A SIMPLE PROOF THAT THE POWER $\frac{2m}{m+1}$ IN THE
BOHNEBBLUST–HILLE INEQUALITIES IS SHARP

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ABSTRACT. The power $\frac{2m}{m+1}$ in the polynomial (and multilinear) Bohnenblust–
Hille inequality is optimal. This result is well-known but its proof highly nontrivial.
In this note we present a quite simple proof of this fact.

1. INTRODUCTION

The polynomial and multilinear Bohnenblust–Hille inequalities were proved by
H.F. Bohnenblust and E. Hille in 1931 and play a crucial role in different fields as
Fourier and Harmonic Analysis and Quantum Information Theory (see [4, 5, 7]).
The polynomial Bohnenblust–Hille inequality proves the existence of a positive
function $C : \mathbb{N} \to [1, \infty)$ such that for every $m$-homogeneous polynomial $P$ on $\mathbb{C}^N$,
the $\ell^{\frac{2m}{m+1}}$-norm of the set of coefficients of $P$ is bounded above by $C_m$ times the sup-
premum norm of $P$ on the unit polydisc. This result has important striking applications
in different contexts (see [4]). The multilinear version of the Bohnenblust–Hille in-
equality asserts that for every positive integer $m \geq 1$ there exists a sequence of positive scalars $(C_m)_{m=1}^{\infty}$ in $[1, \infty)$ such that

$$\left( \sum_{i_1, \ldots, i_m=1}^{N} |T(e_{i_1}, \ldots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \sup_{z_1, \ldots, z_m \in \mathbb{D}^N} |T(z_1, \ldots, z_m)|$$

for all $m$-linear forms $T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to \mathbb{C}$ and every positive integer $N$, where $(e_i)_{i=1}^{N}$ denotes the canonical basis of $\mathbb{C}^N$ and $\mathbb{D}^N$ represents the open unit polydisk in $\mathbb{C}^N$.

The original proof ([3]) that the power $\frac{2m}{m+1}$ is optimal is quite puzzling. Acc-
cording to Defant et al ([4] page 486), Bohnenblust and Hille “showed, through a
highly nontrivial argument, that the exponent $\frac{2m}{m+1}$ cannot be improved” or ac-
cording to Defant and Schwarting ([6] page 90), Bohnenblust and Hille showed “with
a sophisticated argument that the exponent $\frac{2m}{m+1}$ is optimal”. In [2] there is an
alternative proof for the case of multilinear mappings, but the arguments are also
nontrivial, involving $p$-Sidon sets and sub-Gaussian systems. The main goal of this
note is to present a quite elementary proof (which solves simultaneously the cases
of polynomials and multilinear mappings) of the optimality of $\frac{2m}{m+1}$.

Key words and phrases. Bohnenblust–Hille inequalities.
2. The new proof of the sharpness of $\frac{2m}{m+1}$

We will show that the optimality of the power $\frac{2m}{m+1}$ is a straightforward consequence of the following famous result known as Kahane-Salem-Zygmund inequality (see [3, Theorem 4, Chapter 6] or [1, page 21]):

**Theorem 2.1** (Kahane-Salem-Zygmund inequality). Let $m, n$ be positive integers. Then there are signs $\varepsilon_\alpha = \pm 1$ so that the $m$-homogeneous polynomial

$$P_{m,n} : \ell^\infty_n \to \mathbb{C}$$

given by

$$P_{m,n} = \sum_{|\alpha| = m} \varepsilon_\alpha z^\alpha$$

satisfies

$$\|P_{m,n}\| \leq C n^{(m+1)/2} \sqrt{\log m}$$

where $C$ is an universal constant (it does not depend on $n$ or $m$).

**Theorem 2.2.** The power $\frac{2m}{m+1}$ in the Bohnenblust–Hille inequalities is sharp.

**Proof.** Let $m \geq 2$ be a fixed positive integer. For each $n$, let $P_{m,n} : \ell^\infty_n \to \mathbb{C}$ be the $m$-homogeneous polynomial satisfying the Kahane-Salem-Zygmund inequality. For our goals it suffices to deal with the case $n > m$.

Let $q < \frac{2m}{m+1}$. Then a simple combinatorial calculation shows that

$$\left( \sum_{|\alpha| = m} |\varepsilon_\alpha|^q \right)^{1/q} = \left( p(n) + \frac{1}{m!} \prod_{k=0}^{m-1} (n-k) \right)^{\frac{1}{q}},$$

where $p(n) > 0$ is a polynomial of degree $m-1$. If the polynomial Bohnenblust–Hille inequality was true with the power $q$, then there would exist a constant $C_{m,q} > 0$ so that

$$C_{m,q} \geq \frac{1}{n^{(m+1)/2} \sqrt{\log m}} \left( p(n) + \frac{1}{m!} \prod_{k=0}^{m-1} (n-k) \right)^{1/q}$$

for all $n$. If we raise both sides to the power of $q$ and make $n \to \infty$ we obtain

$$(C_{m,q}C)^q \geq \lim_{n \to \infty} \left( \frac{r(n)}{m!n^q(m+1)/2 (\sqrt{\log m})^q} + \frac{p(n)}{n^q(m+1)/2 (\sqrt{\log m})^q} \right),$$

with

$$r(n) = \prod_{k=0}^{m-1} (n-k).$$

Since

$$\deg r = m > \frac{q(m+1)}{2}$$

we have

$$\lim_{n \to \infty} \left( \frac{r(n)}{m!n^q(m+1)/2 (\sqrt{\log m})^q} + \frac{p(n)}{n^q(m+1)/2 (\sqrt{\log m})^q} \right) = \infty,$$

a contradiction. Since the multilinear Bohnenblust–Hille inequality (with a power $q$) implies the polynomial Bohnenblust–Hille inequality with the same power, we conclude that $\frac{2m}{m+1}$ is also sharp in the multilinear case. $\Box$
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