EQUIVARIANT MIRRORS AND THE VIRASORO CONJECTURE FOR FLAG MANIFOLDS

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Abstract. We found an explicit description of all $GL(n, \mathbb{R})$-Whittaker functions as oscillatory integrals and thus constructed equivariant mirrors of flag manifolds. As a consequence we proved the Virasoro conjecture for flag manifolds.

1. Introduction

A general quintic hypersurface in $\mathbb{C}P^4$ is a compact Calabi-Yau threefold. Its rational Gromov-Witten invariants had been predicted by mirror symmetry discovered in string theory [1]. The prediction was proven by Givental [7], which is to be explained as follows. Let $n_d$ be the virtual number of degree $d$ rational curves in the quintic threefold and let $F(q) = 1 + \frac{1}{5} \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d}$. The Givental $J$-function $J$ for the quintic hypersurface, satisfies the so-called quantum differential equation

$$\left(h \frac{d}{dt}\right)^2 \frac{1}{F(q)} \left(h \frac{d}{dt}\right)^2 J = 0$$

where $t = \ln q$ and $h$ is a formal parameter. The prediction was that the solutions to the quantum differential equation coincide with those to the Picard-Fuchs differential equation of the mirror family after the explicit mirror transformation. The periods $I$, the solutions to the Picard-Fuchs differential equation, are

$$I = \int_{\Gamma} d(x_1 + \ldots + x_5 - 1) \wedge d(x_1 \ldots x_5 - q)$$

where $\Gamma$ are real 3-cycles in the affine varieties $\{x_1 + \ldots + x_5 = 1, x_1 \ldots x_5 = q\}$ birational to the mirror manifolds.

In 1993 Givental proposed a generalization of this mirror phenomenon to non-Calabi-Yau manifolds. A mirror family of a Fano manifold
X is by definition a stationary phase integral representation

\[ I = \int_{\Gamma \subset \mathcal{Y}_q} e^{f_q / \hbar \omega_q} \]

of solutions to the quantum differential equations of X up to change of coordinates, where \( q \in H^2(X, \mathbb{C})/2\pi \sqrt{-1}H^2(X, \mathbb{Z}) \). Here \( f_q \) is a holomorphic function on a possibly noncompact variety \( \mathcal{Y}_q \) and \( \omega_q \) is a holomorphic volume form on \( \mathcal{Y}_q \). The mirror theorem has been established for Calabi-Yau or Fano complete intersections \( X \) in a projective space \([7]\) and further generalized to the case when the ambient space is a Fano toric projective manifold \([8]\).

The quantum differential operators for the flag manifolds have been found in \([14, 8]\) to be “quantum Toda operators” which are by definition nonconstant integrals of motions for quantum Toda lattice. Moreover a mirror family of the flag manifold has been constructed in \([8]\). In this paper we build the equivariant mirror of the flag manifold, that is, a stationary phase integral representation of complete spectra of quantum Toda operators (see theorem \([9]\)). Using the equivariant mirror construction we confirm the so-called “R-conjecture” which leads to a proof of a formula for any genus gravitational descendent potential for flag manifolds and as a corollary we obtain the Virasoro conjecture for the manifolds (see theorem \([4]\), theorem \([7]\), and corollary \([3]\)). The equivariant version of the formula was obtained in \([11]\).

Finally we remark that the integral representation that we found gives another explicit description of all \( GL(n, \mathbb{R}) \)-Whittaker functions \([13, 17]\).

2. Quantum Differential Operators for Flag Manifolds

2.1. Quantum Toda Operators. Denote by \( t_i, i = 0, \ldots, n \), the standard coordinate functions on \( \mathbb{C}^{n+1} \), and let \( q_i = e^{t_i-t_{i-1}}, i = 1, \ldots, n \). Consider a matrix

\[
A = \begin{bmatrix}
p_0 & q_1 & 0 & 0 & \cdots \\
-1 & p_1 & q_2 & 0 & \cdots \\
0 & \cdots & 0 & -1 & p_n
\end{bmatrix}
\]

of size \((n + 1) \times (n + 1)\). If

\[
\det(A + xI) = x^{n+1} + \sum_{i=1}^{n+1} D_i(p_0, \ldots, p_n, q_1, \ldots, q_n)x^{n+1-i},
\]
denote $D_i(h \frac{\partial}{\partial x_0}, ..., h \frac{\partial}{\partial x_n}, q_1, ..., q_n)$ simply by $D_i$ unless stated otherwise. Let 

$$H = \frac{\hbar^2}{2} \sum_{i=0}^{n} \frac{\partial^2}{\partial t_i^2} - \sum_{i=1}^{n} e^{t_i-t_{i-1}}$$

which is a quantization of the Hamiltonian of non-periodic Toda lattice. The quantum Toda lattice is a completely integrable system with quantum integrals $D_i$, $i = 1, ..., n + 1$. We include an elementary proof of the commutativity of $D_i$’s which is known.

**Proposition 1.** Let $D$ be a linear holomorphic differential operator on $\mathbb{C}^{n+1}$ with coefficients in the Laurent polynomial ring $\mathbb{C}[h, e^{\pm t_0}, ..., e^{\pm t_n}]$ over $\mathbb{C}[\hbar]$. Suppose that $[H, D] = 0$, then the coefficients of the principal part of $D$ are contained in $\mathbb{C}[\hbar]$.

*Proof.* Let $m$ be the order of the differential operator $D$ and for multiple index $\alpha$, let $\partial^\alpha = \frac{\partial^{\alpha_0}}{\partial x_0^{\alpha_0}} ... \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$. Then $D = \sum_{|\alpha| = m} a_\alpha \partial^\alpha + D'$ where $a_\alpha$ are some polynomials in $e^{\pm t_0}, ..., e^{\pm t_n}$ over $\mathbb{C}[\hbar]$ and $D'$ is a differential operator with order less than $m$. In the bracket $[H, D]$, the degree $m+1$ part is created only in $[\Delta, \sum_{|\alpha| = m} a_\alpha \partial^\alpha]$ where $\Delta$ is the Laplace operator $\sum \frac{\partial^2}{\partial t_i^2}$. It is enough to prove that if $[\Delta, \sum_{|\alpha| = m} a_\alpha \partial^\alpha]$ has no order $m+1$ part then $a_\alpha \in \mathbb{C}[\hbar]$ for all $\alpha$. To prove the claim, use induction on $n$. When $n = 0$, it is clear. Let $n > 0$. Without loss of generality, we may assume that $a_\alpha$ are not in $\mathbb{C}[\hbar]$ for all $\alpha$. Let $k$ be the maximum of $a_\alpha$ for all $\alpha$. Decompose $\sum_{|\alpha| = m} a_\alpha \partial^\alpha = \sum_{\alpha_0 = k} a_\alpha \partial^\alpha +$ the rest. When $\alpha_0 = k$, $a_\alpha$ does not have terms in $e^{\pm t_0}$ since $[\Delta, \sum_{|\alpha| = m} a_\alpha \partial^\alpha]$ has no order $m+1$. If $[\Delta, \text{the rest}] = \sum_{\beta_0 = k} b_\beta \partial^\beta + D''$, then every $b_\beta$ depends on $e^{\pm t_0}$. So, $[\Delta, \sum_{\alpha_0 = k} a_\alpha \partial^\alpha]$ has no order $m+1$ part. Now we apply the induction hypothesis to $\sum_{\alpha_0 = k} a_\alpha \partial^\alpha$ whose coefficients do not depend on $t_0$ and conclude that $a_\alpha \in \mathbb{C}[\hbar]$ if $\alpha_0 = k$. The conclusion is contradictory to the assumption that for all $\alpha$, $a_\alpha$ is not in $\mathbb{C}[\hbar]$. \qed

**Theorem 1.** $[D_i, D_j] = 0$ for all $i = 0, ..., n$.

*Proof.* The commutativity of $H$ and $D_i$ is proven in [8]. Since $[H, [D_i, D_j]] = 0$, by the above proposition, the highest order part of $[D_i, D_j]$ has coefficients in $\mathbb{C}[\hbar]$. Now it is enough to prove the following claim. For any multi-indices $\alpha$ and $\beta$ and any $a$ and $b$ in the polynomial ring $\mathbb{C}[h, q_1, ..., q_n]$, if we let $[a \partial^\alpha, b \partial^\beta] = \sum c_\gamma \partial^\gamma$, then any $c_\gamma$ cannot be in $\mathbb{C}[\hbar]$ unless $c_\gamma = 0$. Notice that $[a \partial^\alpha, b \partial^\beta]$ is

$$\sum_{\alpha' + \alpha'' = \alpha, \alpha' \neq 0} \left( \frac{\alpha'}{\alpha} \right) a(\partial^{\alpha'} b)\partial^{\alpha'' + \beta} - \sum_{\beta' + \beta'' = \beta, \beta' \neq 0} \left( \frac{\beta'}{\beta} \right) b(\partial^{\beta'} a)\partial^{\beta'' + \beta}. $$
However the polynomials \( a(\partial^b) \) and \( b(\partial^a) \) in \( q_i, i = 0, \ldots, n \), over \( \mathbb{C}[\hbar] \) have no constant coefficients of \( \mathbb{C}[\hbar] \). So, \( c_\gamma \) are zeros or in \( \mathbb{C}[\hbar, q_1, \ldots, q_n] - \mathbb{C}[\hbar] \). \( \square \)

2.2. Givental’s \( J \)-functions for flag manifolds. Recall that the (resp. equivariant) quantum differential operators are defined to be operators which annihilate the (resp. equivariant) \( J \)-functions. For the flag manifolds they are generated by the quantum Toda operators.

Let \( Fl(n + 1) \) be the set of all complete flags \( V_1 \subset V_2 \subset \ldots \subset V_n \subset \mathbb{C}^{n+1} \) of subspaces of \( \mathbb{C}^{n+1} \), where \( \dim V_i = i \). It is the flag manifold of dimension \( n(n + 1)/2 \). Consider the torus \( T = (\mathbb{C}^*)^{n+1} \) action on \( Fl(n + 1) \) induced from the standard \( T \) action on \( \mathbb{C}^{n+1} \). Denote by \( \nabla_i \) the universal subbundle over \( Fl(n + 1) \) with fiber \( V_i \) at point \((V_1, \ldots, V_n)\). Let \( p_i \) be the equivariant first Chern class \( c_i^T(\nabla_{i+1}/\nabla_i), i = 0, \ldots, n \). Here \( \nabla_0 \) is by definition rank 0 bundle. Denote \( H^*_T(\text{point}, \mathbb{Z}) = H^*(\mathbb{P}^\infty)^{\otimes n+1} = \mathbb{Z}[\lambda_0, \ldots, \lambda_n] \). Here the following convention is used: the \( i \)-th equivariant Chern class \( c_i^T(\nabla_{n+1}) \) is the \( i \)-th elementary symmetric polynomial \( \sigma_i \) in \( \lambda_0, \ldots, \lambda_n \). Then it is known that the equivariant small quantum cohomology ring of \( Fl(n + 1) \) is generated by \( p_i \) with relations \( D_i(p,q) = \sigma_i, i = 1, \ldots, n + 1 \). From now on take a torus \( T \) as the subgroup \( \{(a_0, \ldots, a_n) \mid \prod_i a_i = 1\} \) of \( (\mathbb{C}^*)^{n+1} \) so that \( \sigma_1 = 0 \).

Let \( \overline{M}_{g,m}(Fl(n+1), d) \) be the moduli of degree \( d \) stable maps \((f, C, x_1, \ldots, x_m)\) to \( Fl(n + 1) \) from \( m \) pointed prestable genus \( g \) curves \( C \) and let \( \text{ev}_i \) be the evaluation map at the \( i \)-th marked point. Thus, \( f : C \to Fl(n + 1) \) is a morphism such that \( f_*[C] = d \in H_2(Fl(n + 1), \mathbb{Z}) \) and \( \text{ev}((f, C, x_1, \ldots, x_m)) \) is by definition \( f(x_i) \). Let \( \psi_i \) be the first Chern class of the universal cotangent orbi-line bundle, whose fiber at \([(f, C, x_1, \ldots, x_m)] \) is \( T^*_{x_i} C \). Fix a basis \( \{\phi_\alpha\} \) of the free \( H^*_T(\text{point}) \)-module \( H^*_T(Fl(n + 1), \mathbb{Z}) \).

Define equivariant Givental’s \( J \)-functions \( J_\alpha(t_0, \ldots, t_n; \lambda_0, \ldots, \lambda_n) \) to be

\[
\sum_{d \in H_2(Fl(n+1), \mathbb{Z})} q^d \int_{\overline{M}_{0,1}(Fl(n+1), d)} \frac{\text{ev}_1^*(\phi_\alpha \wedge \exp(\sum p_i t_i/\hbar))}{\hbar(\hbar - \psi_1)}.
\]

where \( q^d = \prod_{i=1}^n q_i^{e_{d, c_1(\nabla^*)}} \) with \( q_i = e^{t_i - t_{i-1}} \).

**Theorem 2.** (\( [4] \))

\[
D_i J_\alpha = \sigma_i J_\alpha
\]

for all \( i \) and \( \alpha \).
3. Equivariant mirrors of flag manifolds

In this section we find a stationary phase integral representation of all solutions $I$ to $D_i I = \sigma_i I$, where $i = 1, \ldots, n + 1$. To do so, we shall make use of Givental’s construction $I' = \int_{\Gamma \subset Y} e^{f'/\hbar} \omega$ of the mirror of the flag manifold [8]. It satisfies differential equations $D_i I' = 0$. From such $I'$, in order to build spectrum solutions, we shall add appropriate weight factors to the phase function $f'$ and show that if $I = \int_{\Gamma \subset Y} e^{(f' + \text{the weight factors})/\hbar} \omega_q$, then $D_i I = \sigma_i I$, $i = 1, \ldots, n + 1$.

Introduce a graph and coordinates for vertices in the graph:

\[
\begin{array}{ccccc}
(0, 0) & & (1, 0) & & (0, 1) \\
\bullet & \rightarrow & \bullet & \rightarrow & \bullet \\
(2, 0) & \downarrow & (1, 1) & \downarrow & (0, 2) \\
\bullet & \rightarrow & \bullet & \rightarrow & \bullet \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
(n, 0) & \downarrow & (n - 1, 1) & \downarrow & \cdots & (1, n - 1) & \downarrow & (0, n) \\
\bullet & \rightarrow & \bullet & \rightarrow & \cdots & \bullet & \rightarrow & \bullet \\
\end{array}
\]

For each edge introduce edge variables: $u_{ij}$ denotes the vertical edge variable such that whose head vertex has coordinate $(i = \text{diagonal, } j = \"x-axis\")$. Also $v_{ij}$ denotes the parallel edge variable whose tail vertex has coordinate $(i, j)$. Edges will be identified with corresponding variables.

\[
\begin{array}{c}
u_{ij} \\
(i, j) \rightarrow v_{ij}
\end{array}
\]

For each box

\[
\begin{array}{c}
\begin{array}{c}
v_{i,j} \\
u_{i+1,j} \rightarrow \bullet \\
\bullet \rightarrow \bullet \\
v_{i+1,j}
\end{array}
\rightarrow \\
\begin{array}{c}
v_{i,j+1} \\
u_{i+1,j+1} \rightarrow \bullet \\
\bullet \rightarrow \bullet \\
v_{i+1,j+1}
\end{array}
\end{array}
\]

in the graph, impose “box” relation

\[v_{i,j}u_{i,j+1} - u_{i+1,j}v_{i+1,j} = 0.\]
For each roof

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\rightarrow \\
\bullet \\
\end{array}
\]

\[u_{1,i} \rightarrow v_{1,i}\]

in the graph, impose “roof” equation

\[u_{1,i}v_{1,i} = q_i.\]

For nonzero complex numbers \(q_i, \, i = 1, \ldots, n\), define \(Y_q\) to be the affine variety

\[
\text{Spec}\mathbb{C}[u, v]/(..., v_{i,j}u_{i,j+1} - u_{i+1,j}v_{i+1,j}, \ldots, u_{1,i}v_{1,i} - q_i, ...),
\]

given by all box equations and all roof equations. It is a complex torus \((\mathbb{C}^\times)^{n(n+1)/2}\). The fact follows from the interpretation of each edge variable as the fractions of “voltages” at vertices of the edge. To each vertex, assign free vertex variables \(T_{ij}\) with one condition

\[
\sum_{i=0}^{n} T_{0i} = 0.
\]

Then it is natural to let \(u_{i,j} = e^{T_{i,j} - T_{i-1,j}}\) and \(v_{i,j} = e^{T_{i-1,j+1} - T_{i,j}}\).

Now introduce a stationary phase integral

\[
\mathcal{I}_\Gamma(t_0, ..., t_n) = \int_{\Gamma \subseteq Y_q} e^{f_q/\hbar} \omega_q.
\]

It is a complex valued function on \((t_0 = T_{00}, t_1 = T_{01}, ..., t_n = T_{nn})\) with \(t_0 + \ldots + t_n = 0\). Here \(\Gamma\) is a descending Morse cycle of \(\text{Re} f_q\) with a suitable Riemannian metric on \(Y_q\). \(\Gamma\) varies covariant constantly with the Gauss-Manin connection on the relative homology bundle with fibers \(H_{n(n+1)/2}(Y_q, \text{Re} f_q = -\infty)\). Let

\[
\omega_q = \bigwedge_{\text{all vertices not on the main diagonal}} dT_v.
\]

Observe that the form \(\omega_q\) can be defined on \(Y_q\) since it is translation invariant.

To outer edges \(\varepsilon\), that is, edges whose targets are \((i, 0)\) or sources are \((i, n - i)\), assign weights \(\lambda_\varepsilon\) by

\[
\lambda_\varepsilon = \begin{cases} 
-\lambda_{i-1} - \frac{1}{2} \sum_{j<i-1} \lambda_j & \text{if } \varepsilon = v_{i,n-i} \\
\lambda_{i-1} + \frac{1}{2} \sum_{j<i-1} \lambda_j & \text{if } \varepsilon = u_{i,0} 
\end{cases}.
\]
For each edge $\varepsilon$ which are neither at the far left nor at the bottom of the graph, let

$$\lambda_\varepsilon = \begin{cases} -\frac{1}{2} \lambda_{i-1} & \text{if } \varepsilon = u_{i,j} \\ \frac{1}{2} \lambda_{i-1} & \text{if } \varepsilon = v_{i,j} \end{cases}.$$ 

The assignment of weights on edges is given to satisfy the condition that the sum of outgoing weights minus the sum of incoming weights at each vertex at diagonal level $k$ is exactly $\lambda_k - \lambda_{k-1}$, if we set $\lambda_{-1} = 0$. This property will enable us to prove the equivariant mirror theorem for flag manifolds.

Now define a phase function $f_q$ by

$$f_q = \sum_{i>0,j} (u_{i,j} + v_{i,j}) + \sum_{i>0,j} (\lambda_{u_{i,j}} \ln u_{i,j} + \lambda_{v_{i,j}} \ln v_{i,j})$$

$$= \sum_{i>0,j} (e^{T_{i,j}-T_{i-1,j}} + e^{T_{i-1,j+1}-T_{i,j}})$$

$$+ \sum_{i>0,j} (\lambda_{u_{i,j}} (T_{i,j} - T_{i-1,j}) + \lambda_{v_{i,j}} (T_{i-1,j+1} - T_{i,j})).$$

The first and second summation terms are respectively $f_q'$ and the weight factor that we mentioned in the beginning of this section.

Define $\sigma_i$ by equation $x^{n+1} + \sigma_1 x^n + ... + \sigma_{n+1} = \prod_{i=0}^{n} (x - \lambda_i)$. We state the equivariant mirror theorem for flag manifolds and shall prove it in section 3.1 and 3.2.

**Theorem 3.** Let $\Gamma$ be a descending Morse cycle of $\text{Re} f_q$ at a nondegenerate critical point of $f_q$. Then the stationary phase integral

$$\mathcal{I}_\Gamma(t_0, ..., t_n) = \int_{\Gamma \subset Y_q} e^{f_q/\hbar} \omega_q$$
satisfies eigenvalue differential equations \( D_1 \mathcal{I} = \sigma_1 \mathcal{I}, D_2 \mathcal{I} = \sigma_2 \mathcal{I}, \ldots, D_{n+1} \mathcal{I} = \sigma_{n+1} \mathcal{I} \).

**Corollary 1.** Let \( q \) be a general point in \( \mathbb{C}^n \). Then there are \((n+1)!\) critical points of \( f_q \). Furthermore they are all nondegenerate.

**Remark.** When \( \lambda_i = 0 \) for all \( i \), the above theorem is proven in \([3]\).

3.1. **Proof of theorem** \([3]\). Since \( \frac{\partial f_q}{\partial t_i} \) does not depend on \( t_j \) if \( j \neq i \), the amplitude created by operating \( D_i \) on \( \mathcal{I}_\Gamma \) are the corresponding coefficients of the characteristic polynomial of \( A_1 - \lambda_0 I \), where

\[
A_k = \begin{bmatrix}
-u_{k0} & u_{k0}v_{k0} & 0 & \cdots \\
-1 & v_{k0} - u_{k1} & u_{k1}v_{k1} & \cdots \\
& \vdots & \ddots & \vdots \\
& & & u_{k,n-k}v_{k,n-k} - 1 \end{bmatrix}
\]

for \( k = 1, \ldots, n \). Hence

\[
(x^{n+1} + D_1 x^n + \ldots + D_{n+1}) \cdot \int_\Gamma e^{f_q/\hbar \omega_q} = \int_\Gamma \det(A_1 - \lambda_0 I + xI) \cdot e^{f_q/\hbar \omega_q}.
\]

To prove the theorem it is enough to show the following.

**Proposition 2.** For \( k = 1, \ldots, n \)

\[
\int_\Gamma \det(A_k - \lambda_{k-1} I + xI) \cdot e^{f_q/\hbar \omega_q} = (x - \lambda_{k-1}) \int_\Gamma \det(A_{k+1} - \lambda_k I + xI) \cdot e^{f_q/\hbar \omega_q},
\]

where \( A_{n+1} = 0 \).

**Proof.** As noticed in \([3]\), \( A_k = U_k V_k \) where

\[
U_k = \begin{bmatrix}
u_{k0} & 0 & \cdots \\
1 & u_{k1} & 0 & \cdots \\
& \vdots & \ddots & \vdots \\
& & & 1 \end{bmatrix}
\]

\[
V_k = \begin{bmatrix}
1 & u_{k,n-k} & 0 \\
0 & 1 & 0
\end{bmatrix}
\]
and

\[ V_k = \begin{bmatrix} -1 & v_{k0} & 0 & \ldots & 0 \\ 0 & -1 & v_{k1} & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & -1 \end{bmatrix} \]

Since \( V_k \) is invertible, the characteristic polynomial of \( A_k - \lambda_{k-1} I \) is equal to that of \( V_k U_k - \lambda_{k-1} I \). Using the critical point conditions

\[ \frac{\partial f_q}{\partial T_{k,i}} = u_{ki} - v_{ki} + v_{k+1,i-1} - u_{k+1,i} + \lambda_{k-1} - \lambda_k \]

and box relations \( u_{k,i+1} v_{ki} = u_{k+1,i} v_{k+1,i} \), we see that

\[ V_k U_k - \lambda_{k-1} I = \begin{bmatrix} A_{k+1} - \lambda_k I & 0 \\ 0, ..., 0, -1 & -\lambda_{k-1} \end{bmatrix} - \text{diag}(\frac{\partial f_q}{\partial T_{k,0}}, ..., \frac{\partial f_q}{\partial T_{k,n-k}}, 0). \]

Note that the terms in \( \det(V_k U_k - \lambda_{k-1}) \) which involve the derivatives of \( \frac{\partial f_q}{\partial T_{k,i}} \) might have extra multiples of edge variables whose vertices are not \((k, i)\). However if \( P \) is a polynomial of edge variables except whose vertices are not \((k, i)\), the following holds:

\[ \int_{\Gamma} \frac{\partial^2 f_q(h)}{\partial T_{k,i}} P e^{f_q/h} \omega_q = \int_{\Gamma} \frac{\partial}{\partial T_{k,i}} (P e^{f_q/h}) \omega_q = \int_{\Gamma} L \frac{\partial}{\partial T_{k,i}} (P e^{f_q/h}) \omega_q = \int_{\Gamma} d_i \frac{\partial}{\partial T_{k,i}} P e^{f_q/h} \omega_q = 0. \]

This completes the proof.

3.2. Proof of Corollary \([1]\). Let

\[ Y = \{(..., u_{ij}, v_{i,j}, ...) \mid \text{all } u_{ij} \neq 0, \text{all } v_{i,j} \neq 0, \text{all box relations}\}. \]

\( Y \) is a complex torus \((\mathbb{C}^x)^{n(n+1)/2+n}\). For \( \lambda = (\lambda_0, ..., \lambda_n) \) with \( \sum \lambda_i = 0 \), denote by \( Z_\lambda \) the closed subscheme in \( Y \) defined by the ideal generated by \( \frac{\partial f_q}{\partial T_{ij}} \), \( \forall \ i > 0, j \). Let \( X_\lambda = \text{Spec} \mathbb{C}[p_0, ..., p_n, q_1^{\pm 1}, ..., q_n^{\pm 1}] / (D_1(p, q) - \sigma_1, ..., D_{n+1}(p, q) - \sigma_{n+1}) \), which is an irreducible nonsingular rational variety \([10]\). Since \( \det(A_1 - \lambda_0 I + xI) = \prod_{i=0}^n (x - \lambda_i) \) on \( Z_\lambda \) as shown in the proof of proposition \([2]\), a morphism \( \phi : Z_\lambda \rightarrow X_\lambda \) is defined by \( p_0 = -u_1, p_1 = v_1 - u_1, ..., p_n = v_{n-1}, q_1 = u_0 v_1, ..., q_n = u_{n-1} v_{n-1} \). Direct inverting of \( \phi \) shows that \( \phi \) is a birational morphism. Let \( \pi \) be the projection from \( X_\lambda \) to \((\mathbb{C}^x)^n\) defined by \( \pi(p, q) = q \). The projection
is a finite map of degree \((n+1)!\) which is verified by the dimension of the algebra \(\mathbb{C}[p,q]/D_1(p,q) - \sigma_1, ..., D_{n+1}(p,q) - \sigma_{n+1}\) at \(q = 0\). Now the proof of the corollary follows from the irreducibility of \(X_\lambda\), the degree of the projection \(\pi\) and Sard’s lemma.

3.3. **Study of the phase function** \(f_q\). Given a sequence \(\sigma = (k_1, ..., k_n)\) of integers where \(0 \leq k_i \leq n - i + 1\), let \(w^\sigma_{ij} = u_{ij}\) if \(j < k_i\), otherwise let \(w^\sigma_{ij} = v_{ij}\). We shall express \(f_q\) in terms of these independent variables \(w^\sigma_{ij}\).

We associate a sequence \(\rho^\sigma_{ij}\) of weights to \(\sigma\), inductively on \(n - i\) and then \(j\), where \(i = 1, ..., n + 1\) and \(j = -1, 0, ..., n - i + 1\). First, let \(\rho^\sigma_{n,-1} = 0\), \(\rho^\sigma_{n-i+1} = \sum_{k=i-1}^n \lambda_k\). And let \(\rho^\sigma_{ij} = \rho^\sigma_{i+1,j} + \lambda_{i-1}\). It is easy to see inductively that \(\{\rho^\sigma_{ij} - \rho^\sigma_{i,j-1} \mid j = 0, 1, ..., n - i + 1\} = \{\lambda_{i-1}, ..., \lambda_n\}\).

![Diagram](image)

**Proposition 3.** The phase function \(f_q\) is, in terms of \(w^\sigma_{ij}\),

\[
\sum_{i=1,...,n} \rho^\sigma_{i,i-1} \ln q_i + \sum_{i=1,...,n,j=0,...,n-i} (w^\sigma_{ij} + r_{ij}(..., w^\sigma_{ab}, ..., q_1, ..., q_n)) + \sigma(i, j) \ln w^\sigma_{ij},
\]

where \(\sigma(i, j)\) is \(\lambda_{i-1} - (\rho^\sigma_{ij} - \rho^\sigma_{i,j-1})\) if \(j < k_i\) and \(\sigma(i, j)\) is \(-\lambda_{i-1} + (\rho^\sigma_{i,j+1} - \rho^\sigma_{ij})\) otherwise. Here \(r_{ij}\) are monomials in \((w^\sigma_{ab})^{\pm 1}\) and \(q_k\) with at least one factor among \(q_1, ..., q_n\).

**Proof.** To apply induction on \(n\), let \(L\) be the triangular graph introduced in the beginning of section 3 and let \(L_k\) be the triangular sublattice of \(L\) whose vertices \((i, j)\) satisfy inequalities \(k \leq i\). The edges of \(L_k\) are by definition the edges of \(L\), connecting neighboring vertices of \(L_k\).
Note that $L_0$ is the full lattice $L$. Introduce new variables associated to the sublattice $L_k$ as follows,

$$Q_{j+1}^k = u_{k+1,j}v_{k+1,j} \quad \Delta_k = \sum_{j=0}^{k-1} \lambda_j.$$ 

Moreover let $Q_j^0 = q_j, \Delta_0 = 0$. Consider

$$(f_k)Q_k = \sum_{\text{vertices } \in L_k} (u_{i,j} + v_{i,j}) + \sum_{\text{vertices } \in L_k} (\lambda u_{i,j} \ln u_{i,j} + \lambda v_{i,j} \ln v_{i,j}).$$

By the induction hypothesis, we can assume that

$$(f_k)Q_k = \sum_{i=1}^{n-k} (\rho^\sigma_{k+1,i-1} + \frac{\Delta_k}{2}) \ln Q_i^k$$

$$+ \sum_{i=1}^{n-k} (w_{ij} + r^k_{ij}(..., w_{ab}, ..., Q^k_1, ..., Q^k_{n-k}))$$

$$+ \sigma(i, j) \ln w^\sigma_{ij},$$

where $r^k_{ij}$ are monomials in $(w_{ab})^{\pm 1}$ and $Q^k_1, ..., Q^k_{n-k}$. Then substitute $v_{k,i}u_{k,i+1}$ for $Q^k_{i+1}$. Then derive $(f_{k-1})Q_{k-1}$ in terms of $w^\sigma_{ij}$ and $Q_{k-1}^j$. The rest is straightforward. \square

Let $f_q^\sigma = f_q - \sum_i \rho^\sigma_{i-1} \ln q_i$. Then $f_q^\sigma$ is regular at $q_1 = ... = q_n = 0$ and has exactly one simple critical point at the origin $q = 0$. Therefore there is exactly one critical point of $f_q$ whose limit as $q$ goes to 0 is the simple critical point of $f_q^\sigma$. Let us denote by $\Gamma_\sigma$ the descending Morse cycle of $f_q$, whose limit is the Morse cycle $\Gamma_\sigma,0$ of $f_q^\sigma$. Now we obtain a corollary which will be applied to the proof of theorem 4.

**Corollary 2.**

$$\lim_{q \to 0} e^{-\sum_{j=0}^{\sigma} (\rho^\sigma_{i,j-1} - \rho^\sigma_{i,j})t_j} I_{\sigma} = \prod_{i=1}^{n} \int_{C^\sigma(i,j)} e^{w_{ij}/h} \frac{d w_{ij}}{w_{ij}}w^\sigma_{ij},$$

where $\sigma(i, j)$ is defined as in the previous proposition and $C^\sigma(i,j)$ is the descending Morse cycle of an one-variable function $x + \sigma(i, j) \ln x$.

For later use, for a given sequence $\sigma = (k_1, ..., k_n)$ of integers $0 \leq k_i \leq n - i + 1$, we define $\sigma(i) \in \{0, 1, ..., n\}$ by the requirement

$$\lambda_{\sigma(j)} = \rho^\sigma_{1,j} - \rho^\sigma_{1,j-1},$$

so that we may identify $\sigma$ as a permutation element of the symmetric group $S_{n+1}$ of $n + 1$ letters.
4. the Virasoro Conjecture

4.1. The fundamental solution of the form $\Psi Re^{U/h}$. Now we recall the equivariant big quantum cohomology of a projective algebraic manifold $X$ with a Hamiltonian torus $T$-action. Fix a homogeneous basis $\{\phi_\alpha\}$ of $H_T^*(X, \mathbb{C})$. For simplicity assume that $H^*(X, \mathbb{C}) = H^{even}(X, \mathbb{C})$ (i.e. there are no odd classes) and there is a basis $\omega_i$ of $H^2(X, \mathbb{Z})$ such that $\langle \omega_i, [C]\rangle$ is nonnegative for any curve (effective) class $[C]$. Introduce formal parameters $Q_i$ and for $d \in H_2(X, \mathbb{Z})$ let $Q^d = \prod Q_i^d$ where $d_i = < \omega_i, d >$. A potential function $F(t)$ is by definition

$$F(t) = \sum_{m,d} \frac{Q^d}{m!} \int_{\overline{M}_{0,m}(X,d)^{virt}} ev_1^*(t) \cdots ev_m^*(t).$$

Here is an explanation of notation in the equation. First, $\overline{M}_{g,m}(X, d)^{virt}$ denotes the virtual fundamental class of the moduli space $\overline{M}_{g,m}(X, d)$ of stable maps to $X$ with genus $g$, $m$ marked points and degree $d \in H_2(X, \mathbb{Z})$. Secondly, $t \in H_T^*(X)$ is considered as $t = \sum t_\alpha \phi_\alpha$ and the integral is taken as the equivariant pushforward so that the value is in $H^*(BT)$. The potential function defines the equivariant big quantum cohomology of $X$ as following. Let $H = H_T^*(X, \mathbb{C}[[Q]])$. The quantum cohomology is a certain multiplication structure on the tangent space $T_t H$. The quantum product $\frac{\partial}{\partial t_\alpha} \frac{\partial}{\partial t_\beta}$ is defined by equation

$$\left( \frac{\partial}{\partial t_\alpha}, \frac{\partial}{\partial t_\beta}, \frac{\partial}{\partial t_\gamma} \right) = \frac{\partial}{\partial t_\alpha} \frac{\partial}{\partial t_\beta} \frac{\partial}{\partial t_\gamma} F(t).$$

Here $(\cdot, \cdot)$ is the equivariant Poincaré pairing on $T_t H$ defined by $(\frac{\partial}{\partial t_\alpha}, \frac{\partial}{\partial t_\beta}) := \int_X \phi_\alpha \wedge \phi_\beta =: g_{\alpha\beta}$. There is a pencil of flat connections $\nabla_h = h\nabla - \sum_\alpha dt_\alpha \frac{\partial}{\partial t_\alpha} \circ$ with parameter $h$. Here $d$ is the Levi-Civita connection with respect to the vector space structure of $H$. (Strictly speaking, $\nabla_{h/h}$ are connections.)

$\text{Spec}(\text{Vect}(H))$ defines a Lagrangian formal scheme $L \subset T^*H$ over the ground ring $\mathbb{C}[\lambda_0, \ldots, \lambda_n][[Q]]$, where $H^*(BT) = \mathbb{C}[\lambda_0, \ldots, \lambda_n]$. Let $\text{rank} H^*(X) = N$ and let $u$ be a semi-simple point in $H$, that is, $T_u H$ is a semi-simple algebra. Then $L_u \subset T^*_u H$ consists of $N$ many simple points and there are $N$ many local functions $u_1, \ldots, u_N$ at $u$ such that the canonical 1-form $\sum_\alpha P_\alpha dt_\alpha$ restricted to $L_i$ coincides with $du_i := \sum \partial u_i \frac{\partial}{\partial u_i} dt_\alpha$, where $L_i$ are branches of $L$. Since $\{du_i\}$ is linearly independent, $\{u_i\}$ forms a local coordinate system of $H$ and the dual vector fields $\frac{\partial}{\partial u_i}$ can be defined. Notice that $\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_j}$. We
call $u_i$ canonical coordinates. These coordinates are unique up to order and additions by constants.

Let $U$ be diag$(u_1, ..., u_N)$ and $A^1$ be the connection 1-form matrix of $\nabla_h$ so that $\nabla_h \frac{\partial}{\partial u_\alpha} = \sum_\beta A^1_{\beta,\alpha} \frac{\partial}{\partial u_\beta}$. Let $1/\Delta_\beta$ be the length of $\frac{\partial}{\partial u_\beta}$ with respect to the Poincare paring metric and let $\Delta_\alpha \frac{\partial}{\partial u_\alpha} = \sum_\beta \psi_{\beta,\alpha} \frac{\partial}{\partial u_\beta}$. Then the computation of connection 1-form $\hbar \frac{\partial}{\partial u_\alpha}$ using the expression $\nabla_h = h d - \sum du_\beta \frac{\partial}{\partial u_\beta}$ shows that

$$A^1 \Psi = \Psi du.$$ 

Notice that $u_i$ linearly depends on $t_\alpha$’s modulo $Q$ and $\Psi$ is constant modulo $Q$.

**Example.** Let $X = \text{Fl}(2) = \mathbb{CP}^1$. Consider $T = \mathbb{C}^\times$ action as $t[x_0, x_1] = [tx_0, t^{-1}x_1]$ where $t \in T$ and $[x_0, x_1] \in \mathbb{CP}^1$. The Lagrangian scheme $L \subset T^* H$ is defined by equation $p^2 = \lambda^2 + e^t Q$ and $1 = 1$ where $p$ is the cotangent direction variable associated to $-t_0 = t_1$ and $1$ is the cotangent direction variable associated to the identity element of $H^*_T(\mathbb{CP}^1)$. The fiber $L(t_0, t)$ of $L$ over a point $(t_0, t)$ of $H$ consists of two points $\{ (1, \pm \sqrt{\lambda^2 + e^t Q}) \}$. Then $du_\pm = dt_0 + p_\pm dt = dt_0 \pm \sqrt{\lambda^2 + e^t Q} dt$. The canonical vector fields are $\frac{\partial}{\partial u_\pm} = \frac{1}{2p_\pm} (p_\pm \frac{\partial}{\partial t_0} + \frac{\partial}{\partial t})$. Let $v = \sqrt{p_+}$. Then with respect to the ordered basis $\{1, p\}$ of $H^*(\mathbb{CP}^1)$

$$A^1 = \begin{bmatrix} dt_0 & v dt_0 \\ dt_0 & dt_0 \end{bmatrix}, \quad \Psi = \frac{1}{\sqrt{2}} \begin{bmatrix} v & -iv \\ 1/v & i/v \end{bmatrix}.$$

On the other hand, we have shown that $\{(p, q) \mid \frac{\partial f_q}{\partial y} = 0, p = \frac{\partial f_q}{\partial t} \}$ is the Lagrangian scheme $L' = \text{Spec} \mathbb{C}[p, q^{\pm 1}]/(p^2 - q - \lambda^2)$. At critical points of $f_q$, $df_q = pdt$, so that $u_\pm(\lambda, q) = t_0 + f_q(p_\pm(\lambda, q), q)$ (up to constant addition). Thus, $e^{t_0/\hbar} T_{t_0} \sim e^{u_\pm/\hbar}$ (up to constant multiplication). In fact a direct computation shows that at critical points

$$f_q(p_\pm, q) = 2p_\pm + \lambda_0 \ln(1 + p_\pm) + \lambda_1 \ln(\lambda_0 + p_\pm)$$

and $\frac{\partial f_q(p_\pm, q)}{\partial t} = p_\pm$. \square

Near a semi-simple point $u$, due to the papers [10, 11, 12] there is a fundamental solution of form

$$\Psi Re^{U/\hbar}$$

such that $\Psi$ and $U$ are defined as above and

$$R(h) = 1 + R_1 h + R_2 h^2 + ..., \quad \text{for } h \to 0.$$
with $R'(h)R(-h) = 1$. Since $Q_i \frac{\partial}{\partial Q_i} F(t) = q_i \frac{\partial}{\partial q_i} F(t)$, we require that $Q_i \frac{\partial}{\partial Q_i} R = q_i \frac{\partial}{\partial q_i} R$, where $e^t = q_i$. Such matrix $R$ exists uniquely up to right multiplication by diagonal matrices $\exp(\sum_{k=1}^{\infty} a_k h^{2k-1})$ where $a_k = \text{diag}(a_k^{(1)}, ..., a_k^{(N)})$ are constants.

Suppose $T$ acts on $X$ with isolated fixed points and let $w_i, i = 1, ..., N$ be $T$-fixed points of $X$ and let $N_{i}^{(i)} = \sum_{j} \frac{1}{\chi_j(w_i)}$ where $\chi_j(w_i)$ are the weights of the induced torus action on the cotangent space $T_{w_i}^* X$ at $w_i$.

Let $u$ be a semi-simple point of $H^*_T(X)$. There is an unique asymptotic fundamental solution of type $\Psi Re^{U/h}$ of the connection $\nabla_h$ satisfying the following four conditions on $R = 1 + R_1 h + R_2 h^2 + ...$:

0. the divisor condition: $Q_i \frac{\partial}{\partial Q_i} R = q_i \frac{\partial}{\partial q_i} R$ for all $i$,

1. the orthogonal condition: $R'(h)R(-h) = 1$,

2. the classical limit condition: its classical limit of letting $Q \to 0$ is $\exp(\text{diag}(b_1, ..., b_N))$ where

$$b_i(h) = \sum_{k=1}^{\infty} N_{i}^{(i)} \frac{B_{2k}}{2k-1} \frac{h^{2k-1}}{2k-1}.$$ 

Here $B_{2k}$ are Bernoulli numbers defined by $x/(1 - e^{-x}) = 1 + x/2 + \sum_{k=1}^{\infty} B_{2k} x^{2k}/(2k)!$,

3. the equivariant homogeneity condition: $(h \partial_h + \sum_{i} u_i \partial_i + \sum_{j} \lambda_j \partial_{\lambda_j}) R(h) = 0$.

**R-Conjecture** ([12]) The matrix $R$ has the nonequivariant limit of letting $\lambda_0 \to 0, ..., \lambda_n \to 0$ and thus the nonequivariant limit satisfies the homogeneity condition $(h \partial_h + \sum_{i} u_i \partial_i) R(h) = 0$ where $u_i$ denote the corresponding nonequivariant canonical coordinates.

We shall prove the $R$-conjecture restricted to $H^2_T(X)$ when $X$ is a flag manifold. In such restriction the homogeneity condition is replaced by $(h \partial_h + \sum_{i} c_1(T_X), d > \frac{\partial}{\partial q_i}) R(h)|_{H^2_T(X)} = 0$. There are two known ways to obtain fundamental solutions of form $\Psi Re^{U/h}$ restricted to $H^2_T(X)$. One is the localization expansion of the construction by two-pointed gravitational Gromov-Witten invariants and the other one is the stationary phase approximation of the mirror construction [10]. We use both. There is a natural 1-1 correspondence between the fixed points $w_i$ and canonical coordinates $u_i$ since the points can be identified with points $L_t$ of the Lagrangian variety $L$ (first with $Q = 0$ and
general \( \lambda \) and then continuously). So we use the same label. If

\[
S_{\beta,\alpha} = \sum_d \frac{Q^d}{m!} \int_{\{M_{0,m+2}(\mathcal{F}(n+1),d)\}^{\text{virt}}} e_v^*(\phi_{\beta})(\prod_{i=2}^{m+1} e_v^*(t)) \frac{e_{m+2}^*(\phi_{\alpha})}{\hbar - \psi_{m+2}}
\]

then

\[
s_{\alpha} = \sum_{\beta,\gamma} S_{\beta,\alpha} \theta^{\beta,\gamma} \phi_{\gamma}
\]

is a flat section for all \( \alpha \) [7]. Recall if \( \phi_{\beta} = 1 \), \( t = \sum t_i \omega_i \in H^2_T(X) \) then

\[
S_{\beta,\alpha} = \sum_d Q^d q^d \int_{\{M_{0,1}(X,d)\}^{\text{virt}}} \frac{e_v^*(\phi_{\alpha} \wedge \exp(t/\hbar)))}{\hbar(\hbar - \psi_1)}
\]

which is denoted by \( S_{1,\alpha} \). We see the equality using the string and divisor axioms.

Suppose that \( X \) is a Fano manifold. Let \( T \) act on \( X \) and assume that \( H^2_T(X) \) is generated by the second cohomology classes \( \{\omega_i\} \). Here \( T \) could be just the trivial action. Let \( D \) be the algebra of differential operators \( \hbar \partial / \partial t \) over \( \mathbb{C}[\lambda, Q, q, \hbar] \). Then the following lemma holds.

**Lemma 1.** For each \( \phi \in H^*(X) \), there is a differential operator \( D_\phi \) in \( D \) such that \( (s_{\alpha}, \phi) = D_\phi S_{1,\alpha}(t) \) for every \( \alpha, t \in H^2_T(X) \).

**Proof.** We may set \( D_{\omega_i} = h \frac{\partial}{\partial \omega_i} \). If \( D_\phi S_{1,\alpha} = (s_{\alpha}, \phi) \) and \( D_{\phi \omega_i - \phi \wedge \omega_i} S_{1,\alpha} = (s_{\alpha}, \phi \omega_i - \phi \wedge \omega_i) \), then \( (h \frac{\partial}{\partial \omega_i} D_\phi - D_{\phi \omega_i - \phi \wedge \omega_i}) S_{1,\alpha} = (s_{\alpha}, \phi \wedge \omega_i) \) by the flatness of the section \( s_{\alpha} \), where the vector fields on the affine space \( H^*(X) \) are identified with classes in \( H^*(X) \). Notice that the degree of each homogeneous term of the \( q \)-linear combination class \( \phi \omega_i - \phi \wedge \omega_i \) is less than \( \deg \phi \wedge \omega_i \) since \( X \) is a Fano manifold. This completes the proof. \( \square \)

**Remark.** When \( X = \mathcal{F}(n+1) \), since the oscillatory integrals \( \mathcal{I} \) and \( S_{1,\beta} \) with \( Q = 1 \) generated the same \( D \)-module, we conclude that the matrix \( (D_\alpha \mathcal{I}_j) \) is a fundamental solution of \( \nabla_\hbar \) with \( Q = 1 \) for some \( D_\alpha \in D \).

**Theorem 4.** The R-conjecture restricted to \( H^2_T(X) \) is true for \( X = \mathcal{F}(n+1) \) with the \( T \)-action.

**Proof.** We may let \( Q = 1 \) by the requirement 0 of the conjecture. Recall that \( s_i = \sum_{\alpha, \beta} (D_\alpha \mathcal{I}_i) g^{\alpha,\beta} \frac{\partial}{\partial \beta} \) is a flat section by remark above. Since

\[
D_\alpha \mathcal{I}_i = \int_{\Gamma_i} \frac{e_J / \hbar}{\phi_{\alpha,d} \omega}
\]
for some polynomial $\phi_{\alpha,q}$ in $\lambda$, $\hbar$, and $q_i$ (see Proposition 3), we have the asymptotic expansion

$$e^{u_i/\hbar} \frac{\hbar^{(n+1)/2}\phi_{\alpha,q}(\text{crit}_\sigma)}{\sqrt{\det \text{Hessian} f_q(\text{crit}_\sigma)}} (1 + o(\hbar))$$

of the integral. From the expansion we obtain the fundamental solution $\Psi_{osc} R_{osc} e^{U/\hbar}$ where $R_{osc} = 1 + (R_1)_{osc} \hbar + \ldots$ and

$$\Psi_{osc}(\alpha,\sigma) = \frac{\phi_{\alpha,q}(\text{crit}_\sigma)}{\sqrt{\det \text{Hessian} f_q(\text{crit}_\sigma)}}.$$

We shall see that $\Psi_{osc}$ coincides with $\Psi$ up to left multiplication by a constant matrix (not depending on $\hbar$ and $q$). Also we shall show that $R_{osc}$ is the $R$ satisfying all the properties in the theorem/conjecture.

First we prove that $\Psi_{osc}$ coincides with $\Psi$ modulo right multiplication of a constant diagonal matrix, using the symmetry of the differential equation. Since $\Psi_{osc} R_{osc} e^{U/\hbar}$ is a fundamental solution, $\Psi_{osc}$ is an eigenvector matrix of the connection matrix $A^1$. However there is a symmetry of the differential equation:

$$\hbar d(s_i(-\hbar),s_j(\hbar)) = (\hbar ds_i(-\hbar),s_j(\hbar)) + (s_i(-\hbar),\hbar ds_j(\hbar))$$

$$= - (A^1 \Psi_{osc} R_{osc}(-\hbar) e^{-U/\hbar})^t G \Psi_{osc} R_{osc}(\hbar) e^{U/\hbar}$$

$$+ (\Psi_{osc} R_{osc}(-\hbar) e^{-U/\hbar})^t G A^1 \Psi_{osc} R_{osc}(\hbar) e^{U/\hbar} = 0$$

since $GA$ is symmetric, where $G = (g_{\alpha,\beta})$. This shows that $\Psi_{osc}^t G \Psi_{osc}$ is constant in $q$ and so the claim is true. Now considering $c_i D_{\alpha} T_{\Gamma_i}$ for some constant $c_i$ not depending on $\hbar$ and $q$, we conclude that $\Psi R_{osc} e^{U/\hbar}$ is a fundamental solution matrix.

We investigate $R_{osc}$ modulo $q$. In order to select $R_{osc}$ not $\Psi R_{osc}$ from the phase integrals we remove $\Psi$ by expressing the fundamental solution with respect to the basis $\{ \Delta_{\alpha} \frac{\partial}{\partial u_{\alpha}} \}$. If $s$ is a flat section of $\nabla_{\hbar}$, then $(\hbar \frac{\partial}{\partial u_{\alpha}})^t(s,1) = (s, p_i^j)$ modulo $q$. So, if $s = \sum b_i \Delta_{\alpha} \frac{\partial}{\partial u_{\alpha}}$, then

$$b_\sigma = \hat{L}_\sigma(s,1) \text{ modulo } q$$

where

$$\hat{L}_\sigma = \prod_{i>j} \frac{\hbar \frac{\partial}{\partial u_{\alpha(i)}} - \lambda_j}{\sqrt{\lambda_i - \lambda_j}}.$$

(Here $\sigma$ as an element of permutation is defined in the remark below of corollary 4.) Therefore, the entries of $R_{osc}$ that we obtained above satisfy

$$(\hat{L}_\tau c_\sigma \int_{\Gamma_{\sigma}} e^{t \omega})_{\tau,\sigma} \sim R_{osc} e^{U/\hbar} \text{ (mod q)}.$$
By corollary 2
\[
\hat{L}_\tau c_\sigma I_\sigma |_{q=0} \sim \delta_{\sigma,\tau} e^{-\sum \lambda_\sigma(i) t_i / \hbar} \prod_{n \geq i > j \geq 0} \Gamma \left( \frac{1}{\hbar} (\lambda_\sigma(i) - \lambda_\sigma(j)) \right)
\]
up to an irrelevant multiplication factor. However, for large \( z \),
\[
\ln \Gamma(z) \sim (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln 2\pi + \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)} z^{2i-1}
\]
shown in p.252 of [18]. (The convention of Bernoulli numbers in [18] is different from us.) So \( R_{\text{osc}} \) satisfies the classical limit condition.

Also from the symmetry of the differential equation we see that the orthogonal condition \( R_{\text{osc}}(\pm \hbar) R_{\text{osc}}(\pm \hbar) = 1 \) since \( R_{\text{osc}}(\pm \hbar) R_{\text{osc}}(\pm \hbar) = 1 \mod q \).

Since \( \phi_{\alpha,q} \) has the nonequivariant limit obviously, \( R_{\text{osc}} \) has the nonequivariant limit of \( \lambda = 0 \). Finally, we prove the equivariant homogeneity of \( R_{\text{osc}} \). First notice that \( f_q \) is quasi-homogeneous of degree 1, if we assign degree 1 to the integration variables and assign degree 2 (resp. degree 1) to \( q_i \) (resp. \( \lambda_i \)). Expand \( f_q^\sigma \) at critical point \( u_\sigma \) and so that after a coordinate change \( f_q^\sigma = u_\sigma - \sum y_j^2 \) for some local variables \( y_j \).

Now we see that
\[
\int e^{f_q^\sigma / \hbar} = e^{u_\sigma / \hbar} \int e^{-\sum y_j^2 / \hbar (k_0 + k_1 y_j^2 + k_2 y_j^4 \ldots)} \prod_j dy_j
\]
where \( y_j \) has degree 1/2 and \( k_m \) has degree \( -m - \frac{1}{2} \dim X \). Now using the asymptotic formula of the last integral and also of \( \int e^{f_q^\sigma / \hbar} \phi_{\alpha,q} \omega \), we conclude that the degree of \( R_k \) is \( -k \). \( \square \)

4.2. The Virasoro constraints for flag manifolds. In this section we explain in short how to prove the Virasoro conjecture for flag manifolds, applying Givental’s theory [12].

Let \( X \) be a complex projective algebraic manifold with \( H_{\text{odd}}(X) = 0 \). To study the so-called gravitational descendent Gromov-Witten invariants, consider a generating function called the genus \( g \) descendent potential
\[
F_X^g(t := (t_0, t_1, \ldots)) = \sum_{m,d \in H_2(X,\mathbb{Z})} \frac{Q^d}{m!} \int_{\overline{M}_{g,m}(X,d)} \prod_{i=1}^{m} (\sum_{k=0}^{\infty} \psi_i^k e v_i^* t_k)
\]
and the total descendent potential
\[
Z_X = \exp \left( \sum_{g=0,1,\ldots} e^{g-1} F_X^g \right).
\]
Here \( t_i \) are in \( \mathbb{H}^*(X) \). The potential will be considered as a formal function on \( t(h) = t_0 + t_1 h + t_2 h^2 + \ldots \). Define \( q \)-coordinates by the dilaton shift \( q(i/h) = q_0 + q_1 h + q_2 h^2 + \ldots := t(h) - 1 h \). Here 1 is the identity class in \( \mathbb{H}^*(X) \).

When \( X \) is a point, Kontsevich proved the Witten conjecture that the total descendant potential \( Z_{pt} \) is annihilated by specific quadratic differential operators \( \mathcal{L}_m + \delta_{m,0}/16, m = -1, 0, 1, \ldots \) with commuting relation \([\mathcal{L}_m + \delta_{m,0}/16, \mathcal{L}_{m'} + \delta_{m',0}/16] = (m - m')(\mathcal{L}_{m+m'} + \delta_{m+m',0}/16)\). The commutation relation is the Lie algebra of vector fields \(-x^{m+1} \frac{d}{dx}\) on the line. The first four of them are as follows.

\[
\begin{align*}
\hat{L}_{-1} &= \frac{q_0^2}{2\epsilon} + \sum_{m \geq 0} q_{m+1}\partial_m , \\
\hat{L}_0 &= \sum_{m \geq 0} (m + 1/2) q_m \partial_m , \\
\hat{L}_1 &= \epsilon \partial_0^2 / 8 + \sum_{m \geq 0} (m + 1/2)(m + 3/2) q_m \partial_{m+1} , \\
\hat{L}_2 &= 3\epsilon \partial_0 \partial_1 / 4 + \sum_{m \geq 0} (m + 1/2)(m + 3/2)(m + 5/2) q_m \partial_{m+2} .
\end{align*}
\]

Eguchi - Hori - Jinzenji - Xiong and Katz [4, 5] extended the Witten conjecture for Grassmannians \( X \) and for all target spaces \( X \), respectively. The extended conjecture is called the Virasoro conjecture: \( (\mathcal{L}_m^X + \delta_{m,0}/16) Z_X = 0 \) where \( \mathcal{L}_m^X \) are defined by data of cohomology of \( X \) and Chern classes of \( X \), which will be specified later in theorem [3]. However \( \mathcal{L}_m^X = \sum_{\alpha, \beta} \sum_{m \geq 0, \alpha} t_0^\alpha \tilde{t}_0^{\beta} \eta_{\alpha \beta} + \sum_{m \geq 1, \alpha} q_m^{\alpha} \eta_{m, m-1} \) and \( \mathcal{L}_m^X Z_X = 0 \) means the string equation. Here \( \eta_{\alpha, \beta} \) is the Poincare metric and \( t(h) = \sum \alpha t_0^\alpha \phi_\alpha + \sum \alpha t_1^\alpha \phi_\alpha h + \ldots \) with a fixed basis \( \{ \phi_\alpha \} \) of \( \mathbb{H}^*(X) \).

If \( H \) denotes the vector space \( \mathbb{H}^*(X) \) with Poincaré pairing \((,\rangle\), then the quotient ring \( H((h)) \), of formal power series of \( h \) over \( H \), is endowed with a symplectic form \( \Omega \) defined by

\[
\Omega(f, g) = \frac{1}{2\pi i} \oint (f(-h), g(h)) dh
\]

for \( f \) and \( g \) in \( H((h)) \). So, \( \mathcal{H} = H((h)) \) is an infinite dimensional symplectic vector space. Given a transformation \( T \) of \( \mathcal{H} \) which is infinitesimally symplectic, that is \( \Omega(T f, g) + \Omega(f, T g) = 0 \), define a differential operator \( \hat{T} \) as a quantization of \( T \) as follows. First, consider a quadratic function \( \hat{T} \) associated with \( T \) by assignment \( f \mapsto \frac{1}{2} \Omega(f, T f) \). Then take a quantization \( \hat{T} \) of \( \hat{T} \) by the rule: \( p_i p_j \mapsto \epsilon \frac{\partial}{\partial p_i} \frac{\partial}{\partial q_j} , p_i q_j \mapsto q_j \frac{\partial}{\partial q_j} , q_i q_j \mapsto q_i q_j / \epsilon \) with Daboux coordinates \( (p, q) \) of polarization \( \mathcal{H} = \mathcal{H}_+ + \mathcal{H}_- \).
where $\mathcal{H}_+$ is the subspace of nonnegative power series of $\hbar$. For example, if $H = \mathbb{C}^2$ with the standard inner product and $T$ is the multiplication operator by $1/\hbar$, i.e., $Tf = f/\hbar$ with

$$f = ... - (p_2^1 + p_2^2)/\hbar + (p_1^1 + p_1^2)/\hbar^2 - (p_1^1 + p_0^2)/\hbar + (q_1^1 + q_0^2) + (q_1^1 + q_2^2)\hbar + ...$$

(here superscripts are indices), then

$$\hat{T} = \sum_{i=1,2} \left( \frac{(q_i^0)^2}{2\epsilon} + \sum_{m \geq 0} q_{m+1}^i \frac{\partial}{\partial q_m^i} \right).$$

In fact, when $H = \mathbb{C}$, and $D = \hbar \frac{d}{dh} \hbar$, the quantization of $D_m = \hbar^{-1/2} D^{m+1} \hbar^{-1/2}$ are exactly the Virasoro operators $\hat{L}_m$ for $X =$ point.

Define a transformation $S_t$ on $H[[\hbar^{-1}]]$ by $(a, S_t b) = << a, \frac{b}{\hbar - \psi} >>$, where

$$<< a, \frac{b}{\hbar - \psi} >> = (a, b) + \sum_{0 \neq d \in H_2(x), m = 0, l = 0}^{\infty, \infty} \frac{Q^d}{\hbar^{l+1} m!} \int_{[\hbar^{1/2}(x, d)]_{\text{virt}}} (ev^*_i a)(\prod_{i=2}^m ev^{*}_i t)(ev^{*}_{m+2} b) \psi^i_{m+2}.$$

As introduced for $X = Fl(n + 1)$, for general $X$ we have notions of semi-simple quantum cohomology, canonical coordinates $u_\alpha$, a pencil of flat connections and an asymptotic fundamental solution $\Psi \mu_{\hbar + 1/2}$. Here $R$ is form of $1 + R_1 \hbar + ...$ satisfying the orthogonality condition $R^*(\hbar) R(\hbar) = 1$. Such $R$ is unique up to right multiplication by diagonal matrices. However $R$ is uniquely determined if the homogeneity condition is imposed. Now if $T$ is $S$ or $R(\hbar)$, then $T$ could be viewed as a symplectic transformation of suitable completions of $\mathcal{H}$ since $T^*(\hbar) T(\hbar) = 1$. Let $\hat{T} = \exp(\ln T)$. Denote $(q^1(\hbar), ..., q^N(\hbar)) = \Psi^{-1} q(\hbar)$ for $q(\hbar) \in \mathcal{H}_+$ and define an operator $\hat{\Psi}$ by $f(\Psi^{-1} q) \mapsto f(q)$. Notice that $q^i(\hbar)$-coordinate system is based on the orthonormal frame $\Delta_u^a \frac{\partial}{\partial u_a}$. The homogeneity condition of $R$ is $E(\Psi \mu_{\hbar + 1/2} = \mu(\Psi \mu_{\hbar + 1/2})$, where $E = \hbar \partial + \sum u_a \partial u_a$ and $\mu = \text{diag}(\phi_1 - \text{dim}_C X/2, ..., \phi_N - \text{dim}_C X/2)$. On the other hand $S$ satisfies the homogeneity condition of $E S = \mu S + S(\mu + \rho/\hbar)\hbar$, where $\rho$ is the operator of multiplication by $c_1(T_X)$ in ordinary cohomology ring. Define

$$\mathcal{L}_m^X = \hat{S}_u^{-1} \hat{\Psi} \hat{R} \hat{L}_m \hat{R}^{-1} \hat{\Psi}^{-1} \hat{S}_u$$

for $m = -1, 0, 1, 2, ..., \text{where } L_m$ is taken as $D_m$ with $H = H^*(X)$.

The following theorem in [12] explicitly shows that the operator $\mathcal{L}_m^X$ is completely determined by topological terms.
Theorem 5. The operator \( \mathcal{L}_m^X \) is 
\[
\hat{L}_{m}^{\mu, \rho} + \frac{\delta_{m,0}}{4} \text{tr}(\mu \mu^*),
\]
where 
\[
L_{m}^{\mu, \rho} = \hbar^2 h^{-\rho} L_{m} h^\rho h^{-\mu} = h^{-1/2} (\hbar \frac{d}{d\hbar} - \mu h + \rho) m + 1 h^{-1/2}, m \geq -1.
\]

The Virasoro operators \( \hat{L}_{m}^{\mu, \rho} + \frac{\delta_{m,0}}{4} \text{tr}(\mu \mu^*) \) agree with the operators in \([3]\). The theorem holds for a conformal semi-simple Frobenius manifolds.

Let a projective manifold \( X \) have a Hamiltonian torus \( T \) action with isolated fixed points and let \( u \) be a semisimple point of \( H^*_T(X) \). The previous potentials have the obvious equivariant counterpart. The following is shown in \([11]\) and is reformulated in \([12]\) as stated here.

**Theorem 6.** In the equivariant setting of Gromov-Witten theory,
\[
Z_T^X(t(\hbar)) = e^{C(u) \hat{S}^{-1}_u \hat{\Psi} \hat{R} e^{(U/\hbar)}} \prod_{i=1}^{N} Z_{pt}(q^i(\hbar))
\]
if \( R(z) \) is normalized by the classical limit condition in theorem \([4]\) and
\[
C(u) = \frac{1}{2} \int_{u}^{u} \sum_{i} (R_1)_i du^i
\]
is defined up to addition of constant.

**Remark.** According to \([12]\) the right side of the equation of the above theorem does not depend on the choices of a semi-simple point \( u \), even though each term may depend on the choices.

The theorem shows
\[
(\mathcal{L}_m^{X,T} + N \delta_{m,0}/16) Z_X^T = 0,
\]
where \( \mathcal{L}_m^{X,T} \) is the equivariant counterpart of \( \mathcal{L}_m^X \).

**Theorem 7.** The total descendent potential \( Z_X \) of a flag manifold \( X \) coincides with \( e^{C(u) \hat{S}^{-1}_u \hat{\Psi} \hat{R} e^{(U/\hbar)}} \prod_{i=1}^{N} Z_{pt}(q^i(\hbar)) \).

**Proof.** Take the nonequivariant limit of the equation of theorem \([3]\) at a semi-simple point \( u \) in \( H^*_T(X) \). The left side of the equation is specialized to the ordinary total descendent potential. The limit of \( R \) on the right side exits. The limit is the homogeneous ordinary \( R \) due to theorem \([4]\) combined with the uniqueness of homogeneous \( R|_{H^*_T(X)} \).

\( \square \)

Combined with theorem \([5]\) the above theorem shows the following corollary.

**Corollary 3.** The Virasoro conjecture for flag manifolds \( X \) holds:
\[
(\mathcal{L}_m^X + N \delta_{m,0}) Z_X = 0, m \geq -1.
\]
Acknowledgment. B.K. would like to thank A. Givental, D. van Straten for useful discussions and J.-H. Yang for informing the existence of the paper [17]. The authors also thank J. Byeon for numerous discussions on oscillatory integrals. B.K. thanks staffs in ESI for their warm hospitality while his visit to the institute, being writing the paper. D.J. is supported by KOSEF 2000-2-10100-002-3. B.K. is supported by KOSEF 1999-2-102-003-5 and R03-2001-00001.

References

[1] P. Candelas, X. C. de la Ossa, P. S. Green, L. Parkes, A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory, Nucl. Phys. B, 359 (1991), 21-74.
[2] B. Dubrovin, Geometry of 2D topological field theories In Integrable Systems and Quantum Groups, Springer Lecture Notes in Math, 1620 (1996), 120-348.
[3] B. Dubrovin and Y. Zhang, Frobenius manifolds and Virasoro constraints, Selecta Math. (N.S.) 5 (1999), 423-466.
[4] T. Eguchi, K. Hori, C.-S. Xiong Quantum cohomology and Virasoro algebra, Phys. Lett. B 402 (1997) 71-80.
[5] T. Eguchi, M. Jinzenji, C.-S. Xiong, Quantum cohomology and free field representations Nuclear Phys. B 510 1998, 608-622.
[6] A. Givental, Homological geometry and mirror symmetry in Proceedings of the International Congress of Mathematicians, 1994, Zürich, Birkhäuser, Basel, 1995, 472-480.
[7] A. Givental, Equivariant Gromov-Witten invariants, Internat. Math. Res. Notices, 13 (1996), 613-663.
[8] A. Givental, Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture, A.M.S. Transl. (2) 180 (1997), 103-115.
[9] A. Givental, A mirror theorem for toric complete intersections, Topological field theory, primitive forms and related topics (Kyoto, 1996), 141-175, Progr. Math., 160, Birkhäuser Boston, Boston, MA, 1998.
[10] A. Givental, Elliptic Gromov-Witten invariants and the generalized mirror conjecture, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 107-155, World Sci. Publishing, River Edge, NJ, 1998.
[11] A. Givental, Semisimple Frobenius structures at higher genus, Internat. Math. Res. Notices, 23 (2001), 1265-1286.
[12] A. Givental, Gromov-Witten invariants and quantization of quadratic hamiltonians, math.AG/0108100.
[13] H. Jacquet, Fonctions de Whittaker associées aux groupes de Chevalley, Bull. Soc. Math. France 95 (1967), 243-309.
[14] B. Kim, Quantum cohomology of flag manifolds G/B and quantum Toda lattices, Annals of Math. 149 (1999), 129-148.
[15] B. Kim, Quantum hyperplane section theorem for homogeneous spaces, Acta Math. 183 (1999), 71-99.
[16] B. Kostant, The solution to a generalized Toda lattice and representation theory Adv. in Math. 34 (1979), 195-338.
[17] E. Stade, On explicit integral formulas for GL(n, R)-Whittaker functions, Duke Math. J. 60 No. 2 (1990), 313-362.
[18] E. Whittaker and G. Watson, *A course of modern analysis*, Cambridge University Press, 4th edition, Reprinted in 1978.

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