A Theoretical Analysis of Sparse Recovery Stability of Dantzig Selector and LASSO

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Abstract. Dantzig selector (DS) and LASSO problems have attracted plenty of attention in statistical learning, sparse data recovery and mathematical optimization. In this paper, we provide a theoretical analysis of the sparse recovery stability of these optimization problems in more general settings and from a new perspective. We establish recovery error bounds for these optimization problems under a mild assumption called weak range space property of a transposed design matrix. This assumption is less restrictive than the well known sparse recovery conditions such as restricted isometry property (RIP), null space property (NSP) or mutual coherence. In fact, our analysis indicates that this assumption is tight and cannot be relaxed for the standard DS problems in order to maintain their sparse recovery stability. As a result, a series of new stability results for DS and LASSO have been established under various matrix properties, including the RIP with constant $\delta_{2k} < 1/\sqrt{2}$ and the (constant-free) standard NSP of order $k$. We prove that these matrix properties can yield an identical recovery error bound for DS and LASSO with stability coefficients being measured by the so-called Robinson’s constant, instead of the conventional RIP or NSP constant. To our knowledge, this is the first time that the stability results with such a unified feature are established for DS and LASSO problems. Different from the standard analysis in this area of research, our analysis is carried out deterministically, and the key analytic tools used in our analysis include the error bound of linear systems due to Hoffman and Robinson and polytope approximation of symmetric convex bodies due to Barvinok.

Key words. Dantzig selector, LASSO, convex optimization, sparse recovery stability, error bound, polytope approximation

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1 Introduction

Let $\mathbb{R}^n_+$ denote the nonnegative orthant of the Euclidean space $\mathbb{R}^n$. The set of $m \times n$ matrices will be denoted by $\mathbb{R}^{m \times n}$. All vectors are column vectors and the identity matrix of any order will be denoted by $I$, unless otherwise stated. We use $\phi : \mathbb{R}^q \to \mathbb{R}_+$ to denote a general norm on $\mathbb{R}^q$ satisfying that $\phi(e_i) = 1$ for all $i = 1, \ldots, q$, where $e_i \in \mathbb{R}^q$, $i = 1, \ldots, q$, are the standard basis of $\mathbb{R}^q$, i.e., the column vectors of the $q \times q$ identity matrix. In particular, we use $\| \cdot \|_p : \mathbb{R}^q \to \mathbb{R}_+$ to denote the $\ell_p$-norm, i.e., $\|x\|_p = \left( \sum_{i=1}^q |x_i|^p \right)^{1/p}$ for $x \in \mathbb{R}^q$, where $p \in [1, \infty]$. In particular, $\|x\|_1 = \sum_{i=1}^q |x_i|$ and $\|x\|_\infty = \max_{1 \leq i \leq q} |x_i|$.

Given matrices $A \in \mathbb{R}^{m \times n} (m < n)$ and $M \in \mathbb{R}^{m \times q} (m \leq q)$ with rank($A$) = rank($M$) = $m$, we consider the following $\ell_1$-minimization problem:

$$\min_x \{ \|x\|_1 : \phi(M^T(Ax - y)) \leq \tau \},$$

(1)

where $\tau \in \mathbb{R}_+$ is a given parameter and $y \in \mathbb{R}^m$ is a given vector. In signal recovery scenarios, $A$ is often called a design or sensing matrix consisting of a set of known or learned dictionaries, and $y = Ax + \theta$ is a measurement vector acquired for the signal $x$ to recover, and $\theta \in \mathbb{R}^m$ is the measurement error, bounded as $\phi(M^T \theta) \leq \tau$.

Problem (1) includes several important special cases. In fact, when $\tau = 0$, the problem is reduced to the standard $\ell_1$-minimization, i.e., $\min \{ \|x\|_1 : Ax = y \}$, which is a signal recovery problem in noiseless situations (e.g., [6, 24, 28]). When $\tau > 0$, $M = I$ and $\phi(\cdot) = \| \cdot \|_2$, problem (1) becomes a nonlinear $\ell_1$-minimization which has been widely studied in the field of compressed sensing and signal processing (e.g., [17, 20, 24, 25, 28]). Taking $M = A$ in (1) leads to the problem

$$\min_x \{ \|x\|_1 : \phi(A^T(Ax - y)) \leq \tau \},$$

(2)

When $\phi(\cdot) = \| \cdot \|_\infty$, this problem becomes the standard Dantzig selector (DS) introduced by Candès and Tao [12, 13]. In this paper, problem (1) is still referred to as the Dantzig selector (DS) although it is more general than the standard one. Closely related to (1) is the problem

$$\min_x \{ \phi(M^T(Ax - y)) : \|x\|_1 \leq \mu \},$$

(3)

where $\mu > 0$ is a given parameter. This problem also includes several important special cases. When $M = I$ and $\phi(\cdot) = \| \cdot \|_2$, problem (3) becomes the well known LASSO (least absolute shrinkage and selection operator) problem introduced by Tibshirani [49]:

$$\min_x \{ \|Ax - y\|_2 : \|x\|_1 \leq \mu \}.$$  

(4)

In addition, setting $M = A$ in (4) yields the model

$$\min_x \{ \phi(A^T(Ax - y)) : \|x\|_1 \leq \mu \},$$

(5)

which clearly is related to the DS problem (2). In this paper, problem (3) is still called the LASSO problem despite the fact that it is more general than (4).

To possibly recover the sparse data $\hat{x}$ satisfying the bound $\phi(M^T(A\hat{x} - y)) \leq \tau$, problem (1) seeks the $\ell_1$-minimizer $x$ that complies with the same bound $\phi(M^T(Ax - y)) \leq \tau$, while problem (3) minimizes the error $\phi(M^T(Ax - y))$ by assuming that the recovered data $x$ and the original data $\hat{x}$ obey the same $\ell_1$-norm bound. Under the conditions that the optimal solution of a recovery
problem is unique and $A$ satisfies certain strong properties, the current stability analysis checks whether the difference between the original data and the found solution of a recovery problem can stay under control. Given a recovery problem, however, the optimal solution of the problem is not always unique from a mathematical point of view, and more importantly the matrix might satisfy a condition less restrictive than the existing assumptions. This motivates one to carry out the stability analysis of a general recovery problem such as (1) and (3) under a mild assumption, which may apply to a wider range of situations.

DS and LASSO are popular in the statistics literature [12, 13, 23, 26, 4, 7, 32]. As pointed out in [34, 4, 19, 41, 48], DS and LASSO exhibit a similar behavior in many situations, especially in a sparsity scenario. Under the sparsity assumption and certain matrix conditions such as mutual coherence, restricted isometry property (RIP) or null space property (NSP), the standard $\ell_1$-minimization has been shown to be stable in sparse data recovery [29, 20, 21, 15, 14, 27, 9, 11, 10]. These results might be valid for DS and LASSO under suitable assumptions. For instance, the mutual coherence introduced in [29, 51] in signal processing was shown to ensure the recovery stability of LASSO in [12, 53]. Under the RIP assumption, Candès and Tao [12] have shown that if the true vector $\hat{x}$ is sufficiently sparse then $\hat{x}$ can be estimated reliably via DS based on the noise data observation $y$. Cai, Xu and Zhang [11] have shown certain improved stability results for DS and LASSO by slightly weakening the condition in [12]. It is worth mentioning that the stability of LASSO can be guaranteed under the restricted eigenvalue condition (REC) introduced by Bickel, Ritov and Tsybakov [4]. This condition holds with high probability for sub-Gaussian design matrices [17]. Other stability conditions have also been examined in the literature such as the compatibility condition [31], $H_{s,q}$ condition [36], and certain variants of RIP, NSP or REC [19, 35, 37, 40]. A good summary of stability results of LASSO can be found in [7, 32].

A typical feature of the current stability theory for DS and LASSO is that the coefficients in error bounds are usually measured by the RIP, REC or other individual matrix constants. In the current framework, deferent matrix assumptions require distinct analysis and yield different stability constants determined by the assumed matrix constants, such as the RIP constant, which are usually hard to certify (see, e.g., [2, 50]). The purpose of this paper is to develop stability results for general problems (1) and (3) under a constant-free matrix condition which is a very mild assumption. In fact, it turns out that this assumption is both necessary and sufficient for the standard DS to be stable in sparse data recovery. Our stability results and analysis are developed and carried out under less restrictive assumptions than the existing ones and for more general optimization problems. More specifically, our work is carried out differently in three aspects.

(i) The results in this paper are established in a fairly general setting. Our analysis is based on the fundamental Karush-Kuhn-Tucker (KKT) optimality conditions [38, 52] which capture the deep property of the optimal solution of a convex optimization problem. Thus KKT conditions naturally lead to the so-called weak range space property (weak RSP) of order $k$ of $A^T$ (see Definitions 2.6 for details) which turns out to be a very mild assumption guaranteeing the stability of a broad range of sparse optimization problems including the standard DS and LASSO. For the standard DS, we will show that this assumption is tight and cannot be relaxed in order to guarantee its sparse recovery stability.

(ii) The weak RSP of order $k$ of $A^T$ is a constant-free matrix property. A unique feature of the stability results developed in this paper is that the recovery error bounds are measured through so-called Robinson’s constant [45] which depends on the given problem data. Many known matrix conditions, such as the RIP [15], NSP [18], mutual coherence [24], REC [4], and the range space
property (RSP) of $A^T$ [51, 57], imply the weak RSP introduced in this paper. We show that the existing conditions imply an identical recovery error bound measured with Robinson’s constants. Thus our results can be viewed as certain unified stability theorems for DS and LASSO problems.

(iii) The classic error bound of linear systems (developed by Hoffman [33], and refined later by Robinson [45] and Mangasarian [39]) is a major analytic tool used in this paper. It provides a useful means for the study of stability of DS and LASSO problems. The stability of linear DS problems can be established directly by such an error bound of linear systems. Combined with the polytope approximation of symmetric convex bodies developed by Barvinok [3] (see also the earlier work by Dudley [22], Bronshtein and Ivanov [5], and Pisier [13]), the Hoffman’s error bound can also be used to establish the stability of nonlinear convex optimization problems, including the nonlinear DS and LASSO.

This paper is organized as follows. In Section 2, we provide basic facts, definitions, and initial results that will be used in the remainder of the paper. The stability of the linear DS under the weak RSP is proved in Section 3. Based on the polytope approximation of the unit ball, we show in Sections 4 and 5 that the nonlinear problems (1) and (3) are also stable under the weak RSP.

2 Preliminaries

2.1 Notations

In addition to the notations introduced at the beginning of Section 1, the following notations will also be used throughout the paper. $x \in \mathbb{R}^n$ is called a $k$-sparse vector if it admits at most $k$ nonzero entries, i.e., $\|x\|_0 \leq k$, where $\cdot \|_0$ counts the number of nonzero components of $x$. For $x \in \mathbb{R}^n$, let $|x|, (x)^+ \text{ and } (x)^-$ be the vectors with components $|x|_i := |x_i|, [(x)^+)_i := \max\{x_i, 0\}$ and $[(x)^-)_i := \min\{x_i, 0\}$ for $i = 1, \ldots, n$, respectively. For $x, y \in \mathbb{R}^n$, the inequality $x \leq y$ means $x_i \leq y_i$ for all $i = 1, \ldots, n$. Given a set $S \subseteq \{1, \ldots, n\}$, we use $\overline{S} = \{1, \ldots, n\} \setminus S$ to denote the complement of $S$ with respect to $\{1, \ldots, n\}$. We use $x_S$ to denote the subvector of $x \in \mathbb{R}^n$ by deleting the components $x_i$ with $i \in \overline{S}$, and we use $Q_S$ to denote the submatrix of $Q \in \mathbb{R}^{m \times n}$ by removing those columns of $Q$ with column index $i \in \overline{S}$. $Q^T$ denotes the transpose of the matrix $Q$, and $\mathcal{R}(Q^T) = \{Q^Tu : u \in \mathbb{R}^m\}$ denotes the range space of $Q^T$. For any given norms $\phi' : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and $\phi'' : \mathbb{R}^m \rightarrow \mathbb{R}_+$, the induced norm $\|Q\|_{\phi' \rightarrow \phi''}$ of the matrix $Q \in \mathbb{R}^{m \times n}$ is defined as

$$\|Q\|_{\phi' \rightarrow \phi''} = \max_{\phi'(x) \leq 1} \phi''(Qx).$$

In particular, $\|Q\|_{p \rightarrow q} = \max_{\|x\|_p \leq 1} \|Qx\|_q$ where $p, q \geq 1$. Given two sets $\Omega_1, \Omega_2 \subseteq \mathbb{R}^m$, we use $d^H(\Omega_1, \Omega_2)$ to denote the Hausdorff distance of $(\Omega_1, \Omega_2)$, i.e.,

$$d^H(\Omega_1, \Omega_2) = \max \left\{ \sup_{u \in \Omega_1} \inf_{z \in \Omega_2} \|u - z\|_2, \sup_{z \in \Omega_2} \inf_{u \in \Omega_1} \|u - z\|_2 \right\}. \quad (6)$$

For a closed convex set $\Omega \subseteq \mathbb{R}^n$, let $\Pi_\Omega(x)$ denote the orthogonal projection of $x \in \mathbb{R}^n$ into $\Omega$, i.e., $\Pi_\Omega(x) := \arg \min_u \{\|x - u\|_2 : u \in \Omega\}$.

2.2 Robinson’s Constant and Hoffman’s Lemma

Let $M' \in \mathbb{R}^{p_1 \times q}$ and $M'' \in \mathbb{R}^{p_2 \times q}$ be two matrices. Let $\| \cdot \|_{\alpha_1}$ and $\| \cdot \|_{\alpha_2}$, where $\alpha_1, \alpha_2 \geq 1$, be the $\ell_{\alpha_1}$- and $\ell_{\alpha_2}$-norms on $\mathbb{R}^q$ and $\mathbb{R}^{p_1 + p_2}$, respectively. Define

$$\mu_{\alpha_1, \alpha_2}(M', M'') := \max_{(d', d'') \in \mathcal{F}; \|\phi'(d', d'\|_{\alpha_2} \leq 1} \min_{u \in \mathbb{R}^q} \{\|u\|_{\alpha_1} : M'u \leq d', \ M''u = d''\}, \quad (7)$$

where $\mathcal{F}$ is the set of all pairs $(d', d'')$ of vectors in $\mathbb{R}^{p_1 + p_2}$.
where $\mathcal{F}$ is the set defined by

$$\mathcal{F} = \{(d', d'') \in \mathbb{R}^{p_1+p_2} : \text{for some } u \in \mathbb{R}^q \text{ such that } M'u \leq d' \text{ and } M''u = d''\}.$$ 

Note that $\mu_{\alpha_1, \alpha_2}(M', M'')$ is a finite number (see Robinson [45]). Let $M^1 \in \mathbb{R}^{m_1 \times \ell}$ and $M^2 \in \mathbb{R}^{m_2 \times \ell}$ be a pair of matrices and $S$ be a subset of $\{1, \ldots, m_1\}$. We consider the following two matrices:

$$\begin{bmatrix} I_S & 0 \\ -I & 0 \end{bmatrix} \in \mathbb{R}^{(|S|+m_1) \times (m_1+m_2)}, \quad \begin{bmatrix} M^1 \\ M^2 \end{bmatrix}^T \in \mathbb{R}^{\ell \times (m_1+m_2)},$$

where $I_S$ is the submatrix extracted from the identity matrix $I \in \mathbb{R}^{m_1 \times m_1}$ by deleting the rows corresponding to indices not in $S$. Substituting the above pair into (7) and taking the maximum over all possible subsets $S$ leads to

$$\sigma_{\alpha_1, \alpha_2}(M^1, M^2) := \max_{S \subseteq \{1, \ldots, m_1\}} \mu_{\alpha_1, \alpha_2} \left( \begin{bmatrix} I_S & 0 \\ -I & 0 \end{bmatrix}, \begin{bmatrix} M^1 \\ M^2 \end{bmatrix}^T \right),$$

which is a constant introduced by Robinson [45]. We call $\sigma$ the Robinson’s constant in this paper. Using this constant with $(\alpha_1, \alpha_2) = (\infty, 2)$, Robinson [45] proved that the classic Hoffman’s Lemma [33] concerning linear systems can be restated as follows.

**Lemma 2.1.** [33, 45] Let $M^1 \in \mathbb{R}^{m_1 \times \ell}$ and $M^2 \in \mathbb{R}^{m_2 \times \ell}$ be given matrices and $\mathcal{L} = \{u \in \mathbb{R}^\ell : M^1u \leq d^1, M^2u = d^2\}$ where $d^1 \in \mathbb{R}^{m_1}$ and $d^2 \in \mathbb{R}^{m_2}$. Then for any $x \in \mathbb{R}^\ell$, there is a point $x^* \in \mathcal{L}$ such that

$$\|x - x^*\|_2 \leq \sigma_{\infty, 2}(M^1, M^2) \left\| \begin{bmatrix} M^1 - d^1 \\ M^2 - d^2 \end{bmatrix}_+ \right\|_1.$$

### 2.3 Polytope approximation of the unit ball

Given a norm $\phi(\cdot)$ on $\mathbb{R}^q$, let $\phi^*(\cdot)$ be the dual norm of $\phi$, i.e., $\phi^*(u) = \max_{\phi(x) \leq u} u^T x$. From the definition, we see that $x^T u \leq \phi(x) \phi^*(u)$ for any $x, u \in \mathbb{R}^q$. In particular, the dual norm of the $\ell_p$-norm, $p \in [1, \infty]$, is the $\ell_\beta$-norm with $\beta \in [1, \infty]$, where $p$ and $\beta$ satisfy that $1/p + 1/\beta = 1$. For instance, the $\ell_1$-norm and $\ell_\infty$-norm are dual to each other. In this paper, we restrict our attention to the norm $\phi$ with $\phi(e_i) = 1$ and $\phi^*(e_i) = 1$ for all $i = 1, \ldots, q$. Clearly, any $\ell_p$-norm satisfies this property. Let

$$\mathcal{B}_\phi = \{x \in \mathbb{R}^q : \phi(x) \leq 1\}$$

be the unit ball in $\mathbb{R}^q$ defined by the norm $\phi$. It is evident that

$$\mathcal{B}_\phi = \bigcap_{\phi^*(a) = 1} \{x \in \mathbb{R}^q : a^T x \leq 1\}.$$

This means that $\mathcal{B}_\phi$ is the intersection of all half spaces in the form $\{x \in \mathbb{R}^q : a^T x \leq 1\}$ where $a \in \mathbb{R}^q$ and $\phi^*(a) = 1$. Any finite number of the vectors $a^i \in \mathbb{R}^q$ with $\phi^*(a^i) = 1, i = 1, \ldots, k$, yield the following polytope approximation of $\mathcal{B}_\phi$:

$$\mathcal{B}_\phi \subseteq \bigcap_{1 \leq i \leq k} \{x \in \mathbb{R}^q : (a^i)^T x \leq 1\}.$$

Note that $1 = \phi^*(a^i) = \sup_{\phi(u) \leq 1} (a^i)^T u = (a^i)^T u^*$ for some $u^*$ with $\phi(u^*) = 1$. Thus every half space $\{x \in \mathbb{R}^q : (a^i)^T x \leq 1\}$ with $\phi^*(a^i) = 1$ must be a support half space of $\mathcal{B}_\phi$ in the sense that
it contains $\mathfrak{B}^\phi$ and the plane $\{ x : (a^i)^T x = 1 \}$ touches $\mathfrak{B}^\phi$ at a point on its boundary. Conversely, any support half space of $\mathfrak{B}^\phi$ can be represented this way, i.e., $\{ x : (a^i)^T x \leq 1 \}$ with some $a^i$ satisfying $\phi^* (a^i) = 1$. Note that $\mathfrak{B}^\phi$ is a symmetric convex body (i.e., if $x$ is in the set, so is $-x$), to which there is a polytope approximation [22,3], as claimed by the following theorem.

**Theorem 2.2.** (Barvinok [3]) For any constant $\chi > \frac{e}{4\sqrt{2}} \approx 0.48$, there exists an $\epsilon_0 = \epsilon_0(\chi)$ such that for any $0 < \epsilon < \epsilon_0$ and for any symmetric convex body $B$ in $\mathbb{R}^q$, there is a symmetric polytope in $\mathbb{R}^q$, denoted by $P_\epsilon$, with $N$ vertices such that $N \leq \left( \frac{\chi \ln \frac{1}{\epsilon}}{\chi} \right)^q$ and $P_\epsilon \subseteq B \subseteq (1 + \epsilon)P_\epsilon$.

The above theorem indicates that for every sufficiently small $\epsilon > 0$ there exists a polytope $P_\epsilon$ satisfying $P_\epsilon \subseteq \mathfrak{B}^\phi \subseteq (1 + \epsilon)P_\epsilon$. Thus $(1 + \epsilon)P_\epsilon$ is an outer approximation of $\mathfrak{B}^\phi$. To get a tighter approximation of $\mathfrak{B}^\phi$, we compress the polytope $(1 + \epsilon)P_\epsilon$ by shifting all affine planes, expanded by the faces of $(1 + \epsilon)P_\epsilon$, toward $\mathfrak{B}^\phi$ until they touch $\mathfrak{B}^\phi$ on its boundary. By such a compression, the resulting polytope denoted by $\hat{P}_\epsilon$ is then formed by a finite number of support half spaces of $\mathfrak{B}^\phi$. Therefore, there exists a set of vectors $a^i$ for $i = 1, \ldots, J$ with $\phi^* (a^i) = 1$ such that

$$\hat{P}_\epsilon = \bigcap_{i=1}^J \{ x : (a^i)^T x \leq 1 \}.$$ 

We now add the $2m$ half spaces $(\pm e_i)^T z \leq 1$, $i = 1, \ldots, q$, to $\hat{P}_\epsilon$, yielding the following polytope:

$$\mathcal{T}_\epsilon := \hat{P}_\epsilon \cap \left\{ z \in \mathbb{R}^q : e_i^T z \leq 1, \quad -e_i^T z \leq 1, \quad i = 1, \ldots, q \right\}. \tag{9}$$

Clearly, we have the relation: $P_\epsilon \subseteq \mathfrak{B}^\phi \subseteq \mathcal{T}_\epsilon \subseteq \hat{P}_\epsilon \subseteq (1 + \epsilon)P_\epsilon$. Throughout the paper, we let $\epsilon_k \in (0, \epsilon_0)$, where $\epsilon_0$ is the constant specified in Theorem 2.2, be a positive and strictly decreasing sequence satisfying $\epsilon_k \to 0$ as $k \to \infty$. We consider the sequence of polytopes $\{ Q_{\epsilon_j} \}_{j \geq 1}$, where

$$Q_{\epsilon_j} = \bigcap_{k=1}^j \mathcal{T}_{\epsilon_k}. \tag{10}$$

Then for every $\epsilon_j$, $Q_{\epsilon_j}$ satisfies that

$$P_{\epsilon_j} \subseteq \mathfrak{B}^\phi \subseteq Q_{\epsilon_j} \subseteq \mathcal{T}_{\epsilon_j} \subseteq (1 + \epsilon_j)P_{\epsilon_j}. \tag{11}$$

Note that $Q_{\epsilon_j}$ is a polytope formed by a finite number (say, $K$) of half spaces, denoted by $(a^i)^T z \leq 1$ where $\phi^* (a^i) = 1$ for $i = 1, \ldots, K$. We use $\Gamma_{Q_{\epsilon_j}} := [a^1, \ldots, a^K]$ to denote the matrix with the vectors $a^i$’s as its columns, and we use $\mathcal{Y}(\Gamma_{Q_{\epsilon_j}}) = \{ a^1, \ldots, a^K \}$ to denote the set of columns of $\Gamma_{Q_{\epsilon_j}}$. Then $Q_{\epsilon_j}$ can be expressed as

$$Q_{\epsilon_j} = \left\{ z \in \mathbb{R}^q : (a^i)^T z \leq 1, \quad i = 1, \ldots, K \right\} = \left\{ z \in \mathbb{R}^q : (\Gamma_{Q_{\epsilon_j}})^T z \leq \hat{e} \right\}, \tag{12}$$

where $\hat{e}$ denotes the vector of ones in $\mathbb{R}^K$. The following lemma is useful in our later analysis.

**Lemma 2.3.** For any $j \geq 1$, let $Q_{\epsilon_j}$ be constructed as (11). Then for any $a^*$ on the unit sphere $\{ x \in \mathbb{R}^q : \phi(x) = 1 \}$, there exists a vector $a^i \in \mathcal{Y}(\Gamma_{Q_{\epsilon_j}})$ such that $(a^*)^T a^i \geq \frac{1}{1 + \epsilon_j}$.

**Proof.** Let $a^*$ be any point on the unit sphere, i.e., $\phi(a^*) = 1$. Note that $Q_{\epsilon_j}$ satisfies (11). The straight line starting from the origin and passing through the point $a^*$ on the surface of $\mathfrak{B}^\phi$ will shoot a point $z'$ on the boundary of $Q_{\epsilon_j}$ and a point $z''$ on the boundary of $(1 + \epsilon)P_{\epsilon_j}$. From (11), we see that $z'' = (1 + \epsilon''/\epsilon) a^*$ for some number $\epsilon'' \leq \epsilon_j$. Note that $z'$ is situated between $a^*$ and
\[ z'' \text{. This implies that } z' = (1 + \epsilon')a^* \text{ for some } \epsilon' \leq \epsilon'' \text{. Since } z' \text{ is on the boundary of } Q_{\epsilon_j}, \text{ it must be on a face of this polytope and hence there exists a vector } a^i \in \mathcal{Y}(Q_{\epsilon_j}) \text{ such that } (a^i)^T z' = 1. \]

Note that \( z' = (1 + \epsilon')a^* \text{ where } \epsilon' \leq \epsilon'' \leq \epsilon_j. \text{ Thus } 1 = (a^i)^T z' = (1 + \epsilon')(a^i)^T a^* \text{ which implies that } (a^i)^T a^* = \frac{1}{1 + \epsilon'} \geq \frac{1}{1 + \epsilon_j}. \]

### 2.4 Stability and weak RSP condition

Let us first give the definition of the stability of a sparse optimization problem. For a given vector \( x \in \mathbb{R}^n \), we recall that the best \( k \)-term approximation of \( x \) is defined as follows:

\[
\sigma_k(x)_1 := \inf_{u} \{\|x - u\|_1 : \|u\|_0 \leq k\},
\]

where \( k \) is a given integer number and \( \|u\|_0 \) counts the number of nonzero entries of \( u \in \mathbb{R}^n \). For problem (1), the stability can be described as follows.

**Definition 2.4.** Let \( \hat{x} \) be the original data obeying the constraint of (1). Problem (1) is said to be stable in data recovery if there is an optimal solution \( x^* \) of (1) approximating \( \hat{x} \) with error

\[
\|\hat{x} - x^*\|_2 \leq C' \sigma_k(\hat{x})_1 + C'' \tau,
\]

where \( C' \) and \( C'' \) are two constants depending only on the problem data \((M, A, y, \tau)\).

For the LASSO problem (3), we introduce the following definition.

**Definition 2.5.** Let \( \hat{x} \) be the original data obeying the constraint of (3). Problem (3) is said to be stable in data recovery if there is an optimal solution \( x^* \) of (3) approximating \( \hat{x} \) with error

\[
\|\hat{x} - x^*\|_2 \leq C_1 \sigma_k(\hat{x})_1 + C_2 (\mu - \|\hat{x}\|_1) + C_3 \mu \phi(M^T(A\hat{x} - y)),
\]

where \( C_1, C_2 \) and \( C_3 \) are constants depending only on the problem data \((M, A, y, \mu)\).

We now introduce the weak range space property of \( A^T \). By the KKT optimality condition, a \( k \)-sparse vector \( \hat{x} \) is an optimal solution to the standard \( \ell_1 \)-minimization problem \( \min\{\|x\|_1 : Ax = y := A\hat{x}\} \) if and only if there is a vector \( \zeta \in \mathcal{R}(A^T) \) satisfying that \( \zeta_i = 1 \) for \( \hat{x}_i > 0 \), \( \zeta_i = -1 \) for \( \hat{x}_i < 0 \), and \( |\zeta_i| \leq 1 \) for \( \hat{x}_i = 0 \). This property of \( A^T \) depends on the individual vector \( \hat{x} \). To ensure every \( k \)-sparse vector can be exactly recovered by \( \ell_1 \)-minimization, it is necessary to strengthen this property so that it is independent of any individual \( k \)-sparse vector. This naturally leads to the next definition.

**Definition 2.6.** (Weak RSP of order \( k \) of \( A^T \)) Let \( A \) be an \( m \times n \) matrix with \( m < n \). We say that \( A^T \) admits the weak range space property of order \( k \) if for any disjoint subsets \( S_1, S_2 \subseteq \{1, \ldots, n\} \) with \(|S_1| + |S_2| \leq k\), there exists a vector \( \zeta \in \mathcal{R}(A^T) \) obeying

\[
\zeta_i = 1 \text{ for } i \in S_1, \quad \zeta_i = -1 \text{ for } i \in S_2, \text{ and } |\zeta_i| \leq 1 \text{ for } i \notin S_1 \cup S_2.
\]

Slightly strengthening the condition “\(|\zeta_i| \leq 1 \text{ for } i \notin S_1 \cup S_2\)” to “\(|\zeta_i| < 1 \text{ for } i \notin S_1 \cup S_2\),” the above concept becomes the RSP of order \( k \) of \( A^T \) introduced in [54] (some earlier related works can be found in [44, 30]). It was shown in [54] that the RSP of order \( k \) of \( A^T \) is a necessary and sufficient condition for the recovery of every \( k \)-sparse signal via the standard \( \ell_1 \)-minimization. Many sparse recovery conditions must imply the weak RSP. To see this, let us first recall a few existing matrix properties. \( A \in \mathbb{R}^{m \times n} \) is said to satisfy the restricted isometry property (RIP).
of order \(k\) if there exists a constant \(\delta_k \in (0, 1)\) such that \((1 - \delta_k)\|x\|^2 \leq \|Ax\|^2 \leq (1 + \delta_k)\|x\|^2\) for any \(k\)-sparse vector \(x \in \mathbb{R}^n\) (see [15]). \(A\) satisfies the null space property (NSP) of order \(k\) if \(\|\zeta_1\| < \|\zeta_2\|_1\) holds for any \(\zeta \neq 0\) in the null space of \(A\) and any \(S \subseteq \{1, \ldots, n\}\) with \(|S| \leq k\) (see [18, 28]). Strengthening the NSP concept leads to the following stable or robust NSP of order \(k\) (see [18, 28]): (i) \(A \in \mathbb{R}^{m \times n}\) satisfies the stable null space property of order \(k\) if there is a constant \(\rho \in (0, 1)\) such that \(\|\zeta_1\| \leq \rho \|\zeta_2\|_1\) for any \(\zeta \neq 0\) in the null space of \(A\) and any \(S \subseteq \{1, \ldots, n\}\) with \(|S| \leq k\); (ii) \(A\) is said to admit the robust null space property of order \(k\) if there are constants \(\rho' \in (0, 1)\) and \(\rho'' > 0\) such that \(\|\zeta_1\| \leq \rho' \|\zeta_2\|_1 + \rho'' \|A\zeta\|\) for any \(\zeta \neq 0\) in the null space of \(A\) and any \(S \subseteq \{1, \ldots, n\}\) with \(|S| \leq k\). It is well known that the mutual coherence condition \(\mu_1(K) + \mu_1(k - 1) < 1\) introduced in [51, 21, 24] implies the RIP and NSP (see Theorem 5.15 in [28] and Lemma 1.5 in [25]). The NSP is strictly weaker than the RIP (e.g. [27, 8]). Note that NSP of order \(k\) is also a necessary and sufficient condition for the recovery of every \(k\)-sparse signal (see [28]), the NSP of order \(k\) is equivalent to the RSP of order \(k\), and hence each of them implies the weak RSP of order \(k\) of \(A^T\). From the above discussion, we see that the RIP of \(A\) is a strictly stronger than the RSP of \(A^T\). Thus many existing sparse recovery conditions imply the weak RSP of order \(k\) of \(A^T\) which, in fact, is a necessary condition for many recovery problems to be stable as shown by the next theorem.

**Theorem 2.7.** Let \(\varphi : \mathbb{R}^q \rightarrow \mathbb{R}_+\) be a finite convex function on \(\mathbb{R}^q\) satisfying \(\varphi(0) = 0\) and \(\varphi(u) > 0\) for \(u \neq 0\). Let \(A \in \mathbb{R}^{m \times n}\) and \(M \in \mathbb{R}^{m \times q}\), where \(m < n\) and \(m \leq q\), be full-rank matrices. Suppose that for any given \(\varepsilon \geq 0\) and any given \(y \in \mathbb{R}^n\), the vector \(x \in \mathbb{R}^n\) satisfying \(\varphi(M^T(Ax - y)) \leq \varepsilon\) can be approximated by an optimal solution \(x^*\) of the problem

\[
\min_{\hat{x}} \{\|\hat{x}\|_1 : \varphi(M^T(A\hat{x} - y)) \leq \varepsilon\}
\]

with error

\[
\|x - x^*\|_2 \leq C'\sigma_k(x) + C''\gamma(\varepsilon),
\]

where \(C'\) and \(C''\) are constants depending on the problem data \((M, A, y, \varepsilon)\), and \(\gamma(\cdot)\) is a certain continuous function satisfying \(\gamma(0) = 0, \gamma(\varepsilon) > 0\) for \(\varepsilon > 0\), and \(C''\gamma(\varepsilon) \rightarrow 0\) as \(\varepsilon \rightarrow 0\). Then \(A^T\) must satisfy the weak RSP of order \(k\).

**Proof.** Let \(S_1, S_2 \subseteq \{1, \ldots, n\}\) be two arbitrary and disjoint sets with \(|S_1| + |S_2| \leq k\). We prove that there is a vector \(\zeta \in \mathcal{R}(A^T)\) satisfying (15). Let \(\hat{x}\) be a \(k\)-sparse vector in \(\mathbb{R}^n\) with

\[
\{i : \hat{x}_i > 0\} = S_1, \{i : \hat{x}_i < 0\} = S_2.
\]

Consider the small parameter \(\varepsilon\) such that

\[
C''\gamma(\varepsilon) \leq \min_{\hat{x}_i \neq 0} |\hat{x}_i|.
\]

Let the measurements \(y \approx A\hat{x}\) be accurate enough such that \(\varphi(M^T(A\hat{x} - y)) \leq \varepsilon\). For this pair \((y, \varepsilon)\), we consider the problem (16) to which, by the assumption, any feasible point can be approximated by an optimal solution of (16) with error (17). Therefore, there is an optimal solution \(x^*\) of (16) which approximates \(\hat{x}\) with error \(\|\hat{x} - x^*\|_2 \leq C'\sigma_k(\hat{x}) + C''\gamma(\varepsilon)\). Since \(\hat{x}\) is \(k\)-sparse, we have \(\sigma_k(\hat{x})_1 = 0\). Thus \(\|\hat{x} - x^*\|_2 \leq C''\gamma(\varepsilon)\) which, together with (19), implies that for positive components \(\hat{x}_i > 0\), the corresponding components \(x_i^*\) must be positive, and that for negative components \(\hat{x}_i < 0\), the corresponding \(x_i^*\) must be negative. Thus, we have

\[
S_1 = \{i : \hat{x}_i > 0\} \subseteq \{i : x_i^* > 0\}, \ S_2 = \{i : \hat{x}_i < 0\} \subseteq \{i : x_i^* < 0\}.
\]
Note that \( x^* \) is an optimal solution to the convex optimization problem (11), which satisfies a constraint qualification. In fact, if \( \varepsilon = 0 \), the constraint is reduced to the linear equation \( Ax = y \), and if \( \varepsilon > 0 \), the Slater’s constraint qualification is satisfied due to the fact that \( A \) is underdetermined (in which case there is a vector \( z \) such that \( Az = y \), and hence \( \varphi(M^T(Az - y)) < \varepsilon \)). So \( x^* \) must satisfy the KKT optimality condition, i.e.,

\[
0 \in \partial_x \left\{ \|x\|_1 + \lambda \left[ \varphi(M^T(Ax - y)) - \varepsilon \right] \right\}_{x=x^*},
\]

where \( \lambda \) is a Lagrangian multiplier and \( \partial_x \) denotes the subgradient with respect to \( x \). Note that the domains of the functions \( \varphi(M^T(Ax - y)) \) and \( \|x\|_1 \) are \( \mathbb{R}^n \), by Theorem 23.8 in [46], the above optimality condition is equivalent to

\[
0 \in \left\{ \partial \|x^*\|_1 + \lambda A^T M \partial \varphi(M^T(Ax^* - y)) \right\},
\]

where \( \partial \varphi(M^T(Ax^* - y)) \) is the subgradient of \( \varphi \) at \( M^T(Ax^* - y) \), and \( \partial \|x^*\|_1 \) is the subgradient of the \( \ell_1 \)-norm at \( x^* \), i.e.,

\[
\partial \|x^*\|_1 = \{ \zeta \in \mathbb{R}^n : \zeta_i = 1 \text{ for } x^*_i > 0, \zeta_i = -1 \text{ for } x^*_i < 0, |\zeta_i| \leq 1 \text{ for } x^*_i = 0 \}.
\]

Thus there exists a vector \( v \in \partial \varphi(M^T(Ax^* - y)) \) and a vector \( \zeta \in \partial \|x^*\|_1 \) such that \( \zeta + \lambda A^T M v = 0 \). Setting \( u = -\lambda M v \) yields \( \zeta = A^T u \). Since \( \zeta \in \partial \|x^*\|_1 \), we see that \( \zeta_i = 0 \) for \( x^*_i > 0 \), \( \zeta_i = -1 \) for \( x^*_i < 0 \), and \( |\zeta_i| \leq 1 \) for \( x^*_i = 0 \). This, together with (20), implies that the vector \( \zeta \) satisfies (13). Note that \( S_1 \) and \( S_2 \) are arbitrary and disjoint subsets of \( \{1, ..., n\} \) with \( |S_1| + |S_2| \leq k \). By Definition 2.6, \( A^T \) must satisfy the weak RSP of order \( k \). \( \square \).

As shown by Theorem 2.7, the weak RSP of \( A^T \) is a necessary condition for many sparse recovery problems to be stable. It implies that this condition cannot be further relaxed in order to ensure the stability of these problems. In later sections, we show that the weak RSP of \( A^T \) is also a sufficient condition for a wide range of sparse optimization problems, including DS and LASSO, to be stable in sparse data recovery. Under the assumptions of RIP, stable NSP or Robust NSP, the traditional error bounds of a recovery problem are measured in terms of these conventional matrix constants. Different from these assumptions, the weak RSP of \( A^T \) is a constant-free condition in the sense that the definition of this property does not involve any constant, so is the standard NSP of order \( k \) and RSP of order \( k \). Thus an immediately question arises: How to establish the stability of DS and LASSO under a constant-free matrix property? The main purpose of this study is to address this question.

### 3 Dantzig selectors with linear constraints

The constraint of (1) becomes linear when \( \phi \) is the \( \ell_{\infty} \)-norm, \( \ell_1 \)-norm, or their combination. In this case, problem (11) is equivalent to a linear program which can be solved efficiently by simplex methods or interior-point methods. Thus in this section, we focus on the norm \( \phi(\cdot) = \alpha \|\cdot\|_{\infty} + (1 - \alpha) \|\cdot\|_1 \), where \( \alpha \in [0, 1] \) is a fixed constant. Using this norm, the DS is in the form

\[
\min_x \{ \|x\|_1 : \alpha \|M^T(Ax - y)\|_{\infty} + (1 - \alpha) \|M^T(Ax - y)\|_1 \leq \tau \}. \tag{21}
\]

This problem encompasses the following special cases:

\[
\min_x \{ \|x\|_1 : \|M^T(Ax - y)\|_{\infty} \leq \tau \}, \tag{22}
\]
\[
\min_x \{ \|x\|_1 : \|M^T(Ax - y)\|_1 \leq \tau \},
\]
which correspond to the cases \( \alpha = 1 \) and \( \alpha = 0 \) in (21), respectively. Particularly, when \( M = A \), problem (22) is the standard DS proposed by Candès and Tao [12]. The purpose of this section is to establish a stability result for (21) under the weak RSP of order \( k \) of \( AT \).

By introducing two auxiliary variables \((\xi, v)\), the DS problem (21) can be written as

\[
\min_{(x, \xi, v)} \|x\|_1 \\
\text{s.t.} \quad \alpha \xi + (1 - \alpha)e^Tv \leq \tau, \quad \|M^T(Ax - y)\|_\infty \leq \xi, \\
\quad |M^T(Ax - y)| \leq v, \quad \xi \in \mathbb{R}_+, \quad v \in \mathbb{R}_q^q,
\]

where \( e \) is the vector of ones in \( \mathbb{R}^n \). Clearly, at any optimal solution \((x, t, \xi, v)\) of (25), it must hold that \( t = |x|, \ \xi \geq \|M^T(Ax - y)\|_\infty \) and \( v \geq |M^T(Ax - y)| \). There are many ways to obtain the dual problem of (25). For example, we may rewrite (25) as the so-called canonical form

\[
\begin{align*}
\min_{(x, t, \xi, v)} \bar{e}^T t \\
\text{s.t.} \quad x \leq t, \quad -x \leq t, \quad \alpha \xi + (1 - \alpha)e^Tv \leq \tau, \\
\quad -\xi e \leq M^T(Ax - y) \leq \xi e, \quad M^T(Ax - y) \leq v, \\
\quad -M^T(Ax - y) \leq v, \quad t \in \mathbb{R}_+, \quad \xi \in \mathbb{R}_+, \quad v \in \mathbb{R}_q^q,
\end{align*}
\]

where \( \bar{e} \) is the vector of ones in \( \mathbb{R}^n \). By the KKT optimality condition of (25) or (26), we immediately have the following lemma.

Lemma 3.1. \( x^* \) is an optimal solution of the DS problem (21) if and only if there exist vectors \((t^*, \xi^*, v^*, w_1^{(1)}, \ldots, w_7^{(1)})\) such that \((x^*, t^*, \xi^*, v^*, w_1^{(1)}, \ldots, w_7^{(1)}) \in \mathcal{D}\), where \( \mathcal{D} \) is the set of vectors \((x, t, \xi, v, w_1, \ldots, w_7)\) satisfying that

\[
\begin{align*}
&x \leq t, \quad -x \leq t, \quad \alpha \xi + (1 - \alpha)e^Tv \leq \tau, \\
&-\xi e \leq M^T(Ax - y) \leq \xi e, \quad M^T(Ax - y) \leq v, \quad -M^T(Ax - y) \leq v, \\
&-w^{(1)} + w^{(2)} \leq \bar{e}, \\
&w^{(1)} + w^{(2)} + A^TM(w^{(4)} - w^{(5)} - w^{(6)} + w^{(7)}) = 0, \\
&-\alpha w^{(3)} + e^Tw^{(5)} \leq 0, \\
&-\xi e + w^{(6)} + w^{(7)} \leq 0, \\
&w^{(i)} \geq 0, \quad i = 1, \ldots, 7,
\end{align*}
\]

For any \((x, t, \xi, v, w^{(1)}, \ldots, w^{(7)}) \in \mathcal{D}\), we must have that \( t = |x|, \ \xi \geq \|M^T(Ax - y)\|_\infty \) and \( v \geq |M^T(Ax - y)| \).
Note that the conditions \( t \in \mathbb{R}_+^q, \, \xi \in \mathbb{R}_+^p \) and \( v \in \mathbb{R}_+^q \) are implied from other conditions in (27), and thus these conditions can be removed from (27). It is easy to write \( \mathcal{D} \) in the form

\[
\mathcal{D} = \{z = (x, t, \xi, v, w^{(1)}, \ldots, w^{(7)}) : \, M^1 z \leq d^1, \, M^2 z = d^2 \},
\]

where \( d^2 = 0, \, d^1 = (0, 0, \tau, y^T M, -y^T M, y^T M, -y^T M, e^T, 0, 0, 0, 0, 0, 0, 0, 0)^T \) and

\[
M^1 = \begin{bmatrix}
I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha & (1-\alpha)e^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
M^T A & 0 & -e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-M^T A & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
M^T A & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-M^T A & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & I & -\alpha & e^T & e^T & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-\alpha)e & 0 & 0 & I & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 \\
\end{bmatrix},
\]

(29)

where \( I \)'s are the identity matrices and \( 0 \)'s are the zero matrices with suitable sizes. We now prove that the stability of (21) is guaranteed when \( A^T \) admits the weak RSP of order \( k \).

**Theorem 3.2.** Let the problem data \((A, M, y, \tau, \alpha)\) in (27) be given, where \( A \in \mathbb{R}^{m \times n} (m < n) \) and \( M \in \mathbb{R}^{m \times q} (m \leq q) \) with \( \text{rank}(A) = \text{rank}(M) = m \). Suppose that \( A^T \) satisfies the weak RSP of order \( k \). Then for any \( x \in \mathbb{R}^n \), there is an optimal solution \( x^* \) of (21) approximating \( x \) with error

\[
\|x - x^*\|_2 \leq \gamma \left\{ (\alpha\|\vartheta\|_\infty + (1 - \alpha)\|\vartheta\|_1 - \tau)^+ + 2\sigma_k(x_1) + c(\tau + \|\vartheta\|_\infty) \right\},
\]

(31)

where \( \vartheta = M^T (Ax - y) \), \( c \) is the constant given as

\[
c = \max_{G \subseteq \{1, \ldots, q\}, |G| = m} \{\|M_G^{-1}(AA^T)^{-1}A\|_{\infty \to 1} : M_G \in \mathbb{R}^{m \times m} \text{ is an invertible submatrix of } M\},
\]

and \( \gamma = \sigma_\infty (M^1, M^2) \) is the Robinson’s constant with \((M^1, M^2)\) being given in (29) and (30). In particular, for any \( x \) satisfying the constraints of (21) there is a solution \( x^* \) of (21) approximating \( x \) with error

\[
\|x - x^*\|_2 \leq \gamma \left\{ 2\sigma_k(x_1) + c\tau + c\|M^T (Ax - y)\|_\infty \right\} \leq 2\gamma \{\sigma_k(x_1) + c\tau\}.
\]

(32)

**Proof.** Let \( x \) be any given vector in \( \mathbb{R}^n \). We set \((t, \xi, v)\) as follows:

\[
t = |x|, \quad \xi = \|M^T (Ax - y)\|_\infty, \quad v = |M^T (Ax - y)|.
\]

(33)

Let \( S \) be the support of the \( k \)-largest components of \(|x|\), i.e., \( S = \{i_1, \ldots, i_k\} \), when the components of \(|x|\) are sorted into a descending order as \(|x_{i_1}| \geq \cdots \geq |x_{i_k}| \geq \cdots \geq |x_{i_n}| \). We define by \( S' = \{ j \in \)
Thus the vector $(w^{(1)}, \ldots, w^{(7)})$ which satisfies the constraint of problem \((26)\). First, we set \(w^{(1)}\) and \(w^{(2)}\) as follows: \(w^{(1)}_i = 1\) and \(w^{(2)}_i = 0\) for \(i \in \mathcal{S}'\); \(w^{(1)}_i = 0\) and \(w^{(2)}_i = 1\) for \(i \in \mathcal{S}''\); and \(w^{(1)}_i = (1 + \xi_i)/2\) and \(w^{(2)}_i = (1 - \xi_i)/2\) for all \(i \notin \mathcal{S}' \cup \mathcal{S}''\). This choice of \((w^{(1)}, w^{(2)})\) satisfies that \(w^{(1)} - w^{(2)} = \zeta\). Note that \(M\) is a full-row-rank matrix. Thus there exists an invertible \(m \times m\) submatrix of \(M\), denoted by \(M_3\), where \(\mathfrak{I} \subseteq \{1, \ldots, q\}\) with cardinality \(|\mathfrak{I}| = m\). We choose the vector \(h \in \mathbb{R}^q\) as \(h_\mathfrak{I} = M_3^{-1}u^*\) and \(h_\mathfrak{I} = 0\) where \(\mathfrak{I} = \{1, \ldots, q\}\setminus\mathfrak{I}\). Thus \(Mh = u^*\). We now define the nonnegative vectors \(w^{(3)}, \ldots, w^{(7)}\) as follows:

\[
  w^{(3)} = \|h\|_1, \ w^{(4)} = \alpha(h)^+, \ w^{(5)} = -\alpha(h)^-, \ w^{(6)} = -(1 - \alpha)(h)^-, \ w^{(7)} = (1 - \alpha)(h)^+,
\]

which implies that

\[
  e^T(w^{(4)} + w^{(5)}) = \alpha e^T h = \alpha \|h\|_1 = \alpha w^{(3)}, \ w^{(6)} + w^{(7)} = (1 - \alpha)|h| \leq (1 - \alpha)w^{(3)} e.
\]

Therefore,

\[
  A^T M(w^{(4)} - w^{(5)} - w^{(6)} + w^{(7)}) = A^T (Mh) = A^T u^* = \zeta = w^{(1)} - w^{(2)}.
\]

Thus the vector \((w^{(1)}, \ldots, w^{(7)})\) constructed as above satisfies the constraints of \((26)\). Consider the set \(\mathcal{D}\) written as \((28)\). For the vector \((x, t, \xi, v, w^{(1)}, \ldots, w^{(7)})\), where \((t, \xi, v)\) is chosen as \((33)\) and \(w^{(1)}, \ldots, w^{(7)}\) are chosen as above, applying Lemma 2.1 with \((M^1, M^2)\) being given in \((29)\) and \((30)\), there exists a vector \((x^*, t^*, \xi^*, v^*, w^{(1)}_*, \ldots, w^{(7)}_*) \in \mathcal{D}\) such that

\[
  \left| \begin{bmatrix} x & x^* \\ t & t^* \\ \xi & \xi^* \\ v & v^* \\ w^{(1)} & w^{(1)}_* \\ \vdots & \vdots \\ w^{(7)} & w^{(7)}_* \end{bmatrix} \right|_2 \leq \gamma \left| \begin{bmatrix} (x - t)^+ \\ (-x - t)^+ \\ (\alpha \xi + (1 - \alpha) e^T v - \tau)^+ \\ (M^T (Ax - y) - \xi e)^+ \\ (M^T (Ax - y) - \xi e)^+ \\ (M^T (Ax - y) - v)^+ \\ (M^T (Ax - y) - v)^+ \end{bmatrix} \right|_1
\]

\[
  + \left| \begin{bmatrix} A^T M(w^{(4)} - w^{(5)} - w^{(6)} + w^{(7)}) - w^{(1)} + w^{(2)} \\ w^{(1)} + w^{(2)} - 2\tilde{e}^+ \\ -\alpha w^{(3)} + e^Tw^{(4)} + e^Tw^{(5)}^+ \\ -\alpha w^{(3)} + e^Tw^{(4)} + e^Tw^{(5)}^+ \\ \tilde{e}^T t + \tau w^{(3)} - y^T M(w^{(4)} - w^{(5)} - w^{(6)} + w^{(7)}) \end{bmatrix} \right|_1
\]

where \((Y)^+ = ((-t)^+, (-\xi)^+, (-v)^+, (-w^{(1)})^+, \ldots, (-w^{(7)})^+), \) and \(\gamma = \sigma_{\infty, 2}(M^1, M^2)\) is the Robinson’s constant with \((M^1, M^2)\) being given in \((29)\) and \((30)\). It follows from \((33)\) that

\[
  (x - t)^+ = (-x - t)^+ = 0, \ (M^T (Ax - y) - \xi e)^+ = 0,
\]

\[
  (M^T (Ax - y) - \xi e)^+ = 0, \ (M^T (Ax - y) - v)^+ = 0, \ (M^T (Ax - y) - v)^+ = 0.
\]
Since \((w^{(1)}, \ldots, w^{(7)})\) is a feasible point to \((26)\), we also see that
\[
A^T M (w^{(4)} - w^{(5)} - w^{(6)} + w^{(7)}) - w^{(1)} + w^{(2)} = 0, \quad (w^{(1)} + w^{(2)} - \bar{e})^+ = 0,
\]
\[
\left(-\alpha w^{(3)} + e^T w^{(4)} + e^T w^{(5)}\right)^+ = 0, \quad \left(-(1 - \alpha)w^{(3)} + w^{(6)} + w^{(7)}\right)^+ = 0.
\]
Moreover, the nonnegativity of \((t, \xi, v, w^{(1)}, \ldots, w^{(7)})\) implies that \((Y)^+ = 0\). Note that
\[
\|x - x^*\|_2 \leq \|(x, t, \xi, v^{(1)}, \ldots, w^{(7)}) - (x^*, t^*, \xi^*, v^*, w^{(1)}, \ldots, w^{(7)})\|_2.
\]
Thus the inequality \((35)\) is reduced to
\[
\|x - x^*\|_2 \leq \gamma \left\|\left[\tilde{e}^T t + \tau w^{(3)} - y^T M (w^{(4)} - w^{(5)} - w^{(6)} + w^{(7)})\right]_1\right\|.
\]
(36)
Denote by \(\vartheta = M^T (Ax - y)\) which implies that \(y^T M = x^T A^T M - \vartheta^T\). It follows from \((33), (34)\)
and the fact \(A^T M h = A^T u^* = \zeta\) that
\[
\left(\alpha \zeta + (1 - \alpha) e^T v - \tau\right)^+ = (\alpha \|\vartheta\|_\infty + (1 - \alpha) \|\vartheta\|_1 - \tau)^+
\]
and
\[
\left|\tilde{e}^T |x| + \tau w^{(3)} - (x^T A^T M - \vartheta^T) h\right| = \left|\tilde{e}^T |x| + \tau w^{(3)} - x^T \zeta + \vartheta^T h\right| \\
\leq 2\sigma_k (x)_1 + \tau \|h\|_1 + \|\vartheta\|_\infty \|h\|_1,
\]
(38)
where the last inequality follows from the fact \(|\vartheta^T h| \leq \|\vartheta\|_\infty \|h\|_1\) and
\[
|\tilde{e}^T |x| - x^T \zeta| = \left|\tilde{e}^T |x| - x^T \zeta S - x^T \frac{1}{S} S|\right| \leq \|x\|_1 - \|x_S\|_1 + \|x_S^T\|_1 \|\zeta\|_\infty \leq 2\|x^T\|_1 = 2\sigma_k (x)_1.
\]
We define the constant
\[
c = \max_{G \subseteq \{1, \ldots, q\}, |G| = m} \{\|M_G^{-1} (AA^T)^{-1} A\|_\infty \to 1 : M_G \in \mathbb{R}^{m \times m} \text{ is an invertible submatrix of } M\}.
\]
Noting that \(h_T = 0\) and \(M3 h_3 = u^* = (AA^T)^{-1} A \zeta\), we have
\[
\|h\|_1 = \|h_3\|_1 = \|M_3^{-1} (AA^T)^{-1} A \zeta\|_1 \leq \|M_3^{-1} (AA^T)^{-1} A\|_\infty \to 1 \|\zeta\|_\infty \leq c.
\]
(39)
Combining \((36), (39)\) leads to
\[
\|x - x^*\|_2 \leq \gamma \left\{\alpha \|\vartheta\|_\infty + (1 - \alpha) \|\vartheta\|_1 - \tau\right\}^+ + 2\sigma_k (x)_1 + c (\tau + \|\vartheta\|_\infty)
\]
which is exactly the bound given in \((31)\). In particular, if \(x\) satisfies the constraint of \((21)\), then
\[
\|x - x^*\|_2 \leq \gamma \left\{2\sigma_k (x)_1 + c (\tau + \|\vartheta\|_\infty)\right\}.
\]
(40)
Since \(\|\vartheta\|_\infty \leq \alpha \|\vartheta\|_\infty + (1 - \alpha) \|\vartheta\|_1 \leq \tau\), the bound \((32)\) follows from \((40)\) immediately. \(\Box\)

Many existing conditions imply the weak RSP of order \(k\) of \(A^T\). The following result can be immediately obtained from Theorem 3.2.
Corollary 3.3. Let $A$ and $M$ be given as in Theorem 3.2. Suppose that one of the following conditions holds: (a) $A$ (with $\ell_2$-normalized columns) satisfies the mutual coherence property $\mu_1(k) + \mu_1(k-1) < 1$; (b) RIP of order $2k$ with constant $\delta_{2k} < 1/\sqrt{2}$; (c) stable NSP of order $k$ with constant $\rho \in (0, 1)$; (d) robust NSP of order $k$ with constants $\rho' \in (0, 1)$ and $\rho'' > 0$; (e) NSP of order $k$; (f) RSP of order $k$ of $A^T$. Then the conclusions of Theorem 3.2 are valid for (27).

In this corollary, condition (a) implies (e) (see Theorem 5.15 in [25] and the definition of mutual coherence $\mu_1(\cdot)$ therein). Condition (b) implies that every $k$-sparse vector can be exactly recovered by $\ell_1$-minimization (see [10]), and thus implies (e). Each of conditions (c) and (d) implies (e). Conditions (e) and (f) are equivalent. Thus each of conditions (a) – (e) implies (f), and hence they imply the weak RSP of order $k$ of $A^T$. Thus Corollary 3.3 follows immediately from Theorem 3.2. This is a unified result in the sense that any of the above-mentioned conditions implies the same error bounds [31] and [32]. In particular, by setting $\alpha = 1$ and $\alpha = 0$, respectively, Theorem 3.2 claims that under the weak RSP of order $k$, both DS problems [22] and [23] are stable in sparse vector recovery. The results for $M = I$ and $M = A$, respectively, can follow immediately from Theorem 3.2 and Corollary 3.3. As an example, let us state the result for the standard DS, corresponding to the case $M = A$ and $\alpha = 1$ in [21]. Combining Theorems 3.2 and 2.7 as well as Corollary 3.3 leads to the following result.

Corollary 3.4. Let $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = m < n$. Consider the following standard DS problem:

$$\min_x \{\|x\|_1 : \|A^T(Ax - y)\|_\infty \leq \tau\}. \tag{41}$$

Then the following statements hold:

(i) Suppose that $A^T$ satisfies the weak RSP of order $k$. Then for any $x \in \mathbb{R}^n$, there is an optimal solution $x^*$ of (41) approximating $x$ with error

$$\|x - x^*\|_2 \leq \gamma \{\|A^T(Ax - y)\|_\infty - \tau\}^+ + 2\sigma_k(x)_1 + c_A\tau + c_A\|A^T(Ax - y)\|_\infty,$$

where $c_A$ is the constant

$$c_A = \max_{G \subseteq \{1, \ldots, n\}, |G| = m}\{\|A^{-1}_G(AA^T)^{-1}A\|_{\infty \rightarrow 1} : A_G \text{ is an invertible submatrix of } A\},$$

and $\gamma = \sigma_{\infty, 2}(M^1, M^2)$ is the Robinson’s constant determined by $(M^1, M^2)$ given in [22] and [30] with $M = A$ and $\alpha = 1$. In particular, for any $x$ satisfying the constraint of (41), there is a solution $x^*$ of (41) approximating $x$ with error

$$\|x - x^*\|_2 \leq 2\gamma \{\sigma_k(x)_1 + c_A\tau + c_A\|A^T(Ax - y)\|_\infty\} \leq 2\gamma \{\sigma_k(x)_1 + c_A\tau\}. \tag{42}$$

Conversely, if for any given small $\tau \geq 0$ and for any $x$ obeying $\|A^T(Ax - y)\|_\infty \leq \tau$, there is an optimal solution $x^*$ of (41) such that the estimate (42) holds, then $A^T$ must satisfy the weak RSP of order $k$.

(ii) Every matrix condition listed in Corollary 3.3 is sufficient to ensure that, for any $x$ obeying $\|A^T(Ax - y)\|_\infty \leq \tau$, there is an optimal solution $x^*$ of (41) such that (42) holds.

Item (i) above is different from existing stability results for the standard DS in terms of the mild assumption, analytic method, and the way of expression of stability coefficients. Roughly speaking, Corollary 3.4 indicates that the weak RSP of $A^T$ is both necessary and sufficient for
the standard DS to be weakly stable in sparse recovery. Item (ii) above indicates that the error bound \( (42) \) holds under any of the conditions listed in Corollary 3.3. Letting \( \alpha = 0 \) and replacing \( \| \cdot \|_\infty \) by \( \| \cdot \|_1 \), Corollary 3.4 immediately becomes the stability result for the problem (23) with \( M = A \).

**Remark 3.5.** When the matrix \( A \) does not satisfy a desired matrix property like the RIP, NSP, RSP or REC), a scaled version of this matrix, i.e., \( AU \), where \( U \) is a nonsingular matrix, might admit a desired property. This partially explains why a weighted \( \ell_1 \)-minimization algorithm (e.g., [16, 55, 56]) often numerically outperforms the standard \( \ell_1 \)-minimization in sparse data recovery.

The stability theory developed in this paper can be easily generalized to weighted \( \ell_1 \)-minimization problems. Take the following weighted Dantzig selector as an example:

\[
\min \{ \| Wx \|_1 : \| A^T(Ax - y) \|_\infty \leq \tau \},
\]

(43)

where \( W \) is a nonsingular diagonal matrix. We ask whether this problem is weakly stable in sparse data recovery. By the nonsingular transform \( u = Wx \), this problem can be written as

\[
\min \{ \| u \|_1 : \| M^T(A\tilde{u} - y) \|_\infty \leq \tau \},
\]

(44)

with \( \tilde{A} = AW^{-1} \) and \( M = A \). This is the recovery problem (1) with \( \phi = \| \cdot \|_\infty \). Thus it is straightforward to extend the stability results developed in this paper to the weighted Dantzig selector (43) under the weak RSP assumption on the scaled matrix \( \tilde{A} = AW^{-1} \).

## 4 Dantzig selectors with nonlinear constraints

In this section, we deal with the nonlinear problem (1), where the constraint \( \phi(M^T(Ax - y)) \leq \tau \) cannot be represented exactly as a finite number of linear constraints, for example, when \( \phi = \| \cdot \|_p \) with \( p \in (1, \infty) \). In this case, \( \tau \) must be positive, since otherwise if \( \tau = 0 \) the constraint will reduce to the linear system \( Ax = y \). We show that the nonlinear DS problem (1) remains stable in sparse data recovery under the weak RSP assumption.

Let \( \varrho^* \) be the optimal value of (1) and \( S^* \) the set of optimal solutions of (1), which clearly can be written as

\[
S^* = \{ x \in \mathbb{R}^n : \| x \|_1 \leq \varrho^*, \phi(M^T(Ax - y)) \leq \tau \}.
\]

In terms of \( \mathcal{B}^\phi \) in \( \mathbb{R}^q \), the nonlinear problem (1) can be written as

\[
\varrho^* = \min_{(x,u)} \{ \| x \|_1 : u = M^T(Ax - y)/\tau, u \in \mathcal{B}^\phi \}.
\]

(45)

Unlike the linear case examined in Section 3, the nonlinearity of the constraint prohibits applying Lemma 2.1 directly to establish a stability result. A natural idea is to use a certain polytope approximation of the unit ball in \( \mathbb{R}^q \). In this section, we use the polytope \( \mathcal{Q}_\epsilon \), which is defined by (10) and satisfies the relation (11), to approximate the unit ball \( \mathcal{B}^\phi \). Let us first develop a few technical results. The first one below, which is of independent interest, describes certain properties of the projection operator.

**Lemma 4.1.** (i) Let \( \Omega \subseteq T \) be two compact convex sets in \( \mathbb{R}^n \). Then for any \( x \in \mathbb{R}^n \),

\[
\| \Pi_{\Omega}(x) - \Pi_T(x) \|_2^2 \leq d^H(\Omega, T) \| x - \Pi_{\Omega}(x) \|_2.
\]

(46)
(ii) Let $\Omega, U, T$ be three compact convex sets in $\mathbb{R}^n$ satisfying $\Omega \subseteq T$ and $U \subseteq T$. Then for any $x \in \mathbb{R}^n$ and any $u \in U$,

$$\|x - \Pi_\Omega(x)\|_2 \leq d^H(\Omega, T) + 2\|x - u\|_2. \quad (47)$$

**Proof.** By the property of the projection operator, we have $(x - \Pi_T(x))^T(u - \Pi_T(x)) \leq 0$ for all $u \in T$, and $(x - \Pi_\Omega(x))^T(u - \Pi_\Omega(x)) \leq 0$ for all $v \in \Omega$. Since $\Pi_\Omega(x) \in \Omega \subseteq T$ and $\Pi_\Omega(\Pi_T(x)) \in \Omega$, we immediately have the following two equalities:

$$(x - \Pi_T(x))^T[\Pi_\Omega(x) - \Pi_T(x)] \leq 0, \quad (x - \Pi_\Omega(x))^T(\Pi_\Omega(\Pi_T(x)) - \Pi_\Omega(x)) \leq 0. \quad (48)$$

Since $\Omega \subseteq T$ and $\Pi_T(x) \in T$, by the definition of Hausdorff metric, we have

$$d^H(\Omega, T) = \sup_{w \in T} \inf_{z \in \Omega} \|w - z\|_2 \geq \inf_{z \in \Omega} \|\Pi_T(x) - z\|_2 = \|\Pi_T(x) - \Pi_\Omega(\Pi_T(x))\|_2. \quad (49)$$

By (48) and (49), we have

$$\|\Pi_\Omega(x) - \Pi_T(x)\|_2^2 = (\Pi_\Omega(x) - x + x - \Pi_T(x))^T(\Pi_\Omega(x) - \Pi_T(x))$$

$$= (\Pi_\Omega(x) - x)^T(\Pi_\Omega(x) - \Pi_T(x)) + (x - \Pi_T(x))^T(\Pi_\Omega(x) - \Pi_T(x))$$

$$\leq (\Pi_\Omega(x) - x)^T[\Pi_\Omega(x) - \Pi_\Omega(\Pi_T(x)) + \Pi_\Omega(\Pi_T(x)) - \Pi_T(x)]$$

$$\leq (\Pi_\Omega(x) - x)^T[\Pi_\Omega(\Pi_T(x)) - \Pi_T(x)]$$

$$\leq \|x - \Pi_\Omega(x)\|_2^2 \|\Pi_\Omega(\Pi_T(x)) - \Pi_T(x)\|_2$$

$$\leq d^H(\Omega, T)\|x - \Pi_\Omega(x)\|_2.$$

Thus (46) holds. We now prove (47). For any $u \in U \subseteq T$, we clearly have $\|x - \Pi_T(x)\|_2 \leq \|x - u\|_2$. Thus by the triangle inequality and (46), we have

$$\|x - \Pi_\Omega(x)\|_2 \leq \|x - \Pi_T(x)\|_2 + \|\Pi_T(x) - \Pi_\Omega(x)\|_2$$

$$\leq \|x - u\|_2 + \sqrt{d^H(\Omega, T)\|x - \Pi_\Omega(x)\|_2} \quad (50)$$

for any $x \in \mathbb{R}^n$ and $u \in U$. Note that the quadratic equation $t^2 = \alpha + \sqrt{\beta}t$ in $t$, where $\alpha \geq 0$ and $\beta \geq 0$, admits a unique nonnegative root $t^* = \frac{\sqrt{\beta + \sqrt{\beta}}}{\sqrt{\beta + \sqrt{\beta}}}$.

Thus, by setting

$$t = \sqrt{\|x - \Pi_\Omega(x)\|_2}, \quad \alpha = \|x - u\|_2, \quad \beta = d^H(\Omega, T),$$

it immediately follows from (50) that

$$\sqrt{\|x - \Pi_\Omega(x)\|_2} \leq \sqrt{d^H(\Omega, T) + \sqrt{d^H(\Omega, T) + 4\|x - u\|_2}} \leq \frac{\sqrt{d^H(\Omega, T) + \sqrt{d^H(\Omega, T) + 4\|x - u\|_2}}}{2} \leq d^H(\Omega, T) + 2\|x - u\|_2,$$

which implies that

$$\|x - \Pi_\Omega(x)\|_2 \leq \left(\sqrt{d^H(\Omega, T) + \sqrt{d^H(\Omega, T) + 4\|x - u\|_2}}\right)^2 \leq d^H(\Omega, T) + 2\|x - u\|_2,$$

where the last inequality follows from the fact $(t_1 + t_2)^2 \leq 2(t_1^2 + t_2^2)$ for any numbers $t_1$ and $t_2$. 

$\square$
To show the next two technical results, let us first define a set which is a relaxation of the solution set $S^*$ of problem (1). Note that
\[ S^* = \{ x \in \mathbb{R}^n : \|x\|_1 \leq \varrho^*, \ u = M^T(Ax - y)/\tau, \ u \in \mathfrak{B}^\phi \}. \]
Replacing $\mathfrak{B}^\phi$ with the polytope $Q_{\phi'}$, which is an outer approximation of $\mathfrak{B}^\phi$, yields the set
\[ S_{\phi'} = \{ x \in \mathbb{R}^n : \|x\|_1 \leq \varrho^*, \ u = M^T(Ax - y)/\tau, \ u \in Q_{\phi'} \}. \quad (51) \]
Clearly, $S^* \subseteq S_{\phi'}$ due to the fact $\mathfrak{B}^\phi \subseteq Q_{\phi'}$.

**Lemma 4.2.** Let $S_{\phi'}$ be the set defined in (51) where $Q_{\phi'}$ is given in (10). For every $j$, let $x_{\phi'}$ be an arbitrary point in $S_{\phi'}$. Then every accumulation point $\hat{x}$ of the sequence $\{x_{\phi'}\}_{j \geq 1}$ satisfies $\phi(M^T(A\hat{x} - y)) \leq \tau$.

**Proof.** Recall that $Q_{\phi'}$ is represented as (12), i.e.,
\[ Q_{\phi'} = \{ u \in \mathbb{R}^q : (a^i)^T u \leq 1 \text{ for all } a^i \in \mathcal{Y}(\Gamma_{Q_{\phi'}}) \}. \]
Note that $x_{\phi'} \in S_{\phi'}$ for any $j \geq 1$. Then for any $j \geq 1$, we see from (51) that
\[ \|x_{\phi'}\|_1 \leq \varrho^*, \ (a^i)^T M^T(Ax_{\phi'} - y) \leq \tau \text{ for all } a^i \in \mathcal{Y}(\Gamma_{Q_{\phi'}}). \quad (52) \]
The sequence $\{x_{\phi'}\}_{j \geq 1}$ is bounded. Let $\hat{x}$ be any accumulation point of the sequence $\{x_{\phi'}\}_{j \geq 1}$. Passing through to a subsequence if necessary, we may assume that $x_{\phi'} \to \hat{x}$ as $j \to \infty$. We prove the lemma by contradiction. Assume that $\phi(M^T(A\hat{x} - y)) > \tau$. Then we define
\[ \hat{\sigma} := \frac{\phi(M^T(A\hat{x} - y)) - \tau}{\tau}, \]
which is a positive constant under the assumption. Since $\epsilon_j \to 0$ as $j \to \infty$, there exists an integer number $j_0$ such that $\epsilon_j < \hat{\sigma}$ for any $j \geq j_0$. By the definition of $\Gamma_{Q_{\phi'}}$, we see that
\[ \mathcal{Y}(\Gamma_{Q_{\phi'}}) \subseteq \mathcal{Y}(\Gamma_{Q_{\phi'}}) \text{ for any } j \geq j' \geq j_0. \quad (53) \]
Thus for any fixed integer number $j' \geq j_0$, the following holds for all $j \geq j'$:
\[ \sup_{a^i \in \mathcal{Y}(\Gamma_{Q_{\phi'}})} (a^i)^T M^T(Ax_{\phi'} - y) \leq \sup_{a^i \in \mathcal{Y}(\Gamma_{Q_{\phi'}})} (a^i)^T M^T(Ax_{\phi'} - y) \leq \tau, \]
where the first inequality follows from (53) and the second inequality follows from (52). For every fixed $j' \geq j_0$, noting that $x_{\phi'} \to \hat{x}$ as $j \to \infty$, it follows from the above inequality that
\[ \sup_{a^i \in \mathcal{Y}(\Gamma_{Q_{\phi'}})} (a^i)^T M^T(A\hat{x} - y) \leq \tau. \quad (54) \]
Consider the vector $\hat{a} = M^T(A\hat{x} - y)/\phi(M^T(A\hat{x} - y))$ which is on the surface of the unit ball $\mathfrak{B}^\phi$. Note that $\epsilon_{j'} < \hat{\sigma}$. Applying Lemma 2.3 to $Q_{\phi'}$, we conclude that for the vector $\hat{a}$, there is a vector $a^i \in \mathcal{Y}(\Gamma_{Q_{\phi'}})$ such that $(a^i)^T \hat{a} \geq \frac{1}{1 + \epsilon_{j'}} > \frac{1}{1 + \sigma}$, which implies that
\[ (a^i)^T M^T(A\hat{x} - y) > \frac{\phi(M^T(A\hat{x} - y))}{1 + \sigma} = \tau, \quad (55) \]
where the equality follows from the definition of \( \hat{x} \). Since \( a^i \in \U(\Gamma_{Q_{\epsilon_j}}) \), the inequality \([55]\) contradicts \([54]\). Therefore, for any accumulation point \( \hat{x} \) of the sequence \( \{x_{\epsilon_j}\}_{j \geq 1} \), we must have that \( \phi(M^T(A\hat{x} - y)) \leq \tau \). \( \square \)

**Lemma 4.3.** Let \( S^* \) be the solution set of problem \( (1) \) and let \( S_{\epsilon_j} \) be the set given in \([51]\). Then \( d^H(S^*, S_{\epsilon_j}) \to 0 \) as \( j \to \infty \).

**Proof.** Since \( S^* \subseteq S_{\epsilon_j} \), by the definition of Hausdorff metric, we see that

\[
d^H(S^*, S_{\epsilon_j}) = \sup_{x \in S_{\epsilon_j}} \inf_{z \in S^*} \|x - z\|_2 = \sup_{x \in S_{\epsilon_j}} \|x - \Pi_{S^*}(x)\|_2. \tag{56}
\]

Note that \( S^* \) and \( S_{\epsilon_j} \) are compact convex sets and the projection operator \( \Pi_{S^*}(x) \) is continuous in \( \mathbb{R}^n \). For every polytope \( S_{\epsilon_j} \), the supremum in \( [56] \) can be attained. Thus there exists a point in \( S_{\epsilon_j} \), denoted by \( x_{\epsilon_j} \), such that

\[
d^H(S^*, S_{\epsilon_j}) = \|x_{\epsilon_j} - \Pi_{S^*}(x_{\epsilon_j})\|_2. \tag{57}
\]

Note that \( S^* \subseteq S_{\epsilon_j} \) for any \( j \geq 1 \). The sequence \( \{d^H(S^*, S_{\epsilon_j})\}_{j \geq 1} \) is non-increasing and nonnegative. The limit \( \lim_{j \to \infty} d^H(S^*, S_{\epsilon_j}) \) exists. Passing through to subsequence if necessary, we may assume that the sequence \( \{x_{\epsilon_j}\}_{j \geq 1} \) tends to \( \hat{x} \). Note that \( x_{\epsilon_j} \subseteq S_{\epsilon_j} \) which indicates that \( \|x_{\epsilon_j}\|_1 \leq \rho^* \) and hence \( \|\hat{x}\|_1 \leq \rho^* \). By Lemma 4.2, \( \hat{x} \) must satisfy that \( \phi(M^T(A\hat{x} - y)) \leq \tau \) which, together with \( \|\hat{x}\|_1 \leq \rho^* \), implies that \( \hat{x} \in S^* \). As a result, \( \Pi_{S^*}(\hat{x}) = \hat{x} \). Therefore,

\[
\lim_{j \to \infty} d^H(S^*, S_{\epsilon_j}) = \lim_{j \to \infty} \|x_{\epsilon_j} - \Pi_{S^*}(x_{\epsilon_j})\|_2 = \|\hat{x} - \Pi_{S^*}(\hat{x})\|_2 = 0,
\]

as desired. \( \square \)

Throughout the remainder of this section, let \( \delta \) be any fixed small constant (e.g., a sufficiently small constant in \((0, \tau)\)). By Lemma 4.3, there is a \( j_0 \) such that \( S_{\epsilon_{j_0}} \) defined in \([51]\) achieves

\[
d^H(S^*, S_{\epsilon_{j_0}}) \leq \delta. \tag{58}
\]

In the reminder of this section, we focus on the fixed polytope \( Q_{\epsilon_{j_0}} \), as an approximation of \( \mathfrak{B}^\phi \). We use \( \hat{n} \) to denote the number of the columns of \( \Gamma_{Q_{\epsilon_{j_0}}} \) and use \( \hat{e} \) to denote the vector of ones in \( \mathbb{R}^{\hat{n}} \) to distinguish the vector of ones in other spaces. Replacing \( \mathfrak{B}^\phi \) in \([45]\) with \( Q_{\epsilon_{j_0}} \) leads to the following relaxation of problem \( (11) \):

\[
\rho^*_{j_0} : = \min_{(x,u)} \{\|x\|_1 : u = M^T(Ax - y)/\tau, \ u \in Q_{\epsilon_{j_0}}\} = \min_x \{\|x\|_1 : (\Gamma_{Q_{\epsilon_{j_0}}})^T[M^T(Ax - y)] \leq \tau \hat{e}\}, \tag{59}
\]

where \( \rho^*_{j_0} \) denotes the optimal value of the above optimization problem. Clearly, \( \rho^*_{j_0} \leq \rho^* \) due to the fact \( \mathfrak{B}^\phi \subseteq Q_{\epsilon_{j_0}} \). Let \( S^*_{\epsilon_{j_0}} \) denote the set of optimal solutions of \([58]\), i.e.,

\[
S^*_{\epsilon_{j_0}} = \{x \in \mathbb{R}^n : \|x\|_1 \leq \rho^*_{j_0}, \ u = M^T(Ax - y)/\tau, \ u \in Q_{\epsilon_{j_0}}\}. \tag{60}
\]

By \([51]\), we immediately see that \( S^*_{\epsilon_{j_0}} \subseteq S_{\epsilon_{j_0}} \) since \( \rho^*_{j_0} \leq \rho^* \). The problem \([58]\) can be written as

\[
\min_{(x,t)} \{t^T x : x \leq t, \ -x \leq t, \ t \geq 0, \ (\Gamma_{Q_{\epsilon_{j_0}}})^T[M^T(Ax - y)] \leq \tau \hat{e}\}, \tag{61}
\]

where \( \hat{e} \) is the vector of ones in \( \mathbb{R}^n \). Clearly, \( t = \|x\|_1 \) at any optimal solution of the problem. The Lagrangian dual of the above problem is given as

\[
\max_{w_3} - [\tau \hat{e} + (M\Gamma_{Q_{\epsilon_{j_0}}})^T y]^T w_3 \tag{62}
\]

s.t. \( (A^T M\Gamma_{Q_{\epsilon_{j_0}}}) w_3 + w_1 - w_2 = 0, \ w_1 + w_2 \leq \hat{e}, \ w_1, w_2, w_3 \geq 0 \).
By the KKT optimality condition, the solution set of (58) can be completely characterized.

**Lemma 4.4.** $x^* \in \mathbb{R}^n$ is an optimal solution of (58) if and only if there exist vectors $t^*, w_1^*, w_2^* \in \mathbb{R}_+$ and $w_3^* \in \mathbb{R}_+$ such that $(x^*, t^*, w_1^*, w_2^*, w_3^*) \in \mathcal{D}^{(1)}$, where $\mathcal{D}^{(1)}$ is the set of vectors $(x, t, w_1, w_2, w_3)$ satisfying the following system:

\[
\begin{cases}
  x \leq t, & -x \leq t, \\ 
  w_1 + w_2 \leq \tilde{e}, & A^T M \Gamma_{Q_{s_{j_0}}} w_3 + w_1 - w_2 = 0, \\ 
  \tilde{e}^T t = -[\tilde{\tau} \tilde{e} + (M \Gamma_{Q_{s_{j_0}}})^T y] w_3, \\ 
  (t, w_1, w_2, w_3) \geq 0.
\end{cases}
\]  

(60)

By optimality, we see that $t = \|x\|$ for any $(x, t, w_1, w_2, w_3) \in \mathcal{D}^{(1)}$. We write (60) as

$$\mathcal{D}^{(1)} = \{z = (x, t, w_1, w_2, w_3) : \bar{M}^1 z \leq \bar{b}^1, \bar{M}^2 z = \bar{b}^2\},$$

where $\bar{b}^2 = 0$, $\bar{b}^1 = [0, 0, \tilde{e}^T, ((M \Gamma_{Q_{s_{j_0}}})^T y + \tilde{\tau} \tilde{e})^T, 0, 0, 0, 0]^T$ and

$$\bar{M}^1 = \begin{bmatrix}
I & -I & 0 & 0 & 0 \\
-I & -I & 0 & 0 & 0 \\
0 & 0 & I & I & 0 \\
(M \Gamma_{Q_{s_{j_0}}})^T A & 0 & 0 & 0 & 0 \\
0 & -I & 0 & 0 & 0 \\
0 & 0 & -I & 0 & 0 \\
0 & 0 & 0 & -I & 0 \\
0 & 0 & 0 & 0 & -\hat{I}
\end{bmatrix},$$

(61)

$$\bar{M}^2 = \begin{bmatrix}
0 & 0 & I & -I & A^T M \Gamma_{Q_{s_{j_0}}} \\
0 & \tilde{e}^T & 0 & 0 & \tilde{\tau} \tilde{e}^T + y^T M \Gamma_{Q_{s_{j_0}}}
\end{bmatrix},$$

(62)

where $I \in \mathbb{R}^{n \times n}$ and $\hat{I} \in \mathbb{R}^{n \times n}$ are identity matrices and 0’s are zero-matrices with suitable sizes. The main result in this section is given as follows.

**Theorem 4.5.** Given the problem data $(A, M, y, \tau)$, where $A \in \mathbb{R}^{m \times n}$ ($m \leq n$) and $M \in \mathbb{R}^{m \times q}$ ($m \leq q$) with $\text{rank}(A) = \text{rank}(M) = m$. Let $\delta \in (0, \tau)$ be a fixed constant and let $Q_{s_{j_0}}$ be the fixed polytope such that (57) is achieved. Suppose that $A^T$ satisfies the weak RSP of order $k$. Then for any $x \in \mathbb{R}^n$, there is a solution $x^*$ of (1) approximating $x$ with error

$$\|x - x^*\|_2 \leq \delta + 2\tilde{\tau} \left\{\tilde{n} (\phi(M^T (Ax - y)) - \tau)^+ + 2\sigma_k(x) + c\tau + c\phi(M^T (Ax - y))\right\},$$

(63)

where $c$ is the constant given in Theorem 3.2, and $\tilde{\tau} = \sigma_{\infty,2}(\bar{M}^1, \bar{M}^2)$ is the Robinson constant determined by $(\bar{M}^1, \bar{M}^2)$ in (61) and (62). Moreover, for any $x$ satisfying the constraint of (1), there is an optimal solution $x^*$ of (1) approximating $x$ with error

$$\|x - x^*\|_2 \leq \delta + 2\tilde{\tau} \left\{2\sigma_k(x) + c\tau + c\phi(M^T (Ax - y))\right\} \leq \delta + 4\tilde{\tau} \left\{\sigma_k(x) + c\tau\right\}.$$  

(64)

**Proof.** Let $x$ be any given vector in $\mathbb{R}^n$. Let $S$ be the support set of the $k$-largest entries of $|x|$. Let $S' = \{i \in S : x_i > 0\}$ and $S'' = \{i \in S : x_i < 0\}$. Clearly, $|S'| + |S''| \leq |S| \leq k$, and $S'$ and $S''$ are disjoint. Since $A^T$ satisfies the weak RSP of order $k$, there exists a vector $\zeta = A^Tu^*$ for some $u^* \in \mathbb{R}^m$ satisfying $\zeta_i = 1$ for $i \in S'$, $\zeta_i = -1$ for $i \in S''$, and $|\zeta_i| \leq \delta$. By the KKT optimality condition, the solution set of (58) can be completely characterized.
1 for \( i \notin S' \cup S'' \). For the fixed constant \( \delta \in (0, \tau) \), there exists an integer number \( j_0 \) such that the polytope \( \mathcal{Q}_{\epsilon j_0} \), represented as \([12]\), ensures that the set \( S_{\epsilon j_0} \), defined by \([51]\), achieves the bound \([57]\). We now construct a feasible solution \((\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)\) to the problem \([59]\). Set \((\tilde{w}_1)_i = 1\) and \((\tilde{w}_2)_i = 0\) for all \( i \in S' \), \((\tilde{w}_1)_i = 0\) and \((\tilde{w}_2)_i = 1\) for all \( i \in S'' \), and \((\tilde{w}_1)_i = (|\zeta_i| + \zeta_i)/2\) and \((\tilde{w}_2)_i = (|\zeta_i| - \zeta_i)/2\) for all \( i \notin S' \cup S'' \). This choice ensures that \((\tilde{w}_1, \tilde{w}_2, \tilde{w}_3) \geq 0\), \( \tilde{w}_1 + \tilde{w}_2 \leq \bar{e} \) and \( \tilde{w}_1 - \tilde{w}_2 = \zeta \).

We now construct the vector \( \tilde{w}_3 \). By the definition of \( \mathcal{Q}_{\epsilon j_0} \), we see that \( \{ \pm e_i : i = 1, \ldots, q \} \subseteq \mathcal{Y}(\Gamma_{\mathcal{Q}_{\epsilon j_0}}) \), the set of column vectors of \( \Gamma_{\mathcal{Q}_{\epsilon j_0}} \) with cardinality \(|\mathcal{Y}(\Gamma_{\mathcal{Q}_{\epsilon j_0}})| = \tilde{n} \). It is not difficult to show that there exists a vector \( \tilde{w}_3 \in \mathbb{R}^q \) satisfying \( M\Gamma_{\mathcal{Q}_{\epsilon j_0}} \tilde{w}_3 = -u^* \). First, since \( M \) is a full row rank matrix, there exists an \( m \times m \) invertible submatrix \( M_3 \) with \( |\mathcal{J}| = m \) consisting of \( m \) independent columns in \( M \). Then by choosing \( \tilde{h} \in \mathbb{R}^q \) such that \( \tilde{h}_i = 0 \) for all \( i \notin \mathcal{J} \) and \( \tilde{h}_i = -M_3^{-1}u^* \) which implies that \( M\tilde{h} = -u^* \). We now find \( \tilde{w}_3 \) such that \( \Gamma_{\mathcal{Q}_{\epsilon j_0}} \tilde{w}_3 = \tilde{h} \). In fact, without loss of generality, we assume that \( \{ -e_i : i = 1, \ldots, q \} \) are arranged as the first \( q \) columns of \( \Gamma_{\mathcal{Q}_{\epsilon j_0}} \) and \( \{ e_i : i = 1, \ldots, q \} \) are arranged as the second \( q \) columns of \( \Gamma_{\mathcal{Q}_{\epsilon j_0}} \). For every \( i = 1, \ldots, q \), if \( \tilde{h}_i \geq 0 \), then we set \((\tilde{w}_3)_i = \tilde{h}_i \); otherwise, if \( \tilde{h}_i < 0 \), then we set \((\tilde{w}_3)_q+i = -\tilde{h}_i \).

All remaining entries of \( \tilde{w}_3 \in \mathbb{R}^\tilde{n} \) are set to be zero. By this choice of \( \tilde{w}_3 \), we see that \( \tilde{w}_3 \geq 0 \) satisfying that \( \Gamma_{\mathcal{Q}_{\epsilon j_0}} \tilde{w}_3 = \tilde{h} \) and

\[
\|\tilde{w}_3\|_1 = \|\tilde{h}\|_1 = \|\tilde{h}_3\|_1 = \|M_3^{-1}u^*\|_1 = \|M_3^{-1}(AA^T)^{-1}A\zeta\|_1 \leq \|M_3^{-1}(AA^T)^{-1}A\|_{\infty \rightarrow 1} \|\zeta\|_{\infty} \leq c,
\]

where \( c \) is the constant given in Theorem 3.2. By the triangle inequality and the fact \( \phi^*(e_i) = 1, i = 1, \ldots, q \), we have

\[
\phi^*(\tilde{h}) = \phi^*(\sum_{j \in \mathcal{J}} \tilde{h}_j e_j) \leq \sum_{j \in \mathcal{J}} \phi^*(\tilde{h}_j e_j) = \sum_{j \in \mathcal{J}} |\tilde{h}_j| \phi^*(e_j) = \sum_{j \in \mathcal{J}} |\tilde{h}_j| = \|\tilde{h}\|_1 \leq c,
\]

where the last inequality follows from \([65]\).

For the vector \((x, t, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3)\) with \( t = |x| \), applying Lemma 2.1 with \((M^1, M^2) = (\overline{M}^1, \overline{M}^2)\), where \(\overline{M}^1\) and \(\overline{M}^2\) are given in \([61]\) and \([62]\), yields a point in \(\mathcal{D}^{(1)}\), denoted by \((\hat{x}, \hat{t}, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3)\), such that

\[
\left\| \begin{bmatrix} x \\ t \\ \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \end{bmatrix} - \begin{bmatrix} \hat{x} \\ \hat{t} \\ \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \end{bmatrix} \right\|_2 \leq \mathcal{T} \left\| \begin{bmatrix} (x-t)^+ \\ (-x-t)^+ \\ (\Gamma_{\mathcal{Q}_{\epsilon j_0}})[M^T(Ax-y) - \tau\bar{e})^+ \\ (\tilde{w}_1 + \tilde{w}_2 - \bar{e})^+ \\ AT^M\Gamma_{\mathcal{Q}_{\epsilon j_0}}\tilde{w}_3 + \tilde{w}_1 - \tilde{w}_2 \\ \bar{e}^Tt + (\tau\bar{e} + (M\Gamma_{\mathcal{Q}_{\epsilon j_0}})^Ty)^T \tilde{w}_3 \end{bmatrix} \right\}_1,
\]

where \((V)^+ = ((-t)^+, (-\tilde{w}_1)^+, (-\tilde{w}_2)^+, (-\tilde{w}_3)^+)\), and \(\mathcal{T} = \sigma_{\infty, 2}(\overline{M}^1, \overline{M}^2)\) is the Robinson’s constant determined by \((\overline{M}^1, \overline{M}^2)\) in \([61]\) and \([62]\). By the nonnegativity of \((t, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3)\), we have that \((V)^+ = 0\). Since \( t = |x| \), we have \((x-t)^+ = (-x-t)^+ = 0\). Since \((\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)\) satisfies the constraints of \([59]\), we have \((\tilde{w}_1 + \tilde{w}_2 - \bar{e})^+ = 0\) and \(A^T\Gamma_{\mathcal{Q}_{\epsilon j_0}}\tilde{w}_3 + \tilde{w}_1 - \tilde{w}_2 = 0\). Thus the inequality \([67]\) is reduced to

\[
\|x - \hat{x}\|_2 \leq \mathcal{T} \left\{ \left\| (\Gamma_{\mathcal{Q}_{\epsilon j_0}})[M^T(Ax-y) - \tau\bar{e})^+ \right\|_1 + \left\| \bar{e}^Tt + (\tau\bar{e} + (M\Gamma_{\mathcal{Q}_{\epsilon j_0}})^Ty)^T \tilde{w}_3 \right\|_1 \right\}.
\]
Recall that $\phi^*(a^i) = 1$ for every $a^i \in \mathcal{Y}(\Gamma_{Q_{j0}})$. Thus
\[
(a^i)^T(M^T(Ax - y)) \leq \phi^*(a^i)\phi(M^T(Ax - y)) = \phi(M^T(Ax - y)).
\]
This implies that $((a^i)^T(M^T(Ax - y)) - \tau)^+ \leq (\phi(M^T(Ax - y)) - \tau)^+$, and hence
\[
\left(\Gamma_{Q_{j0}}\right)^T(M^T(Ax - y)) - \tau \hat{e} + \phi(M^T(Ax - y)) - \tau)^+ \leq (\phi(M^T(Ax - y)) - \tau)^+ \hat{e}.
\]
Therefore,
\[
\left\|\left(\Gamma_{Q_{j0}}\right)^T(M^T(Ax - y)) - \tau \hat{e}\right\|_1 \leq \tilde{n}(\phi(M^T(Ax - y)) - \tau)^+.
\]
Note that $x^T A^T u^* = x^T = \|x_S\|_1 + x^T \zeta_S$ and $\|x^T \zeta_S\| \leq \|x_S\|_1 \|\zeta_S\|_\infty \leq \|x_S\|_1$. Thus
\[
|\hat{e}^T x| = \|x^T A^T u^* - x^T \zeta_S \|_{\infty} \leq \frac{1}{3}\|x_S\|_1 = \frac{1}{2}\sigma_k(x).
\]
Denote by $\vartheta = M^T(Ax - y)$ and note that $M\Gamma_{Q_{j0}} \tilde{w}_3 = -u^*$ and $\Gamma_{Q_{j0}} \tilde{w}_3 = -\tilde{h}$. We have
\[
|\hat{e}^T t + |\tau \hat{e} + (M\Gamma_{Q_{j0}})^T y)\|_{T \tilde{w}_3}| = |\hat{e}^T t| + |\tau \hat{e} + \vartheta | \hat{e} + (x^T A^T M - \vartheta) \Gamma_{Q_{j0}} \tilde{w}_3|.
\]
Combining of (68), (69) and (70) gives rise to
\[
\|x - \hat{x}\|_2 \leq \overline{\tau} \left\{ \tilde{n}(\phi(M^T(Ax - y)) - \tau)^+ + 2\sigma_k(x) + c\tau + c\phi(M^T(Ax - y)) \right\}.
\]

We now consider the three bounded convex sets $S^*, S_{e_{j0}}^*$ and $S_{\epsilon_{j0}}$. By their definitions, $S^* \subseteq S_{\epsilon_{j0}}$ and $S_{e_{j0}}^* \subseteq S_{\epsilon_{j0}}$. Let $x^* = \Pi_{S^*}(x)$ and $\overline{x} = \Pi_{S_{e_{j0}}}(x)$. Note that $\hat{x} \in S_{e_{j0}}^*$. Applying Lemma 4.1 by setting $S = S^*, U = S_{e_{j0}}^*$ and $T = S_{\epsilon_{j0}}$, we conclude that
\[
\|x - x^*\|_2 \leq d^H(S^*, S_{e_{j0}}) + 2\|x - \hat{x}\|_2 \leq \delta + 2\|x - \hat{x}\|_2.
\]
Combining this inequality with (71) yields (63), i.e.,
\[
\|x - x^*\|_2 \leq \delta + 2\overline{\tau} \left\{ \tilde{n}(\phi(M^T(Ax - y)) - \tau)^+ + 2\sigma_k(x) + c\tau + c\phi(M^T(Ax - y)) \right\}.
\]
When $x$ satisfies $\phi(M^T(Ax - y)) \leq \tau$, the inequality above is reduced to (64). □

Since every condition listed in Corollary 3.5 implies the weak RSP of order $k$, we immediately have the following result for DS with a nonlinear constraint.

**Corollary 4.6.** Let $A$ and $M$ be given as in Theorem 4.5. Let $\delta \in (0, \tau)$ be a fixed constant and let $Q_{j0}$ be the fixed polytope represented as (12) such that (74) is achieved. Suppose that one of the following conditions holds: (a) $A$ (with $\ell_2$-normalized columns) satisfies the mutual coherence property $\mu_1(k) + \mu_1(k - 1) < 1$; (b) RIP of order $2k$ with constant $\delta_{2k} < 1/\sqrt{2}$; (c) stable NSP of order $k$ with constant $\rho \in (0, 1)$; (d) robust NSP of order $k$ with $\rho' \in (0, 1)$ and $\rho'' > 0$; (e) NSP of order $k$; (f) RSP of order $k$ of $A^T$. Then the conclusions of Theorem 4.5 are valid for the DS problem (7).
5 The LASSO problem

In this section, we consider the nonlinear minimization problem (3) which is still called a LASSO problem in this paper since it includes the standard LASSO as a special case. Let \( \rho^* \) denote the optimal value of (3), i.e.,

\[
\rho^* = \min_x \{ \phi(M^T(Ax - y)) : \|x\|_1 \leq \mu \},
\]

(72)

where the problem data \((M, A, y, \mu)\) is given, and \(\phi(\cdot)\) is any norm with \(\phi(e_i) = 1\) and \(\phi^*(e_i) = 1\) for \(i = 1, \ldots, q\). In this section, we focus on the nonlinear norm \(\phi\) in the sense that the inequality \(\phi(x) \leq t\) cannot be represented as a finite number of linear inequalities or equalities, for instance, when \(\phi\) is the \(\ell_p\)-norm with \(p \in (1, \infty)\). We show that problem (3) is also stable under the weak RSP of order \(k\) of \(A^T\). As a result, the stability theorem can be established for LASSO with a broad range of matrix properties. Problem (3), i.e., (72), is equivalent to

\[
\rho^* = \min_{(x, \rho)} \{ \rho : \phi(M^T(Ax - y)) \leq \rho, \|x\|_1 \leq \mu \}.
\]

(73)

Let \(\Lambda^*\) be the set of optimal solutions of (3), which in terms of \(\rho^*\) can be written as

\[
\Lambda^* = \{ x \in \mathbb{R}^n : \|x\|_1 \leq \mu, \phi(M^T(Ax - y)) \leq \rho^* \}.
\]

Since the first constraint in (73) is nonlinear, we use the analytic method in Section 4 to develop the stability result for problem (3).

Recall that \(Q_{\epsilon_j}\), defined in (10), is a polytope approximation of \(\mathfrak{B}_\phi\) in \(\mathbb{R}^q\), and is represented as

\[
Q_{\epsilon_j} = \{ u \in \mathbb{R}^q : (a^i)^T u \leq 1 \text{ for all } a^i \in \mathcal{Y}(\Gamma_{Q_{\epsilon_j}}) \}.
\]

The vectors \(a^i\) of \(\mathcal{Y}(\Gamma_{Q_{\epsilon_j}})\) are drawn on the surface of the dual unit ball, i.e., \(a^i \in \{ a \in \mathbb{R}^q : \phi^*(a) = 1 \}\). Using this approximation, we introduce a relaxation of the solution set \(\Lambda^*\):

\[
\Lambda_{\epsilon_j} = \{ x \in \mathbb{R}^n : \|x\|_1 \leq \mu, (a^i)^T(M^T(Ax - y)) \leq \rho^* \text{ for all } a^i \in \mathcal{Y}(\Gamma_{Q_{\epsilon_j}}) \}.
\]

(74)

Clearly, \(\Lambda^* \subseteq \Lambda_{\epsilon_j}\) for any \(j \geq 1\). Then we have the following lemma.

**Lemma 5.1.** Let \(\Lambda_{\epsilon_j}\) be defined in (74). The following properties hold:

(i) For every \(j\), let \(x_{\epsilon_j}\) be an arbitrary point in \(\Lambda_{\epsilon_j}\). Then any accumulation point \(\hat{x}\) of the sequence \(\{x_{\epsilon_j}\}\) satisfies that \(\|\hat{x}\|_1 \leq \mu\) and \(\phi(M^T(A\hat{x} - y)) \leq \rho^*\), i.e., \(\hat{x} \in \Lambda^*\).

(ii) \(d_H(\Lambda^*, \Lambda_{\epsilon_j}) \to 0\) as \(j \to \infty\).

**Proof.** The proof is similar to that of Lemmas 4.2 and 4.3. Note that \(x_{\epsilon_j} \in \Lambda_{\epsilon_j}\) for any \(j \geq 1\). Thus for every \(j\), we have

\[
\|x_{\epsilon_j}\|_1 \leq \mu, (a^i)^T(M^T(Ax_{\epsilon_j} - y)) \leq \rho^* \text{ for all } a^i \in \mathcal{Y}(\Gamma_{Q_{\epsilon_j}}).
\]

(75)

Let \(\hat{x}\) be any accumulation point which clearly obeys \(\|\hat{x}\|_1 \leq \tau\). Passing through to a subsequence if necessary, we may assume that \(x_{\epsilon_j} \to \hat{x}\) as \(j \to \infty\). Assume that \(\phi(M^T(A\hat{x} - y)) > \rho^*\). We now prove that this assumption leads to a contradiction. Under this assumption, we define

\[
\sigma^* := \begin{cases} \frac{\phi(M^T(A\hat{x} - y)) - \rho^*}{\rho^*} & \rho^* \neq 0, \\ 1 & \rho^* = 0, \end{cases}
\]

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which is a positive constant. By the definition of \(\epsilon_j\), there exists an integer number \(j_0\) such that 
\[ \epsilon_j < \sigma^* \]
for any \(j \geq j_0\). By a similar argument in the proof of Lemma 4.3, for any \(j \geq j' \geq j_0\), it follows from (75) and the fact \(\mathcal{Y}(\Gamma_{Q_j'}) \subseteq \mathcal{Y}(\Gamma_{Q_j})\) that
\[ \sup_{a^i \in \mathcal{Y}(\Gamma_{Q_j})}^{} (a^i)^T (M^T(y - y)) \leq \rho^*. \]  
(76)

Let \(\tilde{a} = M^T(y - y)/\phi(M^T(y - y))\), which is on the surface of \(\mathcal{B}^d\). Applying Lemma 2.3 to \(Q_{\epsilon_j'}\) for \(j' \geq j_0\), we see that for \(\tilde{a}\), there is a vector \(a^i \in \mathcal{Y}(\Gamma_{Q_{\epsilon_j'}})\) such that \((a^i)^T \tilde{a} \geq \frac{1}{1 + \epsilon_j'} > \frac{1}{1 + \sigma^*}\), which implies that
\[ (a^i)^T [M^T(y - y)] > \frac{\phi(M^T(y - y))}{1 + \sigma^*} \geq \rho^*, \]  
where the second inequality follows from the definition of \(\sigma^*\). This contradicts (76). Therefore, we must have that \(\phi(M^T(y - y)) \leq \rho^*\). This, together with \(\|\tilde{a}\| \leq \tau\), implies that \(\tilde{a} \in \Lambda^*\).

We now prove that \(d^H(\Lambda^*, \Lambda_{\epsilon_j}) \to 0\) as \(j \to \infty\). Since \(\Lambda^* \subseteq \Lambda_{\epsilon_j}\), by the continuity of \(\Pi_{\Lambda^*}(\cdot)\) and compactness of \(\Lambda_{\epsilon_j}\), there exists for each \(\epsilon_j\) a point \(\tilde{x}_{\epsilon_j} \in \Lambda_{\epsilon_j}\) such that
\[ d^H(\Lambda^*, \Lambda_{\epsilon_j}) = \sup_{x \in \Lambda_{\epsilon_j}} \inf_{z \in \Lambda^*} \|x - z\|_2 = \sup_{x \in \Lambda_{\epsilon_j}} \|x - \Pi_{\Lambda^*}(x)\|_2 = \|\tilde{x}_{\epsilon_j} - \Pi_{\Lambda^*}(\tilde{x}_{\epsilon_j})\|_2. \]  
(77)

We also note that \(\Lambda^* \subseteq \Lambda_{\epsilon_{j+1}} \subseteq \Lambda_{\epsilon_j}\) for any \(j \geq 1\). Thus \(\{d^H(\Lambda^*, \Lambda_{\epsilon_j})\}_{j \geq 1}\) is a non-increasing nonnegative sequence. Thus \(\lim_{j \to \infty} d^H(\Lambda^*, \Lambda_{\epsilon_j})\) exists. Since the sequence \(\{\tilde{x}_{\epsilon_j}\}_{j \geq 1}\) is bounded, passing through to subsequence of \(\{\tilde{x}_{\epsilon_j}\}\) if necessary, we may assume that \(\tilde{x}_{\epsilon_j} \to \tilde{x}\) as \(j \to \infty\). By result (i), \(\tilde{x}\) must be in \(\Lambda^*\). Therefore \(\Pi_{\Lambda^*}(\tilde{x}) = \tilde{x}\). It follows from (77) that \(\lim_{j \to \infty} d^H(\Lambda^*, \Lambda_{\epsilon_j}) = \|\tilde{x} - \Pi_{\Lambda^*}(\tilde{x})\|_2 = 0\). \(\Box\)

In the reminder of this section, let \(\delta\) be any fixed sufficiently small constant. By Lemma 5.1, there exists an integer number \(j_0\) such that
\[ d^H(\Lambda^*, \Lambda_{\epsilon_{j_0}}) \leq \delta, \]  
(78)
where \(\Lambda_{\epsilon_{j_0}}\) is the set (74) determined by \(Q_{\epsilon_{j_0}}\). We use \(\hat{n} = |\mathcal{Y}(\Gamma_{Q_{\epsilon_{j_0}}})|\) to denote the number of columns of \(\Gamma_{Q_{\epsilon_{j_0}}}\) and \(\hat{\epsilon}\) the vector of ones in \(\mathbb{R}^{\hat{n}}\). Thus \(Q_{\epsilon_{j_0}}\) is represented as
\[ Q_{\epsilon_{j_0}} = \{u \in \mathbb{R}^d : (\Gamma_{Q_{\epsilon_{j_0}}})^T u \leq \hat{\epsilon}\}. \]

We consider the following relaxation of (73):
\[ \rho_{\epsilon_{j_0}}^* := \min_{(x, \rho)} \{\rho : \|x\|_1 \leq \mu, (\Gamma_{Q_{\epsilon_{j_0}}})^T (M^T(y - y)) \leq \rho \hat{\epsilon}\}, \]  
(79)
where \(\rho_{\epsilon_{j_0}}^*\) denotes the optimal value of the above optimization problem. Clearly, \(\rho_{\epsilon_{j_0}}^* \leq \rho^*\) due to the fact that (73) is a relaxation of (73). Since \(\Gamma_{Q_{\epsilon_{j_0}}}\) includes \(\pm e_i, i = 1, \ldots, n\) as its columns, the variable \(\rho\) in (79) must be nonnegative. Let
\[ \Lambda_{\epsilon_{j_0}} = \{x \in \mathbb{R}^n : \|x\|_1 \leq \mu, (\Gamma_{Q_{\epsilon_{j_0}}})^T (M^T(y - y)) \leq \rho_{\epsilon_{j_0}}^* \hat{\epsilon}\} \]
be the set of optimal solutions of (79). Recall that
\[ \Lambda_{\epsilon_{j_0}} = \{x \in \mathbb{R}^n : \|x\|_1 \leq \mu, (\Gamma_{Q_{\epsilon_{j_0}}})^T (M^T(y - y)) \leq \rho^* \hat{\epsilon}\}. \]
Clearly, \( \Lambda_{\epsilon t_0}^* \subseteq \Lambda_{\epsilon t_0} \) due to fact \( \rho_{\epsilon t_0}^* \leq \rho^* \). The problem \((79)\) can be written as
\[
\min_{(x,t,\rho)} \left\{ \rho : x \leq t, -x \leq t, \tilde{e}^T t \leq \mu, (\Gamma_{\epsilon t_0})^T[M^T(Ax - y)] \leq \rho \tilde{e}, (t, \rho) \geq 0 \right\},
\]
where \( \tilde{e} \) is still the vector of ones in \( \mathbb{R}^n \). It is straightforward to verify that the Lagrangian dual of this problem is given as
\[
\max \quad -\mu w_3 - (y^T M \Gamma_{\epsilon t_0}) w_4
\]
\[
\text{s.t.} \quad A^T M \Gamma_{\epsilon t_0} w_4 + w_1 - w_2 = 0, \quad w_1 + w_2 - w_3 \tilde{e} \leq 0, \quad \tilde{e}^T w_4 \leq 1,
\]
\[
w_1 \in \mathbb{R}_+^n, \quad w_2 \in \mathbb{R}_+^n, \quad w_3 \in \mathbb{R}_+, \quad w_4 \in \mathbb{R}_.
\]
The next lemma follows immediately from the KKT optimality condition of \((80)\) or \((81)\).

**Lemma 5.2.** \( x \in \mathbb{R}^n \) is an optimal solution of \((79)\) if and only if there exist vectors \( \tilde{t}, \tilde{w}_1, \tilde{w}_2 \in \mathbb{R}_+^n, \tilde{t} \in \mathbb{R}_+, \tilde{w}_3 \in \mathbb{R}_+ \) and \( \tilde{w}_4 \in \mathbb{R}^n_+ \) such that \( (\tilde{x}, \tilde{t}, \rho, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4) \in \mathcal{D}(2) \), where \( \mathcal{D}(2) \) is the set of vectors \( (x,t,\rho,w_1,w_2,w_3,w_4) \) satisfying the following system:
\[
\begin{align*}
 x \leq t, & \quad -x \leq t, \quad \tilde{e}^T t \leq \mu, \quad (\Gamma_{\epsilon t_0})^T[M^T(Ax - y)] \leq \rho \tilde{e}, \\
 A^T M \Gamma_{\epsilon t_0} w_4 + w_1 - w_2 = 0, & \quad w_1 + w_2 - w_3 \tilde{e} \leq 0, \\
 \tilde{e}^T w_4 \leq 1, & \quad \rho = -\mu w_3 - (y^T M \Gamma_{\epsilon t_0}) w_4,
\end{align*}
\]
(82)

By optimality, it is evident that \( t = |x| \) for any \( (x,t,\rho,w_1,w_2,w_3,w_4) \in \mathcal{D}(2) \). Clearly, \((82)\) can be written as
\[
\mathcal{D}(2) = \{ z = (x,t,\rho,w_1,w_2,w_3,w_4) : \hat{M}^1 z \leq \hat{b}^1, \hat{M}^2 z = \hat{b}^2 \},
\]
where \( \hat{b}^2 = 0, \hat{b}^1 = (0,0,\mu,y^T M \Gamma_{\epsilon t_0}^*,0,1,0,0,0,0,0)^T \) and
\[
\hat{M}^1 = \begin{bmatrix}
 I & -I & 0 & 0 & 0 & 0 & 0 \\
 -I & -I & 0 & 0 & 0 & 0 & 0 \\
 0 & \tilde{e}^T & 0 & 0 & 0 & 0 & 0 \\
 (M \Gamma_{\epsilon t_0})^T A & 0 & -\tilde{e} & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & I & I & -\tilde{e} & 0 \\
 0 & 0 & 0 & 0 & 0 & \tilde{e}^T & \tilde{e} \\
 0 & -I & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & -I & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & -\tilde{I}
\end{bmatrix},
\]
\[
\hat{M}^2 = \begin{bmatrix}
 0 & 0 & 0 & I & -I & 0 & A^T M \Gamma_{\epsilon t_0} \\
 0 & 0 & 1 & 0 & 0 & \mu & y^T M \Gamma_{\epsilon t_0}^*
\end{bmatrix},
\]
(85)

where \( I \in \mathbb{R}^{n \times n} \) and \( \tilde{I} \in \mathbb{R}^{n \times n} \) are identity matrices and 0’s are zero matrices with suitable sizes.

We now prove the main result in this section.

**Theorem 5.3.** Let \( \delta > 0 \) be any fixed sufficiently small constant, and let \( \mathcal{Q}_{\epsilon t_0} \) be the fixed polytope represented as \((12)\) such that \((78)\) is achieved. Let the data \( (M,A,y,\mu) \) in \((3)\) be given,
where $\mu > 0$, $A \in \mathbb{R}^{m \times n}$ ($m < n$) and $M \in \mathbb{R}^{m \times q}$ ($m \leq q$) with $\text{rank}(A) = \text{rank}(M) = m$. Suppose that $A^T$ satisfies the weak RSP of order $k$. Then for any $x \in \mathbb{R}^n$, there is a solution $x^*$ of (3) approximating $x$ with error

$$
\|x - x^*\|_2 \leq \delta + 2\gamma \left[ (\|x\|_1 - \mu)^+ + 2\phi(M^T(Ax - y)) + \frac{\|x\|_1 + 2\sigma_k(x)}{c} \right],
$$

(86)

where $c$ is the constant given in Theorem 3.2, and $\gamma = \sigma_{\infty, 2}(\overline{M}_1, \overline{M}_2)$ is the Robinson’s constant determined by $(\overline{M}_1, \overline{M}_2)$ in [3] and [5]. Moreover, for any $x$ with $\|x\|_1 \leq \tau$, there is an optimal solution $x^*$ of (3) approximating $x$ with error

$$
\|x - x^*\|_2 \leq \delta + 2\gamma \left[ 2\phi(M^T(Ax - y)) + \frac{\|x\|_1 + 2\sigma_k(x)}{c} \right].
$$

(87)

Proof. Let $x$ be any vector in $\mathbb{R}^n$. Then set $t = |x|$ and let $\rho = \phi(M^T(Ax - y))$ which implies from $\mathcal{Y}(\mathcal{G}_{\tau}) \subseteq \{a \in \mathbb{R}^q : \phi^*(a) = 1\}$ that

$$
(\mathcal{G}_{\tau})^T(M^T(Ax - y)) \leq \rho \mathbf{e}.
$$

(88)

Denote by $S$ the support set of the $k$-largest entries of $|x|$. Let $S' = \{i \in S : x_i > 0\}$ and $S'' = \{i \in S : x_i < 0\}$. Since $A^T$ satisfies the weak RSP of order $k$, there exists a vector $\zeta = A^T u^*$ for some $u^* \in \mathbb{R}^m$ satisfying $\zeta_i = 1$ for $i \in S'$, $\zeta_i = -1$ for $i \in S''$, and $|\zeta_i| \leq 1$ for $i \notin S' \cup S''$. Let $c$ be the constant given in Theorem 3.2. We now construct a set of vectors $(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4)$ which satisfies the constraints [81]. First, we set $\tilde{w}_3 = 1/c$. Then we set

$$(\tilde{w}_1)_i = 1/c \text{ and } (\tilde{w}_2)_i = 0 \text{ for all } i \in S',$$

$$(\tilde{w}_1)_i = 0 \text{ and } (\tilde{w}_2)_i = 1/c \text{ for all } i \in S'',$n

$$(\tilde{w}_1)_i = \frac{|\zeta_i| + \zeta_i}{2c} \text{ and } (\tilde{w}_2)_i = \frac{|\zeta_i| - \zeta_i}{2c} \text{ for all } i \notin S' \cup S''.$$n

This choice of $\tilde{w}_1$ and $\tilde{w}_2$ implies that $(\tilde{w}_1, \tilde{w}_2) \geq 0$, $\tilde{w}_1 + \tilde{w}_2 \leq \tilde{w}_3 \mathbf{e}$, and $\tilde{w}_1 - \tilde{w}_2 = \zeta/c$. We now construct the vector $\tilde{w}_4$ as follows. By the definition of $Q_{\tau}$, we see that $\{\pm e_i : i = 1, \ldots, q\} \subseteq \mathcal{Y}(\mathcal{G}_{\tau})$. Since $M$ has a full row rank matrix, there exists an $m \times m$ invertible square submatrix $M_3$, where $\mathfrak{J} \subseteq \{1, \ldots, q\}$ with $|\mathfrak{J}| = m$. We define the vector $\tilde{g} \in \mathbb{R}^q$ as follows: $(\tilde{g})_\mathfrak{J} = M_3^{-1} u^*$ and $(\tilde{g})_i = 0$ for $i \notin \mathfrak{J}$. Clearly, we have $M\tilde{g} = u^*$. It is not difficult to show that there exists a vector $\tilde{w}_4 \in \mathbb{R}^q_+$ satisfying $\mathcal{G}_{\tau} \tilde{w}_4 = -\tilde{g}/c$ and $\|\tilde{w}_4\|_1 \leq 1$. In fact, without loss of generality, we assume that $\{-e_i : i = 1, \ldots, q\}$ are arranged as the first $q$ columns and $\{e_i : i = 1, \ldots, q\}$ as the second $q$ columns in $\mathcal{G}_{\tau}$. For every $i = 1, \ldots, q$, we set

$$
(\tilde{w}_4)_i = \begin{cases} 
(\tilde{g})_i/c & \text{if } (\tilde{g})_i \geq 0, \\
0 & \text{otherwise}, 
\end{cases}
$$

and

$$
(\tilde{w}_4)_{q+i} = \begin{cases} 
-(\tilde{g})_i/c & \text{if } (\tilde{g})_i < 0, \\
0 & \text{otherwise}. 
\end{cases}
$$

All remaining entries of $\tilde{w}_4 \in \mathbb{R}^q$ are set to be zero. By this choice, we see that $\tilde{w}_4 \geq 0$, $\mathcal{G}_{\tau} \tilde{w}_4 = -\tilde{g}/c$ and

$$
\mathbf{e}^T\tilde{w}_4 = \|\tilde{w}_4\|_1 = \|\tilde{g}\|_1/c = \|\tilde{g}\|_1/c = \|M_3^{-1} u^*\|_1/c = \|M_3^{-1}(AA^T)^{-1}A\zeta\|_1/c \leq \|\zeta\|_\infty/c \leq \|\zeta\|_\infty = 1.
$$

(89)
By an argument similar to (65), we see that
\[ \phi^*(\tilde{g}) \leq \|\tilde{g}\|_1 \leq c. \] (90)

We also note that
\[ A^T M \Gamma_{Q_{x_0}} \tilde{w}_1 = A^T M(-\tilde{g}/c) = A^T (-u^*/c) = -\zeta/c = -(\tilde{w}_1 - \tilde{w}_2). \]

Thus the vector \((\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4)\) constructed above satisfies the constraint of (81). Let \(D(2)\) be given as in Lemma 5.2. For the vector \((x, t, \rho, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4)\), by applying Lemma 2.1 with \((M^1, M^2) := (\tilde{M}^1, \tilde{M}^2)\) where \(\tilde{M}^1\) and \(\tilde{M}^2\) are given in (81) and (85), we conclude that there is a point in \(D(2)\), denoted by \((\hat{x}, \hat{t}, \hat{\rho}, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4)\), such that
\[
\begin{bmatrix} x \\ t \\ \rho \\ \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{w}_4 \end{bmatrix} - \begin{bmatrix} \hat{x} \\ \hat{t} \\ \hat{\rho} \\ \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{w}_4 \end{bmatrix} \leq \gamma \begin{bmatrix} (x-t)^+ \\ \nu^T_t \end{bmatrix},
\]
where
\[
(Z)^+ = ((-t)^+, (-\rho)^+, (-\tilde{w}_1)^+, (-\tilde{w}_2)^+, (-\tilde{w}_3)^+, (-\tilde{w}_4)^+),
\]
and \(\gamma = \sigma_{\infty,2}(\tilde{M}^1, \tilde{M}^2)\) is the Robinson’s constant determined by \((\tilde{M}^1, \tilde{M}^2)\) given in (81) and (85). The nonnegativity of \((t, \rho, \tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4)\) implies that \(Z)^+ = 0. The fact \(t = |x|\) implies that \((x-t)^+ = (-x-t)^+ = 0\) and \(\nu^T_t = \|x\|_1\). Since \((\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4)\) is feasible to (81), we have
\[
(\nu^T \tilde{w}_4 - 1)^+ = 0, (\tilde{w}_1 + \tilde{w}_2 - \tilde{w}_3 \nu)^+ = 0, A^T M \Gamma_{Q_{x_0}} \tilde{w}_4 + \tilde{w}_1 - \tilde{w}_2 = 0.
\]
Note that \(\rho = \phi(M^T(Ax - y))\) implies (88), and hence \(\left( (\Gamma_{Q_{x_0}})^T (M^T(Ax - y)) - \rho \nu \right)^+ = 0\). Thus it follows from (81) that
\[
\|x - \hat{x}\|_2 \leq \gamma \left\{ \left( \|x\|_1 + \mu \right)^+ + \rho + \mu \tilde{w}_3 + y^T M \Gamma_{Q_{x_0}} \tilde{w}_4 \right\}.
\]
(92)

From the definition of \(S\) and \(C\), we have \(x^T \zeta = \|x\|_1 + x^T S^T \zeta\) and \(\|x\|_1 = \sigma_k(x_1)\). Thus
\[
\|x\|_1 - x^T \zeta = \|x\|_1 - x^T S^T \zeta \leq \|x\|_1 + \|x\|_1 \|x\|_1 \|C\|_\infty \leq 2\sigma_k(x_1).
\]
From (85), we see that \(\|\tilde{g}\|_1 \leq c\). Note that \(\Gamma_{Q_{x_0}} \tilde{w}_4 = -\tilde{g}/c\) and \(A^T M \Gamma_{Q_{x_0}} \tilde{w}_4 = -(\tilde{w}_1 - \tilde{w}_2) = -\zeta/c\). By letting \(\psi = M^T(Ax - y)\), we have
\[
\begin{align*}
\rho + \mu \tilde{w}_3 + y^T M \Gamma_{Q_{x_0}} \tilde{w}_4 &= \rho + \mu \tilde{w}_3 + (x^T A^T M - \psi^T) \Gamma_{Q_{x_0}} \tilde{w}_4 \\
&= \rho + \mu \tilde{w}_3 + x^T A^T M \Gamma_{Q_{x_0}} \tilde{w}_4 - \psi^T \Gamma_{Q_{x_0}} \tilde{w}_4 \\
&\leq \rho + \frac{1}{c} \left( \mu - \|x\|_1 \right) + \|x\|_1 - x^T \zeta + \psi^T \tilde{g} \\
&\leq \rho + \frac{1}{c} \left( \mu - \|x\|_1 + 2\sigma_k(x_1) \right) + \phi(\psi) \frac{\|\tilde{g}\|_1}{c} \\
&\leq 2\phi(\psi) + \frac{\mu - \|x\|_1 + 2\sigma_k(x_1)}{c},
\end{align*}
\] (93)

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where the last two inequalities follow from the fact \( \rho = \phi(\psi), \ |\psi^T \bar{g}| \leq \phi(\psi) \phi^*(\bar{g}) \) and \((90)\). Combining \((92)\) and \((93)\) leads to

\[
\|x - \hat{x}\|_2 \leq \gamma \left[ (\|x\|_1 - \mu)^+ + 2\phi(\psi) + \frac{|\mu - \|x\|_1| + 2\sigma_k(x)_1}{c} \right].
\]

(94)

Let \( x^* \) and \( \overline{\pi} \) be the projections of \( x \) onto the compact convex sets \( \Lambda^* \) and \( \Lambda_{\epsilon_{\delta_0}} \), respectively, i.e.,

\[
x^* = \Pi_{\Lambda^*}(x) \in \Lambda^*, \quad \overline{\pi} = \Pi_{\Lambda_{\epsilon_{\delta_0}}}(x) \in \Lambda_{\epsilon_{\delta_0}}.
\]

By \((78)\), we have \( d^H(\Lambda^*, \Lambda_{\epsilon_{\delta_0}}) \leq \delta \). Note that \( \hat{x} \in \Lambda^*_{\epsilon_{\delta_0}} \subseteq \Lambda_{\epsilon_{\delta_0}} \). It follows from Lemma 4.1 that \( \|x - x^*\|_2 \leq \delta + 2\|x - \hat{x}\|_2 \). From this inequality and \((94)\), we conclude that

\[
\|x - x^*\|_2 \leq \delta + 2\gamma \left[ (\|x\|_1 - \mu)^+ + 2\phi(\psi) + \frac{|\mu - \|x\|_1| + 2\sigma_k(x)_1}{c} \right].
\]

Particularly, if \( x \) obeys \( \|x\|_1 \leq \mu \), we obtain

\[
\|x - x^*\|_2 \leq \delta + 2\gamma \left[ 2\phi(\psi) + \frac{|\mu - \|x\|_1| + 2\sigma_k(x)_1}{c} \right],
\]

as desired. \( \square \)

When the parameter \( \mu \) is large, the optimal solution \( x^* \) of \((3)\) might be taken in the interior of the feasible set, i.e., \( \|x^*\|_1 < \mu \). When \( \mu \) is small, the optimal solution of \((3)\) usually attains at the boundary of its feasible set, i.e., \( \|x^*\|_1 = \mu \). Thus, in stability analysis of LASSO, we are particularly interested in the gap between \( x^* \) and those vectors satisfying \( \|x\|_1 < \mu \) or \( \|x\|_1 = \mu \). The following result follows immediately from Theorem 5.3.

**Corollary 5.4.** Let \( \delta > 0 \) be any fixed small constant, and let \( Q_{\epsilon_{\delta_0}} \) be the fixed polytope represented as \((12)\) such that \((78)\) is achieved. Let the data \( (M, A, y, \mu) \) in \((3)\) be given, where \( \mu > 0, A \in \mathbb{R}^{m \times n} (m < n) \) and \( M \in \mathbb{R}^{n \times q} (m \leq q) \) with \( \text{rank}(A) = \text{rank}(M) = m \). Suppose that one of the conditions listed in Corollary 3.3 is satisfied. Then the following statements hold:

(i) For any \( x \in \mathbb{R}^n \) with \( \|x\|_1 < \mu \), there is an optimal solution \( x^* \) of \((3)\) approximating \( x \) with error \((77)\).

(ii) For any \( x \in \mathbb{R}^n \) with \( \|x\|_1 = \mu \), there is an optimal solution \( x^* \) of \((3)\) approximating \( x \) with error

\[
\|x - x^*\|_2 \leq \delta + 4\gamma \left[ \phi(M^T(Ax - y)) + \frac{\sigma_k(x)_1}{c} \right],
\]

(95)

where the constants \( c \) and \( \gamma \) are given as in Theorem 5.3.

The stability results for the special cases \( M = I \) or \( M = A \) can be obtained immediately from the above result. The statement of such a result is omitted here.

### 6 Conclusions

We have shown that the general Dantzig selector and LASSO problems are stable in sparse data recovery under the so-called weak range space property of a transposed design matrix. These optimization problems are general enough to include many important special cases. Our stability analysis for these problems is carried out differently from those in the literature in terms of the analytic method, mild assumption and the way of expression of stability coefficients. The classic
Hoffman’s Lemma and a polytope approximation technique of convex bodies are employed as a new deterministic analytic method for the development of a stability theory for Dantzig selector and LASSO problems. The stability coefficients are measured by Robinson’s constants depending on the problem data. The assumption made in this paper is a constant-free matrix condition, which is naturally originated from the fundamental optimality condition of convex optimization. It turns out to be a mild sufficient condition for Dantzig selector and LASSO problems to be stable in sparse data recovery. We have shown that this assumption is also necessary for the standard Dantzig selector to be stable. Many known matrix conditions in compressed sensing such as RIP, NSP and others imply our assumption.

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