Existence of Ground State Eigenvalues for the Spin-Boson Model with Critical Infrared Divergence and Multiscale Analysis

Volker Bach
Institut für Analysis und Algebra
Technische Universität Braunschweig
Germany (v.bach@tu-bs.de)

Miguel Ballesteros
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas (IIMAS).
Universidad Nacional Autónoma de México (UNAM)
Mexico (miguel.ballesteros@iimas.unam.mx)

Martin Könenberg
Institut für Analysis, Dynamik und Modellierung
Universität Stuttgart
Germany (martin.koenenberg@mathematik.uni-stuttgart.de)

Lars Menrath
Institut für Analysis und Algebra
Technische Universität Braunschweig
Germany (l.menrath@tu-bs.de)

Abstract

A two-level atom coupled to the radiation field is studied. First principles in physics suggest that the coupling function, representing the interaction between the atom and the radiation field, behaves like $|k|^{-1/2}$, as the photon momentum $k$ tends to zero. Previous results on non-existence of ground state eigenvalues suggest that in the most general case binding does not occur in the spin-boson model, i.e., the minimal energy of the atom-photon system is not an eigenvalue of the energy operator. Hasler and Herbst have shown [12], however, that under the additional hypothesis that the coupling function be off-diagonal -which is customary to assume-
binding does indeed occur. In this paper an alternative proof of binding in case of off-diagonal coupling is given, i.e., it is proven that, if the coupling function is off-diagonal, the ground state energy of the spin-boson model is an eigenvalue of the Hamiltonian. We develop a multiscale method that can be applied in the situation we study, identifying a new key symmetry operator which we use to demonstrate that the most singular terms appearing in the multiscale analysis vanish.

1 Introduction

The precise description of nonrelativistic matter in interaction with the quantized radiation field has been in the focus of mathematical research ever since the proposal of the quantization of the radiation field by Dirac more than eighty years ago [9].

The invention of the Laser some fifty years ago necessitated the development of a simplified, yet, adequate model for the description of its mechanism in theoretical physics. It proved useful to simplify the model of matter from atom and molecules to two-level atoms. The corresponding model, known as the spin-boson model, became the work horse of quantum optics and is nowadays of key importance for quantum computing, with the interpretation of the two-level atom as a qubit.

Starting more than twenty years ago, the mathematical aspects of the models of nonrelativistic matter coupled to the quantized radiation field -known as non-relativistic quantum electrodynamics, NR QED- were systematically investigated. In contrast to many models from relativistic quantum mechanics or quantum field theory, the models in NR QED are defined by a self-adjoint, semi-bounded Hamiltonian \( H = H^* \geq c > -\infty \) acting on the tensor product \( \mathcal{H} = \mathcal{H}_{at} \otimes \mathcal{F} \) of the Hilbert spaces \( \mathcal{H}_{at} \) of matter and \( \mathcal{F} \) of the radiation field, respectively. During the past two decades or so, for many models of NR QED, basic spectral properties like binding and the existence of resonances have been established. These represent the expected fate of the eigenvalues of the atom as it is coupled to the radiation field: The lowest spectral point persists to be an eigenvalue and all other atomic eigenvalues are unstable and give rise to metastable states, the resonances.

Specifically, binding means that the infimum \( E_{gs} := \inf \sigma(H) > -\infty \) of the spectrum of the Hamiltonian is an eigenvalue, called the ground state energy, with an eigenvector \( \varphi_{gs} \in \mathcal{H} \), called the ground state, i.e., \( H \varphi_{gs} = E_{gs} \varphi_{gs} \).

Binding in NR QED was established for atoms and molecules coupled to the radiation field [4, 5], as well as, for the spin-boson model [1] about twenty years ago under the assumption that the coupling function \( G(k) \) is slightly more regular, \( |G(k)| \leq C |k|^{-\frac{1}{2} + \mu} \), for some \( C < \infty \) and \( \mu > 0 \), in the infrared limit \( k \to 0 \),
than what derives from first principles in physics, namely, $|G(k)| \sim C|k|^{-\frac{3}{2}}$, as $k \to 0$.

For these latter, more singular models, with $|G(k)| \sim C|k|^{-\frac{3}{2}}$, as $k \to 0$, binding was shown to hold true a few years later [7] in the special, but physically most relevant case that the radiation field is minimally coupled to the electrons of the atom. Here, it was used that the model possesses additional symmetries such as the $U(1)$-gauge symmetry. The key identity (in the case of one electron, as for the hydrogen atom) made use of in the proof is

$$\vec{v} = i[H, \vec{x}],$$

where $\vec{v} = -i\nabla x - \vec{A}(x)$ is the velocity operator and $\vec{x}$ the position operator of the electron.

Following an argument of Fröhlich [11] it was assumed for many years [2] that the spin-boson model with singular coupling does not bind in the above sense, but rather possesses a ground state that is revealed by a (non-unitary) change of the representation of the canonical commutation relations. In view of this common belief the recent proof of Hasler and Herbst [12] for binding of the spin-boson model with singular coupling is a remarkable result. Their proof uses the renormalization group based on the isospectral Feshbach-Schur map developed in [5, 6, 3]. Their additional key observation is that since there is no self-interaction of each of the two levels of the atom, but only a coupling to one another, the (discrete) flow equation defined by the renormalization group is more regular than it seems to be at first glance.

In the present paper we give an alternative proof for binding of the spin-boson model with singular coupling. We consider the spin-boson Hamiltonian

$$H := H_{at} + H_{ph} + \Phi(G),$$

(1.1)

where $H_{ph} \equiv 1_{at} \otimes H_{ph}$ is the field Hamiltonian and $H_{at} = \sigma_3 + 1_{at} \equiv H_{at} \otimes 1_F$ is the Hamiltonian of the two-level atom, with $\sigma_\nu$ denoting the Pauli matrices. Furthermore, $\Phi(G)$ is the interaction with field operator $\Phi(G) = a^*(G) + a(G)$, with $G \equiv g\sigma_1 \otimes h(k)$, where $h$ is a compactly supported coupling function obeying $|h(k)| \sim C|k|^{-\frac{3}{2}}$, as $k \to 0$, and $g \geq 0$ is the coupling strength, see Eqs. (1.11)–(1.13). For this Hamiltonian $H$ we prove our main result, Theorem 1.1, which states that the infimum of its spectrum is an eigenvalue.

Our construction is based on Pizzo’s method [15], rather than the renormalization group induced by the Feshbach-Schur map. That is, we consider a sequence $H_n \equiv H(G_n)$ of regularized Hamiltonians whose coupling functions $G_n(k) = 1(|k| \geq \rho_n)G(k)$ are the restrictions of $G$ to photon momenta larger than $\rho_n = \kappa \gamma^n$, for some fixed $\gamma < 1$ and all $n \in \mathbb{N}$. Following the idea originally formulated by Pizzo, we inductively prove that each $H_n$ shows binding with a ground state energy $E_n$ being a non-degenerate eigenvalue with normalized eigenvector $\phi_n$ and rank-one eigenprojection $P_n = |\phi_n\rangle\langle\phi_n|.$

It is fairly easy to establish the existence of these eigenprojections $P_n$, for each $n$, and the principal difficulty of this and all other such constructions lies
in the proof of convergence $P_n \to P_{gs}$ of $P_n$ (here the range of $P_{gs}$ consists of ground state eigenvectors of $H$). The additional property from which we derive this convergence in this seemingly too singular case is that the original Hamiltonian $H$, as well as, all Hamiltonian operators $H_n$, commute with a symmetry $S = \sigma_3 (-1)^{N_{ph}}$, where $N_{ph}$ is the photon number operator. This symmetry induces a decomposition of the Hilbert space into the two subspaces $\mathcal{H}_\pm$ corresponding to the eigenvalues $\pm 1$ of $S$. The operators $H$ and $H_n$ leave these subspaces invariant, and from this we draw the consequence that

$$\text{Tr} \{ P_n \sigma_1 P_n \} = 0,$$

for all $n \in \mathbb{N}$, which enters our proofs at key steps.

### 1.1 The Model

We study a two-level atom interacting with the radiation field. We assume, without loss of generality, that the ground state (free) energy of the atom equals 0 and the excited energy equals 2. In this paper we only consider non-degenerate energies. Therefore, the Hamiltonian of the atom alone is given by the matrix

$$H_{at} := \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

(1.3)

acting on the atom Hilbert space

$$\mathcal{H}_{at} := \mathbb{C}^2.$$  

(1.4)

For every Hilbert space $\mathfrak{h}$, we denote by

$$\mathcal{F}[\mathfrak{h}] := \mathbb{C} \oplus \bigoplus_{\ell=1}^{\infty} S_{\ell} \otimes_{k=1}^{\ell} \mathfrak{h}$$

(1.5)

its associated bosonic (symmetric) Fock space. $\Omega_{\mathfrak{h}} \in \mathcal{F}[\mathfrak{h}]$ denotes the vacuum vector. Here $S_{\ell}$ denotes the orthogonal projection onto the subspace of totally symmetric tensors. The Hilbert space for the radiation field is defined to be

$$\mathcal{F} \equiv \mathcal{F}[L^2[\mathbb{R}^3]].$$

(1.6)

Note that it is not quite adequate to call the quanta of this scalar field photons, as polarization is not taken into account. The free photon energy is given by the operator

$$H_{ph} \equiv H_{ph}(\omega) := d\Gamma(\omega) = \int_{\mathbb{R}^3} \omega(k) a^\dagger(k) a(k) dk,$$

(1.7)
where $\omega(k) := |k|$, and $a^*(k), a(k)$ denote the creation and annihilation operators representing the canonical commutation relations on $\mathcal{F}$, i.e.,

$$[a(k), a^*(k')] = \delta(k - k'), \quad [a(k), a(k')] = [a^*(k), a^*(k')] = 0, \quad a(k)\Omega = 0,$$

(1.8)

for all $k, k' \in \mathbb{R}^3$, in the sense of operator-valued distributions. In Eq. (1.7) we use Nelson’s notation for the second quantization $\Gamma(A)$ of a one-photon operator $A$. We furthermore introduce the photon number operator $N_{\text{ph}}$, defined on $\mathcal{H}$, by the following equation

$$N_{\text{ph}} := H_{\text{ph}}(1_{\mathbb{R}^3}) \equiv d\Gamma(1_{\mathbb{R}^3}).$$

(1.9)

The Hilbert space of the (full) atom-photon system is the tensor product of the atom and the photon Hilbert spaces:

$$\mathcal{H} := \mathcal{H}_{\text{at}} \otimes \mathcal{F}.$$  

(1.10)

The interaction between the atom and the photon field is expressed in terms of the field operator,

$$\Phi(G) := \int_{\mathbb{R}^3} [G(k) \otimes a^*(k) + G^*(k) \otimes a(k)]dk,$$

(1.11)

where we assume that $G$ is of the form

$$G(k) := g \frac{\Lambda(k)}{4\pi \sqrt{\omega(k)}} f(k)\sigma_1, \forall k \in \mathbb{R}^3,$$

(1.12)

with $\Lambda(k) := 1_{\{k : |k| < \kappa\}}$ (the characteristic function of the set $\{k : |k| < \kappa\}$) being an ultraviolet cutoff and the coupling constant $g > 0$ being a small parameter. For convenience (without loss of generality) we choose the UV-cutoff scale as $\kappa < 1$. We assume that $f = f^* \in L^\infty(\mathbb{R}^3)$ is uniformly bounded by 1 and that $\sigma_1$ is the first Pauli matrix (the diagonal entries equal zero and the other entries equal 1). The energy of the full system is the sum of all energies just introduced,

$$H := H_{\text{at}} + \Phi(G) + H_{\text{ph}}.$$  

(1.13)

Here we use the identifications $H_{\text{at}} \equiv H_{\text{at}} \otimes 1_{\mathcal{F}}$, $H_{\text{ph}} \equiv 1_{\mathcal{H}_{\text{at}}} \otimes H_{\text{ph}}$. In general, for pairs of Hilbert spaces $V_1$ and $V_2$ and operators $A_1$ and $A_2$ defined on $V_1$ and $V_2$, respectively, we leave out trivial tensor factors and write

$$A_1 \equiv A_1 \otimes 1_{V_2}, \quad A_2 \equiv 1_{V_1} \otimes A_2.$$  

(1.14)
1.2 Main Theorem and Outline of its Proof

Our main result is proven in Section 3.3, specifically it is restated in Theorems 3.4 and 3.5 (see also Remark 3.6). Here we provide the core of our results in the next

**Theorem 1.1.** For sufficiently small $g > 0$ the bottom of the spectrum,

$$E_{gs} := \inf \sigma(H),$$  \hspace{1cm} (1.15)

is an eigenvalue of $H$.

The proof of Theorem 1.1 uses perturbation theory in a non-trivial way. Notice that the free Hamiltonian

$$H_{\text{Free}} := H_{\text{at}} + H_{\text{ph}}$$  \hspace{1cm} (1.16)

has zero as an eigenvalue at the bottom of its spectrum. As the spectrum of $H_{\text{Free}}$ is $[0, \infty)$, 0 is immersed in the continuum. Thus, standard perturbation theory of isolated eigenvalues of finite multiplicity cannot be applied and multiscale or renormalization techniques must be used, we utilize multiscale analysis. Since the coupling function behaves asymptotically as $\|G(k)\| \sim |k|^{-1/2}$, the interaction $\Phi(G)$ scales like the field Hamiltonian $H_{\text{ph}}$ under unitary dilations, namely, like an inverse length. Consequently, $\Phi(G)$ is a marginal perturbation of $H_{\text{ph}}$, which makes the direct application of renormalization group schemes difficult. In order to prove that $\Phi(G)$ is actually marginally irrelevant, we identify a new symmetry $\mathcal{S}$ of the system which allows us to conclude that the matrix element $\langle \psi | \sigma_1 | \psi \rangle$ vanishes, for any eigenvector of $H$. One of the main purposes of this paper is to demonstrate that this information can be used to show the convergence of the ground state construction.

The multiscale analysis is based on the construction of a sequence of infrared regular Hamiltonians whose ground state projections converge to a projection with range consisting of eigenvectors of $H$. The elements of this sequence of Hamiltonians cut off small momenta, but progressively incorporate ever smaller momenta in such a way that eventually all momenta are taken into account. More specifically, we proceed as follows:

The infrared cutoff functions are characterized by a decreasing sequence $(\rho_n)_{n \in \mathbb{N}_0}$ of numbers

$$\rho_n := \kappa \gamma^n < 1, \ \forall n \in \mathbb{N}_0,$$  \hspace{1cm} (1.17)

for some specifically small parameter $\gamma \in (0, 1)$ that will be conveniently chosen later on (recall that we set $\kappa < 1$ above). Here $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The number $\rho_n$ represents the lowest allowed photon energy at step $n$. We cut off energies below $\rho_n$ in the following manner: Define

$$\omega_n := 1_{\mathbb{R}^3 \setminus B_n} \omega, \hspace{1cm} G_n := 1_{\mathbb{R}^3 \setminus B_n} G,$$  \hspace{1cm} (1.18)
where

\[ B_n := \{ k \in \mathbb{R}^3 : |k| < \rho_n \} \]  \hspace{1cm} (1.19)

is the ball centered at the origin with radius \( \rho_n \). In Eq. (1.18), \( \omega_n \) cuts off the free photon energies below \( \rho_n \). Similarly \( G_n \) cuts off interacting energies. The symbol \( 1_A \) represents the characteristic (or indicator) function of the set \( A \). Notice that \( G_0 = 0 \), since \( G \) is supported in \( B_0 \). Now we define a sequence of infrared-cutoff Hamiltonians \((H_n)_{n \in \mathbb{N}}\). Set

\[ H_n := H_{\text{at}} + \Phi(G_n) + H_{\text{ph}}(\omega_n) \]  \hspace{1cm} (1.20)

on \( \mathcal{H}_n := \mathcal{H}_{\text{at}} \otimes \mathcal{F}_n \), where

\[ \mathcal{F}_n := C[e^{t \mathcal{L}} [L^2(\mathbb{R}^3 \setminus B_n)]] , \]  \hspace{1cm} (1.21)

with vacuum state denoted by \( \Omega_n \). Cutting off the photon energies below \( \rho_n \) implies that, for every \( n \), the Hamiltonian \( H_n \) has an isolated eigenvalue,

\[ E_n := \inf \sigma(H_n), \]  \hspace{1cm} (1.22)

at the bottom of its spectrum, and we prove the gap above the spectrum to be bigger than, or equal to, \( \frac{1}{2} \rho_n \). The idea of our construction is quite natural, we prove that the sequence of eigenvalues converges and that the limit of it is actually an eigenvalue of the spectrum of \( H \), namely, its ground state energy.

Besides the considerations involving the symmetry \( S \), we need some robust estimates which are standard, but included in this paper in Section 2 for convenience of the reader. The principal properties we prove are bounds for the energy differences \( |E_{n+1} - E_n| \) and the distance \( \text{gap}_n > 0 \) of \( E_n \) to the rest of the spectrum of \( H_n \), which we call the gap at step \( n \), for every \( n \): In Proposition 2.4 we show that

\[ |E_{n+1} - E_n| < g \rho_n, \]  \hspace{1cm} (1.23)

and in Lemma 2.5 we prove that, for every \( n \in \mathbb{N}_0 \),

\[ \text{gap}_n := \inf \{ \sigma(H_n) \setminus \{ E_n \} \} \geq \frac{1}{2} \rho_n, \]  \hspace{1cm} (1.24)

for sufficiently small \( g \), uniformly in \( n \). Eq. (1.23) already implies the convergence of the sequence \( \{ E_n \}_{n \in \mathbb{N}} \). We actually prove (see Remark 3.6) that

\[ E_{gs} = \lim_{n \to \infty} E_n, \]  \hspace{1cm} (1.25)

where we recall that \( E_{gs} \) is the infimum of the spectrum of \( H \). The proof that the limit \( \lim_{n \to \infty} E_n \) yields the ground state energy goes along with proving the
convergence of the ground state projections corresponding to $H_n$, for every $n$, is the main part of our proof. These projections, at each step $n$, are proved to exist and to be rank-one, for sufficiently small $g$ uniformly in $n$: Eq. (1.24) permits us to calculate the projection associated to $E_n$ using Riesz integrals, since it implies that $E_n$ is isolated. We actually define, for every $n \in \mathbb{N}$,

$$
\Gamma_n := \left\{ z \in \mathbb{C} \mid |z - E_n| = \frac{1}{8} \rho_n \right\}
$$

and

$$
P_n := -\frac{1}{2\pi i} \int_{\Gamma_n} \frac{dz}{H_n - z}.
$$

It is not difficult to prove that $P_n$ is a rank-one projection, for every $n$. It follows from the fact that, for sufficiently small $g$ (uniformly in $n$), $\|P_{n+1} - P_n\|$ is strictly smaller than 1 and $P_0$ is rank-one – see Corollary 2.9. As mentioned above, this is proved without using that $G$ is off-diagonal.

The most difficult part of the paper is to prove that the sequence of projections $\{P_n\}_{n \in \mathbb{N}}$ converges (the range of the limiting projection actually consists of ground state eigenvectors of the original Hamiltonian $H$). Of course, the projections $P_n$, for $n \in \mathbb{N}$, act on different Hilbert spaces, but we identify them with projections acting on the full Hilbert space $\mathcal{H}$ by applying tensor products with the vacuum state projections on $\mathcal{F}_n := \mathcal{F}[L^2(B_n)]$,

$$
P_n^\infty := P_n \otimes \left( |\Omega_n^\infty\rangle \langle \Omega_n^\infty | \right),
$$

where $\Omega_n^\infty$ is the vacuum in $\mathcal{F}_n$, notice that $\mathcal{H} = \mathcal{H}_n \otimes \mathcal{F}_n^\infty$.

In Theorem 3.4 we prove that, for sufficiently small $g$ and $\gamma$,

$$
\|P_{n+1}^\infty - P_n^\infty\| \leq \left( \frac{1}{2} \right)^n.
$$

Observe that the bound above is exponentially small in $n$. Actually, in the irrelevant case – if the factor is $1/|k|^{1/2-\mu}$, for some $\mu > 0$, instead of $1/|k|^{1/2} - \mu$ a positive power of $\rho_n$ appears multiplying the right side of Eq. (1.30) (see Remark 2.10). Of course, this term makes the calculations much simpler and direct (actually if we had assumed an infrared regular interaction, Section 2 would basically contain the proof of our main result – see Remark 2.10). The present situation is more complicated and a more subtle argument is required. At this point the symmetries $H$ and $H_n$ possess, play a key role. Namely, with the help of a symmetry operator $S$, see (2.84), which we prove to commute with $H_n$ ($n \in \mathbb{N}_0$), i.e.,
we identify a new conserved quantity of the model. The symmetry $S$ is used to prove that the fact that $\sigma_1$ maps the ground state eigenspace corresponding to $H_{\text{at}}$ to its orthogonal complement holds also true for every member of the sequence $\{H_n\}_{n=0}^{\infty}$ of operators, i.e.,

$$P_n \sigma_1 P_n = 0,$$

(1.31)

for every $n$, see Lemma 2.11.

The proof of Eq. (1.30) is technical and concentrated in Section 3.2, see Lemmas 3.1, 3.2, and 3.3. These Lemmas are collected in the proof of Theorem 3.4, which is our principal demonstration.

Eq. (1.30) implies that the sequence of projections converges, provided that we choose $g > 0$ sufficiently small. Setting

$$P_{gs} := \lim_{n \to \infty} P_n^\infty,$$

(1.32)

we observe that $P_{gs}$ is a rank-one projection (being the limit of rank-one projections) and, most importantly: Any non-zero vector in the range of $P_{gs}$ is an eigenvector of $H$ corresponding to the eigenvalue $E_{gs}$. While this is our main result, we do not give a proof of the simplicity of the eigenvalue $E_{gs}$ here. Note, however, that the semigroup $e^{-\beta H}$ generated by $H$ is known to be positivity improving (in a suitable representation) [13], and from this the uniqueness (non-degeneracy) of the ground state follows from a standard Perron-Frobenius argument, see, e.g., [17, Thm. XIII.44].

### 1.3 Prospective Generalizations

In this paper we assume that the interaction between the atom and the photon field is off-diagonal, and we restrict ourselves to a two-level atom. The generalization to an $N$-level atom is, however, not straightforward, because the mere existence of a symmetry $S_N$ similar to the symmetry $S_2 \equiv S$ of the two-level atom implies severe and unphysical restrictions on the structure of the coupling function $G$. Indeed, the transcription of the proof of Lemma 2.11 would require the symmetry $S_N$ to be invertible and commuting with $H$ and the $N \times N$ coupling function $G(k)$ to be similar (as a matrix) to $-G(k) = S_N G(k) S_N^{-1}$ when conjugated with $S_N$. This is not surprising because of several known negative results and strong requirements on putative ground states, see [2].

For a coupling function $G$ with a bipartite structure, these requirements are fulfilled. Bipartiteness means that the atomic energy levels form two disjoint sets, $A$ and $B$, say, and level transitions $A \to A$ and $B \to B$ are forbidden. (For the two-level atom considered here, $A = \{0\}$ and $B = \{2\}$ and bipartiteness simply means that there are no self-interactions of the atomic orbitals.) There is no physical reason that would justify this assumption, in general. Nevertheless, the
proof of binding presently given can be easily transcribed to the general bipartite situation.

As established here for two-level atoms or elsewhere for other models of NR QED, **binding** states that the ground state energy is an eigenvalue and the ground state vector is an element of the Hilbert space $\mathcal{H} = \mathcal{H}_{at} \otimes \mathcal{F}$ which carries a Fock representation of the canonical commutation relation. We believe that, following an argument originating in work by Fröhlich \[10\] and Pizzo \[15, 16\] and further developed by Chen and Fröhlich \[8\] and by Matte and one of us \[14\], it is possible to establish binding in a more general (and weaker) sense, and we now outline how this could be done on the example of the **Generalized Spin-Boson-Hamiltonian**

$$\hat{H} = H_{at} + H_{ph}(\omega) + \Phi(\hat{G}), \quad (1.33)$$

which is an operator on

$$\mathcal{H} = \mathbb{C}^d \otimes \mathcal{F}. \quad (1.34)$$

The atomic Hamiltonian, $H_{at}$, is a diagonal self-adjoint $d \times d$-matrix whose lowest eigenvalue is simple and $\hat{G}$ is of the form

$$\hat{G}(k) := g \frac{\Lambda(k)}{4\pi \sqrt{\omega(k)}} f(k) M, \quad \forall k \in \mathbb{R}^3, \quad (1.35)$$

where $f$ is as before, and $M$ is a self-adjoint $d \times d$-matrix. Similar to the method applied in this paper, we define an infrared-regularized Hamiltonian $\hat{H}_n$ on

$$\mathcal{H}_n = \mathbb{C}^d \otimes \mathcal{F}_n$$

by replacing $\hat{G}$ by $\hat{G}_n := 1_{\mathbb{R}^3 \setminus B_n} \hat{G}$ and $\omega$ by $\omega_n$ in $(1.33)$, Proposition 2.7 and its proof hold for $\hat{H}_n$, mutatis mutandis. In particular, if $g$ is sufficiently small then there exists a unique normalized ground state $\hat{\phi}_n$ of $\hat{H}_n$, for every $n \in \mathbb{N}_0$.

As opposed to the sequence $\{\phi_n\}_{n=0}^\infty$ of ground states analyzed in this paper, the sequence $\{\hat{\phi}_n\}_{n=0}^\infty$ of ground states does not converge (strongly), but $\hat{\phi}_n \to 0$ weakly, as $n \to \infty$. Yet, as a state on

$$\bigcup_{m=1}^\infty B[\mathcal{H}_m] \otimes \mathcal{F}_\infty \ni A,$$

$$\hat{\omega}(A) = \lim_{n \to \infty} \langle \tilde{\phi}_n, A \tilde{\phi}_n \rangle. \quad (1.36)$$

does exist, using $\tilde{\phi}_n = \hat{\phi}_n \otimes \Omega_\infty$, cf. $(1.29)$. This limit state can be represented as the GNS-vector in a non-Fock representation of the CCR-algebra. The absence of binding in the strict sense is reflected here in the fact that there is no vector $\hat{\phi}_{gs}$ (nor density matrix) in the original Hilbert space $\mathcal{H}$ such that $\hat{\omega}(A) = \langle \hat{\phi}_{gs}, A \hat{\phi}_{gs} \rangle$.

Yet, the nature of the limit in $(1.36)$ can be made more precise. Namely, the conjugation of $\hat{H}_n$ by a unitary operator $U_n$,

$$K_n := U_n \hat{H}_n U_n^*, \quad \Omega_n := U_n \hat{\phi}_n \otimes \Omega_\infty, \quad (1.37)$$
for each $n$, yields new sequences $\{K_n\}_{n=0}^{\infty}$ and $\{\Psi_n\}_{n=0}^{\infty}$ of Hamiltonian operators on $\mathcal{H}$ with ground state energies $E_n$ and unique normalized ground state vectors $\Psi_n$. The main point is that there exists a sequence $\{U_n\}_{n=0}^{\infty}$ of suitably chosen Bogolubov transformations (in fact, even Weyl transformations with a fairly explicit description) such that $K_n \to \hat{K}$ converges in strong resolvent sense to a self-adjoint operator $\hat{K}$ on $\mathcal{H}$ and $\Psi_n \to \Psi_{gs} \in \mathcal{H}$, as $n \to \infty$. The sequence $\{U_n\}_{n=0}^{\infty}$ of Bogolubov transformations, however, does not converge, and even though $K$ can be formally obtained from a shift $a(k) \mapsto a(k) + h(k)$, for a suitable function $h : \mathbb{R} \setminus \{0\} \to \mathbb{C}$, this shift is not unitarily implementable, i.e., there is no unitary operator $U$ on $\mathcal{H}$ such that $K = \hat{U} \hat{H} U^*$. Nevertheless, one may argue that $K$ is the new, renormalized Hamiltonian describing the physics (generating the actual dynamics).

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2 The Sequence of Infrared-Regular Ground State Energies and Projections

In this section we study spectral properties of $H_n$, for every $n \in \mathbb{N}_0$. We start with a brief notation section (see Section 2.1 in which we also state some standard results). Then we estimate the distance between consecutive spectral points $E_{n+1}$ and $E_n$, see Proposition 2.4. Right after, we prove that $E_n$ is isolated from the rest of the spectrum of $H_n$, for every $n \in \mathbb{N}_0$. This is achieved in Proposition 2.7 which quantifies the gap above the ground state energy of $H_n$ and is a direct consequence of Lemma 2.5. The main technical tool in the proof of both, Proposition 2.4 and Proposition 2.7, is Lemma 2.2. Of course, the gap estimates ensure the existence of ground state projections, see Eqs. (1.27) and (2.74). An additional effort permits us to estimate the norm difference of projections $P_n$ and $P_{n+1}$, where
$P_n$ denotes the projection onto the ground state eigenspace of $H_n$. This is derived in Proposition 2.8 which due to $\|P_n - P_{n+1}\| < 1$ implies that all projections $P_n$, $n \in \mathbb{N}_0$, are rank-one (see Corollary 2.9). In Section 2.3 we present a new conserved quantity in the spin-boson model with off-diagonal interaction. We prove that the fact that $\sigma_1$ maps the ground state eigenspace corresponding to $H_n$ to its orthogonal complement is preserved by the flow of operators $\{H_n\}_{n \in \mathbb{N}_0}$, i.e., we prove that $P_n \sigma_1 P_n = 0$, for all $n$, see Lemma 2.11. This is achieved with the help of an operator $S$, see (2.84), which we prove to commute with $H_n$ ($n \in \mathbb{N}_0$) and hence identify a new conserved quantity of the model.

2.1 Notation and Standard Results

For every normed vector space $V$, we denote by $\| \cdot \|_V$ its norm. If $V$ is a Hilbert space, we denote by $\langle \cdot | \cdot \rangle_V$ its inner product. If it is clear, however, from the context, we omit the subscripts $V$.

We introduce some basic notation that we use for the construction of the sequence of eigenvalues $\{E_n\}_{n \in \mathbb{N}_0}$ and ground state projections $\{P_n\}_{n \in \mathbb{N}_0}$. Recalling that $B_n := \{k \in \mathbb{R}^3 | ||k|| < \rho_n\} \subset \mathbb{R}^3$, $\rho_n = \kappa \gamma^n$, (2.38)

we introduce

$$\tilde{\omega}_n(k) := 1_{B_n \setminus B_{n+1}} \omega,$$

$$\tilde{G}_n(k) := 1_{B_n \setminus B_{n+1}} G$$

and the Fock spaces

$$\tilde{F}_n := \mathcal{F}[L^2[B_n \setminus B_{n+1}]],$$

(2.40)

with vacuum states

$$\Omega_{L^2[B_n \setminus B_{n+1}]} \equiv \tilde{\Omega}_n.$$

The projections onto the one-dimensional subspaces generated by the vectors $\Omega$, $\Omega_n$ and $\tilde{\Omega}_n$ are denoted by

$$P_{\Omega}, \quad P_{\Omega_n}, \quad P_{\tilde{\Omega}_n},$$

(2.42)

respectively. We define

$$\tilde{H}_n := H_n \otimes 1_{\tilde{F}_n} + 1_{\mathcal{H}_n} \otimes H_{ph}(\tilde{\omega}_n),$$

(2.43)

as operators on (a suitable domain in) $\mathcal{H}_{n+1}$. We observe that $\inf \sigma(H_n) = \inf \sigma(\tilde{H}_n)$ and denote

$$E_n := \inf \sigma(H_n) = \inf \sigma(\tilde{H}_n).$$

(2.44)
The distance (gap) from $E_n$ to the rest of the spectrum of $H_n$ (respectively $\tilde{H}_n$) is given by

$$\text{gap}_n := \inf \{ \sigma(H_n) \setminus \{E_n\} \} - E_n$$

(2.45)

and

$$\tilde{\text{gap}}_n := \inf \{ \sigma(\tilde{H}_n) \setminus \{E_n\} \} - E_n,$$

(2.46)

respectively. The following basic estimate is frequently used in this paper (see [5, 6] for a proof):

**Lemma 2.1.** Let $\rho > 0$ be arbitrary. For all $F \in L^2(\mathbb{R}^3; \mathbb{C})$ with $\omega^{-\frac{1}{2}}F \in L^2(\mathbb{R}^3; \mathbb{R})$,

$$\left\| \Phi(F) \left( H_{ph}(1_{\text{supp}(F)} \omega) + \rho \right)^{-\frac{1}{2}} \right\| \leq 2 \left( \|\omega^{-1/2}F\|_{L^2} + \rho^{-1/2}\|F\|_{L^2} \right),$$

where $\Phi(F)$ is defined as in (1.11).

Since we assume $\|f\|_{L^\infty} \leq 1$, we immediately get by Lemma 2.1 that, for every $n \in \mathbb{N}_0$,

$$\left\| \Phi(\tilde{G}_n) \left( H_{ph}(\tilde{\omega}_n) + \rho_n \right)^{-\frac{1}{2}} \right\| \leq g \rho_n^{\frac{1}{2}}. \quad (2.48)$$

### 2.2 The Sequence of Ground State Eigenvalues and Projections

In this section we estimate the distance between consecutive spectral infima $E_{n+1}$ and $E_n$, see Proposition 2.3. Right after we prove that $E_n$ is isolated from the rest of the spectrum of $H_n$, for every $n \in \mathbb{N}_0$. This is achieved in Proposition 2.7 which is a direct consequence of Lemma 2.5. Of course, the gap estimates ensure the existence of ground state projections, see Eqs. (1.27) and (2.74). An additional effort permits us to estimate the norm-difference of consecutive projections $P_n$ and $P_{n+1}$ in Proposition 2.8. In particular we prove that $\|P_n - P_{n+1}\| < 1$, which implies that all projections $P_n$, $n \in \mathbb{N}_0$, are rank-one (see Corollary 2.9).

**Lemma 2.2.** For every $n \in \mathbb{N}_0$,

$$H_{n+1} + \rho_n \geq H_n + (1 - g) \left( H_{ph}(\tilde{\omega}_n) + \rho_n \right)$$

(2.49)

holds true in the sense of quadratic forms.
Proof: Let $\psi \in \mathcal{H}_{n+1}$ be a normalized vector in the domain of $H_{n+1}$. We calculate

$$\langle \psi | H_{n+1} \psi \rangle = \langle \psi | H_n \psi \rangle + \langle \psi | \Phi(\tilde{G}_n) \psi \rangle + \langle \psi | H_{ph}(\tilde{\omega}_n) \psi \rangle. \tag{2.50}$$

Next, we set

$$A := 1 + \left( H_{ph}(\tilde{\omega}_n) + \rho_n \right)^{-1/2} \Phi(\tilde{G}_n) \left( H_{ph}(\tilde{\omega}_n) + \rho_n \right)^{-1/2} \tag{2.51}$$

and notice that

$$\langle \psi | \Phi(\tilde{G}_n) \psi \rangle + \langle \psi | (H_{ph}(\tilde{\omega}_n) + \rho_n) \psi \rangle \tag{2.52}$$

and

$$= \left( \left( H_{ph}(\tilde{\omega}_n) + \rho_n \right)^{1/2} \psi \right) \left( A \left( H_{ph}(\tilde{\omega}_n) + \rho_n \right)^{1/2} \psi \right).$$

As (see Eq. (2.48))

$$\left\| \left( H_{ph}(\tilde{\omega}_n) + \rho_n \right)^{-1/2} \Phi(\tilde{G}_n) \left( H_{ph}(\tilde{\omega}_n) + \rho_n \right)^{-1/2} \right\| \leq \rho_n^{-1/2} \left\| \Phi(\tilde{G}_n) \left( H_{ph}(\tilde{\omega}_n) + \rho_n \right)^{-1/2} \right\| \leq g, \tag{2.53}$$

we obtain that

$$A \geq 1 - g, \tag{2.54}$$

and, therefore, using (2.51) we get

$$\langle \psi | \Phi(\tilde{G}_n) \psi \rangle + \langle \psi | (H_{ph}(\tilde{\omega}_n) + \rho_n) \psi \rangle \geq (1 - g) \langle \psi | (H_{ph}(\tilde{\omega}_n) + \rho_n) \psi \rangle. \tag{2.55}$$

Eqs. (2.50) and (2.55) imply Eq. (2.49).

The same argument as in the proof of Lemma 2.2 yields the following quadratic form estimate.

**Lemma 2.3.** For every $n \in \mathbb{N}_0$, we have that

$$H + \rho_n \geq H_n + (1 - g) \left( H_{ph}(\tilde{\omega}_n) + \rho_n \right). \tag{2.56}$$

**Proposition 2.4 (Energy Differences).** Suppose $g < 1$. For every $n \in \mathbb{N}_0$, we have that

$$|E_{n+1} - E_n| \leq g \rho_n. \tag{2.57}$$
Proof: First notice that, for every normalized vector \( \phi \in H_n \) in the domain of \( H_n \),
\[
\psi = \phi \otimes \tilde{\Omega}_n \in H_{n+1}
\]
and
\[
E_{n+1} \leq \langle \psi | H_{n+1} \psi \rangle_{H_{n+1}} = \langle \phi | H_n \phi \rangle_{H_n}. \tag{2.58}
\]
Taking the infimum over such \( \phi \)'s we get
\[
E_{n+1} \leq E_n. \tag{2.59}
\]
Now we take a normalized vector \( \psi \) in the domain of \( H_{n+1} \). We notice that (we recall that \( g < 1 \))
\[
(1 - g) \left( H_{ph}(\tilde{\omega}_n) + \rho_n \right) \geq (1 - g) \rho_n \tag{2.60}
\]
and use Lemma 2.2 to obtain
\[
\langle \psi | H_{n+1} \psi \rangle_{H_{n+1}} \geq \langle \psi | H_n \psi \rangle_{H_{n+1}} - g \rho_n \geq E_n - g \rho_n, \tag{2.61}
\]
from which we get
\[
E_{n+1} \geq E_n - g \rho_n. \tag{2.62}
\]
Eqs. (2.59) and (2.62) imply (2.57).

Lemma 2.5. Suppose \( g < \frac{1}{2} \gamma \). For every \( n \in \mathbb{N}_0 \), we have that
\[
\operatorname{gap}_{n+1} \geq \min \left( \operatorname{gap}_n, (1 - g) \rho_{n+1} \right) - g \rho_n. \tag{2.63}
\]
Proof: We use the min-max principle to estimate \( \operatorname{gap}_{n+1} \) as
\[
\operatorname{gap}_{n+1} \geq \sup_{\psi \in H_{n+1} \setminus \{0\}} \inf_{\phi \perp \psi, \| \phi \|=1} \langle \phi | (H_{n+1} - E_{n+1}) \phi \rangle, \tag{2.64}
\]
where \( \phi \) is additionally assumed to lie in the domain of \( H_{n+1} \). We take \( \phi \) as in Eq. (2.64) and utilize Lemma 2.2 to obtain:
\[
\langle \phi | (H_{n+1} - E_{n+1}) \phi \rangle \geq \langle \phi | (H_n - E_n) \phi \rangle + E_n - E_{n+1} - \rho_n \tag{2.65}
\]
\[
+ \left( 1 - g \right) \langle \phi | (H_{ph}(\tilde{\omega}_n) + \rho_n) \phi \rangle
\]
\[
= \langle \phi | \left( H_n - E_n + (1 - g) H_{ph}(\tilde{\omega}_n) \right) \phi \rangle
\]
\[
+ E_n - E_{n+1} - g \rho_n.
\]
We temporarily set
\[
Q := H_n - E_n + (1 - g) H_{ph}(\tilde{\omega}_n) \tag{2.66}
\]
and observe that \( \inf \sigma(Q) = 0 \). Denoting by \( \text{gap}(Q) \) the distance between 0 and the rest of the spectrum of \( Q \), we arrive at

\[
\text{gap}_{n+1} \geq \text{gap}(Q) + E_n - E_{n+1} - g\rho_n, \tag{2.67}
\]

where we use (2.65), (2.66), and the min-max principle applied to \( H_{n+1} \) and \( Q \). Using the fact that \( H_n \) and \( H_{\phi_h(n)} \) act on different factors in the tensor product decomposition \( H_{n+1} = H_n \otimes F_n \), we readily get

\[
\sigma(Q) \subset \{0\} \cup \left[ \min \left( \text{gap}_n, (1-g)\rho_{n+1} \right), \infty \right) \tag{2.68}
\]

(recall that \( g < 1 \)) and, therefore,

\[
\text{gap}(Q) \geq \min \left( \text{gap}_n, (1-g)\rho_{n+1} \right). \tag{2.69}
\]

Eqs. (2.59), (2.67) and (2.69) imply Eq. (2.63), here we use that \( g < \frac{1}{2} \gamma \).

**Remark 2.6.** We notice that the spectral theorem directly implies that, for every \( n \in \mathbb{N}_0 \):

\[
\widetilde{\text{gap}}_n = \min \left( \text{gap}_n, \rho_{n+1} \right). \tag{2.70}
\]

As \( G_0 = 0 \), the spectrum of \( H_0 \) can be calculated explicitly, with the result that

\[
\sigma(H_0) = \{0\} \cup [\rho_0, \infty), \tag{2.71}
\]

and, therefore,

\[
\text{gap}_0 = \rho_0 = \kappa. \tag{2.72}
\]

We simplify Eq. (2.63) by assuming some hypothesis on \( g \) and \( \gamma \). Taking, for example, \( g < \frac{1}{4} \gamma \) and \( \gamma < \frac{1}{4} \), we inductively obtain, from (2.63), that \( \text{gap}_n \geq \frac{1}{2}\rho_n \), for all \( n \in \mathbb{N}_0 \). It also follows, from Remark 2.6, that \( \text{gap}_n = \rho_{n+1} \). Then, we arrive at the following proposition.

**Proposition 2.7 (Gaps).** Suppose that \( g < \frac{1}{4} \gamma \) and \( \gamma < \frac{1}{4} \). Then

\[
\text{gap}_n \geq \frac{1}{2}\rho_n, \quad \widetilde{\text{gap}}_n = \rho_{n+1}, \tag{2.73}
\]

for all \( n \in \mathbb{N}_0 \).
Proposition 2.7 permits us to define $P_n$ as in (1.27). It also allows defining

$$\tilde{P}_n := -\frac{1}{2\pi i} \int_{\Gamma_{n+1}} \frac{dz}{\tilde{H}_n - z},$$

(2.74)

where the contour $\Gamma_n$ is defined in Eq. (1.26), provided $g > 0$ obeys $g < \frac{1}{64} \gamma$, because in this case $E_n$ is the only spectral point of $\tilde{H}_n$ in the interior of $\Gamma_{n+1}$ (see Propositions 2.4 and 2.7). Notice that

$$\tilde{P}_n = P_n \otimes \tilde{P}_{\Omega_n}.$$ (2.75)

In the next lemma we estimate the norm-difference of $P_{n+1}$ and $\tilde{P}_n$.

**Proposition 2.8.** Suppose that $g < \frac{1}{64} \gamma$ and $\gamma < \frac{1}{2}$. Then

$$\| P_{n+1} - \tilde{P}_n \| \leq \frac{16}{\gamma} g \leq \frac{1}{4}.$$ (2.76)

**Proof:** The second inequality in (2.76) is obvious. We use the second resolvent identity and (1.27) and (2.74) to get

$$P_{n+1} - \tilde{P}_n = \frac{1}{2\pi i} \int_{\Gamma_{n+1}} \frac{1}{H_{n+1} - z} \phi(\tilde{G}_n) \frac{1}{H_n - z} dz.$$ (2.77)

Next we estimate

$$\| \phi(\tilde{G}_n) \frac{1}{H_n - z} \| \leq \| \phi(\tilde{G}_n) (H_{ph}(\tilde{\omega}_n) + \rho_n)^{-1/2} \|$$

$$\cdot \frac{1}{\| (H_{ph}(\tilde{\omega}_n) + \rho_n)^{1/2} \frac{1}{H_n - z} \|},$$

(2.78)

where we use (2.48). Functional calculus of self-adjoint operators allows us to compute

$$\frac{1}{H_n - z} = \sup_{s \in \sigma(H_n)} \sup_{r \in (0, \rho_{n+1})} \left| \frac{(r + \rho_n)^{1/2} \frac{1}{s + r - z}}{s + r - z} \right|.$$ (2.79)

The definition of $\Gamma_{n+1}$,

$$\Gamma_{n+1} := \left\{ z \in \mathbb{C} \mid |z - E_{n+1}| = \frac{1}{8} \rho_{n+1} \right\}$$ (2.80)
(see (1.26)), Propositions 2.4 and 2.7 and \( g \rho_n \leq \frac{7}{64} \rho_n = \frac{1}{64} \rho_{n+1} \), then lead us to

\[
|s - z| \geq \frac{1}{16} \rho_{n+1}, \quad \forall s \in \sigma(H_n),
\]

\[
|s + r - z| \geq \frac{r}{2} \geq \frac{1}{2} \left( \frac{r}{2} + \frac{\rho_{n+1}}{2} \right), \quad \forall s \in \sigma(H_n), r \in [\rho_{n+1}, \infty).
\]

Eqs. (2.79) and (2.81) imply that

\[
\left\| (H_{ph}(\tilde{\omega}_n) + \rho_n)^{1/2} \frac{1}{H_n - z} \right\| \leq \frac{16}{\gamma \rho_n^{1/2}}, \quad (2.82)
\]

while (2.78) and (2.82) imply that

\[
\left\| \frac{1}{H_{n+1} - z} \Phi(\tilde{G}_n) \frac{1}{H_n - z} \right\| \leq \frac{128g}{\gamma \rho_{n+1}}. \quad (2.83)
\]

We prove (2.76) using (2.77), (2.83), and the definition of \( \Gamma_{n+1} \).

\( \square \)

**Corollary 2.9.** Suppose that \( g < \frac{1}{64} \gamma \) and \( \gamma < \frac{1}{2} \). Then \( P_n \) is a rank-one orthogonal projection, for every \( n \in \mathbb{N}_0 \).

**Proof:** As \( G_0 = 0 \), it is straightforward to verify that \( P_0 \) and hence \( \tilde{P}_0 \) are of rank-one. Proposition 2.8 implies that \( \| P_1 - \tilde{P}_0 \| < 1 \) and, therefore, \( P_1 \) is of rank-one, too. We proceed inductively to conclude that \( P_n \) is of rank one for every \( n \in \mathbb{N}_0 \). The self-adjointness of \( H_n \) ensures the self-adjointness of \( P_n \).

\( \square \)

**Remark 2.10.** In case that \( G \) was infrared regular, i.e., if \( \| |k|^{1/2-\mu} G(k) \| \) was bounded at \( k = 0 \), for some \( \mu > 0 \), then in Eq. (2.78) and, consequently, in Eq. (2.76) we would gain a positive power of \( \rho_n \). This would immediately imply the convergence of the projections \( \{ P_n \} \) \( n \in \mathbb{N}_0 \), see Eq. (1.29), which in turn implies the existence of the ground state. In other words: If we had an infrared regular interaction, this section would basically contain the proof of the existence of a ground state, our main result. Not assuming infrared regularity complicates the matter significantly. The next section addresses this complication.

### 2.3 Invariant Subspaces Due to the Symmetry

In this section we present a new conserved quantity in the spin-boson model with off-diagonal interaction. We prove that the fact that \( \sigma_1 \) maps the ground state eigenspace corresponding to \( H_{at} \) to its orthogonal complement (it is off-diagonal) is preserved by the flow of operators \( \{ H_n \} \) \( n \in \mathbb{N}_0 \), i.e., we prove that \( P_n \sigma_1 P_n = 0 \), for all \( n \), see Lemma 2.11. This is achieved with the help of a symmetry operator.
\[ S := \sigma_3 (-1)^{N_{ph}}. \] (2.84)

**Lemma 2.11.** For every \( n \in \mathbb{N}_0 \), we have that
\[ P_n \sigma_1 P_n = 0, \quad \tilde{P}_n \sigma_1 \tilde{P}_n = 0. \] (2.85)

**Proof:** The second equality in (2.85) follows from Eq. (2.75) and the first equality. We prove that \( P_n \sigma_1 P_n = 0 \). A direct computation shows that \([\sigma_3, H_{at}] = 0 \) (here \([\cdot, \cdot]\) denotes the commutator). Furthermore, using the pull-through formulae,
\[ a(k)N_{ph} = (N_{ph} + 1)a(k) \quad \text{and} \quad a^*(k)(N_{ph} + 1) = N_{ph}a^*(k), \] (2.86)
we conclude that
\[ [S, H_n] = 0. \] (2.87)

As \( S^2 = 1 \), we observe that
\[ \text{Tr}(SP_n \sigma_1 P_n S) = \text{Tr}(P_n \sigma_1 P_n). \] (2.88)

Eq. (2.87) implies that \( S \) commutes with \( P_n \) and it is straightforward to verify that it anti-commutes with \( \sigma_1 \). Hence we have
\[ \text{Tr}(SP_n \sigma_1 P_n S) = -\text{Tr}(S^2 P_n \sigma_1 P_n) = -\text{Tr}(P_n \sigma_1 P_n). \] (2.89)

Then we obtain from (2.88) and (2.89) that \( \text{Tr}(P_n \sigma_1 P_n) = 0 \). Since \( P_n \) is a rank-one projection, we conclude that \( P_n \sigma_1 P_n = 0 \).

\[ \square \]

### 2.4 Further Estimates

In this section we derive some estimates that are consequences of the computations in the present section, but will be used in our main section, Section 3.

**Lemma 2.12.** Suppose that \( g < \frac{1}{16} \gamma \) and \( \gamma < \frac{1}{2} \). Take \( z \in \Gamma_{n+1} \) (see Eq. (1.26)). The following norm bounds hold true,
\[ \left\| \Phi(\tilde{G}_n) \frac{1}{H_n - z} \right\| \leq \frac{16}{\gamma} g, \] (2.90)
\[ \left\| \Phi(\tilde{G}_n) \frac{1}{H_{n+1} - z} \right\| \leq \frac{32}{\gamma} g. \] (2.91)

**Proof:** We use that, see (2.48),
\[ \left\| \Phi(\tilde{G}_n) \left( H_{\text{ph}}(\omega_n) + \rho_n \right)^{-1/2} \right\| \leq g \rho_n^{1/2}, \] (2.92)
and Eq. (2.82) to prove the first inequality in (2.90). To prove the second inequality we use a Neumann series,
\[ \Phi(\tilde{G}_n) \frac{1}{H_{n+1} - z} = \Phi(\tilde{G}_n) \frac{1}{H_n - z} \sum_{n=0}^{\infty} \left( - \frac{\Phi(\tilde{G}_n)}{H_n - z} \right)^n, \] (2.93)
the first inequality in (2.90), and the fact that \( \frac{16}{\gamma} g \leq \frac{1}{2} \).

**Corollary 2.13.** Suppose that \( g < \frac{1}{64} \gamma \) and \( \gamma < \frac{1}{2} \). Take \( z \) in the interior of \( \Gamma_{n+1} \) (see Eq. (1.26)). Then the following estimates hold true
\[ \left\| \Phi(\tilde{G}_n) \frac{1}{H_n - z} (1 - \tilde{P}_n) \right\| \leq \frac{16}{\gamma} g, \] (2.94)
\[ \left\| \Phi(\tilde{G}_n) \frac{1}{H_{n+1} - z} (1 - P_{n+1}) \right\| \leq \frac{32}{\gamma} g. \] (2.95)

**Proof:** To prove the first inequality notice that \( \frac{1}{H_n - z} (1 - \tilde{P}_n) \) is analytic in the interior of \( \Gamma_{n+1} \). Then the claim follows from the maximum modulus principle. The second inequality is proved in the same way.

### 3 Convergence of the Sequence of Ground State Projections

In this section we prove our main result: We demonstrate that the sequence of ground state projections \( (P_n^{\infty})_{n \in \mathbb{N}_0} \) [see (1.29)], converges and that the limit of it is a rank-one projection whose range consists of ground state eigenvectors corresponding to \( E_{gs} \). The convergence of the ground state projections is the content of Theorem 3.3. The proof that the limit of this sequence corresponds to the ground state projection of \( H \) is derived in Theorem 3.5. The convergence of the sequence of the projections \( P_n \) rests on the fact that \( P_n \sigma_1 P_n = 0 \), established in Lemma 2.11. On the technical level, this property is used in Eqs. (3.111) and (3.127).
below. The special difficulties of our proof come from the fact that the coupling function $G(k)$ behaves as $|k|^{-1/2}$, for small $k$, as explained before. In fact, for a more regular coupling function the results in Section 2 suffice (basically) to prove existence of the ground state. We start this section with introducing new notation in Section 3.1. Most of technical tools we need to prove our main results are collected in Section 3.2. A long line of arguments is split onto three lemmas: Lemma 3.1, Lemma 3.2, and Lemma 3.3. The idea is to bound the norm $\|P_{n+1} - \tilde{P}_n\|$ in terms of the quantity $\|R_n \sigma_1 P_n\|$, that can be recursively estimated in terms of $g$ and $\gamma$. Lemmas 3.1, 3.2, and 3.3 establish the main bounds we need to reach the recursive relation we are looking for. We put together all results of Section 3.2 in the proof of Theorem 3.4 in Section 3.3. Theorem 3.4 gives the convergence of the sequence of ground state projections $\{P_n^{\infty}\}_{n \in \mathbb{N}_0}$. The limit of this sequence is denoted by $P_{gs}$, its range consists of all ground state vectors corresponding to the fully interacting operator $H$, as it is demonstrated in Theorem 3.5.

### 3.1 Notation

For every projection $P$ we denote by $P^\perp := 1 - P$, the complement of $P$. We define

$$R_n(z) := (H_n - z)^{-1}, \quad (3.96)$$

$$\tilde{R}_n(z) := (\tilde{H}_n - z)^{-1}, \quad (3.97)$$

$$R_n(z)^\perp := R_n(z) P_n^\perp, \quad (3.98)$$

$$\tilde{R}_n(z)^\perp := \tilde{R}_n(z) \tilde{P}_n^\perp. \quad (3.99)$$

whenever $z$ is not in the spectrum of the corresponding operator. If we project out the eigenspace corresponding to $E_n$, we can take $z = E_n$ and set

$$R_n(E_n)^\perp \equiv R_n^\perp \equiv R_n P_n^\perp, \quad (3.100)$$

$$\tilde{R}_n(E_n)^\perp \equiv \tilde{R}_n^\perp \equiv \tilde{R}_n \tilde{P}_n^\perp.$$

We finally introduce the function $\eta_n : \mathbb{R}^3 \to \mathbb{C}$ by

$$\eta_n(k) := g 1_{B_n \setminus B_{n+1}} \frac{\Lambda(k)}{4 \pi \sqrt{\omega(k)}} f(k), \quad (3.101)$$

and note that (see (2.9))

$$\tilde{G}_n = \eta_n \sigma_1.$$

We define the field operator $\Phi(\eta_n) := a^* (\eta_n) + a(\eta_n)$, as in (1.11).
3.2 Key Estimates

Most of the technical tools we need to prove our main results are collected in this section. The idea is to bound the norm \( \| P_{n+1} - \tilde{P}_n \| \) in terms of the quantity \( \| R_n^\perp \sigma_1 P_n \| \) that can be recursively estimated in terms of \( g \) and \( \gamma \). Lemmas 3.1, 3.2 and 3.3 establish the main bounds we need to reach the recursive relation we are looking for.

Lemma 3.1. Suppose that \( g < \frac{1}{64} \gamma \) and \( \gamma < \frac{1}{2} \). It follows, for every \( n \in \mathbb{N}_0 \), that

\[
\| P_{n+1} - \tilde{P}_n \| \leq 4 \| P_{n+1}^\perp \tilde{P}_n \|. \quad (3.102)
\]

Proof: A direct computation shows that

\[
P_{n+1} - \tilde{P}_n = (P_{n+1} - \tilde{P}_n)^2 (\tilde{P}_n - \tilde{P}_n) + \tilde{P}_n (\tilde{P}_n - P_{n+1}) \tilde{P}_n + \tilde{P}_n (\tilde{P}_n - P_{n+1}) \tilde{P}_n. \quad (3.103)
\]

In fact, to prove (3.103) we expand the right hand side of (3.103) and use that

\[
\tilde{P}_n P_{n+1} \tilde{P}_n + \tilde{P}_n P_{n+1} \tilde{P}_n = \tilde{P}_n \tilde{P}_n = 0, \quad (3.104)
\]

then we utilize the following identities

\[
P_{n+1} \tilde{P}_n + \tilde{P}_n P_{n+1} \tilde{P}_n - \tilde{P}_n P_{n+1} \tilde{P}_n = P_{n+1}(1 - \tilde{P}_n) + (\tilde{P}_n P_{n+1} - (1 - \tilde{P}_n)(1 - P_{n+1})) \tilde{P}_n = P_{n+1} + (-1 + \tilde{P}_n) \tilde{P}_n = P_{n+1}. \quad (3.105)
\]

Using Proposition 2.8 and (3.103) we obtain

\[
\| P_{n+1} - \tilde{P}_n \| \leq 2 \| (P_{n+1} - \tilde{P}_n)^2 \| + 2 \| \tilde{P}_n (P_{n+1} \tilde{P}_n - P_{n+1}) \tilde{P}_n \| \leq \frac{1}{2} \| P_{n+1} - \tilde{P}_n \| + 2 \| P_{n+1} \tilde{P}_n \|. \quad (3.106)
\]

Solving this for \( \frac{1}{2} \| P_{n+1} - \tilde{P}_n \| \), we get (3.102).

\[ \square \]

Lemma 3.2. Suppose that \( g < \frac{1}{64} \gamma \) and \( \gamma < \frac{1}{2} \). It follows, for every \( n \in \mathbb{N}_0 \), that

\[
\| P_{n+1}^\perp \tilde{P}_n \| \leq 2g \rho_n \| R_{n+1}^\perp (P_n \otimes P_{\tilde{\omega}_n}) \sigma_1 \left( P_n \otimes P_{\tilde{\omega}_n} \right) \|. \quad (3.107)
\]
Proof: Multiplying and dividing by $H_{n+1} - E_{n+1}$ and using that $(\tilde{H}_n - E_n)\tilde{P}_n = 0$, we get

\[
P_{n+1}^\perp \tilde{P}_n = R_{n+1}^\perp [H_{n+1} - E_{n+1}] \tilde{P}_n = R_{n+1}^\perp [\Phi(\tilde{G}_n) + E_n - E_{n+1}] \tilde{P}_n = R_{n+1}^\perp \Phi(\tilde{G}_n) \tilde{P}_n + [E_n - E_{n+1}] R_{n+1}^\perp \tilde{P}_n.
\]  

(3.108)

Propositions 2.4 and 2.7 and the spectral theorem yield

\[
\left\| (E_n - E_{n+1}) R_{n+1}^\perp P_{n+1}^\perp R_{n+1}^\perp \tilde{P}_n \right\| \leq \frac{1}{16} \left\| P_{n+1}^\perp \tilde{P}_n \right\|.
\]  

(3.109)

Eqs. (3.108) and (3.109) imply that

\[
\left\| P_{n+1}^\perp \tilde{P}_n \right\| \leq 2 \left\| R_{n+1}^\perp \Phi(\tilde{G}_n) \tilde{P}_n \right\|.
\]  

(3.110)

The key symmetry property of our model implies that (see Lemma 2.11) $\Phi(\tilde{G}_n) P_n = P_n^\perp \Phi(\tilde{G}_n) P_n$, which in turn, together with the fact that $\Phi(\tilde{G}_n)[1_{\mathcal{H}_n} \otimes P_{\Omega_n}] = [1_{\mathcal{H}_n} \otimes P_{\Omega_n}^\perp] \Phi(\tilde{G}_n)[1_{\mathcal{H}_n} \otimes P_{\Omega_n}^\perp]$, gives (see also Eqs. (2.75) and (1.12), (3.101))

\[
\Phi(\tilde{G}_n) \tilde{P}_n = \left( P_n^\perp \otimes P_{\Omega_n}^\perp \right) \Phi(\tilde{G}_n) \tilde{P}_n = \left( P_n^\perp \otimes P_{\Omega_n}^\perp \right) \sigma_1 \left( P_n \otimes P_{\Omega_n}^\perp \right) \Phi(\eta_n) \tilde{P}_n.
\]  

(3.111)

Finally, we use that $\eta_n$ is supported in $B_n$ and thus $\left\| \Phi(\eta_n) \tilde{P}_n \right\| = \left\| P_n \right\| \cdot \left\| a^*(\eta_n) P_{\Omega_n} \right\| \leq g \rho_n$—see Eq. (1.8)—and Eqs. (3.110) and (3.111) to arrive at Eq. (3.107). 

Lemma 3.3. Suppose that $g < \frac{1}{64} \gamma$ and $\gamma < \frac{1}{2}$. It follows, for every $n \in \mathbb{N}_0$, that

\[
\left\| P_{n+1}^\perp - \tilde{P}_n \right\| \leq 48g \rho_n \left\| R_{n+1}^\perp \sigma_1 P_n \right\|.
\]  

(3.112)

Proof: By the second resolvent identity and Cauchy’s integral formula, we have that

\[
\tilde{P}_n = \frac{-1}{2\pi i} \int_{\Gamma_{n+1}} (E_n - z)^{-1} - (\tilde{H}_n - z)^{-1} dz
\]  

(3.113)

\[
\tilde{P}_n = \frac{-1}{2\pi i} \int_{\Gamma_{n+1}} (E_n - z)^{-1}(\tilde{H}_n - E_n)(\tilde{H}_n - z)^{-1} dz
\]  

\[
\tilde{P}_n = \frac{-1}{2\pi i} \int_{\Gamma_{n+1} - E_n} (-z)^{-1}(\tilde{H}_n - E_n)(\tilde{H}_n - E_n - z)^{-1} dz
\]  

\[
\tilde{P}_n = \frac{-1}{2\pi i} \int_{\Gamma_{n+1} - E_{n+1}} (-z)^{-1}(\tilde{H}_n - E_n)(\tilde{H}_n - E_n - z)^{-1} dz,
\]
where we deform the contour from $\Gamma_{n+1} - E_n$ to $\Gamma_{n+1} - E_{n+1}$ using Propositions 2.4 and 2.7. Eq. (3.113) implies that

$$\tilde{R}_n = -\frac{1}{2\pi i} \int_{\Gamma_{n+1} - E_{n+1}} (-z)^{-1}(\tilde{H}_n - E_n - z)^{-1} dz.$$  

(3.114)

Similarly, we get

$$R_{n+1} = -\frac{1}{2\pi i} \int_{\Gamma_{n+1} - E_{n+1}} (-z)^{-1}(H_{n+1} - E_{n+1} - z)^{-1} dz.$$  

(3.115)

Hence, using the second resolvent identity again, we obtain

$$R_{n+1} - \tilde{R}_n = -\frac{1}{2\pi i} \int_{\Gamma_{n+1} - E_{n+1}} \left[ z^{-1}(H_{n+1} - E_{n+1} - z)^{-1}(\Phi(\tilde{G}_n) + E_n - E_{n+1}) \cdot (\tilde{H}_n - E_n - z)^{-1} \right] dz.$$  

(3.116)

We notice that

$$(H_{n+1} - E_{n+1} - z)^{-1} = (H_{n+1} - E_{n+1} - z)^{-1}P_{n+1} - z^{-1}P_{n+1}$$  

(3.117)

and

$$(\tilde{H}_n - E_n - z)^{-1} = (\tilde{H}_n - E_n - z)^{-1}\tilde{P}_n - z^{-1}\tilde{P}_n.$$  

(3.118)

Inserting Eqs. (3.117) and (3.118) in (3.116) and using the Cauchy's integral formula for the derivative of a function, we arrive at (notice that the first terms in the right hand side of Eqs. (3.117) and (3.118) are analytic in the interior of $\Gamma_{n+1}$)

$$R_{n+1} - \tilde{R}_n = P_{n+1}(\Phi(\tilde{G}_n) - E_{n+1} + E_n)\tilde{R}_n + \tilde{R}_n^2P_{n+1}(\Phi(\tilde{G}_n) - E_{n+1} + E_n)\tilde{P}_n - R_{n+1}(\Phi(\tilde{G}_n) - E_{n+1} + E_n)\tilde{R}_n.$$  

(3.119)

Adding $\tilde{R}_n$ in Eq. (3.119) and applying it to $\left(P_n^\perp \otimes P_{\tilde{\Omega}_n}^\perp\right)\sigma_1\left(P_n \otimes P_{\tilde{\Omega}_n}^\perp\right) = P_n^\perp\sigma_1P_n \otimes P_{\tilde{\Omega}_n}^\perp$ leads us to

$$R_{n+1}(P_n^\perp \otimes P_{\tilde{\Omega}_n}^\perp)\sigma_1(P_n \otimes P_{\tilde{\Omega}_n}^\perp) = \left[ 1 + P_{n+1}(\Phi(\tilde{G}_n) - E_{n+1} + E_n)\tilde{R}_n - R_{n+1}(\Phi(\tilde{G}_n) - E_{n+1} + E_n)\tilde{P}_n \right]$$
where we used that \( \tilde{P}_n\left( P_n^+ \otimes P_{\tilde{\Omega}_n}^\perp \right) = 0 \).

Using Eq. (3.120), together with Corollary 2.13, and Propositions 2.4 and 2.7, we obtain

\[
\|R_{n+1}^+ \left( P_n^+ \otimes P_{\tilde{\Omega}_n}^\perp \right) \sigma_1 \left( P_n \otimes P_{\tilde{\Omega}_n}^\perp \right) \| \leq 3 \|\tilde{R}_n^+ P_n^+ \sigma_1 P_n \otimes P_{\tilde{\Omega}_n}^\perp \| \leq 6 \| R_n^+ \sigma_1 P_n \|, \tag{3.121}
\]

since

\[
\|\tilde{R}_n^+ P_n^+ \sigma_1 P_n \otimes P_{\tilde{\Omega}_n}^\perp \|
\leq (1 + (16 + 32 + 2 + 1) \frac{g}{\gamma}) \|\tilde{R}_n^+ P_n^+ \sigma_1 P_n \otimes P_{\tilde{\Omega}_n}^\perp \|
\leq 3 \|\tilde{R}_n^+ P_n^+ \sigma_1 P_n \otimes P_{\tilde{\Omega}_n}^\perp \| \leq 6 \| R_n^+ \sigma_1 P_n \|, \tag{3.122}
\]

and we conclude by putting together Eqs. (3.102), (3.107), (3.121) and (3.122), which leads us to Eq. (3.112).

\[\Box\]

### 3.3 Main Results: Convergence of the Regularized Ground State Projections and Existence of the Ground State of \( H \)

Here we prove our principal theorems. We collect all results of Section 3.2 (Lemmas 3.1, 3.2, and 3.3) to prove our first main theorem, Theorem 3.4, which is the most difficult and technical result in the present paper. Theorem 3.4 gives the convergence of the sequence of ground state projections \( \{P_n^\infty\}_{n \in \mathbb{N}_0} \). The limit of this sequence is denoted by \( P_{gs} \), its range actually consists of ground state eigenvectors corresponding to the full energy operator \( H \), as it is demonstrated in Theorem 3.5.

#### 3.3.1 Convergence of the regularized Ground State Projections

**Theorem 3.4.** Suppose that \( g < \frac{1}{64} \gamma \) and \( \gamma < \frac{1}{2} \). It follows, for every \( n \in \mathbb{N}_0 \), that

\[
\|P_{n+1} - \tilde{P}_n\| \leq 48g \left( \gamma + 147g \right)^n, \tag{3.123}
\]
In particular, if additionally $\gamma \leq 1/8$, then
\[
\|P_{n+1} - \tilde{P}_n\| \leq \left(\frac{1}{2}\right)^n,
\] (3.124)
and the sequence $(P_n^\infty)_{n \in \mathbb{N}_0}$, see (1.29), converges to a rank-one projection
\[
P_{gs} = \lim_{n \to \infty} P_n^\infty.
\] (3.125)

**Proof:** First we notice that
\[
R_n^\perp \sigma_1 P_n = R_n^\perp \sigma_1 \tilde{P}_{n-1} + R_n^\perp \sigma_1 [P_n - \tilde{P}_{n-1}].
\] (3.126)
Next we consider the following computations, which use Eq. (3.119) with $n - 1$ replacing $n$,
\[
R_n^\perp \sigma_1 \tilde{P}_{n-1} = \left[1 + P_n (\tilde{\Phi}(G_{n-1}) - E_n + E_{n-1}) \tilde{R}_{n-1}^\perp \right.
\]
\[
- R_n^\perp (\tilde{\Phi}(G_{n-1}) - E_n + E_{n-1}) \tilde{P}_{n-1}^\perp \left.\right] 
\]
\[
\cdot \tilde{R}_{n-1}^\perp [P_{n-1} \otimes P_{\Omega_{n-1}}] \sigma_1 \tilde{P}_{n-1} 
\]
\[
= \left[1 + P_n (\tilde{\Phi}(G_{n-1}) - E_n + E_{n-1}) \tilde{R}_{n-1}^\perp \right.
\]
\[
- R_n^\perp (\tilde{\Phi}(G_{n-1}) - E_n + E_{n-1}) \tilde{P}_{n-1}^\perp \left.\right] \cdot [R_{n-1}^\perp \sigma_1 P_{n-1} \otimes P_{\Omega_{n-1}}].
\] (3.127)
Here we use the key symmetry
\[
\tilde{P}_{n-1}^\perp \sigma_1 \tilde{P}_{n-1} = 0,
\] (3.128)
see Lemma 2.11 to drop the contribution from the second line in Eq. (3.119). We also use $\tilde{P}_{n-1}^\perp = P_{n-1}^\perp + P_{n-1} \otimes P_{\Omega_{n-1}}^\perp$ to prove that
\[
\tilde{P}_{n-1}^\perp \sigma_1 \tilde{P}_{n-1} = P_{n-1}^\perp \otimes P_{\Omega_{n-1}}^\perp \sigma_1 \tilde{P}_{n-1}.
\]
With the help of Eq. (3.127), together with Corollary 2.13 and Propositions 2.4 and 2.7 we obtain
\[
\|R_n^\perp \sigma_1 \tilde{P}_{n-1}\| \leq \left(1 + 51 \frac{g}{\gamma}\right) \cdot \|R_{n-1}^\perp \sigma_1 P_{n-1}\|,
\] (3.129)
where we argue as in Eq. (3.121). Lemma 3.3, together with Eqs. (3.126) and (3.129) (see also Proposition 2.7), imply (notice that $\gamma < 1/2$)
\[
\|R_n^\perp \sigma_1 P_n\| \leq \left(1 + 96 + 51 \frac{g}{\gamma}\right) \|R_{n-1}^\perp \sigma_1 P_{n-1}\|.
\] (3.130)
Then we inductively get, using Lemma 3.3 and (3.130), that
\[
\| P_{n+1} - \tilde{P}_n \| \leq 48g \rho_n \left( 1 + 147 \frac{g}{\gamma} \right)^n \| R_0^\perp \sigma_1 P_0 \|
\]
\[
\leq 48 \left( \gamma + 147g \right)^n,
\]
where we use (2.72), recalling that \( \kappa < 1 \). This establishes Eq. (3.123). Eqs. (3.124) and (3.125) are direct consequences of (3.123). Clearly \( P_{gs} \) being limit of rank-one projections (see Corollary 2.9) is rank-one.

3.3.2 Construction of the Ground State Projection of \( H \)

**Theorem 3.5.** Suppose that \( g < \frac{1}{64} \gamma \) and \( \gamma < \frac{1}{8} \). Then the range of \( P_{gs} \) is contained in the domain of \( H \) and
\[
HP_{gs} = E_{gs} P_{gs},
\]
where \( E_{gs} = \lim_{n \to \infty} E_n \).

**Proof.** We denote
\[
\omega_n^\infty(k) := 1_{B_n} \omega, \quad G_n^\infty(k) := 1_{B_n} G,
\]
and define \( H_{ph}(\omega_n^\infty) \) and \( \Phi(G_n^\infty) \) as in Eqs. (1.7) and (1.11) on \( F_n^\infty \) (see (1.28)). Since \( H_{ph}(\omega_n^\infty) P_n^\infty = 0 \), see (1.29), we have that
\[
HP_n^\infty = E_n P_n^\infty + \Phi(G_n^\infty) P_n^\infty.
\]
As
\[
\lim_{n \to \infty} E_n P_n^\infty = E_{gs} P_{gs},
\]
and
\[
\lim_{n \to \infty} \| \Phi(G_n^\infty) P_n^\infty \| = \lim_{n \to \infty} \| G_n^\infty \|_{L^2} = 0,
\]
we obtain that
\[
\lim_{n \to \infty} HP_n^\infty = E_{gs} P_{gs}.
\]
Since the sequence of projections \( (P_n^\infty)_{n \in \mathbb{N}} \) converges, we can find \( N \in \mathbb{N} \) and a vector \( \phi \in \mathcal{H} \) such that \( \phi_n := P_n^\infty \phi \neq 0 \), for all \( n \geq N \), and \( \psi := P_{gs} \phi \neq 0 \). Then we have
\[
\psi = \lim_{n \to \infty} \phi_n, \quad E_{gs} \psi = \lim_{n \to \infty} H \phi_n,
\]
(3.138)
where we use Eq. (3.137). As $H$ is a closed operator, $\psi$ belongs to its domain and

$$H\psi = E_{gs}\psi.$$  \hspace{1cm} (3.139)

The fact that $P_{gs}$ is rank-one and Eq. (3.139) imply that the range of $P_{gs}$ is contained in the domain of $H$ and Eq. (3.132).

\[\square\]

**Remark 3.6.** It is not difficult to prove that $E_{gs} = \lim_{n \to \infty} E_n$ is actually the infimum of the spectrum of $H$. In fact, Lemma 2.3 implies that

$$\inf \sigma(H) + \rho_n \geq E_n + (1 - g)\rho_n,$$

for all $n$. Therefore $\inf \sigma(H) \geq E_{gs}$. As $E_{gs}$ is itself a spectral point of $H$, it equals $\inf \sigma(H)$.

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