Single Machine Weighted Number of Tardy Jobs Minimization With Small Weights

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Abstract

In this paper we prove new results concerning pseudo-polynomial time algorithms for the classical scheduling problem of minimizing the weighted number of jobs on a single machine, the so-called 1 || \sum w_j U_j problem. The previously best known pseudo-polynomial algorithm for this problem, due to Lawler and Moore [Management Science’69], dates back to the late 60s and has running time O(d_{max}n) or O(wn), where d_{max} and w are the maximum due date and sum of weights of the job set respectively. Using the recently introduced “prediction technique” by Bateni et al. [STOC’19], we present an algorithm for the problem running in \tilde{O}(d_{#}(n + dw_{max})) time, where d_{#} is the number of different due dates in the instance, d is the total sum of the d_{#} different due dates, and w_{max} is the maximum weight of any job. This algorithm outperform the algorithm of Lawler and Moore for certain ranges of the above parameters, and provides the first such improvement for over 50 years. We complement this result by showing that 1 || \sum w_j U_j has no \tilde{O}(n + w_{max}^{1-\epsilon}) time algorithms assuming \forall\exists-SETH conjecture, a recently introduced variant of the well known Strong Exponential Time Hypothesis (SETH).

1 Introduction

One of the most fundamental problems in the area of scheduling is the problem of non-preemptively scheduling a set of jobs on a single machine so as to minimize the weighted number of tardy jobs. In this problem, we are given a set of n jobs J = {1,...,n}, where each job j has a processing time p_j \in \mathbb{N}, a weight w_j \in \mathbb{N}, and a due date d_j \in \mathbb{N}. A schedule \sigma for J is a permutation \sigma : \{1,...,n\} \rightarrow \{1,...,n\} specifying the processing order of the jobs. In a given schedule \sigma, the completion time C_j of a job j under \sigma is C_j = \sum_{i : \sigma(i) \leq \sigma(j)} p_i; that is, the total processing time of jobs preceding j in \sigma (including j itself). Job j is tardy in \sigma if C_j > d_j, and early otherwise. Our goal is find a schedule where the total weight of tardy jobs is minimized.

For a given schedule, if we assign a binary indicator variable U_j to each job j, where U_j = 1 if job j is tardy and otherwise U_j = 0, then our goal is to minimize \sum_{j=1}^{n} w_j U_j. This is the reason this problem is denoted by 1 || \sum w_j U_j in the standard three field notation for scheduling problems of Graham [10] – the 1 in the first field indicates a single machine model, the empty second field indicates there are no additional constraints, and the third field indicates the objective function.

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The $1 \mid \sum w_j U_j$ problem is meant to model situations where being tardy incurs a fixed penalty, regardless of how tardy you really are. As a simple example, think of a pizza delivering service offering a free pizza when late on delivery. Such situations are in fact quite common in modern production management where tardiness significantly disrupts various supply chains. Thus, $1 \mid \sum w_j U_j$ models several natural real life scheduling applications. But it also plays a prominent role in the theory of scheduling algorithms, being the first scheduling problem to be shown to be NP-hard [13], as well as one of the first examples of a scheduling problem that admits a fully polynomial time approximation scheme (FPTAS) [20]. For this reason, practically any standard textbook on scheduling (e.g. [6, 18]) devotes some sections to $1 \mid \sum w_j U_j$ and its variants. We refer the reader also to a relatively recent survey by Adamu and Adewumi [2] for further background.

Another reason $1 \mid \sum w_j U_j$ is such a prominent problem in the area of scheduling is that it is a natural generalization of the classical Knapsack problem. Indeed, the special case where all jobs have a common due date (i.e. $d_1 = \cdots = d_n = d$), denoted as the $1 \mid d_j = d \mid \sum w_j U_j$ problem, translates directly to the dual version of Knapsack: If all due dates equal to $d$, our goal is to minimize the total weight of jobs that complete after $d$, where in Knapsack we wish to maximize the total weight of jobs that complete before $d$. (Here $d$ corresponds to the Knapsack size, the processing times correspond to item sizes, and the weights correspond to item values.) This equivalence was already observed in Karp’s classical paper [13], albeit under a slightly different version of Knapsack.

**Theorem 1 ([13]).** Any algorithm for Knapsack running in $T(n)$ time can be used to solve the $1 \mid d_j = d \mid \sum w_j U_j$ problem in $O(T(n))$ time, and vice-versa. In particular, $1 \mid d_j = d \mid \sum w_j U_j$ is (weakly) NP-hard.

Although $1 \mid \sum w_j U_j$ is NP-hard, it is only weakly NP-hard, meaning that it admits pseudopolynomial time algorithms. The famous Lawler and Moore algorithm [14] for $1 \mid \sum w_j U_j$ dates way back to the late 60s, and is one the earliest examples of a dynamic programming algorithm. Its running time is $O(pn)$, where $p$ denotes the total processing time of all jobs in a given instance of $1 \mid \sum w_j U_j$, i.e. $p = p_1 + \cdots + p_n$. A careful look into this algorithm actually shows that its running time can be bounded by $O(d_{\text{max}} n)$, where $d_{\text{max}} = \max_j d_j$ is the maximum due date of all jobs in the instance. Note that we can always assume that $d_{\text{max}} < p$, since otherwise all jobs can be scheduled prior to $d_{\text{max}}$ (and so we can safely remove any job $j$ with $d_j = d_{\text{max}}$). Furthermore, reversing the roles of processing times and weights in the Lawler and Moore algorithm results in an algorithm with $O(w \cdot n)$ running time, where $w = w_1 + \cdots + w_n$.

**Theorem 2 ([14]).** $1 \mid \sum w_j U_j$ can be solved in $O(d_{\text{max}} \cdot n)$ or $O(w \cdot n)$ time.

Despite the simplicity of the Lawler and Moore algorithm, and despite the fact that its generic nature also allows solving a multitude of other problems [14], it is still the fastest algorithm known for $1 \mid \sum w_j U_j$ in the realm of pseudo-polynomial time algorithms. Thus, one of the major open questions in this context is

“Can we obtain faster pseudo-polynomial time algorithms for $1 \mid \sum w_j U_j$, or alternatively, prove that these do not exist under some plausible conjecture?”

This question, which as far as we can tell has never been really addressed, is the main motivation behind this paper.
Consider first the $O(d_{\text{max}} n)$ running time of Theorem 2 Cygan et al. showed that the single due date case $1 | d_j = d | \sum w_j U_j$ can be solved in $\tilde{O}(n + d_{\text{max}}^2)$ time using an elegant layering technique introduced by Bringmann [4]. This improves upon Theorem 2 for the common due date case when $d = d_{\text{max}} \ll n$. Furthermore, they also proved that a significant improvement to this algorithm is unlikely, as an $O(n + d_{\text{max}}^2 \varepsilon)$ time algorithm for $1 | d_j = d | \sum w_j U_j$ will implies that the $(\text{max}, +)$-convolution (see Section 2) between two vectors can be computed in sub-quadratic time. Since the later is believed not to be possible, this gives some evidence that the $O(d_{\text{max}} n)$ running time of Theorem 2 cannot be significantly improved, even in the case where all jobs have common due dates, at least when we insist that the dependency on $n$ remain linear.

So let us consider the $O(w n)$ running time in Theorem 2. Note that when the maximum weight $w_{\text{max}} = \max_j w_j$ of any job is constant, this running time is $O(n^2)$. Can this be improved? Indeed, Moore [16] showed that $1 || \sum U_j$, the special case of $1 || \sum w_j U_j$ with unit weights, is solvable in $O(n \log n)$ time. So perhaps we can generalize this result and obtain an algorithm that can handle non-uniform weights, but still improves upon the Lawler and Moore algorithm when $w_{\text{max}}$ is sufficiently small.

There are encouraging results in this direction. Pisisinger [19] presented an $O(n w_{\text{max}} p_{\text{max}})$ time algorithm for $1 | d_j = d | \sum w_j U_j$ (or rather Knapsack) which improves upon Theorem 2 when $w_{\text{max}} p_{\text{max}} = o(\min\{w, d_{\text{max}}\})$. Furthermore, in a recent breakthrough paper, Bateni et al. [3] showed that this running time can be improved, by introducing the so-called “prediction technique” for speeding up certain $(\text{max}, +)$-convolutions. Using the prediction technique, Bateni et al. proved the following:

**Theorem 3** (3). The $1 | d_j = d | \sum w_j U_j$ problem can be solved in $\tilde{O}(n + d_{\text{max}} w_{\text{max}})$ time.

They also showed that the problem is solvable in $\tilde{O}((n + d_{\text{max}}) \cdot p_{\text{max}})$ time by an alternative method.

### 1.1 Our results

We extend the algorithm of Bateni et al. [3] (Theorem 3) to the case of multiple deadlines. Let $d_{\#}$ denote the number of different due dates in a $1 || \sum w_j U_j$ instance, i.e., $d_{\#} = |\{d_j : j \in J\}|$. Clearly, $1 \leq d_{\#} \leq n$, where the case of $d_{\#} = 1$ is $1 | d_j = d | \sum w_j U_j$. We show that the algorithm of Bateni et al. [3] can be extended to handle $d_{\#}$ different due dates, at a cost of an increase of a factor of $d_{\#}^2$ to its running time. More specifically, let $d^{(1)} < d^{(2)} < \ldots < d^{(d_{\#})} = d_{\text{max}}$ denote the different due dates of a given $1 || \sum w_j U_j$ instance, and let $d = \sum_i d^{(i)}$. We prove the following:

**Theorem 4.** $1 || \sum w_j U_j$ can be solved in $\tilde{O}(d_{\#} (n + dw_{\text{max}}))$ time.

Note that as $d \leq d_{\#} \cdot d_{\text{max}}$, the running time in the theorem above can be rewritten as $\tilde{O}(d_{\#} \cdot (n + dw_{\text{max}})) = \tilde{O}(d_{\#}^2 \cdot (n + d_{\text{max}} w_{\text{max}})) = \tilde{O}(d_{\#}^2 \cdot T)$, where $T$ is the running time of the Bateni et al. algorithm. Furthermore, observe that our algorithm improves upon the Lawler and Moore algorithm (Theorem 2) whenever $d_{\#} = o(w)$ and $d_{\#} d_{\text{max}} = o(\min\{wn, d_{\text{max}} n\})$. As an example, consider the case where $w_j = O(1)$ for all jobs $j \in J$, $d_{\#} = \log^{O(1)} n$, and $d_{\text{max}} = O(n)$. Then the Lawler and Moore approach gives us an algorithm with running time $O(n^2)$ for this case, while Theorem 4 yields an algorithm with a nearly linear $\tilde{O}(n)$ running time.

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1 We use $\tilde{O}$ to suppress polylogarithmic factors.
To complement Theorem 4 we also present a lower bound result concerning the running time of any pseudo-polynomial time algorithm for $1\|\sum w_j U_j$. Our result is based on a recently introduced quantified version of the well known Strong Exponential Time Hypothesis (SETH) [12], called $\forall\exists$-SETH (see Section 4 for a formal definition).

**Theorem 5.** Assuming $\forall\exists$-SETH, there are is no algorithm for the $1\|\sum w_j U_j$ problem running in $\tilde{O}(n + w_{\text{max}}^{1-\varepsilon} \cdot n)$ time, nor $\tilde{O}(n + w_{\text{max}} \cdot n^{1-\varepsilon})$ time, for any $\varepsilon > 0$.

Note that while this lower bound is quite far away from the upper bound given in Theorem 4, it is still the first known lower bound for $1\|\sum w_j U_j$ that relates to the weights of the instance.

### 1.2 Further related work

As mentioned above, $1\|\sum w_j U_j$ is one of the earliest studied problems in the field of combinatorial optimization in general, and in scheduling theory in particular [14]. Karp placed the problem in the pantheon of combinatorial optimization problems by listing it in his landmark 1972 paper [13]. The problem is known to be polynomial-time solvable in a few special cases: Moore [16] provided an $O(n \log n)$ time algorithm for solving the unit weight $1\|\sum U_j$ problem, and Peha [17] presented an $O(n \log n)$ time algorithm for $1 \mid p_j = p \mid \sum w_j U_j$ (the variant where all jobs have equal processing time), see also [7]. Sahni showed that $1\|\sum w_j U_j$ admits an FPTAS, in yet another landmark paper in the area of scheduling [20].

Exact algorithms for the problem based on a branch-and-bound procedure can be found in [15, 21, 22]. The problem is known to be polynomial-time solvable when either the number of different processing times $p_\#$, or the number of different weights $w_\#$, is bounded by a constant, and fixed parameter tractable when parameterized by either $p_\# + w_\#$, $p_\# + d_\#$, or $w_\# + d_\#$ [11]. Finally, the special case where $w_j = p_j$ for all jobs $j$, the $1\|\sum p_j U_j$ problem, was recently shown to be solvable in $\tilde{O}(p^{7/4})$, $\tilde{O}(pd_\#)$, and $\tilde{O}(p + d)$ time [5].

### 2 Algorithm Overview

In this section we present an overview of our algorithm used for proving Theorem 4. In particular, we discuss the prediction technique introduced by Bateni et al. [3] which is also a core element of our algorithm.

A key observation for the $1\|\sum w_j U_j$ problem, used already in the Lawler and Moore [14] algorithm, is that any instance of the problem always has an optimal schedule which is an Earliest Due Date schedule. An Earliest Due Date (EDD) schedule is a schedule $\pi: J \rightarrow \{1, \ldots, n\}$ such that all early jobs are scheduled before all tardy jobs, and all early jobs are scheduled in non-decreasing order of due dates.

**Lemma 1** ([14]). Any instance of $1\|\sum w_j U_j$ has an optimal schedule which is EDD.

The $d_\#$-many due dates in our instance partition the input set of jobs $J$ in a natural manner: Define $J_i = \{j : d_j = d(i)\}$ for each $i \in \{1, \ldots, d_\#\}$. The first step of our algorithm is to treat each of the $J_i$'s as a separate $1\mid d_j = d \mid \sum w_j U_j$ instance, and use the Bateni et al. [3] algorithm running in $\tilde{O}(n + dw_{\text{max}})$ time stated in Theorem 4 to solve each of these instances. Consider the instance corresponding to some $J_i$. The Bateni et al. [3] algorithm has a useful property that it produces an integer vector $A = (A[k])_{k=0}^{d(i)}$ where the $k$th entry $A[k]$ is equal to the maximum weight
of any subset of early jobs in \( J_i \) with total processing time at most \( k \). We call such an integer vector a solution vector for the \( 1 \| d_j = d \| \sum w_j U_j \) instance \( J_i \). We can extend this notion naturally to a solution vector for a general \( 1 \| \sum w_j U_j \) instance.

**Definition 1.** A solution vector for a \( 1 \| \sum w_j U_j \) instance \( J \) is an integer vector \( A = (A[k])_{k=0}^{d_{\text{max}}} \) where the \( k \)th entry \( A[k] \) is equal to the maximum weight of any subset of early jobs in \( J \) with total processing time at most \( k \).

The next step of the algorithm is to combine the solution vectors corresponding to each \( 1 \| d_j = d \| \sum w_j U_j \) instance \( J_i \) into a single solution vector corresponding to our original \( 1 \| \sum w_j U_j \) instance \( J \). Due to Lemma 1 above, we can restrict our attention to EDD schedules. Constructing such a schedule corresponds to choosing a subset \( E_i \subseteq J_i \) of jobs for each due date \( d^{(i)} \) such that \( \sum_{j \in E_i, t \leq p_j} w_j \leq d^{(i)} \) holds for each \( i \in \{1, \ldots, d_\# \} \). Moreover, the optimal EDD schedule maximizes the total weight of all jobs selected in the \( E_i \)’s. Thus, to combine the solution vectors corresponding to each \( J_i \), we use \((\text{max, +})\)-convolutions (Definition 2) between the solution vectors. Such convolutions correspond precisely to combining early jobs of \( J_i \) with early jobs of \( J_{i+1} \), for some \( i \in \{1, \ldots, d_\# - 1\} \). Note that these type of convolutions are also central to the algorithm of Bateni et al. [3].

**Definition 2.** Let \( A = (A[k])_{k=0}^{m} \) and \( B = (B[\ell])_{\ell=0}^{n} \) be two integer vectors with \( m \leq n \). The \((\text{max, +})\)-convolution of \( A \) and \( B \), denoted \( A \oplus B \), is defined as the vector \( C = (C[\ell])_{\ell=0}^{n} \) where

\[
C[\ell] = \max_{0 \leq k \leq \ell} A[k] + B[\ell - k] \quad \text{for each} \quad \ell \in \{0, \ldots, n\}.
\]

Algorithm 1 provides a full description of our algorithm. It successively combines the solution vectors \( A_1, \ldots, A_{d_\#} \) corresponding to \( J_1, \ldots, J_{d_\#} \) into a single solution vector \( A \) for \( J \) using \((\text{max, +})\)-convolutions. The last entry \( A[d_{\text{max}}] \) of \( A \) will contain the maximum weight of early jobs in the instance, so the algorithm returns \( w - A[d_{\text{max}}] \).

**Algorithm 1**

**Input:** A \( 1 \| \sum w_j U_j \) instance.

**Output:** The minimum weighted number of tardy jobs.

1: Compute \( J_1, \ldots, J_{d_\#} \).
2: Compute solution vectors \( A_1, \ldots, A_{d_\#} \) corresponding to \( J_1, \ldots, J_{d_\#} \) using [3].
3: \( A = A_1 \).
4: for \( i = 2, \ldots, d_\# \) do \( A = A \oplus A_i \).
5: return \( w - A[d_{\text{max}}] \).

**Lemma 2.** Algorithm 1 correctly returns the minimum total weight of tardy jobs.

**Proof.** We prove the lemma by induction on the \( d_\# \) iterations preformed in line 4 of of the algorithm (where the first iteration is line 3). For an iteration \( i \in \{1, \ldots, d_\# \} \), we claim that any entry \( A[k], k \in \{0, \ldots, d^{(i)} \} \), at iteration \( i \) contains the maximum total weight of early jobs with total processing time \( k \) in an EDD schedule for \( J_1 \cup \cdots \cup J_i \). According to Lemma 1 such a schedule is optimal for \( J_1 \cup \cdots \cup J_i \) (under the constraint that the total processing time is at most \( k \)), and so proving this claim shows that \( w - A[d_{\text{max}}] \) is indeed the minimum total weight of tardy jobs at the end of the algorithm.
For $i = 1$ the claim is true due to the correctness of the algorithm by Bateni et al. Assume therefore that $i > 1$, and choose some arbitrary $k \in \{1, \ldots, d^{(i)}\}$. Let $E_1, \ldots, E_i$ denote the set of early jobs in an optimal EDD schedule for $J_1 \cup \cdots \cup J_i$ with total processing time at most $k$, and let $a$ denote the total weight of these jobs. Furthermore, let $k'$ and $a'$ denote the total processing time and weighted number of the jobs in $E_1, \ldots, E_{i-1}$ respectively. Then by induction we have $A'[k'] \geq a'$, where $A'$ is the solution vector for iteration $i - 1$, as $E_1, \ldots, E_{i-1}$ is a set of early jobs in $J_1 \cup \cdots \cup J_{i-1}$ with total processing time $k'$. Similarly, by correctness of the algorithm in [3], we know that $A_i[k - k'] \geq a - a'$. Hence, $A[k] \geq A'[k'] + A_i[k - k'] = a' + (a - a') = a$.

Conversely, suppose that $A[k] = A'[k'] + A_i[k - k']$ for some $k' \in \{0, \ldots, k\}$. Then there is a set of early jobs in $J_1 \cup \cdots \cup J_{i-1}$ with $A'[k']$ weighted number of tardy jobs and total processing time $k' \leq d^{(i-1)}$, and a set of early jobs in $J_i$ with $A_i[k - k']$ weighted number of tardy jobs and total processing time $k - k' \leq d^{(i)} - k'$. Combining these two jobs together yields a set of early jobs in $J_1 \cup \cdots \cup J_i$ with $A[k]$ weighted number of early jobs. Hence, by optimality of $E_1, \ldots, E_i$, we have $A[k] \leq a$. All together this shows that $A[k] = a$, and so the lemma holds. 

**Fast (max, +)-convolutions via the prediction technique:** Observe that computing the (max, +)-convolution of two integer vectors of length $m \leq n$ can trivially be done in $O(n^2)$. Unfortunately, this is not fast enough to improve upon the Lawler and Moore algorithm. Moreover, for general integer vectors, an $O(n^2)$ algorithm is believed to be essentially the best we can hope for [3]. However, Bateni et al. [3] showed that for certain types of integer vectors, we can nevertheless improve on the $O(n^2)$ bound via their prediction technique. Roughly speaking, the prediction technique allows for fast (max, +)-convolution of two vectors $A$ and $B$ when we can quickly compute certain intervals in $B$. Formally, we have the following theorem:

**Theorem 6 ([3])**. Let $A = (A[k])_{k=0}^n$ and $B = (B[\ell])_{\ell=0}^n$ be two integer vectors with $m \leq n$, and let $e$ be some positive integer. Assume we have $m$ intervals $[x_k, y_k] \subseteq [0, n]$, $1 \leq k \leq m$, such that

1. $A[k] + B[\ell] \geq (A \oplus B)[k + \ell] - e$ for all $k \in \{1, \ldots, m\}$ and $\ell \in [x_k, y_k]$.

2. For all $\ell \in \{1, \ldots, n\}$ there exists an $k \in \{1, \ldots, m\}$ such that $A[k] + B[\ell - k] = (A \oplus B)[\ell]$ and $\ell - k \in [x_k, y_k]$.

3. $x_k \leq x_{k+1}$ and $y_k \leq y_{k+1}$ for all $0 \leq k < m$.

Then $A \oplus B$ can be computed in $\tilde{O}(ne)$ time.

We refer to a set of $|A|$ intervals that satisfy all requirements of Theorem [3] with parameter $e$ as range intervals of $A$ in $B$ with error $e$.

As Betani et al. [3] show that one can obtain such intervals in $\tilde{O}(n)$ time when both vectors in the (max, +)-convolution correspond to Knapsack (or $1 \mid d_j = d \mid \sum w_j U_j$) instances with bounded weights. Our main technical contribution is to extend this result to the more general $1 \mid \sum w_j U_j$ problem. In Section [3] we prove the following lemma which gives a running time bound for one iteration of Algorithm [4].

**Lemma 3.** Let $i \in \{2, \ldots, d_\#\}$. One can compute the (max, +)-convolution in iteration $i$ of Algorithm [4] above in $\tilde{O}(id^i w_{\max} + n)$ time.

Using Lemma [3] we can easily bound the running time of Algorithm [4]. Combining this with the correctness of the algorithm proven in Lemma [2] completes the proof for Theorem [4].
Proof of Theorem 4. Lemma 2 proves that Algorithm 1 correctly computes the minimum weighted number of tardy jobs in any given \(1 \parallel \sum w_j U_j\) instance. To bound its running time, note that lines 1 and 2 of the algorithm can be performed in \(\tilde{O}(d\#n + d\#d_{\text{max}} w_{\text{max}})\) time due to Theorem 3. Moreover, each iteration \(i \in \{2, \ldots, d\#\}\) in line 4 requires \(\tilde{O}(id(i) w_{\text{max}} + n)\) time according to Lemma 3. This gives us a total of
\[
\sum_{i \in \{2, \ldots, d\#\}} \tilde{O}(id(i) w_{\text{max}} + n) = \tilde{O}(d\# \cdot (n + dw_{\text{max}})).
\]
for all iterations all together (recall that \(d = \sum_i d(i)\)). As this running time dominates the time required to perform lines 1 and 2, this gives us the overall running time of our algorithm. \(\square\)

3 Range Intervals for Solution Vectors

In this section, we formally prove Lemma 3. In each iteration of Algorithm 1 we need to compute a \((\text{max}, +)\)-convolution and our goal is to do this quickly using Theorem 6. Our algorithm for computing such intervals is similar to the one used by Bateni et al. [2] for computing range intervals corresponding to Knapsack solution vectors. However, there are some key differences because our algorithm needs to deal with multiple due dates.

Let \(i^* \in \{2, \ldots, d\#\}\) denote some iteration in Algorithm 1. Let \(A\) be a solution vector for the \(1 \parallel \sum w_j U_j\) instance formed by all jobs in \(J_A = J_1 \cup \ldots \cup J_{i^* - 1} = \{j \in J : d_j \leq d(i^* - 1)\}\), and let \(B\) denote the solution vector for the \(1 | d_j = d | \sum w_j U_j\) instance \(J_B = J_{i^*} = \{j \in J : d_j = d(i^*)\}\). Our main result of this section is the following.

Lemma 4. There is an algorithm running in \(\tilde{O}(i^* d(i^*) + n)\) time that computes range intervals of \(A\) and \(B\) with error \(e \in O(i^* \cdot w_{\text{max}})\).

Proof of Lemma 4. According to Lemma 4 we can compute range intervals of \(A\) in \(B\) with error \(e \in O(i^* w_{\text{max}})\) in \(O(i^* d(i^*) + n)\) time. Using Theorem 3 we can compute \(A \oplus A_i\) in \(O(i^* d(i^*) w_{\text{max}})\) time. Altogether this gives an algorithm running in \(O(i^* d(i^*) w_{\text{max}} + n)\) time for computing the \((\text{max}, +)\)-convolution in iteration \(i^*\) of Algorithm 1. \(\square\)

Roughly speaking, the range intervals for \(A\) in \(B\) tell us which entries \(A[k]\) and \(B[\ell]\) sum up to a value close to \(C[k + \ell]\). To be able to determine this without computing \(C\), we need a good approximation for \(C\). Similar to Bateni et al. [2], we make use of the solution to the fractional version of our problem to obtain an approximation for the (non-fractional) solution.

Definition 3. In the fractional \(1 \parallel \sum w_j U_j\) problem we are given a set of jobs \(J = \{1, \ldots, n\}\), with processing times \((p_j)_{j=1}^n\), weights \((w_j)_{j=1}^n\), and due dates \((d_j)_{j=1}^n\), and our goal is to compute \(n\) real values \(0 \leq x_1, \ldots, x_n \leq 1\) such that

- \(\sum_{d_k \leq d_j} p_k x_k \leq d_j\) for all \(j \in J\), and
- \(\sum_{j \in J} w_j x_j\) is maximized.
Similar to the 0/1 version of 1 \( \sum w_j U_j \), we define a fractional solution vector for a fractional 1 \( \sum w_j U_j \) instance \( J \) as a vector \( A' \) of size \( d_{\text{max}} = \max\{d_j : j \in J\} \), where \( A'[k] \) equals the maximum value \( \sum w_j x_j \) over all feasible solutions that satisfy the additional constraint \( \sum p_j x_j \leq k \), for each \( k \in \{0, \ldots, d_{\text{max}}\} \).

Algorithm 2 below computes a solution vector for a given fractional 1 \( \sum w_j U_j \) instance. It is a modified version of the standard greedy algorithm for computing fractional solutions for Knapsack \( \[3\] \) used by Batani et al. \( \[3\] \), that is able to deal with the presence of more than one due date. The algorithm assumes the jobs are sorted according to the Weighted Shortest Processing Time (WSPT) rule, i.e. in non-increasing values of \( w_j/p_j \). This is a standard technique in many scheduling algorithms, and can be performed by any \( O(n \log n) \) sorting algorithm. Moreover, in the description of the algorithm, we assume that \( w_j/p_j = 0 \) whenever \( j > n \).

In the algorithm we keep track of the total processing times \( p^{(i)} \) of (fractional) jobs that are scheduled early from \( J_1 \cup \cdots \cup J_{i} \). The algorithm then iterates over the WSPT sorted list of jobs, and repeatedly adds a slice of unit processing from the current job to the next entry in the solution vector. Before adding the slice, the algorithm checks that no due dates will be violated by adding the slice using the current values of the \( p^{(i)} \)'s. If adding the current slice violates some due date, the algorithm skips to the next job in the list.

Algorithm 2

Input: WSPT sorted set of jobs \( J = \{1, \ldots, n\} \) with \( i^* \) different deadlines.

Output: A fractional solution vector \( A' \) for \( J \).

1: Let \( \{d^{(1)}, \ldots, d^{(i^*)}\} = \{d_j : j \in J\} \) be the different due dates in \( J \).
2: Set \( p^{(i)} = 0 \) for all \( 1 \leq i \leq i^* \).
3: Set \( A'[0] = 0, \ j = 1, \ \text{and} \ p_j^* = 0 \).
4: for \( k = 1, \ldots, d^{(i^*)} \) do
5: \quad Let \( i \in \{1, \ldots, i^*\} \) be such that \( d_j = d^{(i)} \).
6: \quad if \( \min\{p_j - p_j^*, d^{(i)} - p^{(i)}, d^{(i+1)} - p^{(i+1)}, \ldots, d^{(i^*)} - p^{(i^*)}\} > 0 \) then
7: \quad \quad \( A'[k] = A'[k-1] + w_j/p_j \).
8: \quad \quad \( p_j^* = p_j^* + 1 \).
9: \quad for \( i' = i, \ldots, i^* \) do \( p^{(i')} = p^{(i')} + 1 \).
10: \quad otherwise \( j = j + 1 \) and \( p_j^* = 0 \).
11: return \( A' \).

Lemma 5. Given a fractional 1 \( \| \sum w_j U_j \) instance with \( n \) jobs, \( i^* \) many different due dates, and maximum due date \( d^{(i^*)} \), sorted according to the WSPT rule, Algorithm 2 correctly computes a solution vector for this instance in \( \tilde{O}(i^* d^{(i^*)} + n) \) time.

Proof. It is straightforward to verify that Algorithm 2 runs in \( \tilde{O}(i^* d^{(i^*)} + n) \) time. To prove its correctness, we argue that \( A' \) is a correct solution vector for the fractional 1 \( \| \sum w_j U_j \) instance \( J \). For a given fractional solution \( 0 \leq x_1, \ldots, x_n \leq 1 \), we define the length of this solution to be the value \( \sum_j p_j x_j \). An optimal fractional solution of length \( k \) is a feasible fractional solution with maximum value \( \sum w_j x_j \) among all solutions of length \( k \).

Consider some iteration \( k \in \{0, \ldots, d^{(i^*)}\} \) of the algorithm, and let \( j \) be the current job under consideration. We argue by induction on \( k \) that there exist an optimal fractional solution \( x_1, \ldots, x_n \)
of length \(k\), with value \(A[k]\), such that at iteration \(k\) we have

\[
p^{(i)} = \sum_{\ell \leq j, d_i^\ell \leq d^{(i)}} p^{(i)} x_\ell \leq d^{(i)} \tag{1}
\]

for each \(i \in \{1, \ldots, i^*\}\). For \(k = 0\) this is clearly the case for the solution \(x_1 = \cdots = x_n = 0\), as \(A[0] = 0\). So assume \(k > 0\), and let \(i \in \{1, \ldots, i^*\}\) be such that \(d_j = d^{(i)}\). Moreover, let \(x_1, \ldots, x_n\) be the fractional solution of length at most \(k - 1\) which is guaranteed by induction. Observe that line 6 ensures that we can add extra unit of processing time unit to all \(p^{(i')}'\)'s with \(i' \geq i\) without violating (1). It also ensures that \(x_j + 1/p_j \leq 1\), since \(p_j - p_j^* = p_j - p_j/x_j > 0\). Thus, by setting \(x_j = x_j + 1/p_j\), we obtain a fractional solution that satisfies (1) after the \(p^{(i)}\)'s are updated in line 9 at the end of the iteration. Thus, our new fractional solution remains feasible, and it clearly has length at most \(k\). Thus, what remains to show is that this solution is optimal among all feasible solutions of length \(k\).

First observe that by the description of the algorithm, increasing any value of some \(x_\ell\) with \(\ell < j\) makes the solution \(x_1, \ldots, x_n\) infeasible. Thus, to improve our solution, we can only increase the value of some \(x_\ell\) with \(\ell \geq j\). Since the jobs are ordered according to the WSPT rule, the current job \(j\) has the best weight per processing time ratio among all jobs in \(\{j, \ldots, n\}\). Thus, scheduling a processing time unit of a combination of different jobs in \(\{j+1, \ldots, n\}\) cannot increase the total value of the solution by more than \(w_j/p_j\). Moreover, increasing \(x_j\) by more than \(1/p_j\) results in a solution of length greater than \(k\), and so \(x_\ell > 0\) for some other job \(\ell < j\) must be decreased. However, this cannot increase \(\sum_{j \in J} w_j x_j\) either, because all jobs \(\ell < j\) have at least the same weight per processing time as job \(j\). It follows that \(x_1, \ldots, x_n\) is has the maximum value \(\sum w_j x_j\) among all feasible solutions of length \(k\).

Each entry of the fractional solution vector clearly upper bounds the (integral) solution vector component-wise. That is, if \(A\) is a solution vector of some \(1 \parallel \sum w_j U_j\) instance, and \(A'\) is the corresponding fractional solution vector which is computed by Algorithm 2 on this instance, then \(A'[k] \geq A[k]\) for each entry \(k\). In the next lemma we show that each \(A'[k]\) is not too far away from \(A[k]\).

**Lemma 6.** Let \(J\) be a given \(1 \parallel \sum w_j U_j\) instance with \(i^*\) many different due dates, and maximum weight \(w_{\text{max}}\). Furthermore, let \(A\) a solution vector for this instance, and let \(A'\) be the fractional solution vector for \(J\). Then

\[
A'[k] - A[k] \leq i^* \cdot w_{\text{max}}
\]

for every entry \(k\) in \(A\) and \(A'\).

**Proof.** We claim there exist an optimal fractional solution \((x_j)_{j=1}^n\) for \(J\) such that at most \(i^*\) values of the fractional solution are non-integer. More specifically, we show that the optimal fractional solution \((x_j)_{j=1}^n\) implicitly computed by Algorithm 2 has this property. Note that whenever \(\min\{p_j - p_j^*, d^{(i)} - p^{(i)}, d^{(i+1)} - p^{(i+1)}, \ldots, d^{(i')} - p^{(i')}\} = 0\) for some job \(j\) with deadline \(d_j = d^{(i)}\), we move to the next job. If \(p_j - p_j^* = 0\), then we implicitly have \(x_j = 1\) since all of job \(j\) is scheduled. If this is not the case, we have that \(d^{(i')} - p^{(i')} = 0\) for some \(i' > i\). Assume that \(j\) is the first job with deadline \(d^{(i)}\) where this happens. It follows that no unit of processing time of any other job with deadline \(d^{(i)}\) is scheduled. Consequently we have that for every deadline \(d^{(i)}\), at most one non-integer fraction of a job with that deadline is in the optimal fractional solution. Hence, the
total number of fractional jobs in the optimal fractional solution is at most \( i^* \). Removing these jobs decreases the total weight of early jobs by at most \( i^* \cdot w_{\text{max}} \) and yields an integer solution for \( J \). It follows that \( A'[k] - A[k] \leq i^* \cdot w_{\text{max}} \) for all \( k \).

Having Lemma 6 in place, we can construct the range intervals of \( A \) in \( B \). We begin by first using Algorithm 2 to compute the fractional solution vectors \( A' \), \( B' \), and \( C' \), and considering the set of intervals \( J \) of jobs from \( A \). We define \( x_k \) and \( y_k \) as follows:

\[
x_k = \min \{ \ell : C'[k + \ell] - (A'[k] + B'[\ell]) \leq 2i^* \cdot w_{\text{max}} \}, \quad \text{and}
\]

\[
y_k = \max \{ \ell : C'[k + \ell] - (A'[k] + B'[\ell]) \leq 2i^* \cdot w_{\text{max}} \}.
\]

Using analogous arguments as is done by Bateni et al., we can show that the set of intervals defined above fulfill the requirements from Theorem 6 with an error \( \epsilon \in O(i^* \cdot w_{\text{max}}) \). For completeness, we also provide a proof here.

**Lemma 7.** The set of intervals \( \{[x_k, y_k] : 0 \leq k \leq |A|\} \) defined above are range intervals of \( A \) in \( B \) with an error of \( \epsilon \in O(i^* \cdot w_{\text{max}}) \).

**Proof.** We begin by first showing that set of intervals \( \{[x_k, y_k] : 0 \leq k \leq |A|\} \) satisfy the first two conditions of Theorem 6. Consider some \( k \in \{0, \ldots, |A|\} \), and let \( \Delta_k(\ell) = C'[k + \ell] - (A'[k] + B'[\ell]) \). By Lemma 6, we have that

\[
\Delta_k(\ell) \leq 2i^* \cdot w_{\text{max}} \implies C'[k + \ell] - (A'[k] + B'[\ell]) \leq 4i^* \cdot w_{\text{max}}.
\]

Define \( \delta(k) \) as the smallest integer such that \( C'[k + \delta(k)] = A'[k] + B'[\delta(k)] \). Now consider \( 1 \leq \ell \leq \delta(k) \). We argue that \( \Delta_k(\ell - 1) \geq \Delta_k(\ell) \). Using the definition of \( \Delta_k \) we can rewrite this as \( C'[k + \ell] - C'[k + (\ell - 1)] \leq B'[\ell] - B'[\ell - 1] \). Recall that due to the description of Algorithm 2 we have \( C'[k + \ell] - C'[k + (\ell - 1)] = w_{j_c}/p_{j_c} \) for some \( j_c \in J_A \cup J_B \), and \( B'[\ell] - B'[\ell - 1] = w_{j_b}/p_{j_b} \) for some \( j_b \in J_B \). Now, observe that for \( \delta(k) \), the total processing time of jobs from \( J_A \) (resp. \( J_B \)) scheduled in the fractional solution for \( C'[k + \ell] \) is exactly \( k \) (resp. \( \delta(k) \)). Thus, since Algorithm 2 never removes jobs in its computation, and since \( \ell \leq \delta(k) \), we know that the total processing time of jobs from \( J_A \) scheduled in the fractional solution for \( C'[k + (\ell - 1)] \) is at most \( k \), which means that the total processing time of jobs from \( J_B \) in this solution is at least \( \ell - 1 \). Thus, if \( j_c \in J_B \), we have \( w_{j_c}/p_{j_c} \leq w_{j_b}/p_{j_b} \). This is because in the fractional solution corresponding to \( B'[\ell - 1] \), the total processing time of jobs from \( J_B \) is exactly \( \ell - 1 \), and the jobs are sorted according to the WSPT rule. If, on the other hand, \( j_c \in J_A \), then the total processing time of jobs from \( J_B \) scheduled in the fractional solution for \( C'[k + (\ell - 1)] \) is at least \( \ell \), which means that job \( j_b \) is already scheduled in the fractional solution for \( C'[k + (\ell - 1)] \). So again we get \( w_{j_c}/p_{j_c} \leq w_{j_b}/p_{j_b} \), implying that \( C'[k + \ell] - C'[k + (\ell - 1)] \leq B'[\ell] - B'[\ell - 1] \).

Thus, we have that \( \Delta_k(\ell - 1) \geq \Delta_k(\ell) \) for all \( 1 \leq \ell \leq \delta(k) \). By an analogous argument we can show that \( \Delta_k(\ell - 1) \leq \Delta_k(\ell) \) holds for all \( \delta(k) \leq \ell \leq |B'| \). It follows that \( \Delta_k \) is non-increasing in the interval \([0, \delta(k)]\), and non-decreasing in the interval \([\delta(k), |B'|] \). From this, we immediately get that the first two requirements from Theorem 6 are fulfilled for the interval \([x_k, y_k] \) with an error parameter of \( e = 4i^* \cdot w_{\text{max}} \).

To show that the third requirement from Theorem 6 is fulfilled, we argue that \( \Delta_{k+1}(x_k) \geq \Delta_k(x_k) \). Using the definition of \( \Delta_k \) we can rewrite this as \( C'[k+1+x_k] - C'[k+x_k] \geq A'[k+1] - A'[k] \).
Note that $x_k \leq \delta(k)$. Hence, we know that $C'[k + 1 + x_k]$ and $C'[k + x_k]$ contain jobs from $J_A$ of total processing time at most $k + 1$ and $k$, respectively. Let job $j$ be added by Algorithm 2 when computing $C'[k + 1 + x_k]$ from $C'[k + x_k]$, and let job $j'$ be added Algorithm 2 when computing $A'[k + 1]$ from $A'[k]$. It follows that $w_j/p_j \geq w_{j'}/p_{j'}$. By an analogous argument we get that $\Delta_{k+1}(y_k) \geq \Delta_k(y_k)$. It follows that the third requirement from Theorem 6 is fulfilled.

We can compute range intervals of $A$ in $O(|B|) = O(d^{\ast})$ time using Algorithm 3. To do so, we exploit the third requirement from Theorem 6. This improves the computation time of the range intervals by a factor of $\log(d^{\ast})$ when compared to the approach by Bateni et al. [3].

Algorithm 3

| Input: Vectors $A', B', C'$, number of deadlines $i^\ast$, and the maximum weight $w_{\max}$. |
|---|
| Output: Range intervals $\{[x_k, y_k] : 0 \leq k \leq |A|\}$ of $A$ in $O(d^{\ast})$ time. |

```
1: Set $x = y = 0$.
2: for $k = 1$ to $|A'|$ do
3:   while $C'[k + x] - A'[k] - B'[x] > 2i^\ast \cdot w_{\max}$ do $x = x + 1$.
4:   if $k = 1$ then $y = x$.
5:   while $C'[k + y] - A'[k] - B'[y] \leq 2i^\ast \cdot w_{\max}$ do $y = y + 1$.
6: Set $[x_k, y_k] = [x, y - 1]$.
```

Lemma 8. Algorithm 3 computes the range intervals $[x_i, y_i]_{i=0}^{|A|}$ of $A$ in $B$ in $O(d^{\ast})$ time.

Proof. It is straightforward to verify that Algorithm 3 runs in $O(|A| + |B|) = O(d^{\ast})$ time. For its correctness, recall the function $\Delta_k(\ell) = C'[k + \ell] - A'[k] - B'[\ell]$ used in the proof of Lemma 7. As is shown in this proof, for every $k$ there exist an $\delta(k)$ such that $\Delta_k(\ell)$ is non-increasing in the interval $[0, \delta(k)]$, and $\Delta_k(\ell)$ is non-decreasing in the interval $[\delta(k), |B'|]$. Hence, to find $[x_1, y_1]$, Algorithm 3 first finds $x_1$ by iterating over values for $\ell$ starting from 0, thereby finding the smallest integer $\ell$ such that $\Delta_1(\ell) \leq 2i^\ast \cdot w_{\max}$. Afterwards, Algorithm 3 finds $y_1$ by iterating over values for $\ell$ and finding the largest integer $\ell$ such that $\Delta_1(\ell) \leq 2i^\ast \cdot w_{\max}$. We know that in the interval $[x_1, \delta(1)]$ the function $\Delta_1(\ell)$ is non-increasing, hence the largest integer $\ell$ such that $\Delta_1(\ell) \leq 2i^\ast \cdot w_{\max}$ must lie in the interval $[\delta(1), |B'|]$ and, more specifically, once $\Delta_1(\ell') > 2i^\ast \cdot w_{\max}$ for some $\ell' \in [\delta(1), |B'|]$, we know that $\Delta_1(\ell'') > 2i^\ast \cdot w_{\max}$ for all $\ell'' > \ell'$. Hence, Algorithm 3 can stop the search for $y_1$ after encountering the first value $\ell$ such that $\Delta_1(\ell) > 2i^\ast \cdot w_{\max}$. Furthermore, we know that for all $k$ we have that $x_k \leq x_{k+1}$ and $y_k \leq y_{k+1}$, since the range intervals fulfill the third requirement of Theorem 6. It follows that for any $k > 1$, Algorithm 3 can start the search for $x_k$ at $x_{k-1}$ and the search for $y_i$ at $y_{k-1}$. Thus, Algorithm 3 indeed correctly computes the range intervals of $A$ in $B$, and the lemma follows.

Proof of Lemma 4. Computing all three fractional solution vectors $A'$, $B'$, and $C'$ can be done in $O(i^*d^* + n)$ time according to Lemma 5. From these we can compute the range intervals of $A$ in $B$ in $O(d^{\ast})$ time due to Lemma 8. These intervals satisfy the conditions of Theorem 6 with an error parameter $\epsilon = O(i^\ast \cdot w_{\max})$ as is proven in Lemma 7 (which in turn relies on Lemma 6). All together this gives us an algorithm for computing the range intervals of $A$ in $B$ in $O(i^\ast d^* + n)$ time.
4 Lower Bounds

In this section we present our lower bounds for $1 || \sum w_j U_j$, namely we prove Theorem \ref{thm:lower-bound}. We begin by stating the $\forall \exists$ Strong Exponential Time Hypothesis ($\forall \exists$-SETH) on which our lower bounds are based upon.

**Hypothesis 1**. There is no $0 < \alpha < 1$ and $\varepsilon > 0$ such that for all $k \geq 3$ there is an $O(2^{(1-\varepsilon)n})$ time algorithm for the following problem: Given a $k$-CNF formula $\phi$ on $n$ variables $x_1, \ldots, x_n$, decide whether for all assignments to $x_1, \ldots, x_{\lceil \alpha \cdot n \rceil}$ there exists an assignment to the rest of the variables that satisfies $\phi$, that is, whether:

$$\forall x_1, \ldots, x_{\lceil \alpha \cdot n \rceil} \exists x_{\lceil \alpha \cdot n \rceil} + 1, \ldots, x_n : \phi(x_1, \ldots, x_n) = \text{true}.$$ 

Our lower bounds are provided via a reduction from the AND Subset Sum problem. Recall that in Subset Sum, we are given a set of integers $X$ and a target $t$, where each integer $x \in X$ is in the range $0 < x \leq t$. The goal is to determine whether there is a subset $Y \subseteq X$ whose elements sum up to exactly $t$; i.e. $\sum_{x \in Y} x = t$. If this is in fact the case, $(X, t)$ is a yes-instance. In the corresponding AND Subset Sum problem, we are given $N$ many Subset Sum instances $(X_1, t_1), \ldots, (X_N, t_N)$, and the goal is to determine whether all instances are yes-instances. We have the following relationship between $\forall \exists$-SETH (Hypothesis 1) and the AND Subset Sum problem.

**Theorem 7**. Assuming $\forall \exists$-SETH, there are no $\delta, \varepsilon > 0$ such that the following problem can be solved in $\tilde{O}(N^{1+\delta-\varepsilon})$ time: Given $N$ Subset Sum instances, each with $O(N^\delta)$ integers and target $O(N^\varepsilon)$, determine whether all of these instances are yes-instances.

Before showing our reduction from AND Subset Sum to $1 || \sum w_j U_j$, we will need the following two auxiliary lemmas that will help us with proving the correctness of our construction.

**Lemma 9.** The following equality holds for all $\ell \in \{1, \ldots, N-1\}$:

$$\sum_{i=\ell}^{N-1} 1/(i(i+1)) + 1/N = 1/\ell.$$ 

**Proof.** The proof is by reverse induction on $\ell$. For $\ell = N - 1$ we have that

$$1/(N(N - 1)) + 1/N = (1 + N - 1)/(N(N - 1)) = 1/(N - 1),$$

and so the lemma holds for this case. Assume then that $\ell \leq N - 1$, and that the lemma holds for $\ell + 1$. We have

$$\sum_{i=\ell}^{N-1} 1/(i(i+1)) + 1/N = 1/\ell(\ell + 1) + \sum_{i=\ell+1}^{N-1} 1/(i(i+1)) + 1/N = 1/\ell(\ell + 1) + 1/(\ell + 1) = 1/\ell,$$

where the second equality follows from our inductive assumption. The lemma thus holds. \qed

**Lemma 10.** Consider the following linear program, defined over a set of continuous non-negative variables $x_1, \ldots, x_N \geq 0$, where we wish to maximize

$$\min f(x_1, \ldots, x_N) = \sum_{i=1}^{N} (q_i - x_i)$$

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subject to
\[ g_\ell(x_1, \ldots, x_N) = \sum_{i=1}^{\ell} ix_i - \sum_{i=1}^{\ell} it_i \leq 0 \quad \text{for all} \quad \ell = 1, \ldots, N \]
where \(t_1 \leq q_1, \ldots, t_N \leq q_N\) are given constants. Then the solution \(x_i^\ast = t_i\) for \(i = 1, \ldots, N\) is a unique optimal solution for this program.

Proof. As the program is a linear program, any solution that simultaneously satisfies for some \(\lambda_1, \ldots, \lambda_N \geq 0\) the following two conditions, known as the Karush-Kuhn-Tucker (KKT) conditions, is necessarily optimal:

\[ -1 + \ell \sum_{i=1}^{N} \lambda_i = 0 \quad \text{for} \quad \ell = 1, \ldots, N \quad (2) \]

and

\[ \lambda_\ell \left( \sum_{i=1}^{\ell} ix_i - \sum_{i=1}^{\ell} it_i \right) = 0 \quad \text{for} \quad \ell = 1, \ldots, N \quad (3) \]

From (2) with \(\ell = N\), we have that \(\lambda_N = 1/N\). Let us now prove (by reverse induction) that \(\lambda_\ell = 1/(\ell(\ell + 1))\) for \(\ell = 1, \ldots, N - 1\). Let \(\ell \leq N - 1\), and assume that \(\lambda_i = 1/(i(i + 1))\) for all \(i \in \{\ell + 1, \ldots, N - 1\}\). From (2), we obtain that

\[ \ell\lambda_\ell = 1 - \ell \sum_{i=\ell+1}^{N} \lambda_i = 0, \]

and accordingly

\[ \ell\lambda_\ell = 1 - \ell \left( \sum_{i=\ell+1}^{N-1} 1/(i(i + 1)) + 1/N \right). \]

Thus, plugging in Lemma (3), we get \(\ell\lambda_\ell = 1 - \ell(1/(\ell(\ell + 1)))\), and that \(\lambda_\ell = 1/\ell(\ell + 1)\). It follows that \(\lambda_\ell > 0\) for \(\ell = 1, \ldots, N\), and so from (3), we have that \(\sum_{i=1}^{\ell} ix_i = \sum_{i=1}^{\ell} it_i\) for \(i = 1, \ldots, N\). The lemma thus follows.

We are now in position to present our reduction from AND Subset Sum to \(1 \ \| \ \sum w_j U_j\). Lemma 11 describes such a reduction that constructs a \(1 \ \| \ \sum w_j U_j\) instances with a sufficiently small number of jobs and maximum weight.

Lemma 11. There is an algorithm that takes as input \(N\) Subset Sum instances \((X_1, t_1), \ldots, (X_N, t_N)\), where there exist \(\delta, \varepsilon > 0\) such that \(|X_i| = O(N^\varepsilon)\) and \(t_i = O(N^\delta)\) for all \(1 \leq i \leq N\), and outputs in \(O(N^{1+\varepsilon})\) time a \(1 \ \| \ \sum w_j U_j\) instance \(J\) such that

(i) \(J\) is a yes-instance of \(1 \ \| \ \sum w_j U_j\) iff each \((X_i, t_i)\) is a yes-instances of Subset Sum.

(ii) \(n = O(N^{1+\varepsilon})\) and \(w_{\max} = O(N^\delta)\).

Proof. Let \((X_1, t_1), \ldots, (X_N, t_N)\) be \(N\) instances of Subset Sum as in the lemma statement, and let \(q_i\) denote the sum of all integers in \(X_i\). For each \(i \in \{1, \ldots, N\}\), and for each \(x_{i,j} \in X_i\), we create a job \((i, j)\) with the following parameters:

- processing time \(p_{i,j} = i \cdot x_{i,j}\),

- weight \(w_{i,j} = x_{i,j}\),

- release time \(r_{i,j} = 0\),

- deadline \(d_{i,j} = q_i + t_i\),

- machine \(m_{i,j} = j\),

- profit \(p_{i,j} = 0\).

We then compute the schedule \(S\) of the jobs\((i, j)\) such that the objective function \(\sum_{j=1}^{N} \alpha_j (d_{i,j} - p_{i,j})\) is maximized, where \(\alpha_j\) is the weight of job \((i, j)\). We then replace each job \((i, j)\) by \(j\) jobs \((i, j), (i, j+1), \ldots, (i, j+j-1)\) with the same parameters. Finally, we solve the resulting \(1 \ \| \ \sum w_j U_j\) instance, and output the yes-instance if and only if the optimal solution is positive for each \((i, j)\) job.
• weight \( w_{i,j} = x_{i,j} \),
• and due date \( d_{i,j} = d_i = \sum_{\ell=1}^{i} \ell \cdot t_\ell \).

Let \( J = \{(i,j) : 1 \leq i \leq N, 1 \leq j \leq |X_i|\} \) denote the set of constructed jobs. Observe that \( w_{\text{max}} = \max_{i,j} x_{i,j} \leq \max_i t_i = O(N^\delta) \), and that \( |J| = \sum_i |X_i| = O(N^{1+\epsilon}) \). Also note that \( J \) can be constructed in \( O(N^{1+\epsilon}) \) time. To complete the proof, we argue that there is a schedule for all jobs in \( J \) with total weight of tardy jobs at most \( w^* = \sum_i (q_i - t_i) \) if and only if each Subset Sum instance \((X_i, t_i)\) is a yes-instance.

Consider any feasible (not necessarily optimal) schedule for \( J \). For \( i \in \{1, \ldots, N\} \), let \( E_i \) denote the set of early jobs in the schedule with due date \( d_i \), for \( i = 1, \ldots, N \), and let \( x_i = \sum_{(i,j) \in E_i} x_{i,j} \). Moreover, let \( T = J \setminus \bigcup_i E_i \). Since all elements in \( X_i \) sum up to \( q_i \), the weighted number of tardy jobs is given by
\[
\sum_{(i,j) \in T} w_{i,j} = \sum_{(i,j) \in T} x_{i,j} = \sum_{i=1}^{N} (q_i - x_i).
\]

Due to Lemma 1, we assume that the order in which the set of jobs are scheduled is \( E_1, E_2, \ldots, E_N, T \) (where the order of the jobs in each set is arbitrary). Thus, since each job in \( \bigcup_i E_i \) is early, the following inequality holds for each \( \ell \in \{1, \ldots, N\} \):
\[
\sum_{i=1}^{\ell} \sum_{(i,j) \in E_i} p_{i,j} = \sum_{i=1}^{\ell} \sum_{(i,j) \in E_i} i \cdot x_{i,j} = \sum_{i=1}^{\ell} i \cdot x_i \leq d_\ell = \sum_{i=1}^{\ell} i \cdot t_i.
\]

According to Lemma 10 the problem of minimizing (4) subject to the \( N \) constraints given by (5) has a unique optimal solution of \( x_i = t_i \) for each \( i \in \{1, \ldots, N\} \). This solution has value of \( \sum_{i=1}^{N} (q_i - t_i) = w^* \). This implies that the \( 1 \mid \sum w_j \text{U}_j \) instance has a feasible schedule where the total weight of tardy jobs is \( w^* \) iff \( x_i = \sum_{(i,j) \in E_i} x_{i,j} = t_i \) for each \( i = 1, \ldots, N \).

**Proof of Theorem 2.** Both lower bounds in the theorem follow directly from Lemma 11 together with Theorem 7. In each case, the key is to appropriately choose \( \epsilon \) and \( \delta \) for the given AND Subset Sum instances, and then apply Lemma 11 to obtain bounds on \( n \) and \( w_{\text{max}} \).

1. Assume we are given an algorithm for \( 1 \mid \sum w_j \text{U}_j \) with running time \( \tilde{O}(n + w_{\text{max}} \cdot n^{1-\epsilon_0}) \) for some \( \epsilon_0 > 0 \). Choose \( \delta = \max\{2, 2\epsilon_0\} \) and \( \epsilon = \epsilon_0 \) for the AND Subset Sum instance. By Lemma 11 we have that \( n = O(N^{1+\epsilon}) \) and \( w_{\text{max}} = O(N^\delta) \). Using the assumed algorithm, we can solve AND Subset Sum in \( \tilde{O}(N^{1+\epsilon} + N^{1+\delta+\epsilon-\delta\epsilon_0}) = \tilde{O}(N^{1+\delta-\epsilon}) \) time.

2. Assume we are given an algorithm for \( 1 \mid \sum w_j \text{U}_j \) with running time \( \tilde{O}(n + w_{\text{max}} \cdot n^{1-\epsilon_0}) \) for some \( \epsilon_0 > 0 \). Choose \( \delta = \epsilon_0 \) and \( \epsilon = \epsilon_0/2 \) for the AND Subset Sum instance. By Lemma 11 we have that \( n = O(N^{1+\epsilon}) \) and \( w_{\text{max}} = O(N^\delta) \). Using the assumed algorithm, we can solve AND Subset Sum in \( \tilde{O}(N^{1+\epsilon} + N^{1+\delta+\epsilon-\epsilon_0+\epsilon\epsilon_0}) = \tilde{O}(N^{1+\delta-\epsilon}) \) time.

Thus, in both cases we obtain a \( \tilde{O}(N^{1+\delta-\epsilon}) \) time algorithm for AND Subset Sum which implies that \( \forall \exists \text{-SETH} \) is false according to Theorem 7. \( \square \)
5 Conclusion

We identified a new scenario of $1 || \sum w_j U_j$ where it is possible to improve upon Lawler and Moore’s classic algorithm [14]. Our algorithm is based on $(\max, +)$-convolutions and employs the prediction technique recently developed by Bateni et al. [3]. Intuitively speaking, we show that we can obtain good predictions, or more precisely, range intervals with a small error, when the weights in the $1 || \sum w_j U_j$ instance are relatively small. This leads to a fast algorithm for this case. We complement this result by providing a new conditional lower bound for running times involving the maximum weight for $1 || \sum w_j U_j$ algorithms.

Our work leaves several open questions that are interesting for future research. Below we list four question which we feel are the most important:

1. Can we obtain a fast $1 || \sum w_j U_j$ algorithm for when the maximum processing time $p_{\max}$ is relatively small? More specifically, can we get an algorithm with running time $O(d_{\#}(n + dp_{\max}))$, or perhaps $\tilde{O}(d_{\#}(n + wp_{\max}))$?

2. Can our lower bound be strengthened so as it takes into account also the number of different due dates $d_{\#}$? This is a particularly important parameter, as it quantifies the essential difference between $1 || \sum w_j U_j$ and Knapsack.

3. Can our algorithm be extended to the parallel machine case?

4. Can the prediction technique speed up pseudo-polynomial time algorithm for other scheduling problems apart from $1 || \sum w_j U_j$?

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