DISCRETE TIME-DEPENDENT WAVE EQUATIONS I. SEMICLASSICAL ANALYSIS

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Abstract. In this paper we consider a semiclassical version of the wave equations with singular Hölder time-dependent propagation speeds on the lattice \( \mathbb{hZ}^n \). We allow the propagation speed to vanish leading to the weakly hyperbolic nature of the equations. Curiously, very much contrary to the Euclidean case considered by Colombini, de Giorgi and Spagnolo [2] and by other authors, the Cauchy problem in this case is well-posed in \( \ell^2(\mathbb{hZ}^n) \). However, we also recover the well-posedness results in the intersection of certain Gevrey and Sobolev spaces in the limit of the semiclassical parameter \( \hbar \to 0 \).

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1. INTRODUCTION

In this paper we are interested in the wave equation on the lattice

\[ h\mathbb{Z}^n = \{ x \in \mathbb{R}^n : x = hk, \; k \in \mathbb{Z}^n \}, \]

depending on a (small) parameter \( h > 0 \), and in the behaviour of its solutions as \( h \to 0 \). The Laplacian on \( h\mathbb{Z}^n \) is denoted by \( L_h \) and is defined by

\[ L_h u(k) = \sum_{j=1}^{n} (u(k + h v_j) + u(k - h v_j)) - 2nu(k), \]

where \( v_j \) is the \( j^{th} \) basis vector in \( \mathbb{Z}^n \), having all zeros except for 1 as the \( j^{th} \) component.

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The semiclassical analogue of the classical wave equation with time-dependent coefficients on the lattice $\hbar \mathbb{Z}^n$ is given by the Cauchy problem

$$
\begin{cases}
\partial_t^2 u(t, k) - \hbar^{-2} a(t) \mathcal{L}_n u(t, k) + b(t) u(t, k) = f(t, k), & \text{with } t \in (0, T], \\
u(0, k) = u_0(k), & k \in \hbar \mathbb{Z}^n, \\
\partial_t u(0, k) = u_1(k), & k \in \hbar \mathbb{Z}^n,
\end{cases}
$$

for the time-dependent propagation speed $a = a(t) \geq 0$ and bounded real-valued functions $b = b(t) \geq 0$, as well as $f \in L^\infty([0, T], \ell^2(\hbar \mathbb{Z}^n))$. The equation (1.1) is the semiclassical discretisation of the classical wave equation on $\mathbb{R}^n$ with the Cauchy problem in the form

$$
\begin{cases}
\partial_t^2 v(t, x) - a(t) \Delta v(t, x) + b(t) v(t, x) = f(t, x), & x \in \mathbb{R}^n, \\
v(0, x) = u_0(x), \\
\partial_t v(0, x) = u_1(x),
\end{cases}
$$

where $\Delta$ is the usual Laplacian on $\mathbb{R}^n$. Here we can think of the Cauchy data in (1.1) as the evaluations of the Cauchy data from (1.2) on the lattice $\hbar \mathbb{Z}^n$.

There is an extensive literature concerning (1.2) going back to the seminal paper by Colombini, de Giorgi and Spagnolo [2]. In particular, some very peculiar things can happen with the solvability of (1.2). For example, it has been shown by Colombini and Spagnolo in [5] and by Colombini, Jannelli and Spagnolo in [3] that even in one space dimension, the Cauchy problem (1.2) does not have to be well-posed in $C^\infty$ if $a \in C^\infty$ is not strictly positive or if it is in the Hölder class $a \in C^\alpha$ for $0 < \alpha < 1$. Therefore, to ensure the well-posedness of (1.2), one is forced to work in Gevrey spaces $\gamma^s$.

In the analysis of (1.2), it makes sense to distinguish between the following four cases depending on the properties of the propagation speed $a(t)$, with $C^\alpha([0, T])$ denoting the Hölder class (see (2.1)):

- **Case 1:** $a \in \text{Lip}([0, T]), \min_{[0, T]} a(t) > 0$,
- **Case 2:** $a \in C^\alpha([0, T])$, with $0 < \alpha < 1$, $\min_{[0, T]} a(t) > 0$,
- **Case 3:** $a \in C^l([0, T])$, with $l \geq 2$, $a(t) \geq 0$,
- **Case 4:** $a \in C^\alpha([0, T]), \min_{[0, T]} a(t) > 0$.

Indeed, the proofs of all these cases are different, also yielding results of different type in several cases. Here, Case 1 is the best ‘classical’ situation, Case 2 is lower regularity strictly hyperbolic case, Case 3 is the regular weakly hyperbolic case, and Case 4 is the worst one when $a$ is of low regularity and may also vanish. Let us briefly summarise the known results for these cases, which we formulate, for simplicity, for the case $f \equiv 0$.

**Case 1:** For any $s \in \mathbb{R}$, if the initial Cauchy data $(u_0, u_1)$ are in the Sobolev spaces $H^{s+1}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$, then there exists a unique solution of (1.2) in the space $C([0, T], H^{s+1}(\mathbb{R}^n)) \cap C^1([0, T], H^s(\mathbb{R}^n))$. 
Case 2: If the initial Cauchy data \((u_0, u_1)\) are in \(\gamma^s \times \gamma^s\), then there exists a unique solution of (1.2) in \(C^2([0, T], \gamma^s)\), provided that
\[
1 \leq s < 1 + \frac{\alpha}{1 - \alpha}.
\] (1.3)

Case 3: If the initial Cauchy data \((u_0, u_1)\) are in \(\gamma^s \times \gamma^s\), then there exists a unique solution of (1.2) in \(C^2([0, T], \gamma^s)\), provided that
\[
1 \leq s < 1 + \frac{l}{2}.
\] (1.4)

Case 4: If the initial Cauchy data \((u_0, u_1)\) are in \(\gamma^s \times \gamma^s\), then there exists a unique solution of (1.2) in \(C^2([0, T], \gamma^s)\), provided that
\[
1 \leq s < 1 + \frac{\alpha}{2}.
\] (1.5)

The situation in Case 1 is a classical result. Case 2 was shown in [2] with further extensions in [4] and [7] for equations of higher order on \(\mathbb{R}\) and \(\mathbb{R}^n\), respectively. Case 3 was shown, in particular, in [10], and the higher dimensional case including low order terms was analysed in [8]. For the analysis in Case 4 on \(\mathbb{R}^n\), [7] can be referred.

The situation with (1.1) is in a striking difference with the above results for (1.2), in the sense that (1.1) is always well-posed in \(\ell^2(h\mathbb{Z}^n)\) for all the Cases 1-4. In some sense, it is natural since no regularity issues on the lattice are involved, so there is no noticeable loss of regularity.

**Theorem 1.1.** Let \(T > 0\). Assume \(b \in L^\infty([0, T])\) satisfies \(b(t) \geq 0\) for all \(t \in [0, T]\). Then in all the cases, Case 1-Case 4, the Cauchy problem (1.1) is well-posed in \(\ell^2(h\mathbb{Z}^n)\). In particular, if \(u_0, u_1 \in \ell^2(h\mathbb{Z}^n)\) and \(f \in L^2([0, T], \ell^2(h\mathbb{Z}^n))\), then for every \(t \in [0, T]\), we have \(u(t), \partial_t u(t) \in \ell^2(h\mathbb{Z}^n)\). Moreover, for each \(h > 0\), there exists a constant \(C_{h,T} > 0\) such that
\[
\|u(t)\|_{\ell^2(h\mathbb{Z}^n)}^2 + \|\partial_t u(t)\|_{\ell^2(h\mathbb{Z}^n)}^2 
\leq C_{h,T}\left(\|u_0\|_{\ell^2(h\mathbb{Z}^n)}^2 + \|u_1\|_{\ell^2(h\mathbb{Z}^n)}^2 + \|f\|_{L^2([0,T];\ell^2(h\mathbb{Z}^n))}^2\right),
\] (1.6)

for all \(u_0, u_1 \in \ell^2(h\mathbb{Z}^n)\) and \(f \in L^2([0,T];\ell^2(h\mathbb{Z}^n))\).

At the same time, we can explain why there is such a difference in the well-posedness results between the settings of \(\mathbb{R}^n\) and \(h\mathbb{Z}^n\). For this we need to specify the constants \(C_{h,T}\) appearing in (1.6), especially their dependence on \(h\). Clearly, this constant may go to infinity as \(h \to 0\), especially in Cases 2-4. The following theorem shows that under the assumptions that the solutions in \(\mathbb{R}^n\) exist, they can be recovered in the limit as \(h \to 0\). We require a little additional Sobolev regularity to ensure that the convergence results are global on the whole of \(\mathbb{R}^n\).

**Theorem 1.2.** Let \(u\) and \(v\) be the solutions of the Cauchy problems (1.1) on \(h\mathbb{Z}^n\) and (1.2) on \(\mathbb{R}^n\), respectively, with the same Cauchy data \(u_0, u_1\). Now,

1) for the Case 1, if \((u_0, u_1) \in H^{s+1}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)\) with \(s \geq 5\) for \(n \leq 3\) and \(s > 3 + \frac{n}{2}\) for \(n \geq 4\), and

2) for the Case 2-Case 4, if \((u_0, u_1) \in (\gamma^s(\mathbb{R}^n) \cap H^{p+1}(\mathbb{R}^n)) \times (\gamma^s(\mathbb{R}^n) \cap H^p(\mathbb{R}^n))\) satisfies the assumptions (1.3)-(1.5) and \(p \geq 5\),
then for every \( t \in [0, T] \), we have
\[
\|\partial_t u(t) - \partial_t v(t)\|_{\ell^2(\mathbb{Z}^n)} + \|u(t) - v(t)\|_{\ell^2(\mathbb{Z}^n)} \to 0 \quad \text{as} \quad h \to 0.
\] (1.7)

Thus, when \( h \to 0 \), for the above mentioned space, we actually recover the well-posedness results on \( \mathbb{R}^n \), where the solution \( u \) in the above statement becomes restricted to the lattice \( h\mathbb{Z}^n \).

We note that the symbolic calculus of pseudo-differential operators on lattices (or pseudo-difference operators) has been developed in [1]. In particular, the symbol of \( L_h \) defined by \( \sigma_{L_h}(k, \theta) = e^{-2\pi i k \cdot \theta} L_h(e^{2\pi i k \cdot \theta}) \) is given by
\[
\sigma_{L_h}(k, \theta) = e^{-2\pi i k \cdot \theta} \left( \sum_{j=1}^{n} \left( e^{2\pi i j \cdot (k + hv_j)} - e^{2\pi i j \cdot (k - hv_j)} \right) - 2ne^{2\pi i k \cdot \theta} \right) 
\]
\[
= \sum_{j=1}^{n} \left( e^{2\pi i j \cdot \theta} - e^{-2\pi i j \cdot \theta} \right) - 2n 
\]
\[
= 2 \sum_{j=1}^{n} \cos(2\pi \theta_j) - 2n, \quad \text{for all} \quad (k, \theta) \in h\mathbb{Z}^n \times \mathbb{T}^n.
\] (1.8)

with \((k, \theta) \in h\mathbb{Z}^n \times \mathbb{T}^n\), and it is independent of \( k \) (and of \( h \)).

**Definition 1.3. (Symbol class \( S^m(\mathbb{Z}^n \times \mathbb{T}^n) \)).** Let \( m \in (-\infty, \infty) \). We say that a function \( \sigma : h\mathbb{Z}^n \times \mathbb{T}^n \to \mathbb{C} \) belongs to \( S^m(\mathbb{Z}^n \times \mathbb{T}^n) \) if \( \sigma(k, \cdot) \in C^\infty(\mathbb{T}^n) \) for all \( k \in h\mathbb{Z}^n \), and for all multi-indices \( \alpha, \beta \) there exists a positive constant \( C_{\alpha,\beta,h} \) such that
\[
\left|D^{(\alpha)}_\theta \Delta^\beta_k \sigma(k, \theta)\right| \leq C_{\alpha,\beta,h}(1 + |k|)^{m-|\alpha|}, \quad \text{for all} \quad k \in h\mathbb{Z}^n \quad \text{and} \quad \theta \in \mathbb{T}^n.
\] (1.9)

We denote by \( \text{Op}(\sigma) \) the pseudo-difference operator with symbol \( \sigma \) given by
\[
\text{Op}(\sigma)f(k) := \int_{\mathbb{T}^n} e^{2\pi i k \cdot \theta} \sigma(k, \theta) \hat{f}(\theta) d\theta, \quad k \in h\mathbb{Z}^n,
\] (1.10)

where
\[
\hat{f}(\theta) = \sum_{k \in h\mathbb{Z}^n} e^{-2\pi i k \cdot \theta} f(k), \quad \theta \in \mathbb{T}^n, \quad k \in h\mathbb{Z}^n.
\] (1.11)

We refer to Section 3 for further explanations in this direction, and to [1] for thorough details. The proof of Theorem 1.1, given in Section 3, will rely on some elements of the Fourier analysis on the lattice \( \mathbb{Z}^n \). Difference equations on lattices, including Schrödinger equations have been studied (e.g. in [11–15]) by developing the analysis in terms of kernels. The general symbolic calculus of pseudo-differential operators on \( \mathbb{Z}^n \) has been recently established in [1]. It allows one to refine other previously known results, e.g. conditions for the boundedness of pseudo-differential operators in the lattice as e.g. in [16]. The symbolic calculus on \( \mathbb{Z}^n \) can be thought of as a dual one to the calculus developed on the torus in [18,19,21] in the framework of more general research on pseudo-differential operators on compact Lie groups [20,22].

In Section 2, we establish some results concerning ordinary differential equations that will be instrumental in the proof of Theorem 1.1. Moreover, we establish them with explicit expressions on the appearing constants. While this is not needed for the
proof of Theorem 1.1, this will be useful in the subsequent paper where we will treat the case of distributional $a, b$ and $f$. In Section 3, we prove Theorem 1.1. Finally, in Section 4, we discuss the limiting behaviour of solutions to (1.1) in the limit of the semiclassical parameter $\hbar \rightarrow 0$.

To simplify the notation, throughout the paper we will be writing $A \lesssim B$ if there exists a constant $C$ independent of the appearing parameters such that $A \leq CB$.

2. PREPARATORY ESTIMATES

In the next proposition, we prove certain energy estimates for second order ordinary differential equations with explicit dependence on parameters. This will be crucial in the second part of this paper when we will be considering distributional coefficients. It extends the result obtained in [17] from the point of view of inclusion of the lower order terms and the explicit control on the constants.

Notation-wise, we write that $a \in C^\alpha([0,T])$ if for some constant $L_a$, we have

$$|a(t) - a(s)| \leq L_a|t - s|^{\alpha},$$

(2.1)

for all $t, s \in [0,T]$. The smallest $L_a$ in the above inequality will be denoted by $\|a\|_{C^\alpha([0,T])}$.

**Proposition 2.1.** Let $T > 0$ and $\beta > 0$ be positive constants, let $b(t)$ be a bounded real-valued function and let $a(t)$ be a function satisfying one of the following conditions:

Case 1: $a \in \text{Lip}([0,T]), a_0 := \min_{[0,T]} a(t) > 0$;

Case 2: $a \in C^\alpha([0,T]),$ with $0 < \alpha < 1$, $a_0 := \min_{[0,T]} a(t) > 0$;

Case 3: $a \in C^l([0,T])$, with $l \geq 2$, $a(t) \geq 0$;

Case 4: $a \in C^\alpha([0,T]),$ with $0 < \alpha < 2$, $a(t) \geq 0$.

Consider the following Cauchy problem:

$$\left\{ \begin{array}{l}
 v''(t) + \beta^2 a(t)v(t) + b(t)v(t) = 0, \quad t \in (0,T], \\
 v(0) = v_0 \in \mathbb{C}, \\
 v'(0) = v_1 \in \mathbb{C}.
 \end{array} \right.$$  

(2.2)

Then the following holds:

Case 1: There exist two positive constants $C_1, K_1 > 0$ such that for all $t \in [0,T]$, we have

$$|v(t)|^2 + |v'(t)|^2 \leq C_1 e^{K_1 T \beta^2} (|v_0|^2 + |v_1|^2),$$  

(2.3)

for all $\beta > 0$.

Case 2: There exist two positive constants $C_2, K_2 > 0$ such that for all $t \in [0,T]$, we have

$$|v(t)|^2 + |v'(t)|^2 \leq C_2 e^{K_2 T \beta^2} (|v_0|^2 + |v_1|^2),$$  

(2.4)

for any $0 < s \leq \frac{T}{2}$ and for all $\beta > 0$.

Case 3: There exist two positive constants $C_3, K_3 > 0$ such that for all $t \in [0,T]$, we have

$$|v(t)|^2 + |v'(t)|^2 \leq C_3 e^{K_3 \beta^{\frac{4}{\sigma}}} (|v_0|^2 + |v_1|^2),$$  

(2.5)

with $\sigma = 1 + \frac{1}{\beta}$ and for all $\beta > 0$. 

Case 4: There exist two positive constants $C_4, K_4 > 0$ such that for all \( t \in [0, T] \), we have
\[
|v(t)|^2 + |v'(t)|^2 \leq C_4 e^{K_4 T(\beta)^{\frac{3}{4}}} (|v_0|^2 + |v_1|^2),
\]
for any \( 0 < s \leq \frac{\alpha + 2}{4} \) and for all \( \beta > 0 \).
The constants \( C_j \)'s and \( K_j \)'s for \( j = 1, 2, 3 \) and \( 4 \) may depend on \( T \) but not on \( \beta \).

**Proof.** First, we reduce the problem (2.2) to a first order equation. We define
\[
V(t) := \begin{pmatrix} iv(t) \\ \partial_t v(t) \end{pmatrix}, \quad V_0 := \begin{pmatrix} iv_0 \\ v_1 \end{pmatrix},
\]
and the matrices
\[
A(t) := \begin{pmatrix} 0 & 1 \\ a(t) & 0 \end{pmatrix}, \quad B(t) := \begin{pmatrix} 0 & 1 - \beta^2 \\ b(t) & 0 \end{pmatrix}.
\]
This allows us to reformulate the given second order system (2.2) as the first order system
\[
\begin{cases}
V' = i\beta^2 A(t)V(t) + iB(t)V(t), \\
V(0) = V_0.
\end{cases}
\]
Now we will treat each case separately.

**Case 1:** \( a \in \text{Lip}([0, T]), a_0 := \min_{[0,T]} a(t) > 0 \).

We observe that the eigenvalues of the matrix \( A(t) \) are given by \( \pm \sqrt{a(t)} \). The symmetrizer \( S \) of \( A \), i.e., the matrix such that
\[
SA - A^*S = 0,
\]
is given by
\[
S(t) = \begin{pmatrix} a(t) & 0 \\ 0 & 1 \end{pmatrix}.
\]
Consider
\[
(S(t)V(t), V(t)) = a(t)|v(t)|^2 + |v'(t)|^2 
\leq \max_{t \in [0,T]} \{ a(t), 1 \}|v(t)|^2 + \max_{t \in [0,T]} \{ a(t), 1 \}|v'(t)|^2 
= \max_{t \in [0,T]} \{ a(t), 1 \}|V(t)|^2.
\]
Similarly
\[
(S(t)V(t), V(t)) \geq \min_{t \in [0,T]} \{ a(t), 1 \}|V(t)|^2.
\]
If we now define the energy
\[
E(t) := (S(t)V(t), V(t)),
\]
then from (2.10), and (2.11), it follows that
\[
\min_{t \in [0,T]} \{ a(t), 1 \}|V(t)|^2 \leq E(t) \leq \max_{t \in [0,T]} \{ a(t), 1 \}|V(t)|^2.
\]
Since $a \in \text{Lip}([0, T])$, there exist two constants $a_0$ and $a_1$ such that

$$a_0 = \min_{t \in [0, T]} a(t) \quad \text{and} \quad a_1 = \max_{t \in [0, T]} a(t).$$

Thus, if we set $c_{0,a} = \min\{a_0, 1\}$ and $c_{1,a} = \max\{a_1, 1\}$, then the inequality (2.12) becomes

$$c_{0,a}|V(t)|^2 \leq E(t) \leq c_{1,a}|V(t)|^2. \quad (2.13)$$

Then we can calculate

$$E_t(t) = (S_t(t) V(t), V(t)) + (S(t)V(t), V(t)) + (S(t) V(t), V(t))$$

$$= (S_t(t) V(t), V(t)) + i\beta^2 (S(t) A(t) V(t), V(t)) + i (S(t) B(t) V(t), V(t)) - i\beta^2 (S(t) V(t), A(t) V(t)) - i (S(t) V(t), B(t) V(t))$$

$$= (S_t(t) V(t), V(t)) + i\beta^2 ((SA - A^*S) V(t), V(t)) +$$

$$i ((SB - B^*S) V(t), V(t))$$

$$= (S_t(t) V(t), V(t)) + i ((SB - B^*S) V(t), V(t)). \quad (2.14)$$

From the definition of $S$ and $B$, we get

$$SB - B^*S = \begin{pmatrix} 0 & a(1 - \beta^2) - b \\ b - a(1 - \beta^2) & 0 \end{pmatrix}.$$ 

Then

$$\|SB - B^*S\|_{L^\infty(0,T)} \leq (\beta)^2 \|a\|_{L^\infty(0,T)} + \|b\|_{L^\infty(0,T)}. \quad (2.15)$$

From using (2.15) in (2.14), we get

$$E_t(t) \leq \|S_t(t)\|_{L^\infty(0,T)}|V(t)|^2 + (\beta)^2 \|a\|_{L^\infty(0,T)} + \|b\|_{L^\infty(0,T)}|V(t)|^2$$

$$= \left(\|a\|_{L^\infty(0,T)} + (\beta)^2 \|a\|_{L^\infty(0,T)} + \|b\|_{L^\infty(0,T)}\right)|V(t)|^2. \quad (2.16)$$

Applying (2.13) in (2.16), we get

$$E_t(t) \leq c_{0,a}^{-1} \left(\|a\|_{L^\infty(0,T)} + (\beta)^2 \|a\|_{L^\infty(0,T)} + \|b\|_{L^\infty(0,T)}\right) E(t)$$

$$= \kappa E(t), \quad (2.17)$$

where $\kappa = c_{0,a}^{-1} \left(\|a\|_{L^\infty(0,T)} + (\beta)^2 \|a\|_{L^\infty(0,T)} + \|b\|_{L^\infty(0,T)}\right) > 0$. Applying the Gronwall lemma to (2.17), we deduce that

$$E(t) \leq e^{\kappa T} E(0) = c_{0,a}^{-1} \left(\|a\|_{L^\infty(0,T)} + (\beta)^2 \|a\|_{L^\infty(0,T)} + \|b\|_{L^\infty(0,T)}\right) T E(0). \quad (2.18)$$

Therefore, putting together (2.18) and (2.13) we obtain

$$c_{0,a}|V(t)|^2 \leq E(t) \leq c_{1,a} e^{c_{0,a}^{-1} \left(\|a\|_{L^\infty(0,T)} + (\beta)^2 \|a\|_{L^\infty(0,T)} + \|b\|_{L^\infty(0,T)}\right) T} E(0)$$

$$\leq c_{1,a} e^{c_{0,a}^{-1} \left(\|a\|_{L^\infty(0,T)} + (\beta)^2 \|a\|_{L^\infty(0,T)} + \|b\|_{L^\infty(0,T)}\right) T} |V(0)|^2. \quad (2.19)$$

If we set, $C_1 = c_{0,a}^{-1} c_{1,a} e^{c_{0,a}^{-1} \left(\|a\|_{L^\infty(0,T)} + \|b\|_{L^\infty(0,T)}\right) T}$, and $K_1 = c_{0,a}^{-1} \|a\|_{L^\infty(0,T)}$, then using the definition of $V(t)$ and (2.19), we obtain (2.3), that is,

$$|v(t)|^2 + |v'(t)|^2 \leq C_1 e^{K_1 T (\beta)^2} (|v_0|^2 + |v_1|^2),$$

finishing the proof of Case 1.
Case 2: \( a \in C^\alpha([0,T]) \), with \( 0 < \alpha < 1 \), \( a_0 := \min_{[0,T]} a(t) > 0 \).

Here we follow the method developed by Colombini and Kinoshita in [4]. We look for solutions of the form

\[
V(t) = e^{-\rho(t)\frac{1}{2} \left( \det H(t) \right)^{-1} H(t)W(t)},
\]

(2.20)

where

- \( s \in \mathbb{R} \) depends on \( \alpha \) as will be determined in the argument;
- the function \( \rho \in C^1([0,T]) \) is real-valued with \( \rho(0) = 0 \);
- \( W(t) \) is the energy;
- \( H(t) \) is the matrix defined by

\[
H(t) = \begin{pmatrix}
1 & 1 \\
\lambda_1^\epsilon(t) & \lambda_2^\epsilon(t)
\end{pmatrix},
\]

where for all \( \epsilon \in (0,1] \), \( \lambda_1^\epsilon(t) \) and \( \lambda_2^\epsilon(t) \) are regularisations of the eigenvalues of the matrix \( A(t) \) of the form

\[
\lambda_1^\epsilon(t) := \left( -\sqrt{a^* \phi_\epsilon} \right)(t),
\]

\[
\lambda_2^\epsilon(t) := \left( +\sqrt{a^* \phi_\epsilon} \right)(t),
\]

with \( \{ \phi_\epsilon(t) \}_{\epsilon>0} \), being a family of cut-off functions defined starting from a non-negative even function \( \phi \in C^\infty_0(\mathbb{R}) \), with \( \int_\mathbb{R} \phi = 1 \), by setting \( \phi_\epsilon(t) := \frac{1}{\epsilon} \phi(\frac{1}{\epsilon}) \).

By construction, it follows that \( \lambda_1^\epsilon, \lambda_2^\epsilon \in C^\infty([0,T]) \).

We observe the inequality

\[
\det H(t) = \lambda_2^\epsilon(t) - \lambda_1^\epsilon(t) \geq 2\sqrt{a_0}.
\]

(2.21)

Moreover, using the Hölder regularity of \( a(t) \) of order \( \alpha \) and, for all \( t \in [0,T] \), we have

\[
\left| \lambda_1^\epsilon(t) + \sqrt{a(t)} \right| = \left| \left( -\sqrt{a^* \phi_\epsilon} \right)(t) + \sqrt{a(t)} \right|
\]

\[
= \left| \int_\mathbb{R} \sqrt{a(t-y)} \phi_\epsilon(y) dy - \sqrt{a(t)} \int_\mathbb{R} \phi(y) dy \right|
\]

\[
= \left| \int_\mathbb{R} \sqrt{a(t-\epsilon x)} \phi(x) dx - \sqrt{a(t)} \int_\mathbb{R} \phi(x) dx \right|
\]

\[
\leq \int_\mathbb{R} \frac{|a(t-\epsilon x) - a(t)|}{\sqrt{a(t-\epsilon x)} + \sqrt{a(t)}} \phi(x) dx
\]

\[
\leq \frac{\|a\|_{C^\alpha([0,T])} \epsilon^\alpha}{2\sqrt{a_0}}.
\]

Similarly, we can compute \( |\lambda_2^\epsilon(t) - \sqrt{a(t)}| \) and we get

\[
\left| \lambda_1^\epsilon(t) + \sqrt{a(t)} \right| \leq c(a) \epsilon^\alpha,
\]

\[
\left| \lambda_2^\epsilon(t) - \sqrt{a(t)} \right| \leq c(a) \epsilon^\alpha,
\]

(2.22)
Now substituting the suggested solution (2.20) in (2.9), we get

\[ -\rho'(t)\langle \beta \rangle^\frac{1}{2}e^{-\rho(t)\langle \beta \rangle^\frac{1}{2}} \frac{H(t)W(t)}{\det H(t)} + \rho'(t)\langle \beta \rangle^\frac{1}{2} \frac{H(t)W(t)}{\det H(t)} + e^{-\rho(t)\langle \beta \rangle^\frac{1}{2}} \frac{H(t)W(t)}{\det H(t)} \]

\[ - e^{-\rho(t)\langle \beta \rangle^\frac{1}{2}} \frac{(\det H)(t)}{(\det H(t))^2} H(t)W(t) \]

\[ = i\beta^2 A(t)e^{-\rho(t)\langle \beta \rangle^\frac{1}{2}} \frac{H(t)W(t)}{\det H(t)} + iB(t)e^{-\rho(t)\langle \beta \rangle^\frac{1}{2}} \frac{H(t)W(t)}{\det H(t)}. \]

Multiplying both sides by \( e^{\rho(t)\langle \beta \rangle^\frac{1}{2}} \det H(t)H^{-1}(t) \), we get

\[ W_t(t) = \rho'(t)\langle \beta \rangle^\frac{1}{2}W(t) - H^{-1}(t)H(t)W(t) + (\det H)(t)(\det H(t))^{-1}W(t) \]

\[ + i\beta^2 H^{-1}(t)A(t)H(t)W(t) + iH^{-1}(t)B(t)H(t)W(t). \]

This leads to the equality

\[ \frac{d}{dt} |W(t)|^2 = 2\text{Re}(W_t(t), W(t)) \]

\[ = 2\rho'(t)\langle \beta \rangle^\frac{1}{2}|W(t)|^2 - 2\text{Re} \left( H^{-1}(t)H(t)W(t), W(t) \right) \]

\[ + 2(\det H(t))^{-1}(\det H)(t)|W(t)|^2 + 2\beta^2 \text{Im} \left( H^{-1}(t)A(t)H(t)W(t), W(t) \right) \]

\[ + 2\text{Im} \left( H^{-1}(t)B(t)H(t)W(t), W(t) \right). \]

We note that

\[ 2\text{Im} \left( H^{-1}AHW, W \right) \leq \|H^{-1}AH - (H^{-1}AH)^*\| \|W\|^2. \]

Thus, we obtain

\[ \frac{d}{dt} |W(t)|^2 \leq \left( 2\rho'(t)\langle \beta \rangle^\frac{1}{2} + 2\|H^{-1}(t)H(t)\| + 2(\det H(t))^{-1}(\det H)(t) \right) \]

\[ + \langle \beta \rangle^2 \|H^{-1}AH - (H^{-1}AH)^*\| + \|H^{-1}BH - (H^{-1}BH)^*\| \|W\|^2. \]

Using the techniques from [9, 17], we estimate the above terms as follows:

1. \( \|H^{-1}(t)H(t)\| \lesssim \frac{\|a\|_{C^\alpha([0,T])}}{\alpha_0} e^{\alpha - 1}, \)
2. \( |(\det H(t))^{-1}(\det H)(t)| \lesssim \frac{\|a\|_{C^\alpha([0,T])}}{\alpha_0} e^{\alpha - 1}, \)
3. \( \|H^{-1}AH - (H^{-1}AH)^*\| \lesssim \frac{\|a\|_{C^\alpha([0,T])}}{\alpha_0} \sqrt{|a|} \|L^\infty([0,T])\| e^{\alpha}, \)
4. \( \|H^{-1}BH - (H^{-1}BH)^*\| \lesssim \langle \beta \rangle^2 \frac{\|L^\infty([0,T])\|^2}{\sqrt{|a_0|}}. \)

Indeed, we deal with these four terms as follows:

1. Since \( H^{-1}(t) = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix} \) and \( H_t(t) = \begin{pmatrix} 0 & 0 \\ \partial_t \lambda_1 & \partial_t \lambda_2 \end{pmatrix} \), it follows that

the entries of the matrix \( H^{-1}H_t \) are given by the functions \( \frac{\partial_t \lambda_1}{\lambda_2 - \lambda_1} \). We compute
for \( j = 2 \), i.e., \( \partial_t \lambda_2'(t) \), can be estimated by
\[
|\partial_t \lambda_2'(t)| = |\sqrt{a} \ast \partial_t \phi_c(t)| = \left| \frac{1}{\epsilon} \int_{\mathbb{R}} \sqrt{a(t - \rho \epsilon)} \phi'(\rho) d\rho \right|
\leq \frac{1}{\epsilon} \int_{\mathbb{R}} \left| \sqrt{a(t - \rho \epsilon)} - \sqrt{a(t)} \right| \phi'(\rho) d\rho
= \frac{1}{\epsilon} \int_{\mathbb{R}} \frac{|a(t - \rho \epsilon) - a(t)|}{\sqrt{a(t - \rho \epsilon) + a(t)}} \phi'(\rho) d\rho
\leq c(a)e^{\alpha - 1}, \tag{2.26}
\]
where \( c(a) \) given by (2.23) comes from the Hölder continuity of \( a(t) \), and also using \( \int \phi' = 0 \). Then using (2.21), and (2.26), we get
\[
\|H^{-1}(t)H_t(t)\| \leq \frac{\|a\|_{C^0([0,T])}}{4a_0} \epsilon^{\alpha - 1} \lesssim \frac{\|a\|_{C^0([0,T])}}{a_0} \epsilon^{\alpha - 1}.
\]
(2) We can estimate
\[
|(\det H(t))^{-1}(\det H)_t(t)| = \left| \frac{\partial_t \lambda_2 - \partial_t \lambda_1}{\lambda_2 - \lambda_1^*} \right| \leq \frac{\|a\|_{C^0([0,T])}}{2a_0} \epsilon^{\alpha - 1} \lesssim \frac{\|a\|_{C^0([0,T])}}{a_0} \epsilon^{\alpha - 1}.
\]
(3) In this case, we calculate the matrix that we are interested in, that is,
\[
H^{-1}AH - (H^{-1}AH)^* = \frac{1}{\lambda_2 - \lambda_1^*} \begin{pmatrix}
0 & -2a + (\lambda_1^*)^2 + (\lambda_2^*)^2 \\
2a - (\lambda_1^*)^2 - (\lambda_2^*)^2 & 0
\end{pmatrix}.
\tag{2.27}
\]
Since \((\lambda_1^*)^2 = (\lambda_2^*)^2\), and recalling inequality (2.21), we see that the desired norm can be estimated if we estimate the function \( |a(t) - (\lambda_2^*)^2| \) only. A straightforward calculation using (2.22), shows that
\[
|a(t) - (\lambda_2^*)^2| = \left| \left( \sqrt{a(t)} - (\lambda_2^*) \right) \left( \sqrt{a(t)} + (\lambda_2^*) \right) \right|
\leq c(a)e^{\alpha} \left| \sqrt{a(t)} + \int_{\mathbb{R}} \sqrt{a(t - s)} \phi_c(s) ds \right|
\leq 2c(a)\|\sqrt{a}\|_{L^\infty([0,T])} e^{\alpha}, \tag{2.28}
\]
where \( c(a) \) given by (2.23) comes from the Hölder continuity of \( a(t) \). Using (2.28), and (2.21), it follows that
\[
\|H^{-1}AH - (H^{-1}AH)^*\| \leq \frac{2c(a)\|\sqrt{a}\|_{L^\infty([0,T])}}{\sqrt{a_0}} e^{\alpha}
\lesssim \frac{\|a\|_{C^0([0,T])}\sqrt{a}}{a_0} \|L^\infty([0,T])\| e^{\alpha}. \tag{2.29}
\]
(4) Next we estimate the term \( \|H^{-1}BH - (H^{-1}BH)^*\| \). For that, we explicitly write the matrix we are interested in, that is
\[ H^{-1}BH - (H^{-1}BH)^* = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} 0 & 2(1 - \beta^2)(\lambda_1^2 + 2b) \\ -2(1 - \beta^2)(\lambda_1^2 + 2b) & 0 \end{pmatrix}. \] 

(2.30)

Consequently, using (2.21), we get

\[
\|H^{-1}BH - (H^{-1}BH)^*\| \leq 2 \left( \frac{\|b\|_{L^\infty([0,T])} + \langle \beta \rangle^2 \|\sqrt{a}\|_{L^\infty([0,T])}}{\sqrt{a_0}} \right) 
\]

(2.31)

Combining estimates (1)-(4) above, we get the estimate for the derivative of the energy, that is

\[
\frac{d}{dt}|W(t)|^2 \leq 2\rho(t)\langle \beta \rangle^\frac{1}{2} + \frac{\|a\|_{C^\alpha([0,T])}}{a_0} \epsilon^{\alpha - 1} + \langle \beta \rangle^2 \frac{\|a\|_{C^\alpha([0,T])}}{a_0} \|\sqrt{a}\|_{L^\infty([0,T])} \epsilon^\alpha 
\]

\[
+ \langle \beta \rangle^2 \frac{\|b\|_{L^\infty([0,T])} + \|\sqrt{a}\|_{L^\infty([0,T])}}{\sqrt{a_0}} |W(t)|^2. \] 

(2.32)

We choose \( \epsilon = \langle \beta \rangle^{-2} \) and define \( \rho(t) := \rho(0) - K_2t = -K_2t \), for some \( K_2 > 0 \) to be specified. Substituting it in (2.32), we get

\[
\frac{d}{dt}|W(t)|^2 \leq \left( -2K_2\langle \beta \rangle^\frac{1}{2} + \frac{\|a\|_{C^\alpha([0,T])}}{a_0} \langle \beta \rangle^{2(1 - \alpha)} + \frac{\|a\|_{C^\alpha([0,T])}}{a_0} \|\sqrt{a}\|_{L^\infty([0,T])} \langle \beta \rangle^{2(1 - \alpha)} 
\]

\[
+ \langle \beta \rangle^2 \frac{\|b\|_{L^\infty([0,T])} + \|\sqrt{a}\|_{L^\infty([0,T])}}{\sqrt{a_0}} |W(t)|^2. \] 

(2.33)

Since \( 0 < \alpha < 1 \), this implies

\[
\frac{d}{dt}|W(t)|^2 \lesssim \left( -2K_2\langle \beta \rangle^\frac{1}{2} + \kappa \langle \beta \rangle^2 \right) |W(t)|^2, \] 

(2.34)

where

\[
\kappa = \frac{\|a\|_{C^\alpha([0,T])}}{a_0} + \frac{\|a\|_{C^\alpha([0,T])} \|\sqrt{a}\|_{L^\infty([0,T])}}{a_0} + \frac{\|b\|_{L^\infty([0,T])} + \|\sqrt{a}\|_{L^\infty([0,T])}}{\sqrt{a_0}}. \] 

(2.35)

If we choose \( K_2 = \frac{s}{2} \) and \( \frac{1}{s} \geq 2 \), then for all \( t \in [0,T] \) and \( \beta > 0 \), we have

\[
\frac{d}{dt}|W(t)|^2 \leq 0. \] 

(2.36)

Now by using monotonicity of \( |W(t)| \) and \( \rho(0) = 0 \), we get

\[
|W(t)| \leq \|H(0)\|^{-1} |\det H(0)||V(0)|. \] 

(2.37)

Now from (2.20) and (2.37), we get

\[
|V(t)| \leq \|H(t)\| |\det H(t)|^{-1} e^{K_2t\langle \beta \rangle^\frac{1}{2}} |W(t)| 
\]

\[
\leq \|H(t)\||H(0)\|^{-1} |\det H(t)|^{-1} |\det H(0)||e^{K_2t\langle \beta \rangle^\frac{1}{2}}|V(0)|. \] 

(2.38)
Then (2.38) and (2.39) allow one to estimate
\( \lambda_1 \leq \frac{\|H(t)\|H(0)^{-1} |\text{det } H(t)|^{-1} |\text{det } H(0)|}{\sqrt{a_0}} \lesssim \frac{\|a\|_{L^\infty([0,T])}}{\sqrt{a_0}} = \frac{\|a\|_{L^\infty([0,T])}}{\sqrt{a_0}}. \) (2.39)

Then (2.38) and (2.39) allow one to estimate
\[ |V(t)| \lesssim \frac{\|a\|_{L^\infty([0,T])} e^{K_2T(\beta)^{1/2}}}{\sqrt{a_0}} |V(0)|. \] (2.40)

If we set, \( C_2 = C' \frac{\|a\|_{L^\infty([0,T])}}{\sqrt{a_0}} \), for some absolute constant \( C' > 0 \), then by definition of \( V(t) \), we obtain (2.4), that is,
\[ |v(t)|^2 + |v'(t)|^2 \leq C_2 e^{K_2T(\beta)^{1/2}} (|v_0|^2 + |v_1|^2), \]
finishing the proof of Case 2.

**Case 3:** \( a \in C^l([0,T]) \), with \( l \geq 2 \), \( a(t) \geq 0 \).

Consider the quasi-symmetriser of \( A(t) \), that is, a family of coercive, Hermitian matrices of the form
\[ Q^{(2)}_\epsilon(t) := S(t) + \epsilon^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a(t) & 0 \\ 0 & 1 \end{pmatrix} + \epsilon^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \] (2.41)
for all \( \epsilon \in (0, 1] \), so that \((Q^{(2)}_\epsilon A - A^*Q^{(2)}_\epsilon)\) goes to zero as \( \epsilon \) goes to zero. The associated perturbed energy will be given by
\[ E_\epsilon(t) := (Q^{(2)}_\epsilon V(t), V(t)). \]

We proceed by estimating this energy. We have
\[
\frac{d}{dt} E_\epsilon(t) = \left( \frac{d}{dt} Q^{(2)}_\epsilon(t)V(t), V(t) \right) + (Q^{(2)}_\epsilon(t)V(t), V(t)) + (Q^{(2)}_\epsilon(t)V(t), V(t)) \\
= \left( \frac{d}{dt} Q^{(2)}_\epsilon(t)V(t), V(t) \right) + i\beta^2 \left( Q^{(2)}_\epsilon(t)A(t)V(t), V(t) \right) + \\
i \left( Q^{(2)}_\epsilon(t)B(t)V(t), V(t) \right) - i\beta^2 \left( Q^{(2)}_\epsilon(t)V(t), A(t)V(t) \right) - \\
i \left( Q^{(2)}_\epsilon(t)V(t), B(t)V(t) \right) \\
= \left( \frac{d}{dt} Q^{(2)}_\epsilon(t)V(t), V(t) \right) + i\beta^2 \left( (Q^{(2)}_\epsilon(t)A(t) - A^*(t)Q^{(2)}_\epsilon(t))V(t), V(t) \right) + \\
i \left( (Q^{(2)}_\epsilon(t)B(t) - B^*(t)Q^{(2)}_\epsilon(t))V(t), V(t) \right). \] (2.42)

Now we will estimate the above terms as follows:

1. \( i \left( (Q^{(2)}_\epsilon(t)A(t) - A^*(t)Q^{(2)}_\epsilon(t))V(t), V(t) \right) \leq \epsilon \left( Q^{(2)}_\epsilon(t)V(t), V(t) \right), \)

2. \( i \left( (Q^{(2)}_\epsilon(t)B(t) - B^*(t)Q^{(2)}_\epsilon(t))V(t), V(t) \right) \leq \langle \beta \rangle^2 \left( \|a\|_{L^\infty([0,T])} + \epsilon^2 + \|b\|_{L^\infty([0,T])} \right) |V(t)|^2. \) (2.43)

Indeed, we will deal with these two terms as follows:
(1) Since $Q_{\epsilon}^{(2)}(t)A(t) - A^*(t)Q_{\epsilon}^{(2)}(t) = \epsilon^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, simple calculations will give

$$(Q_{\epsilon}^{(2)}(t)V(t), V(t)) = (a(t) + \epsilon^2)|v_1|^2 + |v_2|^2. \quad (2.44)$$

Now we have

$$i \left( (Q_{\epsilon}^{(2)}(t)A(t) - A^*(t)Q_{\epsilon}^{(2)}(t))V(t), V(t) \right) = i\epsilon^2(v_2\bar{v}_1 - v_1\bar{v}_2)$$

$$= i\epsilon^2 2\text{Im}(v_2\bar{v}_1)$$

$$\leq \epsilon^2|v_1||v_2|$$

$$\leq \epsilon^2|v_1|^2 + |v_2|^2$$

$$\leq \epsilon ((a(t) + \epsilon^2)|v_1|^2 + |v_2|^2). \quad (2.45)$$

From (2.44) and (2.45), we get

$$i \left( (Q_{\epsilon}^{(2)}(t)A(t) - A^*(t)Q_{\epsilon}^{(2)}(t))V(t), V(t) \right) \leq \epsilon (Q_{\epsilon}^{(2)}(t)V(t), V(t)).$$

(2) Since $Q_{\epsilon}^{(2)}(t)B(t) - B^*(t)Q_{\epsilon}^{(2)}(t) = ((1 - \beta^2)(a(t) + \epsilon^2) - b(t)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, this implies

$$i \left( (Q_{\epsilon}^{(2)}(t)B(t) - B^*(t)Q_{\epsilon}^{(2)}(t))V(t), V(t) \right)$$

$$\leq \langle \beta \rangle^2 \left( \|a\|_{L^\infty([0,T])} + \epsilon^2 + \|b\|_{L^\infty([0,T])} \right) |V(t)|^2. \quad (2.46)$$

Next we estimate the perturbed energy. From (2.44), and for $\epsilon \leq 1$, we get

$$(Q_{\epsilon}^{(2)}(t)V(t), V(t)) = (a(t) + \epsilon^2)|v_1|^2 + |v_2|^2$$

$$\leq (\|a\|_{L^\infty([0,T])} + 1)|v_1|^2 + |v_2|^2$$

$$= (\|a\|_{L^\infty([0,T])} + 1)|V(t)|^2, \quad (2.47)$$

and

$$(Q_{\epsilon}^{(2)}(t)V(t), V(t)) = (a(t) + \epsilon^2)|v_1|^2 + |v_2|^2$$

$$\geq \epsilon^2|v_1|^2 + \frac{\epsilon^2}{\|a\|_{L^\infty([0,T])} + 1}|v_2|^2$$

$$\geq \frac{\epsilon^2}{\|a\|_{L^\infty([0,T])} + 1}|v_1|^2 + \frac{\epsilon^2}{\|a\|_{L^\infty([0,T])} + 1}|v_2|^2$$

$$= c_1^{-1}\epsilon^2|V(t)|^2. \quad (2.48)$$

From (2.47) and (2.48), we get

$$c_1^{-1}\epsilon^2|V(t)|^2 \leq (Q_{\epsilon}^{(2)}(t)V(t), V(t)) \leq c_1|V(t)|^2, \quad (2.49)$$

where $c_1 = \|a\|_{L^\infty([0,T])} + 1$. 

Using the above estimates, we have
\[
\frac{d}{dt} E_\epsilon(t) \leq \left( \frac{d}{dt} Q^{(2)}(t)V(t), V(t) \right) + \langle \beta \rangle^2 \epsilon \left( Q^{(2)}(t)V(t), V(t) \right) + \\
\langle \beta \rangle^2 \left( \|a\|_{L^\infty([0,T])} + \epsilon^2 + \|b\|_{L^\infty([0,T])} \right) |V(t)|^2
\]
\[
\leq \left( \frac{d}{dt} Q^{(2)}(t)V(t), V(t) \right) + \langle \beta \rangle^2 \epsilon E_\epsilon(t) + \\
c_1 \epsilon^2 \langle \beta \rangle^2 \left( \|a\|_{L^\infty([0,T])} + \epsilon^2 + \|b\|_{L^\infty([0,T])} \right) E_\epsilon(t)
\]
\[
= \left( \frac{d}{dt} Q^{(2)}(t)V(t), V(t) \right) + \langle \beta \rangle^2 \epsilon E_\epsilon(t) + c_1 \epsilon^2 \langle \beta \rangle^2 \|b\|_{L^\infty([0,T])} E_\epsilon(t) + \\
c_1 \langle \beta \rangle^2 E_\epsilon(t) + c_1 \epsilon^2 \langle \beta \rangle^2 \|b\|_{L^\infty([0,T])} E_\epsilon(t),
\]
which gives
\[
\frac{d}{dt} E_\epsilon(t) \leq \left[ \left( \frac{d}{dt} Q^{(2)}(t)V(t), V(t) \right) Q^{(2)}(t)V(t), V(t) \right) \right] + \langle \beta \rangle^2 \epsilon + c_1 \epsilon^2 \langle \beta \rangle^2 \left( \|a\|_{L^\infty([0,T])} + \|b\|_{L^\infty([0,T])} \right) \\
+ c_1 \langle \beta \rangle^2 \right] E_\epsilon(t),
\]
where \( c_1 = \|a\|_{L^\infty([0,T])} + 1 \). We will use the following estimate:
\[
\int_0^T \left( \frac{d}{dt} Q^{(2)}(t)V(t), V(t) \right) dt \leq c_T c_1 \epsilon^{-\frac{3}{2}} \|a\|^2_{C^1([0,T])},
\]
where \( c_1 = \|a\|_{L^\infty([0,T])} + 1 \) for some \( c_T > 0 \), depending only on \( T \), which we assume for a moment. From (2.51) and (2.52), we get
\[
\frac{d}{dt} E_\epsilon(t) \leq \left( c_T c_1 \epsilon^{-\frac{3}{2}} \|a\|^2_{C^1([0,T])} + \langle \beta \rangle^2 \epsilon + c_1 \epsilon^2 \langle \beta \rangle^2 \left( \|a\|_{L^\infty([0,T])} + \|b\|_{L^\infty([0,T])} \right) + c_1 \langle \beta \rangle^2 \right) E_\epsilon(t)
\]
\[
= \kappa E_\epsilon(t),
\]
where \( \kappa = c_T c_1 \epsilon^{-\frac{3}{2}} \|a\|^2_{C^1([0,T])} + \langle \beta \rangle^2 \epsilon + c_1 \epsilon^2 \langle \beta \rangle^2 \left( \|a\|_{L^\infty([0,T])} + \|b\|_{L^\infty([0,T])} \right) + c_1 \langle \beta \rangle^2 \).

Applying the Gronwall lemma on (2.53), we deduce that
\[
E_\epsilon(t) \leq e^{\kappa T} E_\epsilon(0)
\]
\[
= E_\epsilon(0) e^{ \left( c_T c_1 \epsilon^{-\frac{3}{2}} \|a\|^2_{C^1([0,T])} + \langle \beta \rangle^2 \epsilon + c_1 \epsilon^2 \langle \beta \rangle^2 \left( \|a\|_{L^\infty([0,T])} + \|b\|_{L^\infty([0,T])} \right) + c_1 \langle \beta \rangle^2 \right) T}.
\]
Combining this inequality with (2.49), we get
\[
c_1^{-1} \epsilon^2 \|V(t)\|^2 \leq E_\epsilon(t) \leq \\
c_1 \epsilon \left( c_T c_1 \epsilon^{-\frac{3}{2}} \|a\|^2_{C^1([0,T])} + \langle \beta \rangle^2 \epsilon + c_1 \epsilon^2 \langle \beta \rangle^2 \left( \|a\|_{L^\infty([0,T])} + \|b\|_{L^\infty([0,T])} \right) + c_1 \langle \beta \rangle^2 \right) T |V(0)|^2.
\]
Choosing, \( \epsilon^{-\frac{7}{2}} = \langle \beta \rangle^2 \epsilon \) and setting \( \sigma = 1 + \frac{1}{2} \), we get \( \epsilon = \langle \beta \rangle^{-\frac{1}{2}} \leq 1 \) for all \( \beta > 0 \) and \( \epsilon^{-\frac{7}{2}} = \langle \beta \rangle^2 \), so that

\[
|V(t)|^2 \leq c_1^2 \langle \beta \rangle^4 e \left( (cT+c_1\|a\|_{C^1((0,T),\mathbb{C})}) \right)^{-\frac{1}{2}} \left( \|a\|_{L^\infty((0,T))} + \|b\|_{L^\infty((0,T))} \right)^{T} |V(0)|^2
\]

\[
= c_2^2 \langle \beta \rangle^{4-\frac{1}{2}} e \left( (cT+c_1\|a\|_{C^1((0,T),\mathbb{C})}) \right)^{-\frac{1}{2}} \left( \|a\|_{L^\infty((0,T))} + \|b\|_{L^\infty((0,T))} \right)^{T} |V(0)|^2
\]

\[
\leq c_1^2 \epsilon \left( 1+c_1 \right) \left( cT+c_1\|a\|_{C^1((0,T),\mathbb{C})} \right) \left( \|a\|_{L^\infty((0,T))} + \|b\|_{L^\infty((0,T))} \right)^{T} \langle \beta \rangle^{6-\frac{4}{2}} |V(0)|^2
\]

\[
= c_3 e^{K_3 \langle \beta \rangle^{6-\frac{4}{2}}} |V(0)|^2. \tag{2.56}
\]

If we set \( K_3 = 1 + c_1 \left( cT+c_1\|a\|_{C^1((0,T),\mathbb{C})} \right) \left( \|a\|_{L^\infty((0,T))} + \|b\|_{L^\infty((0,T))} \right) \) and \( C_3 = c_3^2 \), then by the definition of \( V(t) \) we get

\[
|v(t)|^2 + |v'(t)|^2 \leq C_3 e^{K_3 \langle \beta \rangle^{6-\frac{4}{2}}} (|v_0|^2 + |v_1|^2). \tag{2.57}
\]

This gives (2.5). It still remains to show (2.52). We can obtain this estimate by an argument from [6]. Indeed, using (2.41), for \( V = (V_1, V_2)^T \), one readily obtains

\[
|Q^2(t)V, V| = a(t)|V_1|^2 + |V_2|^2 + \epsilon^2 |V_1|^2, \tag{2.58}
\]

and

\[
\left( \frac{d}{dt} Q^2(t)V, V \right) = a'(t)|V_1|^2. \tag{2.59}
\]

From (2.58) we have

\[
c_1^{-\frac{1}{2}} |V(t)|^2 \leq (Q^2(t)V(t), V(t)) \leq c_1 |V(t)|^2, \tag{2.60}
\]

where \( c_1 = \|a\|_{L^\infty((0,T))} + 1 \). At the same time

\[
\int_0^T \frac{\left( \frac{d}{dt} Q^2(t)V(t), V(t) \right)}{\left( Q^2(t)V(t), V(t) \right)} dt = \int_0^T \frac{\left( \frac{d}{dt} Q^2(t)V(t), V(t) \right)}{\left( Q^2(t)V(t), V(t) \right)} \frac{1}{\left( Q^2(t)V(t), V(t) \right)} dt
\]

\[
\leq \int_0^T \left( \frac{d}{dt} Q^2(t)V(t), V(t) \right) \left( Q^2(t)V(t), V(t) \right)^{1-\frac{1}{2}} dt
\]

\[
\leq c_1 e^{-\frac{7}{2} \int_0^T \frac{\left( \frac{d}{dt} Q^2(t)V(t), V(t) \right)}{\left( Q^2(t)V(t), V(t) \right)} \left( Q^2(t)V(t), V(t) \right)^{1-\frac{1}{2}} |V(t)|^2} dt. \tag{2.61}
\]

Next we try to find the estimate of

\[
\int_0^T \frac{\left( \frac{d}{dt} Q^2(t)V(t), V(t) \right)}{\left( Q^2(t)V(t), V(t) \right)^{1-\frac{1}{2}} |V(t)|^2} dt. \tag{2.62}
\]
The above case (2.62), is a special case of a general estimate

$$\int_0^T \frac{|f'(t)|}{|f(t)|^{1-\frac{1}{2}}} dt \leq c_T \|f\|_{C([0,T])}^{\frac{1}{2}},$$

(2.63)

which holds for any real or complex-valued function $f \in C([0,T])$. Since $Q^{(2)}_\epsilon(t)$ is a diagonal matrix, an estimate for (2.62) can be obtained directly by applying (2.63) to each of the entries of $Q^{(2)}_\epsilon(t)$. Then noting that $|V|^2 = |V_1|^2 + |V_2|^2$, we get

$$\int_0^T \left| \frac{d}{dt} Q^{(2)}_\epsilon(t) V(t), V(t) \right| dt \leq \int_0^T \frac{a'(t)|V_1(t)|^2}{(a(t) + |V_1(t)|^{2(1-\frac{1}{2})})|V_1(t)|^{\frac{1}{2}}} dt$$

$$\leq \int_0^T \frac{|a'(t)||V_1(t)|^2}{(a(t) + |V_1(t)|^{2(\frac{1}{2})})|V_1(t)|^{\frac{1}{2}}} dt$$

$$\leq \int_0^T \frac{|a'(t)|}{|a(t)|^{1-\frac{1}{2}}} dt$$

$$\leq c_T \|a\|_{C([0,T])}^{\frac{1}{2}},$$

(2.64)

Using (2.64) in (2.61) we get

$$\int_0^T \left| \frac{d}{dt} Q^{(2)}_\epsilon(t) V(t), V(t) \right| dt \leq c_1 c_T e^{-\frac{1}{2}} \|a\|_{C([0,T])}^{\frac{1}{2}},$$

(2.65)

yielding (2.52) where $c_1 = \|a\|_{L^\infty([0,T])} + 1$.

**Case 4:** $a \in C^\alpha([0,T])$, with $0 < \alpha < 2$, $a(t) \geq 0$.

In this last case we extend the proof of Case 2. However, under these assumptions the eigenvalues of the matrix $A(t)$, that are $\pm \sqrt{a(t)}$ might coincide, and hence they are Hölder continuous of order $\frac{\alpha}{2}$ instead of $\alpha$. In order to adapt this proof to the one for Case 2, and to simplify the notation, we assume without loss of generality that $a \in C^{2\alpha}([0,T])$ with $0 < \alpha < 1$, so that $\sqrt{a} \in C^\alpha([0,T])$. Then we change $\alpha$ into $\frac{\alpha}{2}$ in the final statement.

We look again for solutions of the form

$$V(t) = e^{-\rho(t)(\beta)^{1/2}} (\det H(t))^{-1} H(t) W(t),$$

with the real valued function $\rho(t)$, the exponent $s$ and the energy $W(t)$ to be chosen later, while $H(t)$ is the matrix given by

$$H(t) = \begin{pmatrix} 1 & 1 \\ \lambda_{1,\alpha}(t) & \lambda_{2,\alpha}(t) \end{pmatrix}.$$
where the regularised eigenvalues of $A(t)$ are $\lambda_{1,\alpha}^\epsilon(t)$ and $\lambda_{2,\alpha}^\epsilon(t)$ differ from the ones defined in the previous case in the following way,

$$\lambda_{1,\alpha}^\epsilon(t) := \left(-\sqrt{a} * \phi_\epsilon\right)(t) + \epsilon^a,$$

$$\lambda_{2,\alpha}^\epsilon(t) := \left(+\sqrt{a} * \phi_\epsilon\right)(t) + 2\epsilon^a.$$

Arguing as in Case 2, we have

$$\det H(t) = \lambda_{2,\alpha}^\epsilon(t) - \lambda_{1,\alpha}^\epsilon(t) \geq \epsilon^a,$$

and also, similar to (2.22), we have

$$\left|\lambda_{1,\alpha}^\epsilon(t) + \sqrt{a(t)}\right| = \left|\left(-\sqrt{a} * \phi_\epsilon\right)(t) + \sqrt{a(t)} + \epsilon^a\right|$$

$$= \left|\int_\mathbb{R} \sqrt{a(t-y)}\phi_\epsilon(y)dy - \sqrt{a(t)}\int_\mathbb{R} \phi(y)dy - \epsilon^a\right|$$

$$= \left|\int_\mathbb{R} \left(\sqrt{a(t-\epsilon x)} - \sqrt{a(t)}\right)\phi(x)dx - \epsilon^a\right|$$

$$\leq \left(\|\sqrt{a}\|_{c^\alpha([0,T])} + 1\right)\epsilon^a. \quad (2.67)$$

Similarly, we can compute $\left|\lambda_{2,\alpha}^\epsilon(t) - \sqrt{a(t)}\right|$, and we get

$$\left|\lambda_{1,\alpha}^\epsilon(t) + \sqrt{a(t)}\right| \leq c_1\epsilon^a, \quad (2.68)$$

$$\left|\lambda_{2,\alpha}^\epsilon(t) - \sqrt{a(t)}\right| \leq c_2\epsilon^a, \quad (2.69)$$

with $c_1 = \|\sqrt{a}\|_{c^\alpha([0,T])} + 1$ and $c_2 = \|\sqrt{a}\|_{c^\alpha([0,T])} + 2$. Now we look at the energy estimates:

$$\frac{d}{dt}\|W(t)\|^2 \leq (2\rho(t)\langle\beta\rangle\frac{1}{2} + 2\|H^{-1}(t)H_t(t)\| + 2\|(\det H(t))^{-1}(\det H)_t(t)\| +$$

$$\langle\beta\rangle^2\|H^{-1}AH - (H^{-1}AH)^*\| + \|H^{-1}BH - (H^{-1}BH)^*\|)\|W\|^2. \quad (2.70)$$

In the present setting, inequality (2.21) is replaced by

$$\det H(t) = \lambda_{2,\alpha}^\epsilon(t) - \lambda_{1,\alpha}^\epsilon(t) \geq \epsilon^a. \quad (2.71)$$

Taking this into account, the estimates (1) and (2) in Case 2 are replaced by

1. $\|H^{-1}(t)H_t(t)\| \lesssim \|\sqrt{a}\|_{c^\alpha([0,T])}\epsilon^{-1},$
2. $\|(\det H(t))^{-1}(\det H)_t(t)\| \lesssim \|\sqrt{a}\|_{c^\alpha([0,T])}\epsilon^{-1}.$

We now will estimate $\|H^{-1}AH - (H^{-1}AH)^*\|$. First, we explicitly write this matrix, recalling (2.27), that is

$$H^{-1}AH - (H^{-1}AH)^* = \frac{1}{\lambda_{2,\alpha}^\epsilon - \lambda_{1,\alpha}^\epsilon}\begin{pmatrix} 0 & -2a + (\lambda_{2,\alpha}^\epsilon)^2 + (\lambda_{1,\alpha}^\epsilon)^2 \\ 2a - (\lambda_{2,\alpha}^\epsilon)^2 - (\lambda_{1,\alpha}^\epsilon)^2 & 0 \end{pmatrix}.$$
We consider the functions $|a(t) - (\lambda_{2,\alpha}^t)^2|$ and $|a(t) - (\lambda_{1,\alpha}^t)^2|$. A straightforward estimate, using (2.68) and (2.69), gives

\[
|a(t) - (\lambda_{2,\alpha}^t)^2| = \left| (\sqrt{a(t)} - \lambda_{2,\alpha}^t)(\sqrt{a(t)} + \lambda_{2,\alpha}^t) \right| \\
\leq c_2 \epsilon^\alpha \left| (\sqrt{a(t)} + \lambda_{2,\alpha}^t) \right| \\
= c_2 \epsilon^\alpha \sqrt{a(t)} + \int_{\mathbb{R}} \sqrt{a(t - s \epsilon)} \phi(s) ds + 2 \epsilon^\alpha \\
= c_2 \epsilon^\alpha \int_{\mathbb{R}} \left( \sqrt{a(t)} - \sqrt{a(t + s \epsilon)} \right) \phi(s) ds + 2 \epsilon^\alpha \\
\leq c_2 (2 + \|\sqrt{a}\|_{c^\alpha([0,T])}) \epsilon^2 \alpha \\
= (2 + \|\sqrt{a}\|_{c^\alpha([0,T])})^2 \epsilon^2 \alpha. \tag{2.72}
\]

Similarly, we have

\[
|a(t) - (\lambda_{1,\alpha}^t)^2| \leq (1 + \|\sqrt{a}\|_{c^\alpha([0,T])})^2 \epsilon^2 \alpha. \tag{2.73}
\]

Combining, the above estimates, we have

\[
|a(t) - (\lambda_{1,\alpha}^t)^2| \leq (2 + \|\sqrt{a}\|_{c^\alpha([0,T])})^2 \epsilon^2 \alpha. \tag{2.74}
\]

It follows using (2.71), that

\[
\|(H^{-1}AH) - (H^{-1}AH)^*\| \lesssim (2 + \|\sqrt{a}\|_{c^\alpha([0,T])})^2 \epsilon^\alpha.
\]

Now, we will estimate $\|H^{-1}BH - (H^{-1}BH)^*\|$. Consider the matrix,

\[
H^{-1}BH - (H^{-1}BH)^* = \frac{1}{\lambda_{2,\alpha} - \lambda_{1,\alpha}} \begin{pmatrix} 0 & -2b + (1 - \beta^2)(\lambda_{1,\alpha}^2 + (\lambda_{2,\alpha}^t)^2) \\ 2b - (1 - \beta^2)(\lambda_{1,\alpha} + (\lambda_{2,\alpha}^t)^2) & 0 \end{pmatrix}. \tag{2.75}
\]

Since, $\lambda_{2,\alpha}^t + \lambda_{1,\alpha}^t = 3 \epsilon^\alpha$ and $\lambda_{2,\alpha}^t - \lambda_{1,\alpha}^t \geq \epsilon^\alpha$, this implies $(\lambda_{2,\alpha}^t)^2 \geq (\lambda_{1,\alpha}^t)^2$, and

\[
(\lambda_{1,\alpha}^t)^2 + (\lambda_{2,\alpha}^t)^2 \leq 2 (\lambda_{2,\alpha}^t)^2 \\
= 2 \left( (+\sqrt{a} * \phi) (t) + 2 \epsilon^\alpha \right)^2 \\
= 2 \left( \int \sqrt{a(t - s \epsilon)} \phi(s) ds + 2 \epsilon^\alpha \right)^2 \\
\leq 2 \left( \sqrt{a(t)} + \int \sqrt{a(t - s \epsilon)} \phi(s) ds + 2 \epsilon^\alpha \right)^2 \\
= 2 \left( \int \left( \sqrt{a(t)} - \sqrt{a(t + s \epsilon)} \right) \phi(s) ds + 2 \epsilon^\alpha \right)^2 \\
\leq 2 \left( \|\sqrt{a}\|_{c^\alpha([0,T])} \epsilon^\alpha + 2 \epsilon^\alpha \right)^2 \\
= 2 \epsilon^2 \alpha \left( \|\sqrt{a}\|_{c^\alpha([0,T])} + 2 \right)^2. \tag{2.76}
\]
Using (2.75) and (2.76), we can estimate

\[ \| H^{-1}BH - (H^{-1}BH)^* \| \leq 2 \left( \langle \beta \rangle^2 (\| \sqrt{a} \|_{C^0([0,T])} + 2)^2 \epsilon^\alpha + \| b \|_{L^\infty([0,T])} \epsilon^{-\alpha} \right) \]

\[ \lesssim \langle \beta \rangle^2 (\| \sqrt{a} \|_{C^0([0,T])} + 2)^2 \epsilon^\alpha + \| b \|_{L^\infty([0,T])} \epsilon^{-\alpha}. \]  

(2.77)

Using these estimates in (2.70), we get

\[ \frac{d}{dt} \| W(t) \|^2 \lesssim (2\rho'(t)\langle \beta \rangle \frac{1}{2} + 2\| \sqrt{a} \|_{C^0([0,T])} \langle \beta \rangle^2 \epsilon^{-1} + 2\langle \beta \rangle^2 (2 + \| \sqrt{a} \|_{C^0([0,T])}^2) \epsilon^\alpha + \| b \|_{L^\infty([0,T])} \epsilon^{-\alpha}) \| W \|^2. \]

We choose \( \epsilon^{-1} = \langle \beta \rangle^2 \epsilon^\alpha \), which yields \( \epsilon = \langle \beta \rangle^{\frac{2}{1+\alpha}} \leq 1 \), for all \( \beta > 0 \). Let us now define \( \rho(t):= \rho(0) - K_4T \) with \( K_4 > 0 \), to be chosen later, which gives

\[ \frac{d}{dt} \| W(t) \|^2 \lesssim (-2K_4\langle \beta \rangle \frac{1}{2} + \| \sqrt{a} \|_{C^0([0,T])} \langle \beta \rangle^2 \gamma + (2 + \| \sqrt{a} \|_{C^0([0,T])}^2) \langle \beta \rangle^{2\gamma} + \| b \|_{L^\infty([0,T])} \langle \beta \rangle^{2(1-\gamma)}) \| W \|^2, \]  

(2.78)

where \( \gamma = \frac{1}{\alpha+1} \). Since \( 0 < \alpha < 1 \), this implies \( \frac{1}{2} < \gamma < 1 \). Therefore

\[ \frac{d}{dt} |W(t)|^2 \lesssim \left( -2K_4\langle \beta \rangle \frac{1}{2} + \kappa \langle \beta \rangle^{2\gamma} \right) |W(t)|^2, \]

(2.79)

where \( \kappa = \| \sqrt{a} \|_{C^0([0,T])} + (2 + \| \sqrt{a} \|_{C^0([0,T])}^2) + \| b \|_{L^\infty([0,T])} \).

If we choose \( K_4 = \frac{\alpha}{2} \) and \( \frac{1}{s} \geq 2\gamma - \frac{2}{\alpha+1} \), then for all \( t \in [0,T] \) and \( \beta > 0 \), we have

\[ \frac{d}{dt} |W(t)|^2 \leq 0. \]

Then similar to the Case 2, this monotonicity of the energy \( W(t) \) yields the bound of the solution vector \( V(t) \) as:

\[ |V(t)| \leq \| H(t) \| \| H(0) \|^{-1} | \det H(t) |^{-1} | \det H(0) | e^{|K_4T\langle \beta \rangle \frac{1}{2}} | V(0) |. \]

(2.80)

Using (2.71), we have

\[ \| H(t) \| \| H(0) \|^{-1} | \det H(t) |^{-1} | \det H(0) | \lesssim (2 + \| \sqrt{a} \|_{C^0([0,T])})^2. \]

So we have the inequality

\[ |V(t)| \lesssim (2 + \| \sqrt{a} \|_{C^0([0,T])})^2 e^{|K_4T\langle \beta \rangle \frac{1}{2}} | V(0) |. \]

(2.81)

If we replace \( \alpha \) by \( \frac{\alpha}{2} \) and set \( C_4 = C'(2 + \| \sqrt{a} \|_{C^0([0,T])}^2) \), then by definition of \( V(t) \), we get

\[ |v(t)|^2 + |v'(t)|^2 \leq C_4 e^{|K_4T\langle \beta \rangle \frac{1}{2}} (|v_0|^2 + |v_1|^2). \]

(2.82)

This completes the proof of Case 4, with \( 0 < s \leq \frac{\alpha+2}{4} \). \( \square \)

By the standard methods of ordinary differential equations, Proposition 2.1, has the following corollary for the corresponding inhomogeneous equations.
Corollary 2.2. Let $T > 0$ and $\beta > 0$ be positive constants, let $b = b(t) \geq 0$, and $g(t)$ be bounded real-valued functions in $L^\infty([0, T])$, and let $a \not\equiv 0$ be a function satisfying one of the conditions of Proposition 2.1. Consider the following Cauchy problem:

$$
\begin{aligned}
& w''(t) + \beta^2 a(t) w(t) + b(t) w(t) = g(t), \quad t \in (0, T], \\
& w(0) = w_0 \in \mathbb{C}, \\
& w'(0) = w_1 \in \mathbb{C}.
\end{aligned}
$$

(2.83)

Let $v(t)$ be the solution of (2.83), with $g = 0$, that is, it satisfies (2.2), and the respective estimates in Proposition 2.1. Then we have the estimate

$$
|w(t)|^2 + |w'(t)|^2 \lesssim |v(t)|^2 + |v'(t)|^2 + (1 + \beta^2)^2 \|g\|_{L^2([0,T])}^2,
$$

(2.84)

for all $t \in [0, T]$, with the constant independent of $\beta$.

Proof. Similar to the proof of Proposition 2.1, we denote

$$
W(t) := \begin{pmatrix} iw(t) \\ \partial_t w(t) \end{pmatrix}, \quad W_0 := \begin{pmatrix} iw_0 \\ w_1 \end{pmatrix},
$$

(2.85)

and we will use the matrices

$$
A(t) := \begin{pmatrix} 0 & 1 \\ a(t) & 0 \end{pmatrix}, \quad B(t) := \begin{pmatrix} 0 & 1 - \beta^2 \\ b(t) & 0 \end{pmatrix} \text{ and } G(t) := \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.
$$

(2.86)

In this notation (2.83), becomes

$$
\begin{aligned}
& W' = i\beta^2 A(t) W(t) + iB(t) W(t) + G(t), \\
& W(0) = W_0.
\end{aligned}
$$

(2.87)

Consequently, we consider the following two equations

$$
\begin{aligned}
& V'_I = i (\beta^2 A(t) + B(t)) V_I(t), \\
& V_I(0) = W_0,
\end{aligned}
$$

(2.88)

and

$$
\begin{aligned}
& V''_{II} = i (\beta^2 A(t) + B(t)) V_{II}(t) + G(t), \\
& V_{II}(0) = 0.
\end{aligned}
$$

(2.89)

Then the solution of (2.87), is given by $W = V_I + V_{II}$. From the general theory of ordinary differential equations we can say that if

$$
\Phi(t; s) := \begin{pmatrix} \phi_1(t; s) \\ \phi_2(t; s) \end{pmatrix},
$$

(2.90)
is the solution of the equation
\[
\begin{aligned}
\Phi'(t, s) &= i(\beta^2 A(t) + B(t)) \Phi(t, s), \quad s \leq t, \\
\Phi(s; s) &= G(s),
\end{aligned}
\] (2.91)
then
\[
\Phi(t, s) = e^{i \int_t^s (\beta^2 A(\tau) + B(\tau)) d\tau} G(s).
\]
Then the solution of (2.89) is given by
\[
V_{II}(t) = \int_0^t \Phi(t, s) ds.
\] Consequently, we have
\[
W(t) = V_I + \int_0^t e^{i \int_t^s (\beta^2 A(\tau) + B(\tau)) d\tau} G(s) ds.
\] (2.92)
Here, on the right hand side, we will estimate the matrix term \(e^{i \int_t^s (\beta^2 A(\tau) + B(\tau)) d\tau}\).
Since \(A(t) = \begin{pmatrix} 0 & 1 \\ a(t) & 0 \end{pmatrix}\) and \(B(t) = \begin{pmatrix} 0 & 1 - \beta^2 \\ b(t) & 0 \end{pmatrix}\), this implies that
\[
M(t) := \beta^2 A(t) + B(t) = \begin{pmatrix} 0 & 1 \\ \beta^2 a(t) + b(t) & 0 \end{pmatrix}.
\]
This gives \(e^{i \int_t^s (\beta^2 A(\tau) + B(\tau)) d\tau} = e^{i \tilde{M}}\) where denoting \(\tilde{M} := \int_s^t M(\tau) d\tau\), we can abbreviate writing \(\tilde{M} = \begin{pmatrix} 0 & \tau \\ m & 0 \end{pmatrix}\), where \(\tau = t - s > 0\) and \(m = \int_s^t (\beta^2 a(\tau) + b(\tau)) d\tau \geq 0\), since \(a(t), b(t) \geq 0\) for all \(t \in [0, T]\). For now we will consider the case \(m \neq 0\). It can be readily seen from the matrix multiplication and induction that
\[
\tilde{M}^{2k} = (\tau m)^k I
\]
and
\[
\tilde{M}^{2k+1} = (\tau m)^k \tilde{M},
\]
for \(k = 0, 1, 2, \ldots\). Then we have
\[
e^{i \tilde{M}} = \sum_{n=0}^{\infty} \frac{(i \tilde{M})^n}{n!} = \sum_{k=0}^{\infty} \frac{(i \tilde{M})^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(i \tilde{M})^{2k+1}}{(2k + 1)!} = \sum_{k=0}^{\infty} \frac{i^{2k}(\tau m)^k}{(2k)!} I + \sum_{k=0}^{\infty} \frac{i^{2k+1}(\tau m)^k}{(2k + 1)!} \tilde{M} = \sum_{k=0}^{\infty} \frac{(-1)^k(\sqrt{\tau m})^{2k}}{(2k)!} I + \frac{i}{\sqrt{\tau m}} \sum_{k=0}^{\infty} \frac{(-1)^k(\sqrt{\tau m})^{2k+1}}{(2k + 1)!} \tilde{M} = \cos(\sqrt{\tau m}) I + \frac{i}{\sqrt{\tau m}} \sin(\sqrt{\tau m}) \tilde{M}.
\] (2.93)
Simple calculations will give us
\[ \|\hat{\mathcal{M}}\| \leq T(1 + \beta^2 \|a\|_{L^\infty([0,T])} + \|b\|_{L^\infty([0,T])}). \] (2.94)

From (2.93) and (2.94), we have
\[ \|e^{i\int_s^t (\beta^2 A(r) + B(r))dr}\| \leq 1 + T(1 + \beta^2 \|a\|_{L^\infty([0,T])} + \|b\|_{L^\infty([0,T])}). \] (2.95)

Now, when \( m = 0 \), we have \( \mathcal{M} = \begin{pmatrix} 0 & \tau \\ 0 & 0 \end{pmatrix} \) and \( e^{i\tilde{\mathcal{M}}} = I + i\tau \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), so that in this case we have
\[ \|e^{i\int_s^t (\beta^2 A(r) + B(r))dr}\| \leq 1 + T. \] (2.96)

Thus from (2.95) and (2.96), we get
\[ \|e^{i\int_s^t (\beta^2 A(r) + B(r))dr}\| \leq C(1 + \beta^2), \] (2.97)

where \( C = \max\{1 + T + T\|b\|_{L^\infty([0,T])}, T\|a\|_{L^\infty([0,T])}\}. \) Combining (2.92) and (2.97), we get
\[ |W(t)|^2 \leq 2|V_I(t)|^2 + 2C^2(1 + \beta^2)^2\|g\|^2_{L^2([0,T])} \]
\[ \lesssim |V_I(t)|^2 + (1 + \beta^2)^2\|g\|^2_{L^2([0,T])}, \] (2.98)

for all \( t \in [0,T] \), with the constant independent of \( \beta \). This completes the proof. \( \Box \)

3. PROOF OF THE MAIN RESULT

We now proceed to prove Theorem 1.1. We first recall a few facts concerning the Fourier analysis on the lattice \( h\mathbb{Z}^n \). The Schwartz space \( S(h\mathbb{Z}^n) \) on the lattice \( h\mathbb{Z}^n \) is the space of rapidly decreasing functions \( u : h\mathbb{Z}^n \to \mathbb{C} \), that is, \( u \in S(h\mathbb{Z}^n) \) if for any \( L < \infty \) there exists a constant \( C_{u,L,h} \) such that
\[ |u(k)| \leq C_{u,L,h}(1 + |k|)^{-L}, \quad \text{for all } k \in h\mathbb{Z}^n, \] (3.1)

where \( |k| = h \left( \sum_{j=1}^n k_j^2 \right)^{1/2} \). The Fourier transform \( \hat{u} \) of \( u : h\mathbb{Z}^n \to \mathbb{C} \) is defined as
\[ \hat{u}(\theta) = \sum_{k \in h\mathbb{Z}^n} u(k)e^{-\frac{2\pi i}{h} k \cdot \theta}, \quad \theta \in \mathbb{T}^n, \quad k \in h\mathbb{Z}^n. \] (3.2)

The Plancherel formula takes the form
\[ \int_{\mathbb{T}^n} |\hat{u}(\theta)|^2 \ d\theta = \int_{\mathbb{T}^n} \hat{u}(\theta)\overline{\hat{u}(\theta)} \ d\theta \]
\[ = \int_{\mathbb{T}^n} \sum_{k \in h\mathbb{Z}^n} u(k)e^{-\frac{2\pi i}{h} k \cdot \theta} \sum_{l \in h\mathbb{Z}^n} \overline{u(l)}e^{\frac{2\pi i}{h} l \cdot \theta} \ d\theta \]
\[ = \sum_{k \in h\mathbb{Z}^n} |u(k)|^2. \] (3.3)

The scalar product in the Hilbert space \( \ell^2(h\mathbb{Z}^n) \) is given by
\[ (u, v)_{\ell^2(h\mathbb{Z}^n)} = \sum_{k \in h\mathbb{Z}^n} u(k)\overline{v(k)}. \]
We note that the symbol of $L_h$ defined by $\sigma_{L_h}(k, \theta) = e^{-2\pi i \frac{k}{\hbar} \cdot \theta} \mathcal{L}_h(e^{2\pi i \frac{k}{\hbar} \cdot \theta})$ is given by

$$\sigma_{L_h}(k, \theta) = \sum_{j=1}^{n} \left( e^{2\pi i \theta_j} + e^{-2\pi i \theta_j} \right) - 2n = 2 \sum_{j=1}^{n} \cos(2\pi \theta_j) - 2n, \quad (3.4)$$

with $(k, \theta) \in \mathbb{hZ}^n \times \mathbb{T}^n$. We also note that it is independent of $k$ and $\mathbb{h}$.

**Proof of Theorem 1.1.** Our aim is to reduce the Cauchy problem (1.1) to a form allowing us to apply Corollary 2.2. In order to do this, we take the Fourier transform of (1.1) with respect to $k \in \mathbb{hZ}^n$, which gives

$$\partial_t^2 \mathcal{\hat{u}}(t, \theta) - \mathbb{h}^{-2} a(t)\sigma_{L_h}(k, \theta) \mathcal{\hat{u}}(t, \theta) + b(t)\mathcal{\hat{u}}(t, \theta) = \mathcal{\hat{f}}(t, \theta), \quad \theta \in \mathbb{T}^n. \quad (3.5)$$

Formally recalling the notation used in Proposition 2.1 and Corollary 2.2, we write

$$v(t) := \mathcal{\hat{u}}(t, \theta), \quad \beta^2 := -\mathbb{h}^{-2} \sigma_{L_h}(k, \theta) = \mathbb{h}^{-2} \left( 2n - 2 \sum_{j=1}^{n} \cos(2\pi \theta_j) \right), \quad (3.6)$$

as well as

$$v_0 := \mathcal{\hat{u}}_0(\theta), \quad v_1 := \mathcal{\hat{u}}_1(\theta), \quad g(t) := \mathcal{\hat{f}}(t, \theta). \quad (3.7)$$

Therefore, equation (3.5) becomes

$$v''(t) + \beta^2 a(t)v(t) + b(t)v(t) = g(t), \quad t \in [0, T]$$

with $\beta = \beta_{h, \theta}$ and all other functions depending on $\theta$ as a parameter. We proceed by discussing implications of Corollary 2.2 separately in each case.

**Case 1:** $a \in \text{Lip}([0, T])$, $a(t) \geq a_0 > 0$.

Applying Corollary 2.2 and inequality (2.3) in Proposition 2.1, we get

$$|v(t)|^2 + |v'(t)|^2 \lesssim C_1 e^{K_1 T (\beta)^2} (|v_0|^2 + |v_1|^2) + (1 + \beta^2)^2 \|g\|_{L^2([0, T])}^2. \quad (3.8)$$

Recalling the notation (3.6) and (3.7), in (3.8), we get

$$|\mathcal{\hat{u}}(t, \theta)|^2 + |\partial_t \mathcal{\hat{u}}(t, \theta)|^2 \lesssim C_1 e^{K_1 T (1 - \mathbb{h}^{-2} \sigma_{L_h}(k, \theta))} (|\mathcal{\hat{u}}_0(\theta)|^2 + |\mathcal{\hat{u}}_1(\theta)|^2) + (1 - \mathbb{h}^{-2} \sigma_{L_h}(k, \theta))^2 \|\mathcal{\hat{f}}(\cdot, \theta)\|_{L^2([0, T])}^2. \quad (3.9)$$

Now, by using the Plancherel formula (3.3) and Fourier transform, we get

$$\|u(t)\|_{L^2(\mathbb{hZ}^n)}^2 + \|\partial_t u(t)\|_{L^2(\mathbb{hZ}^n)}^2 \lesssim C_1 \left( \|e^{B_1 T (1 - \mathbb{h}^{-2} \mathcal{L}_h)} u_0\|_{L^2(\mathbb{hZ}^n)}^2 + \|e^{B_1 T (1 - \mathbb{h}^{-2} \mathcal{L}_h)} u_1\|_{L^2(\mathbb{hZ}^n)}^2 \right) + \|(I - \mathbb{h}^{-2} \mathcal{L}_h) f\|_{L^2([0, T]; L^2(\mathbb{hZ}^n))}^2. \quad (3.10)$$

where $B_1 = \frac{K_1}{2} > 0$. We also note that the symbol $\sigma_{L_h}$ is of order zero, therefore $I - \mathbb{h}^{-2} \mathcal{L}_h$ and $e^{B_1 T (1 - \mathbb{h}^{-2} \mathcal{L}_h)}$ are bounded pseudo-difference operators on $\ell^2(\mathbb{hZ}^n)$,
associated with the symbols $1 - h^{-2} \sigma \mathcal{L}_h$ and $e^{B_1 T (1 - h^{-2} \sigma \mathcal{L}_h)}$, respectively. Therefore

$$
\| u(t) \|_{L^2(\mathbb{R}^n)}^2 + \| \partial_t u(t) \|_{L^2(\mathbb{R}^n)}^2 \leq C_1 \| e^{B_1 T (1 - h^{-2} \mathcal{L}_h)} \|^2 \left( \| u_0 \|_{L^2(\mathbb{R}^n)}^2 + \| u_1 \|_{L^2(\mathbb{R}^n)}^2 \right) + \| I - h^{-2} \mathcal{L}_h \|_{L^2(\mathbb{R}^n)}^2 \| f \|_{L^2(\mathbb{R}^n)}^2 
$$

\[
\leq C_{h,T} \left( \| u_0 \|_{L^2(\mathbb{R}^n)}^2 + \| u_1 \|_{L^2(\mathbb{R}^n)}^2 + \| f \|_{L^2(\mathbb{R}^n)}^2 \right),
\]

(3.11)

where $C_{h,T} = \max\{ C_1 \| e^{B_1 T (1 - h^{-2} \mathcal{L}_h)} \|^2 \| L^2(\mathbb{R}^n)), \| I - h^{-2} \mathcal{L}_h \|_{L^2(\mathbb{R}^n)}^2 \}$.

**Case 2:** $a \in C^\alpha([0,T])$, with $0 < \alpha < 1$, $a_0 := \min_{[0,T]} a(t) > 0$.

The application of Corollary 2.2 and inequality (2.4) in Proposition 2.1, gives

$$
|v(t)|^2 + |v'(t)|^2 \lesssim C_2 e^{K_2 T^2 (\beta)^{\frac{1}{2}} (|v_0|^2 + |v_1|^2) + (1 + \beta^2)^2 \| g \|_{L^2(\mathbb{R}^n)}^2},
\]

(3.12)

for $0 < s \leq \frac{1}{2}$. Using (3.6) and (3.7) in (3.12), and similar to previous case, we get

$$
\| u(t) \|_{L^2(\mathbb{R}^n)}^2 + \| \partial_t u(t) \|_{L^2(\mathbb{R}^n)}^2 \lesssim C_2 \| e^{B_2 T (1 - h^{-2} \mathcal{L}_h)} \|^2 \left( \| u_0 \|_{L^2(\mathbb{R}^n)}^2 + \| u_1 \|_{L^2(\mathbb{R}^n)}^2 \right) + \| I - h^{-2} \mathcal{L}_h \|_{L^2(\mathbb{R}^n)}^2 \| f \|_{L^2(\mathbb{R}^n)}^2 
$$

\[
\leq C_{h,T} \left( \| u_0 \|_{L^2(\mathbb{R}^n)}^2 + \| u_1 \|_{L^2(\mathbb{R}^n)}^2 + \| f \|_{L^2(\mathbb{R}^n)}^2 \right),
\]

(3.13)

where $C_{h,T} = \max\{ C_2 \| e^{B_2 T (1 - h^{-2} \mathcal{L}_h)} \|^2 \| L^2(\mathbb{R}^n)), \| I - h^{-2} \mathcal{L}_h \|_{L^2(\mathbb{R}^n)}^2 \}$.

**Case 3:** $a \in C^l([0,T])$, with $l \geq 2$, $a(t) \geq 0$.

Similar to the above, from Corollary 2.2 and inequality (2.5) in Proposition 2.1, we get

$$
|v(t)|^2 + |v'(t)|^2 \lesssim C_3 e^{K_3 T^2 (\beta)^{\frac{1}{2}} (|v_0|^2 + |v_1|^2) + C(1 + \beta^2)^2 \| g \|_{L^2(\mathbb{R}^n)}^2},
\]

(3.14)

for $\sigma = 1 + \frac{1}{2}$. At the same time, recalling the notation (3.6) and (3.7), in (3.14), and similar to Case 1, we get

$$
\| u(t) \|_{L^2(\mathbb{R}^n)}^2 + \| \partial_t u(t) \|_{L^2(\mathbb{R}^n)}^2 \lesssim C_3' \| e^{B_3 (1 - h^{-2} \mathcal{L}_h)} \|^2 \left( \| u_0 \|_{L^2(\mathbb{R}^n)}^2 + \| u_1 \|_{L^2(\mathbb{R}^n)}^2 \right) + \| I - h^{-2} \mathcal{L}_h \|_{L^2(\mathbb{R}^n)}^2 \| f \|_{L^2(\mathbb{R}^n)}^2 
$$

\[
\leq C_{h,T} \left( \| u_0 \|_{L^2(\mathbb{R}^n)}^2 + \| u_1 \|_{L^2(\mathbb{R}^n)}^2 + \| f \|_{L^2(\mathbb{R}^n)}^2 \right),
\]

(3.15)

where $C_{h,T} = \max\{ C_3' \| e^{B_3 (1 - h^{-2} \mathcal{L}_h)} \|^2 \| L^2(\mathbb{R}^n)), \| I - h^{-2} \mathcal{L}_h \|_{L^2(\mathbb{R}^n)}^2 \}$.

**Case 4:** $a \in C^\alpha([0,T])$, with $0 < \alpha < 2$, $a(t) \geq 0$.  

Similar to the previous cases from Corollary 2.2 and inequality (2.6) in Proposition 2.1, we get
\[ |v(t)|^2 + |v'(t)|^2 \leq C_4 e^{K_4 T (2^s)^{\frac{1}{2}}} (|v_0|^2 + |v_1|^2) + C(1 + \beta^2)^2\|g\|_{L^2(0,T)}^2, \]  
for $0 < s \leq 1$. At the same time, recalling the notation (3.6) and (3.7) in (3.16), then similar to the Case 1, we get
\[
\|u(t)\|_{\ell^2(hZ^n)}^2 + \|\partial_t u(t)\|_{\ell^2(hZ^n)}^2 \lesssim C_4 \|\mathcal{E}(I-h^{-2}\mathcal{L}_h)\|_{\ell^2(hZ^n)}^2 \left( \|u_0\|_{\ell^2(hZ^n)}^2 + \|u_1\|_{\ell^2(hZ^n)}^2 \right) + \|I - h^{-2}\mathcal{L}_h\|_{\ell^2([0,T];\ell^2(hZ^n))}^2 \|f\|_{L^2([0,T];\ell^2(hZ^n))}^2 \leq C_{h,T} \left( \|u_0\|_{\ell^2(hZ^n)}^2 + \|u_1\|_{\ell^2(hZ^n)}^2 + \|f\|_{L^2([0,T];\ell^2(hZ^n))}^2 \right),
\]
where $C_{h,T} = \max\{C_4 \|\mathcal{E}(I-h^{-2}\mathcal{L}_h)\|_{\ell^2(hZ^n)}^2, \|I - h^{-2}\mathcal{L}_h\|_{\ell^2(hZ^n)}^2 \}$.

This proves (1.6) in Theorem 1.1. We will consider (1.7) in the next section. \qed

4. LIMIT AS $h \to 0$

In this section, we compare the solutions of (1.1) on $h\mathbb{Z}^n$ as $h \to 0$, with the solutions of (1.2) on $\mathbb{R}^n$. We now proceed to prove Theorem 1.2.

**Proof of Theorem 1.2.** Consider two Cauchy problems:
\[
\begin{align*}
\partial^2_t u(t, k) - a(t)h^{-2}\mathcal{L}_h u(t, k) + b(t)u(t, k) &= g(t, k), \quad k \in h\mathbb{Z}^n, \\
u(0, k) &= u_0(k), \\
\partial_t u(0, k) &= u_1(k),
\end{align*}
\]
and
\[
\begin{align*}
\partial^2_t v(t, x) - a(t)\mathcal{L} v(t, x) + b(t)v(t, x) &= g(t, x), \quad x \in \mathbb{R}^n, \\
v(0, x) &= u_0(x), \\
\partial_t v(0, x) &= u_1(x),
\end{align*}
\]
where $\mathcal{L}$ is the Laplacian on $\mathbb{R}^n$. From the equations (4.1) and (4.2), denoting $w := u - v$, we get
\[
\begin{align*}
\partial^2_t w(t, k) - a(t)h^{-2}\mathcal{L}_h w(t, k) + b(t)w(t, k) &= a(t) (h^{-2}\mathcal{L}_h - \mathcal{L}) v(t, k), \quad k \in h\mathbb{Z}^n, \\
w(0, k) &= 0, \\
\partial_t w(0, k) &= 0.
\end{align*}
\]

Our aim is to reduce the Cauchy problem (4.3) to a form allowing us to apply Corollary 2.2. In order to do this, we take the Fourier transform of (4.3) with respect to $k \in h\mathbb{Z}^n$. This gives
\[
\partial^2_t \hat{w}(t, \theta) - h^{-2}a(t)\sigma_{\mathcal{L}_h}(k, \theta) \hat{w}(t, \theta) + b(t)\hat{w}(t, \theta) = \hat{f}(t, \theta), \quad \theta \in \mathbb{T}^n,
\]
where $\sigma_{\mathcal{L}_h}(k, \theta)$ is the symbol of $\mathcal{L}_h$.
where \( f(t, k) := a(t)(\bar{h}^{-2} \mathcal{L}_h - \mathcal{L})v(t, k) \), and \( \sigma_{\mathcal{L}_h}(k, \theta) = \sigma_{\mathcal{L}_h}(\theta) \) is independent of \( k \in \mathbb{h}\mathbb{Z}^n \). Since \( w_0 = w_1 = 0 \), from Proposition 2.1, it follows that the solution of corresponding homogeneous equation of (4.3) is identically zero. Now applying Corollary 2.2 in (4.4), we get
\[
|\hat{w}(t, \theta)|^2 + |\partial_t \hat{w}(t, \theta)|^2 \lesssim (1 - \bar{h}^{-2}\sigma_{\mathcal{L}_h}(k, \theta))^2 \| \hat{f}(\cdot, \theta) \|^2_{L^2([0, T])}.
\] (4.5)

Similar to the proof of (1.6), by recalling Plancherel formula (3.3) and Fourier transform, we have
\[
\|w(t)\|^2_{\ell^2(\mathbb{h}\mathbb{Z}^n)} + \|\partial_t w(t)\|^2_{\ell^2(\mathbb{h}\mathbb{Z}^n)} \leq C \| (I - h^{-2} \mathcal{L}_h) a(h^{-2} \mathcal{L}_h - \mathcal{L})v \|^2_{L^2([0, T]; \ell^2(\mathbb{h}\mathbb{Z}^n))} \\
\leq C \|a\|^2_{\ell^\infty(0, T)} \sup_{t \in [0, T]} \| (I - h^{-2} \mathcal{L}_h) (h^{-2} \mathcal{L}_h - \mathcal{L})v(t, \cdot) \|^2_{\ell^2(\mathbb{h}\mathbb{Z}^n)},
\] (4.6)
where \( C > 0 \) is independent of \( h \).

Now we will estimate the term \( \| (I - h^{-2} \mathcal{L}_h) (h^{-2} \mathcal{L}_h - \mathcal{L})v(t, \cdot) \|^2_{\ell^2(\mathbb{h}\mathbb{Z}^n)} \). Let \( \phi \in C^4(\mathbb{R}^n) \), then by Taylor’s theorem with the Lagrange’s form of the remainder, we have
\[
\phi(\xi + h) = \sum_{|\alpha| \leq 3} \frac{\partial^\alpha \phi(\xi)}{\alpha!} h^n + \sum_{|\alpha|=4} \frac{\partial^\alpha \phi(\xi + \theta_4 h)}{\alpha!} h^n,
\] (4.7)
for some \( \theta_4 \in (0, 1) \) depending on \( \xi \). Let \( v_j \) be the \( j^{th} \) basis vector in \( \mathbb{Z}^n \), having all zeros except for 1 as the \( j^{th} \) component and then by taking \( h = v_j \) and \(-v_j \) in (4.7), we have
\[
\phi(\xi + v_j) = \phi(\xi) + \phi(v_j) + \frac{1}{2!} \phi^{(2v_j)}(\xi) + \frac{1}{3!} \phi^{(3v_j)}(\xi) + \frac{1}{4!} \phi^{(4v_j)}(\xi + \theta_4 v_j),
\] (4.8)
and
\[
\phi(\xi - v_j) = \phi(\xi) - \phi(v_j) + \frac{1}{2!} \phi^{(2v_j)}(\xi) - \frac{1}{3!} \phi^{(3v_j)}(\xi) + \frac{1}{4!} \phi^{(4v_j)}(\xi - \tilde{\theta}_4 v_j),
\] (4.9)
for some \( \theta_j, \tilde{\theta}_j \in (0, 1) \). Using (4.8) and (4.9), we have
\[
\phi(\xi + v_j) + \phi(\xi - v_j) - 2\phi(\xi) = \phi^{(2v_j)}(\xi) + \frac{1}{4!} \left( \phi^{(4v_j)}(\xi + \theta_j v_j) + \phi^{(4v_j)}(\xi - \tilde{\theta}_j v_j) \right).
\]
Since \( \delta_{\xi_j} \phi(\xi) = \phi(\xi + v_j) + \phi(\xi - v_j) - 2\phi(\xi) \), where \( \delta_{\xi_j} \phi(\xi) := \phi(\xi + \frac{1}{2} v_j) - \phi(\xi - \frac{1}{2} v_j) \), is the usual central difference operator, it follows that
\[
\delta_{\xi_j}^2 \phi(\xi) = \phi^{(2v_j)}(\xi) + \frac{1}{4!} \left( \phi^{(4v_j)}(\xi + \theta_j v_j) + \phi^{(4v_j)}(\xi - \tilde{\theta}_j v_j) \right).
\] (4.10)

Now by adding all the above \( n \)-equations for \( j = 1, \ldots, n \), we get
\[
\sum_{j=1}^n \delta_{\xi_j}^2 \phi(\xi) = \sum_{j=1}^n \phi^{(2v_j)}(\xi) + \frac{1}{4!} \sum_{j=1}^n \left( \phi^{(4v_j)}(\xi + \theta_j v_j) + \phi^{(4v_j)}(\xi - \tilde{\theta}_j v_j) \right).
\] (4.11)

Let us define a function \( V_{\theta_j v_j} \phi : \mathbb{R}^n \to \mathbb{R} \) by \( V_{\theta_j v_j} \phi(\xi) := \phi(\xi - \theta_j v_j) \), then we get
\[
\sum_{j=1}^n \delta_{\xi_j}^2 \phi(\xi) = \sum_{j=1}^n \delta_{\xi_j}^2 \phi(\xi) - \sum_{j=1}^n \delta_{\xi_j}^2 \phi(\xi) = \frac{1}{4!} \sum_{j=1}^n \left( V_{-\theta_j v_j} \phi^{(4v_j)}(\xi) + V_{\theta_j v_j} \phi^{(4v_j)}(\xi) \right).
\] (4.12)
Therefore, the equation (4.13) becomes
\[ \mathcal{L}_1 \phi_h(\xi) - L \phi_h(\xi) = \frac{1}{4!} \sum_{j=1}^{n} \left( V_{-\theta_j \nu_j} \phi^{(4\nu_j)}(\xi) + V_{\theta_j \nu_j} \phi^{(4\nu_j)}(\xi) \right), \] (4.13)
where \( \mathcal{L} \) is the Laplacian on \( \mathbb{R}^n \) and \( \mathcal{L}_1 \) is the discrete difference Laplacian on \( \mathbb{Z}^n \). One can quickly notice that
\[ V_{-\theta_j \nu_j} \phi^{(4\nu_j)}(\xi) = \phi^{(4\nu_j)}(\xi + \theta_j \xi \nu_j) = h^4 \phi^{(4\nu_j)}(h\xi + h\theta_j \xi \nu_j) = h^4 V_{-h\theta_j \nu_j} \phi^{(4\nu_j)}(h\xi). \] (4.14)
Therefore, the equation (4.13) becomes
\[ (L - h^2 \mathcal{L}) \phi(h\xi) = \frac{h^4}{4!} \sum_{j=1}^{n} \left( V_{-h\theta_j \nu_j} \phi^{(4\nu_j)}(h\xi) + V_{h\theta_j \nu_j} \phi^{(4\nu_j)}(h\xi) \right), \] (4.15)
whence we get
\[ (I - h^{-2} L_h) (L - h^2 \mathcal{L}) \phi(h\xi) = \frac{h^4}{4!} \sum_{j=1}^{n} \left( (I - h^{-2} L_h) V_{-h\theta_j \nu_j} \phi^{(4\nu_j)}(h\xi) + \right) \]
\[ \left( I - h^{-2} L_h \right) V_{h\theta_j \nu_j} \phi^{(4\nu_j)}(h\xi) \). \] (4.16)
Hence, it follows that
\[
\left\| (I - h^{-2} L_h) (h^{-2} \mathcal{L} - L) \phi \right\|^2_{\ell^2(h\mathbb{Z}^n)} \lesssim h^4 \left[ \max_{1 \leq j \leq n} \left\| (I - h^{-2} L_h) V_{-h\theta_j \nu_j} \phi^{(4\nu_j)} \right\|_{\ell^2(h\mathbb{Z}^n)}^2 \right]
+ \max_{1 \leq j \leq n} \left\| (I - h^{-2} L_h) V_{h\theta_j \nu_j} \phi^{(4\nu_j)} \right\|_{\ell^2(h\mathbb{Z}^n)}^2 \right]. \] (4.17)
Since in Case 1, \( v(t, \cdot) \in H^{s+1}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n) = H^{s+1}(\mathbb{R}^n) \), by Sobolev embedding theorem (see e.g. [20, Exercise 2.6.17]), we have
\[ s > k + \frac{n}{2} \implies H^s(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n). \] (4.18)
Thus for \( s > 3 + \frac{n}{2} \), we have \( H^{s+1}(\mathbb{R}^n) \subseteq C^4(\mathbb{R}^n) \), and if \( v(t, \cdot) \in H^{s+1}(\mathbb{R}^n) \), then \( v^{(4\nu_j)}(t, \cdot) \in H^{s-3}(\mathbb{R}^n) \subseteq H^2(\mathbb{R}^n) \) whenever \( s \geq 5 \), which gives
\[ \lim_{h \to 0} \left\| (I - h^{-2} L_h) V_{-h\theta_j \nu_j} v^{(4\nu_j)}(t, \cdot) \right\|^2_{\ell^2(h\mathbb{Z}^n)} = \left\| (I - L) v^{(4\nu_j)}(t, \cdot) \right\|^2_{L^2(\mathbb{R}^n)} < \infty. \] (4.19)
Combining the above assumptions, we can choose \( s \geq 5 \) for \( n \leq 3 \) and \( s > 3 + \frac{n}{2} \) for \( n \geq 4 \). On the other hand for Case 2, 3 and 4, we have \( v(t, \cdot) \in \gamma^a(\mathbb{R}^n) \cap H^{p+1}(\mathbb{R}^n) \subseteq C^\infty(\mathbb{R}^n) \), under the assumptions (1.3)-(1.5) and \( p \geq 5 \). Now from (4.17), it follows
that

\[
\| (I - \hbar^{-2} \mathcal{L}_h) (\mathcal{L} - \hbar^{-2} \mathcal{L}_h) v(t, \cdot) \|_{\ell^2(\mathbb{Z}^n)}^2 \lesssim \\
\hbar^4 \max_{1 \leq j \leq n} \left\| \left( I - \hbar^{-2} \mathcal{L}_h \right) \mathcal{L}^{-\hbar \theta_j v_j (4v_j)}(t, \cdot) \right\|_{\ell^2(\mathbb{Z}^n)}^2
\]

\[
+ \max_{1 \leq j \leq n} \left\| \left( I - \hbar^{-2} \mathcal{L}_h \right) \mathcal{L}^{-\hbar \theta_j v_j (4v_j)}(t, \cdot) \right\|_{\ell^2(\mathbb{Z}^n)}^2.
\]

(4.20)

Using (4.6), (4.19) and (4.20), we get \( \| w(t) \|_{\ell^2(\mathbb{Z}^n)}^2 + \| \partial_t w(t) \|_{\ell^2(\mathbb{Z}^n)}^2 \to 0 \) as \( \hbar \to 0 \).

Hence \( \| w(t) \|_{\ell^2(\mathbb{Z}^n)} \to 0 \) and \( \| \partial_t w(t) \|_{\ell^2(\mathbb{Z}^n)} \to 0 \) as \( \hbar \to 0 \). This finishes the proof of Theorem 1.2. \( \square \)

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