The Bañados–Silk–West Effect with Immovable Particles Near Static Black Holes and Its Rotational Counterpart

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Abstract—The BSW effect implies that the energy \( E_{\text{c.m.}} \) in the center of mass frame of two particles colliding near a black hole can become unbounded. Usually, it is assumed that the particles move along geodesics or electrogeodesics. Instead, we consider another version of this effect. One particle is situated at rest near a static, generally speaking, distorted black hole. If another particle (say, coming from infinity) collides with it, the collision energy \( E_{\text{c.m.}} \) in the center of mass frame grows unboundedly (the BSW effect). The force required to keep such a particle near a black hole diverges for nonextremal horizons but remains finite and nonzero for an extremal one and vanishes in the horizon limit for ultraextremal black holes. A generalization to the rotating case implies that a particle corotates with the black hole but does not have a radial velocity. At that, the energy \( E \to 0 \), provided the angular momentum \( L \) is zero. This condition replaces that of fine tuning of the parameters in the standard version of the BSW effect.

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1. INTRODUCTION

High-energy particle collisions near black holes attract much attention after the findings of Bañados, Silk, and West who showed that a collision of freely falling particles in the background of an extremal rotating black hole can produce an indefinitely large energy \( E_{\text{c.m.}} \) in the center of mass frame \([1]\) (the BSW effect, after the names of its authors). This requires one of particles to be fine-tuned in such a way that the Killing energy and angular momentum obey the relation

\[
X \equiv E - \omega_H L = 0, \quad (1)
\]

\( \omega_H \) being the angular velocity of a black hole (it is a so-called critical particle). Such a critical particle collides with a usual (not fine-tuned) one to produce an unbounded growth of \( E_{\text{c.m.}} \).

This is what can be called the basic scenario of the BSW effect. Within this type of scenario, the BSW effect is impossible for static black holes since \( \omega_H = 0 \) for them, and the equality \( (1) \) fails. Meanwhile, there exists another, special type, auxiliary to the basic one. Within this special type, the BSW could be possible, in principle, even near static black holes, say, the Schwarzschild one. Let \( L = 0 \). Then, Eq. (1) requires that \( E = 0 \), or \( E \) should be arbitrarily close to zero. This special type of scenario can be decomposed into the following subcases. (i) A particle with \( E \approx 0 \) moves freely and collides with a usual particle in the vicinity of a Schwarzschild black hole \([2]\). (ii) A particle with \( E \approx 0 \) is kept fixed near the Schwarzschild horizon and collides with a usual particle falling from infinity \([3]\) (see a discussion on p. 3864 before Eq. (35) there). However, each of the two subcases encounters serious difficulties that prevent it from physical realization. As far as Subcase (i) is concerned, it was shown later that such a particle cannot come from infinity and, moreover, it cannot be created near a black hole as a result of previous collisions between particles falling from infinity \([4]\). In Subcase (ii) there is a difficulty of another kind: a force required to keep a particle near the horizon grows indefinitely.

The aim of the present work is to show that there is one more type of special scenario free from the aforementioned difficulties. In this sense, we fill the gap in the classification of BSW scenarios left before, and bring this issue to completion. We also demonstrate an analogue of the special type scenario for rotating black holes. Thus we widen the class of objects and scenarios for which high energy collisions are possible.

There is one more aspect of the issue under discussion. Some other cursory notes were made in the literature on the BSW effect concerning the states with \( E = 0 \). The criticality condition with \( E = 0 \) was considered in Sec. II of \([5]\) for \( 2 + 1 \) black holes with a cosmological constant \([6]\). It appeared as...
a limit of the angular momentum \( L \to 0 \) in a more general condition (2.15) there. The corresponding space-time is not asymptotically flat, but near the horizon the same features manifest themselves, so \( E_{\text{c.m.}} \) grows unboundedly as \( E \to 0 \) for one of two particles. Earlier, it was also pointed out (see, e.g., [8], page 4) that the values \( E = 0 \) and \( L = 0 \) are possible but these special cases were mentioned in the context where freely moving particles were considered. Meanwhile, we show that these cases are relevant for nongeodesic particles which are kept in equilibrium by some force.

2. COLLISIONS: BASIC FORMULAS

Let two particles 1 and 2 collide at some point. Then, the energy \( E_{\text{c.m.}} \) in the center of mass frame is defined according to

\[
E_{\text{c.m.}}^2 = -(m_1 u_1^\mu + m_2 u_2^\mu)(m_1 u_1_\mu + m_2 u_2_\mu) = m_1^2 + m_2^2 + 2m_1m_2\gamma.
\]

(2)

Here, \( m_i \) are masses, \( u_i^\mu \) are four-velocities, \( i = 1, 2 \),

\[
\gamma = -u_1_\mu u_2^\mu \quad (3)
\]
is the Lorentz factor of relative motion.

3. THE BEHAVIOR OF ACCELERATION. A STATIC CASE

For what follows we will need to know the behavior of the acceleration near the horizon. Our main goal is to show that a finite acceleration can be reconciled with the BSW effect. To this end, we enumerate all possible variants. In doing so, we extend the formulas well known for spherically symmetric cases (see, e.g., [10] for the Schwarzschild metric) to more general metrics and other types of horizons.

Let us consider the gravitational field described by a static black hole metric \( g_{\mu\nu} \). We assume no spatial symmetry. It can be represented (at least in some finite region) in the form

\[
d s^2 = -N^2 dt^2 + dn^2 + \gamma_{ab} dx^a dx^b, \quad (4)
\]

\( a, b = 1, 2 \). The lapse function \( N = 0 \) on the horizon. Here, the quantity \( n \) measures the proper distance to the horizon. Thus it is, in general, a distorted black hole.

Let a particle be at rest, having \( n = n_0 = \text{const} \), \( x^a = \text{const} \). Then, it follows from the normalization condition \( u_\mu u^\mu = -1 \) for the four-velocity that \( u^0 = 1/N \), \( u_0 = -N \). It is easy to calculate the components of the acceleration \( a^\mu \): \( a^0 = 0 \), and

\[
a^n = \Gamma^n_{00} u^0 = \frac{1}{N} \frac{\partial N}{\partial n},
\]

(5)

where \( \gamma_{ab} \) is the tensor inverse to \( \gamma_{\mu\nu} \).

Now, we consider three cases separately, depending on the near-horizon behavior of the metric. In particular, the crucial role is played here by the surface gravity \( \kappa \) defined according to

\[
\kappa = \lim_{N \to 0} \sqrt{\left(\nabla N\right)^2};
\]

(7)

where \((\nabla N)^2 = g^{\mu\nu} N_{\mu} N_{\nu}\).

3.1. A Nonextremal Black Hole

For a nonextremal black hole, the regularity conditions near the horizon require [11]

\[
N = \kappa n + b(x^a)n^3 + o(n^3).
\]

(8)

We see that in the horizon limit, as \( n \to 0 \), \( N \to 0 \),

\[
a^n \approx \frac{1}{n}, \quad a^0 = O(n^2)
\]

(9)

\[
a^a = a_\mu a^\mu \approx \frac{1}{n^2},
\]

(10)

so \( a^2 \) diverges. This is a direct generalization of the situation in the spherically symmetric static gravitational field (see Eq. 2.2.6 in [10]).

3.2. An Extremal Black Hole

By definition, this means that near the horizon

\[
N \approx N_0 \exp(-cn),
\]

(11)

where \( N_0 \) and \( c \) are some constants [12],

\[
\frac{\partial \gamma_{ab}}{\partial n} = O \exp(-cn).
\]

(12)

The horizon limit \( N \to 0 \) is achieved when \( n \to \infty \). As a result, \( a^2 \to c^2 \) and remains bounded and nonzero.

3.3. An Ultraextremal Black Hole

By definition, the horizon is called ultraextremal if in the horizon limit \( n \to \infty \),

\[
N \approx \frac{N_0}{n^s}
\]

(13)

with \( s > 0 \), \( N_0 > 0 \) being some constant. This definition becomes clearer in the particular case of a spherically symmetric metric

\[
ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

(14)

where \( f = f(r) \). If

\[
f \approx f_0 (r - r_+)^k,
\]

(15)
where \( r_+ \) is the horizon radius, \( f_0 \) is constant, \( k = 3, 4, 5, \ldots \) Comparing (13) and (15), we see that in this case \( s = \frac{k}{k-2} \). The regularity conditions for metrics of this type were studied in [11, 12]. Now,

\[
a \approx \frac{s}{n} \to 0. \quad (16)
\]

In all three cases, the energy per unit mass \( \varepsilon = \frac{E}{m} \) is equal to

\[
\varepsilon = -u_0 = N. \quad (17)
\]

We see that in the horizon limit \( \varepsilon \to 0 \), thus realizing the special case of the critical condition (1) for static black holes (\( \omega_H = 0 \)).

4. THE BEHAVIOR OF ACCELERATION. A STATIONARY CASE

Let us consider a metric of a stationary axially symmetric black hole. It can be written in the form

\[
ds^2 = -N^2 dt^2 + g_{\phi}(d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_{\theta\theta} \theta^2, \quad (18)
\]

where all metric coefficients do not depend on \( t \) and \( \phi \). Let a particle move over a circle with \( u^r = 0 = u^\theta \),

\[
u^\mu = u^t(1, \Omega, 0, 0), \quad (19)
\]

where

\[
\Omega = \frac{d\phi}{dt}. \quad (20)
\]

From the condition that a trajectory cannot be space-like, \( ds^2 \leq 0 \), we obtain

\[
\omega_- \leq \Omega \leq \omega_+, \quad (21)
\]

where

\[
\omega_\pm = \omega \pm \frac{N}{\sqrt{g_{\phi}}}, \quad (22)
\]

\[
\Omega = \omega + \frac{N}{\sqrt{g_{\phi}}} \alpha, \quad (23)
\]

where \( |\alpha| \leq 1 \). It follows from \( g_{\mu\nu} u^\mu u^\nu = -1 \) that

\[
\nu^t = \frac{1}{\sqrt{N^2 - g_{\phi}(\omega - \Omega)^2}} = \frac{1}{N \sqrt{1 - \alpha^2}}, \quad (24)
\]

\[
\nu^\phi = \Omega u^t = \frac{\Omega}{N \sqrt{1 - \alpha^2}} = \frac{\omega + (N/\sqrt{g_{\phi}}) \alpha}{N \sqrt{1 - \alpha^2}}. \quad (25)
\]

We also have for the angular momentum \( L \) and energy \( E \):

\[
\mathcal{L} \equiv \frac{L}{m} = u_\phi = g_{\phi} \frac{(\Omega - \omega)}{N \sqrt{1 - \alpha^2}} = \frac{\sqrt{g_{\phi}} \alpha}{\sqrt{1 - \alpha^2}}, \quad (26)
\]

\[
\varepsilon = \frac{E}{m} = -u_t = \frac{1}{\sqrt{1 - \alpha^2}}(\omega \sqrt{g_{\phi}} \alpha + N), \quad (27)
\]

\[
\frac{X}{m} \equiv \mathcal{X} = \varepsilon - \omega \mathcal{L} = \frac{N}{\sqrt{1 - \alpha^2}}. \quad (28)
\]

For simplicity, we choose \( \alpha = 0 \). This corresponds to a so-called zero-angular momentum observer (ZAMO), \( L = 0 \) [14].

In the horizon limit, rotation is inevitable there due to a strong frame-dragging effect. Now, instead of being at rest, a particle should corotate with the black hole. Then, \( X \to 0 \), so this is a so-called critical particle according to the standard nomenclature [13].

4.1. Particles Near Rotating Black Holes

Let us consider a collision between particle 1 rotating around a black hole as described above and particle 2 moving freely. For simplicity, we choose particle 2 to move within the equatorial plane.

Then, for particle 2 we have from the equations of motion

\[
u_2^\mu = \left( \frac{X_2}{N^2}, \frac{L}{g_{\phi}} + \frac{\omega X_2}{N^2}, -\sqrt{\frac{A}{N^2}} \sqrt{X_2^2 - N^2(1 + \frac{L^2}{g_{\phi}})}, 0 \right). \quad (29)
\]

Then, we obtain from (3)

\[
\gamma = -u_2^\mu (u_2^\mu)_1 = \frac{X_2}{N}. \quad (30)
\]

In the limit \( N \to 0 \) we see that \( \gamma \sim \frac{1}{N} \to \infty \). Thus the BSW effect does occur.

Now, we calculate the components of the acceleration. For particle 1 with \( L = 0 \) one can find easily \( a^t = 0 \),

\[
a^r = \frac{A}{N} \partial_r N, \quad (31)
\]

where (24) with \( \alpha = 0 \) was used. Then,

\[
a^2 = \frac{A}{N^2} (\partial_r N)^2. \quad (32)
\]

Near the horizon, we assume \( A \sim N^2 \). It is an additional assumption, but it holds for most of the metrics of physical interest. In particular, it is true for the Kerr and Kerr–Newman metrics. (As one can make rescaling of the radial coordinate, this can be violated far from the horizon but we are interested in the vicinity of the horizon only.) In the static limit it holds for the Schwarzschild and Reissner–Nordström metrics everywhere. For nonextremal black holes \( N \sim \sqrt{r-r_+} \), for extremal black holes \( N \sim r-r_+ \), and for ultraextremal ones \( N \sim (r-r_+)^p \), with \( p > 1 \). Using the general formula (7), it is easy to confirm...
that $\kappa \neq 0$ in the first case, and $\kappa = 0$ in the second and third cases.

Thus in the horizon limit $\kappa^2$ diverges in the nonextremal case, is finite and nonzero in the extremal case, and vanishes in the ultraextremal one.

5. A LIMITING TRANSITION AND FAKE ORBITS ON THE HORIZON

Thus we have seen that for extremal and ultraextremal horizons one can keep a particle as close to the horizon as one likes. In the first case the force remains finite and nonzero, in the second case it tends to zero. It is natural to ask: is it possible to bring the limiting procedure to completion and keep a particle on the horizon itself? The answer is negative. At that, the fact that trajectories of particles on the horizon are fake, reveals itself. This is because a problem is connected not only with the dynamics but also with kinematics. As a horizon is a lightlike surface, timelike trajectories of massive particles are impossible on it. Although formally one can put $r = r_+$, the corresponding trajectory is forbidden. The situation is already described in detail in Section III C of [7] (see also a brief discussion after Eq. 4.2 in [8]) and Section VI of [9].

6. DISCUSSION AND CONCLUSIONS

Thus we have shown that, indeed, there is a scenario in which (i) a particle remains in rest near the horizon (in the static case) or corotates with a black hole (stationary case), (ii) as a consequence, its energy is zero in this limit, (iii) the acceleration experienced by a particle is finite and nonzero for an extremal black hole or zero for an ultraextremal one, and (iv) its collision with an infalling particle leads to an indefinite growth of the energy in the center of mass frame.

It was explained earlier that, kinematically, the BSW effect arises due to a collision between a slow target and a rapid particle that hits it [15]. Now, this circumstance reaches its ultimate form for static black holes since one of the particles does not move before collision at all.

It is worth noting that the role of a particle can also be played by a macroscopic body, being slowly lowered by a rope towards a black hole. Models of such a kind are used from time to time in black hole physics. In particular, they were found to be instructive in the discussion of Gedankenexperiments with thermal radiation (see pp. 479–489 in [10] and references therein). Now, as we see, this model appeared in quite a different context.

There is one more aspect. A deviation from geodesic motion was pointed out as one of potential restrictions acting against the BSW effect [16]. Meanwhile, the present work clearly shows that, by itself, the presence of a force can be compatible with the BSW effect, thus realizing, as a simple example, a general scheme suggested in [17].

The scenarios studied in the present work can be considered not only on their own right but also as a qualitative approximation for more complicated processes in the near-horizon plasma, when particles move, say, under the action of the electromagnetic force, being also bombarded by colliding particles.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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