A G-version of Smale’s theorem

Imre Major

ABSTRACT

We will prove the equivariant version of Smale’s transversality theorem: suppose that the compact Lie-group $G$ acts on the compact differentiable manifold $M$ on which an invariant Morse-function $f$ and an invariant vector field $X$ are given so that $X$ is gradient-like with respect to $f$ (i.e. $X(f) < 0$ away from critical orbits and $X$ is the gradient of $f$ w.r.t. a fixed invariant Riemannian metric) on some invariant open subsets about critical orbits of $f$.) Given a bound $\varepsilon > 0$ we will prove the existence of an invariant vector field $Y$ of class $C^1$ for which vector field $X + Y$ is also gradient-like such that:

(a) $\|Y\|_1 < \varepsilon$ ($\|\cdot\|_1$ is the $C^1$ norm).

(b) The intersection of the stable and unstable sets of vector field $X + Y$ taken at a pair of critical orbits of $f$ is transverse when restricted to an orbit type of the action.

Key words: Group actions, transversality.

1 Introduction, basic concepts

Suppose, that compact Lie group $G$ acts on the compact orientable smooth manifold $M^m$ and let $f : M \to \mathbb{R}$ be an invariant function (i.e. $f(gx) = f(x)$ ($g \in G$)). Fix an invariant metric $g$. If an orbit contains a critical point of $f$ then, by invariance, all points of this orbit are critical points and the orbit itself is called a critical orbit of $f$. Fix an invariant Riemannian metric $g$ and at a point $p \in M$ let

$$\perp_p := (T_pG(p))^\perp$$

be the perpendicular complement of the tangent space to the orbit through point $p$ and let

$$U_p := \exp_p(\perp_p(\varepsilon))$$

where $\perp_p(\varepsilon)$ is the $\varepsilon$-disk about the origin in $\perp_p$ on which the exponential map of the Levi-Civita connection of metric $g$ is injective. Then a critical point $x$ of $f$ is also a critical point of $f|_{U_p}$. We say that the $G$-invariant function $f$ is a $G$-Morse function if the Hessian of $f|_{U_p}$ at $x$ is non-degenerate for each critical point $x$. This property does not depend on the choice of metric $g$ (see e.g. Wasserman [7]). Non-degeneracy of the Hessian also ensures that each critical orbit has an invariant...
neighborhood (called tube about the orbit) that does not contain any other critical orbits. We can suppose, that $GU_x$ is such an invariant neighborhood.

The induced action of the isotropy subgroup $G_p$ at a point $p \in M$:

$$G_p \times T_pM/T_pG(p) \rightarrow T_pM/T_pG(p)$$

on the normal space is called the normal action (see e.g. Bredon [1]).

$$pr : M \rightarrow M/G, \quad p \mapsto G(p)$$

is the canonical projection. For a set $N \subset M$ we use notation $N := pr(N)$.

Observe that the relation $$x \sim y \Leftrightarrow \exists (g \in G) \ni G_x = gG_yg^{-1} \quad (x, y \in M)$$ is an equivalence relation. The equivalence classes provide a partition

$$M = \bigcup_{\alpha \in \mathcal{A}} M_{\alpha}$$

of $M$. The index set $\mathcal{A}$ is the set of conjugacy classes of isotropy subgroups of $G$. It is partially ordered by relation $$\alpha < \beta \Leftrightarrow \forall (x \in M_{\alpha}) \exists y \in M_{\beta} \ni G_y \subset G_x$$

(the property on the RHS does not depend on the choice of representative $x \in M_{\alpha}$).

An index $\alpha \in \mathcal{A}$ (or submanifold $M_{\alpha}$) is called an orbit-type and we say that a point $p$ is of type $\alpha$ if $p \in M_{\alpha}$ (in notation, $[p] = \alpha$). Note, that $M_{\alpha} \subset M$ is a $G$-invariant subset. Theorem 4. of Chapter 3. in [1] implies that $M_{\alpha}$ and $M_{\beta}$ are smooth manifolds and the partition in formula (1) is locally finite, thus $\mathcal{A}$ is finite when $M$ is compact (cf. 4.3 Theorem, pg 187 and 10.4 Theorem, pg. 220 in [1]). Let $m_{\alpha} := \dim(M_{\alpha})$, $m_{\perp}^{\alpha} := \codim(M_{\alpha})$ and $o_{\alpha} := \dim(G/G_p) \ (p \in M_{\alpha})$. Notice that then $\dim(M) := m = m_{\alpha} + m_{\perp}^{\alpha} + o_{\alpha}$. An $\alpha$-slice is a disk $D \subset M_{\alpha}$ which intersects an orbit at most once and along which the isotropy subgroup is constant, moreover the union of orbits $GD$ is open in $M_{\alpha}$.

For an arbitrary subset $Q \subset M$

$$Q_{\alpha} := Q \cap M_{\alpha}$$

is the $\alpha$-part of $Q$.

The set of differentials of left translations $\{dL_g \mid g \in G\}$ act on the tangent bundle $TM$. A vector field $X$ is invariant under this action (called an invariant vector field) iff its flow is an equivariant flow (i.e. for trajectory $\lambda_p$ of vector field $X$ through point $p$

$$L_y(\lambda_p) = \lambda_{gy}$$

holds.) This implies that the isotropy group is constant along trajectories, in particular, an invariant vector field is tangent to the orbit types.

**Definition 1.** An invariant vector field $X$ is gradient-like for the $G$-Morse function $f$ if:

(i) Each critical orbit has an invariant neighborhood $U$ such that

$$X|_U = -\nabla G_{f}|_U$$

(ii) $X(f) < 0$ away from critical orbits.
Definition 2. At a critical orbit $O$:

$$W^-_O = \{ p \in M | \lim_{t \to -\infty} \lambda_p(t) \in O \}$$

is called the unstable set and

$$W^+_O = \{ p \in M | \lim_{t \to +\infty} \lambda_p(t) \in O \}$$

is the stable set of the flow (of invariant vector field $X$.) (Notations $W^-_O$, $W^+_O$ are also used in the literature).

Definition 3. A Morse chart about critical orbit $O$ is given by:

(i) A splitting of the normal bundle $\perp_O$ of $O$ into two invariant orthogonal subbundles $\perp_O = \perp^-_O \oplus \perp^+_O$.

(ii) An equivariant diffeomorphism $\eta_O : \perp_O(\epsilon) \to U_O$ from the $\epsilon$-disc bundle of the normal bundle of $O$ (with constant $\epsilon$) onto an invariant open neighborhood $U_O$ of $O$ ($\eta_O$ is the identity on the zero section $O$ of $\perp_O$) such that

$$f \circ \eta_O = -\|P^-\|^2 + \|P^+\|^2 + f(O)$$

where ($P^-, P^+$) : $\perp_O \to (\perp^-, \perp^+_O)$ are the projections that belong to the decomposition in (i). The open set $U_O$ is called a Morse-tube.

Observe that $\perp^-_O \to O$ and $\perp^+_O \to O$ are $G$-vector bundles induced from the orthogonal representations on the Euclidean spaces $\perp^- := \perp \cap \perp_O^-$, $\perp^+ := \perp \cap \perp_O^+$.

The restriction $g|_{U_O}$ is a scalar product on vector bundle $\perp_O$, thus it defines a Riemannian metric $\langle , \rangle$ on $\perp_O$ in the canonical way. The push-forward $\eta_O_*\langle , \rangle$ of this Riemannian metric along map $\eta_O$ can be patched together with original metric $g$ by using an invariant cutoff function. Thus we can presume that for the restrictions we have:

$$g|_{U_O} = \eta_O_*\langle , \rangle|_{U_O}$$

Lemma: (Equivariant Morse Lemma) Let $f : M \to \mathbb{R}$ be a $G$-Morse function on a Riemannian $G$-manifold. Then there is a Morse chart about each critical orbit (see Wasserman [1]).

Note that by the above lemma the stable and unstable sets are, in fact, invariant submanifolds of $M$. Let $O_1, \ldots, O_K$ be the set of critical orbits of $f$, and abbreviate $W^+_j := W^+_O$, etc.

Definition 4. The gradient-like vector field $X$ (or its flow $\Lambda : M \times \mathbb{R} \to M$) is $G$-Morse-Smale if it is of class $C^1$ moreover $W^+_j \cap M_\alpha$ and $W^-_k \cap M_\alpha$ intersect transversely as submanifolds of $M_\alpha$ for each choice of $\alpha \in \mathcal{A}$, $1 \leq j, k \leq K$. (We refer to this property as relative transversality of stable and unstable submanifolds, or $\alpha$-transversality, when the orbit type $\alpha$ is fixed.)

As we wish to perturb a given gradient-like vector field by an invariant vector field (the flow of which thus keeps orbit types), the above definition seems to be the only plausible one for the $G$-version of Morse-Smale property.

Theorem: For any given $\varepsilon > 0$ an invariant gradient-like vector field $X$ can be approximated by a $G$-Morse-Smale vector field $X'$, which is also gradient-like for $f$ such that $\|X - X'\|_1 < \varepsilon$. 

G-SMALE THEOREM
2 Proof of the Theorem

Fix an invariant gradient-like vector field $X$ for $G$-Morse function $f$ on Riemannian $G$-manifold $(M, g)$ (e.g. $-\text{grad}_g(f)$ is such a vector field). In order to ensure that the perturbed vector field is $C^1$-close to the original one, we need a family of special coordinate charts. Invariant charts would serve our purpose the best, however, we might not be able to arrange such charts with invariant domains (by compactness of group $G$, the domain of such a chart can’t be an $m$-ball).

Remark: As we will perturb vector field $X$ within the Morse-tubes (which we have already fixed), we could measure the $C^1$ norm of the perturbing vector field with respect to the Morse-charts (and in some sense, this is what will happen). Still, for completeness of our discussion we wish to include the passage below, in which we introduce a special finite set of charts that cover $M$.

As the boundary of an orbit type $M_\alpha$ is the union of some lower dimensional orbit types that precede $\alpha$ with respect to $\prec$, a level can be associated to each orbit type: the closed ones being at level 0 and inductively, at level $i > 0$ we have the orbit types such that their boundaries contain orbit types of level at most $i-1$ and contains at least one orbit type at level $i-1$. (The level is also called the depth in the literature.)

For each orbit type fix a representative isotropy subgroup $G_\alpha = G_{p_\alpha}$ (where $p_\alpha \in M_\alpha$), a coordinate chart $(E_\alpha, (\tilde{x}_\alpha^{m_\alpha + m_\alpha + 1}, \ldots, \tilde{x}_\alpha^m))$ about the unit element $G_\alpha$ of quotient group $G/G_\alpha$, an open neighborhood $E'_\alpha \subset G/G_\alpha$ of $G_\alpha$ such that $E'_\alpha \subset E_\alpha$ and elements $g_\alpha^1, \ldots, g_\alpha^n \in G$ such that the translates

$$g_\alpha^1 E'_\alpha, \ldots, g_\alpha^n E'_\alpha$$

cover $G/G_\alpha$. Note that each orbit of type $\alpha$ contains a point with isotropy subgroup $G_\alpha$. Let $N_\alpha \to M_\alpha$ be the normal bundle of orbit type $M_\alpha$.

It is a well known fact that $G$ is a principal bundle over quotient group $G/G_\alpha$ by the canonical projection $G \to G/G_\alpha$ with structure group $G_\alpha$, so we can fix a section $\sigma_\alpha : E_\alpha \to G$, $\sigma_\alpha(G_\alpha) = e$ of this bundle over contractible neighborhood $E_\alpha$ (i.e. $\sigma_\alpha(gG_\alpha)G_\alpha = gG_\alpha$).

Definition 5. A coordinate chart $(U, (x^1, \ldots, x^m))$ is adapted to orbit type $\alpha$ if there exists a relatively compact $\alpha$-slice $U^*_\alpha \subset M_\alpha$ with isotropy subgroup $G_\alpha$, a coordinate chart $(U^*_\alpha, x^1_\alpha, \ldots, x^m_\alpha)$ (with respect to the smoothness structure on quotient manifold $M_\alpha$), an index $1 \leq i \leq n_\alpha$ and $\epsilon > 0$ such that:

(i) $U = \exp(N_\alpha(e)U_\alpha^i)$ with $U^i_\alpha = g_\alpha^i E_\alpha U^*_\alpha$, i.e. $U$ is the exp-image of the restriction of the $\epsilon$-disc bundle of $N_\alpha$ to subset $U^i_\alpha$.

(ii) $x^i(q) = x^i_\alpha \circ \text{pr} \circ \Pi_\alpha \circ \exp^{-1}(q)$ ($i = 1, \ldots, n_\alpha$, $q \in U$).

(iii) By clause (i), $\forall r \in U \exists! \text{ pair } (gG_\alpha, q) \in E_\alpha \times U^*_\alpha$ such that:

$$\Pi_\alpha \circ \exp^{-1}(r) = g_\alpha q \cdot q$$

holds. Define

$$x^i(r) = \tilde{x}^i_\alpha(gG_\alpha), \quad (i = m_\alpha + m_\alpha^1 + 1, \ldots, m) \quad (2)$$
(iv) Fix an orthonormal frame bundle \((v_1, \ldots, v_m)\) of trivial bundle \(N_\alpha|_{U_\alpha}\) and extend it to \(U_\alpha^i\) by \(v_j(g_\alpha^igq) = dL_{g_\alpha^i}\sigma_\alpha(g_\alpha^i)(v_j(q))\). Define:

\[
\exp^{-1}(r) = \sum_{i=1}^{m_\alpha} x^{m_\alpha+i}(r)v_i \quad (r \in U) \tag{3}
\]

Note that the first \(m_\alpha\) coordinates do not depend on the choice of the rest of the coordinates. For the definition of the \(C^1\)-norm of a vector field we need to fix a compact subset within each chart, which we can get as follows:

Fix \(K_\alpha^* \subset U_\alpha^*\) compact set, \(0 < \epsilon' < \epsilon\) real number and define

\[
K = \exp(N_\alpha(\epsilon')|_{K_\alpha^*}) \subset U
\]

with \(K_{\alpha}^i = g_\alpha^iE_{\alpha}^*K_{\alpha}^*\).

We will call \(\text{int}(K)\) the strong interior of chart \((U, (x^1, \ldots, x^m))\). In the sequel we will presume that each adapted chart has a fixed compact set in its interior (even if we don’t state this explicitly).

To cover \(M\) choose adapted charts

\[
(U_j, (x_j^1, \ldots, x_j^m)) \quad (j = 1, \ldots, j_0)
\]

so that their strong interior cover the level-0 orbit types. Then the complement of the union of strong interiors of these charts in any of the level-1 strata is compact, so it can be covered by the strong interiors of

\[
(U_j, (x_j^1, \ldots, x_j^m)) \quad (j = j_0 + 1, \ldots, j_1)
\]

so that each chart is adapted to the stratum at issue, a.s.o. Finally we get a finite family of adapted coordinate charts so that their strong interiors \(\text{int}(K_1), \ldots, \text{int}(K_{j_1})\) cover \(M\) (here \(L\) is the highest level).

Given a vector field \(Y\) with local coordinates:

\[
Y|_{U_j} = \sum_{i=1}^{m} y_j^i \frac{\partial}{\partial x_j^i} \quad (j = 1, \ldots, j_L)
\]

its \(C^1\)-norm is defined as:

\[
\|Y\|_1 := \sum_{i,j,k} \left( \sup_{K_j} |y_j^i| + \sup_{K_j} \left| \frac{\partial}{\partial x_j^i} y_j^i \right| \right) \tag{4}
\]

We will perturb vector field \(X\) into a \(G\)-Morse-Smale vector field in succession of increasing order of the level of strata. Although domains \(U_1, \ldots, U_{j_L}\) cover \(M\), each stratum has parts covered by some of the \(U_j^i\)’s that are adapted to a stratum at a lower level. To attain \(\alpha\)-transversality, we should adjust \(X\) in charts that are adapted to orbit type \(\alpha\). It seems so that in order to end up with a \(C^1\) vector field, we need to modify vector field \(X\) in finite steps. To cover a non-compact
stratum, however, we need to use infinitely many adapted charts. To overcome this discrepancy, for each orbit type $\alpha$ we will choose a countable family of adapted coordinate charts with set of domains $Q^{(\alpha)}$ which is a finite union

$$Q^{(\alpha)} = Q^{(\alpha);1} \cup \ldots \cup Q^{(\alpha);k_{\alpha}}$$

of sub-families such that:

**Property I.** For a fixed $i = 1, \ldots, k_{\alpha}$ the closure of domains in

$$Q^{(\alpha);i} = \{Q_{1}^{(\alpha);i}, Q_{2}^{(\alpha);i}, \ldots, Q_{j}^{(\alpha);i}, \ldots\}$$

are pairwise disjoint.

**Property II.** The strong interiors of domains in $Q^{(\alpha)}$ cover $M_{\alpha}$.

**Property III.**

$$Q_{j}^{(\alpha);i} \cap Q_{l}^{(\beta);k} \neq \emptyset \Rightarrow \alpha \text{ and } \beta \text{ can be compared w.r.t. } \prec$$

**Remark:** Given a family of open subsets about strata, it is standard to impose a condition similar to Property III. (see e.g. Mather [3].) It is also shown there, that arbitrary system of tubes about strata can be trimmed down so that Property III holds.

**Definition 6.** A cover of each stratum by adapted charts with the above three properties is called a *stratified cover of $M$*.

**Proposition 1.** A compact $G$-manifold has a stratified cover.

**Proof:** We have already constructed a finite cover of each level-0 strata. For an orbit type $M_{\alpha}$ at the $(k + 1)$th level the complement $M_{\alpha} \setminus \bigcup_{j=1}^{j_{k}} U_{j}$ is compact so it can be covered by the strong interiors of finitely many adapted charts, thus it is enough to choose a stratified cover for each set

$$M_{\alpha} \cap U_{j} \quad 1 \leq j \leq j_{k}$$

separately. For fixed $j$ domain $U_{j}$ is adapted to $M_{\beta}$ for some $\beta \prec \alpha$. Choose $\epsilon' < \epsilon$ and let $S_{\epsilon'} \subset N_{\beta}$ denote the $\epsilon'$-sphere bundle of the normal bundle of stratum $M_{\beta}$. The level of orbit type $(\exp|_{U_{j}})^{-1}(M_{\alpha}) \cap S_{\epsilon'}$ of the normal action is at most $k$, thus by induction we can choose a stratified cover $Q'^{(\alpha)}$ of the subset

$$(S_{\epsilon'})_{\alpha} := (\exp|_{U_{j}})^{-1}(M_{\alpha}) \cap S_{\epsilon'}$$

This means that family $Q'^{(\alpha)}$ is a finite union

$$Q'^{(\alpha)} = \bigcup_{i=1}^{k} Q'^{(\alpha);i}$$

where each subset $Q'^{(\alpha);i}$ consists of relatively compact open sets (in the topology of $S_{\epsilon'}$) with pairwise disjoint closure. Consider the subsets

$$R_{1} := \left\{\left(\frac{1}{2n+1}, \frac{2}{4n-1}\right) \mid n \in \mathbb{Z}^{+}\right\}$$

$$R_{2} := \left\{\left(\frac{1}{2n}, \frac{4}{8n-5}\right) \mid n \in \mathbb{Z}^{+}\right\}$$
Let
\[ Q^{(α):i}_1 := \{(a, b) \times Q \mid (a, b) \in R_1, Q \in Q^{(α):i}\} \]
\[ Q^{(α):i}_2 := \{(a, b) \times Q \mid (a, b) \in R_2, Q \in Q^{(α):i}\}. \]
Then the exp-images of sets \( Q^{(α):i}_1, Q^{(α):i}_2 (i = 1 \ldots k) \) provide a stratified cover for \( M_α \cap U_j \). (Pairwise disjointness and relative compactness follow trivially; intervals \((a, b)\) can serve as new coordinates.)

**Notation:** In each step \( X \) will denote the invariant gradient-like vector field that has already been adjusted along certain strata (so we will not re-denote \( X \) in every single step).

For the proof of our theorem suppose inductively that relative transversality of ascending and descending submanifolds has been attained below critical level \( f(O) = c \). Fix critical orbit \( O \) and a Morse-chart \( η_O : \bot_O(c) \to U_O \) about \( O \). We will perturb \( X \) by an invariant vector field with support contained in the Morse-tube \( U_O \) (in the case when we have more than one critical orbits at level \((f = c)\), we choose disjoint Morse-tubes about them). This way we will not influence relative transverse intersections that have already been established in previous steps.

**Notations:** At a point \( x \in O \) let \( S_x, S_x^−, S_x^+ \) denote the spheres with radius \( ε < ε^* \) (about the origin) in subspaces \( \bot_x, \bot_x^−, \bot_x^+ \) respectively. Let \( B = \bot_x(ε^*) \) denote the \( ε^* \)-disk about the origin of \( \bot_x \). As for the rest of this paper we will work in ball \( B \), we can (ab)use notation by denoting the trajectory of vector field
\[ X|_B := dn_O^{-1}(X|_{U_O}) = \text{grad}(f \circ η_O) \]
through point \( p \in B \) by \( λ_p \).

Fix \( x \in O \) and drop it from the subscripts. Let \( \hat{G} := G_x \) be the isotropy subgroup at \( x \) and choose an invariant open neighborhood
\[ ν := ν_x \subset η(O)^{-1}(f^{-1}(c-ε) \cap U_O) \cap \bot_x \]
of outbound sphere \( S^− := S^−_x \). Then \( ν \) is partitioned by the orbit types of action \( \hat{G} \times ν \to ν \) as
\[ ν = \bigsqcup_{\alpha} ν_α \]
For a subset \( Z \subset ν \) and an orbit type \([x] \leq α \) let \( Z_α := Z \cap ν_α \) be the \( α \)-part of set \( Z \) in orbit type \( ν_α \). Let \( \{O_1, ..., O_k\} \) denote the set of critical orbits that are connected with \( O \) by a trajectory of \( X \) and reside below critical level \( c \). Set
\[ Σ_j := ν \cap η(O)^{-1}(W^+_j \cap U_O), \quad Σ := \bigsqcup_{j=1}^k Σ_j \]

**Observe:** that the intersection \( W^−_O \cap W^+_j \) is relative transverse (with respect to the partition \( M = \bigsqcup_{α} M_α \)) if and only if for all orbit types \( α \) preceded by \([x]\) the intersection \( S^−_O \cap Σ_j \) is transverse in submanifold \( ν_α \). Recall notation \( U_x := η_O(B) \) and choose a point \( p \in U_x \cap W^−_O \cap W^+_j \cap M_α \). We have
\[ T_p(W^−_O)_α + T_p(W^+_j)_α = T_p(W^−_O \cap U_x)_α + T_p(W^+_j \cap U_x)_α + T_pG(p) \quad (5) \]
thus \( T_p(W^-) + T_p(W^+_j) = T_p M \) if and only if
\[
T_p(W^- \cap U_x) + T_p(W^+_j \cap U_x) = T_p(U_x)
\]
This means that it is enough to ensure relative transversality of intersection \( S^- \cap \Sigma j \) with respect to \( \nu \).

Strategy of proof: first we will construct the perturbing vector field on ball \( B \), then we extend it along the \( G \)-action to a vector field on \( \perp O (\epsilon^*) \), finally we push it forward along Morse coordinate system \( \eta_O : \perp O (\epsilon^*) \to U_O \). We will proceed by induction on the level of orbit types of action \( G \times S^- \to S^- \). In the sequel "level" will always be meant in this sense. We will use the same method to perturb vector field \( X \) for both the base step and the induction step, so by induction suppose that we need to define the perturbing vector field for orbit type \( \alpha \) and we are done with all orbit types at lower levels. We can presume \( G_\alpha \subset \tilde{G} \). Then the fixed point set of orthogonal action \( G_\alpha \times \perp_x \to \perp_x \) is a linear subspace
\[
V = V^- \oplus V^+
\]
where \( V^- \subset \perp_x^-, V^+ \subset \perp_x^+ \) and \( \dim(V^-) \neq 0 \).

Note, that the normalizer \( N_\alpha \) of subgroup \( G_\alpha \subset \tilde{G} \) acts on linear subspace \( V \) as well as on
\[
V^-_\alpha := [V^- \less \text{less the points that are fixed by a group strictly larger than } G_\alpha]
\]
Let \( Q^* \) be an \( \alpha \)-slice for action \( N_\alpha \times V^-_\alpha \to V^-_\alpha \) such that it is a union of (open) rays in \( V^-_\alpha \) (thus it is a cone \( C(Q^* \cap S^-) \) over \( \alpha \)-slice \( Q^* \cap S^- \) for action \( N_\alpha \times S^- \to S^- \). Let \( D^+ \subset V^+ \) be a small disc about the origin. Then \( Q^* \oplus D^+ \) is an \( \alpha \)-slice of action \( N_\alpha \times V \to V \) and also for action \( \tilde{G} \times \perp_x \to \perp_x \), thus
\[
(Q^* \oplus D^+) \cap \nu
\]
is an \( \alpha \)-slice for action \( \tilde{G} \times \nu \to \nu \). This leads to the following conclusion, which is crucial for the proof:

**Observation 1.:** For an \( \alpha \)-slice \( Q^{**} \) of action \( \tilde{G} \times S^- \to S^- \) there exists an \( \alpha \)-slice of action
\[
\tilde{G} \times \nu \to \nu
\]
which is a product of \( Q^{**} \) and the fiber. We will call such a slice a *product slice associated* to \( Q^{**} \) and its image under \( G \) (i.e. the union of paths through its points) is the \( G \)-extension of the associated slice.

Level-0 orbit types are compact, so are their projections to orbit space \( \mathcal{M} \). In the base step of induction we define the perturbing vector fields along these projections and then we will lift them into invariant vector fields defined on \( \mathcal{M} \). This is done in complete analogy with the non-equivariant case (see Smale [5]).

**Remark:** We could have chosen to work out the whole reasoning in the orbit space, lifting the resulting vector fields (isotopies) to \( \mathcal{M} \) afterwards. In spite of the simplicity of some parts of this path of reasoning, other technical difficulties would have arisen (e.g. the quotient space \( \mathcal{V} \) is not a linear space, etc).
**Construction of the stratified cover:** first we choose a family of \( \alpha \)-slices

\[
Q_j^{(\alpha);i} \subset S^{-}_\alpha \cap V^- \quad (i = 1, \ldots, k_\alpha, \ j = 1, \ldots, n, \ldots)
\]

for the action

\[
\hat{G} \times S^- \to S^-
\]

together with associated product slices

\[
Q_j^{(\alpha);i} \oplus D^+
\]

so that

\[
\overline{GQ_j^{(\alpha);i}} \cap \overline{GQ_j^{(\alpha);i}} = \emptyset \quad (j \neq j')
\]

Intersections

\[
\nu \cap C(Q_j^{(\alpha);i} \oplus D^+)
\]

will be \( \alpha \)-slices for action

\[
\hat{G} \times \nu \to \nu
\]

so if \( n_\alpha \) is the normal bundle of \( \nu_\alpha \subset \nu \) then with the aid of a small disc bundle of \( n_\alpha |_{\hat{G}C(Q_j^{(\alpha);i} \oplus D^+) \cap \nu} \) we can define a stratified cover

\[
Q^{(\alpha)} = Q^{(\alpha);1} \cup \ldots \cup Q^{(\alpha);k_\alpha}
\]

of the \( \alpha \)-part \( \nu_\alpha \) (the cover is taken in \( \nu \)). Choose a smaller disk \( D^+ \subset D^+ \) and fix compact subsets

\[
K_j^{(\alpha);i} \subset Q_j^{(\alpha);i}, \quad K_j^{(\alpha);i} \subset Q_j^{(\alpha);i}
\]

With \( \epsilon' < \epsilon \) define

\[
\check{Q}_j^{(\alpha);i} := \{ q \in \lambda_p \mid p \in Q_j^{(\alpha);i} \ \text{and} \ c - \epsilon < f(q) < c - \epsilon' \}
\]

\[
\check{K}_j^{(\alpha);i} := \{ q \in \lambda_p \mid p \in K_j^{(\alpha);i} \ \text{and} \ c - \epsilon < f(q) < c - \epsilon' \}
\]

The \( \alpha \)th step will consist of \( k_\alpha \) substeps so that in the \( i \)th substep the support of the perturbing vector field is contained in the extension

\[
G \left( \bigcup_{j=1}^{\infty} \check{Q}_j^{(\alpha);i} \right) \quad (i = 1, \ldots, k_\alpha)
\]

The following observations can be made:

**Observation 2.** The perturbed vector field remains the same on lower level strata, so does the transversality of intersections \( S^-_\beta \cap \Sigma_\beta \) (that have already been established by induction).

**Observation 3.** By the fact that transverse intersections are stable under small perturbations of class \( C^1 \), in each sub-step we can choose the perturbing vector field to be so small that it will not destroy transverse intersections that had already
been established in previous sub-steps. Thus it is enough to describe the \(i\)th sub-step, or, as the domains 
\[\{\hat{Q}^{(\alpha)}_j : j = 1, \ldots, n, \ldots\}\]
are pairwise disjoint, it is enough to describe how to modify \(X\) within one such domain.

**Observation 4.** The bound we imposed in the Theorem divided by the total number of sub-steps (i.e. 
\[\varepsilon' := \frac{\varepsilon}{\sum_{\alpha \in \mathcal{A}} k_\alpha}\]
provides a bound we should use in formula (4) in each sub-step.

**Observation 5.** For each triple \((\alpha, i, j)\) there is a bound \(\varepsilon_j^{(\alpha):i}\) so that if in the \(i\)th sub-step we choose the perturbing vector field with \(C^1\)-norm measured on domain \(\hat{Q}^{(\alpha)}_j\) smaller than \(\varepsilon_j^{(\alpha):i}\), then its \(C^1\)-norm defined by formula (4) is smaller than \(\varepsilon''\) (this is a well-known fact that follows from relative compactness of the domains, see e.g. Sternberg [3].)

**Observation 6.** The major difficulty is to ensure that after the final step we end up with a \(C^1\) vector field. This will give us additional conditions on the size of perturbation we can make in each sub-step. These conditions are described in the Lemma below.

By the above observations it is enough to show that for arbitrary indices \((\alpha, i, j)\) (that we fix and drop, using notation \(\varepsilon := \varepsilon_j^{(\alpha):i}\)) the following holds:

**Proposition 2.** Given a domain \(\hat{Q} = \hat{Q}^{(\alpha)}_j\) with compact subset \(\hat{K} = \hat{K}^{(\alpha)}_j\) and \(\varepsilon'' > 0\) there exists an invariant vector field \(Y\) with support in \(G\hat{Q}\) such that:

(i) Vector field \(X + d\eta_{O}(Y)\) is also gradient like for \(G\)-Morse function \(f\).

(ii) \(\|Y\|_{1,\hat{Q}} < \varepsilon''\) (i.e. the \(C^1\)-norm on \(G\hat{Q}\) is smaller than \(\varepsilon''\).)

(iii) The intersection 
\[\left[ S_\alpha^- \cap \hat{G}K^{(\alpha):i} \right] \cap \left[ \Sigma_\alpha \cap \hat{G}R_j^{(\alpha):i} \right] \]
(notations stand for objects of the flow of vector field \(X|_B + Y\) is transverse in orbit type \(\nu_\alpha\), thus by formula (5) intersection 
\[\left[ W_\alpha^- \cap G\hat{K} \cap M_\alpha \right] \cap \left[ W_\alpha^+ \cap G\hat{K} \cap M_\alpha \right]\]
is transverse in orbit type \(M_\alpha\) for any other critical orbit \(O'\).

**Proof:** \(\dim(V^+) = 0\) implies \(\nu_\alpha = S_\alpha^-\) thus transversality of intersections in formula (8) follow trivially.

Otherwise note that \(V \cap \Sigma_j\) is the fixed point set of subgroup \(G_\alpha \subset \hat{G}\) for the action \(\hat{G} \times \Sigma_j \to \Sigma_j\), thus it is a submanifold of \(\Sigma_j\). Let \(Q^*_p\) be an \(\alpha\)-slice at point \(p \in \Sigma_j \cap V^*_\alpha\) for action \(N_\alpha \times (V \cap \Sigma_j) \to (V \cap \Sigma_j)\). Then \(V^*_p := T_pQ^*_p\) is an affine subspace of \(V\), so it splits into the sum
\[V^*_p = V^*_p^- \oplus V^*_p^+\]
where $V_p^\alpha := V_p^* \cap V^-$. Intersection $(\Sigma_j)_\alpha \cap S^-_\alpha$ is $\alpha$-transverse if $\dim(V_p^\alpha) = \dim(V^-)$, or in other words if the origin is a regular value of the projection to the second factor

$$P^+|_{V \cap \Sigma_j} : V \cap \Sigma_j \to V^+$$

(see also formula (5)).

By Sard’s theorem the set of critical values of the above projection is of measure-0 for $j = 1, \ldots, k$ thus the same holds for their union. This means that we can choose an arbitrarily small vector $v \in D^+$ so that the constant section $C(K_j^{(\alpha);\ast} \oplus v) \cap \nu_\alpha$ intersects $(\Sigma_j)_\alpha$ $\alpha$-transversely (i.e., relative to $\nu_\alpha$) for $j = 1, \ldots, k$. Choose cutoff function

$$\phi : Q_j^{(\alpha);\ast} \to [0, 1], \quad \phi(K_j^{(\alpha);\ast}) = 1, \quad \supp(\phi) \subset Q_j^{(\alpha);\ast}$$

and let $H_D$ be an isotopy of disk $D^+$ which moves the origin into $v$ and fixes a neighborhood of the boundary. Define isotopy

$$H : (Q_j^{(\alpha);\ast} \oplus D^+) \times [0, 1] \to Q_j^{(\alpha);\ast} \oplus D^+, \quad H((p, w), t) = (p, H_D(w, \phi(p)t)$$

Using rays this defines an isotopy of set $C(Q_j^{(\alpha);\ast} \oplus D^+) \cap \nu$. An application of the argument in Milnor ([5], pp. 42-43) to the restriction $X_B|_F$ where

$$F := \{ q \in \lambda_p \mid p \in C(Q_j^{(\alpha);\ast} \oplus D^+) \cap \nu \text{ and } c - \epsilon < f(q) < c - \epsilon' \}$$

produces a vector field $Y^*$ which is tangent to $F$ so that the difference between moving along the flow-lines of vector fields $X_B|_F$ and $X_B|_F + Y^*$ shows up in the application of map $H_1$ on level set $(f = c - \epsilon)$. Let $\tilde{Y}$ be the extension of $Y^*$ along the action onto set $GF$. This set is an open subset of the $\alpha$-part $B_\alpha$ of disk $B = \mathbb{D}(\epsilon^*)$.

For a $\delta > 0$ choose cutoff function

$$\psi : [0, \delta] \to [0, 1], \quad \psi(0) = 1, \quad \psi(\delta) = 0, \quad \frac{d\psi}{dt} \leq 0.$$

Let $N_\alpha$ be the normal bundle of the $\alpha$-part $B_\alpha \subset B$ (taken in ball $B$). Consider the Levi-Civita connection of the Euclidean metric on vector bundle $N_\alpha$ and let $Y'$ be the horizontal lift of $\tilde{Y}$ (i.e. a vector field on $N_\alpha$ with vertical components $= 0$ at each point so that $d\tau(Y'|_p) = \tilde{Y}|_{x(p)}$). Let

$$Y = \exp_\ast(\psi(||v||)Y'|_v) \quad (v \in N_\alpha)$$

and extend $Y$ via the action onto tube $U_O$. Let $Y$ be the 0 vector field outside of set $G\exp[N_\alpha(\delta)]_{GF}$. Then clauses (i) and (iii) hold for vector field $X_B + Y$ (its restriction to a smaller neighborhood of critical orbit $O$ is gradient-like for $f$).

As in Definition 5., choose a coordinate system $(x^2, \ldots, x^{m_\alpha})$ on $\alpha$-slice $Q_j^{(\alpha);\ast} \oplus D^+$. Use $G$-Morse function $f$ as the first coordinate $x^1$. As in clause (iv) of Definition 5., fixing an orthonormal frame bundle on $N_\alpha|_F$ introduces further coordinates $x^{m_\alpha + 1}, \ldots, x^{m_\alpha + m_\alpha^\ast}$. Finally, supplementing these with the coordinates in formula (2) provides a finite family of coordinate charts on invariant open set $GQ$. Note, that only the first $m_\alpha + m_\alpha^\ast$ coordinates of vector field $Y$ are non-zero and the coordinates themselves are bounded by $||v||$. By relative
compactness of set $Q_j^{(\alpha);i}$ we can choose an upper bound $N$ for the absolute value of the derivatives of cutoff functions $\phi$ and $\psi$. Then

$$2 \left( N\|v\| + \frac{2\|v\|}{\epsilon - \epsilon'} \right)$$

will serve as an upper bound for the absolute values of the first derivatives of coordinates of vector field $Y$. This shows that by choosing $\|v\|$ small enough, one can arrange that clause (ii) of our Proposition holds as well.

Differentiability of the final vector field can be ensured by choosing $\|v\|$ according to how close the domain $\tilde{Q}_j^{(\alpha);i}$ is from the boundary of $B_\alpha$. Let

$$\text{Fr}(B_\alpha) := \overline{B_\alpha} \setminus B_\alpha$$

be the frontier of orbit type $B_\alpha$ and let $d$ stand for the Euclidean distance on ball $B$. Then, by relative compactness of set $Q_j^{(\alpha);i}$ in $S_\alpha$, the distance

$$d_{ij} := d(\tilde{GQ}_j^{(\alpha);i}, \text{Fr}(B_\alpha)) > 0$$

In the $i$th substep the components of $\text{supp}(Y)$ are contained in subset

$$\bigcup_{j=1}^\infty G\tilde{Q}_j^{(\alpha);i}$$

By invariance it is enough to prove differentiability of restriction $Y|_B$. By construction, $Y$ is a smooth vector field in a tube about $\alpha$-part $B_\alpha$ (i.e. on a set

$$\exp(N_\alpha(\zeta))$$

where $N_\alpha(\zeta)$ is an appropriate disk bundle of the normal bundle of $B_\alpha \subset B$) and $Y$ is the zero vector field outside of this tube. This shows that we can have problems with differentiability of vector field $Y$ only at the points of the frontier $\text{Fr}(B_\alpha)$. We will prove that imposing an additional condition ensures that vector field $Y$ is of class $C^1$ at points of $\text{Fr}(B_\alpha)$. We will formulate this condition by utilizing the fact that $B$ is a Euclidean disk, thus its tangent bundle is trivial by natural identification

$$TB = B \times \mathbb{R}^n$$

($n=\dim(B)$.) This way we can look at vector field $Y|_B$ as a map

$$Y_B : B \rightarrow \mathbb{R}^n$$

**Lemma.** Vector field $Y$ is of class $C^1$ whenever

$$\|Y(q)\| < d_{ij}^3 \quad (q \in \tilde{Q}_j^{(\alpha);i}, \ i = 1, \ldots, k_\alpha, \ j \in N)$$

Proof A point $q \in \text{Fr}(B_\alpha)$ belongs to $q \in B_\beta$ for some orbit type $\beta < \alpha$. By definition $Y$ is differentiable at point $q$ if

$$\lim_{p \to q} \frac{\|Y(p) - Y(q)\|}{\|p - q\|} = 0$$

(9)
Value $Y(q) = 0$ for points of the frontier, thus $Y(p) \neq 0$ implies that $p \in \tilde{GQ}_j^{(a):i}$ for some indices $i \in \{1, \ldots, k\}$, $j \in \mathbb{N}$. But then
$$
\|Y(p)\| < d_{ij}^3 \leq \|p - q\|^3
$$
thus the limit in formula (9) is indeed 0, together with the limit of the first partial derivatives of $Y$.

**Remark:** The proof of the lemma has been built heavily on the Euclidean structure on disk $B$. In a general setting a somewhat more complicated proof would work: one considers the isotopy induced by time-dependent vector field
$$
Y_t := \xi(t)Y \quad (t \in [0, 1])
$$
where $\xi : [0, 1] \to [0, 1]$, $\xi(0, \epsilon) = 0$ is some cutoff function. One puts the analogous condition on the displacement of this isotopy (measured in the metric distance on $B$). Then a similar argument proves that the isotopy is of class $C^1$, consequently vector field $Y$ is also of class $GSM$.

This paper is an improved version of Chapter 3. of my Ph.D. thesis [2]. I wish to express my gratitude to my advisor, Professor Dan Burghelea for his help.

**References**

[1] Bredon, G.: *Introduction to compact transformation groups*. Academic Press, (1970).

[2] Major, I.: *On Equivariant Morse Theory* (Ph.D. Thesis, The Ohio State University, 1997)

[3] Mather, J.: *Notes on topological stability*. Harvard University, 1970.

[4] Milnor, J.W.: *Lectures on the $H$-cobordism theorem*. Ann. of Math. Studies 51, Princeton Univ. Press, (1965).

[5] Smale, S.: *On gradient dynamical systems* Ann. of Math. 74 (1961), pp. 199-206.

[6] Sternberg, S.: *Lectures on Differential Geometry*. Prentice Hall, 1964.

[7] Wasserman: *Morse theory for $G$-manifolds*. cd majorBull. AMS 71 (1965) 304-388.

Imre Major
Gabor Denes School of Informatics and
Central European University, Budapest