Localisation of perturbations of a constant state in a traffic flow model

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Abstract

We consider, in the Aw-Rascle-Zhang traffic flow model, the problem of the asymptotic stability of constant flows. By using a perturbative approach, we show the stability in a larger space of perturbation than previous results. Furthermore, we are able to compute where the perturbation is mainly localised in space for a given time, based on the localisation of the perturbation initially. These new ideas can be applied to various other models of hyperbolic conservation laws with relaxations.

1 Introduction and presentation of the results

1.1 The Aw-Rascle-Zhang model

We look here at the Aw-Rascle-Zhang traffic flow model introduced in [1] and [17], that is composed of two equations on unknowns $\rho, u : \mathbb{R} \times \mathbb{R}_+^+ \to \mathbb{R}$. The quantity $\rho(x,t)$ for $x \in \mathbb{R}, t \geq 0$ is the density of cars on an infinite highway at position $x$ and time $t$, and $u(x,t)$ is the speed at that same point, and they are solutions of the system

\[(\text{ARZ}) \begin{cases} 
\partial_t \rho + \partial_x (\rho u) = 0 \\
\partial_t (u + h(\rho)) + u \partial_x (u + h(\rho)) = \frac{1}{\tau} (U(\rho) - u).
\end{cases} \tag{1.1}
\]

The first equation encode the fact that the cars move at speed $u$, and implies the conservation of mass. The second equation describes the variation of the speed of the cars. Two effects are taken into account in this model, in addition to the fact that the speed moves with the cars. First, a pressure $h(\rho)$, a strictly increasing function of $\rho$, that makes cars entering an area with higher density slow down. Secondly, an equilibrium speed $U(\rho) = U_f (1 - \rho)$ at which cars try to move at, and they relax to this speed at a caracteristic time $\tau > 0$. $U_f > 0$ is the free speed, at which cars move if alone on the road, and finally $\rho_{\text{max}} = 1$ is the maximum density, at which $U(\rho_{\text{max}}) = 0$, and $h(\rho)$ blows up when $\rho \to \rho_{\text{max}}$. This model is an improvement of previous traffic flow models like [11], [14] that captures many physical phenomenons, see [1].

We consider equation (1.1) with $\tau > 0, U(\rho) = U_f (1 - \rho), U_f > 0$ and $h \in C^\infty([0,1],\mathbb{R})$ with $h' > 0$ and $h(\rho) \to +\infty$ when $\rho \to 1$. There exists constant solutions to this equation: $\rho = \rho_0 \in ]0,1[$ and $u = U(\rho_0)$. We are interested here in their stability.

1.2 Previous stability results on the constant flow

The Aw-Rascle-Zhang model is part of a large class of hyperbolic systems with relaxation of the form

\[
\begin{cases} 
\partial_t \rho + \partial_x (f(\rho, w)) = 0 \\
\partial_t w + \partial_x (g(\rho, w)) = r(\rho, w)
\end{cases}
\]

on the unknown $(\rho, w)$, where $f, g, r$ are known function. the system (ARZ) is of this form, with the quantity for any $c \in ]0,1[$,

$G(\rho) := \int_c^\rho \frac{h'(\nu)}{\nu} d\nu, w := u + G(\rho)$,
if we take
\[ f(\rho, w) = \rho w - \rho G(\rho), g(\rho, w) = \frac{w^2}{2} - \frac{g(\rho)}{2}, r(\rho, w) = \frac{1}{\tau}(U(\rho) + G(\rho) - w). \]

In this kind of model and natural generalisations, the question of the stability of constant flows and small travelling waves have been studied extensively using energy methods. See for instance [13] and reference therein, as well as [4], [10] for the stability of constant flows, [12], [19] for the stability of travelling waves, and [2], [18] for generalisations. See also [6] for a recent work on another model.

Let us summarize the results of [4] here in the particular case of the system (ARZ). Consider an initial data \((\rho_0 + u_0, U(\rho_0) + u_1)\) and let us write the solution of (1.1) on the form \((\rho_0 + \rho, U(\rho_0) + u)\). In [4], it is shown that at leading order, \(\rho\) satisfies an equation of the form
\[ \partial_t \rho - \nu \partial^2_x \rho + \partial_x (f_\ast(\rho)) = 0, \]
where \(\nu \in \mathbb{R}\) and \(f_\ast\) is a smooth function. For this equation to have a solution, it is necessary that \(\nu > 0\). This is called the subcharacteristic condition, and in the case of (ARZ), it is equivalent to \(s_{cc}(\rho_0) := h'(\rho_0) - U_f > 0\). Then, considering that \(\rho\) is small, this equation is developed as
\[ \partial_t \rho - \nu \partial^2_x \rho + \lambda_s \partial_x \rho + \frac{f''(\rho_0)}{2} \partial_x (\rho^2) = 0. \]

In the case of (ARZ), we can compute that \(\lambda_s := U(\rho_0) - \rho_0 U_f\). This Burgers type equation admit self similar solutions \(\theta(x, t)\), that satisfy
\[
\begin{align*}
\left\{ \begin{array}{c}
\partial_t \theta - \nu \partial^2_x \theta + \lambda_s \partial_x \theta + \frac{f''(\rho_0)}{2} \partial_x (\theta^2) = 0 \\
\theta(x, -1) = m \delta(x)
\end{array} \right. \quad (1.2)
\end{align*}
\]
where \(m = \int_\mathbb{R} \rho_i(x) dx\) is the mass of the perturbation (which is conserved with time) and \(\delta\) is the dirac function. In the case \(f''(\rho_0) = 0\), this solution is
\[
\theta(x, t) = \frac{m}{4\pi \nu \sqrt{1 + t}} \exp \left( -\frac{(x - \lambda_s (1 + t))^2}{4\nu (1 + t)} \right),
\]
that is the Gaussian. In the case \(f''(\rho_0) \neq 0\) the solution is also explicit and has similar properties (see [4] below equation (3.1)).

The main result of [4] is the asymptotic stability of this diffusion wave.

**Theorem 1.1 ([4], Theorem 1.1)** Consider equation (1.1) with an initial data \((\rho_0 + \rho_i, U(\rho_0) + u_i)\), \(\rho_i, u_i \in C^2_{loc}\), with \(\rho_0 \in [0, 1]\) and \(s_{cc}(\rho_0) = h'(\rho_0) - U_f > 0\). With \(m = \int_\mathbb{R} \rho_i(x) dx\), we consider \(\theta\) the solution of (1.2) and \(\theta_0\) its value at \(t = 0\). Then, there exists \(\delta_0 > 0\) depending on \(\rho_0, h'(\rho_0), U_f\) such that, if the functions \(z_0, w_0\) defined by
\[ \rho_i = \theta_0 + \partial_x z_0, u_i = U(\theta_0) + w_0 \]
are such that
\[ |m| + \sum_{l=0}^3 \|\partial_x^l z_0\|_{L^2(\mathbb{R})} + \sum_{l=0}^2 \|\partial_x^l w_0\|_{L^2(\mathbb{R})} + \|z_0\|_{L^1(\mathbb{R})} + \|w_0\|_{L^1(\mathbb{R})} \leq \delta_0, \]
then (1.1) admit a classical solution for this initial condition for all positive time. Furthermore, if we decompose this solution as \((\rho_0 + \rho, U(\rho_0) + u)\), then for \(l \in \{0, 1, 2\}\),
\[ \|\partial_x^l (\rho - \theta)(., t)\|_{L^p} = O_{t \to \infty}(1 + t)^{-\frac{1}{2} + \frac{l}{4}} \text{ for } p \in [1, +\infty] \]
and
\[ \|\partial_x^l (u - U(\theta))(., t)\|_{L^p} = O_{t \to \infty}(1 + t)^{-\frac{1}{2} + \frac{l}{4}} \text{ for } p \in [2, +\infty]. \]

This stability result is proven using energy methods. Remark that it requires the initial data to be well localized (the condition \(\|z_0\|_{L^1(\mathbb{R})} < +\infty\) is a lot stronger than \(\rho_i \in L^1(\mathbb{R})\)). For instance, take \(\chi\) a smooth positive function with value 1 in \([-1, 1]\) and 0 outside of \([-2, 2]\). Then Theorem 1.1 can not be applied to the case
\( \rho_i = \varepsilon (\chi(x - A) + \chi(x + A)), u_i = 0 \) for \( \varepsilon > 0 \) small, uniformly in \( A \) (it would require \( A \ll \frac{1}{\varepsilon} \)). The stability has been shown for a larger space of perturbation than Theorem 1.1, but for simpler hyperbolic systems, see [7] and [15].

This localisation on the initial data is necessary to get the asymptotic profile. Our goal here is to use a new approach, that will allow us to extend the space of perturbations. We also want to give a different type of precision on the behavior of the perturbation. Broadly speaking, we want to know where the perturbation is mainly localized, approach, that will allow us to extend the space of perturbations. We also want to give a different type of precision on the behavior of the perturbation. Broadly speaking, we want to know where the perturbation is mainly localized.

\[ \text{Proposition 1.2} \]

Consider the equation (1.1) for the functions \( \rho_0 + \rho, U(\rho_0) + u \), and keep only the linear terms in \( (\rho, u) \). Then, the equations become

\[
\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} + \begin{pmatrix} U(\rho_0) \\ 0 \end{pmatrix} \rho_0 U(\rho_0) - \rho_0 h'(\rho_0) \partial_x \begin{pmatrix} \rho \\ u \end{pmatrix} + \frac{1}{\tau} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix} = 0. \tag{1.3}
\]

This is the linearized equation around a constant flow \( (\rho_0, U(\rho_0)) \) for \( \rho_0 \in [0, 1] \). A useful remark about this linear problem is that the equations on \( \rho \) and \( u \) can be decoupled. In particular, with the speed \( \lambda_* = U(\rho_0) - \rho_0 U_f \), the functions

\[ q(x, t) := \rho(x + \lambda_* t, t), \quad v(x, t) := u(x + \lambda_* t, t) \]

satisfy the equations

\[
\partial_t^2 q + (\lambda_0^1 + \lambda_0^2) \partial_x^2 q + \lambda_1^0 \lambda_0^2 \partial_x^2 q + \frac{1}{\tau} \partial_t q = 0
\]

and

\[
\partial_t^2 v + (\lambda_0^1 + \lambda_0^2) \partial_x^2 v + \lambda_1^0 \lambda_0^2 \partial_x^2 v + \frac{1}{\tau} \partial_t v = 0,
\]

where \( \lambda_0^1 := \rho_0 U_f, \lambda_0^2 := \rho_0 (U_f - h'(\rho_0)) \). See Lemma 5.1 for the proof of these computations. The subcharacteristic condition \( s_{\text{sc}}(\rho_0) > 0 \) implies that \( \lambda_1^0 > 0, \lambda_2^0 < 0 \), and in particular \( \lambda_1^0 \lambda_2^0 < 0 \). The equation satisfied by \( q \) (and \( v \)) is called the damped wave equation. It turns out that we can compute explicitly its solution.

\subsection{1.3.1 The damped wave equation}

In this section, we are interested in the properties of a solution to the problem

\[
\begin{cases}
\partial_t^2 f + (\lambda_1 + \lambda_2) \partial_x^2 f + \lambda_1 \lambda_2 \partial_x^2 f + \delta \partial_t f = S(x, t) \\
f_{t=0} = f_0 \\
\partial_t f_{t=0} = f_1 
\end{cases} \tag{1.4}
\]

where \( \delta > 0, \lambda_1 > 0, \lambda_2 < 0 \) are constants, because \( q \) and \( v \) satisfy this equation with \( S = 0; \delta = \frac{1}{\tau}, \lambda_1 = \lambda_1^0 \) and \( \lambda_2 = \lambda_2^0 \). We can compute explicitly its solution.

\[ \text{Proposition 1.2} \]

For \( \delta > 0, \lambda_1 > 0, \lambda_2 < 0 \), the solution to the problem (1.4) with \( f_0, f_1, S \in C^2_{\text{loc}} \) satisfies

\[
f(x, t) = \int_{\lambda_2 t}^{\lambda_1 t} V(y, t)(\delta f_0 + (\lambda_1 + \lambda_2)f_0'(x-y))dy \\
+ \int_{\lambda_2 t}^{\lambda_1 t} \partial_t V(y, t)f_0(x-y)dy \\
+ \lambda_1 e^{\frac{\lambda_1 - \lambda_2}{\lambda_1^0}} f_0(x - \lambda_1 t) - \lambda_2 e^{\frac{\lambda_2 - \lambda_1}{\lambda_2^0}} f_0(x - \lambda_2 t) \\
+ \int_0^t \int_{\lambda_2 (t-s)}^{\lambda_1 (t-s)} V(y, t-s)S(x-y, s)dyds
\]
Lemma 1.3 For \( k, n \in \mathbb{N}, \lambda_1 > 0, \lambda_2 < 0, \delta > 0 \), the function

\[
V(y, t) := e^{-\frac{2\delta}{\lambda_1 - \lambda_2} y} I_0 \left( \frac{2\delta \sqrt{-\lambda_2} \sqrt{-(y - \lambda_1 t)(y - \lambda_2 t)}}{(\lambda_1 - \lambda_2)^2} \right),
\]

is smooth on \( t \geq 0, y \in [\lambda_2 t, \lambda_1 t] \) and there exists \( C_{k, n}(\lambda_1, \lambda_2, \delta), a_0(\lambda_1, \lambda_2, \delta) > 0 \) such that, for any \( t \geq 0, y \in [\lambda_2 t, \lambda_1 t] \), we have

\[
|\partial_x^k \partial_t^n V(y, t)| \leq C_{k, n}(\lambda_1, \lambda_2, \delta) e^{-a_0 t^{\frac{2}{1 + \frac{n}{2}}}} \frac{1}{(1 + t)^{\frac{k}{2} + \frac{n}{2}}}
\]

We can also show some estimates on the kernel \( V \):

**Lemma 1.3** For \( k, n \in \mathbb{N}, \lambda_1 > 0, \lambda_2 < 0, \delta > 0 \), the function

\[
V(y, t) := e^{-\frac{2\delta}{\lambda_1 - \lambda_2} y} I_0 \left( \frac{2\delta \sqrt{-\lambda_2} \sqrt{-(y - \lambda_1 t)(y - \lambda_2 t)}}{(\lambda_1 - \lambda_2)^2} \right)
\]

is smooth on \( t \geq 0, y \in [\lambda_2 t, \lambda_1 t] \) and there exists \( C_{k, n}(\lambda_1, \lambda_2, \delta), a_0(\lambda_1, \lambda_2, \delta) > 0 \) such that, for any \( t \geq 0, y \in [\lambda_2 t, \lambda_1 t] \), we have

\[
|\partial_x^k \partial_t^n V(y, t)| \leq C_{k, n}(\lambda_1, \lambda_2, \delta) e^{-a_0 t^{\frac{2}{1 + \frac{n}{2}}}} \frac{1}{(1 + t)^{\frac{k}{2} + \frac{n}{2}}}
\]

Section 2 is devoted to the proof of these two results. They follow from the computation of the kernel of the simpler and well-known equation \( \partial_x^2 u - \partial_t^2 u + \partial_t u = 0 \) (see [5]).

### 1.3.2 Statement of the linear stability

With the explicit solution coming from Proposition 1.2 and estimate in Lemma 1.3 on the kernel that appears, we can show a precise estimate on the solution of the linearized problem (1.3).

**Proposition 1.4** Consider the equation (1.3) with \( \rho_0 \in [0, 1] \) and \( s_{cc}(\rho_0) = h'(\rho_0) - U_f > 0 \) and some initial data \( \rho_{i,t=0} = \rho_0 + \rho_i, u_{i,t=0} = U(\rho_0) + u_i \) with \( \rho_i, u_i \in C^j_{loc}(\mathbb{R}, \mathbb{R}) \) for some \( j \geq 2 \). Then, the solution \( (\rho_0 + \rho, U(\rho_0) + u) \) of (1.3) is well defined in \( C^2_{loc}(\mathbb{R}_x \times \mathbb{R}_t, \mathbb{R}) \) and we have the following estimates. There exists \( a_L, b_L > 0 \) depending on \( \rho_0, U_f, h'(\rho_0) \) such that, for any \( k, n \in \mathbb{N} \) with \( k + n \leq j \), defining

\[
\lambda_0^0 = \rho_0 U_f > 0, \lambda_2^0 = \rho_0 (U_f - h'(\rho_0)) < 0, \lambda_* = U(\rho_0) - \rho_0 U_f,
\]

there exists \( K_{n, k} > 0 \) depending on \( n, k, \rho_0, U_f, h'(\rho_0) \) such that

\[
|\partial_x^k \partial_t^n (u(\lambda_0^0 t, t))| + |\partial_x^k \partial_t^n (\rho(\lambda_0^0 t, t))| \leq \frac{K_{n, k}}{(1 + t)^{\frac{k}{2} + \frac{n}{2}}} \int_{\lambda_0^0 t}^{1 \lambda_0^0 t} e^{-a_L t^{\frac{2}{1 + \frac{n}{2}}}} \left| u_i(x - y) \right| + |\rho_i(x - y)| dy
\]

\[
+ \quad K_{n, k} e^{-b_L t} \left( \sum_{j=0}^{n+k} (|\rho_i^{(j)}| + |u_i^{(j)}|)(x - \lambda_0^0 t) + (|\rho_i^{(j)}| + |u_i^{(j)}|)(x - \lambda_2^0 t) \right)
\]

for all \( x \in \mathbb{R}, t \geq 0 \).

Let us first make a few remarks on this result.

- Suppose that \( \rho_i, u_i \in C^j \cap L^1(\mathbb{R}) \). We check that this result implies

\[
|\partial_x^k \partial_t^n (u(\lambda_0^0 t, t))| + |\partial_x^k \partial_t^n (\rho(\lambda_0^0 t, t))| \leq \frac{K_{n, k}(\|u_i\|_{C^{n+k, 1}(\mathbb{R})} + \|\rho_i\|_{C^{n+k, 1}(\mathbb{R})})}{(1 + t)^{\frac{k}{2} + \frac{n}{2}}}
\]

- This formulation allows us to keep information on the localisation of the perturbation when time evolves. That is, to estimate \( u(x - \lambda_0^0 t, t) \), we can only look at the initial data in a neighborhood of \( x \) of size \( \sqrt{t} \), and the error committed by doing so is almost exponentially small in time, while this main term is, if the initial
data is $L^1(\mathbb{R})$, of size $\frac{1}{\sqrt{t}}$. This means for instance that if our initial perturbation is compactly supported in $[-1,1]$ at $t=0$, then for any $\varepsilon > 0$, at time $t \geq 1$ the perturbation outside $[(\lambda_+ - \varepsilon)t, (\lambda_+ + \varepsilon)t]$ is exponentially small in time. This is something that can not be shown easily using energy methods. By linearity of (1.3) this can also be applied to a sum of localizations.

- Another way to say this is the following. Amongst all the speeds between $\lambda_2^0 < 0$ and $\lambda_1^0 > 0$ that the perturbation could take, it only, up to exponentially small error, goes at the speed $\lambda_+$. That is, $\rho(\lambda t, t)$ is exponentially small in time for all $\lambda \neq \lambda_+$.

- If the initial condition is in $L^1(\mathbb{R})$, we get the same decay in time as Theorem 1.1. Remark also that this estimate make sense even if the initial condition are not in $L^1(\mathbb{R})$, but in that case we do not have necessarily a decay in time of the solution.

- The main term in the estimate looks like the one we would get for the heat equation with a transport term, that is the solution of $\left( \partial_t - \frac{1}{a_L} \partial_x^2 + \lambda \partial_x \right) \rho = 0$. It is in fact the dominating effect at first order (which makes sense in regards of Theorem 1.1 from [4]). We see two main difference with the simpler heat + transport system: the integral is only on $[\lambda_2^0, \lambda_1^0]$ instead of $\mathbb{R}$, and we have an additional exponentially small error in time.

- Remark that here, since we only look at the linear problem, we do not need any smallness on the initial data. In fact it does not need to decay at infinity for this result to hold, thanks to the finite speed of propagation. We can also go above the threshold of maximum density $\rho_{\text{max}} = 1$, but this is only true for the linear problem, since in that case the function $h'$, that blows up at 1, is taken in $\rho_0$ and not $\rho_0 + \rho$.

This proposition is a corollary of Proposition 1.2 and Lemma 1.3, see section 5 for its proof.

### 1.4 The nonlinear stability in $L^1(\mathbb{R})$

Our goal here is to show a similar estimate as Proposition 1.4 but in the nonlinear case. Taking equation (1.1) for the functions $u_0 + \rho, U(\rho_0) + u$, and then introducing $q(x,t) = \rho(x-\lambda_+t,t), v(x,t) = u(x-\lambda_+t,t)$ and the non constant characteristic speeds $\lambda_1 := \rho_0 U_f + v, \lambda_2 := \rho_0 U_f - (\rho_0 + q)h'(\rho_0 + q) + v$, we can check (see Lemma 6.1 for the computation) that the functions $q,v$ satisfy the equations

$$\partial_t^2 q + (\lambda_1 + \lambda_2)\partial_x^2 q + \lambda_1 \lambda_2 \partial_x^2 v + \frac{1}{\tau} \partial_v v + \Omega_1 \partial_x v = 0$$

and

$$\partial_t^2 v + (\lambda_1 + \lambda_2)\partial_x^2 v + \lambda_1 \lambda_2 \partial_x^2 v + \frac{1}{\tau} \partial_v v + \Omega_2 \partial_x v = 0,$$

where $\Omega_1, \Omega_2, \Omega_3$ are small if $q,v$ are small. In particular, it is no longer possible to decouple the two equations.

There are several obstacles to do an estimation similar to the linear case, that will force us to reduce the quality of the estimates. The first one is the fact that the characteristic speeds are no longer constant. In the linear setting, they were $\lambda_1^0 = \rho_0 U_f, \lambda_2^0 = \rho_0 (U_f - h'(\rho_0))$, and in the nonlinear setting, they are $\lambda_1 = \rho_0 U_f + v, \lambda_2 = \rho_0 U_f - (\rho_0 + q)h'(\rho_0 + q) + v$. Concerning $\lambda_+$, if we would define it in the nonlinear setting, it would not even have the same value for $q$ and $v$ (it would be $\lambda_+ + \Omega_1$ for $v$ and $\lambda_+ + \Omega_3$ for $q$). A usual method to deal with nonconstant characteristic speed is to do a change of coordinate to make them constant. But here, because there are at least three speeds we would want to make constant at the same time, this is difficult.

The second one is simply to show that the nonlinear terms does not infer growth or blow up. They will be consider as source terms of the linear problem. This is why in Proposition 1.2 we did the computations with a generic source term $S$. This itself pose a difficulty, since terms in the source term will have as many derivatives as linear terms (for instance the equation on $q$ will contain $(\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0)\partial_x^2 q$ in $S$).

Let us infer our main stability result before discussing on how to solve these issues. We recall the norm on the space $W^{2,1}(\mathbb{R})$:

$$\|f\|_{W^{2,1}(\mathbb{R})} := \sum_{j=0}^2 \|f^{(j)}\|_{L^1(\mathbb{R})}$$
and the norm on the space $C^2(\mathbb{R})$:

$$\|f\|_{C^2(\mathbb{R})} := \sum_{j=0}^{2} \|f^{(j)}\|_{L^\infty(\mathbb{R})}.$$ 

**Theorem 1.5** Consider the equation (1.1) with $\rho_0 \in [0, 1]$ and $s_c(\rho_0) = h'(\rho_0) - U_f > 0$ and initial data $\rho(t=0) = \rho_0 + \rho_i, u(t=0) = u(\rho_0) + u_i$ with $\rho_i, u_i \in C^2_{loc}(\mathbb{R}, \mathbb{R})$. Suppose furthermore that $\rho_i, u_i \in W^{2,1}(\mathbb{R}) \cap C^2(\mathbb{R})$ and define

$$\varepsilon := \|\rho_i\|_{W^{2,1}(\mathbb{R})} + \|u_i\|_{W^{2,1}(\mathbb{R})} + \|\rho_i\|_{C^2(\mathbb{R})} + \|u_i\|_{C^2(\mathbb{R})}$$

as well as

$$w_i = \sum_{j=0}^{2} |\rho_i^{(j)}| + |u_i^{(j)}|.$$

Then, for any $\nu > 0$ there exists $\varepsilon_0 > 0$ depending on $\rho_0, h'(\rho_0), U_f, \nu$, such that if $\varepsilon \leq \varepsilon_0$, then the solution $(\rho_0 + \rho, U(\rho_0) + u)$ of (1.1) is well defined in $C^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ and we have the following estimates. There exists $a, b > 0$ depending on $\rho_0, U_f, h$, and $\delta > 0$ depending on $\rho_0, U_f, h, \varepsilon$ with $\delta = o_{\varepsilon \to 0}(1)$ such that, for any $k, n \in \mathbb{N}$ with $k + n \leq 2$, defining

$$\lambda_1^0 = \rho_0 U_f > 0, \lambda_2^0 = \rho_0 (U_f - h'(\rho_0)) < 0, \lambda_* = U(\rho_0) - \rho_0 U_f,$$

$$\gamma(0, 0) = \frac{1}{2}, \gamma(0, 1) = 1 - \nu, \gamma(0, 1) = \frac{3}{2} - \nu, \gamma(k, n) = 1 - \nu$$

otherwise, there exists $K_{n,k} > 0$ depending on $n, k, \rho_i, U_f, h'(\rho_0), \nu$ such that

$$|\partial_t \lambda_k^0 \partial_x^k (u(x - \lambda_* t, t))| + |\partial_t \lambda_k^0 \partial_x^k (\rho(x - \lambda_* t, t))|$$

$$\leq \frac{K_{n,k}}{(1 + t)^{k+\delta}} \sup_{y \in [-\delta, \delta]} (w_i(x - \lambda_1^0 t + y) + w_i(x - \lambda_2^0 t + y))$$

for all $x \in \mathbb{R}, t \geq 0$.

We start by introducing some quantities that we want to use to estimate $u$ and $\rho$. For $\mu_1 > 0, \mu_2 < 0, \gamma > 0, a > 0$ and $w \in L^1_{loc}(\mathbb{R}, \mathbb{R}^+)$, we introduce the quantity

$$F_{a,\mu_1,\mu_2}(x, t) := \frac{1}{(1 + t)^{\gamma}} \int_{\mu_2 t}^{\mu_1 t} e^{-a \frac{y^2}{1+t}} w(x - y) dy.$$ 

Remark that in Proposition 1.4, the functions $\partial_t \lambda_k^0 \partial_x^k (u(x - \lambda_* t, t))$ and $\partial_t \lambda_k^0 \partial_x^k (\rho(x - \lambda_* t, t))$ are estimated in part by $F_L := F_{a,\lambda_1^0,\lambda_2^0}$, where $F_L$ can not estimate these functions in the nonlinear case. For instance, it only “see” the initial data on $[\lambda_2^0 t, \lambda_1^0 t]$, whereas the true characteristic speeds $\lambda_2, \lambda_1$ can for instance be such that $\lambda_2 < \lambda_1^0, \lambda_1 > \lambda_1^0$ and thus the light cone is larger than $[\lambda_2^0 t, \lambda_1^0 t]$. It also does not take into account the fact that the speed $\lambda_*$ is no longer constant. Despite all that, to make the estimate work, we will only need to change the parameters of $F$. In Theorem 1.5, the quantity $F_{a,\lambda_1^0,\lambda_2^0}$ is replaced by $F_{a,\gamma(k,n), w_i}$, where $\delta > 0$ is small, $aL > a > 0, \frac{1}{2} \leq \gamma(k,n) \leq \frac{1}{2} + \frac{1}{2} + n$ and $w_i$ incode the initial value of the perturbation. Taking $\lambda_1^0 + \delta, \lambda_2^0 - \delta$ for the characteristic allows us to have a bigger light cone than the real (non straight) one, and having $a < aL$ will allow us to absorb the error committed by following the speed $\lambda_*$. There will also be a loss in decay in time $(\gamma(k,n))$, on which we will come back later on.

The second quantity we introduce is, for $\mu \in \mathbb{R}, b > 0, \delta > 0, w \in L^1_{loc}(\mathbb{R}, \mathbb{R})$,

$$G_{b,\mu,\delta}(x, t) := e^{-bt} \sup_{y \in [-\delta, \delta]} w(x - \mu t + y).$$
Remark that the estimate of Proposition 1.4 can be written as

\[ |\partial_\nu a^k \partial_\nu b^k (u(x + \lambda_s t, t))| \leq K_{n,k} \left( F_{n,k}^\nu w + G_{n,k}^\nu + G_{n,k}^\nu \right) (x, t) \]

with

\[ w = \sum_{j=0}^{n+k} |\partial_j^2 u(x)|. \]

As for \( F \), we will not be able to take \( \delta = 0 \) in \( G_{n,k}^\nu \), for the estimate in Theorem 1.5 for similar reasons. We will also need to take \( 0 < b < b_L \). To simplify the notation, we introduce finally

\[ F_{a,\gamma,\delta} := F_{a,\gamma,\delta}^\nu \]

and

\[ G_{b,\delta} := G_{b,\delta}^\nu + G_{b,\delta}^\nu. \]

The estimate of Theorem 1.5 can be written with these notations as

\[ |\partial_\nu a^k \partial_\nu b^k (u(x + \lambda_s t, t))| \leq K_{n,k} (F_{a,\gamma,\delta} (x, t) + G_{b,\delta} (x, t)) \]

for \( n + k \leq 2 \).

Let us make a few remarks on Theorem 1.5.

- We keep informations on the localisation of the perturbation. The remarks done in the linear problem also apply here.

- In the case \( k = n = 0 \), since \( w_i \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), we have that

\[ |\rho(x, t)| + |u(x, t)| \leq \frac{K \varepsilon}{\sqrt{1 + t}}. \]

This is the optimal decay, that was already shown in [4]. For first derivatives, we have a small loss in decay with this method (we lose \( t^\nu \)) but \( \nu \) can be made as small as we want when \( \varepsilon \to 0 \). For second order derivatives, we have a bigger loss. See the sketch of the proof in subsection 1.5 for more details about why, but we expect to only get a \( t^\nu \) loss if we ask for the smallness on more derivatives on the initial data.

- We can check easily that Theorem 1.5 implies that \( \rho, u \in W^{2,1} \cap C^2(\mathbb{R}) \) for all positive time and that for \( n + k \leq 2 \) we have

\[ ||\partial_\nu a^k \partial_\nu b^k (u, t)||_{L^\infty} + ||\partial_\nu a^k \partial_\nu b^k (u, t)||_{L^\infty} \leq \frac{K \varepsilon}{(1 + t)^{\gamma(n,k)}}. \]

as well as

\[ ||\partial_\nu a^k \partial_\nu b^k (u, t)||_{L^1} + ||\partial_\nu a^k \partial_\nu b^k (u, t)||_{L^1} \leq \frac{K \varepsilon}{(1 + t)^{\gamma(n,k)-1/2}}. \]

By interpolation we can deduce \( L^p \) estimates on these quantities.

- On the initial perturbation, we only require for it to be small in \( W^{2,1} \cap C^2 \), which is a larger space than the space of perturbations in [4]. However, we do not have an equivalent when \( t \to +\infty \), only a bound on the error. This allows us to sidestep the difficulty of finding the first order, which, for such a general initial perturbation, might be difficult.

- Remark that the estimates are given on \( \rho(x - \lambda_s t, t) \) and \( u(x - \lambda_s t, t) \), while the speed \( \lambda_s \) is connected to the linear problem and not the nonlinear one. The "true" speed of the perturbation should be \( \lambda_s + \Omega_1 \) for \( \rho \), with \( |\Omega_1| \leq \frac{\varepsilon}{\sqrt{1 + t}} \). Such an error is absorbed by the form of the estimate we took. Indeed, we can show that

\[ F_{a,\gamma,\delta} (x + \kappa(x, t), t) \leq K F_{a,\gamma,\delta} (x, t) \]
for some $K, \alpha' < a, \delta > 0$ if $|\alpha| \leq \varepsilon \sqrt{1+t}$. We can check that $a > 0$ defined in Theorem 1.5 will be smaller than $a_L > 0$ from the linear case, to take into account this kind of error. In other words, we had to change the parameters in $F$ and $G$ to take into account the nonlinear characteristic speed, but also to absorb the error committed by the fact that $\lambda_\ast$ is not exactly the right speed of the error.

- We could be more specific in what derivatives are needed on the initial condition to estimate $\rho, u$ and its derivatives. That is, instead of considering $w$, that regroups all derivatives up to 2 of the initial data, we could consider only a part of it in the case $k + n \leq 1$. We do not focus on this here, to simplify some part of the proof and some notations.

We will sketch the proof of Theorem 1.5 in the next subsection. In this paper, we looked specifically at the Aw-Rascle-Zhang model rather than the general case for several reason. The model in itself is interesting and has been studied mathematically with different approach and question. For instance, the limit $\tau \to 0$ has been studied in [8]. This also allow us to simplify some notations without losing too much generality. For other hyperbolic systems, Theorem 1.5 should hold, with some technical conditions. For instance, it will be important that $\lambda_1, \lambda_2$, the two nonlinear characteristic speeds, can depend on $\rho, u$ but not on their derivatives.

### 1.5 Plan of the proofs

We start by solving, in section 2, the equation

$$
\partial_t^2 f + (\lambda_1^2 + \lambda_2^2)\partial_{xx}^2 f + \lambda_1 \lambda_2 \partial_x^2 f + \delta \partial_t f = S,
$$

see Proposition 1.2 and Lemma 1.3. The solution of $\partial_t^2 u - \partial_x^2 v + \partial_v u = S$ is well known, and is connected to (1.5) by a change of variables. The estimates shown in Lemma 1.3 are a consequence of usual computations on Bessel functions.

Section 3 is devoted to estimates on $F_{a, \gamma, w}^{\mu, \nu}$ and $G_{b, \delta, w}^{\mu}$, in particular, their interaction with the kernel of the damped wave equation. The contribution of the source $S$ in the damped wave equation is

$$
\int_0^t \int_{\lambda_1^2(t-s)}^{\lambda_2^2(t-s)} V(y, t-s)S(x-y, s)dyds,
$$

and we compute this quantity and its derivatives with respect to $x$ and $t$ in the case $|S| \leq aF_{a, \gamma, w}^{\lambda_1^2 + \delta, \lambda_2^2 - \delta} + \beta G_{b, \delta, w}^{\mu}$ for $\alpha, \beta > 0, \mu \in \{\lambda_1^0, \lambda_2^0\}$. The key point is that for such a value of $S$, the quantity (1.6) and its derivatives can be estimates by terms that are also of the form $\alpha'F_{a, \gamma, w}^{\lambda_1^0 + \delta, \lambda_2^0 - \delta} + \beta'G_{b, \delta, w}^{\mu}$ with $\alpha', \beta' > 0$.

Then, in section 4, we compute a change of variable to deal with the fact that the characteristic speeds $\lambda_1$ and $\lambda_2$ are not constant. It will be used to estimate second derivatives of the perturbation. However, this will not be used for the estimate of the first and non derivative, and most likely it is not possible to do so. Since $\lambda_1$ and $\lambda_2$ remain close to constants $\lambda_1^0, \lambda_2^0$, we will do a change of coordinate close to the identity, both in time and space, to change the characteristic speeds to $\lambda_1^0, \lambda_2^0$.

Section 5 is devoted to the proof of Proposition 1.4. This is done using Proposition 1.2 and Lemma 1.3.

Finally, section 6 is devoted to the proof of Theorem 1.5. We treat the nonlinear term perturbatively, using Proposition 1.2. We show the estimates by a bootstrap. The result hold for small time by standard hyperbolic theory. Then, using the results in section 3, we show that they hold for all times for $\rho, u$ and their first derivatives.

For the second derivative, we use section 4 to write the equation on a similar form, but this time the terms in $S$ do not contain second derivatives of $\rho$ and $u$. Otherwise, we will have a loss of derivatives. Still using section 3 we compute that the estimates hold for these new variables. Then, we show that we can get these estimates back in the original coordinates with the quantities $F_{a, \gamma, w}^{\mu, \nu}$ and $G_{b, \delta, w}^{\mu}$, up to reducing the value of $a$ and $b$ a little.

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2 Solution to the general damped wave equation

This section is devoted to the proof of Proposition 1.2 and Lemma 1.3. We suppose here that $\lambda_1 > 0, \lambda_2 < 0, \delta > 0$.

2.1 Proof of Proposition 1.2

We recall the equation we are trying to solve, which is

\[
\begin{aligned}
\frac{\partial^2 f}{\partial t^2} + (\lambda_1 + \lambda_2)\frac{\partial^2 f}{\partial x^2} + \lambda_1 \lambda_2 \frac{\partial^2 f}{\partial x^2} + \delta \frac{\partial f}{\partial t} &= S(x,t) \\
f_{t=0} &= f_0 \\
\frac{\partial f}{\partial t}_{t=0} &= f_1.
\end{aligned}
\]

2.1.1 Solution to the classical damped wave equation

We recall the definition of $I_0$, the modified Bessel function of the first kind and of order 0:

\[
I_0(y) := \sum_{n=0}^{+\infty} \frac{1}{n!\Gamma(n+1)} \left(\frac{y}{2}\right)^{2n}.
\]

We will recall some of its properties in subsection 2.2.1. It appears in the solution of the classical damped wave equation on $\mathbb{R}$:

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = S, \quad u_{t=0} = u_0, \quad \frac{\partial u}{\partial t}_{t=0} = u_1. \tag{2.1}
\]

Theorem 2.1 ([5]) Consider the function

\[
J(t, y) := \frac{e^{-t/2}}{2 I_0 \left(\frac{1}{2} \sqrt{t^2 - y^2}\right)}.
\]

The solution to

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = S, \quad u_{t=0} = u_0, \quad \frac{\partial u}{\partial t}_{t=0} = u_1
\]

is

\[
\begin{aligned}
\int_{-t}^{t} J(t, y) u_1(x-y) dy \\
+ \int_{-t}^{t} J(t, y) u_0(x-y) dy + \frac{\partial}{\partial t} \left( \int_{-t}^{t} J(t, y) u_0(x-y) dy \right) \\
+ \int_{0}^{t} \left( \int_{|y|\leq t-s} J(t-s, y) S(x-y, s) dy \right) ds.
\end{aligned}
\]

We recall briefly the proof of Theorem 2.1. We first check that

\[
\frac{\partial^2 J}{\partial t} + \frac{\partial J}{\partial t} - \frac{\partial^2 J}{\partial x^2} = 0.
\]

Then, we can solve the problem if $u_0 = S = 0$, simply by computing

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} \right) \left( \int_{-t}^{t} J(t, y) u_1(x-y) dy \right) = 0.
\]

and checking that at $t = 0$, we have the right initial condition. Then, using this function, we construct the parts coming from $u_0$ and $S$, and we add them together by linearity.
2.1.2 From the classical to the general damped wave equation

Our goal here is to find the function $V$ that satisfies for the general damped wave equation (1.4) the same role that $J$ satisfies for the classical damped wave equation (2.1). We define $a > 0, b \in \mathbb{R}, c > 0$ and the function $V$ by

$$J(y, t) = V(ay, by + ct).$$

We recall that $\partial_t^2 J + \partial_t J - \partial_y^2 J = 0$, and we compute that

$$\partial_t J = c\partial_t V, \partial_t^2 J = c^2 \partial_t^2 V$$
$$\partial_y J = a\partial_y V + b\partial_t V, \partial_y^2 J = a^2 \partial_y^2 V + 2ab \partial_y^4 V + b^2 \partial_t^2 V$$

hence

$$\partial_t^2 V - \frac{2ab}{c^2 - b^2} \partial_y^2 V - \frac{a^2}{c^2 - b^2} \partial_y^2 V + \frac{c}{c^2 - b^2} \partial_t V = 0.$$ 

We then look for solutions of

$$\frac{-2ab}{c^2 - b^2} = \beta(= \lambda_1 + \lambda_2), \quad \frac{a^2}{c^2 - b^2} = \alpha(= -\lambda_1 \lambda_2 > 0), \quad \frac{c}{c^2 - b^2} = \delta > 0.$$ 

We find

$$a = \frac{1}{3} \sqrt{\alpha + \frac{\beta^2}{4}}, \quad b = -\frac{\beta}{2\alpha \delta} \sqrt{\alpha + \frac{\beta^2}{4}}, \quad c = \frac{1}{\alpha \delta} \left(\alpha + \frac{\beta^2}{4}\right).$$

We deduce that the function

$$V(y, t) = \frac{1}{\lambda_1 - \lambda_2} J \left(\frac{y}{a \lambda_1 - \lambda_2}, \frac{at - by}{ac}\right)$$

satisfies the equation

$$\partial_t^2 V + (\lambda_1 + \lambda_2) \partial_y^2 V + \lambda_1 \lambda_2 \partial_y^4 V + \delta \partial_t V = 0.$$ 

The factor $\frac{1}{\lambda_1 - \lambda_2}$ is here to have a cleaner formula in Proposition 1.2. We compute

$$\frac{at - bx}{ac} = \frac{\delta}{\alpha + \frac{\beta^2}{4}} \left(\frac{at + \frac{\beta^2}{4}}{2}\right)$$

and therefore

$$V(y, t) = e^{-\frac{2(\alpha + \frac{\beta^2}{4})}{2\alpha \delta \lambda_1 - \lambda_2}} I_0 \left(\frac{\delta}{2} \sqrt{\left(\frac{1}{\alpha + \frac{\beta^2}{4}} \left(\frac{at + \frac{\beta^2}{4}}{2}\right)\right)^2 - \left(\frac{y}{\sqrt{\alpha + \frac{\beta^2}{4}}}ight)^2}\right).$$

After some simplifications, using $\alpha = -\lambda_1 \lambda_2, \beta = \lambda_1 + \lambda_2, \alpha + \frac{\beta^2}{4} = \frac{1}{4}(\lambda_1 - \lambda_2)^2$, we find that

$$V(y, t) = e^{-\frac{2\delta}{\lambda_1 - \lambda_2} \left(-\lambda_1 \lambda_2 t + \frac{\lambda_1 + \lambda_2 y}{2}\right)} I_0 \left(\frac{2\delta \sqrt{-\lambda_1 \lambda_2}}{(\lambda_1 - \lambda_2)^2} \sqrt{-(y - \lambda_1 t)(y - \lambda_2 t)}\right).$$

Let us summarize our computations so far, and give as well some specific values of $V$.

Lemma 2.2 The function

$$V(y, t) = e^{-\frac{2\delta}{\lambda_1 - \lambda_2} \left(-\lambda_1 \lambda_2 t + \frac{\lambda_1 + \lambda_2 y}{2}\right)} I_0 \left(\frac{2\delta \sqrt{-\lambda_1 \lambda_2}}{(\lambda_1 - \lambda_2)^2} \sqrt{-(y - \lambda_1 t)(y - \lambda_2 t)}\right)$$

satisfies

$$\partial_t^2 V + (\lambda_1 + \lambda_2) \partial_y^2 V + \lambda_1 \lambda_2 \partial_y^4 V + \delta \partial_t V = 0.$$
We also compute

\[ V(\lambda_1 t, t) = \frac{e^{-\lambda_1 t}}{\lambda_1 - \lambda_2}, \quad V(\lambda_2 t, t) = \frac{e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \]

and

\[ \partial_t V(\lambda_1 t, t) = \frac{2\delta \lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2} \left( 1 - \frac{I'''_0(0)\delta \lambda_1 t}{\lambda_1 - \lambda_2} \right) V(\lambda_1 t, t), \]
\[ \partial_t V(\lambda_2 t, t) = \frac{2\delta \lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2} \left( 1 + \frac{I'''_0(0)\delta \lambda_2 t}{\lambda_1 - \lambda_2} \right) V(\lambda_1 t, t) \]
as well as
\[ \partial_y V(\lambda_1 t, t) = \left( -\frac{\delta (\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^2} + \frac{2\delta^2 \lambda_1 \lambda_2 t I''_0(0)}{(\lambda_1 - \lambda_2)^4} \right) V(\lambda_1 t, t), \]
\[ \partial_y V(\lambda_2 t, t) = \left( \frac{-\delta (\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^2} - \frac{2\delta^2 \lambda_1 \lambda_2 t I''_0(0)}{(\lambda_1 - \lambda_2)^4} \right) V(\lambda_2 t, t). \]

Remark that \( \lambda_1 - \lambda_2 > 0 \) and \( \frac{\delta \lambda_2}{\lambda_1 - \lambda_2} < 0. \)

**Proof** We complete the above computations with the following particular values of \( V \). We define

\[ \gamma_1(y, t) := \frac{-2\delta}{(\lambda_1 - \lambda_2)^2} \left( -\lambda_1 \lambda_2 t + \frac{\lambda_1 + \lambda_2}{2} y \right) \]

and

\[ \gamma_2(y, t) := \frac{2\delta \sqrt{-\lambda_1 \lambda_2}}{(\lambda_1 - \lambda_2)^2} \sqrt{-(\lambda_1 t)(\lambda_2 t)} \]

so that

\[ V(y, t) = \frac{e^{\gamma_1(y, t)}}{\lambda_1 - \lambda_2} I_0(\gamma_2(y, t)). \]

We compute

\[ \partial_t \gamma_1 = \frac{2\delta \lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2}, \quad \partial_y \gamma_1 = \frac{-\delta (\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^2} \]

and

\[ \partial_t \gamma_2 = \frac{(\lambda_1 + \lambda_2)y - 2\lambda_1 \lambda_2 t}{2\gamma_2} \left( -\frac{4\delta^2 \lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^4} \right) \]
as well as

\[ \partial_y \gamma_2 = \frac{-2y + (\lambda_1 + \lambda_2) t}{2\gamma_2} \left( -\frac{4\delta^2 \lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^4} \right) . \]

We compute

\[ \gamma_2(\lambda_1 t, t) = \gamma_2(\lambda_2 t, t) = 0, \]
\[ \gamma_1(\lambda_1 t, t) = \frac{-\delta \lambda_1 t}{\lambda_1 - \lambda_2} \leq 0, \quad \gamma_1(\lambda_2 t, t) = \frac{\delta \lambda_2 t}{\lambda_1 - \lambda_2} \leq 0, \]

therefore

\[ V(\lambda_1 t, t) = \frac{e^{\gamma_1(\lambda_1 t, t)}}{\lambda_1 - \lambda_2} I_0(\gamma_2(\lambda_1 t, t)) = \frac{e^{\frac{-\delta \lambda_1 t}{\lambda_1 - \lambda_2}}}{\lambda_1 - \lambda_2} \]

and

\[ V(\lambda_2 t, t) = \frac{e^{\gamma_1(\lambda_2 t, t)}}{\lambda_1 - \lambda_2} I_0(\gamma_2(\lambda_2 t, t)) = \frac{e^{\frac{\delta \lambda_2 t}{\lambda_1 - \lambda_2}}}{\lambda_1 - \lambda_2}. \]

Furthermore, since

\[ V(y, t) = \frac{e^{\gamma_1(y, t)}}{\lambda_1 - \lambda_2} I_0(\gamma_2(y, t)), \]
we have
\[\partial_t V(y, t) = \partial_t \gamma_1(y, t)V(y, t) + \frac{e^{\gamma_1(y, t)}}{\lambda_1 - \lambda_2} \partial_t \gamma_2(y, t)I_0'(\gamma_2(y, t)),\]
and since \(\gamma_2(\lambda_1 t, t) = 0\), we compute that
\[\partial_t V(\lambda_1 t, t) = \partial_t \gamma_1(\lambda_1 t, t)V(\lambda_1 t, t) + \frac{e^{\gamma_1(\lambda_1 t, t)}}{2(\lambda_1 - \lambda_2)} (\partial_t (\gamma_2(y, t)^2))_{|y=\lambda_1 t} I_0''(0)\]
\[= \frac{2\delta_1 \lambda_2}{(\lambda_1 - \lambda_2)^2} V(\lambda_1 t, t) + \frac{I_0''(0)}{2} \left(-\frac{4\delta_1^2 \lambda_1^2 \lambda_2 t}{(\lambda_1 - \lambda_2)^2}\right) V(\lambda_1 t, t)\]
as well as
\[\partial_t V(\lambda_2 t, t) = \partial_t \gamma_1(\lambda_2 t, t)V(\lambda_2 t, t) + \frac{e^{\gamma_1(\lambda_2 t, t)}}{2(\lambda_1 - \lambda_2)} (\partial_t (\gamma_2(y, t)^2))_{|y=\lambda_2 t} I_0''(0)\]
\[= \frac{2\delta_1 \lambda_2}{(\lambda_1 - \lambda_2)^2} V(\lambda_1 t, t) + \frac{I_0''(0)}{2} \left(\frac{4\delta_1^2 \lambda_1^2 \lambda_2 t}{(\lambda_1 - \lambda_2)^2}\right) V(\lambda_1 t, t)\]
Similarly,
\[\partial_y V(\lambda_1 t, t) = \left(\partial_y \gamma_1 + \frac{1}{2} \partial_y (\gamma_2^2)I_0''(0)\right)_{|y=\lambda_1 t} V(\lambda_1 t, t),\]
hence
\[\partial_y V(\lambda_1 t, t) = \left(-\frac{\delta(\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^2} + \frac{2\delta_1^2 \lambda_1 \lambda_2 t I_0''(0)}{(\lambda_1 - \lambda_2)^3}\right) V(\lambda_1 t, t)\]
and we also check that
\[\partial_y V(\lambda_2 t, t) = \left(-\frac{\delta(\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^2} - \frac{2\delta_1^2 \lambda_1 \lambda_2 t I_0''(0)}{(\lambda_1 - \lambda_2)^3}\right) V(\lambda_2 t, t)\]

We are now equipped to compute the solution of the general damped wave equation (1.4).

**Lemma 2.3** Given \(u_1 \in C^2_c(\mathbb{R})\), the function
\[v(x, t) = \int_{\lambda_2 t}^{\lambda_1 t} V(y, t)u_1(x - y)dy\]
satisfies \(v(x, 0) = 0, \partial_t v(x, 0) = u_1\) and the equation
\[\partial_t^2 v + (\lambda_1 + \lambda_2)\partial_{yy}^2 v + \lambda_1 \lambda_2 \partial_y^2 v + \delta \partial_t v = 0.\]

**Proof** We compute
\[\partial_t v = \lambda_1 V(\lambda_1 t, t)u_1(x - \lambda_1 t) - \lambda_2 V(\lambda_2 t, t)u_1(x - \lambda_2 t)\]
\[+ \int_{\lambda_2 t}^{\lambda_1 t} \partial_t V(y, t)u_1(x - y)dy\]
and
\[\partial_t^2 v = \lambda_1 (\partial_t (V(\lambda_1 t, t))) + \partial_t V(\lambda_1 t, t))u_1(x - \lambda_1 t)\]
\[+ \lambda_2 (\partial_t (V(\lambda_2 t, t))) + \partial_t V(\lambda_2 t, t))u_1(x - \lambda_2 t)\]
\[+ \lambda_1^2 V(\lambda_1 t, t)u_1'(x - \lambda_1 t) + \lambda_2^2 V(\lambda_2 t, t)u_1'(x - \lambda_2 t)\]
\[+ \int_{\lambda_2 t}^{\lambda_1 t} \partial_t^2 V(y, t)u_1(x - y)dy.\]
We also have
\[
\partial_{yt}^2 v = \lambda_1 V(\lambda_1 t, t)u'_1(x - \lambda_1 t) - \lambda_2 V(\lambda_2 t, t)u'_1(x - \lambda_2 t) \\
- \partial_t V(\lambda_1 t, t)u_1(x - \lambda_1 t) + \partial_t V(\lambda_2 t, t)u_1(x - \lambda_2 t) \\
+ \int_{\lambda_2 t}^{\lambda_1 t} \partial_y^2 V(y, t)u_1(x - y)dy
\]
and
\[
\partial_y^2 v = -(V(\lambda_1 t, t)u'_1(x - \lambda_1 t) + \partial_y V(\lambda_1 t, t)u_1(x - \lambda_1 t)) \\
+ V(\lambda_2 t, t)u'_1(x - \lambda_2 t) + \partial_y V(\lambda_2 t, t)u_1(x - \lambda_2 t) \\
+ \int_{\lambda_2 t}^{\lambda_1 t} \partial_y^2 V(y, t)u_1(x - y)dy.
\]

We deduce that
\[
d_{t}^2 v + (\lambda_1 + \lambda_2)d_{y}^2 v + \lambda_1 \lambda_2 d_{x}^2 v + \delta \partial_t v \\
= u_1(x - \lambda_1 t)(\lambda_1 (\partial_t V(\lambda_1 t, t)) + \partial_t V(\lambda_1 t, t)) - (\lambda_1 + \lambda_2) \partial_t V(\lambda_1 t, t) - \lambda_1 \lambda_2 \partial_x V(\lambda_1 t, t) + \delta \lambda_1 V(\lambda_1 t, t)) \\
+ u_1(x - \lambda_2 t)(-\lambda_2 (\partial_t V(\lambda_2 t, t)) + \partial_t V(\lambda_2 t, t)) + (\lambda_1 + \lambda_2) \partial_t V(\lambda_2 t, t) + \lambda_1 \lambda_2 \partial_x V(\lambda_2 t, t) - \delta \lambda_2 V(\lambda_2 t, t)) \\
+ u'_1(x - \lambda_1 t)(-\lambda_1^2 V(\lambda_1 t, t) + (\lambda_1 + \lambda_2) \lambda_1 V(\lambda_1 t, t) - \lambda_1 \lambda_2 V(\lambda_1 t, t)) \\
+ u'_1(x - \lambda_2 t)(\lambda_2^2 V(\lambda_2 t, t) - (\lambda_1 + \lambda_2) \lambda_2 V(\lambda_2 t, t) + \lambda_1 \lambda_2 V(\lambda_2 t, t)).
\]

We check easily with Lemma 2.2 that
\[
-\lambda_1^2 V(\lambda_1 t, t) + (\lambda_1 + \lambda_2) \lambda_1 V(\lambda_1 t, t) - \lambda_1 \lambda_2 V(\lambda_1 t, t) = 0
\]
and
\[
\lambda_2^2 V(\lambda_2 t, t) - (\lambda_1 + \lambda_2) \lambda_2 V(\lambda_2 t, t) + \lambda_1 \lambda_2 V(\lambda_2 t, t) = 0,
\]
and similarly for the other boundaries terms. \(\square\)

We can now solve the general damped wave equation with a source and any initial condition. **Proof** [Of Proposition 1.2] We recall that equation (1.4) is linear. We have prove it for the contribution in \(f_1\) in Lemma 2.3. Now consider \(v\) the solution to
\[
\begin{cases}
\partial_{tt}^2 v + (\lambda_1 + \lambda_2)\partial_{yy}^2 v + \lambda_1 \lambda_2 \partial_{xx}^2 v + \delta \partial_t v = 0 \\
v_{t=0} = 0 \\
\partial_t v_{t=0} = f_0.
\end{cases}
\]
and define \(w = \delta v + \partial_t v + (\lambda_1 + \lambda_2) \partial_x v\). Then, \(w_{t=0} = \partial_t v_{t=0} = f_0\) and \(\partial_t w = -\lambda_1 \partial_x \partial_x^2 v\) hence \(\partial_t w_{t=0} = 0\). We deduce that \(w\) solves the problem
\[
\begin{cases}
\partial_{tt}^2 w + (\lambda_1 + \lambda_2)\partial_{yy}^2 w + \lambda_1 \lambda_2 \partial_{xx}^2 w + \delta \partial_t w = 0 \\
w_{t=0} = f_0 \\
\partial_t w_{t=0} = 0.
\end{cases}
\]
Finally, we define
\[
F(t, x, s) = \int_{\lambda_2 t}^{\lambda_1 t} V(z, t)S(x - z, s)dz
\]
and
\[
u(x, t) = \int_{0}^{t} F(t - s, x, s)ds.
\]
We check easily that \(u(x, 0) = \partial_t u(x, 0) = 0\) and since
\[
F(0, x, t) = \partial_x F(0, x, t) = \partial_x F(0, x, t) = 0
\]
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as well as
\[ \partial_t F(0, x, t) = \lambda_1 V(0, 0)S(x, t) - \lambda_2 V(0, 0)S(x, t) = (\lambda_1 - \lambda_2)V(0, 0)S(x, t) = S(x, t) \]
we check easily that
\[ \partial^2 u + (\lambda_1 + \lambda_2)\partial^2 x u + \lambda_1 \lambda_2 \partial^2 u + \delta \partial u = S(x, t). \]
We conclude by linearity of the equation, taking for \( f \) the sum of the functions \( u, w \) and the solution constructed in Lemma 2.3. \( \square \)

2.2 Properties of the function \( V \)

We are interested here in estimates on the function
\[ V(y, t) = e^{\frac{2t}{(\lambda_1 - \lambda_2)^2}} \left( -\frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 - \lambda_2} \right) I_0 \left( \frac{2\sqrt{\lambda_1 \lambda_2}}{(\lambda_1 - \lambda_2)^2} \sqrt{-(y - \lambda_1 t)(y - \lambda_2 t)} \right), \]
with \( \delta > 0, \lambda_1 > 0 \) and \( \lambda_2 < 0. \)

2.2.1 Properties of the Bessel function \( I_0 \)

Let us also recall here some properties of \( I_0 \):

**Lemma 2.4** ([16]) The function \( I_0 \in C^\infty(\mathbb{R}^+, \mathbb{R}) \) satisfies the following properties:

- \( I_0(0) = 1, I_0' \geq 0 \)
- \( I_0(y) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left( \frac{y}{2} \right)^{2n} \)
- \( I_0(x) = \frac{e^x}{\sqrt{2\pi x}} (1 + O_{x \rightarrow \infty} \left( \frac{1}{x} \right)) \).

2.2.2 Estimates on the function \( V \)

**Lemma 2.5** For \( \delta > 0, \lambda_1 > 0, \lambda_2 < 0 \), there exists \( C_{0,0}(\lambda_1, \lambda_2, \delta), a_0(\lambda_1, \lambda_2, \delta) > 0 \) such that the function
\[ V(y, t) = e^{\frac{2t}{(\lambda_1 - \lambda_2)^2}} \left( -\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} \right) I_0 \left( \frac{2\sqrt{\lambda_1 \lambda_2}}{(\lambda_1 - \lambda_2)^2} \sqrt{-(y - \lambda_1 t)(y - \lambda_2 t)} \right) \]
satisfied for \( t \geq 0, y \in [\lambda_2 t, \lambda_1 t] \) that
\[ |V(y, t)| \leq \frac{C_{0,0}(\lambda_1, \lambda_2, \delta)}{(1 + t)^{\frac{1}{2}}} e^{-a_0 \frac{y^2}{(1+t)^2}}. \]

**Proof** First we check easily the estimate if \( 0 \leq t \leq 1, y \in [\lambda_2 t, \lambda_1 t] \) (which is simply the fact that \( V \) is bounded if \( t \) and \( y \) are bounded). We now suppose that \( t \geq 1 \). We define as in the proof of Lemma 2.2 the quantities
\[ \gamma_1(y, t) = \frac{-2\delta}{(\lambda_1 - \lambda_2)^2} \left( -\lambda_1 \lambda_2 + \frac{\lambda_1 + \lambda_2}{2} y \right) \]
and
\[ \gamma_2(y, t) = \frac{2\sqrt{\lambda_1 \lambda_2}}{(\lambda_1 - \lambda_2)^2} \sqrt{-(y - \lambda_1 t)(y - \lambda_2 t)}, \]
so that \( V(y, t) = \frac{e^{\gamma_1(y, t)} I_0(\gamma_2(y, t))}{\lambda_1 - \lambda_2} \). We also write
\[ \gamma_2(y, t) = tG_0 \left( \frac{y}{t} \right) \]
with
\[ G_0(z) := \frac{2\delta \sqrt{-\lambda_1 \lambda_2}}{(\lambda_1 - \lambda_2)^2} \sqrt{-(z - \lambda_2)(z - \lambda_1)}. \]

We also define
\[ H(x) := e^{-x} I_0(x) \]
which is a smooth decreasing function on \(\mathbb{R}^+\) that satisfies, using Lemma 2.4, that
\[ H(0) = 1, H(x) \sim \frac{1}{\sqrt{2\pi x}} \]
when \(x \to +\infty\). Remark that \(0 \leq G_0(x) \leq K(\lambda_1, \lambda_2, \delta)\) for \(x \in [\lambda_2, \lambda_1]\), and that for \(x \in \left[\frac{\lambda_2}{2}, \frac{\lambda_1}{2}\right]\) we have
\[ G_0(x) \geq C_1(\lambda_1, \lambda_2, \delta) > 0, \]
as well as
\[ V(y, t) = \frac{e^{\gamma_1(y,t) + \gamma_2(y,t)}}{\lambda_1 - \lambda_2} H \left( tG_0 \left( \frac{y}{t} \right) \right). \]
We compute that
\[ \gamma_1(y,t) + \gamma_2(y,t) = \frac{-\delta}{(\lambda_1 - \lambda_2)^2} \frac{y^2}{t} - tF_0 \left( \frac{y}{t} \right) \]
where
\[ F_0(x) := \frac{2\delta \lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2} \left( -1 + \frac{1}{2} \left( \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} x - \frac{x^2}{\lambda_1 \lambda_2} \right) + \sqrt{1 - \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} x + \frac{x^2}{\lambda_1 \lambda_2}} \right). \]
Indeed, we have
\[ \gamma_1(y,t) + \gamma_2(y,t) \]
\[ = \frac{2\delta}{(\lambda_1 - \lambda_2)^2} \left( \lambda_1 \lambda_2 t - \frac{\lambda_1 + \lambda_2}{2} y + \sqrt{\lambda_1 \lambda_2(y - \lambda_1 t)(y - \lambda_2 t)} \right) \]
\[ = \frac{2\delta t}{(\lambda_1 - \lambda_2)^2} \left( \lambda_1 \lambda_2 - \frac{\lambda_1 + \lambda_2}{2} y^2 + \sqrt{(\lambda_1 \lambda_2)^2 - \frac{y^2}{t}(\lambda_1 \lambda_2^2 + \lambda_2^2 \lambda_2 + \lambda_1 \lambda_2 \frac{y^2}{t^2})} \right) \]
\[ = \frac{2\delta \lambda_1 \lambda_2 t}{(\lambda_1 - \lambda_2)^2} \left( 1 - \frac{1}{2} \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right) y \right) - \sqrt{1 - \frac{1}{t} \left( \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \right)^2 + \frac{1}{\lambda_1 \lambda_2} \frac{y^2}{t^2}} \]
\[ = -tF_0 \left( \frac{y}{t} \right) + \frac{-\delta}{(\lambda_1 - \lambda_2)^2} \frac{y^2}{t}. \]
Remark that with \(z = \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} x - \frac{x^2}{\lambda_1 \lambda_2}\) we have \(F_0(x) = \frac{2\delta \lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2} \left( -1 + \frac{1}{2} z + \sqrt{1 - z} \right)\). This implies that we have \(F_0(x) \geq 0\) for all \(x \in [\lambda_2, \lambda_1]\) and for \(x \in [\lambda_2, \lambda_1] \setminus \left[\frac{\lambda_2}{2}, \frac{\lambda_1}{2}\right]\),
\[ F_0(x) \geq C_2(\lambda_1, \lambda_2, \delta) > 0. \]
We have the decomposition
\[ V(y, t) = \frac{e^{\lambda_1 \lambda_2 y^2}}{\lambda_1 - \lambda_2} e^{-tF_0 \left( \frac{y}{t} \right)} H \left( tG_0 \left( \frac{y}{t} \right) \right). \]
For \(\frac{y}{t} \in \left[\frac{\lambda_2}{2}, \frac{\lambda_1}{2}\right], \left| e^{-tF_0 \left( \frac{y}{t} \right)} \right| \leq 1\) and \(H \left( tG_0 \left( \frac{y}{t} \right) \right) \leq H(C_1(\lambda_1, \lambda_2, \delta) t) \leq \frac{K(\lambda_1, \lambda_2, \delta)}{\sqrt{1 + t}}\) thus for \(a_0 = \frac{\delta}{(\lambda_1 - \lambda_2)^2} > 0\), we have
\[ |V(y, t)| \leq \frac{C(\lambda_1, \lambda_2, \delta) e^{\frac{-\delta}{(\lambda_1 - \lambda_2)^2} \frac{y^2}{t}}}{(1 + t)^{\frac{3}{2}}} \leq \frac{K_1(\lambda_1, \lambda_2, \delta) e^{a_0 \frac{y^2}{t}}}{(1 + t)^{\frac{3}{2}}}. \]
And for $\pi \in [\lambda_2, \lambda_1] \setminus \left[\frac{\lambda_2}{2}, \frac{\lambda_1}{2}\right]$, we have $e^{-tF_0(\pi)} \leq e^{-C_2(\lambda_1, \lambda_2, \delta)t}, |H(tG_0(\pi))| \leq K$ and thus

$$|V(y, t)| \leq C(\lambda_1, \lambda_2, \delta)e^{\frac{-\delta}{(1 + \lambda_1^2 + \lambda_2^2)\sqrt{\lambda_1 \lambda_2} t}} e^{C_2(\lambda_1, \lambda_2, \delta)t} \leq \frac{K_2(\lambda_1, \lambda_2, \delta)e^{\alpha_0 \frac{\rho^2}{1 + t^2}}}{(1 + t)^2}.$$  

This concludes the estimate for $C_{0,0} = \max(K_1, K_2)$. \hfill $\square$

### 2.2.3 Estimates on derivatives of $V$

**Lemma 2.6** For $\delta > 0, \lambda_1 > 0, \lambda_2 < 0$, the function

$$V(y, t) = \frac{e^{-\frac{2\delta}{(\lambda_1 - \lambda_2)^2}(-\lambda_1 \lambda_2 t + \frac{\lambda_1 + \lambda_2}{2} y)}}{\lambda_1 - \lambda_2} I_0 \left( \frac{2\delta\sqrt{-\lambda_1 \lambda_2}}{(\lambda_1 - \lambda_2)^2} \sqrt{-(y - \lambda_1 t)(y - \lambda_2 t)} \right)$$

is smooth in both $y$ and $t$ on $t \geq 0, y \in [\lambda_2 t, \lambda_1 t]$, including at the boundary.

**Proof** We simply check, with Lemma 2.4, that

$$I_0 \left( \frac{2\delta\sqrt{-\lambda_1 \lambda_2}}{(\lambda_1 - \lambda_2)^2} \sqrt{-(y - \lambda_1 t)(y - \lambda_2 t)} \right) = \sum_{n=0}^{+\infty} \frac{1}{n!\Gamma(n+1)} \left( \frac{\delta^2\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2} (y - \lambda_1 t)(y - \lambda_2 t) \right)^n,$$

which leads to the differentiability of $V$ on $t \geq 0, y \in [\lambda_2 t, \lambda_1 t]$, even at the boundary $y = \lambda_2 t$ or $y = \lambda_1 t$. In fact, by this formula,

$$V(y, t) = e^{-\frac{2\delta}{(\lambda_1 - \lambda_2)^2}(-\lambda_1 \lambda_2 t + \frac{\lambda_1 + \lambda_2}{2} y)} \sum_{n=0}^{+\infty} \frac{1}{n!\Gamma(n+1)} \left( \frac{\delta^2\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2} (y - \lambda_1 t)(y - \lambda_2 t) \right)^n$$

can be extended to $\mathbb{R}_y \times \mathbb{R}_t$ and is smooth on it. \hfill $\square$

We can now conclude the proof of Lemma 1.3.

**Proof** [of Lemma 1.3] We check easily the estimate for $0 \leq t \leq 1$ and we suppose from now on that $t \geq 1$. We have, under the notations of the proof of Lemma 2.5, that

$$V(y, t) = \frac{e^{\gamma_1(y, t)}}{\lambda_1 - \lambda_2} I_0(\gamma_2(y, t))$$

with

$$\gamma_1(y, t) = \frac{-2\delta}{(\lambda_1 - \lambda_2)^2} \left(-\lambda_1 \lambda_2 t + \frac{\lambda_1 + \lambda_2}{2} y\right)$$

and

$$\gamma_2(y, t) = \frac{2\delta\sqrt{-\lambda_1 \lambda_2}}{(\lambda_1 - \lambda_2)^2} \sqrt{-(y - \lambda_1 t)(y - \lambda_2 t)}.$$  

We compute that

$$\partial_t V = V \left( \partial_t \gamma_1 + \partial_t(\gamma_2) \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} \right) = V \left( \partial_t \gamma_1 + \partial_t \gamma_2 + \partial_t \gamma_2 \left( \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} - 1 \right) \right).$$

Under the previous notations, we have

$$\gamma_2(y, t) = tG_0 \left( \frac{y}{t} \right),$$

where

$$G_0(z) = \frac{2\delta\sqrt{-\lambda_1 \lambda_2}}{(\lambda_1 - \lambda_2)^2} \sqrt{-(z - \lambda_1)(z - \lambda_2)}.$$  

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We check that \( \partial_t \gamma_1 = -G_0(0) \) and
\[
\partial_t \gamma_2 = G_0 \left( \frac{y}{t} \right) - \frac{y}{t} G'_0 \left( \frac{y}{t} \right).
\]
Therefore,
\[
\partial_t \gamma_1 + \partial_t \gamma_2 = -G_0(0) + G_0 \left( \frac{y}{t} \right) - \frac{y}{t} G'_0 \left( \frac{y}{t} \right).
\]
We have
\[
\partial_t V = \frac{e^{-(\lambda_1 - \lambda_2) t} y^2}{\lambda_1 - \lambda_2} \left( -G_0(0) + G_0 \left( \frac{y}{t} \right) - \frac{y}{t} G'_0 \left( \frac{y}{t} \right) \right)
+ \frac{e^{-(\lambda - \lambda_2) t} y^2}{\lambda_1 - \lambda_2} \left( \partial_t \gamma_2 \left( \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} - 1 \right) \right).
\]
We have
\[
G''_0(0) = \frac{1}{-\lambda_1 \lambda_2} \left( \sqrt{-\lambda_1 \lambda_2} + \frac{\lambda_1 + \lambda_2}{2} \right) > 0
\]
(it is 0 if and only if \( \lambda_1 - \lambda_2 = 0 \)) and therefore
\[
-G_0(0) + G_0 \left( \frac{y}{t} \right) - \frac{y}{t} G'_0 \left( \frac{y}{t} \right) = \text{O}_0 \left( \frac{y^2}{t^2} \right).
\]
We compute
\[
F''_0(0) = -\delta(\lambda_1 + \lambda_2)^2 \frac{1}{2\lambda_1 \lambda_2 (\lambda_1 - \lambda_2)^2} \geq 0
\]
but it can be 0 if \( \lambda_1 + \lambda_2 = 0 \). We therefore estimate
\[
\left| \partial_t V(y, t) \right| \leq C(\lambda_1, \lambda_2, \delta) e^{-(\lambda_1 - \lambda_2) t} t^2 \left[ -G_0(0) + G_0 \left( \frac{y}{t} \right) - \frac{y}{t} G'_0 \left( \frac{y}{t} \right) \right]
+ C(\lambda_1, \lambda_2, \delta) e^{-(\lambda_1 - \lambda_2) t} t^2 \left[ \partial_t \gamma_2 \left( \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} - 1 \right) \right].
\]
Since \( (F_0(x) + \frac{\delta y^2}{\lambda_1 - \lambda_2})''(0) > 0 \) and \( G''_0(0) > 0 \), we check that for \( \frac{y}{t} \in \left[ \frac{\lambda_1}{2}, \frac{\lambda_2}{2} \right] \),
\[
e^{-t \left( F_0 \left( \frac{y}{t} \right) + \frac{\delta y^2}{\lambda_1 - \lambda_2} \right)} \left| -G_0(0) + G_0 \left( \frac{y}{t} \right) - \frac{y}{t} G'_0 \left( \frac{y}{t} \right) \right|
\leq C(\lambda_1, \lambda_2, \delta) \frac{1}{t}.
\]
This is a consequence of the fact that for \( j \geq 1, t > 0, \)
\[
\sup_{y \geq 0} \{ e^{-t y^2} y^j \} = \frac{1}{(2e)^j} \left( \frac{j}{2e} \right)^{j/2}.
\] (2.4)
Similarly, for \( \frac{y}{t} \in \left[ \frac{\lambda_1}{2}, \frac{\lambda_2}{2} \right], \)
\[
e^{-t \left( F_0 \left( \frac{y}{t} \right) + \frac{\delta y^2}{\lambda_1 - \lambda_2} \right)} \left| \partial_t \gamma_2 \left( \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} - 1 \right) \right|
\leq C(\lambda_1, \lambda_2, \delta) e^{-t \left( F_0 \left( \frac{y}{t} \right) + \frac{\delta y^2}{\lambda_1 - \lambda_2} \right)} \left| \partial_t \gamma_2 \right|
\leq C(\lambda_1, \lambda_2, \delta) \frac{1}{t}.
\]
For $\lambda \in [\lambda_2, \lambda_1] \setminus \left[ \frac{2\lambda_1}{\lambda_2}, \frac{2\lambda_2}{\lambda_1} \right]$, we simply go back to

$$V(y, t) = e^{-\frac{2\delta t}{\lambda_1 - \lambda_2}} \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} I_0 \left( \frac{\frac{1}{\lambda_1} \lambda_2 - y}{\lambda_1 - \lambda_2} \right) \left( \frac{\frac{1}{\lambda_2} \lambda_1 - y}{\lambda_1 - \lambda_2} \right),$$

and remark that there exists $a > 0$ such that

$$e^{-\frac{2\delta t}{\lambda_1 - \lambda_2}} \left( -\lambda_1 \lambda_2 + \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} y \right) \leq e^{-at}.$$ 

With equation (2.3) we check that

$$\left| \partial_t \left( I_0 \left( \frac{\lambda_1 - \lambda_2}{(\lambda_1 - \lambda_2)^2} \sqrt{-(y - \lambda_1 t)(y - \lambda_2 t)} \right) \right) \right| \leq K(\lambda_1, \lambda_2, \delta)(1 + t)$$

if $\frac{y}{t} \in [\lambda_2, \lambda_1]$. We deduce that for $y \in [\lambda_2, \lambda_1] \setminus \left[ \frac{2\lambda_1}{\lambda_2}, \frac{2\lambda_2}{\lambda_1} \right]$, $t \geq 1$, $t \geq \alpha \frac{y^2}{t^2}$ for some $\alpha > 0$, thus

$$|\partial_t V(y, t)| \leq K(1 + t)e^{-\alpha t} \leq K(1 + t)e^{-\frac{y^2}{t^2}} e^{-\frac{\alpha^2}{1 + t}} \leq \frac{K}{t^2} e^{-\frac{\alpha^2}{1 + t}}. \tag{2.5}$$

We now generalize to higher order derivatives in time. We recall that

$$\gamma_2(y, t) = t G_0 \left( \frac{y}{t} \right).$$

We show by induction that

$$\partial_t^n \gamma_2(y, t) = t^{1-n} G_n \left( \frac{y}{t} \right),$$

with

$$G_{n+1}(x) = (1 - n) G_n(x) - x G_n'(x).$$

Furthermore, we have

$$I_0(x), I'_0(x) = e_x \frac{1}{\sqrt{2\pi x}} \left( 1 + O_{x \to +\infty} \left( \frac{1}{x} \right) \right),$$

therefore

$$\frac{I'_0(x)}{I_0(x)} = 1 + O_{x \to +\infty} \left( \frac{1}{x} \right).$$

We can show, using the development of $I_0, I'_0 = I_1$ from [16], that for $x \geq 0, k \geq 1$

$$\left| \partial_x^k \left( \frac{I'_0(x)}{I_0(x)} \right) \right| \leq \frac{K(k)}{(1 + x)^{k+1}}. \tag{2.6}$$

We recall that

$$\partial_t V = V \left( \partial_t \gamma_1 + \partial_t \gamma_2 \frac{I'_0(\gamma_2)}{I_0(\gamma_2)} \right),$$

and $\partial_t^2 \gamma_1 = 0$. For $n \geq 1$ we have

$$\partial_t^{n+1} V = \sum_{k=0}^n \binom{n}{k} \partial_t^{n-k} V \partial_t^k \left( \partial_t \gamma_1 + \partial_t \gamma_2 \frac{I'_0(\gamma_2)}{I_0(\gamma_2)} \right).$$

We check that

$$\left| \partial_t^k \left( \partial_t \gamma_2 \frac{I'_0(\gamma_2)}{I_0(\gamma_2)} \right) \right| \leq \frac{K(k)}{(1 + t)^k}.$$ 

Indeed, each derivatives hit either a term of the form $\partial_t^k \gamma_2$, which yields an additional $\frac{1}{1 + t}$, of it hit a term of the form $\partial_x^k \left( \frac{I'_0}{I_0} \right)(\gamma_2)$, and

$$\partial_t \left( \partial_x^k \left( \frac{I'_0}{I_0} \right)(\gamma_2) \right) = \partial_t \gamma_2 \partial_x^k \left( \frac{I'_0}{I_0} \right)(\gamma_2),$$

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adding a term in \( \frac{1}{(1+t^2)^{\gamma_3}} \sim \frac{1}{1+t^2} \) if \( \gamma_3 \in [\frac{1}{2}, \frac{1}{1}] \). Outside of \( [\frac{1}{2}, \frac{1}{1}] \), we use the exponential decay as for the proof of (2.5).

For the term \( \partial_t^n V \left( \partial_t \gamma_1 + \partial_t \gamma_2 \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} \right) \), we still use the fact that

\[
\left| \partial_t \gamma_1 + \partial_t \gamma_2 \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} \right| \leq O_{\gamma \to 0} \left( \frac{y^2}{t^2} \right) + \frac{K}{t}
\]

and we simply use equation (2.4). By induction we deduce the estimate on \( \partial_t^n V \).

Now, we have

\[
\partial_y V = V \left( \partial_y \gamma_1 + \partial_y \gamma_2 + \partial_y \gamma_2 \left( \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} - 1 \right) \right)
\]

and we compute that

\[
\partial_y \gamma_1 = -\frac{\delta(\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^2} = -G'_0(0)
\]

and

\[
\partial_y \gamma_2 = G'_0 \left( \frac{y}{t} \right),
\]

therefore

\[
\partial_y \gamma_1 + \partial_y \gamma_2 = G'_0 \left( \frac{y}{t} \right) - G'_0(0) = O_{\gamma \to 0} \left( \frac{y}{t} \right).
\]

We conclude that

\[
|\partial_y V| \leq C(\lambda_1, \lambda_2, \delta) e^{\frac{-\delta}{2(\lambda_1 - \lambda_2)^2}} \frac{y^2}{t}.
\]

Now, we have

\[
\partial^{n+1} y V = \sum_{k=0}^{n} \binom{n}{k} \partial_y^{n-k} V \partial_y^k \left( \partial_y \gamma_1 + \partial_y \gamma_2 \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} \right).
\]

We have \( \partial_y \gamma_1 = 0 \) and \( \gamma_2(y, t) = tG_0 \left( \frac{y}{t} \right) \), therefore

\[
\partial_y \gamma_2 = t^{1-j} G_0^{(j)} \left( \frac{y}{t} \right).
\]

In the sum, we deal with the case \( k = 0 \) as in the case of the derivative in time. Otherwise, we estimate that for \( k \geq 1 \),

\[
\left| \partial_y^k \left( \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} \right) \right| \leq \frac{K(k)}{(1+t)^k}.
\]

Indeed, derivatives either fall on term of the form \( \partial_y \gamma_2 \), adding a factor \( \frac{1}{t^k} \), or it falls on a term of the form \( \partial_y \left( \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} \right) \), and we have \( \partial_y \left( \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} \right) = \partial_y \gamma_2 \partial_j^{j+1} \left( \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} \right) \). Since \( \partial_y \gamma_2 \) is bounded in time and \( \gamma_2 \simeq t \), we get an additional factor \( \frac{1}{t} \) from (2.6).

We conclude by induction that

\[
|\partial_y^n V| \leq \frac{K(n, \lambda_1, \lambda_2, \delta)}{(1+t)^{k+1/2+\eta/2}} e^{\frac{-\delta}{(\lambda_1 - \lambda_2)^2}} \frac{y^2}{t}.
\]

Remark that the only term with the decay in time \( \frac{1}{(1+t)^{k+1/2+\eta/2}} \) is the case \( k = 0 \) in the sum (2.7).

We complete with the cross derivatives. Starting from

\[
\partial_t V = V \left( \partial_t \gamma_1 + \partial_t \gamma_2 \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} \right),
\]

we have

\[
\partial^{n+1} t V = \sum_{i=0}^{n} \sum_{j=0}^{k} \binom{k}{i} \binom{n}{j} \partial_y^{k-j} \partial_t^{n-i} \partial_y^j \partial_t^i \left( \partial_t \gamma_1 + \partial_t \gamma_2 \frac{I_0'(\gamma_2)}{I_0(\gamma_2)} \right).
\]

By a similar induction estimate, separating the case \( i = 0, j = 0 \) from the other, and by remarking that \( \partial_t \partial_y \gamma_1 = 0 \), we conclude the proof of this lemma.\( \square \)
3 Estimation on some convolutions

We recall the quantities for \( \mu_1 > 0, \mu_2 < 0, \mu \in \mathbb{R}, a, \delta, \gamma > 0, w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^+) \) defined in the introduction

\[
F^{\mu_1, \mu_2}_{a, \gamma, w}(x, t) = \frac{1}{(1 + t)^{\gamma}} \int_{\mu_2 t}^{\mu_1 t} e^{-\frac{y^2}{a + t}} w(x - y) dy
\]

and

\[
G^{\mu}_{a, \delta, w}(x, t) = e^{-bt} \sup_{z \in [-\delta, \delta]} w(x - \mu t + z).
\]

Remark that since \( w \geq 0 \), we have \( F^{\mu_1, \mu_2}_{a, \gamma, w} \geq 0 \) and \( G^{\mu}_{a, \delta, w} \geq 0 \). In this section, we are going to estimate some convolutions involving these two quantities, that will appear in the Duhamel formulation of the linear and nonlinear equation. They are in subsection 3.1 for the ones concerning \( F^{\mu_1, \mu_2}_{a, \gamma, w} \), then in subsection 3.2 for the ones concerning \( G^{\mu}_{a, \delta, w} \). Furthermore, in subsection 3.3, we will give bounds on \( F^{\mu_1, \mu_2}_{a, \gamma, w} \) and \( G^{\mu}_{a, \delta, w} \) in \( L(L^1(\mathbb{R})) \) and \( L^\infty(\mathbb{R}) \).

3.1 Convolutions involving \( F^{\mu_1, \mu_2}_{a, \gamma, w} \)

Lemma 3.1 Consider, for \( \mu_1 > 0, \mu_2 < 0, \mu \in [\mu_2, \mu_1], a > 0, a_1 > 0, \gamma \geq 0, w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^+) \) the function

\[
H(x, t) = \int_0^t e^{-a_1(t-s)} F^{\mu_1, \mu_2}_{a, \gamma, w}(x - \mu(t-s), s) ds.
\]

Then, there exists \( K(a_1, \gamma) > 0 \) such that, if \( a \leq \frac{\mu_1(\gamma)}{\mu_2} \), then

\[
H(x, t) \leq K(a_1, \gamma) F^{\mu_1, \mu_2}_{a, \gamma, w}(x, t).
\]

Proof We have

\[
H(x, t) = \int_0^t \frac{e^{-a_1(t-s)}}{(1 + s)^{\gamma}} \int_{\mu_2 s}^{\mu_1 s} e^{-\frac{y^2}{a + (t+s)}} w(x - \mu(t-s) - y) dy ds.
\]

We first do the change of variable \( z = y + \mu(t-s) \) leading to

\[
\int_{\mu_2 s}^{\mu_1 s} e^{-\frac{y^2}{a + s}} w(x - \mu(t-s) - y) dy = \int_{\mu_2 s + \mu(t-s)}^{\mu_1 s + \mu(t-s)} e^{-\frac{(z - \mu(t-s))^2}{a + (t+s)}} w(x - z) dz.
\]

Remark that for \( s \in [0, t] \) and \( \mu \in [\mu_2, \mu_1] \), we have

\[
\mu_2 s + \mu(t-s) \geq \mu_2 t
\]

and

\[
\mu_1 s + \mu(t-s) \leq \mu_1 t.
\]

We deduce that

\[
\int_{\mu_2 s + \mu(t-s)}^{\mu_1 s + \mu(t-s)} e^{-\frac{(z - \mu(t-s))^2}{a + (t+s)}} w(x - z) dz \leq \int_{\mu_2 t}^{\mu_1 t} e^{-\frac{(z - \mu(t-s))^2}{a + (t+s)}} w(x - z) dz
\]

and thus

\[
H(x, t) \leq \int_0^t \frac{1}{(1 + s)^{\gamma}} \int_{\mu_2 t}^{\mu_1 t} e^{-\frac{z^2}{a + (t+s)}} e^{f(z, s, t)} |w(x - z)| dz ds
\]

where

\[
f(z, s, t) = -a_1(t-s) - a \frac{(z - \mu(t-s))^2}{(1 + s)} + a \frac{z^2}{(1 + t)}.
\]

For \( s \in [0, t] \), we deduce that if \( a \leq \frac{\mu_1(\gamma)}{\mu_2} \), then

\[
f(z, s, t) \leq a \left( -\frac{(\mu(t-s))^2}{t-s} - \frac{(z - \mu(t-s))^2}{(1 + s)} + \frac{z^2}{(1 + t)} \right) - a_1 \left( t-s \right).
\]
We compute in general that for \( x, y \in \mathbb{R} \),

\[
\frac{y^2}{t-s} + \frac{(x-y)^2}{1+s} - \frac{x^2}{1+t} = \frac{(1+s)(1+t)y^2 + (x-y)^2(t-s)(1+t) - x^2(1+s)(t-s)}{(t-s)(1+s)(1+t)} = \frac{(y^2(1+t)^2 + x^2(t-s)^2 - 2xy(t-s)(1+t))}{(t-s)(1+s)(1+t)} = \frac{(x(t-s) - y(1+t))^2}{(t-s)(1+s)(1+t)}. \tag{3.1}
\]

Applying it to \( x = z, y = \mu(t-s) \), we deduce that for \( s \in [0,t] \),

\[
-\frac{(\mu(t-s))^2}{t-s} - \frac{(z - \mu(t-s))^2}{(1+s)} + \frac{z^2}{(1+t)} = -\frac{(t-s)(z - \mu(1+t))^2}{(1+s)(1+t)} \leq 0.
\]

We deduce that

\[
H(x, t) \leq \int_{\mu_1t}^{\mu_1(t-s)} e^{-a_0s \frac{s^2}{(1+s)^2}} w(x-z) \int_0^t \frac{1}{(1+s)\gamma} e^{-\frac{a_0}{\gamma} (t-s)} ds dz.
\]

\[
\leq \frac{K(a_1, \gamma)}{(1+t)^\gamma} \int_{\mu_2t}^{\mu_1(t-s)} e^{-a_0s \frac{s^2}{(1+s)^2}} w(x-z) dz.
\]

This concludes the proof of this lemma. \( \square \)

**Lemma 3.2** Consider, for \( \mu_1 > 0, \mu_2 < 0, a, a_0, \alpha, \beta, \delta > 0, w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^+) \), with \( \delta \) small compared to \( \mu_1, |\mu_2| \), the function

\[
H(x, t) = \int_0^t \int_{\mu_1(t-s)}^{\mu_1t} e^{-a_0s \frac{s^2}{(1+s)^2}} F_{\mu_1+\delta, \mu_2-\delta}(x-y, s) dy ds.
\]

Then, there exists \( K > 0 \) depending on \( \alpha, \beta, a_0, a, \mu_1, \mu_2 \) and \( \varepsilon(a_0) > 0 \) such that if \( a \leq \varepsilon(a_0) \), then

\[
H(x, t) \leq K \left( \int_{\beta-1/2}(t) \frac{L_{\beta-1/2}(t)}{(1+t)^\alpha} + \int_{\alpha-1/2}(t) \frac{L_{\alpha-1/2}(t)}{(1+t)^\beta} \right) \int_{(\mu_2-\delta)t}^{(\mu_1+\delta)t} w(x-r) e^{-a_0s \frac{s^2}{(1+s)^2}} dr, \tag{3.2}
\]

where

\[
L_{\gamma}(t) := \int_0^t \frac{ds}{(1+s)^\gamma}.
\]

Equation (3.2) is an estimate where the right hand side if \( F_{a, \gamma, w}^{\mu_1+\delta, \mu_2-\delta} \) for some \( \gamma \in \mathbb{R} \). In particular, if \( \alpha, \beta \neq \frac{3}{2} \), then

\[
\frac{L_{\beta-1/2}(t)}{(1+t)^\alpha} + \frac{L_{\alpha-1/2}(t)}{(1+t)^\beta} \leq \frac{K}{(1+t)^{\alpha+\beta-\frac{3}{2}}}
\]

for \( K \) depending on \( \alpha, \beta \) and thus

\[
H(x, t) \leq K F_{a, \alpha+\beta-\frac{3}{2}, w}^{\mu_1+\delta, \mu_2-\delta}(x, t). \tag{3.3}
\]

If \( \alpha = \frac{3}{2} \) or \( \beta = \frac{3}{2} \), then there is an additional factor \( \ln(1+t) \). This is the reason why in the estimates of Theorem 1.5 there is a small loss of decay in time for the derivatives of \( u \) and \( \rho \).

**Proof** We have

\[
H(x, t) = \int_0^t \int_{\mu_2(t-s)}^{\mu_1(t-s)} e^{-a_0s \frac{s^2}{(1+s)^2}} \int_{(\mu_2-\delta)s}^{(\mu_1+\delta)s} e^{-a_0s \frac{s^2}{(1+s)^2}} w(x-y, s) dy ds dy ds.
\]
By the change of variable $r = y + z$, we have
\[
\int_{(\mu_2 - \delta)t}^{(\mu_1 + \delta)t} e^{-a\frac{r^2}{1+t}} w(x - r)dr = \int_{(\mu_2 - \delta)s + y}^{(\mu_1 + \delta)s + y} e^{-a\frac{x - y}{1+t}} w(x - r)dr.
\]
For $y \in [\mu_2(t - s), \mu_1(t - s)]$ and $s \in [0, t]$, we have that
\[
(\mu_2 - \delta)s + y \geq (\mu_2 - \delta)t \quad \text{and} \quad (\mu_1 + \delta)s + y \leq (\mu_1 + \delta)t,
\]
therefore
\[
H(x, t) \leq \int_{(\mu_2 - \delta)t}^{(\mu_1 + \delta)t} w(x - r) e^{-a\frac{r^2}{1+t}} \int_0^t \int_{\mu_2(t - s)}^{\mu_1(t - s)} e^{-\frac{a}{2(1+s)^2}} (1+t-s)^{\alpha}(1+s)^{3\delta} e^{f(r, y, t, s)} dyds dr
\]
with
\[
f(r, y, t, s) = a \frac{r^2}{1+t} - \frac{a_0}{2} \frac{y^2}{1+t-s} - a \frac{(r - y)^2}{1+s}.
\]
If $t \leq 3$, then the domain of integration of $r, y$ and $s$ are all bounded and then for $r \in [(\mu_2 - \delta)t, (\mu_1 + \delta)t]$ we have
\[
\int_{t-2}^{t} \int_{\mu_2(t - s)}^{\mu_1(t - s)} e^{-\frac{a_0}{2} \frac{y^2}{1+t-s} - a \frac{(r - y)^2}{1+s}} e^{f(r, y, t, s)} dyds \leq K(\mu_1, \mu_2, a_0, a),
\]
concluding the proof. Now, if $t > 3$, we look first at the case $s \in [t - 2, t]$. Then,
\[
f(r, y, t, s) \leq a \frac{r^2}{1+t} - \frac{a_0}{6} \frac{y^2}{t-s} - a \frac{(r - y)^2}{1+s} \leq 0
\]
given that $t \geq 3$ and $a \leq \varepsilon(a_0)$ for some function $\varepsilon(a_0) > 0$. In that case, $y \in [2\mu_2, 2\mu_1]$ and therefore
\[
\int_{t-2}^{t} \int_{\mu_2(t - s)}^{\mu_1(t - s)} e^{-\frac{a_0}{2} \frac{y^2}{1+t-s} - a \frac{(r - y)^2}{1+s}} e^{f(r, y, t, s)} dyds \leq \frac{K(\mu_1, \mu_2, a_0, a)}{(1+t)^{3\delta}}.
\]
Now, if $t \geq 3$ and $s \in [0, t - 2]$, which is the main case, we have
\[
f(r, y, t, s) \leq a \frac{r^2}{1+t} - \frac{a_0}{6} \frac{y^2}{t-s} - a \frac{(r - y)^2}{1+s}
\]
and by equation (3.1), we deduce that for $a \leq \varepsilon(a_0)$, we have
\[
f(r, y, t, s) \leq 0.
\]
Therefore, for the case $t - 2 \geq s \geq t/2$,
\[
\int_{t/2}^{t-2} \int_{\mu_2(t - s)}^{\mu_1(t - s)} e^{-\frac{a_0}{2} \frac{y^2}{1+t-s} - a \frac{(r - y)^2}{1+s}} e^{f(r, y, t, s)} dyds \leq \int_{t/2}^{t-2} \int_{\mu_2(t - s)}^{\mu_1(t - s)} e^{-\frac{a_0}{2} \frac{y^2}{1+t-s} - a \frac{(r - y)^2}{1+s}} dyds \leq K(a_0) \int_{t/2}^{t} (1+t-s)^{\alpha - 1/2}(1+s)^{3\delta} ds \leq \frac{K(a_0)}{(1+t)^{3\delta}} L_{\alpha-1/2}(t).
\]
Now, remark that if \( s \leq t/2 \) and \( a \leq \varepsilon(a_0) \), there exists \( a_2(a_0, a) > 0 \) such that

\[
f(r, y, t, s) \leq -a_2 \frac{y^2}{(1 + s)}.
\]

We deduce that

\[
\begin{align*}
\int_0^{t/2} \int_{\mu_2(t-s)}^\mu & e^{-\frac{a_2}{2} (1+t-s)} e^{f(r, y, t, s)} dy ds \\
\leq & \int_0^{t/2} \int_{\mu_2(t-s)}^\mu e^{-a_2 \frac{y^2}{(1 + s)}} dy ds \\
\leq & \int_0^{t/2} \frac{1}{(1 + t - s)^{\alpha} (1 + s)^{\beta-1/2}} ds \\
\leq & \frac{K(\alpha)}{(1 + t)^\alpha} L_{\beta-1/2}(t).
\end{align*}
\]

This concludes the proof of this lemma. \( \Box \)

### 3.2 Convolution with a boundary term

We recall the notation

\[
F_{\alpha, \gamma, \omega, w}^\mu(x, t) = \frac{1}{(1 + t)^\gamma} \int_{\mu(t)}^{\mu_1} e^{-\omega \frac{y^2}{2(t-s)}} w(x-y) dy.
\]

We are interested here in the quantity for \( b, \delta > 0, \mu \in \mathbb{R} \) and \( w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^+)\):

\[
G_{b, \delta, w}^\mu(x, t) = e^{-bt} \sup_{y \in [-\delta t, \delta t]} w(x - \mu t + y) \geq 0.
\]

**Lemma 3.3** Consider \( b, \delta > 0, \mu \in \{\mu_1, \mu_2\}, w \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^+) \) and the function

\[
G_{b, \delta, w}^\mu(x, t) = e^{-bt} \sup_{z \in [-\delta t, \delta t]} w(x - \mu t + z).
\]

Then, for any \( k, n \in \mathbb{N} \), there exists \( C > 0 \) depending on \( \mu_1, \mu_2, b, n, k \) and \( a > 0 \) depending on \( \mu_1, \mu_2, b \) such that

\[
\int_0^t \int_{\mu_2(t-s)}^{\mu_1(t-s)} |\partial_x^k \partial_t^n V(y, t-s) G_{b, \delta, w}^\mu(x-y, s)| dy ds \\
\leq C F_{a, \frac{\delta + \mu_2 - \delta}{\mu_1(t-s) + \frac{\mu_2 - \delta}{\mu_1(t-s) + \mu_2 - \delta}}(x, t).
\]

**Proof** First, by a change of variable on \( y \), we compute that

\[
\begin{align*}
\int_0^t \int_{\mu_2(t-s)}^{\mu_1(t-s)} |\partial_x^k \partial_t^n V(y, t-s) e^{-bs} \sup_{z \in [-\delta s, \delta s]} w(x - y - \mu s + z) | dy ds \\
\leq & \int_0^t \sup_{z \in [-\delta s, \delta s]} \int_{\mu_2(t-s) + \mu s - z}^{\mu_1(t-s) + \mu s - z} |\partial_x^k \partial_t^n V(y - \mu s + z, t-s) e^{-bs} w(x-y) | dy ds.
\end{align*}
\]

Remark that for \( \mu \in \{\mu_1, \mu_2\} \) and \( s \in [0, t] \), we have \( \mu_1(t-s) + \mu s \leq \mu_1 t \) and \( \mu_2(t-s) + \mu s \geq \mu_2 t \). We deduce that, denoting

\[
A = \int_0^t \int_{\mu_2(t-s)}^{\mu_1(t-s)} |\partial_x^k \partial_t^n V(y, t-s) e^{-bs} \sup_{z \in [-\delta s, \delta s]} w(x - y - \mu s + z) | dy ds,
\]

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that
\[ A \leq \int_{(\mu_2-\delta)t}^{(\mu_1+\delta)t} w(x-y) \left( \int_0^t \sup_{z \in [-\delta s, \delta s]} |\partial_x^k \partial_t^n V(y - \mu s + z, t-s)| e^{-bs} \right) dy. \]

We recall from Lemma 1.3 that there exists \( a_0 > 0 \) such that
\[ |\partial_x^k \partial_t^n V(y - \mu s + z, t-s)| \leq \frac{K(k, n, \mu_1, \mu_2)}{(1 + t-s)^{\frac{k}{2} + \frac{n}{2} + \eta}} e^{-a_0 \frac{(y-\mu s+z)^2}{1+t-s^2}}. \]

Now, first for the case \( s \geq t/2 \), we estimate using simply
\[ |\partial_x^k \partial_t^n V(y - \mu s + z, t-s)| \leq K(k,n) \]
\[ \leq K(k,n,\mu_1,\mu_2) e^{-\frac{b}{2}t} \int_{1/2}^t e^{-\frac{b}{2}s} ds \]
and thus, using that for \( y \in [(\mu_2-\delta)t,(\mu_1+\delta)t] \), we have \( t \geq K(\mu_1,\mu_2,\delta) \frac{y^2}{(1+t)} \), we deduce
\[ \int_{(\mu_2-\delta)t}^{(\mu_1+\delta)t} w(x-y) \left( \int_0^t \sup_{z \in [-\delta s, \delta s]} |\partial_x^k \partial_t^n V(y - \mu s + z, t-s)| e^{-bs} \right) dy \]
\[ \leq K(k,n,\mu_1,\mu_2,b) e^{-\frac{b}{2}t} \int_{(\mu_2-\delta)t}^{(\mu_1+\delta)t} w(x-y) e^{-\frac{b}{2}s} dy \]
\[ \leq \frac{K(k,n,\mu_1,\mu_2,b)}{(1+t)^{\frac{k}{2} + \frac{n}{2} + \eta}} \int_{(\mu_2-\delta)t}^{(\mu_1+\delta)t} e^{-\frac{b}{2}s} w(x-y) dy \]
\[ \leq K(k,n,\mu_1,\mu_2,b) \bar{\rho}_n^{\mu_1+\delta,\mu_2-\delta} \gamma_{\bar{a},\frac{k}{2} + \frac{n}{2} + \eta}^{x,y} (x,t) \]
for some small \( \bar{a} > 0 \) depending on \( b, \mu_1, \mu_2 \). Now, for the case \( s \leq t/2 \), we estimate
\[ \int_0^{t/2} \sup_{z \in [-\delta s, \delta s]} |\partial_x^k \partial_t^n V(y - \mu s + z, t-s)| e^{-bs} ds \]
\[ \leq \frac{K(k,n,\mu_1,\mu_2)}{(1+t)^{\frac{k}{2} + \frac{n}{2} + \eta}} \int_0^{t/2} \sup_{z \in [-\delta s, \delta s]} e^{-\frac{a_0}{4} \frac{(y-\mu s+z)^2}{(1+t-s)^2}} e^{-\frac{b}{2} s} ds, \]
and we check that for \( s \in [0,t/2], y \in [\mu_2 t, \mu_1 t], z \in [-\delta s, \delta s] \) there exists \( a' > 0 \) depending on \( \mu_1, \mu_2, b, \delta \) such that
\[ \frac{a_0}{4} \frac{(y-\mu s+z)^2}{(1+t-s)^2} - \frac{b}{2} s \leq -a' \frac{y^2}{1+t}. \]
We deduce that
\[ \int_0^{t/2} \sup_{z \in [-\delta s, \delta s]} |\partial_x^k \partial_t^n V(y - \mu s + z, t-s)| e^{-bs} ds \]
\[ \leq \frac{K(k,n,\mu_1,\mu_2)}{(1+t)^{\frac{k}{2} + \frac{n}{2} + \eta}} e^{-a' \frac{y^2}{1+t}} \int_0^{t/2} e^{-\frac{b}{2} s} ds \]
\[ \leq \frac{K(k,n,\mu_1,\mu_2,b)}{(1+t)^{\frac{k}{2} + \frac{n}{2} + \eta}} e^{-a' \frac{y^2}{1+t}} \]
and therefore
\[
\int_{(\mu_1 - \delta)t}^{(\mu_1 + \delta)t} w(x - y) \left( \int_0^{t/2} \sup_{z \in [-\delta s, \delta s]} |\partial_x^k \partial_t^n V(y - \mu s + z, t - s)| e^{-bs} ds \right) dy \\
\leq \frac{K(k, n, \mu_1, \mu_2, b, \delta)}{(1 + t)^{k + n}} \int_{(\mu_2 - \delta)t}^{(\mu_2 + \delta)t} w(x - y) e^{-\alpha^2 \frac{y^2}{2t}} dy \\
\leq K(k, n, \mu_1, \mu_2, b) F^{\mu_1 + \delta, \mu_2 - \delta}_{a', \frac{1}{2} + \frac{1}{2} + n, w}(x, t).
\]

We conclude that
\[
A \leq K(k, n, \mu_1, \mu_2, b) F^{\mu_1 + \delta, \mu_2 - \delta}_{a', \frac{1}{2} + \frac{1}{2} + n, w}(x, t)
\]
for \( a = \min(a', \tilde{a}) \).

\[\square\]

**Lemma 3.4** Consider \( b, \delta > 0, \mu, \mu' \in \{\mu_1, \mu_2\} \) with \( b < a_0 \) (\( a_0 \) is defined in Lemma 1.3), \( \delta < \frac{1}{2} \min(\mu_1, \mu_2) \) and the function
\[
G^\mu_{b, \delta, w}(x, t) = e^{-bt} \sup_{z \in [-\delta t, \delta t]} w(x - \mu t + z).
\]

Then, there exists \( C > 0 \) depending on \( k, n, \mu_1, \mu_2 \) and \( a > 0 \) depending on \( \mu_1, \mu_2, b \) such that
\[
\int_0^{t} |\partial_x^k \partial_t^n V(\mu'(t - s), t - s)G^\mu_{b, \delta, w}(x - \mu'(t - s), s)| ds \\
\leq C \left( G^\mu_{b, \delta, w}(x, t) + F^{\mu_1 + \delta, \mu_2 - \delta}_{a, \frac{1}{2} + \frac{1}{2} + n, w}(x, t) \right).
\]

**Proof** We recall from Lemma 1.3 that there exists \( a_0 > 0 \) depending on \( \mu_1, \mu_2 \) such that
\[
|\partial_x^k \partial_t^n V(\mu'(t - s), t - s)| \leq K(k, n, \mu_1, \mu_2) e^{-a_0(t - s)}.
\]

We deduce that
\[
\int_0^{t} |\partial_x^k \partial_t^n V(\mu'(t - s), t - s)G^\mu_{b, \delta, w}(x - \mu'(t - s), s)| ds \\
\leq K(k, n, \mu_1, \mu_2) \int_0^{t} e^{-a_0(t - s)} e^{-bs} \sup_{z \in [-\delta s, \delta s]} w(x - \mu s - \mu'(t - s) + z) ds.
\]

If \( \mu = \mu' \), then \( -\mu s - \mu'(t - s) = -\mu't \) and
\[
\int_0^{t} |\partial_x^k \partial_t^n V(\mu'(t - s), t - s)G^\mu_{b, \delta, w}(x - \mu'(t - s), s)| ds \\
\leq K(k, n, \mu_1, \mu_2) \sup_{z \in [-\delta t, \delta t]} w(x - \mu' t + z) \int_0^{t} e^{-a_0(t - s)} e^{-bs} ds \\
\leq K(k, n, \mu_1, \mu_2, b) G^\mu_{b, \delta, w}(x, t).
\]

Now, if \( \mu \neq \mu' \), then we do the change of variable \( y = \mu s + \mu'(t - s) - z \), leading to
\[
\int_0^{t} |\partial_x^k \partial_t^n V(\mu'(t - s), t - s)G^\mu_{b, \delta, w}(x - \mu'(t - s), s)| ds \\
\leq K(k, n, \mu_1, \mu_2) \sup_{z \in [-\delta t, \delta t]} \int_{\mu't + z}^{\mu't + z} e^{-a_0(t - s)} e^{-bs} w(x - y) dy \\
\leq K(k, n, \mu_1, \mu_2) \int_{(\mu_2 - \delta)t}^{(\mu_1 + \delta)t} e^{-a_0(t - s)} e^{-bs} w(x - y) dy.
\]
and there exists $a > 0$ small enough depending on $a_0$ and $b$ such that $e^{-a_0(t-s)}e^{-bs} \leq e^{-at}e^{-a_2/2}$ for all $y \in [\mu_2 t, \mu_1 t]$, therefore

\[
\int_0^t |\partial_k \partial_n V(\mu'(t-s), t-s)G_{b,\delta,w}^\mu(x - \mu'(t-s), s)ds| \\
\leq K(k, n, \mu_1, \mu_2)e^{-at} \int_{(\mu_2 - \delta)t}^{(\mu_1 + \delta)t} e^{-a_2/2} w(x-y)dy \\
\leq K(k, n, \mu_1, \mu_2)F_{a,\gamma,\delta,w}^{\mu_1+\delta,\mu_2-\delta}(x, t),
\]

concluding the proof of this lemma. \hfill \Box

### 3.3 Estimates on $F_{a,\gamma,\delta,w}^{\mu_1,\mu_2}$ and $G_{b,\delta,w}^\mu$

**Lemma 3.5** Consider for $\gamma, a > 0, \mu_1 > 0, \mu_2 < 0, w \in L^1(\mathbb{R}, \mathbb{R}^+)$ the function

\[
F_{a,\gamma,\delta,w}^{\mu_1,\mu_2}(x, t) = \frac{1}{1 + t} \int_{\mu_2}^{\mu_1 t} e^{-a_2/2} w(x-y)dy.
\]

Then, there exists $K > 0$ depending on $\mu_1, \mu_2, a$ such that for all $x \in \mathbb{R}, t \geq 0$,

\[
\|F_{a,\gamma,\delta,w}^{\mu_1,\mu_2}\|_{L^1_x(t)} \leq \frac{K}{(1 + t)^{\gamma - 1/2}} \|w\|_{L^1(\mathbb{R})}
\]

and

\[
|F_{a,\gamma,\delta,w}^{\mu_1,\mu_2}(x, t)| \leq \frac{K}{(1 + t)^{\gamma}} \|w\|_{L^1(\mathbb{R})}.
\]

**Proof** We simply compute

\[
\int_{\mathbb{R}} F_{a,\gamma,\delta,w}^{\mu_1,\mu_2}(x, t)dx \leq \frac{\|w\|_{L^1(\mathbb{R})}}{1 + t} \int_{\mu_2}^{\mu_1 t} e^{-a_2/2} dy \\
\leq K \|w\|_{L^1(\mathbb{R})} \frac{1}{(1 + t)^{\gamma - 1/2}}
\]

and

\[
F_{a,\gamma,\delta,w}^{\mu_1,\mu_2}(x, t) \leq \frac{\|w\|_{L^1(\mathbb{R})}}{1 + t} \left\| e^{-a_2/2} \right\|_{L^\infty(\mathbb{R})} \\
\leq \frac{\|w\|_{L^1(\mathbb{R})}}{(1 + t)^{\gamma}},
\]

concluding the proof. \hfill \Box

For the estimates on

\[ G_{b,\delta,w}^\mu(x, t) = e^{-bt} \sup_{z \in [-\delta t, \delta t]} w(x - \mu t + z), \]

we check easily that

\[
|G_{b,\delta,w}^\mu(x, t)| \leq e^{-bt} \|w\|_{L^\infty(\mathbb{R})}
\]

and

\[
\|G_{b,\delta,w}^\mu(\cdot, t)\|_{L^1(\mathbb{R})} \leq e^{-bt} \|w\|_{L^1(\mathbb{R})},
\]

(3.4)
4 Nonlinear change of variable to straighten the light cone

For the equation
\[ \partial_t^2 f + (\lambda_1 + \lambda_2)\partial_{xt}^2 f + \lambda_1\lambda_2\partial_x^2 f + \delta \partial_t f = 0 \]

with \( \lambda_1 > 0, \lambda_2 < 0, \delta > 0 \), we have an explicit solution thanks to Proposition 1.2 in the case that these three coefficients are constants. However, we see that in the nonlinear stability problem (Lemma 6.1), these coefficients depend on the unknown. We can try to put them as a source term. That is, if \( \lambda_1 \approx \lambda_1^0, \lambda_2 \approx \lambda_2^0, \delta \approx \delta^0 \) where \( \lambda_1^0, \lambda_2^0, \delta^0 \) are constants and we write it
\[ \partial_t^2 f + (\lambda_1^0 + \lambda_2^0)\partial_{xt}^2 f + \lambda_1^0\lambda_2^0\partial_x^2 f + \delta^0 \partial_t f = S \]

with
\[ S = -(\lambda_1 + \lambda_2 - \lambda_1^0 - \lambda_2^0)\partial_{x}^2 f - (\lambda_1\lambda_2 - \lambda_1^0\lambda_2^0)\partial_x^2 f - (\delta - \delta^0)\partial_t f. \]

If we consider \( f_L \) the solution to the linear problem
\[
\begin{align*}
\{ & \partial_t^2 f_L + (\lambda_1^0 + \lambda_2^0)\partial_{xt}^2 f_L + \lambda_1^0\lambda_2^0\partial_x^2 f_L + \delta^0 \partial_t f_L = 0 \\
& f_{L|t=0} = f_0, \partial_t f_{L|t=0} = f_1,
\end{align*}
\]
we have from Proposition 1.2 that
\[
 f(x,t) = f_L(x,t) + \int_0^t \int_{\lambda_1^0(t-s)} \int_{\lambda_2^0(t-s)} V(y,t-s)S(x-y,s)dyds,
\]
where \( V \) is defined with \( \lambda_1^0, \lambda_2^0, \delta^0 \). If we want to do an estimate on \( f \) with a bootstrap, we see that with this formulation, provided some estimates on the second derivatives of \( f \), we can estimate \( f \) and its first derivatives. Indeed, if we differentiate it with respect to \( t \), the derivative never fall on \( S \), and if we differentiate with respect to \( x \), when the derivative fall on \( S \) we can do an integration by parts.

However, it is not (a priori) possible to estimate the second derivatives of \( f \) without informations on its third derivatives. In this sense, we have a loss of derivative in this problem. To solve this issue, we will do a nonlinear change of variable that will change the characteristic speeds \( \lambda_1, \lambda_2 \) to constants \( \lambda_1^0, \lambda_2^0 \). This will be use only to estimate second derivatives.

4.1 Reformulation of the equation

We consider the equation
\[ \partial_t^2 f + (\lambda_1 + \lambda_2)\partial_{xt}^2 f + \lambda_1\lambda_2\partial_x^2 f + \delta \partial_t f + w \partial_x f = 0, \]
where \( \lambda_1, \lambda_2, \delta, w \) depend on \( (x,t) \) and \( f_{L|t=0} = \partial_t f_{L|t=0} = 0 \). In fact, it can depend on them through \( f \), we will precise this later. We suppose that \( \lambda_1, \lambda_2 \) are close respectfully to \( \lambda_1^0 > 0, \lambda_2^0 < 0 \). For now, we just want to define the change of variable. Specific estimate on the functions that will appear will be done later.

We write
\[ f(x,t) = g(h_1(x,t), h_2(x,t)). \]

In the case \( \lambda_1 = \lambda_1^0, \lambda_2 = \lambda_2^0 \) where \( \lambda_1^0, \lambda_2^0 \) are constants, we would take \( h_1(x,t) = x \) and \( h_2(x,t) = t \). Here, we only suppose that
\[ h_1(x,0) = x, h_2(x,0) = 0. \]

We denote these new variables by \( (y, \nu) := (h_1(x,t), h_2(x,t)) \). Here and afterward, \( g \) is taken in \( (y, \nu) = (h_1(x,t), h_2(x,t)) \) and \( \lambda_1, \lambda_2, \delta, w, h_1, h_2 \) are taken in \( (x,t) \). We compute
\[
\begin{align*}
\partial_x g &= \partial_x h_1 \partial_y g + \partial_x h_2 \partial_\nu g, \\
\partial_x f &= (\partial_x h_1)^2 \partial_{y}^2 g + 2 \partial_x h_1 \partial_x h_2 \partial_{y\nu} g + (\partial_x h_2)^2 \partial_\nu^2 g + \partial_x^2 h_1 \partial_y g + \partial_x^2 h_2 \partial_\nu g, \\
\partial_{xx}^2 f &= \partial_x h_1 \partial_y \partial_y h_1 \partial_y^2 g + (\partial_x h_1 \partial_y h_2 + \partial_x h_2 \partial_\nu h_1) \partial_{y\nu}^2 g + \partial_x h_2 \partial_\nu \partial_y h_2 \partial_\nu^2 g \\
&+ \partial_x^2 h_1 \partial_y g + \partial_x^2 h_2 \partial_\nu g,
\end{align*}
\]
\[
\partial f = \partial t h_1 \partial_y g + \partial_t h_2 \partial_y g
\]
and
\[
\partial^2 f = (\partial_t h_1)^2 \partial_y^2 g + 2\partial_t h_1 \partial_t h_2 \partial_y^2 g + (\partial_t h_2)^2 \partial_y^2 g + \partial_t^2 h_1 \partial_y g + \partial_t^2 h_2 \partial_y g.
\]

We deduce that \( g \) satisfies the equation
\[
\begin{align*}
\partial^2 g((\partial_t h_2)^2 + (\lambda_1 + \lambda_2) \partial_x h_2 \partial_t h_2 + \lambda_1 \lambda_2 (\partial_x h_2)^2) \\
+ \partial^2 g(2\partial_t h_1 \partial_t h_2 + (\lambda_1 + \lambda_2)(\partial_t h_1 \partial_x h_2 + \partial_x h_2 \partial_t h_1)) + 2\lambda_1 \lambda_2 \partial_x h_1 \partial_x h_2 \\
+ \partial^2 g((\partial_t h_2)^2 + (\lambda_1 + \lambda_2) \partial_x h_2 \partial_t h_2 + \lambda_1 \lambda_2 (\partial_x h_2)^2) \\
+ \partial_y g((\partial^2_t h_2)^2 + (\lambda_1 + \lambda_2) \partial^2_x h_2 + \lambda_1 \lambda_2 \partial^2_t h_2 + \delta \partial_x h_2 + w \partial_x h_2) \\
+ \partial_y g((\partial^2_t h_1)^2 + (\lambda_1 + \lambda_2) \partial^2_t h_1 + \lambda_1 \lambda_2 \partial^2_x h_1 + \delta \partial_t h_1 + w \partial_x h_1) \\
= 0. 
\end{align*}
\]

(4.1)

Dividing by \((\partial_x h_2)^2 + (\lambda_1 + \lambda_2) \partial_x h_2 \partial_t h_2 + \lambda_1 \lambda_2 (\partial_x h_2)^2\), we therefore want to choose \(h_1, h_2\) such that
\[
\frac{2\partial_t h_1 \partial_t h_2 + (\lambda_1 + \lambda_2)(\partial_t h_1 \partial_t h_2 + \partial_x h_2 \partial_t h_1)}{(\partial_t h_2)^2 + (\lambda_1 + \lambda_2) \partial_x h_2 \partial_t h_2 + \lambda_1 \lambda_2 (\partial_x h_2)^2} = \lambda_1^0 + \lambda_2^0
\]
and
\[
\frac{(\partial_t h_1)^2 + (\lambda_1 + \lambda_2) \partial_x h_1 \partial_t h_1 + \lambda_1 \lambda_2 (\partial_x h_1)^2}{(\partial_t h_2)^2 + (\lambda_1 + \lambda_2) \partial_x h_2 \partial_t h_2 + \lambda_1 \lambda_2 (\partial_x h_2)^2} = \lambda_1^0 \lambda_2^0,
\]
where \(\lambda_1^0, \lambda_2^0\) are constants. We remark that
\[
\begin{align*}
(\partial_t h_2)^2 + (\lambda_1 + \lambda_2) \partial_x h_2 \partial_t h_2 + \lambda_1 \lambda_2 (\partial_x h_2)^2 \\
= (\partial_t h_2 + \lambda_1 \partial_x h_2)(\partial_t h_2 + \lambda_2 \partial_x h_2), \\
(\partial_t h_1)^2 + (\lambda_1 + \lambda_2) \partial_x h_1 \partial_t h_1 + \lambda_1 \lambda_2 (\partial_x h_1)^2 \\
= (\partial_t h_1 + \lambda_1 \partial_x h_1)(\partial_t h_1 + \lambda_2 \partial_x h_1)
\end{align*}
\]
and
\[
2\partial_t h_1 \partial_t h_2 + (\lambda_1 + \lambda_2)(\partial_t h_1 \partial_t h_2 + \partial_x h_2 \partial_t h_1) + 2\lambda_1 \lambda_2 \partial_x h_1 \partial_x h_2 \\
= (\partial_t h_1 + \lambda_1 \partial_x h_1)(\partial_t h_2 + \lambda_2 \partial_x h_2) + (\partial_t h_1 + \lambda_2 \partial_x h_1)(\partial_t h_2 + \lambda_1 \partial_x h_2).
\]

Therefore, if we define
\[
A_1 := \partial_t h_1 + \lambda_1 \partial_x h_1, \quad A_2 = \partial_t h_1 + \lambda_2 \partial_x h_1
\]
as well as
\[
B_1 := \partial_t h_2 + \lambda_1 \partial_x h_2, \quad B_2 = \partial_t h_2 + \lambda_2 \partial_x h_2,
\]
we have
\[
A_1 A_2 = \lambda_1^0 \lambda_2^0 B_1 B_2, \quad A_1 B_2 + A_2 B_1 = (\lambda_1^0 + \lambda_2^0) B_1 B_2,
\]
(4.2)
therefore if \(B_1, B_2 \neq 0\),
\[
\frac{A_1}{B_1} \frac{A_2}{B_2} = \lambda_1^0 \lambda_2^0, \quad \frac{A_1}{B_1} + \frac{A_2}{B_2} = \lambda_1^0 + \lambda_2^0.
\]
Equality (4.2) is satisfied if
\[
A_1 = \lambda_1^0 B_1, \quad A_2 = \lambda_2^0 B_2,
\]
that is
\[\partial_t + \lambda_1 \partial_x)(h_1 - \lambda_1^0 h_2) = (\partial_t + \lambda_2 \partial_x)(h_1 - \lambda_2^0 h_2) = 0.\]
We define \(\tilde{h}_1, \tilde{h}_2\) by
\[
h_1(x, t) = x + \tilde{h}_1(x, t), \quad h_2(x, t) = t + \tilde{h}_2(x, t)
\]
We define \(\tilde{h}_1, \tilde{h}_2\) by
\[
h_1(x, t) = x + \tilde{h}_1(x, t), \quad h_2(x, t) = t + \tilde{h}_2(x, t)
\]
and
\[ H_1 := \tilde{h}_1 - \lambda_1^0 \tilde{h}_2, H_2 := \tilde{h}_1 - \lambda_2^0 \tilde{h}_2, \]
with \( H_1|_{t=0} = H_2|_{t=0} = 0 \) (by 4.1). Then,
\[ (\partial_t + \lambda_1 \partial_x)H_1 = \lambda_1^0 - \lambda_1, (\partial_t + \lambda_2 \partial_x)H_2 = \lambda_2^0 - \lambda_2. \]

(4.3)

We find back \( \tilde{h}_1, \tilde{h}_2 \) by
\[ \tilde{h}_1 = \frac{\lambda_2^0 H_1 - \lambda_1^0 H_2}{\lambda_2^0 - \lambda_1^0}, \tilde{h}_2 = \frac{H_1 - H_2}{\lambda_2^0 - \lambda_1^0}. \]

(4.4)

We assume for now that we are able to show that for some small \( \bar{\varepsilon} > 0 \), we have
\[ |\partial_t H_1(x,t)| + |\partial_x H_1(x,t)| + |\partial_t H_2(x,t)| + |\partial_x H_2(x,t)| \leq \bar{\varepsilon}. \]

This will be done using equation (4.3) and some estimate on \( \lambda_0^0 - \lambda_1, \lambda_0^2 - \lambda_2 \) later on. We deduce in that case that
\[ (x, t) \rightarrow (x + \tilde{h}_1(x,t), t + \tilde{h}_2(x,t)) = (y(x,t), \nu(x,t)) \]
is an invertible function since
\[ |\partial_{\tilde{h}_1} + |\partial_{\tilde{h}_2} + |\partial_{\nu} \tilde{h}_1| + |\partial_{\nu} \tilde{h}_2| \leq K\bar{\varepsilon} \ll 1. \]
It is in fact close to the identity, in particular this function and its inverse have a norm close to one. This allows us to define \( g \) through \( f \).

We check indeed that \( t + \tilde{h}_2(x, t) \geq 0 \) for all \( t \geq 0 \), using the fact that \( \tilde{h}_2(x, 0) = 0 \) and \( |\partial_t \tilde{h}_2| \leq K\bar{\varepsilon} \) which implies that \( |\tilde{h}_2(x,t)| \leq K\bar{\varepsilon}t \leq \frac{1}{2} \) given \( \bar{\varepsilon} \) small enough. We have then
\[ \partial_{\nu} \tilde{h}_2(x, t) = \partial_{\nu}(t + \tilde{h}_2(x, t)) = 1 + \partial_{t} \tilde{h}_2(x, t) \geq 1 - \bar{\varepsilon}. \]

(4.5)

We go back on the equation (4.1) satisfied by \( g \). We have
\[ \begin{align*}
\partial^2_{\nu} g + (\lambda_1^0 + \lambda_2^0) \partial_{\nu}^2 g + \lambda_1^0 \lambda_2^0 \partial^2_{\nu} g \\
+ \frac{\partial^2_{\nu} h_2 + (\lambda_1 + \lambda_2) \partial^2_{\nu} h_2 + \lambda_1 \lambda_2 \partial^2_{x} h_2 + \delta \partial_{\nu} h_2 + w \partial_{x} h_2}{(\partial_{\nu} h_2)^2 + (\lambda_1 + \lambda_2) \partial_{\nu} h_2 \partial_{x} h_2 + \lambda_1 \lambda_2 (\partial_{x} h_2)^2} \partial_{\nu} g \\
+ \frac{\partial^2_{\nu} h_1 + (\lambda_1 + \lambda_2) \partial^2_{\nu} h_1 + \lambda_1 \lambda_2 \partial^2_{x} h_1 + \delta \partial_{\nu} h_1 + w \partial_{x} h_1}{(\partial_{\nu} h_2)^2 + (\lambda_1 + \lambda_2) \partial_{\nu} h_2 \partial_{x} h_2 + \lambda_1 \lambda_2 (\partial_{x} h_2)^2} \partial_{\nu} g \\
= 0.
\end{align*} \]

Remark that \( g \) is taken in \((h_1(x, t), h_2(x, t)) = (y, \nu) \) and \( h_1, h_2, \lambda_1, \lambda_2, \delta, w \) in \((x, t) \). We define \( R_1(x, t), R_2(x, t) \) by
\[ \begin{align*}
\frac{\partial^2_{\nu} h_2 + (\lambda_1 + \lambda_2) \partial^2_{\nu} h_2 + \lambda_1 \lambda_2 \partial^2_{x} h_2 + \delta \partial_{\nu} h_2 + w \partial_{x} h_2}{(\partial_{\nu} h_2)^2 + (\lambda_1 + \lambda_2) \partial_{\nu} h_2 \partial_{x} h_2 + \lambda_1 \lambda_2 (\partial_{x} h_2)^2} (x, t) = \delta^0 + R_1(x, t)
\end{align*} \]

(4.6)

and
\[ \begin{align*}
\frac{\partial^2_{\nu} h_1 + (\lambda_1 + \lambda_2) \partial^2_{\nu} h_1 + \lambda_1 \lambda_2 \partial^2_{x} h_1 + \delta \partial_{\nu} h_1 + w \partial_{x} h_1}{(\partial_{\nu} h_2)^2 + (\lambda_1 + \lambda_2) \partial_{\nu} h_2 \partial_{x} h_2 + \lambda_1 \lambda_2 (\partial_{x} h_2)^2} = R_2(x, t)
\end{align*} \]

We recall that \((x, t) \rightarrow (h_1(x, t), h_2(x, t)) \) is smooth, invertible and close to the identity for \( \bar{\varepsilon} \) small enough. We define \( \pi \) its inverse (that is, \( \pi(y, \nu) = (\pi_1(y, \nu), \pi_2(y, \nu)) \)) is such that \( y = h_1(\pi_1(y, \nu), \pi_2(y, \nu)), \nu = h_2(\pi_1(y, \nu), \pi_2(y, \nu)) \)). Then,
\[ g(y, \nu) = (f \circ \pi)(y, \nu). \]

In particular,
\[ \partial_{\nu} g = \partial_{\nu} \pi_1(\partial_{x} f \circ \pi) + \partial_{\nu} \pi_2(\partial_{t} f \circ \pi) \]
and
\[ \partial_{\nu} g = \partial_{\nu} \pi_1(\partial_{x} f \circ \pi) + \partial_{\nu} \pi_2(\partial_{t} f \circ \pi). \]
Now, letting the variables be \((y, \nu) = (h(x, t), h_2(x, t))\), the equation satisfied by \(g\) is

\[
(\partial^2_y g + (\lambda_1^0 + \lambda_2^0)\partial_{yy}^2 g + \lambda_1^0 \lambda_2^0 \partial_{yy}^2 g + \delta_0 \partial_y g)(y, \nu)
= -(R_1 \nu \partial_{\nu} g - R_2 \nu \partial_{\nu} y g).
\]

At \(\nu = 0\), we have \(t + h_2(x, t) = 0\) and from (4.5) this implies that \(t = 0\), hence \(g(y, 0) = f(y, 0), \partial_{\nu} g(y, 0) = \partial_{\nu} \pi_1(y, 0) \partial_x f(y, 0) + \partial_{\nu} \pi_2(y, 0) \partial_t f(y, 0)\). From Proposition 1.2, this implies that for any \(y \in \mathbb{R}, \nu \geq 0\),

\[
g(y, \nu) = g_L(y, \nu) + \int_0^\nu \int_{\lambda_1^0}^{\lambda_1^0(y-s)} V(z, \nu - s) S(y - z, s) dz ds,
\]

where \(S(y-z, s) = (-R_1 \nu \partial_{\nu} g - R_2 \nu \partial_{\nu} y g)(y-z, s)\) and \(g_L\) is solution to \(\partial^2_y g_L + (\lambda_1^0 + \lambda_2^0)\partial_{yy}^2 g_L + \lambda_1^0 \lambda_2^0 \partial_{yy}^2 g_L + \frac{1}{2} \partial_{\nu} g_L = 0\) with the initial condition \(g_L(y, 0) = f_0, \partial_{\nu} g_L(y, 0) = \partial_{\nu} \pi_1(y, 0) \partial_x f(y, 0) + \partial_{\nu} \pi_2(y, 0) \partial_t f(y, 0)\).

Our goal is to estimate \(R_1, R_2\) with estimates on \(\lambda_1, \lambda_2, \delta, w\) that require the less possible amount of derivatives on these functions. This is because they will depend on \(f\), and to do a bootstrap we will have information only on second derivatives of \(f\).

First, we remark that for any function \(h\), we have

\[
\partial^2_t h + (\lambda_1 + \lambda_2) \partial^2_x h + \lambda_1 \lambda_2 \partial^2_x h = (\partial_t + \lambda_1 \partial_x)(\partial_t + \lambda_2 \partial_x)h - (\partial_t \lambda_2 + \lambda_1 \partial_x \lambda_2) \partial_x h
\]

and

\[
\partial^2_t h + (\lambda_1 + \lambda_2) \partial^2_x h + \lambda_1 \lambda_2 \partial^2_x h = (\partial_t + \lambda_2 \partial_x)(\partial_t + \lambda_1 \partial_x)h - (\partial_t \lambda_1 + \lambda_2 \partial_x \lambda_1) \partial_x h.
\]

Therefore, since \(h_1 = x + \frac{\lambda_0^0 H_1 - \lambda_0^0 H_2}{\lambda_2^0 - \lambda_1^0}\), we compute

\[
\partial^2_t h_1 + (\lambda_1 + \lambda_2) \partial^2_x h_1 + \lambda_1 \lambda_2 \partial^2_x h_1 = \frac{\lambda_0^0}{\lambda_2^0 - \lambda_1^0}(\partial_t^2 + (\lambda_1 + \lambda_2) \partial_{xt}^2 + \lambda_1 \lambda_2 \partial_x^2) H_1
\]

\[
- \frac{\lambda_0^0}{\lambda_2^0 - \lambda_1^0}(\partial_t^2 + (\lambda_1 + \lambda_2) \partial_{xt}^2 + \lambda_1 \lambda_2 \partial_x^2) H_2
\]

\[
= \frac{\lambda_0^0}{\lambda_2^0 - \lambda_1^0}((\partial_t + \lambda_2 \partial_x)(\partial_t + \lambda_1 \partial_x) H_1 - (\partial_t \lambda_1 + \lambda_2 \partial_x \lambda_1) \partial_x H_1)
\]

\[
- \frac{\lambda_0^0}{\lambda_2^0 - \lambda_1^0}((\partial_t + \lambda_1 \partial_x)(\partial_t + \lambda_2 \partial_x) H_2 - (\partial_t \lambda_2 + \lambda_1 \partial_x \lambda_2) \partial_x H_2)
\]

\[
= \frac{\lambda_0^0}{\lambda_2^0 - \lambda_1^0}(-(\partial_t \lambda_1 + \lambda_2 \partial_x \lambda_1) \partial_x H_1)
\]

\[
- \frac{\lambda_0^0}{\lambda_2^0 - \lambda_1^0}(\partial_t \lambda_2 + \lambda_1 \partial_x \lambda_2) \partial_x H_2).
\]

A similar estimate can be made for \(\partial^2_t h_2 + (\lambda_1 + \lambda_2) \partial^2_x h_2 + \lambda_1 \lambda_2 \partial^2_x h_2\). Remark that \(\partial^2_t h_1 + (\lambda_1 + \lambda_2) \partial^2_x h_1 + \lambda_1 \lambda_2 \partial^2_x h_1\) was the only component of \(R_1\) that has two derivatives in \(h_1\) (see (4.6)). The situation is similar for \(\partial^2_t h_2 + (\lambda_1 + \lambda_2) \partial^2_x h_2 + \lambda_1 \lambda_2 \partial^2_x h_2\) and \(R_2\).

### 4.2 Summary

Consider the equation

\[
\partial^2_t f + (\lambda_1 + \lambda_2) \partial^2_x f + \lambda_1 \lambda_2 \partial^2_x f + \delta \partial_t f + w \partial_x f = 0,
\]

where \(\lambda_1, \lambda_2, \delta, w\) depend on \((x, t)\). We suppose that there exists constants \(\lambda_1^0 > 0, \lambda_2^0 < 0, \delta^0 > 0\) that are close respectfully to \(\lambda_1, \lambda_2, \delta\). Then, we define \(H_1, H_2\) the solutions to the problems

\[
(\partial_t + \lambda_1 \partial_x) H_1 = \lambda_1^0 - \lambda_1, (\partial_t + \lambda_2 \partial_x) H_2 = \lambda_2^0 - \lambda_2
\]
with $H_1|_{t=0} = H_2|_{t=0} = 0$, as well as

$$
\tilde{h}_1 = \frac{\lambda_0^2 H_1 - \lambda_0^2 H_2}{\lambda_0^2 - \lambda_1^2}, \tilde{h}_2 = \frac{H_1 - H_2}{\lambda_2^0 - \lambda_1^0}.
$$

We suppose that we can show that

$$
|\partial_t \tilde{h}_1| + |\partial_x \tilde{h}_1| + |\partial_t \tilde{h}_2| + |\partial_x \tilde{h}_2| \ll 1,
$$
in particular, the change of variable

$$(y, \nu) = (x + \tilde{h}_1(x, t), t + \tilde{h}_2(x, t))$$
is invertible and close to the identity. We then define the function $g$ by

$$
g(y, \nu) = f(\pi(y, \nu)) = f(x, t),
$$
where $\pi(y, \nu) = (x, t)$ are obtained by solving $y = x + \tilde{h}_1(x, t), \nu = t + \tilde{h}_2(x, t)$. Then, the function $g$ satisfies the equation

$$
(\partial_t^2 g + (\lambda_0^0 + \lambda_0^2) \partial_{\nu^0}^2 g + \lambda_0^0 \lambda_0^2 \partial_{\nu^0}^2 g + \delta^0 \partial_{\nu^0} g)(y, \nu) = -(R_1 \nu \pi \partial_{\nu^0} g - R_2 \nu \pi \partial_{\nu^0} g),
$$

with $g(y, 0) = f(y, 0), \partial_{\nu^0} g(y, 0) = \partial_{\nu^0} \pi_1(y, 0) \partial_x f(y, 0) + \partial_{\nu^0} \pi_2(y, 0) \partial_x f(y, 0)$, and where $R_1, R_2$ are small functions depending on $H_1, H_2, \partial_x H_1, \partial_x H_2, \partial_t H_1, \partial_t H_2, \partial_x \lambda_1, \partial_x \lambda_2, \partial_t \lambda_1, \partial_t \lambda_2, \delta - \delta^0, w$. In particular, they do not depend on second derivatives of $H_1, H_2$, or $\lambda_2$. It means that $R_1, R_2, \partial_t R_1, \partial_x R_1, \partial_t R_2$ and $\partial_x R_2$ are small if we control $H_1, H_2, \lambda_1, \lambda_2$ and their first and second derivatives, but we do not need any informations on their third or more derivatives. Now, we can use Proposition 1.2 on $g$, and since we have only first derivatives in the right hand side (we are going to be able to control $n$ derivatives of $H_1, H_2, x$ derivatives on our original variables), there is no longer loss of derivatives. However, if we try to go back to $f$ by this method, all the derivatives are “mixed” together. We can not expect to get a better decay than the worst derivative, that is $\partial_x f$, on which the decay in time is $\frac{1}{(1+t)^{\nu}}$, the $\nu$-loss being caused by convolutions (see the remark below Lemma 3.2).

5  Proof of the linear stability (Proposition 1.4)

5.1  The equations of the linear problem

Lemma 5.1  The linear problem around the constant flow is

$$
\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} + \begin{pmatrix} U(\rho_0) \\ 0 \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_0 h'(\rho_0) \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ u \end{pmatrix} + \frac{1}{\tau} \begin{pmatrix} 0 & 0 \\ U_f & 1 \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix} = 0.
$$

In particular, with $\lambda_* = U(\rho_0) - \rho_0 U_f$, the functions

$$
g(x, t) := \rho(x - \lambda_* t, t), v(x, t) := u(x - \lambda_* t, t)
$$
satisfies the equation

$$
\partial_t^2 q + (\lambda_0^0 + \lambda_0^2) \partial_{x t}^2 q + \lambda_0^0 \lambda_0^2 \partial_{x t}^2 q + \frac{1}{\tau} \partial_t q = 0
$$
and

$$
\partial_t^2 v + (\lambda_1^0 + \lambda_2^0) \partial_{x t}^2 v + \lambda_1^0 \lambda_2^0 \partial_{x t}^2 v + \frac{1}{\tau} \partial_t v = 0,
$$

where $\lambda_1^0 = \rho_0 U_f, \lambda_2^0 = \rho_0 U_f - \rho_0 h'(\rho_0)$.

Proof  The linear equations satisfied by $\rho$ and $u$ are

$$
\partial_t \rho + U(\rho_0) \partial_x \rho + \rho_0 \partial_x u = 0
$$
and
\[ \partial_t u + (U(\rho_0) - \rho_0 h'(\rho_0))\partial_x u + \frac{1}{\tau}(U_1 \rho + u) = 0.\]

Taking \( \partial_t + U(\rho_0)\partial_x \) of the second equation, we have
\[
\begin{align*}
\partial_t^2 u + (2U(\rho_0) - \rho_0 h'(\rho_0))\partial_x^2 u + U(\rho_0)(U(\rho_0) - \rho_0 h'(\rho_0))\partial_t^2 u \\
+ \frac{1}{\tau}(\partial_t u + (U(\rho_0) - \rho_0 U_1)\partial_x u)
\end{align*}
= 0.
\]

The equation on \( v \) follows from the change of variable \( v(x, t) = u(x - \lambda_s t, t) \). To get the equation on \( \rho \), we take
\( \partial_t + (U(\rho_0) - \rho_0 h'(\rho_0))\partial_x + \frac{1}{\tau} \) of the first equation and the result follows.

\[ \square \]

### 5.2 Proof of Proposition 1.4

**Proof** From Lemma 5.1, the function
\[ v(x, t) = u(x - \lambda_s t, t) \]
satisfies
\[ \partial_t^2 v + (\lambda_1^0 + \lambda_2^0)\partial_x^2 v + \lambda_1^0 \lambda_2^0 \partial_t^2 v + \frac{1}{\tau} \partial_t v = 0, \]
where \( \lambda_1^0 = \rho_0 U_f > 0, \lambda_2^0 = \rho_0 U_f - \rho_0 h'(\rho_0) < 0 \) since \( s_{cc} \rho_0 = h'(\rho_0) - U_f > 0 \). We have
\[ v_0 = v|_{t=0} = u|_{t=0}, v_1 = \partial_t v|_{t=0} = (\partial_t u + (U(\rho_0) - \rho_0 U_f)\partial_x u)|_{t=0}. \]

From Proposition 1.2, we deduce that
\[
\begin{align*}
v(x, t) &= \int_{\lambda_2^0 t}^{\lambda_1^0 t} V(y, t) \left( \frac{v_0}{\tau} + (\lambda_1 + \lambda_2) v_0' + v_1 \right) (x - y) dy \\
&+ \int_{\lambda_2^0 t}^{\lambda_1^0 t} \partial_t V(y, t) v_0(x - y) dy.
\end{align*}
\]
\[
\begin{align*}
&+ \lambda_1^0 e^{\gamma(x)-\gamma(y)} \frac{\lambda_2^0}{\lambda_1^0 - \lambda_2^0} v_0(x - \lambda_1^0 t) - \lambda_2^0 e^{\gamma(x)-\gamma(y)} \frac{\lambda_1^0}{\lambda_1^0 - \lambda_2^0} v_0(x - \lambda_2^0 t),
\end{align*}
\]
where
\[
V(y, t) = e^{-\gamma(x)-\gamma(y)} \left( -\lambda_1^0 \lambda_2^0 + \lambda_1^0 \lambda_2^0 \right) I_0 \left( \frac{2\sqrt{-\lambda_1^0 \lambda_2^0}}{\tau (\lambda_1^0 - \lambda_2^0)^2} \sqrt{-(y - \lambda_1^0 t)/(y - \lambda_2^0 t)} \right).
\]

We check easily that this quantity is well defined and \( C^2 \) with respect both to the position and derivative, and thus is the classical solution of (1.3).

We recall from Lemma 1.3 that there exists \( a_0 > 0 \) (depending on \( \lambda_1^0, \lambda_2^0, \tau \)) such that for all \( k, n \in \mathbb{N} \), there exists \( C_{k,n}(\lambda_1^0, \lambda_2^0, \tau) > 0 \) such that for \( t \geq 0, y \in [\lambda_2 t, \lambda_1 t] \),
\[
|\partial_t^k \partial_x^n V(y, t)| \leq \frac{C_{k,n}(\lambda_1^0, \lambda_2^0, \tau)e^{-a_0 \gamma(y)}}{(1 + t)^{\frac{1}{2} + \frac{a_0}{2} + n}}, \quad (5.1)
\]

We recall also the notation
\[ F_{a, r, w}(x, t) = \frac{1}{(1 + t)^{\frac{1}{2} + \frac{a_0}{2} + n}} \int_{\lambda_2^0 t}^{\lambda_1^0 t} e^{-a_0 \gamma(y)} w(x - y) dy. \]

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We first estimate with (5.1) that
\[
\left| \int_{\lambda_1^0 t}^{\lambda_2^0 t} V(y, t)v_0(x - y)dy \right|
\leq K(\lambda_1^0, \lambda_2^0, \tau) \int_{\lambda_1^0 t}^{\lambda_2^0 t} e^{-a_0 \frac{t^2}{2}} |v_0(x - y)|dy
\leq K(\lambda_1^0, \lambda_2^0, \tau) F_{\alpha_0, \beta_0, |v_0|}(x, t).
\]
We check easily that there exists \( b_1 > 0 \) depending on \( \lambda_1^0, \lambda_2^0, \tau \) such that
\[
\left| \lambda_1 \frac{e^{-\lambda_1^0 t}}{\lambda_1 - \lambda_2^0} v_0(x - \lambda_1^0 t) - \lambda_2 \frac{e^{-\lambda_2^0 t}}{\lambda_1 - \lambda_2^0} v_0(x - \lambda_2^0 t) \right|
\leq C(\lambda_1^0, \lambda_2^0)e^{-b_1 t}(|v_0(x - \lambda_1^0 t)| + |v_0(x - \lambda_2^0 t)|).
\]
Furthermore, by integration by parts,
\[
\int_{\lambda_1^0 t}^{\lambda_2^0 t} V(y, t)v_0'(x - y)dy
= -\int_{\lambda_1^0 t}^{\lambda_2^0 t} V(y, t)\partial_y(v_0(x - y))dy
= -V(\lambda_1^0 t, t)v_0(x - \lambda_1^0 t) + V(\lambda_2^0 t, t)v_0(x - \lambda_2^0 t)
+ \int_{\lambda_1^0 t}^{\lambda_2^0 t} \partial_y V(y, t)v_0(x - y)dy.
\]
From (5.1) and Lemma 1.3, we check that there exists \( b_2 > 0 \) depending on \( \lambda_1^0, \lambda_2^0, \tau \) such that
\[
\left| \int_{\lambda_1^0 t}^{\lambda_2^0 t} V(y, t)v_0(x - y)dy \right|
\leq K(\lambda_1^0, \lambda_2^0, \tau) F_{\alpha_0, \beta_0, |v_0|}(x, t)
+ C(\lambda_1^0, \lambda_2^0)e^{-b_2 t}(|v_0(x - \lambda_1^0 t)| + |v_0(x - \lambda_2^0 t)|)
\]
for the same \( a_0 \) as equation (5.1). With these estimate and integration by parts, we can check that for the initial datas
\[
u_i = u_{t=0}, \rho_i = \rho_{t=0} \in C_{loc}^0(\mathbb{R}),
\]
we have from Lemma 5.1, \( \partial_t u + (U(\rho_0) - \rho_0 h'(\rho_0))\partial_x u + \frac{1}{2}(U' \rho + u) = 0 \) (that we take at \( t = 0 \) to write \( \partial_t u \) in terms of \( \partial_x u, \rho, u \) at \( t = 0 \)), therefore there exists \( b_0 > 0 \) depending on \( \lambda_1^0, \lambda_2^0, \tau \) such that we have
\[
|v(x, t)|
\leq K(\lambda_1^0, \lambda_2^0, \tau) F_{\alpha_0, \beta_0, |v_0|}(x, t)
+ C(\lambda_1^0, \lambda_2^0)e^{-b_0 t}(|u_i(x - \lambda_1^0 t)| + |u_i(x - \lambda_2^0 t)|)
+ C(\lambda_1^0, \lambda_2^0)e^{-b_0 t}(|\rho_i(x - \lambda_1^0 t)| + |\rho_i(x - \lambda_2^0 t)|).
\]
For \( n, k \in \mathbb{N} \), we now want to estimate \( |\partial_x^n \partial_t^k v(x, t)| \). First, we check that
\[
\left| \partial_x^n \partial_t^k \left( \lambda_1 \frac{e^{-\lambda_1^0 t}}{\lambda_1 - \lambda_2^0} u_i(x - \lambda_1^0 t) - \lambda_2 \frac{e^{-\lambda_2^0 t}}{\lambda_1 - \lambda_2^0} u_i(x - \lambda_2^0 t) \right) \right|
\leq K(n, \lambda_1^0, \lambda_2^0, \tau)e^{-b_0 t} \sum_{j=k}^{n+k} (|u_i^{(j)}(x - \lambda_1^0 t)| + |u_i^{(j)}(x - \lambda_2^0 t)|).
\]
Let us now look at
\[
\partial^n \partial_x^k \left( \int_{\lambda_0^t}^{\lambda_0^t} V(y, t) u_k(x - y) dy \right) = \partial^n \left( \int_{\lambda_0^t}^{\lambda_0^t} V(y, t) u_k^{(k)}(x - y) dy \right).
\]
Either at least one derivative in time fall on the boundaries of the integral. In that case, the term coming from it can be estimated as (5.4). If none does, then we are left with the term
\[
\int_{\lambda_0^t}^{\lambda_0^t} \partial^n \partial_y^k V(y, t) u_k^{(k)}(x - y) dy.
\]
On this term, we do integration by parts, in the spirit of (5.3), to move the k derivatives from \(v_0\) to \(V\). In the process, we create boundary terms that can be estimated as follow for some \(b_2 > 0\):
\[
\left| \int_{\lambda_0^t}^{\lambda_0^t} \partial^n \partial_y^k V(y, t) u_k^{(k)}(x - y) dy - \int_{\lambda_0^t}^{\lambda_0^t} \partial^n \partial_y^k V(y, t) u_k(x - y) dy \right|
\leq K(n, k, \lambda_0^t \lambda_0^t) e^{-b_2 t} \sum_{j=0}^{k-1} (|u_j^{(j)}(x - \lambda_0^t)| + |u_j^{(j)}(x - \lambda_0^t)|). \tag{5.5}
\]
Finally, we estimate with Lemma 1.3 that
\[
\left| \int_{\lambda_0^t}^{\lambda_0^t} \partial^n \partial_y^k V(y, t) u_k(x - y) dy \right|
\leq C_{k,n}(\lambda_0^t \lambda_0^t, \tau) e^{-a_0 \frac{2}{\tau} + \frac{n}{2}} \int_{\lambda_0^t}^{\lambda_0^t} |u_k(x - y)| dy
\leq C_{k,n}(\lambda_0^t \lambda_0^t, \tau) e^{a_0 \frac{2}{\tau} + \frac{n}{2} + |u_k|}(x, t).
\]
We can then estimate every other term by this methods. We put all derivatives on \(V\), and the boundary terms this create can be estimated as in (5.5). At the end, on \(V\) we will have at least \(k\) derivatives in position and \(n\) in time, leading to the estimate on \(\partial^n \partial_x^k V\).

The estimate on \(\partial^n \partial_x^k q\) follows from a similar proof, and this concludes the proof of Proposition 1.4. \(\square\)

6 Proof of the nonlinear stability (Theorem 1.5)

6.1 The equations of the nonlinear problem

Lemma 6.1 Taking the problem (ARZ) for the variable \((\rho_0 + \rho, U(\rho_0) + u)\) yield the equations

\[
\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} + A \partial_x \begin{pmatrix} \rho \\ u \end{pmatrix} + B \begin{pmatrix} \rho \\ u \end{pmatrix} = 0, \tag{6.1}
\]

where
\[
A = \begin{pmatrix} U(\rho_0) + u & \rho_0 + \rho \\ 0 & U(\rho_0) + u - h(\rho_0 + \rho)(\rho_0 + \rho) \end{pmatrix}
\]
and
\[
B = \frac{1}{\tau} \begin{pmatrix} 0 & 0 \\ U_f & 1 \end{pmatrix}.
\]

In particular, with \(\lambda_* = U(\rho_0) - \rho_0 U_f\), the function
\[
q(x, t) := \rho(x - \lambda_* t, t), v(x, t) := u(x - \lambda_* t, t)
\]
satisfies, with the nonlinear characteristic speeds

$$\lambda_1 := \rho_0 U_f + v, \lambda_2 := \rho_0 U_f - (\rho_0 + q)h'(\rho_0 + q) + v,$$

the equations

$$\partial_t^2 v + (\lambda_1 + \lambda_2)\partial_x^2 v + \lambda_1 \lambda_2 \partial_x^2 v + \frac{1}{\tau} \partial_t v + \Omega_1 \partial_x v = 0$$  (6.2)

with

$$\Omega_1 := \partial_t \lambda_2 + \lambda_1 \partial_x \lambda_2 + \frac{1}{\tau} (v - Ufq)$$

and

$$\partial_t^2 q + (\lambda_1 + \lambda_2)\partial_x^2 q + \lambda_1 \lambda_2 \partial_x^2 q + \left(1 + \Omega_2\right) \partial_t q + \Omega_3 \partial_x q = 0, \quad (6.3)$$

with

$$\Omega_2 := \partial_x v, \Omega_3 := \frac{v - Ufq}{\tau} + \lambda_2 \partial_x v.$$

**Proof** We decompose $P = \rho_0 + \rho, V = U(\rho_0) + u$ in

$$\begin{cases} 
\partial_t P + \partial_x (PV) = 0 \\
\partial_t (V + h(P)) + V \partial_x (V + h(P)) = \frac{1}{\tau} (U(P) - V),
\end{cases}$$

where $(\rho_0, U(\rho_0))$ is a constant solution to (ARZ). We have

$$0 = \partial_t P + \partial_x (PV) = \partial_t \rho + (\rho_0 + \rho) \partial_x u + (U(\rho_0) + u) \partial_x \rho.$$  

Furthermore,

$$\begin{align*}
\partial_t (V + h(P)) &= \partial_t u + h'(P) \partial_t \rho \\
&= \partial_t u + h'(P) (- (\rho_0 + \rho) \partial_x u - (U(\rho_0) + u) \partial_x \rho)
\end{align*}$$

and

$$V \partial_x (V + h(P)) = V \partial_x u + V h'(P) \partial_x \rho.$$  

Also, since $U(\rho) = U_f (1 - \rho)$,

$$\frac{1}{\tau} (U(P) - V) = \frac{1}{\tau} (U(\rho_0 + \rho) - U(\rho_0) - u) = -\frac{1}{\tau} (Uf \rho - u),$$

hence

$$0 = \partial_t (V + h(P)) + V \partial_x (V + h(P)) - \frac{1}{\tau} (U(P) - V)$$

$$= \partial_t u + h'(P) (- (\rho_0 + \rho) \partial_x u - (U(\rho_0) + u) \partial_x \rho)$$

$$+ V \partial_x u + V h'(P) \partial_x \rho$$

$$+ \frac{1}{\tau} (Uf \rho + u). \quad (6.4)$$

We conclude the proof of (6.1) by remarking that $V h'(P) \partial_x \rho - (U(\rho_0) + u) h'(P) \partial_x \rho = 0$. We deduce that

$$\begin{cases} 
\partial_t \rho + (U(\rho_0) + u) \partial_x \rho + (\rho_0 + \rho) \partial_x u = 0 \\
\partial_t u + (U(\rho_0) + u - h'(\rho_0 + \rho)(\rho_0 + \rho)) \partial_x u + \frac{1}{\tau} (Uf \rho + u) = 0
\end{cases}$$
and therefore, if we define
\[ v(x, t) := u(x - \lambda_* t, t) \]
and
\[ q(x, t) := \rho(x - \lambda_* t, t), \]
we have the equations
\[ \partial_t q + \lambda_1 \partial_x q + (\rho_0 + q) \partial_x v = 0 \]
and
\[ \partial_t v + \lambda_2 \partial_x v + \frac{1}{\tau}(U_f q + v) = 0, \]
where
\[ \lambda_1 = \rho_0 U_f + v, \lambda_2 = \rho_0 U_f - (\rho_0 + q) h'(\rho_0 + q) + v. \]

Taking \( \partial_t + \lambda_1 \partial_x \) of equation (6.6), we prove equation (6.9) with
\[ \Omega_1 = \partial_t \lambda_2 + \lambda_1 \partial_x \lambda_2 + \frac{1}{\tau}(\lambda_1 - U_f(\rho_0 + q)). \]
Since \( \lambda_1 = \rho_0 U_f + v \), we have
\[ \Omega_1 = \partial_t \lambda_2 + \lambda_1 \partial_x \lambda_2 + \frac{1}{\tau}(v - U_f q). \]
Now, taking \( \partial_t + \lambda_2 \partial_x \) of equation (6.5), we show equation (6.10). \( \square \)

6.2 Proof of Theorem 1.5

**Proof** Let us give here an overview of the proof. In step 1 and 2 we recall briefly the equation satisfied by \( q, v \) and the notations. In step 3 we set up the bootstrap. Step 4 to 6 are devoted respectfully to estimates on \( v \), its first derivatives, then its second derivatives. We conclude in step 7 by explaining how to do similar estimates on \( q \).

**Step 1.** Equation satisfied by \( v(x, t) = u(x + (U(\rho_0) - \rho_0 U_f)t, t) \) and \( q(x, t) = \rho(x + (U(\rho_0) - \rho_0 U_f)t, t) \).

We recall that here, \( \rho_0 \) is a constant. By Cauchy theory, we have the existence at least for small times of a solution of (1.1). It exists in particular as long as \( \rho, u \) and its derivatives in time are bounded.

We recall that (1.1) is equivalent to
\[
\begin{align*}
\partial_t \rho + (U(\rho_0) + u) \partial_x \rho + (\rho_0 + \rho) \partial_x u &= 0 \\
\partial_t u + (U(\rho_0) + u - h'(\rho_0 + \rho)(\rho_0 + \rho)) \partial_x u + \frac{1}{\tau}(U_f \rho + u) &= 0
\end{align*}
\]
if we write the solution \((\rho_0 + \rho, U(\rho_0) + u)\), and therefore, if we define
\[ v(x, t) = u(x - \lambda_* t, t), q(x, t) = \rho(x - \lambda_* t, t), \]
we have the equations
\[ \partial_t q + \lambda_1 \partial_x q + (\rho_0 + q) \partial_x v = 0 \]
and
\[ \partial_t v + \lambda_2 \partial_x v + \frac{1}{\tau}(U_f q + v) = 0, \]
where
\[ \lambda_1 = \rho_0 U_f + v, \lambda_2 = \rho_0 U_f - (\rho_0 + q) h'(\rho_0 + q) + v. \]

We define
\[ \lambda_1^0 = \rho_0 U_f > 0, \lambda_2^0 = \rho_0 U_f - \rho_0 h'(\rho_0) < 0. \]

By Lemma 6.1, we have
\[ \partial_t^2 v + (\lambda_1 + \lambda_2) \partial_{xt}^2 v + \lambda_1 \lambda_2 \partial_x^2 v + \frac{1}{\tau} \partial_t v + \Omega_1 \partial_x v = 0, \]
where
\[ \Omega_1 = \partial_x \lambda_2 + \lambda_1 \partial_x \lambda_2 + \frac{1}{\tau}(\lambda_1 - U_f(\rho_0 + q)). \]

Similarly,
\[ \partial_t^2 q + (\lambda_1 + \lambda_2) \partial^2_{x,t} q + \lambda_1 \lambda_2 \partial_x^2 \partial_t q + \left(\frac{1}{\tau} + \Omega_2\right) \partial_t q + \Omega_3 \partial_x q = 0, \] (6.10)
where
\[ \Omega_2 = \partial_x v, \Omega_3 = \frac{v - U_f q}{\tau} + \lambda_2 \partial_x v. \]

**Step 2.** Linear and nonlinear parts of the equation on \( v \).

We decompose equation (6.9) in
\[
\begin{aligned}
\partial_t^2 v + (\lambda_1^0 + \lambda_2^0) \partial^2_{x,t} v + \lambda_1^0 \lambda_2^0 \partial_x^2 v + \frac{1}{\tau} \partial_t v
&= -\Omega_1 \partial_x v - (\lambda_1 - \lambda_1^0 + \lambda_2 - \lambda_2^0) \partial^2_{x,t} v - (\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0) \partial_x^2 v
\end{aligned}
\]
with \( \lambda_i^0 = \rho_0 U_f, \lambda_{ij}^0 = \rho_0 U_f - \rho_0 h'(\rho_0) \) which are constants. We note \( v_0 = v_{|t=0} = u_{|t=0} \) and \( v_1 = \partial_t v_{|t=0} = ((-U_f \rho_0 - u + (\rho + \rho_0) h'(\rho_0 + \rho)) \partial_x u - \frac{1}{\tau}(U_f \rho + u))_{|t=0}. \)

We define the linear part of the solution, \( v_L \), as the solution to
\[
\begin{cases}
\partial_t^2 v_L + (\lambda_1^0 + \lambda_2^0) \partial^2_{x,t} v_L + \lambda_1^0 \lambda_2^0 \partial_x^2 v_L + \frac{1}{\tau} \partial_t v_L = 0, \\
v_{L,|t=0} = v_0, \quad \partial_t v_{L,|t=0} = v_1.
\end{cases}
\]
Estimates on \( v_L \) follows from Proposition 1.4: there exists \( C_0, a_1, b_1 > 0 \) (depending on \( \lambda_1^0, \lambda_2^0, \tau \)) such that for \( n + k \leq 2 \), we have
\[
|\partial^n_i \partial^k_t v_L(x, t)| \leq \frac{C_0}{(1 + t)^{\frac{n}{2} + \frac{k}{2} + n}} \int_{\lambda_2^0 t}^{\lambda_1^0 t} e^{-a \frac{\int_{y}^{x} |\rho_i(x - y)| + |\rho_i(y)|d\lambda}} dy + C_0 e^{-b_1 t} \sum_{j=0}^{n+k} (|\rho_i^{(j)}| + |u_i^{(j)}|)(|x - \lambda^0_i t| + (|\rho_i^{(j)}| + |u_i^{(j)}|)(x - \lambda^0_2 t))
\]
We define
\[ S := -\Omega_1 \partial_x v - (\lambda_1 - \lambda_1^0 + \lambda_2 - \lambda_2^0) \partial^2_{x,t} v - (\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0) \partial_x^2 v. \]
From Proposition 1.2, we can write \( v = v_L + v_{NL} \) where
\[ v_{NL}(x, t) := \int_0^t \int_{\lambda^0_1(t-s)}^{\lambda^0_2(t-s)} V(y, t-s)S(x, y, s)dyds. \]

**Step 3.** Notations and setting up the bootstrap.

We recall the notation
\[
F_{a, \gamma, \omega}^{\lambda_1, \lambda_2}(x, t) = \frac{1}{(1 + t)^{\gamma}} \int_{\lambda_2 t}^{\lambda_1 t} e^{-a \frac{\int_{y}^{x} |\rho_i(x - y)| + |u_i(x)|d\lambda}} w(x-y) dy.
\]
we define the function
\[ w_i := \sum_{j=0}^{2} |\rho_i^{(j)}| + |u_i^{(j)}| \]
and, to simplify the notations, we define for \( a > 0, \gamma \in \mathbb{R} \) the quantity
\[
F_{a, \gamma, \delta}(x, t) := F_{a, \gamma, \omega_i}^{\lambda_1^0 + \delta, \lambda_2^0 - \delta}(x, t) = \frac{1}{(1 + t)^{\gamma}} \int_{(\lambda_2^0 - \delta) t}^{(\lambda_1^0 + \delta) t} e^{-a \frac{\int_{y}^{x} |\rho_i(x - y)| + |u_i(x)|d\lambda}} w_i(x-y) dy
\]
for some small $\delta > 0$ that will be fixed later on. We also define for some $b, \delta > 0$ the quantity

$$G_{b, \delta, w}(x, t) = e^{-bt} \sup_{y \in [-bt, bt]} w(x - \mu t + y)$$

and

$$G_{b, \delta}(x, t) := G_{b, \delta, w}^0(x, t) + G_{b, \delta, w}^\infty(x, t).$$

Our initial estimate on $v_L$ can be translated using $F_{a, \gamma, \delta}$ and $G_{b, \delta}$: There exists $C_L, a_L, b_L > 0$ depending on $\lambda_1^0, \lambda_2^0, \tau$ such that for $n + k \leq 2$,

$$|\partial_t^n \partial_x^k v_L(x, t)| \leq C_L \left( F_{a_L, \gamma, \delta} + G_{b_L, \delta} \right)(x, t)$$

for all $x \in \mathbb{R}, t \geq 0$. Finally, for small $\nu > 0$ ($\nu < \frac{1}{3}$ will be needed in the proof) and $k + n \leq 2$, we define $\gamma_\nu(k, n)$ by

$$\gamma_\nu(0, 0) = \frac{1}{2}, \gamma_\nu(1, 0) = 1 - \nu, \gamma_\nu(0, 1) = \frac{3}{2} - \nu, \gamma_\nu(k, n) = 1 - \nu$$

for $k + n = 2$. Remark that in all cases,

$$\gamma_\nu(k, n) \leq \frac{1}{2} + \frac{k}{2} + n.$$

We want to bootstrap the following results: there exists $a, b, \delta > 0$ such that for any fixed small $\nu > 0$, there exists $C_0(\nu) > 0$ such that for $k + n \leq 1$,

$$|\partial_t^n \partial_x^k v(x, t)| + |\partial_t^n \partial_x^k q(x, t)| \leq C_0(\nu)(F_{a, \gamma_\nu(k, n), \delta} + G_{b, \delta})(x, t),$$

and for $k + n = 2$, for some $a_2, b_2, \delta_2 > 0$,

$$|\partial_t^n \partial_x^k v(x, t)| + |\partial_t^n \partial_x^k q(x, t)| \leq C_0(\nu)(F_{a_2, \gamma_\nu(k, n), \delta_2} + G_{b_2, \delta_2})(x, t).$$

In the bootstrap, the coefficient $a_2, b_2, \delta_2$ are not the same as $a, b, \delta$ (in face $a_2, b_2$ will be respectively smaller than $a, b$). In the computations, we will first take $b$ small enough to beat some constants depending on $\rho_0, U_f, h$, then $a, a_2, b_2$ will be taken small enough to beat constants depending on $b, \rho_0, U_f, h$.

We denote by $T^* \geq 0$ the maximum time such that the solution exists in the classical sense (that is it is a $C^2$ function) and these estimates hold for given $C_0(\nu)$, $a, b, a_2, b_2, \delta$ and $\delta_2$.

First, from standard local Cauchy theory (see for instance [3]), there exists $\tau_0^* > 0$ such that the solution exists and is $C^2$ on $[0, \tau_0^*]$. Furthermore, still from [3], if the estimates hold on $[0, T^*]$, then there exists $\tau^* > 0$ such that the solution exists and is $C^2$ on $[0, T^* + \tau^*]$. The fact that $\tau^* > 0$ if we take $C_0(\nu)$ large enough is a consequence of the standard estimates from Kato (see [9]).

Our goal is to show that $T^* = +\infty$. We suppose that $T^* < +\infty$. From subsection 3.3, we check that on $[0, T^*]$, for $C_0(\nu)$ large enough,

$$\|\partial_t^n \partial_x^k v(\cdot, t)\|_{L^\infty(\mathbb{R})} + \|\partial_t^n \partial_x^k q(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq C_0(\nu) \frac{\|w_1\|_{L^1(\mathbb{R})} + \|w_2\|_{L^\infty(\mathbb{R})}}{(1 + t)^{\gamma_\nu(k, n)}} (6.11)$$

and

$$\|\partial_t^n \partial_x^k v(\cdot, t)\|_{L^1(\mathbb{R})} + \|\partial_t^n \partial_x^k q(\cdot, t)\|_{L^1(\mathbb{R})} \leq C_0(\nu) \frac{\|w_1\|_{L^1(\mathbb{R})}}{(1 + t)^{\gamma_\nu(k, n)-\frac{k}{2}}} (6.12)$$

We recall that $\|w_1\|_{L^1(\mathbb{R})} + \|w_2\|_{L^\infty(\mathbb{R})} \leq \varepsilon < 1$ by hypothesis. In particular, for $\varepsilon$ small enough, we have $\rho_0 + \rho \in |0, 1|, \lambda_1 > 0, \lambda_2 < 0$ on $[0, T^*]$.

**Step 4. Estimates on $v$**

We recall from step 2. that $v = v_L + v_{NL}$ with

$$v_{NL}(x, t) = \int_0^t \int_{\lambda_2^0(t-s)}^{\lambda_1^0(t-s)} V(y, t-s) S(x-y, s) dy ds$$

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and \(v_L\) satisfies
\[
|v_L(x, t)| \leq C_L \left( F_{a_1, \frac{1}{2}, 0} + G_{b_1, 0} \right)(x, t)
\]
for some fixed values of \(C_L, a_1, b_1\) for all positive times. We will decompose \(S\) in a particular form for this estimate. We recall that
\[
S = -\Omega_t \partial_x v - (\lambda_1 - \lambda_1^0 + \lambda_2 - \lambda_2^0)\partial_x^2 v - (\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0)\partial_x^2 v.
\]
Here, the difficulty is to keep the decay in time and have an estimate with \(F_{a_1, \frac{1}{2}, 0}\). It would be easy to have the result with \(F_{a_1, \frac{1}{2} - \nu, \delta}\) instead (by applying Lemma 3.2), but we need some specific computation to not lose the factor \(t^{2\nu}\).

We start with the estimate of
\[
E_1 := \int_0^t \int_{\lambda_2^0 (t-s)}^{\lambda_2^0 (t-s)} -V(y, t-s)((\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0)\partial_x^2 v)(x-y, s)dyds,
\]
one of the terms appearing in \(v_{NL}\). We have that
\[
\partial_x^2 v(x-y, s) = -\partial_y (\partial_x v(x-y, s)),
\]
therefore, by integration by parts,
\[
E_1(x, t) = \int_0^t V(\lambda_1^0(t-s), t-s)((\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0)\partial_x v)(x - \lambda_1^0(t-s), s)ds
\]
\[
- \int_0^t V(\lambda_2^0(t-s), t-s)((\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0)\partial_x v)(x - \lambda_2^0(t-s), s)ds
\]
\[
- \int_0^t \int_{\lambda_2^0 (t-s)}^{\lambda_2^0 (t-s)} \partial_y V(y, t-s)((\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0)\partial_x v)(x-y, s)dyds
\]
\[
- \int_0^t \int_{\lambda_2^0 (t-s)}^{\lambda_2^0 (t-s)} V(y, t-s)(\partial_y (\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0)\partial_x v)(x-y, s)dyds.
\]
We have
\[
\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0 = (\lambda_1 - \lambda_1^0)\lambda_2 + \lambda_1^0 (\lambda_2 - \lambda_2^0),
\]
and
\[
\lambda_3 - \lambda_1^0 = v, \lambda_2 - \lambda_2^0 = \rho_0 h'(\rho_0) - (\rho_0 + q) h'(\rho_0 + q) + v.
\]
Therefore on \([0, T]\), by equation (6.11), we have
\[
|(\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0)(x - \lambda_1^0(t-s), s)| \leq K C_0(\nu) \varepsilon. \tag{6.13}
\]
This implies that on \([0, T]\),
\[
\left| \int_0^t V(\lambda_1^0(t-s), t-s)((\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0)\partial_x v)(x - \lambda_1^0(t-s), s)ds \right|
\]
\[
\leq \ K C_0(\nu) \varepsilon \int_0^t |V(\lambda_1^0(t-s), t-s)\partial_x v(x - \lambda_1^0(t-s), s)|ds
\]
\[
\leq \ K C_0(\nu)^2 \varepsilon \int_0^t |V(\lambda_1^0(t-s), t-s)F_{a, \gamma_0(1,0), \delta}(x - \lambda_1^0(t-s), s)|ds
\]
\[
+ \ K C_0(\nu)^2 \varepsilon \int_0^t |V(\lambda_1^0(t-s), t-s)G_{b, \delta}(x - \lambda_1^0(t-s), s)|ds.
\]
From Lemmas 1.3 and 3.1, we have (if \( a > 0 \) is small enough)

\[
\int_0^t |V(\lambda^0_1(t-s), t-s)F_{a,\gamma,\nu,(1,0)}(x - \lambda^0_1(t-s), s)| ds \\
\leq K F_{a,\gamma,\nu,(1,0),\delta}(x, t) \\
\leq K F_{a,\frac{1}{2},\delta}(x, t).
\]

From Lemma 3.4, we have (if \( a \) is small enough, depending on \( b \))

\[
\int_0^t |V(\lambda^0_1(t-s), t-s)G_{b,\delta}(x - \lambda^0_1(t-s), s)| ds \\
\leq K \left( G_{b,\delta}(x, t) + F_{a,\frac{1}{2},\delta}(x, t) \right).
\]

We deduce that

\[
\left| \int_0^t V(\lambda^0_1(t-s), t-s)((\lambda_1 \lambda_2 - \lambda^0_1 \lambda^0_2)\partial_x v)(x - \lambda^0_1(t-s), s)| ds \right| \\
\leq KC_0(\nu)^2 \varepsilon \left( G_{b,\delta}(x, t) + F_{a,\frac{1}{2},\delta}(x, t) \right).
\]

The same estimate holds for the second line of \( E_1 \). For the third line, using for \( s \in [0, T^*] \) that

\[
\left| (\lambda_1 \lambda_2 - \lambda^0_1 \lambda^0_2) (x - y, s) \right| \leq \frac{KC_0(\nu)^2 \varepsilon}{(1+s)^{1/2}},
\]

we have (with Lemma 1.3)

\[
\left| \int_0^t \int_{\lambda^0_1(t-s)}^{\lambda^0_1(t)} \partial_y V(y, t-s)((\lambda_1 \lambda_2 - \lambda^0_1 \lambda^0_2)\partial_x v)(x - \lambda^0_1(t-s), s) dy ds \right| \\
\leq KC_0(\nu)^2 \varepsilon \int_0^t \int_{\lambda^0_1(t-s)}^{\lambda^0_1(t)} \frac{e^{-a_0 t^2 y^2}}{(1+t-s)} F_{a,\gamma,\nu,(1,0)+\frac{1}{2},\delta}(x - y, s) dy ds \\
+ KC_0(\nu)^2 \varepsilon \int_0^t \int_{\lambda^0_1(t-s)}^{\lambda^0_1(t)} \frac{e^{-a_0 t^2 y^2}}{(1+t-s)} G_{b,\delta}(x - y, s) dy ds.
\]

From Lemma 3.2, given that \( a \) is small enough, we have

\[
\int_0^t \int_{\lambda^0_1(t-s)}^{\lambda^0_1(t)} \frac{e^{-a_0 t^2 y^2}}{(1+t-s)} F_{a,\gamma,\nu,(1,0)+\frac{1}{2},\delta}(x - y, s) dy ds \\
\leq K F_{a,1-\nu,\delta}(x, t) \\
\leq K F_{a,\frac{1}{2},\delta}(x, t),
\]

and using Lemma 3.3, for \( a \) small enough

\[
\int_0^t \int_{\lambda^0_1(t-s)}^{\lambda^0_1(t)} \frac{e^{-a_0 t^2 y^2}}{(1+t-s)} G_{b,\delta}(x, t) dy ds \\
\leq K F_{a,1,\delta}(x, t) \\
\leq K F_{a,\frac{1}{2},\delta}(x, t).
\]

We deduce that

\[
\left| \int_0^t \int_{\lambda^0_1(t-s)}^{\lambda^0_1(t)} \partial_y V(y, t-s)((\lambda_1 \lambda_2 - \lambda^0_1 \lambda^0_2)\partial_x v)(x - y, s) dy ds \right| \\
\leq KC_0(\nu)^2 \varepsilon F_{a,\frac{1}{2},\delta}(x, t).
\]
A similar computation gives the same estimate on the fourth line of $E_1$. This concludes the proof of

$$|E_1(x, t)| \leq KC_0(\nu)^2 \epsilon \left( F_{a, b, \delta}(x, t) + G_{b, \delta}(x, t) \right).$$

We can do a similar decomposition and estimation for

$$E_2(x, t) = \int_0^t \int_{\lambda_2(t-s)}^{\lambda_0(t-s)} -V(y, t-s)((\lambda_1 - \lambda_2 + \lambda_2)\partial_x^2 v)(x-y, s)dy ds,$$

leading to

$$|E_2(x, t)| \leq KC_0(\nu)^2 \epsilon \left( F_{a, b, \delta}(x, t) + G_{b, \delta}(x, t) \right).$$

Finally, we look at

$$E_3(x, t) = \int_0^t \int_{\lambda_2(t-s)}^{\lambda_0(t-s)} V(y, t-s)(\Omega_1 \partial_x v)(x-y, s)dy ds,$$

where (from step 1)

$$\Omega_1 = \partial_\lambda \lambda_2 + \lambda_1 \partial_x \lambda_2 + \frac{1}{\tau}(v - U_f q).$$

Simply using Lemma 3.2 at this stage would lead to an estimate with $F_{a, b, \delta}$ instead of $F_{a, b, \delta}$. It is critical to get $F_{a, b, \delta}$ here to make the bootstrap work. We had some margin in $E_1$ and $E_2$ but it will not be the case here. From step 1, we recall that

$$\partial v + \lambda_2 \partial_x v + \frac{1}{\tau}(U_f q + v) = 0,$$

which implies that on $[0, T^*]$ we have

$$|(U_f q + v)(x, t)| \leq KC_0(\nu)(G_{b, \delta}(x, t) + F_{a, b, \delta}(x, t)).$$

Remark that this is better than estimates on $q$ or $v$ individually. We decompose

$$\Omega_1 = \frac{2}{\tau} v + \partial_\lambda \lambda_2 + \lambda_1 \partial_x \lambda_2 - \frac{1}{\tau}(U_f q + v).$$

Remark that

$$\left| \partial_\lambda \lambda_2 + \lambda_1 \partial_x \lambda_2 - \frac{1}{\tau}(U_f q + v) \right| (x, t) \leq \frac{KC_0(\nu)\epsilon}{(1 + t)^{\gamma_v(1, 0)}}$$

on $[0, T^*]$. Therefore, using Lemmas 3.2 and 3.3, we estimate

$$\left| \int_0^t \int_{\lambda_2(t-s)}^{\lambda_0(t-s)} V(y, t-s) \left( \left( \partial_\lambda \lambda_2 + \lambda_1 \partial_x \lambda_2 - \frac{1}{\tau}(U_f q + v) \right) \partial_x v \right)(x-y, s)dy ds \right|

\leq \frac{KC_0(\nu)\epsilon}{(1 + t)^{\gamma_v(1, 0)}} \int_0^t \int_{\lambda_2(t-s)}^{\lambda_0(t-s)} |V(y, t-s)|(F_{a, 2, \nu}(1, 0, \delta) + G_{b, \delta})(x-y, s)dy ds

\leq \frac{KC_0(\nu)\epsilon}{(1 + t)^{\gamma_v(1, 0)}} \left( F_{a, b, \delta}(x, t) + G_{b, \delta}(x, t) \right).$$

Finally, by integration by parts,

$$\int_0^t \int_{\lambda_2(t-s)}^{\lambda_0(t-s)} V(y, t-s) \left( \frac{2}{\tau} v \partial_x v \right)(x-y, s)dy ds

= -\frac{1}{\tau} \int_0^t \int_{\lambda_2(t-s)}^{\lambda_0(t-s)} V(y, t-s) \partial_y (v^2(x-y, s))dy ds

= -\frac{1}{\tau} \int_0^t \int_{\lambda_2(t-s)}^{\lambda_0(t-s)} V(\lambda_1(t-s), t-s) v^2(x-\lambda_1(t-s), s)ds

+ \frac{1}{\tau} \int_0^t \int_{\lambda_2(t-s)}^{\lambda_0(t-s)} \partial_y V(y, t-s) v^2(x-y, s)dy ds.$$
These three terms can be estimated as previously. We deduce that
\[ |v_{NL}(x,t)| \leq |E_1(x,t)| + |E_2(x,t)| + |E_3(x,t)| \leq KC_0(\nu)^2 \varepsilon \left( F_{a,\frac{1}{2},\delta}(x,t) + G_{b,\delta}(x,t) \right), \]
and therefore, for \( a < a_L, b < b_L \),
\[ |v(x,t)| \leq |v_{L}(x,t)| + |v_{NL}(x,t)| \leq (C_L + KC_0(\nu)^2 \varepsilon) \left( F_{a,\frac{1}{2},\delta} + G_{b,\delta} \right)(x,t). \]
Taking \( \varepsilon \) small enough (depending on \( \nu \)), we deduce that on \([0,T^*]\), we have
\[ |v(x,t)| \leq \frac{C_0(\nu)}{2} \left( F_{a,\frac{1}{2},\delta} + G_{b,\delta} \right)(x,t). \]
Therefore \( v \) is not the one responsible for the fact that \( T^* \neq +\infty \).

**Step 5.** Estimates on \( \partial_t v \) and \( \partial_x v \)

We compute that
\[
\partial_x v_{NL}(x,t) = \int_0^t \int_{\lambda_1(t-s)}^{\lambda_2(t-s)} V(y, t-s) (-\partial_y) (S(x - y, s)) dy ds \\
= - \int_0^t V(\lambda_1^0(t-s), t-s) S(x - \lambda_1^0(t-s), s) ds \\
+ \int_0^t V(\lambda_2^0(t-s), t-s) S(x - \lambda_2^0(t-s), s) ds \\
+ \int_0^t \int_{\lambda_2(t-s)}^{\lambda_1(t-s)} \partial_y V(y, t-s) S(x - y, s) dy ds.
\]
We estimate as previously that on \([0,T^*]\), estimating the second derivatives of \( v \) and \( q \) in \( L^\infty \) by (6.11), that
\[
|S(x,t)| \leq |\Omega_1 \partial_x v| + |(\lambda_1 - \lambda_1^0 + \lambda_2 - \lambda_2^0) \partial_x^2 v| + |(\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0) \partial_x^2 v| \\
\leq ||\partial_x v||_{L^\infty} |\Omega_1| + ||\partial_x^2 v||_{L^\infty} |(\lambda_1 - \lambda_1^0 + \lambda_2 - \lambda_2^0)| \\
+ ||\partial_x^2 v||_{L^\infty} |(\lambda_1 \lambda_2 - \lambda_1^0 \lambda_2^0)| \\
\leq KC_0(\nu)^2 \varepsilon \left( F_{a,\frac{1}{2},\nu,\delta} + G_{b,\delta} \right)(x,t).
\]
We deduce that on \([0,T^*]\),
\[
|\partial_x v_{NL}(x,t)| \leq KC_0(\nu)^2 \varepsilon (F_{a,\gamma_a(1,0),\delta} + G_{b,\delta})(x,t).
\]
Since
\[
|\partial_x v_L(x,t)| \leq C_L (F_{a_L,1,0} + G_{b_L,0})(x,t) \leq C_L (F_{a,\gamma_a(1,0),\delta} + G_{b,\delta})(x,t),
\]
we deduce that, taking \( \varepsilon \) small enough (depending on \( \nu \)), on \([0,T^*]\) we have
\[
|\partial_x v(x,t)| \leq \frac{C_0(\nu)}{2} (F_{a,\gamma_a(1,0),\delta} + G_{b,\delta})(x,t).
\]
Also,
\[
\partial_t v_{NL}(x,t) = \lambda_1^0 \int_0^t V(\lambda_1^0(t-s), t-s) S(x - \lambda_1^0(t-s), s) ds \\
- \lambda_2^0 \int_0^t V(\lambda_2^0(t-s), t-s) S(x - \lambda_2^0(t-s), s) ds \\
+ \int_0^t \int_{\lambda_2(t-s)}^{\lambda_1(t-s)} \partial_y V(y, t-s) S(x - y, s) dy ds,
\]
Furthermore, by (6.14) we have
\[ \partial_t H = \frac{C_0(\nu)}{2}(F_{\nu,0,0,0} + G_{b,\delta})(x, t). \]

**Step 6.** Estimates on the second derivatives of \( v \)

The method used in the previous steps does not work, as some terms will have three derivatives. That is, if we compute \( \partial^2 \nu \), for instance, a derivative will fall on \( S \) in a boundary term, and it cannot be moved by integration by parts. This difficulty comes from the fact that \( \lambda_1 \) and \( \lambda_2 \) are not constants, and they affect the cone of light. We are going to use section 4 to solve this issue.

We recall that
\[ \partial^2 v + (\lambda_1 + \lambda_2) \partial_x^2 v + \lambda_1 \lambda_2 \partial_t^2 v + \frac{1}{\tau} \partial_t v + \Omega_1 \partial_x v = 0, \]
and we define \( \delta^0 = \frac{1}{\tau}, \omega = \Omega_1 \) to be consistent with the notations of section 4.

Following subsection 4.2, we define the functions \( H_1, H_2 \) by
\[
\begin{cases}
\partial_t H_1 + \lambda_1 \partial_x H_1 = \lambda_1^0 - \lambda_1 \\
H_{1|t=0} = 0 \quad \text{(6.14)}
\end{cases}
\]
and
\[
\begin{cases}
\partial_t H_2 + \lambda_2 \partial_x H_2 = \lambda_2^0 - \lambda_2 \\
H_{2|t=0} = 0.
\end{cases}
\]

We define the new variables
\[ y = x + \tilde{h}_1(x, t), \nu = t + \tilde{h}_2(x, t) \]
where
\[ \tilde{h}_1 = \frac{\lambda_1^0 H_1 - \lambda_2^0 H_2}{\lambda_2 - \lambda_1^0}, \tilde{h}_2 = \frac{H_1 - H_2}{\lambda_2 - \lambda_1} \]

Let us now compute some estimates on \( H_1, H_2 \). First, for \( x \in \mathbb{R} \), we define \( X_x \) on \([0, T^*]\) by
\[
\begin{cases}
X'_x(t) = \lambda_1(X_x(t), t) \\
X_x(0) = x.
\end{cases}
\]

Now, we infer that for any \( y \in \mathbb{R}, t \leq T^* \), there exists \( x \in \mathbb{R} \) such that \( X_x(t) = y \). Indeed, by (6.11) we have with some margin
\[ |\lambda_1 - \lambda_1^0| \leq \varepsilon \]
and we take \( \varepsilon \) small enough such that \( \varepsilon < \frac{\lambda_1^0}{2} \). We deduce that \( |X'_x(t) - \lambda_1^0| \leq \frac{\lambda_1^0}{2} \), thus \( \frac{\lambda_1^0}{2} \geq X'_x(t) \geq \frac{\lambda_1^0}{2} \). In particular, \( X_{y - \lambda_1^0 t}(t) \geq y - \frac{\lambda_1^0}{2} t + \int_0^1 X'_x(s) ds \geq y \) and \( X_{y - \lambda_1^0 t}(t) \leq y \). By continuity of \( x \rightarrow X_x(t) \), we deduce that there exists \( x \in [y - \frac{\lambda_1^0}{2} t, y - \frac{\lambda_1^0}{2} t] \) such that \( X_x(t) = y \).

Now, we have by differentiating the equation on \( H_1 \) that
\[ (\partial_t + \lambda_1 \partial_x)(\partial_t H_1) = -\partial_t \lambda_1 (1 + \partial_x H_1). \]

Furthermore, by (6.14) we have \( \partial_x H_1 = \frac{\lambda_0^0 - \lambda_1}{\lambda_1} - \frac{\partial H_1}{\lambda_1} \), hence
\[ (\partial_t + \lambda_1 \partial_x)(\partial_t H_1) = -\partial_t \lambda_1 \left( 1 + \frac{\lambda_0^0 - \lambda_1}{\lambda_1} - \frac{\partial H_1}{\lambda_1} \right). \]

We deduce that for any \( x \in \mathbb{R} \),
\[ \partial_t (\partial_t H_1(X_x(t), t)) = -\left( \partial_t \lambda_1 \left( 1 + \frac{\lambda_0^0 - \lambda_1}{\lambda_1} - \frac{\partial H_1}{\lambda_1} \right) \right) (X_x(t), t). \]
This implies that
\[
\partial_t H_1(X_x(t), t) = \partial_t H_1(X_x(0), 0) + \int_0^t \left( \partial_t \lambda_1 \left( 1 + \frac{\lambda_0 - \lambda_1}{\lambda_1} - \frac{\partial_t H_1}{\lambda_1} \right) \right) (X_x(s), s) \, ds.
\]
(6.15)

Let us show that for any \( t \in [0, T^*] \), \( y \in \mathbb{R} \), we have
\[
|\partial_t H_1(y, t)| \leq 2\varepsilon.
\]

We denote by \( T \geq 0 \) the maximum time such that this holds. Since \( H_1|_{t=0} = 0 \) and \( H_1 \) satisfies the equation
\[
(\partial_t + \lambda_1 \partial_x)H_1 = \lambda_0 \nu - \lambda_1 \nu
\]
we have
\[
|\partial_t H_1(y, 0)| = |\lambda_0 \nu - \lambda_1(0)| \leq \varepsilon.
\]
By continuity, we have \( T > 0 \). Suppose that \( T < T^* \). Then for any \( x \in \mathbb{R} \), using (6.11) and (6.15), we estimate
\[
|\partial_t H_1(X_x(T^*), T^*)| \leq \varepsilon + C_1 \int_0^{T^*} \left| \left( \partial_t \lambda_1 \left( 1 + \frac{\lambda_0 - \lambda_1}{\lambda_1} - \frac{\partial_t H_1}{\lambda_1} \right) \right) (X_x(s), s) \right| \, ds
\]
\[
\leq \varepsilon + C_1 \int_0^{T^*} \frac{\varepsilon^2}{(1 + s)^{1+\nu}} \, ds
\]
\[
\leq \varepsilon + C_2(\nu)\varepsilon^2
\]
\[
\leq \frac{3}{2}\varepsilon
\]
given that \( \varepsilon > 0 \) is small enough (here appears the fact that \( \varepsilon \) depends on \( \nu \)). by surjectivity of \( x \to X_x(T^*) \) we have a contradiction. This implies that \( T = T^* \) and we have the estimate \( |\partial_t H_1(y, t)| \leq 2\varepsilon \). From (6.14) we also estimate that \( |\partial_y H_1(y, t)| \leq 3\varepsilon \). We have the same estimates on \( H_2 \), and by (4.4), we have similar estimates on the first derivatives of \( \tilde{h}_1, \tilde{h}_2 \), which is what was needed in subsection 4 to complete the change of variable. In particular,
\[
(x, t) \to (x + \tilde{h}_1(x, t), t + \tilde{h}_2(x, t)) = (y(x, t), \nu(x, t))
\]
is an inversible function since
\[
|\partial_t \tilde{h}_1| + |\partial_x \tilde{h}_1| + |\partial_t \tilde{h}_2| + |\partial_x \tilde{h}_2| \leq K\varepsilon \ll 1.
\]
It is in fact close to the identity, in particular this function and its inverse have a norm close to one.

We recall that \( t + \tilde{h}_2(x, t) \geq 0 \) for all \( t \geq 0 \), using the fact that \( \tilde{h}_2(0, x) = 0 \) and \( |\partial_t \tilde{h}_2| \leq K\varepsilon \) which implies that \( |\tilde{h}_2(x, t)| \leq K\varepsilon t \leq \frac{1}{2} \) given \( \varepsilon \) small enough. We have then
\[
\partial_t \tilde{h}_2(x, t) = \partial_t (t + \tilde{h}_2(x, t)) = 1 + \partial_t \tilde{h}_2(x, t) \geq 1 - 2\varepsilon.
\]
(6.16)

We then write
\[
v(x, t) = w(x + h_1(x, t), t + h_2(x, t)) = w(y, \nu)
\]
and
\[
g(x, t) = p(x + h_1(x, t), t + h_2(x, t)) = p(y, \nu).
\]
we recall that the change of variable \( \pi(y, \nu) \to (x, t) \) is invertible and close to the identity for \( \varepsilon \) small enough. Then, by section 4, \( w \) satisfies
\[
(\partial^2_\nu w + (\lambda_0^2 + \lambda_2^2)\partial^2_\nu w + \lambda_0^2 \lambda_2^2 \partial^2_\nu w + \delta^\nu \partial_\nu w)(y, \nu)
\]
\[
= -(R_1\nu\pi)\partial_\nu w - R_2\nu\pi \partial_\nu w
\]
(6.17)
where \( R_1, R_2 \) are small functions depending on \( H_1, H_2, \partial_x H_1, \partial_x H_2, \partial_t H_1, \partial_t H_2, \partial_x \lambda_1, \partial_t \lambda_1, \partial_x \lambda_2, \partial_t \lambda_2, \nu \) but not depending on their derivatives. The same can be said for \( \pi \), that depends on \( H_1, H_2 \) but not its derivatives.

At \( \nu = 0 \), we have \( t = 0 \) from section 4, therefore
\[
w(y, 0) = v(y, 0) = 0, \partial_\nu w(y, 0) = \partial_\nu \pi_1(y, 0) \partial_\nu v(y, 0) + \partial_\nu \pi_2(y, 0) \partial_t v(y, 0) = 0.
\]
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From Proposition 1.2, with \( w_0 = w_{[\nu=0]} \) and \( w_1 = \partial_\nu w_{[\nu=0]} \), as well as 
\[
S = -(R_1 \sigma) \partial_\nu w - R_2 \sigma \partial_\nu w,
\]
we have
\[
w(y, \nu) = w_L(y, \nu) + \int_0^\nu \int_{\lambda^0_1(\nu-s)}^\lambda^0_2(\nu-s) V(z, \nu - s) S(y - z) dz ds.
\]
(6.18)

Let us make a few remarks on the domain of definition of \( w \), that is
\[
\Pi := \pi_{-1}(\mathbb{R}^s \times [0, T^*]).
\]
Since \( \pi \) is \( C^1 \) and close to the identity, we have \( \Pi = \{(z, T_\pi(z), z \in \mathbb{R})\} \) for some smooth function \( T_\pi \) that satisfies \( |T_\pi| \leq K\varepsilon \). In particular, if \( (y, \nu) \in \Pi \) and \( s \in [0, \nu], z \in [\lambda^0_2(\nu - s), \lambda^0_1(\nu - s)] \), then \( (y - z, s) \in \Pi \) if \( \varepsilon \) is small enough. Therefore in equation (6.18) the function \( S \) will always be taken in a point where we have our bootstrap estimate.

Now, let us estimate \( w, \partial_\nu w, \partial_\nu^2 w \) by using
\[
w(y, \nu) = v(\pi_{-1}(y, \nu)).
\]
We recall that \( \pi_{-1}^{-1}(y, \nu) = x, \pi_{-2}^{-1}(y, \nu) = t \). With section 4, we estimate that
\[
(1 + \varepsilon) \nu \geq \pi_2^{-1}(y, \nu) \geq (1 - \varepsilon) \nu,
\]
and that
\[
|y - \pi_1^{-1}(y, \nu)| \leq 2\varepsilon \nu.
\]
Then using the results of steps 4 and 5, we deduce that
\[
|w(y, \nu)| \leq C_0(\nu) \left( F_{a, \frac{1}{2}, \delta} + G_{b, \delta} \right)(\pi_{-1}^{-1}(y, \nu), \pi_{-2}^{-1}(y, \nu)).
\]
First, we compute
\[
G_{b, \delta}(\pi_{-1}^{-1}(y, \nu), \pi_{-2}^{-1}(y, \nu))
\leq e^{-b \pi_{-2}^{-1}(y, \nu)} \sup_{z \in [-\pi_2^{-1}(y, \nu), \pi_2^{-1}(y, \nu)]} w_i(y - z)
\leq e^{-b(1-\varepsilon)\nu} \sup_{z \in [-\pi_2^{-1}(y, \nu), \pi_2^{-1}(y, \nu)]} w_i(y - z)
\leq e^{-b(1+\varepsilon)\nu} \sup_{z \in [-\pi_2^{-1}(y, \nu), \pi_2^{-1}(y, \nu)]} w_i(y - z)
\leq G_{b(1-\varepsilon), \delta(1+2\varepsilon)}(y, \nu).
\]
(6.19)

We also compute that
\[
F_{a, \frac{1}{2}, \delta}(\pi_{-1}^{-1}(y, \nu), \pi_{-2}^{-1}(y, \nu))
\leq \frac{1}{\sqrt{1 + \pi_2^{-1}(y, \nu)}} \int (\lambda^0_2(\lambda^0_1 + \delta)^{-1}(y, \nu) e^{-\frac{a}{1+\pi_2^{-1}(y, \nu)}} w_i(y - z) dz
\leq 2 \frac{2}{\sqrt{1 + \nu}} \int (\lambda^0_2(\lambda^0_1 + \delta)^{-1}(y, \nu) e^{-\frac{a}{\pi_2(1+\varepsilon)^{1+\delta}} w_i(y - z) dz
\leq 2 F_{a, \frac{1}{2}, \delta}(y, \nu)
\]
(6.20)
given that \(a_3, \delta_3\) are small enough (compared to \(a, \delta_1\)). We deduce that on \(\Pi\) for some \(b_3 > 0\),

\[
|w(y, \nu)| \leq 2C_0(\nu) \left( F_{a_3, b_3, \delta_3} + G_{b_3, \delta_3} \right) (y, \nu).
\]

A similar estimate can be done for \(\partial_y w\) and \(\partial_{\nu} w\), leading to (taking \(a_3, b_3, \delta_3\) small enough)

\[
|\partial_y w(y, \nu)| \leq 2C_0(\nu)(F_{a_3, b_3, \delta_3}) (y, \nu)
\]

and

\[
|\partial_{\nu} w(y, \nu)| \leq 2C_0(\nu)(F_{a_3, b_3, \delta_3}) (y, \nu).
\]

Now we estimate the second derivatives with

\[
w(y, \nu) = w_L(y, \nu) + \int_0^\nu \int_{\lambda_0^0(\nu-s)}^{\lambda_0^0(\nu-s)} V(z, \nu-s)S(y - z, s)dzds.
\]

The estimate on \(\partial^n_{\nu} \partial^{k}_{y} w_L\) for \(n + k = 2\) follows from initial conditions and Proposition 1.4. For the nonlinear part, we first compute

\[
\partial_{\nu}^2 \left( \int_0^\nu \int_{\lambda_0^0(\nu-s)}^{\lambda_0^0(\nu-s)} V(z, \nu-s)S(y - z, s)dzds \right)
\]

\[
= \lambda_1^0 \int_0^\nu V(\lambda_0^0(\nu-s), \nu-s)\partial_y S(y - \lambda_1^0(\nu-s), s)ds
\]

\[
- \lambda_2^0 \int_0^\nu V(\lambda_2^0(\nu-s), \nu-s)\partial_y S(y - \lambda_2^0(\nu-s), s)ds
\]

\[
+ \int_0^\nu \int_{\lambda_0^0(\nu-s)}^{\lambda_0^0(\nu-s)} \partial_y V(z, \nu-s)\partial_y S(y - z, s)dzds.
\]

We recall that \(S = -(R_1 \alpha \nu)\partial_y w - R_2 \alpha \nu \partial_{\nu} w\). We check that on \(\Pi\), with our bootstrap estimates, we have

\[
|\partial_y S| \leq K(\nu)\varepsilon(F_{a_3, 1-\nu, \delta_3} + G_{b_3, \delta_3}).
\]

Using Lemmas 3.1, 3.2, 3.3 and 3.4, we can thus show by a bootstrap argument that if we take \(\varepsilon\) small enough, then on \(\Pi\) we have

\[
|\partial^n_{\nu} \partial^{k}_{y} w_L(y, \nu)| \leq C_0(\nu)(F_{a_3, 1-\nu, \delta_3} + G_{b_3, \delta_3}) (y, \nu).
\]

Similarly, we can estimate \(\partial^2_{\nu} w\) and get the same estimate. By (6.17), the equation satisfied by \(w\), we deduce the same estimate on \(\partial^2_{\nu} w\).

Finally, we infer that these estimates holds also in the variables \((x, t) \in \mathbb{R} \times [0, T^*]\) if we consider \(\delta_2, a_2, b_2\) small enough instead of \(\delta_3, a_3, b_3\) by doing the reversed estimates of (6.19) and (6.20) (the computations are similar). This concludes the estimates on \(v\) and its first and second derivatives.

**Step 7.** Conclusion

To conclude the proof, we need to do the same estimates on \(q\). The equation satisfied by \(q\) is (6.10) which is of the same form than the one on \(v\). We infer that a similar proof, with the same steps 4-6 give the same estimate on \(q\). This concludes the proof of the bootstrap and the fact that \(T^* = +\infty\). □

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