Controlled Spin Transport in Planar Systems Through Topological Exciton

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It is shown that a charge-neutral spin-1 exciton, possibly realizable only in planar systems like graphene and topological insulators, can be effectively used for controlled spin transport in such media. The effect of quantum and thermal fluctuations yield a parametric excitation threshold for its realization. This planar exciton differs from the conventional ones, as it owes its existence to the topological Chern-Simons (CS) term. The parity and time-reversal violating CS term can arise from quantum effects in systems with parity-breaking mass-gap. The spinning exciton naturally couples to magnetic field, leading to the possibility of controlled spin transport. Being neutral, it is immune to a host of effect, which afflicts spin transport through charged fermions.

Lower dimensional systems exhibit a host of unique features [1], absent in their higher-dimensional counterparts. Realization of Majorana fermion [2], thin film topological insulators (TIs) [3], charge density wave [4], helical transport [5] and charge-spin separation [6] characterize one-dimensional systems; whereas planar materials have led to physical realization of large spin-orbit coupling [7], spin-momentum locking [8] and low-energy Dirac fermions [9] etc. The planarization of large spin-orbit coupling [7], spin-momentum locking [8] and lattice curvature couple as [32] through the CS term. The fact that the in-plane optical phonon of topological origin, unlike fermionic spin transport, arising from the CS term, is essential for generating the CS term, not present in natural electromagnetism.

Quite some time back, Hagen demonstrated [22] the existence of a spin-1 exciton, a weakly bound fermion-antifermion pair, of topological origin in 2+1 dimensional CS gauge theory, arising due to the CS term. Though excitons have recently been observed in graphene nano-ribbons [42] and in TIs [43], they are not of topological origin. Further, the former being a spin-1 object, naturally couples to magnetic field, leading to possible controlled spin-transport, which is fundamentally different from the spin-transport already realized in graphene [44] and in TIs [45], through spin $\frac{1}{2}$ charged particles.

Interestingly in graphene, in-plane optical phonons [46, 47] and defects induced lattice curvature [47, 48] couple to the low-energy relativistic fermions, as minimally-coupled dynamic $U(1)$ gauge fields. This has led to ‘non-adiabatic’ effects, behind ‘phonons behaving badly’ [49]. Ensuring zero charge-density of the ground state, which can physically manifest as a chemical potential [50], a temporal component can be introduced [51] to this spatial gauge field, as a chemical potential, enabling a conventional covariant description. The physical conclusions, however, are independent of this covariant description. The characteristic structure of gauge self-energy in 2+1 dimensions, with a logarithmic singularity leads to a characteristic ‘Kohn anomaly’ [52]. This has been experimentally observed [53], and realized in graphene through electron-phonon interaction [54].

In this letter, we demonstrate the possible realization of controlled spin transport, through charge-less, spin-1 excitons of topological origin, unlike fermionic spin transport, arising through the CS term. The fact that the in-plane optical phonon and lattice curvature couple as $U(1)$ gauge fields to the emergent Dirac fermions of graphene, leads to possible realization of the topological CS term, along with a dynamic gauge part of the Maxwell type $(-\frac{1}{4} F_{\nu\mu} F^{\mu\nu})$, through quantum corrections. For this purpose, Hagen’s model has been generalized to include the dynamic part, incorporating quantum fluctuations, that leads to a parametric threshold for excitation of this topological quasi-particle. A further finite-temperature analysis tests its stability against thermal fluctuations also, inferring about its melting. This exciton is further shown to affect topo-

\[ \mathcal{L}_{\text{CS}} = -\frac{\mu}{2} \epsilon^{\mu\nu\rho} a_\rho \partial_\nu a_\mu, \]

in the gauge sector, due to which, propagating photons can have gauge-invariant mass [19, 21]. Interacting particles can acquire additional spin [20, 22, 23], leading to a change in their statistics [21, 24–28]. The CS term arises due to quantum effects, in theories with parity-breaking massive fermions [19, 20, 29–31] and bosons [32]. The advent of planar materials like graphene [9, 33, 34] and TIs [15, 17] have led to physical realization of low-energy ‘relativistic’ planar systems. Massive fermions have been experimentally realized by inducing sub-lattice density asymmetry in bi-layer graphene under transverse electric field [35, 36] and in graphene monolayer, through miss-aligning with the substrate [37]. Recently proposed ‘penta-graphene’ displays intrinsic quasi-direct band gap [38], whereas the hexagonal structure of TIs leads to a single Dirac point, with locally inducible gap through magnetic induction [17, 39–41]. The importance of mass term lies in the fact that in 2+1 dimensions, mass-less particles are spin-less. Furthermore, parity-breaking mass term is essential for generating the CS term, not present in natural electromagnetism.

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logical spin transport, controllable through coupling with external magnetic field. Realization of these exclusively planar phenomena, in physical systems like graphene has been discussed in detail. The transport of the proposed charge-neutral exciton remain unaffected by charge-dependent phenomena.

In 2+1 dimensions, the CS term leads to a gauge-invariant massive photon possessing unit spin. The effective gauge action is obtained by integrating out the minimally coupled fermionic field, with the lowest order (1-loop) quantum contribution expressed through the vacuum polarization tensor, split into parity odd and even parts as \([55]\).

\[ \Pi^{\mu\nu}(q) = \Pi_c(q) \left( q^\mu q^\nu - \eta^\mu\nu q^2 \right) + \Pi_o(q) \epsilon^{\mu\nu\rho} q_{\rho}, \]

with corresponding form factors \(\Pi_{c,o}(q)\) respectively. This further represents the linear response of the system \([50,55]\), which is of interest here. In the low-energy scenario \([50]\, and also from the large-N analysis \([59]\), the topological part is the most dominant quantum contribution, and is unaffected beyond 1-loop contribution from fermions \([64]\).

Following proper regularization, the vacuum polarization form factors are found to be \([61,62]\).

\[
\Pi_c(q) = \frac{e^2}{4\pi} \left[ \frac{1}{|q|} \left( \frac{1}{4} + \frac{m^2}{q^2} \right) \log \left( \frac{2|m| + |q|}{2|m| - |q|} \right) - \frac{|m|}{q^2} \right], \\
\Pi_o(q) = -\frac{i}{4\pi} \frac{e^2}{|q|} \log \left( \frac{2|m| + |q|}{2|m| - |q|} \right). \tag{1} \]

Evidently, both the form factors possess logarithmic singularities at the fermionic two-particle threshold, \(q^2 = 4m^2\), and these expressions are valid below the logarithmic singularity, which is the domain of bound-state formation. The even form factor \(\Pi_c(q)\) influences wave-function renormalization, while the odd one, \(\Pi_o(q)\), is the emergent 1-loop topological contribution in the same sense as \(\mu\).

The modification of dynamics at tree-level due to quantum corrections can be represented through the Schwinger-Dyson equation \([63]\). At the 1-loop level, it leads to the full gauge propagator,

\[ [G_F^{\mu\nu}(q)]^{-1} = [G_F^{(0)}]^{-1} + \Pi^{\mu\nu}(q) - \frac{1}{\xi} q^\mu q^\nu. \tag{2} \]

The first term on RHS is the tree-level contribution, while the last one incorporating covariant \((R_x)\) gauge fixing. The tree-level gauge propagator is solely determined by the Lagrangian \(\mathcal{L}_g = -\frac{i}{4} F_{\mu\nu} F^{\mu\nu} - \xi \epsilon^{\mu\nu\rho\sigma} \partial_\mu a_\sigma \partial_\nu a_\rho\), for the topologically massive dynamic gauge field \(a_\mu\), with field strength tensor \(F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu\) and natural units \(\hbar = c = 1\) been adopted. The non-perturbative 1-loop propagator, incorporating contributions from an infinite number of bubble diagrams, is finally obtained as \([64]\).

\[
G_F^{\mu\nu}(q) \equiv \frac{1}{q^2 \{1 + \Pi_c(q)\}^2 + \{\Pi_o(q) + i\mu\}^2} q^2 \times \left[ (q^\mu q^\nu - \eta^\mu\nu q^2) \{1 + \Pi_c(q)\} - \epsilon^{\mu\nu\rho\sigma} \partial_\rho a_\sigma \right] \Pi_o(q) + i\mu - \frac{\eta_{\mu\nu}}{q^2}, \tag{3} \]

with a non-trivial, gauge-invariant pole corresponding to \(q^2 \{1 + \Pi_c(q)\}^2 + \{\Pi_o(q) + i\mu\}^2 = 0\), which can represent a physical state \([55]\). Such a pole was obtained by Hagen \([22]\), without the dynamic term \(-\frac{2}{3} F_{\mu\nu} F^{\mu\nu}\), just below the fermionic two-particle threshold: \(q^2 \approx (2|m| - \epsilon)^2, \ 0 < \epsilon \ll 1\). Interpreted as a fermion-antifermion spin-1 bound state (exciton), as planar particle and antiparticle have the same spin projection \([31]\). It had a ‘binding energy’ \(\epsilon \approx 4|m| \exp \left( -4\pi\mu/e^2 \right)\), with \(\mu = 1\) numerically. In presence of the dynamic term, we obtain a significantly modified expression:

\[
\epsilon \approx 4|m| \exp \left( \frac{4\pi}{e^2} \left( 2|m| - \mu \frac{|m|}{m} \right) \right). \tag{4} \]

In both these cases, presence of a tree-level CS term, in the effective theory, is necessary for attaining a self-consistent (small magnitude) value of the exciton binding energy, subjected to the smallness of the \(U(1)\) coupling \(\epsilon\). As can be checked with different gauge Lagrangians, for \(\mu = 0\), such a value cannot exist, thereby highlighting the intrinsic topological nature of this exciton.

The smallness of the binding energy (\(\epsilon\)) necessitates a negative exponent, thereby fixing a threshold value \(\mu_t = 2|m|\) for bound-state formation, highlighting the destabilizing effect of vacuum fluctuation on the excitation due to dynamical Maxwellian term. This ‘non-perturbative’ scenario, even with the sense of largeness of coupling strength \(\epsilon\), is justified in terms of the large-N suppression of higher-order contributions. For a negative exponent, a shallow bound state is physically meaningful for small coupling \(\epsilon\), and deep otherwise, maintaining the self-consistency regarding \(\epsilon \ll |m|\). The parametric regions for attaining this exciton is shown in Fig. 1.

As a further check on the physicality of the planar exciton, the dynamic CS Lagrangian is found to be dual to the massive pure CS gauge Lagrangian: \(\mathcal{L}_g = \mu^2 \partial^\alpha a_\alpha - (\mu/2) \epsilon^{\mu\nu\rho\sigma} a_\rho \partial_\nu a_\sigma\), with \(\mu\) being the mass also, at the level of equation of motion \([65]\). The later is not gauge-invariant, reflected in the connecting gauge transformations, singular at \(\mu = 0\) \([66]\). The ponding expression for exciton binding energy this dual system, \(\epsilon = 4|m| \exp \left( 2|\mu/(2m - \mu)/e^2|m| \right)\) depicts the same bound-state threshold \(\mu\). This marks the physical constraint, that unless the energy is above the gauge particle rest-mass, a physical state (the exciton) cannot be excited. Again, it can be seen that the exciton is entirely topological as the photon mass is.

To physically realize this topological exciton, one needs to have a planar theory with parity-breaking mass-gap for the
fermionic fields. This is naturally realized in the surface of topological insulators and in graphene through asymmetry in valley fermions. In graphene, the origin of topological term in the effective theory is traceable to the the fundamental electronic Hamiltonian \[ H = \sum_i \left[ \mathcal{E}_A A_i^\dagger A_i + \mathcal{E}_B B_i^\dagger B_i \right] - t \sum_{\langle i,j \rangle} [A_i B_j + \text{h.c.}], \]

with \( \mathcal{E}_A \neq \mathcal{E}_B \) for non-Dirac fermion mass \( M = (\mathcal{E}_A - \mathcal{E}_B)/2v_F^2 \). The electron hopping has strength \( t \), with \( i,j \) denoting nearest neighbor sites in two different sublattices \((A, B)\) and annihilating \((A, B)\) operators anti-commutes and constitutes the staggered magnetization \((\text{SM})\) operator \( S_z = \frac{1}{2} \sum_{\langle i,j \rangle} [A_i^\dagger A_j - B_i^\dagger B_j] \), representing the anti-ferromagnetic order parameter of the system, encapsulating the ‘net’ spin of graphene, as sub-lattices carry opposite electron spins \([47]\). \( S_z \) is also an \( SU(2) \) generator \([58, 64]\), in the \( K \)-space. The inherent non-locality of hopping \((t \neq 0)\) prevents \( S_z \) to be an observable, yielding,

\[ \left[ \mathcal{H}, S_z \right] = -iv_F \left( \vec{\sigma} \times \vec{k} \right)_z ; \quad \mathcal{H} = v_F \vec{\sigma} \cdot \vec{k} + \sigma_z Mv_F^2, \]

in the low momentum limit. The same result was obtained by considering \( L_z = \vec{r} \times \vec{k} \) in Ref. \([57]\), necessitating the existence of ‘lattice spin’ \( \vec{S}_z = \sigma_z/2 \), so that \( [\mathcal{H}, (L + S)_z] = 0 \). The later is intrinsic to the emergent Dirac theory, as the ‘orbital’ part \((S_z)\) is due to lattice geometry. This directly implies \( M \neq 0 \), that breaks chiral symmetry and leads to the CS term as quantum correction in the gauge sector \([31]\). It identifies \( S_z \) as the Dirac spin \( \left( \frac{M}{2M} \right) \), of purely quantum origin and additive in 2+1 dimensions \([31]\), reflected by the common Pauli-Lubanski pseudo-scalar \( W := k_\mu J^\mu = -\frac{1}{2} Mv_F^2 \sigma_z \). Though fundamental excitations in graphene couple to Dirac fermions as effective gauge fields, they do not constitute a tree-level gauge Lagrangian, necessary for the topological exciton. In principle, such a gauge Lagrangian can be generated by integrating out one of the two massive Dirac fermion species due to the broken valley degeneracy. However, both species of Dirac fermion are of same mass in graphene \([47]\), canceling the individual 1-loop topological contributions \([34]\), unlike the even parts of vacuum polarization which add up, yielding the dynamic part of the effective gauge Lagrangian. This warrants the the fermion species to have different masses, with one integrated-out to yield the effective dynamic CS Lagrangian. Another possibility may be of having odd number \((\geq 3)\) of primary Dirac fermions, with individual CS loop contributions canceled-out in pairs but one. Also, emergent bosons can generate the CS term \([32]\), with the dynamic term generated by a massless fermion. Then the remaining massive Dirac fermion can minimally couple to the effective \( \mu_\mu \), yielding a second vacuum polarization tensor to be included in the SDE, and yield the exciton.

With emergent massive Dirac fermions, a systems like graphene displays modified dispersion of the form: \[ E^2 = p^2 v_F^4 + M^2 v_F^4 \] \([47]\) results in scaling of mass \((m \to Mv_F^2)\) and coupling strength \( e \to gv_F \) due to Fermi velocity \( v_F \ll c \). This changes the expression for binding energy to,

\[ \epsilon \approx 4|MV| v_F^2 \exp \left( \frac{4\pi}{g^2 v_F^2} \left( 2|MV|^2 - \mu \frac{|M|}{M} \right) \right), \]

with \( \mu_t \to \mu_t = 2|MV|^2 \). Interestingly, the replacement \( e^2 \to g^2 v_F^2 \) of coupling, in the denominator of the exponent, makes the transition about this critical value more prominent, as \( v_F \ll 1 \) in natural units. Hence, though both exciton binding energy and the threshold for formation are much smaller in graphene, the sensitivity to the bound state formation will be considerably higher, making experimental verification much more likely.

To see the effect of thermal fluctuation on this topological exciton, a finite temperature treatment \([67]\) is carried out, generalizing already known results for massless fermion \([68]\) and induced topological part \([59]\) in QED\(_2\). Adopting the manifestly covariant real-time formulation \([69]\), the temperature dependent extension to \( \Pi^{\mu\nu} \) in 2+1 dimensions has been evaluated, yielding the full temperature-dependent 1-loop propagator through the SDE, leading to two non-trivial pole equations \([64]\),

\[ \Pi_L - q^2 (\Pi_e + 1) = 0 \]

and

\[ \left[ \Pi_T - q^2 (\Pi_e + 1) \right]^2 + q^2 (\Pi_0 + i\mu)^2 = 0, \]

with no gauge dependence to ensure physical states. The finite temperature form-factors, \( \Pi_{L,T}(q, T) \), are free of any divergences \([64]\), but cannot be evaluated exactly, as known \([67]\). The conventional high-temperature approximation \((T \gg q, m)\) yields,
C10, the expressions for same way as in the zero-temperature case, which is most effects the temperature-independent terms in the exponent in the al does not melt first. observation of the phenomena more likely, given that the crys-

\[ \Pi_L(q, T) \approx \frac{-T}{3\pi} e^2 \log(2) \frac{\frac{|q_0|}{q_0}}{1 - \frac{|q_0|^2}{2}}, \]

\[ \Pi_T(q, T) \approx \frac{T}{2\pi} e^2 \log(2) \left[ 1 - \frac{2}{3} \frac{1}{1 - \frac{|q_0|^2}{2}} \right]. \] (6)

which are linear in temperature, as per dimensional arguments [67]. Consequently, the first pole equation unacceptably shows increase of binding energy with temperature, in the physical domain \(|q| \leq |q_0|\), illustrated in Fig. 2 by plotting temperature-independent terms, \(\Pi_{L,T}(q) = \Pi_{L,T}(q, T)/Te^2\).

On the other hand, the second pole equation correctly represents evaporation of the exciton at sufficiently high temperature. The corresponding expression for exciton binding energy, through near two-fermion threshold expansion of \(\Pi_{L,T}(q, T), \Pi_{\mu}(q, T) [56]\) and \(\Pi_{\epsilon}(q)\), is,

\[ \epsilon \approx 4|m| \exp \left[ -T \frac{4\pi}{|m|} \Pi_t - 8\pi \frac{\mu}{e^2} + 16\pi \frac{|m|}{e^2} - 1 \right]. \] (7)

In the \(T = 0\) limit, it smoothly goes to the previous result, modulo a constant. Although the melting temperature of the exciton cannot be determined exactly through this expansion, a physical estimation of the same may be obtained.

The modified dispersion of graphene-like materials effects the temperature-independent terms in the exponent in the same way as in the zero-temperature case, which is most extensive on the temperature-dependent first term. From Eqs. the expressions for \(\Pi_{L,T}\) gets altered as per \(|q|/|q_0| = 2|m/q_0| \rightarrow 2M/q_0|v_F|^2\) in a complicated way. In the feasible limit \(q^2 \rightarrow 0\), the dominant contribution stays immune to this effect. The term \(|m| \rightarrow |M|/v_F^2\) in the denominator of the exponent ensures that the temperature effect to be more prominent in graphene, and the binding energy is more vulnerable to melting effects. This should make the experimental observation of the phenomena more likely, given that the crystal does not melt first.

The topological excitation with unit spin \(\vec{\mu} / |\vec{\mu}| [20]\) is fit for spin-transport, which is unperturbed by local electric fields in the suitable energy limits, as it is charge-neutral. Subsequent dynamics can be controlled by an external magnetic field \(\vec{B}\), subsequently allowing for spintronics. The corresponding 'external' gauge field \((A_\mu)\) couples to the Dirac fermions with the same strength \(g\), thereby extending the 1-loop gauge effective action, along with the usual CS contributions for \(A_\mu\), by the gauge-invariant mixed CS term [19, 59, 70]:

\[ L_{CS}^m = -\tilde{\Gamma} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu \partial_\rho, \]

in each valley, which add-up [47]. This extension to the Lagrangian effectively means current \(j^\mu = \epsilon^{\mu\nu\rho} \partial_\nu A_\rho\), coupled with 'internal' gauge field \(a_\mu\), with strength \(\epsilon\), physically equivalent to a chemical potential [50]. This 'temporal' gauge field component is unlike the 'spatial' ones \((a_x, a_y)\), proportional to components of relative sub-lattice displacement in the same direction [47], which arise from lattice geometry. However, the CS term has no dynamics, and does not appear in the corresponding Hamiltonian [21]. Therefore, although \(a_\mu\) is essentially a Lagrange multiplier for generating the Gauss law constraint, as the fermionic fields are integrated-out, the effective coupling \(\epsilon^{a_0 j_0}\) represents interaction of bound fermion-antifermion pair, of spin \(\vec{\mu} / |\vec{\mu}|\) with \(B_\perp\), responsible for controlling its transport. Additionally, as spin is a pseudo-scalar in 2+1 dimensions, \(\mu a_\mu j_\mu\) equivalently represents interaction of exciton spin with external gauge momentum, akin to a reduced Pauli-Lubanski 'pseudo-scalar'. Further, integrating out \(a_\mu\) yields an effective Lagrangian \(-\frac{\mu}{4\pi} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}\), for \(A_\mu\), with corresponding field-strength tensor \(\tilde{F}_{\mu\nu}\), that can effect resistance-less transport [51, 71].

In summary, we have demonstrated that a novel exciton exists in planar systems with emergent Dirac fermions, that can couple with phonon-induced gauge fields, yielding dynamic CS gauge Lagrangian. These excitons are intrinsically topological in nature. Appearing in the gauge sector, they are charge-neutral spin-1 excitations, ideal for spin-transport in planar materials like graphene an the surface of topological insulators, where massive Dirac modes have been realized. This exciton is characterized by a parametric threshold condition, reflecting the competition of quantum fluctuations with topological stability, and dissociates at high-temperature in a smooth manner. Further, the dynamics of this exciton can be controlled by an external magnetic field, leading to spin-transport. Bound states of fermions, possessing different spins naturally manifest in various branches of physics. Pseudoscalar pions have been realized as composite protons and neutrons in the Nambu-Jona-Lasinio model [72]. The parity and time-reversal breaking low-energy topological Wess-Zumino-Novikov-Witten term [73] correctly describes the interactions of these particles in 3+1 dimensions. The observation of topological exciton in a planar system will not only...
demonstrate the gauge-invariant mass of a propagating spin-1 bound state, but will also physically demonstrate the corresponding spin arising from topology, the fact that, the spin carrying bound state is neutral and its spin is of topological origin, ensures its stability against charge-dependent forces and both quantum and thermal fluctuations.

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Appendix A: Schwinger’s Angular Momentum Algebra for Graphene

In graphene, massive Dirac fermions follows from the fundamental electronic Hamiltonian [57],

\[ H = \sum_i \left[ E_A A_i^\dagger A_i + E_B B_i^\dagger B_i \right] - t \sum_{<i,j>} \left[ A_i^\dagger B_j + \text{h.c.} \right], \]

with \( E_A \neq E_B \) for non-zero Dirac fermion mass \( mv_F^2 = (E_A - E_B)/2 \), where \( v_F \) is the fermi velocity. The electron hopping term has strength \( t \), and \( i, j \) denote nearest neighbour sites at two different sub-lattices \((A, B)\). The respective fermionic creation \((A_i^\dagger, B_i^\dagger)\) and annihilation \((A_i, B_i)\) operators, anti-commute, to yield the following Schwinger’s angular momentum algebra [58]

\[
[S_+, S_-] = 2S_z, \quad [S_z, S_\pm] = \pm S_\pm; \quad \text{where}
L_+ = \sum_i A_i^\dagger B_i, \quad S_- = \sum_i B_i^\dagger A_i &
S_z = \frac{1}{2} \sum_i \left[ A_i^\dagger A_i - B_i^\dagger B_i \right].
\]

With \( S_z \) represents the staggered magnetization (SM) of graphene electrons, which is the anti-ferromagnetic order parameter. The adoption of the nearest neighbour approximation yields,

\[
[H, S_\pm] = \pm 2 \left[ mv_F^2 S_\pm + t S_z \right],
[H, S_z] = t \sum_{<i,j>} \left[ A_i^\dagger B_j - \text{h.c.} \right],
\]

(A1)

where the \( SU(2) \) generators are defined in the product-space of sub-lattice operators. The \( K \)-space forms of these generators yield a \((2 \times 2) SU(2)\) representation. The inherent non-locality of hopping \((t \neq 0)\) prevents \( S_z \) to be an observable, leading to,

\[
[H, S_z] = -i v_F \left( \vec{\sigma} \times \vec{k} \right)_z \quad ;
\]

(A2)

\[
H = v_F \vec{\sigma} \times \vec{k} + \sigma_z mv_F^2, \quad S_z = \frac{1}{2} \sigma_z,
\]

in the low momentum limit. The same result was obtained by considering \( L_z = \vec{r} \times \vec{k} \) in Ref. [57], necessitating the existence of ‘lattice spin’ \( \vec{S}_z = \sigma_z/2 \), so that \( [\mathcal{H}, J_z] = 0, J_z = (L + \vec{S})_z \).

Appendix B: Exciton in Maxwell QED\(_3\) \((T = 0)\)

The role of the tree-level gauge Lagrangian,

\[
\mathcal{L}_g = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{\mu}{2} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho, \quad \text{(B1)}
\]

is to determine the tree-level propagator,

\[
G_{F0}^{\mu\nu} = -\frac{1}{q^2 - \mu^2} \left[ q_\mu q_\nu - \frac{q^2}{q^2} - i \frac{\mu}{q} \epsilon^{\rho\mu\nu} q_\rho \right] - \xi q^\mu q^\nu, \quad \text{(B2)}
\]

with covariant \( R_\xi \) gauge depicted by the last term. It modifies the 1-loop Schwinger-Dyson equation (SDE) [63]:

\[
\left[ G_{F0}^{\mu\nu}(q) \right]^{-1} = \left[ G_{F0}^{\mu\nu}(q) \right]^{-1} + \Pi_0^{\mu\nu}(q) - \frac{1}{2} \xi q^\mu q^\nu. \quad \text{(B3)}
\]

thereby effecting the pole structure of the full propagator \( G_{F0}^{\mu\nu}(q) \). The full propagator gets contribution from a series of infinite vacuum polarization [\( \Pi_0^{\mu\nu}(q) \)] terms, making the result non-perturbative [59]. On integrating out the fermion field from the full Lagrangian,

\[
\mathcal{L} = \bar{\psi}(x) \left( i \gamma^\mu \partial_\mu - m \right) \psi(x) - e \bar{\psi}(x) \sigma^\mu(x) \psi(x) + \mathcal{L}_g, \quad (h = c = 1) \quad \text{(B4)}
\]

, the vacuum polarization contribution to the effective gauge action is obtained as,

\[
\Pi_0^{\mu\nu}(q) = ie^2 Tr_D \int \frac{d^3 p}{(2\pi)^3} \left[ \gamma^\nu S_F(p_+) \gamma^\mu S_F(p_-) + \gamma^\nu \frac{\partial}{\partial p_\nu} S_F(p) \right]; \quad \text{(B5)}
\]

\[
S_F(p) = \left( \frac{\gamma^\mu \mu - m}{p_+} \right)^{-1}, \quad p_\pm = p \pm \frac{q}{2}, \quad \mu \equiv i \partial_\mu,
\]

with the Schwinger regularization [22] adopted in the second term in the integrand that removes the UV divergence for zero gauge momentum \((q = 0)\). The rest of the integral is standard and can be evaluated by the derivative expansion method [56], to obtain [61, 62],

\[
\Pi_0^{\mu\nu}(q) \equiv \Pi_0^{\mu\nu}(q) + \Pi_0^{\mu\nu}(q),
\]

\[
\Pi_0^{\mu\nu}(q) = -\Pi_0(q) Q^{\mu\nu}, \quad \Pi_0^{\mu\nu}(q) = \Pi_0(q) e^{\mu\nu\rho} q_\rho,
\]

\[
Q^{\mu\nu} = \eta^{\mu\nu} q^2 - \eta^{\mu\nu} q^2, \quad \eta^{\mu\nu} = \text{diag}(1, -1, -1),
\]

\[
\Pi_0(q) = \frac{e^2}{4\pi} \left[ \frac{1}{|q|} \left( \frac{m^2 + |q|}{4q^2} \right) \log \left( \frac{2|m| + |q|}{2|m| - |q|} \right) - \frac{m^2}{q^2} \right],
\]

\[
\Pi_0(q) = -i \frac{me^2}{4\pi |q|} \log \left( \frac{2|m| + |q|}{2|m| - |q|} \right). \quad \text{(B6)}
\]
The parity-odd contribution $\Pi_{\mu\nu}^e(q)$, unique to 2+1 dimensions [29] arising due to non-zero trace of three Dirac matrices, is the induced CS contribution. The parity even contribution $\Pi_{\mu\nu}^o(q)$ is responsible for wave-function renormalization [29]. The plots of both the form factors $|\Pi_{e,o}(q)|$ are shown in Fig. 3 with the well-known singularities at the two-particle threshold: $q^2 = 4m^2$. The above results are valid below the same, and requires the replacement:

$$\log\left(\frac{2|m| + |q|}{2|m| - |q|}\right) \rightarrow \log\left(\frac{2|m| + |q|}{2|m| - |q|}\right) - i\pi,$$

above it, owing to the corresponding branch-cut. For the Maxwell CS QED, the tree level and full Following the same steps as before, one obtains [62],

$$G_{F}^{\mu\nu}(q) \equiv \frac{1}{q^2\{1 + \Pi_e(q)\}^2 + \{\Pi_o(q) + i\mu\}^2} q^2 \times \left\{ (q^\mu q^\nu - q_{\mu\nu}^2) \{1 + \Pi_e(q)\} - e^{\mu\nu}\rho \{\Pi_o(q) + i\mu\} \right\} - \epsilon q^\mu q^\nu \frac{\xi q\rho}{q^2}. \quad \text{(B7)}$$

This leads to the non-trivial pole governed by,

$$q^2\{1 + \Pi_e(q)\}^2 + \{\Pi_o(q) + i\mu\}^2 = 0. \quad \text{(B8)}$$

The near two fermion threshold is parametrized as $|m|/|q| = 0.5$. Here $e^2 = 1$.

$\Pi_e(q) \approx \frac{1}{4\pi} \left[ \frac{1}{2|m|} \left( \frac{1}{4} + \frac{1}{4} \right) \log\left(\frac{4|m|}{\epsilon}\right) - \frac{1}{4|m|}\right]$

$$\approx \frac{1}{16\pi|m|} \left[ \log\left(\frac{4|m|}{\epsilon}\right) - 1 \right]$$

and

$$\Pi_o(q) \approx -i \frac{m}{8\pi|m|} \log\left(\frac{4|m|}{\epsilon}\right). \quad \text{(B9)}$$

Then, from Eq. [B8] the binding energy of the exciton is,

$$\epsilon \approx 4|m| \exp \left\{ \frac{4\pi}{\epsilon^2} \left( 2|m| - \frac{m^2}{m} \right) \right\}. \quad \text{(B10)}$$

As expected, the above expression of binding energy yields Hagen’s result as a special case, without the first term in the exponent.

### Appendix C: At finite temperature

The extension of above system to finite temperature has been carried out in this section, both in real [74] and imaginary time [75] formalisms. In 2+1 dimensions, results were known for massless fermion case [68] and induced CS term [56] at low energies. We here obtain the general results, and show the known results at finite temperature and those obtained earlier at zero temperature to be the special cases of the prior in suitable limits.

#### 1. Real time formalism

In the real-time formalism of QFT, the fermionic propagator at $T \neq 0$ is obtained from that at $T = 0$ through the Bogoliubov unitary transformation [74] as,

$$G(q) = U_B(\beta) G_0(q) U_B^\dagger(\beta) = G_0(q) + G_\beta(q)$$

$$\equiv \frac{1}{\gamma.q + m + 2i\pi n_F(q^0) (\gamma.q - m) \delta(q^2 - m^2)}, \quad \beta = 1/T. \quad \text{(C1)}$$

On separating-out the zero-temperature contribution, the finite temperature correction is identified as,

$$\Pi_{\beta}^{\mu\nu}(q) \equiv iT \int_p \left[ \gamma^\mu G_o(p_+) \gamma^\nu G_\beta(p_-) + \gamma^\mu G_\beta(p_+) \gamma^\nu G_o(p_-) \right],$$

$$p_\pm = p \pm \frac{q}{2}, \quad p_\mu = i\partial_\mu. \quad \text{(C2)}$$

We first proceed to obtain the contribution to the even part of vacuum polarization, for which we follow the formalism by Weldon [69], originally carried-out in 3+1 dimensions. Introduction of finite temperature includes the notion of a thermal
bath as a physical reference frame, thus breaking the manifest Lorenz co-variance. However, it is possible to obtain a Lorentz covariant formulation by projecting onto and out of the bath coordinate vector \( \{ u^{\mu} \} \) \textsuperscript{[67]}. Then the temperature dependent part of the even component of vacuum polarization can be expressed as,

\[
\Pi^{\mu \nu} (q, T) = \Pi_T (q, \omega) P^{\mu \nu} + \Pi_L (q, \omega) R^{\mu \nu},
\]

\[
P^{\mu \nu} = \eta^{\mu \nu} - u^{\mu} u^{\nu} + \frac{q^{\mu} q^{\nu}}{Q^2},
\]

\[
R^{\mu \nu} = -\frac{1}{q^2 Q^2} (Q^2 u^{\mu} + q^{\nu}) (Q^2 u^{\nu} + q^{\mu}),
\]

\[
\omega = q \cdot u, \quad Q^2 = \omega^2 - q^2,
\]

\[
\bar{q}^{\mu} = q^{\mu} - \omega u^{\mu}.
\]

(C3)

Here \( P^{\mu \nu} \) is transverse and \( R^{\mu \nu} \) is longitudinal in nature and are orthogonal to each-other. The thermal form-factors are expressed as:

\[
\Pi_L (q, \omega) = -\frac{q^2}{Q^2} \eta_{\mu \nu} u^{\mu} \Pi^{\mu \nu},
\]

\[
\Pi_T (q, \omega) = -\frac{1}{2} \Pi_L (q, \omega) + \frac{1}{2} \eta_{\mu \nu} \Pi^{\mu \nu},
\]

\[
Re (\eta_{\mu \nu} \Pi^{\mu \nu}) = e^2 G_f (q, \omega),
\]

\[
Re (u_{\mu} u_{\nu} \Pi^{\mu \nu}) = e^2 H_f (q, \omega).
\]

(C4)

The real part of \( \Pi^{\mu \nu} \) is of interest here, as the imaginary part of the same corresponds to the induced CS term, which was obtained by Babu et al \textsuperscript{[54]}, along with the \( T = 0 \) contribution, in 2+1 in the high temperature limit \( T \gg q, m \) suitable for present discussion. We will obtain the general result in the imaginary-time formalism shortly, and show the mentioned result as a limiting case.

From Eqs. \textsuperscript{C3} and \textsuperscript{C4} one obtains:

\[
G_f (q, T) \equiv \frac{1}{\pi} \sum_{s=-1}^{1} \int_{0}^{\infty} \frac{q_0 \omega_p + s 2 m^2}{|q|\sqrt{4(\omega_p^2 - s q_0 \omega_p - m^2) + q^2}} \frac{dq_p}{e^{\omega_p/T} + 1} \quad \&
\]

\[
H_f (q, T) \equiv \frac{1}{3 \pi} \sum_{s=-1}^{1} \int_{0}^{\infty} \frac{q_0 \omega_p + s (2 m^2 + \frac{q_2}{4} q^2 - \frac{3}{4} q_0^2)}{|q|\sqrt{4(\omega_p^2 - s q_0 \omega_p - m^2) + q^2}} \frac{dq_p}{e^{\omega_p/T} + 1},
\]

where \( \omega_p^2 = p^2 + m^2 \). The finite temperature contribution does not introduce additional divergences \textsuperscript{[67]}. It is well-known that the above \( \omega_p \)-integrals cannot be solved exactly, a problem with analogue in the imaginary-time formulation too, as we will see shortly. As mentioned before, we ought to take the high temperature (T) limit with \( \omega_p := x T \), finally yielding,

\[
\Pi_L (q) \equiv -\frac{T}{\pi} e^{2} \log(2) \frac{|q| q_0}{q^2} \quad \&
\]

\[
\Pi_T (q) \equiv \frac{T}{2 \pi} e^{2} \log(2) \left[ \frac{|q| q_0}{q^2} + 3 \frac{|q|}{q_0} \right].
\]

(C5)

These contributions are linear in \( T \), as in the hard thermal loop (HTL) approximation for 2+1. This is expected from dimensional arguments as the 3+1 counterparts are quadratic in \( T \) \textsuperscript{[76]}.

To obtain the 1-loop gauge propagator at finite temperature, the obtained finite temperature results are to be moulded in suitable forms. By choosing \( u = (1, 0, 0) \),

\[
\frac{\Pi^{\mu \nu} (q, T)}{q^2} = \frac{1}{q^2} \left( \Pi_T - \frac{q^2}{q^2} \Pi_L \right) q^\mu q^\nu
\]

\[
- \frac{q_0}{q^2} (\Pi_T - \Pi_L) (q^\mu u^\nu + u^\mu q^\nu)
\]

\[
+ q^2 \frac{q^2}{q^2} (\Pi_T - \Pi_L) u^\mu u^\nu,
\]

(C6)

which is to be substituted in the finite temperature 1-loop SDE,

\[
\{ G^{\mu \nu}_F (q, T) \}^{-1} \equiv \{ G^{\mu \nu}_F (0, q, T) \}^{-1} + \Pi^{\mu \nu}_L (q, 0) + \Pi^{\mu \nu}_T (q, T)
\]

\[
+ \Pi^{\mu \nu}_0 (q, T) + \frac{1}{\xi} q^\mu q^\nu,
\]

\[
\Pi^{\mu \nu}_0 (q, T) = -i \frac{m e^2}{8 \pi |m|} \tanh \left( \frac{|m|}{2 T} \right) e^{\mu \rho} q_\rho.
\]

(C7)

Here, \( \Pi^{\mu \nu}_0 (q, T) \) is the complete odd 1-loop contribution at finite temperature \textsuperscript{[56]}, at suitable high temperature and low momentum limit. In that domain, up to \( O(1/T) \), the form factor \( \Pi_0 (q, T) \) is independent of the external momentum \( q \) as has been checked explicitly. The full tree level gauge propagator at finite temperature (including zero temperature contributions), \( G^{\mu \nu}_F (q, T) \), is obtained through the replacement \textsuperscript{[77]},

\[
- \frac{1}{q^2 - m^2} \rightarrow - \frac{1}{q^2 - m^2} - \frac{2 \pi}{q^2} \delta (q^2 - m^2),
\]

in the over-all factor. The finite-temperature contribution vanishes for \( q^2 \neq m^2 \), where \( m_0 \) is the mass of the gauge particle. Since \( q \) is the external momentum, and we eventually are interested in the region just below the two-particle threshold, this contribution can be neglected from the onset as either \( m_0^2 = 0 \) or \( m_0^2 = \mu^2 < 4 m^2 \), with quantum corrections included. Thus, one can work with the zero-temperature tree-level propagator, which have been verified directly too.

The expressions for temperature-dependent exciton binding energy, \( \epsilon \), that we are going to obtain are expected to yield zero temperature results in the smooth limit \( T \rightarrow 0 \).
In presence tree-level propagator, one obtains,

\[ G^{\mu\nu}(q) = a(q)\eta^{\mu\nu} + b(q)q^\mu q^\nu + c(q)u^\mu u^\nu + d(q)(q^\mu u^\nu + u^\mu q^\nu) + e(q)\epsilon^{\mu\nu\rho\sigma}q_\rho; \]

\[ a(q,T) = \frac{\Pi_T - q^2 (\Pi_e + 1)}{[\Pi_T - q^2 (\Pi_e + 1)^2 + q^2 (\Pi_e + i\mu)^2]^2}, \]

\[ b(q,T) = \frac{\Pi_L - q^2 (\Pi_e + 1)}{q^2 [q^2 (\Pi_e + 1) - \Pi_L]} a(q,T) \]

\[ + \frac{\epsilon}{q^2}, \]

\[ c(q,T) = -\frac{q^2}{q^2} \frac{\Pi_T - \Pi_L}{\Pi_L - q^2 (\Pi_e + 1)} a(q,T), \]

\[ d(q,T) = \frac{q_0}{q^2} c(q,T), \]

\[ e(q,T) = -\frac{(\Pi_L + i\mu)}{\Pi_T - q^2 (\Pi_e + 1)} a(q,T), \] (C8)

The gauge dependence retains its covariant form in b(q). Also, c(q) and d(q) vanishes at zero temperature and the corresponding full propagator is retained.

At finite temperature, there are two non-trivial poles, instead of a single one at zero temperature, which are given by,

\[ \Pi_L - q^2 (\Pi_e + 1) = 0 \quad \text{and} \quad [\Pi_T - q^2 (\Pi_e + 1)]^2 + q^2 (\Pi_e + i\mu)^2 = 0. \] (C9)

Near the two-particle threshold, the \(\epsilon\)-expansion yields,

\[ \Pi_e(q,T) \approx \frac{\beta e^2}{8\pi m}, \quad \Pi_L(q,0) \approx \frac{e^2}{16\pi|m|} \log(4|m|/\epsilon) - 1, \]

\[ \Pi_L(q,T) \approx -\frac{T}{3\pi} e^2 \log(2) \left(\frac{q_0}{q^2}\right)^3 \frac{1}{1 - \frac{q_0}{q^2}}, \]

\[ \Pi_T(q,T) \approx \frac{T}{2\pi} e^2 \log(2) \left(\frac{q_0}{q^2}\right)^3 \frac{1}{3 - \frac{q_0}{q^2}} \left[ 1 - \frac{2}{3} \frac{q_0^2}{q^2} \right]. \] (C10)

For Maxwell CS QED\(_3\), the second of the Eqs. [C9] leads to the binding energy:

\[ \epsilon \approx 4|m| \exp \left[ -T \frac{4\pi}{|m|} \Pi_e - 8\pi \frac{\mu}{e^2} + 16\pi \frac{|m|}{e^2} - 1 \right], \]

\[ Te^2\Pi_T = \Pi_{L,T}. \] (C11)

2. The imaginary time formalism

It is instructive to verify finite temperature results in both real and imaginary time formalisms [56], owing to the non-trivial analytic behaviour of the results over wick rotation. The vacuum polarization for QED\(_3\) was carried out first in [68] for massless fermions. We extend the same to the massive case. To this end, the ‘usual’ definition of vacuum polarization is adopted:

\[ \Pi^{\mu\nu}(q) = ie^2 T_\gamma \int \frac{d^3 p}{(2\pi)^3} \gamma^\mu S_F(p) \gamma^\nu S_F(p - q), \]

\[ S_F(p) := [\gamma^\mu p_\mu - m], \quad p_\mu := i\partial_\mu, \] (C12)

where, in contrast with Eq. [B5], we have adopted the definition non-symmetric in \(q\), as per [68], and left out the Schwinger regularization term for brevity. The formalism is as follows: upon the wick rotation, the fermion variables are,

\[ p_E = (p_3, \vec{p}), \quad p_3 = (2n + 1) \frac{\pi}{\beta}, \]

\[ n = 0, \pm 1, \pm 2..., \quad \beta = 1/T, \]

\[ p_3^2 = -p_0^2, \quad p_E^2 = p_3^2 + p^2 = -p^2, \]

and the boson variables are,

\[ q_E = (q_3, \vec{q}), \quad q_3 = 2n \frac{\pi}{\beta}, \]

\[ r = 0, \pm 1, \pm 2..., \quad \beta = 1/T, \]

\[ q_3^2 = -q_0^2, \quad q_E^2 = q_3^2 + q^2 = -q^2. \]

The manifest co-variant projections of vacuum polarization tensor, in Euclidean space, are introduced as,

\[ \Pi^{\mu\nu}(q_E,\beta) = \Pi_A(q_E,\beta) A^{\mu\nu}(q_E) + \Pi_B(q_E,\beta) B^{\mu\nu}(q_E), \]

\[ A^{\mu\nu}(q_E) := \left( \delta^{\mu 3} - \frac{q_0^\mu q_3^\nu}{q_E^2} \right) \left( \delta^{\nu 3} - \frac{q_0^\nu q_3^\mu}{q_E^2} \right), \]

\[ B^{\mu\nu}(q_E) := \delta^{\mu 3} \left( \delta^{\nu 3} - \frac{q_0^\nu q_3^\mu}{q_E^2} \right) \delta^{\mu\nu}, \]

\[ A^{\mu\nu} + B^{\mu\nu} \equiv \delta^{\mu\nu} - \frac{q_0^\mu q_3^\nu}{q_E^2} = \frac{1}{q^2} \delta^{\mu\nu}, \]

\[ \delta^{\mu\nu} = -\eta^{\mu\nu}. \]

The last line of above equations leads to the physical constraint that, at \(T = 0\),

\[ \Pi_A(q_E,\beta = \infty) = \Pi_B(q_E,\beta = \infty) \equiv \Pi_e(q) \] (C13)

From the definitions, it is obtained that,

\[ \Pi_A(q_E,\beta) = \frac{q_0^2}{q_E^2} \Pi^{00}(q_E,\beta) \quad \text{and} \]

\[ \Pi_B(q_E,\beta) = -\Pi^{i i}(q_E,\beta) - \frac{q_0^2}{q_E^2} \Pi^{00}(q_E,\beta), \] (C14)

where repeated indices mean summation, unless mentioned otherwise.
Therefore it boils down to the evaluation of temporal and spatial components of vacuum polarization tensor. First we will obtain those for the even part, which is,

\[
\Pi^{\mu\nu}(q) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2 - a^2)^2} \left[ 2p^\mu p^\nu 
+ 2x(1 - x)Q^{\mu\nu} - \eta^{\mu\nu}(p^2 - a^2) 
- (2x - 1)(q^{\mu\nu} p.q - p^{\mu}q^{\nu} - p^{\nu}q^{\mu}) \right],
\]

where we have utilized the Feynman integration trick with the shift \( p \to p + q x \). Upon continuing to Euclidean space by the rules:

\[
p_0 = ip_3, \quad p^2 = -p^2_E, \quad q^2 = -q^2_E, \\
p.q = -p^*_E q_E, \\
Q^{\mu\nu} = Q^{\mu\nu}_E = \delta^{\mu\nu} q^2_E - q^{\mu\nu}, \\
a^2 = a^2_E = m^2 + x(1 - x)q^2_E, \\
\int \frac{dp_3}{2\pi} \rightarrow \frac{1}{\beta} \sum_{n = -\infty}^{\infty}, \quad p_3 = \frac{2\pi}{\beta} (n + X), \\
X := \frac{1}{2} + xr
\]

With these definitions, we can express \([68]\).

\[
\Pi^{00}(q_E, \beta) = -2q^2_E \frac{e^2}{\beta q^2} \int \frac{d^3p}{(2\pi)^3} \left[ S_1 - 2p^*_3 S_2 
- 2x(1 - x)q^2 S_2 - (2x - 1)q^*_3 S^*_2 \right], \\
\Pi^{ii}(q_E, \beta) = -2q^2_E \frac{e^2}{\beta q^2} \int \frac{d^3p}{(2\pi)^3} \left[ 2x(1 - x) \left( q^2_E + q^2_3 \right) S_2 
- 2a^2_E S_2 - 2p^*_3 S_2 + (2x - 1)q^*_3 S^*_2 \right],
\]

where:

\[
S_i := \sum_{n = -\infty}^{\infty} \frac{1}{p^2_E + a^2_E}, \quad i = 1, 2 \\
S^*_i := \sum_{n = -\infty}^{\infty} \frac{p^*_3}{p^2_E + a^2_E}.
\]

For the massive fermion, the frequency sums followed by 2-momentum integrals yields,

\[
\int \frac{d^2p}{(2\pi)^2} S_1 = -\frac{1}{4\pi} \log \left| \frac{\sin(\pi X^m_m)}{\sin(\pi X^m_E)} \right|, \\
\int \frac{d^2p}{(2\pi)^2} S_2 = \frac{\beta}{16\pi} Im \cot(\pi X^a_a), \\
\int \frac{d^2p}{(2\pi)^2} S^*_a = \frac{\beta}{16\pi} \left[ \sin(2\pi X) - R_e \cot(\pi X^a_a) \right], \\
\int \frac{d^2p}{(2\pi)^2} p^*_3 S^*_a = \frac{1}{4} Im \left[ \frac{\sin(\pi X^m_m) \cos(\pi X^m_a)}{\sin(\pi X^m_E)^2} \right] \\
+ \cot(\pi X^a_a) \right]; \\
X^a,m = \frac{1}{2} + xr \pm \frac{i\beta}{2\pi} (a, m), \quad r \in \mathcal{N}, \\
a^2 = m^2 - x(1 - x)q^2 \equiv a^2_E.
\]

Finally, the imaginary-time form factors are obtained as,

\[
\Pi_A = \frac{e^2 q^2_E}{\beta q^2} \int \frac{1}{2\pi} \log \left| \frac{\sin(\pi X^m_m)}{\sin(\pi X^m_E)} \right| \\
- \frac{1}{2\pi} \log \left| \frac{\sin(\pi X^m_m) \cos(\pi X^m_a)}{\sin(\pi X^m_E)^2} \right| + \cot(\pi X^a_a) \right]\bigg] \\
+ \frac{e^2}{8\pi} \int \frac{1}{2\pi} \log \left| \frac{\sin(\pi X^m_m) \cos(\pi X^m_a)}{\sin(\pi X^m_E)^2} \right| + \cot(\pi X^a_a) \right]\bigg] \\
+ \frac{e^2}{8\pi} \int \frac{1}{2\pi} \log \left| \frac{\sin(\pi X^m_m) \cos(\pi X^m_a)}{\sin(\pi X^m_E)^2} \right| + \cot(\pi X^a_a) \right]\bigg] \\
+ \frac{e^2}{8\pi} \int \frac{1}{2\pi} \log \left| \frac{\sin(\pi X^m_m) \cos(\pi X^m_a)}{\sin(\pi X^m_E)^2} \right| + \cot(\pi X^a_a) \right]\bigg].
\]

The first terms in both \( \Pi_{A,E} \) have been separated-out for clarity. In the high-temperature limit, they carry the only temperature dependence, which is linear in \( T = 1/\beta \), like in the real time formalism. The \( x \)-integrals cannot be evaluated exactly, as is known \([67]\). For \( T = 0 \), \( \Pi_A = \Pi_B \) as required, modulo an additional term in \( \Pi_B \), known to arise for the fermion-antifermion pair of the loop in vacuum polarization having the same mass \([67]\).

The real- and imaginary-time form factors are related through:

\[
\Pi_T = \Pi_A + q^2 \Pi_e, \quad \Pi_L = \Pi_B + q^2 \Pi_e,
\]

leading to the same finite temperature poles are as before, with proper expressions obtained in the limit \( T \gg q, m \).
limit, it yields the corresponding expression in Eqs. 37. This ensures consistency of our treatment. Further, in the high temperature limit (β → 0), the dominant O(β) term is independent of q, and is the same as that for q = 0 60. Thus, one can work with its expression in Eqs. 10 near the two-particle threshold.

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