Some Important Concepts in Nonstandard Analysis Theory of Turbulence (The revised)

Feng Wu

Department of Mechanics and Mechanical Engineering,
University of Science and Technology of China, Hefei 230026, China

Some important concepts in the nonstandard analysis theory of turbulence are presented in this article. The structure of point, on which differential equations are defined, is analyzed. The distinction between the uniform point and the non-uniform point, as well as between the standard point and the nonstandard point, is showed. A new kind of equations, which differ essentially from those in existent theory, is emphasized. These new equations can hold at non-uniform points. The applicability of the Navier-Stokes equations to turbulence is discussed. Some illustrations of the nonstandard analysis theory of turbulence are given too.

PACS 47.27.Ak

1 Introduction

A new approach, the nonstandard analysis picture, to the theory of turbulence was presented in the paper [1]. The nonstandard analysis theory of turbulence (NATT) is based on the nonstandard analysis mathematics and the fact that hierarchical structure is universally existent in the world. The theory shows that in a laminar flow, a particle of fluid is taken as uniform wholly, and there does not exist any structure in the particle; on the other hand, a particle of fluid in a turbulent field should not be uniform wholly and some interior structure occurs.

There are six assumptions in the new theory of turbulence. They are:
Assumption 1: Global turbulent field is composed of standard points, and every standard point is yet a monad. Each monad possesses the internal structure, namely a monad is also composed of infinite nonstandard points (so called interior points of the monad).

Assumption 2: The flows in monad fields are controlled by the Navier-Stokes equations.

Assumption 3: Turbulent field is continuous.

Assumption 4: When a measurement at any point (monad) \((x_1, x_2, x_3, t)\) in a physical field is taken, the operation of the measurement will act randomly on one interior point (nonstandard point) of the point \((x_1, x_2, x_3, t)\).

Assumption 5: When a measurement at any point (monad) of a turbulent field is made, the operation of the measurement will act in equiprobability on various interior points of the monad. This assumption is called the equiprobability assumption.

Assumption 6: In both the value and structure of function, physical function, defined on the interior points of the monads of a turbulent field, is infinitely close between two monads, when these two monads are infinitely close to each other.

Based on these assumptions, it is presented that the real turbulent fluctuation stems from the uncertainty of measurement of turbulence. The average of physical quantities is taken over the point(monad), namely the point(monad)-average is adopted and computed. And the fundamental equations of turbu-
lence are obtained too. The closure problem is overcome easily. The closed equations are:

Choice one:
\[
\frac{\partial \tilde{U}_i}{\partial x_i} = 0, \quad \frac{\partial \tilde{U}_i}{\partial t} + \frac{\partial \tilde{U}_i \tilde{U}_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{P}}{\partial x_i} + \nu \nabla^2 \tilde{U}_i + 0(\varepsilon^2)
\] (1)

Choice two:
\[
\frac{\partial U_i}{\partial x_i} = 0, \quad \frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \nabla^2 U_i
\] (2)
\[
\frac{\partial u_i}{\partial x_i} = 0, \quad \frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} - 2 \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + 0(\varepsilon^3)
\] (3)

Choice three:
\[
\frac{\partial \tilde{U}_i}{\partial x_i} = 0, \quad \frac{\partial \tilde{U}_i}{\partial t} + \frac{\partial \tilde{U}_i \tilde{U}_j}{\partial x_j} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{P}}{\partial x_i} + \nu \nabla^2 \tilde{U}_i + 0(\varepsilon^3)
\] (4)
\[
\frac{\partial u_i}{\partial x_i} = 0, \quad \frac{\partial u_i}{\partial t} + \tilde{U}_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial \tilde{U}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + 0(\varepsilon^3)
\] (5)

Here \( \varepsilon \) is the linear dimension of monad, yet nonstandard number infinitesimal. And

\[
U_i = \tilde{U}_i + u_i, \quad P = \tilde{P} + p,
\]

“~” expresses the average over monad. \( U_i \) and \( P \) are the instantaneous quantities of velocity and pressure respectively.

After further contemplation over the NATT, we can present some important concepts. These concepts are discussed as follows.

2 On the concept of “point”

When physical phenomena (e.g., complex phenomena) are studied, people’s attention is usually concentrated on inquiry into the characteristic of the equations, governing the phenomena, and their solutions. Is it possible that the
phenomena are discussed from other angle? It is well known that differential 
equations are always defined on and applied to points. A point in mathematics 
is absolute geometric point; but a physical point, in fact, is micro-volume. And 
a physical point, in some cases, has the structure in it. Therefore, we could 
observe and comprehend physical phenomena from the angle of analyzing the 
nature of the point, to which the governing equations of the phenomena are 

applied.

Point is an abstract concept of mathematics, surely the concept of point 
is drawn from objective physical reality. In fluid mechanics, for example, a 
particle of fluid is taken, in the abstract, as a point. In a laminar flow, a point(a 
particle of fluid) is uniform and no structure exists through the particle. On 
the other hand, in turbulence, a point (a particle of fluid) possesses interior 
structure and is non-uniform. A particle of fluid in the two cases, laminar flow 
and turbulence, is abstracted as a point. However, the former is uniform point 
and the latter non-uniform point. Every point(particle of fluid), which has 
interior structure under some conditions, is formed of numerous fluid-particles 
in lower level. A fluid-particle in lower level can be thought of as uniform and 
abstracted as a uniform point. Yet every point(particle of fluid) of a flow field 
is called as monad; and a fluid-particle in lower level is called as an interior 
point of the monad in paper [1].

Moreover, a particle of fluid is abstracted as a standard point and a fluid-
particle in lower level is abstracted as a nonstandard point in paper [1]. A 
standard point(monad) corresponds to a real number, and a nonstandard point 
to a nonstandard number. Here we give the other meaning of standard points. 
The dimension of the standard point is an infinitesimal in the level, of which
the characteristic dimension could compare with the dimension of human being self. In that level, people do many practical activities, such as navigation, aviation, spaceflight etc.. The standard point is proper for these practical activities, i.e., the point in the physical models related to these activities corresponds just to the standard point. The dimension of standard points is not determined by people at will, but by the nature of physical laws and the characteristic of the practical activities. Though we can not show exactly how large a standard point is, the dimension of standard points is objective. Similarly, the dimension of the nonstandard point is an infinitesimal in lower level.

The meaning of “point”, in a word, is not fixed and absolute. By the concepts of nonstandard analysis, “point” possesses plentiful and vital content. There is need to make a distinction between different points, i.e., the uniform point and the non-uniform point, the standard point and the nonstandard point.

3 Two kinds of differential equations

Physical laws usually are expressed by differential equations in mathematics. Yet the differential terms in the equations of existent theory are as follows:

\[
\frac{\partial f}{\partial t} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \quad \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\] (6)

Therefore, the limit in mathematics means the fundament of the differential equations in existent theory, namely these equations are in the frame of \( \delta - \varepsilon \). The limits(\( \Delta t \to 0, \Delta x \to 0 \)) in these equations mean that \( \Delta t \) and \( \Delta x \) tend to absolute zero in mathematics, while tend to uniform point(uniform particle) in physics. This fact sets a limit to the nature of the points, to which
these equations are applicable. These equations are applicable only to uniform points, rather than non-uniform points.

On the other hand, the fundamental equations of turbulence were presented in the paper [1]. The equations have the same form as those in existent theory. But the “differential” terms in the equations of the new theory (NATT) are:

\[
\frac{\partial f}{\partial t} = \frac{f(t + \varepsilon_t) - f(t)}{\varepsilon_t}, \quad \frac{\partial f}{\partial x} = \frac{f(x + \varepsilon_x) - f(x)}{\varepsilon_x}
\]

(7)

Here \(\varepsilon_t\) and \(\varepsilon_x\) are the infinitesimals, which are the linear dimensions of monads, rather than arbitrary infinitesimals. In nonstandard analysis, \(\varepsilon_t\) and \(\varepsilon_x\) are certain numbers (nonstandard numbers). So there is no limit term in these equations. Obviously, these “differential equations” are not based on the limits \((\Delta t \to 0, \Delta x \to 0)\) and are out of the frame of \(\delta - \varepsilon\). In fact, a new kind of equations was presented in paper [1]. The new equations differ essentially from those in existent theory. Here the term in new equations, by convention, is still written as the differential in form. But the meaning of the term is different from that of ordinary differential. New kind of equations can hold at non-uniform points. Obviously, these new equations have more natural relation with discretization form of equations than those based on the frame of \(\delta - \varepsilon\) in numerical computation.

Therefore, there are two kinds of equations: One is the equation in existent theory. This equation is based on limits \((\Delta t \to 0, \Delta x \to 0)\) (or so called in the frame of \(\delta - \varepsilon\)) and applicable only to uniform points. The other is the equation in the NATT. The second kind of equations is out of the frame of \(\delta - \varepsilon\), and based on the nonstandard analysis. These new equations can hold
at non-uniform points. When motion of fluid is slow and varies small, the
description of the motion by the first kind of equations is suitable. Otherwise,
when the motion of fluid is very fast and varies drastically, e.g. in the case
of turbulence, the reasonable equations describing the motion are not the first
kind of equations, but the second kind of equations. Surely, the second kind
of equations becomes the first kind of equations, i.e. there is no difference
between the two kinds of equations at the uniform point.

4 Applicability of the Navier-Stokes Equations

Surely to answer whether the Navier-Stokes equations are applicable to the
turbulence is important. Now this question is presented by following state-
ments: “The Navier-Stokes equations hold in laminar flows.”, “The Navier-
Stokes equations hold in turbulence too.”, “Do the Navier-Stokes equations
hold in turbulence?” etc.. Obviously, these statements show that when think-
ing of this question, people always pay their attention to that a flow is laminar
or turbulence. Usually the Navier-Stokes equations are thought of as applicable
to laminar flows. And some think that the Navier-Stokes equations hold
in turbulence, others do not.

The Navier-Stokes equations:

\[
\begin{align*}
\frac{\partial U_i}{\partial x_i} &= 0, \\
\frac{\partial U_i}{\partial t} + \frac{\partial U_i U_j}{\partial x_j} &= -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \nabla^2 U_i
\end{align*}
\] (8)

It is well known that the Navier-Stokes equations are based on the limits(\(\Delta t \to 0, \Delta x \to 0\)). Therefore, the Navier-Stokes equations hold only at uniform
points. In laminar flows, not only nonstandard but also standard points are
uniform. So the Navier-Stokes equations hold in laminar flows. However, in
turbulence, nonstandard points are uniform, while standard points are not uniform and possess interior structure. Hence, in turbulence the Navier-Stokes equations hold only at nonstandard points, rather than at standard points. Therefore, the key of the applicability of the Navier-Stokes equations lies in that the Navier-Stokes equations do not hold at non-uniform points, but only uniform points.

Therefore, if the turbulent field is thought as composed of nonstandard points, i.e., the turbulent field is thought as in one level only rather than possesses hierarchical structure, the Navier-Stokes equations hold in the turbulence. But this viewpoint has only theoretical and abstract meanings. In engineering practice, the points, corresponding to the engineering practice, are the standard points (monads). The Navier-Stokes equations do not hold on these standard points (monads), i.e., the Navier-Stokes equations are not applicable to turbulence. In other word, the Navier-Stokes equations are applicable to monad field rather than the global turbulent field.

5 Some Illustrations of the NATT

The equations (1)-(5) are the new kind of equations, which can hold on the non-uniform points. For example, they can hold on the standard points (monads) in turbulence. But the Navier-Stokes equations hold only on the uniform points. In the case of turbulence, the Navier-Stokes equations hold only on the nonstandard points. It is obvious logically that the discretization of the Navier-Stokes equations must be based on the nonstandard points, while discretization for the equations (1)-(5) are based on the standard points (monads) in numerical computation of turbulence. The standard and
nonstandard points are points in different levels, the scale of standard point is $\sim \varepsilon$ (Here $\varepsilon$ is the scale of the monad of global turbulent field.), but the non-standard point has the scale $\sim \varepsilon^3$. This is the reason why the Navier-Stokes equations must be computed by use of very small grids, while the equations (1)-(5) could be simulated by coarse grids in the case of turbulence.

By the analysis above, we know that when numerical computation is taken for the Navier-Stokes equations, there must be a limit to the grids obtained from discretization of the Navier-Stokes equations. The limit is just that the grids must be uniform, i.e. there is not structure through every grid wholly. The direct numerical simulation (DNS) is generally recognized as a good method of computation for the Navier-Stokes equations in turbulence now. But the grid of discretization in DNS is not so small that there is not structure in the grid. It is known that there still exists structure (so called smallest vortices) in the discretization-grid of DNS. Therefore, the method of DNS is not reasonable from the angle of the Navier-Stokes equations in mathematics. But the results of the computation in DNS are very well, why? The reason lies in that when DNS is made, people do not really compute the Navier-Stokes equations, but those in the NATT (i.e., the equations (1) in Choice one). The results of DNS are not the values of instantaneous quantities at the standard point, but average quantities over the grid. Hence, the success of DNS could be taken for example of the reasonableness of the NATT.

Moreover, it is well known that many researchers have used coarse grids for turbulence simulation. They computed the Navier-Stokes equations by coarse grids in turbulence. Their computations are not reasonable from the angle of the Navier-Stokes equations. But the results of their com-
computations are well agreeable to the measurements. Also this is an example for
the reasonableness of the NATT, because what they have simulated are not
the Navier-Stokes equations but the equations (1). The equations (1) are the
new kind of equations, which can hold on the standard points(monads). The
course grids are obtained from the discretization based on the standard points.

References

[1] F.Wu, *Nonstandard Picture of Turbulence (The Second Revised)*,
physics/0308012(lanl.arXiv), 2003

[2] T.Kawamura and K.Kuwahara, Computation of high Reynolds number flow
around a circular cylinder with surface roughness, AIAA paper, 84-0340

[3] K.Kuwahara and S.Komurasaki, Simulation of high Reynolds number flows
using multidirectional upwind scheme, AIAA paper, 2002-0133

[4] P.N.Shankar and M.D.Deshpande, Fluid mechanics in the driven cavity,
Annu.Rev.Fluid.Mech., 32:93-136,2000

[5] J.Salat,S.Xin,P.Joubert,A.Sergent,F.Penot,P.Le Quere, Experimental and
numerical investigation of turbulent natural convection in a large air-filled
cavity, Int.J.Heat and Fluid Flow, 25,824-832, 2004