The diachromatic number of double star graph

Nilamsari Kusumastuti, Raventino, Fransiskus Fran
Departments of Mathematics, Tanjungpura University, Jalan Prof. H. Hadari Nawawi, Pontianak, Indonesia
E-mail: nilamsari@math.untan.ac.id

Abstract. We are interested in the extension for the concept of complete colouring for oriented graph $\overrightarrow{G}$ that has been proposed in many different notions by several authors (Edwards, Sopena, and Araujo-Pardo in 2013, 2014, and 2018, respectively). An oriented colouring is complete if for every ordered pair of colours, at least one arc in $\overrightarrow{G}$ whose endpoints are coloured with these colours. The diachromatic number, $d_{ac}(\overrightarrow{G})$, is the greatest number of colours in a complete oriented colouring. In this paper, we establish the formula of diachromatic numbers for double star graph, $\overrightarrow{K_{1,n,n}}$, over all possible orientations on the graph. In particular, if $d_{in}(u) = 0$ (resp. $d_{out}(u) = 0$) and $d_{in}(w_i) = 1$ (resp. $d_{out}(w_i) = 1$) for all $i$, then $d_{ac}(\overrightarrow{K_{1,n,n}}) = \lceil \sqrt{n} \rceil + 1$, where $u$ is the internal vertex and $w_i$, $i \in \{1, \ldots, n\}$, is the pendant vertices of the digraph.

1. Introduction
One of the most well-known problems in graph theory is the vertex colouring problem, which traces its origin to 1852 when Francis Guthrie observed a map of England’s countries. As a result, the map can be coloured using four colours so that two countries that share the same boundaries have coloured with different colours. This problem is known as the Four Colour Problem, which questioned whether any map could be coloured only using four colours. This question received much attention over centuries before Appel and Hakes answered with certainty [1] in 1976.

An $n$-colouring of an undirected graph $G$ is a function $\tau$ from $V(G)$ to $\mathbb{N} = \{1, 2, \ldots, n\}$ such that whenever $u, v \in V(G)$ are adjacent then $\tau(u) \neq \tau(v)$. This $n$-colouring is complete if for every pair of distinct colours, there exist two adjacent vertices that are assigned with these colours. The chromatic number $\chi(G)$ of a graph $G$ is the smallest number $n$ such that $G$ has an $n$-colouring. The achromatic number $\psi(G)$ of a graph $G$ is the greatest number $m$ such that $G$ has a complete $m$-colouring. More than half a century since Harary, Hedetniemi, and Prins first introduced it [2], the concept of achromatic numbers has been studied intensively and continuously developed in graph theory. See [3, 4, 5, 6] for more reference results related to this parameter.

The concept of complete colouring has been generalized for an oriented graph or digraph $\overrightarrow{G}$ in recent studies. There are several interesting extensions of complete-oriented colouring. One by Edwards [7] in 2013, one by Sopena [8] in 2014, and another one by Araujo-Pardo [9] in 2018. Edwards considered the extension in terms of its underlying graph, while Sopena proposed another extension in terms of complete homomorphism. Araujo-Pardo introduced the extension in terms of acyclic colouring. In all extensions, it is proven that the interpolation property does not hold. Moreover, Edward stated that a digraph does not necessarily have a completed colouring, and determining whether one exists is an NP-complete problem. See [10, 11] for
more results. If for a digraph $\overrightarrow{G}$ there exists a complete colouring, then the greatest number of colours that suits that colouring is called the \textit{diachromatic number}, denoted by $\text{dac}(\overrightarrow{G})$.

This paper determines the formula for diachromatic numbers in a double star graph over all possible orientations on the graph. Moreover, we only considered the orientation in which the diachromatic number exists.

1.1. General notation and definition

This part gives formal definitions of the terms oriented complete coloring, degree of the vertices in a digraph, and introduces some further notation and terminology.

Let $\overrightarrow{G}$ be an oriented graph of $G$ obtained from an undirected graph by giving each edge in $E(G)$ one of its possible orientation $\overrightarrow{uv}$ or $\overrightarrow{vu}$. The \textit{vertex set} of $\overrightarrow{G}$ is the vertex set of $G$ and the \textit{edge set} of $\overrightarrow{G}$ become the set of oriented edge called by \textit{arc set} is denoted by $A(\overrightarrow{G})$. Element of $A(\overrightarrow{G})$ is called an arc $(u, v)$, that is a directed edge from $u$ to $v$. The vertex $u$ is called the \textit{head}, and $v$ is called the \textit{tail} of the arc. \textit{Order} of digraph $\overrightarrow{G}$ is the number of vertices in $\overrightarrow{G}$, and \textit{size} of digraph $\overrightarrow{G}$ is the number of arc in $\overrightarrow{G}$ [12].

A vertex $w$ is an \textit{in-neighbour} (\textit{out-neighbour}, respectively) of $v$ if $w \rightarrow v$ ($v \rightarrow w$, respectively) is an arc of the digraph. The \textit{in-degree} of $v$, $d_{\text{in}}(v)$, (\textit{out-degree} of $v$, $d_{\text{out}}(v)$, respectively) is the number of in-neighbours (out-neighbours, respectively) that it has [13].

An oriented colouring $\tau$ is complete if for every ordered pair $(i, j)$ there exist an arc $(u, v)$ such that $\tau(u) = i$ and $\tau(v) = j$. A vertex colouring $\sigma$ is an acyclic colouring if every chromatic class $\sigma^{-1}(i)$ induces a sub-digraph with no directed cycles. We define the diachromatic number as the greatest number of colours in a complete acyclic colouring.

In general, a digraph $\overrightarrow{G}$ is not symmetric. Assume that $\omega$ is the number of colours used in complete colouring. Since in a complete colouring, each ordered pair of colours must be assigned to an arc, and then the possible numbers of colour pairs that have to assign, i.e., the permutation $2$ of $\omega$, must be less then the size of $\overrightarrow{G}$ or

$$P_2^\omega \leq |A(\overrightarrow{G})|. \quad (1)$$

1.2. Double star graph

The star graph is the complete bipartite graph $K_{1,n}$ with one internal vertex, i.e. the vertex of degree $n$, and $n$ pendant vertices, i.e. the vertices of degree 1. The double star graph is a star graph $K_{1,n,n}$ obtained $K_{1,n}$ from adding a new pendant edge of the existing $n$ pendant vertices. The graph $K_{1,n,n}$ has $2n + 1$ vertices and $2n$ edges [14]. Let $V(K_{1,n,n}) = \{u\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{w_1, w_2, \ldots, w_n\}$. We represent in Figure 1 the double star graph $K_{1,3,3}$ and $K_{1,1,3}$.

We give an example how to assign a complete oriented colouring for $\overrightarrow{K_{1,3,3}}$ as follows. First we set some direction in $\overrightarrow{K_{1,3,3}}$ as represented in Figure.

The colouring in Figure 2(b) is complete since for every ordered pair of colours $(1, 2)$ and $(2, 1)$; there exists an arc that is assigned to these colours. Conversely, the colouring in Figure 2(c) is not complete since no arc that is assigned to the colour pairs $(3, 2)$. v

2. Method

The starting point for determining the diachromatic number of oriented double star graph $K_{1,n,n}$ is to assign the complete colouring for $n = 2$. We set a certain direction on the graph, apply the complete colouring to the graph, and then analyse whether the applied colour has been maximized. With the same steps, we try for $n = 3$ and so on. From these results, we determine
3. Results and Discussion

Observe that $K_{1,n,n}$ is a tree, and then the condition acyclic colouring is superfluous. Therefore, we need only consider the complete colouring condition to determine the dichromatic number for $\overrightarrow{K_{1,n,n}}$.

Let $\overrightarrow{K_{1,n,n}}$ be an oriented graph of $K_{1,n,n}$ and $\lfloor n \rfloor$ denotes the greatest integer less than or equal to $n$. We obtained the following for one of the orientation cases in $\overrightarrow{K_{1,n,n}}$.

**Theorem 1.** Let $u$ be the internal vertex and $w_i, i \in \{1, \ldots, n\}$, be the pendant vertices of $\overrightarrow{K_{1,n,n}}$. If $d_{in}(u) = 0$ (resp. $d_{out}(u) = 0$) and $d_{in}(w_i) = 1$ (resp. $d_{out}(w_1) = 1$) for all $i$, then

$$dac(\overrightarrow{K_{1,n,n}}) = \lfloor \sqrt{n} \rfloor + 1.$$ 

**Proof.** We give an illustration in Figure 3 for this first case of orientation.

Assume that $\omega$ is the number of colours used in a complete colouring, and we assign colour 1 to the internal vertex $u$. Then all the vertices $v_i$ can be coloured by $j$, with $j = 2, \ldots, \omega$. The ordered pairs that have been used to coloured the arc $(u, v_i)$ are of the form $(1, j), j = 2, \ldots, \omega$, and their number is $\omega - 1$. Those ordered pairs can not be used to colour the arc $(v_i, w_i)$. Consequently, and by (1), the number of ordered colour pairs remaining is $P_T^2 - (\omega - 1)$, and the number of uncoloured arcs is half the size of the digraph. We have
Figure 3. (a) The case for $d_{in}(u) = 0$ and for all $i = 1, \ldots, n$, $d_{in}(w_i) = 1$, (b) The case for $d_{out}(u) = 0$ and for all $i = 1, \ldots, n$, $d_{out}(w_i) = 1$

\[
P_2^ω - (ω - 1) \leq \frac{|A(\overrightarrow{K_{1,n,n}})|}{2}
\]

Suppose that the largest integer value of $ω$ that satisfies the inequality (2) is $t$. Thus,

\[
P_2^t - (t - 1) \leq \frac{2n}{2} \Rightarrow t(t - 1) - (t - 1) \leq n
\]
\[
\Rightarrow (t - 1)^2 \leq n
\]
\[
\Rightarrow (t - 1) \leq \sqrt{n}
\]
\[
\Rightarrow t \leq \sqrt{n} + 1
\]

Since $t$ is the largest integer, then we use the floor function. Hence

\[
t = \lfloor \sqrt{n} + 1 \rfloor = \lfloor \sqrt{n} \rfloor + 1,
\]

whence the statement.

The proof is complete.

Recall that a dipath in a directed graph is a finite sequence of arcs in the same orientation, which joins a sequence of distinct vertices, and the length of dipath is the number of arcs in that dipath. The distance between two vertices is the length of the shortest path between them, and the diameter of a connected digraph the largest distance between pairs of vertices of the digraph [15].

We summarize the results for other orientations in the following theorem.

**Theorem 2.** Let $\overrightarrow{K_{1,n,n}}$ be oriented double star graph of order $2n + 1$ and size $2n$. Let $u$ be the internal vertex and $w_i$, $i \in \{1, \ldots, n\}$, be the pendant vertices of $\overrightarrow{K_{1,n,n}}$.

(i) If the diameter of $\overrightarrow{K_{1,n,n}}$ equal to one, then there is no complete colouring in $\overrightarrow{K_{1,n,n}}$.

(ii) If there is exactly one dipaths of length 2 then $\text{dac}(\overrightarrow{K_{1,n,n}}) = 2$.

(iii) If there are at least $\lfloor \sqrt{n} \rfloor$ dipaths of length 2 that have tail-end or head-end in $u$, then $\text{dac}(\overrightarrow{K_{1,n,n}}) = \lfloor \sqrt{n} \rfloor + 1$.

(iv) If there are $m$, $2 \leq m < \lfloor \sqrt{n} \rfloor$, dipaths of length 2 that have tail-end or head-end in $u$, then $\text{dac}(\overrightarrow{K_{1,n,n}}) = m + 1$. 
**Proof.** Without loss of generality, we give some illustration of the orientation in each case in Figure 4. We can get a similar situation for each case by reversing all the directions.

(i) Suppose that we assign colour 1 to the internal vertex \( u \). Then no arc can be assigned for the ordered pairs of the form \((1, j)\), \( j \neq 1 \), since there is no arc that head-end in \( u \). Thus, complete colouring does not exist in this case. Observe that this is also the case for the opposite direction.

(ii) Suppose that the diachromatic number of this case is equal to three, and we assign colour 1 to the internal vertex \( u \). Then the arc \((u, v_1)\) and \((v_1, w_1)\) are assigned by the pairs \((1, 2)\) and \((2, 1)\), respectively. Observe that no arc can be assigned for the ordered pair \((3, 1)\). Hence, we can not use three colours in this complete colouring, therefore \( \text{dac}(K_{1,n,n}) = 2 \).

(iii) Assume that \( \sigma = \lfloor \sqrt{n} \rfloor \), and we assign colour 1 to the internal vertex \( u \). Then for the vertices \( v_i, i = 1, \ldots, n \) can be coloured by \( j \), with \( j = 2, \ldots, \sigma + 1 \), and the vertices \( w_k, k = 1, \ldots, \sigma \) can be coloured by 1. The ordered pairs that have been used to coloured the arc \((u, v_i)\) and \((v_k, w_k)\) are of the form \((1, j)\) and \((j, 1)\), respectively, with \( j = 2, \ldots, \sigma + 1 \), and their numbers are \(2\sigma\). Observe that the number of ordered colour pairs remaining is \( \sigma^2 - \sigma \), and the number of uncoloured arcs is \( n - \sigma \). It is easy to see that \( \sigma^2 - \sigma \leq n - \sigma \) since we assume that \( \sigma = \lfloor \sqrt{n} \rfloor \). Hence by (1) the colouring with \( \lfloor \sqrt{n} \rfloor + 1 \) in this case is complete colouring.

(iv) By changing \( \sigma = m \) with \( 2 \leq m < \lfloor \sqrt{n} \rfloor \) and with the same kind of reasoning as in case (iii), whence the statement.

The proof is complete.

![Figure 4](https://example.com/figure4.png)
4. Conclusion
In this work, we studied the notions of complete colouring in the oriented graph. We introduced the diachromatic number as the greatest number of colours in a complete oriented. We provided the results on double star graph $K_{1,n,n}$ of order $2n+1$ and size $2n$ over all possible orientation as follows, and we only considered the orientation in which the diachromatic number exists.

Let $u$ be the internal vertex and $w_i$, $i \in \{1,\ldots,n\}$, be the pendant vertices of $K_{1,n,n}$.

(i) If $d_{in}(u) = 0$ (resp. $d_{out}(u) = 0$) and $d_{in}(w_i) = 1$ (resp. $d_{out}(w_1) = 1$) for all $i$, then
$$\text{dac}(K_{1,n,n}) = \lfloor \sqrt{n} \rfloor + 1.$$

(ii) If $d_{out}(u) = 0$ and $d_{out}(w_1) = 1$ for all $i$, then
$$\text{dac}(K_{1,n,n}) = \lfloor \sqrt{n} \rfloor + 1.$$

(iii) If there is exactly one dipaths of length 2 then
$$\text{dac}(K_{1,n,n}) = 2.$$

(iv) If there are at least $\lfloor \sqrt{n} \rfloor$ dipaths of length 2 that have tail end or head end in $u$, then
$$\text{dac}(K_{1,n,n}) = \lfloor \sqrt{n} \rfloor + 1.$$

(v) If there are $m$, $2 \leq m < \lfloor \sqrt{n} \rfloor$, dipaths of length 2 that have tail end or head end in $u$, then
$$\text{dac}(K_{1,n,n}) = m + 1.$$

Acknowledgments
The authors thank the organizing committees of the International Conference on Mathematical and Statistical Sciences (ICMSS) 2021 for accepting this paper and the anonymous referees for their helpful comments.

References
[1] Appel K and Haken W 1976 Bulletin of the American mathematical Society 82 711–712
[2] Harary F and Hedetniemi S 1970 Journal of Combinatorial Theory 8 154–161
[3] Edwards K 1997 The harmonious chromatic number and the achromatic number Surveys in combinatorics, 1997 (Cambridge University Press) pp 13–48
[4] Vernold V J, Venkatachalam M and Akbar A M 2009 Filomat 23 251–255
[5] Arundhadhi R and Sattanathan R 2012 International Journal of Scientific and Research Publications (IJSRP) 2 1–4
[6] Araujo-Pardo G, Montellano-Ballesteros J, Olsen M and Rubio-Montiel C 2019 preprint arXiv:1912.10104
[7] Edwards K J 2013 Discrete Applied Mathematics 161 369–376
[8] Sopena E 2014 Discrete Applied Mathematics 173 102–112
[9] Araujo-Pardo G, Montellano-Ballesteros J J, Olsen M and Rubio-Montiel C 2018 Electron. J. Comb. 25 P3.51
[10] Sopena E 2001 Discrete Mathematics 229 359–369
[11] Sopena E 2016 Discrete Mathematics 339 1993–2005
[12] Chartrand G and Zhang P 2019 Chromatic graph theory (CRC press) ISBN 0-429-79828-8
[13] Chartrand G, Lesniak L and Zhang P 2016 Graphs & digraphs SIXTH EDITION vol 22 (Chapman & Hall London)
[14] Venkatachalam M, Kaliraj K et al. 2012 Tamkang Journal of Mathematics 43 153–158
[15] Bang-Jensen J and Gutin G Z 2008 Digraphs: theory, algorithms and applications (Springer Science & Business Media)