New Model of Higher-Spin Particle

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Abstract

We elaborate on a new model of the higher-spin (HS) particle which makes manifest the classical equivalence of the HS particle of the unfolded formulation and the HS particle model with a bosonic counterpart of supersymmetry. Both these models emerge as two different gauges of the new master system. Physical states of the master model are massless HS multiplets described by complex HS fields which carry an extra $U(1)$ charge $q$. The latter fully characterizes the given multiplet by fixing the minimal helicity as $q/2$. We construct the twistorial formulation of the master model and discuss symmetries of the new HS multiplets within its framework.

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1 Introduction

Concise and suggestive formulations of higher spin (HS) theory make use of (super)spaces with extra bosonic co-ordinates (see e.g. [1]–[5]). The simple and, at the same time, powerful device for the analysis of the geometric structure of such (super)spaces is provided by (super)particles propagating in them.

The so-called unfolded formulation of the HS theory [5] is reproduced by quantizing the tensorial particle [6]–[9], or the equivalent HS particle [5], in which tensorial coordinates were gauged away. Such formulations possess $Sp(8)$ and $OSp(1|8)$ symmetries (which are hidden in the second formulation). There also exists a different formulation of the HS particle exhibiting invariance under the even counterpart of supersymmetry [10]. Quantization of this model produces HS multiplets with all helicities, just as in the case of the unfolded HS particle. In the next Section we briefly review the HS particle models of both types.

Our basic aim in this contribution is to describe new (“master”) model of the HS particle [11]. It yields, on the classical level, both above-mentioned HS particle models as its two appropriate gauges. In Sect. 3 we give a general description of our model. The master HS particle propagates in a space which is parametrized, in addition to the position four-vector, by two pairs of the commuting spinorial variables and an extra complex scalar $\eta$. The set of the first class constraints includes the basic unfolded constraints, the first class generalization of the spinor constraints and a scalar constraint which generates local $U(1)$ transformations of the twistor-like spinor variables and complex scalar co-ordinate.

In Sect. 4 we perform the quantization of the master system. In the relevant set of HS equations for the wave function, the basic unfolded equation [5] proves to be a consequence of the quantum counterparts of the spinor constraints. There is also a scalar $U(1)$ constraint which is an analog of the “spin–shell” constraint present in the model of massless particle with fixed helicity [12], [13]. In our case the degree of homogeneity of the HS field with respect to commuting twistor-like spinors is not fixed due to the presence of complex scalar coordinate $\eta$ with non-zero $U(1)$ charge. The external $U(1)$ charge $q$ of the HS wave function in extended space is defined as a degree of homogeneity with respect to both spinor and scalar co-ordinates. Physical states of the HS fields are massless particles with the helicities ranging from $\frac{q}{2}$ to infinity. Besides the standard HS multiplet [5] corresponding to the choice of $q = 0$, the considered setting implies the existence of new HS multiplets with non-zero minimal helicities $\frac{q}{2}$, $q \neq 0$.

In Sect. 5 we construct a twistor formulation of the master system. The master HS particle propagates in a space parameterized by a unit twistor and an additional complex scalar. We construct a coordinate twistor transform relating different classical formulations of the master system, as well as a field twistor transform which allows one to reconstruct the space–time HS fields by the twistorial “prepotential” which solves the HS equations. Using twistorial formulation of various HS multiplets, we find the HS algebras associated with them.

In Sect. 6 we summarize our results.
2 A survey of the previously known HS particle models

HS particle related to the unfolded formulation

There are two world–line interpretations of the unfolded formulation of the HS field theory [5]. One of them proceeds from the model of tensorial superparticle [6, 7]. Actually, the latter can be equivalently formulated either in a hyperspace containing a ten–dimensional bosonic subspace alongside an extra commuting Weyl spinor, or in superspace with the Grassmann spinor coordinate (quantization of tensorial superparticle and links of it to an unfolded formulation were also studied in [8, 9]).

There also exists a version of the unfolded formulation which makes no use of the tensorial coordinates at all [5]. In the pure bosonic case the basic equation for the HS field \( \Phi(x, y, \bar{y}) \) [5] reads

\[
\left( \partial_{\alpha\dot{\alpha}} + i \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \bar{y}^{\dot{\alpha}}} \right) \Phi = 0 ,
\]  

\[
(2.1)
\]

where \( y^\alpha \) is a commuting Weyl spinor, \( \bar{y}^{\dot{\alpha}} = (y^\alpha) \). Solution of the unfolded equation (2.1) can be found, assuming the polynomial dependence of the wave function on the commuting spinors \( y^\alpha, \bar{y}^{\dot{\alpha}} \)

\[
\Phi(x, y, \bar{y}) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y^{\alpha_1} \ldots y^{\alpha_m} \bar{y}^{\dot{\alpha}_1} \ldots \bar{y}^{\dot{\alpha}_n} \varphi_{\alpha_1 \ldots \alpha_m \dot{\alpha}_1 \ldots \dot{\alpha}_n}(x) .
\]

\[
(2.2)
\]

Independent space–time fields in the expansion of the general field (2.2) are self–dual \( \varphi_{\alpha_1 \ldots \alpha_m} \) and anti–self–dual \( \varphi_{\dot{\alpha}_1 \ldots \dot{\alpha}_n} \) field strengths of all helicities (including the half–integer ones). Basic unfolded equation (2.1) leads to Klein–Gordon and Dirac equations for them. All other component fields are expressed as \( x \)-derivatives of these basic ones. Reality condition for the HS field \( \Phi(x, y, \bar{y}) = \bar{\Phi}(x, y, \bar{y}) \) leads to the reality conditions \( \varphi_{\dot{\alpha}_1 \ldots \dot{\alpha}_n} = \bar{\varphi}_{\alpha_1 \ldots \alpha_m} \) for physical fields. Thus, the massless HS multiplet described by the real HS field \( \Phi(x, y, \bar{y}) \) contains all helicities, each helicity appearing only once.

A classical counterpart of this unfolded formulation is the particle system with the action [5]

\[
S_1 = \int d\tau \left( \lambda_\alpha \bar{\lambda}_\dot{\alpha} \dot{x}^{\alpha \dot{\alpha}} + \lambda_\alpha \dot{y}^\alpha + \bar{\lambda}_\dot{\alpha} \bar{y}^{\dot{\alpha}} \right).
\]

\[
(2.3)
\]

The spinors \( \lambda_\alpha, \bar{\lambda}_\dot{\alpha} \) are canonical momenta for \( y^\alpha, \bar{y}^{\dot{\alpha}} \). The first class constraints

\[
P_{\alpha\dot{\alpha}} - \lambda_\alpha \bar{\lambda}_\dot{\alpha} \approx 0
\]

\[
(2.4)
\]

after quantization reproduce the unfolded equation (2.1).

Model with an even counterpart of N=1 supersymmetry

A different model of the massless HS particle was proposed in [10]. The action resembles that of the usual massless \( N = 1 \) superparticle

\[
S_2 = \int d\tau \left( P_{\alpha\dot{\alpha}} \omega^{\alpha \dot{\alpha}} - e P_{\alpha\dot{\alpha}} P^{\alpha \dot{\alpha}} \right),
\]

\[
(2.5)
\]

\[
\omega^{\alpha \dot{\alpha}} \equiv \dot{x}^{\alpha \dot{\alpha}} - i \bar{\zeta}^{\dot{\alpha}} \dot{\zeta}^\alpha + i \dot{\bar{\zeta}}^{\dot{\alpha}} \zeta^\alpha .
\]

\[
(2.6)
\]
However, the crucial difference from the superparticle case is that the Weyl spinor $\zeta^\alpha$, $\bar{\zeta}^{\dot{\alpha}} = (\bar{\zeta}^{\dot{\alpha}})$, is commuting. The distinguishing feature of this model is its manifest invariance under the even counterpart of 4D supersymmetry (SUSY) [14, 15, 10]

$$\delta x^{\dot{\alpha}} = i(\epsilon^{\dot{\alpha}} \zeta^\alpha - \bar{\zeta}^{\dot{\alpha}} \epsilon^\alpha), \quad \delta \zeta^\alpha = \epsilon^\alpha, \quad \delta \bar{\zeta}^{\dot{\alpha}} = \bar{\epsilon}^{\dot{\alpha}},$$

where $\epsilon^\alpha$ is a commuting Weyl spinor. The detailed analysis of global symmetries of the model (2.5) was fulfilled in [16]. In particular, the even SUSY translations (2.7) are part of the $SU(3,2)$ group symmetry of the system (2.5).

The set of the Hamiltonian constraints of the system includes the mass-shell constraint $P_{\alpha\dot{\alpha}} P^{\alpha\dot{\alpha}} \approx 0$ and the even spinor constraints

$$D_\alpha \equiv \pi^\alpha + i P_{\alpha\dot{\alpha}} \bar{\zeta}^{\dot{\alpha}} \approx 0, \quad \bar{D}_{\dot{\alpha}} \equiv \bar{\pi}_{\dot{\alpha}} - i \zeta^\alpha P^{\alpha\dot{\alpha}} \approx 0,$$

where $\pi^\alpha$ and $\bar{\pi}_{\dot{\alpha}}$ are conjugate momenta for $\zeta^\alpha$ and $\bar{\zeta}^{\dot{\alpha}}$.

The wave function of the particle model with even “supersymmetry” (2.5) was obtained in [10] (see also [16], where a superextension of the model (2.5) was considered), and it is an even counterpart of chiral $\mathcal{N}=1$ superfield

$$\Psi(x_L, \zeta) = \sum_{n=0}^{\infty} \zeta^{\alpha_1} \ldots \zeta^{\alpha_n} \psi_{\alpha_1 \ldots \alpha_n}(x_L),$$

where $x_L^{\dot{\alpha}} = x^{\dot{\alpha}} + i \bar{\zeta}^{\dot{\alpha}} \zeta^\alpha$. Besides the chirality condition $\bar{D}_{\dot{\alpha}} \Psi = 0$, this field is subjected to the equations\(^1\)

$$\partial_L^{\dot{\alpha} \alpha} \partial_\alpha \Psi = 0, \quad \partial_L^{\dot{\alpha} \alpha} \partial_{L \alpha \dot{\alpha}} \Psi = 0,$$

which are quantum counterparts of the first class constraints. Due to eqs. (2.10) the fields in the expansion (2.9) are complex self–dual field strengths of the massless particles of all helicities. As a result, the spectrum contains all helicities, every non-zero helicity appearing only once. In this picture the scalar field is complex, as opposed to the real scalar field present in the HS field (2.2) of the unfolded formulation.

### 3 Master HS particle model

In a recent paper [11] we proposed a new model of the even HS particle which plays the role of the “master system” both for the particle model (2.3) corresponding to the unfolded formulation and for the model (2.5) with the explicit even “supersymmetry”.

The master system involves the variables of both systems (2.3) and (2.5) and also an additional complex scalar $\eta$ ($\bar{\eta} = \eta^\dagger$). The model is described by the following action

$$S = \int dt \left[ \lambda_\alpha \lambda_\dot{\alpha} \omega^{\dot{\alpha} \alpha} + \lambda_\alpha \dot{y}^\alpha + \lambda_\dot{\alpha} \dot{y}^{\dot{\alpha}} + i(\bar{\eta} \eta - \bar{\eta} \bar{\eta}) - 2i \bar{\eta} \eta \lambda_\alpha + 2i \eta \bar{\lambda}_\dot{\alpha} \dot{\zeta}^{\dot{\alpha}} - l (\mathcal{N} - c) \right].$$

The field $l$ acts as a Lagrange multiplier effecting the constraint

$$\mathcal{N} - c \equiv i (\bar{y}^\alpha \lambda_\alpha - \bar{\lambda}_\dot{\alpha} \dot{y}^{\dot{\alpha}}) - 2\bar{\eta} \eta - c \approx 0,$$

\(^1\)Here and lower we use following notation $\partial_\alpha \equiv \frac{\partial}{\partial \zeta^\alpha}$, $\bar{\partial}_{\dot{\alpha}} \equiv \frac{\partial}{\partial \bar{\zeta}^{\dot{\alpha}}}$. 

3
which generates, in the Hamiltonian formalism, local $U(1)$ transformations of the involved complex fields, except for the fields $\zeta, \bar{\zeta}$.

The action (3.1) produces the following primary constraints

$$T_{\alpha\dot{\alpha}} \equiv P_{a\dot{a}} - \lambda_{a} \bar{\lambda}_{\dot{a}} \approx 0, \quad (3.3)$$

$$\mathcal{D}_{\alpha} \equiv D_{\alpha} + 2i\bar{\eta}\lambda_{\alpha} \approx 0, \quad \bar{\mathcal{D}}_{\dot{\alpha}} \equiv \bar{D}_{\dot{\alpha}} - 2i\eta\bar{\lambda}_{\dot{\alpha}} \approx 0 \quad (3.4)$$

and

$$g \equiv p_{\eta} + i\bar{\eta} \approx 0, \quad \bar{g} \equiv \bar{p}_{\eta} - i\eta \approx 0, \quad (3.5)$$

where $D_{\alpha}$ and $\bar{D}_{\dot{\alpha}}$ are defined in (2.8) and $p_{\eta}$ and $\bar{p}_{\eta}$ are the conjugate momenta for $\eta$ and $\bar{\eta}$, respectively. We at once treat the variables $\lambda_{\alpha}$ and $\bar{\lambda}_{\dot{\alpha}}$ as conjugate momenta for $y^{a}$ and $\bar{y}\dot{a}$.

The constraints (3.5) possess a non-vanishing Poisson bracket:

$$[g, \bar{g}]_{\rho} = 2i. \quad (3.6)$$

So they are second class and can be treated in the strong sense by introducing Dirac brackets for them. Then the complex scalar $\eta$ and its complex conjugate form the canonical pair

$$[\eta, \bar{\eta}]_{D} = \frac{i}{2}. \quad (3.7)$$

Let us now explain in which way the master system is gauge-equivalent to the HS particle systems (2.5) and (2.3).

The systems (2.5) and (3.1) are (classically) equivalent to each other in the common sector of their phase space. This sector is singled out by choosing the definite sign of the energy $P_{0}$ which is fixed due to the constraint (3.3). This equivalence becomes evident if one observes that the system (3.1) can be interpreted as the system (2.5) in which the second class constraints are converted into the first class ones by introducing the new canonical pair $\eta, \bar{\eta}$.

To be more precise, we use the covariant conversion method firstly proposed in [12] for the case of usual superparticle. To convert two second class constraints contained in the spinor constraints (2.8) into first class ones, we introduce two additional degrees of freedom carried by the complex scalar field $\eta$. We also introduce a commuting Weyl spinor $\lambda_{\alpha}$ to ensure the Lorentz covariance of the new spinor constraints (3.4). The closure of the algebra of the new spinor constraints

$$[\mathcal{D}_{\alpha}, \bar{\mathcal{D}}_{\dot{\alpha}}]_{D} = 2iT_{a\dot{a}} \quad (3.8)$$

gives just the constraint (3.3) resolving four-momentum in terms of the spinor product. This resolution is defined up to an arbitrary phase transformation of $\lambda_{a} \bar{\lambda}_{\dot{a}}$. In order to ensure this $U(1)$ gauge invariance in the full modified action, we are led to add the constraint (3.2).

A heuristic argument why this equivalence should hold is that both models, (2.5) and (3.1), have the same number $n_{ph} = 8$ of the physical degrees of freedom. The rigorous proof of the equivalence can be achieved by reducing both systems to the physical degrees of freedom. Namely, choosing light–cone gauge and following the gauge-fixing procedure as in [12], [13], we showed in [11] that the actions of the systems (3.1) and (2.5) written in terms of physical variables coincide with each other in the sector with the definite sign of energy.
The world-line particle model (2.3) also follows from the master model (3.1) under a particular gauge choice. The spinor constraints (3.4) and the gauge-fixing condition $\zeta^{\alpha} \approx 0$ together with its complex conjugate can be used to eliminate the variables $\zeta^{\alpha}$, $\pi_{\alpha}$ and their complex conjugates. Then the constraint (3.2) can be used to gauge away the variable $\eta$. The constraint (3.2) is linear in $\rho \equiv \eta \bar{\eta}$ and generates arbitrary local transformations of $\varphi - i \ln(\eta/\bar{\eta})$. This constraint, together with the gauge fixing condition $\chi \equiv \varphi - const \approx 0$, can be used to eliminate the variables $\rho$, $\varphi$ at expense of the variables $\lambda_{\alpha}$, $y^{\alpha}$, $\bar{\lambda}_{\dot{\alpha}}$, $\bar{y}^{\dot{\alpha}}$. Since the gauge fixing condition includes only $\varphi$, the brackets for the remaining variables are not affected. As a result, we arrive at the system (2.3).

4 First-quantized theory

In the representation in which the operators $\hat{P}_{\alpha \dot{\alpha}}$, $\hat{\pi}_{\alpha}$, $\hat{\pi}_{\dot{\alpha}}$, $\hat{\lambda}_{\alpha}$, $\hat{\bar{\lambda}}_{\dot{\alpha}}$ and $\hat{\bar{\eta}}$ are realized by differential operators, the equations on the wave function $F^{(q)}(x, \zeta, \bar{\zeta}, y, \bar{y}, \eta)$ are

\[
\left( \partial_{\alpha \dot{\alpha}} + i \frac{\partial}{\partial y^{\alpha}} \frac{\partial}{\partial \bar{y}^{\dot{\alpha}}} \right) F^{(q)} = 0, \quad (4.1)
\]

\[
\begin{align*}
(a) & \quad \left( D_{\alpha} + \frac{\partial}{\partial \eta} \frac{\partial}{\partial y^{\alpha}} \right) F^{(q)} = 0, \\
(b) & \quad \left( \bar{D}_{\dot{\alpha}} - 2\eta \frac{\partial}{\partial y^{\alpha}} \right) F^{(q)} = 0, \\
& \quad \left( y^{\alpha} \frac{\partial}{\partial y^{\alpha}} - \bar{y}^{\dot{\alpha}} \frac{\partial}{\partial \bar{y}^{\dot{\alpha}}} - \eta \frac{\partial}{\partial \eta} \right) F^{(q)} = q F^{(q)}.
\end{align*} \quad (4.2) \quad (4.3)
\]

Here, the operators $D_{\alpha} = -i(\partial_{\alpha} + i\partial_{\alpha} \bar{\zeta}^{\alpha})$ and $\bar{D}_{\dot{\alpha}} = -i(\bar{\partial}_{\dot{\alpha}} - i\zeta^{\dot{\alpha}} \partial_{\alpha})$ are quantum counterparts of the “covariant momenta” (2.8).

The external $U(1)$ charge $q$ defined by (4.3) is the quantum counterpart of the “classical” constant $c$ present in the constraint (3.2), now with the ordering ambiguities taken into account. Eq. (4.3) implies $U(1)$ covariance of the wave function

\[
F^{(q)}(x, \zeta, \bar{\zeta}, e^{i\varphi} y, e^{-i\varphi} \bar{y}, e^{-i\varphi} \eta) = e^{q \varphi} F^{(q)}(x, \zeta, \bar{\zeta}, y, \bar{y}, \eta). \quad (4.4)
\]

Requiring $F^{(q)}$ to be single-valued restricts $q$ to the integer values. It is important to note that this $U(1)$ covariance just implies that any monomial of the charged coordinates in the series expansion of the wave function $F^{(q)}$ has the same fixed charge $q$ and it does not entail any $U(1)$ transformation of the coefficients fields. In this respect $q$ resembles the “harmonic $U(1)$ charge” of the harmonic superspace approach [17, 18] and the operator in (4.3) is an analog of the $U(1)$ charge-counting operator $D^{0}$ in this approach.

Though the system of eqs. (4.1)–(4.3) involves the Vasiliev-type unfolded vector equation (4.1), the latter follows from the spinorial eqs. (4.2) as their integrability condition. So the basic independent equations of the system (4.1)–(4.3) are the bosonic spinorial equations (4.2) and the $U(1)$ condition (4.3).

Let us solve eqs. (4.2)–(4.3). In the variables $\zeta^{\alpha}$, $\bar{\zeta}^{\dot{\alpha}}$, $y^{\alpha}$, $\eta$ and

\[
x^{\alpha}_{\zeta} = x^{\dot{\alpha}} + i \zeta^{\dot{\alpha}} \zeta^{\alpha}, \quad \bar{y}^{\dot{\alpha}} = \bar{y}^{\dot{\alpha}} + 2i \eta \zeta^{\dot{\alpha}}.
\]

eqs. (4.2)–(4.3) take the form

\[
\begin{align*}
(a) & \quad \left[ -i \left( \partial_{\alpha} + i \frac{\partial}{\partial \eta} \frac{\partial}{\partial y^{\alpha}} \right) + 2 \zeta^{\dot{\alpha}} \left( \partial_{\alpha \dot{\alpha}} + i \frac{\partial}{\partial y^{\alpha}} \frac{\partial}{\partial \bar{y}^{\dot{\alpha}}} \right) \right] F^{(q)} = 0, \\
(b) & \quad \bar{\partial}_{\dot{\alpha}} F^{(q)} = 0.
\end{align*} \quad (4.6)
\]
\[
\left( y^\alpha \frac{\partial}{\partial y^\alpha} - \tilde{y}_L^\alpha \frac{\partial}{\partial \tilde{y}_L^\alpha} - \eta \frac{\partial}{\partial \eta} \right) F^{(q)} = q F^{(q)} .
\]  

(4.7)

Eq. (4.6b) is the bosonic chirality condition stating that \( F^{(q)} \) does not depend on \( \tilde{\zeta}^\alpha \) in the new variables, \( F^{(q)} = F^{(q)}(x_L, \zeta, y, \tilde{y}_L, \eta) \). Then eq. (4.6a) amounts to the equations

\[
\left( \partial_\alpha + i \frac{\partial}{\partial \eta} \frac{\partial}{\partial y^\alpha} \right) F^{(q)} = 0
\]

(4.8)

and

\[
\left( \partial_{\alpha\dot{\alpha}} + i \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \tilde{y}_L^\alpha} \right) F^{(q)} = 0 .
\]

(4.9)

The solutions of eqs. (4.7), (4.8) and (4.9) can be obtained in several equivalent ways.

**The unfolded-type description**

By analogy with (2.9) we assume the polynomial dependence of wave function on \( \zeta^\alpha \)

\[
F^{(q)}(x_L, \zeta, y, \tilde{y}_L, \eta) = \sum_{n=0}^{\infty} \zeta^{\alpha_1} \ldots \zeta^{\alpha_n} \Phi_{\alpha_1 \ldots \alpha_n}^{(q)}(x_L, y, \tilde{y}_L, \eta) .
\]

(4.10)

Eq. (4.8) expresses all the coefficients in this expansion as derivatives of the first coefficient \( \Phi^{(q)}(x_L, y, \tilde{y}_L, \eta) \) satisfying the two equations

(a) \( \left( \partial_{\alpha\dot{\alpha}} + i \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial \tilde{y}_L^\alpha} \right) \Phi^{(q)} = 0 \),  

(b) \( \left( y^\alpha \frac{\partial}{\partial y^\alpha} - \tilde{y}_L^\alpha \frac{\partial}{\partial \tilde{y}_L^\alpha} - \eta \frac{\partial}{\partial \eta} \right) \Phi^{(q)} = q \Phi^{(q)} .
\]

(4.11)

Like in the previous cases, we assume that \( \Phi^{(q)}(x_L, y, \tilde{y}_L, \eta) \) has a non-singular polynomial expansion over the additional coordinates. Then eq. (4.11b) implies

\[
\Phi^{(q)}(x_L, y, \tilde{y}_L, \eta) = \sum_{k=0}^{\infty} \eta^k \varphi^{(q+k)}(x_L, y, \tilde{y}_L) ,
\]

(4.12)

and

\[
\left( y^\alpha \frac{\partial}{\partial y^\alpha} - \tilde{y}_L^\alpha \frac{\partial}{\partial \tilde{y}_L^\alpha} \right) \varphi^{(q+k)} = (q + k) \varphi^{(q+k)} .
\]

(4.13)

The reduced \( U(1) \) condition (4.13) fixes the \( y, \tilde{y} \) dependence of the functions \( \varphi^{(q+k)} \) as

\[
\varphi^{(q+k)}(x_L, y, \tilde{y}_L) = \begin{cases} 
\sum_{n=0}^{\infty} y^{\alpha_1} \ldots y^{\alpha_q+k+n} \tilde{y}_L^{\beta_1} \ldots \tilde{y}_L^{\beta_n} \phi_{\alpha_1 \ldots \alpha_q+k+n}^{\beta_1 \ldots \beta_n}(x_L), & (q + k) \geq 0, \\
\sum_{n=0}^{\infty} y^{\alpha_1} \ldots y^{\alpha_n} \tilde{y}_L^{\beta_1} \ldots \tilde{y}_L^{\beta_{q+k+n}} \phi_{\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_{q+k+n}}(x_L), & (q + k) < 0.
\end{cases}
\]

(4.14)

It remains to find the restrictions imposed on the fields \( \varphi^{(q+k)} \) by the remaining unfolded equation (4.11a).

It is easy to see that in the case \( q = 0 \) eq. (4.11a) expresses all the fields \( \phi_{\alpha_1 \ldots \alpha_k}^{\beta_1 \ldots \beta_n} \) with \( n > 0 \) in \( \varphi^{(k)} \) as \( x \)-derivatives of the lowest component, the self–dual field \( \phi_{\alpha_1 \ldots \alpha_k} \). The latter field satisfies Dirac and Klein–Gordon equations

\[
\partial^{\beta_{\alpha_1}} \phi_{\alpha_1 \ldots \alpha_k} = 0 , \quad \partial^{\beta_{\alpha}} \partial_{\alpha\dot{\alpha}} \phi = 0
\]

(4.15)
also as a consequence of the same eq. (4.11a). Thus the space of physical states of the model is spanned by the complex self–dual field strengths $\phi_{\alpha_1...\alpha_k}$, $k = 0, 1, 2, \ldots$, of the massless particles of all integer and half-integer helicities, and the case of $q = 0$ basically amounts to the standard HS multiplet of ref. [5].

Like in the $q = 0$ case, for $q > 0$ eq. (4.11a) expresses the fields $\phi_{\alpha_1...\alpha_{q+k-1}}\beta_1...\beta_n$ with $n > 0$ in (4.14) in terms of the $\partial_{\alpha_\eta}$-derivatives of the self-dual fields $\phi_{\alpha_1...\alpha_{q+k}}$. Also, the same eq. (4.9) yields Dirac equations for the independent fields

$$\partial^{\beta\alpha_1}\phi_{\alpha_1...\alpha_{q+k}} = 0 , \quad k = 0, 1, 2, \ldots \quad (4.16)$$

Thus the space of physical states of the model is spanned by the self–dual field strengths of the massless particles with helicities $\frac{q}{2}, \frac{q}{2} + \frac{1}{2}, \frac{q}{2} + 1, \ldots$. We observe that the scalar field is absent in the spectrum for non-zero positive $q$. The relevant HS multiplet is fully characterized by the value of $q$.

For $q < 0$ the expansion (4.12) can be conveniently rewritten as

$$\Phi^{(q)}(x_L, y, \bar{y}_L, \eta) = \eta^{[q]} \bar{\Phi}^{(0)}(x_L, y, \bar{y}_L, \eta) +$$

$$+ \sum_{|q|} \sum_{k=1}^{\infty} \eta^{[q]-k} y^{\alpha_1} \ldots y^{\alpha_m} \bar{y}_L^{\beta_1} \ldots \bar{y}_L^{\beta_{m+k}} \phi_{\alpha_1...\alpha_m\beta_1...\beta_{m+k}}(x_L). \quad (4.17)$$

Thus in this case we deal with the HS field $\bar{\Phi}^{(0)}$ having the same helicity contents as the $q = 0$ multiplet and an additional term involving the space–time fields $\phi_{\alpha_1...\alpha_m\beta_1...\beta_{m+k}}$. The set of independent fields in this expansion consists of self–dual fields $\phi_{\alpha_1...\alpha_k}$, $k = 0, 1, 2, \ldots$ present in the HS field $\Phi^{(0)}$ and anti–self–dual fields $\phi_{\beta_1...\beta_k}$, $k = 1, \ldots, |q|$. Eq. (4.11a) expresses all other space–time fields as space–time derivatives of these basic ones. Eq. (4.11a) also implies the Dirac and Klein–Gordon equations for the basic fields. Thus, for $q < 0$ physical fields in the spectrum describe massless particles with positive helicities starting from the zero one, and also a finite number of massless states with negative helicities $-\frac{1}{2}, -1, \ldots, -\frac{|q|}{2}$. This HS multiplet can be naturally called “helicity-flip” multiplet. Note that, being considered together with its conjugate, this multiplet reveals a partial doubling of fields with a given helicity, the phenomenon which is absent in the previous two cases.

The description with explicit even SUSY

We start with the case $q = 0$ and consider at first the expansion of wave function with respect to the coordinates $y$, $\bar{y}_k$ and $\eta$

$$F^{(0)}(x_L, \zeta, y, \bar{y}_L, \eta) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \eta^{k} y^{\alpha_1} \ldots y^{\alpha_{k+n}} \bar{y}_L^{\beta_1} \ldots \bar{y}_L^{\beta_n} \Psi_{\alpha_1...\alpha_{k+n}\beta_1...\beta_n}(x_L, \zeta), \quad (4.18)$$

where we have already taken care of eq. (4.7). It remains to take into account eqs. (4.8) and (4.9). They express all the coefficient fields in (4.18) as derivatives of the lowest coefficient $\Psi(x_L, \zeta)$ which is exactly the chiral HS field (2.9) of ref. [10]. It comprises the same irreducible HS $q = 0$ multiplet as $\Phi^{(0)}(x_L, y, \bar{y}_k, \eta)$. The possibility to describe the same multiplet in two equivalent ways is just the quantum manifestation of the equivalence between the HS particles (2.3) and (2.5) which both follow from the master HS particle. In a sense, the description by the field $\Psi(x_L, \zeta)$ is more economical since this quantity contains, in its $\zeta^\alpha$
expansion, just the independent space-time self-dual fields (anti-self-dual fields are collected by the complex–conjugated function $\bar{\Psi}(x_\alpha, \zeta)$.

The $q > 0$ counterpart of the $q = 0$ expansion is

$$F^{(q)}(x_L, \zeta, y, \bar{y}_L, \eta) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \eta^k y^{\alpha_1} \cdots y^{\alpha_{n+k}} \bar{y}_L^{\alpha_1} \cdots \bar{y}_L^{\alpha_n} \Psi_{\alpha_1 \cdots \alpha_{n+k} \beta_1 \cdots \beta_n}(x_L, \zeta),$$

where we have already taken into account the $U(1)$ condition (4.7). Then eqs. (4.8) and (4.9) lead to the expressions for all component fields in terms of the even counterpart of $N = 1$ chiral field with external indices $\Psi_{\alpha_1 \cdots \alpha_q}(x_L, \zeta)$. As a consequence of eqs. (4.8) and (4.9), this field is subjected to the equations

$$\partial^\alpha L \Psi_{\alpha_1 \cdots \alpha_q} = 0, \quad \bar{\partial}^{\dot{\alpha}} L \Psi_{\alpha_1 \cdots \alpha_q} = 0,$$

Expanding the field $\Psi_{\alpha_1 \cdots \alpha_q}$ in powers of $\zeta$,

$$\Psi_{\alpha_1 \cdots \alpha_q}(x_L, \zeta) = \sum_{n=0}^{\infty} \zeta^{\beta_1} \cdots \zeta^{\beta_n} \psi_{\alpha_1 \cdots \alpha_q \beta_1 \cdots \beta_n}(x_L),$$

we observe that all component fields in this expansion are totally symmetric in the spinor indices due to the first equation in (4.21). As a consequence of eqs. (4.20) and the second equation in (4.21), all component fields satisfy Dirac equation. Therefore, the HS field (4.22) describes the same physical spectrum as in (4.16).

5 Twistorial formulation of the master HS particle

Action of the master HS particle in twistor formulation

The twistorial formulation of the master HS particle (3.1) was constructed in [11] and is a HS generalization of the well–known twistor formulation of massless particles with fixed helicities. It is described by two Weyl spinors $\lambda_\alpha$ and $\bar{\mu}^{\dot{\alpha}}$ and a complex scalar $\xi$ which are introduced by the following twistor transform

$$\mu^\alpha = y^\alpha + \bar{\lambda}_\beta (x^{\dot{\beta}} - i \bar{\zeta}^{\dot{\beta}} \zeta^\alpha) - 2i \bar{\eta} \zeta^\alpha, \quad \bar{\mu}^{\dot{\alpha}} = \bar{y}^{\dot{\alpha}} + (x^{\dot{\alpha}} + i \bar{\xi}^{\dot{\alpha}} \zeta^\beta) \lambda_\beta + 2i \eta \zeta^\beta,$$

$$\xi = \eta + \bar{\zeta}^{\dot{\alpha}} \lambda_\beta, \quad \bar{\xi} = \bar{\eta} + \bar{\lambda}_\beta \bar{\zeta}^{\dot{\beta}}.$$

The spinors $\lambda_\alpha$, $\bar{\mu}^{\dot{\alpha}}$ are the components of the twistor ($SU(2,2)$ spinor) $Z_\alpha = (\lambda_\alpha, \bar{\mu}^{\dot{\alpha}})$, $\alpha = 1, \ldots, 4$, in the basis where it splits into irreps of the Lorentz group $SL(2, C)$ and dilatations $SO(1,1)$.

Up to some boundary terms, the action (3.1) takes the following form in the twistorial variables

$$S^{HS-tw.} = \int d\tau \left[ \lambda_\alpha \dot{\mu}^\alpha + \bar{\lambda}_\dot{\alpha} \dot{\bar{\mu}}^{\dot{\alpha}} + i(\dot{\xi} \bar{\zeta} - \bar{\xi} \dot{\zeta}) - l (U - c) \right].$$

Here, the $U(1)$ constraint

$$U - c \equiv i(\mu^\alpha \lambda_\alpha - \bar{\lambda}_\dot{\alpha} \bar{\mu}^{\dot{\alpha}}) - 2 \xi \bar{\xi} - c \approx 0$$

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is the condition (3.2) rewritten in the twistorial variables (5.1), (5.2).

The twistorial formulation (5.3) of the HS particle reproduces the twistorial HS particle which was considered in [6, 7]. Due to the constraint (5.4), one can gauge away the variables $\xi, \bar{\xi}$ as in Section III. As a result, we obtain the system described by the action of ref. [6, 7]

$$S_{1}^{HS-tw.} = \int d\tau \left( \lambda_{a} \dot{\mu}^{a} + \bar{\lambda}_{a} \dot{\bar{\mu}}^{a} \right).$$

(5.5)

The twistor system (5.5) can be also obtained directly from the system (2.3) via the following standard twistor transform

$$\mu^{a} = y^{a} + \bar{\lambda}_{\beta} \dot{x}^{\beta a}, \quad \bar{\mu}^{a} = \bar{y}^{a} + x^{\alpha \beta} \lambda_{\beta}.$$

This provides one more proof of the equivalence of the systems (2.3) and (3.1).

Quantization in twistor formulation: twistor transform for HS fields

Quantization in the twistorial formulation (5.3) gives rise to the same results as in Section 4. In the “twistorial representation”, when $\lambda_{a}, \bar{\mu}^{a}$ and $\xi$ are diagonal, the twistorial wave function $G^{(q-2)}(\lambda, \bar{\mu}, \xi)$ satisfies the quantum counterpart of the constraints (5.4)

$$\left( -\lambda_{a} \frac{\partial}{\partial \lambda_{a}} - \bar{\mu}^{a} \frac{\partial}{\partial \bar{\mu}^{a}} - \xi \frac{\partial}{\partial \xi} \right) G^{(q-2)} = (q - 2) G^{(q-2)}.$$  

(5.6)

The wave function in the space–time description is derived by analogy with the standard twistor approach [19]. One substitutes the incidence conditions (5.1) and (5.2) for the variables $\bar{\mu}^{a}$ and $\xi$ in the twistorial wave function and performs a Fourier-type integral transformation from $\lambda_{a}$ to its canonically conjugated variable

$$F^{(q)}(x_{L}, \zeta, y, \bar{y}_{L}, \eta) = \int d^{2} \lambda e^{iy^{a} \lambda_{a}} G^{(q-2)}(\lambda_{a}, \bar{y}_{L}^{a} + x^{(a \beta} \lambda_{\beta}, \eta + \zeta^{\beta} \lambda_{\beta}) \cdot$$

(5.7)

Note that the variables $x_{L}$ and $\bar{y}_{L}$ defined in (4.5) already appeared in the twistor transform (5.1) for $\bar{\mu}$. The integrand in (5.7) includes the Fourier exponent in contrast to the Penrose integral transform [19].

Using the particular dependence of the twistorial field $G^{(q-2)}$ on the involved co-ordinates, it is easy to check that the field $F^{(q)}$ defined by (5.7) automatically satisfies eqs. (4.7), (4.8) and (4.9). Thus, the twistorial formulation solves eqs. (4.7), (4.8) and (4.9) in terms of the unconstrained “prepotential” $G^{(q-2)}(\lambda, \bar{\mu}, \xi)$.

Symmetries of HS multiplets

The symmetry analysis is direct in the twistor formulation in which different HS multiplets labelled by the $U(1)$ charge $q$ are specified by the single eq. (5.6). We can find the symmetries following the techniques exploited in ref. [5]. The HS fields depend on the twistor variables $Z_{a} = (\lambda_{a}, \bar{\mu}^{a})$, $a = 1, ..., 4$, and complex scalar $\xi$. The symmetry generators are products of the $Z_{a}$, $\xi$ monomials of arbitrary degree and those of the derivatives $\frac{\partial}{\partial Z_{a}}, \frac{\partial}{\partial \xi}$, such that they preserve eq. (5.6) rewritten in the form

$$\left( \hat{U} - \hat{q} \right) G^{(\hat{q})} = 0,$$

(5.8)

where

$$\hat{U} \equiv -Z_{a} \frac{\partial}{\partial Z_{a}} - \xi \frac{\partial}{\partial \xi}, \quad \hat{q} \equiv q - 2.$$  

(5.9)
Let us consider the generators

\[ T^{(N)}(Z, \xi) \equiv T^{(N;K)}(Z, \xi) = T^{(N)}(Z) \cdot T^{(K)}(\xi), \quad N = N + K, \quad (5.10) \]

where the quantities

\[ T^{(N)}(Z) \equiv T^{(n,m)}(Z) = Z_{a_1} \cdots Z_{a_n} \frac{\partial}{\partial Z_{b_1}} \cdots \frac{\partial}{\partial Z_{b_m}}, \quad N = n + m \quad (5.11) \]

act in the twistor sector, whereas the quantities

\[ T^{(K)}(\xi) \equiv T^{(k,l)}(\xi) = \xi^k \frac{\partial}{\partial \xi^l}, \quad K = k + l \quad (5.12) \]

act on the scalar \( \xi \). The generators (5.10) form an infinite–dimensional algebra (modulo some coefficients)

\[ [T^{(N)}, T^{(M)}] = \sum_{\ell=0}^{N+M-2} T^{(\ell)}. \quad (5.13) \]

The symmetry algebra of the physical states described by HS field \( G(\hat{q}) \) is formed by the generators \( F^{(n,m;k,l)} \) from (5.10) commuting with the operator (5.9):

\[ [F^{(n,m;k,l)}, \hat{U}] = 0. \quad (5.14) \]

Using \([T^{(n,m;k,l)}, \hat{U}] = (n + k - m - l) T^{(n,m;k,l)}\) we find that \( F^{(n,m;k,l)} \) are the generators (5.10) with \( n + k = m + l \). We denote this infinite–dimensional algebra by \( hsc(3,2) \) (with \( hsc \) for higher spin conformal) to emphasize the presence of the finite-dimensional subalgebra \( u(3,2) \) with the generators

\[ F^a = Z_a \frac{\partial}{\partial Z^b}, \quad F_a = Z_a \frac{\partial}{\partial \xi^b}, \quad F^b = \xi^b \frac{\partial}{\partial Z^a}, \quad F = \xi \frac{\partial}{\partial \xi}. \quad (5.15) \]

The generators \( F^a \) form \( u(2,2) \) algebra which is an extension of the conformal algebra \( su(2,2) \) by the generator of phase transformations (5.9) of twistor. The operators \( F_a \) and \( F^b \) generate bosonic supersymmetry translations and bosonic superconformal boosts [16]. The subalgebra \( hsc(2,2) \) formed by the pure twistorial generators (5.11) with \( n = m \) produces higher spin algebras explored in [3, 4, 5].

The algebra \( hsc(3,2) \) is not simple since it contains ideals \( I_\hat{q} \) spanned by the elements of the form

\[ H^{(n,m;k,l)} = (\hat{U} - \hat{q}) F^{(n,m;k,l)}. \quad (5.16) \]

However, the operators (5.16) become trivial on the HS multiplet \( G(\hat{q}) \). Therefore \( G(\hat{q}) \) is associated with the quotient algebra \( hsc_\hat{q}(3,2) = hsc(3,2)/I_\hat{q} \). Different \( \hat{q} \) correspond to different quotients of \( hsc(3,2) \) associated with different HS multiplets. So the quotient algebras \( hsc_\hat{q}(3,2) \) play the role of “primary” symmetry algebras for the HS multiplets \( G(\hat{q}) \) considered as their moduli. Here we do not discuss other symmetries of HS multiplets which are hidden in the twistorial formulation (see a comment in [11]).

\[ ^2\text{For definiteness, we use the } \hat{Q}\hat{P} \text{–ordering with respect to } Z_a, \xi \text{ and their “momenta”}. \]
6 Summary

In this paper we have described a new HS particle model. This model is unifying ("master") for the unfolded HS particle and the HS particle with the even "supersymmetry", and it yields both these models upon choosing the appropriate gauges.

After quantization, the unfolded formulation of HS fields and their formulation with the explicit bosonic "supersymmetry" are equivalent to each other as they correspond to different ways of solving the same master system of HS equations. One of the novel features of this system is that the infinite towers of higher spins in the quantum spectrum are accommodated by some holomorphic functions depending on a new scalar complex bosonic variable $\eta$. These functions are characterized by the "external" $U(1)$ charge number $q$ which fully specifies the corresponding infinite-dimensional multiplet of spins. The HS fields respect a local $U(1)$ symmetry which is similar to the $U(1)$ covariance of the harmonic approach [17, 18]. Crucial for maintaining this covariance is the holomorphic dependence of the HS wave function on the complex coordinate $\eta$.

Depending on their external $U(1)$ charge $q$, the HS fields in the extended space accommodate different HS multiplets of ordinary 4D fields. The all-helicity HS multiplet of the unfolded formulation (with a complex scalar field) is recovered as the $q = 0$ multiplet and its conjugate. Also, some new HS multiplets with $q \neq 0$ emerge. For $q > 0$ they are spanned by the self-dual field strengths of growing positive helicities, starting from $\frac{q}{2}$. The $q < 0$ multiplets show up an interesting "spin–flip" feature: they include self-dual fields of all positive helicities, as well as a finite number of anti-self-dual fields with negative helicities. The complementary helicities are accommodated by the complex conjugate wave functions.

Finally, let us note that the master model presented here can play an important role in supersymmetric extensions of HS theory (i.e. those with standard "odd" supersymmetries). As shown in [16], the supersymmetric $N = 1$ HS theories constructed as extensions of theories with bosonic "supersymmetry" are the simplest HS theories respecting the fundamental notion of chirality which underlies the geometric approach to the ordinary $N = 1$ supergravity [20]. The existence of a chiral limit seems to be crucial for any satisfactory HS superfield theory including HS generalizations of $N = 1$ supergravity [9, 21]. Odd–supersymmetric extensions of the master model could provide further insights into these and related issues.

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