Time limit for using the semi-infinite heat transfer solutions: a novel approach

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Abstract

In the present study, authors are conceptually showing that the classically considered time limit to use the semi-infinite approximate solutions is highly conservative, particularly at the internal location(s) inside the finite heat conduction medium. Accordingly, a new length scale, which accounts the heat propagation from the far-field boundary condition as well, is proposed to ascertain the prolonged time limit. The proposed time limit is obtained by comparing the temperature distribution in a finite heat conduction problem with its equivalent semi-infinite model. Overall, three standard one-dimensional heat conduction problems are analysed and the proposed time limit is found to be valid in all three problems. The new time limit will certainly boost the utility of the semi-infinite solutions and rejuvenate the interest of the scientific community in such solutions.

1. Introduction

Heat propagation in the solids is studied extensively by physicists and thermal engineers. In general, heat propagation takes a finite amount of time, and any change occurring at the boundary is felt inside the medium only after a certain time. This lagging behavior is a manifestation of the thermal diffusivity and is an important parameter in appreciating the time-bound heat transfer solutions. As thermal diffusivity is finite for any given material, along with the time limit, the length scale of the medium becomes imperative to examine the heat propagation. Based on length scale, heat conduction mediums are categorized as ‘finite’ and ‘semi-infinite’. As Semi-Infinite (SI) approximation of the medium simplifies the problems, it is explored extensively for heat conduction in finite mediums as well. The time limit up-to which SI approximation remains valid is classically known as ‘penetration time’. This penetration time \( t_p \) is a function of the thermal diffusivity \( \alpha \) and length of the medium \( L \) \[1\].

\[
t_p = f(\alpha, L)
\]

In a transient heat conduction problem, thermal conditions in the domain are time-dependent. Accordingly, to distinguish the commencement of heat transfer and determine the associated penetration time, one needs to define some mathematical criterion. A specified non-dimensional temperature change from the initial condition is generally used for this purpose \[1, 2\]. Within the penetration time, the SI approximation of the medium can be invoked, and the obtained results are acceptable in most of the practical applications. A conservative time limit to use such approximate solutions is obtained for a temperature jump boundary condition (represents \( h \rightarrow \infty \)), and given as:

\[
t_p = 0.075 \left( \frac{L^2}{\alpha} \right)
\]

In the present paper, authors are proposing a new time limit to use the SI approximate solutions at internal location(s) in the medium in a more effective manner. Unlike the classical approach where the effect of only one boundary condition in one-dimensional heat conduction is considered, the effect of both the boundary conditions on the applicable time limit is accounted in this study. Accordingly, the new time limit is evaluated by
viewing it as a function of local length scale (the length scale \( L \) in equation (2) is modified for this purpose).

Finally, one-dimensional transient heat conduction in a finite medium with no heat generation is considered as an example to develop and validate the stretched applicability of the SI solutions at internal location(s).

2. Applicability of the semi-infinite approximation

Heat transfer in a theoretically infinite medium dictates that the thermal conditions remain unaffected of any change occurring at the opposite boundary of the medium. If similar conditions persist at a particular far-field boundary in the case of a finite medium as well, the medium can be treated as infinite in that specific direction. However, in a finite medium, any change occurring at a particular boundary (front boundary) will modify the thermal conditions at the opposite boundary (rear boundary) after a certain time \( t_p \) (see figure 1). As heat conduction problems are boundary value problems, any change occurring at the boundary will eventually affect the overall solution. In the SI model, the effect of the front boundary is already accounted in the mathematical model. Therefore, any discrepancy in the SI solutions and the exact solution is because of the modification at the rear boundary, the boundary where the condition is originally considered as invariant in the SI model [2]. Therefore, the applicable time of the SI solutions within the medium depends on the propagation of information from the altered rear boundary condition rather than the heat propagation from the front boundary only.

Two important conclusions come-out from the above arguments. Firstly, the applicable time limit for SI approximate solutions is different than the penetration time at internal location(s) and depends on the change occurring at the rear boundary as well. Secondly, the time limit is a function of the local length scale within the medium, unlike a classically defined constant value (refer to equation (2)). One needs to revisit the classical time limit and should adopt a comprehensive approach to account the effect of conditions prevailing at the boundaries on the time limit up-to which the SI approximate solutions remain applicable.

3. Proposed time limit

It is discussed that the time scale for the applicability of the SI approximate solutions depends on the heat propagation from both the boundaries. Many a time, information at one boundary, say at the front boundary, are primarily sought in practical applications with another boundary open to the atmosphere [3]. Accordingly, any change occurring at the front boundary initially propagates through the medium and reaches the rear boundary. Classically defined penetration time is required in this process. Once the heat propagated up-to-the rear boundary, it starts altering the thermal condition there. These altered conditions are not taken into account in the SI model. Therefore, when the effect of the altered condition reaches the location of interest in the medium, the SI approximate solution starts giving erroneous results. This heat propagation phenomenon at three internal locations, \( x_1 \), \( x_2 \), and \( x_3 \), is depicted in figure 1. In this way, the information at the front boundary needs to propagate an additional length from the rear boundary to the location of interest (along with the usual total length of the medium) to start invalidating the SI solutions. Therefore, equation (2) provides the time scale of using the SI approximation up-to-the rear boundary, i.e., throughout the medium. However, in a situation when information is needed up-to-certain internal location(s), then the time scale for using SI approximate solutions should be given as follows:

\[
t_{SI} = 0.075 \left( \frac{L_{eq}^2}{\alpha} \right)
\]

here, \( t_{SI} \) is the time scale for using the SI approximate solutions, and \( L_{eq} \) is the proposed length scale.

Figure 1. Shows heat propagation from the front to the rear boundary, and subsequently after the classical penetration time \( t_p \), backward heat propagation up-to-the point of interest.
\( L_{eq} \) is the sum of length of the medium \( (L) \) and distance of the internal location of interest from the back boundary \( (L_{bs}) \).

\[
L_{eq} = L + L_{bs}
\] (4)

It is important to note that the solution obtained using the SI approximation for the time scale given by equation (3) is applicable in the region \( (L - L_{bs}) \) from the front boundary. Although the solution is restrictive in nature, it is particularly useful in inverse heat transfer techniques [4, 5].

4. Validation of the proposed time limit

Three one-dimensional transient heat transfer problems are considered in a finite medium to validate the proposed time limit. A constant temperature jump is considered at the front boundary in all the three problems \( (T^* = 1) \), whereas a constant temperature \( (T = 0) \), an insulated \( (\dot{q} = 0) \), and a constant heat transfer coefficient type of boundary condition \( (B_i = hL/k = 1) \) (one at a time) are considered at the rear boundary (refer to figure 2). Here, \( B_i \) is Biot number at the rear surface, \( h \) is the heat transfer coefficient, and \( k \) is the thermal conductivity of the solid material. Three distinct conditions are considered at the rear side so that their effect on the time limit of the SI solution can be ascertained. Isotropic medium with a constant initial temperature \( (T_0) \) throughout the medium is assumed to simplify the problem. The same medium is assumed as SI in extent to find the temperature distribution via the SI model. In the SI model, boundary condition only at the front boundary \( (T^* = 1) \) is considered. Analytical solutions of all the above problems for temperature distribution \( T(x, t) \) are readily available in [1–3, 6, 7]. Finally, temperature values from the finite and SI models are compared, and the following equation is used to find the new time limit:

\[
\left| \frac{T_f (\hat{x}, \hat{t}) - T_{SI} (\hat{x}, \hat{t})}{T (x = 0, t) - T (x, t = 0)} \right| \leq 0.01
\] (5)

here, \( T_f (\hat{x}, \hat{t}) \) is the temperature at a distance \( \hat{x} \) and time \( \hat{t} \). Finite and SI mediums are denoted by subscripts \( f \) and \( SI \), respectively.

Till the time inequality in equation (5) satisfies, the SI solutions closely mimic heat transfer phenomena occurring inside the finite medium. Accordingly, the time limit at different internal locations are calculated for the three one-dimensional heat transfer problems and plotted in figure 3. It is found that the proposed time limit better represents the applicability of SI solutions, while the classical time limit is clearly conservative. Improvement in the time limit is particularly prominent nearer to the boundary of interest. These results confirm the reasoning provided in section 3 for the increase in the time limit. Therefore, we recommend using the new time limit proposed in this study to employ the SI approximate solutions at different locations within the finite medium.
5. Conclusions

A new time limit is proposed and validated for using the semi-infinite approximate heat transfer solutions at internal locations in the medium. Unlike the classical penetration time, the proposed time limit accounts the effect of both boundary conditions in one-dimensional heat conduction and found to be a function of local length scale. A significantly higher time limit is obtained near the boundary of interest. The extended time limit will supplement the use of semi-infinite solutions in practical applications, particularly in inverse problems.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Data availability

Upon request, data supporting the study will be made available by the corresponding author.

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