INVERSE PROBLEM FOR A STRUCTURAL ACOUSTIC INTERACTION

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Abstract. In this work, we consider an inverse problem of determining a source term for a structural acoustic partial differential equation (PDE) model, comprised of a two or three-dimensional interior acoustic wave equation coupled to a Kirchoff plate equation, with the coupling being accomplished across a boundary interface. For this PDE system, we obtain the uniqueness and stability estimate for the source term from a single measurement of boundary values of the “structure”. The proof of uniqueness is based on Carleman estimate. Then, by means of an observability inequality and a compactness/uniqueness argument, we can get the stability result. Finally, an operator theoretic approach gives us the regularity needed for the initial conditions in order to get the desired stability estimate.

Keywords: Structural acoustic interaction, inverse problem, Carleman estimate, continuous observability inequality

1. Introduction and Main Results

1.1. Statement of the Problem. Let $\Omega$ be an open bounded subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ with smooth boundary $\Gamma$ of class $C^2$, and we designate a nonempty simply connected segment of $\Gamma$ as $\Gamma_0$ with then $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. We consider here the following system comprised of a “coupling” between a wave equation and an elastic plate-like equation:

$$
\begin{align*}
\frac{\partial^2 z}{\partial t^2}(x,t) &= \Delta z(x,t) + q(x)z(x,t) \quad \text{in } \Omega \times [0,T] \\
\frac{\partial}{\partial \nu} z(x,t) &= 0 \quad \text{on } \Gamma_1 \times [0,T] \\
z_t(x,t) &= -v_{tt}(x,t) - \Delta^2 v(x,t) - \Delta^2 v_t(x,t) \quad \text{on } \Gamma_0 \times [0,T] \\
v(x,t) &= \frac{\partial v_t}{\partial \nu}(x,t) = 0 \quad \text{on } \partial \Gamma_0 \times [0,T] \\
\frac{\partial z_t}{\partial \nu}(x,t) &= v_t(x,t) \quad \text{on } \Gamma_0 \times [0,T] \\
z_t(\cdot, \frac{T}{2}) &= z_0(x) \quad \text{in } \Omega \\
z_t(\cdot, \frac{T}{2}) &= z_1(x) \quad \text{in } \Omega \\
v(\cdot, \frac{T}{2}) &= v_0(x) \quad \text{on } \Gamma_0 \\
v_t(\cdot, \frac{T}{2}) &= v_1(x) \quad \text{on } \Gamma_0
\end{align*}
$$

(1.1)

where the coupling occurs across the boundary interface $\Gamma_0$. $[z_0, z_1, v_0, v_1]$ are the given initial conditions and $q(x)$ is a time-independent unknown coefficient. For this
system, notice that the map \( \{ q \} \rightarrow \{ z(q), v(q) \} \) is nonlinear, therefore we consider the following nonlinear inverse problem: Let \( \{ z = z(q), v = v(q) \} \) be the weak solution to system \((1.1)\). Under suitable geometrical conditions on \( \Gamma_1 = \Gamma \setminus \Gamma_0 \), is it possible to retrieve \( q(x), x \in \Omega \), from measurement of \( v_{tt}(q) \) on \( \Gamma_0 \times [0,T] \)? In other words, is it possible to recover the internal wave potential from the observation of the acceleration of the elastic plate.

Our emphasis here that we determine the interior acoustic property from observing the acceleration of the elastic wall (portion of the boundary), is not only due to physical consideration, but also to the implications of such inverse type analysis related to the coupling nature of the structural acoustic flow. In many structural acoustics applications, the problem of controlling interior acoustic properties is directly correlated with the problem of controlling structural vibrations since the interior noise fields are often generated by the vibrations of an enclosing structure. An important example of this is the problem of controlling interior aircraft cabin noise which is caused by fuselage vibrations that are induced by the low frequency high magnitude exterior noise fields generated by the engines.

The primary goal in this paper is to study the uniqueness and stability of the interior time-independent unknown coefficient \( q(x) \) in some appropriate function space. More precisely, we consider the follow uniqueness and stability problems:

**Uniqueness in the nonlinear inverse problem**

Let \( \{ z = z(q), v = v(q) \} \) be the weak solution to system \((1.1)\). Under geometrical conditions on \( \Gamma_1 \), does the acceleration of the wall \( v_{tt}|_{\Gamma_0 \times [0,T]} \) determine \( q(x) \) uniquely? In other words, does

\[
v_{tt}(q)|_{\Gamma_0 \times [0,T]} = v_{tt}(p)|_{\Gamma_0 \times [0,T]}
\]

imply \( q(x) = p(x) \) in \( \Omega \)?

**Stability in the nonlinear inverse problem**

Let \( \{ z(q), v(q) \}, \{ z(p), v(p) \} \) be weak solutions to system \((1.1)\) with corresponding coefficients \( q(x) \) and \( p(x) \). Under geometric conditions on \( \Gamma_1 \), is it possible to estimate \( \| q - p \|_{L^2(\Omega)} \) by some suitable norms of \( (v_{tt}(q) - v_{tt}(p))|_{\Gamma_0 \times [0,T]} \)?

In order to study the nonlinear inverse problem, we first linearize \((1.1)\) and hence we consider the following system:

\[
\begin{cases}
  w_{tt}(x,t) - \Delta w(x,t) - q(x)w(x,t) = f(x)R(x,t) & \text{in } \Omega \times [0,T] \\
  \frac{\partial w}{\partial \nu}(x,t) = 0 & \text{on } \Gamma_1 \times [0,T] \\
  w_t(x,t) = -u_t(x,t) - \Delta^2 u(x,t) - \Delta^2 u_t(x,t) & \text{on } \Gamma_0 \times [0,T] \\
  u(x,t) = \frac{\partial w}{\partial \nu}(x,t) = 0 & \text{on } \partial \Gamma_0 \times [0,T] \\
  w(\cdot, \frac{T}{2}) = 0 & \text{in } \Omega \\
  w_t(\cdot, \frac{T}{2}) = 0 & \text{in } \Omega \\
  u(\cdot, \frac{T}{2}) = 0 & \text{on } \Gamma_0 \\
  u_t(\cdot, \frac{T}{2}) = 0 & \text{on } \Gamma_0
\end{cases}
\]

\((1.2)\)
where \( q \in L^\infty(\Omega) \) is given, \( R(x,t) \) is fixed suitably while \( f(x) \) is an unknown time-independent coefficient. For this linearized system, we have the advantage that the map \( \{f\} \to \{w(f), u(f)\} \) is linear, hence we consider the corresponding linear inverse problem:

**Uniqueness in the linear inverse problem**

Let \( \{w = w(f), u = u(f)\} \) be the weak solution to system (1.2). Under geometrical conditions on \( \Gamma_1 \), does \( u_{tt}|_{\Gamma_0 \times [0,T]} \) determine \( f(x) \) uniquely? In other words, does

\[
u_{tt}(f)|_{\Gamma_0 \times [0,T]} = 0
\]

imply \( f(x) = 0 \) in \( \Omega \)?

**Stability in the linear inverse problem**

Let \( \{w = w(f), u = u(f)\} \) be the weak solution to system (1.2). Under geometrical conditions on \( \Gamma_1 \), is it possible to estimate \( \|f\|_{L^2(\Omega)} \) by some suitable norms of \( u_{tt}|_{\Gamma_0 \times [0,T]} \)?

**Remark 1.1.** In our models (1.1) and (1.2) we regard \( t = \frac{T}{2} \) as the initial time. This is not essential, because the change of independent variables \( t \to t - \frac{T}{2} \) transforms \( t = \frac{T}{2} \) to \( t = 0 \). However, this is convenient for us to apply the Carleman estimate established in [29]. In fact, one can keep \( t = 0 \) as initial moment by doing an even extension of \( w \) and \( u \) to \( \Omega \times [-T,T] \), but then the Carleman estimate in [29] needs to be modified accordingly.

1.2. **Literature and Motivation.** The PDE system (1.1) is an example of a structural acoustic interaction. It mathematically describes the interaction of a vibrating beam/plate in an enclosed acoustic field or chamber. In this situation, the boundary segment \( \Gamma_1 \) represents the “hard” walls of the chamber \( \Omega \), with \( \Gamma_0 \) being the flexible portion of the chamber wall. The flow with in the chamber is assumed to be of acoustic wave type, and hence the presence of the wave equation in \( \Omega \), satisfied by \( z \) in (1.1), coupled to a structural plate equation (in variable \( v \)) on the flexible boundary portion \( \Gamma_0 \). This type of PDE models has long existed in the literature and has been an object of intensive experimental and numerical studies at the Nasa Langley Research Center [31, 9, 10]. Moreover, recent innovations in smart material technology and the potential applications of these innovations in control engineering design have greatly increased the interest in studying these structural acoustic models. As a result, there has been a lot of recent contributions to the literature deal with various topics; e.g., optimal control, stability, controllability, regularity [1, 2, 3, 4, 5, 6, 7, 8, 14, 23]. However, to the best of our knowledge, there are no results available in the literature for our inverse type analysis on the model.

On the other hand, the interest to the inverse problem has been stimulated by the studies of applied problems such as geophysics, medical imaging, scattering, nondestructive testing and so on. These problems are of the determination of unknown coefficients of differential equations which are the functions depending on the point of the space [11, 15, 16]. For the uniqueness in multidimensional inverse problem with a single boundary observation, the pioneering paper by Bukhgeim and Klibanov [12] provides a methodology based on a type of exponential weighted energy estimate,
which is usually referred as the Carleman estimate since the original work \cite{13} by Carleman. After \cite{12}, several papers concerning inverse problems by using Carleman estimate have been published (e.g. \cite{17, 21}). In particular, for the inverse hyperbolic type problems that is related to our concern in this paper, there has been intensively studies \cite{18, 19, 20, 32, 37}. However, we mentioned again that there is not any such uniqueness and stability analysis for the structural acoustic models or even in general coupled PDE systems. This motivates the work of the present paper.

The usual problem setting for inverse hyperbolic problem includes determining a coefficient from measurements on the whole boundary or part of the boundary, either Dirichlet type \cite{12, 20, 32, 37} or Neumann type \cite{18, 19}. Usually the coefficient describes a physical property of the medium (e.g. the elastic modulus in Hooke’s law), and the inverse problem is to determine such a property. In our formulation of the inverse problem, we need to determine the time-independent wave potential \( q(x) \) by observing the acceleration from the flexible portion of the boundary \( \Gamma_0 \).

The mathematical challenge in this problem stems from the fact that we are dealing with the “coupling” on the part of the boundary and the main technical difficulty associated with this structure is the lack of the compactness of the resolvent. As a result, the space regularity for the solution of the wave equation component is limited by the structure on the plate and hence this will prevent us going to higher dimension \((n > 7)\) no matter how smooth the initial data is. This is a distinguished feature of this structural acoustic model comparing to the purely wave equation model as in that case the solution can be as smooth as we want as long as the initial data is smooth enough. In this present paper, we prove the cases where the dimension \( n = 2 \) and \( 3 \) (physical meaningful cases) by using the Carleman estimate for the Neumann problem in \cite{29} and an operator theoretic formulation. We show that indeed the observation of the acceleration on the plate can determine the potential \( q \) under some restrictions on the initial data and some geometrical conditions on the boundary. As we mentioned, the argument will also work for dimension up to \( n = 7 \).

1.3. Main Assumptions and Preliminaries. In this section we state the main geometrical assumptions throughout this paper. These assumptions are essential in order to establish the Carleman estimate stated in section 2.

Let \( \nu = [\nu_1, \cdots, \nu_n] \) be the unit outward normal vector on \( \Gamma \), and let \( \frac{\partial}{\partial \nu} = \nabla \cdot \nu \) denote the corresponding normal derivative. Moreover, we assume the following geometric conditions on \( \Gamma_1 = \Gamma \setminus \Gamma_0 \):

\[
(A.1) \text{There exists a strictly convex (real-valued) non-negative function } d : \overline{\Omega} \rightarrow \mathbb{R}^+, \text{ of class } C^3(\overline{\Omega}), \text{ such that, if we introduce the (conservative) vector field } h(x) = [h_1(x), \cdots, h_n(x)] = \nabla d(x), x \in \Omega, \text{ then the following two properties hold true:}
\]

\[
(i) \quad \frac{\partial d}{\partial \nu} \bigg|_{\Gamma_1} = \nabla d \cdot \nu = h \cdot \nu = 0; \quad h \equiv \nabla d
\]
(ii) the (symmetric) Hessian matrix $H_d$ of $d(x)$ [i.e., the Jacobian matrix $J_h$ of $h(x)$] is strictly positive definite on $\overline{\Omega}$: there exists a constant $\rho > 0$ such that for all $x \in \overline{\Omega}$:

$$H_d(x) = J_h(x) = \begin{bmatrix} d_{x_1x_1} & \cdots & d_{x_1x_n} \\ \vdots & \ddots & \vdots \\ d_{x_nx_1} & \cdots & d_{x_nx_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial x_1} & \cdots & \frac{\partial h_n}{\partial x_n} \end{bmatrix} \geq \rho I$$

(A.2) $d(x)$ has no critical point on $\overline{\Omega}$:

$$\inf_{x \in \Omega} |h(x)| = \inf_{x \in \Omega} |\nabla d(x)| = s > 0$$

Remark 1.2. One canonical example is that $\Gamma_1$ is flat (not the case in our problem setting here), where then we can take $d(x) = |x - x_0|^2$, with $x_0$ on the hyperplane containing $\Gamma_1$ and outside $\Omega$, then $h(x) = \nabla d(x) = 2(x - x_0)$ is radial. However, in general $h(x)$ is not necessary radial. In particularly in our case where $\Gamma_1$ is convex, the corresponding required $d(x)$ can also be explicitly constructed. For more examples of such function $d(x)$ with different geometries of $\Gamma_1$, we refer to the appendix of [29].

Next we introduce an abstract operator theoretic formulation associated to (1.1) for which we will need the following facts and definitions: Let the operator $A$ be

$$Az = -\Delta z - q(x)z, \quad D(A) = \{ z : \Delta z + q(x)z \in L^2(\Omega), \frac{\partial z}{\partial \nu} |_{\Gamma} = 0 \}$$

Notice the lower-order part is a perturbation which preserves generation of the self-adjoint principle part $A_N$ (e.g. [27]), where $A_N : L^2(\Omega) \supset D(A_N) \to L^2(\Omega)$ is defined by:

$$A_Nz = -\Delta z, \quad D(A_N) = \{ z : \Delta z \in L^2(\Omega), \frac{\partial z}{\partial \nu} |_{\Gamma} = 0 \}$$

Then $A_N$ is positive self-adjoint and

$$D(A_N^{\frac{1}{2}}) = H^1(\Gamma_1, \Omega) = \{ z : z \in H^1(\Omega), \frac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma_1 \}$$

Then we define the Neumann map $N$ by:

$$z = Ng \iff \begin{cases} \Delta z = 0 \quad \text{in} \quad \Omega \\ \frac{\partial z}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_1 \\ \frac{\partial z}{\partial \nu} = g \quad \text{on} \quad \Gamma_0 \end{cases}$$

By elliptic theory

$$N \in \mathcal{L}(L^2(\Gamma_0), H^{3/2}(\Omega))$$

Now we define

$$\mathcal{B} = A_N N : L^2(\Gamma_0) \to D(A_N^{\frac{1}{2}})'$$
via the conjugation $B^* = N^* A_N$. Then with $v \in L^2(\Gamma)$ and for any $y \in D(A_N^{\frac{1}{2}})$ we have
\begin{equation}
(B^* y, v)_\Gamma = -(N^* A_N y, v)_\Gamma = -(A_N y, N v)_\Omega = (\Delta y, N v)_\Omega
\end{equation}
\begin{equation}
= (y, \Delta (N v))_\Omega + (\frac{\partial y}{\partial \nu}, N v)_\Gamma - (y, \frac{\partial (N v)}{\partial \nu})_\Gamma = -(y, v)_\Gamma
\end{equation}

by Green’s theorem, the definition of $N$ and the fact $\frac{\partial y}{\partial \nu} = 0$ on $\Gamma_1$ when $y \in D(A_N^{\frac{1}{2}})$. In other words, we have
\begin{equation}
N^* A_N y = \begin{cases} y, & \text{on } \Gamma_0 \\ 0, & \text{on } \Gamma_1 \end{cases} \quad \text{for } y \in D(A_N^{\frac{1}{2}})
\end{equation}
i.e. $B^* = N^* A_N$ is the restriction of the trace map from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\Gamma_0)$.

Last we set $\tilde{A} : L^2(\Gamma_0) \supset D(\tilde{A}) \to L^2(\Gamma_0)$ to be
\begin{equation}
\tilde{A} = \Delta^2, D(\tilde{A}) = \{ v \in H_0^2(\Gamma_0) : \Delta^2 v \in L^2(\Gamma_0) \}
\end{equation}
where $H_0^2(\Gamma_0) = \{ v \in H^2(\Omega) : v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial \Gamma_0 \}$. $\tilde{A}$ is self-adjoint, positive definite, and we have the characterization
\begin{equation}
D(\tilde{A}^{\frac{1}{2}}) = H_0^2(\Gamma_0)
\end{equation}

Now set
\begin{equation}
A = \begin{bmatrix}
0 & I & 0 & 0 \\
-A_N + q & 0 & 0 & B \\
0 & 0 & 0 & I \\
0 & -B^* & -A & -\tilde{A}
\end{bmatrix}
\end{equation}
on the energy space
\begin{equation}
H = D(A_N^{\frac{1}{2}}) \times L^2(\Omega) \times D(\tilde{A}^{\frac{1}{2}}) \times L^2(\Gamma_0)
\end{equation}
\begin{equation}
= H_1^1(\Omega) \times L^2(\Omega) \times H_0^2(\Gamma_0) \times L^2(\Gamma_0)
\end{equation}

Then we have the domain of the operator $A$
\begin{equation}
D(A) = \{ [z_0, z_1, v_0, v_1]^T \in [D(A_N^{\frac{1}{2}})]^2 \times [D(\tilde{A}^{\frac{1}{2}})]^2 \text{ such that} \\
\quad - z_0 + N v_1 \in D(A_N) \text{ and } v_0 + v_1 \in D(\tilde{A}) \}
\end{equation}
\begin{equation}
\quad = \{ [z_0, z_1, v_0, v_1]^T : z_0 \in H_1^1(\Omega), z_1 \in H_1^1(\Omega), v_0 \in H_0^2(\Gamma_0), v_1 \in H_0^2(\Gamma_0),
\quad \Delta + q)z_0 \in L^2(\Omega), \frac{\partial z_0}{\partial \nu} = v_1 \text{ on } \Gamma_0 \text{ and } v_0 + v_1 \in D(\tilde{A}) \}
\end{equation}
\begin{equation}
\quad = \{ [z_0, z_1, v_0, v_1]^T : z_0 \in H_1^2(\Omega), z_1 \in H_1^1(\Omega), v_0 \in H_0^2(\Gamma_0), v_1 \in H_0^2(\Gamma_0),
\quad \frac{\partial z_0}{\partial \nu} = v_1 \text{ on } \Gamma_0 \text{ and } v_0 + v_1 \in D(\tilde{A}) \}
\end{equation}
where in the last step we get $z_0 \in H^2(\Omega)$ from $q \in L^\infty(\Omega)$ and $(\Delta + q)z_0 \in L^2(\Omega)$ due to elliptic theory. Therefore with these notations, the original system (1.1) becomes to the first order abstract differential equation

$$\frac{dy}{dt} = Ay$$

where $y = [z, z_t, v, v_t]^T$. From semigroup theory, when the initial conditions $[z_0, z_1, v_0, v_1]$ are in $D(A)$ we have that the solution $y$ satisfies

$$y \in D(A), \quad y_t \in H$$

**Remark 1.3.** The structure of $A$ reflects the coupled nature of this structural acoustic system (1.1). One distinguished feature of the system is that the resolvent of $A$ is not compact. However, it can still be shown that $A$ generates a $C_0$-semigroup of contractions $\{e^{At}\}_{t \geq 0}$ which establishes the well-posedness of the system [4].

### 1.4. Main results.

For the inverse problems stated in section 1.1, we have the following results:

**Theorem 1.4.** (Uniqueness for the linear inverse problem) Under the main assumptions (A.1), (A.2) and let

$$T > 2 \sqrt{\max_{x \in \Omega} d(x)}$$

Moreover, let

$$R \in W^{3,\infty}(Q)$$

and

$$|R \left( x, \frac{T}{2} \right) | \geq r_0 > 0, \quad |R_t \left( x, \frac{T}{2} \right) | \geq r_1 > 0$$

for some positive constants $r_0$, $r_1$ and $x \in \overline{\Omega}$. In addition, let

$$q \in L^\infty(\Omega)$$

If the weak solution $\{w = w(f), u = u(f)\}$ to system (1.2) satisfies

$$w, w_t, w_{tt} \in H^2(Q) = H^2(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$$

and

$$u_{tt}(f)(x, t) = 0, \quad x \in \Gamma_0, t \in [0, T]$$

then $f(x) = 0$, $x \in \Omega$.

**Theorem 1.5.** (Uniqueness for the nonlinear inverse problem) Under the main assumptions (A.1), (A.2), assume (1.21) and

$$q, p \in L^\infty(\Omega)$$

Let either of $z(q)$ and $z(p)$ satisfy

$$z \in W^{3,\infty}(Q)$$
Moreover, let
\begin{equation}
|z_0(x)| \geq s_0 > 0, \quad |z_1(x)| \geq s_1 > 0
\end{equation}
for some positive constants \( s_0, s_1 \) and \( x \in \overline{\Omega} \). If the weak solutions \( \{z(q), v(q)\} \) and \( \{z(p), v(p)\} \) to system (1.1) satisfy
\begin{equation}
z(q) - z(p), z_t(q) - z_t(p), z_{tt}(q) - z_{tt}(p) \in H^2(Q)
\end{equation}
and
\begin{equation}
v_{tt}(q)(x,t) = v_{tt}(p)(x,t), \quad x \in \Gamma_0, t \in [0,T]
\end{equation}
then \( q(x) = p(x), x \in \Omega \).

**Theorem 1.6.** (Stability for the linear inverse problem) Under the main assumptions (A.1), (A.2), assume (1.21), (1.22), (1.23) and (1.24). Moreover, let
\begin{equation}
R_t \in H^{\frac{1}{2}+\epsilon}(0,T; L^\infty(\Omega))
\end{equation}
for some \( 0 < \epsilon < \frac{1}{2} \). Then there exists a constant \( C = C(\Omega, T, \Gamma_0, \varphi, q, R) > 0 \) such that
\begin{equation}
\|f\|_{L^2(\Omega)} \leq C \left( \|u_t\|_{L^2(\Gamma_0 \times [0,T])} + \|u_{tt}\|_{L^2(\Gamma_0 \times [0,T])} + \|\Delta^2 u_{tt}\|_{L^2(\Gamma_0 \times [0,T])} \right)
\end{equation}
for all \( f \in L^2(\Omega) \).

**Theorem 1.7.** (Stability for the nonlinear inverse problem) Under the main assumptions (A.1), (A.2), assume (1.21), (1.27), (1.28) and (1.29). Moreover, let the initial data satisfy the compatibility condition

1. When \( n = 2 \), \( [z_0, z_1, v_0, v_1] \in D(A^2) \) where
\[
D(A^2) = \{ [z_0, z_1, v_0, v_1]^T : z_0 \in H^{1}_{\Gamma_1}(\Omega), z_1 \in H^2_{\Gamma_1}(\Omega), v_0 \in H^2_0(\Gamma_0), v_1 \in H^2_0(\Gamma_0),
\]
\[
\bar{A}(v_0 + v_1) + B^* z_1 \in H^2_0(\Gamma_0), v_1 - \bar{A}(v_0 + v_1) - B^* z_1 \in D(\bar{A}),
\]
\[
\frac{\partial z_0}{\partial \nu}|_{\Gamma_0} = v_1, \frac{\partial z_1}{\partial \nu}|_{\Gamma_0} = -\bar{A}(v_0 + v_1) - B^* z_1
\]

2. When \( n = 3 \), \( [z_0, z_1, v_0, v_1] \in D(A^3) \) where
\[
D(A^3) = \{ [z_0, z_1, v_0, v_1]^T : z_0 \in H^{1}_{\Gamma_1}(\Omega), z_1 \in H^3_{\Gamma_1}(\Omega), v_0 \in H^2_0(\Gamma_0), v_1 \in H^2_0(\Gamma_0),
\]
\[
\bar{A}(v_0 + v_1) + B^* z_1 \in H^2_0(\Gamma_0), \frac{\partial z_0}{\partial \nu}|_{\Gamma_0} = v_1, \frac{\partial z_1}{\partial \nu}|_{\Gamma_0} = -\bar{A}(v_0 + v_1) - B^* z_1
\]
\[
\bar{A}(v_0 + v_1) + B^* z_1 + \bar{A}[v_1 - \bar{A}(v_0 + v_1) - B^* z_1] + B^* \{(-A_N + q)z_0 + Bv_1\} \in D(\bar{A})
\]
\[
\bar{A}(v_1 - \bar{A}(v_0 + v_1) - B^* z_1) + B^* \{(-A_N + q)z_0 + Bv_1\} \in H^2_0(\Gamma_0),
\]
\[
\frac{\partial \{(-A_N + q)z_0 + Bv_1\}}{\partial \nu}|_{\Gamma_0} = -\bar{A}[v_1 - \bar{A}(v_0 + v_1) - B^* z_1] - B^* \{(-A_N + q)z_0 + Bv_1\}
\]

Then there exists a constant \( C = C(\Omega, T, \Gamma_0, \varphi, q, p, z_0, z_1, v_0, v_1) > 0 \) such that
\begin{equation}
\|q - p\|_{L^2(\Omega)} \leq C \left( \|v_{tt}(q) - v_{tt}(p)\|_{L^2(\Gamma_0 \times [0,T])} + \|v_{ttt}(q) - v_{ttt}(p)\|_{L^2(\Gamma_0 \times [0,T])} + \|\Delta^2 (v_{tt}(q) - v_{tt}(p))\|_{L^2(\Gamma_0 \times [0,T])} \right)
\end{equation}
for all \( q, p \in W^{1,\infty}(\Omega) \) when \( n = 2 \) and all \( q, p \in W^{2,\infty}(\Omega) \) when \( n = 3 \).

The rest of this paper is organized as follows: In section 2 we give the key Carleman estimate that is used in the proof of uniqueness result. Based on the same Carleman estimate, we also prove an observability inequality that is needed in section 5. Section 3 to 6 are devoted to the proofs of our main results Theorems 1.4 to 1.7. Some concluding remarks will be given in section 7.

2. CARLEMAN ESTIMATE AND OBSERVABILITY INEQUALITY

2.1. Carleman Estimate. In this section, we state a Carleman estimate result that plays a key role in the proof of our uniqueness theorem. The result is due to [29].

We first introduce the pseudo-convex function \( \varphi(x, t) \) defined by

\[
\varphi(x, t) = d(x) - c \left( t - \frac{T}{2} \right)^2; \quad x \in \Omega, t \in [0, T]
\]

where \( T \) is as in (1.21) and \( 0 < c < 1 \) is selected as follows: By (1.21), there exists \( \delta > 0 \) such that

\[
T^2 > 4 \max_{x \in \Omega} d(x) + 4\delta
\]

For this \( \delta > 0 \), there exists a constant \( c, 0 < c < 1 \), such that

\[
cT^2 > 4 \max_{x \in \Omega} d(x) + 4\delta
\]

Henceforth, with \( T \) and \( c \) chosen as described above, this function \( \varphi(x, t) \) has the following properties:

(a) For the constant \( \delta > 0 \) fixed in (2.2) and for any \( t > 0 \)

\[
\varphi(x, t) \leq \varphi(x, \frac{T}{2}), \quad \varphi(x, 0) = \varphi(x, T) \leq d(x) - c\frac{T^2}{4} \leq -\delta
\]

uniformly in \( x \in \Omega \).

(b) There are \( t_0 \) and \( t_1 \), with \( 0 < t_0 < \frac{T}{2} < t_1 < T \), such that we have

\[
\min_{x \in \Omega, t \in [t_0, t_1]} \varphi(x, t) \geq \sigma
\]

where \( 0 < \sigma < \min_{x \in \Omega} d(x) \).

Moreover, if we introduce the space \( Q(\sigma) \) that is defined by the following

\[
Q(\sigma) = \{(x, t) | x \in \Omega, 0 \leq t \leq T, \varphi(x, t) \geq \sigma > 0\}
\]

Then an important property of \( Q(\sigma) \) is that (see [29]):

\[
[t_0, t_1] \times \Omega \subset Q(\sigma) \subset [0, T] \times \Omega
\]

Then for the wave equation of the form

\[
w_{tt}(x, t) - \Delta w(x, t) - q(x)w(x, t) = F(x, t), \quad x \in \Omega, t \in [0, T]
\]

we have the following Carleman-type estimate:
Theorem 2.1. Under the main assumptions (A.1) and (A.2), with \( \varphi(x,t) \) defined in (2.1). Let \( w \in H^2(Q) \) be a solution of the equation (2.8) where \( q \in L^\infty(\Omega) \) and \( F \in L_2(Q) \). Then the following one-parameter family of estimates hold true, with \( \rho > 0, \beta > 0, \) for all \( \tau > 0 \) sufficiently large and \( \epsilon > 0 \) small:

\[
BT|_w + 2 \int_Q e^{2\tau \varphi} |F|^2 dQ + C_{1,T} e^{2\tau \sigma} \int_Q w^2 dQ \geq (\tau \epsilon \rho - 2C_T) \int_Q e^{2\tau \varphi} \left( w_t^2 + |\nabla w|^2 \right) dQ + (2\tau^3 \beta + O(\tau^2) - 2C_T) \int_{Q(\sigma)} e^{2\tau \varphi} w^2 dx dt - c_T \tau^3 e^{-2\tau \delta}[E_w(0) + E_w(T)]
\]

Here \( \delta > 0, \sigma > 0 \) are the constants in (2.2), (2.5), while \( C_T, c_T \) and \( C_{1,T} \) are positive constants depending on \( T \) and \( d \). In addition, the boundary terms \( BT|_w \) are given explicitly by

\[
BT|_w = 2\tau \int_0^T \int_{\Gamma_0} e^{2\tau \varphi} (w_t^2 - |\nabla w|^2) h \cdot \nu d\Gamma dt + 8\tau \int_0^T \int_{\Gamma} e^{2\tau \varphi} \left( t - \frac{T}{2} \right) w_t \frac{\partial w}{\partial \nu} d\Gamma dt + 4\tau \int_0^T \int_{\Gamma} e^{2\tau \varphi} (h \cdot \nabla w) \frac{\partial w}{\partial \nu} d\Gamma dt + 4\tau^2 \int_0^T \int_{\Gamma} e^{2\tau \varphi} \left( |h|^2 - 4c^2(t - \frac{T}{2})^2 + \frac{\alpha}{2\tau} \right) w \frac{\partial w}{\partial \nu} d\Gamma dt + 2\tau \int_0^T \int_{\Gamma_0} e^{2\tau \varphi} \left[ 2\tau^2 \left( |h|^2 - 4c^2(t - \frac{T}{2})^2 \right) + \tau(\alpha - \Delta d - 2c) \right] w^2 h \cdot \nu d\Gamma dt
\]

where \( \alpha = \Delta d - 2c - 1 + k \) for \( 0 < k < 1 \) is a constant and \( E_w \) is defined as follows:

\[
E_w(t) = \int_{\Omega} \left[ w^2(x,t) + w_t^2(x,t) + |\nabla w(x,t)|^2 \right] d\Omega
\]

An immediate corollary of the estimate is the following (Theorem 6.1 in [29])

Corollary 2.2. Under the assumptions in Theorem (2.1), the following one-parameter family of estimates hold true, for all \( \tau \) sufficiently large, and for any \( \epsilon > 0 \) small:

\[
\overline{BT}|_w + 2 \int_0^T \int_{\Omega} e^{2\tau \varphi} F^2 dQ + \text{const.}_\varphi \int_0^T \int_{\Omega} F^2 dQ \geq k_\varphi[E_w(0) + E_w(T)]
\]

for a constant \( k_\varphi > 0 \) while \( \overline{BT}|_w \) is given by:

\[
\overline{BT}|_w = BT|_w + \text{const.}_\varphi \left[ \int_0^T \int_{\Gamma} \left| \frac{\partial w}{\partial \nu} \right| d\Gamma dt + \int_{\Gamma_0} \int_{\Gamma_0} w^2 d\Gamma_0 dt \right]
\]
Remark 2.3. For the proof of the above Carleman estimate and the corollary, we refer to [29] and we omit the details here.

2.2. Continuous Observability Inequality. Using the Carleman estimate in last section, we can prove the following observability inequality:

Theorem 2.4. Under the main assumptions (A.1) and (A.2), for the following initial boundary value problem

\[
\begin{align*}
  w_{tt}(x,t) &= \Delta w(x,t) + q(x)w(x,t) \quad \text{in } \Omega \times [0,T] \\
  w(\cdot, \frac{T}{2}) &= w_0(x) \quad \text{in } \Omega \\
  w_t(\cdot, \frac{T}{2}) &= w_1(x) \quad \text{in } \Omega \\
  \frac{\partial w}{\partial \nu}(x,t) &= 0 \quad \text{on } \Gamma_1 \times [0,T] \\
  \frac{\partial w}{\partial \nu}(x,t) &= g(x,t) \quad \text{on } \Gamma_0 \times [0,T]
\end{align*}
\]

(2.14)

where \( w_0 \in H^1(\Omega) \), \( w_1 \in L^2(\Omega) \), \( g \in L^2(\Gamma \times [0,T]) \) and \( q \in L^\infty(\Omega) \). We have the following continuous observability inequality:

\[
\|w_0\|^2_{H^1(\Omega)} + \|w_1\|^2_{L^2(\Omega)} \leq C \left( \|w\|^2_{L^2(\Omega \times [0,T])} + \|w_t\|^2_{L^2(\Omega \times [0,T])} + \|g\|^2_{L^2(\Gamma \times [0,T])} \right)
\]

where \( T \) is as in (1.21) and \( C = C(\Omega, T, \Gamma_0, \varphi, \tau, q) \) is a positive constant.

Proof. For the case when \( g = 0 \), we refer to [29] where the continuous observability inequality is established for zero Neumann data on the whole boundary. Here we give the proof for the case of general \( g \in L^2(\Gamma_0 \times [0,T]) \), which is still based on the proof in [29]. We first introduce the following result that is from the section 7.2 of [28].

Lemma 2.5. Let \( w \) be a solution of the equation

\[
  w_{tt}(x,t) = \Delta w(x,t) + q(x)w(x,t) + f(x,t) \quad \text{in } Q = \Omega \times [0,T]
\]

(2.15)

with \( q \in L^\infty(\Omega) \) and \( w \) in the following class:

\[
\begin{align*}
  w &\in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega)) \\
  w_t, \frac{\partial w}{\partial \nu} &\in L^2(0,T; L^2(\Gamma))
\end{align*}
\]

(2.16)

Given \( \epsilon > 0, \epsilon_0 > 0 \) arbitrary, given \( T > 0 \), there exists a constant \( C = C(\epsilon, \epsilon_0, T) > 0 \) such that

\[
\int_\epsilon^{T-\epsilon} \int_{\Gamma} |\nabla_{\tan} w|^2 d\Gamma dt \leq C \left( \int_0^T \int_{\Gamma} w_t^2 + \left( \frac{\partial w}{\partial \nu} \right)^2 d\Gamma dt + \|w\|^2_{L^2(0,T; H^{\frac{1}{2} + \epsilon_0}(\Omega))} + \|f\|^2_{H^{\frac{1}{2} + \epsilon_0}(Q)} \right)
\]

(2.17)

Now to prove (2.4), we first establish the following weaker conclusion under the assumptions (A.1) and (A.2)

\[
E \left( \frac{T}{2} \right) \leq C \left( \int_0^T \int_{\Gamma_0} [w^2 + w_t^2 + g^2] d\Gamma_0 dt + \|w\|^2_{L^2(0,T; H^{\frac{1}{2} + \epsilon_0}(\Omega))} \right)
\]

(2.18)
which is the desired inequality (2.4) polluted by the interior lower order term $\|w\|$. To see this, we introduce a preliminary equivalence first. Let $u \in H^1(\Omega)$, then the following inequality holds true: there exist positive constants $0 < k_1 < k_2 < \infty$, independent of $u$, such that

$$k_1 \int_{\Omega} [u^2 + |\nabla u|^2] d\Omega \leq \int_{\Omega} |\nabla u|^2 d\Omega + \int_{\Gamma_0} u^2 d\Gamma \leq k_2 \int_{\Omega} [u^2 + |\nabla u|^2] d\Omega$$

where $\Gamma_0$ is any (fixed) portion of the boundary $\Gamma$ with positive measure. Inequality (2.19) is obtained by combining the following two inequalities:

$$\int_{\Omega} u^2 d\Omega \leq c_1 \left[ \int_{\Omega} |\nabla u|^2 d\Omega + \int_{\Gamma_0} u^2 d\Gamma \right]; \quad \int_{\Gamma_0} u^2 d\Gamma \leq c_2 \int_{\Omega} [u^2 + |\nabla u|^2] d\Omega$$

The inequality on the left of (2.20) replaces Poincaré’s inequality, while the inequality on the right of (2.20) stems from (a conservative version of) trace theory. Thus, for $w \in H^2(Q)$, if we introduce

$$\varepsilon(t) = \int_{\Omega} \left[ |\nabla w(t)|^2 + w^2(t) \right] d\Omega + \int_{\Gamma_0} w^2(t) d\Gamma_0$$

where $\Gamma_0 = \Gamma \setminus \Gamma_1$ is as defined in the main assumptions, then (2.19) yields the equivalence

$$aE(t) \leq \varepsilon(t) \leq bE(t)$$

for some positive constants $a > 0$, $b > 0$.

Now in a standard way, we multiply equation (2.15) by $w_t$ and integrate over $\Omega$. After an application of the first Green’s identity, we have

$$\frac{1}{2} \frac{\partial}{\partial t} \left( \int_{\Omega} [w_t^2 + |\nabla w|^2] d\Omega + \int_{\Gamma_0} w^2 d\Gamma_0 \right) = \int_{\Gamma} \frac{\partial w}{\partial \nu} w_t d\Gamma + \int_{\Gamma_0} w w_t d\Gamma_0 + \int_{\Omega} [q(x) + f] w_t d\Omega$$

Notice that on both sides of (2.23) we have added term $\frac{1}{2} \frac{\partial}{\partial t} \int_{\Gamma_0} w^2 d\Gamma_0 = \int_{\Gamma_0} w w_t d\Gamma_0$. Recalling $\varepsilon(t)$ in (2.22), we integrate (2.23) over $(s, t)$ and obtain

$$\varepsilon(t) = \varepsilon(s) + 2 \int_s^t \left[ \int_{\Gamma} \frac{\partial w}{\partial \nu} w_t d\Gamma + \int_{\Gamma_0} w w_t d\Gamma_0 \right] dr + 2 \int_s^t \int_{\Omega} [q(x) + f] w_t d\Omega dr$$

We apply Cauchy-Schwartz inequality on $[q(x) + f] w_t$, invoke the left hand side $E(t) \leq \frac{1}{a} \varepsilon(t)$ of (2.19), and obtain

$$\varepsilon(t) \leq [\varepsilon(s) + N(T)] + C_T \int_s^t \varepsilon(r) dr$$

$$\varepsilon(s) \leq [\varepsilon(t) + N(T)] + C_T \int_s^t \varepsilon(r) dr$$
where we have set

\begin{equation}
N(T) = \int_0^T \int_\Omega f^2 dQ + 2 \int_0^T \int_\Gamma |\partial w w_t| d\Gamma dt + 2 \int_0^T \int_{\Gamma_0} |ww_t| d\Gamma_0 dt
\end{equation}

Gronwall’s inequality applied on (2.25), (2.26) then yields for $0 \leq s \leq t \leq T$,

\begin{equation}
\varepsilon(t) \leq [\varepsilon(s) + N(T)]e^{Cr(t-s)}; \quad \varepsilon(s) \leq [\varepsilon(t) + N(T)]e^{Cr(t-s)}
\end{equation}

We consider the following three cases here:

**Case 1:** $0 \leq s \leq t \leq \frac{T}{2}$. In this case we set $t = \frac{T}{2}$ and $s = t$ in the first inequality of (2.28); and set $s = 0$ in the second inequality of (2.28), to obtain

\begin{equation}
\varepsilon(\frac{T}{2}) \leq [\varepsilon(t) + N(T)]e^{Cr \frac{T}{2}}; \quad \varepsilon(0) \leq [\varepsilon(t) + N(T)]e^{Cr \frac{T}{2}}
\end{equation}

Summing up these two inequalities in (2.29) yields for $0 \leq t \leq \frac{T}{2}$,

\begin{equation}
\varepsilon(t) \geq \frac{\varepsilon(\frac{T}{2}) + \varepsilon(0)}{2} e^{-Cr \frac{T}{2}} - N(T)
\end{equation}

after recalling the left hand side of the equivalence in (2.22).

**Case 2:** $\frac{T}{2} \leq s \leq t \leq T$. In this case we set $t = T$ and $s = t$ in the first inequality of (2.28); and set $s = \frac{T}{2}$ in the second inequality of (2.28), to obtain

\begin{equation}
\varepsilon(T) \leq [\varepsilon(t) + N(T)]e^{Cr \frac{T}{2}}; \quad \varepsilon(\frac{T}{2}) \leq [\varepsilon(t) + N(T)]e^{Cr \frac{T}{2}}
\end{equation}

Summing up these two inequalities in (2.31) yields for $\frac{T}{2} \leq t \leq T$,

\begin{equation}
\varepsilon(t) \geq \frac{\varepsilon(\frac{T}{2}) + \varepsilon(T)}{2} e^{-Cr \frac{T}{2}} - N(T)
\end{equation}

after recalling the left hand side of the equivalence in (2.22).

**Case 3:** $0 \leq s \leq \frac{T}{2} \leq t \leq T$. In this case we set $t = 0$ and $s = t$ in the first inequality of (2.28); and set $s = \frac{T}{2}$ in the second inequality of (2.28), to obtain

\begin{equation}
\varepsilon(0) \leq [\varepsilon(t) + N(T)]e^{Cr \frac{T}{2}}; \quad \varepsilon(\frac{T}{2}) \leq [\varepsilon(t) + N(T)]e^{Cr \frac{T}{2}}
\end{equation}

Summing up these two inequalities in (2.33) yields for $\frac{T}{2} \leq t \leq T$,

\begin{equation}
\varepsilon(t) \geq \frac{\varepsilon(\frac{T}{2}) + \varepsilon(0)}{2} e^{-Cr \frac{T}{2}} - N(T)
\end{equation}

after recalling the left hand side of the equivalence in (2.22).
In summary, we get for any $0 \leq t \leq T$,

$$
\varepsilon(t) \geq \frac{a}{2} E\left(\frac{T}{2}\right) e^{-C_T^T} - N(T)
$$

We now apply the Corollary 2.2 of the Carleman estimate, except on the interval $[\epsilon, T - \epsilon]$, rather than on $[0, T]$ as in (2.12). Thus, we obtain since $f = 0$:

$$
BT_{|[\epsilon, T - \epsilon] \times \Gamma} \geq k_\varphi E(\epsilon)
$$

where $BT_{|[\epsilon, T - \epsilon] \times \Gamma}$ is given as in (2.13). Since we have $\frac{\partial w}{\partial \nu} = 0$ on $\Gamma_1 \times [0, T]$ and $\frac{\partial w}{\partial \nu} = g(x, t)$ on $\Gamma_0 \times [0, T]$ by (2.14), with the additional information that $h \cdot \nu = 0$ on $\Gamma_1$ by the assumption (A.1). Thus, by using the explicit expression (2.10) for $BT_{|w}$, we have that $BT_{|[\epsilon, T - \epsilon] \times \Gamma}$ is given by:

$$
BT_{|[\epsilon, T - \epsilon] \times \Gamma} = 2\tau \int_{\epsilon}^{T - \epsilon} \int_{\Gamma_0} e^{2\tau \varphi} \left( w^2 - |\nabla w|^2 \right) h \cdot \nu d\Gamma dt
$$

Next, by the right side of equivalences (2.22) and (2.35), we obtain

$$
E(\epsilon) \geq \frac{\varepsilon(\epsilon)}{b} \geq \frac{a}{2b} E\left(\frac{T}{2}\right) e^{-C_T^T} - 2 \int_0^T \int_{\Gamma_0} |gw| d\Gamma dt - 2 \int_0^T \int_{\Gamma_0} |ww| d\Gamma_0 dt
$$

recalling $N(T)$ in (2.27). We use (2.38) in (2.36). Finally, we invoke estimate (2.17) of Lemma 2.5 on the first and the third integral terms of (2.37). This way, we readily obtain (2.18), which is our desired inequality polluted by $\|w\|_{L^2(0,T;H^{1+\alpha}(\Omega))}$. To eliminate this interior lower order term, we can apply the standard compactness/uniqueness argument (e.g. [24]) by invoking the global uniqueness Theorem 7.1 in [29]. \qed
3. **Proof of Theorem 1.4**

We let \( \bar{w} = \bar{w}(f) = w_t(f) \) then from (1.2) we have \( \bar{w}, u \) satisfy

\[
\begin{cases}
\ddot{\bar{w}}(x,t) - \Delta \bar{w}(x,t) - q(x)\bar{w}(x,t) = f(x)R_t(x,t) & \text{in } \Omega \times [0,T] \\
\frac{\partial \bar{w}}{\partial \nu}(x,t) = 0 & \text{on } \Gamma_1 \times [0,T] \\
\bar{w}(x,t) = -u_{tt}(x,t) - \Delta^2 u(x,t) - \Delta^2 u_t(x,t) & \text{on } \Gamma_0 \times [0,T] \\
u(t,\frac{T}{2}) = 0 & \text{on } \partial \Gamma_0 \times [0,T] \\
\bar{w}(\cdot,\frac{T}{2}) = f(x)R(x,\frac{T}{2}) & \text{in } \Omega \\
u(\cdot,\frac{T}{2}) = 0 & \text{on } \Gamma_0 \\
u(\cdot,\frac{T}{2}) = 0 & \text{on } \Gamma_0
\end{cases}
\]

(3.1)

Under the assumptions in Theorem 1.4 we can apply the Carleman estimate to the wave equation in the system (3.1) \( \bar{w}_{tt}(x,t) - \Delta \bar{w}(x,t) - q(x)\bar{w}(x,t) = f(x)R_t(x,t) \) and get

\[
BT|_{\bar{w}} + 2 \int_Q e^{2\tau \varphi} |fR_t|^2 dQ + C_{1,T} e^{2\tau \sigma} \int_Q \bar{w}^2 dQ \geq (\tau \epsilon \rho - 2C_T) \int_Q e^{2\tau \varphi} [\bar{w}_t^2 + |\nabla \bar{w}|^2] dQ
\]

\[
+ [2\tau^3 \beta + O(\tau^2) - 2C_T] \int_{Q(\sigma)} e^{2\tau \varphi} \overline{\bar{w}}^2 dxdt - c_T \tau^3 e^{-2\tau \delta}[E_\varphi(0) + E_\varphi(T)]
\]

where the boundary terms are given explicitly by

\[
BT|_{\bar{w}} = 2\tau \int_0^T \int_{\Gamma_0} e^{2\tau \varphi} (\overline{\bar{w}_t^2} - |\nabla \bar{w}|^2) h \cdot \nu dt
\]

\[
+ 8\tau C \int_0^T \int_\Gamma e^{2\tau \varphi} (t - \frac{T}{2}) \overline{\bar{w}_t} \frac{\partial \bar{w}}{\partial \nu} d\Gamma dt
\]

\[
+ 4\tau \int_0^T \int_\Gamma e^{2\tau \varphi} (h \cdot \nabla \bar{w}) \frac{\partial \bar{w}}{\partial \nu} d\Gamma dt
\]

\[
+ 4\tau^2 \int_0^T \int_\Gamma e^{2\tau \varphi} \left( |h|^2 - 4c^2 (t - \frac{T}{2})^2 + \frac{\alpha}{2\tau} \right) \overline{\bar{w}_t} \frac{\partial \bar{w}}{\partial \nu} d\Gamma dt
\]

\[
+ 2\tau \int_0^T \int_{\Gamma_0} e^{2\tau \varphi} \left[ 2\tau^2 \left( |h|^2 - 4c^2 (t - \frac{T}{2})^2 \right) \right] \overline{\bar{w}_t^2} h \cdot \nu d\Gamma dt
\]

(3.2)

Since we have the extra observation that \( u_{tt}(x,t) = 0 \) on \( \Gamma_0 \times [0,T] \) and note that the initial conditions \( u(x,\frac{T}{2}) = u_t(x,\frac{T}{2}) = 0 \) on \( \Gamma_0 \), thus by the fundamental theorem of calculus we have \( u(x,t) = 0 \) on \( \Gamma_0 \times [0,T] \) and hence from the coupling in the system (3.1) we get

\[
\bar{w}(x,t) = -u_{tt}(x,t) - \Delta^2 u(x,t) - \Delta^2 u_t(x,t) = 0 \text{ on } \Gamma_0 \times [0,T]
\]

(3.3)
and

\[ \frac{\partial \bar{w}}{\partial \nu}(x, t) = u_t(x, t) = 0 \text{ on } \Gamma_0 \times [0, T] \]

Plugging (3.3) and (3.4) into (3.2), note also that \( \frac{\partial \bar{w}}{\partial \nu} = 0 \) on \( \Gamma_1 \times [0, T] \), therefore we get \( BT|\bar{w} \equiv 0 \).

In addition, in view of (1.22), (1.23), we have \( |fR_t| \leq C|f| \) for some positive constant \( C \) depend on \( R_t \). Moreover, notice that \( \lim_{\tau \to \infty} \tau^3 e^{-2\tau \delta} = 0 \). Hence when \( \tau \) is sufficiently large, the above Carleman estimate can be rewritten as the following:

\[ C_{1, \tau} \int_Q e^{2\tau \varphi} \bar{w}_t^2 + |\nabla \bar{w}|^2 dQ + C_{2, \tau} \int_{Q(\sigma)} e^{2\tau \varphi} \bar{w}_l^2 dxdt \leq C \int_Q e^{2\tau \varphi} |f|^2 dQ + C e^{2\tau \sigma} \]

where we set

\[ C_{1, \tau} = \tau \rho - 2C_T, \quad C_{2, \tau} = 2\tau^3 \beta + O(\tau^2) - 2C_T \]

and \( C \) denote generic constants which do not depend on \( \tau \) and henceforth we will use this notation for the rest of this paper. In addition, note that \( f \) is time-independent, so if we differentiate the system (3.1) in time twice, we can get the following wave equations for \( \bar{w}_t \) and \( \bar{w}_{tt} \):

\[ (\bar{w}_t)_{tt}(x, t) - \Delta \bar{w}_t(x, t) - q(x)\bar{w}_t(x, t) = f(x)R_t(x, t) \]

and

\[ (\bar{w}_{tt})_{tt}(x, t) - \Delta \bar{w}_{tt}(x, t) - q(x)\bar{w}_{tt}(x, t) = f(x)R_{tt}(x, t) \]

Notice the assumptions (1.22), (1.23), therefore we have similarly as (3.5) the following estimates for the two new systems:

\[ C_{1, \tau} \int_Q e^{2\tau \varphi} \bar{w}_t^2 + |\nabla \bar{w}|^2 dQ + C_{2, \tau} \int_{Q(\sigma)} e^{2\tau \varphi} \bar{w}_l^2 dxdt \leq C \int_Q e^{2\tau \varphi} |f|^2 dQ + C e^{2\tau \sigma} \]

and

\[ C_{1, \tau} \int_Q e^{2\tau \varphi} \bar{w}_{tt}^2 + |\nabla \bar{w}_{tt}|^2 dQ + C_{2, \tau} \int_{Q(\sigma)} e^{2\tau \varphi} \bar{w}_{tt}^2 dxdt \leq C \int_Q e^{2\tau \varphi} |f|^2 dQ + C e^{2\tau \sigma} \]

where \( \tau \) is sufficiently large and \( C_{1, \tau}, C_{2, \tau} \) are defined as in (3.6).

Adding (3.5), (3.9) and (3.10) together we then have

\[ C_{1, \tau} \int_Q e^{2\tau \varphi} [\bar{w}_t^2 + \bar{w}_{tt}^2 + \bar{w}_l^2 + |\nabla \bar{w}|^2 + |\nabla \bar{w}_t|^2 + |\nabla \bar{w}_{tt}|^2] dQ + C_{2, \tau} \int_{Q(\sigma)} e^{2\tau \varphi} [\bar{w}_l^2 + \bar{w}_t^2 + \bar{w}_{tt}^2] dxdt \leq C \left( \int_Q e^{2\tau \varphi} |f|^2 dQ + e^{2\tau \sigma} \right) \]
Again we use the wave equation \( \ddot{w}_t(x, t) - \Delta \dot{w}(x, t) - q(x)\dot{w}(x, t) = f(x)R_t(x, t) \), plugging in the initial time of \( t = \frac{T}{2} \) and use the zero initial conditions of \( \dot{w}(\cdot, \frac{T}{2}) = 0 \), we have

\[
\begin{align*}
(3.12) \quad \ddot{w}_t(x, \frac{T}{2}) - \Delta \dot{w}(x, \frac{T}{2}) - q(x)\dot{w}(x, \frac{T}{2}) = \ddot{w}_t(x, \frac{T}{2}) = f(x)R_t(x, \frac{T}{2})
\end{align*}
\]

Since \( |R_t(x, \frac{T}{2})| \geq r_1 > 0 \) from (1.23), therefore we have \( |f(x)| \leq C|\ddot{w}_t(x, \frac{T}{2})| \) and hence we have the following estimates on \( \int_Q e^{2\varphi}|f|^2dQ \):

\[
\begin{align*}
(3.13) \quad \int_Q e^{2\varphi}|f|^2dQ &= \int_0^T \int_\Omega e^{2\varphi(x, t)}|f(x)|^2d\Omega dt \\
&\leq C \int_0^T \int_\Omega e^{2\varphi(x, t)}|\ddot{w}_t(x, \frac{T}{2})|^2d\Omega dt \\
&\leq C \int_\Omega e^{2\varphi(x, \frac{T}{2})}|\ddot{w}_t(x, \frac{T}{2})|^2d\Omega \\
= C \left( \int_\Omega \int_0^{\frac{T}{2}} \frac{d}{ds}(e^{2\varphi(x, s)}|\ddot{w}_t(x, s)|^2)dsd\Omega + \int_\Omega e^{2\varphi(x, 0)}|\ddot{w}_t(x, 0)|^2d\Omega \right) \\
= C \left( 4C_T \int_\Omega \int_0^{\frac{T}{2}} (T - s)e^{2\varphi(x, s)}|\ddot{w}_t(x, s)|^2d\Omega \\
+ 2 \int_\Omega \int_0^{\frac{T}{2}} e^{2\varphi}|\ddot{w}_t(x, s)||\ddot{w}_tt(x, s)|dsd\Omega + \int_\Omega e^{2\varphi(x, 0)}|\ddot{w}_t(x, 0)|^2d\Omega \right) \\
\leq C \left( \tau \int_\Omega \int_0^{\frac{T}{2}} e^{2\varphi}|\ddot{w}_t|^2dtd\Omega + \int_\Omega \int_0^{\frac{T}{2}} e^{2\varphi}(|\ddot{w}_t|^2 + |\ddot{w}_tt|^2)dtd\Omega \\
+ \int_\Omega |\ddot{w}_t(x, 0)|^2d\Omega \right) \\
= C \left( \tau \int_Q e^{2\varphi}|\ddot{w}_t|^2dQ + \int_Q e^{2\varphi}(|\ddot{w}_t|^2 + |\ddot{w}_tt|^2)dQ \right) \\
= C \left( (\tau + 1) \int_Q e^{2\varphi}|\ddot{w}_t|^2dQ + \int_Q e^{2\varphi}|\ddot{w}_tt|^2dQ + e^{2\varphi} \right)
\end{align*}
\]

where in the above estimates we use the definition (2.11) and the property (2.4) of \( \varphi \) as well as Cauchy-Schwartz inequality. Collecting (3.13) with (3.11), we have

\[
(3.14) \quad C_{1, \tau} \int_Q e^{2\varphi}|\ddot{w}_t|^2 + \ddot{w}_tt + \ddot{w}_tt + |\nabla \ddot{w}|^2 + |\nabla \ddot{w}_t|^2 + |\nabla \ddot{w}_tt|^2)dQ \\
+C_{2, \tau} \int_{Q(\sigma)} e^{2\varphi}dtdt \leq C \left( (\tau + 1) \int_Q e^{2\varphi}|\ddot{w}_t|^2dQ + \int_Q e^{2\varphi}|\ddot{w}_tt|^2dQ + e^{2\varphi} \right)
\]
Note that in (3.14), the right hand side term \( C \int_Q e^{2\tau \varphi} |\bar{w}_{tt}|^2 dQ \) can be absorbed by the term \( C_1, \tau \int_Q e^{2\tau \varphi} [\bar{w}_t^2 + \bar{w}_{tt}^2 + \bar{w}_{ttt}^2] dQ \) on the left hand side when \( \tau \) is large enough. In addition, since \( e^{2\tau \varphi} < e^{2\tau \sigma} \) on \( Q \setminus Q(\sigma) \) by the definition of \( Q(\sigma) \), we have

\[
C(\tau + 1) \int_Q e^{2\tau \varphi} |\bar{w}_{tt}|^2 dQ = C(\tau + 1) \left( \int_{Q(\sigma)} e^{2\tau \varphi} |\bar{w}_{tt}|^2 dtdx + \int_{Q \setminus Q(\sigma)} e^{2\tau \varphi} |\bar{w}_{tt}|^2 dxdt \right) \\
\leq C(\tau + 1) \left( \int_{Q(\sigma)} e^{2\tau \varphi} |\bar{w}_{tt}|^2 dtdx + e^{2\tau \sigma} \int_{Q \setminus Q(\sigma)} |\bar{w}_{tt}|^2 dxdt \right) \\
\leq C(\tau + 1) \int_{Q(\sigma)} e^{2\tau \varphi} |\bar{w}_{tt}|^2 dtdx + C(\tau + 1) e^{2\tau \sigma}
\]

Again \( C(\tau + 1) \int_{Q(\sigma)} e^{2\tau \varphi} |\bar{w}_{tt}|^2 dtdx \) on the right hand side of (3.15) can be absorbed by \( C_2, \tau \int_{Q(\sigma)} e^{2\tau \varphi} [\bar{w}^2 + \bar{w}_t^2 + \bar{w}_{tt}^2] dxdt \) on the left hand side of (3.14) when taking \( \tau \) large enough. Therefore (3.14) becomes to

\[
C_1, \tau \int_Q e^{2\tau \varphi} [\bar{w}_t^2 + \bar{w}_{tt}^2 + \bar{w}_{ttt}^2 + |\nabla \bar{w}|^2 + |\nabla \bar{w}_t|^2 + |\nabla \bar{w}_{tt}|^2] dQ \\
+ C_2, \tau \int_{Q(\sigma)} e^{2\tau \varphi} [\bar{w}^2 + \bar{w}_t^2 + \bar{w}_{tt}^2] dxdt \leq C \left( (\tau + 1) e^{2\tau \sigma} + e^{2\tau \sigma} + \tau^3 e^{-2\delta} \right)
\]

Where we have

\[
C_1, \tau = \tau \epsilon \rho - C, \quad C_2, \tau = 2\tau^3 \beta + O(\tau^2)
\]

Now we take \( \tau \) sufficiently large such that \( C_1, \tau > 0, C_2, \tau > 0 \). Then in (3.16) we can drop the first term on the left hand side and get

\[
C_2, \tau \int_{Q(\sigma)} e^{2\tau \varphi} [\bar{w}^2 + \bar{w}_t^2 + \bar{w}_{tt}^2] dxdt \leq C [(\tau + 1) e^{2\tau \sigma} + e^{2\tau \sigma}] \\
\leq C(\tau + 2) e^{2\tau \sigma}
\]

Note again from (2.6) the definition of \( Q(\sigma) \), we have \( e^{2\tau \varphi} \geq e^{2\tau \sigma} \) on \( Q(\sigma) \), therefore (3.18) implies

\[
C_2, \tau \int_{Q(\sigma)} [\bar{w}^2 + \bar{w}_t^2 + \bar{w}_{tt}^2] dxdt \leq C(\tau + 2)
\]

Divide \( \tau + 2 \) on both sides of (3.19), we get

\[
\frac{C_2, \tau}{\tau + 2} \int_{Q(\sigma)} [\bar{w}^2 + \bar{w}_t^2 + \bar{w}_{tt}^2] dxdt \leq C
\]

By (3.17), \( \frac{C_2, \tau}{\tau + 2} \rightarrow \infty \) as \( \tau \rightarrow \infty \), thus (3.20) implies that we must have \( \bar{w} \equiv 0 \) on \( Q(\sigma) \) and hence we have

\[
f(x) R_t(x, t) = \bar{w}_{tt}(x, t) - \Delta \bar{w}(x, t) - q(x) \bar{w}(x, t) = 0, \quad (x, t) \in Q(\sigma)
\]
Recall again that $|R_t(x, T)| \geq r_1 > 0$ from (1.23) and the property that $Q \supset Q(\sigma) \supset \{t_0, t_1\} \times \Omega$ from (2.7). Thus we have from (3.21) that $f(x) \equiv 0$, for all $x \in \Omega$. \hfill \Box

4. Proof of Theorem 1.5

Setting $f(x) = q(x) - p(x)$, $w(x, t) = z(q)(x, t) - z(p)(x, t)$, $u(x, t) = v(q)(x, t) - v(p)(x, t)$ and $R(x, t) = z(p)(x, t)$, we then obtain (1.2) after the subtraction of (1.1) with $p$ from (1.1) with $q$. Since $R(x, T) = z(p)(x, T) = z_0(x)$ and $R_t(x, T) = z'(p)(x, T) = z_1(x)$, the conditions (1.29) imply (1.23). In addition, the condition $v(q)(x, t) = v(p)(x, t)$, $x \in \Gamma_0$, $t \in [0, T]$ implies that $u(x, t) = 0$ on $\Gamma_0 \times [0, T]$ and (1.30) implies (1.25). Therefore from the above Theorem 1.4 we conclude $f(x) = q(x) - p(x) = 0$, i.e., $q(x) = p(x)$, $x \in \Omega$. \hfill \Box

5. Proof of Theorem 1.6

In relation with this system (3.1), we define $\psi$ which satisfies the following equation

$$\begin{aligned}
\psi_t(x, t) &= \Delta \psi(x, t) + q(x)\psi(x, t) & \text{in} \ \Omega \times [0, T] \\
\frac{\partial \psi}{\partial n}(x, t) &= 0 & \text{on} \ \Gamma_1 \times [0, T] \\
\psi(\cdot, T) &= 0 & \text{on} \ \Gamma_0 \times [0, T] \\
\psi_t(\cdot, T) &= f(x)R(x, T) & \text{in} \ \Omega
\end{aligned}
$$

Set $y = w - \psi$, then we have $y$ satisfies the following initial-boundary value problem

$$\begin{aligned}
y_t(x, t) - \Delta y(x, t) - q(x)y(x, t) &= f(x)R_t(x, t) & \text{in} \ \Omega \times [0, T] \\
\frac{\partial y}{\partial n}(x, t) &= 0 & \text{on} \ \Gamma \times [0, T] \\
y(\cdot, T) &= 0 & \text{in} \ \Omega \\
y_t(\cdot, T) &= 0 & \text{in} \ \Omega
\end{aligned}
$$

It is easy to see that both (5.1) and (5.2) are well-posed. For the system (5.1), we apply the continuous observability inequality in Theorem 2.4 to get

$$\|fR(\cdot, T)\|_{L^2(\Omega)}^2 \leq C \left( \|\psi\|_{L^2(\Gamma_0 \times [0, T])}^2 + \|\psi_t\|_{L^2(\Gamma_0 \times [0, T])}^2 + \|\frac{\partial \psi}{\partial n}\|_{L^2(\Gamma_0 \times [0, T])}^2 \right)
$$

Notice that $|R(x, T)| \geq r_0 > 0$, $\frac{\partial \psi}{\partial n}(x, t) = u_t(x, t)$ on $\Gamma_0 \times [0, T]$ and $\frac{\partial \psi}{\partial n}(x, t) = 0$ on $\Gamma_1 \times [0, T]$, therefore we have from (5.3)

$$\|f\|_{L^2(\Omega)} \leq C \left( \|\psi\|_{L^2(\Gamma_0 \times [0, T])} + \|\psi_t\|_{L^2(\Gamma_0 \times [0, T])} + \|u_t\|_{L^2(\Gamma_0 \times [0, T])} \right)
$$

On the other hand, for the system (5.2), we have the following lemma:

**Lemma 5.1.** Let $q \in L^\infty(\Omega)$ and $R(x, t)$ satisfies $R_t \in H^{\frac{1}{2}+}(0, T; L^\infty(\Omega))$ for some $0 < \epsilon < \frac{1}{2}$ as in Theorem 1.6. If we define the operators $K$ and $K_1$ by $K, K_1: L^2(\Omega) \rightarrow L^2(\Gamma_0 \times [0, T])$, such that

$$Kf(x, t) = y(x, t), \quad (K_1f)(x, t) = y_t(x, t), \quad x \in \Gamma_0, t \in [0, T]
$$

where $y$ is the unique solution of the equation (5.2). Then $K$ and $K_1$ are both compact operators.
Proof. It suffices to just show that $K_1$ is compact, then it follows similarly that $K$ is also compact. Since $f \in L^2(\Omega)$ and $R_t \in H^{\frac{1}{2}+\epsilon}(0, T; L^\infty(\Omega))$, we have

\begin{equation}
(5.6) \quad f R_t \in H^{\frac{1}{2}+\epsilon}(0, T; L^2(\Omega))
\end{equation}

Therefore we have the solution $y$ satisfies (e.g. Corollary 5.3 in [27])

\begin{equation}
(5.7) \quad y \in C([0, T]; H^{\frac{1}{2}+\epsilon}(\Omega)), \quad y_t \in C([0, T]; H^{\frac{1}{2}+\epsilon}(\Omega))
\end{equation}

Hence by (5.6), $q \in L^\infty(\Omega)$ and $y_{tt} = \Delta y + q(x)y + fR_t$ we can get

\begin{equation}
(5.8) \quad y_{tt} \in L^2(0, T; H^{-\frac{1}{2}+\epsilon}(\Omega))
\end{equation}

In addition, by (5.7) and trace theorem we have $y_t \in C([0, T]; H^{\epsilon}(\Gamma))$. Since the embedding $H^{\epsilon}(\Gamma) \to L^2(\Gamma)$ is compact, we have by Lions-Aubin’s compactness criterion (e.g. Proposition III.1.3 in [33]) that the operator $K_1$ is a compact operator. \qed

Now we have that the inequality (5.4) becomes to

\begin{equation}
\|f\|_{L^2(\Omega)} \leq C \left( \|\psi\|_{L^2(\Gamma_0\times [0,T])} + \|\psi_t\|_{L^2(\Gamma_0\times [0,T])} + \|u_{tt}\|_{L^2(\Gamma_0\times [0,T])} \right)
\end{equation}

\begin{equation}
\leq C \left( \|\bar{w} - y\|_{L^2(\Gamma_0\times [0,T])} + \|\bar{w}_t - y_t\|_{L^2(\Gamma_0\times [0,T])} + \|u_{tt}\|_{L^2(\Gamma_0\times [0,T])} \right)
\end{equation}

\begin{equation}
\leq C \left( \|\bar{w}\|_{L^2(\Gamma_0\times [0,T])} + \|\bar{w}_t\|_{L^2(\Gamma_0\times [0,T])} + \|u_{tt}\|_{L^2(\Gamma_0\times [0,T])} \right)
\end{equation}

\begin{equation}
+ C\|y\|_{L^2(\Gamma_0\times [0,T])} + C\|y_t\|_{L^2(\Gamma_0\times [0,T])}
\end{equation}

\begin{equation}
= C \left( \|\bar{w}\|_{L^2(\Gamma_0\times [0,T])} + \|\bar{w}_t\|_{L^2(\Gamma_0\times [0,T])} + \|u_{tt}\|_{L^2(\Gamma_0\times [0,T])} \right)
\end{equation}

\begin{equation}
+ C\|Kf\|_{L^2(\Gamma_0\times [0,T])} + C\|K_1f\|_{L^2(\Gamma_0\times [0,T])}
\end{equation}

\begin{equation}
\leq C \left( \|u_{tt}\|_{L^2(\Gamma_0\times [0,T])} + \|u_{ttt}\|_{L^2(\Gamma_0\times [0,T])} + \|\Delta^2 u_{tt}\|_{L^2(\Gamma_0\times [0,T])} \right)
\end{equation}

\begin{equation}
+ C\|Kf\|_{L^2(\Gamma_0\times [0,T])} + C\|K_1f\|_{L^2(\Gamma_0\times [0,T])}
\end{equation}

where in the last step we use the coupling $\bar{w}(x, t) = -u_{tt}(x, t) - \Delta^2 u(x, t) - \Delta^2 u_t(x, t)$ on $\Gamma_0 \times [0, T]$ from (3.1) and again the initial conditions $u(\cdot, \frac{T}{2}) = u(\cdot, \frac{T}{2}) = 0$ on $\Gamma_0 \times [0, T]$ so that by the fundamental theorem of calculus, we have

\begin{equation}
(5.10) \quad \|u\|_{L^2(\Gamma_0\times [0,T])} \leq C\|u_{tt}\|_{L^2(\Gamma_0\times [0,T])} \leq C\|u_{tt}\|_{L^2(\Gamma_0\times [0,T])}
\end{equation}

To complete the proof, we need to absorb the last two terms in (5.9). To achieve that, we apply the compactness-uniquness argument. For simplicity we denote

\begin{equation}
\|u\|_X = \|u_{tt}\|_{L^2(\Gamma_0\times [0,T])} + \|u_{ttt}\|_{L^2(\Gamma_0\times [0,T])} + \|\Delta^2 u_{tt}\|_{L^2(\Gamma_0\times [0,T])}
\end{equation}

Suppose contrarily that the inequality (1.33) does not hold. Then there exists $f_n \in L^2(\Omega), n \geq 1$ such that

\begin{equation}
(5.11) \quad \|f_n\|_{L^2(\Omega)} = 1, \quad n \geq 1
\end{equation}

and

\begin{equation}
(5.12) \quad \lim_{n \to \infty} \|u(f_n)\|_X = 0
\end{equation}

From (5.11), there exists a subsequence, denoted again by $\{f_n\}_{n \geq 1}$ such that $f_n$ converges to some $f_0 \in L^2(\Omega)$ weakly in $L^2(\Omega)$. Moreover, since $K$ and $K_1$ are
Therefore by the uniqueness theorem \ref{uniqueness}, we have
\begin{equation}
\|K f_n - K f_m\|_{L^2(\Gamma_0 \times [0,T])} = 0, \quad \lim_{m,n \to \infty} \|K f_n - K f_m\|_{L^2(\Gamma_0 \times [0,T])} = 0
\end{equation}
On the other hand, it follows from \eqref{eq:2} that
\begin{equation}
\|f_n - f_m\|_{L^2(\Omega)} \leq C\|u(f_n) - u(f_m)\|_{X} + C\|K f_n - K f_m\|_{L^2(\Gamma_0 \times [0,T])} + C\|K f_n - K f_m\|_{L^2(\Gamma_0 \times [0,T])}
\end{equation}
\begin{equation}
\leq C\|u(f_n)\|_{X} + C\|u(f_m)\|_{X} + C\|K f_n - K f_m\|_{L^2(\Gamma_0 \times [0,T])} + C\|K f_n - K f_m\|_{L^2(\Gamma_0 \times [0,T])}
\end{equation}
Thus by \eqref{eq:12} and \eqref{eq:13}, we have that
\begin{equation}
\lim_{m,n \to \infty} \|f_n - f_m\|_{L^2(\Omega)} = 0
\end{equation}
and hence $f_n$ converges strongly to $f_0$ in $L^2(\Omega)$. So by \eqref{eq:11} we obtain
\begin{equation}
\|f_0\|_{L^2(\Omega)} = 1
\end{equation}
On the other hand, by \eqref{eq:23} and a usual a-priori estimate, we have that
\begin{equation}
\|\bar{w}(f)\|_{C([0,T];H^1(\Omega))} + \|\bar{w}_t(f)\|_{C([0,T];L^2(\Omega))} \leq C\|R_t\|_{L^1(0,T;L^2(\Omega))}
\end{equation}
\begin{equation}
\leq C\|R_t\|_{L^1(0,T;L^\infty(\Omega))}\|f\|_{L^2(\Omega)}
\end{equation}
Hence trace theorem implies that
\begin{equation}
\|\bar{w}(f)\|_{L^2(\Gamma_0 \times [0,T])} \leq C\|f\|_{L^2(\Omega)}
\end{equation}
where $C > 0$ depends on $\|R_t\|_{L^1(0,T;L^\infty(\Omega))}$. Therefore by \eqref{eq:18} we have
\begin{equation}
\lim_{n \to \infty} \|\bar{w}(f_n) - \bar{w}(f_0)\|_{L^2(\Gamma_0 \times [0,T])} \leq C\lim_{n \to \infty} \|f_n - f_0\|_{L^2(\Omega)} = 0
\end{equation}
Moreover, by \eqref{eq:12} and the coupling $\bar{w}(x,t) = -u_{tt}(x,t) - \Delta^2 u(x,t) - \Delta^2 u_t(x,t)$ on $\Gamma_0 \times [0,T]$, we have
\begin{equation}
\lim_{n \to \infty} \|\bar{w}(f_n)\|_{L^2(\Gamma_0 \times [0,T])} \leq \lim_{n \to \infty} \|u\|_{X} = 0
\end{equation}
Thus by \eqref{eq:19} and \eqref{eq:20}, we obtain
\begin{equation}
\bar{w}(f_0)(x,t) = 0, \quad x \in \Gamma_0, t \in [0,T]
\end{equation}
Therefore from \eqref{eq:3} we have $u = u(f_0)$ satisfies the initial boundary problem:
\begin{equation}
\begin{cases}
-u_{tt}(x,t) - \Delta^2 u(x,t) - \Delta^2 u_t(x,t) = 0 & \text{in } \Gamma_0 \times [0,T] \\
u(x,t) = \frac{\partial u}{\partial t}(x,t) = 0 & \text{on } \partial \Gamma_0 \times [0,T] \\
u(\cdot, \frac{T}{2}) = 0 & \text{in } \Gamma_0 \\
u_t(\cdot, \frac{T}{2}) = 0 & \text{in } \Gamma_0
\end{cases}
\end{equation}
which has only zero solution, namely, we have $u(f_0)(x,t) = 0, x \in \Gamma_0, t \in [0,T]$. Therefore by the uniqueness theorem \ref{uniqueness}, we have $f_0 \equiv 0$ in $\Omega$ which contradicts with \eqref{eq:16}. Thus we must have
\begin{equation}
\|f\|_{L^2(\Omega)} \leq C \left(\|u_{tt}\|_{L^2(\Gamma_0 \times [0,T])} + \|u_{ttt}\|_{L^2(\Gamma_0 \times [0,T])} + \|\Delta^2 u_{tt}\|_{L^2(\Gamma_0 \times [0,T])}\right)
\end{equation}
and the proof of the theorem is complete. 
\hfill \Box
6. Proof of Theorem 1.1

We now go back to the original system (1.1).

Case 1: \( n = 2 \). Let \( \tilde{z} = z_t \) and \( \tilde{v} = v_t \), then the system (1.1) becomes to

\[
\begin{aligned}
&\frac{\partial \tilde{z}}{\partial t} = \Delta \tilde{z} + q \tilde{z} + Bv, \\
&\frac{\partial \tilde{v}}{\partial t} = \Delta \tilde{v} - \Delta^2 \tilde{v} - \Delta^2 v, \\
&\tilde{z} = z_{1t}, \\
&\tilde{v} = v_{1t}.
\end{aligned}
\]

(6.1)

By using the similar operator setting as in section 1.2 and notice the new initial conditions, we can compute the domain of the operator \( A^2 \):

(6.2)

\[
D(A^2) = \left\{ \begin{aligned}
[z_0, z_1, v_0, v_1]^T : & z_1, (-A_N + q)z_0 + Bv_1, v_1, -\tilde{A}(v_0 + v_1) - B^*z_1 \\
& \in D(A)
\end{aligned} \right\}
\]

\[
= \left\{ \begin{aligned}
[z_0, z_1, v_0, v_1]^T : & z_1 \in H^2_1(\Omega), (-A_N + q)z_0 + Bv_1 \in H^1_1(\Omega), v_0 \in H^2_0(\Gamma_0), \\
& v_1 \in H^2_0(\Gamma_0), -\tilde{A}(v_0 + v_1) + B^*z_1 \in H^2_0(\Gamma_0), \\
& v_1 - \tilde{A}(v_0 + v_1) - B^*z_1 \in D(\tilde{A}), \frac{\partial z_1}{\partial \nu} |_{\Gamma_0} = -\tilde{A}(v_0 + v_1) - B^*z_1
\end{aligned} \right\}
\]

where in the last step \( z_0 \in H^3_1(\Omega) \) is from elliptic theory when provided that \( q(x) \in W^{1,\infty}(\Omega) \). Therefore when \( z_0 \in H^3_1(\Omega), z_1 \in H^2_1(\Omega), v_0 \in H^2_0(\Gamma_0), v_1 \in H^2_0(\Gamma_0) \) with compatible conditions as in \( D(A^2) \) and \( q \in W^{1,\infty}(\Omega) \), then from semigroup theory we have that the solution of (6.1) satisfies

(6.3)

\[
\tilde{z} \in C([0, T]; H^1(\Omega)), \quad \tilde{z}_{tt} \in C([0, T]; L^2(\Omega))
\]

Hence we have on the one hand

(6.4)

\[
z_t \in H^1(0, T; H^1(\Omega))
\]
On the other hand, from (6.3) and \( \ddot{z}_{tt}(x, t) = \Delta \ddot{z}(x, t) + q(x) \ddot{z}(x, t) \), we have by elliptic theory that

(6.5) \hspace{1cm} z_t = \ddot{z} \in L^2(0, T; H^2(\Omega))

Interpolate between (6.4) and (6.5), we have for \( 0 < \epsilon < \frac{1}{2} \),

(6.6) \hspace{1cm} z_t \in H^{\frac{1}{2} + \epsilon}(0, T; H^{\frac{3}{2} - \epsilon}(\Omega)) \subset H^{\frac{1}{2} + \epsilon}(0, T; L^\infty(\Omega))

where the inclusion is by Sobolev embedding theorem.

**Case 2: n=3.** We let \( \ddot{z} = z_{tt}, \ddot{v} = v_{tt} \), then we have \( \ddot{z}, \ddot{v} \) satisfy

(6.7)

\[
\begin{align*}
\ddot{z}_{tt}(x, t) &= \Delta \ddot{z}(x, t) + q(x)\ddot{z}(x, t) \quad \text{in } \Omega \times [0, T] \\
\frac{\partial \ddot{z}}{\partial \nu}(x, t) &= 0 \quad \text{on } \Gamma_1 \times [0, T] \\
\ddot{z}_t(x, t) &= -\ddot{v}_{tt}(x, t) - \Delta^2 \ddot{v}(x, t) - \Delta^2 \ddot{v}_t(x, t) \quad \text{on } \Gamma_0 \times [0, T] \\
\ddot{v}(x, t) &= \frac{\partial \ddot{z}}{\partial \nu}(x, t) = 0 \quad \text{on } \partial \Gamma_0 \times [0, T] \\
\frac{\partial \ddot{v}}{\partial \nu}(x, t) &= \ddot{v}_t(x, t) \quad \text{on } \Gamma_0 \times [0, T] \\
\ddot{z}(\cdot, \frac{T}{2}) &= \Delta z_0(x) + q(x)z_0 \quad \text{in } \Omega \\
\ddot{z}_t(\cdot, \frac{T}{2}) &= \Delta z_1(x) + q(x)z_1 \quad \text{in } \Omega \\
\ddot{v}(\cdot, \frac{T}{2}) &= -z_1(x) - \Delta^2 v_0(x) - \Delta^2 v_1(x) \quad \text{on } \Gamma_0 \\
\ddot{v}_t(\cdot, \frac{T}{2}) &= -\Delta z_0(x) - q(x)z_0(x) - \Delta^2 v_1(x) \quad \text{on } \Gamma_0 \\
&\quad + \Delta^2 z_1(x) + \Delta^4 v_0(x) + \Delta^4 v_1(x) \quad \text{on } \Gamma_0 
\end{align*}
\]

Then still using the similarly operator setting as before we can compute the domain of \( A^3 \):

(6.8)

\[
D(A^3) = \{(z_0, z_1, v_0, v_1)^T : (z_1, (-A_N + q)z_0 + Bv_1, v_1, -\tilde{A}(v_0 + v_1) - B^*z_1) \in D(A^2)\}
\]

\[
= \{(z_0, z_1, v_0, v_1)^T : z_1 \in H^3_{\Gamma_1}(\Omega), (\Delta + q)z_0 \in H^2_{\Gamma_1}(\Omega), \frac{\partial z_0}{\partial \nu}|_{\Gamma_0} = v_1, \}
\]

\[
v_0 \in H^2(\Gamma_0), v_1 \in H^2(\Gamma_0), \tilde{A}(v_0 + v_1) + B^*z_1 \in H^2(\Gamma_0), \]

\[
\tilde{A}(v_0 + v_1) + B^*z_1 + \tilde{A}[v_1 - \tilde{A}(v_0 + v_1) - B^*z_1] + B^*[-(A_N + q)z_0 + Bv_1] \in D(\tilde{A})
\]

\[
\frac{\partial z_1}{\partial \nu}|_{\Gamma_0} = -\tilde{A}(v_0 + v_1) - B^*z_1, \frac{\partial[(A_N + q)z_0 + Bv_1]}{\partial \nu}|_{\Gamma_0} =
\]

\[
- \tilde{A}[v_1 - \tilde{A}(v_0 + v_1) - B^*z_1] - B^*[-(A_N + q)z_0 + Bv_1]
\]

\[
= \{(z_0, z_1, v_0, v_1)^T : z_1 \in H^3_{\Gamma_1}(\Omega), z_1 \in H^3_{\Gamma_1}(\Omega), v_0 \in H^2(\Gamma_0), v_1 \in H^2(\Gamma_0), \}
\]

\[
\tilde{A}(v_0 + v_1) + B^*z_1 \in H^2(\Gamma_0), \frac{\partial z_0}{\partial \nu}|_{\Gamma_0} = v_1, \frac{\partial z_1}{\partial \nu}|_{\Gamma_0} = -\tilde{A}(v_0 + v_1) - B^*z_1 \text{ on } \Gamma_0, \]

\[
\tilde{A}(v_0 + v_1) + B^*z_1 + \tilde{A}[v_1 - \tilde{A}(v_0 + v_1) - B^*z_1] + B^*[-(A_N + q)z_0 + Bv_1] \in D(\tilde{A})
\]

\[
\tilde{A}(v_1 - \tilde{A}(v_0 + v_1) - B^*z_1) + B^*[-(A_N + q)z_0 + Bv_1] \in H^2(\Gamma_0),
\]

\[
\frac{\partial[(A_N + q)z_0 + Bv_1]}{\partial \nu}|_{\Gamma_0} = -\tilde{A}[v_1 - \tilde{A}(v_0 + v_1) - B^*z_1] - B^*[-(A_N + q)z_0 + Bv_1] \}
where in the last step \( z_0 \in H^{\frac{7}{4}}_{1,1}(\Omega) \) is from trace theory of solving \( \frac{\partial v_1}{\partial n}|_{\Gamma_0} = v_1 \in H^2(\Gamma_0) \). Therefore when \( z_0 \in H^{\frac{7}{4}}(\Omega) \), \( z_1 \in H^{\frac{3}{2}}(\Omega), v_0 \in H^2(\Gamma_0), v_1 \in H^2(\Gamma_0) \) with compatible conditions as in \( D(\mathcal{A}^3) \) and \( q \in W^{2,\infty}(\Omega) \), then from semigroup theory we have that the solution of (6.1) satisfies
\[
(6.9) \quad \tilde{z}_t \in C([0,T];H^1(\Omega)), \quad \tilde{z}_{tt} \in C([0,T];L^2(\Omega))
\]
Hence we have on the one hand
\[
(6.10) \quad z_t \in H^2(0,T;H^1(\Omega))
\]
On the other hand, from (6.9) and \( \tilde{z}_{tt}(x,t) = \Delta \tilde{z}(x,t) + q(x)\tilde{z}(x,t) \), we have by elliptic theory that
\[
(6.11) \quad z_{tt} = \tilde{z} \in L^2(0,T;H^2(\Omega))
\]
which implies
\[
(6.12) \quad z_t \in H^1(0,T;H^2(\Omega))
\]
Now interpolate between (6.10) and (6.12), we have for \( 0 < \epsilon < \frac{1}{2} \),
\[
(6.13) \quad z_t \in H^{\frac{3}{2}}(0,T;H^{\frac{3}{2}}(\Omega)) \subset H^{\frac{1}{2}+\epsilon}(0,T;L^\infty(\Omega))
\]
where the inclusion is again by Sobolev embedding.

Hence in either case \( n = 2 \) or \( n = 3 \), under the assumptions on the initial data \( [z_0, z_1, v_0, v_1] \) and \( q(x) \), \( p(x) \) in Theorem 1.6 we have that \( z_t \in H^{\frac{1}{2}+\epsilon}(0,T;L^\infty(\Omega)) \). Thus when we again set \( f(x) = q(x) - p(x), w(x,t) = z(q)(x,t) - z(p)(x,t), u(x,t) = v(q)(x,t) - v(p)(x,t) \) and \( R(x,t) = z(p)(x,t) \) as in section 4, we obtain (1.32) that \( R_t \in H^{\frac{1}{2}+\epsilon}(0,T;L^\infty(\Omega)) \) and hence all the assumptions in theorem 1.6 are satisfied. Therefore, we get the desired stability (1.34) from the stability (1.33) of the linear inverse problem in Theorem 1.6.

7. Concluding remark

As we mentioned at the beginning and the calculations of \( D(\mathcal{A}^2) \) and \( D(\mathcal{A}^3) \) show, the lack of compactness of the resolvent limits the space regularity of the solutions for the wave equation parts since we always have the elliptic problem for \( z \) or \( z_t \) such that \( (\Delta + q)z \in L^2(\Omega) \) with \( \frac{\partial v}{\partial n} \in H^2_0(\Gamma_0) \) provided \( q \) in some suitable space. Therefore the best space regularity that \( z \) could get is \( 2 + \frac{3}{4} = \frac{11}{4} \) from elliptic and trace theory. As a result, our argument of the stability in the nonlinear inverse problem will only work for dimension up to \( n = 7 \) as we need the Sobolev embedding \( H^{\frac{11}{4}}(\Omega) \subset L^\infty(\Omega) \) in order to achieve the space regularity of \( z_t \) in \( L^\infty(\Omega) \) which is needed in the proof.

REFERENCES

[1] G. Avalos, The exponential stability of a coupled hyperbolic/parabolic system arising in structural acoustics, Appl. Abstr. Anal. 1 (2), 203-217.
[2] G. Avalos, Exact-approximate boundary reachability of thermoelastic plates under variable thermal coupling, Inverse Problems, 16 (2000), 979-996.
[3] G. Avalos and I. Lasiecka, Differential Riccati equation for the active control of a problem in structural acoustics, J. Optim. Theory. Appl., 91 (3) (2001), 695-728.
[4] G. Avalos and I. Lasiecka, The strong stability of a semigroup arising from a coupled hyperbolic/parabolic system, *Semigroup Forum*, Vol. 57 (1998), 278-292.

[5] G. Avalos and I. Lasiecka, Exact controllability of structural acoustic interactions, *J. Math. Pures. Appl.*, 82 (2003), 1047-1073.

[6] G. Avalos and I. Lasiecka, Exact reachability of finite energy states for an acoustic wave/plate interaction under the influence of boundary and localized controls, *IMA Preprint Series 2017*, (January, 2005).

[7] G. Avalos, I. Lasiecka and R. Rebarber, Well-posedness of a structural acoustics control model with point observation of the pressure, *Journal of Differential Equations*, 173 (2001), 40-78.

[8] G. Avalos, I. Lasiecka and R. Rebarber, Boundary controllability of a coupled wave/Kirchoff system, *System and Control Letters*, 50 (2003), 331-341.

[9] H. T. Banks, W. Fang, R. J. Silcox and R.C. Smith, Approximation methods for control of acoustic/structure models with piezoceramic actuators, *Contract Report 189578 NASA* (1991).

[10] H. T. Banks and R.C. Smith, Feedback control of noise in a 2-D nonlinear structural acoustics model, *Discrete and Continuous Dynamical Systems* Vol 1, No. 1, (1995), 119-149.

[11] A. Bukhgeim, *Introduction to the Theory of Inverse Problems*, VSP, Utrecht, 2000.

[12] A. Bukhgeim and M. Klibanov, Global uniqueness of a class of multidimensional inverse problem, *Sov. Math.-Dokl.* 24 (1981), 244-7.

[13] T. Carleman, Sur un problème d’unicité pour les systèmes d’équations aux derivées partielles à deux variables independantes, *Ark. Mat. Astr. Fys.*, 2B (1939), 1-9.

[14] M. Camurdan and R. Triggiani, Sharp regularity of a coupled system of a wave and a Kirchoff plate with point control arising in noise reduction, *Differential and Integral Equations*, 12 (1999), 101-107.

[15] V. Isakov, *Inverse Source Problems*, American Mathematical Society, 2000.

[16] V. Isakov, *Inverse Problems for Partial Differential Equations*, Second Edition, Springer, New York, 2006.

[17] V. Isakov, Uniqueness and stability in multi-dimensional inverse problems, *Inverse Problems*, 9 (1993), 579-621.

[18] O. Imanuvilov and M. Yamamoto, Global Lipschitz stability in an inverse hyperbolic problem by interior observations, *Inverse Problems*, 17(2001), 717-728.

[19] V. Isakov and M. Yamamoto, Carleman estimate with the Neumann boundary condition and its application to the observability inequality and inverse hyperbolic problems, *Contemp. Math.*, 268(2000), 191-225.

[20] A. Khaidarov, Carleman estimates and inverse problems for second order hyperbolic equations, *Math. USSR Sbornik*, 58 (1987), 267-277.

[21] M. Klibanov, Inverse problems and Carleman estimates, *Inverse Problems*, 8(1992), 575-596.

[22] M. Klibanov, Carleman estimates and inverse problems in the last two decades, *Surveys on Solutions Methods for Inverse Problems*, Springer, Wien, 2000, pp 119-146.

[23] I. Lasiecka, Mathematical Control Theory of Coupled PDE’s, *CBMS-NSF Regional Conference Series in Applied Mathematics*, SIAM Publishing, Philadelphia (2002).

[24] I. Lasiecka and R. Triggiani, Exact controllability of the wave equation with Neumann boundary control, *Appl. Math. & Optimiz.*, 19(1989), 243-290.

[25] I. Lasiecka and R. Triggiani, Carleman estimates and exact boundary controllability for a system of coupled, nonconservative second order hyperbolic equations, in Partial Differential Equations Methods in Control and Shape Analysis, *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, New York, Vol. 188, 215-243.

[26] I. Lasiecka and R. Triggiani, Sharp regularity theory for second order hyperbolic equations of Neumann type. Part I. $L_2$ Nonhomogeneous data, *Ann. Mat. Pura. Appl. (IV)*, CLVII(1990), 285-367.

[27] I. Lasiecka and R. Triggiani, Regularity theory of hyperbolic equations with non-homogeneous Neumann boundary conditions. II. General boundary data, *Journal of Differential Equations*, 94(1991), 112-164.
[28] I. Lasiecka and R. Triggiani, Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions, *Appl. Math. & Optimiz.*, 25(1992), 189-244.

[29] I. Lasiecka, R. Triggiani and X. Zhang, Nonconservative wave equations with unobserved Neumann B.C.: global uniqueness and observability in one shot, *Contemp. Math.*, 268(2000), 227-325.

[30] J.L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, Vol. I, Springer-Verlag, Berlin, 1972.

[31] W. Littman and B. Liu, On the spectral properties and stabilization of acoustic flow, *IMA Preprint Series #1436*, (November, 1996).

[32] J-P Puel and M. Yamamoto, On a global estimate in a linear inverse hyperbolic problem, *Inverse Problems*, 12(1996), 995-1002.

[33] R.E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Mathematical Surveys and Monographs Volume 49, American Mathematical Society, 1997.

[34] D. Tataru, On the regularity of boundary traces for the wave equation, *Annali Scuola Normale di Pisa*, Classe Scienze (4), 26(1998), no. 1, 355-387.

[35] D. Tataru, A priori estimates of Carleman’s type in domains with boundary, *J.Math.Pures Appl.*, 73(1994), no. 4, 185-206.

[36] R. Triggiani, Wave equation on a bounded domain with boundary dissipation: an operator approach, *Journal of Mathematical Analysis and Applications*, 137(1989), 438-461.

[37] M. Yamamoto, Uniqueness and stability in multidimensional hyperbolic inverse problems, *J. Math. Pures Appl.*, 78(1999), 65-98.