Integrable and Superintegrable Potentials of 2d Autonomous Conservative Dynamical Systems

Antonios Mitsopoulos 1,*, Michael Tsamparlis 1 and Andronikos Paliathanasis 2,3

1 Department of Astronomy-Astrophysics-Mechanics, Faculty of Physics, University of Athens, Panepistemiopolis, 157 83 Athens, Greece; mtsampa@phys.uoa.gr
2 Institute of Systems Science, Durban University of Technology, Durban 4000, South Africa; anpaliat@phys.uoa.gr
3 Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia 5090000, Chile
* Correspondence: antmits@phys.uoa.gr

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Abstract: We consider the generic quadratic first integral (QFI) of the form \( I = K_{ab}(t,q)\dot{q}^a\dot{q}^b + K_a(t,q)\dot{q}^a + K(t,q) \) and require the condition \( dI/dt = 0 \). The latter results in a system of partial differential equations which involve the tensors \( K_{ab}(t,q) \), \( K_a(t,q) \), \( K(t,q) \) and the dynamical quantities of the dynamical equations. These equations divide in two sets. The first set involves only geometric quantities of the configuration space and the second set contains the interaction of these quantities with the dynamical fields. A theorem is presented which provides a systematic solution of the system of equations in terms of the collineations of the kinetic metric in the configuration space. This solution being geometric and covariant, applies to higher dimensions and curved spaces. The results are applied to the simple but interesting case of two-dimensional (2d) autonomous conservative Newtonian potentials. It is found that there are two classes of 2d integrable potentials and that superintegrable potentials exist in both classes. We recover most main previous results, which have been obtained by various methods, in a single and systematic way.

Keywords: integrable potentials; superintegrable potentials; Killing tensors; Bertrand-Darboux equation; Lewis invariant; Darboux solution; First integrals; Linear First Integrals; Quadratic First Integrals

1. Introduction

The precise meaning of the solution of a system of differential equations can be cast in several ways [1]. We say that we have determined a closed-form solution for a dynamical system when we have determined a set of explicit functions describing the variation of the dependent variables in terms of the independent variable(s). On the other hand, when we have proved the existence of a sufficient number of independent explicit first integrals and invariants for the dynamical system, we say that we have found an analytic solution of the dynamical equations. In addition, an algebraic solution is found when one has proved the existence of a sufficient number of explicit transformations which permit the reduction of the system of differential equations to a system of algebraic equations. A feature that is central to each of these three equivalent prescriptions of integrability is the existence of explicit functions that are first integrals/invariants or the coefficient functions of the aforementioned transformations.

A first integral (FI) of a dynamical system is a scalar \( I \) defined on the phase space of the system such that \( dI/dt = 0 \). The FIs are classified according to the power of the momenta. The linear FIs (LFIs) are linear in the momenta, the quadratic FIs (QFIs) contain products of two momenta, and so on. A dynamical system of \( n \) degrees of freedom is called integrable if it admits \( n \) (functionally) independent FIs which are in involution [2], that is, their Poisson brackets are zero, i.e., \( \{ I_i, I_j \} = 0 \).
The maximum number of independent FIs that a dynamical system of \( n \) degrees of freedom can have is \( 2^n - 1 \), and when this is the case, an integrable system is called superintegrable. The above apply to all dynamical systems which are described by dynamical equations independently if they are Lagrangian or Hamiltonian. If the dynamical system is Hamiltonian, then the FIs are defined equivalently by the requirement \( \{H, I\} = 0 \), where \( H \) is the Hamiltonian function of the system.

FIs are important for the determination of the solution and the study of dynamical systems. In particular, when a dynamical system is integrable, then (in principle) the solution of the dynamical equations can be found by means of quadratures. Such dynamical systems are characterized as Liouville integrable. For this reason, the systematic computation of FIs is a topic of active interest for a long time, perhaps by the time of the early Mechanics. Originally, the FIs were concerned in the field of the geometry of surfaces (see, for example, in [3]) where an attempt was made to compute all integrable and superintegrable 2d surfaces. A new dimension to the topic gave the introduction of the theorem of Noether in 1918 [4] which prevails the topic since then. More recently, one more systematic method, but less general than the Noether one, was presented in which one assumes a general form of the QFI and then uses the condition \( dI/dt = 0 \), or the \( \{H, I\} = 0 \), to find a system of simultaneous equations involving the coefficients defining \( I \) (see, for instance, in [5–8]). The solution of that system of conditions provides us with all the QFIs admitted by a given dynamical system.

The determination of integrable and superintegrable systems is a topic which is in continuous investigation. Obviously a universal method which computes the FIs for all types of dynamical equations independently of their complexity and degrees of freedom is not available. For this reason, the existing studies restrict their considerations to flat spaces or spaces of constant curvature of low dimension (see, e.g., in [3,9–15] and references therein). The prevailing cases involve the autonomous conservative dynamical systems with two degrees of freedom and the classification of the potential functions in integrable and superintegrable. A comprehensive review of the known integrable and superintegrable 2d autonomous potentials is given in [16].

Besides the two methods mentioned above, other approaches have appeared. For example, Koenigs [17] used coordinate transformations in order to solve the system of equations resulting from the condition \( \{H, I\} = 0 \). This solution of that system of equations gives the general functional form of the QFIs and the superintegrable free Hamiltonians, that is, the ones which possess two more QFIs—in addition to the Hamiltonian—which are functionally independent. Koenig’s method has been generalized in several works (see in [18] and references cited therein) for two dimensional autonomous conservative systems.

In the present work, we follow the method which uses the solution of the simultaneous system of equations resulting from the condition \( dI/dt = 0 \). This approach has been used extensively, see, e.g., in [5–7]; however, always for special cases only. In this work, we use Theorem 1 proved in [19], which gives the general solution of this system in terms of the collineations and the Killing tensors (KTs) of the kinetic metric in the configuration space. This solution is systematic and covariant therefore can be used in higher dimensions and for curved configuration spaces. Furthermore, it is shown that it is directly related to the Noether approach.

Theorem 1 is applied to the case of 2d autonomous conservative dynamical systems in order to determine the integrable and the superintegrable potentials. It is found that the integrable potentials are classified in Class I and Class II and that superintegrable potentials exist in both classes. All potentials together with their QFIs are listed in tables for easy reference. All the results listed in the review paper of [16] as well as in more recent works (see, e.g., in [14,15]) are recovered while some new ones are found which admit time-dependent LFIs and QFIs.

2. Gauged Noether Symmetries and QFIs

We consider an autonomous conservative dynamical system of \( n \) degrees of freedom \( q^a \) with kinetic energy \( T = \frac{1}{2} \gamma_{ab} \dot{q}^a \dot{q}^b \) where \( \dot{q}^a = \frac{dq^a}{dt} \). We define in the configuration space of the system
the kinetic metric $\gamma_{ab}$ by the requirement $\gamma_{ab} = \frac{\partial^2 T}{\partial q^a \partial q^b}$. When the dynamical system is regular, that is, $\det \left( \frac{\partial^2 T}{\partial q^a \partial q^b} \right) \neq 0$, it can be shown that the dynamical equations can be written in the form

$$\ddot{q}^a = -\Gamma^a_{bc} \dot{q}^b \dot{q}^c - V(q)^a. \quad (1)$$

where $\Gamma^a_{bc}$ are the Riemann connection coefficients defined by the kinetic metric $\gamma_{ab}$, $V(q)$ stands for the conservative forces, a comma indicates the partial derivative and the Einstein summation convention is used. Finally, the metric $\gamma_{ab}$ is used for lowering and raising the indices.

The main methods for the determination of the FIs of Equation (1) are

a. The theorem of Noether which is the standard one and requires a Lagrangian.

b. The direct method (see, e.g., in [5–7,18,20]), which is not applied widely, uses only the dynamical equations and involves the solution of the system of equations resulting from the condition $dI/dt = 0$.

The two methods are related as follows.

In the Noether approach the Noether symmetries are generated by vector fields (We restrict our considerations to vector fields in the jet space $J^1(t,q,\dot{q})$).

$$X = \xi(t,q,\dot{q}) \frac{\partial}{\partial t} + \eta^a(t,q,\dot{q}) \frac{\partial}{\partial q^a}$$

whose first prolongation $X^{[1]}$ in the jet space $J^1(t,q,\dot{q})$ is given by

$$X^{[1]} = \xi \frac{\partial}{\partial t} + \eta^a \frac{\partial}{\partial q^a} + (\dot{\eta}^a - \dot{q}^a \dot{\xi}) \frac{\partial}{\partial \dot{q}^a}. \quad (2)$$

Let $L = T - V$ be the Lagrangian of the dynamical system. The Noether symmetries of Equation (1) are its Lie symmetries which satisfy in addition the Noether condition

$$X^{[1]} L + L \xi = \dot{f} \quad (3)$$

where $f(t,q,\dot{q})$ is the Noether or the gauge function. According to the theorem of Noether, a Noether symmetry produces the FI

$$I = f - L \xi - \frac{\partial L}{\partial \dot{q}^a} (\eta^a - \dot{\xi} q^a). \quad (4)$$

The Noether symmetries are classified in a formal way in two classes (The original paper of Noether does not distinguish these classes. For a recent enlightening discussion of Noether theorem see in [21] and references therein.):

(a) The point Noether symmetries whose generators are vector fields on the augmented configuration space $\{t, q^a\}$ and usually lead to LFs.

(b) The generalized Noether symmetries whose generators are vector fields in the jet space $J^1(t,q,\dot{q})$ which produce FIs of higher degree.

In the present work, we restrict our considerations to generalized Noether symmetries in the first jet space $J^1(t,q,\dot{q})$, which produce QFIs.

The 2d autonomous potentials which admit point Noether symmetries have already been classified in [12] and more recently recovered and extended in [13]. Furthermore, in [13] it has been shown that the generators of the point Noether symmetries are the elements of the homothetic algebra of the kinetic metric. Obviously that firm result is not expected to apply in the case of generalized Noether symmetries which form an infinite dimensional Lie group.

It is well known [22] that the generalized Noether symmetries have one extra degree of freedom (being special generalized Lie symmetries) which is removed if we consider the gauge condition
\( \xi = 0 \), which we assume to be the case. Therefore, the (gauged) Noether symmetries we consider are generated by vector fields of the form \( X = \eta^a(q, \dot{q}, \ddot{q}, \ldots) \frac{\partial}{\partial \dot{q}^a} \) and accordingly the Noether condition and the corresponding FI are simplified as follows,

\[
X^{[1]}L = \dot{f}, \quad I = f - \frac{\partial L}{\partial \dot{q}^a} \eta^a. \tag{5}
\]

In the direct method, one assumes for the QFI the generic expression

\[
I = K_{ab}(t, q)\dot{q}^a \dot{q}^b + K_a(t, q) \dot{q}^a + K(t, q)
\]

where \( K_{ab}(t, q), K_a(t, q), \) and \( K(t, q) \) are unknown tensor quantities and demands the condition (Equivalently, if the system is Hamiltonian, one requires \( \{H, I\} = 0 \) where \( \{\cdot, \cdot\} \) is the Poisson bracket.) \( \frac{\partial}{\partial \dot{q}^a} = 0 \). This condition leads to a system of simultaneous equations among the coefficients \( K_{ab}(t, q), K_a(t, q), \) and \( K(t, q) \) whose solution provides all the LFIs and the QFIs of the system of this functional form. The involvement of the specific dynamical system is in the replacement of the term \( \ddot{q}^a \) whenever it appears from the dynamical Equation (1).

The direct approach is related to the Noether symmetries because once one has determined the QFI the generator of the corresponding gauged Noether symmetry and the Noether function follow immediately. Indeed for a gauged Noether symmetry (in the gauge \( \xi = 0 \)) relation Equation (4) becomes

\[
I = f - \frac{\partial L}{\partial \dot{q}^a} \eta^a. \tag{7}
\]

Replacing \( L = T - V(q) \) we find

\[
I = f - \eta^a \gamma_{ab} \dot{q}^b = f - \eta_a \dot{q}^a \tag{8}
\]

and using Equation (6) it follows

\[
\eta_a = -K_{ab} \dot{q}^b - K_a, \quad f = K \tag{9}
\]

that is, we obtain directly the Noether generator and the Noether function from the QFI \( I \) by reading the coefficients \( K_{ab}(t, q), K_a(t, q) \) and \( K(t, q) \), respectively. It can be proved that (a) the set \( \{-K_{ab} \dot{q}^b - K_a; K\} \) does satisfy the gauged Noether condition \( X^{[1]}L = \dot{f} \) and (b) the QFI \( I \) defined in Equation (6) is not in general Noether invariant (as it is the case for the point Noether symmetries—see proposition 2.2 in [23]). Finally, the (gauged) point Noether symmetries, which are defined by the vector \( K_a \) \( (K_{ab} = 0) \), give the LFIs whereas the (gauged) generalized Noether symmetries with \( K_{ab} \neq 0 \) give the QFIs.

### 3. The QFIs of an Autonomous Conservative Dynamical System

It is known (see, e.g., in [5,7]) that condition \( dI/dt = 0 \) leads to the following system of equations (using the dynamical Equations (1) to replace \( \ddot{q}^a \) whenever it appears the condition \( dI/dt = 0 \) is written)

\[
K_{(ab)c} \dot{q}^a \dot{q}^b \dot{q}^c + \left( K_{ab,t} + K_{(ab)} \right) \dot{q}^a \dot{q}^b + \left( K_{a,t} + K_{a} - 2K_{ab} V^b \right) \dot{q}^a + K_{t} - K_{a} V^a = 0.
\]

\[
K_{(ab)c} = 0 \tag{10}
\]

\[
K_{ab,t} + K_{(ab)} = 0 \tag{11}
\]

\[
-2K_{ab} V^b + K_{a,t} + K_{a} = 0 \tag{12}
\]

\[
K_{t} - K_{a} V^a = 0 \tag{13}
\]
Here, round/square brackets indicate symmetrization/antisymmetrization of the enclosed indices and a semicolon denotes the Riemannian covariant derivative. In the special case of a scalar function, for example, the potential \( V \), it holds that \( V^a = V_a \).

Condition \( K_{(ab)c} = 0 \) implies that \( K_{ab} \) is a Killing tensor (KT) of order 2 (possibly zero) of the kinetic metric \( \gamma_{ab} \). Because \( \gamma_{ab} \) is autonomous we assume

\[
K_{ab}(t,q) = g(t)C_{ab}(q)
\]

where \( g(t) \) is an arbitrary analytic function and \( C_{ab}(q) \) (\( C_{ab} = C_{ba} \)) is a KT of order 2 of the metric \( \gamma_{ab} \). This choice of \( K_{ab} \) and Equation (11) indicates that we set

\[
K_a(t,q) = f(t)L_a(q) + B_a(q)
\]

where \( f(t) \) is an arbitrary analytic function and \( L_a(q), B_a(q) \) are arbitrary vectors. With these choices the system of Equations (10)–(13) becomes

\[
\begin{align*}
g(t)C_{(abc)} &= 0 \tag{14} \\
g_cC_{ab} + f(t)L_{(ab)c} + B_{(ab)c} &= 0 \tag{15} \\
-2g(t)C_{ab}V^b + f_cL_a + K_a &= 0 \tag{16} \\
K_t - (fL_a + B_a)V^a &= 0 \tag{17}.
\end{align*}
\]

Conditions Equations (14)–(17) must be supplemented with the integrability conditions \( K_{at} = K_{fa} \) and \( K_{[ab]} = 0 \) for the scalar \( K \). The integrability condition \( K_{at} = K_{fa} \) gives—if we make use of Equations (16) and (17)—the PDE

\[
f_{ct}L_a + f_cL_bA^b_a + f \left( L_bV^b \right)_{\alpha t} + \left( B_bV^b \right)_{\alpha t} - 2g_{c\alpha}C_{ab}V^b = 0. \tag{18}
\]

Condition \( K_{[ab]} = 0 \) gives the equation known as the second-order Bertrand–Darboux PDE

\[
2g \left( C_{[a|c]}V^c \right)_{\beta t} - f_{c}L_{[ab]} = 0 \tag{19}
\]

where indices enclosed between vertical lines are overlooked by symmetrization or antisymmetrization symbols.

Finally, the system of equations which we have to solve consists of Equations (14)–(19). The general solution of that system in terms of the collineations of the kinetic metric is given in the following Theorem (see [19]).

**Theorem 1.** The functions \( g(t), f(t) \) are assumed to be analytic so that they may be represented by polynomial expansion as follows,

\[
g(t) = \sum_{k=0}^{n} c_k t^k = c_0 + c_1 t + ... + c_n t^n \tag{20}
\]

\[
f(t) = \sum_{k=0}^{m} d_k t^k = d_0 + d_1 t + ... + d_m t^m \tag{21}
\]

where \( n, m \in \mathbb{N} \), or may be infinite, and \( c_k, d_k \in \mathbb{R} \). Then, the independent QFIs of an autonomous conservative dynamical system are the following.

**Integral 1.**

\[
I_1 = -\frac{t^2}{2} L_{(ab)c}q^a q^b + C_{ab}q^a q^b + \frac{t^2}{2} a^2 \frac{L_a V^a}{L_a} + G(q)
\]

where \( C_{ab}, L_{(ab)c} \) are KTs, \( \left( L_b V^b \right)_{\alpha t} = -2L_{(ab)}V^b + G_{\alpha} = 2C_{ab}V^b - L_a \).
Integral 2.

\[ I_2 = -\frac{t_3}{2} L_{(a;b)} \ddot{q}^a \dot{q}^b + t^2 L_a \dot{q}^a + \frac{t_3}{3} I_a V^a - t B_{(a;b)} \dot{q}^a \dot{q}^b + B_a \dot{q}^a + t B_a V^a \]

where \( L_a, B_a \) are such that \( L_{(a;b)}, B_{(a;b)} \) are KTIs, \( \left( L_b V^b \right)_\mu = -2L_{(a;b)} V^b \), and \( \left( B_b V^b \right)_\mu = -2B_{(a;b)} V^b - 2I_a \).

Integral 3.

\[ I_3 = -e^{2\lambda} L_{(a;b)} \dot{q}^a \dot{q}^b + \lambda e^{\lambda} L_a \dot{q}^a + e^{\lambda} L_a V^a \]

where \( \lambda \neq 0, L_a \) is such that \( L_{(a;b)} \) is a KT and \( \left( L_b V^b \right)_\mu = -2L_{(a;b)} V^b - \lambda^2 L_a \).

It can be checked that the FIs listed above produce all the potentials which admit a LFI or a QFI given in [13] and are due to point Noether symmetries. Since, as shown above, these FIs also follow form a gauged velocity dependent Noether symmetry we conclude that there does not exist a one-to-one correspondence between Noether FIs and the type of Noether symmetry. For example the FI of the total energy (Hamiltonian) \( E = \frac{1}{2} \gamma_{ab} \dot{q}^a \dot{q}^b + V(q) \) (case Integral 1 for \( L_a = 0 \) and \( C_{ab} = \frac{2\psi}{\lambda} \)) is generated by the point Noether symmetry \( (\xi = 1, \eta_a = 0; f = 0) \) and also by the gauged generalized Noether symmetry \( (\xi = 0, \eta_a = -\frac{1}{2} \gamma_{ab} \dot{q}^b; f = V(q)) \).

The FI \(-I_2(L_a = 0)\) for \( B_a \) be a HV with conformal factor \( \psi = \text{const} \) is generated by the point Noether symmetry \( (\xi = 2\phi t, \eta_a = B_a t; f = ct) \) such that \( B_a V^a + 2\phi V^a = 0 \), and also by the gauged generalized Noether symmetry \( (\xi = 0, \eta_a = -\chi \gamma_{ab} \dot{q}^b + B_a; f = -t B_a V^a) \).

As a final example, we consider the FI \(-\frac{b_3}{\lambda} \) for the gradient HV \( L_a = \Phi(q)_{,a} \) where \( \Phi_{,ab} = \lambda \gamma_{ab} \) with \( \psi = \text{const} \). This FI is generated by the point Noether symmetry

\( (\xi = \frac{2\psi}{\lambda} e^{\lambda t}, \eta_a = e^{\lambda t} \Phi(q)_{,a}; f = \lambda e^{\lambda t} \Phi(q) - \frac{c}{\lambda} e^{\lambda t}) \)

where \( \lambda, c \) are non-zero constants and \( \Phi_{,ab} V^a = -2\psi V - \lambda^2 \Phi + c \), and also by the gauged generalized Noether symmetry

\( (\xi = 0, \eta_a = -\frac{e^{\lambda t}}{\lambda} \psi \gamma_{ab} \dot{q}^b + e^{\lambda t} \Phi_{,a}; f = -\frac{e^{\lambda t}}{\lambda} \Phi_{,a} V^a) \).

4. The Determination of the QFIs

From Theorem 1 follows that for the determination of the QFIs the following problems have to be solved.

a. Determine the KTIs of order 2 of the kinetic metric \( \gamma_{ab} \).

b. Determine the special subspace of KTIs of order 2 of the form \( C_{ab} = L_{(a;b)} \) where \( L^a \) is a vector.

c. Determine the KTIs satisfying the constraint \( G_{,a} = 2C_{ab} V^b \).

d. Find all KVs \( L_a \) of the kinetic metric which satisfy the constraint \( L_a V^a = s \) where \( s \) is a constant, possibly zero.

We note that constraints a. and b. depend only on the kinetic metric. Because the kinetic energy is a positive definite non-singular quadratic 2-form, we can always choose coordinates in which this form reduces either to \( \delta_{ab} \) or to \( A(q) \delta_{ab} \). As we know the KTIs and all the collineations of a conformally flat metric (of Euclidean or Lorentzian character) [24], we already have the results for all autonomous (Newtonian or special relativistic) conservative dynamical systems.

The involvement of the potential function is only in the constraints c. and d. which also depend on the geometric characteristics of the kinetic metric. There are two different ways to proceed.
4.1. The Potential $V(q)$ Is Known

In this case the following procedure is used.

(a) Substitute $V$ in the constraints $L_a V^a = s$ and $G_{ab} = 2C_{ab} V^b$ and find conditions for the defining parameters of $L_a$ and $C_{ab}$.

(b) From these conditions determine $L_a, C_{ab}$.

(c) Substitute $C_{ab}$ in the constraint $G_{ab} = 2C_{ab} V^b$ and find the function $G(q)$.

(d) Using the above results write the FI $L$ in each case and determine directly the gauged Noether generator and the Noether function.

(e) Examine if $L$ can be reduced to simpler independent FIs or if it is new.

4.2. The Potential $V(q)$ Is Unknown

In this case the following algorithm is used.

(a) Compute the KTs and the KV of the kinetic metric.

(b) Solve the PDE $L_a V^a = s$ or the ($I$) the integrability conditions for the scalar $G$ are very general PDEs from which one can find only special solutions by making additional simplifying assumptions (e.g., symmetries) involving $L_a, C_{ab}$, and $V(q)$ itself. Therefore, one does not find the most general solution. For example, in [25] it is required that the QFI $I$ is axisymmetric, that is, $\phi^{[i]} L = 0$, where $\phi^{[i]} = -y \partial_x + x \partial_y - y \partial_x + x \partial_y$ is the first prolongation of the rotation $\phi^i = -y \partial_x + x \partial_y$. It is proved easily that in this case we have also the constraints $L_q K_a = 0$ and $L_q K_{ab} = 0$ and find the possible potentials $V(q)$.

(c) Substitute the potentials and the KTs found in the constraint $G_{ab} = 2C_{ab} V^b$ and compute the function $G(q)$.

(d) Write the FI $L$ for each potential and determine the gauged Noether generator and the Noether function.

(e) Examine if $L$ can be reduced further to simpler independent FIs or if it is a new FI.

In the following sections we assume the potential is not given and apply the second procedure. For that we need first the geometric quantities of the 2d Euclidean plane $E^2$.

5. The Geometric Quantities of $E^2$

Using well-known results (see also in [11,20]), we state the following.

- $E^2$ admits two gradient Killing vectors (KVs) $\partial_x, \partial_y$ whose generating functions are $x, y$ respectively and one non-gradient KV (the rotation) $y \partial_x - x \partial_y$. These vectors can be written collectively

$$L_a = \begin{pmatrix} b_1 + b_3 y \\ b_2 - b_3 x \end{pmatrix}$$

where $b_1, b_2, b_3$ are arbitrary constants, possibly zero.

- The general KT of order 2 in $E^2$ is

$$C_{ab} = \begin{pmatrix} \gamma y^2 + 2\alpha y + A & -\gamma xy - \alpha x - \beta y + C \\ -\gamma xy - \alpha x - \beta y + C & \gamma x^2 + 2\beta x + B \end{pmatrix}$$

from which follows

$$C_{ab}(q) \dot{q}^a \dot{q}^b = (\gamma y^2 + 2\alpha y + A) x^2 + 2(-\gamma xy - \alpha x - \beta y + C) xy + (\gamma x^2 + 2\beta x + B) y^2$$

where $\alpha, \beta, \gamma, A, B, C$ are arbitrary constants.
- The vectors \( L^a \) generating KTs of \( E^2 \) of the form \( C_{ab} = L_{(a,b)} \) are

\[
L^a = \begin{pmatrix}
-2\beta y^2 + 2ax y + A x + a_1 y + a_4 \\
-2ax^2 + 2\beta xy + a_3 x + B y + a_2
\end{pmatrix}
\]  

(25)

where \( a_1, a_2, a_3, a_4 \) are arbitrary constants.

- The KTs \( C_{ab} = L_{(a,b)} \) in \( E^2 \) generated from the vector Equation (25) are

\[
C_{ab} = L_{(a,b)} = \begin{pmatrix}
\frac{1}{2} L_{xx} & \frac{1}{2} (L_{yx} + L_{xy}) \\
\frac{1}{2} (L_{xy} + L_{yx}) & \frac{1}{2} L_{yy}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
2ay + A & -ax - \beta y + C \\
-ax - \beta y + C & 2\beta x + B
\end{pmatrix}
\] 

(26)

where (Note that \( L^a \) in Equation (25) is the sum of the non-proper ACs of \( E^2 \) and not of its KVs which give \( C_{ab} = 0 \). \( 2C = a_1 + a_3 \). Observe that these KTs are special cases of the general KTs Equation (23) for \( \gamma = 0 \). According to Theorem 1 the above are common to all 2d Newtonian systems and what changes in each particular case are the constraints \( G_{ab} = 2C_{ab}V^b \) and \( L_aV^a = s \) which determine the potential \( V(q) \).

6. Computing the Potentials and the Fls

The application of Theorem 1 in the case of \( E^2 \) indicates that there are three different ways to find potentials that admit QFIs (other than the Hamiltonian):

1) The constraint \( L_aV^a = s \), which leads to the PDE

\[
(b_1 + b_3 y)V_x + (b_2 - b_3 x)V_y - s = 0.
\]  

(27)

2) The constraint \( G_{ab} = 2C_{ab}V^b \), which leads to the second order Bertrand–Darboux PDE \( (G_{,xy} = G_{,yx}) \)

\[
0 = (\gamma xy + ax + \beta y - C)(V_{xx} - V_{xy}) + \left[ \gamma(y^2 - x^2) - 2\beta x + 2ay + A - B \right] V_{xy} - 3(\gamma x + \beta)V_y + 3(\gamma y + a)V_x.
\] 

(28)

3) The constraint \( \left( L_bV^b \right)_{,ab} = -2L_{(a,b)}V^b - \lambda^2 L_a \) with (For \( \lambda = 0 \) this constraint is a subcase of \( G_{ab} = 2C_{ab}V^b \), hence only the case \( \lambda \neq 0 \) must be considered.). \( \lambda \neq 0 \) and the integrability condition \( \left( L_bV^b \right)_{,xy} = \left( L_bV^b \right)_{,yx} \), which lead to the PDEs

\[
0 = (-2\beta y^2 + 2ax y + Ax + a_1 y + a_4) V_{xx} + (-2ax^2 + 2\beta xy + a_3 x + B y + a_2) V_{xy} + (-6ax + 2a_3 + a_1) V_y + 3(2ax + A)V_x + \lambda^2(-2\beta y^2 + 2ax y + Ax + a_1 y + a_4)
\] 

(29)

\[
0 = (-2ax^2 + 2\beta xy + a_3 x + B y + a_2) V_{yy} + (-2\beta y^2 + 2ax y + Ax + a_1 y + a_4) V_{xy} + 3(2\beta x + B)V_y + (-6\beta y + 2a_1 + a_3)V_x + \lambda^2(-2ax^2 + 2\beta xy + a_3 x + B y + a_2)
\] 

(30)

\[
0 = (ax + \beta y - C)(V_{xx} - V_{yy}) + (-2\beta x + 2ay + A - B) V_{xy} - 3\beta V_y + 3aV_x + \lambda^2(6ax - 6\beta y + a_1 - a_3), \quad 2C = a_1 + a_3.
\] 

(31)

For \( \alpha = \beta = 0 \) and \( a_1 = a_3 \) Equation (31) reduces to Equation (28). Therefore, in order to find new potentials one of these conditions must be relaxed. This case of finding potentials is the most difficult
because the problem is overdetermined, i.e., we have a system of three PDEs Equations (29)–(31) and only one unknown function, the \( V(x, y) \).

In the following sections, we solve these constraints and find the admitted potentials which, as a rule, are integrable. Subsequently, we apply Theorem 1 to each of these potentials in order to compute the admitted FIs and determine which of those are integrable and in particular superintegrable.

7. The Constraint \( L_\alpha V^\mu = s \)

The constraint \( L_\alpha V^\mu = s \) gives Equation (27) which can be solved using the method of the characteristic equation.

To cover all possible occurrences we have to consider the following cases. (a) \( b_3 = 0 \) and \( b_1 \neq 0 \) (KVs \( \partial_x \) and \( \partial_x, \partial_y \)); (b) \( b_3 = b_1 = 0 \) and \( b_2 \neq 0 \) (KV \( \partial_y \)); and (c) \( b_3 \neq 0 \) (KVs \( y\partial_x - x\partial_y; \partial_x, y\partial_x - x\partial_y; \) and \( \partial_y, y\partial_x - x\partial_y \)). For each case the solution is shown in the following table.

| Case | KV             | \( V(x, y) \)                                      |
|------|----------------|---------------------------------------------------|
| a    | \( b_3 = 0, b_1 \neq 0 \) | \( \frac{1}{b_1} x + F(b_1 y - b_2 x) \)          |
| b    | \( b_3 = b_1 = 0, b_2 \neq 0 \) | \( \frac{1}{b_2} y + F(x) \)                      |
| c    | \( b_3 \neq 0 \) | \( \tan^{-1} \left( \frac{y + b_1}{x + b_2} \right) + F(b_1 y + \frac{b_2}{2} y^2 - b_2 x + \frac{b_1}{2} x^2) \) |

We shall refer to the above solutions as Class I potentials. In order to determine if these potentials admit QFIs, we apply Theorem 1 to the following potentials resulting from the table above.

\[
\begin{align*}
V_1 &= cx + F(y - bx) \\
V_2 &= cy + F(x) \\
V_3 &= c \tan^{-1} \left( \frac{y + b_1}{x + b_2} \right) + F \left( \frac{x^2 + y^2}{2} + b_1 y - b_2 x \right).
\end{align*}
\]

Before we continue we recall that if \( I_1, I_2, ..., I_k \) are FIs of a given dynamical system then any function \( f(I_1, ..., I_k) \) is also a FI of the dynamical system.

7.1. The Potential \( V_1 = cx + F(y - bx) \)

Case a. \( b = 0 \) and \( F = \lambda y \).

The potential reduces to \( V_{1a} = cx + \lambda y \).

The irreducible FIs are

\[
\begin{align*}
L_{11} &= \dot{x} + ct, & L_{12} &= \dot{y} + \lambda t, & Q_{11} &= \frac{1}{2} \dot{x}^2 + cx, & Q_{12} &= \frac{1}{2} \dot{y}^2 + \lambda y.
\end{align*}
\]

We note that \( Q_{11} + Q_{12} = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + V = H \) the Hamiltonian. We compute \( \{Q_{11}, Q_{12}\} = 0 \), \( \{L_{11}, Q_{11}\} = -c \).

The FIs \( I_1 = Q_{11} + Q_{12}, I_2 = \lambda L_{11} - c L_{12} = \lambda \dot{x} - c \dot{y} \) and \( I_3 = Q_{11} \) are functionally independent and satisfy the relations

\[
\{I_1, I_2\} = \{I_1, I_3\} = 0, \quad \{I_2, I_3\} = -c \lambda.
\]

Therefore, the potential \( V_{1a} \) is superintegrable.

We note that the FIs \( I_2, Q_{11} \) are, respectively, the FIs (3.1.4) and (3.2.20) of [16].

Case b. \( \frac{dF}{dx} \neq 0 \) and \( w \equiv y - bx \).
The irreducible FLs are

\[ L_{21} = \dot{x} + by + ct, \quad L_{22} = t(\dot{x} + by) - (x + by) + \frac{c}{2} t^2, \quad Q_{21} = (\dot{x} + by)^2 + 2c(x + by). \]

For \( F(y - bx) = -\frac{1}{2} \lambda^2 y^2 \) and \( b = 0 \), we have the potential \( V_{1b} = cx - \frac{1}{2} \lambda^2 y^2, \lambda \neq 0 \), which admits the additional time-dependent FI \( L_{23} = e^{\lambda t}(\dot{y} - \lambda y) \). Observe also that in this case \( Q_{21} \) reduces to \( Q_{31} = \frac{1}{2} x^2 + cx \) which using the Hamiltonian generates the QFI

\[ Q_{22} \equiv H - Q_{31} = \frac{1}{2} y^2 - \frac{1}{2} \lambda^2 y^2. \]

The LFI \( L_{21}(c = 0) \) is the (3.1.4) of [16].

We compute \( \{ H, L_{21} \} = \frac{\partial L_{22}}{\partial t} = c \) because \( L_{21} \) is a time-dependent FI.

The potential of the case b is integrable because \( \{ H, Q_{21} \} = 0 \).

Moreover,

\[ \{ H, L_{22} \} = L_{21} = \frac{\partial L_{22}}{\partial t}, \quad \{ L_{21}, L_{22} \} = 1 + b^2, \]

\[ \{ Q_{21}, L_{21} \} = 2c(1 + b^2) = 2c(L_{21}, L_{22}), \quad \{ Q_{21}, L_{22} \} = 2(1 + b^2) L_{21} = 2 L_{21} L_{22} L_{21}. \]

For the special case \( V = cx - \frac{1}{2} \lambda^2 y^2 \) we have

\[ \{ H, L_{23} \} = \{ Q_{22}, L_{23} \} = \lambda L_{23} = \frac{\partial L_{23}}{\partial t}, \quad \{ Q_{31}, L_{23} \} = 0. \]

The triplet \( Q_{31}, Q_{22}, L_{23} \) proves that this potential is superintegrable.

We note that in [16] only the Class II potentials (to be considered in the next section) are examined for superintegrability (see in [16] p. 108 (3.2.34)–(3.2.36)).

For \( c \neq 0 \), the potential \( V_1 = cx + F(y - bx) \) is not included in [16] because the author seeks for autonomous FLs of the form (3.1.1) and in that case \( s = 0 \).

7.2. The Potential \( V_2 = cy + F(x) \)

We consider the case \( F'' = \frac{dF}{dx} \neq 0 \), because otherwise we retrieve the potential \( V_{1a} \) discussed above.

The irreducible FLs are

\[ L_{31} = y + ct, \quad Q_{31} = \frac{1}{2} x^2 + F(x), \quad Q_{32} = \frac{1}{2} y^2 + cy. \]

Therefore, the potential \( V_2 \) is integrable. This potential is also of the form \( V = F_1(x) + F_2(y) \), which is the (3.2.20) of [16].

For \( F(x) = -\frac{1}{2} \lambda^2 x^2 \), we obtain the potential \( V_{2a} = cy - \frac{1}{2} \lambda^2 x^2, \lambda \neq 0 \), which admits the additional FI \( L_{32} = e^{\lambda t}(\dot{x} - \lambda x) \). This potential is superintegrable because of the functionally independent triplet \( Q_{31}, Q_{32}, \) and \( L_{32} \).

7.3. The Potential \( V_3 = c \tan^{-1}\left( \frac{y + b_1}{x + b_2} \right) + F\left( \frac{x^2 + y^2}{2} + b_1 y - b_2 x \right) \)

We find the time-dependent LFI

\[ L_{31} = y\dot{x} - x\dot{y} + b_1 \dot{x} + b_2 \dot{y} + ct. \]

For \( c = 0 \) this potential is integrable. For \( c \neq 0 \) we do not know.
- For $c = 0$ and $F = \lambda \left( \frac{x^2+y^2}{2} + b_1 y - b_2 x \right)$, the independent FIs are

$$L_{41} = y\dot{x} - x\dot{y} + b_1 \dot{x} + b_2 \dot{y}, \quad Q_{41} = \frac{1}{2} x^2 + \frac{1}{2} \lambda x^2 - \lambda b_2 x,$$

$$Q_{42} = \frac{1}{2} y^2 + \frac{1}{2} \lambda y^2 + \lambda b_1 y,$$

$$Q_{43} = x\dot{y} + \lambda (xy + b_1 x - b_2 y).$$

Observe that $Q_{41} + Q_{42} = H$ is the energy of the system. The LFI $L_{41}$ is the (3.1.6) of [16]. The functionally independent triplet $H, L_{41}, Q_{41}$ proves that this potential is superintegrable. We have

$$\{ H, L_{41} \} = \{ H, Q_{41} \} = 0, \quad \{ L_{41}, Q_{41} \} = -Q_{43} + \lambda b_1 b_2.$$

If $b_1 = b_2 = 0$ and $\lambda = -b^2 \neq 0$ we obtain the superintegrable (A subcase of the above superintegrable potential is the potential $V_{3a} = \lambda \left( \frac{x^2+y^2}{2} + b_1 y - b_2 x \right)$) potential $V_{3b} = -\frac{1}{2} b^2 (x^2 + y^2)$ which admits the additional time-dependent LFIs

$$L_{42\pm} = e^{\pm bt} (x \pm b x), \quad L_{43\pm} = e^{\pm bt} (y \mp b y).$$

We also compute

$$\{ L_{41}, Q_{42} \} = Q_{43} - \lambda b_1 b_2, \quad \{ L_{41}, Q_{43} \} = 2Q_{41} - 2Q_{42} + \lambda (b_2^2 - b_1^2)$$

$$\{ Q_{41}, Q_{42} \} = 0, \quad \{ Q_{41}, Q_{43} \} = \{ Q_{43}, Q_{42} \} = -\lambda L_{41}.$$

In Section 4 of the work in [26], the author has found the superintegrable Class I potentials $V_{1a}$ and $V_{3a}$.

We note that in the review in [16], the time-dependent LFIs of the potentials $V_{1a}, V_2$ are not discussed. In general, in [16] all the time-dependent FIs are ignored, although they can be used to decide the superintegrability of the system.

### 7.4. Summary

We collect the results for the Class I potentials in the following tables.

| Potential          | Ref. [16] | LFIs and QFIs                                      |
|--------------------|-----------|---------------------------------------------------|
| $V_3(c \neq 0) = c \tan^{-1} \left( \frac{y+by}{x+bx} \right) + F \left( \frac{x+y}{2} + b_1 y - b_2 x \right)$ | $L_{51} = y\dot{x} - x\dot{y} + b_1 \dot{x} + b_2 \dot{y} + ct$ |
|                    |           | $L_{21} = \dot{x} + by + ct,$                     |
|                    |           | $L_{22} = t(\dot{x} + by) - (x + by) + \frac{e}{2} t^2,$ |
|                    |           | $Q_{21} = (x + by)^2 + 2c(x + by)$                 |
| $V_1 = cx + F(y - bx), \frac{dF}{dx} \neq 0, w \equiv y - bx$ | $L_{31} = \dot{y} + ct, Q_{31} = \frac{1}{2} b^2 F(x),$ |
|                    |           | $Q_{32} = \frac{1}{2} y^2 + cy$                    |
| $V_2 = cy + F(x), F'' \neq 0$ (3.20) | $L_{51} = \dot{y} + ct$                            |
| $V_3(c = 0)$       | (3.1.6)   | $L_{51}(c = 0)$                                   |
8. The Constraint $G_{ab} = 2C_{ab}V^{,b}$

In this case, we have the PDE Equation (28)

$$0 = (\gamma xy + ax + \beta y - C)(V_{xx} - V_{yy}) + \left[\gamma(y^2 - x^2) - 2\beta x + 2ay + A - B\right]V_{xy} - 3(\gamma x + \beta)\dot{V}_y + 3(\gamma y + \alpha)V_x. \quad (32)$$

The potentials which follow from this equation we call Class II potentials. This equation cannot be solved in full generality (see also in [16]), therefore we consider various cases which produce the known FIs. We emphasize that the potentials we find in this section are only a subset of the possible potentials which will follow from the general solution of Equation (32). However the important point here is that we recover the known results with a direct and unified approach which can be used in the future by other authors to discover new integrable and superintegrable potentials in $E^2$ and in other spaces.

(1) $\gamma \neq 0$, $A = B$ and $\alpha = \beta = C = 0$. Then $C_{ab} = \left(\begin{array}{cc} \gamma y^2 + A & -\gamma xy \\ -\gamma xy & \gamma x^2 + A \end{array}\right)$ and Equation (32) becomes

$$xy(V_{xx} - V_{yy}) + (y^2 - x^2)V_{xy} - 3xV_y + 3yV_x = 0 \quad (33)$$

whose solution gives

$$V_{21} = \frac{F_1\left(\frac{y}{x}\right)}{d_1x^2 + d_2y^2} + F_2(x^2 + y^2) \quad (34)$$

where $d_1, d_2$ are arbitrary constants.

For the subcase $d_1 = d_2 = 1$ with $A = 0$, we find the QFI

$$L_{11} = (y\dot{x} - x\dot{y})^2 + 2F_1\left(\frac{y}{x}\right) = (r^2\theta)^2 - \Phi(\theta) \quad (35)$$

where $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. This is the well-known Ernkov–Lewis invariant; see also (3.2.11) of the work in [16].

For $d_1 \neq 0$, the potential Equation (34) is written equivalently

$$V_{21} = \frac{F_1\left(\frac{y}{x}\right)}{x^2 + cy^2} + F_2(x^2 + y^2)$$

where $c$ is an arbitrary constant.
This potential admits QFIs for $F_1 = \frac{k y^2 + k x^2}{x^2 + (2 - c) y^2}$. Therefore,

$$V_{21a} = \frac{k}{x^2 + (2 - c) y^2} + F_2(x^2 + y^2) = \frac{k}{x^2 + \ell y^2} + F_2(x^2 + y^2)$$

with the QFI

$$I_{11a} = (y \dot{x} - x \dot{y})^2 + \frac{2k(c - 1)y^2}{x^2 + (2 - c) y^2} = (y \dot{x} - x \dot{y})^2 + \frac{2k(1 - \ell)y^2}{x^2 + \ell y^2}$$

(36)

where $\ell = 2 - c$.

For $d_1 = 0$, $d_2 \neq 0$, the potential $V_{21}$ becomes

$$V_{21} = \frac{F_1(\frac{y}{x})}{y^2} + F_2(x^2 + y^2).$$

This potential admits QFIs for $F_1 = \frac{k y^2}{2x^2 + y^2}$. Then,

$$V_{21b} = \frac{k}{2x^2 + y^2} + F(x^2 + y^2)$$

with the QFI

$$I_{11b} = (y \dot{x} - x \dot{y})^2 + \frac{ky^2}{2x^2 + y^2}.$$  

(37)

Observe that $V_{21b}$ is of the form $V_{21a}(c = 3/2)$ or $V_{21a}(\ell = 1/2)$ with $k \equiv 2k$. Therefore, $V_{21b}$ is included in case $V_{21a}$.

(2) $\gamma = 1$ and $\alpha = \beta = B = C = 0$. Then $C_{\alpha \beta} = \begin{pmatrix} y^2 + A & -xy \\ -xy & x^2 \end{pmatrix}$ and Equation (32) becomes

$$xy(V_{xx} - V_{yy}) + (y^2 - x^2 + A)V_{xy} - 3xV_{y} + 3yV_{x} = 0.$$  

(38)

For $A = 0$ Equation (38) reduces to Equation (33).

For $A \neq 0$ the PDE Equation (38) gives the Darboux solution

$$V_{22} = \frac{F_1(u) - F_2(v)}{u^2 - v^2}$$  

(39)

where $r^2 = x^2 + y^2$, $u^2 = r^2 + A + (r^2 + A)^2 - 4Ax^2)^{1/2}$ and $v^2 = r^2 + A - (r^2 + A)^2 - 4Ax^2)^{1/2}$.

We find the QFI (see (3.2.9) of the work in [16]).

$$I_{21} = (y \dot{x} - x \dot{y})^2 + A x^2 + \frac{v^2 F_1(u) - u^2 F_2(v)}{u^2 - v^2}.$$  

(40)

(3) $\gamma = 1$, $B = -A$, $C = \pm iA \neq 0$ and $\alpha = \beta = 0$. Then,

$$C_{\alpha \beta} = \begin{pmatrix} y^2 + A & -xy \pm iA \\ -xy \pm iA & x^2 - A \end{pmatrix}$$

and Equation (32) gives again a potential of the form Equation (39), but with $u^2 = r^2 + [r^4 - 4A(x \pm iy)]^{1/2}$ and $v^2 = r^2 - [r^4 - 4A(x \pm iy)]^{1/2}$.

We find the QFI (see (3.2.13) in [16])

$$I_{31} = (y \dot{x} - x \dot{y})^2 + A(\dot{x} \pm i\dot{y})^2 + \frac{v^2 F_1(u) - u^2 F_2(v)}{u^2 - v^2}.$$  

(41)
(4a) $\alpha = 1$ and $\beta = \gamma = A = B = C = 0$. Then,

\[ C_{ab} = \begin{pmatrix} 2y & -x \\ -x & 0 \end{pmatrix} \]

and Equation (32) becomes

\[ x(V_{xx} - V_{yy}) + 2yV_{xy} + 3V_x = 0 \]  

(42)

which gives the potential

\[ V_{24} = \frac{F_1(r + y) + F_2(r - y)}{r} \]  

(43)

where $r^2 = x^2 + y^2$.

We find the QFI (see (3.2.15) in [16])

\[ I_{41} = x(y\dot{x} - xy) + \frac{(r + y)F_2(r - y) - (r - y)F_1(r + y)}{r}. \]  

(44)

(4b) $\beta = 1$ and $\alpha = \gamma = A = B = C = 0$. Then,

\[ C_{ab} = \begin{pmatrix} 0 & -y \\ -y & 2x \end{pmatrix} \]

and Equation (32) becomes

\[ y(V_{xx} - V_{yy}) - 2xV_{xy} - 3V_y = 0 \]  

(45)

which gives the potential

\[ V_{24b} = \frac{F_1(r + x) + F_2(r - x)}{r} \]  

(46)

where $r^2 = x^2 + y^2$.

We find the QFI

\[ I_{41b} = y(xy - y\dot{x}) + \frac{(r + x)F_2(r - x) - (r - x)F_1(r + x)}{r}. \]  

(47)

Observe that the potential Equation (46) is just the Equation (43) after the rotation $x \leftrightarrow y$. All the results of the case 4b can be derived from the case 4a if we apply the transformation $x \leftrightarrow y$. For this reason the case 4b is ignored when we search for integrable systems, but in superintegrability the PDE Equation (45) shall be proved useful (see superintegrable potential Equation (74) in Section 8.1).

(5) $\alpha = 1$, $\beta = -i$, $A = -B = \frac{1}{4}$, $C = \frac{1}{4}$ and $\gamma = 0$. Then,

\[ C_{ab} = \begin{pmatrix} 2y + i\frac{1}{4} & -x + iy + i\frac{1}{4} \\ -x + iy + i\frac{1}{4} & -2ix - i\frac{1}{4} \end{pmatrix} \]

and Equation (32) becomes

\[ (x - iy - \frac{1}{4})(V_{xx} - V_{yy}) + 2\left(y + ix + i\frac{1}{4}\right)V_{xy} + 3iV_y + 3V_x = 0. \]  

(48)

This is written equivalently

\[ (x - iy) \left( \partial_x + i\partial_y \right)^2 V - \frac{1}{4}(\partial_x - i\partial_y)^2 V + 3(\partial_x + i\partial_y)V = 0 \]  

(49)

and gives the potential

\[ V_{25} = w^{-1/2} \left[ F_1(z + \sqrt{w}) + F_2(z - \sqrt{w}) \right] \]  

(50)
where \( z = x + iy \) and \( w = x - iy \).

We find the QFI (see (3.2.17) in [16])

\[
I_{51} = (y\dot{x} - x\dot{y})(\dot{x} + iy) + i\frac{1}{8}(\dot{x} - iy)^2 + i\left(1 - \frac{z}{\sqrt{w}}\right) F_1(z + \sqrt{w}) + \\
i\left(1 - \frac{z}{\sqrt{w}}\right) F_2(z - \sqrt{w}).
\]

(6) \( \alpha = 1, \beta = \mp i \) and \( \gamma = A = B = C = 0 \). Then,

\[
\alpha = \frac{2y}{\sqrt{w}} - x \pm iy - x \pm iy \mp 2ix
\]

and Equation (32) becomes

\[
(x \mp iy) (V_{xx} - V_{yy}) + 2(y \pm ix) V_{xy} + 3iV_y + 3V_x = 0
\]

from which follows

\[
V_{26} = \frac{F_1(z)}{r} + F_2(z)
\]

where \( F_2 = \frac{dF_2}{dz} \) and \( z = x \pm iy \).

We find the QFI (see (3.2.18) in [16])

\[
I_{61} = (y\dot{x} - x\dot{y})(\dot{x} \pm iy) - izV + iF_2(z).
\]

(7) \( AB \neq 0, A \neq B \) and \( \alpha = \beta = \gamma = C = 0 \). Then, \( C_{ab} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \).

Equation (32) becomes

\[
(A - B)V_{xy} = 0 \implies V_{xy} = 0
\]

which gives the separable potential

\[
V_{27} = F_1(x) + F_2(y).
\]

We find the irreducible QFIs (see (3.2.20) of [16])

\[
I_{71a} = x^2 + 2F_1(x), \quad I_{71b} = y^2 + 2F_2(y).
\]

It can be shown that there are four special potentials of the potential Equation (55) which admit additional time-dependent QFIs and are superintegrable. These are

7a. The potential

\[
V_{271} = \frac{k_1}{(x + c_1)^2} + \frac{k_2}{(y + c_2)^2}
\]

admits the independent FIs

\[
I_{72a} = -\frac{t^2}{2}y^2 + t(y + c_2)\dot{y} - \frac{k_2}{2(y + c_2)^2} - \frac{1}{2}y^2 - c_2y
\]

\[
I_{72b} = -\frac{t^2}{2}x^2 + t(x + c_1)\dot{x} - \frac{k_1}{2(x + c_1)^2} - \frac{1}{2}x^2 - c_1x.
\]

7b. The potential

\[
V_{272} = F_1(x) + \frac{k_2}{(y + c_2)^2}
\]

admits the FI \( I_{72a} \).
7c. The potential
\[ V_{273} = F_2(y) + \frac{k_1}{(x + c_1)^2} \]
admits the FI \( I_{72b} \).

7d. The potential (see \[8\])
\[ V_{274} = -\frac{\lambda^2}{8}(x^2 + y^2) - \frac{\lambda^2}{4}(c_1 x + c_2 y) - \frac{k_1}{(x + c_1)^2} - \frac{k_2}{(y + c_2)^2} \]
admits the independent FIs.

In all the above relations, \( \lambda, c_1, c_2, k_1, k_2 \) are arbitrary constants.

We note that \( b_0(A, B, C) \). Here, \( A, B, C \) are parameters of the potential and therefore cannot be taken as independent parameters of the QFI.

For \( b_0 = 0 \), we have \( A = B, V_{yy} - V_{xx} = 0 \) and the potential reduces to
\[ V_{28}(b_0 = 0) = F_1(y + x) + F_2(y - x) \]
which is the solution of the 1d-wave equation.

For the potential Equation (59) we find the QFI
\[ I_{82} = \dot{x}\dot{y} + F_1(y + x) - F_2(y - x). \]

(9) \( A = 2, C = \pm i \) and \( \alpha = \beta = \gamma = B = 0 \). Then,
\[ C_{ab} = \begin{pmatrix} 2 & \pm i \\ \pm i & 0 \end{pmatrix} \]
and Equation (32) becomes
\[ \mp i(V_{xx} - V_{yy}) + 2V_{xy} = 0. \]
Solving Equation (60), we find the potential

\[ V_{29} = r^2 F_1''(z) + F_2(z) \]  

(61)

where \( F_1'' = \frac{d^2 F_1}{dz^2} \) and \( z = x \pm iy \).

This potential admits the QFI (see (3.2.21) in [16])

\[ I_{QFI} = \dot{x}(\dot{x} \pm i \dot{y}) + V_{29} + 2z F_1'(z) - 2F_1(z). \]  

(62)

Observe that for the trivial KT \( C_{ab} = A \delta_{ab} \), the condition \( G_{ab}, a = 2 C_{ab} V_{gb} \) gives

\[ G_{ab} = 2AV_{ab} \implies G = 2AV \]

for all potentials \( V(x, y) \). Therefore, we recover the trivial result that all 2d-potentials \( V(x, y) \) admit the QFI

\[ I = A(x^2 + y^2 + 2V) = 2AH. \]

Comparing with previous works, we see that the potentials \( V_{21a} \) and \( V_{28} \) are new. The potential \( V_{274} \) \((c_1 = c_2 = 0)\) is mentioned in [8].

8.1. The Superintegrable Potentials

When a potential belongs to two of the above nine Class II cases simultaneously it is superintegrable (e.g., potentials (3.2.34)–(3.2.36) of [16]), because in that case the potential admits two more autonomous FIs other than the Hamiltonian. From the above results, we find the following Class II superintegrable potentials (see also in [14,15]).

(S1) The potential (see (3.2.34) in [16], case (b) in [8,14])

\[ V_{s1} = k \left( \frac{1}{2}(x^2 + y^2) + \frac{b}{x^2} + \frac{c}{y^2} \right) \]  

(63)

where \( k, b, c \) are arbitrary constants.

This is of the form Equation (34) for \( d_1 = d_2 = 1, \)

\[ F_1 \left( \frac{y}{x} \right) = b \left( \frac{y}{x} \right)^2 + c \left( \frac{x}{y} \right)^2, \quad F_2(x^2 + y^2) = \frac{k}{2}(x^2 + y^2) + \frac{b}{x^2} + \frac{c}{y^2} \]

and also of the separable form Equation (55). Therefore, \( V_{s1} \) admits the additional QFIs

\[ I_{s1a} = (y^2 - x^2)^2 + 2b y^2 + 2c y^2 \]  

(64)

\[ I_{s1b} = \frac{1}{2} x^2 + \frac{k}{2} x^2 + \frac{b}{x^2} \]  

(65)

\[ I_{s1c} = \frac{1}{2} y^2 + \frac{k}{2} y^2 + \frac{c}{y^2} \]  

(66)

We note that \( V_{s1} \left( k = -\frac{\lambda^2}{4}, b = -k_1, c = -k_2 \right), \lambda \neq 0, \) is the \( V_{274} \) for \( c_1 = c_2 = 0, \) and therefore admits also the time-dependent FIs \( I_{s23a}, I_{s23b}. \)

(S2) Potentials of the form Equations (43) and (55). Then we have to solve the systems of PDEs Equation (42) and \( V_{xy} = 0. \) We find

\[ V_{s2} = \frac{k_1}{2}(x^2 + 4y^2) + \frac{k_2}{x^2} + k_3 y \]  

(67)
and the QFIs

\[ I_{s2a} = \dot{x}(y\dot{x} - x\dot{y}) - k_1 y x^2 + \frac{2k_2 y}{x^2} - \frac{k_3}{2} y^2 \]  
\[ I_{s2b} = \frac{1}{2} x^2 + \frac{k_1}{2} x^2 + \frac{k_2}{x^2} \]  
\[ I_{s2c} = \frac{1}{2} y^2 + 2k_1 y^2 + k_3 y \]  

(68)

(69)

(70)

where \( k_1, k_2, k_3 \) are arbitrary constants.

This is the superintegrable potential of case (a) in [14]. Note that the QFI \( I_S^2 \) given in [14] is not correct. The correct is the \( I_{s2a} \) of Equation (68) above.

We remark that the potential (3.2.35) in [16] is superintegrable only for \( b = 4a \) in which case the potential becomes \( V_{s2} \) for \( k_1 = 2a, k_2 = c, k_3 = 0 \).

(S3) Potentials of the form Equations (34) and (43). We solve the system of PDEs Equations (33) and (42). We find

\[ V_{s3} = \frac{k_1}{x^2} + \frac{k_2}{r} + \frac{k_3 y}{r x^2} \]  

(71)

and the QFIs

\[ I_{s3a} = (y\dot{x} - x\dot{y})^2 + 2k_1 \frac{y^2}{x^2} + 2k_3 \frac{r y}{x^2} \]  
\[ I_{s3b} = \dot{x}(y\dot{x} - x\dot{y}) + 2k_1 \frac{y}{x^2} + k_2 \frac{y}{r} + k_3 \frac{x^2 + 2y^2}{r x^2} \]  

(72)

(73)

where \( r^2 = x^2 + y^2 \).

The superintegrable potential Equation (71) is symmetric \((x \leftrightarrow y)\) to the superintegrable potential of case (c) of [14]. Indeed in order to find the superintegrable potential of [14] we simply consider the case leading to the potential of the form \( V_{s24} \) of Equation (43) for \( \beta = 1 \) instead of \( a = 1 \).

We note that if we rename the constants in Equation (71) as \( k_1 = b + c, k_2 = a, k_3 = c - b \) we recover the superintegrable potential (3.2.36) of [16]. Indeed, we have

\[ V_{s3} = \frac{a}{r} + \frac{b}{r + y} + \frac{c}{r - y}. \]

(S4) If we substitute the solution Equation (43) of the PDE Equation (42) in the PDE Equation (45), we find that for

\[ F_1 (r + y) = k_1 + k_2 \sqrt{r + y}, \quad F_2 (r - y) = k_3 \sqrt{r - y} \]

both PDEs Equations (42) and (45) are satisfied simultaneously. Therefore, the potential (see case (d) in [14])

\[ V_{s4} = \frac{k_1}{r} + \frac{k_2 \sqrt{r + y}}{r} + \frac{k_3 \sqrt{r - y}}{r} \]  

(74)

is superintegrable with additional QFIs

\[ I_{s4a} = \dot{x}(y\dot{x} - x\dot{y}) + \frac{k_1 y}{r} + \frac{k_3 (r + y) \sqrt{r - y} - k_2 (r - y) \sqrt{r + y}}{r} \]  
\[ I_{s4b} = y(\dot{x}y - \dot{y}x) + G(x, y) \]

where \( G_x + y V_{s4,y} = 0 \) and \( G_y + y V_{s4,x} - 2x V_{s4,y} = 0 \).

We note that in case (d) in [14] the corresponding QFIs \( I_S^4 \) and \( I_{s4}^4 \) are not correct, because \( \{ H, I_S^4 \} \neq 0 \) and \( \{ H, I_{s4}^4 \} \neq 0 \). Moreover, this superintegrable potential is the case (E20) in [15], and it is not mentioned in the review [16].
In the following tables, we collect the results on **Class II** potentials with the corresponding reference to the review paper [16].

| Potential | Ref. [16] | LFIs and QFIs |
|-----------|-----------|---------------|
| $V_{21} = \frac{f_1(z)}{x+y} + F_2(x^2 + y^2)$ | (3.2.10) | $I_{11} = (y\dot{x} - xy)^2 + 2F_1 \left(\frac{x}{y}\right)$ |
| $V_{21a} = \frac{z}{x+y} + F_2(x^2 + y^2)$ | - | $I_{11a} = (y\dot{x} - xy)^2 + \frac{2(1-iy)^2}{x+y}$ |
| $V_{22} = \frac{f_1(u)-F_2(v)}{u-v}$, $u^2 = r^2 + A + \left[(r^2 + A)^2 - 4Ax^2\right]^{1/2}$, $v^2 = r^2 + A - \left[(r^2 + A)^2 - 4Ax^2\right]^{1/2}$ | (3.2.7,8) | $I_{21} = (y\dot{x} - xy)^2 + Ax^2 + \frac{r^2F_1(u)-u^2F_2(v)}{u-v}$ |
| $V_{23} = \frac{f_1(u)-F_2(v)}{u-v}$, $u^2 = r^2 + \left[r^2 - 4A(x \pm iy)^2\right]^{1/2}$, $v^2 = r^2 - \left[r^2 - 4A(x \pm iy)^2\right]^{1/2}$ | (3.2.7,12) | $I_{31} = (y\dot{x} - xy)^2 + A(x \pm iy)^2 + \frac{r^2F_1(u)-u^2F_2(v)}{u-v}$ |
| $V_{24} = \frac{f_1(r+y)+F_2(r-y)}{x+y}$ | (3.15) | $I_{41} = 1/(y\dot{x} - xy) + \frac{(r+y)F_1(r-y)-(r-y)F_1(r+y)}{x+y}$ |
| $V_{25} = w^{-1/2} \left[F_1(z + \sqrt{w}) + F_2(z - \sqrt{w})\right]$, $z = x + iy$, $w = x - iy$ | (3.2.17) | $I_{51} = (y\dot{x} - xy)(x + iy) + \frac{1}{2}(x - iy)^2 + i\left(1 - \frac{1}{\sqrt{w}}\right)F_1(z + \sqrt{w}) + \frac{1}{2}(1 - 1/\sqrt{w})F_2(z - \sqrt{w})$ |
| $V_{26} = \frac{f_1(z)}{z} + F_2(z)$, $F_2 = \frac{df_2}{dz}$, $z = x + iy$ | (3.18) | $I_{61} = (y\dot{x} - xy)(x \pm iy) - izV + if_2(z)$ |
| $V_{27} = F_1(x) + F_2(y)$ | (3.2.20) | $I_{71a} = \frac{1}{2}x^2 + F_1(x), I_{71b} = \frac{1}{2}y^2 + F_2(y)$ |
| $V_{28} = F_1 \left(y + b_0 x + \sqrt{b_0^2 + 1} x\right) + F_2 \left(y + b_0 x - \sqrt{b_0^2 + 1} x\right), b_0 = \frac{A-B}{2C}$ | - | $I_{81} = Ax^2 + By^2 + 2Cxy + (A + B)V + 2C\sqrt{b_0^2 + 1}(F_1 - F_2)$ |
| $V_{28}(b_0 = 0) = F_1(y + x) + F_2(y - x)$ | - | $I_{82} = x\dot{y} + F_1(y + x) - F_2(y - x)$ |
| $V_{29} = \frac{r^2F_1''(z) + F_2(z)}{x+y}$, $z = x \pm iy$ | (3.2.21) | $I_{91} = x(\dot{x} \pm iy) + V_{29} + 2zF'_1(z) - 2F_1(z)$ |
9. The Constraint \((L_b V^b)_a = -2L_{(ab)} V^b - \lambda^2 L_a\)

The integrability condition of the constraint \((L_b V^b)_a = -2L_{(ab)} V^b - \lambda^2 L_a\) gives the PDE Equation (31).

As mentioned above in Section 6, in order to find new potentials from the PDE Equation (31) one (or both) of the conditions \(\alpha = \beta = 0\) and \(a_1 = a_3\) must be relaxed. However, if we do find a new potential, this solution should satisfy also the remaining PDEs Equations (29) and (30) in order to admit the time-dependent QFI \(I_3\) given in case Integral 3 of Theorem 1. New potentials which admit the QFI \(I_3\) shall be referred as Class III potentials.

We note that the PB \(\{H, I_3\} = \frac{\partial L}{\partial \phi} \neq 0\). Therefore, to find a new integrable potential we should find a Class III potential admitting two independent Fls of the form \(I_3\), say \(I_{3a}\) and \(I_{3b}\), such that \(\{I_{3a}, I_{3b}\} = 0\).

After relaxing one, or both, of the conditions \(\alpha = \beta = 0\) and \(a_1 = a_3\) we found that the only non-trivial Class III potential is the superintegrable potential \(V_{3b} = -\frac{\lambda}{2}(x^2 + y^2)\) (see Section 7.3) found for \(\alpha \neq 0\) or \(\beta \neq 0\) above. Therefore, there are no new Class III potentials.

10. Using Fls to Find the Solution of 2d Integrable Dynamical Systems

In this section we consider examples which show how one uses the 2d (super-)integrable potentials to find the solution of the dynamical equations.

(1) The superintegrable potential \(V_{3b} = -\frac{1}{2}k^2(x^2 + y^2)\) where \(k \neq 0\).
We find the solution by using the time-dependent LFIs $L_{42\pm} = e^{\pm ik} (\dot{x} + kx)$ and $L_{43\pm} = e^{\pm ik} (\dot{y} + ky)$.
Specifically, we have
\[
\begin{align*}
\begin{cases}
  e^{ik} (\dot{x} - kx) = c_{1+} \\
  e^{-ik} (\dot{x} + kx) = c_{1-}
\end{cases} \quad \implies \quad
\begin{cases}
  \dot{x} - kx = c_{1+} e^{-ikt} \\
  \dot{x} + kx = c_{1-} e^{ikt}
\end{cases}
\implies \quad x(t) = \frac{c_{1+} - c_{1-}}{2k} e^{ikt} - \frac{c_{1+} + c_{1-}}{2k} e^{-ikt}.
\end{align*}
\]
Similarly for the LFIs $L_{43\pm}$ we find
\[
y(t) = \frac{c_{2-}}{2k} e^{ikt} - \frac{c_{2+}}{2k} e^{-ikt}.
\]
Here, $c_{1\pm}, c_{2\pm}$ are arbitrary constants.

(2) The integrable potential $V_2 = cy + F(x)$ where $F'' \neq 0$.
Using the LFI $L_{31} = \dot{y} + ct = c_1$, we find directly $y(t) = -\frac{x}{2} t^2 + c_1 t + c_2$ where $c, c_1, c_2$ are arbitrary constants.
Using the QFI $2Q_{31} = \dot{x}^2 + 2F(x) = \text{const} = c_3$, we have
\[
\frac{dx}{dt} = \pm [2F(x) + c_3]^{1/2} \implies dt = \pm [2F(x) + c_3]^{-1/2} dx \implies t = \pm \int [2F(x) + c_3]^{-1/2} dx + c_0
\]
where $c_0$ is an arbitrary constant. The inverse function of $t = t(x)$ is the solution of the system. If the function $F(x)$ is given, the solution can be explicitly determined.

(3) For the integrable potential $V_{27} = F_1(x) + F_2(y)$ by using the QFIs
\[
I_{71a} = \frac{1}{2} x^2 + F_1(x) \quad \text{and} \quad I_{71b} = \frac{1}{2} y^2 + F_2(y)
\]
we find
\[
t = \int [c_1 - 2F_1(x)]^{-1/2} dx + c_0, \quad t = \int [c_2 - 2F_2(y)]^{-1/2} dy + c_3
\]
where $c_0, c_1 = 2I_{71a}, c_2 = 2I_{71b}, c_3$ are constants.

11. Conclusions

Using Theorem 1 we have reproduced in a systematic way most known integrable and superintegrable 2d potentials of autonomous conservative dynamical systems. The method used being covariant it is directly applicable to spaces of higher dimensions and to metrics with any signature and curvature.

We have found two classes of potentials, and in each class we have determined the integrable and the superintegrable potentials together with their QFIs. As the general solution of the PDE Equation (28) is not possible, we have found the potentials due to certain solutions only. New solutions of this equation will lead to new integrable and possibly superintegrable 2d potentials.

It appears that the most difficult part in the application of Theorem 1 to higher dimensions and curved configuration spaces is the determination of the KTs. The use of algebraic computing is limited once one considers higher dimensions since then the number of the components of the KT increases dramatically. Fortunately, today new techniques in Differential Geometry have been developed (see, e.g., in [24,27–30]), especially in the case of spaces of constant curvature and decomposable spaces, which can help to deal with this problem.

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