ON SOME PRODUCTS OF FINITE GROUPS

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Abstract A classical result of Baer states that a finite group $G$ which is the product of two normal supersoluble subgroups is supersoluble if and only if $G'$ is nilpotent. In this article, we show that if $G = AB$ is the product of supersoluble (respectively, $w$-supersoluble) subgroups $A$ and $B$, $A$ is normal in $G$ and $B$ permutes with every maximal subgroup of each Sylow subgroup of $A$, then $G$ is supersoluble (respectively, $w$-supersoluble), provided that $G'$ is nilpotent. We also investigate products of subgroups defined above when $A \cap B = 1$ and obtain more general results.

Keywords: finite groups; residuals; semidirect products; supersoluble groups; direct product

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1. Introduction

All groups considered here will be finite.

A significant number of articles investigating the properties of groups expressible as a product of two supersoluble subgroups were published since the 1957 paper by Baer [2] in which he proved that a normal product $G = AB$ of two supersoluble subgroups...
A and B is supersoluble provided that the derived subgroup $G'$ is nilpotent. There has been many generalizations of this theorem. Instead of having normal subgroups, certain permutability conditions were imposed on the factors. The case in which A permutes with every subgroup of $B$ and $B$ permutes with every subgroup of $A$, that is, when $G$ is a mutually permutable product of $A$ and $B$, is in fact one of the most interesting cases and has been investigated in detail (see [5] for a thorough review of results in this context and also [1] for general results on products).

In this article, we study a weak form of a normal product arising quite frequently in the structural study of mutually permutable products and appears not to have been investigated in detail.

**Definition 1.1.** Let $G = AB$ be a product of subgroups $A$ and $B$. We say that $G$ is a weak normal product of $A$ and $B$ if

(a) $A$ is normal in $G$.

(b) $B$ permutes with all the maximal subgroups of Sylow subgroups of $A$.

As an important first step in the study of weak normal products $G = AB$ and motivated by the mutually permutable case, we analyse the situation $A \cap B = 1$. In this case, they are semidirect products of $A$ and $B$.

**Definition 1.2.** Let the group $G = AB$ be the weak normal product of $A$ and $B$ with $A$ normal in $G$. We say that $G$ is a weak direct product of $A$ and $B$ if $A \cap B = 1$. In this case, we write $G = [A]B$.

We study these products when the factors are supersoluble and widely supersoluble and analyse the behaviour of the residuals associated to these classes of groups. Recall that a widely supersoluble group, or $w$-supersoluble group for short, is defined as a group $G$ such that every Sylow subgroup of $G$ is $\mathbb{P}$-subnormal in $G$ (a subgroup $H$ of a group $G$ is $\mathbb{P}$-subnormal in $G$ whenever either $H = G$ or there exists a chain of subgroups $H = H_0 \leq H_1 \leq \cdots \leq H_{n-1} \leq H_n = G$, such that $|H_i:H_{i-1}|$ is a prime for every $i = 1, \ldots, n$).

The class of $w$-supersoluble groups, denoted $w\mathfrak{U}$, is a subgroup-closed saturated formation containing the subgroup-closed saturated formation $\mathfrak{U}$ of all supersoluble groups. Moreover $w$-supersoluble groups have a Sylow tower of supersoluble type (see [8, Corollary]).

Our first aim is to show that the saturated formations of all supersoluble groups and w-supersoluble groups are closed under the formation of weak direct products.

**Theorem A.** Let $G = [A]B$ be a weak direct product of $A$ and $B$. If $A$ and $B$ belong to $\mathfrak{U}$, then $G$ is also supersoluble.

Theorem A will be very useful in the proofs of Theorem B and Theorem C. Also we obtain the following result as a corollary.

**Corollary A.** Let $G = [A]B$ be a weak direct product of $A$ and $B$. If $A$ and $B$ belong to $w\mathfrak{U}$, then $G$ is $w$-supersoluble.
Our second aim is to show that the product of the supersoluble (respectively, w-supersoluble) residuals of the factors of weak direct products is just the supersoluble (respectively, w-supersoluble) residual of the group.

**Theorem B.** Suppose that $\mathfrak{F} = \mathfrak{U}$ or $\mathfrak{F} = w\mathfrak{U}$. Let $G = [A]B$ be a weak direct product of $A$ and $B$. Then

$$G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}.$$ 

We now analyse the behaviour of weak normal products with respect to the formations of all supersoluble and w-supersoluble groups. Our next result shows that Baer’s theorem can be generalized in this new direction:

**Theorem C.** Let $G = AB$ be a weak normal product of $A$ and $B$. If $G'$ is nilpotent, $A$ is normal in $G$ and $A, B \in \mathfrak{U}$, then $G \in \mathfrak{U}$.

As a corollary, we obtain the result for $w\mathfrak{U}$-groups.

**Corollary B.** Let the group $G = AB$ be a weak normal product of $w\mathfrak{U}$-subgroups $A$ and $B$. If $G'$ is nilpotent and $A$ is normal in $G$, then $G$ belongs to $w\mathfrak{U}$.

Our second objective is to investigate the residuals of weak normal products. Unfortunately, it does not follow that $G^{\mathfrak{U}} = A^{\mathfrak{U}}B^{\mathfrak{U}}$ when $G$ is a weak normal product as the following examples show. Example 1.3(ii) generalizing (i) was communicated to the authors by the referee:

**Example 1.3.**

(i) Let

$$A = \langle g_2, g_4, g_5, g_6, g_7 \mid g_2^3 = g_4^3 = g_5^3 = g_6^3 = g_7^3 = 1, g_4^{g_2} = g_4g_6, g_5^{g_2} = g_5g_7, g_6^{g_2} = g_6, g_7^{g_2} = g_7, g_4^{g_5} = g_5, g_6^{g_5} = g_6, g_7^{g_5} = g_7, g_6^{g_7} = g_6, g_7^{g_7} = g_7 \rangle.$$ 

Let $Q = \langle b \rangle \cong C_4$ act on $A$ via

$$g_2^b = g_2, \quad g_4^b = g_4g_5, \quad g_5^b = g_4g_5^2, \quad g_6^b = g_6g_7, \quad g_7^b = g_6g_7^2.$$ 

Let $G = [A]Q$ be the corresponding semidirect product.

Note that $A' = \Phi(A) = \langle g_6, g_7 \rangle$. Let $A_0 = \langle g_4, g_7 \rangle$. Then $A_0$ is not a normal subgroup of $A$ but is normalized by $Q$. Let $B = A_0(b)$, then $\text{Core}_G(B) = 1$. Furthermore, $B$ permutes with the 13 maximal subgroups of $A$. The supersoluble residual of $G$ is $\langle g_4, g_5, g_6, g_7 \rangle$, giving a quotient isomorphic to $C_{12}$. Consequently, $G^{\mathfrak{U}} \neq A^{\mathfrak{U}}B^{\mathfrak{U}}$. This group corresponds to $\text{SmallGroup}(972, 406)$ of GAP.
(ii) Suppose \( p \) is a prime number and \( n \) is a positive integer, where \( n \) is not a multiple of \( p \), and the order of \( p \) modulo \( n \) is 2. Let \( \mathbb{F}_{p^2} \) be the Galois field of order \( p^2 \), and note that \( n \) is a factor of \( p^2 - 1 \), so there is an element \( \beta \) of multiplicative order \( n \) in \( \mathbb{F}_{p^2} \). Let \( V = U_0 \oplus U_1 \) be a vector space of dimension 2 over \( \mathbb{F}_{p^2} \), where \( U_0 \) and \( U_1 \) are one-dimensional \( \mathbb{F}_{p^2} \)-subspaces of \( V \). Take elements \( a \) and \( b \) in the general linear group \( \text{GL}_2(\mathbb{F}_{p^2}) \), with

\[
\begin{bmatrix}
1 & 1 \\
0 & 1 
\end{bmatrix}, \quad b = \begin{bmatrix}
\beta & 0 \\
0 & \beta 
\end{bmatrix}.
\]

Then \( V \) can be regarded as an \( \mathbb{F}_{p^2}\langle a, b \rangle \)-module. Let \( G = [V]\langle a, b \rangle \) be the corresponding semidirect product. Consider the following subgroups of \( G \):

\[
P = \langle a \rangle \simeq C_p, \quad A = VP, \quad Q = \langle b \rangle \simeq C_n, \quad B = U_0Q.
\]

As in (i), \( A \) is normal in \( AB = G \), \( A' = \Phi(A) = [V, P] = U_1 \), \( U_0 \) is an \( \mathbb{F}_pQ \)-simple module, \( B' = [U_0, Q] = U_0 \), the core of \( B \) in \( G \) is 1, the number of maximal subgroups of \( A \) is \( (p^3 - 1)/(p - 1) = p^2 + p + 1 \), \( A \) is the unique Sylow \( p \)-subgroup of \( G \) and all the maximal subgroups of \( A \) permute with \( B \), while the supersoluble residual of \( G \) is \( G^{\Delta} = V \), with \( G/G^{\Delta} \simeq PQ \simeq C_{pn} \) and \( G^{\Delta} \neq A^{\Delta}B^{\Delta} \). This construction can be carried out when \( p = 2 \) and \( n = 3 \), giving an Example with \( |G| = 2^5 3 = 96 \); moreover, there is a maximal subgroup of \( A \) which does not permute with the Sylow subgroup \( Q \) of \( B \).

We will use Theorem C to prove the following result (note that Example 1.3(ii) suggests the permutability hypothesis in Theorem D and Corollary D):

**Theorem D.** Let the group \( G = AB \) be a product of the subgroups \( A \) and \( B \). Assume that \( A \) is a normal subgroup of \( G \) and every Sylow subgroup of \( B \) permutes with every maximal subgroup of every Sylow subgroup of \( A \). If \( G' \) is nilpotent, then \( G^{\Delta} = A^{\Delta}B^{\Delta} \).

An immediate consequence is:

**Corollary C.** Let the group \( G = AB \) be a product of the subgroups \( A \) and \( B \). Assume that \( A \) is a normal subgroup of \( G \) and every Sylow subgroup of \( B \) permutes with every maximal subgroup of every Sylow subgroup of \( A \). If \( G' \) is nilpotent, then \( G^{\Delta} = A^{\Delta}B^{\Delta} \).

Denote by \( \mathfrak{N} \) the class of all nilpotent groups. A nice result of Monakhov [7, Theorem 1] states that if \( G = AB \) is the mutually permutable product of the supersoluble subgroups \( A \) and \( B \), then \( G^{\Delta} = (G')^{\mathfrak{N}} = [A, B]^{\mathfrak{N}} \). We prove an analogue of this result for weak normal products.

**Corollary D.** Let \( G = AB \) be a weak normal product of the supersoluble subgroups \( A \) and \( B \). If \( A \) is normal in \( G \), we have that \( G^{\Delta} = (G')^{\mathfrak{N}} = [A, B]^{\mathfrak{N}} \).
2. Preliminary results

It is easy to see that factor groups of weak normal products are also weak normal products. For weak direct products, we have the following:

**Lemma 2.1.** Let $G = [A]B$ be a weak direct product of $A$ and $B$.

(a) If $N$ is a normal subgroup of $G$ such that $N \leq A$ or $N \leq B$, then $G/N = [AN/N](BN/N)$ is a weak direct product of $AN/N$ and $BN/N$.

(b) If $K$ is a subgroup of $B$, then $[A]K$ is a weak direct product of $A$ and $K$.

**Proof.** (a) Let $H/N$ be a Sylow $p$-subgroup of $AN/N$. Then $H/N = PN/N$, where $P$ is a Sylow $p$-subgroup of $A$. Let $K/N$ be a maximal subgroup of $H/N$. Then $K = K \cap PN = N(P \cap K)$ and $K/N = N(P \cap K)/N$. Thus,

$$p = |PN/N : (P \cap K)N/N| = |P|/|N| |P \cap K \cap N| / |P \cap N| |P \cap K||N||.$$  

Hence, $P \cap K$ is a maximal subgroup of $P$. Then $B$ permutes with $P \cap K$, and so $BN/N$ permutes with $K/N$. Therefore, $G/N = [AN/N](BN/N)$ is a weak direct product of $AN/N$ and $BN/N$.

(b) Let $K$ be any proper subgroup of $B$ and $H$ be any maximal subgroup of a Sylow subgroup of $A$. By the hypotheses, we have $HB = BH$ and so $H = H(A \cap B) = A \cap HB$. Since $A$ is normal in $G$, it follows that $H$ is normal in $HB$ and so $B$ normalizes $H$. Hence, $K$ permutes with $H$. Therefore, $[A]K$ is a weak direct product of $A$ and $K$.

Our second lemma contains some of the properties of $P$-subnormal subgroups.

**Lemma 2.2.** [8, Lemma 1.4] Let $G$ be a soluble group and $H$ and $K$ two subgroups of $G$. The following properties hold:

(i) If $H$ is $P$-subnormal in $G$ and $N$ is normal in $G$, then $HN/N$ is $P$-subnormal in $G/N$.

(ii) If $N$ is normal in $G$ and $HN/N$ is $P$-subnormal in $G/N$, then $HN$ is $P$-subnormal in $G$.

(iii) If $H$ is $P$-subnormal in $K$ and $K$ is $P$-subnormal in $G$, then $H$ is $P$-subnormal in $G$.

3. Supersoluble and w-supersoluble residuals

We start this section by proving Theorem A.

**Proof of Theorem A.** Assume that the result is false, and let $G$ be a counterexample of minimal order. Clearly, $G$ is soluble and $A$ and $B$ are proper subgroups of $G$. Let $N$ be a minimal normal subgroup of $G$ contained in $A$, then applying Lemma 2.1(a),
\( G/N = [A/N](BN/N) \) is a weak direct product of \( A/N \) and \( BN/N \). By the minimality of \( G \), \( G/N \in \mathfrak{N} \). Since the class of all supersoluble groups is a saturated formation, there exists a unique minimal normal subgroup \( N \) of \( G \) contained in \( A \), \( N \) a \( p \)-group for some prime \( p \), \(|N| > p \) and \( \Phi(A) = 1 \). Since \( A \) is supersoluble, \( A \) has a normal Sylow subgroup, and since \( N \) is the unique minimal normal subgroup of \( G \) contained in \( A \), it follows that \( \Phi(A) \) is a \( p \)-group and \( \Phi(A) \) is an elementary abelian Sylow \( p \)-subgroup of \( A \).

Assume that \( A \) is not a \( p \)-group. Then \( \Phi(A) \) is a completely reducible \( A \)-module, and so \( \Phi(A) = N \times Z \), for some \( A \)-module \( Z \). Let \( L \) be a minimal normal subgroup of \( A \) contained in \( N \). Then \( N = L \times D \), for some \( A \)-module \( D \). Then \( \Phi(A) = L \times DZ \), and \( DZ \) is a maximal subgroup of \( \Phi(A) \) because \( L \) is of prime order. Therefore, \( E = DZ \) permutes with \( B \). Hence, \( DZ = A \cap (DZ)B \), and so \( DZ \) is normalized by \( B \). Since \( DZ \) is also normalized by \( A \), it follows that \( DZ \) is a normal subgroup of \( G \). The minimality of \( N \) forces \( D = 1 \) and so \( N \) is of prime order, which is a contradiction. Consequently, \( A \) is an elementary abelian \( p \)-group. Note that \( A \) cannot be cyclic since \(|N| > p \). Let \( 1 \neq X \) be a maximal subgroup of \( A \). Arguing as above, we have that \( X \) is normal in \( XB \) so that \( X \) is normalised by \( B \). Hence \( X \) is normal in \( G \) because \( A \) is abelian. Therefore, \( N \) is contained in \( X \), and so \( N \leq \Phi(A) = 1 \), our final contradiction. \( \square \)

**Proof of Corollary A.** Assume, by way of contradiction, that the result fails, and let \( G \) be a counterexample of least order. Clearly, \( G \) is soluble and \( A \) and \( B \) are proper subgroups of \( G \). Since the class of all \( w \)-supersoluble groups is a saturated formation, we can argue as in Theorem A to conclude that there exists a unique minimal normal subgroup \( N \) of \( G \) contained in \( A \), and \( N \) is a \( p \)-group for some prime \( p \). Moreover, \( \Phi(A) = 1 \), and \( A_p = \Phi(A) \) is the Sylow \( p \)-subgroup of \( A \). By the minimality of \( G \), \( G/N \) is \( w \)-supersoluble. Let \( P \) be a Sylow \( p \)-subgroup of \( G \). Then \( P/N \) is \( P \)-subnormal in \( G/N \). By Lemma 2.2(ii), \( P \) is \( P \)-subnormal in \( G \). Suppose that for every prime \( q \neq p \) dividing \(|G|\) and every Sylow \( q \)-subgroup \( B_q \) of \( B \), we have that \( AB_q \) is a proper subgroup of \( G \). Let \( A_q \) be a Sylow \( q \)-subgroup of \( A \) such that \( G_q = A_qB_q \) is a Sylow \( q \)-subgroup of \( G \). Since \( G/N \) is \( w \)-supersoluble, it follows that \( G_qN \) is \( P \)-subnormal in \( G \). By Lemma 2.1(b), \( AB_q \) satisfies the hypotheses of the theorem. Hence, \( AB_q \) is \( w \)-supersoluble by the choice of \( G \). Thus, \( G_qN \leq AB_q \) is \( w \)-supersoluble. Consequently, \( G_q \) is \( P \)-subnormal in \( G_qN \), which is \( P \)-subnormal in \( G \). Applying Lemma 2.2(iii), \( G_q \) is \( P \)-subnormal in \( G \). Therefore, the Sylow subgroups of \( G \) are \( P \)-subnormal in \( G \), and so \( G \) is \( w \)-supersoluble, a contradiction.

Thus, we may assume there exists \( q \neq p \) such that \( G = AB_q \). Let \( T = A_pG_q = (A_pA_q)B_q \). Since \( A \) is normal in \( G \), we have that \( A_q \) is normal in \( G_q \) and then \( A_pA_q \) is normalized by \( B_q \). Moreover, \( A_pA_q \) is a \( w \)-supersoluble metanilpotent subgroup of \( G \). By [8, Theorem 2.13(1)], \( A_pA_q \) is supersoluble. It is clear that \( T \) is a weak direct product of the supersoluble subgroups \( A_pA_q \) and \( B_q \). Applying Theorem A, it follows that \( T \) is supersoluble. Therefore, \( T \) is \( w \)-supersoluble. But \( G_qN \leq T \), which is \( w \)-supersoluble. Thus, \( G_q \) is \( P \)-subnormal in \( G_qN \), which is \( P \)-subnormal in \( G \). Again the application of Lemma 2.2(iii) yields \( G_q \) is \( P \)-subnormal in \( G \). If \( G_r \) is a Sylow \( r \)-subgroup of \( G \) for some prime \( r \neq p, q \), then \( G_r \) is contained in \( A \) and so \( G_r \) is \( P \)-subnormal in \( A \). Since \( A \) is also \( P \)-subnormal in \( G \), we have that \( G_r \) is \( P \)-subnormal in \( G \). Consequently, every Sylow subgroup of \( G \) is \( P \)-subnormal in \( G \), and \( G \) is \( w \)-supersoluble. This final contradiction completes the proof of the corollary. \( \square \)
Proof of Theorem B. Suppose that the result is not true, and let $G$ be a minimal counterexample. Then

(i) $A \in \mathfrak{F}$, $B^\delta \neq 1$, $\text{Core}_G(B) = 1$ and $G^\delta = B^\delta N$ for every minimal normal subgroup $N$ of $G$ such that $N \leq A$.

Let $N$ be a minimal normal subgroup of $G$ such that $N \leq A$ or $N \leq B$. Then $G/N = [AN/N](BN/N)$ is a weak direct product of $AN/N$ and $BN/N$ by Lemma 2.1(a).

The minimal choice of $G$ implies that $G^\delta N/N = (A^\delta N/N)(B^\delta N/N)$, that is, $G^\delta N = A^\delta B^\delta N$. Since $G/G^\delta \in \mathfrak{F}$, $AG^\delta /G^\delta$ and $BG^\delta /G^\delta$ also belong to $\mathfrak{F}$ and then $A^\delta \leq G^\delta$ and $B^\delta \leq G^\delta$. If $G^\delta \cap N = 1$, then $G^\delta = A^\delta B^\delta(G^\delta \cap N) = A^\delta B^\delta$, a contradiction. Hence, $G^\delta = A^\delta B^\delta N$ for every minimal normal subgroup $N$ of $G$ such that $N \leq A$ or $N \leq B$. If $A^\delta \neq 1$, then there exists a minimal normal subgroup $N$ of $G$ contained in $A^\delta$ because $A^\delta$ is normal in $G$. This contradiction yields $A \in \mathfrak{F}$ and $G^\delta = B^\delta N$ for every minimal normal subgroup $N$ of $G$ such that $N \leq A$ or $N \leq B$. If $B^\delta \in \mathfrak{F}$, then $G = G^\delta B^\delta(G^\delta \cap N) = A^\delta B^\delta$, a contradiction.

(ii) $F(A)$ is a Sylow $p$-subgroup of $A$, where $p$ is the largest prime dividing $|A|$.

Since $A \in \mathfrak{F}$, it follows that $A$ is a Sylow tower group of supersoluble type. In particular, $1 \neq O_p(A)$ is the Sylow $p$-subgroup of $A$, where $p$ is the largest prime dividing $|A|$. If $F(A)$ is not a $p$-group, then $1 \neq O_p(A) \leq O_q(G)$. Let $N_1$ be a minimal normal subgroup of $G$ contained in $O_p(A)$, and let $N_2$ be a minimal normal subgroup of $G$ contained in $O_q(A)$. Then $G^\delta = B^\delta N_1 = B^\delta N_2$, which is a contradiction since $B^\delta \cap N_1 = B^\delta \cap N_2 = 1$.

Therefore, $F(A) = O_p(A)$ is the Sylow $p$-subgroup of $A$.

(iii) $G$ is soluble, $AK$ belongs to $\mathfrak{F}$ for every proper subgroup $K$ of $B$; in particular, $B$ is a minimal non-supersoluble group and $B^\delta$ is a $q$-subgroup of $B$ for some prime $q$.

Suppose that $K$ is a proper subgroup of $B$. By Lemma 2.1, $AK$ satisfies the hypotheses of the theorem, and so $(AK)^\delta = K^\delta$ by the minimal choice of $G$. Since $(AK^x)^\delta = (K^x)^\delta = (K^\delta)^x$ for any $x \in B$, it follows that $A$ normalizes $(K^\delta)^x$. Thus, $A$ normalizes $(K^\delta)^x \cap x \in B$. Then $(K^\delta)^x \cap x \in B \leq G$, contrary to $\text{Core}_G(B) = 1$. Hence, $(K^\delta)^x = 1$. Consequently, $AK$ belongs to $\mathfrak{F}$. This shows that $B$ is $\mathfrak{F}$-critical, and by [8, Theorem 2.9], we have that $B$ is a minimal non-supersoluble group. By [3, Theorem 10], we have that $B^\delta$ is a $q$-group for some prime $q$. In particular, $B$ and then $G$ are soluble.

(iv) $G^\delta = B^\delta \times N$ is an elementary abelian $p$-group.

Applying (iii), it follows that $B^\delta$ is a $q$-group for some prime $q$. Let $N$ be a minimal normal subgroup of $G$ contained in $A$. Then $G^\delta = B^\delta N$ by (i), and $N$ is a $p$-group by (ii).

Suppose that $B^\delta$ is a normal subgroup of $G^\delta$. Then $G^\delta /B^\delta$ is an elementary abelian $p$-group. Consequently, the residual $X$ of $G^\delta$ associated to the formation of all elementary abelian $p$-groups is a normal subgroup of $G$ contained in $B$. Hence, $X \leq \text{Core}_G(B) = 1$, and $G^\delta$ is an elementary abelian $p$-group.

Assume that $p \neq q$. Let $N$ be a minimal normal subgroup of $G$ contained in $A$. Then $G^\delta = B^\delta N$, and $N$ is a $p$-group by (ii). Hence, $B^\delta$ is a Sylow $q$-subgroup of $G^\delta = B^\delta N$. Applying Frattini’s argument, we have that $G = G^\delta N_G(B^\delta) = NN_G(B^\delta)$. Since
Core$_G(B) = 1$, it follows that $N_G(B^\delta)$ is a proper subgroup of $G$. Hence, $N$ is not contained in $\Phi(G)$ for each minimal normal subgroup $N$ of $G$ contained in $A$. If $\Phi(A) \neq 1$, a minimal normal subgroup of $G$ must be contained in $\Phi(A) \subseteq \Phi(G)$, a contradiction. Therefore, $\Phi(A) = 1$. Let $N$ be a minimal normal subgroup of $G$ contained in $A$. Then $N = N_1 \times N_2 \times \cdots \times N_r$ is a direct product of minimal normal subgroups of $A$, and there exists $i \in \{1, 2, \ldots, r\}$ such that $N_i$ is not contained in $\Phi(A)$. Suppose $i = 1$. Let $M$ be a maximal subgroup of $A$ such that $A = N_1 M$ and $N_1 \cap M = 1$. Assume first that $A$ is a $p$-group. Then $BM$ is a subgroup of $G$, and $M = BM \cap A$ is a normal subgroup of $BM$. Hence, $M$ is normalized by $B$, and so $M$ is a normal subgroup of $G$. Now $N = N_1 (M \cap N)$. But $M \cap N$ is normal in $G$. The minimality of $N$ yields $N = N_1$ and then $|N| = p$. Thus, $G/C_G(N)$ is abelian. Hence, $G^\delta$ centralizes $N$, and $B^\delta$ is a normal subgroup in $G^\delta$, and so $G^\delta$ is an elementary abelian $p$-group. This contradiction implies that $A$ is not a $p$-group. Then $T = F(A)B$ is a proper subgroup of $G$ which is a weak direct product of $F(A)$ and $B$. By the minimality of $G$, $T^\delta = B^\delta$. Then $B^\delta$ is a normal subgroup of $G^\delta$, and so $G^\delta$ is an elementary abelian $p$-group, a contradiction which shows that $p = q$. Then $B^\delta$ is a subnormal subgroup of $G$. By [6, Lemma A.14.3], $N$ normalizes $B^\delta$, and therefore $B^\delta$ is a normal subgroup of the elementary abelian $p$-group $G^\delta$.

**(v) Final contradiction.** By [6, IV, 5.18], since $B^\delta$ is abelian, there exists an $\mathcal{N}$-projector $K$ of $B$ such that $B = B^\delta K$ and $K \cap B^\delta = 1$. Consider the subgroup $Z = AK$ of $G$. Applying (iii), $Z$ belongs to $\mathcal{N}$ and $G = B^\delta Z = F(G)Z$. By [6, III, 3.23(b)], there exists a unique $\mathcal{N}$-projector of $G$ containing $Z$, $E$ say. Hence, $G = B^\delta Z = G^\delta E$ and $G^\delta \cap E = 1$ by (iii) and [6, IV, 5.18]. In particular, $B^\delta \cap Z = 1$. Now $|Z||B^\delta| = |E||G^\delta| = |E||B^\delta||N|$. Hence, $|Z| = |E||N|$. This implies $Z = E$ and then $B^\delta = G^\delta$, a contradiction. 

**Proof of Theorem C.** Assume that the result is false and let $G$ be a minimal counterexample. Then every proper epimorphic image of $G$ is supersoluble, and hence $G$ has exactly one minimal normal subgroup $N$ which is not contained in the Frattini subgroup of $G$. Since $G$ is soluble, it follows that $N$ is abelian, $N = C_G(N) = F(G)$, and there exists a core-free maximal subgroup of $G$ such that $G = NM$ and $N \cap M = 1$. Let $p$ be the prime dividing $|N|$. Then $|N| > p$. Since $1 \neq G'$ is nilpotent, we have that $G' = N$ and $M$ is abelian. However, $O_p(M) = 1$ by [6, Lemma A.13.6]. Hence, $M$ is a $p'$-group and $N$ is the Sylow $p$-subgroup of $G$. Since $B \neq G$, we have that $N \leq A$. Note that $N = C_A(N) = O_{p'}(A)$. Therefore, $A/O_p(A) = A/O_{p}(A) = A/N$ is abelian of exponent $p - 1$ because $A$ is supersoluble. Assume $BN$ is a proper subgroup of $G$. Then, by the minimality of $G$, $BN$ is supersoluble, and so $B_{p'} \cong BN/O_{p'}(BN)$ is abelian of exponent $p - 1$. Consequently, $M$ is abelian of exponent $p - 1$. Since $N$ is an irreducible and faithful module for $M$, we have that $N$ has order $p$ by [6, Theorem B.9.8], a contradiction. Hence, $G = BN$. Now $B \cap N$ is a normal subgroup of $G$ contained in $N$. Thus, $B \cap N = 1$, and $G = BN$ is the weak direct product of $B$ and $N$. By Theorem A, $G$ is supersoluble. This contradiction proves the theorem. 

**Proof of Corollary B.** Note that since $G'$ is nilpotent, $A$ and $B$ are metanilpotent. By [8, Theorem 2.11], $A$ and $B$ are supersoluble. By Theorem C, $G$ is supersoluble, and hence $G \in \omega \mathcal{U}$. 


Proof of Theorem D. Suppose the theorem is not true and let \((G, A, B)\) be a counterexample with \(|G|+|A|+|B|\) as small as possible. Let \(N\) be a minimal normal subgroup of \(G\). It is easy to check that \(G/N\) satisfies the hypotheses of the theorem. By the minimality of \(G\), we have that \(G^\ast N = A^\ast B^\ast N\). Hence, \(G^\ast = A^\ast B^\ast (G^\ast \cap N)\). Consequently, \(\text{Soc}(G)\) is contained in \(G^\ast\) and \(G^\ast = A^\ast B^\ast N\) for every minimal normal subgroup \(N\) of \(G\). Since \(G^\ast\) is contained in \(G^*\), we have that \(G^\ast\) is nilpotent.

Note that \(A^\ast\) is a normal subgroup of \(G\). If \(A^\ast \neq 1\), then there exists a minimal normal subgroup \(N\) of \(G\) such that \(N \leq A^\ast\) and so \(G^\ast = A^\ast B^\ast N = A^\ast B^\ast\), a contradiction. Hence, we may assume that \(A\) is supersoluble and that \(G^\ast = B^\ast N\) for every minimal normal subgroup \(N\) of \(G\). If \(B\) were supersoluble, then \(G\) would be supersoluble by Theorem C, which is a contradiction. Hence, \(B^\ast \neq 1\). Furthermore, \(B^\ast\) cannot contain a normal subgroup of \(G\). Hence, \(\text{Core}_G(B^\ast) = 1\). Let \(p\) be the largest prime dividing \(|A|\). Since \(A\) is a Sylow tower group of supersoluble type, \(A\) has a normal Sylow \(p\)-subgroup, \(A_p\), say, which is also normal in \(G\). Hence, \(G\) has a minimal normal subgroup \(N\) of \(G\) which is a \(p\)-group. Since \(G^\ast\) is nilpotent, we have that \(B^\ast\) is a subnormal subgroup of \(G\). By [6, Lemma A.14.3], \(B^\ast\) is normalized by \(N\). Thus, \(B^\ast\) is a normal subgroup of \(G^\ast\), and \(G^\ast/B^\ast\) is an elementary abelian \(p\)-group. Consequently \(B^\ast\) contains the residual \(X\) of \(G^\ast\) associated to the formation of all elementary abelian \(p\)-groups. Since \(X\) is a normal subgroup of \(G\), it follows that \(X \leq \text{Core}_G(B^\ast) = 1\). Hence, \(G^\ast\) is an elementary abelian \(p\)-group.

Since \(\text{Soc}(G)\) is contained in \(G^\ast\), \(O_{p^r}(G) = 1\) and hence \(F(G) = O_{p^r}(G)\). Therefore \(G' \leq F(G)\) is a \(p\)-group, and \(F(G)\) is the unique Sylow \(p\)-subgroup of \(G\). Moreover, the Hall \(p\)-subgroups of \(G\) are abelian (note that \(G\) is soluble). Assume \(A_p B < G\). Then \(A_p B\) satisfies the hypotheses of the theorem. By the choice of \(G\), we have that \((A_p B)^\ast = B^\ast\). Note that \(G' \leq A_p B\). Hence, \(A_p B\) is a normal subgroup of \(G\). This implies that \(B^\ast\) is normal in \(G\), a contradiction. Hence, \(G = A_p B\) and \(A_p\) and \(B\) satisfy the hypotheses of the theorem. If \(A \neq A_p\), the choice of \((G, A, B)\) implies that \(G^\ast = B^\ast\), a contradiction. Consequently, we have that \(A = A_p\).

Write \(T = A B_p\). By Theorem C, \(T\) is supersoluble. Moreover, since \(G = F(G)B_p\), it follows that every minimal normal subgroup \(N\) of \(G\) contained in \(T\) is a minimal normal subgroup of \(T\). Thus, \(|N| = p\). Consequently, \(N\) is \(\ast\)-central in \(G\). By [6, V, 3.2], \(N\) is contained in every supersoluble normalizer of \(G\). Let \(E\) be one of them. Then \(G = G^\ast E\) and \(G^\ast \cap E = 1\). However, \(N \leq G^\ast \cap E = 1\). This final contradiction proves the theorem.

Proof of Corollary C. Since \(\ast \subseteq w\ast\), we have \(G^{w\ast} \leq G^\ast \leq G'\). Then \(G/G^{w\ast}\) is a metanilpotent \(w\)-supersoluble group. Applying [8, Theorem 2.11], we have that \(G/G^{w\ast}\) is supersoluble. Hence, \(G^\ast \leq G^{w\ast}\), and therefore \(G^\ast = G^{w\ast}\), and the same is true for \(A\) and \(B\). Therefore, by Theorem D, \(G^{w\ast} = A^{w\ast} B^{w\ast}\), as desired.

4. An analogue of Monakhov’s result

The following two results are the key to prove Corollary D.

Lemma 4.1. [4, Theorem A] Let the group \(G = HK\) be the product of the subgroups \(H\) and \(K\). Assume that \(H\) permutes with every maximal subgroup of \(K\) and \(K\) permutes with...
every maximal subgroup of $H$. If $H$ is supersoluble, $K$ is nilpotent and $K$ is $\delta$-permutable in $H$, where $\delta$ is a complete set of Sylow subgroups of $H$, then $G$ is supersoluble.

**Proposition 4.2.** Let $G = AB$ be a weak normal product of $A$ and $B$, with $A$ and $B$ supersoluble and $A$ normal in $G$. Then $B'$ is a subnormal subgroup of $G$.

**Proof.** Assume the result is not true, and let $G$ be a counterexample of minimal order with $|A|$ as small as possible. Let $p$ be the largest prime dividing the order of $A$. Then $A$ has a normal Sylow $p$-subgroup $A_p$, which is also a normal subgroup of $G$. Let $N$ be a minimal normal subgroup of $G$ such that $N \leq A_p$. It is clear that $A_pB$ satisfies the hypotheses of the theorem. Assume that $A_pB$ is a proper subgroup of $G$. By the minimality of $G$, $B'$ is a subnormal subgroup of $A_pB$. Hence, $B' \leq F(A_pB)$. By Lemma 2.1(a), $G/N$ is a weak normal product of $A/N$ and $BN/N$. By the minimality of $G$, we have that $B'N$ is a subnormal subgroup of $G$. Since $N \leq F(A_pB)$, it follows that $B'N \leq F(A_pB)$. Hence, $B'N$ is a subnormal nilpotent subgroup of $G$. Consequently, $B'N \leq F(G)$. Thus, $B'$ is a subnormal subgroup of $G$, a contradiction. Hence, we may assume that $G = A_pB$. The minimality of $|A|$ implies that $A = A_p$. Applying now the above Lemma, we conclude that $G$ is supersoluble and therefore $G'$ is nilpotent. Hence, $B'$ is subnormal in $G$. This final contradiction proves the proposition. □

**Proof of Corollary D.** Arguing as in [7, Theorem 1], we obtain $G^{\text{nil}} = (G')^{\text{nil}}$. Moreover, by [7, Lemma 1(3)], we have that $G' = A'B'_{[A,B]} = (A')^G(B')^G_{[A,B]}$. Since $A$ is a normal subgroup of $G$, then $A'$ is a subnormal subgroup of $G$. Also the application of Proposition 4.2 yields $B'$ is subnormal in $G$ and both $A'$ and $B'$ are nilpotent. Hence, $(A')^G(B')^G_{[A,B]}$ is a normal nilpotent subgroup of $G$. By [6, II, Lemma II.2.12], $(G')^{\text{nil}} = ((A')^G(B')^G)^{\text{nil}}_{[A,B]} = [A, B]^{\text{nil}}$, as desired. □

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