Induced Fractional Zero-Point Angular Momentum for Charged Particles of the Bohm-Aharonov System by means of a “Spectator” Magnetic Field *

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Abstract

An induced fractional zero-point angular momentum of charged particles by the Bohm-Aharonov (B-A) vector potential is realized via a modified combined trap. It explores a “spectator” mechanism in this type of quantum effects: In the limit of the kinetic energy approaching one of its eigenvalues the B-A vector potential alone cannot induce a fractional zero-point angular momentum at quantum mechanical level in the B-A magnetic field-free region; But when there is a “spectator” magnetic field the B-A vector potential induces a fractional zero-point angular momentum. The “spectator” does not contribute to such a fractional angular momentum, but plays essential role in guaranteeing non-trivial dynamics at quantum mechanical level in the required limit. This “spectator” mechanism is significant in investigating the B-A effects and related topics in both aspects of theory and experiment.

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As is well known, quantum states of charged particles can be influenced by electromagnetic effects even if those particles are in a region of vanishing field strength [1, 2]. As predicted by Bohm and Aharonov (B-A) [2], experiments [3] showed that in a multiply connected region where field strength is zero everywhere the interference spectrum suffered a shift according to the amount of the loop integral of magnetic vector potential around an unshrinkable loop. Wu and Yang [4] pointed out that the B-A effects is due to the non-trivial topology of the space where the magnetic field strength is vanishing. The B-A effect is purely quantum mechanical one which explores far-reaching consequences of vector potential in quantum theory. This effect has been received much attention for years [3, 6, 7]. Recently investigations in this topic concentrated on revealing new types of quantum phases: The Aharonov-Casher effect [8], the He-McKellar-Wilkens phase [9] and the Anandan phase [10].

In another aspect a fractional angular momentum originated from the Poynting vector produced by crossing the Coulomb field of a charged particle with an external magnetic field has been predicted by Peshkin, Talmi and Tassie for years [6, 11]. There are lots of works concerning fractional angular momentum in B-A dynamics and their fractional statistics (see the reviews [12–17] and references therein). Spatial noncommutativity also leads to fractional angular momentum [18, 19].

Recently Kastrup [20] considered the question of how to quantize a classical system of the canonically conjugate pair angle and orbital angular momentum. This has been a controversial issue since the founding days of quantum mechanics [21]. The problem is that the angle is a multivalued or discontinuous variable on the corresponding phase space. A crucial point is that the irreducible unitary representations of the euclidean group $E(2)$ or of its covering groups allow for orbital angular momentum $l = h(n + \delta)$ where $n = 0, \pm 1, \pm 2, \cdots$, and $0 \leq \delta < 1$. The case $\delta \neq 0$ corresponds to fractional zero-point angular momentum. Kastrup investigated the physical possibility of fractional orbital angular momentum in connection with the quantum optics of Laguerre-Gaussian laser modes in external magnetic fields, and pointed out that if implementable this would lead to a wealth of new theoretical, experimental and even technological possibilities.

In this paper the induced fractional zero-point angular momentum of charged parti-
cles by the B-A vector potential is realized via a modified combined trap. It explores a “spectator” mechanism in this type of quantum effects: In the limit of the kinetic energy approaching one of its eigenvalues the B-A vector potential alone cannot induce a fractional zero-point angular momentum of charged particles at quantum mechanical level in a region of vanishing B-A field strength; But when there is a “spectator” magnetic field the B-A vector potential induces a fractional zero-point angular momentum in the same region. The “spectator” does not contribute to such a fractional angular momentum, but plays essential role in guaranteeing non-trivial dynamics at quantum mechanical level in the required limit. This type of quantum effects is so remarkable that in quantum mechanics the vector potential itself has physical significant meaning and becomes effectively measurable not only in shifts of interference spectra originated from quantum phases but also in physical observables.

1. Dynamics in a Modified Combined Trap — We consider ions constrained in a modified combined trap including the B-A type magnetic field. The Paul, Penning, and combined traps share the same electrode structure [22]. A combined trap operates in all of the fields of the Paul and Penning traps being applied simultaneously. The trapping mechanism in a Paul trap involves an oscillating axially symmetric electric potential $\tilde{U}(\rho, \phi, z, t) = U(\rho, \phi, z)\cos \tilde{\Omega} t$ with $U(\rho, \phi, z) = V(z^2 - \rho^2/2)/2d^2$ where $\rho$, $\phi$ and $z$ are cylindrical coordinates, $V$ and $d$ are, respectively, characteristic voltage and length, and $\tilde{\Omega}$ is a large radio-frequency. The dominant effect of the oscillating potential is to add an oscillating phase factor to the wave function. Rapidly varying terms of time in Schrödinger equation can be replaced by their average values. Thus for $\tilde{\Omega} \gg \Omega \equiv (\sqrt{2q|V|/\mu d^2})^{1/2}$ we obtain a time-independent effective electric potential [23] $V_{eff} = q^2 \nabla U \cdot \nabla U / 4\mu\tilde{\Omega}^2 = \mu \omega_P^2 (\rho^2 + 4z^2)/2$ where $\mu$ and $q(>0)$ are, respectively, the mass and charge of the trapped ion, and $\omega_P = \Omega^2 / 4\tilde{\Omega}$. A modified combined trap combines the above electrostatic potential and two magnetic fields [24]: a homogeneous magnetic field $B_c$ aligned along the $z$ axis in a normal combined trap and a B-A type magnetic field $B_0$ produced by, for example, an infinitely long solenoid with radius $\rho = (x_1^2 + x_2^2)^{1/2} = a$. Inside the solenoid ($\rho < a$) $B_{0, in} = (0, 0, B_0)$ is homogeneous along the $z$ axis, and outside the solenoid ($\rho > a$) $B_{0, out} = 0$. The vector potential $A_c$ of $B_c$ is chosen as (Henceforth
the summation convention is used) \( A_{c,i} = -B_c \epsilon_{ij} x_j / 2 \), \( A_{c,z} = 0 \), \((i, j = 1, 2)\). The B-A vector potential \( \mathbf{A}_0 \) is: Inside the solenoid \( A_{0,i} = A_{in,i} = -B_0 \epsilon_{ij} x_j / 2 \), \( A_{in,z} = 0 \); Outside the solenoid \( A_{0,i} = A_{out,i} = -B_0 a^2 \epsilon_{ij} x_j / 2 x_k x_k \), \( A_{out,z} = 0 \), \((i, j, k = 1, 2)\). At \( \rho = a \) the potential \( \mathbf{A}_{in} \) passes continuously over into \( \mathbf{A}_{out} \). The Hamiltonian of the modified combined trap is \( H = (p_i - qA_{c,i}/c - qA_{0,i}/c)^2 / 2\mu + p_z^2 / 2\mu + \mu\omega_p^2 (x_i^2 + 4z^2) / 2 \). This Hamiltonian can be decomposed into a one-dimensional harmonic Hamiltonian \( H_z(z) \) along the \( z \)-axis with the axial frequency \( \omega_z = 2\omega_p \) and a two-dimensional Hamiltonian \( H_{\perp}(x_1, x_2) \), \( H = H_z(z) + H_{\perp}(x_1, x_2) \). Inside the solenoid the ion’s motion is the same as the one with a total magnetic field \( \mathbf{B}_c + \mathbf{B}_{0,in} \).

In the following we consider the motion outside the solenoid. The two-dimensional Hamiltonian outside the solenoid is \([22, 23]\)

\[
H_{\perp}(x_1, x_2) = \frac{1}{2\mu} \left( p_i + \frac{1}{2} \mu\omega_c \epsilon_{ij} x_j + \mu\omega_0 a^2 \frac{\epsilon_{ij} x_j}{2 x_k x_k} \right)^2 + \frac{1}{2} \mu\omega_p^2 x_i^2, \tag{1}
\]

where \( \omega_c = qB_c / \mu c \) and \( \omega_0 = qB_0 / \mu c \) are the cyclotron frequencies corresponding to, respectively, the magnetic fields \( \mathbf{B}_c \) and \( \mathbf{B}_{0,in} \). The Hamiltonian \( H_{\perp} \) possess a rotational symmetry in \((x_1, x_2)\) - plane. The \( z \)-component of the orbital angular momentum \( J_z = \epsilon_{ij} x_i p_j \) commutes with \( H_{\perp} \). They have common eigenstates.

**Dynamics in the Limit of the Kinetic Energy Approaching its Lowest Eigenvalue** – In this limit the kinetic energy is \( E_k = \mu \dot{x}_i \dot{x}_i / 2 = (K_1^2 + K_2^2) / 2\mu \) where

\[
K_i \equiv p_i + \frac{1}{2} \mu\omega_c \epsilon_{ij} x_j + \mu\omega_0 a^2 \frac{\epsilon_{ij} x_j}{2 x_k x_k}, \quad [K_i, K_j] = i\hbar \mu\omega_c \epsilon_{ij} \tag{2}
\]

Here \( K_i \) is the mechanical momenta corresponding to the vector potentials \( A_{c,i} \) and \( A_{out,i} \). It is worth noting that the B-A vector potential \( A_{out,i} \) does not contributes to the commutator \([K_i, K_j] \). The canonical momenta \( p_i \) are quantized, \( p_i = -i\hbar \partial / \partial x_i \). They commute each other \([p_i, p_j] = 0 \). We define canonical variables \( Q = K_1 / \mu\omega_c \) and \( \Pi = K_2 \) which satisfy \([Q, \Pi] = i\hbar \delta_{ij} \). The kinetic energy \( E_k \) is rewritten as the Hamiltonian of a harmonic oscillator \( E_k = \Pi^2 / 2\mu + \mu\omega_c^2 Q^2 / 2 \). The lowest eigenvalue \( \mathcal{E}_{k0} \) of the kinetic energy \( E_k \) is \([25]\) \( \mathcal{E}_{k0} = \hbar \omega_c / 2 \).

In a laser trapping field, using a number of laser beams and exploiting Zeeman tuning, the speed of atoms can be slowed to the extent of \( 1 \text{ m/s} \), see \([26]\). Ions are the common
object in cooling and trapping. In order to experimentally realizing the limit of $E_k \to E_{k0}$ through laser cooling in a trap ions are used.

In the limit of the kinetic energy approaching its lowest eigenvalue the Hamiltonian $H_\perp$ in Eq. (1) has non-trivial dynamics \cite{27, 28, 19}. The Lagrangian corresponding to $H_\perp$ is

$$
L = \frac{1}{2} \mu \ddot{x}_i - \frac{1}{2} \mu \omega_c \epsilon_{ij} \dot{x}_i x_j - \mu \omega_0 a^2 \frac{\epsilon_{ij} \dot{x}_i x_j}{2 x_k x_k} - \frac{1}{2} \mu \omega_0^2 x_i x_i. \quad (3)
$$

In the limit of $E_k \to E_{k0}$, the Hamiltonian $H_\perp$ reduces to $H_0 = \hbar \omega_c / 2 + \mu \omega_0^2 x_i x_i / 2$. The Lagrangian corresponds to $H_0$ is

$$
L_0 = -\frac{1}{2} \mu \omega_c \epsilon_{ij} \dot{x}_i x_j - \mu \omega_0 a^2 \frac{\epsilon_{ij} \dot{x}_i x_j}{2 x_k x_k} - \frac{1}{2} \mu \omega_0^2 x_i x_i - \frac{1}{2} \hbar \omega_c. \quad (4)
$$

**Constraints** – For the reduced system $(H_0, L_0)$ the canonical momenta are

$$
p_i = \frac{\partial L_0}{\partial \dot{x}_i} = -\frac{1}{2} \mu \omega_c \epsilon_{ij} x_j - \mu \omega_0 a^2 \frac{\epsilon_{ij} x_j}{2 x_k x_k}. \quad (5)
$$

Eq. (5) does not determine velocities $\dot{x}_i$ as functions of $p_i$ and $x_j$, but gives relations among $p_i$ and $x_j$, that is, such relations are the primary constraints \cite{29, 28, 19}

$$
\varphi_i(x, p) = p_i + \frac{1}{2} \mu \omega_c \epsilon_{ij} x_j + \mu \omega_0 a^2 \frac{\epsilon_{ij} x_j}{2 x_k x_k} = 0. \quad (6)
$$

The physical meaning of Eq. (6) is that it expresses the dependence of degrees of freedom among $p_i$ and $x_j$. The constraints (6) should be carefully treated \cite{30}. The subject can be treated simply by the symplectic method in \cite{31, 32}. In this paper we work in the Dirac formalism. The Poisson brackets of the constraints (6) are

$$
C_{ij} = \{\varphi_i, \varphi_j\} = \mu \omega_c \epsilon_{ij}. \quad (7)
$$

From Eq. (7), $\{\varphi_i, \varphi_j\} \neq 0$, it follows that the conditions of the constraints $\varphi_i$ holding at all times do not lead to secondary constraints.

$C_{ij}$ defined in Eq. (7) are elements of the constraint matrix $C$. Elements of its inverse matrix $C^{-1}$ are $(C^{-1})_{ij} = -\epsilon_{ij} / \mu \omega_c$. The corresponding Dirac brackets of $\{\varphi_i, x_j\}_D$, $\{\varphi_i, p_j\}_D$, $\{x_i, x_j\}_D$, $\{p_i, p_j\}_D$ and $\{x_i, p_j\}_D$ can be defined. The Dirac brackets of $\varphi_i$ with any variables $x_i$ and $p_j$ are zero so that the constraints (6) are strong conditions. It can be
used to eliminate dependent variables. If we select $x_1$ and $x_2$ as the independent variables, from the constraints (6) the variables $p_1$ and $p_2$ can be represented by, respectively, the independent variables $x_2$ and $x_1$ as

$$p_1 = -\frac{1}{2} \mu \omega_c x_2 - \mu \omega_0 a^2 \frac{x_2}{2x_k x_k}, \quad p_2 = \frac{1}{2} \mu \omega_c x_1 + \mu \omega_0 a^2 \frac{x_1}{2x_k x_k}$$  \hfill (8)

The Dirac brackets of $x_1$ and $x_2$ is

$$\{x_1, x_2\}_D = \frac{1}{\mu \omega_c}.$$  \hfill (9)

We introduce new canonical variables $x = x_1$ and $p = \mu \omega_c x_2$. Their Dirac bracket is $\{x, p\}_D = 1$. According to Dirac’s formalism of quantizing constrained systems the corresponding quantum commutation relation is $[x, p] = i \hbar$.

**Quantum Behavior of the Reduced System** – Now we consider quantum behavior of the reduced system $(H_0, L_0)$. By defining the following effective mass and frequency, $\mu^* \equiv \mu \omega_c^2 / \omega_0^2$, $\omega^* \equiv \omega_0^2 / \omega_c$, the Hamiltonian $H_0$ is represented as $H_0 = p^2 / 2\mu^* + \mu^* \omega^2 x^2 / 2 + \hbar \omega_c / 2$. We introduce an annihilation operator $A = \sqrt{\mu^* \omega^* / 2 \hbar} x + i \sqrt{1 / 2 \hbar \mu^* \omega^*} p$ and its conjugate one $A^\dagger$. The operators $A$ and $A^\dagger$ satisfies $[A, A^\dagger] = 1$. The eigenvalues of the number operator $N = A^\dagger A$ is $n = 0, 1, 2, \cdots$. Using $A$ and $A^\dagger$, the reduced Hamiltonian $H_0$ is rewritten as $H_0 = \hbar \omega^* (A^\dagger A + 1 / 2) + \hbar \omega_c / 2$.

Now we consider the angular momentum of the ion. Using Eq. (8) to replace $p_1$ and $p_2$ by, respectively, the independent variables $x_2$ and $x_1$, the orbital angular momentum $J_z = \epsilon_{ij} x_i p_j$ is rewritten as

$$J_z = q \frac{\Phi_0}{2\pi c} + \frac{1}{2} \mu \omega_c (x_1^2 + x_2^2),$$  \hfill (10)

where $\Phi_0 = \pi a^2 B_0$ is the total flux of the magnetic field $B_0$ inside the solenoid. Similarly, using $A$ and $A^\dagger$ to rewrite $J_z$, we obtain $J_z = q \Phi_0 / 2\pi c + \hbar (A^\dagger A + 1 / 2)$, the zero-point angular momentum of $J_z$ is $J_0 = \hbar / 2 + q \Phi_0 / 2\pi c$. In the above the term

$$J_{AB} = \frac{q}{2\pi c} \Phi_0$$  \hfill (11)

is the zero-point angular momentum induced by the AB vector potential. $J_{AB}$ takes fractional values. It is related to the region where the magnetic field $B_{0, out} = 0$ but the corresponding vector potential $A_{out} \neq 0$. 6
2. Dynamics in the Case of $B_c = 0$ – It is worth noting that here $B_c$, like a “spectator”, does not contribute to $J_{AB}$. In order to clarify the role played by $B_c$, we consider the case of $B_c = 0$. In this case the modified combined trap is as stable as a Paul trap. The corresponding kinetic energy reduces to $\tilde{E}_k = \mu \dot{x}_i \dot{x}_i / 2 = (\tilde{K}^2_1 + \tilde{K}^2_2) / 2\mu$ where

$$\tilde{K}_i \equiv p_i + \mu \omega a^2 \frac{\epsilon_{ij} \dot{x}_j \dot{x}_i}{2x_k x_k}, \quad [\tilde{K}_i, \tilde{K}_j] = 0.$$ (12)

In the above $\tilde{K}_i$ is the mechanical momenta corresponding to the B-A vector potential $A_{out,i}$. Unlike the ordinary vector potential, the special feature of the B-A vector potential is that it does not contribute to the commutator $[\tilde{K}_i, \tilde{K}_j]$. Because $\tilde{K}_i$ are commuting, behavior of $\tilde{E}_k$ is similar to a Hamiltonian of a free particle. Its spectrum is a continuous one. When $\tilde{E}_k$ approaching some constant $\tilde{E}_k(\neq 0)$ the Hamiltonian $H_\perp$ reduces to $\tilde{H}_0 = \tilde{E}_k + \mu \omega_2 p x_i x_i / 2$. The Lagrangian corresponding to $\tilde{H}_0$ is

$$\tilde{L}_0 = -\mu \omega_0 a^2 \frac{\epsilon_{ij} \dot{x}_j \dot{x}_i}{2x_k x_k} - \frac{1}{2} \mu \omega_2 p x_i x_i - \tilde{E}_k.$$ (13)

From $\tilde{L}_0$ we obtain the canonical momenta

$$\tilde{p}_i = \frac{\partial \tilde{L}_0}{\partial \dot{x}_i} = -\mu \omega_0 a^2 \frac{\epsilon_{ij} \dot{x}_j \dot{x}_i}{2x_k x_k}. \quad (14)$$

Now we clarify that the case $\tilde{E}_k = 0$ should be excluded. The limit of the kinetic energy $E_k = \mu \dot{x}_i \dot{x}_i / 2 \to 0$ corresponds two possibilities: $\dot{x}_i = 0$ or $\mu \to 0$. In the case $\dot{x}_i = 0$ the Lagrangian $L$ in Eq. (3) reduces to $\tilde{L}_0 = -\mu \omega_2 p x_i x_i / 2$. The corresponding canonical momenta $\tilde{p}_i = \partial \tilde{L}_0 / \partial \dot{x}_i = 0$. Therefore there is no dynamics. According to the definition of the frequency $\Omega$ the other possibility $\mu \to 0$ is forbidden.

Eq. (14) gives the reduced primary constraints

$$\tilde{\varphi}_i = \tilde{p}_0 i + \mu \omega_0 a^2 \frac{\epsilon_{ij} \dot{x}_j \dot{x}_i}{2x_k x_k} = 0.$$ (15)

Here the special feature is that the corresponding Poisson brackets are zero,

$$\tilde{C}_{ij} = \{\tilde{\varphi}_i, \tilde{\varphi}_j\} \equiv 0.$$ (16)

From Eq. (16), $\{\tilde{\varphi}_i, \tilde{\varphi}_j\} \equiv 0$, it follows that the conditions of the constraints $\tilde{\varphi}_i$ holding at all times lead to secondary constraints $\tilde{\varphi}_i^{(2)} = -\mu \omega_2 p x_i$. The Poisson brackets $\{\tilde{\varphi}^{(2)}_i, \tilde{\varphi}^{(2)}_j\} = 0$,
\{\tilde{\varphi}_i^{(2)}, \tilde{\varphi}_j^{(2)}\} = 0,$ and $\{\tilde{\varphi}_i^{(2)}, \bar{H}_0\} = 0,$ so that persistence of the secondary constraints $\tilde{\varphi}_i^{(2)}$ in course of time does not lead to further secondary constraints $\tilde{\varphi}_i^{(3)}$.

Because of $\tilde{C}_{ij} \equiv 0,$ the inverse matrix $\tilde{C}^{-1}$ does not exist. The Dirac brackets $\{\tilde{\varphi}_i, x_j\}_D,$ $\{\tilde{\varphi}_i, p_j\}_D,$ $\{\tilde{\varphi}_i^{(2)}, x_j\}_D,$ $\{\tilde{\varphi}_i^{(2)}, p_j\}_D,$ and $\{x_i, p_j\}_D$ cannot be defined. According to Dirac’s formalism of quantizing constrained systems, there is no way to establish dynamics at quantum mechanical level. This means that the B-A vector potential alone cannot lead to non-trivial dynamics at quantum mechanical level in the required limit, thus does not contribute to the energy spectrum and angular momentum at all.

It is clear that though the vector potential $A_{c,i}$ of the “spectator” magnetic field $B_c$ does not contribute to $J_{AB},$ it plays essential role in guaranteeing non-trivial dynamics at quantum mechanical level in the limit of the kinetic energy approaching one of its eigenvalues. This example reveals that, unlike ordinary vector potential, the physical role played by the B-A vector potential is subtle. This needs to be carefully analyzed at quantum mechanical level.

3. Dynamics in the Case of $B_0 = 0$ – In order to further clarify the essential difference between $A_o$ and $A_c$ in the region of $B_{0,\text{out}} = 0$ we consider the case of $B_0 = 0.$ In this case the modified combined trap reduces to a combined trap. The Hamiltonian $H_\perp(x_1, x_2)$ in Eq. (1) reduces to $\hat{H}_\perp(x_1, x_2) = (p_i + \mu \omega_c \epsilon_{ij} x_j / 2)^2 / 2\mu + \mu \omega_p^2 x_i^2 / 2.$ Its kinetic energy is $\hat{E}_k = (\hat{K}_1^2 + \hat{K}_2^2) / 2\mu$ where

$$\hat{K}_i \equiv p_i + \mu \omega_c \epsilon_{ij} x_j / 2, \quad [\hat{K}_i, \hat{K}_j] = i\hbar \mu \omega_c \epsilon_{ij}. \quad (17)$$

In Eq. (17) $\hat{K}_i$ is the mechanical momenta corresponding to the vector potentials $A_{c,i}.$ The commutation relations between $\hat{K}_i$’s are the same as the ones between $K_i$’s in Eq. (2). The eigenvalues of $\hat{E}_k$ is $\hat{E}_{kn} = \hbar \omega_c (n + 1/2),$ which are just the Landau levels of charged particles in an external magnetic field.

In the following we consider the limit of $\hat{E}_k$ approaching the lowest eigenvalue $\hat{E}_{k0} = \hbar \omega_c / 2.$ The Lagrangian corresponding to $\hat{H}_\perp$ is

$$\hat{L} = \mu \dot{x}_i \dot{x}_i / 2 - \mu \omega_c \epsilon_{ij} \dot{x}_i \dot{x}_j / 2 - \mu \omega_p^2 x_i x_i / 2. \quad (18)$$

In the limit of $\hat{E}_k \to \hat{E}_{k0},$ the Hamiltonian $\hat{H}_\perp$ reduces to $\hat{H}_0 = \hbar \omega_c / 2 + \mu \omega_p^2 x_i x_i / 2$ which
is the same as $H_0$. The Lagrangian corresponds to $\hat{H}_0$ is

$$\hat{L}_0 = -\mu \omega \epsilon_{ij} \dot{x}_i x_j / 2 - \mu \omega^2 p_i x_i / 2 - \hbar \omega / 2.$$  \hspace{1cm} (19)$$

For the reduced system ($\hat{H}_0, \hat{L}_0$) the canonical momenta are $\hat{p}_i = \partial \hat{L}_0 / \partial \dot{x}_i = -\mu \omega \epsilon_{ij} x_j / 2$. It leads to the following constraints

$$\hat{\varphi}_i = p_i + \mu \omega \epsilon_{ij} x_j / 2 = 0.$$  \hspace{1cm} (20)$$

The Poisson brackets of $\hat{\varphi}_i$ are the same as ones of the constraints $\varphi_i$ in Eq. (7):

$$\hat{C}_{ij} = \{ \hat{\varphi}_i, \hat{\varphi}_i \} = \mu \omega \epsilon_{ij}.$$  \hspace{1cm} (21)$$

From Eq. (21), $\{ \hat{\varphi}_i, \hat{\varphi}_i \} \neq 0$, it follows that the conditions of the constraints $\hat{\varphi}_i$ holding at all times do not lead to secondary constraints.

By the similar procedure of treating the constraints (19), we find that the reduced system ($\hat{H}_0, \hat{L}_0$) has non-trivial dynamics at quantum mechanical level in the limit of $\hat{E}_k \to \hat{E}_k_0$. The constraints (20) are strong conditions which can be used to eliminate dependent variables. We select $x_1$ and $x_2$ as the independent variables. The variables $p_1$ and $p_2$ can be represented by, respectively, $x_2$ and $x_1$ as $p_1 = -\mu \omega x_2 / 2$, $p_2 = \mu \omega x_1 / 2$. The Dirac brackets of $x_1$ and $x_2$ is $\{ x_1, x_2 \}_D = 1 / \mu \omega$. We introduce new canonical variables $x = x_1$ and $p = \mu \omega x_2$. Their Dirac bracket is $\{ x, p \}_D = 1$. The corresponding quantum commutation relation is $[x, p] = i \hbar$. Using these results the orbital angular momentum $\hat{J}_z = \epsilon_{ij} x_i p_j$ can be represented by the canonical variables $x$ and $p$ as $\hat{J}_z = (p^2 / 2 \mu + \mu \omega^2 x^2 / 2) / \omega$. The zero-point angular momentum can be read out from this harmonic-like “Hamiltonian”, $\hat{J}_0 = \hbar / 2$. We note that in this case there is no fractional zero-point angular momentum.

The above results elucidate that $A_c$ are essentially different from $A_0$: the $A_c$ alone can lead to non-trivial dynamics at quantum mechanical level in the limit of the kinetic energy approaching its lowest eigenvalue.

4. Gauge Transformation – As is well known, we can perform a gauge transformation $\chi$ so that the resulting vector potential $A_\text{out}' = A_\text{out} + \nabla \chi = 0$. A suitable gauge function is $\chi = -B_0 a^2 \tan^{-1}(x_2 / x_1) / 2$. In the Schrödinger equation the corresponding gauge transformation is $\mathcal{G} = \exp(iq \chi / \hbar c)$. Under this gauge transformation the Hamiltonian
$H_{\perp}(x_1, x_2)$ in Eq. (1) is transformed into $H_{\perp} \rightarrow G H_{\perp} G^{-1} = H'_{\perp} = \left( p_i + \mu \omega_c \epsilon_{ij} x_j / 2 \right)^2 / 2 \mu + \mu \omega_c^2 x_i^2 / 2$. Here $H'_{\perp}$ is the same $\hat{H}_{\perp}$.

In the limit of the kinetic energy approaching its lowest eigenvalue the corresponding reduced constraints are the same $\hat{\phi}_i$ in Eq. (20). Under the gauge transformation $G$ the angular momentum $J_z = \epsilon_{ij} x_i p_j$ is transformed into $J_z \rightarrow G J_z G^{-1} = J'_z = x_1 p_2 - x_2 p_1 + q \Phi_0 / 2\pi c$. Using the constraints $\hat{\phi}_i$ in Eq. (20) to represent $p_1$ and $p_2$ by, respectively, the independent variables $x_2$ and $x_1$, the first term in $J'_z$ reads $x_1 p_2 - x_2 p_1 = \mu \omega_c (x_1^2 + x_2^2) / 2$. Thus we obtain

$$J'_z = \frac{q}{2\pi c} \Phi_0 + \frac{1}{2} \mu \omega_c (x_1^2 + x_2^2). \tag{22}$$

$J'_z$ is the same $J_z$ in Eq. (10). This result shows that the fractional zero-point angular momentum induced by the B-A vector potential is a real physical observable which cannot be gauged away by a gauge transformation.

In summary, this paper explores a “spectator” mechanism in B-A effects. It is clarified that the B-A vector potential alone cannot lead to non-trivial dynamics at quantum mechanical level in the limit of the kinetic energy approaching one of its eigenvalues. In such a limit the B-A vector potential alone cannot induce a fractional zero-point angular momentum. When there is a “spectator” magnetic field the B-A vector potential induces a fractional zero-point angular momentum. The induced effect essentially depends upon the participation of a “spectator” magnetic field. The “spectator” vector potential does not contribute to the fractional angular momentum, but plays essential role in guaranteeing non-trivial dynamics at quantum mechanical level in the required limit. The “spectator” mechanism is significant in both aspects of theory and experiment. In the theoretical aspect, it is revealed that, unlike ordinary vector potentials, the physical role played by the B-A vector potential is subtle. This needs to be carefully analyzed at quantum mechanical level. In the experimental aspect, existence of a “spectator” magnetic field is necessary for inducing the fractional angular momentum by the B-A vector potential. As an example, the modified combined trap provides a realistic way to realize this “spectator” mechanism.

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consists essentially in a shift of the phase of the original wave function. One can adjust the radio-frequency $\tilde{\Omega}$ to compensate the phase shift, therefore for a modified combined trap the derivation in Ref. [23] also remains valid. The modification of $\tilde{\Omega}$ leads to the corresponding modification of the effective frequency $\omega_P = \Omega^2/4\tilde{\Omega}$ of $V_{\text{eff}}$. In Eq. (1) the $\omega_P$ means the modified effective frequency.

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From the Hamiltonian $H_0 = p_i\dot{x}_i - L_0$, using $p_i = \partial L_0/\partial \dot{x}_i$ and the Lagrangian equation $\dot{p}_i = \partial L_0/\partial x_i$, it follows that

$$\delta H_0 = \dot{x}_i \delta p_i - \dot{p}_i \delta x_i.$$  

It indicates that $H_0$ can be expressed as a function of $x_i$ and $p_i$. Thus we obtain

$$\delta H_0(x, p) = \frac{\partial H_0}{\partial x_i} \delta x_i + \frac{\partial H_0}{\partial p_i} \delta p_i.$$  

Because of the constraints $\varphi_i(x, p) = 0$ of Eq. (6), $H_0$ plus any linear combination of $\varphi_i$ is also a Hamiltonian of the system, i.e., the $H_0$ can be replaced by
From the above two equations, including the contributions of \( \delta (\lambda_i(x,p)\varphi_i(x,p)) \), it follows that the Hamiltonian equations read

\[
\dot{p}_i = -\frac{\partial H_0}{\partial x_i} - \lambda_k \frac{\partial \varphi_k}{\partial x_i}, \quad \dot{x}_i = \frac{\partial H_0}{\partial p_i} + \lambda_k \frac{\partial \varphi_k}{\partial p_i}.
\]

Eq. (6) gives \( \partial \varphi_k / \partial p_i = \delta_{ki} \). From the reduced Hamiltonian \( H_0 \) obtained from \( L_0 \) in Eq. (4) it follows that \( \partial H_0 / \partial p_i = 0 \). Thus the second equation reduces to

\[
\dot{x}_i = \lambda_i.
\]

In this example the Lagrange multiplier \( \lambda_i \) is just the velocity \( \dot{x}_i \).

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\[
\mathbf{J} = \frac{1}{4\pi c} \int \mathbf{r} \times [\mathbf{E} \times \mathbf{B}(\mathbf{r})]d^3r.
\]

In cases where the magnetic field is only in the \( z \)-direction, this angular momentum reduces to

\[
J_z = -\frac{q\phi}{2\pi c},
\]

where \( \phi = \int \int B_z(x_1,x_2)dx_1dx_2 \) is the total magnetic flux. \( J_z \) is the angular momentum of the electromagnetic fields. In cases where the magnetic field \( B_z \) is produced by an infinitely long solenoid, this angular momentum exists only inside the solenoid. \( J_z \) should be distinguished from \( J_{AB} \) of Eq. (11). \( J_{AB} \) is the angular momentum of the charged particle. It is worth noting that \( J_{AB} \) is induced by the B-A vector potential outside the solenoid and does not exist inside the solenoid.

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