New treatment of the noncommutative Dirac equation with a Coulomb potential

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Abstract

Using the approach of the modified Euler-Lagrange field equation together with the corresponding Seiberg-Witten maps of the dynamical fields, a noncommutative Dirac equation with a Coulomb potential is derived. We then find the noncommutative modification to the energy levels and the possible new transitions. In the nonrelativistic limit a general form of the hamiltonian of the hydrogen atom is obtained, and we show that the noncommmutativity plays the role of spin and magnetic field which gives the hyperfine structure.

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1 Introduction

The connection between string theory and the noncommutativity [1, 2, 3, 4] motivated a large amount of work to study and understand many physical phenomenon. There is a flurry of activity in analysing divergences [5], unitarity violation [6], causality [7], and new physics at very short distances of the Planck-length order [8].

The noncommutative field theory is characterised by the commutation relations between the noncommutative coordinates themselves; namely:

\[[\hat{x}^\mu, \hat{x}^\nu]_\ast = i\theta^{\mu\nu}, \quad (1)\]

where \(\hat{x}^\mu\) are the coordinate operators and \(\theta^{\mu\nu}\) are the non-commutativity parameters of dimension of area that signify the smallest area in space that can be probed in principle. The Groenewald-Moyal star product of two fields \(f(x)\) and \(g(x)\) is given by

\[f(x) \ast g(x) = \exp \left( \frac{i}{2} \partial_i \partial_j \right) f(x) g(y) \bigg|_{y=x} \quad (2)\]

The most obvious natural phenomena to use in hunting for noncommutative effects are simple quantum mechanics systems, such as the hydrogen atom [9, 10, 11]. In the noncommutative space one expects the degeneracy of the initial spectral line to be lifted, thus one may say that non-commutativity plays the role of spin.

In a previous work [12], by solving the deformed Klein-Gordon equation in canonical non-commutative space, we showed that the energy is shifted: the first term of the energy correction is proportional to the magnetic quantum number, which behavior is similar to the Zeeman effect as applied to a system without spin in a magnetic field; the second term is proportional to \(\theta^2\), thus we explicitly accounted for spin effects in this space.

The purpose of this paper is to study the extension of the Dirac field in the same context by applying the result obtained to a hydrogen atom.

This paper is organized as follows. In section 2 we propose an invariant action of the noncommutative Dirac field in the presence of an electromagnetic field. In section 3, using the generalised Euler-Lagrange field equation, we derive the deformed Dirac equation. In section 4, we apply these results to the hydrogen atom, and by the use of the perturbation theory, we solve the deformed Dirac equation and obtain the noncommutative modification of the energy levels. In section 5, we introduce the non-relativistic limit of the noncommutative Dirac equation and solve it using perturbation theory and deduce that the non-relativistic noncommutative Dirac equation is the same as the Schrödinger equation on noncommutative space. Finally, in section 5, we draw our conclusions.
2 Seiberg-Witten maps

Here we look for a mapping \( \phi^A \rightarrow \hat{\phi}^A \) and \( \lambda \rightarrow \hat{\lambda} (\lambda, A_\mu) \), where \( \phi^A = (A_\mu, \psi) \) is a generic field, \( A_\mu \) and \( \psi \) are the gauge field and spinor respectively (the Greek and Latin indices denote curved and tangent space-time respectively), and \( \lambda \) is the U(1) gauge Lie-valued infinitesimal transformation parameter, such that:

\[
\hat{\phi}^A (A) + \hat{\delta}_\lambda \hat{\phi}^A (A) = \hat{\phi}^A (A + \delta_\lambda A),
\]

where \( \delta_\lambda \) is the ordinary gauge transformation and \( \hat{\delta}_\lambda \) is a noncommutative gauge transformation which are defined by:

\[
\hat{\delta}_\lambda \hat{\psi} = i\hat{\lambda} * \hat{\psi}, \quad \delta_\lambda \psi = i\lambda \psi,
\]

\[
\hat{\delta}_\lambda \hat{A}_\mu = \partial_\mu \hat{\lambda} + i \left[ \hat{\lambda}, \hat{A}_\mu \right]_*, \quad \delta_\lambda A_\mu = \partial_\mu \lambda.
\]

In accordance with the general method of gauge theories, in the noncommutative space, using these transformations one can get at second order in the non-commutative parameter \( \theta^{\mu\nu} \) (or equivalently \( \theta \)) the following Seiberg–Witten maps \([1]\):

\[
\hat{\psi} = \psi + \theta^1 \psi + \mathcal{O} (\theta^2),
\]

\[
\hat{\lambda} = \lambda + \theta^1 (\lambda, A_\mu) + \mathcal{O} (\theta^2),
\]

\[
\hat{A}_\xi = A_\xi + \theta A_\xi (A_\xi) + \mathcal{O} (\theta^2),
\]

\[
\hat{F}_{\mu\xi} = F_{\mu\xi} (A_\xi) + \theta F_{\mu\xi} (A_\xi) + \mathcal{O} (\theta^2),
\]

where

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]

To begin, we consider an action for a non-commutative Dirac field in the presence of an electrodynamic gauge field in a non-commutative space-time. We can write:

\[
S = \int d^4x \left( \overline{\psi} * \left( i\gamma^\nu \hat{D}_\nu - m \right) * \hat{\psi} - \frac{1}{4} \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} \right),
\]

where the gauge covariant derivative is defined as: \( \hat{D}_\mu \hat{\psi} = \left( \partial_\mu + ie\hat{A}_\mu \right) * \hat{\psi} \).

Next we use the generic-field infinitesimal transformations \([4]\) and \([5]\) and the star-product tensor relations to prove that the action in eq. \([11]\) is invariant. By varying the scalar density under the gauge transformation and from the generalised field equation and the Noether theorem we obtain \([13]\):

\[
\frac{\partial L}{\partial \hat{\psi}} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \psi)} \right) + \partial_\mu \partial_\nu \left( \frac{\partial L}{\partial (\partial_\mu \partial_\nu \hat{\psi})} \right) + \mathcal{O} (\theta^2) = 0.
\]
3 Non-commutative Dirac equation

In this section we study the Dirac equation for a Coulomb interaction \((-e/r)\) in the free non-commutative space. This means that we will deal with solutions of the U(1) gauge-free non-commutative field equations [14]. For this we use the modified field equations in eq. (12) and the generic field \(\hat{A}_\mu\) so that:

\[
\delta \hat{A}_\mu = \partial_\mu \hat{\lambda} - ie \hat{A}_\mu * \hat{\lambda} + ie \hat{\lambda} * \hat{A}_\mu, \tag{13}
\]

and the free non-commutative field equation:

\[
\partial^\mu \hat{F}_{\mu\nu} - ie [\hat{A}_\mu, \hat{F}_{\mu\nu}] = 0, \tag{14}
\]

where we assumed the non-commutative current to vanish everywhere in space \((r \neq 0)\) [14]. Using the Seiberg-Witten maps (8)–(9) and the choice (14) (static solution), we can obtain the following deformed Coulomb potential [10]:

\[
\hat{a}_0 = -\frac{e}{r} - \frac{e^3}{r^4} \theta^0 x_j + \mathcal{O} (\theta^2), \tag{15}
\]

\[
\hat{a}_i = \frac{e^3}{4r^4} \theta^{ij} x_j + \mathcal{O} (\theta^2). \tag{16}
\]

Using the modified field equations in eq. (12) and the generic field \(\hat{\psi}\) so that:

\[
\delta \hat{\lambda} \hat{\psi} = i \hat{\lambda} * \hat{\psi}, \tag{17}
\]

the modified Dirac equation in a non-commutative space-time in the presence of the vector potential \(\hat{A}_\mu\) up to the first order of \(\theta\) can be cast into:

\[
(i\gamma^\mu \partial_\mu - m) \hat{\psi} - e\gamma^\mu \hat{A}_\mu \hat{\psi} + \frac{ie}{2} \theta^\rho\sigma \gamma^\mu \partial_\rho \hat{A}_\mu \partial_\sigma \hat{\psi} = 0. \tag{18}
\]

3.1 Non-commutative space-space Dirac equation

For a noncommutative space-space \((\theta^{0i} = 0\) where \(i = 1, 2, 3\), we do not consider a noncommutative space-time \((\theta^{0i} \neq 0)\) since several works have shown that the theory suffers lack of unitarity. See for instance ref [6], it is easy to check that:

\[
i\gamma^\mu \partial_\mu - m = i\gamma^0 \partial_0 + i\gamma^i \partial_i - m, \tag{19}
\]

\[
-e\gamma^\mu \hat{A}_\mu = \frac{e^2}{r} \gamma^0 - \frac{e^4}{4r^4} \gamma^i \theta^{ij} x_j, \tag{20}
\]

\[
\frac{ie}{2} \theta^\rho\sigma \gamma^\mu \partial_\rho \hat{A}_\mu \partial_\sigma = \frac{ie^2}{2v^3} \theta^{ij} \gamma^0 x_i \partial_j = \frac{e^2}{2v^3} \gamma^0 \hat{\theta} \cdot \hat{L}. \tag{21}
\]

Notice that: \(\theta_i = \frac{1}{2} \epsilon_{ijk} \theta_{jk}\). Then the noncommutative Dirac equation (18) up to \(\mathcal{O} (\theta^2)\) takes the following form:

\[
\left[ i\gamma^0 \partial_0 + i\gamma^i \partial_i - m + \frac{e^2}{r} \gamma^0 - \frac{e^4}{4r^4} \gamma^i \theta^{ij} x_j + \frac{e^2}{2v^3} \gamma^0 \hat{\theta} \cdot \hat{L} \right] \hat{\psi} (t, r, \theta, \varphi) = 0. \tag{22}
\]
We can write this equation as:

$$\hat{H} \psi(t, r, \theta, \varphi) = i \partial_t \psi(t, r, \theta, \varphi).$$  \hspace{1cm} (23)$$

Then

$$\hat{H} = H_0 + H_{\text{pert}}^0,$$  \hspace{1cm} (24)$$

where $H_0$ is the relativistic hydrogen atom hamiltonian

$$H_0 = \vec{\alpha} \cdot \left(-\vec{\nabla} \right) + \beta m - \frac{e^2}{r},$$  \hspace{1cm} (25)$$

and $H_{\text{pert}}^0$ is the leading-order perturbation

$$H_{\text{pert}}^0 = -\frac{e^2}{2r^4} \vec{\theta} \cdot \vec{\mathcal{L}} + \frac{e^4}{4} \vec{\theta} \cdot \left(\vec{\alpha} \times \vec{\theta} \right).$$  \hspace{1cm} (26)$$

The first term of (26) which coincides with the one given in [10] describes the interaction spin-orbit where $\theta$ plays the role of spin. The second term is absent in ref [10] and $\theta$ here corresponds to a magnetic field.

In the above the matrices $\vec{\alpha}$ and $\beta$ are given by:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \hspace{1cm} \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix},$$

where $\sigma^i$ are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \hspace{1cm} \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \hspace{1cm} \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (27)$$

To investigate the modification of the energy levels by (26), we use the first-order perturbation theory. The spectrum of $H_0$ and the corresponding wave functions are well known and given by (see [15, 16, 17, 18, 19, 20, 21]):

$$\psi(r, \theta, \varphi) = \begin{pmatrix} \phi(r, \theta, \varphi) \\ \chi(r, \theta, \varphi) \end{pmatrix} = \begin{pmatrix} f(r) \Omega_{JM}(\theta, \varphi) \\ g(r) \Omega_{JM}(\theta, \varphi) \end{pmatrix},$$  \hspace{1cm} (28)$$

where the bi-spinors $\Omega_{JM}(\theta, \varphi)$ are defined by:

$$\Omega_{JM}(\theta, \varphi) = \begin{pmatrix} \frac{\sqrt{(j+1/2)\pm(M+1/2)}}{2j+1/2} Y_{j\pm1/2, M-1/2}(\theta, \varphi) \\ \frac{\sqrt{(j+1/2)\pm(M-1/2)}}{2j+1/2} Y_{j\pm1/2, M+1/2}(\theta, \varphi) \end{pmatrix},$$  \hspace{1cm} (29)$$

with the radial functions $f(r)$ and $g(r)$ are given as:

$$\begin{pmatrix} f(r) \\ g(r) \end{pmatrix} = \frac{(ma)^2}{\nu} \sqrt{\frac{(E \mp m\nu) n!}{m\mu(\nu-\nu)} \Gamma(n+2\nu)} e^{-\frac{x}{2} x^{\nu-1} \times} \left( f_1 x L^{2\nu+1}_{n-1}(x) + f_2 L^{2\nu-1}_{n-1}(x) \right) \left( g_1 x L^{2\nu+1}_{n-1}(x) + g_2 L^{2\nu-1}_{n}(x) \right),$$  \hspace{1cm} (30)$$

where $n$ is the principal quantum number, $J$ the total angular momentum, $M$ the magnetic quantum number, $\nu=\Delta J$, $\mu=m_e/m$, and $x=r/\alpha$. $\phi(r, \theta, \varphi)$ and $\chi(r, \theta, \varphi)$ are the eigenfunctions of relativistic and non-relativistic hydrogen atom.
where the relativistic energy levels are given by

\[
E = E_{n,l} = \frac{m(n + \nu)}{\sqrt{\alpha^2 + (n + \nu)^2}} , \quad n = 0, 1, 2...
\]  

(30)

and \( L^\alpha_n(x) \) are the associated Laguerre polynomials [20], with the following notations:

\[
a = \frac{1}{m} \sqrt{m^2 - E^2}, \quad \kappa = \pm \left( j + \frac{1}{2} \right), \quad \nu = \sqrt{\kappa^2 - \alpha^2}, \quad x = 2\sqrt{m^2 - E^2},
\]

\[
f_1 = \frac{a\alpha}{E\kappa - m\nu}, \quad f_2 = \kappa - \nu, \quad g_1 = \frac{a(\kappa - \nu)}{E\kappa - m\nu}, \quad g_2 = e^2 = \alpha.
\]

3.2 Noncommutative corrections of the energy

Now to obtain the modification to the energy levels as a result of the terms (26) due to the non-commutativity of space-space, we use perturbation theory up to the first order. With respect the selection rules \( \Delta l = 0 \) we have:

\[
\Delta E_{n,l} = \Delta E^{(1)}_{n,l} + \Delta E^{(2)}_{n,l},
\]

(31)

where:

\[
\Delta E^{(1)}_{n,l} = -\frac{e^2}{2} \int_0^{4\pi} d\Omega \int_0^{\infty} dr r^{-1} [\psi^\dagger_{njlM} (r, \theta, \varphi) \left( \overrightarrow{\theta} \cdot \overrightarrow{L} \right) \psi_{nj'l'M'} (r, \theta, \varphi)]
\]

\[
= -\frac{e^2}{2} \varrho^{(1)}_{n,l,M,M'} \Theta^{(1)}_{n,l,M,M'},
\]

(32)

\[
\Delta E^{(2)}_{n,l} = \frac{e^4}{4} \int_0^{4\pi} d\Omega \int_0^{\infty} dr r^{-1} [\psi^\dagger_{njlM} (r, \theta, \varphi) \left| \overrightarrow{\alpha} \cdot ( \overrightarrow{\theta} \times \overrightarrow{r} ) \right| \psi_{nj'l'M'} (r, \theta, \varphi)]
\]

\[
= \frac{e^4}{4} \varrho^{(2)}_{n,l,M,M'} \Theta^{(2)}_{n,l,M,M'},
\]

(33)

where

\[
\varrho^{(1)}_{n,l} = \int_0^{+\infty} r^{-1} (f_1^2 + g_1^2) dr,
\]

(34)

\[
\varrho^{(2)}_{n,l} = \int_0^{+\infty} r^{-1} (f_2^1 - g_2^2) dr,
\]

(35)

\[
\Theta^{(1)}_{n,l,M,M'} = \int_0^{4\pi} d\Omega \Omega^\dagger_{jlM} (\theta, \varphi) \left( \overrightarrow{\theta} \cdot \overrightarrow{L} \right) \Omega_{j'l'M'} (\theta, \varphi),
\]

(36)

\[
\Theta^{(2)}_{n,l,M,M'} = \int_0^{4\pi} d\Omega \Omega^\dagger_{jlM} (\theta, \varphi) \overrightarrow{\theta} \cdot ( \overrightarrow{\theta} \times \overrightarrow{r} ) \Omega_{j'l'M'} (\theta, \varphi).
\]

(37)
where the radial integrals are given by [21]:

\[ 2^{(1)}_{n,l} = (ma)^3 \left[ \frac{3E \kappa (E \kappa - m) - (\nu^2 - 1)}{m^2 \nu (4\nu^2 - 1)(\nu^2 - 1)} \right], \tag{38} \]

\[ 2^{(2)}_{n,l} = \frac{2 (ma)^3 E m + 2m\nu^2 - 3E \kappa}{m^2 \nu (4\nu^2 - 1)(\nu^2 - 1)}. \tag{39} \]

The selection rules for the possible transitions between levels \((NlM \rightarrow Nl'M')\) are \(\Delta l = 0\) and \(\Delta M = 0, \pm 1\), where \(N = n + |\kappa|\) describes the principal quantum number. The \(2P_{1/2}\) and \(2P_{3/2}\) levels correspond respectively to:

\(n = 1, j = 1/2, \kappa = 1, M = \pm 1/2\)

and

\(n = 0, j = 3/2, \kappa = 2, M = \pm 1/2, \pm 3/2\).

The corresponding angular corrections are given by (we take \(\theta_i = \theta_{\delta_3}\)):

\[ \Theta^{(1)}_{2P_{1/2}} = \frac{2}{3} \theta \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda^{(1)}_{2P_{1/2}} = \pm \frac{2}{3} |\theta|, \tag{40} \]

\[ \Theta^{(1)}_{2P_{3/2}} = \frac{1}{3} \theta \begin{pmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda^{(1)}_{2P_{3/2}} = \pm |\theta|, \pm \frac{|\theta|}{3}, \tag{41} \]

\[ \Theta^{(2)}_{2P_{1/2}} = 0, \quad \Theta^{(2)}_{2P_{3/2}} = 0, \tag{42} \]

where \(\lambda^{(1)}_{2P_{1/2}}\) and \(\lambda^{(1)}_{2P_{3/2}}\) are respectively the eigenvalues of the angular part.

From [32], [33], [34] and [41], we can write:

\[ \Delta E_{2P_{1/2}} = -\frac{e^2}{2} \Theta^{(1)}_{2P_{1/2}} \lambda^{(1)}_{2P_{1/2}} = \mp 6.57668 \times 10^6 |\theta| (eV)^3, \tag{43} \]

\[ \Delta E_{2P_{3/2}} = -\frac{e^2}{2} \Theta^{(1)}_{2P_{3/2}} \lambda^{(1)}_{2P_{3/2}} = 1.578 \times 10^6 \left( \pm |\theta|, \pm \frac{1}{3} |\theta| \right) (eV)^3. \tag{44} \]

According to Ref. [22], the current theoretical accuracy on the \(2P\) Lamb shift is about 0.08 kHz. From the splitting [43], we get the bound

\[ \theta \lesssim (4 GeV)^{-2} \]

and from the splitting [44], we get the bound

\[ \theta \lesssim (2 GeV)^{-2} \text{ or } \theta \lesssim (1, 2 GeV)^{-2} \]

It is worth mentioning that the second term of the perturbation in the hamiltonian expression [26] does not remove the degeneracy of the energy levels because it is a non-diagonal matrix. However, for instance, the non-vanishing matrix elements between \(2S_{1/2}\) \((n = 1, j = 1/2, \kappa = -1, M = \pm 1/2)\) and \(2P_{1/2}\)
(n = 1, j = 1/2, \kappa = 1, M = \pm 1/2) states for the selection rules \( \Delta l = 1 \) and \( \Delta M = 0, \pm 1 \) give the possible transition:

\[
\langle 2P_{1/2} | \frac{e^4}{4} \hat{\theta} \cdot (\hat{\alpha} \times \frac{\vec{r}}{r^4}) \rangle 2S_{1/2} = \frac{e^4}{4} \Theta_{2S_{1/2} \rightarrow 2P_{1/2}} \Theta_{2S_{1/2} \rightarrow 2P_{1/2}},
\]

(45)

where

\[
\Theta_{2S_{1/2} \rightarrow 2P_{1/2}} = \frac{2}{3} \theta \left( \begin{array}{cc}
1 & 0 \\
0 & -1 
\end{array} \right),
\]

(46)

and

\[
\Theta_{2S_{1/2} \rightarrow 2P_{1/2}} = \frac{2 (ma_1)^3 E_1}{m^2} \frac{m + 2 \nu_1^2 - 3 E_1 \kappa}{\nu_1 (4 \nu_1^2 - 1)(\nu_1^2 - 1)}.
\]

(47)

From (45), (46) and (47) there is an energy splitting of the levels equal to

\[
\Delta E_{2S_{1/2} \rightarrow 2P_{1/2}} = 2 \frac{e^4}{4} \frac{(ma_1)^3 E_1}{3m^2} \frac{m + 2 \nu_1^2 - 3 E_1 \kappa}{\nu_1 (4 \nu_1^2 - 1)(\nu_1^2 - 1)} |\theta| \approx \alpha |\Delta E_{2P_{1/2}}|
\]

(48)

This splitting is very similar to the anomalous Zeeman effect or Stark effect at second order. Although the transition energy is small compared to the Bohm shift \( \Delta E_{2P_{1/2}} \). However it remains important in the case of treatment of the hydrogen atom in the framework of noncommutative QCD. This term is necessary to maintain the invariance of the modified Dirac equation under the Seiberg-Witten maps.

### 4 Non-relativistic limit of NC Dirac equation

The non-relativistic limit of the noncommutative Dirac equation (23) corresponds to \( \tilde{\chi} \ll \tilde{\phi} \) [19], where, by restoring the constants \( c \) and \( \hbar \), the wave function takes the new form

\[
\hat{\psi}(t, r, \theta, \phi) = \hat{\psi}'(t, r, \theta, \phi) \exp \left( (-imc^2t/\hbar) \right),
\]

(49)

then, the non-relativistic form of the expression (23) is given by the following set of equations

\[
\left( \frac{i\hbar}{\partial t} - e\Phi \right) \hat{\phi} = c \hat{\sigma} \cdot \left( \hat{\vec{p}} - \frac{e}{\hbar} \hat{\vec{A}} \right) \hat{\chi},
\]

(50)

\[
\left( \frac{i\hbar}{\partial t} - e\Phi + 2mc^2 \right) \hat{\chi} = c \hat{\sigma} \cdot \left( \hat{\vec{p}} - \frac{e}{\hbar} \hat{\vec{A}} \right) \hat{\phi},
\]

(51)
where
\[ \vec{A} = \frac{e^3}{4\hbar c} \left( \vec{\theta} \times \vec{r} \right) \], \quad \vec{\Phi} = - \left( \frac{e}{r} + \frac{e}{2\hbar r^3} \vec{\theta} \cdot \vec{L} \right). \quad (52) \]

If we consider the corrections up to the order of $1/c^2$, we can write the Schrödinger equation of the bi-spinor $\tilde{\varphi}$ as
\[ \varepsilon \varphi_{nM}^{sc} (t, r, \theta, \varphi) = \hat{H} \varphi_{nM}^{sc} (t, r, \theta, \varphi), \quad (53) \]
where
\[ \hat{H} = \frac{1}{2m} \left( \vec{p} - \frac{e}{c} \vec{A} \right)^2 + e\vec{\Phi} - \frac{p^4}{8m^3c^2} - \frac{e\hbar}{2mc} \vec{\sigma} \cdot \left( \vec{\nabla} \times \vec{A} \right), \]
\[ - \frac{e\hbar}{4m^2c^2} \vec{\sigma} \cdot \left( \vec{E} \times \vec{P} \right) - \frac{e^2\hbar^2}{8m^2c^2} \vec{\nabla} \cdot \vec{E}, \quad \vec{E} = -\vec{\nabla} \vec{\Phi}, \quad (54) \]
and
\[ \varphi_{nM}^{sc} (t, r, \theta, \varphi) = R_{nl} (r) \Omega_{j=\pm \frac{1}{2}, M} (\theta, \varphi), \quad (55) \]
\[ R_{nl} (r) = \frac{2}{n^2} \sqrt{\frac{(n-l-1)!}{a_0^3 [(n+l)]^3}} x^l e^{-x/2} l_{2l-1}^{2l+1} (x), \quad x = \frac{2r}{na_0}, \quad a_0 = \frac{\hbar^2}{m e^2}, \quad (56) \]
\[ \Omega_{j=\pm \frac{1}{2}, M} (\theta, \varphi) = \left( \pm \sqrt{\frac{i\pm M+\frac{1}{2}}{2l+1}} \left( Y_{l,M+\frac{1}{2}} (\theta, \varphi) \right) \right). \quad (57) \]
The energy corresponding to $\theta = 0$ in the Schrödinger equation (53) is given by
\[ \varepsilon_n^0 = - \left( \frac{e^2}{\hbar c} \right) \frac{2me^2}{2n^2}, \quad n = 1, 2, 3, \ldots \quad (58) \]

After a straightforward calculation, the equation (54) takes the form:
\[ \hat{H} = \frac{p^2}{2m} - \frac{e^2}{r} - \frac{p^4}{8m^3c^2} - \frac{e^4}{hmc^2r^4} \left( \vec{\theta} \cdot \vec{L} \right) - \frac{e^2}{2\hbar r^3} \left( \vec{\theta} \cdot \vec{L} \right) + \]
\[ + \frac{e^4}{8mc^2r^4} \left[ (\vec{\sigma} \cdot \vec{\theta}) - \frac{4}{r^2} (\vec{\sigma} \cdot \vec{r}) (\vec{\theta} \cdot \vec{r}) \right] + \frac{e^2\hbar}{4m^2c^2r^6} \left[ (\vec{\sigma} \cdot \vec{L}) \right] + \]
\[ + \frac{3}{2\hbar r^2} \left[ \left( \vec{\theta} \cdot \vec{L} \right) (\vec{\sigma} \cdot \vec{L}) + \hbar (\vec{\sigma} \cdot \vec{r}) (\vec{\theta} \cdot \vec{r}) - \hbar (\vec{\sigma} \cdot \vec{\theta}) (\vec{r} \cdot \vec{r}) \right] + \]
\[ + \frac{e^2\hbar^2}{8m^2c^2} \left[ 4\pi \delta (r) \frac{(\vec{\theta} \cdot \vec{L})^2}{h\gamma^3} + \frac{3}{\hbar^2 r^5} \left[ 2\hbar (\vec{\theta} \cdot \vec{L}) - \right] \right. \]
\[ - \left. \left( \vec{\theta} \cdot \vec{L} \right) (\vec{p} \cdot \vec{r}) - \vec{r} \cdot \left( \vec{\theta} \cdot \vec{L} \right) \cdot \vec{p} \right] + O \left( \frac{1}{c^3} \right). \quad (59) \]

This Hamiltonian is the non-relativistic limit of the one in eq. (24), and contains new terms involving the parameter $\theta$ that are similar to the ones of the ordinary
hyperfine splitting: we can say that the noncommutativity in this case plays
the same role as the spin interaction between the proton and the electron in
the presence of a magnetic field, which is responsible for the hyperfine
splitting.

Now to obtain the modification of energy levels as a result of the non-
commutative terms in eq. (59), we use the first-order perturbation theory.
The expectation value of non-vanishing terms of the hamiltonian (59) with
respect to the solution in eq. (53) are given by (θi = θδ3 and l ≠ 0):

\[\langle p^4 \rangle = -\frac{4m^2e^4}{n^3a_0^6} \left[ \frac{1}{l+1/2} - \frac{3}{4n} \right],\]
\[\langle \overrightarrow{\theta} \cdot \overrightarrow{L} \rangle = \theta\hbar m_j \left( 1 \mp \frac{1}{2l+1} \right) \langle r^{-4} \rangle,\]  \(60\)
\[\langle \overrightarrow{\theta} \cdot \overrightarrow{L} \rangle = \theta\hbar m_j \left( 1 \mp \frac{1}{2l+1} \right) \langle r^{-3} \rangle ,\]
\[\langle \overrightarrow{\theta} \cdot \overrightarrow{L} \rangle = \pm \theta \frac{2m_j}{2l+1} \langle r^{-4} \rangle,\]
\[\langle (\overrightarrow{\sigma} \cdot \overrightarrow{\theta}) (\overrightarrow{\theta} \cdot \overrightarrow{\sigma}) \rangle = \pm \theta \frac{2m_j}{2l+1} \left[ \frac{(l + m_j + 1/2) (l - m_j + 1/2)}{(2l + 1)^2} + \frac{(l + m_j + 1/2 \pm 1) (l - m_j + 3/2)}{(2l + 1)^2} \right] \langle r^{-4} \rangle,\]  \(61\)
\[\langle \overrightarrow{\sigma} \cdot \overrightarrow{L} \rangle = \hbar \left[ j (j+1) - l (l+1) - \frac{3}{4} \right] \langle r^{-3} \rangle ,\]
\[\langle \overrightarrow{\sigma} \cdot \overrightarrow{\sigma} \rangle = \theta \hbar^2 m_j \left( 1 \mp \frac{1}{2l+1} \right) \times \left[ j (j+1) - l (l+1) - \frac{3}{4} \right] \langle r^{-5} \rangle ,\]

\[\pi \langle \delta(r) \rangle = \frac{(e^2m)^3}{\hbar^6n^3} \quad \text{for } l = 0 \quad \text{and} \quad 0 \quad \text{for } l \neq 0,\]
\[\langle \overrightarrow{\theta} \cdot \overrightarrow{L} \rangle \overrightarrow{p^2} = 2\theta\hbar e^2m_j \left( 1 \mp \frac{1}{2l+1} \right) \times \left[ \frac{1}{2a_0^2n^2} \langle r^{-3} \rangle + \langle r^{-4} \rangle \right],\]
\[\langle \overrightarrow{\theta} \cdot \overrightarrow{L} \rangle \overrightarrow{r^5} = \theta\hbar m_j \left( 1 \mp \frac{1}{2l+1} \right) \langle r^{-5} \rangle .\]

where we have \( \overrightarrow{S} = \frac{L}{2} \overrightarrow{\sigma} \), \( \langle S_z \rangle = \pm \frac{hm_j}{2l+1} \), \( \langle L_z \rangle = \langle J_z - S_z \rangle = hm_j \left( 1 \mp \frac{1}{2l+1} \right) \) and the positive and negative sings correspond to \( j = l + 1/2 \) and \( j = l - 1/2 \) respectively. In the above \( a_0 \) is the Bohr radius.

Finally the first-order energy correction is
\[\Delta \varepsilon (l \neq 0) = \Delta \varepsilon_0 + \Delta \varepsilon_\theta.\]  \(62\)
The first term \( \Delta \varepsilon_0 \) represents the ordinary fine-structure correction and is given by:

\[
\Delta \varepsilon_0 = \frac{e^4}{2mc^2} \frac{1}{n^3 a_0^3} \left[ \frac{1}{l + 1/2} - \frac{3}{4n} \right] + \frac{e^2 \hbar^2}{4m^2 c^2} \left[ j (j + 1) - l (l + 1) - \frac{3}{4} \right] \langle r^{-3} \rangle.
\]

(63)

The last term \( \Delta \varepsilon_\theta \) is very similar to that of the hyperfine structure correction, where \( \theta \) is now replacing spin and magnetic field, and is given by:

\[
\Delta \varepsilon_\theta = \frac{\theta^2 e^2 m_j}{2} \left\{ \left( -1 + \frac{e^4}{4\hbar^2 c^2 n^2} \right) \left( 1 + \frac{1}{2l + 1} \right) \times \langle r^{-3} \rangle - \frac{e^2}{2mc^2} \left[ \left( \frac{5}{2l + 1} \right) \pm \frac{4}{(2l + 1)} \left[ \frac{(l + m_j + 1/2) (l - m_j + 1/2)}{(2l + 1)^2} + \frac{3}{4} \frac{(l + m_j + 1/2) (l - m_j + 3/2)}{(2l + 1)^2} \right] \right] \times \langle r^{-4} \rangle + \frac{3\hbar^2}{4m^2 c^2} \left( 1 + \frac{1}{2l + 1} \right) \left[ j (j + 1) - l (l + 1) - \frac{5}{4} \right] \times \langle r^{-5} \rangle \right\}.
\]

(64)

where

\[
\langle r^{-3} \rangle = \frac{1}{a_0^3} \frac{1}{l (l + 1/2) (l + 1)},
\]

(65)\[
\langle r^{-4} \rangle = \frac{2}{a_0^3} \frac{1}{(2l + 3) (2l - 1) (l + 1/2)} \left[ -\frac{1}{n^2} + \frac{3}{l (l + 1)} \right],
\]

(66)\[
\langle r^{-5} \rangle = \frac{1}{3a_0^3} \frac{1}{(l + 2) (l - 1) (l + 1/2)} \times \left\{ -\frac{2}{n^2} \frac{1}{l (l + 1)} + \frac{5}{(2l + 3) (l - 1/2)} \left[ -\frac{1}{n^2} + \frac{3}{l (l + 1)} \right] \right\},
\]

(67)

This result shows that, in the non-commutative non-relativistic theory, the degeneracy is completely removed and describes the correction of the fine structure of the spectrum, and corresponds to the hyperfine splitting. Thus by comparing to the data one can get an experimental bound on the value of \( \theta \).

For the case \( l = 0 \) all terms in eq. (61) vanish except the fourth and fifth ones. These terms give a divergence and hence we use the \( \Lambda_{\text{QCD}} \) (\( \sim 200 \text{MeV} \)) cutoff (see ref [23]) and obtain the result:

\[
\left\langle \left( (\vec{\sigma} \cdot \vec{\theta})/r^4 - 4 (\vec{\sigma} \cdot \vec{r}) (\vec{\theta} \cdot \vec{r})/r^6 \right) \right\rangle_{1S} = \frac{4\theta}{3} \alpha^3 m^3 \Lambda_{\text{QCD}}.
\]

(68)

From equation (65) we obtain the modified energy level in noncommutative space-space in the non-relativistic limit for the state 1S:

\[
\Delta \varepsilon_\theta = \frac{\theta}{6} \alpha^5 m^2 \Lambda_{\text{QCD}}
\]

(69)
According to Ref. [22] the current theoretical accuracy on the 1S Lamb shift is about 14 kHz. From the splitting (68), the bound is given by

$$\theta \lesssim (5.6 GeV)^{-2}$$ (70)

This value is better than the limit obtained in [14, 23] and it justifies our expansion of the Hamiltonian in eq (26).

5 Conclusions

In this work we proposed an invariant noncommutative action for a Dirac particle under the generalised infinitesimal gauge transformations. Using the Seiberg-Witten maps and the Moyal product, we generalised the equation of motion with a noncommutative space-space and derived the modified Dirac equation for a Coulomb potential to the first order of $\theta$. By perturbation-theory methods in first order, we derived the noncommutative corrections of the energy. In addition to the Hamiltonian given in [10] where the authors have used the noncommutative Bopp-shift, another term appears in the Hamiltonian which is similar to the interaction term describing charged particles in a non-zero magnetic field. This non-diagonal term is a vectorial potential due to the invariance of the modified Dirac equation under the Seiberg-Witten maps. In this case the degeneracy of energy-level states is removed and the lamb-shift is induced. The bound we found on $\theta$ has the same order of magnitude obtained in ref. [10].

In the non-relativistic limit, we have obtained a general modified form of the Hamiltonian of the hydrogen atom with new terms involving the $\theta$ parameter. This expression is similar to the hyperfine structure one. The expression of the Hamiltonian (in the non-relativistic limit) which describes the hyperfine correction in the hydrogen atom imply that the noncommutativity plays the role of the magnetic field (Zeeman effect) and the role of the spin (of the proton or nucleon). Then the interaction electron-nucleon is equivalent to an electron in a noncommutative space-space. In this case, the degeneracy of the energy-level states is completely removed and the bound on $\theta$ was derived.

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