Scale-dependent correction to the dynamical conductivity of a disordered system at unitary symmetry

P Ostrovsky\textsuperscript{1,2}, Tomoyuki Nakayama\textsuperscript{3}, K A Muttalib\textsuperscript{3} and P Wölfle\textsuperscript{4,5}

\textsuperscript{1}Max Planck Institute for Solid State Research, Heisenbergstraße 1, D-70569 Stuttgart, Germany
\textsuperscript{2}L D Landau Institute for Theoretical Physics, 142432 Chernogolovka, Russia
\textsuperscript{3}Department of Physics, University of Florida, Gainesville, FL 32611-8440, USA
\textsuperscript{4}Institute for Condensed Matter Theory and Institute for Nanotechnology, Karlsruhe Institute of Technology, D-76128 Karlsruhe, Germany
E-mail: peter.woelfle@kit.edu

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\textbf{Abstract.} Anderson localization has been studied extensively for more than half a century. However, while our understanding has been greatly enhanced by calculations based on a small $\epsilon$ expansion in $d = 2 + \epsilon$ dimensions in the framework of nonlinear sigma models, those results cannot be safely extrapolated to $d = 3$. Here we calculate the leading scale-dependent correction to the frequency-dependent conductivity $\sigma(\omega)$ in dimensions $d \leq 3$. At $d = 3$, we find a leading correction $\text{Re}\sigma(\omega) \propto |\omega|$, which at low frequency is much larger than the $\omega^2$ correction derived from the Drude law. We also determine the leading correction to the renormalization group $\beta$-function in the metallic phase at $d = 3$. 

\textsuperscript{5}Author to whom any correspondence should be addressed.

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1. Introduction

Anderson localization of quantum particles in a random potential or of classical waves in a
random medium has been studied intensively since the phenomenon was first proposed in
a seminal paper by Anderson in 1958 \[1\]. In these systems localization is a consequence
of interference of multiple scattering processes. The more it affects the motion of particles
or waves, the stronger the disorder or the more restricted the available geometry, leading to
complete localization of quantum particles at all energies in dimensions \(d \leq 2\). In dimensions
\(d > 2\) a quantum phase transition from metallic to insulating behavior takes place as the disorder
strength \(\lambda\) is increased beyond a certain threshold value \(\lambda_c\). There are convincing arguments that
the transition is continuous and that there exists a critical point at \(\lambda = \lambda_c\), characterized by a
diverging correlation length \(\xi\) (the localization length on the insulating side of the transition).
The behavior of physical observables such as the conductivity \(\sigma\) and the dielectric function
\(\epsilon\), as well as \(\xi\), is described in terms of power laws in \(\lambda - \lambda_c\), with critical exponents. It has been
one of the goals of the theory to determine these critical exponents.

Assuming that there is only one length scale (\(\xi\)) in the system, Abrahams et al \[2\] have
proposed a scaling theory for the dimensionless conductance \(g\) as a function of the length \(L\)
of the sample (considering only hypercubic systems), which takes the form of a renormalization
group (RG) equation:

\[
\frac{d \ln g}{d \ln L} = \beta_L(g).
\]

The \(\beta\)-function has been determined in the asymptotic regimes of very large (metallic) and very
small (insulating) \(g\). On the basis of these limiting behaviors, it is plausible to conclude that a
metal–insulator transition occurs in \(d = 3\) dimensions at \(g \approx O(1)\). At these intermediate values
of \(g\), a systematic and controlled calculation of the \(\beta\)-function is not easily possible.

The scaling hypothesis has been justified to a large extent by a mapping of the original
problem onto an effective field theory (nonlinear \(\sigma\)-model) of interacting matrices, first
proposed by Wegner \[4\]. Within that model one confirms the result that all states are localized in
\(d \leq 2\) dimensions. In order to determine critical exponents, the \(\beta\)-function has been calculated
in dimensions \(d = 2 + \epsilon\), \(\epsilon \ll 1\), when the critical point is shifted to large values of conductance,
\(g_c \gg 1\), and a loop expansion in powers of \(1/g\) is feasible. This program has been carried out
for the three main symmetry classes: orthogonal, unitary and symplectic symmetry \[5\]. In this
way the \(\beta\)-function has been found e.g. in the orthogonal case up to terms of fourth order in \(1/g\)
with coefficients calculated to linear order in \(\epsilon\) \[4, 6\].
In the unitary case, the leading correction terms to the $\beta$-function have been calculated [4]:

$$\beta_L(g) = \epsilon - \frac{c_2}{g^2} + O(g^{-5}).$$  \hspace{1cm} (2)

The latter result shows that in the limit of $\epsilon \to 0$ the term linear in $1/g$ is absent. As a consequence, the critical exponent of the conductivity turns out to be $s = 1/2\epsilon$, which extrapolates to $s = 1/2$ at $d = 3$, and is much too small compared to the value of $s \approx 1.3$ obtained in numerical studies [7]. For a review of the scaling theory, see [3].

Here we propose an alternative strategy designed to access the $\beta$-function in $d = 3$ dimensions in the metallic regime: a direct calculation of the leading terms in an expansion in $1/g$ in perturbation theory. Here we present first results for the unitary case. It is well known that the leading (one-loop) contribution is exactly zero in this case as the so-called Cooperon propagator acquires a mass, removing its diffusion pole. As a consequence, no scale-dependent ($L$-dependent) contribution to the conductance $g(L)$ is present in the lowest order. Scale-dependent terms growing with $L$ may only be generated by infrared divergent building blocks, i.e. diffusion propagators. It may be shown that diagrams containing one diffusion pole and any other elements may be combined to cancel out exactly [8]. The argument carries over to higher order terms, involving more than one diffusion pole (but no Cooperon poles, of course), considered in the diffusive regime. This property is a consequence of gauge symmetry.

Early attempts to calculate the quantum correction to the conductivity in the unitary symmetry case in perturbation theory (rather than from the nonlinear sigma model using dimensional regularization) were discouraged by the above-mentioned theorem [8].

However, by keeping track of the contributions from larger momenta $q \gtrsim 1/l$, where $l$ is the mean free path (ballistic regime), the correct result is recovered [9]. This calculation manifestly obeys the gauge symmetry property discussed above and confirms the result obtained earlier within the dimensional regularization scheme. We now generalize the method to arbitrary dimensions $d$, in particular $d = 3$. We determine the scale-dependent contributions to the conductance in the lowest non-zero order in $1/g$ and from there derive corrections to the $\beta$-function in $d$ dimensions.

2. Dynamical conductivity

In order to access the scale-dependent terms in perturbation theory, from which the $\beta$-function may be derived, it is convenient to consider the dynamical conductivity $\sigma(\omega)$ of the infinite system as a function of positive Matsubara frequency $\omega$, rather than the conductance $g(L)$ at $\omega = 0$ as a function of system length $L$. The conductivity of an isotropic disordered system (short-range disorder) is a function of the electron density (characterized by the Fermi wave number $k_F$) and the disorder strength (characterized by the mean free path $l$) and obeys in the scaling regime the scaling property, expressed in terms of the correlation length $\xi$,

$$\sigma(\omega; l, k_F) = \xi^{2-d} G(\omega \xi^z),$$  \hspace{1cm} (3)

where $z$ is the dynamical critical exponent, known to be $z = d$ [10]. We introduce the characteristic length $L_\omega = (D/\omega)^{1/2}$, where $D(\omega) = \sigma(\omega)/\nu_d$ is the diffusion coefficient and $\nu_d$ is the density of states at the Fermi level. In the limit $\omega \to 0$ the scaling function tends to a constant, and therefore $\xi = [\sigma(\omega = 0)]^{1/(2-d)}$. The scaling law equation (3) holds in the neighborhood of the quantum critical point (QCP). There are subdominant corrections to the
scaling form, the most familiar one being generated by the Drude law (using units of electrical charge $e = 1$),

$$\sigma_0(\omega) = \frac{n_d}{m} \frac{\tau}{1 + \omega \tau} = \frac{v_0 D_d}{1 + \omega \tau},$$

(4)

with $n_d$ and $m$ the electron density and mass, $\tau$ the momentum relaxation time and $D_d = v_F^2 \tau / d$ the bare diffusion coefficient. We define the correlation length $\xi$ in the metallic regime as $\sigma(0) = \xi^{2-d}$. One observes that to the leading order $\xi \sim \tau^{1/(2-d)}$, and then $\omega \tau \sim \omega \xi^{2-z}$, in conflict with scaling. However, the dominant (scale-dependent) contributions to $\sigma(\omega)$ in the limit $\omega \to 0$ will obey the scaling property, as shown below.

A RG equation for $\sigma(\omega)$ as a function of $\omega$ is obtained by considering the dimensionless conductance $g_\omega(L_\omega) = L_\omega^{d-2} \sigma(\omega)$ as a function of $L_\omega$:

$$\frac{d \ln g_\omega}{d \ln L_\omega} = \beta(g_\omega).$$

(5)

From a RG perspective, $g_\omega(L_\omega)$ is equivalent to $g(L)$, introduced in equation (1), in the sense that only a different cutoff procedure is used, while the critical properties of $g_\omega(L_\omega)$ and $g(L)$ are identical. In principle, the $\beta$-function may be obtained from a calculation of the conductance in perturbation theory in the disorder. The dependence on length $L_\omega$ may be represented in the form of a power series in $\ln(L_\omega / L_0)$, where $L_0$ is a reference length (later to be taken equal to $l$). One thus has a double power series in the disorder parameter $\lambda = 1/(\pi k_F l)$ and $\ln(L_\omega / L_0)$:

$$g_\omega = g_0(\lambda) + g_1(\lambda) \ln(L_\omega / L_0) + g_2(\lambda) \ln^2(L_\omega / L_0) + \cdots$$

(6)

and

$$g_\nu(\lambda) = \sum_{n=-1}^{\infty} a_{\nu n} \lambda^n.$$  

(7)

The $\beta$-function may be extracted from this power series by taking the derivative of $g_\omega$ with respect to $\ln L_\omega$ and then putting $L_\omega = L_0$:

$$\beta(g_0) = \frac{1}{g_0} g_1(\lambda(g_0)).$$

(8)

Here we defined $g_\omega(L_0) = g_0$, and $\lambda(g_0)$ is the inverse of the function $g_0(\lambda)$.

### 3. Perturbation theory

Scale-dependent quantum corrections to the conductivity are generated by diagrams with infrared divergent diffusion propagators (diffusons). As shown by Hikami [6], the perturbation theory may be organized in terms of diffusons connected by ‘Hikami boxes’. At two-loop order we may distinguish diagrams with two and with three diffusons (see figure 1), with four-vertex and six-vertex Hikami boxes $h_4$ and $h_6$, respectively (see figure 2). In addition, as shown in [9], gauge invariance requires a family of one-diffuson diagrams $h_2$ to be added within the perturbation scheme (see figure 3).
Figure 1. Diagrams contributing to the two-loop corrections to the conductance. Dashed squares represent four-vertex vector Hikami boxes $h_4$, while the hexagon represents the six-vertex scalar Hikami box $h_6$. Wavy lines denote diffusons.

$$h_4 = \begin{array}{c} \begin{array}{c} \text{(a)} \end{array} \\ \begin{array}{c} \text{(b)} \end{array} \end{array}$$

$h_6 = \begin{array}{c} \begin{array}{c} \text{(a)} \end{array} \\ \begin{array}{c} \text{(b)} \end{array} \\ \begin{array}{c} \text{(c)} \end{array} \end{array}$

Figure 2. Diagrams contributing to the four-vertex vector Hikami box $h_4$ (top) and the six-vertex scalar Hikami box $h_6$ (bottom). Dashed lines represent impurity scattering.

Figure 3. Additional two-vertex (one-diffuson) diagrams $h_2$ required by gauge invariance for two-loop corrections to the conductance.
In the lowest order, we then obtain the following contribution to $\sigma(\omega)$:

$$\sigma_{2\text{-loop}}(\omega) = \frac{1}{2\pi} \int \frac{d^d q}{(2\pi)^d} D(q, \omega) \int \frac{d^d Q}{(2\pi)^d} F(q, Q, \omega),$$

(9)

$$F(q, Q, \omega) = D(Q, \omega) \left[ \frac{1}{d} h_4^2(q, Q) D(q + Q, \omega) + h_6(q, Q) + h_2(Q) \right].$$

(10)

In this expression, we have singled out one diffuson with momentum $q$ and collected all the remaining factors in the integral of the function $F(q, Q, \omega)$.

Quite generally the diffusion propagator $D(q, \omega)$ is obtained by summing the particle–hole ladder diagrams as

$$D(q, \omega) = \frac{n_{\text{imp}} u_0^2}{1 - n_{\text{imp}} u_0^2 \Pi(q, \omega)},$$

(11)

where $u_0$ is a short-range impurity scattering potential, $n_{\text{imp}}$ is the impurity density and

$$\Pi(q, \omega) = \frac{1}{V} \sum_k G_{k+q}^R(E) G_k^A(E).$$

(12)

Here the retarded (advanced) Green function of electrons of mass $m$ is given by

$$G^R_k(E) = [G^A_k(E)]^* = \frac{1}{E_F - k^2/2m + i\tau},$$

(13)

with $E_F$ the Fermi energy and $\tau$ the single-particle relaxation time, defined in terms of the disorder potential by $\tau^{-1} = 2\pi v u n_{\text{imp}} u_0^2$.

We are interested in the momentum regime $q < k_F$, anticipating that all momentum integrals are convergent and the contribution from the momenta $q > k_F$ is negligible ($k_F$ is the Fermi wave vector). We consider the weak disorder regime only, where $k_F l \gg 1$. In the limit of small momentum, $q l \ll 1$, the diffusion has the usual diffusion pole form

$$D(q, \omega) \approx \frac{1}{2\pi \tau} \frac{1}{D_0 q^2 + \omega}.$$ 

(14)

In the same diffusive limit $q l, Q l \ll 1$, one has to retain only diagram (a) in the four-vertex Hikami box $h_4$ and diagrams (a)–(c) in the six-vertex Hikami box $h_6$ (see figure 3) while the single diffusion diagrams of figure 2 are not important. This yields

$$h_4^2(Q, q) = -(4\pi v d D_d)^2 \tau^6 (Q + q)^2,$$

(15)

$$h_6(Q, q) = 4\pi v d D_d \tau^4,$$

(16)

$$h_2(Q) = 0.$$ 

(17)

We now substitute these results into the above expression for the conductivity, first concentrating on the contribution from the diffusive regime

$$\sigma_{2\text{-loop}}^d(\omega) = \frac{D_d}{2\pi^2 v d} \int \frac{d^d q d^d Q}{(2\pi)^{2d}} \left[ \frac{1}{D_0 q^2 + \omega} \frac{1}{D_0 Q^2 + \omega} \right] \left[ 1 - \frac{2D_d(q + Q)^2}{D_d(q + Q)^2 + \omega} \right].$$

(18)

In low dimensions, $d < 2$, the integrals are convergent, yielding the scale-dependent result

$$\sigma_{2\text{-loop}}^d(\omega) = C_{d<2} v_d D_d \left( \frac{\omega v_d^2}{\xi^d} \right)^{(2-d)/d} D_d^{d/(2-d)} L_{\omega}^{4-2d},$$

(19)

$$= C_{d<2} \xi^{2-d} \left[ \frac{L_{\omega}}{\xi} \right]^{4-2d}.$$
where the constant is given by

\[
C_d < 2 = \frac{1}{\pi^2} \int \frac{d^d x \, d^d y}{(2\pi)^{2d}} \frac{1}{x^2 + 1} \frac{1}{y^2 + 1} \left[ \frac{d}{2} - 1 + \frac{1}{(x+y)^2 + 1} \right].
\] (20)

The integrals may be reduced to a single integral of a hypergeometric function

\[
C_d < 2 = 2^{-2d} \pi^{d-2} \left[ \frac{1}{2} - \frac{1}{d} \right] \Gamma^2 \left( 1 - \frac{d}{2} \right) + \frac{2\Gamma(d-3)}{d(4-d)} \int_0^1 dt \, F \left( 2 - \frac{d}{2}, \frac{d}{2}, 3 - \frac{d}{2}, 1 - t + t^2 \right)
\] (21)

with \( C_1 = -1/(24\pi^2) \) and \( C_{2-\epsilon} = -1/(16\pi^4 \epsilon) \), at \( \epsilon \ll 1 \). Thus, in the limit \( \epsilon \to 0 \), we find the logarithmic scaling dependence

\[
\sigma_{2-\epsilon}^{d-\text{loop}}(\omega) = -\frac{1}{16\pi^4 v_2 D_2} \frac{\omega^{-\epsilon}}{\epsilon} \to \ln \omega \frac{\ln L_\omega}{16\pi^4 v_2 D_2} = -\frac{\ln L_\omega}{8\pi^4 v_2 D_2}
\] (22)

which is in agreement with the known two-loop correction result to the conductivity in \( d = 2 \) dimensions (here obtained by approaching \( d = 2 \) from below).

The same result can be obtained directly in \( d = 2 \), without resorting to the \( \epsilon \) expansion. If we assume that \( q < Q \), then the infrared-divergent part of equation (9) originates from the integral over \( q \), while the \( Q \)-integral yields the numerical prefactor. The diffusons and Hikami boxes entering the function \( F(q, Q, \omega) \) should be calculated with increased precision taking into account the contribution from high (ballistic) values of \( Q \). This requires taking into account all the diagrams shown in figures 2 and 3. Ballistic contributions make the integral over \( Q \) convergent, leading to the same logarithmic correction (22).

We now turn to dimensions \( d > 2 \), for which the momentum integrals in equation (18) are not converging. In order to regularize these integrals, we first take a logarithmic derivative with respect to frequency \( \omega \):

\[
\frac{d}{d \ln \omega} \sigma_d^{d-\text{loop}}(\omega) = I_1 + I_2,
\] (23)

where

\[
I_1 = -\frac{D_d}{\pi^2 v_d} \int \frac{d^d q \, d^d Q}{(2\pi)^{2d}} \frac{\omega}{(D_d q^2 + \omega)^2} \frac{1}{D_d Q^2 + \omega}
\times \left[ 1 - \frac{2D_d(q + Q)^2/d}{D_d(q + Q)^2 + \omega} \right].
\] (24)

\[
I_2 = \frac{d}{d \pi^2 v_d} \int \frac{d^d q \, d^d Q}{(2\pi)^{2d}} \frac{\omega}{D_d q^2 + \omega}
\times \frac{1}{D_d Q^2 + \omega} \frac{D_d(q + Q)^2}{[D_d(q + Q)^2 + \omega]^2}.
\] (26)

While \( I_2 \) converges provided \( d < 3 \) (for a further regularization, see below), \( I_1 \) is still ultraviolet divergent. Including the contribution to the momentum integral on \( Q \) from the ballistic regime,
C. estimate the crossover frequency as more generally, is a consequence of the gauge invariance. Therefore, replacing $F(q, Q, \omega)$ by $F(q, Q, \omega) - F(0, Q, 0)$ in equation (28), we obtain convergent integrals in the ultraviolet, which allows us to use the diffusive limit expressions for $F(q, Q, \omega)$. Adding the contribution $I_2$, and reformulating the integrals using the symmetry under $q \leftrightarrow Q$, we then find

\[
\frac{d}{d \ln \omega} \sigma^d_{2\text{-loop}}(\omega) = \frac{d}{d \ln \omega} \frac{D_d}{d \pi^2 v_d} \int \omega \frac{d^d q \ d^d Q}{(2\pi)^d D_d q^2 + \omega} \frac{1}{D_d Q^2 + \omega} \times \left[ \frac{1}{D_d(q+Q)^2 + \omega} - \frac{(1-d/2)\omega}{D_d^2 q^2 Q^2} \right].
\]

The scaling contribution to the conductivity in dimensions $2 < d < 3$ takes the same form as in equation (19) where the constant $C$ is now given by

\[
C_{d=2} = \frac{1}{d \pi^2} \int \frac{d^d x \ d^d y}{(2\pi)^d} \frac{1}{x^2 + 1} \frac{1}{y^2 + 1} \left[ \frac{d}{2} - 1 \right] \left[ \frac{1}{x^2 y^2} + \frac{1}{(x+y)^2 + 1} \right].
\]

The integrals may be done to give the same expression as for $d < 2$, namely equation (21). This means that by analytic continuation in the complex $d$-plane one may pass from the regime $d < 2$ to the regime $d > 2$, around the singularity at $d = 2$. For dimensions close to integer values, we obtain $C_{2+\epsilon} = 1/(16\pi^4 \epsilon) = -C_{2-\epsilon}$ and $C_{3-\epsilon} = 1/(96\pi^4 \epsilon)$.

The logarithmic ultraviolet divergence in the case of three dimensions is cut off by the upper limit $q = q_0 \approx 1/\ell$, leading to a logarithmic contribution

\[
\sigma^d_{2\text{-loop}}(\omega) = -\frac{\omega \ln \omega}{96\pi^4 v_3 D_3^2} \rightarrow \frac{1}{48\pi^4} \ell^{-1} \ln(c L_0/\xi) \frac{L_0/\xi}{(L_0/\xi)^2}.
\]

Subleading corrections requiring a precise calculation of the contribution from the ballistic regime lead to the factor $1/\xi$ and the constant $c$ under the logarithm in the last equation. We now analytically continue from the imaginary frequency axis to the real axis by replacing the Matsubara frequency $\omega$ by the real frequency $\Omega = i\omega$. We note that $\text{Re} \sigma^d_{2\text{-loop}}(\Omega) \sim |\Omega|$ and $\text{Im} \sigma^d_{2\text{-loop}}(\Omega) \sim \Omega \ln|\Omega|$ in that case.

It is remarkable that the correction term to $\text{Re} \sigma(\Omega)$ varies linearly with $\Omega$,

\[
\text{Re} \sigma^d_{2\text{-loop}}(\Omega) = \sigma_0 \frac{9\pi^2}{32} |k_F\ell|^{-3} \frac{|\Omega|}{\epsilon_F},
\]

and is thus dominant at frequencies $\Omega < \Omega_c$, compared to the leading finite frequency correction $~\sim -\sigma_0(\Omega \tau)^2$ of the Drude law. Moreover, the scale-dependent term is positive in contrast to the Drude correction, so that we predict that the conductivity has a maximum at $\omega = \Omega_c/2$. We estimate the crossover frequency as

\[
\Omega_c = \frac{9\pi}{8} (k_F\ell)^{-5} \epsilon_F.
\]

At finite temperature the RG flow is cut off by phase relaxation effects. This is because in the presence of interaction the number of particles at a given energy is no longer conserved,
giving rise to a phase relaxation term $1/\tau_\phi$ in the independent particle diffusion pole expression (14). The phase relaxation rate is temperature dependent and vanishes at $T = 0$ (for calculations of $1/\tau_\phi$ in two dimensions, see [12, 13]). At finite $T$ one may replace $\Omega_1$ by $\Omega_1 + i/\tau_\phi$ in equation (32). At $\Omega \ll i/\tau_\phi$, one then finds $\sigma \sim \Omega_1^2 \tau_\phi$, which is distinguished from the Drude contribution by a temperature-dependent prefactor growing with decreasing temperature and is larger than the Drude term if $1/\tau_\phi < \Omega_1$.

4. Renormalization group equation

We are now in a position to determine the leading term in a $1/g_\omega$ expansion of the RG-$\beta$ function for any dimension $d$. For dimensions $2 < d < 3$, we then find, following the steps outlined in the introduction,

$$\frac{d \ln g_\omega}{d \ln L_\omega} = \beta_\omega (g_\omega) = d - 2 - 2(d - 2)C_d \frac{1}{g_\omega^2} + \cdots. \quad (34)$$

In the limit $d \to 2$, putting $d = 2 + \epsilon$, we recover the known result (now approaching $d = 2$ from above)

$$\beta_\omega (g_\omega) = \epsilon - \frac{1}{8\pi^4} \frac{1}{g_\omega^2} + \cdots. \quad (35)$$

The case $d = 3$ requires special attention. As discussed above, in that case a logarithmic correction factor in the scale dependence is found [11]. Consequently, the $\beta$-function also acquires a logarithmic correction factor

$$\beta_\omega (g_\omega) = 1 - a^2 \ln g_\omega + b \frac{g_\omega^2}{g_\omega^2} + \cdots. \quad (36)$$

The prefactor $a^2 = 1/48\pi^4$, while the constant $b$ has not been calculated yet, but preliminary estimates give a value of order unity. Defining a rescaled conductance $\overline{g}_\omega = g_\omega/a$ (observing $\beta_\omega (g_\omega) = \beta_\omega (\overline{g}_\omega)$) and assuming that the constant $c = b + \ln a \simeq 1 - \delta$, $\delta \ll 1$, we find an unstable fixed point of the RG equation at $\overline{g}_\omega^* \simeq 1 - \delta$. The derivative of $\beta_\omega (\overline{g}_\omega)$ at $\overline{g}_\omega^*$ is found as $\frac{\beta_\omega (\overline{g}_\omega^*)}{g_\omega^*} \simeq 1 - \delta$. The critical exponent follows as

$$s = (\overline{g}_\omega^* \beta_\omega (\overline{g}_\omega^*))^{-1} \simeq 1 + 2\delta. \quad (37)$$

It will be interesting to see the result of a careful calculation of the coefficient $b$.

5. Conclusion

In the above we presented a derivation of the leading scale-dependent contribution to the dynamical conductivity of a disordered system in the unitary symmetry class within the model of non-interacting fermions. We showed how the scaling terms may be extracted from the perturbation theory in any dimension $d$. On the basis of these results, we determined the leading term in the renormalization group (RG) $\beta$-function in the regime of dimensions $2 \leq d \leq 3$. It is interesting to note that the leading correction term in $\beta$ at large conductance $g$ varies as $g^{-2}$, with the $d$-dependent coefficient, evolving into a logarithmic correction at $d = 3$. Interestingly enough, the fixed point of the RG equation depends on the numerical coefficient in the argument of the logarithm. A calculation of this coefficient requires a careful
consideration of contributions from the ballistic regime, which has not yet been done. The
dynamical conductivity $\text{Re} \sigma(\omega + i0)$ at low $\omega$, in $d = 3$, is found to vary as $|\omega|$ (rather than
$\omega^2$ as predicted by Drude theory), which should be observable.

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