A note on the Erdős-Faber-Lovász Conjecture: quasigroups and complete digraphs

Gabriela Araujo-Pardo † Christian Rubio-Montiel ‡
Adrián Vázquez-Ávila §
March 3, 2022

Abstract
A decomposition of a simple graph $G$ is a pair $(G, P)$ where $P$ is a set of subgraphs of $G$, which partitions the edges of $G$ in the sense that every edge of $G$ belongs to exactly one subgraph in $P$. If the elements of $P$ are induced subgraphs then the decomposition is denoted by $[G, P]$.

A $k$-P-coloring of a decomposition $(G, P)$ is a surjective function that assigns to the edges of $G$ a color from a $k$-set of colors, such that all edges of $H \in P$ have the same color, and, if $H_1, H_2 \in P$ with $V(H_1) \cap V(H_2) \neq \emptyset$ then $E(H_1)$ and $E(H_2)$ have different colors. The chromatic index $\chi'(\langle G, P \rangle)$ of a decomposition $(G, P)$ is the smallest number $k$ for which there exists a $k$-P-coloring of $(G, P)$.

The well-known Erdős-Faber-Lovász Conjecture states that any decomposition $[K_n, P]$ satisfies $\chi'([K_n, P]) \leq n$. We use quasigroups and complete digraphs to give a new family of decompositions that satisfy the conjecture.

1 Introduction
Erdős, Faber and Lovász, in 1972, conjectured the following (see [2]): “if $|A_i| = n$, 1 $\leq i \leq n$, and $|A_i \cap A_j| \leq 1$, for $1 \leq i < j \leq n$, then one can color the elements of the union $\bigcup_{i=1}^{n} A_i$ by $n$ colors, so that every set has elements of all the colors.” This conjecture is called the Erdős-Faber-Lovász Conjecture (for short EFL), and this can be set in terms of decompositions (see [1, 3]).
Conjecture 1.1. If \([K_n, P]\) is a decomposition, then \(\chi'(\{K_n, P\}) \leq n\).

In the following section we give a family of decompositions using finite quasigroups and complete digraphs satisfying Conjecture 1.1 this is a generalization of a previous result given in [3] and it is related with a result given in [4].

2 Quasigroups and digraphs

To begin with, we introduce definitions related to quasigroups, complete digraphs and linear-factorizations. A digraph \(D\) is a finite, non-empty set \(V\) (the vertices of \(D\)) together with a set \(A\) of ordered pairs of elements of \(V\) (the arcs of \(D\)). We denote by \(|V|\) the order and by \(|A|\) the size of \(D\) respectively.

A digraph \(D\) is called symmetric if whenever \((u, v)\) is an arc of \(D\) then \((v, u)\) is an arc of \(D\) –every graph can be interpreted as a symmetric digraph–. A directed cycle or a \(d\)-gon is a subdigraph with set of vertices \(\{v_1, v_2, \ldots, v_d\}\), such that their arcs are \((v_i, v_{i+1})\) and \((v_i, v_{i+1})\) for \(i \in \{1, \ldots, d - 1\}\) and \(d \geq 2\). A loop or a \(1\)-gon is an arc joining a vertex with itself.

The complete digraph \(K_n^\ast\) has order \(n\) and size \(n^2\) (\(n\) loops and \(\binom{n}{2}\) \(2\)-gons).

A linear-factor of the complete digraph \(K_n^\ast\) is a subdigraph of order \(n\) and size \(n\), such that it is a set of pairwise vertex-disjoint \(d\)-gons. A linear-factorization of \(K_n^\ast\) is a set of pairwise arc-disjoint linear-factors, such that these linear-factors induce a partition of the arcs, see Figure 1: c).

A quasigroup \((Q_n, \cdot)\) is a set \(Q\) of \(n\) elements with a binary operation \(\cdot\), such that for each \(x\) and \(y\) in \(Q\) there exist unique elements \(a\) and \(b\) in \(Q\) with \(x \cdot a = y\) and \(b \cdot x = y\).

Let \((Q_n, \cdot)\) be a quasigroup and the complete digraph \(K_n^\ast\), such that its vertices are the elements of \(Q_n\). Afterwards, we color the arcs of \(K_n^\ast\) by \(n\) colors which are in a one-to-one correspondence with the elements of \(Q_n\) so that for any two vertices \(x\) and \(y\) in \(Q_n\) the arc \((x, y)\) obtains the color corresponding to \(a \in Q_n\) for which \(x \cdot a = y\) holds true. Then the resulting graph with the described coloring of arcs is called the Cayley color graph \(C(Q_n)\) of \(Q_n\). The Cayley color graph of a quasigroup is described in [4].

It is not hard to prove that the arcs colored by the same color in \(C(Q_n)\) induce a linear-factor of this digraph. An arc colored by the color corresponding color to some \(a \in Q_n\) outgoing from the vertex \(x\) leads into \(x \cdot a\) in \(C(Q_n)\). The element \(x \cdot a\) is exactly one for any \(x\) and any \(a\) of \(Q_n\).

Consequently, the Cayley color graph \(C(Q_n)\) can be considered as a linear-factorization \(F\) of \(K_n^\ast\) of \(n\) linear-factors. In [4] it was proved that any linear-factorization \(F\) of the complete digraph \(K_n^\ast\) and any one-to-one mapping of the vertex set of \(K_n^\ast\) onto the set of linear-factors of \(F\) determines a quasigroup \(Q_n\), such that the Cayley color graph \(C(Q_n)\) of \(Q_n\) can be considered \((K_n^\ast, F)\), as described above.

Following, we relate the previous concepts with decompositions of complete graphs. Let \([K_n, P]\) be a decomposition \(P\) of \(K_n\) and let \(\overline{K_n}\) be the symmetric
Figure 1: a) Two elements $p$ and $q$ of a decomposition of $K_{13}$ into triangle arising from the cyclic Steiner System $STS(13)$, b) $K_{13}$ as a symmetric digraph c) A linear-factor $F$ for $n = 13$. The mapping $i \mapsto i + 1$ produces a linear-factorization. d) The restriction of $F$ onto $p$ and $q$.

complete digraph (without loops). We consider the decomposition $[\overrightarrow{K}_n, P]$ induced by $[K_n, P]$, that is, $P$ is a set of subdigraphs of $\overrightarrow{K}_n$, which partitions the arcs of $\overrightarrow{K}_n$ in the sense that every arc of $\overrightarrow{K}_n$ belongs to exactly one subdigraph in $P$ and every element of $P$ is a symmetric complete subdigraph. The digraph $\overrightarrow{K}_n^*$ is $\overrightarrow{K}_n$ with the set $L$ of $n$ loops.

Now, we state and prove the main theorem:

**Theorem 2.1.** Let $[\overrightarrow{K}_n, P]$ be a decomposition $P$ of $\overrightarrow{K}_n$ arising from $[K_n, P]$ and let $(\overrightarrow{K}_n^*, F)$ be a linear-factorization $F$ of $\overrightarrow{K}_n^*$. If there exists a function $h: P \rightarrow F$, such that for any $p \in P$, $(A(p) \cup L) \cap A(h(p))$ is a linear-factor $F_p$ of $p^* - p$ with loops-- and for any $p, q \in P$, $A(F_p) \cap A(F_q) = \emptyset$ then $\chi'(\overrightarrow{K}_n, P) \leq n$.

**Proof.** Color the edges of an element $p$ of $P$ with $f(h(p))$ where $f$ is a one-to-one mapping of a quasigroup $Q$ onto the set of linear-factors of $F$. The $n$-coloring is well-defined due to the fact that for any $p, q \in P$, $A(F_p) \cap A(F_q) = \emptyset$ and the result follows. \hfill \Box

We can explain Theorem 2.1 as following:

Let $(\overrightarrow{K}_n^*, F)$ be a linear-factorization $F$ of $\overrightarrow{K}_n^*$. Then every decomposition $P$ formed by complete subdigraphs obtained via some linear-factor $f_0$ of $F$, meaning, the intersection of the arcs of $p \in P$ with the arcs of $f_0$ is a linear factor of $p$ has a consequence that $\chi'(\overrightarrow{K}_n, P) \leq n$. Figure 1 illustrates Theorem 2.1 with an example for $n = 13$.

**References**

[1] G. Araujo-Pardo and A. Vázquez-Ávila, A note on Erdős-Faber-Lovász conjecture and edge coloring of complete graphs, Ars Combin. (accepted).
[2] P. Erdős, *Problems and results in graph theory and combinatorial analysis*, Proc. Fifth Brit. Comb. Conf. (Univ. Aberdeen, Aberdeen, 1975) (Winnipeg, Man.), Utilitas Math., 1976, pp. 169–192. Cong. Num., No. XV.

[3] D. Romero and A. Sánchez-Arroyo, *Adding evidence to the Erdős-Faber-Lovász conjecture*, Ars Combin. 85 (2007), 71–84.

[4] B. Zelinka, *Quasigroups and factorisation of complete digraphs*, Mat. Časopis Sloven. Akad. Vied 23 (1973), 333–341.