AN EXPLICIT BOUND FOR THE LOG-CANONICAL DEGREE OF CURVES ON OPEN SURFACES.

PIETRO SABATINO

ABSTRACT. Let $X$, $D$ be a smooth projective surface and a simple normal crossing divisor on $X$, respectively. Suppose $x(X, K_X + D) \geq 0$, let $C$ be an irreducible curve on $X$ whose support is not contained in $D$ and $\alpha$ a rational number in $[0,1]$. Following Miyaoka, we define an orbibundle $\mathcal{E}_\alpha$ as a suitable free subsheaf of log differentials on a Galois cover of $X$. Making use of $\mathcal{E}_\alpha$ we prove a Bogomolov-Miyaoka-Yau inequality for the couple $(X, D + \alpha C)$. Suppose moreover that $K_X + D$ is big and nef and $(K_X + D)^2$ is greater than $e_{X,D}$, namely the topological Euler number of the open surface $X \setminus D$. As a consequence of the inequality, by varying $\alpha$, we deduce a bound for $(K_X + D) \cdot C$ by an explicit function of the invariants: $(K_X + D)^2$, $e_{X,D}$ and $e_{C,D}$, namely the topological Euler number of the normalization of $C$ minus the points in the set theoretic counterimage of $D$. We finally deduce that on such surfaces curves with $-e_{C,D}$ bounded form a bounded family, in particular there are only a finite number of curves $C$ on $X$ such that $-e_{C,D} \leq 0$.

1. Introduction and Statement of Results

Let $X$ be a minimal complex projective surface of general type such that $K_X^2 > c_2(X)$, in [Bog77], Bogomolov proved the well known result according to which irreducible curves of fixed geometric genus on $X$ form a bounded family. Since Bogomolov’s argument depended on the analysis of curves contained in a certain closed set (see [Des79] for an exposition), his remarkable result was not effective. Indeed, Bogomolov was able to prove that curves in this closed set form a bounded family by considerations involving algebraic foliations but without providing an explicit bound on their degree. Because of this, in a deformation of the surface $X$, the number of either rational or elliptic curves might in principle tend to infinity. This situation can be ruled out providing an upper bound on the canonical degree of irreducible curves on $X$ by a function of the invariants of $X$ and the geometric genus of the curve. The existence of such a function and its form was then conjectured in various places and in slightly different contexts, see for instance [Tia96, §9], with the function depending only on $K_X^2$, $c_2(X)$ and the geometric genus of the curve. The conjecture was proved with some restrictive hypothesis on the singularities of the curve involved by Langer in [Lan03] and finally in its full generality by Miyaoka in [Miy08]. It is interesting to note that part of Miyaoka’s result can be recovered by methods closer in spirit to the original argument of Bogomolov, see McQuillan [McQ17, Corollary 1.3], though one is able to prove the existence of the aforementioned function no explicit form can be established.

The aim of the present paper is to prove a bound as in [Miy08] but in the contest of open surfaces. Let then $X$ be a smooth projective surface, $C$ be an irreducible curve and $D$ a simple normal crossing divisor on $X$. In what follows we will assume that the curve $C$ is not part of the boundary divisor $D$, even if not explicitly stated. Regarding divisors and line bundles we will generally follow the terminology and notation of [Laz04], in particular a simple normal crossing...
divisor is a reduced divisor whose components are smooth and cross normally. Moreover, given a divisor $D$ we identify its support, $\text{Supp}(D)$ with the underlying effective reduced divisor.

First of all, following Miyaoka [Miy08], we are going to prove a Bogomolov-Miyaoka-Yau inequality for the couple $(X, D + \alpha C)$, $\alpha \in [0, 1]$ a rational number, and then deduce our bound from this inequality. In order to state the result we need a couple of definitions.

**Definition 1.1.** Denote by $e_{X \setminus D}$ the topological Euler number of the open surface $X \setminus D$. Note that $e_{X \setminus D} = e_{\text{orb}}(X \setminus D)$ is the orbifold Euler number of $(X,D)$, see Remark 3.3. Let $\eta$: $\tilde{C} \to C$ be the normalization of $C$, we set

$$e_{C \setminus D} := e_{\text{top}}\left(\tilde{C} \setminus \eta^{-1}(D)\right) = e_{\text{top}}(\tilde{C}) - \frac{1}{2}(\eta^{-1}(D)),$$

namely the topological Euler number of the open set $\tilde{C} \setminus \nu^{-1}(D)$.

**Definition 1.2.** Let $C$ be a curve on $X$ not contained in $D$, we say that $C$ is a smooth $D$-rational curve if $C \cong \mathbb{P}^1$ and $D \cdot C \leq 1$. In other words, $C$ is smooth, it crosses $D$ transversally and $C \setminus D$ contains an open set isomorphic to $\mathbb{A}^1$.

**Theorem 1.1** (Bogomolov-Miyaoka-Yau type inequality). Let $X$ be a smooth projective surface, $D$ be a simple normal crossing divisor on $X$ and $C$ an irreducible curve on $X$ not contained in $D$. Suppose that $K_X + D$ is a $\mathcal{Q}$-effective divisor

(i) If $\alpha$ is a real number $\alpha \in [0, 1]$, then the following inequality holds:

$$\frac{\alpha^2}{2} \left[ C^2 + 3(K_X + D) \cdot C + 3C_{\setminus D} \right] - 2\alpha \left[ (K_X + D) \cdot C + \frac{3}{2} C_{\setminus D} \right] + 3e_{X \setminus D} - (K_X + D)^2 \geq 0. \tag{1.1}$$

(ii) Suppose moreover that $C$ is not a smooth $D$-rational curve and $(K_X + D) \cdot C \geq -\frac{3}{2} e_{X \setminus D}$ then the following inequality holds:

$$2 \left[ (K_X + D) \cdot C + \frac{3}{2} C_{\setminus D} \right]^2 - 3e_{X \setminus D} - (K_X + D)^2 \left[ C^2 + 3(K_X + D) \cdot C + 3C_{\setminus D} \right] \leq 0. \tag{1.2}$$

**Remark 1.1.** Observe that in case $D$ is the zero divisor then $-\frac{1}{2} C_{\setminus D} = g - 1$, where as usual $g$ denotes the geometric genus of $C$. It is clear then that in this case Theorem 1.1.(i) is a generalization of [Miy08, Theorem 1.3 (i), (ii)], in which $-\frac{1}{2} C_{\setminus D}$ plays the role of $g - 1$. In case $\alpha = 0$, Theorem 1.1.(i) coincides with [Sak80, Theorem (7.6)].

It is worth noting that Theorem 1.1.(i) is not a direct consequence of a general Bogomolov-Miyaoka-Yau inequality in the form of [Lan03, Theorem 0.1]. Indeed we do not impose any restriction on the singularities of the curve $C$, hence $(X, D + \alpha C)$ may not be log canonical. Though Theorem 1.1 suits our present needs, it seems then natural to ask whether or not it is possible to prove, by a modification of the argument provided here, a version of Theorem 1.1.(i) for a more general couple $(X, B)$, where $B = \sum_i \beta_i B_i$, $\beta_i \in [0, 1]$ rational, and the couple may have worse singularities then log canonical. We will address this question in a successive paper.

Given Theorem 1.1 a direct argument will lead us to:

**Theorem 1.2.** Let $X$ be a smooth projective surface, $D$ be a simple normal crossing divisor on $X$ and $C$ an irreducible curve on $X$ not contained in $D$. Suppose moreover that $K_X + D$ is $\mathcal{Q}$-effective.

(i) If $K_X + D$ is nef, $(K_X + D)^2 > 0, (K_X + D)^2 > e_{X \setminus D}$ and moreover $C$ is not a smooth $D$-rational curve then the relative canonical degree of $C$ is bounded by:

$$(K_X + D) \cdot C \leq A \left( -\frac{1}{2} e_{C \setminus D} \right) + B, \tag{1.3}$$

In other words the log Kodaira dimension of $X \setminus D$ is greater than or equal to zero, namely $\kappa(X, K_X + D) \geq 0$. 

\[\text{\footnotesize{1}}\]
where $A, B$ depends only on $(K_X + D)^2$, $e_{X \cdot D}$ and can be chosen as:

$$A = \frac{2(K_X + D)^2 + \sqrt{2(K_X + D)^2 \left[ 3e_{X \cdot D} - (K_X + D)^2 \right]}}{(K_X + D)^2 - e_{X \cdot D}},$$

$$B = \frac{(K_X + D)^2 \left[ 3e_{X \cdot D} - (K_X + D)^2 \right] + 2e_{X \cdot D} \sqrt{2(K_X + D)^2 \left[ 3e_{X \cdot D} - (K_X + D)^2 \right]}}{2((K_X + D)^2 - e_{X \cdot D})}.$$

(ii) If $K_X + D$ is nef, $C$ is smooth and $D$ and $C$ intersects transversally then the relative canonical degree of $C$ is bounded by:

$$(1.4) \quad (K_X + D) \cdot C \leq -\frac{3}{2}e_{C \cdot D} + \frac{\sqrt{3e_{X \cdot D} - (K_X + D)^2 \sqrt{-2e_{C \cdot D} + 3e_{X \cdot D} - (K_X + D)^2}}}{2} + \frac{3e_{X \cdot D} - (K_X + D)^2}{2},$$

if $C$ is not a smooth $D$-rational curve. If $C$ is a smooth $D$-rational curve, namely it is isomorphic to $\mathbb{P}^1$ and $D.C \leq 1$, then the relative canonical degree of $C$ is bounded by:

$$(1.5) \quad (K_X + D) \cdot C \leq 3e_{X \cdot D} - (K_X + D)^2 - 3.$$

Theorem 1.2.(i) corresponds to [Miy08, Theorem 1.1] and Theorem 1.2.(ii) corresponds to [Miy08, Corollary 1.4]. In analogy with this last result, for a smooth curve $C$ that meets $D$ transversally and such that $-e_{C \cdot D}$ is very large, the relative canonical degree is bounded by a function that asymptotically behaves like $\frac{3}{2}(-e_{C \cdot D})$. As remarked in [McQ17], considerations of differential geometric nature suggest that, in such bounds on the canonical degree, good choices for the constant in front of $e_{C \cdot D}$ are the reciprocal of either $-\frac{2}{3}$ or $-\frac{1}{2}$, the holomorphic sectional curvature of the Kähler-Einstein metric of balls and bi-discs, respectively. It turns out that in the algebraic geometric setting, see [ACLG12], $-\frac{2}{3}$ is optimal, in particular taking into account singular curves. For smooth curves $-\frac{2}{3}$ seems the right choice, at least asymptotically.

**Corollary 1.3** (Uniform bound of the relative canonical degree). Let $X$ be a smooth projective surface, $D$ be a simple normal crossing divisor on $X$ and $C$ an irreducible curve on $X$ not contained in $D$. If $K_X + D$ is nef and big and moreover $(K_X + D)^2 > e_{X \cdot D}$ then the relative canonical degree of $C$ is bounded by

$$(K_X + D) \cdot C \leq A \left( -\frac{1}{2}e_{C \cdot D} \right) + B$$

where $A, B$ depend only on $e_{X \cdot D}$ and $(K_X + D)^2$.

By the above bound on the canonical degree it follows that curves for which $-e_{C \cdot D}$ is fixed form a bounded family. In particular:

**Corollary 1.4.** Let $X$ be a smooth projective surface and $D$ a simple normal crossing divisor on $X$. Suppose that $K_X + D$ is nef and big and moreover that $(K_X + D)^2 > e_{X \cdot D}$. Then, curves $C$ on $X$ that are not contained in $D$ and such that $-e_{C \cdot D}$ is bounded form a bounded family, where the number of components is bounded by a function that depends on $(K_X + D)^2$ and $e_{X \cdot D}$. In particular on $X$ there are only a finite number of curves $C$ such that $-e_{C \cdot D} \leq 0$ and their number is bounded by a function of $(K_X + D)^2$ and $e_{X \cdot D}$.

**Remark 1.2.** In view of the hypotheses of Corollary 1.4, a bound on the log canonical degree of curves translates in an analogous bound on their degree with respect to any fixed ample divisor. The same bound, given its nature, holds uniformly in a family of deformations of the surface $X$ too. It is worth noting that since surfaces of log general type with $(K_X + D)^2$ bounded are bounded, see [Ale94, Theorem 7.7] for instance, hence the conclusions of Corollary 1.4 hold in the most general sense.
Remark 1.3. Let $\tilde{C}$ be the normalization of $C$, $\eta: \tilde{C} \to C$ the corresponding map and consider $\eta^{-1}(D)$ as a closed set. By definition of $e_{C\setminus D}$, if $-e_{C\setminus D} \leq 0$ then the geometric genus of $C$ is less than or equal to one and there are only four possibilities for the open set $\tilde{C} \setminus \eta^{-1}(D)$, namely

$$\tilde{C} \setminus \eta^{-1}(D) \cong \begin{cases} \mathbb{P}^1 & \\ \mathbb{A}^1 & \\ \mathbb{A}^1 \setminus \{pt\} & \\ \text{Elliptic curve} & \\ \end{cases}$$

hence Corollary 1.4 generalizes [Miy08, Corollary 1.2].

We end the series of results with the following Corollary, where we apply Theorem 1.2.(i) to an elementary (i.e. that can be formulated in elementary terms) situation in $\mathbb{P}^2$.

**Corollary 1.5.** Let $D_1$, $D_2$, $C$ be distinct irreducible curves in $\mathbb{P}^2$ of degree $d_1$, $d_2$ and $d$, respectively, such that $D = D_1 + D_2$ is a simple normal crossing divisor. Denote by $g$ the geometric genus of $C$, consider $\tilde{C}$ the normalization of $C$, $\eta: \tilde{C} \to \mathbb{P}^2$ the induced map and define

$$m := \min_{p \in \tilde{C} \cap \eta^{-1}(D)} \{\text{mult}_p (\eta^*D)\}.$$ 

Suppose that $d \geq d_2 \geq d_1 > 0$ and set

$$\lambda = \frac{d_1}{d_2}, \quad \nu = \frac{d}{d_2}.$$

There exist constants $\lambda_0$, $\frac{2}{3} < \lambda_0 < 1$, $h$ and $k$ such that if $d_2 \geq 6$, $\lambda \geq \lambda_0$ and

$$\nu > \frac{h g + k}{\left(\frac{\lambda_0}{2} - \frac{1}{3}\right)\left(\frac{\lambda_0}{2} + \frac{1}{4}\right)}$$

then

$$m \leq \frac{50}{\left(\frac{1}{2} - \frac{1}{3}\right)}.$$ 

The statement of Corollary 1.5 may result a bit obscure at a first reading, but basically its content is the following. After arranging the degrees of $D$ in such a way that the hypotheses of Theorem 1.2.(i) are satisfied, if the degree of $C$ is sufficiently large, then the order of tangency between $C$ and $D$ can not be everywhere too high. Note moreover that we need $D$ to have at least two components to guarantee the required flexibility in order to obtain a couple $(X, D)$ of log general type, minimal and such that $(K_{\mathbb{P}^2} + D)^2 - e_{\mathbb{P}^2 \setminus D} > 0$. In contrast to the elementary nature of the statement we are not aware of any elementary proof. Details of the proof are provided at the end of §5.

2. Preliminaries

In the present section, for reader’s convenience, we gather a number of results that we will use during the course of our proofs. For the sake of clarity we state them in the form more suitable to our needs. Where it is possible without complicating the discussion, we provide short proofs, if they are not available elsewhere or if they provide a way to quickly gain insight on the particular topic.

2.1. Zariski decomposition with support in a negative cycle. The Zariski decomposition was introduced in [Zar62, §7] and its proof involves a rather elementary although lengthy argument in linear algebra and quadratic forms. Following Miyaoka, see [Miy08, §2], our argument will rely on a slight modification of the classical Zariski decomposition. Basically we require the support of the negative part to be contained in a fixed negative definite cycle. The existence of the Zariski decomposition with support can be proved following, with minor modifications,
the original argument of Zariski. Zariski constructs the negative part of the decomposition and then as a consequence the positive part, but given the fact that the positive part can be interpreted as a solution to a maximization problem (this was already remarked for instance by Kawamata in [Kaw79, Proposition (1.5) and (1.6)], it turns out that it is much easier to start by constructing the positive part and then deduce the existence of the Zariski decomposition, see for instance [Bau09] and [BCK12] for an even more elementary exposition. Following this approach we are going to summarize results regarding Zariski decomposition with support and its relation with the classical one. It is worth noting that a similar discussion is contained in [Laf16], nonetheless we prefer to briefly summarize it here in a way more convenient for our needs.

**Notation 2.1.** We denote by \( \prec \) the partial order on \( \text{Div}_R(X) \) given by \( D_1 \prec D_2 \) if \( D_2 - D_1 \) is effective.

**Definition 2.1.** Let \( E_1, \ldots, E_l \) be irreducible curves on \( X \), the cycle \( E = \sum_{i=1}^l E_i \) is said to be **negative definite** if the intersection matrix relative to \( E \), \( \left( E_i \cdot E_j \right)_{ij} \), is negative definite.\(^2\) In order to simplify the exposition we will consider the trivial cycle negative definite.

**Proposition 2.1.** Let \( D \in \text{Div}_0(X) \) be effective and \( E = \sum_{i=1}^l E_i \) be a negative definite cycle then there exist \( P_E(D), N_E(D) \in \text{Div}_0(X) \) such that:

(i) \( P_E(D) \) and \( N_E(D) \) are effective and \( D = P_E(D) + N_E(D) \).

(ii) \( P_E(D) \) is nef on \( E \) namely \( P_E(D) \cdot E_i \geq 0 \) for every \( i = 1, \ldots, l \).

(iii) The support of \( N_E(D) \) is contained in \( E \), namely \( N_E(D) = \sum_{i=1}^l a_i E_i, \ a_i \geq 0 \) for \( i = 1, \ldots, l \).

(iv) \( P_E(D) \) is numerically trivial on \( N_E(D) \), namely \( P_E(D) \cdot E_i = 0 \) for every prime component \( E_i \) in the support of \( N_E(D) \). It follows that \( P_E(D) \cdot N_E(D) = 0 \) and then \( D^2 = P_E^2(D) + N_E^2(D) \).

(v) If the above properties are satisfied then \( P_E(D) \) and \( N_E(D) \) are unique. In particular, \( P_E(D) \) can be characterized as the largest effective \( \mathcal{Q} \)-divisor such that \( P_E(D) \prec D \) and \( P_E(D) \) is nef on \( E \) (namely if \( P' \) is an effective \( \mathcal{Q} \)-divisor \( P' \prec D \) and \( P_E(D) \) is nef on \( E \) then \( P' \prec P_E(D) \)).

**Proof.** First of all, we are going to prove that there exist unique \( \mathbb{R} \)-divisors \( P_E(D) \) and \( N_E(D) \) that satisfy properties (i)--(iv), that these divisors are rational will follow immediately from unicity, namely they will be the unique solution of a system of linear equations with integral coefficients. Up to reordering, we may suppose that \( E_1, \ldots, E_k, \ k \leq l \), are the components of \( E \) contained in the support of \( D \). Put

\[
D = \sum_{j=1}^{h+k} d_j D_j
\]

where the irreducible components of the support of \( D \) not contained in \( E \) are denoted by \( D_1, \ldots, D_h \), moreover \( D_{h+1} = E_1, \ldots, D_{h+k} = E_k \), and we set \( E' = \sum_{i=1}^k E_i \). Consider the linear space of \( \mathbb{R} \)-divisors with support in \( D \), we can write its elements as \( \sum_{j=1}^{h+k} x_j D_j \). In this linear space consider the compact subset defined by:

\[
0 \leq x_j \leq d_j, \quad j = 1, \ldots, h + k
\]

and

\[
\sum_{j=1}^{h+k} x_j \left( D_j \cdot E_i \right) \geq 0 \quad i = 1, \ldots, k .
\]

This compact set contains at least one point that maximizes the function \( \sum_{j=1}^{h+k} x_j \), let \( P_E(D) \) be the corresponding divisor, \( N_E(D) = D - P_E(D) \), (i) is then satisfied. Moreover since \( P_E(D) \) satisfies (2.2), it is nef on \( E' \) and then on \( E \). Observe that in (2.2) the coefficient \( D_j \cdot E_i \) can be negative only for \( j > h \), hence these inequalities do not impose any restrictions on \( x_j \) for \( j \leq h \). It follows that without loss of generality we can substitute the first \( h \) inequalities in (2.1) with the equalities

\[
x_j = d_j, \quad j = 1, \ldots, h
\]

\(^2\)Note that if \( E \) is negative definite then the curves \( E_i \) must be distinct.
and then $N_E(D)$ satisfies (iii). Since $P_E(D)$ maximizes $\sum_j x_j$, for every fixed $1 \leq i \leq k$ such that $d_{h+i} \neq 0$ and small $\epsilon > 0$, $P_E(D) + \epsilon E_i$ is not nef on $E'$, and then

$$P_E(D) + \epsilon E_i < 0$$

(2.4)

since the other intersections $(P_E(D) + \epsilon E_i) \cdot E_j, j \neq i$, are still non negative. Passing to the limit in (2.4) we get $P_E(D) \cdot E_i \leq 0$ but since $P_E(D)$ is nef on $E'$ we have $P_E(D) \cdot E_i = 0$ and this concludes the proof of (iv).

Observe that in view of (iv) we can now rewrite (2.2) as

$$\sum_{j=1}^{h+k} x_j [D_j \cdot E_i] = 0 \quad i = 1, \ldots, k,$$

and hence $P_E(D)$ is a solution of the $h + k$ equations given by (2.3), (2.5). If we write this system of equations in matrix form we obtain:

$$\begin{bmatrix} I_h & 0 \\ A & E' \end{bmatrix} X = \begin{bmatrix} d \\ 0 \end{bmatrix}$$

where $I_h$ is the $h \times h$ identity matrix, $E'$ is the $k \times k$ negative definite intersection matrix of $E'$ and $A$ is a $k \times h$ matrix with rational integral entries. Since all involved coefficients are rational, then the unique solution of the above system of equations corresponding to $P_E(D)$ is rational.

For the proof of the last part of (v) we are going to use the following Lemma.

Lemma 2.2. Let $P, P'$ be two effective divisors $P, P' \subseteq D$, $P = \sum_{j=1}^{h+k} y_j D_j$ and $P' = \sum_{j=1}^{h+k} y'_j D_j$. If $P, P'$ are both nef on $E$ then $\max(P, P') := \sum_{j=1}^{h+k} \max(y_j, y'_j) D_j$ is nef on $E$.

Proof of Lemma 2.2. Given an element in the linear space of $\mathbb{R}$-divisors with support contained in $D$, it is nef on $E$ if and only if its coordinates $x_1, \ldots, x_{h+k}$ satisfy the inequalities in (2.2). Consider then the $i$-th inequality in (2.2), its left-hand side is a linear polynomial in the $x$’s that has only one negative coefficient, namely $D_{h+i} \cdot E_i$. We may suppose without loss of generality that $y_{h+i} \geq y'_{h+i}$, then

$$\left(\max(P, P') - P\right) \cdot E_i \geq 0$$

from which it follows that $\max(P, P') \cdot E_i \geq 0$. Since the above argument holds for every $i = 1, \ldots, k$, this concludes the proof of the Lemma. □

Let us now complete the proof of (v). Let $P_E'(D) \ll D$ be an effective divisor that is nef on $E$. Since the support of the negative part is contained in $E'$ (see (2.3)) then there exist $x_i \geq 0, i = 1, \ldots, k$, such that

$$\max(P_E(D), P_E'(D)) = P_E(D) + \sum_{i=1}^{k} x_i E_i.$$

Since by Lemma 2.2 $\max(P_E(D), P_E'(D))$ is nef on $E'$ then for $j = 1, \ldots, k$ we have

$$\sum_{i=1}^{k} x_i E_i \cdot E_j \geq 0$$

and then

$$\left(\sum_{i} x_i E_i\right) \cdot \left(\sum_{i} x_i E_i\right) = \sum_{j} \left(\sum_{i} x_i E_i \cdot E_j\right) \geq 0.$$

By hypothesis the intersection matrix of $E$, and then that of $E'$, is negative definite, the above inequality implies $x_i = 0, i = 1, \ldots, k$ and then $\max(P_E(D), P_E'(D)) = P_E(D)$. This concludes the proof of (v) and of the Proposition. □

Definition 2.2. Given an effective $Q$-divisor $D$ and a negative definite cycle $E$ we will call the decomposition $D = P_E(D) + N_E(D)$ of Proposition 2.1 the Zariski decomposition of $D$ with support in $E$. In particular we will call $P_E(D)$ the $E$-nef part of $D$ and $N_E(D)$ the $E$-negative part of $D$. 
Remark 2.1. From the proof of Proposition 2.1 it follows that the Zariski decomposition of a divisor $D$ with support on a particular negative cycle $E$ depends only on the part of $E$ supported in $N_E(D)$.

Remark 2.2. In order to distinguish the Zariski decomposition with support from the classical one, we will refer to the latter as the absolute Zariski decomposition and denote it by $D = P(D) + N(D)$. The reasons behind this terminology as well as the relation between the two decompositions will be clarified by the next Corollary. In particular, the absolute Zariski decomposition coincides with the Zariski decomposition of $D$ with support in $N(D)$, moreover among the various Zariski decompositions of $D$ the absolute one is characterized by having maximal negative part and minimal nef part.

Corollary 2.3. Let $D, D'$ two effective $Q$-divisors and $E, \hat{E}$ negative definite cycles.

(i) If $D \preceq D'$ then $P_E(D) \preceq P_E(D')$.

(ii) If $E \preceq \hat{E}$ then $N_E(D) \preceq N_{\hat{E}}(D)$, $P_E(D) \succ P_{\hat{E}}(D)$, $0 \geq \left( N_E(D) \right)^2 \geq \left( N_{\hat{E}}(D) \right)^2$ and $\left( P_E(D) \right)^2 \leq \left( P_{\hat{E}}(D) \right)^2$.

(iii) Let $N$ be the support of $N(D)$, $N$ is a negative definite cycle, $N(D) = N_N(D)$, $P(D) = P_N(D)$, $N_E(D) \preceq N(D)$, $P_E(D) \succ P(D)$, $0 \geq \left( N_E(D) \right)^2 \geq \left( N(D) \right)^2$ and $\left( P_E(D) \right)^2 \leq \left( P(D) \right)^2$.

Proof. Observe that $P_E(D) \preceq D \preceq D'$ and $P_E(D)$ is nef on $E$, then (i) follows by Proposition 2.1.(v). Similarly, in (ii), $P_E(D) \preceq D$, it is nef on $\hat{E}$ and then on $E$. By Proposition 2.1.(v), $P_E(D) \succ P_{\hat{E}}(D)$ and consequently $N_E(D) = D - P_E(D) \preceq D - P_{\hat{E}}(D) = N_{\hat{E}}(D)$.

Let us prove the inequalities involving intersection numbers in (ii). In the vector space of cycles whose support is contained in $D$ denote by $V$ and $\hat{V}$ cycles whose support are contained in $E$, $\hat{E}$, respectively. Observe that since $V \subseteq \hat{V}$ then $V^\perp \supseteq \hat{V}^\perp$ and we have the two orthogonal decompositions:

$$D = P_E(D) + N_E(D) \in V^\perp \oplus V$$
$$D = P_{\hat{E}}(D) + N_{\hat{E}}(D) \in \hat{V}^\perp \oplus \hat{V}$$

It follows that

$$R = P_E(D) - P_{\hat{E}}(D) = N_E(D) - N_{\hat{E}}(D) \in V^\perp \cap \hat{V}$$
and we have the two orthogonal decompositions

$$P_E(D) = P_{\hat{E}}(D) + R \in \hat{V}^\perp \oplus \hat{V}$$
$$N_E(D) = N_{\hat{E}}(D) + R \in V \oplus V^\perp$$
from which the inequalities involving intersection numbers in (ii) follow directly by observing that since $R \in \hat{V}$ then $R^2 \leq 0$.

Finally by definition of absolute Zariski decomposition of $D$, it coincides with the Zariski decomposition with support contained in the negative part $N(D)$. Since $P(D)$ is nef by Proposition 2.1.(v) we have $P(D) \preceq P_E(D)$, it follows that $N_E(D) \preceq N(D)$. Without loss of generality, see Remark 2.1, we can assume that the support of $E$ is contained in $N$, the remaining inequalities in (iii) then follow by part (ii) of the Corollary. □

Corollary 2.4. Let $D_1, D_2 \in \text{Div}_Q(X)$ be effective divisors and let $E$ be a negative definite cycle.

(i) If $D_1 \equiv_{\text{num}} D_2$ then $N_{E}(D_1) = N_{E}(D_2)$ and $N(D_1) = N(D_2)$.

(ii) If moreover $D_1 \equiv_{\text{lin}} D_2$ then $P_{E}(D_1) \equiv_{\text{lin}} P_{E}(D_2)$ and $P(D_1) \equiv_{\text{lin}} P(D_2)$.

Proof. The second statement follows directly from the first one. Let us prove the first one for the absolute Zariski decomposition, the case of relative Zariski decomposition can be proved by an analogous argument. Observe that, by the maximality of the positive part, a curve $F'$ is contained in the support of the negative part of $D_1$ if and only if $D_1\cdot F' < 0$. It follows that $N(D_1)$ and $N(D_2)$...
are supported on the same negative definite cycle, say \( F = \sum_{j=1}^{k} F_j \). Let \( F \) be the \( k \times k \) negative definite intersection matrix of \( F \) and write \( N(D_i) = \sum_{j=1}^{k} x_j F_j \), then \( X^i = \{ x_1^i, \ldots, x_j^i, \ldots, x_k^i \} \) is the unique solution to the system of linear equations

\[
F \cdot X^i = d_i
\]

where \( d_i = (D_i F_1, \ldots, D_i F_k) \), since \( d_1 = d_2 \), this concludes the proof.

\( \square \)

**Remark 2.3.** In view of Corollary 2.4, it does make sense to consider the Zariski decomposition, either relative or absolute, of a divisor that is linearly equivalent to an effective \( Q \)-divisor.

**2.2. Nef reduction.** Let \( \rho: Z \to Y \) be a surjective morphism between non singular projective surfaces whose exceptional locus \( R \) is a divisor of simple normal crossings. Suppose moreover

\[
R \subseteq \rho^{-1}(\Lambda) \subseteq \Delta,
\]

where \( \Delta \) is an normal crossing divisor on \( Z \) and \( \Lambda \) an effective reduced divisor on \( Y \). Observe that \( \rho(R) \) is a finite set and then we can find an affine open subset \( U \) containing \( \rho(R) \) in which \( \Lambda \) is defined by a single equation, say \( \lambda \). Hence \( \rho^*(\log \lambda) \) is a section of \( \Omega^1_Z(\log \Delta) \) defined around \( R \). Indeed it suffices to check this statement locally at each point \( q \in R \subseteq \rho^{-1}(\Lambda) \). Indeed since \( \rho^{-1}(\Lambda) \) is contained in \( \Delta \) of simple normal crossing, there exist \( z_1, z_2 \) local coordinates around \( q \) such that

\[
\rho^* \lambda = u z_1^a z_2^b,
\]

\( a, b \geq 0, a + b > 0, u \) a unit, and then

\[
(2.6) \quad \rho^*(\log \lambda) = a \frac{dz_1}{z_1} + b \frac{dz_2}{z_2} \tag{regular 1-form},
\]

with \( (a, b) \neq (0, 0) \).

**Proposition 2.5.** Let \( \rho: Z \to Y \), \( R \), \( \Delta \), \( \Lambda \), \( \lambda \) as above and \( E \) a rank two vector bundle on \( Z \) such that:

\( \begin{align*}
(1) & \quad E \text{ is a subsheaf of } \Omega^1_Z(\log \lambda). \\
(2) & \quad D = c_1(E) \text{ is a } Q \text{-effective divisor.} \\
(3) & \quad N_R(D) \text{ is an integral divisor.} \\
(4) & \quad \text{There exist a neighborhood } V \text{ of } R \text{ such that } \rho^*(\log \lambda) \in \Gamma(V, E) \subseteq \Gamma(\Omega^1_Z(V, \log \Delta)).
\end{align*} \)

Then, there exist a rank two vector bundle \( \mathcal{P}_R(E) \subseteq E \) such that:

\( \begin{align*}
(1) & \quad \mathcal{P}_R(E) = E \text{ outside of } R; \\
(2) & \quad c_1(\mathcal{P}_R(E)) = P_R(D); \\
(3) & \quad c_2(\mathcal{P}_R(E)) = c_2(E).
\end{align*} \)

**Remark 2.4.** Proposition 2.5 is basically [Miy08] Lemma 2.3. Though the hypothesis in Miyaoka are slightly different from ours, indeed he supposes that the divisor \( E \) coincides with the exceptional locus of the map \( \rho \), his proof still works without any substantial modification here as well.

For reader’s convenience, we briefly recall:

**Proof of Proposition 2.5.** Since \( \rho^*(\log \lambda) \) is a nowhere vanishing section of \( E \) in \( V \), it induces an injection \( \mathcal{O}_V \to E|_V \). In view of (2.6), after restricting \( V \) if needed, the cokernel of the above injection is locally free of rank one and then isomorphic to \( \det(E)|_V \), summing up we get an exact sequence

\[
0 \to \mathcal{O}_V \to E|_V \to E_V(D) \to 0.
\]

In order to simplify notation set \( N = N_R(D) \) and \( P = P_R(D) \). With a slight abuse of notation we continue to denote by \( N \) the subscheme relative to the sheaf of ideals \( \mathcal{O}_Z(-N) \). Observe that by hypothesis \( N \) is a subscheme of \( V \), consider the map obtained composing the surjections

\[
E \to E|_V \to E_V(D) \to E_N(D)
\]

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and denote by $\mathcal{P}_R(\mathcal{E})$ its kernel. Since $P$ is numerically trivial on $N$, we get the exact sequence
\[ 0 \to \mathcal{P}_R(\mathcal{E}) \to \mathcal{E} \to \mathcal{O}_N(N) \to 0 \]
then
\[ c(\mathcal{P}_R(\mathcal{E}))(1+N) = c(\mathcal{E}) \]
from this equality
\[ c_1(\mathcal{P}_R(\mathcal{E})) = P \]
and finally
\[ c_2(\mathcal{P}_R(\mathcal{E})) = c_2(\mathcal{E}) - c_1(\mathcal{P}_R(\mathcal{E}))N = c_2(\mathcal{E}) - PN = c_2(\mathcal{E}) \].

\[ \square \]

**Definition 2.3.** We call $\mathcal{P}_R(\mathcal{E})$, as in the Proposition 2.5, a $R$-nef reduction of $\mathcal{E}$.

2.3. **Bogomolov-Miyaoka-Yau type inequality for subsheaves of logarithmic forms.** In order to obtain our explicit version of Bogomolov-Miyaoka-Yau inequality, following a quite common path (see for instance, [Meg99], [Lan01], [Miy08], [Lan03], only to name a few) we will start with a generalized inequality applied to a suitable subsheaf of logarithmic forms. The proof will then consist in a straightforward computation of the Chern classes appearing in the inequality and a careful estimation of the contributions due to the singularities involved. In particular we will make use of the following:

**Theorem 2.6.** Let $X$ be a smooth projective surface, $D$ a simple normal crossing divisor on $X$ and $\mathcal{E}$ a locally free rank two subsheaf of $\Omega^1_X(\log D)$ such that $c_1(\mathcal{E})$ is an effective $Q$-divisor. Then
\[ 3c_2(\mathcal{E}) + \frac{1}{4} \left| N(c_1(\mathcal{E})) \right|^2 \geq c_1^2(\mathcal{E}) \).

**Proof.** See Miyaoka [Miy84, Remark 4.18, p. 170], [Lan01, Theorem 0.1]. \[ \square \]

**Remark 2.5.** We did not state Theorem 2.6 in the most general version available, indeed the above form will suffice for most of our purposes. We will need Bogomolov-Miyaoka-Yau inequality in the generality provided by [Lan03, Theorem 0.1] only to prove (1.5). For a generalization to the case of log canonical surfaces see for instance [Lan03].

### 3. Main Construction

In this Section, and for the rest of this paper, we denote by $X$ a smooth projective surface, $D$ a simple normal crossing divisor on $X$ and $C$ an irreducible curve on $X$ not contained in $D$. Let $\alpha \in [0,1] \cap Q$, following Miyaoka [Miy08] §3, after taking a log resolution of $C+D$, we are going to construct a vector bundle $\mathcal{E}_\alpha$ in terms of $X$, $C$, $D$ and $\alpha$. Namely, given a log resolution of $C+D$, say $\tilde{Y}$, we consider logarithmic forms $^3$ with poles on the total transform $\tilde{C} + \tilde{D}$ and, roughly speaking, define $\mathcal{E}_\alpha$ as the kernel of the map
\[ \rho : \Omega^1_{\tilde{Y}}(\log \tilde{C} + D) \to \mathcal{E}_{(1-\alpha)\tilde{C}} \]
induced by the natural residue map
\[ \Omega^1_{\tilde{Y}}(\log \tilde{C} + D) \to \mathcal{O}_{\tilde{C}} \to 0 \]
where $\tilde{C}$ denotes the strict transform of $C$. Since $1 - \alpha$ is not in general integral, in order to formulate this definition in a rigorous way we have implicitly to consider roots of a local equation of $\tilde{C}$. We will then take a suitable “Kawamata covering” $f : Z \to \tilde{Y}$ (see [Kaw81, Theorem 17], [Laz04, Proposition 4.1.12]) in which these roots are defined, or in other words, such that $(1 - \alpha)f^*\tilde{C}$ is an integral divisor. $\mathcal{E}_\alpha$ will be defined on the Galois covering $Z$, by construction it will inherit an equivariant action and it will then be, technically speaking, an “orbibundle” on $\tilde{Y}$.

We are going to detail the construction of $\mathcal{E}_\alpha$, since in the following Section we will perform explicit computations involving its Chern classes, we are going to take some time expressing them in terms of log resolution data.

$^3$For general facts on logarithmic forms used here the reader may consult [EV92].
3.1. Definition of $E_0$. For reader convenience we will adopt, as far as possible, Miyaoka’s notation in [Miy08], the main difference being that we need to distinguish carefully between singular points contained in the boundary divisor $D$ and points that are not. First of all, consider the set of points where $C + D$ fails to be simple normal crossing, say $S$. We can express $S$ as a disjoint union $S = S_1 \cup S_2$, where $S_1$ is the set of singular points of $C$ that are not contained in $D$ and $S_2$ is the set of points of $C \cap D$ where $D + C$ is not simple normal crossing. We are going to consider a log-resolution of $D + C$. Let us start blowing up $X$ with centers in the points of $S$ say $\mu: Y \to X$, moreover observe that we can write $\mu = \mu_2 \circ \mu_1$, where $\mu_1: Y' \to X$ is the blowing up of $X$ with centers in the points of $S_1$ and $\mu_2: Y \to Y'$ the blowing up of $Y'$ with centers in the set $\mu_1^{-1}(S_2)$. Let us denote by $s', s$ the cardinality of the sets $S_1$ and $S$, respectively. Let $E_1, \ldots, E_s$ be the exceptional curves of $\mu$, where $E_1, \ldots, E'_s$ come from $\mu_1$ and $E_{s'}, \ldots, E_s$ come from $\mu_2$. Since $\mu^*(C + D)$ is not necessarily simple normal crossing, we proceed until we get $\pi: \bar{Y} \to Y$ given by a composition

$$\bar{Y} = Y \xrightarrow{\pi_{s'}} Y_{r-1} \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_{r'}} Y_r,$$

where

- At each intermediate step, $\pi_i$ is the blowing up of $Y_{i-1}$ in a point where $\text{Supp} \pi_i^* \cdot \mu^*(C + D)$ is not simple normal crossing.
- The first $r'$ blow-ups are centered in points mapping to $S_1$.
- The remaining $(r - r')$ blow-ups are centered in points mapping to $S_2$.
- In the end, denote by $\pi = \pi_r \circ \cdots \circ \pi_1$, $\text{Supp} \pi^* \mu^*(C + D)$ is a simple normal crossing divisor in $\bar{Y}$ that we denote by $\overline{C + D}$.\(^4\)

We will, moreover, use the following additional notation for the elements of the above resolution.

| Notation | Definition |
|----------|------------|
| $E_{s+i}, i = 1, \ldots, r'$ | Exceptional curve of the blow-up $\pi_i: Y_{i-1} \to Y_i$, $i = 1, \ldots, r'$ centered in points mapping to $S_1$. |
| $E_{s+i}, i = r' + 1, \ldots, r$ | Exceptional curve of the blow-up $\pi_i: Y_{i-1} \to Y_i$, $i = r' + 1, \ldots, r$ centered in points mapping to $S_2$. |
| $F_1, \ldots, F_s', F_{s'+1}, \ldots, F_s \text{Div}(\bar{Y})$ | Strict transforms in $\bar{Y}$ of the exceptional curves $E_1, \ldots, E_s$, respectively. |
| $G_1, \ldots, G_{r'}, G_{r'+1}, \ldots, G_r \in \text{Div}(\bar{Y})$ | Strict transforms in $\bar{Y}$ of the exceptional curves $E_{s+1}, \ldots, E_{s+r}$, respectively. |
| $\overline{E}_1, \ldots, \overline{E}_{s'}, \overline{E}_{s'+1}, \ldots, \overline{E}_s$, $\overline{E}_{s+r}, \ldots, \overline{E}_{s+r'} \in \text{Div}(\bar{Y})$ | Total transforms in $\bar{Y}$ of the exceptional curves $E_1, \ldots, E_{s+r}$, respectively. Note that for the last blow-up $\overline{E}_{s+r} = E_{s+r} = G_{s+r}$. |

\(^4\)A word of caution is in order here. Following Miyaoka, $\overline{C + D}$ does not denote the total transform of the curve $C + D$, and indeed it is a reduced divisor. On the other hand, $\overline{E}_i$ denotes the total transform in $\bar{Y}$ of the exceptional curve $E_i$. 

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The exceptional locus of $\pi \circ \mu$ is equal to $F + G$ and

$$C + D = \tilde{C} + \tilde{D} + F + G$$

where $\tilde{C}, \tilde{D}$ denote the strict transforms of $C$ and $D$, respectively. Given $\alpha \in [0, 1] \cap \mathbb{Q}$, we can consider, see Proposition 4.1.12 in [Laz04], a “Kawamata covering” of the couple $\tilde{Y}, \tilde{C} + \tilde{D}$, namely a finite covering $f : Z \to \tilde{Y}$ such that $A_{\alpha} = (1 - \alpha)f^* C \preceq f^* \tilde{C}$ is an integral divisor, Supp $f^* \tilde{C}$ is smooth and Supp $f^* \tilde{C} + \tilde{D}$ has simple normal crossings. The natural residue map induces a surjection

$$f^* \Omega_{\tilde{Y}}^1(\log \tilde{C} + \tilde{D}) \twoheadrightarrow \Theta_Z/\Theta_Z(-A_\alpha) \to 0$$

define $\mathcal{E}_a$ as the kernel of this map so that it fits in the exact sequence

$$(3.1) \quad 0 \to \mathcal{E}_a \to f^* \Omega_{\tilde{Y}}^1(\log \tilde{C} + \tilde{D}) \to \Theta_Z/\Theta_Z(-A_\alpha) \to 0,$$

it follows from the preceding exact sequence that $\mathcal{E}_a \subseteq f^* \Omega_{\tilde{Y}}^1(\log \tilde{C} + \tilde{D})$ is locally free and of rank two.

**Remark 3.1.** In defining $\mathcal{E}_a$ we have followed closely [Miy08], but it is worth noting that the end result is equal to $f^* \Omega_{\tilde{Y}}^1(\log (a\tilde{C} + \tilde{D} + E))$, according to Langer’s notation in [Lan03, §2, p. 365]. This can be checked in a rather straightforward manner by a computation in local coordinates. To give the flavor, consider a point $p \in \text{Supp } f^* \tilde{C}$ that is also a smooth point of Supp $(a\tilde{C} + \tilde{D} + E)$. Then there exist local coordinates, $z'$ around $f(p)$ such that $\tilde{C}$ has equation $z' = 0$ and $z$ around $p$ such that Supp $f^* \tilde{C}$ has equation $z = 0$. If $t = 0$ is a local equation for $f^* (a\tilde{C})$ around $p$ then we may suppose without loss of generality that $t = z^{am}$, where the integer $m > 0$ is such that $am$ is again integral. By definition of $f^* \Omega_{\tilde{Y}}^1(\log (a\tilde{C} + \tilde{D} + E))$ and $\mathcal{E}_a$ then:

$$f^* \Omega_{\tilde{Y}}^1(\log (a\tilde{C} + \tilde{D} + E)) = \frac{f^* dz'}{t} \Theta_{\tilde{Y}} + f^* \Omega_{\tilde{Y}}^1 \Omega_1 \frac{df^* z'}{t} \Theta_{\tilde{Y}} + f^* \Omega_{\tilde{Y}}^1 = \Omega_1 \frac{dz}{z^{am-1}} \Theta_{\tilde{Y}} + f^* \Omega_{\tilde{Y}}^1 \frac{dz}{z} \Theta_{\tilde{Y}} + f^* \Omega_{\tilde{Y}}^1 = \mathcal{E}_a$$

locally around $p$. The remaining cases are dealt with analogously.

**3.2. Chern classes of $\mathcal{E}_a$.** We are going to express the Chern classes of $\mathcal{E}_a$ in terms of log resolution data. Refer to §3.1 for the notation regarding resolution data. It is worth noting that these computations can be performed directly by using (3.1), but in view of Remark 3.1, $\mathcal{E}_a = f^* \Omega_{\tilde{Y}}^1(\log (a\tilde{C} + \tilde{D} + E))$ according to Langer’s notation, see [Lan03, §2, 3], and then the Chern numbers of $\mathcal{E}_a$ are equal to the Chern numbers of the pair $(\tilde{Y}, a\tilde{C} + \tilde{D} + E)$. In particular by [Lan03, p. 368, Theorem 3.6], the computation of $c_2(\mathcal{E}_a)$ can be reduced to the computation of $e_{orb}(\tilde{Y}, a\tilde{C} + \tilde{D} + E)$, the orbifold Euler number of the pair. In what follows we will pursue this latter approach, tacitly adopting Langer’s notation when necessary. Before proceeding further let us introduce:

**Notation 3.1.** For $i = 1, \ldots, s + r$ let $m_i$, $\delta_i$ and $\epsilon_i$ non negative integers such that

$$(3.2) \quad \tilde{C} = \pi^* \mu^* C - \sum_{i=1}^{s+r} m_i \overline{E}_i,$$

$$\tilde{D} = \pi^* \mu^* D - \sum_{i=1}^{s+r} \delta_i \overline{E}_i,$$

$$E = \sum_{i=1}^{s} \overline{E}_i - \sum_{j=s+1}^{s+r} \epsilon_j \overline{E}_j,$$

respectively. The above integers can be defined recursively, see the proof of Lemma 3.1, and $m_i \geq 1$, $i = 1, \ldots, s + r$, since at each step the blow up occurs in a point of a strict transform of $C$.  

Remark 3.2. Observe that
\[ E' = \sum_{i=1}^{s'} E_i - \sum_{j=s+1}^{s+r} \epsilon_j E_j , \]
and
\[ E'' = \sum_{i=s'+1}^{s} E_i - \sum_{j=s'+1}^{s+r} \epsilon_j E_j , \]
moreover
\[ \delta_j = 0 \text{ for } 1 \leq j \leq s' \text{ and } s+1 \leq j \leq s+r' . \]

Lemma 3.1. There exist integers \( x_i, i = 1, \ldots, s+r \) such that:
\[ K_{\bar{Y}} + \bar{D} + E = \pi^* \mu^*(K_X + D) + \sum_{i=1}^{s} x_i E_i \]
and
\[ x_i \geq 0 \text{ for } i = 1, \ldots, s+r , \]
\[ x_i = 2 \text{ for } 1 \leq i \leq s' , \]
\[ x_i \leq 1 \text{ for } s'+1 \leq i \leq s , \]
\[ x_j + \delta_j + \epsilon_j = 1 \text{ for } s+1 \leq j \leq s+r . \]

Proof. For \( k = 0, \ldots, r \), denote by \( D^k \) the strict transform of \( D \) in \( Y_k \) and by \( E_i^k, i = 1, \ldots, s+k \), the strict transform and the total transform in \( Y_k \) of the exceptional curve \( E_i \), \( i = 1, \ldots, k \) respectively. Where in particular \( E_{s+k}^k = E_{s+k}^0 \) and \( E_i^0 = E_i^0, i = 1, \ldots s \). It follows that for \( k = 0 \)
\[ D^0 = \mu^* D - \sum_{i=1}^{s} \delta_i E_i^0 , \]
and for \( k \geq 1 \)
\[ D^k = \pi_k^* \ldots \pi_1^* \mu^* D - \sum_{i=1}^{s+k} \delta_i E_i^k , \]
moreover
\[ E^k = \sum_{i=1}^{s} E_i^k - \sum_{j=s+1}^{s+k} \epsilon_j E_j^k , \]
where we put \( E^k = \sum_{i=1}^{s+k} E_i^k \). By the formula for the canonical divisor of a blow-up there exist integers \( x_1, \ldots, x_k \) such that:
\[ K_{Y_0} + D^0 + E^0 = \mu^* (K_X + D) + \sum_{i=1}^{s} x_i E_i^0 , \]
and
\[ x_i = \begin{cases} 2 & 1 \leq i \leq s' \\ 0 & s'+1 \leq i \leq s , \text{ and } E_i^0 \text{ maps onto a double point of } D \\ 1 & s'+1 \leq i \leq s , \text{ and } E_i^0 \text{ maps onto a smooth point of } D \end{cases} . \]

Arguing now by induction, put \( \pi_0 := id_Y \), suppose that for \( k \geq 0 \) we have
\[ K_{Y_k} + E_k^k + D^k = \pi_k^* \ldots \pi_1^* (K_X + D) + \sum_{i=1}^{s+k} x_i E_i^k , \]
then pulling back both sides of the above equality by \( \pi_{k+1} \) we obtain
\[ K_{Y_{k+1}} + E_{k+1}^{k+1} + D^{k+1} = \pi_{k+1}^* \ldots \pi_1^* (K_X + D) + \sum_{i=1}^{s+k} x_i E_i^{k+1} , \]
and since $E_{s+k+1}^{k+1} = E_{s+k+1}^{k+1}$
\[ K_{Y^{k+1}} + E_{s+k+1}^{k+1} + D^{k+1} = \pi_{k+1}^* \cdots \mu^*(K_X + D) + \sum_{i=1}^{s+k+1} x_i E_i^{k+1} \]

with $x_{s+k+1} = (1 - \epsilon_{s+k+1} - \delta_{s+k+1})$. Observe that $\epsilon_{s+k+1} + \delta_{s+k+1} \leq 1$, indeed after the first $s$ blow-ups, the center of the blow-up can not be a point where a component of the strict transform of $D$ and two exceptional curves meet since $D$ is assumed to be simple normal crossing. This concludes the proof.

**Notation 3.2.** $g(C)$, $g(D_k)$ denote the geometric genus of $C$, $D_k$, respectively.

**Remark 3.3.** In view of Definition 1.1, since $D$ is simple normal crossing and the topological Euler-Poincaré characteristic is additive under disjoint unions, we have

\[ e_{X \setminus D} = e_{\text{top}}(X) - \sum_i (2 - 2g(D_i)) + \sum_{i < j} D_i \cdot D_j , \]

in other words $e_{X \setminus D} = e_{\text{orb}}(X \setminus D)$, see for instance [Lan03, pag. 359]. Moreover considering the normalization $\eta: \tilde{C} \to C$ obtained by restricting $\pi \circ \mu$ to $\tilde{C}$, then

\[ e_{C \setminus D} = e_{\text{top}}(\tilde{C} \setminus \eta^{-1}(D)) = e_{\text{top}}(\tilde{C}) - \varepsilon(\eta^{-1}(D)) = 2 - 2g(C) - (E'' + D) \cdot \tilde{C} , \]

namely $e_{X \setminus D}$, $e_{C \setminus D}$ are both orbifold Euler numbers.

**Lemma 3.2.**

\[ c_2(\mathcal{E}_a) = d \left( e_{X \setminus D} - a e_{C \setminus D} + \sum_{i=1}^{s'} (am_i - 1) - \sum_{j=s+1}^{s+r'} am_j \right) \]

\[ c_1(\mathcal{E}_a) = f^* \left( \pi^* \mu^*(K_X + D + aC) + \sum_{i=1}^{s+r} (x_i - am_i) E_i \right) \]

and

\[ c_1^2(\mathcal{E}_a) = d \left( (K_X + D)^2 + 2a(K_X + D) \cdot C + a^2 C^2 - \sum_{i=1}^{s+r} (x_i - am_i)^2 \right) , \]

where $d$ denotes the degree of the finite map $f$.

**Proof.** Let us start by computing the second Chern class. By Remark 3.1, $\mathcal{E}_a = f^* \Omega_Y \left[ \log(\alpha \tilde{C} + D + E) \right]$ and then by [Lan03, Theorem 3.6] we have that

\[ c_2(\mathcal{E}_a) = d \cdot e_{\text{orb}}(\tilde{Y}, \alpha \tilde{C} + D + E) . \]

Denote by $\mathcal{I}$ the set of singular points of $\text{Supp} \ a\tilde{C} + D + E$ and observe that in the present situation local orbifold Euler numbers are zero. Indeed, local orbifold Euler numbers depend only on the analytic type of the singularity [Lan03, p. 366, 3.2], and they are zero in the case of normal crossings [Lan03, Lemma 3.3]. Since each blow-up increases the topological Euler number of the surface by one, it follows that [Lan03, Definition 3.1] reads

\[ e_{\text{orb}}(\tilde{Y}, \alpha \tilde{C} + D + E) = e_{\text{top}}(X) + r + s + \sum_k (2 - 2g(D_k)) - 2r - 2s \]

\[ + 2 \cdot (\mathcal{I} \setminus \tilde{C}) + \varepsilon(\mathcal{I} \cap \tilde{C}) - a \left( 2 - 2g(C) - \varepsilon(\mathcal{I} \cap \tilde{C}) - \varepsilon(\mathcal{I} \setminus \tilde{C}) \right) + \varepsilon(\mathcal{I} \setminus \tilde{C}) = \]

\[ e_{\text{top}}(X) + \sum_k (2 - 2g(D_k)) - r - s + \varepsilon(\mathcal{I} \setminus \tilde{C}) - a \left( 2 - 2g(C) - \varepsilon(\mathcal{I} \cap \tilde{C}) \right) . \]

The number of double points of $E$ is equal to $r$ by construction and then

\[ \varepsilon(\mathcal{I} \setminus \tilde{C}) = r + E \cdot \tilde{D} + \sum_{i<j} \tilde{D}_i \cdot \tilde{D}_j , \]

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as a consequence we have
\[ c_2(\mathcal{E}_a) = d \left( e_{\text{top}}(X) - s - \sum_k (2 - 2g(D_k)) + E \cdot \tilde{D} + \sum_{i<j} \tilde{D}_i \cdot \tilde{D}_j + \alpha \left( 2 - 2g(C) \right) + \alpha (E + \tilde{D}) \cdot \tilde{C} \right). \]  
(3.12)

Observe that \( \sum_{i<j} D_i \cdot D_j \) equals the number of double points of \( D \). Each blow-up in a double point of \( D \) increases \( E \cdot \tilde{D} \) by two and a blow-up in a smooth point by one. It follows that
\[ \sum_{i<j} D_i \cdot D_j = E \cdot \tilde{D} - (s - s') + \sum_{i<j} \tilde{D}_i \cdot \tilde{D}_j \]
and then in view of (3.10), (3.11) and (3.12) we finally have
\[ c_2(\mathcal{E}_a) = d \left( e_{X \setminus D} - \alpha e_{C \setminus D} + \alpha E' \cdot \tilde{C} - s' \right). \]

Since
\[ E' \cdot \tilde{C} = \sum_{i=1}^{s'} E_i - \sum_{j=s+1}^{s+s'} e_j E_j \cdot \left( \pi^* \mu^* C - \sum_{i=1}^{s+r} m_i E_i \right) = \sum_{i=1}^{e'} m_i - \sum_{j=s+1}^{s+s'} m_j e_j \]
we can conclude that
\[ c_2(\mathcal{E}_a) = d \left( e_{X \setminus D} - \alpha e_{C \setminus D} + \sum_{i=1}^{e'} (am_i - 1) - \sum_{j=s+1}^{s+s'} am_j e_j \right). \]

Regarding \( c_1(\mathcal{E}_a) \), note that \( c_1(\mathcal{E}_a) = f^* \left( K_{\tilde{Y}} + \tilde{D} + E + \alpha \tilde{C} \right) \). For instance, this can be deduced by a direct computation from (3.1). In view of (3.2) and (3.6) we have that
\[ c_1(\mathcal{E}_a) = f^* \left( K_{\tilde{Y}} + \tilde{D} + E + \alpha \tilde{C} \right) \]
\[ = f^* \left( \pi^* \mu^*(K_X + D) + \sum_{i=1}^{s+r} \pi_i E_i + \alpha \pi^* \mu^* C - \sum_{i=1}^{s+r} am_i E_i \right) \]
\[ = f^* \left( \pi^* \mu^*(K_X + D + \alpha C) + \sum_{i=1}^{s+r} (x_i - am_i) E_i \right) \]
and then
\[ c_1^2(\mathcal{E}_a) = d \left( (K_X + D)^2 + 2\alpha (K_X + D) \cdot C + \alpha^2 C^2 - \sum_{i=1}^{s+r} (x_i - am_i)^2 \right). \]

\[ \square \]

3.3. **Adjunction formula.** Following the notation introduced in the proof of Lemma 3.1, let us denote by \( D^0, C^0 \) the strict transforms by the blow-up \( \mu \) of \( D \) and \( C \), respectively. We have
\[ \tilde{D} = \pi^* D^0 - \sum_{j=s+1}^{s+r} \delta_j E_j = \pi^* D^0 - \sum_{j=s+r+1}^{s+r} \delta_j E_j, \]
(3.13)
\[ \tilde{C} = \pi^* C^0 - \sum_{j=s+1}^{s+r} m_j E_j, \]
(3.14)
where the second equality in (3.13) follows by (3.5).

**Proposition 3.3.**

\[ (K_X + D) \cdot C + C^2 = -e_{C \setminus D} + \sum_{i=1}^{s} m_i (m_i - x_i + 1) + \sum_{j=s+1}^{s+r} m_j (m_j - 1) + \sum_{j=s+r+1}^{s+r} m_j (e_j + \delta_j). \]
Proof. By (3.6) and (3.2) we have

$$\tag{3.15} (K_{\hat{Y}} + \hat{D} + E + \hat{C}) \cdot \hat{C} = (K_X + D) \cdot C + C^2 + \sum_{i=1}^{s+r} m_i(x_i - m_i),$$

observe that by adjunction:

$$2g(C) - 2 = (K_{\hat{Y}} + \hat{C}) \cdot \hat{C},$$

moreover by (3.2) and (3.3):

$$E \cdot \hat{C} = \left( \sum_{i=1}^{g} E_i - \sum_{j=s+1}^{s+r} \epsilon_j E_j \right) \cdot \left( \pi^* \mu^* C - \sum_{i=1}^{s+r} m_i E_i \right) = \sum_{i=1}^{g} m_i - \sum_{j=s+1}^{s+r} m_j \epsilon_j,$$

and by (3.13) and (3.14):

$$\hat{D} \cdot \hat{C} = \left( \sum_{i=s+1}^{g} E_i - \sum_{j=s+1}^{s+r} \epsilon_j E_j \right) \cdot \left( \pi^* C^0 - \sum_{j=s+1}^{s+r} m_j \delta_j E_j \right) = D^0 \cdot C^0 - \sum_{j=s+1}^{s+r} m_j \epsilon_j.$$

Substituting the above equalities and (3.9), (3.11) in (3.15) we get:

$$(K_X + D) \cdot C + C^2 = -e_{C \cdot D} D^0 \cdot C + (\hat{D} \cdot \hat{C}) + \sum_{i=1}^{s} m_i(x_i - m_i) + \sum_{j=s+1}^{s+r} m_j (m_j - 1),$$

but by (3.4), (3.13) and (3.14):

$$\langle E'' + \hat{D} \rangle \cdot \hat{C} = \left( \sum_{i=s+1}^{g} E_i - \sum_{j=s+1}^{s+r} \epsilon_j E_j + \pi^* D^0 - \sum_{j=s+1}^{s+r} \delta_j E_j \right) \cdot \left( \pi^* C^0 - \sum_{j=s+1}^{s+r} \delta_j E_j \right) = D^0 \cdot C^0 - \sum_{j=s+1}^{s+r} m_j (\epsilon_j + \delta_j),$$

and this concludes the proof of the Proposition.  \qed

4. PROOF OF THEOREM 1.1

Given $X$, $D$, $C$ and $\alpha$ as in the statement of Theorem 1.1, let us apply the construction of Section 3, refer to §3.1 for the notation regarding resolution data. We obtain a smooth surface $\hat{Y}$, a finite morphism $f : Z \to \hat{Y}$, an orbibundle $\mathcal{E}_\alpha$ defined on $Z$, etc. Consider the divisor

$$D_a := K_{\hat{Y}} + \hat{D} + E + a\hat{C} = \pi^* \mu^* (K_X + D) + \sum_{i=1}^{s+r} x_i E_i + a\hat{C}$$

$$= \pi^* \mu^* (K_X + D + a C) + \sum_{i=1}^{s+r} (x_i - a m_i) E_i,$$

where the second equality follows by (3.6) and the third one by (3.2). Since by hypothesis $K_X + D$ is $\mathbb{Q}$-effective, it follows by (3.7) and the second equality in (4.1) that $D_a$ is $\mathbb{Q}$-effective, then in turn the integral divisor $c_1(\mathcal{E}_\alpha) = f^* D_a$ is $\mathbb{Q}$-effective and admits Zariski decomposition. In order to prove Theorem 1.1, we might apply Theorem 2.6 to the bundle $\mathcal{E}_\alpha$. Indeed by Lemma 3.2 we have

$$\frac{1}{d} \left( 3e_2(\mathcal{E}_\alpha) - c^2_1(\mathcal{E}_\alpha) \right) = 3e_\mathbb{Q} \cdot D - (K_X + D)^2 - 2a \left[ (K_X + D) \cdot C + \frac{3}{2} e_{C \cdot D} \right] - \alpha^2 C^2$$

$$+ \sum_{i=1}^{s'} (3(\alpha m_i - 1) + (x_i - \alpha m_i)^2) + \sum_{i=s+1}^{g} (x_i - \alpha m_i)^2 + \sum_{j=s+1}^{s+r} \left( -3 \alpha m_j \epsilon_j + (x_i - \alpha m_i)^2 \right)$$

$$+ \sum_{j=s+r+1}^{s+r} (x_i - \alpha m_i)^2$$

and we might find a suitable upper bound for the sums in the right hand of the above inequality and finally replace it with the aid of Proposition 3.3. But from the third equality in (4.1), $D_a$ is
not in general nef on $E$ and analogous considerations hold for $c_1(\mathcal{E}_a)$ and $\text{Supp } f^*(E)$. Moreover, it is not clear how to, if possible at all, bound the summands on the right hand side of (4.2) by the summands on the right hand side of the equality in Proposition 3.3. In order to avoid the first problem, we could add the term $\frac{\ell N_a}{4}$ on the left hand side of (4.2), where $N_a$ stands for the negative part of the absolute Zariski decomposition of $D_a$, and consequently subtract a corresponding quantity to the summands on the right-hand side of the equality. But it turns out that the denominator 4 is too big to obtain the desired bound. In order to get rid of this denominator for as many terms as possible, following Miyaoka, we first perform a nef reduction of $\mathcal{E}_a$ contracting a part of the negative locus and finally apply Theorem 2.6 to the resulting bundle.

Consider the Zariski decomposition of $D_a = P_{G+F'}(D_a) + N_{G+F'}(D_a)$ into its $(G+F')$-nef and $(G+F'')$-negative part, respectively. In order to simplify notation set $P_a = P_{G+F'}(D_a)$ and $N_a = N_{G+F'}(D_a)$. By unicity of the Zariski decomposition we have that $f^*D_a = f^*P_a + f^*N_a$ gives the Zariski decomposition of $c_1(\mathcal{E}_a) = f^*\mu_2(D)$ into its $(\text{Supp } f^*(G+F'))$-nef and $(\text{Supp } f^*(G+F''))$-negative part, respectively. We are going to perform a nef reduction of $\mathcal{E}_a$ according to Proposition 2.5, but observe that the effective Q-divisor $f^*N_a$ with simple normal crossing support in not a priori necessarily integral. Consider then $\tilde{f}$, the composition of $f$ and a Kawamata covering, $[\text{Laz04}, 4.1.12]$, such that $\tilde{f}^*N_a$ is an integral divisor, $\text{Supp } \tilde{f}^*(C + \tilde{D})$ has again simple normal crossings and $\tilde{f}^*\mathcal{E}_a \subseteq \tilde{f}^*\Omega_2(\log C + \tilde{D}) \subseteq \Omega_2(\log(\text{Supp } f^*(C + \tilde{D})))$. With a slight abuse of notation we will still denote $\tilde{f}$ by $f$ and $\tilde{f}^*\mathcal{E}_a$ by $\mathcal{E}_a$ since this does not generate any confusion. Indeed observe that all the results in Section 3 are still valid with $\tilde{f}^*\mathcal{E}_a$ in place of $\mathcal{E}_a$ and $\tilde{f}$ in place of $f$.

Let us apply Proposition 2.5 to the map $\rho := \mu_2 \circ \pi \circ f$, $\rho : Z \to Y'$, and the rank two vector bundle

$$\mathcal{E}_a \subseteq f^*\Omega_Y(\log C + \tilde{D}) \subseteq \Omega_2(\log(\text{Supp } f^*(C + \tilde{D}))).$$

With the notation introduced in the Proposition, if we set

$$\Gamma = E_1 + \cdots E_s,$$

and

$$\Lambda = \Gamma + \text{Supp } \mu_2^s(D)$$

the exceptional locus of $\rho$ is the simple normal crossing divisor

$$R = \text{Supp } f^*(F + G),$$

that in turn is contained in the simple normal crossing divisor

$$\Delta = \text{Supp } f^*(C + \tilde{D})$$

and moreover $\rho(R) \subseteq \Lambda$. Note that

$$\rho^*\Omega_Y(\log \Lambda) \subseteq \Omega_2(\log(\text{Supp } \rho^*(\Lambda)) \subseteq \mathcal{E}_a,$$

since $\text{Supp } \rho^*(\Lambda) = \text{Supp } (\mu_2 \circ \pi \circ f)^*(\Gamma + \text{Supp } \mu_2^s(D))$ does not contain $\text{Supp } f^*\tilde{C}$. We obtain a rank two vector bundle $\mathcal{E}_a = \mathcal{Q}_{\text{Supp } f^*(F + G)}(\mathcal{E}_a)$ such that $c_1(\mathcal{E}_a) = f^*P_a$ and $c_2(\mathcal{E}_a) = c_2(\mathcal{E}_a)$. Since $N_a$ is supported on $G + F''$, we set

$$(4.3) \quad N_a = \sum_{j=s+1}^{s+r} b_j E_j,$$

and we are in position to formulate the following:

---

5Recall that $N_a^2 \neq 0$.

6Observe indeed that the support of $N_a$ is contained in the support of $C + \tilde{D}$. 

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Proposition 4.1.

\[
\frac{1}{d} \left( 3c_2(\bar{\theta}_a) - c_1^2(\bar{\theta}_a) \right) = \frac{1}{d} \left( 3c_2(\theta_a) - c_1^2(\theta_a) + f^*N_a^2 \right) = 3e_X \cdot D - (K_X + D)^2 - 2a \left[ (K_X + D) \cdot C + \frac{3}{2} e_{C \backslash D} \right] - \alpha^2 C^2
\]

\[
+ \sum_{i=1}^{s} \left( 3(am_i - 1) + (x_i - am_i)^2 \right) + \sum_{i=s+1}^{s+r} \left( (x_i - am_i)^2 - b_i^2 \right)
\]

\[
+ \sum_{j=s+1}^{s+r} \left( -3am_j \epsilon_j + (x_j - am_j)^2 - b_j^2 \right) + \sum_{j=s+r+1}^{\infty} \left( (x_i - am_i)^2 - b_i^2 \right)
\]

Proof. In view of (4.3) and Lemma 3.2, the Proposition follows by a straightforward computation.

We are going to find an upper bound for each of the last three summations contained in the right hand of the equality in Proposition 4.1. First of all we have:

Lemma 4.2.

\[ b_j \geq x_j - am_j \text{ for } s' + 1 \leq j \leq s + r, \]

and then

(4.4)

\[ b_j^2 \geq \max(x_j - am_j, 0)^2 \text{ for } s' + 1 \leq j \leq s + r. \]

Proof. Let us prove the first inequality. If \( j \geq s' + 1 \) then the upport of the effective divisor \( \overline{E}_j \) is contained in the support of \( F'' + G \). It follows that:

\[ 0 \leq P_a \cdot \overline{E}_j = (D_a - N_a) \cdot \overline{E}_j = -x_j + am_j + b_j, \]

note indeed that the divisors \( \overline{E}_j, j = 1, \ldots, s + r \), are orthogonal with respect to the intersection product. Regarding (4.4), it is an immediate consequence of the first inequality given that \( b_j^2 \geq 0 \).

From Lemma 4.2 we deduce the following inequalities:

Corollary 4.3.

(4.5) \[(x_i - am_i)^2 - b_i^2 \leq \alpha^2 m_i (m_i - x_i) \text{ for } s' + 1 \leq i \leq s, \]

(4.6) \[-3am_j \epsilon_j + (x_j - am_j)^2 - b_j^2 \leq \alpha^2 m_j (m_j - 1 - \epsilon_j + \delta_j) \text{ for } s + 1 \leq j \leq s + r', \]

(4.7) \[(x_i - am_i)^2 - b_i^2 \leq \alpha^2 m_j (m_j - 1 + \epsilon_j + \delta_j) \text{ for } s + r' + 1 \leq j \leq s + r. \]

Proof. Let us start proving (4.5). If \( x_i - am_i \leq 0 \) then \( x_i \leq am_i \) and by (4.4) we have

\[
(x_i - am_i)^2 - b_i^2 \leq x_i^2 - 2am_i x_i + \alpha^2 m^2 \leq am_i x_i - 2am_i x_i + \alpha^2 m^2 = \alpha^2 m_i (m_i - x_i).
\]

If \( x_i - am_i > 0 \) then by (4.4), \(-b_i^2 \leq -(x_i - am_i)^2 \) and (4.5) becomes \( 0 \leq \alpha^2 m_i (m_i - x_i) \), which is true since \( x_i \leq 1 \) for \( s' + 1 \leq i \leq s \) by (3.8) of Lemma 3.1. Let us consider (4.6). First of all, observe that, for \( s + 1 \leq j \leq s + r' \), by (3.9) of Lemma 3.1 we have that \( x_j + \delta_j + \epsilon_j = 1 \), moreover \( \delta_j = 0 \), see (3.5), since blow-ups occur in points that do not map onto \( D \), it follows that \( x_j = 1 - \epsilon_j \). If \( x_i - am_i \leq 0 \), again by Lemma 4.2 we have that:

\[
-3am_j \epsilon_j + (x_j - am_j)^2 - b_j^2 \leq -3am_j \epsilon_j + (x_j - am_j)^2 = -3am_j \epsilon_j + (1 - \epsilon_j - am_j)^2 = -3am_j \epsilon_j + (1 - \epsilon_j)^2 - 2(1 - \epsilon_j)am_j + \alpha^2 m^2 = \alpha^2 m_j^2 - (2 + \epsilon_j)(1 - \epsilon_j)am_j + (1 - \epsilon_j)am_j = \alpha^2 m_j^2 - (2 + \epsilon_j)(1 - \epsilon_j)am_j \leq \alpha^2 m_j^2 - \alpha^2 m_j = \alpha^2 m_j (m_j - 1).
\]
To conclude the proof of (4.6), observe that if \( x_j - am_j > 0 \) then (4.6) reduces to
\[
-3e_j am_j \leq a^2 m_j (m_j - 1)
\]
that is always true since the left hand of the inequality is always non positive and the right hand is always non negative. Finally let us prove (4.7) beginning again with the case \( x_j - am_j = 0 \). By (4.4) of Lemma 4.2 and Equation (3.9) of Lemma 3.1 we have:
\[
(x_j - am_j)^2 - b_j^2 = (x_j - am_j)^2 = x_j^2 - 2x_j am_j + a^2 m_j^2 \\
2a^2 m_j^2 - 2x_j am_j \leq a^2 m_j^2 - x_j am_j \leq a^2 m_j^2 - a^2 m_j x_j = \\
a^2 m_j (m_j - x_j) = a^2 m_j (m_j - 1 + \epsilon_j + \delta_j) = a^2 m_j (m_j - 1) + a^2 m_j (\epsilon_j + \delta_j).
\]
In case \( x_j - am_j > 0 \), as before, (4.7) reduces to
\[
0 \leq a^2 m_j (m_j - 1) + a^2 m_j (\epsilon_j + \delta_j),
\]
that is trivially true. □

Observe that \( P_a \) may not be nef on \( F + G \). Put \( \tilde{P}_a = P_{F + G}(P_a), \tilde{N}_a = N_{F + G}(P_a) \) and \( \overline{P}_a = P(P_a), \overline{N}_a = N(P_a) \). \( P_a = \tilde{P}_a + \tilde{N}_a \) is then the Zariski decomposition with support in \( F + G \) and \( P_a = \overline{P}_a + \overline{N}_a \) the absolute Zariski decomposition. Let us write \( \tilde{N}_a \) as a sum:\'
\[
\tilde{N}_a = \sum_{i=1}^{s+r} \hat{b}_i E_i.
\]

**Lemma 4.4.**
\[
\hat{b}_i \geq x_i - am_i \quad \text{for} \quad 1 \leq i \leq s',
\]
then
\[
\hat{b}_i^2 \geq (\max(x_i - am_i, 0))^2 \quad \text{for} \quad 1 \leq i \leq s'
\]
and
\[
(4.8) \quad \tilde{N}_a^2 \leq \tilde{N}_a^2 \leq - \sum_{i=1}^{s'} (\max(x_i - am_i, 0))^2.
\]

**Proof.** We follow the same argument as in the proof of Lemma 4.2. We have that
\[
0 \leq \tilde{P}_a \cdot E_i = (P_a - \tilde{P}_a) \cdot E_i = -x_i + am_i + \hat{b}_a,
\]
for \( i = 1, \ldots, s + r \), indeed \( \tilde{P}_a \) is nef on \( F + G \) and the support of the effective divisor \( E_i \), for \( i = 1, \ldots, s + r \), is contained in \( F + G \). It follows that \( \hat{b}_i \geq x_i - am_i \) and moreover, since \( \hat{b}_i^2 \) is non negative, \( \hat{b}_i^2 \geq (\max(x_i - am_i, 0))^2 \). Finally, observe that by Proposition 2.1 we have that \( \overline{N}_a^2 \leq \tilde{N}_a^2 \) and then
\[
\overline{N}_a^2 \leq \tilde{N}_a^2 = - \sum_{i=1}^{s+r} \hat{b}_i^2 \leq - \sum_{i=1}^{s'} \hat{b}_i^2 \leq - \sum_{i=1}^{s'} (\max(x_i - am_i, 0))^2,
\]
that proves (4.8) and conclude the proof of the Lemma. □

We can summarize the above computations in the following:

**Proposition 4.5.**
\[
\frac{1}{d} \left[ 3c_2(\tilde{E}_a) - c_1^2(\tilde{E}_a) + \left( f^* \tilde{N}_a \right)^2 \right] \leq 3e X \cdot D - (K_X + D)^2 \leq 3e C \cdot D - (K_X + D)^2 + \frac{\alpha^2}{2} \left[ C^2 + (3K_X + D) \cdot C + 3e C \cdot D \right].
\]

\( ^7 \)Since by construction \( P_a \) is nef on \( F' + G \), we could write \( \tilde{N}_a = \sum_{i=1}^{s'} \hat{b}_i E_i + \sum_{j=1}^{s} \hat{b}_d E_{d+j} \), indeed the remaining \( \hat{b}_i \) are certainly zero. Since we will need to bound \( \hat{b}_i, i = 1, \ldots, s' \), this would have no impact in what follows.
Proof. Combining Proposition 4.1, Corollary 4.3 and (3.8) we get:

\[
\frac{1}{d} \left( 3c_2(\vec{\eta}_a) - c_1^2(\vec{\eta}_a) + \frac{(f_\ast \bar{N}_a)^2}{4} \right) \leq 3e_{X\setminus D} - (K_X + D)^2 - 2a \left( (K_X + D) \cdot C + \frac{3}{2} e_{C\setminus D} \right) - a^2 C^2
\]

\[
(4.10)
\]

\[+ \sum_{i=1}^{s'} \left( 3(ami - 1) + (2 - am_i)^2 - \frac{1}{4}(\max(2 - am_i, 0))^2 \right) + \sum_{i=s+1}^{s+s'} \alpha^2 \rho_i(m_i - x_i) + \sum_{j=s+1}^{s+r} \alpha^2 \sigma_j(m_j - 1) + \sum_{j=s+r+1}^{s+r} \alpha^2 \sigma_j(m_j + \epsilon_j + \delta_j) .
\]

We are going to bound the terms \(3(ami - 1) + (2 - am_i)^2 - \frac{1}{4}(\max(2 - am_i, 0))^2\) in the above sum. First of all observe that

\[
3(ami - 1) + (2 - am_i)^2 = 1 - am_i + a^2 m_i^2,
\]

following Miyaoka, we claim that

\[
(4.11)
\]

\[
4 \left( 3(ami - 1) + (2 - am_i)^2 \right) - (\max(2 - am_i, 0))^2 = 4(1 - am_i + a^2 m_i^2) - (\max(2 - am_i, 0))^2 \leq 6a^2 m_i(m_i - 1),
\]

for \(i = 1, \ldots, s'.\) If \(2 > am_i, (4.11)\) reduces to \(3a^2 \leq 6a^2 m_i(m_i - 1)\) that it is true thanks to the fact that \(m_i \geq 2\) for \(i = 1, \ldots, s'.\) Consider now the case \(2 \leq am_i,\) then

\[
4(1 - am_i + a^2 m_i^2) - 6a^2 m_i(m_i - 1) = 2am_i - 4am_i + 4a^2 m_i^2 - 6a^2 m_i(m_i - 1) = -2am_i + 4a^2 m_i^2 - 6a^2 m_i(m_i - 1) = 2am_i(-1 + am_i - 3am_i + 3) = 2am_i(3a - 1 - am_i) \leq 2am_i(3 - 1 - am_i) = 2am_i(2 - am_i) \leq 0 ,
\]

and this completes the proof of (4.11). In view of (4.11), the inequality (4.10) becomes

\[
\frac{1}{d} \left( 3c_2(\vec{\eta}_a) - c_1^2(\vec{\eta}_a) + \frac{(f_\ast \bar{N}_a)^2}{4} \right) \leq 3e_{X\setminus D} - (K_X + D)^2 - 2a \left( (K_X + D) \cdot C + \frac{3}{2} e_{C\setminus D} \right) - a^2 C^2 + \sum_{i=1}^{s'} \frac{3}{2} a^2 \rho_i(m_i - 1) + \sum_{i=s+1}^{s+s'} a^2 \rho_i(m_i - x_i) + \sum_{j=s+1}^{s+r} \alpha^2 \sigma_j(m_j - 1) + \sum_{j=s+r+1}^{s+r} a^2 \sigma_j(m_j + \epsilon_j + \delta_j) \leq 3e_{X\setminus D} - (K_X + D)^2 - 2a \left( (K_X + D) \cdot C + \frac{3}{2} e_{C\setminus D} \right) - a^2 C^2 + \frac{3}{2} a^2 \left[ \sum_{i=1}^{s} m_i(m_i - x_i + 1) + \sum_{j=s+1}^{s+r} m_j(m_j - 1) + \sum_{j=s+r+1}^{s+r} m_j(\epsilon_j + \delta_j) \right] ,
\]

combining the above inequality and Proposition 3.3 we obtain (4.9) and this concludes the proof of the Proposition. \[\square\]

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. With the notation introduced above, recall that by construction we have that \(\text{Supp} f_\ast(C + D)\) is a simple normal crossing effective divisor,

\[
\bar{\eta}_a \subseteq \eta_a \subseteq f_\ast \Omega_Z \left( \log C + D \right) \subseteq \Omega_Z \left( \log(\text{Supp} f_\ast(C + D)) \right) ,
\]
moreover \( c_1(\bar{\ell}_a) = P_a \) is \( \mathbb{Q} \)-effective. We can then apply Bogomolov-Miyaoka-Yau inequality, in the form of Theorem 2.6, to \( \bar{\ell}_a \) obtaining
\[
\left( 3c_2(\bar{\ell}_a) - c_1^2(\bar{\ell}_a) + \frac{(f^*N_a)^2}{4} \right) \geq 0.
\]
In view of the above inequality Theorem 1.1.(i) follows directly by Proposition 4.5.

Let us prove Theorem 1.1.(ii). Consider the inequality (1.2), its left-hand side is a quadratic polynomial in \( \alpha \), since \( C \) is not a smooth \( D \)-rational curve and \((K_X + D) \cdot C \geq -\frac{3}{2}e_{X \setminus D}\) the leading coefficient is non positive. Indeed since \( C \) is not a smooth \( D \)-rational curve we have that
\[
C^2 + (K_X + D) \cdot C = 2g(C) - 2 + \Sigma + D \cdot C \geq 0
\]
where \( \Sigma \geq 0 \) is the contribution of the singular points of \( C \) and \( g(C) \) denotes its geometric genus as usual. In view of \((K_X + D) \cdot C \geq -\frac{3}{2}e_{X \setminus D} \) we have then
\[
\tag{4.12} C^2 + 3(K_X + C) \cdot C + 3e_{C \setminus D} > 2(K_X + D) \cdot C + 3e_{C \setminus D} = 2(K_X + D) \cdot C + \frac{3}{2}e_{C \setminus D} > 0.
\]
It follows that the left-hand side of the inequality (1.1) attains its minimum in correspondence of
\[
\alpha_0 = \frac{2\left((K_X + D) \cdot C + \frac{3}{2}e_{C \setminus D}\right)}{C^2 + 3(K_X + C) \cdot C + 3e_{C \setminus D}}
\]
and since \( 0 \leq \alpha_0 \leq 1 \), see (4.12), substituting \( \alpha_0 \) in (1.1) we get
\[
2\left((K_X + D) \cdot C + \frac{3}{2}e_{C \setminus D}\right)^2
\]
\[
\frac{C^2 + 3(K_X + C) \cdot C + 3e_{C \setminus D}}{C^2 + 3(K_X + C) \cdot C + 3e_{C \setminus D}} + 3e_{X \setminus D} - (K_X + D)^2 \geq 0
\]
from which (1.2) and then Theorem 1.1.(ii) follows immediately. \( \square \)

5. PROOF OF THEOREM 1.2 AND COROLLARY 1.5

**Notation 5.1.** Suppose that \( K_X + D \) is a nef and big divisor or equivalently that \( (K_X + D) \) is nef and \((K_X + D)^2 > 0 \). We introduce the following notation:
\[
x = \frac{(K_X + D) \cdot C}{(K_X + D)^2},
\]
\[
\sigma = \frac{e_{X \setminus D}}{(K_X + D)^2},
\]
\[
\gamma = \frac{-\frac{3}{2}e_{C \setminus D}}{(K_X + D)^2},
\]
\[
y^2 = -\frac{(C - x(K_X + D))^2}{(K_X + D)^2} = -\frac{C^2}{(K_X + D)^2} + x^2.
\]

**Remark 5.1.**
(1) \( x \geq 0 \), since we suppose that \( K_X + D \) is nef.
(2) \( \sigma \geq \frac{1}{3} \). Indeed, since we suppose \( K_X + D \) big, it is then linearly equivalent to an effective \( \mathbb{Q} \)-divisor and the inequality follows applying for instance [Miy84, Corollary 1.2].
(3) \( y^2 \geq 0 \). Since \((K_X + D)^2 > 0 \) the inequality follows applying Hodge Index Theorem to \( K_X + D \) and \( E = [(K_X + D) \cdot C][(K_X + D) - (K_X + D)^2C]. \)
(4) \( \gamma \geq -1 \), by definition of \( e_{C \setminus D} \).

The following Proposition is a consequence of Theorem 1.1.

**Proposition 5.1.** Let \( X \) be a smooth projective surface, \( D \) a simple normal crossing divisor on \( X \) and \( C \) an irreducible curve on \( X \). Suppose that \( K_X + D \) is a nef and big divisor, \( C \) is not a smooth \( D \)-rational curve and moreover that \((K_X + D)^2 \) \( e_{X \setminus D} \). With Notation 5.1;
(1) If \((K_X + C) \cdot C > -\frac{3}{2} e_{C \cdot D}\) or equivalently \(x > 3\gamma\), we have that:

\[
\mathcal{P}(x) = (\sigma - 1)x^2 + (4\gamma + 3\sigma - 1)x - 2\gamma(3\gamma + 3\sigma - 1) \geq 0
\]

and \(x \leq R_+(\sigma, \gamma)\), where \(R_+(\sigma, \gamma)\) denotes the largest root of \(\mathcal{P}\):

\[
(5.1) \quad R_+(\gamma, \sigma) = \frac{4\gamma + (3\sigma - 1) + \sqrt{8(3\sigma - 1)^2 + 8\sigma(3\sigma - 1)\gamma + (3\sigma - 1)^2}}{2(1 - \sigma)}.
\]

(2) In general

\[
x \leq \max\{3\gamma, R_+(\sigma, \gamma)\} = R_+(\sigma, \gamma).
\]

**Proof.** Let us start proving statement (1). The hypotheses of Theorem 1.1 are satisfied, consider then (1.2), dividing by \(\left[(K_X + D)^2\right]^2 > 0\) and substituting Notation 5.1 we obtain:

\[
(\sigma - 1)x^2 + (4\gamma + 3\sigma - 1)x - 2\gamma(3\gamma + 3\sigma - 1) \geq \left(\sigma - \frac{1}{3}\right)\gamma^2,
\]

the left-hand side of the above inequality is \(\mathcal{P}(x)\) and the right-hand side is greater than or equal to zero by Remark 5.1. Observe that since the curve \(C\) is not a smooth \(D\)-rational curve \(\gamma \geq 0\) and then by Remark 5.1 the discriminant of the polynomial \(\mathcal{P}\) is greater than or equal to zero:

\[
\Delta = 8(3\sigma - 1)^2 + 8\sigma(3\sigma - 1)\gamma + (3\sigma - 1)^2 \geq 0,
\]

moreover since \(\sigma < 1\) the leading coefficient of \(\mathcal{P}\) is negative, it follows that \(x \leq R_+(\gamma, \sigma)\), and (5.1) follows by a straightforward computation. Statement (2) follows now immediately from part (1). Indeed, since \(\sigma \geq 1/3\) then \(\frac{1}{1 - \sigma} \geq \frac{3}{2}\) and

\[
R_+(\sigma, \gamma) \geq \frac{4\gamma}{2(1 - \sigma)} = \frac{2\gamma}{1 - \sigma} \geq 3\gamma.
\]

\[\square\]

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let us start by proving Theorem 1.2.(i). With Notation 5.1, by Proposition 5.1 and taking into account that \(\frac{3\sigma - 1}{8\gamma^2} - \frac{\sigma^2}{12\gamma^2} \leq 0\) we have that:

\[
x \leq R_+(\sigma, \gamma) = \frac{4\gamma + (3\sigma - 1) + 2\gamma \sqrt{2(3\sigma - 1)}}{2(1 - \sigma)} \leq 2\sigma + \frac{8\gamma}{\sqrt{2(3\sigma - 1)}} \leq \frac{4\gamma + (3\sigma - 1) + 2\gamma \sqrt{2(3\sigma - 1)}}{2(1 - \sigma)} \leq \frac{4\gamma + (3\sigma - 1) + 2\gamma \sqrt{2(3\sigma - 1)}}{2(1 - \sigma)}
\]

Substituting Notation 5.1 in the above inequality we finally obtain part 1.2.(i) of the Theorem.

We are going to prove Theorem 1.2.(ii) now. Let \(t = (K_X + D) \cdot C\), since \(C\) is smooth and \(D\), \(C\) meet transversally, \(C^2 + (K_X + D) \cdot C = -e_{C \cdot D}\), substituting the preceding inequalities in (1.2) we obtain

\[
(5.2) \quad 2t^2 + \left[6e_{C \cdot D} - 2\left(3e_{X \cdot D} - (K_X + D)^2\right)\right] t - 2e_{C \cdot D} \left(3e_{X \cdot D} - (K_X + D)^2\right) + \frac{9}{2} e_{C \cdot D}^2 \leq 0.
\]
As in the proof of the preceding Part of the Theorem, the leading coefficient of the above polynomial in $t$ is greater than or equal to zero, and its discriminant equals 
\[ \left(3e_{X \setminus D} - (K_X + D)^2\right)\left(2e_{C \setminus D} + 3e_{X \setminus D} - (K_X + D)^2\right). \]
If $C$ is not a smooth $D$-rational curve then $-e_{C \setminus D} \geq 0$, moreover by the log Bogomolov-Miyaoka-Yau inequality as in Miyaoka [Miy84, Corollary 1.2], the above discriminant is non-negative and finally $t$ is bounded above by the largest root of the polynomial in $(5.2)$. A straightforward computation gives now (1.4).

The proof of (1.5) follows exactly as per [Miy08, Remark A, pg. 405] but employs [Lan03, Theorem 0.1] so as to have the Bogomolov-Miyaoka-Yau inequality in the generality necessary to deal with $D \cap C \neq \emptyset$, indeed after contracting $C$ we may obtain a quotient singularity on the boundary divisor. Specifically the curve $C$ of (1.5) has self intersection $C^2 = -m$, so that on its contraction $(X_0, D_0)$ taking into account the contribution by the quotient singularity $\,^8$, if any,

\begin{align*}
(e_{X_0 \setminus D_0} &= \begin{cases} e_{X \setminus D} - 2 + \frac{1}{m} & \text{if } C \cap D = \emptyset, \\
e_{X \setminus D} & \text{otherwise} \end{cases} \\
(K_{X_0} + D_0)^2 &= \begin{cases} (K_X + D)^2 + \frac{(m-2)^2}{m} & \text{if } C \cap D = \emptyset, \\
(K_X + D)^2 + \frac{(m-1)^2}{m} & \text{otherwise} \end{cases} \end{align*}

while the canonical degree is given by

\begin{align*}
t = (K_X + D) \cdot C &= \begin{cases} m - 2 & \text{if } C \cap D = \emptyset, \\
m - 1 & \text{otherwise} \end{cases} \\
\end{align*}

so that from (5.3)-(5.4) the orbifold Bogomolov-Miyaoka-Yau inequality,

\[ (K_{X_0} + D_0)^2 \leq 3e_{X_0 \setminus D_0} \]

becomes in the notation of (5.4)

\begin{align*}
3e_{X \setminus D} - (K_X + D)^2 &\geq \begin{cases} \frac{t^2}{t+2} - \frac{3}{t+2} + 6 & \text{if } C \cap D = \emptyset, \\
\frac{t^2}{t+1} + 3 & \text{otherwise} \end{cases} .
\end{align*}

Observe that

\[ \frac{t^2}{t+2} - \frac{3}{t+2} + 6 \geq t + 3 \]

for every $t \geq 0$, moreover since $3e_{X \setminus D} - (K_X + D)^2$ is an integer, we have

\[ \left\lfloor \frac{t^2}{t+1} + 3 \right\rfloor \leq 3e_{X \setminus D} - (K_X + D)^2 , \]

but for $t \geq 0$:

\[ \left\lfloor \frac{t^2}{t+1} + 3 \right\rfloor = \left\lfloor \frac{t^2-1}{t+1} + \frac{1}{t+1} + 3 \right\rfloor = \left\lfloor t + 2 + \frac{1}{t+1} \right\rfloor = t + 3 \]

whence (1.5) from (5.5).

We conclude with

---

\(^8\)See [Lan03, p. 359] for the computation of the relative orbifold Euler numbers.
Proof of Corollary 1.5. First of all, observe that the degree of $K_{p_3} + D$ equals $(d_1 + d_2 - 3)$, since $d_2 = 6 > 3$ then $K_{p_2} + D$ is nef and $(K_{p_2} + D)^2 > 0$. Moreover,

$$D \cdot C = (d_1 + d_2) d > 1$$

$$e_{p_2 \setminus D} = 3 + d_1(d_1 - 3) + d_2(d_2 - 3) + d_1d_2$$

$$(K_{p_2} + D)^2 = (d_1 + d_2)^2 - 6(d_1 + d_2) + 9$$

and

$$(K_{p_2} + D)^2 - e_{p_2 \setminus D} = d_1d_2 - 3(d_1 + d_2) + 6 = \lambda d_2^2 - 3(\lambda + 1)d_2 + 6 = \lambda d_2 (d_2 - 3(\lambda + 1)) + 6 > 0$$

then the hypotheses of Theorem 1.2.(i) are satisfied. Expressing the quantities in Theorem 1.2.(i) in terms of the given data we obtain:

$$e_{p_2 \setminus D} = (\lambda^2 + \lambda + 1)d_2^2 - 3(\lambda + 1)d_2 + 3 \leq (\lambda^2 + \lambda + 1)d_2^2$$

$$e_{p_2 \setminus D} = (\lambda + 1)^2d_2^2 - 6(\lambda + 1)d_2 + 9 \leq (\lambda + 1)^2d_2^2$$

$$3e_{p_2 \setminus D} - (K_{p_2} + D)^2 = (2\lambda^2 + \lambda + 2)d_2^2 - 3(\lambda + 1)d_2 \leq (2\lambda^2 + \lambda + 2)d_2^2$$

$$-\frac{1}{2} e_{C \setminus D} \leq g - 1 + \frac{(d_1 + d_2)d}{2m} = g - 1 + \frac{(\lambda + 1)}{2m} v d_2^2$$

moreover

$$(K_{p_2} + D) \cdot C = (d_1 + d_2 - 3)d \geq \left(\lambda^2 + \frac{1}{2}\right)vd_2^2$$

$$(K_{p_2} + D)^2 - e_{p_2 \setminus D} = \lambda d_2^2 - 3(\lambda + 1)d_2 + 6 \geq \left(\frac{\lambda}{2} - \frac{1}{3}\right)d_2^2$$

and if we fix $\lambda_0 > \frac{2}{3}$, then for $\lambda \geq \lambda_0$ we have $\left(\frac{1}{2} - \frac{1}{3}\right)d_2^2 > 0$. We can now bound $A, B$ in (1.3) in terms of $\lambda$ and $d_2$, indeed substituting the above expressions in the definition of $A$ and $B$, we get

$$A \leq a(\lambda), \quad B \leq d_2^2 b(\lambda)$$

where

$$a(\lambda) = \frac{2(\lambda^2 + \lambda + 1)^2}{\left(\lambda^2 + \lambda + 1\right)^2}$$

and

$$b(\lambda) = \frac{\left[\lambda + 1\right]1^2(2\lambda^2 + \lambda + 2) + (\lambda + 1)^2(\lambda + 1)(\lambda + 1)}{2\left(\lambda^2 + \lambda + 1\right)^2}.$$ 

Summing up, we can rewrite (1.3) as

$$\left(\lambda + 1\right)v d_2^2 \leq ((\lambda + 1)d_2 - 3)v d_2 \leq a(\lambda) \left(g - 1 + \frac{(\lambda + 1)d_2^2}{2m}v\right) + d_2^2 b(\lambda)$$

from which rearranging terms and dividing by $d_2^2$ we get:

$$\left(\lambda + 1\right)v^2 \leq \frac{a(\lambda)}{d_2^2} (g - 1 + b(\lambda)).$$

If $m > \frac{(\lambda + 1)a(\lambda)}{\lambda + 1}$ from (5.6) we have:

$$\left(\frac{\lambda_0}{2} - \frac{1}{3}\right)\left(\frac{\lambda_0}{2} + \frac{1}{4}\right)v \leq \frac{4}{9} g + 22,$$
and this concludes the proof of the Corollary if we set $h = \frac{4}{9}$ and $k = 22$, indeed

$$\frac{(\lambda + 1)a(\lambda)}{(\lambda + \frac{1}{2})} \leq \frac{50}{(\frac{1}{2} - \frac{1}{3})}$$

if $\frac{2}{3} < \lambda \leq 1$. □

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Via Val Sillaro 5 - 00141 Roma

Email address: pietrsabat@gmail.com

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