PHASE TRANSITION FOR THE FROG MODEL
ON BIREGULAR TREES

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Abstract. We study the frog model with death on the biregular tree $\mathbb{T}_{d_1,d_2}$. Initially, there is a random number of awake and sleeping particles located on the vertices of the tree. Each awake particle moves as a discrete-time independent simple random walk on $\mathbb{T}_{d_1,d_2}$ and has a probability of death $(1 - p)$ before each step. When an awake particle visits a vertex which has not been visited previously, the sleeping particles placed there are awakened. We prove that this model undergoes a phase transition: for values of $p$ below a critical probability $p_c$, the system dies out almost surely, and for $p > p_c$, the system survives with positive probability. We establish explicit bounds for $p_c$ in the case of random initial configuration. For the model starting with one particle per vertex, the critical probability satisfies $p_c(\mathbb{T}_{d_1,d_2}) = 1/2 + \Theta(1/d_1 + 1/d_2)$ as $d_1, d_2 \to \infty$.

1. Introduction

This paper concerns the issue of phase transition for the frog model with death, a discrete-time growing system of simple random walks on a rooted graph $G$, which is described as follows. Initially there is an independent random number of particles at each vertex of $G$. All particles are sleeping at time zero, except for those that might be placed at $\emptyset$, the root of $G$. Each awake (active) particle moves as a discrete-time independent simple random walk (SRW) on the vertices of $G$, and has a probability of death $(1 - p)$ before each step. When an awake particle visits a sleeping particle, that particle becomes active and starts to walk, performing exactly the same dynamics, independently of everything else. The particles are referred to as frogs, we will continue the tradition here. This process can be thought as a model for describing rumor (or infection) spreading. Think that every awake particle is an informed (or infected) agent and it shares the rumor with (or infects) a sleeping particle at the first time they meet.

In recent years, there has been considerable interest in the behavior of stochastic systems on more general graphs than the $d$-dimensional integer lattice $\mathbb{Z}^d$ or the

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homogeneous tree $\mathbb{T}_d$ of degree $(d + 1)$. For instance, work has been done on the contact process, percolation models, branching random walks and related models. For the frog model, most of the work has involved studying the process on $\mathbb{Z}^d$, $\mathbb{T}_d$ and recently on $d$-ary trees. As far as we know, only Rosenberg [15] considers another kind of tree, proving the recurrence of the process (without death) in a 3,2-alternating tree (in which the generations of vertices alternate between having 2 and 3 children).

The first published paper dealing with the frog model (with $p = 1$, $G = \mathbb{Z}^d$) is due to Telcs and Wormald [16], where it was referred to as the “egg model”. They proved that, starting from the one-particle-per-vertex initial configuration, almost surely infinitely many frogs will visit the origin for all $d \geq 3$ (that is, although each frog is individually transient, the process is recurrent). Popov [12] exhibits the critical rate at which the frog model with Bernoulli($\alpha/||x||^2$) sleeping frogs at each $x \in \mathbb{Z}^d \setminus \{0\}$ changes from transience to recurrence. A similar result is obtained by Hoffman et al. [8] for the model on $d$-ary trees, with Poisson($\mu$) sleeping frogs at each vertex. More precisely, the authors prove that the model undergoes a phase transition between transience and recurrence, as the initial density $\mu$ of particles increases. For the model starting with one particle per vertex, Hoffman et al. [9] establish that there is a phase transition in the dimension of the tree, by proving recurrence for $d = 2$ and transience for $d \geq 5$. Based on simulations, they conjecture that the model is recurrent for $d = 3$, and transient for $d = 4$.

In Alves et al. [3], for the frog model without death on $\mathbb{Z}^d$, it is proved that, starting from the one-particle-per-vertex initial configuration, the set of the original positions of all awake particles, rescaled by the elapsed time, converges to a nonempty compact convex set. Alves et al. [1] prove the same statement in the case of random initial configuration; these results are known as shape theorems. For a continuous-time version of the frog model, a limiting shape result is stated by Ramírez and Sidoravicius [14]. We refer to Popov [13] for a survey on some results for the model and its variations.

Regarding the frog model with death, the existence of phase transition as $p$ varies was first studied by Alves et al. [2], especially on $\mathbb{Z}^d$ and $\mathbb{T}_d$. As we will detail
later, the occurrence of phase transition means that there is a nontrivial value of $p$ separating the phases of extinction and survival of the process. Lebensztayn et al. [11] prove that the critical probability for the frog model on a homogeneous tree of degree $(d + 1)$ is at most $(d + 1)/(2d)$; that result is an improvement of the upper bound stated by Fontes et al. [5], namely, $(d + 1)/(2d - 2)$. Further improvements on the upper bound for this critical probability were recently obtained by Gallo and Rodríguez [6], using Renewal Theory. For more details on the subject, see these papers and references therein.

The aim of the present paper is to deepen the study of the critical phenomenon of frog model on infinite graphs, particularly on nonhomogeneous trees. We consider the model on a specific class of nonhomogeneous trees, namely, biregular trees. The main results present explicit bounds for the critical probability, in the case of random initial configuration. Since there is no a single parameter measuring the size of such trees, bounds on the critical parameter are harder to get at. We also obtain the asymptotic behavior of the critical probability for large values of the dimension of the tree.

To finish the section, let us present some basic definitions and notations of Graph Theory. Let $G = (V, E)$ be an infinite connected locally finite graph, with vertex-set $V$ and edge-set $E$. A vertex $∅ \in V$ is fixed and called the root of $G$. We denote an unoriented edge with endpoints $x$ and $y$ by $xy$. Vertices $x$ and $y$ are said to be neighbors if they belong to a common edge $xy$; we denote this by $x \sim y$. The degree of a vertex is the number of its neighbors. A path of length $n$ from $x$ to $y$ is a sequence $x = x_0, \ldots, x_n = y$ of vertices such that $x_i \sim x_{i+1}$ for all $i = 0, \ldots, n - 1$. The graph distance $\text{dist}(x, y)$ between $x$ and $y$ is the minimal length of a path connecting the two vertices; the level of $x$ is $\text{dist}(∅, x)$. A tree is a connected graph without loops or cycles, where by a cycle in a graph we mean a sequence of vertices $x_0, \ldots, x_n$, $n \geq 3$, with no repetitions besides $x_n = x_0$. A graph is bipartite if its vertex-set $V$ can be partitioned into two subsets $V_1$ and $V_2$, in such a form that every edge joins a vertex of $V_1$ to a vertex of $V_2$. For $d_1 \geq 1$ and $d_2 \geq 1$, we denote by $T_{d_1, d_2}$ the $(d_1, d_2)$-biregular tree, which is the bipartite tree where the degree of a vertex is $(d_1 + 1)$ or $(d_2 + 1)$, according to the level of the vertex is even or odd. In this case,
the class $V_1 \ [V_2]$ is the set of vertices at even [odd] distance from the root. From now on, a vertex $x \in V_i$ will be called a type $i$ vertex. Notice that $T_{1,1}$ is isomorphic to $Z$. See Figure 1 for a illustration of $T_{2,4}$.

Figure 1. The first three levels of the biregular tree $T_{2,4}$.

2. Formal description of the model and main results

Now we describe the model in a formal way, keeping the notation of Alves et al. \cite{2} and Lebensztayn et al. \cite{11}, whenever possible. We write $\mathbb{N} = \{1, 2, \ldots \}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\eta$ be a random variable assuming values in $\mathbb{N}_0$, and define $\rho_k := P[\eta = k]$, $k \in \mathbb{N}_0$. We suppose that $P[\eta \geq 1] > 0$, that is, $\rho_0 < 1$. For $s \in [0, 1]$, let $\varphi(s) := E[s^\eta]$ be the probability generating function of $\eta$. To define the frog model, let $\{\eta(x) : x \in V\}$, $\{(S^*_n(k))_{n \in \mathbb{N}_0} ; k \in \mathbb{N}, x \in V\}$ and $\{\Xi^*_p(k) ; k \in \mathbb{N}_0, x \in V\}$ be independent sets of random objects defined as follows. For each $x \in V$, $\eta(x)$ has the same law as $\eta$, and gives the initial number of frogs at vertex $x$. If $\eta(x) \geq 1$, then for each $k \in \{1, \ldots, \eta(x)\}$, $(S^*_n(k))_{n \in \mathbb{N}_0}$ is a discrete-time SRW on $T_{d_1,d_2}$ starting from $x$, and $\Xi^*_p(k)$ is a random variable whose law is given by $P[\Xi^*_p(k) = j] = (1 - p)p^{j-1}$, $j \in \mathbb{N}$, where $p \in [0, 1]$ is a fixed parameter. These random objects describe respectively the trajectory and the lifetime of the $k$-th particle placed initially at $x$. Thus, the $k$-th particle at vertex $x$, whenever it is activated, follows the SRW $(S^*_n(k))_{n \in \mathbb{N}_0}$, and disappears at the instant it reaches...
a total of $(\Xi_p^x(k) - 1)$ jumps. At the moment the particle disappears, it is not able to activate other particles (first the particle decides whether or not to survive, and only after that it is allowed to jump). There is no interaction between awake particles. We call this model the frog model on $T_{d_1, d_2}$ with survival parameter $p$ and initial configuration ruled by $\eta$, and denote it by $\text{FM}(T_{d_1, d_2}, p, \eta)$. For $\eta \equiv 1$ (one-particle-per-vertex initial configuration), we simply write $\text{FM}(T_{d_1, d_2}, p)$.

**Definition 2.1.** A particular realization of the frog model survives if for every instant of time there is at least one awake particle. Otherwise, we say that it dies out.

A coupling argument shows that $P[\text{FM}(T_{d_1, d_2}, p, \eta) \text{ survives}]$ is a nondecreasing function of $p$, and therefore we define the critical probability as

$$p_c(T_{d_1, d_2}, \eta) = \inf \{ p : P[\text{FM}(T_{d_1, d_2}, p, \eta) \text{ survives}] > 0 \}.$$  

As usual, we say that $\text{FM}(T_{d_1, d_2}, p, \eta)$ exhibits phase transition if $p_c(T_{d_1, d_2}, \eta) \in (0, 1)$. For the one-particle-per-vertex initial configuration, we drop the dependency on $\eta$, and write $p_c(T_{d_1, d_2})$.

As proved by Alves et al. [2, Theorem 1.1], if $d_1 = d_2 = 1$, then under the condition $E \log (\eta \vee 1) < \infty$, we have that the frog model dies out almost surely for every $p < 1$, that is, $p_c(Z, \eta) = 1$. The picture is quite different for higher degrees. Indeed, Alves et al. [2, Theorems 1.2 and 1.5] prove that the frog model on $T_d$ exhibits phase transition for every $d \geq 2$, provided that $\rho_0 < 1$ and $E\eta^{\delta} < \infty$ for some $\delta > 0$. Here we prove a similar result when the process lives on $T_{d_1, d_2}$.

**Theorem 2.1.** Suppose that $\rho_0 < 1$ and $E\eta < \infty$. If $d_1 \geq 2$ or $d_2 \geq 2$, then

$$0 < p_c(T_{d_1, d_2}, \eta) < 1.$$  

Actually, we prove Theorem 2.1 by stating nontrivial bounds for the critical probability. First we give a sufficient condition for the almost sure extinction of the process.

**Theorem 2.2.** Suppose that $E\eta < \infty$. If $d_1 \geq 2$ or $d_2 \geq 2$, then

$$p_c(T_{d_1, d_2}, \eta) \geq \sqrt{\frac{(d_1 + 1)(d_2 + 1)}{d_1(E\eta + 1) + 1}[d_2(E\eta + 1) + 1]}.$$  


To present the upper bound for $p_c(T_{d_1,d_2}, \eta)$, we need the following definition.

**Definition 2.2.** Let $q = 1 - \rho_0 > 0$, $\kappa = (d_1 + 1)(d_2 + 1)$, $\Delta = \kappa^2 - 2\kappa(d_1 + d_2)p^2 + (d_2 - d_1)^2 p^4$. We also define the functions $\alpha, \beta, f : [0, 1] \to [0, 1]$ given by

$$
\alpha(p) = \alpha^{(d_1,d_2)}(p) = \begin{cases} 
\frac{\kappa + p^2(d_2 - d_1) - \sqrt{\Delta}}{2d_2(d_1 + 1)p} & \text{if } 0 < p \leq 1, \\
0 & \text{if } p = 0,
\end{cases} \quad (2.1)
$$

$$
\beta(p) = \beta^{(d_1,d_2)}(p) = \begin{cases} 
\frac{\kappa + p^2(d_1 - d_2) - \sqrt{\Delta}}{2d_1(d_2 + 1)p} & \text{if } 0 < p \leq 1, \\
0 & \text{if } p = 0,
\end{cases} \quad (2.2)
$$

$$
f^{(d_1,d_2,q)}(p) = \alpha(p) \beta(p)[1 + q(1 - \alpha(p))][1 + q(1 - \beta(p))] - \frac{1}{d_1d_2}.
$$

**Theorem 2.3.** Suppose that $\rho_0 < 1$. If $d_1 \geq 2$ or $d_2 \geq 2$, then

$$
p_c(T_{d_1,d_2}, \eta) \leq \tilde{p}(d_1,d_2,q),
$$

where $\tilde{p}(d_1,d_2,q)$ is the unique root of the function $f^{(d_1,d_2,q)}$ in the interval $(0, 1)$.

**Corollary 2.1.** Suppose that $\eta \equiv 1$. If $d_1 \geq 2$ or $d_2 \geq 2$, then

$$
p_c(T_{d_1,d_2}) \leq \frac{1}{2} \sqrt{\frac{(d_1 + 1)(d_2 + 1)}{d_1d_2}}.
$$

**Remark.** In the case of $d_1 = d_2 = d$ and $\eta \equiv 1$ in Theorem 2.3 (also in Corollary 2.1), we obtain the upper bound for the critical probability of FM($T_d,p$) proved in Lebensztayn et al. [11, Section 4], namely, $(d + 1)/(2d)$.

The following result concerns the asymptotic behavior of $p_c(T_{d_1,d_2})$ as $d_1, d_2 \to \infty$.

The lower and upper bounds given in Theorem 2.2 with $\eta \equiv 1$ and Corollary 2.1 are, respectively,

$$
\frac{1}{2} + \frac{1}{8} \left( \frac{1}{d_1} + \frac{1}{d_2} \right) + O \left( \frac{1}{d_1^2} + \frac{1}{d_2^2} \right) \quad \text{and} \quad \frac{1}{2} + \frac{1}{4} \left( \frac{1}{d_1} + \frac{1}{d_2} \right) + O \left( \frac{1}{d_1^2} + \frac{1}{d_2^2} \right).
$$

As a consequence, we find the correct order of magnitude for the critical probability.

**Corollary 2.2.** For the frog model on $T_{d_1,d_2}$ starting from the one-particle-per-vertex initial configuration, we have

$$
p_c(T_{d_1,d_2}) = \frac{1}{2} + \Theta \left( \frac{1}{d_1} + \frac{1}{d_2} \right) \quad \text{as} \quad d_1, d_2 \to \infty.
$$
For comparison, we include a result proved by Alves et al. [2], that establishes a lower bound for the critical probability of the frog model on bounded degree graphs.

**Proposition 2.1.** Suppose that $G$ is a graph with maximum degree $(D + 1)$, and $E\eta < \infty$. Then,

$$p_c(G, \eta) \geq \frac{D + 1}{D(E\eta + 1) + 1}.$$

Notice that, for $G = T_{d_1,d_2}$ with $d_1 = d_2$, the lower bound given in Theorem 2.2 equals the one provided by Proposition 2.1. However, for $d_1 \neq d_2$, the lower bound stated in Theorem 2.2 is better than that of Proposition 2.1. For a numerical illustration, we consider $\eta \equiv 1$, and compute the bounds established in Theorems 2.2, 2.3 and Proposition 2.1 for some values of the pair $(d_1, d_2)$. The resulting values are given in Table 1.

| $(d_1, d_2)$ | LB Proposition 2.1 | LB Theorem 2.2 | UB Theorem 2.3 |
|-------------|---------------------|---------------|---------------|
| (1, 2)      | 0.6000              | 0.6325        | 0.8588        |
| (1, 3)      | 0.5714              | 0.6172        | 0.8039        |
| (1, 4)      | 0.5556              | 0.6086        | 0.7749        |
| (2, 2)      | 0.6000              | 0.6000        | 0.7500        |
| (2, 3)      | 0.5714              | 0.5855        | 0.7063        |
| (2, 4)      | 0.5556              | 0.5774        | 0.6828        |
| (3, 100)    | 0.5025              | 0.5359        | 0.5771        |
| (3, 1000)   | 0.5002              | 0.5347        | 0.5743        |
| (4, 10000)  | 0.5000              | 0.5271        | 0.5572        |

**Table 1.** Numerical values of the bounds on the critical probability $p_c(T_{d_1,d_2})$.

In the next section, we present the proof of Theorem 2.2. The central idea is to construct a multitype branching process that dominates the frog model on $T_{d_1,d_2}$ (in the sense that the frog model becomes extinct if this process does). The question of survival (Theorem 2.3 and Corollary 2.1) is addressed in Section 4. The main steps to prove Theorem 2.3 are as follows. First, in Subsection 4.1, we describe the frog model on $T_{d_1,d_2}$ as a particular anisotropic percolation model, and we compute its parameters. Then, in Subsection 4.2, we define a sequence of Galton–Watson branching processes dominated by the frog model. Finding $p$ such that each one of these branching processes is supercritical leads us to obtain a sequence of upper.
bounds for the critical probability, which converges to the upper bound \( \tilde{p}(d_1, d_2, q) \) stated in Theorem 2.3. Subsection 4.3 is devoted to the finalization of the proof.

3. Extinction of the process

Our strategy to prove Theorem 2.2 is to compare the frog model on \( T_{d_1, d_2} \) with a suitable subcritical multitype Galton–Watson branching process.

**Proof of Theorem 2.2.** Consider a multitype Galton–Watson branching process with two types, whose progeny distribution is as follows. For \( i = 1, 2 \) and \( k_1, k_2 \in \mathbb{N}_0 \), let \( p^{(i)}(k_1, k_2) \) denote the probability that a type \( i \) particle produces \( k_1 \) particles of type 1, and \( k_2 \) particles of type 2. Recall that \( \rho_k := \mathbb{P}[\eta = k] \), and define

\[
\begin{align*}
    p^{(1)}(0, 0) &= 1 - p, & p^{(1)}(0, 1) &= p + \frac{d_1 \rho_0}{d_1 + 1}, & p^{(1)}(0, k) &= \frac{p d_1 \rho_{k-1}}{d_1 + 1}, & k &= 2, 3, \ldots, \\
    p^{(2)}(0, 0) &= 1 - p, & p^{(2)}(1, 0) &= p + \frac{d_2 \rho_0}{d_2 + 1}, & p^{(2)}(k, 0) &= \frac{p d_2 \rho_{k-1}}{d_2 + 1}, & k &= 2, 3, \ldots,
\end{align*}
\]

Notice that, in the frog model, every vertex with at least one awake particle at time \( n \in \mathbb{N} \) has at least one neighbor vertex whose original particles have been activated prior to time \( n \). Thus, the frog model on \( T_{d_1, d_2} \) is dominated by the multitype Galton–Watson process just defined. Now let \( M := (m_{i,j})_{i,j \in \{1,2\}} \) be the first moment matrix, i.e., \( m_{i,j} \) is the expected number of type \( j \) offspring of a single type \( i \) particle in one generation. Since the number of types is finite, it is well known (cf. Athreya and Ney [4, Chap. V]) that the multitype Galton–Watson process dies out almost surely if and only if the largest eigenvalue \( \lambda(M) \) of the matrix \( M \) is less than 1. An elementary calculation shows that if

\[
p < \sqrt{\frac{(d_1 + 1)(d_2 + 1)}{d_1(\mathbb{E}[\eta] + 1)[d_2(\mathbb{E}[\eta] + 1) + 1]}},
\]

then \( \lambda(M) < 1 \), therefore the multitype Galton–Watson process dies out almost surely. Consequently, the same happens to the frog model. \( \square \)

4. Survival of the process

To prove Theorem 2.3, the key idea is to describe \( \text{FM}(T_{d_1, d_2}, p, \eta) \) as a percolation model which dominates suitably defined Galton–Watson branching processes.
4.1. FM($\mathbb{T}_{d_1,d_2}, p, \eta$) seen as percolation. In the bond percolation model, each edge of an infinite locally finite graph $\bar{G} = (\bar{V}, \bar{E})$ is randomly assigned the value 1 (open) or 0 (closed), according to some probability measure on the product space $\{0,1\}^{\bar{E}}$. Then, one studies the connectivity properties of the random subgraph of $\bar{G}$ which arises by removing closed edges. For a fundamental reference on the subject, we refer to Grimmett [7].

Next, we describe the frog model on $\mathbb{T}_{d_1,d_2}$ as a particular bond percolation model. Indeed, for every $x \in V$ and $1 \leq k \leq \eta(x)$, we define the virtual set of vertices visited by the $k$-th frog located originally at $x$ by

$$R^k_x := \{S^x_n(k) : 0 \leq n < \Xi^x_p(k)\} \subset V.$$  

The set $R^k_x$ becomes real in the case when $x$ is actually visited (and thus the sleeping particles placed there are activated). We define the range of $x$ by

$$R_x := \begin{cases} \bigcup_{k=1}^{\eta(x)} R^k_x & \text{if } \eta(x) \geq 1, \\ \{x\} & \text{otherwise}. \end{cases}$$

Now let $\mathbb{T}_{d_1,d_2} = (\mathbb{V}, \mathbb{E})$ be the oriented graph with vertex-set $\mathbb{V}$ and edge-set $\mathbb{E} := \{xy : (x,y) \in \mathbb{V} \times \mathbb{V}, x \neq y\}$. That is, for every pair of distinct vertices $x$ and $y$, an oriented edge is drawn from $x$ to $y$. Then, we introduce the following notations: for $x, y \in \mathbb{V}$ distinct, $[x \rightarrow y] := [y \in R_x]$, $[x \nrightarrow y] := [y \notin R_x]$. This defines an oriented dependent long range anisotropic percolation model on $\mathbb{T}_{d_1,d_2}$: $x \rightarrow y$ is declared to be open if $[x \rightarrow y]$, and closed otherwise. Observe that the probability that an oriented edge is open depends on its orientation: an edge emanating from a type 1 vertex to a type 2 vertex has probability of being open different from an edge emanating from a type 2 vertex to a type 1 vertex. This feature gives the anisotropy of the model. Our next step is to determine an explicit formula for the probability that an oriented edge is open. Toward this end, let $\alpha^{(d_1,d_2)}(p) [\beta^{(d_1,d_2)}(p)]$ denote the probability that a type 1 [type 2] particle ever visits a type 2 [type 1] neighbor vertex. Avoiding a cumbersome notation, we sometimes write $\alpha(p)$ or simply $\alpha [\beta(p), \beta]$. With a typical abuse of language, we call this model as an anisotropic percolation model on the biregular tree $\mathbb{T}_{d_1,d_2}$, with parameters $\alpha$, $\beta$ and ruled by $\eta$. We denote it by APM($\mathbb{T}_{d_1,d_2}, \alpha, \beta, \eta$).
Notice that the cluster of the root in \( \text{APM}(T_{d_1,d_2}, \alpha, \beta, \eta) \) has infinite size if and only if there exists an infinite sequence of distinct vertices \( \emptyset = x_0, x_1, x_2, \ldots \) such that \( x_j \to x_{j+1} \) for all \( j \geq 1 \). Of course, this event is equivalent to the survival of \( \text{FM}(T_{d_1,d_2}, p, \eta) \).

The following lemma about hitting probabilities of SRWs on \( T_{d_1,d_2} \) provides a formula for \( \alpha \) and \( \beta \). Recall that \( V_1 \) and \( V_2 \) are respectively the set of vertices at even and odd distance from the root of \( T_{d_1,d_2} \).

**Lemma 4.1.** Let \( x \sim y \) be a pair of neighbor vertices, and suppose \( (x,y) \in V_i \times V_j \), with \( i,j \in \{1,2\}, i \neq j \). Then,

\[
\mathbb{P}[y \in R_x^1] = \begin{cases} 
\alpha(p) & \text{if } i = 1, j = 2, \\
\beta(p) & \text{if } i = 2, j = 1,
\end{cases}
\]

where \( \alpha \) and \( \beta \) are given in Definition 2.2.

**Proof.** Let \( \tau_{ij} := \tau_{xy} \) be the first time when the simple random walk on \( T_{d_1,d_2} \) starting from \( x \) visits \( y \). Suppose first that \( p < 1 \), so conditioning on the lifetime of the particle located at \( x \), we have

\[
\mathbb{P}[y \in R_x^1] = \mathbb{E}[p^{\tau_{ij}}].
\]

Now by conditioning on the first jump of the frog at \( x \), we have

\[
\mathbb{E}[p^{\tau_{12}}] = \frac{1}{d_1+1}p + \frac{d_1}{d_1+1}p\mathbb{E}[p^{\tau_{12}}]\mathbb{E}[p^{\tau_{21}}]
\]

\[
\mathbb{E}[p^{\tau_{21}}] = \frac{1}{d_2+1}p + \frac{d_2}{d_2+1}p\mathbb{E}[p^{\tau_{12}}]\mathbb{E}[p^{\tau_{21}}]
\]

By right-continuity of the probability generating functions of \( \tau_{12} \) and \( \tau_{21} \), it follows that \( \lim_{p \to 0^+} \mathbb{E}[p^{\tau_{12}}] = \lim_{p \to 0^+} \mathbb{E}[p^{\tau_{21}}] = 0 \). Therefore, (2.1) and (2.2) are the only possible solutions for the previous system of equations. This concludes the proof for \( p < 1 \). Finally, if \( p = 1 \), then

\[
\mathbb{P}[y \in R_x^1] = \mathbb{P}[\tau_{ij} < \infty] = \lim_{p \to 1^-} \mathbb{E}[p^{\tau_{ij}}] = \frac{d_j + 1}{d_j(d_i + 1)}.
\]

\[\square\]

**Remark.** If \( d_1 \leq d_2 \), then \( \alpha^{(d_1,d_2)}(p) \geq \beta^{(d_1,d_2)}(p) \).
**Lemma 4.2.** Let $x$ and $y$ be two vertices of $\mathbb{T}_{d_1,d_2}$ with $(x,y) \in V_i \times V_j$, $i, j \in \{1, 2\}$, and let $k = \text{dist}(x,y) \geq 1$. Let $\pi_\eta(i,j,k)$ denote the probability that the oriented edge $\overrightarrow{xy}$ is open. Then,

$$
\pi_\eta(i,j,k) = \begin{cases} 
1 - \varphi(1 - \alpha^n\beta^{n-1}) & \text{if } i = 1, j = 2, k = 2n - 1, \\
1 - \varphi(1 - \alpha^{n-1}\beta^n) & \text{if } i = 2, j = 1, k = 2n - 1, \\
1 - \varphi(1 - \alpha^n\beta^n) & \text{if } i = j, k = 2n,
\end{cases}
$$

(4.1)

where $\varphi(s) = \mathbb{E}(s^n)$, $s \in [0,1]$, is the probability generating function of $\eta$.

**Proof.** By conditioning on the initial number of frogs at vertex $x$, it is enough to consider $\eta \equiv 1$, that is, $\varphi(s) \equiv s$. In this case, $\pi_\eta(i,j,k)$ is the probability that a type $i$ particle ever visits a type $j$ vertex at distance $k$. So, we need to prove that for $\eta \equiv 1$,

$$
\pi_\eta(i,j,k) = \begin{cases} 
\alpha^n\beta^{n-1} & \text{if } i = 1, j = 2, k = 2n - 1, \\
\alpha^{n-1}\beta^n & \text{if } i = 2, j = 1, k = 2n - 1, \\
\alpha^n\beta^n & \text{if } i = j, k = 2n.
\end{cases}
$$

(4.2)

Formula (4.2) follows from the fact that if $(x,y) \in V_1 \times V_2$ with $k = 2n - 1$, then the first time when the frog starting from $x$ visits $y$ is equal in distribution to the sum of $2n - 1$ independent random variables, such that $n$ of them are independent copies of $\tau_{12}$ and $n - 1$ of them are independent copies of $\tau_{21}$. Using this fact and Lemma 4.1, we obtain that

$$
\pi_\eta(1,2,2n-1) = \alpha^n\beta^{n-1}.
$$

The other cases are analogous. $\square$

4.2. A sequence of branching processes dominated by the frog model. The central idea is as follows. For each $n \geq 1$, we define a Galton–Watson branching process whose survival implies that the cluster of the root in $\text{APM}(\mathbb{T}_{d_1,d_2}, \alpha, \beta, \eta)$ has infinite size. For each branching process, we find a sufficient condition which guarantees that the process is supercritical. We get in this manner a sequence of upper bounds for the critical probability, which converges to the upper bound given in the statement of Theorem 2.3. This technique of using embedded branching processes is very similar to that used for the contact process on homogeneous trees; see the paper by Lalley and Sellke [10].

Let us now carry out this plan.
Definition 4.1.

(i) We define a partial order on the set $\mathbb{V}$ as follows: for $x, y \in \mathbb{V}$, we say that $x \preceq y$ if $x$ belongs to the path connecting $\emptyset$ and $y$; $x \prec y$ if $x \preceq y$ and $x \neq y$.

(ii) For any vertex $x \neq \emptyset$, let $\mathbb{V}^+(x) = \{ y \in \mathbb{V} : x \preceq y \}$. Also define $\mathbb{V}^+(\emptyset) = \mathbb{V} \setminus \mathbb{V}^+(z)$, where $z$ is a fixed vertex neighbor to $\emptyset$.

(iii) For $x \in \mathbb{V}$ and $k \in \mathbb{N}$, define $L_k(x) = \{ y \in \mathbb{V}^+(x) : \text{dist}(x, y) = k \}$.

Definition 4.2. For $x \in \mathbb{V}$ and $y \in L_k(x)$, $k \in \mathbb{N}$, consider $x_0 = x \prec x_1 \prec \cdots \prec x_{k-1} \prec x_k = y$, the path connecting $x$ and $y$. For each $\ell = 1, 2, \ldots, n - 1$, we denote by $[x_0 \sim x_\ell]$ the event that $[x_0 \rightarrow x_\ell] \cap [x_0 \nrightarrow x_{\ell+1}]$. We define the event $[x_0 \rightarrow x_k]$ inductively on $k$ by:

(i) If $k = 1$, then $[x_0 \rightarrow x_k] := [x_0 \rightarrow x_k]$.

(ii) If $k \geq 2$, then $[x_0 \rightarrow x_k] := [x_0 \rightarrow x_k] \cup \bigcup_{\ell=1}^{k-1} [x_0 \sim x_\ell, x_\ell \rightarrow x_k]$.

Besides, we denote the complement of $[x \rightarrow y]$ by $[x \nrightarrow y]$.

Now we construct a sequence of Galton–Watson branching processes embedded in $\text{APM}(T_{d_1, d_2}, \alpha, \beta, \eta)$. Starting from the root, the potential direct descendants of a vertex $x$ are vertices located in $L_k(x)$, where $k = 2n$ is an even number. Roughly speaking, a vertex $y \in L_{2n}(x)$ is said to be a child of vertex $x$ if and only if $[x_0 \rightarrow y]$. More formally, let $n \in \mathbb{N}$ be fixed. Define $Y_{0,n} := \{ \emptyset \}$, and for $\ell \in \mathbb{N}$ define

$Y_{\ell,n} := \bigcup_{x \in Y_{\ell-1,n}} \{ y \in L_{2n}(x) : x \rightarrow y \}$.

Let $Y_{\ell,n} = |Y_{\ell,n}|$ be the cardinality of $Y_{\ell,n}$.

Lemma 4.3. For every $n \in \mathbb{N}$, $\{ Y_{\ell,n} \}_{\ell \in \mathbb{N}_0}$ is a Galton–Watson branching process whose survival implies the occurrence of percolation. In addition, its mean number of offspring per individual satisfies $E[Y_{1,n}] \geq (d_1 d_2)^n \phi_n^{(d_1, d_2, q)}(p)$, where

$$
\phi_n^{(d_1, d_2, q)}(p) := q[\alpha(p) \beta(p)(1 + q(1 - \beta(p))]^n[1 + q(1 - \alpha(p))]^{n-1}. \tag{4.3}
$$
Since the first claim in Lemma 4.3 is clear, to prove it, we have to compute $E[Y_{1,n}]$.

To accomplish this, we show a recursive formula for the probability of the event $[x \rightarrow y]$. Let us define the domain $A_{d_1,d_2} := \left[0, \frac{d_2+1}{d_2(d_1+1)}\right] \times \left[0, \frac{d_1+1}{d_1(d_2+1)}\right] \subseteq [0,1]^2$. In formulas that appear from this point on, we assume that a summation of the form $\sum^0_1$ equals 0.

Lemma 4.4. For $(x,y) \in \mathbb{V}_i \times \mathbb{V}_j$, $i,j \in \{1,2\}$, with $\text{dist}(x,y) = k \geq 1$, define

$$\nu_n(i,j,k) := \mathbb{P}[x \rightarrow y].$$

For every $n \geq 1$, there exists functions $K_n$, $F_n$, $K^*_n$, $F^*_n$ with domain $A_{d_1,d_2}$, not depending on $d_1, d_2$, such that

$$\nu_n(1,2,2n-1) = K_n(\alpha(p), \beta(p)), \quad \nu_n(1,1,2n) = F_n(\alpha(p), \beta(p)),$$

$$\nu_n(2,1,2n-1) = K^*_n(\alpha(p), \beta(p)), \quad \nu_n(2,2,2n) = F^*_n(\alpha(p), \beta(p)).$$

Proof. For $n \geq 1$ and $(a,b) \in A_{d_1,d_2}$, we define the following sequence of functions recursively:

$$K_n(a,b) = [1 - \varphi(1 - a^n b^{n-1})] + \sum_{\ell=1}^{n-1} [\varphi(1 - a^{\ell+1} b^{\ell}) - \varphi(1 - a^{\ell} b^{\ell})] K_{n-\ell}(a,b)$$

$$+ \sum_{\ell=1}^{n-1} [\varphi(1 - a^{\ell} b^{\ell}) - \varphi(1 - a^{\ell-1} b^{\ell-1})] F^*_{n-\ell}(a,b),$$

$$K^*_n(a,b) = [1 - \varphi(1 - b^n a^{n-1})] + \sum_{\ell=1}^{n-1} [\varphi(1 - b^{\ell+1} a^{\ell}) - \varphi(1 - b^{\ell} a^{\ell})] K^*_{n-\ell}(a,b)$$

$$+ \sum_{\ell=1}^{n-1} [\varphi(1 - a^{\ell} b^{\ell}) - \varphi(1 - a^{\ell-1} b^{\ell-1})] F_{n-\ell}(a,b),$$

$$F_n(a,b) = [1 - \varphi(1 - a^n b^n)] + \sum_{\ell=1}^{n-1} [\varphi(1 - a^{\ell+1} b^{\ell}) - \varphi(1 - a^{\ell} b^{\ell})] F_{n-\ell}(a,b)$$

$$+ \sum_{\ell=1}^{n} [\varphi(1 - a^{\ell} b^{\ell}) - \varphi(1 - a^{\ell-1} b^{\ell-1})] K^*_{n+1-\ell}(a,b),$$

$$F^*_n(a,b) = [1 - \varphi(1 - b^n a^n)] + \sum_{\ell=1}^{n-1} [\varphi(1 - b^{\ell+1} a^{\ell}) - \varphi(1 - b^{\ell} a^{\ell})] F^*_{n-\ell}(a,b)$$

$$+ \sum_{\ell=1}^{n} [\varphi(1 - a^{\ell} b^{\ell}) - \varphi(1 - b^{\ell} a^{\ell-1})] K_{n+1-\ell}(a,b).$$
Next, using that $P[x_0 \sim x_\ell] = P[x_0 \rightarrow x_\ell] - P[x_0 \rightarrow x_{\ell+1}]$, we obtain, for every $n \geq 1$,

$$P[x_0 \rightarrow x_{2n-1}] = P[x_0 \rightarrow x_{2n-1}] + \sum_{\ell=1}^{2n-2} P[x_0 \sim x_\ell]P[x_\ell \rightarrow x_{2n-1}]$$

$$= P[x_0 \rightarrow x_{2n-1}] + \sum_{\ell=1}^{2n-2} \{P[x_0 \rightarrow x_\ell] - P[x_0 \rightarrow x_{\ell+1}]\}P[x_\ell \rightarrow x_{2n-1}],$$

and

$$P[x_0 \rightarrow x_{2n}] = P[x_0 \rightarrow x_{2n}] + \sum_{\ell=1}^{2n-1} P[x_0 \sim x_\ell]P[x_\ell \rightarrow x_{2n}]$$

$$= P[x_0 \rightarrow x_{2n}] + \sum_{\ell=1}^{2n-1} \{P[x_0 \rightarrow x_\ell] - P[x_0 \rightarrow x_{\ell+1}]\}P[x_\ell \rightarrow x_{2n}].$$

We break the proof up into two cases:

(i) If $(x_0, x_{2n-1}) \in \mathbb{V}_i \times \mathbb{V}_j$ with $i \neq j$, then

$$\nu(i,j,2n-1) = \pi(i,j,2n-1) + \sum_{\ell=1}^{n-1} [\pi(i,i,2\ell) - \pi(i,j,2\ell+1)]\nu(i,j,2(n-\ell)-1)$$

$$+ \sum_{\ell=1}^{n-1} \pi(i,j,2\ell-1) - \pi(i,i,2\ell)\nu(j,j,2(n-\ell)).$$

(ii) If $(x_0, x_{2n}) \in \mathbb{V}_i \times \mathbb{V}_i$, then

$$\nu(i,i,2n) = \pi(i,i,2n) + \sum_{\ell=1}^{n-1} [\pi(i,i,2\ell) - \pi(i,j,2\ell+1)]\nu(i,i,2(n-\ell))$$

$$+ \sum_{\ell=1}^{n} \pi(i,j,2\ell-1) - \pi(i,i,2\ell)\nu(j,j,2(n-\ell)+1).$$

Using Equation (4.1), the result follows by induction on $n$. \qed

**Remark.** In general, the functions $K_n$, $F_n$, $K_n^*$ and $F_n^*$ are not polynomial in $(a,b)$.

**Proof of Lemma 4.3.** From Lemma 4.4, the Galton-Watson branching process has mean number of progeny per individual given by

$$E[Y_{1,n}] = (d_1d_2)^n \nu(1,1,2n) = (d_1d_2)^n F_n(\alpha(p), \beta(p)).$$

To obtain a lower bound for this expected value (not depending on the function $\varphi$), we truncate the initial configuration of the frog model. We consider the modified initial configuration $\eta'$ by

$$\eta'(x) = 1_{\{\eta(x) \geq 1\}}, \ x \in \mathbb{V}.$$
Since the initial condition $\eta'$ is dominated by $\eta$ in the usual stochastic order, it follows that

$$\nu_n(1, 1, 2n) \geq \nu_{\eta'}(1, 1, 2n).$$

But for the restricted frog model with initial configuration ruled by $\eta'$,

$$\nu_{\eta'}(1, 1, 2n) = F_n^{(q)}(\alpha(p), \beta(p)),$$

where $F_n^{(q)}(a, b)$ is given by the formulas stated in the proof of Lemma 4.4 with the choice $\varphi(s) = 1 - q(1 - s)$. Using these formulas and induction on $n$, we have that $F_n^{(q)}$ satisfies

$$F_{n+1}^{(q)}(a, b) = ab[1 + q(1 - a)][1 + q(1 - b)]F_n^{(q)}(a, b), \ n \geq 1,$$

with initial condition $F_1^{(q)}(a, b) = q[ab(1 + q(1 - b))]$. Consequently, for every $n \geq 1$,

$$F_n^{(q)}(a, b) = q[ab(1 + q(1 - b))][1 + q(1 - a)]^{n-1}.$$ 

The result follows by noting that $\phi_n^{(d_1, d_2, q)}(p)$ given in (4.3) is simply $F_n^{(q)}(\alpha(p), \beta(p))$. 

\[4.3. \textbf{Proofs of Theorem 2.3 and Corollary 2.1.}\] From Lemma 4.3 it follows that, by solving for each $n \geq 1$ the equation in $p$

$$(d_1 d_2)^n \phi_n^{(d_1, d_2, q)}(p) = 1,$$

we obtain a sequence of upper bounds for the critical probability $p_c(T_{d_1, d_2, \eta})$. So, for every $n \geq 1$, we define the function

$$f_n^{(d_1, d_2, q)}(p) = (\phi_n^{(d_1, d_2, q)}(p))^{1/n} - \frac{1}{d_1 d_2}. \tag{4.4}$$

Notice that, for $p \in [0, 1]$,

$$\lim_{n \to \infty} f_n^{(d_1, d_2, q)}(p) = f^{(d_1, d_2, q)}(p),$$

where $f^{(d_1, d_2, q)}(p)$ is defined in (2.3). To prove the upper bound presented in Theorem 2.3 we use the following fact of Real Analysis (which is stated without proof).

\[\textbf{Lemma 4.5.}\] Let $\{f_n\}$ be a sequence of increasing, continuous real-valued functions defined on $[0, 1]$, such that $f_n(0) < 0$ and $f_n(1) > 0$ for every $n$. Suppose that $\{f_n\}$ converges pointwise as $n \to \infty$ to an increasing, continuous function $f$ defined on
and let \( \tilde{r}_n \) be the unique root of \( f_n \) in \([0, 1] \). Then, there exists \( \tilde{r} = \lim_{n \to \infty} \tilde{r}_n \) and \( f(\tilde{r}) = 0 \).

**Proof of Theorem 2.3.** It is straightforward to prove that there exists a positive integer \( N = N(d_1, d_2) \) such that \( \{f_n^{(d_1, d_2, q)}\}_{n \geq N} \) and \( f^{(d_1, d_2, q)} \) defined in (4.4) and (2.3) satisfy the conditions of Lemma 4.5. Therefore, by defining \( \tilde{p}_n(d_1, d_2, q) \) as the unique root of \( f_n^{(d_1, d_2, q)} \) in the interval \([0, 1] \), it follows that

\[
\lim_{n \to \infty} \tilde{p}_n(d_1, d_2, q) = \tilde{p}(d_1, d_2, q),
\]

where the limit is the unique root of \( f^{(d_1, d_2, q)}(p) \) in \((0, 1)\). Since \( f_n^{(d_1, d_2, q)}(p) > 0 \) for every \( p > \tilde{p}_n(d_1, d_2, q) \), from Lemma 4.3, we have that

\[
p_c(T_{d_1, d_2}, \eta) \leq \tilde{p}_n(d_1, d_2, q).
\]

Taking \( n \to \infty \), the results follows. \( \square \)

**Proof of Corollary 2.1.** If \( \eta \equiv 1 \), then the function \( f^{(d_1, d_2, q)}(p) \) simplifies to

\[
f^{(d_1, d_2)}(p) = \alpha(p)\beta(p)[2 - \alpha(p)][2 - \beta(p)] - \frac{1}{d_1d_2}.
\]

The upper bound \( \bar{p}(d_1, d_2) \) for \( p_c(T_{d_1, d_2}) \) is the unique root in \((0, 1)\) of this continuous and increasing function. Let

\[
\bar{p} = \bar{p}(d_1, d_2) = \frac{1}{2} \sqrt{\frac{(d_1 + 1)(d_2 + 1)}{d_1d_2}}
\]

denote the upper bound given in the statement of Corollary 2.1. The result follows from the fact that \( f^{(d_1, d_2)}(\bar{p}) \geq 0 \). \( \square \)

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