The duality relation between Glauber dynamics and the diffusion-annihilation model as a similarity transformation

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Abstract

In this paper we address the relationship between zero temperature Glauber dynamics and the diffusion-annihilation problem in the free fermion case. We show that the well-known duality transformation between the two problems can be formulated as a similarity transformation if one uses appropriate (toroidal) boundary conditions. This allow us to establish and clarify the precise nature of the relationship between the two models. In this way we obtain a one-to-one correspondence between observables and initial states in the two problems. A random initial state in Glauber dynamics is related to a short range correlated state in the annihilation problem. In particular the long-time behaviour of the density in this state is seen to depend on the initial conditions. Hence, we show that the presence of correlations in the initial state determine the dependence of the long time behaviour of the density on the initial conditions, even if such correlations are short-ranged. We also apply a field-theoretical method to the calculation of multi-time correlation functions in this initial state.
1 Introduction

It has been known for a long time that there is a relation between Glauber dynamics and the symmetric diffusion problem in the presence of annihilation and deposition of pairs of particles for a certain choice of the diffusion constant, which corresponds to the case in which this problem can be solved using free fermions. Using this relation, Family and Amar have computed the time evolution of the particle density in the transformed state of the annihilation problem that corresponds to random initial conditions in Glauber dynamics at $T = 0$. They have shown that their result only agrees with the previously known results by Spouge if one starts with zero initial magnetization in the Glauber problem. In all other cases the long time behaviour of the density shows a dependence on the initial conditions, a result which differs from the well known universal behaviour valid for random initial states in the annihilation problem. This rises the question of the correspondence between initial states in the Glauber and annihilation problems and more generally the relation between observables in the two systems. In this paper we show that the duality transformation between the two systems is really a similarity transformation, if one uses sector dependent, toroidal boundary conditions in the Glauber model (see below). We show that a random initial state in Glauber dynamics is mapped through this similarity transformation to a state with nearest neighbour correlations. Such state is translationally invariant, which allow us to recover the result by Family and Amar and also to compute higher order correlators using a field-theoretic technique. The relation between these quantities in the transformed state and their counterparts in the random initial state with density $1/2$ is explored and some calculations are explicitly done. Finally, we emphasize the role of the correlations in the initial state for the failure of the universality hypothesis.

The structure of this paper is as follows: in section 2, we present Glauber dynamics in terms of a quantum spin chain and the definition of a Temperley-Lieb algebra in terms of the spin $1/2$ operators. The associated Hecke algebra permits the construction of the similarity transformation to the reaction-diffusion model and also determines the boundary conditions of the system. The correspondence between observables in Glauber and annihilation dynamics is also addressed. In section 3, we discuss the relation between the initial states in the two problems. In particular, we study the mapping between the random initial state in Glauber dynamics and a short-range correlated state in annihilation dynamics. Using these results and those of section 2 we establish a correspondence between correlation functions in the two problems. In section 4, we show that
the short range correlated state is a translationally invariant state and we show how to calculate multiple
time correlation functions in this state using a field-theoretic technique. The density is explicitly computed
and shown to agree with Family and Amar’s result. We study the relation between two-point correlation
functions in these states and the random initial state with density 1/2 and recover the zero time correlations
as a special case. Finally, in section 5, we present our conclusions.

For simplicity we discuss only the case of zero temperature Glauber dynamics although our results are
easily generalised to other problems with little modification, e.g. Glauber dynamics at finite temperature
or the model of generalised dynamics considered by Peschel and Emery \(^8\), which maps by a duality trans-
formation to a model of diffusing particles with pair annihilation and creation away from the free fermion
line.

2 The transformation law for the operators

It is well known \(^3\), \(^9\) that certain reaction-diffusion systems provide physical realisations of Hecke algebras.
The time evolution of these processes is described by a master-equation which can be conveniently written
using an operator formalism, in which one assigns to each configuration of the system a state vector in an
Hilbert space \(^10\). The probability distribution is then represented by a state vector obeying a “Schrödinger”
equation \(\partial_t |\Psi\rangle = -H |\Psi\rangle\) with \( |\Psi\rangle = \sum_n P(n, t) |n\rangle \) where \( P(n, t) \) is the probability of finding confi-
guration \( n \) at time \( t \) and the set of different \( |n\rangle \) is supposed to be orthonormal and complete. The operator
\( H \) is a linear and in general non-hermitian operator encoding the rules of the stochastic process. For some
systems of interest this operator can be written \(^3\), \(^9\) as a sum of generators of Hecke algebras. Given that
there exists a relation between reaction-diffusion systems and Glauber dynamics \(^2\) it is natural to ask if the
evolution operator for this system can also be written in terms of Hecke algebra generators and if so, what
conclusions can one draw from it. In order to show this we define the following operators on a lattice of \( L \)
sites

\[
H^\pm = \sum_{j=1}^{L} (1 - e_{2j-1}) (1 - (e_{2j} + e_{2j-2} - 1)) \\
e_{2j-1} = \frac{1}{2} (1 + \hat{\sigma}_j^x) \quad 1 \leq j \leq L \\
e_{2j} = \frac{1}{2} (1 + \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z) \quad 1 \leq j \leq L - 1
\] (1)
and $e_0 = e_{2L}$ which will be given below. The operators $\hat{\sigma}_j^z$, $\hat{\sigma}_j^x$ are Pauli spin matrices at site $j$.

The set $e_j \ (1 \leq j \leq 2L-1)$ forms a Temperley-Lieb algebra [11], characterized by the relations

$$
e_j^2 = e_j
$$

$$
e_j e_{j \pm 1} e_j = \frac{1}{2} e_j
$$

$$
e_j e_i = e_i e_j \quad \text{if} \quad |j-i| \geq 2.
$$

In order to construct $e_{2L}$ explicitly we define the set of operators $g_j \ (1 \leq j \leq 2L-1)$ and the duality operator $D$ by [12]

$$
g_j = (1 + i) e_j - 1
$$

$$
D = \left( \prod_{j=1}^{2L-1} g_j \right) X
$$

where $\prod_{j=1}^{2L-1} g_j$ is the ordered product of the $g_j$’s and $X$ will be either 1 or $\hat{\sigma}_L^z$ in this paper. The important points about these two operators are that both commute with $g_j$ for $1 \leq j \leq 2L-2$ and that $(g_{2L-1} X)^2 = (X g_{2L-1})^2$. When $X = 1$, we will call the corresponding duality operator $D_+$ and when $X = \hat{\sigma}_L^z$, we will call it $D_-$. As we will see below the choice of the operator $D$ is directly related with the different types of toroidal boundary conditions that were referred in the last section. The operators $g_j$, together with the commutation relations with $X$, form an affine Hecke algebra associated with the Temperley-Lieb algebra given above (see [3, 9, 12] and references therein) and one finds $D e_j = e_{j+1} D$ for $1 \leq j \leq 2L-2$. Since $g_j^\dagger = (1-i) e_j - 1$, one finds from (3) that $g_j^\dagger g_j = 1$. Also $X^2 = 1$, and we conclude that $D$ is unitary and hence invertible. We define $e_{2L}$ as

$$
e_{2L} = D e_{2L-1} D^{-1}.
$$

The set of operators $e_j \ (1 \leq j \leq 2L-1)$ and $e_{2L}$ satisfies the relations of a periodic Temperley-Lieb algebra [12] with $2L$ generators which is defined by (2) together with similar relations for $e_{2L}$ ($e_{2L+1} = e_1$, etc). For the particular choices of $X$ given above, it can be shown that [13]

$$
\begin{align*}
e_{2L} &= \frac{1}{2} \left( 1 + \hat{C} \hat{\sigma}_L^z \hat{\sigma}_1^z \right) & \text{if} \quad X = 1 \\
e_{2L} &= \frac{1}{2} \left( 1 - \hat{C} \hat{\sigma}_L^z \hat{\sigma}_1^z \right) & \text{if} \quad X = \hat{\sigma}_L^z
\end{align*}
$$

where $\hat{C} = \prod_{j=1}^{L} \hat{\sigma}_j^x$. If we substitute the definitions of the $e_j$’s in (1) $H^\pm$ can be seen to be the generator of the time evolution for Glauber dynamics. The $+$ ($-$) sign stands when $X = 1$ ($X = \hat{\sigma}_L^z$). This explains the use of the notation $H^\pm$. The choice $X = 1$ ($X = \hat{\sigma}_L^z$), corresponds to the operator $H^+$ ($H^-$) which,
when applied to eigenstates of $\hat{C}$ with eigenvalue 1 ($-1$), generates the time evolution for Glauber dynamics at $T = 0$ with periodic boundary conditions [14]. When applied to eigenstates of $\hat{C}$ with eigenvalue $-1$ ($1$) $H^+$ ($H^-$) generates the time evolution for Glauber dynamics with anti-periodic boundary conditions. Both dynamics are stochastic. We see that one can indeed write the evolution operator for Glauber dynamics in terms of Hecke algebra generators. We now define the following similarity transformation

$$V_\pm = R D_\pm$$  \hspace{1cm} (6)

where $R = \exp \left( i \frac{\pi}{4} \sum_{j=1}^{L} \hat{\sigma}_y^j \right)$ is a global rotation of $\frac{\pi}{2}$ around the y axis. From the relations between the Temperley-Lieb generators $e_{j+1} = D e_j D^{-1}$ it can be easily shown that

$$V_\pm \hat{\sigma}_j^z V_\pm^{-1} = \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z \hspace{1cm} 1 \leq j \leq L - 1$$

$$V_\pm \hat{\sigma}_j^z V_\pm^{-1} = \hat{\sigma}_{j+1}^z \hspace{1cm} 1 \leq j \leq L - 1$$

$$V_\pm \hat{\sigma}_L^z V_\pm^{-1} = \pm \hat{Q}_L \hat{\sigma}_L^z \hat{\sigma}_1^z$$

$$V_\pm \hat{C} \hat{\sigma}_L^z \hat{\sigma}_1^z V_\pm^{-1} = \pm \hat{\sigma}_1^z$$

where $\hat{Q}_L = R \hat{C} R^{-1} = \prod_{j=1}^{L} \hat{\sigma}_j^z$. From [7] it also follows that $V_\pm \hat{C} V_\pm^{-1} = \pm \hat{Q}_L$. Applying $V_+$ to $H^+$ and $V_-$ to $H^-$ gives

$$\hat{H}^\pm = V_\pm H^\pm V_\pm^{-1} = \sum_{j=2}^{L-1} \frac{1}{2} (1 - \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x) (1 - \frac{1}{2} (\hat{\sigma}_j^z + \hat{\sigma}_{j+1}^z))$$

$$+ \frac{1}{2} (1 - \hat{\sigma}_1^z \hat{\sigma}_2^z) \left( 1 - \frac{1}{2} \left( \hat{\sigma}_1^z + \hat{\sigma}_2^z \right) \right)$$

$$+ \frac{1}{2} \left( 1 \mp \hat{Q}_L \hat{\sigma}_L^z \hat{\sigma}_1^x \right) \left( 1 - \frac{1}{2} \left( \hat{\sigma}_1^z + \hat{\sigma}_L^z \right) \right) .$$  \hspace{1cm} (8)

The operators $\hat{H}^\pm$, restricted to the subspaces $\hat{Q}_L = 1$ for $\hat{H}^+$ and $\hat{Q}_L = -1$ for $\hat{H}^-$, are equivalent to the Hamiltonian of the diffusion-annihilation problem with rates of diffusion $1/2$ and rate of annihilation 1 and periodic boundary conditions, which can be solved in terms of free fermions [3]. The other cases correspond to non-stochastic processes. Hence, we obtain a rigorous formulation of the well known duality transformation between the two models [3]. These results are summarized in Table 1. Notice that, although we have not used it explicitly, the similarity transformation preserves the relations [2] so one can also represent $\hat{H}^\pm$ in terms of Hecke algebra generators. However, one has used here an Hermitian quotient of the algebra which is different from the one used in references [3] and which allows the Hamiltonian for the diffusion-annihilation to be written as a linear combination of Hecke algebra generators.
Table 1: Summary of the relations between Glauber and annihilation dynamics. On the left hand side we have the Hamiltonian operator which is equivalent to the Glauber-Ising Hamiltonian and the sector of the Hilbert space where that equivalence holds, indicated by the eigenvalue of $\hat{C}$. On the right hand side we have the transformed Hamiltonian and the sector in which it is equivalent to the diffusion-annihilation process (given by the eigenvalue of $\hat{Q}_L$).

Finally, let us consider the action of $V_\pm$ in a single $\hat{\sigma}_j^z$ operator. Expressing the operators $g_l$ in terms of Pauli spin matrices in (3) one can, using the commutation relations for these operators show that

$$V_\pm \hat{\sigma}_j^z V_\pm^{-1} = -\hat{\sigma}_1^y \hat{\sigma}_2^z \ldots \hat{\sigma}_j^z.$$  

(9)

Using (9), one obtains the following transformation law for a pair of $\hat{\sigma}_k^z \hat{\sigma}_l^z$ $k < l$

$$V_\pm \hat{\sigma}_k^z \hat{\sigma}_l^z V_\pm^{-1} = (V_\pm \hat{\sigma}_k^z V_\pm^{-1})(V_\pm \hat{\sigma}_l^z V_\pm^{-1})$$

$$= (\hat{\sigma}_1^y \hat{\sigma}_2^z \ldots \hat{\sigma}_k^z)(\hat{\sigma}_1^y \hat{\sigma}_2^z \ldots \hat{\sigma}_l^z) = \hat{\sigma}_{k+1}^z \ldots \hat{\sigma}_l^z.$$  

(10)

Note that taking $l = k + 1$ or $k = L$, $l = 1$ we recover the equalities (7) concerning $\hat{\sigma}_{k+1}^z$. Equation (10) will be useful below when we derive equalities concerning correlation functions. We now proceed to study the effect of $V_\pm$ in the states of the theory.

3 The transformation law for the states

The Glauber-Ising Hamiltonian at $T = 0$ has two ground states, the ferromagnetic states with all spins up or down. Since $H^+$ is only equivalent to it in the subspace of the states with $\hat{C} = 1$ we have to find a linear combination of these two states that belongs to this subspace (for simplicity we will specialize in $V_+ \rangle$). This state is simply $| \Psi \rangle = \frac{1}{2} (| \uparrow \ldots \uparrow \rangle + | \downarrow \ldots \downarrow \rangle)$. It is normalized in the sense that $\langle s | \Psi \rangle = 1$, where $\langle s |$ is...
the sum of all configurations of spins with weight one. This expresses the fact that, for a stochastic process, 
the sum of the probabilities of the different configurations accessible to the system (e.g. the configurations 
of spins in the Glauber problem) has to add to one. Also if $H$ is the stochastic Hamiltonian representing 
the dynamics then it follows from conservation of probability that $\langle s | H = 0$. One can easily check that $\hat{C} | \Psi \rangle = | \Psi \rangle$. This equality implies that 

$$
V_+ | \Psi \rangle = V_+ \hat{C} | \Psi \rangle = (V_+ \hat{C} V_+^{-1}) V_+ | \Psi \rangle = \hat{Q}_L V_+ | \Psi \rangle
$$

(11)

where we have used the transformation law for $\hat{C}$ found above. Thus, the transformed state $V_+ | \Psi \rangle$ belongs 
to the eigenspace with $\hat{Q}_L = 1$. In the annihilation language this corresponds to the sector with an even 
number of particles $[4]$. Also, since $H^+ | \Psi \rangle = 0$, one obtains 

$$
V_+ H^+ | \Psi \rangle = (V_+ H^+ V_+^{-1}) V_+ | \Psi \rangle = \hat{H}^+ V_+ | \Psi \rangle = 0
$$

(12)

and hence $V_+ | \Psi \rangle$ is a ground state of the annihilation Hamiltonian (in the subspace $\hat{Q}_L = 1$, $\hat{H}^+$ is 
equivalent to it). The only ground state belonging to the subspace with an even number of particles is the 
vacuum $| 0 \rangle$, i.e. the state with no particles. Hence, we conclude that $V_+ | \Psi \rangle \propto | 0 \rangle$. The proportionality 
constant can be shown, using (10), to be equal to $\sqrt{2}(-1)^{L-1} e^{i\pi(L-1)}$ and can be absorbed in the definition 
of $V_+$. Following the same reasoning that led to (12), and in light that $\langle s | \hat{C} = \langle s |$, one can similarly show 
that $\langle s | V_+^{-1} \hat{H}^+ = 0$ and that $\langle s | V_+^{-1}$ belongs to the even sector. On the same grounds of uniqueness this 
shows that this state is equal (up to a normalisation constant which we absorb in the definition of $V_+^{-1}$) to 
the left ground state of the annihilation Hamiltonian with an even number of particles, i.e. $\langle s |^{even}$, which 
is the sum of all configurations with an even number of particles with weight one. We are using the same 
notation $\langle s |$ for the left ground states of the two Hamiltonians because both describe stochastic processes 
and this is the usual convention.

We have also found that $H^-$ is equivalent to the Glauber-Ising Hamiltonian in the subspace $\hat{C} = -1$. 
Since $V_- \hat{C} V_-^{-1} = - \hat{Q}_L$ this subspace is mapped to the even sector of the annihilation problem. But in this 
sector $\hat{H}^-$ is not equivalent to a stochastic Hamiltonian. If one starts with the subspace $\hat{C} = 1$ then $V_-$ 
effectively maps this sector to the odd sector where $\hat{H}^-$ is equivalent to the annihilation Hamiltonian, but 
in this case $H^-$ is not equivalent to the Glauber-Ising Hamiltonian with periodic boundary conditions, but 
to the Glauber-Ising Hamiltonian with anti-periodic boundary conditions, which is also a stochastic process. 
Using the same argument as above one can show that the ground state of this Hamiltonian is mapped to the
ground state of the odd sector of the annihilation Hamiltonian. This state is just a uniform superposition of states with one particle at each site of the lattice \[ \hat{C} = 1 \] subspace. If we apply \( V_\pm \) to this state we will therefore obtain the ground state of the Glauber-Ising Hamiltonian with anti-periodic boundary conditions (the observation about normalisation factors also applies here). Such state is a uniform superposition of \( 2L - 2 \) states with two domain walls, one at the boundary and one at each site of the lattice, plus the two states with all spins up or down. These two states are, due to the anti-periodic boundary conditions, the image of the state with one particle at the boundary in the diffusion-annihilation model. The previous discussion shows that the transformations \( V_+ \) and \( V_- \) restricted to the \( \hat{C} = 1 \) subspace, map a stochastic process (Glauber-Ising dynamics with periodic or anti-periodic boundary conditions) to a stochastic process (annihilation dynamics), something which we also pointed out in Table 1. In the other cases the mapping is from a stochastic problem to a non-stochastic problem. Nevertheless, these mappings can be considered from a purely formal point of view and examples of similarity transformations between stochastic and non-stochastic processes have already been studied in the literature [14, 15, 16].

Here we shall concentrate in the stochastic-stochastic mapping given by \( V_+ \). We will consider the study of time dependent correlation functions in uncorrelated random initial states evolving in time according to Glauber dynamics. The state

\[
| \Phi \rangle = \prod_{j=1}^{L} \left[ \frac{1 + m^2}{2} + \frac{1 - m^2}{2} \hat{\sigma}_j^z \right] | \Psi \rangle \tag{13}
\]

corresponds to the superposition of two random initial states with initial magnetization \( m \) and \(-m\). From the point of view of the calculation of correlation functions of an even number of \( \hat{\sigma}_j^z \) operators the two states are equivalent and \( | \Phi \rangle \) belongs to the \( \hat{C} = 1 \) subspace. Under the application of \( V_+ \), \( | \Phi \rangle \) transforms to

\[
| \tilde{\Phi} \rangle = \prod_{j=1}^{L} \left[ \frac{1 + m^2}{2} + \frac{1 - m^2}{2} \hat{\sigma}_j^z \hat{\sigma}_{j+1}^x \right] | 0 \rangle \tag{14}
\]

where we have used (7) and the fact that \( V_+ | \Psi \rangle = | 0 \rangle \). This is an initial state with short range correlations and its form will play a crucial role in the determination of the correlation functions in the late time regime. Under \( V_+ \) the multiple time correlation functions of an even number of \( \hat{\sigma}_j^z \) spins at times \( t_1, t_2, \text{etc.} \), transform as

\[
\langle s | \hat{\sigma}_{j_1}^z (t_1) \hat{\sigma}_{j_2}^z (t_2) \hat{\sigma}_{j_3}^z (t_2) \hat{\sigma}_{j_4}^z (t_2) \ldots \hat{\sigma}_{j_{2N-1}}^z (t_N) \hat{\sigma}_{j_{2N}}^z (t_N) | \Phi \rangle 
\]

\[
= \langle s | \hat{\sigma}_{j_1}^z \hat{\sigma}_{j_2}^z e^{-H^+ (t_1 - t_2)} \hat{\sigma}_{j_3}^z \hat{\sigma}_{j_4}^z e^{-H^+ (t_2 - t_3)} \ldots e^{-H^+ (t_{N-1} - t_N)} \hat{\sigma}_{j_{2N-1}}^z \hat{\sigma}_{j_{2N}}^z e^{-H^+ t_N} | \Phi \rangle
\]
for any odd number of $\hat{\sigma}_j$ since $\hat{\sigma}_j$ where we have used equations (8) and (10). We suppose that

$$\langle s | \sum_{j=1}^{n} \hat{\sigma}_j^z (t_1) \cdots \hat{\sigma}_{j_{2N+1}}^z (t_{2N+1}) | \Phi \rangle = \langle s | \hat{\sigma}_1^z (t_1) \hat{\sigma}_2^z (t_1) \cdots \hat{\sigma}_{j_{2N+1}}^z (t_{2N+1}) | \Phi \rangle$$

(15)

where we have used equations (8) and (10). We suppose that $j_1 < j_2, j_3 < j_4$, etc. Also one concludes that for any odd number of $\hat{\sigma}_j$ operators, one has

$$\langle s | \hat{\sigma}_j^z (t_1) \cdots \hat{\sigma}_{j_{2N+1}}^z (t_{2N+1}) | \Phi \rangle = 0,$$

(16)

since $\hat{\sigma}_j^z$ anticommutes with $\hat{C}$.

### 4 The calculation of correlation functions

Using (15) we have, in particular, that

$$\langle s | \frac{1}{2} (1 - \hat{\sigma}_j^z (t) \hat{\sigma}_j^z (t)) | \Phi \rangle = \langle s | \frac{1}{2} (1 - \hat{\sigma}_j^z (t)) | \Phi \rangle = \langle s | \hat{n}_j (t) | \Phi \rangle.$$  

(17)

The operator $\frac{1}{2} (1 - \hat{\sigma}_j^z)$ checks for the existence of a domain wall at site $j$ in the Glauber problem, i.e. it checks if at site $j$ the spins cease to point up and start to point down or vice-versa. The operator $\frac{1}{2} (1 - \hat{\sigma}_j^z)$ checks for the existence of a particle at site $j$, i.e. if spin $j$ is down in the annihilation problem (17).

This indeed corresponds to the well known duality transformation (15). Similar relations hold for higher order correlation functions. Now we will use the fact that the diffusion annihilation Hamiltonian can be completely solved in terms of free fermions by means of the Jordan-Wigner transformation (15). Using it, one is able to write the spin raising and lowering operators $\hat{s}_j^+ , \hat{s}_j^-$ ($\hat{s}_j^\pm = \frac{1}{\sqrt{2}} (\hat{\sigma}_j^x \pm i \hat{\sigma}_j^y)$) in terms of creation and annihilation operators $\hat{a}_j^\dagger, \hat{a}_j$ for spinless fermions. It turns out that the calculation of a multiple time correlation function of the $\hat{n}_j$ operators in state $| \Phi \rangle$ ($\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$ in the fermion language) can be reduced to the calculation of objects like $\langle s | \hat{b}_{p_1} \cdots \hat{b}_{p_{2k}} | \Phi \rangle$ ($k \leq N$) (16) where $\hat{b}_p$ is the Fourier transform of the annihilation operator $\hat{a}_i$. The momentum labels $p_j$ are half-odd integers between $-L/2 + 1$ and $L/2$ (16).

Also, one is able to express the state (14) in terms of fermion operators. The result is

$$| \Phi \rangle = \prod_{j=1}^{L} \left[ \frac{1 + m}{2} \frac{1 - m}{2} \hat{\sigma}_j^x \hat{\sigma}_j^x \right] | 0 \rangle = \exp \left( \beta \sum_{j=1}^{L} (\hat{\sigma}_j^x \hat{\sigma}_j^x - 1) \right) | 0 \rangle$$

$$= \exp \left( \beta \sum_{j=1}^{L} (\hat{a}_j^\dagger \hat{a}_{j+1}^\dagger + \hat{a}_j^\dagger \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \hat{a}_j + \hat{a}_{j+1}^\dagger \hat{a}_j - 1) \right) | 0 \rangle$$

(18)
where \( m = e^{-2\beta} \). The first equality follows from the fact that \((\hat{\sigma}^{+}_j \hat{\sigma}^{-}_{j+1})^2 = 1\) and the second just follows from the rules of the Jordan-Wigner transformation. In terms of the momentum space operators we can write \( |\tilde{\Phi}\rangle \) as

\[
|\tilde{\Phi}\rangle = \exp \left( 2\beta \sum_{p>0} \left[ \cos \left( \frac{2\pi p}{L} \right) (\hat{b}_p \hat{\beta}_p + \hat{b}^\dagger_{-p} \hat{\beta}^{-1}_p) + \sin \left( \frac{2\pi p}{L} \right) (\hat{b}_p \hat{\beta}_{-p} + \hat{b}^\dagger_{-p} \hat{\beta}^{-1}_p) - 1 \right] \right) |0\rangle . \tag{19}
\]

The simplest approach to use if one wants to calculate the density or any equal-time correlators is given in \([4]\).

For the calculation of multiple time correlators, we will follow a different route. We will look for operators \( \hat{c}^\dagger_p, \hat{c}_p \) that diagonalize the quadratic form appearing in the exponent of \((19)\). This can be accomplished by means of a Bogoliubov transformation \([13]\)

\[
\left\{ \begin{aligned}
\hat{c}_p &= \cos \left( \frac{2\pi p}{L} \right) \hat{b}_p - \sin \left( \frac{2\pi p}{L} \right) \hat{b}^\dagger_{-p} \\
\hat{c}^\dagger_p &= \cos \left( \frac{2\pi p}{L} \right) \hat{b}^\dagger_{-p} - \sin \left( \frac{2\pi p}{L} \right) \hat{b}_{-p}
\end{aligned} \right. \tag{20}
\]

and one gets

\[
|\tilde{\Phi}\rangle = \exp \left( 2\beta \sum_p (\hat{c}^\dagger_p \hat{c}_p - 1) \right) |0\rangle . \tag{21}
\]

One then expands the exponential using the anti-commutation relations for the \( \hat{c}^\dagger_p, \hat{c}_p \). Re-expressing these operators in terms of \( \hat{b}^\dagger_{p}, \hat{b}_p \) and applying them to the vacuum, one finally obtains

\[
|\tilde{\Phi}\rangle = e^{-\beta L} \prod_{p>0} [\gamma_p + \delta_p \hat{b}^\dagger_{-p} \hat{b}^{-1}_p] |0\rangle \tag{22}
\]

where \( \gamma_p = \cosh(2\beta) - \sinh(2\beta) \cos \left( \frac{2\pi p}{L} \right) \) and \( \delta_p = \sinh(2\beta) \sin \left( \frac{2\pi p}{L} \right) \). The form \((22)\) expresses the translation invariance of the state \( |\tilde{\Phi}\rangle \). One defines the following object

\[
Z[\eta_p, \eta_{-p}] = \langle 0 | \exp \left( \sum_{p>0} \left( \cot \left( \frac{\pi p}{L} \right) \hat{b}_p \hat{\beta}_p + \eta_p \hat{b}^\dagger_{-p} + \eta_{-p} \hat{b}^{-1}_{-p} \right) \right) |\tilde{\Phi}\rangle \tag{23}
\]

where the quantities \( \eta_p, \eta_{-p} \) are Grassmann variables anticommuting among themselves and with the \( \hat{b}_p \)'s. Their presence is necessary to make the terms in the exponential commute with each other. It can be shown \([13]\) that \( Z[\eta_p, \eta_{-p}] \) is the generating function for the quantities \( \langle s | \hat{b}_{p_1} \ldots \hat{b}^\dagger_{p_{2k}} |\tilde{\Phi}\rangle \) that is

\[
\langle s | \hat{b}_{p_1} \ldots \hat{b}^\dagger_{p_{2k}} |\tilde{\Phi}\rangle = \partial_{\eta_{p_1}} \ldots \partial_{\eta_{p_{2k}}} Z[\eta_p, \eta_{-p}] |_{\eta_p=0} . \tag{24}
\]

Substituting \((22)\) in \((23)\) we find, after a few algebraic manipulations, involving the anticommutation relations between the \( \hat{b}^\dagger_{p}, \hat{b}_p \) \([13]\), the following expression for \( Z[\eta_p, \eta_{-p}] \)

\[
Z[\eta_p, \eta_{-p}] = \exp \left( \sum_{p>0} \sinh(2\beta) e^{-2\beta} \sin \left( \frac{2\pi p}{L} \eta_p \eta_{-p} \right) \right) . \tag{25}
\]
Since $W = \ln Z$ is a quadratic function in the $\eta$’s it immediately follows that a Wick’s decomposition holds for the quantities $\langle s | \hat{b}_{p_1} \ldots \hat{b}_{p_{2k}} | \tilde{\Phi} \rangle$ [18]. In particular, from (24) and (25) one has

$$
\langle s | \hat{b}_{p'} \hat{b}_p | \tilde{\Phi} \rangle = \sinh(2\beta) e^{-2\beta} \sin \left( \frac{2\pi p'}{L} \right) \delta_{p,-p'} . \tag{26}
$$

Given that the expression for the space-dependent average density is [4]

$$
\langle \hat{n}_j(t) \rangle = \frac{1}{L} \sum_{p,p'} e^{2\pi i (p-p') - (\epsilon_p + \epsilon_{p'}) t} \cot \left( \frac{\pi p}{L} \right) \langle s | \hat{b}_{p'} \hat{b}_{-p} | \tilde{\Phi} \rangle \tag{27}
$$

where $\epsilon_p = 1 - \cos p$, one obtains, substituting (26) in (27) and taking the thermodynamic limit $L \to \infty$, the following expression for the density of particles $\rho(t)$ at time $t$

$$
\rho(t) = \frac{1}{2} (1 - m^2) e^{-2t (I_0(2t) + I_1(2t))} \tag{28}
$$

where $I_0(2t)$, $I_1(2t)$ are the modified Bessel functions of order zero and one, and where we have used the identity $m = e^{-2\beta}$. This is the well-known expression obtained by Family and Amar [4] who have also considered a random initial state in Glauber dynamics. They have shown that while the initial distribution of spins is uncorrelated the distribution of domain walls, i.e. the distribution of particles in the annihilation problem, is correlated. This correlated structure is found in the transformed state $| \tilde{\Phi} \rangle$ (14). For $m = 0$, the expression (28) is identical to the one found by Spouge [1] for an uncorrelated random initial state with initial density $1/2$ in the annihilation problem. Indeed, this identity is more general. If $m = 0$ then this means that we have to take the limit $\beta \to \infty$. If we take such a limit in equation (22) then one obtains the exact expression for a random initial state with density $1/2$, projected over the even sector [4]. Therefore, our calculations provide a rigorous framework for the well-known duality transformation. In the long time limit $t \to \infty$ one finds from (28) the leading behaviour $\rho(t) \approx \frac{1-m^2}{2\sqrt{\pi t}}$. It depends on the initial conditions (i.e. magnetization) [3]. The amplitude of $\rho(t)$ differs from the universal result found for uncorrelated random initial states [4,15]. Thus it is seen that the presence of correlations in the initial state breaks the universality of the amplitudes of correlation functions.

Our results, namely the Wick decomposition, also allow the explicit computation of higher order correlation functions. Apart from a factor $2 \sinh(2\beta) e^{-2\beta}$ the result (26) is identical to the results obtained for the random initial state with density $1/2$ [18] and, as stated above, it reduces to it when $m = 0$. Therefore one has

$$
\langle s | \hat{b}_{p_1} \ldots \hat{b}_{p_{2k}} | \tilde{\Phi} \rangle = (1 - e^{-4\beta})^k \langle s | \hat{b}_{p_1} \ldots \hat{b}_{p_{2k}} | 1/2 \rangle^{\text{even}} . \tag{29}
$$
As an example of the above result let us consider the two-point correlation function \( \langle s \mid \hat{n}_j(t) \hat{n}_k(t') \mid \Phi \rangle \). We use (29) with \( k = 2 \) and \( k = 1 \) because this object can be written as a linear combination of the terms \( \langle s \mid \hat{b}_{p_1} \hat{b}_{p_2} \hat{b}_{p_3} \hat{b}_{p_4} \mid \Phi \rangle \) and \( \langle s \mid \hat{b}_{p_1} \hat{b}_{p_2} \mid \Phi \rangle \) [18], which contribute with different powers of \( (1 - e^{-4\beta}) \).

Separating such powers, we obtain

\[
\langle s \mid \hat{n}_j(t) \hat{n}_k(t') \mid \Phi \rangle = (1 - e^{-4\beta})^2 \langle s \mid \hat{n}_j(t) \hat{n}_k(t') \mid 1/2 \rangle_{\text{even}} \\
+ \frac{1}{2L^2} \left[ (1 - e^{-4\beta}) - (1 - e^{-4\beta})^2 \right] \times
\left\{ \sum_{p_1, p_2} e^{\frac{2\pi i}{L}(p_1 - p_2)(j-k) - (e_{p_1} + e_{p_2})l + (e_{p_1} - e_{p_2})l'} \sin \left( \frac{2\pi p_1}{L} \right) \cot \left( \frac{\pi p_1}{L} \right) \\
+ \sum_{p_1, p_2} e^{\frac{2\pi i}{L}(p_1 - p_2)(j-k) - (e_{p_1} + e_{p_2})l - (e_{p_1} - e_{p_2})l'} \sin \left( \frac{2\pi p_1}{L} \right) \cot \left( \frac{2\pi p_1}{L} \right) \right\} .
\]

(30)

In the thermodynamic limit \( L \to \infty \), we have for \( t' = t \)

\[
\langle s \mid \hat{n}_j(t) \hat{n}_k(t) \mid \Phi \rangle = (1 - e^{-4\beta})^2 \langle s \mid \hat{n}_j(t) \hat{n}_k(t) \mid 1/2 \rangle_{\text{even}} \\
+ e^{-4\beta} (1 - e^{-4\beta}) \times \left\{ \frac{1}{2} e^{-2t} (I_0(2t) + I_1(2t)) \delta_{j,k} \\
+ \frac{1}{4} \left[ \theta(j-k) - \theta(k-j) \right] e^{-2t} (I_{j-k-1}(2t) - I_{j-k+1}(2t)) \right\}
\]

(31)

where we have used the integral representation of the modified Bessel functions \( I_j(2t) \) and \( \theta(x) \) is the Heaviside step function. The term \( \theta(j-k) - \theta(k-j) \) is the Fourier transform of \( \cot \left( \frac{2\pi}{L} \right) \) [13]. In particular, when \( t = 0 \), this reduces to

\[
\langle s \mid \hat{n}_j \hat{n}_k \mid \Phi \rangle = \frac{1}{2} (1 - e^{-4\beta}) \delta_{j,k} + \frac{1}{4} (1 - e^{-4\beta})^2 (1 - \delta_{j,k}) \\
+ \frac{1}{4} e^{-4\beta} (1 - e^{-4\beta}) \left( \delta_{j,k+1} + \delta_{j,k-1} \right)
\]

(32)

where we have used the fact that there are no correlations in the state \( 1/2 \) at \( t = 0 \). One clearly recognizes in the first two terms the contribution of the unconnected part of the correlation function. But the third term indeed confirms that even at \( t = 0 \) there are short range correlations. This term is zero when we take \( \beta \to \infty \) and we just obtain the trivial result for the \( 1/2 \) state.

If we use equation (30) to calculate the density-density correlation function in the thermodynamic limit we see that the second term of (30) will vanish. This leaves us with a term with an amplitude proportional to \((1 - m^2)^2\). The ratio of this correlation function with the square of the density [28] is independent of \( m \) in agreement with the general results known from the renormalisation group approach (see for example [20]). So despite the fact that the amplitudes of the various correlation functions are non-universal as emphasized above, their ratios obey the universality hypothesis.
The results discussed above show that this approach allows not only to recover the known results but also provides a convenient way to compute higher order correlation functions that can of course be translated back to the Glauber-Ising language.

5 Conclusions

We investigated the relation between the Glauber-Ising model at zero temperature and the diffusion-annihilation model in the free fermion case. We obtained the following new results

(i) The duality transformation between the two models can be formulated as a similarity transformation if one uses an Hamiltonian with sector-dependent toroidal boundary conditions. The transformation laws for the operators are explicitly given. We also obtain the transformation laws for the states. This permits a one-to-one correspondence between a state of Glauber-Ising dynamics and a state of the diffusion-annihilation problem. In particular an uncorrelated random initial state in Glauber dynamics transforms to a state with short range correlations.

(ii) Using the free fermion solution of the diffusion-annihilation problem we have computed the time dependent behaviour of the density and equal time two point correlation function in this short-range correlated state. For the density, we recover the results of the literature. We show that, surprisingly, the presence of correlations extending over only one lattice site in the initial state leads to a long time behaviour of the density dependent on the initial condition. The field theoretic approach we have used can be applied to the study of higher order correlation functions. Its use depends on the form (22) of the initial state which reflects the translation invariance of this state. An initial state with the Wick’s decomposition property was also discussed by Balboni et al. [21]. They have considered a continuous system described by a boson field theory. The initial state which they have chosen is characterized by the fact that the connected correlation functions of the density operator of order higher than two vanish. They found that for pure annihilation the amplitudes are also non-universal. In the case of the initial state (14) the higher order connected correlation functions of the density operator are non-zero, which shows that this state has a different structure.

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