat quantum differential system in section 11.3. The mirror theorem is stated in section 11.5 and we check in section 11.6 that specialization of the previous results at suitable values of the parameters \(q_1\) et \(q_2\) gives the small quantum orbifold cohomology of \(\mathbb{P}(1, 1, 2)\), an aspect of Ruan’s conjecture (this has been done first in [1], in a slightly different setting). On the way, we use our computations in order to construct a logarithmic Frobenius manifold in section 11.4.
11.1 A Landau-Ginzburg model for the Hirzebruch surface $\mathbb{F}_2$

We will denote by $X$ the Hirzebruch surface $\mathbb{F}_2$ which is a compact and smooth toric variety. We have

$$\text{Pic}(X) = \mathbb{Z}f \oplus \mathbb{Z}H$$

where $f$ is the class of a fiber and $H$ is the class of $\mathcal{O}_{\mathbb{F}_2}(1)$ and the intersection numbers

$$f.f = 0, \ H.H = 2 \text{ and } H.f = 1$$

We consider the standard fan $\Sigma$ of $X$ with

$$\Sigma(1) = \{(1, 0), (0, 1), (-1, 2), (0, -1)\}$$

where we identify elements of $\Sigma(1)$, the one dimensional cones of $\Sigma$, with their primitive generators $v_1, v_2, v_3$ and $v_4$. The latter correspond respectively to the divisors $D_1, D_2$ (the exceptional one), $D_3$ and $D_4$, with intersection numbers

$$D_1.D_1 = 0, \ D_2.D_2 = -2, \ D_3.D_3 = 0, \ D_4.D_4 = 2$$

Recall also the exact sequence

$$0 \rightarrow H_2(X, \mathbb{Z}) \xrightarrow{\psi} \mathbb{Z}^4 \xrightarrow{\varphi} \mathbb{Z}^2 \rightarrow 0 \quad (95)$$

where $\varphi(e_i) = v_i$ for $i = 1, \cdots, 4$ ($(e_i)_i$ is the canonical basis of $\mathbb{Z}^4$) and $\psi(d) = \sum_{i=1}^{4} <D_i, d>e_i$. Applying the functor $\text{Hom}_{\mathbb{Z}}(\_\_ , \mathbb{C}^*)$ to the exact sequence $(95)$, we get the Landau-Ginzburg model for $X$, which the function $F$ defined by

$$w_0 + w_1 + w_2 + w_3$$

restricted to the sub-torus

$$U = \{(w_0, w_1, w_2, w_3) \in (\mathbb{C}^*)^4 | w_1w_3 = q_2 \text{ and } w_0w_1^2w_2 = q_1\}$$

Notice that in the standard basis $(f, H)$ of $H_2(X, \mathbb{Z})$ we have

$$D_1 = f, \ D_2 = H - 2f, \ D_3 = f, \ D_4 = H$$

and thus $D_1 + D_2 + D_3 + D_4 = 2H$. This divisor is Cartier but not ample: $X$ is not Fano.

11.2 The setting

11.2.1 Tameness

Let $F$ be the function defined on $(\mathbb{C}^*)^2 \times \mathbb{C}^2$ by

$$F(u_1, u_2, q_1, q_2) = u_1 + u_2 + q_1\frac{u_2^2}{u_1} + q_2\frac{1}{u_2}.$$ 

It has some tameness properties, depending on the position of the parameters $(q_1, q_2)$. In order to see this, we will need the following combinatorial data: let $\Gamma$ be the convex hull of $(1, 0), (0, 1), (-1, 2), (0, -1)$ in $\mathbb{R}^2$ (the “Newton polyhedron of $f$ at infinity” for $(q_1, q_2) \neq (0, 0)$, cf. [25]) and let us denote

---

24One can also choose the model $G(u_1, u_2, q_1, q_2) = u_1 + u_2 + q_1q_2^2\frac{u_2}{u_1u_2} + q_2\frac{1}{u_2}$.
• Γ₀ the face of Γ whose equation is \( x - y = 1 \),
• Γ₁ the face whose equation is \(-3x - y = 1\),
• Γ₂ the face whose equation is \( x + y = 1 \)

The main difference with the usual fano case is encoded by the following lemma:

**Lemma 11.2.1** The Laurent polynomial function \( f : (u_1, u_2) \mapsto F(u_1, u_2, q_1, q_2) \) is convenient and non-degenerate (in the sense of [25]) for all

\[ (q_1, q_2) \in M := \{(q_1, q_2) \in (\mathbb{C}^*)^2 | q_1 \neq \frac{1}{4}\} \]

In this case it has four non-degenerate critical points and its global Milnor number is equal to 4.

**Proof.** The function \( f \) is convenient for \( q_1q_2 \neq 0 \) because 0 belongs to the interior of Γ. In this case, we define

\[
\begin{align*}
    f_{|\Gamma_0} &= u_1 + q_1 \frac{1}{u_2}, \\
    f_{|\Gamma_1} &= q_1 \frac{u_2}{u_1} + q_2 \frac{1}{u_2}, \\
    f_{|\Gamma_2} &= u_1 + u_2 + q_1 \frac{u_2}{u_1},
\end{align*}
\]

the restrictions of \( f \) to the boundary of Γ. It is easily seen that

\[
    u_1 \frac{\partial f_{|\Gamma_j}}{\partial u_1} = u_2 \frac{\partial f_{|\Gamma_j}}{\partial u_2} = 0 \implies u_1u_2 = 0
\]

if and only if moreover \( q_1 \neq \frac{1}{4} \). Condition (98) means precisely that \( f \) is non-degenerate. The assertion about the global Milnor number then follows from [25], but it can also be directly checked. \( \square \)

Notice that the restriction of \( F \) at \( q_1 = \frac{1}{4} \) has two non-degenerate critical points and its (global) Milnor number is equal to two. This “jump” of Milnor numbers is not so surprising: the restriction of \( f \) at \( q_1 = \frac{1}{4} \) is indeed degenerate.

### 11.2.2 Basic formula

We will use the following lemma:

**Lemma 11.2.2** One has

\[
\begin{align*}
    u_1 du_1 \wedge du_2 &= q_1 \frac{u_2}{u_1} du_1 \wedge du_2 + dF \wedge u_1 du_2 \\
    u_2 du_1 \wedge du_2 &= q_2 \frac{1}{u_2} du_1 \wedge du_2 - 2q_1 \frac{u_2}{u_1} du_1 \wedge du_2 + dF \wedge -u_2 du_1 \\
    \frac{1}{u_2} du_1 \wedge du_2 &= \frac{1}{q_2} du_1 \wedge du_2 + 2q_1 \frac{u_2}{u_1} du_1 \wedge du_2 + dF \wedge \frac{1}{q_2} du_1 \\
    q_1 \frac{1}{u_1} du_1 \wedge du_2 &= q_2 \frac{1}{u_1} (1-4q_1) \frac{u_2}{u_1} du_1 \wedge du_2 + 2q_1 \frac{1}{q_2} du_1 \wedge du_2 + dF \wedge \frac{u_1 - u_2}{u_1} du_2 + dF \wedge \frac{u_1}{q_2} - \frac{2u_1}{q_2} u_2 du_1
\end{align*}
\]
Remark 11.2.4

and, for instance,

\[ q_1(4q_1 - 1)u_1^4 du_1 \wedge du_2 = 2q_1q_2u_1^2u_2 du_1 \wedge du_2 - q_2 du_1 \wedge du_2 + dF \wedge [-2q_1u_1^2 + u_2^2] du_1 + dF \wedge u_2^2 du_2 \]

where \( d \) denotes the relative differential \( d(C^2)_{x/M} \).

Proof. Because \( dF = (1 - q_1u_1^2) du_1 + (1 + 2q_1u_1^2 - q_2u_2^1) du_2 \). \( \square \)

Corollary 11.2.3 The Brieskorn lattice \( G_0 \) of \( F \) is free of finite type over \( \mathbb{C}[q_1, q_2, q_1^{-1}, (4q_1 - 1)^{-1}, q_2^{-1}, \theta] \) and the classes

\[ \Delta := (\Delta_0, \Delta_1, \Delta_2, \Delta_3) := ([du_1 \wedge du_2], [u_2 du_1 \wedge du_2], [u_2 du_1 \wedge du_2], [u_2 du_1 \wedge du_2]) \]

give a basis of it.

Proof. Lemma [11,24] shows that \( G_0 \) is of finite type over \( \mathbb{C}[q_1, q_2, q_1^{-1}, (4q_1 - 1)^{-1}, q_2^{-1}, \theta] \), indeed, it gives a “division” recipe which allows to express, by decreasing the degrees, each class \([u_1^{a_1}u_2^{a_2} du_1 \wedge du_2], (a_1, a_2) \in \mathbb{Z}^2 \), as a linear combination of elements of \( \Delta \), as in [12, Proof of Proposition 3.2]. In order to get the freeness, we can use the following result (peculiar to Brieskorn lattices) which is proven as in the second part of [7, lemma 4.2]: let \( \omega_1, \cdots, \omega_\mu \in G_0 \) such that there are no non-trivial relations between \([\omega_1], \cdots, [\omega_\mu] \) in \( G_0/\theta G_0 \); then, there are no non-trivial relations between \( \omega_1, \cdots, \omega_\mu \) in \( G_0 \). \( \square \)

Remark 11.2.4 We have \( \Delta = (\omega_0, -\theta \nabla_{\partial_{q_1}} \omega_0, \theta \nabla_{\partial_{q_2}} \theta \nabla_{\partial_{q_1}} \omega_0, -\theta \nabla_{\partial_{q_1}} \omega_0) \) where \( \omega_0 := [du_1 \wedge du_2] \) (by the definition of the relative connection).

In order to describe the matrix of the connection in the basis \( \Delta \), we will use formula (99) below and we first need the following data: let us define, for \( j = 0, 1, 2 \),

- \( h_{\Gamma_j} = a_j^1 u_1 \frac{\partial F}{\partial u_1} + a_j^2 u_2 \frac{\partial F}{\partial u_2} - F \) if \( \Gamma_j \) has equation \( a_j^1 x + a_j^2 y = 1 \),
- \( \phi_{\Gamma_j}(u_1^{a_1}u_2^{a_2}) = a_j^1 a_1 + a_j^2 a_2 \) and \( \phi(u_1^{a_1}u_2^{a_2}) = \max_j \phi_{\Gamma_j}(u_1^{a_1}u_2^{a_2}) \).

One has

\[ \begin{align*}
  h_{\Gamma_0} &= -2u_2 - 4q_1 \frac{u_2^2}{u_1}, \\
  h_{\Gamma_1} &= -4u_1 - 2u_2, \\
  h_{\Gamma_2} &= -2q_2 \frac{1}{u_2}
\end{align*} \]

and, for instance,

\[ \begin{align*}
  \phi(1) &= \phi_{\Gamma_2}(1) = 0, \\
  \phi\left(\frac{1}{u_2}\right) &= \phi_{\Gamma_0}\left(\frac{1}{u_2}\right) = 1, \\
  \phi\left(\frac{u_2}{u_1}\right) &= \phi_{\Gamma_1}\left(\frac{u_2}{u_1}\right) = 2, \\
  \phi\left(\frac{u_2^2}{u_1}\right) &= \phi_{\Gamma_2}\left(\frac{u_2^2}{u_1}\right) = 1.
\end{align*} \]

The map \( \phi \) is the Newton degree, giving rise to the Newton filtration, closely related with the \( V \)-filtration and the spectrum at infinity of the function \( f \) for \( (q_1, q_2) \in M \), see [11, Section 4].

Last, we will use the following basic formula, which easily follows from the computation rules in the Gauss-Manin system, see the Appendix:

Lemma 11.2.5 We have, in the Gauss-Manin system \( G \) of \( F \),

\[ (\tau \partial_r + \phi_{\Gamma_j}(g))[g \frac{du_1}{u_1} \wedge \frac{du_2}{u_2}] = \tau h_{\Gamma_j}[g \frac{du_1}{u_1} \wedge \frac{du_2}{u_2}] \]

(99) for any monomial \( g = u_1^{a_1}u_2^{a_2}, (a_1, a_2) \in \mathbb{Z}^2 \), \([ \cdot \cdot \cdot ]\) denoting the class in \( G \).
11.3 A non-resonant logarithmic quantum differential system

11.3.1 A differential system

We solve here the Birkhoff problem for the Brieskorn lattice of $F$ as described in the Appendix. More precisely, we describe here a basis of $G_0$ giving:

- a differential system on $\mathbb{P}^1 \times M$, with logarithmic poles along $\{\tau = 0\} \times M$ and with poles of Poincare rank less or equal to 1 along $\{\theta = 0\} \times M$,

- (canonical) logarithmic extensions of the Brieskorn lattice along $q_2 = 0$ and $q_1 = 0$,

The adjective canonical refers to canonical Deligne’s extensions: we require that the eigenvalues of the residue matrices along $q_2 = 0$ and $q_1 = 0$ do not differ from non-zero integers (in other words, we get a non-resonant logarithmic quantum differential system as defined in section $6$).

Define

$$\omega = (\omega_0, \omega_1, \omega_2, \omega_3) := (\omega_0, -\theta q_2 \nabla_{\partial q_2} \omega_0, \theta q_2 \nabla_{\partial q_2} \theta q_1 \nabla_{\partial q_1} \omega_0, -\theta q_1 \nabla_{\partial q_1} \omega_0)$$ (100)

By remark $11.2.4$ we have $\omega = (\triangle_0, q_2 \triangle_1, q_1 q_2 \triangle_2, q_1 \triangle_3)$ where the basis $\triangle$ is defined in corollary $11.2.3$; $\omega$ is thus a basis of $G_0$. We will make a constant use of the following result which describes the matrix of the connection $\nabla$ in the basis $\omega$:

**Theorem 11.3.1**

1. The matrix of $\tau \nabla_{\partial \tau}$ in the basis $\omega$ takes the form

$$\tau \begin{pmatrix} 0 & -2q_2 & 0 & 0 \\ -2 & 0 & -4q_1q_2 & 0 \\ 0 & -4 & 0 & -2 \\ 0 & 0 & 2q_2(4q_1 - 1) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

2. In the basis $\omega$ the matrix of $q_2 \nabla_{q_2}$ is

$$\tau \begin{pmatrix} 0 & -q_2 & 0 & 0 \\ -1 & 0 & -2q_1q_2 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & q_2(4q_1 - 1) & 0 \end{pmatrix}$$

and the one of $q_1 \nabla_{q_1}$ is

$$\tau \begin{pmatrix} 0 & 0 & 0 & \frac{q_2q_1}{4q_1-1} \\ 0 & 0 & -q_1q_2 & 0 \\ 0 & -1 & 0 & -\frac{2q_1}{4q_1-1} \\ -1 & 0 & 2q_1q_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \frac{q_2}{4q_1-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2q_1}{4q_1-1} \end{pmatrix}$$
Proof. First, using formula (99) and lemma 11.2.2 we get that the matrix of \( \tau \) in the basis \( \triangle \) is
\[
\begin{pmatrix}
0 & -2 & 0 & 0 \\
-2q_2 & 0 & -4q_2 & 0 \\
0 & -4q_1 & 0 & -2q_2 \\
0 & 0 & 2(4q_1 - 1) & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
the one of \( \nabla q_2 \) is
\[
\begin{pmatrix}
0 & -\frac{1}{q_2} & 0 & 0 \\
-1 & 0 & -2 & 0 \\
0 & -2\frac{q_1}{q_2} & 0 & -1 \\
0 & 0 & (4q_1 - 1) & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\frac{1}{q_2} & 0 & 0 \\
0 & 0 & -\frac{1}{q_2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
and the one of \( \nabla q_1 \) is
\[
\begin{pmatrix}
0 & 0 & 0 & \frac{q_2}{q_1(4q_1 - 1)} \\
0 & 0 & -\frac{q_2}{q_1} & 0 \\
0 & -1 & 0 & -2\frac{q_2}{4q_1 - 1} \\
-1 & 0 & 2 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & \frac{q_2}{q_1(4q_1 - 1)} \\
0 & 0 & 0 & \frac{q_2}{q_1(4q_1 - 1)} \\
0 & 0 & -\frac{1}{q_1} & 0 \\
0 & 0 & 0 & -\frac{(4q_1 - 1)}{q_1(4q_1 - 1)}
\end{pmatrix}
\]
The residue matrix of \( \nabla q_2 \) along \( q_2 = 0 \) is resonant (the difference of two of its eigenvalues is a non-zero integer). In order to overcome this problem, we last define
\[
\omega = (\omega_0, \omega_1, \omega_2, \omega_3) := (\triangle_0, q_2 \triangle_1, q_1 q_2 \triangle_2, q_1 \triangle_3)
\]
In the basis \( \omega \), the connection has the expected form. \( \square \)

Remark 11.3.2 It is remarkable to notice that in this example the basis \( \omega \) has a very simple description in terms of the (relative) covariant derivatives: the situation is much more complicated in general, even in the case of weighted projective spaces.

Corollary 11.3.3 The lattice
\[
\mathcal{L} := \sum_{i=0}^{3} \mathbb{C}[q_1, q_2, (4q_1 - 1)^{-1}, \theta] \omega_i \subset G_0
\]
gives a logarithmic extension of the connection at \( q_1 = 0 \) and \( q_2 = 0 \) for which the eigenvalues of the residue matrices are equal to zero.

Remark 11.3.4 Put \( v_1 = q_2^{-1/2}u_1 \) and \( v_2 = q_2^{-1/2}u_2 \). Then
\[
F(v_1, v_2) = q_2^{1/2}L(v_1, v_2, q_1)
\]
where \( L(v_1, v_2, q_1) = v_1 + v_2 + q_1 v_1^2 + \frac{1}{v_1} \). The function \( F \) is thus a “rescaling” of the function \( L \), and these kind of functions produce naturally logarithmic degenerations along \( q_2 = 0 \) (see section 7.1). A connected result is that \( \tau \nabla q_2 = 2q_2 \nabla q_2 \) : in terms of mirror symmetry, this agrees with the fact that the anticanonical class of the Hirzebruch surface is \( -2H \) by equations (96) (see also remark 11.5.5 below).
11.3.2 Monodromies

We are now able to describe the monodromy matrices.

**Lemma 11.3.5** The monodromy matrices around \( \tau = 0, q_2 = 0, q_1 = 0 \) in the basis \( \omega \) are respectively

\[
M\{\tau=0\} := \exp(2i\pi \begin{pmatrix}
0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 \\
0 & -4 & 0 & -2 \\
0 & 0 & 0 & 0
\end{pmatrix}),
M\{q_2=0\} := \exp(2i\pi \tau \begin{pmatrix}
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -2 & 0 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix})
\]

and

\[
M\{q_1=0\} := \exp(2i\pi \tau \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix})
\]

**Proof.** Theorem 11.3.1 gives explicit residue matrices: for the two last assertions we can use the non-resonance condition and for the first one the fact that \([B_\infty, B_0(0)] = -B_0(0)\) if we write the matrix of \( \tau \partial_\tau \) as \( \tau B_0(q) + B_\infty \).

By isomonodromy, \( M\{q_2=0\} \) does not depend on \( \tau \), up to conjugacy. More precisely, we have:

**Corollary 11.3.6** In the basis \( W := (W_0, W_1, W_2, W_3) = (\omega_0, 2i\pi \tau \omega_1, (2i\pi)^2 \tau^2 \omega_2, 2i\pi \tau \omega_3) \) we have

\[
M\{q_2=0\} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
1 & -2 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\text{ and } M\{q_1=0\} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{pmatrix}
\]

Let us notice that these monodromies are not cyclic. We also have \( M^2\{q_2=0\} = M\{\tau=0\} \), as predicted by remark 11.3.4.

11.3.3 Flat bases

In order to get a quantum differential system, we are still looking for a \( \nabla \)-flat “metric” \( S \), see definition 2.1.1. We first define flat bases with respect to the flat residual connection at \( \theta = 0 \): indeed, the flat metric we are looking for should be constant, and therefore easier to describe, in such bases.

First, theorem 11.3.1 gives a flat connection \( \nabla \) on the restriction \( E := G_0 / \theta G_0 \). Its matrix is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4q_1-1} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{4q_1-1}
\end{pmatrix} dq_1
\]

\[\text{Strictly speaking, this flat connection is first defined as the restriction of } \nabla \text{ at } \tau = 0 \text{ and by triviality is shifted at } \theta = 0.\]
in the basis of $E$ induced by $\omega$ and one gets a $\nabla$-flat basis from $\omega$ via a multivalued base change, whose matrix is

$$
\begin{pmatrix}
\frac{-p_{11}}{2} (4q_1 - 1)^{1/2} + p_{21} & \frac{p_{12}}{2} (4q_1 - 1)^{1/2} + p_{22} & \frac{p_{13}}{2} (4q_1 - 1)^{1/2} + p_{23} & \frac{p_{14}}{2} (4q_1 - 1)^{1/2} + p_{24} \\
p_{31} & \frac{-p_{32}}{2} (4q_1 - 1)^{1/2} + p_{41} & \frac{p_{33}}{2} (4q_1 - 1)^{1/2} + p_{42} & \frac{p_{34}}{2} (4q_1 - 1)^{1/2} + p_{44}
\end{pmatrix}
$$

The constants $p_{ij}$ are chosen such that this matrix is invertible. Several normalizations are possible, and we will focus on the following ones: start with the base change

$$
P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & (1 - 4q_1)^{1/2}
\end{pmatrix}
$$

(the parameter $p_{24}$ is after all natural because it corresponds to the choice of a basis of $H^2(F_2)$, see section 11.5 below). Then:

1. one can take for instance $p_{24} = 0$. We will denote by $P^{flat}$ the corresponding matrix: it is the simplest base change.

2. One can consider the case $p_{24} = -\frac{1}{2}$; this is the normalization which fits the best with mirror symmetry, in particular with metrics (see remark 11.3.10 below). We will denote by $P^{can}$ the corresponding matrix.

Let us define accordingly

$$
\omega^{flat} = (\omega_0^{flat}, \omega_1^{flat}, \omega_2^{flat}, \omega_3^{flat}) := (\omega_0, \omega_1, \omega_2, \omega_3) P^{flat} \\
\omega^{can} = (\omega_0^{can}, \omega_1^{can}, \omega_2^{can}, \omega_3^{can}) := (\omega_0, \omega_1, \omega_2, \omega_3) P^{can}
$$

These bases are of course related by a constant linear base change.

**Lemma 11.3.7** 1. In the basis $\omega^{flat}$, the matrix $\tau \nabla_{\partial_r}$ is

$$
\tau \begin{pmatrix}
0 & -2q_2 & 0 & q_2(1 - 4q_1)^{1/2} \\
-2 & 0 & -q_2 & 0 \\
0 & -4 & 0 & 0 \\
0 & 0 & -2q_2(1 - 4q_1)^{1/2} & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
$$

the one of $\nabla_{q_2}$ is

$$
\tau \begin{pmatrix}
0 & -1 & 0 & \frac{(1-4q_1)^{1/2}}{2} \\
-\frac{1}{q_2} & 0 & -\frac{1}{2} & 0 \\
0 & -\frac{2}{q_2} & 0 & 0 \\
0 & 0 & -(1 - 4q_1)^{1/2} & 0
\end{pmatrix},
$$

and the one of $\nabla_{q_1}$ is

66
We are now able to describe the expected flat metric.

### 11.3.4 Flat metric

**Remark 11.3.8** We define naturally the degree of \( \omega_0^{\text{flat,cl}} \) (resp. \( \omega_1^{\text{flat,cl}}, \omega_2^{\text{flat,cl}}, \omega_3^{\text{flat,cl}} \)) to be 0 (resp. 1, 2, 1).

#### 11.3.4 Flat metric

We are now able to describe the expected \( \nabla \)-flat metric. Let us define, for \( a \in \mathbb{C}^* \),

\[
S(\omega_0^{\text{flat}}, \omega_2^{\text{flat}}) := a, \quad S(\omega_1^{\text{flat}}, \omega_1^{\text{flat}}) = 2a, \quad S(\omega_3^{\text{flat}}, \omega_3^{\text{flat}}) = -\frac{a}{2}
\]

and \( S(\omega_i^{\text{flat}}, \omega_j^{\text{flat}}) = 0 \) otherwise. This gives a bilinear form \( S \) on \( G_0 \) as in formula (4) by

\[
S(\omega_0, \omega_2) := a, \quad S(\omega_1, \omega_1) = 2a, \quad S(\omega_3, \omega_3) = \frac{2q_1}{4q_1 - 1}a, \quad S(\omega_1, \omega_3) = a, \quad S(\omega_2, \omega_2) = 2aq_1q_2
\]

and \( S(\omega_i, \omega_j) = 0 \) otherwise.
Lemma 11.3.9 The form $S$ is non-degenerate and $\nabla$-flat.

Proof. This result is directly checked: flatness follows from the symmetry properties of the matrices involved in lemma 11.3.7. $\square$

Remark 11.3.10 The flat form $S$ is unique up to the non zero multiplicative constant $a$. We will choose in the sequel $a = 1$. In this case, we also have

$$S(\omega_{0,\text{can}}, \omega_{1,\text{can}}) := 1, \ S(\omega_{1,\text{can}}, \omega_{1,\text{can}}) = 2, \ S(\omega_{3,\text{can}}, \omega_{1,\text{can}}) = -1$$

(103)

and $S(\omega_{i,\text{can}}, \omega_{j,\text{can}}) = 0$ otherwise. Formula (103) explains why the normalization $p_{24} = -\frac{1}{\pi}$ is the good one to consider in the mirror symmetry framework (see formula (94) and theorem 11.5.2).

11.3.5 Flat coordinates

In the light of mirror symmetry, flat coordinates are central objects: indeed, and as it follows from example 2.1.7, these are the natural coordinates on the $A$-side. In order to get a precise mirror theorem, we first search for such coordinates.

We first define, as in definition 2.3.1, the period map (see section 11.4 below for a logarithmic point of view)

$$\varphi_{\omega_0} : \Theta_M \to G_0/\theta G_0$$

(104)

by

$$\varphi_{\omega_0}(X) = -\Phi_X(\omega_0)$$

(105)

where we use the notations of proposition 2.1.2. We thus have, by theorem 11.3.1,

$$\varphi_{\omega_0}(q_1 \partial_{q_1}) = -\omega_3 \text{ et } \varphi_{\omega_0}(q_2 \partial_{q_2}) = -\omega_1.$$

We use this map to shift the connection $\nabla$ to a flat connection $\nabla^{\omega_0}$ on $\Theta_M$ by

$$\varphi_{\omega_0}(\nabla^{\omega_0} q_1 \partial_{q_1}) = -\nabla \omega_3 \text{ and } \varphi_{\omega_0}(\nabla^{\omega_0} q_2 \partial_{q_2}) = -\nabla \omega_1$$

(106)

(notice that $\varphi_{\omega_0}$ is injective). Similarly, we shift the $\nabla$-flat metric $S$ to a $\nabla^{\omega_0}$-flat metric $S^{\omega_0}$ on $\Theta_M$.

Lemma 11.3.11 1. We have

$$\nabla^{\omega_0}_{\partial_{q_1}} q_1 \partial_{q_1} = (4q_1 - 1)^{-1} q_2 \partial_{q_2} - 2(4q_1 - 1)^{-1} q_1 \partial_{q_1} \text{ and } \nabla^{\omega_0}_{\partial_{q_2}} q_1 \partial_{q_1} = 0,$$

$$\nabla^{\omega_0}_{\partial_{q_1}} q_2 \partial_{q_2} = 0 \text{ and } \nabla^{\omega_0}_{\partial_{q_2}} q_2 \partial_{q_2} = 0.$$

In particular, the vector fields

$$-(1 - 4q_1)^{1/2} q_1 \partial_{q_1} + \left(\frac{1}{2} (1 - 4q_1)^{1/2} - p_{24}\right) q_2 \partial_{q_2} \text{ and } q_2 \partial_{q_2},$$

where $p_{24} \in \mathbb{C}$, are $\nabla^{\omega_0}$-flat.
2. We have
\[ S^{\omega_2}(q_2 \partial_{q_2}, q_2 \partial_{q_2}) = 2a, \quad S^{\omega_0}(q_1 \partial_{q_1}, q_2 \partial_{q_2}) = a \quad \text{and} \quad S^{\omega_0}(q_1 \partial_{q_1}, q_1 \partial_{q_1}) = 2q_1 a/(4q_1 - 1) \]

Proof. Follows from the definition of $\nabla^{\omega_2}$ and theorem 11.3.1.

We will put, according to the cases $p_{24} = 0$ and $p_{24} = -\frac{1}{2}$ described above,

\begin{itemize}
  \item $\xi_1 = - (1 - 4q_1)^{1/2} q_1 \partial_{q_1} + \frac{1}{2} (1 - 4q_1)^{1/2} q_2 \partial_{q_2}$,
  \item $\xi_1^{\text{can}} = - (1 - 4q_1)^{1/2} q_1 \partial_{q_1} + (\frac{1}{2} (1 - 4q_1)^{1/2} + \frac{1}{2}) q_2 \partial_{q_2}$
\end{itemize}

and $\xi_2 = \xi_2^{\text{can}} = q_2 \partial_{q_2}$.

Recall that flat coordinates $\varphi_1$ and $\varphi_2$ are coordinates such that the vector fields $\partial_{\varphi_1}$ and $\partial_{\varphi_2}$ defined by $\partial_{\varphi_i}(d\varphi_j) = \delta_{ij}$, where $\delta_{ij}$ denotes the Kronecker symbol, are $\nabla^{\omega_2}$-flat.

Corollary 11.3.12 1. The functions $\varphi_1$ and $\varphi_2$ defined by
\[ q_1 = \frac{e^{\varphi_1}}{(1 + e^{\varphi_1})^2} \quad \text{and} \quad q_2 = \frac{e^{\varphi_2} e^{\varphi_1} + 1}{e^{\varphi_1/2}} \]
are flat coordinates (case $p_{24} = 0$).

2. The functions $\varphi_1^{\text{can}}$ and $\varphi_2^{\text{can}}$ defined by
\[ q_1 = \frac{e^{\varphi_1^{\text{can}}}}{(1 + e^{\varphi_1^{\text{can}}})^2} \quad \text{and} \quad q_2 = e^{\varphi_2^{\text{can}}} (e^{\varphi_1^{\text{can}}} + 1) \]
are flat coordinates (case $p_{24} = -\frac{1}{2}$).

Proof. Indeed, we have $\partial_{\varphi_1} = \xi_1$ and $\partial_{\varphi_2} = \xi_2$ (resp. $\partial_{\varphi_1^{\text{can}}} = \xi_1^{\text{can}}$ and $\partial_{\varphi_2^{\text{can}}} = \xi_2^{\text{can}}$) and we use the fact that $\xi_1$ and $\xi_1^{\text{can}}$ (resp. $\xi_2$ and $\xi_2^{\text{can}}$) are, by definition, the flat vector fields.

Definition 11.3.13 We define $r_i := e^{\varphi_i}$ and $r_i^{\text{can}} := e^{\varphi_i^{\text{can}}}$ for $i = 1, 2$.

By definition, we have $r_i \partial_{r_i} = \xi_i$ (resp. $r_i^{\text{can}} \partial_{r_i^{\text{can}}} = \xi_i^{\text{can}}$).

Lemma 11.3.14 In the basis $\omega^{\text{can}}$, the matrix $\tau_{\nabla_{\partial_{r_i}}}$ is
\[
\tau = \begin{pmatrix}
0 & -2r_2^{\text{can}}(1 + r_1^{\text{can}}) & 0 & 2r_1^{\text{can}}r_2^{\text{can}} \\
-2 & 0 & -2r_1^{\text{can}}r_2^{\text{can}} & 0 \\
0 & -4 & 0 & 2 \\
0 & 0 & -2(r_1^{\text{can}} - 1)r_2^{\text{can}} & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -2 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]

the one of $r_2^{\text{can}} \nabla_{\varphi_2^{\text{can}}}$ is

\[ \text{[26] Compare with [5] formula (11.94) p. 394} \]

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and the one of $r_{,1}^{can}\nabla_{r_{,1}^{can}}$ is

$$\tau \begin{pmatrix} 0 & -r_{,2}^{can} r_{,1}^{can} & 0 & r_{,2}^{can} r_{,1}^{can} \\ -1 & 0 & -r_{,1}^{can} r_{,2}^{can} & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & -(r_{,1}^{can} - 1) r_{,2}^{can} & 0 \end{pmatrix}$$

Proof. Follows from lemma 11.3.7.

11.3.6 Résumé

The first part of this section gives a trivial bundle $\mathcal{G}$ on $\mathbb{P}^1 \times M$, equipped with a meromorphic connection $\nabla$ with the expected poles and the second part gives a $\nabla$-flat metric $S$, where we choose the normalization $a = 1$ in formulas (101). Summarizing, we get

**Theorem 11.3.15** The tuple $Q_F = (M, \mathcal{G}, \nabla, S, 2)$ is a quantum differential system on $M$.

The quantum differential system $Q_F$ provides

- a Dubrovin connection,
- a canonical pre-primitive section $\omega_0$ (the class of $\frac{du_1}{u_1} \wedge \frac{du_2}{u_2}$ in $G_0$).

giving rise to

- a quantum product and a flat connection $\nabla$ on $E := G_0 / \theta G_0$,
- flat coordinates on $M$.

**Remark 11.3.16** In the same way, using lemma 11.3.14 and the metric defined in remark 11.3.10, we also define a quantum differential system $Q_{F,can}$ on the universal covering of $M$. Of course, we get also a quantum differential system $Q_{F,flat}$.

11.4 Remark : a logarithmic Frobenius manifold

On the way, we show how the previous results provide a logarithmic Frobenius manifold in the sense of [32]. Recall the lattice $\mathcal{L}$ from corollary 11.3.3: this is an extension of $G_0$ along $D := \{(q_1, q_2) \in \mathbb{C}^2 | q_1 q_2 = 0\}$. Define

$$E^{log} := \mathcal{L} / \theta \mathcal{L}$$

and the period map

$$\varphi_{\omega_0} : Der(\log D) \rightarrow E^{log} \quad (107)$$

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\[
\varphi_{\omega_0}^\log (X) = -\Phi_X(\omega_0)
\]

where \(\text{Der}(\log D)\) denotes the module of the logarithmic vector fields along \(D\) (see proposition \ref{prop:mod_log} for the definition of \(\Phi_X\)). By definition \ref{def:mod_log} and theorem \ref{thm:mod_log} we have in particular \(\varphi_{\omega_0}^\log (q_1\partial_{q_1}) = -\omega_3\) and \(\varphi_{\omega_0}^\log (q_2\partial_{q_2}) = -\omega_1\).

As usual, we will denote by \(o\) the fiber at \((q_1, q_2) = (0, 0)\).

**Lemma 11.4.1**

1. The section \(\omega_0\) is \(\nabla\)-flat,
2. the map \(\varphi_{\omega_0}^{\log, o}\) is injective,
3. the vector \(\omega_0^o\) of \(E^{\log, o}\) and its images under iterations of the maps \(\Phi_X : E^{\log, o} \to E^{\log, o}\) generate \(E^{\log, o}\).

**Proof.** The first assertion is clear. For the second one, notice that \(\omega_3^o\) and \(\omega_1^o\) are linearly independent in \(E^{\log, o}\) and, for the third one, notice that \(\Phi_{q_1}\partial_{q_1} \omega_0^o = \omega_3^o, \Phi_{q_2}\partial_{q_2} \omega_0^o = \omega_1^o\) and \(\Phi_{q_1}\partial_{q_1} \circ \Phi_{q_2}\partial_{q_2} \omega_0^o = \omega_2^o\).

\[\square\]

Last, denote by \(G^{\log}\) the extension of \(L\) at \(\mathbb{P}^1 \times N\) where \(N := (\mathbb{C}^2, 0)\) and by \(S\) the bilinear form on \(L\) defined by formula (102) and extended by linearity. Recall the logarithmic quantum differential systems of definition \ref{def:log_qd_system}.

**Corollary 11.4.2** The tuple \(Q^{\log} = (G^{\log}, \nabla, S, N, D)\) is a logarithmic quantum differential system. It has a universal unfolding which defines, together with the \(\nabla\)-flat section \(\omega_0\), a logarithmic Frobenius manifold at the origin of \(\mathbb{C}^2\).

**Proof.** The first assertion follows now from definition \ref{def:log_qd_system} because \(S\) is nondegenerate on \(N\) and the second from \ref{ref:log_qd_system}.

\[\square\]

**Remark 11.4.3** This construction of logarithmic Frobenius manifold gives an intermediate step between the one associated with projective space and the one associated with weighted projective spaces as described in \ref{ref:log_qd_system}: the monodromies are not cyclic, as it is the case for weighted projective spaces (nevertheless the generation condition in \ref{ref:log_qd_system} holds), and the bilinear form \(S\) is nondegenerate, as it is the case for projective spaces.

### 11.5 A mirror theorem for the small quantum cohomology of \(\mathbb{F}_2\)

The goal of this section is to describe a mirror partner of the small quantum cohomology of \(\mathbb{F}_2\), in the sense of section \ref{sec:mirror_theorem}. On the small quantum cohomology side, we keep the notations of \ref{sec:small_qcoh}: the cohomology algebra is \(\mathbb{C}[f, H]/ < f^2, H^2 - 2Hf >\).

Recall the quantum differential system \(Q^{\text{can}}_F\) defined in remark \ref{rmk:can_qd_system} using the flat coordinates \((r_1^{\text{can}}, r_2^{\text{can}})\), and let \(Q_{\mathbb{F}_2}\) be the one associated with the small quantum cohomology of \(\mathbb{F}_2\) by example \ref{ex:small_qcoh}. Let us define the map

\[\gamma : Q_{\mathbb{F}_2} \to (id, \nu)^* Q^{\text{can}}_F\]
of quantum differential systems in the following way:

- the map $\nu$ is the identity,
- the map $\gamma$ is defined by

$$\gamma(1) = \omega_0^{\text{can}}, \quad \gamma(f) = \theta r_1^{\text{can}} \nabla r_1^{\text{can}} \omega_0^{\text{can}}, \quad \gamma(H) = \theta r_2^{\text{can}} \nabla r_2^{\text{can}} \omega_0^{\text{can}}, \quad \gamma(H \circ f) = \theta r_2^{\text{can}} \nabla r_2^{\text{can}} \theta r_1^{\text{can}} \nabla r_1^{\text{can}} \omega_0^{\text{can}}$$

We have

$$\gamma(1) = \omega_0^{\text{can}}, \quad \gamma(f) = \omega_3^{\text{can}}, \quad \gamma(H) = -\omega_1^{\text{can}} \quad \text{and} \quad \gamma(H \circ f) = \omega_2^{\text{can}} + r_1^{\text{can}} r_2^{\text{can}} \omega_0^{\text{can}}$$

as shown by lemma [11.3.14].

**Remark 11.5.1** In the original coordinates $(q_1, q_2)$, the map $\nu$ is given by $\nu(q_1, q_2) = (r_1^{\text{can}}, r_2^{\text{can}})$.

**Theorem 11.5.2** The map $\gamma$ is an isomorphism for which

1. the small quantum product is given by

$$f \circ c = \gamma^{-1}((\theta r_1^{\text{can}} \nabla r_1^{\text{can}} \gamma(c))|_{\theta=0}) \quad \text{and} \quad H \circ c = \gamma^{-1}((\theta r_2^{\text{can}} \nabla r_2^{\text{can}} \gamma(c))|_{\theta=0})$$

for any cohomology class $c$ in $H^*(\mathbb{F}_2)$,

2. the metric $g_{\mathbb{F}_2}$ of $Q_{\mathbb{F}_2}$ is given by

$$g_{\mathbb{F}_2}(a, b) = S(\gamma(a), \gamma(b))$$

for any cohomology classes $a$ and $b$ in $H^*(\mathbb{F}_2)$ where $S$ is defined by formula (103).

**Proof.** By equations (110), the map $\gamma$ is indeed an isomorphism. The matrix of $f \circ$ , the quantum multiplication by $f$, in the basis $1, f, H, f \circ H$ is

$$\begin{pmatrix}
0 & r_1^{\text{can}} r_2^{\text{can}} & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & r_1^{\text{can}} r_2^{\text{can}} \\
0 & 0 & 1 & 0
\end{pmatrix}$$

and the one of $H \circ$ , the quantum multiplication by $H$ in the same basis is

$$\begin{pmatrix}
0 & 0 & r_2^{\text{can}} (1 - r_1^{\text{can}}) & 0 \\
0 & 0 & 0 & r_2^{\text{can}} (1 - r_1^{\text{can}}) \\
1 & 0 & 0 & 2 r_1^{\text{can}} r_2^{\text{can}} \\
0 & 1 & 2 & 0
\end{pmatrix}$$
We have moreover $g_{F^2}(1, H \circ f) = g_{F^2}(f, H) = \frac{1}{2} g_{F^2}(H, H) = 1$, $g_{F^2}(H \circ f, H \circ f) = 2r_1^{can}r_2^{can}$ and $g_{F^2}(a, b) = 0$ otherwise. In other words, the matrix of $g_{F^2}$ in the basis $1, f, H, f \circ H$ is

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 2 & 0 \\
1 & 0 & 0 & 2r_1^{can}r_2^{can}
\end{pmatrix}
\]

We conclude using the definition of $S$ and lemma 11.3.14.

\[\blacksquare\]

**Remark 11.5.3 (Versus I-function)** It follows from theorem 11.3.1 (2) that

\[
\theta q_2 \nabla q_2 (\theta q_2 \nabla q_2 - 2\theta q_1 \nabla q_1) \omega_0^{can} = q_2 \omega_0
\]

This is precisely the differential equation satisfied by Givental’s $I_{F^2}$-function, see f.i [6, page 394, formula 11.96]. This gives

\[
\theta r_2^{can} \nabla r_2^{can} (\theta r_2^{can} \nabla r_2^{can} - 2\theta r_1^{can} \nabla r_1^{can}) \omega_0^{can} = r_2^{can} (1 - r_1^{can}) \omega_0^{can}
\]

in flat coordinates. By definition of the map $\gamma$, it follows directly that $H \circ (H - 2f) = r_2^{can} (1 - r_1^{can})$.

**Remark 11.5.4** Using the definition of $\gamma$ and the properties of $S$ we check directly the Frobenius property $g_{F^2}(a \circ b, c) = g_{F^2}(b, a \circ c)$ for any cohomology classes $a, b, c$ in $H^*(F_2)$.

**Remark 11.5.5** The restriction of the Euler vector field $E$ at $H^2(F_2)$ is equal to $2H$. Together with the mirror correspondence, this explains the last part of remark 11.3.4.

### 11.6 The quantum cohomology of the weighted projective space $\mathbb{P}(1, 1, 2)$ as a classical limit (after [5])

We check here that specialization of the previous results at suitable values of the parameters $q_1$ et $q_2$ gives the small quantum orbifold cohomology of $\mathbb{P}(1, 1, 2)$, an aspect of Ruan’s conjecture. This has been first done in [5], in a slightly different setting.

We will work in the standard basis $(1, p, p^2, 1/2)$ of the orbifold cohomology $H_{orb}^*(\mathbb{P}(1, 1, 2))$ of the weighted projective space $\mathbb{P}(1, 1, 2)$. The matrix of the small quantum multiplication by $p$ in this basis is

\[
\begin{pmatrix}
0 & 0 & 0 & \frac{1}{2}q^{1/2} \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{2}q^{1/2} & 0
\end{pmatrix}
\]

Recall the coordinates $(r_1^{can}, r_2^{can})$ defined by

\[
(q_1, q_2) = \left(\frac{r_1^{can}}{(1 + r_1^{can})^2}, \frac{r_2^{can} (1 + r_1^{can})}{(1 + r_1^{can})^2}\right)
\]

and the flat basis $\omega^{can} = (\omega_0^{can}, \omega_1^{can}, \omega_2^{can}, \omega_3^{can})$.

\[\text{27 Notice however that, in our framework, we get the flat coordinates considered in [5] working with the flat basis obtained putting } p_{24} = \frac{1}{2} \text{ in section 11.3.3}\]
Proposition 11.6.1  1. The matrix of the small quantum multiplication $p_\circ$ in the basis $(1, p, p^2, 1_{1/2})$ of $H^*_\text{orb}(\mathbb{P}(1,1,2))$ is obtained from the matrix of $-\frac{1}{2} \theta r^\text{can}_2 \nabla r^\text{can}_2$ in the basis 

$$(\omega^\text{can}_0, \frac{1}{2} \omega^\text{can}_1, \frac{1}{2} \omega^\text{can}_2, i \omega^\text{can}_3 + i \omega^\text{can}_1)$$

putting $r^\text{can}_1 = -1$ and $r^\text{can}_2 = -iq^{1/2}$.

2. This correspondence associates the section 1 (resp. $H$, $H \circ f$, $f$) of $H^2(\mathbb{P})$ to the section 1 (resp. $-2p$, $2p^2 + q^{1/2} 1$, $-i 1_{1/2} - p$) of $H^*_\text{orb}(\mathbb{P}(1,1,2))$.

Proof. By lemma 11.3.14 the matrix of $-\frac{1}{2} \theta r^\text{can}_2 \nabla r^\text{can}_2$ in the basis $\omega^\text{can}$ is

$$\frac{1}{2} \begin{pmatrix} 0 & r^\text{can}_2 (1 + r^\text{can}_1) & 0 & -r^\text{can}_2 r^\text{can}_1 \\ 1 & 0 & r^\text{can}_1 r^\text{can}_2 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & r^\text{can}_1 (r^\text{can}_1 - 1) & 0 \end{pmatrix}$$

and, in the basis $(\omega^\text{can}_0, \frac{1}{2} \omega^\text{can}_1, \frac{1}{2} \omega^\text{can}_2, i \omega^\text{can}_3 + i \omega^\text{can}_1)$, it takes the form

$$\frac{1}{2} \begin{pmatrix} 0 & \frac{1}{2} r^\text{can}_2 (1 + r^\text{can}_1) & 0 & i \frac{1}{2} r^\text{can}_2 (1 - r^\text{can}_1) \\ 2 & 0 & \frac{1}{2} r^\text{can}_1 (r^\text{can}_1 + 1) & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & i \frac{1}{2} r^\text{can}_2 (1 - r^\text{can}_1) & 0 \end{pmatrix}$$

and thus we get, putting $r^\text{can}_1 = -1$,

$$\begin{pmatrix} 0 & 0 & i \frac{r^\text{can}_2}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \frac{r^\text{can}_2}{2} \end{pmatrix}$$

and the first assertion follows. The second one follows from theorem 11.5.2. $\square$

Remark 11.6.2 Put $v_1 = u_1$ and $v_2 = \frac{q_2}{u_2}$; our Landau-Ginzburg model becomes

$$v_1 + v_2 + \frac{q_2}{v_2} + \frac{q_1 q_2^2}{v_1 v_2}$$

and thus, in the flat coordinates $(r^\text{can}_1, r^\text{can}_2)$,

$$v_1 + v_2 + \frac{r^\text{can}_2 (1 + r^\text{can}_1)}{v_2} + \frac{r^\text{can}_1 (r^\text{can}_2)^2}{v_1 v_2}$$

If we set $r^\text{can}_1 = -1$ and $r^\text{can}_1 (r^\text{can}_2)^2 = q$ we get the usual Landau-Ginzburg model for $\mathbb{P}(1,1,2)$. 74
A Appendix: construction of the quantum differential systems associated with regular tame functions (B-side)

The Laplace transform of the Gauss-Manin connection of a tame regular function on an affine manifold produces quantum differential systems, see [11], [37], [8], [9]. We outline here the construction.

Let \( f : U \to \mathbb{C} \) be a regular function on an affine manifold \( U \), equipped with coordinates \( u = (u_1, \cdots, u_n) \). We consider the differential system satisfied by the Laplace integrals \( \int_{\Gamma} e^{-f/\theta} \omega \) where \( \omega \in \Omega^n(U) \) and \( \Gamma \) is a Lefschetz thimble [31]. This differential system is a meromorphic connection on \( \mathbb{P}^1 \) with poles at \( \theta = 0 \) and \( \theta = \infty \), that is a free \( \mathbb{C}[\theta, \theta^{-1}] \)-module \( G \) of finite rank \( \mu \), equipped with a flat connection \( \nabla \), the Gauss-Manin connection. We have

\[
G = \frac{\Omega^n(U)[\theta, \theta^{-1}]}{(d - \theta^{-1}df) \wedge \Omega^{n-1}(U)[\theta, \theta^{-1}]}
\]

(in other words, we work modulo the exacts forms \( d(e^{-f/\theta} \omega) \)) and the connection \( \nabla \) is defined by

\[
\theta^2 \nabla_{\partial_\theta} \left( \sum_i \omega_i\theta^i \right) = \sum_i f \omega_i\theta^i + \sum_i i\omega_i\theta^{i+1},
\]

taking into account the kernel \( e^{-f/\theta} \).

**Step 1: construction of a trivial (algebraic) bundle on \( \mathbb{P}^1 \).** We need a free \( \mathbb{C}[\theta] \)-submodule \( G_0 \) in \( G \) of maximal rank (in other words, a lattice in \( G \), which gives an extension of \( G \) at \( \theta = 0 \)) and a module opposite to \( G_0 \), that is a free \( \mathbb{C}[\tau] \)-submodule \( G_\infty \) (an extension of \( G \) at \( \theta = \infty \)) such that

\[
G_0 = G_0 \cap G_\infty \oplus \theta G_0
\]

Indeed, we have \( G = G_0[\tau] = G_\infty[\theta] \): the pair \( (G_0, G_\infty) \) defines a bundle \( \mathcal{G} \) on \( \mathbb{P}^1 \) and the decomposition (113) shows that this bundle is trivial, see [35] (Chapitre IV, paragraphe 5). It follows from equation (113) that the restrictions of \( \mathcal{G} \) at \( \theta = 0 \) and \( \theta = \infty \) are isomorphic via the global sections \( G_0 \cap G_\infty \). A natural candidate for \( G_0 \) is

\[
G_0 := \frac{\Omega^n(U)[\theta]}{(\theta d - df) \wedge \Omega^{n-1}(U)[\theta]},
\]

the Brieskorn module of \( f \), the image of \( \Omega^n(U)[\theta] \) in \( G \). Notice the following important two points: by definition we have

\[
G_0/\theta G_0 = \Omega^n(U)/df \wedge \Omega^{n-1}(U)
\]

and

\[
\theta^2 \nabla_{\partial_\theta} G_0 \subset G_0.
\]

However, \( G_0 \) is not always free over \( \mathbb{C}[\theta] \): it will be the case if \( f \) is assumed to be tame [37], [11] (in this situation, we will call \( G_0 \) the Brieskorn lattice of \( f \)). A basic example of such tame functions are the (Laurent) polynomials which are convenient and non-degenerate with respect to their Newton polygons at infinity, for which the freeness follows from a division theorem (essentially due to

\[28\] Prototypes : \( U = \mathbb{C}^n \) and \( U = (\mathbb{C}^*)^n \)
Step 2: adding a connection with prescribed poles. We still need a connection on the trivial bundle $G$ with poles of order less or equal to 2 at $\theta = 0$ and logarithmic poles at $\theta = \infty$. In other words, the matrix of this connection in a basis of global sections should take the form

$$\left(\frac{A_0}{\theta} + A_\infty\right) \frac{d\theta}{\theta} \quad (114)$$

This is the so-called Birkhoff problem for $G_0$. A canonical solution is provided by Hodge theory as follows: first, the general statement is The solutions of the Birkhoff problem are in one-to-one correspondence with the opposite filtrations, stable under the action of the monodromy, to the Hodge filtration defined on the nearby cycles, see [11], [39]. In brief, the opposinness gives decomposition (113) and the stability with respect to the monodromy gives formula (114). Here we use also the classical correspondence between logarithmic lattices and decreasing filtrations, see f.i [35, Theorem III.1.1]. To be precise, let $V_\bullet$ be the Kashiwara-Malgrange filtration of $G$ at $\tau = 0$ and $H_\alpha := V_\alpha G/V_{<\alpha}G$. For $\alpha \in \mathbb{Q} \cap [0, 1]$, we define the (Hodge) filtration $F_\bullet$ by

$$F_p H_\alpha := (V_\alpha G \cap \tau^p G_0 + V_{<\alpha} G)/V_{<\alpha}G$$

where $p \in \mathbb{Z}$. Because $F_\bullet$ is the Hodge filtration of a mixed Hodge structure (see [36]), there exists a decreasing filtration $U_\bullet$ of $H_\alpha$ such that:

- for all $p \in \mathbb{Z}$, $N(U^p H_\alpha) \subset U^{p+1} H_\alpha$, where $N$ denotes the nilpotent endomorphism induced by $\tau \partial_\tau + \alpha$ on $H_\alpha$,
- the filtration $U_\bullet$ is a filtration opposite to the filtration $F_\bullet$, i.e $H_\alpha = \bigoplus q F_q H_\alpha \cap U_q H_\alpha$.

As observed in [39, Lemma 2.8] (a game with Deligne’s $I^{pq}$), we can even choose the filtration $U^\bullet$ such that $N(U^p H_\alpha) \subset U^{p+1} H_\alpha$. In this case, the matrix $A_\infty$ in equation (114) is semi-simple, with the expected eigenvalues. This opposite filtration, built using M. Saito’s method, provides a canonical solution of the Birkhoff problem, see [11, Appendix B].

Step 3: the metric. The Gauss-Manin system $G$ of a tame, regular function, is self-dual (microlocal Poincaré duality, see [37]): if

$$G^* = Hom_{\mathbb{C}[\theta, \theta^{-1}]}(G, \mathbb{C}[\theta, \theta^{-1}])$$

we have an isomorphism of connections $G^* \rightarrow j^* G$ which sends $G_0^*$ onto $\theta^n j^* G_0$. We thus get a non-degenerate bilinear form $S : G \times j^* G \rightarrow \mathbb{C} \theta^n (115)$ such that $S : G_0 \times j^* G_0 \rightarrow \theta^n \mathbb{C}[\theta]$. Let us write $S = \sum_{k \geq 0} S_k \theta^k$ on $G_0$: the pairings $S_k$ are called higher residue pairings (after K. Saito) and $S_n$ is precisely the Grothendieck residue defined on $G_0/\partial G_0$. The form $S$ extends to $G$ if there exists a basis $\omega$ of global sections which is adapted to $S$, i.e

$$S(\omega_i, \omega_j) \in \mathbb{C} \theta^n \quad (115)$$

for all $i, j$. This will be the case if the lattice $G_\infty$ alluded to in step 1 and constructed in step 2 is choosen such that $S : G_\infty \times j^* G_\infty \rightarrow \tau^{-n} \mathbb{C}[\tau]$. But this is again provided by the canonical opposite
filtration.

**Résumé of steps 1-3:** we attach a quantum differential system (on a point) \((G, \nabla, S, n)\) to any regular, tame function on \(U\).

**Step 4 : adding parameters.** In order to get a bundle on \(\mathbb{P}^1 \times M\), we have to extend the previous situation to a situation “with parameters”. We will denote by \(x = (x_1, \cdots, x_r)\) the coordinates on \(M\).

**Method 1:** one can repeat the previous construction, starting with the Gauss-Manin system of an unfolding \(F\) of \(f\) (see for instance [11]) and taking into account (and in addition) the covariant derivative of the Gauss-Manin connection with respect to the parameters. Due to the “critical points vanishing at infinity” (see [11, Examples 2.5]), this method is in general transcendental, in the parameter axis (always) but also in the \(\theta\)-axis. The finitness of \(G_0^F\) follows in this setting from standard results in analytic geometry, as in the local (i.e germ) case. Notice that \(\theta \nabla_{\partial_{x_i}} G_0^F \subset G_0^F\): in a basis of \(G_0^F\) is *a priori*

\[
\frac{C^{(i)}(x)}{\theta} + D^{(i)}(x) + \sum_{r=1}^{p} D^{(i)}_r(x)\theta^r
\]  

and we want the formula \(D^{(i)}(x) + \frac{C^{(i)}(x)}{\theta}\) in order to get a quantum differential system.

**Method 2:** one can use, as in [8, 9] for instance, the Dubrovin-Malgrange-Hertling-Manin reconstruction theorem, see theorem 2.3.2. The idea is to start with a deformation of \(f\) that doesn’t produce vanishing critical points at infinity: this is actually what is done for ”subdiagram deformations” of a convenient and non-degenerate polynomial in [8, 9]. In some cases these deformations (the “initial data”) suffice in order to understand universal ones, thanks to the reconstruction theorem quoted above. The advantage now is that we work algebraically in the variable \(\theta\).

**Résumé of steps 1-4 :** summarizing, one associates a quantum differential system on \(M\) to a tame regular function on the affine manifold \(U\).

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