THE FLAG MANIFOLED OVER THE SEMIFIELD Z

G. Lusztig

INTRODUCTION

0.1. Let $G$ be a connected semisimple simply connected algebraic group over $\mathbb{C}$ with a fixed pinning (as in [L94b, 1.1]). In this paper we assume that $G$ is of simply laced type. Let $B$ be the variety of Borel subgroups of $G$. In [L94b, 2.2, 8.8] a submonoid $G_{\geq 0}$ of $G$ and a subset $B_{\geq 0}$ of $B$ with an action of $G_{\geq 0}$ (see [L94b, 8.12]) was defined. (When $G = SL_n$, $G_{\geq 0}$ is the submonoid consisting of the real, totally positive matrices in $G$.) More generally, for any semifield $K$, a monoid $G(K)$ was defined in [L19b], so that when $K = \mathbb{R}_{>0}$ we have $G(K) = G_{\geq 0}$. (In the case where $K$ is the semifield as in (i),(ii) below, the monoid $G(K)$ already appeared in [L94b, 2.2, 9.10].)

In [L19b, 4.9], for any semifield $K$, a definition of a set $B(K)$ was given (based on ideas of Marsh and Rietsch [MR]) which when $K = \mathbb{R}_{>0}$ gives $B(K) = B_{\geq 0}$; but in that definition the lower and upper triangular part of $G$ play an asymmetric role and as a consequence only a part of $G(K)$ acts on $B(K)$ (unlike the case $K = \mathbb{R}_{>0}$ when the entire $G(K)$ acts). To get the entire $G(K)$ act one needs a conjecture stated in [L19b, 4.9] which is still open. In this paper we get around that conjecture and provide an unconditional definition of the set $B(K)$ with an action of $G(K)$, assuming that $K$ is either

(i) the semifield consisting of all rational functions in $\mathbb{R}(x)$ (with $x$ an indeterminate) of the form $x^e f_1 / f_2$ where $e \in \mathbb{Z}$ and $f_1 \in \mathbb{R}[x], f_2 \in \mathbb{R}[x]$ have constant term in $\mathbb{R}_{>0}$ (standard sum and product); or

(ii) the semifield $\mathbb{Z}$ in which the sum of $a, b$ is $\min(a, b)$ and the product of $a, b$ is $a + b$.

For $K$ as in (i) we give two definitions of $B(K)$; one of them is elementary and the other is less so, being based on the theory of canonical bases (the two definitions are shown to be equivalent). For $K$ as in (ii) we only give a definition based on the theory of canonical bases.

A part of our argument involves a construction of an analogue of the finite dimensional irreducible representations of $G$ when $G$ is replaced by the monoid $G(K_1)$ where $K_1$ is a semifield.

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Let $W$ be the Weyl group of $G$. Now $W$ is naturally a Coxeter group with generators $\{s_i; i \in I\}$ and length function $w \mapsto |w|$. Let $\leq$ be the Chevalley partial order on $W$.

In §3 we prove the following result which is a $\mathbb{Z}$-analogue of a result (for $\mathbb{R}_{>0}$) in [MR].

**Theorem 0.2.** The set $\mathcal{B}(\mathbb{Z})$ has a canonical partition into pieces $P_{v,w}(\mathbb{Z})$ indexed by the pairs $v \leq w$ in $W$. Each such piece $P_{v,w}(\mathbb{Z})$ is in bijection with $\mathbb{Z}^{|w|−|v|}$; in fact, there is an explicit bijection $\mathbb{Z}^{|w|−|v|} \cong P_{v,w}(\mathbb{Z})$ for any reduced expression of $w$.

In §3 we also prove a part of a conjecture in [L19b, 2.4] which attaches to any $v \leq w$ in $W$ a certain subset of a canonical basis, see 3.10.

In §4 we show that our definitions do not depend on the choice of a (very dominant) weight $\lambda$. In §5 we show how some of our results extend to the non-simply laced case and to the infinite dimensional case (including the case of partial flag manifolds).

**Contents**

1. Definition of $\mathcal{B}(\mathbb{Z})$.
2. Preparatory results.
3. Parametrizations.
4. Independence on $\lambda$.
5. Complements.

### 1. Definition of $\mathcal{B}(\mathbb{Z})$

#### 1.1. In this section we will give the definition of the flag manifold $\mathcal{B}(K)$ when $K$ is as in 0.1(i),(ii).

#### 1.2. We fix some notation on $G$. Let $w_I$ be the longest element of $W$. For $w \in W$ let $\mathcal{I}_w$ be the set of all sequences $i = (i_1,i_2,\ldots,i_m)$ in $I$ such that $w = s_{i_1}s_{i_2}\ldots s_{i_m}$, $m = |w|$.

The pinning of $G$ consists of two opposed Borel subgroups $B^+, B^−$ with unipotent radicals $U^+, U^−$ and root homomorphisms $x_i : \mathbb{C} \to U^+$, $y_i : \mathbb{C} \to U^−$ indexed by $i \in I$. Let $T = B^+ \cap B^−$, a maximal torus. Let $\mathcal{Y}$ be the group of one parameter subgroups $\mathbb{C}^* \to T$; let $\mathcal{X}$ be the group of characters $T \to \mathbb{C}^*$. Let $\langle , \rangle : \mathcal{Y} \times \mathcal{X} \to \mathbb{Z}$ be the canonical pairing. The simple coroot corresponding to $i \in I$ is denoted again by $i \in \mathcal{Y}$; let $i' \in \mathcal{X}$ be the corresponding simple root. Let $\mathcal{X}^+ = \{\lambda \in \mathcal{X}; \langle i, \lambda \rangle \geq 0 \quad \forall i \in I\}$, $\mathcal{X}^{++} = \{\lambda \in \mathcal{X}; \langle i, \lambda \rangle \geq 1 \quad \forall i \in I\}$. Let $G(\mathbb{R})$ be the subgroup of $G$ generated by $x_i(t), y_i(t)$ with $i \in I, t \in \mathbb{R}$. Let $\mathcal{B}(\mathbb{R})$ be the subset of $\mathcal{B}$ consisting of all $B \in \mathcal{B}$ such that $B = gB^+g^{-1}$ for some $g \in G(\mathbb{R})$. We have $G_{\geq 0} \subset G(\mathbb{R}), B_{\geq 0} \subset \mathcal{B}(\mathbb{R})$. For $i \in I$ we set $s_i = y_i(1)x_i(-1)y_i(1) \in G(\mathbb{R})$, an element normalizing $T$. For $(B, B') \in \mathcal{B} \times \mathcal{B}$ we write $pos(B, B')$ for the relative position of $B, B'$ (an element of $W$).
1.3. Let $K$ be a semifield. Let $K^\dagger = K \cup \{ \circ \}$ where $\circ$ is a symbol. We extend the sum and product on $K$ to a sum and product on $K^\dagger$ by defining $\circ + a = a$, $a + \circ = a$, $\circ \times a = \circ$, $a \times \circ = \circ$ for $a \in K$ and $\circ + \circ = \circ, \circ \times \circ = \circ$. Thus $K^\dagger$ becomes a monoid under addition and a monoid under multiplication. Moreover the distributivity law holds on $K^\dagger$. When $K$ is $\mathbb{R}_{\geq 0}$ we have $K^\dagger = \mathbb{R}_{\geq 0}$ with $\circ = 0$ and the usual sum and product. When $K$ is as in 0.1(i), $K^\dagger$ can be viewed as the subset of $\mathbb{R}(x)$ given by $K \cup \{ 0 \}$ with $\circ = 0$ and the usual sum and product. When $K$ is as in 0.1(ii) we have $0 \in K$ and $\circ \neq 0$.

1.4. Let $V = \lambda V$ be the finite dimensional simple $G$-module over $\mathbb{C}$ with highest weight $\lambda \in \mathcal{X}^\dagger$. For $\nu \in \mathcal{X}$ let $V_\nu$ be the $\nu$-weight space of $V$ with respect to $T$. Thus $V_\lambda$ is a line. We fix $\xi^+ = \lambda \xi^+$ in $V_\lambda - 0$. For each $i \in I$ there are well defined linear maps $e_i : V \rightarrow V, f_i : V \rightarrow V$ such that $x_i(t)\xi = \sum_{n \geq 0} t^n e_i^{(n)} \xi, y_i(t)\xi = \sum_{n \geq 0} t^n f_i^{(n)} \xi$ for $\xi \in V, t \in \mathbb{C}$. Here $e_i^{(n)} = (n!)^{-1} e_i^n : V \rightarrow V, f_i^{(n)} = (n!)^{-1} f_i^n : V \rightarrow V$ are zero for $n \gg 0$. For an integer $n < 0$ we set $e_i^{(n)} = 0, f_i^{(n)} = 0$.

Let $\beta = \lambda \beta$, $\beta$ be the canonical basis of $V$ (containing $\xi^+$) defined in [L90a]. Let $\xi^-$ be the lowest weight vector in $V - 0$ contained in $\beta$. For $b \in \beta$ we have $b \in V_\nu_b$ for a well defined $\nu_b \in \mathcal{X}$, said to be the weight of $b$. By a known property of $\beta$ (see [L90a, 10.11] and [L90b, 33], or alternatively [L93, 22.1.7]), for $i \in I, b \in \beta, n \in \mathbb{Z}$ we have

$$e_i^{(n)} b = \sum_{b' \in \beta} c_{b,b',i,n} b', \quad f_i^{(n)} b = \sum_{b' \in \beta} d_{b,b',i,n} b'$$

where

$$c_{b,b',i,n} \in \mathbb{N}, \quad d_{b,b',i,n} \in \mathbb{N}.$$ 

Hence for $i \in I, b \in \beta, t \in \mathbb{C}$ we have

$$x_i(t) b = \sum_{b' \in \beta, n \in \mathbb{N}} c_{b,b',i,n} t^n b', \quad y_i(t) b = \sum_{b' \in \beta, n \in \mathbb{N}} d_{b,b',i,n} t^n b'.$$

For any $i \in I$ there is a well defined function $z_i : \beta \rightarrow \mathbb{Z}$ such that for $b \in \beta, t \in \mathbb{C}^*$ we have $i(t) b = t^{z_i(b)} b$.

Let $P = \lambda P$ be the variety of $\mathbb{C}$-lines in $V$. Let $P^\bullet = \lambda P^\bullet$ be the set of all $L \in P$ such that for some $g \in G$ we have $L = gV_\lambda$. Now $P^\bullet$ is a closed subvariety of $P$. For any $L \in P^\bullet$ let $G_L = \{ g \in G; gL = L \}$; this is a parabolic subgroup of $G$.

Let $V^\bullet = \lambda V^\bullet = \bigcup_{L \in P^\bullet} L$, a closed subset of $V$. For any $\xi \in V, b \in \beta$ we define $\xi_b \in \mathbb{C}$ by $\xi = \sum_{b \in \beta} \xi_b b$. Let $V_{\geq 0} = \lambda V_{\geq 0}$ (resp. $V_{\mathbb{R}}$) be the set of all $\xi \in V$ such that $\xi_b \in \mathbb{R}_{\geq 0}$ (resp. $\xi_b \in \mathbb{R}$) for any $b \in \beta$. We have $V_{\geq 0} \subset V_{\mathbb{R}}$. Note that $V_{\mathbb{R}}$ is stable under the action of $G(\mathbb{R})$ on $V$. Let $P_{\geq 0} = \lambda P_{\geq 0}$ (resp. $P_{\mathbb{R}}$) be the set of lines $L \in P$ such that $L \cap V_{\geq 0} \neq 0$ (resp. $L \cap V_{\mathbb{R}} \neq 0$). We have $P_{\geq 0} \subset P_{\mathbb{R}}$.

Let $V_{\geq 0}^\bullet = \lambda V_{\geq 0}^\bullet = V^\bullet \cap V_{\geq 0}, P_{\geq 0}^\bullet = \lambda P_{\geq 0}^\bullet = P^\bullet \cap P_{\geq 0}$. 

Now let $K$ be a semifield. Let $V(K) = V(K)$ be the set of formal sums $\xi = \sum_{b \in \beta} \xi_b b, \xi_b \in K^1$. This is a monoid under addition $(\sum_{b \in \beta} \xi_b b) + (\sum_{b \in \beta} \xi'_b b) = \sum_{b \in \beta} (\xi_b + \xi'_b) b$ and we define scalar multiplication $K^1 \times V(K) \to V(K)$ by $(k, \sum_{b \in \beta} \xi_b b) \mapsto \sum_{b \in \beta} (k \xi_b) b$.

For $\xi = \sum_{b \in \beta} \xi_b b \in V(K)$ we define $\text{supp}(\xi) = \{ b \in \beta; \xi_b \in K \}$.

Let $\text{End}(V(K))$ be the set of maps $\zeta : V(K) \to V(K)$ such that $\zeta(\xi + \xi') = \zeta(\xi) + \zeta(\xi')$ for all $\xi, \xi' \in V(K)$ and $\zeta(k \xi) = k \zeta(\xi)$ for all $k \in K^1, \xi \in V(K)$. This is a monoid under composition of maps. Define $\varnothing \in V(K)$ by $\varnothing b = o$ for all $b \in \beta$. The group $K$ (for multiplication in the semifield structure) acts freely (by scalar multiplication) on $V(K) - \varnothing$; let $P(K) = \lambda P(K)$ be the set of orbits of this action.

For $i \in I, n \in \mathbb{Z}$ we define $e_i^{(n)}, f_i^{(n)} \in \text{End}(V(K))$ by

$$e_i^{(n)}(b) = \sum_{b' \in \beta} c_{b,b',i,n} b', \quad f_i^{(n)}(b) = \sum_{b' \in \beta} d_{b,b',i,n} b',$$

with $b \in \beta$. Here a natural number $N$ (such as $c_{b,b',i,n}$ or $d_{b,b',i,n}$) is viewed as an element of $K^1$ given by $1 + 1 + \cdots + 1$ ($N$ terms, where $1$ is the neutral element for the product in $K$, if $N > 0$) or by $0 \in K^1$ (if $N = 0$).

For $i \in I, k \in K$ we define $i^k \in \text{End}(V(K)), (-i)^k \in \text{End}(V(K))$ by

$$i^k(b) = \sum_{n \in \mathbb{N}} k^n e_i^{(n)} b, \quad (-i)^k(b) = \sum_{n \in \mathbb{N}} k^n f_i^{(n)} b,$$

for any $b \in \beta$. We show:

(a) The map $i^k : V(K) \to V(K)$ is injective. The map $(-i)^k : V(K) \to V(K)$ is injective.

Using a partial order of the weights of $V$, we can write $V(K)$ as a direct sum of monoids $V(K)_s, s \in \mathbb{Z}$ where $V(K)_s = \{ \varnothing \}$ for all but finitely many $s$ and $(-i)^k$ maps any $\xi \in V(K)_s$ to $\xi$ plus an element in the direct sum of $V(K)_{s'}$ with $s' < s$. Then (a) for $(-i)^k$ follows immediately. A similar proof applies to $i^k$.

For $i \in I, k \in K$ we define $\tilde{i}^k \in \text{End}(V(K))$ by $\tilde{i}^k(b) = k^z(i) b$ for any $b \in \beta$. Let $\mathcal{G}(K)$ be the monoid associated to $G, K$ by generators and relations in [L19a, 2.10(i)-(vii)]. (In loc. cit. it is assumed that $K$ is as in 0.1(i) or 0.1(ii) but the same definition makes sense for any $K$.) We have the following result.

Proposition 1.5. The elements $i^k, (-i)^k, \tilde{i}^k$ (with $i \in I, k \in K$) in $\text{End}(V(K))$ satisfy the relations in [L19a, 2.10(i)-(vii)] defining the monoid $\mathcal{G}(K)$ hence they define a monoid homomorphism $\mathcal{G}(K) \to \text{End}(V(K))$.

We write the relations in loc. cit. (for the semifield $\mathbb{R}_{>0}$) for the endomorphisms $x_i(t), y_i(t), i(t)$ of $V$ with $t \in \mathbb{R}_{>0}$. These relations can be expressed as a set of identities satisfied by $c_{b,b',i,n}, d_{b,b',i,n}, z_i(b)$ and these identities show that the endomorphisms $i^k, (-i)^k, \tilde{i}^k$ of $V(K)$ satisfy the relations in loc. cit. (for the semifield $K$). The result follows.
1.6. Consider a homomorphism of semifields \( r : K_1 \to K_2 \). Now \( r \) induces a homomorphism of monoids \( \mathcal{G}_r : \mathcal{G}(K_1) \to \mathcal{G}(K_2) \). It also induces a homomorphism of monoids \( V_r : V(K_1) \to V(K_2) \) given by \( \sum_{b \in \beta} \xi_b b \mapsto \sum_{b \in \beta} r(\xi_b)b \). From the definitions, for \( g \in \mathcal{G}(K_1), \xi \in V(K_1) \), we have \( V_r(g\xi) = \mathcal{G}_r(g)(V_r(\xi)) \) where \( g\xi \) is given by the \( \mathcal{G}(K_1) \)-action on \( V(K_1) \) and \( \mathcal{G}_r(g)(V_r(\xi)) \) is given by the \( \mathcal{G}(K_2) \)-action on \( V(K_2) \). Assuming that \( r : K_1 \to K_2 \) is surjective (so that \( \mathcal{G}_r : \mathcal{G}(K_1) \to \mathcal{G}(K_2) \) is surjective) we deduce:

(a) If \( E \) is a subset of \( V(K_1) \) which is stable under the \( \mathcal{G}(K_1) \)-action on \( V(K_1) \), then the subset \( V_r(E) \) of \( V(K_2) \) is stable under the \( \mathcal{G}(K_2) \)-action on \( V(K_2) \).

1.7. In the remainder of this section we assume that \( \lambda \in \mathcal{X}^{++} \). Then \( L \mapsto G_L \) is an isomorphism \( \pi : P^* \sim \to B \) and

(a) \( \pi \) restricts to a bijection \( \pi_{\geq 0} : P_{\geq 0}^* \sim \to B_{\geq 0} \).

See [L94b, 8.17].

1.8. Let \( \Omega \) be the set of all open nonempty subsets of \( C \). Let \( X \) be an algebraic variety over \( C \). Let \( X_1 \) be the set of pairs \( (U, f_U) \) where \( U \in \Omega \) and \( f_U : U \to X \) is a morphism of algebraic varieties. We define an equivalence relation on \( X_1 \) in which \( (U, f_U), (U', f_U') \) are equivalent if \( f_U|_{U \cap U'} = f_U'|_{U \cap U'} \). Let \( \tilde{X} \) be the set of equivalence classes. An element of \( \tilde{X} \) is said to be a rational map \( f : C \to X \). For \( f \in \tilde{X} \) let \( \Omega_f \) be the set of all \( U \in \Omega \) such that \( f \) contains \( (U, f_U) \in X_1 \) for some \( f_U' \); we shall then write \( f(t) = f_U(t) \) for \( t \in U \). We shall identify any \( x \in X \) with the constant map \( f_X : C \to X \) with image \( \{x\} \); thus \( X \) can be identified with a subset of \( \tilde{X} \). If \( X' \) is another algebraic variety over \( C \) then we have \( \tilde{X} \times X' = \tilde{X} \times \tilde{X}' \) canonically. If \( F : X \to X' \) is a morphism then there is an induced map \( \tilde{F} : \tilde{X} \to \tilde{X}' \); to \( f : C \to X \) it attaches \( f' : C \to X' \) where for some \( U \in \Omega_f \) we have \( f'(t) = F(f(t)) \) for all \( t \in U \). If \( H \) is an algebraic group over \( C \) then \( \tilde{H} \) is a group with multiplication \( \tilde{H} \times \tilde{H} = \tilde{H} \times \tilde{H} \to \tilde{H} \) induced by the multiplication map \( H \times H \to H \). Note that \( H \) is a subgroup of \( \tilde{H} \). In particular, the group \( \tilde{G} \) is defined. Also, the additive group \( \tilde{C} \) and the multiplicative group \( \tilde{C}^* \) are defined. Also \( \tilde{B} \) is defined.

1.9. Let \( X \) be an algebraic variety over \( C \) with a given subset \( X_{\geq 0} \). We define a subset \( \tilde{X}_{\geq 0} \) of \( \tilde{X} \) as follows: \( \tilde{X}_{\geq 0} \) is the set of all \( f \in \tilde{X} \) such that for some \( U \in \Omega_f \) and some \( \epsilon \in \mathbb{R}_{>0} \) we have \( (0, \epsilon) \subset U \) and \( f(t) \in X_{\geq 0} \) for all \( t \in (0, \epsilon) \). (In particular, \( \tilde{G}_{\geq 0} \) is defined in terms of \( G, G_{\geq 0} \) and \( \tilde{B}_{\geq 0} \) is defined in terms of \( B, B_{\geq 0} \).) If \( X' \) is another algebraic variety over \( C \) with a given subset \( X'_{\geq 0} \), then \( X \times X' \) with its subset \( (X \times X')_{\geq 0} = X_{\geq 0} \times X'_{\geq 0} \) gives rise as above to the set \( \tilde{X} \times X'_{\geq 0} \) which can be identified with \( \tilde{X}_{\geq 0} \times X'_{\geq 0} \). If \( F : X \to X' \) is a morphism such that \( F(X_{\geq 0}) \subset X'_{\geq 0} \), then the induced map \( \tilde{F} : \tilde{X} \to \tilde{X}' \) carries \( \tilde{X}_{\geq 0} \) into \( \tilde{X}'_{\geq 0} \) hence it restricts to a map \( \tilde{F}_{\geq 0} : \tilde{X}_{\geq 0} \to \tilde{X}'_{\geq 0} \). From the definitions we see that:

(a) if \( \tilde{F} \) is an isomorphism of \( \tilde{X} \) onto an open subset of \( \tilde{X}' \) and \( F \) carries \( \tilde{X}_{\geq 0} \)
bijectionally onto $\tilde{X}'_0$, then the map $\tilde{F}_{\geq 0}$ is a bijection.

Now the multiplication $G \times G \to G$ carries $G_{\geq 0} \times G_{\geq 0}$ to $G_{\geq 0}$ hence it induces a map $\tilde{G}_{\geq 0} \times \tilde{G}_{\geq 0} \to \tilde{G}_{\geq 0}$ which makes $\tilde{G}_{\geq 0}$ into a monoid; the conjugation action $G \times B \to B$ carries $G_{\geq 0} \times B_{\geq 0}$ to $B_{\geq 0}$ hence it induces a map $\tilde{G}_{\geq 0} \times \tilde{B}_{\geq 0} \to \tilde{B}_{\geq 0}$ which define an action of the monoid $\tilde{G}_{\geq 0}$ on $\tilde{B}_{\geq 0}$. We define $\tilde{C}^*_{\geq 0}$ in terms of $C^*$ and its subset $C^*_{\geq 0} := \mathbb{R}_{\geq 0}$. The multiplication on $C^*$ preserves $C^*_{\geq 0}$ hence it induces a map $\tilde{C}^*_{\geq 0} \times \tilde{C}^*_{\geq 0} \to \tilde{C}^*_{\geq 0}$ which makes $\tilde{C}^*_{\geq 0}$ into an abelian group. We define $\tilde{C}_{\geq 0}$ in terms of $C$ and its subset $C_{\geq 0} := \mathbb{R}_{\geq 0}$. The addition on $C$ preserves $C_{\geq 0}$ hence it induces a map $\tilde{C}_{\geq 0} \times \tilde{C}_{\geq 0} \to \tilde{C}_{\geq 0}$ which makes $\tilde{C}_{\geq 0}$ into an abelian monoid. The imbedding $C^* \subset C$ induces an imbedding $\tilde{C}^*_{\geq 0} \to \tilde{C}_{\geq 0}$; the monoid operation on $\tilde{C}_{\geq 0}$ preserves the subset $\tilde{C}^*_{\geq 0}$ and makes $\tilde{C}^*_{\geq 0}$ into an abelian monoid. This, together with the multiplication on $\tilde{C}^*_{\geq 0}$ makes $\tilde{C}^*_{\geq 0}$ into a semifield. From the definitions we see that this semifield is the same as $K$ in 0.1(i) and that $\tilde{G}_{\geq 0}$ is the monoid associated to $G$ and $K$ in [L94b, 2.2] (which we now call $G(K)$). We define $\mathcal{B}(K)$ to be $\tilde{B}_{\geq 0}$ with the action of $\tilde{G}_{\geq 0} = G(K)$ described above. This achieves what was stated in 0.1 for $K$ as in 0.1(i).

1.10. In the remainder of this section $K$ will denote the semifield in 0.1(i) and we assume that $\lambda \in \mathcal{X}^{++}$. We associate $\tilde{P}_{\geq 0} = \lambda \tilde{P}_{\geq 0}$ to $P$ and its subset $P_{\geq 0}$ as in 1.9. We associate $\tilde{P}^*_{\geq 0} = \lambda \tilde{P}^*_{\geq 0}$ to $P^*$ and its subset $P^*_{\geq 0}$ as in 1.9. We write $P^*(K) = \lambda P^*(K) = \tilde{P}^*_{\geq 0}$.

We associate $\tilde{V}_{\geq 0} = \lambda \tilde{V}_{\geq 0}$ to $V$ and its subset $V_{\geq 0}$ as in 1.9. We can identify $\tilde{V}_{\geq 0} = V(K)$ (see 1.4). We associate $\tilde{V}^*_{\geq 0} = \lambda \tilde{V}^*_{\geq 0}$ to $V^*$ and its subset $V^*_{\geq 0}$ as in 1.9. We write $V^*(K) = \lambda V^*(K) = \tilde{V}^*_{\geq 0}$. We have $V^*(K) \subset \tilde{V}_{\geq 0}$.

The obvious map $a' : V - 0 \to P$ restricts to a (surjective) map $a'_{\geq 0} : V_{\geq 0} - 0 \to P_{\geq 0}$ and it defines a map $\tilde{a}'_{\geq 0} : \tilde{V}_{\geq 0} - 0 \to \tilde{P}_{\geq 0}$. The scalar multiplication $C^* \times (V - 0) \to V - 0$ carries $C^*_{\geq 0} \times (V_{\geq 0} - 0)$ to $V_{\geq 0} - 0$ hence it induces a map $\tilde{C}^*_{\geq 0} \times (\tilde{V}_{\geq 0} - 0) \to \tilde{V}_{\geq 0} - 0$ which is a (free) action of the group $K = \tilde{C}^*_{\geq 0}$ on $\tilde{V}_{\geq 0} - 0 = V(K) - 0$. From the definitions we see that $a'_{\geq 0}$ is surjective and it induces a bijection $(V(K) - 0)/K \sim \tilde{P}_{\geq 0}$. Thus we have $\tilde{P}_{\geq 0} = P(K)$ (notation of 1.4). Note that $P^*(K) \subset P(K)$.

The obvious map $a : V^* - 0 \to P^*$ restricts to a (surjective) map $a_{\geq 0} : V^*_{\geq 0} - 0 \to P^*_{\geq 0}$ and it defines a map $\tilde{a}_{\geq 0} : V^*(K) = \tilde{V}^*_{\geq 0} - 0 \to \tilde{P}^*_{\geq 0} = P^*(K)$. The (free) $K$-action on $\tilde{V}_{\geq 0} - 0$ considered above restricts to a (free) $K$-action on $V^*(K) - 0 = \tilde{V}^*_{\geq 0} - 0$. From the definitions we see that $\tilde{a}_{\geq 0}$ is constant on any orbit of this action. We show:

(a) The map $\tilde{a}_{\geq 0}$ is surjective. It induces a bijection $(V^*(K) - 0)/K \sim P^*(K)$. Let $f \in \tilde{P}^*_{\geq 0}$. We can find $U \in \Omega_f$, $\epsilon \in \mathbb{R}_{>0}$ such that $(0, \epsilon) \subset U$ and $f(t) \in P^*_{\geq 0}$ for $t \in (0, \epsilon)$. Using the surjectivity of $a_{\geq 0}$ we see that for $t \in (0, \epsilon)$ we have $f(t) = a(x_t)$ where $t \mapsto x_t$ is a function $(0, \epsilon) \to V^*_{\geq 0} - 0$. We can assume that
there exists $B \in \mathcal{B}(\mathbb{R})$ such that $\pi(f(t))$ is opposed to $B$ for all $t \in U$. Let $\mathcal{O} = \{B_1 \in \mathcal{B}; B_1$ opposed to $B\}$; thus we have $\pi(f(t)) \in \mathcal{O}$ for all $t \in U$. Let $B' \in \mathcal{O} \cap \mathcal{B}(\mathbb{R})$ and let $\xi' \in V_{\mathcal{R}} - 0$ be such that $\pi(C\xi') = B'$. Let $U_B$ be the unipotent radical of $B$. Then $U_B \to \mathcal{O}$, $u \mapsto uB'u^{-1}$ is an isomorphism. Hence there is a unique morphism $\zeta : \mathcal{O} \to V^\bullet - 0$ such that $\zeta(uB'u^{-1}) = u\xi'$ for any $u \in U_B$. From the definitions we have $\zeta(\mathcal{O} \cap \mathcal{B}(\mathbb{R})) \subset (V_{\mathcal{R}} \cap V^\bullet) - 0$. We define $f' : U \to V^\bullet - 0$ by $f'(t) = \zeta(\pi(f(t)))$. We can view $f'$ as an element of $\tilde{V}^\bullet - 0$ such that $\tilde{a}(f') = f$. Since $\pi(f(t)) \in \mathcal{B}(\mathbb{R})$, we have $f'(t) \in (V_{\mathcal{R}} \cap V^\bullet) - 0$ for $t \in (0, \epsilon)$. For such $t$ we have $a(f'(t)) = f(t) = a(x_t)$ hence $f'(t) = z_t x_t$ where $t \mapsto z_t$ is a (possibly discontinuous) function $(0, \epsilon) \to \mathbb{R} - 0$. Since $x_t \in V_{\geq 0} - 0$ and $\mathbb{R}_{>0}(V_{\geq 0} - 0) = V_{\geq 0} - 0$, we see that for $t \in (0, \epsilon)$ we have $f'(t) \in (V_{\geq 0} - 0) \cup (-1)(V_{\geq 0} - 0)$. Since $(0, \epsilon)$ is connected and $f'$ is continuous (in the standard topology) we see that $f'(0, \epsilon)$ is contained in one of the connected components of $(V_{\geq 0} - 0) \cup (-1)(V_{\geq 0} - 0)$ that is, in either $V_{\geq 0} - 0$ or in $(-1)(V_{\geq 0} - 0)$. Thus there exists $s \in \{1, -1\}$ such that $sf'(0, \epsilon) \subset V_{\geq 0} - 0$ hence also $sf'(0, \epsilon) \subset V^\bullet - 0$. We define $f'' : U \to V^\bullet - 0$ by $f''(t) = sf'(t)$. We can view $f''$ as an element of $\tilde{V}_0^\bullet - 0$ such that $\tilde{a}_{\geq 0}(f'') = f$. This proves that $\tilde{a}_{\geq 0}$ is surjective. The remaining statement of (a) is immediate.

Since $P^\bullet$ and its subset $P^\bullet_{\geq 0}$ can be identified with $\mathcal{B}$ and its subset $\mathcal{B}_{\geq 0}$ (see 1.7(a)), we see that we may identify $P^\bullet(K) = \mathcal{B}(K)$. The action of $\mathfrak{G}(K)$ on $P^\bullet(K)$ induced from that on $V^\bullet(K) - 0$ is the same as the previous action of $G(K)$ on $\mathcal{B}(K)$, since $G(K) = \mathfrak{G}(K)$, by [L19a, 2.13(d)]. This gives a second incarnation of $\mathcal{B}(K)$.

**1.11.** Let $\mathbb{Z}$ be the semifield in 0.1(ii). Following [L94b], we define a (surjective) semifield homomorphism $r : K \to \mathbb{Z}$ by $r(x^e f_1/f_2) = e$ (notation of 0.1). Now $r$ induces a surjective map $V_r : V(K) \to V(\mathbb{Z})$ as in 1.6. Let $V^\bullet(\mathbb{Z}) = \lambda V^\bullet(\mathbb{Z}) \subset V(\mathbb{Z})$ be the image under $V_r$ of the subset $V^\bullet(K)$ of $V(K)$. Then $V^\bullet(\mathbb{Z}) - \varnothing = V_r(V^\bullet(K) - 0)$.

The $\mathbb{Z}$-action on $V(\mathbb{Z}) - \varnothing$ in 1.4 leaves $V^\bullet(\mathbb{Z}) - \varnothing$ stable. (We use the $K$-action on $V^\bullet(K) - 0$.) Let $\tilde{P^\bullet}(\mathbb{Z}) = \lambda P^\bullet(\mathbb{Z})$ be the set of orbits of this action. We have $P^\bullet(\mathbb{Z}) \subset P(Z)$ (notation of 1.4). From 1.6(a) we see that $V^\bullet(\mathbb{Z}) - \varnothing$ is stable under the $\mathfrak{G}(\mathbb{Z})$-action on $V(\mathbb{Z})$ in 1.6. Since the $\mathfrak{G}(\mathbb{Z})$-action commutes with scalar multiplication by $\mathbb{Z}$ it follows that the $\mathfrak{G}(\mathbb{Z})$-action on $V(\mathbb{Z}) - \varnothing$ and $V^\bullet(\mathbb{Z}) - \varnothing$ induces a $\mathfrak{G}(\mathbb{Z})$-action on $P(\mathbb{Z})$ and $P^\bullet(\mathbb{Z})$.

**1.12.** We set $\mathcal{B}(\mathbb{Z}) = \lambda P^\bullet(\mathbb{Z})$. This achieves what was stated in 0.1 for the semifield $\mathbb{Z}$, since $\mathfrak{G}(\mathbb{Z}) = G(\mathbb{Z})$, by [L19a, 2.13(d)]. This definition of $\mathcal{B}(\mathbb{Z})$ depends on the choice of $\lambda \in \lambda^{++}$. In §4 we will show that $\mathcal{B}(\mathbb{Z})$ is independent of this choice up to a canonical bijection. (Alternatively, if one wants a definition without such a choice one could take $\lambda$ such that $\langle i, \lambda \rangle = 1$ for all $i \in I$.)

2. Preparatory results

**2.1.** We preserve the setup of 1.4. As shown in [L94a, 5.3, 4.2], for $w \in W$ and
\( i = (i_1, i_2, \ldots, i_m) \in I_w \), the subspace of \( V \) generated by the vectors

\[ f_{i_1}^{(c_1)} f_{i_2}^{(c_2)} \ldots f_{i_m}^{(c_m)} \xi^+ \]

for various \( c_1, c_2, \ldots, c_m \) in \( N \) is independent of \( i \) (we denote it by \( V^w \)) and \( \beta^w := \beta \cap V^w \) is a basis of it. Let \( V'^i \) be the subspace of \( V \) generated by the vectors

\[ e_{i_m}^{(d_m)} e_{i_{m-1}}^{(d_{m-1})} \ldots e_{i_2}^{(d_2)} b_w \]

for various \( d_1, d_2, \ldots, d_m \) in \( N \), where

\[ b_w = \dot{w} \xi^+ \]

\[ \dot{w} = \dot{s}_{i_1} \dot{s}_{i_2} \ldots \dot{s}_{i_m} \]

We show:

(a) \( V^w = V'^i \).

We show that \( V^w \subset V'^i \). We argue by induction on \( m = |w| \). If \( m = 0 \), the result is obvious. Assume now that \( m \geq 1 \). Let \( c_1, c_2, \ldots, c_m \) be in \( N \). By the induction hypothesis,

(b)

\[ f_{i_1}^{(c_1)} f_{i_2}^{(c_2)} \ldots f_{i_m}^{(c_m)} \xi^+ \]

is a linear combination of vectors of form

\[ f_{i_1}^{(c_1)} e_{i_m}^{(d_m)} e_{i_{m-1}}^{(d_{m-1})} \ldots e_{i_2}^{(d_2)} b_{s_{i_1} w} \]

for various \( d_1, \ldots, d_m \) in \( N \). Using the known commutation relations between \( f_{i_1} \) and \( e_j \) we see that (b) is a linear combination of vectors of form

\[ e_{i_m}^{(d_m)} e_{i_{m-1}}^{(d_{m-1})} \ldots e_{i_2}^{(d_2)} f_{i_1}^{(c_1)} b_{s_{i_1} w} \]

for various \( d_1, \ldots, d_m \) in \( N \). It is then enough to show that

\[ f_{i_1}^{(c_1)} b_{s_{i_1} w} = e_{i_1}^{(d_1)} g_{i_1} b_{s_{i_1} w} \]

for some \( d_1 \in N \). This follows from the fact that

(c) \( e_{i_1} b_{s_{i_1} w} = 0 \) and \( b_{s_{i_1} w} \) is in a weight space of \( V \).

Next we show that \( V'^i \subset V^w \). We argue by induction on \( m = |w| \). If \( m = 0 \) the result is obvious. Assume now that \( m \geq 1 \). Since \( V^w \) is stable under the action of \( e_i, i \in I \) it is enough to show that \( b_w \in V^w \). By the induction hypothesis, \( b_{s_{i_1} w} \in V^{s_{i_1} w} \). Using (c), we see that for some \( c_1 \in N \) we have

\[ b_w = \dot{s}_{i_1} b_{s_{i_1} w} = f_{i_1}^{(c_1)} b_{s_{i_1} w} \in f_{i_1}^{(c_1)} V^{s_{i_1} w} \subset V^w. \]

This completes the proof of (a).
From [L93, 28.1.4] one can deduce that \( b_w \in \beta \). From (a) we see that \( b_w \in V^w \).
It follows that
\[
(d) \quad b_w \in \beta^w.
\]

2.2. For \( v \leq w \) in \( W \) we set
\[
B_{v,w} = \{ B \in B, pos(B^+, B) = w, pos(B^-, B) = w_I v \}
\]
a (locally closed subvariety of \( B \)) and
\[
(B_{v,w})_{\geq 0} = \bigcap_{w \leq w} B_{v,w}, \quad B_{\geq 0} = \bigcap_{w \leq w} W(B_{v,w})_{\geq 0}.
\]

2.3. Recall that there is a unique isomorphism \( \phi : G \rightarrow G \) such that \( \phi(x_i(t)) = y_i(t), \phi(y_i(t)) = x_i(t) \) for all \( i \in I, t \in C \) and \( \phi(g) = g^{-1} \) for all \( g \in T \). This carries Borel subgroups to Borel subgroups hence induces an isomorphism \( \phi : B \rightarrow B \) such that \( \phi(B^+) = B^-, \phi(B^-) = B^+ \). For \( i \in I \) we have \( \phi(s_i) = s_i^{-1} \). Hence \( \phi \) induces the identity map on \( W \). For \( v \leq w \) in \( W \) we have \( wv \leq vw_I \); moreover,
\[
(a) \phi \text{ defines an isomorphism } B_{v,w} \sim B_{v,w}.
\]
(See [L19b, 1.4(a)].) From the definition we have
\[
(b) \phi(G_{\geq 0}) = G_{\geq 0}.
\]
From [L94b, 8.7] it follows that
\[
(c) \phi(B_{\geq 0}) = B_{\geq 0}.
\]
From (a),(c) we deduce:
\[
(d) \phi \text{ defines a bijection } (B_{v,w})_{\geq 0} \sim (B_{v,w})_{\geq 0}.
\]
By [L90b, §3] there is a unique linear isomorphism \( \phi : V \rightarrow V \) such that \( \phi(g) = \phi(g)\phi(\xi) \) for all \( g \in G, \xi \in V \) and such that \( \phi(\xi^+) = \xi^- \); we have \( \phi(\beta) = \beta \) and \( \phi^2(\xi) = \xi \) for all \( \xi \in V \).

2.4. Assume now \( \lambda \in X^{++} \). Let \( B \in B_{v,w} \) and let \( L \in P^* \) be such that \( \pi(L) = B \).
Let \( \xi \in L - 0, b \in \beta \). We show:
\[
(a) \xi_b \neq 0 \implies b \in \beta^w \cap \phi(\beta^{uw}).
\]
We have \( B = gb^w g^{-1} \) for some \( g \in B^+ \). Then \( \xi = cg\xi^+ \) for some \( c \in C^* \).
We write \( g = g'g'' \) with \( g' \in U^+, g'' \in B^+ \). We have \( \xi = c'g'g\xi^+ = c'g'g_b \) where \( c' \in C^* \). By 2.1(d) we have \( b_w \in \beta^w \). Moreover, \( V^w \) is stable by the action of \( U^+ \); we see that \( \xi \in V^w \). Since \( \xi_b \neq 0 \) we have \( b \in \beta^w \). Let \( B' = \phi(B) \). We have \( B' \in B_{uv, uv+} \) (see 2.3(a)). Let \( L' = \phi(L) \in P^* \) and let \( \xi' = \phi(\xi) \in L' - 0, b' = \phi(b) \in \beta \). We have \( \xi'_{b'} \neq 0 \).
Applying the first part of the proof with \( B, L, \xi, v, w, b \) replaced by \( B', L', \xi', v', w', b' \) we obtain \( b' \in \beta^{uw} \). Hence \( b \in \phi(\beta^{uw}) \). Thus, \( b \in \beta^w \cap \phi(\beta^{uw}) \), as required.

2.5. We return to the setup of 1.4. For \( i \in I \) we set
\[
V^{e_i} = \{ \xi \in V; e_i(\xi) = 0 \} = \{ \xi \in V; \sum_{b \in \beta} \xi_b c_b b_{l,i} = 0 \text{ for all } b' \in \beta \},
\]
\[ V^{f_i} = \{ \xi \in V; f_i(\xi) = 0 \} = \{ \xi \in V; \sum_{b \in \beta} \xi_b d_{b,b',i,1} = 0 \text{ for all } b' \in \beta \}. \]

If \( \xi \in V_{\geq 0} \), the condition that \( \sum_{b \in \beta} \xi_b c_{b,b',i,1} = 0 \) is equivalent to the condition that \( \xi_b c_{b,b',i,1} = 0 \) for any \( b, b' \) in \( \beta \). Thus we have

\[ V_{\geq 0} \cap V^{e_i} = \{ \xi \in V_{\geq 0}; \xi = \sum_{b \in \beta^{e_i}} \xi_b b \} \]

where \( \beta^{e_i} = \{ b \in \beta; c_{b,b',i,1} = 0 \text{ for any } b' \in \beta \} \). Similarly, we have

\[ V_{\geq 0} \cap V^{f_i} = \{ \xi \in V_{\geq 0}; \xi = \sum_{b \in \beta^{f_i}} \xi_b b \} \]

where \( \beta^{f_i} = \{ b \in \beta; d_{b,b',i,1} = 0 \text{ for any } b' \in \beta \} \).

Now the action of \( \hat{s}_i \) on \( V \) defines an isomorphism \( T_i : V^{e_i} \rightarrow V^{f_i} \). If \( b \in \beta^{e_i} \) we have \( T_i(b) = f_i^{(i,\nu_b)} b = \sum_{b' \in \beta} d_{b,b',i,(i,\nu_b)} b' \); in particular, we have \( T_i(b) \in V_{\geq 0} \cap V^{f_i} \). Thus \( T_i \) restricts to a map \( \widetilde{T_i} : V_{\geq 0} \cap V^{e_i} \rightarrow V_{\geq 0} \cap V^{f_i} \). Similarly the action of \( \hat{s}_i^{-1} \) restricts to a map \( \widetilde{T}_i'' : V_{\geq 0} \cap V^{f_i} \rightarrow V_{\geq 0} \cap V^{e_i} \). This is clearly the inverse of \( \widetilde{T}_i' \).

2.6. Now let \( K \) be a semifield. Let

\[ V(K)^{e_i} = \{ \sum_{b \in \beta} \xi_b b; \xi_b \in K^1 \text{ if } b \in \beta^{e_i}, \xi_b = o \text{ if } b \in \beta - \beta^{e_i} \}, \]

\[ V(K)^{f_i} = \{ \sum_{b \in \beta} \xi_b b; \xi_b \in K^1 \text{ if } b \in \beta^{f_i}, \xi_b = o \text{ if } b \in \beta - \beta^{f_i} \}. \]

We define \( T_{i,K} : V(K) \rightarrow V(K) \) by

\[ \sum_{b \in \beta} \xi_b b \mapsto \sum_{b' \in \beta} (\sum_{b \in \beta} d_{b,b',i,\nu_b} \xi_b) b' \]

(notation of 1.4). From the results in 2.5 one can deduce that

(a) \( T_{i,K} \) restricts to a bijection \( \widetilde{T}_{i,K} : V(K)^{e_i} \rightarrow V(K)^{f_i} \).

2.7. Let \( K \) be a semifield. We define an involution \( \phi : V(K) \rightarrow V(K) \) by \( \phi(\sum_{b \in \beta} \xi_b b) = \sum_{b \in \beta} \xi_{\phi(b)} b \). (Here \( \xi_b \in K^1 \); we use that \( \phi(\beta) = \beta \).) This restricts to an involution \( V(K) - \underline{o} \rightarrow V(K) - \underline{o} \) which induces an involution \( P(K) \rightarrow P(K) \) denoted again by \( \phi \).
3. Parametrizations

3.1. In this section $K$ denotes the semifield in 0.1(i). For $v \leq w$ in $W$ we define $B_{v,w}(K) = \overline{B_{v,w}}_0$ as in 1.9 in terms of $B_{v,w}$ and its subset $(B_{v,w})_0$. We have

$$B(K) = \sqcup_{v \leq w} w B_{v,w}(K).$$

3.2. We preserve the setup of 1.4. We now fix $v \leq w$ in $W$ and $i = (i_1, i_2, \ldots, i_m) \in T_w$. According to [MR], there is a unique sequence $q_1, q_2, \ldots, q_m$ with $q_k \in \{s_{i_k}, 1\}$ for $k \in [1, m]$, so that $q_1 \leq q_1q_2 \leq \cdots \leq q_1q_2 \cdots q_m$ and $q_1 \leq q_1s_{i_2}q_1q_2 \leq q_1q_2s_{i_3}q_1q_2 \cdots q_m - 1 \leq q_1q_2 \cdots q_m - 1s_{i_m}$. Let $[1, m]' = \{k \in [1, m]; q_k = 1\}$, $[1, m]'' = \{k \in [1, m]; q_k = s_{i_k}\}$. Let $A$ be the set of maps $h : [1, m]' \to C^*$; this is naturally an algebraic variety over $C$. Let $A_{00}$ be the subset of $A$ consisting of maps $h : [1, m]' \to R_{10}$. Following [MR], we define a morphism $s : A \to G$ by $h \mapsto g(h)_{1}g(h)_{2} \cdots g(h)_{m}$ where

(a) $g(h)_{i} = y_{i}(h(k))$ if $k \in [1, m]'$ and $g(h)_{k} = s_{i}(k)$ if $k \in [1, m]'$. We show:

(b) If $h \in A_{00}$, then $s(h)_{i} \in V^{w}$, so that $s(h)$ is a linear combination of vectors $b \in \beta_{W}$. Moreover, $(s(h)_{i})_{b_{w}} \neq 0$.

From the properties of Bruhat decomposition, for any $h \in A_{00}$ we have $s(h) \in B_{w}B_{w}$, so that $s(h)_{i} = cu_{i}w_{i}^{+} = cu_{i}w_{i}$ where $c \in C^+$, $u \in U_{+}$. Since $b_{w} \in V^{w}$ and $V^{w}$ is stable under the action of $U_{+}$, it follows that $cu_{i}w_{i}^{+} \in V^{w}$. More precisely, $ub_{w} = b_{w}$ plus a linear combination of elements $b \in \beta$ of weight other than that of $b_{w}$. This proves (b).

We show:

(c) Let $h \in A_{00}$. Assume that $i \in I$ is such that $|s_{i}w| > |w|$ and that $b \in \beta$ is such that $(s(h)_{i})_{b} \neq 0$. Then $\nu_{b} \neq \nu_{b_{w}} + i'$. Since $|s_{i}w| > |w|$ we have $e_{i}b_{w} = 0$. We write $s(h)_{i} = cu_{i}w_{i}$ with $c, u$ as in the proof of (b). Now $ub_{w}$ is a linear combination of vectors of the form $e_{j_{1}}e_{j_{2}} \cdots e_{j_{k}}b_{w}$ with $j_{t} \in I$. Such a vector is in a weight space $V(\nu)$ with $\nu = \nu_{b_{w}} + j_{1}' + j_{2}' + \cdots + j_{k}'$. If $j_{1}' + j_{2}' + \cdots + j_{k}' = i'$ then $k = 1$ and $j_{1} = i$. But in this case we have $e_{j_{1}}e_{j_{2}} \cdots e_{j_{k}}b_{w} = e_{i}b_{w} = 0$. The result follows.

3.3. Let $h \in A_{00}$. Let $k \in [1, m]'$. The following result appears in the proof of [MR, 11.9].

(a) We have $(g(h)_{k+1}g(h)_{k+2} \cdots g(h)_{m})^{-1}x_{i_{k}}(a)g(h)_{k+1}g(h)_{k+2} \cdots g(h)_{m} \in U_{+}.\

From (a) it follows that for $\xi \in V$ we have

$$e_{i_{k}}(g(h)_{k+1}g(h)_{k+2} \cdots g(h)_{m}\xi) = g(h)_{k+1}g(h)_{k+2} \cdots g(h)_{m}(e'_{i} \xi)$$

where $e' : V \to V$ is a linear combination of products of one or more factors $e_{j}, j \in I$. When $\xi = \xi^{+}$ we have $e'_{i} \xi = 0$ hence $e_{i_{k}}(g(h)_{k+1}g(h)_{k+2} \cdots g(h)_{m}\xi^{+}) = 0$. We can write uniquely

$$g(h)_{k+1}g(h)_{k+2} \cdots g(h)_{m}\xi^{+} = \sum_{\nu \in X} (g(h)_{k+1}g(h)_{k+2} \cdots g(h)_{m}\xi^{+})_{\nu}$$
with \((g(h)_{k+1}g(h)_{k+2} \ldots g(h)_m\xi^+)\) in \(V_{\nu}\). We have
\[
\sum_{\nu \in \mathcal{X}} e_{i_k}((g(h)_{k+1}g(h)_{k+2} \ldots g(h)_m\xi^+)\nu) = 0.
\]

Since the elements \(e_{i_k}((g(h)_{k+1}g(h)_{k+2} \ldots g(h)_m\xi^+)\nu)\) (for various \(\nu \in \mathcal{X}\)) are in distinct weight spaces, it follows that \(e_{i_k}((g(h)_{k+1}g(h)_{k+2} \ldots g(h)_m\xi^+)\nu) = 0\) for any \(\nu \in \mathcal{X}\). If \(\xi \in V_{\nu}\) satisfies \(e_{i_k}\xi = 0\), then

(b) \(\hat{s}_{i_k}\xi = f_{i_k}^{(i_k,\nu)}\xi\).

(If \(\langle i_k, \nu \rangle < 0\) then \(\xi = 0\) so that both sides of (b) are 0.) We deduce

(c) \(g(h)_k((g(h)_{k+1}g(h)_{k+2} \ldots g(h)_m\xi^+)\nu) = f_{i_k}^{(i_k,\nu)}((g(h)_{k+1}g(h)_{k+2} \ldots g(h)_m\xi^+)\nu)\)
for any \(\nu \in \mathcal{X}\).

### 3.4.
Let \(h \in A_{\geq 0}\). For any \(k \in [1, m]\) we set \([k, m]' = [k, m] \cap [1, m]', [k, m]'' = [k, m] \cap [1, m]''). Let \(E_{\geq k}\) be the set of all maps \(\chi : [k, m]' \to \mathbb{N}\). (If \([k, m]' = \emptyset\), \(E_{\geq k}\) consists of a single element.) For \(\chi \in E_{\geq k}\) and \(k' \in [k, m]\) let \(\chi_{\geq k'}\) be the restriction of \(\chi\) to \([k', m]'\).

We now define an integer \(c(k, \chi)\) for any \(k \in [1, m]'\) and any \(\chi \in E_{\geq k}\) by descending induction on \(k\). We can assume that \(c(k', \chi')\) is defined for any \(k' \in [k + 1, m]'\) and any \(\chi' \in E_{\geq k'}\). We set \(c_{k, \chi} = \langle i_k, \nu \rangle\) where

(a) \(\nu = \lambda - \sum_{\kappa \in [k+1, m]'} \chi(\kappa) i^\kappa - \sum_{\kappa \in [k+1, m]' ; c(\chi, \chi_{\geq \kappa}) \geq 0} c(\kappa, \chi_{\geq \kappa}) i^\kappa \in \mathcal{X}\).

This completes the inductive definition of the integers \(c(k, \chi)\).

Next we define for any \(k \in [1, m]\) and any \(\chi \in E_{\geq k}\) an element \(\mathcal{J}_{k, \chi} \in V\) by
\[
\mathcal{J}_{k, \chi} = g(h)^\chi_k g(h)^\chi_{k+1} \ldots g(h)^\chi_m\xi^+.
\]

where
\[
g(h)^\chi_k = h(\kappa)\chi(\kappa) f_{i_\kappa}^{(\chi(\kappa))}\text{ if } \kappa \in [k, m]\',
\]
\[
g(h)^\chi_k = f_{i_\kappa}^{(c(\kappa, \chi_{\geq \kappa})}\text{ if } \kappa \in [k, m]''.
\]

For \(k \in [1, m]\) we show:

(b) \(g(h)_k g(h)_{k+1} \ldots g(h)_m\xi^+ = \sum_{\chi \in E_{\geq k}} \mathcal{J}_{k, \chi}\).

We argue by descending induction on \(k\). Assume first that \(k = m\). If \(k \in [1, m]'\) then
\[
g(h)_k\xi^+ = \sum_{n \geq 0} h(k)^n f_{i_\kappa}^{(n)}\xi^+ = \sum_{\chi \in E_{\geq k}} \mathcal{J}_{k, \chi},
\]
as required. If \( k \in [1, m]^\prime \), then \( g(h)k^\xi = \hat{s}_h^i, \xi = f^{(i_k, \lambda)}_h \xi^+ \), see 3.3(b).

Next we assume that \( k < m \) and that (b) holds for \( k \) replaced by \( k + 1 \). Let \( \chi' = \chi_{\geq k+1} \). By the induction hypothesis, the left hand side of (b) is equal to

\[
(g(h)k + \sum_{\chi \in \mathcal{E}_{\geq k+1}} J_{k+1, \chi}.
\]

If \( k \in [1, m]' \), then clearly (c) is equal to the right hand side of (b). If \( k \in [1, m]' \), then from the induction hypothesis we see that for any \( \nu \in \mathcal{X} \) we have

\[
(g(h)k_{1+} \ldots g(h)m^\xi^+)_{\nu} = \sum_{\chi \in \mathcal{E}_{\geq k+1}} (J_{k+1, \chi})_{\nu} = \sum_{\chi \in \mathcal{E}_{\geq k+1, \nu}} J_{k+1, \chi}
\]

where \( \mathcal{E}_{\geq k+1; \nu} \) is the set of all \( \chi \in \mathcal{E}_{\geq k+1} \) such that

\[
\nu = \lambda - \sum_{\kappa \in [k+1, m]'} \chi(\kappa)\nu'_k - \sum_{\kappa \in [k+1, m]'c(\kappa, \chi_{\geq k}) \geq 0} c(\kappa, \chi_{\geq k})\nu'_k.
\]

Using this and 3.3(c) we see that

\[
g(h)k_{1+} \ldots g(h)m^\xi^+ = \sum_{\nu \in \mathcal{X}} f^{(i_k, \nu)}_h ((g(h)k_{1+} \ldots g(h)m^\xi^+)_{\nu})
= \sum_{\nu \in \mathcal{X}} f^{(i_k, \nu)}_h \sum_{\chi \in \mathcal{E}_{\geq k+1, \nu}} J_{k+1, \chi} = \sum_{\chi \in \mathcal{E}_{\geq k}} f^{(c(\kappa, \chi))}_h J_{k, \chi} = \sum_{\chi \in \mathcal{E}_{\geq k}} J_{k, \chi}.
\]

This completes the inductive proof of (b).

In particular, we have

\[
g(h)k_{1}g(h)_{2} \ldots g(h)m^\xi^+ = \sum_{\chi \in \mathcal{E}} J_{1, \chi},
\]

where \( \mathcal{E} \) is the set of all maps \( \chi : [1, m]' \rightarrow N \). This shows that for any \( b \in B \) there exists a polynomial \( P_b \) in the variables \( x_k; k \in [1, m]' \) with coefficients in \( N \) such that the coefficient of \( b \) in \( g(h)k_{1}g(h)_{2} \ldots g(h)m^\xi^+ \) is obtained by substituting in \( P_b \) the variables \( x_k \) by \( h(k) \in R_{\geq 0} \) for \( k \in [1, m]' \), \( h \in A_{\geq 0} \). Each coefficient of this polynomial is a sum of products of expressions of the form \( d_{b_1, b_2, i, n} \in N \) (see 1.4); if one of these coefficients is \( \neq 0 \) then after the substitution \( x_k \mapsto h(k) \in R_{\geq 0} \) we obtain an element in \( R_{\geq 0} \) while if all these coefficients are 0 then the same substitution gives 0. Thus, there is a well defined subset \( \beta_{v, i} \) of \( B \) such that \( P_b|_{x_k = h(k)} \) is in \( R_{\geq 0} \) if \( b \in \beta_{v, i} \) and is 0 if \( b \in B - \beta_{v, i} \).

For a semifield \( K_1 \) we denote by \( A(K_1) \) the set of maps \( h : [1, m]' \rightarrow K_1 \). For any \( h \in K_1 \) we can substitute in \( P_b \) the variables \( x_k \) by \( h(k) \in K_1 \) for \( k \in [1, m]' \); the result is an element \( P_{b, h, K_1} \in K_1 \). Clearly, we have \( P_{b, h, K_1} \in K_1 \) if \( b \in \beta_{v, i} \) and \( P_{b, h, K_1} = 0 \) if \( b \in B - \beta_{v, i} \).
From 3.2(b) we see that $b_w \in \beta_n$. We see that for a semifield $K_1$, $h \mapsto \sum_{b \in \beta} P_{b,h,K_1} b$ is a map $\theta_{K_1} : A(K_1) \rightarrow V(K_1) - \emptyset$ and

$\theta_{K_1}(A(K_1)) \subset \{\xi \in V(K_1); \text{supp}(\xi) = \beta_n\}$.

(supp(\xi) as in 1.4.) Let $\omega_{K_1} : A(K_1) \rightarrow P(K_1)$ be the composition of $\theta_{K_1}$ with the obvious map $V(K_1) - \emptyset \rightarrow P(K_1)$. From the definitions, if $K_1 \rightarrow K_2$ is a homomorphism of semifields, then we have a commutative diagram

$$
A(K_1) \xrightarrow{\omega_{K_1}} P(K_1) \\
\downarrow \quad \downarrow \\
A(K_2) \xrightarrow{\omega_{K_2}} P(K_2)
$$

where the vertical maps are induced by $K_1 \rightarrow K_2$.

3.5. In this subsection we assume that $m \geq 1$. We will consider two cases:

(I) $t_1 = s_i$,

(II) $t_1 = 1$.

In case (I) we set $(v', w') = (s_i v, s_i w)$, $i' = (i_2, i_3, \ldots, i_m) \in \mathcal{I}_{w'}$. We have $v' \leq w'$ and the analogue of the sequence $q_1, q_2, \ldots, q_m$ in 3.2 for $(v', w', i')$ is $q_2, q_3, \ldots, q_m$.

In case (II) we set $(v', w') = (v, s_i w)$, $i' = i$. We have $v' \leq w'$ and the analogue of the sequence $q_1, q_2, \ldots, q_m$ in 3.2 for $(v', w', i')$ is $q_2, q_3, \ldots, q_m$. For a semifield $K_1$ let $A'(K_1)$ be the set of maps $[2, m]' \rightarrow K_1$ (notation of 3.4) and let $\theta'_{K_1} : A'(K_1) \rightarrow V(K_1) - \emptyset$, $\omega'_{K_1} : A'(K_1) \rightarrow P(K_1)$ be the analogues of $\theta_{K_1}, \omega_{K_1}$ in 3.4 when $v, w$ is replaced by $v', w'$. From the definitions, in case (I), for $h \in A(K_1)$ we have

(a) $\theta'_{K_1}(h) = T_{i_1, K_1}(\theta'_{K_1}(h|_{[2,m]'}))$

(b) $\omega'_{K_1}(h) = [T_{i_1, K_1}](\omega'_{K_1}(h|_{[2,m]'})$

where $[T_{i_1, K_1}]$ is the bijection $(V(K_1)^{s_{i_1}} - \emptyset)/K_1 \rightarrow (V(K_1)^{i_{s_{i_1}}} - \emptyset)/K_1$ induced by $T_{i_1, K_1} : V(K_1)^{s_{i_1}} \rightarrow V(K_1)^{i_{s_{i_1}}}$ (the image of $\omega'_{K_1}(h|_{[2,m]'})$ is contained in $(V(K_1)^{i_{s_{i_1}}} - \emptyset)/K_1$).

From the definitions, in case (II), for $h \in A(K_1)$ we have

(c) $\theta'_{K_1}(h) = (-i_1)^{h(i_1)}(\theta'_{K_1}(h|_{[2,m]'})$

3.6. In the remainder of this section we assume that $\lambda \in \mathcal{X}^{++}$. In the setup of 3.5, let $h, \tilde{h}$ be elements of $A(K_1)$. Let $\xi = \theta'_{K_1}(h|_{[2,m]'})$, $\tilde{\xi} = \theta'_{K_1}(\tilde{h}|_{[2,m]'})$ be such that $(-i_1)^{h(i_1)}(\xi)$, $(-i_1)^{\tilde{h}(i_1)}(\tilde{\xi})$ have the same image in $P(K)$. We show:
(a) \( h(i_1) = \tilde{h}(i_1) \) and \( \xi, \tilde{\xi} \) have the same image in \( P(K) \).

By 3.2(a),(b) (for \( w' \) instead of \( w \)),

(b) \( b_{w'} \) appears in \( \xi \) with coefficient \( c \in K_1 \); if \( b \in \beta \) appears in \( \xi \) with coefficient \( \neq 0 \) then \( v_b \neq v_{b_{w'}} + i' \).

Similarly,

(c) \( b_{w'} \) appears in \( \tilde{\xi} \) with coefficient \( \tilde{c} \in K_1 \); if \( b \in \beta \) appears in \( \tilde{\xi} \) with coefficient \( \neq 0 \) then \( v_b \neq v_{b_{w'}} + i' \).

From our assumption on \( \lambda \) we have \( b_{w'} \neq b_w = f_{i_0}^{(m)} b_{w'} \) and \( f_{i_0}^{(1)} b_{w'} \neq \emptyset \). By (b),(c) we have

\[
(-i_1)^h(i_1)(\xi) = c_{\beta_{w'}} + h(i_1) c f_{i_0}^{(1)} b_{w'} + K_1 \text{-comb. of } b \in \beta \text{ of other weights,}
\]

\[
(-i_1)^{\tilde{h}(i_1)}(\tilde{\xi}) = \tilde{c}_{\beta_{w'}} + \tilde{c} h(i_1) f_{i_0}^{(1)} b_{w'} + K_1 \text{-comb. of } b \in \beta \text{ of other weights.}
\]

We deduce that for some \( k \in K_1 \) we have \( \tilde{c} = kc, \tilde{c} \tilde{h}(i_1) = kch(i_1) \). It follows that \( h(i_1) = \tilde{h}(i_1) \). Using this and our assumption, we see that for some \( k \in K_1 \) we have \( (-i_1)^{h(i_1)}(\xi) = (-i_1)^{\tilde{h}(i_1)}(c\tilde{\xi}) \). Using 1.4(a) we deduce \( \xi = c\tilde{\xi} \). This proves (a).

3.7. In the setup of 3.4 we show:

(a) \( \omega_{K_1} : A(K_1) \rightarrow P(K_1) \) is injective.

We argue by induction on \( m \). If \( m = 0 \) there is nothing to prove. We now assume that \( m \geq 1 \). Let \( \omega'_{K_1} : A'(K_1) \rightarrow P(K_1) \) be as in 3.5. By the induction hypothesis, \( \omega'_{K_1} \) is injective. In case I (in 3.5), we use 3.5(b) and the bijectivity of \([T_{i_1,\nu}]\) to deduce that \( \omega_{K_1} \) is injective. In case II (in 3.5), we use 3.5(c) and 3.6(a) to deduce that \( \omega_{K_1} \) is injective. This proves (a).

3.8. According to [MR],

(a) \( h \mapsto \sigma(h) B^+ \sigma(h)^{-1} \) defines an isomorphism \( \tau \) from \( A \) to an open subvariety of \( B_{v,w} \) containing \( (B_{v,w})_{\geq 0} \) and \( \tau \) restricts to a bijection \( A_{\geq 0} \xrightarrow{\sim} (B_{v,w})_{\geq 0} \).

(The existence of a homeomorphism \( R^{|w| - |v|} \xrightarrow{\sim} (B_{v,w})_{\geq 0} \) was conjectured in [L94b].)

We define \( \tilde{A}_{\geq 0} \) in terms \( A \) and its subset \( A_{\geq 0} \) as in 1.9. Note that \( \tilde{A}_{\geq 0} \) can be identified with the set of maps \( h : [1, m]' \rightarrow \tilde{K} \) that is, with \( A(K) \) (notation of 3.4). Now \( \tau : A \rightarrow B_{v,w} \) (see (a)) carries \( A_{\geq 0} \) onto the subset \( (B_{v,w})_{\geq 0} \) of \( B_{v,w} \) hence it induces a map

(b) \( A(K) = \tilde{A}_{\geq 0} \rightarrow \widetilde{B}_{v,w, \geq 0} \) which is a bijection.

(We use (a) and 1.9(a)).

3.9. From the definition we deduce that we have canonically

(a) \( \tilde{B}_{\geq 0} = \bigcup_{v,w} \tilde{B}_{v,w, \geq 0} \).

The left hand side is identified in 1.10 with \( P^*(K) \), a subspace of \( P(K) \). Hence the subset \( \tilde{B}_{v,w, \geq 0} \) of \( \tilde{B}_{\geq 0} \) can be viewed as a subset \( \tilde{P}_{v,w}(K) \) of \( P(K) \) and 3.8(b) defines a bijection of \( A(K) \) onto \( P_{v,w}(K) \). The composition of this bijection with the imbedding \( P_{v,w}(K) \subset P(K) \) coincides with the map \( \omega_K : A \rightarrow P(K) \) in 3.4.

(This follows from definitions.)
Similarly, the composition of the imbeddings

\[(B_{v,w})_{\geq 0} \subset B_{\geq 0} = P_{\geq 0} \subset P_{\geq 0} = P(R_{>0})\]

(see 1.7(a)) can be identified via 3.8(a) with the imbedding \(\omega_{R_{>0}} : A_{\geq 0} \to P(R_{>0})\) whose image is denoted by \(P_{v,w}(R_{>0})\).

Recall that \(P^{\bullet}(Z)\) is the image of \(P^{\bullet}(K)\) under the map \(P(K) \to P(Z)\) induced by \(r : K \to Z\) (see 1.11). For \(v \leq w\) in \(W\) let \(P_{v,w}(Z)\) be the image of \(P_{v,w}(K)\) under the map \(P(K) \to P(Z)\). We have clearly \(P^{\bullet}(Z) = \cup_{v \leq w} P_{v,w}(Z)\). From the commutative diagram in 3.4 attached to \(r : K \to Z\) we deduce a commutative diagram

\[
\begin{array}{ccc}
A(K) & \longrightarrow & P_{v,w}(K) \\
\downarrow & & \downarrow \\
A(Z) & \longrightarrow & P_{v,w}(Z)
\end{array}
\]

in which the vertical maps are surjective and the upper horizontal map is a bijection. It follows that the lower horizontal map is surjective; but it is also injective (see 3.7(a)) hence bijective.

3.10. We return to the setup of 3.4. If \(K_1\) is one of the semifields \(R_{>0}, K, Z\), then the elements of \(P_{v,w}(K_1)\) are represented by elements of \(\xi \in V(K_1) - \emptyset\) with \(\text{supp}(\xi) = \beta_{v,i}\). In the case where \(K_1 = R_{>0}\), \(P_{v,w}(K_1)\) depends only on \(v, w\) and not on \(i\). It follows that \(\beta_{v,i}\) depends only on \(v, w\) not on \(i\) hence we can write \(\beta_{v,w}\) instead of \(\beta_{v,i}\).

Note that in [L19b, 2.4] it was conjectured that the set \([[v, w]]\) defined in [L19b, 2.3(a)] in type \(A_2\) should make sense in general. This conjecture is now established by taking \([[v, w]] = \beta_{v,w}\).

Using 2.4(a) and the definitions we see that

\[
\text{(a)} \quad \beta_{v,w} \subset \beta^w \cap \phi(\beta^{v \cdot w_i}).
\]

We expect that this is an equality (a variant of a conjecture in [L19b, 2.4], see also [L19b, 2.3(a)]). From 3.4 we see that

\[
\text{(b)} \quad b_w \in \beta_{v,w}.
\]

From 2.3(d) we deduce:

\[
\text{(c)} \quad \phi(\beta_{ww \cdot i, vw \cdot i}) = \beta_{v,w}.
\]

Using (b),(c) we deduce:

\[
\text{(d)} \quad \phi(b_{vw \cdot i}) \in \beta_{v,w}.
\]
3.11. For $K_1$ as in 3.10 and for $v \leq w$ in $W$, $v' \leq w'$ in $W$, we show:

(a) If $P_{v,w}(K_1) \cap P_{v',w'}(K_1) \neq \emptyset$, then $v = v'$, $w = w'$.

If $K_1$ is $\mathbb{R}_{>0}$ or $K$ this is already known. We will give a proof of (a) which applies also when $K_1 = \mathbb{Z}$. From the results in 3.10 we see that it is enough to show:

(b) If $\beta_{v,w} = \beta_{v',w'}$, then $v = v'$, $w = w'$.

From 3.10(b) we have $b_{w'} \in \beta_{v',w'}$ hence $b_{w'} \in \beta_{v,w}$ so that (using 3.10(a)) we have $b_{w'} \in \beta_{v}$. Using 2.1(a) we deduce that $b_{w'} \in V^i$ (with $i$ as in 2.1). It follows that either $b_{w'} = b_w$ or $\nu_{b_{w'}} - \nu_{b_w}$ is of the form $j_1 + j_2 + \cdots + j_k$ with $j_t \in I$ and $k \geq 1$. Interchanging the roles of $w, w'$ we see that either $b_{w'} = b_w$ or $\nu_{b_{w'}} - \nu_{b_w}$ is of the form $\tilde{j}_1 + \tilde{j}_2 + \cdots + \tilde{j}_{k'}$ with $\tilde{j}_t \in I$ and $k' \geq 1$. If $b_{w'} \neq b_w$ then we must have $j_1 + j_2 + \cdots + j_k + \tilde{j}_1 + \tilde{j}_2 + \cdots + \tilde{j}_{k'} = 0$, which is absurd. Thus we have $b_{w'} = b_w$. Since $\lambda \in \mathcal{X}^+$ this implies $w = w'$.

Now applying $\phi$ to the first equality in (a) and using 3.10(c) we see that $\beta_{wuw_1,vuw_1} = \beta_{wuw_1,vuw_1}$. Using the first part of the argument with $v, w, v', w'$ replaced by $w w_1, v w_1, w' w_1, v' w_1$, we see that $vw_1 = v' w_1$ hence $v = v'$. This completes the proof of (b) hence that of (a).

Now the proof of Theorem 0.2 is complete.

3.12. Now $\phi : \mathcal{B} \rightarrow \mathcal{B}$ (see 2.3) induces an involution $\tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}}$ and an involution $\mathcal{B}_{\geq 0} \rightarrow \mathcal{B}_{\geq 0}$ denoted again by $\phi$. From 2.3(a),(d) we deduce that this involution restricts to a bijection $\mathcal{B}_{ww_1,vw_1 \geq 0} \rightarrow \mathcal{B}_{w,w \geq 0}$ for any $v \leq w$ in $W$. The involution $\phi : \tilde{\mathcal{B}}_{\geq 0} \rightarrow \tilde{\mathcal{B}}_{\geq 0}$ can be viewed as an involution of $P^*(K)$ which coincides with the restriction of the involution $\phi : P(K) \rightarrow P(K)$ in 2.7. The last involution is compatible with the involution $\phi : P(Z) \rightarrow P(Z)$ in 2.7 under the map $P(K) \rightarrow P(Z)$ induced by $r : K \rightarrow Z$. It follows the image $P^*(Z)$ of $P^*(K)$ under $P(K) \rightarrow P(Z)$ is stable under $\phi : P(Z) \rightarrow P(Z)$. Thus there is an induced involution $\phi$ on $\mathcal{B}(Z) = P^*(Z)$ which carries $P_{ww_1,vw_1}(Z)$ onto $P_{v,w}(Z)$ for any $v \leq w$ in $W$.

4. Independence on $\lambda$

4.1. Let $K_1$ be a semifield. Let $\lambda \in \mathcal{X}^+, \mathcal{N} \in \mathcal{X}^+$. Let $\mathcal{S} = \lambda \beta \times \lambda' \beta$. Let $\lambda, \lambda' V(K_1)$ be the set of formal sums $u = \sum_{s \in S} u_s s$ where $u_s \in K_1$. This is a monoid under addition (component by component) and we define scalar multiplication

$$K_1 \times \lambda, \lambda' V(K_1) \rightarrow \lambda, \lambda' V(K_1)$$

by $(k, \sum_{s \in S} u_s s) \mapsto \sum_{s \in S} (ku_s)s$. Let $\text{End}(\lambda, \lambda' V(K_1))$ be the set of maps $\zeta : \lambda, \lambda' V(K_1) \rightarrow \lambda, \lambda' V(K_1)$ such that $\zeta(\xi + \xi') = \zeta(\xi) + \zeta(\xi')$ for $\xi, \xi' \in \lambda, \lambda' V(K_1)$ and $\zeta(k\xi) = k\zeta(\xi)$ for $\xi \in \lambda, \lambda' V(K_1), k \in K_1$. This is a monoid under composition of maps.

We define a map

$$E(K_1) : \lambda V(K_1) \times \lambda' V(K_1) \rightarrow \lambda, \lambda' V(K_1)$$
by
\[
(\sum_{b_1 \in \lambda \beta} \xi_{b_1}), (\sum_{b'_1 \in \lambda' \beta'} \xi'_{b'_1}) \mapsto \sum_{(b_1, b'_1) \in S} \xi_{b_1} \xi'_{b'_1} (b_1, b'_1).
\]

We define a map
\[
\text{End}(\lambda V(K_1)) \times \text{End}(\lambda' V(K_1)) \to \text{End}(\lambda,\lambda' V(K_1))
\]
by \((\tau, \tau') \mapsto [(b_1, b'_1) \mapsto E(K_1)(\tau(b_1), \tau'(b'_1))].\) Composing this map with the map
\[
\mathcal{G}(K_1) \to \text{End}(\lambda V(K_1)) \times \text{End}(\lambda' V(K_1))
\]
whose components are the maps
\[
\mathcal{G}(K_1) \to \text{End}(\lambda V(K_1)), \quad \mathcal{G}(K_1) \to \text{End}(\lambda' V(K_1))
\]
in 1.5 we obtain a map \(\mathcal{G}(K_1) \to \text{End}(\lambda,\lambda' V(K_1))\) which is a monoid homomorphism. Thus \(\mathcal{G}(K_1)\) acts on \(\lambda,\lambda' V(K_1)\); it also acts on \(\lambda V(K_1) \times \lambda' V(K_1)\) (by 1.5) and the two actions are compatible with \(E(K_1)\).

Let \(\mathcal{G}(K_1)\) be the element \(u \in \lambda,\lambda' V(K_1)\) such that \(u_s = 0\) for all \(s \in S\). Let \(\lambda,\lambda' P(K_1)\) be the set of orbits of the free \(K_1\) action (scalar multiplication) on \(\lambda,\lambda' V(K_1) - \mathcal{G}\). Now \(E(K_1)\) restricts to a map
\[
(\lambda V(K_1)) - \mathcal{G}) \times (\lambda' V(K_1) - \mathcal{G}) \to \lambda,\lambda' V(K_1) - \mathcal{G}
\]
and induces an (injective) map
\[
\bar{E}(K_1) : \lambda P(K_1) \times \lambda' P(K_1) \to \lambda,\lambda' P(K_1).
\]
Now \(\mathcal{G}(K_1)\) acts naturally on \(\lambda P(K_1) \times \lambda' P(K_1)\) and on \(\lambda,\lambda' P(K_1)\); these \(\mathcal{G}(K_1)\)-actions are compatible with \(\bar{E}(K_1)\).

4.2. There is a unique linear map
\[
\Gamma : \lambda+\lambda' V \to \lambda V \otimes \lambda' V
\]
which is compatible with the \(G\)-actions and takes \(\lambda+\lambda' \xi^+\) to \(\lambda \xi^+ \otimes \lambda' \xi^+\). For \(b \in \lambda+\lambda' \beta\) we have
\[
\Gamma(b) = \sum_{(b_1, b'_1) \in S} e_{b_1, b'_1} b_1 \otimes b'_1
\]
where \(e_{b_1, b'_1} \in \mathbb{N}\). (This can be deduced from the positivity property [L93, 14.4.13(b)] of the homomorphism \(r\) in [L93, 1.2.12].) There is a unique map
\[
\Gamma(K_1) : \lambda+\lambda' V(K_1) \to \lambda,\lambda' V(K_1)
\]
compatible with addition and scalar multiplication and such that for $b \in \lambda + \lambda' \beta$ we have

$$\Gamma(K_1)(b) = \sum_{(b_1, b'_1) \in S} e_{b,b_1,b'_1}(b_1, b'_1)$$

where $e_{b,b_1,b'_1}$ are viewed as elements of $K'_1$. Since $\Gamma$ is injective, for any $b \in \lambda + \lambda' \beta$ we have $e_{b,b_1,b'_1} \in \mathbb{N} - \{0\}$ for some $b_1, b'_1$, hence $e_{b,b_1,b'_1} \in K_1$, when viewed as an element of $K'_1$. It follows that $\Gamma(K_1)$ maps $\lambda + \lambda' V(K_1) - \varnothing$ into $\lambda + \lambda' V(K_1) - \varnothing$.

Hence $\Gamma(K_1)$ defines an (injective) map

$$\bar{\Gamma}(K_1) : \lambda + \lambda' P(K_1) \rightarrow \lambda + \lambda' P(K_1)$$

which is compatible with the action of $\mathfrak{S}(K_1)$ on the two sides.

4.3. We now assume that $K_1$ is either $K$ as in 0.1(i) or $\mathbb{Z}$ as in 0.1(ii) and that $\lambda \in \mathcal{X}^+, \lambda' \in \mathcal{X}^+$ so that $\lambda + \lambda' \in \mathcal{X}^+$. We have the following result.

(a) Let $L \in \lambda + \lambda' P^\bullet(K_1)$. Then $\bar{\Gamma}(K_1)(L) = \bar{E}(K_1)(L_1, L'_1)$ for some $(L_1, L'_1) \in \lambda P^\bullet(K_1) \times \lambda' P(K_1)$ (which is unique, by the injectivity of $\bar{E}(K_1)$). Thus, $L \mapsto L_1$ is a well defined map $H(K_1) : \lambda + \lambda' P^\bullet(K_1) \rightarrow \lambda P^\bullet(K_1)$.

We shall prove (a) for $K_1 = \mathbb{Z}$ assuming that it is true for $K_1 = K$. We can find $\tilde{L} \in \lambda + \lambda' P^\bullet(K)$ such that $L \in \lambda + \lambda' P^\bullet(\mathbb{Z})$ is the image of $\tilde{L}$ under the map $\lambda + \lambda' P^\bullet(K) \rightarrow \lambda + \lambda' P^\bullet(\mathbb{Z})$ induced by $r : K \rightarrow \mathbb{Z}$. By our assumption we have $\bar{\Gamma}(K)(\tilde{L}) = \bar{E}(K)(\tilde{L}_1, \tilde{L}'_1)$ with $(\tilde{L}_1, \tilde{L}'_1) \in \lambda P^\bullet(K) \times \lambda' P(K)$. Let $L_1$ (resp. $L'_1$) be the image of $\tilde{L}_1$ (resp. $\tilde{L}'_1$) under the map $\lambda P^\bullet(K) \rightarrow \lambda P^\bullet(\mathbb{Z})$ (resp. $\lambda' P(K) \rightarrow \lambda' P(\mathbb{Z})$) induced by $r : K \rightarrow \mathbb{Z}$. From the definitions we see that $\bar{\Gamma}(Z)(L) = \bar{E}(\mathbb{Z})(L_1, L'_1)$. This proves the existence of $(L_1, L'_1)$. The proof of (a) in the case where $K_1 = \mathbb{K}$ will be given in 4.6.

Assuming that (a) holds, we have a commutative diagram

$$\begin{array}{ccc}
\lambda + \lambda' P^\bullet(K) & \xrightarrow{H(K)} & \lambda P^\bullet(K) \\
\downarrow & & \downarrow \\
\lambda + \lambda' P^\bullet(\mathbb{Z}) & \xrightarrow{H(\mathbb{Z})} & \lambda P^\bullet(\mathbb{Z})
\end{array}$$

in which the vertical maps are induced by $r : K \rightarrow \mathbb{Z}$.

4.4. We preserve the setup of 4.3. For each $w \in W$ we assume that a sequence $i_w = (i_{1}, i_{2}, \ldots, i_{m}) \in I_w$ has been chosen (here $m = |w|$). Let $Z(K_1) = \sqcup_{v \leq w} A_{v,w}(K_1)$ where $A_{v,w}(K_1)$ is the set of all maps $[1, m]' \rightarrow K_1$ (with $[1, m]'$ defined as in 3.2 in terms of $v, w$ and $i = i_w$). From the results in 3.9 we have a bijection

$$\lambda D(K_1) : Z(K_1) \xrightarrow{\sim} \lambda P^\bullet(K_1)$$
whose restriction to \( A_{v,w}(K_1) \) is as in the last commutative diagram in 3.9 (with \( i = i_w \)). Replacing here \( \lambda \) by \( \lambda + \lambda' \) we obtain an analogous bijection

\[
\lambda + \lambda' D(K_1) : Z(K_1) \overset{\sim}{\to} \lambda + \lambda' P^*(K_1).
\]

From the commutative diagram in 3.4 we deduce a commutative diagram

\[
\begin{array}{ccc}
Z(K) & \xrightarrow{\lambda D(K)} & \lambda P^*(K) \\
\downarrow & & \downarrow \\
Z(Z) & \xrightarrow{\lambda D(Z)} & \lambda P^*(Z)
\end{array}
\]

and a commutative diagram

\[
\begin{array}{ccc}
Z(K) & \xrightarrow{\lambda + \lambda' D(K)} & \lambda + \lambda' P^*(K) \\
\downarrow & & \downarrow \\
Z(Z) & \xrightarrow{\lambda + \lambda' D(Z)} & \lambda + \lambda' P^*(Z)
\end{array}
\]

in which the vertical maps are induced by \( r : K \to Z \).

**4.5.** We preserve the setup of 4.3. We assume that 4.3(a) holds. From the commutative diagrams in 4.3, 4.4 we deduce a commutative diagram

\[
\begin{array}{ccc}
Z(K) & \xrightarrow{(\lambda D(K))^{-1} H(K) \lambda + \lambda' D(K)} & Z(K) \\
\downarrow & & \downarrow \\
Z(Z) & \xrightarrow{(\lambda D(Z))^{-1} H(Z) \lambda + \lambda' D(Z)} & Z(Z)
\end{array}
\]

in which the vertical maps are induced by \( r : K \to Z \). Recall that \( K_1 \) is \( K \) or \( Z \).

We have the following result.

(a) \((\lambda D(K_1))^{-1} H(K_1) \lambda + \lambda' D(K_1)\) is the identity map \( Z(K_1) \to Z(K_1) \).

If (a) holds for \( K_1 = K \) then it also holds for \( K_1 = Z \), in view of the commutative diagram above in which the vertical maps are surjective. The proof of (a) in the case \( K_1 = K \) will be given in 4.7.

From (a) we deduce:

(b) \( H(K_1) \) is a bijection.

**4.6.** In this subsection we assume that \( K_1 = K \). Let \( k = \mathbb{C}(x) \) where \( x \) is an indeterminate. We have \( K^1 \subset k \). For any \( \lambda \in \chi^+ \) we set \( \lambda V_k = k \otimes \lambda V \). This is naturally a module over the group \( G(k) \) of \( k \) points of \( G \). Let \( B(k) \) be the set of subgroups of \( G(k) \) that are \( G(k) \)-conjugate to \( B^+(k) \), the group of \( k \)-points of
$B^+$. We identify $\lambda V(K)$ with the set of vectors in $\lambda V_k$ whose coordinates in the $k$-basis $\lambda\beta$ are in $B^1$. In the case where $\lambda \in \mathcal{X}^{++,+}$, we identify $\lambda V^\bullet(K) = 0$ with the set of all $\xi \in \lambda V(K) - 0$ such that the stabilizer in $G(k)$ of the line $[\xi]$ belongs to $\mathcal{B}(k)$. (For a nonzero vector $\xi$ in a $k$-vector space we denote by $[\xi]$ the $k$-line in that vector space that contains $\xi$.)

Now let $\lambda \in \mathcal{X}^{++,+}, \lambda' \in \mathcal{X}^+$. We show that 4.3(a) holds for $\lambda, \lambda'$. We identify $\lambda' V(K)$ with the set of vectors in $\lambda' V_k \otimes_k \lambda' V_k$ whose coordinates in the $k$-basis $\lambda\beta \otimes \lambda'\beta$ are in $B^1$.

Then $E(K)$ becomes the restriction of the homomorphism of $G(k)$-modules $E' : \lambda V_k \otimes \lambda' V_k \rightarrow \lambda V_k \otimes \lambda' V_k$ given by $(\xi, \xi') \mapsto \xi \otimes_k \xi'$ and $\Gamma(K)$ becomes the restriction of the homomorphism of $G(k)$-modules $\Gamma' : \lambda' V_k \rightarrow \lambda V_k$ obtained from $\Gamma$ by extension of scalars.

Let $L_\lambda = [\lambda\xi^+] \subset \lambda V_k$, $L_{\lambda'} = [\lambda'\xi^+] \subset \lambda' V_k$, $L_{\lambda+\lambda'} = [\lambda+\lambda'\xi^+] \subset \lambda+\lambda' V_k$. Now let $\xi \in \lambda+\lambda' V^\bullet(K) - 0$. Then $[\xi] = gL_{\lambda+\lambda'}$ for some $g \in G(k)$ hence

$$\Gamma'([\xi]) = g(L_\lambda \otimes L_{\lambda'}) = (gL_\lambda) \otimes (gL_{\lambda'}) = E'(gL_\lambda, g(L_{\lambda'})) = E'(g(\lambda\xi^+), g(\lambda'\xi^+)).$$

To prove 4.3(a) in our case it is enough to prove that for some $c, c'$ in $k^*$ we have $cg(\lambda\xi^+ \in \lambda V(K), c'g(\lambda'\xi^+) \in \lambda' V(K).$ We have $\xi = c_0g(\lambda+\lambda'\xi^+)$ for some $c_0 \in k^*$ and $\Gamma'(\xi) = \Gamma(\xi) \in \lambda, \lambda' V(K).$ Thus, $c_0g(\lambda\xi^+) \otimes (\lambda'\xi^+) \in \lambda, \lambda' V(K).$ It is enough to show:

(a) If $z \in \lambda V_k$, $z' \in \lambda' V_k$, $c_0 \in k^*$ satisfy $c_0z \otimes z' \in \lambda, \lambda' V(K) - 0$, then $cz \in \lambda V(K) - 0, c'z' \in \lambda' V(K) - 0$ for some $c, c'$ in $k^*$.

We write $z = \sum_{b \in \lambda, \beta} z_b b$, $z' = \sum_{b' \in \lambda', \beta} z'_b b'$ with $z_b, z'_b$ in $k$. We have $c_0 z_b z'_b \in K^1$ for all $b, b'$. Replacing $z$ by $c_0z$ we can assume that $c_0 = 1$ so that $z_b z'_b \in K^1$ for all $b, b'$ and $z_b z'_b \neq 0$ for some $b, b'$. Thus we can find $b_0 \in \lambda, \beta$ such that $z_b^b \in K$.

We have $z_b z'_b \in K^1$ for all $b$. Replacing $z$ by $z_b z_0$, we can assume that $z_b \in K^1$ for all $b$. We can find $b_0 \in \lambda, \beta$ such that $z_b \in K$. We have $z_b z'_b \in K^1$ for all $b'$. It follows that $z'_b \in K^1$ for all $b'$. This proves (a) and completes the proof of 4.3(a).

4.7. We preserve the setup of 4.3 and assume that $K_1 = K$. We show that 4.5(a) holds in this case. Let $w \leq v, i$ be as in 3.2 and let $A(K_1)$ be as in 3.4. Let $h \in A(K_1)$. We have $\sigma K_1(h) \lambda+\lambda' \xi^+ in \sigma K_1 : A(K_1) \rightarrow G(k)$ is defined by the same formula as $\sigma$ in 3.2. (Note that for $i \in I, y_i(t) \in G(k)$ is defined for any $t \in k$.) Hence

$$\bar{\Gamma}(K_1)(\lambda+\lambda' D(K_1)(h) = [(\sigma K_1(h)\lambda\xi^+ \otimes (\sigma K_1(h)\lambda'\xi^+)]$$

so that

$$H(K_1)(\lambda+\lambda' D(K_1)(h) = [(\sigma K_1(h)\lambda\xi^+ \otimes (\sigma K_1(h)\lambda'\xi^+)]$$

This shows that the map in 4.5(a) takes $h$ to $h$ for any $h \in A(K_1)$. This proves 4.5(a).
4.8. We now assume that $K_1$ is either $K$ as in 0.1(i) or $Z$ as in 0.1(ii) and that $\lambda \in \mathcal{X}^{++}, \lambda' \in \mathcal{X}^{++}$. From 4.3(a), 4.5(a) we have a well defined bijection $H(K_1) : \lambda + \lambda' P^*(K_1) \to \lambda P^*(K_1)$. Interchanging $\lambda, \lambda'$ we obtain a bijection $H'(K_1) : \lambda + \lambda' P^*(K_1) \to \lambda' P^*(K_1)$. Hence we have a bijection

$$\gamma_{\lambda, \lambda'} = H'(K_1)H(K_1)^{-1} : \lambda P^*(K_1) \to \lambda' P^*(K_1).$$

From the definitions we see that $H(K_1)$ is compatible with the $\mathfrak{S}(K_1)$-actions. Similarly, $H'(K_1)$ is compatible with the $\mathfrak{S}(K_1)$-actions. It follows that $\gamma_{\lambda, \lambda'}$ is compatible with the $\mathfrak{S}(K_1)$-actions. From the definitions we see that if $\lambda''$ is third element of $\mathcal{X}^{++}$, we have

$$\gamma_{\lambda, \lambda''} = \gamma_{\lambda', \lambda''} \gamma_{\lambda, \lambda'}.$$

This shows that our definition of $B(K_1)$ is independent of the choice of $\lambda$.

5. Complements

5.1. Let $\delta : G \to G$ be an automorphism of $G$ such that $\delta(B^+) = B^+, \delta(B^-) = B^-$ and $\delta(x_i(t)) = x_i(t), \delta(y_i(t)) = y_i(t)$ for all $i \in I, t \in C$ where $i \mapsto i'$ is a permutation of $I$ denoted again by $\delta$. We define an automorphism of $W$ by $s_i \mapsto s_{\delta(i)}$ for all $i \in I$; we denote this automorphism again by $\delta$. We assume further that $s_is_{\delta(i)} = s_{\delta(i)}s_i$ for any $i \in I$. The fixed point set $G^\delta$ of $\delta : G \to G$ is a connected simply connected semisimple group over $C$. The fixed point set $W^\delta$ of $\delta : W \to W$ is the Weyl group of $G^\delta$ and as such it has a length function $w \mapsto |w|_\delta$.

Now $\delta$ takes any Borel subgroup of $G$ to a Borel subgroup of $G$ hence it defines an automorphism of $B$ by $\delta$, with fixed point set denoted by $B^\delta$. This automorphism restricts to a bijection $B_{\geq 0} \to B_{\geq 0}$. We can identify $B^\delta$ with the flag manifold of $G^\delta$ by $B \mapsto B \cap G^\delta$. Under this identification, the totally positive part of the flag manifold of $G^\delta$ (defined in [L94b]) becomes $B^\delta_{\geq 0} = B_{\geq 0} \cap B^\delta$. For $\lambda \in \mathcal{X}$ we define $\delta(\lambda) \in \mathcal{X}$ by $\langle \delta(i), \delta(\lambda) \rangle = \langle i, \lambda \rangle$ for all $i \in I$. In the setup of 1.4 assume that $\lambda \in \mathcal{X}^{++}$ satisfies $\delta(\lambda) = \lambda$. There is a unique linear isomorphism $\delta : V \to V$ such that $\delta(g\xi) = \delta(g)\delta(\xi)$ for any $g \in G, \xi \in V$ and such that $\delta(\xi^+) = \xi^+$. This restricts to a bijection $\beta \mapsto \beta$ denoted again by $\delta$. For any semifield $K_1$ we define a bijection $V(K_1) \to V(K_1)$ by $\sum_{b \in B} \xi_b b \mapsto \sum_{b \in B} \xi_{\delta(b)} b$ where $\xi_b \in K_1^1$. This induces a bijection $P(K_1) \to P(K_1)$ denoted by $\delta$. We now assume that $K_1$ is as in 0.1(i),(ii). Then the subset $P^*(K_1)$ of $P(K_1)$ is defined and is stable under $\delta$; let $P^*(K_1)^\delta$ be the fixed point set of $\delta : P^*(K_1) \to P^*(K_1)$. Recall that $G(K_1)$ acts naturally on $P(K_1)$. This restricts to an action on $P^*(K_1)^\delta$ of the monoid $G(K_1)^\delta$ (the fixed point set of the isomorphism $G(K_1) \to G(K_1)$ induced by $\delta$) which is the same as the monoid associated in [L19a] to $G^\delta$ and $K_1$.

We set $B^\delta(K_1) = P^*(K_1)^\delta$.

The following generalization of Theorem 0.2 can be deduced from Theorem 0.2.

(a) The set $B^\delta(Z)$ has a canonical partition into pieces $P_{v,w;\delta}(Z)$ indexed by the pairs $v \leq w$ in $W^\delta$. Each such piece $P_{v,w;\delta}(Z)$ is in bijection with $Z^{\mid w|_\delta - |v|_\delta}$; in
fact, there is an explicit bijection $\mathbb{Z}^{v|\delta - |v|\delta} \xrightarrow{\sim} P_{v,w;\delta}(\mathbb{Z})$ for any reduced expression of $w$ in $W^\delta$.  

5.2. In the remaining subsections we no longer assume that $G$ is as in 0.1. We shall instead replace $G$ by the group $G(k)$ of simply connected type associated in [MT], [Ma], [Ti], [PK], to a not necessarily positive definite Cartan matrix of simply laced type and to a field $k$ of characteristic zero. We write $(B^+, B^-, x_i : k \to G(k), y_i : k \to G(k) \ (i \in I))$ for the analogue of the pinning of $G$ in 1.2. Let $\mathcal{X}^+$ be the analogue for $G(k)$ of the set with the same name in 1.2. It is a free abelian group with basis $\{\omega_i; i \in I\}$ consisting of fundamental weights. For $\lambda \in \mathcal{X}$ let $\text{supp}(\lambda)$ be the set of all $i \in I$ such that $\omega_i$ appears with $\neq 0$ coefficient in $\lambda$. We fix $J \subset I$. Let $\mathcal{X}_J^+ = \{\lambda \in \mathcal{X}^+; \text{supp}(\lambda) = J\}$.

The irreducible integrable representations of $G(k)$ are indexed by their highest weight, an elements of $\mathcal{X}^+$. Let $\lambda \in \mathcal{X}_J^+$ and let $\lambda V$ be a $k$-vector space which is an irreducible integrable representation of $G(k)$ indexed by $\lambda$. Let $\lambda P$ be the set of lines in the $k$-vector space $\lambda V$. Let $\lambda \xi^+$ be a highest weight vector of $\lambda V$. Let $\lambda P^\bullet$ be the subset of $\lambda P$ consisting of lines in the $G(k)$-orbit of the line spanned by $\lambda \xi^+$. Let $P(k)$ be the set of subgroups of $G(k)$ which are stabilizers of various lines in $\lambda P$; this set depends only on $J$, not on $\lambda$.

Let $\lambda \beta$ be the canonical basis of $\lambda V$ (see [L93, 11.10]) containing $\lambda \xi^+$.

Let $K_1$ be a semifield. Let $\lambda V(K_1)$ be the set of formal sums $\xi = \sum_{b \in \beta} \xi_b b$ with $\xi_b \in K^1$ for all $b \in \beta$ and $\xi_b = 0$ for all but finitely many $b$. We can define addition and scalar multiplication on $\lambda V(K_1)$ as in 1.4. Let $\varnothing = \sum_{b \in \beta} b \in \lambda V(K_1)$. Let $\text{End}(\lambda V(K_1))$ be the set of maps $\lambda V(K_1) \to \lambda V(K_1)$ which commute with addition and scalar multiplication; this is a monoid under composition of maps. Let $\mathfrak{S}(K_1)$ be the monoid defined by generators and relations as in [L18, 3.1]; as in 1.5 we have a natural homomorphism of monoids $\mathfrak{S}(K_1) \to \text{End}(\lambda V(K_1))$. (We use the positivity property [L93, 22.1.7] of $\lambda \beta$.) Let $\lambda P(K_1)$ be the set of orbits of the obvious $K_1$-action on $\lambda V(K_1) \setminus \varnothing$. Then $\mathfrak{S}(K_1)$ acts naturally on $\lambda V(K_1) \setminus \varnothing$ and on $\lambda P(K_1)$.

If $k = \mathbb{C}$, $K_1 = \mathbb{R}_{>0}$, then $\lambda P(K_1)$ can be viewed as a subset of $\lambda P$. We define $\lambda P^\bullet(K_1) = \lambda P^\bullet \cap \lambda P(K_1)$; the $\mathfrak{S}(K_1)$-action on $\lambda P(K_1)$ restricts to a $\mathfrak{S}(K_1)$-action on $\lambda P^\bullet(K_1)$.

We now assume that $k = \mathbb{C}(x)$ where $x$ is an indeterminate and that $K$ is as in 0.1(i). Then $\lambda P(K)$ can be viewed as a subset of $\lambda P$. We define $\lambda P^\bullet(K) = \lambda P^\bullet \cap \lambda P(K)$; the $\mathfrak{S}(K)$-action on $\lambda P(K)$ restricts to a $\mathfrak{S}(K)$-action on $\lambda P^\bullet(K)$. The semifield homomorphism $r : K \to \mathbb{Z}$ in 1.11 induces a surjective map $\lambda P(K) \to \lambda P(\mathbb{Z})$. Let $\lambda P^\bullet(\mathbb{Z})$ be the image of $\lambda P^\bullet(K)$ under this map. The action of $G(\mathbb{Z})$ on $\lambda P(\mathbb{Z})$ leaves $\lambda P^\bullet(\mathbb{Z})$ stable and it restricts to a action of $G(K)$ on $\lambda P^\bullet(\mathbb{Z})$. When $K_1$ is $\mathbb{R}_{>0}, K$ or $\mathbb{Z}$ we define $\lambda P(K_1) = \lambda P^\bullet(K_1)$.

5.3. We preserve the setup of 5.2. Assume that $K_1$ is $\mathbb{R}_{>0}, K$ or $\mathbb{Z}$. We define a partial order $\leq$ on $\mathcal{X}_J^+$ in which $\lambda_1, \lambda_2$ in $\mathcal{X}_J^+$ satisfy $\lambda_1 \leq \lambda_2$ whenever $\lambda_2 - \lambda_1 \in$
\( \mathcal{X}^+ \). For such \( \lambda_1, \lambda_2 \) we have a canonical map
\[
epsilon_{\lambda_2, \lambda_1} : \lambda_2 \mathcal{P}(K_1) \to \lambda_1 \mathcal{P}(K_1)
\]
defined like \( H(K_1) \) in 4.3(a) with \( \lambda = \lambda_1, \lambda' = \lambda_2 - \lambda_1 \). (A proof entirely similar to that in §4 can be carried out in this case.) This map is compatible with the \( \mathfrak{G}(K_1) \)-actions. From the definitions we see that if \( \lambda_1, \lambda_2, \lambda_3 \) in \( \mathcal{X}_J^+ \) satisfy \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \), we have
\[
epsilon_{\lambda_3, \lambda_1} = \epsilon_{\lambda_2, \lambda_1} \epsilon_{\lambda_3, \lambda_2}.
\]
The system formed by the sets \( \lambda \mathcal{P}(K_1) \) (\( \lambda \in \mathcal{X}_J^+ \)) and the maps \( \epsilon_{\lambda_2, \lambda_1} \) has a projective limit which is denoted by \( \mathcal{P}(K_1) \). Note that \( \mathcal{P}(K_1) \) is nonempty. Indeed, it contains at least one point, namely the point represented for any \( \lambda \in \mathcal{X}_J^+ \) by the point of \( \lambda \mathcal{P}(K_1) \) corresponding to \( \lambda \xi^+ \). Moreover, \( \mathfrak{G}(K_1) \) acts naturally on \( \mathcal{P}(K_1) \).

5.4. Assume now that \( k, K_1 \) is equal to \( \mathbb{C}, \mathbb{R}_{>0} \) or to \( \mathbb{C}(x), K \). Then for any \( \lambda \in \mathcal{X}_J^+ \), \( \lambda \mathcal{P}(K_1) \) can be viewed as a subset of \( \mathcal{P}(k) \) so that each transition map \( \epsilon_{\lambda_2, \lambda_1} \) becomes the inclusion of one subset of \( \mathcal{P}(k) \) into another subset of \( \mathcal{P}(k) \). In particular, the maps \( \epsilon_{\lambda_2, \lambda_1} \) are injective. In this case, \( \mathcal{P}(K_1) \) is the intersection (inside \( \mathcal{P}(k) \)) of all subsets \( \lambda \mathcal{P}(K_1) \) with \( \lambda \in \mathcal{X}_J^+ \).

5.5. In this subsection we assume that we are in the setup of 5.3 and that our Cartan matrix is positive definite. In this case,

(a) the transition maps \( \epsilon_{\lambda_2, \lambda_1} \) are bijections if \( \lambda_1, \lambda_2 \) have large enough coordinates.

This can be deduced from [L98, 3.4] when \( k = \mathbb{C}, K_1 = \mathbb{R}_{>0} \), since by [L98, 3.4], for \( \lambda \in \mathcal{X}_J^+ \) with large enough coordinates, the set \( \lambda \mathcal{P}(K_1) \) coincides with the totally positive part \( \mathcal{P}_{>0} \) of the partial flag manifold \( \mathcal{P}(k) \) defined in [L98]; hence in this case we have \( \mathcal{P}(K_1) = \mathcal{P}_{>0} \).

When \( K_1 = K \) or \( \mathbb{Z} \), (a) can be deduced from the case \( K_1 = \mathbb{R}_{>0} \) by arguments similar to those in §3, §4.

5.6. In this subsection we assume that we are in the setup of 5.3, that our Cartan matrix is positive definite and that \( J = \emptyset \). Then the definition of \( \mathcal{P}(K_1) \) given in this section agrees with the definition in §1; in this case all transition maps \( \epsilon_{\lambda_2, \lambda_1} \) are bijections.

5.7. In the case where \( G = \text{SL}_n \) and \( \mathcal{P} \) is a grassmannian, a definition of \( \mathcal{P}(\mathbb{Z}) \) was given earlier in [SW].

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Department of Mathematics, M.I.T., Cambridge, MA 02139