Quantum expanders and the quantum entropy difference problem

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Abstract
We define quantum expanders in a natural way. We show that under certain conditions classical expander constructions generalize to the quantum setting, and in particular so does the Lubotzky, Philips and Sarnak construction of Ramanujan expanders from Cayley graphs of the group $\mathrm{PGL}(2,q)$. We show that this definition is exactly what is needed for characterizing the complexity of estimating quantum entropies.

1 Introduction

Expanders can be defined either combinatorially or algebraically. In the combinatorial definition a graph $G = (V,E)$ is a $(K_{\text{max}},c)$-expander, if every set $A \subseteq V$ of cardinality at most $K_{\text{max}}$ has at least $c|A|$ neighbors. In the algebraic definition we view $G$ as an operator defined by the normalized adjacency matrix of the graph, and we say $G$ is a $\lambda$-expander if the spectral gap between the first and second largest eigenvalues (in absolute value) of this operator is at least $1 - \lambda$.

We are interested in a sequence of graphs $\{G_n\}$, with an increasing number of vertices, but constant degree $D$. The best possible combinatorial expansion such a family can have is about $D - 2$, and the best possible algebraic expansion is about $\frac{2}{\sqrt{D}}$ (see [Nil91]). The algebraic and combinatorial definitions are closely related. Expanders with constant spectral gap have constant combinatorial expansion and vice versa [AM85, Alo86]. However, this equivalence is not tight, and, in particular, graphs with maximal spectral gap may have combinatorial expansion not more than half the degree [Kah95], and graphs with almost optimal combinatorial expansion (close to the degree) may have non-optimal spectral gap.

Both notions have proven extremely useful in computer science and elsewhere. Often, the spectral gap is used (e.g., whenever a random walk on the expander is used), but sometimes combinatorial expansion close to the degree is needed (e.g., in the error correcting codes of [SS96]).

Thus, for decades, a major goal of computer science has been constructing these marvelous graphs explicitly. Pinsker [Pin73] was the first to observe that non-explicitly, constant degree expanders with very good combinatorial expansion exist. Several explicit constructions of constant degree, algebraic expanders with some constant (bounded away from zero) spectral gap were given, e.g., in [Mar73, GG81, JM87]. Lubotzky, Philips and Sarnak [LPS88] and Margulis [Mar88] gave the first Ramanujan graphs, i.e., a family $\{G_n\}$ of degree $D$ graphs with $\lambda$ approaching the optimal value. All the above graphs are Cayley graphs and their analysis is algebraic. More recently, [RVW00] gave a more combinatorial construction, that was used in [CRWV02] to construct an explicit construction of graphs with almost optimal combinatorial expansion.

We refer the interested reader to the excellent survey paper [HLW06] for a comprehensive treatment of expander graphs, their construction and applications.

1.1 Quantum expanders

Expanders are often thought of as combinatorial objects. In this view, expanders are sparse graphs that have combinatorial expansion properties similar to random graphs. It is difficult to see in this view how to generalize the notion to the quantum world.

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However, most expander constructions, and many of the applications that use expanders, treat expanders as algebraic objects, i.e., the graph $G = (V, E)$ is translated to a linear mapping $\hat{G}$ from some vector space $\mathcal{V}$ to itself. Let us describe how this is done. Say $G = (V, E)$ is a graph. We translate $V$ to a vector space $\mathcal{V}$ of dimension $|V|$, with a basis vector $|v\rangle$ for each $v \in V$. A probability distribution over $V$ then translates to a vector $\sum_v p_v |v\rangle$ in this space, with $0 \leq p_v \leq 1$ and $\sum_v p_v = 1$. The graph $G$ is translated to the linear operator $\hat{G}$ from $\mathcal{V}$ to $\mathcal{V}$ which is defined by the normalized adjacency matrix of $G$. $\hat{G}$ is therefore a linear mapping $\hat{G} : \mathcal{V} \rightarrow \mathcal{V}$ that can be classically implemented, and maps probability distributions to probability distributions.

We extend the algebraic definition to the quantum setting. A general classical state is a classical probability distribution over the standard basis $\{|v\rangle\}$ of $\mathcal{V}$, i.e., vectors of the form $\sum_v p_v |v\rangle$ as above. A general quantum state is a density matrix $\rho = \sum p_v |\psi_v\rangle \langle \psi_v|$, with $0 \leq p_v \leq 1$, $\sum p_v = 1$ and $|\psi_v\rangle$ being some orthonormal basis of $\mathcal{V}$. In the classical world we had a linear operator $G : \mathcal{V} \rightarrow \mathcal{V}$. In the quantum world a feasible quantum state is a matrix over $\mathcal{V}$, i.e., an element of $L(\mathcal{V})$, where $L(\mathcal{V})$ is the set of linear operators (matrices) over $\mathcal{V}$. We look for a linear transformation $E : L(\mathcal{V}) \rightarrow L(\mathcal{V})$. Such a transformation is called a super-operator. We want in addition that $E$ can be implemented by some physical process, and this also ensures that $E$ maps density matrices to density matrices. Such a linear operators $E$ is called in the literature an admissible super-operators.

We now turn to the regularity condition. Any (directed or undirected) $D$-regular graph $G$ can have its edges labeled with 1 to $D$ such that each label $d \in [D]$ defines a permutation mapping. We define:

**Definition 1.** We say an admissible super-operator $E : L(\mathcal{V}) \rightarrow L(\mathcal{V})$ is $D$-regular if $E = \frac{1}{D} \sum_d E_d$, and for each $d \in [D]$, $E_d(X) = U_d X U_d^\dagger$ for some unitary transformation $U_d$ over $\mathcal{V}$.

In fact, for many classical constructions the edge labeling is explicitly described in the construction, and in particular this is always true whenever $G$ is a Cayley graph. This property was also exploited in several constructions (e.g., in [RVW00] [CRVW02]).

Intuitively, a quantum expander is an admissible super-operator $E$ that has a spectral gap. We normalize the operator $E$ so that its largest eigenvalue is 1. As in the classical case we want the eigenvector of eigenvalue one to be the completely mixed state. We require that all other eigenvalues have a much smaller absolute value. In general, however, $E$ need not be normal. This already happens in the classical setting whenever we deal with directed graphs. In such a case we need to replace eigenvalues with singular values. Equivalently, we define:

**Definition 2.** An admissible superoperator $E : L(\mathcal{V}) \rightarrow L(\mathcal{V})$ is a $(D, \overline{\lambda})$ expander if $E$ is $D$-regular and:

- $E(\mathbb{I}) = \mathbb{I}$ and the eigenspace of eigenvalue 1 has dimension 1.
- For any $A \in L(\mathcal{V})$ that is orthogonal to $\mathbb{I}$ (with respect to the Hilbert-Schmidt inner product, i.e. $\text{Tr}(A\mathbb{I}) = 0$) it holds that $\|E(A)\|_2 \leq \overline{\lambda} \|A\|_2$.

A quantum expander is explicit if $E$ can be implemented by a polynomial size circuit.

Equivalently, we could have replaced the second condition with the requirement that all singular values of $T$ other than the largest one (which is 1) are smaller than $\overline{\lambda}$.

### 1.2 Are there any non-trivial quantum expanders?

This is indeed a good question, and a major goal of this paper. A first natural attempt is converting a good classical expander, to a quantum super-operator. This indeed can be done, and the resulting super operator $T : L(\mathcal{V}) \rightarrow L(\mathcal{V})$ is analyzed in Section 2. The analysis there shows that $T$ has $|V|$ eigenspaces, each of dimension $|V|$, with eigenvalues $\overline{\lambda} = (\lambda_1 = 1, \ldots, \lambda_{|V|})$, where $\overline{\lambda}$ is the spectrum of the Cayley graph. In particular, the eigenspace of eigenvalue 1 has dimension $|V|$ instead of dimension 1.
Never the less, Ambainis and Smith obtained the following quantum expander that is implicit in their work:

**Theorem 1.1.** [AS04] There exists an explicit \( (\log^2 N, \lambda) \) quantum expander \( E : L(V) \to L(V) \), where \( N = \dim(V) \).

Their quantum expander is based on the classical Cayley expander over the Abelian group \( \mathbb{Z}_2^n \). As explained before, taking the quantum analogue of the classical expander is not enough, and Ambainis and Smith obtain their result using a clever trick, essentially working over \( \mathbb{F}_4^n \) rather than \( \mathbb{Z}_2^n \).

The main problem with Abelian groups is that it is impossible to get a constant degree Cayley expander over them [Kla84, AR94]. This is reflected in the \( O(\log N) \) term in Theorem 1.1. There are constant degree, Ramanujan Cayley graphs, i.e., Cayley graphs that achieve the best possible relationship between the degree and the spectral gap, but they are built over non-Abelian groups. If one wants to get a constant degree quantum expander, then he is forced to work over non-Abelian groups. Can one get constant degree quantum expanders at all?

Our main construction starts with the constant degree Ramanujan expander of [LPS88]. This expander is a Cayley graph over the non-Abelian group \( \text{PGL}(2,q) \). We prove:

**Theorem 1.2.** There exists a \( (D = O(1), \lambda) \) quantum expander.

Our construction is not explicit in the sense that it uses the Fourier transform over \( \text{PGL}(2,q) \), which is not known to have an efficient implementation (see [LR92] for a non-trivial, but still not fast enough, algorithm).

The \( \text{PGL}(2,q) \) quantum expander is as follows: we take two steps on the classical expander graph, with a basis change between the two steps. The basis change is a carefully chosen refinement of the Fourier transformation that maps the standard basis \( |g\rangle \) to the basis of the irreducible, invariant subspaces of \( \text{PGL}(2,q) \). Intuitively, in the Abelian case this basis change corresponds to dealing with both the bit and the phase levels, and is similar to the construction of quantum error correcting codes by first applying a classical code in the standard basis and then in the Fourier basis. However, this intuition is not as clear in the non-Abelian case. Furthermore, in the non-Abelian case not every Fourier transform is good. In this work we single out a natural algebraic property we need from the underlying group that is sufficient for proving the spectral gap of the construction. We then prove that \( \text{PGL}(2,q) \) respects this property.

We mention that there are also explicit, constant degree (non-Ramanujan) Cayley expanders over \( S_n \) and \( A_n \) [Kas05]. Also, there is an efficient implementation of the Fourier transform over \( S_n \) [Bea97]. We do not know, however, whether \( S_n \) (or \( A_n \)) respect our additional property. We discuss this in more detail in Section A.7.

To summarize, Ambainis and Smith showed that good poly-logarithmic-degree quantum expanders exist, and their construction is explicit. Theorem A.6 shows that good constant degree quantum expanders non-explicitly exist (with a degree that is the square of the degree of a Ramanujan graph). Recently, we showed together with Oded Schwartz [BST07] that one can use Theorem A.6 with a Zig-Zag like construction, to obtain an explicit, constant degree quantum expander.

Finally, we show a lower bound on the best achievable spectral gap of quantum expanders.

**Theorem 1.3.** Any \( (D, \lambda) \) quantum expander satisfies \( \lambda \geq \frac{2}{3\sqrt{3D}} \).

The lower bound differs by a constant from the tight lower bound known on classical expanders.

### 1.3 What are quantum expanders good for?

The first application of quantum expanders was given by Ambainis and Smith themselves. They used these expanders to construct short quantum one-time pads. Loosely speaking, they showed how two parties sharing a random bit string of length \( n + O(\log n) \) can communicate an \( n \) qubit state such that any eavesdropper
cannot learn much about the transmitted state. (A subsequent work by [DN06] showed how to remove the \(O(\log n)\) term.)

In this paper we show another application of quantum expanders. Watrous [Wat02] defined the class of quantum statistical zero knowledge languages (QSZK). QSZK is the class of all languages that have a quantum interactive proof system, along with an efficient simulator that produces transcripts that for inputs in the language are statistically close to the correct ones (for the precise details see [Wat02, Wat06]).

Watrous defined the Quantum State Distinguishability promise problem (QSD\(_{\alpha,\beta}\)):

- **Input:** Quantum circuits \(Q_0, Q_1\).
- **Accept:** If \(\|Q_0\rangle - \|Q_1\rangle\|_{tr} \geq \beta\).
- **Reject:** If \(\|Q_0\rangle - \|Q_1\rangle\|_{tr} \leq \alpha\).

where the notation \(\|Q\rangle\) denotes the mixed state obtained by running the quantum circuit \(Q\) on the initial state \(|0^n\rangle\) and tracing out the non-output qubits \(1\) and \(\|A\|_{tr} = \text{Tr} \|A\|\) is the quantum analogue of the classical \(\ell_1\)-norm (and so in particular \(\|\rho_1 - \rho_2\|_{tr}\) is the quantum analogue of the classical variational distance of two probability distributions).

Watrous showed QSD\(_{\alpha,\beta}\) is complete for honest-verifier-QSZK (QSZK\(_{HV}\)) when \(0 \leq \alpha < \beta^2 \leq 1\). He further showed that QSZK\(_{HV}\) is closed under complement, that any problem in QSZK\(_{HV}\) has a 2 message proof system and a 3 message public-coin proof system and also that QSZK \(\subseteq\) PSPACE. Subsequently, in [Wat06], he showed that QSZK\(_{HV}\) = QSZK.

The above results have classical analogues. However, in the classical setting there is another canonical complete problem, the Entropy Difference problem (ED). There is a natural quantum analogue to ED, the Quantum Entropy Difference problem (QED), that we now define:

- **Input:** Quantum circuits \(Q_0, Q_1\).
- **Accept:** If \(S(\|Q_0\rangle) - S(\|Q_1\rangle) \geq \frac{1}{2}\).
- **Reject:** If \(S(\|Q_1\rangle) - S(\|Q_0\rangle) \geq \frac{1}{2}\).

where \(S(\rho)\) is the Von-Neumann entropy of the mixed state \(\rho\). We show that QED is QSZK-complete. We mention that for this purpose the expanders of Ambainis and Smith given in Theorem 1.1 suffice.

The problem QED is very natural from a physical point of view. For example, a common way of measuring the amount of entanglement between registers \(A\) and \(B\) in a pure state \(\psi\) is by the Von-Neumann entropy of \(\text{Tr}_B(\|\psi\rangle\langle\psi\|)\) [PR97]. Now suppose we are given two circuits \(Q_0\) and \(Q_1\), both acting on the same initial pure-state \(|0^n\rangle\), and we want to know which circuit produces more entanglement between \(A\) and \(B\). Our result shows that this problem is QSZK-complete. This, in particular, shows that the harder problem of estimating the amount of entanglement between two registers in a given pure-state is QSZK-hard.

We believe these two applications are a good indication to the usefulness of this notion. We expect that with time other applications will be found.

Our proof that QED is QSZK-complete uses a quantum variant of classical balanced extractors. We explain this variant in Section 4. We show there that good balanced quantum extractors exist. Surprisingly, we believe that unlike the classical case, unbalanced quantum extractors do not exist.

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1. Here we assume that a quantum circuit also designates a set of output qubits.
2. A density matrix \(\rho\) is positive semi-definite and has trace 1. Therefore its eigenvalues are all non-negative and sum up to 1, and can be thought of as defining a probability distribution. The Von-Neumann entropy of \(\rho\) is the Shannon entropy of the eigenvalues of \(\rho\).
1.4 Summary and organization

In classical computation there is a long line of research studying "conductors": objects that manipulate their source entropy, using few independent random bits. This research resulted in beautiful constructions of expanders and extractors, and an amazing variety of applications. We initiate the study of such "conductors" manipulating the entropy of quantum systems.

On the one hand we show that expander-based constructions generalize to the quantum setting (with effort, and not always, but at least in some important cases). On the other hand, we believe all the huge body of work relating classical extractors, condensers and such that map a huge universe to a much smaller universe, is not likely to have a quantum analogue (see Section 4). We think this study deserves interest at its own right.

We also show two neat applications for quantum expanders. One, that was already given in [AS04] and a new one that we give here: we characterize the complexity of approximating entropies. This proof generalizes classical ideas, together with new technical work that is needed for the quantum setting.

The paper is organized as follows. After the preliminaries (Section 2), we give an intuitive exposition in the main text, with the formal details in the Appendix. A complete treatment is given in Section A in the Appendix. In Section 4 we discuss extractors, and discuss why we believe unbalanced quantum extractors are not useful. The final section is devoted to proving the completeness of QED in QSZK. Here, again, we give an intuitive exposition in the main text, with the formal details in the Appendix.

2 Preliminaries

We first define the classical Renyi entropy. Let $P = (p_1, \ldots, p_m)$ be a classical probability distribution. The Shannon entropy of $P$ is $H(P) = \sum_{i=1}^m p_i \log \frac{1}{p_i}$. The min-entropy of $P$ is $H_\infty(P) = \min_i \lg \frac{1}{p_i}$. The Renyi entropy of $P$ is $H_\alpha(P) = \frac{1}{\alpha - 1} \log \sum (\frac{1}{p_i})^{1-\alpha}$, where $\sum (\frac{1}{p_i})^{1-\alpha}$ is the collision probability of the distribution defined by $\text{Col}(P) = \Pr_{x,y}[x = y]$ when $x, y$ are sampled from $P$.

Now let $\rho \in D(V)$ be a density matrix (where $V$ is a Hilbert space, $L(V)$ is the set of linear operators over $V$ and $D(V)$ is the set of positive semi-definite operators in $L(V)$ with trace 1, i.e., all density matrices over $V$). Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be the set of eigenvalues of $\rho$. Since $\rho$ is positive semi-definite, all these eigenvalues are non-negative. Since $\text{Tr}(\rho^2) = 1$ their sum is 1. Thus we can view $\alpha$ as a classical probability distribution. The von Neumann entropy of $\rho$ is $S(\rho) = H(\alpha)$. The min-entropy of $\rho$ is $H_\infty(\rho) = H_\infty(\alpha)$. The Renyi entropy of $\rho$ is $H_\alpha(\rho) = H_\alpha(\alpha)$. The analogue of the collision probability is simply $\text{Tr}(\rho^2) = \sum_i \alpha_i^2 = \|\rho\|_2^2$. We remark that for any distribution $P$, $H_\infty(P) \leq H_2(P) \leq H(P)$ and $2H_\infty(P) \geq H_2(P)$.

The statistical difference between two classical distributions $P = (p_1, \ldots, p_m)$ and $Q = (q_1, \ldots, q_m)$ is $\text{SD}(P, Q) = \frac{1}{2} \sum_{i=1}^m |p_i - q_i|$, i.e., half the $\ell_1$ norm of $P - Q$. This can be generalized to the quantum world by defining the trace-norm of a matrix $X \in L(V)$ to be $\|X\|_\text{tr} = \text{Tr}(|X|)$, where $|X| = \sqrt{XX^\dagger}$, and defining the trace distance between density matrices $\rho$ and $\sigma$ to be $\frac{1}{2} \|\rho - \sigma\|_\text{tr}$.

3 Quantum expanders from non-Abelian Cayley graphs

As we said before, our quantum expander takes two steps on a Cayley expander (over the group $\text{PGL}(2,q)$) with a basis change between each of the steps, and the basis change is a carefully chosen transformation. In this section we give a bird’s view of the proof. We focus on the ideas, obstacles and solutions, and try to give an informal presentation.

Our starting point is generalizing a single step on a Cayley graph to the quantum setting. We fix an arbitrary (Abelian or non-Abelian) group $G$ of order $N$, and a subset $\Gamma$ of group elements closed under inverse. The Cayley graph associated with $\Gamma$, $C(G, \Gamma)$, is a graph over $N$ vertices, with an edge between $(g_1, g_2)$ iff $g_1 = g_2^\gamma$ for some $\gamma \in \Gamma$. Rather then thinking of the Cayley graph as a graph, we prefer to think
We now define our basic superoperator $T : L(\mathbb{C}[G]) \to L(\mathbb{C}[G])$. The superoperator has a register $R$ of dimension $|\Gamma|$ that is initialized at $|\bar{0}\rangle$. It does the following:

- It first applies Hadamard on register $R$ (getting into the density matrix $\frac{1}{|\Gamma|} \rho \otimes \sum_{\gamma, \gamma' \in \Gamma} |\gamma\rangle\langle \gamma'|$).
- Then, it applies the unitary transformation $Z : |g, \gamma\rangle \to |g \gamma, \gamma\rangle$. This transformation is a permutation over the standard basis, and hence unitary. It is also classically easy to compute in both directions, and therefore has an efficient quantum circuit.
- Finally, it measures register $R$.

Thus we have: $T(\rho) = \text{Tr}\_R[ Z(I \otimes H)(\rho \otimes |\bar{0}\rangle\langle \bar{0}|)(I \otimes H)Z^\dagger ]$. It can be easily checked that over “classical” states (a density matrix $\rho$ that is diagonal in the standard basis) $T$ coincides with $M$. Also, by definition, $T$ is $|\Gamma|$-regular.

The first thing to figure out is the eigenspace structure of the super-operator $T$. This turns out to be as follows. $T$ has $N$ orthogonal eigen-spaces, each of dimension $N$, and the eigenvalues $\lambda_1, \ldots, \lambda_N$ are those of $M$ (the orthogonality is under the inner-product of $L(\mathbb{C}[G])$ defined by $\langle A|B\rangle = \text{Tr}(AB^\dagger)$). In particular, if we start with a good Cayley graph where $\lambda_1 = 1$ and all other eigenvalues have absolute value at most $\overline{\lambda}$, then $T$ has an eigenspace $W_1$ of dimension $N$ with eigenvalue 1, and all other eigenvalues have absolute value at most $\overline{\lambda}$. The fact that the dimension of $W_1$ is larger than 1 is not good for us, because it means that $T$ has no spectral gap.

So, now we take a closer look at $W_1$ and we discover that it is spanned by $\{A_g \mid g \in G\}$ where $A_g = \sum_x |gx\rangle\langle x|$. These operators $A_g$ are what is called the regular representation of $G$. Namely, if we denote $\rho_{\text{reg}}(g) = A_g$, then $\rho_{\text{reg}} : G \to L(\mathbb{C}[G])$ is a group homomorphism (namely, $\rho_{\text{reg}}(g_1 \cdot g_2) = \rho_{\text{reg}}(g_1) \cdot \rho_{\text{reg}}(g_2)$). Furthermore, a basic theorem of representation theory says that there is a basis change under which all the operators $A_g = \rho_{\text{reg}}(g)$ simultaneously block-diagonalize, with the blocks corresponding to the irreducible representations of $G$. This (non-unique) basis change is called the Fourier transform of $G$.

Let us first consider the case where $G$ is Abelian, and let $e$ denote the identity element in $G$. In this case all the irreducible representations of $G$ have dimension one, and the Fourier transform $U$ simultaneously diagonalizes all the operators $A_g = \rho_{\text{reg}}(g)$. The elements $\{A_g = \rho_{\text{reg}}(g)\}$ form an orthonormal basis of $W_1$. Doing the basis change, they all become diagonal, i.e., “classical” states. Furthermore, $A_e = \rho_{\text{reg}}(e) = I$ is mapped to $I$ (as is true in any basis change) and all other basis elements are mapped to orthogonal states (as $U$ is unitary). We therefore expect that applying $T$ again now, is equivalent to applying $M$ on the classical state, and will result in a unique eigenvector of eigenvalue 1, with all other eigenvalues being at most $\overline{\lambda}$.

So our (Abelian) quantum expander is as follows. We let $U$ be the Fourier transform over $G$, and the quantum expander is the superoperator

$$E(\rho) = T(U T(\rho) U^\dagger).$$

A simple check shows that $E$ is indeed a $\overline{\lambda}$-expander, and its spectral gap is the same as that of $G$. Also, clearly, $E$ is $|\Gamma|^2$-regular.

We now turn to the non-Abelian case. Here most irreducible representations have dimension larger than 1, and as a result the basis change does not diagonalize all $A_g = \rho_{\text{reg}}(g)$, but rather just block-diagonalizes them, with blocks corresponding to the irreducible representations. In particular, doing the Fourier transform does not map $A_g = \rho_{\text{reg}}(g)$ to “classical” states. Never the less, this does not necessarily mean that the above approach fails. In fact, it turns out that a sufficient requirement for a good basis change is that for any $g_1 \neq e$ and any $g_2$, it holds that
\[
\text{Tr}(U_{\rho,\text{reg}}(g_1)U^\dagger_{\rho,\text{reg}}(g_2)) = 0.
\] (1)

Intuitively, we can do the analysis separately for elements in \(W_1\) and elements in \(W_1^\perp\) - the space perpendicular to \(W_1\) (this is technically more complicated, see Lemma A.4). Elements in \(W_1^\perp\) are immediately shortened by the first application of \(T\). Elements in \(\text{Span}\{A_g = \rho_{\text{reg}}(g) \mid g \neq e\}\) are kept in place by the first application of \(T\), but are mapped to \(W_1^\perp\) by the basis change, and therefore are shortened by the second application of \(T\). Together, if \(U\) is a good basis change then \(E(\rho) = T(U_T(\rho)U^\dagger)\) is a \(\tilde{\chi}\)-expander.

But does a good basis change always exist?

We consider the dihedral group as an illuminating example. The dihedral group has irreducible representations of dimension 2 (and a few of dimension 1). The dihedral group also has a cardinality two subgroup \(H = \{e, s\}\), where \(s\) is the reflection element. The Fourier transform associates the eigen-spaces of the irreducible representations, to elements of \(G\). Now, imagine that we associate the dimension-2 blocks with cosets of \(H\). A moment of thought reveals that if \(g_2 \not\in H\) then Equation (1) is satisfied! This is because \(A = U\rho_{\text{reg}}(g_1)U^\dagger\) has non-zero elements only on the 2 by 2 blocks, while \(B = \rho_{\text{reg}}(g_2) = \sum_x [g_2x] \langle x|\) has non-zero elements only outside these 2 by 2 blocks, and so the inner product \(\langle A|B\rangle = \text{Tr}(AB^\dagger) = \sum_{i,j} A_{i,j} B_{i,j}\) must be zero.

We need also to consider the case where \(g_2 \in H = \{e, s\}\). If \(g_2 = e\) then \(\text{Tr}(U\rho_{\text{reg}}(g_1)U^\dagger\rho_{\text{reg}}(g_2)) = \text{Tr}(\rho_{\text{reg}}(g_1))\) and the analysis is simple. We are left with the case \(g_2 = s\). Recall that \(\text{Tr}(AB^\dagger) = \sum_{i,j} A_{i,j} B_{i,j}\). We can interpret the expression \(\text{Tr}(U\rho_{\text{reg}}(g_1)U^\dagger\rho_{\text{reg}}(s))\) as the sum of all entries \(i, j\) of \(U\rho_{\text{reg}}(g_1)U^\dagger\) that belong to the set \(P = \{(is, i)\}\). We now use the fact that each irreducible representation appears in the regular representation with multiplicity that equals its dimension. In matrix language this means that for each dimension 2 irreducible representation, there are two corresponding blocks in the decomposition, and the entries in these two blocks can be made identical (see Section A.1 for more background on representation theory). As the blocks correspond to cosets of \(H\), multiplication by \(g_2 = s\) has the same effect in the two cosets. I.e., an entry of one block is in \(P\) and is added to the sum, iff the corresponding entry in the other block is also in \(P\) and is also added to the sum. We can therefore force a zero sum, by forcing one block to be the negative of the other block, which can be done by an easy manipulation of the Fourier transform.

At first, the above solution looks ad hoc, and very specific to the dihedral group. So we try to abstract the ingredients that have been used in the solution.

The Fourier transform is a unitary mapping from the standard basis \(\{|g\}\) of \(\mathbb{C}[G]\), to the Fourier basis. It can be formally defined as follows. Let \(\tilde{G}\) denote the set of all inequivalent irreducible representations of \(G\). For a representation \(\rho\) let \(d_\rho\) denote the dimension of \(\rho\). We define the transformation \(F\) by

\[
F|g\rangle = \sum_{\rho \in \tilde{G}} \sum_{1 \leq i, j \leq d_\rho} \sqrt{\frac{d_\rho}{|G|}} \rho_{i,j}(g) |\rho, i, j\rangle.
\]

It can be checked that \(F\) is unitary and that it indeed block-digaonlizes the regular representations, namely,

\[
F \rho_{\text{reg}}(g) F^\dagger = \sum_{\rho \in \tilde{G}} \sum_{1 \leq j \leq d_\rho} |\rho, j\rangle \langle \rho, j| \otimes \sum_{1 \leq i, i' \leq d_\rho} \rho_{i, i'}(g) |i\rangle \langle i|
\]

I.e., for each \(\rho \in \tilde{G}\) and \(j \leq d_\rho\), we have a \(d_\rho \times d_\rho\) block whose entries are \(\rho(g)\).

\(F\) maps \(\mathbb{C}[G]\) to a vector space of the same dimension that is spanned by \(\{|\rho, i, j\rangle : \rho \in \tilde{G}, 1 \leq i, j \leq d_\rho\}\).

To complete the specification of the Fourier transform we also need to specify a map \(S\) between \(\{|\rho, i, j\rangle\}\)
and \(|g| : g \in G\). In the Abelian case there is a canonical map \(S\) between \(\{\rho, i, j\} : \rho \in \hat{G}, 1 = j = 1\) and \(|g| : g \in G\), because when \(G\) is Abelian \(\hat{G}\) is isomorphic to \(G\). However, when \(G\) is not Abelian things are more complicated. It is always true that \(\sum_{\rho \in \hat{G}} d_{\rho}^2 = |G|\), and so there is always a bijection between \(\{\rho, i, j\}\) and \(|g| : g \in G\). However, it is not known, in general, how to find such a natural bijection.

For example, for the symmetric group \(S_n\) the question takes the following form. We look for bijections \(f\) from pairs \((P, T)\) of standard shapes to \(S_n\) (a shape corresponds to an irreducible representation of \(S_n\), and its dimension is the number of standard shapes of that shape). The question of finding an explicit bijection \(f\) from pairs \((P, T)\) of standard shapes to \(S_n\) is a basic question in the study of the representation theory of \(S_n\). The canonical algorithm doing so is the "Robinson-Schensted" algorithm \cite{Rob38, Sch61} that was extensively studied later on (see \cite{Sag01}, and especially Chapter 3 that is almost completely dedicated to this algorithm).

Looking back at the solution we gave for the dihedral group we see that we can express it as follows. We made sure that a block that corresponds to an irreducible representation is contained in a coset of \(H\), and different copies of the same representation get the same indices within \(S\). Generalizing this further, we see that what we actually used is a mapping \(S : \{\rho, i, j\} \rightarrow G\) that is product, i.e., for every \(\rho \in \hat{G}\), \(S(\rho, i, j) = f_1(i) \cdot f_2(j)\) for some functions \(f_1, f_2 : [d_{\rho}] \times [d_{\rho}] \rightarrow G\) (the functions \(f_1\) and \(f_2\) may be specific to \(\rho\)). In the dihedral group, this amounts to \(f_2\) selecting a coset representative, and \(f_1\) selecting an index inside the coset. But, in fact, any product mapping \(S\) is good.

It is not clear at all that for every group \(G\) such a product mapping exists. It is trivial for Abelian groups, and simple for the dihedral group (using cosets of \(\{e, s\}\) for example). It is not clear what is the situation for \(S_n\) - the Robinson-Schensted is not a product mapping, but using specific information about \(S_n\), for \(n \leq 6\), we found out that a product mapping exists. Never the less, we were able to prove that \(PGL(2, q)\) has a product mapping, using information about its subgroup structure, and its irreducible representations.

Putting these things together, we get a quantum expander \(E(\rho) = T(UT(\rho)U^\dagger)\), with \(T\) being a single quantum step on a the Cayley expander, and \(U\) being a good basis change. \(U\) is obtained by doing the standard Fourier transform \(F\) followed by the a product mapping \(S\), and with adding appropriate phases to the basis vectors, so as different copies of the same irreducible representation cancel out.

Clearly, the above discussion is intuitive, and there are many gaps to fill. This is done in Appendix \(A\) where we repeat everything in a relaxed way and with all the necessary details. In Sec \(A.1\) we give some background on representation theory. Section \(A.2\) analyzes a single quantum step on a Cayley graph and in Section \(A.3\) we analyze the quantum expander over Abelian groups. Section \(A.4\) singles out Property \(I\) as a sufficient condition for a good basis change, and Section \(A.5\) shows that all we need for that is finding a product mapping \(S\). Finally, we prove in Section \(A.6\) that \(PGL(2, q)\) has such a product mapping, completing the correctness proof of our constant degree quantum expander.

4 Quantum extractors

The balanced case. The classical proof that ED is SZK-complete uses balanced extractors. A balanced extractor is a function \(E : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^n\). We say \(E\) is a \((k, \epsilon)\) extractor if for every distribution \(X\) on \(\{0, 1\}^n\) that has \(k\) min-entropy the distribution \(E(X, U_d)\) obtained by sampling \(x \in X\), \(y \in \{0, 1\}^d\) and outputting \(E(x, y)\), is \(\epsilon\)-close to uniform. We now define balanced quantum extractors.

**Definition 3.** Let \(V\) be a Hilbert space of dimension \(N\). A superoperator \(T : L(V) \rightarrow L(V)\) is a \((k, d, \epsilon)\) quantum extractor, if \(T\) is \(2^d\)-regular and for every \(\rho \in D(V)\) with \(H_\infty(\rho) \geq k\) we have \(\| T\rho - \hat{I} \|_{tr} \leq \epsilon\), where \(\hat{I} = \frac{1}{N} I\). We say \(T\) is efficient if \(T\) can be implemented by a polynomial-size quantum circuit.

We mention that if \(T\) is \(2^d\)-regular (and, in particular, if it is a \((k, d, \epsilon)\) quantum extractor) then for any \(\rho \in L(V)\) it holds that \(S(T\rho) \leq S(\rho) + d\), i.e., no matter what, the extractor never adds more than \(d\) entropy.
to any input system.

Classically, balanced extractors are closely related to expanders (e.g., [GW97]). This generalizes to the quantum setting. We prove:

**Lemma 4.1.** If $T : L(V) \to L(V)$ is a $(D = 2^d, \lambda)$ quantum expander, then for every $t > 0$, $T$ is also a $(k = n - t, d, \epsilon)$ quantum extractor with $\epsilon = 2^{t/2} \cdot \lambda$.

We give the easy proof in Section C in the Appendix. In particular, we get an $(n - t, d, \epsilon)$ balanced quantum extractor $T : L(V) \to L(V)$ where $n = \text{dim}(V)$, and $d = 2(t + 2 \log(\frac{1}{\epsilon})) + O(1)$ using Theorem A.6 (or the explicit version given in [BST07]).

We use the last lemma to prove our lower bound on the spectral gap of quantum expanders.

**Theorem 1.3.** Any $(D, \lambda)$ quantum expander satisfies $\lambda \geq \frac{2}{3\sqrt{D}^d}$.

In the classical world a tight bound of about $\frac{2\sqrt{D-1}}{D}$ has been proved [Nil91]. The proof there is both algebraic (using eigenvalues) and combinatorial (using paths in the graph). We do not see how to generalize the combinatorial component of the proof. Instead we give an algebraic proof. The proof idea is to take a density matrix which is uniform on a set of “small size”. Applying the extractor yields a density matrix close to the completely mixed state. Such a matrix must have a high rank. On the other hand, because we started with a low-rank matrix, the resulting density matrix cannot have a too-high rank (since $E$ is $D$-regular). The formal details are given in Section C in the Appendix.

**The unbalanced case.** A natural generalization of Definition 3 is for a superoperator $T : L(V) \to L(W)$ where $V, W$ are Hilbert spaces of dimensions arbitrary dimensions $N$ and $M$. I.e., here we let $W$ be different than $V$, and, in particular, the superoperator $T$ can map a large Hilbert space $V$ to a much smaller Hilbert space $W$. In the classical case this corresponds to hashing a large universe $\{0, 1\}^n$ to a much smaller universe $\{0, 1\}^m$. Indeed, in the classical world highly unbalanced extractors exist with a very short seed length $d$. These (and related objects like dispersers, condensers and unbalanced expanders) have numerous applications. There is also a huge body of work constructing explicitly (most of) these objects. See [CRVW02] for an attempt to put some order in the zoo of definitions, and [Nis96, Sha02] for a survey of applications and constructions.

However, here we see a difference between the classical and the quantum world. In the classical world if $X$ has $k$ entropy, and we add $d$ more uniform bits, then the final output distribution can have at most $k + d$ entropy. If we then “ignore” some of the output bits, we can only decrease the entropy of the output distribution. In particular, if the output distribution has $m$ entropy, then most of it (namely, $m - d$) came from the source $X$. We also had a similar property for balanced quantum extractors: for any input $\rho$ we had $S(T \rho) \leq S(\rho) + d$.

In the unbalanced case, however, we output $m \ll n$ qubits, and so we trace-out (or “ignore”) $d$ qubits. This, by itself, may increase the entropy. For example, a mixed state that is with probability one in some pure-state has entropy zero (it is completely determined). Tracing out $k$ bits of the system, may result in a mixed state having $k$ entropy. If we trace out $n/2$ bits, at least theoretically, it is possible that our extractor starts with a pure state as an input $\rho$ (i.e., $\rho$ has zero entropy) and ends up with $T \rho$ being the completely mixed state. Notice that at most $d$ of this entropy comes from the seed, and the rest comes from the tracing-out. We believe this makes any unbalanced extractor with $m < n/2$ not useful. For example, the property $S(T \rho) \leq S(\rho) + d$ (true for balanced quantum extractors) is crucial for our proof that QED is QSZK-complete. We believe that slightly unbalanced expander constructions (e.g., [Mor95]) can probably be converted to useful, slightly unbalanced quantum extractors.

### 5 The complexity of estimating entropy

In this section we show that the QED problem (as defined in the introduction) is QSZK-complete. We do this by showing that QED reduces to QSD and vice versa, using the already known fact that QSD is QSZK-complete.
Proving $\text{QED} \leq \text{QSD}$ is a bit tricky. We first show a related problem, Quantum Entropy Approximation (QEA), reduces to $\text{QSD}$. QEA is the following promise problem:

| Input: | A quantum circuit $Q$ and a non-negative integer $t$. |
| Accept: | If $S(|Q\rangle) \geq t + \frac{1}{2}$. |
| Reject: | If $S(|Q\rangle) \leq t - \frac{1}{2}$. |

QEA is the problem of comparing the entropy of a given quantum circuit to some known threshold $t$, instead of comparing the entropies of two quantum circuits as in QED. Our proof that $\text{QEA} \leq \text{QSD}$ uses quantum expanders and extractors, and we discuss it next.

We begin with the classical intuition why EA reduces to SD (EA is the same promise problem, but with the input being a classical circuit). We are given a circuit $C$ and we want to distinguish between the cases the distribution it defines has substantially more or less than $t$ entropy. First assume that the distribution is flat, i.e., all elements that have a non-zero probability in the distribution, have equal probability. In such a case we can apply an extractor on the $n$ output bits of $C$, hashing it to about $t$ bits. If the input distribution has high entropy, it also has high min-entropy (because for flat distributions entropy is the same as min-entropy) and therefore the output of the extractor is close to uniform. If, on the other hand, the circuit entropy is less than $t - d - 1$, where $d$ is the extractor seed length, than even after applying the extractor the output distribution has at most $t - 1/2$ entropy, and therefore it must be far away from uniform. We get a reduction to SD.

There are, of course, a few gaps to complete. First, our source is not necessarily flat. This is solved in the classical case by taking many independent copies of the circuit, which makes the output distribution "close" to "nearly-flat". A simple analysis shows that this flattening works also in the quantum setting. Also, we need to amplify the gap we have between entropy $t + 1/2$ and $t - 1/2$ to a gap larger than $d$ (the seed length). This, again, is solved by taking many independent copies of $C$, because $S(C \otimes q) = qS(C)$, and works the same way in the quantum setting.

The interesting question is what is needed in the quantum case from the quantum analogue of classical extractors. As it turns out, what is needed is that sources with high min-entropy are mapped close to the completely mixed state, whereas all sources of low min-entropy are mapped far away from it. The first condition is clearly satisfied by our Definition 3. The second condition is implied by the regularity of the extractor: a $D = 2^d$ regular extractor can never add more than $d$ entropy to a source, and so sources with low min-entropy are mapped to sources with low min-entropy, and such sources (with the right parameters) are far away from uniform. The formal proof is given in Section B.1.

We remark that we believe that exactly this property fails in the unbalanced case, i.e., there are input sources with low min-entropy (e.g., pure states) that are mapped close to the completely mixed state, and this additional entropy is obtained not because of the seed, but rather because we have an unbalanced extractor that traces out registers.

This completes the proof that QEA reduces to $\text{QSD}$. As Watrous showed that $\text{QSD} \leq \text{QSD}$, we get that $\text{QEA} \leq \text{QSD}$. We next show that $\text{QEA} \leq \text{QSD}$ implies $\text{QED} \leq \text{QSD}$ using a standard classical trick. We can express: $\text{QED}(Q_0, Q_1) = \bigvee_{t=1} \left[ ((Q_0, t) \in \text{QEA}_Y) \land ((Q_1, t) \in \text{QEA}_N) \right]$. Thus, if QEA reduces to QSD (as we proved), we can express QED as a formula over QSD. We then take the classical result that any Boolean formula over SD reduces to SD, and generalize it to the quantum setting, concluding that QED reduces to QSD as desired. The full details (and this time just for completeness, because the proof closely follows the classical one) are given in Section B.2. This completes the proof that $\text{QED} \leq \text{QSD}$.

The direction that $\text{QSD} \leq \text{QED}$ follows the classical reduction, but using the Holevo bound from quantum information theory. The details are given in Section B.3. Altogether, we see that QED is QSZK complete.
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A Quantum expanders from non-Abelian Cayley graphs

As we said before, our quantum expander takes two steps on a Cayley expander (over the group $\text{PGL}(2,q)$) with a basis change between each of the steps, and the basis change is a carefully chosen transformation.

First, in Subsection A.2, we define and analyze taking one step on a (Abelian or non-Abelian) Cayley graph. Then, in Subsection A.3, we analyze the Abelian case. We do not use the results of Subsection A.3 for analyzing $\text{PGL}(2,q)$, but nevertheless we recommend reading this section because many of its techniques are later on generalized to the non-Abelian case. Then, we study a general template for constructing quantum expanders over non-Abelian groups with a certain property (Subsections A.4, A.5). Finally, we show that $\text{PGL}(2,q)$ has this required property (Subsection A.6).

We begin with some representation theory background.

A.1 Representation Theory Background

We survey some basic elements of representation theory. For complete accounts, consult the books of Serre [Ser77] or Fulton and Harris [HF91]. The exposition below heavily uses the one given in [HRT00].

A representation $\rho$ of a finite group $G$ is a homomorphism $\rho : G \rightarrow \text{GL}(V)$, where $V$ is a (finite-dimensional) vector space over $\mathbb{C}$ and $\text{GL}(V)$ denotes the group of invertible linear operators on $V$. Fixing a basis for $V$, each $\rho(g)$ may be realized as a $d \times d$ matrix over $\mathbb{C}$, where $d$ is the dimension of $V$. As $\rho$ is a homomorphism, for any $g, h \in G$, $\rho(gh) = \rho(g)\rho(h)$ (this second product being matrix multiplication). The dimension $d_\rho$ of the representation $\rho$ is $d$, the dimension of $V$.

We say that two representations $\rho_1 : G \rightarrow \text{GL}(V)$ and $\rho_2 : G \rightarrow \text{GL}(W)$ of a group $G$ are isomorphic when there is a linear isomorphism of the two vector spaces $\phi : V \rightarrow W$ so that for all $g \in G$, $\phi \rho_1(g) = \rho_2(g)\phi$. In this case, we write $\rho_1 \cong \rho_2$. Up to isomorphism, a finite group has a finite number of irreducible representations; we let $\hat{G}$ denote this collection (of representations).

We say that a subspace $W \subseteq V$ is an invariant subspace of a representation $\rho : G \rightarrow \text{GL}(V)$ if $\rho(g)W \subseteq W$ for all $g \in G$. The zero subspace and the subspace $V$ are always invariant. If nonzero proper subspaces are invariant, the representation is said to be irreducible.

If $\rho : G \rightarrow \text{GL}(V)$ is a representation, $V = V_1 \oplus V_2$ and each $V_i$ is an invariant sub-space of $\rho$, then $\rho(g)$ defines two linear representations $\rho_i : G \rightarrow \text{GL}(V_i)$ such that $\rho(g) = \rho_1(g) + \rho_2(g)$. We then write $\rho = \rho_1 \oplus \rho_2$. Any representation $\rho$ can be written $\rho = \rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_k$, where each $\rho_i$ is irreducible. In particular, there is a basis in which every matrix $\rho(g)$ is block diagonal, the $i$th block corresponding to the $i$th representation in the decomposition. While this decomposition is not, in general, unique, the number of times a given irreducible representation appears in this decomposition (up to isomorphism) depends only on the original representation $\rho$.

A representation $\rho$ of a group $G$ is also automatically a representation of any subgroup $H$. We refer to this restricted representation on $H$ as $\text{Res}_H \rho$. Note that even representations that are irreducible over $G$ may be reducible when restricted to $H$.

The group algebra $\mathbb{C}[G]$ of a group $G$ is a vector space of dimension $|G|$ over $\mathbb{C}$, with an orthonormal basis $\{|g| : g \in G\}$ and multiplication $\sum a_g |g| \cdot \sum b_g' |g'| = \sum_{g,g'} a_g b_{g'} |g \cdot g'|$. The group algebra is isomorphic to the set $\{f : G \rightarrow \mathbb{C}\}$ with the isomorphism being $f \mapsto \sum_g f(g) |g|$. The inner product in $\mathbb{C}[G]$ translates to the familiar inner product $(f,h) = \sum_g f(g) \overline{h(g)}$. The regular representation $\rho_{\text{reg}} : G \rightarrow \text{GL}(\mathbb{C}[G])$ is defined by $\rho_{\text{reg}}(s) : |g| \mapsto |sg|$, for any $g \in G$. Notice that $\rho_{\text{reg}}(s)$ is a permutation matrix for any $s \in G$.

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An interesting fact about the regular representation is that it contains every irreducible representation of $G$. In particular, if $\rho_1, \ldots, \rho_k$ are the irreducible representations of $G$ with dimensions $d_{\rho_1}, \ldots, d_{\rho_k}$, then

$$\rho_{\text{reg}} = d_{\rho_1} \rho_1 \oplus \cdots \oplus d_{\rho_k} \rho_k,$$

so that the regular representation contains each irreducible representation $\rho$ exactly $d_\rho$ times.

The *Fourier transform* over $G$ is a unitary transformation $F$ mapping the standard basis $\{|g\rangle : g \in G\}$ to the basis of the invariant subspaces of $\rho_{\text{reg}}$. That is, for any $g \in G$, the matrix $F \rho_{\text{reg}}(g) F^\dagger$ is a block-diagonal matrix, where each block corresponds to $\rho(g)$ for some irreducible representation $\rho$ of $G$. The Fourier transform is unique, up to a permutation of the blocks and up to a choice of basis for $\rho$ for each irreducible $\rho$.

Let $\hat{G}$ denote the set of all inequivalent irreducible representations of $G$. For a representation $\rho$ let $d_\rho$ denote the dimension of $\rho$. We define a transform $F$ by

$$F|g\rangle = \sum_{\rho \in \hat{G}} \sum_{1 \leq i, j \leq d_\rho} \sqrt{\frac{d_\rho}{|G|}} \rho_{i,j}(g) \langle \rho, i, j | \rho', i', j' \rangle.$$

This transformation is unique up to a choice of a unitary map between $\text{Span} \left\{ |\rho, i, j \rangle : \rho \in \hat{G}, 1 \leq i, j \leq d_\rho \right\}$ and $\text{Span} \left\{ |g\rangle : g \in G \right\}$.

The following analysis shows that $F$ is indeed a Fourier transform, in the sense that it block diagonalizes the regular representations (where each $\{|\rho, i, j \rangle \langle \rho', i', j' | : 1 \leq i, i' \leq d_\rho \}$ corresponds to a block).

$$F \rho_{\text{reg}}(g) F^\dagger = \sum_{x \in G} \sum_{\rho, \rho' \in \hat{G}} \sum_{1 \leq i, j \leq d_\rho; 1 \leq i', j' \leq d_{\rho'}} \frac{\sqrt{d_\rho d_{\rho'}}}{|G|} \rho_{i,j}(g) \rho_{i', j'}(x) \langle \rho, i, j | \rho', i', j' \rangle$$

$$= \sum_{x \in G} \sum_{\rho, \rho' \in \hat{G}} \sum_{1 \leq i, j \leq d_\rho; 1 \leq i', j' \leq d_{\rho'}} \frac{\sqrt{d_\rho d_{\rho'}}}{|G|} \sum_{1 \leq k \leq d_\rho} \rho_{i,k}(g) \rho_{k,j}(x) \rho_{i', j'}(x) \langle \rho, i, j | \rho', i', j' \rangle$$

$$= \sum_{\rho, \rho' \in \hat{G}} \sum_{1 \leq i, j \leq d_\rho; 1 \leq i', j' \leq d_{\rho'}} \rho_{i,k}(g) \delta_{\rho, \rho'} \rho_{i', j'}(x) \langle \rho, i, j | \rho', i', j' \rangle$$

$$= \sum_{\rho, \rho' \in \hat{G}} \sum_{1 \leq i, j \leq d_\rho; 1 \leq i', j' \leq d_{\rho'}} \rho_{i,i'}(g) \langle \rho, i, j | \rho, i', j \rangle$$

In the above analysis we have used the beautiful second orthogonality relation: $\frac{\sqrt{d_\rho d_{\rho'}}}{|G|} \sum_{x \in G} \rho_{i,j}(x) \rho_{i', j'}(x) = \delta_{\rho, \rho'} \delta_{i,i'} \delta_{j,j'}$.

### A.2 A single step on a Cayley graph

We now fix an arbitrary (Abelian or non-Abelian) group $G$ of order $N$, and a subset $\Gamma$ of group elements closed under inverse. The *Cayley graph* associated with $\Gamma, C(G, \Gamma)$, is a graph over $N$ vertices, with an edge between $(g_1, g_2)$ iff $g_1 = g_2 \gamma$ for some $\gamma \in \Gamma$. $C(G, \Gamma)$ is a regular directed graph of degree $|\Gamma|$. Rather then thinking of the Cayley graph as a graph, we prefer to think of it as a linear operator over $\mathbb{C}[G]$.

We associate the graph with the operator that is its normalized adjacency matrix $M$ (the normalization is such that the operator norm is 1). This operator is thus $M = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |x\gamma\rangle \langle x|$.\(^3\)

\(^3\)In our definition the generators act from the right. Sometimes the Cayley graph is defined with left action, i.e., $g_1$ is connected to $g_2$ iff $g_1 = \gamma g_2$. However, note that if we define the invertible linear transformation $F$ that maps the basis vector $|g\rangle$ to the basis
The vectors

\[ \text{Lemma A.1.} \]

Proof. We first notice that $\lambda_i \geq \ldots \geq \lambda_N$ the eigenvalues of $M$ with orthonormal eigenvectors $v_1, \ldots, v_N$ (i.e., $\|v_i\|_2 = 1$). As $M$ is regular, we have $\lambda_1 = 1$ and $\lambda_i \geq 1$.

We now define our basic superoperator $T : L(\mathbb{C}[G]) \to L(\mathbb{C}[G])$. The superoperator has a register $R$ of dimension $|\Gamma|$ that is initialized at $|\emptyset\rangle$. It does the following:

- It first applies Hadamard on register $R$ (getting into the density matrix $\frac{1}{|\Gamma|} \rho \otimes \sum_{\gamma, \gamma'} |\gamma\rangle\langle\gamma'|$).
- Then, it applies the unitary transformation $Z : |g, \gamma\rangle \to |g\gamma, \gamma\rangle$. This transformation is a permutation over the standard basis, and hence unitary. It is also classically easy to compute in both directions, and therefore has an efficient quantum circuit.
- Finally, it measures register $R$.

Thus we have: $T(\rho) = \text{Tr}_R[Z(I \otimes H)(\rho \otimes |\emptyset\rangle\langle\emptyset|)(I \otimes H)Z^\dagger]$.

We begin by identifying the eigenvectors and eigenvalues of $T$. We may think of an eigenvector $v_i \in \mathbb{C}^N$ as an element of $\mathbb{C}[G]$, $|v_i\rangle = \sum_g v_i(g) |g\rangle$. We also define the linear transformation $R : \mathbb{C}[G] \to L(\mathbb{C}[G])$ by $R |g\rangle = |g\rangle\langle g|$. With this notation we define:

\[ \mu_{i,g} = \rho_{\text{reg}}(g)(R |v_i\rangle) = \sum_{x \in G} v_i(x) |gx\rangle\langle x| \]

**Lemma A.1.** The vectors $\{\mu_{i,g} | i = 1, \ldots, N, g \in G\}$ form an orthonormal basis of $L(\mathbb{C}[G])$, and $\mu_{i,g}$ is an eigenvector of $T$ with eigenvalue $\lambda_{i,g} = \lambda_i$.

**Proof.** We first notice that $T(|g_1\rangle\langle g_2|) = \frac{1}{|\Gamma|} \sum_{\gamma_1, \gamma_2} U |g_1, \gamma_1\rangle\langle g_2, \gamma_2| U^\dagger = \frac{1}{|\Gamma|} \sum_{\gamma} |g_1\gamma\rangle\langle g_2\gamma|$.\footnote{We remark that if we think of $T$ as an operator over $\mathbb{C}[G \times G]$ (identifying $|x\rangle\langle y|$ with $|x, y\rangle$) then $T$ itself is a Cayley graph with the set of operators being $\{|\gamma, \gamma\rangle | \gamma \in \Gamma\}$. Furthermore, if we look at $W = \{g, g | g \in G\}$ then $W$ is a subgroup of $G \times G$ and $W$ is invariant under $T$. In general, for every $(g_1, g_2) \in G \times G$, the left coset $(g_1, g_2)W = \{(g_1, g; g_2) | g \in G\}$ is invariant under $T$.}

Now,

\[ T(\mu_{i,g}) = T(\sum_x v_i(x) |gx\rangle\langle x|) = \sum_x v_i(x) T(|gx\rangle\langle x|) = \frac{1}{|\Gamma|} \sum_{x, \gamma} v_i(x) | gx\rangle\langle x | \gamma) = \frac{1}{|\Gamma|} \sum_{x, \gamma} v_i(x) \rho_{\text{reg}}(g) | x\gamma\rangle\langle x | \gamma) = \rho_{\text{reg}}(g) \frac{1}{|\Gamma|} \sum_{x, \gamma} v_i(x) | x\gamma\rangle\langle x | \gamma) = \rho_{\text{reg}}(g) R(\sum_x v_i(x) \frac{1}{|\Gamma|} \sum_{\gamma} | x\gamma\rangle) = \rho_{\text{reg}}(g) R(M | v_i\rangle) = \mu_{i,g} \rho_{\text{reg}}(g) R(|v_i\rangle) = \mu_{i,g}. \]

To see orthonormality notice that for $g_1 \neq g_2$, $\text{Tr}(\mu_{i,g_1}\mu_{i,g_2}^\dagger) = 0$ simply because for all $(k, \ell)$ for at least one of the matrices the $(k, \ell)$ entry is zero. If $g_1 = g_2 = g$ then $\text{Tr}(\mu_{i,g}\mu_{i,g}^\dagger) = |v_i\rangle\langle v_i| = \delta_{i,i'}$. As the number of vectors $\{\mu_{i,g}\} is N^2$ they form an orthonormal basis for $L(\mathbb{C}[G])$. \qed
Given \( v \in \mathbb{C}[G] \) we can decompose it and express it as \( v = v^\parallel + v^\perp \) where \( v^\parallel \in \text{Span} \{ |v_1\rangle \} \) and \( v^\perp \in \text{Span} \{ |v_2\rangle, \ldots, |v_N\rangle \} \). In analogy, for \( A \in L(\mathbb{C}[G]) \) we can decompose it to \( A = A^\parallel + A^\perp \) where \( A^\parallel \in \mu^\parallel = \text{Span} \{ \mu_{1.g} | g \in G \} \) and \( A^\perp \in \mu^\perp = \text{Span} \{ \mu_{i.g} | i \neq 1, g \in G \} \). Notice that \( T \) has eigenvalue \( \lambda_i \) on \( \mu_{i.g} \) and so in particular has eigenvalue \( 1 = \lambda_1 \) on \( \mu^\parallel \). Also, let us denote \( \overline{\lambda} = \max_{i \neq 1} |\lambda_i| \). We have:

Claim A.1. For any \( A \in \mu^\perp \), \( \| T(A) \|^2 \leq \overline{\lambda}^2 \| A \|^2 \).

Proof. Express \( A = \sum_{i \neq 1, g} \beta_{i.g} \mu_{i.g} \). Then \( \| A \|^2 = \sum_{i \neq 1, g} |\beta_{i.g}|^2 \) and \( T(A) = \sum_{i \neq 1, g} \beta_{i.g} \lambda_i \mu_{i.g} \). In particular, \( \| T(A) \|^2 = \sum_{i \neq 1, g} |\beta_{i.g}|^2 |\lambda_i|^2 \leq \overline{\lambda}^2 \| A \|^2 \).

### A.3 The Abelian Expander

In this section we describe a quantum expander based on a Cayley graph of an Abelian group, \( G \). When \( G \) is Abelian, all the irreducible representations are of dimension 1 and these are the group characters.\(^5\) There are exactly \( N \) different characters, and we can associate each character \( \chi \) with the norm one vector \( |\chi_g\rangle = \frac{1}{\sqrt{N}} \sum_x \chi(x) |x\rangle \) in \( \mathbb{C}[G] \). The eigenvectors of the Cayley graph are exactly the set of characters \( |v_\lambda\rangle = |\chi_g\rangle \).

We now describe the quantum expander. We let \( U \) be the Fourier transform over \( G \), i.e., the unitary transformation mapping \( |g\rangle \) to \( |\chi_g\rangle \). Our expander is the superoperator \( E(\rho) = T(U\rho U^\dagger) \).

We claim:

Claim A.2. \( U \mu_{g,i} U^\dagger = \chi_i(g^{-1}) \cdot \mu_{i.g^{-1}} \).

Proof.

\[
U \mu_{g,i} U^\dagger = U \rho_{\text{reg}}(i) R |\chi_g\rangle U^\dagger = \frac{1}{\sqrt{N}} \sum_x \chi_g(x) U |ix\rangle \langle x| U^\dagger \\
= \frac{1}{\sqrt{N}} \sum_x \chi_g(x) |\chi_{ix}\rangle \langle \chi_x| \\
= \frac{1}{N \sqrt{N}} \sum_{x,y,y'} \chi_g(x) \chi_{ix}(y) \overline{\chi_x(y')} |y\rangle \langle y'| \\
= \frac{1}{\sqrt{N}} \sum_{y,y'} \chi_i(y) \left( \frac{1}{N} \sum_x \chi_x(g'y'^{-1}) \right) |y\rangle \langle y'| \\
= \frac{1}{\sqrt{N}} \sum_{y,y'} \chi_i(g^{-1}y') |g^{-1}y'\rangle \langle y'| \\
= \chi_i(g^{-1}) \cdot \rho_{\text{reg}}(g^{-1}) R |\chi_i\rangle = \chi_i(g^{-1}) \cdot \mu_{i.g^{-1}} \]

We claim:

Lemma A.2. \( E \) is a \( (|\Gamma|^2, \overline{\lambda}) \) quantum expander.

Proof. The regularity is clear from the way the superoperator \( E \) is defined. We turn to the spectral gap. It is easy to check that \( E(\tilde{I}) = \tilde{I} \). Furthermore, fix any \( x \in L(\mathbb{C}[G]) \) that is perpendicular to \( \tilde{I} \). Write

\(^5\)A character is a homomorphism from \( G \) to \( \mathbb{C} \), i.e., a function \( \chi : G \rightarrow \mathbb{C} \) such that \( \chi(g_1 g_2) = \chi(g_1) \chi(g_2) \).
that if Lemma A.3.

\[ x = x^\parallel + x^\perp \] where \( x^\parallel \in W = \text{Span} \{ \mu_{1,g} \mid 1 \neq g \in G \} \) and \( x^\perp \in \mu^\perp \). Given Claim A.2 one can verify that \( E(x^\parallel) \perp E(x^\perp) \). In particular

\[
\|E(x)\|_2^2 = \|E(x^\parallel)\|_2^2 + \|E(x^\perp)\|_2^2 \\
\leq \|T(Ux^\parallel U^\dagger)\|_2^2 + \|T(x^\perp)\|_2^2 \\
\leq \lambda^2 \|x^\parallel\|_2^2 + \lambda^2 \|x^\perp\|_2^2 = \lambda^2 \|x\|_2^2.
\]

The first inequality is due to the fact that \( T \) has eigenvalue 1 on \( x^\parallel \) and both \( T \) and \( U \) have operator norm at most 1. The second inequality is by Claims A.2 and A.1.

\[ x^\parallel, x^\perp \]

A.4 Template for a quantum expander over a general group

In this subsection we show how to construct a quantum expander over any group \( G \) that possess some general property. We later show that the PGL\((2,q)\) group possesses this property.

Similar to the Abelian case, the expander will be of the form

\[
E(\rho) = T(U\rho U^\dagger),
\]

where \( U \) will be the Fourier transform over \( G \). Unlike the Abelian case, in the non-Abelian case \( G \) has many representations of dimension greater than 1. Thus, a significant part of describing \( U \) will be to describe the basis for each one of the \( \rho_{\text{reg}} \)-invariant subspaces. The property that we need from the unitary transformation \( U \) is:

**Definition 4.** We say \( U \) is a good basis change if for any \( g_1 \neq e \) (where \( e \) denotes the identity element) it holds that

\[
\text{Tr}(U\rho_{\text{reg}}(g_1)U^\dagger \rho_{\text{reg}}(g_2)) = 0. \tag{2}
\]

The intuition behind this choice is as follows. As before, let \( W = \text{Span} \{ \rho_{\text{reg}}(g) : g \neq e \in G \} \) be the set of eigenvectors of \( T \) with eigenvalue 1 (besides the identity). Since each of these eigenvectors was not shrunk by \( T \) in the first step, it is necessary to move them into a perpendicular subspace, such that the second step will shrink them. If \( U \) is a good basis change this indeed happens as captured in:

**Claim A.3.** If \( \rho \in W \) and \( U \) is a good basis change then \( U\rho U^\dagger \perp \mu^\parallel \) (i.e. \( U\rho U^\dagger \in \mu^\perp \)).

**Proof.** \( \{ \rho_{\text{reg}}(g) : g \in G \} \) is an orthonormal basis for \( \mu^\parallel \). \( \{ \rho_{\text{reg}}(g) : g \neq e \in G \} \) is an orthonormal basis for \( W \). Therefore, it is enough to verify that \( \text{Tr}(U\rho_{\text{reg}}(g_1)U^\dagger \rho_{\text{reg}}(g_2)^\dagger) = 0 \) for any \( g_1 \neq e \) and for any \( g_2 \). Since \( \rho_{\text{reg}}(g_2)^\dagger = \rho_{\text{reg}}(g_2^{-1}) \), this follows directly from Property (2).

\[ g_1, g_2 \]

We claim:

**Lemma A.3.** If \( U \) is a good basis change then \( E \) is a \( ([\Gamma]^2, \lambda) \) quantum expander.

**Proof.** The regularity is clear from the way the superoperator \( E \) is defined. We turn to the spectral gap. It is easy to check that \( E(\bar{I}) = \bar{I} \). Furthermore, fix any \( x \in L(\mathbb{C}[G]) \) that is perpendicular to \( \bar{I} \). Write \( x = x^\parallel + x^\perp \) where \( x^\parallel \in W = \text{Span} \{ \mu_{1,g} \mid e \neq g \in G \} \) and \( x^\perp \in \mu^\perp \). Now it is not true any more that \( E(x^\parallel) \perp E(x^\perp) \). However, \( E(x) = T(\sigma^\parallel + \sigma^\perp) \), where \( \sigma^\parallel = UT(x^\parallel)U^\dagger \) and \( \sigma^\perp = UT(x^\perp)U^\dagger \). We know a few things. First, by Claim A.3 \( \sigma^\parallel \perp \mu^\parallel \). Also, \( T(x^\parallel) \perp T(x^\perp) \), and therefore \( \sigma^\parallel \perp \sigma^\perp \). Finally, by Lemma A.1 we know \( T \) is normal. We soon prove:
Lemma A.4. Let $T$ be a normal linear operator with eigen-spaces $V_1, \ldots, V_n$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ in descending absolute value. Suppose $u$ and $w$ are vectors such that $u \in \text{Span}\{V_2, \ldots, V_n\}$ and $w \perp u$ ($w$ does not necessarily belong to $V_1$). Then

$$||(T(u + w))||_2^2 \leq |\lambda_2|^2 ||u||_2^2 + |\lambda_1|^2 ||w||_2^2.$$ 

Using the lemma we see that:

$$||E(x)||_2^2 = ||T(\sigma^{\perp})||_2^2 \leq \lambda^2 ||\sigma^{\perp}||_2^2 + ||\sigma^{\perp}||_2^2 = \lambda^2 ||UT(x^{\perp})U^\dagger||_2^2 + ||UT(x^{\perp})U^\dagger||_2^2 = \lambda^2 ||(T(x^{\perp}))||_2^2 \leq \lambda^2 ||x||_2^2 = \lambda^2 ||x||_2^2.$$

We are left with the proof of Lemma A.4.

Proof. (Of Lemma A.4) Let $\{v_j\}$ be an eigenvector basis for $T$ with eigenvalues $\delta_j$ (from the set $\{\lambda_1, \ldots, \lambda_n\}$). Writing $u = \sum_j \alpha_j v_j$ and $w = \beta v + \sum_j \beta_j v_j$ with $v_j \in \text{Span}\{V_2, \ldots, V_n\}$ and $v \in V_1$, we get:

$$||T(u + v)||_2^2 = ||\lambda_1 \beta v + \sum_j \delta_j (\alpha_j + \beta_j) v_j||_2^2 = |\lambda_1|^2 |\beta|^2 + \sum_j |\delta_j|^2 |\alpha_j + \beta_j|^2 \leq |\lambda_1|^2 |\beta|^2 + |\lambda_2|^2 \sum_j |\alpha_j + \beta_j|^2 = |\lambda_1|^2 |\beta|^2 + |\lambda_2|^2 (\sum_j |\alpha_j|^2 + \sum_j |\beta_j|^2 + \sum_j (\alpha_j^* \beta_j + \alpha_j \beta_j^*)) = |\lambda_1|^2 |\beta|^2 + |\lambda_2|^2 (\sum_j |\alpha_j|^2 + \sum_j |\beta_j|^2) \leq |\lambda_1|^2 (|\beta|^2 + \sum_j |\beta_j|^2) + |\lambda_2|^2 \sum_j |\alpha_j|^2 = |\lambda_2|^2 ||u||_2^2 + |\lambda_1|^2 ||w||_2^2.$$

where in the calculation we used the fact that $\sum_j \alpha_j^* \beta_j = \langle u | v \rangle = 0$ because of the orthogonality of $u$ and $w$. 

A.5 A sufficient condition that guarantees a good basis change

Definition 5. Let $f$ be a bijection from $\{(\rho, i, j) \mid \rho \in \hat{G}, 1 \leq i, j \leq d_{\rho}\}$ to $G$. We say that $f$ is product if for every $\rho \in \hat{G}$:

$$f(\rho, i, j) = f_1(i) \cdot f_2(j) \quad (3)$$

for some functions $f_1, f_2 : [d_{\rho}] \times [d_{\rho}] \to G$ ($f_1$ and $f_2$ may depend on $\rho$).
We first give two examples.

**Example A.1.** (Abelian groups). All irreducible representations are of dimension one, so just define \( f_1(i) = e \) and \( f_2(j) = f(\rho, 1, 1) \).

**Example A.2.** (The Dihedral group) The Dihedral group \( D_m \) is the group of rotations and reflections of a regular polygon with \( m \) sides. Its generators are \( r \), the rotation element, and \( s \), the reflection element. This group has \( 2m \) elements and the defining relations are \( s^2 = 1 \) and \( sr = r^{-1} \). We shall argue this group has a product mapping for odd \( m \) (although it is true for even \( m \) as well). The Dihedral group has \( \frac{m-1}{2} \) representations \( \rho \) of dimension two and two representations of dimension one \( \tau_1, \tau_2 \).

Our product mapping \( f(\rho, i, j) \) is:

\[
f(\rho, i, j) = \begin{cases} 
1 & \text{if } \rho = \tau_1, i = j = 1 \\
s & \text{if } \rho = \tau_2, i = j = 1 \\
\frac{1}{r^2(\ell-1)+i}s^j & \text{if } \rho = \rho \ell 
\end{cases}
\]

(4)

The product structure is clear from Equation (4).

Our claim is that any group that has a product mapping can be used to construct quantum expanders. The parameters of the expander depend on the parameters of the classical Cayley graph given by the group. Optimally, we will want a group that has:

- A constant degree Cayley expander.
- A product mapping.
- An efficient quantum Fourier transform.

Abelian groups have the last two. In the next section we will show that \( \text{PGL}(2, q) \) has the first two (it is an open problem to find an efficient implementation of the quantum Fourier transform over \( \text{PGL}(2, q) \)).

**Lemma A.5.** Let \( G \) be a group that has a product mapping \( f \), and let \( F \) be the Fourier transform over \( G \), \( F |g\rangle = \sum_{\rho \in \hat{G}} \sum_{1 \leq i, j \leq d_{\rho}} \sqrt{d_{\rho}} \rho_{i,j}(g) |\rho, i, j\rangle \). Define the unitary mapping

\[
S : |\rho, i, j\rangle \mapsto \omega_{d_{\rho}}^{ij} |f(\rho, i, j)\rangle
\]

where \( \omega_{d_{\rho}} = e^{2\pi i/d_{\rho}} \), and set \( U \) to be the unitary transformation \( U = SF \). Then \( U \) has property (2) and is a good basis change.

**Proof.**

\[
\text{Tr} \left( U \rho_{\text{reg}}(g_1) U^\dagger \rho_{\text{reg}}(g_2) \right) = \text{Tr} \left( SF \rho_{\text{reg}}(g_1) F^\dagger S^\dagger \rho_{\text{reg}}(g_2) \right) \\
= \text{Tr} \left( S \sum_{\rho \in \hat{G}} \sum_{1 \leq i, j \leq d_{\rho}} \rho_{i,j}(g_1) |\rho, i, j\rangle \langle \rho, i', j'| \sum_{x} |g_{2x}\rangle \langle x| \right) \\
= \sum_{\rho \in \hat{G}} \sum_{1 \leq i, i' \leq d_{\rho}} \rho_{i,i'}(g_1) \text{Tr} \left( \sum_{j=1}^{d_{\rho}} S |\rho, i, j\rangle \langle \rho, i', j'| \sum_{x} |g_{2x}\rangle \langle x| \right).
\]

Therefore, it suffices to show that for any \( \rho, i, i' \) we have \( \text{Tr} \left( \sum_{j=1}^{d_{\rho}} S |\rho, i, j\rangle \langle \rho, i', j'| \sum_{x} |g_{2x}\rangle \langle x| \right) = 0 \). Fix \( \rho \in \hat{G} \) and \( i, i' \in \{1, \ldots, d_{\rho} \} \). Since \( f \) is product, \( f(\rho, i, j) = f_1(i) \cdot f_2(j) \) for some \( f_1, f_2 : [d_{\rho}] \times [d_{\rho}] \to G \). Denote \( h_i = f_1(i) \) and \( t_j = f_2(j) \). The sum we need to calculate can be written as
\[
\text{Tr} \left( \sum_{j=1}^{d_p} S \rho(i,j) \langle \rho(i,j) | g_2 x \rangle \langle x | \right) = \sum_{j=1}^{d_p} \sum_{x} \omega^i_j \text{Tr} \langle h_i t_j | h_i t_j | g_2 x \rangle \langle x | \)
\]

where the last equality is because we get a non-zero value iff \(x = h_i t_j\) and \(h_i t_j = g_2 x\), which happens iff \(h_i t_j = g_2^{-1} h_i t_j\), i.e., \(g_2 = h'_i h_i^{-1}\). However, when \(g_2 = h'_i h_i^{-1}\) we get the sum \(\sum_{j=1}^{d_p} \omega^i_j\). This expression itself is zero when \(i \neq i'\).

We are therefore left with the case \(i = i'\). In this case \(g_2 = h'_i h_i^{-1} = e\). But then,

\[
\text{Tr} \left( U \rho_{\text{reg}}(g_1) U^\dagger \rho_{\text{reg}}(g_2) \right) = \text{Tr} \left( U \rho_{\text{reg}}(g_1) U^\dagger \right) = \text{Tr} \left( \rho_{\text{reg}}(g_1) \right) = 0,
\]

where the last equality follows because \(g_1 \neq e\).

### A.6 The construction of the PGL(2,q) quantum expander

We now work with the group \(G = \text{PGL}(2, q)\) of all \(2 \times 2\) invertible matrices over \(\mathbb{F}_q\) modulo the group center (the set of scalar matrices). This is one of the groups used by [LPS88] to construct Ramanujan expander graphs. Our goal is to show that PGL(2, q) has a product mapping. We therefore need to show a product bijection between \(G\) and the irreducible representations of \(G\). How can we find such a bijection?

We first describe the well known irreducible representations of this group. These are:

- \(2^{-1/2}\) representations of dimension \(q + 1\).
- \(2^{-1/2}\) representations of dimension \(q - 1\).
- 2 representations of dimension \(q\).
- 2 representations of dimension 1.

We need a clean bijection from \(G\) to the irreducible representations of \(G\). Our approach is to use a tower of subgroups, \(G_3 = G > G_2 = D_{2q} > G_1 = \mathbb{Z}_q > G_0 = \{e\}\), with \(G_2\) and \(G_1\) defined as follows. \(G_2\) is generated by the equivalence classes of \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). \(G_2\) is a Dihedral subgroup of \(G\) with \(2q\) elements. The first matrix is the reflection, denoted by \(s\), and the second is the rotation, denoted by \(r\). This group has a cyclic subgroup \(G_1 = \mathbb{Z}_q\) (the group generated by \(r\)).

In Figure 1 we show the product mapping visually. The figure shows the block-diagonal structure of the regular representation (after applying the Fourier transform). Each rectangle is an irreducible representation. Each color represents a different dimension: black rectangles correspond to irreducible representations of dimension \(q\), gray rectangles correspond to irreducible representations of dimension \(q - 1\) and dotted rectangles correspond to irreducible representations of dimension \(q + 1\). Notice that all rectangles fit into larger block diagonal rectangles of dimension \(2q\), marked with dashed lines. These larger rectangles correspond to cosets of \(G_2\). It is straightforward to verify that for any \(q + 1\) dimensional representation (dotted rectangles in the figure), the product condition is satisfied by \(G_2\), by letting \(f_1(i)\) determine the index in the coset, and
\[ (q+1)(q-3)/2 \times 2q \]

\[ (q-1) \times 2q \]

Figure 1: The product mapping of PGL(2,q).

\[ f(z_j) \text{ determine the coset representative. Similarly, for any other representation (black and gray rectangles in the figure) the product condition is satisfied by } G_1. \]

Formally, our product mapping \( f \) is defined as follows. Let \( \ell = \frac{(q-1)(q+1)}{2} \) and let \( T_2 = \{ t_1, \ldots, t_\ell \} \) be a transversal for \( G_2 \) (its size comes from the fact that \( |G| = q(q-1)(q+1) = 2q\ell \)). \( T_1 = \{ t_1, st_1, \ldots, t_\ell, st_\ell \} \) is a transversal for \( G_1 \). We denote by \( \rho_x^2 \) the \( x \)th representation of dimension \( d \) (these are all non-equivalent irreducible representations).

\[
\begin{align*}
  f(\rho_x^1, 1, 1) &= st_{(x-1)(q-3)/2(q+1)} \\
  f(\rho_x^{q-1}, i, j) &= r^{i} st_{(x-1)(q-1)+j} \\
  f(\rho_x^{q}, i, j) &= \begin{cases} 
    r^{i-1} t_{(x-1)(q+1)+j+\frac{(q-3)(q+1)}{2}} & (x-1)q + j \leq q + 1 \\
    r^{i-1} st_{(x-1)q+j-q+1+\frac{(q-3)(q+1)}{2}} & \text{otherwise}
  \end{cases} \\
  f(\rho_x^{q+1}, i, j) &= \begin{cases} 
    r^{i-1} t_{(x-1)(q+1)+j} & i \leq q \\
    st_{(x-1)(q+1)+j} & i = q + 1
  \end{cases}
\end{align*}
\]

As we see, for every \( \rho \in \hat{G} \), \( f \) has a product structure.

We remark that the two previous examples of product mappings (Examples A.1 and A.2) have also this subgroup structure. In the Abelian case (Example A.1) we use \( G < G_0 = \{ e \} \) and in the dihedral case \( G = D_{2n} < G_1 = Z_n < G_0 = \{ e \} \) (or alternatively, \( G = D_{2n} < G_1 = \{ e, s \} < G_0 = \{ e \} \)).

We are now ready to prove Theorem A.6

**Theorem A.6** There exists a \( (D = O(\frac{1}{\lambda}), \lambda) \) quantum expander.
Proof. By Lemma A.2 and the description of the product mapping above, we know that $E$ is a $(|F|^2, \lambda)$ quantum expander. By the [LPS88] construction we know that there exists a Cayley graph for $\text{PGL}(2,q)$ with $\lambda^2 \leq \frac{4}{|\Gamma|}$. Plugging this Cayley graph gives us a $(\frac{16}{\lambda^2}, \lambda)$ quantum expander. \hfill \QED

A.7 How about an $S_n$ Cayley construction?

There are explicit, constant degree (non-Ramanujan) Cayley expanders over $S_n$ [Kas05]. Also, there is an efficient implementation of the Fourier transform over $S_n$ [Bea97]. We do not know, however, whether $S_n$ has product mappings. In $S_n$ the question takes the following form. We look for bijections $f$ from pairs $(P,T)$ of standard shapes to $S_n$ (a shape corresponds to an irreducible representation of $S_n$, and its dimension is the number standard shapes of that shape), and furthermore we want $f(P,T)$ to equal $f_1(P) \cdot f_2(T)$ for some functions $f_1$ and $f_2$ taking values in $S_n$ (this is the product property).

The question of finding an explicit bijection $f$ from pairs $(P,T)$ of standard shapes to $S_n$ is a basic question in the study of the representation theory of $S_n$. The canonical algorithm doing so is the “Robinson-Schensted” algorithm [Rob38, Sch61] that was extensively studied later on (see [Sag01], and especially Chapter 3 that is almost dedicated to this algorithm). The R-S mapping is not product. However, a manual check revealed that $S_n$ has a product mapping for $n \leq 6$. We think it is a natural question whether product mappings for $S_n$ exist, and if so it is natural to look for an explicit description of the mapping (preferably by an algorithm).

B The complexity of estimating entropy

In this section we show that the QED problem (as defined in the introduction) is QSZK-complete, following the outline in Section 5. We prove that QEA $\leq$ QSD in Section B.1 and that QEA $\leq$ QSD implies QED $\leq$ QSD in Section B.2. This completes the proof that QED $\leq$ QSD. We prove that QSD $\leq$ QED in Section B.3.

Before we begin we need a few things. First we need a well known fact about the trace-norm. In the classical world $SD(P,Q) = \max_S P(S) - Q(S)$, i.e., it describes the maximal probability with which one can distinguish the two distributions. The trace distance achieves the same for density matrices, as is captured in:

Fact B.1. (e.g., [NC00]) Let $\rho_0$ and $\rho_1$ be two density matrices. Then there exists a measurement $\mathcal{O}$ with outcome 0 or 1 such that making the measurement on $\rho_0$ yields the bit $b$ with probability $\frac{1}{2} + \frac{\|\rho_0 - \rho_1\|_{tr}}{2}$. Furthermore, no measurement can distinguish the two density matrices better.

As with classical distributions, the distance between density matrices can only decrease with computation, i.e.,

Fact B.2. ([NC00]) Let $\rho_0$ and $\rho_1$ be two density matrices. Then for any quantum operation $\mathcal{E}$ it holds that $\|\mathcal{E}(\rho_0) - \mathcal{E}(\rho_1)\|_{tr} \leq \|\rho_0 - \rho_1\|_{tr}$.

We also need the polarization lemma [Wat02] (that is based on the work of [SV97]), which is used throughout the section.

Theorem B.1. (Polarization lemma, Theorem 5 at [Wat02]) Let $\alpha$ and $\beta$ satisfy $0 \leq \alpha < \beta^2 \leq 1$. Then there is a deterministic polynomial-time procedure that, on input $(Q_0, Q_1, 1^n)$ where $Q_0$ and $Q_1$ are quantum circuits, outputs descriptions of quantum circuits $(R_0, R_1)$ (each having size polynomial in $n$ and in the size of $Q_0$ and $Q_1$) such that

\[ \| |Q_0\rangle - |Q_1\rangle\|_{tr} \leq \alpha \implies \| |R_0\rangle - |R_1\rangle\|_{tr} \leq 2^{-n}, \]
\[ \| |Q_0\rangle - |Q_1\rangle\|_{tr} \geq \beta \implies \| |R_0\rangle - |R_1\rangle\|_{tr} \geq 1 - 2^{-n}. \]
B.1 QEA ≤ QSD

In Section 5 we gave an intuitive explanation of what follows. We first prove the quantum version of the flattening lemma (Lemma B.1), then describe the reduction, and argue that if the input source had much entropy we are close to uniform, whereas if the input source had few entropy, then applying the extractor does not add much entropy, and that this implies that the output state is far away from the completely mixed state (Lemma B.3).

Definition 6. Let \( \rho \) be a density matrix, \( \lambda \) an eigenvalue of \( \rho \) and \( \Delta \) a positive number. We say that \( \lambda \) is \( \Delta \)-typical if \( 2^{-S(\rho)-\Delta} \leq \lambda \leq 2^{-S(\rho)+\Delta} \). We say \( \rho \) is \( \Delta \)-flat if for every \( t > 0 \), with probability \( \geq 1 - 2^{-t^2 + 1} \), a measurement of \( \rho \) in its eigenvector basis results with an eigenvector whose eigenvalue is \( t \Delta \)-typical.

Lemma B.1. Let \( \rho \) be a density matrix and \( k \) a positive integer. Suppose that every non-zero eigenvalue of \( \rho \) is at least \( 2^{-m} \). Then \( \otimes^k \rho \) is \( \Delta \)-flat for \( \Delta = \sqrt{km} \).

**Proof.** Let \( \{\lambda_1, \ldots, \lambda_n\} \) denote the set of eigenvalues of \( \rho \). This implies the eigenvalues of \( \otimes^k \rho \) are \( \{\lambda_1, \ldots, \lambda_n\} \). The entropy of \( \otimes^k \rho \) is \( S(\otimes^k \rho) = k \cdot S(\rho) \). Let \( A \) denote the set of \( t \Delta \)-typical eigenvalues of \( \otimes^k \rho \). Thus \( A = \left\{ \lambda_i : \sum_{j=0}^k - \log \lambda_i - k \cdot S(\rho) \leq t \Delta \right\} \). Let \( p \) denote the probability that a measurement of \( \otimes^k \rho \) in its eigenvector basis results an eigenvalue which is not \( t \Delta \)-typical. Then by Hoeffding inequality,

\[
p \leq \sum_{x \notin A} x \leq 2 \exp \left( \frac{-2 \cdot k \cdot (t \Delta / k)^2}{m^2} \right) \leq 2 \exp(-2t^2) \leq 2^{-t^2 + 1}.
\]

We now define the reduction. Let \( (Q, t) \) be an input to QEA, where \( Q \) is a quantum circuit with \( n \) input qubits and \( m \) output qubits. We first look at the circuit \( Q^{\otimes q} \) (for some \( q = \text{poly}(n) \) to be specified later). We let \( E \) be a \((qt, q(m - t) + 2 \log(\frac{1}{\epsilon}) + \log(qm) + O(1), \epsilon)\) quantum extractor operating on \( qm \) qubits, where \( \epsilon = 1/\text{poly}(n) \) will be fixed later. Such an extractor exists by Lemma 4.1. Let \( \xi = E(Q^{\otimes q}) \) and let \( \tilde{I} = 2^{-qm} I \). The output of the reduction is \( (\xi, \tilde{I}) \).

To show correctness we prove:

**Lemma B.2.**

- *If \( (Q, t) \in \text{QEA}_Y \) then \( \|\xi - \tilde{I}\|_{tr} \leq 5\epsilon \).*

- *If \( (Q, t) \in \text{QEA}_N \) then \( \|\xi - \tilde{I}\|_{tr} \geq \frac{1}{qm} - \frac{1}{2qm} \).*

**Proof.**

Proof of the first item.

Since \( Q \) traces out at most \( n \) qubits, the eigenvalues of \( |Q\rangle \) are all at least \( 2^{-n} \), and by Lemma B.1 we see that \( |Q\rangle^{\otimes q} \) is \( \Delta \)-flat for \( \Delta = \sqrt{qm} \). Thus, with probability at least \( 1 - 2^{-r^2 + 1} \), a measurement of \( |Q\rangle^{\otimes q} \) in its eigenvector basis results with an eigenvector whose eigenvalue is \( r \Delta \)-typical. Let \( \Lambda \) denote the set of \( r \Delta \)-typical eigenvalues of \( |Q\rangle^{\otimes q} \), for \( r = \sqrt{\log(\frac{1}{\epsilon})} \). We write \( |Q\rangle^{\otimes q} \) in its eigenvector basis \( |Q\rangle^{\otimes q} = \sum_i \lambda_i |v_i\rangle \langle v_i| \). Let \( \sigma_0 = \sum_{\lambda_i \in \Lambda} \lambda_i |v_i\rangle \langle v_i| \), and let \( \sigma_1 = \rho^{\otimes q} - \sigma_0 \). Thus, \( \text{Tr}(\sigma_0) \geq 1 - 2^{-r^2 + 1} \). Therefore,

\[
\|\xi - \tilde{I}\|_{tr} = \left\| E(\sigma_0) + E(\sigma_1) - \text{Tr}(\sigma_0) \tilde{I} - \text{Tr}(\sigma_1) \tilde{I} \right\|_{tr} \leq \left\| E(\sigma_0) - \text{Tr}(\sigma_0) \tilde{I} \right\|_{tr} + \| E(\sigma_1) \|_{tr} + \| \text{Tr}(\sigma_1) \tilde{I} \|_{tr} \leq \left\| E\left( \frac{1}{\text{Tr}(\sigma_0)} \sigma_0 \right) - \frac{1}{\text{Tr}(\sigma_0)} \tilde{I} \right\|_{tr} + 2^{-r^2 + 2}.
\]
Now we use the fact that \( \frac{1}{tr(\sigma_0)}\sigma_0 \) is a density matrix with all its eigenvalues \( \leq 2^{-q}S(\rho)+r\Delta \cdot \frac{1}{tr(\sigma_0)} \leq 2^{-q}S(\rho)+r\Delta+1 \). Thus, \( \frac{1}{tr(\sigma_0)}\sigma_0 \) has min-entropy at least \( q \cdot S(\rho)-r\Delta-1 \geq q \cdot (t+1)-r\Delta-1 \) since we started with a yes instance for QEA\( _Y \). We set the parameters such that \( q \geq r\Delta+1 \), and thus our density matrix has min-entropy at least \( qt \) and by the guarantee of our quantum extractor we get that \( \left\| E(\frac{1}{tr(\sigma_0)}\sigma_0) - \hat{I} \right\|_{tr} \leq \epsilon \). Therefore, \( \left\| \xi - \hat{I} \right\|_{tr} \leq \epsilon + 2^{-r^2+2} \leq 5\epsilon \), where the last inequality holds for \( r \geq \sqrt{\log(\frac{1}{\epsilon})} \).

- **Proof of the second item.**

Suppose that \( (Q,t) \in \text{QEA}_{\mathcal{N}} \). By the definition of quantum extractors we get that

\[
S(\xi) \leq S(|Q|^q) + q(m-t) + 2\log(\frac{1}{\epsilon}) + \log(qm) + O(1) \\
\leq q(t-1) + q(m-t) + 2\log(\frac{1}{\epsilon}) + \log(qm) + O(1) \\
= qm - q + 2\log(\frac{1}{\epsilon}) + \log(qm) + O(1) \leq qm - 1,
\]

where the last inequality follows if we choose the parameters such that \( q > 2\log(\frac{1}{\epsilon}) + \log(qm) + O(1) \).

The constraints we have on the parameters are \( q \geq \sqrt{\log(\frac{1}{\epsilon})}\sqrt{qn} + 1 \) and \( q > 2\log(\frac{1}{\epsilon}) + \log(qm) + O(1) \).

To this we add \( 5\epsilon < \left( \frac{1}{qm} - \frac{1}{2^m} \right)^2 \). This ensures a gap which can be amplified by Theorem \( B.1 \) to any desired gap, and completes the proof. These constraints can be easily satisfied by choosing \( q \) and \( \epsilon^{-1} \) to be appropriately large polynomials in \( n \).

### B.1.1 Relating entropy to trace distance from the completely mixed state

Now we relate the distance of a density matrix from uniform to a bound on its entropy. Consider the following classical random variable \( X \) over \( \{0,1\}^n \): with probability \( \epsilon \), \( X \) samples the fixed string \( 0^n \) and with probability \( 1 - \epsilon \), \( X \) is uniformly distributed over \( \{0,1\}^n \). This \( X \) has distance about \( \epsilon \) from uniform \( (\epsilon + \frac{1}{2\epsilon} - \frac{1}{2^n} \) to be exact) and its entropy is \( S(\rho) \leq (1 - \epsilon)n + H(1 - \epsilon) \). We show that this is essentially the worst possible:

**Lemma B.3.** Let \( \rho \) be a density matrix over \( n \) qubits and \( \epsilon > 0 \). If \( S(\rho) \leq (1 - \epsilon)n \) then \( \| \rho - \frac{1}{2^n} I \|_{tr} \geq \epsilon - \frac{1}{2^n} \).

**Proof.** We prove the contrapositive. Let \( \rho \) be a density matrix with \( \| \rho - \frac{1}{2^n} I \|_{tr} < \epsilon - \frac{1}{2^n} \) and minimal Shannon entropy. Writing \( \rho \) in its eigenvector basis we get \( \rho = \sum_{i=1}^{2^n} \lambda_i |v_i\langle v_i| \). W.l.o.g let us assume \( \lambda_1 \) is the largest eigenvalue of \( \rho \). The trace distance of \( \rho \) from \( \frac{1}{2^n} I \) is \( \frac{1}{2} \sum_{i=1}^{2^n} |\lambda_i - \frac{1}{2^n}| \). For any eigenvalue \( \lambda_i > \frac{1}{2^n} \), where \( i \neq 1 \), we can modify the eigenvalues of \( \rho \) such that \( \lambda_1 \leftarrow \lambda_1 + (\lambda_i - \frac{1}{2^n}) \) and \( \lambda_i \leftarrow \frac{1}{2^n} \). Since both \( \lambda_1 \) and \( \lambda_i \) are \( \geq \frac{1}{2^n} \), this does not affect \( \| \rho - \frac{1}{2^n} I \|_{tr} \). Moreover, we claim this operation only decreases \( S(\rho) \):

**Lemma B.4.** Let \( \rho = \sum_{i=1}^{2^n} \lambda_i |v_i\langle v_i| \) be a density matrix over \( n \) qubits with eigenvalues \( (\lambda_1 \geq \ldots \geq \lambda_{2^n}) \).

Let \( \lambda_j > \epsilon > 0 \) for some \( j > 1 \). Let \( \delta_1 = \lambda_1 + \epsilon \), \( \delta_j = \lambda_j - \epsilon \) and \( \delta_i = \lambda_i \) for \( i \neq 1, j \) and let \( \sigma = \sum_{i=1}^{2^n} \delta_i |v_i\langle v_i| \). Then \( S(\rho) \geq S(\sigma) \).
We prove the lemma shortly. Thus, w.l.o.g. we can assume \( \lambda_i \leq 2^{-n} \) for all \( i > 1 \). Having that
\[
\left\| \rho - \bar{I} \right\|_{tr} = \sum_{i: \lambda_i > 2^{-n}} \lambda_i - 2^{-n} = \lambda_1 - 2^{-n}.
\]
As \( \left\| \rho - \bar{I} \right\|_{tr} \leq \epsilon - 2^{-n} \) we conclude that \( \lambda_1 \leq \epsilon \). It follows that
\[
S(\rho) \geq \sum_{i > 1} \lambda_i \log(\lambda_i^{-1}) \geq \sum_{i > 1} \lambda_i \cdot n > (1 - \epsilon)n.
\]
which completes the proof.

Proof. (Of Lemma B.4) \( f(x) = x \log x^{-1} \) is concave. Therefore, for \( \lambda_j = \delta_j + \epsilon = (1 - \frac{\epsilon}{\delta_1 - \delta_j})\delta_j + \frac{\epsilon}{\delta_1 - \delta_j} \)
we get: \( f(\lambda_j) \geq (1 - \frac{\epsilon}{\delta_1 - \delta_j})f(\delta_j) + \frac{\epsilon}{\delta_1 - \delta_j}f(\delta_1) \).
Similarly, \( f(\lambda_1) \geq \frac{\epsilon}{\delta_1 - \delta_j}f(\delta_j) + (1 - \frac{\epsilon}{\delta_1 - \delta_j})f(\delta_1) \).
Together, \( f(\lambda_j) + f(\lambda_1) \geq f(\delta_j) + f(\delta_1) \). Therefore,
\[
S(\rho) - S(\sigma) = \lambda_1 \log \lambda_1^{-1} + \lambda_j \log \lambda_j^{-1} - \delta_1 \log \delta_1^{-1} - \delta_j \log \delta_j^{-1}
\]
\[
= f(\lambda_1) + f(\lambda_j) - f(\delta_1) - f(\delta_j) \geq 0.
\]

B.2 Closure under boolean formula

In order to prove that QED reduces to QSD we need to generalize another classical result about SZK to QSZK, namely, closure under boolean formula. A special case of this is, e.g., that if \( \Pi \in \text{QSZK} \) then the promise problem that accepts \( (x_1, x_2) \) if \( x_1 \in \Pi_{\text{yes}} \) or \( x_2 \in \Pi_{\text{yes}} \) and rejects if both \( x_i \) are in \( \Pi_{\text{no}} \), is also in QSZK. Notice that as we deal with promise problems we have yes instances and no instances and also ”undefined” instances, and therefore we need to say how to treat those ”undefined” instances in our formula.

We define:

Definition 7. For a promise problem \( \Pi \), the characteristic function of \( \Pi \) is the map \( \chi_\Pi : \{0, 1\}^* \rightarrow \{0, 1, \star\} \) given by
\[
\chi_\Pi(x) = \begin{cases} 
1 & \text{if } x \in \Pi_Y \\
0 & \text{if } x \in \Pi_N \\
\star & \text{otherwise}
\end{cases}
\]

and,

Definition 8. A partial assignment to variables \( v_1, \ldots, v_k \) is \( k \)-tuple \( \overline{\psi} = (a_1, \ldots, a_k) \in \{0, 1, \star\}^k \). For a propositional formula \( \phi \) on variables \( v_1, \ldots, v_k \) the evaluation \( \phi(\overline{\psi}) \) is recursively defined as follows:
\[
v_i(\overline{\psi}) = a_i, \quad (\phi \land \psi)(\overline{\psi}) = \begin{cases} 
1 & \text{if } \phi(\overline{\psi}) = 1 \text{ and } \psi(\overline{\psi}) = 1 \\
0 & \text{if } \phi(\overline{\psi}) = 0 \text{ or } \psi(\overline{\psi}) = 0 \\
\star & \text{otherwise}
\end{cases}
\]
\[
(\neg \phi)(\overline{\psi}) = \begin{cases} 
1 & \text{if } \phi(\overline{\psi}) = 0 \\
0 & \text{if } \phi(\overline{\psi}) = 1 \\
\star & \text{otherwise}
\end{cases}
\]
\[
(\phi \lor \psi)(\overline{\psi}) = \begin{cases} 
1 & \text{if } \phi(\overline{\psi}) = 1 \text{ or } \psi(\overline{\psi}) = 1 \\
0 & \text{if } \phi(\overline{\psi}) = 0 \text{ and } \psi(\overline{\psi}) = 0 \\
\star & \text{otherwise}
\end{cases}
\]

Notice that, e.g., \( 0 \land \star = 0 \) even though one of the inputs is ”undefined” in \( \Pi \).

With that we define:

Definition 9. For any promise problem \( \Pi \), we define a new promise problem \( \Phi(\Pi) \) as follows:
\[
\Phi(\Pi)_Y = \{ (\phi, x_1, \ldots, x_m) : \phi(\chi_\Pi(x_1), \ldots, \chi_\Pi(x_m)) = 1 \}
\]
\[
\Phi(\Pi)_N = \{ (\phi, x_1, \ldots, x_m) : \phi(\chi_\Pi(x_1), \ldots, \chi_\Pi(x_m)) = 0 \}
\]
The following is an adaptation of the classical proof of [SV98] to the quantum setting:

**Theorem B.2.** For any promise problem \( \Pi \in \text{QSZK} \), \( \Phi(\Pi) \in \text{QSZK} \).

**Proof.** Let \( \Pi \) be any promise problem in QSZK. Since QSD is QSZK-complete, \( \Pi \) reduces to QSD. This induces a reduction from \( \Phi(\Pi) \) to \( \Phi(\text{QSD}) \). Thus, it suffice to show that \( \Phi(\text{QSD}) \) reduces to QSD.

**Claim B.1.** \( \Phi(\text{QSD}) \) reduces to QSD.

**Proof.** Let \( w = (\phi, (X^1_0, X^1_1), \ldots, (X^m_0, X^m_1)) \) be an instance of \( \Phi(\text{QSD}) \). By applying De Morgan’s Laws, we may assume that the only negations in \( \phi \) are applied directly to the variables. (Note that De Morgan’s Laws still hold in our extended boolean algebra.) By the polarization lemma (Theorem B.1) and by the closure of QSZK under complement (as was shown by [Wat02]), we can construct in polynomial time pairs of circuits \((Y_0^1, Y_1^1), \ldots, (Y_0^m, Y_1^m)\) and \((Z_0^1, Z_1^1), \ldots, (Z_0^m, Z_1^m)\) such that:

\[
\begin{align*}
(X^i_0, X^i_1) \in \text{QSD}_Y & \Rightarrow \| |Y^i_0\rangle - |Y^i_1\rangle \|_{\text{tr}} \geq 1 - \frac{1}{3|\phi|} \quad \text{and} \quad \| |Z^i_0\rangle - |Z^i_1\rangle \|_{\text{tr}} \leq \frac{1}{3|\phi|} \\
(X^i_0, X^i_1) \in \text{QSD}_N & \Rightarrow \| |Y^i_0\rangle - |Y^i_1\rangle \|_{\text{tr}} \leq \frac{1}{3|\phi|} \quad \text{and} \quad \| |Z^i_0\rangle - |Z^i_1\rangle \|_{\text{tr}} \geq 1 - \frac{1}{3|\phi|}
\end{align*}
\]

The reduction outputs the pair of circuits \((\text{BuildCircuit}(\phi, 0), \text{BuildCircuit}(\phi, 1))\), where \text{BuildCircuit} is the following recursive procedure:

\[
\text{BuildCircuit}(\psi, b)
\]

1. If \( \psi = v_i \), output \( Y^i_b \).
2. If \( \psi = \neg v_i \), output \( Z^i_b \).
3. If \( \psi = \tau \lor \mu \), output \text{BuildCircuit}(\tau, b) \otimes \text{BuildCircuit}(\mu, b).
4. If \( \psi = \tau \land \mu \), output \( \frac{1}{2}(\text{BuildCircuit}(\tau, 0) \otimes \text{BuildCircuit}(\mu, b)) + \frac{1}{2}(\text{BuildCircuit}(\tau, 1) \otimes \text{BuildCircuit}(\mu, 1 - b)) \).

Notice that the number of recursive calls equals the number of sub-formula of \( \phi \), and therefore the procedure runs in time polynomial in \( |\psi| \) and \( |X^i_0| \), i.e., polynomial in its input length.

We now turn to proving correctness by induction. For a sub-formula \( \tau \) of \( \phi \), let

\[
\Delta(\tau) = \| (\text{BuildCircuit}(\tau, 0) - \text{BuildCircuit}(\tau, 1)) |0\rangle \|_{\text{tr}}
\]

We claim:

**Claim B.2.** Let \( \pi = (\chi_{\text{QSD}}(X^1_0, X^1_1), \ldots, \chi_{\text{QSD}}(X^m_0, X^m_1)) \). \( \Box \) For every sub-formula \( \psi \) of \( \phi \), we have:

\[
\begin{align*}
\psi(\pi) = 1 & \Rightarrow \Delta(\psi) \geq 1 - \frac{|\psi|}{3|\phi|} \\
\psi(\pi) = 0 & \Rightarrow \Delta(\psi) \leq \frac{|\psi|}{3|\phi|}
\end{align*}
\]

**Proof.** By induction on the sub-formula of \( \phi \). It holds for atomic sub-formula by the properties of the \( Y \)'s and \( Z \)'s.

\[\text{we remind the reader that } \chi_{\text{QSD}}(C_1, C_2) \text{ was defined in Definition 7.}\]
• The case $\psi = \tau \lor \mu$.

If $\psi(\overline{a}) = 1$ then either $\tau(\overline{a}) = 1$ or $\mu(\overline{a}) = 1$. W.l.o.g., say $\tau(\overline{a}) = 1$. In this case we have for any $i \in \{0, 1\}$ that $\text{BuildCircuit}(\tau, i) = \mathcal{E}(\text{BuildCircuit}(\psi, i))$, where $\mathcal{E}$ is the quantum operation tracing out the registers associated with the $\mu$ sub-formula. Thus, by Fact B.2 and by induction,

$$\Delta(\psi) \geq \Delta(\tau) \geq 1 - \frac{|\tau|}{3|\phi|} \geq 1 - \frac{|\psi|}{3|\phi|}.$$  

If $\psi(\overline{a}) = 0$, then both $\tau(\overline{a}) = \mu(\overline{a}) = 0$.

Using

$$\| \rho_0 \otimes \sigma_0 - \sigma_0 \otimes \rho_1 \|_{\text{tr}} \leq \| \rho_0 \otimes \rho_1 - \sigma_0 \otimes \rho_1 \|_{\text{tr}} + \| \sigma_0 \otimes \rho_1 - \sigma_0 \otimes \sigma_1 \|_{\text{tr}} = \| \rho_0 - \sigma_0 \|_{\text{tr}} + \| \rho_1 - \sigma_1 \|_{\text{tr}},$$

we get

$$\Delta(\psi) \leq \Delta(\tau) + \Delta(\mu) \leq \frac{|\tau|}{3|\phi|} + \frac{|\mu|}{3|\phi|} \leq \frac{|\psi|}{3|\phi|}.$$  

• The case $\psi = \tau \land \mu$.

Using

$$\left\| \frac{1}{2} [\rho_0 \otimes \sigma_0 + \rho_1 \otimes \sigma_1] - \frac{1}{2} [\rho_0 \otimes \sigma_1 + \rho_1 \otimes \sigma_0] \right\|_{\text{tr}} = \frac{1}{2} \| (\rho_0 - \rho_1) \otimes (\sigma_0 - \sigma_1) \|_{\text{tr}} = \| \rho_0 - \rho_1 \|_{\text{tr}} \| \sigma_0 - \sigma_1 \|_{\text{tr}},$$

where the equalities above follow because $2 \| X \otimes Y \|_{\text{tr}} = 2 \| X \|_{\text{tr}} 2 \| Y \|_{\text{tr}}$. We get $\Delta(\psi) = \Delta(\tau) \cdot \Delta(\mu)$.

If $\psi(\overline{a}) = 1$, then, by induction,

$$\Delta(\psi) \geq \left( 1 - \frac{|\tau|}{3|\phi|} \right) \left( 1 - \frac{|\mu|}{3|\phi|} \right) > 1 - \frac{|\tau| + |\mu|}{3|\phi|} \geq 1 - \frac{|\psi|}{3|\phi|}.$$  

If $\psi(\overline{a}) = 0$, then, w.l.o.g., say $\tau(\overline{a}) = 0$. By induction

$$\Delta(\psi) = \Delta(\tau) \cdot \Delta(\mu) \leq \Delta(\tau) \leq \frac{|\tau|}{3|\phi|} \leq \frac{|\psi|}{3|\phi|}.$$

Let $A_b = \text{BuildCircuit}(\phi, b)$. By the above claim if $w \in \Phi(\text{QSD})_Y$ then $\| (A - B) \ket{0} \|_{\text{tr}} \geq 2/3$ and if $w \in \Phi(\text{QSD})_N$ then $\| (A - B) \ket{0} \|_{\text{tr}} \leq 1/3$. Thus the claim follows.

To finish the section we observe that

Claim B.3. $QED \leq \Phi(\text{QEA})$, for some formula $\Phi$.  

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Proof. Let \((Q_0, Q_1)\) be an instance of QED. Let \(\xi_i = \otimes^6 |Q_i\rangle\). The output of the reduction is

\[
\sum_{t=1}^{6n} \left[ (\xi_{0}, t) \in \text{QEA}_Y \right] \land \left[ (\xi_{1}, t) \in \text{QEA}_N \right].
\]

If \((Q_0, Q_1)\) is an instance of QED then \(S(\xi_0) \geq S(\xi_1) + 3\). Thus, there exists an integer \(t\) such that \((\xi_0, t) \in \text{QEA}_Y\) and \((\xi_1, t) \in \text{QEA}_N\). On the other hand, if \((Q_0, Q_1)\) is an instance of QED then \(S(\xi_1) \geq S(\xi_0) + 3\). Thus, every integer \(t\) is either greater than \(S(\xi_0) + 1\) or smaller then \(S(\xi_1) - 1\). That is, for every \(t\), \((\xi_0, t) \in \text{QEA}_N\) or \((\xi_1, t) \in \text{QEA}_Y\).

In particular, the closure under formula implies that if QEA \(\leq\) SD then QED = \(\Phi(\text{QEA}) \leq\) QSD, as desired.

B.3 QSD \(\leq\) QED

B.3.1 Some quantum information background

The proof of the following facts can be found in [NC00].

Fact B.3. (Joint entropy theorem) Suppose \(p_i\) are probabilities, \(|i\rangle\) are orthogonal states for a system \(A\), and \(\rho_i\) is any set of density operators for another system \(B\). Then

\[
S \left( \sum_i p_i |i\rangle \langle i| \otimes \rho_i \right) = H(p_i) + \sum_i p_i S(\rho_i).
\]

Fact B.4. (Fannes’ inequality) Suppose \(\rho\) and \(\sigma\) are density matrices over a Hilbert space of dimension \(d\). Suppose further that the trace distance between them satisfies \(t = \|\rho - \sigma\|_1 \leq 1/e\). Then

\[
|S(\rho) - S(\sigma)| \leq t (\ln d - \ln t).
\]

The following lemma is taken from [ANTV02]. It can be proved using Holevo’s bound.

Lemma B.5. (Lemma 3.2, [ANTV02]) Let \(\rho_0\) and \(\rho_1\) be two density matrices, and let \(\rho = \frac{1}{2}(\rho_0 + \rho_1)\). If there exists a measurement with outcome 0 or 1 such that making the measurement on \(\rho_0\) yields the bit \(b\) with probability at least \(p\), then

\[
S(\rho) \geq \frac{1}{2} \left[ S(\rho_0) + S(\rho_1) \right] + (1 - H(p)).
\]

Combining the lemma with Fact B.1 we get

Lemma B.6. Let \(\rho_0\) and \(\rho_1\) be two density matrices, and let \(\rho = \frac{1}{2}(\rho_0 + \rho_1)\). Then

\[
S(\rho) \geq \frac{1}{2} \left[ S(\rho_0) + S(\rho_1) \right] + (1 - H(\frac{1}{2} + \frac{\|\rho_0 - \rho_1\|_1}{2})).
\]

B.3.2 The proof

Theorem B.3. For any \(0 \leq \alpha < \beta^2 \leq 1\), QSD\(_{\alpha, \beta}\) \(\leq\) QED.

Proof. Given circuits \(Q_0, Q_1\). We first apply the polarization lemma (Theorem B.1) with \(n = m_0\) and obtain circuits \(R_0, R_1\). We then construct two circuits \(Z_0, Z_1\) as follows. \(Z_1\) is implemented by a circuit which first applies a Hadamard gate on a single qubit \(b\), measures \(b\) and then conditioned on the result it applies either \(R_0\) or \(R_1\). The output of \(Z_1\) is \(\frac{1}{2}|0\rangle \langle 0| \otimes |R_0\rangle + \frac{1}{2}|1\rangle \langle 1| \otimes |R_1\rangle\). \(Z_0\) is the same as \(Z_1\) except that \(b\) is
Proof. Of information telling us which circuit was activated. However, the matrix
\( QED \)
Y ES
Lemma C.1.
Claim B.5.
traced out. The output of state.
In particular, 
By the Polarization lemma (Theorem B.1) we get 
\( S(|Z_0\rangle) = \frac{1}{2}(S(|R_0\rangle) + S(|R_1\rangle)) + 1. \)

On the other hand, \( |Z_0\rangle \) is very close both to \( |R_0\rangle \) and to \( |R_1\rangle \). Specifically, \( \| |Z_0\rangle - |R_1\rangle \| _{tr} = \| \frac{1}{2} |R_0\rangle - \frac{1}{2} |R_1\rangle \| _{tr} \leq 2^{-m_0} \) . Therefore, by Fennes inequality (Fact B.4) 

\[ S(|Z_0\rangle) - S(|R_0\rangle) \leq 0.1. \]

Combining the two equations we get \( S(|Z_1\rangle) - S(|Z_0\rangle) \geq 0.9. \) Thus, \( S(|Z_1 \otimes Z_1\rangle) - S(|Z_0 \otimes Z_0 \otimes C_1\rangle) \geq 2 \ast 0.9 - 1 = 0.8. \) Therefore, \( (Z_0 \otimes Z_0 \otimes C, Z_1 \otimes Z_1) \in QED_{NO} \)

Claim B.5. If \((Q_0, Q_1) \in (QSD_{\alpha, \beta})_{YES} \) then \((Z_0 \otimes Z_0 \otimes C, Z_1 \otimes Z_1) \in QED_{YES} \)

Proof. By the Polarization lemma (Theorem B.1) \( \| \rho_0 - \rho_1 \| _{tr} \geq 1 - 2^{-m_0} \). Using the Holevo bound (Lemma B.6) we get that \( S(|Z_0\rangle) \geq \frac{1}{2}[S(\rho_0) + S(\rho_1)] + 1 - H(\frac{1}{2} + \frac{\| \rho_0 - \rho_1 \| _{tr}}{\| \rho_0 \| _{tr}}) \geq \frac{1}{2}[S(\rho_0) + S(\rho_1)] + 1 - H(2^{-m_0}) \). By Fact B.3 we know that \( S(|Z_1\rangle) = \frac{1}{2}(S(\rho_0) + S(\rho_1)) + 1. \) Therefore, \( S(|Z_1\rangle) - S(|Z_0\rangle) = H(2^{-m_0}) < 0.1 \) for sufficiently large \( m_0 \).

In particular, \( S(|Z_1 \otimes Z_1\rangle) - S(|Z_0 \otimes Z_0 \otimes C\rangle) \leq 2 \ast 0.1 - 1 = -0.8 \) and \( (Z_0 \otimes Z_0 \otimes C, Z_1 \otimes Z_1) \in QED_{YES} \)

\( A \)

Quantum extractors

Lemma C.1. If \( T : L(V) \rightarrow L(V) \) is a \( (D = 2^d, \lambda) \) quantum expander, then for every \( t > 0 \), \( T \) is also a \( (k = n - t, d, \epsilon) \) quantum extractor with \( \epsilon = 2^t / 2 \cdot \lambda \).

Proof. \( T \) has a dimension 1 eigenspace \( W_1 \) with eigenvalue 1, spanned by the norm 1 eigenvector \( v_1 = \frac{1}{\sqrt{N}} I \) (where \( \text{dim}(V) = N \) ). Our input \( \rho \) is a density matrix and therefore \( \langle \rho \mid v_1 \rangle = \frac{1}{\sqrt{N}} \text{Tr}(\rho) = \frac{1}{\sqrt{N}}. \) In particular \( \rho - \frac{1}{\sqrt{N}} v_1 = \rho - I \) is perpendicular to \( W_1 \). Therefore,

\[
\| T(\rho) - I \| _2^2 = \| T(\rho - I) \| _2^2 \leq \lambda^2 \| \rho - I \| _2^2 = \lambda^2 [\| \rho \| _2^2 - \langle \rho | I \rangle - \langle I | \rho \rangle + \| I \| _2^2] = \lambda^2 [\| \rho \| _2^2 - \frac{1}{N}] \leq \lambda^2 \| \rho \| _2^2.
\]
Plugging $H_2(\rho) \geq H_\infty(\rho) \geq k = n - t$ we see that $| | T(\rho) - \tilde{I} | |_2^2 \leq \lambda^2 2^{-(n-t)}$. Using Cauchy-Schwartz

$$
\| T(\rho) - \tilde{I} \|_{\text{tr}} \leq \sqrt{N} \| T(\rho) - \tilde{I} \|_2 \leq \sqrt{N} \cdot \lambda \cdot 2^{-\frac{n-t}{2}} = 2^{t/2} \cdot \lambda = \epsilon.
$$

\[ \square \]

**Theorem 1.3** Any $(D, \lambda)$ quantum expander satisfies $\lambda \geq \frac{2}{3\sqrt{3}D}$.

**Proof.** Let $E$ be a $(D, \lambda)$ quantum expander operating on the space of $n$ qubits. Let $d = \log D$, and let $\delta > 0$ be a constant to be fixed later. We first apply Lemma 4.1 with $t = d - 2 \log \delta$ to deduce that $E$ is a $(n - d + 2 \log \delta, d, \delta^{-1} 2^{d/2} \lambda)$ quantum extractor.

The proof idea is to take a density matrix which is uniform on a set of "small size". Applying the extractor yields a density matrix close to the completely mixed state. Such a matrix must have a high rank. On the other hand, because we started with a low-rank matrix, the resulting density matrix cannot have a too-high rank (since $E$ is $D$-regular).

Formally, let $\rho \in D(V)$ be a density matrix of a flat (classical) probability distribution over a set of size $2^n - d + 2 \log \delta$. By definition, $H_\infty(\rho) = n - d + 2 \log \delta$. Also, $\log(\text{rank}(\rho)) = n - d + 2 \log \delta$.

Using the quantum extractor definition we get that $E(\rho)$ is $\delta^{-1} 2^{d/2} \lambda$-close to the completely mixed state. Hence,

$$
\text{rank}(E(\rho)) \geq (1 - \delta^{-1} 2^{d/2} \lambda) 2^n.
$$

On the other hand, since $E$ is $2^d$-regular, $E(\rho)$ is a sum of $2^d$ matrices. Each of these matrices has rank $2^n - d + 2 \log \delta$. Hence,

$$
\text{rank}(E(\rho)) \leq 2^d 2^{n-d+2 \log \delta} = 2^n \delta^2.
$$

Combining the two inequalities gives

$$
\lambda \geq \delta (1 - \delta^2) 2^{-d/2} = \frac{\delta (1 - \delta^2)}{\sqrt{D}}.
$$

Taking $\delta = 1/\sqrt{3}$ completes the proof. \[ \square \]