On the numerical solution of the Laplace-Beltrami problem on piecewise-smooth surfaces

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Abstract

The Laplace-Beltrami problem on closed surfaces embedded in three dimensions arises in many areas of physics, including molecular dynamics (surface diffusion), electromagnetics (harmonic vector fields), and fluid dynamics (vesicle deformation). In particular, the Hodge decomposition of vector fields tangent to a surface can be computed by solving a sequence of Laplace-Beltrami problems. Such decompositions are very important in magnetostatic calculations and in various plasma and fluid flow problems. In this work we present an overview of the $L^2$-invertibility of the Laplace-Beltrami operator on piecewise smooth surfaces, extending earlier weak formulations and integral equation approaches on smooth surfaces. We then provide high-order numerical examples along surfaces of revolution to support our analysis, and discuss numerical extensions to general surfaces embedded in three dimensions.

Keywords: Laplace-Beltrami, harmonic vector field, Lipschitz, surface of rotation, Hodge decomposition.

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1 Introduction

The Laplace-Beltrami operator is a second-order elliptic differential operator defined along general manifolds in arbitrary dimensions. Practically, it can be thought of as the extension of the Laplace operator to curved surfaces [22,39]. For the moment, let \( \Gamma \) be a smooth closed surface embedded in three dimensions, and let \( \nabla_{\Gamma} \) and \( \nabla_{\Gamma} \) be the intrinsic surface divergence and surface gradient operators along \( \Gamma \) (these are defined more carefully later on the manuscript). Then the Laplace-Beltrami operator, also referred to as the surface Laplacian, is given as

\[
\Delta_{\Gamma} = \nabla_{\Gamma} \cdot \nabla_{\Gamma}.
\] (1.1)

The Laplace-Beltrami operator is particularly useful for problems in electromagnetics since it allows for the explicit construction of tangential vector fields along multiply-connected surfaces in terms of their gradient-based, divergence-free, and harmonic components. Any smooth tangential vector field \( \mathbf{F} \) along \( \Gamma \) can be written as

\[
\mathbf{F} = \nabla_{\Gamma} \alpha + \mathbf{n} \times \nabla_{\Gamma} \beta + \mathbf{H},
\] (1.2)

where \( \alpha, \beta \) are smooth scaler-valued functions along \( \Gamma \) and \( \mathbf{H} \) is a harmonic vector field, i.e. \( \nabla_{\Gamma} \cdot \mathbf{H} = 0 \) and \( \nabla_{\Gamma} \cdot (\mathbf{n} \times \mathbf{H}) = 0 \). See [17,18,39,42] for more details regarding such decompositions. Conversely, if the vector field \( \mathbf{F} \) above is known (but not its individual components), then its solenoidal (i.e. divergence-free) component involving \( \beta \), for example, can be computed by taking \( \nabla_{\Gamma} \cdot \mathbf{n} \times \) of each side of (1.2), yielding

\[
\Delta_{\Gamma} \beta = -\nabla_{\Gamma} \cdot \mathbf{n} \times \mathbf{F}.
\] (1.3)

Solving the above PDE along \( \Gamma \) requires inverting the Laplace-Beltrami operator. Applications of this problem are myriad in electromagnetics, as mentioned, as well as plasma physics [37], vesicle deformation in biological flows [45,48], surface diffusion [2,20], and computational geometry [33] and computer vision [1].

Along smooth general surfaces, there have been several numerical methods proposed to solve the Laplace-Beltrami problem: finite element methods [2,4,10,14,15] (including eigenfunction computations [5]), the so-called virtual element method [3,23], differencing methods [49], and integral equation methods [35,42]. Of course, in specialized geometries, such as axisymmetric ones, separation of variables can be used to simplify the problem. In [19,43], after separation of variables, an integral equation approach was used along the one-dimensional generating curve of the axisymmetric surface resulting in high-order convergence for the inversion of the surface Laplacian operator. Boundary-value problems for the Laplace-Beltrami problem were addressed in [34,35] via a parametrix method and associated discretization via projection of the sphere.

However, while the related problem of electromagnetic scattering from non-smooth surfaces has been studied in some detail [8,13], the Laplace-Beltrami problem on piecewise-smooth surfaces (or more generally Lipschitz surfaces) has not received as much attention. In order to extend various numerical methods in computational electromagnetics, namely integral equation methods based on generalized Debye source representations [11,17–19], a thorough understanding of the Laplace-Beltrami problem on non-smooth surfaces is required (as well as high-order accurate methods for solving the problem numerically). This work aims to offer a mathematical discussion of the Laplace-Beltrami problem along piecewise smooth surfaces in the \( L^2 \) setting which is compatible with many modern numerical methods (i.e. PDE vs. integral equation methods, Galerkin vs. Nyström discretizations). In particular, our main purpose for focusing on the \( L^2 \) theory of this problem is to
develop fast robust numerical solvers that can be incorporated into computational electromagnetics codes. These solvers often rely on iterative methods whose behavior is coupled with the spectrum of the operator, which depends strongly on the type of discretization used. A suitable $L^2$ embedding of the problem [6] is frequently the most straightforward way to obtain an accurate approximation to the spectrum of the continuous problem.

Previously, the regularity of solutions to the Laplace-Beltrami problem in the general Lipschitz setting was discussed in [24], and some special-case numerical methods on polyhedral surfaces were presented in [26, 50]. These methods are almost exclusively developed in the finite element setting whereby the use of polyhedral surfaces and linear finite elements reduce the continuous problem to a discrete system that can be studied in detail. Similar techniques have been used in the electromagnetics community for analyzing Hodge decompositions on polyhedra [7, 9].

The outline of our approach and the paper is as follows: in Section 2 we recall some standard weak results regarding the Laplace-Beltrami problem on smooth and Lipschitz surfaces and extend them to an interface formulation on piecewise smooth surfaces. In Section 3 we reformulate the Laplace-Beltrami problem as a sequence of decoupled ODEs along a surface of revolution, and introduce an integral equation reformulation of said ODEs. An associated high-order accurate numerical solver is then discussed. Section 4 contains various numerical experiments validating our formulation and demonstrating the accuracy of our solver. Then, in Section 5, we offer some final remarks and point to outstanding problems related to the Laplace-Beltrami problem and future avenues of research.

2 The Laplace-Beltrami problem

In this section we lay out some standard well-known invertibility results for the Laplace-Beltrami problem on smooth surfaces, and then reformulate some existing results for Lipschitz domains to the case in which the boundary is piecewise smooth and the data lie in $L^2(\Gamma)$, where by $\Gamma$ we denote the boundary. From the context, the assumptions on $\Gamma$ should be clear (i.e. either smooth, general Lipschitz, or piecewise smooth).

2.1 Smooth surfaces

To begin with, let $\Omega \subset \mathbb{R}^3$ denote an open bounded domain whose boundary is given by $\Gamma$. The boundary $\Gamma$ can either be simply or multiply connected, and for now is assumed to be globally smooth. We shall start by recalling the definition of the Laplace-Beltrami operator and the Laplace-Beltrami problem on a smooth surface.

**Definition 1 (Laplace-Beltrami Operator).** If $\Gamma$ is a smooth bounded surface, $x = x(\theta, \varphi)$ is a local parameterization of $\Gamma$ with respect to some set of variables $\theta, \varphi$, the functions $x_\theta$ and $x_\varphi$ are the partial derivatives of $x$, and $g$ is the associated metric tensor,

$$g(\theta, \varphi) = \begin{bmatrix} x_\theta \cdot x_\theta & x_\theta \cdot x_\varphi \\ x_\varphi \cdot x_\theta & x_\varphi \cdot x_\varphi \end{bmatrix},$$

then the Laplace-Beltrami operator $\Delta_{\Gamma}$ acting on a smooth function $f = f(\theta, \varphi)$ is defined by the formula

$$\Delta_{\Gamma} f = \frac{1}{\sqrt{\det g}} \left[ \frac{\partial}{\partial \theta} \frac{\partial}{\partial \varphi} \right] \sqrt{\det g} \, g^{-1} \left[ \frac{\partial}{\partial \theta} \right] f$$

$$= \nabla_{\Gamma} \cdot \nabla_{\Gamma} f.$$
In the above definition, we have introduced the notation $\nabla_F \cdot$ and $\nabla_F$ to denote the surface divergence and surface gradient, respectively. The precise definitions in the case of interest, mainly along piecewise smooth surfaces, are given below in Section 2.2.

With the above definition of $\Delta_F$, the Laplace-Beltrami problem refers to solving the equation

$$\Delta_F u = f \quad \text{on } \Gamma$$  \hspace{1cm} (2.3)

for the unknown function $u$. One must define the domains of the data $f$ and solution $u$ to this problem carefully in order to ensure that the problem is well-posed, as the Laplace-Beltrami operator is neither injective nor surjective. The operator has a well-known one-dimensional null space: the space of constant functions along $\Gamma$ [39]. It is also well known that the range of the Laplace-Beltrami operator omits constant functions (i.e. functions with non-zero mean). A standard well-posed version of this problem is summarized in the following theorem, which is proved in [51]:

**Theorem 1.** Suppose that $\Gamma$ is a closed surface that is $C^k$ for some $k \geq 2$. Then, the Laplace-Beltrami problem (2.3) has a unique mean-zero solution for every continuous mean-zero right hand side. Furthermore, if the right hand side is in $C^\ell(\Gamma)$ for some $\ell \leq k-2$, then the solution $u$ will be in $C^{\ell+2}(\Gamma)$.

We now move onto a practical discussion of the Laplace-Beltrami problem along Lipschitz surfaces, a topic which has only been somewhat studied in the literature.

### 2.2 Lipschitz surfaces

In this section, we shall recall the definitions required to clearly state the Laplace-Beltrami problem on a general Lipschitz surface $\Gamma$, as well as summarize the associated invertibility result in [24]. Equivalent definitions of the relevant functions spaces and operators are also discussed in [24] in greater detail. The first notion we outline is that of a Lipschitz surface. A similar definition to that which we give below is also contained in [38].

**Definition 2 (Lipschitz Surface).** Let $\Gamma$ be a bounded surface embedded in $\mathbb{R}^3$. The surface $\Gamma$ is said to be a Lipschitz surface if there exists a finite open covering $\{O_k\}_{k=1}^N$ and an associated set of rigid rotations $\Sigma_k$ such that $\Sigma_k(\Gamma \cap O_k)$ is the graph of a Lipschitz function. That is to say, for each $k$ there exists an open domain $U_k \subset \mathbb{R}^2$ and a Lipschitz function $\varphi_k : U_k \to \mathbb{R}^3$ such that $\Sigma_k(\Gamma \cap O_k) = \{(x, y, \varphi_k(x, y)) : (x, y) \in U_k\}$.

While the above definition of a Lipschitz surface is straightforward in terms of graphs of Lipschitz functions, it will be simpler for our purposes to define the relevant function spaces and differential operators in terms of local parameterizations. With this in mind, we define Sobolev spaces along $\Gamma$ as follows.

**Definition 3 (Sobolev spaces).** Let $\Gamma$ be a bounded Lipschitz surface with a finite open covering $\{O_i\}_{i=1}^N$. Also, let $\{x_i\}_{i=1}^N$ be a collection of local Lipschitz parameterizations, such that each $x_i$ maps an open neighbourhood $U_i \subset \mathbb{R}^2$ into $O_i$. Finally, let $\{\chi_i\}_{i=1}^N$ be a partition of unity of $\Gamma$ such that for each $i$, $\chi_i$ is supported on $O_i$. We then have the following definitions:

1. For all $0 \leq s \leq 1$ we define the Sobolev space along $\Gamma$ of order $s$ as

   $$H^s(\Gamma) = \{ f \in L^2(\Gamma) : (\chi_i f) \circ x_i \in H^s(U_i) \text{ for all } i = 1, \ldots, N \},$$

   with a norm given by

   $$\| f \|_{H^s(\Gamma)} = \sum_{i=1}^N \| (\chi_i f) \circ x_i \|_{H^s(U_i)}.$$
2. For all \(-1 \leq s < 0\), we define the Sobolev space of order \(s\), \(H^s(\Gamma)\), as the dual space of \(H^{-s}(\Gamma)\).

3. For all \(-1 \leq s \leq 1\), we define \(H^s_0(\Gamma)\) to be the subset of \(H^s(\Gamma)\) with mean-zero. More explicitly, we set
\[
H^s_0(\Gamma) = \{ f \in H^s(\Gamma) \mid \langle f, 1 \rangle_{s, \Gamma} = 0 \},
\]  
(2.4)

where \(\langle \cdot, \cdot \rangle_{s, \Gamma}\) is the duality pairing between \(H^s(\Gamma)\) and \(H^{-s}(\Gamma)\).

4. For all \(-1 \leq s \leq 1\), we define the tangent Sobolev space \(H^s_t(\Gamma)\) as the subset of three dimensional vector fields that are tangential to \(\Gamma\):
\[
H^s_t(\Gamma) = \{ \mathbf{v} \in (H^s(\Gamma))^3 \mid \chi_i \mathbf{v} = v_\theta \mathbf{x}_{i, \theta} + v_\varphi \mathbf{x}_{i, \varphi}, \text{for all } i = 1, \ldots, N \}.  
\]  
(2.5)

It should be noted that the above definitions are almost exactly those that are used when \(\Gamma\) is smooth. The only difference is that the \(\mathbf{x}_i\)'s are assumed to be Lipschitz instead of smooth, and we have restricted \(s\) to \([-1, 1]\). These spaces thus coincide with the usual Sobolev spaces whenever \(\Gamma\) happens to be smooth. We also note that for general Lipschitz surfaces, these Sobolev spaces only make sense for \(|s| \leq 1\): a local Lipschitz parameterization will only have fractional derivatives of order \(s\) for \(s \leq 1\). Lastly, we make the usual identification \(H^0(\Gamma) = L^2(\Gamma)\) since \(\| \cdot \|_{H^0(\Gamma)}\) is equivalent to \(\| \cdot \|_{L^2(\Gamma)}\) [24].

With the above function spaces defined, we may now move onto defining the associated surface differential operators in the usual ways.

**Definition 4.** Let \(\mathbf{x}(\theta, \varphi)\) be a local Lipschitz parameterization of \(\Gamma\) with an associated metric \(g\). Interpreting all partial derivatives in the distributional sense [21], we make the following definitions:

1. For \(0 \leq s \leq 1\), the surface gradient \(\nabla_\Gamma f\) of a function \(f \in H^s(\Gamma)\) is given by the formula
\[
\nabla_\Gamma f = \begin{bmatrix} x_\theta & x_\varphi \end{bmatrix} g^{-1} \begin{bmatrix} \partial_\theta \\ \partial_\varphi \end{bmatrix} f,
\]  
(2.6)

where \(x_\theta, x_\varphi, g\) are the same as in Definition 1, although interpreted weakly. It is clear from this formula that \(\nabla_\Gamma f : H^s(\Gamma) \to H^{s-1}_t(\Gamma)\) for any \(0 \leq s \leq 1\).

2. The surface divergence \(\nabla_\Gamma \cdot\) is defined as the negative adjoint of the surface gradient map above. Since we consider the surface gradient as a map from \(H^{1-s}(\Gamma)\) to \(H^{-s}_t(\Gamma)\), the surface divergence is a map from \(H^s_t(\Gamma)\) to \(H^s(\Gamma)\).

3. The Laplace-Beltrami operator \(\Delta_\Gamma : H^1(\Gamma) \to H^{-1}(\Gamma)\) is defined as the composition \(\Delta_\Gamma := \nabla_\Gamma \cdot \nabla_\Gamma\). The formula for applying \(\Delta_\Gamma\) to a function \(f \in H^1(\Gamma)\) is the same as for the smooth case (2.2), except that the derivatives must be interpreted in a weak sense.

From a practical point of view, the general definition of the surface divergence given above is difficult to implement numerically. This obstacle is most easily overcome by assuming a particular (piecewise) parameterization of \(f\) along \(\Gamma\) and using the following equivalent definition: if \(\mathbf{v} \in H^s_t(\Gamma)\) is written as \(\mathbf{v} = v_\theta \mathbf{x}_\theta + v_\varphi \mathbf{x}_\varphi\), then
\[
\nabla_\Gamma \cdot \mathbf{v} = \frac{1}{\sqrt{\det g}} \begin{bmatrix} \partial_\theta \\ \partial_\varphi \end{bmatrix} \sqrt{\det g} \begin{bmatrix} v_\theta \\ v_\varphi \end{bmatrix}.
\]  
(2.7)
These definitions are easily adapted from the discussion in [39]. While the above formula still uses weak derivatives, we shall be able to reinterpret it in a stronger sense once we make further assumptions on $\Gamma$.

We also note that the Laplace-Beltrami operator can only be defined on $H^s(\Gamma)$ for $s = 1$. For any other value of $s$ either the domain $H^s(\Gamma)$ or the range $H^{s-2}(\Gamma)$ would not be well-defined on a general Lipschitz surface. We will thus only consider right hand sides that are contained in $H^1(\Gamma)$ when attempting to solve the Laplace-Beltrami equation $\Delta_{\Gamma}u = f$ on such surfaces. As in the smooth case, it will be necessary to assume that $f$ is mean-zero, and we will also look for a mean-zero solution. The Laplace-Beltrami problem along a Lipschitz surface therefore becomes:

**Problem 1** (Laplace-Beltrami problem). Let $f \in H_0^1(\Gamma)$. The Laplace-Beltrami problem is to find a $u \in H^1_0(\Gamma)$ such that $\Delta_{\Gamma}u = f$.

Before discussing methods to solve the Laplace-Beltrami problem, the following theorem is needed in order to ensure that the problem is well-posed. A proof can be found in [24].

**Theorem 2.** If $\Gamma$ is a bounded Lipschitz surface then the null-space of the Laplace-Beltrami operator is the set of constant functions and the range is $H_0^1(\Gamma)$. The Laplace-Beltrami problem on $\Gamma$ is thus well-posed.

In summary, we now have that the distributional form of our equation is well-posed on a general Lipschitz surface. If we also assume that $f \in L^2(\Gamma)$, then we may reformulate this problem more explicitly in the following weak form

$$ -\int_{\Gamma} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} v = \int_{\Gamma} f v, \quad \text{for all } v \in H^1(\Gamma). \quad (2.8) $$

In the next section, we will convert the expression above into one with interface conditions along edges in piecewise smooth geometries.

### 2.3 Interface conditions

We shall now further restrict ourselves to the case that the surface $\Gamma$ is a finite union of smooth but curved faces $\Gamma_i$ and that $f \in L^2(\Gamma)$. We will use this added smoothness of the geometry to turn the weak equation (2.8) into a strong form on each face augmented with matching conditions along the edges (but away from any corners). First, we develop some notation for piecewise smooth surfaces.

If it exists, the edge connecting the faces $\Gamma_i$ and $\Gamma_j$ will be denoted by $e_{ij}$. We let $n_i$ be the outward normal along $\Gamma_i$, $\tau_i$ be the tangent vector along the boundary $\partial \Gamma_i$ to face $\Gamma_i$, and $b_i = \tau_i \times n_i$. We will refer to $b_i$ as the binormal vector, and assume that the orientation of $\tau_i$ was chosen so that $b_i$ points away from $\Gamma_i$.

On a piecewise smooth surface, the spaces $H^s(\Gamma_i)$ are well defined for each $i$ and for any $s \in \mathbb{R}$, even though $H^s(\Gamma)$ may not be. As this section will require us to work with functions that have more than one square integrable derivative, we will use this fact to define a class of function spaces of functions with higher order derivatives on each face.

**Definition 5.** For any $s \geq 0$, we define the space

$$ \mathcal{H}^s(\Gamma) := \{ v \in L^2(\Gamma) : v|_{\Gamma_i} \in H^s(\Gamma_i) \text{ for all } i \}. \quad (2.9) $$

We now compute the strong form of the equation on each face using integration by parts. This requires that $u$ happens to be in $\mathcal{H}^2(\Gamma)$. For now, we assume that this condition holds; after
computing the strong form, we will go back and verify that indeed \( u \in \mathcal{H}^2(\Gamma) \). Under this new assumption, the weak form (2.8) becomes

\[
\sum_i \left( \int_{\Gamma_i} v \Delta_i u + \int_{\partial \Gamma_i} v \mathbf{b}_i \cdot \nabla_i u \right) = \sum_i \int_{\Gamma_i} f v, \quad \text{for all } v \in H^1(\Gamma).
\] (2.10)

Through the usual variational arguments [30], this equation tells us that \( \Delta_i u |_{\Gamma_i} = f |_{\Gamma_i} \) almost everywhere for each \( i \). By considering choices of \( v \) that concentrate near \( e_{ij} \) for each \( i \) and \( j \) such that \( e_{ij} \) exists, equation (2.10) also tells us that

\[
\int_{e_{ij}} v (\mathbf{b}_i \cdot \nabla_i u - \mathbf{b}_j \cdot \nabla_j u) = 0 \quad \text{for all } v \in H^1(\Gamma).
\] (2.11)

It is not hard to see that this implies that the binormal derivative of \( u \) agrees from both sides of an edge. We thus have that if the solution \( u \in \mathcal{H}^2(\Gamma) \), then \( u \) solves the following problem:

**Problem 2 (Interface form of the Laplace-Beltrami problem).** Let \( \Gamma \) be a surface composed of smooth faces \( \Gamma_i \) and suppose that \( f \in L^2(\Gamma) \). The interface form of the Laplace-Beltrami problem is to find a \( u \in \mathcal{H}^2(\Gamma) \) such that

\[
\Delta_i u |_{\Gamma_i} = f |_{\Gamma_i}, \quad \text{a.e. on } \Gamma_i,
\]

\[
\frac{\partial u}{\partial \mathbf{b}_i} = -\frac{\partial u}{\partial \mathbf{b}_j} \quad \text{on } e_{ij},
\]

\[
u |_{\Gamma_i} = u |_{\Gamma_j} \quad \text{on } e_{ij},
\]

where the edge conditions are interpreted in the trace sense.

We note here that the interface form does not involve any corner conditions on \( u \) beyond the requirement that \( u \in \mathcal{H}^2(\Gamma) \). This comes from the fact that integration by parts does not introduce any corner conditions on piecewise smooth domains.

Having identified this interface form of the equation that is satisfied when \( u \) is smooth enough, we will now go back and prove that \( u \) is indeed in \( \mathcal{H}^2(\Gamma) \) whenever \( f \) is in \( L^2(\Gamma) \). Our proof that \( u \) is in \( \mathcal{H}^2(\Gamma) \) will require the following theorem which gives the behaviour of the solution for an elliptic interface problem near a corner. This theorem is a special case of Theorem 8.6 in [41]. Subsequently, in the proof of Theorem 4 late on, we restrict the Laplace-Beltrami problem to a single chart and apply Theorem 3 to determine the smoothness of \( u \) in a specific region.

**Theorem 3** (Nicaise and Sändig). Suppose the following assumptions hold:

1. The domain \( \Omega = \bigcup_{j=1}^N \Omega_j \subset \mathbb{R}^2 \) is a bounded domain composed of non-intersecting pieces with piecewise smooth boundaries.

2. The domain \( \Omega \) has a finite number of corners, i.e. points where \( \partial \Omega \) is non-smooth for some \( j \), or where multiple boundaries meet.

3. The second order elliptic operator \( L \) has smooth coefficients on each \( \Omega_j \).

4. The function \( v \) is the solution of \( L v = f \), where \( f \) is in \( L^2(\Omega) \) and \( f |_{\Omega_j} \in H^k(\Omega_j) \) for some \( k \geq 0 \) and every \( j \).

5. The function \( v \) is in the space \( V \) of functions that are in \( H^1(\Omega_j) \) for each \( j \), that vanish on \( \partial \Omega \), and whose values and derivatives agree at the boundaries between sub-domains (see [40] for a more precise definition of \( V \)).
6. The neighborhood $U$ contains exactly one corner of $\Omega$, i.e. a point where $\partial \Omega_j$ is non-smooth for some $j$ or that is contained in $\partial \Omega_j$ for multiple choices of $j$. We denote this corner by $S$ and assume $U$ only intersects $\Omega_j$ if $\Omega_j$ touches $S$.

7. The pairs $\{\lambda_n, \tau_n\}_n$ are eigenvalue-eigenfunction pairs of the angular part of $L$ with its coefficients evaluated at $S$, and the eigenvalues $\lambda_n$ are all real. (See [40].)

If the above hold, then we may define polar coordinates $(r, \theta)$ around the corner $S$ and we may find a function $v_0$ and coefficients $a_n$ such that anywhere in $U$ we can write the function $v$ as

$$v = v_0 + \sum_{|\lambda_n| \leq (k+1)} a_n r_{S}^{|\lambda_n|} \tau_n(\theta).$$

(2.13)

Furthermore, the function $v_0$ is in $H^{k+2}(\Omega_j \cap U)$ for each $j$ such that $\Omega_j$ touches $S$.

Applying the previous theorem to a covering of $\Omega$ we have the following corollary:

**Corollary 3.1.** Let $\{U_\ell\}$ be a collection of neighbourhoods covering $\Omega$ so that each satisfy the assumptions of Theorem 3 for a different corner of $\Omega$ and that each corner is covered. We also let $\{\chi_\ell\}$ be a partition of unity such that each $\chi_\ell$ is supported in $U_\ell$. Theorem 3 then gives that we can write the solution $v$ as

$$v = v_0 + \sum_\ell \chi_\ell \sum_{|\lambda_n, l| \leq (k+1)} a_{n,l} r_{S}^{|\lambda_n, l|} \tau_n(\theta),$$

(2.14)

where the function $v_0$ is in $H^{k+2}(\Omega_j)$ for each $j$.

In order to apply Theorem 3 in a useful manner, we need the eigenvalue-eigenfunction pairs for the Laplace-Beltrami operator around corners of the surface $\Gamma$. An examination of the definition of these pairs, shows that this is equivalent to determining the form of the solution of the Laplace-Beltrami problem near a single corner in the case that the faces are flat and the edges are straight. An example of such a geometry and the coordinate system that we consider is in Figure 1. We summarize the result in the following lemma.
Lemma 1. Suppose that $\hat{\Gamma}$ is an infinite pyramid formed by three intersecting infinite triangles $\hat{\Gamma}_1, \hat{\Gamma}_2,$ and $\hat{\Gamma}_3$. Let each face be parameterized using the polar coordinates $(r_i, \theta_i)$ with $\theta_i = 0$ on the edge $e_{i(i-1)}$ and $\theta_i = \gamma_i$ on $e_{i(i+1)}$. The piecewise parameterization of $\hat{\Gamma}$ is then

$$(r, \theta) = \begin{cases} 
(r_1, \theta_1) & \text{on } \hat{\Gamma}_1, \\
(r_2, \theta_2 + \gamma_1) & \text{on } \hat{\Gamma}_2, \\
(r_2, \theta_3 + \gamma_1 + \gamma_2) & \text{on } \hat{\Gamma}_3.
\end{cases} \quad (2.15)$$

The sum $\gamma := \gamma_1 + \gamma_2 + \gamma_3$ is known as the conic angle of the vertex. If $B_R$ is a ball of radius $R$ centered on the vertex and $u \in H^1(B_R \cap \hat{\Gamma})$ is a solution of $\Delta_F u = 0$, then for all $r < R$ the solution $u$ can be written as

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{i \frac{2\pi}{\gamma} n \theta}, \quad (2.16)$$

for some set of constants $a_n$. Furthermore, if $k < 1 + \frac{2\pi}{\gamma}$, then on each face $\hat{\Gamma}_i$ we have that $u \in H^k(B_R \cap \hat{\Gamma}_i)$.

We note that in the case that $\hat{\Gamma}_1, \hat{\Gamma}_2,$ and $\hat{\Gamma}_3$ are co-planar, $\hat{\Gamma}$ will be a plane in $\mathbb{R}^3$. The conic angle $\gamma$ will thus be $2\pi$ and so (2.16) becomes the usual separation of variables solutions to Laplace’s equation in the plane.

Proof. We first note that since $u$ is assumed to be in $H^1(B_R \cap \hat{\Gamma})$ for all $r < R$, we may write $u$ as a convergent Fourier Series

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} c_n(r) e^{i \frac{2\pi}{\gamma} n \theta}.$$ 

Plugging each term into $\Delta_F u = 0$ gives

$$c_0(r) = a_0 + b_0 \log(r),$$

$$c_n(r) = a_n r^{|n|} e^{i \frac{2\pi}{\gamma} n |n|} + b_n r^{-|n|} e^{-i \frac{2\pi}{\gamma} n |n|}, \quad \text{for } n \neq 0. \quad (2.17)$$

Dropping the terms that are not in $H^1(B_R \cap \hat{\Gamma})$ gives (2.16).

In order to see the higher regularity of $u$ on each face, we note that the most irregular terms in the expansion occur when $n = \pm 1$. These terms will be contained in $H^k$ if and only if

$$\left\| \nabla^k F r^{\frac{|n|}{\gamma}} e^{i \frac{2\pi}{\gamma} n \theta} \right\|_{L^2(B_R \cap \hat{\Gamma}_i)} ^{1/2} \left( \int_0^R r^{2(\frac{|n|}{\gamma} - k) + 1} dr \right)^{1/2} < \infty.$$

This occurs precisely when $k < 1 + \frac{2\pi}{\gamma}.$ \qed

We observe that since $\gamma \leq 2\pi$, the above lemma indicates that the solution to the Laplace-Beltrami problem is indeed in $\mathcal{H}^2$ near a corner formed by straight edges. We now extend this lemma to curved geometries in the following theorem, which addresses the interface form of the Laplace-Beltrami problem stated in Problem 2.

Theorem 4 (Strong form of the Laplace-Beltrami equation). Suppose that $\Gamma$ is a bounded Lipschitz surface and that $u$ is the weak solution of $\Delta_F u = f$, for some $f \in \mathcal{H}^k(\Gamma)$ with $k \geq 0$. Under these assumptions, the following holds.
1. The solution $u$ is in $H^2(\Gamma)$ and the traces of both $u$ and $\nabla u$ agree along $\Gamma_i$ and $\Gamma_j$ which share an edge $e_{ij}$.

2. If we let $C$ be the set of corners of $\Gamma$ and $\{\xi_S\}_{S \in C}$ be a collection of smooth cutoff functions, such that $\xi_S$ is one in a neighbourhood of each corner $S$ and vanishes outside a larger neighbourhood; then there exists a $v \in H^{2+k}(\Gamma)$ such that

$$u = v + \sum_{S \in C} \xi_S \left( \sum_{|n| \leq \frac{2\pi}{\gamma} (k+3)} a_{n,S} r_{S}^{|n|} e^{i \frac{2\pi}{\gamma} n \theta_S} \right), \quad (2.18)$$

where $\gamma_S$ is the conic angle for the corner $S$ and $(r_S, \theta_S)$ is a local polar coordinate system defined around the corner $S$, as described in Lemma 1.

Proof. We suppose $\{(U_j, x_j)\}_{j=1}^N$ is a finite collection of domains and parameterizations such that the boundaries of the $U_j$’s are smooth, that the $x_j$’s are Lipschitz, and that $\{x_j(U_j)\}_{j=1}^N$ covers $\Gamma$. We shall also suppose that the covering is dense enough that there exist open sets $\{V_j\}_{j=1}^N$ such that each $V_j$ is contained in $U_j$, but is separated from $\partial U_j$, and that $\{x_j(V_j)\}_{j=1}^N$ also covers $\Gamma$. We shall restrict our attention to a single local parameterization $x_j$ and note that by repeating the following argument for each parameterization we will have the desired global results.

In the set $V_j$, the Laplace-Beltrami problem is an elliptic interface problem, so we would like to use Theorem 3 to see that $u$ has the desired smoothness. Before we can do this, however, we must modify the problem to ensure that the assumptions of Theorem 3 are satisfied.

We define $L$ to be the pull-back of the Laplace-Beltrami operator onto $U_j$:

$$Lg := (\Delta \Gamma (g \circ x_j^{-1})) \circ x_j,$$

and define a smooth cut-off function $\xi \in C_0^\infty(U_j)$ that is identically one on $V_j$ and zero in a neighborhood of $\partial U_j$. We also define the function $v = \xi(u \circ x_j)$. We will apply Theorem 3 to the equation $Lv = \tilde{f}$ on the domain $U_j$, where $\tilde{f} = \Delta \Gamma((\xi \circ x_j^{-1})u) \circ x_j$. By introducing $\xi$ and extending the domain, we have converted the restricted Laplace-Beltrami problem to a problem with homogeneous Dirichlet boundary conditions. By ensuring that $\xi$ is one on $V_j$, we have ensured that $v$ agrees with $u \circ x_j$ on $V_j$ so that statements about $v$ tell us about $u$.

To make this more precise, we define $U_{j,i}$ as the portion of $U_j$ that maps to $\Gamma_i$, i.e. $U_j \setminus x_j^{-1}(\Gamma_i)$, and define $V_{j,i}$ equivalently. After we show that it may be applied, Theorem 3 will yield that

$$\|v\|_{H^2(U_{j,i})} \leq C_1 \|Lv\|_{L^2(U_j)}. \quad (2.19)$$

The properties of the cutoff function $\xi$ then give that

$$\|u \circ x_j\|_{H^2(U_{j,i})} = \|v\|_{H^2(V_{j,i})} \leq C_1 \|Lv\|_{L^2(U_j)} \leq C_2 \left( \|f \circ x_j\|_{L^2(U_j)} + \|u \circ x_j\|_{H^1(U_j)} \right). \quad (2.20)$$

Summing the above inequality over $j$ says that for each $i$, the quantity $\|u\|_{H^2(\Gamma_i)}$ can be bounded by the sum of $\|f\|_{L^2(\Gamma)}$ and $\|u\|_{H^1(\Gamma)}$. The solution $u$ is this in $H^2(\Gamma)$ and so the above integration by parts calculation is valid and gives the result of part 1 of the theorem.

For Part 2 of the theorem, the expansion of $u$ in each corner $S$ is given directly by Theorem 3 with the eigenfunctions given in Lemma 1. The global form of $u$ in (2.18) can then be found using the argument in Corollary 3.1.
We now go back and verify that we may apply Theorem 3 to this equation. It is not hard to check that that $v$ also vanishes at $\partial U_j$ and that $Lv$ is in $L^2(U_j)$, since $u$ is in $H^1(\Gamma)$ and $f$ is in $L^2(\Gamma)$. Furthermore, due to the piecewise smooth assumptions on $\Gamma$, the pull-back $L$ is an elliptic operator with smooth coefficients aside from jumps occurring at interfaces, which correspond to edges on $\Gamma$.

We must now verify that $v$ is in the space $V$ that we introduced in Theorem 3 and that was defined in [40]. Theorem 4.2 of [40] gives that there exists a unique solution $\tilde{v}$ of $L\tilde{v} = \tilde{f}$ in the space $V$. Since both $v$ and $\tilde{v}$ are solutions of the weak form equation

$$\int_{x_j(U_j)} \nabla \tilde{v} \cdot \nabla \varphi = \int_{x_j(U_j)} \tilde{f} \varphi \quad \text{for all } \varphi \in H^1_0(x_j(U_j)), \quad (2.21)$$

the Lax-Milgram theorem tells us that $v = \tilde{v}$. We have thus seen that Theorem 3 may be applied and so the proof is complete.

We finalize our argument by noting that the reverse of the above integration by parts argument gives that solution of the interface form is also a solution of the usual weak form. We may therefore re-interpret the first claim in Theorem 4 as the following theorem, which is the main result of this paper.

**Theorem 5.** If $\Gamma$ is a piecewise smooth surface composed of faces $\Gamma_i$ and $f$ is a function in $L^2(\Gamma)$, then the solution $u$ of the weak form of the Laplace-Beltrami problem $\Delta u = f$ is a solution of the interface form of the Laplace-Beltrami problem. Furthermore, any solution of that interface form is also a solution of the weak form.

As an example, we consider the case where $\Gamma$ is a cone of height $h$ with a base of radius one. We let $\Gamma_1$ be the top half of the cone and let $\Gamma_2$ be the base of the cone, and parameterize these faces with the functions

$$x_1 = \left(\frac{h-z}{h} \cos(\theta), \frac{h-z}{h} \sin(\theta), z\right), \quad z \in [0, h], \quad \theta \in [0, 2\pi),$$

$$x_2 = (r \cos(\theta), r \sin(\theta), 0), \quad r \in [0, 1] \quad \theta \in [0, 2\pi),$$

respectively. Using this parameterization, Mathematica [29] can analytically solve the Laplace-Beltrami problem with data $f$,

$$f|_{\Gamma_1} = (h-z)^\alpha \, e^{in\theta}, \quad \alpha > -1, \quad n = 1, 2, 3, \ldots,$$

$$f|_{\Gamma_2} = 0. \quad (2.23)$$

The analytic solution is given by

$$u|_{\Gamma_1} = \left(c_1 (h-z)^{2+\alpha} + c_2 (h-z)^{\sqrt{1+h^2}|n|}\right) e^{in\theta},$$

$$u|_{\Gamma_2} = c_3 r^{|n|} e^{in\theta}, \quad (2.24)$$

where $c_1$, $c_2$, and $c_3$ are constants that depend on $h$, $n$, and $\alpha$. We can clearly see that in this case the solution has two more integrable derivatives than $f$, but picks up terms whose higher order derivatives are singular near the corner. If we note that the conic angle for this cone is given by $\gamma = 2\pi / \sqrt{1 + h^2}$, then it becomes even clearer that the solution has the expected behavior near the corner and edge.

Lastly, it is easy to see that if a solution of the interface problem happens to be smooth, then it is a solution of the following problem:
Problem 3 (Strong form of the Laplace-Beltrami problem). Let \( \Gamma \) be a surface composed of smooth faces \( \Gamma_i \) and suppose that \( f \in C(\Gamma) \), the strong form of the Laplace-Beltrami problem is to find a \( u \) that is in \( C^2(\Gamma_i) \) for each \( i \) and that satisfies

\[
\begin{align*}
\Delta_F u|_{\Gamma_i} &= f|_{\Gamma_i}, & \text{on } \Gamma_i, \\
\frac{\partial u}{\partial b_i} &= \frac{\partial u}{\partial b_j}, & \text{on } e_{ij}, \\
\quad u|_{\Gamma_i} = u|_{\Gamma_j} & \text{ on } e_{ij}.
\end{align*}
\] (2.25)

The edge conditions imply that \( u \in C^1(\Gamma) \).

While we do not give a proof of the well-posedness of this problem here, we do give the following remark about what such a proof would require.

Remark 1. It is likely that Theorem 4 holds for \( f \in L^p \) in some range of \( p > 2 \), though the authors are not aware of the required elliptic interface problem results. If it does hold, then the usual Sobolev embedding theorem will give that \( u \in C^{1,\alpha}(\Gamma_i) \) for some \( \alpha > 0 \) and for each \( i = 1, \ldots, N \) provided that \( f \in L^p(\Gamma) \). Such a theorem would therefore be enough to prove a version of Theorem 5 for the strong form of the Laplace-Beltrami problem, instead of the weaker interface form.

The results of this section imply that numerical solvers for the Laplace-Beltrami problem on piecewise smooth surfaces can discretize the differential operator acting on function in \( L^2 \) and obtain convergent results. We demonstrate this in the next section via a high-order numerical solver along surfaces of revolution.

3 Application: Surfaces of revolution with edges

In this section we detail a method for using separation of variables to formulate the Laplace-Beltrami problem on a surface of revolution into a sequence of decoupled periodic ODEs.

3.1 Separation of variables

To begin with, denote by \((r, \theta, z)\) the usual cylindrical coordinate system in three dimensions. If \( \Gamma \) is a piecewise smooth surface of revolution about the \( z \)-axis, then let \( \gamma \) denote its generating curve in the plane \( \theta = 0 \). The generating curve can be parameterized in terms of arclength, \( \gamma : [0, L] \rightarrow \mathbb{R}^3 \); let its cylindrical coordinate components (for \( \theta = 0 \)) be parameterized as

\[
\gamma(s) = (r(s), z(s)),
\] (3.1)

where \( s \in [0, L] \) denotes arclength along the generating curve \( \gamma \). As shown in the previous section, since the Laplace-Beltrami problem \( \Delta_F u = f \) is uniquely solvable on the space of mean-zero square-integrable functions on a piecewise smooth surface, we can write the solution \( u \) in a Fourier expansion in cylindrical coordinates as:

\[
u(x) = u(r, \theta, z) = u(s, \theta) = \sum_{n=-\infty}^{\infty} u_n(s) e^{in\theta},
\] (3.2)

and the right hand side \( f \) as:

\[
f(x) = \sum_{n=-\infty}^{\infty} f_n(s) e^{in\theta}.
\] (3.3)
Furthermore, we note that in the variables \(s\) and \(\theta\), the Laplace-Beltrami operator takes the form:

\[
\Delta_{\Gamma} = \frac{\partial^2}{\partial s^2} + \frac{1}{r} \frac{\partial}{\partial s} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.
\]  

(3.4)

Using the decompositions in (3.2) and (3.3), and the above form of the Laplace-Beltrami operator, we can transform the PDE into a sequence of decoupled periodic ODEs, one for each Fourier mode \(n\):

\[
\frac{d^2u_n}{ds^2} + \frac{1}{r} \frac{dr}{ds} \frac{du_n}{ds} - \frac{n^2}{r^2} u_n = f_n, \quad \text{for } s \in [0, L].
\]  

(3.5)

The solution \(u\) can then be easily synthesized via its Fourier ansatz. Note that the mean-zero condition on \(u\) for solvability of the Laplace-Beltrami problem reduces to a condition on only \(u_0\) since every mode with \(n \neq 0\) integrates to zero:

\[
\int_{\Gamma} u = \int_{0}^{2\pi} \int_{0}^{L} \sum_{n} u_n(s) e^{in\theta} r(s) \, ds \, d\theta = 2\pi \int_{0}^{L} u_0(s) r(s) \, ds.
\]  

(3.6)

Enforcing this mean-zero condition on \(u_0\) is discussed in the next section.

### 3.2 A periodic ODE solver

The separation of variables solution to the Laplace-Beltrami problem requires solving the sequence of periodic ODEs in (3.5) with the usual periodic boundary condition: continuity in the solution and its derivative \([12, 16, 47]\). In order to solve the ODE for each Fourier mode of the Laplace-Beltrami equation, we shall convert it into a second-kind integral equation. Doing this will allow us to easily use adaptive high-order quadrature methods to solve it accurately. This conversion is applicable to a broad class of periodic ODEs, so we shall present the procedure in a general framework. A similar approach for the Laplace-Beltrami problem on smooth surfaces of revolution was used in [19] (using a global trapezoidal discretization scheme); a more general adaptive approach for two-point boundary value problems, coupled with a fast direct solver, was detailed in the widely known work of [36]. To this end, we shall consider a method to solve any ODE on \([0, L]\) of the form:

\[
u'' + pu' + qu = f, \quad u(x + L) = u(x), \quad u''(x + L) = u''(x),\]

(3.7)

where \(f, p,\) and \(q\) are known periodic functions in \(L^r([0, L])\) for some \(r > 1\). The solution \(u \in C^1([0, L])\) will be \(L\)-periodic. If \(q = 0\), then the solution can only be determined up to an additive constant; in this case, an additional constraint must be imposed to ensure well-posedness of the problem. Usually this constraint takes the form of a linear function of \(u\), such as

\[
\int_{0}^{L} u(x) \, w(x) \, dx = A.
\]

(3.8)

We address this special case where \(q = 0\) later on in this section.

In order to convert the ODE (3.7) on \([0, L]\) into an integral equation on the same interval, first consider the kernel \(G_L\),

\[
G_L(x) = \frac{1}{2L} \left( \text{mod}(x, L) - \frac{L}{2} \right)^2 + \frac{L}{24}.
\]

(3.9)

It is not hard to verify that if this kernel is convolved with a function \(f\) that is mean-zero on \([0, L]\), then the resulting function \(v = G_L * f\) solves the one-dimensional periodic Poisson equation \(v'' = f\).
This ν is in fact the unique mean-zero solution with \( \int_0^L \nu = 0 \), since \( G_L \) is also mean-zero on this interval. We next define the “single layer operator” \( S_L \) via the convolution

\[
S_L f(x) = \int_0^L G_L(x - t) f(t) \, dt.
\] (3.10)

Since \( G_L \) is mean-zero on \([0, L]\), \( S_L f \) is also mean-zero on the same interval. We now represent the solution \( u \) to (3.7) as

\[
u = S_L \sigma + C,
\] (3.11)

for some unknown density \( \sigma \) with \( \int_0^L \sigma = 0 \) and some unknown constant \( C \) which, by construction, gives

\[
\int_0^L u = \int_0^L S_L \sigma + \int_0^L C \implies C = \frac{1}{L} \int_0^L u,
\] (3.12)
i.e. \( C \) is the mean of the solution \( u \) on \([0, L]\). We have included the constant \( C \) in the representation in order to ensure that it is a complete representation; \( S_L \sigma \) will always be a mean-zero function on \([0, L]\), but the solution \( u \) may not be. This representation for \( u \) also ensures that the solution is automatically periodic since \( G_L \) is periodic.

With this representation, we have changed the problem of finding \( u \) into the problem of finding \( \sigma \) and \( C \). Inserting (3.11) in (3.7) yields a Fredholm second-kind integral equation for \( \sigma \) and \( C \):

\[
\sigma + p S_L' \sigma + q S_L \sigma + q C = f,
\] (3.13)

where

\[
S_L' \sigma(x) = \int_0^L G_L'(x - t) \sigma(t) \, dt.
\] (3.14)

The above integral equation is indeed invertible when \( q \neq 0 \) because the underlying ODE is invertible and we simply used a complete and unique representation for the solution \( u \). In order to see that our transformation has led to a well-conditioned equation, we note that (3.13) has the form of \((I + K)\sigma + q C = f\), where \( I \) is the identity operator and \( K \) is a compact integral operator. The integral equation is therefore of Fredholm second-kind [32].

If \( q = 0 \), then we must explicitly enforce the additional integral condition \( \int_0^L u w = A \) from (3.8). We shall include this condition by simply adding it to (3.13), giving the integral equation

\[
\sigma + p S_L' \sigma + \int_0^L (S_L \sigma + C) w = f + A.
\] (3.15)

This method of adding a linear constraint to an integral equation is equivalent to adding a rank-one update to the original integral operator. It is not difficult to see that this update results in an invertible and well-conditioned equation provided the range of the update is not contained in the range of the original integral operator. See [46] for a discussion of this method in the matrix equation setting. In our case, this is equivalent to asking if there exists a mean-zero function on \([0, L]\) that is not in the null space of the adjoint, i.e. if there exists a mean-zero and periodic \( f \) such that \( f + S_L'[p f] \neq 0 \). Taking a derivative of this expressions gives the condition that \( f' + p f \neq 0 \). Plugging in \( f = \sin(\frac{2\pi}{L} x) \) or \( f = \cos(\frac{2\pi}{L} x) \) will yield at least one example.

Next, we check that the solution of (3.13) produces a \( u \) that is automatically in \( C^1([0, L]) \). To do this, we first note that for (3.13) to make sense, the solution \( \sigma \) must be in \( L^1([0, L]) \). Applying Young’s inequality then tells us that \( S_L \sigma \) and \( S_L' \sigma \) are uniformly bounded, and therefore (3.13) gives that \( \sigma \) is in \( L'([0, L]) \). Finally, we note that \( S_L \) maps \( L'([0, L]) \to C^1([0, L]) \). This can
easily be proven using Hölder’s inequality and the fact that $G_L$ and $G_L'$ are piecewise continuous and bounded. Knowing that $\sigma \in L’([0, L])$ then gives that $S_L\sigma$, and thus the solution $u$, are in $C^1([0, L])$. An analogous argument holds true in the $q = 0$ case as well.

We now have a suitable second-kind integral equation form of (3.7) and its solution will have the expected smoothness properties. In the Laplace-Beltrami case, we have that

$$p = \frac{1}{r} \frac{dr}{ds}, \quad q = -\frac{n^2}{r^2}, \quad \text{and when } n = 0 \text{ we enforce the additional constraint on } u_0 \text{ of}$$

$$\int_0^L u_0(s) r(s) ds = 0,$$

which, as mentioned before, is equivalent to the mean-zero constraint $\int_f u = 0$ [19]. Since $r$ is a piecewise smooth function bounded away from 0, $p$ and $q$ will be in $L’([0, L])$ for any $r \geq 1$. Lastly, in light of the earlier discussion in the manuscript, the case of interest where $f \in L^2[0, L]$ satisfies the earlier requirements.

**Remark 2.** The above integral equation formulation of (3.7) is not the only possible integral equation formulation. The above derivation could easily be repeated with other kernels (i.e. Green’s functions). One example can be found in [19], where they considered the function

$$G_Y(x) = \frac{1}{2} e^{-|x|} - \frac{e^{-L}}{1 - e^{-L}} \cosh(\tilde{x}), \quad \text{where } \tilde{x} = \mod \left( x - \frac{L}{2}, L \right) + \frac{L}{2},$$

instead of the function $G_L$ used above. This function is the $L$-periodic Green’s function for the Yukawa problem $v'' - v = f$. Since $G_Y$ has the same smoothness properties as $G_L$, the resulting integral equation will still be second-kind and give rise to a unique solution $u$ that is periodic and in $C^1([0, L])$ whenever $p, q,$ and $f$ are in $L^r([0, L])$ for some $r > 1$.

We also note that $G_Y$ has a minor advantage over $G_L$ because it has a non-zero mean. Therefore, there is no need to ensure that $\sigma$ is a priori mean-zero, and we would not need to include the constant $C$ in the representation (but the mean-zero constraint (3.8) would still need to be enforced in the case that $q = 0$). In practice, these advantages are small and we will see later that both kernels ultimately lead to very similarly performing integral equation methods.

### 3.3 A numerical solver

In this section, we shall describe a numerical solver for the integral equations in (3.13). To summarize, our solver is based on a Nyström method for the equation using an adaptive discretization of the interval $[0, L]$ consisting of a piecewise 16th-order Gauss-Legendre quadrature. (A 16th-order quadrature was chosen so as to ensure a high-order discretization, of course our method extends to any other order discretization.) An overview of various methods for discretizing integral equations along curves in two dimensions is given in [25]. For problems in which the surface $\Gamma$ is smooth (and therefore so is the generating curve $\gamma$), we partition the interval $[0, L]$ into uniformly sized panels. For piecewise smooth generating curves $\gamma$, we dyadically refine the panels using knowledge of where the geometric singularities occur as a function of arclength.

Let $\{P_i\}_{i=1}^M$ be a partition of the interval $[0, L]$, and $\{x_{ij}, w_{ij}\}_{j=1}^{16}$, be the 16th-order Gauss-Legendre quadrature nodes and weights on panel $i$. The integral equation in (3.13) is enforced at
each of the nodes $x_{ij}$:

$$\sigma_{ij} + \sum_{k=1}^{M} \left( p(x_{ij}) \int_{P_k} G'_L(x_{ij} - t) \sigma(t) \, dt + q(x_{ij}) \int_{P_k} G_L(x_{ij} - t) \sigma(t) \, dt \right) + q(x_{ij}) C = f(x_{ij}), \quad (3.19)$$

where $\sigma_{ij}$ denotes the solution to the linear system which will be approximately equal to $\sigma(x_{ij})$. We will show how to enforce the a priori mean-zero condition on the $\sigma_{ij}$'s below. It remains now to replace the integrals above with discrete sums.

First, notice that the kernel $G_L$ is piecewise smooth with a discontinuity in $G'_L$ only at the origin. With this in mind, the integrals in (3.19) corresponding to $k \neq i$ can be approximated using 16th-order Gauss-Legendre quadrature nodes, for example:

$$\int_{P_k} G_L(x_{ij} - t) \sigma(t) \, dt \approx \sum_{\ell=1}^{16} w_{k\ell} \, G_L(x_{ij} - x_{k\ell}) \sigma_{k\ell}, \quad \text{for } k \neq i. \quad (3.20)$$

For the “near field” integrals corresponding to when $k = i$ in (3.19), standard Gauss-Legendre quadrature will fail to yield high-order convergence due to the irregularity in the kernel $G_L$. With this in mind, we split the $k = i$ integrals into two pieces precisely at the point of irregularity, $x_{ij}$. On each of these new panels, the integrand is smooth and standard Gauss-Legendre quadrature can be used along with Lagrange interpolation on $\sigma$ to obtain implied values at the extra quadrature support nodes.

To explain in more detail: if $P_i = [a_i, b_i]$, then for each $j$, we approximate this near field integral over $P_i$ as

$$\int_{P_i} G_L(x_{ij} - t) \sigma(t) \, dt = \int_{a_i}^{x_{ij}} G_L(x_{ij} - t) \sigma(t) \, dt + \int_{x_{ij}}^{b_i} G_L(x_{ij} - t) \sigma(t) \, dt \approx \sum_{\ell=1}^{16} u_{ij\ell} \, G_L(x_{ij} - s_{ij\ell}) \tilde{\sigma}(s_{ij\ell}) + \sum_{\ell=1}^{16} v_{ij\ell} \, G_L(x_{ij} - t_{ij\ell}) \tilde{\sigma}(t_{ij\ell}), \quad (3.21)$$

where $(s_{ij\ell}, u_{ij\ell})$ is the $\ell$th Gaussian quadrature node and weight pair on the interval $[a_i, x_{ij}]$, $(t_{ij\ell}, v_{ij\ell})$ is the $\ell$th Gaussian quadrature node and weight pair on the interval $[x_{ij}, b_i]$, and $\tilde{\sigma}(x)$ is the value obtained from the $\sigma_{ij}$'s on $P_i$ via Lagrange interpolation to the point $x \in P_i$. The near field integrals in $S'_L$ can be discretized similarly.

Finally, recall our representation for the solution $u$: $u = S_L \sigma + C$, where $\int_0^L \sigma = 0$. This assumption on $\sigma$ must be explicitly enforced, and can easily be done so in one of two ways: (1) the condition can be discretized as

$$\int_0^L \sigma(s) \, ds = 0 \approx \sum_{i=1}^{M} \sum_{j=1}^{16} w_{ij} \sigma_{ij} \quad (3.22)$$

and appended to the system of equations yielding a square linear system of dimension $16M + 1$ for the unknowns $\sigma_{ij}$ and $C$, or (2) the original representation can be replaced with one of the form

$$u = S_L \left[ \sigma - \int_0^L \sigma \right] + \int_0^L \sigma. \quad (3.23)$$
The above alternative representation ensures that the argument to $S_L$ has mean-zero on $[0, L]$ and $C$ is equated with the integral of $\sigma$. This approach results in a square linear system of size $16M$ for only the unknowns $\sigma_{ij}$. In our subsequent numerical experiments, our solver implements the latter choice of changing the representation to equate the integral of $\sigma$ with the integral of the solution $u$.

Lastly, in the purely azimuthal component to the Laplace-Beltrami problem, i.e. the $n = 0$ mode, we must also enforce the mean-zero condition on the solution $u_0$ along the surface $\Gamma$. A discretization of this condition in (3.8) can be obtained by an identical procedure as to that used in discretizing $S_L \sigma$; the resulting equation can be put in the form

$$
\sum_{i=1}^{M} \sum_{j=1}^{16} c_{ij} \sigma_{ij} = 0 \tag{3.24}
$$

and added to each row of the $16M \times 16M$ linear system.

Remark 3. The continuous Laplace-Beltrami equations are well-conditioned with respect to the $L^2([0, L])$ norm, $\| \cdot \|_{L^2([0, L])}$. In order to ensure that our discretized equations are similarly well-conditioned in $\ell^2$ (as an embedding of the continuous problem), we must use a discretization method that approximates $\| \cdot \|_{L^2([0, L])}$ [6]. This is especially important when we use local adaptive refinement, as that will cause the $\ell^2$ norm of the discretized functions to greatly diverge from their true $L^2([0, L])$ norms. We address this issue by replacing $\sigma_{ij}$ in the discretized linear system with $\sqrt{w_{ij}} \sigma_{ij}$. An equivalent linear system is easily obtained by left and right diagonal preconditioners (as detailed in [6]) with the effect that the $\ell^2$ norm of the discrete unknown approximates the true $L^2$ norm of the solution to the continuous problem:

$$
\| \{ \sqrt{w_{ij}} \sigma_{ij} \} \|^2 = \sum_{ij} w_{ij} \sigma_{ij}^2 \approx \int_0^L \sigma^2(s) \, ds. \tag{3.25}
$$

Furthermore, the resulting linear system has a spectrum which converges to the spectrum of the original continuous system, yielding increased performance when using iterative solvers such as GMRES.

4 Numerical examples

In this section, we give the results of some numerical examples demonstrating the ODE and Laplace-Beltrami solvers detailed above. In order to compute the right hand sides $f_n$ in equation (3.5) we discretize the original function $f$ using equispaced points in the azimuthal and compute $f_n$ using the FFT (as in [19]). In all of the tests, the linear systems were solved using GMRES to a relative tolerance of $10^{-14}$. The linear systems were of modest size, and could easily be applied in a matrix-free fashion. All codes were written in MATLAB and no attempt was made to accelerate the code beyond the use of the FFT.

All reported errors were estimated in the relative $L^2$ sense, e.g. the relative error in the solution $u$ to an ODE was measured as

$$
\varepsilon = \sqrt{\frac{\sum_i w_i (u_{true}(x_i) - u_i)^2}{\sum_i w_i u_{true}^2(x_i)}}, \tag{4.1}
$$

where $u_{true}$ was either known a priori or estimated by using a finer discretization and we have used a single subscript $i$ above to index all $16M$ values on the interval $[0, L]$. For Laplace-Beltrami problems on a surface $\Gamma$, the formula above is modified to include the proper quadrature weight for
Figure 2: Convergence results for our smooth ODE solver test. The number of discretization points, $N_s$, is compared to the relative $L^2$ difference between the computed solution and the solution obtained via Chebfun. The difference plateaus at $10^{-12}$ as that is the accuracy reported by the Chebfun solver.

a surface integral (obtained as a tensor product of piecewise Gauss-Legendre quadrature with the trapezoidal rule).

Lastly, in what follows, we will use $N_s = 16M$ to denote the total number of discretization points on an interval $[0, L]$ and $N_\theta$ to denote the total number of points used in the azimuthal direction for Fourier analysis/synthesis via the FFT.

4.1 Comparing Greens functions

In order to compare the Green’s function $G_L$ discussed in Section 3.3 with the Green’s function $G_Y$ used in [19], we test our ODE solver on a globally smooth problem. We set the coefficient functions in the ODE to

$$p(x) = \sin(3x) - 2, \quad q(x) = 2 \sin(5x) - 3 \quad (4.2)$$

and the right hand side to

$$f(x) = \frac{d^2}{dx^2} e^{\sin(2x)}. \quad (4.3)$$

These functions are smooth, periodic, and the resulting ODE is not trivialized by a simple change of coordinates. Since these functions are smooth, we use uniform panels to discretize the interval.

As the exact solution for this equation is not known, we compare the resulting solutions to the solution obtained with Chebfun, a well-known MATLAB package that uses global Chebyshev approximation to solve ODEs with spectral accuracy [44]. As a further test, we repeated this experiment with a Chebyshev-based discretization (as opposed to the Gauss-Legendre one described earlier). In this test, second-kind Clenshaw-Curtis quadratures were used to compute the required integrals.

Figure 2 depicts the convergence of the solution in this regime. We see that our method indeed converges and gives us an accurate solution to the problem with a moderate number of discretization points. We can also see that both choices of kernels result in comparable accuracy, and neither was consistently better. As such, we chose to use the Poisson kernel $G_L$ for the remainder of this paper. Furthermore, Figure 2 also makes it clear that using Gauss-Legendre points results in a more accurate solution than is achieved with the same number of Chebyshev points, but only slightly so and both solutions converge to near machine precision.
4.2 Laplace-Beltrami on a smooth surface

In order to validate our Laplace-Beltrami solver, we constructed a surface $\Gamma$ and right-hand side $f$ with a known solution. We did this by constructing a smooth surface and choosing the exact solution to be a constant plus the restriction of a smooth function $v$ defined in all of $\mathbb{R}^3$ (following the approach in [42]). We then generated $f$ through the following well-known analytic formula which applies in any neighbourhood that a surface is smooth:

$$f = \Delta_{\Gamma} (v|_{\Gamma}) = \Delta v - 2H \frac{\partial v}{\partial n} - \frac{\partial^2 v}{\partial n^2}.$$  \hfill (4.4)

In the above formula, $H$ is the mean curvature of $\Gamma$ and $n$ is the normal to $\Gamma$. A derivation of this formula may be found in [39]. We evaluate $f$ by analytically computing the terms in (4.4) based on a global parameterization of the surface. In this example, $\Gamma$ is given by a circular torus with inner radius one and outer radius two, and we set $v$ to be the Newtonian potential centered at $x_0 = (0, 0.5, 0.5)$:

$$v(x) = -\frac{1}{|x - x_0|}.$$

The exact solution to this problem is given by $u = v|_{\Gamma} - \frac{1}{|\Gamma|} \int_{\Gamma} v$, where $|\Gamma|$ is the surface area of $\Gamma$. This solution and the corresponding right hand side are shown in Figure 3. We conducted a convergence test by varying $N_\theta$ and $N_s$. We can see from the relative errors in Figure 4 that our method is capable of accurately solving this problem, and therefore the solver is working as expected.

4.3 Singular surface test

In this experiment, we tested our Laplace-Beltrami solver on a non-smooth surface with right-hand sides that are irregular at surface edges. We set the surface $\Gamma$ to be a square toroid: the surface resulting from revolving a unit square about the $z$-axis. Some care must be taken when discretizing this surface because of the geometric singularities. It is necessary to ensure that panel boundaries in our discretization of the generating curve coincided with the surface edges, as the coefficient functions in (3.4) are singular at the edges of $\Gamma$. The coarsest possible discretization thus had one panel per face. In order to resolve the singularity in the right-hand side, we dyadically refined this coarse discretization into the edge where $f$ is singular. We denote the width of the finest panel as $h_{\text{final}}$ and study how the errors depend on it. In the $\theta$-direction, our surface and right hand sides were smooth, so we simply set $N_\theta = 10$.

In this example, we are not able to generate an exact solution and right-hand side through the same method as we used in the smooth surface case, as the restriction of a smooth function to our surface would not have been in $C^1(\Gamma)$. Instead, we specify that the true solution satisfies

$$\partial^2_s u(\theta, s) = \Theta(\theta)S(s).$$  \hfill (4.5)

We then use Chebfun’s anti-differentiation routine to compute $u$ and $\partial_s u$, and then set $f = \Delta_{\Gamma} u$ in $\theta$-Fourier space. Specifically, for each Fourier mode $n$, we perform the following calculations:

$$\partial_s u_n = \int \partial^2_s u_n - \frac{1}{4} \int_0^4 \left( \int \partial^2_s u_n \right),$$  \hfill (4.6)

and then for $n \neq 0$

$$u_n = \int \partial_s u_n - \frac{1}{4} \int_0^4 \left( \int \partial_s u_n \right),$$  \hfill (4.7)
(a) The solution of $\Delta F u = f$.

(b) The right hand side of $\Delta F u = f$.

Figure 3: The solution and right hand side of $\Delta F u = f$, where $u$ was chosen to be the mean-zero restriction of the Newtonian potential to $\Gamma$. The Newtonian potential is centered at the black circledot.

Figure 4: The errors of the computed solution in our circular torus tests. $N_x$ is the number of points around the generating curve and $N_\theta$ is the number of points used in the $\theta$-direction.
Figure 5: This figure shows the four solutions that were used to test our solver in the singular geometry of a square toroid. Each solution is plotted around the cross section $\theta = \frac{2\pi}{5}$.

and for $n = 0$

$$u_0 = \int \partial_s u_0 - \frac{1}{4} \int_0^4 w \left( \int \partial_s u_0 \right).$$

The right hand side is then computed (for all $n$) as

$$f_n = \partial^2_s u_n + \frac{\partial_r r}{r} \partial_r u_n + \frac{n^2}{r^2} u_n.$$  

(4.9)

Above, we use $\int$ to denote anti-derivative. We note that this is different than our methodology in Section 4.1, where we instead specified the form of $f$ and used Chebfun to solve the ODE for $u$. We choose to use anti-differentiation here as it becomes trivial to ensure that $u$ is mean-zero on $\Gamma$. Furthermore, this scheme does not require us to solve a singular ODE in order to compute the reference solution.

We tested the solver with several choices of $S$ in (4.5). As a smooth test we set $S(s) = \cos(\pi s / 2)$. We also tested several singular cases of the form $S(s) = |s - 2|^{\alpha}$, where we set $\alpha = -\frac{1}{4}, -\frac{1}{2}$, and $-\frac{3}{4}$. Here $s = 2$ is the arclength parameter corresponding to the top inner edge, so $u$ is non-smooth at that edge (and the one directly across from it). For all of our tests, we set $\Theta(\theta) = \sin(3\theta)$.

Cross sections of the resulting solutions are displayed in Figure 5. As expected, the solutions are continuously differentiable, with local extrema at $s = 2$. Further, as $\alpha$ approaches $-1$ and the minimum $r$ such that $f_n \in L^\infty([0, L])$ shrinks, the point of the extrema becomes sharper and the solution approaches the boundary of $C^1([0, L])$. This experiment is thus a good stress-test of our solver and will demonstrate how the solver behaves on problems that barely satisfy its basic requirements.

Figure 6a shows how the error in each test problem depended on the extent of the dyadic refinement. We see that the smooth case was resolved with a single panel on each face. From this, we can see that discontinuous coefficient functions did not prevent our method from computing accurate solutions with relatively few unknowns. In the singular tests, we can see that sufficient refinement was all that was necessary to achieve an accurate solution. Thus, the limited resolution of the singularity in the finest panel was the source of the dominant error in the solution. Furthermore, Figure 6b demonstrates that the error in fact decayed as $O(h_\text{final}^{1+\alpha})$, which was the order of accuracy in our evaluation of the left hand side of the integral equation. This fact is further discussed in Appendix A.

As a further verification, we performed a self convergence study that did not make use of Chebfun. To do this, we directly set $f(\theta, s) = \Theta(\theta)S(s)$, where $\Theta$ and $S$ were chosen to be the same as in the previous experiment. Figure 7a shows the relative difference between the solution on the finest grid and the solutions on other grids, for each of the choices of $S$. Most of the observations
(a) The relative $L^2$ errors of the square toroid tests.

(b) The relative $L^2$ errors of the square toroid tests raised to the power $\frac{1}{1+\alpha}$. The dashed line indicates the scaling $O(h_{\text{final}}^{1+\alpha})$.

Figure 6: The figures show how the relative error of the square toroid tests depended on the extent of the refinement. Each marker indicates a trial.

(a) The relative $L^2$ differences in the self-convergence study on the square toroid.

(b) The relative $L^2$ difference raised to the power $\frac{1}{1+\alpha}$. The dashed line indicates the scaling $O(h_{\text{final}}^{1+\alpha})$.

Figure 7: These figures show how the relative difference between the most refined solution and the other solutions depends on their refinement in the self-convergence study.
from the previous experiment apply here, except that in the case where \( S(s) = \cos(\pi s/2) \). In this case, the solution was not smooth, so two panels on each face were necessary in order to accurately resolve it.

### 4.4 Harmonic vector field computation

The surfaces of revolution that we have considered so far are non-trivial and of genus one; it is well-known that they support a two-dimensional linear space of harmonic vector fields, i.e. square-integrable tangential vector fields \( H \in L^2_\Gamma(\Gamma) \) such that \( \nabla_\Gamma \cdot H = 0 \) and \( \nabla_\Gamma \cdot (n \times H) = 0 \) in \( H^{-1}(\Gamma) \).

For all surfaces of revolution, an orthogonal basis for these harmonic vector fields is analytically given by

\[
H_1 = \frac{1}{r} \hat{s}, \quad \text{and} \quad H_2 = \frac{1}{r} \hat{\theta},
\]

where \( \hat{s} \) and \( \hat{\theta} \) denote unit vectors along the generating curve and in the azimuthal direction, respectively [18, 19]. This may be easily verified by direct calculation. As a further test of our Laplace-Beltrami solver, we shall use it to construct a basis for harmonic vector fields along \( \Gamma \).

In order to compute a basis for the harmonic vector fields, we shall make use the Hodge decomposition of a general vector field along \( \Gamma \), which was introduced in section 1. This decomposition splits a tangential vector field \( F \) into a curl-free component \( F_{\text{cf}} \) (i.e. one where \( r_\Gamma \cdot n F_{\text{cf}} = 0 \)), a divergence-free component \( F_{\text{df}} \), and a harmonic component \( H \). The Hodge decomposition of \( F \) can be written explicitly as

\[
F = \nabla_\Gamma \alpha + n \times \nabla_\Gamma \beta + H,
\]

where \( \alpha \) and \( \beta \) are mean-zero scalar functions defined on \( \Gamma \). See [28], for example, for a more detailed discussion of this representation in similar genus one geometries. Taking the surface divergence of (4.11), as well as the surface curl (4.11), shows that \( \alpha \) and \( \beta \) must satisfy

\[
\Delta_\Gamma \alpha = \nabla_\Gamma \cdot F, \quad \text{and} \quad \Delta_\Gamma \beta = -\nabla_\Gamma \cdot (n \times F).
\]

With the above Hodge decomposition in mind, it becomes clear how to compute examples of Harmonic vector fields \( H \): we can simply choose a tangential vector field \( F \) and subtract off the components \( \nabla_\Gamma \alpha \) and \( n \times \nabla_\Gamma \beta \). In order to make this numerically feasible, we shall restrict our choice of \( F \) to be smooth on each face of \( \Gamma \). We may then compute \( \nabla_\Gamma \cdot F \) and \( \nabla_\Gamma \cdot (n \times F) \) pseudo-spectrally by first Fourier decomposing \( F \) as:

\[
F = \sum_n \left( F_s^n \hat{s} + F_\theta^n \hat{\theta} \right),
\]

and then, mode-by-mode, applying the formula:

\[
\nabla_\Gamma \cdot F = \frac{d}{ds} F_s^s + \frac{1}{r} \frac{dr}{ds} F_s + \frac{1}{r} \frac{d}{d\theta} F_\theta.
\]

In order to compute \( dF_s^n/\,ds \), we interpolate \( F_s^n \) onto Chebyshev panels in arclength along the generating curve and use Chebyshev differentiation. Having computed the divergences (i.e. the right hand side to a Laplace-Beltrami problem), we use our method to solve the Laplace-Beltrami equation for \( \alpha \) and \( \beta \). Next, we compute the surface gradient of \( \alpha \) and \( \beta \) (again, mode-by-mode) through the formula

\[
\nabla_\Gamma u = \frac{du}{ds} \hat{s} + \frac{1}{r} \frac{du}{d\theta} \hat{\theta}.
\]
Figure 8: The two basis harmonic vector fields for a square toroid $H_1 = \frac{1}{r} \hat{s}$ (left) and $H_2 = \frac{1}{r} \hat{\theta}$ (right). Color is used to indicate the magnitude of the field.

Note that differentiation with respect to $\theta$ is merely multiplication by $\hat{\theta}$ in Fourier-space. Lastly, to compute $d\alpha/ds$ and $d\beta/ds$ we note that we already know $\alpha$ and $\beta$ as the solution of a system of integral equations with the representation, for example, $\alpha_n = S L \sigma_n + C$. We can therefore use the formula $d\alpha_n/ds = S L' \sigma_n$ to easily compute this quantity via quadrature on the integral representation. Once all terms are computed for each mode, we can synthesize the Fourier series and evaluate the harmonic component as

$$H = F - \nabla_F \alpha - n \times \nabla_F \beta. \quad (4.16)$$

As a measure of accuracy, we then project $H$ onto the basis $H_1$ and $H_2$ in (4.10) and look at the $L^2(\Gamma)$ norm of the remainder relative to the norm of $F$ to determine if the computed field lies in the space of harmonic vector fields.

As a test field, we computed the harmonic component of $F = r \hat{s} + r^{-2} \hat{\theta}$. With two panels per face, the relative $L^2$ norm of the remainder was less than $10^{-14}$. We also validated our code by computing the harmonic components of the exact basis $\{H_1, H_2\}$. We found that the basis is within machine precision of being harmonic.

5 Conclusions

In this work, we reformulated the Laplace-Beltrami problem on piecewise smooth surfaces as a collection of smooth problems on each face combined with continuity conditions at surface edges. To summarize, if the right hand side of a Laplace-Beltrami problem is in $L^2(\Gamma)$, then the solution in the usual weak sense will be sufficiently well-behaved at the surface edges and in $H^2$ near surface corners. Furthermore, we analytically computed an expansion of such solutions in corners having conic angle of $\gamma$; the leading order term is of the form $r^{2\pi \gamma / \pi}$, where $r$ is the distance to the corner along the surface.

Furthermore, we used this reformulation to develop a numerical method that solves the Laplace-Beltrami problem on piecewise smooth surfaces of revolution. The numerical results support the theoretical results of the paper. This method converted each Fourier mode of the Laplace-Beltrami problem into a second-kind integral equation that could be accurately solved using standard numerical techniques for integral equations. The integral equation formulation for solving the associated one-dimensional periodic ODEs can be easily generalized to any second-order periodic ODE with coefficients and right hand side in $L'(\mathcal{L})$ for some $r > 1$, even if they are non-smooth.
This Laplace-Beltrami solver, and the experiments used to verify it, demonstrated the ability to easily obtain high-order accuracy for the problem on piecewise smooth surfaces. However, this specific solver is limited to surfaces of revolution that are separated from their axis of rotation. In future work, we plan to develop a new integral equation based solver that can be applied to a more general classes of surfaces, for example piecewise smooth surfaces specified by a collection of charts with no symmetry assumptions at all. This will likely be based on the Calderon-Identity approach discussed in [42] or be based on a parametrix method [27], similar to the approach in [35]. Work in these directions is ongoing.

A Error bounds for singular data

Here we discuss the error in our method for computing the left hand sides of (3.13) in the singular surface test. In this test, we chose the solution to satisfy 

\[ u''(s) = \Theta(\theta)|s - 2|^{\alpha} \]

We will see that the error is \( O(h_{\text{final}}^{1+\alpha}) \), where \( h_{\text{final}} \) denotes the width of the most refined panel. This will imply that the error in \( \sigma \) has the same order.

In order to evaluate the left hand side of (3.13), we must evaluate integrals of the form

\[ \int_0^4 K \sigma_n ds \]

for various kernels \( K \). This integral is challenging to compute because \( \sigma_n(s) = C|s - 2|^{\alpha} \). We shall assume for this discussion that \( K \) is smooth and bounded on each face. In reality, the kernel will only be piecewise smooth and we will the use panel splitting idea in Section 3.3 to accurately compute the integral. However, for the sake of clarity, we omit these details in this discussion.

To study the error, we consider the integral over an example face: \( s \in (2, 4) \). We split the integral into two pieces, one over the finest panel \( (2, 2 + h_{\text{final}}) \) where \( \sigma_n \) is singular, and one over the rest of the panels where \( \sigma_n \) is smooth. On the finest panel, \( K \) may be approximated as having its value at \( s = 2 \) since \( h_{\text{final}} \) is sufficiently small. The integral thus becomes

\[ \int_2^{2+h_{\text{final}}} K \sigma_n ds \approx h_{\text{final}} \int_0^1 K(2) C |h_{\text{final}}|^{\alpha} d\tilde{s} = CK(2)^+ h_{\text{final}}^\alpha \frac{1}{1 + \alpha}. \]  

If we apply our quadrature rule to the integral we obtain

\[ h_{\text{final}} \sum_{i=1}^{16} w_i CK(2)^+ (h_{\text{final}} x_i)^\alpha = CK(2)^+ h_{\text{final}}^{\alpha+1} \sum_{i=1}^{16} w_i x_i^\alpha, \]  

where the \( x_i \)'s and \( w_i \)'s are the standard 16th-order Gauss-Legendre quadrature points and weights. The error on this panel thus approaches

\[ |CK(2)^+| h_{\text{final}}^{\alpha+1} \left| \frac{1}{1 + \alpha} - \sum_{i=1}^{16} w_i x_i^\alpha \right|. \]  

Since the function \( x^\alpha \) is not integrated exactly by Gauss-Legendre quadrature, the error on the finest panel will be \( O(h_{\text{final}}^{1+\alpha}) \).

On the panels where the function is smooth, we use the standard formula for the error resulting from applying \( k \)th-order Gauss-Legendre quadrature to integrate a function \( f \) on the interval \((a, b)\), see §5.2 of [31]:

\[ \left| \int_a^b f(x) dx - \sum_{j=1}^k w_j f(x_j) \right| = \frac{(b - a)^{2k+1}(k!)^4}{(2k + 1)(2k)!^3} f^{(2k)}(\xi), \]  

for some \( \xi \in (a, b) \).
If we let $h_0, \ldots, h_N = h_{\text{final}}$ be the panel widths in our dyadic refinement, then by definition $h_i = h_0 2^{-i}$. Since $K$ is smooth on the interval $(2, 4)$, the dominant term in $(K\sigma_n)^{(2n)}$ will be $CK(s)|s - 2|^{\alpha-2k}$. On the panel of width $h_i$, this term may be bounded by $|C| \max |K| h_i^{\alpha-2k}$, since that panel is a distance $h_i$ away from the singularity. The error on that panel is thus bounded by

$$|C| \max |K| \frac{h_i^{2k+1}(k!)^4}{(2k+1)(2k)!^3} h_i^{\alpha-2k}. \quad (A.5)$$

Summing our error over all of the smooth panels gives the bound

$$|C| \max |K| \frac{h_0^{1+\alpha}(k!)^4}{(2k+1)(2k)!^3} \sum_{i=0}^{N} 2^{(1+\alpha)(-i)} \leq |C| \max |K| \frac{h_0^{1+\alpha}(k!)^4}{(2k+1)(2k)!^3} \frac{1}{1 - 2^{1+\alpha}}. \quad (A.6)$$

Since we do not apply our scheme to the case where $\alpha$ is exponentially close to -1, and we are using $k = 16$, this error will be well below machine precision. The error from the final panel thus dominates the error in the smooth panels, and therefore the error in computing the left hand side of (3.13) will be $O(h_{\text{final}}^{1+\alpha})$. 

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References

[1] S. Angenent, S. Haker, A. Tannenbaum, and R. Kikinis. On the Laplace-Beltrami Operator and Brain Surface Flattening. *IEEE Trans. Med. Imag.*, 18(8):700–711, 1999.

[2] E. Bänsch, P. Morin, and R. H. Nochetto. A finite element method for surface diffusion: The parametric case. *J. Comput. Phys.*, 203(1):321–343, 2005.

[3] L. Beirão da Veiga, F. Brezzi, L. D. Marini, and A. Russo. The hitchhiker’s guide to the virtual element method. *Math. Models and Meth. Appl. Sci.*, 24(08):1541–1573, 2014.

[4] A. Bonito, A. Demlow, and R. H. Nochetto. *Finite element methods for the Laplace–Beltrami operator*, volume 21. Elsevier B.V., 1st edition, 2020.

[5] A. Bonito, A. Demlow, and J. Owen. A Priori Error Estimates for Finite Element Approximations to Eigenvalues and Eigenfunctions of the Laplace–Beltrami Operator. *SIAM J. Num. Anal.*, 56(5):2963–2988, 2018.

[6] J. Bremer. On the Nyström discretization of integral equations on planar curves with corners. *Appl. Comput. Harm. Anal.*, 32(1):45–64, 2012.

[7] A. Buffa and P. Ciarlet. On traces for functional spaces related to Maxwell’s equations Part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications. *Math. Meth. Appl. Sci.*, 24(1):31–48, 2001.

[8] A. Buffa, M. Costabel, and C. Schwab. Boundary element methods for Maxwell’s equations on non-smooth domains. *Numerische Mathematik*, 92(4):679–710, 2002.

[9] A. Buffa, R. Hiptmair, T. von Petersdorff, and C. Schwab. Boundary Element Methods for Maxwell Transmission Problems in Lipschitz Domains. *Num. Math.*, 95:459–485, 2003.

[10] E. Burman, P. Hansbo, M. G. Larson, and A. Massing. A cut discontinuous Galerkin method for the Laplace-Beltrami operator. *IMA J. Num. Anal.*, 37(1):138–169, 2017.

[11] E. V. Chernokozhin and A. Boag. Method of generalized debye sources for the analysis of electromagnetic scattering by perfectly conducting bodies with piecewise smooth boundaries. *IEEE Transactions on Antennas and Propagation*, 61(4):2108–2115, 2013.

[12] E. A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations*. Krieger, Malabar, FL, 1984.

[13] M. Costabel and M. Dauge. Singularities of Electromagnetic Fields in Polyhedral Domains. *Archive for Rational Mechanics and Analysis*, 151(3):221–276, 2000.

[14] A. Demlow and G. Dziuk. An adaptive finite element method for the Laplace–Beltrami operator on implicitly defined surfaces. *SIAM J. Num. Anal.*, 45(1):421–442, 2007.

[15] G. Dziuk. Finite elements for the Beltrami operator on arbitrary surfaces. In *Partial differential equations and calculus of variations*, pages 142–155. Springer, Berlin, Heidelberg, 1988.

[16] M. S. P. Eastham. *The spectral theory of periodic differential equations*. Scottish Academic Press, London, UK, 1973.
[17] C. L. Epstein and L. Greengard. Debye Sources and the Numerical Solution of the Time Harmonic Maxwell Equations II. *Comm. Pure Appl. Math.*, pages NA–NA, dec 2009.

[18] C. L. Epstein, L. Greengard, and M. O’Neil. Debye Sources and the Numerical Solution of the Time Harmonic Maxwell Equations II. *Comm. Pure Appl. Math.*, 66(5):753–789, 2013.

[19] C. L. Epstein, L. Greengard, and M. O’Neil. A high-order wideband direct solver for electromagnetic scattering from bodies of revolution. *J. Comput. Phys.*, 387:205–229, jun 2019.

[20] J. Escher, U. F. Mayer, and G. Simonett. The surface diffusion flow for immersed hypersurfaces. *SIAM J. Math. Anal.*, 29(6):1419–1433, 1998.

[21] G. B. Folland. *Introduction to Partial Differential Equations*. Princeton University Press, Princeton, NJ, 2nd edition, 1996.

[22] T. Frankel. *The Geometry of Physics*. Cambridge University Press, Cambridge, UK, 3rd edition, 2011.

[23] M. Frittelli and I. Sgura. Virtual element method for the Laplace-Beltrami equation on surfaces. *ESAIM: Mathematical Modelling and Numerical Analysis*, 52(3):965–993, 2018.

[24] F. Gesztesy, I. Mitrea, D. Mitrea, and M. Mitrea. On the nature of the Laplace-Beltrami operator on Lipschitz manifolds. *J. Math. Sci.*, 172(3):279–346, 2011.

[25] S. Hao, A. H. Barnett, P. G. Martinsson, and P. Young. High-order accurate methods for Nyström discretization of integral equations on smooth curves in the plane. *Adv. Comput. Math.*, 40:245–272, 2014.

[26] K. Hildebrandt and K. Polthier. On approximation of the Laplace–Beltrami operator and the Willmore energy of surfaces. *Computer Graphics Forum*, 30(5):1513–1520, 2011.

[27] L. Hörmander. *The Analysis of Linear Partial Differential Operators III*. Springer-Verlag, New York, NY, 1994.

[28] L. M. Imbert-Gérard and L. Greengard. Pseudo-Spectral Methods for the Laplace-Beltrami Equation and the Hodge Decomposition on Surfaces of Genus One. *Num. Meth. for Partial Diff. Eq.*, 33(3):941–955, 2017.

[29] W. R. Inc. *Mathematica*, Version 12.3.1. Champaign, IL, 2021.

[30] F. John. *Partial Differential Equations*. Springer-Verlag, New York, NY, fourth edition, 1982.

[31] D. Kahaner, C. Moler, and S. Nash. *Numerical methods and software*. Prentice-Hall, Inc., 1989.

[32] R. Kress and D. Colton. *Integral Equation Methods in Scattering Theory*. John Wiley and Sons, Inc., New York, 1983.

[33] J. Kromer and D. Bothe. Highly accurate numerical computation of implicitly defined volumes using the Laplace-Beltrami operator. *arXiv:1805.03136*, [physics.flu-dyn]:1–25, 2018.

[34] M. Kropinski, N. Nigam, and B. Quaife. Integral equation methods for the Yukawa-Beltrami equation on the sphere. *Adv. Comput. Math.*, 42(2):469–488, 2016.
[35] M. C. A. Kropinski and N. Nigam. Fast integral equation methods for the Laplace-Beltrami equation on the sphere. *Adv. Comput. Math.*, 40(2):577–596, 2014.

[36] J.-Y. Lee and L. Greengard. A Fast Adaptive Numerical Method for Stiff Two-Point Boundary Value Problems. *SIAM J. Sci. Comput.*, 18(2):403–429, 1997.

[37] D. Malhotra, A. Cerfon, L.-M. Imbert-Gérard, and M. O’Neil. Taylor States in Stellarators: A Fast High-order Boundary Integral Solver. *J. Comput. Phys.*, 397:108791, 2019.

[38] M. Mitrea. The method of layer potentials in electromagnetic scattering theory on nonsmooth domains. *Duke Mathematical Journal*, 77(1):111–133, 1995.

[39] J.-C. Nedéléc. *Acoustic and Electromagnetic Equations*. Springer-Verlag, New York, NY, 2001.

[40] S. Nicaise and A.-M. Sändig. General interface problems – I. *Math. Meth. in the Appl. Sci.*, 17(6):395–429, 1994.

[41] S. Nicaise and A.-M. Sändig. General interface problems – II. *Math. Meth. in the Appl. Sci.*, 17(6):431–450, 1994.

[42] M. O’Neil. Second-kind integral equations for the Laplace-Beltrami problem on surfaces in three dimensions. *Adv. Comput. Math.*, 44(5):1385–1409, 2018.

[43] M. O’Neil and A. J. Cerfon. An integral equation-based numerical solver for Taylor states in toroidal geometries. *J. Comput. Phys.*, 359:263–282, 2018.

[44] R. B. Platte and L. N. Trefethen. Chebfun: A new kind of numerical computing. In *Progress in Industrial Mathematics at ECMI 2008*, pages 69–87. Springer, 2010.

[45] A. Rahimian, I. Lashuk, S. Veerapaneni, A. Chandramowlishwaran, D. Malhotra, L. Moon, R. Sampath, A. Shringarpure, J. Vetter, R. Vuduc, et al. Petascale direct numerical simulation of blood flow on 200k cores and heterogeneous architectures. In *SC’10: Proceedings of the 2010 ACM/IEEE International Conference for High Performance Computing, Networking, Storage and Analysis*, pages 1–11. IEEE, 2010.

[46] J. Sifuentes, Z. Gimbutas, and L. Greengard. Randomized methods for rank-deficient linear systems. *Elec. Trans. Num. Anal.*, 44:177–188, 2015.

[47] L. N. Trefethen, A. Birkisson, and T. A. Driscoll. *Exploring ODEs*. SIAM, Philadelphia, PA, 2018.

[48] S. K. Veerapaneni, A. Rahimian, G. Biros, and D. Zorin. A fast algorithm for simulating vesicle flows in three dimensions. *J. Comput. Phys.*, 230(14):5610–5634, 2011.

[49] M. Wang, S. Leung, and H. Zhao. Modified Virtual Grid Difference for Discretizing the Laplace–Beltrami Operator on Point Clouds. *SIAM J. Sci. Comput.*, 40(1):A1–A21, 2018.

[50] M. Wardetzky. *Discrete Differential Operators on Polyhedral Surfaces - Convergence and Approximation*. PhD thesis, Freie Universität Berlin, 2007.

[51] F. W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Springer, New York, NY, 2013.