On the Unification of Gauge Symmetries in Theories with Dynamical Symmetry Breaking

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We analyze approaches to the partial or complete unification of gauge symmetries in theories with dynamical symmetry breaking. Several types of models are considered, including those that (i) involve sufficient unification to quantize electric charge, (ii) attempt to unify the three standard-model gauge interactions in a simple Lie group that forms a direct product with an extended technicolor group, and, most ambitiously, (iii) attempt to unify the standard-model gauge interactions with (extended) technicolor in a simple Lie group.

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I. INTRODUCTION

The standard model (SM) gauge group $G_{SM} = SU(3)_c \times SU(2)_w \times U(1)_Y$ has provided a successful description of both strong and electroweak interactions. Although the standard model itself predicts zero neutrino masses, its fermion content can be augmented to incorporate neutrino masses and lepton mixing. One of the great triumphs of this model was its unification of weak and electromagnetic interactions. However, as has long been recognized, there are a number of properties that this model does not explain, including the quantization of the electric charges of elementary particles, the ratios of the values of the respective standard-model gauge couplings $g_3, g_2L$, and $g_Y$, and the interconnected manner in which quark and lepton contributions to gauge anomalies cancel each other (separately for each generation). These deficiencies motivated the effort to construct theories with higher unification of gauge symmetries. Almost all of the work toward this goal started with the standard model, including its Higgs mechanism, and subsequently, supersymmetric extensions of this model.

In this paper we shall investigate various approaches to the partial or complete unification of gauge symmetries from a different viewpoint, incorporating the standard-model gauge group but removing the Higgs mechanism of this model and replacing it with ingredients that can produce dynamical electroweak symmetry breaking (EWSB) and also dynamical breaking of higher gauge symmetries. The origin of electroweak symmetry breaking is one of the most important outstanding questions in particle physics, and dynamical EWSB remains an interesting alternate to the Higgs approach. We shall focus, in particular, on models in which electroweak symmetry breaking is due to the formation of a bilinear condensate of new fermions interacting via an asymptotically free, vectorial gauge interaction, generically denoted technicolor (TC) $G_{TC}$, that becomes strong at a scale $\Lambda_{TC} \sim 300$ GeV. To communicate the electroweak symmetry breaking to the quarks and leptons (which are technisinglets) and generate masses for these fermions, one adds to the technicolor theory additional gauge degrees of freedom that transform technifermions into standard-model fermions and vice versa $\mathbf{3} \otimes \mathbf{\bar{3}}$. These are denoted extended technicolor (ETC) gauge bosons.

Here we shall consider several types of unification of gauge symmetries with dynamical symmetry breaking:

1. Models that involve sufficient unification to quantize electric charge without embedding all of the three factor groups of the standard model in a (semi)simple Lie group $\mathbf{3} \otimes \mathbf{\bar{3}}$. Since $Q = T_{SM} + Y/2$ in the standard model, a sufficient condition for this quantization is that the weak hypercharge $Y$ be expressed as a linear combination of generators of nonabelian gauge groups.

2. Models that attempt to unify the three standard-model gauge interactions in a simple grand unified (GU) group $G_{GU}$,

$$G_{GU} \supset G_{SM}$$

(1.1)

and then combine this in a direct product, $G_{ETC} \times G_{GU}$ with the ETC gauge group, $G_{ETC}$. A successful model of this type would explain charge quantization and the relative sizes of SM gauge couplings (but not the relative size of the SM and ETC gauge couplings).

3. Most ambitiously, models that attempt to unify the three SM gauge interactions together with technicolor, or a larger gauge symmetry described by a group $G_{SC} \supset G_{TC}$, in a simple Lie group $G$,

$$G \supset G_{SC} \times G_{GU}$$

(1.2)

In the models that we consider of types (1) and (2) above, the technicolor group $G_{TC}$ is embedded in a larger, extended technicolor group, $G_{ETC}$. As indicated by this subgroup relation, in these types of models, the infinitesimal generators of the Lie algebra of $G_{TC}$ close upon themselves, as do the generators of the Lie algebra of $G_{ETC}$. Furthermore, in these models...
$[G_{ETC}, G_{SM}] = 0$, so that the ETC gauge bosons do not carry any SM quantum numbers. In contrast, in models of type (3), although $[G_{ETC}, G_{SM}] = 0$, the commutators of the (linear combinations of) generators of the Lie algebra of $G$ that transform technicolor indices to $SU(3)_c$ and $SU(2)_L$ indices and vice versa (corresponding to the ETC gauge bosons) generate the full Lie algebra of $G$, so that there is no ETC group, per se, that forms a subgroup of $G$ that is smaller than $G$ itself. In these models these ETC gauge bosons generically do carry SM quantum numbers and the corresponding generators of $G$ do not commute with the generators of $G_{SM}$. As will be seen below, the origin of standard-model fermion generations is different in models of types (1) and (2), on the one hand, and models of type (3) on the other.

The present paper is organized as follows. In section II we review some relevant properties of technicolor and extended technicolor. In Section III we consider a partial unification model of type (1) in which charge is quantized and all symmetry breaking is dynamical, and we address the question of how one might try to unify this model further. Sections IV and V contain analyses and critical assessments of models of type (2) and (3). In Section VI we discuss the issue of how to break a unified gauge symmetry dynamically. Section V contains some concluding remarks and ideas for further work. Certain general formulas that are used throughout the paper are contained in an appendix.

II. TECHNICOLOR AND EXTENDED TECHNICOLOR MODELS

In this section we discuss some relevant properties of technicolor and extended technicolor theories. We take the technicolor group to be $G_{TC} = SU(N_{TC})$. For models of types (1) and (2), the technifermions will comprise a standard-model family, i.e., they transform according to the following representations of $G_{SM}$:

$$Q_L^{ai} = \left( \begin{array}{c} U^{ai} \\ D^{ai} \end{array} \right)_L : (N_{TC}, 3, 2)_1/3, L;$$
$$U_R^{ai} : (N_{TC}, 3, 1)_{4/3, R}, \quad D_R^{ai} : (N_{TC}, 3, 1)_{-2/3, R};$$
$$L_L^i = \left( \begin{array}{c} N_i \\ E_i \end{array} \right)_L : (N_{TC}, 1, 2)_{-1, L};$$
$$\{ N_R^i \} : (N_{TC}, 1, 1)_{0, R}, \quad E_R^i : (N_{TC}, 1, 1)_{-2, R}$$

(2.1)

where here the indices $a$ and $i$ are color and technicolor indices, respectively, and the numerical subscripts refer to weak hypercharge. Since the $N_R^i$ are SM-singlets and the group $SU(2)$ is free of anomalies in gauged currents, it follows that if $N_{TC} = 2$, then there can be more than just one $N_R^i$; we indicate this by the brackets and denote this number $N_{NR}$. In models of type (3) we will encounter different types of technicolor fermion sectors.

For models of types (1) and (2), which have a well-defined ETC gauge group, we take this group to be

$$G_{ETC} = SU(N_{ETC}).$$

(2.2)

For these models, a natural procedure in constructing the ETC theory is to gauge the generation index, assigning the first $N_{gen.}$ components of a fundamental representation of $SU(N_{ETC})$ to be the standard-model fermions of these three generations, followed by $N_{TC}$ components which are the technifermions with the same standard-model quantum numbers. The fact that, a priori, the value of $N_{gen.}$ is arbitrary except for the requirement that the ETC theory be asymptotically free, distinguishes these types of models from models of type (3), where the origin and number of SM fermion generations are highly constrained. Thus, models of types (1) and (2) can automatically accommodate the observed value of SM fermion generations, $N_{gen.} = 3$, whereas, in contrast, this success is not guaranteed for a particular model of type (3). Given the way in which models of types (1) and (2) gauge the generational index, it follows that for these models

$$N_{ETC} = N_{gen.} + N_{TC} = 3 + N_{TC}.$$  

(2.3)

The relation (2.3) and the requirement that $N_{TC} \geq 2$ for a nontrivial nonabelian $SU(N_{TC})$ group (as required for asymptotic freedom) together imply that $N_{ETC} \geq 5$ for models of types (1) and (2). The minimal choice, $N_{TC} = 2$ and hence $N_{ETC} = 5$, has been used for a number of recent studies of ETC models [11, 12]. The choice $N_{TC} = 2$ is motivated for a number of reasons; (a) with the one-family structure of eq. (2.1), amounting to $N_{FF} = 2(N_c + 1) = 8$ vectorially coupled technifermions in the fundamental representation of $SU(2)_{TC}$, it can yield an approximate infrared fixed point and associated slow running (“walking”) of the TC gauge coupling [10] from $N_{TC}$ up to an ETC scale [11, 12]. (b) It minimizes the technicolor contributions to the electroweak $S$ parameter [17], and (c) it makes possible a mechanism to account for light neutrinos in an extended technicolor context [12]. Although the value $N_{TC} = 2$ is thus favored, we often shall let $N_{TC}$ be arbitrary in the present paper (subject to the requirement of asymptotic freedom of the TC and ETC theories) in order to show the generality of certain results.

The condition $[G_{ETC}, G_{SM}] = 0$ in models of type (2) means that all components of a given representation of $G_{SM}$ transform according to the same representation of $G_{ETC}$ and all components of a given representation of $G_{ETC}$ transform according to the same representation of $G_{SM}$. However, this does not imply that all of the representations of $G_{SM}$ transform according to the same representation of $G_{ETC}$. For example, in Ref. [12], we studied a class of ETC models in which $[G_{ETC}, G_{SM}] = 0$ and the left-handed and right-handed representations of the charge $Q = -1/3$ quarks and techniquarks transform according to relatively conjugate representations of $G_{ETC}$. 


(which were the fundamental and conjugate fundamental representations), and similarly with the charged leptons and technileptons, while the charge $Q = 2/3$ quarks and techniquarks of both chiralities transformed according to the same representations of $G_{ETC}$.

ETC models of this type (2) can be classified further according to whether (i) the ETC gauge interactions are vectorial on the SM quarks and charged leptons, or (ii) some ETC gauge interactions are chiral on these SM fermions. In both cases, the TC interaction must be vectorial; this is automatically satisfied by the SU(2)$_{TC}$ group since it has only (pseudo)real representations. (In this SU(2)$_{TC}$ case, the number of chiral doublets is $15 + N_{N_R}$, and this must be even to avoid a global SU(2) anomaly, so $N_{N_R}$ must be odd.) These two options (i) and (ii) were labelled VSM and CSM in Ref. [13], where the V and C referred to the corresponding vectorial and (relatively) conjugate ETC representations of the SM quarks and charged leptons, respectively. While both of these classes of models have promising features, neither is fully realistic. In Ref. [13] it was shown that constraints from neutral flavor-changing current processes were not as severe for VSM-type models as had been previously thought. However, these models require additional ingredients to produce mass splittings within each generation, such as $m_{t} > m_{b}, m_{\tau}$ (without excessive contributions to the parameter $\rho = m_{W}^{2}/(m_{Z}^{2} \sin^{2} \theta_{W})$). The CSM-type models in which charge $-1/3$ quarks and leptons of opposite chiralities transform according to relatively conjugate representations of SU(5)$_{ETC}$ while the charge $2/3$ quarks have vectorial ETC couplings can produce these requisite intragenerational mass splittings and also some CKM mixing; however, these models do have problems with flavor-changing neutral current processes. Hence, we shall concentrate on VSM-type ETC models here. Moreover, additional ingredients are necessary to avoid overly light Nambu-Goldstone bosons.

As an illustration, for VSM models of type (2) the fermions with SM quantum numbers are assigned to representations of the group SU(5)$_{ETC} \times G_{SM}$ which are obtained from those listed in eq. (2.1) by letting the index $i$ range over the full set of ETC indices, $i = 1, ..., N_{ETC}$. Here and below, it will often be convenient to use the compact notation $Q_L, L_L, u_R, d_R$, and $e_R$ to denote these ETC multiplets, so that, for example, $e_R$, written out explicitly, is

$$e_R \equiv (e^1, e^2, e^3, e^4, e^5)_R \equiv (e, \mu, \tau, E^1, E^5)_R. \quad (2.4)$$

Since the fermion content of the SU(5)$_{ETC}$ theory is chosen so that it is asymptotically free [13], as the energy scale decreases from large values, the ETC coupling increases in strength. Eventually, this coupling becomes large enough to produce fermion condensates, and the ETC sector is constructed to be a chiral gauge theory, so that a bilinear fermion condensate generically self-breaks the ETC gauge symmetry. In order to obtain the desired sequential breaking of SU(5)$_{ETC}$ to SU(2)$_{TC}$, Refs. [5-13] incorporated an additional asymptotically free gauge interaction which becomes strongly coupled at roughly the same energy scale as the ETC interaction. This additional interaction was called “hypercolor” (HC) and the corresponding gauge group was chosen to be SU(2)$_{HC}$. The symmetry breaking of SU(5)$_{ETC}$ occurs as a combination of self-breaking and couplings to the auxiliary strongly interacting group, SU(2)$_{HC}$. The fermions involved in producing the ETC symmetry-breaking condensates are nonsinglets under the ETC group and are singlets under the standard-model group; they include both singlets and nonsinglets under the HC group. The determination of which condensation channels are dynamically favored is a difficult, nonperturbative problem involving strong coupling. The procedure makes use of the “most attractive channel” (MAC) criterion [19] and tools such as approximate solutions of the Schwinger-Dyson equation for the relevant fermion propagator [20, 22] (see appendix). As the energy scale decreases, the first breaking occurs at a scale denoted $\Lambda_1$, where SU(5)$_{ETC}$ → SU(4)$_{ETC}$: here the first generation fermions split off from the rest in each ETC multiplet. Since this is the highest ETC symmetry-breaking scale, we shall label it more generally as $\Lambda_{ETC,max}$. Similarly, one has the successive breakings SU(4)$_{ETC}$ → SU(3)$_{ETC}$ at $\Lambda_2$, where the second-generation fermions split off, and SU(3) → SU(2)$_{TC}$ at $\Lambda_3$, where the third-generation fermions split off, leaving the exact residual technicolor gauge symmetry. This can account, at least approximately, for the observed fermion masses while satisfying other constraints such as those from flavor-changing neutral current processes, if one takes $\Lambda_1 \sim 10^3$ TeV, $\Lambda_2 \sim 10^2$ TeV, and $\Lambda_3 \approx 4$ TeV, as in Refs. [5-13]. The generational ETC scales are bounded above by the requirement that the resultant SM quark and lepton masses of the $j$th generation, $m_{f_j} \sim \eta_j \Lambda_{TC}^{3}/\Lambda^2_j$ be sufficiently large (where $\eta_j$ is an enhancement factor present in theories with walking technicolor [16] and can be of order $\Lambda_3/\Lambda_{TC}$).

An important general feature of ETC theories, illustrated in the specific models mentioned above, is that the ETC symmetry-breaking scales are far below the conventional grand unification scale $M_{GU} \sim 10^{16}$ GeV. Recall that, before engaging in detailed calculations of gauge coupling evolution, one knows that it would be difficult to have a generic grand unification scale lower than $10^{15} - 10^{16}$ GeV without producing excessively rapid nucleon decay. Therefore, insofar as one studies the possibility of unifying standard-model and technicolor gauge symmetries, it is not technicolor itself, but rather the higher symmetry associated with extended technicolor, that enters into this unification. That is, one must take account of the fact that the effective theory at energy scales far below the grand unification scale is already invariant under a larger symmetry involving gauge degrees of freedom transforming technicolor indices to standard-model (color and electroweak) indices.
III. EXAMPLE OF PARTIAL UNIFICATION OF SM GAUGE SYMMETRIES AND ATTEMPT AT HIGHER UNIFICATION

In this section we consider the model of type (1) from Ref. [11, 12], which successfully achieves the important goals of electric charge quantization via partial unification of standard-model gauge interactions with all symmetry breaking dynamical, and we investigate how one might try to unify it further to be a model of type (2). A sufficient condition for the quantization of electric charge \( Q \) (or equivalently, the quantization of weak hypercharge \( Y \), given the \( Q = T_{3L} + Y/2 \) relation) is that \( Q \) be expressed as a linear combination of generators of non-abelian gauge groups. An early realization of this condition was provided by the Pati-Salam unification of color \( SU(3)_c \) with \( U(1)_{B-L} \) in \( SU(4)_{PS} \), where \( B \) and \( L \) denote baryon and lepton number [23]. In this type of theory, the strong and electroweak gauge groups are enlarged to the group

\[
G_{422} = SU(4)_{PS} \times SU(2)_L \times SU(2)_R \ .
\]

where \( SU(2)_L = SU(2)_w \) of the SM. The left- and right-handed SM fermions of each generation are assigned to the representations

\[
\left( \frac{u^a}{d^a}, \frac{\nu^i}{e^i} \right)_\chi, \quad \chi = L, R \ ,
\]

transforming as (4,2,1) and (4,1,2), respectively, under the group \( G_{422} \). The superscripts \( a \) and \( i \) in \( \chi = L, R \) refer, as before, to color and generation. This fermion content thus requires the addition of three right-handed neutrinos to the standard model. The \( SU(4)_{PS} \) gauge symmetry is vectorial, while the \( SU(2)_L \) and \( SU(2)_R \) symmetries are chiral. The electric charge operator is given by

\[
Q = T_{3L} + T_{3R} + (1/2)(B - L) \\
= T_{3L} + T_{3R} + (2/3)\sqrt{2} T_{PS,15} \\
= T_{3L} + T_{3R} + (1/6) \text{diag}(1,1,1,-3) \ ,
\]

where \( T_{PS,15} = (2\sqrt{6})^{-1} \text{diag}(1,1,1,-3) \) is the third diagonal generator in the \( SU(4)_{PS} \) Lie algebra. The (quantized) hypercharge generator is \( Y = T_{3R} + (2/3)\sqrt{2} T_{PS,15} \). Since we will analyze the question of unification for the gauge couplings of the group \( G_{422} \), we show their explicit normalization via the covariant derivative,

\[
D_\mu = \partial_\mu - ig_{PS} T_{PS} \cdot A_{PS,\mu} \\
- ig_{2L} T_L \cdot A_{L,\mu} - ig_{2R} T_R \cdot A_{R,\mu} \ .
\]

The model of Refs. [11, 12] uses the gauge group

\[
SU(5)_{ETC} \times SU(2)_{HC} \times G_{422}
\]

with the fermion representations

\[
(5, 1, 4, 2, 1)_L , \quad (5, 1, 4, 1, 2)_R \ .
\]

(This model also contains fermions that are singlets under \( G_{422} \).) In addition to the successful quantization of electric charge and partial unification of quarks with leptons (and techniquarks with technileptons), the \( SU(4)_{PS} \) gauge interactions connecting quarks and leptons gives mass to the \( P^0 \) and \( P^3 \) Nambu-Goldstone bosons corresponding to the generators \( I_{2V} \times T_{PS,15} \) and \((T_{3}2V \times T_{PS,15}) [23] \), where \( I \) denotes the identity and the subscript \( 2V \) refers to vectorial isospin, \( SU(2)_V \).

In the model of Ref. [12], as the energy decreases below a scale \( \Lambda_{PS} \gtrsim \Lambda_1 \approx 10^6 \text{ GeV} \), the \( G_{422} \) gauge symmetry is broken to \( G_{SM} \) by the formation of a bilinear fermion condensate. This value for \( \Lambda_{PS} \) satisfies experimental constraints such as those from upper limits on right-handed charged weak currents and on the branching ratio for the decays \( K_L \to \mu^+\mu^- \). The matching relations for the (running) coupling constants at this scale \( \Lambda_{PS} \) are \( g_3 = g_{PS} \) and [12]

\[
\frac{1}{g_T^2} = \frac{1}{g_{2L}^2} + \frac{2}{3g_{PS}^2} \ .
\]

Further symmetry breaking at lower scales is the same as in the model with gauge group [24]. The relation between the electromagnetic coupling \( e \) and the above gauge couplings is

\[
\frac{1}{e^2} = \frac{1}{g_{2L}^2} + \frac{1}{g_Y^2} = \frac{1}{g_{2L}^2} + \frac{1}{g_{2R}^2} + \frac{2}{3g_{PS}^2} \ .
\]

from which one can calculate the weak mixing angle

\[
\sin^2 \theta_W = e^2/g_{2L}^2 \ .
\]

This partial gauge coupling unification is consistent with the precision determination of the three SM gauge couplings [12]. Evolving the SM couplings from \( m_Z \) to \( \Lambda_{PS} \), one finds, at the latter scale, the values \( \alpha_3 = 0.064, \alpha_{2L} = 0.032, \) and \( \alpha_{PS} = 0.008 \) (where \( \alpha_j = g_j^2/(4\pi) \)), so that the matching equations can be satisfied with \( \alpha_{2R}(\Lambda_{PS}) \approx 0.013 \), i.e., \( g_{2L}/g_{2L} \approx 0.64 \) at this scale.

The results of Refs. [11, 12] motivate one to investigate the possibility of embedding the three factor groups of \( G_{422} \) in a simple group \( G_{CU} \), which could form a direct product with \( G_{ETC} \) (and possible other groups such as an auxiliary hypercolor group). This would promote this model of type (1) to a model of type (2). We observe that \( SU(4) \approx SO(6) \) and \( SU(2) \times SU(2) \approx SO(4) \), so \( SO(10) \) contains, as a maximal subgroup, the direct product \( SO(6) \times SO(4) \). Similarly, there is a natural embedding of the three-fold direct product \( G_{422} \) as a maximal subgroup in \( SO(10) \). A necessary condition for this unification is that the three gauge couplings, \( g_{PS}, g_{2L}, \) and
to the single $SO(10)$ coupling $g$. For an exploratory study of the feasibility of this, it will be sufficient to use one-loop renormalization group evolution equations. Some relevant general formulas are listed in the appendix. We need the leading coefficients in the beta functions for each factor group for the energy interval above $\Lambda_{PS}$. These are, for $SU(4)_P$,\[ b^{(PS)}_0 = \frac{1}{3}[44 - 4(N_{gen} + N_{TC})] \]
\[ = \frac{1}{3}(32 - 4N_{TC}) \quad (3.10) \]
and, for $SU(2)_L$ and $SU(2)_R$,\[ b^{(2L)}_0 = b^{(2R)}_0 = \frac{1}{3}[22 - 4(N_{gen} + N_{TC})] \]
\[ = \frac{1}{3}(10 - 4N_{TC}) \quad (3.11) \]
where the formulas are given for general $N_{TC}$ to show the fact that our conclusions concerning unification hold for arbitrary values of this parameter. We find that, with the fermion content as specified above, the couplings $g_{PS}$, $g_{2L}$, and $g_{2R}$ do not unify at any higher energy scale. In particular, since $\alpha_{2L}$ and $\alpha_{2R}$ are unequal at $\Lambda_{PS}$ and have the same beta functions, the respective $\alpha_{2L}^{-1}$ and $\alpha_{2L}$ evolve as a function of $\ln \mu$ as two parallel lines, which precludes unification. This is still true if one augments the fermion content of the hypothetical $SO(10)$ theory, since the representations of $SO(10)$ treat the $SU(2)_L$ and $SU(2)_R$ subgroups symmetrically. Thus, we find that it appears to be difficult to increase the partial unification of the SM gauge symmetries in this model to a full unification of these symmetries in a direct product group containing $SO(10) \times SU(5)_{ETC}$. Nevertheless, the model of Refs. \[ 11,12 \] does provide an example of partial unification of SM gauge symmetries explaining charge quantization in a fully dynamical framework.

IV. PROSPECTS FOR MODELS WITH A $G_{ETC} \times G_{GU}$ SYMMETRY GROUP

A. Evolution of SM Gauge Couplings in an ETC Framework

In this section we assess the prospects for attempting to unify the three gauge groups of the standard model, $SU(3)_c$, $SU(2)_w$, and $U(1)_Y$, in a simple grand unified group $G_{GU}$ which forms a direct product with the ETC group (and possibly other groups such as hypercolor) at a high scale, $M_{GU}$. In terms of the classification given at the beginning of the paper, these are models of type (2). Here and elsewhere in the paper the adjective “grand unified” is used with its historical meaning, as referring to the unification of the three SM gauge interactions only, not additional interactions such as (extended) technicolor. We assume that at $M_{GU}$, $G_{GU}$ breaks to the

three-fold direct product group comprising $G_{SM}$, so that in the interval of energies extending downward from $M_{GU}$ to the highest ETC scale, $\Lambda_{ETC,3} \approx 10^6$ GeV, the effective field theory is invariant under $G_{ETC} \times G_{SM}$ (times possible auxiliary groups such as hypercolor). As before, we take $G_{ETC} = SU(N_{ETC})$ and, to show the generality of our results, we keep $N_{TC}$ arbitrary (subject to the requirement of asymptotic freedom for the ETC and TC group).

A prerequisite for this unification is the condition that the three SM gauge couplings unify at the hypothetical scale $M_{GU}$. The normalization of the abelian gauge coupling is determined by the embedding of the fermions with SM quantum numbers in the unified group, $G_{GU}$. We shall assume that $G_{GU}$ is either of the well-known unification groups $SU(5)$ \[ 25 \] or $SO(10)$ \[ 26 \], with the usual assignments of SM fermions; in both cases, the $U(1)$ coupling that unifies with $g_3$ and $g_{2L}$ is $g_1 = \sqrt{5/3} \, g_Y$. We will take $\mu = M_Z$ as the starting point for the evolution of the SM couplings to higher scales. For our analysis, it will be adequate to use the one-loop approximations to the respective beta functions, as given in eqs. 8.10 and 8.11 of the appendix; these depend on the leading-order coefficients $b^{(j)}_0$ for each factor group $G_j$. It will also be sufficient to take the top quark to be dynamically at the electroweak scale, i.e., to include its contribution in the calculation of the beta functions. The values of the $b^{(j)}_0$ for the standard model with its Higgs boson are well known: \[ b^{(0)}_0 = (1/3)(33 - 2N_q) = 7 \]
\[ b^{(2)}_0 = (1/3)(22 - N_d - 1/2) = 19/6 \]
\[ b^{(3)}_0 = (3/5)b^{(Y)}_0 = -41/10 \]
where $N_q = 2N_{gen} = 6$ denotes the number of active quarks and $N_d = N_{gen}(N_c + 1) = 12$ denotes the number of $SU(2)_w$ doublets. It is also well known that, if one evolves these couplings individually without further new physics at intermediate scales, they do not unify at any one scale.

To calculate the evolution of the SM gauge couplings in the framework of an ETC theory, we first remove the SM Higgs and, for energies above $\mu \sim \Lambda_{TC} \sim 300$ GeV, where the technifermions are active, we add their contributions to the $b^{(j)}_0$. The deletion of the Higgs boson from the theory removes a term $-1/6$ from $b^{(2)}_0$, which becomes $b^{(2)}_0 = 10/3$, and a term $-1/6$ from $b^{(Y)}_0$, so that $b^{(3)}_0$ becomes $b^{(Y)}_0 = -4$ (and leaves $b^{(3)}_0$ unchanged). A caveat is that, even if the SM gauge couplings are small at a given scale $\mu$, their evolution may still be significantly affected by nonperturbative, strong couplings of the SM nonsinglet technifermions. Since we start our integration of the renormalization group equations at $\mu = M_Z$, which is comparable to, and, indeed, slightly less than, the technicolor scale, $\Lambda_{TC} \sim 300$ GeV, these strong technifermion interactions produce some uncertainty in the evolution of the SM gauge couplings. This is a consequence of the fact that in the beta function calculations, one treats the technifermions as weakly interacting, but this is only a good approximation for $\mu \gg \Lambda_1 \approx 10^6$
Generalizing $N_q$ to refer to both quarks and techniquarks, and letting $N_d$ denote the total number of SU(2)$_L$ doublets, we calculate

$$b_0^{(3)} = \frac{1}{3} (11N_c - 2N_q)$$

$$= \frac{1}{3} [33 - 4(N_{gen.} + N_{TC})]$$

$$= 7 - \frac{4}{3}N_{TC}$$  \hspace{1cm} (4.1)

$$b_0^{(2)} = \frac{1}{3} (11N_w - N_d)$$

$$= \frac{1}{3} [22 - (N_c + 1)(N_{gen.} + N_{TC})]$$

$$= \frac{10}{3} - \frac{4}{3}N_{TC}$$ \hspace{1cm} (4.2)

and

$$b_0^{(1)} = \frac{3}{5} b_0^{(Y)} = -\frac{4}{3} (N_{gen.} + N_{TC})$$

$$= -4 - \frac{4}{3}N_{TC}.$$  \hspace{1cm} (4.3)

As is evident in these results, the respective beta functions, and, in particular, the leading coefficients $b_0^{(j)}$, depend on $N_{TC}$ only through the combination $N_{gen.} + N_{TC} = N_{ETC}$. Consequently, the addition of the one family of technifermions to the fermion content of the standard model is equivalent to the addition of $N_{TC}$ additional generations of SM fermions. Now we recall that the addition of one or more (complete) generations of SM fermions leaves the differences $(\Delta b_0_{ij}) = b_0^{(ij)} - b_0^{(j)}$, $ij = 12, 13, 23$ invariant [22]. (Explicitly, $(\Delta b_0_{32}) = 11/3$, $(\Delta b_0_{13}) = 11$, and $(\Delta b_0_{21}) = 22/3$.) Hence, for a given set of values of $\alpha_j$, $j = 1, 2, 3$ at $\mu = M_Z$, the scales $\mu_{ij}$ where $\alpha_i = \alpha_j$ are independent of $N_{TC}$, in the same way as these differences are independent of $N_{gen.}$ Therefore, just as there was no gauge coupling unification in the SM and the SM without a Higgs, so also, this remains true for the theory with one family of technifermions added. For reference, we note that the scales $\mu_{ij}$ at which pairwise equalities of couplings occur are roughly $\mu_{23} \approx 10^{18}$ GeV, $\mu_{13} \approx 10^{14.5}$ GeV, and $\mu_{12} \approx 10^{12.8}$ GeV.

V. ON THE UNIFICATION OF SM AND TC GAUGE SYMMETRIES IN A SIMPLE GROUP

A. General

Here we consider models of type (3), which attempt to achieve the unification of the three standard-model gauge symmetries with a group $G_{SC}$ which contains technicolor,

$$G_{SC} \supseteq G_{TC},$$  \hspace{1cm} (5.1)

and possibly also some generational symmetries, in a unified gauge symmetry described by the group $G$, as specified in eq. (5.2). The physics is invariant under the symmetry group $G$ at energies above the unification scale, $M_{GU}$, and this symmetry breaks at $M_{GU}$. The subscript SC indicates that the $G_{SC}$ gauge interaction becomes strongly coupled at a scale which is roughly comparable to conventional ETC scales, although it is small and perturbative at the high scale $M_{GU}$. These are the most ambitious of all of the three types of models considered here. They entail the unification of the three SM gauge couplings and the SC gauge coupling at $M_{GU}$. Thus, the absence of this gauge coupling unification would, by itself, be enough to exclude such models. However, the analysis of the evolution of the relevant gauge couplings is more complicated for these models because of the nonperturbative behavior and associated dynamical symmetry.
breaking that occurs at intermediate energy scales between $m_Z$ and the hypothesized $M_{GU}$; as a consequence of this, one cannot use the perturbative evolution equation $(8.11)$ for all of the relevant couplings. We will discuss this further below.

As soon as one hypothesizes a technicolor gauge symmetry as a dynamical mechanism for electroweak symmetry breaking, it is natural to explore the idea of trying to unify technicolor with the three SM factor groups - color, weak isospin, and weak hypercharge - in a simple group. The motivations for this are similar to the motivations for the original grand unification program, including a unified description of the fermion representations and an explanation of the relative coupling strengths at lower energies, including, in particular, the property that the TC interaction becomes strongly coupled at a scale lower than $m_Z$.

For all of these interactions as subgroups, $G \supset G_{TCSM}$, where $G_{TCSM} = G_{TC} \times G_{SM}$, such that $G$ breaks to $G_{TCSM}$ at a high scale $M_{GU}$. This hypothetical theory would be constructed so that the technicolor beta function would be more negative than the SU(3)$_C$ beta function, $\beta_{TC} < \beta_{SU(3)}$, $< 0$, and hence, as the energy scale decreases, the technicolor gauge coupling would become sufficiently large to cause a technifermion condensate at a scale $\Lambda_{TC}$ well above the scale $\Lambda_{QCD}$ at which the SU(3)$_C$ coupling gets large and produces the $\langle q\bar{q} \rangle$ condensate. However, this approach is excluded immediately by the fact that the gauge bosons in $G$ that transform technifermions into technisinglet standard-model fermions and hence communicate the electroweak symmetry breaking to the latter and give them masses lie in the coset $G/G_{TCSM}$ and hence pick up masses of order $M_{GU}$. The effective ETC scale would thus be the grand unification scale, $M_{GU}$, resulting in standard-model fermion masses that are much too small; for example, for the illustrative value $M_{GU} = 10^{16}$ GeV, these fermion masses would be of order $\Lambda_{TC}/M_{GU}^2 \simeq 10^{-25}$ GeV. It should be noted that this early approach to the unification of technicolor and SM gauge symmetries led to the inference that $N_{TC}$ had to be greater than $N_c = 3$. But since this attempt at unification is immediately ruled out by its failure to obtain fermion masses of adequate size, its requirement concerning $N_{TC}$ is only of historical interest, and, indeed, many recent TC models use $N_{TC} = 2$ for the reasons that we have discussed in Section II.

Here we consider a different approach to this goal of unification, in which the ETC gauge bosons have masses in the usual ETC range, and not all of the fermion generations arise from the representations of the unified group $G$ but instead, some arise from sequential symmetry breaking of a smaller subgroup of $G$ at ETC-type scales. Let us denote $N_{gh}$ and $N_{gl}$ as the numbers of standard-model fermion generations arising from these two sources, respectively, where the subscripts $gh$ and $gl$ refer to generations from the representation content of the high-scale symmetry group and from the lower-scale breaking. Together, these equal the observed number of SM fermion generations:

$$N_{gen.} = 3 = N_{gh} + N_{gl} .$$

Note that in this approach involving sequential symmetry breaking, one does not calculate the beta functions of the low-energy SC or TC sectors by enumerating the fermion content at the unification scale since some subset of these fermions would be involved in condensates formed at intermediate energy scales, hence would gain dynamical masses of order these scales, and would be integrated out before the energy decreases to the scale relevant for the evolution of SC or TC gauge couplings. It should also be remarked that at this stage the number $N_{gl}$ is only formal; that is, we set up a given model so that, a priori, it can have the possibility that a subgroup of $G$ such as $G_{SC}$ might break in such a manner as to peel off $N_{gl}$ SM fermion generations. In fact, we will show that, at least in the models that we study, it is very difficult to arrange that this desired symmetry breaking actually takes place.

The requirement that the ETC gauge bosons have masses of the necessary scales means that $G$ cannot break to the direct product group $G_{TCSM}$ at the unification scale $M_{GU}$, and also cannot break at this scale to the larger subgroup $G_{SCSM} = G_{SC} \times G_{SM}$. Instead, $G$ must break to a direct product group such that one or more of the factor groups that are residual symmetries between $M_{GU}$ and $\Lambda_{ETC,max}$ contain gauge bosons that transform technifermions into technisinglet standard-model fermions, i.e. are ETC gauge bosons. As the energy scale decreases, this intermediate symmetry should break at $\Lambda_{ETC,max}$ so that some of the ETC gauge bosons get masses of this order, and so forth for other lower sequential ETC scales.

To provide an explicit context for our analysis, let us consider unifying the SU($N_{SC}$) symmetry containing technicolor with the SM gauge symmetries by using a group $G = SU(N)$ as in eq. (1.2), with

$$N = N_{SC} + N_e + N_w = N_{SC} + 5 .$$

Thus, $G \supset G_{SCSM}$. Here we shall take $G_{TC} = SU(N_{TC})$, $G_{SC} = SU(N_{SC})$, and $G_{GU}$ to be the group $SU(5)_{GU}$ of
Ref. [23]. The fermion representations are determined by
the structure of the fundamental representation, which we
take to be
\[
\psi_R = \begin{pmatrix}
(N^c)^\tau \\
d^\tau \\
e^\tau \\
\nu^\tau \\
\end{pmatrix}_R
\]  
(5.4)
where \(d, e, \text{ and } \nu\) are generic symbols for the fermions with
these quantum numbers. Thus, the indices on \(\psi_R\) are ordered so
that the indices in the SC set, which we shall denote \(\tau\), take on the values \(\tau = 1, \ldots N_{\text{SC}}\) and then
the remaining five indices are those of the \(5_R\) of SU(5)_GU [24].
The components of \(N^c_R\) transform according to the
fundamental representation of SU(\(N_{\text{SC}}\)), are singlets un-
der SU(3)c, and SU(2)_w, and have zero weak hypercharge
(hence also zero electric charge). Our choice to write
these components as \((N^c)^\tau_R\) instead of \(N^c_R\) is a conven-
tion. The quantum numbers of components of any represen-
tation of \(G\) are determined by the structure of the fundamental
representation \(\psi_R\). This structure is con-
cordant with the direct product in eq. 122 and the corresponding commutativity property
\[ [G_{\text{SC}}, G_{\text{GU}}] = 0 \]  
(5.5)
which, since \(G_{\text{SC}} \supseteq G_{\text{TC}}\), implies
\[ [G_{\text{TC}}, G_{\text{GU}}] = 0 . \]  
(5.6)
These properties have important consequences for
fermion masses. We recall the theorem from Ref. [12]
discussed in Section IVB, that \([G_{\text{ETC}}, G_{\text{GU}}] = 0\) implies
that for one or more fermions \(f\), since \(f_L\) and \(f_R^c\) are both
contained in the same representation of \(G_{\text{GU}}\), \(f_L\) and \(f_R^c\) transform according to relatively conjugate representa-
tions of \(G_{\text{ETC}}\). By the same argument, the commutativity
property \(\mathcal{M}\) implies that for one or more fermions \(f\), since \(f_L\) and \(f_R^c\) are both contained in the same represen-
tation of \(G_{\text{GU}}\), \(f_L\) and \(f_R^c\) transform according to relatively conjugate representations of \(G_{\text{SC}}\) and hence
also of \(G_{\text{TC}}\). For the case \(G_{\text{GU}} = \text{SU}(5)_\text{GU}\) on which
we focus here, this includes the charge \(2/3\) techniquarks.
In the models of types (1) and (2) discussed in Section
IVB, this would lead to the strong suppression of the
masses of the TC-singlet SM fermions with these quan-
tum numbers, i.e., the charge \(2/3\) quarks. Here, it will
also lead to the strong suppression of certain SM fermion
masses, but because the ETC vector bosons carry color
and charge in the present type-(3) models, the fermions with
suppressed masses will be leptons.
Corresponding to the subgroup decomposition \(G \supset
G_{\text{SC}} \times G_{\text{GU}}\), the Lie algebra of \(G\) contains subalgebras
for \(G_{\text{SC}}\) and \(G_{\text{GU}}\). There are also (linear combinations of)
generators of \(G\) that transform the \(N_{\text{SC}}\) indices
\(\tau = 1, \ldots N_{\text{SC}}\) to the last five indices in the fundamental
representation, and vice versa. The gauge bosons cor-
sponding to these generators include some of the ETC
gauge bosons and have nontrivial SM quantum numbers.
We label the basic transitions as
\[ (N^c)^\tau_R \to d^a_R + V^\tau_a \]  
(5.7)
\[ (N^c)^\tau_R \to e^\tau_R + (U^-)^\tau \]  
(5.8)
and
\[ (N^c)^\tau_R \to \nu^\tau_R + (U^0)^\tau \]  
(5.9)
where \(\tau = 1, \ldots N_{\text{SC}},\) and the \(V^\tau\), and \((U^0)^\tau\) are ETC
gauge bosons. Under the group \(G_{\text{SCSM}}\) these transform
according to
\[ V^\tau_a : (N_{\text{SC}}, 3, 1/2)_a \]  
(5.10)
and
\[ \left( \begin{array}{c}
U^0 \\
U^- 
\end{array} \right)^\tau : (N_{\text{SC}}, 1, 2)_a \]  
(5.11)
These are thus quite different from the ETC gauge bosons
of models (1) and (2), which carry no SM quantum
numbers. It is important to note that the (commutators of)
generators to which these ETC gauge bosons cor-
spond do not close to yield a subalgebra smaller than
the full Lie algebra of \(G\), so that there is no ETC group
\(G_{\text{ETC}}\), as such. This is analogous to the fact that the
(linear combinations of) generators that transform color
indices to electroweak indices in the conventional SU(5)
grand unified theory do not close to form an algebra
smaller than the Lie algebra of SU(5) itself. This is one
of the features that distinguish these models of type (3)
from the models of types (1) and (2), which do have well-
deﬁned ETC gauge groups.
To delineate the remaining ETC gauge bosons, let
us divide the SU(\(N_{\text{SC}}\)) indices into (a) the ones for
SU(\(N_{\text{TC}}\)), say \(\tau = 1, \ldots N_{\text{TC}}\), and (b) the ones for the
coset \(G_{\text{SC}}/G_{\text{TC}}\). \(\hat{\tau} = N_{\text{TC}} + 1, \ldots N_{\text{SC}}\). Consider the
process \((N^c)^\tau_R \to (N^c)^\tau_R + V^\tau\); the \(V^\tau\)'s constitute the
remaining ETC gauge bosons. These carry no SM quant-
um numbers and are similar in this regard to the ETC
gauge bosons of models (1) and (2).
Given that \(G\) cannot break completely to \(G_{\text{TCSM}}\) or
\(G_{\text{SCSM}}\) at the unification scale \(M_{\text{GU}}\) in a viable model,
we next investigate which subgroup it could break to at
this scale. The breaking pattern must be such as to sat-
sify the upper limits on the decays of protons and other-
wise stable bound neutrons. The gauge bosons of \(G_{\text{GU}}\)
that contribute to these decays are the set \((\bar{V}^\tau_a)^a\), where \(a\) is a color index) which transform as \((\bar{3}, 2)_{3/6}\) under
SU(3)c × SU(2)_w × U(1)_Y (whence \(Q_X = 4/3, Q_Y = 1/3\)), and their adjoints. These 12 gauge bosons span the coset
SU(5)_GU/G_{SM} and must gain masses of order \(M_{\text{GU}}\). A priori,
the breaking at \(M_{\text{GU}}\) could leave an invariant sub-
group SU(3)c × G_{SCW}, where G_{SCW} is a group of trans-
formations on the \(N_{\text{SC}}\) indices of SU(\(N_{\text{SC}}\)) and the two
electroweak indices of SU(2)_w which naturally takes the
form \( \text{SU}(N_{SC} + 2) \). However, it appears quite difficult to construct a model of this sort because of the conflicting requirements that the \( G_{SCW} \) coupling get large, as is necessary for self-breaking (see further below) and that the \( \text{SU}(2)_w \) coupling, which is supposed to have a small, perturbative value of \( \alpha_2 = \alpha_{GU} \) at the presumed unification scale, \( M_{GU} \) and then evolve to its similarly small, perturbative value of \( \alpha_2(m_Z) = 0.034 \) at the electroweak scale.

An alternative is that the breaking of \( G \) at \( M_{GU} \) would leave an invariant subgroup \( \text{SU}(2)_w \times G_{SCC} \), where \( G_{SCC} \) is a group of transformations on the \( N_{SC} \) indices of \( \text{SU}(N_{SC}) \) and the three color indices of \( \text{SU}(3)_c \), which would naturally take the form \( \text{SU}(N_{SCC}) \) with

\[
N_{SCC} = N_{SC} + N_c = N_{SC} + 3 .
\]

Thus,

\[
\text{SU}(N_{SCC}) \supset \text{SU}(N_{SC}) \times \text{SU}(3)_c .
\]

All representations of \( \text{SU}(N_{SCC}) \) are determined by the fundamental representation, which follows directly from eq. (5.12); again, it is convenient to write this as a righthanded field,

\[
\left( \frac{(N_e)^r}{d^8} \right)_R .
\]

As is evident from this, the components of a representation of \( \text{SU}(N_{SCC}) \) do not, in general, have the same weak hypercharge \( Y \); so this representation does not have a well-defined value of \( Y \). Considerations of the relative sizes of gauge couplings would favor this option over the one involving \( G_{SCW} \) because the color \( \text{SU}(3)_c \) coupling \( \alpha_3 \) increases substantially from the presumed unification value at \( M_{GU} \) to \( \alpha_3(m_Z) = 0.118 \) at the electroweak scale. To begin with, several possibilities could be envisioned for the symmetry breaking of \( \text{SU}(N_{SCC}) \). One would be that, as the energy scale decreases from \( M_{GU} \), \( \alpha_{SCC} \) becomes sufficiently large at a scale \( \Lambda_{ETC,max} \) for the breaking \( \text{SU}(N_{SCC}) \to \text{SU}(N_{SC}) \times \text{SU}(3)_c \) to occur, and then, if \( N_{SC} > N_{TC} \), the further sequential breakings of \( \text{SU}(N_{SC}) \), eventually yielding the exact symmetry group \( \text{SU}(N_{TC}) \), peeling off the \( N_{gl} \) SM fermion generations. A different scenario would be one in which, as the energy scale decreases below \( M_{GU} \), \( \text{SU}(N_{SCC}) \) breaks into smaller simple groups in a sequence of \( N_{gl} \) steps, and then the residual smaller simple group finally splits into the direct product \( \text{SU}(N_{TC}) \times \text{SU}(3)_c \). A third possibility would be a combination of these two types of breakings, in which \( \text{SU}(N_{SCC}) \) breaks to smaller simple groups in \( k < N_{gl} \) stages, then splits to a two-fold direct product group one of whose factor groups is \( \text{SU}(3)_c \), and then the other factor group sequentially self-breaks \( N_{gl} - k \) times to the residual exact \( \text{SU}(N_{TC}) \) group. In all of these scenarios, the \( V_{a} \)'s and \( V_{\tau} \) would have masses in the usual ETC range, from \( \Lambda_1 \) down to \( \Lambda_3 \) while the \( (U^0)^r \) and \( (U^-)^r \) would have masses of order \( M_{GU} \). (Hence, although \( (U^0)^r \) and \( (U^-)^r \) are formally ETC gauge bosons, they would play a negligible role in producing masses for SM fermions.)

However, this scenario with a \( G_{SCC} \) gauge symmetry characterizing the effective theory between \( M_{GU} \) and \( \Lambda_{ETC,max} \) also encounters a complication. Consider, for example, the version in which \( \text{SU}(N_{SCC}) \) would break to \( \text{SU}(N_{SC}) \times \text{SU}(3)_c \) at roughly \( \Lambda_{ETC,max} \approx 10^9 \text{ GeV} \). The matching conditions for gauge couplings imply that at this energy scale the \( \text{SU}(3)_c \) and \( G_{SC} \) couplings are equal, since they inherit the value that the \( \text{SU}(N_{SCC}) \) coupling had just above \( \Lambda_{ETC,max} \). But, assuming that the unified theory with gauge group \( G \) is self-contained, i.e., there is no direct product at the high scale \( M_{GU} \) with an auxiliary group like hypercolor, the only mechanism for dynamical symmetry breaking below \( M_{GU} \) is self-breaking. This implies that at the above energy scale of \( \sim 10^9 \text{ GeV} \), \( \alpha_{SCC} \) should be \( O(0.1) - O(1) \), depending on the attractiveness of the relevant fermion condensation channel(s) as measured by the respective values of \( \Delta C_2 \), defined in eq. (5.12) in the appendix. This is difficult to reconcile with the \( \text{SU}(3)_c \) beta function, which has leading coefficient \( b_0(3) = 7 \) due to the SM fermions, and, in the energy interval above \( \Lambda_{TC} \), \( b_0(3) = 7 - (4/3)N_{TC} \) for one family of massless technifermions. That is, at an energy scale of \( \sim 10^9 \text{ GeV} \), the value of \( \alpha_3 \) would already be larger than its value at the electroweak scale, which would imply that it would have to decrease, rather than increasing, as the scale \( \mu \) decreases from \( 10^9 \text{ GeV} \) to \( m_Z \), i.e., that the \( \text{SU}(3)_c \) sector would have to be non-asymptotically free in this interval. Another problem is that in the models that we have constructed and studied, we find that the \( \text{SU}(N_{SCC}) \) theory is unlikely to break in the necessary manner; to explain this, it is first necessary to describe the fermion representations, which we do below.

Let us proceed with the construction and critical evaluation of the prospects for this class of unified models. The following conditions are equivalent:

\[
G_{SC} = G_{TC} \iff N_{gh} = 3, \ N_{gl} = 0 ,
\]

i.e., in this case, all of the fermion generations would arise from the fermion representations of \( G \). The other formal possibility is that \( N_{gl} \geq 1 \) so that the coset \( G_{SC}/G_{TC} \) is nontrivial, with \( G_{SC} \) containing some gauged generational structure. For generality, it should be noted that although we use \( G_{GU} \) to classify fermion representations, the breaking of \( G \) may be such that the lower-energy effective field theory is not actually symmetric under a direct product group in which \( G_{GU} \) occurs as a factor group. From the property (5.2), it follows that the ranks satisfy

\[
r(G) \geq r(G_{SC}) + r(G_{GU})
\]

where, with our assumption that \( G_{SC} = \text{SU}(N_{SC}) \), we have \( r(G_{SC}) = N_{SC} - 1 \). In addition to the subgroup decomposition (5.2), we will also use the subgroup de-
composition
\[ \text{SU}(N) \supset \text{SU}(2) \times \text{SU}(N_{\text{SCC}}). \] (5.17)

We shall assume that at energies below the unification scale \(M_{\text{GU}}\) all subsequent breaking of gauge symmetries is dynamical. Since \(M_{\text{GU}}\) is not very far below the Planck scale, which certainly constitutes an upper limit to the possible validity of the theory, owing to the lack of inclusion of quantum gravity, it is not clear that one needs to assume that the initial breaking of \(G\) is dynamical. We shall comment on this further below. The dynamical symmetry breaking at energies below \(M_{\text{GU}}\) can be classified as being of two general types: (i) self-breaking (“tumbling”), in which an asymptotically free chiral gauge symmetry group has an associated coupling that becomes large enough to produce a fermion condensate that breaks the gauge symmetry, and (ii) induced breaking in which a gauge symmetry is weakly coupled, but is broken by the formation of condensates involving fermions that are nonsinglets under a strongly coupled group (which is the way that electroweak symmetry is broken by technifermion condensates); (iii) a combination of the two, as in the sequential breaking of the SU(5)\(_{ETC}\) symmetry in Refs. 3, 12. Since we only consider unification in a single, simple group \(G\) here, we are led to focus on self-breaking below \(M_{\text{GU}}\). Note that with our choice of the minimal GU group as SU(5) with rank 4, the inequality (5.16) becomes \(r(G) \geq r(G_{\text{SCC}}) + 4\). Our choice \(G = \text{SU}(N)\) with \(N\) given by eq. (8.2) satisfies this inequality as an equality; i.e., we are choosing the minimal \(G\) for a given value of \(N_{\text{SCC}}\). In order to maintain the nonabelian structure of the TC group and hence the asymptotic freedom that leads to confinement and the formation of the EWSB bilinear fermion condensate, we require that \(N_{\text{TC}} \geq 2\). This yields the inequality
\[ N \geq 7 \] (5.18)
and \(r(G) \geq 6\).

For any of the possible types of sequential breakings of \(G_{\text{SCC}}\) and/or \(G_{\text{SC}}\) described above that produce the \(N_{\text{gf}}\) SM fermion generations, one has
\[ N_{\text{gf}} = N_{\text{SCC}} - (N_{\text{TC}} + N_{e}). \] (5.19)

In particular, if \(G_{\text{SCC}}\) first splits to \(G_{\text{SC}} \times \text{SU}(3)_{\text{c}}\) and \(G_{\text{SC}}\) then sequentially breaks to produce these \(N_{\text{gf}}\) generations, then
\[ N_{\text{gf}} = N_{\text{SC}} - N_{\text{TC}}. \] (5.20)

The requirement that \(N_{\text{TC}} \geq 2\) in order for the technicolor interactions to be asymptotically free, combined with eq. (5.20), implies
\[ N_{\text{gf}} \leq N_{\text{SC}} - 2. \] (5.21)
Thus, for \(N_{\text{SC}} = 2\) all of the SM fermion generations must arise via \(N_{\text{gf}}\).

We next specify the fermion representations of \(G = \text{SU}(N)\). Without loss of generality, we shall usually deal with left-handed fermions (or antifermions). In order to avoid fermion representations of SU(3), and \(\text{SU}(2)_{w}\) other than those experimentally observed, namely singlets and fundamental or conjugate fundamental representations, one restricts the fermions to lie in \(k\)-fold totally antisymmetrized products of the fundamental or conjugate fundamental representation of \(\text{SU}(N)\) \([50]\); we denote these as \([k]_{N}\) and \([\bar{k}]_{N} = [\bar{k}]_{N}^{*}\). The notational correspondence with Young tableaux is (suppressing the dependence on \(N\)), \([1] \equiv \begin{array} {c} \hline \end{array}\) \([2] \equiv \begin{array} {c} \hline \hline \end{array}\) etc. Some elementary properties of the representation \([k]_{N}\) are listed in the appendix; these include its dimensionality and expressions for the Casimir invariants \(C_{2}([k]_{N})\) and \(T([k]_{N})\). A set of (left-handed) fermions \(\{f\}\) transforming under \(G\) is thus given by
\[ \{f\} = \sum_{k=1}^{N-1} n_{k} [k]_{N} \] (5.22)
where \(n_{k}\) denotes the multiplicity (number of copies) of each representation \([k]_{N}\). We use a compact vector notation
\[ n \equiv (n_{1}, ..., n_{N-1})_{N}. \] (5.23)

If \(k = N - \ell\) is greater than the integral part of \(N/2\), we shall work with \([\ell]_{N}\) rather than \([k]_{N}\); these are equivalent with respect to SU(\(N\)) (see eq. \([8,2]\) in the appendix). An optional additional constraint would be to require that the numbers in the set \(n_{k}, k = 1, ..., N - 1\) have no common factors greater than unity, i.e., the greatest common divisor \(\text{GCD}(\{n_{k}\}) = 1\). This might be viewed as a kind of irreducibility condition. Although we will not impose this condition here, the two candidate models that we consider that have \(\text{GCD}(\{n_{k}\}) \geq 2\) are excluded anyway because the SCC theory is not asymptotically free. A fermion field corresponding to \([k]_{N}\) is denoted generically by \(\psi^{1,...,k}\). It will sometimes be convenient to deal with the charge-conjugate right-handed field.

Before proceeding, it is appropriate to summarize the requirements on the choice of fermion representations:

1. The theory must contain a mechanism to break the unified \(G\) gauge symmetry, eventually down to the symmetry group operative above the electroweak scale, \(\text{SU}(N_{\text{TC}}) \times G_{\text{SM}}\). The breaking scales must be such as to obey upper bounds on the decay rate for protons and bound neutrons.
2. The contributions from various fermions to the total SU(\(N\)) gauge anomaly must cancel each other, yielding zero gauge anomaly.
3. The resultant TC-singlet, SM-nonsinglet left-handed fermions must comprise a set of generations, i.e., must have the form \(N_{\text{gen}}.[(1, \bar{3})]_{L} + \)}
(1, 10)_L], where the first number in parentheses signifies that these are TC-singlets and the second number denotes the dimension of the SU(5)_{GU} representation.

4. For a fully realistic model, one requires \( N_{\text{gen.}} = 3 \).

5. In order to account for neutrino masses, one needs to have TC-singlet, electroweak-singlet neutrinos to produce Majorana neutrino mass terms that can drive an appropriate seesaw [10]. In the present context, these are also singlets under SU(5)_{GU}.

6. The model must contain ETC gauge bosons with masses in the general range from a few TeV to 10^3 TeV so as to produce acceptable SM fermion masses. As explained above, a plausible way to satisfy this requirement is for \( G \) to break to the subgroup [5.17] containing the factor group \( G_{\text{SCC}} \) which contains both SU\( (N_{TC}) \) and SU\( (3)_c \) and is naturally SU\( (N_{SCC}) \) with \( N_{SCC} \) given by eq. [5.18]. Thus, the effective field theory at energy scales between \( M_{GU} \) and \( \Lambda_{\text{ETC,max}} \sim 10^6 \) GeV is invariant under this direct product [5.17]. The dynamics should be such that SU\( (N_{SCC}) \) breaks at ETC scales, in one of the ways delineated above, eventually yielding the residual exact symmetry group SU\( (2)_{TC} \times \) SU\( (3)_c \), with the requisite three SM fermion generations emerging. If at least some of the stages of this process involve self-breaking, then the SCC sector should be an asymptotically free chiral gauge theory.

7. If \( N_{SC} > N_{TC} \), then there must be a mechanism to break SU\( (N_{SC}) \) down in \( N_{gT} \) stages to SU\( (N_{TC}) \). Again, if this is to be a self-breaking, then the SC sector should be an asymptotically free chiral gauge theory so that the associated coupling will increase sufficiently as the energy scale decreases to produce the requisite condensate(s).

8. The color SU\( (3)_c \) interaction must be asymptotically free in the energy interval at and below the electroweak scale, where the associated coupling has been measured.

9. The technicolor interaction must asymptotically free, so that the associated gauge coupling will increase sufficiently, as the energy scale decreases, to produce a technifermion condensate and break the electroweak symmetry; further, the technicolor symmetry must be vectorial so that the technifermions are confined and the technifermion condensate does not self-break \( G_{TC} \).

10. When evolved down to the low energies, the respective SM gauge couplings must agree with their measured values.

Let us check that the first and sixth of these constraints can be simultaneously satisfied. This requires that one confirm that the masses of the (mass eigenstates corresponding to the interaction eigenstates) \( V^\tau \) needed for their role as ETC vector bosons are consistent with the upper bounds on the decays of protons and bound neutrons. These decays are induced by the s-channel transitions \((1) \ u^a + u^b \to X_c, (2) \ u^a + d^b \to Y_c \) (and, if the theory contained another vector boson, \( \Xi \), with \( Q_\Xi = -2/3 \), also \((3) \ d^c + d^b \to \Xi_c \)), where \( a, b, c \) are (different) color indices. Corresponding transitions in the \( t \) and \( u \) channel also contribute. Among the \( G_{SC}-\)non-singlet gauge bosons, the only ones that transform in the right way under color and electric charge to contribute to these decays are the \( V^\tau \), with charge 1/3. Among these, the subset with SC indices \( \tau \) in \( G_{TC} \) cannot contribute to these decays, since the \( G_{TC} \) technicolor symmetry is exact, but the quarks in a nucleon are technisinglets. If \( N_{gT} = 0 \), then \( G_{SC} = G_{TC} \), so all of the SC indices are in \( G_{TC} \). If \( N_{gT} \geq 1 \), then there is also a subset of \( V^\tau \) with indices \( \tau \) in the coset \( G_{SC}/G_{TC} \). The exact symmetries (color, electric charge, and technicolor) allow these to mix with the \( Y_a \), via one-loop and higher-loop nondiagonal propagator corrections, so that the actual vector boson mass eigenstates would be, for \( N_{gT} = 1 \),

\[
V_{a,\text{heavy}} = \cos \omega \ Y_a + \sin \omega \ V^\tau_a
\]

\[
V_{a,\text{light}} = -\sin \omega \ Y_a + \cos \omega \ V^\tau_a
\]  

and similarly in the case where \( N_{gT} \geq 2 \). Since this mixing would be forbidden at energy scales where \( G_{SC} \) or \( G_{SCC} \) is still an exact symmetry, and since the maximum of the relevant breaking scales is of order the highest ETC scale, \( \Lambda_{\text{ETC,max}} \approx 10^6 \) GeV, it follows that

\[
|\omega| \sim \frac{\Lambda_{\text{ETC,max}}}{M_{GU}} \ll 1 .
\]  

Hence, the mixing would lead to a diagram for nucleon decay with a propagator for \( V_{a,\text{light}} \), \( \sim 1/A_{\text{ETC,max}}^2 \) \( M_{GU} \) multiplied by the mixing factors for each of the two vertices involving SM fermions, which have size \( \lesssim (A_{\text{ETC,max}}/M_{GU})^2 \). The product is then \( \lesssim 1/M_{GU}^2 \), the same as for the usual contribution from \( V_{a,\text{heavy}} \). Hence (given that \( M_{GU} \) is sufficiently large so that the usual contributions to nucleon decay are not excessive) this mixing does not significantly increase the rate of nucleon decay. This shows that these ETC gauge bosons can, indeed, have ETC-scale masses, as required to give SM fermions their masses.

We next consider the constraint that there be no anomaly in gauged currents of the unified theory invariant under the group \( G \). For this purpose, we define a \((N-1)\)-dimensional vector of anomalies

\[
a = (A([1]_N), \ldots, A([N-1]_N))
\]

where \( A([k]_N) \) is given in eq. [8.3] in the appendix. Then the constraint that there be no \( G \) gauge anomaly is the condition

\[
n \cdot a = 0 .
\]
which is a diophantine equation for the components of the vector of multiplicities \( \mathbf{n} \), subject to the constraint that the components \( n_k \) are non-negative integers. Geometrically, if \( \mathbf{a} \) and \( \mathbf{n} \) were vectors in \( \mathbb{R}^{N-1} \), then the solution set of eq. (5.24) would be the \( (N-2) \)-dimensional subspace of \( \mathbb{R}^{N-1} \) orthogonal to the vector \( \mathbf{a} \); the situation here is more complicated because of the diophantine requirement that \( n_k \) is a nonnegative integer. The actual solution is also subject to additional conditions, as we shall discuss shortly.

The most natural way to satisfy the third requirement, that the TC-singlet, SM-nonsinglet fermions should form a well-defined set of SM generations, is to impose this separately on the subset of these fermions that arise from the fermion representations of \( G \) and on the complementary subset that arise from the sequential symmetry breaking of \( SU(N_{SCC}) \). The fermion representations of \( G \) transform according to \( (R_{SC}, R_{GU}) \) with respect to the subgroup decomposition \( 12 \). In these terms, the condition on the former subset yields the two conditions

\[
N_{(1,5)} = N_{(1,10)} \tag{5.28}
\]

and, for nonsinglet \( R_{SU(5)_{GU}} \),

\[
N_{(1,R_{SU(5)_{GU}})} = 0 \text{ if } R_{SU(5)_{GU}} \neq 5 \text{ or } 10 \ , \tag{5.29}
\]

i.e., the number of TC-singlet left-handed (anti)fermions from these representations transforming as 5 and 10 of \( SU(5)_{GU} \) must be equal and the theory must not contain any other TC-singlet, SM-nonsinglet fermions. Regarding the fermions in the complementary subset, we note that the breaking of \( SU(N_{SCC}) \) can be viewed as the breaking of the \( SU(N_{SC}) \) part of this group, since the \( SU(3)_c \) part remains unbroken. For these, the requirements that we impose are the analogues of eqs. (5.24) and (5.29) with the multiplets considered to refer to \( (R_{TC}, R_{GU}) \).

As noted earlier, in models of type (3), for a given choice of fermion representations of the unified group \( G \), it is not guaranteed that the resultant (TC-singlet) SM fermions come in well-defined generations, and even for a choice which does satisfy this constraint, it is not guaranteed that the model can accommodate three such generations. In both respects, these models are different from models of type (1) and (2), where one can automatically satisfy both of these conditions.

We now incorporate the constraint that the standard-model fermions that arise from the fermion representations of \( G \) comprise well-defined generations. Each of these SM fermion generations is equivalent to the set \{1, 5\} of representations of left-handed fermions, under the direct product \( SU(N_{SC}) \times SU(5)_{GU} \).

A \{1, 5\} representation arises in two ways (i) from a \([N_{SC} + 4]_N \cong [N - 1]_N \approx [1]_N \) representation, \( \psi_{L,i} \), when \( i \) takes values in \( SU(5)_{GU} \); and (ii) from a \([4]_N \) representation, \( \psi_{L,12}^{i1234} \), when all of the indices take on values in \( SU(5)_{GU} \). If one were to choose \( N_{SC} = 0 \) and hence \( N = 5 \), these sources would coincide; for the relevant case of a nonabelian TC group, for which \( N_{SC} \geq N_{TC} \geq 2 \), they constitute two different sources. Hence, for \( N_{SC} \geq 2 \),

\[
N_{(1,5)} = n_{N_{SC}+1} + n_4 \tag{5.30}
\]

where, equivalently, \( n_{N_{SC}+1} = n_{N-1} \). A \{1, 10\} representation also arises in two ways: (i) from a \([2]_N \) representation, \( \psi_{L,i}^{i12} \), when both of the indices take on values in \( SU(5)_{GU} \), and (ii) from a \([N_{SC} + 2]_N \) representation, when \( N_{SC} \) of the indices take on values in \( SU(N_{SC}) \), thereby producing a singlet under this group, and the remaining two indices take on values in \( SU(5)_{GU} \). Since \( N_{SC} + 2 \cong [N - 3]_N \), one can equivalently describe source (ii) as arising from \( \psi_{L,12}^{i1234} \) when all of the three indices take on values in \( SU(5)_{GU} \). Again, for \( N_{SC} = 0 \) and hence \( N = 5 \), these sources (i) and (ii) coincide; for the relevant nonabelian case \( N_{TC} \geq 2 \) and hence \( N_{SC} \geq 2 \), they are different, so that

\[
N_{(1,10)} = n_2 + n_{N_{SC}+2} \tag{5.31}
\]

Thus, the requirement that the left-handed SC-singlet, SM-nonsinglet (anti)fermions comprise equal numbers of \( \{1, 5\} \) and \( \{1, 10\} \)'s implies the condition

\[
n_{N_{SC}+1} + n_4 = n_2 + n_{N_{SC}+2} \tag{5.32}
\]

and the number of SM fermion generations \( N_{gh} \) produced by the representations of \( G \) is given by either side of this equation;

\[
N_{gh} = n_2 + n_{N_{SC}+2} \tag{5.33}
\]

The remaining \( N_{dT} \) generations of SM fermions arise via the breaking of \( G_{SCC} \) and/or \( G_{SC} \).

We next determine the implications of the constraint excluding SC-singlet fermions that have unphysical nonsinglet SM transformation properties. These can be identified in terms of their \( SU(5)_{GU} \) representations. Since the only possibilities for nonsinglet \([k]_5 \) are \([1]_5 = 5 \), \([2]_5 = 10 \), \([3]_5 = 10 \), and \([4]_5 = 5 \), these unphysical representations are the \((1,5)\) and \((1,10)\). Now a \([1]_N \) representation, \( \psi_L \), yields a \( (1,5) \) when the index \( i \) takes on values in \( SU(5)_{GU} \). Further, a \([N_{SC} + 1]_N \) representation, \( \psi_{L,12}^{i1234} \), also yields a \( (1,5) \) when \( N_{SC} \) of the indices take on values in \( SU(N_{SC}) \), thereby yielding an SC-singlet, and the one remaining index takes on values in \( SU(5)_{GU} \). The requirement that there be no \( (1,5) \)'s is therefore

\[
n_1 = 0 , \quad n_{N_{SC}+1} = 0 . \tag{5.34}
\]

A \([3]_N \) representation, \( \psi_{L,12}^{i1234} \), yields a \( (1,10) \) if all of the three indices take values in \( SU(5)_{GU} \). In addition, a \([N_{SC} + 3]_N \) representation, \( \psi_{L,12}^{i1234} \), yields a \( (1,10) \) in the case when \( N_{SC} \) of the indices take on values in \( SU(N_{SC}) \) and the remaining three indices take on values in \( SU(5)_{GU} \). Since \([N_{SC} + 3]_N \approx [N - 2]_N \), the latter
source is equivalent to $\psi_{1,12-L}$ with both indices taking on values in SU(5)$_{GU}$. Hence, the requirement that there be no $(1,\bar{1}_0)$ is

$$n_3 = 0, \quad n_{NSC+3} = 0 . \quad (5.35)$$

The representations $[1]_N$ and $[N-1]_N \approx [1]_N$, when decomposed with respect to the subgroup \[ 5.17 \], will yield a term $(2,1)$, i.e., a doublet under SU(2)$_w$ which is a singlet under SU(N$_{SCC}$). This is a lepton doublet, such as $\left(\psi^c\right)_L$ The fact that it is a singlet under SU(N$_{SCC}$) means that neither of the component fermions couples directly to the ETC gauge bosons, and hence both have strongly suppressed masses. This is acceptable for the neutrino, but $m_e$ is only about a factor of 10 less than $m_u$, so this strong mass suppression may be problematic for the electron. In order to prevent this, one could require that $n_1 = 0$ and $n_{N-1} = 0$. We shall not do this here, but it should be borne in mind that models with nonzero values for $n_1$ and/or $n_{N-1}$ will have this property.

In the present type-3 models, SC-singlet, SM-singlet fermions $(1,1)$, which can be identified as electroweak-singlet neutrinos, arise, in general, from two sources: (i) $[N_{SC}]_N$, when all of the $N_{SC}$ indices take values in SU(N$_{SC}$); and (ii) $[5]_N$, when all of the indices take values in SU(5)$_{GU}$. In the special case $N_{SC} = 5$, these each contribute.

$$N_{(1,1)} = n_{NSC} + n_5 . \quad (5.36)$$

If $G_{SC}$ is the same as the TC group, then this is the full set of TC-singlet, electroweak-singlet neutrinos, so that the the right-hand side of eq. \[ 5.36 \] should be nonzero. If $N_{SC} > N_{TC}$, then electroweak-singlet neutrinos can also arise from SC-nonsinglet representations when the SC group breaks to the TC group.

The constraint concerning the breaking of SU(N$_{SCC}$) and the behavior of the SU(N$_{TC}$) technicolor group that is operative below the lowest ETC breaking scale entails several parts. Since the TC theory emerges from the breaking of the SCC theory, and since at the unification scale the squared coupling $\alpha_{SCC} = \alpha$ is small, one wants the SCC theory to be asymptotically free in order for $\alpha_{SCC}$ to increase as the energy scale decreases, yielding, after breakings, a TC coupling that is sufficiently large to produce the eventual technifermion condensate. The asymptotic freedom of the SCC theory is required if, as assumed here, one or more of the sequential breakings of the SU(N$_{SCC}$) theory are self-breakings. The constraint $\beta_{TC} < \beta_{SU(3)_c} < 0$ in the original approach to the unification of TC and SM gauge symmetries does not appear here because the SU(3)$_c$ group is subsumed within the SU(N$_{SCC}$) group in the relevant range of energies $\Lambda_{ETC,max} < \mu < M_{GU}$. Note that, since $N_{SCC} \geq 5$, it follows that $b_0^{(SCC)} - b_0^{(2)} = N_{SCC} - 2 \geq 3$, where here $b_0^{(2)}$ is the leading coefficient of the only other nonabelian subgroup of $G$, namely SU(2)$_w$ (cf. eq. \[ 5.17 \]). Hence, provided that the SU(N$_{SCC}$) theory is asymptotically free, its beta function is more negative than that of the SU(2)$_w$ sector, as should be the case to account for the observed values of the SM gauge couplings at the electroweak scale.

Before analyzing specific models, we mention some challenges that can be anticipated at the outset. First, for models with $N_{gh} = 3$ so that $N_{SC} = N_{TC}$, there is a single effective ETC mass scale that governs the origin of the SM fermion masses, namely the scale at which the breaking

$$SU(N_{SCC}) \rightarrow G_{TC} \times SU(3)_c \quad (5.37)$$

occurs. (There would, in general, also be a U(1) factor; here we concentrate on the nonabelian symmetries.) The only other scale that enters into the generation of the masses for these SM fermions is the technicolor scale. With only a single effective ETC scale to work with, one cannot satisfactorily reproduce the observed SM fermion mass hierarchy. In models with $N_{gh} \geq 1$, there is, a priori, the formal possibility of having enough ETC mass scales to produce the observed SM generational hierarchy, making use of the sequential breaking scales of G$_{SCC}$ and/or G$_{SC}$ that are supposed to yield the $N_{gh}$ additional generations.

However, when we actually examine these models based on a simple unification group $G$ (without auxiliary groups such as hypercolor), we find that the requisite sequential dynamical symmetry breaking of SU(N$_{SCC}$) is unlikely to occur. The breaking of G$_{SCC}$ should eventually yield, after the sequential self-breaking, the residual exact nonabelian symmetry group on the right-hand side of eq. \[ 5.37 \]. Given that the SU(N$_{SCC}$) theory is asymptotically free, the associated gauge coupling $\alpha_{SCC}$ will increase as the energy scale decreases from the unification scale, $M_{GU}$, and, when $\alpha_{SCC}$ is sufficiently large, the theory will form bilinear fermion condensate(s). If the SU(N$_{SCC}$) gauge interaction is vectorlike, i.e., if (neglecting other gauge interactions) the nonsinglet fermion content of SU(N$_{SCC}$) consists of the set of left-handed fermions \{ $\bar{\sum}_R \bar{R} + \bar{R}$ \}, then the most attractive channel for this condensation process is

$$\bar{R} \times \bar{R} \rightarrow 1 , \quad (5.38)$$

i.e., it yields a condensate that is a singlet under SU(N$_{SCC}$). Hence, the model would fail to break SU(N$_{SCC}$) at all, let alone to the residual subgroup \[ 5.38 \]. With the same initial set of representations, one also has the channel

$$\bar{R} \times \bar{R} \rightarrow Adj , \quad (5.39)$$

where here Adj refers to the adjoint representation of SU(N$_{SCC}$). This channel could lead to the desired breaking of SU(N$_{SCC}$) in eq. \[ 5.38 \]. However, channel \[ 5.39 \] is always more attractive than channel \[ 5.38 \]. From our studies of specific models, we find that in most cases where $G_{SCC}$ is asymptotically free, it is vectorlike, and...
hence, applying the MAC criterion, one would conclude that the necessary dynamical symmetry breaking would not take place. Even in a model with \( N_{SC} = 5 \) and with \( \mathbf{n} \) given in eq. 5.114, where \( SU(N_{SC}) \) is an asymptotically free chiral gauge theory, we find that it is unlikely to break in the desired manner.

A related problem with the dynamical symmetry breaking is that in many cases, not only does the condensation in the most attractive channel not break \( SU(N_{SC}) \), it breaks \( SU(2)_w \) at a scale which is higher than the ETC scales where \( SU(N_{SC}) \) should break. A possible way to avoid undesired condensation channels of this sort could be to invoke a “generalized most attractive channel” (GMAC) criterion \( \frac{\left|N_{SCC}\right|}{\left|N_{SC}\right|} \), which makes use of vacuum alignment and related energy minimization arguments to suggest that if the condensate formation can avoid breaking a certain symmetry, it will \( \frac{\left|N_{SCC}\right|}{\left|N_{SC}\right|} \).

Yet another complication can occur in cases where \( N_{SC} \) is odd so that \( N_{SCC} \) is even, say \( N_{SCC} = 2p \). In these cases, there can occur a most attractive channel of the form

\[
[p]_{2p} \times [p]_{2p} \to 1 \quad (5.40)
\]

with

\[
\Delta C_2 = 2C_2([p]_{2p}) = \frac{p(2p + 1)}{2} = \frac{N_{SCC}(N_{SCC} + 1)}{4} . \quad (5.41)
\]

The associated condensate is

\[
\langle \epsilon_{i_1 \ldots i_p} \psi_{L}^{(1 \ldots p)} C \psi_{L}^{(p+1 \ldots 2p)} \rangle . \quad (5.42)
\]

This condensate is symmetric (antisymmetric) under interchange of \( \psi_{L}^{(1 \ldots p)} \) and \( \psi_{L}^{(p+1 \ldots 2p)} \) if \( p \) is even (odd). Since the condensate \( \frac{\left|N_{SC}\right|}{\left|N_{SCC}\right|} \) is invariant under \( G_{SCC} \), it is, a fortiori, invariant under \( SU(N_{SC}) \) and \( SU(3)_{\epsilon} \). The only way to construct the requisite SU(3)\(_{\epsilon}\)-invariant contractions involves product(s) \( \epsilon_{\tau \mu \nu} d^\tau d^\mu d^\nu \), each of which has weak hypercharge \( Y = -2 \) (and electric charge \(-1\)). Hence, this condensate violates weak hypercharge and electric charge. It may be noted that in a hypothetical world in which only the SU(2)\(_w\) interaction was strongly coupled, the same kind of violation would presumably occur. Consider, say, the first two generations of lepton doublets, \( \psi_{1,2L} = (\nu^L_\tau) \) and \( \psi_{1,2L} = (\nu^L_\mu) \). These would form an SU(2)\(_w\)-invariant condensate of type \( 5.24 \) with \( p = 1 \), namely

\[
\langle \epsilon_{j k} \psi_{1,gL}^{\dagger} C \psi_{2,gL} \rangle = 2\langle \nu_{eL}^{\dagger} C \mu_L - e_{eL}^{\dagger} C \nu_{\mu L} \rangle . \quad (5.43)
\]

where \( j, k \) are SU(2)\(_w\) indices.

**B. \( N_{SC} = 2, G = SU(7) \)**

We proceed to study a number of specific models of type (3) to explore their properties. We begin by considering the minimal nontrivial case, \( N_{SC} = 2 \). Since this is the smallest value for a nonabelian group, it follows that \( G_{SC} = G_{TC} \) and hence we denote \( G_{SCC} \equiv G_{TCC} \). Further, it follows that \( N_{\ell \mu} = 0 \), so that an acceptable model would have to have \( N_{\ell \nu} = 3 \). The vector \( \mathbf{n} \) has the form \( \mathbf{n} = (n_1, \ldots, n_6)_7 \). From eqs. 5.34 and 5.35 we have \( n_1 = n_3 = n_5 = 0 \). Equation 5.32 yields \( n_6 = n_2 \). The no-anomaly condition, eq. 5.27, reads \( \delta n_2 - 2n_4 - n_6 = 0 \); substituting \( n_6 = n_2 \) in this equation gives the result \( n_2 = n_4 \). Hence, \( \mathbf{n} = n_2(0, 1, 0, 1, 0, 1) \).

Taking \( n_2 = 1 \) yields

\[
\mathbf{n} = (0, 1, 0, 1, 0, 1) , \quad (5.44)
\]

More generally, for an \( SU(N) \) group with \( N \) odd, say \( N = 2n + 1 \), the chiral fermion content \( \{f\} = \sum_{\ell = 1}^{n} [2\ell]_N \) is anomaly-free \( [30] \). This property was used in Ref. [28] for a study of the possible unification of TC and SM symmetries in SU(7) and SU(9). For the fermion set in eq. 5.44, the number of SM generations is given by eq. 5.33 as \( N_{gen.} = 2n_2 = 2 \), so the requirement that \( N_{gen.} = 3 \) cannot be satisfied, and this model is not acceptable. One could, nevertheless, consider it as a toy model. The simplest special case of this toy model has \( n_2 = 1 \), so that there would be two SM generations. Taking the next higher value, \( n_2 = 2 \) is not acceptable because it would yield the unphysical result of four SM generations. Applying eq. 5.30, we note that there is one electroweak-singlet neutrino.

With respect to the group \( 12D \) namely,

\[
SU(2)_{TC} \times SU(5)_{SU} \quad , \quad (5.45)
\]

the fermions have the following decompositions:

\[
[2]_7 = (1, 1) + (2, 5) + (1, 10) \quad (5.46)
\]

\[
[4]_7 \approx [3]_7 = (1, 5) + (2, 10) + (1, 10) \quad (5.47)
\]

\[
[6]_7 \approx [1]_7 = (2, 1) + (1, 5) \quad (5.48)
\]

where here and below we use the equivalences \( 2 \approx 2 \) for \( SU(2) \) and \( 5 \approx 5 \) for \( SU(N) \).

The technifermions in this model are

\[
U_{\tau L}^{\dagger} , \quad D_{\tau L}^{a} , \quad \left( U_{\tau L}^{a} , \quad D_{\tau L}^{a} \right) \quad R \quad (5.49)
\]

\[
N_{\tau L} , \quad E_{\tau L} , \quad \left( N_{\tau L} , \quad E_{\tau L} \right) \quad R \quad (5.49)
\]

where, as before, \( \tau = 1, 2 \) is the technicolor index and \( a = 1, 2, 3 \) is the color index. Note how, in accordance with our general discussion above, the left- and right-handed chiral components of the charge 2/3 techniquark, \( U_{L} \) and \( U_{R} \), transform according to relatively conjugate representations of \( SU(2)_{TC} \). In this case, since \( SU(2) \) has only (pseudo)real representations, these are equivalent, and the technicolor theory is a vectorial gauge theory.

Since both the subgroup \( 12D \) and the subgroup \( 5.17 \),

\[
SU(2)_{w} \times SU(5)_{TCC} , \quad (5.50)
\]
are abstractly SU(2) × SU(5), the fermions have formally the same decompositions with respect to SU(5), although the component fields are different. It should be noted that a fermion that is a singlet under color and technicolor, and hence is a lepton, can occur as a component of a TCC nonsinglet representation. For example, with respect to the subgroup (5,51), the [2]7 yields a term

\[
(1, 10) = \begin{pmatrix}
0 & \nu c & D^{11} & D^{12} & D^{13} \\
0 & D^{21} & D^{22} & D^{23} \\
0 & u_c^5 & -u_c^c & 0 \\
0 & u_c^1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \ .
\] (5.51)

where the upper indices \( \tau a \) on \( D^{\tau a} \) refer to technicolor and color, and the entries left blank are equal to minus the transposed entries. The \( \nu_L \) field illustrates the general point made above.

The SU(5)\(_{TCC} \) interaction (neglecting other interactions) is vectorial, consisting of the left-handed fermion content \( 2(5 + 5 + 10 + \bar{10}) \). The SU(5)\(_{TCC} \) gauge interaction is asymptotically free, with leading beta function coefficient \( b_{\alpha_{TCC}} = 13 \). Hence, as the energy scale decreases below \( M_{GU} \), \( \alpha_{TCC} \) increases to the point where the theory forms bilinear fermion condensates. Let us consider condensations of these fermions, with the representations classified according to the subgroup of eq. (5.50) as \( (R_{SU(2)w}, R_{SU(5)TCC}) \). In this notation, the most attractive channel (MAC) is

\[
(1, 10) \rightarrow (2, 10) \rightarrow (2, 21) \ .
\] (5.52)

If, indeed, this condensate formed, it would rule out this SU(7) model, since it would break SU(2)\(_w\), at much too high a scale; indeed, this SU(2)\(_w\)-breaking scale would be greater than the ETC scales where SU(5)\(_{TCC} \) should break, clearly an unphysical situation. The attractiveness of the channel, as measured by \( \Delta C_2 \), is given by \( \Delta C_2 = 36/5 \). If one uses as a rough guide the Schwinger-Dyson gap equation (eq. (5.41) in the appendix), one finds the value \( \alpha_{TCC} = 5\pi/54 \approx 0.3 \). As an illustrative value, assume that \( \alpha_{TCC} = \alpha_{GU} \approx 0.04 \) at \( M_{GU} \). Substituting the above critical value into eq. (5.51), one obtains the rough estimate that \( \alpha_{TCC} \) increases to the critical value for the condensate (5.52) to form as the scale \( \mu \) decreases through the value \( \mu_c \approx (3 \times 10^{-5}) M_{GU} \). For the hypothetical unification scale \( M_{GU} = 10^{16} \) GeV, this would mean that electroweak symmetry would be broken at \( \sim 10^{14} \) GeV, clearly far too high a scale. In addition, the channel (5.52) would fail to break the SU(5)\(_{SCC} \) group to SU(2)\(_{TCC} \) × SU(3)\(_c\).

Other condensation channels which are, \textit{a priori} possible, are listed below, together with their \( \Delta C_2 \) values,

\[
(1, 10) \rightarrow (1, 5) \ , \ \Delta C_2 = 24/5 \quad (5.53)
\]

\[
(1, 5) \times (2, 5) \rightarrow (2, 1) \ , \ \Delta C_2 = 24/5 \quad (5.54)
\]

\[
(1, 5) \times (1, 10) \rightarrow (1, 5) \ , \ \Delta C_2 = 18/5 \quad (5.55)
\]

\[
[(2, 5) \times (2, 10)]_a \rightarrow (1, 5) \ , \ \Delta C_2 = 18/5 \quad (5.56)
\]

\[
[(2, 5) \times (2, 10)]_s \rightarrow (3, 5) \ , \ \Delta C_2 = 18/5 \quad (5.57)
\]

where the subscripts \( a \) and \( s \) in eqs. (5.55) and (5.56) refer to antisymmetric and symmetric combinations of representations. None of these channels is acceptable in a viable model. Consider, for example, channel (5.57). The condensate for this channel is

\[
\langle \epsilon_{ijk\ell \alpha} \psi_L^{jk} \, T \, C \psi_{L^\alpha} \rangle \quad (5.58)
\]

where the indices are in SU(5)\(_{TCC} \) (with the ordering as in eq. (5.41)) and \( \psi_{L}^{jk} \) is the fermion field transforming as (1,10) under SU(2)\(_w\)×SU(5)\(_{TCC} \). The free index must take on one of the two SU(2)\(_{TC} \) values, \( i = 1 \) or \( i = 2 \), in order to avoid breaking SU(3)\(_c\); with no loss of generality, we may choose \( i = 1 \). The condensate (5.58) is then proportional to

\[
\langle D_{L}^{21} \, T \, C \, u_{L}^{1} + D_{L}^{22} \, T \, C \, u_{L}^{2} + D_{L}^{23} \, T \, C \, u_{L}^{3} \rangle \quad (5.59)
\]

where the indices on \( D^{\tau a} \) are as in eq. (5.51). This condensate violates weak hypercharge and electric charge, and leaves only one unbroken technicolor index, so that technicolor becomes an abelian symmetry. Channel (5.57) breaks SU(2)\(_w\) at too high a scale and fails to break SU(5)\(_{TCC} \). Channels (5.55) and (5.56) leave technicolor as an abelian symmetry. Channel (5.57) breaks SU(2)\(_w\) in the wrong way and at too high a scale, and leaves technicolor as an abelian symmetry.

Thus, none of these condensation channels produces the desired symmetry breaking of SU(5)\(_{TCC} \) to SU(2)\(_{TC} \) × SU(3)\(_c\). In the absence of an actual mechanism to produce this breaking, it is difficult to analyze the properties of the hypothetical resultant SU(2)\(_{TC} \) theory. With the one SM family of technifermions listed in eq. (5.49), the SU(2)\(_{TC} \) interaction would be asymptotically free, with leading beta function coefficient \( b_{\alpha_{TC}} = 2 \), but since this is smaller than the corresponding \( b_{0}^{(3)} \) for SU(3)\(_c\) (which is \( b_{0}^{(3)} = 25/3 \approx 8.3 \) for this toy two-generation model), and \( \alpha_{TC} = \alpha_{SU(3)_{c}} \) at the energy scale where SU(5)\(_{TCC} \) splits to SU(2)\(_{TC} \) × SU(3)\(_c\), the color coupling would grow considerably faster than the technicolor coupling as the energy scale decreased, leading to the unphysical prediction that \( \Lambda_{QCD} > \Lambda_{TC} \).

C. \( N_{SC} = 3, \ G = SU(8) \)

Next, we consider \( N_{SC} = 3, \ G = SU(8) \), so that \( n = (n_1, ..., n_7)_8 \). In this case, \( a \ priori \), one has the two options \( N_{TC} = N_{SC} = 3 \) with \( N_{gt} = 0 \), or \( N_{TC} = 2 \) with \( N_{gt} = 1 \). In general, eqs. (5.34) and (5.35) yield

\[
n_1 = n_3 = n_4 = n_6 = 0 \quad (5.60)
\]
and eq. (5.32) reads
\[ N_{gh} = n_2 + n_5 = n_4 + n_7 . \] (5.61)

The no-anomaly condition is
\[ 4n_2 - 5n_5 - n_7 = 0 . \] (5.62)

For a given value of \( N_{gh} = 3 - N_{gt} \), there are three nondegenerate linear equations for the three quantities \( n_2 \), \( n_5 \), and \( n_7 \). We display the formal solution, with the understanding that it is physical only for positive nonnegative integer values of the \( n_k \):
\[
\begin{align*}
n_2 &= \frac{2N_{gh}}{3} \\
n_5 &= \frac{N_{gh}}{3} \\
n_7 &= N_{gh} .
\end{align*}
\] (5.63)

In order for \( n_2 \) and \( n_5 \) to be nonnegative integers, \( N_{gh} = 0 \mod 3 \); the value \( N_{gh} = 0 \) is not permitted because this would require \( N_{gt} = 3 \), but \( N_{gt} \leq N_{SC} - (N_{TC})_{\min} = N_{SC} - 2 = 1 \). Hence, the only possibility is
\[ N_{gh} = 3, \quad N_{gt} = 0 , \] (5.64)

whence
\[
\begin{align*}
n_2 &= 2, \\
n_5 &= 1, \\
n_7 &= 3 ,
\end{align*}
\] (5.65)

so that
\[ \mathbf{n} = (0, 2, 0, 0, 1, 0, 3)_s , \] (5.66)
i.e., the fermion content of the model is
\[ \{ f \} = [2]_s + [5]_s + 3[7]_s \approx 2[2]_s + [3]_s + 3[1]_s . \] (5.67)

Further, for this case,
\[ G_{SC} = G_{TC} = SU(3)_{TC} \] (5.68)
and
\[ G_{SCC} = SU(6)_{SCC} = SU(6)_{TCC} , \] (5.69)

where, since \( G_{SC} \) is just the technicolor group, we have indicated this explicitly in the subscript. Note that, by eq. (5.30),
\[ N_{(1,1)} = 1 . \] (5.70)

In Table I we list properties of this model and others that we have studied.

Let us analyze this SU(8) model further. With respect to the subgroup given by (5.17), viz.,
\[ SU(8) \supset SU(3)_{TC} \times SU(5)_{GU} , \] (5.71)
we have the decomposition
\[ [2]_s = (3, 1) + (3, 5) + (1, 10) \] (5.72)
\[ [5]_s \approx [3]_s = (1, 1) + (3, 5) + (3, 10) + (1, 10) \] (5.73)
\[ [7]_s \approx [1]_s = (3, 1) + (1, 5) . \] (5.74)

Since both technicolor and color are described by SU(3) subgroups of SU(8), the theory is formally symmetric under the interchange of technicolor indices \( i = 1, 2, 3 \) and color indices, \( i = 4, 5, 6 \), and eqs. (5.72)-(5.74) also describe the decomposition of the fermion representations with respect to the subgroup SU(3)_c \times SU(5), where this SU(5) involves technicolor and electroweak indices. The nonsinglet fermion content under color or technicolor consists of 15 copies of \{ 3 + 3 \}. Evidently, both color and technicolor are vectorial gauge symmetries.

With respect to the subgroup given by (5.14), namely
\[ SU(8) \supset SU(2)_w \times SU(6)_{TCC} \] (5.75)
the fermion representations have the decompositions
\[ [2]_s = (1, 1) + (2, [1]_6) + (1, [2]_6) \] (5.76)
\[ [5]_s \approx [3]_s = (1, [1]_6) + (2, [2]_6) + (1, [3]_6) \] (5.77)
\[ [7]_s \approx [1]_s = (2, 1) + (1, [1]_6) . \] (5.78)

Here and below, it is convenient to use the \([ k ]_N \) notation for larger Lie groups; the corresponding dimensionalities for representations of SU(6)_{TCC} are \([ 2 ]_s = 15 \) and \([ 3 ]_6 = 27 \). Thus, the SU(6)_{TCC} theory is vectorial, with nonsinglet fermion content (neglecting other interactions) consisting of the set of (left-handed) fermions
\[ 4([1]_6 + [1]_6) + 2([2]_6 + [2]_6 + [3]_6) \] (5.79)
(Here, with respect to SU(6), \([ 3 ]_6 \approx [ 3 ]_6 \).) Let us assume that SU(8) breaks in such a manner as to yield an effective theory at lower energies that has a SU(2)_w \times SU(6)_{TCC} symmetry (ignoring an abelian factor). Since the TCC theory is asymptotically free, with leading beta function coefficient \( b_{TCC} = 12 \), the TCC coupling increases as the energy scale decreases. Because the SU(6)_{TCC} theory is vectorial, when the energy scale decreases sufficiently that \( \alpha_{TCC} \sim O(1) \), the TCC interaction will naturally form SU(6)_{TCC}-invariant fermion condensates rather than breaking to SU(3)_c \times SU(3)_{TC}, as is necessary in order to separate color and the TC interaction. The most attractive channel, written in terms of representations of SU(2)_w \times SU(6)_{TCC}, together with its \( \Delta C_2 \) value, is
\[ (1, [3]_6) \times (1, [3]_6) \rightarrow (1, 1) , \quad \Delta C_2 = \frac{21}{2} . \] (5.80)

This condensation channel is of the form of (the conjugate of) (5.40) with \( p = 3 \), \( N_{SC} = 6 \). By our general argument given above, the associated condensate violates weak hypercharge and electric charge. If it did...
occur, this, by itself, would rule out the present SU(8) model. The rough estimate for the corresponding critical coupling is \( \alpha_{TCC,c} \approx 2\pi/63 \approx 0.1 \) from the Schwinger-Dyson equation. Substituting this into eq. \( \text{(5.10)} \) with \( \alpha_{TCC} = \alpha = \mu = MG_U \) and using the illustrative value for the unified coupling \( \alpha = 0.04 \), we find that \( \alpha_{TCC} \) would be large enough for this condensate to form at \( \mu_c \approx (3 \times 10^{-5})MG_U \), i.e., about \( 3 \times 10^{11} \) GeV for the hypothetical unification scale \( MG_U = 10^{16} \).

Other possible condensation channels have smaller values of the attractiveness measure \( \Delta C_2 \); they include the following, in order of descending \( \Delta C_2 \):

\[
(1, [2]\bar{\alpha}) \times (2, [2]\bar{\alpha}) \rightarrow (2, 1) , \quad \Delta C_2 = \frac{28}{3} \approx 9.3 \quad (5.81) \\
(1, [2]\bar{\alpha}) \times (1, [3]\bar{\alpha}) \rightarrow (1, [1]\bar{\alpha}) , \quad \Delta C_2 = 7 \quad (5.82) \\
(2, [2]\bar{\alpha}) \times (1, [3]\bar{\alpha}) \rightarrow (2, [5]\bar{\alpha}) \approx (2, [1]\bar{\alpha}) , \quad \Delta C_2 = 7 \quad (5.83) \\
(1, [1]\bar{\alpha}) \times (1, [2]\bar{\alpha}) \rightarrow (1, [1]\bar{\alpha}) , \quad \Delta C_2 = \frac{35}{6} \approx 5.8 \quad (5.84) \\
(2, [1]\bar{\alpha}) \times (1, [1]\bar{\alpha}) \rightarrow (2, 1) , \quad \Delta C_2 = \frac{35}{6} \quad (5.85)
\]

Channels \( \text{(5.81)} \) and \( \text{(5.84)} \) fail to break SU(6)\(_{TCC} \) and break SU(2)\(_{w} \) at a scale higher than the ETC scale where SU(6)\(_{TCC} \) should break. Channels \( \text{(5.82)} \) and \( \text{(5.83)} \) break SU(6)\(_{TCC} \) to SU(5)\(_{TCC} \) rather than SU(3)\(_c \) \times SU(3)\(_c \), SU(3)\(_c \) breaks SU(2)\(_w \) at too high a scale and breaks SU(6)\(_{TCC} \) to SU(5)\(_{TCC} \).

It may be noted that even if there were some way to produce a breaking of SU(6)\(_{TCC} \) that yielded a lower-energy theory invariant under SU(3)\(_c \) \× SU(3)\(_c \), the interchange symmetry between of SU(3)\(_T \) \× SU(3)\(_T \) would imply that the respective technicolor and color gauge couplings would evolve in the same way as the energy decreases below the scale at which this condensate occurred. Both of these sectors are asymptotically free, and condensates would form, but the model would still not be realistic, since the scale \( \Lambda_{TC} \) would be the same as \( \Lambda_{QCD} \).

### D. \( \text{ \( N_{SC} = 4, \ G = SU(9) \) } \)

Here we consider \( \text{ \( N_{SC} = 4, \ G = SU(9) \) } \) and \( \mathbf{n} = (n_1, ..., n_8) \). \textit{A priori}, one has three possibilities regarding the origins of the SM fermion generations: (i) \( N_{gh} = 3 \), or equivalently, \( N_{gf} = 0 \), whence \( N_{TC} = 4 \); (ii) \( N_{gh} = 2 \) so that \( N_{gf} = 1 \), which would be associated with a breaking of SU(4)\(_{SC} \) to SU(3)\(_{TC} \); (iii) \( N_{gh} = 1 \), or equivalently, \( N_{gf} = 2 \), so that one SM generation would arise initially from the representations of \( G \), and the other two would arise via the sequential breaking SU(4)\(_{SC} \) \rightarrow SU(3)\(_{SC} \) and then SU(3)\(_{SC} \) \rightarrow SU(2)\(_{TC} \). In general, equations \( \text{(5.81)} \) and \( \text{(5.85)} \) yield

\[
n_1 = n_3 = n_5 = n_7 = 0 \quad (5.86)
\]

and eq. \( \text{(5.32)} \) is

\[
N_{gh} = n_2 + n_6 = n_4 + n_8 . \quad (5.87)
\]

The condition of zero gauge anomaly \( \text{(5.32)} \) is

\[
5(n_2 + n_4) - 9n_6 - n_8 = 0 . \quad (5.88)
\]

For a given value of \( N_{gh} = 3 - N_{gf} \), these are three non-degenerate linear equations for the four quantities \( n_2, n_4, n_6, \) and \( n_8 \). Taking \( n_2, n_4 \), say, as the independent variable, initially free to take on values \( n_2 = 0, 1, ..., N_{gh} \), we find the formal solution of these equations to be

\[
n_6 = N_{gh} - n_2 \\
n_4 = \frac{1}{3}(5N_{gh} - 7n_2) \\
n_8 = \frac{1}{3}(7n_2 - 2N_{gh}) . \quad (5.89)
\]

Consider first the \textit{a priori} possible value \( N_{gh} = 3 \), whence \( n_2 = 5 - (7/3)n_2 \) and \( n_8 = (7/3)n_2 - 2 \). In order for \( n_2 \) and \( n_8 \) to be nonnegative integers, \( n_2 = 0 \mod 3 \). Now \( n_2 \) cannot be zero, because this would make \( n_8 \) negative. But \( n_2 \) also cannot be equal to 3, because this would make \( n_4 \) negative. Hence, the value \( N_{gh} = 3 \) is not allowed.

Consider next the value

\[
N_{gh} = 2 \quad (5.90)
\]

whence \( N_{gf} = 1 \) and

\[
G_{SC} = SU(4)_{SC}, \quad G_{TC} = SU(3)_{TC} . \quad (5.91)
\]

Equations \( \text{(5.88)} \) yield \( n_4 = (1/3)(10 - 7n_2) \) and \( n_8 = (1/3)(7n_2 - 4) \). From its formal range for this case, \( n_2 = 0, 1, 2 \), the only allowed value would be form \( n_2 = 1 \) which gives \( n_4 = n_8 = n_6 = 1 \), so that

\[
\mathbf{n} = (0, 1, 0, 1, 0, 1, 0, 1, 0)_g \quad (5.92)
\]

i.e., the fermion content is given by \( \{ f \} = [2]_g + [4]_g + [6]_g + [8]_g \). For this case \( N_{(1,1)} = 1 \).

Finally, we examine the minimal possible value, \( N_{gh} = 1 \), corresponding to the maximal possible value of \( N_{gf} \), namely \( N_{gf} = N_{SC} - (N_{TC})_{min} = 4 - 2 = 2 \). Here eqs. \( \text{(5.89)} \) read \( n_4 = (1/3)(5 - 7n_2) \) and \( n_8 = (1/3)(7n_2 - 2) \), where, \textit{a priori}, \( n_2 \) can take values in the set \{0, 1\}. Evidently, neither of these values would make \( n_4 \) and \( n_8 \) nonnegative integers and hence neither is allowed.

Let us return to the case with \( N_{gh} = 2 \). An initial comment is that the corresponding value \( N_{TC} = 3 \) is disfavored, relative to \( N_{TC} = 2 \), since it leads to larger
technicolor contributions to precision electroweak quantities and could reduce the likelihood of walking behavior for the technicolor theory. Notwithstanding this concern, let us investigate this case. The decomposition of the fermion representations with respect to the subgroup \( \text{G}_{12} \), which for this case reads

\[
\text{SU}(4)_{SC} \times \text{SU}(5)_{GU},
\]

is listed below:

\[
[2]_9 = (6, 1) + (4, 5) + (1, 10)
\]

\[
[4]_9 = (1, 1) + (4, 5) + (6, 10) + (4, \overline{10}) + (1, 5)
\]

\[
[6]_9 \approx [3]_9 = (4, 1) + (6, 5) + (4, \overline{10}) + (1, 10)
\]

\[
[8]_9 \approx [1]_9 = (4, 1) + (1, 5)
\]  

(Note that \( 6 \approx 6 \), i.e., \([2]_4 \approx [2]_4\), in \( \text{SU}(4)_{SC} \).) This decomposition shows that, neglecting the \( \text{GU} \) couplings relative to those of \( \text{SU}(4)_{SC} \), the latter interaction is vectorial, involving nonsinglet fermions comprising a set of 16 copies of \( \{4 + 4 + 6\} \) left-handed fermions.

As before, we consider the implications of a scenario in which the unified group, here \( \text{SU}(9) \), breaks to a vector at lower energy scales that is invariant under the gauge symmetry (ignoring an abelian factor) of the form \( \text{SU}(2) \times \text{SU}(3)_{c} \times \text{SU}(10) \), which for the present model is

\[
\text{SU}(2)_{w} \times \text{SU}(7)_{SCC}.
\]

With respect to this direct product symmetry group, the fermions have the decomposition

\[
[2]_9 = (1, 1) + (2, [1]_7) + (1, [2]_7)
\]

\[
[4]_9 = (1, [2]_7) + (2, [3]_7) + (1, [3]_7)
\]

\[
[6]_9 \approx [3]_9 = (1, [1]_7) + (2, [2]_7) + (1, [3]_7)
\]

\[
[8]_9 \approx [1]_9 = (2, 1) + (1, [1]_7)
\]  

Here, the dimensionalities include \( \text{dim}([2]_7) = 21 \) and \( \text{dim}([3]_7) = 35 \). Thus, the \( \text{SU}(7)_{SCC} \) theory is vectorial, with nonsinglet fermion content (neglecting other interactions) given by

\[
2\{[1]_7 + [1]_7 + 2[2]_7 + 2[3]_7 + 3[3]_7 + [3]_7\}.
\]

The \( \text{SU}(7)_{SCC} \) gauge interaction is asymptotically free, with leading beta function coefficient \( \beta_0^{(\text{SCC})} = 13/3 \). However, as in the models that we studied above, because of the vectorlike nature of the SCC gauge symmetry, when the energy decreases sufficiently so that \( \alpha_{SCC} \) grows large enough to produce fermion condensates, these will preferentially be in the channels \( \mathcal{R} \times \mathcal{R} \to 1 \), where \( \mathcal{R} \) refers to the \( \text{SU}(7)_{SCC} \) representation. Thus, these preserve the \( \text{SU}(7)_{SCC} \) invariance rather than breaking it down to a direct product which includes \( \text{SU}(3)_{c} \times \text{SU}(4)_{SC} \), as is necessary to separate color from the strongly coupled \( \text{SU}(4)_{SCC} \) group. With respect to the \( \text{SU}(2)_{w} \times \text{SU}(7)_{SCC} \) subgroup, the most attractive channel, with its measure of attractiveness \( \Delta C_2 \), is

\[
(2, [3]_7) \times (1, [3]_7) \to (2, 1), \quad \Delta C_2 = \frac{96}{7} \approx 13.7.
\]

In addition to its failure to break the \( \text{SU}(7)_{SCC} \) symmetry, this MAC breaks \( \text{SU}(2)_w \) at a scale that would be higher than the ETC scales where the \( \text{SU}(7)_{SCC} \) should break. The same problems characterize a channel with a somewhat smaller value of \( \Delta C_2 \), namely,

\[
(1, [2]_7) \times (2, [2]_7) \to (2, 1), \quad \Delta C_2 = \frac{80}{7} \approx 11.4.
\]

As before, one can examine other possible condensation channels with still smaller values of \( \Delta C_2 \), which include

\[
(1, [2]_7) \times (1, [3]_7) \to (1, [1]_7), \quad \Delta C_2 = \frac{74}{7} \approx 10.6
\]

\[
(2, [2]_7) \times (2, [3]_7) \to (1, [1]_7), \quad \Delta C_2 = \frac{74}{7} \approx 10.6
\]

\[
(2, [2]_7) \times (2, [3]_7) \to (3, [1]_7), \quad \Delta C_2 = \frac{74}{7} \approx 10.6
\]

Channel 5.108 is unacceptable because it breaks \( \text{SU}(2)_w \) in the wrong way and at too high a scale. Channels 5.106 and 5.107 are not forbidden but would only break \( \text{SU}(7)_{SCC} \) to \( \text{SU}(6)_{SCC} \), thereby necessitating a further breaking to \( \text{SU}(3)_{c} \times \text{SU}(3)_{PC} \); moreover, because of their subdominant \( \Delta C_2 \) values, it is difficult to make a convincing argument that they would predominate. Another conceivable breaking pattern is \( \text{SU}(7)_{SCC} \to \text{SU}(3)_{c} \times \text{Sp}(4) \), but it is not clear what dynamical fermion condensation channel could produce this breaking.

**E.  \( N_{SC} = 5, G = \text{SU}(10) \)**

We proceed to examine the case where \( N_{SC} = 5 \), so that \( G = \text{SU}(10) \) and \( n = (n_1, ..., n_9)_{10} \). A priori, one has four possibilities for the manner in which the SM fermion generation arise, as specified by \( (N_{gh}, N_{gt}, N_{TC}) \), namely (i) \((3,0,5)\), (ii) \((2,1,4)\), (iii) \((1,2,3)\), and (iv) \((0,3,2)\). The minimization of technicolor contributions to electroweak corrections favor the last of these options. The conditions \( 5.33 \) and \( 5.35 \) forbidding \( 5_L \) and \( \overline{10}_L \) yield

\[
n_1 = n_3 = n_6 = n_8 = 0
\]  

\[
(5.109)
\]
and eq. (5.32) is

\[ N_{gh} = n_2 + n_7 = n_4 + n_9 . \]  \quad (5.110)

The condition of zero gauge anomaly is

\[ 6n_2 + 14(n_4 - n_7) - n_9 = 0 . \]  \quad (5.111)

For a given value of \( N_{gh} = 3 - N_{gl} \), these are three nondegenerate linear equations for the five quantities \( n_2, n_4, n_5, n_7, \) and \( n_9 \). Taking \( n_5 \) and \( n_7 \), say, as the two independent variables, with \( n_7 \) constrained by eq. (5.110) to take on values in the set \( \{0, 1, ..., N_{gh}\} \), we find the formal solution of these equations to be

\[ n_4 = \frac{1}{3} (4n_7 - N_{gh}) \]  \quad (5.112)

\[ n_9 = \frac{4}{3} (N_{gh} - n_7) . \]  \quad (5.113)

From eq. (5.113) and the requirement that \( n_9 \) be a non-negative integer, it follows that \( N_{gh} - n_7 = 0 \mod 3 \). This could be satisfied for \( N_{gh} = 3 \) and \( n_7 = 0 \), but this choice is excluded because it would produce a negative value for \( n_4 \). The other choice is \( N_{gh} = n_7 \), which gives \( n_9 = 0 \) and \( n_4 = n_7 \). Substituting \( n_9 = 0 \) in eq. (5.110) yields \( n_4 = N_{gh} \), so that \( n_7 = N_{gh} \) also; substituting the latter in eq. (5.110) then gives \( n_9 = 0 \). These conditions leave \( n_5 \) free, subject to the additional requirement that \( SU(5)_{SC} \) be asymptotically free. Thus, we have

\[ n = (0, 0, 0, N_{gh}, n_5, 0, N_{gh}, 0, 0)_{10} . \]  \quad (5.114)

The number of SC-singlet, \( SU(5)_{GU} \)-singlet fermions is \( N_{(1,1)} = 2n_5 \).

Let us consider first the value \( n_5 = 0 \), so that

\[ n = (0, 0, 0, N_{gh}, 0, 0, N_{gh}, 0, 0)_{10} . \]  \quad (5.115)

This allows one to choose \( (N_{gh}, N_{gl}) = (3, 0), (2, 1), \) or \( (1, 2) \). If one were to apply the irreducibility condition that \( GCD(\{n_k\}) = 1 \), it would imply that \( N_{gh} = 1 \) in eq. (5.116), which requires \( N_{gl} = 2 \), i.e., the TC group is \( SU(3)_{TC} \) and the SC group should undergo two sequential breakings, \( SU(5)_{SC} \rightarrow SU(4)_{SC} \), followed by \( SU(4)_{SC} \rightarrow SU(2)_{TC} \). Although we will not impose the irreducibility here, we will exclude all of the reducible solutions because they lead to excessively many fermions, which render the \( SU(8)_{SCC} \) theory non asymptotically free.

The fermions in each type of representation have the decomposition, with respect to the subgroup \( (1,2) \) for this case,

\[ SU(5)_{SC} \times SU(5)_{GU} , \]  \quad (5.116)

of

\[ [4]_{10} = (5, 1) + (\overline{10}, 5) + (10, 10) + (5, \overline{10}) + (1, 5) \]  \quad (5.117)

\[ [7]_{10} = [\bar{3}]_{10} = (10, 1) + (\overline{10}, 5) + (5, \overline{10}) + (1, 10) . \]  \quad (5.118)

Thus, the \( SU(5)_{SC} \) sector forms a chiral gauge theory, with left-handed (nonsinglet) fermion content consisting of

\[ \{f\} = N_{gh}[10(5 + \overline{10}) + 11(\overline{5} + 10)] . \]  \quad (5.119)

With respect to the subgroup \( (1,2) \),

\[ SU(2)_w \times SU(8)_{SCC} , \]  \quad (5.120)

the fermions in each representation have the decomposition

\[ [4]_{10} = (1, [2]_8) + (2, [3]_8) + (1, [4]_8) \]  \quad (5.121)

\[ [7]_{10} \approx [\bar{3}]_{10} = (1, [\bar{1}]_8) + (2, [\bar{2}]_8) + (1, [\bar{3}]_8) . \]  \quad (5.122)

Some dimensionalities of relevant \( SU(8) \) representations are \( \dim([2]_8) = 28, \dim([\bar{3}]_8) = 56, \) and \( \dim([4]_8) = 70; \) note that \( [4]_8 = [\bar{4}]_8 \). Thus, assuming that the \( SU(10) \) unified theory breaks in such a way as to yield, for a range of lower energies, an effective theory with \( SU(8)_{SCC} \) gauge symmetry, this SCC sector is a chiral gauge theory. The \( SU(8)_{SCC} \) gauge interaction has leading beta function coefficient \( b_0^{(SCC)} = 4(22 - 21N_{gh})/3 \), so it is asymptotically free only for the choice \( N_{gh} = 1 \), for which \( b_0^{(SCC)} = 4/3 \). As the energy scale decreases and the coupling \( \alpha_{SCC} \) becomes sufficiently large, this theory will thus form bilinear fermion condensates. The most attractive channel is

\[ (1, [4]_8) \times (1, [4]_8) \rightarrow (1, 1) , \quad \Delta C_2 = 18 . \]  \quad (5.123)

This is of the form of channel (5.40) with \( p = 4 \) and thus violates weak hypercharge and electric charge.

Other channels with smaller values of \( \Delta C_2 \) include

\[(2, [\bar{3}]_8) \times (1, [\bar{3}]_8) \rightarrow (2, 1) , \quad \Delta C_2 = \frac{135}{8} \approx 16.9 \]  \quad (5.124)

\[ (1, [2]_8) \times (2, [\bar{2}]_8) \rightarrow (2, 1) , \quad \Delta C_2 = \frac{63}{4} = 15.75 \]  \quad (5.125)

\[ (1, [\bar{3}]_8) \times (1, [4]_8) \rightarrow (1, [\bar{1}]_8) , \quad \Delta C_2 = \frac{27}{2} = 13.5 \]  \quad (5.126)

\[ (1, [2]_8) \times (1, [\bar{3}]_8) \rightarrow (1, [\bar{1}]_8) , \quad \Delta C_2 = \frac{45}{4} = 11.25 . \]  \quad (5.127)

Channels (5.124) and (5.126) break \( SU(2)_w \) at too high a scale and fail to break \( SU(8)_{SCC} \). Channels (5.120) and (5.127) are allowed by symmetry considerations. However, since their \( \Delta C_2 \) values are smaller than those of the leading channels, one cannot make a persuasive case that they would occur. Moreover, because of the relatively
small value of $b^{(SCC)}$, estimates based on eq. (5.10) indicate that for a hypothetical unified coupling $g_{GU} \sim 0.04$ at $M_{GU} \sim 10^{16}$ GeV, these condensates would form at much too small a scale for a viable model.

We consider next the choice $N_{gh} = 1$, $n_5 = 1$, so that

$$\mathbf{n} = (0, 0, 0, 1, 1, 0, 1, 0, 0, 0)_10 .$$  \hspace{1cm} (5.128)

For this choice the decomposition of the fermions with respect to the subgroup $\mathbf{5}, \mathbf{10}$ is

$$[4]_{10} = (5, 1) + (\bar{10}, 5) + (10, 10) + (5, \bar{10}) + (1, 5)$$  \hspace{1cm} (5.129)

$$[5]_{10} = 2(1, 1) + (\bar{5}, 5) + (\bar{10}, 10) + (10, \bar{10})$$  \hspace{1cm} (5.130)

$$[7]_{10} \approx 3]_{10} = (10, 1) + (\bar{10}, 10) + (\bar{1}, 10) .$$  \hspace{1cm} (5.131)

The SU(5)$_{SC}$ theory is a chiral gauge theory, consisting of the nonsinglet fermion content

$$\{ f \} = 16(\bar{5}) + 15(5) + 21(10) + 20(\bar{10}) .$$  \hspace{1cm} (5.132)

With respect to the subgroup $\mathbf{5}, \mathbf{10}$, the fermions in the $[4]_{10}$ and $[7]_{10}$ have the decompositions given in eqs. (5.129) and (5.130), and

$$[5]_{10} = (1, 3) + (2, 4) + (1, 1) .$$  \hspace{1cm} (5.133)

Although the $[5]_{10}$ is self-conjugate, the $[4]_{10}$ and $[7]_{10}$ make this SU(8)$_{SCC}$-invariant chiral. However, it is non-asymptotically free, with leading beta function coefficient $b_0^{(SCC)} = -22$.

Finally, we consider the choice $N_{gh} = 0$, $n_5 = 1$, so that

$$\mathbf{n} = (0, 0, 0, 1, 1, 0, 1, 0, 0, 0)_10 .$$  \hspace{1cm} (5.134)

For this choice the decomposition of the fermions in the $[5]_{10}$ with respect to the subgroup $\mathbf{5}, \mathbf{10}$ is given by eq. (5.133) and with respect to the subgroup $\mathbf{5}, \mathbf{10}$ by eq. (5.130). Assuming that the breaking of SU(10) is such as to yield an SU(8)$_{SCC}$-invariant theory for a range of lower energies, its coupling does grow as the energy decreases, as governed by the leading beta function coefficient $b_0^{(SCC)} = 6$. However, as is evident from eq. (5.133), this theory is vectorial, so that when the SU(8)$_{SCC}$ coupling grows sufficiently large to produce a fermion condensate, this condensate will preferentially preserve the SU(8)$_{SCC}$ symmetry rather than breaking it, as is necessary, to SU(5)$_{TC}$ × SU(3)$_c$ (and hence, sequentially, breaking SU(5)$_{TC}$ into SU(2)$_{TC}$). We have investigated higher values of $N_{SC}$ and thus $N$ but have found that they exhibit problems similar to those of the models above.

**F. Assessment**

Thus we find several general problems with the unification approach embodied in models of type 3. One, pertaining to the mechanism for breaking the unified gauge group $G$ in a weak-coupling framework, also applies to models of type (1) and (2) and will be discussed in the next section. A second problem is that, even if one could arrange some mechanism to break the $G$ symmetry in the desired manner to yield a lower-energy theory invariant, presumably, under an SCC gauge symmetry combining the strongly coupled group $G_{SC}$ with the color group, it appears very difficult to get this SCC symmetry to break in the requisite manner. This is especially true when, as is often the case, the SU($N_{SCC}$) gauge interaction is vectorial. Even when it is chiral (and asymptotically free), as in the model with $N_{SC} = 5$ and $\mathbf{n}$ given by eq. (5.115) with $N_{gh} = 1$, the most attractive condensation channels do not lead to the requisite breaking.

Since the determination of the resultant SC and TC sectors depends on having a viable SCC breaking pattern, this prevents one from proceeding very far with the analysis of these lower-energy effective field theories in the context of these models. However, we note that in cases where $N_{gh} \geq 1$, it could also be challenging to get the SC symmetry to break sequentially down to the resultant exact TC symmetry. Because of the previous problems, one cannot obtain very definite predictions for fermion masses. A general concern pertains to reproducing the observed mass hierarchy of the three SM fermion generations. In order to do this, one tends to need three different ETC mass scales, essentially the $\Lambda_j$, $j = 1, 2, 3$ discussed in Section II. In the present approach, one has $N_{gh} + 1$ ETC-type mass scales. This is illustrated by the scenario in which the SU($N_{SCC}$) symmetry first breaks to SU($N_{SC}$) × SU(3)$_c$ and then SU($N_{SC}$) breaks sequentially at $N_{gh}$ lower scales, finally yielding the residual exact SU($N_{TC}$) symmetry. Consequently, unless $N_{gh} \geq 2$, one does not have enough mass scales to account for the SM fermion mass hierarchy. This problem reaches its most acute form when $N_{gh} = 0$, $N_{gh} = 3$.

Because of the presence of intermediate scales between $m_Z$ and $M_{GU}$ with nonperturbative behavior and the feature that the ETC gauge bosons involved in symmetry breakings at these scales carry SM quantum numbers, the calculation of the evolution of the SM gauge couplings is more complicated in models of type (3) than in models of types (1) or (2). However, exploratory analysis of plausible evolution of gauge couplings for the various scenarios that we have examined indicate that satisfying the constraint of gauge coupling unification is still a very restrictive requirement.

**VI. DYNAMICAL BREAKING OF UNIFIED GAUGE SYMMETRIES**

In addition to other issues that we have addressed concerning prospects for unification of gauge symmetries in a dynamical context, there is another one which is quite general. For the present discussion let us assume that one has a model that does achieve gauge coupling unifi-
cation. The resultant value of the unified gauge coupling at $M_{GU}$ is generically expected to be small. But if one is trying to construct a theory in which all gauge symmetry breaking is dynamical, this would normally require there to be some strongly coupled gauge interaction at the relevant scale. For example, the dynamical breaking of the electroweak symmetry in a technicolor theory requires that technicolor be an asymptotically free gauge interaction that becomes strongly coupled at the electroweak scale. Although the ETC interaction is strongly coupled at the scale $\Lambda_1 = \Lambda_{ETC,max} \approx 10^6$ GeV, in typical models of type (1) and (2) the ETC coupling evolves to relatively small values as the energy scale ascends to the region of $M_{GU}$, so one could not use ETC interactions to break $G_{GU}$ at $M_{GU}$ in these models. Moreover, in specific models such as those of Refs. [3, 12], the ETC (and HC) condensates involve SM-singlet fields which, in the present context, would naturally be $G_{GU}$-singlet fields, so their condensates would not break $G_{GU}$.

One way to break a hypothetical symmetry $G_{GU}$ unifying SM gauge interactions at a high scale $M_{GU}$ would be to expand the theory to include an additional gauge interaction, associated with a group $G_a$, that is strongly coupled at this scale, together with fermions that transform as nonsinglets under both $G_{GU}$ and $G_a$. The great disparity between the coupling strengths of the $G_{GU}$ and $G_{ETC}$ gauge bosons, on the one hand, and the assumed $G_a$ gauge bosons, on the other, is a striking property of such a model. Another approach would be to envision a nonperturbative unification of gauge symmetries, but by its very nature, this is difficult to study reliably using tools such as perturbative evolution of SM couplings [33].

For completeness, one should note that a major objection to the use of Higgs fields for symmetry breaking at lower scales is the instability of a Higgs sector to large radiative corrections, necessitating fine tuning to keep the Higgs bosons light compared with an ultraviolet cutoff. But from considerations of nucleon stability alone, not to mention gauge coupling evolution, one knows that in the (four-dimensional) theories considered here, $M_{GU}$ is generically quite high, not too far below the Planck scale where the theories are certainly incomplete, since they do not include quantum gravity. Hence, this objection would not be as strong for the breaking of $G_{GU}$ as for symmetry breakings that occur at substantially lower scales.

VII. CONCLUSIONS

In this paper we have analyzed approaches to the partial or complete unification of gauge symmetries in theories with dynamical symmetry breaking. We considered three main types of models with progressively greater degrees of unification, including those that (1) involve sufficient unification to quantize electric charge, (2) attempt to unify the three standard-model gauge interactions in a simple group that forms a direct product with an extended technicolor group, and, (3) attempt to unify the standard-model gauge interactions with (extended) technicolor in a simple group. The model of Refs. [11, 12] is a successful example of theories of type (1). Models of type (3) provide an interesting contrast to those of types (1) and (2) in the different way in which standard-model fermion generations are produced and in the property of having ETC gauge bosons that carry standard-model quantum numbers. We have pointed out a number of challenges that one faces in trying to construct viable models of types (2) and (3). There are certainly further avenues for research in this area. For example, one could investigate direct product groups involving auxiliary hypercolor-type groups. Another idea would be to study ways to unify top-color [34] with standard-model gauge symmetries. In conclusion, it is possible that electroweak symmetry breaking is dynamical, involving a new strongly coupled gauge symmetry, technicolor, embedded in an extended technicolor theory to give masses to standard-model fermions. There is a strong motivation to understand how the associated symmetries can be unified with the color and weak isospin and hypercharge gauge symmetries. We hope that the results of the present paper will be of use in the further study of this unification program.

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VIII. APPENDIX

We gather in this appendix some standard formulas that are used in the calculations reported in the text.

A. Some Group-Theoretic Properties of Representations $[k]_N$ of SU($N$)

We denote the completely antisymmetric $k$-fold products of the fundamental and conjugate fundamental representation of SU($N$) as $[k]_N$ and $[\bar{k}]_N$, respectively. These can be displayed as tensors with $k$ upper indices, $\psi^{i_1 \cdots i_k}$ and $k$ lower indices, $\psi_{i_1 \cdots i_k}$, and have the (same) dimension

$$\dim([k]_N) = \binom{N}{k}. \quad (8.1)$$

These representations satisfy the equivalence property

$$[N-k]_N \approx [\bar{k}]_N \quad (8.2)$$

under SU($N$), as follows by contraction with the totally antisymmetric tensor density $\epsilon_{i_1 \cdots i_N}$. The fact that these
representations have the same dimension is evident from the identity \( \binom{N}{k} = \binom{N}{N-k} \). Further, if \( N \) is even, say \( N = 2k \), then \([N/2]_N\) is self-conjugate with respect to SU(\(N\)).

The contribution of the (left-handed) fermions in the representation \([k]_N\) to the gauge anomaly for SU(\(N\)) is

\[
A([k]_N) = \frac{(N - 2k)(N - 3)!}{(N - k - 1)!}(k - 1)! , \tag{8.3}
\]

where the normalization is such that contribution of the fundamental representation is 1. The property \( A([k]_N) = A([k]_N) \) for \( 1 \leq k \leq N - 1 \) is evident from eq. (8.3).

The quadratic Casimir invariant \( C_2(\mathcal{R}) \) for the representation \( \mathcal{R} \) is defined by

\[
\sum_{a=1}^{o(G)} \sum_{j=1}^{\dim(\mathcal{R})} \mathcal{D}_\mathcal{R}(T_a)_j^i \mathcal{D}_\mathcal{R}(T_a)_i^j = C_2(\mathcal{R}) \delta_{ab} , \tag{8.4}
\]

where \( \mathcal{D}_\mathcal{R}(T_a) \) is the \( \mathcal{R} \)-representation of the generator \( T_a \) and \( o(G) \) is the order of the group \( G \).

The contribution of a fermion loop, for fermions of representation \( \mathcal{R} \) of SU(\(N\)), to the beta function coefficient \( b_0^{(j)} \), involves the invariant \( T(\mathcal{R}) \) defined by

\[
\sum_{i,j=1}^{\dim(\mathcal{R})} (\mathcal{D}_\mathcal{R}(T_a)_i^j)(\mathcal{D}_\mathcal{R}(T_b)_i^j) = T(\mathcal{R}) \delta_{ab} . \tag{8.5}
\]

These invariants satisfy the elementary relation

\[
C_2(\mathcal{R}) \dim(\mathcal{R}) = T(\mathcal{R}) o(G) . \tag{8.6}
\]

For SU(\(N\)),

\[
C_2([k]_N) = \frac{k(N + 1)(N - k)}{2N} \tag{8.7}
\]

and

\[
T([k]_N) = \frac{1}{2} \binom{N - 2}{k - 1} . \tag{8.8}
\]

As the value of \( N \) (in eq. (8.3)) increases, the number of (left-handed) fermions in a generic set \( \mathbf{n} \) tends to increase rapidly. For example, for even \( N_{SC} \) and hence odd \( N = N_{SC} + 5 \equiv 2m + 1 \), the set \( \mathbf{n} \) with \( n_{2\ell} = 1 \) for \( \ell = 1, .., m \) and \( n_{2\ell+1} = 0 \) for \( \ell = 0, .., m - 1 \), has a total number of fermions given by

\[
\sum_{\ell=1}^{m} \binom{2m + 1}{2\ell} = 2^{2m} - 1 = 2^{N-1} - 1 . \tag{8.9}
\]

Evidently, this grows exponentially rapidly with \( N_{SC} \) and \( N \), which quickly renders the SCC theory non-asymptotically free. Note that for this choice of \( \mathbf{n} \), since \( n_2 = n_{N_{SC}+2} \), it follows that \( N_{gh} = 2 \).

B. Formulas for the Evolution of Gauge Couplings

We consider a factor group \( G \) with gauge coupling \( g_j \) and denote \( \alpha_j = g_j^2/(4\pi) \). The evolution of the gauge couplings as a function of the momentum scale \( \mu \) is given by the renormalization group equation

\[
\beta_j = \frac{d\alpha_j}{dt} = \frac{\alpha_j^2}{2\pi} \left( b_0^{(j)} + \frac{b_1^{(j)}}{2\pi} \alpha_j + O(\alpha_j^2) \right) , \tag{8.10}
\]

where \( t = \ln \mu \), and the first two terms \( b_0^{(j)} \) and \( b_1^{(j)} \) are scheme-independent. The beta function with perturbatively calculated coefficients is appropriate to describe the running of the respective couplings in the energy ranges where the respective gauge fields are dynamical (i.e., above corresponding scales at which \( G_j \) is broken) and where the couplings \( \alpha_j \) are not too large. For our analyses of the perturbative evolution of gauge couplings, it will be sufficient to keep only the \( b_0^{(j)} \) term; the well-known solution of eq. (8.10) is then given by

\[
\alpha_j^{-1}(t_2) = \alpha_j^{-1}(t_1) + \frac{b_0^{(j)}}{2\pi} (t_2 - t_1) . \tag{8.11}
\]

If an effective field theory involves the direct product of two gauge groups \( G_j \) and \( G_\ell \) for energy scales between \( \mu \) and a larger scale \( \mu_k \), where the associated couplings \( \alpha_j \) and \( \alpha_k \) are equal, one has, to this order,

\[
\ln \left( \frac{\mu_{jk}}{\mu_\ell} \right) = \frac{2\pi}{b_0^{(k)} - b_0^{(j)}} \left[ \alpha_j^{-1}(\mu_k) - \alpha_k^{-1}(\mu_\ell) \right] . \tag{8.12}
\]

The three SM gauge couplings are accurately determined at \( \mu = m_Z \), with the results \( \alpha_3(m_Z) \approx 0.118 \), \( \alpha_{em}(m_Z)^{-1} \approx 128 \), and \( \sin^2 \theta_W \). The SU(2) \( L \) and U(1) \( Y \) couplings \( g \equiv g_L \) and \( g' \equiv g_Y \) are given by \( c = g \sin \theta_W = g' \cos \theta_W \) and have the values (quoted to sufficient accuracy for our present purposes) \( \alpha_3(m_Z) = 0.033 \) and \( \alpha_Y(m_Z) = 0.010 \). The evolution of these couplings to scales \( \mu > m_Z \) depends on the type of gauge symmetry unification that one is considering.

C. Fermion Condensation

Consider massless fermions that transform according to some representations \( \mathcal{R} \) of a nonabelian gauge group \( G \). In the approximation of a single-gauge boson exchange, the critical coupling for condensation in the channel

\[
\mathcal{R}_1 \times \mathcal{R}_2 \to \mathcal{R}_{\text{cond}} . \tag{8.13}
\]

given by \( \alpha_c \)

\[
\alpha_c = \frac{2\pi}{3\Delta C_2} . \tag{8.14}
\]
where
\[ \Delta C_2 = C_2(\mathcal{R}_1) + C_2(\mathcal{R}_2) - C_2(\mathcal{R}_{\text{cond}}). \]  
(8.15)

Because \( \alpha \sim O(1) \) where fermion condensation occurs, the one-gauge boson approximation is only a rough guide to the actual critical value of \( \alpha \). Corrections to this have been estimated in Ref. \[20\]. In addition to gauge boson exchange diagrams, nonperturbative processes involving instantons are also important \[21\].

For condensation due to the asymptotically free gauge interaction with gauge group \( G_j \) and associated squared coupling \( \alpha_j \) obeying a renormalization group equation with leading beta function coefficient \( b^{(j)}_\alpha \), the solution in eq. (8.11) together with the condition \( \Delta \alpha \), yield, for the mass scale at which the condensation takes place, the rough estimate
\[ \mu_{c,j} \approx M_{GU} \exp \left[ -\frac{2\pi}{b^{(j)}_\alpha} \left( \alpha_j(M_{GU})^{-1} - \frac{3\Delta C_2}{2\pi} \right) \right]. \]  
(8.16)

where \( \Delta C_2 \) is the value appropriate for this channel, as given by eq. (8.11).

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As will be discussed later in the text, the requirement in this old approach that $\beta_{TC} < \beta_{SU(3)} < 0$ is replaced in our analysis by the requirement of asymptotic freedom of the SCC sector unifying the SC and color interaction.

Related possibilities arise in models with large extra dimensions, where gauge couplings have power-law, rather than logarithmic, dependence on scale; we do not consider these here.

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TABLE I: Some properties of the various models of type (3) discussed in the text with $G_{SC}$ and $G_{SM}$ unified in a simple group $G$. Here, $G_{SC} = SU(N_{SC})$, $G_{TC} = SU(N_{TC})$, and $G_{SC} \supset G_{TC}$. The column marked “SCC” lists some properties of the $SU(N_{SCC})$ theory combining the $SU(N_{SC})$ and $SU(3)_{c}$ groups. See text for further definitions and discussion. The fermion content is indicated by the vector $n$ (with subscript omitted for brevity). The notation “no sol.” means that (in the dynamical framework used) there is no solution to the requirements of anomaly freedom, well-defined SM fermion generations, and $N_{gen.} = 3$. The notation VGT and CGT indicate that the gauge interaction is vectorial and chiral, respectively; AF and NAF mean asymptotically free and non-asymptotically free, respectively. The number $N_{(1,1)}$ in column 9, given by eq. (5.36), is the number of electroweak-singlet neutrinos.

| $N$ | $N_{SC}$ | $N_{SCC}$ | $N_{TC}$ | $N_{gt}$ | $N_{dlt}$ | $n$ | SCC | $N_{(1,1)}$ |
|-----|---------|---------|---------|---------|---------|-----|-----|---------|
| 7   | 2       | 5       | 2       | 0       | 3       | no sol. | –   | –       |
| 8   | 3       | 6       | 3       | 0       | 3       | (0200103) | VGT, AF | 1       |
| 8   | 3       | 6       | 2       | 1       | 2       | no sol. | –   | –       |
| 9   | 4       | 7       | 4       | 0       | 3       | no sol. | –   | –       |
| 9   | 4       | 7       | 3       | 1       | 2       | (01010101) | VGT, AF | 1       |
| 9   | 4       | 7       | 2       | 2       | 1       | no sol. | –   | –       |
| 10  | 5       | 8       | 5       | 0       | 3       | (000300300) | CGT, NAF | 0       |
| 10  | 5       | 8       | 4       | 1       | 2       | (000200200) | CGT, NAF | 0       |
| 10  | 5       | 8       | 3       | 2       | 1       | (000101010) | CGT, AF | 0       |
| 10  | 5       | 8       | 3       | 2       | 1       | (000110100) | CGT, NAF | 2       |
| 10  | 5       | 8       | 2       | 3       | 3       | (000010000) | VGT, AF | 2       |