High dimensional finite elements for multiscale Maxwell wave equations

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Abstract

We develop an essentially optimal numerical method for solving multiscale Maxwell wave equations in a domain $D \subset \mathbb{R}^d$. The problems depend on $n+1$ scales: one macroscopic scale and $n$ microscopic scales. Solving the macroscopic multiscale homogenized problem, we obtain the desired macroscopic and microscopic information. This problem depends on $n+1$ variables in $\mathbb{R}^d$, one for each scale that the original multiscale equation depends on, and is thus posed in a high dimensional tensorized domain. The straightforward full tensor product finite element (FE) method is exceedingly expensive. We develop the sparse tensor product FEs that solve this multiscale homogenized problem with essentially optimal number of degrees of freedom, that is essentially equal to that required for solving a macroscopic problem in a domain in $\mathbb{R}^d$ only, for obtaining a required level of accuracy. Numerical correctors are constructed from the FE solution. For two scale problems, we derive a rate of convergence for the numerical corrector in terms of the microscopic scale and the FE mesh width. Numerical examples confirm our analysis.

1 Introduction

We study the high dimensional finite element (FE) method for solving multiscale Maxwell wave equations in a domain $D \subset \mathbb{R}^d$. The equation depends on the macroscopic scale and $n$ microscopic scales, and is locally periodic. We study the problem via multiscale convergence. In the limit where all the microscopic scales converge to zero, we obtain the multiscale homogenized equation. This equation contains the solution to the homogenized equation which approximates the solution of the original multiscale equation macroscopically, and the scale interacting corrector terms which provide the microscopic behaviour of the solution. Solving the equation, we obtain all the necessary information. However, the multiscale homogenized equation is posed in a high dimensional tensorized domain. It depends on $n+1$ variables in $\mathbb{R}^d$, one for each scale. The direct full tensor product FE method is highly expensive. We develop the sparse tensor product FEs to solve this problem which requires only essentially equal number of degrees of freedom as for solving an equation posed in $\mathbb{R}^d$ for obtaining a required level of accuracy. The complexity is thus essentially optimal.

As for any other multiscale problems, a direct numerical method using fine mesh to capture the microscopic scales is prohibitively expensive. There have been attempts to develop numerical methods for solving multiscale wave equations, and multiscale Maxwell equations with reduced complexity, though comparing to other types of multiscale equations, multiscale wave and multiscale Maxwell equations have been paid far less attention.

For multiscale wave equations, in [28] Owhadi and Zhang build a set of basis functions that contain microscopic information from the solutions of $d$ multiscale equations. These equations are solved using fine mesh to capture the microscopic scales. In [23], Jiang et al. employ the Multiscale Finite Element method ([22], [15]) to solve wave equations that depend on a continuum spectrum of scales, using limited global information. The Heterogeneous Multiscale Method (HMM) ([14], [11]) is employed by Engquist et al. using finite differences to solve multiscale wave equations that show the dispersive behaviour at large time. Abdulle and Grote [2] employ the Heterogeneous Multiscale Method (HMM) to solve multiscale
equation using finite elements. The approaches in these papers are general, but the complexity at each time step grows superlinearly with respect to the optimal complexity level. In [31], Xia and Hoang develop the essentially optimal sparse tensor product FE method for locally periodic multiscale wave equations; the complexity of the method only grows log-linearly at each time step. The method is employed successfully for multiscale elastic wave equations in [33].

There has not been much research on efficient numerical methods for multiscale Maxwell equations. The traditional method that constructs the homogenized equation by solving cell problems is considered in [34] (see also the related references therein) where a set of cell problems are solved at each macroscopic points. The complexity is thus very high. The HMM method is applied for multiscale Maxwell equations in frequency domain in Ciarlet et al. [11]. Ohlberger et al. considered a locally periodic two scale harmonic Maxwell equation in [17] though the problem is assumed uniformly coercive with respect to the microscopic scale. The HMM method is analyzed for the two scale homogenized problem using the approach in [27]. The complexity of the method is equivalent to that of a full tensor product FE method for solving the two scale homogenized equation. In [8], Chu and Hoang develop the sparse tensor product edge FE method for locally periodic stationary multiscale Maxwell equations. The method requires only a number of degrees of freedom that is essentially equivalent to that needed for solving a macroscopic scale Maxwell equation in a domain in \( \mathbb{R}^d \), and is therefore optimal. Chu and Hoang [8] construct numerical correctors from the finite element solutions. For two scale problems, an explicit error in terms of the FE error and the homogenization error is deduced for the numerical corrector.

We develop the sparse tensor product FE approach for multiscale Maxwell wave equations in this paper using edge FEs. We show that the complexity of the method is essentially optimal. The sparse tensor product FE approach for multiscale problems is initiated by Hoang and Schwab in [20] for elliptic equations, and is applied for other types of equations in [19], [31], [32], [33].

In the next section, we set up the multiscale Maxwell wave equation and derive the multiscale homogenized equation. We will only summarize the results and refer to [9] for detailed derivation. In Section 3, we study FE approximation for the multiscale homogenized Maxwell wave equation using general FE spaces. In Subsection 3.1, we study the spatially semidiscrete problem where only the spatial variable is discretized. We follow the framework of Dupont [13] for wave equations. The approach has been applied for the multiscale homogenized equations of scalar multiscale wave equations in Xia and Hoang [31]. However, the application of the framework to multiscale homogenized Maxwell wave equations requires substantial modification for the analysis of the convergence due to the corrector terms in (2.6). In Subsection 3.2, we consider the fully discrete problem where both the temporal and spatial variables are discretized. The convergence of the general discretization schemes in Section 3, and the full and sparse tensor product FE approximations in Section 5 require regularity for the solution of the multiscale homogenized Maxwell wave equation. In Section 4, we prove that the required regularity hold under mild conditions. In Section 5, we apply the discretization schemes in Section 3 for the full tensor product and the sparse tensor product edge FEs. We prove that the sparse tensor product FE method obtains an approximation with essentially the same level of accuracy as the full tensor product FEs but requires only essentially the same number of degrees of freedom as for solving a macroscopic Maxwell equation in \( \mathbb{R}^d \), and is thus essentially optimal. In Section 6, we construct numerical correctors from the FE solutions. For two scale problems, an explicit homogenization error in terms of the microscopic scale is available. The derivation is complicated, especially due to the low regularity of the solution of the homogenized Maxwell wave equation. We therefore only summarize the theoretical results and refer the reader to [9] for details. From this, we derive a numerical corrector with an explicit error in terms of the homogenization error, and the FE error. For general multiscale problems, such a homogenization error is not available. We thus derive a general numerical corrector without an explicit error. Section 7 presents some numerical examples in two dimensions that confirm our analysis.

Throughout the paper, by curl and \( \nabla \) without explicitly indicating the variable, we mean the curl and the gradient of a function of \( x \) with respect to \( x \), and by curl \( x \) and \( \nabla_x \) we denote the partial curl and partial gradient of a function that depends on \( x \) and other variables. We denote by \( \langle \cdot, \cdot \rangle_{X',X} \) the duality pairing of a Banach space \( X \) and its dual \( X' \). Repeated indices indicate summation. The notation \( \# \) denotes spaces of periodic functions with the period being the unit cube in \( \mathbb{R}^d \).
2 Multiscale Maxwell wave problems

We set up the multiscale Maxwell wave equation and use multiscale convergence to homogenize it in this section.

2.1 Problem setting

Let $D$ be a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$). Let $\mathbf{Y}$ be the unit cube in $\mathbb{R}^d$. By $Y_1, \ldots, Y_n$ we denote $n$ copies of $Y$. We denote by $\mathbf{Y}$ the product set $Y_1 \times Y_2 \times \cdots \times Y_n$ and by $\mathbf{y} = (y_1, \ldots, y_n)$. For $i = 1, \ldots, n$, we denote by $\mathbf{Y}_i = Y_1 \times \cdots \times \hat{Y}_i \times \cdots \times Y_n$. Let $a$ and $b$ be functions from $D \times \mathbf{Y}_1 \times \cdots \times \mathbf{Y}_n$ to $\mathbb{R}^{d \times d}$ and $\mathbb{R}^{n \times d}$ respectively. We set up the multiscale Maxwell wave equation and use multiscale convergence to homogenize it in this section.

In variational form, this problem becomes: Find $\mathbf{u} \in \mathbb{H}^2(D) \cap H^1(D)$ such that

$$\int_D \mathbf{u} \cdot \nabla \phi \, dx = \int_D \mathbf{f} \cdot \phi \, dx$$

for all $\phi \in H^1(D)$.

2.2 Problem setting

Let $\omega \in H^1(D)$ be a small positive value, and $\varepsilon_1, \ldots, \varepsilon_n$ be $n$ functions of $\varepsilon$ that denote the $n$ microscopic scales that the problem depends on. We assume the following scale separation properties: for all $i = 1, \ldots, n - 1$

$$\lim_{\varepsilon \to 0} \varepsilon_{i+1}(\varepsilon) = 0.$$  

Without loss of generality, we assume that $\varepsilon_1(\varepsilon) = \varepsilon$. We define the multiscale coefficients of the Maxwell equation $a^\varepsilon$ and $b^\varepsilon$ which are functions from $D$ to $\mathbb{R}^{d \times d}$ as

$$a^\varepsilon(x) = a(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}), \quad b^\varepsilon(x) = b(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}).$$

When $d = 3$ we define the space

$$W = H_0(\text{curl}, D) = \{u \in L^2(\Omega)^3, \ \text{curl} \ u \in L^2(\Omega)^3, \ u \times \nu = 0\}, \quad H = L^2(D)^3$$

and when $d = 2$

$$W = H_0(\text{curl}, D) = \{u \in L^2(D)^2, \ \text{curl} \ u \in L^2(D), \ u \times \nu = 0\}, \quad H = L^2(D)^2$$

where $\nu$ denotes the outward normal vector on the boundary $\partial D$. These spaces form the Gelfand triple $W \subset H \subset W'$. We note that when $d = 3$, $u^\varepsilon$ is a vector function in $L^2(D)^3$ and when $d = 2$, $u^\varepsilon$ is a scalar function in $L^2(D)$. Let $f \in L^2(0, T; H)$, $g_0 \in W$ and $g_1 \in H$. We consider the problem: Find $u^\varepsilon(t, x) \in L^2(0, T; W)$ so that

\[
\begin{cases}
\frac{\partial^2 u^\varepsilon(t, x)}{\partial t^2} + \text{curl}(a^\varepsilon(x)\text{curl} u^\varepsilon(t, x)) = f(t, x), & (0, T) \times D \\
u^\varepsilon(0, x) = g_0(x) \\
u_1^\varepsilon(0, x) = g_1(x)
\end{cases}
\]

with the boundary condition $u^\varepsilon \times \nu = 0$ on $\partial D$. We will mostly present the analysis for the case $d = 3$ and only discuss the case $d = 2$ when there is significant difference. For notational conciseness, we denote by

$$H_i = L^2(\mathbf{Y}_i).$$

In variational form, this problem becomes: Find $u^\varepsilon \in L^2(0, T; W) \cap H^1(0, T; H)$ so that

\[
\int_D \frac{\partial^2 u^\varepsilon}{\partial t^2} \phi(x) \, dx + \int_D a^\varepsilon(x)\text{curl} u^\varepsilon \cdot \text{curl} \phi(x) \, dx = \int_D f(t, x) \cdot \phi(x) \, dx
\]

for all $\phi \in W$ when $d = 3$; and when $d = 2$ we need to replace the vector product for curl by the scalar multiplication. Problem (2.3) has a unique solution $u^\varepsilon \in L^2(0, T; W) \cap H^1(0, T; H) \cap L^2(0, T; W')$ that satisfies

\[
\|u^\varepsilon\|_{L^2(0, T; W)} + \|u^\varepsilon\|_{H^1(0, T; H)} + \|u^\varepsilon\|_{H^2(0, T; W')} \leq c(\|f\|_{L^2(0, T; H)} + \|g_0\|_W + \|g_1\|_H)
\]

where the constant $c$ only depends on the constants $\alpha$ and $\beta$ in (2.1) and $T$ (see Wloka [30]).

We will study this problem via multiscale convergence.
2.2 Multiscale convergence

We study homogenization of problem (2.4) via multiscale convergence. We therefore recall the definition of multiscale convergence (seeNguetseng [26], Allaire [3] and Allaire and Briane [4]).

Definition 2.1 A sequence of functions $\{u^\varepsilon\}_\varepsilon \subset L^2(0,T;H)$ $(n+1)$-scale converges to a function $w_0 \in L^2(0,T;D \times Y)$ if for all smooth functions $\phi(t,x,y)$ which are $Y$ periodic w.r.t $y_i$ for all $i=1, \ldots, n$:

$$\lim_{\varepsilon \to 0} \int_0^T \int_D w^\varepsilon(t,x)\phi(t,x,\frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n})dxdt = \int_0^T \int_Y w_0(t,x,y)\phi(t,x,y)dydxt.$$

We have the following result.

Proposition 2.2 From a bounded sequence in $L^2(0,T;H)$ we can extract an $(n+1)$-scale convergent subsequence.

We note that the definition above for functions which depend also on $t$ is slightly different from that in [26] and [3] as we take also the integral with respect to $t$. However, the proof of Proposition 2.2 is similar.

From (2.5) and Proposition (2.3), we can extract a subsequence (not renumbered), a function $w_0 \in L^2(0,T;W)$, $n$ functions $w_i \in L^2((0,T) \times D \times Y_1 \times \ldots \times Y_{i-1}, H^1_{Y_i}(Y_i)/\mathbb{R})$ such that

$$w^\varepsilon \rightarrow_{(n+1)\text{-scale}} w_0 + \sum_{i=1}^n \nabla_{y_i} w_i.$$

Further, there are $n$ functions $w_i \in L^2((0,T) \times D \times \ldots \times Y_{i-1}, \tilde{H}_{Y_i}(\text{curl}, Y_i))$ such that

$$\text{curl } w^\varepsilon \rightarrow_{(n+1)\text{-scale}} \text{curl } w_0 + \sum_{i=1}^n \text{curl}_{y_i} w_i.$$

From (2.6) and Proposition (2.3), we can extract a subsequence (not renumbered), a function $u_0 \in L^2(0,T;W)$, $n$ functions $u_i \in L^2(0,T) \times D \times Y_1 \times \ldots \times Y_i-1, H^1_{Y_i}(Y_i)/\mathbb{R})$ and $n$ functions $u_i \in L^2(0,T;D \times Y_1 \times \ldots \times Y_i-1, \tilde{H}_{Y_i}(\text{curl}, Y_i))$ such that

$$u^\varepsilon \rightarrow_{(n+1)\text{-scale}} u_0 + \sum_{i=1}^n \nabla_{y_i} u_i,$$  

and

$$\text{curl } u^\varepsilon \rightarrow_{(n+1)\text{-scale}} \text{curl } u_0 + \sum_{i=1}^n \text{curl}_{y_i} u_i.$$

For $i=1, \ldots, n$, let $W_i = L^2(D \times Y_1 \times \ldots \times Y_{i-1}, \tilde{H}_{Y_i}(\text{curl}, Y_i))$ and $V_i = L^2(D \times Y_1 \times \ldots \times Y_{i-1}, H^1_{Y_i}(Y_i)/\mathbb{R})$. We define the space $V$ as

$$V = W \times W_1 \times \ldots \times W_n \times V_1 \times \ldots \times V_n.$$  

For $\mathbf{v} = (v_0, \{v_i\}, \{v_i\}) \in V$, we define the norm

$$|||\mathbf{v}||| = ||v_0||_{H_{(\text{curl},D)}} + \sum_{i=1}^n ||v_i||_{L^2(D \times Y_{i-1}, \tilde{H}_{Y_i}(\text{curl}, Y_i))} + \sum_{i=1}^n ||v_i||_{L^2(D \times Y_1 \times \ldots \times Y_{i-1}, H^1_{Y_i}(Y_i))}.$$

Let $\mathbf{u} = (u_0, \{u_i\}, \{u_i\}) \in V$. We define the function

$$\int_Y b(x,y) \left( u_0(t,x) + \sum_{i=1}^n \nabla_{y_i} u_i(t,x,y_i) \right) dy$$

4
in $W'$ so that
\[
\left\langle \int_Y b(x, y) \left( u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i) \right) dy, v_0 \right\rangle_{W', W} = \int_D \int_Y b(x, y) \left( u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i) \right) v_0 dy dx;
\]
and the function $b(x, y)(u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i))$ in $W'_j$ as
\[
\left\langle b(x, y) \left( u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i) \right), v_j \right\rangle_{W'_j, W'_j} = \int_D \int_Y b(x, y) (u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i)) \nabla_y v_j dy dx.
\]
We then have the following result.

**Proposition 2.4** The function $u = (u_0, \{u_i\}, \{v_i\})$ satisfies
\[
\left\langle \frac{\partial^2}{\partial t^2} \int_Y b(x, y) \left( u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i) \right) dy, v_0 \right\rangle_{W', W} + \sum_{j=1}^n \left\langle \frac{\partial^2}{\partial t^2} b(x, y) \left( u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i) \right), v_j \right\rangle_{W'_j, W'_j} + \int_D \int_Y a(x, y) \left( \text{curl} u_0 + \sum_{i=1}^n \text{curl}_y u_i \right) \cdot \left( \text{curl} v_0 + \sum_{i=1}^n \text{curl}_y v_i \right) dy dx = \int_D f(t, x) \cdot v_0(x) dx \quad (2.9)
\]
for all $v = (v_0, \{v_i\}, \{v_i\}) \in V$.

For the initial conditions, we have

**Proposition 2.5** We have $u_0 \in H^1(0, T; H)$, $\nabla_y u_i \in H^1(0, T; H_i)$ for all $i = 1, \ldots, n$. Further
\[
\frac{\partial}{\partial t} \int_Y b(x, y) u_0(0, x) + \sum_{i=1}^n \nabla_y u_i(0, x, y_i) dy \bigg|_{t=0} = \int_Y b(x, y) g_1(x) dy, \quad \text{in } W' \quad (2.10)
\]
and for $j = 1, \ldots, n$
\[
\frac{\partial}{\partial t} \left( b(x, y)(u_0(t, x) + \sum_{i=1}^n \nabla_y u_i(t, x, y_i)) \right) \bigg|_{t=0} = b(x, y) g_1(x), \quad \text{in } W'_j. \quad (2.11)
\]
We then have

**Proposition 2.6** With the initial conditions (2.10), (2.11) and (2.12), problem (2.9) has a unique solution.

The proofs of Propositions 2.4, 2.5 and 2.6 can be found in [9].

### 3 Finite element discretization

We study finite element approximation for problem (2.9) in this section. We first consider the semidiscrete problem where we discretize the spatial variables. We then consider the fully discrete problem where both the temporal and spatial variables are discretized.
3.1 Spatially semidiscrete problem

We consider in this section the spatial semidiscretization of the homogenized problem (2.9). For approximating \( u_0 \), we suppose that there is a hierarchy of finite dimensional subspaces

\[
W^1 \subset W^2 \subset \ldots \subset W^L \subset W;
\]

to approximate \( u_i, i = 1, 2, \ldots, n \), we assume a hierarchy of finite dimensional subspaces

\[
W_i^1 \subset W_i^2 \subset \ldots \subset W_i^L \subset W_i;
\]

and to approximate \( u_i, i = 1, 2, \ldots, n \), we assume a hierarchy of finite dimensional subspaces

\[
V_i^1 \subset V_i^2 \subset \ldots \subset V_i^L \subset V_i.
\]

Let

\[
V^L = W^L \times W_1^L \times \ldots \times W_n^L \times V_1^L \times \ldots \times V_n^L
\]

which is a finite dimensional subspace of \( V \) defined in (2.3). We consider the spatially semidiscrete approximating problems: Find \( u^L(t) = (u^L_0, u^L_1, \ldots, u^L_n, u^L_{11}, \ldots, u^L_{nn}) \in V^L \) so that

\[
\int_D \int_Y b(x, y) \left( \frac{\partial^2}{\partial t^2} u^L_i(t, x) + \sum_{i=1}^{n} \nabla_y \frac{\partial^2}{\partial t^2} u^L_i(t, x, y) \right) \cdot \left( v^L_i + \sum_{i=1}^{n} \nabla_y v^L_i \right) + a(x, y) \left( \text{curl } u^L_0 + \sum_{i=1}^{n} \text{curl}_y u^L_i \right) \cdot \left( \text{curl } v^L_0 + \sum_{i=1}^{n} \text{curl}_y v^L_i \right) \right) dydx
\]

\[
= \int_D f(t, x) \cdot v^L_i(x) dx \quad (3.1)
\]

for all \( u^L = (u^L_0, u^L_1, \ldots, u^L_n, w^L_0, \ldots, w^L_n) \in V^L \). Let \( g^L_0 \in W^L, g^L_i \in W^L \) which are approximations of \( g_0 \) and \( g_i \) in \( W \) and \( H \) respectively. The initial conditions (2.10) are approximated by:

\[
u^L_0(0, \cdot) = g^L_0, \quad \text{curl}_y u^L_i(0, \cdot) = 0.
\]

(3.2)

We approximate the initial conditions (2.11) and (2.12) by

\[
\int_D \int_Y b(x, y) \left( \frac{\partial u^L_0}{\partial t}(0) + \sum_{i=1}^{n} \frac{\partial}{\partial t} \nabla_y u^L_i(0) \right) \cdot \left( v^L_0 + \sum_{i=1}^{n} \nabla_y v^L_i \right) dydx
\]

\[
= \int_D \int_Y b(x, y) g^L_0(x) \cdot \left( v^L_0 + \sum_{i=1}^{n} \nabla_y v^L_i \right) dydx
\]

(3.3)

for all \( v^L_0 \in W^L \) and \( v^L_i \in V^L_i \), i.e.,

\[
\int_D \int_Y b(x, y) \left( \frac{\partial u^L_0}{\partial t}(0) - g^L_0 + \sum_{i=1}^{n} \frac{\partial}{\partial t} \nabla_y u^L_i(0) \right) \cdot \left( v^L_0 + \sum_{i=1}^{n} \nabla_y v^L_i \right) dydx = 0.
\]

Using the coercivity of the matrix \( b(x, y) \), we get

\[
\frac{\partial u^L_0}{\partial t}(0) = g^L_0, \quad \frac{\partial}{\partial t} \nabla_y u^L_i(0) = 0.
\]

(3.3)

For \( v = (v_0, v_1, \ldots, v_n) \) and \( w = (w_0, w_1, \ldots, w_n) \) in \( V = W \times W_1 \times \ldots \times W_n \times V_1 \times \ldots \times V_n \), we define the bilinear forms

\[
A(v, w) = \int_D \int_Y a(x, y) \left( \text{curl } v_0 + \sum_{i=1}^{n} \text{curl}_y v_i \right) \cdot \left( \text{curl } w_0 + \sum_{i=1}^{n} \text{curl}_y w_i \right) dydx,
\]

and

\[
B(v, w) = \int_D \int_Y b(x, y) \left( v_0 + \sum_{i=1}^{n} \nabla_y v_i \right) \cdot \left( w_0 + \sum_{i=1}^{n} \nabla_y w_i \right) dydx.
\]
Proposition 3.1 Problem (3.1) together with the initial conditions (3.2) and (3.3) has a unique solution.

Proof. In the bilinear form $B$, let $R$ be the matrix that describes the interaction of the basis functions of $W^L$ with themselves, let $N$ be the matrix that describes the interaction of the basis functions of $V^L_1 \times \ldots \times V^L_n$ with themselves, and let $S$ be the matrix that describes the interaction of the basis functions of $W^L$ and the basis functions of $V^L_1 \times \ldots \times V^L_n$. For the bilinear form $A$, let $Q$ be the matrix that describes the interaction of the basis functions of $W^L$ with themselves, let $P$ be the matrix describing the interaction of the basis functions of $W^L$ and $W^L_1 \times \ldots \times W^L_n$, and let $M$ be the matrix describing the interactions of the basis functions of $W^L_1 \times \ldots \times W^L_n$ and themselves. Let $F$ be the column vector describing the interaction of $f$ and the basis functions of $W^L$. Let $C_0$ be the coefficient vector in the expansion of $u^L_0$ with respect to the basis functions of $W^L$. Let $C_1$ be the coefficient vector in the expansion of $(u^L_1, \ldots, u^L_n)$ with respect to the basis functions of $W^L_1 \times \ldots \times W^L_n$. We have the following equations

$$
R \frac{d^2 C_0}{dt^2} + S \frac{d^2 \xi_1}{dt^2} + QC_0 + PC_1 = F, \quad P^T C_0 + MC_1 = 0, \quad S^T \frac{d^2 C_0}{dt^2} + N \frac{d^2 \xi_1}{dt^2} = 0.
$$

Using $C_1 = -M^{-1}P^T C_0$, we deduce the system

$$
\begin{bmatrix}
R & S \\
S^T & N
\end{bmatrix}
\begin{bmatrix}
\frac{d^2 C_0}{dt^2} \\
\frac{d^2 \xi_1}{dt^2}
\end{bmatrix} +
\begin{bmatrix}
Q - PM^{-1}P^T & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
C_0 \\
\xi_1
\end{bmatrix} =
\begin{bmatrix}
F \\
0
\end{bmatrix}.
$$

We note that $\begin{bmatrix}
R & S \\
S^T & N
\end{bmatrix}$ is the Gram matrix for the interaction of the basis of $W^L$ and $V^L_1 \times \ldots \times V^L_n$ in the bilinear form $B$ so is positive definite. The system thus has a unique solution. □

For each $t \in (0, T)$, let $w^L(t) = (w^L_0, w^L_1, \ldots, w^L_n, w^L_1, \ldots, w^L_n) \in V^L$ be the solution of the problem

$$
B(w^L(t) - u(t), v^L) + A(w^L(t) - u(t), v^L) = 0 \quad (3.4)
$$

for all $v^L \in V^L$. As the coefficients $a$ and $b$ in (2.1) are both uniformly bounded and coercive for all $x \in D$ and $y \in Y$, problem (3.4) has a unique solution. Let $q^L = w^L - u$. We then have the following estimate.

Lemma 3.2 For the solution $w^L$ of problem (3.4)

$$
\|q^L(t)\|_V \leq c \inf_{v^L \in V^L} \|u(t) - v^L\|_V.
$$

Proof. From (3.4), we have

$$
B(w^L - u, w^L - u) + A(w^L - u, v^L - v^L) = B(w^L - u, v^L - u) + A(w^L - u, v^L - u)
$$

for all $v^L \in V^L$. From the coerciveness and boundedness of the matrices $a$ and $b$ we get the conclusion. □

When $u$ is sufficiently regular with respect to $t$, we have the following estimates.

Lemma 3.3 If $\frac{\partial u}{\partial t} \in C([0, T], V)$, then

$$
\left\| \frac{\partial q^L}{\partial t} \right\|_{L^\infty(0, T; V)} \leq c \sup_{t \in [0, T]} \inf_{v^L \in V^L} \left\| \frac{\partial u}{\partial t} - v^L \right\|_V.
$$

If $\frac{\partial^2 u}{\partial t^2} \in L^2(0, T; V)$, then

$$
\left\| \frac{\partial^2 q^L}{\partial t^2} \right\|_{L^2(0, T; V)} \leq c \inf_{v^L \in L^2(0, T; V^L)} \left\| \frac{\partial^2 u}{\partial t^2} - v^L \right\|_{V^L}.
$$
Proposition 3.4

The definition of the spaces for all \( v^L \in V^L \). We then proceed as in the proof of Lemma 3.2 to show the first inequality. The proof for the second inequality is similar.

Let \( p^L = u^L - w^L \), i.e., for \( i = 1, \ldots, n, p_i^L = u_i^L - w_i^L, p_i^L = u_i^L - w_i^L \) and \( p_0^L = u_0^L - w_0^L \). We recall the definition of the spaces \( H_1 \) in (2.3).

Proposition 3.4

Assume that \( \frac{\partial^2 u}{\partial t^2} \in L^2(0, T; V) \). Then there is a constant \( c \) depending on \( T \) such that for all \( t \in (0, T) \)

\[
\left\| \frac{\partial p_0^L}{\partial t}(t) + \sum_{i=1}^n \nabla y_i \frac{\partial p_i^L}{\partial t}(t) \right\|_{H_n} + \left\| \text{curl } p_0^L(t) + \sum_{i=1}^n \text{curl } p_i^L(t) \right\|_{H_n} \\
\leq c \left[ \left\| \frac{\partial^2 p_0^L}{\partial t^2} + \sum_{i=1}^n \nabla y_i \frac{\partial^2 p_i^L}{\partial t^2} - q_0^L - \sum_{i=1}^n \nabla y_i q_i^L \right\|_{L^2(0, T; H_n)} \right. \\
+ \left. \left\| \frac{\partial p_0^L}{\partial t}(0) + \sum_{i=1}^n \nabla y_i \frac{\partial p_i^L}{\partial t}(0) \right\|_{H_n} + \| \text{curl } p_0^L(0) \|_H \right].
\]

Proof

Since \( \frac{\partial^2 u}{\partial t^2} \in L^2(0, T; V) \), from (2.3) and (3.1) we have for all \( v^L = (v_0^L, v_1^L, \ldots, v_n^L, v_1^L, \ldots, v_n^L) \in V^L \)

\[
\int_D \int_Y \left[ b(x, y) \left( \frac{\partial^2 p_0^L}{\partial t^2} + \sum_{i=1}^n \nabla y_i \frac{\partial^2 p_i^L}{\partial t^2} \right) \right] v_0^L + \sum_{i=1}^n \nabla y_i v_i^L \\
+ a(x, y) \left( \text{curl } p_0^L + \sum_{i=1}^n \text{curl } p_i^L \right) \left( \text{curl } v_0^L + \sum_{i=1}^n \text{curl } v_i^L \right) \right] dydx \\
= -\int_D \int_Y \left[ b(x, y) \left( \frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^n \nabla y_i \frac{\partial^2 q_i^L}{\partial t^2} \right) \right] v_0^L + \sum_{i=1}^n \nabla y_i v_i^L \\
+ a(x, y) \left( \text{curl } q_0^L + \sum_{i=1}^n \text{curl } q_i^L \right) \left( \text{curl } v_0^L + \sum_{i=1}^n \text{curl } v_i^L \right) \right] dydx - A(q^L, v^L).
\]

From (3.4) we have \( A(q^L, v^L) = -B(q^L, v^L) \). Thus

\[
\int_D \int_Y \left[ b(x, y) \left( \frac{\partial^2 p_0^L}{\partial t^2} + \sum_{i=1}^n \nabla y_i \frac{\partial^2 p_i^L}{\partial t^2} \right) \right] v_0^L + \sum_{i=1}^n \nabla y_i v_i^L \\
+ a(x, y) \left( \text{curl } p_0^L + \sum_{i=1}^n \text{curl } p_i^L \right) \left( \text{curl } v_0^L + \sum_{i=1}^n \text{curl } v_i^L \right) \right] dydx \\
= -\int_D \int_Y \left[ b(x, y) \left( \frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^n \nabla y_i \frac{\partial^2 q_i^L}{\partial t^2} - q_0^L - \sum_{i=1}^n \nabla y_i q_i^L \right) \right] v_0^L + \sum_{i=1}^n \nabla y_i v_i^L \right] dydx. \quad (3.5)
\]
Let \( v^L = \frac{\partial p^L}{\partial t} \). We then have

\[
\frac{1}{2} \frac{d}{dt} \int_D \int_Y \left[ b(x, y) \left( \frac{\partial p^L_0}{\partial t} + \sum_{i=1}^n \nabla x_i \frac{\partial p^L_i}{\partial t} \right) \left( \frac{\partial p^L_0}{\partial t} + \sum_{i=1}^n \nabla x_i \frac{\partial p^L_i}{\partial t} \right) \right] + a(x, y) \left( \text{curl} \, p^L_0 + \sum_{i=1}^n \text{curl} \, y_i \, p^L_i \right) \left( \text{curl} \, p^L_0 + \sum_{i=1}^n \text{curl} \, y_i \, p^L_i \right) d\mu d\nu dx
\]

\[
\leq c \left\| \frac{\partial^2 q^L_0}{\partial t^2} + \sum_{i=1}^n \nabla y_i \frac{\partial^2 q^L_i}{\partial t^2} - q^L_0 - \sum_{i=1}^n \nabla y_i \, q^L_i \right\|_{H_n} \left\| \frac{\partial p^L_0}{\partial t} + \sum_{i=1}^n \nabla y_i \frac{\partial p^L_i}{\partial t} \right\|_{H_n}
\]

\[
\leq c \frac{1}{\gamma} \left\| \frac{\partial^2 q^L_0}{\partial t^2} + \sum_{i=1}^n \nabla y_i \frac{\partial^2 q^L_i}{\partial t^2} - q^L_0 - \sum_{i=1}^n \nabla y_i \, q^L_i \right\|_{L^2(0, T; H_n)}^2 + c \gamma T \sup_{t \in [0, T]} \left\| \frac{\partial p^L_0}{\partial t} + \sum_{i=1}^n \nabla y_i \frac{\partial p^L_i}{\partial t} \right\|_{H_n}^2
\]

for a constant \( \gamma > 0 \). Integrating both sides on \((0, t)\) for \(0 < t < T\), and using the coercivity of the matrices \(a\) and \(b\), we have

\[
\left\| \frac{\partial p^L_0}{\partial t}(t) + \sum_{i=1}^n \nabla y_i \frac{\partial p^L_i}{\partial t}(t) \right\|_{H_n}^2 + \left\| \text{curl} \, p^L_0(t) + \sum_{i=1}^n \text{curl} \, y_i \, p^L_i(t) \right\|_{H_n}^2 \leq c \gamma T \sup_{t \in [0, T]} \left\| \frac{\partial p^L_0}{\partial t}(t) + \sum_{i=1}^n \nabla y_i \frac{\partial p^L_i}{\partial t}(t) \right\|_{H_n}^2 + c \left\| \text{curl} \, p^L_0(0) + \sum_{i=1}^n \text{curl} \, y_i \, p^L_i(0) \right\|_{H_n}^2.
\]

Choosing a sufficiently small constant \( \gamma \), there is a constant \( c \) depending on \( T \) so that for all \( t \in (0, T) \)

\[
\left\| \frac{\partial p^L_0}{\partial t}(t) + \sum_{i=1}^n \nabla y_i \frac{\partial p^L_i}{\partial t}(t) \right\|_{H_n}^2 + \left\| \text{curl} \, p^L_0(t) + \sum_{i=1}^n \text{curl} \, y_i \, p^L_i(t) \right\|_{H_n}^2 \leq c \left\| \frac{\partial^2 q^L_0}{\partial t^2} + \sum_{i=1}^n \nabla y_i \frac{\partial^2 q^L_i}{\partial t^2} - q^L_0 - \sum_{i=1}^n \nabla y_i \, q^L_i \right\|_{L^2(0, T; H_n)}^2 + \left\| \text{curl} \, p^L_0(0) + \sum_{i=1}^n \text{curl} \, y_i \, p^L_i(0) \right\|_{H_n}^2.
\]

Consider equation (3.3) for \( t = 0 \). Let \( v^L_0 = 0 \), \( v^L_t = 0 \) and \( v^L = p^L_t \). We then have

\[
\int_D \int_Y a(x, y) \left( \text{curl} \, p^L_0(0) + \sum_{i=1}^n \text{curl} \, y_i \, p^L_i(0) \right) \left( \sum_{i=1}^n \text{curl} \, y_i \, p^L_i(0) \right) d\mu d\nu dx = 0,
\]

i.e.,

\[
\int_D \int_Y a(x, y) \left( \sum_{i=1}^n \text{curl} \, y_i \, p^L_i(0) \right) \left( \sum_{i=1}^n \text{curl} \, y_i \, p^L_i(0) \right) d\mu d\nu dx = - \int_D \int_Y a(x, y) \text{curl} \, p^L_0(0) \left( \sum_{i=1}^n \text{curl} \, y_i \, p^L_i(0) \right) d\mu d\nu dx.
\]
Using (2.4), we deduce that
\[ \left\| \sum_{i=1}^{n} \text{curl}_{y_i} p_i^L(0) \right\|_{H_n} \leq c \left\| \text{curl} p_0^L(0) \right\|_{H}. \]
We then get the conclusion.

**Proposition 3.5** Assume that \( \frac{\partial^2 u}{\partial x^2} \in L^2(0, T; \mathbf{V}) \), and that
\[
\lim_{L \to \infty} \| g_0^L - g_0 \|_W = 0 \quad \text{and} \quad \lim_{L \to \infty} \| g_1^L - g_1 \|_H = 0.
\]
Then
\[
\lim_{L \to \infty} \left\{ \left\| \frac{\partial (u_0^L - u_0)}{\partial t} \right\|_{L^\infty(0, T; H)} + \sum_{i=1}^{n} \left\| \text{curl}_{y_i} \frac{\partial (u_i^L - u_i)}{\partial t} \right\|_{L^\infty(0, T; H_n)} \right. \\
+ \left\| \text{curl} (u_0^L - u_0) \right\|_{L^\infty(0, T; H)} + \sum_{i=1}^{n} \left\| \text{curl}_{y_i} (u_i^L - u_i) \right\|_{L^\infty(0, T; H_n)} \right\} = 0.
\]

**Proof** From Proposition 3.4 as \( u^L = p^L + q^L \), we have
\[
\left\| \frac{\partial (u_0^L - u_0)}{\partial t} \right\|_{L^\infty(0, T; H)} + \sum_{i=1}^{n} \left\| \text{curl}_{y_i} \frac{\partial (u_i^L - u_i)}{\partial t} \right\|_{L^\infty(0, T; H_n)} \leq c \left[ \left\| \frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^{n} \nabla_{y_i} \frac{\partial^2 q_i^L}{\partial t^2} - q_0^L - \sum_{i=1}^{n} \nabla_{y_i} q_i^L \right\|_{L^2(0, T; H_n)} \right. \\
\left. + \left\| \frac{\partial q_0^L}{\partial t}(0) \right\|_{H_n} + \sum_{i=1}^{n} \left\| \nabla_{y_i} \frac{\partial q_i^L}{\partial t}(0) \right\|_{H_n} \right] + \|q^L\|_{L^\infty(0, T; \mathbf{V})} \right\} = 0.
\]

We show that \( \lim_{L \to \infty} \|q^L\|_{L^\infty(0, T; \mathbf{V})} = 0 \). As \( u \in C([0, T]; \mathbf{V}) \), \( u \) is uniformly continuous as a function from \([0, T]\) to \( \mathbf{V} \). For \( \varepsilon > 0 \), there is a piecewise constant (with respect to time) function \( \bar{u} \in L^\infty(0, T; \mathbf{V}) \) such that \( \| u - \bar{u} \|_{L^\infty(0, T; \mathbf{V})} < \varepsilon \). As \( \bar{u}(t) \) obtains only a finite number of \( \mathbf{V} \)-values, when \( L \) is sufficiently large, there is \( v^L \in L^\infty(0, T; \mathbf{V}) \) such that \( \| \bar{u} - v^L \|_{L^\infty(0, T; \mathbf{V})} < \varepsilon \). Thus
\[
\lim_{L \to \infty} \sup_{t \in (0, T)} \inf_{v^L \in \mathbf{V}} \| u(t) - v^L \|_\mathbf{V} = 0.
\]
We then apply Lemma 3.2. Similarly, we have from Lemmas 3.2 and 3.3
\[
\lim_{L \to \infty} \left\| \frac{\partial^2 q_0^L}{\partial t^2} + \sum_{i=1}^{n} \nabla_{y_i} \frac{\partial^2 q_i^L}{\partial t^2} - q_0^L - \sum_{i=1}^{n} \nabla_{y_i} q_i^L \right\|_{L^2(0, T; H_n)} = 0
\]
and
\[
\lim_{L \to \infty} \left\| \frac{\partial q_0^L}{\partial t} + \sum_{i=1}^{n} \nabla_{y_i} \frac{\partial q_i^L}{\partial t} \right\|_{L^\infty(0, T; H_n)} = 0.
\]
Furthermore, we have that
\[
\| \text{curl} p_0^L(0) \|_{H} \leq \| \text{curl} u_0^L(0) - \text{curl} u_0(0) \|_{H} + \| \text{curl} u_0(0) - \text{curl} u_0^L(0) \|_{H},
\]
which converges to 0 due to \(3.6\) and Lemma 3.2. Similarly, we have
\[
\lim_{L \to \infty} \left\| \frac{\partial p^L_0}{\partial t}(0) + \sum_i \nabla y_i \frac{\partial p^L_i}{\partial t}(0) \right\|_{H_n} = 0.
\]
We then get the conclusion. \(\square\)

### 3.2 Fully discrete problem

Following the scheme of Dupont [10], we discretize problem 3.1 in both spatial and temporal variables. Let \(\Delta t = \frac{T}{M}\) where \(M\) is a positive integer. Let \(t_m = m \Delta t\). We employ the following notations of Dupont for a function \(r \in C([0, T]; X)\) where \(X\) is a Banach space and \(r_m = r(t_m, \cdot)\)

\[
\begin{align*}
    r_{m+1/2} &= \frac{1}{2} (r_{m+1} + r_m), & r_{m, 0} &= \theta r_{m+1} + (1 - 2 \theta) r_m + \theta r_{m-1}, \\
    \partial_t r_{m+1/2} &= (r_{m+1} - r_m)/\Delta t, & \partial_t^2 r_m &= (r_{m+1} - 2 r_m + r_{m-1})/(\Delta t)^2, \\
    \delta r_m &= (r_{m+1} - r_{m-1})/(2 \Delta t).
\end{align*}
\]

We consider the following fully discrete problem:

For \(m = 1, ..., M\) find \(u^L_m = (u^L_{0,m}, u^L_{1,m}, ..., u^L_{n,m}, u^L_{n+1,m}, ..., u^L_{nm, m}) \in \mathbf{V}^L\) such that for \(m = 1, ..., M - 1\)

\[
\int_D \int_Y \left[ b(x, y) \left( \partial_t^2 u^L_{0,m} + \sum_{i=1}^n \nabla y_i \partial_t^2 u^L_{i,m} \right) \cdot \left( v^L_{0} + \sum_{i=1}^n \nabla y_i v^L_{i} \right) + a(x, y) \left( \text{curl} u^L_{0,m,1/4} + \sum_{i=1}^n \text{curl} y_i u^L_{i,m,1/4} \right) \cdot \left( \text{curl} v^L_{0} + \sum_{i=1}^n \text{curl} y_i v^L_{i} \right) \right] dy dx = \int_D f_{m,1/4}(t, x) \cdot v^L_{0}(x) dx,
\]

for all \(v^L = (v^L_{0,0}, v^L_{1,0}, ..., v^L_{n,0}, v^L_{1,1}, ..., v^L_{nm,0}) \in \mathbf{V}^L\).

For continuous functions \(r : [0, T] \to X\), let

\[
\|r\|_{L^\infty(0, T; X)} := \max_{0 \leq m < M} \|r_{m+1/2}\|_X.
\]

We also denote by

\[
\|\partial_t r\|_{L^\infty(0, T; X)} := \max_{0 \leq m < M} \|\partial_t r_{m+1/2}\|_X.
\]

Let

\[
p^L_m := u^L_m - u^L_{m-1}.
\]

**Lemma 3.6** Assume that \(u \in H^2(0, T; \mathbf{V}), \frac{\partial^2 u^L}{\partial t^2} \in L^2(0, T; H), \frac{\partial^2}{\partial t^2} \nabla y_i u^L_i \in L^2(0, T; H_i)\). If \(\frac{\partial^2}{\partial t^2} \nabla y_i u^L_i \in L^2(0, T; H_i)\), then there exists a constant \(c\) independent of \(\Delta t\) and \(u\) such that for each \(j = 1, 2, ..., M - 1\)

\[
\begin{align*}
    \|\partial_t p^L_{0,j+1/2}\|^2_{L^2(H)} + \sum_{i=1}^n \|\partial_t \nabla y_i p^L_{i,j+1/2}\|^2_{L^2(H)} + \|\text{curl} p^L_{0,j+1/2}\|^2_{L^2(H)} + \sum_{i=1}^n \|\text{curl} y_i p^L_{i,j+1/2}\|^2_{L^2(H)} & \\
    \leq c \left[ (\Delta t)^2 \left\| \frac{\partial^2 u^L_0}{\partial t^2} \right\|_{L^2(H)}^2 + (\Delta t)^2 \sum_{i=1}^n \left\| \frac{\partial^3 \nabla y_i u^L_i}{\partial t^3} \right\|_{L^2(H_i)}^2 + \left\| \frac{\partial^2 q^L_i}{\partial t^2} \right\|^2_{L^2(0, T; H_i)} + \sum_{i=1}^n \left\| \nabla y_i q^L_i \right\|^2_{L^2(0, T; H_i)} \right] \\
    + c \left( \|\partial_t p^L_{0,1/2}\|^2_{L^2(H)} + \sum_{i=1}^n \|\partial_t \nabla y_i p^L_{i,1/2}\|^2_{L^2(H_i)} + \|\text{curl} p^L_{0,1/2}\|^2_{L^2(H)} + \sum_{i=1}^n \|\text{curl} y_i p^L_{i,1/2}\|^2_{L^2(H_i)} \right).
\end{align*}
\]
Further, if $\frac{\partial^4 u_0}{\partial t^4} \in L^2(0, T; H)$ and $\frac{\partial^4}{\partial t^4} \nabla y \cdot u_i \in L^2(0, T; H)$, then there exists a constant $c$ independent of $\Delta t$ and $u$ such that for each $j = 1, 2, \ldots, M - 1$

$$
\|\partial_t p_{0,j+1/2}\|^2_H + \sum_{i=1}^n \|\partial_t \nabla y_i \cdot p_{j+1/2}\|^2_H + \|\text{curl} p_{0,j+1/2}\|^2_H + \sum_{i=1}^n \|\text{curl} y_i \cdot p_{j+1/2}\|^2_H \\
\leq c \left[ (\Delta t)^4 \left\| \frac{\partial^4 u_0}{\partial t^4} \right\|^2_H + (\Delta t)^4 \sum_{i=1}^n \left\| \frac{\partial^4 \nabla y_i \cdot u_i}{\partial t^4} \right\|^2_H + \left\| \frac{\partial^2 q_i^L}{\partial t^2} \right\|^2_{L^2(0, T; H)} \\
+ \sum_{i=1}^n \left\| \frac{\partial^2}{\partial t^2} \nabla y_i q_i^L \right\|^2_{L^2(0, T; H)} + \|q_i^L\|^2_{L^2(0, T; H)} + \sum_{i=1}^n \left\| \nabla y_i q_i^L \right\|^2_{L^2(0, T; H)} \right] \\
+ c \left( \|\partial_t p_{0,1/2}\|^2_H + \sum_{i=1}^n \|\partial_t \nabla y_i \cdot p_{1/2}\|^2_H + \|\text{curl} p_{0,1/2}\|^2_H + \sum_{i=1}^n \|\text{curl} y_i \cdot p_{1/2}\|^2_H \right).
$$

Proof From (5.3) and (3.8), we have

$$
A(w^L, v^L) = A(u, v^L) - B(u^L - u, v^L) = \int_D f(x, t) \cdot v_0^L(x) dx \\
- \int_D \int_Y b(x, y) \left( \frac{\partial^2 u_0}{\partial t^2} + \sum_{i=1}^n \frac{\partial^2}{\partial t^2} \nabla y_i, u_i \right) \cdot \left( v_0^L + \sum_{i=1}^n \nabla y_i v_i^L \right) dy dx - B(q_i^L, v^L).
$$

Averaging this equation at $t_{m+1}$, $t_m$ and $t_{m-1}$ with weights $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ respectively, and using (3.8), we get

$$
\int_D \int_Y b(x, y) \left( \frac{\partial^2 u_{0,m}}{\partial t^2} + \sum_{i=1}^n \frac{\partial^2}{\partial t^2} \nabla y_i, u_{i,m} \right) \cdot \left( v_0^L + \sum_{i=1}^n \nabla y_i v_i^L \right) dy dx + A(p_{0,m/4}^L, v^L) \\
= \int_D \int_Y b(x, y) \left( \frac{\partial^2 u_{0,m/4}}{\partial t^2} + \sum_{i=1}^n \frac{\partial^2}{\partial t^2} \nabla y_i, u_{i,m/4} \right) \cdot \left( v_0^L + \sum_{i=1}^n \nabla y_i v_i^L \right) dy dx \\
+ B(q_{0,m/4}^L, v^L).
$$

Thus

$$
\int_D \int_Y b(x, y) \left( \frac{\partial^2 u_{0,m}}{\partial t^2} + \sum_{i=1}^n \frac{\partial^2}{\partial t^2} \nabla y_i, u_{i,m} \right) \cdot \left( v_0^L + \sum_{i=1}^n \nabla y_i v_i^L \right) dy dx + A(p_{0,m/4}^L, v^L) \\
= \int_D \int_Y b(x, y) \left( \frac{\partial^2 u_{0,m/4}}{\partial t^2} + \sum_{i=1}^n \frac{\partial^2}{\partial t^2} \nabla y_i, u_{i,m/4} \right) \cdot \left( v_0^L + \sum_{i=1}^n \nabla y_i v_i^L \right) dy dx \\
- \int_D \int_Y b(x, y) \left( \frac{\partial^2 u_{0,m}}{\partial t^2} - \frac{\partial^2}{\partial t^2} u_{0,m} - \sum_{i=1}^n \left( \frac{\partial^2}{\partial t^2} \nabla y_i, u_{i,m/4} - \nabla y_i \frac{\partial^2}{\partial t^2} u_{i,m} \right) \right) \\
\cdot \left( v_0^L + \sum_{i=1}^n \nabla y_i v_i^L \right) dy dx \\
+ B(q_{0,m/4}^L, v^L).
$$

We denote by

$$
s_{0,m} = \frac{\partial^2 u_{0,m/4}}{\partial t^2} - \frac{\partial^2}{\partial t^2} u_{0,m}, \quad s_{i,m} = \frac{\partial^2}{\partial t^2} \nabla y_i, u_{i,m/4} - \frac{\partial^2}{\partial t^2} \nabla y_i, u_{i,m}.
$$

Let $v^L = \delta_t p^L_m$. Using the following relationships:

$$
\frac{\partial^2}{\partial t^2} r_{m} = \frac{1}{\Delta t} (\partial_t r_{m+1/2} - \partial_t r_{m-1/2}), \quad r_{m,1/4} = \frac{1}{2} (r_{m+1/2} + r_{m-1/2}) \\
\delta tr_{m} = \frac{1}{2} (\partial_t r_{m+1/2} + \partial_t r_{m-1/2}) = \frac{1}{\Delta t} (r_{m+1/2} - r_{m-1/2}),
$$

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we have

\[
\frac{1}{2\Delta t} \int_D \int_Y b(x, y) \left( \partial_t p_{0,m+1/2}^L - \partial_t p_{0,m-1/2}^L + \sum_{i=1}^n \nabla y_i \left( \partial_t p_{i,m+1/2}^L - \partial_t p_{i,m-1/2}^L \right) \right) \\
\cdot \left( \partial_t p_{0,m+1/2}^L + \partial_t p_{0,m-1/2}^L + \sum_{i=1}^n \nabla y_i \left( \partial_t p_{i,m+1/2}^L + \partial_t p_{i,m-1/2}^L \right) \right) \, dy \, dx
\]

\[
+ \frac{1}{2\Delta t} \int_D \int_Y \alpha(x, y) \left( \text{curl} \left( p_{0,m+1/2}^L + p_{0,m-1/2}^L \right) + \sum_{i=1}^n \text{curl} y_i \left( p_{i,m+1/2}^L + p_{i,m-1/2}^L \right) \right) \\
\cdot \left( \text{curl} \left( p_{0,m+1/2}^L - p_{0,m-1/2}^L \right) + \sum_{i=1}^n \text{curl} y_i \left( p_{i,m+1/2}^L - p_{i,m-1/2}^L \right) \right) \, dy \, dx
\]

\[
= \frac{1}{2} \int_D \int_Y b(x, y) \left( s_{0,m} - \partial_t^2 q_{0,m} + q_{0,m+1/4} + \sum_{i=1}^n \left( s_{i,m} - \nabla y_i \partial_t^2 q_{i,m} + \nabla y_i q_{i,m+1/4} \right) \right) \\
\cdot \left( \partial_t p_{0,m+1/2}^L + \partial_t p_{0,m-1/2}^L + \sum_{i=1}^n \left( \nabla y_i \partial_t p_{i,m+1/2}^L + \nabla y_i \partial_t p_{i,m-1/2}^L \right) \right) \, dy \, dx.
\]

We thus have

\[
\frac{1}{2\Delta t} \left[ B \left( \partial_t p_{m+1/2}^L, \partial_t p_{m-1/2}^L \right) - B \left( \partial_t p_{m-1/2}^L, \partial_t p_{m-1/2}^L \right) \right] \\
+ A \left( \partial_t p_{m+1/2}^L, \partial_t p_{m-1/2}^L \right) - A \left( \partial_t p_{m-1/2}^L, \partial_t p_{m-1/2}^L \right) \leq \frac{c}{7} \left( \left\| s_{0,m} \right\|_H^2 + \sum_{i=1}^n \left\| s_{i,m} \right\|_H^2 \right) + \sum_{i=1}^n \left\| \nabla y_i \partial_t q_{i,m}^L \right\|_H^2
\]

\[
+ \left\| q_{0,m+1/4}^L \right\|_H^2 + \sum_{i=1}^n \left\| \nabla y_i q_{i,m+1/4}^L \right\|_H^2 \right)
\]

\[
+ c_7 \left( \left\| \partial_t p_{0,m+1/2}^L \right\|_H^2 + \left\| \partial_t p_{0,m-1/2}^L \right\|_H^2 \right)
\]

\[
+ \sum_{i=1}^n \left\| \nabla y_i \partial_t p_{i,m+1/2}^L \right\|_H^2 + \sum_{i=1}^n \left\| \nabla y_i \partial_t p_{i,m-1/2}^L \right\|_H^2 \right).
\]

Summing this up for all \( m = 1, \ldots, j \), we deduce

\[
B(\partial_t p_{j+1/2}^L, \partial_t p_{j+1/2}^L) - B(\partial_t p_{j+1/2}^L, \partial_t p_{j+1/2}^L) + A(p_{j+1/2}^L, p_{j+1/2}^L) - A(p_{j+1/2}^L, p_{j+1/2}^L) \leq \frac{c}{7} 2\Delta t \sum_{m=1}^M \left( \left\| s_{0,m} \right\|_H^2 + \sum_{i=1}^n \left\| s_{i,m} \right\|_H^2 \right) + \sum_{i=1}^n \left\| \nabla y_i \partial_t q_{i,m}^L \right\|_H^2
\]

\[
+ \left\| q_{0,m+1/4}^L \right\|_H^2 + \sum_{i=1}^n \left\| \nabla y_i q_{i,m+1/4}^L \right\|_H^2 \right)
\]

\[
+ c_7 4\Delta t M \left( \max_{1 \leq m \leq M} \left\| \partial_t p_{0,m+1/2}^L \right\|_H^2 + \sum_{i=1}^n \max_{1 \leq m \leq M} \left\| \partial_t \nabla y_i p_{i,m+1/2}^L \right\|_H^2 \right)
\]

\[
+ c_7 2\Delta t \left( \left\| \partial_t p_{0,1/2}^L \right\|_H^2 + \sum_{i=1}^n \left\| \partial_t \nabla y_i p_{i,1/2}^L \right\|_H^2 \right).
\]

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Following Dupont [13], using the integral formula of the remainder of Taylor expansion, we have,

\[
\|\partial_t p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_i \nabla_y p_{0,j+1/2}^L\|_H^2 + \|\text{curl} p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\text{curl}_y p_{i,j+1/2}^L\|_H^2 \\
\leq \frac{c}{\gamma} 2\Delta t \sum_{m=1}^M \left( \|s_{0,m}\|_H^2 + \sum_{i=1}^n \|s_{i,m}\|_H^2 + \|\partial_t^2 q_{0,m}^L\|_H^2 + \sum_{i=1}^n \|\nabla_y \partial_t^2 q_{i,m}^L\|_H^2 \\
+ \|q_{0,m,1/4}\|_H^2 + \sum_{i=1}^n \|\nabla_y q_{i,m,1/4}\|_H^2 \right) \\
+ c\gamma 4\Delta t M \max_{1 \leq m \leq M} \|\partial_t q_{0,m+1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_y q_{i,m+1/2}^L\|_H^2 + \|\text{curl} q_{0,m+1/2}^L\|_H^2 + \sum_{i=1}^n \|\text{curl}_y q_{i,m+1/2}^L\|_H^2 \\
+ c \left( \|\partial_t q_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_y q_{i,1/2}^L\|_H^2 + \|\text{curl} q_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\text{curl}_y q_{i,1/2}^L\|_H^2 \right).
\]

Choosing \(\gamma\) sufficiently small, we deduce that

\[
\|\partial_t p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_i \nabla_y p_{0,j+1/2}^L\|_H^2 + \|\text{curl} p_{0,j+1/2}^L\|_H^2 + \sum_{i=1}^n \|\text{curl}_y p_{i,j+1/2}^L\|_H^2 \\
\leq \frac{c}{2\gamma} 2\Delta t \sum_{m=1}^M \left( \|s_{0,m}\|_H^2 + \sum_{i=1}^n \|s_{i,m}\|_H^2 + \|\partial_t^2 q_{0,m}^L\|_H^2 + \sum_{i=1}^n \|\nabla_y \partial_t^2 q_{i,m}^L\|_H^2 \\
+ \|q_{0,m,1/4}\|_H^2 + \sum_{i=1}^n \|\nabla_y q_{i,m,1/4}\|_H^2 \right) \\
+ c \left( \|\partial_t q_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_y q_{i,1/2}^L\|_H^2 + \|\text{curl} q_{0,1/2}^L\|_H^2 + \sum_{i=1}^n \|\text{curl}_y q_{i,1/2}^L\|_H^2 \right).
\]

Following Dupont [13], using the integral formula of the remainder of Taylor expansion, we have,

\[
\partial_t^2 q_{0,m}^L = (\Delta t)^{-2} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \frac{\partial^2 q_{0}^L}{\partial t^2} (t_m + \tau) d\tau,
\]

and similarly, for \(i = 1, \ldots, n\)

\[
\partial_t^2 (\nabla_y q_{i,m}^L) = (\Delta t)^{-2} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \frac{\partial^2 \nabla_y q_{i,m}^L}{\partial t^2} (t_m + \tau) d\tau.
\]

Using Cauchy-Schwarz inequality, we have

\[
\sum_{m=1}^M \|\partial_t^2 q_{0,m}^L\|_H^2 \Delta t \leq \frac{4}{3} \left\| \frac{\partial^2 q_{0}^L}{\partial t^2} \right\|_{L^2(0,T;H)}^2;
\]

and similarly, we have

\[
\sum_{m=1}^M \|\partial_t^2 \nabla_y q_{i,m}^L\|_H^2 \Delta t \leq \frac{4}{3} \left\| \frac{\partial^2}{\partial t^2} \nabla_y q_{i,m}^L \right\|_{L^2(0,T;H)}^2.
\]

We write

\[
s_{0,m} = \frac{1}{4} \int_0^{\Delta t} \left( 1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 u_0}{\partial t^3} (t_m + \tau) d\tau - \frac{1}{4} \int_{-\Delta t}^{0} \left( 1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 u_0}{\partial t^3} (t_m + \tau) d\tau
\]

and

\[
s_{i,m} = \frac{1}{4} \int_0^{\Delta t} \left( 1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 \nabla_y u_i}{\partial t^3} (t_m + \tau) d\tau - \frac{1}{4} \int_{-\Delta t}^{0} \left( 1 - 2 \left(1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^3 \nabla_y u_i}{\partial t^3} (t_m + \tau) d\tau.
\]
Therefore
\[
\|s_{0,m}\|_H^2 \leq c \Delta t \int_{t_{m-1}}^{t_{m+1}} \left\| \frac{\partial^3 u_0}{\partial \tau^3} (\tau) \right\|_H^2 \, d\tau, \quad \|s_{1,m}\|_H^2 \leq c \Delta t \int_{t_{m-1}}^{t_{m+1}} \left\| \frac{\partial^3 \nabla_y u_t}{\partial \tau^3} (\tau) \right\|_H^2 \, d\tau.
\]
We also have
\[
\|q_{0,m,1/4}\|_H \leq \max_{t \in [0,T]} \|q_{0}^L(t)\|_H \quad \text{and} \quad \|\nabla_y q_{i,m,1/4}\|_H \leq \max_{t \in [0,T]} \|\nabla_y q_i^L(t)\|_H.
\]
We thus deduce
\[
\|\partial_t p_{0,j+1/2}\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_y p_{i,j+1/2}\|_H^2 + \|\text{curl} p_{0,j+1/2}\|_H^2 + \sum_{i=1}^n \|\text{curl} p_{i,j+1/2}\|_H^2 \leq c \left[ (\Delta t)^2 \left\| \frac{\partial^3 u_0}{\partial \tau^3} \right\|_H^2 + (\Delta t)^2 \sum_{i=1}^n \left\| \frac{\partial^3 \nabla_y u_t}{\partial \tau^3} \right\|_H^2 + \left\| \frac{\partial^2 q_{0}^L}{\partial \tau^2} \right\|_L^2 (0,T,H) \right. \\
\left. + \sum_{i=1}^n \left\| \frac{\partial^2}{\partial \tau^2} \nabla_y q_i^L \right\|_{L^2(0,T,H)}^2 + \|q_{0}^L\|_{L^\infty(0,T,H)} + \sum_{i=1}^n \|\nabla_y q_i^L\|_{L^\infty(0,T,H)} \\
\right] + c \left( \|\partial_t p_{0,1/2}\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_y p_{i,1/2}\|_H^2 + \|\text{curl} p_{0,1/2}\|_H^2 + \sum_{i=1}^n \|\text{curl} p_{i,1/2}\|_H^2 \right).
\]
When
\[
\frac{\partial^4 u_0}{\partial \tau^4} \in L^2(0,T;H) \quad \text{and} \quad \frac{\partial^4}{\partial \tau^4} \nabla_y u_t \in L^2(0,T;H),
\]
we have
\[
s_{0,m} = \frac{1}{12} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \left( 3 - 2 \left( 1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^4 u_0}{\partial \tau^4} (t_m + \tau) d\tau.
\]
and
\[
s_{1,m} = \frac{1}{12} \int_{-\Delta t}^{\Delta t} (\Delta t - |\tau|) \left( 3 - 2 \left( 1 - \frac{|\tau|}{\Delta t} \right)^2 \right) \frac{\partial^4 \nabla_y u_t}{\partial \tau^4} (t_m + \tau) d\tau.
\]
Therefore
\[
\|s_{0,m}\|_H^2 \leq c (\Delta t)^3 \int_{t_{m-1}}^{t_{m+1}} \left\| \frac{\partial^4 u_0}{\partial \tau^4} (\tau) \right\|_H^2 \, d\tau
\]
and
\[
\|s_{1,m}\|_H^2 \leq c (\Delta t)^3 \int_{t_{m-1}}^{t_{m+1}} \left\| \frac{\partial^4 \nabla_y u_t}{\partial \tau^4} (\tau) \right\|_H^2 \, d\tau.
\]
Thus we have
\[
\|\partial_t p_{0,j+1/2}\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_y p_{i,j+1/2}\|_H^2 + \|\text{curl} p_{0,j+1/2}\|_H^2 + \sum_{i=1}^n \|\text{curl} p_{i,j+1/2}\|_H^2 \leq c \left[ (\Delta t)^4 \left\| \frac{\partial^4 u_0}{\partial \tau^4} \right\|_H^2 + (\Delta t)^4 \sum_{i=1}^n \left\| \frac{\partial^4 \nabla_y u_t}{\partial \tau^4} \right\|_H^2 + \left\| \frac{\partial^2 q_{0}^L}{\partial \tau^2} \right\|_L^2 (0,T,H) \right. \\
\left. + \sum_{i=1}^n \left\| \frac{\partial^2}{\partial \tau^2} \nabla_y q_i^L \right\|_{L^2(0,T,H)}^2 + \|q_{0}^L\|_{L^\infty(0,T,H)} + \sum_{i=1}^n \|\nabla_y q_i^L\|_{L^\infty(0,T,H)} \\
\right] + c \left( \|\partial_t p_{0,1/2}\|_H^2 + \sum_{i=1}^n \|\partial_t \nabla_y p_{i,1/2}\|_H^2 + \|\text{curl} p_{0,1/2}\|_H^2 + \sum_{i=1}^n \|\text{curl} p_{i,1/2}\|_H^2 \right).
\]
We then have the following error estimates.
Proposition 3.7 Assume that $u \in H^2(0,T;\mathbf{V})$. If $\frac{\partial^2 u_0}{\partial t^2} \in L^2(0,T;H)$ and $\frac{\partial^3 \nabla_y u_0}{\partial t^3} \in L^2(0,T;H^2)$, then there is a constant $c$ such that

$$\| \partial_t u_0^L - \partial_t u_0 \|_{L^\infty(0,T;H)} + \sum_{i=1}^n \| \partial_t \nabla_y u_i^L - \partial_t \nabla_y u_i \|_{L^\infty(0,T;H)}$$

$$+ \| \text{curl } u_0^L - \text{curl } u_0 \|_{L^\infty(0,T;H)} + \sum_{i=1}^n \| \text{curl } u_i^L - \text{curl } u_i \|_{L^\infty(0,T;H)}$$

$$\leq c \left[ (\Delta t)^2 \left\| \frac{\partial^2 u_0}{\partial t^2} \right\|_{L^2(0,T,H)} + (\Delta t)^2 \sum_{i=1}^n \left\| \frac{\partial^3 \nabla_y u_i}{\partial t^3} \right\|_{H_i} + \sum_{i=1}^n \left\| \frac{\partial^2 q_i}{\partial t^2} \right\|_{L^2(0,T,H)}$$

$$+ \sum_{i=1}^n \left\| \frac{\partial^2 \nabla_y q_i^L}{\partial t^2} \right\|_{L^2(0,T,H)} + \sum_{i=1}^n \left\| \nabla_y q_i^L \right\|_{L^\infty(0,T;H)} + \sum_{i=1}^n \| \nabla_y \nabla_y q_i^L \|_{L^\infty(0,T;H)} \right]$$

$$+ c \left[ \| \partial_t p_{0,1/2} \|_H + \sum_{i=1}^n \| \partial_t \nabla_y p_{1/2}^L \|_H + \| \text{curl } p_{0,1/2}^L \|_H + \sum_{i=1}^n \| \text{curl } p_{1,1/2}^L \|_{H_i} \right]$$

$$+ \| \partial_t q_{0,1/2} \|_{L^\infty(0,T;H)} + \| \text{curl } q_{0,1/2} \|_{L^\infty(0,T;H)}$$

$$+ \sum_{i=1}^n \left( \| \partial_t \nabla_y q_i^L \|_{L^\infty(0,T;H)} + \| \text{curl } q_i^L \|_{L^\infty(0,T;H)} \right).$$

If $\frac{\partial^2 u_0}{\partial t^2} \in L^2(0,T;H)$ and $\frac{\partial^3 \nabla_y u_0}{\partial t^3} \in L^2(0,T;H^2)$, then there is a constant $c$ such that

$$\| \partial_t u_0^L - \partial_t u_0 \|_{L^\infty(0,T;H)} + \sum_{i=1}^n \| \partial_t \nabla_y u_i^L - \partial_t \nabla_y u_i \|_{L^\infty(0,T;H)}$$

$$+ \| \text{curl } u_0^L - \text{curl } u_0 \|_{L^\infty(0,T;H)} + \sum_{i=1}^n \| \text{curl } u_i^L - \text{curl } u_i \|_{L^\infty(0,T;H)}$$

$$\leq c \left[ (\Delta t)^2 \left\| \frac{\partial^2 u_0}{\partial t^2} \right\|_{L^2(0,T,H)} + (\Delta t)^2 \sum_{i=1}^n \left\| \frac{\partial^3 \nabla_y u_i}{\partial t^3} \right\|_{H_i} + \sum_{i=1}^n \left\| \frac{\partial^2 q_i}{\partial t^2} \right\|_{L^2(0,T,H)}$$

$$+ \sum_{i=1}^n \left\| \frac{\partial^2 \nabla_y q_i^L}{\partial t^2} \right\|_{L^2(0,T,H)} + \sum_{i=1}^n \left\| \nabla_y q_i^L \right\|_{L^\infty(0,T;H)} + \sum_{i=1}^n \| \nabla_y \nabla_y q_i^L \|_{L^\infty(0,T;H)} \right]$$

$$+ c \left[ \| \partial_t p_{0,1/2} \|_H + \sum_{i=1}^n \| \partial_t \nabla_y p_{1/2}^L \|_H + \| \text{curl } p_{0,1/2}^L \|_H + \sum_{i=1}^n \| \text{curl } p_{1,1/2}^L \|_{H_i} \right]$$

$$+ \| \partial_t q_{0,1/2} \|_{L^\infty(0,T;H)} + \| \text{curl } q_{0,1/2} \|_{L^\infty(0,T;H)}$$

$$+ \sum_{i=1}^n \left( \| \partial_t \nabla_y q_i^L \|_{L^\infty(0,T;H)} + \| \text{curl } q_i^L \|_{L^\infty(0,T;H)} \right).$$

Proof We note that $v^L - u = p^L + q^L$. The conclusions follow from Lemma 3.8. From this, we deduce

Proposition 3.8 If $u \in H^2(0,T;\mathbf{V})$, $\frac{\partial^2 u_0}{\partial t^2} \in L^2(0,T;H)$ and $\frac{\partial^3 \nabla_y u_0}{\partial t^3} \in L^2(0,T;H^2)$, and if we choose $u_0^L$ and $u_i^L$ such that

$$\lim_{L \to 0} \| \partial_t p_{0,1/2} \|_H + \sum_{i=1}^n \| \partial_t \nabla_y p_{1/2}^L \|_H + \| \text{curl } p_{0,1/2}^L \|_H + \sum_{i=1}^n \| \text{curl } p_{1,1/2}^L \|_{H_i} = 0,$$
for all $A$.

Further, from (3.4), we have that

$$
\lim_{L \to \infty} \| \partial_t u^L_0 - \partial_t u_0 \|_{L^\infty(0,T;H)} + \sum_{i=1}^n \| \partial_t \nabla_y u^L_i - \partial_t \nabla_y u_i \|_{L^\infty(0,T;H)} + \| \text{curl} u^L_0 - \text{curl} u_0 \|_{L^\infty(0,T;H)} + \sum_{i=1}^n \| \text{curl}_y u^L_i - \text{curl}_y u_i \|_{L^\infty(0,T;H)} = 0.
$$

Proof From the hypothesis and Lemma 3.3, we have that

$$
\lim_{L \to \infty} \left\| \frac{\partial^2 q^L}{\partial t^2} \right\|_{L^2(0,T;V)} = 0.
$$

As $u \in C([0, T], V)$, from the proof of Proposition 3.3, $\lim_{L \to \infty} \| q^L \|_{L^\infty(0,T;V)} = 0$. We have that

$$
\| q^L \|_{L^\infty(0,T;V)} \leq \| q \|_{L^\infty(0,T;V)}
$$

so

$$
\lim_{L \to \infty} \| q^L \|_{L^\infty(0,T;V)} = 0.
$$

Further, from (3.4), we have that

$$
B(\partial_t w^L_{m+1/2} - \partial_t u_{m+1/2}, v^L) + A(\partial_t u^L_{m+1/2} - \partial_t u_{m+1/2}, v^L) = 0
$$

for all $v^L \in V^L$. We thus have

$$
\| \partial_t w^L_{m+1/2} - \partial_t u_{m+1/2} \leq c \inf_{v^L \in V^L} \| v^L - \partial_t u_{m+1/2} \| V.
$$

As $\partial_t u_{m+1/2} = \frac{\partial u}{\partial t} (\xi)$ for $\xi \in (0, T)$, we deduce that

$$
\| \partial_t q^L \|_{L^\infty(0,T;V)} \leq c \sup_{t \in (0,T)} \inf_{v^L \in V^L} \| v^L - \frac{\partial u}{\partial t} (t) \| V.
$$

As $\frac{\partial u}{\partial t} \in C([0, T]; V)$, a proof identical to that for $\| q^L \|_{L^\infty(0,T;V)}$ in Proposition 3.3 shows that

$$
\lim_{L \to \infty} \| \partial_t q^L \|_{L^\infty(0,T;V)} = 0.
$$

We thus get the conclusion.

4 Regularity of the solution

To derive an explicit error estimate for the full and sparse tensor product finite element approximating problems in the next section, we now establish the regularity of $u_0$ and $\nabla_y u_i$ with respect to $t$. The function $u_0$ and $u_i$ can be written in terms of $u_0$ from the solution of the cell problems. Let $b^0(x, y_n) = b(x, y)$. Recursively, for all $i = 0, \ldots, n$, let $w^L_i \in V_i$ be the solution of the cell problem

$$
\nabla_{y_i} \cdot (b^i(x, y_i)(e^k + \nabla_{y_i} w^L_i)) = 0
$$

where $e^k$ is the $k$th unit vector with every component equals 0, except the $k$th component which equals 1. For $i = 1, \ldots, n$, the positive definite matrix function $b^{i-1}(x)$ is defined as

$$
b^{i-1}(x, y_{i-1}) = \int_{Y_i} b_{ij}^i(x, y_i) \left( \delta_{ik} + \frac{\partial u^L_i}{\partial y_{ik}} \right) \left( \delta_{pk} + \frac{\partial u^L_j}{\partial y_{ik}} \right) dy_i;
$$

$b^0$ is the homogenized coefficient.

Let $a^0 = a$. Let $N^k_i \in W_i$ be the solution of the cell problem

$$
\text{curl}_{y_i} (a^i(x, y_i)(e^k + \text{curl}_{y_i} N^k_i)) = 0.
$$

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For $i = 1, \ldots, n$, the positive definite coefficient $a^{i-1}$ is defined as

$$a^{i-1}_{pq}(x, y_{i-1}) = \int_{Y_i} a^{i-1}_{kl}(x, y_i) \left( \delta_{ql} + (\text{curl}_y N^p_{l})_k \right) \left( \delta_{pk} + (\text{curl}_y N^p_{k})_l \right) dy,$$ (4.4)

$a^0$ is the homogenized coefficient. The homogenized equation is

$$\int_0^T \int_D b^0 u_0 \cdot v_0 q''(t) dx dt + \int_0^T \int_D a^0 \nabla u_0 \cdot \nabla v_0 q(t) dx dt = \int_0^T \int_D f(t, x) \cdot v_0(x) q(t) dx dt$$

for all $q \in D(0, T)$ and $v_0 \in W$, i.e.

$$b^0(x) \frac{\partial^2 u_0}{\partial t^2}(t, x) + \nabla (a^0(x)) \nabla u_0(t, x) = f(t, x).$$ (4.5)

The solution $u$ is written in terms of $u_0$ as

$$u_i = N_i^{r_i-1}(\delta_{r_i-1} + (\text{curl}_y N_i^{r_i-2})_{r_i-1})(\delta_{r_i-2} + (\text{curl}_y N_i^{r_i-3})_{r_i-2}) \ldots (\delta_{r_1} + (\text{curl}_y N_i^{r_{1-1}})_{r_1})(\text{curl} u_0)_{r_1},$$ (4.6)

and

$$\nabla y_i u_i = u_0 y_i \left( \frac{\partial u_0}{\partial y_{i_1}} \right)(\delta_{i_1} + \frac{\partial u_0}{\partial y_{i_2}})(\delta_{i_2} + \ldots) \nabla y_i w_i^{r_i-1} = g_1 y_i + \frac{\partial u_0}{\partial y_{i_1}}(\delta_{i_1} + \frac{\partial u_0}{\partial y_{i_2}})(\delta_{i_2} + \ldots) \nabla y_i w_i^{r_i-1}.$$ (4.7)

We refer to [8] for detailed derivation. We make the following assumption on the smoothness of the matrix functions $a(x, y)$ and $b(x, y)$.

**Assumption 4.1** The matrix functions $a$ and $b$ belong to $C^1(\bar{D}, C^2(\bar{Y}_1, \ldots, C^2(\bar{Y}_n)))$.

With this assumption, we have

**Proposition 4.2** Under Assumption 4.1, for all $i, r = 1, \ldots, d$, $\text{curl}_y N_i^{r} \in C^1(\bar{D}, C^2(\bar{Y}_1, C^2(\bar{Y}_1, H^2(\bar{Y})) \ldots))$.

We refer to [8] for a proof of this proposition. We have the following regularity results for the solution $u_0$ of the homogenized equation (4.5).

**Proposition 4.3** Under Assumption 4.1, assume

$$\begin{align*}
&f \in H^2(0, T; H), \\
g_1 \in W, \\
&((b^0)^{-1}[f(0) - \text{curl}(a^0(x))\text{curl} g_0]) \in W, \\
&((b^0)^{-1}[f(0) - \text{curl}(a^0(x))\text{curl} g_1]) \in H,
\end{align*}$$ (4.8)

then

$$\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; W), \quad \frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0, T; H), \quad \text{and} \quad \frac{\partial^4}{\partial t^4} \nabla y_i u_i \in L^\infty(0, T; L^2(D \times Y)).$$ (4.9)

Further, if

$$\begin{align*}
&f \in H^3(0, T; H), \\
g_1 \in W, \\
&((b_0)^{-1}[f(0) - \text{curl}(a^0(x))\text{curl} g_0]) \in W, \\
&((b_0)^{-1}[f(0) - \text{curl}(a^0(x))\text{curl} g_1]) \in W, \\
&((b_0)^{-1}[f(0) - \text{curl}(a^0(x))\text{curl} ((b_0)^{-1}(f(0) - \text{curl}(a^0(x))\text{curl} g_0))]) \in H,
\end{align*}$$ (4.10)

then

$$\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; W), \quad \frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0, T; H), \quad \text{and} \quad \frac{\partial^4}{\partial t^4} \nabla y_i u_i \in L^\infty(0, T; L^2(D \times Y)).$$ (4.11)
From (4.8) we have that

\[
\frac{\partial^2 u_0}{\partial t^2} + \text{curl} \left( a^0 \text{curl} \frac{\partial u_0}{\partial t} \right) = \frac{\partial f}{\partial t}
\]

(4.12)

with compatibility initial conditions

\[
\frac{\partial u_0}{\partial t}(0) = g_1 \in W, \quad \frac{\partial}{\partial t} \frac{\partial u_0}{\partial t}(0) = (b^0)^{-1} [f(0) - \text{curl} (a^0 \text{curl} g_0)] \in W
\]

and

\[
\frac{\partial^2 u_0}{\partial t^2}(0) = (b^0)^{-1} [f(0) - \text{curl} (a^0 \text{curl} g_0)] \in W \quad \text{and} \quad \frac{\partial}{\partial t} \frac{\partial^2 u_0}{\partial t^2}(0) = (b^0)^{-1} \left[ \frac{\partial f}{\partial t}(0) - \text{curl} (a^0 \text{curl} g_1) \right] \in H.
\]

We thus derive that

\[
\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; W) \quad \text{and} \quad \frac{\partial^3 u_0}{\partial t^3} \in L^\infty(0, T; H).
\]

From (4.7) and Proposition 4.2, we deduce that the regularity space \( \bar{L} \) and

\[
L
\]

functions belonging to \( \mathbb{D}_{\alpha} \) for \( \alpha \leq 1 \) and \( \alpha \leq 1 \) for \( \alpha \leq 1 \). We define \( \tilde{\bar{L}} \) as the space of functions \( \tilde{\bar{L}} \) which consists of functions \( w \) that belongs to \( L^2(0, T; \bar{L}^\infty(0, T; \bar{L})) \) and \( L^2(0, T; \bar{L}^\infty(0, T; \bar{L})) \) and for all \( j = 1, \ldots, i - 1 \), \( w \in L^2(0, T; \bar{L}^\infty(0, T; \bar{L})) \). We equip \( \tilde{\bar{L}} \) with the norm

\[
\| w \|_{\tilde{\bar{L}}^i} = \| w \|_{L^2(0, T; \bar{L}^\infty(0, T; \bar{L}))} + \sum_{j=1}^{i-1} \| w \|_{L^2(0, T; \bar{L}^\infty(0, T; \bar{L}))}
\]

We define \( \tilde{\bar{L}}^i \) as the space of functions \( w \in L^2(0, T; \bar{L}^\infty(0, T; \bar{L})) \) such that \( w \in L^2(0, T; \bar{L}^\infty(0, T; \bar{L})) \) and for all \( j = 1, \ldots, i - 1 \), \( w \in L^2(0, T; \bar{L}^\infty(0, T; \bar{L})) \). We equip this space with the norm

\[
\| w \|_{\tilde{\bar{L}}^i} = \| w \|_{L^2(0, T; \bar{L}^\infty(0, T; \bar{L}))} + \sum_{j=1}^{i-1} \| w \|_{L^2(0, T; \bar{L}^\infty(0, T; \bar{L}))}
\]

We define the regularity space \( \tilde{\bar{H}}^i \) as

\[
\tilde{\bar{H}}^i = H^i(\text{curl}, D) \times \tilde{\bar{H}}^{i-1} \times \tilde{\bar{L}}^i \times \tilde{\bar{L}}^{i-1} \times \tilde{\bar{L}}^{i-1}
\]

We define \( \tilde{\bar{H}}_i \) as the space of functions \( w \in L^2(0, T; \bar{L}^\infty(0, T; \bar{L})) \) which are periodic with respect to \( y_j \) for the period being \( Y_j \) (\( j = 1, \ldots, i - 1 \)) such that for any \( \alpha_0, \alpha_1, \ldots, \alpha_{i-1} \in \mathbb{R}^d \) with \( |\alpha_0| \leq 1 \) for \( k = 0, \ldots, i - 1 \),

\[
\frac{\partial^{|\alpha_0|+|\alpha_1|+\ldots+|\alpha_{i-1}|}}{\partial x^{\alpha_0} \partial y_1^{\alpha_1} \ldots \partial y_{i-1}^{\alpha_{i-1}}} w \in L^2(0, T; \bar{L}^\infty(0, T; \bar{L}))
\]
We equip \( \hat{\mathcal{H}}_s \) with the norm
\[
\|w\|_{\hat{\mathcal{H}}_s} = \sum_{\alpha_j \in \mathbb{Z}, |\alpha_j| \leq 1} \left\| \frac{\partial |\alpha_0|+|\alpha_1|+...+|\alpha_{i-1}|}{\partial y_1^{\alpha_1}...\partial y_{i-1}^{\alpha_{i-1}}} w \right\|_{L^2(D \times Y_1 \times \ldots \times Y_{i-1}, H^s(D, \text{curl}, Y_1))}.
\]

We can write \( \hat{\mathcal{H}}_s \) as \( H^1(\overline{D}, H^s_#(Y_1, \ldots, H^1_#(Y_{i-1}, H^1_#(\text{curl}, Y_i))) \) for \( 0 < s < 1 \).

By interpolation, we define \( \hat{\mathcal{H}}^s_1 = H^s(\overline{D}, H^s_#(Y_1, \ldots, H^1_#(Y_{i-1}, H^1_#(\text{curl}, Y_i))) \) for \( 0 < s < 1 \).

We define \( \tilde{\mathcal{H}}_s \) as the space of functions \( w \in L^2(D \times Y_1 \times \ldots \times Y_{i-1}, H^2_#(Y_i)) \) that are periodic with respect to \( y_j \) with the period being \( Y_j \) for \( j = 1, \ldots, i-1 \) such that \( \alpha_0, \alpha_1, \ldots, \alpha_{i-1} \in \mathbb{R}^d \) with \( |\alpha_k| \leq 1 \) for \( k = 0, \ldots, i-1 \),
\[
\frac{\partial |\alpha_0|+|\alpha_1|+...+|\alpha_{i-1}|}{\partial y_1^{\alpha_1}...\partial y_{i-1}^{\alpha_{i-1}}} w \in L^2(D \times Y_1 \times \ldots \times Y_{i-1}, H^2_#(Y_i)).
\]

The space \( \tilde{\mathcal{H}}_s \) is equipped with the norm
\[
\|w\|_{\tilde{\mathcal{H}}_s} = \sum_{\alpha_j \in \mathbb{Z}, |\alpha_j| \leq 1} \left\| \frac{\partial |\alpha_0|+|\alpha_1|+...+|\alpha_{i-1}|}{\partial y_1^{\alpha_1}...\partial y_{i-1}^{\alpha_{i-1}}} w \right\|_{L^2(D \times Y_1 \times \ldots \times Y_{i-1}, H^2_#(Y_i))}.
\]

We can write \( \tilde{\mathcal{H}}_s \) as \( H^1(\overline{D}, H^s_#(Y_1, \ldots, H^1_#(Y_{i-1}, H^2_#(\text{curl}, Y_i))) \). By interpolation, we define the space \( \tilde{\mathcal{H}}^s_1 := H^s(\overline{D}, H^s_#(Y_1, \ldots, H^1_#(Y_{i-1}, H^2_#(\text{curl}, Y_i))) \). The regularity space \( \mathcal{H}^s \) is defined as
\[
\mathcal{H}^s = H^s(\text{curl}, D) \times \hat{\mathcal{H}}^s_1 \times \ldots \tilde{\mathcal{H}}^s_1 \times \ldots \tilde{\mathcal{H}}^s_1.
\]

For the regularity of \( u_0 \), we have the following result.

**Proposition 4.4** Under Assumption 4.3 if \( D \) is a Lipschitz polygonal domain, \( f \in H^1(0,T;H), \) \( g_0 \in H^1(\text{curl}, D) \) and \( g_1 \in W, \) \( \text{div } f \in L^\infty(0,T;L^2(D)), \) \( \text{div}(b^0 g_0) \in L^2(D) \) and \( \text{div}(b^0 g_1) \in L^2(D), \) there is a constant \( c \in (0,1) \) such that \( u_0 \in L^\infty(0,T;H^s(\text{curl}, D)). \)

**Proof** Using Proposition 7.10, equations (4.4) and (4.6), we have that \( a^0, b^0 \in C^1(\overline{D})^{d \times d}. \) If \( f \in H^1(0,T;H) \) and \( g_0 \in H^1(\text{curl}, D), \) we have that \( (b^0)^{-1}[f - \text{curl}(a^0 \text{curl} g^0)] \in H. \) The compatibility initial conditions hold so that \( \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0,T;H). \) Thus
\[
\text{curl}(a^0 \text{curl} u_0) = f - b^0 \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0,T;H).
\]

Let \( U(t) = a^0 \text{curl} u_0(t). \) As \( \text{div}((a^0)^{-1} U(t)) = 0 \) and \((a^0)^{-1} U(t) \cdot \nu = 0, \) there is a constant \( c \) and a constant \( s \in (0,1] \) which depend on \( a^0 \) and the domain \( D \) so that
\[
\|U(t)\|_{H^s(D)} \leq c(\|U(t)\|_{L^2(D)}^s + \|U(t)\|_{L^2(D)}^s)
\]
so \( U \in L^\infty(0,T;H^s(D)). \) As \( \text{curl} u_0(t) = (a^0)^{-1} U(t) \) and \((a^0)^{-1} \in C^1(\overline{D})^{d \times d}, \) \( \text{curl} u_0 \in L^\infty(0,T;H^s(D)). \)

We note that
\[
\text{div}\left(b^0 \frac{\partial^2 u_0}{\partial t^2}\right) = \text{div} f,
\]
so
\[
\text{div}(b^0 u_0(t)) = \int_0^t \int_0^s \text{div} f(r)drds + t \text{div}(b^0 g_1) + \text{div}(b^0 g_0) \in L^\infty(0,T;L^2(D)).
\]

From Theorem 4.1 of Hiptmair [13], we deduce that there is a constant \( c \in (0,1] \) (we take it as the same constant as above), so that
\[
\|u_0(t)\|_{H^s(D)} \leq c(\|u_0(t)\|_{H(\text{curl}, D)} + \|\text{div}(b^0 u_0(t))\|_{L^2(D)}^s).
\]

Thus \( u_0 \in L^\infty(0,T;H^s(\text{curl}, D)). \)

Similarly, we can deduce the regularity for \( \frac{\partial^2 u_0}{\partial t^2}. \)
Proposition 4.5 \textit{Under Assumption 4.7}, if $D$ is a Lipschitz polygonal domain, if the compatibility conditions 4.10 hold, and if $\mathbf{f} \in L^\infty(0, T; L^2(D))$, then there is a constant $s \in (0, 1]$ such that $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H^s(\text{curl}, D))$.

\textbf{Proof} From equation (4.13), we have
\begin{equation}
\text{curl} \left( \alpha \text{ curl} \frac{\partial^2 u_0}{\partial t^2} \right) = \frac{\partial^2 f}{\partial t^2} - b \frac{\partial^4 u_0}{\partial t^4} \in L^\infty(0, T; H)
\end{equation}
as $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; H)$ due to 4.10. Following a similar argument as in the proof of Proposition 4.4, we deduce that $\text{curl} \frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H^s(\text{curl}, D))$. We note that
\begin{equation}
\text{div} b \frac{\partial^2 u_0}{\partial t^2} = \text{div} f \in L^\infty(0, T; L^2(D)).
\end{equation}

From Theorem 4.1 of [18], we deduce that $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; H^s(\text{curl}, D))$.

From these we deduce

Proposition 4.6 \textit{Under Assumption 4.2 and the hypothesis of Proposition 4.4}, there is a constant $s \in (0, 1]$ so that $u \in L^\infty(0, T; \mathcal{H})$.

\textbf{Proof} From Proposition 4.2, we have that $N^T_1$ and $\text{curl} N^T_1$ belong to $C^1(\hat{D}, C^2(Y_1, \ldots, C^2(Y_{i-1}, H_F^s(Y_i) \ldots))$. Together with $u_0 \in L^\infty(0, T; H^s(\text{curl}, D))$, this implies $u_i \in L^\infty(0, T; \mathcal{H})$. Similarly, we have $u_i \in L^\infty(0, T; \mathcal{H})$.

Similarly, we have:

Proposition 4.7 \textit{Under Assumption 4.2 and the hypothesis of Proposition 4.3}, there is a constant $s \in (0, 1]$ so that $\frac{\partial^2 u_0}{\partial t^2} \in L^\infty(0, T; \mathcal{H})$.

Remark 4.8 We have
\begin{equation}
\text{curl} \left( \frac{\partial u_0(t)}{\partial t} \right) = \int_0^t \text{curl} \frac{\partial^2 u_0}{\partial t^2}(s) ds + \text{curl} g_1, \quad \text{and} \quad \frac{\partial u_0(t)}{\partial t} = \int_0^t \frac{\partial^2 u_0}{\partial t^2}(s) ds + g_1,
\end{equation}
\begin{equation}
\text{curl} u_0(t) = \int_0^t \int_0^s \text{curl} \frac{\partial^2 u_0}{\partial t^2}(r) dr ds + t \text{curl} g_1 + \text{curl} g_0, \quad \text{and} \quad u_0(t) = \int_0^t \int_0^s \frac{\partial^2 u_0}{\partial t^2}(r) dr ds + t g_1 + g_0.
\end{equation}

Thus with the hypothesis of Proposition 4.3, together with $g_0 \in H^s(\text{curl}, D)$ and $g_1 \in H^s(\text{curl}, D)$, we deduce that $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; H^s(\text{curl}, D))$ and $u_0 \in L^\infty(0, T; H^s(\text{curl}, D))$. This implies also that $u \in L^\infty(0, T; \mathcal{H})$.

5 Full and sparse tensor product approximations

We consider the approximations of problem 2.9 using the full and sparse tensor product FE. We assume that the domain $D$ is a polygon in $\mathbb{R}^3$. Let $T^l$ ($l = 0, 1, \ldots$) be the sets of simplices in $D$ with mesh size $h_l = O(2^{-l})$ which are determined recursively where $T^{l+1}$ is obtained from $T^l$ by dividing each simplex in $T^l$ into 8 tetrahedra. For a tetrahedron $T \in T^l$, we consider the edge finite element space
\begin{equation}
R(T) = \{ v : \quad v = \alpha + \beta \times x, \quad \alpha, \beta \in \mathbb{R}^3 \}.
\end{equation}

When $D$ is a polygon in $\mathbb{R}^2$, $T^{l+1}$ is obtained from $T^l$ by dividing each simplex in $T^l$ into 4 congruent triangles. For each triangle $T \in T^l$, we consider the edge finite element space
\begin{equation}
R(T) = \left\{ v : \quad v = \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \beta \end{array} \right) \right\}
\end{equation}
where $\alpha_1, \alpha_2$ and $\beta$ are constants. Alternatively, when $D$ is partitioned into cubic meshes, we can use edge finite element on cubic mesh instead (see [25]). For each simplex $T \in T^l$, we denote by $\mathcal{P}_1(T)$ the
5.1 Full tensor product finite elements

The spatially semidiscrete full tensor product finite element approximating problem is: Find \( \bar{u} \) for the cube \( Y \), we consider a hierarchy of simplices \( T'_k \) that are distributed periodically. We consider the space of functions

\[
W^T = \{ v \in H_0^i(D), \ v|_T \in R(T) \ \forall T \in T^T \},
\]
\[
V^T = \{ v \in H^i(D), \ v|_T \in P_1(T) \ \forall T \in T^T \}.
\]

For the cube \( Y \), we consider a hierarchy of simplices \( T'_k \) that are distributed periodically. We consider the space of functions

\[
W^T = \{ v \in H_0^i(curl,Y), \ v|_T \in R(T) \ \forall T \in T^T \}
\]
and

\[
V^T = \{ v \in H^i(Y), \ v|_T \in P_1(T) \ \forall T \in T^T \}.
\]

We then have the following standard estimates (see Monk [25] and Ciarlet [10])

\[
\forall v \in H_0^i(curl,Y) \cap H^s(curl,D); \quad \inf_{v_2 \in W^T} \| v - v_2 \|_{H^s(curl,D)} \leq c_{h}^s \| v \|_{H^s(Y)} \quad (5.1)
\]
\[
\forall v \in H^s(D); \quad \inf_{v_2 \in V^T} \| v - v_2 \|_{L^2(D)} \leq c_{h} \| v \|_{H^s(Y)}
\]
\[
\forall v \in H^s(Y); \quad \inf_{v_2 \in V^T} \| v - v_2 \|_{H^s(Y)} \leq c_{h} \| v \|_{H^s(Y)}.
\]

5.1 Full tensor product finite elements

As \( L^2(D \times Y_{i-1}, \tilde{H}_k(curl,Y)) \cong L^2(D) \otimes L^2(Y_1) \otimes \ldots \otimes L^2(Y_{i-1}) \otimes \tilde{H}_k(curl,Y) \) we use the tensor product finite element space

\[
W_i^T = V^T \otimes V^T \otimes \ldots \otimes V^T \otimes W^T
\]

\( i \) times

to approximate \( u_i \). Similarly, as \( u_i \in L^2(D \times Y_{i-1}, H^1_{\#}(Y)) \), we use the finite element space

\[
V_i^T = V^T \otimes V^T \otimes \ldots \otimes V^T \otimes V^T
\]

\( i \) times

to approximate \( u_i \). We define the space

\[
V^L = W^L \otimes W^L \otimes \ldots \otimes W^L \otimes V^L \otimes \ldots \otimes V^L.
\]

The spatially semidiscrete full tensor product finite element approximating problem is: Find \( \bar{u}^L(t) \in V^L \) so that for all \( \tilde{u}^L \in V^L \):

\[
\int_D \int_Y b(x,y) \left( \frac{\partial^2 u^L_0}{\partial t^2}(t) + \sum_{i=1}^n \frac{\partial^2}{\partial t^2} \nabla y_i u^L_i(t) \right) \cdot \left( \tilde{u}^L_0 + \sum_{i=1}^n \nabla y_i \tilde{u}^L_i \right) dy dx
\]
\[
+ a(x,y) \left( \text{curl} \bar{u}^L(t) + \sum_{i=1}^n \text{curl} y_i \bar{u}^L_i(t) \right) \cdot \left( \text{curl} \tilde{u}^L_0 + \sum_{i=1}^n \text{curl} y_i \tilde{u}^L_i \right) dy dx
\]
\[
= \int_D f(t,x) \cdot \bar{u}^L_0(x) dx
\]

(5.1)
for all $\theta^L = (\bar{\theta}^L_0, \{\bar{\theta}^L_i\}, \{\bar{\theta}^L_{m_{i,m}}\}) \in \bar{V}^L$.

To deduce an error estimate for the full tensor product approximations of (2.9), we note the following approximations

**Lemma 5.1** For $w \in \mathcal{H}^s$,

$$\inf_{w^f \in W^f_i} \| w - w^f \|_{L^2(D \times \Omega)} \leq ch^s_w \| w \|_{\mathcal{H}^s}.$$ 

For $w \in \bar{H}^s$,

$$\inf_{w^f \in W^f_i} \| w - w^f \|_{L^2(D \times \Omega)} \leq ch^s_w \| w \|_{\bar{H}^s}.$$ 

The proofs of these results are similar to those for full tensor product finite elements in [20] and [17], using orthogonal projection. We refer to [20] and [17] for details. From this we deduce that for $w \in \mathcal{H}^s$

$$\inf_{w^f \in \mathcal{V}^L} \| w - w^f \|_{\mathcal{V}^L} \leq ch^s_L \| w \|_{\mathcal{H}^s}.$$ 

We then have the following result for the spatially semidiscrete approximation.

**Proposition 5.2** Assume that condition [4,10] and Assumption [4.1] hold, $D$ is a Lipschitz polygonal domain, $\text{div} f \in L^\infty(0,T; L^2(D))$ and $g_0, g_1$ belong to $H^s(\text{curl}, D)$. If $g_0^L$ and $g_1^L$ are chosen so that

$$\| g_0^L - g_0 \|_{L^2} \leq ch^s_L \text{ and } \| g_1^L - g_1 \|_{L^2} \leq ch^s_L,$$

where $s \in (0,1]$ is the constant in Proposition [4.2]. Then

$$\| \partial (u_0^L - u_0) \|_{L^\infty(0,T; H^s)} + \sum_{i=1}^n \| \nabla \partial (u_i^L - u_i) \|_{L^\infty(0,T; H^s)} + \| \text{curl} (u_0^L - u_0) \|_{L^\infty(0,T; H^s)} \leq ch^s_L,$$

with $s \in (0,1]$, and

$$\| q^L \|_{L^\infty(0,T; \mathcal{V})} \leq ch^s_L, \quad \| \partial q^L \|_{L^\infty(0,T; \mathcal{V})} \leq ch^s_L, \quad \text{and} \quad \| \partial^2 q^L \|_{L^2(0,T; \mathcal{V})} \leq ch^s_L.$$ 

These together with

$$\frac{\partial p^L_0}{\partial t}(0) = \partial (u_0^L(0) - u(0)) - \frac{\partial q^L}{\partial t}(0),$$

$$\nabla \partial p^L_0(0) = \nabla \partial (u_0^L(0) - u(0)) - \nabla \partial q^L(0),$$

$$\text{curl} p^L_0(0) = \text{curl} (u_0^L(0) - u(0)) - \text{curl} q^L(0),$$

and [5.2] and [5.5], we have that

$$\| \partial p^L_0(0) + \sum_{i=1}^n \nabla \partial p^L_i(0) \|_{H^s} \leq ch^s_L,$$

and

$$\| \text{curl} p^L_0(0) \|_{H^s} \leq ch^s_L.$$ 

Thus the right hand side of [3.7] is not more than $ch^s_L$. We thus get the conclusion.

The fully discrete problem now becomes: For $m = 1, \ldots, M$ find

$$\tilde{u}^L_m = (\tilde{u}^L_{0,m}, \tilde{u}^L_{1,m}, \ldots, \tilde{u}^L_{n,m}, \tilde{u}^L_{m_{i,m}}, \ldots, \tilde{u}^L_{n_{m_{i,m}}}) \in \bar{V}^L.$$
such that for $m = 1, \ldots, M - 1$

\[
\int_D \int_Y \left[b(x, y) \left( \partial_t \bar{u}_{0,m}^L + \sum_{i=1}^n \nabla_y \partial_t \bar{u}_{i,m}^L \right) \cdot \left( \bar{v}_0^L + \sum_{i=1}^n \nabla_y \bar{v}_i^L \right) \right. \\
+ a(x, y) \left( \text{curl} \; \bar{u}_{0,m,1/4}^L + \sum_{i=1}^n \text{curl}_y \bar{u}_{i,m,1/4}^L \right) \cdot \left( \text{curl} \; \bar{v}_0^L + \sum_{i=1}^n \text{curl}_y \bar{v}_i^L \right) \left] \, dy \, dx \right.
\]

\[
= \int_D f_{m,1/4}(x) \cdot \bar{v}_0^L(x) \, dx
\]  

(5.4)

for all $\bar{\psi}^L = (\bar{v}_0^L, \bar{v}_1^L, \ldots, \bar{v}_n^L, \bar{v}_1^L, \ldots, \bar{v}_n^L) \in V^L$.

**Proposition 5.3** Assume that condition (4.10) and Assumption 4.1 hold, $D$ is a Lipschitz polygonal domain, $\text{div} f \in L^\infty(0, T; L^2(D))$ and $g_0, g_1$ belong to $H^s(\text{curl}, D)$ where $s \in (0, 1]$ is the constant in Proposition 4.2. If the initial value $\bar{u}_t^L$ is chosen so that

\[
\|\partial_t \bar{u}^L_{0,1/2}\|_H + \sum_{i=1}^n \|\partial_t \nabla_y \bar{u}^L_{i,1/2}\|_H + \|\text{curl} \bar{u}^L_{0,1/2}\|_H + \sum_{i=1}^n \|\text{curl}_y \bar{u}^L_{i,1/2}\|_H \leq c((\Delta t)^2 + h_L^2),
\]

then

\[
\|\partial_t \bar{u}^L_0 - \partial_t u_0\|_{L^\infty(0,T;H)} + \sum_{i=1}^n \|\partial_t \nabla_y (\bar{u}^L_i - u_i)\|_{L^\infty(0,T;H)}
\]

\[
+ \|\text{curl} (\bar{u}^L_0 - u_0)\|_{L^\infty(0,T;H)} + \sum_{i=1}^n \|\text{curl}_y (\bar{u}^L_i - u_i)\|_{L^\infty(0,T;H)} \leq c((\Delta t)^2 + h_L^2).
\]

**Proof** The proof is similar to that of Proposition 5.2. We note that

\[
\|\partial_t \bar{u}^L_{0,m+1}\|_H = \left\| \frac{\bar{u}^L_{0,m+1} - \bar{u}^L_{0,m}}{\Delta t} \right\|_H \leq \sup_{t \in (0,T)} \left\| \frac{\partial_t \bar{u}^L_i}{\partial t} \right\|_H \leq c h_L^2
\]

due to (5.3). Similarly

\[
\|\partial_t \nabla_y \bar{u}^L_{i,m+1}\|_H \leq c h_L^2.
\]

We then get the conclusion. \qed

### 5.2 Sparse tensor product finite elements

To define the sparse tensor product finite element spaces, we employ the following orthogonal projection

\[ P^{10} : L^2(D) \to V^l, \quad P^{0\#} : L^2(Y) \to V^l_{\#}. \]

with the convention $P^{-10} = 0$, $P^{-10\#} = 0$. The detail spaces are defined as

\[ Y^l = (P^{10} - P^{(l-1)0})V^l, \quad V^l_{\#} = (P^{0\#} - P^{(l-1)0\#})V^l. \]

We note that

\[ V^l = \bigoplus_{0 \leq i \leq l} Y^i \quad \text{and} \quad V^l_{\#} = \bigoplus_{0 \leq i \leq l} Y^i_{\#}. \]

Therefore the full tensor product spaces $W^L_i$ and $V^L_i$ can be written as

\[ W^L_i = \left( \bigoplus_{0 \leq l_0, \ldots, l_{i-1} \leq L} Y^{l_0} \otimes Y^{l_1} \otimes \ldots \otimes Y^{l_{i-1}} \right) \otimes W^L_{\#}. \]
and
\[ \hat{V}^L_i = \bigoplus_{0 \leq i_0, \ldots, i_{L-1} \leq L} V^{i_0} \otimes V_{i_1}^L \otimes \cdots \otimes V_{i_{L-1}}^L \otimes V_{i_L}^L. \]

We define the sparse tensor product finite element spaces as
\[ \hat{W}^L_i = \bigoplus_{l_0 + \ldots + l_{L-1} \leq L} V^{l_0} \otimes V_{l_1}^L \otimes \cdots \otimes V_{l_{L-1}}^L \otimes W_{i_L}^L, \]
\[ \hat{V}^L_i = \bigoplus_{l_0 + \ldots + l_{L-1} \leq L} V^{l_0} \otimes V_{l_1}^L \otimes \cdots \otimes V_{l_{L-1}}^L \otimes V_{i_L}^L, \]
and
\[ \hat{V}^L = W^L \otimes \hat{W}^L_1 \otimes \cdots \otimes \hat{W}^L_n \otimes \hat{V}^L_1 \otimes \cdots \otimes \hat{V}^L_n. \]

The spatially semidiscrete sparse tensor product finite element approximating problem is: Find \( \hat{u}^L(t) \in \hat{V}^L \) such that:
\[ \int_D \int_Y \left[ b(x,y) \left( \frac{\partial^2 \hat{u}^L}{\partial t^2} (t) + \sum_{i=1}^n \nabla_y \frac{\partial^2 \hat{v}^L}{\partial t^2} (t) \right) \cdot \left( \hat{v}^L_0 + \sum_{i=1}^n \nabla_y \hat{v}^L_i \right) \right. \]
\[ + a(x,y) \left( \text{curl} \hat{u}^L_0 (t) + \sum_{i=1}^n \text{curl}_y \hat{u}^L_i (t) \right) \cdot \left( \text{curl} \hat{v}^L_0 + \sum_{i=1}^n \text{curl}_y \hat{v}^L_i \right) \] \[ = \int_D f(x) \cdot \hat{v}^L_0 (x) dx \] \[ (5.5) \]
for all \( \hat{v}^L = (\hat{v}^L_0, \hat{v}^L_1, \ldots, \hat{v}^L_n, \hat{v}^L_1, \ldots, \hat{v}^L_n) \in \hat{V}^L \). To find an error estimate for the sparse tensor product finite element approximation we note the following results

**Lemma 5.4** For \( w \in \hat{H}^s \),
\[ \inf_{w^L \in \hat{W}^L} \| w - w^L \|_{L^2(D \times Y; \hat{V}^L)} \leq c L^{1/2} h^s \| w \|_{\hat{H}^s}, \]
for \( w \in \hat{H}^s \),
\[ \inf_{w^L \in \hat{V}^L} \| w - w^L \|_{L^2(D \times Y; H^s_w(Y)))} \leq c L^{1/2} h^s \| w \|_{\hat{H}^s}. \]

The proof of these results follow from that for sparse tensor product approximation in [7] and [20]. Therefore, for \( w \in \hat{H}^s \),
\[ \inf_{w^L \in \hat{V}^L} \| w - w^L \|_{L^2} \leq c L^{1/2} h^s \| w \|_{\hat{H}^s}. \]

We then have the following result.

**Proposition 5.5** Assume that condition (4.10) and Assumption (4.1) hold, \( D \) is a Lipschitz polygonal domain, \( \text{div} f \in L^\infty(0, T; L^2(D)) \) and \( g_0, g_1 \) belong to \( H^s(\text{curl}, D) \). If \( g_0^L \) and \( g_1^L \) are chosen so that
\[ \| g_0^L - g_0 \|_{L^2} \leq c L^{n/2} h^s \] and \( \| g_1^L - g_1 \|_{L^\infty} \leq c L^{n/2} h^s \),
where \( s \in (0, 1] \) is the constant in Proposition 4.2, then the solution of the spatially semidiscrete approximating problem (5.3) satisfies
\[ \left\| \frac{\partial (\hat{u}^L_0 - u_0)}{\partial t} \right\|_{L^\infty(0,T;H)} + \sum_{i=1}^n \left\| \nabla_y \frac{\partial (\hat{u}^L_i - u_i)}{\partial t} \right\|_{L^\infty(0,T;H_i)} \]
\[ + \| \text{curl}(\hat{u}^L_0 - u_0) \|_{L^\infty(0,T;H)} + \sum_{i=1}^n \| \text{curl}_y (\hat{u}^L_i - u_i) \|_{L^\infty(0,T;H_i)} \leq c L^{n/2} h^s. \]
The proof of this proposition is identical to that of Proposition 5.2.

The fully discrete sparse tensor finite element product problem is: For \( m = 1, \ldots, M \) find \( \tilde{u}_m^L = (\tilde{u}_{0,m}^L, \tilde{u}_{1,m}^L, \ldots, \tilde{u}_{n,m}^L) \in \mathbb{V}^L \) such that

\[
\int_D \int_Y b(x, y) \left( \partial_t^2 \tilde{u}_{0,m}^L + \sum_{i=1}^n \nabla_{y_i} \partial_t^2 \tilde{u}_{i,m}^L \right) \cdot \left( \tilde{v}_0^L + \sum_{i=1}^n \nabla_{y_i} \tilde{v}_i^L \right) + a(x, y) \left( \text{curl} \tilde{u}_{0,m,1/4}^L + \sum_{i=1}^n \text{curl}_{y_i} \tilde{u}_{i,m,1/4}^L \right) \cdot \left( \text{curl} \tilde{v}_0^L + \sum_{i=1}^n \text{curl}_{y_i} \tilde{v}_i^L \right) dy dx = \int_D f_{m,1/4}(x) \cdot \tilde{v}_0^L(x) dx
\]

(5.6)

for all \( \tilde{v}_L = (\tilde{v}_0^L, \tilde{v}_1^L, \ldots, \tilde{v}_n^L) \in \mathbb{V}^L \).

For the fully discrete problem, we have the following result

**Proposition 5.6** Assume that condition \([4.10]\) and Assumption \([4.4]\) hold, \( D \) is a Lipschitz polygonal domain, \( \text{div} f \in L^\infty(0,T; L^2(D)) \) and \( g_0, g_1 \) belong to \( H^s(\text{curl}, D) \) where \( s \in (0,1] \) is the constant in Proposition 4.2. If the initial value \( \tilde{u}_0^L \) is chosen so that

\[
\| \partial_t p_{0,1/2}^L \|_{L^2(D)} + \sum_{i=1}^n \| \partial_t \nabla_{y_i} p_{i,1/2}^L \|_{L^2(D \times Y)} + \| \text{curl} p_{0,1/2}^L \|_{L^2(D)} + \sum_{i=1}^n \| \text{curl}_{y_i} p_{i,1/2}^L \|_{L^2(D \times Y)} \leq c((\Delta t)^2 + L^{n/2} h_L^s),
\]

then

\[
\| \partial_t \tilde{u}_0^L - \partial_t u_0 \|_{L^\infty(0,T; H^1)} + \sum_{i=1}^n \| \nabla_{y_i} (\tilde{u}_0^L - u_i) \|_{L^\infty(0,T; H^1)} + \| \text{curl} (\tilde{u}_0^L - u_0) \|_{L^\infty(0,T; H^1)} + \sum_{i=1}^n \| \text{curl}_{y_i} (\tilde{u}_0^L - u_i) \|_{L^\infty(0,T; H^1)} \leq c((\Delta t)^2 + L^{n/2} h_L^s).
\]

The proof is identical to that of Proposition 5.3.

### 6 Numerical correctors

We construct numerical correctors in this section. For two scale problems, we derive an explicit error for the corrector in terms of the microscale \( \varepsilon \) and the FE meshsize. For general multiscale problems, as a homogenization error is not available, we derive a corrector without an error estimate. We first review some results for analytic correctors.

#### 6.1 Analytic homogenization errors and correctors

For two scale problems, for conciseness of notations, we denote the solutions to cell problems \( \tilde{N}_r^\varepsilon \) and \( w_r^\varepsilon \) as \( N^\varepsilon \) and \( w^\varepsilon \). We have the following homogenization error for two scale problems. This result generalizes the well known \( O(\varepsilon^{1/2}) \) homogenization error in [5] and [24] to the case where the solution \( u_0 \) of the homogenized equation possesses low regularity. We derive this error for two scale Maxwell wave equations, but the proof works verbatim for the two scale elliptic equations in [5] and [24]. The proof is lengthy and complicated so we refer to [11] for details.

**Proposition 6.1** Assume that \( g_0 = 0, g_1 \in H^1(D) \cap W, f \in H^1(0,T; H), u_0, \frac{\partial u_0}{\partial t} \) and \( \frac{\partial^2 u_0}{\partial x^2} \) belong to \( L^\infty(0,T; H^s(\text{curl}, D)) \) for \( 0 < s \leq 1, N^\varepsilon \in C^1(\overline{D}, C(\overline{\Omega}))^3, \text{curl}_{y} N^\varepsilon \in C^1(\overline{D}, C(\overline{\Omega})), w^\varepsilon \in C^1(\overline{D}, C(\overline{\Omega})) \) for all \( r = 1,2,3 \). There exists a constant \( c \) that does not depend on \( \varepsilon \) such that

\[
\left\| \frac{\partial u^\varepsilon}{\partial t} - \left[ \frac{\partial u_0}{\partial t} \right] + \frac{\partial}{\partial y} \frac{\partial u_1}{\partial t} \left( \cdot, \cdot, \varepsilon \right) \right\|_{L^\infty(0,T; H)} + \left\| \text{curl} w^\varepsilon - \left[ \text{curl} u_0 + \text{curl}_{y} u_1 \left( \cdot, \cdot, \varepsilon \right) \right] \right\|_{L^\infty(0,T; H)} \leq c\varepsilon^{1/2}.
\]
For the case of more than two scales, an explicit homogenization error is not available. However, we can deduce correctors when $\varepsilon_{i+1}/\varepsilon_i$ is an integer for all $i = 2, \ldots, n$. We define the operator $\mathcal{U}_n^g$ as

$$
\mathcal{U}_n^g(\Phi)(x) = \int_{Y_1} \cdots \int_{Y_n} \Phi \left( \frac{x}{\varepsilon_1} + \varepsilon_1 t_1, \frac{x}{\varepsilon_2} + \varepsilon_2 t_2, \cdots, \frac{x}{\varepsilon_n} + \varepsilon_n t_n \right) \frac{\varepsilon_n}{\varepsilon_{n-1}} \frac{\varepsilon_{n-1}}{\varepsilon_{n-2}} \cdots \frac{\varepsilon_2}{\varepsilon_1} dt_1 \cdots dt_n,
$$

for all functions $\Phi \in L^1(D \times Y)$. In the two scale case, we denote $\mathcal{U}_n^g$ by $\mathcal{U}^g$. We note the following property.

**Lemma 6.2** For each function $\Phi \in L^1(D \times Y)$ we have

$$
\int_{D^{\varepsilon_1}} \mathcal{U}_n^g(\Phi) dx = \int_D \int_Y \Phi(x, y) dy dx,
$$

where $D^{\varepsilon_1}$ is the $2\varepsilon_1$ neighbourhood of $D$.

We refer to [12] for a proof. We have the following corrector result for multiscale problems.

**Proposition 6.3** Assume that $g_0 = 0$, $g_1 \in W$ and $f \in H^1(0, T; H)$. We have

$$
\lim_{\varepsilon \to 0} \left\| \frac{\partial u^g}{\partial t} - \mathcal{U}_n^g \left( \frac{\partial u_0}{\partial t} + \sum_{i=1}^{n} \varepsilon_i \nabla g_i \frac{\partial u_i}{\partial t} \right) \right\|_{L^\infty(0, T; H)} + \left\| \text{curl } u^g - \mathcal{U}_n^g \left( \text{curl } u_0 + \sum_{i=1}^{n} \text{curl } g_i u_i \right) \right\|_{L^\infty(0, T; H)} = 0.
$$

The proof of these corrector results can be found in [9].

**Remark 6.4** Generally, the energy of a multiscale wave equation does not always converge to the energy of the homogenized wave equation when $g_0 \neq 0$. We therefore restrict our consideration to the case where $g_0 = 0$. As shown in [9], the corrector of a general two scale wave equation involves the solution of another multiscale equation in the domain $D$. However, the scale interacting terms in [23] always form a part of the corrector.

### 6.2 Numerical correctors for two-scale problems

We now establish numerical correctors with an explicit error estimate for two scale problems. We first note the following result.

**Lemma 6.5** Assume that $\frac{\partial u_0}{\partial t} \in L^\infty(0, T; H^1(D)^3)$ and $\text{curl } u_0 \in L^\infty(0, T; H^1(D)^3)$, $N' \in C^1(D)$ and $w^g \in C^1(D, C_B^r(\bar{Y}))$, $r = 1, 2, 3$. Then

$$
\sup_{t \in [0, T]} \int_D \left| \text{curl } u_1(t, x, \frac{x}{\varepsilon}) - \mathcal{U}^g \left( \text{curl } u_1(t, \cdot, \cdot) \right)(x) \right|^2 dx \leq c\varepsilon^{2s}
$$

and

$$
\sup_{t \in [0, T]} \int_D \left| \frac{\partial}{\partial t} \nabla u_1(t, x, \frac{x}{\varepsilon}) - \mathcal{U}^g \left( \frac{\partial}{\partial t} \nabla u_1(t, \cdot, \cdot) \right)(x) \right|^2 dx \leq c\varepsilon^{2s}.
$$

The proof of this result is similar to that for the time independent case in Appendix B of [8], which utilizes the ideas of the proof of Lemma 5.5 in [21]. We then have the following numerical corrector results.

**Theorem 6.6** Assume that condition [11] and Assumption [4] hold, with $g_0 = 0$ and $\text{div } f \in L^\infty(0, T; L^2(D))$, $D$ is a Lipschitz polygonal domain, and that $g_1^L$ is chosen so that $\|g_1^L - g_1\|_H \leq c\varepsilon^s$ where $s \in (0, 1]$ is the constant in Proposition 4.2. Then for the solution of the semidiscrete problem (6.1) using the full tensor product FEs, we have

$$
\left| \frac{\partial u^g}{\partial t} - \mathcal{U}^g \left( \frac{\partial u^L_0}{\partial t} + \mathcal{U}^g \left( \frac{\partial}{\partial t} \nabla u^L_1 \right) \right) \right|_{L^\infty(0, T; H)} + \left| \text{curl } u^g - \left( \text{curl } u^L_0 + \mathcal{U}^g \left( \text{curl } g^L u^L_1 \right) \right) \right|_{L^\infty(0, T; H)} \leq c \left( \varepsilon \|h^L_1\| + \varepsilon^s \right).
$$
For the semidiscrete problem using the sparse tensor product FEs, if \( \|g^1 - g_1\|_H \leq cL^{1/2}h_L^s \), we have

\[
\left\| \frac{\partial u^\varepsilon}{\partial t} - \left( \frac{\partial \tilde{u}_0^L}{\partial t} + \mathcal{U}^\varepsilon \left( \frac{\partial}{\partial t} \nabla_y \bar{u}_1^T \right) \right) \right\|_{L^\infty(0,T;H)} \\
+ \|\text{curl} \ u^\varepsilon - (\text{curl} \ \tilde{u}_0^L + \mathcal{U}^\varepsilon (\text{curl}_y \bar{u}_1^T)) \|_{L^\infty(0,T;H)} \leq c \left( L^{1/2}h_L^s + \varepsilon \bar{\varepsilon} \right).
\]

**Proof.** With the hypothesis of the theorem, from Propositions 4.1 and 4.5, the conditions of Theorem 6.1 hold. We then have from (6.1)

\[
\left\| \mathcal{U}^\varepsilon \left( \frac{\partial}{\partial t} \nabla_y u_1(t) - \frac{\partial}{\partial t} \nabla_y \bar{u}_1^T(t) \right) \right\|_H \leq \left\| \frac{\partial}{\partial t} \nabla_y u_1(t) - \frac{\partial}{\partial t} \nabla_y \bar{u}_1^T(t) \right\|_{L^2(D \times Y)^3}
\]
and

\[
\|\mathcal{U}^\varepsilon (\text{curl} \ u_0 + \text{curl}_y u_1 - \text{curl} \ \tilde{u}_0^L - \text{curl}_y \bar{u}_1^T)\|_H \leq \|\text{curl} \ u_0 + \text{curl}_y u_1 - \text{curl} \ \tilde{u}_0^L - \text{curl}_y \bar{u}_1^T\|_{L^2(D \times Y)^3}.
\]

We note that

\[
\left\| \frac{\partial u^\varepsilon}{\partial t} - \left( \frac{\partial u_0}{\partial t} + \mathcal{U}^\varepsilon \left( \frac{\partial}{\partial t} \nabla_y u_1 \right) \right) \right\|_{L^\infty(0,T;H)} \leq c \varepsilon \bar{\varepsilon} \bar{\varepsilon}.
\]

From Proposition 5.2 we have

\[
\left\| \frac{\partial u_0}{\partial t} - \frac{\partial \tilde{u}_0^L}{\partial t} \right\|_{L^\infty(0,T;H)} \leq ch_L^s.
\]

From Lemma 6.3 we have

\[
\left\| \frac{\partial}{\partial t} \nabla_y u_1 (\cdot, \cdot, \cdot, \varepsilon) \right\|_{L^\infty(0,T;H)} \leq c \varepsilon^s.
\]

We note that

\[
\left\| \mathcal{U}^\varepsilon \left( \frac{\partial}{\partial t} \nabla_y u_1 \right) - \mathcal{U}^\varepsilon \left( \frac{\partial}{\partial t} \nabla_y \bar{u}_1^T \right) \right\|_{L^\infty(0,T;H)} \leq \left\| \frac{\partial}{\partial t} \nabla_y u_1 - \frac{\partial}{\partial t} \nabla_y \bar{u}_1^T \right\|_{L^\infty(0,T;L^2(D \times Y)^3)} \leq ch_L^s.
\]

Thus

\[
\left\| \frac{\partial u^\varepsilon}{\partial t} - \left( \frac{\partial u_0}{\partial t} + \mathcal{U}^\varepsilon \left( \frac{\partial}{\partial t} \nabla_y \bar{u}_1^T \right) \right) \right\|_{L^\infty(0,T;H)} \leq c \varepsilon \bar{\varepsilon} \bar{\varepsilon} + ch_L^s + c \varepsilon^s + ch_L^s \leq c(h_L^s + \varepsilon \bar{\varepsilon} \bar{\varepsilon}).
\]

Similarly,

\[
\|\text{curl} \ u^\varepsilon - (\text{curl} \ \tilde{u}_0^L + \mathcal{U}^\varepsilon (\text{curl}_y \bar{u}_1^T))\|_{L^\infty(0,T;H)} \leq \|\text{curl} \ u^\varepsilon - \text{curl} \ u_0 - \text{curl}_y \tilde{u}_0^L\|_{L^\infty(0,T;H)} + \|\text{curl} \ u_0 - \text{curl} \ \tilde{u}_0^L\|_{L^\infty(0,T;H)} + \|\mathcal{U}^\varepsilon (\text{curl}_y \tilde{u}_0^L)\|_{L^\infty(0,T;H)} \leq c \varepsilon \bar{\varepsilon} \bar{\varepsilon} + ch_L^s + c \varepsilon^s + ch_L^s \leq c(h_L^s + \varepsilon \bar{\varepsilon} \bar{\varepsilon}).
\]
We then have the desired estimate.

The proof for the semidiscrete sparse tensor finite element solution is similar.

For fully discrete problems, we have the following results.

**Theorem 6.7** Assume that condition (4.10) and Assumption 4.1 hold, with $g_0 = 0$, $g_1 \in H^s(\text{curl}, D)$ and $\text{div} f \in L^\infty(0, T; L^2(D))$, $D$ is a Lipschitz polygonal domain ($s \in (0, 1]$ is the constant in Proposition 4.7). For the fully discrete full tensor product FE problem (5.4), assume that $\tilde{u}_1^L$ is chosen so that

$$
\left\| \partial_t p_{0,1/2}^L \right\|_{L^2(D)^3} + \left\| \partial_t \nabla_y p_{1,1/2}^L \right\|_{L^2(D \times Y)^3} + \left\| \text{curl} p_{0,1/2}^L \right\|_{L^2(D)^3} + \left\| \text{curl}_y p_{1,1/2}^L \right\|_{L^2(D \times Y)^3} \leq c((\Delta t)^2 + h_L^4),
$$

then

$$
\Delta t \max_{0 \leq m < M} \left\| \frac{\partial u^e}{\partial t}(t_m) - \partial_t \tilde{u}_0^L, m+1/2 - U^e(\partial_t \nabla_y \tilde{u}_1^L(\tau_{m+1/2})) \right\|_H + \left\| \text{curl} u^e - \text{curl} \tilde{u}_0^L - U^e(\text{curl}_y \tilde{u}_1^L) \right\|_{L^\infty(0,T;H)} \leq c \left( (\Delta t)^2 + h_L^4 + \varepsilon + \varepsilon \pi^2 \right).
$$

For the sparse tensor product FE problem (5.6), if $\tilde{u}_1^L$ is chosen so that

$$
\left\| \partial_t p_{0,1/2}^L \right\|_{L^2(D)^3} + \left\| \partial_t \nabla_y p_{1,1/2}^L \right\|_{L^2(D \times Y)^3} + \left\| \text{curl} p_{0,1/2}^L \right\|_{L^2(D)^3} + \left\| \text{curl}_y p_{1,1/2}^L \right\|_{L^2(D \times Y)^3} \leq c \left( (\Delta t)^2 + L^{1/2} h_L^4 \right),
$$

then

$$
\Delta t \max_{0 \leq m < M} \left\| \frac{\partial u^e}{\partial t}(t_m) - \partial_t \tilde{u}_0^L, m+1/2 - U^e(\partial_t \nabla_y \tilde{u}_1^L(\tau_{m+1/2})) \right\|_H + \left\| \text{curl} u^e - \text{curl} \tilde{u}_0^L - U^e(\text{curl}_y \tilde{u}_1^L) \right\|_{L^\infty(0,T;H)} \leq c \left( (\Delta t)^2 + L^{1/2} h_L^4 + \varepsilon \pi^2 \right).
$$

**Proof** From the compatibility condition $\frac{\partial u_m}{\partial t} \in L^\infty(0, T; H)$ so $\frac{\partial u_m}{\partial t} \in C([0, T]; H)$. To use the homogenization error in Theorem 6.1, we estimate

$$
\frac{1}{\Delta t} (u_{0, m+1} - u_{0, m}) - \frac{\partial u_0}{\partial t}(t_m) \leq \frac{\partial u_0}{\partial t}(\tau_m) - \frac{\partial u_0}{\partial t}(t_m) = \int_{t_m}^{\tau_m} \frac{\partial^2 u_0}{\partial \tau^2}(\sigma) d\sigma,
$$

for a value $t_m \leq \tau \leq t_{m+1}$. With the compatibility condition (4.10), we have that $\frac{\partial u_m}{\partial t} \in L^\infty(0, T; H)$. Thus

$$
\sup_{0 \leq m < M} \left\| \partial_t u_{0,m+1/2} - \frac{\partial u_0}{\partial t}(t_m) \right\|_H \leq c \Delta t.
$$

Similarly, using the smoothness of $N^r$ and $u^e$ for $r = 1, 2, 3$, we have that $\frac{\partial^2 u}{\partial \tau^2} \in L^\infty(0, T; L^2(D \times Y))$. We note that

$$
\partial_t \nabla_y u_{1, m+1/2} - \frac{\partial}{\partial \tau} \nabla_y u_1(t_m) = \nabla_y u_{1, m+1} - \nabla_y u_{1, m} \Delta t
$$

$$
\frac{\partial}{\partial \tau} \nabla_y u_1(t_m) = \int_{t_m}^{\tau_m} \frac{\partial^2}{\partial \tau^2} \nabla_y u_1(\sigma) d\sigma,
$$

for a value $t_m \leq \tau \leq t_{m+1}$. Thus

$$
\sup_{0 \leq m < M} \left\| \partial_t \nabla_y u_{1, m+1/2} - \frac{\partial}{\partial \tau} \nabla_y u_1(t_m) \right\|_{L^2(D \times Y)} \leq c \Delta t.
$$

We then get the result from Proposition 6.6 and Theorem 6.1.\qed
6.3 Numerical correctors for multiscale problems

As an explicit homogenization error is not available for the case of more than two scales, we do not distinguish the full and sparse tensor FE. We work with general FE spaces instead. For the semidiscrete problem (3.1) we have:

**Theorem 6.8** Assume that condition (1.8) holds with \( g_0 = 0 \), and \( g_1^L \) is chosen so that \( \lim_{L \to \infty} \| g_1^L - g_1 \|_H = 0 \). Then the solution of problem (3.1) satisfies

\[
\lim_{\varepsilon \to 0} \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0^L}{\partial t} - U_n^\varepsilon \left( \sum_{i=1}^n \frac{\partial}{\partial t} \nabla y_i u_i^L \right) \right\|_{L^\infty(0,T;H)}
\]

\[
+ \left\| \text{curl} u^\varepsilon - \text{curl} u_0^L - U_n^\varepsilon \left( \sum_{i=1}^n \text{curl} y_i u_i^L \right) \right\|_{L^\infty(0,T;H)} = 0.
\]

**Proof** The result is a direct consequence of Propositions 3.5 and 6.3. Indeed,

\[
\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0^L}{\partial t} - U_n^\varepsilon \left( \sum_{i=1}^n \frac{\partial}{\partial t} \nabla y_i u_i^L \right) \right\|_{L^\infty(0,T;H)}
\]

\[
\leq \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t} - U_n^\varepsilon \left( \sum_{i=1}^n \frac{\partial}{\partial t} \nabla y_i u_i \right) \right\|_{L^\infty(0,T;H)}
\]

\[
+ \left\| \frac{\partial u_0}{\partial t} - \frac{\partial u_0^L}{\partial t} \right\|_{L^\infty(0,T;H)} + \left\| U_n^\varepsilon \left( \sum_{i=1}^n \frac{\partial}{\partial t} \nabla y_i u_i \right) - U_n^\varepsilon \left( \sum_{i=1}^n \frac{\partial}{\partial t} \nabla y_i u_i^L \right) \right\|_{L^\infty(0,T;H)}.
\]

From Proposition 6.3, we deduce that

\[
\left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t} - U_n^\varepsilon \left( \sum_{i=1}^n \frac{\partial}{\partial t} \nabla y_i u_i \right) \right\|_{L^\infty(0,T;H)} \to 0
\]
as \( \varepsilon \to 0 \). From Proposition 3.5, we deduce that \( \frac{\partial u_0}{\partial t} - \frac{\partial u_0^L}{\partial t} \|_{L^\infty(0,T;H)} \to 0 \) as \( L \to \infty \). The last term

\[
\left\| U_n^\varepsilon \left( \sum_{i=1}^n \frac{\partial}{\partial t} \nabla y_i u_i \right) - U_n^\varepsilon \left( \sum_{i=1}^n \frac{\partial}{\partial t} \nabla y_i u_i^L \right) \right\|_{L^\infty(0,T;H)} \leq \left\| \sum_{i=1}^n \frac{\partial}{\partial t} \nabla y_i u_i - \sum_{i=1}^n \frac{\partial}{\partial t} \nabla y_i u_i^L \right\|_{L^\infty(0,T;H)} \to 0
\]
as \( L \to \infty \). Thus

\[
\lim_{L \to \infty} \left\| \frac{\partial u^\varepsilon}{\partial t} - \frac{\partial u_0}{\partial t} - U_n^\varepsilon \left( \sum_{i=1}^n \frac{\partial}{\partial t} \nabla y_i u_i^L \right) \right\|_{L^\infty(0,T;H)} = 0.
\]

Similarly, we have

\[
\left\| \text{curl} u^\varepsilon - \text{curl} u_0^L - U_n^\varepsilon \left( \sum_{i=1}^n \text{curl} y_i u_i^L \right) \right\|_{L^\infty(0,T;H)}
\]

\[
\leq \left\| \text{curl} u^\varepsilon - \text{curl} u_0 - U_n^\varepsilon \left( \sum_{i=1}^n \text{curl} y_i u_i \right) \right\|_{L^\infty(0,T;H)} + \left\| \text{curl} u_0 - \text{curl} u_0^L \right\|_{L^\infty(0,T;H)}
\]

\[
+ \left\| U_n^\varepsilon \left( \sum_{i=1}^n \text{curl} y_i u_i \right) - U_n^\varepsilon \left( \sum_{i=1}^n \text{curl} y_i u_i^L \right) \right\|_{L^\infty(0,T;H)}
\]

which tends to 0 as \( L \to \infty \) and \( \varepsilon \to 0 \). We then get the conclusion. □

For the fully discrete problem (5.3) we have:
\textbf{Theorem 6.9} Assume that condition (4.8) holds with \(g_0 = 0\), \(u^L_1\) is chosen such that

\[
\lim_{L \to \infty} \|\partial_t p^L_{0,1/2}\|_H + \sum_{i=1}^{n} \|\partial_t \nabla_{y_i} p^L_{i,1/2}\|_H + \|\text{curl}\ p^L_{0,1/2}\|_H + \sum_{i=1}^{n} \|\text{curl}_{y_i} p^L_{i,1/2}\|_H = 0.
\]

Then

\[
\lim_{L \to \infty} \sup_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t u^L_{0,m+1/2} - U_n^\varepsilon \left( \sum_{i=1}^{n} \partial_t \nabla_{y_i} u^L_{i,m+1/2} \right) \right\|_H + \|\text{curl}^\varepsilon - \text{curl} u^L_0 - U_n^\varepsilon \left( \sum_{i=1}^{n} \text{curl}_{y_i} u^L_i \right) \right\|_{L^\infty(0,T;H_i)} = 0.
\]

\textbf{Proof} We have

\[
\left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \partial_t u^L_{0,m+1/2} - U_n^\varepsilon \left( \sum_{i=1}^{n} \partial_t \nabla_{y_i} u^L_{i,m+1/2} \right) \right\|_H 
\leq \left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \frac{\partial u^0}{\partial t}(t_m) - U_n^\varepsilon \left( \sum_{i=1}^{n} \partial_t \nabla_{y_i} u^L_{i,m+1/2} \right) \right\|_H + \left\| \frac{\partial u^0}{\partial t}(t_m) - \partial_t u^L_{0,m+1/2} \right\|_H 
+ \left\| U_n^\varepsilon \left( \sum_{i=1}^{n} \partial_t \nabla_{y_i} u^L_{i,m+1/2} \right) - U_n^\varepsilon \left( \sum_{i=1}^{n} \partial_t \nabla_{y_i} u^L_{i,m+1/2} \right) \right\|_H 
+ \left\| \partial_t u^L_{0,m+1/2} - \partial_t u^L_{0,m+1/2} \right\|_H.
\]

From Proposition 6.3 we deduce that

\[
\left\| \frac{\partial u^\varepsilon}{\partial t}(t_m) - \frac{\partial u^0}{\partial t}(t_m) - U_n^\varepsilon \left( \sum_{i=1}^{n} \partial_t \nabla_{y_i} \partial_t (t_m) \right) \right\|_H \to 0
\]
as \(\varepsilon \to 0\). As \(\frac{\partial^2 u^0}{\partial t^2} \in L^\infty(0,T;H),\)

\[
\lim_{\Delta t \to 0} \sup_{0 \leq m < M} \left\| \partial_t u^0_{0,m+1/2} - \partial_t u^L_{0,m+1/2} \right\|_H = 0;
\]

and from (4.7) we have \(\frac{\partial^2 u^0}{\partial t^2} \nabla_{y_i} u_t \in L^\infty(0,T;H),\)

\[
\lim_{\Delta t \to 0} \sup_{0 \leq m < M} \left\| \partial_t \nabla_{y_i} u^L_{0,m+1/2} - \partial_t \nabla_{y_i} u^L_{0,m+1/2} \right\|_H = 0.
\]

Thus

\[
\sup_{0 \leq m < M} \left\| U_n^\varepsilon \left( \sum_{i=1}^{n} \partial_t \nabla_{y_i} \partial_t (t_m) \right) - U_n^\varepsilon \left( \sum_{i=1}^{n} \partial_t \nabla_{y_i} u^L_{i,m+1/2} \right) \right\|_H 
\leq \sup_{0 \leq m < M} \left\| \sum_{i=1}^{n} \left( \partial_t \nabla_{y_i} (t_m) - \partial_t \nabla_{y_i} u^L_{i,m+1/2} \right) \right\|_{L^2(D \times Y)} \to 0
\]
as \(\Delta t \to 0\). From the FE convergence, we have

\[
\sup_{0 \leq m < M} \left\| \partial_t u^0_{0,m+1/2} - \partial_t u^L_{0,m+1/2} \right\|_H \to 0
\]

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as \( L \to \infty, \Delta t \to 0 \), and
\[
\sup_{0 \leq m < M} \left\| U_n^n \left( \sum_{i=1}^n \partial_i \nabla y_i u_{i,m+1/2}^L \right) - U_n^n \left( \sum_{i=1}^n \partial_i \nabla y_i u_{i,m+1/2}^L \right) \right\|_H \\
\leq \sup_{0 \leq m < M} \left\| \sum_{i=1}^n \left( \partial_i \nabla y_i u_{i,m+1/2}^L - \nabla y_i u_{i,m+1/2}^L \right) \right\|_{L^2(D \times Y)^3} \to 0
\]
as \( L \to \infty, \Delta t \to 0 \). Thus
\[
\lim_{\Delta t \to 0, \varepsilon \to 0} \sup_{0 \leq m < M} \left\| \frac{\partial u^\varepsilon}{\partial t} (t_m) - \partial_t u^L_{0,m+1/2} - U_n^n \left( \sum_{i=1}^n \partial_i \nabla y_i u_{i,m+1/2}^L \right) \right\|_H = 0.
\]
Using similar argument, we have
\[
\lim_{\Delta t \to 0, \varepsilon \to 0} \left\| \text{curl} u^\varepsilon - \text{curl} u^L_0 - U_n^n \left( \sum_{i=1}^n \text{curl}_y u_{i}^L \right) \right\|_{\tilde{L}^2(0,T;\tilde{H})} = 0.
\]
We then get the conclusion. \( \square \)

## 7 Numerical results

We present in this section some numerical examples for two scale problems that confirm our analysis.

To identify the detailed spaces defined in Subsection 6.2, we employ Riesz basis and define the equivalent norms in the spaces \( L^2(D) \) and \( L^2(Y) \). The Riesz basis functions satisfy:

**Assumption 7.1** (i) For all vectors \( j \in \mathbb{N}_0^d \), there exists an index set \( I^j \subset \mathbb{N}_0^d \) and a set of basis functions \( \phi_{jk}^i \in L^2(D) \) for \( k \in I^j \), such that \( V^j = \text{span} \{ \phi_{jk}^i : |j| \leq l \} \). For all \( \phi = \sum_{|j| \leq l, k \in I^j} \phi_{jk}^i c_{jk} \in V^j \)
\[
c_1 \sum_{|j| \leq l} |c_{jk}|^2 \leq \| \phi \|^2_{L^2(D)} \leq c_2 \sum_{|j| \leq l} |c_{jk}|^2,
\]
where \( c_1 > 0 \) and \( c_2 > 0 \) are independent of \( \phi \) and \( l \).

(ii) For the space \( L^2(Y) \), for all \( j \in \mathbb{N}_0^d \), there exists an index set \( I^j \subset \mathbb{N}_0^d \) and a set of periodic basis functions \( \phi_{jk}^i \in L^2(Y) \), \( k \in I^j \), such that \( V^j_y = \text{span} \{ \phi_{jk}^i : |j| \leq l \} \). For all \( \phi = \sum_{|j| \leq l, k \in I^j} \phi_{jk}^i c_{jk} \in V^j_y \)
\[
c_3 \sum_{|j| \leq l} |c_{jk}|^2 \leq \| \phi \|^2_{L^2(Y)} \leq c_4 \sum_{|j| \leq l} |c_{jk}|^2
\]
where \( c_3 > 0 \) and \( c_4 > 0 \) are independent of \( \phi \) and \( l \).

With respect to the norm equivalence, we define the detailed spaces as \( V^j = \text{span} \{ \phi_{jk}^i : |j| = l \} \) and \( V^j_y = \text{span} \{ \phi_{jk}^i : |j| = l \} \).

**Example** (i) For the space \( L^2(0,1) \), a Riesz basis can be constructed as follows. Level 0 contains three piecewise linear basis functions: \( \psi^{01} \) obtains values \( (0,1) \) at \( (0,1/2) \) and is 0 in \( (1/2,1) \), \( \psi^{02} \) obtains values \( (0,1,0) \) at \( (0,1/2) \), and \( \psi^{03} \) obtains values \( (0,1) \) at \( (1/2,1) \) and is 0 in \( (0,1/2) \). For other levels, the basis functions are constructed from the function \( \psi \) that takes values \( (0,-1,2,-1,0) \) at \( (0,1/2,1,3/2,2) \), the left boundary function \( \psi^{left} \) taking values \( (-2,-2,-1,0) \) at \( (0,1/2,1,3/2,2) \), and the right boundary function \( \psi^{right} \) taking values \( (0,-1,2,-2) \) at \( (1/2,1,3/2,2) \). For levels \( j \geq 1 \) with \( I^j = \{ 1,2,\ldots,j \} \), the basis functions are \( \psi^{1x}(x) = 2^{j/2} \psi^{left}(2^j x) \), \( \psi^{2x}(x) = 2^{j/2} \psi(2^j x - k + 3/2) \) for \( k = 0,1,\ldots,k-1 \) and \( \psi^{j+1} = 2^{j/2} \psi^{right}(2^j x - 2^j + 2) \). This basis satisfies Assumption 7.1 (i).
(ii) For $Y = (0, 1)$, a periodic Riesz basis for $L^2(Y)$ can be constructed by modifying the basis in (i). Level 0 contains the periodic piecewise linear function that takes values $(1, 0, 1)$ at $(0, 1/2, 1)$ respectively. At other levels, the functions $\psi^{left}$ and $\psi^{right}$ are replaced by the piecewise linear functions that take values $(0, 2, -1, 0)$ at $(0, 1/2, 1, 3/2)$ and values $(0, -1, 2, 0)$ at $(1/2, 1, 3/2, 2)$ respectively.

A Riesz basis for the space $L^2((0, 1)^d)$ can be constructed by taking the tensor products of the basis functions in $(0, 1)$ with an appropriate scaling, see [16].

Remark 7.2 We note that the norm equivalences above are not necessary for the approximations in Lemma 5.4 to hold, as explained in [19] and [8].

In the first example, we consider a two scale Maxwell wave equation in the two dimension domain $D = (0, 1)^2$.

$$b^\varepsilon \frac{\partial^2 u^\varepsilon}{\partial t^2} + \text{curl}(a^\varepsilon \text{curl } u^\varepsilon) = f(t, x), \quad \text{in } D$$

$$u^\varepsilon(t, \cdot) \times \nu = 0, \quad \text{on } \partial D$$

$$u^\varepsilon(0, x) = 0$$

$$u^\varepsilon_t(0, x) = 0$$

The coefficients are

$$a(x, y) = \frac{1}{(1 + x_1)(1 + x_2)(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)},$$

and

$$b(x, y) = \frac{(1 + x_1)(1 + x_2)}{(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)}.$$

The exact homogenized coefficients are

$$a^0(x) = \frac{4}{9(1 + x_1)(1 + x_2)}$$

and

$$b^0(x) = \frac{\sqrt{2}(1 + x_1)(1 + x_2)}{3}.$$

We choose

$$f(x_1, x_2) = \begin{pmatrix} 2\sqrt{2}(1 + x_1)(1 + x_2)x_1x_2(1 - x_2)t + \frac{4t^3}{9(1 + x_2)^2} \\ 2\sqrt{2}(1 + x_1)(1 + x_2)x_1x_2(1 - x_1)t + \frac{4t^3}{9(1 + x_1)^2} \end{pmatrix}$$

so that the solution to the homogenized equation is

$$u_0 = \begin{pmatrix} x_1x_2(1 - x_2)t^3 \\ x_1x_2(1 - x_1)t^3 \end{pmatrix}.$$ 

From the relation (4.6), we compute the solution curl $u_1$ exactly as

$$\text{curl } u_1 = \left( \frac{4(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)}{9} - 1 \right) (x_2 - x_1)t^3.$$

In Figure 1 we plot the errors $||u_0 - u_1||_{H^1(D)}$ and $||\text{curl } u_1 - \text{curl } u_1^\varepsilon||_{L^2(D)^3}$ versus the mesh size for the sparse tensor product FEs for $(\Delta t, h) = (1/4, 1/4), (1/6, 1/8), (1/8, 1/12)$ and $(1/16, 1/32)$. The result confirm our analysis.

In the second example, we choose

$$a(x, y) = \frac{(1 + x_1)(1 + x_2)}{(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)}.$$
and
\[ b(x,y) = \frac{1}{(1 + x_1)(1 + x_2)(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)}. \]

In this case, the homogenized coefficients are
\[ a^0(x) = \frac{4(1 + x_1)(1 + x_2)}{9} \]
and
\[ b^0(x) = \frac{\sqrt{2}}{3(1 + x_1)(1 + x_2)}. \]

We choose
\[ f(x_1, x_2) = \left( \frac{2\sqrt{2}x_2(1 - x_2)}{1 + x_2} + \frac{3}{4t^3(1 + x_1)(2x_2 - x_1 + 1)} \right) \left( \frac{2\sqrt{2}x_1(1 - x_1)}{1 + x_1} + \frac{3}{4t^3(1 + x_1)(2x_1 - x_2 + 1)} \right) \]
so that the solution to the homogenized problem is
\[ u_0 = \frac{(1 + x_1)x_2(1 - x_2)t^3}{(1 + x_2)x_1(1 - x_1)t^3}. \]
and
\[ \text{curl } u_1 = \left( \frac{4(1 + \cos^2 2\pi y_1)(1 + \cos^2 2\pi y_2)}{3} - 1 \right)(x_2 - x_1)t^3. \]

In Figure 1 we plot the errors \( \| u_0 - u_0^t \|_{H(\text{curl}, D)} \) and \( \| \text{curl } u_1 - \text{curl } u_1^t \|_{L^2(D)} \) versus the mesh size for the sparse tensor product FEs for \((\Delta t, h) = (1/4, 1/4), (1/6, 1/4), (1/8, 1/12)\) and \((1/16, 1/32)\). The result once more confirms our analysis.

Acknowledgement The authors gratefully acknowledge a postgraduate scholarship of Nanyang Technological University, the AcRF Tier 1 grant RG30/16, the Singapore A*Star SERC grant 122-PSF-0007 and the AcRF Tier 2 grant MOE 2013-T2-1-095 ARC 44/13.
Figure 2: The sparse tensor errors $\|u_0 - u_0^h\|_{H(\text{curl},D)}$ and $\|\text{curl} \ u_1 - \text{curl} \ u_1^h\|_{L^2(D)}$. 

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