Hermitian realizations of $\kappa$–Minkowski spacetime

Domagoj Kovačević‡, Stjepan Meljanac †, Andjelo Samsarov † and Zoran Škoda †

‡ Faculty of Electrical Engineering and Computing, Unska 3, HR-10000 Zagreb, Croatia
E-mail: domagoj.kovacevic@fer.hr

† Division for Theoretical Physics, Institute Rudjer Bošković, Bijenička 54, P.O.Box 180, HR-10002 Zagreb, Croatia
E-mail: meljanac@irb.hr, zskoda@irb.hr, asamsarov@irb.hr

doi:10.3842/SIGMA.201*.***

Abstract. General realizations, star products and plane waves for $\kappa$–Minkowski spacetime are considered. Systematic construction of general hermitian realization is presented, with special emphasis on noncommutative plane waves and hermitian star product. Few examples are elaborated and possible physical applications are mentioned. It is pointed out that different realizations of $\kappa$–Minkowski spacetime might have different physical consequences.

2010 Mathematics Subject Classification: 81R60,16A58, 81T75
PACS 02.20.Sv, 11.10.Nx

Key words: star product, kappa-Minkowski spacetime, hermitian realization, plane wave

1 Introduction

It has been argued that the description of spacetime as a smooth manifold is not adequate at the very small distances such as those of the order of the Planck length. Instead, in this region, the spacetime structure might require a description in terms of noncommutative geometry. This may result in numerous consequences, some already seen at the theoretical level, affecting much of the high energy physics, including implications for astrophysics. The change in the nature of the spacetime structure requires the modification of a mathematical setup and eventually leads to a need of rewriting the quantum field theory in the presence of quantum group symmetries.

Since the Poincaré group describes the symmetry inherent to the quantum field theory, it is plausible to assume that the symmetry adequate to such a theory on a noncommutative space might be described by a quantum deformation of the original Poincaré group. One way to accommodate this idea is within the framework of Hopf algebras. One of the most extensively studied Hopf algebras in the role of such a deformation is the $\kappa$-Poincaré algebra [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], with $\kappa$ denoting a masslike deformation parameter, usually associated with the Planck mass.

It is known for some time that the coproduct of the quantum group codifies the curvature in the momentum space and that a noncommutative spacetime algebra multiplication appears as dual to the coproduct on the momentum algebra; moreover the two dual Hopf algebras together form a noncommutative phase space algebra built through a cross (smash) product algebra construction, more specifically the Heisenberg double [15, 16, 17]. Curiously, the idea that the curvature of momentum space could imply the noncommutativity of spacetime does not appear to be a recent one, since it can be traced back to the first half of the past century [18]. Later on, this led to a quantum geometric picture in which curved momentum space is dual to noncommutative spacetime and where the relation to quantum groups is made explicit [19].

The authors of the present paper find $\kappa$-Poincaré algebra and the emergent $\kappa$-Minkowski spacetime interesting from many reasons. However, the following two may be particularly highlighted. Already in the early study of certain limits of quantum gravity coupled to matter in [20, 21, 22], possible effective quantum field theories on noncommutative $\kappa$-Minkowski space were found by integrating out the gravitational degrees of freedom from the generating functional.

1The thesis reported in [22] is somewhat different from the ones advocated in [20, 21] and has been partly challenged by [23, 24].
The \( \kappa \)-Minkowski spacetime and \( \kappa \)-Poincaré group may also provide a setting appropriate for trapping the signals of quantum gravity effects, what might be an explanation for the observations of ultrahigh energy cosmic rays, contradicting the usual understanding of electron-positron pair production in collisions of high energy photons and other high energy astrophysical processes alike. One possible explanation of the deviations of this kind may rely on the modified dispersion relations \[25, 26, 27, 28\], whose distortion can eventually be traced back to noncommutative structure of the spacetime, particularly in the case of \( \kappa \) type. All these may be the relics of the quantum gravity effects and in turn might allow a description in terms of the effective noncommutative quantum field theories (NCQFT). The latter is particularly interesting in light of the arguments supporting the assertion of NCQFT on a \( \kappa \)-Minkowski space emerging from the quantum gravity in a certain low energy limit \[20\].

In \[29\] the properties of the star product in the natural realization of the \( \kappa \)-Minkowski algebra are investigated, together with the properties of the resulting free scalar field theory constructed from this particular star product. The above mentioned realization is the type of a differential representation of the \( \kappa \)-Minkowski algebra that is intimately related to the so called classical basis of \( \kappa \)-Poincaré \[16, 17, 30\]. These properties have been further studied in \[31\] and the analysis has been extended to include the implications of the hermitization process on the star product and the resulting scalar field theory. In this paper, we extend this analysis to include an arbitrary realization of \( \kappa \)-Minkowski algebra, investigate the corresponding star products and the effects of hermitization on their properties.

These issues are particularly important when one seeks to find out how the physical results depend on the choice of a realization of the spacetime algebra. In this direction, the dependence on the realization has been studied for the electrodynamics \[32\] and the geodesics \[33\] in \( \kappa \)-Minkowski spacetime. Furthermore, the time delay phenomenon for photons having different energies has been investigated in the context of relative locality \[34\] to see how it varies with realizations \[35\]. The properties of the scalar field propagation in the noncommutative \( \phi^4 \) model are investigated in \[36\] in the case of the natural realization. It would be interesting to compare these results with the calculations for other realizations. While it is desirable to identify the properties universal to all realizations of \( \kappa \)-Minkowski spacetime, those properties which are not invariant, may, after comparison with the experimental data, serve to narrow down the set of allowed realizations. We hope that the experiments may reach the necessary level of sensitivity for this in near future (see astrophysical review \[37\]). To address these issues, it is necessary to set up the adequate framework and investigate the mathematical properties of the star products related to each of the realizations of \( \kappa \)-Minkowski. The present paper serves this purpose.

After reviewing in Section 2 the general notions used in the analysis, in Sections 3 and 4 we introduce the star products corresponding to various realizations and present their properties. Then, in Section 5, we describe the process of hermitization and its accompanying impact on the properties of the star product. Up to the Section 6 the analysis is general, while in Section 7 we review the most common concrete examples.

2 \( \kappa \)-Minkowski spacetime and \( \kappa \)-Poincaré algebra

We consider the \( \kappa \)-Minkowski space \( m_\kappa \), for which the following commutation relations

\[
[x_\mu, x_\nu] = i (a_\mu x_\nu - a_\nu x_\mu)
\]

are satisfied. Latin indices are used for the set \( \{1, \ldots, n-1\} \) and Greek indices for the set \( \{0, \ldots, n-1\} \). The metric of the \( \kappa \)-Minkowski space is given by \( \eta_{\mu\nu} = \text{diag}(-1,1,\ldots,1) \).

The Lorentz algebra, \( \mathfrak{l} \), is generated by \( M_{\mu\nu} \) \( (\nu_\mu = -M_{\mu\nu}) \) satisfying the usual commutation relations

\[
[M_{\mu\nu}, M_{\lambda\rho}] = M_{\mu\rho} \eta_{\nu\lambda} - M_{\nu\rho} \eta_{\mu\lambda} - M_{\mu\lambda} \eta_{\nu\rho} + M_{\nu\lambda} \eta_{\mu\rho}.
\]

The \( \kappa \)-Minkowski space \( m_\kappa \) can be enlarged to the Lie algebra \( \mathfrak{g}_\kappa \) which is generated by \( M_{\mu\nu} \) and \( x_\lambda \), and where the commutator \([M_{\mu\nu}, x_\lambda]\)

\[
[M_{\mu\nu}, x_\lambda] = x_\mu \eta_{\nu\lambda} - x_\nu \eta_{\mu\lambda} - i (a_\mu M_{\nu\lambda} - a_\nu M_{\mu\lambda}).
\]

In \[38\], it is shown that the relation \[31\] is the unique way to obtain a Lie algebra from \( m_\kappa \) and \( \mathfrak{l} \), so that \( m_\kappa \) and \( \mathfrak{l} \) are complementary Lie subalgebras.
It remains to add the momentum generators \( p_0, \ldots, p_{n-1} \) satisfying the commutation relations

\[
[p_\mu, p_\nu] = 0.
\]  

(4)

Our goal is to enlarge \( U(\mathfrak{g}_n) \) by the momentum generators \( p_\mu \) to the algebra \( \hat{\mathcal{H}} \) (see [39] for a similar approach). For the beginning, let us complete the set of commutation relations for elements \( M_{\mu\nu}, \hat{x}_\mu \) and \( p_\mu \):

\[
[p_\mu, \hat{x}_\nu] = -i\hbar_{\mu\nu}(p)
\]  

(5)

and

\[
[M_{\mu\nu}, p_\lambda] = g_{\mu\nu\lambda}(p).
\]  

(6)

Functions \( h_{\mu\nu} \) and \( g_{\mu\nu\lambda} \) are real, satisfying \( \lim_{\alpha \to 0} h_{\mu\nu} = \eta_{\mu\nu} \cdot 1, \det h \neq 0 \) and \( \lim_{\alpha \to 0} g_{\mu\nu\lambda} = p_\mu \eta_{\nu\lambda} - p_\nu \eta_{\mu\lambda} \). Now, we require that all Jacobi identities are satisfied. This gives conditions on \( h_{\mu\nu} \) and \( g_{\mu\nu\lambda} \). If exactly one of the three generators appearing in a Jacobi identity is \( p_\mu \), the remaining two generators can either be \( M_{\mu\nu} \) and \( M_{\lambda\rho} \), or \( \hat{x}_\mu \) and \( \hat{x}_\nu \), or \( M_{\mu\nu} \) and \( \hat{x}_\lambda \). These three cases lead to the respective system of differential equations:

\[
\frac{\partial g_{\lambda\rho\sigma}}{\partial p_\alpha} g_{\mu\nu\alpha} - \frac{\partial g_{\mu\nu\sigma}}{\partial p_\alpha} g_{\lambda\rho\alpha} = g_{\mu\rho\sigma} \eta_{\lambda\nu} - g_{\nu\rho\sigma} \eta_{\lambda\mu} - g_{\mu\lambda\sigma} \eta_{\nu\rho} + g_{\nu\lambda\sigma} \eta_{\mu\rho},
\]

\[
\frac{\partial h_{\lambda\nu}}{\partial p_\alpha} h_{\alpha\mu} - \frac{\partial h_{\lambda\mu}}{\partial p_\alpha} h_{\alpha\nu} = a_\mu h_{\lambda\nu} - a_\nu h_{\lambda\mu},
\]

\[
\frac{\partial g_{\mu\nu\lambda}}{\partial p_\alpha} h_{\alpha\rho} - \frac{\partial h_{\lambda\nu}}{\partial p_\alpha} g_{\mu\rho\alpha} = h_{\lambda\nu} \eta_{\mu\rho} - h_{\lambda\rho} \eta_{\mu\nu} + a_\mu g_{\nu\rho\lambda} - a_\nu g_{\mu\rho\lambda},
\]

putting the constraints on the possible choices of the functions \( h_{\mu\nu} \) and \( g_{\mu\nu\lambda} \). For more details see also [40].

The system of differential equations \((7)\cdot(9)\) has infinitely many solutions. In Section 6, it is shown how one constructs each of these solutions from a particular one. It is important to mention that functions \( g_{\mu\nu\lambda}(p) \) are completely determined by the functions \( h_{\mu\nu}(p) \) (see Section 6 for details). Generators \( M_{\mu\nu} \) and \( p_\mu \) form \( \kappa \)-Poincaré algebra. This algebra is denoted by \( \mathfrak{p} \).

Let us define the invertible element \( Z \) (see [38]) in \( \hat{\mathcal{H}} \) by the shift property

\[
[Z, \hat{x}_\mu] = ia_\mu Z
\]

(10)

and

\[
[Z, p_\mu] = 0.
\]

(11)

Also, it is useful to introduce the element \( \Box \) in \( \hat{\mathcal{H}} \), defined by the requirements

\[
[\Box, M_{\mu\nu}] = 0,
\]

(12)

\[
[\Box, p_\mu] = 0,
\]

(13)

\[
[\Box, \hat{x}_\mu] = 2i p_\mu,
\]

(14)

and which in the classical limit gives the standard d’Alambertian. Elements \( Z \) and \( \Box \) are in \( \hat{\mathcal{H}} \) and we emphasize them because of their properties. Equations \((1)\cdot(10)\) show that \( \hat{x}_\mu \) and \( Z \) generate the Lie algebra.

In the next section we present how \( M_{\mu\nu} \) can be expressed as a function of \( \hat{x}_\mu \) and \( p_\mu \), while \( Z \) and \( \Box \) can be expressed as functions of \( p_\mu \). Hence, \( \hat{\mathcal{H}} \) is generated by \( \hat{x}_\mu \) and \( p_\mu \) only. To be more precise, \( \hat{\mathcal{H}} \) is the quotient of the free algebra generated by \( \hat{x}_\mu \) and \( p_\mu \) and the ideal generated by the relations \((1)\cdot(11)\) and \((14)\). Let us denote by \( \hat{\mathcal{A}} \) the subalgebra of \( \hat{\mathcal{H}} \) generated by \( \hat{x}_\mu \). Then we can introduce the action \( \triangleright \) of \( \hat{\mathcal{H}} \) on \( \hat{\mathcal{A}} \) defined by

\[\hat{\mathcal{H}} \text{ should be generated by } \hat{x}_\mu \text{ and } p_\mu, \text{ but we simplify the notation by writing } \hat{p}_\mu \equiv p_\mu.\]
1. \( \hat{g}(\hat{x}) | 1 = \hat{g}(\hat{x}), \ x_{\mu} | \hat{g}(\hat{x}) = \hat{x}_{\mu} \hat{g}(\hat{x}), \ \hat{g}(\hat{x}) \in \mathcal{A}, \)

2. \( p_{\mu} | 1 = 0, \ M_{\mu\nu} | 1 = 0, \)

3. \( p_{\mu} | \hat{g}(\hat{x}) = [p_{\mu}, \hat{g}(\hat{x})] | 1 = p_{\mu} \hat{g}(\hat{x}) | 1, \)
\( M_{\mu\nu} | \hat{g}(\hat{x}) = [M_{\mu\nu}, \hat{g}(\hat{x})] | 1 = M_{\mu\nu} \hat{g}(\hat{x}) | 1. \)

Let us further introduce the anti-involution operator \( ^\dagger \) by \( \lambda^\dagger = \overline{\lambda} \), for \( \lambda \in \mathbb{C} \) and bar denoting the ordinary complex conjugation, \( \hat{x}_{\mu}^{\dagger} = \hat{x}_{\mu}, \ M_{\mu\nu}^{\dagger} = -M_{\mu\nu} \) and \( p_{\mu}^{\dagger} = p_{\mu}. \) Then \( Z^{\dagger} = Z \) and \( \square^{\dagger} = \square. \) Since functions \( h_{\mu\nu} \) and \( g_{\mu\nu\lambda} \) are real and \( [a, b]^\dagger = -[a^\dagger, b^\dagger] \) \( \forall a, b \in \mathcal{H}, \) relations \( (11)-(16) \) which define the algebra \( \mathcal{H} \) remain unchanged under the action of \( ^\dagger. \) Also, relations \( (10)-(14) \) which define \( Z \) and \( \square \) remain unchanged. Thus, the commutation relations defining \( \mathcal{H} \) are invariant under operation \( ^\dagger. \)

3 General realizations and the star product

3.1 General considerations

Let us consider realizations of \( \mathcal{H}. \) We want to express noncommutative coordinates \( \hat{x}_{\mu} \) in terms of commutative coordinates \( x_{\mu} \) and momenta \( p_{\mu}. \) Hence, let us consider the Weyl algebra \( \mathcal{H}, \) generated by \( x_{\mu} \) and \( p_{\mu} \) satisfying

\[ [x_{\mu}, x_{\nu}] = [p_{\mu}, p_{\nu}] = 0 \]  

(15)

and

\[ [p_{\mu}, x_{\nu}] = -i \eta_{\mu\nu} \cdot 1. \]  

(16)

Generators \( x_{\mu} \) and \( p_{\mu} \) satisfy \( x_{\mu}^{\dagger} = x_{\mu} \) and \( p_{\mu}^{\dagger} = p_{\mu}. \) Now, \( \hat{x}_{\mu}(x, p) \) has the form

\[ \hat{x}_{\mu} = x^{\alpha} h_{\alpha\mu}(p) + \chi_{\mu}(p). \]  

(17)

Note that this is somewhat more general form of the differential representation than those used in \( [38, 41, 42] \), where it was assumed that \( \chi_{\mu} = 0. \)

The relation \( (11), \) upon substitution \( (17), \) gives a differential equation for \( \chi: \)

\[ \frac{\partial \chi_{\nu}}{\partial p_{\alpha}} h_{\alpha\mu} - \frac{\partial \chi_{\mu}}{\partial p_{\alpha}} h_{\alpha\nu} = a_{\mu} \chi_{\nu} - a_{\nu} \chi_{\mu}. \]  

(18)

For a given \( h_{\mu\nu}, \) this differential equation has infinitely many solutions, each of which generates a realization, which is in general nonhermitian. If we require that \( \hat{x}_{\mu} \) be the hermitian operator, then again there will be an infinite subfamily of solutions. In Section \( [5] \) it is shown how to obtain all possible (hermitian) solutions from one particular solution.

We further introduce the angular momentum operator \( M_{\mu\nu} \) in the following way. Denote by \( P_{\mu} \) a momentum operator \( p_{\mu}, \) Eq. \( (5), \) satisfying the particular commutation relation, namely \( [P_{\mu}, \hat{x}_{\nu}] = -i (Z^{-1}\eta_{\mu\nu} - a_{\mu} P_{\nu}). \) By comparison with the Eq. \( (5), \) this choice for the commutator corresponds to choosing the function \( h_{\mu\nu} = Z^{-1}\eta_{\mu\nu} - a_{\mu} P_{\nu}. \) Then the angular momentum operator can be introduced as

\[ M_{\mu\nu} = i (\hat{x}_{\mu} P_{\nu} - \hat{x}_{\nu} P_{\mu}) Z \]  

(19)

(see also [39, 40, 38]). It is easy to check that \( M_{\mu\nu}^{\dagger} = -M_{\mu\nu} \) if \( \hat{x}_{\mu}^{\dagger} = \hat{x}_{\mu}. \)

Let \( \mathcal{A} \) be the subalgebra of \( \mathcal{H} \) generated by \( x_{\mu}. \) Similarly to \( \mathcal{H} \) and the action \( \triangleright, \) we define the action \( \triangleright \) of \( \mathcal{H} \) on \( \mathcal{A} \) by

1. \( g(x) \triangleright 1 = g(x), \ x_{\mu} \triangleright g(x) = x_{\mu} g(x), \ g(x) \in \mathcal{A} \)
2. \( p_{\mu} \triangleright g(x) = [p_{\mu}, g(x)] \triangleright 1 = p_{\mu} g(x) \triangleright 1. \)
Since \( g(x) \rightarrow 1 = \hat{g}(\hat{x}) \) and \( \hat{g}(\hat{x}) \rightarrow 1 = g(x) \), for \( g(x) \in \mathcal{A} \), the action \( \rightarrow \) is in a certain way the inverse of the action \( \rightarrow \). This relationship between the two operations is a natural consequence of the fact that the algebras \( \hat{\mathcal{A}} \) and \( \mathcal{A} \) are isomorphic as the vector spaces. Moreover, they are also isomorphic at the algebra level if the pointwise multiplication on \( \mathcal{A} \) is replaced by the star product. Let us denote by \( \mathcal{A}_* \) the algebra \( \mathcal{A} \) in which the star product is used instead of pointwise multiplication. Then the isomorphism \( T : \hat{\mathcal{A}} \rightarrow \mathcal{A}_* \) is given by

\[
T(\hat{g}(\hat{x})) = \hat{g}(\hat{x}) \rightarrow 1 = g(x). \tag{20}
\]

In this setting, the star product is defined by

\[
(f \star g)(x) = \hat{f}(\hat{x})\hat{g}(\hat{x}) \rightarrow 1 = \hat{f}(\hat{x}) \rightarrow g(x). \tag{21}
\]

Suppose we have two arbitrary sets \( \{X_\mu, P_\nu\} \) and \( \{x_\mu, p_\nu\} \) of canonical coordinates with their respective derivatives, each forming a set of canonical variables, \([P_\mu, X_\nu] = -i\eta_{\mu\nu}\), and \([p_\mu, x_\nu] = -i\eta_{\mu\nu}\), respectively. Then, there always exists a similarity transformation that maps one set to another:

\[
X_\mu = S x_\mu S^{-1} = x^\alpha \Psi_{\alpha\mu}(p) + \chi'^\mu(p),
\]

\[
P_\mu = \Lambda_\mu(p). \tag{22}
\]

For simplicity, we now assume \( \chi'^\mu(p) = 0 \). Then we can write

\[
\hat{x}_\mu = X_\alpha \Phi_{\alpha\mu}(P) = x_\alpha \Psi_{\alpha\nu}(p) \Phi_{\alpha\mu}(\Lambda(p)) = x_\nu \varphi_{\nu\mu}(p),
\]

implying that

\[
\varphi_{\nu\mu}(p) = \Psi_{\alpha\nu}(p) \Phi_{\alpha\mu}(\Lambda(p)). \tag{23}
\]

Since \([P_\mu, X_\nu] = -i\eta_{\mu\nu}\), after introducing \( P_\mu \) and \( X_\nu \) from (22) into this relation, we obtain

\[
\frac{\partial \Lambda_\mu}{\partial P_\alpha} \Psi_{\alpha\nu}(p) = \eta_{\nu\mu}, \tag{24}
\]

where we have taken into account that the other set of commutative variables is canonical as well, \([p_\mu, x_\nu] = -i\eta_{\mu\nu}\). Relation (24) then means that \( \Psi_{\alpha\nu}(p) \) is the inverse of the matrix \( \left( \frac{\partial \Lambda_\mu}{\partial P_\alpha} \right) \),

\[
\Psi_{\mu\nu}(p) = \left( \frac{\partial \Lambda_\mu}{\partial P_\nu} \right)^{-1}. \tag{25}
\]

Also, it is possible to make a correspondence between the function \( \Lambda(p) \) and the function \( K_h(k) \) introduced in the next section, namely,

\[
K_h(k) = -i\Lambda(ik). \tag{26}
\]

### 3.2 Plane waves

It is clear that the differential realization (17) need not be hermitian in general, in the sense that the condition \((\hat{x}_\mu)^\dagger = \hat{x}_\mu \) may fail. However, for a given \( h_{\mu\nu}(p) \), it may be made hermitian by cleverly choosing the function \( \chi_\mu(p) \). Since for each realization, there exists a corresponding star product, this is the case for the hermitian realizations as well. The star products corresponding to hermitian realizations are called here the hermitian star products. Note that we do not use this term in the usual sense that the property \( f \star g = \bar{f} \star \bar{g} \) holds. Indeed, the hermitian star products in our sense do not need to satisfy this property in general. However, for some very common orderings (realizations) including the Weyl symmetric and left-right symmetric ordering, this property is satisfied. It is also noteworthy that for each \( h_{\mu\nu}(p) \) there exists a whole family of hermitian realizations (infinitely many) and consequently there exists a whole family of star products corresponding to these hermitian realizations. Each star product is thus characterized...
by specifying two functions, $h_{\mu\nu}(p)$ and $\chi_{\mu}(p)$. In what follows we therefore use the symbol $\ast_{h,\chi}$ to denote the star product and also add the subscripts $h$ and $\chi$ to various quantities that are to be introduced in the following, to reflect their dependence on the particular realization in the class \((\mathbb{I})\). The analysis is first carried out for the most general situation of an arbitrary $h_{\mu\nu}(p)$ and $\chi_{\mu}(p)$, not necessarily corresponding to a hermitian realization, and then only after making this case clear, we specialize it to the hermitian realization and the hermitian star product.

Let us define functions $P_h$ and $C_{h,\chi}$ by

$$e^{ik\hat{x}} \ast e^{iq\hat{x}} = C_{h,\chi}(k,q)e^{iP_h(k,q)x},$$

where $\hat{x}$ on the left hand side is given by \((\mathbb{I})\), with arbitrary $h_{\mu\nu}(p)$ and $\chi_{\mu}(p)$. Similarly the function $K_h$ can be defined as the special case of the above,

$$e^{ik\hat{x}} \ast 1 = C_{h,\chi}(k)e^{iK_h(k)x},$$

taking into account that $K_h(k) = P_h(k,0)$ and $C_{h,\chi}(k) = C_{h,\chi}(k,0)$. It is shown in Appendix A that

$$C_{h,\chi}(k,q) = e^{i\int_0^1(k\chi(P_h(ik,q)))dt}.$$  

(29)

This expression is valid for any $h_{\mu\nu}(p)$ and $\chi_{\mu}(p)$. While the dependence of $C$ on $\chi$ is obvious from \((29)\), the dependence on $h$ enters through the function $P_h$. If $\chi = 0$, then $C_{h,\chi}(k,q) = 1$ (see \([38, 41]\) for details). It is convenient to use this special case when $\chi = 0$ to define the function $D_h$ by

$$e^{ik\hat{x}} \ast_{h,\chi=0} e^{iq\hat{x}} = e^{iD_{h,\chi=0}(k,q)x},$$

(30)

where it is understood that $h$ and $\chi$ refer to the functions $h = h_{\mu\nu}(p)$ and $\chi_{\mu} = \chi_{\mu}(p)$, respectively, appearing in the Eqs.\((23)\) and \((\mathbb{I})\). We emphasize here that for a given $h_{\mu\nu}(p)$, the function $D_{h,\chi}(k,q)$ is the same, no matter which form $\chi_{\mu}$ takes. That is, $D_{h,\chi}(k,q) = D_{h,\chi=0}(k,q) \equiv D_h(k,q)$. It is related to $P_h(k,q)$ by $D_h(k,q) = P_h(K_h^{-1}(k), q)$. More details can be found in \([29, 31]\).

We can now introduce the noncommutative plane waves $\hat{e}_k^+(\hat{x})$ with label $+\by$

$$\hat{e}_k^+(\hat{x}) = \frac{1}{C_{h,\chi}(K_h^{-1}(k))}e^{ik\hat{x}}.$$  

(31)

Relation \((28)\) then shows that this noncommutative plane wave can be related to a commutative plane wave $e^{ikx}$ through the isomorphism \((20)\), namely,

$$\hat{e}_k^+(\hat{x}) \triangleright 1 = e^{ikx},$$

(32)

showing that $\hat{e}_k^+(\hat{x})$ is in a one-to-one correspondence with the commutative plane wave $e^{ikx}$.

The star product corresponding to a generic $h_{\mu\nu}(p)$ and $\chi_{\mu}(p)$ is readily deduced from \((21)\) and \((32)\) as

$$e^{ik\hat{x}} \ast_{h,\chi} e^{iq\hat{x}} = \frac{1}{C_{h,\chi}(K_h^{-1}(k))}e^{ik\hat{x}} \frac{1}{C_{h,\chi}(K_h^{-1}(q))}e^{iK_h^{-1}(q)\hat{x}} \triangleright 1.$$  

(33)

Particularly interesting may be a star product between two plane waves with 'opposite' momenta, i.e. $e^{iS_h(k)x} \ast_{h,\chi} e^{iqx}$. Here $S_h(k)$ is the antipode defined in the current context by $D_h(S_h(k), k) = D_h(k, S_h(k)) = 0$. We come back to this particular form of the star product below when considering the hermitian realizations and the star products corresponding to them.
For the time being it remains to finalize the calculation for $e^{ikx} \star_{h,\chi} e^{iqx}$,

$$
e^{ikx} \star_{h,\chi} e^{iqx} = \frac{1}{C_{h,\chi}(K_{h}^{-1}(k))} \frac{1}{C_{h,\chi}(K_{h}^{-1}(q))} e^{K_{h}^{-1}(k) \hat{x} e^{K_{h}^{-1}(q) \hat{x}}} > 1 =
\frac{1}{C_{h,\chi}(K_{h}^{-1}(k))} \frac{1}{C_{h,\chi}(K_{h}^{-1}(q))} e^{\mathcal{D}_{x}(K_{h}^{-1}(k)K_{h}^{-1}(q)) \hat{x}} > 1
= \frac{C_{h,\chi}(\mathcal{D}_{x}(K_{h}^{-1}(k),K_{h}^{-1}(q)))}{C_{h,\chi}(K_{h}^{-1}(k))C_{h,\chi}(K_{h}^{-1}(q))} e^{\mathcal{D}_{x}(K_{h}^{-1}(k),K_{h}^{-1}(q)) x}.
$$
\[ (34) \]

\[ (B.1) \]

\[ (35) \]

where the identity \[ (B.1) \] is again applied in the numerator to simplify the argument of the $C$ factor.

If we restrict ourselves to the class of realizations where the plane waves $e^{ikx}$ and $e^{iS(k)x}$ are orthogonal and normalized, that is

$$
\int d^{m}x e^{iS(k)x} \star e^{iqx} = (2\pi)^{n} \delta^{(n)}(k - q),
$$

than the factors $C$ are calculated in a particularly simple way as (see Appendix B for details)

$$
C_{h,\chi}(K_{h}^{-1}(k)) = \frac{1}{\sqrt{\text{det} \left( \frac{\partial \mathcal{D}_{x}(S_{x}(k),q)}{\partial q} \right)_{\mu}}}.
$$
\[ (36) \]

Otherwise, it is calculated as a special case of the more general factor \[ (29) \] by setting $q = 0$, i.e. $C_{h,\chi}(k) = C_{h,\chi}(k,0)$. In \[ (31) \], the factor $C_{h,\chi}(K_{h}^{-1}(k))$ has been calculated with the help of Eq.\[ (30) \], for one special hermitian version of the natural realization and it can be shown that this result is consistent with that obtained through Eq.\[ (29) \]. We recapitulate this result among other results related to the natural realization in Section 5.

4 **Hermitian realizations**

We now turn to the problem of hermitization of the realizations considered so far. One way to obtain the hermitian realization is to start from the nonhermitian realization for which $\chi = 0$ and then consider $\hat{x}_{\mu}(\hat{x}) = \frac{1}{2}(\hat{x}_{\mu} + \hat{x}^{\dagger}_{\mu})$ (see \[ (31) \] \[ (42) \]). The operators $\hat{x}(\hat{x})$ are hermitian and it remains to prove that they satisfy (11). Actually, the operators $\hat{x}_{\mu}(\hat{x})$ of the form $\hat{x}_{\mu}(\hat{x}) = \beta \hat{x}_{\mu} + (1 - \beta)\hat{x}^{\dagger}_{\mu}$ satisfy the relation (11). The left hand side of (11) has the form

$$
[\hat{x}_{\mu}(\hat{x}) \cdot \hat{x}_{\nu}(\hat{x})] = \beta^{2}[\hat{x}_{\mu}, \hat{x}_{\nu}] + \beta(1 - \beta) \left([\hat{x}_{\mu}, \hat{x}^{\dagger}_{\nu}] + [\hat{x}^{\dagger}_{\mu}, \hat{x}_{\nu}]\right) + (1 - \beta)^{2}[\hat{x}^{\dagger}_{\mu}, \hat{x}^{\dagger}_{\nu}].
$$
\[ (37) \]

The important observation is the identity $[\hat{x}_{\mu}, \hat{x}^{\dagger}_{\nu}] + [\hat{x}^{\dagger}_{\mu}, \hat{x}_{\nu}] = [\hat{x}_{\nu}, \hat{x}^{\dagger}_{\mu}] + [\hat{x}^{\dagger}_{\nu}, \hat{x}_{\mu}]$. Hence, \[ (37) \] transforms to

$$
[\hat{x}_{\mu}(\hat{x}) \cdot \hat{x}_{\nu}(\hat{x})] = \beta[\hat{x}_{\mu}, \hat{x}_{\nu}] + (1 - \beta)[\hat{x}^{\dagger}_{\mu}, \hat{x}^{\dagger}_{\nu}].
$$
\[ (38) \]

Since $\hat{x}_{\mu}$ and $\hat{x}^{\dagger}_{\mu}$ satisfy (11), $\hat{x}_{\mu}(\hat{x})$ also satisfies (11).

For the nonhermitian realization $\hat{x}_{\mu} = x^{\alpha}h_{\alpha\mu}(p)$, the corresponding hermitian realization $\hat{x}(\hat{x})_{\mu}$ has the form

$$
\hat{x}_{\mu}(\hat{x}) = x^{\alpha}h_{\alpha\mu}(p) + \frac{1}{2}[h_{\alpha\mu}(p), x^{\alpha}].
$$
\[ (39) \]

Since $[h_{\alpha\mu}, x^{\alpha}] = -i \frac{\partial h_{\alpha\mu}}{\partial p_{\alpha}}$, \[ (39) \] transforms to

$$
\hat{x}_{\mu}(\hat{x}) = \hat{x}_{\mu} - i \frac{\partial h_{\alpha\mu}}{2 \partial p_{\alpha}}.
$$
\[ (40) \]
The last expression is more convenient for calculation. A large class of covariant realizations can be found in [38] [41]. The noncovariant realizations are analyzed in [40]. Let \( A = -(ap) \) and \( B = -a^2p^2 \). For the natural realization \( \hat{x}_\mu = X_\mu Z^{-1} - (aX)P_\mu \) and \( \hat{x}_\mu(\hat{\phi}) = \hat{x}_\mu + \frac{i}{2} a_\mu \), where \( Z^{-1} = -A + \sqrt{1 - B} \). For the left covariant realization \( \hat{x}_\mu = x_\mu (1 - A) \) and \( \hat{x}_\mu(\hat{\phi}) = \hat{x}_\mu + \frac{i}{2} a_\mu \). For the right covariant realization \( \hat{x}_\mu = x_\mu - a_\mu(xp) \) and \( \hat{x}_\mu(\hat{\phi}) = \hat{x}_\mu + \frac{n+1}{2} i a_\mu \), with \( n \) being the spacetime dimension. Finally, for the realization \( \hat{x}_\mu = x_\mu - (ax)p_\mu - a_\mu(xp) \), we have \( \hat{x}_\mu(\hat{\phi}) = \hat{x}_\mu + \frac{n+1}{2} i a_\mu \).

Let us mention that in the class of realizations with \( \chi = 0 \) there exists one which is hermitian (see also [44]). This one corresponds to the case when \([x, h_{\mu\nu}(p)] = 0\) and is determined by setting

\[
\frac{\partial h_{\alpha\beta}}{\partial p_\alpha} = 0.
\]

(41)

5 Noncommutative plane waves in the hermitian realization and the hermitian star product

In Section 3 the noncommutative plane wave \( \hat{e}^+_k(\hat{x}) \) has been introduced, which, through the isomorphism (20), corresponds to the ordinary commutative plane wave \( e^{ikx} \). Analogously, one can ask for the form of the noncommutative plane wave that corresponds to the ordinary commutative plane wave through the same isomorphism (20), but this time with the 'opposite' or 'inverse' momentum, \( e^{iS(k)x} \). The noncommutative plane wave with this property we label with superscript \( \dagger \), thus writing \( \hat{e}^-_k(\hat{x}) \). To answer the foregoing question and to find the required correspondence, it is necessary to work with the hermitian realizations. In the case that the realization for \( \hat{x} \) is hermitian, the noncommutative plane wave \( \hat{e}^-_k(\hat{x}) \) is a unitary operator, so that the hermitian conjugation operation (anti-involution) \( \dagger \) is well defined when applied to it. Proceeding along these lines and using the identity (B.10), it is straightforward to establish that the noncommutative plane wave \( \hat{e}^-_k(\hat{x}) \), may be introduced via the formula

\[
\hat{e}^-_k(\hat{x}) = \frac{C_{h,\chi}(K^{-1}_h(k))}{C_{h,\chi}(S_h(k))}\hat{e}^+_k(\hat{x})^\dagger, \quad \hat{e}^-_k(\hat{x}) \triangleright 1 = e^{iS_h(k)x}.
\]

(42)

With this and [31], we have defined a complete set of noncommutative plane waves, appearing in the decomposition of any element in \( \hat{A} \). They correspond to the plane waves on the commutative space, \( e^{ikx} \), i.e. \( e^{iS(k)x} \), to which any element of \( A \) is reduced when shattered into pieces.

Before we proceed and decompose generic elements of \( A \) and \( \hat{A} \) into their respective Fourier modes, we need to introduce a generalization of the ordinary complex conjugation, which serves to define a scalar product on \( A \), in a similar way as the ordinary complex conjugation (designated by the bar) serves to define the usual scalar product on \( A \), namely \( (f, g) = \int d^n x f(x) \overline{g(x)} \). It should be noted that the operation \( \dagger \) introduced in the Section 2 is the adjoint operator with respect to this scalar product. Operators \( x_\mu \) and \( p_\mu \) act on \( A \) and \( x_\mu^\dagger \) and \( p_\mu^\dagger \) considered in the previous sections were calculated with respect to this scalar product. Beside that the generalized complex conjugation operation * is used to define the scalar product on \( A \)

\[
(f, g)_\kappa = \int d^n x f^* \ast g,
\]

(43)

for any \( f, g \in A_\kappa \), it is also required to have a smooth limit to the ordinary bar, when the deformation parameter goes to zero. The operation which is the adjoint operation with respect to the scalar product (43) we label by \( \dagger \). It is applied to the operators on \( A \) and for any operator \( A \) on \( A \), it satisfies \( (f, Ag)_\kappa = (\hat{A}^\dagger f, g)_\kappa \), as usual. Since the class of hermitian realizations is only a subclass of the general type of realizations that are considered in the paper, the star product corresponding to any hermitian realization is also given by the expression (35), with the additional remark that \( \chi \) has to be of a particular form to provide for the hermiticity of \( \hat{x} \).

Let us consider the following Fourier decompositions

\[
\phi(x) = \int \frac{d^n k}{(2\pi)^n} \tilde{\phi}(k) e^{ikx},
\]

(44)
for a generic element of the commutative algebra $A$ and
\[
\hat{\phi}(\hat{x}) = \int \frac{d^n k}{(2\pi)^n} \hat{\phi}(k) \hat{\epsilon}_k^+(\hat{x}),
\]  
(45)
for the general element of $\hat{A}$. Then the generalized complex conjugation acts according to
\[
\phi^*(x) = \int \frac{d^n k}{(2\pi)^n} \overline{\hat{\phi}(k)} (e^{ikx})^*,
\]  
(46)
for $f \in A_\ast$.

Having this, it is now straightforward to establish what object does the following map correspond to
\[
\hat{\phi}^\dagger(\hat{x}) \triangleright 1 = \int \frac{d^n k}{(2\pi)^n} \overline{\hat{\phi}(k)} (\hat{\epsilon}_k^+(\hat{x}))^\dagger \triangleright 1
\]  
\[= \int \frac{d^n k}{(2\pi)^n} \overline{\phi(k)} \frac{C_{h,\chi}(K^{-1}_h(S_h(k)))}{C_{h,\chi}(K^{-1}_h(1))} \hat{\epsilon}_k^-(\hat{x}) \triangleright 1.
\]  
(47)
Due to relation (42), this can be written as
\[
\hat{\phi}^\dagger(\hat{x}) \triangleright 1 \quad \int \frac{d^n k}{(2\pi)^n} \overline{\phi(k)} \frac{C_{h,\chi}(K^{-1}_h(S_h(k)))}{C_{h,\chi}(K^{-1}_h(1))} e^{iS_h(k)x}. \quad (48)
\]
Since we require the isomorphism (29) to be realized at the level of adjoint operators as well, that is $\hat{\phi}^\dagger(\hat{x}) \triangleright 1 = \phi^*(x)$, a direct comparison of this requirement with the relation (48) gives
\[
(e^{ikx})^* = \frac{C_{h,\chi}(K^{-1}_h(S_h(k)))}{C_{h,\chi}(K^{-1}_h(1))} e^{iS_h(k)x}. \quad (49)
\]
Note that there is a class of star products where the prefactor appearing in (49) reduces to 1. One example of the star product with such a property is the hermitian star product corresponding to the natural realization.

With the correspondence just established, the scalar product on $A$ can be transformed to the scalar product on $A_\ast$:
\[
\int d^n x \hat{x}^\dagger(\hat{x}) \hat{g}(\hat{x}) \triangleright 1 = \int d^n x f^*(x) * g(x).
\]  
(50)
The last expression can be calculated on $A$ for different types of star products on $\kappa$-Minkowski spacetime [29, 31]. For a certain class of star products it can be shown to reduce to $\int d^n x f(x)g(x)$.

For $f(x) = e^{ikx}$ and $g(x) = e^{iqx}$, one calculates
\[
\int d^n x f(x) * (\partial_\mu g(x)) = \int d^n x (S\partial_\mu f(x) * g(x)). \quad (51)
\]
This statement can be generalized to any pair of two functions $f(x)$ and $g(x)$ and then to any polynomial in $\partial_\mu$.

It turns out that $g^\dagger$ can be calculated as
\[
g^\dagger = S(\gamma). \quad (52)
\]
where $g$ represents any element of $U(\mathfrak{sp}(1, n - 1) \rtimes \mathbb{R}^n)$.

In summarizing the set of identities holding on the $\kappa$-Minkowski spacetime for a general type of the star products, we conclude with the formula
\[
\int d^n x f(x) * g(x) = \int d^n x (Z^\alpha g(x)) * (Z^{-n+1+\alpha}g(x)), \quad (53)
\]
involving also the shift operator $Z$. It can easily be proved for the Fourier modes, i.e. for $f(x) = e^{ikx}$ and $g(x) = e^{iqx}$.
6 General hermitian realization

Generally, starting from a given realization, other realizations can be obtained by the unitary similarity transformations. Let
\[
\begin{align*}
P_\mu &= U^\dagger p_\mu U, \\
X_\mu &= U^\dagger x_\mu U,
\end{align*}
\]
for the unitary operator \( U \) of the form
\[
U = \exp\{i(x_\alpha \Sigma_\alpha + \tilde{\Sigma} - \Sigma_\alpha^\dagger x_\alpha - \tilde{\Sigma}^\dagger)\},
\]
where \( \Sigma_\alpha \) and \( \tilde{\Sigma} \) have the form
\[
\Sigma_\alpha = \Sigma_\alpha(A, B) = \partial_\alpha \sigma_1(A, B) + i a_\alpha \theta^2 \sigma_2(A, B)
\]
and \( \tilde{\Sigma} = \sigma_3(A, B) \) for some functions \( \sigma_i(A, B), i = 1, 2, 3 \). If one starts with a hermitian realization and performs a similarity transformation according to Eq.\( (54) \), then this leads to another realization, which is also hermitian under the assumption that \( U \) is a unitary operator. Thus, we demand that \( \Sigma_\alpha \) in \( (55) \) are hermitian operators (\( \sigma_1 \) and \( \sigma_2 \) are antihermitian). In that case, \( U \) tends to \( I \) as \( a \) tends to 0.

\[
P_\mu = \exp\{-(x_\alpha \Sigma_\alpha + \tilde{\Sigma} - \Sigma_\alpha^\dagger x_\alpha - \tilde{\Sigma}^\dagger)\} p_\mu \exp\{x_\alpha \Sigma_\alpha + \tilde{\Sigma} - \Sigma_\alpha^\dagger x_\alpha - \tilde{\Sigma}^\dagger\} =
\]
\[
= p_\mu + \left(\Sigma_\mu - \Sigma_\mu^\dagger\right) + \frac{1}{2!} \left(\Sigma_\alpha - \Sigma_\alpha^\dagger\right) \frac{\partial}{\partial \sigma_\alpha} \left(\sigma_\mu - \sigma_\mu^\dagger\right) + \ldots =
\]
\[
= p_\mu + \left(\frac{\exp O - 1}{O}\right) \left(\Sigma_\mu - \Sigma_\mu^\dagger\right),
\]
where
\[
O = \left(\Sigma_\alpha - \Sigma_\alpha^\dagger\right) \left(\frac{\partial}{\partial \sigma_\alpha}\right).
\]

Suppose we want to find all hermitian realizations that are obtained by means of the unitary transformations \( (54) \) from a given hermitian realization, say \( \hat{x}_{\alpha}^{(1)} \), characterized by \( h_{\mu\nu} \). Then, starting from \( \hat{x}_{\alpha}^{(1)} \) for a fixed \( h_{\mu\nu} \), one can generate an infinite family of hermitian realizations with the same \( h_{\mu\nu} \), but different \( \chi_\mu \). The condition on the unitary transformations \( U \) that perform this map and give a set of hermitian realizations with the same \( h_{\mu\nu} \) is \( \Sigma_\alpha = \Sigma_\alpha^\dagger = 0 \). Hence, it is possible to introduce the set of unitary transformations \( (55) \) that, after performing \( (54) \), do not change \( h_{\mu\nu} \) in \( (17) \), but change only \( \chi_\mu \); those transformations are specified by \( \Sigma_\alpha = \Sigma_\alpha^\dagger = 0 \).

7 Examples

In this Section we apply the general results developed so far by working them out on the four characteristic and important examples that realize the \( \kappa \)-Minkowski spacetime. Four instances in question are the following: left-covariant, right-covariant, Weyl totally symmetric and natural realization.

It is of interest to see how the star product looks like in each of the four realizations mentioned. Crucial to this is to know the form of the \( C(k, q) \) factors in each of them. The knowledge of these factors also makes possible to calculate the integral \( \int d^nx \ e^{i S_{h}(k)x} \star_{h, \chi} e^{iqx} \) for every and each of the realizations. We particularly consider the hermitian realizations corresponding to \( \beta = \frac{1}{2} \) (that is, hermitian realizations obtained through the standard symmetrization, \( \hat{x}_{\alpha}^{(1)} = \frac{1}{2}(\hat{x}_{\mu} + \hat{x}_{\mu}^\dagger) \) (see Section 4)).

Before going to special cases, we restate the general results for the star product and a scalar product corresponding to realization \( h, \chi \):
\[
e^{ikx} \star_{h, \chi} e^{iqx} = \frac{C_{h, \chi}(K_{h}^{-1}(D_{h}(k, q)))}{C_{h, \chi}(K_{h}^{-1}(k))C_{h, \chi}(K_{h}^{-1}(q))} e^{iD_{h}(k, q)x},
\]
(59)
This can also be expressed as

\[ e^{ikx} \star_{h,\chi} e^{iqx} = \frac{C_{h,\chi}(K_{h}^{-1}(k), q)}{C_{h,\chi}(K_{h}^{-1}(k))} e^{iD_{h}(k,q)x}. \]  

(60)

The scalar product between two plane waves, with momenta \( k \) and \( q \), respectively is thus proportional to

\[
(e^{ikx}, e^{iqx})_{\kappa} \sim \int d^{n}x \ e^{iS_{h}(k)x} \star_{h,\chi} e^{iqx} = (2\pi)^{n} \ \frac{C_{h,\chi}(K_{h}^{-1}(D_{h}(S_{h}(k), q))}{C_{h,\chi}(K_{h}^{-1}(S_{h}(k)))} \ \frac{\delta^{(n)}(k - q)}{\text{det} \left( \frac{\partial(D_{h}(S_{h}(k), q))}{\partial q} \right)}_{q=k} \]  

\[
= (2\pi)^{n} \ \frac{C_{h,\chi}(K_{h}^{-1}(S_{h}(k)), q)}{C_{h,\chi}(K_{h}^{-1}(S_{h}(k)))} \ \frac{\delta^{(n)}(k - q)}{\text{det} \left( \frac{\partial(D_{h}(S_{h}(k), q))}{\partial q} \right)}_{q=k} \]

(61)

### 7.1 Left covariant realization

For the left covariant realization, \( \hat{x}_{\mu} = x_{\mu}(1 - A) = x_{\mu}(1 + ap) \), which means that

\[ h_{\mu\alpha}(p) = (1 + ap)\eta_{\mu\alpha} \]

(62)

and in this case one gets

\[ \mathcal{P}_{h,\mu}(k,q) = k_{\mu} e^{ak} - \frac{1}{ak} (1 + aq) + q_{\mu}, \quad \mathcal{K}_{h,\mu}(k) = k_{\mu} e^{ak} - \frac{1}{ak} . \]

(63)

Consequently, \( K_{h,\mu}^{-1}(k) = k_{\mu} \frac{\ln(1 + ak)}{ak} \) and

\[ \mathcal{D}_{h,\mu}(k,q) = \mathcal{P}_{h,\mu}(K_{h,\mu}^{-1}(k), q) = k_{\mu} Z^{-1}(q) + q_{\mu}, \]

where \( Z^{-1}(q) = 1 + aq \). The antipode in turn can be calculated by setting \( k = S(q) \) and \( D_{h,\mu}(k,q) = 0 \) in the previous expression:

\[ S(q)_{\mu} Z^{-1}(q) + q_{\mu} = 0 \Leftrightarrow S(q)_{\mu} = - q_{\mu} Z(q). \]

(65)

Since \( Z^{-1}(q) = 1 + aq \) for our realization,

\[ Z(q) = \frac{1}{1 + aq}. \]

(66)

Next we calculate a determinant appearing in (61) for this particular case of left covariant realization. In order to simplify calculations, it is convenient to take \( a = (a_{0}, 0) \) and then, due to covariance, to generalize the result to a four-vector \( a_{\mu} \) oriented in an arbitrary direction,

\[
J(k) = \left| \text{det} \left( \frac{\partial D_{h,\mu}(S(k), q)}{\partial q_{\nu}} \right)_{q=k} \right| = \left| \text{det} \left( \frac{\partial(S(k)_{\mu} Z^{-1}(q) + q_{\nu})}{\partial q_{\nu}} \right)_{q=k} \right| = \left| \text{det} \left( (S(k)_{\mu} a_{\nu} + \eta_{\mu\nu}) \right)_{q=k} \right|. \]

(67)

This leads to

\[
\text{det} \left( \frac{\partial D_{h,\mu}(S(k), q)}{\partial q_{\kappa}} \right)_{q=k} = \begin{vmatrix}
\frac{1}{k_{1}+ak} & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
1 & 0 & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\frac{k_{n-1}+ak}{1+ak} & 0 & 0 & \ldots & 1 \\
\end{vmatrix} = \frac{1}{1 + ak} .
\]

(68)
Following the hermitization procedure for the value $\beta = \frac{1}{2}$, it is easy to get $\hat{x}_\mu^{(h)} = \hat{x}_\mu + \frac{i}{2}a_\mu$ and thus read out the quantity $\chi_\mu(p) = \frac{i}{2}a_\mu$, which is important for determining the $C(k, q)$ factors in Eq. (61). Due to $\chi$ being constant, the integration in (29) becomes trivial, giving $C_{h, \chi}(k, q) = C_{h, \chi}(k) = e^{-\frac{1}{2}ak}$ and consequently

$$C_{h, \chi}(K^{-1}_h(k)) = \frac{1}{\sqrt{1 + ak}}, \quad C_{h, \chi}(K^{-1}_h(S_h(k)), q) = \sqrt{1 + ak}. \quad (69)$$

With these results we have for the hermitized ($\beta = \frac{1}{2}$) left covariant realization

$$e^{ikx} \star_{h, \chi} e^{iqx} = e^{iD_h(k, q)x} \quad (70)$$

and the scalar product between two plane waves, with momenta $k$ and $q$, respectively is thus proportional to

$$(e^{ikx}, e^{iqx})_{\kappa} \sim \int d^n x \, e^{iS_h(k)x} \star_{h, \chi} e^{iqx} = (2\pi)^n (1 + ak)\delta^{(n)}(k - q). \quad (71)$$

### 7.2 Right covariant realization

For the right covariant realization, $\hat{x}_\mu = x_\mu - a_\mu x^n p_\alpha$, which means that

$$h_{\mu\alpha}(p) = \eta_{\mu\alpha} - a_\alpha p_\mu. \quad (72)$$

A similar analysis as before, when applied to this particular situation gives

$$D_{h, \mu}(k, q) = k_\mu + Z(k)q_\mu, \quad (73)$$

with $Z(k) = 1 - ak$ ( $Z^{-1}(k) = 1/(1 - ak)$). The opposite momentum, or the antipode is readily extracted from the function $D_h$ to be $S(k) = -k_\mu Z^{-1}(k)$. One can easily check that $Z(S(k)) = Z(k)$. Similarly as before the determinant for the case of right covariant realization can be calculated as

$$J(k) = \left| \det \left( \frac{\partial D_\mu(S(k), q)}{\partial q_\nu} \right)_{q=k} \right| = Z^n(S(k)) = Z^n(k). \quad (74)$$

Hermitization procedure for $\beta = \frac{1}{2}$ identifies the $\chi$ to be $P(k, q) = \frac{i}{2}a_\mu$. This leads to $C_{h, \chi}(k, q) = C_{h, \chi}(k) = e^{-\frac{1}{2}ak}$ and consequently the orthonormality property of the plane waves in the right covariant realization can be expressed as

$$(e^{ikx}, e^{iqx})_{\kappa} \sim \int d^n x \, e^{iS_h(k)x} \star_{h, \chi} e^{iqx} = (2\pi)^n Z^{-n}(k)\delta^{(n)}(k - q)$$

$$= \frac{(2\pi)^n}{(1 - ak)^n} \delta^{(n)}(k - q). \quad (75)$$

### 7.3 Weyl symmetric realization

For the Weyl symmetric realization $\hat{x}_\mu = x_\mu h_s - a_\mu x p_\gamma_1$, that is

$$h_{\mu\alpha}(p) = \eta_{\mu\alpha} h_s - a_\alpha p_\mu \gamma_1, \quad (76)$$

where

$$h_s(A) = \frac{A}{e^A - 1}; \quad \gamma_1(A) = \frac{1}{A} \left(1 - \frac{A}{e^A - 1}\right) = \frac{1 - h_s(A)}{A}, \quad A = -ak. \quad (77)$$

By repeating the same steps as before, we get

$$P_{h, \mu}(k, q) = D_{h, \mu}(k, q) = h_s(k + q) \left(\frac{k_\mu}{h_s(k)} + Z(k)\frac{q_\mu}{h_s(q)}\right), \quad (78)$$
where \( Z(k) = e^{A} = e^{-ak} \), \( Z(S(q)) = e^{-aS(k)} \). Again, the antipode can be deduced by setting \( k = S(q) \) and \( \mathcal{D}_{h,\mu}(k, q) = 0 \).

\[
\mathcal{D}_{h,\mu}(S(k), q) = h_s(S(k) + q) \left( \frac{S(k)}{h_s(S(k))} + Z(S(k)) \frac{q_\mu}{h_s(q)} \right).
\]

(79)

This implies that the antipode \( S(q) \) has to satisfy the relation

\[
\frac{S(q_\mu)}{h_s(q)} + Z(S(q)) \frac{q_\mu}{h_s(q)} = 0.
\]

(80)

The solution is \( S(k) = -k_\mu \), in consistency with the results obtained in Appendix B. The calculation of the determinant in (13) gives

\[
J(k) = \left| \text{det} \left( \frac{\partial \mathcal{D}_{\mu}(S(k), q)}{\partial q_\nu} \right) \right| = \left( \frac{Z^{-1}(k)}{h_s(k)} \right)^{n-1}, \quad h_s(k) = \frac{ak}{1 - e^{-ak}}
\]

(81)

where \( h_s(k) = \frac{ak}{1 - e^{-ak}} \). Again, the hermitization procedure for \( \beta = \frac{1}{2} \) gives rise to \( \chi \), which is this time nontrivial:

\[
\chi_\mu(p) = \frac{i}{2} a_\mu \left( \frac{\partial h_s(A)}{\partial A} + n\gamma_1(A) - a p \frac{\partial \gamma_1(A)}{\partial A} \right) = \frac{i}{2} a_\mu (n - 1) \gamma_1(A).
\]

This leads to

\[
\mathcal{C}_{h,\chi}(k, q) = \left( \frac{ak + aq (e^{-aq} - 1) e^{-ak}}{aq e^{-ak}} \right)^{\frac{n-1}{2}} = \left( \frac{h_s(k + q)}{h_s(q)} Z(k) \right)^{\frac{n-1}{2}}
\]

and consequently the orthonormality property for the plane waves in the Weyl symmetric realization can be expressed as

\[ (e^{ikx}, e^{iqx})_\kappa \sim \int d^n x \; e^{iS_h(k)x} \star_{h,\chi} e^{iqx} = (2\pi)^n Z^{\frac{n-1}{2}}(k) \delta^{(n)}(k - q), \quad Z(k) = e^{-ak}. \]

However, it is interesting to note that the ordinary plane waves are orthonormalized for the scalar product \((,)_\kappa\), in the case of Weyl symmetric ordering,

\[ (e^{ikx}, e^{iqx})_\kappa = (2\pi)^n \delta^{(n)}(k - q). \]

(82)

It is seen from (43) and (49) that, in the case of the Weyl symmetric star product, the additional factor \( \frac{\mathcal{C}_{h,\chi}(K^{-1}_{h}(S_k)))}{\mathcal{C}_{h,\chi}(K^{-1}_{h}(k))} \) in (49) cancels out the corresponding factor in (82), giving rise to the orthonormalization property. In the next subsection we shall see that the same property holds when the scalar product is introduced in terms of the natural realization. Therefore, as it was the case with the Weyl symmetric realization, the plane waves are also orthonormal in the natural realization. This feature was noticed for the first time in (31). In contrast, for the left and right covariant realizations our analysis shows that this property does not hold.

### 7.4 Natural realization

The analysis for the natural realization has already been carried out in (31), so here we only recapitulate the results obtained there and compare them with other realizations. It is specified by the following differential representation for the generators of the \( \kappa \)-Minkowski algebra,

\[
x^h_\mu = x_\mu (a_\alpha p^\alpha + \sqrt{1 + a^2 p^2}) - (ax)p_\mu,
\]

(83)

or in another way, it is specified by

\[
h_{\alpha\mu}(p) = \eta_{\alpha\mu} (ap + \sqrt{1 + a^2 p^2}) - a_\alpha p_\mu.
\]

(84)
This leads to \( \mathcal{D}_h(k, q) \) with the form
\[
(\mathcal{D}_h(k, q))_\mu = k_\mu Z^{-1}(q) + a_\mu - a_\mu(kq)Z(k) + \frac{1}{2}a_\mu(aq)\Box(k)Z(k),
\] (85)
where
\[
Z^{-1}(k) \equiv ak + \sqrt{1 + a^2k^2}, \quad \Box(k) \equiv \frac{2}{a^2} \left[ 1 - \sqrt{1 + a^2k^2} \right].
\] (86)
The determinant in (61) was found to be
\[
J(k) = \left| \det \left( \frac{\partial (\mathcal{D}_h(S(k), q))_\mu}{\partial q_\nu} \right) \right|_{q=k} = \frac{1}{\sqrt{1 + a^2k^2}}
\] (87)
so that \( \delta^{(n)}(\mathcal{D}(S(k), q)) = \sqrt{1 + a^2k^2} \delta^{(n)}(q - k) \). The hermitization procedure, similarly to the previous cases, produces \( \chi \) congruent to the \( \beta = \frac{1}{2} \) hermitization to be
\[
\chi_\mu(p) = -\frac{i}{2} \frac{a^2p_\mu}{\sqrt{1 + a^2p^2}}.
\] (88)
Using the formula for the additional factor \( \mathcal{C}_{h,\chi} \) given by (29), in the case of natural realization we obtain
\[
\mathcal{C}_{h,\chi}(k^{-1}_h(k)) = \sqrt{1 + a^2k^2},
\] (89)
\[
\mathcal{C}_{h,\chi}(k^{-1}_h(k), q) = \sqrt[\sqrt{1 + a^2k^2}]^{1 + a^2D^2_h(k, q)}.
\] (90)
Also, the star product (35) transforms to
\[
e^{ikX} \star_{h,\chi} e^{iqX} = \frac{\mathcal{C}_{h,\chi}(k^{-1}_h(k), q)}{\mathcal{C}_{h,\chi}(k^{-1}_h(k))} \mathcal{C}_{h,\chi}(k^{-1}_h(k)) e^{i\mathcal{D}_h(k, q)X}
\] (91)
\[
= \frac{\sqrt[\sqrt{1 + a^2k^2}]^{1 + a^2D^2_h(k, q)}}{\sqrt[\sqrt{1 + a^2k^2}]^{1 + a^2k^2}} e^{i\mathcal{D}_h(k, q)X},
\] (92)
which in turn gives rise to
\[
(e^{ikx} e^{iqx})_\kappa = \int d^n x e^{iS_h(k)x} \star_{h,\chi} e^{iqx} = (2\pi)^n \delta^{(n)}(k - q).
\]
In summary we arrange the results in the tabular form

|                  | Left covariant | Right covariant |
|------------------|----------------|-----------------|
| \( h_{\mu\alpha} \) | \( (1 + ak)\eta_{\mu\alpha} \) | \( \eta_{\mu\alpha} - a_\alpha k_\mu \) |
| \( \chi_\mu(k) \) | \( t a_\mu \) | \( \frac{t a_\mu}{1 - ak} \) |
| \( Z^{-1}(k) \) | \( 1 + ak \) | \( 1 - ak \) |
| \( J(k) \) | \( \frac{1 + ak}{ak} k_\mu \) | \( \frac{1}{a^2} \frac{1}{a^2} k_\mu \) |
| \( k^{-1}_h(k) \) | \( \frac{\ln(1 + ak)}{ak} k_\mu \) | \( \frac{1}{a^2} \frac{1}{a^2} k_\mu \) |
| \( C_{h,\chi}(k, q) \) | \( e^{-\frac{t}{2}ak} \) | \( e^{-\frac{t}{2}ak} \) |
| \( C_{h,\chi}(k) \) | \( e^{-\frac{t}{2}ak} \) | \( e^{-\frac{t}{2}ak} \) |
| \( C_{h,\chi}(k^{-1}_h(S_h(k)), q) \) | \( \sqrt{1 + ak} \) | \( \frac{1}{(1 - ak)^{\frac{t}{2}}} \) |
| \( C_{h,\chi}(k^{-1}_h(k)) \) | \( \frac{1}{\sqrt{1 + ak}} \) | \( (1 - ak)^{\frac{t}{2}} \) |
| \( I(k) \) | \( 1 + ak \) | \( \frac{1}{(1 - ak)^{\frac{t}{2}}} \) |
| \( (e^{ikx} e^{iqx})_\kappa \) | \( (2\pi)^n (1 + ak)^{\frac{t}{2}} \delta^{(n)}(k - q) \) | \( (2\pi)^n \frac{1}{(1 - ak)^{2\kappa}} \delta^{(n)}(k - q) \) |
In the above table $h_s = \frac{ak}{1-e^{-\frac{ak}{s}}}$ and $\gamma_1 = \frac{h_s}{ak}$.

It has to be emphasized that in all four cases presented in the tables, the function $\chi_\mu(k)$ is chosen so that it corresponds to the hermitization procedure determined by the parameter $\beta = \frac{1}{2}$ (see Section 4). On the other side, the scalar product (the last line in the table) is calculated by using Eqs. (43) and (49) as

$$\langle e^{i k x}, e^{i q x} \rangle_s = \frac{C_{h, \chi}(K^{-1}(h, k))}{C_{h, \chi}(K^{-1}(k))} \int d^4 x \ e^{i S_h(k) x} \ast_{h, \chi} e^{i q x},$$

$$\int d^4 x \ e^{i S_h(k) x} \ast_{h, \chi} e^{i q x} = (2\pi)^4 I(k) \delta^{(4)}(k - q).$$

(93)

> From the table it is readily seen that the plane waves with different momenta constitute an orthonormal set of vectors in Weyl symmetric and natural basis, while in left covariant and right covariant do not. This property is rather appealing because it directly leads to the trace property of the action integral. When introducing a gauge field into the theory, then one might have this property satisfied by the gauge field integral because it ensures a gauge invariance of the action and a theory. We have here identified at least two star products on $\kappa$-Minkowski spacetime that have this desired property and can consequently be considered good candidates for building gauge theory on $\kappa$-space. We point out in general that different realizations of $\kappa$-Minkowski spacetime might have different physical consequences (see also [35, 15]).

### Acknowledgements

We are very thankful to Andrzej Borowiec for the numerous comments on the text. Discussions with Jerzy Lukierski were very valuable. Also, Tajron Jurić is acknowledged for remarks. A.S. thanks the Galileo Galilei Institute for Theoretical Physics (Florence) for the hospitality and the INFN for partial support during which a part of this work was done. This work was also supported by the Ministry of Science, Education and Sports of the Republic of Croatia under contract No. 098-0000000-2865.

### A Calculation of $C(k, q)$

In this section we prove the relation (29). Let us start with the defining relation for $C_{h, \chi}(k, q)$ and $\mathcal{P}_h(k, q)$,

$$e^{i k x} \ast e^{i q x} = C_{h, \chi}(k, q) e^{i \mathcal{P}_h(k, q) x},$$

(A.1)
where \((\hat{x}_{h,\chi})_{\mu} = \hat{x}_{\mu} + \chi_{\mu}(p)\), with \(\hat{x}_{\mu} = x^\alpha h_{\alpha\mu}(p)\). For \(\chi = 0\), \((\hat{x}_{h,\chi})_{\mu} = \hat{x}_{\mu}\) and \(e^{i\hat{k}\hat{x}} > e^{iqx} = e^{i\mathcal{P}_h(k,q)x}\), implying that \(C_{h,\chi} = 0(k,q) = 1\). If we define the family of operators

\[
P^{(t)}(k, -i\partial) = e^{-i\hat{k}\hat{x}}(-i\partial) e^{i\hat{k}\hat{x}}, \quad 0 \leq t \leq 1,
\]

(A.2) parametrized by the free parameter \(t\), then it can be shown (see [43]) that they satisfy the differential equation

\[
\frac{dP^{(t)}(k, -i\partial)}{dt} = h_{\mu\alpha}(P^{(t)}(k, -i\partial))k^\alpha,
\]

(A.3) and that they are related to \(\mathcal{P}_h(k,q)\) as \(\mathcal{P}_{h,\mu}(k, -i\partial) = P^{(1)}(k, -i\partial)\). Hence, the link of the operator family \(\mathcal{A.2}\) with \(\mathcal{P}_h(k,q)\) is obvious after making the identification \(q = -i\partial\). One just needs to take care about the boundary condition for the solution of the differential equation \(\mathcal{A.3}\), which can be put in the form \(P^{(0)}(k, -i\partial) = -i\partial_{\mu} \equiv q_{\mu}\). Note also that \(P^{(t)}(k,q) = P^{(1)}(tk,q) = \mathcal{P}_{h,\mu}(tk,q)\).

To determine the factor \(C_{h,\chi}(k,q)\), act from the left on both sides of the Eq.\(\mathcal{A.1}\), first with the operator \(\partial_{\mu}\) and then with the operator \(e^{-i\hat{k}\hat{x}}\) to yield

\[
e^{-i\hat{k}\hat{x}} \partial_{\mu} e^{i\hat{k}\hat{x}} = C_{h,\chi}(k,q) i\mathcal{P}_h(k,q).
\]

(A.4) Make the exchange \(k \rightarrow tk\) in \(\mathcal{A.4}\) and then differentiate both sides of the resulting expression with respect to \(t\) to get

\[
e^{-i\hat{k}\hat{x}} \left[ i\kappa \left[ \partial_{\mu}, \hat{x}_{\alpha} \right] + \partial_{\mu} i\kappa \chi_{\alpha}(p) \right] e^{i\hat{k}\hat{x}} = i\frac{dC_{h,\chi}(tk,q)}{dt}\mathcal{P}_{h,\mu}(tk,q) + iC_{h,\chi}(tk,q) \frac{d\mathcal{P}_h(tk,q)}{dt}.
\]

(A.5) Due to \([p_{\mu}, \hat{x}_{\alpha}] = [p_{\mu}, (\hat{x}_{h,\chi})_{\alpha}] = -ih_{\mu\alpha}(p)\), the last expression turns into

\[
i\kappa \left[ e^{-i\hat{k}\hat{x}} h_{\mu\alpha}(p)e^{i\hat{k}\hat{x}} + e^{-i\hat{k}\hat{x}} \partial_{\mu} \chi_{\alpha}(p)e^{i\hat{k}\hat{x}} \right] = i\frac{dC_{h,\chi}(tk,q)}{dt}\mathcal{P}_{h,\mu}(tk,q) + iC_{h,\chi}(tk,q) \frac{d\mathcal{P}_h(tk,q)}{dt}.
\]

(A.6) It has left to calculate each of the terms in square brackets. To obtain the first one, we proceed by applying the operator \(h_{\mu\alpha}(p)\) to both sides of Eq.\(\mathcal{A.1}\) from the left (note that the argument \(p\) in \(h_{\mu\alpha}(p)\) is the operator)

\[
h_{\mu\alpha}(p)e^{i\hat{k}\hat{x}} = \mathcal{C}_{h,\chi}(tk,q) h_{\mu\alpha}(\mathcal{P}_h(tk,q)) e^{i\hat{k}\hat{x}} = \mathcal{C}_{h,\chi}(tk,q) h_{\mu\alpha}(\mathcal{P}_h(tk,q)) e^{i\hat{k}\hat{x}}
\]

(A.7) which after multiplying by \(e^{-i\hat{k}\hat{x}}\) gives

\[
e^{-i\hat{k}\hat{x}} h_{\mu\alpha}(p)e^{i\hat{k}\hat{x}} = \mathcal{C}_{h,\chi}(tk,q) h_{\mu\alpha}(\mathcal{P}_h(tk,q)) e^{-i\hat{k}\hat{x}} > e^{i\mathcal{P}_h(tk,q)x}.
\]

(A.8) Since \(e^{-i\hat{k}\hat{x}} > e^{i\mathcal{P}_h(tk,q)x} = e^{iqx}\), we finally have

\[
e^{-i\hat{k}\hat{x}} h_{\mu\alpha}(p)e^{i\hat{k}\hat{x}} = \mathcal{C}_{h,\chi}(tk,q) h_{\mu\alpha}(\mathcal{P}_h(tk,q)).
\]

(A.9) Analogously, for the second term in \(\mathcal{A.6}\) one gets

\[
e^{-i\hat{k}\hat{x}} \partial_{\mu} \chi_{\alpha}(-i\partial) e^{i\hat{k}\hat{x}} = i\mathcal{C}_{h,\chi}(tk,q) \chi_{\alpha}(\mathcal{P}_h(tk,q)) \mathcal{P}_{h,\mu}(tk,q).
\]

(A.10) Plugging last two expressions back to Eq.\(\mathcal{A.6}\) gives

\[
i\kappa \left[ \mathcal{C}_{h,\chi}(tk,q) h_{\mu\alpha}(\mathcal{P}_h(tk,q)) + i\mathcal{C}_{h,\chi}(tk,q) \chi_{\alpha}(\mathcal{P}_h(tk,q)) \mathcal{P}_{h,\mu}(tk,q) \right] = i\frac{dC_{h,\chi}(tk,q)}{dt}\mathcal{P}_{h,\mu}(tk,q) + iC_{h,\chi}(tk,q) \frac{d\mathcal{P}_h(tk,q)}{dt}.
\]

(A.11) Due to the differential equation \(\mathcal{A.3}\), the first term on the left-hand side and the second term on the right-hand side of the above relation cancel each other, producing the following differential equation

\[
\frac{dC_{h,\chi}(tk,q)}{dt} = iC_{h,\chi}(tk,q) k_{\alpha} \chi_{\alpha}(\mathcal{P}_h(tk,q)).
\]

(A.12) This finally integrates to desired result, Eq.\(\mathcal{29}\), which we have set to prove.
B Useful identities

We prove the following identity:

\[ K_h(D_s(K^{-1}_h(k), K^{-1}_h(q))) = D_h(k, q). \]  

(B.1)

It should first be noted that to each realization there is an ordering prescription corresponding to it. Thus, for the realization specified by the functions \( h \) and \( \chi \), Eq. (17), the corresponding ordering we designate by the symbol : \( h, \chi \). This correspondence is realized through the requirement

\[ e^{ik \hat{x}_h, \chi : h, \chi \triangleright 1} = C_{h, \chi}(K_{h, \chi}(k))e^{ikx} \]  

(B.2)

where \( C_{h, \chi} \) is defined in relation (28) and the function \( K_{h, \chi}(k) \) specifies the ordering prescription in such a way that

\[ e^{ik \hat{x}_h, \chi : h, \chi = \epsilon^{iK_{h, \chi}(k)} \hat{x}}. \]  

(B.3)

Under the substitution \( p = K_{h, \chi}(k) \) and relabeling \( p \rightarrow k \) relation (B.2) takes the form

\[ e^{ik \hat{x}_h, \chi : h, \chi \triangleright 1} = C_{h, \chi}(k)e^{iK^{-1}_{h, \chi}(k)x}. \]  

(B.4)

A direct comparison of this with (28) gives the relationship

\[ K_h(k) = K^{-1}_{h, \chi}(k), \]  

showing that \( K_{h, \chi}(k) \equiv K_h(k) \) also does not depend on \( \chi \).

The definition of the star product (21) makes possible to calculate \( e^{ikx} \star_{h, \chi} e^{iqx} \), but this time with the help of the isomorphism (20) specified by the correspondence (B.2),

\[ e^{ikx} \star_{h, \chi} e^{iqx} = \frac{1}{C_{h, \chi}(K_{h, \chi}(k))} : e^{ik \hat{x}_h, \chi : h, \chi \triangleright 1} \frac{1}{C_{h, \chi}(K_{h, \chi}(q))} : e^{iq \hat{x}_h, \chi : h, \chi \triangleright 1} = \frac{1}{C_{h, \chi}(K_{h, \chi}(k))C_{h, \chi}(K_{h, \chi}(q))} : e^{iD_h(k,q) \hat{x}_h, \chi : h, \chi \triangleright 1}. \]  

(B.6)

Finally, due to (B.2), the last expression transforms into

\[ e^{ikx} \star_{h, \chi} e^{iqx} = \frac{C_{h, \chi}(K_{h, \chi}(D_h(k, q)))}{C_{h, \chi}(K_{h, \chi}(k))C_{h, \chi}(K_{h, \chi}(q))} e^{iD_h(k,q)x}. \]  

(B.7)

This is the same star product as that given in Eq. (31), but only calculated in a different way. Hence, a direct comparison of these two, together with (B.5), gives directly the relation (B.1).

The identity (B.1) can be specialized to \( k \rightarrow S(k) \) to yield

\[ e^{iD_h(S_h(k), q)}x = e^{iK_h(D_s(K^{-1}_h(S_h(k)), K^{-1}_h(q)))x}. \]  

(B.8)

By letting \( k = q \) we have \( D_h(S_h(k), k) = 0 \), thus obtaining 1 on the left-hand side of (B.8). This means that the right hand side also has to be equal to 1, yielding

\[ K_h(D_s(K^{-1}_h(S_h(k)), K^{-1}_h(k))) = 0. \]  

(B.9)

Since \( K^{-1}_h(0) = 0 \) and \( D_s(S_h(k), k) = 0 \), relation (B.9) in turn implies

\[ K^{-1}_h(S_h(k)) = S_s(K^{-1}_h(k)) = -K^{-1}_h(k), \]  

(B.10)

where in the final step the obvious property, \( S_s(k) = -k \), has been used. Here \( S_s(k) \) is the antipode corresponding to the momentum addition rule in the Weyl totally symmetric ordering, \( S_s(k) \oplus_s k \equiv D_s(S_s(k), k) = 0 \) in the same way as \( S_h(k) \), satisfying \( S_h(k) \oplus_{h, \chi} k \equiv D_h(S_h(k), k) = 0 \), defines the antipode corresponding to a generic ordering prescription : \( h, \chi \).
C Calculation of $\mathcal{C}(k)$

In order to calculate $\mathcal{C}(k)$, we restrict to the set of realizations whose corresponding star product satisfies

$$\int d^n x e^{iS_h(k)x} *_{h,\chi} e^{iqx} = (2\pi)^n \delta^{(n)}(k - q). \quad (C.1)$$

Now, the general expression for the star product \[35\], applied for $k \rightarrow S_h(k)$, shows that

$$\int d^n x e^{iS_h(k)x} *_{h,\chi} e^{iqx} = \frac{\mathcal{C}_{h,\chi}(K_h^{-1}(D_h(S_h(k), q)))}{\mathcal{C}_{h,\chi}(K_h^{-1}(S_h(k)))}\frac{(2\pi)^n \delta^{(n)}(D_h(S_h(k), q))}{\mathcal{C}_{h,\chi}(K_h^{-1}(q))}. \quad (C.2)$$

The last expression gives a $\delta$-function which is readily calculated as

$$\int d^n x e^{iS_h(k)x} *_{h,\chi} e^{iqx} = \frac{\mathcal{C}_{h,\chi}(K_h^{-1}(D_h(S_h(k), q)))}{\mathcal{C}_{h,\chi}(K_h^{-1}(S_h(k))\mathcal{C}_{h,\chi}(K_h^{-1}(q))\frac{(2\pi)^n \delta^{(n)}(D_h(S_h(k), q))}{\mathcal{C}_{h,\chi}(K_h^{-1}(q))}} \bigg| \frac{\partial \mathcal{C}(D_h(S_h(k), q))}{\partial q_x} \bigg|.$$ 

Since this is equal to $(2\pi)^n \delta^{(n)}(k - q)$, we get a condition

$$\mathcal{C}_{h,\chi}(K_h^{-1}(S_h(k)))\mathcal{C}_{h,\chi}(K_h^{-1}(k)) = \frac{1}{\det \left( \frac{\partial \mathcal{C}(D_h(S_h(k), q))}{\partial q_x} \right)_{k=q}}, \quad (C.3)$$

after setting $k = q$. This is because $D_h(S_h(k), k) = 0$, and consequently $\mathcal{C}_{h,\chi}(K_h^{-1}(D_h(S_h(k), k))) = 1$. If further $\mathcal{C}_{h,\chi}(K_h^{-1}(S_h(k))) = \mathcal{C}_{h,\chi}(K_h^{-1}(k))$, then

$$\mathcal{C}_{h,\chi}(K_h^{-1}(k)) = \frac{1}{\sqrt{\det \left( \frac{\partial \mathcal{C}(D_h(S_h(k), q))}{\partial q_x} \right)_{k=q}}} \quad (C.4)$$

References

[1] J. Lukierski, A. Nowicki, H. Ruegg, V. N. Tolstoy, Phys. Lett. B 264 (1991) 331.
[2] J. Lukierski, A. Nowicki, H. Ruegg, Phys. Lett. B 293 (1992) 344.
[3] J. Lukierski, H. Ruegg, Phys. Lett. B 329 (1994) 189 \href{https://arxiv.org/abs/hep-th/9310117}{[hep-th/9310117]}.
[4] J. Lukierski, H. Ruegg, W. J. Zakrzewski, Ann. Phys. 243 (1995) 90.
[5] S. Zakrzewski, J. Phys. A 27 (1994) 2075.
[6] P. Kosiński, J. Lukierski, P. Maślanka, Phys. Rev. D 62 (2000) 025004 \href{https://arxiv.org/abs/hep-th/9902037}{[hep-th/9902037]}.
[7] M. Dimitrijevic, F. Meyer, L. Möller, J. Wess, Eur.Phys.J. C36 (2004) 117, \href{https://arxiv.org/abs/hep-th/0310116}{[hep-th/0310116]}.
[8] G. Amelino-Camelia, S. Majid, Int.J.Mod.Phys. A15 (2000) 4301, \href{https://arxiv.org/abs/hep-th/9907110}{[hep-th/9907110]}.
[9] T. R. Govindarajan, K. S. Gupta, E. Harikumar, S. Meljanac, D. Meljanac, Phys. Rev. D 77 (2008) 105010 \href{https://arxiv.org/abs/0802.1576}{[arXiv:0802.1576]}.
[10] T. R. Govindarajan, K. S. Gupta, E. Harikumar, S. Meljanac, D. Meljanac, Phys. Rev. D 80 (2009) 025014 \href{https://arxiv.org/abs/0903.2355}{[arXiv:0903.2355]}.
Hermitian realizations

[11] C. A. S. Young, R. Zegers, Nucl. Phys. B 809:439-451 (2009) [arxiv/0807.2745]
[12] D. Kovačević, S. Meljanac, A. Pachol, R. Štrajn, Phys. Lett. B 711, 122-127 (2012) [arxiv/1202.3305].
[13] S. Meljanac, A. Samsarov, R. Štrajn, JHEP 08 (2012) 127 [arxiv/1204.4324].
[14] T. Jurić, S. Meljanac, R. Štrajn, Physics Letters A 377 (2013) 2472-2476, arXiv:1303.0994.
[15] S. Majid, H. Ruegg, Phys. Lett. B 334 (1994) 348 [hep-th/9405107].
[16] J. Kowalski-Glikman, S. Nowak, Phys. Lett. B 539 (2002) 126 [hep-th/0203040].
[17] J. Kowalski-Glikman, S. Nowak, Int. J. Mod. Phys. D 12 (2003) 299 [hep-th/0204245].
[18] H.S. Snyder, Phys. Rev. 71, 38 (1947).
[19] S. Majid, Lect. Notes. Phys. 541 (2000) 227.
[20] G. Amelino-Camelia, L. Smolin, A. Starodubtsev, Class. Quant. Grav. 21 (2004) 3095 [hep-th/0306134].
[21] L. Freidel, J. Kowalski-Glikman, L. Smolin, Phys. Rev. D 69 (2004) 044001 [hep-th/0307085].
[22] L. Freidel, E. R. Livine, Phys. Rev. Lett. 96 (2006) 221301 [hep-th/0512113].
[23] P. K. Osei, B. J. Schroers, J.Math.Phys. 53 (2012) 073510 [arXiv:1109.4086].
[24] C. Meusburger, B. Schroers, Nucl.Phys. B806 (2009) 462, [arXiv:0805.3318].
[25] G. Amelino-Camelia, John Ellis, N.E. Mavromatos, D.V. Nanopoulos, Subir Sarkar, Nature 393 (1998) 763, astro-ph/9712103.
[26] R. Gambini, J. Pullin, Phys. Rev. D 59 (1999) 124021 gr-qc/9809038.
[27] G. Amelino-Camelia, T. Piran, Phys. Rev. D 64 (2001) 036005 astro-ph/0008107.
[28] T. Jacobson, S. Liberati, D. Mattingly, Nature 424 (2003) 1019, astro-ph/0212190.
[29] S. Meljanac, A. Samsarov, M. Stojić, K. S. Gupta, Eur. Phys. J. C 53 (2008) 295 [arXiv:0705.2471].
[30] A. Borowiec, A. Pachol, J. Phys. A 43 (2010) 045203 [arXiv:0903.5251].
[31] S. Meljanac, A. Samsarov, Int. J. Mod. Phys. A 26 (2011) 1439-1468 [arXiv:1007.3943].
[32] E. Harikumar, T. Jurić, S. Meljanac, Phys. Rev. D 84 (2011) 085020 [arXiv:1107.3936].
[33] E. Harikumar, T. Jurić, S. Meljanac, Phys. Rev. D 86 (2012) 045002 [arXiv:1203.1564].
[34] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, L. Smolin, Phys. Rev. D 84 (2011) 084010 [arXiv:1101.0931].
[35] S. Meljanac, A. Pachol, A. Samsarov, K. S. Gupta, Phys. Rev. D 87, 125009 (2013) [arXiv:1210.6814].
[36] S. Meljanac, A. Samsarov, J. Trampetić, M. Wohlgenannt, JHEP 1112 (2011) 010 [arXiv:1111.5553].
[37] G. Amelino-Camelia, Living Rev.Rel. 16 (2013) 5 [arXiv:0806.0339].
[38] S. Meljanac, S. Krešić–Jurić, M. Stojić, Eur. Phys. J. C 51 (2007) 229 [hep-th/0702215].
[39] A. Borowiec, A. Pachol, SIGMA 6 (2010) 086 [arXiv:1005.4429].
[40] S. Meljanac, S. Krešić–Jurić, Int. J. Mod. Phys. A 26 (20) (2011) 3385-3402 [arXiv:1004.4647].
[41] D. Kovačević, S. Meljanac, J. Phys. A: Math. Theor. 45 (2012) 135208 [arXiv:1110.0944].
[42] S. Meljanac, M. Stojić, Eur. Phys. J. C 47 (2006) 531 [hep-th/0605133].
[43] S. Meljanac, D. Meljanac, A. Samsarov, M. Stojić, Mod. Phys. Lett. A 25:579-590 (2010) [arXiv:0912.5087]; Phys.Rev. D 83:065009 (2011) [arXiv:1102.1655].
[44] A. Borowiec, A. Pachoł, Phys. Rev. D 79 (2009) 045012. [arXiv:0812.0576].
[45] A. Borowiec, K. S. Gupta, S. Meljanac, A. Pachoł, Europhys. Lett. 92 (2010) 20006. [arXiv:0912.3299].