HS-integral and Eisenstein integral normal mixed Cayley graphs

Monu Kadyan
Department of Mathematics
Indian Institute of Technology Guwahati, India
monu.kadyan@iitg.ac.in

Abstract

A mixed graph is said to be HS-integral if the eigenvalues of its Hermitian-adjacency matrix of the second kind are integers. A mixed graph is called Eisenstein integral if the eigenvalues of its (0, 1)-adjacency matrix are Eisenstein integers. We characterize the set $S$ for which the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is HS-integral for any finite group $\Gamma$. We further show that a normal mixed Cayley graph is HS-integral if and only if it is Eisenstein integral. This paper generalizes the results of [M. Kadyan, B. Bhattacharjya. HS-integral and Eisenstein integral mixed Cayley graphs over abelian groups. Linear Algebra Appl. 645:68-90, 2022].

Keywords. integral graphs; HS-integral mixed graph; Eisenstein integral mixed graph; normal mixed Cayley graph.

Mathematics Subject Classifications: 05C50, 20C15.

1 Introduction

A mixed graph $G$ is a pair $(V(G), E(G))$, where $V(G)$ and $E(G)$ are the vertex and edge sets of $G$, respectively. Here $E(G) \subseteq V(G) \times V(G) \setminus \{(u, u) : u \in V(G)\}$. If $G$ is a mixed graph, then $(u, v) \in E(G)$ need not imply that $(v, u) \in E(G)$; see [18] for further information. If both $(u, v)$ and $(v, u)$ are members of $E(G)$, then $(u, v)$ is referred to as an undirected edge. If only one of $(u, v)$ and $(v, u)$ is a member of $E(G)$, then it is called a directed edge. As a result, both undirected and directed edges can exist simultaneously in a mixed graph. If all of the edges of $G$ are undirected (resp. directed), we refer to $G$ as a simple graph (resp. an oriented graph). Some definitions and results of this paper have similarities with those in the paper [12]. Throughout the paper, we consider $i = \sqrt{-1}$ and $\omega_n := \exp(\frac{2\pi i}{n})$.

Assume that $G$ is a mixed graph with $n$ vertices. The (0,1)-adjacency matrix and the Hermitian-adjacency matrix of the second kind of $G$ are denoted by $A(G) = (a_{uv})_{n \times n}$ and $H(G) = (h_{uv})_{n \times n}$,
respectively, where

\[
a_{uv} = \begin{cases} 
1 & \text{if } (u, v) \in E \\
0 & \text{otherwise},
\end{cases}
\quad \text{and} \quad
h_{uv} = \begin{cases} 
1 & \text{if } (u, v) \in E \text{ and } (v, u) \in E \\
\frac{1 + iv\sqrt{3}}{2} & \text{if } (u, v) \in E \text{ and } (v, u) \not\in E \\
\frac{1 - iv\sqrt{3}}{2} & \text{if } (u, v) \not\in E \text{ and } (v, u) \in E \\
0 & \text{otherwise}.
\end{cases}
\]

The Hermitian-adjacency matrix of the second kind was presented by Bojan Mohar [20]. An eigenvalue of \( H(G) \) is referred to an HS-eigenvalue of \( G \). An eigenvalue of \( A(G) \) is known as an eigenvalue of \( G \). Similarly, the HS-spectrum of \( G \) is the multi-set of the HS-eigenvalues of \( G \), and the spectrum of \( G \) is the multi-set of the eigenvalues of \( G \). The Hermitian-adjacency matrix of the second kind of a mixed graph is a Hermitian matrix, so its HS-eigenvalues are real numbers. However, if a mixed graph \( G \) has at least one directed edge, then \( A(G) \) is not a Hermitian matrix (or symmetric). As a result, the eigenvalues of \( G \) need not be real numbers.

A mixed graph \( G \) is said to be HS-integral if all of its HS-eigenvalues are integers. A mixed graph \( G \) is said to be Eisenstein integral if all of its eigenvalues are Eisenstein integers. Note that complex numbers of the form \( a + b\omega_3 \), where \( a, b \in \mathbb{Z} \), are known as Eisenstein integers. Note that \( A(G) = H(G) \) for a simple graph \( G \). Therefore, the term integral graph refers to an HS-integral simple graph. As a result, the words HS-eigenvalue, HS-spectrum and HS-integrality of a simple graph \( G \) have the same meaning with that of eigenvalue, spectrum and integrality of \( G \), respectively.

In 1974, Harary and Schwenk [10] raised the question of characterization of integral graphs. This problem has inspired a lot of interest over the last half-century. For more information on integral graphs, we refer the reader to [1, 3, 6, 23, 24].

Throughout the paper, we consider \( \Gamma \) to be a finite group and \( 1 \) to be the identity element of \( \Gamma \). Let \( S \) be a subset of \( \Gamma \) that does not contain the identity element, that is, \( 1 \not\in S \). If \( S \) is closed under inverse (resp. \( a^{-1} \not\in S \) for all \( a \in S \)), it is said to be symmetric (resp. skew-symmetric). Define \( \overline{S} = \{ u \in S : u^{-1} \not\in S \} \). Then \( S \setminus \overline{S} \) is symmetric, while \( \overline{S} \) is skew-symmetric. The mixed Cayley graph \( G = Cay(\Gamma, S) \) is a mixed graph with \( V(G) = \Gamma \) and \( E(G) = \{(a, b) : a, b \in \Gamma, ba^{-1} \in S\} \). If \( S \) is symmetric (resp. skew-symmetric), we refer \( G \) to be a simple Cayley graph (resp. oriented Cayley graph). A mixed Cayley graph \( Cay(\Gamma, S) \) is called normal if \( S \) is the union of some conjugacy classes of the group \( \Gamma \).

In 1982, Bridge and Mena [4] presented a characterization of integral Cayley graphs over abelian groups. Later on, same characterization was obtained by [2, 15, 21]. For results on integral Cayley graphs over non-abelian groups, we recommend the reader to [5, 16, 19]. The HS-integrality and Eisenstein integrality of mixed Cayley graphs over abelian groups and cyclic groups are characterized in [13] and [14], respectively. In 2014, Godsil et al. [9] characterized integral normal Cayley graphs.
The paper is organized as follows. In Section 2, we present some preliminary notions and known results. We also express the HS-eigenvalues of a normal mixed Cayley graph Cay(Γ, S) in terms of the irreducible characters of Γ. In section 3, we find a characterization of HS-integral normal oriented Cayley graphs. In section 4, we extend the characterization obtained in Section 3 to normal mixed Cayley graphs. In the last section, we show that a normal mixed Cayley graph is HS-integral if and only if it is Eisenstein integral.

2 Preliminaries

For \( x \in \Gamma \), let \( \text{ord}(x) \) denote the order of \( x \). If \( g \) and \( h \) are elements of the group \( \Gamma \), then we call \( h \) a conjugate of \( g \) if \( g = x^{-1}hx \) for some \( x \in \Gamma \). The conjugacy class of \( g \), denoted \( \text{Cl}(g) \), is the set of all conjugates of \( g \) in \( \Gamma \). Define \( C_\Gamma(g) \) to be the set of all elements of \( \Gamma \) that commute with \( g \). We denote the group algebra of \( \Gamma \) over a field \( \mathbb{F} \) by \( \mathbb{F}\Gamma \). That is, \( \mathbb{F}\Gamma \) is the set of all formal sums \( \sum_{g \in \Gamma} a_g g \), where \( a_g \in \mathbb{F} \), and we assume \( 1 \cdot g = g \) to have \( \Gamma \subseteq \mathbb{F}\Gamma \).

A representation of a finite group \( \Gamma \) is a homomorphism \( \rho: \Gamma \to \text{GL}_n(\mathbb{C}) \), where \( \text{GL}_n(\mathbb{C}) \) is the set of all \( n \times n \) invertible matrices with complex entries. Here, the number \( n \) is called the degree of \( \rho \). Two representations \( \rho_1 \) and \( \rho_2 \) of \( \Gamma \) of degree \( n \) are equivalent if there is a \( T \in \text{GL}_n(\mathbb{C}) \) such that \( T \rho_1(x) = \rho_2(x)T \) for each \( x \in \Gamma \).

Let \( \rho: \Gamma \to \text{GL}_n(\mathbb{C}) \) be a representation of \( \Gamma \). The character \( \chi_\rho: \Gamma \to \mathbb{C} \) of \( \rho \) is defined by setting \( \chi_\rho(x) := \text{Tr}(\rho(x)) \) for \( x \in \Gamma \), where \( \text{Tr}(\rho(x)) \) is the trace of \( \rho(x) \). By degree of \( \chi_\rho \), we mean the degree of \( \rho \), which is simply \( \chi_\rho(1) \). If \( W \) is a \( \rho(x) \)-invariant subspace of \( \mathbb{C}^n \) for each \( x \in \Gamma \), then we say that \( W \) is a \( \rho(\Gamma) \)-invariant subspace of \( \mathbb{C}^n \). If \( \{0\} \) and \( \mathbb{C}^n \) are the only \( \rho(\Gamma) \)-invariant subspaces of \( \mathbb{C}^n \), then we say \( \rho \) an irreducible representation of \( \Gamma \), and the corresponding character \( \chi_\rho \) an irreducible character of \( \Gamma \).

For a group \( \Gamma \), we denote by \( \text{IRR}(\Gamma) \) and \( \text{Irr}(\Gamma) \) the complete set of non-equivalent irreducible representations of \( \Gamma \) and the complete set of non-equivalent irreducible characters of \( \Gamma \), respectively. For \( z \in \mathbb{C} \), let \( \overline{z} \) denote the complex conjugate of \( z \) and \( \Re(z) \) (resp. \( \Im(z) \)) denote the real part (resp. imaginary part) of the complex number \( z \).

Theorem 2.1 (\[22\]). Let \( \Gamma \) be a finite group and \( \rho \) be a representation of \( \Gamma \) of degree \( k \) with corresponding character \( \chi \). If \( x \in \Gamma \) and \( \text{ord}(x) = m \), then the following assertions hold.

(i) \( \rho(x) \) is similar to a diagonal matrix with diagonal entries \( \epsilon_1, \ldots, \epsilon_k \), where \( \epsilon_i^m = 1 \) for each \( i \in \{1, \ldots, k\} \).

(ii) \( \chi(x) = \sum_{i=1}^k \epsilon_i \), where \( \epsilon_i^m = 1 \) for each \( i \in \{1, \ldots, k\} \).

(iii) \( \chi(x^{-1}) = \overline{\chi(x)} \).
Proof. Note that $\rho(x)^m$ is an identity matrix. Therefore, $\rho(x)$ is diagonalizable, and that its eigenvalues are $m$-th roots of unity. Thus the proofs of Part (i) and Part (ii) follow.

Again, $xx^{-1} = 1$ gives that $\rho(x^{-1}) = \rho(x)^{-1}$. Therefore if $\chi(x) = \sum_{i=1}^{k} \epsilon_i$, then we have that $\chi(x^{-1}) = \sum_{i=1}^{k} \epsilon_i^{-1} = \sum_{i=1}^{k} \bar{\epsilon}_i = \overline{\chi(x)}$. \hfill \Box

For a representation $\rho: \Gamma \to \text{GL}_n(\mathbb{C})$ of $\Gamma$, define $\overline{\rho}: \Gamma \to \text{GL}_n(\mathbb{C})$ by $\overline{\rho}(x) := \overline{\rho(x)}$, where $\overline{\rho(x)}$ is the matrix whose entries are the complex conjugates of the corresponding entries of $\rho(x)$. Note that if $\rho$ is irreducible, then $\overline{\rho}$ is also irreducible. Hence we have the following lemma. See Proposition 9.1.1 and Corollary 9.1.2 in [22] for details.

**Lemma 2.2** ([22]). Let $\Gamma$ be a finite group and $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. If $j \in \{1, \ldots, h\}$, then there exists $k \in \{1, \ldots, h\}$ satisfying $\chi_k = \chi_j$, where $\chi_k: \Gamma \to \mathbb{C}$ such that $\chi_k(x) = \overline{\chi_j(x)}$ for each $x \in \Gamma$.

**Theorem 2.3** ([22]). Let $\Gamma$ be a finite group and $x, y \in \Gamma$. If $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$, then

(i) \[\sum_{x \in \Gamma} \chi_j(x) \overline{\chi_k(x)} = \begin{cases} |\Gamma| & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}\]

(ii) \[\sum_{j=1}^{h} \chi_j(x) \overline{\chi_j(y)} = \begin{cases} |C_{\Gamma}(x)| & \text{if } x \text{ and } y \text{ are conjugates to each other} \\ 0 & \text{otherwise.} \end{cases}\]

For a function $f: \Gamma \to \mathbb{C}$, let $[f(yx^{-1})]_{x,y \in \Gamma}$ be the matrix whose rows and columns are indexed by the elements of $\Gamma$, and for $x, y \in \Gamma$, the $(x, y)$-th entry of the matrix is $f(yx^{-1})$.

**Theorem 2.4** ([8]). Let $\Gamma$ be a finite group and $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. If $f: \Gamma \to \mathbb{C}$ is a class function, then the spectrum of the matrix $[f(yx^{-1})]_{x,y \in \Gamma}$ is $\{[\gamma_1]^{d_1}, \ldots, [\gamma_h]^{d_h}\}$, where $\gamma_j = \frac{1}{\chi_j(1)} \sum_{x \in \Gamma} f(x) \chi_j(x)$ and $d_j = \chi_j(1)$ for each $j \in \{1, \ldots, h\}$.

**Lemma 2.5.** Let $\Gamma$ be a finite group. If $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$, then the HS-spectrum of the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is $\{[\gamma_1]^{d_1}, \ldots, [\gamma_h]^{d_h}\}$, where $\gamma_j = \lambda_j + \mu_j$,

$\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in S \setminus S} \chi_j(s)$, \hspace{0.5cm} $\mu_j = \frac{1}{\chi_j(1)} \sum_{s \in S} (\omega_s \chi_j(s) + \omega_{s^{-1}} \chi_j(s^{-1}))$,

and $d_j = \chi_j(1)$ for each $j \in \{1, \ldots, h\}$.
Proof. Let \( f : \Gamma \to \{0, 1, \omega_6, \omega_6^5\} \) be defined by
\[
f(s) = \begin{cases} 
1 & \text{if } s \in S \setminus \overline{S} \\
\omega_6 & \text{if } s \in \overline{S} \\
\omega_6^5 & \text{if } s \in \overline{S}^{-1} \\
0 & \text{otherwise.}
\end{cases}
\]

Since \( S \) is a union of some conjugacy classes of \( \Gamma \), \( f \) is a class function. The Hermitian adjacency matrix of the second kind of \( \text{Cay}(\Gamma, S) \) is given by \( [f(yx^{-1})]_{x,y \in \Gamma} \). By Theorem 2.4,
\[
\gamma_j = \frac{1}{\chi_j(1)} \left( \sum_{s \in S \setminus \overline{S}} \chi_j(s) + \sum_{s \in \overline{S}} \omega_6 \chi_j(s) + \sum_{s \in \overline{S}^{-1}} \omega_6^5 \chi_j(s) \right),
\]
and the result follows. \qed

As special cases of Lemma 2.5, we have the following two corollaries.

**Corollary 2.5.1.** Let \( \Gamma \) be a finite group. If \( \text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\} \), then the HS-spectrum (or spectrum) of the normal simple Cayley graph \( \text{Cay}(\Gamma, S) \) is \( \{[\lambda_1]^d, \ldots, [\lambda_h]^d\} \), where
\[
\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s) \text{ and } d_j = \chi_j(1) \text{ for each } j \in \{1, \ldots, h\}.
\]

**Corollary 2.5.2.** Let \( \Gamma \) be a finite group. If \( \text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\} \), then the HS-spectrum of the normal oriented Cayley graph \( \text{Cay}(\Gamma, S) \) is \( \{[\mu_1]^d, \ldots, [\mu_h]^d\} \), where
\[
\mu_j = \frac{1}{\chi_j(1)} \sum_{s \in S} (\omega_6 \chi_j(s) + \omega_6^5 \chi_j(s^{-1})) \text{ and } d_j = \chi_j(1) \text{ for each } j \in \{1, \ldots, h\}.
\]

Let \( n \geq 2 \) be a positive integer. For a divisor \( d \) of \( n \), define \( G_n(d) = \{k : 1 \leq k \leq n-1, \gcd(k, n) = d\} \).

It is clear that \( G_n(d) = dG_{\frac{n}{d}}(1) \).

Let \( B(\Gamma) \) be the boolean algebra generated by the subgroups of \( \Gamma \). That is, \( B(\Gamma) \) is the set whose elements are obtained by intersections, unions and complements of subgroups of \( \Gamma \). Define an equivalence relation \( \sim \) on \( \Gamma \) such that \( x \sim y \) if and only if \( y = x^k \) for some \( k \in G_m(1) \), where \( m = \text{ord}(x) \). For \( x \in \Gamma \), let \([x]\) denote the equivalence class of \( x \) with respect to the relation \( \sim \). Note that minimal non-empty sets in a boolean algebra are called its **atoms**.

**Theorem 2.6** ([2]). The atoms of the boolean algebra \( B(\Gamma) \) are the sets \([x]\) for each \( x \in \Gamma \).

By Theorem 2.6, we observe that each element of \( B(\Gamma) \) can be expressed as a disjoint union of the equivalence classes of the relation \( \sim \) on \( \Gamma \). Thus
\[
B(\Gamma) = \{[x_1] \cup \cdots \cup [x_k] : x_1, \ldots, x_k \in \Gamma, k \in \mathbb{N}\}.
\]

**Theorem 2.7** ([9]). Let \( \Gamma \) be a finite group and \( \text{Cay}(\Gamma, S) \) be a normal simple Cayley graph. Then \( \text{Cay}(\Gamma, S) \) is integral if and only if \( S \in B(\Gamma) \).
Let $n \equiv 0 \pmod{3}$. For a divisor $d$ of $\frac{n}{3}$ and $r \in \{1, 2\}$, define
\[
G_{n,3}(d) = \{dk : k \equiv r \pmod{3}, \gcd(dk, n) = d\}.
\]
It is easy to see that $G_n(d) = G_{n,3}(d) \cup G_{n,3}(d)$ is a disjoint union and $G_{n,3}(d) = dG_{n}(1)$ for $r = 1, 2$.

Let $\Gamma(3)$ be the set of all $x \in \Gamma$ satisfying $\ord(x) \equiv 0 \pmod{3}$. That is, $\Gamma(3) := \{x \in \Gamma : \ord(x) \equiv 0 \pmod{3}\}$. Define an equivalence relation $\simeq$ on $\Gamma(3)$ such that $x \simeq y$ if and only if $y = x^k$ for some $k \in G_{m,3}(1)$, where $m = \ord(x)$. Observe that if $x, y \in \Gamma(3)$ and $x \simeq y$ then $x \sim y$, but the converse need not be true. For example, consider $x = 5 \pmod{12}$, $y = 7 \pmod{12}$ in $\mathbb{Z}_{12}$. Here $x, y \in \mathbb{Z}_{12}(3)$ and $x \sim y$, but $x \not\simeq y$. For $x \in \Gamma(3)$, we denote the equivalence class of $x$ with respect to the relation $\simeq$ by $\langle x \rangle$. For $\Gamma(3) \neq \emptyset$, define $E(\Gamma)$ to be the set of all skew-symmetric subsets $S$, where $S = \langle x_1 \rangle \cup \cdots \cup \langle x_k \rangle$ for some $x_1, \ldots, x_k \in \Gamma(3)$. For $\Gamma(3) = \emptyset$, define $E(\Gamma) := \{\emptyset\}$. Thus
\[
E(\Gamma) = \begin{cases} 
\{\langle x_1 \rangle \cup \cdots \cup \langle x_k \rangle : x_1, \ldots, x_k \in \Gamma(3), k \in \mathbb{N}\} & \text{if } \Gamma(3) \neq \emptyset \\
\{\emptyset\} & \text{if } \Gamma(3) = \emptyset.
\end{cases}
\]

3 HS-integral normal oriented Cayley graphs

Let $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. Let $E$ be the matrix $[E_{ij}]$ of size $h \times n$, whose rows are indexed by $1, \ldots, h$, and columns are indexed by the elements of $\Gamma$ such that $E_{ij} = \chi_j(g)$. Note that $EE^* = nI_h$ and the rank of $E$ is $h$, where $E^*$ is the conjugate transpose of $E$.

It is well known that $\text{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}) = \{\sigma_r : r \in G_m(1), \sigma_r(\omega_m) = \omega_m^r\}$. For example, see Section 14.5 in [7]. If $m \equiv 0 \pmod{3}$, then $\mathbb{Q}(\omega_3, \omega_m) = \mathbb{Q}(\omega_m)$. Therefore, the Galois group $\text{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3))$ is a subgroup of $\text{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q})$. Thus $\text{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3))$ contains those automorphisms in $\text{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q})$ that fix $\omega_3$. Note that $G_m(1) = G_{m,3}(1) \cup G_{m,3}(1)$, a disjoint union. Using $\sigma_r(\omega_3) = \omega_3$ for all $r \in G_{m,3}(1)$ and $\sigma_r(\omega_3) = \omega_3^2$ for all $r \in G_{m,3}(1)$, we get
\[
\text{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3)) = \text{Gal}(\mathbb{Q}(\omega_3)/\mathbb{Q}(\omega_3)) = \{\sigma_r : r \in G_{m,3}(1), \sigma_r(\omega_m) = \omega_m^r\}.
\]

If $m \not\equiv 0 \pmod{3}$, then $|\mathbb{Q}(\omega_3, \omega_m) : \mathbb{Q}(\omega_3)| = \varphi(m)$. Thus the field $\mathbb{Q}(\omega_3, \omega_m)$ is a Galois extension of $\mathbb{Q}(\omega_3)$ of degree $\varphi(m)$. Any automorphism of the field $\mathbb{Q}(\omega_3, \omega_m)$ is uniquely determined by its action on $\omega_m$. Hence
\[
\text{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3)) = \{\tau_r : r \in G_m(1), \tau_r(\omega_m) = \omega_m^r \text{ and } \tau_r(\omega_3) = \omega_3\}.
\]

Let $g \in \Gamma$, $m = \ord(g)$ and $\chi$ be a character of $\Gamma$. By Theorem 2.1, $\chi(g) = \sum_{i=1}^k \epsilon_i$, where $\epsilon_1, \ldots, \epsilon_k$ are some $m$-th roots of unity. If $m \equiv 0 \pmod{3}$ and $\sigma_r \in \text{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3))$, then
\[
\sigma_r(\chi(g)) = \sigma_r\left(\sum_{i=1}^k \epsilon_i\right) = \sum_{i=1}^k \sigma_r(\epsilon_i) = \sum_{i=1}^k \epsilon_i^r = \chi(g^r).
\]

Similarly, if $m \not\equiv 0 \pmod{3}$ and $\tau_r \in \text{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3))$, then also $\tau_r(\chi(g)) = \chi(g^r)$. 

6
Theorem 3.1. Let \( \Gamma \) be a finite group and \( \text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\} \). If \( x = \sum_{g \in \Gamma} c_g g \in \mathbb{Q}(\omega_3)\Gamma \), then \( \chi_j(x) \) is rational for each \( j \in \{1, \ldots, h\} \) if and only if the following conditions hold:

(i) \( \sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s \) for each \( g_1, g_2 \in \Gamma(3) \) and \( g_1 \simeq g_2 \);

(ii) \( \sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s \) for each \( g_1, g_2 \in \Gamma \setminus \Gamma(3) \) and \( g_1 \sim g_2 \);

(iii) \( \sum_{s \in \text{Cl}(g)} c_s = \sum_{s \in \text{Cl}(g^{-1})} \tau_s \) for each \( g \in \Gamma \).

Proof. Let \( L \) be a set of representatives of the conjugacy classes in \( \Gamma \). Since characters are class functions, we have

\[
\chi_j(x) = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \right) \chi_j(g) \text{ for each } j \in \{1, \ldots, h\}.
\] (1)

Assume that \( \chi_j(x) \in \mathbb{Q} \) for each \( j \in \{1, \ldots, h\} \). Let \( g_1, g_2 \in \Gamma(3) \), \( g_1 \simeq g_2 \) and \( m = \text{ord}(g_1) \). Therefore, there exist \( r \in G_{m,3}^1(1) \) and \( \sigma_r \in \text{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}(\omega_3)) \) such that \( g_2 = g_1^r \) and \( \sigma_r(\omega_m) = \omega_m^r \). Note that \( \sigma_r(\chi_j(g_1)) = \chi_j(g_1^r) \) for each \( j \in \{1, \ldots, h\} \). For \( t \in \Gamma \), let \( \theta_t = \sum_{j=1}^{h} \chi_j(t) \chi_j \), where \( \chi_j(g) = \overline{\chi_j(g)} \) for each \( g \in \Gamma \). By Theorem 2.3, we have

\[
\theta_t(u) = \begin{cases} 
|C_{\Gamma}(t)| & \text{if } u \text{ and } t \text{ are conjugates to each other} \\
0 & \text{otherwise}.
\end{cases}
\]

So \( \theta_t(x) = |C_{\Gamma}(t)| \sum_{s \in \text{Cl}(t)} c_s \in \mathbb{Q}(\omega_3) \), and it gives that \( \sigma_r(\theta_t(x)) = \theta_t(x) \). Since \( \chi_j(x) \) is assumed to be a rational number, we have \( \sigma_r(\chi_j(x)) = \chi_j(x) \) for each \( j \in \{1, \ldots, h\} \). Thus

\[
|C_{\Gamma}(g_1)| \sum_{s \in \text{Cl}(g_1)} c_s = \theta_{g_1}(x) = \sigma_r(\theta_{g_1}(x)) = \sum_{j=1}^{h} \sigma_r(\chi_j(g_1)) \sigma_r(\chi_j(x))
\]

\[
= \sum_{j=1}^{h} \chi_j(g_1^r) \overline{\chi_j(x)}
\]

\[
= \theta_{g_1^r}(x) = \theta_{g_2}(x) = |C_{\Gamma}(g_2)| \sum_{s \in \text{Cl}(g_2)} c_s.
\] (2)

Since \( g_1 \simeq g_2 \), we have \( C_{\Gamma}(g_1) = C_{\Gamma}(g_2) \). So Equation (2) implies that \( \sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s \). Hence condition (i) holds.

Now let \( g_1, g_2 \in \Gamma \setminus \Gamma(3) \), \( g_1 \sim g_2 \), and \( m = \text{ord}(g_1) \). Then there is \( r \in G_{m,3}(1) \) and \( \sigma_r \in \text{Gal}(\mathbb{Q}(\omega_3, \omega_m)/\mathbb{Q}(\omega_3)) \) such that \( g_2 = g_1^r \), \( \sigma_r(\omega_m) = \omega_m^r \) and \( \sigma_r(\omega_3) = \omega_3 \). Now proceeding as in the proof of condition (i), we have \( \sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s \). Thus condition (ii) also holds.
Again

\[ 0 = \chi_j(x) - \overline{\chi_j(x)} = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \chi_j(g) - \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} \overline{\tau_s} \right) \chi_j(g) \right) \]

\[ = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \chi_j(g) - \sum_{s \in \text{Cl}(g^{-1})} \tau_s \right) \chi_j(g), \]

and so

\[ \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s - \sum_{s \in \text{Cl}(g^{-1})} \tau_s \right) \begin{bmatrix} \chi_1(g) \\ \vdots \\ \chi_h(g) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3} \]

Note that the number of irreducible characters of \( \Gamma \) is equal to the number of conjugacy classes of \( \Gamma \), that is, \(|L| = h\). Since characters are class functions and rank of \( E \) is \( h \), the columns of \( E \) corresponding to the elements of \( L \) are linearly independent. Thus by Equation (3), \( \sum_{s \in \text{Cl}(g)} c_s - \sum_{s \in \text{Cl}(g^{-1})} \tau_s = 0 \) for all \( g \in L \), and so condition (iii) holds.

Conversely, assume that the three conditions of the theorem hold. Let \( n \) be the number of elements of \( \Gamma \). We have the following two cases.

**Case 1.** Assume that \( n \equiv 0 \pmod{3} \). Let \( \sigma_k \in \text{Gal}(\mathbb{Q}(\omega_3, \omega_n)/\mathbb{Q}(\omega_3)) \). Then \( \sigma_k(\omega_n) = \omega_n^k \) and \( k \in G_{n,3}^1(1) \), and so \( \sigma_k(\chi_j(g)) = \chi_j(g^k) \) for each \( j \in \{1, \ldots, h\} \). Thus

\[ \sigma_k(\chi_j(x)) = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \right) \sigma_k(\chi_j(g)) = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \right) \chi_j(g^k). \tag{4} \]

In the sum of Equation (4) we have two possible cases, namely, \( g \in \Gamma(3) \) or \( g \in \Gamma \setminus \Gamma(3) \). If \( g \in \Gamma(3) \), then using the fact \( g \cong g^k \) and condition (i), we get \( \sum_{s \in \text{Cl}(g)} c_s = \sum_{s \in \text{Cl}(g^k)} c_s \). Similarly, if \( g \in \Gamma \setminus \Gamma(3) \), then using the fact \( g \cong g^k \) and condition (ii), we get \( \sum_{s \in \text{Cl}(g)} c_s = \sum_{s \in \text{Cl}(g^k)} c_s \). Therefore, we have \( \sum_{s \in \text{Cl}(g)} c_s = \sum_{s \in \text{Cl}(g^k)} c_s \) for each \( g \in \Gamma \). Now from Equation (4), we get

\[ \sigma_k(\chi_j(x)) = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g^k)} c_s \right) \chi_j(g^k) = \chi_j(x). \tag{5} \]

The second equality in Equation (5) holds, because \( \{g^k : g \in L\} \) is also a set of representatives of conjugacy classes of \( \Gamma \). Now since \( \sigma_k(\chi_j(x)) = \chi_j(x) \) for each \( k \in G_{n,3}^1(1) \), we have that \( \chi_j(x) \in \mathbb{Q}(\omega_3) \).

**Case 2.** Assume that \( n \not\equiv 0 \pmod{3} \). Let \( \tau_r \in \text{Gal}(\mathbb{Q}(\omega_3, \omega_n)/\mathbb{Q}(\omega_3)) \). Then we have \( \tau_r(\chi_j(g)) = \chi_j(g^r) \).
for each $j \in \{1, \ldots, h\}$. Note that $g \sim g'$. Therefore using Equation (1) and condition (ii), we have

$$
\tau_r(\chi_j(x)) = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \right) \tau_r(\chi_j(g))
= \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} c_s \right) \chi_j(g')
= \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g')} c_s \right) \chi_j(g')
= \chi_j(x).
$$

This gives that $\chi_j(x) \in \mathbb{Q}(\omega_3)$. Thus in both the cases, we get $\chi_j(x) \in \mathbb{Q}(\omega_3)$. Taking complex conjugates in Equation (1), we get

$$
\overline{\chi_j(x)} = \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} \overline{\tau_s} \right) \overline{\chi_j(g)}
= \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g)} \tau_s \right) \chi_j(g^{-1})
= \sum_{g \in L} \left( \sum_{s \in \text{Cl}(g^{-1})} c_s \right) \chi_j(g^{-1})
= \chi_j(x). \tag{6}
$$

Equation (6) implies that $\chi_j(x) \in \mathbb{Q}$ for all $j \in \{1, \ldots, h\}$. Indeed, we can replace condition (i) of Theorem 3.1 by $\sum_{s \in \text{Cl}(x)} c_s = \sum_{s \in \text{Cl}(y)} c_s$ for all $x, y \in \langle g \rangle$ and $g \in \Gamma(3)$.

**Theorem 3.2.** Let $\Gamma$ be a finite group and $\text{Cay}(\Gamma, S)$ be a normal oriented Cayley graph. Then $\text{Cay}(\Gamma, S)$ is HS-integral if and only if $S \in E(\Gamma)$.

**Proof.** Let $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$ and $x = \sum_{g \in \Gamma} c_g g$, where

$$
c_g = \begin{cases} 
-\omega_3^2 & \text{if } g \in S \\
-\omega_3 & \text{if } g \in S^{-1} \\
0 & \text{otherwise.}
\end{cases}
$$

Note that $-\omega_3^2 = \omega_6$ and $-\omega_3 = \omega_9^2$. Thus $\chi_j(x) = \sum_{s \in S} (-\omega_3^2 \chi_j(s) - \omega_3 \chi_j(s^{-1}))$, and so $\frac{\chi_j(x)}{\chi_j(1)}$ is an HS-eigenvalue of $\text{Cay}(\Gamma, S)$. Assume that the normal oriented Cayley graph $\text{Cay}(\Gamma, S)$ is HS-integral. Thus $\chi_j(x)$ is an integer for each $j \in \{1, \ldots, h\}$, and therefore the three conditions of Theorem 3.1 are satisfied for $x$. Using the fact that $g \sim g^{-1}$, and conditions (ii) and (iii) of Theorem 3.1, we get $\Im \left( \sum_{s \in \text{Cl}(g)} c_s \right) = 0$ for all $g \in \Gamma \setminus \Gamma(3)$. Note that $S$ is a union of some conjugacy classes of $\Gamma$. Therefore, if $g \in S$ then $\text{Cl}(g) \subseteq S$, and so by the definition of $c_g$, we get $\Im \left( \sum_{s \in \text{Cl}(g)} c_s \right) = \frac{\sqrt{3} \text{Cl}(g)}{2} \neq 0$. Thus $S \cap (\Gamma \setminus \Gamma(3)) = \emptyset$,
that is, $S \subseteq \Gamma(3)$. Again, let $g_1 \in S$, $g_2 \in \Gamma(3)$ and $g_1 \simeq g_2$. By the first condition of Theorem 3.1, we get $0 \neq \sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s$, which implies that $g_2 \in S$. Thus $g_1 \in S$ gives $\langle g_1 \rangle \subseteq S$. Hence $S \in \mathbb{E}(\Gamma)$.

Conversely, assume that $S \in \mathbb{E}(\Gamma)$. Let $\text{Cay}(\Gamma, S)$ be a normal oriented Cayley graph, so that $S$ is a union of some conjugacy classes of $\Gamma$. Let

$$S = \langle x_1 \rangle \cup \cdots \cup \langle x_r \rangle = \text{Cl}(y_1) \cup \cdots \cup \text{Cl}(y_k) \subseteq \Gamma(3)$$

for some $x_1, \ldots, x_r, y_1, \ldots, y_k \in \Gamma(3)$. We have

$$S^{-1} = \langle x_1^{-1} \rangle \cup \cdots \cup \langle x_r^{-1} \rangle = \text{Cl}(y_1^{-1}) \cup \cdots \cup \text{Cl}(y_k^{-1}) \subseteq \Gamma(3).$$

Now for $g_1, g_2 \in \Gamma(3)$, if $g_1 \simeq g_2$ then $\text{Cl}(g_1), \text{Cl}(g_2) \subseteq S$ or $\text{Cl}(g_1), \text{Cl}(g_2) \subseteq S^{-1}$ or $\text{Cl}(g_1), \text{Cl}(g_2) \subseteq (S \cup S^{-1})^c$. Note that $|\text{Cl}(g_1)| = |\text{Cl}(g_2)|$. For all the cases, using the definition of $c_g$, we find

$$\sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s.$$

Thus condition (i) of Theorem 3.1 holds. If $g_1, g_2 \in \Gamma \setminus \Gamma(3)$ and $g_1 \sim g_2$, then clearly $\text{Cl}(g_1), \text{Cl}(g_2) \subseteq \Gamma \setminus \Gamma(3)$. Therefore $\text{Cl}(g_1), \text{Cl}(g_2) \subseteq (S \cup S^{-1})^c$. Accordingly,

$$\sum_{s \in \text{Cl}(g_1)} c_s = 0 = \sum_{s \in \text{Cl}(g_2)} c_s.$$

Hence condition (ii) of Theorem 3.1 also holds.

Again for $g \in \Gamma$, we have $\text{Cl}(g) \subseteq S$ if and only if $\text{Cl}(g^{-1}) \subseteq S^{-1}$. Therefore we have $\sum_{s \in \text{Cl}(g)} c_s = \sum_{s \in \text{Cl}(g^{-1})} \tau_s$, and so condition (iii) of Theorem 3.1 also holds. Thus by Theorem 3.1, $\chi_j(x)$ is a rational number for each $j \in \{1, \ldots, h\}$. Consequently, the HS-eigenvalue $\mu_j := \frac{\chi_j(x)}{\chi(1)}$ of $\text{Cay}(\Gamma, S)$ is a rational algebraic integer, and hence an integer for each $j \in \{1, \ldots, h\}$. \hfill \qed

In the following example, we illustrate an use of Theorem 3.2.

**Example 3.1.** Consider $S = \{(1, 2, 3), (4, 2, 1), (2, 4, 3), (3, 4, 1)\}$ in the alternating group $A_4$. The conjugacy classes of $A_4$ are $\{I\}, \text{Cl}((1, 2)(3, 4)), \text{Cl}((1, 2, 3))$ and $\text{Cl}((1, 3, 2))$, where

$$I = (1)(2)(3)(4),$$

$$\text{Cl}((1, 2)(3, 4)) = \{(1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\},$$

$$\text{Cl}((1, 2, 3)) = \{(1, 2, 3), (4, 2, 1), (2, 4, 3), (3, 4, 1)\}$$

and

$$\text{Cl}((1, 3, 2)) = \{(1, 3, 2), (4, 1, 2), (2, 3, 4), (3, 1, 4)\}.$$
Figure 1: The oriented graph Cay($A_4$, \{(1, 2, 3), (4, 2, 1), (2, 4, 3), (3, 4, 1)\})

|     | $I$                | $\text{Cl}((1, 2)(3, 4))$ | $\text{Cl}((1, 2, 3))$ | $\text{Cl}((1, 3, 2))$ |
|-----|---------------------|---------------------------|------------------------|------------------------|
| $\chi_1$ | 1                  | 1                         | 1                      | 1                      |
| $\chi_2$ | 1                  | 1                         | $\omega_3$             | $\omega_3^2$           |
| $\chi_3$ | 1                  | 1                         | $\omega_3^2$           | $\omega_3$             |
| $\chi_4$ | 3                  | $\mu_1$ = 4               | 0                      | 0                      |

Table 1: Character table of $A_4$

HS-integral by Theorem 3.2. The character table of the group $A_4$ is given in Table 1 [11], where $\text{Irr}(A_4) = \{\chi_1, \chi_2, \chi_3, \chi_4\}$. Further, using Corollary 2.5.2, the HS-spectrum of Cay($A_4, S$) is obtained as $\{[\mu_1]^1, [\mu_2]^1, [\mu_3]^1, [\mu_4]^9\}$, where $\mu_1 = 4(\omega_6 + \omega_6^2) = 4$, $\mu_2 = 4(\omega_6 \omega_3 + \omega_6^2 \omega_3^2) = -8$, $\mu_3 = 4(\omega_6 \omega_3^2 + \omega_6^2 \omega_3) = 4$ and $\mu_4 = 0$.

4 HS-integral normal mixed Cayley graphs

In this section, we extend Theorem 3.2 to normal mixed Cayley graphs.

Lemma 4.1. Let $S$ be a skew-symmetric subset of a finite group $\Gamma$ and $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. Let $S$ be expressible as a union of some conjugacy classes of $\Gamma$ and $t(\neq 0) \in \mathbb{Q}$. If

$$\frac{1}{\chi_j(1)} \sum_{s \in S} \text{hv} \sqrt{3} (\chi_j(s) - \chi_j(s^{-1}))$$

is an integer for each $j \in \{1, \ldots, h\}$, then $S \in E(\Gamma)$. 

11
Proof. Let \( x = \sum_{g \in \Gamma} c_g g \in \mathbb{Q}(\omega_3)\Gamma \), where

\[
c_g = \begin{cases} 
  \text{it} \sqrt{3} & \text{if } g \in S \\
  -\text{it} \sqrt{3} & \text{if } g \in S^{-1} \\
  0 & \text{otherwise.}
\end{cases}
\]

Note that \( \frac{\chi_j(x)}{\chi_j(1)} = \frac{1}{\chi_j(1)} \sum_{s \in S} \text{it} \sqrt{3} (\chi_j(s) - \chi_j(s^{-1})) \). Assume that \( \frac{\chi_j(x)}{\chi_j(1)} \) is an integer for each \( j \in \{1, \ldots, h\} \). Therefore, all the three conditions of Theorem 3.1 are satisfied for \( x \). Using the fact that \( g \sim g^{-1} \), and conditions (ii) and (iii) of Theorem 3.1, we get \( \sum_{s \in \text{Cl}(g)} c_s = 0 \) for all \( g \in \Gamma \setminus \Gamma(3) \), and so we must have \( S \cup S^{-1} \subseteq \Gamma(3) \). Again, let \( g_1 \in S, g_2 \in \Gamma(3) \) and \( g_1 \sim g_2 \). The first condition of Theorem 3.1 gives

\[
\sum_{s \in \text{Cl}(g_1)} c_s = \sum_{s \in \text{Cl}(g_2)} c_s.
\]

Note that \( \sum_{s \in \text{Cl}(g_1)} c_s = \text{it} \sqrt{3} |\text{Cl}(g_1)| \). Therefore \( \sum_{s \in \text{Cl}(g_2)} c_s = \text{it} \sqrt{3} |\text{Cl}(g_1)| \), and so \( g_2 \in S \). Thus \( g_1 \in S \) implies \( \langle g_1 \rangle \subseteq S \). Hence \( S \in E(\Gamma) \).

In [13], the authors proved that if \( \Gamma \) is an abelian group, then \( \langle x \rangle \cup \langle x^{-1} \rangle = [x] \) for each \( x \in \Gamma(3) \). Note that this result and its proof also hold good for non-abelian group. In the subsequent discussion, we use this fact for non-abelian group.

Lemma 4.2. Let \( S \) be a skew-symmetric subset of a finite group \( \Gamma \) and \( \text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\} \). Let \( S \) be expressible as a union of some conjugacy classes of \( \Gamma \) and \( t(\neq 0) \in \mathbb{Q} \). If

\[
\frac{1}{\chi_j(1)} \sum_{s \in S} \text{it} \sqrt{3} (\chi_j(s) - \chi_j(s^{-1}))
\]

is an integer for each \( j \in \{1, \ldots, h\} \), then \( \frac{1}{\chi_j(1)} \sum_{s \in S \cup S^{-1}} \chi_j(s) \) is also an integer for each \( j \in \{1, \ldots, h\} \).

Proof. Assume that \( \frac{1}{\chi_j(1)} \sum_{s \in S} \text{it} \sqrt{3} (\chi_j(s) - \chi_j(s^{-1})) \) is an integer for each \( j \in \{1, \ldots, h\} \). By Lemma 4.1 we have \( S \in E(\Gamma) \), and so \( S = \langle x_1 \rangle \cup \cdots \cup \langle x_k \rangle \) for some \( x_1, \ldots, x_k \in \Gamma(3) \). Therefore, we get

\[
S \cup S^{-1} = (\langle x_1 \rangle \cup \cdots \cup \langle x_k \rangle) \cup (\langle x_1^{-1} \rangle \cup \cdots \cup \langle x_k^{-1} \rangle) = [x_1] \cup \cdots \cup [x_k] \in B(\Gamma).
\]

Thus by Theorem 2.7, \( \text{Cay}(\Gamma, S \cup S^{-1}) \) is integral, that is, \( \frac{1}{\chi_j(1)} \sum_{s \in S \cup S^{-1}} \chi_j(s) \) is an integer for each \( j \in \{1, \ldots, h\} \).

In the next result, we use the fact that the HS-eigenvalues of a mixed Cayley graph are algebraic integers. See Theorem 2.6 of [17] for details.

Lemma 4.3. If \( \Gamma \) is a finite group, then the normal mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is HS-integral if and only if \( \text{Cay}(\Gamma, S \setminus \overline{S}) \) is integral (or HS-integral) and \( \text{Cay}(\Gamma, \overline{S}) \) is HS-integral.
Proof. Let \( \text{Irr}(\Gamma) = \{\chi_1, \cdots, \chi_h\} \). By Lemma 2.5, the HS-spectrum of the normal mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is \([\gamma_1]^d, \cdots, [\gamma_h]^d \) where \( \gamma_j = \lambda_j + \mu_j \),

\[
\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in S \setminus \mathcal{S}} \chi_j(s), \quad \mu_j = \frac{1}{\chi_j(1)} \sum_{s \in \mathcal{S}} (\omega_6 \chi_j(s) + \omega_6^5 \chi_j(s^{-1})),
\]

and \( d_j = \chi_j(1) \) for each \( j \in \{1, \ldots, h\} \). Note that \([\lambda_1]^d, \cdots, [\lambda_h]^d \) is the spectrum of \( \text{Cay}(\Gamma, S \setminus \mathcal{S}) \) and \([\mu_1]^d, \cdots, [\mu_h]^d \) is the HS-spectrum of \( \text{Cay}(\Gamma, \mathcal{S}) \).

Assume that the mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is HS-integral. Let \( j \in \{1, \ldots, h\} \). By Lemma 2.2, there exists \( k \in \{1, \ldots, h\} \) such that \( \chi_k = \chi_j \). Therefore, \( \chi_j(1) = \chi_k(1) \) and

\[
\lambda_j = \frac{1}{\chi_j(1)} \sum_{s \in S \setminus \mathcal{S}} \chi_j(s^{-1}) = \frac{1}{\chi_j(1)} \sum_{s \in \mathcal{S}} \chi_j(s) = \frac{1}{\chi_k(1)} \sum_{s \in \mathcal{S}} \chi_k(s) = \lambda_k.
\]

Now we have

\[
\gamma_j - \gamma_k = \frac{1}{\chi_j(1)} \sum_{s \in S} (\omega_6 \chi_j(s) + \omega_6^5 \chi_j(s^{-1})) - \frac{1}{\chi_k(1)} \sum_{s \in \mathcal{S}} (\omega_6 \chi_k(s) + \omega_6^5 \chi_k(s^{-1})) = \frac{1}{\chi_j(1)} \sum_{s \in S} (\omega_6 \chi_j(s) + \omega_6^5 \chi_j(s^{-1})) - \frac{1}{\chi_j(1)} \sum_{s \in \mathcal{S}} (\omega_6 \chi_j(s) + \omega_6^5 \chi_j(s)) = \frac{1}{\chi_j(1)} \sum_{s \in S} (\omega_6 - \omega_6^5) \chi_j(s) + (\omega_6^5 - \omega_6) \chi_j(s^{-1}))
= \frac{1}{\chi_j(1)} \sum_{s \in S} i\sqrt{3} (\chi_j(s) - \chi_j(s^{-1})).
\]

By assumption \( \gamma_j, \gamma_k \in \mathbb{Z} \), and so \( \frac{1}{\chi_j(1)} \sum_{s \in S} i\sqrt{3} (\chi_j(s) - \chi_j(s^{-1})) \in \mathbb{Z} \) for each \( j \in \{1, \ldots, h\} \). Therefore by Lemma 4.2, we get \( \frac{1}{\chi_j(1)} \sum_{s \in S \setminus \mathcal{S}} \chi_j(s) \in \mathbb{Z} \) for each \( j \in \{1, \ldots, h\} \). Since

\[
\mu_j = \frac{1}{2\chi_j(1)} \sum_{s \in S \setminus \mathcal{S}} \chi_j(s) + \frac{1}{2\chi_j(1)} \sum_{s \in \mathcal{S}} i\sqrt{3} (\chi_j(s) - \chi_j(s^{-1})),
\]

\( \mu_j \) is a rational algebraic integer, and hence it is an integer for each \( j \in \{1, \ldots, h\} \). Thus \( \text{Cay}(\Gamma, \mathcal{S}) \) is HS-integral. Now we have \( \gamma_j, \mu_j \in \mathbb{Z} \), and so \( \lambda_j = \gamma_j - \mu_j \in \mathbb{Z} \) for each \( j \in \{1, \ldots, h\} \). Hence \( \text{Cay}(\Gamma, S \setminus \mathcal{S}) \) is also integral.

Conversely, assume that \( \text{Cay}(\Gamma, S \setminus \mathcal{S}) \) is integral and \( \text{Cay}(\Gamma, \mathcal{S}) \) is HS-integral. Then Lemma 2.5 implies that \( \text{Cay}(\Gamma, S) \) is HS-integral. \( \square \)

Theorem 4.4. Let \( \Gamma \) be a finite group and \( \text{Cay}(\Gamma, S) \) be a normal mixed Cayley graph. Then \( \text{Cay}(\Gamma, S) \) is HS-integral if and only if \( S \setminus \mathcal{S} \in \mathbb{B}(\Gamma) \) and \( \mathcal{S} \in \mathbb{E}(\Gamma) \).
Figure 2: The mixed graph $\text{Cay}(A_4, S)$

**Proof.** By Lemma 4.3, $\text{Cay}(\Gamma, S)$ is HS-integral if and only if $\text{Cay}(\Gamma, S \setminus \overline{S})$ is integral and $\text{Cay}(\Gamma, \overline{S})$ is HS-integral. Now the proof follows from Theorem 2.7 and Theorem 3.2.

We give the following example to illustrate Theorem 4.4.

**Example 4.1.** Consider

$$S = \{(1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,2,3), (4,2,1), (2,4,3), (3,4,1)\}$$

in the alternating group $A_4$. The normal mixed Cayley graph $\text{Cay}(A_4, S)$ is shown in Figure 2. We find that

$$\overline{S} = \langle (1,2)(3,4) \rangle \cup \langle (1,3)(2,4) \rangle \cup \langle (1,4)(2,3) \rangle \cup \langle (1,2,3) \rangle \cup \langle (4,2,1) \rangle = \text{Cl}((1,2,3)) \in \mathbb{E}(\Gamma)$$

and

$$S \setminus \overline{S} = [(1,2)(3,4)] \cup [(1,3)(2,4)] \cup [(1,4)(2,3)] = \text{Cl}((1,2)(3,4)) \in \mathbb{B}(\Gamma).$$

Using Theorem 4.4, $\text{Cay}(A_4, S)$ is HS-integral. The character table of $A_4$ is given in Table 1. Further, using Lemma 2.5, the HS-spectrum of $\text{Cay}(A_4, S)$ is obtained as $\{[\gamma_1]^1, [\gamma_2]^1, [\gamma_3]^1, [\gamma_4]^9\}$, where $\gamma_1 = 3 + 4(\omega_6 + \omega_6^5) = 7$, $\gamma_2 = 3 + 4(\omega_6\omega_3 + \omega_6^5\omega_2^3) = -5$, $\gamma_3 = 3 + 4(\omega_6^2\omega_3^2 + \omega_6^5\omega_3) = 7$ and $\gamma_4 = -1$.

## 5 Eisenstein integral normal mixed Cayley graphs

Assume that $S$ is a union of some conjugacy classes of a finite group $\Gamma$, $1 \notin S$ and $\text{Irr}(\Gamma) = \{\chi_1, \ldots, \chi_h\}$. Using the function $f: \Gamma \to \{0, 1\}$ defined by

$$f(s) = \begin{cases} 1 & \text{if } s \in S \\ 0 & \text{otherwise} \end{cases}$$
in Theorem 2.4, we find that \( \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s) \) is an eigenvalue of the normal mixed Cayley graph \( \text{Cay}(\Gamma, S) \) for each \( j \in \{1, \ldots, h\} \). Indeed, all the eigenvalues of \( \text{Cay}(\Gamma, S) \) are of this form.

For each \( j \in \{1, \ldots, h\} \), define

\[
f_j(S) := \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s) \quad \text{and} \quad g_j(S) := \frac{1}{\chi_j(1)} \sum_{s \in S} (\omega \chi_j(s) + \sqrt{3} \chi_j(s^{-1})),
\]

where \( \omega = \frac{1}{2} - \frac{i \sqrt{3}}{6} \). Let \( j \in \{1, \ldots, h\} \). By Lemma 2.2, there exists \( k \in \{1, \ldots, h\} \) such that \( \chi_k = \overline{\chi}_j \).

Note that

\[
g_j(S) + \omega (g_j(S) - g_k(S)) = (1 + \omega) g_j(S) - \omega g_k(S)
\]

\[
= \frac{1 + i \sqrt{3}}{2 \chi_j(1)} \sum_{s \in S} \left[ \left( \frac{1}{2} - \frac{i \sqrt{3}}{6} \right) \chi_j(s) + \left( \frac{1}{2} + \frac{i \sqrt{3}}{6} \right) \chi_j(s^{-1}) \right]
\]

\[
+ \frac{1 - i \sqrt{3}}{2 \chi_j(1)} \sum_{s \in S} \left[ \left( \frac{1}{2} - \frac{i \sqrt{3}}{6} \right) \chi_k(s) + \left( \frac{1}{2} + \frac{i \sqrt{3}}{6} \right) \chi_k(s^{-1}) \right]
\]

\[
= \frac{1 + i \sqrt{3}}{2 \chi_j(1)} \sum_{s \in S} \left[ \left( \frac{1}{2} - \frac{i \sqrt{3}}{6} \right) \chi_j(s) + \left( \frac{1}{2} + \frac{i \sqrt{3}}{6} \right) \chi_j(s^{-1}) \right]
\]

\[
+ \frac{1 - i \sqrt{3}}{2 \chi_j(1)} \sum_{s \in S} \left[ \left( \frac{1}{2} - \frac{i \sqrt{3}}{6} \right) \chi_j(s) + \left( \frac{1}{2} + \frac{i \sqrt{3}}{6} \right) \chi_j(s) \right]
\]

\[
= \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s).
\]

Therefore

\[
\frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s) = f_j(S) + g_j(S) + \omega (g_j(S) - g_k(S)).
\] (7)

Note that if \( \chi_k = \overline{\chi}_j \), then \( f_j(S) = f_k(S) \) and \( g_j(S) - g_k(S) = [f_j(S) + g_j(S)] - [f_k(S) + g_k(S)] \).

Therefore if \( f_j(S) + g_j(S) \) is an integer for each \( j \in \{1, \ldots, h\} \), then \( g_j(S) - g_k(S) \) is also an integer for each \( j \in \{1, \ldots, h\} \). Hence the normal mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is Eisenstein integral if and only if \( f_j(S) + g_j(S) \) is an integer for each \( j \in \{1, \ldots, h\} \).

**Lemma 5.1.** If \( \Gamma \) is a finite group, then the normal mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is Eisenstein integral if and only if \( 2f_j(S) \) and \( 2g_j(S) \) are integers of the same parity for each \( j \in \{1, \ldots, h\} \).

**Proof.** Assume that the normal mixed Cayley graph \( \text{Cay}(\Gamma, S) \) is Eisenstein integral. Then \( f_j(S) + g_j(S) \) and \( g_j(S) - g_k(S) \) are integers for each \( j \in \{1, \ldots, h\} \), where \( \chi_k = \overline{\chi}_j \). Note that

\[
g_j(S) - g_k(S) = \frac{1}{\chi_j(1)} \sum_{s \in S} \frac{-i \sqrt{3}}{3} (\chi_j(s) - \chi_j(s^{-1})).
\]

Therefore by Lemma 4.2,

\[
2g_j(S) = \frac{1}{\chi_j(1)} \sum_{s \in S} \chi_j(s) - \frac{1}{\chi_j(1)} \sum_{s \in S} \frac{i \sqrt{3}}{3} (\chi_j(s) - \chi_j(s^{-1}))
\]

15
we find that $2g_j(S)$ is an integer. Since $2f_j(S) = 2(f_j(S) + g_j(S)) - 2g_j(S)$, we see that $2f_j(S)$ is also an integer of the same parity with $2g_j(S)$.

Conversely, assume that $2f_j(S)$ and $2g_j(S)$ are integers of the same parity for each $j \in \{1, \ldots, h\}$. Then $f_j(S) + g_j(S)$ is an integer for each $j \in \{1, \ldots, h\}$. Hence the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is Eisenstein integral.

**Lemma 5.2.** The normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is Eisenstein integral if and only if $f_j(S)$ and $g_j(S)$ are integers for each $j \in \{1, \ldots, h\}$.

**Proof.** Let $j \in \{1, \ldots, h\}$. Due to Lemma 5.1, it is enough to prove that $2f_j(S)$ and $2g_j(S)$ are integers of the same parity if and only if $f_j(S)$ and $g_j(S)$ are integers. If $f_j(S)$ and $g_j(S)$ are integers, then clearly $2f_j(S)$ and $2g_j(S)$ are even integers. Conversely, assume that $2f_j(S)$ and $2g_j(S)$ are integers of the same parity. Since $f_j(S)$ is an algebraic integer, the integrality of $2f_j(S)$ implies that $f_j(S)$ is an integer. Thus $2f_j(S)$ is an even integer, and so by assumption $2g_j(S)$ is also an even integer. Hence $g_j(S)$ is an integer.

**Theorem 5.3.** Let $\Gamma$ be a finite group. If the normal mixed Cayley graph $\text{Cay}(\Gamma, S)$ is Eisenstein integral, then $\text{Cay}(\Gamma, S)$ is HS-integral.

**Proof.** Assume that $\text{Cay}(\Gamma, S)$ is Eisenstein integral. By Lemma 5.2, we find that $f_j(S)$ and $g_j(S)$ are integers for each $j \in \{1, \ldots, h\}$. Note that $f_j(S)$ is an eigenvalue of the normal simple Cayley graph $\text{Cay}(\Gamma, S \setminus S)$. By Theorem 2.7, $f_j(S)$ is an integer for each $j \in \{1, \ldots, h\}$ if and only if $S \setminus S \in \mathcal{B}(\Gamma)$. Further, $\frac{1}{\chi_j(1)} \sum_{s \in S} \frac{-i\sqrt{3}}{3} (\chi_j(s) - \chi_j(s^{-1})) = g_j(S) - g_k(S)$, and that $g_j(S) - g_k(S)$ is an integer for each $j \in \{1, \ldots, h\}$, where $\chi_k = \chi_j$. Using Lemma 4.1, we see that $S \in \mathcal{E}(\Gamma)$. Thus by Theorem 4.4, $\text{Cay}(\Gamma, S)$ is HS-integral.

**Lemma 5.4.** Let $x \in \Gamma$ and $\text{ord}(x) = 3'm$. If $m \not\equiv 0 \pmod{3}$, then the following assertions hold.

(i) If $t = 1$, then $[x] = x^m[x^3] \cup x^{2m}[x^3]$.

(ii) If $t = 1$, then

$$\langle [x] \rangle = \begin{cases} x^m[x^3] & \text{if } m \equiv 1 \pmod{3} \\ x^{2m}[x^3] & \text{if } m \equiv 2 \pmod{3}. \end{cases}$$

(iii) If $t \geq 2$, then

$$[x] = \begin{cases} x^m[x^3] \cup x^{2m}[x^3] \cup x^{4m} \langle x^{-3} \rangle \cup x^{5m} \langle x^{-3} \rangle & \text{if } m \equiv 1 \pmod{3} \\ x^m[x^3] \cup x^{2m}[x^3] \cup x^{4m} \langle x^3 \rangle \cup x^{5m} \langle x^3 \rangle & \text{if } m \equiv 2 \pmod{3}. \end{cases}$$
(iv) If \( t \geq 2 \), then
\[
[x] = \begin{cases}
  x^{7m}[x^3] \cup x^{8m}[x^3] \cup x^{4m}[x^3] & \text{if } m \equiv 1 \pmod{3} \\
  x^{7m}[x^3] \cup x^{8m}[x^3] \cup x^{4m}[x^3] & \text{if } m \equiv 2 \pmod{3}.
\end{cases}
\]

(v) If \( t \geq 2 \), then \( [x] = x^m[x^3] \cup x^{2m}[x^3] \cup x^{4m}[x^3] \cup x^{5m}[x^3] \cup x^{7m}[x^3] \cup x^{8m}[x^3] \).

(vi) If \( t \geq 2 \), then
\[
\langle x \rangle = \begin{cases}
  x^m[x^3] \cup x^{4m}[⟨ x^3 ⟩] & \text{if } m \equiv 1 \pmod{3} \\
  x^{2m}[x^3] \cup x^{5m}[⟨ x^3 ⟩] & \text{if } m \equiv 2 \pmod{3}.
\end{cases}
\]

(vii) If \( t \geq 2 \), then
\[
\langle x \rangle = \begin{cases}
  x^{7m}[x^3] \cup x^{4m}[⟨ x^3 ⟩] & \text{if } m \equiv 1 \pmod{3} \\
  x^{8m}[x^3] \cup x^{5m}[⟨ x^3 ⟩] & \text{if } m \equiv 2 \pmod{3}.
\end{cases}
\]

(viii) If \( t \geq 2 \), then
\[
\langle x \rangle = \begin{cases}
  x^m[x^3] \cup x^{4m}[x^3] \cup x^{7m}[x^3] & \text{if } m \equiv 1 \pmod{3} \\
  x^{2m}[x^3] \cup x^{5m}[x^3] \cup x^{8m}[x^3] & \text{if } m \equiv 2 \pmod{3}.
\end{cases}
\]

Proof. (i) Assume that \( \text{ord}(x) = 3m \) and \( m \not\equiv 0 \pmod{3} \). Let us take \( x^{m+3r} \in x^m[x^3] \) for some \( r \in G_m(1) \). Then \( \gcd(r, m) = 1 \), and so \( \gcd(m + 3r, 3m) = 1 \). Therefore \( x^m[x^3] \subseteq [x] \). Similarly, we have \( x^{2m}[x^3] \subseteq [x] \). Therefore \( x^m[x^3] \cup x^{2m}[x^3] \subseteq [x] \). Note that \( |[x]| = \varphi(3m) = 2\varphi(m) \), \( |x^m[x^3]| = \varphi(m) = |x^{2m}[x^3]| \), and that \( x^m[x^3] \cup x^{2m}[x^3] \) is a disjoint union. Thus, the sizes of \([x]\) and \(x^m[x^3] \cup x^{2m}[x^3]\) are equal, and therefore \([x] = x^m[x^3] \cup x^{2m}[x^3]\).

(ii) Assume that \( \text{ord}(x) = 3m \) and \( m \not\equiv 0 \pmod{3} \). Let \( m \equiv 1 \pmod{3} \). We see that \( \gcd(r, m) = 1 \) if and only if \( \gcd(m + 3r, 3m) = 1 \). Also \( m + 3r \equiv 1 \pmod{3} \). Therefore \( x^m[x^3] = \{x^{m+3r} : r \in G_m(1)\} \subseteq \{x^k : k \in G^1_{3m,3}(1)\} = \langle x \rangle \).

Since the sets \( x^m[x^3] \) and \( \langle x \rangle \) are of equal size, we get \( x^m[x^3] = \langle x \rangle \). Similarly, if \( m \equiv 2 \pmod{3} \), we have \( x^{2m}[x^3] = \langle x \rangle \).

(iii) Assume that \( p = 3^t m \), \( t \geq 2 \) and \( m \equiv 1 \pmod{3} \). Let \( x^{m+3r} \in x^m[x^3] \) for some \( r \in G_{2^t}(1) \). Then \( \gcd(r, \frac{3}{2}) = 1 \), and so \( \gcd(m + 3r, p) = 1 \). Thus \( x^m[x^3] \subseteq [x] \). Similarly, \( x^{2m}[x^3] \subseteq [x] \). Now let \( x^{4m+3r} \in x^{4m}[⟨ x^3 ⟩] \) for some \( r \in G_{2^{t+1}}(1) \). Again, \( \gcd(r, \frac{3}{2}) = 1 \) implies that \( \gcd(4m + 3r, p) = 1 \). Therefore \( x^{4m}[⟨ x^3 ⟩] \subseteq [x] \). Similarly, \( x^m[⟨ x^3 ⟩] \subseteq [x] \). Thus \( x^m[x^3] \cup x^{2m}[x^3] \cup x^{4m}[⟨ x^3 ⟩] \cup x^{5m}[⟨ x^3 ⟩] \subseteq [x] \). Note that \( |[x]| = 2 \times 3^{t-1} \varphi(m) \). Also, \( |x^m[x^3]| = 2 \times 3^{t-2} \varphi(m) = |x^{2m}[x^3]| \), \( |x^{4m}[⟨ x^3 ⟩]| = 3^{t-2} \varphi(m) = |x^{5m}[⟨ x^3 ⟩]| \), and that \( x^m[x^3] \cup x^{2m}[x^3] \cup x^{4m}[⟨ x^3 ⟩] \cup x^{5m}[⟨ x^3 ⟩] \) is a disjoint union. Thus, the sizes of \([x]\) and \(x^m[x^3] \cup x^{2m}[x^3] \cup x^{4m}[⟨ x^3 ⟩] \cup x^{5m}[⟨ x^3 ⟩] \) are equal, and hence these two sets are equal. For \( m \equiv 2 \pmod{3} \), the proof follows the similar steps as in the case of \( m \equiv 1 \pmod{3} \).
Lemma 5.6. The proof is similar to the proof Part (iii). For the sake of completeness, we provide the proof for the case \( m \equiv 1 \pmod{3} \). Assume that \( p = 3^t m, t \geq 2 \) and \( m \equiv 1 \pmod{3} \). Let \( x^{7m+3r} \in x^{7m}[x^3] \) for some \( r \in G_{p,3}(1) \). Then \( \gcd(r, \frac{7}{3}) = 1 \), and so \( \gcd(7m + 3r, p) = 1 \). Thus \( x^{7m}[x^3] \subseteq [x] \). Similarly, \( x^{8m}[x^3] \subseteq [x] \). Now let \( x^{4m+3r} \in x^{4m}[x^3] \) for some \( r \in G_{p,3}(1) \). Again, \( \gcd(r, \frac{4}{3}) = 1 \) gives \( \gcd(4m + 3r, p) = 1 \). Thus, \( x^{4m}[x^3] \subseteq [x] \). Similarly, \( x^{5m}[x^3] \subseteq [x] \). Thus \( x^{7m}[x^3] \cup x^{8m}[x^3] \cup x^{4m}[x^3] \cup x^{5m}[x^3] \subseteq [x] \). Note that \( x^{7m}[x^3] \cup x^{8m}[x^3] \cup x^{4m}[x^3] \cup x^{5m}[x^3] \) is a disjoint union, and so its size is equal to \( 2 \times 3^{t-2} \varphi(m) + 2 \times 3^{t-2} \varphi(m) + 3^{t-2} \varphi(m) + 3^{t-2} \varphi(m) \), which is equal to the size \( 2 \times 3^{t-1} \varphi(m) \) of \([x]\). Hence we have the desired equality.

(v) Combine Part (iii) and Part (iv), and use \([x^3] = \langle x^3 \rangle \cup \langle x^{-3} \rangle\) to get the proof of this part.

(vi) Assume that \( p = 3^t m, t \geq 2 \) and \( m \equiv 1 \pmod{3} \). We see that if \( r \in G_{p,3}(1) \), then \( m + 3r \in G_{p,3}(1) \).

Similarly, if \( r \in G_{p,3}(1) \), then \( 4m + 3r \in G_{p,3}(1) \). Thus we have \( x^{3m}[x^3] \cup x^{4m}[x^3] \subseteq \langle x \rangle \).

Since the sizes of \( x^{m}[x^3] \cup x^{4m}[x^3] \) and \( \langle x \rangle \) are equal, we find that \( x^{m}[x^3] \cup x^{4m}[x^3] = \langle x \rangle \).

Similarly, we have \( x^{2m}[x^3] \cup x^{5m}[x^3] = \langle x \rangle \) for \( m \equiv 2 \pmod{3} \).

(vii) The proof of this part follows similar steps as in Part (vi). For the sake of completeness, we provide the proof for the case \( m \equiv 2 \pmod{3} \). Assume that \( p = 3^t m, t \geq 2 \) and \( m \equiv 2 \pmod{3} \). We see that if \( r \in G_{p,3}(1) \), then \( 8m + 3r \in G_{p,3}(1) \). Also, if \( r \in G_{p,3}(1) \), then \( 5m + 3r \in G_{p,3}(1) \). Thus \( x^{3m}[x^3] \cup x^{5m}[x^3] \subseteq \langle x \rangle \). Since the sizes of \( x^{sm}[x^3] \cup x^{5m}[x^3] \) and \( \langle x \rangle \) are equal, we find that \( x^{sm}[x^3] \cup x^{5m}[x^3] = \langle x \rangle \).

Combine Part (vi) and Part (vii), and use \([x^3] = \langle x^3 \rangle \cup \langle x^{-3} \rangle\) to get the proof of this part.

For \( x \in \Gamma \), define \( \mathcal{S}_x^1 := \bigcup_{s \in \text{Cl}(x)} \{ s \} \). We see that if \( m = \text{ord}(x) \), then

\[ \mathcal{S}_x^1 = \{ g^{-1}x^r g : g \in \Gamma, r \in G_m(1) \} = \bigcup_{s \in [x]} \text{Cl}(s). \]

The set \( \mathcal{S}_x^1 \) is also known as the rational conjugacy class of \( x \). See [8] for details. For each \( y \in \mathcal{S}_x^1 \), it is clear that \( \text{Cl}(y), [y] \subseteq \mathcal{S}_x^1 \). Now let \( A \) be a symmetric subset of \( \Gamma \) such that \( x \in A \), and \( \text{Cl}(a), [a] \subseteq A \) for each \( a \in A \). Let \( g^{-1}x^r g \in \mathcal{S}_x^1 \), where \( g \in \Gamma \), \( r \in G_m(1) \) and \( m = \text{ord}(x) \). As \( [x] \subseteq A \), we have \( x^r \in A \). Now \( \text{Cl}(x^r) \subseteq A \), and so \( g^{-1}x^r g \in A \). Thus \( \mathcal{S}_x^1 \subseteq A \), and therefore \( \mathcal{S}_x^1 \) is the smallest symmetric subset of \( \Gamma \) containing \( x \) that is closed under both conjugacy and the equivalence relation \( \sim \). Considering each of the repeated equivalence classes, if any, only once in \( \bigcup_{s \in \text{Cl}(x)} \{ s \} \), we can write \( \mathcal{S}_x^1 = \bigcup_{i=1}^{t} [x_i] \), where the equivalence classes \([x_1], \ldots, [x_t]\) are distinct. We state this fact in the next lemma.

**Lemma 5.5.** If \( x \in \Gamma \), then there exist distinct equivalence classes \([x_1], \ldots, [x_t]\) such that \( \mathcal{S}_x^1 = \bigcup_{i=1}^{t} [x_i] \), where \( x_1, \ldots, x_t \in \text{Cl}(x) \).

**Lemma 5.6.** If \( y \in \mathcal{S}_x^1 \), then \( \mathcal{S}_y^1 = \mathcal{S}_x^1 \).

18
Proof. Let $y \in S_1^1$, so that $y = g^{-1}x'y$ for some $g \in \Gamma$ and $r \in G_m(1)$, where $m = \text{ord}(x)$. We see that $\text{ord}(y) = \text{ord}(x) = m$. Now let $z \in S_1^1$. Then $z = h^{-1}y'h$ for some $h \in \Gamma$ and $t \in G_m(1)$. This gives $z = h^{-1}y'h = h^{-1}g^{-1}x'r'th \in S_1^1$. Conversely, let $w \in S_1^1$ so that $w = h^{-1}x'th$ for some $h \in \Gamma$ and $t \in G_m(1)$. Therefore

$$w = h^{-1}x'th = (h^{-1}g)^{-1}(x'r't)^{-1}(g^{-1}h) = (h^{-1}g)y'^{-1}(g^{-1}h) \in S_1^1.$$ 

Here $r^{-1}$ is the multiplicative inverse of $r$ in the group $G_m(1)$. Hence we conclude that $S_1^1 = S_2^1$. \qed

Due to Lemma 5.6, the sets $S_1^1$ and $S_2^1$ are either disjoint or equal. Hence the class of distinct subsets of $\Gamma$ of the form $S_1^1$ is a partition of $\Gamma$.

Let $x \in \Gamma(3)$ be an element of order $m$. The element $x$ is said to be tolerable if $x^r \not\in \text{Cl}(x)$ for all $r \in G_{m,3}(1)$. The following lemma characterizes tolerable elements in terms of skew-symmetric sets.

**Lemma 5.7.** If $x \in \Gamma(3)$, then $x$ is tolerable if and only if the set $\bigcup_{s \in \text{Cl}(x)} \langle s \rangle$ is skew-symmetric.

**Proof.** We see that if $m = \text{ord}(x)$, then

$$\bigcup_{s \in \text{Cl}(x)} \langle s \rangle = \{g^{-1}x^r g : g \in \Gamma, r \in G_{m,3}(1)\} = \bigcup_{s \in \langle x \rangle} \text{Cl}(s).$$

Assume that $x$ is not tolerable, so that $x^r \in \text{Cl}(x)$ for some $r \in G_{m,3}(1)$. As $m - r \in G_{m,3}(1)$ and $\text{Cl}(x) \subseteq \bigcup_{s \in \text{Cl}(x)} \langle s \rangle$, we find that $x^r, x^{m-r} \in \bigcup_{s \in \text{Cl}(x)} \langle s \rangle$. Hence $\bigcup_{s \in \langle x \rangle} \langle s \rangle$ is not skew-symmetric.

On the other hand, assume that $\bigcup_{s \in \text{Cl}(x)} \langle s \rangle$ is not a skew-symmetric set. Then there is an $y = g^{-1}x'y \in \bigcup_{s \in \text{Cl}(x)} \langle s \rangle$ for some $r \in G_{m,3}(1)$ such that $y^{-1} \in \bigcup_{s \in \text{Cl}(x)} \langle s \rangle$. Therefore we have $g^{-1}x^{-r}g = y^{-1} = h^{-1}x^kh$ for some $h \in \Gamma, k \in G_{m,3}(1)$. Let $t \in G_m(1)$ be the multiplicative inverse of $m - r$. We have $g^{-1}x^{(m-r)t}g = h^{-1}x^kth$, and it gives $x^{kt} = hg^{-1}xgh^{-1} \in \text{Cl}(x)$. Since $(m - r)t \equiv 1 \mod 3$ and $m - r \in G_{m,3}(1)$, we have that $t \in G_{m,3}(1)$. Thus $kt \in G_{m,3}(1)$ with $x^{kt} \in \text{Cl}(x)$, giving that $x$ is not tolerable. \qed

Let $x \in \Gamma(3)$ be tolerable, and define $S_x^3 := \bigcup_{s \in \text{Cl}(x)} \langle s \rangle$. The structure and properties of the set $S_x^3$ are similar to those of $S_1^1$ and $S_1^1$. If $\Gamma$ is abelian, then $S_x^3 = \langle x \rangle$ for each $x \in \Gamma(3)$. For each $y \in S_2^3$, it is clear that $\text{Cl}(y), \langle y \rangle \subseteq S_x^3$. Now let $A$ be a skew-symmetric subset of $\Gamma$ containing a tolerable element $x$, and $\text{Cl}(a), \langle a \rangle \subseteq A$ for each $a \in A$. It is easy to see that $S_x^3 \subseteq A$. Thus, $S_x^3$ is the smallest skew-symmetric subset of $\Gamma$ containing $x$ that is closed under both conjugacy and the equivalence relation $\simeq$. Considering each of the repeated equivalence classes, if any, only once in $\bigcup_{s \in \text{Cl}(x)} \langle s \rangle$, we can write $S_x^3 = \bigcup_{i=1}^k \langle y_i \rangle$, where the equivalence classes $\langle y_1 \rangle, \ldots, \langle y_r \rangle$ are distinct. We state this fact in the next lemma.

**Lemma 5.8.** If $x$ is a tolerable element in $\Gamma(3)$, then there are distinct equivalence classes $\langle x_1 \rangle, \ldots, \langle x_r \rangle$ such that $S_x^3 = \bigcup_{i=1}^r \langle x_i \rangle$, where $x_1, \ldots, x_r \in \text{Cl}(x)$. 

19
Lemma 5.9. If \( y \in S^3_x \), then \( S^3_y = S^3_x \).

Proof. Let \( y \in S^3_x \), so that \( y = g^{-1}x^rg \) for some \( g \in \Gamma \) and \( r \in G^1_{m,3}(1) \), where \( m = \text{ord}(x) \). We see that \( \text{ord}(y) = \text{ord}(x) = m \). Now let \( z \in S^3_y \). Then \( z = h^{-1}y^rh \) for some \( h \in \Gamma \) and \( t \in G^1_{m,3}(1) \). This gives \( z = h^{-1}y^rh = h^{-1}g^{-1}x^rgh \in S^3_x \). Conversely, let \( w \in S^3_x \) so that \( w = h^{-1}x^r \) for some \( h \in \Gamma \) and \( t \in G^1_{m,3}(1) \). Therefore

\[
 w = h^{-1}x^r = (h^{-1}g)^{-1}(x^r)^{-1}g(g^{-1}h) = (h^{-1}g)y^{-1}t(g^{-1}h) \in S^3_y.
\]

Here \( r^{-1} \) is the multiplicative inverse of \( r \) in the subgroup \( G^1_{m,3}(1) \). Thus we conclude that \( S^3_y = S^3_x \). □

Due to Lemma 5.9, the sets \( S^3_x \) and \( S^3_y \) are either disjoint or equal.

Lemma 5.10. Let \( x \in \Gamma(3) \). If \( S^1_x = [x_1] \cup \cdots \cup [x_k] \) for some \( x_1, \ldots, x_k \in \text{Cl}(x) \), then \( S^1_{x^3} = [x^3_1] \cup \cdots \cup [x^3_k] \).

Proof. Let \( m = \text{ord}(x) \) and \( S^1_{x^3} = [x_1] \cup \cdots \cup [x_k] \) for some \( x_1, \ldots, x_k \in \text{Cl}(x) \). Assume that the sets \( [x_1], \ldots, [x_k] \) are all distinct. We see that

\[
 S^1_{x^3} = \{ g^{-1}x^3g : g \in \Gamma, r \in G^1_{x^3}(1) \}
\]

\[
 = \{ g^{-1}x^3g : g \in \Gamma, r \in G^1_{x^3}(1) \} \cup \{ g^{-1}x^{3(\text{ord}+r)}g : g \in \Gamma, r \in G^1_{x^3}(1) \}
\]

\[
 \cup \{ g^{-1}x^{3(\text{ord}+r)}g : g \in \Gamma, r \in G^1_{x^3}(1) \}
\]

\[
 = \{ g^{-1}x^3g : g \in \Gamma, r \in G^1_{m,3}(1), r < \frac{m}{3} \} \cup \{ g^{-1}x^3g : g \in \Gamma, t \in G^1_{m,3}(1), \frac{m}{3} < t < \frac{2m}{3} \}
\]

\[
 \cup \{ g^{-1}x^3g : g \in \Gamma, t \in G^1_{m,3}(1), \frac{2m}{3} < t \}
\]

\[
 = \{ g^{-1}x^3g : g \in \Gamma, r \in G^1_{m,3}(1) \}
\]

\[
 = \{ y^3 : y \in S^1_x \}.
\]

Now noting that \( \{ s^3 : s \in [x] \} = [x^3] \) and \( S^1_x = [x_1] \cup \cdots \cup [x_k] \), we have \( S^1_{x^3} = [x^3_1] \cup \cdots \cup [x^3_k] \). □

Lemma 5.11. If \( x \in \Gamma(3) \) is tolerable, then \( S^1_x \cup S^3_{x^{-1}} = S^1_x \).

Proof. Let \( m = \text{ord}(x) \). We have

\[
 S^1_x \cup S^3_{x^{-1}} = \{ g^{-1}x^rg : g \in \Gamma, r \in G^1_{m,3}(1) \} \cup \{ g^{-1}x^rg : g \in \Gamma, r \in G^1_{m,3}(1) \}
\]

\[
 = \{ g^{-1}x^rg : g \in \Gamma, r \in G^1_{m,3}(1) \} \cup \{ g^{-1}x^rg : g \in \Gamma, r \in G^2_{m,3}(1) \}
\]

\[
 = \{ g^{-1}x^rg : g \in \Gamma, r \in G^1_{m,3}(1) \}
\]

= \( S^1_x \). □

Lemma 5.12. Let \( x \in \Gamma(3) \) be a tolerable element. If \( S^1_x = \langle x_1 \rangle \cup \cdots \cup \langle x_k \rangle \) for some \( x_1, \ldots, x_k \in \text{Cl}(x) \), then \( S^1_{x^3} = [x^3_1] \cup \cdots \cup [x^3_k] \).

20
Proof. Assume that $S^3_x = \langle x_1 \rangle \cup \cdots \cup \langle x_k \rangle$ for some $x_1, \ldots, x_k \in \text{Cl}(x)$. Then we have $S^3_{x-1} = \langle x_1^{-1} \rangle \cup \cdots \cup \langle x_k^{-1} \rangle$. Therefore

$$S^1_x = S^3_x \cup S^3_{x-1} = (\langle x_1 \rangle \cup \langle x_1^{-1} \rangle) \cup \cdots \cup (\langle x_1 \rangle \cup \langle x_k^{-1} \rangle) = [x_1] \cup \cdots \cup [x_k].$$

Now the result follows from Lemma 5.10. \hfill \square

For $x \in \Gamma$ and $j \in \{1, \ldots, h\}$, define

$$C_x(j) := \frac{1}{\chi_j(1)} \sum_{s \in S^1_x} \chi_j(s).$$

Note that $S^1_x \in B(\Gamma)$ and $C_x(j)$ is an eigenvalue of the normal undirected Cayley graph Cay($\Gamma, S^1_x$). As a consequence of Theorem 2.7, $C_x(j)$ is an integer for each $x \in \Gamma$ and $j \in \{1, \ldots, h\}$.

**Lemma 5.13.** Let $x \in \Gamma$ and ord$(x) = 3^m 7^t$. If $m \not\equiv 0 \pmod{3}$ and $t \geq 2$, then

$$2C_x(j) = \left( \sum_{s \in G_0(1)} \chi_j(x^{3m}) \right) C_x^3(j).$$

Moreover, $\frac{C_x(j)}{3}$ is an integer for each $j \in \{1, \ldots, h\}$.

**Proof.** Let $S^1_x = [x_1] \cup \cdots \cup [x_k]$ for some $x_1, \ldots, x_k \in \text{Cl}(x)$ and $j \in \{1, \ldots, h\}$. We use the fact that each $[x_i]$ can be written as disjoint unions in two different ways using Part (iii) and Part (iv) of Lemma 5.4. For $m \equiv 1 \pmod{3}$, using Part (iii) and Part (iv) of Lemma 5.4, we have

$$2 \sum_{s \in [x_i]} \chi_j(s) = \sum_{s \in [x_i]} \chi_j(s) + \sum_{s \in [x_i]} \chi_j(s)$$

$$= \sum_{s \in x_i^m [x_i^3]} \chi_j(s) + \sum_{s \in x_i^{2m} [x_i^3]} \chi_j(s) + \sum_{s \in x_i^{4m} \langle x_i^{-3} \rangle} \chi_j(s) + \sum_{s \in x_i^{7m} \langle x_i^{-3} \rangle} \chi_j(s)$$

$$+ \sum_{s \in x_i^{2m} [x_i^3]} \chi_j(s) + \sum_{s \in x_i^{4m} \langle x_i^{-3} \rangle} \chi_j(s) + \sum_{s \in x_i^{7m} \langle x_i^{-3} \rangle} \chi_j(s)$$

$$= \sum_{s \in [x_i^3]} \chi_j(x_i^m) \chi_j(s) + \sum_{s \in [x_i^3]} \chi_j(x_i^{2m}) \chi_j(s) + \sum_{s \in [x_i^3]} \chi_j(x_i^{4m}) \chi_j(s)$$

$$+ \sum_{s \in [x_i^3]} \chi_j(x_i^{5m}) \chi_j(s) + \sum_{s \in [x_i^3]} \chi_j(x_i^{7m}) \chi_j(s) + \sum_{s \in [x_i^3]} \chi_j(x_i^{8m}) \chi_j(s) \tag{8}$$

for each $i \in \{1, \ldots, k\}$. Similarly, for $m \equiv 2 \pmod{3}$, using Part (iii) and Part (iv) of Lemma 5.4, we have

$$2 \sum_{s \in [x_i]} \chi_j(s) = \sum_{s \in [x_i^3]} \chi_j(x_i^m) \chi_j(s) + \sum_{s \in [x_i^7]} \chi_j(x_i^{2m}) \chi_j(s) + \sum_{s \in [x_i^7]} \chi_j(x_i^{4m}) \chi_j(s)$$

$$+ \sum_{s \in [x_i^7]} \chi_j(x_i^{5m}) \chi_j(s) + \sum_{s \in [x_i^7]} \chi_j(x_i^{7m}) \chi_j(s) + \sum_{s \in [x_i^7]} \chi_j(x_i^{8m}) \chi_j(s) \tag{9}$$

21
for each $i \in \{1, \ldots, k\}$. Thus using Equations (8) and (9), we get

$$2C_x(j) = \frac{1}{\chi_j(1)} \sum_{i=1}^{k} \left( \sum_{s \in [x_i]} \chi_j(x_i^{s_1}) \chi_j(s) + \sum_{s \in [x_i]} \chi_j(x_i^{s_2}) \chi_j(s) + \sum_{s \in [x_i]} \chi_j(x_i^{s_3}) \chi_j(s) + \sum_{s \in [x_i]} \chi_j(x_i^{s_4}) \chi_j(s) \right)$$

$$+ \sum_{s \in [x_i]} \chi_j(x_i^{s_5}) \chi_j(s) + \sum_{s \in [x_i]} \chi_j(x_i^{s_6}) \chi_j(s) + \sum_{s \in [x_i]} \chi_j(x_i^{s_7}) \chi_j(s)$$

$$= \left( \sum_{r \in G_0(1)} \chi_j(x^r) \right) C_{x^3}(j).$$

(10)

Here the third equality in Equation (10) follows from the fact that $x_1, \ldots, x_k \in \text{Cl}(x)$, and the fourth equality in Equation (10) follows from Lemma 5.10.

Let $d_j = \chi_j(1)$. We apply induction on $t$ to prove that $\frac{C_x(j)}{d_j}$ is an integer. Let $t = 2$, so that ord$(x) = 9m$ with $m \not\equiv 0 \pmod{3}$. By Theorem 2.1, we have $\chi_j(x^m) = \sum_{\ell=1}^{d_j} \epsilon_{j,\ell}$, where $\epsilon_{j,1}, \ldots, \epsilon_{j,d_j}$ are some 9-th roots of unity. We have

$$\sum_{r \in G_0(1)} \chi_j(x^r) = \sum_{r \in G_0(1)} \sum_{\ell=1}^{d_j} \epsilon_{j,\ell} = \sum_{\ell=1}^{d_j} \sum_{r \in G_0(1)} \epsilon_{j,\ell}.$$

(11)

Note that $\sum_{r \in G_0(1)} \epsilon_{j,\ell} = (\epsilon_{j,\ell} + \epsilon_{j,\ell}^2)(1 + \epsilon_{j,\ell}^3 + \epsilon_{j,\ell}^6)$. Since $\epsilon_{j,\ell} \in \{1, \omega_9, \omega_9^2, \ldots, \omega_9^8\}$, we have

$$\sum_{r \in G_0(1)} \epsilon_{j,\ell} = \begin{cases} 6 & \text{if } \epsilon_{j,\ell} = 1 \\ -3 & \text{if } \epsilon_{j,\ell} \in \{\omega_9^3, \omega_9^6\} \\ 0 & \text{otherwise}. \end{cases}$$

Thus, $\sum_{r \in G_0(1)} \epsilon_{j,\ell}$ is an integer multiple of 3 for each $\ell \in \{1, \ldots, d_j\}$. Therefore by Equation (11), $\sum_{r \in G_0(1)} \chi_j(x^r)$ is an integer multiple of 3. Now Equation (10) gives that $\frac{2C_x(j)}{d_j}$ is an integer. Since $C_x(j)$ is an integer, integrality of $\frac{2C_x(j)}{d_j}$ gives that $\frac{C_x(j)}{d_j}$ is also an integer.

Assume that $\frac{C_x(j)}{d_j}$ is an integer for each $j \in \{1, \ldots, k\}$ whenever ord$(y) = 3^{t-1}m$ with $m \not\equiv 0 \pmod{3}$ and $t \geq 3$. Let ord$(x) = 3^t m$ with $m \not\equiv 0 \pmod{3}$ and $t \geq 3$. Note that ord$(x^3) = 3^{t-1}m$. Therefore by induction hypothesis, $\frac{C_x(j)}{d_j}$ is an integer. By Equation (10), $\sum_{s \in G_0(1)} \chi_j(x^{sm})$ is a rational algebraic integer whenever $C_{x^3}(j) \neq 0$. Thus, if $C_{x^3}(j) \neq 0$ then $\sum_{s \in G_0(1)} \chi_j(x^{sm})$ is an integer. Therefore by Equation (10), $\frac{2C_x(j)}{3}$ is an integer, and accordingly $\frac{C_x(j)}{3}$ is an integer. Hence the proof is complete by induction. \hfill \square
Let $x \in \Gamma(3)$ be tolerable. For each $j \in \{1, \ldots, h\}$, define
\[
T_x(j) := \frac{1}{\chi_j(1)} \sum_{s \in S^3_x} i\sqrt{3}(\chi_j(s) - \chi_j(s^{-1})).
\]

Let $j \in \{1, \ldots, h\}$. Using $S^1_x = S^3_x \cup S^3_{x^{-1}}$, we see that
\[
\frac{C_x(j) + T_x(j)}{2} = \frac{1}{2\chi_j(1)} \left[ \sum_{s \in S^3_x} \chi_j(s) + \sum_{s \in S^3_{x^{-1}}} i\sqrt{3}(\chi_j(s) - \chi_j(s^{-1})) \right]
\]
\[
= \frac{1}{2\chi_j(1)} \left[ \sum_{s \in S^3_x} \chi_j(s) + \sum_{s \in S^3_{x^{-1}}} \chi_j(s) + \sum_{s \in S^3_{x^{-1}}} i\sqrt{3}(\chi_j(s) - \chi_j(s^{-1})) \right]
\]
\[
= \frac{1}{\chi_j(1)} \left[ \sum_{s \in S^3_x} (\omega_6 \chi_j(s) + \omega_6^5 \chi_j(s^{-1})) \right].
\]

Thus $\frac{C_x(j) + T_x(j)}{2}$ is an HS-eigenvalue of the normal oriented Cayley graph $\text{Cay}(\Gamma, S^3_x)$. Therefore by Theorem 3.2, $\frac{C_x(j) + T_x(j)}{2}$ is an integer. Since $C_x(j)$ is an integer (by Theorem 2.7), $T_x(j)$ is also an integer for each $j \in \{1, \ldots, h\}$.

**Lemma 5.14.** Let $x \in \Gamma(3)$ be tolerable and $\text{ord}(x) = 3m$. If $m \not\equiv 0 \pmod{3}$, then
\[
T_x(j) = \begin{cases} 
-2\sqrt{3}(\chi_j(x^m))C_{x^3}(j) & \text{if } m \equiv 1 \pmod{3} \\
-2\sqrt{3}(\chi_j(x^{2m}))C_{x^3}(j) & \text{if } m \equiv 2 \pmod{3}.
\end{cases}
\]

Moreover, $\frac{T_x(j)}{3}$ is an integer for each $j \in \{1, \ldots, h\}$. 

23
Moreover, \( \langle \langle \rangle \rangle \) for some \( x_1, \ldots, x_k \in \text{Cl}(x) \) and \( j \in \{1, \ldots, h\} \). We get

\[
T_x(j) = \frac{1}{\chi_j(1)} \sum_{i=1}^k \sum_{s \in \langle \langle \rangle \rangle} i\sqrt{3}(\chi_j(s) - \chi_j(s^{-1}))
\]

\[
= \frac{1}{\chi_j(1)} \sum_{i=1}^k \sum_{s \in \langle \langle \rangle \rangle} i\sqrt{3} \chi_j(x_i s) - \chi_j(x_i^{-m} s) \chi_j(s^{-1}) \quad \text{if } m \equiv 1 \pmod{3}
\]

\[
= \frac{1}{\chi_j(1)} \sum_{i=1}^k \sum_{s \in \langle \langle \rangle \rangle} i\sqrt{3} \chi_j(x_i 2m) - \chi_j(x_i^{-2m} s) \chi_j(s^{-1}) \quad \text{if } m \equiv 2 \pmod{3}
\]

\[
\leq \frac{1}{\chi_j(1)} \sum_{i=1}^k \chi_j(x_i s) - \chi_j(x_i m) \sum_{s \in \langle \langle \rangle \rangle} \chi_j(s) \quad \text{if } m \equiv 1 \pmod{3}
\]

\[
= \frac{1}{\chi_j(1)} \sum_{i=1}^k \chi_j(x_i 2m) \sum_{s \in \langle \langle \rangle \rangle} \chi_j(s) \quad \text{if } m \equiv 2 \pmod{3}
\]

\[
\leq -2\sqrt{3}(\chi_j(x_i s)^{-1}) \sum_{i=1}^k \sum_{s \in \langle \langle \rangle \rangle} \chi_j(s) \quad \text{if } m \equiv 1 \pmod{3}
\]

\[
= -2\sqrt{3}(\chi_j(x_i 2m)) \sum_{i=1}^k \sum_{s \in \langle \langle \rangle \rangle} \chi_j(s) \quad \text{if } m \equiv 2 \pmod{3}
\]

Here the second equality follows from Part (ii) of Lemma 5.4, and the fourth equality follows from Lemma 5.12. Let \( d_j = \chi_j(1) \). By Theorem 2.1, we have \( \chi_j(x_i m) = \sum_{\ell=1}^{d_j} \epsilon_{j\ell} \), where \( \epsilon_{j1}, \ldots, \epsilon_{jd_j} \) are cube roots of unity. Therefore, \( 2\sqrt{3}(\chi_j(x_i m)) \) is an integer multiple of 3. Similarly, \( 2\sqrt{3}(\chi_j(x_i 2m)) \) is also an integer multiple of 3. Hence \( \frac{T_x(j)}{3} \) is an integer for each \( j \in \{1, \ldots, h\} \).

\[\boxed{\text{Lemma 5.15.} \quad \text{Let } x \in \Gamma \text{ be tolerable and } \text{ord}(x) = 3^t m. \text{ If } m \not\equiv 0 \pmod{3} \text{ and } t \geq 2, \text{ then}}\]

\[
2T_x(j) = \begin{cases} 
-2\sqrt{3} \left( \sum_{s \in G_{h,3}^2(1)} \Im(\chi_j(x_i sm)) \right) C_{x^3}(j) & \text{if } m \equiv 1 \pmod{3} \\
-2\sqrt{3} \left( \sum_{s \in G_{h,3}^2(1)} \Im(\chi_j(x_i sm)) \right) C_{x^3}(j) & \text{if } m \equiv 2 \pmod{3}
\end{cases}
\]

Moreover, \( \frac{T_x(j)}{3} \) is an integer for each \( j \in \{1, \ldots, h\} \).

\[\text{Proof.} \quad \text{Let } S_x^3 = \langle \langle \rangle \rangle \cup \cdots \cup \langle \langle \rangle \rangle \text{ for some } x_1, \ldots, x_k \in \text{Cl}(x) \text{ and } j \in \{1, \ldots, h\}. \text{ We use the fact that each } \langle \langle \rangle \rangle \text{ can be written as disjoint unions in two different ways using Part (vi) and Part (vii) of}\]

\[\text{24}\]
Lemma 5.4. For \( m \equiv 1 \) (mod 3), using Part (vi) and Part (vii) of Lemma 5.4, we have

\[
2 \sum_{s \in \langle s \rangle} i \sqrt{3}(\chi_j(s) - \chi_j(s^{-1})) = \sum_{s \in \langle s \rangle} i \sqrt{3}(\chi_j(s) - \chi_j(s^{-1})) + \sum_{s \in \langle s \rangle} i \sqrt{3}(\chi_j(s) - \chi_j(s^{-1}))
\]

\[
= \sum_{s \in \langle s \rangle} i \sqrt{3}(\chi_j(s) - \chi_j(s^{-1})) + \sum_{s \in \langle s \rangle} i \sqrt{3}(\chi_j(s) - \chi_j(s^{-1}))
\]

\[
= \sum_{s \in \langle s \rangle} i \sqrt{3}(\chi_j(s) - \chi_j(s^{-1})) + \sum_{s \in \langle s \rangle} i \sqrt{3}(\chi_j(s) - \chi_j(s^{-1}))
\]

\[
= \sum_{s \in \langle s \rangle} i \sqrt{3}(\chi_j(s) - \chi_j(s^{-1}))
\]

\[
= -2\sqrt{3}(\chi_j(x_1^m)) \sum_{s \in \langle s \rangle} \chi_j(s) - 2\sqrt{3}(\chi_j(x_1^m)) \sum_{s \in \langle s \rangle} \chi_j(s)
\]

\[
= -2\sqrt{3}(\chi_j(x_1^m)) \sum_{s \in \langle s \rangle} \chi_j(s)
\]

\[
= -2\sqrt{3}(\chi_j(x_1^m)) \sum_{s \in \langle s \rangle} \chi_j(s)
\]

(12)

for each \( i \in \{1, \ldots, k\} \). Similarly, for \( m \equiv 2 \) (mod 3) we have

\[
2 \sum_{s \in \langle s \rangle} i \sqrt{3}(\chi_j(s) - \chi_j(s^{-1})) = -2\sqrt{3}(\chi_j(x_1^m)) \sum_{s \in \langle s \rangle} \chi_j(s)
\]

(13)

for each \( i \in \{1, \ldots, k\} \). Using Equation (12) and Equation (13), we get

\[
2T_x(j) = \frac{1}{\chi_j(1)} \sum_{i=1}^{k} \sum_{s \in \langle s \rangle} i \sqrt{3}(\chi_j(s) - \chi_j(s^{-1}))
\]

\[
= \begin{cases} 
-2\sqrt{3}(\chi_j(x_1^m)) \sum_{i=1}^{k} \sum_{s \in \langle s \rangle} \chi_j(s) & \text{if } m \equiv 1 \text{ (mod 3)} \\
-2\sqrt{3}(\chi_j(x_1^m)) \sum_{i=1}^{k} \sum_{s \in \langle s \rangle} \chi_j(s) & \text{if } m \equiv 2 \text{ (mod 3)} \\
-2\sqrt{3}(\chi_j(x_1^m)) C_{x^{3}}(j) & \text{if } m \equiv 1 \text{ (mod 3)} \\
-2\sqrt{3}(\chi_j(x_1^m)) C_{x^{3}}(j) & \text{if } m \equiv 2 \text{ (mod 3)} 
\end{cases}
\]

(14)

The last equality in the preceding equations follows from Lemma 5.12.

Let \( d_j = \chi_j(1) \). Assume that \( t = 2 \). By Theorem 2.1, we have \( \chi_j(x^m) = \sum_{\ell=1}^{d_j} \epsilon_{j\ell} \), where \( \epsilon_{j1}, \ldots, \epsilon_{jd_j} \)
are some 9-th roots of unity. We have

\[-2\sqrt{3} \sum_{r \in G_{9,3}^1} \Im(\chi_j(x^{rm})) = i\sqrt{3} \sum_{r \in G_{9,3}^1} (\chi_j(x^{rm}) - \chi_j(x^{-rm})) \]

\[= i\sqrt{3} \sum_{r \in G_{9,3}^1} \left( \sum_{\ell=1}^{d_j} \varepsilon_{j\ell}^r - \sum_{\ell=1}^{d_j} \varepsilon_{j\ell}^{-r} \right) \]

\[= \sum_{\ell=1}^{d_j} \sum_{r \in G_{9,3}^1} i\sqrt{3}(\varepsilon_{j\ell}^r - \varepsilon_{j\ell}^{-r}). \quad (15)\]

Note that \(\sum_{r \in G_{9,3}^1} i\sqrt{3}(\varepsilon_{j\ell}^r - \varepsilon_{j\ell}^{-r}) = i\sqrt{3}(\varepsilon_{j\ell}^r - \varepsilon_{j\ell}^{-r})(1 + \varepsilon_{j\ell}^3 + \varepsilon_{j\ell}^6)\). Since \(\varepsilon_{j\ell} \in \{1, \omega^2, \omega_0^2, \ldots, \omega_8^6\}\), we see that

\[\sum_{r \in G_{9,3}^1} i\sqrt{3}(\varepsilon_{j\ell}^r - \varepsilon_{j\ell}^{-r}) = \begin{cases} 
9 & \text{if } \varepsilon_{j\ell} \in \{\omega_0^3, \omega_9^6\} \\
0 & \text{otherwise.} \end{cases} \]

Thus \(\sum_{r \in G_{9,3}^1} i\sqrt{3}(\varepsilon_{j\ell}^r - \varepsilon_{j\ell}^{-r})\) is an integer multiple of 3. Therefore by Equation (15), we find that

\[-2\sqrt{3} \sum_{r \in G_{9,3}^1} \Im(\chi_j(x^{rm}))\] is an integer multiple of 3. Similarly, \(-2\sqrt{3} \sum_{r \in G_{9,3}^1} \Im(\chi_j(x^{rm}))\) is also an integer multiple of 3. Using Equation (14), we find that \(\frac{2T_x(j)}{3}\) is an integer. Since \(T_x(j)\) is an integer, integrality of \(\frac{2T_x(j)}{3}\) gives that \(\frac{T_x(j)}{3}\) is also an integer for each \(j \in \{1, \ldots, h\}\).

Now assume that \(t \geq 3\) and \(j \in \{1, \ldots, h\}\). Let

\[A_x(j) := \begin{cases} 
-2\sqrt{3} \left( \sum_{r \in G_{9,3}^1} \Im(\chi_j(x^{rm})) \right) & \text{if } m \equiv 1 \pmod{3} \\
-2\sqrt{3} \left( \sum_{r \in G_{9,3}^1} \Im(\chi_j(x^{rm})) \right) & \text{if } m \equiv 2 \pmod{3}. \end{cases} \]

By Equation (14), we find that \(2T_x(j) = A_x(j)C_x(j)\). Therefore \(A_x(j)\) is a rational algebraic integer whenever \(C_x(j) \neq 0\). Thus, if \(C_x(j) \neq 0\) then \(A_x(j)\) is an integer. Now by Lemma 5.13 and Equation (14), \(\frac{2T_x(j)}{3}\) is an integer, and hence \(\frac{T_x(j)}{3}\) is also an integer.

Let \(S\) be a nonempty set in \(E(\Gamma)\) and \(S\) be expressible as a union of some conjugacy classes of \(\Gamma\). Then \(S\) is a skew-symmetric subset of \(\Gamma\) that is closed under both conjugacy and the equivalence relation \(\simeq\). Let \(S = \text{Cl}(x_1) \cup \cdots \cup \text{Cl}(x_k) = \langle \langle y_1 \rangle \rangle \cup \cdots \cup \langle \langle y_r \rangle \rangle\) for some \(x_1, \ldots, x_k, y_1, \ldots, y_r \in \Gamma(3)\). We see that

\[S = \text{Cl}(x_1) \cup \cdots \cup \text{Cl}(x_k) = \bigcup_{s \in \text{Cl}(x_1)} \langle s \rangle \cup \cdots \cup \bigcup_{s \in \text{Cl}(x_k)} \langle s \rangle = S_{x_1}^3 \cup \cdots \cup S_{x_k}^3. \]

Due to Lemma 5.9, we can assume that the sets \(S_{x_1}^3, \ldots, S_{x_k}^3\) are all distinct. In the following result, we also prove the converse of Theorem 5.3.

**Theorem 5.16.** If \(\Gamma\) is a finite group, then the normal mixed Cayley graph \(\text{Cay}(\Gamma, S)\) is Eisenstein integral if and only if it is HS-integral.
Proof. Assume that Cay(Γ, S) is HS-integral and \( j \in \{1, \ldots, h\} \). Then Cay(Γ, \( S \setminus \overline{S} \) ) is integral, and so \( f_j(S) \) is an integer. By Theorem 4.4, \( \overline{S} \in E(\Gamma) \), which implies that \( \overline{S} = S_{x_1}^{3} \cup \cdots \cup S_{x_k}^{3} \) for some \( x_1, \ldots, x_k \in \Gamma(3) \), where the sets \( S_{x_1}^{3}, \ldots, S_{x_k}^{3} \) are all distinct. Using the fact that \( S_{x_i}^{3} \cup S_{x_i}^{3} = S_{x_i}^{1} \), we have \( \overline{S} \cup \overline{S}^{-1} = S_{x_1}^{1} \cup \cdots \cup S_{x_k}^{1} \). Therefore

\[
g_j(S) = \frac{1}{2} \chi_j(1) \sum_{s \in S \cup S^{-1}} \chi_j(s) - \frac{1}{6} \chi_j(1) \sum_{s \in S} i\sqrt{3} (\chi_j(s) - \chi_j(s^{-1}))
\]

\[
= \frac{1}{2} \chi_j(1) \sum_{s \in S_{x_1}^{1}} \chi_j(s) - \frac{1}{6} \chi_j(1) \sum_{s \in S_{x_1}^{3}} i\sqrt{3} (\chi_j(s) - \chi_j(s^{-1}))
\]

\[
= \frac{1}{2} \sum_{s \in S_{x_1}^{1}} C_{x_1}(j) - \frac{1}{6} \sum_{s \in S_{x_1}^{3}} T_{x_1}(j)
\]

\[
= \frac{1}{2} \sum_{s \in S_{x_1}^{1}} \left( C_{x_1}(j) - \frac{1}{3} T_{x_1}(j) \right).
\] (16)

Let \( 1 \leq \ell \leq k \). Since \( \frac{C_{x_\ell}(j) + T_{x_\ell}(j)}{3} \) is an HS-eigenvalue of the normal oriented Cayley graph Cay(Γ, \( S_{x_\ell}^{3} \)), the numbers \( C_{x_\ell}(j) \) and \( T_{x_\ell}(j) \) are integers of the same parity. By Lemma 5.14 and Lemma 5.15, \( \frac{T_{x_\ell}(j)}{3} \) is an integer. Therefore, \( C_{x_\ell}(j) \) and \( \frac{T_{x_\ell}(j)}{3} \) are integers of the same parity. Thus \( C_{x_\ell}(j) - \frac{1}{3} T_{x_\ell}(j) \) is an even integer, and so \( g_j(S) \) is an integer by Equation (16). Hence by Lemma 5.2, Cay(Γ, S) is Eisenstein integral. The other part of the theorem is proved in Theorem 5.3. □

The following example illustrates an use of Theorem 5.16.

Example 5.1. Consider the mixed graph Cay(A_4, S) of Example 4.1. We have already seen that it is HS-integral, and hence it must be Eisenstein integral. We find that the spectrum of Cay(A_4, S) is \( \{σ_1^1, σ_2^1, σ_3^1, σ_4^9\} \), where \( γ_1 = 7, γ_2 = 3 + 4ω_3, γ_3 = -1 - 4ω_3, \) and \( γ_4 = -1 \). It is clear that the eigenvalues of Cay(A_4, S) are Eisenstein integers.

Acknowledgements

I sincerely thank Dr. Bikash Bhattacharjya for his guidance, enthusiastic encouragement and useful critiques of this research work.

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