A THEORY OF FERROMAGNETISM BY ETTORE MAJORANA

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ABSTRACT. We present and analyze in detail an unknown theory of ferromagnetism developed by Ettore Majorana as early as the beginnings of 1930s, substantially different in the methods employed from the well-known Heisenberg theory of 1928 (and from later formulations by Bloch and others). Similarly to this, however, it describes successfully the main features of ferromagnetism, although the key equation for the spontaneous mean magnetization and the expression for the Curie temperature are different from those deduced in the Heisenberg theory (and in the original phenomenological Weiss theory). The theory presented here contains also a peculiar prediction for the number of nearest neighbors required to realize ferromagnetism, which avoids the corresponding arbitrary assumption made by Heisenberg on the basis of known (at that time) experimental observations. Some applications of the theory (linear chain, triangular chain, etc.) are, as well, considered.

1. Introduction

The microscopic interpretation of the magnetic properties of bodies, related to the motion of electrons in atoms, appeared as early as the beginning of the XX century, when P. Langevin [1] provided a successful theory of diamagnetism, by evaluating the induced magnetization in presence of an applied magnetic field in the framework of the Lorentz theory of the electron. Subsequently the paramagnetism of materials was rightly related to the fact that atoms or molecules can have permanent dipole moments arising from the current loops generated by orbiting electrons (proportional to the total angular momentum), and thus acting as tiny magnetic shells. The correct interpretation of the ferromagnetic phenomena, instead, did not come out soon, and their understanding on an atomic basis was achieved only after the advent of quantum mechanics.

The striking property to be explained was that the ferromagnetic substances (like iron, cobalt, nickel, etc.) develop a spontaneous magnetization only when cooled below a certain temperature $T_c$, called the Curie point. Differently from an ordinary paramagnet, above this critical value the temperature variation of the magnetic susceptibility $\chi$ of a ferromagnet was found empirically to follow the Curie-Weiss law, $\chi = C/(T - T_c)$ (where $C$ is a constant), whereas for a paramagnet the Curie law $\chi = C/T$ holds.

Early works by J.A. Ewing, K. Honda and J. Okubo, and others[1] showed that many of the peculiarities exhibited by ferromagnetic bodies could be described by assuming a large potential energy between adjacent microscopic (atomic or molecular) magnets. However, a first important step towards the understanding of the phenomenon was performed by P. Weiss in 1907 [3], when he postulated that the

\footnote{For a description of these and other earlier models see the review in Ref. [2].}
atom magnets tend to be brought into parallel orientation not only by an applied, external magnetic field $H_e$ but also by an inner field $H_i$, called by Weiss the “molecular field”, which is proportional to the magnetization $M$ of the material: $H_i = \alpha M$. By introducing such an idea of an inner field into the Langevin theory of the magnetism of classical dipoles, Weiss was in fact able to account for most of the phenomenology of ferromagnets \[4\]. If a dipole is brought from outside and placed in the molecular field, it will be aligned in the direction of this field (i.e. in the direction of the magnetization itself), giving rise to a spontaneous ordering of the elementary magnets. Such an ordering effect competes with random thermal effects, with energy of the order of $kT$, trying to flip the dipole away from the ordered state, but Weiss showed that (in the simple case that dipoles can align only in two opposite direction), even in the absence of an external magnetic field ($H_e = 0$), a non-vanishing magnetization may arise, satisfying the equation:

$$M = M_0 \tanh \frac{\mu \alpha M}{kT}$$

(1)

($\mu$ is the magnetic moment of the $N$ atoms per unit volume of the body, and $M_0 = N\mu$) \[2\]. In fact, apart from the trivial solution $M = 0$, the implicit equation \[1\] admits also another non-zero solution for the magnetization $M$, provided that $T < T_c = N\mu^2\alpha/k$, as it was seen graphically (see Fig. 1). The magnetic susceptibility was as well calculated, obtaining the experimentally observed Curie-Weiss law (with $C = N\mu^2/k$), thus finally showing the good agreement of the phenomenological Weiss theory with the known properties of ferromagnetism.

However, despite such a success, the question about the origin of the local field remained unanswered, and Weiss himself realized that it could not arise from the magnetic interactions of the magnetic moments of the atoms or molecules, these

\[2\] Actually, in the original formulation, the Langevin function $\cosh x - 1/x$ appeared in Eq. \[1\] instead of the function $\tanh x$, which was considered only later by W. Heisenberg (see below). However, as it is easily recognizable (as done by Heisenberg himself), the physical interpretation of the phenomena does not change and no alteration in the reasoning is required, so that we prefer to use the mathematical form of Eq. \[1\] for later convenience.
being extremely small. In fact, the value of the proportionality coefficient $\alpha$ should be assumed as high as of the order of $10^4$, against the value of $4\pi/3$ calculated under ordinary electromagnetic assumption [5]. On the other hand, classical electrostatic forces may lead to interactions of the right order of magnitude, but do not give a linear proportionality between the Weiss molecular field $H_i$ and the magnetization $M$ [6].

The solution of this puzzle had to wait until the end of 1920s, when W. Heisenberg [7] showed that the forces leading to ferromagnetism are purely quantum mechanical in nature, being due to electron exchange. The key idea was to show how strong non-magnetic forces between electrons that favor spin alignment (and, thus, the alignment of the elementary magnets) arise from the quantum mechanical exchange interactions, in the same way as they produce level splittings between electrons in singlet (para-) and triplet (ortho-) states of two-electron atoms: “the empirical phenomenon that ferromagnetism presents is very similar to the situation we met earlier in the case of the helium atom.” [7]. Heisenberg, indeed, began in 1926 [8] to explore how the so-called “exchange term” appears in a system of atoms with many electrons satisfying the Pauli exclusion principle. Exchange forces are electrostatic in nature but, because of the constraints imposed by the Pauli principle, they are formally equivalent to a tremendously large coupling between spins, as seemingly required by the Weiss phenomenological theory. With a suitable assumption (a positive exchange integral), Heisenberg showed that the state in which spins are aligned in the same direction is energetically favorable, helping in the ordering of the spins in a certain direction, thus proving his model to be well fitted to describe ferromagnetism.

This result was, actually, achieved by following the statistical model built up by E. Ising few years before [9]. In the Ising model (or, rather, the Lenz-Ising model [10]) of a ferromagnetic body, $N$ elementary magnets of moment $\mu$ are assumed to be arranged into a regular lattice, each of them having only two possible orientations, as in the Weiss theory. The magnets experience only short-range interactions on each other, the interaction energy (assumed to be a constant) being non-vanishing and negative (positive) only for each pair of neighboring magnets of same (opposite) direction. This interaction tends, then, to make neighboring spins the same (an additional energy $\pm \mu H_e$ is present for an applied magnetic field $H_e$), so that in principle this model could be suitable for describing ferromagnetic phenomena. The problem is just to find an analytic expression for the statistical mechanical partition function

$$Z = \sum_{\text{config.}} e^{-\frac{E}{kT}}, \quad (2)$$

where $E$ is the total energy of the system (and the sum is over any configuration of the system), from which all the thermodynamic quantities of interest (including the mean magnetization) can be derived. This was effectively done by Ising for magnets arranged into a linear chain, but he “succeeded in showing that the assumption of directed, sufficiently great forces between two neighboring atoms of a chain is not sufficient to explain ferromagnetism” [7], since no non-zero magnetization is predicted at any temperature. This occurs since the ordered state is unstable against random thermal fluctuations tending to destroy spin alignment, and Ising showed (incorrectly) that such negative results applies also in three dimensions.
This problem was not properly solved by Heisenberg in 1928, although his different (non constant) form of the interaction energy depends not only on the arrangement of the elementary magnets, but also on the speed with which they exchange their places. It was correctly addressed only in 1936 by R. Peierls [11] who, against quite a commonly expressed opinion “that the solution of the three-dimensional problem could be reduced to that of the linear model and would lead to similar results”, finally showed that the above-mentioned statistical model of ferromagnetism gives non-zero spontaneous magnetization at finite temperature in two (or more) dimensions.

Returning to the Heisenberg theory, his derivation of the Weiss formula (1) proceeded through standard calculations on the partition function (2). The first step was to find the distribution of the energy levels of $2n$ localized electrons (one valence electron per atom) to first order in the coupling by evaluating the exchange integral between each electron and its nearest neighbors. Given the exponential decrease of the exchange integrals, the couplings among more distant electrons are, in fact, negligible. Formal group-theoretic methods were used to evaluate the energy levels and, in order to calculate the partition function, their fluctuations were neglected. This last assumption effectively meant the replacement of the energy levels of the spin multiplets of the microscopic magnets forming a given total spin $S$ by the average energy corresponding to $S$. Subsequently, with a “somewhat arbitrary” assumption, Heisenberg introduced a Gaussian distribution of energies for $S$ around its mean energy $E_0$ with variance $\Delta E_0^2$; as he himself wrote, “it is of course only to be expected that the temporary theory sketched here offers but a qualitative scheme into which ferromagnetic phenomena will perhaps be later incorporated” [7].

Subsequent improvements, both on the mathematical formalism and on the comprehension of several physical effects, came out between the end of 1920s and the beginning of 1930s, mainly due to F. Bloch, J. Slater and H.A. Bethe. Firstly, Bloch and Slater concentrated (independently) on the understanding of the physical basis of the Heisenberg model. While considering a free electron gas (instead of the bound electrons contemplated by Heisenberg in his Heitler-London-like [12] approach) for studying the role of conduction electrons in ferromagnets (for obtaining its thermal properties), Bloch avoided Heisenberg’s Gaussian assumption and found a contribution coming out “when two electrons are close together, in the neighborhood of the same atom” [13]. Such an important effect (the exchange integral can become negative, this decreasing the possibility to have ferromagnetism), in fact, cannot be taken into account in the Heitler-London-like approach used by Heisenberg. Furthermore Bloch replaced the difficult group-theoretic method employed by Heisenberg with the simpler determinantal method introduced shortly before by Slater in his theory of complex spectra [14]. The great advantage of writing a many-electron wavefunction in the form of a Slater determinant was just to take into account from the very beginning the correct antisymmetry properties imposed by the Pauli principle, thus making sure that no two electrons in the system can be found in identical quantum states. This novel approach directly led Bloch to discover the so-called “spin waves” [15], i.e. the states of the system corresponding to single or few spin flips in the fully aligned ground state, in the region of low temperatures, where both the Weiss phenomenological theory and the Heisenberg calculations couldn’t be applied.
All these results by Heisenberg, Bloch and Slater were reviewed (and criticized) by W. Pauli in his comprehensive review of magnetism delivered at the October 1930 Solvay conference in Paris [16].

This meeting was also attended by E. Fermi who, very likely, was responsible of the spreading of the novel results about ferromagnetism within his group in Rome. Although Fermi and his close collaborators never worked and published on this argument, several people visiting the Institute of Physics in Rome in the early 1930s did, including, primarily, H.A. Bethe [17]. In Rome, he calculated eigenvalues and eigenfunctions (at first approximation) for any value of the resulting magnetic moment of the whole crystal (in the one-dimensional case of a linear chain), by introducing the famous Bethe ansatz for the case of two interacting spin waves. Other minor (to some extent) contributions came out from D.R. Inglis [18]-[20], who visited Rome in 1932, and from a member of the extended Fermi group, G. Gentile, who provided [21] a thorough justification of Bloch’s results on elementary groups (Uebergangsgebiet) of aligned spins inside the ferromagnet [22]. Inglis, instead, considered the problems about the applicability of Heisenberg’s approximation on the Gaussian distribution of energy eigenvalues [18] and, later, the effect of the spin-orbit interactions in ferromagnetic anisotropy [19] and the problem of the lack of orthogonality for the electronic wavefunctions of two neighboring atoms [20].

Here we are not interested to the quite irrelevant question about the possible origin of such works within the Fermi group in Rome [3] but rather report on an unknown but very interesting contribution on ferromagnetism by another distinguished member of that group, E. Majorana, who much contributed to the achievements of several results in different fields of research, results that were often related to other people (for a recent review, see [23] and references therein). As quite usual for Majorana, many of these achievements were never published by him (and, in most cases, even his friends and colleagues were not aware of them), so that they are practically unknown. Fortunately enough, however, Majorana’s calculations have been preserved in his personal notebooks [24] [25] and we are, thus, able to reconstruct, in particular, Majorana’s contribution on ferromagnetism [25]. This is presented in detail in the following section, while we comment on its relevance and some intriguing aspects in Section 3. Finally, our conclusions and main results are summarized in Section 4.

2. Majorana theory of ferromagnetism

The Majorana contribution begins with focusing on a set of \( n \) atoms, considered as magnetic dipoles (or, briefly, spins), arranged into a given geometric array at locations \( q_1, q_2, \ldots q_n \). Among these \( n \) elements, he assumes that a number \( i \) of them has a spin parallel to a given direction (spin-up), while the remaining \( n - i \) elements are antiparallel to that direction (spin-down). Spin-up elements are characterized by the quantum numbers:

\[
r_1, r_2, \ldots r_i \quad \uparrow \uparrow \ldots \uparrow ,
\]

while those with spin-down are characterized by:

\[
r_{i+1}, r_{i+2}, \ldots r_n \quad \downarrow \downarrow \ldots \downarrow .
\]

\[\text{Just to quote a different example, we point out that Bloch and Gentile worked together in Leipzig when, in 1931, they were both at the Institute headed by Heisenberg}\]
The exchange energy between states characterized by \( r \) and \( s \) quantum numbers (or, according to Majorana’s language, the exchange energy between orbitals) is given by:

\[
V_{rs} = \int \frac{e^2}{|q_1 - q_2|} \psi_r(q_1) \psi_s(q_1) \overline{\psi_r(q_2)} \overline{\psi_s(q_2)} \, dq_1 \, dq_2,
\]

with \( V_{rs} = V_{sr} \). The energy eigenvalues are then:

\[
H_{mm} = V_0 - \sum_{r<s} V_{rs} + \sum_{r_1=r}^{r_m} \sum_{s=r+1}^{r_n} V_{rs}
\]

The wavefunctions \( A(r_1 \ldots r_i | r_{i+1} \ldots r_n) \) describing such a system have the form of a Slater determinant; in the following expression, the \( \delta \)-functions (or, more appropriately, the Kronecker symbols) \( \delta(s_k \mp 1) \) guarantee that spin-up/spin-down elements have \( s = \pm 1 \) spin (in some unit), respectively:

\[
A(r_1 \ldots r_i | r_{i+1} \ldots r_n) = \begin{vmatrix}
\psi_{r_1}(q_1) \delta(s_1 - 1) & \ldots & \psi_{r_i}(q_n) \delta(s_n - 1) \\
\vdots & \ddots & \vdots \\
\psi_{r_{i+1}}(q_1) \delta(s_1 + 1) & \ldots & \psi_{r_{i+1}}(q_n) \delta(s_n + 1) \\
\psi_{r_n}(q_1) \delta(s_1 + 1) & \ldots & \psi_{r_n}(q_n) \delta(s_n + 1)
\end{vmatrix}.
\]

Since there are

\[
\tau = \binom{n}{i} = \frac{n!}{i!(n-i)!}
\]

ways to have \( i \) spin-up and \( n - i \) spin-down on a total of \( n \) elements (the order of \( r_1 \ldots r_i \) or \( r_{i+1} \ldots r_n \) is not important), the wavefunctions describing the system are the following:

\[
A_i(r_1^1, r_2^1 \ldots r_i^1 | r_{i+1}^1, r_{i+2}^1 \ldots r_n^1), \\
\ldots
\]

\[
A_\tau(r_1^\tau, \ldots r_i^\tau | r_{i+1}^\tau, r_n^\tau).
\]

Denoting with \( H \) the Hamiltonian operator describing the interaction acting on each element, the electrostatic interaction potential \( V_0 \) is given by:

\[
V_0 = \int H \psi_1(q_1) \overline{\psi_1(q_1)} \psi_2(q_2) \overline{\psi_2(q_2)} \ldots \psi_n(q_n) \overline{\psi_n(q_n)} \, dq_1 \ldots dq_n.
\]

The exchange energy between states characterized by \( r \) and \( s \) quantum numbers (or, according to Majorana’s language, the exchange energy between orbitals and \( r \) and \( s \) orbits) is instead:

\[
V_{rs} = \int \frac{e^2}{|q_1 - q_2|} \psi_r(q_1) \psi_s(q_1) \overline{\psi_r(q_2)} \overline{\psi_s(q_2)} \, dq_1 \, dq_2,
\]

with \( V_{rs} = V_{sr} \). The energy eigenvalues are then:

\[
H_{mm} = V_0 - \sum_{r<s} V_{rs} + \sum_{r_1=r}^{r_m} \sum_{s=r+1}^{r_n} V_{rs}
\]

(the second term describing the action of given elements on themselves, while the third one that of given elements on other elements), while for \( m \neq n \):

\[
H_{mn} = \begin{cases} 
-V_{rs}, & \text{for a transition from } A_m \text{ to } A_n \text{ by exchanging the opposite intrinsic orientation in the orbits } \psi_r \text{ and } \psi_s, \\
0, & \text{for the other cases.}
\end{cases}
\]

Majorana now assumes that (\( \epsilon \) is a constant)

\[
V_{rs} = \begin{cases} 
\epsilon, & \text{for neighbor atoms,} \\
0, & \text{for distant atoms,}
\end{cases}
\]
each atom having a number \( a \) of neighbor atoms. The energy change in the ferromagnetic case is:

\[
E = H - V_0 + \sum_{r<s} V_{rs} \tag{10}
\]

whose eigenvalues are:

\[
E_{mm} = \sum_{r,m} \sum_{r',m'} V_{rs} \tag{11}
\]

and, for \( m \neq n \):

\[
E_{mn} = \begin{cases} 
-V_{rs}, & \text{if } m = n \\
0, & \text{otherwise}
\end{cases} \tag{12}
\]

Let \( y = y(N) \) be the number of configurations in which \( N \) spins can be arranged (at a given temperature) among each other, subject to the normalization condition:

\[
\sum_N y(N) = \binom{n}{i}. \tag{13}
\]

In order to evaluate the partition function of the system, the precise determination of the energy levels (10), corresponding to a given spin state of the system, is required, but this is virtually impossible due to the large number of elements involved. In his theory, Heisenberg therefore made the approximation that the discrete succession of such energy levels can be replaced by a continuum, and that such a continuum is distributed according to the Gaussian error law around the average energy for the given spin state of the system. In his notebooks, Majorana makes a similar assumption: “Can we consider \( E \) as diagonal, in a statistical sense? Let us assume that it can be.” The distribution function, now subject to the normalization condition

\[
\int_{-\infty}^{+\infty} y \, dN = \binom{n}{i}, \tag{13}
\]

is then assumed to be picked around a pronounced maximum

\[
y_0 = y(N_0) \tag{14}
\]

as in the case of a Gaussian function. This means that, with a maximum probability, we have \( y_0 \) quantities \( A \) in \( \{1\} \) corresponding to \( N_0 \).

Then, Majorana continues: “In each of the quantities \( A \) we exchange randomly” (due to temperature effects) “an orbit ↑ with an orbit ↓; the quantities \( A \) change into quantities \( B \):

\[
A_1 \longrightarrow B_1, \quad \ldots \quad A_r \longrightarrow B_r. \tag{15}
\]

Statistically, the set of quantities \( B \) coincides with that of the quantities \( A \). If we perform the transformation (15), the quantities \( B \) corresponding to the \( y_0 \) quantities \( A \) will be distributed between \( N_0 - 2a \) and \( N_0 + 2a \).”
Figure 2. A sketch of the continuous distribution function $y(N)$.

Let $p_{2r}$ be the probabilities that one among the mentioned $B$ quantities corresponds to $N = N_0 + 2r$, with $-a \leq r \leq a$. The “average increment” in the number of aligned spin is

$$
\Delta N_0 = \sum_{r=-a}^{a} 2r \ p_{2r}
$$

and can be evaluated as follows. On average, one of the $i$ spin-up elements has $N_0/i$ spin-down and $a - N_0/i$ spin-up elements as neighbors, while one of the $n - i$ spin-down elements has $N_0/(n - i)$ spin-up and $a - N_0/(n - i)$ spin-down elements as neighbors. By performing the mentioned transformation (15), we have

$$
\Delta N_0 = 2a - 2 \left( \frac{N_0}{i} + \frac{N_0}{n - i} \right).
$$

Since we have a total of $\ell = 2a$ spins around $N_0$, the probability that a given element be a spin-up is assumed by Majorana to be

$$
q = \frac{1}{2} + \frac{\Delta N_0}{4a},
$$

while for the probability to have a spin-down:

$$
1 - q = \frac{1}{2} - \frac{\Delta N_0}{4a}.
$$

Thus, the probability to find $m = a + r$ spin-up elements that alternate every $\ell - m = a - r$ spin-down elements is given by

$$
\binom{\ell}{m} \ q^m (1 - q)^{\ell-m},
$$

since $q$ is evidently a random binomial variable. By replacing in (20) the corresponding quantities in (18), (19), we then get:

$$
p_{2r} = \left( \frac{1}{2} + \frac{\Delta N_0}{4a} \right)^{a+r} \left( \frac{1}{2} - \frac{\Delta N_0}{4a} \right)^{a-r} \frac{(2a)!}{(a - r)! \ (a + r)!}
$$

(such a law for the probability is called “normal” by Majorana).

According to statistical mechanics, the ratio of the probabilities (18) and (19) is just given by the Boltzmann factor:

$$
\frac{q}{1 - q} = e^{-\frac{E}{kT}}.
$$
However, Majorana proceeds in a slightly different way. Assuming that, for a restricted range,

\[ y(N_0 + 1) = y_0 e^K, \]
\[ ... \]
\[ y(N_0 \pm a) = y_0 e^{\pm K a}, \]

the condition that \( y(N) \) does not change while we pass from quantities \( A \) to quantities \( B \) can be expressed as:

\[ \sum_{r=-a}^{a} p_{2r} e^{-2Kr} = 1, \]

that is solved by:

\[ e^K = \frac{1 + \frac{\Delta N_0}{2a}}{1 - \frac{\Delta N_0}{2a}}, \]

which is just Eq. (22), \( e^K \) being the Boltzmann factor. In the continuum limit, \( K \) may be replaced by \( y' / y \). It follows that, by considering \( y \) as a continuous function of \( N \):

\[ \frac{y'}{y} = \log \left( \frac{2 - \frac{N}{a} \left( \frac{1}{i} + \frac{1}{n-i} \right)}{\frac{N}{a} \left( \frac{1}{i} + \frac{1}{n-i} \right)} \right), \]

and setting

\[ \alpha = \frac{1}{a} \left( \frac{1}{i} + \frac{1}{n-i} \right) = \frac{n}{a} \left( \frac{i}{n-i} \right), \]

we have:

\[ \frac{y'}{y} = \log \left( \frac{2}{\alpha N - 1} \right). \]

Such a differential equation is solved by means of separation of variables, with the change of variable

\[ t = \frac{2}{\alpha N} - 1. \]

After few calculations, Majorana finds:

\[ y = c \left( \frac{2}{\alpha N} \right)^{\frac{2}{\alpha} \left( \frac{2}{\alpha N} - 1 \right)^{-\frac{2}{\alpha} + N}} \]

or

\[ y = c \left( \frac{2}{2 - \alpha N} \right)^{\frac{2}{\alpha} \left( \frac{2}{\alpha N} - 1 \right)^N}. \]

The constant \( c \) corresponds to:

\[ y(0) = y \left( \frac{2}{\alpha} \right) = c. \]
(a limit procedure is, obviously, adopted), and may be calculated from the normalization condition (13). Here, in order to evaluate the integral, Majorana approximates the function $y(N)$ to a Gaussian curve,

$$y \simeq y \left( \frac{1}{\alpha} \right) e^{-\alpha(N - \frac{1}{\alpha})^2}$$

(note that $y(N)$ takes its maximum at $N = 1/\alpha$, and $y(1/\alpha) = 2^{2/\alpha} c$), obtaining ($N' = N - 1/\alpha$):

$$\int_{-\infty}^{+\infty} y \, dN \simeq \int_{-\infty}^{+\infty} y \left( \frac{1}{\alpha} \right) e^{-\alpha N'^2} \, dN' = \sqrt{\frac{\pi}{\alpha}} y \left( \frac{1}{\alpha} \right) = \sqrt{\frac{\pi}{\alpha}} c \, 2^{2/\alpha}.$$

From this, we get:

$$c = \left( \frac{n}{i} \right) \left( \frac{1}{2} \right)^{\frac{n}{2}} \sqrt{\frac{\alpha}{\pi}}.$$

The solution found is, then, worked out with a specific numerical example, for $n = 10$, $i = 3$, $a = 4$ (corresponding to $\alpha = 5/42$, $2/\alpha = 16.8$ and $c = 0.0002046$), and a numerical tabulation for it is reported in the notebook. Majorana also pushes further the approximation (33) by expanding, in a Taylor series, log $y$ rather than directly $y$, finally obtaining:

$$\log y = k - \frac{1}{6} \alpha^3 N'^4 - \frac{1}{15} \alpha^5 N'^6 - \frac{1}{28} \alpha^7 N'^8 - \frac{1}{45} \alpha^9 N'^{10} - \ldots$$

with $k$ a given constant.

At a given temperature $T$, the partition function of the system is evaluated from the statistical term

$$y \, e^{-\frac{E}{kT}},$$

which now replaces the distribution function $y$. The Boltzmann factor may be written as

$$e^{-\frac{E}{kT}} = e^{-\frac{N'}{kT}} = e^{-LN}$$

with

$$L = \frac{e}{kT}.$$

The maximum of the probability distribution function (corresponding to $N = N_0$) may be evaluated by taking

$$\frac{d}{dN} \log \left( y \, e^{-\frac{E}{kT}} \right) = 0$$

(and checking, as done by Majorana, the sign of the second derivative). From (as above, $N' = N - 1/\alpha$):

$$\log \left( y \, e^{-\frac{w}{kT}} \right) = k - \frac{1}{\alpha} - LN' - \frac{1}{\alpha} \log(1 - \alpha^2 N'^2) + N' \log \frac{1 - \alpha N'}{1 + \alpha N'},$$

$$\frac{d}{dN} \log \left( y \, e^{-\frac{w}{kT}} \right) = -L + \log \left( \frac{2}{\alpha N} - 1 \right),$$

we obtain:

$$\frac{2}{\alpha N_0} - 1 = e^L$$

or

$$N_0 = \frac{2}{\alpha(e^L + 1)}$$

(39)
(note that for $L = 0$ one re-obtains $N_0 = 1/\alpha$). By setting $N = N_0$ in (31) we have
\[ y_0 = c \left( \frac{e^L + 1}{e^L} \right)^{\frac{2}{\alpha}} e^{LN_0} \tag{40} \]
and
\[ y_0 e^{-\frac{E_0}{kT}} = c \left( \frac{e^L + 1}{e^L} \right)^{\frac{2}{\alpha}} e^{LN_0} e^{-LN_0} = c \left( \frac{e^L + 1}{e^L} \right)^{\frac{2}{\alpha}}. \tag{41} \]
More in general, by expanding Eq. (38) in a Taylor series up to second order terms in $(N - N_0)$ and using (39) (41) we get:
\[ y e^{-\frac{E}{kT}} = c \left( \frac{e^L + 1}{e^L} \right)^{\frac{2}{\alpha}} e^{-\frac{(e^L + 1)^2}{4e^L} \alpha(N - N_0)^2}. \tag{42} \]
From the expression (34) we finally obtain:
\[ \int y e^{-\frac{E}{kT}} dN = \left( \begin{array}{c} n \\ i \end{array} \right) \left( 1 + e^{-L} \right)^{\frac{2}{\alpha}} \frac{2\sqrt{e^L}}{e^L + 1} \tag{43} \]
\[ = \left( \begin{array}{c} n \\ i \end{array} \right) \left( 1 + e^{-L} \right)^{\frac{2(\alpha - 1)}{\alpha}} \frac{2\sqrt{e^L}}{e^L + 1} \]
(in the last passage, Eq. (27) has been used).
At this point, Majorana considers the case where a magnetic field $H$ is applied to the system, and puts
\[ M = \frac{\mu H}{kT} \tag{44} \]
($\mu$ denoting the magnetic moment of a given atom). The additional magnetic energy is ruled by the net number of spins aligned along the direction of $H$ ($n_\uparrow$) or counter-aligned along the same direction ($n_\downarrow$):
\[ -(n_\uparrow - n_\downarrow) \mu H = -(2i - n) \mu H. \tag{45} \]
Note that the local magnetization, in units such that the absolute saturation value is one, is given by
\[ S = \frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow} = \frac{2i - n}{n}, \tag{46} \]
so that the mean magnetization of the system, i.e. “the ratio $S$ between the magnetic moment under the influence of the field $H$ and the saturation magnetic moment”, is:
\[ S = \frac{\sum_i 2i - n}{n} \int y e^{-\frac{E}{kT}} e^{M(2i - n)} dN}{\sum_i y e^{-\frac{E}{kT}} e^{M(2i - n)} dN} \]
\[ = \frac{\sum_i 2i \left( \begin{array}{c} n \\ i \end{array} \right) \left( 1 + e^{-L} \right)^{\frac{2(\alpha - 1)}{\alpha}} e^{M(2i - n)}}{\sum_i \left( 1 + e^{-L} \right)^{\frac{2(\alpha - 1)}{\alpha}} e^{M(2i - n)}} - 1. \tag{47} \]
The (logarithm of the) partition function
\[ Z = \sum_{n_1, n_1} p(S), \]
where \( p(S) \) is the probability to have the magnetization \( S \), in the continuum limit may then be evaluated from
\[ \log \int e^{-\frac{E}{kT}} e^{M(2i-n)} y dN = \frac{2i(n-i)}{n} \log \frac{1 + e^{-L}}{2} + 2Mi - i \log i
- (n-i) \log (n-i) + \text{constant terms}, \]
where the Stirling approximation for the factorials (in the binomial coefficients)
\[ \log n! \approx n \log n - n \]
has been used. Here the omitted terms do not depend on the number \( i \) of spin-up elements. The condition for having a maximum in the magnetization (in the case of ferromagnetism) may then be obtained “by taking the derivative of (49) with respect to \( i \) and equating the result to 0”:
\[ a \frac{2(n-2i)}{n} \log \frac{1 - e^{-L}}{2} + 2M - \log i + \log (n-i) = 0, \]
\[ \log \frac{i}{n-i} = 2M + a \frac{2(n-2i)}{n} \log \frac{1 + e^{-L}}{2}, \]
or, by using (46),
\[ \log \frac{1+S}{1-S} = 2\frac{\mu H}{kT} + 2aS \log \frac{2}{1 + e^{-\frac{2}{\mu H/kT}}}. \]
For small \( H \) and large \( T \), this reduces to:
\[ 2S = 2\frac{\mu H}{kT} + 2aS \log \frac{2}{1 + e^{-\frac{2}{\mu H/kT}}}. \]
“For \( T \) lower than the Curie point, for a given value of \( H \) there exist 2 values of \( S \) which, for not extremely high \( H \), are practically equal and opposite.”

Let now \( s > 0 \) be the local magnetization of the system for \( H = 0 \):
\[ \log \frac{1+s}{1-s} = 2as \log \frac{2}{1 + e^{-\frac{2}{\mu H/kT}}}. \]
The influence of the magnetic field \( H \) on the magnetization \( S \) may be evaluated by putting \( S = s + \Delta s \) in Eq. (50); at first order in \( \Delta s \) we have:
\[ \Delta s = \frac{\mu H}{kT} \frac{2}{1 - s^2 - a \log \frac{2}{1 - e^{-\frac{2}{\mu H/kT}}}}. \]

3. Some implications and further contributions
The main body of the Majorana contribution about ferromagnetism, present in his Quaderni, is all that reported in the previous section (apart from some omissions of scratch calculations and few other interesting points; see below). Here we point out some peculiar features of the Majorana theory, directly obtainable from the calculations reported but not explicitly mentioned in the notebooks. Nevertheless, even though in a form probably different from that given below, it is likely that Majorana was aware of them.
Figure 3. Graphical solution of the implicit equation (54), obtained from the intersection of the two curves $y = \log \frac{1+S}{1-S}$ and $y = \eta x$ for different values of the parameter $\eta$. A non-vanishing spontaneous magnetization (corresponding to the point P) exists only for $\eta > 2$, that is $a \geq 2$ and $T < T_c$.

Let us start with stressing that, besides the trivial solution $S = 0$, Eq. (50) admits also another non-vanishing solution for $H = 0$, depending on the temperature. In fact, the equation:

$$\log \frac{1+S}{1-S} = \eta S, \quad \eta = 2a \log \frac{2}{1+e^{-\frac{\epsilon}{kT}}}$$  \hspace{1cm} (54)

has a $S \neq 0$ solution if the following condition is fulfilled:

$$2a \log \frac{2}{1+e^{-\frac{\epsilon}{kT}}} > 2, \quad \text{or} \quad e^{-\frac{\epsilon}{kT}} < 2e^{-\frac{\epsilon}{kT}} - 1. \quad (55)$$

This condition is non trivially satisfied only if $2e^{-\frac{\epsilon}{kT}} - 1 > 0$, that is

$$a > \frac{1}{\log 2} \simeq 1.44. \quad (56)$$

In such a case, the condition (55) gives:

$$T < \frac{\epsilon}{k \log \left(2e^{-\frac{\epsilon}{kT}} - 1\right)} \equiv T_c. \quad (57)$$

Thus, if the temperature is lower than the critical value given in Eq. (57), the model studied by Majorana predicts a spontaneous ($H = 0$) magnetization $S \neq 0$, i.e. ferromagnetism, exactly as in the phenomenological Weiss theory, although in a form different from that coming out from Eq. (1). In addition to this, and as a necessary condition for it to occur, the inequality (56) must be fulfilled, that is the integer number of nearest neighbor atoms has to be at least 2, a prediction not contained in the Heisenberg model (see the Introduction). Note also that, in the Majorana view, the quantity $a$ effectively may count the number of physical dimensions (see below) of the crystal structure of the ferromagnetic body and, in this respect, such a result anticipates by several years what definitively proved later by Peierls [11].
Contrary to several other topics, none about ferromagnetism was reported in the Volumetti [24], this being an indication of the fact that the author did not regard his calculations as conclusive (see the Preface of Ref. [24]).

For some unknown reason, Majorana did not study further his model of ferromagnetism, probably regarding it incomplete [3] but he nevertheless considered some applications to ferromagnetic materials with different geometries, corresponding to different numbers $i$ of aligned spins on a total of $n$ and different numbers $a$ of nearest neighbors. Here we only mention these peculiarities, leaving aside less interesting numerical calculations of several quantities appearing in the model [25].

The simplest example is that of a linear chain but, contrarily to what usually considered (with $a = 2$), Majorana used $a = 1$ as depicted in Fig. 4a. Explicit numerical calculations for this case referred to $n = 60$ and $i = 10$.

The two dimensional case, with $a = 2$, was instead implemented by means of a triangular geometry, as showed in Fig. 4b, with $n = 24$ and $i = 6$.

Finally Majorana also considered the case with $a = 3$ (and $n = 6$, $i = 3$), realized with two possible, different geometries as illustrated in Fig. 4c.

While it is unquestionable that such examples are the simplest ones in the Majorana model, it is unknown to what particular materials Majorana thought when discussing them. Note, in fact, that the only elements known at that time to exhibit ferromagnetic properties were iron, cobalt and nickel, whose crystal structures are apparently different from those considered by Majorana (body centered cubic for Fe, face centered cubic for Fe and Ni, hexagonal for Co). However, the realization

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4Contrary to several other topics, none about ferromagnetism was reported in the Volumetti [24], this being an indication of the fact that the author did not regard his calculations as conclusive (see the Preface of Ref. [24]).
of unusual geometric arrangements, even in usual materials, could be well explained in terms of different interactions responsible of the effective equilibrium positions of the elementary magnets. In this respect it is quite illuminating the case of a “triatomic molecule forming an equilateral triangle” in a state analogous to that of ferromagnetism, studied by Inglis [20] and, especially, that of pyrrhotite (an iron sulfide mineral with a variable iron content). The same author modelled this element in terms of “one \( d \)-electron (or several independent \( d \)-electrons) per atom, with fairly large spin-orbit interaction (but not as large as the interatomic exchange energies), in a hexagonal crystal which has the iron atoms in successive principal planes not directly opposite one another, but staggered, or else far apart” [19]. If the reasoning of Inglis is really similar to that of his friend Majorana, the model prepared by this scholar was probably intended for providing a suitable framework for explaining peculiar phenomena (such as ferromagnetic anisotropy and others) not considered in the bare Heisenberg model. However, since no further certain indications exist in the Majorana notebooks, we prefer to not lucubrate on this, but rather stop here our analysis.

4. Conclusion

In this paper we have set forth and analyzed a theory of ferromagnetism developed by Majorana as early as the beginnings of 1930s, but never published by him. Such a theory differs substantially (in the methods employed and in the results obtained) from the well-known Heisenberg theory of 1928, although the general framework is the same. In fact, in both cases, the guidelines are those of the statistical Ising model of 1925, and the forces responsible of ferromagnetism are just derived from the quantum mechanical exchange interactions. Differently from Heisenberg, however, Majorana described the system of spin-up/spin-down elementary magnets in terms of wavefunctions written in the form of a Slater determinant (as did by Bloch and by Slater in 1929), in order to take directly into account the constraints imposed by the Pauli exclusion principle.

The distribution function \( y(N) \) for the configurations of the system of spins modelling the ferromagnet is, then, explicitly evaluated in the continuum limit (that is, for a continuous distribution of the energy levels of the system, as in the Heisenberg model). Indeed, from the observation that given configurations of spin-up/spin-down elementary magnets are statistically equivalent to those where a spin-up is replaced by a spin-down (or viceversa) due to thermal fluctuations, Majorana calculates the probability for those configurations of spins and, by solving a differential equation, he obtains the expression for \( y(N) \), from which the partition function of the system may be deduced. It is also interesting to mention that Majorana gives a series expansion of the distribution function obtained which goes well beyond the usual Gaussian approximation (with terms up to order 10).

The introduction of an external magnetic field acting on the elementary magnets allows the calculation of the appropriate partition function of the system and, from a maximum problem (on the probability for realizing a given magnetization), a non-linear equation for the mean magnetization \( S \) of the system is derived. Although such equation is different from that deduced by Heisenberg (and that deduced early by Weiss), it nevertheless admits a non trivial solution \( S \neq 0 \) even in the absence of an external magnetic field, provided that the temperature of the body is lower than a critical (non-vanishing) value. The main features of ferromagnetism are, thus,
obtained in a more transparent manner with respect to Heisenberg and subsequent (for instance, Bloch) derivations.

In addition, and differently from these previous models, a condition on the number of nearest neighbors directly follows in the Majorana theory (instead, in the Heisenberg theory, for example, a condition on this number is only supposed on the basis of available experimental data), anticipating in some respects a later key result by Peierls.

Finally, some applications of the theory to peculiar arrangements of the elementary magnets (linear chain, triangular chain, etc.) are as well considered by Majorana.

From what presently discussed, then, it comes out clear the relevance of the contribution by Majorana to the theory of ferromagnetism which, as happened for several other contributions by this scholar in different fields of physics research [23], unfortunately remained unknown until now. This evidently urges to look into the several thousands of pages of his personal research notes [24] [25], whose careful study has been undertaken only recently and which will certainly deserve, once more, further interesting results.

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