Berry phase in the simple harmonic oscillator

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Abstract

Berry phase of simple harmonic oscillator is considered in a general representation. It is shown that, Berry phase which depends on the choice of representation can be defined under evolution of the half of period of the classical motions, as well as under evolution of the period. The Berry phases do not depend on the mass or angular frequency of the oscillator. The driven harmonic oscillator is also considered, and the Berry phase is given in terms of Fourier coefficients of the external force and parameters which determine the representation.

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1. Introduction

The (time-dependent) harmonic oscillator gives a system whose quantum states are described by solutions of the classical equation of motion (classical solutions). This fact has been recognized by Lewis [1] who found, in an application of asymptotic theory of Kruskal [2], that there exists a quantum mechanically invariant operator. This invariant operator determined by the classical solutions has then been used to construct wave function of (generalized) harmonic oscillator systems [3, 4, 5]. An alternative and simple (at least in conceptually) way to find the wave functions in terms of classical solutions is to use the Feynman path integral method [6]. As observed by Feynman and Hibbs [7], the kernel (propagator) for a general quadratic system is almost determined by classical action. The classical action can be given in terms of two linearly independent classical solutions and, for a general quadratic system, the exact kernel has been found by requiring the kernel to satisfy the initial condition and Schrödinger equation [6]. Further, it has been shown that [8], the wave functions of a general quadratic system can be obtained from those of unit mass harmonic oscillator system through unitary transformations.

A perceptive and interesting observation made by Berry [9] is that, if an eigenstate of Hamiltonian is adiabatically carried around by cyclic Hamiltonian, the change of the phase of wave function separates into the obvious dynamical part and an additional geometric part. That the geometric change (Berry Phase) still remains some naturalness even if the cycled wave function is not an eigenstate of the Hamiltonian and even if the carrying is not adiabatic was pointed out by Aharonov and Anandan [10]. In a recent paper by one of
the authors [11], it was shown that the Berry phase for harmonic oscillator of \( \tau \)-periodic Hamiltonian can be defined only if the two linearly independent classical solutions are finite all over the time. Moreover, in the cases of the two linearly independent solutions finite all over the time, it was shown that there exists at least a representation where the Berry phase can be defined under the \( \tau \)- or \( 2\tau \)-evolution.

The Berry phase is known to be closely related to the classical Hannay angle [12]. By adopting the fact that a Gaussian wave packet could be a wave function of a harmonic oscillator system, the approach of Aharonov and Anandan [10] has been used to calculate Berry’s phase, and the relation between the phase and Hannay angle is given for the wave packet [13].

For the simple harmonic oscillator (SHO) of the equation of motion

\[
M(\ddot{x} + \omega^2 x) = 0
\]  

(1)

with constant mass \( M \) and constant angular frequency \( \omega \), the wave functions become stationary if we choose \( \cos \omega t \) and \( \sin \omega t \) as two solutions, while in general they describe the states of pulsating probability distribution [8].

In this paper, we will calculate Berry phase of the SHO system in general representations, and will show that Berry phase of the SHO system indeed depends on the choice of representation. We will consider a general representation made from the classical solutions \( \cos \omega t \) and \( C \sin(\omega t + \beta) \) of (1), where \( C \) is a nonzero constant and the constant \( \beta \) is not one of \( (2n + 1)\pi/2 \) \( (n = 0, \pm 1, \pm 2, \cdots) \). The period of the classical solutions is \( \tau_0 \) \((=2\pi/\omega)\). It will be shown that, due to a quasiperiodicity of all classical solution of SHO,
Berry phase can be defined under the $\tau_0/2$-evolution as well as $\tau_0$-evolution, which may not be possible for a general oscillator system considered in [11]. The Berry phase will be evaluated in terms of the parameters $C$ and $\beta$; it turns out that the Berry phase does not depend on the mass $M$ or angular frequency $w$ of the oscillator. For the presence of external force, as already noted in [14, 5], the phase can not be defined if a particular solution of the equation of motion diverges as the time goes to infinity. For the cases where the particular solution is periodic, the Berry phase whose leading order is order of $1/\hbar$ will be evaluated in terms of the Fourier coefficients of the external force. In view of view analyses on general harmonic oscillator of [11], the SHO is rather special in the sense that any number can be the period of the Hamiltonian. As will be mentioned in detail, we thus need care in applying the formulas to find Berry phase of the SHO system.

2. The wave functions and (quasi)periodicity

As is well-known [8, 15, 3, 4, 5], the wave functions of SHO can be written as

$$\psi_n(x,t) = \frac{1}{\sqrt{2^n n!}} \left( \frac{\Omega}{\pi \hbar} \right)^{\frac{1}{4}} \left[ \frac{u(t) - iv(t)}{\rho(t)} \right]^{n+\frac{1}{2}} \exp \left[ \frac{x^2}{2\hbar} \left( \frac{\Omega}{\rho^2(t)} + iM(t) \frac{\dot{\rho}(t)}{\rho(t)} \right) \right] \times H_n\left( \sqrt{\frac{\Omega}{\hbar}} \frac{x}{\rho(t)} \right),$$

where the $u(t), v(t)$ are two linearly independent solutions of (1) and $\rho(t)$ is defined as

$$\rho(t) = \sqrt{u^2(t) + v^2(t)}.$$
Without losing generality, the two linearly independent solutions $u(t), v(t)$ can be written as

$$u(t) = \cos wt \quad v(t) = C \sin(wt + \beta).$$  \hfill (4)

If we choose $C = 1$ and $\beta = 0$, it gives the ”stationary representation” where the wave functions are of stationary probability distribution

$$\tilde{\psi}_n(x, t) = \frac{1}{\sqrt{2^n n!}} \left( \frac{Mw}{\pi \hbar} \right)^{1/4} \exp[-i(n + 1/2)wt - \frac{Mw}{\hbar}x^2]H_n\left(\frac{\sqrt{Mw}}{\hbar}x\right).$$

The $\tilde{\psi}_n(x, t)$ is the eigenstate of the Hamiltonian of SHO

$$H = \frac{p^2}{2M} + \frac{Mw^2}{2}x^2,$$ \hfill (5)

with eigenvalue $E_n = (n + 1/2)\hbar w$. The Hamiltonian does not depend on time, so the Hamiltonian is periodic with any period. The analyses in [11] might thus suggest that, in the stationary representation where the Berry phase can be defined for an evolution of any time, the phase is zero. Indeed, phase change of $\tilde{\psi}_n$ over $t$ evolution is simply equals to the dynamical phase change $-E_n t/\hbar$, so that Berry phase is zero.

In the figure 1, some trajectories of $(u(t), v(t))$ which would depict classical motions of SHO are given. They make closed curves, since both of $u(t)$ and $v(t)$ are periodic. Different closed curves in the figure 1 give different sets of wave functions, while it is not possible to construct a representation corresponding to the dashed line in b of the figure. Classical motion of SHO may be depicted as the mass $M$ circulating along a closed curve with uniform angular velocity $w$, so that it needs a period $\tau_0$ for a complete circulation. As noted in [4, 5, 11], if $\rho(t)$ is periodic with some period, the wave functions
Figure 1: The representation space of SHO: the horizontal and vertical axes denote $u$ and $v$, respectively. Some of the curves which would depict classical motions of the SHO are given. Though the classical motion for the dashed line may be allowed, representation corresponding to the line does not exist.

are (quasi)periodic with the period of $\rho(t)$. Since $\rho(t)$ is the distance from the origin to a point of a curve, as in the figure 1, it is clear that in general $\rho(t)$ is periodic with the half of the period of classical motion. The circle of radius 1 in a gives rise to the stationary representation. $\rho(t)$ is constant along the circle and thus any number can be a period of $\rho$, which is compatible with that Berry phase in the stationary representation must be 0.

From the continuity, one may find the relation

$$u(\tau_0/2 + t) - iv(\tau_0/2 + t) = \exp(-i\pi)[u(t) - iv(t)],$$

which gives the quasiperiodicity relation of the wave functions

$$\psi_n(x, \tau_0/2 + t) = \exp[i\chi_n(\tau_0/2)]\psi_n(x, t)$$
with the overall phase change

$$\chi_n(\tau_0/2) = -(1/2 + n)\pi.$$  \hfill (8)

Since a phase is defined only up to an additive constant $2\pi$, (8) can be written in different ways. For example, a relation $\chi_n(\tau_0/2) = (n - 1/2)\pi$ is equivalent to (8).

### 3. Berry Phase

The Berry phase $\gamma_n$ is given from the overall phase change by subtracting the dynamical phase change

$$\gamma_n(\tau_0/2) = \chi_n(\tau_0/2) - \delta_n(\tau_0/2),$$  \hfill (9)

and the dynamical phase change $\delta_n$ is given as

$$\delta_n(\tau') = -\frac{i\hbar}{\hbar} \int_{0}^{\tau'} \int_{-\infty}^{\infty} \psi_n^*(t)H\psi_n(t)dxdt$$

$$= -\frac{1}{2}(n + \frac{1}{2}) \int_{0}^{\tau'} \left[ \frac{\Omega}{M\rho^2} + \frac{M\rho^2}{\Omega} + \frac{\rho^2(t)Mw^2}{\Omega} \right] dt.$$  \hfill (10)

For the classical solutions of (4), $\Omega$ defined as $M(u\dot{v} - v\dot{u})$ is

$$\Omega = MCw \cos \beta,$$  \hfill (11)

and thus the $\delta_n(\tau_0/2)$ is written as

$$\delta_n(\tau_0/2) = -(n + \frac{1}{2})\pi \frac{1 + C^2}{2C \cos \beta}.$$  \hfill (12)

The Berry phase of the wave function $\psi_n$ for the $\tau_0/2$-evolution, therefore, can be given as

$$\gamma_n(\tau_0/2) = (n + \frac{1}{2})\pi[-1 + \frac{1 + C^2}{2C \cos \beta}].$$  \hfill (13)
The Berry phase under $\tau_0$-evolution can be obtained from $\gamma_n(\tau_0/2)$

$$\gamma_n(\tau_0) = 2\gamma_n(\tau_0/2) = (n + \frac{1}{2})\pi\left[\frac{1 - 2C\cos\beta + C^2}{C\cos\beta}\right].$$

(14)

The $\gamma_n$ depends on $C$ and $\beta$, but does not depend on the parameters of the Hamiltonian $M$ and $w$.

The fact that the phase $\gamma_n$ is defined up to an additive constant $\pm 2\pi$ says that there are infinitely many representation which give the same Berry phase. For example, if $\cos\beta = 1/2$, the values of $C$ which gives $\gamma_n(\tau_0/2) = 0$ can be written as

$$(\cdots, 9/2 \pm \sqrt{(9/2)^2 - 1}, 5/2 \pm \sqrt{(5/2)^2 - 1}, -3/2 \pm \sqrt{(3/2)^2 - 1}, -7/2 \pm \sqrt{(7/2)^2 - 1}, \cdots).$$

In order to compare with the the result of Ge and Child [13], by defining

$$\alpha(t) = \frac{1}{\hbar}\left(\frac{\Omega}{\rho^2} - iM\frac{\dot{\rho}}{\rho}\right),$$

one may find the relation

$$-\frac{i}{2} \int_0^{\tau_0} \frac{\dot{\alpha}}{\alpha + \alpha^*} dt = \frac{1 - 2C\cos\beta + C^2}{2C\cos\beta}\pi$$

(15)

which gives the Berry phase $\gamma_0(\tau_0)$ in agreement with the result in (14).

In order to compare the results here with the formulae in [11], we need to consider several cases differently. If we take $\tau$ as one of $(2m - 1)\tau_0/2$ ($m = 1, 2, \cdots$), the Floquet theorem [16] is satisfied by that classical solutions are $2\tau$-periodic. If we take $\tau$ as one of $m\tau_0$ ($m = 1, 2, \cdots$), the Floquet theorem is satisfied by that classical solutions are $\tau$-periodic. In the above cases, the corresponding formula for Berry phases [11] can be used to obtain the Berry phase in (14) or its integral multiples. In a general oscillator,
Berry phase may not be defined under \(\tau/2\)-evolution, and thus the equation (13) has no corresponding formula in [11]. If we take \(\tau\) as being not one of \(m\tau_0/2\) \((m = 1, 2, \cdots)\), the fact that Berry phase can be defined for any evolution in stationary representation is in agreement with the analyses in [11]. Again, a formula corresponding to this case can be used to show that Berry phase is 0 in the stationary representation.

4. Driven harmonic oscillator

For the SHO with a driving force \(F(t)\), the wave function is given as [3, 8, 11, 3]

\[
\psi_n^F = \frac{1}{\sqrt{2^n n!}} \left( \frac{MwC \cos \beta}{\pi \hbar} \right)^{1/2} \frac{1}{\sqrt{\rho(t)}} \left[ \cos wt - iC \sin wt \rho(t) \right]^{n+1/2} \exp \left[ \frac{i}{\hbar} (M \dot{x}_p x + \delta(t)) \right] \\
\times \exp \left[ \frac{(x - x_p)^2}{2 \hbar} \left( -\frac{\Omega}{\rho^2(t)} + iM(t) \dot{\rho}(t) \right) \right] H_n \left( \sqrt{\frac{\Omega \hbar}{2 \hbar}} (x - x_p) \right),
\]

where

\[
\delta(t) = \frac{M}{2} \int_{t_0}^t \left[ w^2 x_p^2(z) - \dot{x}_p^2(z) \right] dz
\]

with arbitrary constant \(t_0\). The \(x_p\) defined by the relation

\[
\ddot{x}_p + w^2 x_p = \frac{F(t)}{M}
\]

may be the classical coordinate of \(x\). \(F\) denotes the driving force, and we only consider the periodic \(F\) satisfying \(F(t + \tau_f) = F(t)\).

If \(x_p\) is finite all over the time and there exist two positive integers \(N, p\) of no common divisor except 1 such that \(\tau_0/\tau_f = p/N\), the Berry phase can be defined under \(N\tau_0\)-evolution in a general representation. The periodic \(F(t)\) can be written as

\[
F(t) = \sum_{n=-\infty}^{\infty} f_n e^{in\omega_f t},
\]
where
\[ f_n = \frac{1}{\tau_f} \int_0^{\tau_f} F(t)e^{-iwn_f t} dt \]  

(20)

with \( w_f = 2\pi/\tau_f \). The \( x_p(t) \) finite all over the time can be written as
\[
x_p = \sum_{n=\infty}^{\infty} \frac{f_n}{M(-n^2 w_f^2 + w^2)} e^{inw_f t} + De^{iwt} + D^*e^{-iwt}.
\]  

(21)

If \( p = 1 \), \( f_N \) must be zero for the finiteness of \( x_p \), which will be assumed from now on. In (21), complex number \( D \) is a free parameter and different choice of \( D \) gives different wave functions of the driven system. The minimum period of \( x_p \) is \( N\tau_0 (= p\tau_f) \) in general so that
\[
x_p(t + N\tau_0) = x_p(t).
\]  

(22)

It has been known that [11] the Berry phase of a driven system separates into the contribution from undriven system and that from the presence of driving force. The contribution from the driving force is written as
\[
\frac{1}{\hbar} \int_0^{\tau'} M\dot{x}_p^2 dt,
\]

where \( \tau' \) is the period needed for the Berry phase. The Berry phase \( \gamma_n^F \) of driven SHO system under \( N\tau_0 \) evolution is thus given as
\[
\gamma_n^F(N\tau_0) = N\gamma_n(\tau_0) + \frac{1}{\hbar} \int_0^{N\tau_0} M\dot{x}_p^2 dt.
\]  

(23)

After some algebra, from (13) and (20) one may find the relation
\[
\gamma_n^F(N\tau_0) = \pi(n + \frac{1}{2})N[1 - 2C\cos\beta + C^2] + 2\pi \frac{N^3p^2}{\hbar Mw^3} \sum_{n=\infty}^{\infty} \frac{n^2|f_n|^2}{(p^2n^2 - N^2)^2} + 2\pi \frac{MNw}{\hbar} |D|^2.
\]  

(24)
There exist the representations where Berry phase can not be defined even with periodic $x_p$. For example, in the representation of $D \neq 1$, if $\tau_f/\tau_0$ is an irrational number Berry phase can not defined under any evolution of finite time. However, for the SHO with driving force, there is a special representation of $C = 1, \beta = 0$ and $D = 0$ where Berry phase can be defined for any periodic $x_p$. In this representation, Berry phase under $\tau_f$-evolution is written as

$$\gamma_n^F(\tau_f) = \frac{2\pi w_f}{\hbar M} \sum_{n=-\infty}^{\infty} \frac{n^2|f_n|^2}{(n^2w_f^2 - w^2)^2}.$$  \hspace{1cm} (25)

The presence of this special representation is due to the fact that, Berry phase can be defined for any evolution in the stationary representation of the (undriven) SHO system.

5. Conclusion

We have calculated Berry phase of the SHO system in terms of classical solutions, and have explicitly shown that the phase depends on the choice of representations. The Berry phase of undriven system does not depend on the mass or angular frequency of the SHO, while it depends on two parameters which comes from choosing classical solutions.

For a driven system of periodic $x_p$, the Berry phase, if it exists, separates into the contribution from the undriven system and that from the presence of driving force. For the SHO system with driving force, the contribution from the presence of of driving force which is the order of $1/\hbar$ depends on mass, angular frequency, and two more real parameters coming from the way of choosing classical solutions. There is a special representation where Berry phase can be defined for any periodic $x_p$, while in some representations
Berry phase can not be defined. For a general harmonic oscillator of time-dependent mass and frequency with driving force, it may not be the case that there exists such a special representation since stationary representation for the system without driving force may not exist in general.

While classical solutions of the SHO system is $\tau_0$-periodic, the solutions have an additional property that absolute value of any classical solution is $\tau_0/2$-periodic. This fact enables one to define the Berry phase under $\tau_0/2$-evolution as well as $\tau_0$-evolution. For a general harmonic oscillator of time-dependent mass and frequency, if there exists such an additional periodicity in the absolute value of all classical solution, making use of the periodicity, it will be possible to define the Berry phase for the evolution of shorter period than that of the Hamiltonian.

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References

[1] Lewis H R (1968) J. Math. Phys. 9 1976
     Lewis H R (1968) 1967 Phys. Rev. Lett. 18 510

[2] Kruskal R M (1957) Phys. Rev. 106 205

[3] Yeon K H, Lee K K, Um C I, George T F and Pandey L N 1993 Phys.Rev. A 48 2716
     Yeon K H, Kim D H, Um C I, George T F and Pandey L N 1997 Phys.Rev. A 55 4023

[4] Ji J-Y, Kim J K, Kim S P and Soh K-S 1995 Phys. Rev. A 52 3352

[5] Lee M-H, Kim H-C and Ji J-Y 1997 J. Korean Phys. Soc. 31 560
     Kim H-C, Lee M-H, Ji J Y and Kim J K 1996 Phys. Rev. A 53 3767

[6] Song D-Y 1999 Phys. Rev. A 59 2616

[7] R.P. Feynman and A.R. Hibbs 1965 Quantum Mechanics and Path Integrals
     (McGraw-Hill Inc: New York) pp 58-60

[8] Song D-Y 1999 J. Phys. A: Math. Gen. 32 3449

[9] Berry M V 1984 Proc. R. Soc. London Ser. A 392 45

[10] Aharonov A and Anandan J 1987 Phys. Rev. Lett. 58 1593

[11] Song D-Y 1999 Phys. Rev. A submitted (= quant-ph/9907062)
[12] Berry M V and Hannay J H 1988 \textit{J. Phys. A: Math. Gen.} \textbf{21} L325

[13] Ge Y C and Child M S 1997 \textit{Phys. Rev. Lett.} \textbf{78} 2507

[14] Moore D J 1990 \textit{J. Phys. A: Math. Gen.} \textbf{23} 5523

Moore D J 1991 \textit{Phys. Rep.} \textbf{210} 1

[15] Khandekar D C and Lawande S V 1975 \textit{J. Math. Phys.} \textbf{16} 384

[16] W. Magnus and S. Winkler, \textit{Hill’s equation} (Dover, New York, 1979) pp4-8