The Viability Property for Path-dependent SDE under Open Constraints

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Abstract

This paper is devoted to the viability of a bounded open domain in $\mathbb{R}^n$ for a process driven by a path-dependent stochastic differential equation with Lipschitz data. We extend the relevant results Theorem 3.2 in \cite{11} obtained by Cannarsa, Da. Prato and Frankowska \textit{Indiana Univ. Math. J.} \textbf{59} (2010) 53-78 to non-Markovian setting.

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1 Introduction

Let $v = \left( \Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P, W \right)$ be a reference probability system composed of a completed probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual assumptions of right-continuity and completeness, and a $d$-dimensional $(\mathcal{F}_t)$-Brownian motion $W$ defined on $(\Omega, \mathcal{F}, P)$.

Recently, Dupire \cite{18} 2009 defined a notion of directional derivative for functionals. Using this notion, Cont and Fournié \cite{13} extended Föllmer’s (see \cite{19, 20}) pathwise change of variable formula to non-anticipative functionals on the space of cadlag paths. Their results lead to functional extension of the Itô’s formula for continuous semimartingale on open set (see Proposition 7, in \cite{13}).

Let $U$ denote an open domain in $\mathbb{R}^n$. In particular, consider a continuous martingale $X$ governed by a stochastic differential equation (SDE in short) in the sense that the coefficients of the SDE are allowed to depend on the path of the process which, of course, is resolutely in a non-Markovian setting. More
previously, the SDE is defined as follows:

\[
\begin{aligned}
&dX^x_t(r) = \mu(X^x_t)\,dr + \sigma(X^x_t)\,dW(r), \\
&X^x_t(s) = \chi_t(s), \quad \text{if } 0 \leq s \leq t,
\end{aligned}
\]

(1.1)

where \( \chi_t \in \Lambda U_t \) which denotes all the right-continuous functions with left limits defined on \([0, t]\) valued in \( U \). \( \mu \) and \( \sigma \) are functionals on \( \Lambda \) which denotes all the right-continuous functions with left limits defined on \([0, t]\) for \( \forall t > 0 \), valued in \( \mathbb{R}^n \). Besides, denote by \( \chi(t) \) the value of \( \chi \) at \( t \) and by \( \chi_t \) the restriction on \([0, t]\), the same for \( X^x_t \).

A natural question arises, if \( \chi_t \in \Lambda U_t \), under what assumptions can one claim

\[ X^x_t(r) \in U, \quad r \in [t, +\infty) \text{ } P\text{-a.s.} \]

We study this problem by means of viability theory (see [1] [2] for more details).

Let \( K \) be a closed subset of \( \mathbb{R}^n \). We say that \( K \) is viable for (1.1) if and only if for any \( \chi_t \in \Lambda K_t \), the solution to (1.1) satisfies

\[ X^x_t(s) \in K, \quad \forall s \geq t, \text{ } P\text{-almost surely.} \]

(1.2)

In fact, the property of viability of \( K \) for the classical systems have been extensively studied. We refer the reader to the monographs [1] for the deterministic case. For the stochastic case, several results have been obtained: through stochastic tangent cones in [11]-[15], through viscosity solutions of partial differential equations in [6]-[10] (for more information see references therein). We mention that in [11], in order to study the existence and uniqueness of the invariant measure associated to the transition semigroup of a diffusion process in a bounded open subset of \( \mathbb{R}^n \), the authors investigated the invariance of a bounded open domain with piecewise smooth boundary for a Markov system. Motivated by this paper, we consider the viability for (1.1) under some open smooth domain.

The rest of this paper is organized as follows: after some preliminaries in the second section, we are devoted the third section to developing the viability result for some smooth domain.

### 2 Preliminaries and Notations

Throughout this paper, the notations are mainly taken from Cont and Fournié [12] [13] [14] and Dupire [18]. For a cadlag path \( x \in D([0, T], \mathbb{R}^n) \), denote by \( x(t) \) the value of \( x \) at \( t \) and by \( x_t = (x(u), 0 \leq u \leq t) \) the restriction of \( x \) to \([0, t]\). Thus \( x_t \in D([0, t], \mathbb{R}^n) \). Similarly, for a stochastic process \( X \) we shall denote \( X(t) \) its value at \( t \) and \( X_t = (X(u), 0 \leq u \leq t) \) its path on \([0, t]\).

Let \( T > 0 \) be a fixed time horizon and \( U \subset \mathbb{R}^n \) an open subset of \( \mathbb{R}^n \) and \( S \subset \mathbb{R}^m \) be a Borel subset of \( \mathbb{R}^m \). We denote the boundary of \( U \) by \( \partial U \), the closure of \( U \) by \( \overline{U} = U \cup \partial U \). We call "\( U \)-valued cadlag function" a right continuous \( f : [0, T] \mapsto U \) with left limits such that for each \( t \in [0, T] \), \( f(t-) \in U \). Denote by \( U_t = D([0, t], U) \) (resp. \( S_t = D([0, t], S) \)) the space of \( U \)-valued cadlag functions (resp. \( S \)), and \( C_0([0, t], U) \) the set of continuous functions with values in \( U \).

**Definition 2.1** (Non-anticipative functionals on path space). A non-anticipative functional on \( U_T \) is a family \( F = (F_t)_{t \in [0, T]} \) of maps

\[ F_t : U_t \rightarrow \mathbb{R}. \]
We consider throughout this paper non-anticipative functionals

\[ F = (F_t)_{t \in [0,T]}, \quad F_t : \mathcal{U}_t \times \mathcal{S}_t \to \mathbb{R}, \]

where \( F \) has a "predictable" dependence with respect to the second argument:

\[ \forall t \in [0,T], \forall (x, v) \in \mathcal{U}_t \times \mathcal{S}_t, \quad F_t (x_t, v_t) = F_t (x_t, v_{t-}). \quad (2.1) \]

\( F \) can be view as a functional on the vector bundle \( \mathcal{Y}^{U \times S} = \cup_{t \in [0,T]} \mathcal{U}_t \times \mathcal{S}_t. \)

For each \( \gamma_t \in \mathcal{U}_t \) denote

\[ \gamma_{t,\delta} (s) = \begin{cases} \gamma_t (s), & \text{for } 0 \leq s < t, \\ \gamma_t (t), & \text{for } t \leq s \leq t + \delta, \end{cases} \]

and for \( x \in \mathbb{R}^n \) small enough,

\[ \gamma_{t}^{x} (s) = \begin{cases} \gamma (s), & \text{for } 0 \leq s < t, \\ \gamma (t) + x, & \text{for } s = t. \end{cases} \]

Apparently, \( \gamma_{t,\delta} \in \mathcal{U}_{t+\delta} \). We also denote

\[ (\gamma_{t}^{x})_{t,\delta} (s) = \begin{cases} \gamma (s), & \text{for } 0 \leq s < t, \\ \gamma (t) + x, & \text{for } t \leq s \leq t + \delta. \end{cases} \]

Let \( \bar{\gamma}_t, \gamma_t \in \mathcal{U}_t \) be given with \( \bar{t} \leq t \), we denote \( \bar{\gamma}_t \otimes \gamma_t \in \mathcal{U}_t \) by

\[ \bar{\gamma}_t \otimes \gamma_t = \begin{cases} \bar{\gamma}_t (s), & \text{for } 0 \leq s < \bar{t}, \\ \gamma_t (s), & \text{for } \bar{t} \leq s < t. \end{cases} \]

We now introduce a distance between two paths, not necessarily defined on the same time interval. For \( T \geq t' = t + h \geq t \geq 0 \), \( (x, v) \in \mathcal{U}_t \times \mathcal{S}_t \) and \( (x', v') \in D([0, t+h], \mathbb{R}^n) \times \mathcal{S}_{t+h} \) define

\[ ||x|| = \sup_{0 \leq s \leq t} |x_t (s)|, \]

\[ d_\infty ((x, v), (x', v')) = \sup_{r \in [0, t+h]} |x_{t+h} (r) - x' (r)| + \sup_{r \in [0, t+h]} |v_{t+h} (r) - v' (r)| + h. \]

(\( \mathcal{Y}^{U \times S}, d_\infty \)) is a metric space.

**Definition 2.2** (Continuous at fixed times). A non-anticipative functional \( F = (F_t)_{t \in [0,T]} \) is said to be continuous at fixed times if for any \( t \leq T, F_t : \mathcal{U}_t \times \mathcal{S}_t \to \mathbb{R} \) is continuous for the supremum norm.

**Definition 2.3** (Left-continuous functionals). Define \( \mathbb{F}_t^\infty \) as the set of functionals \( F = (F_t, t \in [0,T]) \) which verify:

\[ \forall t \in [0,T], \epsilon > 0, \forall (x, v) \in \mathcal{U}_t \times \mathcal{S}_t, \exists \eta > 0, \forall h \in [0, t], \]

\[ \forall (x', v') \in \mathcal{U}_{t-h} \times \mathcal{S}_{t-h}, d_\infty ((x, v), (x', v')) < \eta \Rightarrow |F_t (x, v) - F_{t-h} (x', v')| < \epsilon. \]

**Definition 2.4** (Right-continuous functionals). Define \( \mathbb{F}_r^\infty \) as the set of functionals \( F = (F_t, t \in [0,T]) \) which verify:

\[ \forall t \in [0,T], \epsilon > 0, \forall (x, v) \in \mathcal{U}_t \times \mathcal{S}_t, \exists \eta > 0, \forall h \in [0, T-t], \]

\[ \forall (x', v') \in \mathcal{U}_{t+h} \times \mathcal{S}_{t+h}, d_\infty ((x, v), (x', v')) < \eta \Rightarrow |F_t (x, v) - F_{t+h} (x', v')| < \epsilon. \]

Denote \( \mathbb{F}^\infty = \mathbb{F}_t^\infty \cap \mathbb{F}_r^\infty \) the set of continuous non-anticipative functionals.
Definition 2.5 (Boundedness-preserving functionals). Define $\mathcal{B}$ as the set of non-anticipative functionals $F$ such that for any compact $K \subset U$ and any $R > 0$, there exists $C_{K,R} > 0$ such that:

$$\forall t \leq T, \forall (x, v) \in D([0, t], K) \times S_t, \sup_{s \in [0,t]} |v(s)| < R \Rightarrow |F_t(x, v)| < C_{K,R}.$$ 

Next we introduce two notions of pathwise derivatives for a non-anticipative functional $F = (F_t)_{t \in [0,T]}$: the horizontal derivative and vertical derivative, respectively.

Definition 2.6 (Horizontal derivative). The horizontal derivative at $(x, v) \in U_t \times S_t$ of non-anticipative functional $F = (F_t)_{t \in [0,T]}$ is defined as

$$\mathcal{D}_t F (x, v) = \lim_{h \to 0^+} \frac{F_{t+h} (x_{t,h}, v_{t,h}) - F_t (x, v)}{h}, \quad (2.2)$$

if the corresponding limit exists. If (2.2) is defined for all $(x, v) \in \mathcal{Y}$ the map

$$\mathcal{D}_t F : U_t \times S_t \mapsto \mathbb{R}^d, \quad (x, v) \mapsto \mathcal{D}_t F (x, v)$$

defines a non-anticipative functional $\mathcal{D} F = (\mathcal{D}_t F)_{t \in [0,T]}$, the horizontal derivative of $F$.

Definition 2.7 (Dupire derivative). A non-anticipative functional $F = (F_t)_{t \in [0,T]}$ is said to be vertically differentiable at $(x, v) \in D([0, t], \mathbb{R}^n) \times D([0, t], S^n_t)$ if

$$\mathbb{R}^n \to \mathbb{R} \quad e \mapsto F_t(x^e_t, v_t)$$

is differentiable at 0. Its gradient at 0

$$\nabla_x F_t (x, v) = (\partial_i F_t (x, v), i = 1, \cdots, d)$$

where

$$\partial_i F_t (x, v) = \lim_{h \to 0} \frac{F_t \left( x^i_{t,h}, v \right) - F_t (x, v)}{h}, \quad (2.3)$$

is called the vertical derivative of $F_t$ at $(x, v)$. If (2.3) is defined for all $(x, v) \in \mathcal{Y}$, the vertical derivative

$$\nabla_x F : U_t \times S_t \mapsto \mathbb{R}^n, \quad (x, v) \mapsto \nabla_x F_t (x, v)$$

defines a non-anticipative functional $\nabla_x F = (\nabla_x F_t)_{t \in [0,T]}$ with value in $\mathbb{R}^n$.

Remark 2.1. If a vertically differentiable functional satisfies (2.7), its vertical derivative also satisfies (2.1).

Remark 2.2. If $F_t(x, v) = f (t, x(t))$ with $f \in C^{1,1}([0, T] \times \mathbb{R}^n)$ then we retrieve the usual partial derivatives:

$$\mathcal{D}_t F_t (x, v) = \partial_t f (t, x(t)), \quad \nabla_x F_t (x, v) = \nabla_x f (t, x(t)).$$

Remark 2.3. Note that if $F$ is predictable with respect to the second variable entails that for any $t \in [0, T], F_t(x_t, v^e_t) = F_t(x_t, v_t)$ so an analogous notion of derivative with respect to $v$ would be identically zero under (2.1).

Definition 2.8. Let $I \subset [0, T]$ be a subinterval of $[0, T]$. Define $C^{j,k}(I)$ as the set of non-anticipative functionals $F = (F_t)_{t \in I}$ such that
• $F$ is continuous at fixed times: $F_t : \mathcal{U}_t \times \mathcal{S}_t \mapsto \mathbb{R}$ is continuous for the supremum norm.
• $F$ admits $j$ horizontal derivative and $k$ vertical derivatives at all $(x,v) \in \mathcal{U}_t \times \mathcal{S}_t$, $t \in I$.
• $D^m F$, $m \leq j$, $\nabla^n_x F$, $n \leq k$ are continuous at fixed times $t \in I$.

We now introduce the following result (taken from Proposition 7 in [13]):

**Proposition 2.1** (Functional Itô formula for a continuous semimartingale on open set). Let $X$ be a continuous semimartingale valued in an open domain $U$ subset of $\mathbb{R}^n$ with quadratic variation process $[X]$, and $A$ a continuous adapted process, on some filtered probability space $\left( \tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{P} \right)$. Then for any non-anticipative functional $F \in \mathcal{C}^{1,2}([0,T])$ satisfying

(a) $F$ has a predictable dependence with respect to the second variable, i.e. verifies (2.1),
(b) $\nabla_x F, \nabla^2_x F, DF \in \mathbb{B}$,
(c) $F \in \mathcal{F}^\infty_t$,
(d) $\nabla_x F, \nabla^2_x F \in \mathcal{F}^\infty_r$,

we have

$$F_s(X_s, A_s) - F_t(X_t, A_t) = \int_t^s D_u F (X_u, A_u) \, du + \frac{1}{2} \int_t^s \text{Tr} \left[ \nabla^2_x F_u (X_u, A_u) \, d[X] (u) \right]$$
$$+ \int_t^s \nabla_x F_u (X_u, A_u) \, dX (u), \quad P\text{-a.s.,}$$

where last term is the Itô’s stochastic integral with respect to the $X$.

Now consider the continuous semimartingale $X$ driven as in (1.1) and denote

$$[X](s) = \int_t^s \sigma (X_r) \sigma^* (X_r) \, dr = \int_t^s A (r) \, dr$$

its quadratic variation process. From now on, let $A (r) = \sigma (X_r) \sigma^* (X_r)$, which takes values in the set $\mathbb{S}^+_n$ of symmetric positive $n \times n$ matrices, has cadlag paths. We define the pair $(X_r, A_r) = (X_r, \sigma (X_r) \sigma^* (X_r))$. Note that $A$ needs not to be a semimartingale. In particular, $F(X_u, A_u) = F(X_u, A_{u-})$ where $A_{u-}$ denotes the path defined on $[0, u]$ by

$$A_{u-} (r) = A (r), \quad r \in [0, u), \quad A_{u-} (u) = A (u-).$$

For our purpose, let $U$ be an open subset of $\mathbb{R}^n$ with nonempty boundary $\partial U$ and closure $\overline{U}$. We introduce some useful notations which are mainly taken and amended from [21].

$$\Lambda_{\mathcal{U}_t} \triangleq \{ \gamma_t \in D ([0,t], \mathbb{R}^n) : \gamma (s) \in \mathcal{U}, s \in [0,t] \},$$
$$\Lambda_U \triangleq \bigcup_{t \in [0, +\infty)} \Lambda_{\mathcal{U}_t},$$
$$\Lambda_{\overline{\mathcal{U}}_t} \triangleq \{ \gamma_t \in D ([0,t], \mathbb{R}^n) : \gamma (s) \in \mathcal{U}, s \in [0,t], \gamma (t) \in \overline{\mathcal{U}} \},$$
$$\Lambda_{\overline{U}} \triangleq \bigcup_{t \in [0, +\infty)} \Lambda_{\overline{\mathcal{U}}_t},$$
$$\Lambda_{\partial \mathcal{U}_t} \triangleq \{ \gamma_t \in D ([0,t], \mathbb{R}^n) : \gamma (s) \in \mathcal{U}, s \in [0,t], \gamma (t) \in \partial \mathcal{U} \},$$
$$\Lambda_{\partial U} \triangleq \bigcup_{t \in [0, +\infty)} \Lambda_{\partial \mathcal{U}_t},$$
$$\Lambda \triangleq \bigcup_{t \in [0, +\infty)} D ([0,t], \mathbb{R}^n).$$
We now make the following assumptions for SDE (1.1).

\((H1)\) \(\mu\) and \(\sigma\) are two functionals satisfying

\[
\begin{align*}
\mu : \Lambda &\to \mathbb{R}^n, \\
\sigma : \Lambda &\to \mathbb{R}^{n \times d}.
\end{align*}
\]

\((H2)\) \(\mu\) and \(\sigma\) are Lipschitz in \(x \in \Lambda\), i.e., there is some constant \(C_1 > 0\) such that

\[
|\mu (x_t) - \mu (x'_t)| + |\sigma (x_t) - \sigma (x'_t)| \leq C_1 \|x_t - x'_t\|,
\]

for \(x_t, x'_t \in \Lambda\), and satisfy linear growth, i.e., there is some constant \(C_2 > 0\) such that

\[
|\mu (x_t)| + |\sigma (x_t)| \leq C_2 (1 + \|x_t\|),
\]

for \(x_t \in \Lambda\).

Remark 2.4. Note that \(\mu\) and \(\sigma\) are not necessary non-anticipative functionals.

Lemma 2.1. Assume that the assumptions \((H1)-(H2)\) hold. Then, there exists a unique strong solution to (1.1).

The proof can be found in [23].

In this paper, we also use the following notations mainly taken from [11].

- Let \(C^{2,1} (A)\) denote all twice differentiable functions on \(A\), with bounded Lipschitz second order derivatives, where \(A\) is an open subset of \(\mathbb{R}^n\).

- Let \(S \subset \mathbb{R}^n\) be a nonempty set. We denote by \(d_S\) the Euclidean distance function from \(S\), that is,

\[
d_S (x) = \inf_{y \in S} |x - y|, \quad \forall x \in \mathbb{R}^n.
\]

If \(S\) is closed, then the above infimum is a minimum, which is attained on a set that will be called the projection of \(x \in \mathbb{R}^n\) onto \(S\), i.e.

\[
\text{Proj}_S (x) = \{y \in S : |x - y| = d_S (x)\}, \quad \forall x \in \mathbb{R}^n.
\]

For every \(t \in [0, +\infty)\), \(x \in S\), the hitting time of \(\partial S\) is the random variable defined by

\[
\tau_S (\chi_t) = \inf \{s \geq t : X^{\chi_t} (s) \in \partial S\}, \quad \forall \chi_t \in \Lambda.
\]

- Let \(\mathcal{K}\) be a closed subset of \(\mathbb{R}^n\) with nonempty interior \(\overset{\circ}{\mathcal{K}}\) and boundary \(\partial \mathcal{K}\). We introduce the oriented distance function from \(\partial \mathcal{K}\), i.e., the function

\[
b_{\mathcal{K}} (x) = \begin{cases} 
    d_{\partial \mathcal{K}} (x) & \text{if } x \in \overset{\circ}{\mathcal{K}}, \\
    0 & \text{if } x \in \partial \mathcal{K}, \\
    -d_{\partial \mathcal{K}} (x) & \text{if } x \in \mathcal{K}^c,
\end{cases}
\]

where \(\mathcal{K}^c\) is the complimentary of \(\mathcal{K}\). In what follows, we will use the following sets, defined for any
$\varepsilon > 0$

\[
\begin{align*}
N_{\varepsilon} & \triangleq \{ x \in \mathbb{R}^n : |b_K(x)| < \varepsilon \}, \\
K_{\varepsilon} & \triangleq K \cap N_{\varepsilon}, \\
\overset{\circ}{K}_{\varepsilon} & \triangleq \overset{\circ}{K} \cap N_{\varepsilon}.
\end{align*}
\]

3 Main Result

In this section, we study the viability properties of a compact $C^{2,1}$-smooth domain $K$ defined as following with respect to the flow $X^{\chi_t}(\cdot)$ associated with the SDE (1.1) (with Lipschitz continuous coefficients $\mu$ and $\sigma$ satisfying (H1) and (H2)).

Keep in mind that $K$ is a compact domain of class $C^{2,1}$. From Theorem 5.6 in [17], we know that

\[
K \text{ compact domain of class } C^{2,1} \iff \exists \varepsilon_0 > 0 : b_K \in C^{2,1}(N_{\varepsilon_0}). \tag{3.1}
\]

A useful consequence of the above property is that

\[
\forall x \in K_{\varepsilon_0}, \left\{ \begin{array}{l}
\exists! \bar{x} \in \partial K : b_K(x) = |x - \bar{x}|, \\
D_x b_K(x) = D_x b_K(\bar{x}) = -\bar{n}_K(\bar{x}),
\end{array} \right.
\]

where $\bar{n}_K(\bar{x})$ stands for the outward unit normal to $K$ at $\bar{x}$. See, e.g. [17]. Besides, if $K$ is a compact domain of class $C^{2,1}$, one can see that there is a sequence $\{Q_i\}$ of compact domains of class $C^{2,1}$ such that

\[
Q_i \subset \overset{\circ}{Q}_{i+1} \text{ and } \bigcup_{i=1}^{\infty} Q_i = \overset{\circ}{K}. \tag{3.2}
\]

Indeed, it suffices to take that, for all $i$ large enough,

\[
Q_i = \left\{ x \in K : b_K(x) \geq \frac{1}{i} \right\}.
\]

Now suppose that

\[
0 \notin \text{co} \{ \nabla_x b_K(x) \}, \quad \forall x \in \partial K. \tag{3.3}
\]

Then, Clarke’s tangent cone to $K$ at every $x \in K$ has nonempty interior. For this reason, $K$ coincides with the closure of $\overset{\circ}{K}$ (for more details see [2][11]). Finally, we note that the existence of a sequence $\{Q_i\}$ of compact domain of class $C^{2,1}$ satisfying (3.2) is also guaranteed when $K$ is compact set with above properties (3.3).

We now give our main result as follows:

**Theorem 3.1.** Suppose that assumptions (H1), (H2), and (3.3) hold. Then the following three statements are equivalent:

(i) $K$ enjoys the viability with respect to (1.1);

(ii) For $\forall \chi_t \in \Lambda_{\partial K}$, we have

\[
\begin{align*}
\mathcal{L}_{(\chi_t, A_t)} (-b_K)(\chi_t(t)) & \leq 0, \\
\sigma^* (\chi_t) \nabla_x b_K (\chi_t(t)) & = 0,
\end{align*}
\]

(3.4)
where

\[ L_{(\chi_t, A_t)} \varphi_t = D_t \varphi + \langle \nabla_x \varphi_t , \mu \rangle + \frac{1}{2} \text{Tr} \left( \nabla^2_x \varphi_t A(t) \right), \]

and \( A(t) = \sigma (\chi_t) \sigma^* (\chi_t) \), for \( \varphi \in C^{2,1} ([0, T]) \);

(iii) \( \mathcal{K} \) enjoys the viability with respect to \( (\mathcal{D}, \mathcal{A}) \).

For the proof of this theorem we need two auxiliary lemmata.

**Lemma 3.1.** Assume that (H1) and (H2) hold. Then for any compact domain \( \mathcal{D} \) of \( \mathbb{R}^n \) and for any \( \chi_t \in \Lambda_{\mathcal{D}_t} \), the solution \( X^{\chi_t} \) to (1.1) satisfies

\[ X^{\chi_t} (s) \in \mathcal{D}, \ s \in [t, +\infty), \ a.s. \quad (3.5) \]

Then, for any \( C^{2,1} ([0, T]) \) non-anticipative functional \( \varphi \) satisfying (a)-(d) in Proposition 2.1 with a left frozen maximum (see Lemma 6 in [21]) on \( \mathcal{Y}^{\mathcal{D} \times \mathcal{S}^+_n} \) at \( (\chi_t, A_t) \), we have

\[
\begin{cases}
D_t \varphi (\chi_t, A_t) + \langle \nabla_x \varphi_t (\chi_t, A_t) , \mu (\chi_t) \rangle + \frac{1}{2} \text{Tr} \left( \nabla^2_x \varphi_t (\chi_t, A_t) A(t) \right) \leq 0, \\
\sigma^* (\chi_t) \nabla_x \varphi_t (\chi_t, A_t) = 0.
\end{cases}
\]

**Proof.** Suppose that \( \varphi \) is the non-anticipative test functional with a left frozen maximum on \( \mathcal{Y}^{\mathcal{D} \times \mathcal{S}^+_n} \) at \( (\chi_t, A_t) \) with \( t \in [0, T) \). More precisely, for any given but fixed \( (y_s, z_s) \in \mathcal{Y}^{\mathcal{D} \times \mathcal{S}^+_n} \), \( s \in [t, T) \), we have

\[ \varphi_s (\chi_t \otimes y_s, A_t \otimes z_s) \leq \varphi_t (\chi_t, A_t). \quad (3.6) \]

Immediately, from (3.5) and (3.6), we have

\[ \forall s \geq t, \ \varphi_s (X^{\chi_t}, A_s) \leq \varphi_t (\chi_t, A_t), \quad P\text{-a.s.} \]

From standard arguments, the fact that this inequality holds for any \( s \geq t \), implies that, by virtue of Proposition 2.1

\[ D_t \varphi (\chi_t, A_t) + \langle \nabla_x \varphi_t (\chi_t, A_t) , \mu (\chi_t) \rangle + \frac{1}{2} \text{Tr} \left( \nabla^2_x \varphi_t (\chi_t, A_t) A(t) \right) \leq 0. \quad (3.7) \]

Let \( \beta : \mathbb{R} \to \mathbb{R} \) be an increasing function such that \( \beta' (\varphi_t (\chi_t, A_t)) = 1 \) and \( \beta'' (\varphi_t (\chi_t, A_t)) = \lambda \) where \( \lambda > 0 \) is arbitrary. It is easy to check that \( \beta \circ \varphi_t \) also attains a left frozen maximum on \( \mathcal{Y}^{\mathcal{D} \times \mathcal{S}^+_n} \) at \( (\chi_t, A_t) \). Hence, from (3.7), we have at \( (\chi_t, A_t) \) that

\[
D_t \varphi + \langle \mu, \nabla_x \varphi_t \rangle + \frac{1}{2} \text{Tr} \left( \nabla^2_x \varphi_t \sigma^* \sigma \right) + \frac{\lambda}{2} \text{Tr} \left( \sigma^* \sigma \nabla_x \varphi_t \nabla^*_x \varphi_t \right) \leq 0. \quad (3.8)
\]

where we have omitted the dependence in \( (\chi_t, A_t) \) for simplicity. Obviously,

\[ \lambda \text{Tr} \left( \sigma^* \sigma (\chi_t) \nabla_x \varphi_t (\chi_t, A_t) D^*_x \varphi_t (\chi_t, A_t) \right) = \lambda |\sigma^* (\chi_t) \nabla_x \varphi_t (\chi_t, A_t)|^2. \]

Noting that (3.8) is still bounded as \( \lambda \to +\infty \), we imply that

\[ \sigma^* (\chi_t) \nabla_x \varphi_t (\chi_t, A_t) = 0. \quad (3.9) \]

The proof is complete. \( \square \)

**Remark 3.1.** We do not impose the assumption (compact \( C^{2,1} \)-smooth domain) on \( \mathcal{D} \) in Lemma 3.1.
As a consequence of Lemma 3.1, we have the following:

**Corollary 3.1.** Suppose the same assumptions in Lemma 3.1 but take $K$ instead of $D$. Set
\[
\varphi (x_t, A_t) = -b_K (x_t (t)), \quad x_t \in \Lambda_{K_1}, \quad t \in [0, +\infty).
\]
Then we have, for $\forall x_t \in \Lambda_{\partial K_1}$
\[
\begin{cases}
\mathcal{L}_{(\chi_t, A_t)} (-b_K) (x_t (t)) \leq 0, \\
\sigma^* (x_t) \nabla x b_K (x_t (t)) = 0.
\end{cases}
\tag{3.10}
\]

**Proof.** Let us check that $\varphi (x_t, A_t) = -b_K (x_t (t))$ satisfies (b), (c), (d) in Proposition 2.1. First, (b) can be verified by assumption on $K$ which is a compact domain of class $C^{2,1}$. For (c), let $x \in \Lambda_{K_1}$, $x' \in \Lambda_{K_{1-h}}$. Then, we have
\[
\left| b_K (x_t (t) - b_K (x'_{t-h} (t-h)) \right| \leq \left| x_t (t) - x'_{t-h} (t-h) \right| \leq d_{\infty} (x_t, x'_{t-h} ),
\]
Similarly, let $x \in \Lambda_{K_{1+h}}$, $x' \in \Lambda_{K_{1+h}}$. We also have
\[
\left| b_K (x_t (t) - b_K (x'_{t+h} (t+h)) \right| \leq \left| x_t (t) - x'_{t+h} (t+h) \right| \leq d_{\infty} (x_t, x'_{t+h} ),
\]
since the Lipschitz constant of $b_K$ is one. It remains to check (d). In fact, by (3.1) and the definition of $C^{2,1} (A)$, $\nabla x b_K$, $\nabla^2 x b_K$ can be also proved to be locally Lipschitz for the metric $d_{\infty}$. \hfill \square

**Remark 3.2.** In Theorem 3.2 of [11], condition (b) says $\forall x_0 \in \partial K$, $\sigma^* (x_0) \nabla x b_K (x_0) = 0$ which means that $\langle \sigma^* (x_0), \nabla x b_K (x_0) \rangle = 0$, $i = 1, \ldots, d$, where $\sigma^* (x_0)$ denotes the $i$th row of $\sigma^* (x_0)$. In our paper, for $\forall \bar{x}_t \in \Lambda_{\partial K_1}$, $\langle \sigma^* (\bar{x}_t), \nabla x b_K (\bar{x} (t)) \rangle = 0$, $i = 1, \ldots, d$, which contain more information, in the sense that, for any $x \in \Lambda_{\partial K_1}$, $x' \in \Lambda_{\partial K_{1-h}}$ with $t < t'$, but $x_t (t) = x'_{t'} (t') = x_0 \in \partial K$, we may have $\sigma^*_i (x_t) \neq \sigma^*_i (x'_{t'})$, but $\sigma^*_i (x_t), \sigma^*_i (x'_{t'}) \in \{ p \in \mathbb{R}^n : \langle p, \nabla x b_K (x_0) \rangle = 0 \}$.

Now let us turn back to compact $C^{2,1}$-smooth domain $K$. The following approach is mainly borrowed from [11]. It is not restrictive to assume that $\varepsilon_0 > 0$ is such that, there exists a function $g \in C^{2,1} (\mathbb{R}^n)$ satisfying
\[
\begin{cases}
0 \leq g \leq 1, \quad \text{on } K, \\
0 < g, \quad \text{on } K \setminus \kappa_{\varepsilon_0}, \\
g \equiv b_K, \quad \text{on } \kappa_{\varepsilon_0}.
\end{cases}
\]

Now define
\[
\Psi (x_t, A_t) = -\log (g (x_t (t))), \quad \forall x_t \in \Lambda_{\kappa_{\varepsilon_0}}.
\]

**Lemma 3.2.** Assume that (H1), (H2), and (3.3) hold. Then, there exists a positive constant $M > 0$, such that
\[
\mathcal{L}_{(\chi_t, A_t)} \Psi (x_t, A_t) \leq M, \quad \forall x_t \in \Lambda_{\kappa_{\varepsilon_0}}, \quad \forall t \in \mathbb{R}^+, \tag{3.11}
\]
where
\[
\mathcal{L}_{(\chi_t, A_t)} \Psi (x_t, A_t) = \frac{1}{g^2 (x_t (t))} \left| \sigma^* (x_t) \nabla x g (x_t (t)) \right|^2 - \frac{1}{g (x_t (t))} \mathcal{L}_{(\chi_t, A_t)} g (x_t (t)).
\]

**Proof.** To begin with, we first give some estimations which will be very useful in the sequel. We first notice that
\[
\frac{\partial}{\partial t} \Psi (x_t, A_t) \equiv 0 \quad \text{and} \quad \Psi (x_t, A_t) = \lim_{\Lambda_{\kappa_{\varepsilon_0}} d_{\infty} y_t \to x_t} \Psi (y_t, A_t) = +\infty, \quad \forall x_t \in \Lambda_{\partial K_1}.
\]
Then, after a simple calculation, we have
\[
\mathcal{L}_{(\chi_t, A_t)} \Psi (\chi_t, A_t) = \frac{1}{g(\chi_t(t))} |\sigma^* (\chi_t) D_x g (\chi_t (t))|^2 - \frac{1}{g(\chi_t(t))} \mathcal{L}_{(\chi_t, A_t)} g (\chi_t (t)) , \quad \chi_t \in \Lambda_{K_t}.
\]
We claim that, for any \( \chi_t \in \Lambda_{K_t} \),
\[
\frac{1}{g^2(\chi_t(t))} |\sigma^* (\chi_t) D_x g (\chi_t (t))|^2 - \frac{1}{g(\chi_t(t))} \mathcal{L}_{(\chi_t, A_t)} g (\chi_t (t)) \leq M , \quad \chi_t (t) \in \mathcal{K}, \quad (3.12)
\]
for some constant \( M \geq 0 \). Indeed, the above estimate holds true, when \( \chi_t (t) \in \mathcal{N}_{\varepsilon_0}^c , \ b_K (x) \geq \varepsilon_0 \) since \( g \) is strict positive on \( \mathcal{N}_{\varepsilon_0}^c \) and the assumption (H2), where \( \mathcal{N}_{\varepsilon_0}^c = \left\{ x \in \mathcal{K} : b_K (x) \geq \varepsilon_0 \right\} \). Hence, we have to show \( (3.12) \) holds for all \( \chi_t \in \Lambda_{K_t} \) satisfying \( \chi_t (t) \in \mathcal{K} \cap \mathcal{K}_{\varepsilon_0} \), i.e.,
\[
\frac{1}{b_K^2(\chi_t(t))} |\sigma^* (\chi_t) D_x b_K (\chi_t (t))|^2 - \frac{1}{b_K(\chi_t(t))} \mathcal{L}_{(\chi_t, A_t)} b_K (\chi_t (t)) \leq M , \quad \forall \chi_t (t) \in \mathcal{K} \cap \mathcal{K}_{\varepsilon_0}.
\]

Given an \( \chi_t \in \Lambda_{K_t} \) satisfying \( \chi_t (t) \in \mathcal{K} \cap \mathcal{K}_{\varepsilon_0} \), let \( \bar{x} (t) \) denote the unique projection of \( \chi_t (t) \in \mathcal{K} \cap \mathcal{K}_{\varepsilon_0} \) on the boundary of \( \mathcal{K} \) since \( (3.1) \). Then owning to \( (3.10) \), we have
\[
\sigma^* \left( \chi_t \bar{x}(t) \right) \nabla_x b_K (\bar{x} (t)) = 0 , \quad (3.13)
\]
since \( \chi_t \bar{x}(t) \in \Lambda_{0\mathcal{K}_t} \). Therefore, we have
\[
|\sigma^* (\chi_t) \nabla_x b_K (\chi_t (t))| = \left| \left( \sigma^* (\chi_t) - \sigma^* (\chi_t \bar{x}(t)) \right) \nabla_x b_K (\chi_t (t)) + \sigma^* (\chi_t \bar{x}(t)) \nabla_x b_K (\chi_t (t)) \right|
\leq \left| \sigma^* (\chi_t) - \sigma^* (\chi_t \bar{x}(t)) \right|
\leq C \left| \chi_t - \chi_t \bar{x}(t) \right| = C \left| \chi_t (t) - \bar{x} (t) \right| = C b_K (\chi_t (t)) ,
\]
where \( C \) is a Lipschitz constant for \( \sigma \) in the metric \( ||\cdot|| \). Consequently,
\[
\frac{1}{b_K^2(\chi_t(t))} |\sigma^* (\chi_t) \nabla_x b_K (\chi_t (t))|^2 \leq \left( \frac{C b_K (\chi_t (t)))^2}{b_K^2(\chi_t(t))} \right) \leq C^2 . \quad (3.14)
\]
Also, noting \( (3.1) \), we observe that,
\[
\mathcal{L}_{(\chi_t, A_t)} b_K (\chi_t (t)) = \frac{1}{2} \text{Tr} \left( \sigma^* (\chi_t \bar{x}(t)) A(t) \right) + \langle \mu (\chi_t), \nabla_x b_K (\chi_t (t)) \rangle
\]
is Lipschitz continuous in \( \mathcal{K} \cap \mathcal{K}_{\varepsilon_0} \) for the metric \( ||\cdot|| \). Thus
\[
\mathcal{L}_{(\chi_t \bar{x}(t), A_t)} (-b_K) (\bar{x} (t)) \leq 0 \quad (3.15)
\]
yields that
\[
\begin{align*}
  &- \frac{1}{b_K(\chi_t(t))} \mathcal{L}(\chi_t, A_t) b_K(\chi_t(t)) \\
  & \leq \frac{1}{b_K(\chi_t(t))} \left| \mathcal{L}(\bar{\chi}_t(t), A_t) b_K(\bar{\chi}(t)) - \mathcal{L}(\chi_t, A_t) b_K(\chi_t(t)) \right|
\end{align*}
\]
(3.16)
\[
\begin{align*}
  &\leq C \left| \frac{\chi_t(t) - \bar{\chi}(t)}{b_K(\chi(t))} \right| = C \left| \frac{\chi_t(t) - \bar{\chi}(t)}{b_K(\chi(t))} \right| = C \frac{b_K(\chi(t))}{b_K(\chi(t))} \leq C,
\end{align*}
\]
(3.17)
for all \( \chi_t \in \Lambda_{\phi} \) satisfying \( \chi_t(t) \in \mathcal{K} \cap \mathcal{K}_{\epsilon_0} \), where \( C \) is a Lipschitz constant for \( \mathcal{L}(\chi_t, A_t) b_K \). Consequently, combining (3.14) and (3.17), we get desired result. \( \square \)

**Remark 3.3.** Clearly, the estimations of (3.14) and (3.17) indicate that the initial condition \( \chi_t \) must be chosen in \( D([0, t], \mathcal{K}) \) in Corollary 3.1.

Now we are able to give the proof of Theorem 3.1 similarly as in [11].

**Proof.** We first prove (i)\( \Rightarrow \) (ii). By Corollary 3.1, the first assertion holds. Next, we are going to show (ii)\( \Rightarrow \) (iii). Consider the stopping time \( \tau_{Q_i}(\chi_t) \) where \( \{Q_i\} \) is the sequence of compact domain of class \( C^2,1 \) satisfying (3.2). Applying the generation of Itô's formula (Proposition 2.1) we get, for all \( \chi_t \in \Lambda_{Q_i,t} \) and \( 0 \leq t \leq s \),
\[
\begin{align*}
  \Psi \left( X^{\chi_t}_{s \wedge \tau_{Q_i}(\chi_t)}, A_{s \wedge \tau_{Q_i}(\chi_t)} \right) - \Psi (\chi_t, A_t) &= \int_t^{s \wedge \tau_{Q_i}(\chi_t)} \left( \mathcal{L}(X^{\chi_t}_{r}, A_r) \Psi (X^{\chi_t}_{r}, A_r) \right) dr \\
  & \quad + \int_t^{s \wedge \tau_{Q_i}(\chi_t)} \left( \nabla \Psi (X^{\chi_t}_{r}, A_r), \sigma (X^{\chi_t}_{r}) \right) dW (r).
\end{align*}
\]
(3.18)

Then, taking expectation and noting (3.11), we get
\[
\begin{align*}
  \mathbb{E} \left[ \Psi \left( X^{\chi_t}_{s \wedge \tau_{Q_i}(\chi_t)}, A_{s \wedge \tau_{Q_i}(\chi_t)} \right) \right] &= \Psi (\chi_t, A_t) + \mathbb{E} \left[ \int_t^{s \wedge \tau_{Q_i}(\chi_t)} \left( \mathcal{L}(X^{\chi_t}_{r}, A_r) \Psi (X^{\chi_t}_{r}, A_r) \right) dr \right] \\
  &\leq \Psi (\chi_t, A_t) + Ms.
\end{align*}
\]

By Faton's Lemma, the above inequality yields that
\[
\mathbb{E} \left[ \Psi \left( X^{\chi_t}_{s \wedge \tau_{\mathcal{K}}(\chi_t)}, A_{s \wedge \tau_{\mathcal{K}}(\chi_t)} \right) \right] \leq \Psi (\chi_t, A_t) + Ms, \quad \forall s \geq t \geq 0, \quad \forall \chi_t \in \Lambda_{\mathcal{K}_t}.
\]

Observing that the function in the right-hand above is finite on \( \Lambda_{\mathcal{K}_t} \), we deduce that
\[
P (\tau_{\mathcal{K}}(\chi_t) \leq s) = P \left( \Psi \left( X^{\chi_t}_{s \wedge \tau_{\mathcal{K}}(\chi_t)}, A_{s \wedge \tau_{\mathcal{K}}(\chi_t)} \right) = +\infty \right) = 0, \quad s \geq t \geq 0, \quad \forall \chi_t \in \Lambda_{\mathcal{K}_t}.
\]

Take a sequence \( t_k \uparrow +\infty \) and observe that
\[
0 = P (\tau_{\mathcal{K}}(\chi_t) \leq t_k) \uparrow P (\tau_{\mathcal{K}}(\chi_t) < +\infty), \quad \forall \chi_t \in \Lambda_{\mathcal{K}_t}.
\]
We complete the proof of (ii)⇒(iii).

Next we are going to prove (iii)⇒(i). Assume that \( \mathcal{K} \) is viable and fix \( \chi \in \Lambda_{\mathcal{K}_t} \). Let \( \{\chi_{t,k}\} \) be a sequence in \( \Lambda_{\mathcal{K}_t} \) such that \( \chi_{t,k} \xrightarrow{\|\cdot\|} \chi \) since \( \mathcal{K} \) coincides with the closure of \( \mathcal{K} \). Then, we have \( X^{\chi_{t,k}}(s) \in \mathcal{K} \), \( P \)-a.s. for all \( s \geq t \). Noting \( X^{\chi_{t,k}}(s) \to X^{\chi_t}(s) \), \( P \)-a.s. for \( s \geq t \), we deduce that

\[ X^{\chi_t}(s) \in \mathcal{K}, \ P \text{-a.s.}, \text{for all } s \geq t. \]

From the arbitrary point \( \chi_t \) in \( \Lambda_{\mathcal{K}_t} \), we get the desired result.

\[ \square \]

**Remark 3.4.** If we suppose that \( t = 0 \), \( \chi(0) = x \), \( k(\gamma_s) = k(\gamma_s(s)) \) where \( \gamma \in \Lambda \), \( x \in \mathcal{K} \), \( s \in [0, +\infty) \), while \( k = \mu, \sigma \), respectively. Then we recover Theorem 3.2 in [11].

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