On the Geometry of the Berry-Robbins Approach to Spin-Statistics

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Abstract Within a geometric and algebraic framework, the structures which are related to the spin-statistics connection are discussed. A comparison with the Berry-Robbins approach is made. The underlying geometric structure constitutes an additional support for this approach. In our work, a geometric approach to quantum indistinguishability is introduced which allows the treatment of singlevaluedness of wave functions in a global, model independent way.

Keywords Spin-statistics · Berry-Robbins

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1 Introduction

It is by now widely accepted that the status of the relation between spin and statistics in non-relativistic quantum mechanics is quite unsatisfactory for several reasons. Instead of repeating all these arguments, we would like to add our motivation for dealing with this problem. For us, there are mainly four reasons:

- The influential approach of Berry-Robbins [1].
- The $G$-Theory Principle, as formulated in [2,3].
- We believe that our current understanding of the spin-statistics connection is not complete and that an alternative explanation of it could shed new light into our understanding of quantum theory itself.
- A re-examination of the spin-statistics connection might be required, in view of new developments in theoretical physics as, for instance, in the context of higher dimensional theories, quantum gravity, algebraic quantum field theory or non-commutative quantum field theory.

The first two motivations need some explanation.

In 1999, one of us (N.P.) had the opportunity to attend a seminar given by Sir Michael Berry in Mainz. At that time, the impression was that a proof of the Spin-Statistics-Theorem within the framework of non-relativistic quantum mechanics had been established and the intention was to also find a geometric formulation of this proof. In addition, this seemed to be a very good application of the $G$-Theory Principle. This was

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formulated in 1987 by a group in Mainz in connection with the reduction of the Kaluza-Klein Theory in higher dimensions. The $G$-Theory Principle emphasizes the role of group actions and symmetries in a maximal way. Therefore, it seemed to be well-suited for the purpose of studying the spin-statistics connection. The principle has been applied to the study of anomalies [3] and generalized some years later by J. Sladkowski [4]. In the present context, one encounters two groups and one manifold: The permutation group $S_N$ for $N$ particles, the rotation group $SU(2)$ for spin, and the configuration space $Q$ for $N$ identical (indistinguishable) particles. In this situation, we first have to clarify the role of the group actions and manifolds involved, before we proceed.

Regarding the fourth motivation, it comes from previous joint work of one of us (A.R.) with M. Paschke. In [14], Paschke proposed to study the spin-statistics connection using the tools of noncommutative geometry. The idea has been pursued further [6, 7], in the hope of eventually establishing a link with quantum field theory. The idea that this might be possible is based on the fact that the algebraic language of noncommutative geometry has many features in common with that of quantum field theory [13].

Let us now describe the content of the rest of the paper. After some preparations in section 2 in order to fix the notation, we will proceed with the geometric formulation of the problem in section 3 and with an equivalent algebraic formulation in section 4. The connection with the Berry-Robbins (BR) approach will be given in section 5. This includes a comment on the singlevaluedness condition from the geometric point of view. Finally, in section 6, we will present some conclusions.

Lastly, we would like to stress the fact that the entire information of this paper is completely covered by [5, 6]. Therefore, we will be referring mainly to the three papers [1, 5, 6] and references therein, and apologize for not explicitly mentioning a significant number of other interesting works.

2 Preparation

We consider $N$ identical particles moving in $\mathbb{R}^3$. The unrestricted configuration space is given by

$$ Q_N = \left\{ (r_1, \ldots, r_N) \in \mathbb{R}^{3N} : r_i \neq r_j \right\}. $$

For $N$ identical particles with the permutation group $G = S_N$ the constrained configuration space is given by the quotient space $Q_N = \widetilde{Q}_N / G$. In the present discussion we restrict ourselves to the two particle case, i.e. $N = 2$, referring the reader to [7] for the case of general $N$. For $N = 2$ the effective non-constrained configuration space is given by the sphere $\tilde{Q} = \tilde{Q}_2 \cong S^2 = \{ r \}$. The exchange of the particles 1 and 2 corresponds to $r \mapsto -r$ and the underlying permutation group is now given by $G = \mathbb{Z}_2$. Therefore, the constrained configuration space is given by

$$ Q = Q_2 = \tilde{Q}_2 / \mathbb{Z}_2 = S^2 / \mathbb{Z}_2 = \mathbb{R}P^2 = \{ [x] \equiv \{ x, -x \} : x \in S^2 \} $$

with the injective inclusion $S^2 \hookrightarrow \mathbb{R}^3$, $r \mapsto x = (x_1, x_2, x_3)$. In the standard formalism for two spin $\frac{1}{2}$ particles ($s = \frac{1}{2}$) the two spin basis which is fixed is given by

$$ \{ |s, m_1 \rangle \otimes |s, m_2 \rangle \}_{m_1, m_2} \quad m_1, m_2 \in \{ \pm \frac{1}{2} \} \quad \text{or, equivalently, by} \quad \{ |j, m \rangle \}_{j, m} \quad j \in \{ 0, 1 \}, \ m \in \{ -j, \ldots, +j \}. $$

In addition, the wave function is given by

$$ \Psi(r) = \sum_{m_1, m_2} f_{m_1 m_2}(r) |s, m_1 \rangle \otimes |s, m_2 \rangle = \sum_{j, m} f_{jm}(r) |j, m \rangle. $$

The symmetrization postulate corresponds to the choice of the coefficients: The antisymmetric case $f_{jm}(r)$ and the symmetric case $f_{00}(r)$ respectively.

In the BR approach, the spin basis is moving, i.e. $\{ |s, s, m_1, m_2(r) \rangle \}$, or $\{ |j, m(r) \rangle \}$. For this reason, the wave function looks like

$$ |\Psi(r)\rangle = \sum_{m_1, m_2} \psi_{m_1 m_2}(r) |s, s, m_1, m_2(r) \rangle = \sum_{j, m} \psi_{jm}(r) |j, m(r)\rangle,$$

with the coefficient functions $\psi_{m_1 m_2}(r)$ and $\psi_{jm}(r)$. As discussed in BR, $\psi_{m_1 m_2}$ and $f_{m_1 m_2}$ are the same functions.
The state $|\Psi\rangle$ lives in a two-spin-bundle over $S^2 (\cong \tilde{Q}_2)$ along with the singlevaluedness constraint given by $|\Psi(r)\rangle = |\Psi(-r)\rangle$.

This is the arena where the BR approach takes place. We denote this by I. We may also think about a second point of view II, for which the wave function lives in a two-spin-bundle over $\mathbb{R}P^2 (\cong Q_2)$ without any constraint. We denote this wave function with $\Phi([x])$. We expect of course an equivalent situation between the wave functions $|\Psi(r)\rangle \equiv \Psi_I(r)$ and $\Phi([x]) \equiv \Psi_{II}(r)$ and similarly an equivalency for the two vector bundles $\eta \equiv \xi_I \equiv S^2 \times V$ respectively $\xi \equiv \xi_{II} \equiv \mathbb{R}P^2 \times V$, where $V = \mathbb{C}^4$ (for $s = \frac{1}{2}$). We expect $\xi$ to be a non-trivial bundle (the symbol $\times$ will be used to denote a bundle $M \times V$ with basis $M$ and fiber $V$ which may or may not be a trivial bundle). But what is the precise connection between I and II? This will be clarified in the next section, leading us to a geometric formulation of the given problem.

### 3 Geometric formulation

#### 3.1 The Two Points of view (I and II)

As we already saw, two formulations are possible. It is important to realize that in the first case (I) the configuration space $Q_I$ is not constrained, unlike the associated wave function $\Psi_I$, which is constrained by the singlevaluedness condition. In contrast, in the second case (II) the configuration space $Q_{II}$ is constrained, but the wave function $\Psi_{II}$ is not! For further discussion, it is necessary to keep these two points of view in mind.

In each formulation, we consider three objects:

- **I**
  - The basis manifold: $Q_I \equiv \tilde{Q}$
  - The two-spin vector bundle: $\xi_I \equiv \eta$
  - The wave function: $\Psi_I(r) \equiv |\Psi(r)\rangle$

- **II**
  - The basis manifold: $Q_{II} \equiv Q$
  - The two-spin vector bundle: $\xi_{II} \equiv \xi$
  - The wave function: $\Psi_{II}(r) \equiv \Phi([x])$

In order to find the equivalence between I and II, we start from I and construct II. In case I we deal with an action of the permutation group $G$. The manifold $\tilde{Q}$ is a $G$-space, $G$ is finite and the action of $G$ is free:

$$\rho : \ G \times \tilde{Q} \to \tilde{Q}, \quad (g, r) \mapsto \rho_g(r) = gr,$$

(6)

$\eta$ is also a $G$-space, more precisely a $G$-vector bundle.

**Definition 1 (G-Vector bundle)** A $G$-Vector bundle $(\eta, \tau)$ is given by the following data and properties: In an obvious notation we have $\eta = (E(\eta), \pi, \tilde{Q})$ with fibres $\pi^{-1}(r) = E(\eta)_r \cong V \cong \mathbb{C}^n$, the bundle projection

$$\pi : \ E(\eta) \to \tilde{Q}, \quad z \mapsto \pi(z) = r$$

(7)

and an action $\tau$ of $G$ on the bundle $\eta$

$$\tau : \ G \times \eta \to \eta, \quad (g, z) \mapsto \tau_g(z) = gz.$$

(8)

(i) The projection $\pi$ is an equivariant map (consistent with a $G$-action), i.e. $\pi \circ \tau_g = \rho_g \circ \pi$. This is illustrated by the following commutative diagram:

$$\begin{array}{ccc}
E(\eta) & \xrightarrow{\tau_g} & E(\eta) \\
\pi \downarrow & & \pi \downarrow \\
\tilde{Q} & \xrightarrow{\rho_g} & \tilde{Q}
\end{array}$$

(ii) The map $\tau_g : E(\eta)_r \to E(\eta)_{gr}$ is a vector space isomorphism.
An equivalence (isomorphism) of two $G$-bundles $(\eta, \tau)$ and $(\eta', \tau')$ is a vector bundle isomorphism $\phi : \eta \to \eta'$ that commutes with the two actions $\tau$ and $\tau'$, i.e., such that $\tau_\phi \circ \phi(z) = \phi \circ \tau(z)$. In this category, we denote a $G$-bundle isomorphism by $\eta \cong_G \eta'$. The wave function $\Psi_I$ is an element of the space of sections $\Gamma(\eta)$ in $\eta$, i.e. $\Psi_I \in \Gamma(\eta)$. As explained below, it turns out that in our case $\Psi_I$ is a $G$-invariant section in $\eta$: $\Psi_I \in \Gamma^{inv}(\eta)$.

At this stage, it becomes evident that we may obtain case II from I by a quotining procedure. Consequently we have:

$$Q_{II} = Q_I / G \quad \text{with the projection} \quad q : Q_I \to Q_I / G \ (= Q_{II}) \quad \text{and} \quad \xi_{II} = \xi_I / G \quad \text{with the projection} \quad \eta : \xi_I \to \xi_I / G \ (= \xi_{II}) .$$

The wave function $\Psi_{II}([\cdot]) \in \Gamma(\xi_{II})$ is now completely unconstrained. From the knowledge of $\xi_{II}$ we can take its pull-back and obtain $\xi_I$ from it, as follows. The bundle $q^* \xi_{II}$ will inherit, in a natural way, a structure of $G$-bundle. We will call $\tilde{\tau}$ the corresponding action. It is precisely this $G$-bundle structure that lies behind the constraint condition for the wave function $\Psi_I$.

Remark 1 It is important to realize that the $G$-bundle structure induced on $q^* \xi_{II}$ depends on the topology of the bundle $\xi_{II}$: Two non-isomorphic bundles $\xi_{II}$ and $\tilde{\xi}_{II}$ will necessarily give rise to non-isomorphic $G$-bundles $(q^* \xi_{II}, \tilde{\tau})$ and $(q^* \tilde{\xi}_{II}, \tau')$, even if the bundles $q^* \xi_{II}$ and $q^* \tilde{\xi}_{II}$ happen to be isomorphic bundles. The natural action induced by the pull-back is given by

$$\xi_{I}(x, z) := (\rho_{\tilde{\tau}}(x), z),$$

where $(x, z) \in E(q^* \xi_{II}) \subset Q_I \times Q_{II}$.

In order to simplify, we introduce the notation: $Q := Q_{II}$, $\xi := \xi_{II}$, $\tilde{\xi} = (E(\xi), \pi, Q)$. Notice that as a result of the quotient operation, the group $G$ is not directly acting on $Q$ and $\tilde{\xi}$ (the original action of $G$ on $Q_I$ will, nevertheless, be related to the holonomy of the bundle $\xi$). The precise connection between I and II is given by the following well-known theorem.

Theorem 1 (cf. [8]) If $G$ acts freely on $Q_I$ then there is a bijective correspondence between $G$-bundles $(\xi_I)$ over $Q_I$ and bundles $\xi_{II} = \xi_I / G$ over $Q_{II} = Q_I / G$.

Therefore, we can finally express the correspondence between I and II by means of quotient and pull-back operations, i.e. $\xi_{II} \cong \xi_I / G$ and $\xi_I \cong_G q^*(\xi_{II})$.

If we start with case I we have:

$$\xi_I \xrightarrow{q} \xi_I / G \ (= \xi_{II}) \quad \text{and} \quad Q_I \xrightarrow{q} Q_I / G \ (= Q_{II}) .$$

So we construct the bundle $\xi_{II}$ by taking the quotient with respect to the original $G$-action $\pi$ on $\xi_I$. Again $\xi_I$ can be obtained by a pull-back: $(\xi_I, \pi) \cong_G (q^*(\xi_{II}), \tilde{\tau})$.

If we start with case II, i.e. $\xi_{II} = (E(\xi_{II}), \pi, Q_{II})$ we obtain the $G$-bundle $(q^*(\xi_{II}), \tilde{\tau})$ by a pull-back. Furthermore by taking the quotient (with respect to $\tilde{\tau}$) we achieve again $q^*(\xi_{II}) / G \cong \xi_{II}$. This pull-back construction is expressed by the following diagrams:

$$\begin{array}{ccc}
Q_I & \xrightarrow{q} & Q_{II} \\
\xi_I & \xrightarrow{\pi} & \xi_{II} \\
q(x) & = & [x] \\
E(q^*(\xi_{II}))_{x} & := & E(\xi_{II})_{[x]} \\
\end{array}$$

Remark 2 At this point it may be useful to enumerate all the actions of the permutation group $G$ we use. $G$ acts on $\tilde{Q} \equiv Q_I$, $\eta \equiv \xi_I$, $C(\tilde{Q})$ -the space of continuous functions on $\tilde{Q}$- and $\Gamma(\eta)$ -the space of sections
Γ into η. Taking into account that in our case the bundle η is a trivial bundle (since we are dealing with flat bundles and \(\tilde{Q}\) is simply-connected), we have, in an obvious notation:

\[
\rho : \ G \times \tilde{Q} \rightarrow \tilde{Q}
\]

\[
\tau : \ G \times \eta \rightarrow \eta \quad (g, z) \mapsto \tau_g(z) := \tau_g(x, y) \equiv (\rho_g, R(x, g) y) \quad \text{(with } z = (x, y))
\]

\[
G \times C(\tilde{Q}) \rightarrow C(\tilde{Q}) \quad (g, a) \mapsto (ga)(x) := a(\rho_g^{-1}(x)) \equiv a(g^{-1}x)
\]

\[
G \times \Gamma(\eta) \rightarrow \Gamma(\eta) \quad (g, s) \mapsto (g\tilde{s})(x) := \tau_g(s(g^{-1}x)) \equiv \tau_g(g^{-1}x, s(g^{-1}x)).
\]

Furthermore, as will become apparent below, the section \(s (s(x) \equiv (x, |s(x)|) \) in η is invariant: \(\tilde{g}s = s (s \in \Gamma^{inv}(\eta)) \) if \(|s(gx)| = R(x, g)|s(x)| \) is valid.

**Remark 3** Since in our case the bundle η is trivial, it is legitimate to express elements \(z \in \eta \) (globally) in the form \(z = (x, y)\). It then follows from Definition 1 that \(\tau_g\) must take the form given above, i.e. \(\tau_g(x, y) = (\rho_g, R(x, g) y)\), with \(R\) a linear mapping -that in general depends on both \(x\) and \(g\)- taking the fibre over \(x\) onto the fibre over \(\rho_g(x)\).

### 3.2 The Spin zero case as example

We consider two spin \(S = 0\) particles. The permutation group is \(G = Z_2\). We start with the point of view II. For scalar particles, we expect line bundles \((V = \mathbb{C})\) over the projective space \(\mathbb{R}P^2\). It is a well known fact that there are two line bundles over \(\mathbb{R}P^2\) [9], a trivial \(\xi_+ = \mathbb{R}P^2 \times V\) and a non-trivial one: \(\xi_- = \mathbb{R}P^2 \times V\). This corresponds, as we shall see, to symmetric and antisymmetric functions on the sphere \(S^2\). In this way, we obtain as a direct consequence of the non-trivial topology of the configuration space \(Q = Q_U\) the Bose-Fermi alternative for scalar particles, a well known result [10,11].

From the point of view I, we now have two \(G\)-line bundles: \(\eta_+ = (\eta, \tau_+\) and \(\eta_- = (\eta, \tau_-)\), both with the underlying trivial bundle \(\eta = S^2 \times V\). The permutation group \(G = Z_2\) acts non-trivially only on the second line bundle. We denote this action with \(\tau_-\). Explicitly, we have:

\[
\tau_- : \ Z_2 \times \eta \rightarrow \eta \quad (g, (x, y)) \mapsto (\rho_g(x), \text{sign}(g)y).
\]

(12)

The action \(\tau_+\) is trivial. An explicit construction of the non-trivial bundle \(\xi_-\) follows. This is also needed in the next section.

For the projective space \(\mathbb{R}P^2 = S^2 / G = \{[x]\}\) we choose the three local charts \((U_\alpha, h_\alpha), \alpha \in \{1, 2, 3\}\) as follows:

\[
S^2 = \{x = (x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3
\]

(13)

with the canonical projection \(q : S^2 \rightarrow \mathbb{R}P^2, x \mapsto [x] = \{x, -x\} = \mathbb{R}x\) and the following open covering of \(\mathbb{R}P^2\):

\[
U_\alpha = \{[x] : x_\alpha \neq 0\}. \text{ For example, we may take a look at the first chart, i.e. } \alpha = 1:
\]

\[
h_1 : U_1 \rightarrow \mathbb{R}^2, [x] \mapsto \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right)
\]

(14)

It is convenient to construct the bundle \(\xi_- = (E(\xi_-), \pi, \mathbb{R}P^2)\) as a non trivial sub-bundle of a higher rank trivial bundle \(\mathbb{R}P^2 \times \mathbb{C}^k\). In the construction that follows, we will choose \(k = 3\). Consider now a function

\[
|\chi(\cdot)| : S^2 \rightarrow \mathbb{C}^3,
\]

(15)

with the following properties:

(i) \(|\chi(x)|\chi(x) = 1\) for all \(x\) in \(S^2\).

(ii) \(|\chi(-x)| = -|\chi(x)|\) for all \(x\) in \(S^2\).
Two possible choices for such a function are \(|\chi(x)| = x\) and \(|\chi(x)| = \left(e^{-i\theta\sin\theta/\sqrt{2}}, -\cos\theta, e^{i\theta\sin\theta/\sqrt{2}}\right)\). For the explicit computations that follow, we will stick to the first choice. Define now the total space of the bundle as the subset of \(\mathbb{R}P^2 \times \mathbb{C}^3\) given by

\[
E(\xi_-) = \left\{ ([x], \lambda|\chi(x)) \in \mathbb{R}P^2 \times \mathbb{C}^3 : \lambda \in \mathbb{C}, x \in [x] \right\}.
\] (16)

Notice that, because of properties (i) and (ii), the fibre over \([x]\) is the complex line in \(\mathbb{C}^3\) generated by the vector \(|\chi(x)|\), independently of the choice of representative \(x \in [x]\). However, one must be aware of the fact that, in order to explicitly exhibit an element \(z \in E(\xi_-)\), a choice of representative must be made. If we make the choice \(|\chi(x)|\), then \(z\) is exactly one \(\lambda \in \mathbb{C}\) such that \(z = ([x], \lambda|\chi(x))\). On the other hand, if we choose to express \(z\) in terms of \(|\chi(-x)|\), we will find a unique \(\lambda' \in \mathbb{C}\) such that \(z = ([x], \lambda'|\chi(-x))\).

From the definition of \(E(\xi_-)\) and the properties of \(|\chi|\), it follows that \(\lambda' = -\lambda\). Therefore, we can indistinctly write \(z = ([x], \lambda|\chi(x))) = ([x], -\lambda|\chi(-x)))\). The bundle projection is of course defined by \(\pi(z) = [x]\).

A description of this bundle in terms of transition functions is now easy to obtain. For this we define the following local trivializations:

\[
\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{C}, ([x], \lambda|\chi(x)) \mapsto ([x], \text{sign}(x_\alpha)\lambda) \equiv ([x], v)
\] (17)

and the corresponding transition functions:

\[
\phi_\beta \circ \phi_\alpha^{-1} : ([x], v) \mapsto ([x], g_{\beta\alpha}v) \quad \text{with} \quad g_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow \mathbb{C}^\times, [x] \mapsto \text{sign}(x_\alpha x_\beta).
\] (18)

It will be convenient, for the discussion that follows, to have an explicit description, in this geometric setting, of the space of sections of the bundle just described. First define, for \(\alpha \in \{1, 2, 3\}\):

\[
e_\alpha([x]) := \begin{pmatrix} x_\alpha x_1 \\ x_\alpha x_2 \\ x_\alpha x_3 \end{pmatrix}.
\] (19)

These maps can be used to define local sections:

\[
s_\alpha^\omega : U_\alpha \rightarrow U_\alpha \times \mathbb{C}^3, [x] \mapsto s_\alpha([x]) := ([x], e_\alpha([x])).
\] (20)

Each \(s_\alpha^\omega\) can be smoothly extended to a global section \(s_\alpha \in \Gamma(\xi_-)\). Observe that \(s_\alpha\) is non-vanishing inside \(U_\alpha\), but vanishes exactly outside it, reflecting the fact that \(\xi_-\) is not a trivial (line) bundle. We therefore see that the three sections \(s_1, s_2\) and \(s_3\), act as generators of the space of all sections, i.e., every global section \(s \in \Gamma(\xi_-)\) can be written in the form

\[
s = \sum_{\alpha=1}^{3} f_\alpha s_\alpha,
\] (21)

with \(f_\alpha \in C(\mathbb{R}P^2)\).

Let us now consider the pull-back bundle \(q^*\xi_-\). Its total space is given by the set of all pairs \((x,z)\) in \(S^2 \times E(\xi_-)\) such that \(q(x) = \pi(z)\). Given a section on \(\xi_-\), \(s \in \Gamma(\xi_-)\), we can define the following section on the pull-back bundle \((q^*s) \in \Gamma(q^*\xi_-)):

\[
(q^*s)(x) := (x, s([x])).
\] (22)

Referring back to (19) and (20), we then have

\[
q^*s_\alpha(x) = (x, s_\alpha([x])) = (x, ([x], e_\alpha([x]))) \equiv (x, e_\alpha([x])),
\] (23)

where, in last step, we choose a description of the pull-back bundle as a sub-bundle of a trivial bundle. The definition of these “induced” sections will be needed in the next section.
4 Algebraic formulation

4.1 Geometric - algebraic correspondence

As our example with spin zero particles in the last section shows, in general it is also possible to proceed in our discussion on a general level within the geometric framework. We consider in this sense manifolds and vector bundles as geometric objects. There is, however, an equivalent algebraic framework with applications in physics, e.g. in the non-commutative geometric approach to elementary particle physics. To put it simply, the algebraic object which corresponds to a topological space $M$ (compact and Hausdorff) as is shown by the Gelfand-Naimark theorem, is the algebra $C(M)$ of complex continuous functions on $M$. Similarly, for a vector bundle over a space $M$, the corresponding algebraic object as shown by the Serre-Swan theorem, is a finitely generated projective module over $C(M)$ [12, 13].

A merit of the algebraic formulation is that point sets are treated in a completely global way, and this allows a clear structural analysis of the physical problem under consideration. The price for that is one has to deal (in the present case) with a projective module over an algebra of functions, an object which is not at all common in physics. For example, a vector bundle is described analytically by a projector in this formalism.

Furthermore, it was shown that every $A_{R,i}$ is a finitely generated and projective $A$-module.

Theorem 2 (Reyes-Lega, cf. [6, 7]) There is an integer $N_R$ and a projector $p_R$ so that $A_{R,i}$ can be obtained from $A^{N_R}$, i.e.,

$$A_{R,i} \cong p_R(A^{N_R})$$

In the following diagram, we exhibit the bijective correspondence between the geometric and algebraic formulations, as given by the Serre-Swan theorem.

| Geometric formulation | Algebraic formulation |
|-----------------------|-----------------------|
| Objects: Vector bundle $\xi = (E(\xi), \pi, M)$ | $\rightarrow$ Space $\Gamma(\xi)$ of sections of the bundle $\xi$ over $M$. |
| Data: - Transition functions $\{g_{\beta\alpha}\}$, - Partition of unity of $M$ $\{\varphi_{\alpha}\}$, with $\sum_{\alpha} |\varphi_{\alpha}|^2 = 1$, subordinate to the cover $\{U_{\alpha}\}$ of $M$. | - Algebra of functions on $M$: $A = C(M)$. - There is a free $A$-module $\mathcal{E}$ of the form $\mathcal{E} = A^n$ and a projector $p_{\xi}: \mathcal{E} \rightarrow \mathcal{E}$, with $\Gamma(\xi) \cong p_{\xi}(\mathcal{E})$. (Here $n$ is the number of open sets in the covering times the rank of the bundle). -Projector is given by the $A$-valued block-matrix: $(p_{\xi})_{\alpha\beta} = |\varphi_{\alpha}|g_{\alpha\beta}|\varphi_{\beta}|$. |

Serre-Swan Theorem: Bijective correspondence between geometric and algebraic formulation.
In other words, this shows (in a way which, as shown below, is very well-suited for comparison with the BR approach) that the $G$-actions on $\hat{Q} (= Q_1)$ and $\eta$ ($= \xi_I$) give us all possible (flat) vector bundles which appear in the point of view II:

$$\xi (= \xi_I) \in \{ \xi_k \}_{R \in \text{In}(G)}.$$  

(26)

4.2 Spin zero case, algebraically

Here we essentially consider the algebraic version of subsection 3.2. We use the result of the previous subsection for the permutation group $G = \mathbb{Z}_2 = \{ +1, -1 \}$ and the algebras $\hat{A} = C(\hat{Q}) \equiv C(S^2)$ and $A = C(Q) \equiv C(\mathbb{R}P^2) \equiv A_+$. The irreducible representations of $G$ are given by $\text{Irr}(G) = \{ R_+, R_- \}$, where $R_+$ is the trivial representation and $R_-$ is given by $R_- : g \mapsto (-1)^{\xi}$.

The module decomposition of the previous subsection now takes a simple form:

$$\hat{A} = A_+ \oplus A_- \equiv C_+(S^2) \oplus C_-(S^2) \equiv \text{symmetric} \oplus \text{antisymmetric}. \quad (27)$$

$\hat{A}$, $A_+$ and $A_-$ are $A$-modules. The algebraic objects $A_+$ and $A_-$ represent, as we may infer from the module decomposition and the table in the previous subsection, spaces of sections and correspond to the vector bundles $\xi_+$ and $\xi_-$ (geometric objects). This connection is made explicit by the projectors $p_+$ and $p_-$. It is of course the non-trivial bundle $\xi_-$ which deserves our attention. As already mentioned, the full information about the bundle $\xi_-$ is contained in the projector $p_-$. We shall use the geometric information of $\xi_-$ as given in 3.2 to obtain $p_-$. In a self-explanatory notation we have with $\alpha, \beta \in \{ 1, 2, 3 \}$

$$\varphi_{\alpha}(x) := \left\{ \begin{array}{ll} \sqrt{x_{\alpha}}, & \text{if} \ x \in U_+, \\
0, & \text{otherwise} \end{array} \right. \quad , \quad \sum_{\alpha} x_{\alpha}^2 = x^2 = 1 \quad (28)$$

and the projector which can be written in components with $\chi : \chi(x) = (x_1, x_2, x_3)$:

$$p_- = (p_-)_{\alpha \beta} = g_{\alpha \beta} \varphi_{\alpha} \varphi_{\beta} = \text{sign}(x_{\alpha} x_{\beta}) |x_{\alpha}| |x_{\beta}| = x_{\alpha} x_{\beta} = |\chi| |\chi| \quad . \quad (29)$$

Having the projector $p_-$, we obtain the $A$-module $p_-(A^3)$ from the free module $A^3 = \{ f = (f_{\alpha}) : f_{\alpha} \in A \equiv C(\mathbb{R}P^2), \alpha = 1, 2, 3 \}$ by taking those $f \in A^3$ which obey the relation $p_- f = f$:

$$p_-(A^3) = \{ f : p_- f = f, f \in A^3 \} \quad (30)$$

The relation between the vector bundle $\xi_-$ and the projector $p_-$ is given by the following isomorphism:

$$p_-(A^3) \cong \Gamma(\xi_-), \quad (31)$$

i.e., the space of sections $\Gamma(\xi_-) = \{ s \}$ is isomorphic (as a module over $A$) to $p_-(A^3)$. Moreover, from theorem 2 it follows that $p_-(A^3)$ is also isomorphic to the $A$-module $A_- \equiv C_-(S^2)$. An explicit description of these bijective correspondences follows.

- $p_-(A^3) \hookrightarrow \Gamma(\xi_-)$:

Recall that any section $s \in \Gamma(\xi_-)$ can be written in the form $s = \sum_{\alpha} f_{\alpha} s_{\alpha}$, with $f_{\alpha} \in A \equiv C(\mathbb{R}P^2) \cong C_+(S^2)$, and $s_{\alpha}$ as defined in (20). So, if we start with $s$, we obtain three functions $f_1, f_2$ and $f_3$. Setting $f = (f_1, f_2, f_3)$, one can check, using (19), (20) and (29), that $p_- f = f$ holds. Hence, the map $s = \sum_{\alpha} f_{\alpha} s_{\alpha} \mapsto f$ gives the bijective correspondence between $\Gamma(\xi_-)$ and $p_-(A^3)$.

- $A_- \hookrightarrow p_-(A^3)$:

Consider now an odd function $a \in A_- = C_-(S^2)$. Using $x_1^2 + x_2^2 + x_3^2 = 1$, we can write $a = \sum_{\alpha} (x_{\alpha} a) x_{\alpha}$. Defining the even functions $f_{\alpha}(x) := x_{\alpha} a(x)$, we see that $a$ can be written as $a(x) = \sum_{\alpha} x_{\alpha} f_{\alpha}(x)$. Since the $f_{\alpha}$ are even, we can regard them as elements of $A$. Therefore, the bijective map between $A_-$ and $p_-(A^3)$ is given by $a(x) = \sum_{\alpha} x_{\alpha} f_{\alpha}(x) \mapsto f = (f_1, f_2, f_3)$. 

Remark 4 The proof that the module $A_-$ can be interpreted as the space of sections on the non-trivial line bundle over $\mathbb{R}P^2$ was first given by Paschke [14], using the $SU(2)$ symmetry of the sphere. In the present paper, the explicit form of the projector follows from the proof of theorem 2, for which the permutation group plays a prominent role. The equivalence of the two projectors is explained in [6].

We now come to a crucial point: If there is a bijective correspondence between $\xi_-$ and $(q^*\xi_-, \hat{\xi})$, how does this correspondence look like in the algebraic framework? The answer is obtained from the following isomorphism of $C(\mathbb{R}P^2)$-modules (cf.[13], proposition 2.12):

$$\tilde{T} : C(S^2) \otimes_{C(\mathbb{R}P^2)} \Gamma(\xi_-) \longrightarrow \Gamma(q^*\xi_-)$$

$$\sum_{\alpha,k} b_{\alpha,k} \otimes s_{\alpha} \longrightarrow \sum_{\alpha,k} b_{\alpha,k} q^* s_{\alpha}. \quad (32)$$

In this case, theorem 2 tells us that $\tilde{A} \equiv C(S^2) = A_+ \oplus A_-$. From $A \equiv C(\mathbb{R}P^2) \cong A_+$ we then obtain:

$$\Gamma(q^*\xi_-) \cong C(S^2) \otimes_{C(\mathbb{R}P^2)} \Gamma(\xi_-) \cong \tilde{A} \otimes_A \Gamma(\xi_-) = (A_+ \oplus A_-) \otimes_A \Gamma(\xi_-)$$

$$\cong (A_+ \otimes_A \Gamma(\xi_-)) \oplus (A_- \otimes_A \Gamma(\xi_-)) \cong \Gamma(\xi_-) \oplus (A_- \otimes_A \Gamma(\xi_-)). \quad (33)$$

This means that it is possible to find an isomorphic copy of $\Gamma(\xi_-)$ inside $\Gamma(q^*\xi_-)$ or, in simpler words, every section of $\xi_-$, which is a bundle over $\mathbb{R}P^2$, can be expressed as a certain section on a bundle (the pull-back of $\xi_-$) over $S^2$. All we have to do is to restrict the domain of $\tilde{T}$ to the submodule $\Gamma(\xi_-)$.

With $T := \tilde{T}|_{\Gamma(\xi_-)}$ we then obtain:

$$T : \Gamma(\xi_-) \longrightarrow \tilde{T}(\Gamma(\xi_-)) \subset \Gamma(q^*\xi_-)$$

$$s = \sum_{\alpha} f_\alpha s_{\alpha} \longrightarrow T(s) = \sum_{\alpha} f_\alpha q^* s_{\alpha}. \quad (34)$$

Since sections on $\xi_-$ are unconstrained, we expect to be able to find the correct constraint condition on an arbitrary section of $q^*\xi_-$, in order to be able to regard it as a section on $\xi_-$. From (32) and (34) it is clear that the constraint is the following: A section $\sum_{\alpha} b_\alpha q^* s_{\alpha} \in \Gamma(q^*\xi_-)$ is the isomorphic image of a section in $\Gamma(\xi_-)$ if and only if the $b_{\alpha}$ are even functions. This can be recast in terms of the induced $G$-action $\hat{\xi}$ (cf.[6,7]):

A section $\sigma \in \Gamma(q^*\xi_-)$ belongs to the image of $T$ if, and only if, it is $G$-invariant: $\hat{\xi} \sigma = \sigma. \quad (35)$

4.3 Results

With the above information, it is not difficult to obtain the following results which were derived and discussed in more detail in [6,14] (see also [7], for the general case).

- The isomorphism $A_- \cong p_-(A^3) \cong \Gamma(\xi_-)$:

As already seen, the space of antisymmetric functions on the sphere $A_- = C_-(S^2)$ can also be described with the projector $p_-$ and the vector-space-like space $A^3$ (i.e. free module) by means of the projector $p_-(A^3)$. As a consequence, we have the isomorphisms:

$$A_- \cong p_-(A^3) \cong \Gamma(\xi_-) \cong T(\Gamma(\xi_-)) \subset \Gamma(q^*\xi_-). \quad (36)$$

These isomorphisms can be described with the help of the corresponding generators. In the case of the above $A$-modules, we do not have a basis at our disposal. Hence, we obtain for the generators and the elements the following expressions:

$$A\text{-module: } \quad \begin{array}{ccc} A_- & \longrightarrow & p_-(A^3) & \longrightarrow & \Gamma(\xi_-) & \longrightarrow & T(\Gamma(\xi_-)) \\ \{x_\alpha\} & \alpha & \quad \{x_\alpha \chi(x)\} & \alpha & \quad \{s_\alpha\} & \alpha & \quad \{q^* s_\alpha\} \\ \text{Generators:} & & & & & & \\ \text{Elements:} & a = \sum_{\alpha} (x_\alpha a_\alpha) & \longrightarrow & f = (f_1, f_2, f_3) & \longrightarrow & s = \sum_{\alpha} f_\alpha s_{\alpha} & \longrightarrow & T(s) = \sum_{\alpha} f_\alpha q^* s_{\alpha}, \quad (37) \\ & \alpha & \longrightarrow & & \alpha & \longrightarrow & & \alpha \\ \end{array}$$

with $a \in A_-, f_\alpha \in A_+ \cong A, f_\alpha(x) = x_\alpha a_\alpha(x) \leftrightarrow a(x) = \sum_\alpha x_\alpha f_\alpha(x)$ and $p_- f = f.$
The connection in $\xi^-$:
A natural connection $\nabla$ in $\xi^-$ is the Grassmann connection which we can also express with the help of the projector $p_-$. So we have for a section $s \in \Gamma(\xi^-)$ as denoted above the relation $\nabla s \leftrightarrow p_- df$ $(\nabla s = \nabla p_- A^3)$.

This connection is flat and its holonomy group is $\mathbb{Z}_2$.

The bundle $\xi^-$ is a $SU(2)$ bundle:
It can be shown that the group $SU(2)$ is acting on $\xi^-$ and we have a $SU(2)$ bundle in the sense of the definition in subsection 3.1. Parallel transport by means of the above connection $\nabla$ is consistent with the $SU(2)$ action. This point is important for the exchange mechanism of Berry-Robbins. Similar conditions were demanded and discussed in some detail in [1].

5 Connection with the Berry-Robbins approach

The aim of this section is to discuss the geometric structure which is behind the Berry-Robbins approach. In our opinion, this strengthens the relevance of the Berry-Robbins approach to the spin-statistics problem. For our discussion we need some preparation, we therefore start first with a short review of this approach. In subsection 5.2, we give an explicit construction of the two-spin bundle over the projective space $\mathbb{R}P^2$ which corresponds to the point of view II of subsection 3.2. In subsection 5.3 we comment on the singlevaluedness condition, which plays a central role in the Berry-Robbins approach, from the geometric point of view.

5.1 Short review of the exchange mechanism in the Berry-Robbins approach

We consider two spin $s$ particles. As we already discussed in section 2, the BR approach refers to the point of view I, which means that the wave function is essentially defined on the two sphere $Q_I \equiv \tilde{Q} = S^2$ or equivalently on the corresponding trivial vector bundle $\xi_I \equiv \eta = S^2 \times V$.

The standard spin basis (fixed) is given by:

$$|sm_1\rangle \otimes |sm_2\rangle \equiv |m_1m_2\rangle =: |M\rangle$$

The permutation of the particles 1 and 2 ($(1,2) \mapsto (2,1)$) leads to

$$|M\rangle = |m_1m_2\rangle \leftrightarrow |m_2m_1\rangle =: |M\rangle$$

In order to perform the permutation in a continuous way, an exchange group $G' = SU(2)$ was introduced by BR. The exchange rotation is then represented by $U$. In the parametrization of BR this is given by the map

$$U : S^2 \to GL(V) , \quad r \mapsto U(\mathbf{r}) = \exp(-\theta \mathbf{n}(\mathbf{r}) \cdot \mathbf{E}) \quad \text{with} \quad \mathbf{n} = e_3 \times r , \quad \mathbf{r} = r(\theta, \varphi),$$

where $\mathbf{E}$ is a vector operator constructed from the Schwinger representation of spin (cf.[1]). With $U(\mathbf{r})$ defined this way, the transported spin basis $|M(\mathbf{r})\rangle$ was defined by

$$|M(\mathbf{r})\rangle := U(\mathbf{r})|M\rangle$$

From equation (40) and (41) the relation

$$|M(-\mathbf{r})\rangle = (-)^{2s}|M(\mathbf{r})\rangle$$

for the transported spin basis was obtained. The properties of the transported spin basis of BR can be summarized as follows:

- The smooth map for all $M$:

$$S^2 \to \mathbb{C}^{N_s} , \quad \mathbf{r} \mapsto |M(\mathbf{r})\rangle := U(\mathbf{r})|M\rangle$$

- The following exchange rule:

$$|M(-\mathbf{r})\rangle = (-)^{2s}|M(\mathbf{r})\rangle$$
The parallel transport condition:
\[ \langle M'(r(t)) \frac{d}{dt} M(r(t)) \rangle = 0 \]
for all \( M \) and \( M' \), and for every smooth curve \( t \mapsto r(t) \).

The wave function is given by
\[ |\Psi(r)\rangle = \sum_M \psi_M(r) |M(r)\rangle \]
(43)

In addition, since we have here the point of view I, the following singlevaluedness condition is imposed in order to incorporate the indistinguishability of the particles in the formalism:
\[ |\Psi(-r)\rangle = |\Psi(r)\rangle \]
(44)

Assuming the above properties, a direct consequence of equation (44) is the relation
\[ \psi_{rM}(-r) = (-)^{2s} \psi_M(r). \]
(45)
This is the correct relation between spin and statistics.

5.2 Construction of the two-spin bundle

The relevance of the two-spin bundle over the projective space \( \mathbb{R}P^2 \) was pointed out in BR. Here we give an explicit construction of it for \( s = 1/2 \), using the geometric and algebraic formulations discussed in the previous sections. This construction allows the clarification of the geometric structure which is behind the BR approach. In particular, the relation of the exchange mechanism as given by the exchange rotation \( U(r) \) to topology and geometry (connection and parallel transport) of the system will become transparent. The exchange matrix \( U(r) \), as explained in BR, acts on a 10-dimensional space \( V \). A basis of \( V \) in terms of creation and annihilation operators (Schwinger representation) is given in an obvious notation:

- \( |e_1\rangle := a_1^\dagger a_2^\dagger |0\rangle = |+,+,\rangle \)
- \( |e_2\rangle := b_1^\dagger b_2^\dagger |0\rangle = |-,+,\rangle \)
- \( |e_3\rangle := a_1^\dagger b_2^\dagger |0\rangle = |+,-,\rangle \)
- \( |e_4\rangle := a_2^\dagger b_1^\dagger |0\rangle = |-,+,\rangle \)
- \( |e_5\rangle := \frac{(a_1^\dagger)^2}{\sqrt{2}} |0\rangle \)
- \( |e_6\rangle := \frac{(a_2^\dagger)^2}{\sqrt{2}} |0\rangle \)
- \( |e_7\rangle := \frac{(b_1^\dagger)^2}{\sqrt{2}} |0\rangle \)
- \( |e_8\rangle := \frac{(b_2^\dagger)^2}{\sqrt{2}} |0\rangle \)
- \( |e_9\rangle := \frac{(a_1^\dagger)^2}{\sqrt{2}} |0\rangle \)
- \( |e_{10}\rangle := \frac{(a_2^\dagger)^2}{\sqrt{2}} |0\rangle \)

So we have for the vector space \( V \):
\[ V = \text{span}(|e_1\rangle, |e_2\rangle, \ldots, |e_{10}\rangle) \]
(47)

The four vectors \( |e_1\rangle, |e_2\rangle, |e_3\rangle \) and \( |e_4\rangle \) correspond to the usual two-spin basis \( |M\rangle = |m_1 m_2\rangle \). The transported spin vectors \( |M(r)\rangle \) are maps:
\[ S^2 \rightarrow V, \ r \mapsto |M(r)\rangle \]
(48)

The exchange matrix \( U(r) \) significantly simplifies if we use instead of \( |m_1 m_2\rangle \) the total spin basis \( |jm\rangle \). For \( j = 1 \) we use the notation \( |m\rangle = |j m\rangle \) for \( m \in \{-1,0,+1\} \) and for \( j = 0 \) we take \( |00\rangle \). Now we define a new basis of the space \( V \): For every fixed \( m \) we consider the corresponding exchange triplet:
\[ B_m := \left\{|m\rangle^{(-1)}, |m\rangle^{(0)}, |m\rangle^{(+1)}\right\} \]
(49)
The standard (usual) basis vectors $|m\rangle$ are identified by $|m\rangle^{(0)} \equiv |m\rangle$. Hence, we have with $V_m := \text{span}(B_m)$ an exchange triplet space. There are of course three such subspaces $V_m$ with $m \in \{-1,0,+1\}$. The new basis of $V$ is given in the following basis scheme:

$$ j = 1 : \begin{cases} |1\rangle : |1\rangle^{(-1)} = |e_3\rangle , |1\rangle^{(0)} = |e_2\rangle , |1\rangle^{(+1)} = |e_1\rangle \\ |0\rangle : |0\rangle^{(-1)} = |e_5\rangle , |0\rangle^{(0)} = \frac{1}{\sqrt{2}}(|e_3\rangle + |e_4\rangle) , |0\rangle^{(+1)} = |e_6\rangle \\ |+1\rangle : |+1\rangle^{(-1)} = |e_7\rangle , |+1\rangle^{(0)} = |e_1\rangle , |+1\rangle^{(+1)} = |e_9\rangle \end{cases} \quad (50) $$

Note that the second column represents the standard triplet and singlet. In this basis, the matrix $U(r)$ takes a block diagonal form. The restriction of the $U(r)$ to the space $V_m$ is easily obtained by standard procedures [6] so we have for $U(r) \in GL(V_m)$ the matrix:

$$ U(r) = \begin{pmatrix} \cos^2 \frac{\theta}{2} & -e^{-i\varphi} \sin \frac{\theta}{2} & e^{-2i\varphi} \sin^2 \frac{\theta}{2} \\ e^{i\varphi} \sin \frac{\theta}{2} & \cos \theta & -e^{i\varphi} \sin \frac{\theta}{2} \\ e^{2i\varphi} \sin^2 \frac{\theta}{2} & e^{i\varphi} \sin \frac{\theta}{2} & \cos^2 \frac{\theta}{2} \end{pmatrix} \quad (51) $$

and for the transported vectors we obtain:

$$ |jm(r)\rangle = U(r)|jm\rangle = -e^{-i\varphi} \sin \frac{\theta}{\sqrt{2}} |m\rangle^{(-1)} + \cos \theta |m\rangle^{(0)} + e^{i\varphi} \sin \frac{\theta}{\sqrt{2}} |m\rangle^{(+1)} \quad (52) $$

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From this we again immediately obtain for $\theta = \pi$ the exchange rule expressed now for the total spin basis:

$$ j = 1 : |jm(-r)\rangle = -|jm(r)\rangle \quad , \quad j = 0 : |00(-r)\rangle = |00(r)\rangle $$

For $j = 0$ we had already $|00\rangle = \text{constant}$. It follows also from (52) that every vector $|jm(r)\rangle$ is not vanishing for all $r$ and $m$. Therefore, we can consider the mapping: $r \mapsto |jm(r)\rangle$ as a non-vanishing section in the trivial bundle $S^2 \times V_m$. This leads to a definition of a line bundle. In this way, we obtain the four line bundles $\eta_{jm}$ as given by

$$ \eta_{jm} := \{ |jm(r)\rangle | r \in S^2 \} \cong S^2 \times \mathbb{C} $$

This determines the trivial two-spin bundle over the sphere $S^2$:

$$ \eta = \bigoplus_{j,m} \eta_{jm} \cong S^2 \times \mathbb{C}^4 \quad (54) $$

**Remark 5** Since we are assuming point of view I, in order to complete the description of wave functions one should:

(i) Indicate the action $\tau_{jm}$ of $G$ on each bundle $\eta_{jm}$. Although this is not explicitly done in the BR formalism, it seems natural to assume that the necessary information is “hidden” in the exchange properties of the transported spin basis. We will comment this in more detail in the next subsection.

(ii) Once the correct action $\tau_{jm}$ has been found, one should regard as physical wave functions only those sections of $\eta$ that are invariant with respect to the given $G$-action. This corresponds to the assertion that the physical configuration space is $\mathbb{R}P^2$. The way this is done in the BR formalism is by imposing the single-valuedness condition $|\Psi'(r)\rangle = |\Psi(-r)\rangle$. Our main concern here will be to interpret this condition in terms of our formalism (the details are explained in the next subsection).

We proceed one step further and construct the two-spin bundle over the projective space $\mathbb{R}P^2$ which corresponds to the point of view II as discussed in sections 2 and 3. From the above considerations, it is clear that we expect the bundle $\xi$ to be a direct sum of four line bundles. In analogy to equation (54) we have:

$$ \xi = \bigoplus_{j,m} \xi_{jm} \quad (55) $$
What is left is the determination of the line bundles $\xi_{jm}$ for $j = 1$. In order to achieve this, according to the algebraic formulation in section 4, we only have to determine the projector $P(r)$ which corresponds to the bundle $\xi_{jm}$ ($j = 1, m \in \{-1, 0, +1\}$). This information is, as expected, hidden in the exchange matrix $U(r)$. Within every subspace $V_m$, we consider the projection onto the space generated by $|m\rangle = |jm\rangle \equiv |m\rangle^{(0)}$. Its matrix form, in terms of the basis $B_m$, is:

$$P_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{56}$$

From this and $U(r)$ we may define $P(r)$ [6] on each subspace $V_m$ ($j = 1$):

$$P_m(r) := U(r)P_0U(r)^\dagger \equiv |jm(r)\rangle\langle jm(r)|. \tag{57}$$

From the explicit construction of the projector $P_m(r)$, we see that the transported spin basis gives rise to a projector that is exactly the direct sum of three copies of $p_-$, plus a trivial projector corresponding to the singlet state (this one describes a trivial line bundle over $\mathbb{R}P^2$). Therefore we may write $P = |jm(r)\rangle\langle jm(r)|$ ($j = 1$). It is important to note that the components of $P$ are even functions, so that we can also regard these projectors as describing bundles over $\mathbb{R}P^2$: $P(|x\rangle) = P_{ij}(x)$ with $P_{ij} \in A \equiv C(\mathbb{R}P^2) \equiv C_+(S^2)$ is valid. Therefore, we have $P(|x\rangle) = P(r)$. Its connection to the bundle $\xi_{jm}$ is given by

$$P(A^3) \cong \Gamma(\xi_{jm}) \tag{58}$$

From $P(A^3) \cong p_-(A^3)$ we also see that the isomorphism $\xi_{jm} \cong \xi_-$ holds.

Thus, taking into account the results of subsections 3.2 and 4.2, the determination of the two-spin bundle is completed and we have

$$\xi \cong \xi_{-1} \oplus \xi_{10} \oplus \xi_{11} \oplus \xi_{00} \cong \xi_- \oplus \xi_+ \oplus \xi_- \oplus \xi_+. \tag{59}$$

All this was achieved based on the information which was contained in the transported spin basis $|jm(r)\rangle$ using the geometric and algebraic considerations of the previous sections.

As can be explicitly shown, this basis is, in addition, parallel with respect to the Grassmann connection.

5.3 On the singlevaluedness condition

The singlevaluedness condition $|\Psi(-r)\rangle = |\Psi(r)\rangle$ seems directly evident as it is a geometric condition. In spite of this it seems also necessary to examine this condition in the light of the geometric formulation in section 3, and especially in the light of theorem 1. For this purpose, a short recapitulation of the results in section 3 will be useful. The two points of view I and II may now be summarized as follows:

$$\begin{array}{ll}
\text{I} & \text{II} \\
\text{Configuration space:} & \tilde{Q} (= S^2), \ x = r \\
\text{Bundle:} & Q (= \mathbb{R}P^2), \ [x] \\
\text{Sections:} & \Gamma(\eta) \\
\text{Wave function:} & \Psi(\eta) \equiv \Psi(x) = (x, |\Psi(x)\rangle) \\
& \Psi(|x\rangle) \text{ (a section in } \xi) \\
& \text{with } \hat{g}\Psi = \tilde{\Psi} \\
& \text{with } \Psi(|gx\rangle) = \Psi(|x\rangle) \\
\end{array}$$

The important fact is that in the point of view II the wave function $\Psi \in \Gamma(\xi)$ is unconstrained. The direct physical wave function corresponds in the point of view I to the constrained wave function

$$\tilde{\Psi} \in \Gamma(\eta) \subset \Gamma(\eta). \tag{60}$$

As shown above, the condition on $\tilde{\Psi}$ is given by the action $\hat{g}$ of the permutation group $G$: $\hat{g}\tilde{\Psi} = \tilde{\Psi}$.

The relation between singlevaluedness (as proposed in BR) and invariance of the wave function (as proposed in the present work) is a very subtle issue. Therefore, we will spell out this relation in detail, for the case
$s = 0$, taking advantage of the results presented in the previous sections. In the next paragraphs, we follow the notation and conventions of sections 3 and 4.

Let us start by considering an arbitrary section $\sigma$ of the pull-back bundle $q^*\xi_-$. We have seen that, in view of (32), it can be written in the form $\sigma = \sum b_{\alpha} q^* s_{\alpha}$, with $b_{\alpha} \in A = C(S^2)$ (here, the sections $s_{\alpha}$ denote the generating sections defined in (20)). We have seen that $\sigma$ lies in the image of $T$ if and only if the functions $b_{\alpha}$ are even. The relation with invariance of the section is a consequence of the following calculation:

$$
g\sigma(x) = \tilde{\tau}_x \sigma(g^{-1}x) = \tilde{\tau}_x \left( \sum_{\alpha} b_{\alpha}(g^{-1}x) q^* s_{\alpha}(g^{-1}x) \right)$$

$$= \tilde{\tau}_x \left( g^{-1}x, \sum_{\alpha} b_{\alpha}(g^{-1}x) e_{\alpha}(\|g^{-1}x\|) \right) = [x]$$

$$= \left( x, \sum_{\alpha} b_{\alpha}(g^{-1}x) e_{\alpha}(\|x\|) \right).$$

We thus see that

$$g\sigma = \sigma \iff b_{\alpha} \in A_+ \equiv A \iff \sigma \in T(\Gamma(\xi_-)).$$

Therefore, if a section $\sigma$ is the image of some $s = \sum_{\alpha} f_{\alpha} s_{\alpha} \in \Gamma(\xi_-)$ ($f_{\alpha}$ must be even), then we can express it as follows:

$$\sigma(x) = \left( x, \sum_{\alpha} f_{\alpha}(x) q^* s_{\alpha}(x) \right)$$

$$= \left( x, \sum_{\alpha} f_{\alpha}(x) e_{\alpha}(\|x\|) \right)$$

$$= \left( x, \sum_{\alpha} f_{\alpha}(x) x_{\alpha} |\chi(x)\rangle \right)$$

$$= \left( x, a(x) |\chi(x)\rangle \right).$$

Here we have made use of the fact that we are working on the pull-back bundle, and in this case it is possible to express $e_{\alpha}$ as the product $e_{\alpha}(\|x\|) = x_{\alpha} |\chi(x)\rangle$ and then to “absorb” the term $x_{\alpha}$ into the function $f_{\alpha}$, giving place to the odd function $a_{\alpha} = \sum_{\alpha} f_{\alpha} x_{\alpha}$. This is in full agreement with the bijections described in (37). We may conclude: The section $\sigma$ in (63) is an invariant section (and hence represents a physical wave function) if and only if each $f_{\alpha}$ is an even function or, equivalently, if the function $a$ is an odd function. The fact that we can factor out this odd function is due to our choice of $|\chi(x)\rangle$ with the property $\langle \chi(-x)\rangle = -\langle \chi(x)\rangle$. Notice that we are now working on the pull-back bundle, which is a trivial bundle. Whereas being an invariant section is something independent of the way the pull-back bundle is represented, the fact that $a$ must be odd in order for $\sigma$ in (63) to be invariant is something that depends on our specific construction of the bundle (there are infinitely many bundles that are isomorphic to $q^*\xi_-$, but “look” differently).

In order to distinguish the features that depend on a choice from those that do not, we will proceed in the following way.

1. Let us assume that the physical wave functions for spin zero particles are sections on the bundle $\xi_-$ over $\mathbb{R}P^2$ (this gives of course the wrong connection between spin and statistics, but the same exercise could be done with the trivial bundle). In this case we would have, assuming point of view II: $\Psi_f \equiv s \in \Gamma(\xi_-)$.

2. In order to obtain the description of this wave function using point of view I, we take the pull-back of $\xi_-$ and consider (as we are forced to) the $G$-action $\tilde{\tau}$ on $q^*\xi_-$ naturally induced by the pull-back operation (cf. (9)). From the definition of pull-back, the description of $q^*\xi_-$ as a sub-bundle of a trivial bundle leads naturally to a description of the bundle in terms of the map $|\chi\rangle$.

3. Study the invariance of the section $\Psi_f \equiv T(s)$ using the $G$-bundle $(q^*\xi_-, \tilde{\tau})$ and compare with the singlevaluedness condition.

4. Construct an isomorphism of $G$-bundles $(q^*\xi_-, \tilde{\tau}) \cong_G (\eta, \tau')$, with $\eta$ described in terms of a map $|\chi'\rangle$ having the property $|\chi'(x)\rangle = |\chi'(-x)\rangle$.

5. Study the invariance of the section $\Psi_f \equiv T(s)$ using the bundle $(\eta, \tau')$ and compare with the singlevaluedness condition.

Let us now go through these five steps:
1. We start with \( \Psi_I = s \in \Gamma(\xi_-) \). As explained before, there must be some functions \( f_\alpha \in A \cong A_+ \) such that \( s = \sum f_\alpha s_\alpha \).

2. From (34) we obtain \( T(s) = \sum f_\alpha q^\ast s_\alpha \in T(\Gamma(\xi_-)) \subset \Gamma(q^\ast \xi_-) \). From the definition of pull-back, we obtain:

\[
E(q^\ast \xi_-) = \{(x, (|x|, \lambda \chi(x))) \} \subset S^2 \times E(\xi_-) : \lambda \in \mathbb{C}\} \equiv \{(x, \lambda \chi(x)) \} \subset S^2 \times E(\xi_-) : \lambda \in \mathbb{C}\}. \tag{64}
\]

Using this and (9), we obtain the explicit form of \( \tilde{\tau} \) (for \( g = -1 \)):

\[
\tilde{\tau}_g(x, \lambda \chi(x)) = (\rho_x x, \lambda \chi(x)) = (-x, -\lambda \chi(-x)), \quad \text{i.e., } \tau \equiv \tau_- \tag{65}
\]

3. We must have \( \Psi_I = T(s) = T(\Psi_I) \). Writing \( \Psi_I(x) = (x, |\Psi_I(x)|) \), we conclude, from (64), that there must be a function \( a \in C(S^2) \) such that

\[
\Psi_I(x) = (x, a(x) |\chi(x)|). \tag{66}
\]

From our previous computations it then follows that \( \Psi_I = T(s) \) if and only if \( a(x) \) is odd. This in turn implies:

\[
|\Psi_I(-x)| = a(-x) |\chi(-x)| = (-a(x)) (-|\chi(x)|) = |\Psi_I(x)|. \tag{67}
\]

4. From (64) it is clear that if \( |\chi'(x)| \) is any non-vanishing, normalized and smoothly varying vector, then replacing \( |\chi| \) by \( |\chi'| \) in (64) we obtain a bundle \( \eta \) which is isomorphic to \( q^\ast \xi_- \) (this is a quite obvious fact, because both bundles are trivial). Now, as we have seen, the \( G \)-action \( \tilde{\tau} \) on \( q^\ast \xi_- \) is equivalent to \( \tau_- \).

Since the equivalence class of this action is completely determined by the (now fixed) \( \xi_- \), the action \( \tau' \) must also be equivalent to \( \tau_- \). Therefore, we define

\[
\tau'_g(x, \lambda |\chi'(x)|) := (g x, \text{sign}(g) |\chi'(g x)|). \tag{68}
\]

It is easy to check that \( (q^\ast \xi_-, \tilde{\tau}) \cong_G (\eta, \tau') \). Although this result is independent of the specific choice of \( |\chi'| \), in the next step we will assume that \( |\chi'(-x)| = |\chi'(x)| \).

5. Again, we must have \( \Psi_I(x) = (x, |\Psi_I(x)|) = (x, a(x) |\chi'(x)|) \), for some \( a \in C(S^2) \). We know that \( \Psi_I = T(s) \) if and only if \( \Psi_I \) is an invariant section in \( (\eta, \tau') \). In this case, the requirement of invariance leads to \( (g = -1) \):

\[
\tilde{g} \Psi_I(x) = \tau'_g(-x, a(-x) |\chi'(-x)|)
= (x, -a(-x) |\chi'(-x)|). \tag{69}
\]

Hence, \( \Psi_I \) is invariant if and only if \(-a(-x) = a(x)\), i.e., if and only if \( a \) is odd, as expected from the bijections in (37). The unexpected result is the following:

\[
|\Psi_I(-x)| = a(-x) |\chi'(-x)| = (-a(x)) |\chi'(x)| = -|\Psi_I(x)|. \tag{70}
\]

The result is that from the above considerations, it is not possible to justify the single-valuedness condition. Whereas the single-valuedness condition can be imposed when the transported vector is chosen to be \( |\chi| \), it cannot be imposed if choose to work with \( |\chi'| \). On the other hand, in both cases the invariance condition leads to a bijective correspondence between the section \( s \) and the same odd function \( a \). Our argument can be easily generalized to deal with the general spin case, or with more than two particles, but we have chosen the spin zero case because of its simplicity and because it already contains the essential idea.
6 Discussion

There is no doubt that the understanding of indistinguishability in quantum mechanics is a very subtle problem. It is not difficult to accept that the role of indistinguishable particles in the formulation and interpretation is very important and lies in the heart of quantum mechanics itself. In this sense, it is not understandable why, for instance, quantum field theory, from an outside perspective, should explain the spin-statistics connection and not quantum mechanics itself.

In the present contribution we analyzed within a geometric framework (section 3) and in addition within an equivalent algebraic framework (section 4) the structures related to the spin-statistic connection. Although our approach, particularly in sections 3, 4 and even subsection 5.3 on the singlevaluedness condition, is quite general and independent of the work of Berry-Robbins, we chose a close reference to it since we find it very interesting and inspiring.

In the geometric formulation we point out that there are two points of view when dealing with indistinguishable particles in quantum mechanics. From the point of view I, which is in essence the usual point of view, the configuration space $\tilde{Q}$ is unconstrained whereas the wave function has to be constrained. Here e.g. the singlevaluedness condition or another condition may be imposed. From the point of view II we have the opposite situation: the effective configuration space $Q$ is constrained by identification as imposed by the permutation group whereas the wave function is now completely unrestricted. By a wave function in the case of a non-trivial spin bundle we mean a section in a bundle. This dual situation may cause a lot of confusion. We believe that this was entirely clarified with the help of theorem 1 and the considerations in section 3.

We expect that our approach, both geometric and algebraic, will help to clarify in general the spin-statistics problem. In particular in section 5, the connection with the Berry-Robbins approach allowed us to clarify the geometric structure and underlines in this sense the relevance of the Berry-Robbins approach.

The concept of the singlevaluedness of the wave function under particle exchange is a subtle one. In this work, the geometric approach to quantum indistinguishability allowed us to treat the singlevaluedness of the wave function in a global, model independent way. The result is that we cannot justify this condition from the geometric framework, but have to replace it by a less stringent condition: The global invariance of the wave functions. This does not mean that this condition is wrong. From our experience with anomalies [3] we may expect that there are other physical conditions, not known at the moment, which demand and justify the singlevaluedness condition.
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