Topological spin torque emerging in classical-spin systems with different time scales

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In classical spin systems with two largely different inherent time scales, the configuration of the fast spins almost instantaneously follows the slow-spin dynamics. We develop the emergent effective theory for the slow-spin degrees of freedom and demonstrate that this generally includes a topological spin torque. This torque gives rise to anomalous real-time dynamics. It derives from the holonomic constraints defining the fast-spin configuration space and is given in terms of a topological charge density which becomes a quantized homotopy invariant when integrated.

INTRODUCTION

Introduction. Topological charges are homotopy invariants [1], which can take discrete values only and which are used in theoretical physics to discriminate between topologically different states of physical systems. They have supplemented the more established concept of classifying states of matter based on symmetry, spontaneous symmetry breaking and order parameters. Well known topological invariants for gapped quantum systems are Chern numbers or \( \mathbb{Z}_2 \) invariants classifying, e.g., topological insulators [2, 3]. For classical systems, e.g., for anisotropic classical-spin systems, skyrmion numbers [4–6] are used to characterize topologically different magnetic states.

Quantum-mechanically as well as classically, these topological charges derive from \textit{locally} defined and gauge invariant quantities, i.e., from topological charge \textit{densities} describing local properties of vector bundles. A prime example is given by the Berry curvature, which is a phase 2-form in parameter space [7]. In particular, these topological charge densities describe the effect on the system’s state when steering the system along a closed path in parameter space. A quantum state picks up a geometrical phase, the Berry phase [8, 9], which is obtained by integrating the topological charge density, the Berry curvature, over a surface enclosed by the path. An analog for classical systems is Hannay’s angle variable holonomy [10], which arises for integrable systems when the Hamiltonian is adiabatically taken around a closed path in parameter space.

While the effects of slowly varying parameters on the local and the global topological properties of the system’s state have been studied extensively, for gapped quantum condensed-matter systems [2, 3, 11, 12], for discrete quantum systems with degenerate eigenstates [13], for quantum-spin [14–17] and classical-spin models [4–6, 18], in the context of classical phase transitions [19, 20] and in molecular dynamics [9, 21], etc., the feedback of the local topological charge densities \textit{on the state of the parameters} has not so much been in the focus. This feedback is meaningful, if the “parameters” are actually treated as classical dynamical degrees of freedom with a real-time dynamics that is slow compared to the fast degrees of freedom of the “system”.

Anomalous slow-spin dynamics. With the present study we would like to adopt this change of the perspective. We consider the real-time dynamics of a purely classical system which is governed by two largely different intrinsic time scales. The role of the “parameters” is played by “slow” spins. Their states define a base manifold. Assuming that the fast-spin subsystem follows the slow spins \textit{adiabatically} when the slow-spin state evolves in time, we can define a topological charge density which is reminiscent of a skyrmion density [4–6] but for skyrmions living on the base manifold, which is given by a Cartesian product of Bloch spheres rather than by Euclidean space. The corresponding topological charge is quantized.

More importantly, however, as we can demonstrate very generally, there is an effective theory involving the slow-spin degrees of freedom only, and here the topological charge density gives rise to an unconventional topological spin torque. This torque can lead to sizeable anomalous effects as is explicitly demonstrated for a simple toy model with a single slow classical impurity spin coupled to a classical Heisenberg model. We also check the effective theory against the numerical solution of the full set of dynamical canonical equations. This pinpoints the model parameter range where the dynamics is adiabatic, i.e., Born-Oppenheimer-like [21].

There are a few earlier studies of the dynamical role of the Berry curvature all addressing, however, quantum-classical hybrid systems [22, 23], particularly semiclassical electron dynamics in crystals [24] and adiabatic long-wavelength magnon dynamics [25–27]. Let us emphasize that the present work discusses a conceptually much simpler class of systems, namely systems with entirely classical spin degrees of freedom as given, e.g., by standard classical Heisenberg-type models. Such models are ubiquitously employed, e.g., in the field of classical atomistic spin dynamics [28–31] modelling a great variety of magnetic phenomena, where separation of time scales is frequently caused by weakly coupled spins or by strong
FIG. 1: L “fast” spins \( s_i \) (\( i = 1, \ldots, L \)) on a lattice, mutually interacting via a Heisenberg coupling \( J \) and interacting via a local coupling \( K \) with \( R \) “slow” spins \( S_j \) (\( j = 1, \ldots, R \)). \( s_i \) and \( S_j \) are coupled to the corresponding unit vectors. The fast spins are coupled to \( R \) slow spins \( S_j = S_j \mathbf{m}_j \) via exchange couplings \( K_{ij} \). The slow unit vector at site \( j \) is \( \mathbf{m}_j = S_j/S_j \). The equations of motion,

\[
\mathbf{s}_i = \frac{\partial H}{\partial \mathbf{s}_i} \times \mathbf{s}_i, \quad S_j = \frac{\partial H}{\partial S_j} \times S_j ,
\]

are obtained from the system’s Hamiltonian

\[
H = \frac{1}{2} \sum_{i,i'} J_{ii'} \mathbf{s}_i \mathbf{s}_{i'} + \sum_{ij} K_{ij} \mathbf{s}_i S_j - \sum_j S_j B .
\]

(Fig. 1) sketches a possible realization with a one-dimensional lattice of fast spins and nearest-neighbor antiferromagnetic (AF) Heisenberg coupling \( J \) and with local AF coupling \( K \) to the slow spins. Here, the characteristic time scale of the fast-spin subsystem is given by \( J^{-1} (\hbar = 1) \), and the slow spins are subjected to an external magnetic field driving the slow-spin subsystem on a time scale \( B^{-1} \). Note that the equations of motion preserve the lengths of \( \mathbf{s}_i \) and of \( S_j \) which allows us to absorb constants, like gyromagnetic ratios, in \( s_i \) and \( S_j \).

The considered setup could mimic the magnetic properties of, e.g., magnetic atoms with magnetic moments \( S_j \) on a magnetic solid surface [32], or magnetic molecules [33], etc. The Hamiltonian could be extended by additional couplings between the slow spins or by anisotropic terms, and various alternative geometries are conceivable. With Eq. (2) we focus on a concrete Hamiltonian just to be specific, while our arguments are general.

The time evolution of an initial spin configuration is governed by the coupled nonlinear system of ordinary differential equations of motion (1). It is typically exponentially sensitive to perturbations and quickly gets chaotic [34]. Here, our goal is to study a parameter regime, where the system exhibits two very different “fast” and “slow” time scales and where the fast spins (almost) instantaneously follow the motion of the slow ones. In this adiabatic limit, one can expect a strong conceptual simplification, providing us with an effective theory for the slow degrees of freedom only. As we will argue below, however, the slow-spin dynamics is additionally affected by local topological properties of the fast-spin system, which give rise to unconventional effects.

Adiabatic limit. The adiabatic limit is defined by a parameter range of the Hamiltonian where, at any instant of time \( t \), the configuration of the fast spins \( s(t) \equiv (s_1(t), \ldots, s_L(t)) \) is the ground-state configuration \( s(t) = s_0(S(t)) \), for the present configuration \( S(t) \equiv (S_1(t), \ldots, S_R(t)) \) of the slow spins given at the respective time \( t \). Using “fast” and “slow” unit-vector configurations \( \mathbf{n}(t) \equiv (\ldots, \mathbf{n}_i(t), \ldots) \) and \( \mathbf{m}(t) \equiv (\ldots, \mathbf{m}_j(t), \ldots) \), respectively, we have

\[
\mathbf{n}(t) = \mathbf{n}_0(\mathbf{m}(t)) .
\]

The (approximate) realization of the adiabatic limit and the question, in which parameter regime adiabatic spin dynamics is observed, will strongly depend on the specific system considered. Realizations for a simple toy model will be discussed below. When approaching the adiabatic limit in parameter space, the fast-spin dynamics will be more and more constrained to the time-dependent hyper-surfaces (3) in \( \mathbf{n} \)-space, i.e., in the product of Bloch spheres, \( \mathbf{n} \in \prod_{i=1}^{L} S^2 \). In this limit we can employ Eq. (3) for a strongly simplified description of adiabatic spin dynamics (ASD).

It is tempting to derive the slow-spin dynamics solely from the effective Hamiltonian \( H(\mathbf{s}, \mathbf{S}) \rightarrow H(s_0(\mathbf{m}), S(\mathbf{m})) \equiv H_{\text{eff}}(\mathbf{m}) \) that is obtained using the constraints (3). This naïve ASD thus amounts to solving the \( R \) remaining equations of motion \( S_j \mathbf{m}_j = \partial H_{\text{eff}}/\partial \mathbf{m}_j \times \mathbf{m}_j \) for \( \mathbf{S} \), while \( \mathbf{n}(t) \) can be obtained from Eq. (3). We will demonstrate below, however, that this may lead to incorrect results.

Adiabatic spin dynamics. A correct strategy towards ASD can be based on the action principle

\[
\delta \int dt \mathcal{L}(\mathbf{n}, \dot{\mathbf{n}}, \mathbf{m}, \dot{\mathbf{m}}) = 0
\]

with the Lagrangian [35]

\[
\mathcal{L} = \sum_i A(n_i) \dot{n}_i + \sum_j A(m_j) S_j \dot{m}_j - H(\mathbf{n}, \mathbf{m}) .
\]

Here, the function \( A(\mathbf{r}) \) is the vector potential of a unit magnetic (Dirac) monopole, i.e., \( \nabla \times A(\mathbf{r}) = -r/r^3 \), located at \( \mathbf{r} = 0 \). In the standard gauge [36]:

\[
A(\mathbf{r}) = -\frac{1}{r^2} \frac{\mathbf{r}_z}{1 + \mathbf{e}_z \cdot \mathbf{r}/r}
\]
The straightforward calculation [see the Supplementary Material (SM) [37], section A] shows that the Lagrangian equations, $(d/dt)(\partial L/\partial \dot{m}_i) = \partial L/\partial m_i$ (and analogously for $m_j$), are equivalent with the Hamiltonian Eqs. (1, 2). This justifies the Lagrangian (5). We note in passing that for classical spin dynamics, the Lagrangian and Hamiltonian formulation of the real-time dynamics are not related via a Legendre transformation since this is singular (see the SM [37], Sec. B).

To describe spin dynamics in the adiabatic limit, i.e., spin dynamics constrained to the manifold specified by Eq. (3), one may employ the action principle, Eq. (4). To this end, the holonomic constraints (3) are used to reduce the number of “generalized coordinates” and to define an effective Lagrangian for the slow-spin degrees of freedom only and is thus not related via a Legendre transformation since this is singular (see the SM [37], Sec. B). The straightforward calculation [see the Supplementary Material (SM) [37], section A] shows that the Lagrangian equations, $(d/dt)(\partial L/\partial \dot{m}_i) = \partial L/\partial m_i$ (and analogously for $m_j$), are equivalent with the Hamiltonian Eqs. (1, 2). This justifies the Lagrangian (5). We note in passing that for classical spin dynamics, the Lagrangian and Hamiltonian formulation of the real-time dynamics are not related via a Legendre transformation since this is singular (see the SM [37], Sec. B).

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which is higher than the Larmor frequency $\omega_L = B$ for $S > s$, e.g., $\omega = 2B$ for $S = 1$ and $s = 1/2$, while for $S < s$, the orientation of the precession is inverted.

We conclude that, already for the $R = 1$ case, there are nontrivial effects of the anomalous spin torque. Furthermore, the specialization to $R = 1$ is instructive as this allows for comparison with a system, where the fast spins are replaced by conduction electrons, see Ref. [42]:
For the quantum-classical system, the same renormalization of the precessional motion has been found, but the role of the topological charge density $e^{(i)}_{\mu}(m)$ is played by the (spin) Berry curvature. This underpins the large independency of the novel spin torque from microscopic details of the fast subsystem.

Realization of the adiabatic limit. An important question concerns the conditions for which the constraints (3) are (approximately) satisfied. Previous work on spin dynamics in the $s$-$d$ impurity model [43–46] has shown that the real-time evolution of the classical spin is almost adiabatic in large regions of parameter space. It is thus tempting to expect almost adiabatic motion in a certain parameter range and hence an anomalous precession frequency, as predicted by Eq. (11), for the present case of a purely classical spin system as well. Note that for $s = S$ in Eq. (11), $\omega$ diverges, indicating that adiabatic motion is not possible in this case.

We have checked the predictions of ASD by numerically solving the full equations of motion (1) using a high-order Runge-Kutta technique for a slow spin of length $S = 1$. As expected, for a system with even $L$, the observed precession frequency is very close to the standard Larmor frequency, i.e., $\omega \approx \omega_L = B$, including those parameter regimes where the fast spins almost instantaneously follow the slow-spin dynamics, but with one exception discussed below. For systems with odd $L$, on the other hand, we in fact find an anomalous frequency $\omega \approx 2\omega_L$ resulting from the topological spin torque, see Eq. (9), for model parameters where the dynamics is adiabatic. As is seen in Fig. 2, this is the case whenever the field strength $B$ is small compared to $K$ and $J$.

In the simple two-spin system for $L = 1$ or, equivalently, for $J = 0$, we have $\omega \approx 2\omega_L$ if the two spins are strongly coupled, i.e., for $K/B \gg 1$. An additional weak coupling to $L > 1$ fast spins, i.e., $J/K \ll 1$, then only slightly perturbs the tightly bound two-spin subsystem and the corresponding anomalous dynamics. This explains the increase of $\omega$ with decreasing $J$ and the narrow red-colored anomalous range visible in Fig. 2 for extremely small $J/B$. Since this effect results from the proximity to the $L = 1$ case, it also shows up for even $L$ (not shown).

With increasing system size, the characteristic time scale $1/B$ of the slow spin must increase as well to keep the dynamics in the adiabatic limit. This is obvious as the information on the state of the impurity spin must propagate through the whole system to allow the fast spins to align. Assuming that the propagation time grows at least linearly with system size $L$, we must have $1/B \sim L/J$ for large $K/B$. We find that the numerical results can be fitted nicely assuming $1/B \propto 1/K + (L - 1)/J$ (see the line in the main panel of Fig. 2). The right panel demonstrates the linear growth of the necessary time scale with $L$.

This implies that, for the model considered here, nonadiabatic dynamics must be expected in the thermodynamic limit $L \to \infty$. At the same time the observed odd-even effect becomes irrelevant and we must generalize the theory to mixed states. Hence, there is a big de facto but also a big conceptual difference compared to quantum-classical systems [42–44], where the adiabatic limit can be controlled by the gap in the electronic structure and the adiabatic theorem [47].

Conclusions. Classical spin systems are frequently employed in various contexts, such as atomistic spin dynamics of condensed-matter, nanostructured and molecular systems. Here, we have considered prototypical classical Heisenberg-like models and have demonstrated that time-scale separation generally leads to the emergence of a topological spin torque. This may profoundly affect the real-time spin dynamics as could be seen al-
ready for a simplistic toy model. The full implications of the adiabatic spin-dynamics theory, however, are yet to be worked out. Slow dynamics of weakly coupled spins, e.g., via an indirect (RKKY-like) exchange mechanism, spin dynamics which is slow due to strong magnetic anisotropies, or slow collective dynamics of spin degrees of freedom represent a few promising examples for future applications of ASD theory.

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Topological spin torque emerging in classical-spin systems with different time scales

— Supplemental Material —

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Section A: Lagrangian equations of motion. Consider a Lagrangian of the form

\begin{equation}
L = \sum_i A(n_i) s_i \dot{n}_i - H(n) - \sum_i \lambda_i (n_i^2 - 1), \tag{12}
\end{equation}

as given by Eqs. (5) and (6) but, for simplicity, for a single type of spins only. We have \( s = (s_1, ..., s_L) \), \( s_i = s_i n_i \), \( H(n) \equiv H(s) \) and we have explicitly added Lagrange-multiplier terms \( \propto \lambda_i \) to keep track of the constraints \( n_i^2 = 1 \). The spin directions \( n_i(t) \) and the Lagrange parameters \( \lambda_i(t) \) are obtained from the condition that the action corresponding to \( L \) be stationary. The Lagrangian equations of motion, derived from the action principle (4), read

\begin{equation}
0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{n}_{i\alpha}} - \frac{\partial L}{\partial n_{i\alpha}} = s_i \sum_{\beta} \left( \frac{\partial A_\alpha(n_i)}{\partial n_{i\beta}} - \frac{\partial A_\beta(n_i)}{\partial n_{i\alpha}} \right) \dot{n}_{i\beta} + \frac{\partial H(n)}{\partial n_{i\alpha}} + 2\lambda_i n_{i\alpha}, \tag{13}
\end{equation}

with \( \alpha = x, y, z \). Using the vector notation, we get

\begin{equation}
-s_i \dot{n}_i \times (\nabla \times A(n_i)) + \frac{\partial H(n)}{\partial n_i} + 2\lambda_i n_i = 0. \tag{14}
\end{equation}

Inserting the curl of the vector potential, \( \nabla \times A(n_i) = -n_i/n_i^3 \), yields:

\begin{equation}
s_i \dot{n}_i \times n_i/n_i^3 + \frac{\partial H(n)}{\partial n_i} + 2\lambda_i n_i = 0. \tag{15}
\end{equation}

Taking the cross product from the right, \( \times n_i \), as well as the dot product \( \cdot n_i \), provides us with the following two equations:

\begin{equation}
s_i (\dot{n}_i \times n_i) \times n_i/n_i^3 + \frac{\partial H(n)}{\partial n_i} \times n_i = 0, \quad \frac{\partial H(n)}{\partial n_i} \dot{n}_i + 2\lambda_i n_i^2 = 0 \tag{16}
\end{equation}

which are equivalent with Eq. (15).

The constraint \( n_i^2 = 1 \) and the second equation fix the Lagrange parameter as: \( \lambda_i = -(\partial H(n)/\partial n_i) n_i/2 \). Using the constraint to simplify the first equation, results in:

\begin{equation}s_i \dot{n}_i = \frac{\partial H(n)}{\partial n_i} \times n_i \tag{17}\end{equation}

or, equivalently,

\begin{equation}\dot{s}_i = \frac{\partial H(s)}{\partial s_i} \times s_i, \tag{18}\end{equation}

i.e., the Hamilton equations of motion for \( s \), see Eq. (1).
**Section B: Hamiltonian vs. Lagrangian spin dynamics.** One way to see that the Hamiltonian and the Lagrangian formulation of spin dynamics are not related via a Legendre transformation is the following: Starting from a generic Hamilton function for classical spins $s_i$, e.g.,

$$ H(s) = \frac{1}{2} \sum_{ij} J_{ij} s_i s_j - \sum_i B_i s_i , $$

and introducing coordinates $q_i$ and momenta $p_i$, such that $s_i = q_i \times p_i$, one gets the Hamiltonian

$$ H(q,p) = H(s) \bigg|_{s=q\times p} $$

and the resulting canonical equations of motion for $(q_i, p_i)$:

$$ \dot{q}_i = \left. \frac{\partial H(q,p)}{\partial p_i} \right|_{s=q\times p} = \frac{\partial H(s)}{\partial s_i} \Big|_{s=q\times p} \times q_i , \quad \dot{p}_i = -\left. \frac{\partial H(q,p)}{\partial q_i} \right|_{s=q\times p} = \frac{\partial H(s)}{\partial s_i} \Big|_{s=q\times p} \times p_i . \tag{21} $$

One easily verifies: $\dot{s}_i = q_i \times p_i + \dot{q}_i \times p_i = (\partial H(s)/\partial s_i) \times s_i$. Now, it is tempting to define a Lagrangian via

$$ L(q,\dot{q}) = \sum_i p_i (q,q)\dot{q}_i - H(q,p(q,\dot{q})) , \tag{22} $$

where the function $p(q,\dot{q})$ would be obtained by solving Eq. (21) for $p_i$. However, for a Hamiltonian as given by Eq. (19), the equations,

$$ \dot{q}_i = \left. \frac{\partial H(s)}{\partial s_i} \right|_{s=q\times p} \times q_i = \sum_j J_{ij} (q_j \times p_j) \times q_i - B_i \times q_i , \tag{23} $$

form an inhomogeneous linear system of equations $A \cdot p = b$ for the unknowns $p_i$, which is necessarily singular as the coefficient matrix $A$ with elements $(\alpha, \beta \in \{x, y, z\})$,

$$ A_{\alpha\beta, j} = J_{ij}(\delta_{\alpha\beta} \sum_\gamma q_\gamma q_{j\beta} - q_{j\alpha} q_{i\beta}) , \tag{24} $$

is singular since $A \cdot q = 0$, irrespective of the interaction parameters $J_{ij}$.

**Section C: ASD equation of motion.** The slow-spin dynamics is derived from the effective Lagrangian

$$ L_{\text{eff}}(m,\dot{m}) \equiv L(n_0(m), (d/dt)n_0(m), m, \dot{m}) . \tag{25} $$

With

$$ \frac{d}{dt} n_{0,i}(m) = \sum_j (\dot{m}_j \nabla_j) n_{0,i}(m) \tag{26} $$

we find:

$$ L_{\text{eff}}(m,\dot{m}) = \sum_j A(m_j) S_j \dot{m}_j + \sum_i A(n_{0,i}(m)) s_i \sum_j (\dot{m}_j \nabla_j) n_{0,i}(m) - H_{\text{eff}}(m) , \tag{27} $$

where $i = 1, ..., L$ and $j = 1, ..., R$, and where $H_{\text{eff}}(m) = H(sn_0(m), Sm)$. To get the Lagrange equations of motion, we first compute

$$ \frac{\partial}{\partial m_k} L_{\text{eff}}(m,\dot{m}) = \sum_\beta S_\beta \nabla_k A_{\beta}(m_k) \dot{m}_k \beta + \sum_i s_i A_{\beta}(n_{0,i}(m)) \sum_j (\dot{m}_j \nabla_j) \nabla_k n_{0,i}(m) $$

$$ + \sum_i s_i \frac{\partial A_{\beta}(n_{0,i}(m))}{\partial n_{0,i}(m)} \nabla_k n_{0,i}(m) \sum_j (\dot{m}_j \nabla_j) n_{0,i}(m) - \nabla_k H_{\text{eff}}(m) . \tag{28} $$

Here, $\nabla_j = \partial/\partial m_j$, and Greek indices $\alpha, \beta, ... \in \{x, y, z\}$. Next,

$$ \frac{\partial}{\partial \dot{m}_k} L_{\text{eff}}(m,\dot{m}) = S_k A(m_k) + \sum_i s_i A_\alpha(n_{0,i}(m)) \nabla_k n_{0,i}(m) , \tag{29} $$

...
where

\[
\frac{d}{dt} \frac{\partial L_{\text{eff}}(m, \dot{m})}{\partial \dot{m}_k} = \sum_{\alpha} S_k(\nabla_k A_\alpha(m_k) \dot{m}_k)e_\alpha + \sum_{ij\alpha\beta} s_{ij} \frac{\partial A_\alpha(n_{0,i}(m))}{\partial n_{0,i\beta}} (\dot{m}_j n_{0,i\beta}(m)) \nabla_k n_{0,\alpha}(m)
\]

\[
+ \sum_{ij\alpha} s_{ij} A_{\alpha}(n_{0,i}(m)) \sum_j \nabla_k (\dot{m}_j \nabla_j n_{0,\alpha}(m)).
\]

(30)

The last term equals the second term on the right-hand side of Eq. (28) in the Lagrange equations, since \(\nabla_k\) and \(\dot{m}_j \nabla_j\) commute, such that we are left with:

\[
0 = \frac{d}{dt} \frac{\partial L_{\text{eff}}}{\partial \dot{m}_k} = \sum_{\alpha} S_k \dot{m}_k \nabla_k A_\alpha(m_k) e_\alpha - \sum_{\beta} S_k \dot{n}_{k\beta} \nabla_k A_\beta(m_k) + \nabla_k H_{\text{eff}}(m)
\]

\[
+ \sum_{ij\alpha\beta} s_{ij} \frac{\partial A_\alpha(n_{0,i}(m))}{\partial n_{0,i\beta}} (\dot{m}_j n_{0,i\beta}(m)) \nabla_k n_{0,\alpha}(m) - \sum_{ij\alpha\beta} s_{ij} \frac{\partial A_\beta(n_{0,i}(m))}{\partial n_{0,i\alpha}} \nabla_k n_{0,\alpha}(m) (\dot{m}_j \nabla_j n_{0,i\beta}(m))
\]

\[
= S_k(\nabla_k \times A(m_k)) \times \dot{m}_k + \nabla_k H_{\text{eff}}(m) + T_k,
\]

(31)

where \(T_k\) stands for the last two terms. Taking the cross product from the right, \((...) \times m_k\), we find

\[
S_k((\nabla_k \times A(m_k)) \times \dot{m}_k) \times m_k + \nabla_k H_{\text{eff}}(m) \times m_k + T_k \times m_k = 0.
\]

(32)

Using \(\nabla_k \times A(m_k) = -m_k/m_k^3\), expanding the remaining double cross product and exploiting that \(m_k\) is a unit vector, yields:

\[
S_k \dot{m}_k = \nabla_k H_{\text{eff}}(m) \times m_k + T_k \times m_k,
\]

(33)

which, apart from the extra term involving \(T_k\), just recovers the “naive” adiabatic spin dynamics.

Note that actually we should have added Lagrange-multiplier terms, \(L_{\text{eff}}(m, \dot{m}) \rightarrow L_{\text{eff}}(m, m) - \sum_k \lambda_k (m_k^2 - 1)\), to account for the normalization conditions \(m_k^2 = 1\). However, this would have resulted in an additional summand \(2\lambda_k m_k\) on the r.h.s. of Eq. (31) only, which does not contribute after taking the cross product \((...) \times m_k\). On the other hand, taking the dot product, \((...) \cdot m_k\), of Eq. (31), just yields the necessary conditional equation for \(\lambda_k\), if this was required.

\(T_k\), if nonzero, gives rise to an additional spin torque \(T_k \times m_k\). From Eq. (32) we can read off:

\[
T_k = \sum_{ij\alpha\beta} s_{ij} \left( \frac{\partial A_\alpha(n_{0,i}(m))}{\partial n_{0,i\beta}} - \frac{\partial A_\beta(n_{0,i}(m))}{\partial n_{0,i\alpha}} \right) (\dot{m}_j \nabla_j n_{0,i\beta}(m)) \nabla_k n_{0,\alpha}(m).
\]

(34)

Exploiting once more the defining property of the vector potential, \(\nabla_k \times A(m_k) = -m_k/m_k^3\), and using the normalization \(m_j = 1\) in the end, we find:

\[
T_k = \sum_{ij\alpha\beta} s_{ij} \epsilon_{\alpha\beta\gamma} \nabla_k n_{0,\alpha\beta}(m) (\dot{m}_j \nabla_j n_{0,i\beta}(m)) n_{0,i\gamma}(m)
\]

\[
= \sum_i s_i \sum_l \sum_{\mu\nu} \nabla_{k\mu} n_{0,i}(m) \times \nabla_{l\nu} n_{0,i}(m) \cdot n_{0,i}(m) \dot{m}_{l\nu} e_\mu
\]

(35)

The scalar triple product defines an antisymmetric tensor of rank two:

\[
\Omega_{k\mu,l\nu} = \sum_i s_i \frac{\partial n_{0,i}(m)}{\partial m_k} \times \frac{\partial n_{0,i}(m)}{\partial m_l} \cdot n_{0,i}(m) = -\Omega_{l\nu,k\mu}.
\]

(36)

Hence:

\[
T_k = \sum_l \sum_{\mu\nu} \Omega_{k\mu,l\nu} \dot{m}_{l\nu} e_\mu.
\]

(37)

Note the following sum rule:

\[
\sum_k T_k \dot{m}_k = 0.
\]

(38)
Section D: Total-energy conservation. As the constraints (3) are time-independent and holonomic, total-energy conservation within the effective adiabatic theory is actually ensured by the general Lagrange formalism but can also be verified explicitly by computing the time derivative of the total energy:

$$\frac{dE}{dt} = \frac{dH_{\text{eff}}(m)}{dt} = \frac{d}{dt} \sum_j A(m_j)S_j \dot{m}_j + \frac{d}{dt} \sum_i A(n_0(m))s_i - \frac{d}{dt} L_{\text{eff}}(m, \dot{m}).$$

(39)

Using the Lagranian equations of motion, $\frac{d}{dt}(\partial L_{\text{eff}}/\partial \dot{m}_k) = \partial L_{\text{eff}}/\partial m_k$, and Eq. (29), we have

$$\frac{d}{dt} L_{\text{eff}}(m, \dot{m}) = \sum_j \frac{\partial L_{\text{eff}}}{\partial \dot{m}_j} \dot{m}_j + \sum_j \frac{\partial L_{\text{eff}}}{\partial \dot{m}_j} \ddot{m}_j = \sum_j \left( \frac{d}{dt} \frac{\partial L_{\text{eff}}}{\partial \dot{m}_j} \right) \dot{m}_j + \sum_j \frac{\partial L_{\text{eff}}}{\partial \dot{m}_j} \ddot{m}_j = \frac{d}{dt} \left( \sum_j \frac{\partial L_{\text{eff}}}{\partial \dot{m}_j} \dot{m}_j \right)$$

(40)

Inserting this in Eq. (39) yields $dE/dt = 0$.

Alternatively, total-energy conservation can be verified by using the adiabatic equation of motion (7) and the sum rule (38):

$$\frac{d}{dt} H_{\text{eff}}(m) = \sum_j \frac{\partial H_{\text{eff}}}{\partial m_j} \dot{m}_j = \sum_j \frac{\partial H_{\text{eff}}}{\partial m_j} \frac{1}{S_j} \left( \frac{\partial H_{\text{eff}}}{\partial m_j} \times m_j + T_j \times m_j \right) = \sum_j \frac{1}{S_j} \frac{\partial H_{\text{eff}}}{\partial m_j} \cdot T_j \times m_j$$

$$= - \sum_j \frac{1}{S_j} \frac{\partial H_{\text{eff}}}{\partial m_j} \times m_j \cdot T_j = - \sum_j \left( \dot{m}_j - \frac{1}{S_j} T_j \times m_j \right) \cdot T_j = 0.$$

(41)