Unconventional quantum phase transition in the finite-size Lipkin–Meshkov–Glick model

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Abstract. The Lipkin model of arbitrary particle number $N$ is studied in terms of exact differential-operator representation of spin-operators from which we obtain the low-lying energy spectrum with the instanton method of quantum tunnelling. Our new observation is that the well-known quantum phase transition (QPT) can also occur in the finite-$N$ model only if $N$ is an odd number. We furthermore demonstrate a new type of QPT characterized by level-crossing which is induced by the geometric phase interference and is marvellously periodic with respect to the coupling parameter. Finally, the conventional QPT is understood intuitively from the tunnelling formulation in the thermodynamic limit.

The Lipkin–Meshkov–Glick (LMG) model of many-body two-level systems [1]–[3], originally invented for the study of nuclear giant monopoles [4] is exactly solvable [5]–[7] and has found applications in the statistical mechanics of mutually interacting spins [8, 9] and Bose–Einstein condensates [10]. Recently, it was used to explore the novel relation between quantum entanglement and the quantum phase transition (QPT) [11]–[15]. QPT describing structural change of the ground-state energy-spectrum of many-body systems at a critical value of the coupling parameter has attracted considerable attention in the modern theoretical and experimental communities [16]. As a matter of fact, QPT originates from singularity of the energy spectrum [17] and any non-analyticity point in the ground-state energy of the infinite-lattice system can be identified as a QPT where a level-crossing, i.e., the interchange of excited-state and ground-state levels at the critical point may exist and create non-analyticity.
of the ground-state energy [16]. It is demonstrated that the LMG model described by the Hamiltonian [18]

\[ \hat{H} = -\lambda \sum_{i < j} (\hat{\sigma}_i^x \hat{\sigma}_j^x \pm \gamma \hat{\sigma}_i^y \hat{\sigma}_j^y) - h \sum_i \hat{\sigma}_i^z, \]  

undergoes a QPT at the critical value of coupling parameter \( h_c = \lambda' = N\lambda_c \), where the parameter \( \lambda' \) is used to avoid the formal infinity [18, 19] in the thermodynamic limit \( N \to \infty \) and \( \hat{\sigma}_\alpha \) are the Pauli matrices. When \( h \leq h_c \) (called the deformed phase) the ground state is degenerate, and it is non-degenerate when \( h \geq h_c \) (called the normal phase). For our convenience the parameter values are confined by \( \lambda > 0, 0 < \gamma < 1 \) and we consider two cases denoted by ‘±’ signs. In spin language, the deformed phase corresponds to a long-range magnetic order and may be regarded as a ferromagnet, while the transverse magnetic field \( h \), which lowers the potential barrier, induces quantum tunnelling between two degenerate states and eventually destroys the magnetic order at the critical point \( h = h_c \). A similar QPT was indeed observed with the ferromagnetic moment vanishing continuously at a quantum critical point of the transverse field in the low-lying magnetic excitations of the insulator LiHoFe\(_4\) [20].

The study of scaling properties at the critical point for a finite-\( N \) model is of primary interest and has become a hot topic recently [18, 19]. Based on mean-field approximation, it is found that particle-number dependence of the energy gap between the ground state and first excited state behaves as \( \exp[-N] \) in the deformed phase [18, 19]. However QPT from the deformed phase to the normal phase, which we called the conventional QPT (CQPT), actually does not exist for the finite-\( N \) case [8] since quantum tunnelling can remove the degeneracy leading to tunnel splitting \( \delta E \) even in the absence of external field (\( h = 0 \)) and thus the ground-state energy structure, strictly speaking, does not undergo macroscopic change. The energy gap \( \delta E \) vanishes in the thermodynamic limit \( N \to \infty \) and the ground state becomes degenerate. The quantum tunnelling effect of the finite-\( N \) model, which gives rise to a rich structure of low-lying energy spectra, has not yet been studied explicitly. Based on the tunnelling effect obtained in this paper, we demonstrate that the CQPT can occur in the finite-\( N \) model if \( N \) is an odd number due to the quenching of quantum tunnelling induced by quantum-phase interference and the long-range magnetic order survives from the tunnelling.

Moreover we find a new type of QPT characterized by the level-crossing which is induced also by quantum-phase interference. The ground state and first excited state interchange alternately with the increase of \( h \) resulting in periodic QPT. The finite-\( N \) Lipkin model is studied in terms of the exact differential-operator representation of the spin-operators which gives rise to a potential-barrier tunnelling Hamiltonian and the low-lying energy spectrum is obtained by means of the instanton method. From the viewpoint of tunnelling the CQPT can be understood intuitively since the potential-barrier height decreases with the increase of \( h \) and finally disappears at the critical field value. The level crossing, however, does not appear at all in the CQPT.

We adopt the giant-spin operators defined by

\[ \hat{S}_l = \sum_i \hat{\sigma}_i^l / 2 (l = x, y, z), \]

to convert the \( N \)-particle model into an anisotropic giant-spin Hamiltonian

\[ \hat{H} = -\lambda (\hat{S}_x^2 \pm \gamma \hat{S}_y^2) - h \hat{S}_z \]  

(2)
with the total spin quantum number \( s = N/2, N/2 - 1, N/2 - 2 \cdots, 1/2, \) or 0 depending on odd or even \( N \) and a factor 2 absorbed into the parameters \( \lambda \) and \( h \). Because of the ferromagnetic interaction between spins \( \lambda > 0 \) the ground state and the first excited state lie always in the subspace of maximum spin \([18]\), i.e. \( s = N/2 \). The Hamiltonian of ‘+’ sign describes a giant-spin with an easy axis \( x \) (lowest energy) and a hard axis \( z \) (highest energy). The external magnetic field \( h \) is applied along the hard axis, which is crucial in generating the QPT of level-crossing. For the ‘−’ sign case the \( y \)-axis becomes the hard axis and the magnetic field is then along the medium axis.

We start from stationary Schrödinger equation

\[
\hat{H} \Phi(\varphi) = E \Phi(\varphi),
\]

with the Hamiltonian of ‘−’ sign and a coordinate-frame rotation such that the hard axis and easy axis are along \( z \) and \( x \) respectively for the sake of convenience, where \( \varphi \) is the azimuthal angle of the giant-spin vector. The generating function \([21]\) \( \Phi(\varphi) \) can be constructed in terms of the conventional eigenstates of \( \hat{S}_z \) such that

\[
\Phi(\varphi) = \sum_{m=-s}^{s} C_m \frac{\exp(im\varphi)}{\sqrt{(s-m)!(s+m)!}},
\]

which obviously satisfies the boundary condition,

\[
\Phi(\varphi + 2\pi) = \exp(2\pi i s) \Phi(\varphi).
\]

Thus, we have periodic wavefunctions for even \( N \) and anti-periodic wavefunctions for odd \( N \). The explicit form of the spin operator acting on the wavefunction \( \Phi(\varphi) \) is seen \([21]\) to be

\[
\hat{S}_x = s \cos \varphi - \sin \varphi \frac{d}{d\varphi}, \quad \hat{S}_y = s \sin \varphi + \cos \varphi \frac{d}{d\varphi}, \quad \text{and} \quad \hat{S}_z = -i \frac{d}{d\varphi}.
\]

Substituting these differential operators into the Schrödinger equation we obtain

\[
\left\{-\alpha(1 - \delta \sin^2 \varphi) \frac{d^2}{d\varphi^2} - \left[ \lambda \left( s - \frac{1}{2} \right) \sin 2\varphi - h \sin \varphi \right] \frac{d}{d\varphi} + \left[ \lambda \left( s^2 \cos^2 \varphi + s \sin^2 \varphi \right) - hs \cos \varphi \right] \right\} \Phi(\varphi) = E \Phi(\varphi),
\]

where \( \alpha = \lambda(1 + \gamma), \delta = 1/(1 + \gamma) \). In a new variable \( x \) defined as

\[
x(\varphi) = \int_{0}^{\varphi} \frac{d\varphi'}{\sqrt{1 - \delta \sin^2 \varphi'}} = F(\varphi, \delta),
\]

which is the incomplete elliptic integral of the first kind with modulus \( \sqrt{\delta} \), the trigonometric functions \( \sin \varphi \) and \( \cos \varphi \) become the Jacobian elliptic functions \( \text{sn}(x) \) and \( \text{cn}(x) \) with the same modulus, respectively \([22]\). Using the transformation such that

\[
\Phi[\phi(x)] = \text{dn}'(x) \exp \left\{-\frac{h}{2\lambda \sqrt{\gamma}} \tan^{-1} \left[ \frac{\text{cn}(x)}{\sqrt{\gamma}} \right] \right\} \psi(x),
\]

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Figure 1. The potential barriers and clockwise and anticlockwise tunnel paths for \( h = 0 \) (red line) and 5 with \( \gamma = 0.5 \) in the unit \( \lambda = 1 \).

where \( \text{dn}(x) = \sqrt{1 - \delta \text{sn}^2(x)} \) is also a Jacobian elliptic function, the stationary Schrödinger equation is seen to be

\[
\left[ -\alpha \frac{d^2}{dx^2} + V(x) \right] \psi(x) = E \psi(x),
\]

where the scalar potential is given by

\[
V(x) = \frac{1}{\text{dn}^2(x)} \left[ \zeta \text{cn}^2(x) - h \left( s + \frac{1}{2} \right) \text{cn}(x) + \frac{h^2}{4\alpha} \right]
\]

and \( \zeta = \lambda s(s+1) - \frac{h^2}{4\alpha} \). The boundary condition of the wavefunction becomes

\[
\psi(x + 4K) = \exp(2\pi is)\psi(x),
\]

where \( K \) denotes the complete elliptic integral of the first kind. In the absence of external field \( (h = 0) \) the scalar potential \( V(x) \) is symmetric (see figure 1, red line) with potential minima located at \( x_\pm = \pm K \) corresponding to the original azimuthal angles \( \varphi_\pm = 0, \pi \) which are two equilibrium orientations of the giant spin. Disregarding the negligibly small tunnelling effect the two ground states are degenerate. The external field suppresses the potential barrier and thus increases the tunnelling rate. We now calculate the tunnelling rate between the two degenerate states. To this end, we begin with the tunnelling-induced Feynman propagator through the potential barrier

\[
\langle x_-, \beta | x_+, -\beta \rangle = \langle x_- | \exp^{-2\beta \hat{H}} | x_+ \rangle = \int \mathcal{D}\{x\} \exp^{-S},
\]

where

\[
S = \lim_{\beta \to \infty} \int_{-\beta}^{\beta} \mathcal{L}_c \, d\tau
\]
is the Euclidean action evaluated along the tunnelling trajectory of a pseudoparticle in the barrier region called an instanton and the Euclidean Lagrangian is given by

\[ L_e = \frac{\dot{x}^2}{4\alpha} + U(x) \]

where \( \dot{x} = dx/d\tau \) denotes the imaginary-time derivative and \( \tau = it \) is the imaginary time. The quantum tunnelling removes the degeneracy of ground states \( |\psi_{c\pm}\rangle \) in two potential wells located at \( x_\pm \) respectively. \( |\psi_{c\pm}\rangle \) describe the two equilibrium orientations of the giant spin and thus may be called the Schrödinger cat states. In the two-level approximation we have \( \hat{H}|1\rangle = E_1|1\rangle \), \( \hat{H}|0\rangle = E_0|0\rangle \), where

\[ |0\rangle, |1\rangle = \frac{1}{\sqrt{2}}(|\psi_{c+}\rangle \pm |\psi_{c-}\rangle) \]

are respectively the ground state and the first excited state. The low-lying energy spectrum can be evaluated as \( E_{1,0} = \bar{E} \pm \delta E/2 \) where \( \bar{E} = \langle \psi_{c\pm}|\hat{H}|\psi_{c\pm}\rangle \), and

\[ \delta E = E_1 - E_0 = -\langle \psi_{c+}|\hat{H}|\psi_{c-}\rangle + \langle \psi_{c-}|\hat{H}|\psi_{c+}\rangle \].

Inserting the complete set \( |\psi_{c\pm}\rangle \) in the transition amplitude equation (4) the tunnel splitting can be derived from the path integral

\[ \delta E \sim \frac{e^{2\beta \bar{E}}}{2\beta |\psi_{c+}(x_+)| |\psi_{c-}^*(x_-)|} \int \mathcal{D}[x] e^{-S}, \tag{5} \]

which can be evaluated in terms of the stationary-phase approximation as

\[ \int \mathcal{D}[x] e^{-S} \sim I e^{-S_c}, \]

where \( S_c \) is the action along the classical trajectory of instanton \( x_c(\tau) \) which the solution of the classical equation of motion: \( \delta S = 0 \):

\[ I = \int \mathcal{D}[\eta] e^{-(1/2)\eta^2(\delta^2 S/\delta x\delta x)|_c} \]

denotes the contribution of quantum fluctuation around the classical trajectory such that \( x(\tau) = x_c(\tau) + \eta(\tau) \) with \( \eta \) being the small fluctuation. Working out the path integral for both the clockwise and counterclockwise rotations under the barrier (see figure 1) and wavefunctions \( \psi_{c\pm}(x_\pm) \) we obtain the level splitting

\[ \delta E = Pe^{-W}\sqrt{\cosh \chi + \cos 2\pi s} \tag{6} \]

with

\[ P = 2^{7/2} \left\{ \frac{U_0^3\lambda(1 + \gamma)(1 - \varrho^2)^3}{|\gamma/(1 + \gamma) + \varrho^2/(1 + \gamma)|^2\pi^2} \right\}^{1/4}, \quad W = \sqrt{U_0(1 + \gamma)/\lambda \ln \frac{1 + \sqrt{(1 - \varrho^2)/(1 + \gamma)}}{1 - \sqrt{(1 - \varrho^2)/(1 + \gamma)}}, \]

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Figure 2. The numerical simulation of the first-excited (red line) and ground-state (blue line) energy levels as a function of $h$ for $N = 11$ and 10 (insert) with $\gamma = 0.95$.

and

$$\chi = 2Q \sqrt{\frac{U_0}{\lambda \gamma}} \tan^{-1} \left[ \frac{1}{Q} \sqrt{\frac{\gamma(1 - \rho^2)}{(1 + \gamma)}} \right],$$

where

$$U_0 = \frac{1}{2} \left[ \zeta_1 - \frac{\zeta_3}{\gamma} + \sqrt{\left( \zeta_1 - \frac{\zeta_3}{\gamma} \right)^2 + \frac{\zeta_2^2}{\gamma}} \right]$$

and $Q = \zeta_2/2U_0$ with $\zeta_1 = \lambda s(s + 1) - h^2/4\lambda(1 + \gamma)$, $\zeta_2 = h(s + 1/2)$ and $\zeta_3 = h^2/4\lambda(1 + \gamma)$. When $h = 0$ we have $\chi = 0$ and the square root in equation (6) becomes $\sqrt{2}$ for even $N$. The quantum tunnelling destroys the magnetic order and the ground state $|0\rangle$ is paramagnetic. There is no QPT for finite $N$. However, the situation is different for the odd-$N$ case where the square root in equation (6) becomes zero leading to vanishing tunnel splitting. The degeneracy of ground states in the absence of external field ($h = 0$) cannot be removed by tunnelling similar to the Kramer’s degeneracy in spin systems and the ferromagnetic ground states $|\psi_{c\pm}\rangle$ remain. The magnetic field suppresses the potential barrier seen in figure 1 and enhances the tunnelling rate which destroys the magnetic order gradually along with the continuous increase of the field. The system thus undergoes a smooth second order QPT. The critical transition point

$$h_c = \lambda(1 - \gamma)(s + 1/2) + \lambda\sqrt{(1 - \gamma)^2(s + 1/2)^2 + 4\gamma s(s + 1)}$$
determined precisely from the condition that the central barrier-height vanishes and the double-well of the potential becomes a single one (see figure 1) reduces exactly to $h_c^0 = \lambda'$ in the thermodynamic limit [18, 19]. The numerical simulation of the low-lying levels is shown in figure 2, the energy gap obtained from which agrees with our gap formula equation (6) perfectly for any $s$. From equation (6) the finite-size scaling behaviour of the energy gap is seen to be

$$\delta E \sim \exp \left[ -N \right]$$

which vanishes in the thermodynamic limit no matter whether $N$ is even or odd.
We now consider the most interesting case of the ‘+’ sign where the external field is along the hard axis (z-axis). The Hamiltonian operator does not possess a satisfactory differential-operator representation in this case [23]. The Feynman propagator equation (4) may be evaluated with the spin-coherent state path integrals. The spin-coherent state is defined by \( \hat{S} \cdot n |n⟩ = s |n⟩ \) where 
\[ n = (\sin θ \cos φ, \sin θ \sin φ, \cos θ) \]

is a unit vector [24, 25]. Regarding \( φ \), \( p = s \cos θ \) as canonical variables and integrating over variable \( p \) we obtain the effective Lagrangian given by

\[ L = \frac{1}{2}m(φ)\dot{φ}^2 + A(φ)\dot{φ} - V(φ), \]

where

\[ m(φ) = \frac{1}{2} \lambda (1 - (1 - γ) \sin^2 φ), \quad A(φ) = s [1 - h/2λs(1 - (1 - γ) \sin^2 φ)], \]

and

\[ V(φ) = \lambda (1 - γ)s(s + 1) \sin^2 φ - \frac{h^2(1 - γ) \sin^2 φ}{4λ(1 - (1 - γ) \sin^2 φ)}. \]

The periodic potential \( V(φ) \) has degenerate vacua located at \( φ = 0, π \) corresponding to the equilibrium orientations of the giant spin along the ±x-directions. We again consider the tunnelling-induced propagator between two degenerate vacua. The total time derivative term \( A(φ)\dot{φ} \) coming from the geometric phase of the path integral does not affect the classical equation of motion but gives rise to the quantum phase interference between the clockwise and counterclockwise tunnelling paths. The tunnel splitting is found as

\[ δE = Pe^{-W} \cos [π(s - θ_h)], \quad (7) \]

where

\[ P = 2^4 s^3 λ^2 (1 - γ)^3/2(1 - η)^3/2/π(γ - η), \]

\[ W = \sqrt{s(s + 1)} \left\{ \ln \frac{\sqrt{1 - η} + \sqrt{1 - γ}}{\sqrt{1 - η} - \sqrt{1 - γ}} - \frac{η}{γ} \ln \frac{1 + \sqrt{ς}}{1 - \sqrt{ς}} \right\} \]

and \( θ_h = h/2λ\sqrt{γ} \), with \( η = h^2/4λ^2s(s + 1) \) and \( ζ = (1 - γ)η/γ(1 - η) \). We now come to the most interesting observation of the paper: that the energy gap is a periodic function of \( h \) and the two states interchange periodically. The QPT of level crossing arises at the critical points

\[ h_c(n) = 2λn \sqrt{γ} \]

for odd \( N \), and

\[ (2n + 1)λ\sqrt{γ} \]

for even \( N \), where \( n \) is integer. Figure 3 shows the numerical result of the first excited state (red line) and ground state (blue line) energy levels for odd \( N \) (a) and even \( N \) (b) respectively. The absolute value of the numerical energy gap as a function of \( h \) agrees qualitatively with equation (7)
Figure 3. The numerical simulation of the first-excited and ground-state energy levels as a function of $h$ for $N = 11$ (a) and $N = 10$ (b). Inset: the analytical (solid line) and numerical (dotted line) absolute values of the energy gap for $N = 41$ (a) and $N = 40$ (b).

and the agreement becomes perfect for large $N$ (see figure 3 (inset)) since the spin-coherent state path integral method is only valid for large $s$. For the odd-$N$ case the ground state in the absence of external field ($h = 0, \theta_h = 0$) is degenerate since $\cos(\pi s) = 0$. When $0 < h \leq h_c(1)$ quantum tunnelling destroys the degeneracy and the ground state becomes quantum paramagnetic while the first excited state is ferromagnetic. This can be explained in terms of the Schrödinger cat states $|\psi_\pm\rangle = |n_\pm\rangle$ where

$$\hat{S}_x|n_\pm\rangle = \pm s|n_\pm\rangle$$

and the unit vector of the spin-coherent state is along the $x$-axis. We have $\langle 0|\hat{S}_x|0\rangle = 0$, and $\langle 1|\hat{S}_x|1\rangle = s$. Increasing the field $h$ to cross over the first critical point $h_c(1)$ the first excited state and ground state interchange and the system becomes the ferromagnetic phase. The interchange takes place again at the second critical point and is particularly useful in quantum computing since the two phases can be used to realize a qubit in some practical spin system.
Figure 4. The first-excited (red line) and ground-state (blue line) magnetizations $M$ as a function of $h$ with $\gamma = 0.95, N = 10$.

The magnetization as a function of the magnetic field can be evaluated as

$$M_{0,1} = -\langle 0, 1 | \frac{\partial \hat{H}}{\partial h} | 0, 1 \rangle = -\frac{\partial E_{0,1}}{\partial h}.$$ 

The numerical simulation of the magnetization is shown in figure 4. The critical behaviour of the energy gap $\delta E$ at the vicinity of critical point $h_c$ is found as

$$\delta E(h \rightarrow h_c) = (\pi/\lambda' \sqrt{\gamma}) P(h_c) e^{-W(h_c)} |h - h_c|. \quad (8)$$

In the thermodynamic limit, the tunnel splitting vanishes as

$$\lim_{N \rightarrow \infty} \delta E \sim \exp [-N] \rightarrow 0 \quad (9)$$

and the QPT of level crossing disappears. A critical value

$$h_c^{th} = \lambda' \sqrt{\gamma} \quad (10)$$

of CQPT can be determined again from the vanishing barrier height, which differs from the mean-field value by a factor $\sqrt{\gamma}$.

In conclusion, with explicit calculation of the tunnelling effect we show that QPT can occur in a finite-size LMG model. The QPT of level crossing discovered in this paper may provide a possible technique to realize a qubit in a quantum computer and a paper explaining this procedure will be published elsewhere.

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