Stationary underdispersed INAR(1) models based on the backward approach

Emad-Eldin Aly Ahmed Aly
Department of Statistics and Operations Research,
Faculty of Science, Kuwait University,
P. O. Box 5969, Safat 13060, Kuwait
Email: eealy50@gmail.com

Nadjib Bouzar
Department of Mathematical Sciences, University of Indianapolis,
Indianapolis, IN 46227, USA
Email: nbouzar@uindy.edu

Abstract

Most of the stationary first-order autoregressive integer-valued (INAR(1)) models were developed for a given thinning operator using either the forward approach or the backward approach. In the forward approach the marginal distribution of the time series is specified and an appropriate distribution for the innovation sequence is sought. Whereas in the backward setting, the roles are reversed. The common distribution of the innovation sequence is specified and the distributional properties of the marginal distribution of the time series are studied. In this article we focus on the backward approach in presence of the Binomial thinning operator. We establish a number of theoretical results which we proceed to use to develop stationary INAR(1) models with finite mean. We illustrate our results by presenting some new INAR(1) models that show underdispersion.

Key words and phrases: Integer-valued time series, The Binomial thinning operator, Poissonian Binomial distribution, Heine distribution.

2020 Mathematics Subject Classifications: Primary 62M10; Secondary 60E99.
1 Introduction

The area of integer-valued time series has attracted a lot of interest in research and practice during the last 35 years. It started with the pioneering work of McKenzie (1985), Al-Osh and Alzaid (1987) and McKenzie (1988). The first models are based on the Binomial thinning operator of Steutel and van Harn (1979). Since then, many families of new thinning-based first-order autoregressive integer-valued models (INAR(1)) have been proposed and studied in the literature. Al-Osh and Aly (1992), Aly and Bouzar (1994), Latour (1998), Ristic et al. (2009) and Aly and Bouzar (2019) proposed and studied new INAR(1) models developed by replacing the Binomial thinning operator by other types of thinning operators. Additional references can be found in the review article by Scotto et al. (2015).

The Binomial thinning (Steutel and van Harn (1979)) of $X$, denoted by $\alpha \odot X$, is defined as

$$\alpha \odot X = \sum_{i=1}^{X} Y_i,$$

where $X$ is a $\mathbb{Z}_+$-valued random variable (rv), $\alpha \in (0, 1)$ and $\{Y_i\}$ is a sequence of independent identically distributed (iid) Bernoulli($\alpha$) rv’s independent of $X$. The operation $\odot$ incorporates the discrete nature of the variates and acts as the analogue of the standard multiplication used in the standard ARMA models.

Assume that $0 < \alpha < 1$, and $(\varepsilon_t, t \geq 1)$ is an iid sequence of $\mathbb{Z}_+$-valued rv’s. A sequence $(X_t, t \geq 0)$ of $\mathbb{Z}_+$-valued rv’s is said to be an INAR (1) process if

$$X_t = \alpha \odot X_{t-1} + \varepsilon_t \quad (t \geq 1),$$

such that the binomial thinning $\alpha \odot X_{t-1}$ in (3) is performed independently for each $t$. More precisely, we assume the existence of an array $(Y_{i,t}, i \geq 1, t \geq 0)$ of iid Bernoulli($\alpha$) rv’s, independent of $\{\varepsilon_t\}$, such that

$$\alpha \odot X_{t-1} = \sum_{i=1}^{X_{t-1}} Y_{i,t-1}.$$ 

In (3), $\{\varepsilon_t\}$ is referred to as the innovation sequence and $\alpha$ as the coefficient of the process $\{X_t\}$.

If $\{X_t\}$ is an INAR (1) process of (3), then the pgf $\varphi_{X_t}(z)$ of $X_t$ and the common pgf $\Psi(z)$ of the innovation sequence $\{\varepsilon_t\}$ must satisfy the functional equation

$$\varphi_{X_{t+1}}(z) = \varphi_{X_t}(1 - \alpha + \alpha z)\Psi(z).$$

If one further assumes that $\{X_t\}$ is stationary, then the common pgf $\varphi_X(z)$ of $\{X_t\}$ satisfies

$$\varphi_X(z) = \varphi_X(1 - \alpha + \alpha z)\Psi(z).$$

The next proposition states that equation (4) is a sufficient condition for the existence of stationary INAR (1) processes. For a proof see for example Bouzar and Jayakumar (2008) or, in a more general setting, Aly and Bouzar (1994).
Proposition 1 Let $\alpha \in (0, 1)$ and let $\varphi_X(z)$ and $\Psi(z)$ be pgf’s that satisfy the functional equation (4). Then, there exists a stationary INAR(1) process $\{X_t\}$ on some probability space such that its marginal distribution and that of its innovation sequence $\{\varepsilon_t\}$ have respective pgf’s $\varphi_X(z)$ and $\Psi(z)$.

As a functional equation with two unknown pgf’s ($\varphi_X(\cdot)$ and $\Psi(\cdot)$), (4) can be solved in two different ways. The forward approach: Fix a pgf, $\varphi_X(\cdot)$, and find $\Psi(\cdot)$, as the solution of

$$\frac{\varphi_X(z)}{\varphi_X(1 - \alpha + \alpha z)} = \Psi(z)$$

provided that $\Psi(z)$ is a pgf. The backward approach: Fix a pgf, $\Psi(\cdot)$, and find the pgf $\varphi_X(\cdot)$ that satisfies (4). It can be shown that in this case

$$\varphi_X(z) = \lim_{n \to \infty} \prod_{i=0}^{n} \Psi(1 - \alpha^i + \alpha^i z)$$

provided that the limit exists and is a pgf.

Note that the forward approach is most useful if a researcher is interested in a specific marginal distribution. The backward approach is most useful if a researcher is interested in a specific distribution for the innovations $\varepsilon_t$.

The forward approach has been widely used in the literature. In addition to the above mentioned references, we cite McKenzie (2003), a review article, Joe (1996, 2019), Zhu and Joe (2003 and 2010), and the monograph by Weiß (2018). For results and references on the backward approach we note the work of Jung et al. (2005), Pedeli and Karlis (2011), Weiß (2013), Schweer and Weiß (2014) and Schweer and Wichelhaus (2015). We note that Guerrero et al. (2020) proposed a third approach in which both the innovation distribution and the marginal distribution of the stationary INAR(1) process are specified in advance. The thinning operator specific to these distributions is identified by solving a functional equation.

In the current work, we adopt the backward approach to develop stationary INAR(1) models with finite mean using the Binomial thinning operator. In Section 2, we prove a number of foundational results in the context of the backward approach. These results are then used to obtain most of the needed distributional properties of the marginal distribution of the model under minimal assumptions on the distribution of the innovation sequence. In Sections 3-7, we illustrate our results of Section 2 by introducing and studying in details the important underdispersed models when the innovations follow the logarithmic distribution, the Bernoulli distribution, the Binomial distribution, the Poissonian Binomial distribution and the Heine distribution, respectively. In Section 8, we give some extensions of the previous models via convolution.

We will assume throughout the rest of this paper that $\alpha \in (0, 1)$ and that $\{f_k\}$ is a pmf with pgf $\Psi(z)$ such that $\Psi'(1) < \infty$. We will also be using the notation $\bar{a} = 1 - a$ for $a \in (0, 1)$.

2 Foundational results of the backward approach

The main results of this Section are given in Theorems 1-3 below.
Theorem 1. The function

$$\varphi(z) = \prod_{i=0}^{\infty} \Psi(1 - \alpha^i + \alpha^i z)$$  \hspace{1cm} (5)$$

is a pgf. Moreover, the convergence of the infinite product is uniform over the interval [0, 1] and \(\varphi(z)\) satisfies

$$\varphi(z) = \varphi(1 - \alpha + \alpha z)\Psi(z), \quad z \in [0, 1].$$  \hspace{1cm} (6)$$

Proof: First, we recall some basic results on pgf’s (Feller, Vol I, 1968, is an excellent reference). Let \(\{q_k = \sum_{i=k+1}^{\infty} f_i\}\) be the sequence of the tail probabilities corresponding to \(\{f_k\}\) and let

$$Q(z) = \sum_{k=0}^{\infty} q_k z^k,$$

be the generating function of \(\{q_k\}\). We have

$$1 - \Psi(z) = (1 - z)Q(z), \quad z \in [0, 1]$$  \hspace{1cm} (7)$$

and

$$Q(1) = \sum_{k=0}^{\infty} q_k = \sum_{k=0}^{\infty} k f_k = \Psi'(1) < \infty.$$  \hspace{1cm} (8)$$

Define

$$h_i(z) = 1 - \Psi(1 - \alpha^i + \alpha^i z).$$

We have by (7),

$$h_i(z) = \alpha^i(1 - z)Q(1 - \alpha^i + \alpha^i z).$$

Noting that \(Q\) is increasing over [0, 1], 0 \leq 1 - z \leq 1, and \(Q(1)\) is finite (cf. (8)), it follows that 0 \leq h_i(z) \leq Q(1)\alpha^i and

$$\sum_{i=n+1}^{\infty} h_i(z) \leq Q(1) \sum_{i=n+1}^{\infty} \alpha^i.$$$$

This implies \(\sum_{i=n+1}^{\infty} h_i(z)\) converges uniformly to 0 over the interval [0, 1], which in turn implies, by Theorem 1, p. 381, in Knopp (1990), that

$$\varphi_{n+1}(z) = \prod_{i=0}^{n} \Psi(1 - \alpha^i + \alpha^i z) = \prod_{i=0}^{n} (1 - h_i(z)), n \geq 0$$  \hspace{1cm} (9)$$

converges uniformly over the interval [0, 1] to

$$\varphi(z) = \prod_{i=0}^{\infty} (1 - h_i(z)) = \prod_{i=0}^{\infty} \Psi(1 - \alpha^i + \alpha^i z).$$

Next, we show that \(\lim_{z \uparrow 1} \varphi(z) = 1\). Define

$$r_n(z) = \prod_{i=n+1}^{\infty} \Psi(1 - \alpha^i + \alpha^i z).$$
Let $\delta > 0$ be arbitrary. By the uniform convergence of $\{\varphi_{n+1}(z)\}$ to $\varphi(z)$, there exists a positive integer $N(\delta)$ such that for any $n > N(\delta)$,

$$\sup_{z \in [0,1]} |r_n(z) - 1| < \delta.$$ 

Note that $\varphi_{n+1}(\cdot)$ of (3) satisfies $\varphi_{n+1}(1) = 1$ and $\varphi_{n+1}(z) \leq 1$. Since

$$|\varphi(z) - 1| = |\varphi_{n+1}(z)(r_n(z) - 1) + \varphi_{n+1}(z) - 1|,$$

it follows that for any $n > N(\delta)$

$$|\varphi(z) - 1| \leq \delta + |\varphi_{n+1}(z) - 1|,$$

which in turn implies that

$$\limsup_{z \uparrow 1} |\varphi(z) - 1| = \limsup_{z \uparrow 1} (1 - \varphi(z)) \leq \delta + \liminf_{z \uparrow 1} (1 - \varphi_{n+1}(z)) \leq \delta.$$

Since $\varphi(z)$ is the limit of the sequence of pgf’s $\{\varphi_{n+1}(z)\}$, we conclude by the Continuity Theorem that $\varphi(z)$ is a pgf. Equation (6) is easily shown to hold. ■

Stirling numbers of the second kind, denoted by $S(r, j)$, are defined (see Abramowitz and Stegun (1965) and Goldberg et al. (1976)) as $S(0, 0) = 1, S(0, k) = S(r, 0) = 0$ and

$$S(r, j) = \frac{1}{j!} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} k^r. \tag{10}$$

**Theorem 2.** We use the notation of Theorem 1. Let $\{p_r\}$ be the pmf with pgf $\varphi(z)$ and let $\{f_r^{(i)}\}$ be the pmf with pgf $\Psi(1 - \alpha^i + \alpha^i z)$. Let $\kappa_r^{(f)}$ and $\kappa_r^{(p)}$ be the $r$-th factorial moments of $\{f_r\}$ and $\{p_r\}$, respectively, and assume that $\kappa_r^{(f)}, r \geq 1$, are finite. Then,

1. $f_r^{(0)} = f_r$ and

$$f_r^{(i)} = \begin{cases} f_0 + \sum_{n=1}^{\infty} (1 - \alpha^i)^n f_n, & \text{if } r = 0 \\ \alpha^r \sum_{n=\infty}^{\infty} \binom{n}{r} f_n (1 - \alpha^i)^{n-r}, & \text{if } r \geq 1. \end{cases} \tag{11}$$

2. 

$$p_r = \lim_{k \to \infty} \left( f_r^{(0)} * f_r^{(1)} * \cdots * f_r^{(k-1)} \right)_r, \tag{12}$$

where $f_r^{(0)} * f_r^{(1)} * \cdots * f_r^{(k-1)}$ designates the $k$-factor convolution of the pmf’s

$$\{f_r^{(0)}\}, \{f_r^{(1)}\}, \ldots, \{f_r^{(n-1)}\}.$$

3. For every $r \geq 1$, $\kappa_r^{(p)}$ and $\kappa_r^{(f)}$ are finite and are given by

$$\kappa_r^{(p)} = \frac{\kappa_r^{(f)}}{1 - \alpha^r} \tag{13}$$

and

$$\kappa_r^{(f)} = \sum_{j=1}^{r} S(r, j) \frac{\kappa_{[j]}^{(f)}}{1 - \alpha^j}. \tag{14}$$
Proof: The proof of (11) is straightforward. Since
\[ \varphi(z) = \lim_{k \to \infty} \prod_{i=0}^{k-1} \Psi(1 - \alpha^i + \alpha^i z), \]
we obtain (12) by the Continuity Theorem and (11).
Recall that the factorial cumulants, \( \{\kappa_r \}, r \geq 1 \), and the cumulants, \( \{\kappa_r \}, r \geq 1 \), of a pmf are the coefficients of \( t_r \) in the power series expansions of the factorial cumulant generating function (fcgf) \( \ln \varphi(1 + t) \) and the cumulant generating function (cgf) \( \ln \varphi(e^t) \), respectively. A general formula that links \( \{\kappa_r \} \) and \( \{\kappa_r \} \) (see Johnson et al. (2005), Sections 1.2.7 and 1.2.8) is given by
\[ \kappa_r = \sum_{j=1}^{r} S(r, j) \kappa_{[j]}, \] (15)
where \( S(r, j) \) are the Stirling numbers of the second kind of (10). By (5),
\[ \ln \varphi(1 + t) = \sum_{i=0}^{\infty} \ln \Psi(1 + \alpha^i u) = \sum_{i=0}^{\infty} \sum_{r=1}^{\infty} \alpha^r \kappa_r(f) \frac{t^r}{r!}, \]
or
\[ \ln \varphi(1 + t) = \sum_{r=1}^{\infty} \frac{\kappa_r(f) t^r}{1 - \alpha^r r!}. \]
Hence we obtain (13). Equation (14) for the \( r \)-th cumulant \( \kappa_r(p) \) follows from (13) and (15).

The backward approach is based on the following result which is an immediate consequence of (2), Theorem 1 and Proposition 1. The proof is omitted.

Theorem 3. Any pgf \( \Psi(z) \) such that \( \Psi'(1) < \infty \), gives rise to a stationary INAR (1) process \( \{X_t\} \) defined on some probability space \( (\Omega, \mathcal{F}, P) \) and driven by equation (3). Its marginal pgf is
\[ \varphi_X(z) = \prod_{i=0}^{\infty} \Psi(1 - \alpha^i + \alpha^i z). \] (16)

Remarks:
1. Equation (13) was derived by Weiß (2013) using a different approach.
2. We give a couple of possibly useful formulas for \( \kappa_{[r]}(p) \) and \( \kappa_r(p) \) in terms of the sequences \( \{\kappa_{[i]}(p)\}, i \geq 0 \) and \( \{\kappa_r(i)\}, i \geq 0 \) of the pmf’s \( \{\{f_r^{(i)}\}, i \geq 0\} \),
\[ \kappa_{[r]}(p) = \sum_{i=0}^{\infty} \kappa_{[i]}(p) \quad \text{and} \quad \kappa_r(p) = \sum_{i=0}^{\infty} \kappa_r(i), \] (17)
provided the two series converge. The proof is straightforward.
3. Provided they are finite, the first and second cumulants of a pmf, are its mean and variance, respectively. It follows that the mean, \( \mu(p) \), and the variance, \( (\sigma(p))^2 \), of \( \{p_r\} \) can be obtained from their \( \{f_r\} \) counterparts, \( \mu(f) \), (\( \sigma(f) \))^2:
\[ \mu(p) = \frac{\mu(f)}{1 - \alpha} \quad \text{and} \quad (\sigma(p))^2 = \frac{(\sigma(f))^2 + \alpha \mu(f)}{1 - \alpha^2}. \] (18)
Using (18), it is easily seen, as noted in Weiß (2013), that \( \{ p_r \} \) of (12) is underdispersed (i.e., \((\sigma^{(p)})^2 < \mu^{(p)}\)) if and only if \( \{ f_r \} \) is underdispersed.

4. There are no simple formulas linking the \( r \)-th moment \( \mu^{(p)}_r \) and the \( r \)-th factorial moment \( \mu^{(p)}_{[r]} \) of \( \{ p_r \} \) to their \( \{ f_r \} \) counterparts. However, if either \( \kappa^{(p)}_r \) or \( \kappa^{(p)}_{[r]} \) can be calculated for every \( r \geq 1 \), then one can compute \( \mu^{(p)}_r \) and \( \mu^{(p)}_{[r]} \) using standard formulas that link moments and cumulants (see Johnson et al. (2005), Sections 1.2.7 and 1.2.8 and Smith (1995)).

5. The following additional results (see Al-Osh and Alzaid (1987) and McKenzie (1988)) are needed in the sequel. An INAR (1) model driven by (3) is necessarily a homogeneous Markov chain with the 1-step transition probabilities,

\[
P(X_t = k | X_{t-1} = l) = \sum_{j=0}^{\min(l,k)} \binom{l}{j} \alpha^j (1 - \alpha)^{l-j} P(\varepsilon = k - j).
\]

The \( k \)-step-ahead version of (3) for \( k \geq 1 \) is given by

\[
X_{t+k} \overset{D}{=} \alpha^k \circ X_t + \sum_{j=1}^{k} \alpha^{j-1} \circ \varepsilon_{t+k-j+1}.
\]

It follows from (20) that the conditional pgf of \( X_{t+k} \) given \( X_t \) satisfies

\[
\varphi_{X_{t+k}|X_t}(z) = (1 - \alpha^k + \alpha^k z)^{X_t} \times \prod_{i=0}^{k-1} \Psi(1 - \alpha^i + \alpha^i z).
\]

### 3 Stationary INAR (1) models with logarithmic innovations

We start out by recalling a few facts about the logarithmic distribution (see Johnson et al., 2005). The pmf of the logarithmic(\( p \)) distribution is given by

\[
f_r = p r \ln p, \quad r \geq 1,
\]

where \( p \in (0, 1) \). Its pgf, mean, variance and dispersion index are given respectively by

\[
\Psi(z) = \frac{\ln(1 - pz)}{\ln p},
\]

\[
\mu^{(f)} = -\frac{p}{p \ln p} \quad (\sigma^{(f)})^2 = \frac{p + \ln p}{(p \ln p)^2} \quad \text{and} \quad I^{(f)} = \frac{p + \ln p}{p \ln p}.
\]

Note that the logarithmic distribution is underdispersed if \( p < 1 - 1/e \), equidispersed if \( p = 1 - 1/e \) and overdispersed if \( p > 1 - 1/e \).

The factorial moments of \( \{ f_r \} \) are

\[
\mu^{(f)}_{[r]} = -\frac{p^r (r - 1)!}{(1 - p)^r \ln p} \quad (r \geq 1).
\]

Recall that the moments \( \{ \mu_r \} \) of a random variable can be derived from their factorial counterparts \( \{ \mu_{[r]} \} \) via the equation (see Sections 1.2.7 and 1.2.8 in Johnson et al. (2005))

\[
\mu_r = \sum_{j=1}^{r} S(r,j) \mu_{[j]} \quad (r \geq 1),
\]
with $S(r, j)$ of (10). By (23) and (24), the moments \( \mu_r^{(j)} \) of a logarithmic distribution are given by

\[
\mu_r^{(j)} = -\frac{1}{\ln p} \sum_{j=1}^{r} S(r, j) \frac{p^j (j - 1)!}{(1 - p)^j} \quad (r \geq 1).
\]

**Lemma 1.** Let $p \in (0, 1)$ and let \( \{f_r\} \) be the pmf of a logarithmic($p$) distribution with pgf $\Psi(z)$ of (22). Then for every $i \geq 0$, the pmf, \( \{f_r^{(i)}\} \) of (11) (with pgf $\Psi(1 - \alpha^i + \alpha^i z)$) is a two-mixture of the Dirac measure $\delta_0$ sitting at 0 and the logarithmic($q_i$) distribution, with $q_i = \frac{p \alpha^i}{1 - p(1 - \alpha^i)}$, and with respective mixing probabilities $b_i = 1 - \frac{\ln q_i}{\ln p}$ and $1 - b_i = \frac{\ln q_i}{\ln p}$, i.e.,

\[
f_r^{(i)} = b_i \delta_0(\{r\}) + (1 - b_i) \frac{q_i^r}{-r \ln q_i}.
\]

Note that, $q_0 = p, b_0 = 0$ and $f_r^{(0)} = f_r$. Moreover, the $k$-factor convolution of the pmf’s $\{f_r^{(0)}\}, \{f_r^{(1)}\}, \ldots, \{f_r^{(k-1)}\}$, $k \geq 2$, is a finite mixture of convolutions of logarithmic distributions, namely,

\[
(f^{(0)} * f^{(1)} * \cdots * f^{(k-1)})_r = C_{1, 0} g_r^{(0)} \sum_{l=1}^{k-1} \sum_{j \in J_l} C_{j, l} \left( g^{(0)} * g^{(j_1)} * g^{(j_2)} \cdots g^{(j_l)} \right)_r,
\]

where $\{g_r^{(j)}\}$ is the pmf of the logarithmic($q_j$), $I = \{1, 2, \ldots, k-1\}$, $J_l$ is the collection of ordered $l$-tuples $j = (j_1, j_2, \ldots, j_l)$, $1 \leq j_1 < j_2 < \cdots < j_l \leq k - 1$ and $J_u = \{j_1, j_2, \ldots, j_l\}$ is the corresponding unordered $l$-tuple. The mixing probabilities are

\[
C_{1, 0} = \prod_{j=1}^{k-1} b_j \quad \text{and} \quad C_{j, l} = \left( \prod_{j \in I \setminus J_u} b_j \right) \left( \prod_{h=1}^{l} (1 - b_{j_h}) \right).
\]

**Proof:** If $i = 0$, (26) is true since $\{f_r^{(0)}\} = \{f_r\}$. Assume $i \geq 1$. By (11),

\[
f_r^{(i)} = -\frac{1}{\ln p} \sum_{n=1}^{\infty} \frac{p(1 - \alpha^i)}{n} = \frac{\ln(1 - p(1 - \alpha^i))}{\ln p},
\]

and for $r \geq 1$,

\[
f_r^{(i)} = -\frac{(p \alpha^i)^r}{r \ln p} \sum_{n=r}^{\infty} \left( \frac{n - 1}{r - 1} \right) (p(1 - \alpha^i))^{n-r}.
\]

Using the power series expansion

\[
(1 - t)^{-r-1} = \sum_{n=r}^{\infty} \binom{n}{r} t^{n-r},
\]

with $t = p(1 - \alpha^i)$, it follows that

\[
f_r^{(i)} = -\frac{1}{r \ln p} \left[ \frac{p \alpha^i}{1 - p(1 - \alpha^i)} \right]^r \quad (r \geq 1).
\]
Setting \( q_i = \frac{p \alpha^i}{1 - p (1 - \alpha)} \), it is easily verified that \( f^{(i)} \) satisfies (26). The second part of the Lemma and equations (27) and (28) are proved by a tedious but straightforward induction argument. The details are omitted. ■

**Theorem 4.** Let \( \{X_t\} \) be the stationary INAR(1) process driven by (3) and with a logarithmic(p) innovation sequence for some \( p \in (0, 1) \). Then,

(i) the marginal distribution \( \{p_r\} \) of \( \{X_t\} \) is given by (12), where \( f^{(0)} * f^{(1)} * \cdots * f^{(k - 1)} \) is described by equations (27) and (28).

(ii) the marginal pgf \( \varphi(z) \) of \( \{X_t\} \) admits the representation

\[
\varphi(z) = \prod_{i=0}^{\infty} \left[ 1 - \frac{1}{\ln p} \cdot \ln \frac{q_i}{1 - q_i z} \right].
\]

**Proof:** Part (i) is a direct consequence of (12) and Lemma 1. For (ii), we note the pgf of \( \{f^{(i)}\} \) of (26) is

\[
\Psi(1 - \alpha^i + \alpha^i z) = 1 - \frac{1}{\ln p} \cdot \ln \frac{q_i}{1 - q_i z},
\]

which implies that the pgf of \( f^{(0)} * f^{(1)} * \cdots * f^{(k - 1)} \) is

\[
\varphi_k(z) = \prod_{i=0}^{k-1} \left[ 1 - \frac{1}{\ln p} \cdot \ln \frac{q_i}{1 - q_i z} \right],
\]

which in turn implies (30). ■

We provide additional properties for a stationary INAR(1) process \( \{X_t\} \) with a logarithmic(p) innovation sequence.

By (19), the 1-step transition probability is given by

\[
P(X_t = k | X_{t-1} = l) = -\frac{p}{\ln p} \sum_{j=0}^{\min(l, k-1)} \binom{l}{j} (\alpha/p)^j \alpha^{l-j}.
\]

By (21), the conditional pgf of \( X_{t+k} \) given \( X_t \) satisfies

\[
\varphi_{X_{t+k}, X_t}(z) = (1 - \alpha^k + \alpha^k z)^{X_t} \times \prod_{i=0}^{k-1} \left[ 1 - \frac{1}{\ln p} \cdot \ln \frac{q_i}{1 - q_i z} \right].
\]

Therefore, given \( X_t = n \), the distribution of \( X_{t+k} \) is the convolution of a Binomial(n, \( \alpha^k \)) distribution and the finite mixture of convolutions of logarithmic distributions described by (27) and (28).

By (18) and (23), the mean \( \mu_X \) and variance \( \sigma_X^2 \) of the marginal distribution of \( \{X_t\} \) are given by

\[
\mu_X = -\frac{p}{\ln p} \frac{\alpha}{1 - \alpha} \ln \frac{p}{\alpha} \quad \text{and} \quad \sigma_X^2 = -\frac{p}{\ln p} \frac{\alpha}{1 - \alpha} \ln \left( \frac{\alpha + (p + \ln p)}{\alpha \ln \frac{p}{\alpha}} \right).
\]

Note that the distribution of \( X_t \) is underdispersed if \( p < 1 - 1/e \).

The moments and cumulants of the marginal distribution of \( \{X_t\} \) are computed as follows:
1. Compute the $r$-th cumulant $\kappa^{(f)}_r$ of $\varepsilon_t$ using the formula (due to Smith, 1995)

$$\kappa^{(f)}_r = \mu^{(f)}_r - \sum_{i=1}^{r-1} \binom{r-1}{i} \kappa^{(f)}_{r-i} \mu^{(f)}_i,$$  \hspace{1cm} (31)

along with (25) (recall $\kappa^{(f)}_1 = \mu_\varepsilon$ and $\kappa^{(f)}_2 = \sigma^2_\varepsilon$).

2. Compute the $r$-th factorial cumulant $\kappa^{(f)}_{[r]}$ of $\varepsilon_t$ using the formula (see Johnson et al. (2005), Sections 1.27 and 1.2.8)

$$\kappa^{(f)}_{[r]} = \sum_{j=0}^{r} s(r, j) \kappa^{(f)}_j,$$  \hspace{1cm} (32)

where $s(r, j)$ is the Stirling number of the first kind satisfying the recurrence relation

$$s(r + 1, j) = s(r, j - 1) - ns(r, j),$$  \hspace{1cm} (33)

with $s(n, 0) = 0$ and $s(1, 1) = 1$ (see Johnson et al., 2005).

3. Compute the $r$-th factorial cumulant $\kappa^{(p)}_{[r]}$ of $X_t$ using (13). Use the approach described in Theorem 2 to obtain the cumulant, moment, and factorial moment of $X_t$.

4 Stationary INAR (1) models with Bernoulli innovations

We start out by recalling a result and a definition that will be needed in the sequel. Let $q, c \in (0, 1)$ and $m \geq 2$. Kemp (1987) (see also Johnson et al., 2005, p. 467) introduced and studied the Poissonian Binomial $(m, q, c)$ distribution as the distribution of a finite convolution of Bernoulli$(cq^i)$ distributions, $i = 0, 1, 2, \ldots, m - 1$ with pgf

$$\Psi(z) = \prod_{i=0}^{m-1} (1 - cq^i(1 - z))$$  \hspace{1cm} (34)

and pmf

$$q_r(m, q, c) = \sum_{k=r}^{m} (-1)^{k-r} \binom{k}{r} c^k q^{k(2)} \prod_{l=0}^{k-1} \frac{1 - q^{m-l}}{1 - q^{l+1}}, r = 0, 1, \ldots, m.$$  \hspace{1cm} (35)

A distribution on $\mathbb{Z}_+$ is said to have a discrete pseudo compound Poisson distribution, $PCPD(\lambda, \{a_k\})$, if its pgf can be written as

$$P(z) = \exp \left\{ \lambda \left( \sum_{k=1}^{\infty} a_k (z^k - 1) \right) \right\}$$  \hspace{1cm} (36)

for some $\lambda > 0$ and some sequence of real numbers $(a_k, k \geq 1)$ such that $\sum_{k=1}^{\infty} a_k = 1$ and $\sum_{k=1}^{\infty} |a_k| < \infty$.

**Theorem 5.** Let $\{X_t\}$ be the stationary INAR (1) process driven by (3) and with a Bernoulli$(p)$ innovation sequence for some $p \in (0, 1)$. Then,
1. The marginal pmf \( \{p_r\} \) of \( \{X_t\} \) is the weak limit of Poissonian Binomial \((n, \alpha, p)\) (see (34) and (35)) as \( n \to \infty \) and is given by

\[
p_r = \lim_{n \to \infty} q_r(n, \alpha, p) = \sum_{k=r}^{\infty} (-1)^{k-r} \binom{k}{r} \frac{p^k \alpha^l}{\prod_{l=1}^{k} (1 - \alpha^l)}, r \geq 0.
\]

(37)

2. The tail probabilities \( P(X_t \geq r) = \sum_{j=r}^{\infty} p_j \) of \( X_t \) are obtained by the formula

\[
P(X_t \geq r) = \sum_{k=r}^{\infty} (-1)^{k-r} \binom{k-1}{r-1} \frac{p^k \alpha^l}{\prod_{l=1}^{k} (1 - \alpha^l)}, \quad r \geq 1.
\]

(38)

3. The marginal pgf \( \phi_X(z) \) of \( \{X_t\} \) admits two useful representations:

\[
\phi_X(z) = 1 + \sum_{n=1}^{\infty} p^n (z - 1)^n \frac{\alpha^l}{\prod_{l=1}^{n} (1 - \alpha^l)}
\]

and

\[
\phi_X(z) = \exp \left\{ - \sum_{n=1}^{\infty} \frac{p^n}{n(1 - \alpha^n)} (1 - z)^n \right\}.
\]

(39) (40)

4. If \( p < 1/2 \), then \( \{X_t\} \) has a \( PCPD(\lambda, \{a_k(p, \alpha)\}) \) marginal distribution, where

\[
\lambda = \sum_{n=1}^{\infty} \frac{p^n}{n(1 - \alpha^n)} \quad \text{and} \quad a_n(p, \alpha) = \frac{(-1)^{n+1}}{\lambda} \sum_{j=n}^{\infty} \binom{j}{n} \frac{p^j}{j(1 - \alpha^j)}.
\]

(41)

**Proof:** It is long. We defer it to the Appendix Section.

Next, we establish several properties of the marginal distribution of the stationary INAR (1) process \( \{X_t\} \) with Bernoulli \((p)\) innovations.

By (19), the 1-step transition probability is given by

\[
P(X_t = k|X_{t-1} = l) = \begin{cases} 
0, & k > l + 1 \\
p \alpha^{k-1}, & k = l + 1 \\
\alpha^{k-1} \binom{k}{l} p^{l} \left( 1 - \alpha \right)^{-k} \left( p(l) \alpha + p(l) \alpha \right), & k \leq l 
\end{cases}.
\]

(42)

By (21), the conditional pgf of \( X_{t+k} \) given \( X_t \) satisfies

\[
\phi_{X_{t+k}|X_t}(z) = \left( 1 - \alpha^k z \right)^{X_t} \times \prod_{i=0}^{k-1} \left( 1 - p^n (1 - \alpha) \right).
\]

Therefore, given \( X_t = n \), the distribution of \( X_{t+k} \) is the convolution of a Binomial \((n, \alpha^k)\) distribution and the Poissonian Binomial \((k, \alpha, p)\) distribution of (35).

By (18), the mean, the variance and the index of dispersion of \( X_t \) are

\[
\mu_X = \frac{p}{1 - \alpha}, \quad \sigma_X^2 = \frac{p(1 - p)}{1 - \alpha^2} \quad \text{and} \quad I_X = 1 - \frac{p}{1 + \alpha}.
\]

Clearly, \( \{X_t\} \) is underdispersed.
We derive the factorial moments \((\mu_{[r]}, n \geq 1)\) of \(X_t\). Using the version [39] of \(\varphi_X(z)\), we deduce that

\[
\varphi_X(1 + t) = 1 + \sum_{r=1}^{\infty} \frac{r!p^r \alpha^{(r)}}{\prod_{i=1}^{r}(1 - \alpha^i)} \cdot t^r.
\]

Since the series converges everywhere, the factorial moments of \(X_t\) of all orders are finite and are given by

\[
\mu_{[r]} = \frac{r!p^r \alpha^{(r)}}{\prod_{i=1}^{r}(1 - \alpha^i)} \quad (r \geq 1).
\]  \((43)\)

For example, we have

\[
\mu_{[1]} = \frac{p}{1 - \alpha} \quad \text{and} \quad \mu_{[2]} = \frac{2\alpha p^2}{(1 - \alpha)(1 - \alpha^2)}.
\]  \((44)\)

The moments \(\mu_r\) of \(X_t\), \(r \geq 1\), can be obtained from their factorial counterparts via the formula (see [24] and [10])

\[
\mu_r = \sum_{j=1}^{r} S(r, j) \frac{j!p^j \alpha^{(j)}}{\prod_{i=1}^{j}(1 - \alpha^i)} \quad (r \geq 1).
\]  \((45)\)

By [11], the fcgf of \(X_t\) is given by

\[
\ln \varphi_X(1 + t) = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}(r - 1)!p^r}{(1 - \alpha^r)} \cdot t^r.
\]

Since the series above converges everywhere, the factorial cumulants of all orders are finite and are given by

\[
\kappa_{[r]} = (-1)^{r+1}(r - 1)!p^r \quad (r \geq 1).
\]  \((46)\)

The cumulants of \(X_t\) \((\kappa_r, r \geq 1)\) can be obtained from the factorial cumulants via the formula [15] and [10]

\[
\kappa_r = \sum_{j=0}^{r} S(r, j)(-1)^{j+1} \frac{(j - 1)!p^j}{(1 - \alpha^j)} \quad (r \geq 1).
\]  \((47)\)

For example,

\[
\kappa_1 = \frac{p}{1 - \alpha},
\]

\[
\kappa_2 = -\frac{p^2}{1 - \alpha^2} + \frac{p}{1 - \alpha},
\]

\[
\kappa_3 = 2 \cdot \frac{p^3}{1 - \alpha^3} - 3 \cdot \frac{p^2}{1 - \alpha^2} + \frac{p}{1 - \alpha},
\]

and

\[
\kappa_4 = -6 \cdot \frac{p^4}{1 - \alpha^4} + 12 \cdot \frac{p^3}{1 - \alpha^3} - 7 \cdot \frac{p^2}{1 - \alpha^2} + \frac{p}{1 - \alpha}.
\]
5 Stationary INAR (1) models with Binomial innovations

The treatment is essentially similar to the Bernoulli case \((m = 1)\). We summarize the main results with minimal justifications for the most part.

**Theorem 6.** Let \(\{X_t\}\) be the stationary INAR (1) process driven by \((3)\) and with a Binomial\((m, p)\) innovation sequence for some positive integer \(m\) and some \(p \in (0, 1)\). Then

1. the marginal pmf \(\{p_r\}\) of \(\{X_t\}\) is the \(m\)-fold convolution of the marginal distribution \((37)\) of the INAR (1) process with a Bernoulli\((p)\) innovation, or

\[
p_r = \left[ \sum_{k=r}^{\infty} (-1)^{k-r} \binom{k}{r} p^k \alpha^r \prod_{l=1}^{k-1} (1 - \alpha^l) \right]^m (r \geq 0).
\] (48)

2. the marginal pgf \(\varphi_X(z)\) of \(\{X_t\}\) admits two representations:

\[
\varphi_X(z) = \left[ 1 + \sum_{n=1}^{\infty} \frac{p^n (z - 1)^n \alpha^n}{\prod_{n=1}^{\infty} (1 - \alpha^n)} \right]^m
\] (49)

and

\[
\varphi_X(z) = \exp \left\{ -m \sum_{n=1}^{\infty} \frac{p^n}{n(1 - \alpha^n)} (1 - z)^n \right\}.
\] (50)

3. If \(p < 1/2\), then \(\{X_t\}\) has a PCPD\((\lambda, \{a_k(p, \alpha)\})\) marginal distribution, where

\[
\lambda = m \sum_{n=1}^{\infty} \frac{p^n}{n(1 - \alpha^n)} \quad \text{and} \quad a_n(p, \alpha) = \frac{(-1)^{n+1}}{\lambda} \sum_{j=n}^{\infty} \binom{j}{n} \frac{p^j}{j(1 - \alpha^j)}.
\] (51)

**Proof:** Omitted.

We proceed to give additional properties for the marginal distribution of the stationary INAR (1) process \(\{X_t\}\) with Binomial\((m, p)\) innovations.

By \((19)\), the 1-step transition probability is given by

\[
P(X_t = k | X_{t-1} = l) = p^k \prod_{j=\max(k-m, 0)}^{\min(l,k)} \binom{l}{j} \binom{m}{k-j} \left( \frac{\alpha p^j}{p^j} \right)^j.
\] (52)

By \((21)\), the conditional pgf of \(X_{t+k}\) given \(X_t\) satisfies

\[
\varphi_{X_{t+k}|X_t}(z) = (1 - \alpha^k + \alpha^k z)^X_t \times \left[ \prod_{i=0}^{k-1} (1 - p\alpha^i(1 - z)) \right]^m.
\]

Therefore, the conditional distribution of \(X_{t+k}\) given \(X_t = n\) is the convolution of a Binomial\((n, \alpha^k)\) distribution and the \(m\)-fold convolution of the Poissonian Binomial \((k, \alpha, p)\) distribution of \((35)\).
By (13), the mean, the variance and the index of dispersion of $X_t$ are

$$
\mu_X = \frac{mp}{1 - \alpha}, \quad \sigma^2_X = \frac{mp(1 + \alpha - p)}{1 - \alpha^2} \quad \text{and} \quad I_X = 1 - \frac{p}{1 + \alpha}.
$$

Clearly $\{X_t\}$ is underdispersed.

The factorial moments $(\mu_{[r]}, n \geq 1)$ of $X_t$ can be obtained from the version (49) of $\varphi_X(z)$. In this case,

$$
\varphi_X(1 + t) = \left[1 + \sum_{r=1}^{\infty} \frac{p^r \alpha(1)}{\prod_{i=1}^{r} (1 - \alpha^i)} \cdot t^r \right]^m,
$$

is a power of a power series. Therefore, it admits a power series expansion whose coefficients can be determined relatively easily for small exponents of $t^r$, but not for large values of $r$. These coefficients can also be derived via recurrence formulas (see Knopp, 1990). We won’t pursue this approach. Instead, we proceed to derive simpler recurrence formulas for $(\mu_{[r]}, n \geq 1)$ by using the representation (50) of $\varphi_X(z)$.

Let

$$
\phi(z) = \sum_{n=1}^{\infty} \frac{p^n}{n(1 - \alpha^n)} (1 - z)^n.
$$

The series (53) converges uniformly over the interval $(0, 1)$ due to the fact

$$
\frac{p^n}{n(1 - \alpha^n)} (1 - z)^n \leq \frac{p^n}{n(1 - \alpha^n)}
$$

for every $z \in (0, 1)$ and that $\sum_{n=0}^{\infty} \frac{p^n}{n(1 - \alpha^n)}$ converges. It follows that $\phi'(z)$ and subsequent higher order derivatives exist and converge uniformly over $(0, 1)$ (see Knopp. 1990). The $r$-th derivative of $\phi(z)$ admits the representation

$$
\phi^{(r)}(z) = (-1)^r \sum_{n=r}^{\infty} \frac{p^n}{1 - \alpha^n} \frac{(n - 1)!}{(n - r)!} (1 - z)^{n-r} \quad (r \geq 1).
$$

Uniform convergence allows for the interchange of limit (as $z \uparrow 1$) and summation in (54). Hence,

$$
\phi^{(r)}(1) = (-1)^r \frac{(r - 1)! p^r}{1 - \alpha^r} \quad (r \geq 1).
$$

Since $\ln \varphi_X(z) = -m\phi(z)$ (by (50)), it follows that $\varphi_X'(z) = -m\varphi_X(z)\phi'(z)$. An induction argument shows that the $r$-th derivative, $\varphi_X^{(r)}(z)$, of $\varphi_X(z)$ can be obtained by the following forward recursion (with $\varphi_X^{(0)}(z) = \varphi_X(z)$ and $\varphi^{(0)}_X(1) = 1$):

$$
\varphi_X^{(r)}(z) = -m \sum_{j=0}^{r-1} \binom{r-1}{j} \varphi_X^{(j)}(z)\phi^{(r-j)}(z).
$$

Therefore, the factorial moments $\mu_{[r]} = \varphi^{(r)}(1), \ r \geq 1,$ are finite and satisfy the recurrence relation (with $\mu_{[0]} = 1$),

$$
\mu_{[r]} = \varphi_X^{(r)}(1) = -m \sum_{j=0}^{r-1} \binom{r-1}{j} \mu_{[j]} \phi^{(r-j)}(1) \quad (r \geq 1).
$$

14
For example, we have

$$\mu[1] = \frac{mp}{1 - \alpha} \quad \text{and} \quad \mu[2] = \frac{m[(m+1)\alpha + m-1)p^2}{(1 - \alpha)(1 - \alpha^2)}.$$  \hspace{1cm} (58)

Similarly to the Bernoulli case, the moments of $X_t$, $\mu_r = E(X_t^r)$, $r \geq 1$, are finite and can be obtained from their factorial counterparts via (24) and (10).

By (50), the fcgf of $X_t$ is

$$\ln \varphi_X(1 + t) = m \sum_{r=1}^{\infty} \frac{(-1)^{r+1}(r-1)!p^r}{(1 - \alpha^r)} \frac{t^r}{r!},$$

which leads to the following formula for the factorial cumulants ($\kappa_r$, $r \geq 1$) of $\{X_t\}$:

$$\kappa_r = (-1)^{r+1}m(r-1)!p^r(1 - \alpha^r) \quad (r \geq 1).$$  \hspace{1cm} (59)

The cumulants of $X_t$, $\kappa_r$, $r \geq 1$ are given by (see (15) and (10))

$$\kappa_r = m \sum_{j=0}^{r} S(r, j)(-1)^{j+1}(j-1)!p^j(1 - \alpha^j) \quad (r \geq 1)$$  \hspace{1cm} (60)

6 Stationary INAR (1) models with Poissonian Binomial innovations

In this section, we develop a stationary INAR (1) process with a Poissonian Binomial innovation sequence with pgf and pmf given in (34) and (35), respectively. It is a generalization of the stationary INAR (1) process with binomial innovations seen in the previous Section.

**Theorem 7.** Let $\{X_t\}$ be the stationary INAR (1) process driven by (3) and with a Poissonian Binomial$(m, q, c)$ innovation sequence for some positive integer $m$ and some real numbers $q, c \in (0, 1)$. Then the marginal pgf $\varphi_X(z)$ of $\{X_t\}$ admits the following representations:

1. $\varphi_X(z) = \prod_{j=0}^{m-1} \left[ 1 + \sum_{n=1}^{\infty} \frac{(cq^j)^n(z-1)^n\alpha^n}{\prod_{i=1}^{n}(1 - \alpha^i)} \right].$  \hspace{1cm} (61)

2. $\varphi_X(z) = \exp \left\{ -\sum_{n=1}^{\infty} \frac{1 - qmn}{1 - q^n} \frac{e^n}{n(1 - \alpha^n)}(1 - z)^n \right\}.$  \hspace{1cm} (62)

**Proof:** Let $\Psi(z)$ be the pgf of the Poissonian Binomial $(m, q, c)$ distribution as given in (34). Then,

$$\Psi(1 - \alpha^i + \alpha^i z) = \prod_{j=0}^{m-1} (1 + c\alpha^i q^j(z - 1)).$$
which is the pgf of a Poissonian Binomial($m, q, c\alpha$). By Theorem 1, the marginal pgf $\varphi_X(z)$ is
\[
\varphi_X(z) = \prod_{j=0}^{m-1} \prod_{i=0}^{\infty} (1 + c\alpha^j q^i (z - 1)).
\]
Noting that $\prod_{j=0}^{\infty} (1 + c\alpha^j q^j (z - 1))$ is the marginal pgf of a stationary INAR (1) process with Bernoulli($cq^j$) innovations, representations (61) and (62) follow from (59) and (10), respectively.

For each $j \geq 0$, we denote by $\{q_r^{(j)}\}$ the pmf with pgf
\[
\varphi_j(z) = 1 + \sum_{n=1}^{\infty} \frac{(cq^j)^n(z - 1)^n\alpha_j^n}{\prod_{l=1}^{n}(1 - \alpha^l)}.
\]
By Theorem 5 and (37),
\[
q_r^{(j)} = \sum_{k=r}^{\infty} (-1)^{k-r} \binom{k}{r} \frac{(cq^j)^k\alpha_j^k}{\prod_{l=1}^{k}(1 - \alpha^l)}, \quad r \geq 0. \tag{63}
\]
It follows by Theorem 7 and (61) that the marginal pmf $\{q_r\}$ of the stationary INAR (1) process with a Poisson Binomial($m, q, c$) innovation results from the convolution of the pmf’s $\{q_r^{(j)}\}, 0 \leq j \leq m - 1$, i.e.,
\[
q_r = (q^{(0)} * q^{(1)} * \cdots * q^{(m-1)})_r \quad (r \geq 0). \tag{64}
\]
We proceed to give additional properties for the marginal distribution of the stationary INAR (1) process $\{X_t\}$ with Poisson Binomial($m, q, c$) innovations.

By (19), the 1-step transition probability is given by
\[
P(X_t = k|X_{t-1} = l) = \sum_{j=\max(k-m,0)}^{\min(l,k)} \binom{l}{j} \alpha_j^j (1-\alpha)^{l-j} q_{k-j}(m, q, c). \tag{65}
\]
By (21), the conditional pgf of $X_{t+k}$ given $X_t$ satisfies
\[
\varphi_{X_{t+k}|X_t}(z) = (1 - \alpha^k + \alpha^k z)^{X_t} \times \prod_{j=0}^{m-1} \prod_{i=0}^{k-1} (1 - (cq^j)\alpha^i(1 - z)).
\]
Therefore, the conditional distribution of $X_{t+k}$ given $X_t = n$ is the convolution of a Binomial($n, \alpha^k$) distribution and the Poissonian Binomial ($k, \alpha, cq^j$) distributions, $j = 0, 1, \cdots, m - 1$.

The innovation sequence of $\{X_t\}$ being Poissonian Binomial($m, q, c$) is underdispersed with mean and variance (see Kemp, 1987) given by
\[
\mu_z = \frac{(1 - q^n)c}{1 - q} \quad \text{and} \quad \sigma_z^2 = \frac{(1 - q^n)c}{1 - q} - \frac{(1 - q^{2n})c^2}{1 - q^2}.
\]
Therefore, the marginal distribution of $\{X_t\}$ is also underdispersed with mean and variance and dispersion index given by (18):
\[
\mu_X = \frac{(1 - q^n)c}{(1 - \alpha)(1 - q)}, \quad \sigma_X^2 = \frac{(1 - q^n)c}{(1 - \alpha)(1 - q)} - \frac{(1 - q^{2n})c^2}{(1 - \alpha^2)(1 - q^2)}.
\]
and
\[ I_X = 1 - \frac{(1 + q^n)c}{(1 + \alpha)(1 + q)}. \]

The factorial moments \( \mu_{[r]}, n \geq 1 \) of \( X_t \) can be obtained from the version (61) of \( \varphi_X(z) \). In this case,
\[ \varphi_X(1 + t) = \prod_{j=0}^{m-1} \left[ 1 + \sum_{n=1}^{\infty} \frac{(cq^j)^n\alpha^n_z}{\prod_{l=1}^{n}(1 - \alpha^n_l)} t^n \right]. \]
is a finite product of a power series. Therefore, it admits a power series expansion whose coefficients have rather complex expressions, even via recurrence formulas.

We proceed to derive simpler recurrence formulas for \( \mu_{[r]}, n \geq 1 \) by using the representation (62) of \( \varphi_X(z) \). Let
\[ \phi_1(z) = -\ln \varphi_X(z) = \sum_{n=1}^{\infty} \frac{1 - q^{mn}}{1 - q^n} \frac{c^n}{n(1 - \alpha^n)} (1 - z)^n. \] (66)
The argument we used to derive (54)–(57) in Section 5 carries over almost verbatim. We state the main results without further explanations.

The \( r \)-th derivative of \( \phi_1(z) \) admits the representation
\[ \phi_1^{(r)}(z) = (-1)^r \sum_{n=r}^{\infty} \frac{(1 - q^{mn})c^n}{(1 - q^n)(1 - \alpha^n)} \frac{(n - 1)!}{(n - r)!} (1 - z)^{n-r} \quad r \geq 1. \] (67)
Hence,
\[ \phi_1^{(r)}(1) = (-1)^r \frac{(1 - q^{mn})c^r}{(1 - q^r)(1 - \alpha^r)} (r - 1)! \quad (r \geq 1). \] (68)
Since \( \ln \varphi_X(z) = -\phi_1(z) \), the \( r \)-th derivative, \( \varphi_X^{(r)}(z) \), of \( \varphi_X(z) \) can be obtained by the following forward recursion (with \( \varphi_X^{(0)}(z) = \varphi_X(z) \) and \( \varphi_X^{(0)}(1) = 1 \)):
\[ \varphi_X^{(r)}(z) = -\sum_{j=0}^{r-1} \binom{r - 1}{j} \varphi_X^{(j)}(z) \phi_1^{(r-j)}(z). \] (69)
The factorial moments \( \mu_{[r]} = \varphi_X^{(r)}(1), r \geq 1 \), are finite and satisfy the recurrence relation (with \( \mu_{[0]} = 1 \)),
\[ \mu_{[r]} = -\sum_{j=0}^{r-1} \binom{r - 1}{j} \mu_{[j]} \phi_1^{(r-j)}(1) \quad (r \geq 1). \] (70)
The moments of \( X_t, \mu_r = E(X_t^r), r \geq 1 \), are finite and can be obtained from their factorial counterparts via (24) and (10).

We see by (62) that the fcgf of \( X_t \) is
\[ \ln \varphi_X(1 + t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n - 1)!(1 - q^{mn})c^n t^n}{(1 - q^n)(1 - \alpha^n)} \frac{t^n}{n!}, \]
which yields the following formula for the factorial cumulants \((\kappa_r, r \geq 1)\) of \(\{X_t\}\):

\[
\kappa_r = \frac{(-1)^{r+1}(r-1)!}{(1-q^r)}(1-q^r)(1-\alpha r)
\]

\[r \geq 1.
\]

(71)

The cumulants of \(X_t, (\kappa_r, r \geq 1)\) are given by (see (15) and (10))

\[
\kappa_r = \sum_{r=0}^{\infty} S(r, j)\frac{(-1)^{j+1}(j-1)!}{(1-q^r)}(1-q^r)(1-\alpha r)
\]

\[r \geq 1.
\]

(72)

7 Stationary INAR (1) processes with Heine innovations

A distribution on \(\mathbb{Z}_+\) is said to have the Heine distribution (Heine(\(\lambda, q\))) with parameters \(\lambda > 0\) and \(q \in (0, 1)\) if its pgf and pmf are respectively given by

\[
\Psi(z) = \prod_{j=0}^{\infty} (1 - \beta_j + \beta_j z)
\]

(73)

and

\[
f_r = \frac{\lambda^r q^{r(r-1)/2}}{\prod_{i=1}^{\infty} (1-q^i)} f_0, r \geq 1 \quad \text{and} \quad f_0 = \prod_{j=0}^{\infty} (1-\lambda q^j)^{-1},
\]

(74)

where \(\beta_j = \frac{\lambda q^j}{1+\lambda q^j}\) for \(j \geq 0\).

The Heine distribution was introduced by Benkherouf and Bather (1988). Kemp (1992) studied many of its properties. More details can be found in these references and in Johnson et al. (2005, Section 10.8.2). The Heine distribution is underdispersed and its mean and variance are

\[
\mu = \sum_{r=0}^{\infty} \frac{\lambda q^r}{1+\lambda q^r} \quad \text{and} \quad \sigma^2 = \sum_{r=0}^{\infty} \frac{\lambda q^r}{(1+\lambda q^r)^2}.
\]

(75)

We recall a few facts about double infinite products. Let \(\{a_{mn}\}\) be a double sequence. The double infinite product \(\prod_{i=0}^{\infty} \prod_{j=0}^{\infty} (1+a_{ij})\) is defined as the limit of the double sequence

\[
P_{mn} = \prod_{i=0}^{m} \prod_{j=0}^{n} (1+a_{ij}) \quad \text{as} \quad m, n \to \infty. \quad \text{If} \quad \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |a_{ij}| < \infty, \quad \text{then the double infinite product} \quad \prod_{i=0}^{\infty} \prod_{j=0}^{\infty} (1+a_{ij}) \quad \text{converges}. \quad \text{Moreover, if} \quad \prod_{i=0}^{\infty} \prod_{j=0}^{\infty} (1+a_{mn}) \quad \text{and} \quad \text{the iterated infinite products}
\]

\[
\prod_{i=0}^{\infty} \left[ \prod_{j=0}^{\infty} (1+a_{ij}) \right] \quad \text{and} \quad \prod_{j=0}^{\infty} \left[ \prod_{i=0}^{\infty} (1+a_{ij}) \right]
\]

converge, then they converge to the same limit.

**Theorem 8.** Let \(\{X_t\}\) be the stationary INAR (1) process driven by (3) and with a Heine(\(\lambda, q\)) innovation sequence for some \(\lambda > 0\) and \(0 < q < 1\). Then the marginal pgf \(\varphi_X(z)\) of \(\{X_t\}\) admits the following representations:
\[ \varphi_X(z) = \prod_{j=0}^{\infty} \left[ 1 + \sum_{n=1}^{\infty} \frac{\beta_j^n (z-1)^n \alpha_j(z)}{\prod_{l=1}^{n}(1-\alpha^l)} \right], \]  

where \( \beta_j \) is as in (73).

2. \[ \varphi_X(z) = \exp \left\{ -\sum_{n=1}^{\infty} \frac{B_n}{n(1-\alpha^n)} (1-z)^n \right\} \]  

with \( B_n = \sum_{j=0}^{\infty} \beta_j^n, \ n \geq 1. \)

**Proof:** First, we note that \( 0 < \beta_j < 1 \) for any \( j \geq 0. \) Moreover, for any \( n \geq 1, \)

\[ B_n = \sum_{j=0}^{\infty} \frac{\lambda^n (q^n)^j}{(1+\lambda q^n)^n} \leq \frac{\lambda^n}{1-q^n} < \infty. \]  

The pgf \( \Psi(z) \) of the innovation sequence of \( \{X_t\} \) is given by the right hand side of (73). Noting that for every \( i \geq 0, \)

\[ \Psi(1-\alpha^i + \alpha^i z) = \prod_{j=0}^{\infty} (1-\beta_j \alpha^i (1-z)), \]

it follows by Theorem 3 and (16) that

\[ \varphi_X(z) = \prod_{i=0}^{\infty} \prod_{j=0}^{\infty} (1 - \beta_j \alpha^i (1-z)). \]  

Clearly, the right hand side of (79) converges. A straightforward argument shows that the double infinite product \( \prod_{i=0}^{\infty} \prod_{j=0}^{\infty} (1 - \beta_j \alpha^i (1-z)) \) converges. In order to be able to interchange the order of the infinite products in (79), it remains to show that the iterated infinite product

\[ \prod_{j=0}^{\infty} \left[ \prod_{i=0}^{\infty} (1 - \beta_j \alpha^i (1-z)) \right] = \prod_{j=0}^{\infty} P_j(z) \]

converges, where for each \( j \geq 0, \)

\[ P_j(z) = \prod_{i=0}^{\infty} (1 - \beta_j \alpha^i (1-z)). \]

Note that for each \( j \geq 0, \) \( P_j(\cdot) \) has the form of the pgf of the marginal of an INAR (1) process with a Bernoulli(\( \beta_j \)) innovation (see (95) in Appendix). Therefore, by Theorem 5 and (39),

\[ P_j(z) = 1 + \sum_{n=1}^{\infty} \frac{\beta_j^n (z-1)^n \alpha_j(z)}{\prod_{l=1}^{n}(1-\alpha^l)}. \]  

For \( j \geq 0, \) denote

\[ \zeta_j(z) = \sum_{n=1}^{\infty} \frac{\beta_j^n (z-1)^n \alpha_j(z)}{\prod_{l=1}^{n}(1-\alpha^l)}. \]
By (78) and 0 ≤ z ≤ 1, we have
\[ \sum_{j=0}^{\infty} |\zeta_j(z)| \leq \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} \frac{\beta_j^n \alpha(z)}{\prod_{l=1}^{n} (1 - \alpha^l)} = \sum_{n=1}^{\infty} B_n \alpha(z) \leq \sum_{n=1}^{\infty} a_n, \]
where
\[ a_n = \frac{\lambda^n \alpha(z)}{1 - q^n \prod_{l=1}^{n} (1 - \alpha^l)}. \]
Recall that \( \alpha, q \in (0, 1) \) and hence
\[ \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{\lambda(1 - q^n) \alpha^n}{(1 - q^{n+1})(1 - \alpha^{n+1})} = 0. \]

By the ratio test, \( \sum_{j=0}^{\infty} |\zeta_j(z)| \) converges uniformly over \( z \in [0, 1] \). This in turn implies (see Knopp, 1990) that \( \prod_{j=0}^{\infty} P_j(z) \) converges uniformly over \( z \in [0, 1] \). The representation (76) then follows by interchanging the order of the infinite products in (79) and using (80). We now prove (77). By the first part of the proof, we have
\[ \varphi_X(z) = \prod_{j=0}^{\infty} \left[ \prod_{i=0}^{\infty} (1 - \beta_j \alpha^i(1 - z)) \right]. \]
Applying the representation (40) to \( \prod_{i=0}^{\infty} (1 - \beta_j \alpha^i(1 - z)) \) with \( p = \beta_j \), we have
\[ \varphi_X(z) = \exp \left\{ -\sum_{j=0}^{\infty} \sum_{n=1}^{\infty} \frac{\beta_j^n}{n(1 - \alpha^n)(1 - z)^n} \right\}. \]

This implies that the double series in (81) is convergent. Since its terms are non-negative (as 0 ≤ z ≤ 1), the order of summation can be interchanged (by Cauchy’s criterion for double series). This establishes the representation (77).

For each \( j \geq 0 \), we denote by \( \{p_r^{(j)}\} \) the pmf with pgf
\[ \varphi_j(z) = 1 + \sum_{n=1}^{\infty} \frac{\beta_j^n (z - 1)^n \alpha(z)}{\prod_{l=1}^{n} (1 - \alpha^l)}. \]

By Theorem 5 and (37),
\[ p_r^{(j)} = \sum_{k=r}^{\infty} (-1)^{k-r} \frac{k^r}{r!} \frac{\beta_j^k \alpha(z)}{\prod_{l=1}^{k} (1 - \alpha^l)}, \quad r \geq 0. \]

It follows by Theorem 8, (76) and the Continuity Theorem that the marginal pmf \( \{p_r\} \) of the stationary INAR (1) process with a Heine(\( \lambda, q \)) innovation is
\[ p_r = \lim_{j \to \infty} (p_r^{(0)} * p_r^{(1)} * \cdots * p_r^{(j)}), \quad (r \geq 0). \]

Next, we discuss several properties of the marginal distribution of the stationary INAR (1) process \( \{X_t\} \) with Heine(\( \lambda, q \)) innovations.
The 1-step transition probability can be obtained from (19). Given there are no notable simplifications of the formulas, we omit the details.

By (21), the conditional distribution of \(X_{t+k}\) given \(X_t = n\) results from the convolution of \(k + 1\) distributions, namely a Binomial\((n, \alpha^k)\) distribution and the distributions \(\{g^{(i)}_r\}, 0 \leq i \leq k - 1\) defined as follows:

\[
g^{(i)}_r = \alpha^i \sum_{l=0}^{\infty} \binom{r + l}{r} (1 - \alpha)^l f_{r+l},
\]

where \(\{f_r\}\) is the pmf of the Heine\((\lambda, q)\) distribution (74).

By (18) and (75), the mean, the variance and the index of dispersion of \(X_t\) are given by

\[
\mu_X = \frac{1}{1 - \alpha} \sum_{r=0}^{\infty} \frac{\lambda q^r}{1 + \lambda q^r}, \quad \sigma_X^2 = \frac{1}{1 - \alpha^2} \sum_{r=0}^{\infty} \frac{\lambda q^r}{(1 + \lambda q^r)^2}
\]

and

\[
I_X = \sum_{r=0}^{\infty} \frac{\lambda q^r}{1 + \lambda q^r} \left( [1 + \lambda q^r]^{-1} + \alpha \right) \left( [1 + \lambda q^r]^{-1} + \alpha \right).
\]

Since the Heine distribution is underdispersed, the INAR(1) process with a Heine innovation is underdispersed.

The factorial cumulants \((\kappa_r), r \geq 1\) of \(X_t\) are easily obtained. Indeed, by (77), the fcgf of \(X_t\) is

\[
\ln \varphi_X(1 + t) = \sum_{r=1}^{\infty} \frac{(-1)^{r+1} (r - 1)! B_r t^r}{1 - \alpha^r} r!.
\]

Since the series above converges everywhere, the factorial cumulants of all orders are finite and are given by

\[
\kappa_r = (-1)^{r+1} \frac{(r - 1)! B_r}{1 - \alpha^r} \quad (r \geq 1).
\]

The cumulants of \(X_t\), \((\kappa_r), r \geq 1\), can be obtained from the factorial cumulants via (15) and (10)

\[
\kappa_r = \sum_{j=0}^{r} S(r, j) (-1)^{j+1} (j - 1)! p^j \left( 1 - \alpha^j \right)
\]

The moments \(\mu_r, r \geq 1\) can be computed using the formula due to Smith (1995):

\[
\mu_r = \sum_{j=0}^{r-1} \binom{r - 1}{j} \kappa_{r-j} \mu_j
\]

with initial conditions \(\mu_0 = 1\) and \(\mu_1 = \kappa_1\). In turn, the factorial moments \(\mu_{[r]}, r \geq 1\), of \(X_t\) can be obtained via the formula (see Johnson et al. (2005), Section 1.2.7):

\[
\mu_{[r]} = \sum_{j=0}^{r} S(r, j) \mu_j,
\]

where \(\{s(r, j)\}\) are the Stirling numbers of the first kind of (33).
8 Extensions via convolution

Let \( \nu \geq 1 \) be a positive integer. Assume that \( \Psi(z) \) in Theorem 1 is the pgf of a finite convolution of \( \nu \) pmf’s \( \{f_{1r}\}, \{f_{2r}\}, \ldots , \{f_{\nu r}\} \) with respective pgf’s \( \Psi_1(z), \Psi_2(z), \ldots , \Psi_\nu(z) \) and \( \Psi'_k(1) < \infty \) for every \( 1 \leq k \leq \nu \). We denote by \( \{f_{kr}^{(i)}\} \) the pmf with pgf \( \Psi_k(1-\alpha^i + \alpha^i z) \) (with \( \{f_{kr}^{(0)}\} = \{f_{kr}\} \)). Note that the pmf \( \{f_r^{(i)}\} \) of \( \Psi(1-\alpha^i + \alpha^i z) \) is the convolution

\[
 f_r^{(i)} = \left( f_1^{(i)} * f_2^{(i)} * \cdots * f_\nu^{(i)} \right)_r, \quad r \geq 0. \tag{89}
\]

Applying Theorem 1 to \( \Psi_k(z) \) for each \( k \in \{1, 2, \ldots , \nu\} \), the function

\[
 \varphi^{(k)}(z) = \prod_{i=0}^\infty \Psi_k(1-\alpha^i + \alpha^i z) \tag{90}
\]

is a pgf and its pmf is

\[
 p_r^{(k)} = \lim_{n \to \infty} \left( f_k^{(0)} * f_k^{(1)} * \cdots * f_k^{(n-1)} \right)_r, \quad (r \geq 0), \tag{91}
\]

The pgf \( \varphi(z) \) of (5) with \( \Psi(z) = \prod_{k=1}^\nu \Psi_k(z) \) is \( \varphi(z) = \prod_{k=1}^\nu \varphi^{(k)}(z) \). Therefore, its pmf

\[
 p_r = \left( p^{(1)} * p^{(2)} * \cdots * p^{(\nu)} \right)_r \tag{92}
\]

results from the convolution of \( \{p_r^{(k)}\}, 1 \leq k \leq \nu \).

The mean, the variance, the cumulants and the factorial cumulants of \( \{p_r\} \) are the sums of their \( \{p_r^{(k)}\}, 1 \leq k \leq \nu \) counterparts assuming the latter are finite. The moments and the factorial moments could possibly be computed using the equations (57) and (88).

The Binomial and Poissonian distributions being finite convolutions of Bernoulli distributions, the INAR(1) processes introduced in Sections 5 and 6 could have been developed using the approach described above in conjunction with the Bernoulli INAR(1) of Section 3. However, the authors deemed these two models important enough to be treated separately and more thoroughly.

Next, we discuss some simple examples of stationary INAR(1) processes whose innovation is the convolution of a Poisson(\( \lambda \)) and the underdispersed distributions discussed in Sections 3-7. In the enumeration that follows, \( \{X_t\} \) designates a stationary INAR(1) process of (3).

1. If the innovation \( \{\varepsilon_t\} \) admits marginal law \( \varepsilon_t \sim Pois(\lambda) * Logarithmic(p) \), \( 0 < p < 1 - 1/e \), then its marginal distribution will result from the convolution of a Poisson(\( \lambda/(1-\alpha) \)) and the pmf \( \{p_r\} \) in Theorem 4.

2. If the innovation \( \{\varepsilon_t\} \) has the Power-Law distribution of the first kind \( (PL_1(\lambda, p)) \), i.e., \( \varepsilon_t \sim Pois(\lambda) * Bernoulli(p) \), \( 0 < p < 1 \), then its marginal distribution will result from the convolution of a Poisson(\( \lambda/(1-\alpha) \)) and the pmf \( \{p_r\} \) of (37) in Theorem 5. Note that this model was applied in Section 2.3 of Weiß (2013).

3. If the innovation \( \{\varepsilon_t\} \) has a Power-Law distribution of order \( m \) \( (PL_m^*(\lambda, p)) \), i.e., \( \varepsilon_t \sim Pois(\lambda) * Binomial(m, p) \), then the marginal of the corresponding stationary INAR(1) process is the convolution of Poisson(\( \lambda/(1-\alpha) \)) and the pmf \( \{p_r\} \) of (18) in Theorem 6.
4. If the innovation \( \{ \varepsilon_t \} \) admits marginal law \( \varepsilon_t \sim \text{Pois}(\lambda_1) * \text{Heine}(\lambda, q) \), \( \lambda > 0 \) and \( 0 < q < 1 \), then its marginal distribution will result from the convolution of a Poisson\( (\frac{\lambda_1}{1-\alpha}) \) and the pmf \( \{ p_r \} \) of (83) of Theorem 8.

Appendix

This section is devoted to the proof of Theorem 5. We start out with a Lemma.

**Lemma 2**

Assume \( n \geq 2 \) and \( a_i \in (0, 1) \) for \( i = 0, 1, 2, \cdots, n - 1 \). Then,

1. \[
\prod_{i=0}^{n-1} (1 - a_i) = 1 + \sum_{k=1}^{n} (-1)^k \sum_{0 \leq j_1 < j_2 < \cdots < j_k \leq n-1} \prod_{l=1}^{k} a_{j_l}.
\] (93)

2. \[
\sum_{0 \leq j_1 < j_2 < \cdots < j_k \leq n-1} \alpha^{j_1 \alpha^{j_2} \cdots \alpha^{j_k}} = \alpha^{k \choose 2} \prod_{l=0}^{k-1} \frac{1 - \alpha^{n-l}}{1 - \alpha^{l+1}}.
\] (94)

for every \( k \in \{1, \cdots, n\} \).

**Proof:** (1) follows by a straightforward induction.

(2) We also proceed by induction. The result is trivially true for \( n = 2 \) (forces \( k = 1 \)). Assume the assertion is true up to \( n \). It is clear that (94) holds for \( n+1 \) and \( k = n+1 \).

As in this case,

\[
\sum_{0 \leq j_1 < j_2 < \cdots < j_{n+1} \leq n} \alpha^{j_1 \alpha^{j_2} \cdots \alpha^{j_k}} = \alpha^{n+1 \choose 2} \prod_{l=0}^{n} \frac{1 - \alpha^{n+1-l}}{1 - \alpha^{l+1}}.
\]

Assume now \( k \in \{1, 2, \cdots, n\} \). Setting \( J = (j_1, j_2, \cdots, j_k) \in \mathbb{N}^k \), it is clear that

\[
\{ J \in \mathbb{N}^k : 0 \leq j_1 < j_2 < \cdots < j_k \leq n \} = A \cup B,
\]

where \( A = \{ J \in \mathbb{N}^k : 0 \leq j_1 < j_2 < \cdots < j_k \leq n - 1 \} \) and \( B = \{ J \in \mathbb{N}^k : 0 \leq j_1 < j_2 < \cdots < j_{k-1} \leq n - 1, j_k = n \} \). Therefore,

\[
\sum_{0 \leq j_1 < j_2 < \cdots < j_k \leq n} \prod_{l=1}^{k} \alpha^{j_l} = \sum_{J \in A} \prod_{l=1}^{k} \alpha^{j_l} + \sum_{J \in B} \alpha^n \prod_{l=1}^{k-1} \alpha^{j_l}.
\]

Using the induction hypothesis, it follows that

\[
\sum_{J \in A} \prod_{l=1}^{k} \alpha^{j_l} = \alpha^{k \choose 2} \prod_{l=0}^{k-1} \frac{1 - \alpha^{n-l}}{1 - \alpha^{l+1}}
\]

and

\[
\sum_{J \in B} \alpha^n \prod_{l=1}^{k-1} \alpha^{j_l} = \alpha^n \alpha^{k-1 \choose 2} \prod_{l=0}^{k-2} \frac{1 - \alpha^{n-l}}{1 - \alpha^{l+1}}.
\]
which implies

\[ \sum_{0 \leq j_1 < j_2 < \cdots < j_k \leq n} \prod_{l=1}^{k} \alpha^{j_l} = \frac{\prod_{l=0}^{k-2} (1 - \alpha^{n-l}) \left[ (1 - \alpha^{n-k+1}) \alpha^{(\frac{k}{2})} + (1 - \alpha^{k}) \alpha^{n} \alpha^{(\frac{k-1}{2})} \right]}{\prod_{l=0}^{k-1} (1 - \alpha^{l+1})}. \]

Now, noting that \( \binom{k}{q} = \binom{k-1}{2} + k - 1 \), it is easily seen that

\[ (1 - \alpha^{n-k+1}) \alpha^{(\frac{k}{2})} + (1 - \alpha^{k}) \alpha^{n} \alpha^{(\frac{k-1}{2})} = \alpha^{(\frac{k}{2})}(1 - \alpha^{n+1}). \]

Therefore, \((94)\) holds for \( n + 1 \).

**Proof of Theorem 5:**

Let \( \{X_t\} \) be the stationary INAR (1) process with a Bernoulli\((p)\) innovation sequence. By Theorem 3 and \((16)\), its marginal pgf is

\[ \varphi_X(z) = \prod_{i=0}^{\infty} (1 - pa^i(1 - z)). \quad (95) \]

Since \( \varphi_X(z) = \lim_{n \to \infty} \prod_{i=0}^{n-1} (1 - pa^i(1 - z)) \), we conclude by the continuity theorem that the marginal pmf \( \{p_r\} \) of \( \{X_t\} \) is the weak limit of a sequence of Poissonian Binomial distributions of \((34)\) and \((35)\), with \( m = n, q = \alpha \) and \( c = p \). Let \( r \geq 0 \). We define a purely atomic measure, we denote \( \text{meas}_r \), on \( \mathbb{N}_r = \{r, r+1, r+2, \cdots \} \) and its power set \( \mathcal{P}(\mathbb{N}_r) \) as follows:

\[ \text{meas}_r(\{k\}) = \frac{p^k \alpha^{(\frac{k}{2})}}{\prod_{i=1}^{k} (1 - a^i)}, \quad (k \geq r), \quad (96) \]

with \( \text{meas}_0(\{0\}) = 1 \). It is clear that \( \sum_{k=r}^{\infty} \text{meas}_r(\{k\}) < \infty \). Therefore, \( \text{meas}_r \) is a finite measure. Define now the sequence of functions \( \{f_n(.)\} \) on \( \mathbb{N}_r \) by

\[ f_n(k) = \begin{cases} (-1)^{k-r} \binom{k}{r} \prod_{l=0}^{k-1} (1 - \alpha^{n-l}) & \text{if } k = r, r+1, \cdots, n \\ 0 & \text{if } k > n. \end{cases} \]

Define \( h(k) = \binom{k}{r} \) on \( \mathbb{N}_r \). It is clear that \( |f_n(k)| \leq h(k) \) (recall \( \alpha \in (0, 1) \)) and that \( \sum_{k=r}^{\infty} h(k) \text{meas}_r(\{k\}) < \infty \) (by the ratio test). Moreover, for every \( k \in \mathbb{N}_r \),

\[ f(k) = \lim_{n \to \infty} f_n(k) = (-1)^{k-r} \binom{k}{r}. \]

Rewriting \( p_r^{(n)} \) in terms of the discrete integral of \( f_n(k) \) on the measure space \( (\mathbb{N}_r, \mathcal{P}(\mathbb{N}_r), \text{meas}_r) \) and calling on the dominated convergence theorem, we have

\[ p_r = \lim_{n \to \infty} \int_{\mathbb{N}_r} f_n(k) \text{meas}(dk) = \int_{\mathbb{N}_r} f(k) \text{meas}(dk), \]

\[ \text{by the dominated convergence theorem.} \]
which is precisely (37) and thus part (1) of the Theorem is established. To show (2), note that

\[ P(X_t \geq r) = \sum_{j=r}^{\infty} \sum_{k=r}^{\infty} (-1)^{k-j} \binom{k}{j} \frac{p^k \alpha^{(k)}}{\prod_{i=1}^{k} (1 - \alpha^i)}. \]

Since the double series above converges absolutely, interchanging summations is allowed, leading to

\[ P(X_t \geq r) = \sum_{k=r}^{\infty} \left( \sum_{j=r}^{k} (-1)^{k-j} \binom{k}{j} \frac{p^k \alpha^{(k)}}{\prod_{i=1}^{k} (1 - \alpha^i)} \right). \]

We have by induction on \( k \) that \( \sum_{j=r}^{k} (-1)^{k-j} \binom{k}{j} = (-1)^{k-r} \binom{k-1}{r-1} \), establishing (38). For part (3), we note first that \( \phi_X(z) \) of (95) can be rewritten as

\[ \phi_X(z) = \exp \left\{ \sum_{i=0}^{\infty} \ln(1 - p\alpha^i(1 - z)) \right\}. \]

The representation (40) of \( \phi_X(z) \) follows by way of the power series expansion of

\[-\ln(1 - x) = \sum_{n=1}^{\infty} x^n/n, \quad 0 \leq x < 1\]

applied to \( x = p\alpha^i(1 - z) \) in (97).

To prove (39), we first note that by letting \( a_i = p\alpha^i(1 - z) \) in (93) and using (94), we obtain the following expression for \( \phi_n(z) \) of (34):

\[ \phi_{n-1}(z) = 1 + \sum_{k=1}^{n} p^k(z-1)k\alpha^{(k)} \prod_{l=0}^{k-1} \frac{1 - \alpha^{n-l}}{1 - \alpha^{l+1}}. \]

and therefore,

\[ \phi_X(z) = \lim_{n \to \infty} \left[ 1 + \sum_{k=1}^{n} (z-1)^k \prod_{l=0}^{k-1} (1 - \alpha^{n-l}) \frac{p^k \alpha^{(k)}}{\prod_{i=1}^{k} (1 - \alpha^i)} \right]. \]

We proceed as in the proof of (37). We define a sequence of functions \( g_n(k) \) on the finite measure space \( (\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{meas}_0) \), where \( \text{meas}_0 \) is defined in (96):

\[ g_n(k) = \begin{cases} 
1 & \text{if } k = 0 \\
(z-1)^k \prod_{l=0}^{k-1} (1 - \alpha^{n-l}) & \text{if } 1 \leq k \leq n \\
0 & \text{if } k > n.
\end{cases} \]

It is easily seen that \( |g_n(k)| \leq 1 \) (recall \( \alpha \in (0, 1) \) and \( z \in [0, 1] \)) and that

\[ g(k) = \lim_{n \to \infty} g_n(k) = \begin{cases} 
1 & \text{if } k = 0 \\
(z-1)^k & \text{if } 1 \leq k \leq n \\
0 & \text{if } k > n.
\end{cases} \]
Rewriting (99) in terms of the discrete integral on the measure space \((\mathbb{N}, \mathcal{P}(\mathbb{N}), \text{meas}_0)\) and calling on the Dominated Convergence Theorem, we have

\[
\varphi_X (z) = \lim_{n \to \infty} \int_{\mathbb{N}} g_n(k) \text{meas}(dk) = \int_{\mathbb{N}} g(k) \text{meas}(dk),
\]

which is precisely (39).

Lastly, we prove part (3). We need to show that \(\varphi_X(z)\) admits the representation (36). Recall \(\varphi_X(z) = \exp \{-\phi(z)\}\) with \(\phi(z)\) of (53). Note that

\[
\varphi_X(z) = \exp \left\{ \phi(0) \left( \frac{\phi(0) - \phi(z)}{\phi(0)} - 1 \right) \right\}
\]

and

\[
\frac{\phi(0) - \phi(z)}{\phi(0)} = \frac{1}{\phi(0)} \sum_{n=1}^{\infty} \frac{p^n}{n(1 - \alpha^n)} \{1 - (1 - z)^n\}
\]

\[
= \frac{1}{\phi(0)} \sum_{n=1}^{\infty} (-1)^{n+1} a_n(p, \alpha) a_n^z,
\]

where

\[
a_n(p, \alpha) = \frac{(-1)^{n+1}}{\phi(0)} \sum_{j=n}^{\infty} \left( \frac{j}{n} \right) \frac{p^j}{j(1 - \alpha^j)}.
\]

Hence,

\[
\varphi_X(z) = \exp \left\{ \phi(0) \left( \sum_{n=1}^{\infty} a_n(p, \alpha) a_n^z - 1 \right) \right\}.
\]

Clearly

\[
\sum_{n=1}^{\infty} a_n(p, \alpha) = 1.
\]

Assume that \(p < 0.5\) and note that

\[
\sum_{n=1}^{\infty} |a_n(p, \alpha)| \leq \frac{1}{\phi(0)} \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \left( \frac{j}{n} \right) \frac{p^j}{j(1 - \alpha^j)}
\]

\[
= \frac{1}{\phi(0)} \sum_{j=1}^{\infty} \frac{p^j}{j(1 - \alpha^j)} \sum_{n=1}^{j} \left( \frac{j}{n} \right)
\]

\[
= \frac{1}{\phi(0)} \sum_{j=1}^{\infty} \frac{p^j}{j(1 - \alpha^j)} (2^j - 1)
\]

\[
= \frac{1}{\phi(0)} \sum_{j=1}^{\infty} \frac{(2p)^j}{j(1 - \alpha^j)} - 1 < \infty.
\]

Noting \(\phi(0) = \lambda\) (of (41)). \(\blacksquare\)

**Remark**

We note that Lemma 2 and the representation (99) of the pgf of the Poissonian Binomial distribution (35) were known to Kemp (1987), but not stated in her paper. We chose to include the technical details for the sake of completeness. For example, the pmf of the Poissonian Binomial distribution can also be obtained by applying the Binomial theorem to \((z - 1)^k\) in (98) and switching the order of summation. The factorial moments and factorial cumulants of (35) can be obtained via the expansions of \(\varphi_{n-1}(1 + t)\) and \(\ln \varphi_{n-1}(1 + t)\) in the same way that led to (43) and (46).
References

[1] Abramowitz, M. and Stegun, I.A. (1965). *Handbook of Mathematical Functions*, New York: Dover.

[2] Al-Osh, M.A. and Alzaid, A.A. (1987). First-order integer valued autoregressive INAR(1) process. Journal of Time Series Analysis, 8, 261–275.

[3] Al-Osh, M.A., and Aly, E.-E.A.A. 1992. First order autoregressive time-series with negative binomial and geometric marginals. Communications in Statistics—Theory and Methods, 21, 2483–92.

[4] Aly, E.-E.A.A and Bouzar, N. (1994). Explicit stationary distributions for some Galton-Watson processes with immigration. Communications in Statistics—Stochastic Models, 10, 499–517.

[5] Aly, E.-E.A.A., and Bouzar, N. (2019). Expectation thinning operators based on linear fractional probability generating functions. Journal of the Indian Society for Probability and Statistics, 20, 89–107.

[6] Benkherouf, L. and Bather, J.A. (1988). Oil exploration: Sequential decisions in the face of uncertainty. Journal of Applied Probability, 25, 529–543.

[7] Bouzar, N. and Jayakumar, K. (2008). Time series with discrete semi-stable marginals. Statistical Papers, 49, 619–635.

[8] Feller, W. (1968). *An Introduction to Probability Theory and its Applications*, Vol. 1, John Wiley & Sons Inc.

[9] Goldberg, K., Leighton, F.T., Newman, M., and Zuckerman, S.L. (1976). Tables of binomial coefficients and Stirling numbers. J. of Research of the National Bureau of Standards, 80B, 99–171.

[10] Guerrero, M.B., Barreto-Souza, W., and Ombao, H. (2020). Integer-valued autoregressive process with flexible marginal and innovation distributions. arXiv:2004.08667 [stat.ME].

[11] Joe, H. (1996). Time series models with univariate margins in the convolution-closed infinitely divisible class. Journal of Applied Probability, 33, 664–77.

[12] Joe, H. (2019). Likelihood inference for generalized Integer autoregressive time series models. Econometrics, 7, 1-13.

[13] Johnson, L.N., Kemp, A.W., and Kotz, S. (2005). *Univariate Discrete Distributions*, Third Ed., John Wiley & Sons Inc.

[14] Jung, R.C., Ronning, G., Tremayne, A.R. (2005). Estimation in conditional first order autoregression with discrete support. Statistical Papers, 46, 195–224.

[15] Kemp, A. W. (1987). A Poissonian binomial model with constrained parameters. Naval Research Logistics, 34, 853–858.

[16] Kemp, A.W. (1992). Heine-Euler extensions of the Poisson distribution. Communications in Statistics—Theory and Methods, 21, 571–588.
[17] Knopp, K. (1990). *Theory and Applications of Infinite Series*. Dover, New York.

[18] Latour, A. (1998). Existence and stochastic structure of a non-negative integer-valued autoregressive process. Journal of Time Series Analysis, **19**, 439–455.

[19] McKenzie, E. (1985). Some simple models for discrete variate time series. Water Resources Bulletin, **21**, 645–650.

[20] McKenzie, E. (1988). Some ARMA models for dependent sequences of Poisson counts. Advances in Applied Probability, **20**, 822–835.

[21] McKenzie, E. (2003). Discrete variate time series In: *Handbook of Statistics: Stochastic Processes: Modelling and Simulation*, C.R. Rao and D. Shanbhag, Eds., Elsevier Science, Amsterdam.

[22] Pedeli, X., Karlis, D. (2011). A bivariate INAR(1) process with application. Statistical Modelling, **11**, 325–349.

[23] Ristic, M., Bakouch, H., & Nastic, A. (2009). A new geometric first-order integer-valued autoregressive (NGINAR(1)) process. Journal of Statistical Planning and Inference, **139**, 2218–2226.

[24] Schweer, S. and Weiß, C.H. (2014). Compound Poisson INAR(1) processes: Stochastic properties and testing for overdispersion. Computational Statistics and Data Analysis, **77**, 267–284

[25] Schweer, S., Wichelhaus, C. (2015). Queuing systems of INAR(1) processes with compound Poisson arrival distributions. Stochastic Models, **31**, 618-635.

[26] Scotto, M.G., Weiß, C.H., and Gouveia, S. (2015). Thinning-based models in the analysis of integer-valued time series: a review. Statistical Modelling, **15**, 590–618.

[27] Smith, P.J. (1995). A recursive formulation of the old problem of obtaining moment from cumulants and vice versa. American Statistician, **49**, 217–218.

[28] Steutel, F.W. and van Harn, K. (1979). Discrete analogues of self-decomposability and stability. Annals of Probability, **7**, 893–899.

[29] Weiß, C.H. (2013). Integer-valued autoregressive models for counts showing underdispersion. Journal of Applied Statistics, **40**, 1931–1940.

[30] Weiß, C.H. (2018). *An Introduction to Discrete-Valued Time Series*. Hoboken: Wiley.

[31] Zhu, R. and Joe, H. (2003). A new type of discrete self-decomposability and its application to continuous-time Markov processes for modeling count data time series. Stochastic Models, **19**, 235–254.

[32] Zhu, R. and Joe, H. (2010). Negative binomial time series models based on expectation thinning operators. Journal of Statistical Planning and Inference, **140**, 1874–1888.