Reducing quadrangulations of the sphere and the projective plane

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Abstract

We show that every quadrangulation of the sphere can be transformed into a 4-cycle by deletions of degree-2 vertices and by $t$-contractions at degree-3 vertices. A $t$-contraction simultaneously contracts all incident edges at a vertex with stable neighbourhood. The operation is mainly used in the field of $t$-perfect graphs.

We further show that a non-bipartite quadrangulation of the projective plane can be transformed into an odd wheel by $t$-contractions and deletions of degree-2 vertices.

We deduce that a quadrangulation of the projective plane is (strongly) $t$-perfect if and only if the graph is bipartite.

1 Introduction

For characterising quadrangulations of the sphere, it is very useful to transform a quadrangulation into a slightly smaller one. Such reductions are mainly based on the following idea: Given a class of quadrangulations, a sequence of particular face-contractions transforms every member of the class into a 4-cycle; see eg Brinkmann et al [5], Nakamoto [23], Negami and Nakamoto [24], and Broersma et al. [6]. A face-contraction identifies two non-adjacent vertices $v_1, v_3$ of a 4-face $v_1, v_2, v_3, v_4$ in which the common neighbours of $v_1$ and $v_3$ are only $v_2$ and $v_4$. A somewhat different approach was made by Bau et al. [2]. They showed that any quadrangulation of the sphere can be transformed into a 4-cycle by a sequence of deletions of degree-2 vertices and so called hexagonal contractions. The obtained graph is a minor of the previous graph. Both operations can be obtained from face-contractions.

We provide a new way to reduce arbitrary quadrangulations of the sphere to a 4-cycle. Our operations are minor-operations — in contrast to face-contractions. We use deletions of degree-2 vertices and $t$-contractions. A $t$-contraction simultaneously contracts all incident edges of a vertex with stable neighbourhood and deletes all multiple edges. The operation is mainly used in the field of $t$-perfection. Face-contractions cannot be obtained from $t$-contractions. We restrict ourselves to $t$-contractions at vertices that are only contained in 4-cycles whose interior does not contain a vertex.

These $t$-contractions and deletions of degree-2 vertices can be obtained from a sequence of face-contractions. (1)
Figure 1 illustrates this. The restriction on the applicable $t$-contractions makes sure that all face-contractions can be applied, i.e., that all identified vertices are non-adjacent and have no common neighbours besides the two other vertices of their 4-face.

![Figure 1: Face-contractions that give a deletion of a degree-2 vertex, a $t$-contraction at a degree-3 and a degree-6 vertex](image)

We prove:

**Theorem 1.** Let $G$ be a quadrangulation of the sphere. Then, there is a sequence of $t$-contractions at degree-3 vertices and deletions of degree-2 vertices that transforms $G$ into a 4-cycle. During the whole process, the graph remains a quadrangulation.

The proof of Theorem 1 can be found in Section 2. It is easy to see that both operations used in Theorem 1 are necessary. By (1), Theorem 1 implies:

Any quadrangulation of the sphere can be transformed into a 4-cycle by a sequence of face-contractions.

Via the dual graph, quadrangulations of the sphere are in one-to-one correspondence with planar 4-regular (not necessarily simple) graphs. Theorem 1 thus implies a method to reduce all 4-regular planar graphs to the graph on two vertices and four parallel edges. Broersma et al. [6], Lehel [16], and Manca [17] analysed methods to reduce 4-regular planar graphs to the octahedron graph.

In the second part of this paper, we consider quadrangulations of the projective plane. We use Theorem 1 to reduce all non-bipartite quadrangulations of the projective plane to an odd wheel. A $p$-wheel $W_p$ is a graph consisting of a cycle $(w_1, \ldots, w_p, w_1)$ and a vertex $v$ adjacent to all vertices of the cycle. A wheel $W_p$ is an odd wheel, if $p$ is odd. Figure 2 shows some odd wheels.

**Theorem 2.** Let $G$ be a non-bipartite quadrangulation of the projective plane. Then, there is a sequence of $t$-contractions and deletions of degree-2 vertices that transforms $G$ into an odd wheel. During the whole process, the graph remains a non-bipartite quadrangulation.

The proof of this theorem can be found in Section 2. It is easy to see that both operations used in this theorem are necessary.

Negami and Nakamoto [24] showed that any non-bipartite quadrangulation of the projective plane can be transformed into a $K_4$ by a sequence of face-contractions. This result can be deduced from Theorem 2. By (1), Theorem 2 implies that any non-bipartite quadrangulation of the projective plane can be transformed into an odd wheel by a sequence of face-contractions. The odd
wheel $W_{2k+1}$ can now be transformed into $W_{2k-1}$ — and finally into $W_3 = K_4$ — by face-contractions (see Figure 3).

Nakamoto [23] gave a reduction method based on face-contractions and so called 4-cycle deletions for non-bipartite quadrangulations of the projective plane with minimum degree 3. Matsumoto et al. [18] analysed quadrangulations of the projective plane with respect to hexagonal contractions while Nakamoto considered face-contractions for quadrangulations of the Klein bottle [21] and the torus [22]. Youngs [26] and Esperet and Strehlik [9] considered non-bipartite quadrangulations of the projective plane with regard to vertex-colourings and width-parameters.

Theorem 2 allows an application to the theory of $t$-perfection. A graph $G$ is $t$-perfect if its stable set polytope $SSP(G)$ equals the polyhedron $TSTAB(G)$. The stable set polytope $SSP(G)$ is the convex hull of stable sets of $G$; the polyhedron $TSTAB(G)$ is defined via non-negativity-, edge- and odd-cycle inequalities (see Section 3 for a precise definition).

If the system of inequalities defining $TSTAB(G)$ is totally dual integral, the graph $G$ is called strongly $t$-perfect. Evidently, strong $t$-perfection implies $t$-perfection. It is not known whether the converse is also true.

Theorem 3. For every quadrangulation $G$ of the projective plane the following assertions are equivalent:

(a) $G$ is $t$-perfect
(b) $G$ is strongly $t$-perfect
(c) $G$ is bipartite

See Section 3 for precise definitions and for the proof.

A general treatment on $t$-perfect graphs may be found in Grötschel, Lovász and Schrijver [14, Ch. 9.1] as well as in Schrijver [25, Ch. 68]. We showed that triangulations of the projective plane are (strongly) $t$-perfect if and only
if they are perfect and do not contain the complete graph $K_4$ \[10\]. Bruhn and Benchetrit analysed $t$-perfection of triangulations of the sphere \[3\]. Boulala and Uhry \[4\] established the $t$-perfection of series-parallel graphs. Gerards \[11\] extended this to graphs that do not contain an odd-$K_4$ as a subgraph (an odd-$K_4$ is a subdivision of $K_4$ in which every triangle becomes an odd circuit). Gerards and Shepherd \[12\] characterised the graphs with all subgraphs $t$-perfect, while Barahona and Mahjoub \[1\] described the $t$-imperfect subdivisions of $K_4$. Bruhn and Fuchs \[7\] characterised $t$-perfection of $P_5$-free graphs by forbidden $t$-minors.

2 Quadrangulations

All the graphs mentioned here are finite and simple. We follow the notation of Diestel \[8\]. We begin by recalling several useful definitions related to surface-embedded graphs. For further background on topological graph theory, we refer the reader to Gross and Tucker \[13\] or Mohar and Thomassen \[20\].

An embedding of a simple graph $G$ on a surface is a continuous one-to-one function from a topological representation of $G$ into the surface. For our purpose, it is convenient to abuse the terminology by referring to the image of $G$ as the the graph $G$. The faces of an embedding are the connected components of the complement of $G$. An embedding $G$ is even if all faces are bounded by an even circuit. A quadrangulation is an embedding where each face is bounded by a circuit of length 4. A cycle $C$ is contractible if $C$ separates the surface into two sets $S_C$ and $\overline{S}_C$ where $S_C$ is homeomorphic to an open disk in $\mathbb{R}^2$. Note that for the sphere, $S_C$ and $\overline{S}_C$ are homeomorphic to an open disk. In contrast, for the plane and the projective plane, $\overline{S}_C$ is not homeomorphic to an open disk. For the plane and the projective plane, we call $S_C$ the interior of $C$ and $\overline{S}_C$ the exterior of $C$. Using the stereographic projection, it is easy to switch between embeddings in the sphere and the plane. In order to have an interior and an exterior of a contractible cycle, we will concentrate on quadrangulations of the plane (and the projective plane). Note that by the Jordan curve theorem,

all cycles in the plane are contractible. \[2\]

A cycle in a non-bipartite quadrangulation of the projective plane is contractible if and only if it has even length (see e.g. \[15\] Lemma 3.1). As every non-bipartite even embedding is a subgraph of a non-bipartite quadrangulation, one can easily generalise this result.

**Observation 4.** A cycle in a non-bipartite even embedding in the projective plane is contractible if and only if it has even length.

An embedding is a 2-cell embedding if each face is homeomorphic to an open disk. It is well-known that embeddings of 2-connected graphs in the plane are 2-cell embeddings. A non-bipartite quadrangulation of the projective plane contains a non-contractible cycle; see Observation \[4\]. The complement of this cycle in the projective plane is homeomorphic to an open disk. Thus, we observe:

**Observation 5.** Every quadrangulation of the plane and every non-bipartite quadrangulation of the projective plane is a 2-cell embedding.

This observation makes sure that we can apply Euler’s formula to all the considered quadrangulations. A simple graph cannot contain a 4-circuit that is
not a 4-cycle. Thus, note that every face of a quadrangulation is bounded by a cycle.

It is easy to see that

all quadrangulations of the plane are bipartite. \hfill (3)

We first take a closer look at deletions of degree-2 vertices in graphs that are not the 4-cycle $C_4$.

**Observation 6.** Let $G \neq C_4$ be a quadrangulation of the plane or the projective plane that contains a vertex $v$ of degree 2. Then, $G - v$ is again a quadrangulation.

**Proof.** Let $u$ and $u'$ be the two neighbours of $v$. Then, there are distinct vertices $s, t$ such that the cycles $(u, v, u', s, u)$ and $(u, v, u', t, u)$ are bounding a face. Thus, $(u, s, u', t, u)$ is a contractible 4-cycle whose interior contains only $v$ and $G - v$ is again a quadrangulation.

We now take a closer look at $t$-contractions.

**Lemma 7.** Let $G$ be a quadrangulation of the plane or a non-bipartite quadrangulation of the projective plane. Let $G'$ be obtained from $G$ by a $t$-contraction at $v$. If $v$ is not a vertex of a contractible 4-cycle with some vertices in its interior, then $G'$ is again a quadrangulation.

**Proof.** Let $G''$ be obtained from $G$ by the operation that identifies $v$ with all its neighbours but does not delete multiple edges. This operation leaves every cycle not containing $v$ untouched, transforms every other cycle $C$ into a cycle of length $|C| - 2$, and creates no new cycles. Therefore, all cycles bounding faces of $G''$ are of size 4 or 2. The graphs $G'$ and $G''$ differ only in the property that $G''$ has some double edges. These double edges form 2-cycles that arise from 4-cycles containing $v$. As all these 4-cycles are contractible (see (2) and Observation 4) with no vertex in their interior, the 2-cycles are also contractible and contain no vertex in its interior. Deletion of all double edges now gives $G'$ — an embedded graph where all faces are of size 4.

Lemma 7 enables us to prove the following statement that directly implies Theorem 1.

**Lemma 8.** Let $G$ be a quadrangulation of the plane. Then, there is a sequence of

- $t$-contractions at degree-3 vertices that are only contained in 4-cycles whose interior does not contain a vertex.

- deletions of degree-2 vertices

that transforms $G$ into a 4-cycle. During the whole process, the graph remains a quadrangulation.

**Proof.** Let $\mathcal{C}$ be the set of all contractible 4-cycles whose interior contains some vertices of $G$. Note that $\mathcal{C}$ contains the 4-cycle bounding the outer face unless $G = C_4$.
Let $C \in \mathcal{C}$ be a contractible 4-cycle whose interior does not contain another element of $\mathcal{C}$. We will first see that the interior of $C$ contains a vertex of degree 2 or 3: Deletion of all vertices in the exterior of $C$ gives a quadrangulation $G'$ of the plane. As $G$ is connected, one of the vertices in $C$ must have a neighbour in the interior of $C$ and thus must have degree at least 3. Euler's formula now implies that $\sum_{v \in V(G')} \deg(v) = 2|E(G')| \leq 4|V(G')| - 8$. As no vertex in $G'$ has degree 0 or 1, there must be a vertex of degree 2 or 3 in $V(G') - V(C)$. This vertex has the same degree in $G$ and is contained in the interior of $C$.

We use deletions of degree-2 vertices and $t$-contractions at degree-3 vertices in the interior of the smallest cycle of $\mathcal{C}$ to successively get rid of all vertices in the interior of 4-cycles. By Observation 6 and Lemma 7 the obtained graphs are quadrangulations. Now, suppose that no more $t$-contraction at a degree-3 vertex and no more deletion of a degree-2 vertex is possible. Assume that the obtained graph is not a 4-cycle. Then, there is a cycle $C' \in \mathcal{C}$ whose interior does not contain another cycle of $\mathcal{C}$. As we have seen above, $C' \in \mathcal{C}$ contains a vertex $v$ of degree 3. Since no $t$-contraction can be applied to $v$, the vertex $v$ has two adjacent neighbours. This contradicts (3).

In the rest of the paper, we will consider the projective plane.

A quadrangulation of the projective plane is nice if no vertex is contained in the interior of a contractible 4-cycle.

**Lemma 9.** Let $G$ be a non-bipartite quadrangulation of the projective plane. Then, there is a sequence of $t$-contractions and deletions of vertices of degree 2 that transforms $G$ into a nice quadrangulation. During the whole process, the graph remains a quadrangulation.

**Proof.** Let $C$ be a contractible 4-cycle whose interior contains at least one vertex. Delete all vertices that are contained in the exterior of $C$. The obtained graph is a quadrangulation of the plane. By Lemma 8 there is a sequence of $t$-contractions (as described in Lemma 7) and deletions of degree-2 vertices that eliminates all vertices in the interior of $C$. With this method, it is possible to transform $G$ into a nice quadrangulation.

Similar as in the proof of Theorem 1, Euler’s formula implies that a non-bipartite quadrangulation of the projective plane contains a vertex of degree 2 or 3. As no nice quadrangulation has a degree-2 vertex (see Observation 6), we deduce:

**Observation 10.** Every nice non-bipartite quadrangulation of the projective plane has minimal degree 3.

In an even embedding of an odd wheel $W$, every odd cycle must be non-contractible (see Observation 4). Thus, it is easy to see that there is only one way (up to topological isomorphy) to embed an odd wheel in the projective plane. (This can easily be deduced from [19] — a paper dealing with embeddings of planar graphs in the projective plane.) The embedding is illustrated in Figure 3. Noting that this embedding is a quadrangulation, we observe:

**Observation 11.** Let $G$ be a quadrangulation of the projective plane that contains an odd wheel $W$. If $G$ is nice, then $G$ equals $W$. 




Note that every graph containing an odd wheel also contains an induced odd wheel. Now, we consider even wheels.

**Lemma 12.** Even wheels $W_{2k}$ for $k \geq 2$ do not have an even embedding in the projective plane.

The statement follows directly from [19]. We nevertheless give an elementary proof of the lemma.

**Proof.** First assume that the 4-wheel $W_4$ has an even embedding. As all triangles of $W_4 - w_3w_4$ must be non-contractible by Observation 4, it is easy to see that the graph must be embedded as in Figure 4. Since the insertion of $w_3w_4$ will create an odd face, $W_4$ is not evenly embeddable.

Now assume that $W_{2k}$ for $k \geq 3$ is evenly embedded. Delete the edges $vw_i$ for $i = 5, \ldots, 2k$ and note that $w_5, \ldots, w_{2k}$ are now of degree 2, ie the path $P = (w_4, w_5, \ldots, w_{2k}, w_1)$ bounds two faces or one face from two sides. Deletion of the edges $vw_i$ preserve the even embedding: Deletion of an edge bounding two faces $F_1, F_2$ merges the faces into a new face of size $|F_1| + |F_2| - 2$. Deletion of an edge bounding a face $F$ from two sides leads to a new face of size $|F| - 2$. In both cases, all other faces are left untouched.

Next, replace the odd path $P$ by the edge $w_4w_1$. The two faces $F_3, F_4$ adjacent to $P$ are transferred into two new faces of size $|F_3| - (2k - 3) + 1$ and $|F_4| - (2k - 3) + 1$. This yields an even embedding of $W_4$ which is a contradiction.

![Figure 4](image)

Figure 4: The only even embedding of $W_4 - w_3w_4$ in the projective plane. Opposite points on the dotted cycle are identified.

Note that a $t$-contraction at a vertex $v$ is only allowed if its neighbourhood is stable, that is, if $v$ is not contained in a triangle. The next lemma characterises the quadrangulations to which no $t$-contraction can be applied.

**Lemma 13.** Let $G$ be a non-bipartite nice quadrangulation of the projective plane where each vertex is contained in a triangle. Then $G$ is an odd wheel.

**Proof.** By Observation 10 there is a vertex $v$ of degree 3 in $G$. Let $\{x_1, x_2, x_3\}$ be its neighbourhood and let $x_1, x_2$ and $v$ form a triangle.

Recall that each two triangles are non-contractible (see Observation 4). Consequently each two triangles intersect. As $x_3$ is contained in a triangle intersecting the triangle $(v, x_1, x_2)$ and as $v$ has no further neighbour, we can suppose without loss of generality that $x_3$ is adjacent to $x_1$. The graph induced by the two triangles $(v, x_1, x_2)$ and $(x_1, v, x_3)$ is not a quadrangulation. Further, addition of the edge $x_2x_3$ yields a $K_4$. By Observation 11 $G$ then equals the odd wheel $W_3 = K_4$. 

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Otherwise, the graph contains a further vertex and this vertex is contained in a further triangle $T$. Since the vertex $v$ has degree 3, it is not contained in $T$. If further $x_1 \notin V(T)$, then the vertices $x_2$ and $x_3$ must be contained in $T$. But then $x_2x_3 \in E(G)$ and, as above, $v$, $x_1$, $x_2$ and $x_3$ form a $K_4$. Therefore, $x_1$ is contained in $T$ and consequently in every triangle of $G$. Since every vertex is contained in a triangle, $x_1$ must be adjacent to all vertices of $G - x_1$. As $|E(G)| = 2|V(G)| - 2$ by Euler’s formula, the graph $G - x_1$ has $2|V(G)| - 2 - (|V(G)| - 1) = |V(G)| - 1 = |V(G - x_1)|$ many edges. By Observation 10, no vertex in $G$ has degree smaller than 3. Consequently, no vertex in $G - x_1$ has degree smaller than 2. Thus, $G - x_1$ is a cycle and $G$ is a wheel. By Lemma 12, $G$ is an odd wheel.

Finally, we can prove our second main result:

**Proof of Theorem 3**. Transform $G$ into a nice quadrangulation (Lemma 9). Now, consecutively apply $t$-contractions (as described in Lemma 17) as long as possible. In each step, the obtained graph is a quadrangulation. By Lemma 10, we can assume that the quadrangulation is nice. If no more $t$-contraction can be applied, then every vertex is contained in a triangle. By Lemma 13, the obtained quadrangulation is an odd wheel.

### 3 (Strong) $t$-perfection

The **stable set polytope** SSP($G$) $\subseteq \mathbb{R}^V$ of a graph $G = (V,E)$ is defined as the convex hull of the characteristic vectors of stable, independent, subsets of $V$. The characteristic vector of a subset $S$ of the set $V$ is the vector $\chi_S \in \{0,1\}^V$ with $\chi_S(v) = 1$ if $v \in S$ and 0 otherwise. We define a second polytope TSTAB($G$) $\subseteq \mathbb{R}^V$ for $G$, given by

\[
\begin{align*}
x & \geq 0, \\
x_u + x_v & \leq 1 \text{ for every edge } uv \in E, \\
\sum_{v \in V(C)} x_v & \leq \left\lfloor \frac{|C|}{2} \right\rfloor \text{ for every induced odd cycle } C \text{ in } G.
\end{align*}
\]

These inequalities are respectively known as non-negativity, edge and odd-cycle inequalities. Clearly, SSP($G$) $\subseteq$ TSTAB($G$). The graph $G$ is called $t$-perfect if SSP($G$) and TSTAB($G$) coincide. Equivalently, $G$ is $t$-perfect if and only if TSTAB($G$) is an integral polytope, i.e., if all its vertices are integral vectors. The graph $G$ is called strongly $t$-perfect if the system of inequalities is totally dual integral. That is, if for each weight vector $w \in \mathbb{Z}^V$, the linear program of maximizing $w^T x$ over (4) has an integer optimum dual solution. This property implies that TSTAB($G$) is integral. Therefore, strong $t$-perfection implies $t$-perfection. It is an open question whether every $t$-perfect graph is strongly $t$-perfect. The question is briefly discussed in Schrijver [23, Vol. B, Ch. 68].

It is easy to see that all bipartite graphs are (strongly) $t$-perfect (see e.g., Schrijver [25, Ch. 68]) and that vertex deletion preserves (strong) $t$-perfection. Another operation that keeps (strong) $t$-perfection (see e.g., [25, Vol. B, Ch. 68.4]) was found by Gerards and Shepherd [12]: the $t$-contraction.

Odd wheels $W_{2k+1}$ for $k \geq 1$ are not (strongly) $t$-perfect. Indeed, the vector $(1/3, \ldots, 1/3)$ is contained in TSTAB($W_{2k+1}$) but not in SSP($W_{2k+1}$).
With this knowledge, the proof of Theorem 3 follows directly from Theorem 2.

**Proof of Theorem 3.** If \( G \) is bipartite, the \( G \) is (strongly) \( t \)-perfect.

Let \( G \) be non-bipartite. Then, there is a sequence of \( t \)-contractions and deletions of vertices that transforms \( G \) into an odd wheel (Theorem 2). As odd wheels are not (strongly) \( t \)-perfect and as vertex deletion and \( t \)-contraction preserve (strong) \( t \)-perfection, \( G \) is not (strongly) \( t \)-perfect.

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