Lamination exact relations and their stability under homogenization

Yury Grabovsky

May 3, 2014

Abstract

Relations between components of the effective tensors of composites that hold regardless of composite’s microstructure are called exact relations. Relations between components of the effective tensors of all laminates are called lamination exact relations. The question of existence of sets of effective tensors of composites that are stable under lamination, but not homogenization was settled by Milton with an example in 3D elasticity. In this paper we discuss an analogous question for exact relations, where in a wide variety of physical contexts it is known (a posteriori) that all lamination exact relations are stable under homogenization. In this paper we consider 2D polycrystalline multi-field response materials and give an example of an exact relation that is stable under lamination, but not homogenization. We also shed some light on the surprising absence of such examples in most other physical contexts (including 3D polycrystalline multi-field response materials). The methods of our analysis are algebraic and lead to an explicit description (up to orthogonal conjugation equivalence) of all representations of formally real Jordan algebras as symmetric $n \times n$ matrices. For each representation we examine the validity of the 4-chain relation—a 4th degree polynomial identity, playing an important role in the theory of special Jordan algebras.

1 Introduction

The study of effective behavior of composite materials abounds with beautiful formulas linking the components of the effective tensor and the tensor of material properties of its constituents, when virtually nothing is known about the microstructure of the composite [22, 27, 31, 43, 20, 47, 34, 35, 12, 13, 11, 5, 6, 33] (see also a review by Milton [39]). The general theory of such formulas was developed in [15, 18, 16]. The idea was to identify all relations that are preserved in all laminate microstructures, where two constituent materials are combined in layers perpendicular to a given unit vector (lamination direction). According to the general theory, reviewed in Section 2, each relation preserved under lamination corresponds to a Jordan multialgebra—a subspace of the space of symmetric matrices closed with respect to several Jordan multiplications. Mutations of Jordan algebras [28] are particular examples of Jordan multialgebras, where all multiplications are “internal”, i.e. coming from a single Jordan product in a classical Jordan algebra. In the context of exact relations not every Jordan multialgebra is a mutant of a classical one. The fundamental problem in the theory of exact relations is whether every Jordan multialgebra gives rise to a microstructure-independent relation. If this is the case, then all lamination exact relations are stable under homogenization. In [18, 16] we obtained an algebraic condition on Jordan multialgebras sufficient for stability under homogenization. In order to describe it we define completion of a Jordan multialgebra to be the set of all symmetric matrices in the smallest associative multialgebra containing the original Jordan multialgebra. If the Jordan multialgebra is complete, i.e. equal to its completion, then the corresponding lamination exact relation is stable under homogenization. Mutations of classical special Jordan algebras do not seem to posses superior completion properties compared to the general Jordan multialgebras, and hence, will not be investigated as a special subclass in this paper. The completeness for classical special Jordan algebras is related to the 4th degree polynomial identity, called the 4-chain relation, via Cohn’s theorem [7]. It is well-known that classical special Jordan algebras do not have to be complete. However, from the point of view of the theory of composite materials the situation may be reversed if we restrict our attention only to those Jordan multialgebras that arise in physics. In particular,
we will focus on the Jordan multialgebras corresponding to the polycrystalline composites \[15, 19, 18\]. These possess additional structure of a representation space of \(SO(3)\) or \(SO(2)\) (for 2D or fiber-reinforced composites, whose microgeometry is completely determined by any 2D cross-section). The interaction between the group representation structure of \(SO(d)\) and the multiplicative structure of Jordan multialgebras has been studied in a series of papers \[44, 46, 45, 30\] inspired by the theory of exact relations.

While there is no general theorem that guarantees completeness for \(SO(3)\) or \(SO(2)\)-invariant Jordan multialgebras, all known physically relevant examples of \(SO(3)\) or \(SO(2)\)-invariant Jordan multialgebras are complete. Is it a coincidence or a new fundamental property specific to rotationally invariant Jordan multialgebras? In this paper we exhibit a physically motivated example in 2D featuring an \(SO(2)\)-invariant incomplete Jordan multialgebra and the corresponding lamination exact relation that is not stable under homogenization (see (8.4)). This needs to be compared with an earlier result in \[18\] that all \(SO(3)\)-invariant Jordan multialgebras in the 3D analog of our example are complete, giving further support for the conjecture that all physically relevant \(SO(3)\)-invariant Jordan multialgebras are complete (see Conjecture 2.5 below).

We remark that our example of \(SO(2)\)-invariant incomplete Jordan multialgebra could lead to another example of a rank-1 convex function that is not quasiconvex. While such an example has already been produced both in Calculus of Variations \[48\] and the theory of composite materials \[40\, Section 39.9\], our example, coming from the theory of exact relations would possess additional \(SO(2)\) symmetry absent in the existing examples.

While the 3D version of our example, described in Section 3, can be completely analyzed (see Theorem 3.1), the 2D version is more involved. In the more difficult 2D case we only exhibit a specific family of Jordan multialgebras generated by the subalgebras of \(\text{Sym}(\mathbb{R}^n)\)— the Jordan algebra of all real symmetric \(n \times n\) matrices. The question of completeness of these subalgebras has not been studied. The answer to this question represents an algebraic contribution of this paper, which includes a complete characterization of all faithful representations of all formally real Jordan algebras in \(\text{Sym}(\mathbb{R}^n)\) up to orthogonal conjugation. We note that all formally real Jordan algebras, up to a Jordan algebra isomorphism, were described in \[26\], where an attempt was made to build an algebraic foundation of quantum mechanics, where the observables would be defined axiomatically, and not as operators on a Hilbert space with the extraneous structure of associative multiplication.

The explicit characterization of Jordan subalgebras of \(\text{Sym}(\mathbb{R}^n)\), reveals that completeness can indeed fail, but only in a few exceptional cases. Adding more multiplications with respect to which the algebras have to be closed makes these cases even more exceptional, so that in the relatively low dimensions completeness holds in general. Specifically, all Jordan subalgebras of \(\text{Sym}(\mathbb{R}^n)\) are complete, if \(n < 8\). Even for general \(n\) completeness is generic. A generic symmetric \(n \times n\) matrix has \(n\) distinct eigenvalues. Any subalgebra of \(\text{Sym}(\mathbb{R}^n)\) containing such a matrix must necessarily be complete. This sheds light on the conspicuous absence of incomplete Jordan multialgebras from all physically relevant examples so far, since all of them, with the exception of the example in Section 3, are relatively low dimensional.

2 General theory of exact relations

The standard references \[4, 25\] for the mathematical theory of composite materials emphasize homogenization theorems and deal primarily with conducting and elastic composites. In the case of periodic composites there is an abstract Hilbert space framework \[29, 10, 41, 36, 37, 38\] that treats all coupled field composites (e.g. piezo-electric, thermo-elastic, etc.) from the common point of view. We refer to \[40\] for a systematic treatment of composites and exact relations based on these ideas. In this paper we adopt the abstract Hilbert space formalism. Since our specific question is algebraic in nature we give here an algebra-centered overview of the general theory of exact relations. We refer the reader to \[18, 40\] for a composite material-centered exposition.

A linear material response to applied fields (electric, elastic, thermal, etc.) is described by a symmetric positive definite operator \(L\) on a finite dimensional inner product space \(\mathcal{T}\). If we are interested in polycrystalline composites the space \(\mathcal{T}\) is also a representation space of the rotation group \(SO(3)\) or \(SO(2)\), if we deal with fiber-reinforced composites, whose microstructure is determined by any cross-section perpendicular to
the chosen “fiber direction”. In this case the inner product on $T$ is $SO(d)$-invariant, $d = 2, 3$.

In making a composite we choose a compact set of admissible materials $U \subset \text{Sym}^+(T)$, where $\text{Sym}^+(T)$ is the set of symmetric positive definite operators on $T$. In the case of polycrystalline composites the set $U$ must be $SO(d)$-invariant. The effective tensor $L^*$ of a composite depends on the microstructure. Varying the microstructure over all possible ones we obtain the set $G(U)$ of corresponding effective tensors $L^*$ of composites, called the G-closure of $U$.

Generically the set $G(U)$ has a non-empty interior in $\text{Sym}^+(T)$, even if $U$ consists of only 2 points. We are interested in special situations, where $G(U)$ is a submanifold of $\text{Sym}^+(T)$ of non-zero co-dimension.

**Definition 2.1.** The submanifold $\mathcal{M}$ of $\text{Sym}^+(T)$ is called an exact relation if the effective tensor $L^*$ of a composite will lie in $\mathcal{M}$, whenever constituent materials lie in $\mathcal{M}$, regardless of the microstructure.

### 2.1 Lamination exact relations and Jordan multialgebras

In order to identify these special exact relations cases we test G-closure set by taking two arbitrary points $\{L_1, L_2\} \subset G(U)$ and forming a laminate—a composite consisting of layers of material $L_1$ alternating with layers of material $L_2$. Such a laminate is described by the direction of the lamination $\mathbf{n} \in \mathbb{S}^{d-1}$ ($d = 2$ or 3) and the volume fractions $\theta_1, \theta_2 = 1 - \theta_1$ of $L_1$ and $L_2$, respectively. For the simple laminate microstructure there exists a beautiful explicit formula for the effective tensor [38, 40]

$$W_n(L^*; L_0) = \theta_1 W_n(L_1; L_0) + \theta_2 W_n(L_2; L_0),$$

(2.1)

where $L_0$ is an arbitrary “reference material” and the map $W_n(L, L_0)$ has the form

$$W_n(L; L_0) = [(L_0 - L)^{-1} - \Gamma_{L_0}(\mathbf{n})]^{-1} = [I_T - (L_0 - L)\Gamma_{L_0}(\mathbf{n})]^{-1}(L_0 - L),$$

where $I_T$ is the identity map on $T$, and $\Gamma_{L_0}(\mathbf{n}) \in \text{Sym}(T)$ is an explicitly known function of $\mathbf{n} \in \mathbb{S}^{d-1}$, whose specific definition is not important for our purposes. We remark, that even though the formula (2.1) involves $L_0$, the effective tensor $L^*$ defined by it is independent of $L_0$. Restricting attention only to laminate microstructures we formulate the notion of lamination exact relation.

**Definition 2.2.** A lamination exact relation is a submanifold $\mathcal{M}$ of positive co-dimension in $\text{Sym}^+(T)$ such that the effective tensor $L^*$ of a laminate made with $\{L_1, L_2\} \subset \mathcal{M}$ is in $\mathcal{M}$ for any choice of laminate direction $\mathbf{n} \in \mathbb{S}^{d-1}$ and volume fraction $\theta_1 \in [0, 1]$.

It is clear from (2.1) that $W_n(\mathcal{M}; L_0)$ is a convex subset of $\text{Sym}(T)$ and at the same time a submanifold of $\text{Sym}(T)$ of the same co-dimension as $\mathcal{M}$. Therefore, $W_n(\mathcal{M}; L_0)$ is an affine subspace of $\text{Sym}(T)$. However, choosing $L_0 \in \mathcal{M}$ and observing that $W(L_0; L_0) = 0$ implies that the affine subspaces $\Pi_\mathbf{n} = W_n(\mathcal{M}; L_0)$ are linear subspaces of $\text{Sym}(T)$. The function $\Lambda_{\mathbf{m,n}} = W_n \circ W_n^{-1}$ maps $\Pi_\mathbf{n}$ into $\Pi_\mathbf{m}$ diffeomorphically (at least in the neighborhood of 0). We easily compute

$$\Lambda_{\mathbf{m,n}}(K) = [K^{-1} + \Gamma_{L_0}(\mathbf{n}) - \Gamma_{L_0}(\mathbf{m})]^{-1} = [I_T - (\Gamma_{L_0}(\mathbf{m}) - \Gamma_{L_0}(\mathbf{n}))K]^{-1}K.$$ 

We see that the differential of $\Lambda_{\mathbf{m,n}}$ at $K = 0$ is the identity map: $d\Lambda_{\mathbf{m,n}}(0)\xi = \xi$ for any $\xi \in T_0\Pi_\mathbf{n}$. That means that the tangent spaces of $\Pi_\mathbf{n}$ and $\Pi_\mathbf{m}$ are the same. It follows that $\Pi_\mathbf{n} = \Pi_\mathbf{m}$, since the submanifolds $\Pi_\mathbf{n}$ and $\Pi_\mathbf{m}$, being subspaces of $\text{Sym}(T)$, coincide with their tangent spaces. Thus, $\Pi = W_n(\mathcal{M}; L_0)$ is a well-defined subspace of $\text{Sym}(T)$, independent of $\mathbf{n}$. Expanding the map $\Lambda_{\mathbf{n,m}} : \Pi \to \Pi$ in powers of $K$ we obtain

$$\Lambda_{\mathbf{n,m}}(K) = K\Lambda_{\mathbf{m,n}}(K) + K\Lambda_{\mathbf{m,n}}K\Lambda_{\mathbf{m,n}}(K) + K\Lambda_{\mathbf{m,n}}K\Lambda_{\mathbf{m,n}}K + \cdots ,$$

where $\Lambda_{\mathbf{m,n}} = \Gamma_{L_0}(\mathbf{m}) - \Gamma_{L_0}(\mathbf{n})$. We conclude that for $\mathcal{M}$ to be a lamination exact relation it is necessary and sufficient that

$$KAK \in \Pi, \quad K \in \Pi, \quad A \in \mathcal{A} = \text{Span}\{\Gamma_{L_0}(\mathbf{m}) - \Gamma_{L_0}(\mathbf{n}) : |\mathbf{n}| = 1\}. $$

(2.2)
It is easy to check that the subspace $A$ does not depend on $m$ (regardless of what $\Gamma_{L_0}(m)$ is). The subspaces $\Pi$ satisfying (2.2) are Jordan multialgebras, since they are closed with respect to a family of Jordan multiplications

$$K_1 *_A K_2 = \frac{1}{2}(K_1AK_2 + K_2AK_1), \quad A \in A.$$  

To avoid any ambiguity we give the following definition.

**Definition 2.3.** A subspace $\Pi$ of an associative algebra $B$ is called a **Jordan $A$-multialgebra** if it is closed with respect to a collection of Jordan multiplications

$$X *_A Y = \frac{1}{2}(XAY + YAX), \quad A \in A,$$  

(2.3)

where $A$ is a subspace of $B$. A classical definition of a special Jordan algebra corresponds to 1-dimensional subspaces $A$. A subspace $\Pi'$ of $B$ is called an **associative $A$-multialgebra** if it is closed with respect to a collection of multiplications $(X,Y) \mapsto XAY$, $A \in A$. The associative $A$-multialgebra $\Pi'$ is called **symmetric** if $X \in \Pi'$ implies $X^T \in \Pi'$.

We remark that the notion of mutation of a Jordan algebra [28] provides a family of examples of Jordan multialgebras (they can be called “inner multialgebras”). If $\Pi_0$ is closed with respect to a single product $X *_{A_0} Y$ then it will always be closed with respect to a family of multiplications (2.3), where $A = \{A_0BA_0 : B \in \Pi_0\}$. The question of comparing algebraic structures of an algebra and its mutation is studied in [28], and is not the object of our investigation.

### 2.2 Exact relations and complete Jordan multialgebras

Recall that Jordan multialgebras $\Pi$ correspond to the lamination exact relations (see Definition 2.2). The fundamental question in the theory of exact relations is whether or not all lamination exact relations are exact relations in the sense of Definition 2.1. This question was investigated in [15, 18, 16, 40], where simple algebraic sufficient conditions have been formulated. To state this condition we define the notion of complete Jordan $A$-multialgebra. Starting with a Jordan $A$-multialgebra $\Pi$ we define $A(\Pi)$ to be the smallest associative $A$-multialgebra in the sense of Definition 2.3 containing $\Pi$. The associative $A$-multialgebra $A(\Pi)$ is necessarily symmetric in the sense of Definition 2.3. We then define a completion $\overline{\Pi}$ of $\Pi$ by

$$\overline{\Pi} = A(\Pi)_{\text{sym}},$$  

(2.4)

where

$$V_{\text{sym}} \overset{\text{def}}{=} V \bigcap \text{Sym}(T) = \{K + K^T : K \in V\}$$

for a subspace $V \subset \text{End}(T)$, such that $V^T = V$.

**Definition 2.4.** A Jordan $A$-multialgebra $\Pi \subset \text{Sym}(T)$ is called **complete** if

$$\overline{\Pi} = \Pi,$$  

(2.5)

We will say that $\Pi$ is **incomplete**, if $\Pi$ is not complete.

It is easy to see that the completion $\overline{\Pi}$ of $\Pi$ is the smallest complete Jordan $A$-multialgebra containing $\Pi$. According to the general theory [18] [16], complete Jordan $A$-multialgebras $\Pi$ correspond to exact relations $M = W_n^{-1}(\Pi; L_0)$.

It is easy to show that $\Pi$ is complete if and only if 3 and 4-chain identities

$$K_1A_1K_2A_2K_3 + K_3A_2K_2A_1K_1 \in \Pi, \quad K_1A_1K_2A_2K_3A_3K_4 + K_4A_3K_3A_2K_2A_1K_1 \in \Pi$$  

(2.6)

It would be more proper to use the term special Jordan $A$-multialgebra. In this paper we omit the qualifier “special”, since we do not deal with the most general Jordan algebras over a general field.
hold for all \( \{K_1, K_2, K_3, K_4\} \subset \Pi \) and all \( \{A_1, A_2, A_3\} \subset A \). The proof is a straightforward modification of the proof of Cohn’s theorem \([7, 24]\). We remark that in the case of classical Jordan algebras (one-dimensional subspace \( A \)) the 3-chain relations hold automatically, while this is not so for Jordan multialgebras. This is a crucial distinction between classical special Jordan algebras and Jordan multialgebras, since a significant part of the classical Jordan algebra theory is based on the triple product \([24, 28, 32]\)

\[
\{K_1, K_2, K_3\} = K_1K_2K_3 + K_3K_2K_1.
\]

It is well-known that not every classical special Jordan algebra is complete. However, the examples given in textbooks are not subalgebras of \( \text{Sym}(\mathbb{R}^n) \)—the space of all symmetric \( n \times n \) matrices. Additionally, even if there are incomplete subalgebras of \( \text{Sym}(\mathbb{R}^n) \), it is still not clear if the presence of several multiplications would not change the situation. In fact, we have already shown \([18]\) that all Jordan \( SO(2) \) and \( SO(3) \)-invariant multialgebras in important physical contexts, such as conductivity, elasticity, piezo-electricity, etc. are complete. In an attempt to resolve this apparent contradiction between classical theory of Jordan algebras and physically relevant examples we describe a class of \( SO(2) \) and \( SO(3) \)-invariant Jordan multialgebras.

### 2.3 \( SO(d) \)-invariant Jordan multialgebras

In this section it is important that \( T \) be a real finite dimensional representation of \( SO(d) \) (\( d = 2 \) or \( 3 \)). The irreducible representations (irreps) are parametrized by non-negative integers, called weights. In the theory of composite materials, where one is interested in coupled thermal, electric and elastic properties of materials, the space \( T \) can contain only the irreps of weights 0, 1 and 2. The weight 0 irrep \( \Lambda_0 \) is a 1-dimensional space with trivial group action. The weight 1 irrep \( \Lambda_1 \) is \( \mathbb{R}^d \) with the natural action of \( SO(d) \), while the weight 2 irrep \( \Lambda_2 \) is the space \( \text{Sym}_0(\mathbb{R}^d) \) of symmetric trace-free \( d \times d \) matrices with \( SO(d) \) acting by conjugation \( A \mapsto RAR^{-1} \). Thus, all conceivable coupled field physical problems would be accommodated by

\[
\mathcal{T} = \Lambda_0 \oplus \mathbb{R}^n_0 \oplus \Lambda_1 \oplus \mathbb{R}^n_1 \oplus \Lambda_2 \oplus \mathbb{R}^{n_2}, \quad n_0 \geq 0, \; n_1 \geq 0, \; n_2 \geq 0
\]

(2.7)

for an appropriate choice of \( n_0, n_1 \) and \( n_2 \). The notation in (2.7) emphasizes that the group acts trivially on the second factors in tensor products above. We assume that the representation \( \mathcal{T} \) is equipped with a fixed \( SO(d) \)-invariant inner product. The space \( \text{Sym}(\mathcal{T}) \) of symmetric maps on \( \mathcal{T} \) has a natural action of \( SO(d) \) and splits into the direct sum of irreps up to weight 4. In addition to the representation \( \mathcal{T} \) the algebraic structure of our problem is determined by the choice of a subrepresentation \( A \subset \text{Sym}(\mathcal{T}) \), which must be isomorphic to \( \Lambda_2 \oplus \Lambda_4 \) as an \( SO(d) \) module (or only \( \Lambda_2 \), if \( n_2 = 0 \)). The choice of such a subrepresentation is highly non-unique, and we assume that a generic choice is made. The goal is to answer the fundamental question of completeness for \( SO(d) \)-invariant Jordan \( A \)-multialgebras.

**Conjecture 2.5.** Assume that \( \mathcal{T} \) is given by (2.7) and \( A \subset \text{Sym}(\mathcal{T}) \) is isomorphic to \( \Lambda_2 \oplus \Lambda_4 \) as an \( SO(3) \) module. We conjecture that any rotationally invariant \( A \)-multialgebra is complete.

The rationale behind this conjecture is a positive result established in \([18]\) (see also Section 3), when \( n_0 = n_2 = 0 \) and \( A = \Lambda_2 \oplus \mathbb{R}I_{n_1} \), where \( I_n \) denotes the \( n \times n \) identity matrix. The conjecture is stated only for \( SO(3) \)-invariant Jordan \( A \)-multialgebras, since this paper settles the \( SO(2) \)-invariant version of the conjecture in the negative (see Section 3). In addition to the results of Section 3 we have also computed a complete list of Jordan \( A \)-multialgebras for \( n_0 = n_1 = n_2 = 1 \) in \([18]\). The conjecture was then verified by hand for each individual Jordan \( A \)-multialgebra.

### 3 Case study: Multifield response composite materials

Multifield linear response materials were considered in \([35, 34]\). In this context \( n \) coupled potential fields \( E = (\nabla \phi_1, \ldots, \nabla \phi_n) \) induce \( n \) conjugate fluxes \( J = (j_1, \ldots, j_n) \) satisfying

\[
\nabla \cdot j_1 = \ldots = \nabla \cdot j_n = 0.
\]

5
Thermoelectric properties of composites would fit in this context with \( n = 2 \). For general values of \( n \) the space \( \mathcal{T} \) is a direct sum of \( n \) copies of \( \mathbb{R}^d \):
\[
\mathcal{T} = \mathbb{R}^d \oplus \ldots \oplus \mathbb{R}^d = V_1 \otimes \mathbb{R}^n, \quad d = 2, 3,
\]
where \( V_1 \) is the \( SO(d) \) irrep of weight 1, i.e. \( V_1 = \mathbb{R}^d \) with standard action of \( SO(d) \). The space \( \mathcal{A} \) was computed in \cite{18}.
\[
\mathcal{A} = W_2 \otimes I_n \subset \text{Sym}(\mathcal{T}),
\]
where \( W_2 \subset \text{Sym}(V_1) \) is the \( SO(d) \) irrep of weight 2, denoting the space of symmetric, trace-free \( d \times d \) matrices where \( SO(d) \) acts by conjugation. For the case \( d = 3 \) all \( SO(3) \)-invariant Jordan \( \mathcal{A} \)-multialgebras have been computed in \cite{18}. The result is given in Theorem 3.1 below.

**Theorem 3.1.** Let \( \Pi \) be an \( SO(3) \)-invariant Jordan \( \mathcal{A} \)-multialgebra in \( \text{Sym}(V_1 \otimes \mathbb{R}^n) \), where \( \mathcal{A} = W_2 \otimes I_n \). Then there exists an associative subalgebra \( \mathcal{B} \) of \( \text{End}(\mathbb{R}^n) \), such that \( \mathcal{B}^T = \mathcal{B} \) and \( \Pi = (\text{End}(W_1) \otimes \mathcal{B})_{\text{sym}} \).

An immediate consequence of this theorem is that all \( SO(3) \)-invariant Jordan \( \mathcal{A} \)-multialgebras are complete.

When \( d = 2 \) we can label points in \( V_1 = \mathbb{R}^2 \) with standard action of \( SO(2) \), by complex numbers, so that the action of \( R_\theta \in SO(2) \) on \( z \in \mathbb{C} \) is given by \( e^{i\theta}z \). Here \( R_\theta \) denotes the rotation through the angle \( \theta \) in counterclockwise (positive) direction. We write
\[
\mathcal{T} = \mathbb{C} \otimes \mathbb{R}^2 = \mathbb{C}^n, \quad R_\theta \cdot u = e^{i\theta}u, \quad u \in \mathbb{C}^n, \quad R_\theta \in SO(2).
\]
The \( SO(2) \)-invariant inner product on \( \mathcal{T} \) is \( \langle u, v \rangle_\mathcal{T} = \Re \langle u, v \rangle \), where \( \langle \cdot, \cdot \rangle \) is the standard Hermitean inner product on \( \mathbb{C}^n \). Every \( K \in \text{End}(\mathcal{T}) = \text{End}_\mathbb{R}(\mathbb{C}^n) \) is uniquely determined by two complex \( n \times n \) matrices \( X \) and \( Y \) via its action on \( u \in \mathbb{C}^n \):
\[
Ku = Xu + Y\mathcal{I}.
\]
Therefore, we will write \( K(X, Y) \) to identify elements of \( \text{End}(\mathcal{T}) \). We easily compute
\[
\text{Sym}(\mathcal{T}) = \{ K(X, Y) : X \in \mathcal{H}(\mathbb{C}^n), \ Y \in \text{Sym}(\mathbb{C}^n) \},
\]
where \( \mathcal{H}(\mathbb{C}^n) \) denotes the set of all complex Hermitean \( n \times n \) matrices. In this notation
\[
\mathcal{A} = \{ K(0, zI_n) : z \in \mathbb{C} \}, \quad R_\theta \cdot K(X, Y) = K(X, e^{i\theta}Y).
\]
Therefore, an arbitrary \( SO(2) \) submodule of \( \text{Sym}(\mathcal{T}) \) is given by
\[
\Pi = \Pi_{\mathcal{L}, \mathcal{M}} = \{ (X, Y) : X \in \mathcal{L} \subset \mathcal{H}(\mathbb{C}^n), \ Y \in \mathcal{M} \subset \text{Sym}(\mathbb{C}^n) \},
\]
where \( \mathcal{L} \) can be any real subspace of \( \mathcal{H}(\mathbb{C}^n) \) and \( \mathcal{M} \) can be any complex subspace of \( \text{Sym}(\mathbb{C}^n) \). If an \( SO(2) \) submodule \( \Pi \) is also an \( \mathcal{A} \)-multialgebra, then the subspaces \( \mathcal{L} \) and \( \mathcal{M} \) have to satisfy
\[
Y^2 + XX^T \in \mathcal{M}, \quad XY + YX^* \in \mathcal{L} \quad \text{for all } X \in \mathcal{L}, \ Y \in \mathcal{M}. \tag{3.1}
\]
The 3-chain condition is equivalent to
\[
cX_1X_2^T X_3 + \tau X_3X_2^T X_1 \in \mathcal{L}, \quad \text{for all } c \in \mathbb{C}, \text{ and all } \{ X_1, X_2, X_3 \} \subset \mathcal{L}. \tag{3.2}
\]
The 4-chain condition is a lot more complicated.

The complete characterization of all solutions of (3.1) is unknown. However, some families of solutions can be easily identified. We will focus on one such family, since it will lead us to incomplete \( SO(2) \)-invariant Jordan \( \mathcal{A} \)-multialgebras, in contrast to Theorem 3.1 in 3D.
Let \( \Pi_0 \subset \text{Sym}(\mathbb{R}^n) \) be a Jordan subalgebra of \( \text{Sym}(\mathbb{R}^n) \), i.e. \( \Pi_0 \) is a subspace of \( \text{Sym}(\mathbb{R}^n) \) closed with respect to the Jordan product
\[
A \ast B = \frac{1}{2}(AB + BA).
\]
Then \( L = \{0\} \), \( M = \Pi_0 \otimes \mathbb{C} \) solves (3.4). The 3-chain condition (2.2) is therefore trivially satisfied. The 4-chain condition from (2.6) reduces to the classical 4-chain condition
\[
Y_1 Y_2 Y_3 Y_4 + Y_4 Y_3 Y_2 Y_1 \in \Pi_0 \quad \text{for all} \quad \{Y_1, Y_2, Y_3, Y_4\} \subset \Pi_0
\]
(3.5)

Our next goal is to characterize all Jordan subalgebras \( \Pi_0 \) of \( \text{Sym}(\mathbb{R}^n) \) up to orthogonal conjugation. We observe that all Jordan subalgebras of \( \text{Sym}(\mathbb{R}^n) \) are formally real, i.e. if
\[
X_1^2 + \ldots + X_m^2 = 0, \quad m \geq 1, \quad \{X_1, \ldots, X_m\} \subset \text{Sym}(\mathbb{R}^n)
\]
then \( X_1 = \ldots = X_m = 0 \). The complete classification of all formally real Jordan algebras, up to an isomorphism, has been achieved by Jordan, von Neumann and Wigner in [26]. Our goal is to go a little further and classify all faithful representations of formally real Jordan algebras in \( \text{Sym}(\mathbb{R}^n) \). We note that completeness in the sense of Definition 2.4 is a property of the representation of an algebra, not of the algebra itself. In fact, our results will produce an example of two different faithful representations of the same algebra in \( \text{Sym}(\mathbb{R}^n) \), one of which is complete, while the other is not.

4 Formally real Jordan algebras

In this section we review the complete classification by Jordan, von Neumann and Wigner of all formally real Jordan algebras for the convenience of the reader. In addition to the classification itself, we will also need many of their intermediate results in [26].

The first set of statements (Theorem 4.1) refers to an arbitrary formally real Jordan algebra.

**Theorem 4.1.**

(a) There is a unique element \( \mathbb{I}_\Pi \in \Pi \) such that \( \mathbb{I}_\Pi \ast A = A \) for all \( A \in \Pi \).

(b) The subalgebra \( \langle \mathbb{I}_\Pi, A \rangle \) generated by \( A \in \Pi \) and \( \mathbb{I}_\Pi \) contains pairwise orthogonal non-zero idempotents \( E_1, \ldots, E_s \) such that
\[
E_1 + \ldots + E_s = \mathbb{I}_\Pi, \quad \lambda_1 E_1 + \ldots + \lambda_s E_s = A
\]
for some \( \{\lambda_1, \ldots, \lambda_s\} \subset \mathbb{R} \).

(c) An idempotent is called unresolvable, if it cannot be written as a sum of two orthogonal non-zero idempotents. There exist unresolvable, pairwise orthogonal idempotents \( E_1, \ldots, E_r \) such that
\[
E_1 + \ldots + E_r = \mathbb{I}_\Pi.
\]
The resolution of unity (4.2) is not unique but the number \( r \) of unresolvable pairwise orthogonal idempotents in it is always the same.

(d) If \( A \subset \Pi \) is a proper ideal then \( \Pi = A \oplus B \) as a direct sum of algebras, where \( B = \{B \in \Pi : A \ast B = 0, \quad \text{for all} \quad A \in A\} \) is the complementary ideal.

\[\text{i.e.} \quad E_\nu \ast E_\sigma = \delta_{\nu\sigma} E_\nu, \quad \nu = 1, \ldots, s, \quad \sigma = 1, \ldots, s.\]
(e) Any algebra \( \Pi \) is a direct (orthogonal) sum of simple algebras

\[
\Pi = \Pi_1 \oplus \ldots \oplus \Pi_m,
\]

where each component \( \Pi_j \) is identified with a minimal proper ideal in \( \Pi \).

The second set of results (Theorem 4.2) refers to a simple formally real Jordan algebra \( \Pi \), in which the decomposition of unity (4.2) has been chosen. Let

\[
\mathcal{M}^{\rho\sigma} = \left\{ A \in \Pi : E_\tau * A = \frac{1}{2} (\delta_{\rho\tau} + \delta_{\sigma\tau})A \text{ for all } \tau = 1, \ldots, r \right\}, \quad \rho, \sigma = 1, \ldots, r.
\]

Clearly, \( \mathcal{M}^{\rho\sigma} = \mathcal{M}^{\sigma\rho} \).

**Theorem 4.2.**

(a) \( \Pi = \bigoplus_{1 \leq \rho \leq \sigma \leq r} \mathcal{M}^{\rho\sigma} \), as a direct sum of vector spaces.

(b) \( \dim \mathcal{M}^{\rho\sigma} = 1 \), \( p = \dim \mathcal{M}^{\rho\sigma} \) is independent of \( \sigma \) and \( \rho \), as long as \( \sigma \neq \rho \).

(c) The spaces \( \mathcal{M}^{\rho\sigma} \) have bases \( \{ X_1^{\rho\sigma}, \ldots, X_p^{\rho\sigma} \} \) such that

\[
X_\mu^{\rho\sigma} * X_\nu^{\rho\sigma} = \delta_{\mu
u}(E_\rho + E_\sigma), \quad \mu, \nu = 1, \ldots, p, \quad 1 \leq \rho < \sigma \leq r.
\]

The third result (Theorem 4.3) is the well-known characterization of all formally real simple Jordan algebras.

**Theorem 4.3.** If \( \Pi \) is a formally real simple Jordan algebra then it is isomorphic to one of the following algebras classified according to the number of indecomposable idempotents in the resolution of unity (4.2)

- \( r = 1 \), \( \mathbb{R} \)
- \( r = 2 \), “spin factors” \( \mathfrak{S}_n \), \( n \geq 3 \) defined by \( \mathfrak{S}_n = \text{Span}\{I_{\mathfrak{S}_n}, s_1, \ldots, s_{n-1}\} \), where \( I_{\mathfrak{S}_n} * I_{\mathfrak{S}_n} = I_{\mathfrak{S}_n} \), \( I_{\mathfrak{S}_n} * s_j = s_j \), \( s_i * s_j = \delta_{ij}I_{\mathfrak{S}_n} \).
- \( r \geq 3 \), \( \text{Sym}(\mathbb{R}^r) \), \( \mathfrak{H}(\mathbb{C}^r) \), \( \mathfrak{H}(\mathbb{H}^r) \), denoting real symmetric \( r \times r \) matrices, complex Hermitean \( r \times r \) matrices and quaternionic Hermitean \( r \times r \) matrices, respectively.
- 27-dimensional exceptional Albert algebra \( \mathfrak{M}_8^3 \) \( (r = 3) \). It has no non-trivial representations in \( \text{Sym}(\mathbb{R}^n) \).

It was shown by Albert [3] that \( \mathfrak{M}_8^3 \) cannot be identified as a subspace of an associative algebra that is closed under the multiplication \( \star \).

**Remark 4.4.** It will be useful to list the value of the invariant \( p = \dim \mathcal{M}^{\rho\sigma} \), defined in Theorem 4.2, for all the simple algebras from Theorem 4.3

- \( p = 1 \) for \( \text{Sym}(\mathbb{R}^r) \), \( r \geq 1 \) \( (\text{Sym}(\mathbb{R}^2) \cong \mathfrak{S}_3) \),
- \( p = 2 \) for \( \mathfrak{H}(\mathbb{C}^r) \), \( r \geq 2 \) \( (\mathfrak{H}(\mathbb{C}^2) \cong \mathfrak{S}_4) \),
- \( p = 3 \) for \( \mathfrak{S}_5 \)
- \( p = 4 \) for \( \mathfrak{H}(\mathbb{H}^r) \), \( r \geq 2 \) \( (\mathfrak{H}(\mathbb{H}^2) \cong \mathfrak{S}_6) \),
- \( p \geq 5 \) for \( \mathfrak{S}_{p+2} \).

For \( \mathfrak{M}_3^3 \) \( p = 8 \).
5 The structure of Jordan subalgebras of $\text{Sym}(\mathbb{R}^n)$

5.1 The non-singular Jordan algebras

From now on $\Pi$ refers to a subalgebra of $\text{Sym}(\mathbb{R}^n)$, i.e. a subspace closed with respect to the multiplication. Let

$$\mathcal{N}(\Pi) = \{ u \in \mathbb{R}^n : Au = 0 \text{ for all } A \in \Pi \}. $$

Then, in the basis which is the union of the bases for $V = \mathcal{N}(\Pi)^\perp$ and $\mathcal{N}(\Pi)$

$$\Pi = \left\{ \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : A \in \Pi_0 \subset \text{Sym}(V) \right\},$$

where $\Pi_0$ is a Jordan subalgebra of $\text{Sym}(V)$, for which $\mathcal{N}(\Pi_0) = \{0\}$.

**Definition 5.1.** We say that the Jordan algebra $\Pi$ is **non-singular** if $\mathcal{N}(\Pi) = \{0\}$.

Any algebra $\Pi$ is non-singular on $V = \mathcal{N}(\Pi)^\perp$. Hence, without loss of generality, we may assume that $\Pi$ is non-singular.

**Lemma 5.2.** Suppose that the Jordan algebra $\Pi \subset \text{Sym}(\mathbb{R}^n)$ is non-singular. Then, $I_n \in \Pi$, where $I_n$ is the $n \times n$ identity matrix.

**Proof.** By Theorem 4.1 there exists the algebra identity $I_\Pi \in \Pi$. Then $I_\Pi^2 = I_\Pi$. Therefore, the symmetric matrix $I_\Pi$ may have eigenvalues that are either 1 or 0. Suppose $u \in \mathbb{R}^n$ is an eigenvector of $I_\Pi$ with eigenvalue zero. Then, by assumption, there exists $A \in \Pi$, such that $v = Au \neq 0$. Applying the equality $2A = A I_\Pi + I_\Pi A$ to the vector $u$ we obtain $2v = I_\Pi v$. This contradicts the fact that 2 is not an eigenvalue for $I_\Pi$. Thus, $I_\Pi$ may not have an eigenvalue 0. Therefore, $I_\Pi = I_n$. \qed

Let $A \in \Pi$. Suppose that its eigenvalues are $\{\lambda_1, \ldots, \lambda_s\} \subset \mathbb{R}$ and the corresponding eigenspaces are $V_\alpha$, $\alpha = 1, \ldots, s$. By Theorem 4.1 the idempotents $E_\alpha$ in (4.1) must be orthogonal projections $P_{V_\alpha}$ onto the eigenspaces $V_\alpha$ of $A$. We will call them spectral projections. Thus, for any $A \in \Pi$ all of its spectral projections $P_{V_\alpha}$ must also be in $\Pi$, and

$$A = \sum_{\alpha=1}^{s} \lambda_\alpha P_{V_\alpha}, \quad V_1 \oplus \ldots \oplus V_s = \mathbb{R}^n,$$

(5.2)

5.2 Splitting of Jordan algebras

**Definition 5.3.** We say that the Jordan algebra $\Pi \subset \text{Sym}(\mathbb{R}^n)$ **splits** over the orthogonal decomposition $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_m$, if $\Pi = \Pi_1 \oplus \ldots \oplus \Pi_m$, is a direct sum of Jordan algebras, where

$$\Pi_\alpha = \{ A \in \Pi : Aw = 0 \text{ for all } w \in V_\alpha^\perp \}, \quad \alpha = 1, \ldots, m.$$ 

If $\Pi$ splits over $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_m$, then in the basis, which is the union of bases for $V_\alpha$, $\alpha = 1, \ldots, m$ the algebra $\Pi$ has the form

$$\Pi = \left\{ \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix} : A_\alpha \in \Pi_\alpha^0 \subset \text{Sym}(V_\alpha), \quad \alpha = 1, \ldots, m \right\}.$$ 

In other words all matrices in $\Pi$ have block-diagonal structure, with independent blocks $A_\alpha$ taken from subalgebras $\Pi_\alpha^0 \subset \text{Sym}(V_\alpha)$ that are isomorphic to $\Pi_\alpha$. Clearly, each $\Pi_\alpha \subset \Pi$ is an ideal. Conversely, if $\mathcal{A} \subset \Pi$ is an ideal, then, according to Theorem 4.1 there exists a complementary ideal

$$\mathcal{B} = \{ B \in \Pi : A \ast B = 0 \text{ for all } A \in \mathcal{A} \},$$

such that $\Pi = \mathcal{A} \oplus \mathcal{B}$, as a direct sum of algebras.
Lemma 5.4. Let $A$ and $B$ be the complementary pair of ideals in $\Pi$. Then there exists $V \subset \mathbb{R}^n$ such that $\Pi = A \oplus B$ is the splitting of $\Pi$ over $\mathbb{R}^n = V \oplus V^\perp$.

Proof. Let $V = \mathcal{N}(B)$. The subspace $V$ cannot be all of $\mathbb{R}^n$, since otherwise $B = \{0\}$ and $A = \Pi$. If $V = \{0\}$, then $B$ is a non-singular algebra and $I_n \in B$, by Lemma 5.2. But then $B = \Pi$ and $A = \{0\}$. The subspace $V^\perp$ is invariant for $B$, since all matrices in $B$ are symmetric. Thus, $B$ is isomorphic to a subalgebra $B_0$ of $\text{Sym}(V^\perp)$ defined by the restriction of $B$ on $V^\perp$. By our construction $\mathcal{N}(B_0) = \{0\}$, and applying Lemma 5.2 to $B_0$ we conclude that $P_{V^\perp} \in B$. Now, let $B \in \Pi$ be such that $Bv = 0$ for all $v \in V$. Then $P_{V^\perp} \neq B$. Thus, $B \in B$, since $B$ is an ideal. We have now proved that $B = \{B \in \Pi : B = 0 \text{ for all } v \in V(B)\}$. To finish the proof of the lemma we need to show that $\mathcal{N}(A) = V^\perp$. First we observe that $V$ (and therefore $V^\perp$) is an invariant subspace for $A$. Indeed, for any $v \in V$ and any $A \in \mathcal{A}$ we have

$$P_{V^\perp}(Av) = 2(P_{V^\perp} * A)v - A(P_{V^\perp}v) = 0,$$

since $P_{V^\perp} \in B$. Also, for any $w \in V^\perp$ and any $A \in \mathcal{A}$ we have

$$Aw = P_{V^\perp}(Aw) = 2(P_{V^\perp} * A)w - A(P_{V^\perp}w) = -Aw.$$

Thus, $V^\perp \subset \mathcal{N}(A)$. To prove the reverse inclusion we observe that $P_{V} * B = 0$ for all $B \in B$. Indeed, $B * P_{V^\perp} = B$ for all $B \in B$, therefore $B * P_{V} = B * (I_n - P_{V^\perp}) = B - B = 0$. It follows that $P_{V} \in \mathcal{A}$. Thus, if $Ax = 0$ for all $A \in \mathcal{A}$, then $P_{V}x = 0$ and $x \in V^\perp$. \hfill $\square$

Corollary 5.5. Let $(4.3)$ be the decomposition of $\Pi$ into a direct sum of simple algebras. Then, there exist pairwise orthogonal subspaces $V_1, \ldots, V_m$ of $\mathbb{R}^n$ such that $(4.3)$ is the splitting of $\Pi$ over the orthogonal decomposition $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_m$. Thus,

$$\Pi = \left\{ \begin{bmatrix} A_1 & 0 \\ \vdots & \ddots \\ 0 & A_m \end{bmatrix} : A_\alpha \in \Pi_\alpha \subset \text{Sym}(V_\alpha), \; \alpha = 1, \ldots, m \right\}, \quad (5.3)$$

where $\Pi_\alpha \subset \text{Sym}(V_\alpha), \; \alpha = 1, \ldots, m$ are simple Jordan algebras.

5.3 Irreducible Jordan algebras

According to Corollary 5.5 we need to understand the structure of simple non-singular Jordan algebras $\Pi \subset \text{Sym}(\mathbb{R}^n)$. As in the theory of associative algebras it is important whether or not there is a common invariant subspace for all matrices $A \in \Pi$. Any Jordan algebra is completely reducible in the sense that there exists an orthogonal decomposition $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k$, such that all subspaces $V_\alpha$ are invariant for all matrices $A \in \Pi$ and they do not contain any smaller proper invariant subspaces for $\Pi$.

Definition 5.6. A Jordan subalgebra of $\text{Sym}(\mathbb{R}^n)$ is called irreducible if it does not have any proper invariant subspaces in $\mathbb{R}^n$.

We remark that an irreducible Jordan subalgebra of $\text{Sym}(\mathbb{R}^n)$ is just a faithful irreducible representation of a simple formally real Jordan algebra. Our goal is to describe the structure of an arbitrary simple non-singular subalgebra of $\text{Sym}(\mathbb{R}^n)$ in terms of irreducible algebras.

Lemma 5.7. Let $\Pi$ be a simple non-singular subalgebra of $\text{Sym}(\mathbb{R}^n)$ and $\mathbb{R}^n = V_1 \oplus \ldots \oplus V_k$ be an orthogonal decomposition of $\mathbb{R}^n$ into the sum of irreducible invariant subspaces. Let

$$\Pi_\alpha = \{ P_{V_\alpha}AP_{V_\alpha} : A \in \Pi \} \subset \text{Sym}(V_\alpha), \; \alpha = 1, \ldots, k.$$
Then $\Pi_\alpha \subset \text{Sym}(V_\alpha)$, $\alpha = 1, \ldots, k$ are irreducible Jordan algebras in the sense of Definition 5.6. Moreover, the maps $T_{\alpha\beta} : \Pi_\alpha \to \Pi_\beta$ defined as

$$T_{\alpha\beta}A = P_{V_\beta}KP_{V_\beta}, \quad A = P_{V_\alpha}KP_{V_\alpha}, \quad K \in \Pi$$

are Jordan algebra isomorphisms.

Proof. Let $A \in \Pi_\alpha$ and suppose $\{K_1, K_2\} \subset \Pi$ are such that $A = P_{V_\alpha}K_1P_{V_\alpha} = P_{V_\alpha}K_2P_{V_\alpha}$. Then $P_{V_\alpha}(K_1 - K_2)P_{V_\alpha} = 0$. Let

$$\mathcal{A} = \{K \in \Pi : P_{V_\alpha}KP_{V_\alpha} = 0\}.$$ 

Let us show that $\mathcal{A}$ is an ideal in $\Pi$. Let $X \in \Pi$ and $K \in \mathcal{A}$ be arbitrary. For any $x \in \mathbb{R}^n$ we have $x = v + w$, where $v \in V_\alpha$ and $w \in V_\alpha^\perp$. Obviously $P_{V_\alpha}(X * K)P_{V_\alpha}w = 0$. We also have

$$2P_{V_\alpha}(X * K)P_{V_\alpha}v = P_{V_\alpha}KXv,$$

since $Kv = 0$ for any $v \in V_\alpha$. But $Xv \in V_\alpha$, since $V_\alpha$ is an invariant subspace for $\Pi$, and thus, $KXv = 0$. It follows that $X * K \in \mathcal{A}$. The ideal $\mathcal{A}$ cannot be $\Pi$, since in that case $V_\alpha \subset \mathcal{N}(\Pi)$, contrary to the assumption that $\Pi$ is a non-singular algebra. Hence, $\mathcal{A} = \{0\}$. Hence, for any $A \in \Pi_\alpha$ there is a unique $K \in \Pi$ for which $A = P_{V_\alpha}KP_{V_\alpha}$. Moreover, for any $\{K_1, K_2\} \subset \Pi$ we have

$$(P_{V_\alpha}K_1P_{V_\alpha}) \ast (P_{V_\alpha}K_2P_{V_\alpha}) = P_{V_\alpha}(K_1 \ast K_2)P_{V_\alpha}.$$ 

Indeed, the invariance of $V_\alpha$ implies that for any $v \in V_\alpha$

$$P_{V_\alpha}K_1P_{V_\alpha}P_{V_\alpha}K_2v = P_{V_\alpha}K_1K_2v = K_1K_2v.$$ 

It follows that $\Pi_\alpha \subset \text{Sym}(V_\alpha)$ are irreducible Jordan algebras. The maps $T_{\alpha\beta}$ are well-defines and preserve Jordan multiplication

$$T_{\alpha\beta}(A_1 \ast A_2) = P_{V_\beta}(K_1 \ast K_2)P_{V_\beta} = (P_{V_\beta}K_1P_{V_\beta}) \ast (P_{V_\beta}K_2P_{V_\beta}) = (T_{\alpha\beta}A_1) \ast (T_{\alpha\beta}A_2).$$ 

By construction, the maps $T_{\alpha\beta}$ are surjective. Let us show that they are also injective. If $A \in \Pi_\alpha$ and $A \neq 0$, then there is a unique $K \in \Pi$ such that $A = P_{V_\alpha}KP_{V_\alpha}$. Clearly, $K \neq 0$. If $T_{\alpha\beta}A = P_{V_\beta}KP_{V_\beta} = 0$, then $K' = 0$ and $K \neq K'$ satisfy $P_{V_\beta}KP_{V_\beta} = P_{V_\beta}K'P_{V_\beta}$ in contradiction of uniqueness. \hfill \Box

Hence, we have proved the structure theorem for simple non-singular Jordan subalgebras of $\text{Sym}(\mathbb{R}^n)$.

**Theorem 5.8.** Let $\Pi \subset \text{Sym}(\mathbb{R}^n)$ be a non-singular simple Jordan algebra. Then there exists an orthonormal basis (o.n.b.) of $\mathbb{R}^n$ in which the algebra $\Pi$ has the form

$$\Pi = \left\{ \begin{bmatrix} A & 0 \\ T_1A & \ddots \\ 0 & T_{k-1}A \end{bmatrix} : A \in \Pi_0 \subset \text{Sym}(\mathbb{R}^N) \right\}, \quad (5.4)$$

where $\Pi_0 \subset \text{Sym}(\mathbb{R}^N)$ is an irreducible Jordan algebra and $T_1, \ldots, T_{k-1}$ are Jordan algebra isomorphisms.

Thus, we have reduced the problem of description of all subalgebras of $\text{Sym}(\mathbb{R}^n)$ to the problem of characterization of all irreducible representations of simple Jordan algebras and their isomorphisms.

The problem of completeness of $\Pi$ can also be restated. It is now clear that for a Jordan algebra to be complete in the sense of Definition 2.4 it is necessary and sufficient for each of its simple components $\Pi_\alpha$, to be complete. The latter condition will be satisfied if and only if the irreducible algebras $\Pi_0$ and $\Pi_\alpha = T_\alpha\Pi_0$, $\alpha = 1, \ldots, k - 1$, in the representation (5.4) are complete and Jordan isomorphisms $T_\alpha$ satisfy

$$T_\alpha(A_1A_2A_3A_4 + A_4A_3A_2A_1) = T_\alpha A_1T_\alpha A_2T_\alpha A_3T_\alpha A_4 + T_\alpha A_4T_\alpha A_3T_\alpha A_2T_\alpha A_1.$$ 

$$T_\alpha(A_1A_2A_3A_4 + A_4A_3A_2A_1) = T_\alpha A_1T_\alpha A_2T_\alpha A_3T_\alpha A_4 + T_\alpha A_4T_\alpha A_3T_\alpha A_2T_\alpha A_1.$$ 

11
6 The characterization of irreducible Jordan algebras

To complete the characterization of all Jordan subalgebras \( \text{Sym}(\mathbb{R}^n) \) we need to describe all irreducible Jordan algebras up to an orthogonal equivalence and compute their Jordan isomorphisms. In this section we take on the former problem.

6.1 Block decomposition

Let \( \mathfrak{J} \) be a simple formally real Jordan algebra. Let \( \Phi : \mathfrak{J} \to \text{Sym}(\mathbb{R}^n) \) be an irreducible representation of \( \mathfrak{J} \). According to Theorem 12.2 there exist \( r \) indecomposable orthogonal idempotents \( \{E_1, \ldots, E_r\} \subset \mathfrak{J} \). Their images under the representation \( \Phi \) must necessarily be orthogonal projectors onto the mutually orthogonal subspaces \( W_1, \ldots, W_r \) such that \( \mathbb{R}^n = W_1 \oplus \ldots \oplus W_r \). The algebra \( \Pi = \Phi(\mathfrak{J}) \) can be represented as a direct sum \( \Pi = \bigoplus_{1 \leq \rho \leq \sigma \leq r} \mathcal{M}_{\rho\sigma}^\alpha \) (block decomposition), where

\[
\mathcal{M}^\alpha = \mathbb{R}P_{W_\alpha}, \quad P_{W_\alpha} = \Phi(E_\alpha), \quad \alpha = 1, \ldots, r,
\]

and

\[
\mathcal{M}^{\alpha\beta} = \{P_{W_\alpha}AP_{W_\beta} + P_{W_\beta}AP_{W_\alpha} : A \in \Pi\}, \quad 1 \leq \alpha < \beta \leq r.
\]

The spaces \( \mathcal{M}^{\alpha\beta} \) have the same dimension \( p \) for all \( 1 \leq \alpha < \beta \leq r \) and have the basis

\[
X^{\alpha\beta}_\mu = P_{W_\mu}A_{\mu}P_{W_\alpha} + P_{W_\alpha}A_{\mu}P_{W_\beta}, \quad (6.1)
\]

such that

\[
X^{\alpha\beta}_\mu \ast X^{\alpha\beta}_\nu = \delta_{\mu\nu}(P_{W_\alpha} + P_{W_\beta}), \quad \mu, \nu = 1, \ldots, p. \quad (6.2)
\]

In particular,

\[
(X^{\alpha\beta}_\mu)^2 = P_{W_\mu}A_{\mu}P_{W_\alpha}P_{W_\beta} + P_{W_\alpha}A_{\mu}P_{W_\beta} = P_{W_\alpha} + P_{W_\beta}.
\]

It follows that

\[
P_{W_\mu}A_{\mu}P_{W_\alpha} = P_{W_\alpha}, \quad P_{W_\beta} = P_{W_\mu}A_{\mu}P_{W_\beta} = P_{W_\beta}.
\]

From the first equality we obtain that

\[
\dim(W_\alpha) = \text{rank}(P_{W_\alpha}) = \text{rank}(P_{W_\alpha}A_{\mu}P_{W_\alpha}A_{\mu}P_{W_\alpha}) \leq \dim(W_\beta),
\]

while from the second one we obtain

\[
\dim(W_\beta) = \text{rank}(P_{W_\beta}) = \text{rank}(P_{W_\beta}A_{\mu}P_{W_\alpha}A_{\mu}P_{W_\beta}) \leq \dim(W_\alpha).
\]

Thus, \( \dim(W_\alpha) = d = n/r \) for all \( \alpha = 1, \ldots, r \).

6.2 Structure of \( \mathcal{M}^{\alpha\beta} \)-blocks

In order to continue, it will be convenient to fix arbitrary orthonormal bases for spaces \( W_\alpha \) and work with the space \( U^{\alpha\beta} \) of \( d \times d \) matrices representing the \( \alpha\beta \)-block of \( \mathcal{M}^{\alpha\beta} \). The \( n \times n \) symmetric matrices \( X^{\alpha\beta}_\mu \in \mathcal{M}^{\alpha\beta} \) given by \( (6.1) \) can be described by their \( \alpha\beta \) \( d \times d \) blocks \( \tilde{X}^{\alpha\beta}_\mu \in U^{\alpha\beta} \). When \( \alpha \neq \beta \) the relation \( (6.2) \) for \( X^{\alpha\beta}_\mu \) can be written in terms of matrices \( \tilde{X}^{\alpha\beta}_\mu, \ldots, \tilde{X}^{\alpha\beta}_p \) as follows

\[
\tilde{X}^{\alpha\beta}_\mu(\tilde{X}^{\alpha\beta}_\nu)^T = I_d, \quad \tilde{X}^{\alpha\beta}_\mu(\tilde{X}^{\alpha\beta}_\nu)^T + \tilde{X}^{\alpha\beta}_\nu(\tilde{X}^{\alpha\beta}_\mu)^T = 0, \quad \mu \neq \nu.
\]

It follows that \( \{\tilde{X}^{\alpha\beta}_1, \ldots, \tilde{X}^{\alpha\beta}_p\} \subset O(d) \). Let \( Y^{\alpha\beta}_\mu = \tilde{X}^{\alpha\beta}_\mu(\tilde{X}^{\alpha\beta}_\nu)^T, \mu = 1, \ldots, p \). Then \( Y^{\alpha\beta}_\mu = I_d \) and

\[
(Y^{\alpha\beta}_\mu)^2 = -I_d, \quad Y^{\alpha\beta}_\mu + (Y^{\alpha\beta}_\nu)^T = 0, \quad Y^{\alpha\beta}_\mu Y^{\alpha\beta}_\nu + Y^{\alpha\beta}_\nu Y^{\alpha\beta}_\mu = 0, \quad 1 \leq \mu < \nu \leq p - 1. \quad (6.3)
\]

We consider two cases \( r = 2 \) and \( r > 2 \).
6.2.1 Case \( r > 2 \)

When \( r > 2 \) Remark 4.4 implies that \( \mathfrak{J} \) can only be \( \text{Sym}(\mathbb{R}^r) \), for which \( p = 1 \), \( \mathfrak{J}(\mathbb{C}^r) \), for which \( p = 2 \), or \( \mathfrak{J}(\mathbb{H}^r) \), for which \( p = 4 \). The explicit description of the irreducible representations of these three algebras will be written in terms of the canonical matrix representations of complex numbers and quaternions. Specifically we define the maps \( \varphi, \psi : \mathbb{C} \rightarrow \text{End}_d(\mathbb{R}^2) \) and \( Q : \mathbb{H} \rightarrow \text{End}_d(\mathbb{R}^4) \) as follows

\[
\varphi(x + iy) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}, \quad \psi(x + iy) = \begin{bmatrix} x & y \\ y & -x \end{bmatrix}, \quad (6.4)
\]

\[
Q(q_0 + iq_1 + iq_2 + kq_3) = \begin{bmatrix} \varphi(q_0 + iq_1) & -\psi(q_2 + iq_3) \\ \psi(q_2 + iq_3) & \varphi(q_0 + iq_1) \end{bmatrix}. \quad (6.5)
\]

These functions have the properties

\[
\varphi(a)z = az, \quad \psi(a)z = a\bar{z}, \quad Q(q)h = qh.
\]

Additionally

\[
\varphi(a)^T = \varphi(a), \quad \psi(a)^T = \psi(a), \quad Q(q)^T = Q(\overline{q}).
\]

For a collection of \( d \times d \) matrices \( M_{\alpha\beta} \), \( 1 \leq \alpha \leq r, 1 \leq \beta \leq r \) the notation \( (M_{\alpha\beta}) \) stands for the \( dr \times dr \) matrix given in block-form by the \( r \times r \) block-matrix, whose \( \alpha\beta \)-blocks are \( d \times d \) matrices \( M_{\alpha\beta} \).

**Theorem 6.1.** Up to an orthogonal conjugation a simple formally real special Jordan algebra \( \mathfrak{J} \) with \( r \geq 3 \) has a unique irreducible representation by matrices in \( \text{Sym}(\mathbb{R}^n) \), given explicitly by the following formulas

(a) \( \mathfrak{J} = \text{Sym}(\mathbb{R}^r) \). Then \( n = r \) and \( \Phi(J) = J, \ J \in \text{Sym}(\mathbb{R}^r) \)

(b) \( \mathfrak{J} = \mathfrak{J}(\mathbb{C}^r) \). Then \( n = 2r \) and \( \mathfrak{J}(\mathbb{C}^r) \supseteq J \rightarrow \Phi(J) = (\varphi(J_{\alpha\beta})) \)

(c) \( \mathfrak{J} = \mathfrak{J}(\mathbb{H}^r) \). Then \( n = 4r \) and \( \mathfrak{J}(\mathbb{H}^r) \supseteq J \rightarrow \Phi(J) = (Q(J_{\alpha\beta})) \).

**Proof.** Let \( K \) denote the division algebra \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \) in the definition of the algebra \( \mathfrak{J} \). For an irreducible representation \( \Phi : \mathfrak{J} \rightarrow \text{Sym}(\mathbb{R}^n) \) \( (n = dr) \) and \( \alpha \neq \beta \) let \( \Phi_{\alpha\beta} : K \rightarrow U^{\alpha\beta} \), while \( \Phi_{\alpha\alpha}(1) = I_d \), so that \( \Phi(J) = (\Phi_{\alpha\beta}(J_{\alpha\beta})) \). The maps \( \Phi_{\alpha\beta} \) satisfy

\[
\Phi_{\alpha\beta}(q)^T = \Phi_{\beta\alpha}(\overline{q}), \quad \Phi_{\alpha\beta}(q)\Phi_{\alpha\beta}(q)^T = |q|^2 I_d, \quad \Phi_{\alpha\beta}(q)\Phi_{\beta\gamma}(h) = \Phi_{\alpha\gamma}(qh), \quad \alpha \neq \gamma
\]

In particular, \( \Phi_{\alpha\beta}(1) \in O(d) \). Let

\[
R = \begin{bmatrix} I_d & 0 \\ \Phi_{12}(1) & \ddots \\ 0 & \Phi_{1r}(1) \end{bmatrix}.
\]

Then \( R \in O(n) \) and \( \Psi(J) = R\Phi(J)R^T \) is an orthogonally equivalent irreducible representation of \( \mathfrak{J} \) in \( \text{Sym}(\mathbb{R}^n) \), such that \( \Psi_{1\alpha}(1) = I_d, \ \alpha = 1, \ldots, r \). Let \( \alpha \neq \beta \) and \( \beta \neq 1 \). Then

\[
\Psi_{\alpha\beta}(q) = \Psi_{1\alpha}(1)\Phi_{\alpha\beta}(q) = \Psi_{1\beta}(q).
\]

If in addition, \( \alpha \neq 1 \) (which is possible if and only if \( r \geq 3 \)), then

\[
\Psi_{\alpha\beta}(q) = \Psi_{\beta\alpha}(\overline{q})^T = \Psi_{1\alpha}(\overline{q})^T = \Psi_{\alpha\beta}(q)
\]

Hence, when \( \alpha \neq \beta, \ \alpha \neq 1, \ \beta \neq 1 \) we have

\[
\Psi_{\alpha\beta}(q)\Psi_{\alpha\beta}(h) = \Psi_{\alpha\beta}(q)\Psi_{\alpha\beta}(h) = \Psi_{\alpha\beta}(qh).
\]
For any $\beta \neq 1$ there is $\alpha \neq 1$ and $\alpha \neq \beta$, since $r \geq 3$. Then $\Psi_{1\beta}(q) = \Psi_{\alpha\beta}(q)$. Similarly, for any $\alpha \neq 1$ there is $\beta \neq 1$ and $\alpha \neq \beta$, and it follows that $\Psi_{\alpha 1}(q) = \Psi_{\alpha\beta}(q)$. Hence, for any $\alpha \neq \beta$ the functions $\Psi_{\alpha\beta}$ are algebra homomorphisms. In particular, if $\alpha \neq \beta$ we have

$$\Psi_{\alpha\beta}(q) = \Psi_{\alpha\beta}(q^T) = |q|^2(\Psi_{\alpha\beta}(q))^{-1} = \Psi_{\alpha\beta}(|q|^2 q^{-1}) = \Psi_{\alpha\beta}(q).$$

In particular, $\Psi_{21}(q) = \Psi_{12}(q)$. Also, for any $\alpha \neq 1, 2$

$$\Psi_{1\alpha}(q) = \Psi_{\alpha 1}(q) = \Psi_{\alpha 2}(q) = \Psi_{12}(q).$$

Finally, if $\alpha \neq \beta$, $\alpha \neq 1$, $\beta \neq 1$ then

$$\Psi_{\alpha\beta}(q) = \Psi_{1\beta}(q) = \Psi_{12}(q).$$

Thus, there is a division algebra homomorphism $\Psi_{12} : \mathbb{K} \to \text{End}_\mathbb{R}(\mathbb{R}^d)$ such that $\Psi_{12}(q) = \Psi_{12}(q^T)$ and $\Psi_{\alpha\beta}(q) = \Psi_{12}(q)$ for all $\alpha \neq \beta$ and all $q \in \mathbb{K}$. The homomorphism $\Psi_{12}$ is non-zero and hence injective (since $\mathbb{K}$ is a simple algebra over $\mathbb{R}$). If the associative algebra $U = \Psi_{12}(\mathbb{K}) \subset \text{End}_\mathbb{R}(\mathbb{R}^d)$ has a proper invariant subspace $V \subset \mathbb{R}^d$ then the subspace

$$V \oplus \ldots \oplus V \subset \mathbb{R}^d \oplus \ldots \oplus \mathbb{R}^d = \mathbb{R}^n$$

is a proper invariant subspace of the representation $\Psi(\mathbb{J})$. We conclude that $\Psi_{12}$ must be an irreducible representation of $\mathbb{K}$ for an irreducible representation $\Psi$ of $\mathbb{J}$. It remains to note that there is a unique4 (up to an orthogonal conjugation) irreducible representation $\Psi_{12}$ of $\mathbb{K}$ on $\mathbb{R}^d$ satisfying $\Psi_{12}(q^T) = \Psi_{12}(q^T)$. It is given explicitly by the following formulas

- If $\mathbb{K} = \mathbb{R}$ then $d = 1$ and $\Psi_{12}(q) = q$.
- If $\mathbb{K} = \mathbb{C}$ then $d = 2$ and $\Psi_{12}(q) = \varphi(q)$, given by (6.4).
- If $\mathbb{K} = \mathbb{H}$ then $d = 4$ and $\Psi_{12}(q) = Q(q)$, given by (6.5).

\[\square\]

### 6.2.2 Case $r = 2$

When $r = 2$ we have

$$\Pi = \left\{ \begin{bmatrix} \lambda I_{n/2} & A \\ A^T & \mu I_{n/2} \end{bmatrix} : \{\lambda, \mu\} \subset \mathbb{R}, A \in U \right\}. $$

The conditions (6.3) on the basis $(I,Y_1,\ldots,Y_{r-1})$ of $U^{12} = U$ are both necessary and sufficient for $\Pi$ to be closed with respect to the Jordan product (3.3). The irreducibility condition is equivalent to the requirement that all matrices in $U$ have no common proper invariant subspace. Indeed, if $U$ has a proper invariant subspace $V$ then the space $V = \{(v_1,v_2) : \{v_1,v_2\} \subset V\}$ is a proper invariant subspace for $\Pi$. If $U$ has no proper invariant subspaces and if $V$ is a proper invariant subspace of $\Pi$ then $(v,w) \in V$ implies that $(v,0) = K_1(v,w) \in V$ and $(0,w) = K_2(v,w) \in V$, where $K_1 \in \Pi$ corresponds to $\lambda = 1$, $\mu = 0$, $A = 0$ and $K_2 \in \Pi$ corresponds to $\lambda = 0$, $\mu = 1$, $A = 0$. Then there are subspaces $V_1$ and $V_2$ of $\mathbb{R}^{n/2}$ such that $V = \{(v_1,v_2) : v_1 \in V_1, v_2 \in V_2\}$. The invariance of $V$ is then equivalent to the condition that $U$ maps $V_1$ into $V_2$ and $V_2$ into $V_1$. However, $I_{n/2} \in U$ and hence, if $v_1 \in V_1$ then $v_1 \in V_2$ and conversely, if $v_2 \in V_2$ then $v_2 \in V_1$. It follows that $V_1 = V_2 = V$. But then $V$ must be an invariant subspace for $U$. Therefore, $V$ is either $\{0\}$ or $\mathbb{R}^{n/2}$, in which case $V$ is also either $\{0\}$ or $\mathbb{R}^n$.

The problem of identifying subspaces $U$ as described above is called a real Radon-Hurwitz problem [23][42], who studied it in connection with the question of composition of quadratic forms. This problem has

4For the sake of completeness this statement will be a consequence of our analysis of the case $r = 2$. See Remarks 6.3 and 6.5 below.
connections to self-dual 2-forms [8], linearly independent vector fields on spheres [1], system of hyperbolic conservation laws [2], Clifford algebras [9], etc. In particular the matrices $Y_1, \ldots, Y_{p-1}$ in (6.3) are the generators of the $2^{p-1}$-dimensional Clifford algebra $C\ell_{p-1,0}(\mathbb{R})$. We are interested in an explicit and complete characterization of all possible subspaces $U$ as above. While Hurwitz solved this problem in the complex case, his solution is difficult to adapt to the real case. Here we present the alternative solution based on the representation theory of finite groups [14]. Following [14] we let $G_p$ be a finite group with generators $\varepsilon, a_1, \ldots, a_{p-1}$, $p \geq 2$, satisfying the relations

$$\varepsilon^2 = 1, \quad a_k^2 = \varepsilon, \quad \varepsilon a_k = a_k \varepsilon, \quad a_k a_l = \varepsilon a_l a_k, \quad k, l = 1, \ldots, p - 1, \quad k \neq l.$$ 

Then, the matrices $Y_1, \ldots, Y_{p-1}$ satisfying (6.3) describe an irreducible orthogonal representation of the group $G_p$ that sends $\varepsilon$ to $-I_n/2$. Eckmann [14] has computed the number, type, and dimensions of such irreps of $G_p$ (see Appendix A). His results are summarized in the following theorem.

**Theorem 6.2 (Eckmann).** All non-isomorphic irreducible real representations of $G_p$ that map $\varepsilon$ to $-I$ are characterized as follows.

- $p = 1, 7 \mod 8$. Then the unique real irrep $V_p$ is a representation of real type and $d(p) = \dim V_p = 2^{\frac{p-1}{2}}$.
- $p = 3, 5 \mod 8$. Then the unique real irrep $V_p$ is a representation of quaternionic type and $d(p) = \dim V_p = 2^{\frac{p+1}{2}}$.
- $p = 2, 6 \mod 8$. Then the unique real irrep $V_p$ is a representation of complex type and $d(p) = \dim V_p = 2^\frac{p}{2}$.
- $p = 0 \mod 8$. Then there are two distinct irreps $V_p^+$ and $V_p^-$ both of real type and $d(p) = \dim V_p^\pm = 2^{\frac{p-2}{2}}$.
- $p = 4 \mod 8$. Then there are two distinct irreps $V_p^+$ and $V_p^-$ both of quaternionic type and $d(p) = \dim V_p^\pm = 2^\frac{p}{2}$.

We note that the function $d(p)$ given explicitly in Theorem 6.2 can also be defined recursively by

$$d(p + 8) = 16d(p), \quad d(1) = 1, \quad d(2) = 2, \quad d(3) = d(4) = 4, \quad d(5) = d(6) = d(7) = d(8) = 8.$$  

(6.6)

It remains to construct the representations $V_p$ explicitly. It is sufficient to indicate the images of $a_1, \ldots, a_{p-1}$ (since it is required that $\varepsilon \mapsto -I_{d(p)}$). To describe the answer we need to introduce the following notation.

$$\hat{Q}(q_0 + iq_1 + iq_2 + kq_3) = \begin{bmatrix} \varphi(q_0 + iq_1) & \varphi(q_2 + iq_3) \\ -\varphi(q_2 - iq_3) & \varphi(q_0 - iq_1) \end{bmatrix},$$  

(6.7)

where $\varphi$ was given in (6.4). The function $\hat{Q}$ satisfies

$$\hat{Q}(q_1)\hat{Q}(q_2) = \hat{Q}(q_1q_2), \quad \hat{Q}(q)^T = \hat{Q}(\overline{q}), \quad \hat{Q}(q_1)\hat{Q}(q_2) = \hat{Q}(q_2)\hat{Q}(q_1).$$

For $\{q, h\} \subset \mathbb{H}$ we also define

$$\Theta(q, h) = \begin{bmatrix} Q(q) & \hat{Q}(h) \\ -\hat{Q}(\overline{h}) & Q(\overline{q}) \end{bmatrix},$$  

(6.8)

where the maps $Q$ and $\hat{Q}$ are defined in (6.5) and (6.7), respectively. The map $\Theta$ has the following properties

$$\Theta(q, h)^T = \Theta(\overline{q}, -h), \quad \Theta(q, h)\Theta(q, h)^T = (|q|^2 + |h|^2)I_8.$$ 

For $2 \leq p \leq 9$ we have the following explicit representations, which are slightly modified versions of the ones in [8].
List 6.3.

- $p = 2$, $d(p) = 2$, $\rho_2(a_1) = \varphi(i)$.

**Remark 6.4.** The uniqueness of the irreducible representation $V_2$ implies that up to an orthogonal conjugation the irreducible representation of $\mathbb{C}$ on $\text{End}_G(\mathbb{R}^d)$ satisfies $d = 2$ and is given by $\mathbb{C} \ni c \mapsto \varphi(c)$.

- $p = 3$, $d(p) = 4$, $\rho_3(a_1) = Q(i)$, $\rho_3(a_2) = Q(j)$
- $p = 4$, $d(p) = 4$. There are two non-isomorphic representations:
  \[ \rho_4^+(a_1) = \pm Q(i), \quad \rho_4^+(a_2) = \pm Q(j), \quad \rho_4^+(a_3) = \pm Q(k). \]
  Indeed, \[ \rho_4^+(a_1 a_2) = \rho_4^+(a_3), \quad \rho_4^+(a_1 a_2) = -\rho_4^+(a_3). \]

**Remark 6.5.** Let $\Psi_0 : H \to \text{End}_G(\mathbb{R}^d)$ be a representation of $H$. If we define $\rho(\varepsilon) = -I_d$, $\rho(a_1) = \Psi_0(i)$, $\rho(a_2) = \Psi_0(j)$, $\rho(a_3) = \Psi_0(k)$ then $\rho$ will be a representation of $G_4$ on $\mathbb{R}^d$. Clearly, $\rho$ is irreducible if and only if $\Psi_0$ is irreducible. Thus, $d = 4$ and, up to the orthogonal conjugation, either
  \[ \Psi_0(i) = Q(i), \quad \Psi_0(j) = Q(j), \quad \Psi_0(k) = Q(k), \]
  or
  \[ \Psi_0(i) = -Q(i), \quad \Psi_0(j) = -Q(j), \quad \Psi_0(k) = -Q(k). \]
  However, the latter choice results in $\Psi_0(ij) = -\Psi_0(i)\Psi_0(j)$. Hence, up to an orthogonal conjugation $\Psi_0(q) = Q(q)$.

- $p = 5$, $d(p) = 8$,
  \[ \rho_5(a_1) = \bigcirc(0, 1), \quad \rho_5(a_2) = \bigcirc(0, i), \quad \rho_5(a_3) = \bigcirc(0, j), \quad \rho_5(a_4) = \bigcirc(0, k). \]

- $p = 6$, $d(p) = 8$,
  \[ \rho_6(a_1) = \bigcirc(0, i), \quad \rho_6(a_2) = \bigcirc(0, j), \quad \rho_6(a_3) = \bigcirc(0, k), \quad \rho_6(a_4) = \bigcirc(i, 0), \quad \rho_6(a_5) = \bigcirc(j, 0). \]

- $p = 7$, $d(p) = 8$,
  \[ \rho_7(a_1) = \bigcirc(0, i), \quad \rho_7(a_2) = \bigcirc(0, j), \quad \rho_7(a_3) = \bigcirc(0, k), \quad \rho_7(a_4) = \bigcirc(i, 0), \quad \rho_7(a_5) = \bigcirc(j, 0), \quad \rho_7(a_6) = \bigcirc(k, 0). \]

- $p = 8$, $d(p) = 8$: There are two non-isomorphic representations: $\rho_8^-(a_i) = -\rho_8^+(a_i)$, $i = 1, \ldots, 7$
  \[ \rho_8^-(a_1) = \bigcirc(0, 1), \quad \rho_8^-(a_2) = \bigcirc(0, i), \quad \rho_8^-(a_3) = \bigcirc(0, j), \quad \rho_8^-(a_4) = \bigcirc(0, k), \quad \rho_8^-(a_5) = \bigcirc(i, 0), \quad \rho_8^-(a_6) = \bigcirc(j, 0), \quad \rho_8^-(a_7) = \bigcirc(k, 0). \]
  The irrep $\rho_8^-$ is not isomorphic to $\rho_8^+$ because
  \[ \rho_8^+(a_2 a_3 a_4 a_5 a_6 a_7) = \rho_8^+(a_1), \]
  whereas
  \[ \rho_8^-(a_2 a_3 a_4 a_5 a_6 a_7) = -\rho_8^-(a_1). \]
\( \bullet p = 9, \ d(p) = 16. \)

\[
\rho_9(a_1) = \psi(i) \otimes \mathbb{O}(i, 0), \quad \rho_9(a_2) = \psi(i) \otimes \mathbb{O}(j, 0), \quad \rho_9(a_3) = \psi(i) \otimes \mathbb{O}(k, 0),
\]
\[
\rho_9(a_4) = \psi(i) \otimes \mathbb{O}(0, i), \quad \rho_9(a_5) = \psi(i) \otimes \mathbb{O}(0, j), \quad \rho_9(a_6) = \psi(i) \otimes \mathbb{O}(0, k),
\]
\[
\rho_9(a_7) = \psi(i) \otimes \mathbb{O}(0, 1), \quad \rho_9(a_8) = \varphi(i) \otimes \mathbb{O}(1, 0).
\]

For \( m_1 \times n_1 \) matrix \( A \) and \( m_2 \times n_2 \) matrix \( B \) the tensor product notation \( A \otimes B \) denotes \( m_1m_2 \times n_1n_2 \) matrix written in block-form as

\[
A \otimes B = \begin{bmatrix}
a_{11}B & \ldots & a_{1n_1}B \\
\ldots & \ldots & \ldots \\
a_{m_11}B & \ldots & a_{m_1n_1}B
\end{bmatrix}.
\]

Suppose that representations \( \rho_p \) of \( G_p \) have been constructed for \( 2 \leq p \leq 9 \). For \( p \geq 10 \) we define

\[
\rho_p(a_i) = \rho_9(a_i) \otimes I_{d(p-8)}, \quad i = 1, \ldots, 8, \quad \rho_p(a_i) = B \otimes \rho_{p-8}(a_i), \quad i = 9, \ldots, p-1,
\]

where \( B = \psi(1) \otimes I_8 \in \text{Sym}(\mathbb{R}^{16}). \) We have \( B^2 = I_{16}. \) Therefore, \( \rho_p(a_i)^2 = -I_p \) and \( \rho_p(a_i)\rho_p(a_j) = -\rho_p(a_j)\rho_p(a_i) \) if \( i \neq j \) and both \( i \) and \( j \) are either below 9 or above 8. We need to check the anticommutativity property for \( i \leq 8 \) and \( j \geq 9: \)

\[
\rho_9(a_i)B \otimes \rho_{p-8}(a_j) = -B\rho_9(a_i) \otimes \rho_{p-8}(a_j).
\]

This will be satisfied if

\[
\rho_9(a_i)B = -B\rho_9(a_i).
\]

We can verify (6.10) explicitly via the formulas for \( \rho_9 \) below.

In addition to the explicit representations \( V_p \) it is also convenient to have an explicit form of the subspaces

\[
W_p = \text{Span}\{\rho_p(a_1), \ldots, \rho_p(a_{p-1})\} \subset \text{Skew}(\mathbb{R}^{d(p)}), \quad U = U_p = \mathbb{R}I_{d(p)} \oplus W_p.
\]

**List 6.6.**

\( \bullet p = 1, \ W_1 = \{0\} \subset \mathbb{R}, \ U_1 = \mathbb{R} \)

\( \bullet p = 2, \)

\[
W_2 = \mathbb{R}\varphi(i) = \text{Skew}(\mathbb{R}^2), \quad U_2 = \{\varphi(z) : z \in \mathbb{C}\}.
\]

\( \bullet p = 3, \)

\[
W_3 = \{Q(q) : q \in \mathbb{H}, \ q = q_1i + q_2j\},
\]
\[
U_3 = \{Q(q) : q \in \mathbb{H}, \ q = q_0 + q_1i + q_2j\}.
\]

\( \bullet p = 4, \)

\[
W_4 = \{Q(q) : q \in \mathbb{H}, \ \Re(q) = 0\},
\]
\[
U_4 = \{Q(q) : q \in \mathbb{H}\}.
\]

\( \bullet p = 5, \)

\[
W_5 = \{\mathbb{O}(0, h) : h \in \mathbb{H}\},
\]
\[
U_5 = \{\mathbb{O}(\lambda, h) : h \in \mathbb{H}, \ \lambda \in \mathbb{R}\}.
\]

\( \bullet p = 6, \)

\[
W_6 = \{\mathbb{O}(q, h) : \{q, h\} \subset \mathbb{H}, \ \Re(h) = 0, \ q = q_1i + q_2j\},
\]
\[
U_6 = \{\mathbb{O}(q, h) : \{q, h\} \subset \mathbb{H}, \ \Re(h) = 0, \ q = q_0 + q_1i + q_2j\}.
\]
• $p = 7$,  
\[
\mathcal{W}_7 = \{Q(q, h) : \{q, h\} \subset \mathbb{H}, \Re(q) = 0, \Re(h) = 0\}, \quad \mathcal{U}_7 = \{Q(q, h) : \{q, h\} \subset \mathbb{H}, \Re(h) = 0\}.
\]

• $p = 8$,  
\[
\mathcal{W}_8 = \{Q(q, h) : \{q, h\} \subset \mathbb{H}, \Re(q) = 0\}, \quad \mathcal{U}_8 = \{Q(q, h) : \{q, h\} \subset \mathbb{H}\}.
\]

The formula (6.11) results in  
\[
\mathcal{W}_p = \left\{ \begin{bmatrix} I_p \otimes A & \mathcal{O}(q, h) \otimes I_{d(p-8)} \\ -\mathcal{O}(q, h)^T \otimes I_{d(p-8)} & -I_p \otimes A \end{bmatrix} : A \in \mathcal{W}_{p-8}, \{q, h\} \subset \mathbb{H}\right\}, \quad p \geq 9, \tag{6.11}
\]
where the spaces $\mathcal{W}_1, \ldots, \mathcal{W}_8$ are given in the List (6.6) and $\mathcal{O}(q, h)$ is defined in (6.8).

We remark that the formulas for $\mathcal{W}_p$ and $\mathcal{U}_p$, $p = 1, \ldots, 8$ are somewhat arbitrary, since conjugation by any orthogonal transformation would produce equally valid formulas. However, when $p = 1, 2, 4$ and $8$ the spaces $\mathcal{W}_p$ and $\mathcal{U}_p$ are $O(d(p))$-invariant and therefore represent the canonical forms. The formula (6.11) implies that all spaces $\mathcal{W}_{2^k}$ and $\mathcal{U}_{2^k}$ are canonical.

It is important to note that the two different representations of $G_p$ when $p = 0 \mod 4$ result in different representations of the spin factors $\mathfrak{S}_N$, when $N = 2 \mod 4$ ($N \geq 6$), even though the images of $\Pi_{\mathfrak{S}_N}$ of $\mathfrak{S}_N$ under both representations are the same:

\[
\Pi_{\mathfrak{S}_N} = \left\{ \begin{bmatrix} \lambda I_{d(N-2)} & A \\ A^T & \mu I_{d(N-2)} \end{bmatrix} : \lambda, \mu \in \mathbb{R}, A \in \mathcal{U}_{N-2} = \mathbb{R}I_{d(N-2)} \oplus \mathcal{W}_{N-2}\right\}, \tag{6.12}
\]
where $\mathcal{W}_p$ is given by (6.11). The existence of the two non-isomorphic representations of $G_p$ is reflected in the existence of the map $T : \Pi_{\mathfrak{S}_N} \rightarrow \Pi_{\mathfrak{S}_N}$, which maps $\rho^+$ to $\rho^-$. As such it is a Jordan algebra automorphism that cannot be written as an orthogonal conjugation. Conversely, every Jordan algebra automorphism maps one representation of $G_p$ into another. For those $p$ for which such a representation is unique the automorphism must be an orthogonal conjugation. The Jordan algebra automorphism $T : \Pi_{\mathfrak{S}_N} \rightarrow \Pi_{\mathfrak{S}_N}$ can be written explicitly as

\[
T \begin{bmatrix} \lambda I_{d(N-2)} & A \\ A^T & \mu I_{d(N-2)} \end{bmatrix} = \begin{bmatrix} \lambda I_{d(N-2)} & A^T \\ A & \mu I_{d(N-2)} \end{bmatrix}, \quad A \in \mathcal{U}_{N-2}, \{\lambda, \mu\} \subset \mathbb{R}. \tag{6.13}
\]

Thus, we have proved the following characterization of all irreducible representations of spin factors $\mathfrak{S}_N$, $N \geq 3$ ($\mathfrak{S}_1 = \mathbb{R}$ and $\mathfrak{S}_2$ is not simple) in $\text{Sym}(\mathbb{R}^n)$.

**Theorem 6.7.** Each spin factor $\mathfrak{S}_N$, $N \geq 3$ is represented by a unique (up to the orthogonal conjugation) irreducible subalgebra $\Pi_{\mathfrak{S}_N} \subset \text{Sym}(\mathbb{R}^{2d(N-2)})$ given by (6.12). Moreover, each Jordan algebra automorphisms of $\Pi_{\mathfrak{S}_N}$ can be represented by an orthogonal conjugation, unless $N = 2 \mod 4$ ($N \geq 3$), in which case each Jordan algebra automorphisms of $\Pi_{\mathfrak{S}_N}$ can be represented by a composition of the map $T$ given by (6.13) and an orthogonal conjugation.

### 7 Structure theorem and completness

We can now summarize all our results and describe explicitly, up to an orthogonal conjugation all Jordan subalgebras of $\text{Sym}(\mathbb{R}^n)$.

**Theorem 7.1** (Jordan subalgebras of $\text{Sym}(\mathbb{R}^n)$).
(i) Let $\Pi$ be a Jordan subalgebra of $\text{Sym}(\mathbb{R}^n)$. Then there exists an o.n.b. of $\mathbb{R}^n$ in which

$$\Pi = \left\{ \begin{array}{ccc} A_1 & \cdots & 0 \\ & & \vdots \\ & & A_m \\ 0 & & 0 \end{array} \right\} : A_1 \in \Pi_1, \ldots, A_m \in \Pi_m,$$

where each $\Pi_\alpha$ is a simple non-singular Jordan subalgebra of $\text{Sym}(\mathbb{R}^{d_\alpha})$, for some $d_\alpha \geq 1$ for which $d_1 + \ldots + d_m \leq n$.

(ii) Let $\Pi$ be a simple non-singular Jordan subalgebra of $\text{Sym}(\mathbb{R}^n)$. Then there exits an o.n.b. in $\mathbb{R}^n$ in which $\Pi$ has one of the following forms

- $(a)$ $\Pi = \{I_{n/r} \otimes A : A \in \text{Sym}(\mathbb{R})\}$, $r \geq 1$;
- $(b)$ $\Pi = \{I_{n/2r} \otimes A : A \in \mathcal{J}(\mathbb{C}^r)\}$, $r \geq 2$;
- $(c)$ $\Pi = \{I_{n/4r} \otimes A : A \in \mathcal{J}(\mathbb{H}^r)\}$, $r \geq 2$;
- $(d)$ $\Pi = \{I_{n/2d(N-2)} \otimes A : A \in \Pi_{\mathcal{S}_N}\}$, $N = 5, 7, 8, \ldots$, where $\Pi_{\mathcal{S}_N}$ is given by (6.12) and function $d(\cdot)$ is defined in (6.14);
- $(e)$ $\Pi = \left\{ \begin{array}{ccc} I_{s_1} \otimes A & 0 \\ 0 & I_{s_2} \otimes TA \end{array} \right\} : A \in \Pi_{\mathcal{S}_N}, s_1 + s_2 = n/2d(N-2), s_1 > 0, s_2 > 0, N = 2 \mod 4$,

$N \geq 6$, where the Jordan automorphism $T : \Pi_{\mathcal{S}_N} \to \Pi_{\mathcal{S}_N}$ is given by (6.13).

The explicit characterization of all Jordan subalgebras of $\text{Sym}(\mathbb{R}^n)$ in Theorem 7.1 allows us to answer a question about completeness of a subalgebra $\Pi \subset \text{Sym}(\mathbb{R}^n)$. By part (i) of Theorem 7.1 we can write $\Pi = \Pi_1 \oplus \ldots \oplus \Pi_m$, where $\Pi_\alpha$ is a simple non-singular subalgebra of $\text{Sym}(\mathbb{R}^{d_\alpha})$. The algebra $\Pi$ will be complete if and only if each of the algebras $\Pi_\alpha$ is orthogonally equivalent to one of the algebras in cases (a), (b) or (c) in part (ii) of Theorem 7.1.

We remark on a single exception to the statement that completeness is determined by the isomorphism class of $\Pi$. The algebras $\Pi$ in part (ii)(e) $N = 6$ and part (ii)(c) $r = 2$ are isomorphic to $\mathcal{J}(\mathbb{H}^2)$. However, the algebras in part (ii)(c) $r = 2$ are complete, while the ones in part (ii)(e) $N = 6$ are not.

Suppose now that we are looking for all subalgebras of $\text{Sym}(\mathbb{R}^n)$ that are closed not only with respect to the product but also with respect to the products, where the subspace $\mathcal{A}$ of multiplications is spanned by finitely many matrices $A_1 = I_n, A_2, \ldots, A_s$. Let $\Pi$ be a non-singular $\mathcal{A}$-multialgebra. Then, by Lemma 5.2, $I_n \in \Pi$. Therefore, $A_j = I_n A_j I_n \in \Pi$, $j = 2, \ldots, s$, and hence, each multiplication $X \ast A_j$ $Y$ is a mutation of the standard multiplication (3.3) [28]. We first choose a basis in which $\Pi$ has the form described in the Structure Theorem 7.1. In that basis the submatrices of $A_2, \ldots, A_s$ corresponding to $\mathcal{N}(\Pi)^+$ must be block-diagonal, with each diagonal block being a member of a simple component $\Pi_\alpha$ of $\Pi$. That is also sufficient, since the triple product $K_\alpha A_\alpha L_\alpha + L_\alpha A_\alpha K_\alpha$ always belongs to $\Pi_\alpha$. If we exclude the special case when the $\alpha$-block $A_\alpha$ of $\mathcal{A}$ has the form $A_\alpha = I_k \otimes A_0$ we will exclude all cases in which the simple algebra $\Pi_\alpha$ is reducible. In particular, this would exclude all part (ii)(e) cases. Hence, the question of completeness of $\mathcal{A}$-multialgebras reduces to the question of completeness of $\mathcal{A}$-multialgebras on the invariant subspaces of $\mathcal{A}$. If $\mathcal{A}$ has no invariant subspaces then $\Pi$ must be irreducible. An irreducible algebra is always complete if $n$ is not a power of 2 or is less than 8.

Even though possibilities like $A_\alpha = I_k \otimes A_0$ cannot be ruled out in general, we may try to understand when incomplete algebras can arise generically. The subspace of $\text{Sym}(\mathbb{R}^n)$ of all matrices whose upper left $8 \times 8$ submatrix is in $\Pi_{\mathcal{S}_8}$ has co-dimension 31. Therefore, the space $\mathcal{V}_0$ of $s$-tuples of symmetric matrices (one of which is $I_n$), all of whose upper left $8 \times 8$ submatrices is in $\Pi_{\mathcal{S}_8}$ has co-dimension $31(s-1)$. The dimension of the $O(n)$ orbit of a generic $s$-tuple of matrices is $n(n-1)/2$. The subspace $\mathcal{V}_0$ will intersect this $O(n)$ orbit generically only if $n(n-1) \geq 62(s-1)$. For example, for a generic 5-dimensional space $\mathcal{A}$ (containing $I_n$), all Jordan multialgebras will be complete, if $n < 17$. These dimensional considerations explain the reason for completeness in all of the examples in [18, 16, 17, 21]. Conjecture 2.5 suggests that another way to eliminate failure of completeness may be to restrict the Jordan multialgebras to $SO(3)$-invariant ones.
8 Lamination exact relation that is not closed under homogenization

Returning to the physical example of multifield response composite materials in Section 3 we see that the smallest \( n \) for which the incomplete Jordan subalgebra of \( \text{Sym}(\mathbb{R}^n) \) provides an example on incomplete Jordan \( \mathcal{A} \)-multialgebra is \( n = 8 \). However, the quaternionic formalism of Section 6 suggests an example of incomplete Jordan \( \mathcal{A} \)-multialgebra when \( n = 4 \). Let

\[
\mathcal{L} = \{ iQ(q) : q \in \mathbb{H}, \quad \Re(q) = 0 \} \subset \mathcal{S}_1(\mathbb{C}^4), \quad \mathcal{M} = \{ aI_4 : a \in \mathbb{C} \} \subset \text{Sym}(\mathbb{C}^4),
\]

where \( Q(q) \) is defined in (5.5). Let us verify that the 3-chain condition (8.2) fails. Indeed, let \( c = i \), \( X_1 = iQ(q_1) \), \( X_2 = iQ(q_2) \), \( X_3 = iQ(q_3) \), where it will be convenient to identify the purely imaginary quaternions \( q_1, q_2 \) and \( q_3 \) with vectors in \( \mathbb{R}^3 \). Let us assume that \( \{q_1, q_2, q_3\} \) form a basis of \( \mathbb{R}^3 \), so that the mixed product \( (q_1, q_2, q_3) = q_1 \cdot (q_2 \times q_3) \) is non-zero. We compute

\[
X_1 X_2^T X_3 = iQ(q_1q_2q_3).
\]

Then

\[
iX_1 X_2^T X_3 - iX_3 X_2^T X_1 = Q(2(q_1, q_2, q_3)) = 2(q_1, q_2, q_3)I_4 \notin \mathcal{L}.
\]

If we add real multiples of \( I_4 \) to \( \mathcal{L} \), we will obtain a completion of our incomplete Jordan \( \mathcal{A} \)-multialgebra:

\[
\mathcal{T} = \{ \alpha I_4 + iQ(q) : \alpha \in \mathbb{R}, \quad q \in \mathbb{H}, \quad \Re(q) = 0 \}, \quad \mathcal{M} = \{ aI_4 : a \in \mathbb{C} \}, \quad (8.2)
\]

Indeed, the Jordan \( \mathcal{A} \)-multialgebra (8.2) consists of all elements in \( \text{Sym}(\mathcal{T}) \) that belong to the associative \( \mathcal{A} \)-multialgebra

\[
\mathcal{L}' = \mathcal{M} = \{ aQ(q) : a \in \mathbb{C}, \quad q \in \mathbb{H} \}.
\]

(8.3)

It is easy to check that the associative \( \mathcal{A} \)-multialgebra (8.3) is symmetric in the sense of Definition 2.3.

In [10] it was shown that the 3-chain relation property is necessary for the \( SO(2) \)-invariant Jordan multialgebra to correspond to an exact relation. Hence, the incomplete Jordan multialgebra (8.1) provides the first example of a rotationally invariant lamination exact relation in 2D that is not stable under homogenization. For the sake of reference we compute the image of both (8.1) and its completion (8.2) in physical variables. In order to formulate the results it will be convenient to identify \( \mathcal{T} = \mathbb{R}^2 \otimes \mathbb{R}^4 \) with \( \mathbb{H} \), via the isomorphisms

\[
\mathcal{T} = \mathbb{R}^2 \otimes \mathbb{R}^4 \cong \mathbb{R}^4 \oplus \mathbb{R}^4 \cong \mathbb{H} \oplus \mathbb{H} \cong \mathbb{H}^2,
\]

where we have identified \( \mathbb{R}^4 \) with \( \mathbb{H} \). The quaternionic materials are identified with \( \mathcal{H}^+(\mathbb{H}^2) \)—the set of positive definite quaternionic-Hermitean 2 × 2 matrices

\[
L = \begin{bmatrix} \lambda & h \\ \overline{h} & \mu \end{bmatrix}, \quad \lambda > 0, \quad \mu > 0, \quad h \in \mathbb{H}, \quad \det L = \lambda \mu - |h|^2 > 0.
\]

If

\[
E = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in \mathbb{H}^2 = \mathcal{T}, \quad J = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in \mathbb{H}^2 = \mathcal{T}, \quad J = LE,
\]

then

\[
\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \lambda q_1 + hq_2 \\ \overline{h}q_1 + \mu q_2 \end{bmatrix}.
\]

\footnote{We have identified all solutions of (6.1) for \( n = 2 \) by hand and verified that all the Jordan \( \mathcal{A} \)-multialgebras are complete in this case. The case \( n = 3 \) remains unexplored.}
We may also represent the multifield response of quaternionic materials by the conventional $8 \times 8$ real symmetric matrix

$$
L = \begin{bmatrix}
\lambda I_4 & Q(h) \\
Q(\bar{h}) & \mu I_4
\end{bmatrix},
$$

where $Q(q)$ is defined in \[ 6.5 \].

Finally, the lamination exact relation that is not stable under homogenization corresponding to \[ 8.1 \] is given by

$$
\mathbb{M} = \{ L \in \mathcal{H}^+(\mathbb{H}^2) : \det L = 1 \}.
$$

Obviously, $\det L$ can be any positive constant in the definition of $\mathbb{M}$. The constant is set to 1 for simplicity.

### 9 Acknowledgement

This material is based upon work supported by the National Science Foundation under Grant No. 1008092.

### A Summary of [14]

Let $G_p$ be a finite group with generators $a_1, \ldots, a_{p-1}$, $p \geq 2$ and $\varepsilon$ satisfying the relations

$$
\varepsilon^2 = 1_{G_p}, \quad a_k^2 = \varepsilon, \quad \varepsilon a_k = a_k \varepsilon, \quad a_k a_l = \varepsilon a_l a_k, \quad k, l = 1, \ldots, p-1, \quad k \neq l.
$$

Let $S = \{i_1, \ldots, i_k\}$ be a subset of $\{1, \ldots, p-1\}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq p-1$. Let $a_S = a_{i_1} a_{i_2} \cdots a_{i_k}$, where $a_0 = 1_{G_p}$. Then $G_p = \{a_S, \varepsilon a_S : S \subseteq \{1, \ldots, p-1\}\}$. Hence, $|G_p| = 2^p$. The first observation is that the commutator subgroup of $G_p$ is $K = \{1_{G_p}, \varepsilon\}$. It follows that $G_p$ has exactly $|G_p|/|K| = 2^{p-1}$ non-isomorphic complex 1D representations. In each of them $\varepsilon$ gets sent to 1, since $\varepsilon$ belongs to a commutator subgroup.

The second observation is that it is easy to list all conjugacy classes of $G_p$ explicitly. They are

$$
\begin{align*}
\{1_{G_p}\}, & \{\varepsilon\}, \{a_S, \varepsilon a_S\}, \quad S \subsetneq \{1, \ldots, p-1\}, \quad \{a_{\{1,\ldots,p-1\}}\}, \quad \{\varepsilon a_{\{1,\ldots,p-1\}}\}, \quad p \text{ is even}, \\
\{1_{G_p}\}, & \{\varepsilon\}, \{a_S, \varepsilon a_S\}, \quad S \subset \{1, \ldots, p-1\}, \quad S \neq \emptyset, \quad \{a_{\{1,\ldots,p-1\}}, \varepsilon a_{\{1,\ldots,p-1\}}\}, \quad p \text{ is odd}.
\end{align*}
$$

The total number of non-isomorphic complex irreps of $G_p$ equals to the number of conjugacy classes of $G_p$. We see that when $p$ is odd there is exactly one non-1D irrep, while when $p$ is even there are exactly two. The sum of squares of the dimensions of all the irreps of $G_p$ equals to the order of $G_p$. Hence, when $p$ is odd the dimension $d$ of the non-1D irrep is $d = 2^{\frac{p+1}{2}}$. When $p$ is even we denote the dimensions of the two non-1D irreps by $d_1$ and $d_2$. Recall that $d_1$ and $d_2$ must divide the order of the group. Hence, $d_1 = 2^a$, $d_2 = 2^b$ and $2^a + 2^b + 2^{p-1} \cdot 1^2 = 2^p$. Thus, $d_1 = d_2 = 2^{\frac{p-1}{2}}$. We remark that $\varepsilon$ cannot get sent to the identity matrix $I$ in a non-1D irrep, since in that case the commutator $G_p$ gets mapped to $I$ and all images of elements of $G_p$ would commute with one another. Since the image of $\varepsilon$ must commute with the images of all elements in the group, it must be mapped (by Schur’s lemma) into a multiple of the identity $\lambda I$ for some $\lambda \in \mathbb{C}$. However, $\lambda^2 = 1$, since $\varepsilon^2 = 1_{G_p}$. Therefore, $\varepsilon$ must be sent to $-I$.

The third observation is that $g^2 = 1_{G_p}$ or $\varepsilon$ for any $g \in G_p$. In fact, $(\varepsilon a_S)^2 = a_S^2 = \varepsilon^{r(r+1)/2}$, where $r = |S|$. Therefore, $a_S^2$ gets mapped to $(-1)^{r(r+1)/2}I$. This allows an explicit computation of the Frobenius-Schur indicator

$$
S = \frac{1}{|G_p|} \sum_{g \in G_p} \chi(g^2),
$$

where $\chi$ is the character of the representation. The result is

$$
S = \text{sign} \left( \cos \left( \frac{\pi p}{4} \right) \right).
$$
This shows that the non-1D irreps of $G_p$ are of real type, when $p = 0, 1, 7 \mod 8$, complex type, when $p = 2, 6 \mod 8$ and quaternionic type when $p = 3, 4, 5 \mod 8$. In particular, when $p = 2, 6 \mod 8$ the representations $U$ and $\overline{U}$ are not isomorphic and hence exhaust the list of two non-isomorphic irreps for $G_p$.

Thus, in order to obtain real irreps of $G_p$ in which $\varepsilon$ gets sent to $-I$ we take real parts of the representations $U$ for $p = 0, 1, 7 \mod 8$, real parts of the representations $U \oplus U$ for $p = 3, 4, 5 \mod 8$ and real parts of the representations $U \oplus \overline{U}$ for $p = 2, 6 \mod 8$. Theorem 6.2 summarizes the results.

References

[1] J. F. Adams. Vector fields on spheres. *The Annals of Mathematics*, 75(3):603–632, 1962.

[2] J. F. Adams, P. D. Lax, and R. S. Phillips. On matrices whose real linear combinations are nonsingular. *Proceedings of the American Mathematical Society*, 16(2):318–322, 1965.

[3] A. A. Albert. On a certain algebra of quantum mechanics. *The Annals of Mathematics*, 35(1):65–73, Jan 1934.

[4] A. Bensoussan, J. L. Lions, and G. Papanicolaou. *Asymptotic analysis of periodic structures*. North-Holland Publ., 1978.

[5] Y. Benveniste. Exact results in the micromechanics of fibrous piezoelectric composites exhibiting pyroelectricity. *Proc. Roy. Soc. Lond. A*, 441:59–81, 1993.

[6] Y. Benveniste. Exact connections between polycrystal and crystal properties in two-dimensional polycrystalline aggregates. *Proc. Roy. Soc. Lond. A*, 447:1–22, 1994.

[7] P. M. Cohn. On homomorphic images of special Jordan algebras. *Canadian J. Math.*, 6:253–264, 1954.

[8] N. Değirmenci and Ş. Koçak. Generalized self-duality of 2-forms. *Advances in applied Clifford algebras*, 13(1):107–113, 2003.

[9] N. Değirmenci and N. Ozdemir. The construction of maximum independent set of matrices via clifford algebras. *TURKISH JOURNAL OF MATHEMATICS*, 31(2):193–205, 2007.

[10] G. F. Dell’Antonio, R. Figari, and E. Orlandi. An approach through orthogonal projections to the study of inhomogeneous random media with linear response. *Ann. Inst. Henri Poincaré*, 44:1–28, 1986.

[11] M. L. Dunn. Exact relations between the thermoelectroelastic moduli of heterogeneous materials. *Proc. Roy. Soc. London Ser. A*, 441(1913):549–557, 1993.

[12] G. J. Dvorak. On uniform fields in heterogeneous media. *Proc. Roy. Soc. London Ser. A*, 431(1881):89–110, 1990.

[13] G. J. Dvorak. On some exact results in thermoplasticity of composite materials. *J. Thermal Stresses*, 15(2):211–228, 1992. Sixty-fifth Birthday of Bruno A. Boley Symposium, Part 2 (Atlanta, GA, 1991).

[14] B. Eckmann. Gruppentheoretischer beweis des satzes von hurwitz-radon über die komposition quadratischer formen. *Commentarii Mathematici Helvetici*, 15(1):358–366, 1942.

[15] Y. Grabovsky. Exact relations for effective tensors of polycrystals. I: Necessary conditions. *Arch. Ration. Mech. Anal.*, 143(4):309–330, 1998.

[16] Y. Grabovsky. Algebra, geometry and computations of exact relations for effective moduli of composites. In G. Capriz and P. M. Mariano, editors, *Advances in Multifield Theories of Continua with Substructure*, Modeling and Simulation in Science, Engineering and Technology, pages 167–197. Birkhäuser, Boston, 2004.
[17] Y. Grabovsky. Exact relations for effective conductivity of fiber-reinforced conducting composites with the Hall effect via a general theory. SIAM J. Math Anal., 41(3):973–1024, 2009.

[18] Y. Grabovsky, G. W. Milton, and D. S. Sage. Exact relations for effective tensors of polycrystals: Necessary conditions and sufficient conditions. Comm. Pure. Appl. Math., 53(3):300–353, 2000.

[19] Y. Grabovsky and D. S. Sage. Exact relations for effective tensors of polycrystals. II: Applications to elasticity and piezoelectricity. Arch. Ration. Mech. Anal., 143(4):331–356, 1998.

[20] Z. Hashin. Thermal expansion of polycrystalline aggregates: I. Exact results. J. Mech. Phys. Solids, 32:149–157, 1984.

[21] M. Hegg. Exact Relations And Links For Fiber-Reinforced Elastic Composites. PhD thesis, Temple University, Philadelphia, Pennsylvania, May 2012.

[22] R. Hill. Elastic properties of reinforced solids: Some theoretical principles. J. Mech. Phys. Solids, 11:357–372, 1963.

[23] A. Hurwitz. Über die komposition der quadratischen formen. Mathematische Annalen, 88(1):1–25, 1922.

[24] N. Jacobson. Structure and representations of Jordan algebras. American Mathematical Society, Providence, R.I., 1968. American Mathematical Society Colloquium Publications, Vol. XXXIX.

[25] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik. Homogenization of differential operators and integral functionals. Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian.

[26] P. Jordan, J. v. Neumann, and E. Wigner. On an algebraic generalization of the quantum mechanical formalism. The Annals of Mathematics, 35(1):29–64, 1934.

[27] J. B. Keller. A theorem on the conductivity of a composite medium. J. Math. Phys., 5:548–549, 1964.

[28] M. Koecher, A. Krieg, and S. Walcher. The Minnesota notes on Jordan algebras and their applications. Springer, 1999.

[29] W. Kohler and G. C. Papanicolaou. Bounds for effective conductivity of random media. In R. Burridge, S. Childress, and G. Papanicolaou, editors, Macroscopic properties of disordered media, pages 111–130, Berlin, 1982. Springer.

[30] N. Kwon and D. S. Sage. Subrepresentation semirings and an analog of 6j-symbols. J. Math. Phys., 49(6):063503, 21, 2008.

[31] V. M. Levin. Thermal expansion coefficients of heterogeneous materials. MTT, 2(1):88–94, 1967.

[32] K. McCrimmon. A taste of Jordan algebras. Springer, 2004.

[33] M. Milgrom. Some more exact results concerning multifield moduli of two-phase composites. J. Mech. Phys. Solids, 45(3):399–404, 1997.

[34] M. Milgrom and S. Shtrikman. Linear response of polycrystals to coupled fields: Exact relations among the coefficients. Physical Review B (Solid State), 40(9):5991–5994, 1989.

[35] M. Milgrom and S. Shtrikman. Linear response of two-phase composites with cross moduli: Exact universal relations. Physical Review A (Atomic, Molecular, and Optical Physics), 40(3):1568–1575, 1989.

[36] G. W. Milton. Multicomponent composites, electrical networks and new types of continued fraction. I. Comm. Math. Phys., 111(2):281–327, 1987.

[37] G. W. Milton. Multicomponent composites, electrical networks and new types of continued fraction. II. Comm. Math. Phys., 111(3):329–372, 1987.
[38] G. W. Milton. On characterizing the set of possible effective tensors of composites: the variational method and the translation method. *Comm. Pure Appl. Math.*, 43:63–125, 1990.

[39] G. W. Milton. Composites: A myriad of microstructure independent relations. In T. Tatsumi, E. Watanabe, and T. Kambe, editors, *Theoretical and applied mechanics (Proc. of the XIX International Congress of Theoretical and Applied mechanics, Kyoto, 1996)*, pages 443–459. Elsevier, Amsterdam, 1997.

[40] G. W. Milton. *The theory of composites*, volume 6 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 2002.

[41] G. W. Milton and R. V. Kohn. Variational bounds on the effective moduli of anisotropic composites. *J. Mech. Phys. Solids*, 36(6):597–629, 1988.

[42] J. Radon. Lineare scharen orthogonalen matrizen. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 1:1–14, 1922.

[43] B. W. Rosen and Z. Hashin. Effective thermal expansion coefficients and specific heats of composite materials. *Int. J. Engng. Sci.*, 8:157–173, 1970.

[44] D. S. Sage. Group actions on central simple algebras. *J. Algebra*, 250:18–43, 2002.

[45] D. S. Sage. Quantum Racah coefficients and subrepresentation semirings. *J. Lie Theory*, 15(1):321–333, 2005.

[46] D. S. Sage. Racah coefficients, subrepresentation semirings, and composite materials. *Adv. in Appl. Math.*, 34(2):335–357, 2005.

[47] K. Schulgasser. Thermal expansion of polycrystalline aggregates with texture. *J. Mech. Phys. Solids*, 35(1):34–42, 1987.

[48] V. Šverák. Rank-one convexity does not imply quasiconvexity. *Proc. Roy. Soc. Edinburgh Sect. A*, 120(1-2):185–189, 1992.