The mixed scalar curvature flow on a fiber bundle

Vladimir Rovenski∗ and Leonid Zelenko†
Mathematical Department, University of Haifa

Abstract

We apply conformal flows of metrics restricted to the orthogonal distribution $D$ of a foliation to study the question: Which foliations admit a metric such that the leaves are totally geodesic and the mixed scalar curvature is positive? Our evolution operator includes the integrability tensor of $D$, and for the case of integrable orthogonal distribution the flow velocity is proportional to the mixed scalar curvature. We observe that the mean curvature vector $H$ of $D$ satisfies along the leaves the forced Burgers equation, this reduces to the linear Schrödinger equation, whose potential function is a certain “non-umbilicity” measure of $D$. On order to show convergence of the solution metrics $g_t$ as $t \to \infty$, we normalize the flow, and instead of a foliation consider a fiber bundle $\pi : M \to B$ of a Riemannian manifold $(M, g_0)$. In this case, if the “non-umbilicity” of $D$ is smaller in a sense then the “non-integrability”, then the limit mixed scalar curvature function is positive. For integrable $D$, we give examples with foliated surfaces and twisted products.

Keywords: Riemannian metric; foliation; fiber bundle; totally geodesic; conformal; mixed scalar curvature; Burgers equation; Schrödinger operator; twisted product

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1 Introduction

1.1. Totally geodesic foliations. Let $M^{n+p}$ be a connected manifold, endowed with a $p$-dimensional foliation $\mathcal{F}$, i.e., a partition of $M$ into $p$-dimensional submanifolds. A foliation $\mathcal{F}$ on a Riemannian manifold $(M, g)$ is totally geodesic if the leaves (of $\mathcal{F}$) are totally geodesic submanifolds. A Riemannian metric $g$ on $(M, \mathcal{F})$ is called totally geodesic if $\mathcal{F}$ is totally geodesic w. r. t. $g$.

The simple examples of totally geodesic foliations are parallel circles or winding lines on a flat torus, and a Hopf field of great circles on the sphere $S^3$. Totally geodesic foliations appear naturally as null-distributions (or kernels) in the study of manifolds with degenerate curvature-like tensors and differential forms. Totally geodesic foliations of codimension-one on closed non-negatively curved space forms are completely understood: they are given by parallel hyperplanes in the case of a flat torus $T^n$ and they do not exist for spheres $S^n$. However, if the codimension is greater than one, examples of geometrically distinct totally geodesic foliations are abundant, see survey in [10].

One of the principal problems of geometry of foliations reads as follows: Given a foliation $\mathcal{F}$ on a manifold $M$ and a geometric property $(P)$, does there exist a Riemannian metric $g$ on $M$ such that $\mathcal{F}$ enjoys $(P)$ with respect to $g$? Such problems (first posed by H. Gluck for geodesic foliations) were studied already in the 1970’s when D. Sullivan provided a topological condition (called topological tautness) for a foliation, equivalent to the existence of a Riemannian metric making all the leaves minimal. In recent decades, several tools providing results of this sort have been developed. Among them, one may find Sullivan’s foliated cycles and new integral formulae, [11], Part 1, [16] etc.

1.2. The mixed scalar curvature. There are three kinds of curvature for a foliation: tangential, transversal, and mixed (a plane that contains a tangent vector to the foliation and a vector orthogonal to it is said to be mixed). The geometrical sense of the mixed curvature follows

∗E-mail: rovenski@math.haifa.ac.il. Supported by the Marie-Curie actions grant EU-FP7-P-2010-RG, No. 276919.
†E-mail: zelenko@math.haifa.ac.il
from the fact that for a totally geodesic foliation, certain components of the curvature tensor (i.e., contained in the Riccati equation for the conullity tensor, see Section 3.1), regulate the deviation of the leaf geodesics. In general relativity, the geodesic deviation equation is an equation involving the Riemann curvature tensor, which measures the change in separation of neighboring geodesics or, equivalently, the tidal force experienced by a rigid body moving along a geodesic. In the language of mechanics it measures the rate of relative acceleration of two particles moving forward on neighboring geodesics. Let \( \{e_i, \varepsilon_\alpha\}_{1 \leq i \leq n, \alpha \leq \rho} \) be a local orthonormal frame on \( TM \) adapted to \( D \) and \( D_F \). Tracing the Riccati equation yields the equation with the mixed scalar curvature that is the following function on \( M \), see [10, 11, 16]:

\[
\text{Sc}_{\text{mix}} = \sum_{i=1}^{n} \sum_{\alpha=1}^{p} K(e_i, \varepsilon_\alpha),
\]

where \( K(e_i, \varepsilon_\alpha) \) is the sectional curvature of the mixed plane spanned by the vectors \( e_i \) and \( \varepsilon_\alpha \). For example, \( \text{Sc}_{\text{mix}} \) of a foliated surface \((M^2, g)\) is the gaussian curvature \( K \).

Denote \((\cdot)^F\) and \((\cdot)^\perp\) projections onto \(D_F\) and \(D\), respectively. The second fundamental tensor \(b\) and the integrability tensor \(T\) of \(D\) are given by

\[
2b(X, Y) = (\nabla_X Y - \nabla_Y X)^F, \quad 2T(X, Y) = [X, Y]^F \quad (X, Y \in D),
\]

where \(\nabla\) is the Levi-Civita connection of \(g\). For general (i.e., non-integrable) distribution \(D\), define the domain \(U_T = \{x \in M : T(x) \neq 0\}\). The mean curvature vector of \(D\) is given by \(H = \text{Tr}_g b\).

For a totally geodesic foliation \(F\) we have, see [9] and [16]:

\[
\text{Sc}_{\text{mix}} = \text{div} H + |H|^2 + \|T\|^2 - \|b\|^2. \tag{1}
\]

By the Divergence Theorem, (1) yields the integral formula with \(\text{Sc}_{\text{mix}}\) on a closed manifold \(M\).

A foliation \(F\) is conformal, transversely harmonic, or Riemannian, if \(b = \frac{1}{n} H\hat{g}, \quad H = 0\) or \(b = 0\), respectively. In these cases, the distribution \(D\) is called totally umbilical, harmonic or totally geodesic, respectively. Conformal foliations were introduced by Vaisman [15] as foliations admitting a transversal conformal structure. Molino developed a theory of Riemannian foliations on compact manifolds, such foliations form a subclass of conformal foliations.

**Remark 1.** Formula (1) gives us decomposition criteria for foliated manifolds (with an integrable orthogonal distribution) under the constraints on the sign of \(\text{Sc}_{\text{mix}}\), see [16] and a survey in [10]:

(1) If \(F\) and \(F^\perp\) are complementary orthogonal totally umbilical and totally geodesic foliations on a closed oriented Riemannian manifold \(M\) with \(\text{Sc}_{\text{mix}} \geq 0\), then \(M\) splits along the foliations.

(2) A compact minimal foliation \(F\) on a Riemannian manifold \(M\) with an integrable orthogonal distribution and \(\text{Sc}_{\text{mix}} \geq 0\) splits along the foliations.

(3) A minimal foliation \(F\) on a Riemannian manifold \(M\) with the integrable orthogonal distribution and \(\text{Sc}_{\text{mix}} > 0\) has no compact leaves.

The basic question that we want to address in the paper is the following.

**Question 1:** Which foliations admit a totally geodesic metric of positive mixed scalar curvature?

**Example 1.** (a) A change of initial metric along orthogonal distribution \(D\) preserves the property \(\"F\"\) is totally geodesic”, see Lemma 2 in Section 5.2. Let \(\pi : M \rightarrow B\) be a fiber bundle with compact fibers. One may deform the metric \(g\) along \(D\) to obtain a bundle-like totally geodesic metric \(\hat{g}\) (which in general is not \(D\)-conformal to \(g\)) on a fiber or its neighborhood. If there is a section \(\xi : B \rightarrow M\) transversal to fibers, then the deformation can be done globally, and \(\pi\) becomes a Riemannian submersion with totally geodesic fibers. In this case, the mixed sectional curvature is non-negative (due to the formula \(K(X, V)[X, V]^2 = |A_X V|^2\) for the mixed sectional curvature by O’Neill), moreover, \(\text{Sc}_{\text{mix}}\) with respect to \(\hat{g}\) is positive on \(U_T\) (\(\text{Sc}_{\text{mix}} = 0\) when \(D\) is integrable).

(b) For any \(n \geq 2\) and \(p \geq 1\) there exists a fiber bundle with a closed \((n + p)\)-dimensional total space and a compact \(p\)-dimensional fiber, having a totally geodesic metric of positive mixed scalar
curvature. To show this, consider the Hopf fibration $\tilde{\pi} : S^3 \to S^2$ of a unit sphere $S^3$ by great circles (closed geodesics). Let $(\tilde{F}, g_1)$ and $(\tilde{B}, g_2)$ be closed Riemannian manifolds with dimensions, respectively, $(p - 1)$ and $(n - 2)$. Let $M = \tilde{F} \times S^3 \times \tilde{B}$ be the metric product, and $B = S^2 \times \tilde{B}$. Then $\pi : M \to B$ is a fibration with a totally geodesic fiber $F = \tilde{F} \times S^3$. Certainly, $S_{\text{mix}} = 2 > 0$.

Motivating by Remark 1, we ask the following (more particular than Question 1).

**Question 2:** Given a Riemannian manifold $(M, g)$ with a totally geodesic foliation $\mathcal{F}$, does there exist a $D$-conformal to $g$ metric $\tilde{g}$ on $M$ such that $S_{\text{mix}}$ is positive?

In the paper (at least in main results) we impose the additional restrictions:

- instead of a foliation, $M$ is a total space of a smooth fiber bundle $\pi : M \to B$,
- the fibers (leaves) are compact.

Although a fiber bundle is locally a product (of the base and the fiber), this is not true globally. Meanwhile, in Section 2 we have example of solutions in the class of twisted products.

1.3. $D$-conformal flows of metrics. We attack the Question 2 using evolution PDEs. Evolution equations are important tool to study physical and natural phenomena. The prototype for non-linear advection-diffusion processes is the Burgers equation $v_t + (v^2)_x = \nu v_{xx}$ for a scalar function $v$ (a constant $\nu > 0$ is the kinematic viscosity). It serves as the simplest model equation for solitary waves, and is used for describing wave processes in gas and fluid dynamics [14].

A geometric flow (GF) of metrics on a manifold $M$ is a solution $g_t$ of an evolution equation

$$\partial_t g = S(g),$$

where a geometric functional $S$ (a symmetric $(0,2)$-tensor) is usually related to some kind of curvature. The theory of GFs is a new subject, of common interest in mathematics and physics. GFs (e.g., the Ricci flow and the Yamabe flow), correspond to dynamical systems in the infinite-dimensional space of all possible Riemannian metrics on a manifold.

The notion of the $D$-truncated $(r, k)$-tensor $\hat{S}$ ($r = 0, 1$) will be helpful:

$$\hat{S}(X_1, \ldots, X_k) = S(X_1^\perp, \ldots, X_k^\perp) \quad (X_i \in TM).$$

Rovenski and Walczak [11] introduced $D$-truncated flows of metrics on codimension-one foliations, depending on the extrinsic geometry of the leaves and posed the question:

**Given a foliation $(M, \mathcal{F})$ and a geometric property $(P)$, does there exist a $D$-truncated tensor $S$ such that solution metrics $g_t$ ($t \geq 0$) to $\partial_t g = \hat{S}(g)$ converge to a metric $g_\infty$ for which $\mathcal{F}$ enjoys $(P)$?**

Some of results in [11] were extended by the first author [13] for GFs related to parabolic PDEs, applications to the problem of prescribing the mean curvature function of a codimension-one foliation, and examples with harmonic and totally umbilical foliations are given.

Rovenski and Wolak [12] studied the $D$-conformal flow of metrics on a foliation with the speed proportional to the $\mathcal{F}$-divergence of $H$. Based on known long-time existence results for the heat flow they showed convergence of a solution to a metric for which $H = 0$; actually under some topological assumptions they prescribe the mean curvature $H$. The conditions in their results are rather different and seem to be stronger of known hypotheses to guarantee taughtness in non-constructive results. This makes sense because it is much harder to provide the good GF.

For $D$-conformal flows of metrics on foliations, we have $\hat{S}(g) = s(g) \hat{g}$, where $s(g)$ is a smooth function on the space of metrics on $M$, and the $D$-truncated metric tensor $\hat{g}$ is given by $\hat{g}(N, \cdot) = 0$ and $\hat{g}(X_1, X_2) = g(X_1, X_2)$ for all $X_i \in D, \ N \in D_\mathcal{F}$. By Lemma 2 in Section 3.2 $D$-conformal variations of a metric $g$ preserve the property “$\mathcal{F}$ is totally geodesic”.

In the paper we continue studying $D$-conformal GFs on a foliated manifold $(M, \mathcal{F})$ and, in order to prescribe the positive $S_{\text{mix}}$, introduce the following flow of totally geodesic metrics:

$$\partial_t g = -2 (S_{\text{mix}} - \|T\|^2 - \Phi) \hat{g}. \quad (2)$$

Here $\Phi : M \to \mathbb{R}$ is an arbitrary function constant along the leaves, it is used to normalize the flow equation in order to obtain the convergence of solution metrics $g_t$ as $t \to \infty$. For integrable distribution $D$, we have the PDE (2) with $T = 0$. 

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In order to prescribe the positive mixed scalar curvature, we examine the following.

**Question 3:** Given a Riemannian manifold \((M, g)\) with a totally geodesic foliation \(\mathcal{F}\), when do the solution metrics \(g_t\) of (2) converge as \(t \to \infty\) to a metric \(\tilde{g}\) for which \(\text{Sc}_{\text{mix}}\) is positive?

**Example 2.** (a) For a Hopf fibration \(\pi : S^{2n+1} \to \mathbb{C}P^n\) with fiber \(S^1\), the orthogonal distribution \(D\) is non-integrable while it is totally geodesic \((T \neq 0, b = 0)\). The standard metric on \(S^{2n+1}\) is a fixed point of (2) with \(\Phi = 0\), and we have \(\text{Sc}_{\text{mix}} = \|T\|^2 > 0\) (see Lemma 1 in Section 3.1).

(b) Let \((M^2, g_0)\) be a surface (i.e., \(\dim M = 2\)) foliated by geodesics, and \(K\) its Gaussian curvature. Since one-dimensional distribution \(D\) is integrable, we have \(T = 0\). Then one may take \(\Phi = 0\), and reduce (2) to the following form:

\[
\partial_t g = -2 K \tilde{g}.
\]

The known Cole-Hopf transformation reduces the non-linear Burgers equation to the linear heat equation. We observe that (3) yields both above mentioned PDEs for foliated surfaces. Let \(M^2 \subset \mathbb{R}^3\) be a smooth family of surfaces of revolution about the \(Z\)-axis,

\[
r(x, \theta, t) = [\rho(x, t) \cos \theta, \rho(x, t) \sin \theta, h(x, t)] \quad (0 \leq x \leq l, -\pi \leq \theta \leq \pi).
\]

Let the profile curves be the leaves of \(\mathcal{F}\) (geodesics). Denote by \(k\) the geodesic curvature of parallels (circles orthogonal to the leaves). The following properties are equivalent (see also Theorem 2):

(i) The induced metrics \(g_\beta\) (on \(M^2\)) are the solution of (3).

(ii) The distance \(\rho > 0\) from the profile curve to the axis satisfies the heat equation \(\rho_{t,xx} = \rho_{xx}\).

(iii) The geodesic curvature \(k\) of parallels satisfies the Burgers equation \(k_{t,xx} + (k^2)_{,x} = k_{xx}\).

**1.4. The structure of the paper.** Section 1 introduces the \(D\)-conformal flow of metrics and Section 2 collects main results concerning Questions 2–3. Section 3 represents \(D\)-conformal variations of geometric quantities and Section 4 proves main results and applications to foliated surfaces. Section 5 (Appendix) deals with linear parabolic PDEs on a closed Riemannian manifold (its results seem to be known, but for the convenience of a reader we prove them).

**2 Main results**

An important foundational step in the study of any system of evolutionary PDEs is to show short-time existence and uniqueness.

**Proposition 1.** Let \(\mathcal{F}\) be a totally geodesic foliation of a closed Riemannian manifold \((M, g_0)\). Then (2) has a unique (smooth along the leaves) solution \(g_t\) defined on a positive time interval \([0, \varepsilon)\).

We denote \(\nabla^F f := (\nabla f)^\mathcal{F}\). Given a vector field \(X\) and a function \(F\) on \(M\), define the functions using the derivatives along the leaves: the divergence \(\text{div}^\mathcal{F} X = \sum_{\alpha=1}^{n} g(\nabla_\alpha X, \varepsilon_\alpha)\) and the Laplacian \(\Delta^\mathcal{F} F = \text{div}^\mathcal{F}(\nabla^\mathcal{F} F)\). Remark that \(\nabla^\mathcal{F}, \text{div}^\mathcal{F}\) and \(\Delta^\mathcal{F}\) (i.e., along the leaves) are \(t\)-independent.

Notice the “linear algebra” inequality \(n \|b\|^2 \geq |H|^2\) with the equality when the distribution \(D\) is totally umbilical. Consider the following non-negative measure of “non-umbilicity” of \(D\):

\[
\beta_D := n^{-2} \left( n \|b\|^2 - |H|^2 \right).
\]

For \(p = 1\), we have \(\beta_D = n^{-2} \sum_{i < j} (k_i - k_j)^2\), where \(k_i\) are principal curvatures of \(D\), see [17].

By Proposition 1 in Section 3.2, (2) preserves \(\beta_D\) and the Schrödinger operator \(\mathcal{H} = -\Delta^\mathcal{F} - \beta_D\) id on the leaves. Recall that \(-\Delta^\mathcal{F} \geq 0\). Let \(\lambda_0\) be the smallest real eigenvalue of \(\mathcal{H}\) with a positive eigenfunction \(e_0\) of unit \(L^2\)-norm. Notice that \(\lambda_0\) is constant on the leaves, and if the leaves are compact then \(\lambda_0(x) \geq -\max_{F(x)} \beta_D\), where \(F(x)\) is the leaf through \(x\). To show this, define on \(F(x)\) the operator \(\mathcal{H} = -\Delta^\mathcal{F} - \left( \max_{F(x)} \beta_D \right)\) id. From \(\mathcal{H} \geq \mathcal{H} \geq -\left( \max_{F(x)} \beta_D \right)\) id the estimate of \(\lambda_0\) follows.
Denote by \([g_0]\) the \(D\)-conformal class of the metric \(g_0\) on a foliated manifold \((M,F)\). Certainly, \(\beta_D\) and the operator \(\mathcal{H}\) depend on \([g_0]\) only.

In Proposition 2 we observe that (2) yields the multidimensional Burgers equation (for the mean curvature vector \(H\)) on any leaf, and has a single-point global attractor when the leaf is compact.

**Proposition 2.** Let \(F\) be a totally geodesic foliation of a Riemannian manifold \((M,g_0)\), and \(H_0 = -n\nabla^F(\log u_0)\) for some positive function \(u_0\) on \(M\). If the metrics \(g_t\) \((t \geq 0)\) satisfy (4) then

(i) The mean curvature vector of \(D\) w.r.t. \(g_t\) is a unique solution of the forced Burgers PDE

\[
\partial_t H + \nabla^F g(H,H) = n \nabla^F (\text{div}^F H) - n^2 \nabla^F \beta_D,
\]

(ii) \(H\) converges exponentially as \(t \to \infty\) on compact leaves to the vector field \(\bar{H} = -n\nabla^F(\log e_0)\).

The central result of the work is Theorem 1 (and its applications), where for convergence of solution \(g_t\) as \(t \to \infty\) on a fiber bundle, the flow of metrics (2) is normalized (taking \(\Phi = n \lambda_0\)).

**Theorem 1.** Let \(\pi : (M,g_0) \to B\) be a fibration with compact totally geodesic fibers, and \(H_0 = -n\nabla^F(\log u_0)\) for a positive function \(u_0\) on \(M\) (the potential). Then

(i) The PDE (2) admits a unique global smooth solution \(g_t\) \((t \geq 0)\) on \(M\).

(ii) Let \(\Phi = n \lambda_0\). Then the metrics \(g_t\) converge as \(t \to \infty\) to the limit metric \(\bar{g}\). There exists real \(\tilde{C} > 0\) depending on \([g_0]\) such that \(\text{Sc}_{\text{mix}} > 0\) when \(\|T(x)\|_{g_t}^2 > \tilde{C} \max_{\forall x} F(x) \beta_D\) for all \(x \in M\).

**Corollary 1.** In conditions of Theorem 1 we have the following:

(a) If \(\nabla^F \beta_D = 0\) then \(F\) is \(\bar{g}\)-transversely harmonic and \(\text{Sc}_{\text{mix}} \geq 0\), while \(\text{Sc}_{\text{mix}} > 0\) on \(U_T\).

(b) If \(\beta_D = 0\) then \(F\) is \(\bar{g}\)-Riemannian and \(\text{Sc}_{\text{mix}}\) is constant on the fibers; moreover, if \(D\) is integrable, then \(\text{Sc}_{\text{mix}} = 0\) and \(M\) is locally the product with respect to \(\bar{g}\) (the product globally for simply connected \(M\)).

**Remark 2.** The condition \(H = \nabla^F(\log u_0)\) of Theorem 1 and Proposition 2 is satisfied for twisted products (see Theorem 2 in what follows).

We consider applications of Theorem 1 to the twisted products. The particular case of surfaces of revolution is studied in Section 3.3 (see also Example 3 in Section 3.3).

**Definition 1 (see 3).** Let \((M_1,g_1)\) and \((M_2,g_2)\) be Riemannian manifolds, and \(f \in C^\infty(M_1 \times M_2)\) a positive function. The twisted product \(M_1 \times_f M_2\) is the manifold \(M = M_1 \times M_2\) with the metric \(g = (f^2g_1) \oplus g_2\). If the warping function \(f\) depends on \(M_2\) only, then we have a warped product.

**Remark 3.** The fibers \(M_1 \times \{y\}\) of a twisted product are umbilical with the mean curvature vector \(H = -\nabla^F(\log f)\), while \(\{x\} \times M_2\) are totally geodesic. The fibers of \(M_1 \times \{y\}\) have

(a) constant mean curvature if and only if \(\|\nabla^F(\log f)\|\) is a function of \(M_1\), and

(b) parallel mean curvature vector if and only if \(f = f_1f_2\) for some \(f_i : M_i \to \mathbb{R}^+\) \((i = 1,2)\).

If on a simply connected complete Riemannian manifold \((M,g)\) two orthogonal foliations with the properties (a)–(b) are given, then \(M\) is a twisted product, see 3.

If \(D\) is integrable and totally umbilical (i.e., \(\beta_D = T = 0\)), then \(\Phi = 0\), hence (2) is reduced to the \(D\)-conformal flow of metrics with the speed proportional to \(\text{Sc}_{\text{mix}}\): \(\partial_t g = -2\text{Sc}_{\text{mix}} \bar{g}\).

**Theorem 2.** Let \((M,g_t) = M_1 \times_f M_2\) be a family of twisted products of Riemannian manifolds \((M_1,g_1)\) and \((M_2,g_2)\). Then the following properties are equivalent:

(i) The metrics \(g_t\) satisfy the evolution equation (2).

(ii) The mean curvature vector of fibers \(M_1 \times \{y\}\) satisfies the Burgers type PDE

\[
\partial_t H + \nabla^F g(H,H) = n \nabla^F (\text{div}^F H).
\]

(iii) The warping function satisfies the heat equation \(\partial_t f = n \Delta^F f\).

**Corollary 2.** Let \(M_1 \times_f M_2\) be a twisted product of closed Riemannian manifolds \((M_1,g_1)\) and \((M_2,g_2)\) for a positive warping function \(f \in C^\infty(M_1 \times M_2)\). Then (2) admits a unique smooth solution \(g_t\) \((t \geq 0)\), consisting of twisted product metrics on \(M_1 \times_f M_2\). As \(t \to \infty\), the metrics \(g_t\) converge to the metric \(\bar{g}\) of the product \((M_1,f^2g_1) \times (M_2,g_2)\), where \(f(x) = \int_{M_2} f(0,x,y) \, dy\).
3 Auxiliary results

3.1 Preliminaries

For the convenience of a reader, we recall some definitions.

Definition 2 (see [10]). A family $\mathcal{F} = \{F_\alpha\}_{\alpha \in B}$ of connected subsets of a manifold $M^{n+p}$ is said to be a $p$-dimensional foliation, if 1) $\bigcup_{\alpha \in B} F_\alpha = M^{n+p}$, 2) $\alpha \neq \beta \Rightarrow F_\alpha \cap F_\beta = \emptyset$, 3) for any point $x \in M$ there exists a $C^r$-chart $\varphi_x : U_x \to \mathbb{R}^p$ such that $y \in U_x$, $\varphi_x(y) = 0$, and if $U_x \cap F_\alpha \neq \emptyset$ the connected components of the sets $\varphi_x(U_x \cap F_\alpha)$ are given by equations $x_{p+1} = c_{p+1}, \ldots, x_{n+p} = c_{n+p}$, where $c_j$’s are constants. The sets $F_\alpha$ are immersed submanifolds of $M$ called leaves of $\mathcal{F}$.

Definition 3. Let $F$ and $B$ be smooth manifolds. A fiber bundle over $B$ with fiber $F$ is a smooth manifold $M$, together with a surjective submersion $\pi : M \to B$ satisfying a local triviality condition: For any $x \in B$ there exists an open set $U$ in $B$ containing $x$, and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$ (called a local trivialization) such that $\pi = \pi_1 \circ \phi$ on $\pi^{-1}(U)$, where $\pi_1(x, y) = x$ is the projection on the first factor. The fiber at $x$, denoted by $F_x$, is the set $\pi^{-1}(x)$, which is diffeomorphic to $F$ for each $x$. We call $M$ the total space, $B$ the base space and $\pi$ the projection.

The Weingarten operator $A_N$ of $D$ w. r. t. to $N \in DF$ and the operator $T^d_N$ are given by

$$g(A_N(X), Y) = g(b(X, Y), N), \quad g(T^d_N(X), Y) = g(T(X, Y), N), \quad (X, Y \in D).$$

The co-nullity operator $C : DF \times D \to D$ (see for example [10]) is defined by

$$C_N(X) = -(\nabla_X N)\perp \quad (X \in D, \; N \in DF).$$

Hence $C_N = A_N + T^d_N$ – the linear operator on orthogonal distribution $D$. The equality $C = 0$ means that $D$ is integrable and the integral manifolds are totally geodesic in $M$.

Define the self-adjoint $(1, 1)$-tensor $R_N = R(N, \cdot)N$ ($N \in DF$) on $D$, called Jacobi operator. Next lemma represents [11] in equivalent form as [7].

Lemma 1. For a totally geodesic foliation, we have

$$\nabla_X C_N = C_N^2 + R_N \quad (N \in DF),$$

$$\text{Sc}_{\text{mix}} - \|T\|^2 = \text{div}^F H - g(H, H)/n - n \beta_D. \quad (7)$$

Proof. For Riccati equation (6) see [10]. Substituting $C_N = A_N + T^d_N$ into (6) and taking the symmetric and skew-symmetric parts, yield a pair of equations

$$\nabla_X A_N = A_N^2 + (T^d_N)^2 + R_N, \quad \nabla_X T^d_N = A_N T^d_N + T^d_N A_N. \quad (8)$$

By (5), we also have

$$-N(\|T\|^2) = 2 \text{Tr}(T^d_N(\nabla_N T^d_N)) = 4 \text{Tr}(A_N(T^d_N)^2). \quad (9)$$

Let $e_i$ be a local orthonormal frame of $D$. Since the leaves are totally geodesic, we have $\nabla_N e_i \in D$ for any vector $N \in DF$. Then, using $g(A_N(e_i), \nabla_N e_i) = 0$, we find

$$N(\text{Tr} A_N) = \sum_i \left[g(\nabla_N A_N(e_i), e_i) + g(A_N(\nabla_N e_i), e_i) + g(A_N(e_i), \nabla_N e_i)\right] = \text{Tr}(\nabla_N A_N).$$

Hence, the contraction of (5) over $D$ yields the formula

$$N(\text{Tr} A_N) = \text{Tr}(A_N^2) + \text{Tr}((T^d_N)^2) + \sum_{j=1}^n K(e_j, N) \quad (10)$$

for any unit vector $N \in DF$. Note that $\text{Tr} A_N = g(H, N)$. We have

$$\sum_{\alpha=1}^p \varepsilon_\alpha(\text{Tr} A_{\alpha N}) = \text{div}^F H, \quad \sum_{\alpha=1}^p \text{Tr}((T^d_{\alpha N})^2) = -\|T\|^2, \quad \sum_{\alpha=1}^p \text{Tr}(A^2_{\alpha N}) = \|b\|^2 = n \beta_D + \frac{1}{n} g(H, H).$$

Hence, the contraction of (10) over $D_F$ yields (7).
Remark 4. By the Divergence Theorem, from (17) and \( \text{div} \, H = \text{div}^F H - g(H, H) \) we obtain
\[
n \int_M \beta_D \, d \text{vol} = (1 - \frac{1}{n}) \int g(H, H) \, d \text{vol} - \int (\text{Sc}_{\text{mix}} - \|T\|^2) \, d \text{vol} \geq - \int \text{Sc}_{\text{mix}} \, d \text{vol}
\]
on a closed \( M \). Hence the inequality \( \text{Sc}_{\text{mix}} < 0 \) yields that \( \beta_D \) is somewhere positive.

3.2 \( D \)-truncated families of metrics

Since the difference of two connections is a tensor, \( \partial_t \nabla^t \) is a \((1,2)\)-tensor on \((M, g_t)\). Differentiating the known formula for the Levi-Civita connection with respect to \( t \) yields
\[
2 g_t((\partial_t \nabla^t)(X, Y), Z) = (\nabla_X^t S)(Y, Z) + (\nabla_Y^t S)(X, Z) - (\nabla_Z^t S)(X, Y)
\]
for all \( t \)-independent vector fields \( X, Y, Z \in \Gamma(TM) \), see (11).

Lemma 2. \( D \)-truncated variations of metrics preserve the property “\( F \) is totally geodesic”.

Proof. Let \( g_t \) \((t \geq 0)\) be a family of metrics on a foliation \((M, F)\) such that the tensor \( S_t = \partial_t g_t \) is \( D \)-truncated. We claim that the second fundamental tensor of \( F \) is evolved as \( \partial_t b^\perp = -S^2 \circ b^\perp \).

Using (11) and that the tensor \( S \) is \( D \)-truncated, we find for \( X \in D \) and \( \xi, \eta \in D_F \),
\[
2 g_t(\partial_t b^\perp(\xi, \eta), X) = g_t((\partial_t \nabla^t \eta) \circ \partial_t, \nabla^t \xi, X)
\]
\[
= (\nabla^t \xi S)(X, \eta) + (\nabla^t \eta S)(X, \xi) - (\nabla^t S)(X, \xi) - (\nabla^t S)(X, \xi)
\]
\[
= -2 S(b^\perp(\xi, \eta), X).
\]
From this the claim follows. By the theory of ODEs, if \( b^\perp = 0 \) at \( t = 0 \) then \( b^\perp = 0 \) for all \( t \geq 0 \). \( \square \)

Let a family of metrics \( g_t \) \((0 \leq t < \varepsilon)\) on \((M, F)\) satisfy the equality
\[
\partial_t g = s(g) \hat{g},
\]
where \( s(g) \) can be considered as a \( t \)-dependent function on \( M \). Notice that the volume form \( \text{vol} \)
of \( g_t \) is evolved as \( (d/dt) \text{vol} = (n/2) s(g_t) \text{vol}_{t=0} \), see (11).

The proof of the next lemma is based on (11) with \( S(g) = s(g) \hat{g} \).

Lemma 3 (see (11) and (12)). For a totally geodesic foliation with (12) and \( N \in D_F \), we have
\[
\partial_t b = s b - \frac{1}{2} \hat{g} \nabla^F s, \quad \partial_t T = 0,
\]
\[
\partial_t A_N = -\frac{1}{2} N(s) \hat{1}, \quad \partial_t T^k_N = -s T^k_N, \quad \partial_t C_N = -\frac{1}{2} N(s) \hat{1} - s T^k_N, \quad (13)
\]
\[
\partial_t (\|T\|^2) = -2 s \|T\|^2, \quad \partial_t H = -\frac{n}{2} \nabla^F s, \quad \partial_t (\text{div}^F H) = -\frac{n}{2} \Delta F s. \quad (14)
\]

Remark 5. For any function \( f \in C^1 \) and \( N \in D_F \), using \((\partial_t g)(\cdot, N) = 0\), we find
\[
g(\nabla^F(\partial_t f), N) = N(\partial_t f) = \partial_t N(f) = \partial_t g(\nabla^F f, N) = g(\partial_t (\nabla^F f), N).
\]

Lemma 3 and \( \nabla^F(\partial_t \log \|T\|) = -\nabla^F s = 2 \partial_t H \) yield the following conservation law for the evolution (12) on the domain \( U_T \): \( \partial_t \left( 2 H - n \nabla^F \log \|T\| \right) = 0 \).

Proposition 3 (Conservation of “non-umbilicity”). Let \( g_t \) \((t \geq 0)\) be a \( D \)-conformal family of metrics on a foliated manifold \((M, F)\). Then the function \( \beta_D \) doesn’t depend on \( t \).

Proof. Using Lemma 3 we calculate
\[
\partial_t \|b\|_g^2 = \partial_t \sum_{\alpha} \text{Tr}(A_a^2) = 2 \sum_{\alpha} \text{Tr}(A_{\alpha}^2 \partial_\alpha A_{\alpha}) = -\sum_{\alpha} \varepsilon_\alpha(s) \text{Tr} A_{\alpha} = -g(\nabla s, H),
\]
\[
\partial_t g(H, H) = s \hat{g}(H, H) + 2g(\partial_t H, H) = -n g(\nabla s, H).
\]
Hence, \( n \partial_t \beta_D = \partial_t \|b\|_g^2 - \frac{1}{n} \partial_t g(H, H) = 0 \). \( \square \)
If one has a solution \( u_0 \) to a given non-linear PDE, it is possible to linearise the equation by considering a smooth family \( u = u(t) \) of solutions with a variation \( v = \partial_t u |_{t=0} \). Differentiation the PDE by \( t \), yields a linear PDE in terms of \( v \). The next lemma concerns the linearisation of the differential operator \( g \to -2(Sc_{\text{mix}} - \|T\|^2 - \Phi) \dot{g} \), see [2].

**Lemma 4.** For a totally geodesic foliation with (12), the mixed scalar curvature is evolved by Lemma 4.

**Proof.** Differentiating (7) by \( t \), and using \( \partial_t \beta_D = 0 \) of Proposition 3 and \( \dot{g}(H,H) = 0 \), we obtain

\[
\partial_t(Sc_{\text{mix}} - \|T\|^2) = \partial_t(\text{div}^F H) - \frac{2}{n} g(\partial_t H, H).
\]

By the above, using Lemma 3 we rewrite the above equation as (16).

The following proposition shows that (12) preserves certain geometric properties of \( D \).

**Proposition 4.** Let \( F \) be a totally geodesic foliation of a Riemannian manifold \( (M, g_0) \), and \( g_t \) be a family of Riemannian metrics (12) on \( M \). If \( D \) is either totally umbilical, or harmonic, or totally geodesic with respect to \( g_0 \) then \( D \) is the same for all \( g_t \).

**Proof.** If \( D \) is \( g_0 \)-umbilical then we have \( b = H \dot{g} \) at \( t = 0 \), where \( H \) is the mean curvature vector of \( D \). Applying to (13) the theorem on existence/uniqueness of a solution of ODEs, we conclude that \( b_t = H_t \dot{g}_t \) for all \( t \), for some \( H_t \in \Gamma(D_F) \). Tracing this, we see that \( H_t \) is the mean curvature vector of \( D \) w.r.t. \( g_t \), hence \( D \) is umbilical for any \( g_t \). The proof of other cases is similar.

Assume that \( \int_0^\infty u_0(t) \, dt < \infty \), where \( u_0(t) = \sup_M |s(g_t)|_{g(t)} \). Then the metrics (12) are uniformly equivalent, i.e., there exists a constant \( c > 0 \) such that \( c^{-1} \|X\|^2_{g_0} \leq \|X\|^2_{g_t} \leq c \|X\|^2_{g_0} \) for all points \( (x,t) \in M \times [0, \infty) \) and all vectors \( X \in T_x M \).

We will use the following condition for proving convergence of evolving metrics, see [2].

**Proposition 5.** \( \pi : M \to B \) be a fiber bundle with compact totally geodesic fibers of a closed Riemannian manifold \( (M, g_0) \). Suppose that \( g_t (t \geq 0) \) is the solution of (12). Define functions \( u_j(t) = \sup_{M} |(\nabla^F)^j s(g_t)|_{g(t)} \) and assume that \( \int_0^\infty u_j(t) \, dt < \infty \) for all \( j \geq 0 \). Then, as \( t \to \infty \), the metrics \( g_t \) converge in \( C^\infty \)-topology to a smooth Riemannian metric.

### 3.3 Evolving of geometric quantities

By Proposition 3 the measure of non-umbilicity of \( D \) (see Introduction), is preserved by \( (2) \).

From Lemmas 3 and 4 with \( s = -2(Sc_{\text{mix}} - \|T\|^2 - \Phi) \), we obtain the following.

**Lemma 5.** For a totally geodesic foliation with (2), we have \( \partial_t T = 0 \) and

\[
\begin{align*}
\partial_t b &= -2(Sc_{\text{mix}} - \|T\|^2 - \Phi)b + \dot{g} \nabla^F (Sc_{\text{mix}} - \|T\|^2), \\
\partial_t(Sc_{\text{mix}} - \|T\|^2) &= n \Delta_F (Sc_{\text{mix}} - \|T\|^2) - 2 \nabla_H (Sc_{\text{mix}} - \|T\|^2), \\
\partial_t(\|T\|^2) &= 4 (Sc_{\text{mix}} - \|T\|^2) \|T\|^2.
\end{align*}
\]

**Example 3** \((n = 1)\). Consider a surface \((M^2, g)\) with a geodesic unit vector field \( N \). Let \( k, K \in C^2(M) \) be the curvature of \( N \)-curves and the gaussian curvature of \( M^2 \), respectively. We have

\[
C(X) = k \cdot X, \quad R_N(X) = K \cdot X \quad \text{for} \quad X \perp N.
\]

Take \( \Phi = 0 \). The equation (2) takes the form (3). By Lemma 5 we obtain the PDEs

\[
\begin{align*}
\partial_t K &= N(N(K)) - 2k N(K), \\
\partial_t k &= N(K).
\end{align*}
\]
For \( n = 1 \), (16) reads as the Riccati equation
\[
N(k) = k^2 + K. \tag{18}
\]
Substituting \( K \) from (18) into (17), we obtain the Burgers equation
\[
\partial_t k = N(N(k)) - N(k^2), \tag{19}
\]
it also follows from (4) with \( \beta_D = 0 \). If the solution \( k_t \) of (19) is known, then by (18) we find \( K_t = N(k_t) - k_t^2 \). Finally, we recover the metric by \( \hat{g}_t = \hat{g}_0 \exp(-2\int_0^t K_t \, dt) \).

4 Proof of main results and applications

4.1 The behavior of the mean curvature vector \( H \)

Let \((F, g)\) be a Riemannian manifold, e.g., a leaf of \( F \), or a fiber of \( \pi : M \to B \).

Example 4. Consider the Burgers equation for a potential vector field \( H \) on \((F, g)\)
\[
\partial_t H + a \nabla g(H, H) = \nu \nabla (\text{div } H)
\]
where \( a \in \mathbb{R} \) and \( \nu > 0 \). Using the scaling of independent variables \( x = z a^{-1} \) and \( t = \tau \frac{a^2}{\nu} \), the above equation reduces to the normalized Burgers equation
\[
\partial_t H + \frac{1}{2} \nabla g(H, H) = \nabla (\text{div } H). \tag{20}
\]
To show this, we compute
\[
\partial_t H = \left( a^2/\nu \right) \partial_\tau H, \quad \nabla_z H = (a/\nu) \nabla_x H, \quad (\nabla \text{div})_z H = (a^2/\nu^2) (\nabla \text{div})_x H.
\]
Solutions of (20) correspond to solutions of the homogeneous heat equation on \((F, g)\),
\[
\partial_t u = \Delta u, \tag{21}
\]
using the well-known Cole-Hopf transformation \( H = -2 \nabla (\log u) \).

The forced (or inhomogeneous) Burgers equation, see [5], [14], has attached some attention as an analogue of the Navier-Stokes equations. For a potential vector field \( H \) and a function \( f \in C^\infty(F) \), it can be viewed as the following equation on \( F \):
\[
\partial_t H + \frac{1}{2} \nabla g(H, H) = (\nabla \text{div}) H - 2 \nabla f. \tag{22}
\]
Since the function \( f \) is defined modulo a constant, one may assume \( f \geq 0 \).

By the maximum principle, see [2], we have the following.

Lemma 6. Let \( f \in C^1(F) \) be an arbitrary function and \( u_0 \in C^2(F) \) on a closed Riemannian manifold \((F, g)\). Then the Cauchy’s problem for the heat equation with a linear reaction term
\[
\partial_t u = \Delta u + f u, \quad u(\cdot, 0) = u_0, \tag{23}
\]
has a unique solution \( u(\cdot, t) \) (\( t \geq 0 \)), and if \( u(\cdot, 0) \geq c \) for some \( c \in \mathbb{R} \) then \( u(\cdot, t) \geq c \) for all \( t \).

Let \( \lambda_0 \) be the smallest eigenvalue (with a positive eigenfunction \( e_0 \)) of the Schrödinger operator \( \mathcal{H} = -\Delta - f \text{id} \) on \( F \). One may find the estimate \( \lambda_0 \geq -\max f \).
Proposition 6. Let \((F,g)\) be a closed Riemannian manifold, and \(f \in C^\infty(F)\).

(a) If \(u(x,t)\) is any positive solution of the linear PDE \((22)_1^\) on \(F\) then the vector field \(H = -2\nabla(\log u)\) solves \((22)\). Every solution of \((22)\) converges absolutely and uniformly for all \(t\). Hence \(\text{div} \bar{\mathbf{j}}\) comes by this way.

(b) Let \(u(\cdot,t)\) \((t \geq 0)\) be a global solution of \((22)\) on \(F\) with \(u_0 > 0\) and \(f \geq 0\). Then \(u(\cdot,t) > 0\) for all \(t \geq 0\), and the solution vector-field \(H = -2\nabla(\log u)\) of \((22)\) approaches exponentially as \(t \to \infty\) to a smooth vector-field \(\bar{H} = -2\nabla(\log e_0)\) on \(F\) – a unique conservative solution of the PDE

\[
\text{div} \bar{H} = g(\bar{H}, \bar{H})/2 + 2(f + \lambda_0).
\] (24)

Proof. (a) Notice that \(\partial_t \circ \nabla = \nabla \circ \partial_t\). We rewrite \((22)\) in a form of a conservation law,

\[
\partial_t H = \nabla \left( \text{div} H - g(H,H)/2 - 2f \right).
\]

This can be regarded as the compatibility condition for a function \(\psi\) to exist, such that

\[
\nabla \psi = H, \quad \partial_t \psi = \text{div} H - g(H,H)/2 - 2f.
\] (25)

Substituting \(H\) from \((25)_1^\) into \((25)_2^\), and using the definition \(\Delta = \text{div} \nabla\) for functions, we obtain the following PDE:

\[
\partial_t \psi + \frac{1}{2} g(\nabla \psi, \nabla \psi) = -\psi - 2f - \Delta \psi + 2f = -\frac{2}{u} (\partial_t u - \Delta u - fu).
\]

(b) Using Fourier method and Theorem 5 (Section 3), we represent a solution of \((24)\) as series

\[
u(\cdot, t) = \sum_{j \geq j_0} c_j e^{-\lambda_j t} e_j, \quad c_{j_0} \neq 0
\] (26)

by eigenfunctions of \(H\). The terms with \(e^{-\lambda_j t}\) in \((26)\) dominate as \(t \to \infty\), and can be represented in one-term form as \(\tilde{c} e^{-\lambda_0 t}\tilde{e}\), where \(\tilde{c} \neq 0\) and the eigenfunction \(\tilde{e}\) (for \(\lambda_0\)) has unit \(L_2\)-norm. By the maximum principle (see Lemma 2) we conclude that \(u > 0\) for all \(t \geq 0\). Hence \(\tilde{c} > 0\).

The eigenspace, corresponding to \(\lambda_0\), is one-dimensional, see [7, Theorem 4.8]. Moreover, from [6, Chapt. 2, Theorem 2.13] we conclude that \(j_0 = 0\), hence \(\lambda_{j_0} = \lambda_0\) and \(\tilde{e} = e_0\). Since the series \((26)\) converges absolutely and uniformly for all \(t\), there exists the limit vector field \(\lim_{t \to \infty} H(\cdot,t) = \bar{H}\),

\[
\bar{H}(x) = -2 \lim_{t \to \infty} \frac{\nabla u(x,t)}{u(x,t)} = -2 \sum_{j \geq 0} c_j e^{-\lambda_j t} \nabla e_j(x)/\sum_{j \geq 0} c_j e^{-\lambda_j t} e_j(x) = -2 \frac{\nabla e_0(x)}{e_0(x)},
\]

see [6] in Section 4 and convergence to \(\bar{H}(x)\) is exponential. By the above we find

\[
\text{div} \bar{H} = -2 \Delta (\log e_0) = -2 (\Delta e_0)/e_0 + 2 g(\nabla e_0, \nabla e_0)/g(e_0, e_0).
\]

Hence \(\text{div} \bar{H} = g(\bar{H}, \bar{H})/2 = -2 (\Delta e_0)/e_0 = 2(f + \lambda_0)\), that proves \((24)\).

To prove uniqueness, assume the contrary, that \((24)\) has another conservative solution \(\tilde{H} = \nabla(-2 \log \tilde{e}_0)\), where \(\tilde{e}_0\) is a positive function of unit \(L_2\)-norm. Next, we calculate \((24)\):

\[
\text{div} \tilde{H} = -\frac{1}{2} g(\tilde{H}, \tilde{H}) = -2 (f(x) + \lambda_0) = -\frac{2}{\tilde{e}_0} [\Delta \tilde{e}_0 + (f + \lambda_0) \tilde{e}_0]
\]

and get \(\Delta \tilde{e}_0 + (f + \lambda_0) \tilde{e}_0 = 0\). Since \(\lambda_0\) is a simple eigenvalue of \(H\), we have \(\tilde{e}_0 = e_0\) and \(\bar{H} = \tilde{H}\). \(\square\)
4.2 Proofs

Proof. (of Proposition 1) Let \( g = g_0 + h \), where \( h = s \hat{g}_0 \) and \( s \in C^1(M) \). Notice that \( \nabla^F \hat{g}_0 = 0 \). By Lemma 4, the linearization of (2) at \( g_0 \) is the linear PDE on the leaves:

\[
\partial_t s = n \Delta_F s - 2 g(\nabla s, H(g_0)) - 2 (\text{Sc}_{\text{mix}} - \|T\|^2 - \Phi)_{g_0} s, -\|T\|^2 - n \lambda_0)_{g_0} h.
\]

and \( s|_{t=0} \) is bounded. The result follows from the theory of linear parabolic PDEs.

Proof. (of Proposition 2) By Proposition 1, there exists a local solution \( g_t \) \( (0 \leq t < \varepsilon) \) to (2).

(i) By (15) of Lemma 3 with \( s = -2(\text{Sc}_{\text{mix}} - \|T\|^2 - \Phi) \) and using (7), we obtain (4) for \( H \).

(ii) As in the proof of Proposition 6, a unique global solution \( u \) admits a unique solution \( u > 0 \) for all \( t \to \infty \) as \( t \to \infty \) to \( \partial_t u = n \Delta_F u + n \lambda_0 \) for \( t > 0 \). Hence there exists a unique global solution \( u \) to (2).

Proof. (of Theorem 1) (i) By Proposition 2(i), the Cauchy’s problem for (24) with \( u(\cdot, 0) = u_0 \) admits a unique solution \( u(x, t) \) \( (t \geq 0) \) on any fiber. By the maximum principle (see Lemma 6) we conclude that \( u > 0 \) for all \( t \geq 0 \). Hence there exists a unique global solution \( g_t \) to (2).

(ii) Let \( \Phi = n \lambda_0 \). By Proposition 2(ii), the limit vector field \( \tilde{H} \) depends only on \( \|g_0\| \), and is the unique stationary solution of (24):

\[
\text{div}^F \tilde{H} = g(\tilde{H}, \tilde{H})/n + n (\lambda_D + \lambda_0).
\]

We have \( \lambda_0 \geq -\max_{\tilde{M}} \beta_D \) and \( \nabla^F \lambda_0 = 0 \). By Lemma 1 and (28), we obtain

\[
\lim_{t \to \infty} (\text{Sc}_{\text{mix}} - \|T\|^2) = \text{div}^F \tilde{H} - g(\tilde{H}, \tilde{H})/n - n \beta_D = n \lambda_0.
\]

Since the convergence \( \text{Sc}_{\text{mix}} - \|T\|^2 \to n \lambda_0 \) as \( t \to \infty \) has the exponential velocity \( e^{(\lambda_0 - \lambda_1) t} \), by Proposition 1 a unique global solution \( g_t \) of (2) converge in \( C^\infty \)-topology as \( t \to \infty \) to a smooth Riemannian metric \( \tilde{g} \). By the above, \( \|T\|_t \geq c_2 \|T\|_0 \) for some \( c_2 \). Thus, if \( \tilde{C} = n^2 c_4 \) then

\[
\text{Sc}_{\text{mix}} = \|T\|_\tilde{g}^2 + n \lambda_0 \geq c^4 \|T\|_0^2 - n^2 \max_{F(x)} \beta_D > 0.
\]

By \( \partial_t T = 0 \), see (13) 2, we have \( \tilde{T}(q) \neq 0 \) for \( q \in U_T \) (remark that \( U_T \) is \( t \)-independent).

Proof. (of Corollary 1) (a) By the above, if \( \nabla^F \beta_D = 0 \) then \( e_0 = \text{const} \) on the fibers and \( \lambda_0 = -\beta_D \), hence \( \tilde{H} = 0 \). (b) In particular, \( \lambda_0 = 0 \) when \( \beta_D = 0 \), hence \( \tilde{b} = 0 \). By (4) and the conditions, we have \( \nabla^F \|T\|^2_{\tilde{g}} = 0 \). Since \( \text{Sc}_{\text{mix}} = \|T\|^2_{\tilde{g}} \), the mixed scalar curvature of \( \tilde{g} \) is constant on the fibers. For \( T = 0 \), \( M \) splits along \( D \) and \( D_F \) (de-Rham decomposition).
Proof. (of Theorem 2) By Proposition 1 the flow (2) preserves the twisted product structure. (i) \(\Rightarrow\) (ii), (iii): (other implications can be shown similarly). By Proposition 2 with \(\beta_D = 0\), the mean curvature \(H\) of the fibers \(M_1 \times \{y\}\) satisfies (3). By (27) with \(\beta_D = 0\) and \(H = -n \nabla^2 f(\log f)\), we obtain \(\partial_t f = n \Delta f\), that is the heat equation for the function \(f > 0\) on the fibers. \(\square\)

Proof. (of Corollary 2) We apply Theorem 2 for the fiber bundle \(\pi : M_1 \times M_2 \to M_1\) with totally geodesic fibers \(F_x = \{x\} \times M_2\) and the potential function \(\psi_0 = -\log f\) (at \(t = 0\)). As in the proof of Theorem 2 (see also Section 4.1), we reduce (3) for \(H\) to the heat equation for \(f\) along the fibers, and conclude that \(H = 0\) for the limit metric \(g\). Since the canonical foliation \(M_1 \times \{y\}\) is \(g\)-umbilical, by the above we have \(\bar{b} = 0\) (i.e., \(M_1 \times \{y\}\) is \(g\)-totally geodesic). Thus, \(M_1 \times M_2\) is the metric product with respect to \(g = \hat{f}^2 g_1 \times g_2\) (de-Rham decomposition). \(\square\)

4.3 Applications to surfaces

The results of the section can be easily generalized for hypersurfaces of revolution.

Example 5. Let \(\pi : M^2 \to B\) be a fiber bundle of a two-dimensional torus \((M^2, g_0)\) with Gaussian curvature \(K\), and the fibers are closed geodesics. Let the geodesic curvature \(k\) of orthogonal (to fibers) curves obeys \(k = N(\psi_0)\) for a smooth function \(\psi_0\) on \(M^2\). By Theorem 4 the equation (3) admits a unique solution \(g_t\) converging as \(t \to \infty\) to a flat metric, and the fibers of \(\pi\) compose a rational linear foliation.

Metric on a surface of revolution is a special class of warped products (see Definition 1). The equation (3) on a surface of revolution provides fruitful geometrical interpretation of the classical relation between Burgers and heat equations. We are looking for a one-parameter family of surfaces of revolution, which are foliated by profile curves, and the induced metric \(g_t\) obeys (3). The profile of \(M_0^2\) parameterized as in Example 2(b) is \(XZ\)-plane curve \(\gamma_0 = [\rho(\cdot), 0, h(\cdot, 0)]\) (the fiber), and \(\theta\)-curves are circles in \(\mathbb{R}^3\). Let \(x\) be the natural parameter of \(\gamma_t = \tau(\cdot, t)\), i.e.,

\[
(\rho_x)^2 + (h_x)^2 = 1.
\]

Thus \(N = r_x\) is the unit normal to \(\theta\)-curves on \(M_0^2\). The geodesic curvature, \(k\), of \(\theta\)-curves obeys Burgers equation, while the radius \(\rho\) of \(\theta\)-curves (as Euclidean circles) satisfies the heat equation, see Example 2(b): both functions are related by the Cole-Hopf transformation \(k = -(\log \rho)_x\).

It is known that the gaussian curvature is \(K = -\rho_{xx}/\rho\), and one may assume \(\rho > 0\) for \(t = 0\). Notice that (18), \(k_x = k^2 + K\) for \(t\), is satisfied. The induced metric on \(M_t\) has the rotational symmetric form \(g_t = dx^2 + \rho^2 d\theta^2\). The equation PDE for metrics reads as \(\partial_t g = -2 K \hat{g} \implies \partial_t \rho = -K \rho\). Thus, (4) yields the Burgers equation (19) for \(k\) and the heat equation for \(\rho\),

\[
k_t = k_{xx} - (k^2)_x, \quad \rho_t = \rho_{xx}.
\]

Differentiating (30), by \(x\), we find \((\rho_x)_t = (\rho_x)_{xx}\). Since \(|\rho_x| \leq 1\) holds for \(t \geq 0\). When such a solution \(\rho\) \((t \geq 0)\) is known, we find \(h\) from (29) as \(h = \int \sqrt{1 - (\rho_x)^2} \, dx\). For example, suppose that the boundary conditions are \(\rho(0, t) = \rho_0, \rho(l, t) = \rho_1, \rho(0, t) = 0_t (t \geq 0)\), where \(\rho_1 > \rho_0 > 0\). By the theory of heat equation, the solution \(\rho\) approaches as \(t \to \infty\) to a linear function \(\rho = x\rho_0 + (l - x)\rho_1 > 0\). Also, \(h\) approaches as \(t \to \infty\) to a linear function \(h = xz_0 + (l - x)z_1\), where \(z_1\) may be calculated from the equality \((\rho_1 - \rho_0)^2 + (z_1 - z_0)^2 = l^2\). The curves \(\gamma\) are isometric each to other for all \(t\) (with the same arc-length parameter \(x\)). The limit curve \(\lim_{t \to \infty} \gamma_t = \bar{\gamma} = [\bar{\rho}, \bar{h}]\) is a line segment of length \(l\). Thus, \(M_t\) approach as \(t \to \infty\) to the flat surface of revolution \(\bar{M}\) – the patch of a cone generated by \(\bar{\gamma}\).
5 Appendix: Parabolic PDEs on closed Riemannian manifolds

Let \((F^p, g)\) be a \(C^\infty\)-smooth closed (i.e., compact without a boundary) Riemannian manifold. If \(H\) is a bounded linear operator acting from a Banach space \(E_1\) to a Banach space \(E_2\), we shall write \(H : E_1 \to E_2\). The resolvent set of \(H : E \to E\), is defined by \(\rho(H) = \{\lambda \in \mathbb{C} : H - \lambda \text{ id is invertible and } (H - \lambda \text{ id})^{-1} \text{ is bounded}\}\). The resolvent of \(H\) is the operator \(R_\lambda(H) = (H - \lambda \text{ id})^{-1}\) for \(\lambda \in \rho(H)\), and the spectrum of \(H\) is the set \(\sigma(H) := \mathbb{C} \setminus \rho(H)\), see [3, Chapt. VII, Sect. 9]. Let \(H^1(F)\) be the Hilbert space of differentiable by Sobolev real functions on a manifold \(F\), of order \(l\); with the inner product \((\cdot, \cdot)\) and the norm \(\| \cdot \|_l\). In particular, \(H^0(F) = L^2(F)\) with the inner product \((\cdot, \cdot)_0\) and the norm \(\| \cdot \|_0\). We shall denote \(\| \cdot \|_c\) the norm in \(C^k(F)\) \((\| \cdot \|_c\) when \(k = 0\)). Consider the following operator acting in the Hilbert space \(L^2(F)\):

\[
H(u) = -\Delta u - f(x) u,
\]

(31)
defined on the domain \(D = H^2(F)\). The operator \(H\) is self-adjoint, bounded from below (but it is unbounded). Its resolvent is compact, i.e., for some \(\lambda \in \rho(H)\) the operator \(R_\lambda(H)\) maps any bounded in \(L^2(F)\) set onto a set, whose closure is compact in \(L^2(F)\).

**Proposition 7 (Elliptic regularity, see [1]).** If the operator \(H\) is defined by (77) and \(\gamma \notin \sigma(H)\), then for any nonnegative integer \(k\) we have \((H - \gamma \text{ id})^{-1} : H^k(F) \to H^{k+2}(F)\).

**Proposition 8 (Sobolev embedding Theorem, see [1]).** If a nonnegative \(k \in \mathbb{Z}\) and \(l \in \mathbb{N}\) are such that \(2l > p + 2k\), then \(H^1(F)\) is continuously embedded into \(C^k(F)\).

**Proposition 9.** The spectrum \(\sigma(H)\) consists of an infinite sequence of isolated real eigenvalues \(\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \lambda_n \leq \ldots\) (counting their multiplicities), \(\lambda_n \to \infty\) as \(n \to \infty\). If we fix the orthonormal basis \(\{e_n\}\) in \(L^2(F)\) of the corresponding eigenfunctions (i.e., \(He_n = \lambda_n e_n\), \(\|e_n\|_0 = 1\)), then any function \(u \in L^2(F)\) is expanded into the series (converging to \(u\) in the \(L^2(F)\)-norm)

\[
u(x) = \sum_{n=0}^{\infty} c_n e_n(x), \quad c_n = (u, e_n)_0 = \int_F u(x) e_n(x) \, dx.
\]

(32)

The claim of Proposition 9 follows from the following facts. Since by Proposition 7 we have \((H - \gamma \text{ id})^{-1} : L^2(F) \to H^2(F)\) for \(\gamma \notin \sigma(H)\), and the embedding of \(H^2(F)\) into \(L^2(F)\) is continuous and compact, see [1], then the operator \((H - \gamma \text{ id})^{-1} : L^2(F) \to L^2(F)\) is compact. This means that the spectrum \(\sigma(H)\) of the operator \(H\) is discrete, hence by the spectral expansion theorem for self-adjoint operators, the functions \(\{e_n\}_{n \geq 0}\) form an orthonormal basis in \(L^2(F)\), see [3, Part I, Chapt VII, Sect. 4, and Part II, Chapt. XII, Sect. 2].

**Example 6.** The Cauchy’s problem (21) with \(u(0, \cdot) = u_0 \in H^2(F)\) has a unique solution in the class of functions \(C([0, \infty), H^2(F)) \cap C^1([0, \infty), L^2(F))\). The solution has the property \(u(\cdot, t) \in C^\infty(F)\) for \(t > 0\). Moreover, \(\lim_{t \to \infty} u(\cdot, t) = \bar{u}_0 = \frac{1}{(2\pi)^p} \int_F u_0(x) \, dx\) and \(\|u_t - \bar{u}_0\| \leq e^{-t\|u_0 - \bar{u}_0\|}\) for \(t > 0\). The eigenvalue problem \(-\Delta u = \lambda u\) on \((F, g)\) has solution with a sequence of eigenvalues with repetition (each one as many times as the dimension of its finite dimensional eigenspace) \(0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty\). Let \(\phi_j\) be an eigenfunction with eigenvalue \(\lambda_j\), satisfying \(\int_F \phi_i(x) \phi_j(x) \, dx_g = \delta_{ij}\). For \(\lambda_0\), the eigenfunction is the constant \(\phi_0 = \text{vol}(F, g)^{-1/2}\).

Our goal is to formulate conditions under which this series converges uniformly to \(u\) and it is possible to differentiate it. For this we need estimates for the eigenvalues and the eigenfunctions of \(H\). Denote the distribution function of eigenvalues of \(H\) by \(N(\lambda) = \# \{\lambda_n : \lambda_n \leq \lambda\}\).

Hörmander [4] obtained an asymptotic formula for the kernel \(e(x, y, \lambda)\) of the spectral projection \(E(\lambda)\) (see [3, Part II, Chapt. XII]), which for compact \(F\) has the form \(e(x, y, \lambda) = \sum_{\lambda_n \leq \lambda} e_n(x)e_n(y)\). In our case, this formula is represented by \(e(x, x, \lambda) = \alpha(x)\lambda\delta^2(1 + o(1))\) for
\[ \lambda \to \infty \text{ uniformly w.r.t. } x \in F, \text{ where the function } \alpha(x) \text{ belongs to } C^\infty(F) \text{ and depends only on } (F,g). \]

Integrating the formula for \( e(x,x,\lambda) \) over \( F \), we obtain the formula of Weyl asymptotics

\[ N(\lambda) = \theta \lambda^{\frac{p}{2}} (1 + o(1)) \quad \text{as} \quad \lambda \to \infty, \quad (33) \]

where the constant \( \theta > 0 \) depends only on \( (F,g) \).

**Lemma 7.** There exists \( \delta > 0 \) and \( \gamma_0 \in \mathbb{R} \) such that for any \( n \in \mathbb{N} \cup \{0\} \) we have \( e_n \in C(F) \) and

\[ \|e_n\|_c \leq \delta(\lambda_n + \gamma_0)^{[p/4]} + 1. \quad (34) \]

**Proof.** If we take \( \gamma > -\lambda_0 \), then the operator \( H - \gamma \text{id} \) is invertible in \( L_2(F) \) and its inverse \( (H + \gamma \text{id})^{-1} \) is bounded in \( L_2(F) \). By Proposition \( \ref{prop:invertibility} \), \( (H + \gamma \text{id})^{-1} : H^k(F) \to H^{k+2}(F) \) holds for \( k = 0, 1, 2, \ldots \). Then for any \( l \in \mathbb{N} \) we have

\[ (H + \gamma \text{id})^{-l} : L_2(F) \to H^{2l}(F). \quad (35) \]

As is easy to check, for any nonnegative integer \( n \) we have \( e_n = (\lambda_n + \gamma)^l e_n \). In view of \( (35) \), \( e_n \in H^{2l}(F) \) holds, and we have \( \|e_n\|_{2l} \leq \delta(\lambda_n + \gamma)^l \) for some \( \delta > 0 \) and \( n = 0, 1, 2, \ldots \).

On the other hand, by Proposition \( \ref{prop:embedding} \) with \( k = 0 \), for \( 4l > p \) the space \( H^{2l}(F) \) is continuously embedded into \( C(F) \). Hence \( e_n \in C(F) \), and we have \( \|e_n\|_c \leq \delta \|e_n\|_{2l} \) for some \( \delta > 0 \) and \( n = 0, 1, 2, \ldots \). The above estimates imply the desired inequality \( (34) \) with \( \delta = \delta \delta \).

**Theorem 3.** Let \( (F,g) \) be a closed Riemannian manifold, and \( f \in C^\infty(F) \). Then for the operator \( H = -\Delta u - f(x) \text{id} \), see \( (37) \), any eigenfunction \( e_n \) belongs to class \( C^\infty(F) \), and

(i) the expansion \( (32) \) converges to \( u \) absolutely and uniformly on \( F \);

(ii) for any multi-index \( \alpha \) with \( |\alpha| \geq 1 \) we have

\[ D^\alpha u = \sum_{n=0}^{\infty} (u, e_n)_0 D^\alpha e_n, \quad (36) \]

and this series converges to \( D^\alpha u \) absolutely and uniformly on \( F \).

**Proof.** (i) Since \( u \in C^\infty(F) \), for any \( m \in \mathbb{N} \) and \( \gamma \in \mathbb{R} \) the function \( h = (H + \gamma \text{id})^m u \) is continuous on \( F \), hence \( h \in L_2(F) \). For \( \gamma > -\lambda_0 \), the operator \( H + \gamma \text{id} \) is invertible and the operator \( (H + \gamma \text{id})^{-1} \) is defined on the whole \( L_2(F) \), hence \( u = (H + \gamma \text{id})^{-m} h \). Therefore,

\[ (u, e_n)_0 = ((H + \gamma \text{id})^{-m} h, e_n)_0 = (h, (H + \gamma \text{id})^{-m} e_n)_0 = (\lambda_n + \gamma)^{-m} (h, e_n)_0. \]

Hence in view of Lemma \( \ref{lem:finite} \) we get for \( l > \frac{p}{2} \) the following estimate for the terms of the series \( (32) \):

\[ \|(u, e_n)_0 e_n\|_c \leq \delta(\lambda_n + \gamma)^{-m+1} \|h\|_c. \]

In view of \( (33) \), there exists \( \delta_1 > 0 \) such that the counting function is estimated as \( N(\lambda) \leq \delta_1 \lambda^{\frac{p}{2}} \) for any \( \lambda \in \mathbb{R} \). If we take \( m > \frac{p}{2} + l \), then we get, using integration by parts in the Stilties integral:

\[ \sum_{n=0}^{\infty} (\lambda_n + \gamma)^{-s} = \int_{-\infty}^{\infty} \frac{dN(\lambda)}{(\lambda + \gamma)^s} = \frac{N(\lambda)}{(\lambda + \gamma)^s} |_{\lambda_0 - 1} + s \int_{\lambda_0 - 1}^{\infty} \frac{N(\lambda) d\lambda}{(\lambda + \gamma)^{s+1}} \]

\[ = s \int_{\lambda_0 - 1}^{\infty} \frac{N(\lambda) d\lambda}{(\lambda + \gamma)^{s+1}} \leq s \delta_1 \theta \int_{\lambda_0 - 1}^{\infty} \frac{d\lambda}{(\lambda + \gamma)^{s+1-p/2}}, \]

where \( s = m - l \). The last integral converges, hence the series \( (32) \) converges absolutely and uniformly on \( F \). Since this series converges to \( u \) in \( L_2(F) \), then it converges uniformly to \( u \).

(ii) Let \( k \in \mathbb{N} \) and \( 4l > p + 2k \). By Proposition \( \ref{prop:embedding} \) the space \( H^{2l}(F) \) is continuously embedded into \( C^k(F) \). As in the proof of Lemma \( \ref{lem:finite} \) we obtain that there exists \( \delta_k > 0 \) such that for any integer \( n \geq 0 \) we have \( e_n \in C^k(F) \) and

\[ \|e_n\|_{C^k} \leq \delta_k (\lambda_n + \gamma)^l. \quad (37) \]
Since $k$ is arbitrary, we conclude that any eigenfunction $e_n$ of the operator $\mathcal{H}$ belongs to class $C^\infty(F)$. Similarly as in the proof of claim (i), for $4l > p + 2k$ and $m \in \mathbb{N}$, using (37), we obtain
\[ \| (u, e_n) \|_c \leq \delta_k (\lambda_n + \gamma)^{-m+l} \| h \|_0, \]
where $h = (\mathcal{H} + \gamma id)^m u$. Hence, for $|\alpha| \leq k$, we obtain
\[ \| (u, e_n) D^\alpha e_n \|_c \leq \delta_k (\lambda_n + \gamma)^{-m+l} \| h \|_0. \]

Then, as in the proof of claim (i), we obtain that if $m > \frac{p}{2} + l$, then the series in (33) converges absolutely and uniformly. Since by claim (i), the series (32) converges uniformly to $u$, then by the standard argument of Analysis, the series (36) converges uniformly to the derivative $D^\alpha u$.

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