A STACKY APPROACH TO CRYSTALS

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To Professor C. N. Yang with deepest admiration

Abstract. Inspired by a theorem of Bhatt-Morrow-Scholze, we develop a stacky approach to crystals and isocrystals on “Frobenius-smooth” schemes over $\mathbb{F}_p$. This class of schemes goes back to Berthelot-Messing and contains all smooth schemes over perfect fields of characteristic $p$.

To treat isocrystals, we prove some descent theorems for sheaves of Banachian modules, which could be interesting in their own right.

1. Introduction

Fix a prime $p$.

1.1. A theorem of Bhatt-Morrow-Scholze. Let $X$ be a smooth scheme over a perfect field $k$ of characteristic $p$. Let $X_{\text{perf}}$ be the perfection of $X$, i.e.,

$$X_{\text{perf}} := \lim (\ldots \xrightarrow{\text{Fr}} X \xrightarrow{\text{Fr}} X \xrightarrow{\text{Fr}} X).$$

Let $W(X_{\text{perf}})$ be the $p$-adic formal scheme whose underlying topological space is that of $X_{\text{perf}}$ (or of $X$) and whose structure sheaf is obtained by applying to $\mathcal{O}_{X_{\text{perf}}}$ the functor of $p$-typical Witt vectors. Theorem 1.10 of [BMS] can be reformulated as a canonical isomorphism

$$R\Gamma_{\text{cris}}(X, \mathcal{O}) = R\Gamma(W(X_{\text{perf}})/\mathcal{G}, \mathcal{O}),$$

where $\mathcal{G}$ is a certain flat affine groupoid (in the category of $p$-adic formal schemes) acting on $W(X_{\text{perf}})$, and $W(X_{\text{perf}})/\mathcal{G}$ is the quotient stack. If $X$ is affine this gives an explicit complex computing $R\Gamma_{\text{cris}}(X, \mathcal{O})$ (the complex is constructed using the nerve of $\mathcal{G}$).

One can define $\mathcal{G}$ to be the unique groupoid acting on $W(X_{\text{perf}})$ such that the morphism $\mathcal{G} \to W(X_{\text{perf}}) \times W(X_{\text{perf}})$ is obtained by taking the divided power envelope of the ideal of the closed subscheme

$$X_{\text{perf}} \times_X X_{\text{perf}} \subset W(X_{\text{perf}}) \times W(X_{\text{perf}}).$$

We prefer to define the groupoid $\mathcal{G}$ by describing its nerve $\mathcal{A}_\bullet$ using Fontaine’s functor $A_{\text{cris}}$ (see [2.2.3 in which we follow [BMS]).

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The stacks $(W(X_{\text{perf}})/\mathcal{G}) \otimes_{\mathbb{Z}_p} (\mathbb{Z}/p^n\mathbb{Z})$ are usually not Artin stacks because the two canonical morphisms $\mathcal{G} \to W(X_{\text{perf}})$ usually have infinite type.
1.2. **A generalization.** Our first main result is Theorem 2.4.2. It says that if $X$ is as above then a crystal of quasi-coherent $\mathcal{O}$-modules on the absolute crystalline site of $X$ is the same as an $\mathcal{O}$-module $M$ on the stack $W(X_{\text{perf}})/\mathcal{G}$, and the complex $R\Gamma(W(X_{\text{perf}})/\mathcal{G}, M)$ identifies with the cohomology of the corresponding crystal.

We think of crystalline cohomology not in terms of the de Rham complex but in terms of the bigger and more tautological Čech-Alexander complex. This approach makes Theorem 2.4.2 almost obvious.

Following Berthelot-Messing [BM, §1], we allow $X$ to be any $\mathbb{F}_p$-scheme $X$ which is **Frobenius-smooth** in the sense of §2.1. E.g., if $k$ is a perfect $\mathbb{F}_p$-algebra then Spec $k[[x_1, \ldots, x_n]]$ or any smooth scheme over Spec $k$ is allowed.

Remark 1.2.1. As explained in Appendix A, for any Frobenius-smooth $\mathbb{F}_p$-scheme $X$, the stack $W(X_{\text{perf}})/G$ canonically identifies with the prismatization of $X$. The notion of prismatization is not used in the main body of this article (in fact, it did not exist when the original version of the article was written).

1.3. **Isocrystals.** Now let $X$ be a Noetherian $\mathbb{F}_p$-scheme. For such schemes Frobenius-smoothness is equivalent to $\Omega^1_X$ being locally free and coherent (see §3.1); let us assume this. E.g., if $k$ is a perfect field of characteristic $p$ then Spec $k[[x_1, \ldots, x_n]]$ or any smooth scheme over Spec $k$ is allowed.

1.3.1. **What we mean by an isocrystal.** Consider the category of crystals of finitely generated quasi-coherent $\mathcal{O}$-modules on the absolute crystalline site of $X$. Tensoring it by $\mathbb{Q}$, one gets a category denoted by $\text{Isoc}(X)$, whose objects are called **isocrystals**. (Thus our isocrystals are not necessarily convergent in the sense of [O].)

1.3.2. **The result on isocrystals.** Our second main result (Theorem 3.5.6) provides a canonical equivalence

$$\text{Isoc}(X) \xrightarrow{\sim} \text{Bun}_\mathbb{Q}(W(X_{\text{perf}})/\mathcal{G}),$$

where the right-hand side is, so to say, the category of vector bundles on $(W(X_{\text{perf}})/\mathcal{G})\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (more details are explained in §1.3.3 below).

1.3.3. **“Banachian games” and $\text{Bun}_\mathbb{Q}$**. For any $\mathbb{Z}_p$-flat $p$-adic formal scheme $\mathcal{Y}$, let $\text{Bun}_\mathbb{Q}(\mathcal{Y})$ denote the category of finitely generated locally projective $(\mathcal{O}_\mathcal{Y} \otimes \mathbb{Q})$-modules. We prove that the assignment $\mathcal{Y} \mapsto \text{Bun}_\mathbb{Q}(\mathcal{Y})$ satisfies fpqc descent (see Proposition 3.5.4). Because of this, the meaning of $\text{Bun}_\mathbb{Q}(W(X_{\text{perf}})/\mathcal{G})$ is clear.

More generally, in §3.6 we prove fpqc descent for the category of sheaves of Banachian $(\mathcal{O}_\mathcal{Y} \otimes \mathbb{Q})$-modules, denoted by $\text{QCoh}^b(\mathcal{Y}) \otimes \mathbb{Q}$ (see Corollary 3.6.6).

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2See §2.3.1 for the precise meaning of these words. Note that $\mathcal{Y}$ does not have to be Noetherian.

3By a Banachian space over $\mathbb{Q}_p$ we mean a complete topological vector space such that the topology comes from a non-Archimedean norm.
2. Crystals and crystalline cohomology

2.1. A class of schemes.

**Lemma 2.1.1.** The following properties of an $\mathbb{F}_p$-scheme $X$ are equivalent:

(i) the Frobenius morphism $\text{Fr} : X \to X$ is syntomic (i.e., flat, of finite presentation, and with each fiber being a locally complete intersection);

(ii) every point of $X$ has a neighborhood isomorphic to $\text{Spec } B$ for some $\mathbb{F}_p$-algebra $B$ with a finite $p$-basis in the sense of [BM, Def. 1.1.1] (i.e., there exist $x_1, \ldots, x_d \in B$ such that every element of $B$ can be written uniquely as $\sum_{\alpha \in J} b_\alpha x_\alpha^\alpha$, $b_\alpha, x_\alpha \in B$, where $J := \{0, 1, \ldots, p-1\}^d$ and for $\alpha = (\alpha_1, \ldots, \alpha_d) \in J$ one sets $x_\alpha := x_1^{\alpha_1} \cdot \ldots \cdot x_d^{\alpha_d}$).

**Proof.** Let us only explain why (i) implies that the cotangent complex of $X$ over $\mathbb{F}_p$ is a vector bundle. The relative cotangent complex for $\text{Fr} : X \to X$ is a perfect complex in degrees 0, -1. On the other hand, it is the cone of the map $\text{Fr}^* L_X \to L_X$, where $L_X$ is the cotangent complex of $X$ over $\mathbb{F}_p$. But this map is zero. So we see that $L_X \oplus \text{Fr}^* L_X[1]$ is a perfect complex in degrees 0, -1. So $L_X$ is a vector bundle. \(\square\)

$\mathbb{F}_p$-schemes satisfying the equivalent conditions of Lemma 2.1.1 will be called **Frobenius-smooth**. We are mostly interested in the following two classes of such schemes.

**Example 2.1.2.** $X$ is a smooth scheme over a perfect field $k$ of characteristic $p$.

**Example 2.1.3.** $X = \text{Spec } k[[x_1, \ldots, x_n]]$, where $k$ is a perfect field of characteristic $p$.

**Remark 2.1.4.** The equivalent conditions of Lemma 2.1.1 still hold if in Examples 2.1.2-2.1.3 one replaces “perfect field” with “perfect $\mathbb{F}_p$-algebra”.

**Remark 2.1.5.** In [BM, §1] Berthelot and Messing study $\mathbb{F}_p$-algebras with a not necessarily finite $p$-basis; they prove that in some respects they are as good as smooth $\mathbb{F}_p$-algebras. More results of this type can be found in [dJ, §1]. In particular, any $\mathbb{F}_p$-algebra with a $p$-basis is formally smooth (see [BM, §1.1.1]), and moreover, its cotangent complex is a free module concentrated in degree 0 (see [dJ, Lemma 1.1.2]).

**Lemma 2.1.6.** Let $X$ be an $\mathbb{F}_p$-scheme whose Frobenius endomorphism is finite. Let $n \in \mathbb{N}$. Let $\Delta \subset X^n$ be the diagonal $X$ and $\Delta' := (\text{Fr}_{X^n})^{-1}(\Delta) \subset X^n$. Let $I \subset O_{\Delta'}$ be the ideal of $\Delta \subset \Delta'$. Then $I$ is finitely generated.

**Proof.** Let us recall that according to the definition from EGA, “finitely generated” really means “locally finitely generated”.

Let us only explain why (i) implies that the cotangent complex of $X$ over $\mathbb{F}_p$ is a vector bundle. The relative cotangent complex for $\text{Fr} : X \to X$ is a perfect complex in degrees 0, -1. On the other hand, it is the cone of the map $\text{Fr}^* L_X \to L_X$, where $L_X$ is the cotangent complex of $X$ over $\mathbb{F}_p$. But this map is zero. So we see that $L_X \oplus \text{Fr}^* L_X[1]$ is a perfect complex in degrees 0, -1. So $L_X$ is a vector bundle. \(\square\)

**Corollary 2.1.7.** In the situation of Lemma 2.1.6 the ideal $I$ is nilpotent on every quasi-compact open subset of $X$. \(\square\)

2.2. Some simplicial formal schemes. Let $X$ be an $\mathbb{F}_p$-scheme.
2.2.1. The simplicial scheme $\mathcal{P}_\bullet$. Let $X_{\text{perf}}$ be the perfection of $X$, i.e.,

$$X_{\text{perf}} := \lim_{\leftarrow} \ldots \xrightarrow{\text{Fr}} X \xrightarrow{\text{Fr}} X \xrightarrow{\text{Fr}} X.$$ 

For an integer $n \geq 0$, let $[n] := \{0, 1, \ldots, n\}$ and let $\mathcal{P}_n \subset X^{[n]}_{\text{perf}}$ be the preimage of the diagonal $X \hookrightarrow X^{[n]}$; equivalently, $\mathcal{P}_n$ is the fiber product (over $X$) of $n + 1$ copies of $X_{\text{perf}}$. The schemes $\mathcal{P}_n$ form a simplicial scheme $\mathcal{P}_\bullet = \mathcal{P}_\bullet(X)$ over $X$.

The scheme $\mathcal{P}_0 = X_{\text{perf}}$ is perfect. For any $n$, the scheme $\mathcal{P}_n$ is semiperfect in the sense of [SW], i.e., the Frobenius endomorphism of $\mathcal{P}_n$ is a closed embedding. Note that the underlying topological space of $\mathcal{P}_n$ is that of $X$. As usual, the structure sheaf of $\mathcal{P}_n$ is denoted by $\mathcal{O}_{\mathcal{P}_n}$.

2.2.2. The simplicial formal scheme $\mathcal{F}_\bullet$. Let $\mathcal{F}_n$ be the “Fontainization” of $\mathcal{P}_n$, i.e.,

$$\mathcal{F}_n = \lim_{\leftarrow} (\mathcal{P}_n \xrightarrow{\text{Fr}} \mathcal{P}_n \xrightarrow{\text{Fr}} \mathcal{P}_n \xrightarrow{\text{Fr}} \ldots).$$

Each $\mathcal{F}_n$ is a perfect formal scheme over $\mathbb{F}_p$, whose underlying topological space is that of $X$. These formal schemes form a simplicial formal scheme $\mathcal{F}_\bullet = \mathcal{F}_\bullet(X)$.

Corollary 2.1.7 implies that if $\text{Fr}_X$ is finite then $\mathcal{F}_n$ is equal to the formal completion of $X^{[n]}_{\text{perf}} = X^{n+1}_{\text{perf}}$ along the closed subscheme $\mathcal{P}_n \subset X^{[n]}_{\text{perf}}$.

2.2.3. The simplicial formal scheme $\mathcal{A}_\bullet$. Let $\mathcal{A}_n$ be the $p$-adic formal scheme obtained from $\mathcal{P}_n$ by applying Fontaine’s functor $A_{\text{cris}}$ (see [F94] §2.2 or [Dr1] §2.4]). So the underlying topological space of $\mathcal{A}_n$ is that of $X$, and the structure sheaf $\mathcal{O}_{\mathcal{A}_n}$ is the $p$-adic completion of the PD hull\(^4\) of the surjection $W(\mathcal{O}_{\mathcal{P}_n}) \to \mathcal{O}_{\mathcal{P}_n}$ (as usual, PD stands for “divided powers” and $W$ for the $p$-typical Witt vectors). The formal schemes $\mathcal{A}_n$ form a simplicial formal scheme $\mathcal{A}_\bullet = \mathcal{A}_\bullet(X)$.

By functoriality, the morphism $\text{Fr}_X : X \to X$ induces a canonical simplicial morphism $F : \mathcal{A}_\bullet(X) \to \mathcal{A}_\bullet(X)$. The morphism $F : \mathcal{A}_n \to \mathcal{A}_n$ is usually not an isomorphism if $n > 0$ (if $n = 0$ it is an isomorphism because $\mathcal{A}_0 = W(X_{\text{perf}})$).

2.3. Notation and terminology related to quasi-coherent sheaves.

2.3.1. $p$-adic formal schemes and stacks. By a $p$-adic formal scheme (resp. $p$-adic formal stack) $\mathcal{Y}$ we mean a sequence of schemes (resp. stacks) $\mathcal{Y}_n$ equipped with isomorphisms $\mathcal{Y}_{n+1} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z} \sim \mathcal{Y}_n$. (Then $\mathcal{Y}_n$ is over $\mathbb{Z}/p^n\mathbb{Z}$ and $\mathcal{Y}_{n+1} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z} = \mathcal{Y}_{n+1} \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$). In this situation we often write $\mathcal{Y} \otimes \mathbb{Z}/p^n\mathbb{Z}$ instead of $\mathcal{Y}_n$.

$\mathcal{Y}$ is said to be $\mathbb{Z}_p$-flat if each $\mathcal{Y}_n$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$.

2.3.2. The notation $\text{QCoh}(\mathcal{Y})$. If $\mathcal{Y}$ is a scheme or stack we write $\text{QCoh}(\mathcal{Y})$ for the category of quasi-coherent sheaves on $\mathcal{Y}$.

Now suppose that $\mathcal{Y}$ and $\mathcal{Y}_n$ are as in §2.3.1 Then we write $\text{QCoh}(\mathcal{Y})$ for the projective limit of the categories $\text{QCoh}(\mathcal{Y}_n)$ with respect to the pullback functors $\text{QCoh}(\mathcal{Y}_{n+1}) \to \text{QCoh}(\mathcal{Y}_n)$ (so an object of $\text{QCoh}(\mathcal{Y})$ is a sequence of objects $\mathcal{F}_n \in \text{QCoh}(\mathcal{Y}_n)$ with isomorphisms $\mathcal{F}_{n+1} \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \sim \mathcal{F}_n$). Note that $\text{QCoh}(\mathcal{Y})$ is a Karoubian additive category (but usually not an abelian one). We often write $\mathcal{F} / p^n \mathcal{F}$ instead of $\mathcal{F}_n$.

\(^4\)The definition of PD hull is given in [BP] §1.2.3. Throughout this article, PD hulls are taken in the category of PD algebras over $(\mathbb{Z}_p, p\mathbb{Z}_p)$ (rather than over $(\mathbb{Z}_p, 0)$). In other words, we assume that $\gamma_n(p)$ is equal to $p^n/n! \in p\mathbb{Z}_p$. 

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2.3.3. Z_p-flatness. Let \( \Y \) and \( \Y_n \) be as in §2.3.1 and let \( \mathcal{F} \in \text{QCoh}(\Y) \). We say that \( \mathcal{F} \) is Z_p-flat if each \( \mathcal{F}_n \) is flat over \( \mathbb{Z}/p^n\mathbb{Z} \). Let \( \text{QCoh}^\flat(\Y) \subset \text{QCoh}(\Y) \) be the full subcategory of Z_p-flat objects.

2.3.4. Finite generation. We say that \( \mathcal{F} \in \text{QCoh}(\Y) \) is finitely generated if \( \mathcal{F}_n \) is finitely generated\(^5\) for each \( n \) (or equivalently, for \( n = 1 \)). Let \( \text{QCoh}_\text{fin}(\Y) \subset \text{QCoh}(\Y) \) be the full subcategory of finitely generated objects. Let \( \text{QCoh}_\text{fin}^\flat(\Y) := \text{QCoh}_\text{fin}(\Y) \cap \text{QCoh}^\flat(\Y) \).

2.3.5. Cohomology. Let \( \Y \) and \( \Y_n \) be as in §2.3.1. Let \( \mathcal{F} \) and \( \mathcal{F}_n \) be as in §2.3.2. Then we define \( R\Gamma(\Y, \mathcal{F}) \) to be the homotopy projective limit of \( R\Gamma(\Y_n, \mathcal{F}_n) \).

2.3.6. Equivariant objects. Let \( \Gamma \rightrightarrows \Y \) be a groupoid in the category of p-adic formal schemes. Assume that the canonical morphisms \( \Gamma \to \Y \) are flat, quasi-compact, and quasi-separated. Then for each \( n \) we have the quotient stack \( \Y_n/\Gamma_n \) (the words “quotient” and “stack” are understood in the sense of the fpqc topology). The stacks \( \Y_n/\Gamma_n \) form a p-adic formal stack \( \Y/\Gamma \). In this situation objects of \( \text{QCoh}(\Y/\Gamma) \) are also called \( \Gamma \)-equivariant objects of \( \text{QCoh}(\Y) \).

2.3.7. Objects of \( \text{QCoh}(\Y) \) as sheaves. Let \( \Y \) be a p-adic formal scheme and \( \mathcal{F} \in \text{QCoh}(\Y) \). Let \( \mathcal{F}_\infty \) be the projective limit of the sheaves \( \mathcal{F}_n = \mathcal{F}/p^n\mathcal{F} \) in the category of presheaves on the topological space \( \Y \); it is clear that \( \mathcal{F}_\infty \) is a sheaf. As explained to me by M. Temkin,

\[
(2.1) \quad \mathcal{F}_\infty/p^n\mathcal{F}_\infty = \mathcal{F}_n
\]

(this will be proved in §2.3.8). Moreover,

\[
(2.2) \quad R\Gamma(\Y, \mathcal{F}_\infty) = R\Gamma(\Y, \mathcal{F}),
\]

where \( R\Gamma(\Y, \mathcal{F}) \) was defined in §2.3.5. Indeed, for any open affine \( U \subset \Y \) the derived projective limit of \( H^0(U, \mathcal{F}_n) \) equals the usual one because the maps \( H^0(U, \mathcal{F}_{n+1}) \to H^0(U, \mathcal{F}_n) \) are surjective.

Because of (2.1) and (2.2), we will usually not distinguish \( \mathcal{F}_\infty \) from \( \mathcal{F} \).

Let \( \mathcal{O}_\Y \) denote the projective limit of the sheaves \( \mathcal{O}_{\Y_n} \). Then the ringed space \( (\Y, \mathcal{O}_\Y) \) is a formal scheme in the sense of EGA, and \( \mathcal{F}_\infty \) is a sheaf of \( \mathcal{O}_\Y \)-modules.

2.3.8. Proof of (2.1). It suffices to show that for every open affine \( U \subset \Y \) the sequence

\[
H^0(U, \mathcal{F}_\infty) \xrightarrow{p^n} H^0(U, \mathcal{F}_\infty) \to H^0(U, \mathcal{F}_n) \to 0
\]

is exact. Let \( M := H^0(U, \mathcal{F}_\infty) \) and \( M_n := H^0(U, \mathcal{F}_n) \). Then \( M \) is the projective limit of \( M_n \). The transition maps \( M_{n+1} \to M_n \) are surjective, so the map \( M \to M_n \) is surjective for each \( n \).

Let \( M^n := \text{Ker}(M \to M_n) \), so \( M_n = M/M^n \). We have to show that \( M^n = p^nM \).

Note that \( M \) is separated and complete with respect to the filtration formed by \( M^i, \ i \in \mathbb{N} \). Since \( M_i = M_{i+1}/p^iM_{i+1} \), we have

\[
(2.3) \quad M^i = p^iM + M^{i+1}.
\]

\(^5\)Let us recall that according to the definition from EGA, “finitely generated” really means “locally finitely generated”.
So \( M^n = p^n M + M^{n+1} \supset p^n M \). On the other hand, if \( x \in M^n \) then applying (2.3) for \( i = n, n+1, n+2, \ldots \), we get
\[
x = p^n m_0 + x_1, \quad x_1 = p^{n+1} m_1 + x_2, \quad x_2 = p^{n+2} m_2 + x_3, \ldots
\]
for some elements \( m_j \in M \) and \( x_j \in M^{n+j} \). Then \( x = p^n m \), where \( m = \sum_{j=0}^{\infty} p^j m_j \) (this infinite series converges because \( p^j m_j \in M^j \) and \( M \) is complete).

2.4. Formulation of the results.

2.4.1. Convention. By a crystal on an \( \mathbb{F}_p \)-scheme \( X \) we mean a crystal of quasi-coherent \( \mathcal{O} \)-modules on the absolute crystalline site \( \text{Cris}(X) \). For the definition of this site, see the end of §III.1.1.3 of [B74].

The proof of the following theorem will be given in §2.5-2.7.

**Theorem 2.4.2.** Let \( X \) be a Frobenius-smooth scheme (i.e., an \( \mathbb{F}_p \)-scheme satisfying the equivalent conditions of Lemma 2.1.1).

(i) The simplicial formal scheme \( \mathcal{A}_\bullet \) from §2.2.3 is the nerve\(^6\) of a flat affine\(^7\) groupoid \( \mathcal{G} \) acting on \( \mathcal{A}_0 = \mathcal{W}(X_{\text{perf}}) \).

(ii) A crystal on \( X \) is the same as an object \( M \in \text{QCoh}(\mathcal{W}(X_{\text{perf}})/\mathcal{G}) \) (or equivalently, a \( \mathcal{G} \)-equivariant object of \( \text{QCoh}(\mathcal{W}(X_{\text{perf}})) \).

(iii) For \( M \) as above, \( R\Gamma_{\text{cris}}(X, M) = R\Gamma(\mathcal{W}(X_{\text{perf}})/\mathcal{G}, M) \).

In the case \( M = \mathcal{O} \) statement (iii) is equivalent to [BMS, Thm. 1.10], but our proof is different (we think of crystalline cohomology in terms of the Čech-Alexander complex\(^8\) rather than the de Rham complex).

**Remark 2.4.3.** An object \( M \in \text{QCoh}(\mathcal{W}(X_{\text{perf}})/\mathcal{G}) \) defines a collection of objects \( M^n \in \text{QCoh}(\mathcal{A}_n) \). For each \( r \in \mathbb{N} \), the sheaves \( M^n/p^r M^n \) form a cosimplicial sheaf\(^9\) on the topological space \( X \). Statement (iii) of the theorem can be reformulated as a canonical isomorphism
\[
R\Gamma_{\text{cris}}(X, M) \xrightarrow{\sim} \lim_{\leftarrow r} R\Gamma(X, \text{Tot}(M^\bullet/p^r M^\bullet)).
\]

**Remark 2.4.4.** The stacks \( (\mathcal{W}(X_{\text{perf}})/\mathcal{G}) \otimes_{\mathbb{Z}_p} (\mathbb{Z}/p^r \mathbb{Z}) \) are usually not Artin stacks because the two canonical morphisms \( \mathcal{G} \to \mathcal{W}(X_{\text{perf}}) \) usually have infinite type.

2.5. Proof of Theorem 2.4.2(i). We can assume that \( X = \text{Spec} B \), where \( B \) is an \( \mathbb{F}_p \)-algebra with a finite \( p \)-basis.

2.5.1. The simplicial formal scheme \( \mathcal{X}_\bullet \). By [BM, Prop. 1.1.7] or by [dJ, Remark 1.2.3(a)], \( X \) admits a lift \( \mathcal{X} = \text{Spf} \mathcal{B} \), where \( \mathcal{B} \) is a flat \( p \)-adically complete \( \mathbb{Z}_p \)-algebra with \( \mathcal{B}/p \mathcal{B} = B \). Fix \( \mathcal{X} \). Let \( \mathcal{X}_n \) be the \( p \)-adically completed PD hull of the (ideal of the) diagonal
\[
X \hookrightarrow X^{n+1} \subset \mathcal{X}^{n+1} = \mathcal{X}^{[n]}.
\]

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\(^{6}\)Recall that a groupoid (and more generally, a category) is uniquely determined by its nerve.

\(^{7}\)This means that the two canonical morphisms \( \mathcal{G} \to \mathcal{W}(X_{\text{perf}}) \) are flat and affine.

\(^{8}\)The Čech-Alexander complex is discussed in [Gr68, §5.1, 5.5], [BhdJ, §2], and [L13, §1.7-1.8].

\(^{9}\)More precisely, a cosimplicial sheaf of modules over the cosimplicial sheaf of rings formed by the structure sheaves of the formal schemes \( \mathcal{A}_n \).

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Then $\mathcal{X}_n$ is a $p$-adic formal scheme whose underlying topological space is $X$. The formal schemes $\mathcal{X}_n$ form a simplicial formal scheme $\mathcal{X}_\bullet$.

Fix a $p$-basis $x_1, \ldots, x_d \in \mathcal{B}$. Let $\tilde{x}_j \in \mathcal{B}$, $\tilde{x}_j \mapsto x_j$. We have canonical embeddings $i_m : \mathcal{B} = H^0(\mathcal{X}, \mathcal{O}_X) \hookrightarrow H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$.

**Proposition 2.5.2.** As an $i_m(\mathcal{B})$-algebra, $H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ is the $p$-adically completed algebra of PD polynomials over $i_m(\mathcal{B})$ with respect to the elements $i_r(\tilde{x}_j) - i_m(\tilde{x}_j)$, where $r \in \{0, \ldots, n\}$, $r \neq m$, $1 \leq j \leq d$.

**Proof.** For $n = 2$ this is [BM, Cor. 1.3.2(i)]. The general case follows. □

**Corollary 2.5.3.** $\mathcal{X}_\bullet$ is the nerve of a flat affine groupoid $\mathcal{R}$ acting on $\mathcal{X}$. □

**Lemma 2.5.4.** The formation of PD hulls commutes with flat base change.

**Proof.** This was proved by Berthelot [B74, Prop. I.2.7.1]. □

2.5.5. **End of the proof.** By [BM, Prop. 1.2.6], the morphism $\pi : X_{\text{perf}} \to X$ lifts to a morphism $\tilde{\pi} : W(X_{\text{perf}}) \to \mathcal{X}$. Since $\text{Fr}_X : X \to X$ is flat, so are $\pi$ and $\tilde{\pi}$.

Let $\mathcal{P}_n$ be as in §2.2.1. The formal scheme $\mathcal{A}_n$ from §2.2.3 is the $p$-adically completed PD hull of $\mathcal{P}_n$ in $W(X_{\text{perf}})^[n]$. So by Lemma 2.5.4 and flatness of $\tilde{\pi}$, the diagram

\[
\begin{align*}
\mathcal{A}_n & \xrightarrow{\tilde{\pi}_n} \mathcal{X}_n & \mathcal{A}_n \to \mathcal{X}_n \\
W(X_{\text{perf}})^[n] & \xrightarrow{} \mathcal{X}_n^+[n]
\end{align*}
\]

is Cartesian. So Corollary 2.5.3 implies that $\mathcal{A}_\bullet$ is the nerve of a flat groupoid acting on $W(X_{\text{perf}})$. □

**Remark 2.5.6.** In the situation of $\mathcal{X}_\bullet$, $\tilde{\pi} : W(X_{\text{perf}}) \to \mathcal{X}$ induces an isomorphism

\[
W(X_{\text{perf}})/\mathcal{G} \sim \mathcal{X}/\mathcal{R},
\]

where $\mathcal{G}$ and $\mathcal{R}$ are as in Theorem 2.4.2(i) and Corollary 2.5.3, respectively. This follows from faithful flatness of $\tilde{\pi}$ and the fact that the diagram (2.5) is Cartesian.

**Remark 2.5.7.** The isomorphism (2.6) does not depend on the choice of $\tilde{\pi}$. Indeed, if $\tilde{\pi}' : W(X_{\text{perf}}) \to \mathcal{X}$ is another lift of $\pi$ then the morphism $W(X_{\text{perf}}) \to \mathcal{X}_1 \times \mathcal{X}_1$ factors through $\mathcal{X}_1$; to see this, use the PD structure on the ideal of the subscheme $X_{\text{perf}} \subset W(X_{\text{perf}})$.

2.6. **Proof of Theorem 2.4.2(ii).**

2.6.1. **General remark.** Recall that if $Y$ is a semiperfect $F_p$-scheme then $A_{\text{cris}}(Y) \otimes (\mathbb{Z}/p^r\mathbb{Z})$ is the final object of the crystalline site of $Y$ over $\mathbb{Z}/p^r\mathbb{Z}$ (see [F94, §2.2] or [Dr1, Prop. 2.2.1]). So a crystal on $Y$ is the same as an object $N \in \text{QCoh}(A_{\text{cris}}(Y))$, and the crystalline cohomology of the crystal is just $R\Gamma(A_{\text{cris}}(Y), N)$. 7
2.6.2. The functor in one direction. Let us apply §2.6.1 to the semiperfect schemes $\mathcal{P}_n$ from §2.2.1. Recall that $\mathcal{A}_n := \mathcal{A}_{\text{cris}}(\mathcal{P}_n)$. So given a crystal on $X$, its pullback to $\mathcal{P}_n$ can be viewed as an object $M^n \in \text{QCoh}(\mathcal{A}_n)$. Moreover, the collection of objects $M^n$ is compatible via $\ast$-pullbacks. By the definition of the groupoid $\mathcal{G}$ (see Theorem 2.4.2(i)), such a compatible collection is the same as a $\mathcal{G}$-equivariant object of $\text{QCoh}(\mathcal{A}_n) = \text{QCoh}(W(X_{\text{perf}}))$ or equivalently, an object of $\text{QCoh}(W(X_{\text{perf}})/\mathcal{G})$. Thus we have constructed a functor

\begin{equation}
\{\text{Crystals on } X\} \to \text{QCoh}(W(X_{\text{perf}})/\mathcal{G}).
\end{equation}

It remains to prove that the functor (2.7) is an equivalence. The question is local, so we can assume that $X$ is the spectrum of an $\mathbb{F}_p$-algebra with a finite $p$-basis.

2.6.3. Factorizing the functor (2.7). Let $\mathcal{X}$ and $\mathcal{X}_\bullet$ be as in §2.5.1. Each scheme of the form $\mathcal{X}_n \otimes (\mathbb{Z}/p^r\mathbb{Z})$ is a PD thickening of $X$, so a crystal $M$ on $X$ defines for each $n$ an object $M_{\mathcal{X}_n} \in \text{QCoh}(\mathcal{X}_n)$. This collection of objects is compatible via $\ast$-pullbacks. By the definition of the groupoid $\mathcal{R}$ (see Corollary 2.5.3), such a compatible collection is the same as an $\mathcal{R}$-equivariant object of $\text{QCoh}(\mathcal{X})$ or equivalently, an object of $\text{QCoh}(\mathcal{X}/\mathcal{R})$. Thus we have constructed a functor

\begin{equation}
\{\text{Crystals on } X\} \to \text{QCoh}(\mathcal{X}/\mathcal{R}).
\end{equation}

The functor (2.7) is the composition of (2.8) and the equivalence

$$\text{QCoh}(\mathcal{X}/\mathcal{R}) \xrightarrow{\sim} \text{QCoh}(W(X_{\text{perf}})/\mathcal{G})$$

corresponding to (2.6). It remains to show that (2.8) is an equivalence. This is a consequence of the next lemma.

Lemma 2.6.4. Suppose that $X$ is the spectrum of a $\mathbb{F}_p$-algebra with a finite $p$-basis. Let $\mathcal{X}$ be as in §2.5.1. Then every object of the absolute crystalline site of $X$ admits a morphism to $\mathcal{X}_n \otimes (\mathbb{Z}/p^r\mathbb{Z})$ for some $r$.

Proof. Follows from [BM, Prop. 1.2.6].

2.7. Proof of Theorem 2.4.2(iii). Let $M$ be a crystal on $X$. We can also think of $M$ as an object of $\text{QCoh}(W(X_{\text{perf}})/\mathcal{G})$.

2.7.1. General remark. Let $X_{\text{cris}}$ be the absolute crystalline topos of $X$ (i.e., the category of sheaves on $\text{Cris}(X)$). Let $Y_\bullet$ be a simplicial ringed topos over $X_{\text{cris}}$. Then one has a canonical morphism

\begin{equation}
R\Gamma_{\text{cris}}(X, M) \to \text{Tot}(R\Gamma(Y_\bullet, M^\bullet)),
\end{equation}

where $M^n$ is the pullback of $M$ to $Y_n$. Moreover, this construction is functorial in $Y_\bullet$.

---

10By this we mean that for every map $f : [m] \to [n]$ one has a canonical isomorphism $(f^+)^*M^n \xrightarrow{\sim} M^m$, where $f^+ : X^n \to X^m$ is induced by $f$, and these isomorphisms are compatible with composition of $f$'s.

11Here we use that the map $\mathcal{X}_n \otimes (\mathbb{Z}/p^r\mathbb{Z}) \to \mathcal{X}_m \otimes (\mathbb{Z}/p^r\mathbb{Z})$ corresponding to any map $[m] \to [n]$ is a PD morphism.
2.7.2. The map in one direction. Apply (2.9) for $Y_\bullet = (P\circ)_\text{cris}$, where $P\circ$ is as in §2.2.1. As explained in §2.6.1, we can think of $M^n$ as an object of $\text{QCoh}(A_n)$, where $A_n := A\text{cris}(P^n)$; it is this object that was denoted by $M^n$ in Remark 2.4.3. Moreover, $R\Gamma\text{cris}(P_n, M^n) = R\Gamma(A_n, M^n)$ by §2.6.1. Thus we can rewrite (2.9) as a morphism

$$R\Gamma\text{cris}(X, M) \to \text{Tot}(R\Gamma(A_\circ, M^\circ)) = R\Gamma(W(X_{\text{perf}})/\mathcal{G}, M).$$

It remains to prove that the morphism (2.10) is an isomorphism. The question is local, so we can assume that $X$ is the spectrum of an $\mathbb{F}_p$-algebra with a finite $p$-basis.

2.7.3. End of the proof. Let $\mathcal{X}$ and $\mathcal{X}_\circ$ be as in §2.5.1. By Lemma 2.6.4, we get an isomorphism

$$R\Gamma\text{cris}(X, M) \xrightarrow{\sim} \text{Tot}(R\Gamma(\mathcal{X}_\circ, M_{\mathcal{X}_\circ})) = \text{Tot}(\Gamma(\mathcal{X}_\circ, M_{\mathcal{X}_\circ}))$$

where $M_{\mathcal{X}_\circ}$ is as in §2.6.3. Tot$(\Gamma(\mathcal{X}_\circ, M_{\mathcal{X}_\circ}))$ is called the Čech-Alexander complex.

We have

$$\text{Tot}(\Gamma(\mathcal{X}_\circ, M_{\mathcal{X}_\circ})) = R\Gamma(\mathcal{X}/\mathcal{R}, M),$$

where $\mathcal{R}$ is the groupoid from Corollary 2.5.3 and $\mathcal{X}/\mathcal{R}$ is the quotient stack (just as in §2.6.3, we can think of $M$ as an object of $\text{QCoh}(\mathcal{X}/\mathcal{R})$). It remains to show that (2.10) is the usual isomorphism

$$R\Gamma(\mathcal{X}/\mathcal{R}, M) \to R\Gamma(W(X_{\text{perf}})/\mathcal{G}, M)$$

corresponding to the isomorphism of stacks (2.6). This follows from the next lemma.

**Lemma 2.7.4.** Choose $\tilde{\pi} : W(X_{\text{perf}}) \to \mathcal{X}$ as in §2.5.5 and let $\tilde{\pi}_n : A_n \to \mathcal{X}_n$ be as in diagram (2.5). Then the map (2.10) equals the composition of (2.11) and the morphism

$$\text{Tot}(R\Gamma(\mathcal{X}_\circ, M_{\mathcal{X}_\circ})) \to \text{Tot}(R\Gamma(A_\circ, M^\circ))$$

that comes from the maps $\tilde{\pi}_n^* : R\Gamma(\mathcal{X}_n, M_{\mathcal{X}_n}) \to R\Gamma(A_n, M^n)$.

**Proof.** For each $n$, $\mathcal{X}_n$ is an object of the crystalline topos $X_{\text{cris}}$. Let $X_{\text{cris}}/\mathcal{X}_n$ be the category of objects of $X_{\text{cris}}$ over $\mathcal{X}_n$; this category is a ringed topos over $X_{\text{cris}}$. As $n$ varies, we get a simplicial ringed topos $X_{\text{cris}}/\mathcal{X}_\circ$ over $X_{\text{cris}}$. Moreover, we have a morphism

$$(P\circ)_{\text{cris}} \to X_{\text{cris}}/\mathcal{X}_\circ$$

of simplicial ringed topoi over $X_{\text{cris}}$.

The morphism (2.11) is the morphism (2.9) for $Y_\bullet = X_{\text{cris}}/\mathcal{X}_\circ$. So the lemma follows from functoriality of the map (2.9) with respect to $Y_\bullet$. □

2.8. $H^0_{\text{cris}}(X, \mathcal{O})$ and the ring of constants. As before, we assume that $X$ is Frobenius-smooth in the sense of §2.1. Let us compute $H^0_{\text{cris}}(X, \mathcal{O})$, where $\mathcal{O}$ is the structure sheaf on the absolute crystalline site of $X$.  

9
2.8.1. The ring of constants. Let $A$ be the ring of regular functions on $X$ and $k := \bigcap_{n=1}^{\infty} A^{p^n}$. We call $k$ the ring of constants of $X$.

Frobenius-smoothness implies that $X$ is reduced. So $A$ is reduced, and $k$ is a perfect $\mathbb{F}_p$-algebra.

Lemma 2.8.2. Let $X = \text{Spec } B$, where $B$ is an $\mathbb{F}_p$-algebra with a $p$-basis $x_1, \ldots, x_m$. Let $\pi : X \to \mathcal{G}_a^m$ be the morphism defined by $x_1, \ldots, x_m$. Let $\hat{\mathcal{G}}_a$ be the formal additive group over $\mathbb{F}_p$.

(i) There is a unique action of $\hat{\mathcal{G}}_a$ on $X$ such that $\pi : X \to \mathcal{G}_a^m$ is $\hat{\mathcal{G}}_a$-equivariant.

(ii) The ring of constants on $X$ is equal to the ring of all $\hat{\mathcal{G}}_a$-invariant regular functions on $X$.

(iii) A closed subscheme $Y \subset X$ is preserved by the $\hat{\mathcal{G}}_a$-action if and only if for each $n \in \mathbb{N}$ there exists a closed subscheme $Z \subset X$ such that $Y = (\text{Fr}_X^n)^{-1}(Z)$.

Proof. (i) $\hat{\mathcal{G}}_a$ is the inductive limit of the finite group schemes $\text{Ker}(\mathcal{G}_a^m \xrightarrow{\text{Fr}_n} \mathcal{G}_a^m)$, $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\text{Fr}_n} & X \\
\downarrow \pi & & \downarrow \pi \\
\mathcal{G}_a^m & \xrightarrow{\text{Fr}_n} & \mathcal{G}_a^m
\end{array}
$$

is Cartesian (for $n = 1$ by the definition of $p$-basis, the general case follows). Any action of $\text{Ker}(\mathcal{G}_a^m \xrightarrow{\text{Fr}_n} \mathcal{G}_a^m)$ on $X$ is fiberwise with respect to the morphism $\text{Fr}_n : X \to X$. So there is one and only one such action with the property that $\pi : X \to \mathcal{G}_a^m$ is $\hat{\mathcal{G}}_a$-equivariant.

(ii-iii) The morphism $\text{Fr}_n : X \to X$ is a torsor with respect to $\text{Ker}(\mathcal{G}_a^m \xrightarrow{\text{Fr}_n} \mathcal{G}_a^m)$. □

Proposition 2.8.3. Let $X$ be a Frobenius-smooth $\mathbb{F}_p$-scheme. Let $k$ be the ring of constants of $X$. Then the map $W(k) \to \text{H}^0_{\text{cris}}(X, \mathcal{O})$ induced by the morphism $X \to \text{Spec } k$ is an isomorphism.

Proof. We can assume that $X = \text{Spec } B$, where $B$ is an $\mathbb{F}_p$-algebra with a $p$-basis $x_1, \ldots, x_m$. By [BM Prop. 1.1.7], there exists a flat $p$-adically complete $\mathbb{Z}_p$-algebra $\hat{B}$ with $\hat{B}/p\hat{B} = B$.

For each $i$ choose a lift of $x_i$ to $\hat{B}$; this lift will still be noted by $x_i$. By [BM Prop. 1.3.1], for each $n \in \mathbb{N}$ the module of differentials of $\hat{B}/p^n\hat{B}$ is a free $(\hat{B}/p^n\hat{B})$-module with basis $dx_i$, $1 \leq i \leq m$. So for each $i \leq m$ there is a unique derivation $D_i : \hat{B} \to \hat{B}$ such that $D_i(x_j) = \delta_{ij}$. One has

$$
\text{H}^0_{\text{cris}}(X, \mathcal{O}) = \bigcap_{i=1}^{m} \text{Ker}(\hat{B} \xrightarrow{D_i} \hat{B});
$$

this follows, e.g., from [BM Prop. 1.3.3] (because $\text{H}^0_{\text{cris}}(X, \mathcal{O})$ is the $\mathbb{Z}_p$-module of endomorphisms of the crystal $\mathcal{O}$).

Since $x_1, \ldots, x_m$ form a $p$-basis in $B$, for each $r \in \mathbb{N}$ the ring $\hat{B}$ is topologically generated by $x_1, \ldots, x_m$ and elements of the form $\hat{f}^p$, $\hat{f} \in \hat{B}$. This implies that for each $i \leq m$ and $l \in \mathbb{N}$ one has

$$
D_l^i(\hat{B}) \subset l! \cdot \hat{B},
$$

The definition of $p$-basis was given in Lemma 2.1.
so one has the commuting operators \( D_i^{(l)} := (l!)^{-1}D_i^l \) acting on \( \tilde{B} \) and satisfying the Leibniz formula

\[
D_i^{(l)}(\tilde{f}\tilde{g}) = \sum_{a+b=l} D_i^{(a)}(\tilde{f})D_i^{(b)}(\tilde{g}), \quad \tilde{f}, \tilde{g} \in \tilde{B}
\]

and the relation \( D_i^{(r)}D_j^{(s)} = \binom{r+s}{r}D_i^{(r+s)} \). These operators define an action of \( \hat{G}_a^m \) on \( \tilde{B} \), where \( \hat{G}_a \) is the additive formal group over \( \mathbb{Z}_p \). The corresponding action of \( \hat{G}_a^m \) on \( B \) is the one from Lemma 2.8.2(i).

To prove that the map \( W(k) \to H^0_{\text{cris}}(X, \mathcal{O}) \) is an isomorphism, it suffices to check that if \( f \in \tilde{B} \) is killed by \( D_1, \ldots, D_m \) and \( f \in B \) is the image of \( f \) then \( f \in k \). It is clear that \( f \) is \( \hat{G}_a^m \)-invariant. So \( f \) is \( \hat{G}_a^m \)-invariant. By Lemma 2.8.2(ii), this means that \( f \in k \).  \( \square \)

### 3. Isocrystals

#### 3.1. A class of schemes.

**Proposition 3.1.1.** The following properties of an \( \mathbb{F}_p \)-scheme \( X \) are equivalent:

(i) \( X \) is Noetherian and Frobenius-smooth (see Lemma 2.1.1 and the sentence after it);

(ii) \( X \) is Noetherian, reduced, and \( \Omega^1_X \) is locally free and finitely generated.

These properties imply that the scheme \( X \) is regular and excellent.

**Proof.** It is clear that (i)\( \Rightarrow \) (ii). The implication (ii)\( \Rightarrow \) (i) follows from Theorem 1 of \cite{ly}, whose proof is based on \cite{fog} Prop. 1; the reducedness assumption in (ii) is necessary because the definition of “having a \( p \)-basis” used in \cite{ly} is different from the one from Lemma 2.1.1. E. Kunz proved that property (i) implies regularity and excellence, see Theorems 107-108 of \cite{m} §42.  \( \square \)

**Throughout §3 we assume that \( X \) has the equivalent properties of Proposition 3.1.1.** For instance, if \( k \) is a perfect field of characteristic \( p \) one can take \( X \) to be either \( \text{Spec } k[[x_1, \ldots, x_n]] \) or a quasi-compact smooth \( k \)-scheme.

#### 3.1.2. The ring of constants.

In \cite{2.8.1} we defined the ring of constants \( k \) by \( k := \bigcap_{n=1}^{\infty} A^{p^n} \), where \( A \) is the ring of regular functions on \( X \). If \( X \) has the properties of Proposition 3.1.1 then \( k = \bigcap_{n=1}^{\infty} E^{p^n} \), where \( E \) is the ring of rational functions on \( X \); indeed, if \( f \in \bigcap_{n=1}^{\infty} E^{p^n} \) and \( D \) is the divisor of poles of \( f \) then \( D \) is divisible by \( p^n \) for all \( n \in \mathbb{N} \), so \( D = 0 \). Thus if \( X \) is irreducible then the ring of constants is a perfect field; in general, it is a product of perfect fields.

#### 3.2. Coherent crystals and isocrystals.

Let \( X \) be an \( \mathbb{F}_p \)-scheme that has the equivalent properties of Proposition 3.1.1. By a coherent crystal on an \( \mathbb{F}_p \)-scheme \( X \) we mean a crystal of finitely generated quasi-coherent \( \mathcal{O} \)-modules on the absolute crystalline site of \( X \).

Let \( \mathcal{G} \) be the groupoid on \( W(X_{\text{perf}}) \) constructed in Theorem 2.4.2(i). We have the categories

\[
\text{QCoh}_{\text{fin}}^b(W(X_{\text{perf}})/\mathcal{G}) \subset \text{QCoh}_{\text{fin}}(W(X_{\text{perf}})/\mathcal{G}) \subset \text{QCoh}(W(X_{\text{perf}})/\mathcal{G}),
\]

(we are using the notation of \cite{2.8.2}, \cite{2.8.4}.)

\(^{13}\)Let us recall that according to the definition from EGA, “finitely generated” really means “locally finitely generated”.

11
Lemma 3.2.1. (i) The equivalence (2.6.2) identifies the category of coherent crystals on $X$ with $\text{QCoh}_{\text{fin}}(W(X_{\text{perf}})/\mathcal{G})$.

(ii) The category $\text{QCoh}_{\text{fin}}(W(X_{\text{perf}})/\mathcal{G})$ is abelian.

(iii) $\text{QCoh}_{\text{fin}}(W(X_{\text{perf}})/\mathcal{G}) \otimes \mathbb{Q} = \text{QCoh}_{\text{fin}}(W(X_{\text{perf}})/\mathcal{G}) \otimes \mathbb{Q}$

Proof. By Remark 2.5.6, $W(X_{\text{perf}})/\mathcal{G}$ is locally isomorphic to a quotient of a Noetherian formal scheme by a flat groupoid. This implies (ii-iii). Statement (i) follows from Lemma 2.6.1.

Definition 3.2.2. Objects of the category

$$\text{Isoc}(X) := \text{QCoh}_{\text{fin}}(W(X_{\text{perf}})/\mathcal{G}) \otimes \mathbb{Q} = \text{QCoh}_{\text{fin}}^b(W(X_{\text{perf}})/\mathcal{G}) \otimes \mathbb{Q}$$

will be called isocrystals on $X$.

(Thus our isocrystals are not necessarily convergent in the sense of [O].)

3.3. Local projectivity.

Proposition 3.3.1. Let $\mathcal{X}$ be a $\mathbb{Z}_p$-flat $p$-adic formal scheme with $\mathcal{X} \otimes_{\mathbb{Z}_p} \mathbb{F}_p = X$. Let $M$ be a coherent crystal on $X$ and $M_\mathcal{X}$ the corresponding coherent $\mathcal{O}_\mathcal{X}$-module. Then the $(\mathcal{O}_\mathcal{X} \otimes \mathbb{Q})$-module $M_\mathcal{X} \otimes \mathbb{Q}$ is locally projective.

A proof will be given in §3.4.

Corollary 3.3.2. Let $M \in \text{QCoh}_{\text{fin}}(W(X_{\text{perf}})/\mathcal{G})$. Let $M_{W(X_{\text{perf}})}$ be the corresponding module over $\mathcal{O}_{W(X_{\text{perf}})}$. Then $M_{W(X_{\text{perf}})} \otimes \mathbb{Q}$ is a locally projective $\mathbb{Q}$-module over $\mathcal{O}_{W(X_{\text{perf}})} \otimes \mathbb{Q}$.

Proof. The statement is local on $X$. So by [BM, Prop. 1.2.6], we can assume that the canonical morphism $X_{\text{perf}} \to X$ lifts to a morphism $W(X_{\text{perf}}) \to \mathcal{X}$, where $\mathcal{X} \in \text{FrmSch}^b$, $\mathcal{X} \otimes_{\mathbb{Z}_p} \mathbb{F}_p = X$. It remains to apply Proposition 3.3.1.

Corollary 3.3.3. If $X$ is irreducible then $\text{Isoc}(X)$ is a Tannakian category over $\text{Frac} W(k)$, where $k$ is the field of constants of $X$ (in the sense of §7.1.2).

Proof. The fact that $\text{Isoc}(X)$ is abelian follows from Lemma 3.2.1 ii).

$W(X_{\text{perf}})/\mathcal{G}$ is locally isomorphic to a quotient of a Noetherian formal scheme by a flat groupoid, so the tensor category $\text{QCoh}_{\text{fin}}(W(X_{\text{perf}})/\mathcal{G})$ has internal Hom’s. Proposition 3.3.1 implies that for any $M, N \in \text{QCoh}_{\text{fin}}(W(X_{\text{perf}})/\mathcal{G})$, the canonical morphism

$$\text{Hom}(M, \mathcal{O}) \otimes N \to \text{Hom}(M, N)$$

becomes an isomorphism in the category $\text{Isoc}(X) := \text{QCoh}_{\text{fin}}(W(X_{\text{perf}})/\mathcal{G}) \otimes \mathbb{Q}$. So $\text{Isoc}(X)$ is rigid by [De90, Prop. 2.3].

By Proposition 2.8.3, the endomorphism ring of the unit object of $\text{Isoc}(X)$ equals $\text{Frac} W(k)$. Let $\alpha : \text{Spec } E \to X$ be a point with $E$ being a perfect field. Let $K := \text{Frac} W(E)$. Let

$$\Phi_{\alpha} : \text{Isoc}(X) \to \{K\text{-vector spaces}\}$$

be the functor obtained by composing $\alpha^* : \text{Isoc}(X) \to \text{Isoc}(\text{Spec } E)$ with the obvious fiber functor $\text{Isoc}(\text{Spec } E) \to \{K\text{-vector spaces}\}$. Then $\Phi_{\alpha}$ is an exact tensor functor. Since $X$ is connected, Proposition 3.3.1 implies that if $M \in \text{Isoc}(X)$ and $\Phi_{\alpha}(M) = 0$ then $M = 0$. So $\Phi_{\alpha}$ is a fiber functor. 

A module over a sheaf of rings is said to be locally projective if it can be locally represented as a direct summand of a free module.
3.4. **Proof of Proposition 3.3.1** The proposition is well known if $X$ is a scheme of finite type over a perfect field, see [O, Cor. 2.9] or [B96, §2.3.4]. The proof given in loc. cit. uses the following fact: if $E$ is a field of characteristic 0 and $N$ is a finitely generated module over $E[[t_1, \ldots, t_n]]$ which admits a connection, then $N$ is free. The proof given below is essentially the same (but organized using Fitting ideals).

3.4.1. **Strategy.** We can assume that $X = \text{Spec} \ B$, $\mathcal{X} = \text{Spf} \ ¹B$, where $B$ and $¹B$ are as in the proof of Proposition 2.8.3. Choose an exact sequence of $¹B$-modules

$$0 \to N \to ¹B^l \to M_\mathcal{X} \to 0.$$ 

For $r \in \{0, \ldots, l\}$, let $I_r \subset ¹B$ be the image of the canonical map

$$(3.1) \quad \bigwedge^{l-r} N \otimes \text{Hom}(\bigwedge^{l-r} ¹B^l, ¹B) \to ¹B$$

(so the ideals $I_0 \subset \ldots I_{l-1} \subset I_l = ¹B$ are the Fitting ideals of $M_\mathcal{X}$). Let

$$J_r := \{u \in ¹B \mid p^j u \in I_r \text{ for some } j\}.$$ 

Define the ideals $J_0' \subset \ldots J_{l-1}' \subset J_l' = B$ by $J_i' := J_i/pJ_i$. The $(\mathcal{O}_\mathcal{X} \otimes \mathbb{Q})$-module $M_\mathcal{X} \otimes \mathbb{Q}$ is locally projective of constant rank $r$ if and only if $J_r' = B$ and $J_i' = 0$ for $i < r$. So to prove Proposition 3.3.1, it suffices to show that the closed subscheme $\text{Spec}(B/J_l') \subset \text{Spec} \ B$ is open for each $r$.

We will use the notation from the proof of Proposition 2.8.3 in particular, we have the derivations $D_i : ¹B \to ¹B$ and the differential operators $D_i^{(l)} : ¹B \to ¹B$ and $D_i^{(l)} : B \to B$.

**Lemma 3.4.2.** $D_i(I_r) \subset I_r$ for all $i$ and $r$.

**Proof.** Our $M_\mathcal{X}$ is a module over the ring of differential operators $B[D_1, \ldots, D_m]$. For each $i$ there exists a $¹B$-linear operator $L_i : ¹B^l \to ¹B^l$ such that the map $f : ¹B^l \to M_\mathcal{X}$ satisfies the identity

$$D_i \circ f = f \circ \nabla_i, \text{ where } \nabla_i := D_i + L_i.$$ 

Then $\nabla_i(N) \subset N$. Think of $\nabla$ as a (not necessarily integrable) connection on $¹B^l$ and $N$. Equip $¹B$ with the trivial connection. Then $(3.1)$ becomes a horizontal morphism, so its image $I_r$ is preserved by the operators $D_i$. \hfill $\square$

Our goal is to show that the closed subscheme $\text{Spec}(B/J_l') \subset \text{Spec} \ B$ is open for each $r$. Since $B$ is Noetherian, it suffices to prove the following

**Lemma 3.4.3.** For each $r, n \in \mathbb{N}$ there exists a closed subscheme $Z \subset \text{Spec} \ B = X$ such that $\text{Spec}(B/J_l') = (\text{Fr}_X^n)^{-1}(Z)$.

**Proof.** Lemma 3.4.2 implies that $D_i^{(l)}(J_r) \subset J_r$ for all $i$ and $l$ (recall that $D_i^{(l)} = D_i^{(l)}/l!$). So $D_i^{(l)}(J_r') \subset J_r'$. This means that the closed subscheme $\text{Spec}(B/J_r') \subset \text{Spec} \ B$ is preserved by the $\mathbb{G}_m$-action from Lemma 2.8.2(i). It remains to use Lemma 2.8.2(iii). \hfill $\square$

3.5. **Isocrystals as vector bundles.**
3.5.1. The category $\text{Bun}_Q(\mathcal{Y})$. Let $\text{FrmSch}^b$ denote the category of $\mathbb{Z}_p$-flat $p$-adic formal schemes. For $\mathcal{V} \in \text{FrmSch}^b$, let $\text{Bun}_Q(\mathcal{V})$ denote the category of finitely generated locally projective $(\mathcal{O}_\mathcal{V} \otimes \mathbb{Q})$-modules.

We secretly think of $\text{Bun}_Q(\mathcal{Y})$ as the category of vector bundles on the “generic fiber” $\mathcal{V} \otimes \mathbb{Q}$, which can hopefully be defined as some kind of analytic space (maybe in the sense of R. Huber, see [Hub] and [SW, §2.1]). This is true if $\mathcal{V}$ is a formal scheme of finite type over $W(k)$, where $k$ is a perfect field of characteristic $p$ (moreover, in this case $\mathcal{V} \otimes \mathbb{Q}$ can be understood as an analytic space in the sense of Tate or Berkovich). But we need non-Noetherian formal schemes (e.g., the formal scheme $W(X_{\text{perf}})$ from §3.2).

**Proposition 3.5.2.** Suppose that $\mathcal{V} \in \text{FrmSch}^b$ is affine. Then any object of $\text{Bun}_Q(\mathcal{V})$ is a direct summand of $\mathcal{O}_\mathcal{V}^n \otimes \mathbb{Q}$ for some $n \in \mathbb{N}$.

The proof is given in §3.7.

3.5.3. Flat descent for $\text{Bun}_Q(\mathcal{V})$. Let $C^\bullet$ be a cosimplicial category, i.e., a functor from the simplex category $\Delta$ to the 2-category of categories. Then the projective limit of this functor is denoted by $\text{Tot}(C^\bullet)$. So an object of $\text{Tot}(C^\bullet)$ is an equivalence.

In particular, for a simplicial object $\mathcal{V}_\bullet$ in $\text{FrmSch}^b$ we have the category $\text{Tot}(\text{Bun}_Q(\mathcal{V}_\bullet))$.

**Proposition 3.5.4.** Let $f : \mathcal{V} \to \mathcal{Z}$ be a faithfully flat quasi-compact morphism in $\text{FrmSch}^b$. Let $\mathcal{V}_n$ be the fiber product (over $\mathcal{Z}$) of $n + 1$ copies of $\mathcal{V}$ (in particular, $\mathcal{V}_0 = \mathcal{V}$). Then the functor

$$\text{Bun}_Q(\mathcal{Z}) \to \text{Tot}(\text{Bun}_Q(\mathcal{V}_\bullet))$$

is an equivalence.

**Proof of full faithfulness.** If $\mathcal{V}$ and $\mathcal{Z}$ are affine then the sequence

$$0 \to H^0(\mathcal{Z}, \mathcal{O}_\mathcal{Z}) \to H^0(\mathcal{V}_0, \mathcal{O}_{\mathcal{V}_0}) \to H^0(\mathcal{V}_1, \mathcal{O}_{\mathcal{V}_1})$$

is exact by usual flat descent. This implies that the functor $\text{Bun}_Q(\mathcal{V}_\bullet)$ is fully faithful. Essential surjectivity will be proved in §3.7. \(\square\)

Let us note that Proposition 3.5.4 was substantially generalized by A. Mathew [Mat].

3.5.5. Equivariant objects of $\text{Bun}_Q(\mathcal{V})$. Suppose we have a groupoid $\Gamma \Rightarrow \mathcal{V}$ in $\text{FrmSch}^b$ such that the two morphisms $\Gamma \to \mathcal{V}$ are flat and quasi-compact. Let $\mathcal{V}_\bullet$ be its nerve (this is a simplicial formal scheme with $\mathcal{V}_0 = \mathcal{V}$ and $\mathcal{V}_1 = \Gamma$). Set

$$\text{Bun}_Q(\mathcal{V}/\Gamma) := \text{Tot}(\text{Bun}_Q(\mathcal{V}_\bullet)).$$

This notation is legitimate because by Proposition 3.5.4 the category $\text{Tot}(\text{Bun}_Q(\mathcal{V}_\bullet))$ depends only on the quotient stack $\mathcal{V}/\Gamma$: indeed,

$$\text{Tot}(\text{Bun}_Q(\mathcal{V}_\bullet)) = \lim_{\mathcal{Z}} \text{Bun}_Q(\mathcal{Z}),$$

---

*A morphism of $p$-adic formal schemes $f : \mathcal{V} \to \mathcal{Z}$ is said to be flat if for every $r \in \mathbb{N}$ it induces a flat morphism $\mathcal{V} \otimes (\mathbb{Z}/p^r \mathbb{Z}) \to \mathcal{Z} \otimes (\mathbb{Z}/p^r \mathbb{Z})$ (if $\mathcal{V}$ and $\mathcal{Z}$ are $\mathbb{Z}_p$-flat it suffices to check this for $n = 1$). We do not require $f$ to induce flat morphisms $H^0(\mathcal{Z}', \mathcal{O}_{\mathcal{Z}'}) \to H^0(\mathcal{V}', \mathcal{O}_{\mathcal{V}'})$, where $\mathcal{V}' \subset \mathcal{V}$ and $\mathcal{Z}' \subset \mathcal{Z}$ are open affines such that $\mathcal{V}' \subset f^{-1}(\mathcal{Z}')$. ""
where \( \mathcal{F} \) runs through the category of objects of FrmSch\(^b\) equipped with a morphism to \( \mathcal{U}/\Gamma \). Objects of \( \text{Bun}_{\mathbb{Q}}(\mathcal{U}/\Gamma) \) are called \( \Gamma\)-equivariant objects of \( \text{Bun}_{\mathbb{Q}}(\mathcal{U}) \).

Now let \( \mathcal{U} = W(X_{\text{perf}}) \), \( \Gamma = \mathcal{G} \). Let \( \mathcal{G}_n \) be the nerve of \( \mathcal{G} \). By Corollary 3.3.2 the essential image of the functor

\[
\text{Isoc}(X) = \text{Tot}(\text{QCoh}_{\text{fin}}^b(\mathcal{G}_n)) \otimes \mathbb{Q} \to \text{Tot}(\text{QCoh}_{\text{fin}}^b(\mathcal{G}_n)) \otimes \mathbb{Q}
\]

is contained in \( \text{Tot}(\text{Bun}_{\mathbb{Q}}(\mathcal{G}_n)) = \text{Bun}_{\mathbb{Q}}(W(X_{\text{perf}})/\mathcal{G}) \). So we get a tensor functor

\[
\text{Isoc}(X) \to \text{Bun}_{\mathbb{Q}}(W(X_{\text{perf}})/\mathcal{G}).
\]

**Theorem 3.5.6.** The functor (3.4) is an equivalence.

The proof will be given in 3.6.7.

### 3.6. Banachian games

We will use the notation and terminology of §2.3. As explained in §2.3.7 for a \( p \)-adic formal scheme \( \mathcal{U} \) it is harmless to identify an object \( \mathcal{F} \in \text{QCoh}(\mathcal{U}) \) with the corresponding sheaf \( \mathcal{F}_{\infty} \) on \( \mathcal{U} \), where \( \mathcal{F}_{\infty} \) is the projective limit of the sheaves \( \mathcal{F}_n = \mathcal{F}/p^n\mathcal{F} \). We will do it sometimes.

#### 3.6.1. One of the goals.

Let \( \text{FrmSch}_{\text{qcqs}}^b \subset \text{FrmSch}^b \) be the full subcategory of quasi-compact quasi-separated formal schemes. We will study the presheaves of categories

\[
\mathcal{U} \mapsto \text{QCoh}^b(\mathcal{U}) \otimes \mathbb{Q} \quad \text{and} \quad \mathcal{U} \mapsto \text{QCoh}_{\text{fin}}^b(\mathcal{U}) \otimes \mathbb{Q}, \mathcal{U} \in \text{FrmSch}_{\text{qcqs}}^b.
\]

One of our goals is to prove that these presheaves are fpqc sheaves (see Corollary 3.6.6 below).

**Remark 3.6.2.** \( \text{QCoh}^b(\text{Spf} \mathbb{Z}_p) \otimes \mathbb{Q} \) is the category of Banachian spaces over \( \mathbb{Q}_p \). By a Banachian space over a non-Archimedean field we mean a complete topological vector space such that the topology comes from a non-Archimedean norm. (Note that the topology determines an equivalence class of norms but not a particular norm.)

**Remark 3.6.3.** If \( \mathcal{U} \in \text{FrmSch}^b \) is affine then \( \text{QCoh}^b(\mathcal{U}) \otimes \mathbb{Q} \) is the category of Banachian modules over the Banach \( \mathbb{Q}_p \)-algebra \( H^0(\mathcal{U}, \mathcal{O}_\mathcal{U}) \otimes \mathbb{Q} \). For any \( \mathcal{U} \in \text{FrmSch}_{\text{qcqs}}^b \), Corollary 3.6.6 proved below allows one to identify \( \text{QCoh}^b(\mathcal{U}) \otimes \mathbb{Q} \) with the category of “quasi-coherent sheaves of Banachian \((\mathcal{O}_\mathcal{U} \otimes \mathbb{Q})\)-modules”.

**Proposition 3.6.4.** Let \( f : \mathcal{U}' \to \mathcal{U} \) be a faithfully flat morphism in \( \text{FrmSch}_{\text{qcqs}}^b \). Let \( M \in \text{QCoh}^b(\mathcal{U}) \otimes \mathbb{Q} \). Then \( M \in \text{QCoh}_{\text{fin}}^b(\mathcal{U}) \otimes \mathbb{Q} \) if and only if \( f^*M \in \text{QCoh}_{\text{fin}}^b(\mathcal{U}') \otimes \mathbb{Q} \).

**Proof.** The “only if” statement is obvious.

Say that \( M_1, M_2 \in \text{QCoh}^b(\mathcal{U}) \) are isogenous if they are isomorphic in \( \text{QCoh}^b(\mathcal{U}) \otimes \mathbb{Q} \). To prove the “if” statement, we have to show that if \( M \in \text{QCoh}^b(\mathcal{U}) \) and \( f^*M \) is isogenous to some \( N' \in \text{QCoh}_{\text{fin}}^b(\mathcal{U}') \) then \( M \) is isogenous to some \( N \in \text{QCoh}_{\text{fin}}^b(\mathcal{U}) \). We have \( f^*M \supset N' \supset p^n f^*M \) for some \( n \). The submodule \( N'/p^{n+1} f^*M \subset f^*(M/p^{n+1}M) \) is finitely generated. Since \( \mathcal{U} \otimes (\mathbb{Z}/p^{n+1} \mathbb{Z}) \) is a quasi-compact quasi-separated scheme, \( M/p^{n+1}M \) equals the sum of its finitely generated submodules. So there exists a finitely generated submodule \( L \subset M/p^{n+1}M \) such that \( f^*L \supset N'/p^{n+1} f^*M \). Since \( N' \supset p^n f^*M \), we have \( f^*L \supset f^*(p^n M/p^{n+1}M) \), so \( L \supset p^n M/p^{n+1}M \) by fully faithfulness. Now let \( N \subset M \) be the preimage of \( L \). Then \( N \supset p^n M \), and \( N/p^{n+1}M = L \) is finitely generated. So \( N \) is finitely generated. It is clear that \( N \) is isogenous to \( M \).
Proposition 3.6.5. Let $\Gamma \to \mathcal{V}$ be a flat groupoid in $\text{FrmSch}_{qcqs}^b$. Let $\mathcal{V}_*$ be its nerve. Then the functor
\begin{equation}
\text{Tot}(\text{QCoh}^b(\mathcal{V}_*)) \otimes \mathbb{Q} \to \text{Tot}(\text{QCoh}^b(\mathcal{V}_*) \otimes \mathbb{Q})
\end{equation}
is an equivalence.

The proof given below is similar to that of [O] Thm. 1.9.

Proof. Full faithfulness is clear. Let us prove essential surjectivity.

Let $\pi_0, \pi_1 : \mathcal{V}_1 \to \mathcal{V}_0$ be the face maps and $e : \mathcal{V}_0 \to \mathcal{V}_1$ the degeneration map (i.e., the unit of the groupoid). An object of $\text{Tot}(\text{QCoh}^b(\mathcal{V}_*) \otimes \mathbb{Q})$ is given by a pair $(M, \alpha)$, where $M \in \text{QCoh}^b(\mathcal{V}_0)$ and $\alpha \in \text{Hom}(\pi_0^* M, \pi_1^* M) \otimes \mathbb{Q}$ satisfies the cocycle condition and the condition $e^*(\alpha) = \text{id}_M$. Let $n \geq 0$ be such that $p^n \alpha \in \text{Hom}(\pi_0^* M, \pi_1^* M)$. We will construct an $O_{\mathcal{V}_*}$-submodule $M' \subset M$ containing $p^n M$ such that $M' \in \text{QCoh}(\mathcal{V})$ and the morphism $\alpha : \pi_0^* M \to \pi_1^* M'$ maps $\pi_0^* M'$ to $\pi_1^* M'$. This is clearly enough.

Let us introduce some notation. For $i \leq j$ we have the morphism
\begin{equation}
\alpha : \pi_0^*(p^i M/p^j M) \to \pi_1^*(p^{i-n} M/p^{j-n} M).
\end{equation}

It induces a morphism
\begin{equation}
\beta : p^i M/p^j M \to \Phi(p^{i-n} M/p^{j-n} M), \quad \Phi := (\pi_0)_* \pi_1^*.
\end{equation}

Now let $M' \subset M$ be the submodule containing $p^n M$ such that
\begin{equation}
M' / p^n M = \ker(M/p^n M \xrightarrow{\beta} \Phi(p^{-n} M/M)).
\end{equation}

Then $\alpha(\pi_0^* M') \subset \pi_1^* M$, and $M'$ is the biggest submodule of $M$ with this property. Our goal is to prove that
\begin{equation}
\alpha(\pi_0^* M') \subset \pi_1^* M'.
\end{equation}

Let us make some general remarks. Since $\mathcal{V}_*$ is the nerve of a flat groupoid, $\Phi$ is a left exact comonad acting on the category $\{N \in \text{QCoh}(\mathcal{V}) | p^r N = 0 \text{ for some } r\}$. In particular, we have the coproduct $\mu : \Phi \to \Phi^2$. The morphisms (3.6) satisfy the following identity:\footnote{If $n = 0$ this identity means that $p^i M/p^j M$ is a comodule over $\Phi$.}

which follows from the coproduct property of $\alpha$: the composite maps
\begin{align*}
p^i M/p^j M &\xrightarrow{\beta} \Phi(p^{i-n} M/p^{j-n} M) \xrightarrow{\Phi(\beta)} \Phi^2(p^{j-2n} M/p^{j-2n} M), \\
p^i M/p^j M &\xrightarrow{\beta} \Phi(p^{i-n} M/p^{j-n} M) \to \Phi(p^{j-2n} M/p^{j-2n} M) \to \Phi^2(p^{j-2n} M/p^{j-2n} M)
\end{align*}
are equal to each other. In particular, the composite maps
\begin{align*}
M/p^{2n} M &\xrightarrow{\beta} \Phi(p^{-n} M/p^n M) \xrightarrow{\Phi(\beta)} \Phi^2(p^{-2n} M/M), \\
M/p^{2n} M &\xrightarrow{\beta} \Phi(p^{-n} M/p^n M) \to \Phi(p^{-2n} M/M) \to \Phi^2(p^{-2n} M/M)
\end{align*}
are equal to each other.

Let us now prove (3.8). By (3.7), $M'/p^{2n} M$ is equal to the preimage of $\Phi(M/p^n M)$ under the map
\begin{equation}
M/p^{2n} M \xrightarrow{\beta} \Phi(p^{-n} M/p^n M).
\end{equation}
So the first two arrows of (3.10) kill $M'/p^{2n}M$. Therefore the composite map (3.9) kills $M'/p^{2n}M$. So the morphism (3.11) maps $M'/p^{2n}M$ to

$$\text{Ker}(\Phi(M/p^nM) \xrightarrow{\Phi(\beta)} \Phi^2(p^{-n}M/M)) = \Phi(M'/p^nM).$$

This means that the composite map

$$\pi_0^*(M'/p^{2n}M) \xrightarrow{\alpha} \pi_1^*(p^{-n}M/p^nM) \to \pi_1^*(p^{-n}M/M')$$

is zero, which is equivalent to (3.8).

**Corollary 3.6.6.** Let $f : \mathcal{Y} \to \mathcal{Z}$ be a faithfully flat morphism in FrmSch$^\flat$$_{qcqs}$. Let $\mathcal{Y}_n$ be the fiber product (over $\mathcal{Z}$) of $n + 1$ copies of $\mathcal{Y}$. Then the functors

\begin{align*}
(3.12) & \quad \text{QCoh}^\flat(\mathcal{Z}) \otimes \mathbb{Q} \to \text{Tot}(\text{QCoh}^\flat(\mathcal{Y}_n) \otimes \mathbb{Q}), \\
(3.13) & \quad \text{QCoh}^\flat_{\text{fin}}(\mathcal{Z}) \otimes \mathbb{Q} \to \text{Tot}(\text{QCoh}^\flat_{\text{fin}}(\mathcal{Y}_n) \otimes \mathbb{Q}),
\end{align*}

are equivalences.

**Proof.** Full faithfulness is clear. To prove essential surjectivity of (3.12), combine usual flat descent with Proposition 3.6.5 applied to the groupoid with nerve $\mathcal{Y}_n$. Essential surjectivity of (3.13) follows from Proposition 3.6.4 and essential surjectivity of (3.12). □

3.6.7. Proof of Theorem 3.5.6. By Corollary 3.6.6, the functor (3.3) is an equivalence. Theorem 3.5.6 follows. □

**Proposition 3.6.8.** The categories $\text{QCoh}^\flat(\mathcal{Y}) \otimes \mathbb{Q}$ and $\text{QCoh}^\flat_{\text{fin}}(\mathcal{Y}) \otimes \mathbb{Q}$ are Karoubian for every $\mathcal{Y} \in \text{FrmSch}^\flat_{qcqs}$.

**Proof.** Let $M \in \text{QCoh}^\flat(\mathcal{Y})$, $r \geq 0$ an integer, $\pi \in p^{-r} \cdot \text{End} M$, $\pi^2 = \pi$. Set

$$M' := p^r M + (p^r \cdot \pi)(M) \subset M.$$ 

Then $M' \in \text{QCoh}^\flat(\mathcal{Y})$, $p^r M \subset M' \subset M$. Moreover, $\pi(M') \subset M'$, so we have a direct sum decomposition

$$M' = \pi(M') \oplus (1 - \pi)(M')$$

in $\text{QCoh}^\flat(\mathcal{Y})$. It gives the desired direct sum composition of $M$ in $\text{QCoh}^\flat(\mathcal{Y}) \otimes \mathbb{Q}$.

If $M$ is in $\text{QCoh}^\flat_{\text{fin}}(\mathcal{Y})$ then so are $M'$ and $\pi(M')$. □

3.7. Proof of Propositions 3.5.2 and 3.5.3

**Lemma 3.7.1.** Let $\varphi : A \to B$ be a homomorphism of $\mathbb{Z}_p$-flat $p$-adically complete rings. Assume that the homomorphism $A/p^r A \to B/p^r B$ induced by $\varphi$ is faithfully flat for all $r$ (or equivalently, for $r = 1$). Let $I_A \subset A \otimes \mathbb{Q}$ be an ideal, and let $I_B \subset B \otimes \mathbb{Q}$ be the ideal generated by $I_A$. If $I_B$ is generated by an idempotent in $B \otimes \mathbb{Q}$ then $I_A$ is generated by an idempotent in $A \otimes \mathbb{Q}$.

**Proof.** Let $e \in B \otimes \mathbb{Q}$ be the idempotent that generates $I_B$. Then

$$e \in \text{Ker}(B \Rightarrow B \hat{\otimes}_A B) \otimes \mathbb{Q} = A \otimes \mathbb{Q}.$$ 

Let us show that $I_A = e \cdot (A \otimes \mathbb{Q})$.
Since $A \subset B$ and $(1 - e) \cdot I_B = 0$, we have $(1 - e) \cdot I_A = 0$, so $I_A \subset e \cdot (A \otimes \mathbb{Q})$. It remains to show that
\begin{equation}
I_A + (1 - e) \cdot (A \otimes \mathbb{Q}) = A \otimes \mathbb{Q}.
\end{equation}

Let $J := A \cap (I_A + (1 - e) \cdot (A \otimes \mathbb{Q}))$. Then $J$ generates the unit ideal in $B \otimes \mathbb{Q}$, so $p^r \in JB$ for some $r$. But the map
\[ A/(p^{r+1}A + J) \to B/(p^{r+1}B + J \cdot B) \]
is injective by the faithful flatness assumption. So $p^r \in J + p^{r+1}A$. Therefore $p^r \in J$, which is equivalent to (3.14).

**Corollary 3.7.2.** Let $\varphi : A \to B$ be as in Lemma 3.7.1. Let $f : A^l \otimes \mathbb{Q} \to A^m \otimes \mathbb{Q}$ be an $A$-module homomorphism. Let $f_B : B^l \otimes \mathbb{Q} \to B^m \otimes \mathbb{Q}$ be the base change of $f$. If $\text{Coker } f_B$ is a projective $(B \otimes \mathbb{Q})$-module then $\text{Coker } f$ is a projective $(A \otimes \mathbb{Q})$-module.

**Proof.** For each $d$ apply Lemma 3.7.1 to the ideal of $A \otimes \mathbb{Q}$ generated by all minors of $f$ of order $d$. \[ \square \]

**Lemma 3.7.3.** For every $\mathcal{Y} \in \text{FrmSch}^\flat_{\text{qcqs}}$, the canonical functor
\begin{equation}
\text{QCoh}^\flat_{\text{fin}}(\mathcal{Y}) \otimes \mathbb{Q} \to \{\text{all } (\mathcal{O}_\mathcal{Y} \otimes \mathbb{Q})\text{-modules}\}
\end{equation}
is fully faithful, and its essential image contains $\text{Bun}_\mathbb{Q}(\mathcal{Y})$.

**Proof.** Full faithfulness is clear. Let us show that every object $M \in \text{Bun}_\mathbb{Q}(\mathcal{Y})$ belongs to the essential image of (3.15). If $M$ is a direct summand of $(\mathcal{O}_\mathcal{Y} \otimes \mathbb{Q})^n$ this is true by Proposition 3.6.8. The general case follows by Corollary 3.6.6. \[ \square \]

Lemma 3.7.3 provides a fully faithful embedding $\text{Bun}_\mathbb{Q}(\mathcal{Y}) \hookrightarrow \text{QCoh}^\flat_{\text{fin}}(\mathcal{Y}) \otimes \mathbb{Q}$. It commutes with pullbacks.

The following lemma implies Propositions 3.5.2 and 3.5.4.

**Lemma 3.7.4.** Let $\mathcal{Y}, \mathcal{Z} \in \text{FrmSch}^\flat$ be affine and $f : \mathcal{Y} \to \mathcal{Z}$ a faithfully flat morphism. Let $M \in \text{QCoh}^\flat_{\text{fin}}(\mathcal{Z}) \otimes \mathbb{Q}$. Suppose that $f^*M \in \text{Bun}_\mathbb{Q}(\mathcal{Y})$. Then $M$ is a direct summand of $\mathcal{O}_\mathcal{Z}^m \otimes \mathbb{Q}$ for some $n$.

**Proof.** Let $M_0 \in \text{QCoh}^\flat_{\text{fin}}(\mathcal{Z})$ be a representative for $M$. Since $\mathcal{Z}$ is affine, $M_0$ is generated by finitely many global sections, so we get an exact sequence
\[ 0 \to N_0 \to \mathcal{O}_\mathcal{Z}^m \to M_0 \to 0 \]
in $\text{QCoh}^\flat(\mathcal{Z})$. Let $N \in \text{QCoh}^\flat(\mathcal{Z}) \otimes \mathbb{Q}$ correspond to $N_0$. Then
\[ f^*N \in \text{Bun}_\mathbb{Q}(\mathcal{Y}) \subset \text{QCoh}^\flat_{\text{fin}}(\mathcal{Y}) \otimes \mathbb{Q}, \]
so $N \in \text{QCoh}^\flat_{\text{fin}}(\mathcal{Z})$ by Proposition 3.6.4. Thus we have an exact sequence
\[ (\mathcal{O}_\mathcal{Z} \otimes \mathbb{Q})^l \xrightarrow{\varphi} (\mathcal{O}_\mathcal{Z} \otimes \mathbb{Q})^m \to M \to 0 \]
inducing an exact sequence $(\mathcal{O}_\mathcal{Y} \otimes \mathbb{Q})^l \to (\mathcal{O}_\mathcal{Y} \otimes \mathbb{Q})^m \to f^*M \to 0$. It remains to apply Corollary 3.7.2. \[ \square \]
Appendix A. The isomorphism between \( W(X_{\text{perf}})/G \) and the prismatization of \( X \)

A.1. The goal. Prismatization is a certain functor \( X \mapsto \text{WCart}_X \) from the category of \( p \)-adic formal schemes to the category of stacks, see [BL] or [Dr2, §1]. Let us note that in [Dr2] this functor is denoted by \( X \mapsto X^\Delta \).

The stack \( \text{WCart}_X \) can always be presented (in many different ways) as a quotient of a formal scheme by a flat groupoid. The goal of this Appendix is to deduce from [BL] that if \( X \) is a Frobenius-smooth \( \mathbb{F}_p \)-scheme then \( \text{WCart}_X \) has a canonical presentation of this type; namely, one has a canonical isomorphism

\[
\text{WCart}_X \simto W(X_{\text{perf}})/G,
\]

as promised in Remark 1.2.1.

A.2. Prismatization of semiperfect \( \mathbb{F}_p \)-schemes.

A.2.1. Perfect case. According to [BL, Example 3.12], for any perfect \( \mathbb{F}_p \)-scheme \( X \) one has a canonical isomorphism \( W(X) \simto \text{WCart}_X \), where \( W(X) \) is the \( p \)-adic formal scheme whose underlying topological space is that of \( X \) and whose structure sheaf is obtained by applying to \( \mathcal{O}_X \) the functor of \( p \)-typical Witt vectors.

A.2.2. General case. More generally, for any semiperfect \( \mathbb{F}_p \)-scheme \( X \) one has a canonical isomorphism \( A_{\text{cris}}(X) \simto \text{WCart}_X \), where \( A_{\text{cris}}(X) \) is the \( p \)-adic formal scheme whose underlying topological space is that of \( X \) and whose structure sheaf is obtained by applying Fontaine’s functor \( A_{\text{cris}} \) to \( \mathcal{O}_X \). This is [BL, Lemma 6.1] in the case that \( X \) is regular semiperfect (the only one that we need) and [BL, Cor. 7.18] for arbitrary semiperfect schemes.

In these statements from [BL] \( A_{\text{cris}} \) is not mentioned explicitly. Instead, it is proved that if \( X \) is affine then \( \text{WCart}_X \) is the Spf of the prismatic cohomology of \( X \). But it is known from [BS, §5] that the prismatic cohomology of an \( \mathbb{F}_p \)-scheme identifies with its crystalline cohomology; on the other hand, if \( R \) is a semiperfect \( \mathbb{F}_p \)-algebra then the crystalline cohomology of \( \text{Spec } R \) equals \( A_{\text{cris}}(R) \) (in particular, it lives in degree zero).\(^{17}\)

A.3. The morphism \( W(X_{\text{perf}}) \to \text{WCart}_X \). By A.2.1 For any \( \mathbb{F}_p \)-scheme \( X \), the morphism

\[
X_{\text{perf}} \to X
\]

induces a morphism

\[
W(X_{\text{perf}}) = \text{WCart}_{X_{\text{perf}}} \to \text{WCart}_X
\]

(we have used A.2.1).

From now on, suppose that \( X \) is Frobenius-smooth in the sense of §2.1 Then (A.2) is a quasi-syntomic cover, so by [BL, Lemma 6.3], the map (A.3) is an fpqc cover (by this we mean that for every scheme \( S \), every morphism \( S \to \text{WCart}_X \) lifts to a morphism \( S \to W(X_{\text{perf}}) \) fpqc-locally on \( S \)).

\(^{17}\)In general, the “true” objects are the derived crystalline cohomology of \( \text{Spec } R \) and the derived version of \( A_{\text{cris}}(R) \). They are canonically isomorphic and live in nonpositive degrees; if \( R \) is regular semiperfect then they live in degree zero and are equal to their classical prototypes.
A.4. The Čech nerve of (A.3). By definition, \( G \) is the groupoid acting on \( W(X_{\text{perf}}) \) whose Čech nerve is the simplical formal scheme \( A_\bullet \) from \( \mathbf{2.2.3} \). So to finish constructing the isomorphism (A.1), it remains to identify the Čech nerve of (A.3) with \( A_\bullet \).

Since \( X \) is Frobenius-smooth, the morphism (A.2) is flat. So by [BL, Rem. 3.5], the Čech nerve of (A.3) is \( W\text{Cart}_{\mathcal{P}_\bullet} \), where \( \mathcal{P}_\bullet \) is the Čech nerve of (A.2). By \( \mathbf{A.2.2} \) we have \( W\text{Cart}_{\mathcal{P}_\bullet} = A_{\text{cris}}(\mathcal{P}_\bullet) \). Finally, \( A_{\text{cris}}(\mathcal{P}_\bullet) =: A_\bullet \).

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