ON AUTOEQUIVALENCES OF THE \((\infty, 1)\)-CATEGORY OF \(\infty\)-OPERADS

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ABSTRACT. We study the \((\infty, 1)\)-category of autoequivalences of \(\infty\)-operads. Using techniques introduced by Toën, Lurie, and Barwick and Schommer-Pries, we prove that this \((\infty, 1)\)-category is a contractible \(\infty\)-groupoid. Our calculation is based on the model of complete dendroidal Segal spaces introduced by Cisinski and Moerdijk. Similarly, we prove that the \((\infty, 1)\)-category of autoequivalences of non-symmetric \(\infty\)-operads is the discrete monoidal category associated to \(\mathbb{Z}/2\mathbb{Z}\). We also include a computation of the \((\infty, 1)\)-category of autoequivalences of \((\infty, n)\)-categories based on Rezk’s \(\Theta_n\)-spaces.

CONTENTS

Introduction

1. Review of quasi-categories
2. Autoequivalences and dense functors
3. Autoequivalences of the \((\infty, 1)\)-category of \((\infty, n)\)-categories
4. Autoequivalences of the \((\infty, 1)\)-category of \(\infty\)-operads
5. Autoequivalences of the \((\infty, 1)\)-category of non-symmetric \(\infty\)-operads

References

Introduction

Higher category theory and higher operad theory can be formalized by means of a plethora of different explicit approaches, all of them having their merits and drawbacks. These theories have applications in fields as diverse as algebraic topology, (derived) algebraic geometry, representation theory and homological algebra; see for instance the foundational work of Toën–Vezzosi [TV04, TV05, TV08] and Lurie [Lur09a, Lur12, Lur13]. Having a specific problem at hand, we are hence able to choose an approach accordingly, and it is thus important for practical and theoretical purposes to know how to compare these different formulations.

In the case of \((\infty, 1)\)-categories, the state of the art is very satisfactory. By now, there are many different approaches to the theory of \((\infty, 1)\)-categories, including quasi-categories [Joy02], simplicial categories [Ber07], Segal categories [HS01] and complete Segal spaces [Rez01]. Each of these theories is organized in a Quillen model category and these are related by a web of Quillen equivalences; see [Ber10] for a survey on these Quillen equivalences.

In [Toë05], Toën took this one step further and, based on earlier work of Simpson [Sim01], offered an axiomatization of the theory of \((\infty, 1)\)-categories. Moreover, he showed that the \((\infty, 1)\)-category of autoequivalences of one such theory is the
discrete category on the cyclic group $\mathbb{Z}/2\mathbb{Z}$ of order two, the non-trivial element being the passage to the opposite $(\infty, 1)$-category. More precisely, he computed the (derived) autoequivalences of the simplicial category obtained as the Dwyer–Kan localization of complete Segal spaces. A similar calculation was given by Lurie in [Lur09b, Section 4.4] using the language of quasi-categories. These computations imply that any two possibly different ways of comparing two models for $(\infty, 1)$-categories differ at most by the passage to opposites.

In this paper, we study the $(\infty, 1)$-category of autoequivalences of $\infty$-operads. As in the case of $(\infty, 1)$-categories, there are many different approaches to $\infty$-operads, including simplicial operads [CM13b], $\infty$-operads in the sense of Lurie [Lur13, Chapter 2], dendroidal sets [MW07, CM11] and complete dendroidal Segal spaces [CM13a] (which will be called $\Omega$-spaces in this paper). Again, there are Quillen model categories in the background, and thanks to recent work of Cisinski–Moerdijk [CM11, CM13a, CM13b] and Heuts–Hinich–Moerdijk [HHM13], it is known that all these model structures are connected by Quillen equivalences.

We show that the $(\infty, 1)$-category of autoequivalences of the $(\infty, 1)$-category of $\infty$-operads is a contractible $\infty$-groupoid. More precisely, we prove that the quasi-category of autoequivalences of $\Omega$-spaces is a contractible Kan complex. This implies that if there is a way to compare two models for $\infty$-operads, then this can be done in an essentially unique way. Similarly, we show that the $(\infty, 1)$-category of autoequivalences of the $(\infty, 1)$-category of non-symmetric $\infty$-operads is the discrete category on the cyclic group $\mathbb{Z}/2\mathbb{Z}$ of order two, the non-trivial element being the “mirror autoequivalence”.

One general strategy to compute the autoequivalences of an $(\infty, 1)$-category $\mathcal{C}$, following Toën [Toë05], Lurie [Lur09b, Section 4.4], and Barwick–Schommer-Pries [BSP13], is the following. One first identifies a small category $A$ inside $\mathcal{C}$ such that

(i) the inclusion functor $A \hookrightarrow \mathcal{C}$ is dense;

(ii) the autoequivalences of $\mathcal{C}$ restrict to autoequivalences of $A$.

It then follows formally that the autoequivalences of $\mathcal{C}$ sit fully faithfully in the autoequivalences of $A$. The problem is thus reduced to computing the autoequivalences of $A$ (up to the question of essential surjectivity, which is easy in our cases).

If the $(\infty, 1)$-category $\mathcal{C}$ is a localization of an $(\infty, 1)$-category $\mathcal{P}(A)$ of simplicial presheaves on a small category $A$, then one might hope that, in good cases, $A$ would satisfy the two conditions above. However, in practice it is hard to show the second point directly. For this purpose, Toën introduced the idea of using an intermediate (large) category: the so-called 0-truncated objects of $\mathcal{C}$. This category is trivially stable under autoequivalences of $\mathcal{C}$. Thus, if we assume that the objects of $A$ are 0-truncated, the verification of the second point is reduced to showing that autoequivalences of 0-truncated objects of $\mathcal{C}$ fix the small category $A$. In our cases this turns out to be a much simpler problem. This strategy is formalized by our Proposition 2.8. (For the case where $\mathcal{C}$ is the $(\infty, 1)$-category of $(\infty, 1)$-categories, Lurie [Lur09b, Section 4.4] uses a similar strategy, considering the category of posets as an intermediate (large) category.)

In order to apply this proposition to compute the autoequivalences of $\infty$-operads, we need a model defined as a localization of simplicial presheaves. The only model of this kind for $\infty$-operads proposed so far are the $\Omega$-spaces of Cisinski–Moerdijk [CM13a]. These are defined as a localization of simplicial presheaves on the small category $\Omega$ of trees introduced by Moerdijk and Weiss [MW07]. It is not hard
to identify the 0-truncated Ω-spaces: they are the so-called rigid (strict) operads, i.e., the operads whose underlying category contains no non-trivial isomorphisms. It thus suffices to show that autoequivalences of the category of rigid operads restrict to the small category Ω and to compute the autoequivalences of Ω. Most of our section on ∞-operads is dedicated to the proofs of these two statements. Although this is not formally needed, we also include a similar computation for the autoequivalences of (strict) operads.

We also compute the autoequivalences of non-symmetric ∞-operads using the obvious planar variant of complete dendroidal Segal spaces. The proofs are quite similar to the symmetric case although the combinatorics differ at some points. The difference is mainly due to the fact that objects of the planar version of Ω have no non-trivial automorphisms.

Finally, we include a calculation of the (∞,1)-category of autoequivalences of (∞, n)-categories. This problem has already been solved by Barwick and Schommer-Pries in [BSP13] using a new model for (∞, n)-categories called Υ_n-spaces. Here we provide an alternative calculation using instead the model of Θ_n-spaces introduced by Rezk in [Rez10], based on the category Θ_n introduced by Joyal in [Joy97]. Our choice of Θ_n-spaces rather than Υ_n-spaces is dictated by the simpler combinatorics of the category Θ_n.

Organization of the paper. In Section 1, we recall some facts about quasi-categories. In Section 2, we study restriction functors induced by dense functors and we formalize the general strategy for calculating autoequivalences of certain quasi-categories. In Section 3, we prove that the quasi-category of autoequivalences of Θ_n-spaces is the discrete category \((\mathbb{Z}/2\mathbb{Z})^n\). Along the way, we compute the autoequivalences of the categories of strict n-categories, rigid strict n-categories, and of the category Ω. In Section 4, we prove that the quasi-category of autoequivalences of Ω-spaces is a contractible Kan complex. We also calculate the autoequivalences of the categories of operads, rigid operads, and of the category Ω. Finally, in Section 5, we turn to the quasi-category of planar Ω-spaces and show that its quasi-category of autoequivalences is the discrete category \(\mathbb{Z}/2\mathbb{Z}\). We also describe the autoequivalences of non-symmetric operads, non-symmetric rigid operads and planar trees.

Notation and terminology. If A is a small category, we will denote by Pr(A) the category of (set-valued) presheaves on A, and by sPr(A) the category of simplicial presheaves on A. If C is a category, we will denote by aut(C) the set of autoequivalences of C and by Aut(C) the category of autoequivalences of C. The set aut(C) and the category Aut(C) will sometimes be considered as a monoid and a strict monoidal categories, respectively, the additional structure being given by composition. We will say that a morphism of a category (or more generally of a strict n-category) is non-trivial if it is not an identity.

If C and D are quasi-categories, we will denote by Fun(C, D) the quasi-category of functors from C to D. The full subcategory of Fun(C, C) spanned by the equivalences will be denoted by Aut(C).

We will assume for simplicity (contrary to the strong opinions of the first two authors) that our model categories have functorial factorizations. If \(\mathcal{M}\) is a model category and S is a class of morphisms of \(\mathcal{M}\), we will denote by \(S^{-1}\mathcal{M}\) the left Bousfield localization of \(\mathcal{M}\) with respect to S (if it exists).
We will neglect the usual set theoretic issues related to category theory. In particular, we will shamelessly apply the nerve functor to non-small categories.

1. Review of quasi-categories

In this section, we recall some facts about quasi-categories, mostly about the relation between model categories and quasi-categories, and about localizations of locally presentable quasi-categories. We assume known the basics of the theory of quasi-categories and we refer the reader to the foundational work of Joyal [Joy02, Joy08a, Joy08b] and Lurie [Lur09a, Lur13]. For an introduction to this theory emphasizing the philosophy, see [Gro10].

1.1. We will denote by \( N: \text{Cat} \to \text{SSet} \) the nerve functor from (small) categories to simplicial sets. Since the nerve of a category is a quasi-category, this functor induces a fully faithful functor from categories to quasi-categories. We will often consider this functor as an inclusion.

**Definition 1.2** (Lurie). The *underlying quasi-category* of a model category \( \mathcal{M} \) is a quasi-category \( \mathcal{U}(\mathcal{M}) \) endowed with a functor \( f: \mathcal{M} \to \mathcal{U}(\mathcal{M}) \) such that for any quasi-category \( \mathcal{D} \), the induced functor

\[
\mathcal{F}^*: \text{Fun}(\mathcal{U}(\mathcal{M}), \mathcal{D}) \to \text{Fun}(\mathcal{M}, \mathcal{D})
\]

is fully faithful with essential image the functors \( \mathcal{M} \to \mathcal{D} \) which send weak equivalences of \( \mathcal{M} \) to equivalences in \( \mathcal{D} \).

**Remark 1.3.** This definition differs slightly from the original definition of Lurie [Lur13, Definitions 1.3.4.1 and 1.3.4.15] in which one restricts to cofibrant objects of \( \mathcal{M} \). Nevertheless, the two definitions are equivalent by [Lur13, Remark 1.3.4.16].

**Remark 1.4.** The definition of \( \mathcal{U}(\mathcal{M}) \) only depends on the underlying category of \( \mathcal{M} \) and the weak equivalences of \( \mathcal{M} \). In particular, if \( \mathcal{M} \) and \( \mathcal{N} \) are two model categories on the same underlying category with same weak equivalences, then \( \mathcal{U}(\mathcal{M}) \) and \( \mathcal{U}(\mathcal{N}) \) are canonically equivalent.

**Proposition 1.5** (Lurie). *Every model category has an underlying quasi-category.*

**Proof.** This follows from [Lur13, Remark 1.3.4.2]. □

1.6. We will denote by \( N_\Delta: \text{Cat}_\Delta \to \text{SSet} \) Cordier’s coherent nerve functor from simplicial categories to simplicial sets (see [Lur09a, Definition 1.1.5.5]). If \( \mathcal{M} \) is a simplicial model category, we will denote by \( \mathcal{M}^o \) the full simplicial subcategory of \( \mathcal{M} \) spanned by the cofibrant and fibrant objects. This simplicial category \( \mathcal{M}^o \) is locally fibrant in the sense that all its mapping spaces are Kan complexes. It follows from [CP86, Theorem 2.1] that its coherent nerve \( N_\Delta(\mathcal{M}^o) \) is a quasi-category.

**Theorem 1.7** (Lurie). *If \( \mathcal{M} \) is a simplicial model category, then \( N_\Delta(\mathcal{M}^o) \) is the underlying quasi-category of \( \mathcal{M} \).*

**Proof.** This is [Lur13, Theorem 1.3.4.20]. □

1.8. Recall from [Lur09a, Section 5.5] that the classical notion of locally presentable category can be generalized to the notion of locally presentable quasi-categories. By [Lur09a, Proposition A.3.7.6] and [Lur13, Proposition 1.3.4.22], these quasi-categories can be characterized as those being the underlying quasi-category of a combinatorial model category.
1.9. Denote by $S$ the quasi-category of spaces, that is, the underlying quasi-category of the Kan–Quillen model structure on simplicial sets. If $A$ is a small category, then the quasi-category of presheaves $\mathcal{P}(A)$ on $A$ is the quasi-category $\text{Fun}(A^{\text{op}}, S)$.

**Proposition 1.10** (Heller, Bousfield–Kan). Let $A$ be a small category. We have the following two simplicial proper combinatorial model structures on the category $\text{sPr}(A)$ of simplicial presheaves on $A$:

(i) the injective model structure, whose weak equivalences and cofibrations are the objectwise weak equivalences and the objectwise cofibrations, respectively;

(ii) the projective model structure, whose weak equivalences and fibrations are the objectwise weak equivalences and the objectwise fibrations, respectively.

**Proof.** It seems that the first appearances of the injective and projective model structures are [Hel88, Theorem 4.5] and [BK72, Chapter XI, §8], respectively. See [Lur09a, Proposition A.2.8.2] for a more general statement. The fact that these model structures are simplicial follows easily from the fact that the Kan–Quillen model structure on simplicial sets is simplicial. The left properness is obvious for the injective model structure and right properness for the projective model structure follows easily from right properness of the Kan–Quillen model structure. Since left and right properness only depend on the class of weak equivalences, it follows that these structures are both proper. □

We will denote by $\text{sPr}(A)_{\text{inj}}$ and $\text{sPr}(A)_{\text{proj}}$ these two model structures.

**Proposition 1.11** (Lurie). The quasi-category $\mathcal{P}(A)$ of presheaves on a small category $A$ is canonically equivalent to the underlying quasi-category of the projective model structure on $\text{sPr}(A)$.

**Proof.** This is a special case of [Lur13, Proposition 1.3.4.25]. □

1.12. If $\mathcal{C}$ is a quasi-category and $X, Y$ are two objects of $\mathcal{C}$, we will denote by $\text{Map}_\mathcal{C}(X, Y)$ the space of morphisms from $X$ to $Y$. See [Lur09a, Section 1.2.2] or [DS11] for various approaches to define this space. Recall from [Lur09a, Section 2.2] that if $\mathcal{M}$ is a simplicial model category, then the mapping spaces of the underlying quasi-category of $\mathcal{M}$ can be computed using the simplicial enrichment of $\mathcal{M}$ (when restricted to cofibrant fibrant objects).

1.13. Let $\mathcal{C}$ be a quasi-category and let $S$ be a class of morphisms of $\mathcal{C}$. An object $Y$ of $\mathcal{C}$ is $S$-local if for every map $f: X \to X'$ in $S$, the induced map

$$f^*: \text{Map}_\mathcal{C}(X', Y) \to \text{Map}_\mathcal{C}(X, Y)$$

is a weak equivalence. The full subcategory of the quasi-category $\mathcal{C}$ spanned by the $S$-local objects is called the localization of $\mathcal{C}$ by $S$. We will denote it by $S^{-1}\mathcal{C}$.

**Proposition 1.14** (Lurie). If $\mathcal{C}$ is a locally presentable quasi-category and $S$ is a set of morphisms of $\mathcal{C}$, then the inclusion $i: S^{-1}\mathcal{C} \hookrightarrow \mathcal{C}$ admits a left adjoint $L$. In other words, we have a reflective localization

$$L: \mathcal{C} \rightleftarrows S^{-1}\mathcal{C}: i.$$ 

**Proof.** This is [Lur09a, Proposition 5.5.4.15.(3)]. □
Remark 1.15. When the quasi-category \( \mathcal{C} \) is an ordinary category \( A \), the mapping space \( \text{Map}_A(X, Y) \) is simply the discrete simplicial set \( A(X, Y) \). In particular, an object \( Y \) in \( A \) is \( S \)-local if and only if, for all \( f: X \to X' \) in \( S \), the induced map

\[
\phi^*: A(X', Y) \to A(X, Y)
\]

is a bijection, or, in other words, if and only if \( Y \) is right orthogonal to \( S \).

Proposition 1.16. Let \( M \) be a left proper combinatorial model category and let \( S \) be a set of maps of \( M \). There is a canonical equivalence of quasi-categories

\[
S^{-1}(\mathcal{U}(M)) \simeq \mathcal{U}(S^{-1}M).
\]

Proof. The functor \( \mathcal{U}(M) \to \mathcal{U}(S^{-1}M) \) sends \( S \) to equivalences and hence, by the universal property of the localization, we obtain a functor

\[
\phi: S^{-1}(\mathcal{U}(M)) \to \mathcal{U}(S^{-1}M).
\]

Let us show this functor is an equivalence.

We first assume that \( M \) is a simplicial left proper combinatorial model category. By Theorem 1.7, the underlying quasi-categories of \( M \) and \( S^{-1}M \) are \( N_\Delta(M^\circ) \) and \( N_\Delta((S^{-1}M)^\circ) \), respectively. These two quasi-categories sit fully faithfully in \( N_\Delta(M^\circ) \). Moreover, using the compatibility between the mapping spaces of the model category \( M \) and those of the quasi-category \( N_\Delta(M^\circ) \), it is easy to check that these two quasi-categories are equal as subcategories of \( N_\Delta(M^\circ) \). The identity functor \( S^{-1}N_\Delta(M^\circ) \to N_\Delta((S^{-1}M)^\circ) \) is easily seen to be canonically equivalent to the functor \( \phi \), thereby concluding the proof of the simplicial case.

We now want to drop the additional assumption that \( M \) is simplicial by an application of a well-known result of Dugger. For that purpose, let us assume that we have a Quillen equivalence \( M_1 \rightleftarrows M_2 \) between left proper combinatorial model categories and a set \( S_2 \) of maps of \( M_2 \). Let \( S_1 \) be the derived image of \( S_2 \) under the right adjoint of the Quillen pair. By [Hir03, Theorem 3.3.20], there is an induced Quillen equivalence \( (S_1)^{-1}M_1 \rightleftarrows (S_2)^{-1}M_2 \). Recall that Quillen equivalences between combinatorial model categories induce equivalences between the underlying quasi-categories [Lur13, Lemma 1.3.4.21]. We thus get a commutative diagram of quasi-categories

\[
\begin{array}{ccc}
\mathcal{U}(M_1) & \xrightarrow{\simeq} & (S_1)^{-1}\mathcal{U}(M_1) \\
\phi_1 & & \phi_2 \\
\mathcal{U}(M_2) & \xrightarrow{\simeq} & (S_2)^{-1}\mathcal{U}(M_2)
\end{array}
\]

in which the vertical maps are equivalences. This implies that \( \phi_1 \) is an equivalence if and only if \( \phi_2 \) is an equivalence.

Finally, let \( M \) be a left proper combinatorial model category. Then by [Dug01], there is a left Quillen equivalence \( N \to M \) such that \( N \) is a simplicial left proper combinatorial model category. The statement now follows easily from the previous two paragraphs.

1.17. Let \( \mathcal{C} \) be a quasi-category. An object \( Y \) of \( \mathcal{C} \) is said to be 0-truncated if for every object \( X \) of \( \mathcal{C} \), the mapping space \( \text{Map}_\mathcal{C}(X, Y) \) is discrete. We will denote by \( \tau_{\leq 0}\mathcal{C} \) the full subcategory of \( \mathcal{C} \) spanned by the 0-truncated objects of \( \mathcal{C} \).
Proposition 1.18 (Lurie). If $\mathcal{C}$ is a locally presentable quasi-category, then the inclusion $i: \tau_{\leq 0}\mathcal{C} \hookrightarrow \mathcal{C}$ admits a left adjoint $L$. In other words, we have a reflective localization

$$L: \mathcal{C} \overset{\cong}{\longrightarrow} \tau_{\leq 0}\mathcal{C}: i.$$ 

Proof. This is [Lur09a, Proposition 5.5.6.18]. □

2. Autoequivalences and dense functors

In this section, we formalize the general strategy described in the introduction for computing certain quasi-categories of autoequivalences. This is made precise by Proposition 2.8 and will be used in the following three sections to determine the autoequivalences of $(\infty, n)$-categories, $\infty$-operads and non-symmetric $\infty$-operads.

2.1. Let $A$ be a small category. By Proposition 1.11, the quasi-category $\mathcal{P}(A)$ is canonically equivalent to $N_{\Delta}(sPr(A)^{\text{proj}}_\infty)$. Let us choose a functorial cofibrant replacement functor $Q$ for $sPr(A)^{\text{proj}}_\infty$ with a natural trivial fibration $\phi: Q \to 1$. Since every discrete presheaf is fibrant in $sPr(A)^{\text{proj}}_\infty$, applying the functor $Q$ to such a presheaf yields a cofibrant fibrant object of $sPr(A)^{\text{proj}}_\infty$. Thus the functor $Q$ induces a morphism of simplicial categories $\Pr(A) \to sPr(A)^{\text{proj}}_\infty$ and hence a morphism of quasi-categories $\Pr(A) \to \mathcal{P}(A)$.

Proposition 2.2. Let $A$ be a small category. Then the functor $\Pr(A) \to \mathcal{P}(A)$ is fully faithful, factors through the subcategory $\tau_{\leq 0}\mathcal{P}(A)$ of 0-truncated objects, and the restricted functor $\Pr(A) \to \tau_{\leq 0}\mathcal{P}(A)$ is an equivalence of quasi-categories.

Proof. We begin by showing that the morphism $\Pr(A) \to \mathcal{P}(A)$ is fully faithful. Denote by $\delta: \Pr(A) \to sPr(A)$ the functor sending a presheaf to the associated discrete simplicial presheaf. We have to show that if $X$ and $Y$ are two presheaves on $A$, then the map

$$Q \circ \delta: \text{Map}_{\Pr(A)}(X,Y) \longrightarrow \text{Map}_{sPr(A)}(Q\delta X, Q\delta Y)$$

is a weak equivalence. Since the functor $\delta$ is fully faithful, this amounts to saying that the map

$$\text{Map}_{\Pr(A)}(\delta X, \delta Y) \longrightarrow \text{Map}_{sPr(A)}(Q\delta X, Q\delta Y)$$

induced by $Q$ is a weak equivalence. But this map sits in the commutative diagram

$$\begin{array}{ccc}
\text{Map}_{sPr(A)}(\delta X, \delta Y) & \overset{\cong}{\longrightarrow} & \text{Map}_{\Pr(A)}(\pi_0 \delta X, Y) \\
Q \downarrow & & \downarrow \pi_0(\phi_{\delta Y})^* \\
\text{Map}_{sPr(A)}(Q\delta X, Q\delta Y) & \overset{\cong}{\longrightarrow} & \text{Map}_{\Pr(A)}(\pi_0 Q\delta X, Y),
\end{array}$$

where the top and the bottom maps are isomorphisms coming from the adjunction $(\pi_0, \delta)$. Since $Q\delta X$ is cofibrant and $Q\delta Y \to \delta Y$ is a trivial fibration, the map $\phi_{\delta Y}$ is a weak equivalence. The map $\pi_0(\phi_{\delta X})^*$ is even an isomorphism. It follows that the functor of the statement is fully faithful.
The fully faithfulness of the functor $\Pr(A) \to \mathcal{P}(A)$ easily implies that it factors through 0-truncated objects and it remains to show that the induced functor is essentially surjective. Given $X$ in $\tau_{\leq 0}\mathcal{P}(A)$, we claim that the canonical map $\eta: X \to \delta \pi_0(X)$ is a natural equivalence. Obviously, $\eta$ induces an isomorphism on $\pi_0$. To conclude, it thus suffices to show that for every object $a$ in $A$, the space $X_a$ is discrete. But we have $\text{Map}_{\tau(A)}(a, X) \simeq X_a$, where $a$ denotes the discrete simplicial presheaf associated to $a$. This shows the result since $X$ is 0-truncated. \qed

Remark 2.3. It follows from the previous proposition that if $S$ is a set of morphisms of $\Pr(A)$, then an object of $\Pr(A)$ is $S$-local (i.e., right orthogonal with respect to $S$) if and only if it is $S$-local when considered as an object of $\mathcal{P}(A)$.

Proposition 2.4. Let $\mathcal{C}$ be a locally presentable quasi-category and let $S$ be a set of morphisms between 0-truncated objects of $\mathcal{C}$. Then the quasi-categories $S^{-1}(\tau_{\leq 0}\mathcal{C})$ and $\tau_{\leq 0}(S^{-1}\mathcal{C})$ are equal as subcategories of $\mathcal{C}$.

Proof. We have the following compositions of reflective localizations

$$\mathcal{C} \xrightarrow{S^{-1}} \tau_{\leq 0}(S^{-1}\mathcal{C}), \quad \mathcal{C} \xrightarrow{\tau_{\leq 0}} S^{-1}(\tau_{\leq 0}\mathcal{C}),$$

and it hence suffices to show that the two quasi-categories have the same objects. But if $Y$ is an object of $S^{-1}(\tau_{\leq 0}\mathcal{C})$, then $\text{Map}_{\mathcal{C}}(X, Y)$ is discrete for all $X$ in $\mathcal{C}$. In particular, $\text{Map}_{S^{-1}\mathcal{C}}(X, Y) = \text{Map}_{\mathcal{C}}(X, Y)$ is discrete for all $X$ in $S^{-1}\mathcal{C}$ and hence $Y$ lies in $\tau_{\leq 0}(S^{-1}\mathcal{C})$. Conversely, if $Y$ is an object of $\tau_{\leq 0}(S^{-1}\mathcal{C})$, then $\text{Map}_{\tau_{\leq 0}(S^{-1})\mathcal{C}}(X, Y)$ is discrete for all $X$ in $S^{-1}\mathcal{C}$. Given an arbitrary $X$ in $\mathcal{C}$, then, using the localization functor $L: \mathcal{C} \to S^{-1}\mathcal{C}$ and the $S$-locality of $Y$, the mapping space $\text{Map}_{\mathcal{C}}(X, Y)$ is weakly equivalent to the discrete space $\text{Map}_{S^{-1}\mathcal{C}}(LX, Y)$, and so $Y$ lies in $S^{-1}(\tau_{\leq 0}\mathcal{C})$. \qed

Corollary 2.5. Let $A$ be a small category and let $S$ be a set of morphisms of $\Pr(A)$. The morphism $\Pr(A) \to \mathcal{P}(A)$ induces an equivalence from the category $S^{-1}\Pr(A)$ to the quasi-category $\tau_{\leq 0}(S^{-1}\mathcal{P}(A))$.

Proof. By Proposition 2.2, the morphism $\Pr(A) \to \mathcal{P}(A)$ induces an equivalence $\Pr(A) \to \tau_{\leq 0}\mathcal{P}(A)$ and hence an equivalence $S^{-1}\Pr(A) \to S^{-1}(\tau_{\leq 0}\mathcal{P}(A))$. But by the previous proposition, $S^{-1}(\tau_{\leq 0}\mathcal{P}(A))$ is nothing but $\tau_{\leq 0}(S^{-1}\mathcal{P}(A))$. \qed

2.6. Following [Lur09b, Definition 4.4.2], we will say that a functor $f: \mathcal{C} \to \mathcal{D}$ between quasi-categories is dense if the identity transformation $f \to f$ exhibits $1: \mathcal{D} \to \mathcal{D}$ as a left Kan extension of $f$ along $f$. When $\mathcal{C}$ is small and $\mathcal{D}$ is cocomplete, this amounts to saying the associated nerve functor (that is, the right adjoint to the canonical functor $\mathcal{P}(\mathcal{C}) \to \mathcal{D}$) is fully faithful [Lur09b, Remark 4.4.4]. In particular, if $A$ is a small category and $S$ is a set of morphisms of $\mathcal{P}(A)$ such that representable presheaves are $S$-local, then the canonical functor $A \to S^{-1}\mathcal{P}(A)$ is dense.

Proposition 2.7 (Lurie). Let $i: \mathcal{A} \to \mathcal{B}$ be a dense inclusion of quasi-categories such that $\mathcal{A}$ is small and $\mathcal{B}$ admits small colimits. If any autoequivalence of $\mathcal{B}$ restricts to an autoequivalence of $\mathcal{A}$, then the inclusion $i$ induces a fully faithful functor $i^*: \text{Aut}(\mathcal{B}) \to \text{Aut}(\mathcal{A})$. 
Proof. Denote by \( \text{Fun}^L(-, -) \) the quasi-category of colimit-preserving functors between two quasi-categories. The dense inclusion \( i \) induces a fully faithful functor \( \text{Fun}^L(\mathcal{B}, \mathcal{B}) \to \text{Fun}(A, \mathcal{B}) \) by \cite[Remark 4.4.5]{Lur09b}. Consider the diagram

\[
\begin{array}{ccc}
\text{Fun}^L(\mathcal{B}, \mathcal{B}) & \xrightarrow{i^*} & \text{Fun}(A, \mathcal{B}) \\
\downarrow & & \downarrow \\
\text{Aut}(\mathcal{B}) & \xrightarrow{i^*} & \text{Aut}(A),
\end{array}
\]

where the left vertical map is the canonical inclusion and the right vertical map is the fully faithful map given by postcomposing by \( i \). By our assumption on autoequivalences of \( \mathcal{B} \), the diagonal map factors over \( \text{Aut}(A) \), giving rise to the dotted morphism. Since all remaining maps are fully faithful, the same is true for \( i^* : \text{Aut}(\mathcal{B}) \to \text{Aut}(A) \). \qed

Proposition 2.8. Let \( A \) be a small category and let \( S \) be a set of morphisms of \( \text{Pr}(A) \). Assume that the following conditions are satisfied:

(i) Representable presheaves in \( \text{Pr}(A) \) are \( S \)-local.
(ii) Any autoequivalence of \( S^{-1}\text{Pr}(A) \) restricts to an autoequivalence of \( A \) (the category \( A \) being included in \( S^{-1}\text{Pr}(A) \) because of (i)).

Then \( A \to S^{-1}\mathcal{P}(A) \) induces a fully faithful functor \( \text{Aut}(S^{-1}\mathcal{P}(A)) \to \text{Aut}(A) \). In particular, the quasi-category \( \text{Aut}(S^{-1}\mathcal{P}(A)) \) is discrete.

Proof. By Corollary 2.5, we know that there is an equivalence of quasi-categories \( \tau_{\leq 0}(S^{-1}\mathcal{P}(A)) \simeq S^{-1}\text{Pr}(A) \). Since 0-truncated objects are stable under equivalences, we obtain a functor \( \text{Aut}(S^{-1}\mathcal{P}(A)) \to \text{Aut}(S^{-1}\text{Pr}(A)) \). Using assumption (ii), we get a functor \( \text{Aut}(S^{-1}\text{Pr}(A)) \to \text{Aut}(A) \). Obviously, the composition

\[
\text{Aut}(S^{-1}\mathcal{P}(A)) \to \text{Aut}(S^{-1}\text{Pr}(A)) \to \text{Aut}(A)
\]

of these two functors is induced by \( A \to S^{-1}\text{Pr}(A) \to S^{-1}\mathcal{P}(A) \). But this functor is dense (see the end of paragraph 2.6) and the result thus follows from the previous proposition. \qed

3. Autoequivalences of the \((\infty, 1)\)-category of \((\infty, n)\)-categories

The aim of this section is to show that the quasi-category of autoequivalences of the quasi-category of \( \Theta_n \)-spaces, which is a model for \((\infty, n)\)-categories, is the discrete category \((\mathbb{Z}/2\mathbb{Z})^n\). This calculation is a consequence of two results of Barwick and Schommer-Pries \cite{BSP13}, namely the computation of the autoequivalences of \( \Upsilon_n \)-spaces and the comparison of \( \Upsilon_n \)-spaces and \( \Theta_n \)-spaces. Here we give a direct proof of this fact using the general strategy outlined in the introduction and formalized by Proposition 2.8. In particular, we reduce to the computation of the autoequivalences of Joyal’s category \( \Theta_n \).

Throughout the section, we fix \( n \) such that \( 0 \leq n \leq \infty \).

3.1. Preliminaries on strict \( n \)-categories and the category \( \Theta_n \)

3.1.1. We will denote by \( \text{Cat} \) the category of small categories and by \( n\text{-Cat}_{\text{str}} \) the category of small strict \( n \)-categories. Recall that these categories are complete and cocomplete. We refer the reader to \cite{Ara13a} for details on strict \( n \)-categories compatible with the notation we will use in this section.
3.1.2. Let $k$ be such that $0 \leq k \leq n$. A strict $k$-category can be considered as a strict $n$-category whose $i$-arrows are identities for $i > k$. This defines a fully faithful functor $k\text{-}\mathcal{C}at_{str} \to n\text{-}\mathcal{C}at_{str}$ which we will consider as an inclusion. In particular, we will say that a strict $n$-category is a $k$-category if it is in the image of this functor.

This inclusion functor admits both a left adjoint and a right adjoint. In this paper, we will only consider the right adjoint. It sends a strict $n$-category $C$ to the strict $k$-category $\text{tr}_k(C)$ obtained by throwing out the $i$-arrows of $C$ for $i > k$. This $k$-category $\text{tr}_k(C)$ will be called the $k$-truncation of $C$.

3.1.3. The $k$-disk $D_k$, where $k \geq 0$, is the strict $\infty$-category corepresenting the functor “set of $k$-arrows” $\text{Ar}_k: \infty\text{-}\mathcal{C}at_{str} \to \text{Set}$. The strict $\infty$-category $D_k$ is actually a $k$-category. Here are pictures of (the underlying $\infty$-graphs without the identities of) $D_k$ in low dimension:

$$
\begin{array}{c}
D_0 = 
\begin{array}{c}
. \\
\end{array}&
\begin{array}{c}
D_1 = 
\begin{array}{c}
\bullet
\end{array} \rightarrow 
\begin{array}{c}
. \\
\end{array}
\end{array}&
\begin{array}{c}
D_2 = 
\begin{array}{c}
\bullet
\end{array} \rightarrow 
\begin{array}{c}
\bullet
\end{array} \rightarrow 
\begin{array}{c}
\bullet
\end{array}
\end{array}
\end{array}
\text{and}
\begin{array}{c}
D_3 = 
\begin{array}{c}
\bullet
\end{array} \rightarrow 
\begin{array}{c}
\bullet
\end{array} \rightarrow 
\begin{array}{c}
\bullet
\end{array} \rightarrow 
\begin{array}{c}
\bullet
\end{array}
\end{array}.
\end{array}
$$

For $k > 0$, we have two morphisms $\sigma, \tau: D_{k-1} \to D_k$ corepresenting the natural transformations source and target $\text{Ar}_k \to \text{Ar}_{k-1}$, respectively. Concretely, $\sigma$ (respectively $\tau$) sends the unique non-trivial $(k-1)$-arrow of $D_{k-1}$ to the source (respectively to the target) of the unique non-trivial $k$-arrow of $D_k$.

For $k > k' \geq 0$, we will also denote by $\sigma$ and $\tau$ the morphisms $D_{k'} \to D_k$ obtained by composing

$$
D_{k'}\xrightarrow{\sigma} D_{k'+1} \cdots D_{k-1} \xrightarrow{\tau} D_k \quad \text{and} \quad D_{k'} \xrightarrow{\tau} D_{k'+1} \cdots D_{k-1} \xrightarrow{\tau} D_k,
$$

respectively. Note that $\sigma, \tau: D_{k'} \to D_k$ are the only monomorphisms $D_{k'} \to D_k$.

3.1.4. A table of dimensions is a table

$$
\begin{pmatrix}
k_1 & k_2 & \cdots & k_m \\
k_1' & k_2' & \cdots & k_{m-1}'
\end{pmatrix},
$$

where $m \geq 1$, filled with non-negative integers satisfying $k_i > k_i'$ and $k_{i+1} > k_i'$ for every $i$ such that $0 < i < m$. The greatest integer appearing in the table is called the height of the table.

To such a table $T$, we can associate the following diagram

$$
\begin{array}{c}
\begin{array}{c}
D_{k_1} \leftarrow \\
\sigma
\end{array} &
\begin{array}{c}
D_{k_2} \leftarrow \\
\sigma \\
\tau
\end{array} &
\begin{array}{c}
D_{k_3} \leftarrow \\
\tau
\end{array} &
\cdots &
\begin{array}{c}
D_{k_{m-1}} \leftarrow \\
\sigma
\end{array} &
\begin{array}{c}
D_{k_m} \leftarrow \\
\tau
\end{array}
\end{array}
$$

in $\infty\text{-}\mathcal{C}at_{str}$. We will denote this diagram by $D_T$. The colimit of $D_T$ will be denoted by $\Theta(T)$. We will also sometimes denote it by

$$
D_{k_1} \amalg D_{k_1'} \cdots \amalg D_{k_{m-1}'} \amalg D_k.
$$

This is an $n$-category, where $n$ is the height of $T$. Note that the diagram $D_T$ actually comes from a diagram of $n$-graphs and it follows that the $n$-category $\Theta(T)$ is the free strict $n$-category on an $n$-graph $\Theta_0(T)$.

It is easy to see that the strict $\infty$-category $\Theta(T)$ does not admit any non-trivial automorphisms. This means, in particular, that there is a unique cocone making $\Theta(T)$ the colimit of the diagram $D_T$. 

Example 3.1.5. If $T$ is the following table of dimensions
\[
\begin{bmatrix}
2 & 2 & 0 & 2 & 3 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 1
\end{bmatrix},
\]
then $\Theta_0(T)$ is the following 3-graph:

and the 3-category $\Theta(T)$ associated to $T$ is the strict 3-category freely generated by this 3-graph.

3.1.6. Joyal’s category $\Theta_n$ of $n$-cells is defined in the following way. The objects of $\Theta_n$ are the tables of dimensions of height at most $n$. If $S$ and $T$ are two objects of $\Theta_n$, then the set of morphisms from $S$ to $T$ is given by

\[\Theta_n(S, T) = n\text{-Cat}_{str}(\Theta(S), \Theta(T)).\]

By definition, we have a fully faithful functor $\Theta_n \to n\text{-Cat}_{str}$. It is easy to see that this functor is also injective on objects and we will consider it as an inclusion. In particular, we will not make a difference between the table $T$ and the associated strict $\infty$-category $\Theta(T)$.

Remark 3.1.7. Here are some comments on $\Theta_n$ for $n = 0, 1, \infty$:

(i) The category $\Theta_0$ is the terminal category. It is not the category $\Theta_0$ introduced by Berger in [Ber02], which corresponds to $\Theta_{\infty,0}$ with the notation of our paragraph 3.1.9.

(ii) The category $\Theta_1$ is canonically isomorphic to the category $\Delta$ of simplices.

(iii) The category $\Theta_{\infty}$ is canonically isomorphic to the category $\Theta$ introduced by Joyal in [Joy97] by combinatorial means.

Remark 3.1.8. The category $\Theta_n$ has a universal property relating it to strict $n$-categories. Roughly speaking, it is the free category (having certain colimits) endowed with a strict $n$-co-category object. See Propositions 3.11 and 3.14 of [Ara12] for the case $n = \infty$.

3.1.9. We define a category $\Theta_{n,0}$ in the following way: the objects of $\Theta_{n,0}$ are the same as the ones of $\Theta_n$ and the set of morphisms in $\Theta_{n,0}$ from an object $S$ to an object $T$ is given by

\[\Theta_{n,0}(S, T) = n\text{-Graph}(\Theta_0(S), \Theta_0(T)),\]

where $n\text{-Graph}$ denotes the category of $n$-graphs. The free strict $n$-category functor induces a canonical functor from $\Theta_{n,0}$ to $\Theta_n$ which is obviously faithful.

We will say that a morphism of $\Theta_n$ is inert if it comes from a morphism of $\Theta_{n,0}$. It is easy to see that inert morphisms are monomorphisms. A morphism $S \to T$ of $\Theta_n$ is said to be active if it does not factor through any non-trivial inert morphism $T' \to T$, that is, if every time it can be written as $if$, where $i$ is inert, then $i$ is an identity.

Active morphisms can be described more concretely in the following way. Let $f : D_k \to T$ be a morphism of $\Theta_n$. Such a morphism corresponds to a $k$-arrow
of $\Theta(T)$. This $k$-arrow can be expressed using compositions and identities from some generators of the free strict $n$-category $\Theta(T)$, that is, from the arrows of $\Theta_0(T)$. It is not hard to see that the morphism $f$ is active if and only if all the generators of $\Theta(T)$ are needed to express this $k$-arrow. In other words, $D_k \to T$ is active if and only if it corresponds to (an identity in dimension $k$ of) the total composition of $\Theta(T)$. This means that such an active morphism exists if and only if $k$ is greater than or equal to the height of $T$, and that in this case, it is unique.

More generally, a morphism $S \to T$ of $\Theta_n$ corresponds to a pasting scheme of shape $\Theta_0(S)$ in $\Theta(T)$ and it is active if and only if all the generators of $\Theta(T)$ are needed to express the arrows of this pasting scheme.

**Proposition 3.1.10** (Berger, Weber). Every morphism of $\Theta_n$ can be written in a unique way as a composition of an active morphism followed by an inert morphism.

**Proof.** This follows from the general machinery of [Web07] (see Example 4.21). This was first proved in [Ber02] (see Lemma 1.11) using a different but equivalent definition of $\Theta_n$. See also [Ara10, Proposition 3.3.10].

**3.1.11.** The category of $n$-cellular sets is the category $\text{Pr}(\Theta_n)$ of presheaves on $\Theta_n$. The inclusion $\Theta_n \hookrightarrow n\text{-}\text{Cat}_{\text{str}}$ induces a functor $N_n : n\text{-}\text{Cat}_{\text{str}} \to \text{Pr}(\Theta_n)$ sending a strict $n$-category $C$ to the $n$-cellular set $T \mapsto n\text{-}\text{Cat}_{\text{str}}(T, C)$. This functor $N_n$ is called the $n$-cellular nerve.

**3.1.12.** Let

$$T = \left( \begin{array}{cccc} k_1 & k_2 & \cdots & k_m \\ k'_1 & k'_2 & \cdots & k'_{m-1} \end{array} \right)$$

be an object of $\Theta_n$. By definition, we have

$$T = D_{k_1} \amalg D_{k'_1} \amalg \cdots \amalg D_{k'_{m-1}} \amalg D_{k_m}$$

in $\Theta_n$. The spine $I_T$ of $T$ is the $n$-cellular set

$$I_T = D_{k_1} \amalg D_{k'_1} \amalg \cdots \amalg D_{k'_{m-1}} \amalg D_{k_m},$$

where the colimit is now taken in the category $\text{Pr}(\Theta_n)$. There is a canonical morphism

$$i_T : I_T \to T.$$ 

It is not hard to check that this morphism is a monomorphism. We will denote by $J$ the set

$$J = \{ i_T \mid T \in \text{Ob}(\Theta_n) \}$$

of spine inclusions.

Let now $X$ be an $n$-cellular set. For any object $T$ of $\Theta_n$, the map $i_T$ induces a Segal map

$$X(T) \simeq \text{Pr}(\Theta_n)(T, X) \to \text{Pr}(\Theta_n)(I_T, X) \simeq X_{k_1} \times X_{k'_1} \times \cdots \times X_{k'_{m-1}} \times X_{k_m},$$

where $X_l$ means $X(D_l)$. We will say that $X$ satisfies the Segal condition if all the Segal maps are bijections. This exactly means that $X$ is $J$-local.

**Proposition 3.1.13** (Berger, Weber). The $n$-cellular nerve functor is fully faithful. Moreover, its essential image consists of the $n$-cellular sets satisfying the Segal condition.

**Proof.** This follows from the general machinery of [Web07] (see Example 4.24). This was first proved in [Ber02] (see Theorem 1.12).
Remark 3.1.14. The first assertion of the previous proposition means precisely that the inclusion functor $\Theta_n \hookrightarrow n\text{-}\text{Cat}_{str}$ is dense.

3.2. The quasi-category of $\Theta_n$-spaces.

3.2.1. The category of $n$-cellular spaces is the category $sPr(\Theta_n) \simeq Pr(\Theta_n \times \Delta)$ of simplicial presheaves on $\Theta_n$. The first projection $p: \Theta_n \times \Delta \to \Theta_n$ induces a fully faithful functor $p^*: Pr(\Theta_n) \to Pr(\Theta_n \times \Delta)$ sending an $n$-cellular set to the corresponding discrete $n$-cellular space. We will always consider $n$-cellular sets as a full subcategory of $n$-cellular spaces using this functor.

3.2.2. Let $k \geq 1$. We will denote by $J_k$ the strict $\infty$-category corepresenting the functor $Ar_k^* : \infty\text{-}\text{Cat}_{str} \to \text{Set}$ sending a strict $\infty$-category to its set of strictly invertible $k$-arrows. The $\infty$-category $J_k$ is actually a $k$-category. Here are pictures of (the underlying $\infty$-graphs without the identities of) $J_k$ in low dimension:

$$J_1 = \bullet \xrightarrow{\bullet} \bullet, \quad J_2 = \bullet \xrightarrow{\bullet} \bullet \xrightarrow{\bullet} \bullet \quad \text{and} \quad J_3 = \bullet \xrightarrow{\bullet} \bullet \xrightarrow{\bullet} \bullet \xrightarrow{\bullet} \bullet.$$

There is a canonical morphism $j_k : J_k \to D_{k-1}$ corepresenting the natural transformation $Ar_{k-1} \to Ar_k^*$ sending a $(k-1)$-arrow to its identity. Concretely, the morphism $j_k$ is the unique morphism $J_k \to D_{k-1}$ sending the two non-trivial $k$-arrows of $J_k$ to the identity of the only non-trivial $(k-1)$-arrow of $D_{k-1}$.

3.2.3. Recall from paragraph 3.1.12 that we denote by $J$ the set

$$J = \{i_T \mid T \in \text{Ob}(\Theta_n)\}$$

of spine inclusions.

We will denote by $J$ and $\bar{J}$ the sets

$$J = \{j_k \mid 1 \leq k \leq n\} \quad \text{and} \quad \bar{J} = \{N_n(j_k) \mid 1 \leq k \leq n\}.$$

The sets $J$ and $\bar{J}$ will be considered as sets of maps of $n$-cellular sets or $n$-cellular spaces depending on the context.

For the next two definitions and the proposition that follows, we will assume that $n$ is finite.

Definition 3.2.4 (Rezk). The model category for $\Theta_n$-spaces is the left Bousfield localization of the injective model structure on $n$-cellular spaces by the set $J \cup \bar{J}$. We will denote this model structure by $sPr(\Theta_n)_{\Theta_n-Sp}$.

Remark 3.2.5. The original definition of $\Theta_n$-spaces in [Rez10] proceeds by induction on $n$. Here we are following the (trivially equivalent) definition given in [Ara13b, Section 7].

Definition 3.2.6. The quasi-category of $\Theta_n$-spaces is the localization of the quasi-category $P(\Theta_n)$ by the set $J \cup \bar{J}$. We will denote it by $\Theta_n^{\text{Sp}}$.

Proposition 3.2.7. The quasi-category underlying the model category of $\Theta_n$-spaces is canonically equivalent to the quasi-category of $\Theta_n$-spaces.

Proof. We indeed have

$$\mathcal{U}(sPr(\Theta_n)_{\Theta_n-Sp}) = \mathcal{U}((J \cup \bar{J})^{-1} sPr(\Theta_n)_{inj})$$

$$\simeq (J \cup \bar{J})^{-1} \mathcal{U}(sPr(\Theta_n)_{inj})$$
(by Proposition 1.16)
\[ \simeq (\mathcal{J} \cup \mathcal{J})^{-1}\mathcal{U}(\text{pr}(\Theta_n)_{\text{proj}}) \]
(since $\mathcal{U}$ only depends on the weak equivalences)
\[ \simeq (\mathcal{J} \cup \mathcal{J})^{-1}\mathcal{P}(\Theta_n) \]
(by Proposition 1.11)
\[ = \Theta_n\mathcal{S}\mathcal{P}. \]

### 3.3. Autoequivalences of the category of strict $n$-categories.

#### 3.3.1. Recall that for every $i$ such that $1 \leq i \leq n$, there exists an autoequivalence $\text{op}_i$ of the category $n\text{-Cat}_{\text{str}}$ sending a strict $n$-category $C$ to the strict $n$-category obtained from $C$ by reversing the orientation of the $i$-arrows. The functor $\text{op}_i$ is obviously its own inverse. The mapping $i \mapsto \text{op}_i$ extends formally to a monoid morphism
\[ \left(\mathbb{Z}/2\mathbb{Z}\right)^n \longrightarrow \text{aut}(n\text{-Cat}_{\text{str}}), \]
where for $n = \infty$, we set $\left(\mathbb{Z}/2\mathbb{Z}\right)^\infty = \prod_{k \geq 1} \mathbb{Z}/2\mathbb{Z}$. For $\delta = (\delta_i)_{1 \leq i \leq n}$ an element of $\left(\mathbb{Z}/2\mathbb{Z}\right)^n$, we will denote by $\text{op}_\delta$ the associated autoequivalence of $n\text{-Cat}_{\text{str}}$. Concretely, if $C$ is a strict $n$-category, then $\text{op}_\delta(C)$ is the strict $n$-category obtained from $C$ by reversing the orientation of the $i$-arrows for all $i$ such that $\delta_i = 1$.

Note that the monoid morphism $\left(\mathbb{Z}/2\mathbb{Z}\right)^n \rightarrow \text{aut}(n\text{-Cat}_{\text{str}})$ defines a strict monoidal functor
\[ \left(\mathbb{Z}/2\mathbb{Z}\right)^n_{\text{disc}} \longrightarrow \text{Aut}(n\text{-Cat}_{\text{str}}), \]
where $\left(\mathbb{Z}/2\mathbb{Z}\right)^n_{\text{disc}}$ denotes the discrete category on the set $\left(\mathbb{Z}/2\mathbb{Z}\right)^n$ endowed with the strict monoidal structure given by the group law of $\left(\mathbb{Z}/2\mathbb{Z}\right)^n$, and where the category $\text{Aut}(n\text{-Cat}_{\text{str}})$ is endowed with the strict monoidal structure given by composition of functors.

**Remark 3.3.2.** If $T$ is an object of $\Theta_n$, then it is easy to see that for any $\delta$ in $\left(\mathbb{Z}/2\mathbb{Z}\right)^n$, we have $\text{op}_\delta(T) \simeq T$.

#### 3.3.3. Let $k \geq 0$. The $k$-sphere $S^k$ is the $k$-truncation of the $(k+1)$-disk $D_{k+1}$ (say in $\infty\text{-Cat}_{\text{str}}$). By definition, we have a canonical monomorphism $S^k \hookrightarrow D_{k+1}$. It is easy to prove by induction, setting $S^{-1} = \emptyset$, that
\[ S^k \simeq D_k \coprod_{S^{k-1}} D_k. \]

**Lemma 3.3.4.** Let $k$ be an integer such that $0 < k \leq n$. The $k$-disk $D_k$ is the unique strict $n$-category $C$ satisfying the following three properties:

(i) The $n$-category $C$ is not a $(k-1)$-category.

(ii) Any proper sub-$n$-category of $C$ is a $(k-1)$-category.

(iii) The $(k-1)$-truncation $\text{tr}_{k-1}(C)$ of $C$ is isomorphic to $S^{k-1}$.

**Proof.** It is obvious that $D_k$ satisfies these properties. Suppose $C$ is a strict $n$-category satisfying these three properties. The first property exactly means that $C$ has at least one non-trivial arrow in dimension at least $k$. Every such non-trivial arrow of $C$ defines a sub-$n$-category which is not a $(k-1)$-category. The second property implies that this sub-$n$-category has to be $C$. This means that there can exist only one non-trivial arrow in $C$ in dimension at least $k$. Let $l$ be the dimension of this arrow. By the third property, the $(k-1)$-truncation of $C$ is $S^{k-1}$. Since the $n$-category generated by the unique non-trivial $l$-arrow of $C$ is $C$, the...
two non-trivial \((k-1)\)-arrows of \(S^{k-1}\) have to be iterated sources or targets of this unique non-trivial \(l\)-arrow. This means that \(l\) has to be equal to \(k\) and that the \(n\)-category \(C\) has to be the \(k\)-disk \(D_k\).

**Lemma 3.3.5.** Let \(k\) be such that \(0 \leq k \leq n\). A strict \(n\)-category \(C\) is a \(k\)-category if and only if there exists an extremal epimorphism of the form

\[
P \prod_{E} D_k \rightarrow C,
\]

for some set \(E\), that is, if there exists an epimorphism of the above form that does not factor through any proper subobject of \(C\).

**Proof.** Clearly, such an extremal epimorphism exists if and only if \(C\) is generated by its set of \(k\)-arrows, that is, if and only if \(C\) is a \(k\)-category. \(\square\)

**Proposition 3.3.6.** Every autoequivalence \(F\) of \(n\)\text{-Cat}_{str} preserves the disks. In other words, we have \(F(D_k) \simeq D_k\) for \(0 \leq k \leq n\).

**Proof.** We are going to prove by induction on \(k\) that \(F\) preserves both \(D_k\) and \(S^{k-1}\). For \(k = 0\), the objects \(D_0\) and \(S^1\) are terminal and initial objects of \(n\)\text{-Cat}_{str}, respectively, and hence they are preserved by \(F\).

Suppose the result is true for \(k - 1 \geq 0\) and let us prove it for \(k\). The equivalence \(F\) respects monomorphisms and reflects isomorphisms. It thus respects proper subobjects. It also reflects them since a quasi-inverse of \(F\) respects them. Moreover, since \(F\) preserves sums, extremal epimorphisms and the \((k-1)\)-disk by induction, it preserves \((k-1)\)-categories by Lemma 3.3.5. This shows that \(F\) preserves the two first conditions of Lemma 3.3.4.

We will now check that \(F\) preserves \(S^{k-1}\). There are exactly two monomorphisms \(S^{k-2} \hookrightarrow D_{k-1}\), say \(i_1\) and \(i_2\). By induction, \(F\) preserves \(S^{k-2}\) and \(D_{k-1}\). It thus either fixes \(i_1\) and \(i_2\) or exchanges them. But for any value of \(\varepsilon\) in \(\{1, 2\}\), we have a pushout square

\[
\begin{array}{ccc}
S^{k-2} & \xrightarrow{i_\varepsilon} & D_{k-1} \\
\downarrow & & \downarrow \\
D_{k-1} & \xrightarrow{i_\varepsilon} & S^{k-1}.
\end{array}
\]

Since \(F\) preserves pushouts, we deduce that \(S^{k-1}\) is preserved.

Let us now prove that \(F\) is in some sense compatible with \((k-1)\)-truncation. First, note that for any strict \(n\)-category \(C\), the \((k-1)\)-categories \(\text{tr}_{k-1}(F(C))\) and \(F(\text{tr}_{k-1}(C))\) (recall that we proved that \(F\) preserves \((k-1)\)-categories) have the same arrows. Indeed, the set of \(i\)-arrows for \(i < k\) is corepresented by \(D_i\), which is preserved by \(F\) by induction. Now for every \(l\) such that \(0 < l < k\), the only monomorphisms from \(D_{l-1}\) to \(D_l\) are \(\sigma\) and \(\tau\). Since the disks \(D_{l-1}\) and \(D_l\) are preserved by induction, the equivalence \(F\) either fixes these two morphisms or exchanges them. Define \(\delta(l)\) in \(\mathbb{Z}/2\mathbb{Z}\) to be 0 if \(F\) fixes them and 1 otherwise. For \(l\) such that \(k \leq l \leq n\), set \(\delta(l) = 0\). The \(\delta(l)\) define an element \(\delta(F)\) in \((\mathbb{Z}/2\mathbb{Z})^n\). It is immediate that the underlying \(n\)-graphs of \(\text{tr}_{k-1}(F(C))\) and \(\text{op}_{\delta(F)}(\text{tr}_{k-1}(C))\) are equal.

In particular, if \(C\) is such that \(\text{tr}_{k-1}(C)\) is isomorphic to \(S^{k-1}\), we get that the underlying \(n\)-graph of \(\text{tr}_{k-1}(F(C))\) is the underlying \(n\)-graph of

\[
\text{op}_{\delta(F)}(\text{tr}_{k-1}(C)) \simeq \text{op}_{\delta(F)}(S^{k-1}) \simeq S^{k-1}.
\]
But since there exists only one structure of strict $n$-category on the underlying $n$-graph of $S^{k-1}$, we get that $tr_{k-1}(F(C))$ is isomorphic to $S^{k-1}$. This shows that $F$ preserves the last condition of Lemma 3.3.4.

We thus have proved that the three properties characterizing $D_k$ are stable under $F$ and hence we get that $F$ preserves $D_k$.

Remark 3.3.7. The analogous statement for rigid strict $n$-categories appears as [BSP13, Lemma 4.5]. The proof, which is based on a different characterization of the $n$-disk, also adapts to the case of strict $n$-categories.

3.3.8. Let $F$ be an autoequivalence of $n\text{-Cat}_{str}$. We saw in the proof of the previous proposition that for every $k$ such that $1 \leq k \leq n$, the functor $F$ either fixes $\sigma, \tau: D_{k-1} \to D_k$ or exchanges them, and that this defines an element $\delta(F)$ in $(\mathbb{Z}/2\mathbb{Z})^n$. We thus get a map

$$\text{aut}(n\text{-Cat}_{str}) \to (\mathbb{Z}/2\mathbb{Z})^n.$$ 

It is immediate that this map is a monoid morphism and a retraction of the map $(\mathbb{Z}/2\mathbb{Z})^n \to \text{aut}(n\text{-Cat}_{str})$. In other words, if $F$ and $G$ are autoequivalences of $n\text{-Cat}_{str}$, we have $\delta(GF) = \delta(G)\delta(F)$, and if $\delta$ is an element of $(\mathbb{Z}/2\mathbb{Z})^n$, we have $\delta(\text{op}) = \delta$.

Proposition 3.3.9. If $F$ is an autoequivalence of $n\text{-Cat}_{str}$, then we have $F(T) \simeq T$ for every object $T$ of $\Theta_n$. In particular, $F$ induces an autoequivalence of $\Theta_n$.

Proof. By postcomposing $F$ with $\text{op}_{\delta(F)}$ (which preserves the objects of $\Theta_n$, as we already observed), we can suppose that $\delta(F) = 1$. Let $T$ be an object of $\Theta_n$. The object $T$ seen as a strict $n$-category is the colimit of the diagram $D_T$ of paragraph 3.1.4. But since $\delta(F) = 1$, the functor $F$ preserves the diagram $D_T$ and since $F$ commutes with colimits, we indeed have $F(T) \simeq T$.

Corollary 3.3.10. The dense inclusion $\Theta_n \hookrightarrow n\text{-Cat}_{str}$ induces a fully faithful functor $\text{Aut}(n\text{-Cat}_{str}) \to \text{Aut}(\Theta_n)$.

Proof. This is immediate from the previous proposition and Proposition 2.7.

We will show in Section 3.5 that the monoidal category $\text{Aut}(\Theta_n)$ is isomorphic to the discrete monoidal category $(\mathbb{Z}/2\mathbb{Z})^{\text{disc}}$. As a corollary, we will obtain the following theorem:

Theorem. The functor $(\mathbb{Z}/2\mathbb{Z})^{\text{disc}} \to \text{Aut}(n\text{-Cat}_{str})$ is an equivalence of monoidal categories.

3.4. Autoequivalences of the category of rigid strict $n$-categories.

3.4.1. A strict $n$-category $C$ is said to be rigid if it contains no non-trivial isomorphisms, that is, if for any $k$ such that $1 \leq k \leq n$, any strictly invertible $k$-arrow of $C$ is the identity of a $(k-1)$-arrow. For a fixed $k$, this condition amounts to saying that $C$ is local with respect to the map $j_k: J_k \to D_{k-1}$ of paragraph 3.2.2. In particular, a strict $n$-category $C$ is rigid if and only if it is $\mathcal{J}^n$-local, where $\mathcal{J}^n$ is the set defined in paragraph 3.2.3.

We will denote by $n\text{-Cat}_{str}$ the full subcategory of the category of strict $n$-categories whose objects are the rigid strict $n$-categories.
3.4.2. Let \( \delta \) be an element of \( (\mathbb{Z}/2\mathbb{Z})^n \). It is clear that if \( C \) is a rigid strict \( n \)-category, then so is \( \text{op}_\delta(C) \). In other words, the autoequivalence \( \text{op}_\delta \) induces an autoequivalence of the category \( n\text{-}\text{Cat}_{\text{str,r}} \). We will also denote this equivalence by \( \text{op}_\delta \). We thus get, as in the non-rigid case, a monoidal functor

\[
(\mathbb{Z}/2\mathbb{Z})^n_{\text{disc}} \to \text{Aut}(n\text{-}\text{Cat}_{\text{str,r}}).
\]

Proposition 3.4.3. If \( F \) is an autoequivalence of \( n\text{-}\text{Cat}_{\text{str,r}} \), then \( F(T) \simeq T \) for every object \( T \) of \( \Theta_n \). In particular, \( F \) induces an autoequivalence of \( \Theta_n \).

Proof. The proof used for \( n\text{-}\text{Cat}_{\text{str}} \) in the previous subsection adapts trivially. One only has to observe that the objects of \( \Theta_n \) and the spheres are rigid \( n \)-categories. \( \square \)

Corollary 3.4.4. The dense inclusion \( \Theta_n \hookrightarrow n\text{-}\text{Cat}_{\text{str}} \) induces a fully faithful functor \( \text{Aut}(n\text{-}\text{Cat}_{\text{str,r}}) \to \text{Aut}(\Theta_n) \).

Proof. This is immediate from the previous proposition and Proposition 2.7. \( \square \)

We will show in Section 3.5 that the monoidal category \( \text{Aut}(\Theta_n) \) is isomorphic to the discrete monoidal category \( (\mathbb{Z}/2\mathbb{Z})^n_{\text{disc}} \). As a corollary, we will obtain the following theorem, first proved by Barwick and Schommer-Pries [BSP13, Section 4] by different methods:

Theorem. The functor \( (\mathbb{Z}/2\mathbb{Z})^n_{\text{disc}} \to \text{Aut}(n\text{-}\text{Cat}_{\text{str,r}}) \) is an equivalence of monoidal categories.

3.5. Autoequivalences of the category \( \Theta_n \).

3.5.1. Let \( \delta \) be an element of \( (\mathbb{Z}/2\mathbb{Z})^n \). As already observed, if \( T \) is an object of \( \Theta_n \), then we have \( \text{op}_\delta(T) \simeq T \). This means that the autoequivalence \( \text{op}_\delta \) induces an autoequivalence of the category \( \Theta_n \). We will still denote this equivalence by \( \text{op}_\delta \). We thus get, as in the previous subsections, a monoidal functor

\[
(\mathbb{Z}/2\mathbb{Z})^n_{\text{disc}} \to \text{Aut}(\Theta_n).
\]

Proposition 3.5.2. Every autoequivalence of \( \Theta_n \) is the identity on objects.

Proof. The strategy is similar to the one used for \( n\text{-}\text{Cat}_{\text{str}} \) but has to be adapted since the spheres are not objects of \( \Theta_n \). Nevertheless, for \( k \) such that \( 0 < k < n \), the characterization of the \( k \)-disk given by Lemma 3.3.4 can be replaced by the following: the \( k \)-disk \( D_k \) is the unique object \( T \) of \( \Theta_n \) satisfying the following two properties:

(i) Every proper subobject of \( T \) is an \( l \)-disk for \( l < k \).

(ii) For every \( l \) such that \( 0 \leq l < k \), there are exactly two monomorphisms \( D_l \to T \).

Using this characterization, we get by induction on \( k \) that \( D_k \) is preserved (starting from \( D_0 \) which is the terminal object of \( \Theta_n \)). We then obtain that every object is preserved by expressing an object \( T \) of \( \Theta_n \) as the colimit of the diagram \( D_T \) (using the same argument as in the proof of Proposition 3.3.9). \( \square \)

Proposition 3.5.3. The monoid morphism \( (\mathbb{Z}/2\mathbb{Z})^n \to \text{aut}(\Theta_n) \) is an isomorphism.
Proof. It suffices to show that the retraction $F \mapsto \delta(F)$ of the morphism of the statement is injective, that is, that if $F$ is an autoequivalence of $\Theta_n$ such that $\delta(F) = 1$, then $F$ is the identity.

Let us fix such an $F$. We know that $F$ is the identity on objects by the previous proposition. Since morphisms of strict $n$-categories are determined by their action on arrows, it suffices to show that morphisms of the form $D_k \rightarrow T$, where $k$ is such that $0 \leq k \leq n$ and $T$ is any object of $\Theta_n$, are preserved.

Let $D_k \rightarrow T$ be such a morphism. By Proposition 3.1.10, this morphism factors as a composite

$$D_k \xrightarrow{a} S \xrightarrow{i} T,$$

where $a$ is active and $i$ is inert. Let us first prove that $i$ is preserved by $F$. Recall that $S$ and $T$ are colimits of diagrams $D_S$ and $D_T$ involving disks only. Consider the cocone $D_S \rightarrow T$ associated to $i$. Since $i$ is inert, each of the components $D_t \rightarrow T$ of this cocone corresponds to an $l$-cell of the $n$-graph $\Theta_0(T)$ generating $\Theta(T)$ and can thus be written (in a non-canonical way) as a composite

$$D_t \rightarrow D_{\nu} \rightarrow \mathrm{colim} D_T = T,$$

where the first map is either $\sigma$ or $\tau$, and the second map is one of the canonical morphisms associated to $D_T$. Since $\delta(F) = 1$, the morphism $D_t \rightarrow D_{\nu}$, the diagram $D_S$ and the diagram $D_T$ are preserved by $F$. Since moreover, as already observed, the cocone making $T$ a colimit of the diagram $D_T$ is unique, the canonical morphism $D_{\nu} \rightarrow T$ is also preserved. This shows that $i$ is preserved.

Finally, let us prove that $a: D_k \rightarrow S$ is preserved by $F$. Since there is at most one active morphism from a fixed disk to a fixed object of $\Theta_n$ (see paragraph 3.1.9), it suffices to show that $F(a)$ is active. This follows from the fact, just proved, that any autoequivalence of $\Theta_n$ preserves inert morphisms. Indeed, by Proposition 3.1.10, $F(a)$ factors uniquely as $ib$, where $i$ is inert and $b$ is active. This means that $a = F^{-1}(i)F^{-1}(b)$. By decomposing $F^{-1}(b)$, we get that $a = F^{-1}(i)jc$, where $j$ is inert and $c$ is active. Since $F^{-1}$ preserves inert morphisms, $F^{-1}(i)$ is also inert. By uniqueness of the decomposition of $a$, we obtain that $F^{-1}(i)j$ is an identity. Since inert morphisms are monomorphisms and $\Theta_n$ has no non-trivial isomorphisms, this implies that $F^{-1}(i)$ and hence $i$ are identities, and therefore that $F(a)$ is active. $\square$

Theorem 3.5.4. The functor $(\mathbb{Z}/2\mathbb{Z})^n_{\text{disc}} \rightarrow \text{Aut}(\Theta_n)$ is an isomorphism of monoidal categories.

Proof. The previous proposition states that this functor is bijective on objects. To conclude, it suffices to show that $\text{Aut}(\Theta_n)$ is a discrete category. Let $\delta$ and $\delta'$ be two elements of $(\mathbb{Z}/2\mathbb{Z})^n$, and let $\gamma: \text{op}_{\delta} \rightarrow \text{op}_{\delta'}$ be a natural transformation. For every object $T$ of $\Theta_n$, we thus have an endomorphism $\gamma_T$ of $T$.

Let us show by induction that for every $0 \leq k \leq n$, we have $\delta_k = \delta'_k$ and $\gamma_{D_k} = 1_{D_k}$. (A priori, $\delta_0$ and $\delta'_0$ are not defined and we define them to be both equal to $0$ for the purpose of starting our induction.) The case $k = 0$ is obvious. For $k \geq 1$, consider the naturality squares associated to $\sigma, \tau: D_{k-1} \rightarrow D_k$:
These squares determine the value of \( \gamma_{D_k} \) on the \((k-1)\)-truncation \( S^{k-1} \) of \( D_k \): it is one of the two automorphisms of \( S^{k-1} \). But only the trivial automorphism can be lifted to an endomorphism of \( D_k \) and the unique lift is then the identity of \( D_k \). This exactly means that \( \delta_k \) and \( \delta'_k \) are equal and that \( \gamma_{D_k} \) is the identity of \( D_k \).

This shows that if such a \( \gamma \) exists, then \( \delta = \delta' \). To conclude, we have to show that the identity is the unique natural transformation \( \gamma : op_\delta \to op_\delta \). Fix an object \( T \) of \( \Theta_n \) and consider the naturality squares

\[
\begin{array}{c}
D_k \xrightarrow{\text{op}_\delta(u)} T \\
\downarrow 1_{D_k} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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(i) objects of $\Theta_n$ are rigid strict $n$-categories;
(ii) autoequivalences of $n\text{-Cat}_{str}$ restrict to autoequivalences of $\Theta_n$.

The first point is obvious and the second point is Proposition 3.4.3. We can thus apply the proposition and we get that $\text{Aut}(\Theta_n\text{-Sp})$ is a full subcategory of $\text{Aut}(\Theta_n)$. But $\text{Aut}(\Theta_n)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n_{\text{disc}}}$ by Theorem 3.5.4. To conclude, it thus suffices to show that every autoequivalence of $\Theta_n$ lifts to an autoequivalence of $\Theta_n\text{-Sp}$. This follows from paragraph 3.6.1.

Remark 3.6.4. The above result is a consequence of previous work of Barwick and Schommer-Pries. Indeed, by [BSP13, Theorem 11.15] the quasi-category $\Theta_n\text{-Sp}$ is equivalent to the quasi-category of $\Upsilon_n$-spaces, and by [BSP13, Theorem 8.12] the quasi-category of autoequivalences of $\Upsilon_n$-spaces is $(\mathbb{Z}/2\mathbb{Z})^n$.

Remark 3.6.5. Let $\mathcal{C}$ be a quasi-category and let $\text{Fun}(\mathcal{C},\mathcal{C})$ be the associated quasi-category of endofunctors. Recall from [Lur07, Proposition 3.1.7] that $\text{Fun}(\mathcal{C},\mathcal{C})$ is a monoidal quasi-category, the monoidal structure being given by the composition of endofunctors (for the general theory of non-symmetric monoidal quasi-categories, see [Lur07]). Moreover, the full subcategory $\text{Aut}(\mathcal{C}) \subseteq \text{Fun}(\mathcal{C},\mathcal{C})$ spanned by the autoequivalences inherits a monoidal structure.

Using these monoidal structures, Theorem 3.6.3 admits the following refinement: the quasi-categories $\text{Aut}(\Theta_n\text{-Sp})$ and $(\mathbb{Z}/2\mathbb{Z})^{n_{\text{disc}}}$ are equivalent as monoidal quasi-categories.

4. Autoequivalences of the $(\infty, 1)$-category of $\infty$-operads

In this section, we show that the quasi-category of autoequivalences of the quasi-category of $\Omega$-spaces, which is a model for $\infty$-operads, is a contractible Kan complex. We follow the general strategy described in the introduction and formalized by Proposition 2.8. In particular, we reduce to the computation of the autoequivalences of the category $\Omega$ of trees.

4.1. Preliminaries on operads and the category of trees.

4.1.1. We will denote by $\mathcal{O}p$ the category of (small) symmetric coloured operads (see for instance [EM06, Section 2] where they are called multicategories). We will refer to its objects simply as operads.

If $P$ is an operad and $c_1, \ldots, c_n$ and $d$ are colours of $P$, we will denote by $P(c_1, \ldots, c_n; d)$ the set of operations of $P$ whose input colours are given by the $n$-tuple $(c_1, \ldots, c_n)$ and whose output colour is $d$. If $p \in P(c_1, \ldots, c_n; d)$ is an $n$-ary operation and $\sigma$ is an element of the symmetric group $\Sigma_n$, we will denote by $p\sigma$ the induced operation in $P(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; d)$.

We will denote by $\text{Coll}$ the category of symmetric (coloured) collections and by $\text{Coll}_{ns}$ the category of non-symmetric (coloured) collections. We have forgetful functors

$$\mathcal{O}p \rightarrow \text{Coll} \rightarrow \text{Coll}_{ns}$$

and these functors admit left adjoints.

If $p$ and $q$ are two operations of an operad $P$, we will write $p \sim_{\Sigma} q$ if $p$ and $q$ have the same arity $n$ and there exists an element $\sigma$ in $\Sigma_n$ such that $q = p\sigma$.

We will say that an operation of an operad is non-trivial if it is not the identity of a colour.
We will identify the category $\mathsf{Cat}$ of small categories with the full subcategory of $\mathsf{Op}$ consisting of operads having only unary operations. The inclusion functor $\mathsf{Cat} \hookrightarrow \mathsf{Op}$ admits a right adjoint sending an operad to its so-called underlying category. Concretely, the underlying category of an operad $P$ is the suboperad of $P$ obtained by throwing out the non-unary operations (in particular, its objects are the colours of $P$).

4.1.2. A combinatorial tree is a non-empty finite connected graph with no loops. A vertex of a combinatorial tree is said to be outer if it has only one edge attached to it.

An operadic tree is a combinatorial tree endowed with the choice of an outer vertex called the output and of a (possibly empty) set of outer vertices not containing the output called the set of inputs. The choice of the output induces an orientation of the tree “from the inputs to the output”. A vertex of an operadic tree is defined as a vertex of the underlying combinatorial tree which is neither an input nor the output. An edge of an operadic tree is an edge of the underlying combinatorial tree. The edge attached to the output will be called the root and the edges attached to the inputs will be called the leaves.

We will follow the usual conventions when drawing operadic trees: the output will be drawn at the bottom of the tree and the vertices corresponding to the inputs and the output will be deleted. Here is an example of such a tree:

From now on, by a tree, we will mean an operadic tree.

A planar structure on a tree $T$ consists of the data, for any vertex $v$ of $T$, of an ordering on the input edges of $v$. A planar tree is a tree endowed with a planar structure. We will sometimes use non-planar tree as a synonym of tree in order to emphasize that a given tree is not planar.

4.1.3. To every tree $T$, we can associate a symmetric collection $K(T)$. The colours of $K(T)$ are the edges of $T$ and for every vertex $v$ of $T$ with $n$ inputs and every choice of an order $e_1 < \cdots < e_n$ on the input edges of $v$, there is an operation in $K(T)(e_1, \ldots, e_n; e)$, where $e$ is the output edge of $v$. The group $\Sigma_n$ acts on the operations associated to $v$ in the obvious way.

If $T$ is a tree, we will denote by $\Omega(T)$ the free symmetric operad on the symmetric collection $K(T)$.

Similarly, to every planar tree $\overline{T}$, we can associate a non-symmetric collection $K_p(\overline{T})$. Its colours are the edges of $\overline{T}$ and for every vertex $v$ in $\overline{T}$ with input edges $e_1 < \cdots < e_n$ and output edge $e$, there is an operation in $K_p(\overline{T})(e_1, \ldots, e_n; e)$.

Note that if $T$ is the underlying non-planar tree of a planar tree $\overline{T}$, then $K(T)$ is the free symmetric collection on $K_p(\overline{T})$. (Another way to put this is to say that the choice of a planar structure on $T$ corresponds to a choice of generators of $K(T)$.) In particular, $\Omega(T)$ is the free symmetric operad on $K_p(\overline{T})$.

4.1.4. The category of trees $\Omega$, introduced by Moerdijk and Weiss in [MW07], is defined as follows: the objects of $\Omega$ are trees up to isomorphism (that is, up
to renaming of their vertices and edges) and the set of morphisms in \( \Omega \) from an object \( S \) to an object \( T \) is given by

\[
\Omega(S, T) = \text{Op}(\Omega(S), \Omega(T)).
\]

By definition, there is a canonical fully faithful functor \( \Omega \hookrightarrow \text{Op} \) and we will always consider \( \Omega \) as a full subcategory of \( \text{Op} \) using this functor.

We will denote by \( \eta \) the tree with one edge and no vertices. For \( n \geq 0 \), we will denote by \( C_n \) the \( n \)-corolla, that is, the tree with one vertex and \( n \) leaves. Note that \( C_n \), seen as an object of \( \text{Op} \), corepresents the functor “set of \( n \)-ary operations”. Similarly, \( \eta \) corepresents the functor “set of colours”. In particular, for any tree \( T \), we have a root map \( \eta \to T \) and leaf maps \( \eta \to T \).

We now fix for every object \( T \) of \( \Omega \) the choice of a planar structure on \( T \). We will denote the resulting planar tree by \( \overline{T} \). The purpose of these choices is to make precise the idea that every tree can be obtained by glueing corollas (see Proposition 4.1.8).

**4.1.5.** Given a tree \( T \), we will denote by \( \Omega_0(T) \) the non-symmetric collection \( K_p(\overline{T}) \). We define a category \( \Omega_0 \) in the following way: the objects of \( \Omega_0 \) are the same as the ones of \( \Omega \) and the set of morphisms in \( \Omega_0 \) from an object \( S \) to an object \( T \) is given by

\[
\Omega_0(S, T) = \text{Coll}_{in}(\Omega_0(S), \Omega_0(T)).
\]

The free symmetric operad functor on a non-symmetric collection induces a canonical functor from \( \Omega_0 \) to \( \Omega \) which is obviously faithful. We will always consider \( \Omega_0 \) as a subcategory of \( \Omega \) using this functor.

**Remark 4.1.6.** The subcategory \( \Omega_0 \) of \( \Omega \) does depend on the choice of the planar structures. One way to avoid these choices is to replace \( \Omega \) by the following equivalent category \( \overline{\Omega} \): an object of \( \overline{\Omega} \) is an object of \( \Omega \) endowed with the choice of a planar structure and the set of morphisms in \( \overline{\Omega} \) from an object \( \overline{S} \) to an object \( \overline{T} \) is given by

\[
\overline{\Omega}(\overline{S}, \overline{T}) = \text{Op}(\Omega(S), \Omega(T)),
\]

where \( S \) and \( T \) denote the respective underlying non-planar trees of \( \overline{S} \) and \( \overline{T} \). It is then possible to define a canonical subcategory \( \overline{\Omega}_0 \) of \( \overline{\Omega} \) by setting

\[
\overline{\Omega}_0(\overline{S}, \overline{T}) = \text{Coll}_{in}(K_p(\overline{S}), K_p(\overline{T})).
\]

Note that the choice of planar structures we made corresponds to the choice of a section of the equivalence of categories given by the forgetful functor \( \overline{\Omega} \to \Omega \).

**4.1.7.** We will denote by \( \mathbb{C} \) the full subcategory of \( \Omega_0 \) whose objects are \( \eta \) and the corollas. The morphisms of \( \mathbb{C} \), besides the identities, are exactly the root and the leaf maps of corollas. We will denote by \( j \) the inclusion functor \( \mathbb{C} \hookrightarrow \Omega_0 \). Note that the category of non-symmetric collections can be identified with the category of presheaves on \( \mathbb{C} \).

If \( T \) is an object of \( \Omega \), we will denote by \( \mathbb{C}/T \) the comma category \( j \downarrow T \), where \( T \) is seen as an object of \( \Omega_0 \). An object of \( \mathbb{C}/T \) is hence a pair \((C, C \to T)\), where \( C \) is an object of \( \mathbb{C} \) and \( C \to T \) is a morphism of \( \Omega_0 \). A morphism from \((C, C \to T)\) to \((C', C' \to T)\) is a morphism \( C \to C' \) in \( \mathbb{C} \) (i.e., in \( \Omega_0 \)) making the obvious triangle commute. Thus, an object of \( \mathbb{C}/T \) is just a colour or a generating operation of \( T \).

We will denote by \( \mathcal{D}_T \) the functor \( \mathbb{C}/T \to \mathbb{C} \leftarrow \Omega \).
Proposition 4.1.8. For every object $T$ of $\Omega$, the canonical morphism
\[
\colim D_T = \colim_{(C,C \to T) \in C/T} C \to T
\]
is an isomorphism of $\Omega$. Moreover, the inclusion functor $\Omega \hookrightarrow \Op$ preserves this colimit.

Proof. Since $\Omega$ is a full subcategory of $\Op$, it suffices to prove that we have a canonical isomorphism when the colimit is taken in $\Op$. The canonical decomposition of the non-symmetric collection $K_p(T)$, seen as a presheaf over $C$, as a colimit of representable presheaves gives a canonical isomorphism of non-symmetric collections
\[
\colim_{(C,C \to T) \in C/T} K_p(C) \to K_p(T).
\]
The result then follows from the fact that the free symmetric operad functor commutes with colimits. □

Example 4.1.9. Let $T$ be the tree

endowed with the planar structure given by the picture. We define $T_i$ as the corolla associated to the vertex $v_i$, i.e., $T_i$ has the input edges of $v_i$ as leaves and the output edge of $v_i$ as root. Then, the diagram $D_T$ has the corollas $T_i$ and a copy $\eta_e$ of $\eta$ for every edge $e$ in $T$ as objects. The morphisms in the diagram are the morphisms from each $\eta_e$ to the corresponding edge of $T_i$. Thus, the diagram $D_T$ is the following:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
T_1 \quad T_2 \quad T_3
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

4.1.10. The category of dendroidal sets is the category $\Pr(\Omega)$ of presheaves on $\Omega$. The inclusion $\Omega \hookrightarrow \Op$ induces a functor $N_d: \Op \to \Pr(\Omega)$ sending an operad $P$ to the dendroidal set $T \mapsto \Op(T, P)$. This functor $N_d$ is called the dendroidal nerve.

4.1.11. Let $T$ be a tree. The spine of $T$ is the dendroidal set
\[
I_T = \colim_{(C,C \to T) \in C/T} C
\]
where the colimit is taken in $\Pr(\Omega)$. By Proposition 4.1.8, there is a canonical morphism of dendroidal sets $i_T: I_T \to T$. It is not hard to check that this morphism
is a monomorphism. We will denote by \( J \) the set
\[
J = \{ i_T \mid T \in \text{Ob}(\Omega) \}
\]
of spine inclusions.

Let now \( X \) be a dendroidal set. For any tree \( T \), the map \( i_T \) induces a Segal map
\[
X(T) \simeq \Pr(\Omega)(T, X) \to \Pr(\Omega)(I_T, X) \simeq \lim_{(C, C \to T) \in C/T} X(C).
\]
We will say that \( X \) satisfies the Segal condition if all the Segal maps are bijections. This exactly means that \( X \) is \( J \)-local.

**Proposition 4.1.12** (Cisinski–Moerdijk, Weber). The dendroidal nerve functor is fully faithful. Moreover, its essential image consists of the dendroidal sets satisfying the Segal condition.

**Proof.** This is \([CM13a, Corollary 2.6]\). It also follows from the general machinery of \([Web07]\) (see Example 4.27). \(\square\)

**Remark 4.1.13.** The first assertion of the previous proposition precisely means that the inclusion functor \( \Omega \hookrightarrow \mathcal{O} \) is dense.

### 4.2. The quasi-category of complete dendroidal Segal spaces.

#### 4.2.1. The category of dendroidal spaces is the category \( sPr(\Omega) \simeq \Pr(\Omega \times \Delta) \) of simplicial presheaves on \( \Omega \). The first projection \( p: \Omega \times \Delta \to \Omega \) induces a fully faithful functor \( p^* : \Pr(\Omega) \to \Pr(\Omega \times \Delta) \) sending a dendroidal set to the corresponding discrete dendroidal space. We will always consider dendroidal sets as a full subcategory of dendroidal spaces using this functor.

#### 4.2.2. Let \( X \) be a dendroidal set and let \( T \) be a tree. Then, by functoriality, the group \( \text{aut}_\Omega(T) \) acts on \( X(T) \). More generally, if \( f : X \to Y \) is a monomorphism of dendroidal sets, then \( \text{aut}_\Omega(T) \) acts on the difference \( Y(T) \setminus f_T(X(T)) \). Such a monomorphism is said to be normal if this action is free for any \( T \).

More generally, a monomorphism of dendroidal spaces \( f : X \to Y \) will be called normal if the monomorphism of dendroidal sets \( f_{\bullet,n} : X_{\bullet,n} \to Y_{\bullet,n} \) is normal for every \( n \geq 0 \).

**Proposition 4.2.3** (Cisinski–Moerdijk). There is a simplicial proper combinatorial model structure on \( sPr(\Omega) \) whose weak equivalences are the objectwise simplicial weak homotopy equivalences and whose cofibrations are the normal monomorphisms.

**Proof.** The existence of this model structure is given by \([CM13a, Proposition 5.2]\), as well as the fact that it is proper and combinatorial.

Let us prove it is simplicial. We have to show that the pushout product \( f \square g \) of a monomorphism \( f \) of simplicial sets and a normal monomorphisms \( g \) of dendroidal spaces is again a normal monomorphisms of dendroidal sets, and that moreover, if either \( f \) or \( g \) is a weak equivalence, so is \( f \square g \). Using the fact that pushouts in a presheaf category are computed objectwise, the first statement amounts to showing that the pushout product of a map of sets and a normal monomorphism of dendroidal sets is again a normal monomorphism of dendroidal sets. This follows at once by inspection. The second statement follows immediately from the fact that the Kan–Quillen model structure is cartesian closed. \(\square\)
Following \cite{CM13} and \cite{BM11}, the model structure of the previous proposition will be called the \textit{generalized Reedy model structure} on dendroidal spaces. We will denote it by $\text{sPr}(\Omega)_{\text{Reedy}}$.

\textbf{4.2.4.} Recall that the category of operads is endowed with a symmetric monoidal closed structure given by the so-called Boardman–Vogt tensor product \cite[Definition 2.14]{BV73} (see also \cite[Part I, Section 4.1]{MT10}). We will denote this tensor product by $\otimes_{\text{op}}$. The associated internal hom will be denoted by $\text{Hom}_{\text{op}}$. The unit of this tensor product is the operad $\eta$. In particular, for any operad $P$, we have a canonical isomorphism $\text{Hom}_{\text{op}}(\eta, P) \cong P$.

The only thing we will need to know about $\text{Hom}_{\text{op}}$ is the description of its underlying category: if $P$ and $Q$ are two operads, then the set of objects of the underlying category of $\text{Hom}_{\text{op}}(P, Q)$ is the set of maps of operads from $P$ to $Q$; if $f, g$ are two such maps, a morphism from $f$ to $g$ is a natural transformation $\alpha$ from $f$ to $g$, that is, the data of an operation $\alpha_c$ in $Q(f(c), g(c))$ for every colour $c$ of $P$ such that for every operation $p$ in $P(c_1, \ldots, c_n; d)$ we have $g(p) \circ (\alpha_{c_1}, \ldots, \alpha_{c_n}) = \alpha_d \circ f(p)$.

\textbf{4.2.5.} Recall from paragraph 4.1.11 that we denote by $I$ the set $I = \{i_T \mid T \in \text{Ob}(\Omega)\}$ of spine inclusions.

Let $J$ be the simply connected groupoid on two objects and let $j: J \to \eta$ be the unique map of operads. For any tree $T$, by tensoring $j$ with $T$ we obtain an induced map of operads $j_T: J \otimes_{\text{op}} T \to \eta \otimes_{\text{op}} T \simeq T$.

Note that $j_\eta$ is canonically isomorphic to $j$. We will denote by $\beta^p$ and $\beta$ the sets

\[ \beta^p = \{j_T \mid T \in \text{Ob}(\Omega)\} \quad \text{and} \quad \beta = \{N_d(j_T) \mid T \in \text{Ob}(\Omega)\}. \]

The sets $\beta$ and $\beta^p$ will be considered as sets of maps in dendroidal sets or dendroidal spaces depending on the context.

\textbf{Definition 4.2.6 (Cisinski–Moerdijk).} The \textit{model category for complete dendroidal Segal spaces} or $\Omega$-\textit{spaces} is the left Bousfield localization of the generalized Reedy model structure on dendroidal spaces by the set $I \cup \beta$. We will denote this model structure by $\text{sPr}(\Omega)_{\Omega-\text{Sp}}$.

\textbf{Remark 4.2.7.} This definition is slightly different from the one appearing in \cite[Definition 6.2]{CM13}, which uses the tensor product of dendroidal sets. The two definitions are immediately seen to be equivalent using that $N_d(J \otimes_{\text{od}} T) \simeq N_d(J) \otimes T$ (here $\otimes$ denotes the symmetric monoidal structure on dendroidal sets induced by the Boardman–Vogt tensor product \cite[Section 5]{MW07}); see \cite[Lemma 4.3.3]{Wei07}.

\textbf{Definition 4.2.8.} The \textit{quasi-category of $\Omega$-spaces} is the localization of the quasi-category $\mathcal{P}(\Omega)$ by the set $I \cup \beta$. We will denote it by $\Omega-\text{Sp}$.

\textbf{Proposition 4.2.9.} The quasi-category underlying the model category of $\Omega$-spaces is canonically equivalent to the quasi-category of $\Omega$-spaces.

\textbf{Proof.} We indeed have

\[ \mathcal{U}(\text{sPr}(\Omega)_{\Omega-\text{Sp}}) = \mathcal{U}((I \cup \beta)^{-1} \text{sPr}(\Omega)_{\text{Reedy}}) \]
\[(I \cup J)^{-1}U(sPr(\Omega)_{\text{Reedy}})\]
(by Proposition 1.16)
\[(I \cup J)^{-1}U(sPr(\Omega)_{\text{proj}})\]
(since \(U\) only depends on the weak equivalences)
\[(I \cup J)^{-1}P(\Omega)\]
(by Proposition 1.11)
= \(\Omega-\text{Sp}.\)

4.3. Autoequivalences of the category of operads.

**Lemma 4.3.1.** Let \(P\) be an operad. The following are equivalent:

(i) The operad \(P\) is a category.

(ii) There exists a morphism from \(P\) to \(\eta\).

(iii) There exists exactly one morphism from \(P\) to \(\eta\).

(iv) For every non-empty operad \(Q\), there exists a morphism from \(P\) to \(Q\).

**Proof.** The equivalence of the three first properties is obvious. It is clear that (iv) implies (ii). Conversely, if \(P\) satisfies (ii) and \(Q\) is a non-empty operad, we obtain a map from \(P\) to \(Q\) by composing \(P \to \eta \to Q\), where the second map corresponds to the choice of a colour of \(Q\). \(\square\)

**Proposition 4.3.2.** Let \(F\) be an autoequivalence of \(\mathcal{Op}\). If \(C\) is a category, then \(F(C)\) is a category. In particular, \(F\) restricts to an equivalence of \(\mathcal{Cat}\).

**Proof.** Since the empty operad is the initial object of \(\mathcal{Op}\), it has to be preserved by \(F\). By applying this remark to a quasi-inverse of \(F\), we immediately get that property (iv) of the previous lemma is preserved by \(F\). The first assertion thus follows from the lemma. This shows that \(F\) induces an endofunctor of \(\mathcal{Cat}\). This functor is immediately seen to be an equivalence of categories by applying the first assertion to a quasi-inverse of \(F\). \(\square\)

**Corollary 4.3.3.** For every autoequivalence \(F\) of \(\mathcal{Op}\) we have \(F(\eta) \simeq \eta\). In particular, if \(P\) is an operad, then \(P\) and \(F(P)\) have the same number of colours.

**Proof.** Since \(\eta\) is the terminal object of \(\mathcal{Cat}\), the first assertion follows from the fact that \(F\) restricts to an equivalence of \(\mathcal{Cat}\). The second assertion follows since \(\eta\) corepresents the functor “set of colours”. \(\square\)

**Definition 4.3.4.** An operad is said to be **discrete** if it can be written as a coproduct of copies of \(\eta\). An operad is said to be **non-discrete** if it is not discrete, i.e., if it has at least one non-trivial operation.

**Definition 4.3.5.** A **pseudo-corolla** is an operad \(P\) having a non-trivial operation \(p\) satisfying the following two properties:

(i) For every non-trivial operation \(q\) of \(P\) we have \(p \sim \Sigma q\).

(ii) The only colours of \(P\) are the inputs and the output of \(p\).

**Remark 4.3.6.** Roughly speaking, a pseudo-corolla is a corolla where the input and output colours do not have to be distinct. More precisely, a pseudo-corolla is a corolla if and only if the input and output colours of its unique non-trivial operation (up to permutation) are distinct.
Note that a pseudo-corolla is not determined by an arity and its colours. Indeed, there are two pseudo-corollas $P$ with colours $a, b$ and having an operation $p$ in $P(a, a; b)$. One has no other operations and a trivial action of $\Sigma_2$ and the other one has another operation $q$ in $P(a, a; b)$ and the transposition of $\Sigma_2$ acts by exchanging $p$ and $q$.

Note also that if $P$ is a pseudo-corolla, then any non-trivial operation of $P$ satisfies the conditions (i) and (ii) of the definition.

Remark 4.3.7. The purpose of the second condition of the definition is to exclude operads like $C_n \sqcup \eta$.

Lemma 4.3.8. Let $P$ be an operad. Then the following are equivalent:

(i) The operad $P$ is a pseudo-corolla.

(ii) The operad $P$ is non-discrete and every proper suboperad of $P$ is discrete.

Proof. Every pseudo-corolla satisfies (ii) by definition. Conversely, suppose $P$ satisfies (ii). Since $P$ is non-discrete, it has at least one non-trivial operation $p$. If $q$ is another non-trivial operation of $P$, then we must have $p \sim_{\Sigma} q$, for otherwise $q$ would generate a non-discrete proper suboperad of $P$. If $c$ is a colour of $P$, then $c$ is necessarily an input or the output of $p$, for otherwise $p$ would generate a non-discrete proper suboperad of $P$. This shows that $P$ is a pseudo-corolla. □

Proposition 4.3.9. Every autoequivalence of $\mathcal{O}P$ preserves pseudo-corollas.

Proof. Let $F$ be an autoequivalence of $\mathcal{O}P$. Since $F$ preserves $\eta$ (Corollary 4.3.3) and coproducts, it also preserves discrete operads. Applying the same argument to a quasi-inverse of $F$, we get that $F$ preserves non-discrete operads. It is then immediate to see that the characterization of pseudo-corollas given by the previous lemma is stable under $F$. □

Lemma 4.3.10. Let $P$ be an operad. Then the following are equivalent:

(i) The operad $P$ is the $n$-corolla.

(ii) The operad $P$ is a pseudo-corolla satisfying the following property: for every pseudo-corolla $Q$, if there exists a map $Q \to P$, then $Q$ has exactly $n + 1$ colours.

Proof. If $P = C_n$ is the $n$-corolla, then it is a pseudo-corolla and if there is a map $f$ from a pseudo-corolla $Q$ to $C_n$, then $Q$ has to have exactly $n + 1$ colours, since $f$ sends operations of arity $n$ to operations of arity $n$ and is surjective on the colours.

Conversely, let $P$ be a pseudo-corolla satisfying (ii). Note that taking $Q = P$ and the identity map, we get that $P$ has exactly $n + 1$ colours. If $P$ has arity $n$, then $P = C_n$ and we are done. If $P$ had arity $m \neq n$, then there would be a map from $C_m$ to $P$. But $C_m$ has $m + 1$ colours and $P$ would thus not satisfy (ii). □

Proposition 4.3.11. Every autoequivalence $F$ of $\mathcal{O}P$ preserves corollas. In particular, if $P$ is an operad, then $P$ and $F(P)$ have the same number of $n$-ary operations for all $n$.

Proof. Using the characterization of $n$-corollas given by the previous lemma, this immediately follows from the fact that $F$ preserves pseudo-corollas and the number of colours (Proposition 4.3.9 and Lemma 4.3.10). □
Proposition 4.3.12. Let $F$ be an autoequivalence of $\mathcal{O}p$. Then, for every $n \geq 0$, $F$ preserves the root map of the corolla $C_n$.

Proof. Let $r: \eta \to C_n$ be the root map of $C_n$. By Corollary 4.3.3, $F(\eta)$ is a terminal operad and hence corepresents colours. Let $r': F(\eta) \to F(C_n)$ be the corresponding root map of $F(C_n)$. We have to prove that $F(r) = r'$. Recall moreover that by Proposition 4.3.11, $F(C_n)$ is an $n$-corolla.

The case $n = 0$ being obvious, let us assume $n \geq 1$. Suppose on the contrary that $F(r) \neq r'$. This means that $F(r): F(\eta) \to F(C_n)$ is a leaf map. Since $F$ is faithful, only one colour $\eta \to C_n$ can be sent to the root map. In particular, when $n = 2$, there exists a leaf map $d: \eta \to C_2$ which is not sent to the root map. Let us now consider the following pushout diagram

\[ \begin{array}{ccc} \eta & \xrightarrow{r} & C_n \\ \downarrow{d} & & \downarrow{C_2 \circ_d C'_n,} \\ C_2 & \xrightarrow{d} & C_2 \circ_d C'_n, \end{array} \]

where $C_2 \circ_d C_n$ is the tree obtained by grafting $C_n$ on the leaf $d$ of $C_2$. Since $F$ preserves pushouts, we obtain a pushout diagram

\[ \begin{array}{ccc} F(\eta) & \xrightarrow{F(r)} & F(C_n) \\ F(d) \downarrow & & \downarrow \\ F(C_2) & \xrightarrow{F(d)} & F(C_2 \circ_d C'_n). \end{array} \]

A description of this pushout (keeping in mind that $F(r)$ and $F(d)$ are leaf maps) reveals that $F(C_2 \circ_d C_n)$ has only non-trivial operations in arity 2 and $n$. If $n \geq 2$, this is a contraction, since $C_2 \circ_d C_n$ has an operation of arity $n+1$ and $F$ preserves this property by Proposition 4.3.11. Similarly, if $n = 1$, we get a contraction since $C_2 \circ_d C_1$ has strictly more binary operations than $F(C_2 \circ_d C_1)$. \(\square\)

4.3.13. Let $F$ be an autoequivalence of $\mathcal{O}p$ and let $T$ be a tree. Recall from paragraph 4.1.7 that we have a diagram $\mathcal{D}_T: \mathcal{C}/T \to \Omega$. We will now consider it as a diagram in $\mathcal{O}p$ using the inclusion functor $\Omega \hookrightarrow \mathcal{O}p$. The purpose of this paragraph is to construct a canonical natural isomorphism $\phi: \mathcal{D}_T \to F\mathcal{D}_T$.

Let $(C, C \to T)$ be an object of $\mathcal{C}/T$. Suppose first $C = \eta$. By Corollary 4.3.3, $F(\eta)$ is a terminal operad and we define $\phi_{(\eta, \eta \to T)}$ to be the unique morphism $\eta \to F(\eta)$.

Suppose now $T = C_n$ for $n \geq 0$. By Proposition 4.3.11, $F(C_n)$ is an $n$-corolla. Since $F$ is fully faithful, it induces a bijection $\mathcal{O}p(\eta, C_n) \to \mathcal{O}p(F(\eta), F(C_n))$ or, in other words, a bijection between the colours of $C_n$ and the colours $F(C_n)$. By Proposition 4.3.12, this bijection sends the root of $C_n$ to the root of $F(C_n)$ and we hence get a bijection between the leaves of the $n$-corolla $C_n$ and the leaves of the $n$-corolla $F(C_n)$. This bijection determines a unique morphism $C_n \to F(C_n)$ and we define $\phi_{(C_n, C_n \to T)}$ to be this morphism.
By definition, \( \phi(C_n, C_n \to T) \) is the unique morphism from \( C_n \) to \( F(C_n) \) such that the square

\[
\begin{array}{ccc}
\eta & \xrightarrow{c} & C_n \\
\downarrow & & \downarrow \\
F(\eta) & \xrightarrow{F(c)} & F(C_n)
\end{array}
\]

commutes for every colour \( c: \eta \to C_n \). This precisely means that \( \phi \) is indeed a natural transformation and hence a natural isomorphism since its components are obviously isomorphisms.

**Remark 4.3.14.** The natural isomorphism \( \phi \) is actually the unique natural transformation from \( D_T \) to \( F D_T \), as the components of \( \phi \) are determined by the naturality squares.

**Proposition 4.3.15.** If \( F \) is an autoequivalence of \( \mathcal{O}p \), then \( F(T) \simeq T \) for every object \( T \) of \( \Omega \). In particular, \( F \) induces an autoequivalence of \( \Omega \).

**Proof.** Using the canonical decomposition of \( T \) (Proposition 4.1.8), the canonical isomorphism \( D_T \simeq F D_T \) of the previous paragraph and the fact that \( F \) commutes with colimits, we obtain a chain of canonical isomorphisms

\[
T \simeq \text{colim } D_T \simeq \text{colim } F D_T \simeq F(\text{colim } D_T) \simeq F(T),
\]

thereby proving the result. \( \square \)

**Corollary 4.3.16.** The dense inclusion \( \Omega \hookrightarrow \mathcal{O}p \) induces a fully faithful functor \( \text{Aut}(\mathcal{O}p) \to \text{Aut}(\Omega) \).

**Proof.** This is immediate from the previous proposition and Proposition 2.7. \( \square \)

We will show in Section 4.5 that the category \( \text{Aut}(\Omega) \) is a contractible groupoid. As a corollary, we will obtain the following theorem:

**Theorem.** The category \( \text{Aut}(\mathcal{O}p) \) is a contractible groupoid.

4.4. Autoequivalences of the category of rigid operads.

4.4.1. An operad is *rigid* if its underlying category is rigid, that is, if every invertible unary operation is the identity of a colour. In other words, an operad \( P \) is rigid if it is \( j \)-local, where \( j: J \to \eta \) is the map of paragraph 4.2.5.

We will denote by \( \mathcal{O}p_r \) the full subcategory of \( \mathcal{O}p \) whose objects are the rigid operads. Note that the operads induced by trees are rigid. We hence get a canonical inclusion functor \( \Omega \hookrightarrow \mathcal{O}p_r \). We also have a canonical inclusion functor \( \text{Cat}_r \hookrightarrow \mathcal{O}p_r \).

**Proposition 4.4.2.** If \( F \) is an autoequivalence of \( \mathcal{O}p_r \), then \( F(T) \simeq T \) for every object \( T \) of \( \Omega \). In particular, \( F \) induces an autoequivalence of \( \Omega \).

**Proof.** The strategy of the proof is similar to the one used for \( \mathcal{O}p \) in the previous subsection. Each of the results of that subsection has an obvious variant for \( \mathcal{O}p_r \) obtained by inserting the adjective “rigid” at appropriate places. We leave it to the reader to check that the proofs of these results adapt trivially using the fact that the empty operad, operads induced by trees, pseudo-corollas, discrete operads and the operad \( C_2 \circ_d C_n \) appearing in the proof of Proposition 4.3.12 are rigid. \( \square \)
Corollary 4.4.3. The dense inclusion $\Omega \hookrightarrow \mathcal{O}_p$ induces a fully faithful functor $\text{Aut}(\mathcal{O}_p) \to \text{Aut}(\Omega)$.

Proof. This is immediate from the previous proposition and Proposition 2.7. □

We will show in Section 4.5 that the category $\text{Aut}(\Omega)$ is a contractible groupoid. As a corollary, we will obtain the following theorem:

Theorem. The category $\text{Aut}(\mathcal{O}_p)$ is a contractible groupoid.

4.5. Autoequivalences of the category of trees.

4.5.1. Let $A$ be a small category. Let us denote by $\Sigma_A$ the group $\prod_{a \in \text{Ob}(A)} \text{aut}_A(a)$. To any element $\sigma = (\sigma_a) \in \Sigma_A$, we associate an endofunctor $F_\sigma$ of $A$ in the following way:

(i) For any object $a$ in $A$, we set $F_\sigma(a) = a$.

(ii) For any morphism $f : a \to b$ in $A$, we set $F_\sigma(f) = \sigma_b f \sigma_a^{-1}$.

This assignment defines a monoid morphism $\Sigma_A \to \text{aut}(A)$. In general, this morphism is neither injective nor surjective. Furthermore, for any $\sigma$ and $\sigma'$ in $\Sigma_A$, we have a canonical natural isomorphism from $F_\sigma$ to $F_{\sigma'}$ whose $a$-th component is $\sigma_a' \sigma_a^{-1}$.

Let us denote by $(\Sigma_A)_{sc}$ the simply connected groupoid on $\Sigma_A$, that is, the category whose objects are the elements of $\Sigma_A$ and with a unique morphism between any two objects. The groupoid $(\Sigma_A)_{sc}$ is canonically endowed with the strict monoidal structure given by the group structure of $\Sigma_A$. By the previous paragraph, we have a canonical strict monoidal functor $(\Sigma_A)_{sc} \to \text{Aut}(A)$, where $\text{Aut}(A)$ is endowed with the strict monoidal structure given by composition.

The purpose of this subsection is to show that for $A = \Omega$, the canonical functor $(\Sigma_\Omega)_{sc} \to \text{Aut}(\Omega)$ is an isomorphism of strict monoidal categories.

Proposition 4.5.2. Every autoequivalence of $\Omega$ is the identity on objects.

Proof. The strategy of the proof is similar to the one used for $\mathcal{O}_p$ and $\mathcal{O}_p^r$ in the previous subsections. Note that $\Omega$ is a skeletal category and it thus suffices to show that $F(T) \simeq T$ for every tree $T$.

Lemma 4.3.1 can be adapted to $\Omega$ by saying that a tree $T$ is linear if and only if, for every tree $S$, there exists a least one map from $S \to T$. This shows that $F$ restricts to an equivalence of the full subcategory of $\Omega$ whose objects are linear trees (which is nothing but the category $\Delta$). Since $\eta$ is the terminal object of this category, it has to be preserved by $F$. In particular, for every tree $T$, the objects $T$ and $F(T)$ have the same number of colours. Furthermore, the $n$-corolla can be characterized in $\Omega$ as the unique tree which only has $\eta$ as a proper subobject and which has $n + 1$ colours. It has therefore to be preserved by $F$.

The proof of Proposition 4.3.12 can be adapted to show that $F$ also preserves root maps of corollas (one has to observe that a diagram $C_2 \leftarrow \eta \to C_n$, where the arrows are leaf maps, does not admit a pushout in $\Omega$). If now $T$ is an arbitrary tree, then using the canonical decomposition of $T$ (Proposition 4.1.8) and the natural isomorphism $\phi : D_T \to F_D T$ of functors $\mathcal{C}/T \to \Omega$ constructed in paragraph 4.3.13, we get a chain of isomorphisms

$$T \simeq \text{colim} D_T \simeq \text{colim} F D_T \simeq F(\text{colim} D_T) \simeq F(T),$$

thereby proving the result. □
4.5.3. Let $F$ be an autoequivalence of $\Omega$ and let $T$ be an object of $\Omega$. Recall from paragraphs 4.1.7 and 4.3.13 that we have a diagram $D_T: \mathbb{C}/T \to \Omega$ and a canonical natural isomorphism $\phi: D_T \to F^D_T$. We will denote by $\sigma(F)_T$ the automorphism of $T$ given by the composition of the canonical isomorphisms

$$T = \text{colim} D_T \to \text{colim} F^D_T \to F(\text{colim} D_T) = F(T) = T.$$ 

Unravelling the definitions, this means that $\sigma(F)_T$ is the unique endomorphism of $T$ such that for any object $(C, i: C \to T)$ of $\mathbb{C}/T$, we have

$$\sigma(F)_T \circ i = F(i) \circ \phi(C, i: C \to T).$$

Since maps of $\Omega$ are uniquely determined by their action on the colours, $\sigma(F)_T$ is uniquely determined by the following property: for every morphism $c: \eta \to T$ of $\Omega$, we have

$$(\ast) \quad \sigma(F)_T \circ c = F(c).$$

(We are using here the fact that $\phi(\eta, c: \eta \to T) = 1_\eta$) Note that by definition of $\phi$, if $(C_n, C_n \to T)$ is an object of $\mathbb{C}/T$, we have $\phi(C_n, C_n \to T) \circ c = F(c)$ and hence

$$\phi(C_n, C_n \to T) = \sigma(F)_{C_n}.$$ 

We will denote by $\sigma(F)$ the element of $\Sigma_\Omega$ whose component at a tree $T$ is given by $\sigma(F)_T$.

**Lemma 4.5.4.** The assignment $F \mapsto \sigma(F)$ satisfies the following properties:

(i) For any $\sigma$ in $\Sigma_\Omega$, we have $\sigma(F_\sigma) = \sigma$.

(ii) For any autoequivalence $F$ of $\Omega$, we have $F = F_{\sigma(F)}$.

**Proof.** Let $\sigma$ be an element of $\Sigma_\Omega$. To prove the first point, it suffices to check that for any colour $c: \eta \to T$ of $\Omega$, we have $\sigma(F_\sigma)_T \circ c = \sigma T \circ c$. But we have

$$\sigma(F_\sigma)_T \circ c = F_\sigma(c) = \sigma_T \circ c,$$

where the first equality holds by $(\ast)$ and the second one by definition.

Let us prove the second point. By Proposition 4.5.2, $F$ is the identity on objects. The same is true for $F_{\sigma(F)}$ by definition. Since maps of $\Omega$ are determined by their action on the colours, any autoequivalence of $\Omega$ which is the identity on objects is determined by its action on the maps whose source is $\eta$. We thus have to check that for any colour $c: \eta \to T$ of a tree $T$, we have $F_{\sigma(F)}(c) = F(c)$. This is indeed the case since

$$F_{\sigma(F)}(c) = \sigma(F)_T \circ c = F(c),$$

where the first equality holds by definition and the second one by $(\ast)$. \qed

**Proposition 4.5.5.** The monoid morphism $\Sigma_\Omega \to \text{aut}(\Omega)$ is an isomorphism.

**Proof.** The previous lemma precisely says that the map $F \mapsto \sigma(F)$ is an inverse to the map of the statement. \qed

**Theorem 4.5.6.** The functor $(\Sigma_\Omega)_{sc} \to \text{Aut}(\Omega)$ is an isomorphism of categories. In particular, the category $\text{Aut}(\Omega)$ is a contractible groupoid.

**Proof.** The previous proposition states that this functor is bijective on objects. To conclude, it suffices to show that there exists a unique isomorphism between $F_\sigma$ and $F_{\sigma'}$, where $\sigma$ and $\sigma'$ are two elements of $\Sigma_\Omega$. The functor of the statement gives a map from $F_\sigma$ to $F_{\sigma'}$. Let us prove its uniqueness. Let $\gamma: F_\sigma \to F_{\sigma'}$ be a
natural transformation. For any tree $T$ and any colour $c: \eta \to T$, the naturality square

$$
\begin{array}{ccc}
F_{\sigma}(\eta) = \eta & \xrightarrow{\gamma_{\eta}} & F_{\sigma}(T) = T \\
\gamma_{\eta} = 1_{\eta} & & \gamma_{T} \\
F_{\sigma'}(\eta) = \eta & \xrightarrow{\gamma_{\eta}} & F_{\sigma'}(T) = T \\
\end{array}
$$

shows that the action of $\gamma_T$ on the colours is uniquely determined. The morphism $\gamma_T$ is hence uniquely determined, thereby proving the result. \hfill \Box

**Theorem 4.5.7.** The categories $\text{Aut}(\mathcal{O}p)$ and $\text{Aut}(\mathcal{O}p_r)$ are contractible groupoids.

**Proof.** By Corollaries 4.3.16 and 4.4.3, the categories $\text{Aut}(\mathcal{O}p)$ and $\text{Aut}(\mathcal{O}p_r)$ are both full subcategories of the category $\text{Aut}(\Omega)$. The result then follows immediately from the previous theorem, since full subcategories of contractible groupoids are again contractible groupoids. \hfill \Box

### 4.6. Autoequivalences of the quasi-category of $\infty$-operads.

**Proposition 4.6.1.** Let $P$ be an operad. Then the following are equivalent:

(i) The operad $P$ is rigid.

(ii) For any tree $T$, the operad $\text{Hom}_{\mathcal{O}p}(T, P)$ is rigid.

(iii) For any operad $Q$, the operad $\text{Hom}_{\mathcal{O}p}(Q, P)$ is rigid.

**Proof.** Recall that $\eta$ is the unit of the Boardman–Vogt tensor product, hence $\text{Hom}_{\mathcal{O}p}(\eta, P)$ is canonically isomorphic to $P$ for every operad $P$. This shows that (ii) implies (i). Clearly, (iii) implies (ii). Let us prove that (i) implies (iii). Recall that we have a concrete description of the underlying category of $\text{Hom}_{\mathcal{O}p}(Q, P)$ (see paragraph 4.2.4). If $f$ and $g$ are two objects of this category, then an isomorphism between them is given by isomorphisms $\alpha_c$ in $P(f(c), g(c))$, where $c$ ranges through the colours of $Q$, such that $g(q) \circ (\alpha_{c_1}, \ldots, \alpha_{c_n}) = \alpha_d \circ f(q)$. Since $P$ is rigid, these isomorphisms have to be the identity. This implies that $f$ has to be equal to $g$, and that $\alpha$ is an identity, thereby proving the result. \hfill \Box

The sets $I, J$ and $\flat$ appearing in the remainder of the section are those introduced in paragraph 4.2.5.

**Proposition 4.6.2.** An operad is $J^{\flat}$-local if and only if it is rigid.

**Proof.** Let $P$ be an operad. Denote by $e: P \to 1$ the unique map of operads from $P$ to the terminal operad. For any tree $T$, we have

$$
\begin{align*}
\gamma_{T} \perp e & \iff j \otimes_{\mathcal{O}p} T \perp e \iff j \perp \text{Hom}_{\mathcal{O}p}(T, e),
\end{align*}
$$

where $f \perp g$ is notation for saying that $f$ has the unique left lifting property with respect to $g$. Since $1$ is the terminal operad, so is $\text{Hom}_{\mathcal{O}p}(T, 1)$ and $\text{Hom}_{\mathcal{O}p}(T, e)$ is the unique map from $\text{Hom}_{\mathcal{O}p}(T, P)$ to the terminal operad. This means that $P$ is $\gamma_T$-local if and only if $\text{Hom}_{\mathcal{O}p}(T, P)$ is $j$-local, that is, if and only if $\text{Hom}_{\mathcal{O}p}(T, P)$ is rigid (see paragraph 4.4.1). The result then follows from Proposition 4.6.1. \hfill \Box

**Proposition 4.6.3.** A dendroidal set is $(I \cup \flat)$-local if and only if it is the nerve of a rigid operad.
Proof. Proposition 4.1.12 precisely says that a dendroidal set is $\mathcal{J}$-local if and only if it is the nerve of an operad. The previous proposition shows that such a dendroidal set is $\mathcal{J}$-local if and only if the operad of which it is the nerve is rigid, thereby proving the result. □

Theorem 4.6.4. The quasi-category $\text{Aut}(\Omega{-}\text{Sp})$ is a contractible Kan complex.

Proof. We are going to apply Proposition 2.8 to $A = \Omega$ and $S = \mathcal{J} \cup \mathcal{J}$. Let us check that the hypotheses are fulfilled. Using the previous proposition, this amounts to verifying that

(i) objects of $\Omega$ are rigid operads;
(ii) autoequivalences of $\mathcal{O}_\mathcal{P}$ restrict to autoequivalences of $\Omega$.

The first point is obvious and the second point is Proposition 4.4.2. We can thus apply the proposition and we get that $\text{Aut}(\Omega{-}\text{Sp})$ is a full subcategory of $\text{Aut}(\Omega)$. But $\text{Aut}(\Omega)$ is a contractible groupoid by Theorem 4.5.6 and the result follows. □

5. Autoequivalences of the $(\infty, 1)$-category of non-symmetric $\infty$-operads

The purpose of this section is to show that the quasi-category of autoequivalences of the quasi-category of planar $\Omega$-spaces, which we use as a model for non-symmetric $\infty$-operads, is the discrete category $\mathbb{Z}/2\mathbb{Z}$. The combinatorics are similar to the symmetric case, the main differences being due to the fact that a planar tree has no non-trivial automorphisms.

5.1. Preliminaries on non-symmetric operads and planar trees.

5.1.1. We will denote by $\mathcal{O}_\mathcal{P}_{\text{ns}}$ the category of (small) non-symmetric coloured operads, i.e., operads without an action of the symmetric group. There is a forgetful functor from $\mathcal{O}_\mathcal{P}_{\text{ns}}$ to the category $\text{Coll}_{\text{ns}}$ of non-symmetric collections, and this functor admits a left adjoint.

5.1.2. Recall from paragraph 4.1.3 that to every planar tree $T$, we can associate a non-symmetric collection $K_{T}(T)$. We define the non-symmetric operad $\Omega_{\text{ns}}(T)$ as the free non-symmetric operad on $K_{T}(T)$.

5.1.3. The category of planar trees $\Omega_{\text{ns}}$, introduced by Moerdijk in [MT10], is defined as follows: the objects of $\Omega_{\text{ns}}$ are planar trees (up to isomorphism) and the set of morphisms in $\Omega_{\text{ns}}$ from an object $S$ to an object $T$ is given by

$$\Omega_{\text{ns}}(S, T) = \mathcal{O}_\mathcal{P}_{\text{ns}}(\Omega_{\text{ns}}(S), \Omega_{\text{ns}}(T)).$$

By definition, there is a canonical fully faithful functor $\Omega_{\text{ns}} \hookrightarrow \mathcal{O}_\mathcal{P}_{\text{ns}}$ and we will always consider $\Omega_{\text{ns}}$ as a full subcategory of $\mathcal{O}_\mathcal{P}_{\text{ns}}$ using this functor.

Remark 5.1.4. One important difference between $\Omega_{\text{ns}}$ and $\Omega$ is that $\text{aut}\Omega_{\text{ns}}(T)$ is trivial for every object $T$ of $\Omega_{\text{ns}}$.

5.1.5. As in paragraph 4.1.7, for any planar tree $T$, we have a category $\mathcal{C}/T$ and a diagram $\mathcal{D}_T : \mathcal{C}/T \to \Omega_{\text{ns}}$. A similar proof as the one of Proposition 4.1.8 shows that for every object $T$ of $\Omega_{\text{ns}}$, the canonical morphism

$$\text{colim} \mathcal{D}_T = \text{colim}_{(C, C \to T) \in \mathcal{C}/T} C \to T$$

is an isomorphism in $\Omega_{\text{ns}}$, and that, moreover, the inclusion functor $\Omega_{\text{ns}} \hookrightarrow \mathcal{O}_\mathcal{P}_{\text{ns}}$ preserves this colimit.
5.1.6. The category of planar dendroidal sets \( \Pr(\Omega_{\text{ns}}) \) is the category of presheaves on \( \Omega_{\text{ns}} \). The inclusion \( \Omega_{\text{ns}} \hookrightarrow \mathcal{O}_{\text{ns}} \) induces a planar dendroidal nerve functor \( N_{\text{ns},d} : \mathcal{O}_{\text{ns}} \to \Pr(\Omega_{\text{ns}}) \).

5.1.7. The spine of a planar tree \( T \) is the planar dendroidal set

\[
I_T = \text{colim} \left( C, C \to T \right) \in C/T \subset \mathcal{C} \to T
\]

where the colimit is taken in \( \Pr(\Omega_{\text{ns}}) \). There is a canonical monomorphism of planar dendroidal sets \( i_T : I_T \to T \). We will denote by \( \mathcal{I} \) the set

\[
\mathcal{I} = \{ i_T \mid T \in \text{Ob}(\Omega_{\text{ns}}) \}
\]

of spine inclusions.

Let now \( X \) be a planar dendroidal set. For any planar tree \( T \), the map \( i_T \) induces a Segal map

\[
X(T) \simeq \Pr(\Omega_{\text{ns}})(T, X) \to \Pr(\Omega_{\text{ns}})(I_T, X) \simeq \lim_{(C, C \to T) \in C/T} X(C).
\]

We will say that \( X \) satisfies the Segal condition if all the Segal maps are bijections. This exactly means that \( X \) is \( \mathcal{I} \)-local.

**Proposition 5.1.8** (Cisinski–Moerdijk, Weber). The planar dendroidal nerve functor is fully faithful. Moreover, its essential image consists of the planar dendroidal sets satisfying the Segal condition.

**Proof.** The proof of [CM13a, Corollary 2.6] can be adapted to the case of planar dendroidal sets. It also follows from the general machinery of [Web07]. \( \square \)

**Remark 5.1.9.** The first assertion of the previous proposition precisely means that the inclusion functor \( \Omega_{\text{ns}} \hookrightarrow \mathcal{O}_{\text{ns}} \) is dense.

5.2. The quasi-category of complete planar dendroidal Segal spaces.

5.2.1. The category of planar dendroidal spaces is the category \( \text{sPr}(\Omega_{\text{ns}}) \) of simplicial presheaves on \( \Omega_{\text{ns}} \). We will consider the category of planar dendroidal sets as a full subcategory of the category of planar dendroidal spaces (as in paragraph 4.2.1).

5.2.2. In the definition of the Boardman–Vogt tensor product of operads, it is crucial that the operads under consideration are symmetric. However, the tensor product still makes sense without the symmetries when at least one of the operads involved is a category.

More precisely, the category of non-symmetric operads is tensored over the category of categories. We will denote by \( \otimes_{\mathcal{O}_{\text{ns}}} \) this tensor. The tensor \( \otimes_{\mathcal{O}_{\text{ns}}} \) is closed and we will denote by \( \text{Hom}_{\mathcal{O}_{\text{ns}}} \) the associated enrichment over categories. This means that if \( C \) is a category, and \( P \) and \( Q \) are non-symmetric operads, then there is a canonical bijection

\[
\mathcal{O}_{\text{ns}}(C \otimes_{\mathcal{O}_{\text{ns}}} P, Q) \simeq \text{Cat}(C, \text{Hom}_{\mathcal{O}_{\text{ns}}}(P, Q)).
\]

The category \( \text{Hom}_{\mathcal{O}_{\text{ns}}}(P, Q) \) can be described in the same way as the underlying category of the operad of morphisms between operads described in paragraph 4.2.4.

5.2.3. Recall from paragraph 5.1.7 that we denote by \( \mathcal{I} \) the set

\[
\mathcal{I} = \{ i_T \mid T \in \text{Ob}(\Omega_{\text{ns}}) \}
\]

of spine inclusions.
As in the non-planar case, for every planar tree $T$, we have a canonical map

$$j_T: J ⊗ \mathcal{Op}_{ns} T → T$$

of non-symmetric operads, where $J$ denotes the simply connected groupoid on two objects. We will denote by $\mathcal{J}$ and $\mathcal{J}$ the sets

$$\mathcal{J} = \{j_T | T ∈ \text{Ob}(\Omega_{ns})\} \quad \text{and} \quad \mathcal{J} = \{N_{ns,d}(j_T) | T ∈ \text{Ob}(\Omega_{ns})\}.$$ 

The sets $\mathcal{J}$ and $\mathcal{J}$ will be considered as sets of maps in planar dendroidal sets or planar dendroidal spaces depending on the context.

**Definition 5.2.4.** The model category for complete planar dendroidal Segal spaces or $\Omega_{ns}$-spaces is the left Bousfield localization of the injective model structure on planar dendroidal spaces by the set $\mathcal{J} ∪ \mathcal{J}$.

**Definition 5.2.5.** The quasi-category of $\Omega_{ns}$-spaces is the localization of the quasi-category $\mathcal{P}(\Omega_{ns})$ by the set $\mathcal{J} ∪ \mathcal{J}$. We will denote it by $\Omega_{ns}$-$\text{Sp}$.

**Proposition 5.2.6.** The quasi-category underlying the model category of $\Omega_{ns}$-spaces is canonically equivalent to the quasi-category of $\Omega_{ns}$-spaces.

**Proof.** The proof is the same as the one of Proposition 4.2.9. □

### 5.3. Autoequivalences of the category of non-symmetric operads.

**5.3.1.** For each $n ≥ 1$, we denote by $\mu_n$ the mirror permutation in $\Sigma_n$, that is, the permutation defined by

$$\mu_n(i) = n - i + 1, \quad 1 ≤ i ≤ n.$$ 

We define an endofunctor $M$ of the category $\mathcal{Op}_{ns}$ in the following way:

(i) Given an operad $P$ in $\mathcal{Op}_{ns}$, the operad $M(P)$ has the same colours as $P$ and, for colours $c_1, \ldots, c_n$ and $c$ of $M(P)$, we set

$$M(P)(c_1, \ldots, c_n; c) = P(c_{\mu_n(1)}, \ldots, c_{\mu_n(n)}; c).$$

(ii) For a map of operads $f: P → Q$ in $\mathcal{Op}_{ns}$, the map $M(f)$ is defined on components by

$$M(f)(c_1, \ldots, c_n; c)(p) = f(c_{\mu_n(1)}, \ldots, c_{\mu_n(n)}; c)(p),$$

where $p$ is an operation in $M(P)(c_1, \ldots, c_n; c)$. It is easy to check that this functor is indeed well-defined. Obviously, $M ∘ M$ is the identity and $M$ is hence an autoequivalence. We will call $M$ the mirror autoequivalence. Note that the mirror autoequivalence sends a planar tree to the planar tree obtained by reversing the orientation of the plane.

The autoequivalence $M$ defines a monoid morphism

$$\mathbb{Z}/2\mathbb{Z} → \text{aut}(\mathcal{Op}_{ns})$$

and hence a strict monoidal functor

$$(\mathbb{Z}/2\mathbb{Z})_{\text{disc}} → \text{Aut}(\mathcal{Op}_{ns}),$$

where $(\mathbb{Z}/2\mathbb{Z})_{\text{disc}}$ denotes the discrete category on the set $\mathbb{Z}/2\mathbb{Z}$ endowed with the strict monoidal structure given by the group law of $\mathbb{Z}/2\mathbb{Z}$, and the category $\text{Aut}(\mathcal{Op}_{ns})$ is endowed with the strict monoidal structure given by composition of functors.
Proposition 5.3.2. Let $F$ be an autoequivalence of $\mathcal{O}_{\text{ns}}$. Then $F$ preserves $\eta$, the corollas and the root maps of the corollas.

Proof. The proofs of Corollary 4.3.3, Proposition 4.3.11 and Proposition 4.3.12 are easily adapted using the following notion of non-symmetric pseudo-corollas: a non-symmetric pseudo-corolla is a non-symmetric operad $P$ with a unique non-trivial operation $p$ such that the only colours of $P$ are the inputs and the output of $p$. Non-symmetric pseudo-corollas can be characterized as in Lemma 4.3.8, and non-symmetric $n$-corollas can be characterized in terms of non-symmetric pseudo-corollas as in Lemma 4.3.10.

5.3.3. We will denote by $\Sigma$ the group $\prod_{n \geq 0} \Sigma_n$. (Note that $\Sigma_0$ and $\Sigma_1$ are trivial.) We will associate to every autoequivalence $F$ of $\mathcal{O}_{\text{ns}}$ an element $\sigma(F)$ in $\Sigma$. Let us define its components $\sigma_n(F)$.

Let thus $F$ be an autoequivalence of $\mathcal{O}_{\text{ns}}$ and let $n \geq 0$. Since $F$ is fully faithful, it induces a bijection $\mathcal{O}_{\text{ns}}(\eta, C_n) \rightarrow \mathcal{O}_{\text{ns}}(F(\eta), F(C_n))$. By the previous proposition, we have $F(\eta) \simeq \eta$ and $F(C_n) \simeq C_n$. Moreover, since $\eta$ and $C_n$ have no non-trivial automorphisms, these isomorphisms are canonical. We thus obtain an automorphism of the set $\mathcal{O}_{\text{ns}}(\eta, C_n)$ and, since by the previous proposition the root map of $C_n$ is preserved, an automorphism of the input colours of $C_n$, that is, an element in $\Sigma_n$. This permutation is by definition the component $\sigma_n(F)$ of $\sigma(F)$.

Clearly, if $F = M$ is the mirror autoequivalence, we have $\sigma(M)_n = \mu_n$. In particular, $\sigma(M)_2$ is the transposition $\tau$ of $\Sigma_2$.

5.3.4. Given any autoequivalence $F$ of $\mathcal{O}_{\text{ns}}$, we will denote by $\overline{F}$ the autoequivalence

$$
\overline{F} = \begin{cases} 
F & \text{if } \sigma(F)_2 = 1, \\
M \circ F & \text{if } \sigma(F)_2 = \tau.
\end{cases}
$$

Note that $\sigma(\overline{F})_2 = 1$ for every autoequivalence $F$.

5.3.5. For every $n \geq 2$, we will denote by $B_n$ the planar binary tree with $n$ leaves of the following shape:

In particular, $B_2$ is the 2-corolla $C_2$.

Lemma 5.3.6. Let $F$ be an autoequivalence of $\mathcal{O}_{\text{ns}}$. Then $\overline{F}(B_n) \simeq B_n$ and $\overline{F}$ preserves all maps from $\eta$ to $B_n$ for every $n \geq 2$.

Proof. We prove it by induction on $n \geq 2$. If $n = 2$, then $B_2$ is the 2-corolla $C_2$ which is preserved by Proposition 5.3.2. Moreover, since $\sigma(\overline{F})_2 = 1$, we know that $\overline{F}$ preserves all maps from $\eta$ to $C_2 = B_2$. 

Suppose now that the result is true for some \( n \geq 2 \). We have the following pushout square

\[
\begin{array}{ccc}
\eta & \rightarrow & B_n \\
\downarrow & & \downarrow \\
B_2 & \rightarrow & B_n+1,
\end{array}
\]

where the top map is the root map, the map on the left corresponds to the left leaf of \( B_2 \), \( u \) is the inclusion of \( B_n \) into the upper part of \( B_{n+1} \) and \( l \) is the inclusion of \( B_2 \) into the lower part of \( B_{n+1} \). By induction hypothesis, the maps from \( \eta \) to \( B_2 \) and \( B_n \) are preserved by \( \tilde{F} \). Since \( \tilde{F} \) preserves pushouts, it has to preserve \( B_{n+1} \). Moreover, since \( B_{n+1} \) has no non-trivial automorphisms, it also has to preserve \( l \) and \( u \).

Let now \( f : \eta \rightarrow B_{n+1} \) be any morphism. Such a map corresponds to a colour of \( B_{n+1} \) which has to belong either to \( B_2 \) or to \( B_n \). More precisely, either there exists a map \( g : \eta \rightarrow B_2 \) such that \( f = lg \), or there exist a map \( h : \eta \rightarrow B_n \) such that \( f = uh \). Since \( l \) and \( u \) are preserved and any map from \( \eta \) to \( B_2 \) or \( B_n \) is preserved by induction hypothesis, we obtain that \( \tilde{F} \) preserves \( f \).

**Proposition 5.3.7.** If \( F \) is an autoequivalence of \( \mathcal{O}_{\text{NS}} \), then \( \sigma(\tilde{F}) = 1 \). In other words, \( \tilde{F} \) preserves all maps from \( \eta \) to the \( n \)-corolla for every \( n \geq 0 \).

**Proof.** The cases \( n = 0 \) and \( n = 1 \) are trivial, and the case \( n = 2 \) is true by definition of \( \tilde{F} \). Let \( n \geq 3 \) and consider a map \( f : \eta \rightarrow C_n \). We have a unique map \( t : C_n \rightarrow B_n \) from \( C_n \) to \( B_n \). This map corresponds to the total composition of \( B_n \). Set \( g = tf \) and consider the commutative triangle

\[
\begin{array}{ccc}
\eta & \rightarrow & B_n \\
\downarrow & & \downarrow \\
C_n & \rightarrow & B_n,
\end{array}
\]

Since \( t \) is injective on colours, the map \( f \) is the only map making this triangle commute. But the map \( g \) and \( t \) are both preserved by \( \tilde{F} \) (the first one by Lemma 5.3.6 and the second one by uniqueness). It follows that \( f \) is also preserved by \( \tilde{F} \). \( \square \)

**Proposition 5.3.8.** If \( F \) is an autoequivalence of \( \mathcal{O}_{\text{NS}} \), then \( \tilde{F}(T) \simeq T \) for every object \( T \) of \( \Omega_{\text{NS}} \). In particular, \( F \) sends planar trees to planar trees and thus induces an autoequivalence of \( \Omega_{\text{NS}} \).

**Proof.** Recall from paragraph 5.1.5 that every planar tree \( T \) is the colimit of a diagram \( D_T : \mathcal{C}/T \rightarrow \Omega_{\text{NS}} \rightarrow \mathcal{O}_{\text{NS}} \). By Propositions 5.3.2 and 5.3.7, \( \tilde{F} \) preserves \( \eta \), \( C_n \) and all the maps from \( \eta \rightarrow C_n \). Moreover, it preserves them up to a canonical isomorphism since these objects do not have non-trivial automorphisms. We thus have a canonical isomorphism \( D_T \simeq \tilde{F}D_T \) and hence a chain of canonical isomorphisms

\[
T \simeq \text{colim} D_T \simeq \text{colim} \tilde{F}D_T \simeq \tilde{F}(\text{colim} D_T) \simeq \tilde{F}(T),
\]

thereby proving the first assertion.

The second assertion follows from the fact that the mirror autoequivalence sends planar trees to planar trees. \( \square \)
Corollary 5.3.9. The dense inclusion $\Omega_{ns} \hookrightarrow \mathcal{O}_{ns}$ induces a fully faithful functor $\text{Aut}(\mathcal{O}_{ns}) \to \text{Aut}(\Omega_{ns})$.

Proof. This is immediate from the previous proposition and Proposition 2.7. □

We will show in Section 5.6 that the monoidal category $\text{Aut}(\Omega_{ns})$ is isomorphic to the discrete monoidal category $(\mathbb{Z}/2\mathbb{Z})_{\text{disc}}$. As a corollary, we will obtain the following theorem:

Theorem. The functor $(\mathbb{Z}/2\mathbb{Z})_{\text{disc}} \to \text{Aut}(\mathcal{O}_{ns})$ is an equivalence of monoidal categories.

5.4. Autoequivalences of the category of rigid non-symmetric operads.

5.4.1. A non-symmetric operad is rigid if its underlying category is rigid. As in the symmetric case, an operad is rigid if and only if it is $j$-local, where $j : J \to \eta$ is the map $j_\eta$ of paragraph 5.2.3.

We will denote by $\mathcal{O}_{ns,r}$ the full subcategory of $\mathcal{O}$ whose objects are the rigid non-symmetric operads. Note that the non-symmetric operads induced by planar trees are rigid. Hence, as in the symmetric case, we get canonical inclusion functors $\Omega_{ns} \hookrightarrow \mathcal{O}_{ns,r}$ and $\text{Cat}_r \hookrightarrow \mathcal{O}_{ns,r}$.

5.4.2. The mirror autoequivalence obviously preserves rigid non-symmetric operads. It thus induces a mirror autoequivalence of $\mathcal{O}_{ns,r}$, which in turn induces a strict monoidal functor $(\mathbb{Z}/2\mathbb{Z})_{\text{disc}} \to \text{Aut}(\mathcal{O}_{ns,r})$.

In particular, if $F$ is an autoequivalence of $\mathcal{O}_{ns,r}$, we can define an autoequivalence $\tilde{F}$ of $\mathcal{O}_{ns,r}$ as in paragraph 5.3.4.

Proposition 5.4.3. If $F$ is an autoequivalence of $\mathcal{O}_{ns,r}$, then $\tilde{F}(T) \simeq T$ for every object $T$ of $\Omega_{ns}$. In particular, $F$ sends planar trees to planar trees and thus induces an autoequivalence of $\Omega_{ns}$.

Proof. The strategy of the proof is basically the same as in the case of $\mathcal{O}_{ns}$ but taking into account the pertinent modifications for rigid non-symmetric operads as explained in the proof of Proposition 4.4.2 for the case of rigid operads. □

Corollary 5.4.4. The dense inclusion $\Omega_{ns} \hookrightarrow \mathcal{O}_{ns,r}$ induces a fully faithful functor $\text{Aut}(\mathcal{O}_{ns,r}) \to \text{Aut}(\Omega_{ns})$.

Proof. This is immediate from the previous proposition and Proposition 2.7. □

We will show in Section 5.6 that the monoidal category $\text{Aut}(\Omega_{ns})$ is isomorphic to the discrete monoidal category $(\mathbb{Z}/2\mathbb{Z})_{\text{disc}}$. As a corollary, we will obtain the following theorem:

Theorem. The functor $(\mathbb{Z}/2\mathbb{Z})_{\text{disc}} \to \text{Aut}(\mathcal{O}_{ns,r})$ is an equivalence of monoidal categories.
5.5. Autoequivalences of the category of planar trees.

5.5.1. As already observed, the mirror autoequivalence sends planar trees to planar trees. It thus induces a mirror autoequivalence of $\Omega_{\text{ns}}$ that we will still denote by $M$, which in turn induces a strict monoidal functor

$$(\mathbb{Z}/2\mathbb{Z})_{\text{disc}} \rightarrow \text{Aut}(\Omega_{\text{ns}}),$$

as in the previous two subsections.

In particular, if $F$ is an autoequivalence of $\Omega_{\text{ns}}$, we can define an autoequivalence $\tilde{F}$ of $\Omega_{\text{ns}}$ as in paragraph 5.3.4.

**Proposition 5.5.2.** If $F$ is an autoequivalence of $\Omega_{\text{ns}}$, then $\tilde{F}$ is the identity on objects.

**Proof.** The proof of Proposition 4.5.2 can be easily adapted using the canonical isomorphism $D_T \simeq \tilde{F} D_T$ appearing in the proof of proposition 5.3.8. □

**Proposition 5.5.3.** The monoid morphism $\mathbb{Z}/2\mathbb{Z} \rightarrow \text{aut}(\Omega_{\text{ns}})$ is an isomorphism.

**Proof.** Let $F$ be an autoequivalence of $\Omega_{\text{ns}}$. It clearly suffices to show that the autoequivalence $\tilde{F}$ is the identity. By the previous proposition, we know it is the identity on objects. To prove it is the identity on morphisms, we can reduce, as in the symmetric case, to the case of maps from $\eta$ to corollas and the result thus follows from Proposition 5.3.7. □

**Theorem 5.5.4.** The functor $(\mathbb{Z}/2\mathbb{Z})_{\text{disc}} \rightarrow \text{Aut}(\Omega_{\text{ns}})$ is an isomorphism of monoidal categories.

**Proof.** The previous proposition states that this functor is bijective on objects. To conclude, it suffices to show that $\text{Aut}(\Omega_{\text{ns}})$ is a discrete category. Let $F$ and $G$ be two autoequivalences of $\Omega_{\text{ns}}$ and let $\gamma: F \rightarrow G$ be a natural transformation. For every planar tree $T$, we have a morphism $\gamma_T: F(T) \rightarrow G(T)$. If $T$ is the planar tree $B_3$ described in paragraph 5.3.5, then there are no morphisms $T \rightarrow M(T)$ or $M(T) \rightarrow T$. This implies that $F$ and $G$ have to be both equal to the identity autoequivalence or to the mirror autoequivalence.

So let $F$ be either the identity autoequivalence or $M$, and let $\gamma: F \rightarrow F$ be a natural transformation. Let $T$ be a planar tree and $c: \eta \rightarrow T$ any colour. Then, by naturality, we have a commutative square

$$
\begin{array}{ccc}
F(\eta) = \eta & \xrightarrow{F(c)} & F(T) \\
\gamma_{\eta} = 1_{\eta} & & \gamma_T \\
\end{array}
$$

This shows that $\gamma_T$ is the identity on colours and hence the identity map, thereby proving that $\gamma$ is the identity natural transformation. □

**Theorem 5.5.5.** The monoidal categories $\text{Aut}(\mathcal{O}_{\text{ns}})$ and $\text{Aut}(\mathcal{O}_{\text{ns},r})$ are equivalent to the discrete monoidal category $(\mathbb{Z}/2\mathbb{Z})_{\text{disc}}$.

**Proof.** By Corollaries 5.3.9 and 5.4.4, the categories $\text{Aut}(\mathcal{O}_{\text{ns}})$ and $\text{Aut}(\mathcal{O}_{\text{ns},r})$ are both full (monoidal) subcategories of the category $\text{Aut}(\Omega_{\text{ns}})$. To conclude, it thus suffices to show that every autoequivalence of $\Omega_{\text{ns}}$ lifts to autoequivalences of $\mathcal{O}_{\text{ns}}$.
and \(\mathcal{O}_\text{ns,r}\). This is obvious since, by Proposition 5.5.3, the autoequivalences of \(\Omega_{\text{ns}}\) are the identity and the mirror autoequivalence. □

5.6. Autoequivalences of the quasi-category of non-symmetric \(\infty\)-operads.

5.6.1. The autoequivalence \(M\) of \(\Omega_{\text{ns}}\) extends formally to an autoequivalence of the quasi-category \(\mathcal{P}(\Omega_{\text{ns}})\). It is easy to see that the sets \(\mathcal{I}\) and \(\mathcal{J}\) of paragraph 5.2.3 are stable under this autoequivalence and we thus get an induced autoequivalence \(M\) of \(\Omega_{\text{ns}}-\text{Sp} = (\mathcal{I} \cup \mathcal{J})^{-1}\mathcal{P}(\Omega_{\text{ns}})\).

The sets \(\mathcal{I}^{\flat}\) and \(\mathcal{J}\) appearing in the remainder of the section are those introduced in paragraph 5.2.3.

**Proposition 5.6.2.** A non-symmetric operad is \(\mathcal{I}^{\flat}\)-local if and only if it is rigid.

*Proof.* The proof is a trivial adaptation of the proof of Proposition 4.6.2 □

**Proposition 5.6.3.** A planar dendroidal set is \((\mathcal{I} \cup \mathcal{J})\)-local if and only if it is the nerve of a rigid non-symmetric operad.

*Proof.* This follows from the previous proposition and Proposition 5.1.8 as in the proof of Proposition 4.6.3. □

**Theorem 5.6.4.** The quasi-category \(\text{Aut}(\Omega_{\text{ns}}-\text{Sp})\) is canonically equivalent to the discrete category \((\mathbb{Z}/2\mathbb{Z})_{\text{disc}}\).

*Proof.* We are going to apply Proposition 2.8 to \(A = \Omega_{\text{ns}}\) and \(S = \mathcal{I} \cup \mathcal{J}\). Let us check that the hypotheses are fulfilled. Using the previous proposition, this amounts to verifying that

(i) objects of \(\Omega_{\text{ns}}\) are rigid non-symmetric operads;
(ii) autoequivalences of \(\mathcal{O}_\text{ns,r}\) restrict to autoequivalences of \(\Omega_{\text{ns}}\).

The first point is obvious and the second point is Proposition 5.4.3. We can thus apply the proposition and we get that \(\text{Aut}(\Omega_{\text{ns}}-\text{Sp})\) is a full subcategory of \(\text{Aut}(\Omega_{\text{ns}})\). But \(\text{Aut}(\Omega_{\text{ns}})\) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})_{\text{disc}}\) by Theorem 5.5.4. To conclude, it thus suffices to show that every autoequivalence of \(\Omega_{\text{ns}}\) lifts to an autoequivalence of \(\Omega_{\text{ns}}-\text{Sp}\). This follows from paragraph 5.6.1. □

**Remark 5.6.5.** Using the monoidal structures described in Remark 3.6.5, one can show that \(\text{Aut}(\Omega_{\text{ns}}-\text{Sp})\) and \((\mathbb{Z}/2\mathbb{Z})_{\text{disc}}\) are equivalent as monoidal quasi-categories.

References

[Ara10] D. Ara. *Sur les \(\infty\)-groupoïdes de Grothendieck et une variante \(\infty\)-catégorique*. PhD thesis, Université Paris Diderot – Paris 7, 2010. Supervised by G. Maltsiniotis.

[Ara12] D. Ara. The groupoidal analogue \(\tilde{\Theta}\) to Joyal’s category \(\Theta\) is a test category. *Appl. Categ. Structures*, 20(6):603–649, 2012.

[Ara13a] D. Ara. Strict \(\infty\)-groupoids are Grothendieck \(\infty\)-groupoids. *J. Pure Appl. Algebra*, 217(12):1237–1278, 2013.

[Ara13b] D. Ara. Higher quasi-categories vs higher Rezk spaces. *J. K-Theory*, 2013. to appear.

[Ber02] C. Berger. A cellular nerve for higher categories. *Adv. Math.*, 169(1):118–175, 2002.

[Ber07] J. E. Bergner. A model category structure on the category of simplicial categories. *Trans. Amer. Math. Soc.*, 359(5):2043–2058, 2007.

[Ber10] J. E. Bergner. A survey of \((\infty,1)\)-categories. In *Towards higher categories*, volume 152 of IMA Vol. Math. Appl., pages 69–83. Springer-Verlag, 2010.

[BK72] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, 1972.

[BM11] C. Berger and I. Moerdijk. On an extension of the notion of Reedy category. *Math. Z.*, 269(3-4):977–1004, 2011.
ON AUTOEQUIVALENCES OF THE $(\infty,1)$-CATEGORY OF $\infty$-OPERADS

[BSP13] C. Barwick and C. Schommer-Pries. On the unicity of the homotopy theory of higher categories. arXiv:1112.0040v4 [math.AT], 2013.

[BV73] J. M. Boardman and R. M. Vogt. Homotopy invariant algebraic structures on topological spaces. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, 1973.

[CM11] D.-C. Cisinski and I. Moerdijk. Dendroidal sets as models for homotopy operads. J. Topol., 4(2):257–299, 2011.

[CM13a] D.-C. Cisinski and I. Moerdijk. Dendroidal Segal spaces and $\infty$-operads. J. Topol., 6(3):675–704, 2013.

[CM13b] D.-C. Cisinski and I. Moerdijk. Dendroidal sets and simplicial operads. J. Topol., 6(3):705–756, 2013.

[CP86] J.-M. Cordier and T. Porter. Vogt’s theorem on categories of homotopy coherent diagrams. Math. Proc. Cambridge Philos. Soc., 100(1):65–90, 1986.

[DS11] D. Dugger and D. I. Spivak. Mapping spaces in quasi-categories. Algebr. Geom. Topol., 11(1):263–325, 2011.

[EM06] A. D. Elmendorf and M. A. Mandell. Rings, modules, and algebras in infinite loop space theory. Adv. Math., 205(1):163–228, 2006.

[Gro10] M. Groth. A short course on infinity-categories. arXiv:1007.2925 [math.AT], 2010.

[Hel88] A. Heller. Homotopy theories. Mem. Amer. Math. Soc., 71(383):vi+78, 1988.

[HHM13] G. Heuts, V. Hinich, and I. Moerdijk. The equivalence between Lurie’s model and the dendroidal model for infinity-operads. arXiv:1305.3658v1 [math.AT], 2013.

[Hir03] P. S. Hirschhorn. Model categories and their localizations, volume 99 of Mathematical Surveys and Monographs. American Mathematical Society, 2003.

[HS01] A. Hirschowitz and C. Simpson. Descente pour les $n$-champs. arXiv:math/9807049v3 [math.AG], 2001.

[Joy97] A. Joyal. Disks, duality and $\Theta$-categories. Preprint, 1997.

[Joy02] A. Joyal. Quasi-categories and Kan complexes. J. Pure Appl. Algebra, 175(1-3):207–222, 2002.

[Joy08a] A. Joyal. Notes on quasi-categories. Preprint, 2008.

[Joy08b] A. Joyal. The theory of quasi-categories and its applications. Lectures at the CRM (Barcelona). Preprint, 2008.

[Lur07] J. Lurie. Derived Algebraic Geometry II: Noncommutative Algebra. arXiv:math/0702299v5 [math.CT], 2007.

[Lur09a] J. Lurie. Higher topos theory, volume 170 of Annals of Mathematics Studies. Princeton University Press, 2009.

[Lur09b] J. Lurie. (Infinity, 2)-categories and the Goodwillie calculus I. Preprint, 2009.

[Lur12] J. Lurie. Derived Algebraic Geometry V, VII–XIV. Preprints, 2012.

[Lur13] J. Lurie. Higher algebra. Preprint, 2013.

[MT10] I. Moerdijk and B. Toën. Simplicial methods for operads and algebraic geometry. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser/Springer Basel AG, Basel, 2010. Edited by Carles Casacuberta and Joachim Kock.

[MW07] I. Moerdijk and I. Weiss. Dendroidal sets. Algebr. Geom. Topol., 7:1441–1470, 2007.

[Rez01] C. Rezk. A model for the homotopy theory of homotopy theory. Trans. Amer. Math. Soc., 353(3):973–1007, 2001.

[Rez10] C. Rezk. A Cartesian presentation of weak $n$-categories. Geom. Topol., 14(1):521–571, 2010.

[Sim01] C. Simpson. Some properties of the theory of $n$-categories. arXiv:math/0110273v1 [math.CT], 2001.

[Toën05] B. Toën. Vers une axiomatisation de la théorie des catégories supérieures. K-Theory, 34(3):233–263, 2005.

[TV04] B. Toën and G. Vezzosi. From HAG to DAG: derived moduli stacks. In Azmiotic, enriched and motive homotopy theory, volume 131 of NATO Sci. Ser. II Math. Phys. Chem., pages 173–216. Kluwer Acad. Publ., 2004.

[TV05] B. Toën and G. Vezzosi. Homotopical algebraic geometry. I. Topos theory. Adv. Math., 193(2):257–372, 2005.

[TV08] B. Toën and G. Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. Mem. Amer. Math. Soc., 193(902):x+224, 2008.
[Web07] M. Weber. Familial 2-functors and parametric right adjoints. Theory Appl. Categ., 18:No. 22, 665–732, 2007.

[Wei07] I. Weiss. Dendroidal sets. PhD thesis, Utrecht University, 2007. Supervised by I. Moerdijk.

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