HOMOGENEOUS QUANTUM SYMMETRIES OF FINITE SPACES OVER THE CIRCLE GROUP

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Abstract. Suppose $D$ is a finite dimensional $C^*$-algebra carrying a continuous action $\Pi$ of the circle group $\mathbb{T}$. We study the quantum symmetry group of $D$, taking $\Pi$ into account. We show that they are braided compact quantum groups $G$ over $\mathbb{T}$. Here, the $R$-matrix, $\mathbb{Z} \times \mathbb{Z} \ni (m,n) \to \zeta^{-m \cdot n} \in \mathbb{T}$, for a fixed $\zeta \in \mathbb{T}$, governs the braided structure. In particular, if $\Pi$ is trivial, $\zeta = 1$ or $D$ is commutative, then $G$ coincides with Wang’s quantum group of automorphisms of $D$. Moreover, we show that the bosonisation of $G$ corresponds to the quantum symmetry group of the crossed product $C^*$-algebra $D \rtimes \mathbb{Z}$, where the $\mathbb{Z}$-action is generated by $\Pi_{\zeta^{-1}}$.

1. Introduction

In the realm of compact quantum groups of Woronowicz [23,24], the inception of the theory of quantum symmetry groups of classical and quantum space goes back to the seminal paper of Wang [22]. In that article, Wang studied quantum symmetry groups of finite sets and finite dimensional noncommutative $C^*$-algebras or finite quantum spaces equipped with a reference state. They are extended to several discrete structures by Banica, Bichon and several others ([1,2,7,11,21] and the references therein). On the other hand, Goswami [8] introduced the continuous counterpart, namely, the quantum isometry group for spectral triples. He and his collaborators studied it extensively [3,5,9,10]. Goswami’s work motivated Banica and Skalski to develop a general framework of quantum symmetry groups based on orthogonal filtrations of unital $C^*$-algebras [4].

Semidirect product construction of groups is a fundamental way of extending some homogeneous symmetries to inhomogeneous symmetries of a physical system. This perspective suggests that, within the framework of the noncommutative geometry, semidirect product construction for quantum groups might be a natural way to obtain inhomogeneous symmetries of quantum spaces or $C^*$-algebras from their homogeneous symmetries. Several interesting examples of unital $C^*$-algebras $D$ naturally carry a continuous action $\gamma$ of the circle group $\mathbb{T}$, that preserves a distinguished faithful state, say $\phi$, on $D$. Fix $\zeta \in \mathbb{T}$. We view the circle group $\mathbb{T}$ a quasitriangular quantum group with respect to the $R$-matrix $R$, which is a bicharacter on $\mathbb{Z}$, defined by

$$R(l,m) = \zeta^{-lm}, \quad \text{for all } l, m \in \mathbb{Z}. \quad (1.1)$$

The quantum symmetries of compact type of $D$ encoding and respecting the information of the action $\gamma$ are the $\mathbb{T}$-homogeneous compact quantum symmetries of the system $(D, \gamma, \phi)$. If such an object exists, it becomes a braided compact quantum
group $G$ over $\mathbb{T}$ (see [17, Definition 6.1]). The $\mathbb{T}$-action $\gamma$ and the above $R$-matrix govern the braided structure of $G$. Moreover, we expect to capture the inhomogeneous compact quantum symmetries of $(D, \gamma, \phi)$ in terms of the bosonisation of $\mathbb{G}$ (see [17, Theorem 6.4]).

This article aims to support the above model for finite dimensional $C^*$-algebras $D$ carrying an action of $\mathbb{T}$. If $D$ is commutative with $\dim(D) = n$ for some natural number $n$, then it is generated by $n$ many commuting projections $p_1, \ldots, p_n$ satisfying $\sum_{i=1}^n p_i = 1$. If $\gamma$ is a continuous action of $\mathbb{T}$ on $D$ then $\gamma_z(p_i)$ is self adjoint and $\gamma_z(p_i) = \gamma_z(p_i)^2$ for all $z \in \mathbb{T}$ and $1 \leq i \leq n$. This forces $\gamma$ to become trivial. Therefore, the quantum symmetry group of $D$ corresponds to Wang’s quantum permutation group [22, Theorem 3.1]. So, we consider $D$ to be noncommutative, that is, $D$ is isomorphic to a finite direct sum of matrix algebras.

In particular, let $D = M_n(\mathbb{C})$ for a fixed number $n$. The matrix algebra $M_n(\mathbb{C})$ is the universal $C^*$-algebra generated by $\{E_{ij}\}_{1 \leq i, j \leq n}$ subject to the following conditions:

$$E_{ij}E_{kl} = \delta_{j,k} \cdot E_{il}, \quad E_{ij}^* = E_{ji}, \quad \sum_{i=1}^n E_{ii} = 1. \quad (1.2)$$

Let $d_1 \leq d_2 \leq \cdots \leq d_n$ be integers. Then it is easy to verify that $\Pi$, defined by

$$\Pi_z(E_{ij}) := z^{d_i - d_j} E_{ij} \quad \text{for all } z \in \mathbb{T}, \quad (1.3)$$

is a continuous action of $\mathbb{T}$ on $M_n(\mathbb{C})$. Let $\phi$ be the canonical normalised trace on $M_n(\mathbb{C})$, that is, $\phi(E_{ij}) = \frac{1}{n} \cdot \delta_{i,j}$ for all $1 \leq i, j \leq n$. The action $\Pi$ of $\mathbb{T}$ on $M_n(\mathbb{C})$ preserves $\phi$, that is, $\phi \circ \Pi_z(\cdot) = \phi(\cdot) 1$.

Consider the braided tensor product $\boxtimes_\zeta$ of unital $C^*$-algebras with $\mathbb{T}$-action discussed in Section 2.2. The braided compact quantum symmetries of the triple $(M_n(\mathbb{C}), \Pi, \phi)$ are $\phi$ preserving actions $\eta: M_n(\mathbb{C}) \to M_n(\mathbb{C}) \boxtimes_\zeta B$ of braided compact quantum groups $\mathbb{G} = (B, \beta, \Delta_B)$ over $\mathbb{T}$ on $(M_n(\mathbb{C}), \Pi)$ in the sense of the Definitions 3.3.1 and 3.3.18 respectively. In fact, $\eta$ is uniquely determined by its matrix coefficients $\{b_{ij}^{kl}\}_{1 \leq i,j,k,l \leq n} \subseteq B$:

$$\eta(E_{ij}) = \sum_{k,l=1}^n E_{kl} \boxtimes_\zeta b_{ij}^{kl}, \quad \text{for all } 1 \leq i, j \leq n. \quad (1.4)$$

We construct a braided compact quantum group, denoted by $\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)$, in Theorem 3.3.1. Subsequently, we observe in Proposition 3.3.18 that it acts faithfully on $(M_n(\mathbb{C}), \Pi)$ and preserves $\phi$.

According to [17, Section 6], every braided compact quantum group over $\mathbb{T}$ uniquely corresponds to an ordinary compact quantum group. In the purely algebraic setting, this was discovered by Radford [18] and extensively studied by Majid [13] under the name bosonisation. In Section 4 we construct the bosonisation of $\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)$, denoted by $\text{Bos}(\text{Aut}(M_n(\mathbb{C}), \Pi, \phi))$.

In Proposition 5.3.1 we observe that $\text{Bos}(\text{Aut}(M_n(\mathbb{C}), \Pi, \phi))$ acts on $M_n(\mathbb{C}) \boxtimes_\zeta C(\mathbb{T})$, extending the (braided) action of $\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)$ on $(M_n(\mathbb{C}), \Pi)$ and the action of $\mathbb{T}$ on $C(\mathbb{T})$ by translation. Then we use it to prove the universal property of $\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)$ in Theorem 5.3.1.

The $R$-matrix [14] induces an action of $\mathbb{Z}$ on $M_n(\mathbb{C})$ generated by $\Pi_{-1}$. Then the associated crossed product $M_n(\mathbb{C}) \rtimes \mathbb{Z}$ becomes isomorphic to $C(\mathbb{T}) \boxtimes_\zeta M_n(\mathbb{C})$.

Consider the orthogonal filtrations $\widetilde{M_n(\mathbb{C})}$ and $\tilde{C}(\mathbb{T})$ of $M_n(\mathbb{C})$ and $C(\mathbb{T})$ given in Example 5.3.15 and Example 5.3.16 respectively. Then we apply [6, Proposition 4.9] to construct an orthogonal filtration of $C(\mathbb{T}) \boxtimes_\zeta M_n(\mathbb{C})$. Consequently, in Proposition 5.3.19 we prove that the corresponding quantum symmetry group of $C(\mathbb{T}) \boxtimes_\zeta M_n(\mathbb{C})$ is isomorphic to $\text{Bos}(\text{Aut}(M_n(\mathbb{C}), \Pi, \phi))$. This is the universal property of the bosonisation of $\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)$. Furthermore, combining
it with [8] Theorem 5.1 we also observe Bos(Aut(Mn(C), Π, ϕ)) ∼= Dv in Corollary 5.21, where Dv is the Drinfeld’s double of the quantum symmetry groups QISO(C(T)) and QISO(Mn(C)) with respect to some bicharacter V suitably chosen.

Within the scope of this article, the bosonisation picture has significant advantages over the Drinfeld’s double, both computationally and conceptually. Indeed, the concrete computation of the bosonisation of a braided compact quantum group over T is more straightforward than the computation of Drinfeld’s double. Secondly, in Remark 5.21 we observe that Aut(Mn(C), Π, ϕ) is the homogeneous quantum symmetry group of the C∗-dynamical system (Mn(C, Z, Π, · −1)) obtained from (Mn(C), Π). At the same time, its bosonisation (semidirect product) corresponds to the inhomogeneous (extending T-homogeneous) quantum symmetry group of the same system. Conceptually, this approach is more natural while computing the quantum symmetry of the given the C∗-dynamical system (Mn(C, Z, Π, · −1)).

Finally, in Section 6 we employ these analyses to study similar results for the direct sum of matrix algebras. As a byproduct, we observe in Remark 6.10 that a finite classical space, that is, D ∼= Cm for any natural number m, has no braided quantum symmetries (over T), as expected.

2. Preliminaries

All Hilbert spaces and C∗-algebras (which are not explicitly multiplier algebras) are assumed to be separable. For any subsets X, Y of a given unital C∗-algebra D, the norm closure of \(\{xy \mid x \in X, y \in Y\}\) ⊆ D is denoted by XY. We use ⊗ for both the tensor product of Hilbert spaces and the minimal tensor product of C∗-algebras, which is well understood from the context.

2.1. Compact quantum groups and their actions. A compact quantum group is a pair G = (A, ∆A) consisting of a unital C∗-algebra A and a unital ∗-homomorphism ∆A : A → A ⊗ A such that

1. ∆A is coassociative: (idA ⊗ ∆A) ∘ ∆A = (∆A ⊗ idA) ∘ ∆A;
2. ∆A satisfies the cancellation conditions: ∆A(A)(1A ⊗ A) = A ⊗ A = ∆A(A)(A ⊗ 1A).

Let D be a unital C∗-algebra. An action of a compact quantum group G = (A, ∆A) on D is a unital ∗-homomorphism γ : D → D ⊗ A such that

1. γ satisfies the action equation, that is, (idD ⊗ ∆A) ∘ γ = (γ ⊗ idA) ∘ γ;
2. γ satisfies the Podleś condition: γ(D)(1D ⊗ A) = D ⊗ A.

Suppose τ : D → C is a state. Then γ is said to be τ-preserving if (τ ⊗ idA)γ(d) = τ(d) · 1A for all d ∈ D. In fact, γ is injective whenever τ is faithful.

Example 2.1. Represent C(T) on L2(T) faithfully by pointwise multiplication. Let z ∈ C(T) be the unitary operator of pointwise multiplication with the identity function on T. Then ∆C(T)(z) = z ⊗ z defines the comultiplication map on C(T). The pair (C(T), ∆C(T)) is T viewed as a compact quantum group. Suppose D is a unital C∗-algebra and γ : T → Aut(D) is a continuous action. An element d ∈ D is a homogeneous element of degree deg(d) ∈ Z if γz(d) = zdeg(d) · d for all z ∈ T. Equivalently, γ(d) = d ⊗ zdeg(d) while γ is viewed as an action of (C(T), ∆C(T)) on D.

2.2. Braided tensor product of C∗-algebras. Let L1 and L2 be separable Hilbert spaces. Let πi : T → U(Li) be continuous representations for i = 1, 2. Suppose, \(\{λ^i_m\}_{m \in \mathbb{N}}\) is an orthonormal basis of \(L_i\) consisting of the eigenvectors for \(π_i\), that is, \((π_i)_z(λ^i_m) = z^m λ^i_m\) for some \(t^i_m \in \mathbb{N}\), for i = 1, 2.
By [17] Equation (3.2) & Proposition 3.2, the braiding unitary $\mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{L}_2 \otimes \mathcal{L}_1$ and its inverse $\mathcal{L}_2 \times \mathcal{L}_1 \to \mathcal{L}_1 \otimes \mathcal{L}_2$ associated to the R-matrix are defined on the basis elements by

$$\mathcal{L}_1 \times \mathcal{L}_2 (\lambda_m^1 \otimes \lambda_n^2) = \zeta_{m,n} \lambda_n^1 \otimes \lambda_m^2, \quad \mathcal{L}_2 \times \mathcal{L}_1 (\lambda_m^1 \otimes \lambda_n^2) = \zeta_{m,n}^{-1} \lambda_n^2 \otimes \lambda_m^1. \quad (2.2)$$

Consider the category $\mathcal{C}^*\text{alg}(\mathbb{T})$ consisting of unital $C^*$-algebras with a continuous action of $\mathbb{T}$ as objects and $\mathbb{T}$-equivariant unital $^*$-homomorphisms as arrows.

Let $(D_i, \gamma_i)$ are objects of $\mathcal{C}^*\text{alg}(\mathbb{T})$ and $\psi_i : D_i \to B(L_i)$ be faithful $\mathbb{T}$-equivariant representations of $(D_i, \gamma_i)$, respectively, for $i = 1, 2$. For any homogeneous elements $d_i \in D_i$ we have

$$\begin{align*}
(\pi_i)_z(d_i^1 \lambda_m^i) &= ((\psi_i)_z(d_i))((\pi_i)_z(\lambda_m^i)) = z^{\deg(d_i)}(d_i^1 \lambda_m^i), \quad (2.3) \\
(\pi_i)_z(d_i^2 \lambda_m^i) &= ((\pi_i)_z(d_i^2))((\pi_i)_z(\lambda_m^i)) = z^{-\deg(d_i)}(d_i^2 \lambda_m^i).
\end{align*}$$

for all $z \in \mathbb{T}$ and $i = 1, 2$.

A unital $^*$-homomorphism $f : D_1 \to D_2$ is an arrow $D_1 \to D_2$ in $\mathcal{C}^*\text{alg}(\mathbb{T})$ if it is $\mathbb{T}$-equivariant: $f \circ (\gamma_1)_z = (\gamma_2)_z \circ f$ for all $z \in \mathbb{T}$. The set of arrows $D_1 \to D_2$ in $\mathcal{C}^*\text{alg}(\mathbb{T})$ is denoted by $\text{Mor}^*(D_1, D_2)$.

The R-matrix defines the braided tensor product $D_1 \boxtimes D_2 = j_1(D_1)j_2(D_2)$, where $j_i : D_i \to D_1 \boxtimes D_2$ are the canonical embeddings, for $i = 1, 2$. The concrete descriptions of these maps are given by

$$j_1(d_1) = \psi_1(d_1) \otimes 1_{B(L_2)}, \quad j_2(d_2) = \mathcal{L}_1 \times \mathcal{L}_2 (\psi_2(d_2) \otimes 1_{B(L_1)})\mathcal{L}_2 \times \mathcal{L}_1,$$

for all $d_i \in D_i$, where $i = 1, 2$. In particular, if $d_i \in D_i$ are the homogeneous elements, then the commutation relation between the elementary tensor factors $j_i(d_1) := d_1 \boxtimes 1_{D_2}$ and $j_2(d_2) := 1_{D_1} \boxtimes d_2$ is given by

$$d_1 \boxtimes d_2 := (d_1 \boxtimes 1_{D_2}) \cdot (1_{D_1} \boxtimes d_2) = \zeta^{-\deg(d_1)\deg(d_2)}(1_{D_1} \boxtimes d_2) \cdot (d_1 \boxtimes 1_{D_2}). \quad (2.5)$$

The braided tensor product $D_1 \boxtimes D_2$ carries the continuous diagonal action $(\gamma_1 \otimes \gamma_2)_z(d_1 \boxtimes d_2) := (\gamma_1)_z(d_1) \boxtimes (\gamma_2)_z(d_2)$ for all $z \in \mathbb{T}$. Thus $(D_1 \boxtimes D_2, \alpha \otimes \beta)$ is an object of $\mathcal{C}^*\text{alg}(\mathbb{T})$. Furthermore, $(\mathcal{C}^*\text{alg}(\mathbb{T}), \boxtimes)$ is a monoidal category with $\mathbb{C}$ along with the trivial action of $\mathbb{T}$ as monoidal unit. We refer to [16][17] for more details of this construction. In particular, $\boxtimes_1$ coincides with the minimal tensor product $\otimes$ of $C^*$-algebras. Also, $A \boxtimes B \cong A \otimes B$ if either $A$ or $B$ is trivial.

Suppose $(B, \beta)$ and $(D, \gamma)$ are objects of the category $\mathcal{C}^*\text{alg}(\mathbb{T})$. By virtue of [17] Proposition 3.6, there exists an injective unital $^*$-homomorphism $\Psi^{D,B} : C(T) \boxtimes B \to (C(T) \boxtimes D) \otimes (C(T) \boxtimes B)$. For $a \in C(T)$, $d \in D$, $b \in B$ it is defined by

$$\begin{align*}
\Psi^{D,B}_D(j_1(a)) &= (j_1 \otimes j_1)D_{C(T)}(a), \quad \Psi^{D,B}_D(j_2(b)) = (j_2 \otimes j_1)\gamma(d), \\
\Psi^{D,B}_D(j_2(b)) &= 1_{C(T)\boxtimes D} \otimes j_2(b).
\end{align*}$$

Here we view the action $\beta$ as a unital $^*$-homomorphism $\beta : B \to B \otimes C(T)$. These maps will play crucial roles in Sections 4 and 5.

3. Action of Braided Quantum Groups on Matrix Algebras

Fix $\zeta \in \mathbb{T}$ and recall the monoidal category $(\mathcal{C}^*\text{alg}(\mathbb{T}), \boxtimes)$ from the previous section.

**Definition 3.1** (compare with [17] Definition 6.1). A triple $\mathbb{G} = (A, \alpha, \Delta_A)$ is a braided compact quantum group in $(\mathcal{C}^*\text{alg}, \boxtimes)$ if $(A, \alpha)$ is an object of $\mathcal{C}^*\text{alg}(\mathbb{T})$ and $\Delta_A \in \text{Mor}^*(A, A \boxtimes A)$ such that

1. $\Delta_A$ is coassociative

$$\Delta_A \boxtimes \text{id}_A \circ \Delta_A = (\text{id}_A \boxtimes \Delta_A) \circ \Delta_A; \quad (3.2)$$


There exists a unique continuous action $\Delta_A$ of $A \otimes \zeta$ on $A$ satisfying
\[
\Delta_A((1_A \otimes \zeta) A) = A \otimes \zeta A = \Delta_A(A) (A \otimes \zeta 1_A).
\]
(3.3)
Then we say $G = (A, \alpha, \Delta_A)$ is a braided compact quantum group over $\mathbb{T}$. We refer to [12, 14] for genuine examples of braided compact quantum groups over $\mathbb{T}$. Indeed, $G$ is an ordinary compact quantum group if either $\zeta = 1$ or the action $\alpha$ is trivial.

**Theorem 3.4.** For a given natural number $n$, let $d_1 \leq d_2 \leq \cdots \leq d_n$ be integers. Let $A$ be the universal $C^*$-algebra generated by $(u_{ij}^{kl})_{1 \leq i,j,k,l \leq n}$ subject to the following relations:
\[
\sum_{t=1}^n (\zeta^d d_k - d_{l-1} d_t) u_{ik}^{nt} = 0,
\]
\[
\sum_{t=1}^n (\zeta^d d_k - d_{l-1} d_t) u_{jk}^{nt} = 0,
\]
\[
\sum_{t=1}^n (\zeta^d d_k - d_{l-1} d_t) u_{kl}^{nt} = 0.
\]
(3.5)
Thus, $(A, \alpha)$ is an object of $C^*\text{alg}(\mathbb{T})$. There exists a unique $\Delta_A \in \text{Mor}^\pi (A, A \otimes \zeta A)$ such that
\[
\Delta_A(u_{ij}^{kl}) = \sum_{t=1}^n u_{it}^{kl} \otimes \zeta u_{tj}^{kl},
\]
\[
\Delta_A u_{ij}^{kl} = \sum_{r,s=1}^n u_{ir}^{kl} \otimes \zeta u_{sj}^{kl},
\]
(3.10)
Moreover, $\Delta_A$ satisfies [32]-[33]. Then $(A, \alpha, \Delta_A)$ is a braided compact quantum group over $\mathbb{T}$ denoted by $\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)$.

**Proof.** Let $W = \mathbb{C}^n$ and $\rho$ is a unitary representation of $\mathbb{T}$ on $W$. Let $\{ e_1, e_2, \cdots, e_n \}$ be the eigenbasis for $\rho$ such that $\rho_z(e_i) = z^i e_i$ for all $1 \leq i \leq n$.

We identify $\text{End}(W)$ with $M_n(\mathbb{C})$ with respect to this ordered basis. Then $M_n(\mathbb{C})$ is generated by the rank one matrices $\{ E_{ij} \}_{1 \leq i,j \leq n}$, defined by $E_{ij} e_k = \delta_{j,k} e_i$ for all $1 \leq k \leq n$. In fact, $\rho$ induces the action $\Pi$ in (1.3) of $T$ on $M_n(\mathbb{C})$.

Similarly, we identify $\text{End}(M_n(\mathbb{C})) \cong M_n(\mathbb{C}) \otimes M_n(\mathbb{C})^* \cong M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. Subsequently, $\Pi$ induces a continuous action $\tilde{\Pi}$ of $\mathbb{T}$ on $\text{End}(\text{M}_n(\mathbb{C}))$. On the basis vectors $E_{ij} \otimes E_{kl}$ it acts in the following way:
\[
\tilde{\Pi}_z(E_{ij} \otimes E_{kl}) := z^{d_i - d_i - d_j - d_k} E_{ij} \otimes E_{kl} \quad \text{for all } z \in \mathbb{T}.
\]
(3.12)
Suppose $A$ is the universal unital $*$-algebra generated by $u_{ij}^{kl}$ for $1 \leq i,j \leq n$ satisfying (3.10)-(3.11). Define $u \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes A$ by
\[
u = \sum_{i,j,k,l=1}^n E_{ik} \otimes E_{jl} \otimes u_{ij}^{kl}.
\]
(3.13)
Then we use conditions (3.7), (3.14), (3.8) and the fact that ∆ of the action $u \zeta$ Since $\sum_{k=1}^{n} E_{kk} = 1_{A}$ to show $u^* u = 1_{M_n(\mathbb{C})} \otimes 1_{A}$:

$$\sum_{i,j,k,l=1}^{n} E_{ik} \otimes E_{jl} \otimes u_{kl}^{ij} = \left( \sum_{p,q,r,s=1}^{n} E_{pr} \otimes E_{qs} \otimes u_{rs}^{pq} \right)$$

$$= \left( \sum_{p,q,r,s=1}^{n} E_{pr} \otimes E_{qs} \otimes u_{pq}^{rs} \right)$$

Similarly, we can verify that for all $z \in T$ the elements $(\alpha_{z}(u_{kl}^{ij}))_{1 \leq i,j,k,l \leq n}$ in $M_n(\mathbb{C})$ satisfy the relations (3.5)-(3.9). This ensures the existence and uniqueness of the action $\alpha$ of $T$ on $A$. Also, it is easy to verify that any $C^*$-seminorm $\|\cdot\|$ on $A$ will satisfy $\|u_{kl}^{ij}\| \leq 1$ for all $1 \leq i,j \leq n$ and $A$ is the completion of $A$ with respect to the largest $C^*$-seminorm on $A$.

Next, we consider the diagonal action $\alpha \bowtie \alpha$ of $T$ on $\mathbb{C} \otimes A$: $(\alpha \bowtie \alpha)_{z} (u_{kl}^{ij} \otimes u_{rs}^{pq}) = \alpha_{z} (u_{kl}^{ij} \otimes u_{rs}^{pq}) = \sum_{k=1}^{n} E_{kk} \otimes E_{ts} \otimes u_{rs}^{pq}$ for all $1 \leq i,j,k,l \leq n$. This shows that $\Delta_A(u_{kl}^{ij})$ is also a homogeneous element of $A \otimes \mathbb{C}$ A with degree $d_i - d_j + d_k - d_l$. Hence $\Delta_A$ is $T$-equivariant on the generators of $A$. Next, we recall the commutation relations (3.5) and (3.14) to verify the latter for $\Delta_A(u_{kl}^{ij})$:

$$\sum_{t=1}^{n} (\alpha \bowtie \alpha)_{\zeta \cdot t} (\Delta_A(u_{kl}^{ij})) = \sum_{t=1}^{n} \alpha_{\zeta \cdot t} (u_{pq}^{rs} \otimes \alpha_{\zeta \cdot t} (u_{kl}^{ij})) \left( \alpha_{\zeta \cdot t} (u_{kl}^{ij}) \otimes \alpha_{\zeta \cdot t} (u_{xy}^{yz}) \right)$$

Since $\zeta \in T$, we can rewrite (3.7) using the formula (3.10) in the following compact form:

$$\sum_{t=1}^{n} \alpha_{\zeta \cdot t} (u_{pq}^{rs} \otimes \alpha_{\zeta \cdot t} (u_{kl}^{ij})) = \delta_{s,m} \cdot \alpha_{\zeta \cdot t} (u_{kl}^{ij}), \quad \text{for all } 1 \leq i,j,k,l,m,n \leq n. \quad (3.14)$$
Similarly, we can verify that \((\Delta_A(u_{kl}^{ij}))^n\) for \{\Delta_A(u_{kl}^{ij})\}_{i,j,k,l=1}^n. Then, by the universal property of \(A, \Delta_A \) extends to a unique \(\mathbb{T}\)-equivariant unital *-homomorphism \(\Delta_A: A \to A \boxtimes \mathbb{C} A\). The coassociativity for \(\Delta_A\) is a routine check.

In order to verify the cancellation conditions (3.17) for \(\Delta_A\), we are going to employ the argument same as in [12] Section 4. Consider the set \(S := \{a \in A \mid a \boxtimes \mathbb{C} A \in \Delta_A(A)(1 \boxtimes \mathbb{C} A)\}\). Recall the canonical inclusions \(j_1, j_2: A \to A \boxtimes \mathbb{C} A\). Then we observe that

\[
(id_{M_{n \times n}^{\mathbb{C}}} \otimes \Delta_A)u = ((id_{M_{n \times n}^{\mathbb{C}}} \otimes j_1)u)((id_{M_{n \times n}^{\mathbb{C}}} \otimes j_2)u).
\]

Since \(u\) is unitary, we have \(u_{kl}^{ij}, u_{kl}^{ij*} \in S\) for all \(1 \leq i, j, k, l \leq n\). By virtue of [12] Proposition 3.1, for any two homogeneous elements \(x, y \in S\) we have

\[
j_1(xy) = j_1(x)j_1(y) \in \Delta_A(A)j_2(A)j_1(y) = \Delta_A(A)j_1(y)j_2(A) \subseteq \Delta_A(A)\Delta_A(A)j_2(A)j_2(A)
\]

\[
= \Delta_A(A)j_2(A).
\]

So, \(xy \in S\). Therefore, all the monomials in \(u_{kl}^{ij}\) and \(u_{kl}^{ij*}\) belongs to \(S\). Then \(S\) is dense in \(A\). Consequently, \(A \boxtimes \mathbb{C} A = j_1(A)j_2(A) \subseteq \Delta_A(A)j_2(A) \subseteq A \boxtimes \mathbb{C} A\). Similarly, we can show \(\Delta_A(A)j_1(A) = A \boxtimes \mathbb{C} A\) by proving that \(\{a \in A \mid j_2(a) \in j_1(A)\Delta_A(A)\}\) is dense in \(A\).

**Definition 3.16.** An action of a braided compact quantum group \(G = (B, \beta, \Delta_B)\) over \(\mathbb{T}\) on an object \((D, \gamma)\) of \(\mathcal{C}'\boxtimes\mathfrak{qg}(\mathbb{T})\) is an element \(\eta \in \operatorname{Mor}^\mathbb{T}(D, D \boxtimes \mathbb{C} B)\) such that

\[
(1) \ \eta \text{ is a comodule structure:} \quad (id_D \boxtimes \eta) \circ \eta = (\eta \boxtimes id_B) \circ \eta
\]

\[
(2) \ \eta \text{ satisfies the Podlés condition:} \ \eta(D)(id_B \boxtimes \beta) = D \boxtimes B.
\]

Also, \(\eta\) is said to preserve a state \(f \in \operatorname{Mor}^\mathbb{T}(D, \mathbb{C})\) if \((f \boxtimes id_A)\eta(d) = f(d)id_B\) for all \(d \in D\).

Consider faithful covariant representations of \((D, \gamma)\) and \((B, \beta)\) on some separable Hilbert spaces \(L_1\) and \(L_2\) carrying \(\mathbb{T}\)-actions, respectively, \(\eta\) defines a faithful representation \(\eta): D \to D \boxtimes \mathbb{C} B \subseteq B(L_1 \otimes L_2)\).

The action \(\eta\) is called faithful if the norm closure of the linear span of \(\{\omega \otimes id_{\mathbb{B}(L_2)}\eta(d) \mid \omega \in B(L_1)\}\) coincides with image of \(B\) inside \(\mathbb{B}(L_2)\).

If \(\eta\) preserves a faithful state \(f\) on \(D\), then \(\eta\) is injective.

**Proposition 3.18.** The map \(\eta: M_n(\mathbb{C}) \to M_n(\mathbb{C}) \boxtimes \mathbb{C} A\) defined by

\[
\eta(E_{ij}) = \sum_{k,l=1}^n E_{kl} \boxtimes \mathbb{C} A^{ij}_{kl} \quad \text{for all } 1 \leq i, j \leq n,
\]

is a faithful action of \(\operatorname{Aut}(M_n(\mathbb{C}), \Pi, \phi)\) on \((M_n(\mathbb{C}), \Pi)\) and preserves \(\phi\).
Proof. It is easy to verify that $\eta$ is $T$-equivariant, $\eta(E_{ij}^*) = \eta(E_{ij})^*$ and $\sum_{i=1}^{n} \eta(E_{ii}) = 1_{M_n(C)} \otimes \varepsilon 1_A$ for all $1 \leq i, j \leq n$. Also, using the relation (5.14) and the fact $\phi$ is $T$-equivariant, we easily verify that $\eta$ preserves $\phi$.

To verify that $\eta$ is multiplicative, first we recall the commutation relation (2.5) and the condition (5.14). Then for any $1 \leq i, j, k, l \leq n$ we compute

$$\eta(E_{ij})\eta(E_{kl}) = \sum_{r,s,x,y=1}^{n} (E_{rs} \otimes_{\varepsilon} u_{ij}^r)(E_{xy} \otimes_{\varepsilon} u_{kl}^{xy})$$

$$= \sum_{r,s,x,y=1}^{n} \zeta^{(d_r - d_s)(d_i - d_j + d_l - d_k)} (E_{xy} \otimes_{\varepsilon} u_{ij}^r u_{kl}^{xy})$$

$$= \sum_{r,s,x,y=1}^{n} \zeta^{d_r (d_i - d_j + d_l - d_k)} E_{xy} \otimes_{\varepsilon} \left( \alpha_{\varepsilon, \eta}(u_{ij}^r) (u_{kl}^{xy}) \right)$$

$$= \delta_{j,k} \sum_{r,s,x,y=1}^{n} \zeta^{d_r (d_i - d_j + d_l - d_k)} E_{xy} \otimes_{\varepsilon} \alpha_{\varepsilon, \eta}(u_{kl}^{xy}) = \delta_{j,k} \cdot \eta(E_{kl}).$$

Thus $\eta: M_n(C) \to M_n(C) \otimes_{\varepsilon} A$ is a well-defined unital $^*$-homomorphism. Verification of Definition (3.10) (1) is a routine check. Finally, consider the set $S := \{ M \in M_n(C) \mid M \otimes_{\varepsilon} 1_A \in \eta(M_n(C))(1_{M_n(C)} \otimes_{\varepsilon} A) \}$. Since $u$ is unitary, in particular, we have $\sum_{r,s=1}^{n} u_{ij}^r u_{kl}^{rs*} = \delta_{i,k} \cdot \delta_{j,s} \cdot 1_A$ for all $1 \leq i, j, k, l \leq n$. This condition implies

$$\sum_{r,s=1}^{n} \eta(E_{rs})(1 \otimes_{\varepsilon} u_{kl}^{rs*}) = \sum_{i,j,r,s=1}^{n} E_{ij} \otimes_{\varepsilon} (u_{ij}^r u_{kl}^{rs*}) = \sum_{i,j=1}^{n} \delta_{i,k} \cdot \delta_{j,l} \cdot (E_{ij} \otimes_{\varepsilon} 1_A)$$

$$= E_{kl} \otimes_{\varepsilon} 1_A,$$

for all $1 \leq k, l \leq n$. So $E_{kl} \otimes_{\varepsilon} 1_A \in S$. Then using arguments similar to the proof of the first equality in (3.13), as in Theorem 4.4 we can verify Definition (5.14) (2).

Let us consider faithful covariant representations of $(M_n(C), \Pi)$ and $(A, \alpha)$ on some Hilbert spaces $L_1$ and $L_2$, respectively. Then we fix an orthonormal basis $\{ \lambda_i \}_{i \in \mathbb{N}}$ consisting of eigenvectors for the representation of $T$ on $L_i$, for $i = 1, 2$. We fix $i, j \in \{ 1, \cdots, n \}$. Then we use (2.3) and compute

$$\eta(E_{ij})(\lambda_{i}^r \otimes \lambda_{s}^s) = \sum_{k,l=1}^{n} \zeta^{\deg(u_{ij}^r)} (E_{kl} \lambda_{i}^r \otimes u_{ij}^{kl} \lambda_{s}^s), \quad \text{where } r, s \in \mathbb{N}.$$  

Next we fix $a, b \in \{ 1, \cdots, n \}$. Define $f: \{ 1, \cdots, n \} \to \mathbb{C}$ by $f(y) = \zeta^{- \deg(E_{ba}) \deg(u_{ij}^{ab})}$. For any $\xi_1, \xi_2 \in L_1$, the vector functionals $\omega_{\xi_1, \xi_2}$ are defined by $\omega_{\xi_1, \xi_2}(T) = \langle \xi_1, T \xi_2 \rangle$ for all $T \in B(L_1)$. For a fixed $r \in \mathbb{N}$, define $\omega := \sum_{y=1}^{n} f(y) \omega_{E_{ba} \lambda_{i}^r, E_{ba} \lambda_{j}^r}$. A simple computation using the last equation gives

$$((\omega \otimes \text{id}_{B(L_2)}) \eta(E_{ij})) \lambda_{j}^2 = \sum_{y=1}^{n} \lambda_{i}^r, E_{yy} \lambda_{j}^r \right) \zeta^{\deg(u_{ij}^{ab})} a_{ij}^a \lambda_{j}^2 = \zeta^{\deg(u_{ij}^{ab})} a_{ij}^a \lambda_{j}^2 \lambda_{j}^2. \quad \square$$

4. The bosonisation

The bosonisation of a braided compact quantum group over $T$ is an ordinary compact quantum group together with an idempotent quantum group homomorphism ("projection") with image $(\text{C}(T), \Delta_{\text{C}(T)})$. Conversely, every compact quantum group with projection having $(\text{C}(T), \Delta_{\text{C}(T)})$ as image is the bosonisation of a braided compact quantum group over $T$ (7) Theorem 6.4). We construct the bosonisation of $\text{Aut}(M_n(C), \Pi, \phi)$ in the following theorem.
Theorem 4.1. Suppose \((C, \Delta_C)\) is the bosonisation of \(\text{Aut}(\mathcal{M}_n(C), \Pi, \phi)\). Then \(C\) is isomorphic to the universal \(C^*\)-algebra generated by the elements \(z\) and \(u_{ij}^{kl}\) for \(1 \leq i, j, k, l \leq n\) subject to the relations \(z^*z = zz^* = 1\), \((3.5)-(3.9)\) and \(zu_{ij}^{kl}z^* = \zeta^{d_i - d_k + d_j - d_l} u_{ij}^{kl}\). The comultiplication map \(\Delta_C\) is given by
\[
\Delta_C(z) = z \otimes z, \quad \Delta_C(u_{ij}^{kl}) = \sum_{r,s=1}^{n} u_{rs}^{ij} \otimes z^{d_r - d_i + d_j - d_s} u_{kl}^{rs}.
\]

In particular, \(C \cong A \rtimes \mathbb{Z}\), where \(\alpha_{\zeta^{-n}}\) generates the action of \(\mathbb{Z}\) on \(A\). We denote \((C, \Delta_C)\) by \(\text{Bos}(\text{Aut}(\mathcal{M}_n(C), \Pi, \phi))\).

Proof. Let \(\mathcal{L}\) be a separable Hilbert space with a continuous representation \(\pi\) of \(T\). Suppose \(\{\lambda_m\}_{m \in \mathbb{N}}\) is an orthonormal basis of \(\mathcal{L}\) consisting of the eigenvectors for the \(T\)-action, that is, \(\pi_z(\lambda_m) = z^{l_m} \lambda_m\) for some \(l_m \in \mathbb{N}\). The braiding unitary \(L^X_L(T): \mathcal{L} \otimes L^2(\mathbb{T}) \to L^2(\mathbb{T}) \otimes \mathcal{L}\) and its inverse \(L^X_L(T)^*: L^2(\mathbb{T}) \otimes \mathcal{L} \to \mathcal{L} \otimes L^2(\mathbb{T})\) are similar to \((2.3)\). On the basis vectors they are defined by
\[
L^X_L(T)(\lambda_m \otimes z^r) = \zeta^{r l_m}(z^r \otimes \lambda_m), \quad L^X_L(T)^*(z^r \otimes \lambda_m) = \zeta^{-r l_m}(\lambda_m \otimes z^r).
\]
These are the same braiding unitaries in \([17 \text{ Section 6.1]}\). Here \(A\) is viewed as a \(T\)-Yetter-Drinfeld \(C^*\)-algebra with respect to the \(T\)-action \(\alpha\) and the \(\mathbb{Z}\)-action generated by \(\alpha_{\zeta^{-n}}\). The \(\mathbb{Z}\)-action is the composition of \(\alpha\) with the group homomorphism induced by the \(R\)-matrix \((1.1), \mathbb{T} \to \mathbb{T}, n \to \zeta^{-n}\) as observed in \([19 \text{ Appendix A}]\).

Consider a faithful \(T\)-equivariant representation \(A \hookrightarrow \mathcal{B}(\mathcal{L})\). Then \((2.3)\) and \((2.4)\) give
\[
\pi_z(u_{ij}^{kl} \lambda_m) = z^{d_k - d_i + d_j - d_l + \lambda_m} u_{ij}^{kl} \lambda_m, \quad \pi_z(u_{ij}^{kl} \alpha_{\zeta^{-n}} \lambda_m) = z^{d_k - d_i + d_j - d_l + \lambda_m} u_{ij}^{kl} \lambda_m.
\]

Therefore, the underlying \(C^*\)-algebra \(C\) is the braided tensor product:
\[
C = C(\mathbb{T}) \boxtimes_\zeta A = (C(\mathbb{T}) \otimes 1_{\mathcal{B}(\mathcal{L})}) \cdot L^X_L(T)(A \otimes 1_{\mathcal{B}(L^2(\mathbb{T}))})^{-1} \subseteq \mathcal{B}(L^2(\mathbb{T}) \otimes \mathcal{L}).
\]

In fact, \(C\) is generated by \(z \boxtimes_\zeta 1_A = z \otimes 1\) and \(1_{C(\mathbb{T})} \boxtimes_\zeta u_{ij}^{kl} = L^X_L(T)(u_{ij}^{kl} \otimes 1_{\mathcal{B}(L^2(\mathbb{T}))})\). \(L^X_L(T)^*\) acts on \(L^2(\mathbb{T}) \otimes \mathcal{L}\) by
\[
L^X_L(T)^*(u_{ij}^{kl} \otimes 1_{\mathcal{B}(L^2(\mathbb{T}))}), \quad L^X_L(T)^*(z^r \otimes \alpha_{\zeta^{-n}} \lambda_m) = \zeta^{(d_k - d_i + d_j - d_l)}(z^r \otimes u_{ij}^{kl} \lambda_m),
\]
and consequently we get the commutation relation \((3.5)-(3.9)\)
\[
(z \boxtimes_\zeta 1_A)(1_{C(\mathbb{T})} \boxtimes_\zeta u_{ij}^{kl})(z^* \boxtimes_\zeta 1_A) = \zeta^{-(d_k - d_i + d_j - d_l)}(1_{C(\mathbb{T})} \boxtimes_\zeta u_{ij}^{kl}) = 1_{C(\mathbb{T})} \boxtimes_\zeta \alpha_{\zeta^{-n}}(u_{ij}^{kl}).
\]

The above equation shows that the unitaries \(z^r \boxtimes_\zeta 1_A\) and the faithful representation \(A \hookrightarrow C(\mathbb{T}) \boxtimes_\zeta A \subseteq \mathcal{B}(L^2(\mathbb{T}) \otimes \mathcal{L})\) form a covariant representation for the \(\mathbb{Z}\)-action on \(A\) generated by \(\alpha_{\zeta^{-n}}\). Moreover, it is unitarily equivalent to the regular representation that defines the reduced crossed product for the \(\mathbb{Z}\)-action on \(A\) generated by \(\alpha_{\zeta^{-n}}\). So, \(C = C(\mathbb{T}) \boxtimes_\zeta A \cong A \rtimes \mathbb{Z}\). Then, \(C\) is the universal \(C^*\)-algebra generated by a unitary \(z\) and \(\{u_{ij}^{kl}\}_{1 \leq i,j,k,l \leq n}\) subject to the relations \((3.5)-(3.9)\) and the commutation relation
\[
zu_{ij}^{kl} = \zeta^{d_k - d_i + d_j - d_l} u_{ij}^{kl} z.
\]

The comultiplication map \(\Delta_C: C \to C \otimes C\) is defined by \(\Delta_C = \Psi^{A,A} \circ (id_{C(\mathbb{T})} \boxtimes_\zeta \Delta_A)\), where \(\Psi^{A,A}: C(\mathbb{T}) \boxtimes_\zeta A \boxtimes_\zeta A \to (C(\mathbb{T}) \boxtimes_\zeta A) \otimes (C(\mathbb{T}) \boxtimes_\zeta A) = C \otimes C\) is given by \((2.6)\) for \(B = D = A\) and \(\beta = \gamma = \alpha\). Using \(\Delta_{C(\mathbb{T})}(z) = z \otimes z\) and \((3.11)\), we compute
\[
\Delta_C(j_1(z)) = (j_1 \otimes j_1)\Delta_{C(\mathbb{T})}(z) = j_1(z) \otimes j_1(z),
\]
\[
\Delta_C(j_2(u_{ij}^{kl})) = \Psi^{A,A}(\sum_{r,s=1}^{n} j_2(u_{ij}^{rs}) j_3(u_{kl}^{rs})).
\]
\[ = \sum_{r,s=1}^{n} (j_2(u_{rs}^{ij}) \otimes j_1(z^{d_r-d_i+d_j-d_s})) \cdot (1_C \otimes j_2(u_{rs}^{ij})) \]
\[ = \sum_{r,s=1}^{n} j_2(u_{rs}^{ij}) \otimes j_1(z^{d_r-d_i+d_j-d_s}) j_2(u_{rs}^{ij}). \]

After dropping the inclusion maps \(j_1\) and \(j_2\) from our notation above we get the formulas for \(\Delta_C\).

\[ \square \]

5. **The universal property of \(\text{Aut}(\mathcal{M}_n(\mathbb{C}), \Pi, \phi)\) and its bosonisation**

Let \((D, \gamma)\) be an object of the category \(\mathcal{C}^*\text{alg}(\mathbb{T})\). Let \(G = (B, \beta, \Delta_B)\) be a braided compact quantum group over \(\mathbb{T}\) and let \((C', \Delta_{C'})\) be its bosonisation. Suppose \(\eta \in \text{Mor}^\gamma(D, D \boxtimes B)\) is an action of \(G\) on \(D\). Then \(C' = C(\mathbb{T}) \boxtimes_{\eta} B\) and the comultiplication map \(\Delta_{C'} = \Psi^{B,B} \circ (\text{id}_{C(\mathbb{T})} \boxtimes_{\eta} \Delta_B)\), where \(\Psi^{B,B}\) is given by (2.6) for \((D, \gamma) = (B, \beta)\).

**Proposition 5.1.** Define \(\tilde{\eta}: C(\mathbb{T}) \boxtimes_{\eta} D \to (C(\mathbb{T}) \boxtimes_{\eta} D) \otimes (C(\mathbb{T}) \boxtimes_{\eta} B) = (C(\mathbb{T}) \boxtimes_{\eta} D) \otimes C'\) by \(\tilde{\eta} = \Psi^{D,B} \circ (\text{id}_{C(\mathbb{T})} \boxtimes_{\eta} \eta)\), where \(\Psi^{D,B}\) is given by (2.6). Then \(\tilde{\eta}\) is an action of the ordinary compact quantum group \((C', \Delta_{C'})\) on \(C(\mathbb{T}) \boxtimes_{\eta} D\).

**Proof.** Since \(\Psi^{D,B}\) and \(\eta\) are injective, so is \(\tilde{\eta}\). Next, we consider \((D, \gamma), (B \boxtimes \xi, B, \beta \boxtimes \beta)\) and \(D \boxtimes B, \gamma \boxtimes \beta, (B, \beta)\) in (2.6). Then we get the following injective unital \(\ast\)-homomorphisms:

\[ \Psi^{D,B,\boxtimes B}: C(\mathbb{T}) \boxtimes_{\eta} D \boxtimes_{\eta} (B \boxtimes B) \to (C(\mathbb{T}) \boxtimes_{\eta} D) \otimes (C(\mathbb{T}) \boxtimes_{\eta} B \boxtimes B), \]
\[ \Psi^{D,\boxtimes B, B}: C(\mathbb{T}) \boxtimes_{\eta} (D \boxtimes B) \boxtimes_{\eta} B \to (C(\mathbb{T}) \boxtimes_{\eta} D \boxtimes B) \otimes (C(\mathbb{T}) \boxtimes_{\eta} B), \]

satisfying

\[ \Psi^{D,B,\boxtimes B}(j_1(a)) = (j_1 \otimes j_1)\Delta_{C(\mathbb{T})}(a), \quad \Psi^{D,\boxtimes B, B}(j_1(a)) = (j_1 \otimes j_1)\Delta_{C(\mathbb{T})}(a), \]
\[ \Psi^{D,B,\boxtimes B}(j_2(d)) = (j_2 \otimes j_1)\gamma(d), \quad \Psi^{D,\boxtimes B, B}(j_2(d)) = (j_2 \otimes j_1)\gamma(d), \]
\[ \Psi^{D,B,\boxtimes B}(j_3(b)) = 1_{C(\mathbb{T})} \boxtimes_{\eta} D \otimes j_2(b), \quad \Psi^{D,\boxtimes B, B}(j_3(b)) = 1_{C(\mathbb{T})} \boxtimes_{\eta} D \otimes j_2(b), \]
\[ \Psi^{D,B,\boxtimes B}(j_4(b)) = 1_{C(\mathbb{T})} \boxtimes_{\eta} D \otimes j_3(b), \quad \Psi^{D,\boxtimes B, B}(j_4(b)) = 1_{C(\mathbb{T})} \boxtimes_{\eta} D \otimes j_3(b), \]

for all \(a \in C(\mathbb{T}), b \in B\) and \(d \in D\). Using these maps and the equivariance of \(\Delta_B\) with respect to the \(\mathbb{T}\)-actions \(\beta\) and \(\beta \boxtimes \beta\) on \(B\) and \(B \boxtimes B\), we obtain the following identities:

\[ ((\text{id}_{C(\mathbb{T})} \boxtimes_{\eta} \eta) \otimes \text{id}_{C(\mathbb{T})} \boxtimes_{\eta} \Delta_B) \circ \Psi^{D,B} = \Psi^{D,B,\boxtimes B} \circ (\text{id}_{C(\mathbb{T})} \boxtimes_{\eta} \Delta_B), \]
\[ (\text{id}_{C(\mathbb{T})} \boxtimes_{\eta} \eta) \otimes \text{id}_{C(\mathbb{T})} \boxtimes_{\eta} \Delta_B) \circ \Psi^{D,B} = \Psi^{D,B} \circ (\text{id}_{C(\mathbb{T})} \boxtimes_{\eta} \eta \boxtimes \text{id}_{C(\mathbb{T})} \boxtimes_{\eta} \Delta_B). \]

Using the condition \((\gamma \otimes \text{id}_{C(\mathbb{T})}) \circ \gamma = (\text{id}_D \otimes \Delta_{C(\mathbb{T})}) \circ \gamma\) we can verify the following equation

\[ \Psi^{D,\boxtimes B, B}(j_l) = (\Psi^{D,B} \otimes \text{id}_{C(\mathbb{T})} \boxtimes_{\eta} B) \circ j_l, \]

for \(l = 1, 2, 3, 4\). Combining it with (5.2) and using (3.16) we get

\[ \tilde{\eta} \circ \eta \circ j_l = (\text{id}_{C(\mathbb{T})} \boxtimes_{\eta} D \otimes \Delta_{C'}) \circ \eta \circ j_l \quad \text{for} \ l = 1, 2. \]

Recall the (braided) Podlěš condition Definition (3.16) for \(\eta\), the (ordinary) Podlěš condition \(\gamma(D)(1_D \otimes C(\mathbb{T})) = D \otimes C(\mathbb{T})\) for \(\gamma\) and the cancellation condition \(\Delta_{C(\mathbb{T})}C(\mathbb{T})(1 \otimes C(\mathbb{T})) = C(\mathbb{T}) \otimes C(\mathbb{T})\) for \(\Delta_{C(\mathbb{T})}\). Using them we verify the Podlěš condition for \(\tilde{\eta}\):

\[ \tilde{\eta}(j_1(C(\mathbb{T})))j_2(D)) (1_{C(\mathbb{T})} \boxtimes_{\eta} D \otimes C') \]
\[ = (\Psi^{D,B}(C(\mathbb{T}) \boxtimes_{\eta} \eta(D))) (1_{C(\mathbb{T})} \boxtimes_{\eta} D \otimes j_2(B))j_1(C(\mathbb{T}))) \]
\[ = (\Psi^{D,B}(C(\mathbb{T}) \boxtimes_{\eta} (\eta(D)j_2(B))) (1_{C(\mathbb{T})} \boxtimes_{\eta} D \otimes j_1(C(\mathbb{T}))) \]
\[ = (\Psi^{D,B}(C(\mathbb{T}) \boxtimes_{\eta} D \boxtimes_{\eta} B)) (1_{C(\mathbb{T})} \boxtimes_{\eta} D \otimes j_1(C(\mathbb{T}))) \]
\[ = (\Psi^{D,B}(C(\mathbb{T}) \boxtimes_{\eta} D \boxtimes_{\eta} B)) (1_{C(\mathbb{T})} \boxtimes_{\eta} D \otimes j_1(C(\mathbb{T}))) \]
\[ = (\Psi^{D,B}(C(\mathbb{T}) \boxtimes_{\eta} D \boxtimes_{\eta} B)) (1_{C(\mathbb{T})} \boxtimes_{\eta} D \otimes j_1(C(\mathbb{T}))) \]}
= (j_1 \otimes j_1) \Delta_{C(T)}(C(T)) \langle (j_2 \otimes j_1) \gamma(D) \rangle \langle 1_{C(T)} \otimes j_2(B) \rangle j_1(C(T))
= (j_1 \otimes j_1) \Delta_{C(T)}(C(T)) \langle (j_2 \otimes j_1) \gamma(D) \rangle \langle 1_{C(T)} \otimes j_2(B) \rangle j_1(C(T)) j_2(D) \otimes j_2(B)
= j_1(C(T)) j_2(D) \otimes j_1(C(T)) j_2(B) = C(T) \otimes C(T) \otimes C'
\square

5.1. Universal property of Aut(M_n(C), II, \phi).

**Definition 5.3.** Let G_1 = (B_1, \beta_1, \Delta_{B_1}) and G_2 = (B_2, \beta_2, \Delta_{B_2}) be braided compact quantum groups over T. A braided compact quantum groups homomorphism f: G_1 \rightarrow G_2 is an element f \in \text{Mor}_T(B_1, B_2) such that (f \otimes \xi) \circ \Delta_{B_1} = \Delta_{B_2} \circ f.

Let D be a unital C*-algebra carrying an action \gamma of \mathbb{T}. Suppose \gamma preserves a faithful state \phi on D. Let C^D(D, \gamma, \phi) be the category with the pairs (G, \eta) consisting of a braided compact quantum groups G over \mathbb{T} and \eta is a \phi preserving action of G on (D, \gamma) as objects. An arrow between two objects (G_1, \eta_1) and (G_2, \eta_2) is a braided compact quantum group homomorphism f: G_1 \rightarrow G_2 such that (id_{M_n(C)} \otimes \xi) f = \eta_1 = \eta_2 f.

**Theorem 5.4.** Aut(M_n(C), II, \phi) is the universal, initial object of the category C^D(D, \gamma, \phi).

Let G = (B, \beta, \Delta_B) be a braided compact quantum group over \mathbb{T} and let \eta: M_n(C) \rightarrow \otimes \xi B be an action of \text{G} on \text{M_n(C)}. Since \{E_{ij}\}_{i,j=1}^n is basis of \text{M_n(C)}, \eta is uniquely determined by its matrix coefficients \{b_{kl}^{ij}\}_{1 \leq i,j,k,l \leq n} \subseteq B given by (4).

Now \eta is \mathbb{T}-equivariant for the actions II and II \otimes \beta, and E_{ij} \in \text{M_n(C)} is a homogeneous element of degree d_i - d_j. Then \eta(E_{ij}) \in M_n(C) \otimes \xi B must also be a homogeneous element of degree d_i - d_j. Therefore, each term on the right hand side of (4) appearing under summation must also be a homogeneous element of degree d_i - d_j. Equivalently,

\[ z^{d_i - d_j}(E_{kl} \otimes \xi b_{kl}^{ij}) = (II \otimes \beta)_z(E_{kl} \otimes \xi b_{kl}^{ij}) = \Pi_z(E_{kl} \otimes \xi \beta_z(b_{kl}^{ij})) = z^{d_k - d_i} E_{kl} \otimes \xi \beta_z(b_{kl}^{ij}) \]

Therefore, the restriction of \beta on the elements \{b_{kl}^{ij}\}_{1 \leq i,j,k,l \leq n} \subseteq B is given by

\[ \beta_z(b_{kl}^{ij}) = z^{d_i - d_j + d_k - d_l} b_{kl}^{ij}, \quad \text{for all } z \in \mathbb{T}. \] (5.5)

Next we determine the restriction of \Delta_B on \{b_{kl}^{ij}\}_{1 \leq i,j,k,l \leq n}. Let L' be a separable Hilbert space with a continuous representation \pi' of \mathbb{T}. Suppose \{\lambda_m\}_{m \in \mathbb{N}} is an orthonormal basis of eigenvectors for the \mathbb{T}-action: \pi'_t(\lambda_m) = z^{t'} \lambda_m, for some t' \in \mathbb{N}. Let B \rightarrow \mathbb{B}(L') be a faithful, \mathbb{T}-equivariant representation.

Also, recall the \text{n-dimensional vector space} W with the basis \{e_1, e_2, \cdots, e_n\} equipped with \mathbb{T}-action \rho considered in the proof of the Theorem 5.3. Then \text{E_{ij} e_k = d_{j,k}} for all 1 \leq i, j, k \leq n is a faithful and \mathbb{T}-equivariant representation of \text{M_n(C)} on W.

The braiding unitary \text{E'XW}: L' \otimes W \rightarrow W \otimes L' and its inverse \text{WX'X}: W \otimes L' \rightarrow L' \otimes W are similar to (2.2). On the basis vectors they are defined by

\[ \text{E'XW}(e_i \otimes \lambda_m e_{t'}) = \zeta^{d_i t'} \cdot (e_i \otimes \lambda_m e_{t'}), \quad \text{WX'X}(e_i \otimes \lambda_m e_{t'}) = \zeta^{-d_i t'} \cdot (e_i \otimes \lambda_m e_{t'}). \]

The elementary braided tensors of \text{M_n(C) \otimes \xi B} are defined by

\[ a \otimes 1_B = a \otimes 1_{\mathbb{B}(L')}, \quad 1_{M_n(C) \otimes \xi b} = \text{E'XW}(b \otimes 1_{\mathbb{B}(W)})^{WX'X}. \]

In particular, we note that

\[ (1_{M_n(C) \otimes \xi b}(e_k \otimes \lambda_m))^{WXW}(b_{kl}^{ij} \otimes e_k) = \zeta^{-d_k d_i + d_j d_l - d_i} e_k \otimes b_{kl}^{ij} \lambda_m, \]
Thus we obtain the restriction of \(\Delta_B\):

\[
(1 \boxtimes b_{kl}^{ij})(E_{rs} \otimes 1) = \zeta^{(d_i - d_j)(d_k - d_l - d_j - d_l)}(E_{rs} \otimes b_{kl}^{ij}), \quad \text{for } 1 \leq i, j, k, l, r, s \leq n. \tag{5.7}
\]

The second condition in Definition 3.16 implies

\[
\sum_{k,l,r,s=1}^{n} E_{rs} \otimes b^r_k \otimes b^l_j = (\alpha \otimes \text{id}_B)\alpha(E_{ij}) = \sum_{r,s=1}^{n} E_{rs} \otimes \Delta_B(b^r_s). \tag{5.8}
\]

Next we compute the action of the first and the third terms appearing in the above equation on the basis vectors of \(W \otimes \mathcal{L} \otimes \mathcal{L}'\). For all \(1 \leq p \leq n\) and \(x, y \in \mathbb{N}\), we compute

\[
\left( \sum_{k,l,r,s=1}^{n} E_{rs} \otimes b^r_k \otimes b^l_j \right) (e_p \otimes \lambda'_x \otimes \lambda'_y) = \sum_{k,l,r=1}^{n} \zeta^{d_p(d_i - d_j - d_k - d_l - d_j - d_l)} e_r \otimes b^r_p \lambda'_x \otimes b^l_j \lambda'_y,
\]

and

\[
\left( \sum_{k,l,r,s=1}^{n} E_{rs} \otimes \Delta_B(b^r_s) \right) (e_p \otimes \lambda'_x \otimes \lambda'_y) = \sum_{r=1}^{n} \zeta^{d_p(d_i - d_j - d_k - d_l - d_j - d_l)} e_r \otimes \Delta_B(b^r_p)(\lambda'_x \otimes \lambda'_y).
\]

Thus we obtain the restriction of \(\Delta_B\) on \(\{b_{kl}^{ij}\}_{k,l,j,i=1}^{n} \subseteq B\) given by

\[
\Delta_B(b_{kl}^{ij}) = \sum_{r,s=1}^{n} b_{rs}^{ij} \otimes \zeta b^r_s.
\]

**Proof of Theorem 5.3.** Suppose \(G = (B, \beta, \Delta_B)\) is an object of \(\mathcal{C}^+(M_n(\mathbb{C}), \Pi, \phi)\). Denote the action of \(G\) on \(M_n(\mathbb{C})\) by \(\eta\): \(M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes B\). Then \(\eta\) satisfies the following condition:

\[
(\phi \otimes \text{id}_B)\eta(M) = \phi(M)1_B, \quad \text{for all } M \in M_n(\mathbb{C}). \tag{5.9}
\]

Suppose \(H = (\mathcal{C}', \Delta_{\mathcal{C}'})\) is the bosonisation of \(G = (B, \beta, \Delta_B)\) and \(\tilde{\eta}\) be the corresponding action of \(H\) on \(C(T) \boxtimes \mathcal{M}_n(\mathbb{C})\), in Proposition 5.4 for \((M_n(\mathbb{C}), \Pi)\). On the elementary braided tensors \(z^r \boxtimes \zeta E_{ij}\) it is defined by

\[
\tilde{\eta}(z^r \boxtimes \zeta E_{ij}) = \sum_{k,j=1}^{n} (z^r \boxtimes \zeta E_{ki}) \otimes (z^{d_k - d_i + r} \boxtimes \zeta b^k_j), \quad \text{for all } r \in \mathbb{Z}, 1 \leq i, j \leq n, \tag{5.10}
\]

where \(\{b_{kl}^{ij}\}_{i,j,k,l=1}^{n}\) are the matrix elements of \(\eta\) given by (1.4). Also, Theorem 4.1 gives us the following commutation relation in \(D = j_1(C(T))j_2(B) = C(T) \boxtimes B\):

\[
(z \boxtimes \zeta 1_B)(1_{C(T)} \boxtimes \zeta b^k_j)(z \boxtimes \zeta 1_B)^* = \zeta^{-(d_k - d_i + d_j - d_l)} \cdot (1_{C(T)} \boxtimes \zeta b^k_j). \tag{5.11}
\]

for all \(1 \leq i, j, k, l \leq n\). The restriction of \(\tilde{\eta}\) on \(M_n(\mathbb{C})\) becomes

\[
\tilde{\eta}(1_{C(T)} \boxtimes \zeta E_{ij}) = \sum_{k,j=1}^{n} (1_{C(T)} \boxtimes \zeta E_{ki}) \otimes (z^{d_k - d_i} \boxtimes \zeta b^k_j), \quad \text{for all } 1 \leq i, j \leq n. \tag{5.12}
\]

Then we can verify that \(\tilde{\eta}|_{M_n(\mathbb{C})}\) defines an action of \(H\) on \(M_n(\mathbb{C})\) and preserves \(\phi\). So, the matrix elements \(\{z^{d_i - d_j} \boxtimes \zeta b^k_j\}_{i,j,k,l=1}^{n}\) of \(\tilde{\eta}|_{M_n(\mathbb{C})}\) must satisfy the relations (5.3)-(5.9) with \(\zeta = 1\) by [22, Theorem 4.1]. Subsequently, using (5.11) we can easily show that \(\{b_{kl}^{ij}\}_{i,j,k,l=1}^{n}\) satisfy the relations (5.3)-(5.9) in Theorem 5.3. Then the universal property of \(A\) gives a unique \({}^*\)-homomorphism \(f: A \rightarrow B\) such that \(f(\eta^k_j) = b_{kl}^{ij}\) for all \(i, j, k, l \in \{1, \cdots, n\}\). Clearly, \(f\) is \(T\)-equivariant and satisfies \((f \boxtimes f) \circ \Delta_A = \Delta_B \circ f\); hence \(f: \text{Aut}(M_n(\mathbb{C}), \Pi, \phi) \rightarrow G\) is a braided compact quantum group homomorphism. \(\Box\)
5.2. Bosonisation as the inhomogeneous quantum symmetry group.

Definition 5.13 ([11 Definition 2.1]). Let $D$ be a unital C*-algebra and let $\tau_D$ be a faithful state on $D$. An orthogonal filtration for the pair $(D, \tau_D)$ is a sequence of finite dimensional subspaces $\{D_i\}_{i \in I}$ of $D$, where $I$ is the index set with a distinguished element 0 such that $D_0 = \mathbb{C} \cdot 1_D$, $\text{Span}(\cup_{i \in I} D_i)$ is dense in $D$ and $\tau_D(a^*b) = 0$ if $a \in D_i$, $b \in D_j$ and $i \neq j$. We denote the triple $(D, \tau_D, \{D_i\}_{i \in I})$ by $\hat{D}$.

Let $\mathbb{H} = (Q, \Delta_Q)$ be a compact quantum group and let $\hat{D}$ be an orthogonal filtration of $D$. An action $\eta: D \to D \otimes Q$ is said to be filtration preserving if $\eta(D_i) \subseteq D_j \otimes_{\text{alg}} Q$ for all $i \in I$. Let $C(\hat{D})$ be the category with objects as pairs $(\mathbb{H}, \eta)$, where $\mathbb{H}$ is a compact quantum group, $\eta$ is a filtration preserving action of $\mathbb{H}$ on $D$, and arrows are morphisms of compact quantum groups intertwining the respective actions. In [41] it was also observed that if $(\mathbb{H}, \varrho)$ is an object of $C(\hat{D})$, then $\varrho$ preserves $\tau_D$.

Theorem 5.14 ([11 Theorem 2.7]). There exists a universal initial object in the category $C(\hat{D})$, called the quantum symmetry group of the filtration $\hat{D}$ and denoted by $\text{QISO}(\hat{D})$.

Example 5.15. Define $M_n(\mathbb{C})_0 = \text{Span}\{1_{M_n(\mathbb{C})}\}$ and $M_n(\mathbb{C})_1 = \text{Span}\{E_{ij} \mid 1 \leq i, j \leq n \text{ and } i \neq j\}$. Then $M_n(\mathbb{C}) = (M_n(\mathbb{C}), \varphi, \{M_n(\mathbb{C})\}_{i=0,1})$ is an orthogonal filtration. Let $\mathbb{H} = (Q, \Delta_Q)$ be a compact quantum group and let $\varrho: M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes Q$ be an action of $G$ on $M_n(\mathbb{C})$. Then $\varrho$ is uniquely determined by the set elements $\{q_{kl}^{ij}\}_{1 \leq i, j, k, l \leq n} \subseteq Q$ such that

$$\varrho(E_{ij}) = \sum_{k,l=1}^{n} E_{kl} \otimes q_{ij}^{kl}, \quad \text{for all } 1 \leq i, j \leq n.$$  

If $\varrho$ preserves the orthogonal filtration $\hat{M}_n(\mathbb{C})$ then $\varrho$ preserves the trace $\varphi$. Conversely, if we assume that $\varrho$ preserves the canonical trace $\varphi$, then $\sum_{k,l=1}^{n} q_{kl}^{ij} = \delta_{i,l}$ for all $1 \leq k, l \leq n$. Thus $\varrho$ preserves the orthogonal filtration $\hat{M}_n(\mathbb{C})$.

Hence, $\text{QISO}(\hat{M}_n(\mathbb{C}))$ coincides with Wang’s $\text{Aut}(M_n(\mathbb{C}), \varphi)$ in [22 Theorem 4.1].

Example 5.16. Let $z$ be the unitary generator of $C(\mathbb{T})$. Then the continuous linear extension of the map $z^r \to \delta_{r,0}$ for all $r \in \mathbb{Z}$ is the canonical trace $\tau_{C(\mathbb{T})}$ on $C(\mathbb{T}) \cong C^*(\mathbb{Z})$ and the subspaces $C(\mathbb{T})_n := \text{Span}\{z^r\}$ for all $i \in \mathbb{Z}$ form an orthogonal filtration for $(C(\mathbb{T}), \tau_{C(\mathbb{T})})$. Let $\mathbb{H} = (Q, \Delta_Q)$ be a compact quantum group and let $\varrho: C(\mathbb{T}) \to C(\mathbb{T}) \otimes Q$ be an action of $G$ on $C(\mathbb{T})$. Suppose, $(\mathbb{H}, \varrho)$ is an object of the category $C(\mathbb{H})$. Then $\varrho(z^r) = z^r \otimes q_r$, where $q_r \in Q$ for all $r \in \mathbb{Z}$. Since, $\varrho$ is a $^*$-homomorphism then $q_0 = 1_Q$, $q_1$ is unitary and $q_r = q_1^r$ for all $r \in \mathbb{Z}$. Moreover, $\Delta_Q(q_1) = q_1 \otimes q_1$. Thus $\text{QISO}(\mathbb{H}) \cong (C(\mathbb{T}), \Delta_{C(\mathbb{T})})$.

Example 5.17. The braided tensor product $C(\mathbb{T}) \boxtimes_r M_n(\mathbb{C})$ is isomorphic to the (reduced) crossed product $M_n(\mathbb{C}) \rtimes \mathbb{Z}$ for the $\mathbb{Z}$-action on $M_n(\mathbb{C})$ generated by $\Pi_{-1}$. Thus $C(\mathbb{T}) \boxtimes_r M_n(\mathbb{C})$ is the universal C*-algebra generated by a unitary $v$ and $\{E_{ij}\}_{i,j=1}^{n}$ subject to the relations [12] and the commutation relation

$$v E_{ij} v^* = \xi^{d_i - d_j} E_{ij}. \quad (5.18)$$

Composing $\varphi$ with the conditional expectation $M_n(\mathbb{C}) \rtimes \mathbb{Z} \to M_n(\mathbb{C})$ gives a state $\tau$ on $M_n(\mathbb{C}) \rtimes \mathbb{Z}$. Therefore, $\tau$ is the continuous linear extension of the map $v^* E_{ij} \to \frac{1}{n+1} \delta_{i,j} \cdot \delta_{i,j}$. By [6] Proposition 4.9 & Lemma 6.1 the triple $M_n(\mathbb{C}) \rtimes \mathbb{Z} := (M_n(\mathbb{C}) \rtimes \mathbb{Z}, \tau, \{v^* \cdot 1_{M_n(\mathbb{C})}\}_{r \in \mathbb{Z}} \cup \{v^* \cdot E_{ij}\}_{i \leq j, i, j \leq n, i \neq j})$ is an orthogonal filtration.
Recall the bosonisation $\text{Bos}(\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)) = (C, \Delta_C)$ constructed in Theorem [1.1]. Then we apply Proposition [5.1] to the canonical action $\eta$ of $\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)$ on $(M_n(\mathbb{C}), \Pi)$ in Proposition [3.18]. The resulting action $\tilde{\eta}$: $M_n(\mathbb{C}) \times \mathbb{Z} \to M_n(\mathbb{C}) \times \mathbb{Z} \otimes C$ of $(C, \Delta_C)$ on $M_n(\mathbb{C}) \times \mathbb{Z}$ is defined by

$$\tilde{\eta}(v) = v \otimes z, \quad \tilde{\eta}(E_{ij}) = \sum_{k,l=1}^{n} E_{kl} \otimes z^{d_k-d_i} u_{ij}^{kl}.$$ 

Since $\tilde{\eta}$ preserves the filtration $M_n(\mathbb{C}) \times \mathbb{Z}$ the pair $(\text{Bos}(\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)), \tilde{\eta})$ is an object of $\mathcal{C}(M_n(\mathbb{C}) \times \mathbb{Z})$.

**Proposition 5.19.** $\text{QISO}(\widehat{M_n(\mathbb{C})} \times \mathbb{Z}) \cong \text{Bos}(\text{Aut}(M_n(\mathbb{C}), \Pi, \phi))$.

**Proof.** Let $\mathbb{H} = (Q, \Delta_Q)$ be a compact quantum group. Suppose $(\mathbb{H}, \varrho)$ is an object of $\mathcal{C}(M_n(\mathbb{C}) \times \mathbb{Z})$. Then $\varrho$ preserves the filtration described above. There exists a unitary $u \in Q$ such that $\varrho(v^r) = v^r \otimes u^r$ and $\Delta_Q(u^r) = u^r \otimes u^r$ for all $r \in \mathbb{Z}$. On the other hand, the restriction of $\varrho$ on $E_{ij}$ is given by $\varrho(E_{ij}) = \sum_{k,l=1}^{n} E_{kl} \otimes q_{ij}^{kl}$, where $q_{ij}^{kl} \in Q$ for all $1 \leq i, j, k, l \leq n$. Since $\varrho$ preserves $\tau$, the restriction $\varrho|_{M_n(\mathbb{C})}$ preserves $\phi$. By [22] Theorem 4.1], the elements $q_{ij}^{kl}$ satisfy (3.5)-(3.9) with $\zeta = 1$ and

$$\Delta_Q(q_{ij}^{kl}) = \sum_{r,s=1}^{n} q_{rs}^{kl} \otimes q_{ij}^{rs}, \quad \text{for all } 1 \leq i, j, k, l \leq n.$$ 

Using the commutation relation (5.18) and the condition that $\tilde{\eta}$ is a $^*$-homomorphism we compute

$$\zeta^{d_i-d_j} \varrho(E_{ij}) = \varrho(v E_{ij} v^*) = \sum_{k,l=1}^{n} v E_{kl} v^* \otimes u q_{ij}^{kl} u^* = \sum_{k,l=1}^{n} E_{kl} \otimes \zeta^{d_k-d_i} u q_{ij}^{kl} u^*.$$ 

This implies $u q_{ij}^{kl} u^* = \zeta^{d_k-d_i+d_k-d_i} q_{ij}^{kl}$ for all $1 \leq i, j, k, l \leq n$. Therefore, $\text{QISO}(\widehat{M_n(\mathbb{C})}) \cong (Q, \Delta_Q)$, where $Q$ is the universal $C^*$-algebra generated by the elements $u, \{q_{ij}^{kl}\}_{i,j,k,l=1}^{n}$ and $\Delta_Q$ is the restriction of $\Delta_Q$ to $Q$. Then it is a routine check that $Q$ is also generated by elements $u$ and $w_{ij}^{kl} := u^{d_i-d_j} q_{ij}^{kl}$ for all $1 \leq i, j, k, l \leq n$ such that $u$ is unitary, $\{w_{ij}^{kl}\}$ satisfy the relations (3.5)-(3.9) and $u w_{ij}^{kl} u^* = \zeta^{d_k-d_i+d_k-d_i} w_{ij}^{kl}$. Moreover, $\Delta_Q(w_{ij}^{kl}) = \sum_{r,s=1}^{n} w_{rs}^{kl} \otimes u^{d_r-d_i+d_r-d_i} w_{ij}^{rs}$ for all $1 \leq i, j, k, l \leq n$. Thus $(Q, \Delta_Q)$ is the bosonisation $(C, \Delta_C)$ of $\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)$ in Theorem [1.1]. \hfill $\square$

The action $\Pi(E_{ij}) = E_{ij} \otimes z^{d_i-d_j}$ of $\mathbb{T}$ on $M_n(\mathbb{C})$ preserves $\phi$, it also preserves $\widehat{M_n(\mathbb{C})}$ in Example [6.10]. Thus $(\mathbb{C}(\mathbb{T}), \Delta_{\mathbb{C}(\mathbb{T})}, \Pi)$ is an object of $\mathcal{C}(\widehat{M_n(\mathbb{C})})$. Hence, there exists a unique compact quantum group homomorphism $f: \text{QISO}(\widehat{M_n(\mathbb{C})}) \to \mathbb{C}(\mathbb{T})$ such that $f(q_{ij}^{kl}) = \delta_{k,l} \cdot \delta_{i,j} \cdot z^{d_i-d_j}$, where $\{q_{ij}^{kl}\}_{1 \leq i,j,k,l \leq n}$ is a generating set of Wang’s $\text{Aut}(M_n(\mathbb{C}), \phi)$.

Now we view the $R$-matrix in (1.1) as a unitary element of the multiplier algebra of $\mathbb{C}_0(\mathbb{Z}) \otimes C_0(\mathbb{Z})$. Then $V := (\text{id}_{\mathbb{C}_0(\mathbb{Z})} \otimes \tilde{f}) R$ is a bicharacter, where $\tilde{f}$ is the dual quantum group homomorphism from $\mathbb{C}_0(\mathbb{Z})$ to the dual of $\text{QISO}(\widehat{M_n(\mathbb{C})})$ (see [13] Section 3 & 4). By [6] Theorem 5.1], $\text{QISO}(\widehat{M_n(\mathbb{C})} \times \mathbb{Z}) \cong \mathcal{D}_V$, where $\mathcal{D}_V$ is the generalised Drinfeld’s double of $\text{QISO}(\widehat{M_n(\mathbb{C})})$ and $\text{QISO}(\mathbb{C}(\mathbb{T}))$ (see [20]). Combining it with Proposition [5.19] we obtain the following corollary.

**Corollary 5.20.** $\mathcal{D}_V \cong \text{Bos}(\text{Aut}(M_n(\mathbb{C}), \Pi, \phi))$. 
Remark 5.21. For $i, j, k, l \in \{1, \ldots, n\}$, the elements $w_{ij}^{kl}$ generate the copy of $\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)$ inside $\text{Bos}(\text{Aut}(M_n(\mathbb{C}), \Pi, \phi))$, whereas the elements $q_{ij}^{kl} = u^{d_k - d_i} w_{ij}^{kl}$ generate the copy of $\text{Aut}(M_n(\mathbb{C}), \phi)$ inside $\mathcal{D}_V$.

Then $q_{ij}^{kl} = u^{d_k - d_i} w_{ij}^{kl} \in C$ is a homogeneous element of degree $(d_k - d_i) + (d_k - d_i + d_j - d_j)$ for the diagonal action $\Delta_{\mathbb{C}(T)} \bowtie \alpha$. Similarly, an element $v^r E_{ij} \in M_n(\mathbb{C}) \rtimes \mathbb{Z} \cong \mathbb{C}(T) \rtimes_{\phi^r} M_n(\mathbb{C})$ is homogeneous of degree $r + d_i - d_j$ for the diagonal action $\Delta_{\mathbb{C}(T)} \bowtie \Pi$.

The restriction of the action $\theta$ on the factors $\mathbb{C}(T)$ and $M_n(\mathbb{C})$ in the above Proposition 5.19 is precisely the action of $\mathcal{D}_V = (\mathcal{Q}, \Delta_\mathcal{Q})$ on those factors (see [5, Theorem 3.5]). The latter one is given by

$$\theta|_{M_n(\mathbb{C})}(E_{ij}) = \sum_{k,l=1}^{n} E_{kl} \otimes q_{ij}^{kl} = \sum_{k,l=1}^{n} E_{kl} \otimes u^{d_k - d_i} w_{ij}^{kl}.$$ 

However, $\theta|_{M_n(\mathbb{C})}$ is not $T$-equivariant for the actions $\Pi$ and $\Delta_{\mathbb{C}(T)} \bowtie \alpha$. This happens because $\text{Aut}(M_n(\mathbb{C}), \phi)$ fails to capture $T$-homogeneous compact quantum symmetries of the dynamical system $(M_n(\mathbb{C}), Z, \Pi_{-i})$.

On the other hand, if we realise $\theta$ as an action of $\text{Bos}(\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)$ on $M_n(\mathbb{C}) \rtimes \mathbb{Z}$, then it coincides with $\eta$ in Proposition 5.1. Consequently, $\theta|_{M_n(\mathbb{C})}$ gets identified with $\eta$ in Proposition 5.18. In this sense, $\text{Aut}(M_n(\mathbb{C}), \Pi, \phi)$ is the $T$-homogeneous quantum symmetry group of the $\mathbb{C}^*$-dynamical system $(M_n(\mathbb{C}), Z, \Pi_{-i})$ and $\text{Bos}(\text{Aut}(M_n(\mathbb{C}), \Pi, \phi))$ is the inhomogeneous quantum symmetry group of the same system.

6. TOWARDS THE DIRECT SUM OF MATRIX ALGEBRAS

For a fixed $m \in \mathbb{N}$, let $n_1, n_2, \ldots, n_m \in \mathbb{N}$. Then $D = \bigoplus_{x=1}^{m} M_{n_x}(\mathbb{C})$ is the universal $\mathbb{C}^*$-algebra generated by $\{E_{ij,x}\}_{1 \leq i, j \leq n_x, 1 \leq x \leq m}$ subject to the following conditions:

$$E_{ij,x} E_{kl,y} = \delta_{j,k} \cdot \delta_{x,y} E_{il,x}, \quad E_{ij,x}^* E_{ij,x} = E_{ij,x}, \quad \sum_{x=1}^{m} \sum_{i=1}^{n_x} E_{ii,x} = 1. \quad (6.1)$$

For each $x \in \{1, \ldots, m\}$, we fix $d_x^1 \leq d_x^2 \leq \cdots \leq d_{x_{n_x}} \in \mathbb{Z}$ and define an action $\Pi^x$ of $T$ on the $x^{th}$ component of $D$ by $\Pi^x_z(E_{ij,x}) := u^{d_x^i - d_x^j} E_{ij,x}$ for $z \in T$. Extending them, we obtain the following $T$-action on $D$: $\Pi_z(\bigoplus_{x=1}^{m} M_{n_x}) = \bigoplus_{x=1}^{m} \Pi_{z}^x(M_{n_x})$ for all $M_x \in M_{n_x}(\mathbb{C})$, and $z \in T$. The map $\phi(E_{ij,x}) := \delta_{i,j}$ is a positive linear functional on $D$ and $\Pi$ preserves $\phi$.

Theorem 6.2. Consider the monoidal category $(\mathcal{C}'\text{alg}(T), \boxtimes, \zeta)$ for a fixed $\zeta \in T$.

Let $A$ be the universal $\mathbb{C}^*$-algebra with generators $u_{kl,xy}^{ij}$ for $1 \leq i, j \leq n_x, 1 \leq k, l \leq n_y$, $1 \leq x, y \leq m$ and the following relations:

$$\sum_{t=1}^{n_x} (\varepsilon^{d_x^t} (d_x^s - d_x^t + d_x^k - d_x^i) u_{kl,xy}^{ij}) \cdot (\varepsilon^{d_x^t} (d_x^j - d_x^t + d_x^l - d_x^j) u_{lk,wx}^{ij}) \cdot \delta_{a,w} \cdot \delta_{s,m} \cdot \varepsilon^{d_x^t} (d_x^s - d_x^t + d_x^k - d_x^i) u_{kl,xy}^{ij}, \quad (6.3)$$

for all $1 \leq k, s \leq n_y, 1 \leq m, l \leq n_x, 1 \leq i, j \leq n_x, 1 \leq x, y, w \leq m$;

$$\sum_{t=1}^{n_x} (\varepsilon^{d_x^t} (d_x^s - d_x^t + d_x^k - d_x^i) u_{kl,xy}^{ij}) \cdot (\varepsilon^{d_x^t} (d_x^j - d_x^t + d_x^l - d_x^j) u_{lk,wx}^{ij}) \cdot \delta_{a,w} \cdot \delta_{s,m} \cdot \varepsilon^{d_x^t} (d_x^s - d_x^t + d_x^k - d_x^i) u_{kl,xy}^{ij}, \quad (6.4)$$

for all $1 \leq i, s \leq n_y, 1 \leq m, j \leq n_w, 1 \leq k, l \leq n_x, 1 \leq x, y, w \leq m$;

$$u_{kl,xy}^{ij} = \varepsilon^{d_x^t} (d_x^t - d_x^j)(d_x^t - d_x^l + d_x^i - d_x^k)(u_{kl,xy}), \quad (6.5)$$
for all $1 \leq i, j \leq n_x$, $1 \leq k, l \leq n_y$, $1 \leq x, y \leq m$;
\[
\sum_{y=1}^{m} \sum_{r=1}^{n_y} u_{r,x,y}^{ij} = \delta_{i,j}, \quad \text{for all } 1 \leq i, j \leq n_x, \ 1 \leq x \leq m; \tag{6.6}
\]
\[
\sum_{x=1}^{m} \sum_{r=1}^{n_x} u_{r,x,y}^{kl} = \delta_{k,l}, \quad \text{for all } 1 \leq k, l \leq n_y, \ 1 \leq y \leq m. \tag{6.7}
\]

There exists a unique continuous action $\alpha$ of $\mathbb{T}$ on $A$ satisfying
\[
\alpha_z(u_{kl,xy}^{ij}) = z^d_i - d_i^d + d_j^d - d^d_i u_{kl,xy}^{ij}, \quad \text{for all } z \in \mathbb{T}. \tag{6.8}
\]

Thus $(A, \alpha)$ is an object of $\mathcal{C}^{\mathrm{alg}}(\mathbb{T})$. Moreover, there exists a unique $\Delta_A \in \text{Mor}^\circ (\mathbb{D}, \mathbb{A} \otimes \mathbb{C} A)$ such that $(A, \alpha, \Delta_A)$ is a braided compact quantum group over $\mathbb{T}$. We denote it by $\text{Aut}(\mathbb{D}, \mathbb{P}, \phi)$. The map $\eta: \mathbb{D} \to \mathbb{D} \otimes \mathbb{C} A$, defined by
\[
\eta(E_{ij,xy}) = \sum_{y=1}^{m} \sum_{r=1}^{n_y} E_{rs,y} \otimes \zeta u_{r,j,yz}^s,
\]
extends to a faithful $\phi$-preserving action of $\text{Aut}(\mathbb{D}, \mathbb{P}, \phi)$ on $(\mathbb{D}, \mathbb{P})$.

**Proof.** We will essentially use the same methods as in Section 5 to prove this result. Let $\mathcal{A}$ be the $^*$-algebra generated by the elements $u_{kl,xy}^{ij}$ satisfying (6.3)-(6.7). We identify $\text{End}(\mathbb{D})$ with $V := \oplus_{y=1}^{m} \text{End}(\mathbb{M}_{n_x \times n_y}(\mathbb{C}))$ via the vector space isomorphism $E_{ij,xy} \otimes E_{kl,xy} \to E_{ik} \otimes E_{jl} \otimes E_{xy}$. The $\mathbb{T}$-action $\mathbb{P}$ on $\mathbb{D}$ induces a $\mathbb{T}$-action on $V$ defined by $E_{ik} \otimes E_{jl} \otimes E_{xy} \to z^{d_i^d} - d_i^d + d_j^d E_{ik} \otimes E_{jl} \otimes E_{xy}$ for all $z \in \mathbb{T}$. Define $u \in V \otimes \mathcal{A}$ by
\[
u = \sum_{x,y=1}^{m} \sum_{k,l=1}^{n_x} \sum_{i,j=1}^{n_y} E_{ik} \otimes E_{jl} \otimes E_{xy} \otimes u_{kl,xy}^{ij}.
\]
Using the relations (6.3)-(6.7), we can show that $u$ is unitary, and $\mathcal{A}$ is the completion of $\mathcal{A}$ with respect to the largest $C^*$-seminorm on $\mathcal{A}$. Similarly, we can show that $\alpha$ in (6.8) defines a continuous action of $\mathbb{T}$ on $\mathcal{A}$, and the following formulae define $\Delta_A$:
\[
\Delta_A(u_{kl,xy}^{ij}) = \sum_{w=1}^{m} \sum_{r=1}^{n_y} u_{kl,sw}^{ij} \otimes \zeta u_{rk,wy}^{ij}, \quad 1 \leq i, j, k, l \leq n, \tag{6.9}
\]
such that $\text{Aut}(\mathbb{D}, \mathbb{P}, \phi) = (A, \alpha, \Delta_A)$ is a braided compact quantum group over $\mathbb{T}$, and $\eta$ defines a faithful $\phi$-preserving action of $\text{Aut}(\mathbb{D}, \mathbb{P}, \phi)$ on $(\mathbb{D}, \mathbb{P})$. \quad \square

**Remark 6.10.** In particular, if $m = 1$ then the Theorem 6.2 above reduces to Theorem 3.4. On the other hand, if $n_x = 1$ for $1 \leq x \leq m$, then $\mathbb{D} \cong \mathbb{C}^m$ and the action $\mathbb{P}$ is trivial. Define $a_{xy} = u_{11,xy}^{ij}$ for all $1 \leq x, y \leq m$. Indeed, $A$ is generated by the entries of the matrix $u = (a_{xy})$ and the relations (6.3)-(6.7) show that $\text{Aut}(\mathbb{D}, \mathbb{P}, \phi)$ becomes isomorphic to the Wang’s quantum permutation group of $m$ points.

Following Example 5.13, we observe that $\tilde{D} = (D, \phi, \{D_0\} \cup \{D_{1,x}\}_{x=1}^{m})$ is an orthogonal filtration for $(D, \phi)$, where $D_0 = \text{Span}\{1_D\}$ and $D_{1,x} = \text{Span}\{E_{ij,x} \mid 1 \leq i, j \leq n, \ i \neq j\}$. Combining it with Example 5.16 and we construct an orthogonal filtration $\tilde{D} \times \mathbb{Z}$ on the crossed product $D \rtimes \mathbb{Z}$, for the action of $\mathbb{Z}$ on $D$ generated by $\mathbb{P}_{\mathbb{Z}^{-1}}$, as in Example 5.16. Then by the arguments analogous to those of Section 3 and Section 5 we have the following theorem.

**Theorem 6.11.** Consider the categories $\mathcal{C}^\circ (\mathbb{D}, \mathbb{P}, \phi)$ and $\mathcal{C}(\tilde{D} \rtimes \mathbb{Z})$. Then the following statements hold true.
(1) \( \text{Aut}(D, \mathbb{P}, \phi) \) is the universal initial object of the category \( C^*(D, \mathbb{P}, \phi) \);

(2) Suppose \( (C, \Delta_C) \) is the bosonisation of \( \text{Aut}(D, \mathbb{P}, \phi) \). Then \( C \) is the universal \( C^* \)-algebra generated by the elements \( z \) and \( u_{kl,xy}^{ij} \) for \( 1 \leq i, j \leq n_x, 1 \leq k, l \leq n_y, 1 \leq x, y \leq m \) subject to the relations \( z^*z = zz^* = 1 \), and \( z u_{kl,xy}^{ij} z^* = z^* u_{kl,xy}^{ij} z \).

The comultiplication map \( \Delta_C : C \rightarrow C \otimes C \) is given by

\[
\Delta_C(z) = z \otimes z, \quad \Delta_C(u_{kl,xy}^{ij}) = \sum_{u=1}^m \sum_{r,s=1}^n u_{rs,wx}^{ij} \otimes z^{d^r_i - d^r_j + d^r_s} u_{kl,wy}^{ij},
\]

We denote \( (C, \Delta_C) \) by \( \text{Bos}(\text{Aut}(D, \mathbb{P}, \phi)) \).

(3) \( \text{Aut}(D, \mathbb{P}, \phi) \rtimes \mathbb{Z} \) is the universal, initial object of the category \( C(\mathbb{D} \rtimes \mathbb{Z}) \), that is, \( \text{QISO}(\mathbb{D} \rtimes \mathbb{Z}) \cong \text{Bos}(\text{Aut}(D, \mathbb{P}, \phi)) \).

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