Fan, splint and branching rules.

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Abstract
Splint of root system for simple Lie algebra appears naturally in studies of (regular) embeddings of reductive subalgebras. Splint can be used to construct branching rules. We demonstrate that splint properties implementation drastically simplify calculations of branching coefficients.

1 Introduction

Embedding $\phi$ of a root system $\Delta_1$ into a root system $\Delta$ is a bijective map of roots of $\Delta_1$ to a (proper) subset of $\Delta$ that commutes with vector composition law in $\Delta_1$ and $\Delta$.

$$\phi : \Delta_1 \rightarrow \Delta$$

$$\phi (\alpha + \beta) = \phi \circ \alpha + \phi \circ \beta, \ \alpha, \beta \in \Delta_1$$

Note that the image $\text{Im}(\phi)$ must not inherit the root system properties except the addition rules equivalent to the addition rules in $\Delta_1$ (for pre-images). Two embeddings $\phi_1$ and $\phi_2$ can splinter $\Delta$ when the latter can

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be presented as a disjoint union of images $\text{Im}(\phi_1)$ and $\text{Im}(\phi_2)$. The term *splint* was introduced by D. Richter in [?] where the classification of splints for simple Lie algebras was obtained. There was also mentioned that splint must have tight connections with the injection fan construction. The fan $\Gamma \subset \Delta$ was introduced in [?] as a subset of root system describing recurrent properties of branching coefficients for maximal embeddings. Injection fan is an efficient tool to study branching rules. Later this construction was generalized to non-maximal embeddings and infinite-dimensional Lie algebras in [?, ?].

In the present paper we study connections between splint and injection fan for regular embedding of reductive subalgebras $\mathfrak{a}$ in simple Lie algebra $\mathfrak{g}$. We show that (under certain conditions described in section 3) splint is a natural tool to study reduction properties of $\mathfrak{g}$-modules with respect to a subalgebra $\mathfrak{a} \longrightarrow \mathfrak{g}$. Using this tool we obtain the main result – the one-to-one correspondence between weight multiplicities in irreducible modules of splint and branching coefficients for a reduced module $L^\mu_{\mathfrak{g} \downarrow \mathfrak{a}}$.

## 2 Injections and splints

Consider a simple Lie algebra $\mathfrak{g}$ and its regular subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^*_\mathfrak{a} \subset \mathfrak{h}^*_\mathfrak{g}$. Let $\mathfrak{a}^s$ be a semisimple summand of $\mathfrak{a}$, this means that $\mathfrak{a} = \mathfrak{a}^s \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \ldots$. We shall consider $\mathfrak{a}^s$ to be a proper regular subalgebra and $\mathfrak{a}$ to be the maximal subalgebra with $\mathfrak{a}^s$ fixed that is the rank $r$ of $\mathfrak{a}$ is equal to that of $\mathfrak{g}$.

The following notations are used:
- $r$, $(r_{\mathfrak{a}^s})$ — the rank of $\mathfrak{g}$ (resp. $\mathfrak{a}^s$);
- $\Delta$, $(\Delta_{\mathfrak{a}})$ — the root system; $\Delta^+$ (resp. $\Delta^+_{\mathfrak{a}}$) — the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);
- $S$, $(S_{\mathfrak{a}})$ — the system of simple roots (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);
- $\alpha_i$, $(\alpha_{\mathfrak{a}j})$ — the $i$-th (resp. $j$-th) simple root for $\mathfrak{g}$ (resp. $\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}^s})$;
- $\omega_i$, $(\omega_{\mathfrak{a}j})$ — the $i$-th (resp. $j$-th) fundamental weight for $\mathfrak{g}$ (resp. $\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}^s})$;
- $W$, $(W_{\mathfrak{a}})$ — the corresponding Weyl group;
- $C$, $(C_{\mathfrak{a}})$ — the fundamental Weyl chamber;
- $\bar{C}$, $(\bar{C}_{\mathfrak{a}})$ — the closure of the fundamental Weyl chamber;
- $\epsilon(w) := (-1)^{\text{length}(w)}$.
\( \rho, (\rho_a) \) — the Weyl vector;

\( L^\mu (L_a^\nu) \) — the integrable module of \( g \) with the highest weight \( \mu \); (resp. integrable \( a \) -module with the highest weight \( \nu \));

\( N^\mu, (N_a^\nu) \) — the weight diagram of \( L^\mu \) (resp. \( L_a^\nu \));

\( P \) (resp. \( P_a \)) — the weight lattice;

\( P^+ \) (resp. \( P_a^+ \)) — the dominant weight lattice;

\( \mathcal{E} \) (resp. \( \mathcal{E}_a \)) — the formal algebra;

\( m^{(\mu)}_\xi, (m^{(\nu)}_\xi) \) — the multiplicity of the weight \( \xi \in P \) (resp. \( \xi \in P_a \)) in the module \( L^\mu \) (resp. \( L_a^\nu \));

\( \chi(L^\mu) \) (resp. \( \chi(L_a^\nu) \)) — the formal character of \( L^\mu \) (resp. \( L_a^\nu \));

\( \chi(L^\mu) = \frac{\Psi^{(\mu)}(\mu)}{\Psi^{(\mu)}(0)} = \frac{\Psi^{(\mu)}}{R} \) — the Weyl formula;

\( R := \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) \) (resp. \( R_a := \prod_{\alpha \in \Delta_a^+} (1 - e^{-\alpha}) \)) — the Weyl denominator.

Let \( L^\mu \) be completely reducible with respect to \( a \),

\[
L^\mu_{g\downarrow a} = \bigoplus_{\nu \in P_a^+} b_{\nu}^{(\mu)} L_a^\nu.
\]

\[
\pi_a \chi(L^\mu) = \sum_{\nu \in P_a^+} b_{\nu}^{(\mu)} \chi(L_a^\nu).
\] (1)

For the modules we are interested in the Weyl formula for \( \chi(L^\mu) \) can be written in terms of singular elements [?]

\[
\Psi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w(\mu + \rho) - \rho},
\]

namely,

\[
\chi(L^\mu) = \frac{\Psi^{(\mu)}}{\Psi^{(\mu)}(0)} = \frac{\Psi^{(\mu)}}{R}.
\] (2)

The same is true for submodules \( \chi(L_a^\nu) \) in (1)

\[
\chi(L_a^\nu) = \frac{\Psi_a^{(\nu)}}{\Psi_a^{(\nu)}(0)} = \frac{\Psi_a^{(\nu)}}{R_a},
\]

with

\[
\Psi_a^{(\nu)} := \sum_{w \in W_a} \epsilon(w) e^{w(\nu + \rho_a) - \rho_a}.
\]
Applying formula (2) to the branching rule (1) we get a relation connecting the singular elements \( \Psi(\mu) \) and \( \Psi_a(\nu) \):

\[
\sum_{w \in W} \epsilon(w) e^{w(\mu+\rho)-\rho} \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) = \sum_{\nu \in P_a^+} b(\mu) \sum_{w \in W_a} \epsilon(w) e^{w(\nu+\rho_a)-\rho_a} \prod_{\beta \in \Delta_a^+} (1 - e^{-\beta}),
\]

\[
\Psi(\mu) = \sum_{\nu \in P_a^+} b(\mu) \frac{\Psi_a(\nu)}{R_a}.
\]  

(3)

In [?], it was proven that singular branching coefficients \( k(\mu) \) corresponding to the injection \( a \hookrightarrow g \) are subject to the set of recurrent relations:

\[
k(\mu) = -\frac{1}{s(\gamma_0)} \left( \sum_{u \in W/W_\perp} \epsilon(u) \dim \left( L_{\Delta^+_a}(u) \right) \right)_{\delta_{\xi-\gamma_0}} + \sum_{\gamma \in \Gamma_{a \hookrightarrow g}} s(\gamma + \gamma_0) k(\mu)_{\xi+\gamma}.
\]  

(4)

where \( a_\perp \) is the subalgebra determined by the roots of \( g \) orthogonal to roots of \( a \) and \( W_\perp \) is a Weyl group of \( a_\perp \)

\[
\Delta_{a_\perp} := \{ \beta \in \Delta | \forall h \in h_a; \beta(h) = 0 \},
\]  

(5)

\[
\widetilde{a}_\perp := a_\perp \oplus h_\perp \quad \tilde{a} := a \oplus h_\perp
\]  

(6)

and \( \pi \) is the projection operator. Inside the main Weyl chamber \( C_a \) singular branching coefficients coincide with branching coefficients: \( b(\mu) = k(\mu) \) \( \forall \xi \in C_a \). When an injection is maximal the projection becomes trivial and the relation (4) is simplified:

\[
k(\mu) = -\frac{1}{s(\gamma_0)} \left( \sum_{u \in W} \epsilon(u) \delta_{\xi-\gamma_0} \right)_{\delta_{\mu+\rho}} + \sum_{\gamma \in \Gamma_{a \hookrightarrow g}} s(\gamma + \gamma_0) k(\mu)_{\xi+\gamma}.
\]  

(7)

The recursion is governed by the set \( \Gamma_{a \hookrightarrow g} \) called the injection fan. The latter is defined by the carrier set \( \{ \xi \}_{a \hookrightarrow g} \) for the coefficient function \( s(\xi) \)

\[
\{ \xi \}_{a \hookrightarrow g} := \{ \xi \in P_a | s(\xi) \neq 0 \}
\]

appearing in the expansion

\[
\prod_{\alpha \in \Delta^+ \setminus \Delta_a^+} (1 - e^{-\alpha}) = -\sum_{\gamma \in P_a} s(\gamma) e^{-\gamma};
\]

(8)

Now we remind two definitions introduced in [?]
Definition 2.1. Suppose $\Delta_0$ and $\Delta$ are root systems with corresponding weight lattices $P_0$ and $P$. Then $\phi$ is an “embedding”,

$$
\phi : \begin{cases}
\Delta_0 \hookrightarrow \Delta, \\
P_0 \hookrightarrow P,
\end{cases}
$$

if

(a) it injects $\Delta_0$ in $\Delta$, and

(b) acts homomorphically with respect to the vector groups in $P_0$ and $P$:

$$
\phi(\gamma) = \phi(\alpha) + \phi(\beta)
$$

for any triple $\alpha, \beta, \gamma \in P_0$ such that $\gamma = \alpha + \beta$.

$\phi$ induces an injection of formal algebras : $E_0 \hookrightarrow E$ and for the image $E_i = Im(\phi(E_0))$ one can consider its inverse $\phi^{-1} : E_i \twoheadrightarrow E_0$.

Notice that one must distinguish two classes of embeddings: when the scalar product (defined by the Killing form) in the root space $P_0$ is invariant with respect to $\phi$ and when it is not $\phi$-invariant. The first embedding is called ”metric”, the second – ”nonmetric”.

Definition 2.2. A root system $\Delta$ ”splinters” as $(\Delta_1, \Delta_2)$ if there are two embeddings $\phi_1 : \Delta_1 \hookrightarrow \Delta$ and $\phi_2 : \Delta_2 \hookrightarrow \Delta$ where (a) $\Delta$ is the disjoint union of the images of $\phi_1$ and $\phi_2$ and (b) neither the rank of $\Delta_1$ nor the rank of $\Delta_2$ exceeds the rank of $\Delta$.

It is equivalent to say that $(\Delta_1, \Delta_2)$ is a ”splint” of $\Delta$ and we shall denote this by $\Delta \approx (\Delta_1, \Delta_2)$. Each component $\Delta_1$ and $\Delta_2$ is a ”stem” of the splint $(\Delta_1, \Delta_2)$.

To study relations between injection fan technique and splint let us consider the case when one of the stems $\Delta_1 = \Delta_a$ is a root subsystem.

Splint $\Delta \approx (\Delta_1, \Delta_2)$ is called ”injective” if $\Delta_1$ is a root subsystem in $\Delta$ corresponding to a regular reductive subalgebra $a \hookrightarrow g$.

In case of injective splint the second stem $\Delta_s := \Delta_2 = \Delta \setminus \Delta_a$ can be translated into a product (8) and it defines an injection fan $\Gamma_{a \hookrightarrow g}$. Denote by $\Delta_{s0}$ the coimage of the second embedding $\phi : \Delta_{s0} \rightarrow g$. The following conjecture follows.

Conjecture 2.3. Each injective splint $\Delta \approx (\Delta_a, \Delta_s)$ defines an injection fan with the carrier $\{\xi\}_{a \hookrightarrow g}$ fixed by the product

$$
\prod_{\beta \in \Delta_a^+} (1 - e^{-\beta}) = - \sum_{\gamma \in P} s(\gamma)e^{-\gamma}
$$

(10)
In case of injective splint we say that subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$ splinters $\Delta$ (and call $\mathfrak{a}$ the "splinting subalgebra" of $\mathfrak{g}$). In [?] splints are classified (see Appendix there) and the first three types of them are injective.

### 3 How stems define multiplicity functions

In this Section we study properties of injective splints $\Delta \approx (\Delta_\mathfrak{a}, \Delta_\mathfrak{s})$. It will be demonstrated that under certain conditions to find branching coefficients for a splinting injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$ means to find weight multiplicities of an irreducible $\mathfrak{s}$-module $L_\nu^\mathfrak{s}$ with fixed highest weight $\nu$. Notice that $\mathfrak{s}$ must not be a subalgebra of $\mathfrak{g}$.

Let us return to relation (3) and multiply both sides by $R_\mathfrak{a}$:

$$\frac{1}{\prod_{\beta \in \Delta_+}(1 - e^{-\beta})} \Psi_{\mathfrak{g}}^{(\mu)} = \sum_{\nu \in P_+^+} b_{\nu}^{(\mu)} \Psi_{\mathfrak{a}}^{(\nu)}. \tag{11}$$

Here the first factor in the l.h.s. is the inverse of the fan $\Gamma_{\mathfrak{a} \rightarrow \mathfrak{g}}$. Consider the highest weight module $L_\nu^\mathfrak{s}$. The embedding $\phi : \Delta_\mathfrak{s}0 \rightarrow \Delta_\mathfrak{g}$ sends the singular element $\Psi_{\mathfrak{s}}^{(\nu)}$ into $\Psi_{\mathfrak{g}}^{(\mu)}$. Applying the inverse morphism $\phi^{-1}$ to the product $\left(\prod_{\beta \in \Delta_+}(1 - e^{-\beta})\right) \phi \left(\Psi_{\mathfrak{s}}^{(\nu)}\right)$ one gets the character of the module $L_\nu^\mathfrak{s}$,

$$\phi^{-1}\left(\frac{1}{\prod_{\beta \in \Delta_+}(1 - e^{-\beta})} \phi \left(\Psi_{\mathfrak{s}}^{(\nu)}\right)\right) = \frac{1}{\prod_{\beta \in \Delta_+}(1 - e^{-\beta})} \Psi_{\mathfrak{s}}^{(\nu)} = \text{ch} \left(L_\nu^\mathfrak{s}\right). \tag{12}$$

Our task is to find the singular element $\Psi_{\mathfrak{s}}^{(\xi)}$ for the module $L_\xi^\mathfrak{s}$ as a component in $\Psi_{\mathfrak{g}}^{(\mu)}$ and to prove that $L_\xi^\mathfrak{s}$ is uniquely defined by $L_\xi^\mathfrak{g}$ and that the branching coefficients $b_{\nu}^{(\mu)}$ in the r.h.s. of (11) coincide with multiplicities $m_{\xi}^{(\mu)}$ of the corresponding weights in $\mathcal{N}_\mathfrak{s}^\xi$.

For a highest weight irreducible module $L_\mu^\mathfrak{g}$ the singular element $\Psi_{\mathfrak{g}}^{(\mu)}$ is an element of $\mathcal{E}$ corresponding to the shifted Weyl-orbit of the weight $(\mu + \rho) \in P^+$ with the sign function $\epsilon (w)$. It is convenient to use also unshifted singular elements

$$\Phi^{(\mu)} := \Psi^{(\mu)} e^\rho. \tag{13}$$
In these terms the relation (11) looks like

\[
\prod_{\beta \in \Delta_+} \left( 1 - e^{-\beta} \right) \Phi_0^{(\mu)} = \sum_{\nu \in P^+_a} b^{(\mu)}_{\nu} \Phi^{(\nu)}_a.
\] (14)

The orbit related to \( \Phi^{(\mu)}_a \) is completely defined by the set of edges \( \{ \lambda_i \}_{i=1, \ldots, r} \) adjusted to the end of the highest weight vector \( \mu + \rho \). For \( \mu = \sum m_i \omega_i \) these edges are

\[
\lambda_i = -(m_i + 1) \alpha_i, \quad i = 1, \ldots, r.
\] (15)

Each formal exponent \( e^{\mu + \rho + \lambda_i} \) in \( \Phi^{(\mu)}_a \) bears the sign coefficient \( \epsilon = (-) \). The defining property of \( \Phi^{(\mu)}_a \) is as follows. Consider any pair of edges \( \lambda_i, \lambda_j \) and the corresponding weights \( \mu + \rho, \mu + \rho + \lambda_i \) and \( \mu + \rho + \lambda_j \). Apply the reflection \( s_{\alpha_i} \) (or \( s_{\alpha_j} \)),

\[
s_{\alpha_i} \circ \left\{ \begin{array}{l}
(\mu + \rho) \\
(\mu + \rho + \lambda_i) \\
(\mu + \rho + \lambda_j)
\end{array} \right\} = \left\{ \begin{array}{l}
(\mu + \rho + \lambda_i) \\
(\mu + \rho) \\
(\mu + \rho + \lambda_i - (m_j + 1)s_{\alpha_i} \circ \alpha_j)
\end{array} \right\}
\] (16)

**Property 3.1.** The edge \( \lambda_{i,j} \) of \( \Phi^{(\mu)}_a \) starting at the weight \( \mu + \rho + \lambda_i \) along the root \(-s_{\alpha_i} \circ \alpha_j\) has the same length in \( (s_{\alpha_j} \circ \alpha_i) \) as \( \lambda_j \) has in \( \alpha_j \). (The same is true for the edge \( \lambda_{j,i} \), its length in \( (s_{\alpha_i} \circ \alpha_j) \) is equal to the length of \( \lambda_i \) in \( \alpha_i \).)

In \( \Phi^{(\mu)}_a \) the elements \( e^{(\mu + \rho + \lambda_i - (m_j + 1)s_{\alpha_i} \circ \alpha_j)} \) and \( e^{(\mu + \rho + \lambda_j - (m_i + 1)s_{\alpha_i} \circ \alpha_i)} \) have the sign coefficient \( \epsilon = (+) \).

Remember that only three types of splints are injective and thus are naturally connected with branching. Below we reproduce the part of the splints table from [?] corresponding to injective splints:

| type | \( \Delta \) | \( \Delta_a \) | \( \Delta_s \) |
|------|------|------|------|
| (i)  | \( G_2 \) | \( A_2 \) | \( A_2 \) |
|      | \( F_4 \) | \( D_4 \) | \( D_4 \) |
| (ii) | \( B_r \) \((r \geq 2)\) | \( D_r \) | \( \oplus^r A_1 \) |
| (*)  | \( C_r \) \((r \geq 3)\) | \( \oplus^r A_1 \) | \( D_r \) |
| (iii)| \( A_r \) \((r \geq 2)\) | \( A_{r-1} \oplus u (1) \) | \( \oplus^r A_1 \) |
|      | \( B_2 \) | \( A_1 \oplus u (1) \) | \( A_2 \) |

(17)

Each row in the table gives a splint \( (\Delta_a, \Delta_s) \) of the simple root system \( \Delta \). In the first two types both \( \Delta_a \) and \( \Delta_s \) are embedded metrically. Stems in
the first type splints are equivalent and in the second are not. In the third type splints only $\Delta_a$ is embedded metrically. The summands $u(1)$ are added to keep $r_a = r$. This does not change the principle properties of branching but makes it possible to use the multiplicities of $\mathfrak{s}$ -modules without further projecting their weights. The second injective splint of type (ii) (marked by a star) does not generate a unique auxiliary $\mathfrak{s}$ -module and in this case branching is related to splint in a more complicated form. We will not study this case here.

Splints induce a decomposition of the set $S = S_c \cup S_d$ with $S_c = S \cap S_a$ and $S_d = S \cap S_s$. It is easy to check that for any injective splint the subset $S_d$ is nonempty. It follows that in the set $\{\lambda_i\}_{i=1,...,r}$ one can always find simple roots $\beta_k \in \Delta_s$ and that the orbit corresponding to $\Phi_0(\mu)$ contains the edges

$$\lambda_k = - (m_k + 1) \beta_k$$

(18)

attached to the weight $\mu + \rho$. As far as $\Delta_a$ is a root system and for any pair of simple roots from $S_c$ the property [3.1] is fulfilled, the element $\Phi_0(\mu)$ being a singular element for a set of $\mathfrak{a}$-modules. Consider $\beta_l \in \Delta_s$ whose coimage in $\Delta_{s0}$ is simple. In Appendix it is shown that for any such $\beta_l$ there exists a root $\alpha_l \in S_c$ such that $\beta_l = \alpha_l + \beta_k$. It is easily seen that the corresponding edge intersects the boundary plane of the fundamental chamber $\overline{C_a}$ orthogonal to the root $\alpha_l$,

$$s_{\alpha_l}(\mu + \rho - p\beta_l) = s_{\alpha_l}(\mu + \rho) - ps_{\alpha_l}\beta_l = \mu + \rho - p\beta_l,$$

(19)

$$\mu + \rho - s_{\alpha_l}(\mu + \rho) = (m_l + 1) \alpha_l = (m_l + 1) \beta_l - (m_l + 1) \beta_k = p\beta_l - ps_{\alpha_l}\beta_l.$$  

(20)

It follows that $p = (m_l + 1)$ and $s_{\alpha_l}\beta_l = \beta_k$. Now apply the operator $s_{\beta_k}$ and find that the edge along the root $s_{\beta_k} \alpha_l$ attached at the weight $s_{\beta_k}(\mu + \rho)$ is also equal to $-ps_{\beta_k}\alpha_l$. This means that for the triple of roots $\beta_k, \beta_l$ and $s_{\beta_k} \alpha_l$ in $\Delta_s$ the edges $\lambda_k = - (m_k + 1) \beta_k, \lambda_l = - (m_l + 1) \beta_l$ and $\lambda_{kl} = - (m_l + 1) s_{\beta_k} \alpha_l$ demonstrate the property [3.1]. One can continue this procedure further in the 2-dimensional subspace fixed by the roots $\beta_k$ and $\beta_l$ and find the set of formal exponents that being supplied with the corresponding sign factors compose the coimage of the singular element of a module for the subalgebra in $\mathfrak{s}$ (this subalgebra has rank $r = 2$).

The same can be proven for any positive root $\beta_l \in \Delta$ that is simple in $\Delta_{s0}$ and correspondingly for any $r = 2$ subalgebra in $\mathfrak{s}$. The latter means that
to "find" a singular element of $\mathfrak{s}$-module in $\Phi_{0}^{(\mu)}$ it is necessary to incorporate in it additional formal elements $\{-e^{\mu+\rho-(m_l+1)\beta_l}|\beta_l \in S_\epsilon\}$. This fixes the starting edges of the diagram $\phi^{-1}(\Phi_{0}^{\mu})$. As it follows from the reconstruction procedure the highest weight $\tilde{\mu}$ is totally defined by the weight $\mu$, they have the same Dynkin numbers:

$$\mu = \sum m_k \omega_k \implies \tilde{\mu} = \sum m_k \tilde{\omega}_k.$$  \hspace{1cm} (21)

The next step is to check whether the image $\phi(\Phi_{0}^{\tilde{\mu}})$ belongs to $\bar{C}_a$ and the set $\phi(\Phi_{0}^{\tilde{\mu}} \setminus \Phi_{0}^{(\mu)})|_{\bar{C}_a}$ corresponds to the weights in the boundary $\bar{C}_a$ (including the subset $\{-e^{\mu+\rho-(m_l+1)\beta_l}|\beta_l \in S_\epsilon\}$). Provided this condition is fulfilled let us return to relation (14). One can add to $\Phi_{0}^{(\mu)}$ pairs of formal elements constructed above with the opposite signs: $\epsilon(w)|_{w \in W_a}$ and $-\epsilon(w)|_{w \in W_a}$. Attribute the signs $\epsilon(w)|_{w \in W_a}$ to the elements whose weights we shall attribute to $\bar{C}_a$. The same elements with the opposite signs are to be referred to the neighboring Weyl chambers of $\bar{C}_a(i)$ (the latter are connected with the main one via simple reflections $s_{\alpha_i}$ so the signs $-\epsilon(w)|_{w \in W_a}$ are natural for them). In fact one can repeat the procedure and find additional singular weights in any Weyl chamber $\bar{C}_a^{(m)}$ and in them additional singular weights always have the signs opposite to that in their nearest neighbors. Thus without changing the element $\Phi_{0}^{(\mu)}$ one can present it as a sum

$$\Phi_{0}^{(\mu)} = \sum_{w \in W_a} \epsilon(w) w \circ (e^{\rho_a \Psi_{\tilde{\mu}+\rho_a}})$$  \hspace{1cm} (22)

where the weight $\tilde{\mu} = \sum m_k \omega_k^k$ was defined above. As far as the second condition is fulfilled (i.e. $\phi(\Phi_{0}^{\tilde{\mu}}) \subset \bar{C}_a$) the decomposition (22) provides the possibility to apply the factor $\left(\prod_{\beta \in \Delta_a^+}(1-e^{-\beta})\right)^{-1}$ to each summand of the singular element $\Phi_{0}^{(\mu)}$ separately because the sets of weights from different Weyl summands do not intersect. Taking into account the isomorphism $\phi$ one can see that in the main Weyl chamber $\bar{C}_a$ the set of weights generated by the factor $\left(\prod_{\beta \in \Delta_a^+}(1-e^{-\beta})\right)^{-1}$ is isomorphic to the weight diagram $\mathcal{W}_{\tilde{\mu}}^\mu$ of the $\mathfrak{s}$-module $L_{\tilde{\mu}}^\mu$. Now one can restrict relation (14) to $\bar{C}_a$ and obtain the main result:
Property 3.2.

\[
\frac{e^{\rho_0}}{\prod_{\beta \in \Delta_+} (1 - e^{-\beta})} (\Psi_{\tilde{\mu} + \rho_0}) = \sum_{\tilde{\nu} \in N_{\tilde{\mu}}^\Delta} M_{(\tilde{\mu})\tilde{\nu}} e^{(\mu - \phi(\tilde{\mu} - \tilde{\nu}))} = \sum_{\nu \in P_{\tilde{\nu}}^\Delta} b_{(\mu)}^{(\nu)} e^{\nu}. \quad (23)
\]

Any weight with nonzero multiplicity in the right-hand side is equal to one of the highest weights in the decomposition. The multiplicity \(M_{(\tilde{\mu})\tilde{\nu}}\) of the weight \(\tilde{\nu} \in N_{\tilde{\mu}}^\Delta\) defines the branching coefficient \(b_{(\mu)}^{(\nu)}\) for the highest weight \(\nu = (\mu - \phi(\tilde{\mu} - \tilde{\nu}))\):

\[
b_{(\mu)}^{(\nu)} = M_{(\tilde{\mu})\tilde{\nu}}.
\]

4 Examples

Example 4.1. Consider the Lie algebra \(A_2 = \mathfrak{sl}(3)\) and branching of its irreducible module \(L^{[3,2]}_{A_2}\) with respect to the reductive subalgebra \(A_1 \oplus u(1)\). The root system \(\Delta_a = \Delta_{A_1 \oplus u(1)}\) contains the simple root of \(\alpha_1 = e_1 - e_2\) of \(A_2\). The singular element \(\Psi^{[3,2]}_a\) is decomposed into a sum of splint images of singular elements of \(A_1 \oplus A_1\)-modules. Branching coefficients \(b^{[3,2]}_a\) coincide with weight multiplicities of \(L^{[3,2]}_{A_1 \oplus A_1}\)-module (see Fig. 1).

Example 4.2. For the Lie algebra \(B_2 = \mathfrak{so}(5)\) branching of its irreducible module \(L^{[3,2]}_{B_2}\) into modules of a reductive subalgebra \(A_1 \oplus u(1)\) with the root system spanned by the first simple root \(\alpha_1 = e_1 - e_2\) of \(B_2\). Singular element of \(\Psi^{[3,2]}_{B_2}\) is decomposed into the sum of splint images of singular elements of \(A_2\)-modules and branching coefficients coincide with weight multiplicities of \(A_2\)-module (see Fig. 2).

Example 4.3. Lie algebra \(G_2\) has a regular subalgebra \(A_2\) with root system \(\Delta_a = \Delta_{A_2}\) containing the \(G_2\) long roots. Consider branching of an irreducible module \(L^{(3,2)}_{G_2}\) into the \(A_2\)-modules. Singular element \(\Psi_{G_2}(L^{[3,2]}_{A_2})\) is decomposed into the sum of splint images of singular elements \(\Psi_{A_2}(L^{[3,2]}_{A_2})\) and the corresponding branching coefficients coincide with weight multiplicities of \(L^{[3,2]}_{A_2}\)-module (see Fig. 3).
Figure 1: Weyl group orbit (dotted) producing singular element of $L_{A_2}^{[3,2]}$ and its decomposition into the sum of splint images of singular elements of modules $L_{A_1 \oplus A_1}^{[3,2]}$ (dashed). Weight multiplicities of $L_{A_1 \oplus A_1}^{[3,2]}$-module coincide with branching coefficients for the reduction $L_{A_2 \downarrow A_1 \oplus u(1)}^{[3,2]}$.

5 Conclusions

It is explicitly demonstrated that splint presents a very effective tool to find branching coefficients. In particular the injective splints that have the property $\phi \left( \Phi_{s}^{(0)} \right) \subset C_{g}^{(0)}$ provide the possibility to reduce branching rules calculations for the highest weight modules to a determination of weight multiplicities for a module with the same Dynkin labels referred to the Lie algebra $s$. This algebra $s$ must not be a subalgebra in the initial $g$, it has the same rank $r_s = r$, but obviously is less ”complicated” than $g$ – only a subset of the initial root system is involved in the second stem $\Delta_s$.

It is significant that for the injections $D_r \hookrightarrow B_r$ and $A_{r-1} \oplus u(1) \hookrightarrow A_r$ the splint technique shows transparently Gelfand-Tseytlin rules for branching: the reduction is multiplicity free (all nonzero branching coefficients are equal to 1). Here it is an immediate consequence of the structure of the second stem being a direct sum of $A_1$ algebras and the fact that the corresponding module $L_{s}^{\mu}$ is irreducible.
Figure 2: Weights of the $B_2$-module $L^{[3,2]}$ are indicated by dots in the left picture (their multiplicities are also indicated). Contour of the singular element is shown by dotted line. The right picture presents the decomposition of $\Psi_{B_2}(L^{[3,2]}_{B_2})$-singular element into the sum of splint images of singular elements $\Psi_{A_2}(L^{[3,2]}_{A_2})$ (dashed). Weight multiplicities of $L^{[3,2]}_{A_2}$-module coincide with branching coefficients for the reduction $L^{[3,2]}_{B_2\downarrow A_1}\oplus u(1)$.

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Appendix

Let us demonstrate that for injective splints of classical Lie algebras the following property is valid:

**Property 5.1.** Let $\Delta \approx (\Delta_s, \Delta_d)$ be an injective splint with the decomposition of simple roots $S = S_c \cup S_d$ with $S_c = S \cap S_\alpha$ and $S_d = S \cap S_\delta$. For any simple root $\beta \in S_d$ there exists the pair of roots $(\alpha, \beta')$ with $\alpha \in S_c, \beta' \in S_d$.  


Figure 3: Weyl group orbit (dotted) for the singular element $\Psi_{G_2}(L^{[3,2]})$ and its decomposition into the sum of splint images of singular elements of $A_2$-modules (dashed). Weight multiplicities of $L^{[3,2]}_{A_2}$-module coincide with branching coefficients for the reduction $L^{[3,2]}_{G_2 \downarrow A_2}$.

such that $\alpha = \beta - \beta'$.

• Type 1. $\Delta_{G_2} \approx (\Delta_{A_2}, \Delta_{A_2})$.

Here both stems are metric and the corresponding root systems are equivalent. In Figure 4 a part of the singular element $\Phi_{G_2}^{(0)}$ is presented. The boundaries of $C_2$ are the dashed lines starting at the center of the singular element. It contains the edge $\lambda_2 = -\alpha_2 = -\beta_2$ and the roots $-\beta_1 = -s_{\alpha_2} \circ \beta_3$ and $-\beta_3$ ($\beta_3$ is indicated as $\beta_3$). For the root
Figure 4: Positive roots of $G_2$ and formation of singular element $\Phi_s^{(0)}$ in the main Weyl chamber of $\alpha = A_2$.

$\beta_1$ the necessary pair is $(\alpha_1, \beta_2)$: $\alpha_1 = \beta_1 - \beta_2$. The $\lambda^{s}_{2,3} = \beta_3$ edge is equal to $\lambda^{s}_1 = \beta_1 = s_{\alpha_2} \circ \beta_3$ and $m_1$ index is aquired by the $s$-module that also inherit the second index $m_2$. In this particular case they are $m_1 = m_2 = 0$. The general case with the initial module $L^\mu$ and $\mu = m_1 \omega_1 + m_2 \omega_2$ can be treated in the same way: one finds an edge $\lambda_2 = - (m_2 + 1) \beta_2$ and put $\lambda^{s}_1 = - (m_1 + 1) \beta_1$, its end belongs to the boundary $\bar{C}_a$. The reflection $s_{\beta_2}$ sends $\beta_1$ to $\beta_3$ and the corresponding edge $\lambda^{s}_{2,3} = - (m_1 + 1) \beta_3$ has the length $(m_1 + 1)$. Now consider $\lambda^{s}_1$ (or $\lambda^{s}_{2,3}$) and $\lambda^{s}_{1,3}$ (or $\lambda^{s}_{2,3,1}$) edges to find that they belong to the boundary $\bar{C}_a$ and the Weyl symmetry predicts that $\lambda^{s}_{1,3} = - (m_2 + 1) \beta_3$ ($\lambda^{s}_{2,3,1} = - (m_2 + 1) \beta_1$). Finally the edge $\lambda^{s}_{1,3,2} = - (m_1 + 1) \beta_2$ closes the polytope. Its vertices correspond to weights of the singular element $\Phi_s^{(2)} = \sum_{w \in W_s} (w) e^{w(\tilde{\mu} + \rho_s)}$ of the module $L^{(\tilde{\mu})}_s$ with $\tilde{\mu} = m_1 \tilde{\omega}_1 + m_2 \tilde{\omega}_2$.

Notice that in this case the sign factors can be obtained directly in
the initial weight system as far as the stem is metric.

- Type 1. \( \Delta_{F_4} \approx (\Delta_{D_4}, \Delta_{D_4}) \).

Both stems are metric here and the corresponding root systems are equivalent. The system \( \Delta_{D_4} \) of the subalgebra \( \mathfrak{a} = D_4 \) is formed by the set \( \{ \pm e_i \pm e_j \}_{i,j=1,\ldots,4, i\neq j} \). The simple roots \( S = \{ e_2 - e_3, e_3 - e_4 \} \) and \( S = \{ e_4, \frac{1}{2} (e_1 - e_2 - e_3 - e_4) \} \). For a module \( L^\mu \) with \( \mu = \sum m_k \omega_k \) consider the edge \( \lambda_3 = -(m_3 + 1) e_4 = -(m_3 + 1) \beta_3 \). Compose an edge \( \lambda_3 = -(m_2 + 1) \beta_2 \). The necessary pair of roots is \( (\alpha_2 = e_3 - e_4, \beta_3). \) The intersection of \( \lambda_2 \) with the \( \alpha_2 \)-boundary of \( C_a \) fixes its length to be \( \lambda_2 = -(m_2 + 1) \beta_2 \) and the length of the edge \( \lambda_3, \beta_3 \) is equal to that of \( \lambda_2 \). Next consider the edge \( \lambda_2 = -(m_2 + 1) \beta_2 \) and the pair \( (\alpha_1 = e_2 - e_3, \beta_1 = e_2) \). The length of \( \lambda_2 \) becomes equal to \( \lambda_1 = -(m_1 + 1) \beta_1 \). Proceed further till the closure of the polytope. The edges looking along the roots of the \( \alpha_4 \)-type, \( \alpha_4 = \beta_4 = \frac{1}{2} (e_1 - e_2 - e_3 - e_4) \), are treated similarly and finally the singular element \( \Phi_s(\tilde{\mu}) = \sum w \in W_s e^\omega (\tilde{\mu} + \rho_s) \) for the module \( L_s(\tilde{\mu}) \) with \( \tilde{\mu} = \sum m_k \tilde{\omega}_k \) is formed in \( C_a \).

- Type 2. \( \Delta_{B_r} \approx (\Delta_{D_r}, \Delta_{\theta^r A_1}) \).

Both stems are metric. An injection is fixed by the stem \( \Delta_{D_r} \) simple roots \( S = \{ e_1 - e_2, e_2 - e_3, \ldots, e_{r-1} - e_r, e_{r-1} + e_r \} \). The second stem corresponds to a direct sum of algebras \( A_1 \) with the simple roots \( S = \{ e_1, e_2, \ldots, e_{r-1}, e_r \} \). Consider the edge \( \lambda_r = -(m_r + 1) \beta_r \) (here \( \beta_r = e_r \) and \( \lambda_r = -(\tilde{m}_r + 1) \beta_r \) attached to it (here \( \beta_r = e_{r-1} \)). The corresponding pair is \( (\alpha_r = e_r - e_r, \beta_{r-1} = e_{r-1}) \). The intersection condition fixes the second edge to be \( \lambda_{r-1} = -(m_{r-1} + 1) \beta_{r-1} \), it is orthogonal to \( \beta_r \), so the opposite edge has the same length. The Dynkin index \( m_{r-1} \) now refers also to the simple root \( \beta_{r-1} \). Next consider the obtained edge \( \lambda_{r-1} = -(m_{r-1} + 1) \beta_{r-1} \) and \( \lambda_{r-2} = -(\tilde{m}_{r-2} + 1) \beta_{r-2} \) to fix the index \( \tilde{m}_{r-2} = m_{r-2} \) and the edge \( \lambda_{r-2} = -(m_{r-2} + 1) \beta_{r-2} \) and so on till all the pairs of edges are properly fixed. Finally in \( C_{D_r} \) the element \( \Phi_s(\tilde{\mu})_{\theta^r A_1} = \sum w \in W_s e^\omega (\tilde{\mu} + \frac{1}{2} \Sigma e_k) \) can be constructed for the module \( L_s(\tilde{\mu})_{\theta^r A_1} \) with \( \tilde{\mu} = \sum m_k \frac{1}{2} e_k \).

- Type 2. \( \Delta_{C_r} \approx (\Delta_{\theta^r A_1}, \Delta_{D_r}) \).

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The situation in this case is analogous to the previous one and the additional edges are constructed similarly. However in this case the property $\phi \left( \Phi_s^{(0)} \right) \subset C_a^{(0)}$ is violated. The set $\phi \left( \Phi_s^{(0)} \right)$ contains weights in several bordering Weyl chambers $C_a$. The decomposition (22) cannot be performed. The injective splint $\Delta_{C_r} \approx (\Delta_{\oplus^r A_1}, \Delta_{D_r})$ does not induce the property (23).

- Type 3 $\Delta_{A_r} \approx (\Delta_{A_{r-1} \oplus u_1}, \Delta_{\oplus^r A_1})$.

Here only the first stem is metric and it fixes the injection with simple roots $S_a = \{e_1 - e_2, e_2 - e_3, \ldots, e_r - e_{r-1} \}$. The second stem corresponding to a direct sum of $r$ copies of $A_1$ has the simple roots $S_b = \{e_1 - e_{r+1}, e_2 - e_{r+1}, \ldots, e_r - e_{r+1} \}$. Consider the edge $\lambda_r = -(m_r + 1) \beta_r$ with $\beta_r = e_r - e_{r-1}$ and $\lambda_{r-1} = -(\tilde{m}_{r-1} + 1) \beta_{r-1}$ with $\beta_{r-1} = e_{r-1} - e_{r-1}$ attached to it. Then the corresponding pair is $(\alpha_{r-1} = e_{r-1} - e_r, \beta_{r-1} = e_{r-1} - e_{r+1})$. The intersection with the boundary of $\tilde{C}_{A_{r-1}}$ orthogonal to $\alpha_{r-1}$ fixes the second edge to be $\lambda_{r-1} = -(m_{r-1} + 1) \beta_{r-1}$. The Dynkin index $m_{r-1}$ is to be used for the fundamental weight $\omega_{r-1}$. The reflection $s_{\beta_r}$ sends $\lambda_{r-1} = -(m_{r-1} + 1) \beta_{r-1}$ to $\lambda_{r,r-1} = -(m_r - 1) \beta_r$. Next consider the obtained edge $\lambda_{r-2} = -(m_{r-1} + 1) \beta_{r-1}$ and $\lambda_{r-2} = -(\tilde{m}_{r-2} + 1) \beta_{r-2}$ with $\beta_{r-2} = e_{r-2} - e_{r+1}$ to obtain the index $\tilde{m}_{r-2} = m_{r-2}$ and the edge $\lambda_{r-2} = -(m_{r-2} + 1) \beta_{r-2}$ and so on till all the pairs of edges are properly fixed. Finally in $\tilde{C}_{D_r}$ the element $\Phi_{\oplus^r A_1}^{(\tilde{\mu})} = \sum_{w \in W_{\oplus^r A_1}} \varepsilon(w) e^{w(\tilde{\mu} + \tilde{\rho})}$ can be constructed for the module $L_{\oplus^r A_1}^{(\tilde{\mu})}$ with $\tilde{\mu} = \sum m \beta_k$. The simplest case $\Delta_{A_2} \approx (\Delta_{A_1 \oplus u_1}, \Delta_{A_1 \oplus A_1})$ is presented in Example 4.1 and Figure 1.

- Type 3 $\Delta_{B_2} \approx (\Delta_{A_1}, \Delta_{A_2})$.

This splint is illustrated in Example 4.1 and Figure 1, $S_{A_1} = \{e_1 - e_2\}$, $S_{A_2} = \{e_1, e_2\}$. The edge $\lambda_{\beta_2} = - (m_2 + 1) \beta_2$ is followed by $\lambda_{\beta_1} = -(\tilde{m}_1 + 1) \beta_1$. Consider the pair $(\alpha_1 = e_1 - e_2, \beta_1 = e_1)$. The end of the edge $\lambda_{\beta_1}$ must indicate a weight invariant under the reflection $s_{\alpha_1}$. Its length is thus fixed: $\lambda_{\beta_1} = -(m_1 + 1) \beta_1$. In the coincidence of the second stem, that is in the root system $\Delta_{A_2}$, the reflection $s_{\beta_2}$ sends $\lambda_{\beta_1} = -(m_1 + 1) \beta_1$ to $\lambda_{\beta_2} = - (m_2 + 1) \beta_2$, thus the latter edge has the same length in $\beta_3 = e_1 + e_2$, we have $\lambda_{\beta_3} = - (m_1 + 1) \beta_3$ with $\beta_3 = e_1 + e_3$. The irreducible $s$-module has the highest weight $\tilde{\mu} = m_1 \tilde{\omega}_1 + m_2 \tilde{\omega}_2$. In Figure 1 we see the details of these relations in a particular case where
$L^{[3,2]}_{B_2}$ is reduced to a subalgebra $A_1 \oplus u(1)$ and the corresponding highest weights (with their multiplicities) form the diagram $\mathcal{N}^{[3,2]}_{A_2}$. 